A Unified Stability Analysis Approach for A Class of Interconnected System

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Abstract: From the structural perspective, this paper investigates a new formulation of the concept of input-to-state stability (ISS), and based on this formulation, proposes a new stability analysis approach for a class of interconnected system. The new formulation of ISS is better able to reflect the tendency of the state $x(t)$ tracking the input $u(t)$ and weakens the conservative of the original form. The stability analysis method which transforms the interconnected system into the equivalent cascade form, does not depend on the Lyapunov function, breaks through the limitation of the small-gain theorem and extends the application of ISS. As its applications in three typical kinds of interconnected systems, this method is used to prove the small-gain theorem again and analyzes the stability of a class of interconnected system and the consensus of the multi-agent system (MAS).

Key Words: input-to-state stability, interconnected system, cascade system, small-gain theorem, stability analysis, multi-agent system.

1 Introduction

1.1 Background

The concept of the input-to-state stability (ISS) was introduced by E.D. Sontag in 1989 in the well-known paper[1], which becomes a popular method to study the input-output property of nonlinear systems later. Generally, a system $\dot{x} = f(t, x, u)$ is said to be ISS if there exist class $KL$ function $\beta$ and class $K$ function $\gamma$, such that for any initial value in the closed set $D$ and the bounded input $u(t)$, the following is satisfied:

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0) + \gamma(\sup_{\tau \in [t_0, t]} \|u(\tau)\|)$$

ISS describes the evolution of the state of a stable system when it is driven by the external input. Based on the concept, many researchers proposed some other concepts, such as IOS (input to output stable), OLIOS (output-Lagrange input to output stable), SIOS (state-independent IOS), ROS (robustly output stable), iISS (integral input-to-state stable) [3], and ISDS (input-to-state dynamically stable) [4] which describes the dynamic process of a stable system. The introduction of ISS gives us a new way to study the stability of a system. For more complicated systems, which contain many subsystems that interconnect with each other, if every subsystem is ISS, we can take a structural perspective, ignore their internal details and take full advantage of the input-output property and interconnected relationship between subsystems to study the stability problem. In this way, we only need to focus on the relationship between all subsystems and need not care about all details, which is greatly different from the Lyapunov function based method. Therefore, the concept of ISS greatly simplifies the stability analysis of the complex interconnected system.

A good example of the above idea is the well-known small-gain theorem. In short, if two subsystems are ISS and interconnected with each other, if the composition of the gain function (the quantitative expression of the input-output property of each subsystem) along the closed cycle is less than the identity function, the entire system is stable. Later on, with ISS as tool, many researchers extended the small-gain theorem from linear systems to nonlinear systems and proposed its various forms and the associated proofs [5]-[7]. Furthermore, Jiang and etc. extend the small-gain theorem to the case of the interconnected system with more than two subsystems, and give the sufficient condition that ensures stability in [6], i.e., if the composition of the gain function along every closed cycle is less than the identity function, the entire system is stable. Because the structural perspective is brief and intuitionistic, it becomes an important method to design the controller [8] and analyze the stability.

However, when facing some new problems, the above concept and approach meet across some difficulties.

1) The form of ISS is too conservative to describe the corresponding change of the state $x(t)$ of system tracking the input $u(t)$.

$$\sup_{\tau \in [t_0, t]} \|u(\tau)\|$$ just denotes the maximum of $u(t)$ in a certain time interval, but in fact, $x(t)$ keeps changing the change of the input $u(t)$ all the time so as to be kept in a neighboring area of it. Therefore $\sup_{\tau \in [t_0, t]} \|u(\tau)\|$ is too conservative to describe the fact, especially, when $u(t)$ converges towards a constant.

2) Some systems, only one of whose subsystems is ISS, are still stable in fact.

The small-gain theorem requires all subsystems should be ISS, but in fact, some systems like the one given by equation (1) below, are also stable even though the first subsystem is not ISS.
Example 1. 
\[
\begin{cases}
\dot{x} = z \\
\dot{z} = -z - x
\end{cases}
\] 
(1)

where \(x, z \in \mathbb{R}\). The \(z\)-subsystem, i.e., the second equation, is ISS, but the \(x\)-subsystem, i.e., the first equation, is not.

3) Even though every agent is ISS, the consensus problem of multi-agent systems is not explained via the small-gain theorem. Since the composition of the gain function along the closed cycle equals the identity function, which conflicts with the small-gain theorem.

Therefore, from the structural perspective, the tools in the current literature cannot solve these new problems, which urgently requires to develop a new formulation of ISS and a new approach.

1.2 Problem Statement

In this paper, we investigate a new formulation of ISS, and based on it, propose a stability analysis approach for a class of interconnected system. The contribution of this paper includes three parts. Firstly, a new formulation of ISS is proposed, which is better able to reflect the corresponding change of \(x(t)\) tracking \(u(t)\) than the original form, and using this new formulation, we uncover the essential relationship between the interconnected system and the cascade system. Secondly, based on the research of ISS, a unified stability analysis approach is proposed, which does not depend on the construction of a Lyapunov function. Thirdly, as the application of this approach, we analyze the stability of three typical kinds of interconnected systems, i.e., repeating to prove the small-gain theorem in this new framework and analyzing the stability of a class of interconnected system like equation (1) and the consensus of a multi-agent system.

