CIRCLE-VALUED MORSE THEORY FOR FRAME SPUN KNOTS AND SURFACE-LINKS

HISAAKI ENDO AND ANDREI PAJITNOV

ABSTRACT. Let $N^k \subset S^{k+2}$ be a closed oriented submanifold, denote its complement by $C(N) = S^{k+2} \setminus N$. Denote by $\xi \in H^1(C(N))$ the class dual to $N$. The Morse-Novikov number of $C(N)$ is by definition the minimal possible number of critical points of a regular Morse map $C(N) \to S^1$ belonging to $\xi$. In the first part of this paper we study the case when $N$ is the twist frame spun knot associated to an $m$-knot $K$. We obtain a formula which relates the Morse-Novikov numbers of $N$ and $K$ and generalizes the classical results of D. Roseman and E.C. Zeeman about fibrations of spun knots. In the second part we apply the obtained results to the computation of Morse-Novikov numbers of surface-links in 4-sphere.

CONTENTS

1. Introduction 1
2. Twist frame spun knots 3
3. Rotation 5
4. Surface-links 9
5. Acknowledgements 12
References 12

1. Introduction

1.1. Overview of the article. Let $N^k \subset S^{k+2}$ be a closed oriented submanifold, let $C(N) = S^{k+2} \setminus N$ be its complement. The orientation of $N$ determines a cohomology class $\xi \in H^1(C(N)) \cong [C(N), S^1]$. We say that $N$ is fibred if there is a Morse map $f : C(N) \to S^1$ homotopic to $\xi$ which is regular nearby $N$ (see Def [1.1]) and has no critical points. In general a Morse map $C(N) \to S^1$ has some critical points, the minimal number of these critical points will be called the Morse-Novikov number of $N$ and denoted $\text{MN}(C(N))$.

In the first part of this paper we study this invariant in relation with constructions of spinning. The classical Artin’s spinning construction [2] associates to each knot $K \subset S^3$ a 2-knot $S(K) \subset S^4$. A twisted version of this construction is due to E.C. Zeeman [14]. In [12] D. Roseman introduced a frame spinning construction, and G. Friedman [4] gave a twisted version of generalized Roseman’s construction to include twisting.

The input data for twist frame spinning construction is:

(TFS1) A closed manifold $M^k \subset S^{m+k}$ with trivial (and framed ) normal bundle.

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An $m$-knot $K^m \subset S^{m+2}$.

A smooth map $\lambda : M \to S^1$.

To these data one associates an $n$-knot $\sigma(M, K, \lambda)$, where $n = k + m$ (see Section 2). We prove in Section 2 the following formula:

$$\mathcal{MN}(C(\sigma(M, K, \lambda))) \leq \mathcal{MN}(C(K)) \cdot \mathcal{MN}(M, [\lambda])$$

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Definition 1.2. The minimal number \( m(f) \) where \( f : C(N) \to S^1 \) is a regular Morse map is called the Morse-Novikov number of \( N \) and denoted by \( \mathcal{MN}(C(N)) \).

To obtain lower bounds for numbers \( m_p(f) \) one uses the Novikov homology. Let \( L = \mathbb{Z}[t, t^{-1}] \); denote by \( \hat{L} = \mathbb{Z}((t)) \) and \( \hat{L}_\mathbb{Q} = \mathbb{Q}((t)) \) the rings of all series in one variable \( t \) with integer (respectively rational) coefficients and finite negative part. Recall that \( \hat{L} \) is a PID, and \( \hat{L}_\mathbb{Q} \) is a field. Consider the infinite cyclic covering \( \hat{C}(N) \to C(N) \); the Novikov homology of \( C(N) \) is defined as follows:

\[
\hat{H}_*(C(N)) = H_*(\hat{C}(N)) \otimes \hat{L}.
\]

The rank and torsion number of the \( \hat{L} \)-module \( \hat{H}_*(C(N)) \) will be denoted by \( \hat{b}_k(C(N)) \), respectively \( \hat{q}_k(C(N)) \). For any regular Morse function \( f \) there is a Novikov complex \( \mathcal{N}_*(f, v) \) over \( \hat{L} \) generated in degree \( k \) by critical points of \( f \) of index \( k \) and such that \( H_*(\mathcal{N}_*(f, v)) \approx \hat{H}_*(C(N)) \). Therefore we have the Novikov inequalities

\[
\sum_k \left( \hat{b}_k(C(N)) + \hat{q}_k(C(N)) + \hat{q}_{k-1}(C(N)) \right) \leq \mathcal{MN}(C(N)).
\]

These inequalities, which are far from being exact in general, are however very useful in particular in the case of surface-links (see Section 4).

