Eliminating useless portfolios in constrained financial economies

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Abstract When financial investors’ portfolio holdings are unconstrained, financial economies are assumed, w.l.o.g., to have no redundant assets. Indeed, eliminating redundant assets allows to replace the initial financial structure by an equivalent one, i.e., one that has the same consumption equilibria. Moreover, at the end of the process, absence of redundant assets guarantees that the set of admissible portfolio allocations is bounded, a fundamental property for the existence of equilibria. In the presence of institutional (exogenous) portfolio constraints, eliminating redundant assets is not innocuous anymore since bounded arbitrage may persist at equilibrium, the law of one price does not hold, and some zero-income portfolios may not be free. The goal of the paper is to replace the elimination of redundant assets by the elimination of useless portfolios, a process that eliminates in particular Werner useless portfolios, but needs to go beyond to obtain the boundedness of the set of admissible portfolio allocations at the end of the purification process. Moreover, the elimination process is carried out without affecting the set of consumption equilibria, hence replacing at each step the financial structure by an equivalent one.
1 Introduction

Absence of redundant assets\(^1\) is commonly assumed in the literature on financial economics, and indeed, it is innocuous when there are no restrictions on consumers’ participation in financial markets since the financial possibilities offered by the set of all assets are the same as those offered by the smaller set of assets after eliminating (literally deleting) all or some of the redundant ones. Furthermore, eliminating redundant assets plays an important role in the existence of equilibria since it does not change consumption equilibria\(^2\) and allows to bound the set of admissible portfolio allocations and thus apply Radner’s 1972 existence result of financial equilibria when portfolio sets are bounded.

Eliminating redundant assets is no longer innocuous, however, when some consumers’ portfolio holdings are restricted. As mentioned by Balasko et al. (1990) “one significant source of restricted participation is financial intermediation […] which typically involves redundancy” and according to LeRoy and Werner (2001) “the term [redundant] is a misnomer in the presence of portfolio restrictions: that the payoff of a security can be duplicated by a portfolio of other securities does not mean that it is redundant because the duplicating portfolio may be infeasible owing to portfolio restrictions.” Hence, in the presence of constraints on consumers’ participation to financial markets, there are no a priori grounds for the “non-redundancy” assumption and deleting a “redundant” security from the model may change the set of equilibria.

The following example shows that, under portfolio restrictions, “naively” eliminating “redundant” assets may drastically hamper investors’ ability to transfer wealth between dates and/or states of the world. Consider a financial model \(\mathcal{F}\) with two dates, two states at the second date, and three assets: the two Arrow securities and the risk-free asset, i.e., with payoffs \((1, 0)\), \((0, 1)\), and \((1, 1)\), respectively. Moreover, there are three consumers with restricted portfolio sets \(\Theta_1\), \(\Theta_2\), and \(\Theta_3\), defined by:

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\(^1\) A security is redundant if its payoff can be generated as the (portfolio) combination of the payoffs of the other securities.

\(^2\) A consumption equilibrium is defined as the “real” part of an equilibrium in the financial exchange economy, i.e., the pair consisting of the equilibrium commodity price and the equilibrium consumption allocation.

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\[ \Theta_1 = \{ \theta = (\theta^1, \theta^2, \theta^3) \in \mathbb{R}^3 : \theta^1 + \theta^3 = 0, \theta^2 + \theta^3 = 0 \}, \]
\[ \Theta_2 = \{ \theta = (\theta^1, \theta^2, \theta^3) \in \mathbb{R}^3 : \theta^1 = \theta^2 = 0 \}, \text{ and} \]
\[ \Theta_3 = \{ \theta = (\theta^1, \theta^2, \theta^3) \in \mathbb{R}^3 : \theta^3 = 0 \}. \]

Then, deleting any of the three assets (that would be redundant in the absence of portfolio restrictions), puts an end to any financial activity.\(^3\)

The previous example gives a hint about how to eliminate *useless* portfolios, the approach proposed in this paper, as an alternative to the elimination of redundant assets (in the presence of portfolio restriction). Heuristically, the portfolio \( \bar{\theta} := \frac{(1, 1, -1)}{\sqrt{3}} \) is useless for consumer 1, in the sense that it pays \( 1(1, 0) + 1(0, 1) - 1(1, 1) = 0 \) at the second date and thus will be eliminated as follows. Formally, define the new financial structure \( F' \) with the same assets as \( F \) and new portfolio sets defined by \( \Theta'_i = \{ \theta_i - (\theta_i \cdot \bar{\theta}) \bar{\theta} : \theta_i \in \Theta_i \} \) for \( i = 1, 2, 3 \). We thus have \( \Theta'_1 = \{ 0 \} \), and the other two consumers have modified portfolio sets that allow them to trade as before, but without consumer 1 who has been put out of the market. In other words, initially, consumer 1 played only the role of an intermediary in the trade process, to allow consumers 2 and 3 to engage in trade, without benefiting from it; the new market organization eliminates the useless portfolios of consumer 1 (thus putting her out of the trade in this particular case), though changing the portfolio sets of the other two consumers to allow them to continue to trade and realize the same financial possibilities as before. This paper will show that (1) the new market organization is equivalent to the initial one in the sense that it does not affect the set of consumption equilibria, (2) the set of useless portfolios is decreasing after each elimination iteration, (3) the elimination process converges to a situation that is useless-free (no useless portfolios), and (4) in this latter case the set of admissible portfolio allocations is bounded; a key property for the existence of equilibria.

The *elimination process* of useless portfolios that was hinted at in the previous example will be defined and studied in the framework of a two-date stochastic financial model with portfolio constraints defined by general closed, convex, portfolio sets as in Siconolfi (1989), Cornet and Gopalan (2010), Aouani and Cornet (2011), and thus goes beyond the case of linear equality constraints, as in Balasko et al. (1990), and linear inequality constraints, as in Aouani and Cornet (2009). The elimination process will make the equilibrium existence problem work better with the transformed financial structure than with the initial one, and thus will adequately replace the elimination of redundant assets that can no longer be used in the presence of portfolio constraints.

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\(^3\) For example, deleting the risk-free asset, amounts to prohibiting each consumer from buying and selling this asset, that is, \( \theta^3_i = 0 \) for each consumer \( i \). But doing so “kills” the market since no financial activity can take place: both consumers 1 and 2 have no trade possibilities (their portfolio can only be \((0, 0, 0)\)) and consumer 3 cannot trade with anybody.
The paper is organized as follows. First (Sect. 3.1), we introduce the class of useless portfolios that is larger than the class of Werner useless portfolios defined in this paper (Sect. 3.4) by transposing the notion of useless commodity bundles by Werner (1987) to the setting of financial structures. Absence of useless portfolios guarantees the boundedness of the set of admissible portfolio allocations (Theorem 1.a), a crucial property for the use of Radner’s 1972 existence result of financial equilibria when portfolio sets are bounded (Theorem 1.b). An interpretation of useless portfolios in terms of individual demand is given in the companion paper Aouani and Cornet (2014) together with a thorough study of both notions of useless portfolios; a characterization of useless-free financial structures (i.e., financial structures that have no useless portfolios) is provided by Aouani and Cornet (2016). In Sect. 3.4, we show that elimination of all Werner useless portfolios does not guarantee, in general, the boundedness property of Theorem 1.a, justifying the need for our larger class of useless portfolios. Finally, the two notions of useless portfolios coincide under Hart (1974)’s assumption of Weak No Market Arbitrage (Theorem 4.b) but may not coincide if the portfolio sets are polyhedral (Example 1).

The elimination process of useless portfolios is defined in Sect. 3.2. It can be carried out in several ways: one-shot elimination, where all the useless portfolios are eliminated in a single shot, or sequentially with the elimination of one or several useless portfolios at each step. The elimination process decreases the space of useless portfolios (Theorem 2.a) at each step and is independent of the order in which the useless portfolios are eliminated (Theorem 3.e). Most importantly, the elimination process can be executed without affecting the set of consumption equilibria (Theorem 2.c), i.e., at each step, the modified financial structure \( F' \) (after elimination) is equivalent to the initial one \( F \), in the sense that, for every exchange economy \( E \) satisfying standard assumptions, the two financial exchange economies \( (E, F) \) and \( (E, F') \) have the same set of consumption equilibria. Finally, the sequential elimination process eventually converges to the same outcome as the one obtained through one-shot elimination (Theorem 3.f). This one-shot process, therefore, defines a unique equivalent useless-free financial structure \( F_* \) (whose sets \( A_{F_*}(p, v) \) of admissible portfolio allocations are bounded) and whose arbitrage-free asset prices are “essentially” arbitrage-free asset prices of the initial one (Corollary 1). The boundedness property of \( A_{F_*}(p, v) \), together with the equivalence property of \( F \) and \( F_* \) are the key ingredients that allow to deduce directly the existence of financial equilibria of \( (E, F) \) from the study of \( (E, F_*) \) (Corollary 2), what will be performed in the general setting of incomplete and non-transitive preferences. The proofs of the paper’s main results are gathered in Sect. 4, while proofs for intermediary lemmas are deferred to the “Appendix”. 
2 The model

2.1 Financial exchange economies

We consider the basic stochastic model with two dates: \( t = 0 \) (today) and \( t = 1 \) (tomorrow). At the second date, there is a non-empty finite set \( S := \{1, \ldots, S\} \) of states of the world, one (and only one) of which prevails at time \( t = 1 \) and is only known at time \( t = 1 \). For convenience, \( s = 0 \) denotes the state of the world (known with certainty) at \( t = 0 \) and we let \( \hat{S} := \{0\} \cup S = \{0, 1, \ldots, S\} \). At each state \( s \in \hat{S} \), today and tomorrow, there is a spot market for a positive number \( \ell \) of perfectly divisible priceable physical commodities. For every physical commodity \( h = 1, \ldots, \ell \), and every state \( s = 0, 1, \ldots, S \), the state-contingent commodity \((h, s)\), simply called commodity when there is no risk of confusion, is a contract that promises the delivery of the physical commodity \( h \) if state \( s \) prevails and nothing otherwise. Thus, the commodity space is \( \mathbb{R}^{L} \), where \( L = \ell (1 + S) \), and we will use the notation \( x = (x(s))_{s \in \hat{S}} \in \mathbb{R}^{L} \), (resp. \( p = (p(s))_{s \in \hat{S}} \)), where \( x(s) = (x_1(s), \ldots, x_\ell(s)) \in \mathbb{R}^\ell \) (resp. \( p(s) = (p_1(s), \ldots, p_\ell(s)) \)) denotes the spot consumption (resp. price) at node \( s \in \hat{S} \).

There is a non-empty finite set \( I := \{1, \ldots, I\} \) of consumers, each of whom is endowed with a consumption set \( X_i \subseteq \mathbb{R}^J \), a preference correspondence \( P_i \), from \( \prod_{i' \in I} X_{i'} \) to \( X_i \), and a state-contingent endowment vector \( e_i \in \mathbb{R}^{J} \). The set \( X_i \) is the set of her possible consumptions, and for \( x \in \prod_{i' \in I} X_{i'} \), \( P_i (x) \subseteq X_i \) is the set of consumption plans in \( X_i \) which are strictly preferred to \( x_i \) by consumer \( i \), given the consumption plans \((x_{i'})_{i' \neq i}\) of the other consumers.

The exchange economy is summarized by

\[ \mathcal{E} = \left( S, I, (X_i, P_i, e_i)_{i \in I} \right). \]
We make the following standard assumptions C1-C6 on the economy $\mathcal{E}$ and we denote by $A_\mathcal{E}$ the set of attainable consumption allocations of $\mathcal{E}$, i.e.,

$$A_\mathcal{E} = \left\{ (x_i)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} X_i : \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} e_i \right\}.$$ 

**Consumption assumption C** For all $i \in \mathcal{I}$ and for all $x \in \prod_{i' \in \mathcal{I}} X_{i'}$.

**C1 Consumption sets**: $X_i \subseteq \mathbb{R}^L$ is closed, convex, and bounded below;

**C2 Continuity**: The correspondence $P_i$, from $\prod_{i' \in \mathcal{I}} X_{i'}$ to $X_i$, is lower semicontinuous with open values in $X_i$ (for the relative topology of $X_i$);

**C3 Convexity**: $P_i(x)$ is convex;

**C4 Irreflexivity**: $x_i \notin P_i(x)$;

**C5 Local non-satiation LNS**: $\forall x \in A_\mathcal{E}$:

(a) $\forall s \in \tilde{S}, \exists x_j'(s) \in \mathbb{R}^L, (x_j'(s), x_j(-s)) \in P_i(x), \quad (b) \forall y_i \in P_i(x), (x_i, y_i) \subseteq P_i(x)$;

**C6 Consumption survival CS**: $e_i \in \text{int} X_i$.

We note that these assumptions are standard in a model with non-ordered preferences; the assumptions on $P_i$ are satisfied in particular when consumers’ preferences are represented by utility functions that are continuous, strongly monotonic, and quasi-concave. An exchange economy $\mathcal{E}$ satisfying Assumption C will be called **standard**.

Consumers may operate financial transfers across states in $\tilde{S}$ (i.e., across the two dates and across the states of the second date) by exchanging finitely many assets $j \in \mathcal{J} := \{1, \ldots, J\}$. Each asset $j$ is traded at $t = 0$ and yields payoffs $V_j^s(p)$ (for a given commodity price $p \in \mathbb{R}^L$) at $t = 1$, contingent on the realization of the state of the world $s \in S$. So, the payoff of asset $j$ across tomorrow states is described by the mapping $p \mapsto V_j^s(p) := (V_j^s(p))_{s \in S} \in \mathbb{R}^S$. The financial structure is described by the payoff matrix mapping $V : p \mapsto V(p)$, where $V(p)$ is the $S \times J$ matrix, whose columns are the payoffs $V_j^s(p)$ ($j = 1, \ldots, J$) of the $J$ assets. A portfolio $\theta = (\theta_j)_{j \in \mathcal{J}} \in \mathbb{R}^J$ specifies the quantities $|\theta_j| (j \in \mathcal{J})$ of each asset $j$, with the convention that the asset $j$ is bought if $\theta_j > 0$ and sold if $\theta_j < 0$; thus, $V(p)\theta$ is its random payoff across states at $t = 1$, and $V_0(p) \cdot \theta$ is its payoff if state $s$ prevails. Each consumer $i$ is endowed with a portfolio set $\Theta_i \subseteq \mathbb{R}^J$, which specifies the exogenous portfolio constraints on the consumer’s portfolio holdings. The financial characteristics, referred to as the financial structure, are summarized by

$$\mathcal{F} = \left( S, \mathcal{I}, \mathcal{J}, V, (\Theta_i)_{i \in \mathcal{I}} \right).$$

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6. Let $\Phi$ be a correspondence from $X$ to $Y$, that is, $\Phi$ is a mapping from $X$ to $2^Y$. Then, $\Phi$ is said to be lower semicontinuous (l.s.c.) at $x_0 \in X$, if for every open set $Y \subseteq Y$ such that $\Phi(x_0) \cap Y \neq \emptyset$, there exists an open neighborhood $U$ of $x_0$ in $X$ such that $\Phi(x) \cap Y \neq \emptyset$ for every $x \in U$. The correspondence $\Phi$ is said to be l.s.c. if it is l.s.c. at every point of $X$. Finally, we denote by $G(\Phi) := \{(x, y) \in X \times Y : y \in \Phi(x)\}$ the graph of $\Phi$.

7. Given $x_i = (x_i(s))_{s \in S} \in \mathbb{R}^{\ell(S)}$ and $s \in S$, we let $x_i(-s) := (x_i(s'))_{s' \neq s} \in \mathbb{R}^{\ell(S)}$ and without any risk of confusion we will write $x_i = (x_i(s), x_i(-s))$. 

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We say that the financial structure $\mathcal{F}$ is standard if it satisfies the following Assumption $\mathbf{F0}$.

$\mathbf{F0}$: For every $i \in \mathbf{I}$, $\Theta_i$ is closed, convex, contains 0, and $V : \mathbb{R}^L \to \mathbb{R}^{S \times J}$ is continuous.

