Quantum no-go theorems in causality respecting systems in the presence of closed timelike curves: Tweaking the Deutsch condition

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Abstract – We consider causality respecting (CR) quantum systems interacting with closed timelike curves (CTCs), within the Deutsch model. We introduce the concepts of popping-up and elimination of quantum information and use them to show that no-cloning and no-deleting, which are true in CR quantum systems, are no more valid in the same that are interacting with CTCs. We also find limits on the possibility of creation of entanglement between a CR system and a CTC, and the same between two CR systems in the presence of a CTC. We prove that teleportation of quantum information, even in its approximate version, from a CR region to a CTC is disallowed. Interestingly, we find that tweaking the Deutsch model, by allowing the input and output not to be the same, leads to a nontrivial approximate teleportation beyond the classical limit.

Introduction. – No-go theorems play an important role in quantum information science [1]. They are crucial both in nonclassical applications of quantum states and operations as resources, and towards a better understanding of quantum concepts. A good example is the no-cloning theorem [2–4], which states that nonorthogonal quantum states cannot be cloned perfectly, and can be seen as an underlying feature of the security of quantum cryptography [5]. Other examples of no-go theorems can be found in refs. [6–12].

Recently, there have been some exciting developments in quantum information theory in the presence of CTCs. It is known that the general theory of relativity does allow the existence of a CTC, a worldline which connects back on itself [13–22] (see [17,21] however). An important question is whether one can formulate a consistent theory of quantum mechanics in the presence of CTCs. Such a formalism was developed by Deutsch who proposed a model of quantum theory in the presence of CTCs [23] (cf. [24–30]). From the perspective of quantum computation, it has, e.g., been shown that access to a CTC would allow factorization of composite numbers efficiently with the help of a classical computer [31], a CTC-assisted quantum computer would be able to solve NP-complete problems [32], both classical and quantum computers under CTC belong to the same complexity class [33], etc. On the other hand, Brun et al. [34] have shown that if one has access to CTCs, then one can perfectly distinguish nonorthogonal quantum states, having possible implications for the security of quantum cryptography [5]. In the presence of CTCs, information flow of quantum states [35], purifications of mixed states [36], and nonlocal boxes [37] have also been addressed. Bennett et al. [38] have argued that the implications obtained by assuming the existence of CTCs need to be revisited when we assume that the input state in any protocol is a mixture of the possible inputs.

We begin with the Deutsch formalism [23]. It involves a unitary interaction, \( U \), between a quantum system in a CR region with another system that has a worldline in a region where CTCs exist. It is assumed that the states of these systems are density matrices in standard quantum mechanics. The combined state of the CR and CTC systems before the interaction is a product state, \( \rho_{\text{CR}} \otimes \rho_{\text{CTC}} \).
The unitary transformation on the joint system changes the composite state as
\[ \rho_{CR} \otimes \rho_{CTC} \rightarrow U(\rho_{CR} \otimes \rho_{CTC})U^\dagger. \] (1)

For self-consistency, the Deutsch model requires
\[ \text{Tr}_{CTC}(U \rho_{CR} \otimes \rho_{CTC} U^\dagger) = \rho_{CTC}, \] (2)

the “Deutsch self-consistency condition”. Therefore the Deutsch formalism demands that the state of the CTC system before and after the evolution be the same. Mathematically, the state of the CTC system is a fixed point solution of the self-consistency equation. In case there are multiple fixed points, one chooses their maximum-entropy solution. On the other hand, the final state of the CR system, which does not have such restriction imposed on it, is given by
\[ \text{Tr}_{CTC}(U \rho_{CR} \otimes \rho_{CTC} U^\dagger) = \rho'_{CR}. \] (3)

Equations (1), (2), and (3) form the basic equations that provide the entire dynamics of the composite CTC and CR systems. It is important to note that \( \rho'_{CR} \) depends nonlinearly on \( \rho_{CR} \). Thus, it is evident that the output of the CR system is a nonlinear function of the input CR system density matrix [39].

