Non-Archimedean Mathematics and the formalism of Quantum Mechanics

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Abstract

This paper is divided in four parts. In the introduction, we discuss the program and the motivations of this paper. In section 2 we introduce the non-Archimedean field of Euclidean numbers $E$ and we present a summary of the theory of $\Lambda$-limits which can be considered as a different approach to nonstandard methods. In the third part (section 3), we define axiomatically the space of ultrafunctions which are a kind of generalized function based on the field of Euclidean numbers $E$. Finally, we describe an application of the previous theory to the formalism of classical Quantum Mechanics.

Keywords. Ultrafunctions, Delta function, non Archimedean Mathematics, Non Standard Analysis, Quantum Mechanics, self-adjoint operators.

Sommario

Questo articolo è diviso in quattro parti. Nell’introduzione discutiamo il programma e le motivazioni di questo lavoro. Nella sezione 2 introduciamo il campo non-Archimedeico dei numeri Euclidei $E$ e presentiamo un riassunto della teoria dei $\Lambda$-limiti che può essere considerata come un approccio diverso ai metodi non standard. Nella terza parte (sezione 3), definiamo assiomaticamente lo spazio delle ultrafunzioni che sono una sorta di funzione generalizzata basate sul campo dei numeri euclidei $E$. Infine, descriviamo un’applicazione della teoria precedente al formalismo della meccanica quantistica classica.

Parole chiave. Ultrafunzioni, funzione Delta, matematica non archimedea, analisi non standard, meccanica quantistica, operatori autoaggiunti.

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1 Introduction

This is not an article of Mathematics, but rather about Mathematics, or more precisely about non-Archimedean Mathematics, namely the Mathematics based on the use of infinitesimal and infinite numbers (NAM). We are convinced that NAM deserves more attention than that which is generally attributed to it. In this paper, we will expose some recent results on Euclidean numbers, A-theory and ultrafunctions emphasizing the ideas, but we have not given the proofs of most results. In any case we have referred to the original papers for the interested reader. Moreover, we will give an original application to the formalism of Quantum Mechanics (QM).

1.1 The Non-Archimedean Mathematics

La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi a gli occhi (io dico l’universo), ma non si può intendere se prima non s’impara a intender la lingua, e conoscere i caratteri, ne’quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi, spazi di Hilbert, varietà simplicative, frattali, infinitesimi ed altre entità matematiche senza i quali mezzi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro labirinto.

Galileo Galilei, Il Saggiatore (1623), Cap. VI

We begin this article by recalling this passage by Galileo which highlights one of the most fascinating aspects of mathematics: its ability to construct models
that allow us to understand, at least in part, the natural world. Obviously, the words written in bold have been added to the original text in order to emphasize the discovery of new mathematical entities that have developed Galileo’s ideas up to the current progress of science. Moreover, these added words remind another aspect of the history of science (which we do not fully understand): the ousting of infinitesimal quantities. The scientific community has always accepted new mathematical entities, especially if these are useful in the modeling of natural phenomena and in solving the problems posed by the technique. Some of these entities are the infinitesimals that, by the discovery of the infinitesimal calculus, have been a carrier of the modern science. But despite the successes achieved with their employment, they have been opposed and even fought by a considerable part of the scientific community. We refer to the essay by Amir Alexander 2 (see also 4) which tells how the Jesuits in Italy and part of the Royal Society in England fought the spread of these ”subversive notions”. Sometimes it is said that people opposed infinitesimals because of their lack of rigor, but this argument convinces me little (see also 26). When at the end of the 19th century they were placed on a more rigorous basis thanks to the works of Du Bois-Reymond 28, Veronese 15,16, Levi-Civita 36 and others ??,30,31, nevertheless they were fought (and defeated) by the likes of Russell (see e.g. 40) and Peano 38. Their defeat was so complete that many of the mathematicians of my generation even ignore the existence of the studies of the above mentioned scholars. Probably, even in the world of science, history is written only by the winners 3. To understand, at least in part, the cultural dynamics of this historical period in relation to the infinitesimal, we suggest some works by Ehrlich 29. Not much better was the reception of the Non Standard Analysis created in the ’60s by Robinson 39,32, which arouses the contempt of (almost) all those who do not know it, even though a minority of mathematicians of the highest level has elaborated on it interesting theories (see e.g. 1,37,44). A brief historical survey on NAM can be found e.g. in 17,20.

Personally, I am convinced that the NAM is a very rich branch of mathematics and allows to construct models of the real world in a more efficient way. In this paper, we will show how NAM allows to construct a formalism for QM which is closer to the ideas of Born, Heisenberg and Bohr than the formalism of Von Neumann. This formalism is based on the theory of ultrafunctions which are functions defined over a non Archimedean field, namely a field which does not satisfy the Principle of Archimedes, and hence it contains infinitesimal and infinite numbers.

1.2 Ultrafunctions and Euclidean numbers

The ultrafunctions can be considered as a kind generalized functions. In many circumstances, the notion of real function is not sufficient to the needs of a theory and it is necessary to extend it. The intensive use of the Laplace transform in engineering led to the heuristic use of symbolic methods, called operational calculus. An influential book on operational calculus was Oliver Heaviside’s
Electromagnetic Theory of 1899. However his methods had a bad reputation among pure mathematicians because they were not rigorous. When the Lebesgue integral was introduced, for the first time a notion of generalized function became central to mathematics since the notion of function was replaced by something defined almost everywhere and not pointwise. During the late 1920s and 1930s further steps were taken, very important to future work. The Dirac delta function was boldly defined by Paul Dirac as a central aspect of his scientific formalism. Jean Leray and Sergei Sobolev, working in partial differential equations, defined the first adequate theory of generalized functions in order to work with weak solutions of partial differential equations. Sobolev’s work was further developed in an extended form by Laurent Schwartz. To day, among people working in partial differential equations, the theory of distributions of L. Schwartz is the most commonly used, but also other notions of generalized functions have been introduced by J.F. Colombeau and M. Sato.

The ultrafunctions can be considered as a new kind of generalized functions. They have been recently introduced and developed in. They provide generalized solutions to certain equations which do not have any solution, not even among the distributions. Actually, the ultrafunctions are pointwise defined functions on a suitable subset of $\mathbb{E}^N$, and take their value in $\mathbb{E}$ where $\mathbb{E}$ is a non-Archimedean field which contains the real numbers. This fact allows to define the Dirac delta ultrafunction $\delta_a$ as a function which takes an infinite value in the point $a$ and vanishes in the other points. So, in this context, expression such as $\sqrt{\delta_a}$ or $\delta_a^2$ make absolutely sense. The field $\mathbb{E}$, introduced in, is called field of Euclidean numbers and it is a particular field of hyperreal numbers. We recall that the fields of hyperreal numbers are the basic fields on which nonstandard analysis (NSA in the sequel) is based. In fact, in the theory of ultrafunctions, a large use of the techniques of NSA is employed even if sometimes it is hidden by the formalism which we have used. This formalism is based on the notion of $\Lambda$-limit. Before ending this introduction, we want to emphasize the differences by our approach to nonstandard methods and the usual one: there are two main differences, one in the aims and one in the methods.