Assumption $\mathbf{F0}$ is standard in the study of financial economies with portfolio constraints, see, e.g., Siconolfi (1989), Cass (2006), Angeloni and Cornet (2006), Aouani and Cornet (2009), and Aouani and Cornet (2011).

2.2 Financial equilibria

Given commodity and asset prices $(p, q) \in \mathbb{R}^L \times \mathbb{R}^J$, the budget set of consumer $i$, denoted $B_i(p, q, \mathcal{E}, \mathcal{F})$ or simply $B_i(p, q)$ when there is no risk of confusion, is:

$$B_i(p, q) = \left\{(x_i, \theta_i) \in X_i \times \Theta_i : p(0) \cdot x_i(0) + q \cdot \theta_i \leq p(0) \cdot e_i(0) \quad \forall s \in S \right\} = \left\{(x_i, \theta_i) \in X_i \times \Theta_i : p \square (x_i - e_i) \leq W(p, q)\theta_i \right\},$$

where $W(p, q) := \begin{bmatrix} -q^T \\ V(p) \end{bmatrix}$ denotes the $(1 + S) \times J$-total payoff matrix.

We now introduce the standard equilibrium notion in this model.

**Definition 1** An equilibrium of the financial exchange economy $(\mathcal{E}, \mathcal{F})$ is a list $(\tilde{\rho}, \tilde{x}, \tilde{q}, \tilde{\theta}) \in \mathbb{R}^L \times (\mathbb{R}^L)^I \times \mathbb{R}^J \times (\mathbb{R}^J)^I$ such that

(i) for every $i$, $(\tilde{x}_i, \tilde{\theta}_i)$ maximizes the preference $P_i$ in the budget set $B_i(\tilde{\rho}, \tilde{q})$, in the sense that

$$(\tilde{x}_i, \tilde{\theta}_i) \in B_i(\tilde{\rho}, \tilde{q}) \quad \text{and} \quad B_i(\tilde{\rho}, \tilde{q}) \cap \left( P_i(\tilde{x}) \times \Theta_i \right) = \emptyset,$$

(ii) [Market Clearing] $\sum_{i \in I} \tilde{x}_i = \sum_{i \in I} e_i$ and $\sum_{i \in I} \tilde{\theta}_i = 0$.

- A consumption equilibrium of $(\mathcal{E}, \mathcal{F})$ is a list $(\tilde{\rho}, \tilde{x}) \in \mathbb{R}^L \times (\mathbb{R}^L)^I$ such that there exist $(\tilde{q}, \tilde{\theta}) \in \mathbb{R}^J \times (\mathbb{R}^J)^I$ and $(\tilde{\rho}, \tilde{x}, \tilde{q}, \tilde{\theta})$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$.

- Let $\mathcal{F} = (\mathbf{I}, \mathbf{S}, \mathbf{J}, V, (\Theta_i)_i)$ and $\mathcal{F}' = (\mathbf{I}, \mathbf{S}, J', V', (\Theta_i')_i)$ be two financial structures (with the same sets $\mathbf{I}$ and $\mathbf{S}$), let $\mathcal{E}$ be a standard exchange economy (defined on $(\mathbf{I}, \mathbf{S})$), then the financial exchange economies $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}, \mathcal{F}')$ are said to be equivalent, denoted $(\mathcal{E}, \mathcal{F}) \sim (\mathcal{E}, \mathcal{F}')$, if they have the same consumption equilibria.

- Two financial structures $\mathcal{F} = (\mathbf{I}, \mathbf{S}, \mathbf{J}, V, (\Theta_i)_i), \mathcal{F}' = (\mathbf{I}, \mathbf{S}, J', V', (\Theta_i')_i)$ are said to be equivalent, denoted $\mathcal{F} \sim \mathcal{F}'$, if for every standard exchange economy $\mathcal{E}$, $(\mathcal{E}, \mathcal{F}) \sim (\mathcal{E}, \mathcal{F}')$.

The binary relation “being equivalent to,” taken from Aouani and Cornet (2011), is clearly an equivalence relation on the set of financial structures defined on $(\mathbf{I}, \mathbf{S})$.

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8 For every $p = (p(s))_{s \in S}, x = (x(s))_{s \in S}$ in $\mathbb{R}^L$, we denote by $p \square x$ the vector $(p(s) \cdot x(s))_{s \in S}$.  

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Two such financial structures are equivalent if they are indistinguishable in terms of consumption equilibria associated with each standard exchange economy $E$ defined also on $(I, S)$. In particular if $F$ and $F'$ are two equivalent financial structures and $E$ is a standard economy, the existence of equilibria of $(E, F)$ is equivalent to the existence of equilibria of $(E, F')$.

We now recall that equilibrium asset prices preclude unbounded arbitrage opportunities under local non-satiation (LNS). We denote by $A$ the asymptotic cone of a non-empty convex set $\Theta \subseteq \mathbb{R}^J$.

**Proposition 1** Assume LNS and that the portfolio sets $\Theta_i$ $(i \in I)$ are closed and convex. If $(\tilde{p}, \tilde{x}, \tilde{q}, \tilde{\theta})$ is an equilibrium of the economy $(E, F)$, then $\tilde{q}$ is arbitrage-free at $\tilde{p}$, in the sense that there does not exist a consumer $i$ and $\xi_i \in A\Theta_i$ such that $W(\tilde{p}, \tilde{q})\xi_i > 0$, that is

$$W(\tilde{p}, \tilde{q}) \left( \bigcup_i A\Theta_i \right) \cap \mathbb{R}^S_+ = \{0\}.$$ 

We denote by $Q_F(p)$ the set of arbitrage-free asset prices at $p \in \mathbb{R}^L$.

3 The main results

3.1 Useless portfolios

We consider the class of useless portfolios introduced by Aouani and Cornet (2011), Aouani and Cornet (2016) which is the key concept of the paper and that is larger than the class of Werner useless portfolios defined in Sect. 3.4 by transposing the notion of useless commodity bundles (Werner 1987) to the setting of financial structures. As we shall see later, elimination of all Werner useless portfolios does not guarantee, in general, the boundedness property in Theorem 1 below (see Example 2), justifying the need for this larger class of useless portfolios. However, the two notions of useless portfolios coincide under Hart (1974)’s assumption of Weak No Market Arbitrage (see Theorem 4) but may not coincide if the portfolio sets are polyhedral (see Example 1).

**Definition 2** Let $F$ be a standard financial structure.

- A portfolio $\zeta \in \mathbb{R}^J$ is said to be useless at price $p$ (resp. useless) if $\zeta \in U_F(p) := A_F(p) \cap -A_F(p)$ (resp. $\zeta \in U_F := \bigcup_{p \in \mathbb{R}^L} A_F(p) \cap -A_F(p)$), where $A_F(p) := A(\bigcup_{i \in I}(\Theta_i \cap \{V(p) \leq 0\}))$
- $F$ is said to be useless-free at $p$ if there is no nonzero useless portfolio, i.e., $U_F(p) = \{0\}$. It is said to be useless-free if $U_F(p) = \{0\}$ for all $p$.

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9 The asymptotic cone of a non-empty convex subset $\Theta$ of $\mathbb{R}^J$ is the set $A\Theta := \{\xi \in \mathbb{R}^J : \xi + \text{cl}\Theta \subseteq \text{cl}\Theta\}$ which is a closed convex cone with vertex $0$. As a direct consequence from the definition, one has $A(\text{cl}\Theta) = A\Theta$ and $A\Theta = 0^+(\text{cl}\Theta)$, where $0^+(C) := \{\xi \in \mathbb{R}^J : \xi + C \subseteq C\}$ is the recession cone of the convex set $C \subseteq \mathbb{R}^J$ [see Rockafellar (1997)]; thus, $A\Theta = 0^+(\Theta)$ when $\Theta$ closed and convex. When $\Theta$ is convex and $0 \in \Theta$ we will use extensively the property that $A\Theta \subseteq \text{cl}\Theta$. We will also use the following result: $A\Theta = \{\lim_n \lambda^n\theta^n : (\lambda^n)_n \downarrow 0 \text{ and } \theta^n \in \Theta \text{ for all } n\}$, and we refer to Debreu (1959) for a general reference.
The interpretation of useless portfolios in terms of individual demand is given in the companion paper (Aouani and Cornet 2014), together with a thorough study of the two notions of useless portfolios and Werner useless portfolios (see Sect. 3.4). The concept is also related to the notion of link portfolios by Hahn and Won (2012) in the context of purely financial assets. At this stage, we note that \( U_\mathcal{F}(p) \) is a vector space and that it is a subset of the set of zero-income portfolios at \( p \), i.e., \( U_\mathcal{F}(p) \subseteq \{ V(p) = 0 \} \) (see Lemma 9), and the inclusion may be strict, implying that not all zero-income portfolios are useless.

In the unconstrained case, i.e., if \( \Theta_{i_o} = \mathbb{R}^J \) for some \( i_o \in \mathbb{I} \), then \( U_\mathcal{F}(p) = \{ V(p) = 0 \} \). Hence, the useless-free assumption \( U_\mathcal{F}(p) = \{ 0 \} \) is equivalent to \( \{ V(p) = 0 \} = \{ 0 \} \) which is the standard non-redundancy assumption. In other words, the useless-free assumption can be interpreted as a non-redundancy assumption in the constrained portfolios case.

Before we state our first result, we need the following definition.

**Definition 3** Let \( \mathcal{F} = (V, (\Theta_i)_i) \) be a financial structure. The truncature (sequence) \( (\mathcal{F}^k)_k \) of \( \mathcal{F} \) is defined as the sequence of \( k \)-truncatures (with bounded portfolio sets) keeping the same payoff matrix \( V \). More precisely,

\[
\mathcal{F}^k = (V, (\Theta^k_i)) \quad \text{where} \quad \Theta^k_i = \Theta_i \cap B_J(0, k).
\]

The following theorem states that if \( \mathcal{F} \) is useless-free, then \( \mathcal{F} \) is “equivalent” to its truncature \( (\mathcal{F}^k)_k \) in the sense that for every standard exchange economy \( \mathcal{E} \), the financial exchange economy \( (\mathcal{E}, \mathcal{F}) \) has the same consumption equilibria as \( (\mathcal{E}, \mathcal{F}^k) \) for \( k \) large enough.

**Theorem 1** Let \( \mathcal{F} = (V, (\Theta_i)_i) \) be a standard financial structure.

(a) If \( \mathcal{F} \) is useless-free at \( p \), then \( \mathcal{F} \) satisfies the boundedness property: \( B(p) \): For all \( v = (v_i)_{i \in \mathbb{I}} \in (\mathbb{R}^S)^I \), the set of admissible portfolio allocations

\[
\mathcal{A}_\mathcal{F}(p, v) := \left\{ (\theta_i)_{i \in \mathbb{I}} \in \prod_{i \in \mathbb{I}} \Theta_i : \forall i, \ V(p)\theta_i \geq v_i, \ \sum_{i \in \mathbb{I}} \theta_i = 0 \right\} \text{is bounded.}
\]

(b) Assume that \( \mathcal{F} \) satisfies \( B(p) \) for all \( p \). Then, \( \mathcal{F} \) is equivalent\(^{10} \) to its truncature \( (\mathcal{F}^k)_k \), in the sense that for every standard exchange economy \( \mathcal{E} \), \( (\mathcal{E}, \mathcal{F}) \sim (\mathcal{E}, \mathcal{F}^k) \) for \( k \) large enough.

The proof of Theorem 1 is given in Sect. 4.1. As a consequence of Theorem 1, the existence of equilibria for \( (\mathcal{E}, \mathcal{F}) \), with \( \mathcal{F} \) useless-free, is reduced to the case of bounded portfolio sets as in Radner (1972) and Aouani and Cornet (2009), Aouani and Cornet (2011) in the case of incomplete and intransitive preference. The existence problem when \( \mathcal{F} \) is not useless-free, see Corollary 2 below, will rely on the elimination

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\(^{10}\) The word “equivalent” takes all its meaning if we identify a financial structure \( \mathcal{F} \) with the constant sequence of financial structures \( (\mathcal{F}^k)_k \) where \( \mathcal{F}^k = \mathcal{F} \) for all \( k \), and extend the notion of equivalence to sequences of financial structures as follows: \( (\mathcal{F}^k_1)_k \) is said to be equivalent to \( (\mathcal{F}^k_2)_k \), denoted \( (\mathcal{F}^k_1)_k \sim (\mathcal{F}^k_2)_k \), if \( \mathcal{F}^1_1 \sim \mathcal{F}^k_2 \) for \( k \) large enough.
The portfolio sets $\Theta_1$ and $\Theta_2$.

$$a = -a + b = -2\alpha$$

$$b = -b$$

$$\Theta_1 \cap \{ V \geq 0 \}$$

$$\Theta_2 \cap \{ V \geq 0 \}$$

$$\Theta^\perp := \Theta \cap \{ V \geq 0 \}$$

$$A_F = A(\Theta^+_1 + \Theta^+_2) = \{ V \geq 0 \}$$

$$U_F = A_F \cap -A_F = \{ V = 0 \}$$

Fig. 1 Calculating $\Theta_{1L}$ and $\Theta_{2L}$

The process of useless portfolios defined in the next section (instead of the elimination of redundant assets as in the case of unrestricted portfolio sets).

### 3.2 Eliminating useless portfolios

Let $\mathcal{F}$ be a standard financial structure that is not necessarily useless-free. This section defines a class of equivalent financial structures that have “less” useless portfolios than $\mathcal{F}$ and one (and only one) of which (denoted $\mathcal{F}_*$) is useless-free.

The elimination process of useless portfolios in the standard financial structure $\mathcal{F} = (V, (\Theta_i)_i)$ proceeds as follows. We denote by $\mathcal{F}_*$ the financial structure which has the same payoff matrix as $\mathcal{F}$ and the price-dependent portfolio sets $\Theta_{i*}$ defined by

$$\Theta_{i*}(p) := \text{cl } \text{proj}_{U_F(p)^\perp}(\Theta_i), \text{ for } i \in \mathcal{I},$$

where $\text{proj}_{U_F(p)^\perp}$ denotes the orthogonal projection mapping from $\mathbb{R}^J$ to $U_F(p)^\perp$, the orthogonal space of $U_F(p)$. Then, $\mathcal{F}_*$ is summarized by $(V, (\Theta_{i*})_i)$. A graphical illustration is presented in Example 1. Note that when the portfolio sets $(\Theta_i)_i$ are polyhedral, then they are closed and their orthogonal projection are also closed; hence, there is no need for the closure operation in Eq. (3.1).
The following Example 1 and Fig. 1 provide an illustration of the elimination process.

**Example 1** Consider $I = 2, S = 2, J = 2$, let $a, \alpha, b, \beta$ be all positive and 

$$V = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \Theta_1 = \{(a, b) : a \geq -a, a + b \geq -2\alpha\},$$

and $\Theta_2 = \{(a, b) : b \geq -b, a + b \geq -2\beta\}$. □

The elimination process will be carried out under the following two assumptions that we maintain throughout the rest of the paper.

**Price-invariance** The space $\mathcal{U}_F(p)$ is independent of $p \in \mathbb{R}^L$ (simply denoted $\mathcal{U}_F$ hereafter).

**Closedness:** $\mathcal{F}$ is said to be closed if, for all $p \in \mathbb{R}^L$, the set:

$$\mathcal{G}_F(p) := \left\{(V(p)\theta_1, \ldots, V(p)\theta_I, \sum_{i \in I} \theta_i) \in \mathbb{R}^S \times \cdots \times \mathbb{R}^S \times \mathbb{R}^J : \forall i \in I, \theta_i \in \Theta_i \right\}$$

is closed.

The first assumption, price-invariance, allows to cover the cases of two very widely used types of financial structures in the literature, namely nominal and numéraire financial structures (up to a modification of the payoff matrix; see Sect. 3.5 for details). This assumption also guarantees that the financial structure $\mathcal{F}_*$ has price-independent portfolio sets; hence, $\mathcal{F}_*$ is also standard if $\mathcal{F}$ is standard.

