LIMITING DYNAMICAL BEHAVIOR OF RANDOM FRACTIONAL FITZHUGH-NAGUMO SYSTEMS DRIVEN BY A WONG-ZAKAI APPROXIMATION PROCESS

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ABSTRACT. In this paper, we study the long term behavior of non-autonomous fractional FitzHugh-Nagumo systems with random forcing given by an approximation of white noise, called Wong-Zakai approximation. We first prove the existence and uniqueness of tempered pullback attractors for the Wong-Zakai approximation fractional FitzHugh-Nagumo systems, and then establish the upper semicontinuity of attractors of system driven by a linear multiplicative Wong-Zakai approximations as random forcing approaches white noise in some sense.

1. Introduction. Let $w$ be a two-sided real-valued Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. Given a nonzero number $\delta$, define a stochastic process $\eta_\delta$ on $(\Omega, \mathcal{F}, P)$ by, for every $t \in \mathbb{R}$,

$$
\eta_\delta(t) = \frac{w(t + \delta) - w(t)}{\delta}.
$$

As $\delta \to 0$, the process $\eta_\delta$ can be considered as an approximation of the white noise $\frac{dw}{dt}$. This observation motivates us to study the dynamics of stochastic fractional FitzHugh-Nagumo systems driven by white noise via the corresponding random systems driven by the approximate process $\eta_\delta$. More precisely, let $s \in (0, 1)$ and $\mathcal{O}$

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be a smooth bounded domain of $\mathbb{R}^n$. Given $\tau \in \mathbb{R}$, we will study the dynamics of the following system defined for $t > \tau$,

$$
\begin{aligned}
\frac{\partial u}{\partial t} + (-\Delta)^s u + \lambda u &= -\alpha v + f(t, x, u) + g_1(t, x, u) \eta_\delta(t), \quad x \in \mathcal{O}, \\
\frac{\partial v}{\partial t} &= -\gamma v + \beta u + g_2(t, x, x) + \sigma v \eta_\delta(t), \quad x \in \mathcal{O}, \\
u(t, x) = v(t, x) &= 0, \quad x \in \mathbb{R}^n \setminus \mathcal{O},
\end{aligned}
$$

(1.1)

where $\alpha, \beta, \gamma, \lambda$ and $\sigma$ are positive constants, $f$ and $h$ are nonlinear functions satisfying certain conditions, and $g_1, g_2 \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathcal{O}))$. Under certain circumstances, we will compare the long term behavior of solutions of (1.1) with the stochastic FitzHugh-Nagumo system driven by white noise for $t > \tau$,

$$
\begin{aligned}
\frac{\partial u}{\partial t} + (-\Delta)^s u + \lambda u &= -\alpha v + f(t, x, u) + g_1(t, x, u) \circ d\omega(t), \quad x \in \mathcal{O}, \\
\frac{\partial v}{\partial t} &= -\gamma v + \beta u + g_2(t, x, x) + \sigma v \circ d\omega(t), \quad x \in \mathcal{O}, \\
u(t, x) = v(t, x) &= 0, \quad x \in \mathbb{R}^n \setminus \mathcal{O},
\end{aligned}
$$

(1.2)

where the symbol $\circ$ indicates that the equation is understood in the sense of Stratonovich integration.

The operator $(-\Delta)^s$ is the so-called fractional Laplacian for $s \in (0, 1)$. Fractional partial differential equations arise from many applications in physics, biology and probability theory, see [17, 18, 19, 21] and the references therein. The solutions of deterministic fractional equations have been extensively studied by many experts, see, [10, 17, 18, 19, 21, 25]. Recently, the random attractors of fractional stochastic equations have been studied by several authors, see, [14, 26, 27, 28, 37] and the references therein. If $s = 1$, then the fractional operator $(-\Delta)^s$ reduces to the standard Laplacian operator $-\Delta$. It is worth mentioning that there are numerous works in the literature on the existence of attractors for standard stochastic equations driven by a white noise, see, [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 20, 22, 23, 24, 33, 34, 35, 36, 38, 41]. The dynamical behavior of standard stochastic equations driven by colored noise or Wong-Zakai approximation process have been examined in [15, 16, 29, 30, 31, 32, 40]. Recently, the random dynamics is investigated for fractional nonclassical diffusion equations driven by colored noise [39]. We here want to investigate the Wong-Zakai approximations given by a stationary process via the Wiener shift and their associated long term behavior of the fractional stochastic equation driven by a linear multiplicative white noise for $s \in (0, 1)$.

As demonstrated in later sections, the random system (1.1) possesses some striking advantages over the stochastic system (1.2). For instance, we will show, in this paper, the random equation (1.1) generates a continuous cocycle for a wide class of nonlinearity $h$ and it has a tempered random attractor in $L^2(\mathcal{O}) \times L^2(\mathcal{O})$. In contrast, for the stochastic equation (1.2), one can only prove it defines a random dynamical system when $h$ is linear in its third argument. Of course, the existence of random attractors for (1.2) can only be established for such a particular $h$. Indeed, the existence of random dynamical systems is unknown in general for stochastic PDEs (see, [12]). Despite the significant difference of equations (1.1) and (1.2), we are still able to find close relations between their solutions under certain conditions. Actually, for linear multiplicative noise, we will prove the solutions of (1.1) converge to that of (1.2) in $L^2(\mathcal{O}) \times L^2(\mathcal{O})$ when $\delta \to 0$. Based on this result, we will further
prove the convergence of random attractors of (1.1) as \( \delta \to 0 \), and show their limit is the attractor of (1.2) in an appropriate sense.

Throughout this paper, we use \( \| \cdot \| \) and \( (\cdot, \cdot) \) to denote the norm and the inner product of \( L^2(O) \) and use \( \| \cdot \|_{L^p} \) to denote the norm in \( L^p(O) \). The norm in Banach space \( X \) is represented by \( \| \cdot \|_X \). The letters \( c \) and \( c_i \) \((i = 1, 2, \ldots)\) are generic positive constants which may change their values from line to line.

2. Random FitzHugh-Nagumo systems driven by approximate white noise. In this section, we first define a continuous cocycle for random fractional FitzHugh-Nagumo systems driven by approximate white noise, and then prove the existence of pullback random attractors.

2.1. Continuous cocycles. Let \( \tau, \delta \in \mathbb{R} \) with \( \delta \neq 0 \) and \( s \in (0, 1) \). Consider the following non-autonomous random fractional FitzHugh-Nagumo system for \( t > \tau \)

\[
\begin{aligned}
\frac{\partial u}{\partial t} + (\Delta)^s u + \lambda u &= -\alpha v + f(t, x, u) + g_1(t, x) + h(t, x, u)\eta_3(t), \quad x \in O, \\
\frac{\partial v}{\partial t} &= -\gamma v + \beta u + g_2(t, x) + \sigma \nu \eta_3(t), \quad x \in O, \\
u(t, x) &= 0, \quad x \in \mathbb{R}^n \setminus O, \\
u(\tau, x) &= u_\tau(x), \quad v(\tau, x) = v_\tau(x), \quad x \in O,
\end{aligned}
\]

where \( g_1 \in L^2_{\text{loc}}(\mathbb{R}, L^2(O)) \) and \( g_2 \in L^1_{\text{loc}}(\mathbb{R}, H^s(O)) \). The nonlinearity function \( f : \mathbb{R} \times O \times \mathbb{R} \to \mathbb{R} \) is continuous such that for all \( t, s \in \mathbb{R} \) and \( x, y \in O \),

\[
f(t, x, s) \leq -\alpha_1 |s|^p + \varphi_1(t, x), \quad (2.2)
\]

\[
|f(t, x, s)| \leq \alpha_2 |s|^{p-1} + \varphi_2(t, x), \quad (2.3)
\]

\[
\frac{\partial}{\partial s} f(t, x, s) \leq -\alpha_3 |s|^{p-2} + \varphi_3(t, x), \quad (2.4)
\]

where \( p > 2 \), \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are positive constants, \( \varphi_1 \in L^1_{\text{loc}}(\mathbb{R}, L^1(O)) \), \( \varphi_2 \in L^p_{\text{loc}}(\mathbb{R}, L^p(O)) \) with \( \frac{1}{p_1} + \frac{1}{p} = 1 \) and \( \varphi_3 \in L^\infty_{\text{loc}}(\mathbb{R}, L^\infty(O)) \). Let \( h : \mathbb{R} \times O \times \mathbb{R} \to \mathbb{R} \) be continuous such that for all \( t, s \in \mathbb{R} \) and \( x \in O \),

\[
|h(t, x, s)| \leq \alpha_4 |s|^{q-1} + \psi_1(t, x), \quad (2.5)
\]

\[
\frac{\partial}{\partial s} h(t, x, s) \leq \alpha_5 |s|^{q-2} + \psi_2(t, x), \quad (2.6)
\]

where \( 2 \leq q < p \), \( \alpha_4 \) and \( \alpha_5 \) are positive constants, and \( \psi_1, \psi_2 \in L^\infty_{\text{loc}}(\mathbb{R}, L^\infty(O)) \).

Note that given \( t, s \in \mathbb{R} \), \( f(t, \cdot, s) : O \to \mathbb{R} \) is only defined on \( O \). But we can extend \( f(t, \cdot, s) \) to the entire space \( \mathbb{R}^n \) by setting \( f(t, x, s) = 0 \) for all \( x \in \mathbb{R}^n \setminus O \). Such an extension is always used in this paper. In other words, any function defined on \( O \) will be identified with its trivial extension to \( \mathbb{R}^n \).

Next, we review the concept of fractional derivatives. For \( 0 < s < 1 \), the fractional Laplace operator \((-\Delta)^s\) is given by, for \( x \in \mathbb{R}^n \),

\[
(-\Delta)^s u(x) = C(n, s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy
\]

\[
= -\frac{1}{2} C(n, s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2s}} dy.
\]
provided the integral exists, where P.V. means the principal value of the integral, and \( C(n, s) \) is a positive constant given by

\[
C(n, s) = \frac{s^{4s} \Gamma \left( \frac{n+2s}{2} \right)}{\pi^s \Gamma(1-s)}.
\] (2.7)

For every \( s \in (0, 1) \), denote by

\[
H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx\, dy < \infty \right\}.
\]

Then \( H^s(\mathbb{R}^n) \) is a Hilbert space with inner product given by

\[
(u, v)_{H^s(\mathbb{R}^n)} = (u, v)_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx\, dy,
\]

for \( u, v \in H^s(\mathbb{R}^n) \). For convenience, we will also use the notation:

\[
(u, v)_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx\, dy, \quad u, v \in H^s(\mathbb{R}^n),
\]

and

\[
\|u\|^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx\, dy, \quad u \in H^s(\mathbb{R}^n).
\]

Then we have

\[
(u, v)_{H^s(\mathbb{R}^n)} = (u, v)_{L^2(\mathbb{R}^n)} + (u, v)_{H^s(\mathbb{R}^n)}, \quad u, v \in H^s(\mathbb{R}^n),
\]

and

\[
\|u\|^2_{H^s(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{H^s(\mathbb{R}^n)}, \quad u \in H^s(\mathbb{R}^n).
\]

By [10], we find that

\[
\|(-\Delta)^{\frac{s}{2}} u\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{2} C(n, s) \|u\|^2_{H^s(\mathbb{R}^n)}, \quad \text{for all } u \in H^s(\mathbb{R}^n),
\]

where \( C(n, s) \) is the same as in (2.7), and hence

\[
H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^n) \right\}.
\]

Denote by

\[
V = \left\{ u \in H^s(\mathbb{R}^n) : u = 0 \text{ a.e. on } \mathbb{R}^n \setminus \mathcal{O} \right\}
\]

and

\[
H = \left\{ u \in L^2(\mathbb{R}^n) : u = 0 \text{ a.e. on } \mathbb{R}^n \setminus \mathcal{O} \right\}.
\]

Note that a function \( u \in H^s(\mathcal{O}) \) (or \( L^2(\mathcal{O}) \)) can be considered as an element of \( V \) (or \( H \)) by setting \( u = 0 \) a.e. on \( \mathbb{R}^n \setminus \mathcal{O} \).

Denote by \( \Omega = \left\{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \right\} \). Let \( \mathcal{F} \) be the Borel \( \sigma \)-algebra induced by the compact-open topology of \( \Omega \), and \( P \) the corresponding Wiener measure on \( (\Omega, \mathcal{F}) \). Let \( \{\theta_t\}_{t \in \mathbb{R}} \) be the group on \( (\Omega, \mathcal{F}, P) \) given by

\[
\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}.
\]

Then \( (\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}}) \) is a metric dynamical system and there exists a \( \{\theta_t\}_{t \in \mathbb{R}} \)-invariant subset \( \bar{\Omega} \subseteq \Omega \) of of full measure such that for each \( \omega \in \bar{\Omega} \),

\[
\frac{\omega(t)}{t} \to 0 \quad \text{as} \quad t \to \pm \infty.
\] (2.8)
For the sake of convenience, from now on, we will abuse the notation slightly and write the space $\Omega'$ as $\Omega$. Given $\delta \neq 0$, define a random variable $G_\delta$ by
\[ G_\delta(\omega) = \frac{\omega(\delta)}{\delta}, \quad \text{for all } \omega \in \Omega. \tag{2.9} \]

From (2.9) we find
\[ G_\delta(\theta_1 \omega) = \frac{\omega(t + \delta) - \omega(t)}{\delta}, \quad \int_0^t G_\delta(\theta_1 \omega)ds = \int_t^{t+\delta} \frac{\omega(s)}{\delta}ds + \int_0^\delta \frac{\omega(s)}{\delta}ds. \tag{2.10} \]

By (2.10) and the continuity of $\omega$ we get for all $t \in \mathbb{R}$,
\[ \lim_{\delta \to 0} \int_0^t G_\delta(\theta_1 \omega)ds = \omega(t). \tag{2.11} \]

Note that this convergence is uniform on a finite interval $[\tau, \tau + T]$ as stated below.