Let us examine the difference in the aims. We think that infinitesimal and infinite numbers should not be considered just as entities living in a parallel universe (the nonstandard universe) which are only a tool to prove some statement relative to our universe (the standard universe), but rather that they should be considered mathematical entities which have the same status of the others and can be used to build models as any other mathematical entity. Actually, the advantages of a theory which includes infinitesimals rely more on the possibility of making new models rather than in the proving techniques. This paper, as well as, are inspired by this principle.

As far as the methods are concerned, we introduce a non-Archimedean field via a new notion of limit (see section and we use a language closer to analysis and to applied Mathematics rather than to Logic.
1.3 Notations

Let $\Omega$ be an open subset of $\mathbb{R}^N$: then

- $C(\Omega)$ denotes the set of continuous functions defined on $\Omega \subset \mathbb{R}^N$;
- $C_c(\Omega)$ denotes the set of continuous functions in $C(\Omega)$ having compact support in $\Omega$;
- $C^k(\Omega)$ denotes the set of functions defined on $\Omega \subset \mathbb{R}^N$ which have continuous derivatives up to the order $k$;
- $C_c^k(\Omega)$ denotes the set of functions in $C^k(\Omega)$ having compact support;
- $L^2(\Omega)$ denotes the space of square integrable functions defined almost everywhere in $\Omega$;
- $\text{mon}(x) = \{y \in \mathbb{E}^N \mid x \sim y\}$ (see Def. 6);
- $\text{gal}(x) = \{y \in \mathbb{E}^N \mid x - y$ is a finite number\} (see Def. 6);
- given any set $E \subset X$, $\chi_E : X \to \mathbb{R}$ denotes the characteristic function of $E$, namely
  \[
  \chi_E(x) := \begin{cases} 
  1 & \text{if } x \in E \\
  0 & \text{if } x \notin E 
  \end{cases}
  \]
- with some abuse of notation we set $\chi_a(x) := \chi_{\{a\}}(x)$;
- $\partial_i = \frac{\partial}{\partial x_i}$ denotes the usual partial derivative; $D_i$ denotes the generalized derivative (see section 3.1);
- $\int$ denotes the usual Lebesgue integral; $\oint$ denotes the pointwise integral (see section 3.1);
- if $E$ is any set, then $|E|$ denotes its cardinality.

2 $\Lambda$-theory

As we have already remarked in the introduction, $\Lambda$-theory can be considered a different approach to Nonstandard Analysis. It can be introduced via the notion of $\Lambda$-limit, and it can be easily used for the purposes of this paper. We introduce the Euclidean numbers (which are the basic object of $\Lambda$-theory) via an algebraic approach as in [5]. An elementary presentation of (part of) this theory can be found in [10] and [8].

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1In [2] and [3], the reader can find several other approaches to NSA and an analysis of them.
2.1 Non Archimedean Fields

Here, we recall the basic definitions and facts regarding Non Archimedean fields. In the following, $K$ will denote an ordered field. We recall that such a field contains (a copy of) the rational numbers. Its elements will be called numbers.

**Definition 1.** Let $K$ be an ordered field. Let $\xi \in K$. We say that:

- $\xi$ is infinitesimal if, for all positive $n \in \mathbb{N}$, $|\xi| < \frac{1}{n}$;
- $\xi$ is finite if there exists $n \in \mathbb{N}$ such that $|\xi| < n$;
- $\xi$ is infinite if, for all $n \in \mathbb{N}$, $|\xi| > n$ (equivalently, if $\xi$ is not finite).

**Definition 2.** An ordered field $K$ is called Non-Archimedean if it contains an infinitesimal $\xi \neq 0$.

It’s easily seen that all infinitesimal are finite, that the inverse of an infinite number is a nonzero infinitesimal number, and that the inverse of a nonzero infinitesimal number is infinite.

**Definition 3.** A superreal field is an ordered field $K$ that properly extends $\mathbb{R}$.

It is easy to show that any superreal field contains infinitesimal and infinite numbers. Infinitesimal numbers can be used to formalize a new notion of ”closeness”:

**Definition 4.** We say that two numbers $\xi, \zeta \in K$ are infinitely close if $\xi - \zeta$ is infinitesimal. In this case we write $\xi \sim \zeta$.

Clearly, the relation ”$\sim$” of infinite closeness is an equivalence relation.

**Theorem 5.** If $K$ is a superreal field, every finite number $\xi \in K$ is infinitely close to a unique real number $r \sim \xi$, called the the standard part of $\xi$.

Given a finite number $\xi$, we denote it standard part by $st(\xi)$, and we put $st(\xi) = +\infty$ ($st(\xi) = -\infty$) if $\xi \in K$ is a positive (negative) infinite number.

**Definition 6.** Let $K$ be a superreal field, and $\xi \in K$ a number. The monad of $\xi$ is the set of all numbers that are infinitely close to it:

$$\text{mon}(\xi) = \{ \zeta \in K : \xi \sim \zeta \},$$

and the galaxy of $\xi$ is the set of all numbers that are finitely close to it:

$$\text{gal}(\xi) = \{ \zeta \in K : \xi - \zeta \text{ is finite} \}.$$ 

By definition, it follows that the set of infinitesimal numbers is $\text{mon}(0)$ and that the set of finite numbers is $\text{gal}(0)$. Moreover, the standard part can be regarded as a function:

$$st : \text{gal}(0) \to \mathbb{R}. \quad (1)$$
Any ordered field $K$ can be complexified to get a new field

$$K + iK$$

namely a field of numbers of the form

$$a + ib, \ a, b \in K.$$ 

Thus also the complexification of non-Archimedean fields does not present any particular peculiarity.

### 2.2 The Euclidean numbers

Let $\Lambda$ be an infinite set containing $\mathbb{R}$ and let $\mathcal{L}$ be the family of finite subsets of $\Lambda$. A function $\varphi : \mathcal{L} \to E$ will be called net (with values in $E$). The set of such nets is denoted by $\mathfrak{F}(\mathcal{L}, \mathbb{R})$. Such a set is a real algebra equipped with the natural operations

$$(\varphi + \psi)(\lambda) = \varphi(\lambda) + \psi(\lambda);$$

$$(\varphi \cdot \psi)(\lambda) = \varphi(\lambda) \cdot \psi(\lambda);$$

and the partial order relation:

$$\varphi \geq \psi \iff \forall \lambda \in \mathcal{L}, \varphi(\lambda) \geq \psi(\lambda).$$

**Definition 7.** The set of Euclidean numbers $E \supset \mathbb{R}$ is a field such that there is a surjective homomorphism

$$J : \mathfrak{F}(\mathcal{L}, \mathbb{R}) \to E$$

or, more exactly a map which satisfies the following properties:

- $J (\varphi + \psi) = J (\varphi) + J (\psi)$;
- $J (\varphi \cdot \psi) = J (\varphi) \cdot J (\psi)$;
- if $\varphi(\lambda) \geq r$, then $J (\varphi) \geq r$.

