Bimetric gravity is cosmologically viable

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Introduction. — The Standard Model of particle physics contains fields with spins 0, 1/2, and 1, describing matter as well as the strong and electroweak forces. General relativity (GR) extends this to the gravitational interactions by introducing a massless spin-2 field. There is theoretical and observational motivation to seek physics beyond the Standard Model and GR. In particular, GR is nonrenormalizable and is associated with the cosmological constant, dark energy, and dark matter problems. To compound the puzzle, the GR-based Λ-cold dark matter (LCDM) model provides a very good fit to observational constraints, including the origin describes nonlinear interactions of the gravitational constant, dark energy, and dark matter problems. To make interacting spin-2 fields especially attractive from a theoretical point of view.

A natural possibility for extending the set of known classical field theories is to include additional spin-2 fields and interactions. While “massive” and “bimetric” theories of gravity have a long history, nonlinear theories of interacting spin-2 fields were found, in general, to suffer from the Boulware-Deser (BD) ghost instability. Recently a particular bimetric theory (or bigravity) has been shown to avoid this ghost instability. This theory describes nonlinear interactions of the gravitational metric with an additional spin-2 field. It is an extension of an earlier ghost-free theory of massive gravity (a massive spin-2 field on a nondynamical flat background) for which the absence of the BD ghost at the nonlinear level was established in Refs.

Including spin-2 interactions modifies GR, inter alia at large distances. Bimetric theory is therefore a candidate to explain the accelerated expansion of the Universe. Indeed, bigravity has been shown to possess Friedmann-Lemaître-Robertson-Walker (FLRW) solutions which can match observations of the cosmic expansion history, even in the absence of vacuum energy.

Linear perturbations around these cosmological backgrounds have also been studied extensively. The epoch of acceleration is set by the mass scale of the spin-2 interactions. Unlike a small vacuum energy, m is protected from large quantum corrections due to an extra diffeomorphism symmetry that is recovered in the limit m → 0, just as fermion masses are protected by chiral symmetry in the Standard Model (see Ref. for an explicit analysis in the massive gravity setup). This makes interacting spin-2 fields especially attractive from a theoretical point of view.

Cosmological solutions lie on one of two branches, called the finite and infinite branches. The infinite-branch models can have sensible backgrounds, but the perturbations have been found to contain ghosts in both the scalar and tensor sectors. Most viable background solutions lie on the finite branch.

1 Stable FLRW solutions do not exist in massive gravity.
2 There is a third branch containing bouncing solutions, but these tend to have pathologies.
While these avoid the aforementioned ghosts, they contain a scalar instability at early times \cite{29,32,33} that invalidates the use of linear perturbation theory and could potentially rule these models out. For parameter values thought to be favored by data, this instability was found to be present until recent times (i.e., a similar time to the onset of cosmic acceleration) and thus seemed to spoil the predictivity of bimetric cosmology.

In this Letter we study a physically well-motivated region in the parameter space of bimetric theory that has been missed in earlier work due to an ubiquitous choice of parameter rescaling. We demonstrate how in this region the instability problem in the finite branch can be resolved while the model still provides late-time acceleration in agreement with observations.

Our search for viable bimetric cosmologies will be guided by the precise agreement of GR with data on all scales, which motivates us to study models of modified gravity which are close to their GR limit. Often this limit is dismayingly trivial; if a theory of modified gravity is meant to produce late-time self-acceleration in the absence of a cosmological constant degenerate with vacuum energy, then we would expect that self-acceleration to disappear as the theory approaches GR. We will see, however, that there exists a GR limit of bimetric gravity which retains its self-acceleration, leading to a GR-like universe with an effective cosmological constant produced purely by the spin-2 interactions.

**Bimetric gravity.**— The ghost-free action for bimetric gravity containing metrics \(g_{\mu\nu}\) and \(f_{\mu\nu}\) is given by \cite{4,13}

\[
S = \int d^4x \left[ -\frac{M_{Pl}^2}{2}\sqrt{g}R(g) - \frac{M_f^2}{2}\sqrt{f}R(f) 
+ m^2 M_{Pl}^2 \sqrt{g}V(X) + \sqrt{g}\mathcal{L}_m(g,\Phi_t) \right].
\]

Here \(M_{Pl}\) and \(M_f\) are the Planck masses for \(g_{\mu\nu}\) and \(f_{\mu\nu}\), respectively, and we will frequently refer to their ratio,

\[
\alpha \equiv \frac{M_f}{M_{Pl}}.
\]