1.3 Organization of the Paper

The rest of the paper is organized as follows. Section 2 gives some notations and briefly recalls some basic background knowledge. Section 3 presents the framework of this approach. Section 4 presents a new formulation of ISS. Section 5 gives its applications for three typical interconnected systems. Section 6 is the conclusion and talks about other applications.

2 Notation and Preliminaries

Classes of \(K, K_\infty, K_L\) and positive definite function follow the definition in [10], which are extensively used in the field. \(D \in \mathbb{R}^n\) denotes a domain containing the origin. Now, we recall the traditional concept of ISS again. Consider the general nonlinear system

\[
\dot{x} = f(x, u)
\]
(2)

where \(f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n\) is the continuous function and local Lipschitz function w.r.t. \(x(t)\) and \(u(t)\).

The Lyapunov-like theorem that follows gives a sufficient and necessary condition for ISS.

**Proposition 1.** [10] Let \(V : [0, \infty] \times \mathbb{R}^n \to \mathbb{R}\) be a continuously differentiable function such that

\[
\alpha_1(||x||) \leq V \leq \alpha_2(||x||)
\]
(3)

\[
\dot{V} \leq -W_3(x), \forall \|x\| \geq \rho(\|u\|) > 0
\]
(4)

where \(\alpha_1, \alpha_2 \in K_\infty, \rho \in K_\infty\), and \(W_3\) is continuous positive definite function on \(\mathbb{R}^n\). Then, the system (1) is input-to-state stable with \(\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho\).

3 Problem Statement and Analysis Framework

In this paper, we investigate the stability problem of a class of interconnected system described by the following

Subsystem \(x\):

\[
\dot{x} = f_1(x, z),
\]
(5)

Subsystem \(z\):

\[
\dot{z} = f_2(z, x),
\]
(6)

where \(x \in \mathbb{R}^n, z \in \mathbb{R}^m, f_1\) and \(f_2\) are continuous functions. Assume that at least one subsystem is ISS, and without loss of generality, suppose \(x\)-subsystem is ISS.

Naturally, Lyapunov function is the most general choice for the stability analysis of such system. However, it is not easy to find an appropriate candidate function, especially when the system becomes more and more complex and a lot of subsystems are strongly coupled with each other. If every subsystem is ISS, we can resort to the small-gain theorem for analysis, which uses the gain of each subsystem to check the stability of the interconnected system. Essentially, such a way represents a structural perspective and is more suitable for interconnected systems than the Lyapunov function based approach. Inspired by this idea, this paper will propose a new structural stability analysis approach for \(5\) and \(6\) in the following.

**Stability Analysis Procedure 1.**

**Step 1.** Transform the interconnected form into a cascade form via the ISS property of \(x\)-subsystem.

The solution of \(x\)-subsystem can be written as a function of the initial value \(x(t_0)\), \(t\) and input \(u(t)\), i.e., \(x(t) = \phi(x_0, t, z)\). Substituting this equation into the \(z\)-subsystem yields \(\dot{z} = f_2(z, \phi(x_0, t, z))\). Thus, the interconnected system becomes a cascade system of the following form

\[
\dot{z} = f_2(z, \phi(x_0, t, z))
\]
(7)

\[
\dot{x} = f_1(x, z)
\]
(8)

**Remark 1.** Using the ISS property, the above process has an intuitive explanation. Since the \(x\)-subsystem is ISS w.r.t. \(z(t)\) and suppose the gain function from \(z(t)\) to \(x(t)\) is \(\gamma\), let \(\gamma(z(t))\) be the input, then the \(x\)-subsystem corresponds to a filter whose gain is 1. Its output is not arbitrary but keeps tracking \(\gamma(z(t))\), and in fact, it is kept in a neighboring area of \(\gamma(z(t))\). Therefore, equation (8) can be written as the following formulation of ISS

\[
x(t) = \beta(x_0, t) + \gamma(z(t)) + \Delta
\]
(9)

where \(\Delta\) denotes a static error with \(\gamma(z(t))\) whose specific form will be given later. Therefore, using \(9\), we can construct a feedback loop as follows

\[
\dot{z} = f_2(z, \beta(x_0, t) + \gamma(z) + \Delta)
\]
(10)
and the cascade form of the interconnected system

Fig. 1: The cascade form of the interconnected system

and the cascade system can be described in the Fig.1.

Step 2. Analyze the stability of the feedback loop of z-subsystem.

Since \( \beta(x_0, t) \) is convergent, the stability of feedback loop of z-subsystem depends on the function \( f_2 \) and \( \gamma(z) + \Delta \). It is necessary to study the new formulation of ISS.

Remark 2. Compare with the original formulation of ISS, \( \gamma(z) + \Delta \) replaces \( \gamma(\sup_{\tau \in [t_0, t]} \|u(\tau)\|) \). It should note that the former represents the current value of the input \( z(t) \), but the later represents its history. Besides, in various proofs of the small-gain theorems (5) and (7), only the form of the gain \( \gamma \) is needed, but \( \Delta \) is not cared absolutely. But later, we will show what is \( \Delta \) and what does it function in the new problem.

Step 3. Analyze the stability of the cascade system.

After transformed into the cascade system, by the stability theorems in [11] about cascade systems and the ISS property of x-subsystem, if and only if the z-subsystem is stable, the entire system is stable, so is the original interconnected system.

