Linearized Einstein theory via null surfaces

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DRAFT

Recently there has been developed a reformulation of General Relativity - referred to as the null surface version of GR - where instead of the metric field as the basic variable of the theory, families of three-surfaces in a four-manifold become basic. From these surfaces themselves, a conformal metric, conformal to an Einstein metric, can be constructed. A choice of conformal factor turns them into Einstein metrics. The surfaces are then automatically characteristic surfaces of this metric.

In the present paper we explore the linearization of this null surface theory and compare it with the standard linear GR. This allows a better understanding of many of the subtle mathematical issues and sheds light on some of the obscure points of the null surface theory. It furthermore permits a very simple solution generating scheme for the linear theory and the beginning of a perturbation scheme for the full theory.

I. INTRODUCTION

It is our intention to introduce a new point of view to the Einstein theory of gravity. We switch our attention from the metric tensor, conventionally viewed as the fundamental field of the theory, to a different set of quantities that are to be more basic than the metric itself, from which the metric can be extracted as a derived concept. The quantities that are our candidates to serve this purpose are null (hyper)surfaces of the spacetime. Null surfaces encode the information of a Lorentzian metric up to a conformal factor and the surfaces are simply sets of points, that do not carry the elaborate behavior of tensor fields on a manifold, therefore being (perhaps) suitable for a more fundamental description of the spacetime from which additional structure can be derived, if desired. Our idea is to obtain a description of gravity in which we are able to understand Einstein spacetimes in terms of null surfaces. In this sense, we depart from the more established view of thinking of GR as a theory for the metric as a field tensor $g_{ab}$ on a manifold. We refer to this new approach as a “surface theory” of General Relativity.

The essence of the null surface approach to gravity (first introduced in 1983 [1]) is the use of a sphere’s worth of null coordinate systems which allows (up to a conformal factor) for all the components of the metric tensor to be obtained by only differential and algebraic manipulations on a single function. The idea is to introduce three parameters to label null surfaces in the manifold, i.e. a two parameter family of foliations. Since every one-parameter family of surfaces can be used to define a coordinate system (e.g. the past light cones emanating from a world line), then we are actually enlarging our set of variables by
adding two more variables to the coordinates. These two additional variables can be taken as coordinates on the unit sphere.

More specifically, we begin with a Lorentzian manifold $M$ with a metric $g_{ab}$, a given three parameter family of null hypersurfaces on the manifold, namely a function of the form $F(x^a, u, \zeta, \bar{\zeta}) = 0$, the three parameters being $u$, $\zeta$ and $\bar{\zeta}$, or alternatively by $u = Z(x^a, \zeta, \bar{\zeta})$ with $g^{ab}Z_aZ_b = 0$. Then every set of null surfaces with fixed value of $\zeta$ and $\bar{\zeta}$ (complex stereographic coordinates on $S^2$), a foliation, can be chosen as the level surfaces of a null coordinate $u$. If we apply an $\partial$ operation (essentially $\frac{\partial}{\partial \zeta}$) to the function $Z$, we obtain another function of $x^a$, $\zeta$ and $\bar{\zeta}$ which for fixed $\zeta$ and $\bar{\zeta}$ is in general independent of the previous one. We define this as a second coordinate $\omega$. In the same fashion, the remaining two coordinates of a new coordinate system for fixed $\zeta$ and $\bar{\zeta}$ can be defined by applying $\bar{\partial}$ and $\partial\bar{\partial}$ to the original function $Z$, and will be denoted $\bar{\omega}$ and $\bar{\Omega}$ respectively. What we obtain is a new coordinate system for every value of the parameters $\zeta$ and $\bar{\zeta}$, i.e., a sphere’s worth of coordinate systems. The structure we have is that of the bundle of null directions over $M$, with each fiber an $S^2$ and a coordinate system associated with each fiber point. There is a major advantage to having this “bundle” of coordinate systems: one can differentiate the components of any tensor with respect to the parameters, i.e. vertically, and obtain relationships among the components themselves. In other words, the components of a tensor will not be independent of each other, and it will be feasible to obtain most of them by knowledge of only one for all values of $(\zeta, \bar{\zeta})$. This and the fact that one of the coordinates is null provide the basis for describing the metric tensor completely in terms of the single function $Z(x^a, \zeta, \bar{\zeta})$, which becomes a basic variable of null surface theory.

The dynamics of the function $Z$ has been a troublesome issue ever since its appearance. In principle, all 10 of Einstein equations are encoded in only one equation [2] for the conformal factor $\Omega$ and $Z(x^a, \zeta, \bar{\zeta})$ by virtue of the sphere’s worth of coordinate systems. There are, in addition, several “auxiliary” conditions that guarantee that the metrics, eventually extracted from $Z(x^a, \zeta, \bar{\zeta})$, are all diffeomorphic to a metric $g_{ab}(x^c)$. Considerable success was originally attained by introducing the holonomy operator around closed loops as an auxiliary variable [3], though the resulting theory was quite complicated. Recently a great simplification was achieved where the full GR was expressed as equations for $Z$ and $\Omega$. There however still remained certain difficult conceptual issues, which led us to study the linearized version developed below. This gives a new insight into the structure of the full equations.

We first review, in Sec.II, the conventional approach (conventional as opposed to the null surface approach) to linearized gravity, showing the Einstein equations in terms of the first order correction to an assumed underlying flat space. We make a distinction between trace and trace free part of the linearized metric and give the equations in terms of these two quantities instead of the full first order tensor. We then show how a choice of gauge, in the conventional sense, can lead to a new coordinate system that is similar to the one described above, except for the critical $(\zeta, \bar{\zeta})$ dependence. We believe that this comparison of the conventional view with the null surface view clarifies many of the issues of null surface theory in first order.

A complete discussion of first order null surface theory for gravity with sources is found in sections III and IV. The attention is focused on the role of sources and a possible cosmological constant on the two relevant quantities. We find that all the components of the metric can be
expressed algebraically in terms of derivatives of a single spin-weight-2 function denoted $\Lambda$, which is defined by $\Lambda \equiv \partial^2 Z$, and the trace of the metric $\nu$ which to first order is equivalent to a small correction to the conformal factor $\Omega$. It is remarkable that the metric does not depend directly on $Z$ but only through $\Lambda$, which means that the metric does not know of the hypersurfaces themselves, but of a related quantity $\Lambda$ whose geometrical meaning will be discussed elsewhere.

II. THE LINEARIZED FIELD EQUATIONS

We begin with a spacetime and a metric $g_{ab}$ that differs from a flat metric by a small term $\gamma_{ab}$ with $g_{ab} = \eta_{ab} - \gamma_{ab}$. In a perturbation expansion to first order, the Einstein equations for $g_{ab}$ yield a system of linear second order equations for $\gamma_{ab}$ on the flat background $\eta_{ab}$. In particular, the linearized Ricci tensor becomes [4]:

$$R_{ab} = -\partial^c \partial_{(b} \gamma_{a)c} + \frac{1}{2} \Box \gamma_{ab} + \frac{1}{2} \partial_a \partial_b \gamma$$  \hfill (1)

$$R = -\partial^c \partial^d \gamma_{cd} + \Box \gamma$$  \hfill (2)

where $\partial_a$ is the flat derivative, $\Box \equiv \partial^c \partial_c$, $R$ and $\gamma$ are the (linearized) traces of the Ricci tensor and the first order correction to the flat metric. Indices are raised and lowered with $\eta_{ab}$. We decompose the first order metric into its trace-free part and its trace: $\gamma_{ab} = \frac{1}{4} \Box \eta_{ab} + q_{ab}$ and redefine the trace $\gamma$ in terms of a scalar $\nu$ in the following way: $\gamma = 8\nu$. The Ricci tensor, in terms of the redefined “trace” $\nu$ and the trace-free metric $q_{ab}$, becomes

$$R_{ab} = 2\partial_a \partial_b \nu + \eta_{ab} \Box \nu - \partial^c \partial_{(b} q_{a)c} + \frac{1}{2} \Box q_{ab}$$  \hfill (3)

and the Ricci scalar becomes:

$$R = 6\Box \nu - \partial^c \partial^d q_{cd}.$$  \hfill (4)

We assume that the source of the Einstein equations $R_{ab} - \frac{1}{2} g_{ab} R = 2T_{ab}$ is a first order stress-energy tensor that is conserved, and does not depend on the first order correction to the metric, but on the flat metric at most. As defined here, $T_{ab}$ absorbs the factor $\kappa/2$ that is usually explicit. $T_{ab}$ can be written as $T_{ab} = t_{ab} + \frac{1}{4} \eta_{ab} T$ with $t_{ab}$ trace free and $T$ its trace. The conservation law $\partial^a T_{ab} = 0$ for the stress energy tensor in terms of $t_{ab}$ and $T$ takes the form:

$$\partial_b T = -4\partial^a t_{ab}.$$  \hfill (5)

We will refer to $q_{ab}$ loosely as the metric and to $\nu$ as the trace, to avoid unnecessary complexity in the terminology which may obscure the content of the work.