The potential \(V(X)\) is constructed from the elementary symmetric polynomials \(e_n(X)\) of the eigenvalues of the matrix \(X \equiv \sqrt{g^{-1}f}\), defined by

\[
X^{\mu\alpha}_\alpha X^{\nu\beta}_\beta \equiv g^{\mu\alpha}f_{\alpha\nu},
\]

and has the form \cite{8,43,14}

\[
\sqrt{g}V(X) = \sqrt{g}\beta_0 + \sqrt{g} \sum_{n=1}^{3} \beta_n e_n(X) + \sqrt{f}\beta_4.
\]

In the above, \(m\) is a mass scale and \(\beta_n\) are dimensionless interaction parameters. \(\beta_0\) and \(\beta_4\) parameterize the vacuum energies in the two sectors. Guided by the absence of ghosts and the weak equivalence principle, we take the matter sector to be coupled to \(g_{\mu\nu}\). Then the vacuum-energy contributions from the matter sector \(\mathcal{L}_m\) are captured in \(\beta_0\). We can interpret \(g_{\mu\nu}\) as the space-time metric used for measuring distance and time, while \(f_{\mu\nu}\) is an additional symmetric tensor that mixes nontrivially with gravity. As we discuss further below, the two metrics do not correspond to the spin-2 mass eigenstates but each contain both massive and massless components. Even before fitting to observational data, the parameters in the bimetric action are subject to several theoretical constraints. For instance, the squared mass of the massive spin-2 field needs to be positive, it must not violate the Higuchi bound \cite{59,60}, and ghost modes should be absent.

In terms of the Einstein tensor, \(G_{\mu\nu}\), the equations of motion for the two metrics take the form

\[
G_{\mu\nu}(g) + m^2 V^g_{\mu\nu} = \frac{1}{M_{Pl}^2} T_{\mu\nu},
\]

\[
\alpha^2 G_{\mu\nu}(f) + m^2 V^f_{\mu\nu} = 0,
\]

where \(V^g_{\mu\nu}\) are determined by varying the interaction potential, \(V\). Taking the divergence of eq. \cite{5} and using the Bianchi identity leads to the Bianchi constraint,

\[
\nabla^\mu V^g_{\nu\mu} = 0.
\]

The analogous equation for \(f_{\mu\nu}\) carries no additional information due to the general covariance of the action.

Finally, note that the action \cite{10} has a status similar to Proca theory on curved backgrounds. It is therefore expected to require an analogue of the Higgs mechanism, with new degrees of freedom, in order to have improved quantum behavior. The search for a ghost-free Higgs mechanism for gravity is still in progress \cite{61}.

**The GR limit.**— When bimetric gravity is linearized around proportional backgrounds \(f_{\mu\nu} = c^2 g_{\mu\nu}\) with constant \(c^2\)

\[
g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{M_{Pl}} \delta g_{\mu\nu},
\]

\[
f_{\mu\nu} = c^2 \bar{g}_{\mu\nu} + \frac{c}{M_f} \delta f_{\mu\nu},
\]

the canonically-normalized perturbations can be diagonalized into massless modes \(\delta G_{\mu\nu}\) and massive modes \(\delta M_{\mu\nu}\) as \cite{4,62}

\[
\delta G_{\mu\nu} \propto (\delta g_{\mu\nu} + c\alpha \delta f_{\mu\nu}),
\]

\[
\delta M_{\mu\nu} \propto (\delta f_{\mu\nu} - c\alpha \delta g_{\mu\nu}).
\]

\[\text{4 More general matter couplings not constrained by these requirements have been studied in Refs.}\ 29,14,58.\]

\[\text{5 These correspond to Einstein spaces and, for nonvanishing } \alpha, \text{ solve the field equations only in vacuum. A quartic equation determines } c = c(\beta_4, \alpha).\]
Notice that when $\alpha \to 0$ (or $M_{P1} \gg M_f$), the massless state aligns with $\delta g_{\mu\nu}$, i.e., up to normalization,

$$\delta G_{\mu\nu} \to \delta g_{\mu\nu} + \mathcal{O}(\alpha^2).$$

(12)

Because $g_{\mu\nu}$ is the physical metric, this suggests that $\alpha \to 0$ is the general-relativity limit of bigravity. We will see below that the nonlinear field equations indeed reduce to Einstein’s equations for $\alpha = 0$ and that the limit is continuous. Thus $g_{\mu\nu}$ is close to a GR solution for sufficiently small values of $\alpha$. We therefore identify $M_{P1}$ with the measured physical Planck mass whenever $\alpha \ll 1$, holding it fixed while making $M_f$ smaller. Interestingly, in the bimetric setup a large physical Planck mass is correlated with the fact that gravity is approximated well by a massless field. In other words, when bimetric theory is close to GR, the gravitational force is naturally weak.