The closedness assumption was introduced and thoroughly discussed by Aouani and Cornet (2011). The next proposition recalls that it covers several independent conditions that are widely used in the literature.

**Proposition 2** (Aouani and Cornet 2011, Proposition A.1) Let $\mathcal{F}$ be a standard financial structure. Then, $\mathcal{F}$ is closed if one of the following five assertions is satisfied:

(i) For all $i \in I$, $\Theta_i = P_i + K_i$, $P_i$ is polyhedral convex and $K_i$ convex, compact.

(ii) Weak No Market Arbitrage (WNMA) (Hart 1974): $\forall p \in \mathbb{R}^L$, $(\forall i \in I, \zeta_i \in A\Theta_i \cap \{V(p) \geq 0\}) \land \sum_{i \in I} \zeta_i = 0 \implies \forall i \in I, \zeta_i \in -A\Theta_i$.

(iii) No Unbounded Arbitrage (NUBA) (Page 1987): $\forall p \in \mathbb{R}^L$, $(\forall i \in I, \zeta_i \in A\Theta_i \cap \{V(p) \geq 0\}) \land \sum_{i \in I} \zeta_i = 0 \implies \forall i \in I, \zeta_i = 0$.

(iv) No Half Lines (NHL) (Siconolfi 1989): $\forall p \in \mathbb{R}^L$, $\forall i \in I, A\Theta_i \cap \text{ker } V(p) = \{0\}$, i.e., portfolio sets do not contain half lines spanned by zero-income portfolios.

(v) For all $p \in \mathbb{R}^L$, the two following conditions hold:
1. $\forall i \in I, \exists i, p \in A\Theta_i, V(p)\zeta_i, p \gg 0$ [Positive Payoff Portfolio];
2. $\forall v = (v_i)i \in I, \mathcal{S}^I, \text{the set } \sum_{i \in I}(\Theta_i \cap \{V(p) \geq v_i\}) \text{ is closed}$.

We are now able to state the main result of this section. Given the financial structure $\mathcal{F} = (V, (\Theta_i)i \in I)$, we denote $\Theta_* := \cup_{i \in I}\Theta_i >$ the vector space spanned by the portfolio sets $\Theta_i$ ($i \in I$, $I$). The set $\Theta_*$ is the space where financial activity takes place. As a consequence, in the following, we will mainly consider asset prices in the set $\mathcal{Q}_F(p) \cap \Theta_*$, which are the only ones that “matter.” More precisely, if $\bar{q}$ is an equilibrium asset price (resp. arbitrage-free asset price), then $\text{proj}_{\Theta_*}\bar{q}$ is also an
equilibrium asset price (resp. arbitrage-free asset price); notice that \( \bar{q} \cdot \theta_i = \text{proj}_{\Theta_F} \bar{q} \cdot \theta_i \) for every \( i \in I \) and for every \( \theta_i \in \Theta_i \).

The elimination process, defined by Eq. (3.1), of useless portfolios in the standard financial structure \( \mathcal{F} = (V, (\Theta_i)_i) \) can be generalized as follows to allow for partial and sequential elimination. Given a linear subspace \( L \) such that \( L \subseteq U_{\mathcal{F}} \), we denote by \( \mathcal{F}_L \), the standard financial structure which has the same payoff matrix as \( \mathcal{F} \) and the price-independent portfolio sets \( \Theta_i_L \) defined by

\[
\Theta_i_L := \text{cl} \text{proj}_{L^\perp} (\Theta_i), \quad \text{for } i \in I,
\]

where \( \text{proj}_{L^\perp} \) denotes the orthogonal projection mapping from \( \mathbb{R}^J \) to \( L^\perp \), the orthogonal space of \( L \). Then, \( \mathcal{F}_L \) is summarized by \( (V, (\Theta_i_L)_i) \).

**Theorem 2** Let \( \mathcal{F} \) be a standard closed financial structure satisfying price-invariance, let \( L \) be a linear subspace of \( U_{\mathcal{F}} \), and let \( \mathcal{F}_L = (V, (\Theta_i_L)_i) \) be defined as above. We have

(a) \( \mathcal{F}_L \) is standard and closed;
(b) \( U_{\mathcal{F}_L} = U_{\mathcal{F}} \cap L^\perp \) or equivalently \( \text{cal} U_{\mathcal{F}} = L \oplus U_{\mathcal{F}_L} \),\(^{11}\) hence \( \mathcal{F}_L \) satisfies price-invariance;
(c) \( \mathcal{F} \) and \( \mathcal{F}_L \) are equivalent;
(d) For all \( p \in \mathbb{R}^L \), \( Q_{\mathcal{F}_L}(p) \cap \Theta_{\mathcal{F}_L} \subseteq Q_{\mathcal{F}}(p) \cap \Theta_{\mathcal{F}} \).

The proof of Theorem 2 is given in Sect. 4.2. We draw some immediate conclusions from the previous theorem.

**Remark 1** From Part (b) of Theorem 2, the following properties hold:

- the financial structure \( \mathcal{F}_L \) has “less” useless portfolios than \( \mathcal{F} \), i.e.,
  \( U_{\mathcal{F}_L} \subseteq U_{\mathcal{F}} \);
- \( \dim U_{\mathcal{F}_L} = \dim U_{\mathcal{F}} - \dim L \);
- the only \( \mathcal{F}_L \) that leaves \( \mathcal{F} \) invariant corresponds to \( L = \{0\} \), formally \( U_{\mathcal{F}_L} = U_{\mathcal{F}} \) if and only if \( L = \{0\} \);
- the only \( \mathcal{F}_L \) that is useless-free corresponds to \( L = U_{\mathcal{F}} \), formally\(^{12}\) \( U_{\mathcal{F}_L} = \{0\} \) if and only if \( L = U_{\mathcal{F}} \).

We define the useless-free reduced form of \( \mathcal{F} \), also simply called reduced form of \( \mathcal{F} \), by taking \( L = U_{\mathcal{F}} \), that is,

\[
\mathcal{F}_* := \mathcal{F}_{U_{\mathcal{F}}}.
\]

The following result is an improvement on [Aouani and Cornet 2011, Theorem 1, (a)].

---

\(^{11}\) Let \( E, L, \) and \( U \) be vector spaces, then \( E = L \oplus U \) means that \( E = L + U \) and \( L \) and \( U \) are orthogonal, i.e., \( U \subseteq L^\perp \). Note that given \( L \subseteq E \), there is a unique \( U \) such that \( E = L \oplus U \), namely \( U = E \cap L^\perp \).

\(^{12}\) Note that the two conditions (i) \( \mathcal{F}' \) is useless-free and (ii) \( \mathcal{F}' \sim \mathcal{F} \), characterize \( \mathcal{F}_* \) if \( \mathcal{F}' \) belongs to the class of \( \{ \mathcal{F}_L \mid L \) is a linear subspace of \( U_{\mathcal{F}} \} \). The result may not be true for a larger class. Indeed, as indicated in the below corollary, \( \mathcal{F} \) is equivalent to \( \mathcal{F}_* \)’s \( k \)-truncatures for \( k \) large enough. By definition, these \( k \)-truncatures have bounded portfolio sets therefore are useless-free and are generally different from \( \mathcal{F}_* \).
Corollary 1 Let \( \mathcal{F} \) be a standard closed financial structure satisfying price-invariance. Then,

(a) \( \mathcal{F} \sim \mathcal{F}_* \),
(b) \( \mathcal{U}_{\mathcal{F}_*} = \{0\} \) hence \( \mathcal{F}_* \) satisfies \( B(p) \) for all \( p \),
(c) \( \mathcal{F} \sim (\mathcal{F}_*)^k \), and
(d) For all \( p \in \mathbb{R}^L \), \( \mathcal{Q}_{\mathcal{F}_*}(p) \cap \Theta_{\mathcal{F}_*} \subseteq \mathcal{Q}_{\mathcal{F}}(p) \cap \Theta_{\mathcal{F}} \).

Proof The facts that \( \mathcal{F} \) and \( \mathcal{F}_* \) are equivalent and \( \mathcal{U}_{\mathcal{F}_*} = \{0\} \) follow immediately from parts (c) and (a) of Theorem 2, respectively. The fact that \( \mathcal{F}_* \) satisfies \( B(p) \) for all \( p \), is a direct consequence of Theorem 1.(a) bearing in mind that \( \mathcal{F}_* \) is useless-free, i.e., \( \mathcal{U}_{\mathcal{F}_*} = \{0\} \). By Theorem 1.(b), \( \mathcal{F}_* \) is therefore equivalent to its truncature \( (\mathcal{F}_*)^k \), i.e., \( \mathcal{F}_* \sim (\mathcal{F}_*)^k \). Hence, using the first part of this corollary yields \( \mathcal{F} \sim (\mathcal{F}_*)^k \).

The last part of the corollary is an immediate consequence of Theorem 1.(d). \( \square \)

Corollary 2 (Existence of equilibria) Let \( \mathcal{E} = (\mathbf{S}, \mathbf{I}, (X_i, P_i, e_i)_{i \in \mathbb{I}}) \) be a standard exchange economy and let \( \mathcal{F} = (\mathbf{S}, \mathbf{I}, \mathbf{J}, V, (\Theta_i)_{i \in \mathbb{I}}) \) be a standard closed financial structure satisfying price-invariance and the following financial survival assumption

FS [Financial Survival].

\[ \forall i \in \mathbb{I}, \forall p \in \mathbb{R}^L, p(0) = 0, \forall q \in \mathcal{Q}_{\mathcal{F}}(p) \cap \Theta_{\mathcal{F}}^{13}, q \neq 0, \exists \theta_i \in \Theta_i, q \cdot \theta_i < 0. \]

Then, the economy \( (\mathcal{E}, \mathcal{F}) \) admits an equilibrium \( (\bar{p}, \bar{x}, \bar{q}, \bar{\theta}) \) such that \( ||\bar{p}(0)|| + ||\bar{q}|| = 1 \) and \( ||\bar{p}(s)|| = 1 \) for all \( s \in \mathcal{S} \). Topic 2, the standard useless-free financial structure \( \mathcal{F}_* \) which is equivalent to the financial structure \( \mathcal{F} \). By Theorem 2, \( \mathcal{F}_* \) satisfies price-invariance and FS.

Second, by (Aouani and Cornet 2009, Theorem 2, p. 777), noticing that Assumption F3 in that paper is equivalent to useless-freeness; see (Aouani and Cornet 2016, Theorem 2, p. 153), the economy (\( \mathcal{E}, \mathcal{F}_* \)) admits an equilibrium \( (p^*, \bar{x}, q^*, \bar{\theta}) \) such that \( ||p^*(s)|| = 1 \) for all \( s \in \mathcal{S} \). The consumption equilibrium \( (p^*, \bar{x}) \) of \( (\mathcal{E}, \mathcal{F}_*) \) is a consumption equilibrium of \( (\mathcal{E}, \mathcal{F}) \) by Theorem 2. Hence there exists \( (q^*, \bar{\theta}) \) such that \( (p^*, \bar{x}, q^*, \bar{\theta}) \) is an equilibrium of \( (\mathcal{E}, \mathcal{F}) \). The end of the proof consists in modifying the prices \( p^*, q^* \) into \( \bar{p}, \bar{q} \) satisfying \( ||\bar{p}(0)|| + ||\bar{q}|| = 1 \) and \( ||\bar{p}(s)|| = 1 \) for all \( s \in \mathcal{S} \) so that \( (\bar{p}, \bar{x}, \bar{q}, \bar{\theta}) \) is also an equilibrium of \( (\mathcal{E}, \mathcal{F}) \). \( \square \)

13 In Aouani and Cornet (2009), the financial survival assumption is made for prices \( q \in \mathcal{Q}_{\mathcal{F}}(p) \cap \Theta_{\mathcal{F}} \), and we notice that the two conditions are actually equivalent since \( \mathcal{Q}_{\mathcal{F}}(p) \cap \Theta_{\mathcal{F}} = \mathcal{Q}_{\mathcal{F}}(p) \cap \Theta_{\mathcal{F}} \).

Note that the inclusion \( \mathcal{Q}_{\mathcal{F}}(p) \cap \Theta_{\mathcal{F}} \subseteq \mathcal{Q}_{\mathcal{F}}(p) \cap \Theta_{\mathcal{F}} \) is immediate and we now show the converse inclusion. Let \( q \in \mathcal{Q}_{\mathcal{F}}(p) \cap \Theta_{\mathcal{F}} \). Then, \( q = \lim_n q^n \) for some sequence \((q^n)\) \( \subseteq \mathcal{Q}_{\mathcal{F}}(p) \). Since \( q \in \Theta_{\mathcal{F}} \), one has \( q = \lim_n q^n \) for some sequence \((q^n)\) \( \subseteq \mathcal{Q}_{\mathcal{F}}(p) \). Clearly, \( \lim_n \mathcal{Q}_{\mathcal{F}}(q^n) \subseteq \Theta_{\mathcal{F}} \) and \( \mathcal{Q}_{\mathcal{F}}(p) \cap \Theta_{\mathcal{F}} \). Hence, \( q = \lim_n q^n \) for some sequence \((q^n)\) \( \subseteq \mathcal{Q}_{\mathcal{F}}(p) \cap \Theta_{\mathcal{F}} \).

14 Let \( \tilde{p} = (\lambda p^*(0), (p^*(s))_{s \in \mathcal{S}}) \) and \( \tilde{q} = \lambda q^* \) with \( \lambda = 1/(||p^*(0)|| + ||q^*||) \) (so that \( ||\tilde{p}(0)|| + ||\tilde{q}|| = 1 \). From Local Non-Satiation LNS (at \( s = 0 \), we deduce that \( ||p^*(0)|| + ||q^*|| > 0 \), thus \( \lambda = 1/(||p^*(0)|| + ||q^*||) \) is well-defined; moreover, \( (\lambda p^*(0), (p^*(s))_{s \in \mathcal{S}}), \tilde{q}, \lambda q^* \) is an equilibrium of \( (\mathcal{E}, \mathcal{F}) \) (from the positive homogeneity property in \((p(0), q)\) of the budget constraint at \( t = 0 \)).
3.3 Sequential elimination of useless portfolios

Elimination of all useless portfolios can be carried out in several ways. The first one considered in the previous section was completed in one step by taking $L = U_F$ in Theorem 2. Alternatively, elimination of useless portfolios can be done sequentially by decreasing the dimension of the space of useless portfolios in each step. We define a sequence of financial structures $F_0, \ldots, F_n, \ldots$ by induction as follows:

**Step 0.** $F_0 := F$.

**Step n** ($n \geq 1$). Choose a vector space $L_n \subseteq U_{F_{n-1}}$ (possibly $L_n = \{0\}$). Let

$$F_n := (F_{n-1})_{L_n}.$$  

The following theorem shows that proceeding sequentially in the above elimination process defines a sequence of financial structures $F_0, \ldots, F_n, \ldots$ each of which is equivalent to $F$ and with decreasing sets of useless portfolios, i.e., $U_{F_{n-1}} \supset U_{F_n}$. Moreover, for the elimination process to be complete (or total), that is, $U_{F_n} = \{0\}$ for $n$ large enough, we need to additionally assume that $L_n \neq \{0\}$ whenever possible. In that case, the elimination process “converges” to $F_\star$, in the sense that $F_n = F_\star$ for $n$ large enough.

**Theorem 3** Assume $F$ is standard, closed, and satisfies price-invariance. Then, the following hold for $n \geq 1$:

(i) $F_n$ is standard, closed, and satisfies price-invariance;
(ii) $F \sim F_n$;
(iii) $U_{F_{n-1}} \supset U_{F_n} = U_{F_{n-1}} \cap L_n^\perp$;
(iv) $F_n = F_{L_1 + \cdots + L_n}$.

If we additionally assume that $L_n \neq \{0\}$ if $U_{F_{n-1}} \neq \{0\}$, then

(v) $U_F = L_1 + \cdots + L_n$ and $L_{n+1} = U_{F_n} = \{0\}$ for $n \geq \dim U_F$.
(vi) $F_n = F_\star$ for $n \geq \dim U_F$.