The complete linearized Einstein equations in terms of $\nu$ and $q_{ab}$ read:

$$\partial_a \partial_b \nu - \eta_{ab} \Box \nu - \frac{1}{2} \partial^c \partial_{(b} q_{a)c} + \frac{1}{4} \Box q_{ab} + \frac{1}{4} \eta_{ab} \partial^c \partial^d q_{cd} = T_{ab}.$$  \hfill (6)
It is possible to choose coordinate systems so that the flat metric has only the following nonvanishing components: \( \eta_{00} = 2 \), \( \eta_{01} = 1 \), and \( \eta^{+} = -1 \). A way to do this is by choosing a null tetrad \((l_a, n_a, m_a, \bar{m}_a)\) where \( l_a \) and \( n_a \) are real, \( m_a = a_a + ib_a \) with \( a_a \) and \( b_a \) real spacelike vectors, and all the scalar products being zero except \( l_a n_a = -m_a \bar{m}_a = 1 \). Then we would take \( l_a, m_a, \bar{m}_a \) and the combination \( n_a - l_a \) as our coordinate basis vectors. The inverse \( \eta^{ab} \) has the following non-vanishing elements: \( \eta^{11} = -2 \), \( \eta^{01} = 1 \) and \( \eta^{+-} = -1 \). We will understand the equations (6) as equations for the two quantities \( \nu \) and \( q_{ab} \) in these particular flat coordinates, so that both of them are functions of the coordinates \((x^0, x^1, x^+, x^-)\) and partial derivatives are taken with respect to these coordinates. An example of these coordinates would be \( x^0 = z - t \), \( x^1 = z \), \( x^+ = x + iy \) and \( x^- = x - iy \).

The linearized equations (6) admit gauge freedom for both the metric and the trace in the following form:

\[
q'_{ab} = q_{ab} + \partial_a \xi_b + \partial_b \xi_a - \frac{1}{2} \eta_{ab} \partial^c \xi_c \tag{7}
\]

\[
\nu' = \nu - \frac{1}{4} \partial^a \xi_a. \tag{8}
\]

where \( \xi_a \) is a free vector field.

Using this gauge freedom we can make \( q'_{1b} = 0 \) for \( b = 1, +, - \) and \( q'_{01} + q'_{+-} = 0 \). Notice that since \( q'_{ab} \) is trace free then \( 0 = \eta^{ab} q'_{ab} = 2(q'_{01} - q'_{11} - q'_{+-}) \) implies \( q'_{+-} = q'_{01} = 0 \) as well. There are no restrictions on the trace \( \nu \). These conditions on the gauge can be expressed as:

\[
\partial_a \xi_b + \partial_b \xi_a - \frac{1}{2} \eta_{ab} \partial^c \xi_c + q_{1b} = 0,
\]

which can be solved explicitly with a given \( q_{1b} \). There is still the remaining gauge freedom of the homogeneous system:

\[
\partial_a \xi_b + \partial_b \xi_a - \frac{1}{2} \eta_{ab} \partial^c \xi_c = 0.
\]

The solutions to this system leave four unspecified functions of the coordinates \( x^0 \), \( x^+ \) and \( x^- \), while the dependence on \( x^1 \) is at most quadratic. The corresponding divergence \( \partial^c \xi_c \) is linear in \( x^1 \) with two free functions of the remaining coordinates as the coefficients. We will return to the issue of the gauge freedom in Sec.IV when we introduce a special family of coordinate systems and restrict to the transformations within the family.

Summarizing, the linearized metric has the form:

\[
g_{ab} = \Omega^{-2}(\eta_{ab} - q_{ab}) = (1 - 2\nu) \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} q_{00} & 0 & q_{0+} & q_{0-} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{-} \end{pmatrix}
\]

with inverse:

\[
g^{ab} = \Omega^{2}(\eta^{ab} + q^{ab}) = (1 + 2\nu) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{11} & q^{1+} & q^{1-} \\ 0 & q^{1+} & q^{++} & 0 \\ 0 & q^{1-} & 0 & q^{--} \end{pmatrix}
\]
If we now look at the $a = 1, b = 1$ component of the Einstein equations (6) we obtain a single equation that relates the sources to only the trace of the first order metric, since all the terms that involve the trace free part $q_{ab}$ are identically zero by virtue of our choice of gauge. The equation will be:

$$\partial_1^2 \nu = t_{11}. \tag{9}$$

This equation tells us that the source directly drives $\nu$, i.e. $\nu$ is given in terms of a double integral of the source $t_{11}$ plus any solution of the homogeneous equation. Note that the trace free component $t_{11}$ is equal to the full $T_{11}$.

The homogeneous solutions to (9) are all functions that are linear in the coordinate $x^1$, with two free functions of the remaining coordinates as the coefficients, i.e. $\nu_H = \alpha + \beta x^1$. We will return to this later.

The motivation for the discussion up to now has been to establish a basis for the understanding of first order null surface theory from a more conventional point of view. We will see that, in a sense to be discussed, Eq.(9) is the single equation that encodes the dynamics of first order Einstein fields from the null surface point of view. The new ingredient in null surface theory that explains why the remaining nine equations do not give any other relevant information is that it makes exhaustive use of the introduction, not of a single coordinate system with these properties, but a smooth two-parameter family of them. We will devote the following sections to a rather complete discussion of the ideas of null surface theory to first order.

III. A SPHERES WORTH OF COORDINATE SYSTEMS

We begin with a manifold $M$ with a Lorentzian metric $g_{ab}$ (we might also want a boundary to $M$ if we wish to discuss asymptotically flat spaces). A choice of coordinates based on null surfaces is made in the following fashion. We choose a family of null surfaces that are labeled by points of a three parameter space: $R \times S^2$, with a real parameter $u$ on $R$, and two stereographic coordinates $(\zeta, \bar{\zeta})$ on $S^2$. This is expressed by $F(x^a, u, \zeta, \bar{\zeta}) = 0$, or preferably by $u = Z(x^a, \zeta, \bar{\zeta})$. An immediate meaning to the function $Z$ from this point of view is that for every fixed value of the parameters $(\zeta, \bar{\zeta})$ the level surfaces of $u = Z$ are null surfaces i.e. $g^{ab} \partial_a Z \partial_b Z = 0$, labeled by the parameter $u$.

For the construction of our coordinate system we choose four functions of the points $x^a$ that are simply $Z, \partial Z, \bar{\partial} Z$, and $\bar{\partial} \partial Z$ at an arbitrary but fixed value of the parameters $(\zeta, \bar{\zeta})$. We choose to work with the $\partial$ and $\bar{\partial}$ operators instead of ordinary partial derivatives because of their covariant nature $^1$. We thus define an $S^2$’s set of coordinate systems for the spacetime in the following way:

$$u = Z(x^a, \zeta, \bar{\zeta})$$
$$\omega = \partial Z(x^a, \zeta, \bar{\zeta})$$
$$\bar{\omega} = \bar{\partial} Z(x^a, \zeta, \bar{\zeta})$$
$$R = \bar{\partial} \partial Z(x^a, \zeta, \bar{\zeta}) \tag{10}$$

$^1$We assume familiarity with the $\partial$ and $\bar{\partial}$ operators. See reference [5].
where the parameters \((\zeta, \bar{\zeta})\) have fixed values (after differentiation). As a matter of notation we refer to them as \(\theta^i = (u, R, \omega, \bar{\omega})\), with \(i = 0, 1, +, −\). We thus have \(\theta^i = \theta^i(x^a, \zeta, \bar{\zeta})\).

As an example of this construction in an asymptotically flat spacetime we can think of the interior spacetime points as being \(x^a\), while the boundary is a three-dimensional space (the null surface \(J^+)\) which can be given the coordinates \(u, \zeta, \bar{\zeta}\). From a fixed value of \((u, \zeta, \bar{\zeta})\), the past light cone is the set of points \(x^a\) that satisfy \(Z(x^a, \zeta, \bar{\zeta}) = u\), thus \(u\) labels past light cones from fixed null generators of \(J^+\). Early history of the null surface theory gave \(Z\) the name light cone cut function \([1]\); we will sometimes refer to \(Z\) as the cut function. On the given past light cone, the values of \((\omega, \bar{\omega})\) label the null generators, thus a particular value of \((u, \omega, \bar{\omega})\) is a null geodesic emanating from the point \((u, \zeta, \bar{\zeta})\) at \(J^+\). The last coordinate \(R\) is a parameter along the null geodesics (in general not affine).