**Lemma 2.1** ([29]). Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$. Then for every $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\varepsilon, \tau, \omega, T) > 0$ such that for all $0 < |\delta| < \delta_0$ and $t \in [\tau, \tau + T]$,
\[ \left| \int_0^t G_\delta(\theta_1 \omega)ds - \omega(t) \right| < \varepsilon. \]

By Lemma 2.1 we find that there exist $\delta = \delta(\tau, \omega, T) > 0$ and $c = c(\tau, \omega, T) > 0$ such that for all $0 < |\delta| < \delta$ and $t \in [\tau, \tau + T]$,
\[ \left| \int_0^t G_\delta(\theta_1 \omega)ds \right| \leq \left| \int_0^t G_\delta(\theta_1 \omega)ds - \omega(t) \right| + |\omega(t)| \leq c. \tag{2.12} \]

By (2.9) we may rewrite system (2.1) as: for $t > \tau$
\[
\begin{aligned}
&\frac{\partial u}{\partial t} + (-\Delta)^s u + \lambda u = -\alpha v + f(t, x, u) + g_1(t, x) + h(t, x, u)G_\delta(\theta_1 \omega), \quad x \in \mathcal{O}, \\
&\frac{\partial v}{\partial t} = -\gamma v + \beta u + g_2(t, x) + \sigma vG_\delta(\theta_1 \omega), \quad x \in \mathcal{O}, \\
&u(t, x) = 0, \quad x \in \mathbb{R}^n \setminus \mathcal{O}, \\
&u(\tau, x) = u_\tau(x), \quad v(\tau, x) = v_\tau(x), \quad x \in \mathcal{O}.
\end{aligned}
\tag{2.13}
\]

We now consider the existence and uniqueness of solutions of equation (2.13). Let $E = L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ be a Hilbert space with inner product
\[ \langle \varphi_1, \varphi_2 \rangle_E = \beta(u_1, u_2) + \alpha(v_1, v_2), \quad \forall \varphi_i = (u_i, v_i) \in E, \quad i = 1, 2. \]

Under assumptions (2.2)-(2.6), we can show that for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\varphi_\tau = (u_\tau, v_\tau) \in E$, problem (2.13) has a unique solution
\[ \varphi(\cdot, \tau, \omega, \varphi_\tau) = (u(\cdot, \tau, \omega, u_\tau), v(\cdot, \tau, \omega, v_\tau)) \in C([\tau, +\infty), E). \]

Moreover, for any $T > 0$, $u \in L^p(\tau, \tau + T; L^p(\mathbb{R}^n)) \cap L^2(\tau, \tau + T; V)$. In addition, the solution $\varphi(t, \tau, \omega, \varphi_\tau)$ is continuous in $E$ with respect to $\varphi_\tau$ and is $(\mathcal{F}, \mathcal{B}(E))$-measurable with respect to $\omega \in \Omega$.

We now define a mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E \to E$ such that for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\varphi_\tau = (u_\tau, v_\tau) \in E$,
\[ \Phi(t, \tau, \omega, \varphi_\tau) = \varphi(t + \tau, \tau, \theta_{-\tau} \omega, \varphi_\tau) \]
\[ = (u(t + \tau, \tau, \theta_{-\tau} \omega, u_\tau), v(t + \tau, \tau, \theta_{-\tau} \omega, v_\tau)), \tag{2.14} \]
where $\varphi = (u, v)$ is a solution of problem (2.13). Then $\Phi$ is a continuous cocycle on $E$ over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$. 


We will study tempered pullback attractors for $\Phi$ in $E$. To this end, we first need to specify a collection of families of subsets of $E$. Given a bounded nonempty subset $D$ of $E$, the Hausdorff semi-distance between $D$ and the origin in $E$ is denoted by $\|D\|_E = \sup_{\psi \in D} \|\psi\|_E$. Let $\mathcal{D}$ be the collection of all families of tempered nonempty subsets of $E$, i.e.,

$$\mathcal{D} = \{ D = D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega : D \text{ is tempered} \}.$$ 

It is clear that $\mathcal{D}$ is neighborhood closed.

Let $\lambda_1 = \min \{ \lambda, \gamma \}$ and $g = (g_1, g_2)$. The following condition will be needed for $g_1, g_2$ and $\varphi_1$ when deriving uniform estimates of solutions:

$$\int_{-\infty}^{\tau} e^{\frac{\lambda_1}{2} s} (\| g(s, \cdot) \|_{L^2}^2 + \| \varphi_1(s, \cdot) \|_{L^1}^2) ds < \infty, \quad \forall \tau \in \mathbb{R}, \quad (2.15)$$

and

$$\int_{-\infty}^{0} e^{\frac{\lambda_1}{2} s} \| g_2(s + \tau, \cdot) \|_{H^1(O)}^2 ds < + \infty, \quad \text{for every } \tau \in \mathbb{R}. \quad (2.16)$$

When constructing tempered pullback attractors, we will assume that

$$\lim_{t \to -\infty} e^{ct} \int_{-\infty}^{0} e^{\frac{\lambda_1}{2} s} (\| g(s + t, \cdot) \|_{L^2}^2 + \| \varphi_1(s + t, \cdot) \|_{L^1}^2) ds = 0, \quad \forall c > 0. \quad (2.17)$$

2.2. Pullback random attractors. This subsection is devoted to the proof of existence and uniqueness of pullback random attractors for the random FitzHugh-Nagumo system (2.13) in $E$. We first derive uniform estimates of solutions of (2.13) and give the asymptotic compactness of the solutions. We finally prove the existence and uniqueness of tempered random attractors for the system (2.13).

**Lemma 2.2.** Suppose (2.2)–(2.44) and (2.15) hold. Then for every $\tau \in \mathbb{R}, \omega \in \Omega,$ and $D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D},$ there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$,

$$\| \varphi(\tau, t, \theta_{-\tau} \omega, \varphi_{t-\tau}) \|_{L^2}^2$$

$$+ \beta \int_{\tau-t}^{\tau} e^{\int_s^\tau (\lambda_1 - 2s \sigma \delta_r(\theta_{t-\tau} \omega)) dr} \| u(s, \tau - t, \theta_{-\tau} \omega, u_{t-\tau}) \|_{H^1(\mathbb{R}^n)}^2 ds$$

$$+ \alpha_1 \beta \int_{\tau-t}^{\tau} e^{\int_s^\tau (\lambda_1 - 2s \sigma \delta_r(\theta_{t-\tau} \omega)) dr} \| u(s, \tau - t, \theta_{-\tau} \omega, u_{t-\tau}) \|_{L^p}^p ds$$

$$+ \frac{\lambda_1}{2} \int_{\tau-t}^{\tau} e^{\int_s^\tau (\lambda_1 - 2s \sigma \delta_r(\theta_{t-\tau} \omega)) dr} \| \varphi(s, \tau - t, \theta_{-\tau} \omega, \varphi_{t-\tau}) \|_{L^2}^2 ds$$

$$\leq M R(\tau, \omega), \quad (2.18)$$

where $\varphi_{t-\tau} = (u_{t-\tau}, v_{t-\tau}) \in D(\tau - t, \theta_{-\tau} \omega)$, $M$ is a positive constant independent of $\tau, \omega$ and $D$, and $R$ is given by

$$R(\tau, \omega) = \int_{-\infty}^{0} e^{\int_s^0 (\lambda_1 - 2s \sigma \delta_r(\theta_{\tau} \omega)) dr} (\| g(s + \tau, \cdot) \|_{L^2}^2 + \| \varphi_1(s + \tau, \cdot) \|_{L^1}^2) ds$$

$$+ \int_{-\infty}^{0} e^{\int_s^0 (\lambda_1 - 2s \sigma \delta_r(\theta_{\tau} \omega)) dr} (| g_3(\theta_{\tau} \omega) |^{\frac{p}{p-1}} + | g_3(\theta_{\tau} \omega) |^{\frac{p}{p-2}} + | g_3(\theta_{\tau} \omega) |^{p_4}) ds. \quad (2.19)$$
Proof. Taking the inner product of (2.13) with \((\beta u, \alpha v)\) we obtain
\[
\frac{1}{2} \int \frac{d}{dt}(\beta \|u\|^2 + \alpha \|v\|^2) + \frac{1}{2} \beta C(n, s) ||u||^2_{H^{1}(\mathbb{R}^n)} + \lambda \beta \|u\|^2 + \alpha(\gamma - \sigma G(\theta, \omega))\|v\|^2 \\
= \beta \int \varphi(t, x, u)udx + \beta G(\theta, \omega) \int \varphi(t, x, u)dx \\
+ \beta \int g_1(t, x)udx + \alpha \int g_2(t, x)vdx.
\]
For the first term on the right-hand side (2.20), by (2.2) we obtain that
\[
\beta \int \varphi(t, x, u)udx \leq -\alpha_1 \beta \int |u|^p dx + \beta \int \varphi_1(t, x)dx.
\]
By (2.5) and Young’s inequality, we get
\[
\beta G(\theta, \omega) \int h(t, x, u)udx \leq \beta |G(\theta, \omega)| \int (|u|^q + |\psi_1(t, x)|)u dx \\
\leq \frac{\alpha_1 \beta}{4} \int |u|^p dx + c|G(\theta, \omega)|^{p_1} + c|G(\theta, \omega)|^{p_1} \|\psi_1(t, \cdot)\|_{||L^{p_1}}^{p_1}.
\]
Finally, Young’s inequality implies that
\[
\beta \int g_1(t, x)udx + \alpha \int g_2(t, x)vdx \leq \frac{\lambda_\theta}{4} \|u\|^2 + \frac{\alpha_\theta}{4} \|v\|^2 + c\|g(t, \cdot)\|_{E}^{2}.
\]
and
\[
2\sigma \int \|G(\theta, \omega)\|_{E}^2 dx \leq \frac{\alpha_1 \beta}{2} \|u\|_{L^p} + c|G(\theta, \omega)|^{p_1} + \|\varphi(t, \cdot)\|_{L^1}.
\]
Note that \(\lambda_1 = \min\{\lambda, \gamma\}\). Thus it follows (2.20)-(2.24) that for \(t > \tau\)
\[
\frac{d}{dt} \|\varphi(t, t, \theta, \omega, \varphi_{\tau-\omega})\|_{E}^{2} \\
+ \beta C(n, s) \int_{\tau-t}^{\tau} e^{-\gamma(s, t, \theta, \omega)} \|u(s, t, \theta, \omega, u_{\tau-\omega})\|_{H^{1}(\mathbb{R}^n)}^{2} ds \\
+ \alpha_1 \beta \int_{\tau-t}^{\tau} e^{-\gamma(s, t, \theta, \omega)} ||u(s, t, \theta, \omega, u_{\tau-\omega})||_{L^{p}}^{p} ds \\
+ \frac{\lambda_1}{2} \int_{\tau-t}^{\tau} e^{-\gamma(s, t, \theta, \omega)} \|\varphi(s, t, \theta, \omega, \varphi_{\tau-\omega})\|_{E}^{2} ds \\
\leq e^{-\gamma(t, \tau-\omega)} \|\varphi(t, \tau, \theta, \omega, \varphi_{\tau-\omega})\|_{E}^{2} \\
+ c \int_{-\infty}^{0} e^{\gamma(s, t, \theta, \omega)} \|\varphi(s, t, \theta, \omega, \varphi_{\tau-\omega})\|_{E}^{2} + \|\varphi_{1}(s, t, \cdot)\|_{L^{1}} ds \\
+ c \int_{-\infty}^{0} e^{\gamma(s, t, \theta, \omega)} \|\varphi_{1}(s, t, \cdot)\|_{L^{1}} ds.
\]
(2.26)
By (2.9) and the ergodic theory, we have
\[
\lim_{s \to \pm \infty} \frac{1}{s} \int_{0}^{s} G_{\delta}(\theta_{s}, \omega) dr = \mathbb{E}(G_{\delta}(\omega)) = 0, \tag{2.27}
\]
which implies that there exists \(T_1 = T_1(\omega) > 0\) such that for all \(s \leq -T_1\),
\[
\int_{0}^{s} (\lambda_1 - 2\sigma G_{\delta}(\theta_s, \omega)) dr \leq \frac{\lambda_1}{2}s. \tag{2.28}
\]
By (2.15) and (2.28), we have
\[
\int_{0}^{0} e^{\int_{0}^{s} \left(\lambda_1 - 2\sigma G_{\delta}(\theta_s, \omega)\right) dr} \left(\|g(s, \tau, \cdot)\|_{L^2}^2 + \|\varphi_1(s, \tau, \cdot)\|_{L^1}\right) ds \nonumber
\]
\[
= \left\{ \begin{array}{ll}
\int_{-\infty}^{-T_1} + \int_{-T_1}^{0} \end{array} \right. e^{\int_{0}^{s} \left(\lambda_1 - 2\sigma G_{\delta}(\theta_s, \omega)\right) dr} \left(\|g(s, \tau, \cdot)\|_{L^2}^2 + \|\varphi_1(s, \tau, \cdot)\|_{L^1}\right) ds \nonumber
\]
\[
\leq \int_{-\infty}^{-T_1} e^{\lambda_1 s} \left(\|g(s, \tau, \cdot)\|_{L^2}^2 + \|\varphi_1(s, \tau, \cdot)\|_{L^1}\right) ds \nonumber
\]
\[
+ \int_{-T_1}^{0} e^{\int_{0}^{s} \left(\lambda_1 - 2\sigma G_{\delta}(\theta_s, \omega)\right) dr} \left(\|g(s, \tau, \cdot)\|_{L^2}^2 + \|\varphi_1(s, \tau, \cdot)\|_{L^1}\right) ds \nonumber
\]
\[
< +\infty. \tag{2.29}
\]
On the other hand, the last integral in (2.26) is well defined due to (2.8). Since \(\varphi_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)\) and \(D\) is tempered, we have
\[
\lim_{t \to +\infty} \sup_{t \to +\infty} e^{\int_{0}^{s} \left(\lambda_1 - 2\sigma G_{\delta}(\theta_s, \omega)\right) dr} \|\varphi_{\tau-t}\|_{E}^2 = \mathbb{E}(G_{\delta}(\omega)) = 0, \tag{2.28}
\]
which shows that there exists \(T = T(\tau, \omega, D) > 0\) such that for all \(t \geq T\),
\[
e^{\int_{0}^{s} \left(\lambda_1 - 2\sigma G_{\delta}(\theta_s, \omega)\right) dr} \|\varphi_{\tau-t}\|_{E}^2 \nonumber
\]
\[
\leq \int_{-\infty}^{-T_1} e^{\int_{0}^{s} \left(\lambda_1 - 2\sigma G_{\delta}(\theta_s, \omega)\right) dr} \left(\|G_{\delta}(\theta_s, \omega)\|_{F^n}^2 + \|G_{\delta}(\theta_s, \omega)\|_{P^1}\right) ds. \tag{2.30}
\]
Therefore (2.18) follows from (2.26) and (2.30). This ends the proof.

