WEll-posedness for a system of quadratic derivative nonlinear Schrödinger equations with low regularity periodic initial data

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Abstract. We consider the Cauchy problem of a system of quadratic derivative nonlinear Schrödinger equations which was introduced by M. Colin and T. Colin (2004) as a model of laser-plasma interaction. For the nonperiodic case, the author proved the small data global well-posedness and the scattering at the scaling critical regularity for $d \geq 2$ when the coefficients of Laplacian satisfy some condition. In the present paper, we prove the well-posedness of the system for the periodic case. In particular, well-posedness is proved at the scaling critical regularity for $d \geq 3$ under some condition for the coefficients of Laplacian.

1. Introduction

We consider the Cauchy problem of the system of nonlinear Schrödinger equations:

$$
\begin{align*}
(i\partial_t + \alpha \Delta) u &= -(\nabla \cdot w)v, \quad (t, x) \in (0, T) \times \mathbb{T}^d, \\
(i\partial_t + \beta \Delta) v &= -(\nabla \cdot \mathbf{w})u, \quad (t, x) \in (0, T) \times \mathbb{T}^d, \\
(i\partial_t + \gamma \Delta) w &= \nabla (u \cdot v), \quad (t, x) \in (0, T) \times \mathbb{T}^d, \\
(u(0, x), v(0, x), w(0, x)) &= (u_0(x), v_0(x), w_0(x)), \quad x \in \mathbb{T}^d,
\end{align*}
$$

where $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$ and the unknown functions $u, v, w$ are $\mathbb{C}^d$ valued. The system (1.1) was introduced by Colin and Colin in [10] as a model of laser-plasma interaction. (1.1) is invariant under the following scaling transformation:

$$
A_\lambda(t, x) = \lambda^{-1} A(\lambda^{-2} t, \lambda^{-1} x) \quad (A = (u, v, w)),
$$

and the scaling critical regularity is $s_c = d/2 - 1$. The aim of this paper is to prove the well-posedness of (1.1) in the scaling critical Sobolev space.

First, we introduce some known results for related problems. The system (1.1) has quadratic nonlinear terms which contains a derivative. A derivative loss arising
from the nonlinearity makes the problem difficult. In fact, Christ ([9]) proved that the flow map of the Cauchy problem:
\[
\begin{cases}
i \partial_t u - \partial_x^2 u = u \partial_x u, & t \in \mathbb{R}, \ x \in \mathbb{T}, \\
u(0, x) = u_0(x), & x \in \mathbb{T}
\end{cases}
\]
is not continuous on $H^s(\mathbb{T})$ for any $s \in \mathbb{R}$. While, there are positive results for the Cauchy problem:
\[
\begin{cases}
i \partial_t u - \Delta u = \partial_j (|u|^2 u), & t \in \mathbb{R}, \ x \in \mathbb{T}^d, \\
u(0, x) = u_0(x), & x \in \mathbb{T}^d,
\end{cases}
\tag{1.2}
\]
where $\partial_j = \partial/\partial x_j \ (j = 1, \cdots, d)$. Grünrock ([14]) proved that (1.2) is globally well-posed in $L^2(\mathbb{T}^d)$ for $d = 1$ and locally well-posed in $H^s(\mathbb{T}^d)$ for $d \geq 2$ and $s > s_c \ (= d/2 - 1)$. For the Cauchy problem of the one dimensional derivative nonlinear Schrödinger equation:
\[
\begin{cases}
i \partial_t u + \partial_x^2 u = i \lambda \partial_x (|u|^2 u), & t \in \mathbb{R}, \ x \in \mathbb{T}, \\
u(0, x) = u_0(x), & x \in \mathbb{T},
\end{cases}
\]
Herr ([18]) proved the local well-posedness in $H^s(\mathbb{T})$ for $s \geq \frac{1}{2}$ by using the gauge transform and Win ([28]) proved the global well-posedness in $H^s(\mathbb{T})$ for $s > 1/2$. For the nonperiodic case, there are many results for the well-posedness of the nonlinear Schrödinger equations with derivative nonlinearity ([1], [2], [3], [7], [8], [12], [13], [20], [21], [23], [24], [25], [26]).

Next, we introduce some known results for (1.1). For the nonperiodic case, Colin and Colin ([10]) proved the local existence of the solution of (1.1) in $H^s(\mathbb{R}^d)$ for $s > d/2 + 3$. Recently, the author ([19]) proved that (1.1) for the nonperiodic case is globally well-posed and the solution scatters for small data in $H^\infty(\mathbb{R}^d)$ under the condition $(\alpha - \beta)(\alpha - \gamma)(\beta + \gamma) \neq 0$ if $d \geq 4$ and $\alpha \beta \gamma(1/\alpha - 1/\beta - 1/\gamma) > 0$ if $d = 2, 3$. The author also obtained some well-posedness results at the scaling subcritical regularity under the other condition for $\alpha, \beta$ and $\gamma$. For the periodic case, Colin, Colin and Métivier ([11]) proved that the coupled system of (1.1) and the nonlinear wave equation is locally well-posed in $H^s(\mathbb{T}^d)$ for large enough $s$.

Now, we give the main results in the present paper. For a Banach space $\mathcal{H}$ and $r > 0$, we define $B_r(\mathcal{H}) := \{f \in \mathcal{H} \mid \|f\|_{\mathcal{H}} \leq r\}$.

**Theorem 1.1** (Critical case). Let $s_c = d/2 - 1$. We assume that $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$ satisfy $\alpha/\beta, \beta/\gamma \in \mathbb{Q}$.

(i) If $d \geq 3$ and $\alpha \beta \gamma(1/\alpha - 1/\beta - 1/\gamma) > 0$, then (1.1) is locally well-posed for small
data in $H^{s_c}(\mathbb{T}^d)$. More precisely, there exist $r > 0$ and $T > 0$ such that for all initial data $(u_0, v_0, w_0) \in B_r(H^{s_c}(\mathbb{T}^d) \times H^{s_c}(\mathbb{T}^d) \times H^{s_c}(\mathbb{T}^d))$, there exists a solution

$$(u,v,w) \in X^s_r([0,T)) \subset C \left( [0,T); H^{s_c}(\mathbb{T}^d) \right)$$

of the system (1.1) on $(0,T)$. Such solution is unique in $X^s_r([0,T))$ which is a closed subset of $X^s([0,T))$ (see [5,4], (5.5) and Definition 4). Moreover, the flow map

$$S : B_r(H^{s_c}(\mathbb{T}^d) \times H^{s_c}(\mathbb{T}^d) \times H^{s_c}(\mathbb{T}^d)) \ni (u_0, v_0, w_0) \mapsto (u,v,w) \in X^s([0,T))$$

is Lipschitz continuous.

(ii) If $d \geq 5$, $\alpha \beta \gamma(1/\alpha - 1/\beta - 1/\gamma) \leq 0$ and $(\alpha - \beta)(\beta + \gamma)(\gamma - \alpha) \neq 0$, then (1.1) is locally well-posed for small data in $H^{s_c}(\mathbb{T}^d)$.

Theorem 1.2 (Subcritical case). Let $d \geq 1$ and $\alpha, \beta, \gamma \in \mathbb{R} \backslash \{0\}$.

(i) If $\alpha \beta \gamma(1/\alpha - 1/\beta - 1/\gamma) > 0$ and $s > \max\{s_c,0\}$, then (1.1) is locally well-posed in $H^s(\mathbb{T}^d)$. More precisely, for any $r > 0$ and for all initial data $(u_0, v_0, w_0) \in B_r(H^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d))$, there exist $T = T(r)$ and a solution

$$(u,v,w) \in X^s_r([0,T)) \subset C \left( [0,T); H^s(\mathbb{T}^d) \right)$$

of the system (1.1) on $(0,T)$. Such solution is unique in $X^s_r([0,T))$ which is a closed subset of $X^s([0,T))$. Moreover, the flow map

$$S : B_r(H^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d)) \ni (u_0, v_0, w_0) \mapsto (u,v,w) \in X^s([0,T))$$

is Lipschitz continuous.

(ii) If $\alpha \beta \gamma(1/\alpha - 1/\beta - 1/\gamma) \leq 0$, $(\alpha - \beta)(\beta + \gamma)(\gamma - \alpha) \neq 0$ and $s > \max\{s_c,1\}$, then (1.1) is locally well-posed in $H^s(\mathbb{T}^d)$.

Remark 1.1. (i) $\alpha \beta \gamma(1/\alpha - 1/\beta - 1/\gamma) > 0$ is the nonresonance condition for High-High interaction and $(\alpha - \beta)(\beta + \gamma)(\gamma - \alpha) \neq 0$ is the nonresonance condition for High-Low interaction. Oh ([22]) also studied the resonance and the nonresonance for the system of KdV equations. He proved that if the coefficient of the linear term of the system satisfies the nonresonance condition, then the well-posedness of the system is obtained at lower regularity than the regularity for the coefficient satisfying the resonance condition.

(ii) If $\alpha, \beta, \gamma \in \mathbb{R} \backslash \{0\}$ satisfy $\alpha \beta \gamma(1/\alpha - 1/\beta - 1/\gamma) > 0$, then $(\alpha - \beta)(\beta + \gamma)(\gamma - \alpha) \neq 0$ holds.

(iii) $\alpha/\beta, \beta/\gamma \in \mathbb{Q}$ is the condition that the products $e^{it\alpha \Delta}u \cdot e^{it\beta \Delta}v$, $e^{it\beta \Delta}v \cdot e^{it\gamma \Delta}w$ and $e^{it\gamma \Delta}w \cdot e^{it\alpha \Delta}u$ are periodic with respect to time variable $t$. We will use this condition to obtain the bilinear Strichartz estimate (3.12).
System (1.1) has the following conservation quantities (see [19] Proposition 7.1):

\[
M(u,v,w) := 2|||u|||_{L^2}^2 + |||v|||_{L^2}^2 + |||w|||_{L^2}^2,
\]
\[
H(u,v,w) := \alpha||\nabla u||_{L^2}^2 + \beta||\nabla v||_{L^2}^2 + \gamma||\nabla w||_{L^2}^2 + 2\text{Re}(w, \nabla(u \cdot v))_{L^2}.
\]

By using the conservation law for \(H\), we obtain the following result.

