Chains of extended Jordanian twists for
Lie superalgebras

V.N. Tolstoy
Institute of Nuclear Physics, Moscow State University
119992 Moscow & Russia (e-mail: tolstoy@nucl-th.sinp.msu.ru)

Abstract

Two type of superization of the Jordanian $r$-matrix for the Lie algebra $\mathfrak{sl}(2)$ are considered. One type is associated with the Lie superalgebra $\mathfrak{sl}(1|1)$ and another type is associated with the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$. Extended Jordanian $r$-matrices of maximal order are obtained for the basic complex Lie superalgebras $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(M|2n)$, and a general procedure for construction of corresponding chains of extended Jordanian twists is given. We also find a relation between the extended Jordanian twist and automorphism which gives trivial coproduct for a subalgebra provided the subalgebra is a kernel of the cobracket for the corresponding $r$-matrix.

1 Introduction

The Drinfeld’s quantum group theory roughly includes two classes of Hopf algebras: quasitriangular and triangular. The (standard) $q$-deformation of simple Lie algebras belongs to the first class. The simplest example of the triangular (non-standard) deformation is the Jordanian deformation of $\mathfrak{sl}(2)$ (e.g., see [1]). In the case of simple Lie algebras of rank $\geq 2$ some non-standard deformations were constructed by Kulish, Lyakhovsky et al. [2]–[6]. These deformations are described by chains of twists which are an extension of the Jordanian twist. Full chains of extended Jordanian twists were constructed for all complex Lie algebras of the classical series $A_n$, $B_n$, $C_n$ and $D_n$.

In this paper a generalization of these results on the supercase is given. Namely, we consider two type of superization of the Jordanian $r$-matrix for the Lie algebra $\mathfrak{sl}(2)$. One type is associated with the Lie superalgebra $\mathfrak{sl}(1|1)$ and another type is associated with the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$. Extended Jordanian $r$-matrices of maximal order are obtained for the basic complex Lie superalgebras $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(M|2n)$, and a general procedure for construction of corresponding chains of the extended Jordanian twists is given. The super-Jordanian deformation of the Lie superalgebra $\mathfrak{osp}(1|2)$ was found in [7]. We also find a relation between the extended Jordanian twist and automorphism which gives trivial coproduct for a subalgebra provided the subalgebra is a kernel of the cobracket for the corresponding $r$-matrix.
2 Classical $r$-matrices of Jordanian type

First we consider some notations and definitions concerning classical $r$-matrices.

Let $\mathfrak{g}$ be any finite-dimensional simple Lie superalgebra then $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_+$ are maximal nilpotent subalgebras and $\mathfrak{h}$ is a Cartan subalgebra. The subalgebra $\mathfrak{n}_+$ ($\mathfrak{n}_-$) is generated by the positive (negative) root vectors $e_\beta$ ($e_{-\beta}$) for all $\beta \in \Delta_+(\mathfrak{g})$. The symbol $\mathfrak{b}_+$ will denote the Borel subalgebra of $\mathfrak{g}$, $\mathfrak{b}_+ := \mathfrak{h} \oplus \mathfrak{n}_+$. Let a Cartan element $h_\theta \in \mathfrak{h}$ and a homogeneous element $e_\theta \in \mathfrak{n}_+$ satisfies the relation

$$[h_\theta , e_\theta ] = e_\theta \quad (\deg(h_\theta ) = 0, \quad \deg(e_\theta ) = 0, \text{or } 1) .$$

Consider the skew-symmetric two-tensor

$$r_\theta (\xi ) = \xi h_\theta \wedge e_\theta := \xi (h_\theta \otimes e_\theta - e_\theta \otimes h_\theta ) , \quad r_{\theta 21}^2 (\xi ) = - r_\theta (\xi ) ,$$

where symbol $\xi$ is called a deformation parameter. We demand that the two-tensor is even, $\deg(r_\theta (\xi )) = 0$. It means that the deformation parameter $\xi$ also should be homogeneous and we have

$$\deg(\xi ) = \deg(e_\theta ) , \quad \xi e_\theta = (-1)^{\deg(\xi ) \deg(e_\theta )} e_\theta \xi .$$

It easy to check that the two-tensor (2.2) with the conditions (2.3) is a classical $r$-matrix, i.e. it satisfies the classical Yang-Baxter equation (CYBE)

$$[r_{\theta 12}^2 (\xi ) , r_{\theta 13}^3 (\xi ) + r_{\theta 23}^3 (\xi )] + [r_{\theta 13}^3 (\xi ) , r_{\theta 23}^3 (\xi )] = 0 .$$

In the case, when the element $e_\theta$ is even, $\deg(e_\theta ) = 0$, and $\xi$ is a complex number, $\xi \in \mathbb{C}$, the $r$-matrix (2.2) is called a Jordanian classical $r$-matrix and it was intensively discussed in the literature. In the supercase, when $\deg(e_\theta ) = \deg(\xi ) = 1$, the $r$-matrix (2.2) will be also called a Jordanian classical $r$-matrix.

Remarks. (i) The condition $\deg(r_\theta (\xi )) = 0$ is natural since $R = 1 + r_\theta (\xi ) + \ldots$, where $R$ is the universal $R$-matrix. (ii) The second condition (2.3) is consequence of the fact that the co-bracket $\delta(x)$ belongs to $g \otimes g$, where $\delta(x)$ is defined by $\xi \delta(x) := [x \otimes 1 + 1 \otimes x , r_\theta (\xi )]$. (iii) In the case, when the deformation parameter $\xi$ is odd ("fermion"), there are two possibilities: $\xi^2 = 0$ (Grassmann variable), and $\xi^2 \neq 0$ (Clifford variable). In this paper we shall not make this specialization.