2. Twist frame spun knots

We start with a recollection of the twist frame spinning construction following \([12], [5], [4]\). See the input data (TFS1) – (TFS3) for this construction on the page 1. Let \( a \in K \). Removing a small open disk \( D(a) \) from \( S^{m+2} \) we obtain an embedded (knotted) disk \( K_0 \) in the disk \( D^{m+2} \approx S^{m+2} \setminus D(a) \). We identify \( D^{m+2} \) with the standard Euclidean disk of radius 1 and center 0 in \( \mathbb{R}^{m+2} \). We have the usual diffeomorphism

\[
\chi : S^{m+1} \times [0, 1[ \to D^{m+2} \setminus \{0\}, \quad (x, t) \to tx.
\]

We can assume that \( K_0 \cap \partial D^{m+2} \) is the standard sphere \( S^{m-1} \) in \( \partial D^{m+2} = S^{m+1} \). Moreover, we can assume that the intersection of \( K_0 \) with a neighbourhood of \( \partial D^{m+2} \) is also standard, that is,

\[
K_0 \cap \chi(S^{m+1} \times [1 - \epsilon, 1]) = \chi(S^{m-1} \times [1 - \epsilon, 1]).
\]

We have a framing of \( M \) in \( S^n \); combining this with the standard framing of \( S^n \) in \( S^{n+2} \) we obtain a diffeomorphism

\[
\Phi : N(M, S^{n+2}) \to M \times D^m \times D^2
\]

where \( N(M, S^{n+2}) \) is a regular neighbourhood of \( M \) in \( S^{n+2} \). We can assume that the restriction of \( \Phi \) to \( N(M, S^n) \) gives a diffeomorphism

\[
\Phi : N(M, S^n) \to M \times D^m \times \{0\},
\]

induced by the given framing of \( M \). The Euclidean disc \( D^{m+2} \) is a subset of \( D^m \times D^2 \), so that \( K_0 \subset D^m \times D^2 \).

For \( \theta \in S^1 \) denote by \( R_\theta \) the rotation of \( D^2 \) around its center. The disc \( D^{m+2} \subset D^m \times D^2 \) is invariant with respect to this rotation as well as the intersection
of $K_0$ with a small neighbourhood of $\partial D^{m+2}$. We have $\Phi(S^n \cap N(M, S^{n+2})) = M \times D^m \times \{0\}$. Let

$$Z = \{(x, y, z) \mid (y, z) \in R_{\lambda(x)}(K_0)\}.$$ 

This is an $m$-dimensional submanifold of $M \times D^m \times D^2$. We define $\sigma(M, K, \lambda)$ as follows

$$\sigma(M, K, \lambda) = \left(S^{n+2} \setminus N(M, S^{n+2})\right) \cup \Phi^{-1}(Z).$$

This is the image of an embedded $n$-sphere, knotted in general.

**Examples and particular cases.**

1) Let $\dim M = 0$, so that $M$ is a finite set; denote by $p$ its cardinality. Then the $n$-knot $\sigma(M, K, \lambda)$ is equivalent to the connected sum of $p$ copies of $K$.

2) If $M$ is the equatorial circle of the sphere $S^2$, which is in turn considered as an equatorial sphere of $S^4$, and $\lambda(x) = 1$, we obtain the classical Artin’s construction. If $\lambda : S^1 \to S^1$ is a map of degree $d$, we obtain the Zeeman’s twist-spinning construction [13].

3) If $\lambda(x) = 1$ for all $x \in M$ we obtain the Roseman’s construction of spinning around the manifold $M$ [12]. In this case we will denote $\sigma(M, K, \lambda)$ by $\sigma(M, K)$.

**Theorem 2.1.**

$$\mathcal{MN}(\sigma(M, K, \lambda)) \leq \mathcal{MN}(K) \cdot \mathcal{MN}(M, [\lambda]).$$

(where $[\lambda] \in H^1(M, \mathbb{Z}) \approx [M, S^1]$ is the homotopy class of $\lambda$).

**Proof.** We will be using the terminology from the above construction of $\sigma(M, K, \lambda)$. We have the standard fibration

$$\psi_0 : S^{n+2} \setminus S^n \to S^1$$

obtained from the canonical framing of $S^n$ in $S^{n+2}$. Observe that the map $\alpha = \psi_0 \circ \Phi^{-1}$ is defined by the following formula

$$\alpha(x, y, z) = \frac{z}{|z|}.$$ 

Let $f : S^{m+2}, K \to S^1$ be a Morse map. The restriction of $f$ to the subset $D^{m+2}, K_0$ will be denoted by the same letter $f$. We can assume that the function $f$ equals $\alpha$ in a neighbourhood of $\partial D^{m+2} = S^{m+1}$. In particular in a neighbourhood of $\partial D^{m+2}$ we have

$$f(R_0(p)) = f(p) + \theta, \quad \text{for } p \in S^{m+1} \setminus K_0.$$ 

Define a function $g$ on $M \times D^{m+2} \setminus Z$ by the following formula:

$$g(x, \xi) = f(R_{-\lambda(x)}(\xi)) + \lambda(x),$$

(where $x \in M$, $\xi \in D^{m+2}$). Define a function $\psi$ on the complement $S^{n+2} \setminus \sigma$ by the following formula:

1) If $p \notin N(M, S^{n+2})$, then $\psi(p) = \psi_0(p)$.