Remark 3. In this approach, the concept of ISS bridges the gap of the interconnected system and the cascade system. In this way, the stability of the complex interconnected system is equivalent to the stability of its equivalent cascade system, and further the stability of a feedback loop. Through transforming the stability of the interconnected system into the stability of one of subsystems, this approach greatly simplifies the analysis.

It should be mentioned that this approach just requires one subsystem should be ISS and need not construct an overall Lyapunov function which used to consider all details and too depends on the specific form of the system.

4 New Formulation of ISS

In this section, we present a new formulation of ISS concept to weaken the conservative of the original one. Consider the general system (2) and suppose it satisfies the following assumption.

Assumption 1. The general system (2) satisfies the proposition 1 and the gain function \( \gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho \) is differentiable.

Then we have the new formulation of ISS in the following.

Theorem 1. Suppose the system (2) satisfies the assumption 1 in \( D \), if there exist a class KL function \( \beta \), a class K function \( \gamma \), and a constant \( L > 0 \) such that for any initial state \( x(t_0) \) and any bounded input \( u(t) \), the solution of \( x(t) \) exists for all \( t \geq t_0 \) and satisfies

\[
\|x\| \leq \beta(x_0, u_0, t) + \gamma(\|u\|) - L \int_{t_0}^{t} e^{-\int_{s}^{t} \kappa(\tau)d\tau} \alpha_4(u) d\tau
\]

where \( \alpha_4(u) = \frac{d_\alpha \rho(\|u\|)}{du} \) and \( k(t) \) is continuous positive definite on \( R \).

proof. By the proposition 1, there exists the Lyapunov function \( V(x) \) such that

\[
\alpha_1(\|x\|) \leq V \leq \alpha_2(\|x\|)
\]

\[
\dot{V} \leq -W(V), \forall \|x\| \geq \rho(\|u\|) > 0.
\]

where \( W \) is positive definite on \( R \), and \( W(V) \) can be obtained by (11). According to (11), the condition of (12) \( \|x\| \geq \rho(\|u\|) \) can be strengthened as \( V \geq \alpha_2(\rho(\|u\|)) \), and define the error \( e = V - \alpha_2(\rho(\|u\|)) \), then we obtain the error system of equation (12)

\[
\dot{\varepsilon} \leq -W(e + \alpha_2(\rho(\|u\|)) - \alpha_4(u) \dot{u}, \forall \varepsilon \geq 0
\]

Solving it and by the comparison theorem in [10] yields

\[
e(t) \leq e^{-\int_{t_0}^{t} k(s)ds} e_0 - \int_{t_0}^{t} e^{-\int_{s}^{t} \kappa(\tau)d\tau} \alpha_4(e) \dot{u} d\tau
\]

Due to \( \int_{t_0}^{t} k(s)ds > 0 \), we define the class KL function \( \beta_1(x_0, u_0, t) = -\int_{0}^{t} k(s)ds e_0 \). In view of \( e = V - \alpha_2(\rho(\|u\|)) \), equation (15) can be written as

\[
V(t) \leq \beta_1(x_0, u_0, t) + \alpha_2(\rho(\|u\|)) + \int_{0}^{t} e^{-\int_{s}^{t} \kappa(\tau)d\tau} \alpha_4(u) \dot{u} d\tau
\]

That is

\[
\|x\| \leq \alpha_1^{-1}(\beta_1(x_0, u_0, t) + \alpha_2(\rho(\|u\|)) - \int_{0}^{t} e^{-\int_{s}^{t} \kappa(\tau)d\tau} \alpha_4(u) \dot{u} d\tau)
\]

By the Lagrange median theorem, we have

\[
\alpha_1^{-1}(\beta_1(x_0, u_0, t) + \alpha_2(\rho(\|u\|)) - \int_{0}^{t} e^{-\int_{s}^{t} \kappa(\tau)d\tau} \alpha_4(u) \dot{u} d\tau) = \frac{d_{\alpha}^{-1}(x)}{dx} |_{x = \xi} (\beta_1(x_0, u_0, t) - \int_{0}^{t} e^{-\int_{s}^{t} \kappa(\tau)d\tau} \alpha_4(u) \dot{u} d\tau)
\]

where \( x \) denotes \( \alpha_2(\rho(\|u\|)) \). Due to \( \alpha_1^{-1}(x) \in K \) and \( d_{\alpha}^{-1}(x) > 0 \), for \( x \in D \), there exists a constant \( L > 0 \), such that

\[
\|x\| \leq \alpha_1^{-1}(\alpha_2(\rho(\|u\|)) + L \beta_1(x_0, u_0, t)
\]

\[
- L \int_{0}^{t} e^{-\int_{s}^{t} \kappa(\tau)d\tau} \alpha_4(u) \dot{u} d\tau.
\]
At last, equation (18) can be written as

\[ \|x\| \leq \beta(x_0, u_0, t) + \gamma(\|u\|) - L \int_0^t e^{-\int_0^t k(\tau) d\tau} \alpha_4(u) \dot{uds} \]

(19)

where \( \beta = L \beta_1, \gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho \).

**Remark 4.** Compare with the old form, \( \gamma(\|u\|) - L \int_0^t e^{-\int_0^t k(\tau) d\tau} \alpha_4(u) \dot{uds} \) is more accurate than \( \gamma(\sup_{\tau \in [t_0, t]} \|u(\tau)\|) \).

