Discontinuous transition in an equilibrium percolation model with suppression

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(Dated: January 18, 2018)

Discontinuous transition is observed in the equilibrium cluster properties of a percolation model with suppressed cluster growth as the growth parameter $g_0$ is tuned to the critical threshold at sufficiently low initial seed concentration $\rho$ in contrast to the previously reported results on non-equilibrium growth models. In the present model, the growth process follows all the criteria of the original percolation model except continuously updated occupation probability of the lattice sites that suppresses the growth of a cluster according to its size. As $\rho$ varied from higher values to smaller values, a line of continuous transition points encounters a coexistence region of spanning and non-spanning large clusters. At sufficiently small values of $\rho$ ($\leq 0.05$), the growth parameter $g_0$ exceeds the usual percolation threshold and generates compact spanning clusters leading to discontinuous transitions.

PACS numbers: 64.60.ah 05.70.Fh 64.60.De

Recently, percolation transition (PT) has been reported as a first order discontinuous transition in a model of explosive percolation (EP) by Achlioptas et al [1]. However, percolation is well known as a model of second order continuous phase transition (CPT) and widely applied in a variety of problems ranging from sol-gel to metal insulator transition [2,3]. Instead of the original equilibrium percolation model, a series of non-equilibrium growth models [4] then proposed to demonstrate first order PT. In these models, imposing the product (or sum) rule to occupy a bond in growing a cluster, a discontinuous jump in the size of the largest cluster at a delayed percolation threshold is characterized as discontinuous phase transition (DPT). However, soon a controversy that the EP is a DPT or not erupts on the basis of slow convergence of asymptotic cluster properties in the $L \to \infty$ limit [4,8]. For example, some of the Euclidean lattice models [6,11] of EP were found inconclusive in their nature of transition [12,13]. In the spanning cluster avoiding (SCA) model of EP, it was claimed that there exists an upper critical dimension below which the transitions will be discontinuous [14] though in this model the transition occurs at unit probability, a trivial percolation threshold. A few growth models [15,16] in Euclidean space, however, are found to display first order DPT. It seems beside CPT and true first order DPT there exists a mixed DPT in which characteristics of both first order and second order transitions appear [20,21] and the system possess unusual finite size scaling (FSS) [22]. In most of the cases, except the jump in the order parameter the other aspects of first order transition such as phase co-existence, nucleation, etc. are ignored [12,23]. More importantly, not only the understanding of the origin of DPT remains incomplete but also it is not yet demonstrated in the context of equilibrium percolation model.

In this letter, we propose a two parameter equilibrium percolation model keeping nucleation and growth as the main ingredient. The parameters are the initial seed concentration $\rho$ and a growth parameter $g_0$. The model plays CPT, mixed DPT and finally true first order DPT at suitable range of parameter values. The DPT in this model is not only characterized by the jump in the order parameter but also supported by the presence of phase co-existence. The model not only distinguishes clearly the features of different PTs but also captures most of the essential features of several different EP models.

The model is developed on a 2-dimensional (2d) square lattice of size $L \times L$ occupying the lattice sites randomly with an initial seed concentration $\rho$. Clusters of occupied sites, connected by nearest neighbor (NN) bonds, are formed. The initial cluster size distribution is determined identifying the clusters by Hoshen-Kopelman algorithm [24]. The clusters are then arranged in an ascending order according to their sizes $s$. These finite clusters are then grown sequentially starting from the smallest cluster with a size dependent probability. At a Monte Carlo (MC) time step $t$, the growth probability $g_s(t)$ of a cluster of size $s$ is given by

$$g_s(t) = g_0 \exp[-\{s(t) - 1\}/s_{\text{large}}(t)]$$ (1)

where the growth parameter $g_0$ is a constant between $[0,1]$ and $s_{\text{large}}(t)$ is the size of the largest cluster present at that time. At any time $t$, the value of $g_s(t)$ is the smallest ($g_0/e$) for the largest cluster and it is largest ($g_0$) for the smallest cluster ($s=1$). Accordingly, the model is called suppressed cluster growth percolation (SCGP) which is quite different from the controlled largest cluster growth model of EP [10]. In a single MC step, only a single layer of empty NN perimeter (both internal and external) sites of a cluster are occupied with its growth probability $g_s(t)$. Once a site is rejected with probability ($1 - g_s(t)$), the site remains unoccupied throughout the growth process as in the original percolation model (OPM) and which is not the case in most of the EP models. An empty lattice site may be a common NN site of more than one cluster. Since we occupy the empty sites of the smallest cluster first, the status of occupation or rejection of such sites cannot be altered in future at the
PT in SCGP is characterized by the properties of the final equilibrium spanning/large clusters. The order parameter, the probability to find a lattice site in the spanning cluster, is defined as $P_\infty = S_{max}/L^2$, where $S_{max}$ is the size of the spanning cluster. The FSS form of $P_\infty$ is then expected to be