**Theorem 1.3.** Let \(d \geq 1\). We assume that \(\alpha, \beta, \gamma \in \mathbb{R}\setminus\{0\}\) have the same sign. There exists \(r > 0\) such that for every \((u_0, v_0, w_0) \in B_r(H^1(T^d) \times H^1(T^d) \times H^1(T^d))\), we can extend the local \(H^1(T^d)\) solution of Theorem 1.1 and Theorem 1.2 globally in time.

**Remark 1.2.** Theorem 1.3 follows from the a priori estimate which is obtained by the conservation law for \(H\). Proof of the a priori estimate is the same as the nonperiodic case (see [19] Proposition 7.2).

Furthermore, for the equation (1.2), we obtain the following result.

**Theorem 1.4.** Let \(d \geq 3\) and \(s_c = d/2 - 1\). Then, the equation (1.2) is locally well-posed for small data in \(H^{s_c}(T^d)\).

**Remark 1.3.** The results by Grünrock (14) do not contain the critical case \(s = s_c\).

The main tools of our results are the Strichartz estimate and the spaces \(U^p\) and \(V^p\). The Strichartz estimate on torus was proved by Bourgain ([4]) and improved by Bourgain ([5]) and Bourgain and Demeter ([6]). The spaces \(U^p\) and \(V^p\) are applied to prove the well-posedness and the scattering for KP-II equation at the scaling critical regularity by Hadac, Herr and Koch ([15], [16]). For the periodic case, the spaces \(U^p\) and \(V^p\) are used to prove the well-posedness of the nonlinear Schrödinger equations with power type nonlinearity at the scaling critical regularity by Herr, Tataru and Tzvetkov ([17]) and Wang ([27]).

To obtain the well-posedness of (1.1) at the scaling critical regularity, we will show the following bilinear estimate.

**Proposition 1.5.** Let \(d \geq 3\) and \(\sigma_1, \sigma_2 \in \mathbb{R}\setminus\{0\}\) satisfy \(\sigma_1 + \sigma_2 \neq 0\) and \(\sigma_1/\sigma_2 = m_1/m_2\) for some \(m_1, m_2 \in \mathbb{Z}\setminus\{0\}\). Assume \(s \geq s_c (= d/2 - 1)\). For any dyadic numbers \(N_1, N_2, N_3 \geq 1\), we have

\[
|||P_{N_3}(P_{N_1}u_1 \cdot P_{N_2}u_2)|||_{L^2(T^d)} \lesssim N_{\min}^s \left( \frac{N_{\min}}{N_{\max}} + \frac{1}{N_{\min}} \right)^\delta ||P_{N_1}u_1||_{Y^\sigma_{1/2}} ||P_{N_2}u_2||_{Y^\sigma_{1/2}}
\]

for some \(\delta > 0\), where \(\sigma := \sigma_1/m_1 = \sigma_2/m_2\), \(N_{\max} := \max_{1 \leq j \leq 3} N_j\), \(N_{\min} := \min_{1 \leq j \leq 3} N_j\).
Wang ([27]) proved (1.3) for the case $N_1 \sim N_3 \gtrsim N_2$. Therefore, Proposition 1.5 is the extension of his estimate. To prove Proposition 1.5, we will show the new bilinear estimate (Proposition 3.6) which is the estimate (1.3) for the case $N_1 \sim N_2 \gg N_3$.

Furthermore, we will show the trilinear estimate

$$
\left| N_{\text{max}} \int_0^T \int_{\mathbb{T}^d} \left( \prod_{j=1}^3 P_{N_j} u_j \right) \, dx dt \right| \lesssim N_{\text{min}}^s \left( \frac{N_{\text{min}}}{N_{\text{max}}} + \frac{1}{N_{\text{min}}} \right) \prod_{j=1}^3 \|P_{N_j} u_j\|_{Y^2_j}. \tag{1.4}
$$

by using Proposition 1.5. For the nonperiodic case in [19], we divided the integrals on the left-hand side of (1.4) into 8 piece of the form

$$
\int_{\mathbb{R}} \int_{\mathbb{R}} \left( \prod_{j=1}^3 Q_{\Delta}^{\sigma_j}(1_{[0,T]} P_{N_j} u_j) \right) \, dx dt
$$

with $Q_{\Delta}^{\sigma_j} \in \{Q_{\Delta}^{\sigma_j} \geq M, Q_{\Delta}^{\sigma_j} < M\}$ ($j = 1, 2, 3$) to recover the derivative loss. But for the periodic case, if we use such division, then we cannot apply the bilinear estimate (1.3) since $Q_{\Delta}^{\sigma_j}(1_{[0,T]} P_{N_j} u_j)$ is not localized at the time interval $[0, 2\pi|\sigma|^{-1})$. Therefore, we will use the division (see Formula (4.3))

$$
\int_0^T \int_{\mathbb{T}^d} \left( \prod_{j=1}^3 P_{N_j} u_j \right) \, dx dt = J_1 + J_2 + J_31 + J_32 + J_33.
$$

The integrands of the terms $J_31$, $J_32$ and $J_33$ are localized at the time interval $[0, 2\pi|\sigma|^{-1})$.

**Notation.** For $\lambda \in \mathbb{R}$, we define the integral on $\mathbb{T}_\lambda$:

$$
\int_{\mathbb{T}_\lambda} f(t) dt := \int_0^{2\pi \lambda^{-1}} f(t) dt.
$$

and the integral on $\mathbb{T}^d$:

$$
\int_{\mathbb{T}^d} f(x) dx := \int_{[0,2\pi]^d} f(x) dx.
$$

We denote the spatial Fourier coefficients for the function on $\mathbb{T}^d$ as

$$
\mathcal{F}_x[f](\xi) := \widehat{f}(\xi) := \int_{\mathbb{T}^d} f(x) e^{-i\xi \cdot x} dx, \quad \xi \in \mathbb{Z}^d
$$

and the space time Fourier transform as

$$
\mathcal{F}[f](\tau, \xi) = \widehat{f}(\tau, \xi) := \int_{\mathbb{R}} \int_{\mathbb{T}^d} f(t, x) e^{-i\tau \cdot x} e^{-ix \cdot \xi} dx dt, \quad \tau \in \mathbb{R}, \xi \in \mathbb{Z}^d.
$$

For $\sigma \in \mathbb{R}$, the free evolution $e^{it\sigma \Delta}$ on $L^2$ is given as a Fourier multiplier

$$
\mathcal{F}_x[e^{it\sigma \Delta} f](\xi) = e^{-it|\sigma|^2} \widehat{f}(\xi).
$$
We will use \( A \lesssim B \) to denote an estimate of the form \( A \leq CB \) for some constant \( C \) and write \( A \sim B \) to mean \( A \lesssim B \) and \( B \lesssim A \). We will use the convention that capital letters denote dyadic numbers, e.g. \( N = 2^n \) for \( n \in \mathbb{N} \) and for a dyadic summation we write \( \sum_N a_N := \sum_{n \in \mathbb{N}} a_{2^n} \) and \( \sum_{N \geq M} a_N := \sum_{n \in \mathbb{N}, 2^n \geq M} a_{2^n} \) for brevity. Let \( \chi \in C_0^\infty((-2, 2)) \) be an even, non-negative function such that \( \chi(s) = 1 \) for \( |s| \leq 1 \).

We define \( \psi_1(s) := \chi(s) \) and \( \psi_N(s) := \psi_1(N^{-1}s) - \psi_1(2N^{-1}s) \) for \( N \geq 2 \). We define frequency and modulation projections

\[
\hat{P}_S u(\xi) := 1_S(\xi) \hat{u}(\xi), \quad \hat{P}_N u(\xi) := \psi_N(|\xi|) \hat{u}(\xi), \quad \hat{Q}^\Delta_M u(\tau, \xi) := \psi_M(\tau + \sigma|\xi|^2) \hat{u}(\tau, \xi).
\]

for the set \( S \subset \mathbb{Z}^d \) and the dyadic numbers \( N, M \), where \( 1_S \) is the characteristic function of \( S \). Furthermore, we define \( Q^\Delta_M := \sum_{N \geq M} Q^\Delta_N \) and \( Q^\Delta_M := \text{Id} - Q^\Delta_M \).

For \( s \in \mathbb{R} \), we define the Sobolev space \( H^s(\mathbb{T}^d) \) as the space of all \( L^2(\mathbb{T}^d) \) functions for which the norm

\[
||f||_{H^s} := \left( \sum_{\xi \in \mathbb{Z}^d} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2 \right)^{1/2} \sim \left( \sum_{N \geq 1} N^{2s} ||P_N f||_{L^2(\mathbb{T}^d)}^2 \right)^{1/2}
\]

is finite.

The rest of this paper is planned as follows. In Section 2, we will give the definition and properties of the \( U^p \) space and \( V^p \) space. In Section 3, we will give the Strichartz estimates on torus and the bilinear estimates. In Section 4, we will give the trilinear estimates. In Section 5, we will give the proof of the well-posedness (Theorems [1.1, 1.2] and 1.4).

## 2. \( U^p, V^p \) Spaces and Their Properties

In this section, we define the \( U^p \) space and the \( V^p \) space, and introduce the properties of these spaces which are proved in [15], [16] and [17]. Throughout this section let \( \mathcal{H} \) be a separable Hilbert space over \( \mathbb{C} \).

We define the set of finite partitions \( \mathcal{Z} \) as

\[
\mathcal{Z} := \left\{ \{t_k\}_{k=0}^K | K \in \mathbb{N}, -\infty < t_0 < t_1 < \cdots < t_K \leq \infty \right\}
\]

and \( t_K = \infty \), we put \( v(t_K) := 0 \) for all functions \( v : \mathbb{R} \to \mathcal{H} \).