Recall equation (9), \( \Delta = -L \int_0^t e^{-\int_0^t k(\tau) d\tau} \alpha_4(u) \dot{uds} \), and its convergence depends on the existence of \( \lim_{t \to \infty} \int_0^t e^{-\int_0^t k(\tau) d\tau} \alpha_4(u) \dot{uds} \).

**Remark 5.** The new formulation reflects the tendency of \( x(t) \) tracking \( u(t) \). When the influence of \( x(t_0) \), the neighboring area is determined by \( \Delta \), that is when \( \dot{u} \to 0, \Delta \to 0 \).

**Remark 6.** Compare two formulations (19) and (20), the formulation (19) containing \( \dot{u} \) is suitable for the analysis of the consensus of multi-agent systems, while (20) only containing \( u \) is convenient for the general stable system.

The following simple example can illustrate the theorem 1. Now, let us consider a simple case. Let \( u \in R \) in the system (2), we obtain the system that follows

\[ \dot{x} = f(x, u) \]

(24)

It satisfies the assumption.

**Assumption 2.** System (24) is ISS, and there exists differentiable function \( \gamma : R^n \to R^n \) whose element \( \gamma_i \) belongs to the class \( K \) function, such that

\[ 0 = f(\gamma(u), u) \]

(25)

**Remark 7.** The assumption represents a class of system whose equilibrium point is a class \( K \) function of input \( u(t) \), that means its static gain can be obtained by solving the algebraic equation (25). There are many examples, e.g. all linear systems and the following nonlinear systems \( \dot{x} = -x^3 + u \) and \( \dot{x} = -\tan x + x + u \).

Before moving on, we introduce a lemma first.

**Lemma 1.** If the function matrix \( A(x) = \{a_{ij}(x)\}, a_{ij} : R^n \to R, x \in D, i, j = 1, ..., n \), and \( a_{ij}(x) \) is negative at \( \lambda_1(x) < \lambda_2(x) < 0 \) (or \( \lambda_2(x) > \lambda_1(x) > 0 \)) such that \( \lambda_1(x) I < A(x) < \lambda_2(x) I \).

**Proof.** Since for any square matrix \( A = \{a_{ij}\}, a_{ij} \), there exists a non-singular matrix \( P \) and a Jordan form \( J = diag(J_i) \), \( \forall i = 1, ..., m, m \leq n \) such that \( A = PJP^{-1} \). Then for \( A(x) \), by the continuity, we have

\[ A(x) = P(x)J(x)P(x)^{-1} \]

(26)

When \( A(x) \) is negative definite in \( D \), its eigenvalue \( \lambda_{i,j}(x) \) of \( J_i \) is negative, where \( i = 1, ..., rank(J_i) \).

Then by the continuity, there exists \( \lambda_{min}(x) < \lambda_{max}(x) \) such that \( \lambda_{min}(x) < \lambda_{ij}(x) < \lambda_{max}(x) \), \( x \in D \), e.g., we can let \( \lambda_{min}(s) = \inf_{s \leq \|x\| \leq r} (\lambda_i(x)), 0 \leq s \leq r \), and \( \lambda_{max}(s) = \sup_{s \leq \|x\| \leq r} (\lambda_i(x)), 0 \leq s \leq r, \forall i = 1, ..., m, m = 1, ..., rank(J_i) \). It should be mentioned that if \( D = R^n \), \( r \) can be ignored.

Thus, every diagonal element of \( J(x) - \lambda_{min}(x)I \) can be written as \( \lambda_{ij}(x) - \lambda_{min}(x)I \) is positive definite, therefore, we have \( J(x) > \lambda_{min}(x)I \). Similarly, we obtain \( J(x) < \lambda_{max}(x)I \). Combining two inequalities yields

\[ \lambda_{min}(x)I < J(x) < \lambda_{max}(x)I \]

(27)

Multiply \( P(x) \) and \( P(x)^{-1} \) on the both sides of (27), and define \( \lambda_1(x) = \lambda_{min}(x) \) and \( \lambda_2(x) = \lambda_{max}(x) \) yields \( \lambda_1(x)I < \lambda_2(x)I \).
\( A(x) < \lambda_2(x)I \) Similarly, in the same way, we can obtain above result when \( A(x) \) is positive definite.

Based on above lemma, we have the following theorem.

**Theorem 2.** Suppose system (24) satisfies the assumption 2, if there exist class \( K \) functions \( \beta_i \), \( \gamma_i \) such that for any initial state \( x(t_0) \) and any bounded input \( u(t) \), the solution of \( x(t) \) exists for all \( t \geq t_0 \) and satisfies

\[
\begin{align*}
\dot{x}_i(t) &\leq \beta_{2i}(x_0, u_0, t) + \gamma_i(u) - f_0^t e^{-\int_0^s \lambda_i(s) ds} \min(\gamma_i', u) ds \\
\dot{x}_i(t) &\geq \beta_{1i}(x_0, u_0, t) + \gamma_i(u) - f_0^t e^{-\int_0^s \lambda_i(s) ds} \max(\gamma_i, u) ds
\end{align*}
\]

where \( i = 1, ..., n, \lambda_3(u, e) \geq \lambda_1(u, e) > 0 \).