The GR limit can be directly realized at the nonlinear level [64, 65]. The metric potentials satisfy the identity

$$\sqrt{g}g^{\mu\alpha}V_{\alpha\nu} + \sqrt{f}f^{\mu\alpha}V_{\alpha\nu} = \sqrt{g}V\delta \mu\nu,$$

(13)

where $V$ is the potential in the action [11]. For $M_f = 0$, the $f_{\mu\nu}$ equation [9] gives $V_{\alpha\nu} = 0$, an algebraic constraint on $f_{\mu\nu}$. Then, using the above identity, the $g_{\mu\nu}$ equation [9] becomes

$$G_{\mu\nu}(g) + m^2Vg_{\mu\nu} = \frac{1}{M_{P1}^2}T_{\mu\nu}.$$  

(14)

Since $T_{\mu\nu}$ is conserved, taking the divergence gives

$$\partial_{\mu}V = 0.$$  

(15)

We see that eq. (13) is the Einstein equation for $g_{\mu\nu}$ with cosmological constant $m^2V$. Remarkably, because $V$ depends on $f_{\mu\nu}$ and all the $\beta_n$, this effective cosmological constant is generically not simply the vacuum energy from matter loops (which is parameterized by $\beta_0$). Even in the GR limit, the impact of the spin-2 interactions remains and bigravity’s self-acceleration survives.

It is straightforward to see that, unlike the $m \to 0$ limit, the $\alpha \to 0$ limit is not affected by the van Dam-Veltman-Zakharov (vDVZ) discontinuity [66, 67]. The cause of this discontinuity is the Bianchi constraint [17] which constrains the solutions even when $m = 0$. On the contrary, when $\alpha \to 0$, the Bianchi constraint simply reduces to eq. (15) and is automatically satisfied.

The conditions $V_{\mu\nu} = 0$ and $\partial_{\mu}V = 0$ determine $f_{\mu\nu}$ algebraically in terms of $g_{\mu\nu}$, generically as $f_{\mu\nu} = c^2g_{\mu\nu}$. In the limit $M_f = 0$, the $f$ sector is infinitely strongly coupled [1]. Due to the nontrivial potential, this causes the $f$ metric to exactly follow the $g$ metric (both at the background and perturbative levels), while the $g$ sector remains weakly coupled.

**Strong-coupling scales.**— We now argue that at energy scales relevant to cosmology, this model avoids known strong-coupling issues, sometimes contrary to intuition gained from massive gravity.

There are several strong-coupling scales one might expect to arise. At an energy scale $k$, the $f$ sector has an effective coupling $k/M_f$, as can be seen from expanding the Einstein-Hilbert action in $\delta f_{\mu\nu}/M_f$ just as in GR. Then, for small but nonzero $\alpha$, which is the case of interest here, one might worry that perturbations of $f_{\mu\nu}$ with momentum $k$ become strongly coupled at low scales $k \sim M_f$. However, we have seen that in the limit of infinite strong coupling, $M_f = 0$, $f_{\mu\nu}$ becomes nondynamical and is entirely determined in terms of $g_{\mu\nu}$, while the $g_{\mu\nu}$ equation is degenerate with GR and its perturbations remain weakly coupled. Due to the continuity of the limit, we expect that, for small enough $\alpha$, strong-coupling effects will continue to not affect the $g$ sector, even when perturbations of $f_{\mu\nu}$ are strongly coupled at relatively small energy scales. In practice, however, since the measured value of $M_{P1}$ is very large, even reasonably high values of $M_f$ can still lead to small $\alpha$. In cosmological applications, all observable perturbations satisfy $k/M_f \ll 1$ for $M_f \gg 100H_0 \sim 10^{-31}$ eV, roughly the scale at which linear cosmological perturbation theory breaks down at recent times, so that perturbations of $f_{\mu\nu}$ remain weakly coupled in any case.

Another potentially-problematic scale is associated with the helicity-0 mode of the massive graviton. In massive gravity, this mode becomes strongly coupled at the scale

$$\Lambda_3 \equiv \left(m^2M_{P1}\right)^{1/3},$$

(16)

where $m$ is defined to coincide with the Fierz-Pauli mass [1] on flat backgrounds. This scale is rather small, $\Lambda_3 \sim 10^{-13}$ eV $\sim (1000 \text{ km})^{-1}$ for $m \sim H_0 \sim 10^{-33}$ eV, and severely restricts the applicability of massive gravity [74]. The same scale also appears in the decoupling-limit analysis of bimetric theory [37], where $m$ is now the parameter in front of the potential in the action [11]. In the limit $\alpha \to 0$, the $f$ sector approaches massive gravity [65] and one might worry that the strong-coupling problem persists or becomes worse with the emergence of an even lower scale $(m^2M_f)^{1/3}$. This is not the case. In the bimetric context, the scale defined in eq. (16) is not physical, since $m^2$ is degenerate with the $\beta_n$. The physically relevant strong-coupling scale must be defined with respect to the bimetric Fierz-Pauli mass [62],

$$m_{FP}^2 = m^2\left(\frac{1}{c^2\alpha^2} + 1\right)\left(c\beta_1 + 2c^2\beta_2 + c^3\beta_3\right),$$