Example 1. Consider the simplest Lie superalgebra $\mathfrak{gl}(1|1)$. It is generated by the Cartan-Weyl elements $h_1 := e_{1-1} , h_2 := e_{2-2} , e_{1-2}$ and $e_{2-1}$ with the relations

$$[h_i , e_{k-l}] = (\delta_{ik} - \delta_{il}) e_{k-l} , \quad \{ e_{1-2} , e_{2-1} \} = h_1 + h_2 , \quad e_{1-2}^2 = e_{2-1}^2 = 0 ,$$

$$\deg(h_1) = \deg(h_2) = 0 , \quad \deg(e_{1-2}) = \deg(e_{2-1}) = 1 .$$

It is well-known the following classical Drinfeld-Jimbo $r$-matrix for $\mathfrak{gl}(1|1)$:

$$r_{D.J}(h) = h ( e_{1-2} \otimes e_{2-1} + e_{2-1} \otimes e_{1-2} ) ,$$

where the parameter $h$ is even, $\deg(h) = 0$.

Let $\eta$ be an odd parameter then we can write down one more additional Jordanian solution:

$$r_1(\eta) = \eta ( h_1 - h_2 ) \wedge e_{1-2} ,$$

where $\deg(\eta) + \deg(e_{1-2}) = 0$ and $\eta e_{1-2} = - e_{1-2} \eta$. 

2
**Example 2.** Consider the Lie superalgebra $\mathfrak{osp}(1|2)$. It is generated by the Cartan-Weyl elements $h, v_\pm$ and $e_\pm$ with the relations

\[
[h, v_\pm] = \pm \frac{1}{2} v_\pm, \quad \{v_+, v_-\} = -\frac{1}{2} h, \quad e_\pm = \pm 4 (v_\pm)^2, \quad \text{deg}(h) = \text{deg}(e_\pm) = 0, \quad \text{deg}(v_\pm) = 1. \tag{2.8}
\]

It is well-known the following classical $r$-matrices for $\mathfrak{osp}(1|2)$ \cite{8,9}:

\[
r_{Dj}(h) = h (e_+ \wedge e_- + 2v_+ \otimes v_- + 2v_- \otimes v_+),
\]

\[
r_1(\xi) = \xi h \wedge e_+, \quad r_2(\xi) = \xi (h \wedge e_+ - 2v_+ \otimes v_+), \tag{2.9}
\]

where the parameters $h$ and $\xi$ are even, $\text{deg}(h) = \text{deg}(\xi) = 0$.

Let $\eta$ be an odd parameter then we can write down two more additional solutions:

\[
r_3(\eta) = \eta h \wedge v_+, \quad r_4(\eta) = \eta v_+ \wedge e_+, \tag{2.11}
\]

where $\text{deg}(\eta) + \text{deg}(v_+) = 0$ and $\eta v_+ = -v_+ \eta$.

Again, let a Cartan element $h_\theta \in \mathfrak{h}$ and a homogeneous root vector $e_\theta \in \mathfrak{n}_+$ satisfy the relation (2.1). Moreover, let homogeneous elements $e_{\gamma_{\pm i}}$ indexed by the symbols $i$ and $-i$, $i \in I = \{1, 2, \ldots, N\}$ satisfy the relations

\[
[h_\theta, e_{\gamma_i}] = (1 - t_{\gamma_i}) e_{\gamma_i}, \quad [h_\theta, e_{\gamma_{-i}}] = t_{\gamma_i} e_{\gamma_{-i}}, \quad (t_{\gamma_i} \in \mathbb{C}),
\]

\[
[e_{\gamma_{\pm i}}, e_{\gamma_{\pm j}}] = 0, \quad [e_{\gamma_{\pm i}}, e_{\gamma_{\pm j}}] = \delta_{k-l} e_{\theta} \quad (k > l \in I \cup (-I)), \tag{2.12}
\]

\[
\text{deg}(e_\theta) = \text{deg}(e_{\gamma_i}) + \text{deg}(e_{\gamma_{-i}}) \pmod{2}.
\]

For the Lie superalgebra $\mathfrak{g}$ the brackets $[\cdot, \cdot]$ always denote the super-commutator:

\[
[x, y] := xy - (-1)^{\text{deg}(x)\text{deg}(y)}yx \tag{2.13}
\]

for any homogeneous elements $x$ and $y$. Consider the even skew-symmetric two-tensor

\[
r_{\theta,N}(\xi) = \xi \left( h_\theta \wedge e_\theta + \sum_{i=1}^{N} (-1)^{\text{deg}(e_{\gamma_i}) \text{deg}(e_{\gamma_{-i}})} e_{\gamma_i} \wedge e_{\gamma_{-i}} \right), \tag{2.14}
\]

where

\[
\text{deg}(\xi) = \text{deg}(e_\theta) = \text{deg}(e_{\gamma_i}) + \text{deg}(e_{\gamma_{-i}}) \pmod{2}. \tag{2.15}
\]

Moreover we assume that the operation ”$\wedge$” in (2.14) is graded:

\[
e_{\gamma_i} \wedge e_{\gamma_{-i}} := e_{\gamma_i} \otimes e_{\gamma_{-i}} - (-1)^{\text{deg}(e_{\gamma_i}) \text{deg}(e_{\gamma_{-i}})} e_{\gamma_{-i}} \otimes e_{\gamma_i}. \tag{2.16}
\]