2) If $p \in N(M, S^{n+2})$, then $\psi(p) = g(\Phi^{-1}(p))$.

We will now prove that if $\lambda$ is a Morse map (this can be achieved by a small perturbation of $\lambda$), then $\psi$ is also a Morse map, and the number $m(\psi)$ of its critical points satisfy

$$m(\psi) = m(\lambda) \cdot m(f).$$
All the critical points of $\psi$ are in $N(M, S^{n+2})$. In this domain the function $\psi$ is diffeomorphic to $g$, and the count of critical points of $g$ is easily achieved with the help of the next lemma.

**Lemma 2.2.** Let $g_1 : N_1 \to S^1$, $g_2 : N_2 \to S^1$ be Morse functions on manifolds $N_1, N_2$. Let $F : N_1 \times N_2 \to N_2$ be a map, such that for each $a \in N_2$ the map $x \mapsto F(a, x)$ is a diffeomorphism $N_2 \to N_2$. Define a function $g : N_1 \times N_2 \to S^1$ by the following formula:

$$g(x_1, x_2) = g_1(x_1) + g_2(F(x_1, x_2)).$$

Then $g$ is a Morse function, $\text{Crit}(g) = \text{Crit}(g_1) \times \text{Crit}(g_2)$ and for every $a_1 \in \text{Crit}(g_1)$, $a_2 \in \text{Crit}(g_2)$ we have $\text{ind}(a_1, a_2) = \text{ind}(a_1) + \text{ind}(a_2)$.

**Proof.** Define a function $g_0$ on $N_1 \times N_2$ by the following formula

$$g_0(x_1, x_2) = g_1(x_1) + g_2(x_2).$$

The conclusions of our Lemma hold obviously if we replace $g$ by $g_0$ in the statement of the Lemma. Observe now that the function $g$ is diffeomorphic to $g_0$ via the diffeomorphism

$$(x_1, x_2) \mapsto (x_1, F(x_1, x_2)).$$

The lemma follows. \hfill \Box

**Corollary 2.3.** Let $K \subset S^3$ be a classical knot, denote by $S(K)$ the spun knot of $K$. Then

$$\mathcal{MN}(S(K)) \leq 2\mathcal{MN}(K)$$

**Proof.** In this case $M = S^1$ and $|\lambda| = 0$. We have $\mathcal{MN}(S^1, 0) = 2$ and the result follows. \hfill \Box

The classical theorems concerning fibrations of spun knots follow from Theorem 2.1.

**Corollary 2.4.** (D. Roseman [12]) If $K$ is fibred, then $\mathcal{MN}(\sigma(M, K))$ is fibred.

**Proof.** Since $\mathcal{MN}(K) = 0$, Theorem 2.1 implies $\mathcal{MN}(\sigma(M, K)) = 0$. \hfill \Box

**Corollary 2.5.** (E.C. Zeeman [13]) The $d$-twist spun knot of any classical knot $K$ is fibred for $d \geq 1$.

**Proof.** Consider a great circle $\Sigma$ in $S^2$. The $d$-twist spun knot of $K$ is by definition the $n + 1$-knot $\sigma(\Sigma, K, \lambda)$ in $S^3$ where $\Sigma \to \Sigma$ is a map of degree $d$. The assertion follows, since $\mathcal{MN}(S^1, \lambda) = 0$. \hfill \Box

**Remark 2.6.** The Zeeman’s theorem above generalizes immediately to the following statement: If $\mathcal{MN}(M, \lambda) = 0$, then the knot $\sigma(M, K, \lambda)$ is fibred for any knot $K$.

3. Rotation

Let $\Sigma$ be an equatorial sphere of $S^{n+1}$. We can view the sphere $S^{n+1}$ as the union of two discs $D_+ \cup D_-$ intersecting by $\Sigma$. Consider $S^{n+2}$ as the equatorial sphere of $S^{n+2}$. The sphere $S^{n+2}$ can be considered as the result of rotation of the disc $D_+$ around its boundary $\Sigma$. We have the (linear orthogonal) action of $S^1$ on $S^{n+2}$, such that $\Sigma$ is the fixed point set of the action, and the action is free on the rest of the sphere $S^{n+2}$. Let $K^{n-1}$ be an $(n-1)$-knot in $S^{n+1}$. We can assume
that $K^{n-1} \subset \text{Int } D_+$. Rotation of $K^{n-1}$ around $\Sigma$ gives a submanifold $R(K)$ of codimension 2 in $S^{n+2}$. The manifold $R(K)$ is diffeomorphic to $S^1 \times K$. We call this construction rotation. When $\dim K = 1$, the manifold $R(K)$ is sometimes called the spun torus of $K$.

In this section we relate the Morse-Novikov numbers of $R(K)$ with those of $K$. The main aim of this section is to prove the following theorem.