The proof of the existence of such a field is an easy consequence of the Krull-Zorn theorem. It can be found, e.g. in [13, 14, 23, 8]. In this paper, we use also the complexification of $E$, denoted by

$$\mathbb{C}^* = E + iE.$$  \hfill (2)

The number $J (\varphi)$ is called the $\Lambda$ limit of the net $\varphi$ and will be denoted by

$$J (\varphi) = \lim_{\lambda \uparrow \Lambda} \varphi(\lambda)$$

The reason of this name/notation is that the operation

$$\varphi \mapsto \lim_{\lambda \uparrow \Lambda} \varphi(\lambda)$$

satisfies many of the properties of the usual limit, more exactly it satisfies the following properties:
• **(Λ-1) Existence.** Every net \( \varphi : \mathcal{L} \to \mathbb{R} \) has a unique limit \( L \in \mathbb{E} \).

• **(Λ-2) Constant.** If \( \varphi(\lambda) \) is eventually constant, namely \( \exists \lambda_0 \in \mathcal{L}, r \in \mathbb{R} \) such that \( \forall \lambda \supset \lambda_0, \varphi(\lambda) = r \), then
  \[
  \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) = r.
  \]

• **(Λ-3) Sum and product.** For all \( \varphi, \psi : \mathcal{L} \to \mathbb{R} \):
  \[
  \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) + \lim_{\lambda \uparrow \Lambda} \psi(\lambda) = \lim_{\lambda \uparrow \Lambda} (\varphi(\lambda) + \psi(\lambda));
  \]
  \[
  \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \cdot \lim_{\lambda \uparrow \Lambda} \psi(\lambda) = \lim_{\lambda \uparrow \Lambda} (\varphi(\lambda) \cdot \psi(\lambda)).
  \]

Now let us see the main differences between the usual limit (which we will call Cauchy limit) and the \( \Lambda \)-limit. We recall the definition of Cauchy limit (as formalized by Weierstrass):
\[
L = \lim_{\lambda \to \Lambda} \varphi(\lambda)
\]
if and only if, \( \forall \varepsilon \in \mathbb{R}^+ \), \( \exists \lambda_0 \in \mathcal{L} \), such that \( \forall \lambda \supset \lambda_0, |\varphi(\lambda) - L| \leq \varepsilon \)

The classical example of Cauchy limit of a net is provided by the definition of the Cauchy integral:
\[
\int_{a}^{b} f(x)dx = \lim_{\lambda \to \Lambda} \sum_{x \in [a,b] \cap \lambda} f(x)(x^+ - x); \quad x^+ = \min \{ y \in \mathbb{R} \cap \lambda \mid y > x \}
\]

Notice that in order to distinguish the two kind of limits we have used the symbols "\( \lambda \uparrow \Lambda \)" and "\( \lambda \to \Lambda \)" respectively. Since also the Cauchy limit (when it exists) satisfies (Λ-2) and (Λ-3) the only difference between the the Cauchy limit and the \( \Lambda \)-limit is that the latter always exists. This fact implies that \( \mathbb{E} \) must be larger than \( \mathbb{R} \) since otherwise a diverging net cannot have a limit. In the case in which the Cauchy limit exists the relation between the two limits is given by the following identity:
\[
\lim_{\lambda \to \Lambda} \varphi(\lambda) = st \left( \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \right)
\]

(3)

In order to give a feeling of the Euclidean number, we will describe a possible interpretation of some of them. If \( E \subset \mathbb{R} \subset \Lambda \), we set
\[
n(E) = \lim_{\Lambda \uparrow \Lambda} |E \cap \lambda|
\]
where \( |F| \) denotes the number of elements of the finite set \( F \). Notice that \( E \cap \lambda \) is a finite set since \( \lambda \in \mathcal{L} \). Then the above limit makes sense since for every \( \lambda \in \mathcal{L}, |E \cap \lambda| \in \mathbb{N} \subset \mathbb{R} \). If \( E \) is a finite set, the sequence is eventually constant,
namely, \( \forall \lambda \supset E, \ |E \cap \lambda| = |E| \) and hence \( n(E) = |E| \). If \( E \) is an infinite set, the above limit gives an infinite number. Hence \( n(E) \) extends the "measure of the size of a set" to infinite sets. The Euclidean number \( n(E) \) is called \textit{numerosity} of \( E \). For example, the number \( \alpha \) defined by

\[
\alpha := \lim_{\lambda \uparrow \Lambda} |N \cap \lambda| 
\]

is a measure of the size of \( N = \{1, 2, 3, \ldots\} \). The theory of numerosity can be considered as an extension of the Cantorian theory of cardinal and ordinal numbers. The reader interested in the details and the developments of this theory is referred to [9, 12, 13, 15].

2.3 Extension of functions and grid functions

If \( \varphi(\lambda) = (\varphi_1(\lambda), ..., \varphi_N(\lambda)) \in \mathbb{R}^N \), we set

\[
\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) = \left( \lim_{\lambda \uparrow \Lambda} \varphi_1(\lambda), ..., \lim_{\lambda \uparrow \Lambda} \varphi_N(\lambda) \right) \in \mathbb{E}^N
\]

Given a set \( A \subset \mathbb{R}^N \), we define

\[
A^* = \left\{ \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \mid \forall \lambda, \ \varphi(\lambda) \in A \right\};
\]

following Keisler [35], \( A^* \) will be called the \textit{natural extension} of \( A \). Clearly we have that \( \mathbb{R}^* = \mathbb{E} \). This fact justifies the notation (2) to denote the complexification of \( \mathbb{E} \):

\[
\mathbb{C}^* = \mathbb{E} + i\mathbb{E} = \mathbb{R}^* + i\mathbb{R}^*.
\]

Any function

\[
f : A \to \mathbb{R}, \ A \subset \mathbb{R}^N
\]

can be univocally extended to \( A^* \) by setting

\[
f^* \left( \lim_{\lambda \uparrow \Lambda} x_\lambda \right) = \lim_{\lambda \uparrow \Lambda} f^* (x_\lambda);
\]

the function

\[
f^* : A^* \to \mathbb{E},
\]

will be called \textit{natural extension} of \( f \). More in general, if

\[
u_\lambda : A \to \mathbb{R}, \ A \subset \mathbb{R}^N
\]

is a net of functions, we define the \( \Lambda \)-limit

\[
u = \lim_{\lambda \uparrow \Lambda} u_\lambda : A^* \to \mathbb{E},
\]

as follows: for any \( x = \lim_{\lambda \uparrow \Lambda} x_\lambda \in \mathbb{E}^N \), we set

\[
u(x) = \lim_{\lambda \uparrow \Lambda} u_\lambda (x_\lambda)
\]
In particular, if, for all \( x \in \mathbb{R}^N \)
\[ v(x) = \lim_{\lambda \to \Lambda} u_\lambda(x) \]
by (3), it follows that
\[ \forall x \in \mathbb{R}^N, \ v(x) = st[u(x)]. \]

**Definition 8.** We say that a set \( F \subset \mathcal{E} \) is **hyperfinite** if there is a net \( \{F_\lambda\}_{\lambda \in \Lambda} \) of finite set such that
\[ F = \left\{ \lim_{\lambda \uparrow \Lambda} x_\lambda \mid x_\lambda \in F_\lambda \right\} \]

The hyperfinite sets share many properties of finite sets. For example, it is possible to "add" the elements of an hyperfinite set of numbers. If \( F \) is an hyperfinite set of numbers, the **hyperfinite sum** of the elements of \( F \) is defined in the following way:
\[ \sum_{x \in F} x = \lim_{\lambda \uparrow \Lambda} \sum_{x \in F_\lambda} x. \]

**Definition 9.** A hyperfinite set \( \Gamma \) such that \( \mathbb{R}^N \subset \Gamma \subset \mathcal{E}^N \) is called **hyperfinite grid**.