As an immediate consequence of Lemma 2.2, we obtain the existence of \(D\)-pullback absorbing set for system (2.13).

**Corollary 1 (2.1).** Suppose (2.2)-(2.6), (2.15) and (2.17) hold. Then the continuous cocycle \(\Phi\) associated with system (2.13) has a closed measurable \(D\)-pullback absorbing set \(K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D\), which is given by for each \(\tau \in \mathbb{R}\)
\[
K(\tau, \omega) = \{\varphi \in E : \|\varphi\|_{E}^2 \leq MR(\tau, \omega)\}, \tag{2.31}
\]
where \(M\) is a positive constant independent of \(\tau, \omega\) and \(D\), and \(R(\tau, \omega)\) is given by (2.19).

**Proof.** Given \(\tau \in \mathbb{R}, \omega \in \Omega\) and \(D \in D\), it follows from Lemma 2.2 that there exists \(T = T(\tau, \omega, D) > 0\) such that for all \(t \geq T\)
\[
\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega). \tag{2.32}
\]
This shows that $K$ pullback-attracts all elements in $D$. We now verify that $K$ given by (2.31) is tempered. Let $\hat{\gamma}$ be an arbitrary positive number. Then for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have by (2.31)
\[
e^\hat{\gamma}t \|K(\tau + t, \theta(t))\|^2_{E} \leq M e^{\hat{\gamma}t} R(\tau + t, \theta(t))
\]
\[
= M e^{\hat{\gamma}t} \int_{-\infty}^{t} e^{\hat{\gamma}s} (\lambda_1-2\sigma \delta_t(\theta_{t+s}+\omega))ds \left(\|g(s + \tau + t, \cdot)\|^2_{E} + \|\varphi_1(s + \tau + t, \cdot)\|_{L^1}\right)
\]
\[
+ |\mathcal{G}_\delta(\theta_{t+s}+\omega)|^2 + |\mathcal{G}_\delta(\theta_{t+s}+\omega)|^2 + |\mathcal{G}_\delta(\theta_{t+s}+\omega)|^p_1 \right) ds.
\]
Note that
\[
\int_{s}^{s+\delta} \mathcal{G}_\delta(\theta_{t+s}+\omega)ds = \int_{s}^{s+\delta} \frac{\omega(r + t)}{\delta} dr - \int_{s}^{s+\delta} \frac{\omega(r + t)}{\delta} dr.
\]
Let $c = \min\{\frac{\lambda_1}{4\sigma}, \frac{\hat{\gamma}}{8\sigma}\}$. By (2.8) we find that there exists $T_1 = T_1(\omega) < 0$ such that
\[
|\omega(t)| \leq -ct, \quad \text{for all } t \leq T_1.
\]
By the mean value theorem we infer that for every $s \leq 0$, there exists $r_0$ between $s$ and $s + \delta$ such that $\int_{s}^{s+\delta} \omega(r + t)dr = \delta \omega(r_0 + t)$. In this case, we have $r_0 \leq |\delta|$, and therefore from (2.35) we get for all $t \leq T_1 - |\delta|$,
\[
|\omega(r_0 + t)| \leq -c(r_0 + t).
\]
Since $s - r_0 \leq |\delta|$, then by (2.36) we have for all $t \leq T_1 - |\delta|$,
\[
- \frac{2\sigma}{\delta} \int_{s}^{s+\delta} \omega(r + t)dr = -2\sigma \omega(r_0 + t) \leq -2c\sigma s - 2c\sigma t + 2c\sigma |\delta|.
\]
Similarly, we can show that for all $t \leq T_1 - |\delta|$,
\[
\frac{2\sigma}{\delta} \int_{0}^{\delta} \omega(r + t)dr \leq -2c\sigma t + 2c\sigma |\delta|.
\]
Consequently, for all $t \leq T_1 - |\delta|$,
\[
- 2\sigma \int_{0}^{\delta} \mathcal{G}_\delta(\theta_{t+s}+\omega)ds \leq -2c\sigma s - 4c\sigma t + 4c\sigma |\delta|.
\]
For all $t \leq T_1 - |\delta|$,
\[
e^{\hat{\gamma}t} \int_{-\infty}^{0} e^{\hat{\gamma}s} (\lambda_1-2\sigma \delta_t(\theta_{t+s}+\omega))ds \left(\|g(s + \tau + t, \cdot)\|^2_{E} + \|\varphi_1(s + \tau + t, \cdot)\|_{L^1}\right) ds
\]
\[
\leq e^{4\sigma |\delta|} e^{\hat{\gamma}t} \int_{-\infty}^{0} e^{\hat{\gamma}s} (\lambda_1-2\sigma \delta_t(\theta_{t+s}+\omega))ds \left(\|g(s + \tau + t, \cdot)\|^2_{E} + \|\varphi_1(s + \tau + t, \cdot)\|_{L^1}\right) ds
\]
\[
\leq e^{4\sigma |\delta|} e^{\hat{\gamma}t} \int_{-\infty}^{0} e^{\hat{\gamma}s} (\|g(s + \tau + t, \cdot)\|^2_{E} + \|\varphi_1(s + \tau + t, \cdot)\|_{L^1}) ds,
\]
which along with (2.17) implies that for all $\hat{\gamma} > 0$ and $\delta \neq 0$
\[
\lim_{t \to -\infty} e^{\hat{\gamma}t} \int_{-\infty}^{0} e^{\hat{\gamma}s} (\lambda_1-2\sigma \delta_t(\theta_{t+s}+\omega))ds \left(\|g(s + \tau + t, \cdot)\|^2_{E} + \|\varphi_1(s + \tau + t, \cdot)\|_{L^1}\right) ds = 0.
\]
Let $\hat{\gamma} = \min\{\lambda_1, \hat{\gamma}\}$. Similarly, we find for all $t \leq T_1 - |\delta|$,
\[
e^{\hat{\gamma}t} \int_{-\infty}^{0} e^{\hat{\gamma}s} (\lambda_1-2\sigma \delta_t(\theta_{t+s}+\omega))ds \left(\|\mathcal{G}_\delta(\theta_{t+s}+\omega)\|^2_{E} + \|\mathcal{G}_\delta(\theta_{t+s}+\omega)\|^2 + \|\mathcal{G}_\delta(\theta_{t+s}+\omega)\|^p_1 \right) ds
\]
\[
\leq e^{4c|\delta|} e^{\frac{3}{2} t} \int_{-\infty}^{0} e^{\frac{3}{2} s} (|G_0(\theta_{s+t}\omega)|^{\frac{p}{p-2}} + |G_0(\theta_{s+t}\omega)|^{\frac{p}{p-2}} + |G_0(\theta_{s+t}\omega)|^{p_1}) \, ds
\]

By (2.8) and (2.10) we know that

\[
\int_{-\infty}^{0} e^{\frac{3}{2} s} (|G_0(\theta_{s}\omega)|^{\frac{p}{p-2}} + |G_0(\theta_{s}\omega)|^{\frac{p}{p-2}} + |G_0(\theta_{s}\omega)|^{p_1}) \, ds \leq +\infty,
\]

and hence by (2.41) we have

\[
\lim_{t \to -\infty} e^{\frac{3}{2} t} \int_{-\infty}^{0} e^{\frac{3}{2} s} (|G_0(\theta_{s+t}\omega)|^{\frac{p}{p-2}} + |G_0(\theta_{s+t}\omega)|^{\frac{p}{p-2}} + |G_0(\theta_{s+t}\omega)|^{p_1}) \, ds = 0.
\]

It follows from (2.33), (2.40) and (2.42) that \( K \) belongs to \( D \). Moreover, since for each \( \tau \in \mathbb{R}, R(\tau, \cdot) : \Omega \to \mathbb{R} \) is \((\mathcal{F}, \mathcal{B}(\mathbb{R}))\)-measurable, then \( K \) given by (2.31) is also measurable. Thus, \( K \in D \) is a closed measurable \( D \)-pullback absorbing set for \( \Phi \), as desired.

Next, we derive uniform estimates for \( u \) in \( \dot{H}^s(\mathbb{R}^n) \) for which we further assume that

\[
|f(t, x, s) - f(t, y, s)| \leq |\varphi_4(x) - \varphi_4(y)|,
\]

and

\[
|h(t, x, s) - h(t, y, s)| \leq |\psi_3(x) - \psi_3(y)|,
\]

where \( \psi_3, \varphi_4 \in H^s(\mathcal{O}) \).

**Lemma 2.3.** Suppose (2.2)-(2.6), (2.15), (2.43) and (2.44) hold. Then for every \( \tau \in \mathbb{R}, \omega \in \Omega, D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D, there exists \( T = T(\tau, \omega, D) \geq 1 \) such that for all \( t \geq T \),

\[
\|u(\tau, \tau - t, \theta_{-\tau}\omega, \omega_{-\tau+t})\|_{\dot{H}^s(\mathbb{R}^n)}^2 \leq M_2(1 + R(\tau, \omega)),
\]

where \( \varphi_{-t} \in D(\tau - t, \theta_{-\tau}\omega), M_2 \) is a positive constant independent of \( D \), and \( R(\tau, \omega) \) is given by (2.19).

**Proof.** Take the inner product of the first equation in (2.13) with \( (-\Delta)^s u \) in \( L^2(\mathbb{R}^n) \), we get

\[
\frac{d}{dt} \|(-\Delta)^s u\|^2 + 2\|(-\Delta)^s u\|^2 + 2\lambda \|(-\Delta)^s u\|^2
\]

\[
= 2(f(t, x, u), (-\Delta)^s u) - 2\alpha(v, (-\Delta)^s u)
\]

\[
+ 2(g_1, (-\Delta)^s u) + 2g_\delta(\theta_{t}\omega)(h(t, x, u), (-\Delta)^s u).
\]

(2.46)

For the first term on the right side of (2.46), by (2.4) and (2.43) we get

\[
2(f(t, x, u), (-\Delta)^s u) = C(n, s)(f(t, x, u), u)_{\dot{H}^s(\mathbb{R}^n)}
\]

\[
+ C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(t, x, u(x)) - f(t, y, u(x))) (u(x) - u(y))}{|x - y|^{n+2s}} \, dx \, dy.
\]

(2.47)
\[ \leq C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\varphi_4(x) - \varphi_4(y)| |u(x) - u(y)|}{|x - y|^{n+2s}} \, dx \, dy \\
+ C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^1 \frac{(-\alpha_3 |\tilde{u}|^{p-2} + \varphi_3(t, y))(u(x) - u(y))^2}{|x - y|^{n+2s}} \, dx \, dy \, d\xi \\
\leq C(n, s) \| \varphi_4 \|_{\dot{H}^{-s}(\mathbb{R}^n)} \| u \|_{\dot{H}^{-s}(\mathbb{R}^n)} + C(n, s) \| \varphi_3(t, \cdot) \|_{L_{loc}^\infty(\mathbb{R}, L^\infty(\mathbb{O}))} \| u \|_{\dot{H}^{-s}(\mathbb{R}^n)}^2 \\
- \alpha_3 C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^1 \frac{|\tilde{u}|^{p-2}(u(x) - u(y))^2}{|x - y|^{n+2s}} \, dx \, dy \, d\xi \\
\leq C(n, s)(1 + 2\| \varphi_3(t, \cdot) \|_{L_{loc}^\infty(\mathbb{R}, L^\infty(\mathbb{O}))}) \| (-\Delta)^{\frac{s}{2}} u \|_2^2 + \frac{1}{4} C(n, s) \| \varphi_4 \|_{\dot{H}^{-s}(\mathbb{R}^n)}^2, \tag{2.47} \]

where \( \tilde{u} = \xi u(x) + (1 - \xi) u(y) \) for \( \xi \in (0, 1) \) and here we use the mean value theorem

\[
\frac{1}{2}(u(x) - u(y))^2 = \frac{1}{2} \int_0^1 \frac{\partial f}{\partial s}(t, y, \xi u(x) + (1 - \xi) u(y)) \, d\xi \\
\leq \frac{1}{2} (u(x) - u(y))^2 \int_0^1 (-\alpha_3 |\tilde{u}|^{p-2} + \varphi_3(t, y)) \, d\xi. \]