Two different values of \((\zeta, \bar{\zeta})\) give two different coordinate systems. However, whatever coordinate conditions are implied by (10) will be true for all values of \((\zeta, \bar{\zeta})\). Since, by assumption, the dependence on \((\zeta, \bar{\zeta})\) is smooth, one can go back and forth from one coordinate system to a neighboring one by differentiating with respect to the parameters \((\zeta, \bar{\zeta})\), or equivalently, by applying \(\partial\) and \(\bar{\partial}\) operations.

From (10) we obtain a sphere’s worth of coordinate tetrads. The gradients \(\theta^i_{a}\) give us a coordinate set of covectors:

\[
\begin{align*}
\theta^0_{a} &= Z_{a} \\
\theta^+_{a} &= \partial Z_{a} \\
\theta^-_{a} &= \bar{\partial} Z_{a} \\
\theta^-_{a} &= \bar{\partial} Z_{a}.
\end{align*}
\]

The coordinate tetrad vectors \(\theta^i_{a}\) dual to \(\theta^i_{a}\) are defined by \(\theta^i_{a} \theta^j_{a} = \delta^i_j\).

We will be concerned with the linearized version of the Einstein equations, in which all terms are flat second derivatives of the first order correction to the flat metric. The function \(Z\) to first order consists of a zeroth order \(Z_M\) and a first order term \(z\), i.e.

\[
Z(x^a, \zeta, \bar{\zeta}) = Z_M(x^a, \zeta, \bar{\zeta}) + z(x^a, \zeta, \bar{\zeta})
\]

where \(Z_M\) is supposed to be known and \(z\) becomes the first order variable. The role of \(Z_M\) is to fix the zeroth order flat background: the flat intrinsic coordinates and tetrad. It will not be necessary to keep first order terms in the tetrad, since, in what follows, they will always appear contracted with tensors that are already first order (see remark below). We will work with the flat tetrad obtained from the function \(Z_M\) for flat space defined by a sphere’s worth of null planes, which is easily seen to be [1]:

\[
Z_M = x^a l_a(\zeta, \bar{\zeta})
\]

where \(x^a\) are standard Minkowskian coordinates and

\[
l^a(\zeta, \bar{\zeta}) = \frac{1}{\sqrt{2(1 + \zeta \bar{\zeta})}} \left( (1 + \zeta \bar{\zeta}), \zeta + \bar{\zeta}, \bar{\zeta} (\bar{\zeta} - \zeta), -1 + \zeta \bar{\zeta} \right)
\]

is a null vector for all \((\zeta, \bar{\zeta})\) with respect to the flat \(\eta_{ab}\). The index is raised with \(\eta_{ab}\) with signature \((+−−−)\). It is customary to define a null tetrad out of this function where the remaining vectors are:
\[ m_a = \delta l_a \]
\[ \bar{m}_a = \bar{\delta} l_a \]
\[ n_a = \bar{\delta} \delta l_a + l_a \]  
(15)

so that \( n^a l_a = 1, m^a \bar{m}_a = -1 \) and the remaining scalar products are zero. Following our approach, we define \( \theta^i = (u, R, \omega, \bar{\omega}) \) as in (10). Then:

\[ u = x^a l_a(\zeta, \bar{\zeta}) \]
\[ \omega = x^a m_a(\zeta, \bar{\zeta}) \]
\[ \bar{\omega} = x^a \bar{m}_a(\zeta, \bar{\zeta}) \]
\[ R = x^a (n_a(\zeta, \bar{\zeta}) - l_a(\zeta, \bar{\zeta})). \]
(16)

Our tetrad is then:

\[ \theta^i_{,a} = (l_a, n_a - l_a, m_a, \bar{m}_a) \]  
(17)

and dual:

\[ \theta^a_i = (n^a + l^a, l^a, -\bar{m}^a, -m^a) \]  
(18)

and the partial derivatives with respect to the coordinates \( \theta^i \) are \( \partial_i = \theta^a_i \partial_a \).

The flat metric \( \eta_{ij} \) in these coordinates has three non-vanishing elements: \( \eta_{00} = 2, \eta_{01} = 1, \) and \( \eta_{+-} = -1 \) with the inverse \( \eta^{ij} \) having the elements: \( \eta^{11} = -2, \eta^{01} = 1 \) and \( \eta^{+\pm} = -1 \) non-vanishing, i.e., is of the form we used in Sec.II.

We end this section by pointing out that this \((\zeta, \bar{\zeta})\)-dependent set of coordinate systems has the properties (i.e. satisfies the same set of coordinate conditions as in Sec.II.) required to obtain the equation for the trace \( \nu \) of the first order metric from the Einstein system of equations, namely Eq.(9), \( \partial^2_{\nu} \nu = t_{11} \), for each value of \((\zeta, \bar{\zeta})\). The advantage now is that we have a smooth two-parameter family of such coordinate systems, so we are allowed to perform differentiations with respect to \((\zeta, \bar{\zeta})\) without changing the coordinate conditions. These remarks will be amplified upon in the next section.

**IV. FIRST ORDER NULL SURFACE THEORY**

**A. The null theory variables \( \nu \) and \( \Lambda \) and kinematic equations**

The components of any tensor referred to the new coordinates are obtained by contraction of the old tensor with the basis or dual tetrad. In particular, the metric components with respect to the new coordinates denoted by \( g_{ij} \) are given by \( g_{ij} = g_{\alpha \beta} \theta^\alpha_i \theta^\beta_j \), with inverse \( g^{ij} = g^{\alpha \beta} \theta_\alpha^i \theta_\beta^j \).

There is a distinction between \( ij \) and \( ab \) indices. Quantities like \( g_{\alpha \beta} \) or \( T_{\alpha \beta} \) are “local” in the sense that they depend only on the local coordinates of the spacetime \( x^a \). However \( g_{ij} \) or \( T_{ij} \) depend on \( \theta^i \) and \((\zeta, \bar{\zeta})\) as well. Similarly, the partial derivatives \( \partial_a \) commute with \( \bar{\delta} \) and \( \delta \), whereas \( \partial_i \) do not in general. The important commutation relations of \( \bar{\delta} \) and \( \delta \) with \( \partial_i \) are given later.
In these coordinates [1], all the components of the metric can be expressed in terms of two functions. One of these two functions is defined as follows:

$$\Lambda \equiv \bar{\Lambda}^2 Z. \quad (19)$$

All the components of the metric get expressed as $g^{01}$ times a simple algebraic function of $\Lambda$ and its derivatives. Therefore $g^{01}$ can be taken as a conformal factor $\Omega^2 = g^{01} = g^{ab}\theta_a \theta_b = g^{ab} Z_{,a} \bar{\partial} Z_{,b}$. Thus the metric is conformally a function of $\Lambda$, the conformal factor being $g^{01}$. This follows essentially from the fact that $g^{00} = g^{ab} Z_{,a} Z_{,b}$ is zero by assumption, since $u = Z = constant$ is a null surface, and the remainder of the components can be computed by applying $\partial$ and $\bar{\partial}$ to $g^{00} = 0$ an appropriate number of times, using the non-trivial fact that $\partial g^{ab} = \bar{\partial} g^{ab} = 0$. For example, from

$$g^{00} = g^{ab} Z_{,a} Z_{,b} = 0 \quad (20)$$

and

$$g^{0+} = g^{ab} Z_{,a} \partial Z_{,b}, \quad (21)$$

by applying $\partial$ to $g^{00} = 0$ we obtain

$$\partial g^{00} = 2g^{0+} \quad (22)$$

and therefore $g^{0+} = 0$. Applying $\partial \bar{\partial}(g^{ab} Z_{,a} Z_{,b}) = 0$ we get $\partial \bar{\partial} g^{00} = 2g^{01} + 2g^{++} = 0$ and therefore

$$g^{++} = -g^{01}. \quad (23)$$

Taking $\partial \bar{\partial} g^{00} = 0$ gives the expression of $g^{--}$ in terms of $g^{01}$ and $\Lambda$. Taking $\partial \bar{\partial}$ of $g^{00} = 0$ gives $g^{1+}$. Finally, by taking $\partial \bar{\partial} \partial \bar{\partial}$ of $g^{00}$ we obtain $g^{11}$. The complex conjugates of all these operations yield the remaining components of the metric.