It is not hard to check that the element (2.14) satisfies CYBE and it will be called the extended Jordanian $r$-matrix of $N$-order. Let $N$ be maximal order, i.e. we assume that another elements $e_{\gamma_j}, j > N$, which satisfy the relations (2.12), do not exist. Such element (2.14) will be called the extended Jordanian $r$-matrix of maximal order. It is evident that the extended Jordanian $r$-matrix of maximal order is defined by the elements $h_\theta \in \mathfrak{h}$, $e_\theta \in \mathfrak{n}_+$ and the Borel subalgebra $\mathfrak{b}_+$. We shall here consider a special (“canonical”) case when $e_\theta$ and $e_{\gamma_{\pm i}}$ ($i = 1, 2, \ldots, N$) are weight elements with respect to the Cartan subalgebra $\mathfrak{h}$:

\[
[h, e_\theta] = (h, \theta) e_\theta, \quad [h, e_{\gamma_{\pm i}}] = (h, \gamma_{\pm i}) e_{\gamma_{\pm i}}. \tag{2.17}
\]
for any $h \in \mathfrak{h}$ and for all $i = 1, 2, \ldots, N$. Analyzing the structure of the positive root systems of the complex simple Lie superalgebras we can see that the very maximal order $N$ of the extended Jordanian $r$-matrix is associated with the maximal root, i.e. the root $\theta$ is maximal.

Consider a maximal subalgebra $\mathfrak{b}_+ \subset \mathfrak{b}_+$ which co-commutes with the maximal extended Jordanian $r$-matrix (2.14), $\mathfrak{b}_+ := \text{Ker} \, \delta \in \mathfrak{b}_+$:

$$\xi \delta(x) := [\Delta(x), r_{\theta, N}(\xi)] = [x \otimes 1 + 1 \otimes x, r_{\theta, N}(\xi)] = 0 \quad (\forall x \in \mathfrak{b}_+). \quad (2.18)$$

Let $r_{\theta_1, N_1}(\xi_1) \in \mathfrak{b}_+ \otimes \mathfrak{b}_+$ is also a extended Jordanian $r$-matrix of the form (2.14) with the maximal root $\theta_1 \in \mathfrak{b}'$ and maximal order $N_1$. Then the sum

$$r_{\theta, N; \theta_1, N_1}(\xi_1, \xi_1) := r_{\theta, N}(\xi) + r_{\theta_1, N_1}(\xi_1) \quad (2.19)$$

is also a classical $r$-matrix, i.e. it satisfies CYBE. This proposition for superalgebras can be check by direct calculations. For the case of Lie algebras the proposition was first formulated in [10].

Again, we consider a maximal subalgebra $\mathfrak{b}_+ \subset \mathfrak{b}_+$ which co-commutes with the maximal extended Jordanian $r$-matrix $r_{\theta_1, N_1}(\xi_1)$ and we construct a extended Jordanian $r$-matrix of maximal order, $r_{\theta_2, N_2}(\xi_2)$. Continuing this process as result we obtain a canonical chain of subalgebras

$$\mathfrak{b}_+ \supset \mathfrak{b}_+ \supset \mathfrak{b}_+ \supset \cdots \supset \mathfrak{b}_+ \quad (2.20)$$

and the resulting $r$-matrix

$$r_{\theta, N; \theta_1, N_1; \cdots; \theta_k, N_k}(\xi_1, \cdots, \xi_k) = r_{\theta, N}(\xi) + r_{\theta_1, N_1}(\xi_1) + \cdots + r_{\theta_k, N_k}(\xi_k) \quad (2.21)$$

is a solution of CYBE. If the chain (2.20) is maximal, i.e. it is constructed in corresponding with the maximal orders $N, N_1, \ldots, N_k$, then the $r$-matrix (2.21) is called the maximal classical $r$-matrix of Jordanian type for the Lie algebra $\mathfrak{g}$.

Now we consider examples of maximal classical $r$-matrices of Jordanian type for the classical Lie superalgebras $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(M|2n)$.

**Example 3. Maximal classical $r$-matrix of Jordanian type for the Lie superalgebra $A(m|n-1) \simeq \mathfrak{sl}(m|n)$.** Let $\epsilon_i$ ($i = 1, 2, \ldots, N := m + n$) be an orthonormalized basis of a $N$–dimensional super-Euclidian space $\mathbb{R}^{(m|n)}$: $(\epsilon_i, \epsilon_j) = \pm \delta_{ij}$. In the terms of $\epsilon_i$ the systems of positive roots, $\Delta_+$, for $\mathfrak{sl}(m|n)$ are presented as follows:

$$\Delta_+(\mathfrak{sl}(m|n)) = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq N\}. \quad (2.22)$$

Here some roots are even and another roots are odd. The root $\epsilon_i - \epsilon_N$ is maximal. Let us write down the positive root system $\Delta_+(\mathfrak{sl}(m|n))$ in the following normal ("convex") ordering

$$\begin{align*}
(\epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_3, \ldots, \epsilon_1 - \epsilon_{N-1}, \epsilon_1 - \epsilon_N, & \epsilon_N - \epsilon_1, \epsilon_N - \epsilon_2, \ldots, \epsilon_N - \epsilon_{N-1}), \\
(\epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_4, \ldots, \epsilon_2 - \epsilon_{N-2}, & \epsilon_2 - \epsilon_{N-1}, \epsilon_{N-2} - \epsilon_1, \epsilon_{N-2} - \epsilon_2, \ldots, \epsilon_{N-2} - \epsilon_{N-1}).
\end{align*} \quad (2.23)$$

The underlined roots are maximal on their lines. Each line of (2.23) is corresponding to an extended Jordanian classical $r$-matrix of maximal order with its parameter deformation. Thus we have the set of such $r$-matrices:

$$\begin{align*}
r_1(\xi_1) & := \xi_1 \left[ \frac{1}{2}(\epsilon_1 - \epsilon_{N-1}) + \epsilon_{1} - \epsilon_{N} + \sum_{i=2}^{N-1} \prod_{j=0}^{i-1} (-1)^{\text{deg} \, \epsilon_{i-j}} \epsilon_{i-j} \right] \left[ \epsilon_{1} - \epsilon_{N} \right], \\
r_2(\xi_2) & := \xi_2 \left[ \frac{1}{2}(\epsilon_2 - \epsilon_{N-1} - N+1) + \epsilon_{2} - \epsilon_{N+1} + \sum_{i=2}^{N-2} \prod_{j=0}^{i-1} (-1)^{\text{deg} \, \epsilon_{i-j}} \epsilon_{i-j} \right] \left[ \epsilon_{2} - \epsilon_{N+1} \right].
\end{align*} \quad (2.24)$$