By Young’s inequality, we find

\[
-2\alpha(v, (-\Delta)^s u) + 2(g_1, (-\Delta)^s u) \leq \| (-\Delta)^s u \|_2^2 + 2\alpha^2 \| v \|_2^2 + 2\| g_1(t, \cdot) \|_2^2. \tag{2.48} \]

For the last term in (2.46), we have from (2.6) and (2.44)

\[
2G_\beta(\theta \omega) \langle h(t, x, u), (-\Delta)^s u \rangle \\
\leq \alpha_3 C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^1 \frac{|\tilde{u}|^{p-2}(u(x) - u(y))^2}{|x - y|^{n+2s}} \, dx \, dy \, d\xi \\
+ \frac{1}{2} C(n, s) \| G_\beta(\theta \omega) \|_{H^{-s}(\mathbb{O})} + c \| G_\beta(\theta \omega) \|_{\dot{H}^{\frac{p-2}{2}}(\mathbb{O})} \| (-\Delta)^{\frac{s}{2}} u \|_2^2. \tag{2.49} \]

Then, it follows from (2.46)-(2.49) that

\[
\begin{align*}
\frac{d}{dt} \| (-\Delta)^{\frac{s}{2}} u \|_2^2 + 2\lambda \| (-\Delta)^{\frac{s}{2}} u \|_2^2 \\
\leq \rho(t, \omega) \| (-\Delta)^{\frac{s}{2}} u \|_2^2 + c \| v \|_2^2 + c \| g_1(t, \cdot) \|_2^2 + |G_\beta(\theta \omega)| + 1, \tag{2.50} \\
\end{align*}
\]

where \( \rho_1(t, \omega) = 1 + |G_\beta(\theta \omega)| + |G_\beta(\theta \omega)|^2 + |G_\beta(\theta \omega)|^{\frac{p-2}{2}}. \) It follows from (2.50) that

\[
\begin{align*}
\frac{d}{dt} \| (-\Delta)^{\frac{s}{2}} u \|_2^2 + (2\lambda_1 - \sigma G_\beta(\theta \omega)) \| (-\Delta)^{\frac{s}{2}} u \|_2^2 \\
\leq \rho(t, \omega) \| (-\Delta)^{\frac{s}{2}} u \|_2^2 + c \| v \|_2^2 + c \| g_1(t, \cdot) \|_2^2 + |G_\beta(\theta \omega)| + 1, \tag{2.51} \\
\end{align*}
\]

where \( \rho(t, \omega) = \rho_1(t, \omega) + |G_\beta(\theta \omega)|. \) Given \( t \geq 1, \tau \in \mathbb{R} \) and \( r \in (\tau - 1, \tau). \) Multiplying (2.51) by \( e^{\int_\tau^t (\lambda_1 - 2\sigma G_\beta(\theta \omega)) \, ds} \) and integrating on \( (r, \tau) \) we know that

\[
\begin{align*}
\| (-\Delta)^{\frac{s}{2}} u(r, \tau - t, \theta_{-t} \omega, u_{\tau - t}) \|_2^2 \\
\leq e^{\int_\tau^r (\lambda_1 - 2\sigma G_\beta(\theta \omega)) \, ds} \| (-\Delta)^{\frac{s}{2}} u(r, \tau - t, \theta_{-t} \omega, u_{\tau - t}) \|_2^2 + c \int_{r}^{\tau} e^{\int_\tau^s (\lambda_1 - 2\sigma G_\beta(\theta \omega)) \, ds} \rho(s, \omega) \| (-\Delta)^{\frac{s}{2}} u(s, \tau - t, \theta_{-t} \omega, u_{\tau - t}) \|_2^2 \, ds, \\
\end{align*}
\]
such that for all $t \geq T$. Applying Gronwall’s inequality to (2.57) on $[\tau, \omega]$, we write

$$\omega \in \Omega$$

when $v \in D$ and $\Omega$. It is evident that $v_1$ satisfies:

$$\|v_1(\tau, t, \theta - \tau)\| \leq e^{-\gamma t + 2\sigma \int_0^t \bar{g}^t(\theta, \omega)d\tau} \|v_{t}\| \rightarrow 0$$

when $t \rightarrow +\infty$.

**Lemma 2.4.** Suppose (2.2)-(2.6) and (2.15)-(2.16) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, there exists $T = T(\tau, \omega, D) > T$ such that for all $t \geq T$,

$$\|(\Delta)^{1/2} v_2(\tau, t, \theta - \tau, \omega, 0)\|^2$$

$$+ \int_{\tau-t}^t e^{\int_{\tau-t}^t (\lambda-2\sigma \bar{g}_k(\theta (\omega)))d\xi} \|(-\Delta)^{1/2} v_2(\tau, t, \theta - \tau, \omega, 0)\|^2 dr$$

$$\leq MR(\tau, \omega) + \int_{-\infty}^0 e^{\int_{-\infty}^0 (\lambda-2\sigma \bar{g}_k(\theta (\omega)))d\xi} \|(-\Delta)^{1/2} g_2(\tau + t, \cdot, \omega)\|^2 dr,$$

where $M$ is a positive constant independent of $\tau, \omega$ and $D$, and $R(\tau, \omega)$ is given by (2.19).

**Proof.** Take the inner product of (2.54) with $(-\Delta)^{1/2} v_2$ in $L^2(\mathbb{R}^n)$, we get

$$\frac{d}{dt} \|(-\Delta)^{1/2} v_2(t, \tau, t, \theta - \tau, \omega, 0)\|^2$$

$$+ \int_{\tau-t}^t e^{\int_{\tau-t}^t (\lambda-2\sigma \bar{g}_k(\theta (\omega)))d\xi} \|(-\Delta)^{1/2} v_2(t, \tau, t, \theta - \tau, \omega, 0)\|^2 dr$$

$$\leq c \|(-\Delta)^{1/2} u(t, \tau, t, \theta - \tau, \omega, 0)\|^2$$

and

$$\leq c \|(-\Delta)^{1/2} v_2(t, \tau, t, \theta - \tau, \omega, 0)\|^2.$$

Applying Gronwall’s inequality to (2.57) on $[\tau - t, \tau]$ for $t \in \mathbb{R}^+$, we find that

$$\|(-\Delta)^{1/2} v_2(t, \tau, t, \theta - \tau, \omega, 0)\|^2$$

$$+ \frac{\gamma}{2} \int_{\tau-t}^t e^{\int_{\tau-t}^t (\lambda-2\sigma \bar{g}_k(\theta (\omega)))d\xi} \|(-\Delta)^{1/2} v_2(t, \tau, t, \theta - \tau, \omega, 0)\|^2 dr$$

\[\text{Integrating with (2.52) respect to } r \text{ over } (\tau - 1, \tau), \text{ we can find}
\]

\[\|(-\Delta)^{1/2} u(\tau, \tau - t, \theta - \tau, u_{\tau-t})\|^2
\]

\[\leq c \int_{\tau-t}^\tau e^{\int_{\tau-t}^\tau (\lambda-2\sigma \bar{g}_k(\theta (\omega)))d\xi} \|(-\Delta)^{1/2} u(\tau, \tau - t, \theta - \tau, u_{\tau-t})\|^2 dr
\]

\[+ c \int_{\tau-t}^\tau e^{\int_{\tau-t}^\tau (\lambda-2\sigma \bar{g}_k(\theta (\omega)))d\xi} \|v(\tau, \tau - t, \theta - \tau, v_{\tau-t})\|^2 dr
\]

\[+ c \int_{\tau-t}^\tau e^{\int_{\tau-t}^\tau (\lambda-2\sigma \bar{g}_k(\theta (\omega)))d\xi} \|u(\tau, \tau - t, \theta - \tau, u_{\tau-t})\|^2 dr,
\]

which along with Lemma 2.2 implies the desired estimates. \[\square\]
Note that Corollary 2.1 shows that $\Phi$ has a closed measurable attractor.

The continuous cocycle $\Phi$ can be split as follows, for every $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$ and $\varphi_\tau = (u_\tau, v_\tau) \in E$, we define

$$\Phi_1(t, \tau, \omega, \varphi_\tau) = (0, v_1(t + \tau, \tau, \theta_{-\tau} \omega, v_\tau)) \quad (2.59)$$

and

$$\Phi_2(t, \tau, \omega, \varphi_\tau) = (u(t + \tau, \tau, \theta_{-\tau} \omega, u_\tau), v_2(t + \tau, \tau, \theta_{-\tau} \omega, 0)), \quad (2.60)$$

where $v_1$ and $v_2$ are the solutions of equations (2.53) and (2.54), respectively. Therefore, the cocycle $\Phi$ can be split as follows,

$$\Phi(t, \tau, \omega, \varphi_\tau) = \Phi_1(t, \tau, \omega, \varphi_\tau) + \Phi_2(t, \tau, \omega, \varphi_\tau). \quad (2.61)$$

**Lemma 2.5.** Suppose (2.2)-(2.6), (2.15)-(2.17), (2.43) and (2.44) hold. Then the continuous cocycle $\Phi$ associated with problem (2.13) is $D$-pullback asymptotically compact in $E$, that is, for every $t \in \mathbb{R}, \omega \in \Omega$, $D \in \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$ and $t_n \rightarrow \infty$, $\varphi_{\tau-t_n} \in D(\tau-t_n, \theta_{-t_n} \omega)$, the sequence $\Phi(t_n, \tau - t_n, \theta_{-t_n} \omega, \varphi_{\tau-t_n})$ has a convergent subsequence in $E$.

**Proof.** By (2.61) we have

$$\Phi(t_n, \tau - t_n, \theta_{-t_n} \omega, \varphi_{\tau-t_n}) = \Phi_1(t_n, \tau - t_n, \theta_{-t_n} \omega, \varphi_{\tau-t_n}) + \Phi_2(t_n, \tau - t_n, \theta_{-t_n} \omega, \varphi_{\tau-t_n}). \quad (2.62)$$

By (2.55) we get that

$$\|\Phi_1(t_n, \tau - t_n, \theta_{-t_n} \omega, \varphi_{\tau-t_n})\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \quad (2.63)$$

Form (2.62) and (2.63) it follows that the sequence $\Phi(t_n, \tau - t_n, \theta_{-t_n} \omega, \varphi_{\tau-t_n})$ will have a convergent subsequence in $L^2(\Omega) \times L^2(\Omega)$ as long as $\Phi_2(t_n, \tau - t_n, \theta_{-t_n} \omega, \varphi_{\tau-t_n})$ is precompact. By Lemma 2.2, Lemma 2.3 and Lemma 2.4, there exist $N_1 = N_1(\tau, \omega, D) \geq 1$ and $c_1 = c_1(\tau, \omega, D) > 0$ such that for all $n \geq N_1$,

$$\|\Phi_2(t_n, \tau - t_n, \theta_{-t_n} \omega, \varphi_{\tau-t_n})\|_{H^s(\Omega) \times H^s(\Omega)} \leq c_1. \quad (2.64)$$

Thus the sequence $\Phi_2(t_n, \tau - t_n, \theta_{-t_n} \omega, \varphi_{\tau-t_n})$ is precompact in $L^2(\Omega) \times L^2(\Omega)$ by the compactness of embedding $H^s(\Omega) \hookrightarrow L^2(\Omega)$. The proof is completed. \qed

We are now ready to present the existence of $D$-pullback attractors for $\Phi$.

**Theorem 2.6.** Suppose (2.2)-(2.6), (2.15)-(2.17), (2.43) and (2.44) hold. Then the continuous cocycle $\Phi$ associated with problem (2.13) has a unique $D$-pullback attractor.