In this work, we uncover two exotic features namely the “popping-up” and “elimination” of quantum information in the presence of CTCs. We show that the information present in the CR system “pops up” in the CTC qubit when it is allowed to interact with the CR system. We also show that it is possible to destroy or completely “eliminate” the information of the state of the CR system in the presence of CTCs. Using these twin features, we prove the violations of two fundamental theorems of quantum mechanics, namely, the no-cloning [2–4, 40–45] and the no-deleting [8, 46, 47] theorems. Note that cloning of an arbitrary quantum state of a CR system when in contact with a CTC is already known [23, 48–50]. We provide an independent proof of the same, which, we believe, gives a fresh perspective on the mechanism of the phenomenon of cloning. While cloning and deleting of quantum systems become possible in CR systems when interactions with CTCs are allowed, teleportation [51], a phenomenon that plays an important role in several aspects in the quantum mechanics of CR systems, becomes impossible. We first show the limits within which entanglement can be generated between a CR system and a CTC, and subsequently prove that teleportation is always disallowed. In contrast to, e.g., the no-cloning theorem in quantum theory of CR systems, the “no teleportation theorem” here does not allow even an approximate teleportation. We, however, show that “tweaking” the Deutsch criterion that allows deviations of the final states from their corresponding initial states, can result in nonclassical teleportation fidelities.

**Popping-up and elimination of quantum information.** – There exists a notion of “permanence” of quantum information in the sense that information in a quantum world can neither be created nor destroyed [52]. In this section, we show that in the presence of a CTC, the information present in a CR system “pops up” in the CTC, and on the other hand, the information of a state present in a CR system can be completely destroyed (“eliminated”). These results suggest that quantum information in the presence of CTCs does not respect permanence of information.

**Popping-up of quantum information.** Let \( |\psi\rangle_{CR} \) be an arbitrary state of the CR system and \( \rho_{CTC} \) be the state of the CTC. Let us suppose that they interact via the unitary, \( U_{SWAP} \), the swap operation, defined, for the bi-orthonormal product basis, \( \{|i\rangle_{CR}|j\rangle_{CTC}\}_{i,j} \) as \( U_{SWAP}|i\rangle_{CR}|j\rangle_{CTC} = |j\rangle_{CR}|i\rangle_{CTC}. \) We will often denote \( |\psi\rangle_{CR} \) as \( \psi \). See fig. 1(a) for the circuit diagram of the popping-up of quantum information. Hence, we have
\[ \psi_{CR} \otimes \rho_{CTC} \rightarrow \psi_{CR} \otimes \psi_{CTC}. \] (5)

Now, if we trace out the CR system, the final state of the CTC is \( |\psi\rangle_{CTC} = |\psi\rangle_{CTC} \). By the consistency condition of Deutsch, this should be same as the original state of the CTC, i.e., \( \rho_{CTC} = |\psi\rangle_{CTC} \). This equation is now a mathematical relation on the space of operators on the CTC Hilbert space. As a mathematical relation, it remains valid in the CR system also. We therefore have that the joint CR-CTC system evolves as
\[ \psi_{CR} \otimes \rho_{CTC} \rightarrow \psi_{CR} \otimes \psi_{CTC}, \]

implying that the information in the CR system is “popping up” in the CTC system. What is more, considering \( \rho_{CTC} = |\psi\rangle_{CTC} \) to be a valid mathematical relation after the swap operation has been applied, we can also say that to start with we had the CR and the CTC systems in the state \( |\psi\rangle_{CR} \otimes |\psi\rangle \). Since this holds for an arbitrary input state of the CR system, the CTC has knowledge about all possible states in the CR system. Thus the mere possibility that a CTC will be interacted with a CR system can result in the popping-up of the quantum information of the CR system in the CTC. Note that the popping-up happens even before the interaction is implemented. Hence a CTC has knowledge about the past and future of all CR quantum systems in the universe, provided we assume that there exists a CTC that can be interacted with the relevant CR system with the swap operation.

**Remark 0.1.** Note that the conclusion that \( \rho \) and \( \psi \) are equal is a result of the Deutsch consistency equation, and our choice of SWAP as the interaction between the CR system and CTC. Also, the above discussion of ours is for the situation when we choose SWAP as our interaction unitary. The discussion is not valid in general. If another unitary is chosen, the conclusion could be different. Also
note that, while the discussions in the preceding paragraph were when the CR system is in a pure state, the same line of reasoning goes through if the CR system is in a mixed state, which in turn can be part of an entangled state [53].