If \( \Gamma_\lambda \) is a family of finite subsets of of \( \mathbb{R} \) which satisfy the following property:
\[ \mathbb{R}^N \cap \lambda \subset \Gamma_\lambda \]
it is not difficult to prove that the the set
\[ \Gamma = \left\{ \lim_{\lambda \uparrow \Lambda} x_\lambda \mid x_\lambda \in \Gamma_\lambda \right\} \]
is a hyperfinite grid. From now on \( \Gamma \) will denote a hyperfinite grid fixed once forever.

**Definition 10.** A space of grid functions is a family \( \mathcal{G}(\Gamma) \) of functions
\[ u : \Gamma \to \mathbb{R} \]
such that, for every \( x = \lim_{\lambda \uparrow \Lambda} x_\lambda \in \Gamma \), we have that
\[ u(x) = \lim_{\lambda \uparrow \Lambda} u_\lambda(x_\lambda). \]

If \( f \in \mathcal{F}(\mathbb{R}^N) \), and \( x = \lim_{\lambda \uparrow \Lambda} x_\lambda \in \Gamma \), we set
\[ f^\circ(x) := \lim_{\lambda \uparrow \Lambda} f(x_\lambda) \tag{5} \]

namely \( f^\circ \) is the restriction to \( \Gamma \) of the natural extension \( f^* \) which is defined on all \( \mathcal{E}^N \).
It is easy to check that, for every \(a \in \Gamma\), \(\chi_a(x)\) is a grid function, and hence every grid function can be represented by the following hyperfinite sum:

\[
u(x) = \sum_{a \in \Gamma} u(a) \chi_a(x)
\]

namely \(\{\chi_a(x)\}_{a \in \Gamma}\) is a basis for \(\mathcal{G}(\mathbb{R}^N)\). If a function, such as \(1/|x|\) is not defined in some point, we put \((1/|x|)^{\circ}\) equal to 0 for \(x = 0\); in general, if \(E\) is a subset of \(\mathbb{R}^N\) and \(f\) is defined on \(E\), we set

\[
f^{\circ}(x) = \sum_{a \in \Gamma \cap E^*} f^*(a) \chi_a(x).
\]

Before ending this section we need an other definition. Given two function spaces \(V\) and \(W\), we set

\[
V^* = \left\{ \lim_{\lambda \uparrow \Lambda} u_\lambda \mid u_\lambda \in V \right\}
\]

\[
W^* = \left\{ \lim_{\lambda \uparrow \Lambda} u_\lambda \mid u_\lambda \in W \right\}
\]

**Definition 11.** An operator

\[F : V^* \rightarrow W^*\]

is called **internal** if there exists a net of operators \(\{F_\lambda\}_{\lambda \in \mathcal{L}}\), such that

\[Fu := \lim_{\lambda \uparrow \Lambda} F_\lambda (u_\lambda)\]

where \(u = \lim_{\lambda \uparrow \Lambda} u_\lambda\), \(u_\lambda \in V\).

In general any mathematical entity is called "internal" if it is the \(\Lambda\)-limit of some other entities. In this section we have introduced internal function (e.g. the grid functions, internal sets, hyperfinite sets and internal operators). We do not need to develop the full theory which is a basic tool in NSA. We have just introduced explicitly the objects which are needed in this exposition.

### 3 Ultrafunctions

#### 3.1 Axiomatic definition of ultrafunctions

Let \(V = C_c(\mathbb{R}^N)\) denote the space of continuous functions with compact support. We will denote by \(\{V_\lambda\}_{\lambda \in \mathcal{L}}\) a directed set of all finite dimensional subspaces of \(V(\Omega)\), namely for every couple of spaces \(V_{\lambda_1}, V_{\lambda_2}\) there exists \(\lambda_3 \supseteq \lambda_1 \cup \lambda_2\) such that

\[V_{\lambda_1} + V_{\lambda_2} \subset V_{\lambda_3}\]
A space of ultrafunctions $V^\circ$ modelled on $\{V_\lambda \}_{\lambda \in \mathcal{L}}$ is a family of grid functions
$$u : \Gamma \to E$$
equipped with an internal functional
$$\oint : V^\circ \to E$$
called \textit{pointwise integral} and an $N$ internal operators
$$D_i : V^\circ \to V^\circ$$
called \textit{generalized partial derivative} which satisfy the following axioms:

**Axiom 1.** For any $u \in V^\circ$, there exists a net $u_\lambda$ such that
$$u_\lambda \in V_\lambda$$
and
$$u = \lim_{\lambda \uparrow \Lambda} u_\lambda.$$

**Axiom 2.** If $u = \lim_{\lambda \uparrow \Lambda} u_\lambda$, $u_\lambda \in V_\lambda$, then
$$\oint u(x) dx = \lim_{\lambda \uparrow \Lambda} \int u_\lambda(x) dx.$$  \text{(7)}

**Axiom 3.** If $a \in \Gamma$,
$$\oint \chi_a(x) dx > 0.$$

**Axiom 4.** If $u = \lim_{\lambda \uparrow \Lambda} u_\lambda$, $u_\lambda \in V_\lambda \cap C^1(\mathbb{R}^N)$, and and $x = \lim_{\lambda \uparrow \Lambda} x_\lambda \in \Gamma$, then
$$D_i u(x) = \lim_{\lambda \uparrow \Lambda} \partial_i u_\lambda(x_\lambda).$$ \text{(8)}

**Axiom 5.** For every $u \in V^\circ$
$$D_i u = 0 \iff u = 0.$$

**Axiom 6.** If we set
$$\text{supp}(u) = \{x \in \Gamma \mid u(x) \neq 0\},$$
then
$$\text{supp}(D_i \chi_a(x)) \subset \text{mon}(a).$$

**Axiom 7.** For every $u, v \in V^\circ$
$$\oint D_i u(x) v(x) dx = -\oint u(x) D_i v(x) dx.$$ \text{(9)}
In the literature several spaces of ultrafunctions have been introduced and developed (see [6, 21, 22, 23, 20, 24] and the references therein). However the proof of a model of ultrafunctions which satisfies all the above seven axioms is a delicate matter and we refer to [7].