**Theorem 3.1.**

$$\mathcal{M}\mathcal{N}(R(K)) \leq 2\mathcal{M}\mathcal{N}(K) + 2.$$

To prove the theorem we associate to each given regular Morse function $\phi : S^{n+1} \setminus K^{n-1} \to S^1$ a regular Morse function $R(\phi) : S^{n+2} \setminus R(K^{n-1}) \to S^1$ such that $m(R(\phi)) = 2m(\phi) + 2$.

We begin by an outline of this construction for the simplest case when $n = 1$ and $K$ consists of two points in $S^2$ (Subsection 3.1). In Subsection 3.2 we give a detailed proof of the assertion of the theorem in full generality.

### 3.1. Rotation of $S^0$.

Let $K^0 = \{a, b\} \subset S^2$. The manifold $S^2 \setminus \{a, b\}$ is fibered over $S^1$, and the structure of the level lines of this fibration is shown on the figure 1 (left).

Let $D_-$ be a small 2-disc around any regular point $a$ of $f$. Denote by $D_+$ the complement $S^3 \setminus \text{Int } D_-$, so that $S^3 = D_+ \cup D_-$ and the discs $D_+$ intersect by their common boundary $\Sigma$. Removing $D_-$ we obtain a map $f : D_+ \setminus \{a, b\} \to S^1$; the structure of its level lines is shown on the figure 1 (middle).

The restriction $f \mid \Sigma$ has two non-degenerate critical points: $N$ and $S$. The vector $v$ in the figure depicts the gradient of the map $f$. Applying the rotation construction to $K_0$ we obtain a trivial 2-component link $R(K^0)$ in $S^3$. Let $F_0 : S^3 \setminus R(K_0) \to S^1$ be the unique $S^1$-invariant function such that $F_0 \mid D_+ = f$. This function is continuous, but not smooth, since its level surfaces have conical singularities in the points of $\Sigma$. To repair this, we will modify the function $f$ in a neighbourhood of $\Sigma$ so that the level lines of the modified function $g : D_+ \setminus \{a, b\} \to S^1$ are as depicted on the figure 1 (right).

Each non-singular level line intersecting $\Sigma$ is orthogonal to $\Sigma$ at the intersection point. Let $G_0 : S^3 \setminus R(K_0) \to S^1$ be the unique $S^1$-invariant function such that $G_0 \mid D_+ = g$. Then $G_0$ is a $C^\infty$ function having two critical points $N$ and $S$. Observe that the descending disc of the critical point $S$ of the function $G_0 \mid \Sigma$ is
in $\Sigma$, therefore the descending discs of $G_0$ will have the same dimension 1, and $\text{ind}_{G_0} S = 1$. The same reasoning holds for the ascending disc of the critical points $N$, therefore $\text{ind}_{G_0} N = 2$.

### 3.2. The general case.

Let $\Sigma$ be the unit sphere in $\mathbb{R}^{n+2}$, that is,

$$\Sigma = \{(x_0, \ldots, x_{n+1}) \mid x_0^2 + \ldots + x_{n+1}^2 = 1\}.$$ 

Denote by $\Sigma'$ its intersection with the hyperplane $x_{n+1} = 0$. Let $a = (0, \ldots, 0, 1)$; for each point $z \in \Sigma'$ denote by $C(z)$ the great circle through $a, -a, z$, and by $C'(z)$ the closed semicircle containing these three points. The projection $p$ onto the $(n+1)$-th coordinate gives the bijection of $C'(z)$ onto the closed interval $[-1, 1]$; this bijection is a diffeomorphism when restricted to $C'(z) \setminus\{a, -a\}$. Let $\beta : [-1, 1] \to [-1, 1]$ be a diffeomorphism such that $\beta(x) = x$ for $x$ in a neighbourhood of $\pm 1$. Then there is a unique diffeomorphism $\bar{\beta}$ of $\Sigma$ onto itself such that for every $z$ the curve $C'(z)$ is $\bar{\beta}$-invariant and $p(\bar{\beta}(v)) = \bar{\beta}(p(v))$ for every $v$. The diffeomorphism $\bar{\beta}$ will be called the sliding, associated to $\beta$. Observe that every sliding is isotopic to the identity map.

Let $D_\rho \subset \Sigma$ be the geodesic disc of radius $\rho$ centered in $-a$. Let $D_- = D_{\pi/2} = \{(x_0, \ldots, x_{n+1}) \mid x_{n+1} \leq 0\}$, $D_+ = \{(x_0, \ldots, x_{n+1}) \mid x_{n+1} \geq 0\}$. Put $\Sigma_\rho = \partial D_\rho$. Let $N(\Sigma_\rho, \epsilon)$ denote the geodesic tubular neighbourhood of $\Sigma_\rho$. For a given $\rho$ and $\epsilon > 0$ sufficiently small there is a sliding $\sigma$ sending $D_\rho$ to $D_-$ and sending each normal geodesic segment of length $2\epsilon$ in $N(\Sigma_\rho, \epsilon)$ isometrically to the corresponding normal geodesic segment in $N(\Sigma, \epsilon)$. We have therefore a commutative diagram

$$N(\Sigma_\rho, \epsilon) \xrightarrow{\sigma} N(\Sigma, \epsilon)$$

$$\Phi \downarrow \quad \quad \downarrow \Psi$$

$$\Sigma_\rho \times [-\epsilon, \epsilon] \xrightarrow{\bar{\sigma}} \Sigma \times [-\epsilon, \epsilon]$$

where the vertical arrows are diffeomorphisms and $\bar{\sigma}(x, \tau) = (\sigma(x), \tau)$.