The linearization can be carried out by noticing that $Z_M = x^a l_a$ satisfies

$$\bar{\partial}^2 Z_M = \bar{\partial}^2 Z_M = 0.$$

Thus the zeroth order does not contribute to $\Lambda$ and therefore $\Lambda$ is a first order quantity. By dividing the $g^{ij}$ by $g^{01}$ and linearizing in $\Lambda$ we have $g^{ij}/g^{01} = \eta^{ij} + f^{ij}$ where $f^{ij}$ is first order and has the following five non-vanishing elements:

$$f^{11} = -\frac{1}{2} \bar{\partial}^2 \Lambda_1 + \bar{\partial} \Lambda_+ = -\frac{1}{2} \frac{1}{2} \bar{\partial}^2 \Lambda_1 + \bar{\partial} \Lambda_+$$
$$f^{1+} = -\frac{1}{2} \bar{\partial} \Lambda_1$$
$$f^{1-} = -\frac{1}{2} \bar{\partial} \Lambda_1$$
$$f^{++} = -\Lambda_1$$
$$f^{--} = -\bar{\Lambda}_1. \quad (24)$$

The non-vanishing $f^{ij}$ are the following:
\[ f_{00} = -\frac{1}{2} \tilde{\partial}^2 \Lambda_1 + \hat{\partial} \Lambda_+ = -\frac{1}{2} \partial^2 \Lambda_1 + \tilde{\partial} \Lambda_- \]
\[ f_{0+} = \frac{1}{2} \tilde{\partial} \Lambda_1 \]
\[ f_{0-} = \frac{1}{2} \tilde{\partial} \Lambda_1 \]
\[ f_{++} = -\Lambda_1 \]
\[ f_{--} = -\Lambda_1 \]  
\( (25) \)

where we have adopted the notation \( \partial_i \Lambda \equiv \Lambda_i \). Note that the first order trace-free metric is given in terms of \( \Lambda \) and its derivatives.

In deriving the components of the metric from \( g^{00} = 0 \) we used repeatedly \( \tilde{\partial} g^{ab} = \hat{\partial} g^{ab} = 0 \). This is the statement that all the \((\zeta, \bar{\zeta})\)-dependent metrics obtained by \( g^{ij} = g^{ab} \theta^i_a \theta^j_b \) are in fact diffeomorphic to \( g^{ab} \), i.e., they are all equivalent. The conditions \( \tilde{\partial} g^{ab} = \hat{\partial} g^{ab} = 0 \) constitute a set of 20 equations of which some, when expressed in these coordinates, give the metric (Eq.(25)). Some are identities. The remaining equations, given below, are referred to as the “metricity conditions”. We discuss their meaning and significance in [6], including an explicit derivation of the metricity conditions from a slightly different point of view.

Notice that \( f = \eta^{ij} f_{ij} = 0 \) automatically. On the other hand, the conformal factor \( g^{01} = 1 + 2\nu \) is equal to \( \eta^{01} = 1 \) to zeroth order plus a first order correction \( 2\nu \) which can be identified with the trace of the full first order departure from flat space, since \( g^{ij} = g^{01}(\eta^{ij} + f^{ij}) = \eta^{ij} + 2\nu \eta^{ij} + f^{ij} \) and therefore the trace of the first order correction is indistinguishable from the first order conformal factor. In fact, we will make no distinction between the trace \( \nu \) of section II and the first order conformal factor of the current section.

We can continue to take \( \bar{\partial} \) and \( \tilde{\partial} \) of the components of the metric that we have already obtained. The only new relations (the metricity conditions) that we can obtain in this way [6] are the following:

\( (*) \) By writing \( g^{01} = g^{ab} Z_a \bar{\partial} Z_b \), applying \( \bar{\partial} \) to \( g^{01} \) and linearizing in \( \Lambda \) we obtain the relationship between \( \nu \) and \( \Lambda \):

\[ 2\bar{\partial} \nu = \Lambda_+ + \frac{1}{2} \tilde{\partial} \Lambda_1. \]  
\( (26) \)

(A) If we apply \( \bar{\partial}^3 \) to \( g^{00} = g^{ab} Z_a Z_b = 0 \) (the \( \theta^+_a \theta^+_b \)-component of \( \bar{\partial} g^{ab} = 0 \)) and linearize in \( \Lambda \) we obtain the following relation for \( \Lambda \) only:

\[ 0 = \Lambda_- - \frac{1}{2} \tilde{\partial} \Lambda_1. \]  
\( (27) \)

(B) We can now take \( \bar{\partial} \bar{\partial}^3 g^{00} = \bar{\partial} \bar{\partial}^3 (g^{ab} Z_a Z_b) = 0 \) (the \( \theta^+_a \theta^+ b \)-component of \( \bar{\partial} g^{ab} = 0 \)). After linearizing in \( \Lambda \) we obtain:

\[ 0 = \bar{\partial}(\bar{\partial} \Lambda_1 - 2\Lambda_-). \]  
\( (28) \)

(C) Finally, by taking \( \bar{\partial}^2 \bar{\partial}^3 g^{00} = 0 \) (the \( \theta^+_a \theta^+_b \)-component of \( \bar{\partial} g^{ab} \)) we obtain another linearized condition on \( \Lambda \):

\[ 0 = \bar{\partial}^3 \Lambda_1 - 2\bar{\partial} \tilde{\partial} \Lambda_+ + 4\tilde{\partial} \Lambda_0 - 4\Lambda_+ + 2\tilde{\partial} \Lambda_1 \]  
\( (29) \)
which, with the help of (26) gives:

\[
0 = \partial^3 \Lambda_1 + \partial \partial^2 \Lambda_1 + 4 \partial \Lambda_0 - 8 \partial \nu - 4 \partial^2 \partial \nu. \tag{30}
\]

These four first order relations (*), (A), (B) and (C), are of a kinematical nature in the sense that they hold for any linearized Lorentzian metric independent of the Einstein equations. They are very important to the null surface approach, and as such they deserve a name of their own: we refer to them as the metricity conditions. If the null surfaces \( u = Z(x^a, \zeta, \bar{\zeta}) \) of a given metric are known and \( \Lambda \) is computed, then the metricity conditions are identities. But if the null surfaces are yet to be found, then the metricity conditions become the equations that yield \( \Lambda \), from which the metric and, if desired, the null surfaces as well, can be computed (the metricity conditions are the requirement that the sphere’s worth of metrics \( g^{ij} \) be really equivalent). It should be noted that in first order theory Eq.(B) follows from Eq.(A) \(^2\); this leaves only three relevant metricity equations (*), (A) and (C). They play an important role later.

Before we further analyze these kinematic relations and properties of the \( f_{ij} \) we need the commutation relations between \( \partial \) and the \( \partial_i \). Since \( \partial \partial_i F = \partial (\theta^a \partial_a F) = \partial \theta^a \partial_a F + \theta^a \partial_a \partial F \) and since \( F \) will be chosen as first order then:

\[
\begin{align*}
\partial_0 \partial & = \partial \partial_0 \\
\partial_+ \partial & = \partial \partial_+ \\
\partial_0 \partial & = \partial \partial_0 + \partial_0 - 2 \partial_1 \\
\partial_1 \partial & = \partial \partial_1 + \partial_-
\end{align*}
\tag{31}
\]

with the complex conjugate relations between \( \bar{\partial} \) and the \( \partial_i \). In obtaining these relations we used the following:

\[
\begin{align*}
\partial \theta^0 & = \partial Z \equiv \theta^+ \\
\partial \theta^+ & = \partial^2 Z = 0 \\
\partial \theta^- & = \bar{\partial} \bar{\partial} Z = \theta_1 \\
\partial \theta^1 & = \partial^2 \bar{\partial} Z = -2 \theta^+.
\end{align*}
\tag{32}
\]

or alternatively relations (15), (17) and (18).

We mention here some very useful results that we will need later, namely the divergence \( \partial^j f_{ij} \) and the double divergence \( \partial^j \partial^i f_{ij} \). With the use of the commutation relations and the kinematic relation (*) we can evaluate \( \partial^j f_{ij} \). First we notice that since \( f_{ij} = 0 \) then the contraction \( \partial^j f_{ij} \) has only three components. Also, since \( \partial^j f_{+j} \) is the complex conjugate of \( \partial^j f_{-j} \) we only need to compute the components \( i = 0 \) and \( i = - \). We display below the results and leave the detailed calculation to an appendix:

\(^2\)B remains an identity even in the full theory (see [6])
\[ \partial^j f_{-j} = 2 \partial_1 \bar{\partial} \nu \]  

with the complex conjugate:

\[ \partial^j f_{+j} = 2 \partial_1 \bar{\partial} \nu \]  

and finally

\[ \partial^j f_{0j} = -2 \partial_1 \bar{\partial} \nu + 2 \partial_+ \bar{\partial} \nu + 2 \partial_- \bar{\partial} \nu. \]  

The double divergence \( \partial^i \partial^j f_{ij} \) turns out to be simply:

\[ \partial^i \partial^j f_{ij} = -2 \partial_1^2 \bar{\partial} \bar{\partial} \nu. \]  

Note that although \( f_{ij} \) are functions of \( \Lambda \) the divergences are functions of only \( \nu \).