...
where the root vectors \( e_{i-k} := e_{\epsilon_i - \epsilon_k} \) \((i < k)\) are chosen such that they satisfy the relations \( (2.12) \). The resulting maximal \( r \)-matrix is the sum of these matrices:

\[
r_{1,2,\ldots,[N/2]} (\xi_1, \xi_2, \cdots, \xi_{[N/2]}) = r_1(\xi_1) + r_2(\xi_2) + \cdots + r_{[N/2]}(\xi_{[N/2]}) \tag{2.25}
\]

**Proposition 2.1** The elements of the subalgebra \( \mathfrak{gl}(m-i|n-i) \) co-commute with the \( r \)-matrix

\[
r_{1,2,\ldots,i} (\xi_1, \xi_2, \cdots, \xi_i) := r_1(\xi_1) + r_2(\xi_2) + \cdots + r_i(\xi_i) \tag{2.26}
\]

\( i.e. \)

\[
[x \otimes 1 + 1 \otimes x, r_{1,2,\ldots,i} (\xi_1, \xi_2, \cdots, \xi_i) ] = 0 \quad (\forall x \in \mathfrak{gl}(m-i|n-i)). \tag{2.27}
\]

Thus the constructed extended Jordanian \( r \)-matrices \( r_1(\xi_1), r_2(\xi_2), \ldots, r_{[N/2]}(\xi_{[N/2]}) \) are associated with the following reduction chain

\[
\mathfrak{gl}(m|n) \supset \mathfrak{gl}(m-1|n-1) \supset \mathfrak{gl}(m-2|n-2) \cdots \supset \mathfrak{gl}(k) \quad (k = 3 \text{ or } 2|1, \text{ or } 2|1) \tag{2.28}
\]

**Example 4. Maximal classical \( r \)-matrix of Jordanian type for the Lie superalgebra \( C(n) \simeq \mathfrak{osp}(1|2n) \).** In the terms of the orthonormalized basis \( \epsilon_i \) \((i = 1, 2, \ldots, n)\) the systems of positive roots, \( \Delta_+ \), for \( \mathfrak{osp}(1|2n) \) are given as follows:

\[
\Delta_+(\mathfrak{osp}(1|2n)) = \{ \epsilon_i \pm \epsilon_j, 2\epsilon_k \mid 1 \leq i < j \leq n; k = 1, 2, \ldots, n \} \tag{2.29}
\]

Let us write down the positive root system \( \Delta_+(\mathfrak{osp}(1|2n)) \) in the following ordering

\[
(\epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_3, \ldots, \epsilon_1 - \epsilon_n, \epsilon_1 + \epsilon_n, \ldots, \epsilon_1 + \epsilon_3, \epsilon_1 + \epsilon_2),
(\epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_4, \ldots, \epsilon_2 - \epsilon_n, \epsilon_2 + \epsilon_n, \ldots, \epsilon_{n-1} + \epsilon_4, \epsilon_2 + \epsilon_3),
\]

\[
\epsilon_n, 2\epsilon_2.
\tag{2.30}
\]

Each line of \( (2.30) \) is corresponding to an extended Jordanian classical \( r \)-matrix of maximal order with its parameter deformation, and we have the set of such \( r \)-matrices:

\[
r_1(\xi_1) := \xi_1 \left( \frac{1}{2} \epsilon_{1-1} \wedge \epsilon_{11} - \epsilon_1 \wedge \epsilon_1 + \sum_{i=2}^{n} \epsilon_{1-i} \wedge \epsilon_{1i} \right),
\]

\[
r_2(\xi_2) := \xi_2 \left( \frac{1}{2} \epsilon_{2-2} \wedge \epsilon_{22} - \epsilon_2 \wedge \epsilon_2 + \sum_{i=3}^{n} \epsilon_{2-i} \wedge \epsilon_{2i} \right),
\]

\[
\cdots
\]

\[
r_n(\xi_n) := \xi_n \left( \frac{1}{2} \epsilon_{n-n} \wedge \epsilon_{nn} - \epsilon_n \wedge \epsilon_n \right),
\tag{2.31}
\]

where the root vectors \( e_{i-k} := e_{\epsilon_i - \epsilon_k}, \) \( e_{ik} := e_{\epsilon_i + \epsilon_k}, \) \( e_i := e_{\epsilon_i} \) are chosen such that they satisfy the relations \( (2.12) \). The resulting maximal \( r \)-matrix is the sum of these matrices:

\[
r_{1,2,\ldots,n} (\xi_1, \xi_2, \cdots, \xi_n) = r_1(\xi_1) + r_2(\xi_2) + \cdots + r_n(\xi_n) \tag{2.32}
\]
Proposition 2.2  The elements of the subalgebra $\mathfrak{osp}(1|2n - 2i)$ co-commute with the $r$-matrix

$$r_{1,2,\ldots,i}(\xi_1, \xi_2, \ldots, \xi_i) := r_1(\xi_1) + r_2(\xi_2) + \cdots + r_i(\xi_i),$$
i.e.