Now, we will briefly discuss these axioms. The first axiom characterizes the ultrafunctions with respect to other internal functions. The second axiom is nothing else but the definition of the pointwise integral. By its definition, for every function \( f \in V \),

\[
\oint f(x)dx = \int f(x)dx
\]

Then it extends the usual Riemann integral from \( V = C_0(\mathbb{R}^N) \) to \( V^\circ \). Axiom 3 shows that the above inequality cannot hold for all the Riemann integrable function since

\[
\int \chi_a(x)dx = 0 \neq \oint \chi_a(x)dx
\]

This Axiom is natural, since when we work in a non-Archimedan world the infinitesimals matter and cannot be forgotten as the Riemann integral does. Also the above inequality shows that it is necessary to use a different symbol (namely \( \oint \)) to distinguish the pointwise integral from the Riemann or the Lebesgue integral. Axiom 4 shows that the generalized derivative extends the usual derivative; in fact if \( f \in C^1(\mathbb{R}^N) \) and \( x \in \mathbb{R}^N \), then

\[
D_i f^\circ(x) = \lim_{\lambda \uparrow \Lambda} \partial_i f(x) = \partial_i f(x).
\]

On the other hand the operator \( D_i \) is defined on all the functions and it must be defined in such a way that the most useful property of the usual derivative be satisfied, and this is the content of the last three axiom. Axiom 5 says that the ultrafunctions behave as compactly supported \( C^1 \) functions. Axiom 6 states that the derivative is a local operator. Axiom 7 states a formula which is of primary importance in the theory of weak derivatives, distribution, calculus of variations etc. Usually this formula is deduced by the Leibniz rule

\[
D(fg) = Dfg + fDg
\]

However, the Leibniz rule cannot be satisfied by every ultrafunction by the Schwartz impossibility theorem (see [12], [20]; see also the footnote at p. 23). Nevertheless the identity (11) holds for all the ultrafunctions.

### 3.2 Structure of the space of ultrafunctions

By (6) and Axiom 3 it follows that, for every \( u \in V^\circ \),

\[
\oint u(x)dx = \sum_{a \in \Gamma} u(a)d(a)
\]
where
\[ \forall a \in \Gamma, \quad d(a) = \oint \chi_a(x)dx, \]
namely the pointwise integral reduces to a hyperfinite sum. This fact justifies its name.

In view of our application to QM, from now on, we will consider complex valued ultrafunctions, namely ultrafunctions in the space

\[ H^\circ := V^\circ \oplus iV^\circ \]

Also we shall use the notations
\[ H := V \oplus iV = \mathbb{C}_c(\mathbb{R}^N, \mathbb{C}) \]
\[ H_\lambda := V_\lambda \oplus iV_\lambda \]

The pointwise integral allows to define the following sesquilinear form on \( H^\circ \):
\[ \oint u(x)v(x)dx = \sum_{x \in \Gamma} u(x)v(x)d(x). \] (10)
Here \( \overline{z} \) represents the complex conjugate of \( z \). By virtue of Axiom 3, such a form is scalar product. The norm of an ultrafunction will be given by
\[ \|u\| = \left( \oint |u(x)|^2 dx \right)^{\frac{1}{2}}. \]

Also, the pointwise integral allows us to define the \textbf{delta (or the Dirac) ultrafunction} as follows: for every \( a \in \Gamma \),
\[ \delta_a(x) = \frac{\chi_a(x)}{d(a)}. \]

In fact, for every \( u \in V^\circ \), we have that
\[ \oint \delta_a(x)u(x)dx = \sum_{x \in \Gamma} u(x)\delta_a(x)d(x) = \sum_{x \in \Gamma} u(x)\frac{\chi_a(x)}{d(a)}d(x) = u(a). \]

The delta functions are orthogonal with each other with respect to the scalar product (10): hence, if normalized, they provide an orthonormal basis, called \textbf{delta-basis}, given by
\[ \left\{ \delta_a\sqrt{d(a)} \right\}_{a \in \Gamma} = \left\{ \frac{\chi_a}{\sqrt{d(a)}} \right\}_{a \in \Gamma}. \] (11)

Hence, every ultrafunction can be represented as follows:
\[ u(x) = \sum_{a \in \Gamma} \left( \oint u(\xi)\delta_a(\xi)d\xi \right)\chi_a(x). \] (12)

The scalar product allows the following proposition:
Proposition 12. If \( \Phi : H \to \mathbb{C} \) is a linear internal functional, then there exists \( u_\Phi \) such that, for all \( v = \lim_{\lambda \uparrow \Lambda} v_\lambda \in H^\circ \)

\[
\int u_\Phi v \, dx = \lim_{\lambda \uparrow \Lambda} \Phi (v_\lambda)
\]

and for every \( f \in V \),

\[
\int u_\Phi f^o \, dx = \Phi (f)
\]

Proof: If \( v \in H_\Lambda \), then the map

\[
v \mapsto \Phi (v)
\]

is a linear functional over \( H_\Lambda \) and hence, since there exists \( u_\lambda \in H_\Lambda \) such that

\[
\int u_\lambda v \, dx = \Phi (v)
\]

If we set

\[
u_\Phi = \lim_{\lambda \uparrow \Lambda} u_\lambda
\]

the conclusion follows.

□

If we want to apply the theory of ultrafunctions to QM, one of the most important thing to analyze is their relation to the \( L^2 \)-functions; in fact in the usual formalism of QM a physical state is described by a \( L^2 \)-function \( \psi \), but we cannot associate an ultrafunction \( u \) to a function \( \psi \) by using eq.(5) since the \( L^2 \)-functions are not pointwise defined. Then we need a new definition.

Definition 13. Given a function \( \psi \in L^2 (\Omega) \), we denote by \( \psi^o \) the unique ultrafunction such that, for every \( v = \lim_{\lambda \uparrow \Lambda} v_\lambda (x) \in H^\circ \),

\[
\int \psi^o v \, dx = \lim_{\lambda \uparrow \Lambda} \int_\Omega \psi v_\lambda dx.
\]

The above definition is well posed since the map

\[
\Phi : v \mapsto \int \psi v dx
\]

is a functional on the space \( H^\circ \) an hence, by Prop. 12 there exists an ultrafunction \( \psi^o \) such that

\[
\int \psi^o v \, dx = \Phi (v).
\]
3.3 Self-adjoint operators on ultrafunctions

If

$$L : H^0 \rightarrow H^0, \quad H^0 = V^o \oplus iV^o$$

is an internal linear operator, it can be regarded as an infinite matrix since, by Def. 11,

$$Lu := \lim_{\lambda \uparrow \Lambda} L_\lambda u_\lambda$$

where

$$L_\lambda : H_\lambda \rightarrow H_\lambda; \quad H_\lambda = V_\lambda \oplus iV_\lambda$$

can be represented by a matrix since $H_\lambda$ is a finite dimensional space over $\mathbb{C}$. Then if $L$ is a selfadjoint operator, namely

$$\int Lu \overline{v} \, dx = \int u \overline{Lv} \, dx$$

the matrices $L_\lambda$ are Hermitian. $L$ can be regarded as an infinite dimensional Hermitian matrix. Hence $\sigma(L)$, the spectrum of $L$ consists of eigenvalues only, more exactly

$$\sigma(L) = \left\{ \lim_{\lambda \uparrow \Lambda} \mu_\lambda \in \mathbb{E} \mid \forall \lambda, \mu_\lambda \in \sigma(L_\lambda) \right\}.$$  

The corresponding normalized eigenfunctions form an orthonormal basis of $V^o$. So, in the ultrafunction approach, as in the finite dimensional vector-spaces, the distinction between self-adjoint operators and Hermitian operators is not needed since every Hermitian operator is self-adjoint.

