THE KATO SMOOTHING EFFECT FOR THE NONLINEAR REGULARIZED SCHRÖDINGER EQUATION ON COMPACT MANIFOLDS

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ABSTRACT. We establish Strichartz estimates for the regularized Schrödinger equation on a two dimensional compact Riemannian manifold without boundary. As a consequence we deduce global existence and uniqueness results for the Cauchy problem for the nonlinear regularized Schrödinger equation and we prove under the geometric control condition the Kato smoothing effect for solutions of this equation in this particular geometries.

1. Introduction and results. This paper is devoted to the study of a smoothing effect for a nonlinear regularized Schrödinger equation on compact manifolds without boundary. In order to formulate the results, we shall begin by recalling some results for Schrödinger equation linking the regularity of solutions and the geometry of domain where these equations are posed.

It is well known that the solution of the linear Schrödinger equation in \( \mathbb{R}^d \)

\[
\begin{aligned}
&i \partial_t u + \Delta u = 0 \\
&u(0, \cdot) = u_0
\end{aligned}
\]  

satisfies the Kato Smoothing effect: For every data \( u_0 \in L^2(\mathbb{R}^d) \), the solution \( u(t, \cdot) \) belongs, for almost all \( t \in \mathbb{R} \), to the local Sobolev space \( H^1_{loc}(\mathbb{R}^d) \). More precisely, for every \( R > 0 \), there exists \( c_R \) such that

\[
\int_{\mathbb{R}} \int_{|x|<R} |(1 - \Delta)^{\frac{1}{4}} u(t, x)|^2 \, dx \, dt \leq c_R \|u_0\|_{L^2(\mathbb{R}^d)}^2.
\]

This property of gain of regularity has been first observed in the case of \( \mathbb{R}^d \) in the works of Constantin-Saut \([11]\), Sjölin \([24]\) and Vega \([26]\) and it was later generalized to different perturbations of the flat Laplacian (see Ben Artzi-Klainerman \([6]\), Ben Artzi-Devinatz \([5]\), Constantin-Saut \([10]\) and Doï \([17, 15]\) ).

In the case of domains with boundary, Burq, Gerard and Tzvetkov \([8]\) proved a local smoothing effect in the exterior domains with non-trapping assumption.

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Doi [16] has proved that, for Schrödinger operators in $\mathbb{R}^d$, the non trapping assumption is necessary for the $H^{d/2}$ smoothing effect. For further results in this direction see Burq [7], Robbiano-Zuily [23], Szeftel [25] and Wang [27].

As a conclusion, we can assert that the smoothing property for Schrödinger equation is equivalent to the non trapping condition.

Furthermore, the non-trapping assumption is also equivalent to the uniform decay of the local energy for the wave equation (see [21],[19]). For the trapping domains, when no such decay is hoped, the idea of stabilization for the wave equation is to add a dissipative term of type $a(x)\partial_t u$ to the equation to force the energy of the solution to decrease uniformly under a microlocal geometric assumption called “geometric control” (Every ray of geometric optics enters the region where the damping term is effective or leak to infinity)(see (4), [22]).

By analogy to the stabilization problem for waves, Aloui ([1],[2]) has introduced the “forced” smoothing effect for Schrödinger equation in bounded domains; it consists to act on the equation to produce some smoothing effects. More precisely, he considered under the geometric control condition (G.C.C) on a bounded domain $\Omega \subset \mathbb{R}^n$

$$
\begin{cases}
  i\partial_t u - \Delta D u + i a(x)(-\Delta D)^{1/2} a(x) u = 0 & \text{in } \mathbb{R}^+ \times \Omega \\
  u(0,.) = u_0 & \text{in } \Omega \\
  u = 0 & \text{on } \mathbb{R}^+ \times \partial \Omega
\end{cases}
$$

the following smoothing effect

$$
\|u\|_{L^2([\varepsilon,T],H^{s+1}_0(\Omega))} \leq c \|u_0\|_{H^s(\Omega)},
$$

where $0 < \varepsilon < T < \infty$, $u_0 \in H^s_0(\Omega)$ (See [2] for the definition of $H^s_0(\Omega)$), $a(x) \in C^\infty(\Omega)$ and $\Delta_D$ is the Dirichlet-Laplace operator on $\Omega$.

On the other hand, Aloui, Khenissy and Robbiano [3] improved this result in exterior domains $\Omega \subset \mathbb{R}^n$ and they proved, for the equation (2) with a source term $f$, the following smoothing effect

$$
\|\chi u\|_{L^2([0,T],H^{s+1}((\Omega)))} \leq C \left( \|u_0\|_{H^s(\Omega)} + \|<x>^\rho f\|_{L^2([0,T],H^{s-1/2}(\Omega))} \right),
$$

where $T > 0$, $s \in (-\frac{1}{2},\frac{1}{2})$, $\rho \in (\frac{1}{2},1]$ and $\chi$ is compactly supported.

Our goal here is to generalize this type of result for the nonlinear Schrödinger equation

$$
\begin{cases}
  i\partial_t u + \Delta u - a(x)(1-\Delta)^{-\frac{1}{2}} a(x) \partial_t u = P'(|u|^2) u & \text{in } [0,+) \times M \\
  u(0,.) = u_0 \in H^1(M),
\end{cases}
$$

where $M$ is a compact Riemannian manifold of dimension 2, without boundary, $\Delta$ is the Laplace-Beltrami operator on $M$ and $P$ is a polynomial function with real coefficients, satisfying $P(0) = 0$ and the following defocusing assumption,

$$
\lim_{r \to +\infty} P'(r) = +\infty.
$$

The first main result of this paper is the Strichartz estimate theorem for the regularized equation (5) without the nonlinear term.

**Theorem 1.1.** Let $M$ be a compact riemannian manifold of dimension $d \geq 1$. Let $(p,q) \in [2,\infty[ \times [2,\infty[$ satisfying the admissibility condition

$$
\frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \quad (p,q) \neq (2,\infty).
$$

$$
\begin{cases}
  \frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \quad (p,q) \neq (2,\infty).
\end{cases}
$$
The solution $v = e^{itA}v_0 \in C([0, +\infty[, L^2(M))$ of
\[
\begin{align*}
\begin{cases}
i\partial_t v + \Delta v + ia(x)(1 - \Delta)^{1/2}a(x)v = 0 & \text{in } [0, +\infty[ \times M \\
v(0,.) = v_0 \in L^2(M)
\end{cases}
\end{align*}
\tag{7}
\]
satisfies, for every finite time interval $I$
\[
\|v\|_{L^p(I,L^q(M))} \leq C(I) \|v_0\|_{H^{1/p}(M)},
\tag{8}
\]
where $A = \Delta + ia(x)(1 - \Delta)^{1/2}a(x)$.

We introduce the following notation.

**Notation.** For any positive $A$ and $B$ the notation $A \lesssim B$ (resp. $A \gtrsim B$) means that there exists a positive constant $c$ such that $A \leq cB$ (resp. $A \geq cB$).