(17)

which is only defined around proportional backgrounds, $f_{\mu\nu} = c^2g_{\mu\nu}$. In the massive-gravity limit, $\alpha \to \infty$, the

\footnote{See Ref. [63] for an early discussion of such a limit.}
\footnote{Strongly-coupled gravity in the context of GR has been studied, for instance, in Refs. [68, 71] and has been argued to allow for a simplified quantum-mechanical treatment.}
helicity-0 mode is mostly contained in \( g \) with a strong-coupling scale

\[ \Lambda_3 \equiv \left( \frac{m_{2\text{FP}}^2 M_{\text{Pl}}}{f} \right)^{1/3}, \tag{18} \]

consistent with eq. (14) for appropriately restricted parameters. However, in the GR limit, \( \alpha \to 0 \), the helicity-0 mode resides mostly in \( f \), where the strong-coupling scale is

\[ \hat{\Lambda}_3 \equiv \left( \frac{m_{2\text{FP}}^2 M_{\text{Pl}}}{f} \right)^{1/3} \to \left( \frac{m_{2\text{FP}}^2 M_{\text{Pl}}}{\alpha} \mathcal{O}(\beta_n) \right)^{1/3}, \tag{19} \]

which is no longer small. Note that for solutions that admit this limit, \( \beta \) becomes independent of \( \alpha \). We can also consider the \( \alpha \to 0 \) limit of eq. (18), to verify that the small part of the helicity-0 mode in \( g \) is not strongly coupled,

\[ \Lambda_3 \to \left( \frac{m_{2\text{FP}}^2 M_{\text{Pl}}}{\alpha^2} \mathcal{O}(\beta_n) \right)^{1/3}, \tag{20} \]

This is even higher than \( \hat{\Lambda}_3 \). Therefore the strong-coupling issues with the helicity-0 mode are alleviated, rather than exacerbated, when \( \alpha \to 0 \).

**Cosmology.**— We now proceed to apply the above arguments to the particular example of a homogeneous and isotropic universe. We will take both metrics to be of the diagonal FLRW form \([14–16]\)

\[
g_{\mu \nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j, \tag{21}
\]

\[
f_{\mu \nu} dx^\mu dx^\nu = -X^2(t) dt^2 + Y^2(t) \delta_{ij} dx^i dx^j, \tag{22}
\]

where we can freely choose the cosmic-time coordinate for \( g_{\mu \nu} \) (\( g_{00} = -1 \)) because of general covariance. Because matter couples minimally to \( g_{\mu \nu} \), this choice is physical, and \( a(t) \) corresponds to the scale factor inferred from observations. We furthermore take the matter source to be a perfect fluid, \( T^\mu_\nu = \text{diag}(-\rho, p, p, p) \). The \( g \)-metric equation (27) leads to the Friedmann equation

\[ 3 H^2 = \frac{\rho}{M_{\text{Pl}}^2} + m^2 \left( \beta_0 + 3 \beta_1 y + 3 \beta_2 y^2 + 3 \beta_3 y^3 \right), \tag{23} \]

where the Hubble rate is defined as \( H \equiv \dot{a}/a \) and the ratio of the scale factors is

\[ y \equiv \frac{Y}{a}. \tag{24} \]

The analogous equation for the \( f \) metric is

\[ 3 K^2 = m^2 \frac{X^2}{\alpha^2} \left( \frac{\beta_1}{y^2} + \frac{\beta_2}{y^3} + \frac{\beta_3}{y} + \beta_4 \right), \tag{25} \]

with \( K \equiv \dot{Y}/Y \). The final ingredient is the Bianchi constraint (7), which yields

\[ (H X - K y) \left( \beta_1 + 2 \beta_2 y + \beta_3 y^2 \right) = 0. \tag{26} \]

Taking the first or second term of eq. (26) to vanish selects the so-called dynamical or algebraic branches, respectively. Perturbations in the algebraic branch are pathological \([29]\), so we will consider the dynamical branch in which the \( f \)-metric lapse is fixed,

\[ X = \frac{K y}{H}. \tag{27} \]

Inserting this into the \( f_{\mu \nu} \) equation (28) transforms it into an “alternate” Friedmann equation,

\[ 3 \alpha^2 H^2 = m^2 \left( \frac{\beta_1}{y} + 3 \beta_2 + 3 \beta_3 y + \beta_4 y^2 \right). \tag{28} \]

We take at least two of the \( \beta_n \) for \( n \geq 1 \) to be nonzero in order to ensure the existence of interesting solutions in the GR limit \( \alpha \to 0 \). The solutions to eq. (28) in the GR limit are always on the “finite” branch, i.e., \( y \) evolves from 0 to a finite late-time value. The perturbations on this branch are healthy except for a scalar instability, which we discuss below.