**B. The dynamics of \( \nu \) and \( \Lambda \)**

So far the description of null surface theory has been completely kinematical. Our variables (\( \nu \) and \( \Lambda \)) must satisfy the metricity conditions \( (*) \), \( (A) \) and \( (C) \) to define for us a Lorentzian metric. We now address the problem of finding a metric that in addition satisfies the Einstein equations. The Einstein equations are implemented in the following way. We contract the linearized Einstein equations (6) with \( \ell^a \ell^b \) (= \( \theta^a \theta^b \)):

\[ l^a l^b \left( \partial_a \partial_b \nu - \eta_{ab} \nabla \nu - \frac{1}{2} \partial^c \partial_d q_{cd} + \frac{1}{4} q_{ab} + \frac{1}{2} \eta_{ab} \partial^c \partial^d q_{cd} \right) = T_{ab} l^a l^b. \]  

Because our gauge is the one described in section II, and because \( l^a \) is null, we immediately obtain:

\[ l^a l^b \partial_a \partial_b \nu = T_{ab} l^a l^b \]

or

\[ \partial_1^2 \nu(\theta^i, \zeta, \bar{\zeta}) = t_{11}(\theta^i, \zeta, \bar{\zeta}). \]  

This is the dynamical equation for \( \Lambda \) and \( \nu \). Though it resembles Eq.(9), its meaning is different. Because of the \( (\zeta, \bar{\zeta}) \) dependence of \( l^a \), it is not a single equation, but an \( S^2 \)'s worth of equations instead. It contains all the equations that make up the trace-free part of the Einstein system.

[Another way to see that Eq.(38) encodes the trace-free part of the Einstein system is the following. We can apply \( \bar{\partial} \) and \( \bar{\partial} \) to Eq.(38) an appropriate number of times so that we obtain the remaining eight of the trace-free equations. It is clear that we can do so because \( t_{11} = T_{ab} l^a l^b \), and thus an \( \bar{\partial} \) operation on \( t_{11} \) goes through \( T_{ab} \) and is applied to \( l^a \) directly, giving a different tetrad vector \( (m^a) \), and therefore giving a different component of \( T_{ab} \). See the appendix for a more complete discussion of this issue.]

It may appear that we are lacking a tenth equation, the trace of the Einstein system. It is true that the trace is lost when we contract the Einstein tensor with \( l^a l^b \). However, the
information contained in the trace of the Einstein system is redundant (see Eq.(5)). If we put the Einstein equations in terms of the trace and trace-free part of the Einstein tensor
\[ G_{ab} = g_{ab} + \frac{1}{4} G \]
they become:
\[ G_{ab} = t_{ab} \]
and
\[ G = T. \]
In addition, the Bianchi identities \( \nabla^a G_{ab} = 0 \) become:
\[ \nabla_b G = -4 \nabla^a G_{ab}. \]
Therefore the trace of the Einstein tensor is determined (up to a constant) by the Bianchi identities. However, the Bianchi identities are of a kinematical nature, in that they are true for any Lorentzian metric. This means that the trace of the Einstein system is redundant when \( \mathcal{G}_{ab} \) and \( G \) are expressed as functions of a metric. Now returning to the null surface point of view, we claim that any solutions \( \nu = \nu(\theta^i, \zeta, \bar{\zeta}) \) to Eq.(38) and the metricity conditions as well (since the metricity conditions guarantee a metric), will provide us with a Lorentzian metric that satisfies the complete set of Einstein equations. It is in this sense that we say that Eq.(38) encodes the full Einstein equations.

[We mention that the only piece of information that we lack in the null surface approach is the integration constant of the Bianchi identities. This integration constant is the cosmological constant in the vacuum case, or part of the stress tensor in the case of non-vanishing sources.]

In the remainder of this section we will clarify the issues just raised. First, we will return to the conventional form of the Einstein equations and calculate the trace of the Einstein Eqs.(6). Then we will show that in the null surface approach, this trace equation is satisfied if Eq.(38) and the metricity conditions are satisfied. Thus, we will have proved our claim that we do not need the trace equation. In the null surface approach, the tenth equation is thus encoded into the metricity conditions and (38).

Contracting Eq.(6) with \( \eta^{ab} \) we obtain:
\[ -3 \Box \nu + \frac{1}{2} \partial^i \partial^j f_{ij} = T. \]
By virtue of our previous calculation for the double divergence of the metric \( f_{ij} \), i.e. Eq.(36), Eq.(39) turns out to be an equation for only the conformal factor \( \nu \), decoupled from \( \Lambda \):
\[ -3 \Box \nu - \partial^2 \partial \partial \nu = T \]
where \( T \) is uniquely given by \( \partial_a T = -4 \partial^b t_{ab} \) up to a constant. This would be the tenth Einstein equation. We now show that (40) is satisfied by virtue of (38). With some work (by essentially using the commutation relations (31)) it can be shown that Eq.(38) implies the following:
\[ \partial_1 (-3 \Box \nu - \partial^2 \partial \partial \nu) = \partial_1 T \]
(where $\partial_1 T$ is equal to $-4\partial^i t_{ij}$). Furthermore, we show in the appendix that the metricity conditions (*) and (A) imply:

\[
\bar{\partial}(-3\Box \nu - \partial^2_1 \bar{\partial} \nu) = 0 \tag{42}
\]

\[
\bar{\partial}(-3\Box \nu - \partial^2_1 \bar{\partial} \nu) = 0. \tag{43}
\]

These are sufficient conditions to guarantee that $\partial_i (-3\Box \nu - \partial^2_1 \bar{\partial} \nu) = \partial_i T$ for all $i$ because the application of $\bar{\partial}$ and $\bar{\partial}$ on $\partial_1 (-3\Box \nu - \partial^2_1 \bar{\partial} \nu) = \partial_1 T$ yields the remaining equations if $(-3\Box \nu - \partial^2_1 \bar{\partial} \nu)$ is a function only of $x^a$, as implied by (42) and (43). Therefore any solution to Eq.(38) and the metricity conditions satisfies the tenth equation automatically. Note that (42) and (43) imply severe conditions on $\alpha$ and $\beta$ in the solution to (38) (see Sec.IV.D).

C. The solution to the metricity conditions

Our point of view is now as follows: without yet worrying about the details (see discussion at the end of this Sec.) the conformal factor $\nu$ can be found explicitly as a double integral in $R$ of the given stress tensor $t_{111}$. Once $\nu$ has been solved for, it enters the equations for $\Lambda$ (Eq.(26), (27) and (30)) as a known source only. Our task is then to solve the three kinematic equations for $\Lambda$. [It should be noted, however, that the decoupling of the equations for $\nu$ and $\Lambda$ can not be carried out in this way in the full theory, in which the metricity conditions and the dynamical equation are strongly coupled [6].]

We begin by manipulating (*) and (A) to obtain two very important relations for $\Lambda$. The following is merely the derivation of Eqs.(45) and (46) which we include at this point for the sake of completeness (too frequently in this work the results are not easy to reproduce, due in part to the non trivial commutation relations (31)).

By applying $\bar{\partial}$ to (27) and commuting $\bar{\partial} \partial$ on the right side we obtain:

\[
\begin{align*}
\bar{\partial} \Lambda_- &= \frac{1}{2} \bar{\partial} \partial \Lambda_1 + 2 \Lambda_1 \\
\end{align*}
\]

By applying now $\partial_1$ to both sides and commuting $\partial_1 \bar{\partial}$ in the left side we obtain:

\[
\begin{align*}
\bar{\partial} \partial_1 \Lambda_- + \partial_+ \Lambda_- &= \frac{1}{2} \bar{\partial} \partial_1 \partial \Lambda_1 + 2 \partial_1 \Lambda_1.
\end{align*}
\]

If we commute now $\bar{\partial} \partial_-$ in the left side we obtain:

\[
\begin{align*}
\partial_- \bar{\partial} \Lambda_1 - \partial_0 \Lambda_1 + 2 \partial_1 \Lambda_1 + \partial_+ \Lambda_- &= \frac{1}{2} \bar{\partial} \partial_1 \partial \Lambda_1 + 2 \partial_1 \Lambda_1.
\end{align*}
\]

The $\partial_1 \Lambda_1$ cancel out from both sides. We now use the basic equation (26) to eliminate $\bar{\partial} \Lambda_1$ in the left side and obtain:

\[
\begin{align*}
\partial_+ \Lambda_- &= -\partial_1 \Lambda_0 - \frac{1}{2} \bar{\partial} \partial \Lambda_1 + 4 \partial_- \bar{\partial} \nu. \tag{44}
\end{align*}
\]