$$P_\infty = L^{-\beta/\nu} \tilde{P}_\infty[(g_0 - g_{0c})L^{1/\nu}]$$

where $g_{0c}$ is the critical value of growth parameter at which the PT occurs. The average value of $S_{max}$ at the threshold scales as $\langle S_{max}\rangle \approx L^{d_f}$, where $d_f = d - \beta/\nu$ is the fractal dimension of the spanning cluster. Following the formalism of analyzing thermal critical phenomena \textsuperscript{26,27}, the distribution of $P_\infty$ is taken as

$$P(P_\infty) = L^{\beta/\nu} \tilde{P}[P_\infty L^{\beta/\nu}]$$

where $\tilde{P}$ is a universal scaling function. Such a distribution function of $P_\infty$ is also used in the context of PT recently \textsuperscript{12}. With such scaling form of $P_\infty$ distribution, one could easily show that $\langle S_{max}\rangle$ as well as $\langle P_\infty\rangle'$ scale as $\sim L^{-2\beta/\nu}$. The susceptibility is defined in terms of the fluctuation in $P_\infty$ as

$$\chi_\infty = [\langle S_{max}^2 \rangle - \langle S_{max}\rangle^2]/L^2.$$  (4)

Following the hyper-scaling relation $d\nu = \gamma + 2\beta$, the FSS form of $\chi_\infty$ is obtained as

$$\chi_\infty = L^{\gamma/\nu} \tilde{\chi}[(g_0 - g_{0c})L^{1/\nu}]$$

where $\tilde{\chi}$ is the scaling function. Studying FSS of $P_\infty$ and its fluctuation $\chi_\infty$, the critical thresholds $g_{0c}(L)$ are identified and the values of $\beta/\nu$, $\gamma/\nu$ are estimated. The order of transition is verified by estimating higher order Binder cumulant (BC) \textsuperscript{28,29}. Below we present data for two extreme values of $\rho$, 0.50 and 0.02, and we comment on data for the intermediate range of $\rho$.

In Fig. 2 $\chi_\infty/L^2$ is plotted against $g_0$ for two different values of $\rho$: 0.50 (a) and 0.02 (b). As expected, $\chi_\infty$ is found to have a maximum for a particular value of $g_0$ for a given $L$. The positions of these maxima $g_{0c}(L)$ correspond to the percolation thresholds in this model and are marked by crosses on the $g_0$ axis. For $\rho = 0.02$ and $L = 2048$, it is found that $g_{0c}(L) = 0.6536(2)$ which is higher than $p_c$ of OPM, the critical occupation probability for growing the percolation clusters from a single seed following Leath algorithm \textsuperscript{30}, as it happens in most of the EP growth models \textsuperscript{1,10,21}. Note that, the threshold here is a non-trivial finite value in contrast to the trivial threshold value in SCA \textsuperscript{14}. As in the case of explosive electric breakdown model \textsuperscript{31}, the values of $g_{0c}(L)$ are found to decrease with increasing $L$ for $\rho \leq 0.4$. The maximum values of the susceptibility $\chi_{max}$ are expected to follow a scaling relation $\chi_{max} \sim L^{\gamma/\nu}$. Values of $\chi_{max}$ for different $L$ at their respective $g_{0c}(L)$ are plotted against $L$ in the insets of Fig. 2(a) and (b).
for $\rho = 0.50$ and $0.02$ respectively. By linear least square fit to the data points, the values of $\gamma/\nu$ are extracted. For $\rho = 0.50$, it is found that $\gamma/\nu = 1.80 \pm 0.01$, that of the OPM ($\approx 1.792$) within error bar. The value of $\gamma/\nu$ remains unaltered within $\pm 0.02$ for $\rho \geq 0.45$. Hence, the transitions for $\rho \geq 0.45$ belong to the same universality class of OPM. Whereas for $\rho = 0.02$, $\gamma/\nu$ is found $2.00 \pm 0.01$ as it occurs in a first order DPT. The value of $\gamma/\nu \approx 2$ is also found to occur for $\rho \leq 0.05$ within error bar. In the intermediate region $0.05 < \rho < 0.45$, the value of $\gamma/\nu$ is found to decrease continuously from 2.0 to 1.80 as $\rho$ changes from 0.05 to 0.45. Such continuously varying exponents are also observed in a hybrid PT model \[22, 32\]. To confirm the nature of PT in SCGP, the 4th order BC

$$B_{\rho,L}(g_0) = \frac{(3/2)[1 - \langle S_{\max}^2 \rangle/3\langle S_{\max}^2 \rangle^2]}{\rho}$$

(6)

is studied. In Fig. 3, $B_{\rho,L}(g_0)$ is plotted against $g_0$ for different $L$ for $\rho = 0.50$ (a) and $0.02$ (b). For $\rho = 0.50$, the plots of $B_{\rho,L}(g_0)$ for different $L$ cross at a point corresponding to the critical percolation threshold of SCGP, $g_{\rho,c}(\infty) \approx 0.1895$ as it occurs for a CPT. Such crossing of BCs are also observed for $\rho \geq 0.45$. For $\rho = 0.02$, however, no such crossing of BCs for different $L$ is found to occur as expected in first order transitions. Non-crossing of BCs are also observed for $\rho \leq 0.05$. In the intermediate region $0.05 < \rho < 0.45$, BCs cross over a range of $g_0$ values indicating no precise crossover value. The FSS form of BC, $B_{\rho,L}(g_0) = \tilde{B}[(g_0 - g_{\rho,c}(L))L^{1/\nu(\rho)}]$, where $\tilde{B}$ is a scaling function is verified in the insets of Fig. 3, plotting BC against $(g_0 - g_{\rho,c}(L))L^{1/\nu(\rho)}$.