Equation (28) has two features which are useful for our purposes. First, in the limit \( \alpha \to 0 \) it tends to a polynomial constraint that leads to a constant solution for \( y \), so that the potential term in the Friedmann equation (23) becomes a cosmological constant. This provides an explicit example of the statement above that as \( \alpha \to 0 \), the theory approaches general relativity with an effective cosmological constant. This is pathological \([29]\), so we will consider the dynamical branch in which the \( f \)-metric lapse is fixed.

The effective cosmological constant. — Let us illustrate the new viable bimetric cosmologies qualitatively by selecting the model with \( \beta_0 = \beta_3 = \beta_4 = 0 \) \([14]\) which we will refer to as the \( \beta_1 \beta_2 \) model. The Friedmann and “alternate” Friedmann equations (23) and (28) are

\[ 3 H^2 = \frac{\rho}{M_{\text{Pl}}^2} + 3 m^2 \left( \beta_1 y + \beta_2 y^2 \right), \tag{29} \]

\[ 3 \alpha^2 H^2 = m^2 \left( \frac{\beta_1}{y} + 3 \beta_2 \right). \tag{30} \]

One can also combine eqs. (23) and (28) to obtain a quartic equation for \( y \) involving \( \rho \) \([14, 15, 31]\), but this is more cumbersome as it involves higher powers of \( y \) than eq. (28) does.

Since we are interested in finding self-accelerating solutions in the absence of vacuum energy, we will set \( \beta_0 = 0 \) herein, but emphasize that this is not necessary.

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8 See Ref. [72] and references therein for other possible metrics in bimetric cosmology.
The stability of cosmological constant is instructive to work in the GR limit where eq. (30) gives
\[ \Lambda_{\text{eff}} = -\frac{2 \beta_1^2}{3 \beta_2} m^2. \] (31)

The effective cosmological constant is
\[ \Lambda_{\text{eff}} = -\frac{2 \beta_1^2}{3 \beta_2} m^2. \] (33)

Late-time acceleration requires \( \beta_2 < 0 \).

When we are not exactly in the GR limit, we should consider corrections to eq. (32),
\[ 3H^2 = \rho_{\text{eff}} - \frac{2 \beta_1^2}{3 \beta_2} m^2. \] (32)

This expansion is valid as long as
\[ H^2 \lesssim \frac{\beta_2 m^2}{\alpha_0}. \] (35)

Rearranging and again keeping terms up to \( \mathcal{O}(\alpha^2) \), we find a standard Friedmann equation with a time-varying effective cosmological constant given by
\[ \Lambda_{\text{eff}} = -\frac{2 \beta_1^2}{3 \beta_2} m^2 - \frac{2 \beta_1^2}{9 \beta_2^3} \alpha_0^2 \left( \rho_{\text{eff}} - \frac{2 \beta_1^2}{3 \beta_2} m^2 \right) + \mathcal{O}(\alpha^4). \] (36)

Because matter is coupled minimally to gravitational waves, it will have the standard behavior \( \rho \sim a^{-3(1+w)} \), where \( w = \rho/\rho \) is the equation-of-state parameter, allowing \( \alpha \) to stand in for time. This captures the first hint of the dynamical dark energy that is typical of bigravity [16, 20].

These results generalize easily to other parameter combinations. We list the effective cosmological constant up to \( \mathcal{O}(\alpha^2) \) for all the two-parameter models (setting \( \beta_0 = 0 \) ) in table [1]. We remind the reader that, in order for the \( \alpha \to 0 \) limit to be well-behaved, at least two of the \( \beta_n \) parameters (excluding the vacuum energy contribution, \( \beta_0 \)) must be nonzero.