**Proof.** Note that Corollary 2.1 shows that $\Phi$ has a closed measurable $D$-pullback absorbing set $K$, and Lemma 2.5 implies that $\Phi$ is asymptotically compact in $E$ with respect to $D$. Therefore, the existence of $D$-pullback attractor $A(\tau, \omega)$ follows from Proposition 2.1 in [36] immediately. ☐
3. FitzHugh–Nagumo system driven by multiplicative noise. In this section, we consider FitzHugh–Nagumo system (1.2) driven by a linear multiplicative noise: for any given $\tau \in \mathbb{R}$ and $t > \tau$

$$\begin{cases}
\frac{\partial u}{\partial t} + (-\Delta)^{\alpha} u + \lambda u = -\alpha v + f(t, x, u) + g_1(t, x) + u \circ \frac{dw(t)}{dt}, & x \in \mathcal{O}, \\
\frac{\partial v}{\partial t} = -\gamma v + \beta u + g_2(t, x) + v \circ \frac{dw(t)}{dt}, & x \in \mathcal{O}, \\
u(t, x) = v(t, x) = 0, & x \in \mathbb{R}^n \setminus \mathcal{O}, \\
u(\tau, x) = u_\tau(x), & x \in \mathcal{O}.
\end{cases} \quad (3.1)$$

We need to convert (3.1) into a pathwise deterministic equation that can be done by the standard transformation $(\tilde{u}(t, \tau, \omega), \tilde{v}(t, \tau, \omega)) = e^{\omega(t)}(u(t, \tau, \omega), v(t, \tau, \omega))$. Then $\tilde{\varphi} = (\tilde{u}, \tilde{v})$ satisfies for $t > \tau$

$$\begin{cases}
\frac{\partial \tilde{u}}{\partial t} + (-\Delta)^{\alpha} \tilde{u} + \lambda \tilde{u} = -\alpha \tilde{v} + e^{\omega(t)} f(t, x, e^{\omega(t)} \tilde{u}) + e^{\omega(t)} g_1(t, x), & x \in \mathcal{O}, \\
\frac{\partial \tilde{v}}{\partial t} = -\gamma \tilde{v} + \beta \tilde{u} + e^{\omega(t)} g_2(t, x), & x \in \mathcal{O}, \\
\tilde{u}(t, x) = \tilde{v}(t, x) = 0, & x \in \mathbb{R}^n \setminus \mathcal{O}, \\
\tilde{u}(\tau, x) = \tilde{u}_\tau(x) = e^{\omega(\tau)} u_\tau(x), & \tilde{v}(\tau, x) = \tilde{v}_\tau(x) = e^{\omega(\tau)} v_\tau(x), & x \in \mathcal{O}.
\end{cases} \quad (3.2)$$

Given $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $\tilde{\varphi} = (\tilde{u}_\tau, \tilde{v}_\tau) \in E$, system (3.2) is a deterministic system. We can prove that if $f$ satisfies all the assumptions in the previous section then system (3.1) has a unique solution $\varphi(\cdot, \tau, \omega, \varphi_\tau) = (u(\cdot, \tau, \omega, u_\tau), v(\cdot, \tau, \omega, v_\tau)) \in C([\tau, \infty), E)$. Moreover, for any $T > 0$, $u \in L^p(\tau, \tau + T; L^p(\mathbb{R}^n)) \cap L^q(\tau, \tau + T; V)$. In addition, $\varphi(\cdot, \tau, \omega, \varphi_\tau)$ is continuous in $\varphi_\tau$ with respect to the norm of $E$ and is $(\mathcal{F}, \mathcal{B}(E))$-measurable in $\omega \in \Omega$. This enables us to define a continuous cocycle $\Phi_0 : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E \to E$ over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ in the following way

$$\Phi_0(t, \tau, \omega, \varphi_\tau) = \varphi(t + \tau, \tau, \theta_{-\tau} \omega, \varphi_{-\tau}) = e^{\omega(t) - \omega(-\tau)} \tilde{\varphi}(t + \tau, \tau, \theta_{-\tau} \omega, \tilde{\varphi}_{-\tau}), \quad (3.3)$$

where $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

We first will show that system (3.1) has a $D$-pullback attractor in $E$. To this end, we must derive uniform estimates of the solutions which are given below.

**Lemma 3.1.** Suppose that (2.2)-(2.4) and (2.15) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$,

$$\begin{align*}
\|\varphi(\tau, t - \tau, \omega, \varphi_{t-\tau})\|_E^2 & + \int_{t-\tau}^t e^{\lambda_1(r-\tau) - 2\omega(r-\tau)} \|(-\Delta)^{\alpha} u(r, \tau - t, \theta_{-\tau} \omega, u_{t-\tau})\|_E^2 \, dr \\
& + \int_{t-\tau}^t e^{\lambda_1(r-\tau) - 2\omega(r-\tau)} \|\varphi(r, \tau - t, \theta_{-\tau} \omega, \varphi_{t-\tau})\|_E^2 \, dr \leq MR_0(\tau, \omega),
\end{align*} \quad (3.4)$$

where $\varphi_{t-\tau} \in D(\tau - t, \theta_{-\omega})$, $M$ is a positive constant independent of $\tau, \omega$ and $D$ and

$$R_0(\tau, \omega) = \int_{-\infty}^{0} e^{\lambda_1(r-\tau) - 2\omega(r)} (\|g(r + \tau, \cdot)\|_E^2 + \|\varphi_1(r + \tau, \cdot)\|_L^1) \, dr. \quad (3.5)$$

**Proof.** It follows from (3.2) that

$$\frac{d}{dt}(\beta \|\tilde{u}\|^2 + \alpha \|\tilde{v}\|^2) + 2\beta \|(-\Delta)^{\alpha} \tilde{u}\|^2 + 2\beta \lambda \|\tilde{u}\|^2 + 2\alpha \gamma \|\tilde{v}\|^2$$
\[= 2\beta e^{-\omega(t)} \int f(t, x, e^{\omega(t)} \hat{u}) \hat{u} dx + 2\beta e^{-\omega(t)} (g_1(t, x), \hat{u}) + 2\alpha e^{-\omega(t)} (g_2(t, x), \hat{v}). \]

(3.6)

By (2.2) we obtain that
\[
2\beta e^{-\omega(t)} \int f(t, x, e^{\omega(t)} \hat{u}) \hat{u} dx \leq -2\alpha_1 \beta e^{-2\omega(t)} \|u\|_{L^p}^2 + 2\beta e^{-\omega(t)} \varphi_1(t, \cdot) \|L^1. \]

(3.7)

For the last term in (3.6) Young’s inequality implies that
\[
2\beta e^{-\omega(t)} (g_1(t, x), \hat{u}) + 2\alpha e^{-\omega(t)} (g_2(t, x), \hat{v}) \leq \frac{\beta \lambda}{2} \|\hat{u}\|^2 + \frac{\alpha \gamma}{2} \|\hat{v}\|^2 + \frac{\beta}{\lambda} e^{-2\omega(t)} \|g_1(t, \cdot)\|^2 + \frac{2\alpha}{\gamma} e^{-\omega(t)} \|g_2(t, \cdot)\|^2. \]

(3.8)

Thus it follows (3.6)-(3.8) that for \(t > \tau\)
\[
\frac{d}{dt} \|\hat{\varphi}\|^2_E + \frac{3\lambda}{2} \|\hat{\varphi}\|^2_E + 2\beta \|(-\Delta)^{\frac{1}{2}} \hat{u}\|^2 + 2\alpha_1 \beta e^{-\omega(t)} \|u\|_{L^p}^2 \leq c e^{-2\lambda(t)} (\|g(t, \cdot)\|^2_E + \|\varphi_1(t, \cdot)\|_{L^1}). \]

(3.9)

By Gronwall’s inequality to (3.9) over \((\tau - t, \tau)\), we get for every \(\omega \in \Omega\)
\[
\|\hat{\varphi}(\tau, \tau - t, \theta_{-\tau} \omega, \hat{\varphi}_{\tau - t})\|^2_E + 2\beta \int_{\tau - t}^\tau e^{\lambda_1(r - \tau)} \|(-\Delta)^{\frac{1}{2}} \hat{u}(r, \tau - t, \theta_{-\tau} \omega, \hat{u}_{\tau - t})\|^2 dr
\]
\[
+ 2\alpha_1 \beta \int_{\tau - t}^\tau e^{\lambda_1(r - \tau)} e^{-2\omega(r - \tau) + 2\omega(-\tau)} \|u(r, \tau - t, \theta_{-\tau} \omega, \hat{u}_{\tau - t})\|^2_E dr
\]
\[
+ \int_{\tau - t}^\tau e^{\lambda_1(r - \tau)} \|\varphi(r, \tau - t, \theta_{-\tau} \omega, \hat{\varphi}_{\tau - t})\|^2_E dr
\]
\[
\leq e^{-\lambda_1 t} \|\hat{\varphi}_{\tau - t}\|^2_E + c \int_{\tau - t}^\tau e^{\lambda_1(r - \tau)} e^{-2\omega(r - \tau) + 2\omega(-\tau)} (\|g(r, \cdot)\|^2_E + \|\varphi_1(r, \cdot)\|_{L^1}) dr. \]

(3.10)

Notice the fact that for any \(\sigma \geq \tau - t\)
\[
\varphi(\sigma, \tau - t, \theta_{-\tau} \omega, \hat{\varphi}_{\tau - t}) = e^{\omega(-\tau) - \omega(-\tau)} \varphi(\sigma, \tau - t, \theta_{-\tau} \omega, \hat{\varphi}_{\tau - t}), \]

(3.11)

which together with (3.10) yields that for every \(\omega \in \Omega\)
\[
\|\varphi(\tau, \tau - t, \theta_{-\tau} \omega, \hat{\varphi}_{\tau - t})\|^2_E
\]
\[
+ 2\beta \int_{\tau - t}^\tau e^{\lambda_1(r - \tau) - 2\omega(r - \tau)} \|(-\Delta)^{\frac{1}{2}} \hat{u}(r, \tau - t, \theta_{-\tau} \omega, \hat{u}_{\tau - t})\|^2 dr
\]
\[
+ \int_{\tau - t}^\tau e^{\lambda_1(r - \tau) - 2\omega(r - \tau)} \|\varphi(r, \tau - t, \theta_{-\tau} \omega, \hat{\varphi}_{\tau - t})\|^2_E dr
\]
\[
\leq e^{-\lambda_1 t - 2\omega(-t)} \|\hat{\varphi}_{\tau - t}\|^2_E + c \int_{\tau - t}^\tau e^{\lambda_1(r - \tau) - 2\omega(r - \tau)} (\|g(r, \cdot)\|^2_E + \|\varphi_1(r, \cdot)\|_{L^1}) dr
\]
\[
\leq e^{-\lambda_1 t - 2\omega(-t)} \|\hat{\varphi}_{\tau - t}\|^2_E + c \int_{-\infty}^0 e^{\lambda_1(r - \tau) - 2\omega(r)} (\|g(r + \tau, \cdot)\|^2_E + \|\varphi_1(r + \tau, \cdot)\|_{L^1}) dr. \]

(3.12)

By (2.8) and (2.15), we see that \(R_0\) given by (3.5) is bounded. Since \(\varphi_{\tau - t} \in D(\tau - t, \theta_{-\tau} \omega)\) and \(D\) is tempered, we find that
\[
\limsup_{t \to \infty} e^{-\lambda_1 t - 2\omega(-t)} \|\varphi_{\tau - t}\|^2 \leq \limsup_{t \to \infty} e^{-\lambda_1 t - 2\omega(-t)} \|D(\tau - t, \theta_{-\tau} \omega)\|^2 = 0, \]
which shows that there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$,
\[
e^{-\lambda_1 t - 2\omega(t)} \| \varphi_{\tau - t} \|_{L^2(t)}^2 \leq R_0(\tau, \omega).
\] (3.13)

Then (3.12) and (3.13) can imply the desired estimates. □

Next, we derive uniform estimates for $u$ in $V$.

**Lemma 3.2.** Suppose that (2.2)-(2.4), (2.15) and (2.43) hold. Then there for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, there exists $T = T(\tau, \omega, D) \geq 1$ such that for all $t \geq T$,
\[
\| u(\tau, t, \theta, \omega, u_{\tau - t}) \|_{V'}^2 \leq M(1 + R_0(\tau, \omega)),
\] (3.14)

where $u_{\tau - t} \in D(\tau - t, \theta, \omega)$ and $M$ is a positive constant depending on $\lambda_1$, but independent of $D$, and $R_0(\tau, \omega)$ is given by (3.5).