Now, let us analyze the main selfadjoint/Hermitian operators of QM in the frame of ultrafunctions. The **position operator**

$$q : V^o \rightarrow (V^o)^N$$

is defined by

$$(qu)(x) = xu(x).$$  \hspace{1cm} (13)

It is immediate to check that $\sigma(p) = \Gamma$ and that the corresponding orthonormal basis is given by the $\delta$-basis \(\{\delta\}\); in fact

$$(q\delta_q)(x) = x\delta_q(x) = q\delta_q(x)$$

since for $x \neq q$, $\delta_q(x) = 0$.

The **momentum operator**

$$p : V^o \rightarrow (V^o)^N$$

is defined by

$$(pu)(x) = -iDu(x) = -i(D_1u(x), ..., D_Nu(x))).$$  \hspace{1cm} (14)

\footnote{Here and in the rest of this paper, we assume $\hbar = 1$.}
Its spectrum $\sigma(p)$ cannot be computed explicitly, but it is possible to prove that infinitely close to any vector $v \in \mathbb{R}^N$ there is at least an eigenvalue (actually infinitely many eigenvalues) $k \in \sigma(p)$: $k \sim v$. The corresponding orthonormal basis is given by

$$\left\{ \frac{e^{ik \cdot x}}{\sqrt{\beta}} \right\}_{k \in \sigma(p)} \tag{15}$$

where $\sqrt{\beta}$ is a normalization factor, namely

$$\beta = \int |e^{ik \cdot x}| \, dx = \int dx = \sum_{x \in \Gamma} d(x)$$

is an infinite number. Notice that $e^{ik \cdot x}$ is an eigenvalue for the operator $-iD$ only for some particular values of $k \in \mathbb{E}^N$. If you take $k \in \mathbb{R}^N$ arbitrarily, it might happen that $(e^{ik \cdot x})^\circ$ is not an eigenfunction of $-iD$, since the equality

$$D_i f^\circ = (\partial_i f) ^\circ,$$

holds only for functions $f \in V \cap C^1(\mathbb{R}^N) = C^1_c(\mathbb{R}^N)$ (see Axiom 4). So, for a generic $k$ the equality

$$D \left( e^{ik \cdot x} \right)^\circ = ik \left( e^{ik \cdot x} \right)^\circ$$

might be violated in some point $x$ "at infinity" even if it holds for every $x \in \mathbb{R}^N$. In particular, by Axiom 5, it follows that $0 \notin \sigma(p)$.

Next let us consider the generalization of the Laplacian operator

$$\Delta : C^2(\mathbb{R}^N) \rightarrow C^0(\mathbb{R}^N)$$

defined by

$$D^2 = \sum_{j=1}^N D_j^2 : V^\circ \rightarrow V^\circ.$$ 

Probably the most important operator in classical QM is the following Hamiltonian operator

$$H u(x) = -\frac{1}{2} \Delta u(x) + V(x) u(x) \tag{16}$$

While in the $L^2$-theory a very delicate question is to choose an appropriate potential $V$ such that (16) make sense and to define an appropriate selfadjoint realization of $H$, in the theory of ultrafunctions any internal function $V : \Gamma \rightarrow \mathbb{E}$ provides a selfadjoint operator on $H^\circ$ given by

$$H^\circ u(x) := -\frac{1}{2} D^2 u(x) + V(x) u(x) \tag{17}$$

with a spectrum consisting only of eigenvalues.

\footnote{Here, for simplicity, we assume that $m_1 = 1$; $m_i$ is the mass of the $i$-th particle.}
In particular it is possible to consider "very singular potential" such as
\[ V(x) = k\delta_a(x), \quad k \in \mathbb{E} \]  
(18)
\[ V(x) = \alpha\chi_{\Omega}(x) \]  
(19)
where \( \alpha \), defined by (4), is an infinite number. These potentials cannot be defined outside of NAM; nevertheless they have an interesting physical meaning.

3.4 The heat equation
In QM and more in general in Mathematical Physics, an expression such as (16) does not define a selfadjoint operator; it is necessary to construct a selfadjoint realization of it; sometimes it is a very delicate question and it has relevant physical meaning. How can we compare this fact with the NAM-theory where (17) always describe a selfadjoint operator with a spectrum of only eigenvalues? As matter of fact NAM-theory can overcome these difficulties using a larger set of numbers (and hence of operators) to describe the different physical phenomena. For example in (19) the infinite number \( \alpha \) is present.

In order to clarify this issue, let us consider a very simple example: the heat equation in dimension 1
\[ \partial_t u = \frac{1}{2}\partial_x^2 u \]
In this case we have that \( H = \frac{1}{2}\partial_x^2 \) and \( V(x) = 0 \). Suppose that we want to describe the diffusion of heat in a bar modelled by the interval \([0, 1] \); we suppose that the initial condition is given by a function \( \psi \in L^2([0, 1]) \). This problem is not well posed because there are different selfadjoint realizations of \( H \) in \( L^2([0, 1]) \) which describe different physical situations. These selfadjoint realizations are determined by imposing boundary conditions (BC): the most important ones are:

- the Dirichlet BC:
  \[ u(t, 0) = u(t, 1) = 0 \]

- the Neumann BC:
  \[ \partial_x u(t, 0) = \partial_x u(t, 1) = 0 \]

These two kind of conditions provide two different selfadjoint realizations of \( H \) which we will call \( H_D \) and \( H_N \) respectively. The spectrum of the first one is given by
\[ \sigma(H_D) = \pi\mathbb{N} = \{\pi, 2\pi, 3\pi, \ldots\} \]
and the eigenfunctions are
\[ \{\sin(\pi x), \sin(2\pi x), \sin(3\pi x), \ldots\} \]
The spectrum of the \( H_N \) is given by
\[ \sigma(H_N) = \pi\mathbb{N}_0 = \{0, \pi, 2\pi, 3\pi, \ldots\} \]
and the eigenfunctions are

\[ \{1, \cos(\pi x), \cos(2\pi x), \cos(3\pi x), \ldots\} \]

In the ultrafunction theory, the two physical phenomena are described by choosing different Hamiltonian defined on all \( V^\circ \):

\[
    H^\circ_D u = -\frac{1}{2}D^2 u + \alpha \chi^\circ_{[0,1]}(x)u; \quad [0,1]^c = \mathbb{R}\setminus[0,1] \quad (20)
\]

\[
    H^\circ_N u = -\frac{1}{2}D \left( \chi^\circ_{[0,1]}(x)Du \right) \quad (21)
\]

The physical meaning of these two Hamiltonians is evident: in the first case we have that, outside \([0,1]\), the heat is absorbed with an infinite strength \(-\alpha u\) so that it can reach only infinitesimal values. In the second case we have that the diffusion coefficient \(k(x)\chi^\circ_{[0,1]}\) vanishes out of \([0,1]\) and hence, all the heat is kept inside our bar. In both cases the evolution is given by the exponential matrix

\[
    u(t,x) = e^{-tH^\circ_D}\psi^\circ; \quad u(t,x) = e^{-tH^\circ_N}\psi^\circ.
\]

Concluding, in classical mathematics we have to choose different self-adjoint realization of a given differential operator in order to describe a given phenomenon. In ultrafunction theory, it is sufficient to choose the appropriate operator since you have a larger set of operators.