As a consequence of Theorem 1.1, we will show that the system (5) is well-posed in $C([0, +\infty[, H^1(M))$ and therefore we establish the Kato Smoothing effect Theorem for the nonlinear Schrödinger equation (5).

**Theorem 1.2.** Let $(M, g)$ be a Riemannian compact surface.

(i) For every initial data $u_0 \in H^1(M)$, the system (5) admits a unique solution $u \in C(\mathbb{R}_+, H^1(M))$. Moreover, $u \in L^\infty(\mathbb{R}_+, H^1(M))$ and $u \in \cap_{p < \infty} L^p_{loc}(\mathbb{R}_+, L^\infty(M))$.

(ii) Assume that $\omega = \{x \in M, a(x) \neq 0\}$ controls geometrically $M$, i.e. every geodesic of $M$ enters the set $\omega$. For every $T > 0$, there exists a constant $c > 0$ such that
\[
\|u\|_{L^2([0,T], H^\beta_2(M))} \leq c \left(\|u_0\|_{H^1(M)} + \|u_0\|_{H^2(M)}^\beta\right),
\tag{9}
\]
where $\beta = 2d^0(P) - 1$ and if $u$ is solution of (5).

**Remark 1.** 1. Dehman et al. [13] have studied the stabilization property for the Schrödinger equation (5), where the dissipative term is of type $a(x)(1 - \Delta)^{-1}a(x)\partial_t u$, and they proved that solution decreases exponentially in the energy space.

2. The equation (5) displays two energy functionals, namely: the $L^2$ energy (or mass), defined as $\|u(t)\|_{L^2}^2$, and the nonlinear energy or $H^1$ energy given by
\[
\text{Eu}(t) = \int_M |\nabla u(t,x)|^2 dx + \int_M P(|u(t,x)|^2)dx
\tag{10}
\]
by multiplying the equation by $\overline{u}(\text{resp. } \partial_t \overline{u})$, integrating and taking the imaginary (resp. real) part and it easy to check that its unique solution $u$ satisfies the energy identity
\[
\text{Eu}(t_2) - \text{Eu}(t_1) = -2 \int_{t_1}^{t_2} \| (1 - \Delta)^{-\frac{1}{2}} a(x) \partial_t u \|^2 dt.
\tag{11}
\]

3. The solution $v$ of (7) satisfies
\[
\|e^{itA}v_0\|_{L^2(M)} \leq \|v_0\|_{L^2(M)}.
\tag{12}
\]
This can be easily seen by multiplying the equation (7) by $\overline{v}$, integrating and taking the imaginary part.

The rest of the paper is organized as follows: In Section 2, we consider semi-classical problems which will be needed to prove Theorem 1.1. In Section 3, we prove that problem (5) is well posed in $H^1(M)$ which heavily relies on the fixed point method with the result of Strichartz estimates established in Theorem 1.1 rather than Sobolev embedding. In Section 4, we prove, under the “geometric control”
condition and the result of Aloui, Khenissi and Robbiano in [3], the Kato smoothing effect for the linear Schrödinger equation. Finally, we consider the nonlinear term as the source term and we deduce the smoothing effect for the nonlinear Schrödinger equation. Finally, we consider the nonlinear

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effect for the linear Schrödinger equation.

2. Strichartz estimates. In this section, we prove Strichartz estimates with fractional loss of derivatives for the regularized Schrödinger equation on any Riemannian compact manifold which play a crucial role in the whole work.

Let \( \varphi \in C_0^\infty(\mathbb{R}) \). We introduce the operator \( \varphi(h^2\Delta) \) using the Helffer-Sjöstrand formula [12] and refer to [20], [12], or [18] for a complete overview of its properties.

**Definition 2.1.** Let \( \varphi \in C_0^\infty(\mathbb{R}) \). We have

\[
\varphi(h^2\Delta) = -\frac{1}{\pi} \int_\mathbb{C} \overline{\varphi}(z) (z - h^2\Delta)^{-1} dL(z),
\]

where \( dL(z) \) denotes the Lebesgue measure on \( \mathbb{C} \) and \( \varphi \) is an almost analytic extension of \( \varphi \) of degree \( n \),

\[
\varphi(z) = \left( \sum_{k=0}^{n} \varphi^{(k)}(Rez) \frac{(iImz)^k}{k!} \right) \zeta(IImz),
\]

with \( \zeta \in C_0^\infty(\mathbb{R}) \), \( \zeta = 1 \) near zero and \( |\overline{\varphi}(z)| \leq C|Imz|^{n+1} \).

Notice that we can find a simpler construction of almost analytic extensions in the book of Dimassi and Sjöstrand ([14] p.93).

To prove the Theorem 1.1, we need the following proposition of Burq, Gérard and Tzvetkov [9]:

**Proposition 1.** Under the assumptions of Theorem 1.1, for any \( \varphi \in C_0^\infty(\mathbb{R}) \) there exist \( C > 0 \) and \( \alpha > 0 \) such that for any \( h \in [0,1] \) and any interval \( J \) of length \( |J| \leq \alpha h \),

\[
\left( \int_J \|e^{it\Delta} \varphi(h^2\Delta)v_0\|_{L^p_x}^p \, dt \right)^{1/p} \leq C \|v_0\|_{L^2(M)},
\]

(15)

Notice that \( \|v_0\|_{L^2(M)} \) can be replaced in the right side of (15) by \( \|\varphi(h^2\Delta)v_0\|_{L^2(M)} \).

Thanks to Proposition 1 and the decrease of the \( L^2 \) norm (12), we have the following proposition:

**Proposition 2.** Under the assumptions of Theorem 1.1, for any \( \varphi \in C_0^\infty(\mathbb{R}) \) there exists \( \tilde{C} > 0 \) such that for any \( h \in [0,1] \) and any interval \( J = [0, \alpha h] \), \( \alpha > 0 \),

\[
\left( \int_J \|\varphi(h^2\Delta)e^{itA}v_0\|_{L^p_x}^p \, dt \right)^{1/p} \leq \tilde{C} \|v_0\|_{L^2(M)}.
\]

(16)

**Proof.** Consider the regularized Schrödinger equation (7), then the solution \( v \) can be written as

\[
e^{itA}v_0 = e^{it\Delta}v_0 - \int_0^t e^{i(t-\tau)\Delta} B v(\tau) \, d\tau,
\]

(17)

when \( A = \Delta + ia(1 - \Delta)^{1/2}a \) and \( B = a(1 - \Delta)^{1/2}a \).