**Proof.** Define the error \( e = x - \gamma(u) \), then the error system of system (24) can be written as

\[
\dot{e} = f(e + \gamma(u), u) - \gamma'(u) \dot{u}
\]

where \( \gamma'(u) = \frac{\partial \gamma(u)}{\partial u} \).

Expanding \( f(e + \gamma(u), u) \) yields

\[
f(e + \gamma(u), u) = f(\gamma(u), u) + \nabla f'(\xi + \gamma(u), u) e
\]

where \( \xi = \theta e, \theta \in [0, 1] \). Due to \( f(\gamma(u), u) = 0 \), then (29) can be written as

\[
\dot{e} = A(u, e) e - \gamma'(u) \dot{u}
\]

where \( A(u, e) = \nabla f'(\xi + \gamma(u), u) \).

Because when \( u = 0 \), system (24) will converge to the constant \( u, \dot{e} = f + \gamma(u), \dot{u} \) is asymptotically stable. By proposition 1, there exists Lyapunov function \( V(e) \) such that

\[
\begin{align*}
\dot{V} &= \frac{\partial V(e)}{\partial e} A(u, e) e \leq -W(e), \forall \|e\| \geq \rho (\|\dot{u}\|) \\
\dot{V} &= \frac{\partial V(e)}{\partial e} A(u, e) e \leq -W(e), \forall \|e\| \geq \rho (\|\dot{u}\|)
\end{align*}
\]

where \( \alpha_1, \alpha_2 \in K_\infty \) and \( \rho \in K \), \( W \) is a continuous positive definite function on \( R^n \).

Calculating the derivative of (32) yields

\[
\begin{align*}
\frac{\partial \alpha_1(\|e\|)}{\partial \|e\|} \frac{d \|e\|}{de} \leq \frac{\partial V(e)}{\partial e} \leq \frac{\partial \alpha_2(\|e\|)}{\partial \|e\|} \frac{d \|e\|}{de}
\end{align*}
\]

By the equivalence of the norms, there exists \( k_1, k_2 > 0 \), such that for any norm \( \|e\| \), the following equation is true \( k_2 e^T e \leq \|e\| \leq k_1 e^T e \). Calculating the derivative of above equation yields

\[
k_2 e^T e \leq \frac{d \|e\|}{de} \leq k_1 e^T e
\]

In view of (35), (36) can be written as

\[
\begin{align*}
\frac{\partial \alpha_1(\|e\|)}{\partial \|e\|} k_2 e^T &\leq \frac{\partial V(e)}{\partial e} \leq \frac{\partial \alpha_2(\|e\|)}{\partial \|e\|} k_1 e^T
\end{align*}
\]

Multiplying \( A(u, e) e \) to the both sides of (36) yields

\[
\begin{align*}
\frac{\partial \alpha_1(\|e\|)}{\partial \|e\|} k_2 e^T A(u, e) e &\leq \frac{\partial V(e)}{\partial e} A(u, e) e \leq \frac{\partial \alpha_2(\|e\|)}{\partial \|e\|} k_1 e^T A(u, e) e
\end{align*}
\]

Consider (33), we obtain \( \frac{\partial \alpha_1(\|e\|)}{\partial \|e\|} k_2 e^T A(u, e) e \leq -W(e) \).

Due to \( \alpha_1 \in K, \frac{\partial \alpha_1(\|e\|)}{\partial \|e\|} k_2 > 0 \), then for any \( e \in R^n, e^T A(u, e) e < 0 \), that is \( A(u, e) \) is negative definite.

By Lemma 1, there exist scalar functions \( \lambda_1(u, e) > \lambda_2(u, e) > 0 \) such that \( -\lambda_1(u, e) I < A(u, e) < -\lambda_2(u, e) I \).

Then there exists derivative inclusive

\[
\dot{e} = A_s(u, e) e - b_s(u, \dot{u})
\]

where \( A_s(u, e) \in \{ -\lambda_1(u, e) I, -\lambda_2(u, e) I \} \).

Solving (38) yields

\[
\begin{align*}
e_i(t) &\leq e^-f_1' \lambda_2(u, e) ds e_{i, \dot{e}} - \int_0^{t} e^-f_1' \lambda_2(u, e) ds \min(\gamma'_i, u) ds \\
\dot{e}_i(t) &\leq e^-f_1' \lambda_2(u, e) ds e_{i, \dot{e}} - \int_0^{t} e^-f_1' \lambda_2(u, e) ds \max(\gamma'_i, u) ds
\end{align*}
\]

In view of \( e = x - \gamma(u) \) and define \( \beta_{1i}(x_0, u_0, t) = e^-f_1' \lambda_2(u, e) ds e_{i, \dot{e}} \) and \( \beta_{2i}(x_0, u_0, t) = e^-f_1' \lambda_2(u, e) ds e_{i, \dot{e}} \), we obtain

\[
\begin{align*}
\dot{x}_i(t) &\leq \beta_{2i}(x_0, u_0, t) + \gamma_i(u) - \int_0^{t} e^-f_1' \lambda_2(u, e) ds \min(\gamma'_i, u) ds \\
\dot{x}_i(t) &\geq \beta_{1i}(x_0, u_0, t) + \gamma_i(u) - \int_0^{t} e^-f_1' \lambda_2(u, e) ds \max(\gamma'_i, u) ds
\end{align*}
\]

\( \forall i = 1, ..., n \).