**Proof.** Taking the inner product of the first equation in (3.2) with $(-\Delta)^s \hat{u}$ in $L^2(\mathbb{R}^n)$, we find that
\[
\frac{d}{dt} \| (-\Delta)^{\frac{1}{2}} \hat{u} \|^2 + 2\| (-\Delta)^{\frac{1}{2}} \hat{u} \|^2 + 2\lambda \| (-\Delta)^{\frac{1}{2}} \hat{u} \|^2 = -2\alpha(\hat{v}, (-\Delta)^s \hat{u})
\]
\[+ 2e^{-\omega(t)}(f(\tau, t, \epsilon \omega(t) \hat{u}), (-\Delta)^s \hat{u}) + 2e^{-\omega(t)}(g_1(\tau, t, \hat{u}), (-\Delta)^s \hat{u}).
\] (3.15)

We first estimate the nonlinear term in (3.15). Similarly as (2.47), we have
\[
2e^{-\omega(t)}(f(\tau, t, \epsilon \omega(t) \hat{u}), (-\Delta)^s \hat{u}) = C(n, s)e^{-\omega(t)} \left( f(\tau, \epsilon \omega(t) \hat{u}), \hat{u} \right)_{H^s(\mathbb{R}^n)}
\]
\[\leq (1 + 2\| \varphi_3(t, \cdot) \|_{L^\infty(\mathbb{R}, L^\infty(\mathbb{R}^n))})(-\Delta)^{\frac{1}{2}} \hat{u} \|^2 + ce^{-2\omega(t)}.
\] (3.16)

On the other hand, the third term and last term on the right-hand side of (3.15) are bounded by
\[
-2\alpha(\hat{v}, (-\Delta)^s \hat{u}) + 2e^{-\omega(t)}(g_1(\tau, t, \hat{u}), (-\Delta)^s \hat{u})
\]
\[\leq \| (-\Delta)^s \hat{u} \|^2 + c\| \hat{v} \|^2 + ce^{-2\omega(t)}\| g_1(\tau, t, \cdot) \|^2.
\] (3.17)

By (3.15)-(3.17) we get
\[
\frac{d}{dt} \| (-\Delta)^{\frac{1}{2}} \hat{u} \|^2 + \lambda_1 \| (-\Delta)^{\frac{1}{2}} \hat{u} \|^2
\]
\[\leq c\| (-\Delta)^{\frac{1}{2}} \hat{u} \|^2 + c\| \hat{v} \|^2 + ce^{-2\omega(t)} + ce^{-2\omega(t)}\| g_1(\tau, t, \cdot) \|^2.
\] (3.18)

Given $t \geq 1, \tau \in \mathbb{R}, \omega \in \Omega$ and $r \in (\tau - 1, \tau)$. Multiplying (3.18) by $e^{\lambda_1 t}$ and integrating on $(r, \tau)$ we know that
\[
\| (-\Delta)^{\frac{1}{2}} \hat{u}(\tau, t, \omega, \hat{u}_{\tau - t}) \|^2 \leq e^{\lambda_1 (r - \tau)}\| (-\Delta)^{\frac{1}{2}} \hat{u}(\tau) \|^2
\]
\[+ c \int_r^\tau e^{\lambda_1 (\xi - \tau)}\| (-\Delta)^{\frac{1}{2}} \hat{u}(\xi) \|^2 d\xi + c \int_r^\tau e^{\lambda_1 (\xi - \tau)}\| \hat{v}(\xi) \|^2 d\xi
\]
\[+ c \int_r^\tau e^{\lambda_1 (\xi - \tau)}e^{-2\omega(\xi)} d\xi + c \int_r^\tau e^{\lambda_1 (\xi - \tau)} e^{-2\omega(\xi)}\| g_1(\xi, \cdot) \|^2 d\xi.
\] (3.19)

Now integrating the above with respect to $s$ over $(\tau - 1, \tau)$ and replacing $\omega$ by $\theta_{-\tau}$, we find that
\[
\| (-\Delta)^{\frac{1}{2}} \hat{u}(\tau, t, \theta_{-\tau}, \hat{u}_{\tau - t}) \|^2
\]
\[\leq (c + 1) \int_{\tau - 1}^{\tau} e^{\lambda_1 (r - \tau)}\| (-\Delta)^{\frac{1}{2}} \hat{u}(\tau) \|^2 dr + c \int_{\tau - 1}^{\tau} e^{\lambda_1 (r - \tau)}\| \hat{v}(\tau) \|^2 dr
\]
\[ + c \int_{-\infty}^{\tau} e^{\lambda_1(r-\tau)} e^{-2\omega(r-\tau)+2\omega(-\tau)} (1 + \|g_1(r, \cdot)\|^2) dr. \]  

(3.20)

It follows from (3.11) and (3.20) that

\[ \|(-\Delta)^{\frac{1}{2}} u(r, \tau-t, \theta_{-\tau} \omega, u_{\tau-t})\|^2 \]

\[ \leq c \int_{-\infty}^{\tau} e^{\lambda_1(r-\tau)-2\omega(r-\tau)} \|(-\Delta)^{\frac{1}{2}} u(r, \tau-t, \theta_{-\tau} \omega, v_{\tau-t})\|^2 dr \]

\[ + c \int_{-\infty}^{\tau} e^{\lambda_1(r-\tau)-2\omega(r-\tau)} \|v(r, \tau-t, \theta_{-\tau} \omega, v_{\tau-t})\|^2 dr \]

\[ + c \int_{-\infty}^{\tau} e^{\lambda_1(r-\tau)-2\omega(r-\tau)} (1 + \|g_1(r, \cdot)\|^2) dr. \]  

(3.21)

Note that for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[ \int_{\tau-1}^{\tau} e^{\lambda_1(r-\tau)-2\omega(r-\tau)} (1 + \|g_1(r, \cdot)\|^2) dr \leq \int_{-\infty}^{0} e^{\lambda_1 r - 2\omega(r)} (1 + \|g_1(r + \tau, \cdot)\|^2) dr. \]  

(3.22)

Then the desired estimate follows from (3.21), (3.22) and Lemma 3.1. \( \square \)

Next, we will show the asymptotic compactness of solutions of system (3.1). To that end, we write \( v = v_1 + v_2 \) where \( v_1 \) and \( v_2 \) solve the following systems

\[ \frac{\partial v_1}{\partial t} = -\gamma v_1 + v_1 \circ \frac{du(t)}{dt}, \quad t > \tau, \ x \in \Omega, \quad v_1(\tau) = v_\tau \]  

(3.23)

and

\[ \frac{\partial v_2}{\partial t} = -\gamma v_2 + \beta u + g_2(t, x) + v_2 \circ \frac{dw(t)}{dt}, \quad t > \tau, \ x \in \Omega, \quad v_2(\tau) = 0. \]  

(3.24)

Let

\[ \tilde{v}_1(t) = e^{-\omega(t)} v_1(t) \quad \text{and} \quad \tilde{v}_2(t) = e^{-\omega(t)} v_2(t). \]  

(3.25)

Then \( \tilde{v}_1 \) and \( \tilde{v}_2 \) solve the following systems

\[ \frac{\partial \tilde{v}_1}{\partial t} = -\gamma \tilde{v}_1, \quad t > \tau, \ x \in \Omega, \quad \tilde{v}_1(\tau) = \tilde{v}_\tau \]  

(3.26)

and

\[ \frac{\partial \tilde{v}_2}{\partial t} = -\gamma \tilde{v}_2 + \beta \tilde{u} + e^{-\omega(t)} g_2(t, x), \quad t > \tau, \ x \in \Omega, \quad \tilde{v}_2(\tau) = 0. \]  

(3.27)

It is evident that \( \tilde{v}_1 \) satisfies:

\[ \|\tilde{v}_1(\tau, \tau-t, \theta_{-\tau} \omega, \tilde{v}_{\tau-t})\| \leq e^{-\gamma t} \|\tilde{v}_{\tau-t}\|. \]  

(3.28)

We have from (3.25) and (3.28)

\[ \|v_1(\tau, \tau-t, \theta_{-\tau} \omega, v_{\tau-t})\| \leq e^{-\gamma t} \|v_{\tau-t}\| \to 0, \]  

(3.29)

when \( t \to \infty \). Next, we derive uniform estimates for \( v_2 \) in \( V \).

**Lemma 3.3.** Suppose (2.2)-(2.4) and (2.15)-(2.16) hold. Then for every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \), and \( D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D} \), there exists \( T = T(\tau, \omega, D) > 1 \) such that for all \( t \geq T \),

\[ \|v_2(\tau, \tau-t, \theta_{-\tau} \omega, 0)\|_{V}^2 \leq MR_0(\tau, \omega) + M \int_{-\infty}^{0} e^{\lambda_1 r - 2\omega(r)} \|(-\Delta)^{\frac{1}{2}} g_2(r+\tau, \cdot)\|^2 dr, \]  

(3.30)
where \( R_0(\tau, \omega) \) is given by (3.5), \( M \) is a positive constant depending on \( \lambda_1 \), but independent of \( \tau, \omega \) and \( D \).

**Proof.** The proof is similar to that of Lemma 2.4 and hence omitted here.

We now establish the \( D \)-pullback asymptotic compactness of \( \Phi_0 \) in \( E \). For this purpose, we need to split it as follows. For every \( \tau \) independent of \( \tau \), we define

\[
\Phi_0(t, \tau, \omega, \varphi_\tau) = (0, v_1(t + \tau, \tau, \theta_{-\tau} \omega, v_\tau))
\]

and

\[
\Phi_{02}(t, \tau, \omega, \varphi_\tau) = (u(t + \tau, \tau, \theta_{-\tau} \omega, u_\tau), v_2(t + \tau, \tau, \theta_{-\tau} \omega, 0)),
\]

where \( v_1 \) and \( v_2 \) are the solutions of equations (3.23) and (3.24), respectively. Therefore, the cocycle \( \Phi_0 \) can be split as follows,

\[
\Phi_0(t, \tau, \omega, \varphi_\tau) = \Phi_{01}(t, \tau, \omega, \varphi_\tau) + \Phi_{02}(t, \tau, \omega, \varphi_\tau).
\]

**Lemma 3.4.** Suppose (2.2)-(2.4), (2.15)-(2.17) and (2.43) hold. Then the continuous cocycle \( \Phi_0 \) associated with problem (3.1) is \( D \)-pullback asymptotically compact in \( E \).

**Proof.** The proof is similar to that of Lemma 2.5, and hence omitted here.

We are now ready to present the existence of \( D \)-pullback attractors for \( \Phi_0 \).

**Theorem 3.5.** Suppose (2.2)-(2.4), (2.15)-(2.17) and (2.43) hold. Then the continuous cocycle \( \Phi_0 \) associated with equation (3.1) has a unique \( D \)-pullback attractor \( \mathcal{A}_0 = \{A_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D \) in \( E \).

**Proof.** For every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), denote \( K_0(\tau, \omega) \) by

\[
K_0(\tau, \omega) = \{ \varphi \in E : \|\varphi\|^2_E \leq M R_0(\tau, \omega) \},
\]

where \( M \) is a positive constant as in (3.4) and \( R_0(\tau, \omega) \) is given by (3.5). For every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( D \in D \), it follows from Lemma 3.1 that there exists \( T = T(\tau, \omega, D) > 0 \) such that for all \( t \geq T \)

\[
\Phi_0(t, \tau - t, \theta_{-t} \omega, D(\tau - t, \theta_{-t} \omega)) \subseteq K_0(\tau, \omega).
\]

By (2.8) and (2.17), we can verify that \( K_0 \) given by (3.34) is tempered. Consequently, \( K_0 \in D \) is a closed measurable \( D \)-pullback absorbing set for \( \Phi_0 \). By Lemma 3.4, \( \Phi_0 \) is \( D \)-pullback asymptotically compact in \( E \). Then the proof of this theorem follows from Proposition 2.1 in [36] immediately.

We now propose to approximate the solution of equation (3.1) by the following pathwise deterministic equation: for any \( \tau \in \mathbb{R} \),

\[
\begin{aligned}
\frac{\partial u_\delta}{\partial t} + (-\Delta)^s u_\delta + \lambda u_\delta &= -\alpha v_\delta + f(t, x, u_\delta) + g_1(t, x) + G_\delta(\theta_{i} \omega) u_\delta, \quad t > \tau, \ x \in \mathcal{O}, \\
\frac{\partial v_\delta}{\partial t} &= -\gamma v_\delta + \beta u_\delta + g_2(t, x) + G_\delta(\theta_{i} \omega) v_\delta, \quad t > \tau, \ x \in \mathcal{O}, \\
u_\delta(t, x) &= v_\delta(t, x) = 0, \quad t > \tau, \ x \in \mathbb{R}^n \setminus \mathcal{O}, \\
u_\delta(\tau, x) &= u_{\delta, \tau}(x), \ v_\delta(\tau, x) = v_{\delta, \tau}(x), \ x \in \mathcal{O}.
\end{aligned}
\]

(3.36)

To indicate the dependence of solutions on the parameter \( \delta \), we write the solution of equation (3.36) as \( \varphi_\delta = (u_\delta, v_\delta) \). From the previous section, we know for every \( \delta \neq 0 \), equation (3.36) defines a continuous cocycle as \( \Phi_\delta \) in \( E \) which possesses a unique
D-pullback attractor $A_δ(τ, ω)$. In what follows, we will discuss the convergence of solutions of (3.36) as $δ → 0$. Furthermore, we will establish the upper semicontinuity of random attractors $A_δ(τ, ω)$ as $δ → 0$. These results partially justify the idea to approximate a stochastic FitzHugh-Nagumo system by replacing the white noise by a process $G_δ(θ, ω)$ with small $δ$.

To better understand the relations between the solutions of (3.1) and (3.36), we introduce a similar transformation for (3.36) as we did for (3.1). Denote by

$$\tilde{φ}_δ(t, τ, ω, \tilde{φ}_δ, τ) = (\tilde{u}_δ(t, τ, ω, u_δ, τ), \tilde{v}_δ(t, τ, ω, \tilde{v}_δ, τ))$$

$$= e^{-\int_0^t \tilde{G}_δ(θ, ω)dr} \phi_δ(t, τ, ω, \tilde{φ}_δ, τ) \text{ and } \tilde{φ}_δ, τ = e^{-\int_0^τ \tilde{G}_δ(θ, ω)dr} \phi_δ, τ. \quad (3.37)$$

Then we get from (3.36) and (3.37) that

$$\left\{ \begin{array}{l}
\frac{∂\tilde{u}_δ}{∂t} + (-Δ)^{σ/2} \tilde{u}_δ + λ\tilde{u}_δ = -α\tilde{v}_δ + e^{-\int_0^t \tilde{G}_δ(θ, ω)dr} f(t, x, e^{\int_0^t \tilde{G}_δ(θ, ω)dr} \tilde{u}_δ) \\
+ e^{-\int_0^t \tilde{G}_δ(θ, ω)dr} g_1(t, x), \quad t > τ, \ x ∈ \mathcal{O}, \\
\frac{∂\tilde{v}_δ}{∂t} = -γ\tilde{v}_δ + β\tilde{u}_δ + e^{-\int_0^t \tilde{G}_δ(θ, ω)dr} g_2(t, x), \quad t > τ, \ x ∈ \mathcal{O}, \\
\tilde{u}_δ(t, x) = \tilde{v}_δ(t, x) = 0, \quad t > τ, \ x ∈ \mathbb{R}^n \setminus \mathcal{O}, \\
\tilde{u}_δ(τ, x) = \tilde{u}_δ, τ(x), \quad \tilde{v}_δ(τ, x) = \tilde{v}_δ, τ(x), \quad x ∈ \mathcal{O}.
\end{array} \right. \quad (3.38)$$

We have the following estimates on the solutions of system (3.38) on a finite time interval.