For any \( \varphi \in C_0^\infty(\mathbb{R}) \) and \( h \in [0,1] \),

\[
\varphi(h^2\Delta)e^{itA}v_0 = e^{it\Delta}\varphi(h^2\Delta)v_0 - \int_0^t e^{i(t-\tau)\Delta} \varphi(h^2\Delta)Be^{irA}v_0 \, d\tau.
\]

(18)
If $J = [0, \alpha h]$, then we have
\[
\|\varphi(h^2\Delta)e^{itA}v_0\|_{L^p(J, L^q(M))} \leq \|e^{it\Delta}\varphi(h^2\Delta)v_0\|_{L^p(J, L^q(M))} \tag{19}
\]
\[
+ \left\| \int_{J} 1_{\tau \leq t} e^{i(t-\tau)\Delta}\varphi(h^2\Delta) B \, v(\tau) \, d\tau \right\|_{L^p(J, L^q(M))}.
\]

Let $\tilde{G}_\tau(t) = 1_{\tau \leq t} e^{i(t-\tau)\Delta}\varphi(h^2\Delta) B \, v(\tau)$. Therefore the Minkowski integral inequality gives
\[
\|\varphi(h^2\Delta)e^{itA}v_0\|_{L^p(J, L^q(M))} \leq \|e^{it\Delta}\varphi(h^2\Delta)v_0\|_{L^p(J, L^q(M))} + \int_{J} \|\tilde{G}_\tau\|_{L^p(J, L^q(M))} d\tau
\]
\[
\leq \|e^{it\Delta}\varphi(h^2\Delta)v_0\|_{L^p(J, L^q(M))} + \int_{J} \|e^{i(t-\tau)\Delta}\varphi(h^2\Delta) B \, v(\tau)\|_{L^p(J, L^q(M))} d\tau.
\]

From Proposition 1, we obtain
\[
\|\varphi(h^2\Delta)e^{itA}v_0\|_{L^p(J, L^q(M))} \lesssim \|v_0\|_{L^2(M)} + \int_{J} \|\varphi(h^2\Delta) B \, v(\tau)\|_{L^2(M)} d\tau
\]
\[
\lesssim \|v_0\|_{L^2(M)} + \frac{1}{h} \|\varphi(h^2\Delta) B \, v(\tau)\|_{L^1(J, L^2(M))}
\]
\[
\lesssim \|v_0\|_{L^2(M)} + \frac{1}{h} \|v(\tau)\|_{L^1(J, L^2(M))}.
\]

Taking advantage of the decay of the $L^2$ norm (12), we have
\[
\|\varphi(h^2\Delta)e^{itA}v_0\|_{L^p(J, L^q(M))} \lesssim \|v_0\|_{L^2(M)} + \frac{1}{h} \|v_0\|_{L^1(J, L^2(M))}.
\]

Using advantage of the length of the $J$, we conclude
\[
\|\varphi(h^2\Delta)e^{itA}v_0\|_{L^p(J, L^q(M))} \lesssim \|v_0\|_{L^2(M)} + \left(\frac{1}{h}\right) \alpha h \|v_0\|_{L^2(M)}
\]
\[
\leq \tilde{C} \|v_0\|_{L^2(M)}.
\]

From the above proposition, we can deduce the following estimates:

**Lemma 2.2.** Under the assumptions of Theorem 1.1, for any $\varphi \in C_0^\infty(\mathbb{R})$, there exist $\tilde{C} > 0$ and $C' > 0$ such that for any $h \in [0, 1]$ and any interval $J = [0, \alpha h]$, $\alpha > 0$,
\[
\|w\|_{L^p(J, L^q(M))} \leq \tilde{C} \|\varphi(h^2\Delta)v_0\|_{L^2(M)} + C' h \|v_0\|_{L^2(M)},
\]
where $w = \varphi(h^2\Delta)v$ and $v = e^{itA}v_0$.

**Proof.** Let $\varphi \in C_0^\infty(\mathbb{R})$ and $\varphi^* \in C_0^\infty(\mathbb{R})$ such that $\varphi^* \varphi = \varphi$. If $w = \varphi(h^2\Delta)v$ where $v = e^{itA}v_0$, then we have
\[
\begin{align*}
\partial_t w + \Delta w + ia(x)(1 - \Delta)^{1/2}a(x)w &= ia(x)(1 - \Delta)^{1/2} [a, \varphi(h^2\Delta)] v \\
+ i \left[ a, \varphi(h^2\Delta) \right] (1 - \Delta)^{1/2}a(x)v.
\end{align*}
\]

Hence, Duhamel’s formula gives
\[
w = e^{itA} \varphi(h^2\Delta)v_0 + \int_0^t e^{i(t-\tau)A} g(\tau) \, d\tau, \tag{21}
\]
where $g = ia(x)(1 - \Delta)^{1/2} [a, \varphi(h^2\Delta)] v + i \left[ a, \varphi(h^2\Delta) \right] (1 - \Delta)^{1/2}a(x)v.$
Since \( w = \varphi^*(h^2 \Delta)w \), then \( w \) can be written as follows
\[
    w = \varphi^*(h^2 \Delta)e^{itA} \varphi(h^2 \Delta)v_0 + \int_0^t \varphi^*(h^2 \Delta)e^{i(t-\tau)A} g(\tau) \, d\tau. \tag{22}
\]

Using estimates (16) and \( J = [0, \alpha h] \), we obtain,
\[
    \|w\|_{L^p(J, L^q(M))} \lesssim \|\varphi(h^2 \Delta)v_0\|_{L^2(M)} + \|g\|_{L^1(J, L^2(M))}.
\]

Since the map \( v \mapsto g \) is bounded on \( L^2 \), then we have
\[
    \|w\|_{L^p(J, L^q(M))} \lesssim \|\varphi(h^2 \Delta)v_0\|_{L^2(M)} + \|v\|_{L^1(J, L^2(M))}.
\]

Taking advantage of the decay of the \( L^2 \) norm of \( v \), we conclude
\[
    \|w\|_{L^p(J, L^q(M))} \lesssim \tilde{C} \|\varphi(h^2 \Delta)v_0\|_{L^2(M)} + C' h \|v_0\|_{L^2(M)}.
\]

**Proof of Theorem 1.1.** Let \( w = \varphi(h^2 \Delta)v \) and \( v = e^{itA}v_0 \). According to the previous lemma, we have
\[
    \|w\|_{L^p(J, L^q(M))} = \|\varphi(h^2 \Delta)e^{itA}v_0\|_{L^p(J, L^q(M))}
    \lesssim \tilde{C} \|\varphi(h^2 \Delta)v_0\|_{L^2(M)} + C' h \|v_0\|_{L^2(M)}.
\]

Writting \( I \) as a union of \( N \) intervals \( J_k \) of length \( \leq \alpha h \) with \( N \lesssim \frac{1}{h} \)
\[
    \int_I \|\varphi(h^2 \Delta)e^{itA}v_0\|_{L^q(M)}^p \, dt \leq \sum_{k=1}^N \int_{J_k} \|\varphi(h^2 \Delta)e^{itA}v_0\|_{L^q(M)}^p \, dt
    \lesssim N \left[ \|\varphi(h^2 \Delta)v_0\|_{L^2(M)} + h\|v_0\|_{L^2(M)} \right]^p.
\]

Then
\[
    \|\varphi(h^2 \Delta)e^{itA}v_0\|_{L^p(I, L^q(M))} \lesssim \frac{1}{h^{1/p}} \|\varphi(h^2 \Delta)v_0\|_{L^2(M)} + h^{1-1/p}\|v_0\|_{L^2(M)}. \tag{23}
\]