The proof of Theorem 3 is given in Sect. 4.3. The following remark, which is a direct consequence of Theorem 3, shows some robustness properties of the elimination process with respect to the choice of the sequence of spaces $L_1, \ldots, L_n$.

**Remark 2**  
(a) Note that the spaces $L_1, \ldots, L_n$ are in orthogonal direct sum, i.e., $L_m \subseteq L_k^\perp$ for every $m \neq k$. So $L_1 + \cdots + L_n = L_1 \oplus \cdots \oplus L_n$.
(b) Changing the order in which the spaces $L_k, k = 1, \ldots, n$ are chosen does not change the process after $n$. Formally, let $\sigma$ be a permutation of $\{1, \ldots, n\}$, then the financial structure that is reached after using the vector spaces $L_1, \ldots, L_n$ (in this order) is the same as the one that is reached after using the vector spaces $L_{\sigma(1)}, \ldots, L_{\sigma(n)}$ (in this order). This is an immediate consequence of part (iv) of the above theorem since $F_n = F_{L_1 + \cdots + L_n} = F_{L_{\sigma(1)} + \cdots + L_{\sigma(n)}}$.
(c) The “limit” $F_\star$ does not depend on the choice of the sequence of vector spaces $L_1, \ldots, L_N$ that sum up to $U_F$. That is, consider two sequences of vector

\[ U_{F_n} = \sum_{j \geq n+1} L_j \text{ for all } n; \text{ note that the infinite sum is well defined since } L_j = \{0\} \text{ for } j \text{ large enough.} \]

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spaces $L_1, \ldots, L_N$ and $L'_1, \ldots, L'_N$ such that $L_1 + \cdots + L_N = L'_1 + \cdots + L'_N = U_F$, then $F_N = F'_N = F^\star$.

### 3.4 When does Werner elimination lead to $F^\star$?

To answer this question, we first need to formally define Werner useless portfolios and their elimination process. For simplicity, we will work in this section with nominal financial structures. However, all our results hold true under the following assumption, strong price-invariance, which is a slight strengthening of price-invariance.

**Strong price-invariance:** The cone $\{ V(p) \geq 0 \} := \{ \theta \in \mathbb{R}^J : V(p)\theta \geq 0 \}$ is independent of $p \in \mathbb{R}^L$.

Note that strong price-invariance entails that the space $\{ V(p) = 0 \}$ is independent of $p \in \mathbb{R}^L$ (hence simple denoted $\{ V = 0 \}$) and also allows to cover both cases of nominal and numéraire financial structures.

**Definition 4** Let $F = (V, (\Theta_i)_{i \in I})$ be a standard financial structure satisfying strong price-invariance.

- A portfolio $\zeta \in \mathbb{R}^J$ is said to be Werner useless if $\zeta \in U^W_F := \sum_{i \in I} A \Theta_i \cap -A \Theta_i \cap \{ V = 0 \}$.
- $F$ is said to be Werner useless-free if there is no nonzero Werner useless portfolio, i.e., $U^W_F = \{ 0 \}$.

An interpretation of Werner useless portfolios in terms of individual demand is provided in the companion paper (Aouani and Cornet 2014). We note that $U^W_F$ is a vector space and that all Werner useless portfolios are useless, i.e., $U^W_F \subseteq U_F$ (see Lemma 9) and the inclusion can be strict (see Example 2 below).

It is worth pointing out that the “boundedness” property $[B(p)$ for all $p]$ may not hold if $F$ is only Werner useless-free; see the following example. This is the main reason why we need to enlarge the class of Werner useless portfolios.

**Example 2** An example of $F$ such that $U^W_F = \{ 0 \}$, $[B(p)$ for all $p]$ is not satisfied, and $U_F \neq \{ 0 \}$.

The process of eliminating Werner useless portfolios can also be carried out in several ways, i.e., in one-shot or sequentially. This section will show that neither one nor the other will lead to $F^\star$ in general (i.e., under the closedness assumption). Moreover, the result may not be true even if we assume that portfolio sets are polyhedral: a standard assumption in economics. More precisely, in contrast to the case of useless portfolios, eliminating Werner useless portfolios at once yields a financial structure that might still have Werner useless portfolios, moreover sequentially eliminating Werner useless portfolios yields a financial structure that is free of Werner useless portfolios.
but which may still have useless portfolios. The answer, however, will be positive if we assume WNMA because Werner useless portfolios and useless portfolios coincide, i.e., \( U_F = U_F^W \).

Let \( F = (V, (\Theta_i)_{i \in I}) \) be a standard financial structure. One-shot elimination of Werner useless portfolios consists in considering the (obviously standard) financial structure \( F_{1,w} := F_{U_F^W} \) obtained from \( F \) by taking \( L = U_F^W \) in the useless portfolios elimination process defined in Sect. 3.2. This amounts to eliminating the entire package of Werner useless portfolios of \( F \) at once. However, we have seen previously that we may have \( F_{1,w} \neq F \) or equivalently, \( U_{F_{1,w}} \neq \{0\} \) (there may still exist useless portfolios in \( F_{1,w} \)); see Example 2.

In fact, as established by Example 3 below, \( F_{1,w} \) may still have Werner useless portfolios which are also useless portfolios for the original financial structure \( F \) (by Part (b) of Theorem 2 and Lemma 9) and we can carry on the process by eliminating, in each step, the entire package of those new Werner useless portfolios as follows: Let \( F_{0,w} = F \) and, for \( n \geq 0 \), define

\[
F_{n+1,w} = (F_{n,w})_{U_{F_n,w}^W}.
\]

Thus

\[
\begin{align*}
\bullet \ U_{F_n,w}^W &= \{0\} \Rightarrow F_{n+1,w} = F_{n,w}. \\
\bullet \ U_{F_{n+1,w}} = U_{F_{n,w}}^W \oplus U_{F_{n+1,w}} (\text{by Part (a) of Theorem 2}). \\
\bullet \ U_F = U_{F_{0,w}} = U_{F_{1,w}}^W \oplus U_{F_{1,w}}^W \oplus \cdots \oplus U_{F_{n,w}}^W \oplus U_{F_{n+1,w}} (\text{by induction}).
\end{align*}
\]

Given that the spaces \( U_{F_n,w}^W \) (for \( n \geq 0 \)) are in orthogonal direct sum and are all subspaces of the finite dimensional space \( U_F \), this process converges in the sense that there exists \( N \) such that \( U_{F_n,w}^W = \{0\} \) for \( n > N \). We define the Werner useless-free reduced form of \( F \), also simply called Werner reduced form of \( F \), by

\[
F_w := \lim_{n} F_{n,w} = F_{N,w}.
\]

Note that \( F_w \) can also be obtained by taking \( L = U_{F_{0,w}}^W \oplus U_{F_{1,w}}^W \oplus \cdots \oplus U_{F_{N,w}}^W \) in the elimination process defined in Sect. 3.2.

**Theorem 4** Let \( F \) be a standard financial structure satisfying strong price-invariance.

(a) If \( F \) is closed then \( F \sim F_w \), which is Werner useless-free, i.e., \( U_{F_w}^W = \{0\} \), but we may have \( U_{F_w} \neq \{0\} \) and (therefore) \( F_w \neq F_* \).

(b) If \( F \) satisfies WNMA, then \( U_F^W = U_F \) and \( F_w = F_* \).

The proof of Theorem 4 is given in Sect. 4.4. According to the above result, WNMA is the appropriate environment in which elimination of Werner useless portfolios would work best. In fact the two notions of useless portfolios coincide under WNMA. Note, however, that WNMA is sufficient but not necessary for the equality \( F_* = F_w \) to hold, as shown by the following example.
Fig. 2 $\Theta_1 = \{(a, b, c) \in \mathbb{R}^3 : b \geq (\max\{0, -a\})^2\}$ and $\Theta_2 = \{(a, b, c) \in \mathbb{R}^3 : a \geq b\}$

Example 3 Let $I = 2$, $S = 1$, $J = 3$, $V = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, and

$$\Theta_1 := \{(a, b, c) \in \mathbb{R}^3 : b \geq (\max\{0, -a\})^2\} = \{(a, b) \in \mathbb{R}^2 : b \geq (\max\{0, -a\})^2\} \times \mathbb{R},$$

$$\Theta_2 := \{(a, b, c) \in \mathbb{R}^3 : a \geq b\} = \{(a, b) \in \mathbb{R}^2 : a \geq b\} \times \mathbb{R}.$$

See Fig. 2. Then, $\mathcal{F}$ satisfies the three following properties.

- $\mathcal{F}_{1,w} \neq \mathcal{F}_w$ and $\mathcal{F}_{2,w} = \mathcal{F}_w$, i.e., it takes more than one step in the elimination process of Werner useless portfolios to reach $\mathcal{F}_w$,
- $\mathcal{F}_w = \mathcal{F}_*$, and
- $\mathcal{F}$ does not satisfy WNMA.

We now prove the above properties. It is easy to check that

$$\{V = 0\} = \mathbb{R} \times \mathbb{R} \times \{0\} \text{ and } \{V \geq 0\} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+,$$

$$A\Theta_1 \cap -A\Theta_1 \cap \{V = 0\} = \{0\},$$

$$A\Theta_2 \cap -A\Theta_2 \cap \{V = 0\} = \{(a, b) \in \mathbb{R}^2 : a = b\} \times \{0\},$$

$$\mathcal{U}_F^W = \{(a, a) \in \mathbb{R}^2 \} \times \{0\} \text{ and } (\mathcal{U}_F^W)^\perp = \{(a, -a) \in \mathbb{R}^2 \} \times \mathbb{R}.$$
\[ F_{1,w} \neq F_w \text{ and } F_{2,w} = F_w. \text{ Indeed, } \mathcal{U}^W_{F_{1,w}} := \sum_{i \in \mathcal{I}} (A\Theta_i^W \cap -A\Theta_i^W \cap \{ V = 0 \}) = \{(a, -a) \in \mathbb{R}^2 \times \{0\} \neq \{0\} \text{ implying that } F_{1,w} \neq F_w \text{ (since } \mathcal{U}^W_{F_{1,w}} = \{0\} \text{ and } \mathcal{U}^W_{F_{1,w}} \neq \{0\}). \text{ In fact to obtain } F_w, \text{ we need to eliminate Werner useless portfolios in } F_{1,w}. \text{ Given that } (\mathcal{U}^W_{F_{1,w}})^\perp = \{(a, a) \in \mathbb{R}^2 \times \mathbb{R}, \text{ it is easy to check that } \]

\[
\begin{align*}
(\Theta_1^W)^W & := \text{proj}_{(\mathcal{U}^W_{F_{1,w}})^\perp} \Theta_1^W = \{0\} \times \{0\} \times \mathbb{R}, \\
(\Theta_2^W)^W & := \text{proj}_{(\mathcal{U}^W_{F_{1,w}})^\perp} \Theta_2^W = \{0\} \times \{0\} \times \mathbb{R}.
\end{align*}
\]

\[
\mathcal{U}^W_{F_{2,w}} := \sum_{i \in \mathcal{I}} A(\Theta_i^W)^W \cap -A(\Theta_i^W)^W \cap \ker V = \{0\}, \text{ that is, } F_{2,w} = F_w. \]

• \( F_* = F_w. \text{ Indeed, } \mathcal{U}_F = \mathbb{R} \times \mathbb{R} \times \{0\}, \text{ hence } \Theta_{i,*} := \text{proj}_{(\mathcal{U}_F)^\perp} \Theta_i = \{0\} \times \{0\} \times \mathbb{R} = \Theta_{i,w}. \)

• \( F \) does not satisfy WNMA. Indeed, \( \zeta_1 := (1, 2, 0) \in A\Theta_1 \cap \{ V \geq 0 \} \) and \( -\zeta_1 \in A\Theta_2 \cap \{ V \geq 0 \} \) but \( \zeta_1 \notin -A\Theta_1 \) (for, if \( \zeta_1 \in -A\Theta_1 \) then \( \zeta_1 \in A\Theta_1 \cap -A\Theta_1 \cap \{ V = 0 \} = \{0\} \)). \qed

### 3.5 Nominal and numéraire assets

The financial structure \( F \) is said to be nominal, if assets are purely financial, that is, their payments are made in units of account. Hence, the matrix \( V(p) \) of financial payoffs does not depend on the commodities price vector \( p \) in which case it is denoted \( R \). Nominal financial structures obviously fall under the umbrella of the price-invariance assumption.

A numéraire financial structure \( F \) consists only of numéraire assets whose payments are made in terms of a commodity bundle \( v \in \mathbb{R}^\ell \) called numéraire. An important example is the case where payments are made in terms of a single commodity, typically the last good, in which case \( v = (0, \ldots, 0, 1) \). Hence, there exists a \( S \times J \) matrix \( R \) such that

\[
V(p) := \begin{pmatrix}
p(1) \cdot v & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \\
\vdots & \ddots & 0 & \\
0 & \cdots & 0 & p(S) \cdot v
\end{pmatrix} R.
\]

Note that the financial structure \( F \) may not satisfy price-invariance, and there are two ways to overcome this difficulty. The first one (not used here) is to restrict the set of commodity prices so that \( p(s) \cdot v > 0 \) for every \( s \in S \) since at equilibrium this property will be satisfied under a standard desirability assumption of the numéraire (see below). The second one, adopted hereafter, is to modify the payoff matrix as follows: Define the modified financial structure \( F^\varepsilon = (V^\varepsilon, (\Theta_i)_i) \), for \( \varepsilon > 0 \), by taking the same portfolio sets \( \Theta_i \) as for \( F \) and defining the modified payoff matrix \( V^\varepsilon \) by
Then, \( \{ V^\varepsilon(p) \geq 0 \} = \{ R \geq 0 \} \) for every \( p \in \mathbb{R}^L \); hence, \( V^\varepsilon \) satisfies price-invariance. In the context of equilibrium existence, the above modification can be made without loss of generality since, for \( \varepsilon \) small enough, equilibria associated with \( \mathcal{F}^\varepsilon \) are also equilibria associated with \( \mathcal{F} \). See (Aouani and Cornet 2009, Lemma 3, page 778) for a proof.

**Lemma 1** Let \( \mathcal{F} \) be a numéraire financial structure. Let \( \mathcal{E} \) be a standard exchange economy such that for every consumer \( i \) the correspondence \( P_i \) has an open graph, and the commodity bundle \( v \in \mathbb{R}^L \) is desirable at every state \( s \in \mathcal{S} \), i.e., for all \( x \in \mathcal{A}\mathcal{E} \), for all \( t > 0 \), \( (x_i(s) + tv, x_i(-s)) \in P_i(x) \). Then, for \( \varepsilon > 0 \) small enough, every equilibrium \( (\bar{p}, \bar{q}, \bar{x}, \bar{\theta}) \) of \( (\mathcal{E}, \mathcal{F}^\varepsilon) \) such that \( ||\bar{p}(s)|| = 1 \) for \( s \in \mathcal{S} \) is an equilibrium of the economy \( (\mathcal{E}, \mathcal{F}) \).

## 4 Proofs of the main results

### 4.1 Proof of Theorem 1

We prepare the proof of Theorem 1 by the following lemma whose proof is given in Sect. 5.1.