$$[x \otimes 1 + 1 \otimes x, r_{1,2,\ldots,i}(\xi_1, \xi_2, \ldots, \xi_i)] = 0 \quad (\forall x \in \mathfrak{osp}(1|2n - 2i)).$$
Thus the constructed extended Jordanian classical $r$-matrices $r_1(\xi_1)$, $r_2(\xi_2)$, \ldots, $r_n(\xi_n)$ are associated with the following reduction chain

$$\mathfrak{osp}(1|2n) \supset \mathfrak{osp}(1|2n - 2) \supset \mathfrak{osp}(1|2n - 4) \cdots \supset \mathfrak{osp}(1|2).$$

Example 5. Maximal classical $r$-matrix of Jordanian type for the Lie superalgebras

$B(m|n) \simeq \mathfrak{osp}(2m + 1|2n)$ and $D(m|n) \simeq \mathfrak{osp}(2m|2n)$. In the terms of the orthonormalized basis $e_i$ ($i = 1, 2, \ldots, m + n$) the systems of positive roots for $\mathfrak{osp}(2m + 1|2n)$ are given as follows:

$$\Delta_+(\mathfrak{osp}(2m + 1|2n)) = \{e_i \pm e_j, e_k, e_{2l} | 1 \leq i < j \leq N; 1 \leq k \leq N; m + 1 \leq l \leq N\}. \quad (2.36)$$
where $N := m + n$. Let us write down the positive root system in the following ordering

$$(\epsilon_1 - \epsilon_2), (\epsilon_1 - \epsilon_3), \ldots, (\epsilon_1 - \epsilon_N), \epsilon_1, \epsilon_1 + \epsilon_N, \ldots, \epsilon_1 + \epsilon_3, \epsilon_1 + \epsilon_2, \epsilon_2 - \epsilon_3,$$

$$\epsilon_2 - \epsilon_4, \ldots, \epsilon_2 - \epsilon_N, \epsilon_2 + \epsilon_N, \ldots, \epsilon_2 + \epsilon_4, \epsilon_2 + \epsilon_3,$$

$$(\epsilon_3 - \epsilon_4), (\epsilon_3 - \epsilon_5), \ldots, (\epsilon_3 - \epsilon_N), \epsilon_3, \epsilon_3 + \epsilon_N, \ldots, \epsilon_3 + \epsilon_5, \epsilon_3 + \epsilon_4, \epsilon_4 - \epsilon_5,$$

$$\epsilon_4 - \epsilon_6, \ldots, \epsilon_4 - \epsilon_N, \epsilon_4 + \epsilon_N, \ldots, \epsilon_4 + \epsilon_5, \epsilon_4 + \epsilon_3,$$

$$(\epsilon_m - \epsilon_{m' + 1}), \ldots, (\epsilon_m - \epsilon_N), \epsilon_m, 2\epsilon_m, \epsilon_{m'} + \epsilon_N, \ldots, \epsilon_m + \epsilon_{m' + 1},$$

$$\epsilon_{N'}, 2\epsilon_{N'},$$
where $m' = m + 1$ if $m$ is an even positive integer and $m' = m + 2$ if $m$ is an odd positive integer. Each set of roots in the brackets $(\ldots)$ is corresponding to an extended Jordanian classical $r$-matrix of maximal order with its parameter deformation:

$$r_1(\xi_1, \xi_1) := \xi_1 \left(\frac{1}{2}(e_{1-1} + e_{2-2}) \wedge e_{12} + (-1)^{\deg e_1 \deg e_2} e_1 \wedge e_2ight)$$
$$+ \sum_{i=1}^{N} (-1)^{\deg e_1 - i \deg e_{2i}} e_{1-1} \wedge e_{12} + \sum_{i=3}^{N} (-1)^{\deg e_{1i} \deg e_{2-1}} e_{1i} \wedge e_{2-1}),$$
$$+ \xi_1 \left(\frac{1}{2}(e_{1-1} - e_{2-2}) \wedge e_{1-2},ight.$$\n
$$r_2(\xi_2, \xi_2) := \xi_2 \left(\frac{1}{2}(e_{3-3} + e_{4-4}) \wedge e_{34} + (-1)^{\deg e_3 \deg e_4} e_3 \wedge e_4ight)$$
$$+ \sum_{i=1}^{N} (-1)^{\deg e_3 - i \deg e_{4i}} e_{3-i} \wedge e_{3i} \wedge e_{4i} + \sum_{i=5}^{N} (-1)^{\deg e_{3i} \deg e_{4-i}} e_{3i} \wedge e_{4-i})$$
$$+ \xi_2 \left(\frac{1}{2}(e_{3-3} - e_{4-4}) \wedge e_{3-4},ight.$$\n
$$\cdots$$

$$r_{m'}(\xi_{m'}) := \xi_{m'} \left(\frac{1}{2}(e_{m'-m'} \wedge e_{m'm'} - e_{m'} \wedge e_{m'} + \sum_{i=m'+1}^{N} e_{m'-i} \wedge e_{m'i})ight),$$
$$\cdots$$

$$r_N(\xi_N) := \xi_N \left(\frac{1}{2} e_{N'-N} \wedge e_{NN} - e_N \wedge e_N \right).$$
The resulting maximal $r$-matrix is the sum of these matrices:

$$
r_{1,2,...,N}(\xi_1, \xi_1', \xi_2, \xi_2', \ldots, \xi_{m'}, \ldots, \xi_N) =
= r_1(\xi_1, \xi_1') + r_2(\xi_2, \xi_2') + \cdots + r_m(\xi_{m'}) + \cdots + r_N(\xi_N) .
$$

(2.39)

**Proposition 2.3** The elements of the subalgebra $\mathfrak{so}(3) \oplus \mathfrak{osp}(2(m - i) - 1|2n)$ co-commute with the $r$-matrix

$$
r_{1,2,...,i}(\xi_1, \xi_1', \xi_2, \xi_2', \ldots, \xi_i, \xi_i') := r_1(\xi_1, \xi_1') + r_2(\xi_2, \xi_2') + \cdots + r_i(\xi_i, \xi_i') ,
$$

i.e.