Applying Corollary 2.3 [9] with \( m = 2 \), \( P = \Delta \) to \( f = e^{itA}v_0 = v \) and take the \( L^p \) norm for \( t \in I \), we obtain
\[
    \|v\|_{L^p(I, L^q(M))} \lesssim \left( \|v_0\|_{L^2(M)} + \left[ \sum_{k=1}^\infty \|\varphi(2^{-2k}\Delta)v\|_{L^q(M)}^2 \right]^{1/2} \right)_{L^p(I)}.
\]

which by the Minkowski inequality leads to
\[
    \|v\|_{L^p(I, L^q(M))} \lesssim \left( \|v_0\|_{L^2(M)} + \left[ \sum_{k=1}^\infty \|\varphi(2^{-2k}\Delta)e^{itA}v_0\|_{L^q(M)}^2 \right]^{1/2} \right).
\]

Finally, we have by using (23)
\[
    \|v\|_{L^p(I, L^q(M))}
    \lesssim \left( \|v_0\|_{L^2(M)} + \left[ \sum_{k=1}^\infty \left( (2^k/p)^2 \|\varphi(2^{-2k}\Delta)v_0\|_{L^q(M)}^2 + (2^{-k(1-\frac{1}{p})})^2\|v_0\|_{L^2(M)}^2 \right]^{1/2} \right) \right).
\]
Proof. Let \( \| v_0 \|_{L^2(M)} + \left[ \sum_{k=1}^{\infty} (2^{k/p})^2 \| \phi(\frac{2^{-2k} \Delta} t) v_0 \|_{L^2(M)}^2 \right]^{1/2} \)
\[+ \left[ \sum_{k=1}^{\infty} 2^{2k(1-\frac{1}{p})} \| v_0 \|_{L^2(M)}^2 \right]^{1/2} \]
\[\lesssim \left( \| v_0 \|_{L^2(M)} + \| v_0 \|_{H^{1/p}(M)} + \| v_0 \|_{L^2(M)} \right) \]
\[\lesssim \| v_0 \|_{H^{1/p}(M)} \]
Which complete the proof of Theorem 1.1. \( \square \)

As for the regularized Strichartz estimates, we infer estimates for the nonhomogeneous equation from estimates for the homogeneous one.

**Corollary 1.** For every \( T > 0 \) there exists \( C > 0 \) such that
\[
\left\| \int_0^t e^{i(t-\tau)A} f(\tau) \, d\tau \right\|_{L^p([0,T],L^q(M))} \leq C_T \| f \|_{L^1([0,T],H^{1/p}(M))},
\]
where \((p,q), p \geq 2 \) and \( q < \infty \), satisfy (6).

**Proof.** Let \( I = \left\| \int_0^T \mathbf{F}_\tau \, d\tau \right\|_{L^p([0,T],L^q(M))} \).

Using the Minkowski integral inequality and Strichartz estimates (8), we obtain
\[
I \leq \int_0^T \| \mathbf{F}_\tau \|_{L^p([0,T],L^q(M))} \, d\tau
\leq \int_0^T \| e^{i(t-\tau)A} f(\tau) \|_{L^p([0,T],L^q(M))} \, d\tau
\leq \int_0^T \| f(\tau) \|_{H^{1/p}(M)} \, d\tau.
\]
\( \square \)

**Corollary 2.** For \( T \) small enough and for \( \sigma \in [0,1] \), there exists \( C > 0 \)
\[
\| e^{itA} v_0 \|_{L^p([0,T],W^{\sigma,q}(M))} \leq C \| v_0 \|_{H^{\sigma+1/p}(M)},
\]
where \((p,q), p \geq 2 \), and \( q < \infty \) satisfy (6).

Moreover
\[
\| (\Lambda f) \|_{L^p([0,T],W^{\sigma,q}(M))} \leq C \| f \|_{L^1([0,T],H^{\sigma+1/p}(M))},
\]
where \((\Lambda f)(t) = \int_0^t e^{i(t-\tau)A} f(\tau) \, d\tau \), \((p,q), p \geq 2 \) and \( q < \infty \) satisfies (6).

**Proof.** Let \( \sigma \in [0,1] \), we set \( r = (1-\Delta)^{\sigma/2}(e^{itA} v_0) \) then \( r \) satisfies the following system
\[
\begin{cases}
    i \partial r + \Delta r + i a(x)(1-\Delta)^{1/2} a(x) r = H(v) & \text{in } ]0, +\infty[ \times M \\
    r(0,.) = (1-\Delta)^{\sigma/2} v_0 & \text{in } M,
\end{cases}
\]
where \( H(v) = [i a(x)(1-\Delta)^{1/2} a(x), (1-\Delta)^{\sigma/2}] v \). Therefore, Duhamel formula gives
\[
r(t, x) = e^{itA}(1-\Delta)^{\sigma/2} v_0 - i \int_0^t e^{i(t-\tau)A} H(v) \, d\tau.
\]
Then
\[ \|r\|_{L^p([0, T], L^q(M))} \leq \left\| e^{itA}(1 - \Delta)^{\sigma/2}v_0 \right\|_{L^p([0, T], L^q(M))} + \left\| \int_0^t e^{i(t-\tau)A}H(v)\,d\tau \right\|_{L^p([0, T], L^q(M))}. \]

Assume that
\[ K = \left\| \int_0^t e^{i(t-\tau)A}H(v)\,d\tau \right\|_{L^p([0, T], L^q(M))} \]
\[ = \left\| \int_0^T Q_\tau \,d\tau \right\|_{L^p([0, T], L^q(M))}, \]
where \( Q_\tau(t) = 1_{\tau \leq t} e^{i(t-\tau)A}H(v)(\tau) \). Using the Minkowski integral inequality and the Strichartz estimates (8), we have
\[ K = \left\| \int_0^T Q_\tau \,d\tau \right\|_{L^p([0, T], L^q(M))} \leq \int_0^T \left\| Q_\tau \right\|_{L^p([0, T], L^q(M))} \,d\tau, \]
\[ \leq \int_0^T \left\| e^{i(t-\tau)A}H(v)(\tau) \right\|_{L^p([0, T], L^q(M))} \,d\tau \]
\[ \leq \int_0^T \left\| H(v)(\tau) \right\|_{H^{s+1/p}(M)} \,d\tau \]
\[ \leq \int_0^T \left\| v(\tau) \right\|_{H^{s+1/p}(M)} \,d\tau. \]

Then
\[ \|r\|_{L^p([0, T], L^q(M))} \leq \left\| e^{itA}(1 - \Delta)^{\sigma/2}v_0 \right\|_{L^p([0, T], L^q(M))} + \int_0^T \left\| v(\tau) \right\|_{H^{s+1/p}(M)} \,d\tau \]
\[ \lesssim \|v_0\|_{H^{s+1/p}(M)} + T \|v\|_{L^\infty([0, T], H^{s+1/p}(M))} \]
\[ \lesssim \|v_0\|_{H^{s+1/p}(M)} + T \|v_0\|_{H^{s+1/p}(M)} \]
\[ \lesssim \|v_0\|_{H^{s+1/p}(M)}. \]

This complete the proof of (25). Estimate (26) follows from (25) and the Minkowski integral inequality applied in time variable.