Next we carry out a similar procedure on (26). First we apply $\bar{\partial}$ and then $\partial_1$ to both sides. By commuting $\partial_1 \bar{\partial}$ on the left side obtain:
$$\partial\partial_1\Lambda_+ + \partial_-\Lambda_+ = -\frac{1}{2}\partial_1\ddbar\Lambda_1 + 2\partial_1\bar{\sigma}\nu.$$ 

Further, by commuting $\partial\partial_+$ on the left side we obtain:

$$\partial_+\partial_1\Lambda_1 - \partial_0\Lambda_1 + 2\partial_1\Lambda_0 + \partial_-\Lambda_+ = -\frac{1}{2}\partial_1\ddbar\Lambda_1 + 2\partial_1\bar{\sigma}\nu.$$ 

Using (27) to eliminate $\partial\Lambda_-$ on the left side we are left with:

$$3\partial_-\Lambda_+ - \partial_0\Lambda_1 + 2\partial_1\Lambda_1 = -\frac{1}{2}\partial_1\ddbar\Lambda_1 + 2\partial_1\bar{\sigma}\nu.$$ 

Using $\partial_+\Lambda_- = \partial_-\Lambda_+$ and inserting $\partial_+\Lambda_-$ from (44), we obtain our first basic result:

$$4\partial_1\Lambda_0 + \partial_1\ddbar\Lambda_1 - 2\partial_1\Lambda_1 = 12\partial_-\partial\nu - 2\partial_1\bar{\sigma}\nu \quad (45)$$

Eliminating $\partial_1\ddbar\Lambda_1$ from (45) via (44) we obtain our second basic result:

$$\square\Lambda \equiv 2(\partial_0\partial_1 - \partial_1\partial_1 - \partial_+\partial_-)\Lambda = 4\partial_-\partial\nu - 2\partial_1\bar{\sigma}\nu. \quad (46)$$

$\Lambda$ thus satisfies the wave equation with the conformal factor playing the role of the source. [This equation, (46), has been derived here just for completeness and will not be used. Its use will be presented in a future paper where $\Lambda$ will be the basic variable and $Z$ purely auxiliary. In the present paper we use $\Lambda$ as auxiliary while $Z$ plays the primary role.]

Equation (45) can be integrated on $R$, assuming asymptotically flat boundary conditions, to yield the following:

$$\Lambda_0 = -\frac{1}{4}\ddbar\Lambda_1 + \frac{1}{2}\Lambda_1 + 3\int^R_\infty \partial_-\partial\nu d\tilde{R} - \frac{1}{2}\partial\nu + \sigma_{,0}(u, \zeta, \tilde{\zeta}) \quad (47)$$

where $\sigma(u, \zeta, \tilde{\zeta})$ is free radiation data.

Taking $\ddbar$ of $\Lambda_0$ and inserting it into the third metricity condition (30) we find:

$$-\frac{1}{4}\ddbar^3\Lambda_1 = \ddbar\sigma_{,0} + 3\ddbar\int^R_\infty \partial_-\partial\nu d\tilde{R} - 2\partial\nu - \frac{3}{2}\partial\bar{\sigma}\nu \quad (48)$$

and its complex conjugate.

By applying $\ddbar$ to (30) we obtain:

$$\ddbar^2\Lambda_0 = -\frac{1}{4}\ddbar^3\Lambda_1 - \frac{1}{4}\ddbar^3\Lambda_1 + \ddbar\partial\nu + \ddbar\ddbar\ddbar\nu.$$ 

Now using (48) and its complex conjugate in the previous equation leads to:

$$\ddbar^2\Lambda_0 = \ddbar^2\sigma_{,0} + \ddbar^2\sigma_{,0} + 3\ddbar^2\int^R_\infty \partial_-\partial\nu d\tilde{R} + 3\ddbar^2\int^R_\infty \partial_+\partial\nu d\tilde{R} - 3\ddbar\partial\nu - 2\ddbar^2\bar{\sigma}\nu \quad (49)$$

Integrating once in $u$ and using $\Lambda = \ddbar^2Z$ finally yields the fundamental equation for the cut function $Z:$
\[\bar{\partial}^2 \bar{\partial}^2 Z = \bar{\partial}^2 \sigma + \bar{\partial}^2 \tilde{\sigma} + \int_u^{\infty} \left( 3 \bar{\partial}^2 \int_{\infty}^{R} \partial_{-} \bar{\nu} d\tilde{R} + 3 \bar{\partial}^2 \int_{R}^{\infty} \partial_{+} \bar{\nu} d\tilde{R} - 3 \bar{\partial} \bar{\partial} \nu - 2 \bar{\partial} \bar{\partial}^2 \nu \right) d\tilde{u}. \] (50)

Equation (50), our final equation, equivalent to the linearized Einstein equations, can be solved by means of a simple Green’s function for the operator \( \bar{\partial}^2 \bar{\partial}^2 \) (essentially the double laplacian on the sphere). The Green’s function is given explicitly in appendix C [7]. As complicated as it appears, the right hand side is simply the source, a known function of \((x^a, \zeta, \bar{\zeta})\) once the conformal factor has been solved for.

To solve for the conformal factor in terms of only the source \( t_{11} \) is a task more subtle than it appears at first sight. Formally, one can write the solution to Eq.(38) as the double integral of \( t_{11} \) plus a homogeneous solution that is linear in \( R \) and that contains the values of \( \nu \) and \( \partial_1 \nu \) at a fixed value \( R = R_0 \), as shown earlier. The homogeneous part is severely restricted, however, by the trace equation. The trace equation, in terms of \( \alpha \) and \( \beta \) becomes quite cumbersome in the case of general (arbitrary) sources, involving the values of \( t_{11} \) and different derivatives of \( t_{11} \) in a non-trivial manner at the surface \( R = R_0 \). We will bypass the problem of solving for the homogeneous part of the conformal factor in the general case by restricting ourselves to the case where there exists a surface \( R = R_0 \) on which \( t_{11} \) vanishes as a function of \((u, R_0, \omega, \bar{\omega}, \zeta, \bar{\zeta})\). This includes the asymptotically flat case (where \( t_{11} = 0 \) at \( J^+ \)) but is slightly more general.

For the particular case of sources that vanish at a surface \( R = R_0 \), the conformal factor can be solved for unambiguously by:

\[ \nu = \int_{R_0}^{R} \int_{R_0}^{\tilde{R}} t_{11} dR' d\tilde{R} \] (51)

in the gauge such that \( \alpha = \beta = 0 \) (see next section for a discussion of gauge).

The corresponding equation for the cut function is the same as (50) with the lower limit of the integrals being replaced by \( R_0 \).

D. Gauge conditions and remaining freedom

This section is supplementary to the linearized null surface approach described in the previous sections. We justify particular choices of the conformal factor \( \nu \) made in past works [3] (and in (51)) and provide some insights into the issue of the gauge freedom.

We mentioned before, Sec.II, that by imposing \( g_{ib} = 0 \) we were still left with gauge freedom of the form:

\[ \partial_1 \xi_i + \partial_b \xi_1 + \frac{1}{2} \eta_{ik} \partial^k \xi_k = 0. \]

For \( i = 1 \) we have:

\[ \xi_{1,1} = 0 \]

therefore \( \xi_1 = \phi(u, \omega, \bar{\omega}, \zeta, \bar{\zeta}) \). For \( i = + \) we have:

\[ \xi_{1,+} + \xi_{+,1} = 0 \]
therefore $\xi_+ = -\phi_+ R + \tau_+$ where $\tau_+$ does not depend on $R$. Similarly for $i= -$ we obtain $\xi_- = -\phi_- R + \tau_-$ where $\tau_-$ does not depend on $R$. The component $\xi_0$ can be solved for in a similar fashion from $\partial_1 \xi_0 + \partial_0 \xi_1 - \frac{1}{2} \partial^k \xi_k = 0$ but it will not be needed in what follows. The divergence $\partial^k \xi_k$ is equal to $-2(\xi_{+,+} + \xi_{-,+})$ and therefore $\partial^k \xi_k = 4\phi_{+,+} R - 2(\tau_{+,+} + \tau_{-,+})$.

At this stage it is clear that the gauge freedom in $\partial_1 \text{div} \xi$ is equal to $4\phi_{+,+} R$ and $\partial_0 \xi$, corresponds to taking a total derivative with respect to $(\theta, \phi)$ and $(\zeta, \bar{\zeta})$ as the coefficients.