**Exorcising the instability.** — The stability of cosmological perturbations in bigravity was investigated in Ref. [22] by determining the full solutions to the linearized Einstein equations in the subhorizon regime. The perturbations were shown to obey a WKB solution given by
\[ \Phi \sim e^{i\omega N}, \] (37)

TABLE I. The effective cosmological constant and lowest-order corrections (which are time-dependent through \( \rho \) ) for a variety of two-parameter models. We have chosen solution branches which lead to positive \( \Lambda_{\text{eff}} \) for appropriate signs of the \( \beta_n \), and generally take \( \beta_1 \geq 0 \) based on viability conditions [19]. The \( \beta_1, \beta_4 \neq 0 \) model does not possess a finite-branch solution [11].

| Model | \( \Lambda_{\text{eff}} (\alpha \to 0) \) | \( \mathcal{O}(\alpha^2) \) correction |
|-------|---------------------------------|----------------------------------|
| \( \beta_1, \beta_2 \neq 0 \) | \( -\frac{2 \beta_1^2}{3 \beta_2} m^2 \) | \( \frac{2 \beta_1^2}{3 \beta_2} \alpha \left( \frac{\rho_{\text{eff}}}{2 M_{\text{Pl}}^2} - \frac{2 \beta_1^2}{3 \beta_2} m^2 \right) \) |
| \( \beta_1, \beta_4 \neq 0 \) | \( -\frac{8 \beta_1^2}{3 \beta_2^3} m^2 \) | \( \frac{8 \beta_1^2}{3 \beta_2^3} \alpha \left( \frac{\rho_{\text{eff}}}{2 M_{\text{Pl}}^2} - \frac{8 \beta_1^2}{3 \beta_2^3} m^2 \right) \) |
| \( \beta_1, \beta_4 \neq 0 \) | \( -\frac{2 \beta_1^2}{3 \beta_2} m^2 \) | \( \frac{2 \beta_1^2}{3 \beta_2} \alpha \left( \frac{\rho_{\text{eff}}}{2 M_{\text{Pl}}^2} - \frac{2 \beta_1^2}{3 \beta_2} m^2 \right) \) |
| \( \beta_1, \beta_4 \neq 0 \) | \( -\frac{2 \beta_1^2}{3 \beta_2} m^2 \) | \( \frac{2 \beta_1^2}{3 \beta_2} \alpha \left( \frac{\rho_{\text{eff}}}{2 M_{\text{Pl}}^2} - \frac{2 \beta_1^2}{3 \beta_2} m^2 \right) \) |

where \( \Phi \) represents any of the scalar perturbation variables, \( \tilde{\gamma} \equiv \ln a \), and we have taken the limit \( k \gg aH \) where \( k \) is the comoving wavenumber. The eigenfrequencies \( \omega \) were presented for particular models in Ref. [32], where it was found that all models with viable backbones have \( \omega^2 < 0 \) at early times, revealing a gradient instability that only ends at a very low redshift. Using the formulation of the linearized equations of motion presented in Ref. [33], we can write the eigenfrequencies for general \( \beta_n \) and \( \alpha \) in the compact form [11],

\[ \left( \frac{aH}{k} \right)^2 \omega^2 = 1 + \frac{(\beta_1 + 4 \beta_2 y + 3 \beta_3 y^2)}{3y (\beta_1 + 2 \beta_2 y + \beta_3 y^2)} \frac{y'}{y} - \frac{1}{3 \alpha^2 y^2 \rho (1 + w)}. \] (38)

where \( \rho \equiv \rho/\rho_{\text{eff}} M_{\text{Pl}}^2 \) and primes denote d/d ln a.

We apply this to the \( \beta_1 \beta_2 \) model. Assuming a universe dominated by dust \( (w = 0) \), \( \omega^2 \) crosses zero when [13]

\[ 18 \alpha^2 \beta_2 (\alpha^2 \beta_1^2 + 4 \beta_2^2) y^3 + 9 \alpha^2 \beta_1 (\alpha^2 \beta_1^2 + 10 \beta_2^2) y^4 + 48 \alpha^2 \beta_1^2 \beta_2 y^3 + 6 \beta_2 (2 \alpha^2 \beta_1^2 - \beta_2^2) y^2 - 6 \beta_1^2 \beta_2 y - \beta_1^3 = 0. \] (39)

Solving this for \( y \), we can then use eq. (30) to determine the value of Hubble rate at the transition era, before which the gradient instability is present and after which it vanishes. While this solution is too complicated to write down explicitly, in the limit \( \alpha \to 0 \) the leading-

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11 We have used eqs. [24] and [25] and their derivatives to solve for \( y' \) and \( \rho \) in eq. [31] in terms of \( \beta_n \) and \( y \) [20].
TABLE II. The values of $\alpha$ and $M_f$ for a few choices of the era at which perturbations become stable.