3. Well-posedness. In this section, we establish that the Cauchy problem of non-linear Schrödinger equation (5) is globally well posed in \( H^1(M) \).

For the proof of Theorem 1.2 (i) we need the following proposition.

**Proposition 3.** Let \( a = a(x) \in C_\infty(M) \) be a real valued function and consider the system
\[
\begin{cases}
    i\partial_t u + \Delta u - a(x)(1 - \Delta)^{-1/2}a(x)\partial_x u = 0 & \text{in } [0, +\infty[ \times M \\
    u(0, \cdot) = u_0 \in H^s(M),
\end{cases}
\]
where \( s \in \mathbb{R} \), \( M \) is a compact Riemannian manifold of dimension 2, without boundary, \( \Delta \) is the Laplace-Beltrami operator on \( M \).
The operator $\mathcal{J}$ defined by $\mathcal{J}v = (1 + ia(1 - \Delta)^{-1/2})v$ is an isomorphism on $H^s$ and on $L^p$ ($s \in \mathbb{R}, 1 \leq p \leq +\infty$). Then, the system (28) can be written as

$$\begin{align*}
\partial_v - i\Delta v + a(x)(1 - \Delta)^{1/2}a(x)v &= R_0v, \\
v &= \mathcal{J}u, \\
v(0,.) &= v_0 = \mathcal{J}u_0 \in H^s(M),
\end{align*}$$

(29)

where

$$R_0 = -[(1 - \Delta), a](1 - \Delta)^{-\frac{1}{2}}a + i(1 - \Delta)a(1 - \Delta)^{-\frac{1}{2}}a^2(1 - \Delta)^{-\frac{1}{2}}aJ^{-1} + a(1 - \Delta)^{-\frac{1}{2}}aJ^{-1}$$

(30)

is pseudo-differential operator of order 0.

**Proof.** It is easy to check that the system (28) can be written as

$$\begin{align*}
\partial_v - i\Delta v &= R_1v, \\
v &= \mathcal{J}u, \\
v(0,.) &= v_0 = \mathcal{J}u_0 \in H^s(M),
\end{align*}$$

(31)

where $R_1 = -i\Delta + i\Delta J^{-1}$ is pseudo-differential operator of order 1.

Let us also remark that $R_1 = \Delta a(1 - \Delta)^{-1/2}aJ^{-1}$ and $J^{-1} = 1 - ia(1 - \Delta)^{-1/2}aJ^{-1}$. Then we have

$$\begin{align*}
R_1 &= -(1 - \Delta)a(1 - \Delta)^{-\frac{1}{2}}aJ^{-1} + a(1 - \Delta)^{-\frac{1}{2}}aJ^{-1} \\
&= -(1 - \Delta)a(1 - \Delta)^{-\frac{1}{2}}a \\
&\quad + i(1 - \Delta)a(1 - \Delta)^{-\frac{1}{2}}a^2(1 - \Delta)^{-\frac{1}{2}}aJ^{-1} + a(1 - \Delta)^{-\frac{1}{2}}aJ^{-1} \\
&= -[(1 - \Delta), a](1 - \Delta)^{-\frac{1}{2}}a - a(1 - \Delta)^{\frac{3}{2}}a \\
&\quad + i(1 - \Delta)a(1 - \Delta)^{-\frac{1}{2}}a^2(1 - \Delta)^{-\frac{1}{2}}aJ^{-1} + a(1 - \Delta)^{-\frac{1}{2}}aJ^{-1}.
\end{align*}$$

This completes the proof of Proposition 3. \qed

By the work of Aloui [1], the system (29) admits a unique solution in the space $C(\mathbb{R}^+, H^s(M))$ for all $s \in \mathbb{R}$.

**Proof of Theorem 1.2 (i).** The proof is similar to the proof of Proposition 8 in [13] and heavily relies on the fixed point argument with the result of Strichartz estimates (8) and embeddings of Sobolev.

Firstly, using the Proposition 3, the system (5) can be written as

$$\begin{align*}
i\partial_v + \Delta v + ia(x)(1 - \Delta)^{1/2}a(x)v &= P'(|u|^2)u + iR_0v, \\
v &= \mathcal{J}u, \\
v(0,.) &= v_0 = \mathcal{J}u_0 \in H^1(M),
\end{align*}$$

(32)

where $R_0$ is defined in (30).

Therefore, it is enough to prove that the system (32) is well posed in $H^1(M)$.

For $T > 0$, we denote by $Y_T$ the Banach space

$$Y_T = C([0,T], H^1(M)) \cap L^p([0,T], W^{\sigma,q}(M))$$

equipped with the natural norm

$$\|u\|_{Y_T} = \|u\|_{L^\infty([0,T], H^1(M))} + \|u\|_{L^p([0,T], W^{\sigma,q}(M))}.$$

Moreover, we choose $p > max(\beta - 1, 2)$ with $\beta = 2d^*(P) - 1 \geq 3$, $\frac{2}{p} + \frac{2}{q} = 1$, $p < \infty$, and $\sigma = 1 - \frac{1}{p}$.

Since $\sigma > \frac{2}{q}$, then by Sobolev embedding we have $Y_T \subset L^p([0,T], L^\infty(M))$. 
We first show that for sufficiently small \( T > 0 \), the functional
\[
\Phi(v)(t) = e^{itA}v_0 + \int_0^t e^{i(t-\tau)A} (R_0v - iP^\prime(|u|^2)u)(\tau) \, d\tau,
\]
(33)
where \( A = \Delta + ia(x)(1 - \Delta)^{1/2}a(x) \), is a contraction on a some ball of \( Y_T \).

It is easy to verify that
\[
\|\Phi(v)\|_{H^1(M)} \lesssim \|v_0\|_{H^1(M)}
\]
and
\[
\|\Phi(v)\|_{L^p([0,T],W^{1,q}(M))} \leq \|e^{itA}v_0\|_{L^p([0,T],W^{1,q}(M))}
\]
(34)

Using Corollary 2, we have
\[
\|\Phi(v)\|_{L^p([0,T],W^{1,q}(M))} \leq \|e^{itA}v_0\|_{L^p([0,T],W^{1,q}(M))}
\]
(35)

Next, applying the Hölder inequality in the right hand side of inequality, with \( \gamma + \frac{\beta - 1}{p} = 1 \), we deduce that
\[
\|\Phi(v)\|_{Y_T} \lesssim \|v_0\|_{H^1(M)} + \int_0^T \|R_0v\|_{H^1(M)} \, d\tau + \int_0^T \|P'(|u(\tau)|^2)u(\tau)\|_{H^1(M)} \, d\tau.
\]

Similarly,
\[
\|\Phi(v) - \Phi(\tilde{v})\|_{Y_T} \leq C \|v - \tilde{v}\|_{Y_T} (1 + \|v\|_{Y_T} + \|\tilde{v}\|_{Y_T})^{\beta - 1}.
\]
(36)
We choose \( R = C_1 \|v_0\|_{H^1(M)} \), \( C_1 = 2C_0 \), and \( T \) such that
\[
C \, T^\gamma (1 + 2R)^{\beta - 1} < 1 \quad \text{and} \quad T < \left( \frac{1}{C_1 + C_1 \|v_0\|_{H^1}^{\beta - 1}} \right)^{1/\gamma}.
\]
Therefore, \( P \) and as a consequence of Corollary 2, we have \( q < L^\infty \).