**Definition 1.** Let \( 1 \leq p < \infty \). For \( \{t_k\}_{k=0}^K \in \mathcal{Z} \) and \( \{\phi_k\}_{k=0}^{K-1} \subset \mathcal{H} \) with \( \sum_{k=0}^{K-1} ||\phi_k||_{\mathcal{H}}^p = 1 \) we call the function \( a : \mathbb{R} \to \mathcal{H} \) given by

\[
a(t) = \sum_{k=1}^K 1_{[t_{k-1}, t_k)}(t) \phi_{k-1}
\]
a \textquotedblleft U^p\textnormal{-atom}\textquotedblright. Furthermore, we define the atomic space

\[ U^p(\mathbb{R}; \mathcal{H}) := \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j \mid a_j : U^p\textnormal{-atom}, \lambda_j \in \mathbb{C} \text{ such that } \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\} \]

with the norm

\[ \|u\|_{U^p(\mathbb{R}; \mathcal{H})} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \mid u = \sum_{j=1}^{\infty} \lambda_j a_j, \text{ } a_j : U^p\textnormal{-atom}, \lambda_j \in \mathbb{C} \right\}. \]

\textbf{Definition 2.} Let \( 1 \leq p < \infty \). We define the space of the bounded \( p \)-variation

\[ V^p(\mathbb{R}; \mathcal{H}) := \{ v : \mathbb{R} \to \mathcal{H} \mid \|v\|_{V^p(\mathbb{R}; \mathcal{H})} < \infty \} \]

with the norm

\[ \|v\|_{V^p(\mathbb{R}; \mathcal{H})} := \sup_{\{t_k\}_{k=0}^{K}} \left( \sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{H^p}^p \right)^{1/p}. \]

Likewise, let \( V^p_{-rc}(\mathbb{R}; \mathcal{H}) \) denote the closed subspace of all right-continuous functions \( v \in V^p(\mathbb{R}; \mathcal{H}) \) with \( \lim_{t \to -\infty} v(t) = 0 \), endowed with the same norm \( \| \cdot \|_{V^p(\mathbb{R}; \mathcal{H})} \).

\textbf{Proposition 2.1} (\cite{15} Proposition 2.2, 2.4, Corollary 2.6). Let \( 1 \leq p < q < \infty \).

(\text{i}) \( U^p(\mathbb{R}; \mathcal{H}), V^p(\mathbb{R}; \mathcal{H}) \) and \( V^p_{-rc}(\mathbb{R}; \mathcal{H}) \) are Banach spaces.

(\text{ii}) The embeddings \( U^p(\mathbb{R}; \mathcal{H}) \hookrightarrow V^p_{-rc}(\mathbb{R}; \mathcal{H}) \hookrightarrow U^q(\mathbb{R}; \mathcal{H}) \hookrightarrow L^\infty(\mathbb{R}; \mathcal{H}) \) are continuous.

\textbf{Definition 3.} Let \( 1 \leq p < \infty \), \( s \in \mathbb{R} \) and \( \sigma \in \mathbb{R}\setminus\{0\} \). We define

\[ U^p_{\sigma \Delta} H^s := \{ u : \mathbb{R} \to H^s \mid e^{-it\sigma \Delta} u \in U^p(\mathbb{R}; H^s(\mathbb{T}^d)) \} \]

denoted by \( ||\cdot||_{U^p_{\sigma \Delta} H^s} \) with the norm \( ||u||_{U^p_{\sigma \Delta} H^s} := ||e^{-it\sigma \Delta} u||_{U^p(\mathbb{R}; H^s)} \).

\[ V^p_{\sigma \Delta} H^s := \{ v : \mathbb{R} \to H^s \mid e^{-it\sigma \Delta} v \in V^p_{-rc}(\mathbb{R}; H^s(\mathbb{T}^d)) \} \]

denoted by \( ||\cdot||_{V^p_{\sigma \Delta} H^s} \) with the norm \( ||v||_{V^p_{\sigma \Delta} H^s} := ||e^{-it\sigma \Delta} v||_{V^p(\mathbb{R}; H^s)} \).

\textbf{Remark 2.1.} We note that \( ||\mathcal{T}||_{U^p_{\sigma \Delta} H^s} = ||u||_{U^p_{\sigma \Delta} H^s} \) and \( ||\mathcal{T}||_{V^p_{\sigma \Delta} H^s} = ||v||_{V^p_{\sigma \Delta} H^s} \).

\textbf{Proposition 2.2} (\cite{15} Corollary 2.18). Let \( 1 < p < \infty \) and \( \sigma \in \mathbb{R}\setminus\{0\} \). We have

\[ ||Q^\Delta_{\geq M} u||_{L^2(\mathbb{R}; L^2)} \lesssim M^{-1/2} ||u||_{V^p_{\sigma \Delta} L^2}, \]  
\[ ||Q^\Delta_{< M} u||_{V^p_{\sigma \Delta} L^2} \lesssim ||u||_{V^p_{\sigma \Delta} L^2}, \]
\[ ||Q^\Delta_{\geq M} u||_{V^p_{\sigma \Delta} L^2} \lesssim ||u||_{V^p_{\sigma \Delta} L^2} \]  
\[ ||Q^\Delta_{< M} u||_{V^p_{\sigma \Delta} L^2} \lesssim ||u||_{V^p_{\sigma \Delta} L^2}. \]
Proposition 2.3 ([15] Proposition 2.19). Let
\[ T_0 : L^2(\mathbb{T}^d) \times \cdots \times L^2(\mathbb{T}^d) \to L^1_{\text{loc}}(\mathbb{T}^d) \]
be a m-linear operator and \( I \subset \mathbb{R} \) be an interval. Assume that for some \( 1 \leq p, q < \infty \),
\[ \| T_0(e^{it\sigma_1 \Delta} \phi_1, \ldots, e^{it\sigma_m \Delta} \phi_m) \|_{L^p(I; L^q_\ell)} \lesssim \prod_{i=1}^m \| \phi_i \|_{L^2}. \]
Then, there exists \( T : U^p_{\sigma_1 \Delta} L^2 \times \cdots \times U^p_{\sigma_m \Delta} L^2 \to L^p(I; L^q_\ell) \) satisfying
\[ \| T(u_1, \ldots, u_m) \|_{L^p(I; L^q_\ell)} \lesssim \prod_{i=1}^m \| u_i \|_{U^p_{\sigma_i \Delta} L^2} \]
such that \( T(u_1, \ldots, u_m)(t)(x) = T_0(u_1(t), \ldots, u_m(t))(x) \) a.e.

Proposition 2.4 ([15] Proposition 2.20). Let \( q > 1 \), \( E \) be a Banach space and \( T : U^q_{\sigma \Delta} L^2 \to E \) be a bounded, linear operator with \( \| Tu \|_E \leq C_q \| u \|_{U^q_{\sigma \Delta} L^2} \) for all \( u \in U^q_{\sigma \Delta} L^2 \). In addition, assume that for some \( 1 \leq p < q \) there exists \( C_p \in (0, C_q] \) such that the estimate \( \| Tu \|_E \leq C_p \| u \|_{U^p_{\sigma \Delta} L^2} \) holds true for all \( u \in U^p_{\sigma \Delta} L^2 \). Then, \( T \) satisfies the estimate
\[ \| Tu \|_E \lesssim C_p \left( 1 + \ln \frac{C_q}{C_p} \right) \| u \|_{U^p_{\sigma \Delta} L^2}, \quad u \in V^p_{\sigma \Delta} L^2, \]
where implicit constant depends only on \( p \) and \( q \).

Next, we define the function spaces which will be used to construct the solution.

Definition 4. Let \( s, \sigma \in \mathbb{R} \).

(i) We define \( Z^s_\sigma \) as the space of all functions \( u : \mathbb{R} \to H^s(\mathbb{T}^d) \) such that for every \( \xi \in \mathbb{Z}^d \) the map \( t \mapsto e^{it\sigma|\xi|^2} \widehat{u(t)}(\xi) \) is in \( U^2(\mathbb{R}; \mathbb{C}) \), and for which the norm
\[ \| u \|_{Z^s_\sigma} := \left( \sum_{\xi \in \mathbb{Z}^d} |\xi|^{2s} \| e^{it\sigma|\xi|^2} \widehat{u(t)}(\xi) \|_{U^2(\mathbb{R}; \mathbb{C})}^2 \right)^{1/2} \]
is finite.

(ii) We define \( Y^s_\sigma \) as the space of all functions \( u : \mathbb{R} \to H^s(\mathbb{T}^d) \) such that for every \( \xi \in \mathbb{Z}^d \) the map \( t \mapsto e^{it\sigma|\xi|^2} \widehat{u(t)}(\xi) \) is in \( V^2_{-|\xi|}(\mathbb{R}; \mathbb{C}) \), and for which the norm
\[ \| u \|_{Y^s_\sigma} := \left( \sum_{\xi \in \mathbb{Z}^d} |\xi|^{2s} \| e^{it\sigma|\xi|^2} \widehat{u(t)}(\xi) \|_{V^2(\mathbb{R}; \mathbb{C})}^2 \right)^{1/2} \]
is finite.
Remark 2.2 ([15] Remark 2.23). Let $E$ be a Banach space of continuous functions $f : \mathbb{R} \to \mathcal{H}$, for some Hilbert space $\mathcal{H}$. We also consider the corresponding restriction space to the interval $I \subset \mathbb{R}$ by

$$E(I) = \{ u \in C(I, \mathcal{H}) | \exists v \in E \text{ s.t. } v(t) = u(t), \ t \in I \}$$

endowed with the norm $\| u \|_{E(I)} = \inf \{ \| v \|_{E} | v(t) = u(t), \ t \in I \}$. Obviously, $E(I)$ is also a Banach space.

Proposition 2.5 ([17] Proposition 2.8, Corollary 2.9). The embeddings

$$U^2_{\sigma} H^s \hookrightarrow Z^s_{\sigma} \hookrightarrow Y^s_{\sigma} \hookrightarrow V^2_{\sigma} H^s$$

are continuous. Furthermore if $Z^d = \cup C_k$ be a partition of $Z^d$, then

$$\left( \sum_k \| P_{C_k} u \|_{Y^2_{\sigma} H^s}^2 \right)^{1/2} \lesssim \| u \|_{Y^s_{\sigma}}. \quad (2.3)$$

For $f \in L^1_{loc}(\mathbb{R}; L^2(\mathbb{T}^d))$ and $\sigma \in \mathbb{R}$, we define

$$I_{\sigma}[f](t) := \int_0^t e^{i(t-t')\sigma} f(t') dt'.$$

for $t \geq 0$ and $I_{\sigma}[f](t) = 0$ for $t < 0$.