**Elimination of quantum information.** Just like the creation of quantum information, CTCs can assist in the complete destruction of quantum information in a CR system. However, unlike in popping-up, here we require two CR systems $|\psi\rangle_{CR(1)} \otimes |0\rangle_{CR(2)}$ and a CTC system $\rho_{CTC}$, which interact with the unitary $U_{SWAP}$. Note that $|0\rangle$ is a fixed state. We denote $|\psi\rangle_{CR}$ as $|\psi_{CR}\rangle$, and so on. See fig. 1(b) for the circuit diagram of elimination of quantum information. Applying $U_{SWAP}$ between CR(2) and CTC, we have

$$\psi_{CR(1)} \otimes 0_{CR(2)} \otimes \rho_{CTC} \rightarrow \psi_{CR(1)} \otimes \rho_{CR(2)} \otimes 0_{CTC}. \quad (6)$$

Applying the Deutsch self-consistency condition, we get $\rho = 0$. Therefore, after the action of the swap operation and the application of the Deutsch condition, we have

$$\psi_{CR(1)} \otimes 0_{CR(2)} \otimes \rho_{CTC} \rightarrow \psi_{CR(1)} \otimes 0_{CR(2)} \otimes 0_{CTC}. \quad (7)$$

We now apply $U_{SWAP}$ between CR(1) and CTC to obtain

$$\psi_{CR(1)} \otimes 0_{CR(2)} \otimes 0_{CTC} \rightarrow 0_{CR(1)} \otimes 0_{CR(2)} \otimes 0_{CTC}. \quad (8)$$

If we are allowed to apply the Deutsch condition once more (see footnote 1 for our viewpoint), we obtain $0 = \psi$, so that the final state of the three-party system is

$$0_{CR(1)} \otimes 0_{CR(2)} \otimes 0_{CTC}. \quad (9)$$

1The paper of Deutsch [23] does not rule out this belief. Rather to the contrary, it seems that this is a natural way to use the Deutsch-CTC evolution. The Deutsch formalism of CTCs does not define the “point-of-exit” of a time traveler from such a curve. Therefore, the Deutsch self-consistency condition should be applicable also for intermediate times during the evolution. A typical scenario can be the following. An observer, Ayesha, applies the joint unitary $U_{A}$ on the joint CR-CTC system. If she wishes to know the output, she must apply the Deutsch consistency equation. (Indeed, the output should not depend on whether or not she “wishes” to know the output.) It is not yet known whether another observer will have her/his hands on the joint system. However, it so happens that another observer, Badal, gets hold of the joint systems, and promptly applies the joint unitary $U_{B}$ on the joint CR-CTC system. If he too wishes to know the output, he needs to apply the Deutsch equation again. There are certainly other options to proceed. In one, Ayesha needs to ascertain that no other observer is going to apply a joint unitary on the CR-CTC system before she considers the Deutsch condition. In another, after Badal gets hold of the joint system, he needs to know whether Ayesha had applied the Deutsch condition beforehand, and whether later observers (after Badal) will apply any joint unitary. We believe that this situation is too artificial for any evolution. It is difficult to ascertain when “all” interactions have been performed, and why a future observer should agree to such a restriction. Even if we decide today that all operations are complete (“we” being the experimenters), and if we are allowed to take this decision, we can decide tomorrow that it is not. If this change of decision is not allowed, that would be an extra restriction on the Deutsch evolution rules. We have therefore taken the avenue that intermediate application of the Deutsch criterion is allowed. Please also look at the Remarks 1.1, 2.1 and 2.2 made in the main text, where we clearly state the exact form in which the Deutsch condition is applied in the corresponding proofs.

**Cloning and deleting of an arbitrary quantum state in the presence of CTCs.** – Cloning and deleting of arbitrary quantum states are not possible in the quantum world [2–4,8,40–47]. In this section, we use the phenomena of popping-up and elimination of quantum information to prove that it is always possible to clone and delete quantum states in presence of CTCs. The possibility of cloning of quantum states in the presence of a CTC is known [23,48–50], and ref. [49], in particular, uses a sequence of unitaries including the swap operation to achieve the same. Here we present an independent proof that reveals an interesting way in which cloning can take place in a world with CTCs. It is to be noted here that the phenomena of popping-up or elimination by themselves are not cloning or deleting (respectively), as we want the entire phenomena of cloning or deleting to occur only in CR systems.