Moreover, \( E_v(t) \leq C E_v(0) \),
where \( E_v \) is the energy of \( v \) at time \( t \geq 0 \) which is defined in (10). The decay of energy for \( t \geq 0 \) show that the energy does not blow up in finite time. This allows to extend the solution for all times and thus the solution \( v \) is global in time.

Now, we check that if \( v \in L^\infty([0,T],H^1(M)) \) for all \( T > 0 \), then \( v \in \bigcap_{p<\infty} L^p_{loc}(\mathbb{R}_+,L^\infty(M)) \).

Let \( u \in L^\infty([0,T],H^1(M)) \) solution of (32). By Sobolev embedding, we have \( v \in L^\infty([0,T],L^{r^*}(M)) \) for all \( r^* < \infty \). Moreover, using the Hölder inequality with \( \frac{1}{q^*} = \frac{1}{2} + \frac{1}{2q} \), for all \( q^* < 2 \), we can deduce that \( \dot{P}'(|u|^2)u \in L^\infty([0,T],W^{1,q^*}(M)) \).

Therefore \( \dot{P}'(|u|^2)u \in L^\infty([0,T],H^s(M)) \) for all \( s < 1 \). Since
\[
v(t) = e^{itA}v_0 + \int_0^t e^{i(t-\tau)A} [R_0v - iP'(|u|^2)u] \, d\tau
\]
and as a consequence of Corollary 2, we have
\[
\|v\|_{L^p([0,T],W^{s-\frac{1}{p}-\frac{1}{q^*}}(M))} \lesssim \|v_0\|_{H^s(M)} + \|R_0v - iP'(|u|^2)u\|_{L^1([0,T],H^s(M))}.
\]

Therefore \( v \in L^p([0,T],W^{s-\frac{1}{p}-\frac{1}{q}}(M)), \frac{1}{p} < s < 1 \) and for all \( p, q \) satisfying (6), \( q < \infty \). But
\[
s - \frac{1}{p} - \frac{2}{q} = s - \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{2}{q} = s - \frac{1}{2} - \frac{1}{q} > 0 \text{ if } s \approx 1 \text{ and } q > 2,
\]
then Sobolev estimate gives \( v \in \bigcap_{p<\infty} L^p([0,T],L^\infty(M)) \).

We now turn to the uniqueness issue. Assume that \( v \) and \( \tilde{v} \) are two solution both of (32) such that \( \Phi(v) = v \) and \( \Phi(\tilde{v}) = \tilde{v} \).

\[
\|\Phi(v) - \Phi(\tilde{v})\|_{L^2(M)} \lesssim \int_0^T \|R_0(v - \tilde{v})\|_{L^2(M)} \, d\tau + \int_0^T \|P'(|u|^2)u - P'(|\tilde{u}|^2)\tilde{u}\|_{L^2(M)} \, d\tau
\]
\[
\lesssim \int_0^T \|R_0(v - \tilde{v})\|_{L^2(M)} \, d\tau
\]
\[
+ \|\mathcal{J}^{-1}(v - \tilde{v})\|_{L^\infty([0,T],L^2(M))} \int_0^T \left( 1 + \|\mathcal{J}^{-1}u\|_{L^\infty}^{\beta-1} + \|\mathcal{J}^{-1}\tilde{u}\|_{L^\infty}^{\beta-1} \right) \, d\tau.
\]

Since \( R_0 \) and \( \mathcal{J}^{-1} \) are both operators of order zero, we have
\[
\|\Phi(v) - \Phi(\tilde{v})\|_{L^2(M)} \lesssim \int_0^T \|v - \tilde{v}\|_{L^2(M)} \, d\tau
\]
\[
+ \|v - \tilde{v}\|_{L^\infty([0,T],L^2(M))} \int_0^T \left( 1 + \|v\|_{L^\infty}^{\beta-1} + \|\tilde{v}\|_{L^\infty}^{\beta-1} \right) \, d\tau
\]
\[ \lesssim \|v - \tilde{v}\|_{L^\infty(L^2(M))}^\gamma \left( 1 + \|v\|_{L^p([0,T],L^\infty(M))}^{\frac{\beta-1}{p}} + \|\tilde{v}\|_{L^p([0,T],L^\infty(M))}^{\frac{\beta-1}{p}} \right) \]

with \( \gamma = 1 - \frac{\beta-1}{p} > 0 \). Then

\[ \|v - \tilde{v}\|_{L^\infty([0,T],L^2(M))} \leq C \|v - \tilde{v}\|_{L^\infty(L^2)}^\gamma \left( 1 + \|v\|_{L^p([0,T],L^\infty)} + \|\tilde{v}\|_{L^p([0,T],L^\infty)} \right)^{\frac{\beta-1}{p}}. \]

We take \( T \) small enough such that

\[ C T^\gamma \left( 1 + \|v\|_{L^p([0,T],L^\infty)} + \|\tilde{v}\|_{L^p([0,T],L^\infty)} \right)^{\frac{\beta-1}{p}} < 1. \]

We infer that \( v = \tilde{v} \).

This completes the proof (i) of Theorem 1.2. \( \Box \)

4. The Kato Smoothing effect. This section is devoted to the proof of (9). The first part is devoted to proving the Kato Smoothing effect for the linear equation on local time interval. As a consequence, we establish in the second part the Smoothing effect for the nonlinear Schrödinger equation (5).

4.1. Linear case. In order to state our results, we first recall the definition of Exterior Geometric Control condition [3]:

Let \( K \) be a compact obstacle in \( \mathbb{R}^d \) whose complement \( \Omega \) an open set with \( C^\infty \) boundary \( \partial \Omega \).

**Definition 4.1.** Let \( R > 0 \) be such that \( K \subset B_R = \{ |x| < R \} \) and \( w \) be a subset of \( \Omega \). We say that \( w \) verifies the Exterior Geometric Control condition (E.G.C.) if there exists \( T_R > 0 \) such that every generalized bicharacteristic \( \gamma \) starting from \( B_R \) at time \( t = 0 \), is such that :

\( \star \gamma \) leaves \( \mathbb{R}^+ \times B_R \) before the time \( T_R \), or \( \star \gamma \) meets \( \mathbb{R}^+ \times w \) between the times \( 0 \) and \( T_R \).