Let $K$ be an $(n-1)$-knot in $S^{n+1}$ and $\phi : S^{n+1} \setminus K \to S^1$ a Morse map. We can assume that

1) $K \subset \text{Int } D_+$,

2) $-a \notin \text{Crit } f$,

3) the submanifold $\phi^{-1}(\phi(-a))$ is tangent to the hyperplane defined by the equation $x_n = 0$.

The restriction $\bar{\phi} = \phi \mid \partial D_\rho$ can be considered as a real-valued Morse map. Choosing $\rho$ sufficiently small we can assume that $\phi$ on $\Sigma_\rho$ is a Morse map having one maximum and one minimum. Denote the function $\phi \circ \Phi$ by $h : \Sigma_\rho \times [-\epsilon, \epsilon] \to \mathbb{R}$. For $\rho$ sufficiently small, this function has the following property:

$$\text{If } \frac{\partial h}{\partial t}(x, t) = 0, \text{ then } \frac{\partial h}{\partial x}(x, t) \equiv 0, \text{ where } x \in \Sigma_\rho, \ t \in [-\epsilon, \epsilon].$$

Consider the restriction of $\phi$ to the subset $S^{n+1} \setminus (K \cup D_\rho)$. Composing $\phi$ with $\sigma^{-1}$ we obtain a function

$$\phi_0 : D_+ \setminus \sigma(K) \to S^1.$$
This is a Morse map which extends to a geodesic tubular neighbourhood of $\Sigma = \partial D^+$, and can be considered as a real-valued Morse function in this neighbourhood. The restriction $\phi_0 \mid \Sigma$ has two critical points of indices $n$ and 0. Denote these critical points by $N$ and $S$, so that $\text{ind} \phi_0 N = n$, $\text{ind} \phi_0 S = 0$. The function $h_0 = \phi_0 \circ \Psi$ has the following property:

\begin{equation}
\label{eq:7}
\text{If } \frac{\partial h_0}{\partial t}(x, t) = 0, \text{ then } \frac{\partial h_0}{\partial x}(x, t) \neq 0, \text{ where } x \in \Sigma, \ t \in (-\epsilon, \epsilon].
\end{equation}

Now we will modify the function $\phi_0$ nearby $\Sigma$. Let $\lambda : [-\epsilon, \epsilon] \to \mathbb{R}$ be a $C^\infty$ function such that $\lambda(t) = |t|$ for $t$ in a neighbourhood of $[-\epsilon, \epsilon]$ and $\lambda(t) = t^2$ for $|t| \leq \epsilon/2$. Define a function $h_1$ by the following formula:

$$h_1(x, t) = h_0(x, \lambda(t)),$$

and define a function

$$\phi_1 : D_+ \setminus \sigma(K) \to S^1$$

as follows:

1) if $v \notin N(\Sigma, \epsilon)$, put $\phi_1(v) = \phi_0(v)$.
2) if $v \in N(\Sigma, \epsilon)$, $v = \Psi(x, t)$ with $x \in \Sigma$, $t \in (-\epsilon, \epsilon]$, put $\phi_1(v) = h_1(x, t)$.

**Proposition 3.2.** The function $\phi_1$ has two critical points in $N(\Sigma, \epsilon)$, namely $N$ and $S$. Their indices are equal, respectively, to $n$ and 1.

**Proof.** The partial derivatives of $h_1$ are equal to $\frac{\partial h_0}{\partial x}(x, \lambda(t))(x, t)$ and $\frac{\partial h_0}{\partial t}(x, \lambda(t)) \cdot \lambda'(t)$. For $t = 0$ the second derivative equals 0, and $\frac{\partial h_0}{\partial x}(x, \lambda(t))(x, 0)$ vanishes in $N$ and $S$. If $t \neq 0$, then $\lambda'(t) \neq 0$. and for $(x, t)$ to be a critical point of $\phi_1$ it is necessary that $\frac{\partial h_0}{\partial x}(x, \lambda(t))(x, t)$ vanish, which implies that $\frac{\partial h_0}{\partial t}(x, \lambda(t))(x, t) \neq 0$ (see the property (7)).