Proposition 2.6 ([17] Proposition 2.10). For $s \geq 0$, $T > 0$, $\sigma \in \mathbb{R}\{0\}$ and $f \in L^1([0, T); H^s(\mathbb{T}^d))$ we have $I_{\sigma}[f] \in Z^s_{\sigma}([0, T))$ and

$$\| I_{\sigma}[f] \|_{Z^s_{\sigma}([0, T])} \leq \sup_{v \in Y^{-s}_{\sigma}([0, T))} \left\| v \|_{Y^{-s}_{\sigma}} = 1 \right\|_0^T \int_{\mathbb{T}^d} f(t, x) \overline{v(t, x)} dt.$$

3. STRICHARTZ AND BILINEAR STRICHARTZ ESTIMATES

In this section, we introduce some Strichartz estimates on torus proved in [6, 17, 27] and the bilinear estimate proved in [27]. Furthermore, we show the new bilinear estimate (Proposition 3.6).

For a dyadic number $N \geq 1$, we define $C_N$ as the collection of disjoint cubes $C \subset \mathbb{Z}^d$ of side-length $N$ with arbitrary center and orientation.

Proposition 3.1 ([6] Theorem 2.3, 2.4, Remark 2.5). Let $d \geq 1$, $\sigma \in \mathbb{R}\{0\}$ and $m \in \mathbb{Z}\{0\}$. Assume $s \geq d/2 - (d+2)/p$ if $p > 2(d+2)/d$ and $s > d/2 - (d+2)/p$ if $p = 2(d+2)/d$.

(i) For any dyadic number $N \geq 1$, we have

$$\| P_N e^{it\sigma} \varphi \|_{L^p(T_{[\sigma/m]} \times \mathbb{T}^d)} \lesssim N^s \| P_N \varphi \|_{L^2(\mathbb{T}^d)}. \quad (3.1)$$
(ii) For any $C \in C_N$ with $N \geq 1$, we have
\[
||P_C e^{it\sigma \Delta} \varphi||_{L^p(T_{|\sigma/m|} \times \mathbb{T}^d)} \lesssim N^s ||P_C \varphi||_{L^2(T^d)}.
\] (3.2)

Remark 3.1. (i) The estimate (3.2) follows from (3.1) and the Galilean transformation (see Formula (5.7), (5.8) in [4]).
(ii) Implicit constants in the estimates (3.1), (3.2) depend on $m$ and $\sigma$.
(iii) The estimates (3.1), (3.2) also hold for $s > 0$ and $p < 2(d+2)/d$ since $L^2(d+2)/d(T_{|\sigma/m|} \times \mathbb{T}^d) \subset L^p(T_{|\sigma/m|} \times \mathbb{T}^d)$ holds for $p < 2(d+2)/d$.

For dyadic numbers $N \geq 1$ and $M \geq 1$, we define $\mathcal{R}_M(N)$ as the collection of all sets of the form
\[
\{\xi_0 + [-N, N]^d \cap \{\xi \in \mathbb{Z}^d \mid a \cdot \xi - A \leq M\}
\]
with some $\xi_0 \in \mathbb{Z}^d$, $a \in \mathbb{R}^d$, $|a| = 1$ and $A \in \mathbb{R}$.

Proposition 3.2 ([17] Proposition 3.3, [27] Formula (19)). Let $d \geq 1$, $\sigma \in \mathbb{R} \setminus \{0\}$ and $m \in \mathbb{Z} \setminus \{0\}$. For any $R \in \mathcal{R}_M(N)$ with $N \geq M \geq 1$, we have
\[
||P_R e^{it\sigma \Delta} \varphi||_{L^\infty(T_{|\sigma/m|} \times \mathbb{T}^d)} \lesssim M^{1/2} N^{(d-1)/2} ||P_R \varphi||_{L^2(T^d)}. \tag{3.3}
\]

By using the Hölder inequality with (3.2) for $p < 4$ and (3.3), we have the following $L^4$-Strichartz estimate.

Proposition 3.3. Let $d \geq 1$, $\sigma \in \mathbb{R} \setminus \{0\}$ and $m \in \mathbb{Z} \setminus \{0\}$. Assume $s \geq d/4 - 1/2$ if $d \geq 3$ and $s > 0$ if $d = 1, 2$. For any $R \in \mathcal{R}_M(N)$ with $N \geq M \geq 1$, we have
\[
||P_R e^{it\sigma \Delta} \varphi||_{L^4(T_{|\sigma/m|} \times \mathbb{T}^d)} \lesssim N^s \left(\frac{M}{N}\right)^{\delta} ||P_R \varphi||_{L^2(T^d)} \tag{3.4}
\]
for some $\delta > 0$.

By Propositions 2.3 and 5.1, we have following:

Corollary 3.4. Let $\sigma \in \mathbb{R} \setminus \{0\}$ and $m \in \mathbb{Z} \setminus \{0\}$. Assume $s \geq d/2 - (d+2)/p$ if $p > 2(d+2)/d$, $s > d/2 - (d+2)/p$ if $p = 2(d+2)/d$ and $s > 0$ if $p < 2(d+2)/d$.

For any dyadic number $N \geq 1$ and $C \in C_N$, we have
\[
||P_N u||_{L^p(T_{|\sigma/m|} \times \mathbb{T}^d)} \lesssim N^s ||P_N u||_{U^p_{\sigma \Delta} L^2}, \tag{3.5}
\]
\[
||P_C u||_{L^p(T_{|\sigma/m|} \times \mathbb{T}^d)} \lesssim N^s ||P_C u||_{U^p_{\sigma \Delta} L^2}. \tag{3.6}
\]

Next, we give the bilinear estimates.
Proposition 3.5 ([27] Proposition 4.2). Let \( d \geq 1 \) and \( \sigma_1, \sigma_2 \in \mathbb{R} \setminus \{0\} \) satisfy \( \sigma_1/\sigma_2 = m_1/m_2 \) for some \( m_1, m_2 \in \mathbb{Z} \setminus \{0\} \). Assume \( s \geq s_c (= d/2 - 1) \) if \( d \geq 3 \) and \( s > 0 \) if \( d = 1, 2 \). For any dyadic numbers \( H \) and \( L \) with \( H \leq L \geq 1 \), we have

\[
\|P_H u_1 \cdot P_L u_2\|_{L^2(\mathbb{T}^d)} \lesssim L^s \left( \frac{L}{H} + \frac{1}{L} \right)^\delta \|P_H u_1\|_{Y^0_{s_1}} \|P_L u_2\|_{Y^0_{s_2}} \tag{3.7}
\]

for some \( \delta > 0 \), where \( \sigma := \sigma_1/m_1 = \sigma_2/m_2 \).

Remark 3.2. Wang proved (3.7) only for \( d \geq 5 \) (and \( \sigma_1 = \sigma_2 = 1 \)). To obtain (3.7) for \( 1 \leq d \leq 4 \), we choose \( p = q = 4 \) and use (3.4) as above in the proof of [27] Proposition 4.2 for \( k = 1, n \geq 5 \). The other parts of the proof are the same way.

We get the following bilinear estimate.

Proposition 3.6. Let \( d \geq 1 \) and \( \sigma_1, \sigma_2 \in \mathbb{R} \setminus \{0\} \) satisfy \( \sigma_1 + \sigma_2 \neq 0 \) and \( \sigma_1/\sigma_2 = m_1/m_2 \) for some \( m_1, m_2 \in \mathbb{Z} \setminus \{0\} \). Assume \( s \geq s_c (= d/2 - 1) \) if \( d \geq 3 \) and \( s > 0 \) if \( d = 1, 2 \). For any dyadic numbers \( L, H, H' \) with \( H \sim H' \gg L \geq 1 \), we have

\[
\|P_L (P_H u_1 \cdot P_{H'} u_2)\|_{L^2(\mathbb{T}^d)} \lesssim L^s \left( \frac{L}{H} + \frac{1}{L} \right)^\delta \|P_H u_1\|_{Y^0_{s_1}} \|P_{H'} u_2\|_{Y^0_{s_2}} \tag{3.8}
\]

for some \( \delta > 0 \), where \( \sigma := \sigma_1/m_1 = \sigma_2/m_2 \).

Proof. We decompose \( P_H u_1 = \sum_{C_1 \in \mathcal{C}_L} P_{C_1} P_H u_1 \). For fixed \( C_1 \subset C_L \), let \( \xi_0 \in C_1 \) be the center of \( C_1 \). Since \( \xi_1 \subset C_1 \) and \( |\xi_1 + \xi_2| \leq 2L \) imply \( |\xi_2 + \xi_0| \leq 3L \), we obtain

\[
\|P_L (P_{C_1} P_H u_1 \cdot P_{H'} u_2)\|_{L^2(\mathbb{T}^d)} \leq \|P_{C_1} P_H u_1 \cdot P_{C_2(C_1)} P_{H'} u_2\|_{L^2(\mathbb{T}^d)}.
\]

where \( C_2(C_1) := \{\xi_2 \in \mathbb{Z}^d | |\xi_2 + \xi_0| \leq 3L\} \). If we prove

\[
\|P_{C_1} P_H u_1 \cdot P_{C_2(C_1)} P_{H'} u_2\|_{L^2(\mathbb{T}^d)} \lesssim L^s \left( \frac{L}{H} + \frac{1}{L} \right)^\delta \|P_{C_1} P_H u_1\|_{Y^2_{s_1 \Delta L^2}} \|P_{C_2(C_1)} P_{H'} u_2\|_{Y^2_{s_2 \Delta L^2}} \tag{3.9}
\]

for some \( \delta > 0 \), then we obtain

\[
\|P_L (P_H u_1 \cdot P_{H'} u_2)\|_{L^2(\mathbb{T}^d)} \lesssim \sum_{C_1 \in \mathcal{C}_L} L^s \left( \frac{L}{H} + \frac{1}{L} \right)^\delta \|P_{C_1} P_H u_1\|_{Y^2_{s_1 \Delta L^2}} \|P_{C_2(C_1)} P_{H'} u_2\|_{Y^2_{s_2 \Delta L^2}} \lesssim L^s \left( \frac{L}{H} + \frac{1}{L} \right)^\delta \left( \sum_{C_1 \in \mathcal{C}_L} \|P_{C_1} P_H u_1\|_{Y^2_{s_1 \Delta L^2}}^2 \right)^{1/2} \left( \sum_{C_1 \in \mathcal{C}_L} \|P_{C_2(C_1)} P_{H'} u_2\|_{Y^2_{s_2 \Delta L^2}}^2 \right)^{1/2}
\]
and the proof is complete by (2.3). The estimate (3.9) follows by interpolation between

\begin{equation}
\| P_{C_1} P_{H} u_1 \cdot P_{C_2(C_1)} P_{H'} u_2 \|_{L^2(T|\sigma| \times T^d)} \lesssim L^s \| P_{C_1} P_{H} u_1 \|_{U^{\sigma_1}_s L^2} \| P_{C_2(C_1)} P_{H'} u_2 \|_{U^{\sigma_2}_s L^2} \end{equation}