Under the assumption of the Exterior Geometric Control condition on \( w = \{ x \in \Omega; \ a^2(x) > 0 \} \), Aloui, Khenissi and Robbiano have proved in [3] the Kato Smoothing effect and the non homegenous bound for the regularized Schrödinger equation in exterior domains \( \Omega \). More precisely, we have the following results [3] :

**Proposition 4.** Let \( T > 0 \), \( s \in (-\frac{1}{2}, 1) \), \( \rho \in (\frac{1}{2}, 1) \) and \( \chi \in C^\infty_c(\mathbb{R}^d) \). Then under the (E.G.C.) on \( w \), there exists a constant \( c > 0 \) such that inequality

\[ \|\chi u\|_{L^2([0,T],H^{s+\frac{1}{2}}(\Omega))} \leq C \left( \|u_0\|_{H^s(\Omega)} + \|u_0\|_{H^s(\Omega)} + \|u_0\|_{H^s(\Omega)} + \|u_0\|_{H^s(\Omega)} \right) \] (35)

holds for every solution \( u(t,x) \) of the system

\[
\begin{cases}
\partial_t u + \Delta u + iBu = f & \text{in } \mathbb{R}^+ \times \Omega \\
u_0(\cdot, \cdot) = u_0 & \text{in } L^2(\Omega),
\end{cases}
\] (36)

where \( B = a(x)(-\Delta)^{1/2}a(x), (u_0,f) \in C^\infty_0(\Omega) \times C^\infty_0(\mathbb{R}^+ \times \Omega) \) and \( a \in C^\infty_0(\Omega) \).

This result remains true in the case of bounded domains (see Remarks 1.3 [3]). So we have the following results in manifolds compact without boundary \( M \):
Corollary 3. Assume that \( w = \{ x \in M, a(x) \neq 0 \} \) controls geometrically \( M \), i.e. every geodesic of \( M \) enters the set \( w \). For every \( T > 0 \), \( s \in \mathbb{R} \), there exists a constant \( c > 0 \) such that inequality
\[
\| u \|_{L^2([0, T], H^{s+\frac{1}{2}}(M))} \leq c \left( \| au \|_{L^2([0, T], H^{s+\frac{1}{2}}(M))} + \| u_0 \|_{H^s(M)} + \| f \|_{L^2([0, T], H^{s-\frac{1}{2}}(M))} \right)
\]  
holds for every solution \( u(t, x) \) of the system
\[
\begin{cases}
  i\partial_t u + \Delta u = f & \text{on } [0, +\infty[ \times M \\
  u(0, \cdot) = u_0 \in H^s(M)
\end{cases}
\]  

Proof. We add on both sides of the linear equation (39) the term \( ia(-\Delta)^{1/2}au \), we multiply by \((1 - \Delta)^{s/2}\) and we let \( v = (1 - \Delta)^{s/2}u \). Then the problem (39) becomes
\[
\begin{cases}
  i\partial_t v + \Delta v + iBv = (1 - \Delta)^{s/2}f + iBv & \text{on } [0, +\infty[ \times M \\
  u(0, \cdot) = u_0 \in H^s(M)
\end{cases}
\]  

From estimate (35) we deduce
\[
\| v \|_{L^2([0, T], H^{\frac{s}{2}}(M))} \leq \| v_0 \|_{L^2(M)} + \| (1 - \Delta)^{s/2}f \|_{L^2([0, T], H^{-\frac{1}{2}}(M))} + \| Bv \|_{L^2([0, T], H^{-\frac{1}{2}}(M))}
\]  
which completes the proof.

Now, we come to the nonlinear case. The key tool is to consider the nonlinear term as the source term in the linear Schrödinger equation.

4.2. Nonlinear case.

Lemma 4.2. Let \( u_0 \in H^s(M) \), \( s \in \mathbb{R} \), \( u \in L^2([0, T], H^{s+\frac{1}{2}}(M)) \), \( f \in L^2([0, T], H^{s-\frac{1}{2}}(M)) \) and \( \epsilon > 0 \). If \( u \) is solution of
\[
\begin{cases}
  i\partial_t u + \Delta u + ia(x)(1 - \Delta)^{1/2} a(x)u = f & \text{on } [0, +\infty[ \times M \\
  u(0, \cdot) = u_0
\end{cases}
\]  
then there exists a constant \( c > 0 \) such that
\[
\| au \|_{L^2([0, T], H^{s+\frac{1}{2}}(M))} \leq c \left( \| u_0 \|_{H^s(M)} + \frac{1}{\epsilon} \| f \|_{L^2([0, T], H^{s-\frac{1}{2}}(M))} + \epsilon \| u \|_{L^2([0, T], H^{s+\frac{1}{2}}(M))} \right).
\]  

Proof. The proof is similar to the one in [1]. In fact, we multiply (41) by \((1 - \Delta)^{s/2}\), we integrate over \([0, T] \times M\) and take the imaginary part. We obtain
\[
\int_0^T \| au \|_{H^{s+\frac{1}{2}}(M)}^2 = \frac{1}{2} \| u(0, \cdot) \|_{H^s(M)}^2 - \| u(T, \cdot) \|_{H^s(M)}^2
\]  
\[
- \text{Re} \left( \int_0^T ((1 - \Delta)^{\frac{s}{2}} a(x)u, (1 - \Delta)^{-\frac{s}{2}} \frac{1}{(x, 1 - \Delta)^{s/2}} u)_{L^2(M)} \right)
\]  
\[
+ \text{Im} \left( \int_0^T (f, (1 - \Delta)^s u)_{L^2(M)} \right).
\]
The notation \( Re \) and \( Im \) stand for the real and imaginary part respectively. Then we have
\[
\left| \int_0^T ((1 - \Delta)^{\frac{1}{2}} a(x) u, (1 - \Delta)^{-\frac{1}{2}} u)_{L^2(M)} \right| \leq c \| u_0 \|^2_{H^s(M)}
\]
and
\[
\left| \int_0^T (f, (1 - \Delta)^{\frac{1}{2}} u)_{L^2(M)} \right| = \left| \int_0^T (f, (1 - \Delta)^{-\frac{1}{2}} f)_{L^2(M)} \right| \\
\leq \frac{1}{2} \left( \frac{1}{\epsilon^2} \| f \|_{L^2([0,T], H^{s-\frac{1}{2}}(M))}^2 + \epsilon^2 \| u \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))}^2 \right) \\
\leq \frac{1}{\epsilon} \left( \| f \|_{L^2([0,T], H^{s-\frac{1}{2}}(M))}^2 + \epsilon \| u \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))}^2 \right). 
\]
Therefore
\[
\| u \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))} \lesssim \| u_0 \|_{H^s(M)} + \frac{1}{\epsilon} \| f \|_{L^2([0,T], H^{s-\frac{1}{2}}(M))} + \epsilon \| u \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))}. 
\]
This completes the proof of Lemma 4.2. \( \square \)

4.2.1. Proof of (9). Firstly we show the result for the equivalent problem (32) then we return to our system (5) to provide the desired result.