Now we are ready to construct a Morse function on the complement to $R(K)$. Observe that the knot $K$ is equivalent to the knot $\sigma(K)$. By a certain abuse of notation we will replace $\sigma(K)$ by $K$, so in particular, $K \subset \text{Int} D_+$. Add one more coordinate $x_{n+2}$ and consider the sphere

$$\Sigma = \{(x_0, \ldots, x_{n+2}) \mid x_0^2 + \cdots + x_{n+2}^2 = 1\}.$$

We have $D_+ \subset S^{n+2}$. The knot $R(K)$ is defined by the following formula:

$$R(K) = \{(x_0, \ldots, x_{n+2}) \mid (x_0, \ldots, x_n, \sqrt{x_{n+1}^2 + x_{n+2}^2}) \subset K\}.$$

The circle $S^1$ acts on $S^{n+2}$ by rotations in the two last coordinates. Define the Morse function $\phi_2$ on the complement to $R(K)$ by the two following properties:

1) $\phi_2 \mid D_+ \setminus K = \phi_1$.
2) $\phi_2$ is $S^1$-invariant.

The second property implies that

$$\phi_2(x_0, \ldots, x_{n+2}) = \phi_1\left(x_0, \ldots, x_n, \sqrt{x_{n+1}^2 + x_{n+2}^2}\right).$$

Observe that the property 2) of the function $\phi_1$ guarantees that $\phi_2$ is $C^\infty$ on the subset $S^{n+2} \setminus R(K)$.

**Proposition 3.3.**

1) $\text{Crit}(\phi_2) = S^1 \cdot \text{Crit}(\phi_1) \cup \{N, S\}$.
2) The critical points \( N \) and \( S \) are non-degenerate, and

\[
\text{ind}_{\phi_2} N = \text{ind}_{\phi_1} N = n, \quad \text{ind}_{\phi_2} S = \text{ind}_{\phi_1} S + 1 = 2.
\]

Proof. The point 1) is easy to deduce from the definition of \( \phi_2 \). As for the indices of the critical points observe that the descending disc of the critical point \( N \) in \( N(\Sigma, \epsilon) \) belongs to the sphere \( \Sigma \) which is fixed by the action of \( S^1 \). Thus the index of \( N \) does not change when we replace \( \phi_1 \) by \( \phi_2 \). A similar argument applies to the ascending disc of \( S \), and this implies the rest of the proposition.

Each critical point of \( \phi_1 \) gives rise to a circle of critical points of \( \phi_2 \). Using the same method, as in the previous work of the authors, we have perturb the function \( \phi_2 \) in a neighbourhood of each of these critical circles, and obtain finally a regular Morse function \( R(\phi) \) on the complement to \( R(K) \) such that

\[
\#\text{Crit}(R(\phi)) = 2\#\text{Crit}(\phi_2) + 2.
\]

This completes the proof of Theorem 3.1. 

\[ \square \]

3.3. 4-thread spinning. In this subsection we give a brief description of one more construction of surface-links. Let \( L \subset S^3 \) be a classical link and \( \phi : S^3 \setminus L \to S^1 \) a Morse map. Let \( p, q \in L \) and let \( \gamma : [0, 1] \to S^3 \) be a \( C^\infty \) curve joining \( p \) and \( q \) and belonging entirely to one of the regular level surfaces \( \phi^{-1}(\lambda) \) of the map \( \phi \). We assume moreover that \( \text{Im} \gamma \cap L = \{p, q\} \), and that \( \gamma'(0) \) and \( \gamma'(1) \) are not tangent to \( L \). Let \( D \) be a small neighbourhood of \( \text{Im} \gamma \) diffeomorphic to a 3-disc. Denote by \( \Sigma \) its boundary. We can assume that \( L \cap \Sigma \) consists of four points and that \( L \) is orthogonal to \( \Sigma \) at each of these points. Denote by \( S^2_0 \) the 2-sphere with 4 points removed. Recall that there is a standard Morse function \( \phi_0 \) on \( S^2_0 \) having 2 critical points of indices 1. We can assume that the restriction of \( \phi \) to \( \Sigma \setminus L \) is diffeomorphic to \( \phi_0 \).

Remove the interior of \( D \) from \( S^3 \) and rotate the remaining manifold \( S^3 \setminus \text{Int} D \) around \( \Sigma \). We obtain the sphere \( S^4 \); the subset which is spun by \( L \setminus \text{Int} D \) during the rotation is an embedded 2-surface in \( S^4 \).

We call this construction 4-thread spinning to distinguish it from the usual spinning, and denote the resulting surface-link by \( S'(L) \). If \( p \) and \( q \) are on different connected components of \( L \), then the number of connected components of \( S'(L) \) is the same as for \( L \). If \( p \) and \( q \) are in different connected components of \( L \), then the number of connected components of \( S'(L) \) equals that of \( L \) increased by 1. Applying the same method as in the Subsection 3.2 we can construct a Morse function \( \bar{\phi} \) on \( S^4 \setminus S'(L) \to S^1 \) such that \( m(\bar{\phi}) = m(\phi) + 2 \).

Corollary 3.4.

\[
\mathcal{MN}(S'(L)) \leq 2\mathcal{MN}(L) + 2.
\]

\[ \square \]

4. Surface-links

In this section we develop circle-valued Morse theory for surface-links.