**Theorem 1 (CTC-assisted cloning).** If $|\psi\rangle_{CR(1)}$ is an arbitrary quantum state in a CR region, then there exists a map $|\psi\rangle_{CR(1)} \rightarrow |\psi\rangle_{CR(1)}|\psi\rangle_{CR(2)}$ from one CR region to another, in the presence of a CTC satisfying the Deutsch condition for the swap operator.

**Proof.** Consider two CR systems, one in an arbitrary quantum state, $|\psi\rangle_{CR(1)}$, (the state to be cloned), and another in a fixed “blank” quantum state, $|B\rangle_{CR(2)}$. Consider also a CTC in the state $\rho_{CTC}$. The CTC system can be thought of as the “machine”. We will apply two swap gates to prove the possibility of cloning in this model. Let

![Fig. 1: (Colour online) Circuit diagrams of (a) popping-up and (b) elimination, of quantum information. The broken lines with arrow in opposite direction depicts the dynamics of a CTC, satisfying the Deutsch self-consistency condition eq. (2). Here SWAP is the required unitary operation.](image-url)
us begin with the first, which is applied on the systems CR(1) and CTC, whence

\[ \psi_{\text{CR}(1)} \otimes B_{\text{CR}(2)} \otimes \rho_{\text{CTC}} \]
\[ t_{\text{SWAP}}^{\text{CR(1)-CTC}} \quad \rho_{\text{CR}(1)} \otimes B_{\text{CR}(2)} \otimes \psi_{\text{CTC}}. \quad (10) \]

Applying the Deutsch condition, we have \( \rho = \psi \) as a mathematical relation which can be applied to the CR system, so that the state of the tripartite system becomes \( \psi_{\text{CR}(1)} \otimes B_{\text{CR}(2)} \otimes \psi_{\text{CTC}} \). This is the phenomenon of popping-up between the systems CR(1) and CTC. We now apply the second swap operation on the systems CR(2) and CTC, whereby

\[ \psi_{\text{CR}(1)} \otimes B_{\text{CR}(2)} \otimes \psi_{\text{CTC}} \]
\[ t_{\text{SWAP}}^{\text{CR(2)-CTC}} \quad \psi_{\text{CR}(1)} \otimes \psi_{\text{CR}(2)} \otimes B_{\text{CTC}}. \quad (11) \]

\[ \square \]

**Remark 1.1.** After the first swap operation in the proposed cloning process, the CTC is in a fixed state \( \psi \) (which is the initial state of the system whose state is to be cloned). At this point, there are at least two options to take. We may assume that when we apply the swap operation for the second time, we do not apply the Deutsch consistency condition. In that case, the right-hand side of eq. (11) will be \( \psi \otimes \psi \otimes B \), and this is what it is now. However, we can choose a second option. We can assume that the Deutsch condition has to be applied every time we consider a dynamics including a CTC. In that case, applying the Deutsch condition after we act with the swap operator for the second time, we obtain \( \psi = B \). Thereby, the right-hand side of eq. (11) could be any one of the eight possible combinations from the set \( \{ \psi \otimes \psi \otimes \psi, \psi \otimes \psi \otimes B, \ldots, B \otimes B \otimes B \} \). We choose \( \psi \otimes \psi \otimes B \), as this achieves our goal of cloning.

In the case of deleting, we are given two copies of an arbitrary state \( |\psi\rangle \), and we ask if there can be a physical (quantum) operation in a closed system which can take \( |\psi\rangle|\psi\rangle|A\rangle \rightarrow |\psi\rangle|0\rangle|A'\rangle \), where \( |0\rangle, |A\rangle, \) and \( |A'\rangle \) are fixed states independent of \( |\psi\rangle \). The no-deleting theorem says that this operation is impossible in the quantum world, where the third subsystem consists of the entire universe outside the first two subsystems. However, it follows from the discussion above on the elimination of quantum information that in the presence of CTC, deleting is possible simply by elimination of quantum information in the second copy of \( \psi \), while the first copy stands idle. More precisely, we have the following result.