4 Ultrafunctions and Quantum Mechanics

In this section we will describe an application of the previous theory to the formalism of Quantum Mechanics. In the usual formalism, a physical state is described by a unit vector \(\psi\) in a Hilbert space \(\mathcal{H}\) and an observable by a self-adjoint operator defined on it. In the ultrafunctions formalism, a physical state is described by an ultrafunction \(\psi\) in \(H^\circ = V^\circ + iV^\circ\) and an observable by a Hermitian operator defined on it.

The ultrafunctions approach to the QM-formalism presents the following peculiarities:

- once you have learned the basic facts of \(\Lambda\)-theory, the formalism which you get is easier to handle since it is based on Hermitian matrices rather than on unbounded self-adjoint operators in Hilbert spaces;
- this approach is closer to the "infinite" matrix approach of the beginning of QM before the work of von Neumann and also closer to the way of thinking of the theoretical physicists and chemists;
- all observables (hyperfine matrices) have infinitely many eigenvectors; so the continuous spectrum can be considered as a set of eigenvalues infinitely close to each other;
• the operator
\[ H^\circ u(x) = -\frac{1}{2} D^2 u(x) + V(x) u(x) \]
defines a selfadjoint operator on \( H^\circ = V^\circ + iV^\circ \) for any ultrafunction \( V(x) \).

• the dynamics does not present any difficulty since it is given by the exponential matrix relative to the Hamiltonian matrix \( H^\circ \);

• the ideal ultrafunctions, namely the ultrafunctions which are not close to any "classical state" such as the Dirac ultrafunctions, represent ideal states which cannot be reproduced in laboratory.

4.1 The axioms of Quantum Mechanics

We start giving a list of the main axioms of quantum mechanics as it is usually given in any textbook and then we will compare it with the alternative formalism based on ultrafunctions.

Von Neumann Axioms of QM

\textbf{Axiom C1.} A physical state is described by a unit vector \( \psi \) in a Hilbert space \( \mathcal{H} \).

\textbf{Axiom C2.} An observable is represented by a self-adjoint operator \( A \) on \( \mathcal{H} \).

(a) The set of observable outcomes is given by the eigenvalues \( \mu_j \) of \( A \).

(b) After an observation/measurement of an outcome \( \mu_j \), the system is left in a eigenstate \( \psi_j \) associated with the detected eigenvalue \( \mu_j \).

(b) In a measurement the transition probability \( P \) from a state \( \psi \) to an eigenstate \( \psi_j \) is given by
\[ P = |(\psi, \psi_j)|^2. \]

\textbf{Axiom C3.} The evolution of a state is given by the Schrödinger equation
\[ i \frac{\partial \psi}{\partial t} = H \psi \]
where \( H \), the Hamiltonian operator, is a self-adjoint operator representing the energy of the system.

Axioms of QM based on ultrafunctions

\textbf{Axiom U1.} A physical state is described by a unit complex-valued ultrafunction \( \psi \).
**Axiom U2.** An observable is represented by a Hermitian operator $A$ on $H^\circ$.

(a) The set of observable outcomes is given by

$$st(\mu_j)$$

where $\mu_j$ is an eigenvalue of $A$.

(b) After an observation/measurement of an outcome $st(\mu_j)$, the system is left in an eigenstate $\psi_j$ associated with the detected eigenvalue $\mu_j$.

(b) In a measurement the transition probability $P$ from a state $\psi$ to an eigenstate $\psi_j$ is given by

$$P = |(\psi, \psi_j)|^2.$$

**Axiom U3.** The evolution of the state of a system is given by the Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = H^\circ \psi \quad (22)$$

where $H^\circ$, the Hamiltonian operator, representing the energy of the system.

**Axiom U4.** In laboratory you can realize only the states which correspond to a finite expectation value of the energy (and/or the other physically relevant quantities). We will call them physical states and the others will be called ideal states.

### 4.2 Discussion of the axioms

**AXIOM 1.** In the classical formalism, a physical system is described by a vector in an Hilbert space. In particular, taking the Schrödinger representation of $H$, $\psi$ can be represented by a function $\psi \in L^2(\Omega)$, $\Omega \subset \mathbb{R}^N$; so, by Def. [13] there is the following canonical embedding

$\circ : \mathcal{H} \rightarrow H^\circ$

$\psi \mapsto \psi^\circ$

Since $H^\circ$ is much richer than $\mathcal{H}$, in the ultrafunction framework there exist more possible states; in particular the ideal states, see Axiom **U4**.

**AXIOM 2.** In the ultrafunction formalism, the Von Neuman notion of self-adjoint operator is not needed. In fact observables can be represented by internal Hermitian operators which are trivially self-adjoint. It follows that any observable has exactly $\kappa = \dim^*(H^\circ)$ eigenvalues (of course, if you take account of their multiplicity). No essential distinction between eigenvalues and continuous spectrum is required. For example, consider the eigenvalues of the position operator $q$ of a free particle. The eigenfunction relative to an eigenvalue $q \in \mathbb{R}$ is the Dirac ultrafunction $\delta_q$. 
The eigenvalues of an internal Hermitian operator \( A \) are Euclidean numbers, and hence, assuming that a measurement gives a real number, we have imposed in Axiom 2 that the outcome of an experiment is \( st(\mu) \). However, we think that the probability is better described by the Euclidean number \( \left| \langle \psi, \psi_j \rangle \right|^2 \) rather than the real number \( st(\left| \langle \psi, \psi_j \rangle \right|^2) \). For example, let \( \psi \in H^o \) be the state of a system; the probability of that a particle is in the position \( q \) is given by

\[
\left| \int \psi(x)\delta_q(x)\sqrt{d(a)}dx \right| = |\psi(q)| \sqrt{d(q)}
\]

which is is an infinitesimal number. We refer to [18, 19] for a presentation and discussion of the Non Archimedean Probability.

AXIOM 3. Since \( H^o \) is an internal operator defined on a hyperfinite vector space, it can be represented by an Hermitian hyperfinite matrix and hence the evolution operator of (22) is described by the exponential matrix \( e^{itH^o} \).