| Era of transition to stability | $H_\star$ | $\alpha$ | $M_f$ |
|--------------------------------|----------|----------|-------|
| BBN                           | $10^{-16}$ eV | $10^{-13}$ | 100 GeV |
| $\Lambda_3 = (m^2 M_{Pl}/\alpha)^{1/3}$ | $10^{-3}$ eV | $10^{-31}$ | $10^{-3}$ eV |
| GUT-scale inflation           | $10^{13}$ GeV | $10^{-55}$ | $10^{-27}$ eV |
| $M_{Pl}$                      | $10^{19}$ GeV | $10^{-61}$ | $10^{-33}$ eV |

We pick the negative branch of eq. (40) for physical rea-

A natural requirement would be to push the instability
outside the range of the effective field theory, i.e., above
either the cut-off scale where new physics must enter, or
the strong-coupling scale where tree-level unitary breaks
down. The cut-off scale in massive and bimetric gravity
is not known. The strong-coupling scale, to the extent it
is understood, was discussed above. Here we focus on ob-
servational constraints. It is natural to demand that the
instability lie beyond some important cosmic era which we
can indirectly probe, such as big-bang nucleosynthesis
(BBN) or inflation. Both of these possibilities are then
likely to be observationally safe as long as the Universe is
decelerating (e.g., is radiation-dominated) after inflation,
because the instability is only a problem for subhorizon
modes with large $k/aH$, and during a decelerating epoch
modes with fixed comoving wavelength always become
smaller with respect to the horizon. Consider, as an ex-
ample, that the transition to stability occurs between
inflation and BBN. During that period, modes will grow
rapidly on small scales, but those will be far, far smaller
than the modes relevant for the cosmic microwave back-
ground or large-scale structure. One might worry that in-
flation’s ability to set initial conditions is spoiled in this
scenario (assuming that the linear theory is even valid
during inflation, which is not guaranteed due to the argu-
ments above). However, the instability should be absent
during inflation; notice from eq. (35) that $\omega^2$ generically
becomes large and positive for $w$ close to $-1$. Therefore
the instability would not affect the generation of primor-
dial perturbations during inflation. If the instability later
appears with the onset of radiation domination, it would
only affect small scales which are irrelevant for present-
day cosmology.

If the instability ends at the time of BBN, $M_f$ can be
as high as about 100 GeV, far larger than the wavenum-
bers probed by cosmological observations. We remind
the reader that for such a “large” $M_f$, perturbations in
the Einstein-Hilbert term for $f_{\mu\nu}$ remain weakly-coupled
for all observationally-relevant $k$.

While analytic results like eq. (40) cannot be obtained
for most of the other two-parameter models, we have
checked that in each case the relevant behavior, $H_\star \sim
H_\Lambda/\alpha$, holds. The values given in table II are there-
fore fairly model-independent.

The other pathology that is typical of massive and bi-
metric gravity, the Higuchi ghost, is not present in these

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12 While eq. (40) only holds exactly in the presence of dust, $w = 0$,
for other reasonable equations of state, such as radiation ($w = 1/3$), it will only be modified by an $O(1)$ factor. Since we will be
using this analysis only to make order-of-magnitude estimates,
the exact factors are unimportant.

13 These two are not always the same, and may not be in massive
and bimetric gravity.

14 This depends on the exact $\beta_3$ parameters and the evolution of $y$.
Background viability requires $\beta_3 > 0$ so as long as $\beta_3 \leq 0$,
at the very least the last term in eq. (35) is large and manifestly
positive.

15 Specifically, this holds in the models with $\beta_3 \neq 0$. The gradient
instability is absent from the $\beta_3 \beta_3$ and $\beta_2 \beta_4$ models at early
times. These were shown in Ref. 14 to have problematic
background behavior at early times, but these again can be made
unobservably early in the GR limit.
models. There is a simple condition for the absence of this ghost, \( dp/dy < 0 \) [25, 36] (see also Refs. [33, 57]). Because for normal matter \( \rho \) is always decreasing with time, this amounts to demanding that \( y \) be increasing. In the “finite-branch” solutions which we are considering, \( y \) evolves monotonically from 0 at early times to a fixed positive value at late times, and so the Higuchi bound is always satisfied [41].