4.1. Motion pictures and saddle numbers. Let \( F \) be a surface-link, that is, a closed oriented 2-dimensional \( C^\infty \) submanifold of \( S^4 \). We can assume \( F \subset \mathbb{R}^4 \).

Choose a projection \( p \) of \( \mathbb{R}^4 \) onto a line. Assume that the critical points of the function \( p|F \) are non-degenerate. Denote by \( \text{sd}(F) \) the minimal number of saddle points of \( p|F \) over all the projections \( p \).
**Definition 4.1.** A saddle number $sd(F)$ is the minimum of numbers $sd(F')$ where $F'$ ranges over all surface-links ambiantly isotopic to $F$.

The invariant $sd(F)$ is closely related to the *ch-index* of $F$, introduced and studied by K. Yoshikawa in [13]. In particular, we have $sd(F) \leq \text{ch}(F)$. In order to relate the number $sd(F)$ to $M\mathcal{N}(K)$ we will reformulate the definition of the saddle number.

Let $F \subset S^4$ be a surface-link. The equatorial 3-sphere $S^3$ of the standard Euclidean sphere $S^4$ divides $S^4$ into two parts:

$$S^4 = D^4_+ \cup D^4_-, \text{ with } D^4_+ \cap D^4_- = \Sigma^3.$$

We assume that $F$ is included in $\text{Int}(D^4)$ and $F$ does not include the centre of $D^4$. Perturbing the embedding $F \subset D^4$ if necessary, we can assume that the restriction $\rho = r|_F$ of the radius function $r : D^4 \to [0,1]$ is a Morse function. The family $\{(r^{-1}(t), \rho^{-1}(t))\}_{t \in [0,1]}$ of possibly singular links can be drawn as a *motion picture* (see [8], Chapter 8). Each singularity of a link in the family corresponds to a critical point of $\rho$. A critical point of $\rho$ of index 0 (1, 2, respectively) is called a *minimal point* (saddle point, maximal point, respectively) of $\rho$, which is represented by a *minimal band* (saddle band, maximal band, respectively) in (a modification of) the motion picture.

It is clear that the minimal number of the saddle points for all such Morse functions $\rho$ is equal to $sd(F)$.

**Theorem 4.2.** $M\mathcal{N}(F) \leq 2sd(F) + \chi(F) - 2$.

Proof. Since $\rho$ is a Morse function, the manifold $D^4 \setminus \text{Int} N(F)$ admits a handle decomposition with one 0-handle and $m_i(\rho)$ ($i + 1$)-handles for $i \in \{0,1,2\}$ (see [7], and also [6], Proposition 6.2.1).

The exterior $E(F) = S^4 \setminus \text{Int} N(F)$ of $F$ is obtained by attaching a 4-handle $D^4_+ \setminus \text{Int} N(F)$ to $D^4 \setminus \text{Int} N(F)$. Since $D^4 \setminus \text{Int} N(F)$ is connected, there is a 3-handle in $D^4 \setminus \text{Int} N(F)$ which connects $\partial N(F)$ with $\partial D^4_+$. Thus the 3-handle cancels the 4-handle $D^4_+$ (see [9], Section 5). Turning the handlebody upside down, we obtain a dual decomposition of $E(F)$ and a corresponding Morse function $f : E(F) \to \mathbb{R}$ which is constant on $\partial E(F)$ and the following Morse numbers: $m_1(f) = m_2(\rho) - 1$, $m_2(f) = m_1(\rho)$, $m_3(f) = m_0(\rho)$, $m_4(f) = 1$.

Using the argument from work of the second author [10], p. 629, we can deform the real-valued Morse function $f$ to a circle-valued regular function $\phi : E(F) \to S^1$, such that $m_k(f) = m_k(\phi)$ for every $k$. Consider the function $-\phi$, which has one critical point of index 0. Applying the cancellation of this local minimum, we obtain a Morse function $\psi : E(F) \to S^1$ belonging to the class $-\xi$, and such that $m_0(\psi) = 0$, $m_1(\psi) = m_3(f) - 1$, $m_2(\psi) = m_2(f)$, $m_3(\psi) = m_1(f)$, $m_4(\psi) = 0$. Put $g = -\psi$. Then we have

$$m_0(g) = m_4(g) = 0, \text{ } m_1(g) = m_2(\rho) - 1, \text{ } m_2(g) = m_1(\rho), \text{ } m_3(g) = m_0(\rho) - 1.$$  

Observe that $m_0(\rho) - m_1(\rho) + m_2(\rho) = \chi(S^2) = 2$, therefore the total number of critical points of $g$ equals $2m_1(\rho)$. Choosing the function $\rho$ with $m_1(\rho) = sd(F)$ we accomplish the proof. \hfill $\square$

**Corollary 4.3.** Let $K \subset S^4$ be a 2-knot. Then $M\mathcal{N}(C_K) \leq 2sd(K)$.

\hfill $\square$
Proposition 4.4. \textit{Let }$F \subset S^4$\textit{ be the trivial }$k$\textit{-component surface-link. Then }
\[ \mathcal{MN}(F) = 4k - 2 - \chi(F). \]