Additional gauge conditions arise from our $(\zeta, \bar{\zeta})$-dependent coordinate systems; see Eqs.(10). The field $\xi^i$ is just an infinitesimal variation of the coordinates $\theta^i$ to neighboring coordinates $\theta^i = \theta^i + \xi^i$. Explicitly:

$$
u^i = u + \delta u = u + \xi^0$$
$$\omega^i = \omega + \delta \omega = \omega + \xi^+$$
$$\bar{\omega}^i = \bar{\omega} + \delta \bar{\omega} = \bar{\omega} + \xi^-$$
$$R' = R + \delta R = R + \xi^1. \tag{52}$$

Since $\theta^i$ and $\theta^i$ for $i = 1,+,-$ are obtained from $\theta^0$ by the application of $\delta$, $\bar{\delta}$ and $\delta \delta$, the components $\xi^+$, $\xi^-$ and $\xi^1$ can also be obtained by applying $\delta$ and $\bar{\delta}$ to $\xi^0$, i.e.:

$$\xi^+ = \delta \xi^0$$
$$\xi^- = \bar{\delta} \xi^0$$
$$\xi^1 = \delta \bar{\delta} \xi^0.$$ 

Thus, $\tau_+$, $\tau_-$ and $\phi$ are not independent of each other. If we lower the index with $\xi_i = \eta_{ij} \xi^j$ we find the following:

$$\xi_1 = \xi^0 = \phi(u, \omega, \bar{\omega}, \zeta, \bar{\zeta})$$
$$\xi_+ = -\xi^- = -\bar{\delta} \xi^0 = -\bar{\delta} \xi_1$$
$$\xi_- = -\xi^+ = -\delta \xi^0 = -\delta \xi_1.$$ 

Since $\xi_1 = \phi(u, \omega, \bar{\omega}, \zeta, \bar{\zeta})$ and the coordinates $u$, $\omega$ and $\bar{\omega}$ depend on $(\zeta, \bar{\zeta})$ then taking $\delta$ corresponds to taking a total derivative with respect to $(\zeta, \bar{\zeta})$. Assuming that $\phi$ is a first order quantity we have:

$$\delta \xi_1 = \delta \phi = \phi_{,0} \partial \theta^0 + \phi_{,+} \partial \theta^+ + \phi_{,-} \partial \theta^- + \partial' \phi$$
$$\bar{\delta} \xi_1 = \bar{\delta} \phi = \bar{\phi}_{,0} \partial \bar{\theta}^0 + \bar{\phi}_{,+} \partial \bar{\theta}^+ + \bar{\phi}_{,-} \partial \bar{\theta}^- + \bar{\partial}' \phi$$

where $\partial'$ and $\bar{\partial}'$ are taken holding $\theta^i$ fixed and the definitions of $\omega$, $\bar{\omega}$ and $R$ have been used. Therefore

$$\tau_+ = -\phi_{,0} \bar{\omega} - \bar{\partial}' \phi.$$
\[ \tau_\nu = -\phi \omega - \phi' \phi. \]

With these expressions the divergence \( \partial^k \xi_k \) is given as:

\[
\partial^k \xi_k = 4\phi_{+,+} + 4\phi_{,0} + 2\phi_{,0} \bar{\omega} + 2\phi_{,0} \omega + 2\phi'_{-,+} + 2\phi'_{+,+}. \tag{53}
\]

Here \( \phi \), an arbitrary function of \((u, \omega, \bar{\omega}, \zeta, \bar{\zeta})\), represents our full gauge freedom. The divergence \( \partial^k \xi_k \) can be used to modify \( \nu \), as noted in Sec.II. Recall (8):

\[
\nu' = \nu - \frac{1}{4} \partial^a \xi_a.
\]

By virtue of Eq.(38), \( \nu \) is determined by \( t_{11} \) up to a linear function of \( R \). We are interested in seeing whether we can gauge away the linear (homogeneous) part of \( \nu \). We claim that if \( \nu \) satisfies Eq.(38) and the metricity conditions, then we have just enough freedom to gauge away the homogeneous part of \( \nu \), in the case where the cosmological constant vanishes and there is a surface \( R = R_0 \) on which \( t_{11} \) vanishes. In particular, \( \nu \) would be pure gauge for the vacuum case with vanishing cosmological constant. This would justify the choice \( \nu = 0 \) made in earlier works on the linearized vacuum null surface theory [3].

The proof of our claim is in the following argument. As was previously shown, any \( \nu \) that satisfies both Eq.(38) and the metricity conditions will consequently satisfy

\[
-3\Box \nu - \partial^a \partial \phi \nu = T,
\]

the trace equation. Although not needed in general, it proves particularly useful in this section. If there exists a surface \( R = R_0 \) in which \( t_{11} \) vanishes, when we substitute \( \nu = \int_{R_0}^{R} \int_{R_0}^{t_{11}} dR' d\bar{R} + \alpha + \beta R \) (the general solution to Eq.(38)) into the trace equation, the non-homogeneous part of \( \nu \) drops out and we obtain an equation for only \( \alpha \) and \( \beta \) that has the following form:

\[
8\dot{\beta} - 4\alpha_{+,+} + 2\dot{\beta}_{-,+} \omega + 2\dot{\phi}'_{-,+} \omega + 2\phi'_{-,+} \omega + 2\partial' \beta_{-,+} = -\lambda \tag{54}
\]

where \( \phi' \) and \( \bar{\phi}' \) are taken by holding the coordinates \( \theta^i \) fixed, \( \beta \equiv \beta_{,0} \) and \( \lambda \) is the cosmological constant.

One technical point that was needed in the derivation of (54) is the commutator of \( \bar{\phi} \) through the integral sign, given by (boundary terms vanish for our particular restriction on \( t_{11} \)):

\[
\partial \int_{R_0}^{R} F d\bar{R} = \int_{R_0}^{R} \partial F d\bar{R} + \int_{R_0}^{R} \int_{R_0}^{R} \partial_\nu F d\bar{R}'.
\]

In this sense, we think of the trace equation as an equation for just the homogeneous part of \( \nu \). It is remarkable that \( \partial^k \xi_k \) (given above (53) of the form \( \dot{\alpha} + \dot{\beta} R \)) is of the form so that \( \dot{\alpha} \) and \( \dot{\beta} \) solve the trace equation. In other words, the expression for \( \partial^k \xi_k \) can be chosen as the solution to the trace equation (with vanishing cosmological constant). Therefore, we can always choose a gauge so that the conformal factor \( \nu \) has no homogenous part, if the cosmological constant is zero and the source \( t_{11} \) vanishes on a surface \( R = R_0 \).

The solution to the trace equation with non-vanishing \( \lambda \), modulo gauge, is

\[
\nu = \lambda (uR - R^2 - \omega \bar{\omega}). \tag{55}
\]

This solution can not be gauged away. We are free to choose \( \lambda \) to vanish or not to vanish. A general way to rule out a possible cosmological constant is by requiring asymptotically flat boundary conditions for the conformal factor.
V. CONCLUDING REMARKS

The null surface theory approach to GR to first order just described is meant to serve as a reliable test of the concepts of the full null surface theory. We found it extremely useful in clarifying the full version of the theory, which will be published elsewhere [6]. As a restriction to the linearized regime, however, it’s utility is limited and some results can not be carried over to the full theory. For example, we found that condition (B) in Sec.(IV.A.) is an identity in first order, but the fact that it remains an identity to all orders can not be established from the linearization. Nevertheless, it gave a hint as to what to expect from the full theory with respect to the relevance of (B).

The linearized null surface theory departs conceptually from the full theory in that it is a theory for the quantity \( \Lambda \) on a fixed known background. In this sense, many of the complications of the full theory do not have a place here, in particular, the difficulty that the coordinates and \( \Lambda \) are conceptually intertwined in the full theory, which makes the equations extremely lengthy (and the ideas at the start quite confusing). It’s a fact that the linearization of the null surface theory becomes itself a separate theory, easier to handle, where the methods used do not need to agree completely with the ones needed in the full theory. For example, the conformal factor \( \nu \) can actually be solved for without knowing \( \Lambda \), and then it can be used as a known source to find \( \Lambda \), a fact that does not apply to the full theory.

Throughout this work, the null surfaces are known to zeroth order and get corrected by the first order \( \Lambda \), i.e., \( \Lambda \) gives a first order correction to the null planes introduced in Sec.II. One can develop a perturbative scheme in which the surfaces are null planes in flat space, and get corrected successively with every higher order step in the hierarchy. The complications in the equations increase considerably at the second order level, making it, at least for the moment, a very difficult task to accomplish explicitly. A formal procedure can be outlined which shows no particular complications in the development of a perturbative scheme around the flat null planes.