AXIOM 4. In ultrafunction theory, the mathematical distinction between physical eigenstates and the ideal eigenstates is intrinsic and it does not correspond to anything in the usual formalism. The point is to know if it corresponds to something physically meaningful. Basically, we can say that the physical states can be prepared and measured in a laboratory, while the ideal states represent "extreme" situations useful in the foundations of the theory, in thought experiments (gedankenexperiment) and in the computations. For example the Dirac ultrafunction is not a physical state but an ideal state and it represents a situation in which the position of a particle is perfectly determined. Clearly this state cannot be produced in a laboratory since it requires infinite energy (see section 4.3), but nevertheless it is useful in our description of the physical world. This situation makes more explicit something which is already present in the classical approach. For example, in the Shroedinger representation of a free particle in \( \mathbb{R}^3 \), consider the state

\[
\psi(x) = \frac{\varphi(x)}{|x|}, \quad \varphi \in C(\mathbb{R}^3), \quad \varphi(0) > 0.
\]

We have that \( \psi \in L^2(\mathbb{R}^3) \) but this state cannot be produced in a laboratory, since the expected value of its energy

\[
\langle H\psi, \psi \rangle = \frac{1}{2} \int |\nabla \psi|^2 dx
\]

is infinite (even if the result of a single experiment is a finite number). In other words, Axiom \textbf{U4} makes formally precise something which, in some sense, is already present (but hidden) in the classical formalism.
4.3 The Heisenberg algebra

In this section, we will apply ultrafunction theory to the description of a quantum particle via the algebraic approach. For simplicity here we consider the one-dimensional case. The states of a particle are defined by the observables $q$ and $p$ which represent the position and the momentum respectively. A quantum particle is described by the algebra of observables generated by $p$ and $q$ according to the following commutation rules:

$$[p, q] = i, \quad [p, p] = 0, \quad [q, q] = 0$$

The algebra generated by $p$ and $q$ with the above relations is called the Heisenberg algebra and denoted by $\mathfrak{A}_H$. The Heisenberg algebra does not fit in the general theory of $C^*$-algebras since both $p$ and $q$ are not bounded operators. The usual technical solution to this problem is done via the Weyl operators and the Weyl algebra (for more details and a discussion on this point we refer to [43]).

Let us see an alternative approach via the ultrafunction theory. First of all we take the following representation of $\mathfrak{A}_H$:

$$J : \mathfrak{A}_H \rightarrow \mathfrak{A}(H^\circ)$$

where $\mathfrak{A}(H^\circ)$ is the algebra of the Hermitian internal operators on $H^\circ$; $J$ is defined by

$$J(p) = p = -iD; \quad J(q) = q = x.$$ (see (14) and (13)). $p$ and $q$ are Hermitian operators and hence $H^\circ$ has an orthonormal basis generated by the eigenfunctions of $p$ or $q$. A very interesting fact is that the ultrafunction model violate the Heisenberg uncertainty relations $[p, q] = i$. To see this fact, we argue indirectly. Assume that the Heisenberg relation holds; then

$$\langle [p, q] \delta_a, \delta_a \rangle = i \| \delta_a \|^2.$$

On the other hand, by a direct computation\(^4\) we get:

$$\oint [p, q] \delta_a(x)\overline{\delta_a(x)} \, dx = \oint -iD(x\delta_a(x)) \delta_a(x) \, dx + \oint x(iD\delta_a(x)) \delta_a(x) \, dx$$

$$= -ia \oint [D\delta_a(x)] \delta_a(x) \, dx + i \oint x[D\delta_a(x)] \delta_a(x) \, dx$$

$$= -iaD\delta_a(a) + iaD\delta_a(a) = 0$$

This fact is consistent with the Axiom $U4$ which establishes that the ideal states cannot be produced in laboratory. According to this description of QM,

\(^4\) Notice that this computation implies the violation of the Leibniz rule of the differentiation of a product. This fact is consistent with the Schwartz impossibility theorem [42] which states that there is not a differential algebra containing a set isomorphic to the space of distribution. In our case, we have that $V^\circ$ is an algebra “containing” the distributions and hence, it cannot be a differential algebra; in fact the Leibniz rule, in some cases, is violated.
the uncertainty relations hold only for the limitation of the experimental apparatus. In a laboratory you can prepare a state corresponding to a function

\[ \psi(x) = \sum_{a \in \Gamma} \psi(a) \chi_a(x) = \sum_{a \in \Gamma} \frac{\psi(a)}{d(a)} \delta_a(x) \]

for which the expectation value of the energy is finite. But the eigenfunction

\[ \psi_a(x) = \frac{\delta_a(x)}{\|\delta_a\|} = \sqrt{d(a)} \delta_a(x) \]

has an infinite expectation value of the energy as the following computation shows:

\[ \langle H^0 \psi, \psi \rangle = \frac{1}{2} \int |D\psi_a|^2 \, dx = \frac{1}{2} d(a) \int |D\delta_a|^2 \, dx = \frac{1}{2} d(a) \|D\delta_a\|^2 = \|D\delta_a\|^2 \|\delta_a\|^2 \]

The conclusion follows from the following proposition:

**Proposition 14.** \( \frac{\|D\delta_a\|}{\|\delta_a\|} \) is an infinite number.

**Idea of the proof:** The Poincaré inequality states that

\[ \forall u \in C^1_c(a, b), \quad \int_a^b |u|^2 \, dx \leq (b - a)^2 \int_a^b |\partial u|^2 \, dx \]

Since \( \delta_a \) is the \( \Lambda \)-limit of functions having support in an interval whose length tends to 0, then

\[ \int_a^b |\delta_q|^2 \, dx \leq \varepsilon^2 \int_a^b |D\delta_q|^2 \, dx \]

where \( \varepsilon \) is an infinitesimal.

\( \square \)

In conclusion we get the following picture. The ultrafunction Heisenberg algebra \( \mathfrak{A}(H^0) \) is an algebra over the field \( \mathbb{C}^* \) and hence it is also an algebra over \( \mathbb{C} \) and \( \mathfrak{A}_H \) is a subalgebra of \( \mathfrak{A}(H^0) \) over the field \( \mathbb{C} \). One of the advantages in using \( \mathfrak{A}(H^0) \) is that all the operators in \( \mathfrak{A}(H^0) \) are bounded and the spectrum consists of eigenvalues, namely it behaves as a matrix algebra. The price that we pay for this is the existence of ideal states which are not is feasible in a laboratory. With this respect, the Heisenberg uncertainty relations take a somewhat different philosophical meaning.

## 5 Conclusion

If we live in a larger space there are more things to see and more things we can do. This is the palpable truth that we tried to illustrate in this work.
If we work in a mathematical universe containing infinitesimal, we can have a greater choice for models able to describe natural phenomena. In particular, we can describe the physical space as a set of points wider than those that can be described by real coordinates. Furthermore we can make space models whose points are represented by a hyperfinite grid. In this way we can have a set of functions much wider than the real functions, but at the same time they are easier to handle because they satisfy the same (internal) rules of the vectors in a finite dimensional vector space. The use of this space model has many advantages in a wide range of problems. In particular, the axioms of the QM that we have illustrated here allow considerable technical simplifications by reducing the self-adjoint operators to Hermitian matrices of hyperfinite size. Among the possible eigenstates of an unbounded operator, there are also those who have infinite energy, which although not achievable in a laboratory allow a broader vision and an alternative description of the phenomenon. For example, Heisenberg’s relations are reformulated as follows: in order to determine exactly the position of a particle, an experimental apparatus is needed that can provide the particle with an infinite impulse and energy.

In conclusion, we belive that NAM in the future can have more uses than commonly think, both in foundational and applicative.

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