(3.10)

and

\begin{equation}
\| P_{C_1} P_{H} u_1 \cdot P_{C_2(C_1)} P_{H'} u_2 \|_{L^2(T|\sigma| \times T^d)} \lesssim L^s \left( \frac{L}{H} + \frac{1}{L} \right)^{\delta'} \| P_{C_1} P_{H} u_1 \|_{U^{\sigma_1}_s L^2} \| P_{C_2(C_1)} P_{H'} u_2 \|_{U^{\sigma_2}_s L^2} \end{equation}

(3.11)

via Proposition 2.3. The estimate (3.10) is proved by the Cauchy-Schwartz inequality and (3.6). While the estimate (3.11) follows from the bilinear Strichartz estimate

\begin{equation}
\| P_{C_1} P_{H}( e^{it\sigma_1} \phi_1 ) \cdot P_{C_2(C_1)} P_{H'}( e^{it\sigma_2} \phi_2 ) \|_{L^2(T|\sigma| \times T^d)} \lesssim L^s \left( \frac{L}{H} + \frac{1}{L} \right)^{\delta'} \| P_{C_1} P_{H} \phi_1 \|_{L^2(T^d)} \| P_{C_2(C_1)} P_{H'} \phi_2 \|_{L^2(T^d)}. \end{equation}

(3.12)

and Proposition 2.3.

Now, we prove the estimate (3.12) for some $\delta' > 0$. Put $u_j = e^{it\sigma_j} \phi_j$ ($j = 1, 2$). We note that $u_1 u_2$ is periodic function with period $2\pi |\sigma|^{-1}$ with respect to $t$ since $\sigma_1/\sigma_2 = m_1/m_2 \in \mathbb{Q}$. We partition $C_1 = \cup_k R_{1,k}$ and $C_2(C_1) = \cup_l R_{2,l}$ into almost disjoint strips as

- $R_{1,k} = \{ \xi \in C_1 | \xi_1 \in [\xi_0 |Mk|, |\xi_0|M(k+1)] \}, \ k \sim H/M$,
- $R_{2,l} = \{ \xi \in C_2(C_1) | \xi_2 \in [-|\xi_0|M(l+1), -|\xi_0|Ml] \}, \ l \sim H/M$, 

where $M = \max\{L^2/H, 1\}$. The condition for $k$ and $l$ as above follows from $|\xi_0| \sim H \gg L$. To obtain the almost orthogonality for the summation

$$P_{C_1} P_{H} u_1 \cdot P_{C_2(C_1)} P_{H'} u_2 = \sum_k \sum_l P_{R_{1,k}} P_{H} u_1 \cdot P_{R_{2,l}} P_{H'} u_2,$$

we use the argument in [17] Proposition 3.5. Since $L^2 \lesssim M^2 k \sim M^2 l$, we have

$$|\xi_1|^2 = \frac{|\xi_1 \cdot \xi_0|^2}{|\xi_0|^2} + |\xi_1 - \xi_0|^2 - \frac{|(\xi_1 - \xi_0) \cdot \xi_0|^2}{|\xi_0|^2} = M^2 k^2 + O(M^2 k)$$

and

$$|\xi_2|^2 = \frac{|\xi_2 \cdot \xi_0|^2}{|\xi_0|^2} + |\xi_2 + \xi_0|^2 - \frac{|(\xi_2 + \xi_0) \cdot \xi_0|^2}{|\xi_0|^2} = M^2 l^2 + O(M^2 k)$$

for any $\xi_1 \in R_{1,k}$ and $\xi_2 \in R_{2,l}$. More precisely, there exist the constants $A_1, A_2 > 0$ which do not depend on $k$ and $l$, such that $\xi_1 \in R_{1,k}$ and $\xi_2 \in R_{2,l}$ satisfy

$$M^2 k^2 - A_1 M^2 k \leq |\xi_1|^2 \leq M^2 k^2 + A_1 M^2 k, \ M^2 l^2 - A_2 M^2 k \leq |\xi_2|^2 \leq M^2 l^2 + A_2 M^2 k.$$
Furthermore, \( \sigma_1 k^2 + \sigma_2 l^2 \neq 0 \) because
\[
|\sigma_1 \xi_1|^2 + \sigma_2 |\xi_2|^2 = |(\sigma_1 + \sigma_2)\xi_1 - \sigma_2 (\xi_1 - \xi_2) \cdot (\xi_1 + \xi_2)| \sim H^2,
\]
where we used the condition \( \sigma_1 + \sigma_2 \neq 0 \). Therefore, the expression \( P_{R_{1,k}} P_H u_1 \cdot P_{R_{2,l}} P_H u_2 \) are localized at time frequency \(-M^2 (\sigma_1 k^2 + \sigma_2 l^2) + O(M^2 k)\). This implies the almost orthogonality
\[
||P_{C_1} P_H u_1 \cdot P_{C_2(C_1)} P_H u_2 ||_{L^2(T|_{\sigma}| \times T^d)} \lesssim \sum_k \sum_l ||P_{R_{1,k}} P_H u_1 \cdot P_{R_{2,l}} P_H u_2 ||_{L^2(T|_{\sigma}| \times T^d)}^2
\]
since \( P_{C_1} P_H u_1 \cdot P_{C_2(C_1)} P_H u_2 \) is periodic function with period \( 2\pi|\sigma|^{-1} \) with respect to \( t \). By the Cauchy-Schwartz inequality and (3.4), we have
\[
||P_{R_{1,k}} P_H u_1 \cdot P_{R_{2,l}} P_H u_2 ||_{L^2(T|_{\sigma}| \times T^d)} \lesssim ||P_{R_{1,k}} P_H u_1 ||_{L^4(T|_{\sigma}| \times T^d)} ||P_{R_{2,l}} P_H u_2 ||_{L^4(T|_{\sigma}| \times T^d)}
\]
\[
\lesssim L^s(M/L)^{-\delta'} ||P_{R_{1,k}} P_H \phi_1 ||_{L^2(T^d)} ||P_{R_{2,l}} P_H \phi_2 ||_{L^2(T^d)}
\]
for some \( \delta' > 0 \) and any \( s \geq s_c \) if \( d \geq 3 \) and \( s > 0 \) if \( d = 1, 2 \) since \( R_{1,k} \in \mathcal{R}_M(L) \), \( R_{2,l} \in \mathcal{R}_M(2L) \). Therefore, we obtain (3.12) by the \( L^2 \)-orthogonality and \( M \leq L^2/H + 1 \).

**Remark 3.3.** Proposition 3.5 is implied from Propositions 3.5 and 3.6

To deal with large data at the scaling subcritical regularity, we show the following.

**Proposition 3.7.** Let \( d \geq 1 \) and \( \sigma_1, \sigma_2 \in \mathbb{R}\{0\} \). Assume \( s > s_0 := \max\{s_c, 0\} \).

For any dyadic numbers \( N_1, N_2, N_3 \geq 1 \) and \( 0 < T \leq 2\pi|\sigma|^{-1} \), we have
\[
||P_{N_3}(P_{N_1} u_1 \cdot P_{N_2} u_2) ||_{L^2([0,T) \times T^d)} \lesssim T^s N_{\min}^s \left( \frac{N_{\min}}{N_{\max}} + \frac{1}{N_{\min}} \right)^{\delta} ||P_{N_1} u_1 ||_{Y_{\sigma_1}^0} ||P_{N_2} u_2 ||_{Y_{\sigma_2}^0}
\]
(3.13)
for some \( \delta > 0 \) and \( \epsilon > 0 \), where \( \sigma := \max\{|\sigma_1|, |\sigma_2|\} \), \( N_{\max} := \max N_j \), \( N_{\min} := \min_{1 \leq j \leq 3} N_j \).

**Proof.** We first prove the case \( N_1 \sim N_2 \gg N_3 \). By (3.6) and the embedding \( V_{\sigma_1}^2 L^2 \hookrightarrow U_{\sigma_1} p L^2 \), we have
\[
||P_C u ||_{L^p(T^d \times T^d)} \lesssim N^{\max\{d/2 - (d+2)/p, a\}} ||P_C u ||_{U_{\sigma_1}^p L^2} \lesssim T^{\epsilon'/4} N^{s_0/2 + \epsilon''} ||P_C u ||_{V_{\sigma_1}^2 L^2}
\]
for any \( p > 4, a > 0, C \in \mathcal{C}_N \) with \( N \geq 1 \), and \( u \in V_{\sigma_1}^2 L^2 \). Therefore, for sufficient small \( \epsilon'' > 0 \), there exists \( 0 < \epsilon' < 1 \) such that
\[
||P_C u ||_{L^4([0,T) \times T^d)} \lesssim ||1||_{L^{4/(1-\epsilon')}([0,T) \times T^d)} ||P_C u ||_{L^4(1-\epsilon')(T^d \times T^d)} \lesssim T^{\epsilon'/4} N^{s_0/2 + \epsilon''} ||P_C u ||_{V_{\sigma_1}^2 L^2}
\]
(3.14)
for any $0 < T \leq 2\pi \sigma^{-1}$. The Cauchy-Schwartz inequality and (3.14) with $\epsilon'' = (s - s_0)/3$ imply

$$
||P_{C_1} P_{N_1} u_1 \cdot P_{C_2(C_1)} P_{N_2} u_2||_{L^2(0, T) \times \mathbb{T}^d}
\lesssim T^{\epsilon'/2} N_3^{(s - s_0)/3} ||P_{C_1} P_{N_1} u_1||_{V^2_{\sigma_1 \Delta} L^2} ||P_{C_2(C_1)} P_{N_2} u_2||_{V^2_{\sigma_2 \Delta} L^2},
$$

(3.15)

where $C_1$ and $C_2(C_1)$ are same as in the proof of Proposition 3.6. Therefore, we have (3.13) by using (3.15) instead of (3.12) in the proof of Proposition 3.6.