**Lemma 2** Let \( \mathcal{F} \) be a standard financial structure. Then, \( \{ B(p) \) for all \( p \in B_L(0, 1) \} \) is equivalent to the following property B.

\begin{itemize}
  \item [B.] For all \( v = (v_i)_{i \in \mathcal{I}} \in (\mathbb{R}^S)^\mathcal{I} \), the set \( \mathcal{A}\mathcal{F}(v) \) of admissible portfolio allocations and admissible commodity prices is bounded, where

\[ \mathcal{A}\mathcal{F}(v) := \left\{ (p, (\theta_i)_{i \in \mathcal{I}}) \in B_L(0, 1) \times \left( \prod_{i \in \mathcal{I}} \Theta_i \right) : \forall i, \ V(p)\theta_i \geq v_i, \sum_{i \in \mathcal{I}} \theta_i = 0 \right\}. \]

\end{itemize}

**Proof of Part (a) of Theorem 1.** Let \( \mathcal{F} \) be a standard financial structure. Assume that, for some \( p \in B_L(0, 1) \), \( \mathcal{F} \) is useless-free at \( p \) and \( B(p) \) does not hold, i.e., there exists \( v \in (\mathbb{R}^S)^\mathcal{I} \) such that \( \mathcal{A}\mathcal{F}(p, v) \) is not bounded. Then, there exist sequences \( (\theta^n_i)_{i \in \mathcal{I}} \subseteq \Theta_i \) (for \( i \in \mathcal{I} \)) such that for all \( n \) and all \( i \), \( V(p)\theta^n_i \geq v_i, \sum_{i \in \mathcal{I}} \theta^n_i = 0 \), and \( \sum_{i \in \mathcal{I}} ||\theta^n_i|| \to +\infty \). Passing to subsequences if necessary, we can assume that, for each \( i \), the bounded sequence \( (\theta^n_i/\sum_{k \in \mathcal{I}} ||\theta^n_k||)_{n} \) converges to some \( \zeta_i \). The vector \( \zeta_i \) belongs to \( A\Theta_i \) since \( \theta^n_i \in \Theta_i \) for every \( n \) and \( 1/\sum_{k \in \mathcal{I}} ||\theta^n_k|| \to 0 \). Moreover, for every \( n \), we have \( V(p)(\theta^n_i/\sum_{k \in \mathcal{I}} ||\theta^n_k||) \geq v_i/\sum_{k \in \mathcal{I}} ||\theta^n_k|| \). Passing to the limit, we obtain \( V(p)\zeta_i \geq 0 \). Recalling that, for every \( n \), \( \sum_{i \in \mathcal{I}} \theta^n_i = 0 \), we get \( 0 = \sum_{i \in \mathcal{I}} (\theta^n_i/\sum_{k \in \mathcal{I}} ||\theta^n_k||) \to \sum_{i \in \mathcal{I}} \zeta_i = 0 \). Recalling that for each \( i, \zeta_i \in A\Theta_i \cap \{ V(p) \geq 0 \} \subseteq \mathcal{A}\mathcal{F}(p) \) we conclude that \( \zeta_i \in \mathcal{A}\mathcal{F}(p) \) for each \( i \) and \( \sum_{i \in \mathcal{I}} \zeta_i = 0 \).
which implies that \( \zeta_1 = - \sum_{i \neq 1} \zeta_i \in A_F(p) \cap - A_F(p) = U_F(p) = \{0\} \) (since \( F \) is uselessly free at \( p \)), and similarly \( \zeta_i = 0 \) for every \( i \). But \( \theta^n_{i} / \sum_{k \in I} ||\theta^n_{k}|| \xrightarrow{n \to \infty} \zeta_i \), hence

\[
I = \sum_{i \in I} ||\theta^n_{i}|| \xrightarrow{n \to \infty} \sum_{i \in I} ||\zeta_i|| = 0; \text{ a contradiction.} \quad \square
\]

**Proof of Part (b) of Theorem 1.** Let \( E \) be a standard exchange economy and let \( F \) be a financial structure satisfying \( F_0 \) and \( B(p) \) for all \( p \). Then, by Lemma 2, \( F \) satisfies \( B \). We denote by \( \tilde{X}_i \) the projection of the set of attainable allocations \( A_E \) on \( X_i \). Since \( A_E \) is bounded (by Assumption \( C1 \)), the sets \( \tilde{X}_i \) are also bounded for every \( i \in I \). Consequently, one can choose \( r_1 > 0 \) large enough such that

\[
\tilde{X}_i \subseteq \text{int} B_L(0, r_1) \quad \text{for every} \quad i \in I.
\]

For \( i \in I \), let \( \psi_i = (\psi_i(s)) \in \mathbb{R}^S \), where for \( s \in S \),

\[
\psi_i(s) := -1 + \min \{ p(s) \cdot (x_i(s) - e_i(s)) : p(s) \in B_E(0, 1), x_i \in B_L(0, r_1) \}.
\]  

(4.1)

which is well defined from the compactness of the closed balls \( B_E(0, 1) \) and \( B_L(0, r_1) \). We denote by \( \tilde{\Theta}_i \) the projection of \( A_F(\psi) \) on \( \Theta_i \). The sets \( \tilde{\Theta}_i \) are bounded for every \( i \in I \), since \( A_F(\psi) \) is bounded (by property \( B \)). Consequently, one can choose \( r_2 > 0 \) large enough such that

\[
\tilde{\Theta}_i \subseteq \text{int} B_J(0, r_2) \quad \text{for every} \quad i \in I.
\]

Consider an integer \( n := n(E) \geq \max(r_1, r_2) \) and for every \( i \in I \), let \( \Theta^n_{i} = \Theta_i \cap B_J(0, n) \). We claim that \( (E, F) \sim (E, F^n) \). For if \( (\tilde{p}, \tilde{x}, \tilde{q}, \tilde{\theta}) \) is an equilibrium of \( (E, F) \), then market clearing conditions are satisfied by the consumption allocation \( \tilde{x} \) and the portfolio allocation \( \tilde{\theta} \) and since \( \Theta^n_{i} \subseteq \Theta_i \) for all \( i \in I \), one has \( B_i(\tilde{p}, \tilde{q}, F^n) \subseteq B_i(\tilde{p}, \tilde{q}, F) \) which implies

\[
B_i(\tilde{p}, \tilde{q}, F^n) \cap (P_i(\tilde{x}) \times \Theta^n_{i}) \subseteq B_i(\tilde{p}, \tilde{q}, F) \cap (P_i(\tilde{x}) \times \Theta_{i}) \quad \text{for all} \quad i \in I.
\]

Given that for every \( i \in I \), \( B_i(\tilde{p}, \tilde{q}, F) \cap (P_i(\tilde{x}) \times \Theta_{i}) = \emptyset \) (because \( (\tilde{p}, \tilde{x}, \tilde{q}, \tilde{\theta}) \) is an equilibrium of \( (E, F) \)), we conclude that \( B_i(\tilde{p}, \tilde{q}, F^n) \cap (P_i(\tilde{x}) \times \Theta^n_{i}) = \emptyset \). Hence, \((\tilde{p}, \tilde{x}, \tilde{q}, \tilde{\theta}) \) is an equilibrium of \( (E, F) \).

Conversely, let \((\tilde{p}, \tilde{x}, \tilde{q}, \tilde{\theta})\) be an equilibrium of \((E, F^n)\). To show that it is an equilibrium of \((E, F)\), we only have to prove that

\[
B_i(\tilde{p}, \tilde{q}, F) \cap (P_i(\tilde{x}) \times \Theta_{i}) = \emptyset \quad \text{for every} \quad i \in I.
\]

Assume that it is not true, then for some \( i \), there exists \((x_i, \theta_{i}) \in B_i(\tilde{p}, \tilde{q}, F) \cap (P_i(\tilde{x}) \times \Theta_{i}) \). From the choice of \( n(E) \), for all \( i, \tilde{\theta}_{i} \in \tilde{\Theta}_i \subseteq \text{int} B_J(0, n) \). Thus, for \( t > 0 \) small enough, \((x'_i, \theta'_i) := (\tilde{x}_i + t(x_i - \tilde{x}_i)), \tilde{\theta}_i + t(\theta_i - \tilde{\theta}_i) \) \( \in X_i \times \Theta^n_{i} \) (recall that \( X_i \) is convex by Assumption \( C1 \) and \( \Theta_i \) is convex by Assumption \( F0 \)), hence \((x'_i, \theta'_i) \in B_i(\tilde{p}, \tilde{q}, F^n) \) (since \((x_i, \theta_{i}) \in B_i(\tilde{p}, \tilde{q}, F)\) and \((\tilde{x}_i, \tilde{\theta}_i) \in B_i(\tilde{p}, \tilde{q}, F^n)) \). On the other hand, from Assumption \( C5 \) (Local non-satiation \( LNS \)), for every \( t \in \)}
(0, 1], \bar{x}_i + t (x_i - \bar{x}_i) \in P_i(\bar{x}). Hence \((x^i_1, \theta^i_1) \in B_i(\bar{\rho}, \bar{q}, \mathcal{F}^n) \cap (P_i(\bar{x}) \times \Theta^n_i) \neq \emptyset, in contradiction with the fact that \((\bar{\rho}, \bar{x}, \bar{q}, \bar{\theta})\) is an equilibrium of \((\mathcal{E}, \mathcal{F}^n)\). \qed

4.2 Proof of Theorem 2

Let \(\mathcal{F} = (V, (\Theta_i)_{i \in I})\) be a standard closed financial structure satisfying price-invariance and let \(L \subseteq \mathcal{U}_\mathcal{F}\) be a linear subspace of \(\mathcal{U}_\mathcal{F}\). We consider the financial structure \(\mathcal{F}_L\) which has the same payoff matrix as \(\mathcal{F}\) and the portfolio sets \(\text{cl} \pi_L \Theta_i (i \in I)\) where \(\pi_L := \text{proj}_{L^\perp}\) is the orthogonal projection mapping from \(\mathbb{R}^J\) to \(L^\perp\), the orthogonal space of \(L\). We will use extensively the following properties\(^{16}\) for all \((p, q, \theta) \in \mathbb{R}^L \times \mathbb{R}^J \times \mathbb{R}^J\),

\[
q \cdot \pi_L \theta = \pi_L q \cdot \pi_L \theta = \pi_L q \cdot \theta, \\
\ker \pi_L = L \subseteq \mathcal{U}_\mathcal{F} \subseteq \ker V(p), \\
V(p) \pi_L \theta = V(p) \theta, \\
W(p, q) \pi_L \theta = W(p, \pi_L q) \pi_L \theta = W(p, \pi_L q) \theta.
\]

\[(4.2)\]

We will need the following two lemmas. See (Aouani and Cornet 2011, Proposition A.1., page 325) for a proof for Lemma 3. The proof of Lemma 4 is given in Sect. 5.2.

**Lemma 3** Let \(\mathcal{F} = (V, (\Theta_i)_{i \in I})\) be a financial structure and let \(p \in \mathbb{R}^L\) be given. The set \(\mathcal{G}_\mathcal{F}(p)\) is closed if and only if the set \(\mathcal{G}_\mathcal{F}^L(p)\) is closed, where

\[
\mathcal{G}_\mathcal{F}^L(p) := \left\{(v_1, \ldots, v_I, \sum_{i \in I} \theta_i) \in \mathbb{R}^S \times \cdots \times \mathbb{R}^S \times \mathbb{R}^J : \forall i \in I, \theta_i \in \Theta_i, V(p) \theta_i \geq v_i \right\}.
\]

**Lemma 4** Let \(\mathcal{F} = (V, (\Theta_i)_{i \in I})\) be a standard closed financial structure and let \(L\) be a linear subspace of \(\mathcal{U}_\mathcal{F}\). We have

\[
\sum_{i \in I} (\text{cl} \pi_L \Theta_i \cap \{V(p) \geq v_i\}) \subseteq \sum_{i \in I} (\Theta_i \cap \{V(p) \geq v_i\}), \forall p \in \mathbb{R}^L, \forall (v_i)_{i \in I} \in (\mathbb{R}^S)^I.
\]

4.2.1 Proof of part (a) of Theorem 2

\(\mathcal{F}_L\) is obviously standard, and we prove that it is closed: Let \(p \in \mathbb{R}^L\), we show that \(\mathcal{G}_\mathcal{F}_L^L(p)\) is closed (from Lemma 3, the set \(\mathcal{G}_\mathcal{F}_L^L(p)\) is then closed). Consider a sequence \((w^n)_{n \in \mathbb{N}} \subseteq \mathcal{G}_\mathcal{F}_L^L(p)\), converging to some \(w\), i.e.,

\[
\lim_{n \to \infty} w^n = (v^n_1, \ldots, v^n_I, \sum_{i \in I} y^n_i) \quad \Rightarrow \quad w = (v_1, \ldots, v_I, y),
\]

\(^{16}\) The first equality comes from the fact that \(\pi_L q \cdot \pi_L \theta = \pi_L q \cdot \theta\), since \(\pi_L q \in \text{Im} \pi_L\) and \(\theta - \pi_L \theta \in \ker \pi_L = (\text{Im} \pi_L)^\perp\) since \(\pi_L\) is an orthogonal projection mapping; then by symmetry \(q \cdot \pi_L \theta = \pi_L q \cdot \pi_L \theta = \pi_L q \cdot \theta\). The last inclusion holds since \(\mathcal{U}_\mathcal{F} := A_\mathcal{F}(p) \cap -A_\mathcal{F}(p) \subseteq \{V(p) \geq 0\} \cap \{-V(p) \geq 0\} = \ker V(p)\). The second set of equalities holds since \(\theta - \pi_L \theta \in \ker \pi_L = L \subseteq \ker V(p)\).
with \( y^n_i \in \text{cl}(\pi_L \Theta_i) \) and \( V(p)y^n_i \geq v^n_i \) for all \( i \in I \) and all \( n \in \mathbb{N} \). By Lemma 4, \( \sum_{i \in I} y^n_i \in \sum_{i \in I} (\Theta_i \cap \{ V(p) \geq v^n_i \}) \). Write \( \sum_{i \in I} y^n_i = \sum_{i \in I} \theta^n_i \) with \( \theta^n_i \in \Theta_i \) and \( V(p)\theta^n_i \geq v^n_i \) for all \( i \in I \) and all \( n \in \mathbb{N} \). Hence \( (v^n_1, \ldots, v^n_i, \sum_{i \in I} y^n_i) = (v^n_1, \ldots, v^n_i, \sum_{i \in I} \theta^n_i) \in \mathcal{G}_{\mathcal{F}}(p) \). Since \( \mathcal{F} \) is closed, we have \( (v_1, \ldots, v_i, y) \in \mathcal{G}_{\mathcal{F}}(p) \), hence \( y = \sum_{i \in I} \theta_i \) with \( \theta_i \in \Theta_i \) and \( V(p)\theta_i \geq v_i \) for all \( i \in I \). Now, since \( y \in \text{cl}(\sum_{i \in I} \text{cl}(\pi_L \Theta_i)) \subseteq L^\perp = \text{Im} \pi_L \), we have \( y = \pi_L y = \sum_{i \in I} \pi_L \theta_i \). Moreover, since ker \( \pi_L = L \subseteq \mathcal{U}_\mathcal{F} \subseteq \{ V(p) = 0 \} \), we have \( V(p)\pi_L \theta_i = V(p)\theta_i \geq v_i \). Therefore, \( y = \sum_{i \in I} \pi_L \theta_i \in \sum_{i \in I} (\pi_L \Theta_i \cap \{ V(p) \geq v_i \}) \subseteq \sum_{i \in I} (\text{cl}(\pi_L \Theta_i) \cap \{ V(p) \geq v_i \}) \), that is, \( (v_1, \ldots, v_i, y) \in \mathcal{G}_{\mathcal{F}_L}(p) \).

4.2.2 Proof of part (b) of Theorem 2

- First we show that for all \( p \in \mathbb{R}^L \),
  \[
  \pi_L(A_{\mathcal{F}}(p)) = A_{\mathcal{F}_L}(p),
  \]
  (4.3)
  We claim that
  \[
  \pi_L(A_{\mathcal{F}}(p)) \subseteq \pi_L \sum_{i \in I} (\Theta_i \cap \{ V(p) \geq 0 \}) \subseteq \sum_{i \in I} (\pi_L \Theta_i \cap \{ V(p) \geq 0 \}) \subseteq A_{\mathcal{F}_L}(p).
  \]
  Indeed, the first inclusion follows from the fact that if \( \zeta \in A_{\mathcal{F}}(p) \) then \( \zeta = \lim_k \lambda^k \theta^k \) for some \( \lambda^k \downarrow 0 \), \( (\theta^k)_k \subseteq \sum_{i \in I} (\Theta_i \cap \{ V(p) \geq 0 \}) \); hence, \( \pi_L \zeta = \pi_L (\lim_k \lambda^k \theta^k) = \lim_k \lambda^k \pi_L (\theta^k) \in \pi_L \sum_{i \in I} (\Theta_i \cap \{ V(p) \geq 0 \}) \). The second inclusion comes from (4.2), and the last inclusion is immediate.