$$
[x \otimes 1 + 1 \otimes x, r_{1,2,...,i}(\xi_1, \xi_2, \xi_2', \ldots, \xi_i, \xi_i')] = 0 \quad (\forall x \in \mathfrak{so}(3) \oplus \mathfrak{osp}(2(m - i) - 1|2n)) .
$$

(2.41)

Thus the constructed extended Jordanian classical $r$-matrices $r_1(\xi_1, \xi_1'), r_2(\xi_2, \xi_2'), \ldots, r_N(\xi_N)$ are associated with the following reduction chain

$$
\mathfrak{osp}(2m + 1|2n) \supset \mathfrak{so}(3) \oplus \mathfrak{osp}(2m - 3|2n) \supset \ldots \supset \mathfrak{osp}(1|2n') \supset \ldots \supset \mathfrak{osp}(1|2)
$$

(2.42)

where $n' = n$ if $m$ is an even positive integer, and $n' = n - 1$ if $m$ is an odd positive integer.

We obtain the results for the Lie superalgebra $D(m|n) \simeq \mathfrak{osp}(2m|2n)$ if we remove all roots $\varepsilon_i$ and the root vectors $e_i$ ($i = 1, \ldots, N$) in the formulas (2.37), (2.38).

## 3 Chains of extended Jordanian twists

Basic elements of Drinfeld’s theory of twisting quantization for Hopf algebras [11] is easy generalized to the case of Hopf superalgebras. Indeed, formulae describing twist quantization for Lie algebras and Lie superalgebras are the same, provided in the second case the grading is properly taken into account.

Let $\mathfrak{U} := \mathfrak{U}(m, \Delta, S, \varepsilon)$ be a Hopf superalgebra with graded structure ($\mathfrak{U} = \mathfrak{U}_0 \oplus \mathfrak{U}_1$) and with the grading preserving operations, a multiplication $m : \mathfrak{U} \otimes \mathfrak{U} \to \mathfrak{U}$, a coproduct $\Delta : \mathfrak{U} \to \mathfrak{U} \otimes \mathfrak{U}$, an antipode $S : \mathfrak{U} \to \mathfrak{U}$, and a counit $\varepsilon : \mathfrak{U} \to \mathbb{C}$. Let there exists an invertible even element $F = \sum_i f_i^{(1)} \otimes f_i^{(2)}$ of some extension of $\mathfrak{U} \otimes \mathfrak{U}$, such that it satisfies the cocycle equation

$$
F^{12}(\Delta \otimes \text{id})(F) = F^{23}(\text{id} \otimes \Delta)(F) ,
$$

(3.1)

and the "unital" normalization condition

$$
(\varepsilon \otimes \text{id})(F) = (\text{id} \otimes \varepsilon)(F) = 1 .
$$

(3.2)

Then the new Hopf superalgebra $\mathfrak{U}^{(F)} := \mathfrak{U}^{(F)}(m, \Delta^{(F)}, S^{(F)}, \varepsilon)$ with the same multiplication $m$ and the counit $\varepsilon$ but with the twisted coproduct and antipode

$$
\Delta^{(F)}(a) = F\Delta(a)F^{-1}, \quad S^{(F)} = u S(a) u^{-1}, \quad u = \sum_i f_i^{(1)} S(f_i^{(2)}) \quad (\forall a \in \mathfrak{U})
$$

(3.3)

is called the twisted Hopf superalgebra or twist quantization of the Hopf superalgebra $\mathfrak{U}$.

The Hopf superalgebra $\mathfrak{U}$ is called quasitriangular if it has an additional invertible element (universal $R$-matrix) $R$ which relates the coproduct $\Delta$ with its opposite coproduct $\tilde{\Delta}$ by the similarity transformation

$$
\tilde{\Delta}(a) = R \Delta(a) R^{-1} \quad (\forall a \in \mathfrak{U}) ,
$$

(3.4)
and $R$ satisfies the quasitriangularity conditions
\[
(\Delta \otimes \text{id})(R) = R^{13}R^{23} \ 	ext{,} \quad (\text{id} \otimes \Delta)(R) = R^{13}R^{12} \ .
\] (3.5)

The twisted ("quantized") Hopf algebra $\mathfrak{U}^{(F)}$ is also quasitriangular with the universal $R$-matrix $R^{(F)}$ defined as follows
\[
R^{(F)} = F^{21}RF^{-1} \ ,
\] (3.6)
where $F^{21} = \sum_i f_i^{(2)} \otimes f_i^{(1)}$ provided $F = \sum_i f_i^{(1)} \otimes f_i^{(2)}$. For the non-deformed, classical case $\mathfrak{U} = U(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie superalgebra, the universal $R$-matrix is trivial, $R = 1$.