For the case $N_1 \sim N_3 \gtrsim N_2$, we also obtain (3.13) by using above argument in the proof of [27] Proposition 4.2.

Remark 3.4. Since the bilinear Stricartz estimate (3.12) is not used to obtain (3.15), the conditions $\sigma_1 + \sigma_2 \neq 0$ and $\sigma_1/\sigma_2 = m_1/m_2$ for some $m_1$, $m_2 \in \mathbb{Z} \setminus \{0\}$ are not necessary for (3.13).

4. Trilinear estimates and the proof of the well-posedness

In this section, we give the trilinear estimates which will be used to prove the well-posedness.

Lemma 4.1 ([19] Lemma 4.1). Let $d \geq 1$. We assume that $\sigma_1$, $\sigma_2$, $\sigma_3 \in \mathbb{R} \setminus \{0\}$ satisfy $(\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) \neq 0$ and $(\tau_1, \xi_1)$, $(\tau_2, \xi_2)$, $(\tau_3, \xi_3) \in \mathbb{R} \times \mathbb{R}^d$ satisfy $\tau_1 + \tau_2 + \tau_3 = 0$, $\xi_1 + \xi_2 + \xi_3 = 0$.

(i) If there exist $1 \leq i, j \leq 3$ such that $|\xi_i| \ll |\xi_j|$, then we have

$$
\max_{1 \leq j \leq 3} |\tau_j + \sigma_j |\xi_j|^2 \gtrsim \max_{1 \leq j \leq 3} |\xi_j|^2.
$$

(4.1)

(ii) If $|\xi_1| \sim |\xi_2| \sim |\xi_3|$ and $\sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$, then we have (4.1).

Now, we give the trilinear estimates under the condition $\sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$.

Proposition 4.2. Let $d \geq 3$ and $\sigma_1$, $\sigma_2$, $\sigma_3 \in \mathbb{R} \setminus \{0\}$ satisfy $\sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$ and $\sigma_1/\sigma_2 = m_1/m_2$, $\sigma_2/\sigma_3 = m_2/m_3$ for some $m_1$, $m_2$, $m_3 \in \mathbb{Z} \setminus \{0\}$. Assume $s \geq s_c (= d/2 - 1)$. For any dyadic numbers $N_1$, $N_2$, $N_3 \geq 1$ with $N_{\max} \geq 2$, $0 < T \leq 2\pi |\sigma|^{-1}$ and $P_{N_j} u_j \in V^2_{\sigma_j} L^2$ ($j = 1, 2, 3$), we have

$$
N_{\max} \int_0^T \int_{\mathbb{T}^d} \left( \prod_{j=1}^3 P_{N_j} u_j \right) \, dx \, dt \lesssim N_{\min} \left( \frac{N_{\min}}{N_{\max}} + \frac{1}{N_{\min}} \right)^{\delta} \prod_{j=1}^3 ||P_{N_j} u_j||_{V^{\delta}_{\sigma_j}}
$$

(4.2)

for some $\delta > 0$, where $\sigma := \sigma_1/m_1 = \sigma_2/m_2 = \sigma_3/m_3$, $N_{\max} = \max_{1 \leq j \leq 3} N_j$, $N_{\min} = \min_{1 \leq j \leq 3} N_j$. 

Proof. We define \( u_{j,T} := 1_{[0,T]}u_j \) (\( j = 1, 2, 3 \)). Furthermore for sufficiently large constant \( C \), we put \( M := C^{-1}N_{\max}^2 \) and decompose \( Id = Q_{\leq M}^\sigma + Q_{\geq M}^\sigma \) (\( j = 1, 2, 3 \)). We divide the integral on the left-hand side of (1.2) into

\[
\int_0^T \int_{\mathbb{T}^d} \left( \prod_{j=1}^3 P_{N_j} u_j \right) \, dx \, dt = J_1 + J_2 + J_{31} + J_{32} + J_{33},
\]

where

\[
J_1 = \int_\mathbb{R} \int_{\mathbb{T}^d} \left( \prod_{j=1}^3 Q_{\leq M}^\sigma P_{N_j} u_{j,T} \right) \, dx \, dt, \quad J_2 = \int_\mathbb{R} \int_{\mathbb{T}^d} \left( \prod_{j=1}^3 Q_{\geq M}^\sigma P_{N_j} u_{j,T} \right) \, dx \, dt,
\]

\[
J_{31} = \int_\mathbb{R} \int_{\mathbb{T}^d} \left( Q_{\geq M}^\sigma P_{N_1} u_{1,T} \right) \left( Q_{\leq M}^\sigma P_{N_2} u_{2,T} \right) P_{N_3} u_{3,3T} \, dx \, dt,
\]

\[
J_{32} = \int_\mathbb{R} \int_{\mathbb{T}^d} \left( Q_{\geq M}^\sigma P_{N_1} u_{1,T} \right) \left( Q_{\leq M}^\sigma P_{N_2} u_{2,T} \right) P_{N_3} u_{3,3T} \, dx \, dt,
\]

\[
J_{33} = \int_\mathbb{R} \int_{\mathbb{T}^d} \left( Q_{\leq M}^\sigma P_{N_1} u_{1,T} \right) \left( Q_{\geq M}^\sigma P_{N_2} u_{2,T} \right) P_{N_3} u_{3,3T} \, dx \, dt.
\]

By Plancherel’s theorem, Lemma [1.1] and \( N_{\max} \geq 2 \), we have

\[
J_1 = 0.
\]

**Estimate for \( J_2 \)**

We can assume \( N_1 \geq N_2 \geq N_3 \) by the symmetry. By the Hölder inequality, we have

\[
|J_2| \lesssim \|Q_{\leq M}^\sigma P_{N_1} u_{1,T}\|_{L^2(\mathbb{R} \times \mathbb{T}^d)} \|Q_{\geq M}^\sigma P_{N_2} u_{2,T}\|_{L^2(\mathbb{R} \times \mathbb{T}^d)} \|Q_{\geq M}^\sigma P_{N_3} u_{3,T}\|_{L^\infty(\mathbb{R} \times \mathbb{T}^d)}.
\]

Furthermore by (2.1), \( M \sim N_{\max}^2 \) and the embedding \( Y_{\sigma_j}^0 \hookrightarrow V_{\sigma_j}^2 L^2 \), we have

\[
\|Q_{\leq M}^\sigma P_{N_j} u_{j,T}\|_{L^2(\mathbb{R} \times \mathbb{T}^d)} \lesssim N_{\max}^{-1} \|P_{N_j} u_{j,T}\|_{V_{\sigma_j}^{2,\Delta} L^2} \lesssim N_{\max}^{-1} \|P_{N_j} u_{j}\|_{Y_{\sigma_j}^0}
\]

for \( j = 1, 2 \) since \( \|1_{[0,T]}f\|_{Y_{\sigma_j}^0} \lesssim \|f\|_{Y_{\sigma_j}^0} \) for any \( T > 0 \) and \( f \in Y_{\sigma_j}^0 \). While by the Sobolev inequality, the embeddings \( Y_{\sigma_3}^0 \hookrightarrow V_{\sigma_3}^2 L^2 \hookrightarrow L^\infty(\mathbb{R}; L^2(\mathbb{T}^d)) \) and (2.2), we have

\[
\|Q_{\geq M}^\sigma P_{N_3} u_{3,T}\|_{L^\infty(\mathbb{R} \times \mathbb{T}^d)} \lesssim N_3^{d/2} \|P_{N_3} u_{3,T}\|_{L^\infty(\mathbb{R}; L^2(\mathbb{T}^d))} \lesssim N_3^{d/2} \|P_{N_3} u_{3,T}\|_{Y_{\sigma_3}^0}
\]

since \( \|1_{[0,T]}f\|_{Y_{\sigma_3}^0} \lesssim \|f\|_{Y_{\sigma_3}^0} \) for any \( T > 0 \) and \( f \in Y_{\sigma_3}^0 \). Therefore, we obtain

\[
|N_{\max} J_2| \lesssim N_{\min}^{d/2-1} N_{\max}^{d/2} \prod_{j=1}^3 \|P_{N_j} u_j\|_{Y_{\sigma_j}^0}.
\]

**Estimate for \( J_{31} \)**
By the Cauchy-Schwartz inequality, we have
\[ |J_{31}| \leq ||Q_{> M}^{1/2} P_{N_1} u_{1,T}||_{L^2(\mathbb{R} \times T^d)} \|	ilde{P}_{N_1} (Q_{< M}^{1/2} P_{N_2} u_{2,T} \cdot P_{N_3} u_{3,T})||_{L^2(\mathbb{R} \times T^d)} \]
\[ \leq ||Q_{> M}^{1/2} P_{N_1} u_{1,T}||_{L^2(\mathbb{R} \times T^d)} \|	ilde{P}_{N_1} (Q_{< M}^{1/2} P_{N_2} u_{2,T} \cdot P_{N_3} u_{3,T})||_{L^2(T|\sigma| \times T^d)} \]
since \(0 < T \leq 2\pi|\sigma|^{-1}\) and \(P_{N_1} = \tilde{P}_{N_1} P_{N_1} + P_{N_1} + P_{2N_1}\). Furthermore by (2.1), \(M \sim N_{\text{max}}^2\) and the embedding \(Y^0_{\sigma_1} \hookrightarrow V^2 \mathcal{T}^L\), we have
\[ ||Q_{> M}^{1/2} P_{N_1} u_{1,T}||_{L^2(\mathbb{R} \times T^d)} \leq N_{\text{max}}^{-1} ||P_{N_1} u_{1,T}||_{Y^0_{\sigma_1} L^2} \leq N_{\text{max}}^{-1} ||P_{N_1} u_{1,T}||_{Y^0_{\sigma_1}} \]
since \(|1_{[0,T]} f||_{Y^0_{\sigma_1}} \lesssim ||f||_{Y^0_{\sigma_1}}\) for any \(T > 0\) and \(f \in Y^0_{\sigma_1}\). While by (1.3) and (2.2), we have
\[ ||\tilde{P}_{N_1} (Q_{< M}^{1/2} P_{N_2} u_{2,T} \cdot P_{N_3} u_{3,T})||_{L^2(T|\sigma| \times T^d)} \leq N_{\text{min}} \left( \frac{N_{\text{min}}}{N_{\text{max}}} + \frac{1}{N_{\text{min}}} \right)^{\delta} ||P_{N_2} u_{2,T}||_{Y^0_{\sigma_2}} ||P_{N_3} u_{3,T}||_{Y^0_{\sigma_3}} \]
for some \(\delta > 0\) since \(|1_{[0,T]} f||_{Y^0_{\sigma_2}} \lesssim ||f||_{Y^0_{\sigma_2}}\) for any \(T > 0\) and \(f \in Y^0_{\sigma_2}\). Therefore, we obtain
\[ |N_{\text{max}} J_{31}| \leq N_{\text{max}} \left( \frac{N_{\text{min}}}{N_{\text{max}}} + \frac{1}{N_{\text{min}}} \right)^{\delta} \prod_{j=1}^{3} ||P_{N_j} u_j||_{Y^0_{\sigma_j}} \]
The estimates for \(J_{32}\) and \(J_{33}\) can be obtained by the same way as the estimate for \(J_{31}\). \(\square\)