**Theorem 2 (CTC-assisted deleting).** In a closed CR quantum system, deleting is possible provided it is allowed to interact with a CTC system satisfying the Deutsch condition for the swap operator.

**Proof.** We begin with two copies of \( |\psi\rangle \) and a single copy of \( |0\rangle \) in CR regions, and CTC in the state \( \rho \). We denote \( |\psi\rangle_{\text{CR}}(|\psi\rangle \) as \( \psi_{\text{CR}} \), and so on. We again assume that the Deutsch self-consistency condition can be called more than once. The proof can be gathered in the following lines:

\[ \psi_{\text{CR}(1)} \otimes \psi_{\text{CR}(2)} \otimes \rho_{\text{CTC}} \]
\[ t_{\text{SWAP}}^{\text{CR(1)-CTC}} \quad \psi_{\text{CR}(1)} \otimes \psi_{\text{CR}(2)} \otimes \rho_{\text{CTC}} \]
\[ \rho = 0 \]
\[ \psi_{\text{CR}(1)} \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \]
\[ t_{\text{SWAP}}^{\text{CR(2)-CTC}} \quad \psi_{\text{CR}(1)} \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \]
\[ \rho = 0 \]
\[ \psi_{\text{CR}(1)} \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \] (12)

where the equalities follow due to the Deutsch self-consistency condition. Therefore, we have been able to take the CR(2) system to a standard state, independent of its input state \( |\psi\rangle \), while still keeping the state of the CR(1) system at its pre-interaction state. Furthermore, the CTC system is finally in the state \( |0\rangle \), which is independent of \( |\psi\rangle \), so that there is no information that is leaked out from the CR systems, CR(1) and CR(2), into the CTC.

\[ \square \]

**Remark 2.1.** More specifically, the discussion on elimination of quantum information implies that if \( |\psi\rangle \) and \( |0\rangle \) are respectively arbitrary and fixed quantum states, then there exists a map \( |\psi\rangle_{\text{CR}(1)}|\psi\rangle_{\text{CR}(2)} \rightarrow |\psi\rangle_{\text{CR}(1)}|0\rangle_{\text{CR}(2)} \) of two CR regions, in presence of a CTC satisfying the Deutsch condition for the swap operator and an additional CR system initially in the state \( |0\rangle \). The action can be considered to be closed as the final state of the CTC system, the additional CR system, or any other system does not retain any information of the initial states of the CR systems, CR(1) and CR(2). This is important as deleting must be considered in a system from which information is not leaking into the environment \([8,54,55]\).

**Remark 2.2.** We note here that both in the case of cloning and deleting, we require to apply the swap operator twice. However, while we call the Deutsch self-consistency condition once for cloning, it has to be used twice for deleting.

**Impossibility of teleporting a quantum state from CR to CTC.**

Creating entanglement between CR and CTC. Before considering the question of teleportation between CR and CTC worlds, let us briefly consider whether entanglement can be created between these two worlds, a question addressed before in ref. [36]. Suppose that we wish to create the state \( \rho_{\text{CR}(1),\text{CTC}} \), which may be entangled, between a CR system, CR(1), and a CTC. For this purpose, we consider two CR systems and a CTC to be initially in the state \( \rho_{\text{CR}(1),\text{CTC}} \). Applying the swap operator, \( t_{\text{SWAP}}^{\text{CR(2)-CTC}} \), we will be able to have the state \( \rho_{\text{CR}(1),\text{CTC}} \), irrespective of whether the bipartite state \( \rho \) is entangled or separable. Note that for the Deutsch condition to be satisfied for the evolution considered, one must...
have \( \hat{\rho}_{\text{CTC}} = \text{Tr}_{\text{CR}(1)}\rho_{\text{CR}(1),\text{CR}(2)} \). In spite of this possibility, we have the following no-go theorems.

**Theorem 3 (No Entanglement between CR and CTC).** A CR system and a CTC cannot get correlated, neither classically nor quantumly, if the CTC is allowed to interact with another CR system in a fixed pure state, and if the CTC follows the Deutsch condition for the swap operator.