The twisting two-tensor $F_{\theta,N}(\xi)$ corresponding to the $r$-matrix (2.14) has the form
\[
F_{\theta,N}(\xi) = \mathfrak{F}_N(\xi)F_J
\] (3.7)
where the two-tensor $F_J$ is the Jordanian twist corresponding to the Jordanian $r$-matrix (2.2) and $\mathfrak{F}_N$ is extension of the Jordanian twist. These two-tensors are given by the formulas
\[
F_J = \exp(2h_\theta \otimes \sigma_\theta) \ ,
\] (3.8)
\[
\mathfrak{F}_N(\xi) = \left( \prod_{i=1}^{N'} \exp \left( \xi(-1)^{\deg e_{\gamma_i} \deg e_{\gamma_i}} e_{\gamma_i} \otimes e_{\gamma_i} \exp(-2t_{\gamma_i} \sigma_\theta) \right) \right) \mathfrak{F}_S
\] (3.9)
\[
= \exp \left( \xi \sum_{i=1}^{N'} (-1)^{\deg e_{\gamma_i} \deg e_{\gamma_i}} e_{\gamma_i} \otimes e_{\gamma_i} \exp(-2t_{\gamma_i} \sigma_\theta) \right) \mathfrak{F}_S \ ,
\]
where
\[
\mathfrak{F}_S = \left( 1 - \frac{e_{\theta/2}}{\exp \sigma_\theta + 1} \otimes \frac{e_{\theta/2}}{\exp \sigma_\theta + 1} \right) \sqrt{\frac{(e_\sigma + 1) \otimes (e_\sigma + 1)}{2(e_\sigma \otimes e_\sigma + 1)}} \ ,
\] (3.10)
if $\theta/2$ is a root (i.e. $\theta/2 = \gamma_i = \gamma_{-i}$ for some $i$), $e_{\theta/2} = e_\theta$, $N' = N - 1$, and
\[
\mathfrak{F}_S = 1 \ .
\] (3.11)
if $\theta/2$ is not any root, $N' = N$. Moreover
\[
\deg(\xi) = \deg(e_\theta) = \deg(e_{\gamma_i}) + \deg(e_{\gamma_{-i}}) \quad (\text{mod} \ 2) \ ,
\] (3.12)
\[
\sigma_\theta := \frac{1}{2} \ln(1 + \xi e_\theta) \ .
\] (3.13)

It should be noted that if the root vector $e_\theta$ is odd then $\sigma_\theta = \frac{1}{2} \xi e_\theta$.

The twisted coproduct $\Delta_\xi(\cdot) := F_{\theta,N}(\xi)\Delta(\cdot)F_{\theta,N}^{-1}(\xi)$ and the corresponding antipode $S_\xi$ for elements in (2.12) are given by the formulas
\[
\Delta_\xi(\exp(\pm \sigma_\theta)) = \exp(\pm \sigma_\theta) \otimes \exp(\pm \sigma_\theta) \ , \quad \Delta_\xi(e_{\theta/2}) = e_{\theta/2} \otimes 1 + \exp(\sigma_\theta) \otimes e_{\theta/2} \ ,
\] (3.14)
\[
\Delta_\xi(h_\theta) = h_\theta \otimes \exp(-2\sigma_\theta) + 1 \otimes h_\theta + \xi \frac{e_{\theta/2}}{4} \exp(-\sigma_\theta) \otimes e_{\theta/2} \exp(-2\sigma_\theta)
\]
\[
- \xi \sum_{i=1}^{N'} (-1)^{\deg e_{\gamma_i} \deg e_{\gamma_i}} e_{\gamma_i} \otimes e_{\gamma_i} \exp(-2(t_{\gamma_i} + 1)\sigma_\theta) \ ,
\] (3.15)
\[
\Delta_\xi(e_{\gamma_i}) = e_{\gamma_i} \otimes \exp(-2t_{\gamma_i}\sigma_\theta) + 1 \otimes e_{\gamma_i} \ ,
\] (3.16)
\[
\Delta_\xi(e_{\gamma_{-i}}) = e_{\gamma_{-i}} \otimes \exp(2t_{\gamma_i}\sigma_\theta) + \exp(2\sigma_\theta) \otimes e_{\gamma_{-i}} \ ,
\] (3.17)
\[ S_\xi(\exp(\pm \sigma \theta)) = \exp(\mp \sigma \theta), \quad S_\xi(e_{\theta/2}) = -e_{\theta/2} \exp(-\sigma \theta), \quad (3.18) \]
\[ S_\xi(h_\theta) = -h_\theta \exp(2\sigma \theta) + \frac{1}{4} \left( \exp(2\sigma \theta) - 1 \right) - \xi \sum_{i=1}^{N'} (-1)^{\deg e_{\gamma_i}} e_{\gamma_i} e_{\gamma_{-i}}, \quad (3.19) \]
\[ S_\xi(e_{\gamma_i}) = -e_{\gamma_i} \exp(2t_{\gamma_i} \sigma \theta), \quad S_\xi(e_{\gamma_{-i}}) = -e_{\gamma_{-i}} \exp(-2(t_{\gamma_i} + 1)\sigma \theta). \quad (3.20) \]

If \( \theta/2 \) is not any root, the third term in (3.18) and the second term in (3.19) should be removed.

**Proposition 3.1** (i) Let \( g' \) be a subalgebra of \( g \), which co-commutes with the r-matrix (2.14), \( \delta_\xi(g') = 0 \) (see [2.18]). If an invertible element \( w_\xi \) of some extension of \( U(g) \) satisfies the equations
\[ [\Delta(x),(w_\xi^{-1} \otimes w_\xi^{-1})\Delta_\xi(w_\xi)F_{\theta,N}(\xi)] = 0 \quad (\forall x \in g'), \quad w_\xi \equiv 1 \mod \xi, \quad \varepsilon(w_\xi) = 1, \quad (3.21) \]
then the automorphism \( w_\xi x w_\xi^{-1} \) simplifies (makes trivial) the twisted coproduct \( \Delta_\xi(\cdot) \) in \( g' \):
\[ \Delta_\xi(w_\xi x w_\xi^{-1}) := w_\xi x w_\xi^{-1} \otimes 1 + 1 \otimes w_\xi x w_\xi^{-1}, \quad x \in g'. \quad (3.22) \]