**Remark 8.** Since \( \gamma(u) \) is the equilibrium point of (24), \( x(t) \) can converge towards \( \gamma(u) \) at last when \( \dot{u} \rightarrow 0 \), which is different with theorem 1. The following two cases should be noted. If \( x \in R, (28) \) can be written as

\[
x(t) = \beta(x_0, u_0, t) + \gamma(u) - \int_0^{t} e^-f_1' \lambda_2(u, e) ds \gamma'_i(\dot{u}) ds
\]

If \( \gamma_i(u) = \gamma_j(u), \forall i, j = 1, ..., n, (28) \) can be written as

\[
\begin{align*}
x_i(t) &\leq \beta_{2i}(x_0, u_0, t) + \gamma_i(u) - \int_0^{t} e^-f_1' \lambda_2(u, e) ds \gamma'_i(\dot{u}) ds \\
x_i(t) &\geq \beta_{1i}(x_0, u_0, t) + \gamma_i(u) - \int_0^{t} e^-f_1' \lambda_2(u, e) ds \gamma'_i(\dot{u}) ds
\end{align*}\]

**Example 3.** Some simple examples are \( \dot{x}_1 = -\tan x_1 + u \) and

\[
\dot{x}_1 = -3x_1^3 + 3x_2^2 + u \dot{x}_2 = -2x_3^2 + 2x_1^3
\]

If let system (24) be a scalar system, we have

\[
\dot{x} = f(x, u)
\]

where \( x \in R, f : R \times R \rightarrow R \) is the continuous and local Lipschitz function w.r.t. \( x \) and \( u \).

Then like the corollary 1, we have the similar result.

**Corollary 2.** Suppose the condition of theorem 2 is satisfied, the solution of system (41) can be written as

\[
x(t) = \beta(x_0, u_0, t) + (1 - e^-f_1' u(s) ds) \gamma(u(t))
\]

where \( \xi \in [0, t] a(x) > 0 \), and \( \beta \in KL \).

**Proof.** Based on (39) and following the way of Corollary 1 we can prove the corollary 2.
5 Stability Analysis of Interconnected Systems

In this section, we analyze three typical kinds of interconnected systems via the approach proposed in the section III.

5.1 A New Proof of Small-gain Theorem

Consider the following system

\[ \dot{x} = f_1(x, z) \quad (43) \]

\[ \dot{z} = f_2(z, x) \quad (44) \]

where \( x \in R^{n_1}, z \in R^{n_2} \), \( f_1 \) and \( f_2 \) are continuous functions similar with \( \xi \).

x-subsystem and z-subsystem satisfy the proposition 1 and the following assumption.

Assumption 3. The gain \( \gamma_z \) of x-subsystem is differentiable. \( \gamma_z \) and the gain \( \gamma_z \) of z-subsystem satisfies \( \gamma_z \circ \gamma_z(s) < s, \forall s \in [0, \infty] \).

Then we have the following theorem.

Theorem 3. If the nonlinear system \( (43)(44) \) are ISS and satisfy the assumption 3. Then the interconnected system is asymptotically stable.

Proof. Step 1. Transform the interconnected system into the cascade system.

According to proposition 1, for z-subsystem there exists the Lyapunov function \( V(z) \) such that

\[ \alpha_{1z}(\|z\|) \leq V \leq \alpha_{2z}(\|z\|) \quad (45) \]

\[ \dot{V} \leq -W_3(z), \forall \|z\| \geq \rho_z(\|x\|) \]

where \( \alpha_{1z}, \alpha_{2z} \in K_{\infty}, \rho_z \in K, \) and \( W_3 \) is the continuous positive definite function.

Consider equation \( (45) \), the condition \( \|z\| \geq \rho_z(\|x\|) \) can be strengthened as \( V \geq \alpha_{2z}(\rho_z(\|x\|)) \).

For the x-subsystem, by corollary 1, we have

\[ \|x\| \leq \beta(x_0, z_0, t) + \alpha_{1x}^{-1}(1 - e^{-\int_0^t \alpha_1 z(k(s))ds} \alpha_{2x}(\rho_z(\|z\|))). \quad (46) \]

where \( k(t) > 0 \).

Substituting above equation into \( V \geq \alpha_{2z}(\rho_z(\|x\|)) \)

\[ V \geq \alpha_{2z}(\rho_z(x_0, z_0, t) + \alpha_{1x}^{-1}(1 - e^{-\int_0^t \alpha_1 z(k(s))ds} \alpha_{2x}(\rho_z(\|z\|)))) \]

Consider equation \( (45) \), above equation is strengthened as

\[ \|z\| \geq \alpha_{1z}^{-1}(\alpha_{2z} \circ \alpha_{2x} \circ \beta(x_0, z_0, t) + \alpha_{1x}^{-1}(1 - e^{-\int_0^t \alpha_1 z(k(s))ds} \alpha_{2x}(\rho_z(\|z\|)))) \quad (47) \]

Due to \( \beta(x_0, z_0, t) \in KL_{\infty} \) is equivalent to

\[ \|z\| \geq \alpha_{1z}^{-1} \circ \alpha_{2z} \circ \rho(\alpha_{1x}^{-1}(1 - e^{-\int_0^t \alpha_1 z(k(s))ds} \alpha_{2x}(\rho_z(\|z\|)))). \quad (48) \]