**Proof.** Suppose that we are given a CR system and a CTC in the state \( \rho_{\text{CR}(1),\text{CTC}} \) and another CR system in the pure state \( |0\rangle_{\text{CR}(2)}|0\rangle \) is allowed to interact with CTC by the swap operator. Post-interaction, we apply the Deutsch condition. This implies that the local state of CTC must be (both before and after the interaction) in the pure state \( |0\rangle \). Thus, \( \rho_{\text{CR}(1),\text{CTC}} \) must have already been in a product state, precluding any correlation between the CR system, \( \text{CR}(1) \), and the CTC. \( \square \)

**Theorem 4 (No entanglement between two CRs).** Two CR systems, \( \text{CR}(1) \) and \( \text{CR}(2) \), cannot get correlated, neither classically nor quantumly, if a) a third CR system in a fixed pure state is allowed to interact with a CTC via the swap operator, and if b) after the operation in a), either \( \text{CR}(1) \) or \( \text{CR}(2) \) is allowed to interact with the CTC via the swap operator, and if we are allowed to apply the Deutsch condition for the second swap operation.

**Proof.** Suppose that initially two CR systems are in the state \( \rho_{\text{CR}(1),\text{CR}(2)} \), a CTC system is in the state \( \hat{\rho}_{\text{CTC}} \), and a third CR system is in the state \( |0\rangle_{\text{CR}(3)}|0\rangle \). Now the CTC system is allowed to interact with the swap operator with \( \text{CR}(3) \), followed by with \( \text{CR}(2) \). Post-interaction, if we apply the Deutsch condition, we get \( |0\rangle_{\text{CTC}}|0\rangle = \text{Tr}_{\text{CR}(1)}\rho_{\text{CR}(1),\text{CTC}} \). This implies that there is no correlation between two CR systems, \( \text{CR}(1) \) and \( \text{CR}(2) \). \( \square \)

**Remark 4.1.** While we refer to the above theorems (Theorems 3 and 4) as the “no entanglement theorems”, they are in reality “no correlation theorems”, as neither classical nor quantum correlations are allowed in the situations described. We call them as the “no entanglement theorems”, since we are using them in the context of the no teleportation theorem below. We would also like to mention that the popping-up of information in a CR-CTC system, if it were possible in a situation where the dynamics is linear, would create classical correlations between the CR system and the CTC. Because, in such a (hypothetical) case, one can use a mixture of two states, \( |\psi\rangle \) and \( |\phi\rangle \) (say), as the input for the CR system, and the output of the joint CR-CTC system will then be in a classically correlated state (mixture of \( |\psi\rangle|\psi\rangle \) and \( |\phi\rangle|\phi\rangle \)). Note that we have assumed that the two copies of \( |\phi\rangle \) (or \( |\psi\rangle \)) are created by the nonlinear evolution via the Deutsch condition, while the mixing is done linearly. The evolution here is however not linear, and hence the proof of the implication is not valid. By this argument, we do not, however, claim that nonlinearity implies no correlations.

**Remark 4.2.** We have discussed a method of creating entanglement between a CR system and a CTC, above Theorem 3. Theorem 3 states that entanglement between a CR system and a CTC can exist only if the premises of this theorem are not satisfied. Similarly, Theorem 4 states that the entanglement between the two CR systems can exist only if the premises of the theorem are not satisfied.

No teleportation of quantum states from CR to CTC. Assuming that creation of an entangled state between a CR system and a CTC is possible, a next question is whether the same can be put to use. In particular, can we use the entangled state to teleport quantum information from the CR system to the CTC. The following result on teleportation is actually independent of whether the shared state is entangled or not.

**Theorem 5 (No teleportation theorem).** It is not possible to teleport, even approximately, an arbitrary quantum state from a CR system to a CTC, if after the teleportation, the CTC is allowed to interact with another CR system in a fixed state via the swap operator, and if the Deutsch condition can be applied for the swap operator.