**Lemma 3.6.** Suppose (2.2)-(2.4) and (2.15) hold. Then for every $τ ∈ \mathbb{R}$, $ω ∈ Ω$, and $T > 0$, there exist $δ_0 = δ_0(τ, ω, T) > 0$ and $c(τ, ω, T) > 0$ such that for all $0 < |δ| < δ_0$ and $t ∈ [τ, τ + T]$,

$$\|\phi_δ(t, τ, ω, \phi_δ, τ)\|^2_E + \int_τ^t \|\phi_δ(r, τ, ω, \phi_δ, τ)\|^2_E dr$$

$$+ \int_τ^t \|u_δ(r, τ, ω, u_δ, τ)\|^2_{H^σ(\mathbb{R}^n)} dr + \int_τ^t \|u_δ(r, τ, ω, u_δ, τ)\|^p_{L^p} dr$$

$$≤ c\|\phi_δ, τ\|^2_E + c \int_τ^t \left( \|g(r, \cdot)\|^2_E + \|g(r, \cdot)\|^p_{L^p} \right) dr. \quad (3.39)$$

**Proof.** It follows from (3.38) that

$$\frac{d}{dt} (β\|\tilde{u}_δ\|^2 + α\|\tilde{v}_δ\|^2) + 2β\|(-Δ)^{σ/2} \tilde{u}_δ\|^2 + 2λβ\|\tilde{u}_δ\|^2 + 2αγ\|\tilde{v}_δ\|^2$$

$$= 2βe^{-\int_0^t \tilde{G}_δ(θ, ω)dr} \int_Ω f(t, x, e^{\int_0^t \tilde{G}_δ(θ, ω)dr} \tilde{u}_δ) \tilde{u}_δ dx$$

$$+ 2βe^{-\int_0^t \tilde{G}_δ(θ, ω)dr} (g_1(t, x), \tilde{u}_δ) + 2αe^{-\int_0^t \tilde{G}_δ(θ, ω)dr} (g_2(t, x), \tilde{v}_δ). \quad (3.40)$$

By (2.2) we obtain that

$$2βe^{-\int_0^t \tilde{G}_δ(θ, ω)dr} \int_Ω f(t, x, e^{\int_0^t \tilde{G}_δ(θ, ω)dr} \tilde{u}_δ) \tilde{u}_δ dx$$

$$≤ -2α_1 βe^{-2\int_0^t \tilde{G}_δ(θ, ω)dr} \|u_δ\|^p_{L^p} + 2βe^{-2\int_0^t \tilde{G}_δ(θ, ω)dr} \|\phi_δ(t, \cdot\)\|^p_{L^p}. \quad (3.41)$$

For the last term in (3.40), Young’s inequality implies that

$$2βe^{-\int_0^t \tilde{G}_δ(θ, ω)dr} (g_1(t, x), \tilde{u}_δ) + 2αe^{-\int_0^t \tilde{G}_δ(θ, ω)dr} (g_2(t, x), \tilde{v}_δ)$$

$$≤ \frac{λβ}{2} \|\tilde{u}_δ\|^2 + \frac{αγ}{2} \|\tilde{v}_δ\|^2 + ce^{-2\int_0^t \tilde{G}_δ(θ, ω)dr} \|g(t, \cdot\)\|^2_E. \quad (3.42)$$
It follows from (3.40)-(3.42) that for \( t > \tau \)
\[
\frac{d}{dt} \|\tilde{\varphi}_\delta\|_E^2 + \frac{3}{2} \lambda_1 \|\tilde{\varphi}_\delta\|_E^2 + 2\beta \|(-\Delta)^\frac{3}{2} \tilde{u}_\delta\|_E^2 + 2\alpha_1 \beta e^{-2 \int_{\tau}^t \|(-\Delta)^\frac{3}{2} \tilde{u}_\delta\|_E^2} \|\tilde{\varphi}_\delta\|_{L^p}^p \\
\leq e^{-2 \int_{\tau}^t \|\varphi(t, \cdot)\|_E^2} (\|g(t, \cdot)\|_E^2 + \|\varphi_1(t, \cdot)\|_{L^1}).
\]  
(3.43)

By Gronwall’s inequality to (3.43) over \((\tau, t)\), we get for every \( \omega \in \Omega \)
\[
\|\tilde{\varphi}_\delta(t, \tau, \omega, \tilde{\varphi}_{\delta, \tau})\|_E^2 + 2\beta \int_\tau^t e^{\lambda_1 (r-t)} \|(-\Delta)^\frac{3}{2} \tilde{u}_\delta(r, \tau, \omega, \tilde{u}_{\delta, \tau})\|_E^2 dr \\
+ 2\alpha_1 \beta \int_\tau^t e^{\lambda_1 (r-t)} e^{-2 \int_{\tau}^r \|\varphi_1(r, \cdot, \omega)\|_{L^1}^2} \|\tilde{u}_\delta(r, \tau, \omega, u_{\delta, \tau})\|_{L^p}^p dr \\
+ \frac{\lambda_1}{2} \int_\tau^t e^{\lambda_1 (r-t)} \|\tilde{\varphi}_\delta(r, \tau, \omega, \tilde{\varphi}_{\delta, \tau})\|_E^2 dr \\
\leq e^{-\lambda_1 (t-\tau)} \|\tilde{\varphi}_{\delta, \tau}\|_E^2 + c \int_\tau^t e^{\lambda_1 (r-t)} e^{-2 \int_{\tau}^r \|\varphi_1(r, \cdot, \omega)\|_{L^1}^2} (\|g(r, \cdot, \omega)\|_E^2 + \|\varphi_1(r, \cdot)\|_{L^1}) dr.
\]  
(3.44)

Then (3.37) and (3.44) yields that for every \( \omega \in \Omega \)
\[
\|\varphi_\delta(t, \tau, \omega, \varphi_{\delta, \tau})\|_E^2 + 2\beta \int_\tau^t e^{\lambda_1 (r-t)} e^{-2 \int_{\tau}^r \|\varphi_1(r, \cdot, \omega)\|_{L^1}^2} \|(-\Delta)^\frac{3}{2} \tilde{u}_\delta(r, \tau, \omega, \tilde{u}_{\delta, \tau})\|_E^2 dr \\
+ 2\alpha_1 \beta \int_\tau^t e^{\lambda_1 (r-t)} e^{-2 \int_{\tau}^r \|\varphi_1(r, \cdot, \omega)\|_{L^1}^2} \|\tilde{u}_\delta(r, \tau, \omega, u_{\delta, \tau})\|_{L^p}^p dr \\
+ \frac{\lambda_1}{2} \int_\tau^t e^{\lambda_1 (r-t)} e^{-2 \int_{\tau}^r \|\varphi_1(r, \cdot, \omega)\|_{L^1}^2} \|\varphi_\delta(r, \tau, \omega, \varphi_{\delta, \tau})\|_E^2 dr \\
\leq e^{-\lambda_1 (t-\tau)} \|\varphi_{\delta, \tau}\|_E^2 + c \int_\tau^t e^{\lambda_1 (r-t)} e^{-2 \int_{\tau}^r \|\varphi_1(r, \cdot, \omega)\|_{L^1}^2} (\|g(r, \cdot, \omega)\|_E^2 + \|\varphi_1(r, \cdot)\|_{L^1}) dr.
\]  
(3.45)

Then by (2.12) and (3.45) we can get (3.39), and the proof completes. \( \square \)

Next, we derive uniform estimates on the solutions of (3.36) when time is sufficiently large.

**Lemma 3.7.** Suppose (2.2)-(2.4) and (2.15) hold. Then for every \( \delta \neq 0, \tau \in \mathbb{R} \), \( \omega \in \Omega \), and \( D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \), there exists \( T = T(\tau, \omega, D, \delta) > 0 \) such that for all \( t \geq T \),
\[
\|\varphi_\delta(t, \tau - t, \theta \omega, \varphi_{\delta, \tau-t})\|_E^2 \leq M R_\delta(\tau, \omega),
\]  
(3.46)

where \( \varphi_{\delta, \tau-t} \in D(\tau - t, \theta \omega, \omega) \), \( M \) is positive constants independent of \( \tau, \omega \) and \( D \), and \( R_\delta \) is given by
\[
R_\delta(\tau, \omega) = \int_{-\infty}^0 e^{\lambda_1 r + 2 \int_0^r \|G(\theta \omega)\|_{L^1}^2} (\|g(r + \tau, \cdot)\|_E^2 + \|\varphi_1(r + \tau, \cdot)\|_{L^1}) dr.
\]  
(3.47)

**Proof.** By Gronwall’s inequality to (3.43) over \((t - \tau, t)\), we get for every \( \omega \in \Omega \)
\[
\|\tilde{\varphi}_\delta(t, \tau - t, \theta \omega, \tilde{\varphi}_{\delta, \tau-t})\|_E^2 + 2\beta \int_{t-t}^t e^{\lambda_1 (r-t)} \|(-\Delta)^\frac{3}{2} \tilde{u}_\delta(r, \tau - t, \theta \omega, \tilde{u}_{\delta, \tau-t})\|_E^2 dr \\
+ 2\alpha_1 \beta \int_{t-t}^t e^{\lambda_1 (r-t)} e^{-2 \int_{t-t}^r \|\varphi_1(r, \cdot, \omega)\|_{L^1}^2} \|\tilde{u}_\delta(r, \tau - t, \theta \omega, u_{\delta, \tau-t})\|_{L^p}^p dr \\
+ \frac{\lambda_1}{2} \int_{t-t}^t e^{\lambda_1 (r-t)} e^{-2 \int_{t-t}^r \|\varphi_1(r, \cdot, \omega)\|_{L^1}^2} \|\tilde{\varphi}_\delta(r, \tau - t, \theta \omega, \tilde{\varphi}_{\delta, \tau-t})\|_E^2 dr \\
\leq e^{-\lambda_1 (t-\tau)} \|\tilde{\varphi}_{\delta, \tau-t}\|_E^2 + c \int_{t-t}^t e^{\lambda_1 (r-t)} e^{-2 \int_{t-t}^r \|\varphi_1(r, \cdot, \omega)\|_{L^1}^2} (\|g(r, \cdot, \omega)\|_E^2 + \|\varphi_1(r, \cdot)\|_{L^1}) dr.
\]
Then (3.37) and (3.48) yields that for every \( \omega \in \Omega \)

\[
\| \varphi_\delta(\tau, t - \lambda, \omega, \varphi_{\delta, t - \lambda}) \|_E^2 \leq e^{-\lambda t} \| \varphi_{\delta, t - \lambda} \|_E^2 + c \int_{\tau-t}^\tau e^{-\lambda (\tau - t)} e^{-2 \int_0^\tau \varphi_{\delta} \delta(\tau_t - \omega, \varphi_{\delta, \tau_t - \lambda}) d\xi} (\| g(r, \cdot) \|_E^2 + \| \varphi_1(r, \cdot) \|_{L^1}) \, dr.
\]  

(3.48)

which implies the desired estimates. \( \square \)

As a consequence of Lemma 3.7, we find that equation (3.36) has a \( D \)-pullback absorbing set in \( E \).

**Lemma 3.8.** Suppose (2.2)-(2.4) and (2.15)-(2.17) hold. Then the continuous cocycle \( \Phi_\delta \) associated with system (3.36) has a closed measurable \( D \)-pullback absorbing set \( K_\delta = \{ K_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \subset D \), which is given by for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \)

\[
K_\delta(\tau, \omega) = \{ \varphi_\delta \in E : \| \varphi_\delta \|_E^2 \leq M R_\delta(\tau, \omega) \},
\]  

(3.50)

where \( M \) is positive constant as in (3.46), and \( R_\delta(\tau, \omega) \) is given by (3.47). In addition, we have for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \)

\[
\lim_{\delta \to 0} R_\delta(\tau, \omega) = R_0(\tau, \omega),
\]  

(3.51)

where \( R_0(\tau, \omega) \) is defined in (3.5).

**Proof.** Note \( K_\delta \) given by (3.50) is a closed measurable random set in \( E \). Given \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( D \in D \), it follows from Lemma 3.7 that there exists \( T_0 = T_0(\tau, \omega, D, \delta) \) such that for all \( t \geq T_0 \)

\[
\Phi_\delta(t, \tau - t, \omega, D(\tau - t, \omega - t)) \subseteq K_\delta(\tau, \omega).
\]  

(3.52)

This shows that \( K_\delta \) pullback attracts all elements in \( D \). By (2.8) one may verify that \( K_\delta \) is tempered. The convergence (3.51) can be obtained by the Lebesgue dominated convergence as in [29]. The details are omitted here. \( \square \)

For the attractor \( A_\delta \) of \( \Phi_\delta \), we have the following uniform compactness.

**Lemma 3.9.** Suppose (2.2)-(2.4) and (2.15)-(2.17) hold. Then for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), there exists \( \delta_0 = \delta_0(\omega) > 0 \) such that \( \bigcup_{0 < |\delta| < \delta_0} A_\delta(\tau, \omega) \) is precompact in \( E \).