Due to \( (1 - e^{-\int_0^t \alpha_1 z(k(s))ds}) \in (0, 1) \) and by the monotonically increasing property of class \( K \) function, we have

\[ \alpha_{1z}^{-1} \circ \alpha_{2z} \circ \rho(\alpha_{1x}^{-1}(1 - e^{-\int_0^t \alpha_1 z(k(s))ds} \alpha_{2x}(\rho_z(\|z\|)))) \]

That is \( \gamma_z \circ \gamma_z(\|z\||) \geq \alpha_{2z}^{-1} \circ \alpha_{2z} \circ \rho(\alpha_{1x}^{-1}(1 - e^{-\int_0^t \alpha_1 z(k(s))ds} \alpha_{2x}(\rho_z(\|z\|)))). \]

Then the original interconnected system is transformed into the following cascade form

\[ \begin{cases} \dot{V} \leq -W_3(z), \forall \|z\| \geq \gamma_z \circ \gamma_z(\|z\||) \quad (49) \\ \dot{x} = f_1(x, z) \end{cases} \]

Step 2. Analyze the stability of the feedback loop.

By the assumption 2, there exists \( \|z\| \geq \gamma_z \circ \gamma_z(\|z\||) \), such that z-subsystem is stable.

Step 3. Analyze the stability of cascade system.

By the lemma 4.7 in [10] and the ISS property of x-subsystem, the cascade system \( (49) \) is stable, so is the interconnected system \( (43)(44) \).

5.2 A Special Interconnected System

If at least one of subsystems in an interconnected system is not ISS, the small-gain theorem is not used directly, e.g., the following system.

x-subsystem:

\[ \dot{x} = f_1(x, -z) \quad (50) \]

z-subsystem:

\[ \dot{z} = f_2(t, x) \quad (51) \]

where \( x \in R, z \in R, f_1 \) and \( f_2 \) are continuous functions, x-subsystem is ISS and z-subsystem satisfies the following assumption

Assumption 4. \( \dot{z} = f_2(t, -k(t)\rho(z)) \) is stable, where \( z \in D, 1 > k(t) > 0 \) and \( \gamma \in K \) is the differentiable gain function of x-subsystem.

There exists the following theorem.

Theorem 4. If the nonlinear system \( (50) \) is ISS and \( (51) \) satisfies the assumption 4. Then the interconnected system is asymptotically stable in \( D \).

Proof. Step 1. Transform the interconnected system into the cascade system.

By corollary 2, equation \( (50) \) can be written as

\[ x(t) = \beta(x_0, u_0, t) - (1 - e^{-\int_0^t \alpha_2 z(s)ds}) \gamma(z(t)) \quad (52) \]

where \( \gamma \in K \) is the gain of \( (50) \) and \( a(t) > 0 \). Taking the Lyapunov function \( V(z) \) for \( (51) \) yields

\[ \dot{V} = \frac{\partial V}{\partial z} f_2(t, x) \quad (53) \]

Substituting \( (52) \) into \( (53) \) yields the cascade system as follows

\[ \begin{cases} \dot{V} = \frac{\partial V}{\partial z} f_2(t, \beta(x_0, u_0, t) - (1 - e^{-\int_0^t \alpha_2 z(s)ds}) \gamma(z(t))) \\ \dot{x} = f_1(x, -z) \end{cases} \quad (54) \]

Step 2. Analyze the stability of the feedback loop.

By the theorem 3.4 in [10], ignore \( \beta(x_0, u_0, t) \), we have

\[ \dot{V} = \frac{\partial V}{\partial z} f_2(t, - (1 - e^{-\int_0^t \alpha_2 z(s)ds}) \gamma(z(t))) \]

By the assumption \( 3 \) and \( 1 > 1 - e^{-\int_0^t \alpha_2 z(s)ds} > 0, z \)-subsystem is stable.

Step 3. Analyze the stability of cascade system.
By the lemma 4.7 in [10] and the ISS property of x-subsystem, the cascade system (54) is stable, so is the interconnected system (50)(51).

**Example 4.** Other simple examples are in the following.
\[
\begin{cases}
\dot{x}_1 = x_2^3 \\
\dot{x}_2 = -(1 + x_1^2)x_1 - x_2
\end{cases}
\]
and
\[
\begin{cases}
\dot{x}_1 = -x_1 + x_2^2 \\
\dot{x}_2 = -x_1
\end{cases}
\]

5.3 The Consensus Analysis of Multi-agent Systems—Simple Case

Traditionally, Lyapunov function based method is the popular tool to analyze the consensus. Here, we will present another way to use the method proposed in section III to analyze the consensus problem, we consider the following nonlinear multi-agent system.

**x-subsystem:**
\[
\dot{x} = (z - x)^3
\]

**z-subsystem:**
\[
\dot{z} = (x - z)^3
\]

where \(z, x \in R\), suppose \(\gamma_x\) and \(\gamma_z\) are the gains of the x-subsystem and z-subsystem respectively.

Even though every system is ISS, the composition of the gain \(\gamma_x(s) \circ \gamma_z(s) = s\) causes the small-gain theorem is not applied directly in the situation. Following the method proposed in this paper, we have the following steps.