**Proof.** Consider first the case of exact teleportation. Suppose Alice and Bob share the state \( \rho_{\text{CR}(2),\text{CTC}} \), of a CR system and a CTC. Let us suppose that Alice possesses the state \( \hat{\rho}_{\text{CR}(1)} \) to be teleported. In addition, assume that Bob has another CR system, in a fixed state \( \rho'_{\text{CR}(3)} \), in his possession. Irrespective of whether the shared state between Alice and Bob is entangled or separable, let us assume that it is possible to exactly teleport the arbitrary state of the CR(1) system to the CTC, by using the shared state. That is

\[
\hat{\rho}_{\text{CR}(1)} \otimes \rho_{\text{CR}(2)} \otimes \rho'_{\text{CR}(3)} \xrightarrow{\text{Teleportation}} \hat{\rho}_{\text{CR}(1),\text{CR}(2),\text{CTC}} \otimes \rho'_{\text{CR}(3)}. \tag{13}
\]

If we apply the Deutsch condition after the teleportation has been performed, we obtain

\[
\text{Tr}_{\text{CR}(2)}\rho_{\text{CR}(2),\text{CTC}} = \text{Tr}_{\text{CR}(1),\text{CR}(2)}\hat{\rho}_{\text{CR}(1),\text{CR}(2),\text{CTC}}. \tag{14}
\]

However, we will not be using this equation. Though exact teleportation seems feasible here, it requires that post-teleportation, CTC part of the teleported state must be kept isolated, i.e., it must be prohibited to interact with any CR system in a fixed state because it would then render teleportation impossible, even approximately, as we show now. Let us assume that after teleportation, the CTC is allowed to interact with the CR(3) system via the swap operator, \( U_{\text{SWAP}} \), which yields \( \hat{\rho}_{\text{CR}(1),\text{CR}(2),\text{CTC}} \otimes \rho'_{\text{CTC}} \). If we apply Deutsch condition here, we get

\[
\text{Tr}_{\text{CR}(1),\text{CR}(2)}\hat{\rho}_{\text{CR}(1),\text{CR}(2),\text{CTC}} = \rho'_{\text{CTC}}. \tag{15}
\]

So the state obtained after teleportation is a fixed state. But the success of the teleportation protocol, in the exact case, implies that this fixed state must be equal
to the arbitrary state of CR(1). Therefore, exact teleportation is not possible. Moreover, since the output CTC state, for the teleportation protocol, must be in a fixed state, irrespective of the input at CR(1), we cannot even have an approximate teleportation in the scenario considered.

**Tweaking the Deutsch condition and approximate teleportation.** – The no teleportation theorem, derived in the preceding section, is similar to the no-go theorems of quantum mechanics in CR systems. However, it has a distinct character of its own. In particular, both exact and approximate teleportations are not allowed from CR systems to CTC. This is in sharp contrast to, e.g., the quantum no cloning theorem in CR systems, which disallows exact cloning but allows approximate cloning [40–45]. In a quest for attaining approximate teleportation, here we address the question whether the same can be achieved by some change in the Deutsch condition (cf. [34]). Through this exercise, we may also gain insight about physical phenomena when the Deutsch condition is not stringent. The immediate question, however, is whether a nonstringent Deutsch condition allows us to attain approximate teleportation of an arbitrary state $|\psi\rangle$ in a CR system to a CTC, whereby an approximate copy, $\rho'_\text{CTC}(\psi)$, is created in the CTC, which is as close as possible to the arbitrary state $|\psi\rangle$. We demand that the fidelity of the approximate copy (averaged over the input state space), defined as $F = \int \langle \psi | \rho'_\text{CTC}(\psi) | \psi \rangle \, d\psi$, be higher than the classically achievable fidelity, i.e., the fidelity that is possible to achieve without the use of entangled shared states [56]. For the case of teleportation of a qubit, the classically achievable fidelity is $(2/3)$ in the quantum domain. Let us now introduce a deviation, which is just sufficient for the intended purpose of attaining nonclassical teleportation, of the Deutsch self-consistency condition. To do that, we suppose that the initial and final states of the CTC, respectively denoted by $\rho^f_\text{CTC}$ and $\rho'_\text{CTC}$, are such that

$$\|\rho'_\text{CTC} - \rho^f_\text{CTC}\| \leq \epsilon,$$

for an $\epsilon$ in $[0,1]$. The norm is chosen in such a way that both $\rho^f_\text{CTC} - (1-\epsilon)\rho^f_\text{CTC}$ and $(1+\epsilon)\rho^f_\text{CTC} - \rho^f_\text{CTC}$ are positive semidefinite (see [57,58]). We refer to this as the $\epsilon$-close Deutsch model of CTC, or, for short, $\epsilon$-CTC.