Let \( v \) be the solution of (32). By using Corollary 3, we have
\[
\| v \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))} \lesssim \| av \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))} + \| v_0 \|_{H^s(M)} \\
+ \| a(1 - \Delta)^{1/2} v \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))} \\
+ \| P'(|u|^2) u \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))} + \| R_0 v \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))}. 
\]
Then
\[
\| v \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))} \lesssim \| av \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))} + \| v_0 \|_{H^s(M)} \\
+ \| P'(|u|^2) u \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))} + \| R_0 v \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))}. 
\]
By Lemma 4.2, the first term in the right hand side of inequality (42) can be estimated by
\[
\| av \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))} \lesssim \| v_0 \|_{H^s(M)} + \epsilon \| v \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))} \\
+ \frac{1}{\epsilon} \left( \| R_0 v \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))} + \| P'(|u|^2) u \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))} \right). 
\]
Then, we obtain
\[
(1 - \epsilon) \| v \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))} \lesssim \| v_0 \|_{H^s(M)} + \| P'(|u|^2) u \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))} \\
+ \frac{1}{\epsilon} \| v \|_{L^\infty([0,T], H^s(M))}. 
\]
Next we choose \( \epsilon \) small enough, hence we have
\[
\| v \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))} \lesssim \| v_0 \|_{H^s(M)} + \| P'(|u|^2) u \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))}. 
\]
To estimate the second term \( \| P'(|u|^2) u \|_{L^2([0,T], H^{s+\frac{1}{2}}(M))} \), we will use Hölder inequality with \( \frac{1}{q^*} = \frac{1}{2} + \frac{2-q^*}{2q^*}, q^* < 2 \), rather than Sobolev embedding.
For every $2 \leq i \leq \beta - 1$, where $\beta = 2 \deg(P) - 1$, we have

$$\|u^i u\|_{L^2([0,T],H^\frac{1}{2}(M))} \leq \|u^i u\|_{L^2([0,T],W^{1,q^*}(M))}$$

for all $q^* < 2$.

Therefore

$$\|u^i u\|_{L^2([0,T],H^\frac{1}{2}(M))} \leq \|u^i u\|_{L^2([0,T],L^{q^*}(M))} + \|\nabla(|u|^i u)\|_{L^2([0,T],L^{q^*}(M))}. \quad (43)$$

The first term of $(43)$ can be estimated by

$$\|u^i u\|_{L^2([0,T],L^{q^*}(M))} \leq \|u^i\|_{L^2([0,T],L^2(M))} \|u\|_{L^\infty([0,T],L^{\frac{2q^*}{q^*}}(M))}$$

$$\leq \|u^i\|_{L^2([0,T],L^{2\beta^*}(M))} \|u\|_{L^\infty([0,T],H^\beta(M))}$$

$$\leq \|u^i\|_{L^\infty([0,T],H^\beta(M))}$$

and the second term of $(43)$ can be estimated by

$$\|\nabla(|u|^i u)\|_{L^2([0,T],L^{q^*}(M))} \leq \|\nabla(|u|^i u)\|_{L^2([0,T],L^{q^*}(M))} + \||u|^i \nabla u\|_{L^2([0,T],L^{q^*}(M))}.$$

Using the fact that $\nabla(|u|^i u) = 2Re(\nabla u)|u|^{i-2}u$, we obtain

$$\|\nabla(|u|^i u)\|_{L^2([0,T],L^{q^*}(M))} \leq 2\|\nabla u\|_{L^\infty([0,T],L^{2\beta^*}(M))} \||u|^i\|_{L^2([0,T],L^{\frac{2q^*}{q^*}}(M))}$$

$$\leq \|\nabla u\|_{L^\infty([0,T],L^{2\beta^*}(M))} \||u|^i\|_{L^2([0,T],L^{\frac{2q^*}{q^*}}(M))}$$

$$\leq \|u^i\|_{L^\infty([0,T],H^\beta(M))},$$

Consequently,

$$\|P'(|u|^2)u\|_{L^2([0,T],H^\frac{1}{2}(M))} \lesssim \|u\|_{L^\infty([0,T],H^\beta(M))} + \||u|^2\|_{L^\infty([0,T],H^\beta(M))}$$

$$+ \ldots + \||u|^{2\beta - 1}\|_{L^\infty([0,T],H^{\beta - 1}(M))} \left(1 + \||u|^{2\beta - 1}\|_{L^\infty([0,T],H^{\beta - 1}(M))}\right).$$

According to the problem $(32)$ and taking advantage of the decay of the energy $H^1$ $(11)$, we have

$$\|u\|_{L^2([0,T],H^\frac{1}{2}(M))} = \|J^{-1}u\|_{L^2([0,T],H^\frac{1}{2}(M))}$$

$$\lesssim \|v_0\|_{H^1(M)} + \|P'(|u|^2)u\|_{L^2([0,T],H^\frac{1}{2}(M))}$$

$$\lesssim \|u_0\|_{H^1(M)} + \|u_0\|_{H^1(M)}.$$
M. Ben-Artzi and S. Klainerman, Decay and regularity for the Schrödinger equation, *J. Anal. Math.*, 58 (1992), 25–37.

N. Burq, Smoothing effect for Schrödinger boundary value problems, *Duke Math. J.*, 123 (2004), 403–427.

N. Burq, P. Gérard and N. Tzvetkov, On nonlinear Schrödinger equations in exterior domain, *Ann. Inst. H. Poincaré Anal. Non. Linéaire*, 21 (2004), 295–318.

N. Burq, P. Gérard and N. Tzvetkov, Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds, *Amer. J. Math.*, 126 (2004), 569–605.

P. Constantin and J.-C. Saut, Local smoothing properties of dispersive equations, *J. Amer. Math. Soc.*, 1 (1988), 413–439.

P. Constantin and J.-C. Saut, Local smoothing properties of Schrödinger equations, *Indiana Univ. Math. J.*, 38 (1989), 791–810.

B. Dehman, P. Gérard and G. Lebeau, Stabilization and control for the nonlinear Schrödinger equation on a compact surface, *Math. Z.*, 254 (2006), 729–749.

M. Dimassi and J. Sjöstrand, *Spectral Asymptotics in the Semi-Classical Limit*, London Mathematical Society Lecture Note Series, 268. Cambridge University Press, Cambridge, 1999.

S. Doi, Remarks on the Cauchy problem for Schrödinger type equations, *Communications in Partial Differential Equations*, 21 (1996), 163–178.

S. Doi, Smoothing effects for Schrödinger evolution equation and global behaviour of geodesic flow, *Math. Ann.*, 318 (2000), 355–389.

J. V. Ralston, Solutions of the wave equation with localized energy, *Comm. Pure Appl. Math.*, 22 (1969), 807–823.

J. Rauch and M. Taylor, Exponential decay of solutions of hyperbolic equations in bounded domains, *Indiana Univ. Math. J.*, 24 (1974), 79–86.

L. Robbiano and C. Zuily, Microlocal analytic smoothing effect for the Schrödinger equation, *Duke Mathematical Journal*, 100 (1999), 93–129.

P. Sjölin, Regularity of solutions to the Schrödinger equation, *Duke Math. J.*, 55 (1987), 699–715.

J. Szefetl, Microlocal dispersive smoothing for the nonlinear Schrödinger equation, *SIAM Journal on Mathematical Analysis*, 37 (2005), 549–597.

L. Vega, Schrödinger equations: Pointwise convergence to the initial data, *Proc. Amer. Math. Soc.*, 102 (1988), 874–878.

X. P. Wang, Time-decay of scattering solutions and classical trajectories, *Ann. I.H.P. Phys. Théor.*, 47 (1987), 25–37.

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