**Proof.** Let \( t_n \to \infty \) and \( \omega \in A_\delta(\tau, \omega) \) for some \( \delta \in (0, \delta_0) \). By the invariance of \( A_\delta \), for each \( n \in \mathbb{N} \), there exists \( \omega_n \in A_\delta(\tau - t_n, \theta_{-t_n} \omega) \) such that

\[
\omega = \Phi_\delta(t_n, \tau - t_n, \theta_{-t_n} \omega, \omega_n) = \Phi_{\delta, 1}(t_n, \tau - t_n, \theta_{-t_n} \omega, \omega_n) + \Phi_{\delta, 2}(t_n, \tau - t_n, \theta_{-t_n} \omega, \omega_n).
\]  

(3.53)
Since $A_δ ∈ D$ and $t_n → ∞$, by Lemma 2.2, Lemma 2.3 and Lemma 2.4 we see that there exists $δ_0 = δ_0(ω)$ and $N_1 = N_1(τ, ω, δ_0) ≥ 1$ such that for all $n ≥ N_1$ and $δ ∈ (0, δ_0)$,
\[ \|Φ_{δ,2}(t_n, τ - t_n, θ - t_n, ω, u_n)\|_{H^s(O) × H^s(O)} ≤ c_1, \]
where $c_1 = c_1(τ, ω) > 0$. By (3.54), we find that there exists $w ∈ H^s(O) × H^s(O)$ such that, up to a subsequence,
\[ Φ_{δ,2}(t_n, τ - t_n, θ - t_n, ω, u_n) → w \text{ weakly in } H^s(O) × H^s(O). \]

On the other hand, by (2.55) we get
\[ Φ_{δ,1}(t_n, τ - t_n, θ - t_n, ω, u_n) → 0 \text{ in } E. \]
By (3.55) and (3.56) we obtain $w = w^*$ and
\[ \|w\|_{H^s(O) × H^s(O)} ≤ c_1, \quad \text{for all } w ∈ A_δ(τ, ω) \text{ with } 0 < δ ≤ δ_0. \] (3.57)
By the compactness of embedding $H^s(O) ↪ L^2(O)$, the set $\bigcup_{0 < |δ| < δ_0} A_δ(τ, ω)$ is precompact in $E$. The proof is completed. \[ \square \]

4. Upper semicontinuity. In this section, we establish the convergence of solutions of (3.36) as $δ → 0$. To this end, we further assume the following condition on $f$: there exists $ϕ_δ(t, x) ∈ L^∞(R, L^∞(R^n))$ such that for all $t, s ∈ R$ and $x ∈ R^n$
\[ \frac{∂f}{∂s}(t, x, s) ≤ ϕ_δ(t, x)(1 + |s|^{p-2}). \] (4.1)

**Lemma 4.1.** Suppose (2.2)-(2.4) and (4.1) hold. Let $ϕ$ and $ϕ_δ$ be the solutions of (3.1) and (3.36), respectively. Then for every $r ∈ R, ω ∈ Ω, T > 0$ and $ε ∈ (0, 1)$, there exists $δ_0 = δ_0(τ, ω, T, ε) > 0$ and $c = c(τ, ω, T) > 0$ such that for all $0 < |δ| < δ_0$ and $t ∈ [τ, τ + T],$
\[ \|ϕ_δ(t, τ, ω, ϕ_δ, ω) - ϕ(t, τ, ω, ϕ_δ, ω)\|_E^2 \]
\[ ≤ c\|ϕ_δ, ω - ϕ, ω\|_E^2 + c\int_0^T (\|g(s, ω)\|_E + \|ϕ_δ, ω\|_L^1 + \|ϕ_δ, ω\|_L^{p_1})ds. \] (4.2)

**Proof.** Let $ϕ = (u, v) = ϕ_δ - ϕ$, where $ϕ$ and $ϕ_δ$ are the solutions of (3.2) and (3.38), respectively. By (3.2) and (3.38) we get
\[ \frac{d}{dt} \|ϕ\|_E^2 + 2β\|(-Δ)^{δ}u\|_E^2 + 2β\|ϕ\|_E^2 \]
\[ ≤ 2(e^{-\int_0^t G_δ(θ, ω)dr} - e^{-ω(t)})[β(g_1(t, x), u) + α(g_2(t, x), v)] \]
\[ + 2β \int_R^n \left( e^{-\int_0^t G_δ(θ, ω)dr}f(t, x, e^{-\int_0^t G_δ(θ, ω)dr}u_δ) - e^{-ω(t)}f(t, x, e^{ω(t)}u) \right)udx. \] (4.3)

Simple computation yields that
\[ \int_R^n \left( e^{-\int_0^t G_δ(θ, ω)dr}f(t, x, e^{-\int_0^t G_δ(θ, ω)dr}u_δ) - e^{-ω(t)}f(t, x, e^{ω(t)}u) \right)udx \]
\[ = \int_R^n \left( e^{-\int_0^t G_δ(θ, ω)dr}f(t, x, e^{-\int_0^t G_δ(θ, ω)dr}u_δ) - f(t, x, e^{-\int_0^t G_δ(θ, ω)dr}u_δ) \right)udx \]
\[ + \int_R^n \left( e^{-\int_0^t G_δ(θ, ω)dr} - e^{-ω(t)} \right)f(t, x, e^{-\int_0^t G_δ(θ, ω)dr}u)udx \]
By Lemma 2.1, we find that for every \( \varepsilon \in (0, 1) \), there exists \( \delta_1 = \delta_1(\tau, \omega, T, \varepsilon) > 0 \) such that for all \( 0 < |\delta| < \delta_1 \) and \( t \in [\tau, \tau + T] \)

\[
|e^{-\int_0^t G_s(\theta, \omega) \, ds} - e^{-\omega(t)}| < \varepsilon \quad \text{and} \quad |e^{\int_0^t G_s(\theta, \omega) \, ds} - e^{-\omega(t)} - 1| < \varepsilon. \tag{4.5}
\]

By (4.4) and (4.5), we have for all \( 0 < |\delta| < \delta_1 \) and \( t \in [\tau, \tau + T] \)

\[
\int_{\mathbb{R}^n} \left( e^{-\int_0^t G_s(\theta, \omega) \, ds} f(t, x, e^{\int_0^t G_s(\theta, \omega) \, ds} \tilde{u}) - e^{-\omega(t)} f(t, x, e^{\omega(t)} \tilde{u}) \right) \tilde{u} \, dx \\
\leq \| \varphi_3(t) \|_{L_{\infty}} \| \tilde{u} \|_2^2 + 2\varepsilon e^{\int_0^t G_s(\theta, \omega) \, ds} \int_{\mathbb{R}^n} |\tilde{u}|^{p-1} |\tilde{u}| \, dx + \varepsilon \int_{\mathbb{R}^n} \varphi_2(t, x) |\tilde{u}| \, dx \\
+ c \varepsilon \int_{\mathbb{R}^n} \varphi_2(t, x) \left( e^{-\int_0^t G_s(\theta, \omega) \, ds} + e^{\omega(t)} \right) |\tilde{u}|^{p-2} |\tilde{u}|^{p-1} |\tilde{u}| + |\tilde{u}| \, dx \\
\leq c_1 \| \tilde{u} \|_2^2 + c_1 \varepsilon \left( 1 + \| \tilde{u} \|_{L^p}^p + \| \tilde{u} \|_{L^p}^p + \| \varphi_2(t) \|_{L^p}^{p_1} \right). \tag{4.6}
\]

By (4.5) again, we get for all \( 0 < |\delta| < \delta_1 \) and \( t \in [\tau, \tau + T] \)

\[
2(e^{-\int_0^t G_s(\theta, \omega) \, ds} - e^{-\omega(t)}) \left[ \beta(g_1(t, x), \tilde{u}) + \alpha(g_2(t, x), \tilde{u}) \right] \leq \varepsilon \left( \| \tilde{\varphi} \|_E^2 + \| g(t, \cdot) \|_E^2 \right). \tag{4.7}
\]

Therefore (4.3), (4.6) and (4.7) imply that there is a \( c_2 = c_2(\tau, \omega, T) \) such that for all \( \varepsilon \in (0, 1), 0 < |\delta| < \delta_1 \) and \( t \in [\tau, \tau + T] \)

\[
\frac{d}{dt} \| \tilde{\varphi} \|_E^2 \leq c_2 \| \tilde{\varphi} \|_E^2 + c_2 \varepsilon \left( 1 + \| \tilde{u} \|_{L^p}^p + \| \tilde{u} \|_{L^p}^p + \| g(t, \cdot) \|_E^2 + \| \varphi_2(t) \|_{L^p}^{p_1} \right). \tag{4.8}
\]

By Gronwall’s inequality, we have for all \( 0 < |\delta| < \delta_1 \) and \( t \in [\tau, \tau + T] \)

\[
\| \tilde{\varphi}(t) \|_E^2 \leq e^{c_2(t-\tau)} \| \tilde{\varphi}(\tau) \|_E^2 + c_2 \varepsilon e^{c_2(t-\tau)} \int_\tau^t \left( 1 + \| \tilde{u}(s) \|_{L^p}^p + \| g(s, \cdot) \|_E^2 + \| \varphi_2(s, \cdot) \|_{L^p}^{p_1} \right) \, ds. \tag{4.9}
\]

Then by (4.9), (3.10) and Lemma 3.6 we find that there exist \( \delta_2 \in (0, \delta_1) \) and \( c_3 = c_3(\tau, \omega, T) > 0 \) such that for all \( 0 < |\delta| < \delta_2 \) and \( t \in [\tau, \tau + T] \)

\[
\| \varphi_3(t, \tau, \omega, \varphi_3, \varphi_\tau) \|_E^2 \leq e^{c_2(t-\tau)} \| \varphi_3(\tau, \omega, \varphi_\tau) \|_E^2 + c_3 \varepsilon e^{c_2(t-\tau)} \left( 1 + \| \varphi_3(t, \tau, \omega, \varphi_\tau) \|_{L^p}^p + \| \varphi_2(t, \tau, \omega, \varphi_\tau) \|_{L^p}^{p_1} \right) \right). \tag{4.10}
\]

Note that

\[
\varphi_3(t, \tau, \omega, \varphi_3, \varphi_\tau) - \varphi(t, \tau, \omega, \varphi_\tau) = e^{\int_0^t G_s(\theta, \omega) \, ds} \varphi_3(t, \tau, \omega, \varphi_3, \varphi_\tau) - e^{\omega(t)} \varphi(t, \tau, \omega, \varphi_\tau) \\
= e^{\int_0^t G_s(\theta, \omega) \, ds} \left( \varphi_3(t, \tau, \omega, \varphi_3, \varphi_\tau) - \varphi(t, \tau, \omega, \varphi_\tau) \right) \\
+ \left( e^{\int_0^t G_s(\theta, \omega) \, ds} - e^{\omega(t)} \right) \varphi(t, \tau, \omega, \varphi_\tau), \tag{4.11}
\]

where \( \varphi_3(t, \tau, \omega, \varphi_3, \varphi_\tau) \) and \( \varphi(t, \tau, \omega, \varphi_\tau) \) are \( \varphi_3(t, \tau, \omega, \varphi_3, \varphi_\tau) \). It follows from and (4.11) that there exist \( \delta_3 \in (0, \delta_2) \) and \( c_4 = c_4(\tau, \omega, T) > 0 \) such that for all \( 0 < |\delta| < \delta_3 \) and \( t \in [\tau, \tau + T] \)

\[
\| \varphi_3(t, \tau, \omega, \varphi_3, \varphi_\tau) - \varphi(t, \tau, \omega, \varphi_\tau) \|_E^2 \leq c_4 \| \varphi_3(t, \tau, \omega, \varphi_3, \varphi_\tau) - \varphi(t, \tau, \omega, \varphi_\tau) \|_E^2.
\]
\[ + c_4 \int_0^t G_\delta(t, \omega) d\omega(t) - \omega(t) - 1^2 \| \tilde{\varphi}(t, \tau, \omega, \varphi_{\tau}) \|^2 \|_E, \]

which along with (4.5), (3.10) and (4.10) implies (4.2). \[ \Box \]

Finally, we present the upper semicontinuity of random attractors as \( \delta \to 0 \).

**Theorem 4.2.** Suppose (2.2)-(2.4), (2.15)-(2.17), (2.43) and (4.1) hold. Then for every fixed \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[ \lim_{\delta \to 0} d_E(\mathcal{A}_\delta(\tau, \omega), \mathcal{A}_0(\tau, \omega)) = 0. \]  

**Proof.** Let \( \delta \to 0 \) and \( u_{\delta_n, \tau} \to u_\tau \) in \( E \). Then by Lemma 4.1, we find that for all \( \tau \in \mathbb{R}, t \geq 0 \) and \( \omega \in \Omega \),

\[ \Phi_{\delta_n}(t, \tau, \omega, \varphi_{\delta_n, \tau}) \to \Phi_0(t, \tau, \omega, \varphi_\tau) \]  

in \( E \). (4.13)

By Lemma 3.8 we have for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \)

\[ \lim_{\delta \to 0} \| K_\delta(\tau, \omega) \| \leq K_0(\tau, \omega), \]  

which along with (4.13) and Lemma 3.9 shows that all conditions (2.5) and (2.7)-(2.8) in Proposition 2.2 in [40] are fulfilled, and thus (4.12) follows. \[ \Box \]

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