  We now prove the converse inclusion \( A_{\mathcal{F}_L}(p) \subseteq \pi(A_{\mathcal{F}}(p)) \). From Lemma 4, taking \( v_i = 0 \) (\( i \in I \)) and then the asymptotic cones of both sides of the inclusion, we get \( A_{\mathcal{F}_L}(p) \subseteq A_{\mathcal{F}}(p) \). Therefore, \( \pi_L(A_{\mathcal{F}_L}(p)) \subseteq \pi_L(A_{\mathcal{F}}(p)) \). Since that \( A_{\mathcal{F}_L}(p) \subseteq \text{Im} \pi_L \), we conclude that \( A_{\mathcal{F}_L}(p) \subseteq \pi_L(A_{\mathcal{F}}(p)) \).

- Second, we show that \( \pi_L(\mathcal{U}_\mathcal{F}) \subseteq \mathcal{U}_{\mathcal{F}_L} \subseteq \mathcal{U}_\mathcal{F} \cap \mathcal{L} \subseteq \pi_L(\mathcal{U}_\mathcal{F}) \), which entails that the inclusions are in fact equalities proving \( \mathcal{U}_{\mathcal{F}_L}(p) = \mathcal{U}_\mathcal{F} \cap \mathcal{L} \) and that \( \mathcal{F}_L \) satisfies price-invariance.

- We begin by proving \( \pi_L(\mathcal{U}_\mathcal{F}) \subseteq \mathcal{U}_{\mathcal{F}_L}(p) \). From Eq. (4.3), we conclude that
  \[
  \pi_L(\mathcal{U}_\mathcal{F}) = \pi_L(A_{\mathcal{F}}(p) \cap -A_{\mathcal{F}}(p)) \subseteq \pi_L(A_{\mathcal{F}}(p)) \cap -\pi_L(A_{\mathcal{F}}(p)) = A_{\mathcal{F}_L}(p) \cap -A_{\mathcal{F}_L}(p) = \mathcal{U}_{\mathcal{F}_L}(p).
  \]

- We turn now to the proof of \( \mathcal{U}_{\mathcal{F}_L}(p) \subseteq \mathcal{U}_\mathcal{F} \cap \mathcal{L} \). Clearly \( \mathcal{U}_{\mathcal{F}_L}(p) \subseteq \text{Im} \pi_L = \mathcal{L} \) and it remains to show \( \mathcal{U}_{\mathcal{F}_L}(p) \subseteq \mathcal{U}_\mathcal{F} \). Again, from Lemma 4, taking \( v_i = 0 \) (\( i \in I \)) and then the asymptotic cones of both sides of the inclusion and taking into account that \( \mathcal{F} \) satisfies price-invariance, we get \( A_{\mathcal{F}_L}(p) \subseteq A_{\mathcal{F}}(p) \), and therefore, \( \mathcal{U}_{\mathcal{F}_L}(p) = A_{\mathcal{F}_L}(p) \cap -A_{\mathcal{F}_L}(p) \subseteq A_{\mathcal{F}}(p) \cap -A_{\mathcal{F}}(p) = \mathcal{U}_\mathcal{F} \).
The equality $\mathcal{U}_F \cap L^\perp = \pi_L(\mathcal{U}_F)$ obviously holds since $L$ is a linear subspace of $\mathcal{U}_F$.

Third, since $\mathcal{U}_{F_L} = \mathcal{U}_F \cap L^\perp$, one has $\dim \mathcal{U}_{F_L} = \dim (\mathcal{U}_F \cap L^\perp) = \dim \mathcal{U}_F - \dim L$ (the last equality holds since $L$ is a linear subspace of $\mathcal{U}_F$).

Finally, since $\pi_L(\mathcal{U}_F) = \mathcal{U}_{F_L}$, one has $\mathcal{U}_{F_L} = \{0\}$ if and only if $\mathcal{U}_F \subseteq \ker \pi_L = L$. Since $L \subseteq \mathcal{U}_F$ (by assumption), we conclude that $\mathcal{U}_{F_L} = \{0\}$ if and only if $L = \mathcal{U}_F$. □

4.2.3 Proof of part (c) of Theorem 2

We prepare the proof by the following claim and we recall that $(q, \theta) \in \mathbb{R}^J \times (\prod_i \Theta_i)$ is arbitrage-free at $p$ in $\mathcal{F}$ if for every $i \in I$, there is no $\tilde{\theta}_i \in \Theta_i$ such that $W(p, q, \theta)_i \geq W(p, q)\tilde{\theta}_i$.

Claim 4.1 Let $\mathcal{F}$ be a standard closed financial structure and let $L$ be a linear subspace of $\mathcal{U}_F$. Let $(q, y) \in \mathbb{R}^J \times (\prod_i \Theta_i)$ be arbitrage-free at $p$ in $\mathcal{F}$, and $\sum_{i \in I} y_i = 0$. Then, there exists $\tilde{\theta} \in \prod_{i \in I} \Theta_i$ such that $\sum_{i \in I} \tilde{\theta}_i = 0$, and $W(p, q, \theta)_{\tilde{\theta}_i} = W(p, q, \theta)_{\tilde{\theta}_i}$ for all $i \in I$.

Proof of Claim 4.1. Let $(q, (y_i)_i)$ be arbitrage-free at $p$ in $\mathcal{F}$ and $\sum_{i \in I} y_i = 0$. Then, by Lemma 4,

$$0 = \sum_{i \in I} y_i \in \sum_{i \in I} \left( \left. \left( \Theta_i \cap \{ V(p) \geq V(p) y_i \} \right) \right\} \sum_{i \in I} \left( \left. \left( \Theta_i \cap \{ V(p) \geq V(p) y_i \} \right) \right\} \right) .$$

Hence, $0 = \sum_{i \in I} \tilde{\theta}_i$ for some $\tilde{\theta}_i \in \Theta_i (i \in I)$ such that $V(p, \theta)_{\tilde{\theta}_i} \geq V(p) y_i$ for every $i$. Noticing that $\sum_{i \in I} \tilde{\theta}_i = \sum_{i \in I} y_i = 0$, we conclude that $V(p, \theta)_{\tilde{\theta}_i} = V(p) y_i$ for every $i$. We end the proof by showing that $-q \cdot y_i = -\pi_L q \cdot \tilde{\theta}_i$ for all $i \in I$. Since $\sum_{i \in I} -q \cdot y_i = 0 = \sum_{i \in I} -\pi_L q \cdot \tilde{\theta}_i$, it suffices to show that $-\pi_L q \cdot \tilde{\theta}_i \leq -q \cdot y_i$ for all $i \in I$. If it is not true, there exists $i_o \in I$, $-\pi_L q \cdot \tilde{\theta}_i > -q \cdot y_i$. Recalling that $V(p, \theta)_{\tilde{\theta}_i} = V(p) y_i$, and using (4.2), we get $W(p, q, \pi_L \tilde{\theta})_{\tilde{\theta}} > W(p, q, \theta)_{\tilde{\theta}}$, which contradicts that $(q, (y_i)_i)$ is arbitrage-free in $\mathcal{F}_L$. □

First, we show that for every standard economy $\mathcal{E}$, for every equilibrium $(\tilde{p}, \tilde{x}, \tilde{q}, \tilde{y})$ of $\mathcal{E}$, there exists $\tilde{\theta} \in \prod_i \Theta_i$ such that $(\tilde{p}, \tilde{x}, \pi_L \tilde{q}, \tilde{y})$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$.

Let $(\tilde{p}, \tilde{x}, \tilde{q}, \tilde{y})$ be an equilibrium of $(\mathcal{E}, \mathcal{F}_L)$, then $(\tilde{q}, \tilde{y})$ is arbitrage-free at $\tilde{p}$ in $\mathcal{F}_L$, under Local Non-Satiation (LNS) (see Angeloni and Cornet 2006). From the above Claim 4.1, there exists $\tilde{\theta}_i \in \Theta_i (i \in I)$ such that $W(\tilde{p}, \pi_L \tilde{q} \tilde{\theta})_{\tilde{\theta}} = W(\tilde{p}, \tilde{q}) \tilde{y}_i$ for all $i$, $\sum_{i \in I} \tilde{\theta}_i = 0$. We show that $(\tilde{p}, \tilde{x}, \pi_L \tilde{q}, \tilde{y})$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$. First, we have $(\tilde{x}_i, \tilde{\theta}_i) \in B_i(\tilde{p}, \pi_L \tilde{q}, \mathcal{F})$ since $(\tilde{x}_i, \tilde{y}_i) \in B_i(\tilde{p}, \tilde{q}, \mathcal{F}_L)$ and $W(\tilde{p}, \tilde{q}) \tilde{y}_i = W(\tilde{p}, \pi_L \tilde{q}) \tilde{\theta}_i$ for each $i \in I$.

We complete the proof by showing that $B_i(\tilde{p}, \pi_L \tilde{q}, \mathcal{F}) \cap (P_i(\tilde{x}) \times \Theta_i) = \emptyset$ for all $i \in I$. Suppose it is not true, then there exist $i \in I$ and $(\tilde{x}_i, \tilde{\theta}_i) \in B_i(\tilde{p}, \pi_L \tilde{q}, \mathcal{F}) \cap (P_i(\tilde{x}) \times \Theta_i)$. Consequently, $(\tilde{x}_i, \pi_L \tilde{\theta}_i) \in B_i(\tilde{p}, \tilde{q}, \mathcal{F}_L) \cap (P_i(\tilde{x}) \times \pi_L \Theta_i)$ since $W(\tilde{p}, \tilde{q}) \pi_L \theta_i = W(\tilde{p}, \pi_L \tilde{q}) \tilde{\theta}_i$ by (4.2). This contradicts the fact that $(\tilde{p}, \tilde{x}, \tilde{q}, \tilde{y})$ is an equilibrium of $(\mathcal{E}, \mathcal{F}_L)$.
Second, we show that for every standard economy \( \mathcal{E} \), if \((\bar{p}, \bar{x}, \bar{q}, \bar{\theta})\) is an equilibrium of \((\mathcal{E}, \mathcal{F})\), then \((\bar{p}, \bar{x}, \bar{q}, \pi_L \bar{\theta})\) is an equilibrium of \((\mathcal{E}, \mathcal{F}_L)\).

We will use the following two lemmas whose proofs are given in Sects. 5.3 and 5.4, respectively.

**Lemma 5** Let \( \mathcal{F} = (V, (\Theta_i)_i) \) be a standard closed financial structure. If \((\bar{q}, \bar{\theta})\) is arbitrage-free at \( \bar{p} \) in \( \mathcal{F} \), and \( \sum_{i \in I} \bar{\theta}_i = 0 \) then \( \bar{q} \in \mathcal{U}_F^+ \).

**Lemma 6** Assume that \( e_i \in \text{int} X_i \) and \( \bar{p}(s) \neq 0 \) for all \( s \in \bar{S} \), then

\[
B_i(\bar{p}, \bar{q}, \mathcal{F}_L) = \text{cl}\{(x_i, y_i) \in X_i \times \pi_L \Theta_i : \bar{p} \square (x_i - e_i) \ll W(\bar{p}, \bar{q})y_i\}.
\]

We have \( \sum_{i \in I} \pi_L \bar{\theta}_i = 0 \) as a direct consequence of \( \sum_{i \in I} \bar{\theta}_i = 0 \), the asset market clearing condition in \((\mathcal{E}, \mathcal{F})\).

We show that \((\bar{x}_i, \pi_L \bar{\theta}_i) \in B_i(\bar{p}, \bar{q}, \mathcal{F}_L)\) for all \( i \in I \). Since \((\bar{x}_i, \bar{\theta}_i) \in B_i(\bar{p}, \bar{q}, \mathcal{F})\), it suffices to show that \( W(\bar{p}, \bar{q}) \bar{\theta}_i = W(\bar{p}, \bar{q}) \pi_L \bar{\theta}_i \). By (4.2) we have \( V(\bar{p}) \bar{\theta}_i = V(\bar{p}) \pi_L \bar{\theta}_i \) and it remains to show that \(-\bar{q} \cdot \bar{\theta}_i = -\bar{q} \cdot \pi_L \bar{\theta}_i \). Since \((\bar{p}, \bar{x}, \bar{q}, \bar{\theta})\) is an equilibrium of \((\mathcal{E}, \mathcal{F})\), \((\bar{q}, \bar{\theta})\) is arbitrage-free at \( \bar{p} \) in \( \mathcal{F} \) (see Angeloni and Cornet 2006) and \( \sum_{i \in I} \bar{\theta}_i = 0 \). Hence, by Lemma 5, we have \(-\bar{q} \cdot \bar{\theta}_i = -\bar{q} \cdot \pi_L \bar{\theta}_i = 0 \) (since clearly, \( \bar{\theta}_i - \pi_L \bar{\theta}_i \in \ker \pi_L = L \subseteq \mathcal{U}_F \) and from the lemma \( \bar{q} \in \mathcal{U}_F^+ \)). Therefore \(-\bar{q} \cdot \bar{\theta}_i = -\bar{q} \cdot \pi_L \bar{\theta}_i \).

We now show that for each \( i \in I \), \((\bar{x}_i, \pi_L \bar{\theta}_i)\) solves consumer \( i \)'s problem in \((\mathcal{E}, \mathcal{F}_L)\). Suppose on the contrary that there exist \( i \) and \((x_i, \theta_i) \in B_i(\bar{p}, \bar{q}, \mathcal{F}_L), x_i \in P_i(\bar{x}) \). Recall that by LNS one has \( \bar{p}(s) \neq 0 \) for all \( s \in \bar{S} \). From Lemma 6, \((x_i, \theta_i) = \lim_n (x^n_i, \pi_L \theta^n_i) \subseteq X_i \times \Theta_i \) such that

\[
\bar{p} \square (x^n_i - e_i) \ll W(\bar{p}, \bar{q}) (\pi_L \theta^n_i) \ll 0.
\]

We have \( W(\bar{p}, \bar{q}) (\pi_L \theta^n_i) = W(\bar{p}, \bar{q}) \theta^n_i \) (from (4.2) and Lemma 5, as proved above). Consequently,

\[
\bar{p} \square (x^n_i - e_i) \ll W(\bar{p}, \bar{q}) (\pi_L \theta^n_i) = \bar{p} \square (x^n_i - e_i) \ll W(\bar{p}, \bar{q}) (\pi_L \theta^n_i) \ll 0.
\]

Thus \((x^n_i, \theta^n_i) \in B_i(\bar{p}, \bar{q}, \mathcal{F})\). Recalling that \( x_i \in P_i(\bar{x}), x_i = \lim_n x^n_i \) and using the fact that \( P_i(\bar{x}) \) is open (by Assumption C2), we deduce that for \( n \) large enough \( x^n_i \in P_i(\bar{x}) \). Together, the two assertions \((x^n_i, \theta^n_i) \in B_i(\bar{p}, \bar{q}, \mathcal{F})\) and \( x^n_i \in P_i(\bar{x}) \) contradict the fact that \((\bar{x}_i, \bar{\theta}_i)\) solves consumer \( i \)'s problem in \((\mathcal{E}, \mathcal{F})\).

\[\square\]

**4.2.4 Proof of part (d) of Theorem 2**

We will show successively that

(i) \( \mathcal{Q}_{\mathcal{F}_L}(p) \cap \Theta_{\mathcal{F}_L} \subseteq \mathcal{Q}_{\mathcal{F}_L}(p) \cap \text{Im} \pi_L \subseteq \mathcal{Q}_{\mathcal{F}_L}(p) \), and

(ii) \( \Theta_{\mathcal{F}_L} \subseteq \Theta_{\mathcal{F}} \).