We note that
\[ ||Q_{> M} (1_{[0,T]} u)||_{L^2(\mathbb{R} \times T^d)} \leq T^{1/2} ||u||_{L^\infty(\mathbb{R}; L^2(\mathbb{R}))} \lesssim T^{1/2} ||u||_{V^2_{\sigma} L^2} \]  \hspace{1cm} (4.4)
by the embedding \(V^2_{\sigma} L^2 \hookrightarrow L^\infty(\mathbb{R}; L^2(\mathbb{R}))\). Therefore for any \(\epsilon > 0\), we have
\[ ||Q_{> M} (1_{[0,T]} u)||_{L^2(\mathbb{R} \times T^d)} \lesssim T^\epsilon M^{-1/2+\epsilon} ||u||_{V^2_{\sigma} L^2} \]  \hspace{1cm} (4.5)
by the interpolation between (2.1) and (4.4). By using (4.5) instead of (2.1) for \(J_2\) and (3.13) instead of (1.3) for \(J_{31}\) in the proof of Proposition 4.2, we get the following.

**Proposition 4.3.** Let \(d \geq 1\) and \(\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R} \setminus \{0\}\) satisfy \(\sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0\). Assume \(s > \max \{s_c, 0\}\). For any dyadic numbers \(N_1, N_2, N_3 \geq 1\) with \(N_{\text{max}} \geq 2, 0 < T \leq 2\pi|\sigma|^{-1}\) and \(P_{N_j} u_j \in V^2_{\sigma_j} L^2\) \((j = 1, 2, 3)\), we have
\[ \left| \max_{N_{\text{max}}} \int_0^T \int_{T^d} \left| \prod_{j=1}^{3} P_{N_j} u_j \right| \, dx dt \right| \lesssim T^\epsilon N_{\text{min}}^{-\delta} \left( \frac{N_{\text{min}}}{N_{\text{max}}} + \frac{1}{N_{\text{min}}} \right)^{\delta} \prod_{j=1}^{3} ||P_{N_j} u_j||_{Y^0_{\sigma_j}} \]  \hspace{1cm} (4.6)
for some \(\delta > 0\) and \(\epsilon > 0\), where \(\sigma := \max_{1 \leq j \leq 3} |\sigma_j|, N_{\text{max}} = \max_{1 \leq j \leq 3} N_j, N_{\text{min}} = \min_{1 \leq j \leq 3} N_j\).
Next, we give the trilinear estimates under the condition $\sigma_1\sigma_2\sigma_3(1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) \leq 0$.

**Proposition 4.4.** Let $d \geq 5$ and $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\setminus\{0\}$ satisfy $\sigma_1\sigma_2\sigma_3(1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) \leq 0$, $(\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) \neq 0$ and $\sigma_1/\sigma_2 = m_1/m_2$, $\sigma_2/\sigma_3 = m_2/m_3$ for some $m_1, m_2, m_3 \in \mathbb{Z}\setminus\{0\}$. Assume $s \geq s_c$ ($= d/2 - 1$). For any dyadic numbers $N_1, N_2, N_3 \geq 1$ with $N_{\text{max}} \geq 2$, $0 < T \leq 2\pi|\sigma|^{-1}$ and $P_{N_j}u_j \in V^2_{\sigma_j}L^2$ ($j = 1, 2, 3$), we have (4.2) for some $\delta > 0$ if $N_i \ll N_j$ holds for some $1 \leq i, j \leq 3$ and for $\delta = 0$ if $N_1 \sim N_2 \sim N_3$ holds.

**Proof.** If $N_i \ll N_j$ holds for some $1 \leq i, j \leq 3$, we can obtain (4.2) since (4.1) holds. Therefore, we assume $N_1 \sim N_2 \sim N_3$. By the Hölder inequality, we have

$$
\left| N_{\text{max}} \int_0^T \int_{\mathbb{T}^d} \left( \prod_{j=1}^3 P_{N_j}u_j \right) \, dxdt \right| \leq N_{\text{max}} \prod_{j=1}^3 \|P_{N_j}u_j\|_{L^3(\mathbb{T}|\sigma| \times \mathbb{T}^d)}.
$$

Furthermore, by (3.5) with $p = 3$ and the embeddings $Y^0_{\sigma_j} \hookrightarrow V^2_{\sigma_j} \hookrightarrow U^3_{\sigma_j} \Delta L^2$, we have

$$
\|P_{N_j}u_j\|_{L^3(\mathbb{T}|\sigma| \times \mathbb{T}^d)} \lesssim N_j^d/6 - 2/3 \|P_{N_j}u_j\|_{Y^0_{\sigma_j}}
$$

since $3 > 2(d + 2)/2$ holds for $d \geq 5$. Therefore, we obtain (4.2) with $\delta = 0$. \qed

**Proposition 4.5.** Let $d \geq 1$ and $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\setminus\{0\}$ satisfy $\sigma_1\sigma_2\sigma_3(1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) \leq 0$ and $(\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) \neq 0$. Assume $s > \max\{s_c, 1\}$. For any dyadic numbers $N_1, N_2, N_3 \geq 1$ with $N_{\text{max}} \geq 2$, $0 < T \leq 2\pi|\sigma|^{-1}$ and $P_{N_j}u_j \in V^2_{\sigma_j}L^2$ ($j = 1, 2, 3$), we have (4.6) for some $\delta > 0$ if $N_i \ll N_j$ holds for some $1 \leq i, j \leq 3$ and for $\delta = 0$ if $N_1 \sim N_2 \sim N_3$ holds.

**Proof.** If $N_i \ll N_j$ holds for some $1 \leq i, j \leq 3$, we can obtain (4.6) since (4.1) holds. Therefore, we assume $N_1 \sim N_2 \sim N_3$. By (3.5), we have

$$
\|P_{N_j}u_j\|_{L^p(\mathbb{T}|\sigma| \times \mathbb{T}^d)} \lesssim N_j^{\max\{d/2 - (d+2)/p, a\}} \|P_{N_j}u_j\|_{U^p_{\sigma_j} \Delta L^2}
$$

for any $p > 3$, $a > 0$. Therefore, for sufficient small $\epsilon'' > 0$, there exists $0 < \epsilon' < 1$ such that

$$
\|P_{N_j}u_j\|_{L^3(0,T) \times \mathbb{T}^d)} \lesssim \left| 1 \right| \|P_{N_j}u_j\|_{L^3(0,T) \times \mathbb{T}^d)} \|P_{N_j}u_j\|_{U^p_{\sigma_j} \Delta L^2} \lesssim T^{\epsilon'/3} N_{\text{max}}^{d/6 - 2/3} \|P_{N_j}u_j\|_{Y^0_{\sigma_j}}
$$

for any $0 < T \leq 2\pi|\sigma|^{-1}$ since the embeddings $Y^0_{\sigma_j} \hookrightarrow V^2_{\sigma_j} \Delta L^2 \hookrightarrow U^p_{\sigma_j} \Delta L^2$ hold for $p > 3$. Therefore, we obtain (4.6) with $\delta = 0$ by the same way as the proof of Proposition 4.4. \qed
5. Proof of the well-posedness

In this section, we give the trilinear estimates and prove Theorems 1.1, 1.2 and 1.4. We define the map

$$\Phi(u, v, w) = (\Phi^{(1)}_{\alpha, u_0}(w, v), \Phi^{(1)}_{\beta, v_0}(w, v), \Phi^{(2)}_{\gamma, u_0}(u, \overline{v}))$$

as

$$\Phi^{(1)}_{\sigma, \varphi}(f, g)(t) := e^{it\sigma \Delta} \varphi - I^{(1)}_{\sigma}(f, g)(t),$$

$$\Phi^{(2)}_{\sigma, \varphi}(f, g)(t) := e^{it\sigma \Delta} \varphi + I^{(2)}_{\sigma}(f, g)(t),$$

where

$$I^{(1)}_{\sigma}(f, g)(t) := \int_0^t 1_{[0, \infty)}(t') e^{i(t-t')\sigma \Delta} (\nabla \cdot f(t')) g(t') dt',$$

$$I^{(2)}_{\sigma}(f, g)(t) := \int_0^t 1_{[0, \infty)}(t') e^{i(t-t')\sigma \Delta} \nabla (f(t') \cdot g(t')) dt'.$$

To obtain the well-posedness of (1.1), we prove that $\Phi$ is a contraction map on a closed subset of $Z^s_\alpha([0, T]) \times Z^s_\beta([0, T]) \times Z^s_\gamma([0, T])$. Key estimates are the followings:

**Proposition 5.1.** Let $s_c = d/2 - 1$ and $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$ satisfy $\alpha/\beta, \beta/\gamma \in \mathbb{Q}$. Assume $d \geq 3$ if $\alpha \beta \gamma (1/\alpha - 1/\beta - 1/\gamma) > 0$ and $d \geq 5$ if $\alpha \beta \gamma (1/\alpha - 1/\beta - 1/\gamma) \leq 0$ and $(\alpha - \beta)(\beta + \gamma)(\gamma - \alpha) \neq 0$. Then there exists $T > 0$, such that

$$\|I^{(1)}_\alpha(w, v)\|_{Z^s_\alpha([0, T])} \lesssim \|w\|_{Y^s_\alpha([0, T])} \|v\|_{Y^s_\beta([0, T])},$$

$$\|I^{(1)}_\beta(\overline{w}, u)\|_{Z^s_\beta([0, T])} \lesssim \|w\|_{Y^s_\alpha([0, T])} \|u\|_{Y^s_\beta([0, T])},$$

$$\|I^{(2)}_\gamma(u, \overline{v})\|_{Z^s_\gamma([0, T])} \lesssim \|u\|_{Y^s_\alpha([0, T])} \|v\|_{Y^s_\beta([0, T])},$$

(5.1)

(5.2)

(5.3)

hold.