**Step 1:** Transform the interconnected system into the cascade system.

By the theorem 2, the solution of x-subsystem is
\[
x(t) = \beta(x_0, u_0, t) + z - \int_0^t e^{-\int_0^s a(\tau) d\tau} \dot{z} ds
\]
where \(a(t) > 0\). Substituting above equation into z-subsystem yields
\[
\dot{z} = L(z, t)(\beta(x_0, u_0, t) - \int_0^t e^{-\int_0^s a(\tau) d\tau} \dot{z} ds)
\]
where \(L(z, t) = (\beta(x_0, u_0, t) - \int_0^t e^{-\int_0^s a(\tau) d\tau} \dot{z} ds)^2\). Then the original system is transformed into the following cascade system
\[
\dot{z} = L(z, t)\beta(x_0, u_0, t) - L(z, t) \int_0^t e^{-\int_0^s a(\tau) d\tau} \dot{z} ds
\]
\[
\dot{x} = (z - x)^3
\]

**Step 2:** Analyze the stability of the feedback loop.

For z-subsystem, define \(q(t) = e^{\int_0^t a(s) ds} e^{\int_0^t L(z, s) ds}\) Multiplying both sides of (55) by \(q(t)\) and rearrange the equation we obtain
\[
q(t)\dot{z} + q(t)L(z, t) \int_0^t e^{-\int_0^s a(\tau) d\tau} \dot{z} ds
= q(t)L(z, t)\beta(x_0, u_0, t)
\]
\[
= \frac{d}{dt}(e^{\int_0^t L(z, s) ds} \int_0^t e^{-\int_0^s a(\tau) d\tau} \dot{z} ds)
\]
(61)

Using (61) in (60) and integrating both sides of (60), we obtain
\[
e^{\int_0^t L(z, s) ds} \int_0^t e^{-\int_0^s a(\tau) d\tau} \dot{z} ds = \int_0^t q(s)L(z, s)\beta(x_0, u_0, s) ds
\]

Therefore,
\[
e^{\int_0^t L(z, s) ds} \int_0^t e^{-\int_0^s a(\tau) d\tau} \dot{z} ds
= \int_0^t e^{\int_0^s L(z, s) ds} \int_0^t e^{-\int_0^s a(\tau) d\tau} \dot{z} ds
\]

By theorem 2, due to \(\beta(x_0, u_0, t) = e^{-\int_0^t a(s) ds}\), then
\[
e^{\int_0^t L(z, s) ds} \int_0^t e^{-\int_0^s a(\tau) d\tau} \dot{z} ds
= \int_0^t e^{\int_0^s L(z, s) ds} \int_0^t e^{-\int_0^s a(\tau) d\tau} \dot{z} ds
\]

Since \(L(z, s) = \frac{1}{s^2}(\int_0^s -L(z, \tau) d\tau)\), we have
\[
\int_0^t e^{\int_0^s L(z, s) ds} \int_0^t e^{-\int_0^s a(\tau) d\tau} \dot{z} ds
= e^{\int_0^s -L(z, s) ds} |_{s=0}^{s=t} = 1 - e^{\int_0^t -L(z, \tau) d\tau}
\]

Due to \(e^{\int_0^t -L(z, \tau) d\tau} \in KL\), we have
\[
\int_0^t e^{\int_0^s -L(z, \tau) d\tau} \dot{z} ds \rightarrow 1\]
and
\[
e^{-\int_0^t a(s) ds} \int_0^t e^{\int_0^s -L(z, \tau) d\tau} L(z, s) \rightarrow 0, \forall t \rightarrow \infty
\]
Namely,
\[
\int_0^t e^{-\int_0^s a(\tau) d\tau} \dot{z} ds \rightarrow 0, \text{ and } L \rightarrow 0, \forall t \rightarrow \infty
\]
Thus \(\dot{z} \rightarrow 0, \forall t \rightarrow \infty\). Because \(L(z, t)\beta(x_0, u_0, t) > 0\), \(\int_0^t L(z, t)\beta(x_0, u_0, t) ds\) is integrable, so is \(\int_0^t L(z, t)e^{-\int_0^s a(s) ds} \int_0^s e^{\int_0^\tau L(z, s) ds} ds L(z, s) ds\), therefore
\[
z(t) = z(0) + \int_0^t L(z, t)\beta(x_0, u_0, t) ds
\]
\[
- L(z, t)e^{-\int_0^s a(s) ds} \int_0^s e^{\int_0^\tau L(z, s) ds} ds L(z, s) ds
\]
(62)

exists.

**Step 3:** Analyze the stability of cascade system.

By the theorem 2, for x-subsystem, when \(z(t) \rightarrow c\) where \(c\) is a constant, \(x(t) \rightarrow z(t), \forall t \rightarrow \infty\). Therefore, the system is consensus.

6 CONCLUSION

In this paper, based on the structural perspective, we propose a new framework for the stability analysis of interconnected systems. This method bases on a new formulation of ISS concept which weakens the conservative of the original form and can analyze some problems which can not be done by the small-gain theorem. As its applications, we also investigates the three kinds of typical interconnected systems respectively. It should be mentioned that we use a unifying approach
to treat three different problems and provide a deep insight for the common essence of them. Further more this method can be used to design the lower-order controller for the high-order minimum-phase systems e.g. [13][14][15]. In the future, we will combine this method and the idea in [12] to analyze the consensus of MAS with nonlinear protocol on any topology and with the time-delay in the communication.

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