The first inclusion of (i) is a consequence of the fact that \( \Theta_{\mathcal{F}_L} \subseteq \text{Im} \pi_L \). We prove the second inclusion of (i) by contradiction. Assume that there is some \( q \in \]
4.3 Proof of Theorem 3

We will use the following two lemmas. The proof of Lemma 7 is immediate, and the proof of Lemma 8 is given in Sect. 5.5.

Lemma 7 Let $E$ and $F$ be two topological spaces, $g : E \rightarrow F$ a continuous map, and $B \subseteq E$. Then, $\text{cl} g(B) = \text{cl} g(\text{cl} B)$.

Lemma 8 Consider vector spaces $L_1, \ldots, L_n$ such that $L_k \subseteq (L_1 + \cdots + L_{k-1})^\perp$ for all $k \leq n$. Then,

$$\text{proj}_{L_n^\perp} \circ \cdots \circ \text{proj}_{L_1^\perp} = \text{proj}_{(L_1 + \cdots + L_n)^\perp}.$$

We now come back to the proof of Theorem 3.

Proof of (i)-(iv). By induction. First, it is true for $n = 1$. Indeed, since $\mathcal{F}_0 = \mathcal{F}$ is standard closed and satisfies price-invariance, and $L_1$ is a linear subspace of $\mathcal{U}_{\mathcal{F}_0}$, Theorem 2 applies. Therefore

(i) $\mathcal{F}_1 = (\mathcal{F}_0)_{L_1} = \mathcal{F}_{L_1}$ is standard closed and satisfies price-invariance by Theorem 2.

(ii) $\mathcal{F} \sim \mathcal{F}_{L_1}$ by Theorem 2, hence $\mathcal{F} \sim \mathcal{F}_1$ (since $\mathcal{F}_1 = \mathcal{F}_{L_1}$).

(iii) By Theorem 2, $\mathcal{U}_{\mathcal{F}_1} = \mathcal{U}_{\mathcal{F}} \cap L_1$, implying $\mathcal{U}_{\mathcal{F}_1} \subseteq \mathcal{U}_{\mathcal{F}} = \mathcal{U}_{\mathcal{F}_0}$ (since $\mathcal{F}_0 = \mathcal{F}$).

(iv) $\mathcal{F}_1 = (\mathcal{F}_0)_{L_1} = \mathcal{F}_{L_1}$.

Second, assume that properties (i)–(iv) of Theorem 3 hold true for $n - 1$ ($n \geq 2$). We will prove that they hold true for $n$. Since $\mathcal{F}_{n-1}$ is standard, closed, and satisfies price-invariance (by the induction hypothesis), and $L_n$ is a linear subspace of $\mathcal{U}_{\mathcal{F}_{n-1}}$, Theorem 2 applies. Hence,

(i) $\mathcal{F}_n = (\mathcal{F}_{n-1})_{L_n}$ is standard and, by Theorem 2, is closed and satisfies price-invariance.

(ii) By Theorem 2, $\mathcal{F}_n \sim \mathcal{F}_{n-1}$, and by the induction hypothesis $\mathcal{F} \sim \mathcal{F}_{n-1}$. Hence, $\mathcal{F} \sim \mathcal{F}_n$.

(iii) By Theorem 2, $\mathcal{U}_{\mathcal{F}_n} = \mathcal{U}_{\mathcal{F}_{n-1}} \cap L_n^\perp$, implying $\mathcal{U}_{\mathcal{F}_n} \subseteq \mathcal{U}_{\mathcal{F}_{n-1}}$. \textcopyright Springer
Denote by $\Theta_{i,n}$ consumer $i$'s portfolio set in the financial structure $\mathcal{F}_n$ (with $\Theta_{i,0} = \Theta_i$). Since $\mathcal{F}_n = (\mathcal{F}_{n-1})_{L_n}$, we have

$$\Theta_{i,n} = \text{clproj}_{L_n} \Theta_{i,n-1}.$$ 

On the other hand, from the induction hypothesis $\mathcal{F}_{n-1} = \mathcal{F}_{L_1 + \cdots + L_{n-1}}$, we get

$$\Theta_{i,n-1} = \text{clproj}_{(L_1 + \cdots + L_{n-1})} \Theta_i.$$ 

Therefore, we successively have:

$$\Theta_{i,n} = \text{clproj}_{L_n} \Theta_{i,n-1} = \text{clproj}_{L_n} \text{clproj}_{(L_1 + \cdots + L_{n-1})} \Theta_i = \text{clproj}_{L_n} \circ \text{proj}_{(L_1 + \cdots + L_{n-1})} \Theta_i = \text{clproj}_{(L_1 + \cdots + L_n)} \Theta_i.$$ 

The third equality follows from Lemma 7 given that projections are continuous, and the last equality is a consequence of Lemma 8 given that $L_n \subseteq \mathcal{U}_{\mathcal{F}_{n-1}} = \mathcal{U}_{\mathcal{F}_{L_1 + \cdots + L_{n-1}}} = (L_1 + \cdots + L_{n-1})^\perp$, where the first inclusion holds by construction, the equality holds by the induction hypothesis, and the last inclusion follows from Part (a) of Theorem 2. Recalling that the payoff matrix is the same for all the $\mathcal{F}_n$'s we conclude that $\mathcal{F}_n = \mathcal{F}_{L_1 + \cdots + L_n}$.

For the proof of (v)−(vi) given below, we additionally assume that $L_n \neq \{0\}$ if $\mathcal{U}_{\mathcal{F}_{n-1}} \neq \{0\}$.

Proof of (v). By property (iii) of Theorem 3, $\mathcal{U}_{\mathcal{F}_n} = \mathcal{U}_{\mathcal{F}_{n-1}} \cap L_n^\perp$. Given that $L_n \subseteq \mathcal{U}_{\mathcal{F}_{n-1}}$, we conclude that $\dim \mathcal{U}_{\mathcal{F}_n} = \dim \mathcal{U}_{\mathcal{F}_{n-1}} - \dim L_n$ (for $n \geq 1$). Since $\dim L_n > 0$ whenever $\dim \mathcal{U}_{\mathcal{F}_{n-1}} > 0$, it follows that $\dim \mathcal{U}_{\mathcal{F}_n} < \dim \mathcal{U}_{\mathcal{F}_{n-1}}$ whenever $\dim \mathcal{U}_{\mathcal{F}_{n-1}} > 0$. Hence, there exists an integer $N$ such that $\dim \mathcal{U}_{\mathcal{F}_N} = 0$. By property (iv) of Theorem 3, $\mathcal{F}_n = \mathcal{F}_{L_1 + \cdots + L_n}$ for $n \geq 1$. Part (a) of Theorem 2 then yields $\mathcal{U}_{\mathcal{F}_n} = \mathcal{U}_{\mathcal{F}} \cap (L_1 + \cdots + L_n)^\perp$, which in turn implies $\dim \mathcal{U}_{\mathcal{F}} = \dim L_1 + \cdots + \dim L_n + \dim \mathcal{U}_{\mathcal{F}_n}$ for $n \geq 1$. Therefore, $\dim \mathcal{U}_{\mathcal{F}_N} = 0$ implies that $N \leq \dim \mathcal{U}_{\mathcal{F}}$, which in turn implies $\mathcal{U}_{\mathcal{F}_n} = \{0\} = L_{n+1}$ for $n \geq \dim \mathcal{U}_{\mathcal{F}}$. Hence, $\mathcal{U}_{\mathcal{F}} = L_1 + \cdots + L_n$ for $n \geq \dim \mathcal{U}_{\mathcal{F}}$ (follows immediately from the last point in Part (a) of Theorem 2).

Proof of (vi). Combining properties (iv) and (v) of Theorem 3 when $n \geq \dim \mathcal{U}_{\mathcal{F}} \geq 1$ we get $\mathcal{F}_n = \mathcal{F}_{L_1 + \cdots + L_n} = \mathcal{U}_{\mathcal{F}_n} = \mathcal{U}_{\mathcal{F}} = \mathcal{F}_\ast$. \qed

4.4 Proof of Theorem 4

We prepare the proof by a lemma whose proof is given in Sect. 5.6.
Lemma 9 Let $\mathcal{F}$ be a standard financial structure satisfying strong price-invariance. Then, $\mathcal{U}_W^\mathcal{F} \subseteq \mathcal{U}_\mathcal{F} \subseteq \{V = 0\}$. We now give the proof of Theorem 4.

Proof of Part (a) of Theorem 4. Since $\mathcal{F}$ is standard closed and satisfies strong price-invariance, and $\mathcal{U}_W^\mathcal{F} \subseteq \mathcal{U}_\mathcal{F}$ (by Lemma 9), the equivalence $\mathcal{F} \sim \mathcal{F}_w$ follows immediately from Part (c) of Theorem 2. By definition of $\mathcal{F}_w$, we have $\mathcal{U}_W^\mathcal{F}_w = \{0\}$. Finally, Example 2 provides a financial structure $\mathcal{F}$ such that $\mathcal{F}_w = \mathcal{F}$ (because $\mathcal{U}_W^\mathcal{F}_w = \{0\}$) and $\mathcal{U}_\mathcal{F}_w \neq \{0\}$, hence $\mathcal{F} \neq \mathcal{F}_w$ (since, by Corollary 1, $\mathcal{U}_\mathcal{F}_w = \{0\}$), and therefore $\mathcal{F}_w \neq \mathcal{F}_w$. □

Proof of Part (b) of Theorem 4. Clearly, the equality $\mathcal{F}_w = \mathcal{F}_w$ follows from $\mathcal{U}_W^\mathcal{F} = \mathcal{U}_\mathcal{F}$. Indeed, $\mathcal{F}_w = \mathcal{F}_W^\mathcal{F}_w = \mathcal{F}_\mathcal{F}_w = \mathcal{F}_w$. It remains to show that $\mathcal{U}_W^\mathcal{F} = \mathcal{U}_\mathcal{F}$ under WNMA. If $\mathcal{F}$ satisfies WNMA, then it follows from (Rockafellar 1997, Theorem 9.1 page 73) (applied to $f : (\mathbb{R}^J)^I \to \mathbb{R}^J$, $(\theta_i)_{i \in I} \mapsto \sum_{i \in I} \theta_i$ and $C = \prod_{i \in I}(\theta_i \cap \{V \geq 0\})$) that $A_\mathcal{F} = \sum_{i \in I}(A\theta_i \cap \{V \geq 0\})$, hence

$$A_\mathcal{F} \cap -A_\mathcal{F} = \sum_{i \in I}(A\theta_i \cap \{V \geq 0\}) \cap - \sum_{i \in I}(A\theta_i \cap \{V \geq 0\}).$$

To conclude that $\mathcal{U}_\mathcal{F} = \mathcal{U}_W^\mathcal{F}$, i.e., $A_\mathcal{F} \cap -A_\mathcal{F} = \sum_{i \in I}(A\theta_i \cap \{V \geq 0\}) \cap - \sum_{i \in I}(A\theta_i \cap \{V \geq 0\})$, it suffices now to prove that (it is clear that the RHS is a subset of the LHS in the below equation)

$$\sum_{i \in I}(A\theta_i \cap \{V \geq 0\}) \cap - \sum_{i \in I}(A\theta_i \cap \{V \geq 0\}) \subseteq \sum_{i \in I} A\theta_i \cap -A\theta_i \cap \ker V.$$

Let $\zeta \in \sum_{i \in I}(A\theta_i \cap \{V \geq 0\}) \cap - \sum_{i \in I}(A\theta_i \cap \{V \geq 0\})$. Write

$$\zeta = \sum_{i \in I} \zeta_i = \sum_{i \in I} \zeta_i', \text{ with } \zeta_i, \zeta_i' \in A\theta_i \cap \{V \geq 0\} \text{ for all } i \in I.$$

Then, $0 = \sum_{i \in I}(\zeta_i + \zeta_i')$ and for each $i \in I$, $\zeta_i + \zeta_i' \in A\theta_i \cap \{V \geq 0\}$ which implies (under WNMA) that for every $i$, $\zeta_i + \zeta_i' \in A\theta_i \cap -A\theta_i \cap \ker V$. Hence

$$\zeta_i = -\zeta_i' + (\zeta_i + \zeta_i') \in -(A\theta_i \cap \{V \geq 0\}) + [A\theta_i \cap -A\theta_i \cap \ker V] \subseteq -(A\theta_i \cap \{V \geq 0\}).$$

Therefore, for every $i$,

$$\zeta_i \in (A\theta_i \cap \{V \geq 0\}) \cap - (A\theta_i \cap \{V \geq 0\}) \subseteq A\theta_i \cap -A\theta_i \cap \ker V.$$

Hence, $\zeta \in \sum_{i \in I} \zeta_i \in \sum_{i \in I} A\theta_i \cap -A\theta_i \cap \ker V$. □
5 Appendix

For \( p \in \mathbb{R}^L \), we let

\[
\mathcal{V}(p) := \{ v = (v_i)_{i \in I} \in (\mathbb{R}^S)^I : \Theta_i \cap \{ V(p) \geq v_i \} \neq \emptyset \ \text{for all} \ i \in I \}.
\]

The following result will be used to establish Claim 5.1 (needed for the proof of Lemma 4) and Lemma 5. We refer the reader to (Aouani and Cornet 2011, Claim A.1) for a proof.

**Proposition 3** Let \( \mathcal{F} \) be standard and closed, and let \( p \in \mathbb{R}^L \) be given. Then, \( A_{\mathcal{F}}(p, v) := A(\sum_{i \in I} (\Theta_i \cap \{ V(p) \geq v_i \})) = A_{\mathcal{F}}(p) \) for all \( v \in \mathcal{V}(p) \).

### 5.1 Proof of Lemma 2

Clearly,

\[
A_{\mathcal{F}}(v) = \bigcup_{p \in B_L(0,1)} (A_{\mathcal{F}}(p, v) \times \{ p \}) \subseteq A_{\mathcal{F}}(B_L(0,1), v) \times B_L(0,1).
\]

Hence, \( B \Rightarrow [B(p)] \) for all \( p \in B_L(0,1) \) follows from the above equality. To prove that \( [B(p)] \) for all \( p \in B_L(0,1) \) \( \Rightarrow B \), we show that the correspondence \( A_{\mathcal{F}}(\cdot, v) \) has bounded image and we conclude using the above inclusion. For that it suffices to show that the correspondence

\[
A_{\mathcal{F}}(\cdot, v) : B_L(0,1) \rightarrow (\mathbb{R}^J)^I, p \mapsto A_{\mathcal{F}}(p, v)
\]

is upper semicontinuous (because \( B_L(0,1) \) is metric compact, \((\mathbb{R}^J)^I \) is metric, and \( A_{\mathcal{F}}(\cdot, v) \) has compact values by Property \( B \), upper semicontinuity of \( A_{\mathcal{F}}(\cdot, v) \) would then imply that the image \( A_{\mathcal{F}}(B_L(0,1), v) \) is compact).