**Proof.** We prove only (5.3) for the case $d \geq 3$, $\alpha \beta \gamma (1/\alpha - 1/\beta - 1/\gamma) > 0$ since the other cases and the estimates (5.1), (5.2) are proved by the same way. Let $(u_1, u_2) := (u, \overline{v})$ and $(\sigma_1, \sigma_2, \sigma_3) := (\alpha, -\beta, -\gamma)$. Since $\sigma_1/\sigma_2, \sigma_2/\sigma_3 \in \mathbb{Q}$, there exist $m_1, m_2, m_3 \in \mathbb{Z} \setminus \{0\}$ such that $\sigma_1/\sigma_2 = m_1/m_2, \sigma_2/\sigma_3 = m_2/m_3$. We choose $T > 0$ satisfying $T \leq 2\pi|\sigma|^{-1}$, where $\sigma := \sigma_1/m_1 = \sigma_2/m_2 = \sigma_3/m_3$. We define

$$S_j := \{(N_1, N_2, N_3) | N_{\text{max}} \sim N_{\text{med}} \gtrsim N_{\text{min}} \geq 1, N_{\text{min}} = N_j \} \quad (j = 1, 2, 3)$$

and $S := \bigcup_{j=1}^3 S_j$, where $(N_{\text{max}}, N_{\text{med}}, N_{\text{min}})$ be one of the permutation of $(N_1, N_2, N_3)$ such that $N_{\text{max}} \geq N_{\text{med}} \geq N_{\text{min}}$. Since

$$\hat{u}_1(t, 0) \hat{u}_2(t, 0) \overline{\nabla \cdot u_3(t, 0)} = 0$$
for any $t > 0$, we can assume $N_{\text{max}} \geq 2$. Therefore we have

$$
\left| \frac{D}{D_s^t} (u_1, u_2) \right|_{Z_{\alpha e_3} \cap [0, T]} \leq \sup_{||u_3||_{Y_{s_3}^{c}}} \left| \int_0^T \int_\mathbb{T} u_1 u_2 (\nabla \cdot u_3) dx dt \right|
$$

by Proposition 2.6 and Proposition 4.2. Furthermore, we have

$$
\sum_{s_1} N_{s_1}^{s_{e_3}} \left( \frac{N_{\min}}{N_{\max}} + \frac{1}{N_{\min}} \right)^{\delta} \prod_{j=1}^{3} ||P_N u_j||_{Y_{s_j}^0}
$$

and

$$
\sum_{s_3} N_{s_3}^{s_{e_3}} \left( \frac{N_{\min}}{N_{\max}} + \frac{1}{N_{\min}} \right)^{\delta} \prod_{j=1}^{3} ||P_N u_j||_{Y_{s_j}^0}
$$

by the Cauchy-Schwartz inequality for the dyadic sum. By the same way as the estimate for the summation of $S_1$, we have

$$
\sum_{s_2} N_{s_2}^{s_{e_3}} \left( \frac{N_{\min}}{N_{\max}} + \frac{1}{N_{\min}} \right)^{\delta} \prod_{j=1}^{3} ||P_N u_j||_{Y_{s_j}^0} \leq ||u_1||_{Y_{s_1}^e} ||u_2||_{Y_{s_2}^e} ||u_3||_{Y_{s_3}^{e}}.
$$

Therefore, we obtain 5.3 since $||u_1||_{Y_{s_1}^e} = ||u||_{Y_{s}^e}$ and $||u_2||_{Y_{s_2}^e} = ||v||_{Y_{s}^e}$.

By using Proposition 4.3 instead of Proposition 4.2 in the proof of Proposition 5.1, we get the following.

**Proposition 5.2.** Let $d \geq 1$ and $\alpha$, $\beta$, $\gamma \in \mathbb{R}\setminus\{0\}$. Assume $s > \max\{s_c, 0\}$ if $\alpha \beta \gamma (1/\alpha - 1/\beta - 1/\gamma) > 0$ and $s > \max\{s_c, 1\}$ if $\alpha \beta \gamma (1/\alpha - 1/\beta - 1/\gamma) \leq 0$ and
\((\alpha - \beta)(\beta + \gamma)(\gamma - \alpha) \neq 0\). Then there exists \(T > 0\) and \(\epsilon > 0\), such that
\[
\|I_{\alpha}^{(1)}(w, v)\|_{Z_{\alpha}^{s_c}([0, T])} \lesssim T^{\epsilon} \|w\|_{Y_{\alpha}^{s_c}(0, T)} \|v\|_{Y_{\beta}^{s_c}(0, T)},
\]
\[
\|I_{\beta}^{(1)}(w, v)\|_{Z_{\beta}^{s_c}([0, T])} \lesssim T^{\epsilon} \|w\|_{Y_{\alpha}^{s_c}(0, T)} \|u\|_{Y_{\gamma}^{s_c}(0, T)},
\]
\[
\|I_{\gamma}^{(2)}(u, v)\|_{Z_{\gamma}^{s_c}([0, T])} \lesssim T^{\epsilon} \|u\|_{Y_{\alpha}^{s_c}(0, T)} \|v\|_{Y_{\beta}^{s_c}(0, T)}
\]
hold.

**Proof of Theorem 1.1.** We consider only for small data at the scaling critical regularity since for large data at the scaling subcritical regularity is similar argument. For an interval \(I \subset \mathbb{R}\), we define
\[
X^{s_c}(I) := Z_{\alpha}^{s_c}(I) \times Z_{\beta}^{s_c}(I) \times Z_{\gamma}^{s_c}(I).
\] (5.4)
Furthermore for \(r > 0\), we define
\[
X^{s_c}(I) := \left\{ (u, v, w) \in X^{s_c}(I) \mid \|u\|_{Z_{\alpha}^{s_c}(I)}, \|v\|_{Z_{\beta}^{s_c}(I)}, \|w\|_{Z_{\gamma}^{s_c}(I)} \leq 2r \right\}
\] (5.5)
which is a closed subset of \(X^{s_c}(I)\). Let \((u_0, v_0, w_0) \in B_r(H^{s_c}(\mathbb{T}^d) \times H^{s_c}(\mathbb{T}^d) \times H^{s_c}(\mathbb{T}^d))\) be given and \(T > 0\) be given in the proof of Proposition 5.1. Then for \((u, v, w) \in X^{s_c}([0, T])\), we have
\[
\|\Phi_{\alpha, u_0}^{(1)}(w, v)\|_{Z_{\alpha}^{s_c}([0, T])} \leq \|u_0\|_{H^{s_c}} + C \|w\|_{Z_{\alpha}^{s_c}(0, T)} \|v\|_{Z_{\beta}^{s_c}(0, T)} \leq r(1 + 4Cr),
\]
\[
\|\Phi_{\beta, v_0}^{(1)}(w, u)\|_{Z_{\beta}^{s_c}([0, T])} \leq \|v_0\|_{H^{s_c}} + C \|w\|_{Z_{\alpha}^{s_c}(0, T)} \|u\|_{Z_{\gamma}^{s_c}(0, T)} \leq r(1 + 4Cr),
\]
\[
\|\Phi_{\gamma, w_0}^{(2)}(u, v)\|_{Z_{\gamma}^{s_c}([0, T])} \leq \|w_0\|_{H^{s_c}} + C \|u\|_{Z_{\alpha}^{s_c}(0, T)} \|v\|_{Z_{\beta}^{s_c}(0, T)} \leq r(1 + 4Cr)
\]
and
\[
\|\Phi_{\alpha, u_0}^{(1)}(w_1, v_1) - \Phi_{\alpha, u_0}^{(1)}(w_2, v_2)\|_{Z_{\alpha}^{s_c}([0, T])} \leq 2Cr \left( \|w_1 - w_2\|_{Z_{\alpha}^{s_c}(0, T)} + \|v_1 - v_2\|_{Z_{\beta}^{s_c}(0, T)} \right),
\]
\[
\|\Phi_{\beta, v_0}^{(1)}(w_1, u_1) - \Phi_{\beta, v_0}^{(1)}(w_2, u_2)\|_{Z_{\beta}^{s_c}([0, T])} \leq 2Cr \left( \|w_1 - w_2\|_{Z_{\alpha}^{s_c}(0, T)} + \|u_1 - u_2\|_{Z_{\gamma}^{s_c}(0, T)} \right),
\]
\[
\|\Phi_{\gamma, w_0}^{(2)}(u_1, v_1) - \Phi_{\gamma, w_0}^{(2)}(u_2, v_2)\|_{Z_{\gamma}^{s_c}([0, T])} \leq 2Cr \left( \|u_1 - u_2\|_{Z_{\alpha}^{s_c}(0, T)} + \|v_1 - v_2\|_{Z_{\beta}^{s_c}(0, T)} \right)
\]
by Proposition 5.1 and
\[
\|e^{i\sigma t \Delta} \varphi\|_{Z_{\beta}^{s_c}(0, T)} \leq \|1_{[0, T]} e^{i\sigma t \Delta} \varphi\|_{Z_{\beta}^{s_c}} \leq \|\varphi\|_{H^{s_c}},
\]
where \(C\) is an implicit constant in (5.1)–(5.3). Therefore if we choose \(r > 0\) satisfying
\[
r < (4C)^{-1},
\]
then \(\Phi\) is a contraction map on \(X^{s_c}([0, T])\). This implies the existence of the solution of the system (1.1) and the uniqueness in the ball \(X^{s_c}([0, \infty))\). The Lipschitz continuously of the flow map is also proved by similar argument. \(\Box\)

Theorem 1.4 is proved by using the estimate (5.1) for \((\alpha, \beta, \gamma) = (-1, 1, 1)\).
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