(ii) The element \( w_\xi \equiv \sqrt{u(\delta_N(\xi))} \) satisfies the equations (3.21), where \( u(\delta_N(\xi)) \) is the Hopf "folding" of the two-tensor (3.4):
\[ u(\delta_N(\xi)) = ((S_\xi \otimes \text{Id})\delta_N(\xi)) \circ 1 = ((\text{Id} \otimes S_j)\delta_N(\xi)) \circ 1 \quad (3.23) \]
\[ = \left( \prod_{i=1}^{N'} \left( \sum_{n=0}^{\infty} \frac{(-\xi)^n}{n!} (-1)^{n \deg e_{\gamma_i}} e_{\gamma_i} e_{\gamma_{-i}} \right) \right) \exp \left( \frac{-2\xi \sigma \theta}{\exp(2\sigma \theta) - 1} \sum_{i=1}^{N'} (-1)^{\deg e_{\gamma_i}} e_{\gamma_i} e_{\gamma_{-i}} \right) \quad (3.24) \]
\[ = \exp \left( \frac{-2\xi \sigma \theta}{\exp(2\sigma \theta) - 1} \sum_{i=1}^{N'} (-1)^{\deg e_{\gamma_i}} e_{\gamma_i} e_{\gamma_{-i}} \right) u_{S1}. \quad (3.25) \]

Here \( u_S \) is the folding of the super-tensor \( \delta_S \): \( u_S = \exp(\frac{1}{2} \sigma) \) if \( \theta/2 \) is a root, and \( u_S = 1 \) if \( \theta/2 \) is not any root; \( S_j \) is the antipode after the Jordanian twist (3.8); the operation "\( \circ \)" means \( (a \otimes b) \circ x = axb. \) The inverse element \( u^{-1}(\delta_N(\xi)) \) is given the following explicit formula
\[ u^{-1}(\delta_N(\xi)) = ((\text{Id} \otimes S_\xi)\delta_N^{-1}(\xi)) \circ 1 = ((S_j \otimes \text{Id})\delta_N^{-1}(\xi)) \circ 1 \quad (3.26) \]
\[ = \left( \prod_{i=1}^{N'} \left( \sum_{n=0}^{\infty} \frac{(-\xi)^n}{n!} \exp \left( -2n\sigma \theta \right) (-1)^{n \deg e_{\gamma_i}} e_{\gamma_i} e_{\gamma_{-i}} \right) \right) u_{S1}^{-1} \quad (3.27) \]
\[ = \exp \left( \frac{2\xi \sigma \theta}{\exp(2\sigma \theta) - 1} \sum_{i=1}^{N'} (-1)^{\deg e_{\gamma_i}} e_{\gamma_i} e_{\gamma_{-i}} \right) u_{S1}^{-1}. \quad (3.28) \]

Moreover
\[ \sqrt{u(\delta_N(\xi))} = \left( \prod_{i=1}^{N'} \left( \sum_{n=0}^{\infty} \frac{(-\xi)^n}{n! (\exp(\sigma \theta) + 1)^n} e_{\gamma_i} e_{\gamma_{-i}} \right) \right) \sqrt{u_S} \quad (3.29) \]
\[ = \exp \left( \frac{-\xi \sigma \theta}{\exp(2\sigma \theta) - 1} \sum_{i=1}^{N'} (-1)^{\deg e_{\gamma_i}} e_{\gamma_i} e_{\gamma_{-i}} \right) \sqrt{u_S}, \quad (3.30) \]
\[ \sqrt{u^{-1}(\mathcal{F}_N(\xi))} = \left( \prod_{i=1}^{N} \left( \sum_{n=0}^{\infty} \frac{\xi^n \exp(-n\sigma)}{n!(\exp(\sigma_{\theta}) + 1)^n} (-1)^n \deg_{\gamma_i} \deg_{\gamma_{i-1}} \exp_{\gamma_i} \exp_{\gamma_{i-1}} \right) \right) \sqrt{u^{-1}} \] (3.31)

\[ = \exp \left( \frac{\xi \sigma_{\theta}}{\exp(2\sigma_{\theta}) - 1} \right) \sum_{i=1}^{N} (-1)^{\deg_{\gamma_i} \deg_{\gamma_{i-1}}} \exp_{\gamma_i} \exp_{\gamma_{i-1}} \right) \sqrt{u^{-1}} \]. (3.32)

**Remarks.** The formula (3.30) for the case \( g = sp(2n) \) was found in [6] by fitting.

With the help of the elements \( w_{\xi} = \sqrt{u(\mathcal{F}_N(\xi))}, w_{\xi_1} = \sqrt{u(\mathcal{F}_{N_1}(\xi_1))}, \ldots , w_{\xi_k} = \sqrt{u(\mathcal{F}_{N_k}(\xi_k))} \)

the total twist chain corresponding to the \( r \)-matrix (2.21) can be presented as follows

\[
F_{\theta,N;\theta_1,N_1;\ldots;\theta_k,N_k}(\xi, \xi_1, \ldots, \xi_k) = F_{\theta_k,N_k}(\xi, \xi_1, \ldots, \xi_{k-1}; \xi_k) \prod_{i=1}^{k-1} \prod_{i=1}^{k-1} \frac{w_{\xi_{i}} \otimes w_{\xi_{i-1}}} {w_{\xi_{i}} \otimes w_{\xi_{i-1}}} \] (3.33)

where

\[
F_{\theta_i,N_i}(\xi, \xi_1, \ldots, \xi_{i-1}; \xi_i) := (w_{\xi_{i-1}} \otimes w_{\xi_{i-1}}) \cdots (w_{\xi_1} \otimes w_{\xi_1})(w_{\xi} \otimes w_{\xi}) F_{\theta_i,N_i}(\xi_i) \times (w_{\xi_{i-1}}^{-1} \otimes w_{\xi_{i-1}}^{-1}) (w_{\xi_1}^{-1} \otimes w_{\xi_1}^{-1}) \cdots (w_{\xi_{i-1}}^{-1} \otimes w_{\xi_{i-1}}^{-1}) \] (3.34)