Clearly, \( A_{\mathcal{F}}(\cdot, v) \) is closed (i.e., has a closed graph) so to prove that it is upper semicontinuous, we prove that it is locally bounded, that is,

\[
\forall p_o \in B_L(0,1), \exists \epsilon, \exists K \text{ compact} \subseteq (\mathbb{R}^J)^I, \forall p \in \text{int} B_L(p_o, \epsilon), A_{\mathcal{F}}(p, v) \subseteq K.
\]

Assume, by contradiction, that there exist \( p_o, (p^n)_n \) in \( B_L(0,1) \), \( (\theta^n)_n \) in \( \mathbb{R}^J \), such that

\[
p_n \overset{n \rightarrow \infty}{\longrightarrow} p_o, \ \theta^n = (\theta^n_i)_{i \in I} \in A_{\mathcal{F}}(p^n, v), \text{ and } ||\theta^n||_{n \rightarrow \infty} \rightarrow +\infty.
\]

Passing to a subsequence if necessary, we can assume that, for each \( i \), the bounded sequence \( (\theta^n_i / \sum_{k \in I} ||\theta^n_k||)_n \) converges to some \( \zeta_i \). For every \( i \), the vector \( \zeta_i \) belongs to \( A \Theta_i \) since \( \theta^n_i \in \Theta_i \) for every \( n \) and \( 1 / \sum_{k \in I} ||\theta^n_k|| \overset{n \rightarrow \infty}{\longrightarrow} 0 \). Moreover, \( \sum_{i \in I} \zeta_i = 0 \) and \( \sum_{i \in I} ||\zeta_i|| = 1 \) hence \( \xi := (\zeta_i)_{i \in I} \neq 0 \). Furthermore, for every \( n \), we have \( V(p^n)(\theta^n_i / \sum_{k \in I} ||\theta^n_k||) \geq v_i / \sum_{k \in I} ||\theta^n_k|| \). Passing to the limit, we obtain \( V(p_o)\zeta_i \geq 0 \).
(using the fact that the sequence \((\theta^n_i / \sum_{k \in I} ||\theta^n_k||)\) is bounded and that \(V\) is continuous by \(F0\). Now, let \(\theta \in A_{\mathcal{F}}(p_0, v)\), then \(\theta + t \zeta \in A_{\mathcal{F}}(p_0, v)\) for every \(t > 0\), which yields a contradiction to \(B\) (since \(\zeta \neq 0\)). Therefore \(A_{\mathcal{F}}(\cdot, v)\) is upper semicontinuous on its domain. \(\square\)

5.2 Proof of Lemma 4

We prepare the proof with a claim.

Claim 5.1 Let \(\mathcal{F} = (V, (\Theta_i)_{i})\) be standard closed and let \(L\) be a linear subspace of \(\mathcal{U}_{\mathcal{F}}\). One has

\[
\sum_{i \in I} (\pi_L \Theta_i \cap \{V(p) \geq v_i\}) \subseteq \sum_{i \in I} (\Theta_i \cap \{V(p) \geq v_i\}), \quad \forall p \in \mathbb{R}^L, \forall (v_i)_{i \in I} \in (\mathbb{R}^S)^I.
\]

Proof of Claim 5.1. Assume that \(\sum_{i \in I} (\pi_L \Theta_i \cap \{V(p) \geq v_i\}) \neq \emptyset\), otherwise the proof is immediate. We show successively that

\[
\sum_{i \in I} (\pi_L \Theta_i \cap \{V(p) \geq v_i\}) \subseteq \sum_{i \in I} \pi_L (\Theta_i \cap \{V(p) \geq v_i\}) \subseteq \ker \pi_L + \sum_{i \in I} (\Theta_i \cap \{V(p) \geq v_i\}) \subseteq \sum_{i \in I} (\Theta_i \cap \{V(p) \geq v_i\}).
\]

To prove (5.1), it suffices to notice that \(\pi_L \Theta_i \cap \{V(p) \geq v_i\} \subseteq \pi_L (\Theta_i \cap \{V(p) \geq v_i\})\) for all \(i \in I\); indeed, let \(y_i \in \pi_L \Theta_i \cap \{V(p) \geq v_i\}\), then \(y_i = \pi_L \theta_i\) for some \(\theta_i \in \Theta_i\), and \(V(p) y_i \geq v_i\). But \(V(p) \theta_i = V(p) (\pi_L \theta_i) = V(p) y_i\), by (4.2). Therefore \(\theta_i \in \Theta_i \cap \{V(p) \geq v_i\}\) and \(y_i = \pi_L \theta_i \in \pi_L (\Theta_i \cap \{V(p) \geq v_i\})\).

To prove (5.2), let \(y = \sum_{i \in I} \pi_L \theta_i\) with \(\theta_i \in \Theta_i \cap \{V(p) \geq v_i\}\). Then, \(y = \pi_L \theta = (\pi_L \theta - \theta) + \theta\) with \(\pi_L \theta - \theta \in \ker \pi_L\) and \(\theta = \sum_{i \in I} \theta_i \in \sum_{i \in I} (\Theta_i \cap \{V(p) \geq v_i\})\). This ends the proof of (5.2).

The last inclusion (5.3) comes from the fact that

\[
\ker \pi_L = L \subseteq \mathcal{U}_{\mathcal{F}} \subseteq \mathcal{A}_{\mathcal{F}} = A \left( \sum_{i \in I} (\Theta_i \cap \{V(p) \geq v_i\}) \right),
\]

where the first equality holds by definition of \(\pi_L\), the first inclusion holds by assumption, the second inclusion is immediate, and the second equality holds by Proposition 3 since \(\sum_{i \in I} (\Theta_i \cap \{V(p) \geq v_i\}) \neq \emptyset\) (by (5.1), (5.2), and the assumption that \(\sum_{i \in I} (\pi_L \Theta_i \cap \{V(p) \geq v_i\}) \neq \emptyset\)). Consequently,
Proof of Lemma 4. If the left-hand side of the inclusion is empty, then the result is trivial. Otherwise, let \( y_i \in (\text{cl} \pi_L \Theta_i) \cap \{ V(p) \geq v_i \} (i \in I) \). Take \( v^n_i \uparrow v_i \) such that \( v_i \gg v^n_i \) for every \( n \). Pick \( \bar{y}_i \in r_i \pi_L \Theta_i \) and consider \( y^n_i = (1 - \lambda^n)y_i + \lambda^n \bar{y}_i \) with \( 0 < \lambda^n < \frac{1}{n} \) small enough so that \( V(p)y^n_i \gg v^n_i \). Then, \( y^n_i \in [\bar{y}_i, y_i) \subseteq r_i \pi_L \Theta_i \) since \( y_i \in \text{cl} \pi_L \Theta_i \) and \( \bar{y}_i \in r_i \pi_L \Theta_i \) [Theorem 6.1 page 45 in Rockafellar (1997)]. Thus \( y^n_i \in \pi_L \Theta_i \cap \{ V(p) \geq v^n_i \} \) and, by Claim 5.1,

\[
\sum_{i \in I} y^n_i \in \sum_{i \in I} (\Theta_i \cap \{ V(p) \geq v^n_i \}) \quad \text{hence} \quad (v^n_1, \ldots, v^n_i, \sum_{i \in I} y^n_i) \in G'_F(p).
\]

Since \( F \) is closed, the set \( G'_F(p) \) is closed (by Lemma 3). Thus

\[
\left( v_1, \ldots, v_I, \sum_{i \in I} y_i \right) = \lim_n \left( v^n_1, \ldots, v^n_i, \sum_{i \in I} y^n_i \right) \in \text{cl} G'_F(p) = G'_F(p),
\]

and \( \sum_{i \in I} y_i \in \sum_{i \in I} (\Theta_i \cap \{ V(p) \geq v_i \}) \). \( \square \)

5.3 Proof of Lemma 5

We show that\(^{17}\) \( \bar{q} \in -(A_F)^o \) and we then deduce the desired result from the fact that \( -(A_F)^o \subseteq (\mathcal{U}_F)^\perp \) since \( \mathcal{U}_F \subseteq A_F \) and \( A_F \) is a linear space.

We prove that \( \bar{q} \in -(A_F)^o \) by contradiction. Indeed, if it is not true, then there exists \( \zeta \in A\sum_{i \in I} (\Theta_i \cap \{ V(\bar{p}) \geq 0 \}) \) such that \( -\bar{q} \cdot \zeta > 0 \). We have \( \sum_{i \in I} (\Theta_i \cap \{ V(\bar{p}) \geq 0 \}) \neq \emptyset \) since \( 0 = \sum_{i \in I} \bar{\theta}_i \in \sum_{i \in I} (\Theta_i \cap \{ V(\bar{p}) \geq 0 \}) \).

Hence, by Proposition 3, \( \zeta \in A\sum_{i \in I} (\Theta_i \cap \{ V(\bar{p}) \geq V(\bar{p})\bar{\theta}_i \}) \). Again, because \( 0 = \sum_{i \in I} \bar{\theta}_i \in \sum_{i \in I} (\Theta_i \cap \{ V(\bar{p}) \geq V(\bar{p})\bar{\theta}_i \}) \), we have \( A(\sum_{i \in I} \Theta_i \cap \{ V(\bar{p}) \geq V(\bar{p})\bar{\theta}_i \}) \subseteq \sum_{i \in I} \Theta_i \cap \{ V(\bar{p}) \geq V(\bar{p})\bar{\theta}_i \} \), and thus \( \zeta \in \sum_{i \in I} \Theta_i \cap \{ V(\bar{p}) \geq V(\bar{p})\bar{\theta}_i \} \). Write \( \zeta = \sum_{i \in I} \bar{\theta}_i \) with \( \bar{\theta}_i \in \Theta_i \) and \( V(\bar{p})\bar{\theta}_i \geq V(\bar{p})\bar{\theta}_i \) for all \( i \). Now, since \( -\bar{q} \cdot \zeta > 0 \) then \( -\bar{q} \cdot \sum_{i \in I} \bar{\theta}_i > 0 \) or \( -\bar{q} \cdot \sum_{i \in I} \bar{\theta}_i > 0 \), which implies that there exists a consumer \( i_o \) such that \( -\bar{q} \cdot \bar{\theta}_{i_o} > -\bar{q} \cdot \bar{\theta}_{i_o} \). Recalling that \( V(\bar{p})\bar{\theta}_{i_o} \geq V(\bar{p})\bar{\theta}_{i_o} \), we see that \((\bar{q}, \bar{\theta}_{i_o}) \) is not arbitrage-free at \( \bar{p} \); a contradiction. This ends the proof of the lemma. \( \square \)

\(^{17}\) If \( A \subseteq \mathbb{R}^J \), we denote by \( A^o := \{ q \in \mathbb{R}^J : q \cdot a \leq 0 \text{ for all } a \in A \} \) the negative polar of \( A \).
5.4 Proof of Lemma 6

We first choose \( \delta = (\delta(s))_{s \in \mathcal{S}} \in \mathbb{R}^L \) such that (i) \( e_i - \delta \in X_i \) and (ii) \( \bar{p}(s) \cdot \delta(s) > 0 \) for every \( s \in \mathcal{S} \); indeed, take \( \delta = \lambda \bar{p} \) for \( \lambda > 0 \) small enough so that \( e_i - \delta \in X_i \) (since \( e_i \in \text{int}X_i \)) and \( \bar{p}(s) \cdot \delta(s) = \lambda \bar{p}(s) \cdot \bar{p}(s) > 0 \), since \( \bar{p}(s) \neq 0 \) for all \( s \in \mathcal{S} \).

Let \((x_i, y_i) \in B_i(\bar{p}, \bar{q}, \mathcal{F}_L)\). Let \( \alpha \in (0, 1) \). Then, \( x_i^\alpha := \alpha x_i + (1 - \alpha)(e_i - \delta) \in X_i \) since \( x_i \in X_i \), \( e_i - \delta \in X_i \) and \( X_i \) is convex, and \( \alpha y_i \in \text{cl} \pi_L \Theta_i \) since \( 0 \in \text{cl} \pi_L \Theta_i \), \( y_i \in \text{cl} \pi_L \Theta_i \), and \( \text{cl} \pi_L \Theta_i \) is convex. We claim that,

\[
\bar{p} \diamond (x_i^\alpha - e_i) - W(\bar{p}, \bar{q})(\alpha y_i) \ll 0.
\]

Indeed, \( \bar{p} \diamond (x_i^\alpha - e_i) - W(\bar{p}, \bar{q})(\alpha y_i) = \alpha \left( \bar{p} \diamond (x_i - e_i) - W(\bar{p}, \bar{q})y_i \right) - (1 - \alpha) p \diamond \delta. \) Since \((x_i, y_i) \in B_i(\bar{p}, \bar{q}, \mathcal{F}_L)\), i.e., \( \bar{p} \diamond (x_i - e_i) - W(\bar{p}, \bar{q})y_i \leq 0 \), and \( \alpha > 0 \), the first term is non-positive. Since \( \bar{p} \diamond \delta \gg 0 \) (from above) and \( \alpha < 1 \), the second term satisfies \(-(1 - \alpha)\bar{p} \diamond \delta \ll 0 \). This ends the proof of the claim.

Consequently, there exists \( y_i^\alpha \in \pi_L \Theta_i \) such that \( ||y_i^\alpha - y_i|| \leq (1 - \alpha)||y_i|| \) and

\[
\bar{p} \diamond (x_i^\alpha - e_i) - W(\bar{p}, \bar{q})y_i^\alpha \ll 0.
\]

Noticing that, \((x_i^\alpha, y_i^\alpha) \to (x_i, y_i)\) when \( \alpha \to 1 \), we get the desired result. \( \square \)

5.5 Proof of Lemma 8

We prove, by induction, that

\[
\text{proj}_{L_k^\perp} \circ \cdots \circ \text{proj}_{L_1^\perp} = \text{proj}_{(L_1 + \cdots + L_k)^\perp} \quad \text{for all } k \leq n.
\]

First, this is true for \( k = 1 \). Second, we prove it for \( k = 2 \). Indeed let \( x \in X \), then \( x = b_1 + a_1, b_1 \in L_1, a_1 \in L_1^\perp \), and \( \text{proj}_{L_1^\perp} x = a_1 = b_2 + a_2 \) with \( b_2 \in L_2, a_2 \in L_2^\perp \). So \( \text{proj}_{L_2^\perp} \circ \text{proj}_{L_1^\perp} x = a_2. \) But \( x = b_1 + a_1 = b_1 + b_2 + a_2 \) with \( b_1 + b_2 \in L_1 + L_2 \) and \( a_2 = a_1 - b_2 \in L_1^\perp \cap (L_1^\perp - L_2) \subseteq L_2^\perp \cap L_1^\perp = (L_1 + L_2)^\perp. \) This proves that \( a_2 = \text{proj}_{(L_1 + L_2)^\perp} x. \)

Third, assume now that it is true for \( i \leq k \). We prove it is true for \( k + 1 \). Indeed, from the induction hypothesis and the fact that it is true for \( 2, \)

\[
\text{proj}_{L_{k+1}^\perp} \circ \left[ \text{proj}_{L_k^\perp} \circ \cdots \circ \text{proj}_{L_1^\perp} \right] = \text{proj}_{L_k^\perp} \circ \text{proj}_{(L_1 + \cdots + L_k)^\perp}.
\]

Hence \( \text{proj}_{L_{k+1}^\perp} \circ \text{proj}_{(L_1 + \cdots + L_k)^\perp} = \text{proj}_{(L_1 + \cdots + L_k + L_{k+1})^\perp} \) since \( L_{k+1} \subseteq (L_1 + \cdots + L_k)^\perp. \) \( \square \)
5.6 Proof of Lemma 9

We first prove that $U_W \subseteq U_F$. Since $0 \in \Theta_i$ for all $i$, we have $\Theta_k \cap \{V \geq 0\} \subseteq \sum_{i \in I}(\Theta_i \cap \{V \geq 0\})$, for every $k \in I$. Thus $A(\Theta_k \cap \{V \geq 0\}) \subseteq A\sum_{i \in I}(\Theta_i \cap \{V \geq 0\})$. Given that $A(\Theta_k \cap \{V \geq 0\}) = A\Theta_k \cap \{V \geq 0\}$ we get $A\Theta_k \cap -A\Theta_k \cap \{V = 0\} \subseteq A_F \cap -A_F = U_F$. Since $U_F$ is a linear space, the inclusion $U_W = \sum_{i \in I}A\Theta_i \cap -A\Theta_i \cap \{V = 0\} \subseteq U_F$ follows.

Finally, we prove $U_F \subseteq \{V = 0\}$. Clearly, $\sum_{i \in I}(\Theta_i \cap \{V \geq 0\}) \subseteq \{V \geq 0\}$. Hence $A\sum_{i \in I}(\Theta_i \cap \{V \geq 0\}) \subseteq A\{V \geq 0\} = \{V \geq 0\}$. Therefore, $U_F \subseteq \{V \geq 0\} \cap -\{V \geq 0\} = \{V = 0\}$. $\square$

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18 See for example Debreu (1959), pp 21-24, for elementary properties of asymptotic cones.