To realize the presence of co-existing phases in SCGP, an ensemble of large clusters, the spanning or the largest if no spanning cluster appears, at $g_{\rho,c}(L)$ are generated. The probability to find a lattice site in a largest cluster of size $S_{\text{large}}$ is $P_{\text{large}} = S_{\text{large}}/L^2$. The distribution of $P_{\text{large}}$ is expected to be

$$P_{\ell}(P_{\text{large}}) \sim L^{\beta/\nu} \tilde{P}_l[P_{\text{large}}L^{\beta/\nu}]$$

(7)
where $\tilde{P}_c$ is a scaling function. In Fig. 5(a), the distribution $P_c(P_{\text{large}})$, interpolated through 1000 equally spaced bins of data points, are plotted against $P_{\text{large}}$ for a wide range of $\rho$. Whenever there is a crossing point in the BCs, the distributions are obtained at $g_{0c}$ corresponding to that crossing otherwise they are obtained at $g_{0c}(L)$. For $\rho \geq 0.45$, only the distributions are found single-humped but also the scaled distributions $P_c L^{-\beta/\nu}$ collapse onto a single curve when plotted against $P_{\text{large}}, L^{\beta/\nu}$. The transitions in this region are thus CPT which follow usual percolation scaling. On the other hand, for $\rho \leq 0.05$, the distributions are found double-humped bimodal distributions as it appears in thermal phase transitions [34] and also reported in some of the EP models [12, 35–37]. The appearance of bimodal distribution indicates the coexistence of the spanning cluster with the large (non-spanning) clusters. No suitable scaling exponent is found to collapse either of the humps of these bimodal distributions. The heights of the humps are found increasing with $L$ for a given $\rho$. Though a leftward shift of the distribution is found to occur with $L$, the hump to hump separation $\Delta P_{\text{large}}$ is found either constant or increasing with $L$ as shown in the inset of Fig. 5(a) for different values of $\rho \leq 0.05$, as in some of the EP models [31, 32]. It is important to note that $\Delta P_{\text{large}}$ is also increasing with decreasing $\rho$ for a large $L$ making the jump more drastic in the dilute limit of $\rho$. Not only the compact spanning clusters appear in this region due to nucleation of finite compact clusters but also the number of clusters merged to form the spanning cluster become non-extensive with $L$. All these features provide a strong evidence of true first order DPT. For $0.05 < \rho < 0.45$, $P_c(P_{\text{large}})$ becomes broader as well as double humps start developing and becomes prominent as $\rho$ decreases. Thus, in this region, the model exhibits non-universal critical behavior accompanied by unusual FSS beside a finite jump in the order parameter like mixed DPT in some of the EP models [32]. However, $\Delta P_{\text{large}}$ is found to decrease with $L$ in the intermediate range of $\rho$. Thus the apparent DPT in this region may disappear in $L \to \infty$ limit and one may find the line of CPT is extended further down to lower values of $\rho$. In Fig. 5(b), a phase diagram for SCGP is presented in the $P_c - \rho$ parameter plane for $L = 2048$ where $P_c$ is the critical area fraction of spanning (or largest) cluster, the mean value of $P_{\text{large}}$ distribution in the case of a single hump otherwise the hump positions. Though the boundaries of the regions are not sharp, it can be seen that a line of second order transition ($\rho \geq 0.45$) bifurcates at a tricritical point into two lines of first order transitions ($\rho < 0.45$) enclosing a coexistence region which ultimately represent true first order DPT for $\rho \leq 0.05$. The existence of a tricritical point is also observed in a growth model with modified product rule incorporating a dilution parameter $20$.

In conclusion, an equilibrium SCGP with two parameters is developed which clearly distinguishes CPT, DPT and mixed DPT in one of its phase plane. The usual equilibrium spanning cluster approach demonstrates CPT for $\rho \geq 0.45$ and strong first order DPT for $\rho \leq 0.05$. The CPTs are found to belong to the same universality class of OPM. The DPT, however, is characterized by a discontinuous jump in the order parameter, coexistence of spanning and non-spanning large clusters and appearance of compact spanning cluster. A compact spanning cluster in this region is an outcome of merging of the compact finite clusters that were grown with high $g_{0c}$. The region of coexistence is found to be confined within a double-humped bimodal distribution of the order parameter. In the intermediate range of $\rho$, the nature of PT still remains inconclusive as characteristic features of both CPT and DPT appear concurrently and can only be resolved in true thermodynamic limit.

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