For simplicity, let us suppose that the final state, $\rho^f_\text{CTC}$, of the CTC, after the teleportation protocol has been carried out, is of the form

$$\rho^f_\text{CTC} = (1-\epsilon)\rho^f_\text{CTC} + \epsilon \rho,$$

where $\rho$ is the arbitrary state to be teleported. For $\epsilon = 0$, the state is not teleported at all, and the original Deutsch self-consistency condition is exactly satisfied, while when $\epsilon = 1$, the state is perfectly teleported. Note that in all cases where $\epsilon$ is nonzero, the Deutsch self-consistency is violated. Now, while $\rho^f_\text{CTC} - (1-\epsilon)\rho^f_\text{CTC} = \epsilon \rho \geq 0$, $(1+\epsilon)\rho^f_\text{CTC} - \rho^f_\text{CTC} = \epsilon(2\rho^f_\text{CTC} - \rho)$ is non-negative when $2\rho^f_\text{CTC} - \rho \geq 0$. Thus, $\|\rho'_\text{CTC} - \rho^f_\text{CTC}\| \leq \epsilon$ holds when $2\rho^f_\text{CTC} - \rho_\text{CTC} \geq \rho$. We refer to this relation as the “approximate teleportation condition”. In the special case that we are working in, which is when eq. (17) is valid, the approximate teleportation condition implies that $\rho^f_\text{CTC} = \frac{1}{2} I$, since $\rho$ is an arbitrary state, and where we have additionally assumed that the input state to be teleported is a qubit. Here $I$ is the identity operator on the qubit Hilbert space. Let $\rho = |\psi\rangle \langle \psi|$ be an arbitrary single-qubit pure state, with $|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$, on the Bloch sphere, so that the average fidelity is

$$\int \langle \psi | \rho'_\text{CTC} | \psi \rangle \, d\psi = \frac{1}{2} (1 + \epsilon),$$

which exceeds the classical fidelity of two-thirds, when $\epsilon > 1/3$.

We note here that when ($\epsilon = 0$), the combined CR-CTC system is not quantum mechanical. It may however be expected that for nonzero ($\epsilon$), quantum mechanics will be regained. Our results show that quantum mechanics is at least beyond ($\epsilon = 1/3$).

**Conclusion.** – In summary, we find that cloning and deleting of quantum information are possible in CR quantum systems provided we allow the CR system to interact with a CTC. We show that the phenomena of cloning and deleting are possible in the presence of CTCs due to two other phenomena, viz., popping-up and elimination of quantum information, that seem to be natural to CR systems in the presence of CTCs. We, moreover, prove that teleportation of an arbitrary quantum state, even approximately, from a CR system to a CTC cannot occur, although teleportation between two CR systems is known to be possible. We also find limits to the creation of entangled states between a CR system and a CTC, and between two CRs in the presence of a CTC.

It is interesting to mention here a result by Hardy [59], which shows that it is possible to build a toy model of the physical world in which teleportation is possible and no-cloning is valid, while the system has a local realistic description. It was later shown [60] that if one considers an arbitrary set of commuting quantum states, then any member of the set can be teleported through a separable shared quantum state. The shared quantum state, being separable, has a local hidden variable description, and an arbitrary set of commuting quantum states, since it may contain nonorthogonal quantum states, that cannot be cloned (in CR quantum systems), although can be (quantum mechanically) broadcast [6]. In this paper, we found that a CR system in the presence of a CTC has the opposite behavior: teleportation is disallowed, while cloning is possible.

We subsequently show that a modification in the Deutsch model of CTCs, which we call the $\epsilon$-close Deutsch model, allows for nonclassical teleportation fidelities. We believe that our results show the power as

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well as the limitations of CTC in quantum information processing.

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