Parameter rescalings. We have presented a physically well-motivated region of bimetric parameter space, near the GR limit, in which observable cosmological perturbations are stable and yet self-acceleration remains. One is naturally led to ask how this has been missed by the many previous studies of bimetric cosmology. The issue lies in a rescaling which leaves the action (1) invariant [28, 62],

\[
f_{\mu\nu} \rightarrow \Omega^2 f_{\mu\nu}, \quad \beta_n \rightarrow \frac{1}{\Omega^n} \beta_n, \quad M_f \rightarrow \Omega M_f, \tag{43}
\]

and hence gives rise to a redundant parameter. It has become common to let \( \alpha \) play this role and perform the rescaling \( \Omega = 1/\alpha \) such that \( \alpha \) is set to unity. While our results do not invalidate this rescaling, they do show that it picks out a particular region of parameter space which may not capture all physically-meaningful situations. In particular, the \( \alpha \rightarrow 0 \) limit, in which the theory approaches GR—the behavior at the heart of our removing the gradient instability—would look extremely odd after this rescaling: the \( \beta_n \) would not only be very large, but each \( \beta_{n+1} \) would be parametrically larger than \( \beta_n \).[16] Therefore, studies which set \( \alpha \) to unity could in principle have found the GR-like solutions which we study here, but only by looking at what would have appeared to be a highly unnatural and tuned set of parameters, even though they have a simple and sensible physical explanation. Without performing this rescaling, we can simply take the nonzero \( \beta_n \) to be \( \mathcal{O}(1) \) and consider that we are in the small-\( M_f \) régime.

It is clear that in phenomenological studies of bigravity, \( \alpha \) must not automatically be set to unity. When working with a two-\( \beta_n \) model, perhaps a more sensible rescaling would be one such that the two \( \beta_n \) are equal to each other (up to a possible sign). They can further be absorbed into \( m^2 \). In this case, the free parameters are effectively the spin-2 interaction scale, \( m^2 \), and the \( f \)-metric Planck mass, \( M_f \). Their effects decouple nicely: \( M_f \) controls the earliness of the instability, while \( m \) sets the acceleration scale. Alternatively, one may consider that the rescaling [19] simply tells us that rather different regions of parameter space happen to have the same solutions, and therefore not perform any rescaling a priori at all.

Summary and discussion. We have shown that a well-motivated but heretofore underexplored region of parameter space in bimetric gravity can lead to cosmological solutions which are observationally viable and close to general relativity, with an effective cosmological constant that is set by the spin-2 interaction scale \( m \). In this limit, obtained by taking a small \( f \)-metric Planck mass, the gradient instability that seems to generically plague bimetric models at late times is relegated to the very early Universe, where it can be either made unobservable or pushed outside the régime of validity of the effective theory. This instability had been considered in previous work to make bimetric cosmologies nonpredictive even at late times. Furthermore, in this limit the theory avoids the usual low-scale strong-coupling issue that affects the helicity-0 sector in the massive-gravity limit.

What is encouraging is that the one property of bigravity which survives in the small-\( \alpha \) limit is its cosmologically most useful feature, the technically-natural dark energy scale. In other words, the effective cosmological constant of bigravity in a region close to GR is not just the vacuum-energy contribution and can give rise to self-acceleration in its absence.

The model we have presented is expected to be extremely close to GR at all very high energy scales. In particular the Newtonian limit is well-behaved; unlike \( m^2 \rightarrow 0 \), which suffers from the vDVZ discontinuity, the GR limit \( \alpha \rightarrow 0 \) is completely smooth because all the helicity states of the massive spin-2 mode decouple from matter. Note also that massive gravity does not possess such a continuous GR limit.

It is worth emphasizing that the \( \alpha \rightarrow 0 \) limit brings bimetric theory arbitrarily close to GR even for a large value of the spin-2 mass scale, \( m \gg H_0 \). The presence of heavy spin-2 fields in the Universe is therefore not excluded as long as their self-interaction scale (set by \( M_f \)) is sufficiently small compared to \( M_{Pl} \). In this case, however, the naturalness argument is lost and fine-tuning of the \( \beta_n \) parameters is needed to make the effective cosmological constant small enough to be compatible with observations.[17]

Finally we comment on the potential observable signatures of this theory. While at low energies, corresponding to recent cosmological epochs, this limit of bigravity is extremely close to GR, there may be observable effects at early times when the effects of strong coupling become important. In this case, given by \( H > H_\star \), the small-\( \alpha \) approximation breaks down and modified-gravity effects must be taken into account. This may be particularly important for inflation, which will see such effects unless \( M_f \) is extraordinarily small. A better understanding of strong coupling in the \( f_{\mu \nu} \) sector will therefore point the

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16 We can recast this as a large \( m^2 \), but there would remain a specific tuning among the \( \beta_n \) of the form \( \beta_n/\beta_{n+1} \sim \epsilon \), where \( \epsilon \) is the value of \( \alpha \) before the rescaling.

17 Indeed, without this fine-tuning of the \( \beta_n \), the interaction term would lead to acceleration at an unacceptably early epoch. This scenario is related to the findings of Ref. [52], where it was shown that the instability becomes negligible for large values of \( m \).
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