\textit{Proof}. It is not difficult to show that $\hat{b}_1(C(F)) \geq k - 1$, $\hat{b}_3(C(F)) \geq k - 1$. Therefore for every regular Morse map $f : C(F) \to S^1$ we have $m_1(f) + m_3(f) \geq 2(k - 1)$. Assuming $m_0(f) = m_4(f) = 0$ we have $m_1(f) - m_2(f) + m_3(f) = 2 - \chi(F)$, and $\mathcal{MN}(C(F)) \geq 4k - 2 - \chi(F)$; this lower bound coincides with the upper bound derived from Theorem 4.2. \hfill $\square$

4.2. Spin knots. Let $K$ be a classical knot in $S^3$ denote by $S(K)$ the corresponding spun knot.

Proposition 4.5. \textit{If }$K$\textit{ is a non-fibered knot of tunnel number }$1$\textit{, then }$\mathcal{MN}(S^4 \setminus S(K)) = 4$.

\textit{Proof}. Recall that $\mathcal{MN}(S^4 \setminus S(K)) \leq 2\mathcal{MN}(K)$ (Corollary 2.3). In the paper [10] of the second author it is shown that $\mathcal{MN}(K) \leq 2t(K)$, hence $\mathcal{MN}(S(K)) \leq 4$ by Corollary 2.3. Put $G = \pi_1(S^3 \setminus K)$, then $\pi_1(S^4 \setminus S(K))$; let $H = [G, G]$. Let $f : S^4 \setminus S(K) \to S^1$ be a regular Morse map without minima and maxima. If $m_1(f) = 0$, then a standard Morse-theoretic argument applied to the infinite cyclic cover of $S^4 \setminus S(K)$ implies that $H$ is finitely generated, which is impossible, since $K$ is not fibered. Therefore $m_1(f) \geq 1$, and similarly, $m_3(f) \geq 1$, hence $m_2(f) \geq 2$ and the proposition is proved. \hfill $\square$

4.3. Surface-links of Yoshikawa’s table. Yoshikawa [13] suggested a method for enumerating surface-links. To each surface-link $F$ he associated a natural number $ch(F)$. His methods allowed him to make a list of all (weakly prime) surface-links $F$ with $ch(F) \leq 10$. It is clear from the definition of the invariant $ch(F)$ that we have $sd(F) \leq ch(F)$. In the rest of this section we assume that the reader is familiar with Yoshikawa’s work, and with his terminology. There are 6 two-knots in Yoshikawa’s table, namely

\[ 0_1, 8_1, 9_1, 10_1, 10_2, 10_3. \]

The trivial 2-knot $0_1$ is obviously fibered. The knots $8_1$ and $10_1$ are spun knots of the trefoil knot and respectively of the figure 8 knot, thus both $8_1$ and $10_1$ are fibered by [11].

The case of $9_1$ is more complicated. The saddle number of this 2-knot is 2. Therefore $\mathcal{MN}(9_1) \leq 4$. Using the presentation of the fundamental group of the complement to $9_1$ (see [13]) and Poincaré duality properties it is easy to compute the Novikov numbers of $9_1$. Namely we have $q_1 = 1$, $q_2 = q_3 = 0$. Therefore

\[ 2 \leq \mathcal{MN}(9_1) \leq 4. \]

The 2-knot $10_2$ is the 2-twist-spun knot of the trefoil knot, hence fibered by Zeeman’s theorem [14]. Similarly, $10_3$ is fibered, being the 3-twist spin of the trefoil knot.

The surface-link $6_1^{0,1}$ is the result of spinning of the Hopf link which is fibered (see the left of Figure 2) therefore $\mathcal{MN}(6_1^{0,1}) = 0$.

The surface-link $8_1^{1,1}$ is the spin torus of the Hopf link. Applying Theorem 3.1 we get the upper bound $\mathcal{MN}(8_1^{1,1}) \leq 2$. Computing the Euler charactaristic implis the inverse inequality, so $\mathcal{MN}(8_1^{1,1}) = 2$. 
The same argument applies to the surface-link $10_1^{1,1}$, which is the spun torus of the trefoil knot, see the figure 2 (middle), so that $\mathcal{MN}(10_1^{1,1}) = 2$.

The surface-link $10_1^{0,1}$ is the result of spinning of the link $4_1^2$ which is fibred, therefore $\mathcal{MN}(10_1^{0,1}) = 0$.

The case of the surface-link $F = 10_1^{0,0,1}$ is more complicated. This surface-link is the result of 4-threaded spinning of the connected sum $L$ of two copies of the Hopf link, see Figure 2 (right) and applying Corollary 3.4 we deduce $\mathcal{MN}(F) \leq 2$. The computation of Euler characteristic gives the lower bound 2 for the Morse-Novikov number, thus $\mathcal{MN}(10_1^{0,0,1}) = 2$.

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