First order vacuum theory can be obtained by setting the sources equal to zero, and taking \( \nu = 0 \). It is interesting to notice that under these conditions, \( f_{ij} \) satisfies both the following: \( \eta^{ij} f_{ij} = 0 \) and \( \partial^j f_{ij} = 0 \), from equations (35) and (33). This means that in vacuum our gauge conditions are a null version of the harmonic gauge. Furthermore, equation (46) reduces to the homogenous wave equation \( \Box \Lambda = 0 \). Since all the components of \( f_{ij} \) can be found by applying partial derivatives and \( \delta \) or \( \bar{\delta} \) operations to \( \Lambda \) (all of which commute with \( \Box \)) we have the result that every component of the metric satisfies the homogeneous wave equation, \( \Box f_{ij} = 0 \). We are currently exploring the possibility of finding the general solution to (*), (A) and (C) in vacuum directly for \( \Lambda \) without the use of \( Z \). This approach, which reverses the roles of \( Z \) and \( \Lambda \) as primary versus auxiliary quantities, and which yields new insights into the metricity conditions, will be presented elsewhere.

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APPENDIX A: THE SOLUTION OF THE CUT FUNCTION EQUATION BY MEANS OF GREEN’S FUNCTIONS

The equation for the linearized cut function (50) is of the form:

\[ \bar{\partial}^2 \bar{\partial} Z = S(x^a, \zeta, \bar{\zeta}) \]  \hspace{1cm} (A1)

where \( Z \) is a function of \((\zeta, \bar{\zeta})\) and the four parameters \(x^a\). \( S \) is a given source. The solution to this equation is of the form:

\[ Z(\zeta, \bar{\zeta}, x^a) = \int G(\zeta, \bar{\zeta}, \zeta', \bar{\zeta}') S(\zeta', \bar{\zeta}') dS' \]  \hspace{1cm} (A2)

where \( G(\zeta, \bar{\zeta}, \zeta', \bar{\zeta}') \) is the appropriate Green’s function for the operator \( \bar{\partial}^2 \bar{\partial} \) on the sphere. This function, which has been found [3], is a rather simple expression in terms of the vector \( l^a \) introduced earlier in this work (see (14):

\[ G(\zeta, \bar{\zeta}, \zeta', \bar{\zeta}') = \frac{1}{4\pi} l^a(\zeta, \bar{\zeta}) l_a(\zeta', \bar{\zeta}') \ln \left( l^b(\zeta, \bar{\zeta}) l_b(\zeta', \bar{\zeta}') \right). \]  \hspace{1cm} (A3)

It is also useful to have the explicit version in terms of \((\zeta, \bar{\zeta})\), which is found from:

\[ l^a(\zeta, \bar{\zeta}) l_a(\zeta', \bar{\zeta}') = \frac{(\zeta - \zeta')(\bar{\zeta} - \bar{\zeta}')}{(1 + \zeta\bar{\zeta})(1 + \zeta'\bar{\zeta}')} \]

There is a method for explicitly evaluating integrals of this type on the sphere for a large class of sources \( S \) [8]. This method consists of rewriting the integral as a sum of contour integrals around simple poles. In this way the integral is reduced in a large part to the calculation of residues. The method works for singular integrands as well, our present case. Explicit results have already been obtained by the authors for the vacuum case with arbitrary quadrupole radiation data of the form \( \sigma(u, \zeta, \bar{\zeta}) = a(u) \sigma_2 Y_{2m}(\zeta, \bar{\zeta}) \), where \( 2Y_{2m}(\zeta, \bar{\zeta}) \) is a spin-weight-2 spherical harmonic and \( a(u) \) is arbitrary. These results will appear elsewhere.

APPENDIX B: CALCULATION OF THE DIVERGENCE AND DOUBLE DIVERGENCE OF THE METRIC

For the component \( i = - \) only the use of (26) is needed:

\[ \partial^i f_{-j} = \partial^0 f_{-0} + \partial^- f_{--} \]
\[ = \partial^0 \left( \frac{1}{2} \bar{\partial} \Lambda_1 \right) + \partial^- (-\Lambda_1) \]
\[ = \partial_t \left( \frac{1}{2} \bar{\partial} \Lambda_1 \right) - \partial_t (-\Lambda_1) \]
\[ = \partial_t \left( \frac{1}{2} \bar{\partial} \Lambda_1 + \Lambda_+ \right) \]
\[ = 2 \partial_t \bar{\partial} \nu. \]  \hspace{1cm} (B1)

The component \( i = 0 \) requires the use of (26) and the commutation relations:
\[ \partial^j f_{0j} = \partial^0 f_{00} + \partial^- f_{0-} + \partial^+ f_{0+} \]

\[ = \partial^0 \left( -\frac{1}{2} \bar{\delta}^2 \Lambda_1 \right) + \partial^- f_{0-} + \partial_1 \bar{\delta} \nu + \text{complex conjugate.} \]

\[ = \partial^0 \left( -\frac{1}{2} \bar{\delta}^2 \Lambda_1 \right) + \partial^- \left( \frac{1}{2} \bar{\delta} \Lambda_1 \right) + \partial_1 \bar{\delta} \nu + \text{complex conjugate.} \]

\[ = \left( -\partial^0 \bar{\delta} + \partial^- \right) \frac{1}{2} \bar{\delta} \Lambda_1 + \partial_1 \bar{\delta} \nu + \text{complex conjugate.} \]

\[ = -\left( \partial_1 \bar{\delta} + \partial_+ \right) \frac{1}{2} \bar{\delta} \Lambda_1 + \partial_1 \bar{\delta} \nu + \text{complex conjugate.} \]

\[ = -2(\partial_1 \bar{\delta} + \partial_+) \bar{\delta} \nu + \partial_+ (\partial_1 \bar{\delta} + \partial_+) \Lambda + \partial_1 \bar{\delta} \nu + \text{complex conjugate.} \]

\[ = -2(\partial_1 \bar{\delta} + \partial_+) \bar{\delta} \nu + \partial_+ (\bar{\delta} \partial_1 + 2 \partial_+ \nu) + \partial_1 \bar{\delta} \nu + \text{complex conjugate.} \]

\[ = -2\partial_1 \bar{\delta} \nu + 2 \partial_+ \bar{\delta} \nu + \partial_1 \bar{\delta} \nu + \text{complex conjugate.} \]

\[ = -2\partial_1 \bar{\delta} \nu + 2 \partial_+ \bar{\delta} \nu + \text{complex conjugate.} \]  \hspace{1cm} (B2)

The double divergence \( \partial^i \partial^j f_{ij} \) becomes:

\[ \partial^i \partial^j f_{ij} = \partial^0 \partial^j f_{0j} + \partial^+ \partial^j f_{+j} + \partial^- \partial^j f_{-j} \]

\[ = \partial_1 \partial^j f_{0j} - \partial_- \partial^j f_{-j} - \partial_+ \partial^j f_{+j} \]

\[ = \partial_1 (-2 \partial_1 \bar{\delta} \nu + 2 \partial_- \bar{\delta} \nu + 2 \partial_+ \bar{\delta} \nu) - 2 \partial_- (\partial_1 \bar{\delta} \nu) - 2 \partial_+ (\partial_1 \bar{\delta} \nu) \]

\[ = -2 \partial_1^2 \bar{\delta} \nu. \]  \hspace{1cm} (B3)

**APPENDIX C: THE STRESS ENERGY TENSOR FROM THE NULL SURFACE POINT OF VIEW**

Since we have an explicit \((\zeta, \bar{\zeta})\) dependence of the component \(t_{11}\) of a given symmetric trace-free tensor \(t_{ij}\), the remaining components can be obtained by an appropriate combination of successive applications of \(\overline{\delta}\) and \(\delta\) to \(t_{11}\). The following have been obtained with the help of the vanishing trace condition, \(t_{11} - t_{01} + t_{+-} = 0\):

\[ t_{1-} = -\frac{1}{2} \bar{\delta} t_{11} \]

\[ t_{1+} = -\frac{1}{2} \bar{\delta} t_{11} \]

\[ t_{-} = \frac{1}{2} \delta^2 t_{11} \]

\[ t_{+} = \frac{1}{2} \delta^2 t_{11} \]
\begin{align*}
t_{++} &= \frac{1}{4} \bar{\partial} \bar{\partial} t_{11} + \frac{1}{2} t_{11} \\
t_{0-} &= -\frac{1}{4} \bar{\partial} \delta^2 t_{11} - \partial t_{11} \\
t_{0+} &= -\frac{1}{4} \bar{\partial} \delta^2 t_{11} - \bar{\partial} t_{11} \\
t_{00} &= \frac{1}{4} \bar{\partial} \delta^2 t_{11} + \frac{3}{2} \bar{\partial} \bar{\partial} t_{11} + 3 t_{11} \\
t_{01} &= \frac{1}{4} \bar{\partial} \partial t_{11} + \frac{3}{2} t_{11} \tag{C1}
\end{align*}

As an example of the calculation, the first of (C1) was obtained in the following manner:

\[ t_{11} = \theta^a \theta^b t_{ab} = l^a l^b t_{ab} \text{ therefore } \bar{\partial} t_{11} = 2 \bar{\partial} l^a l^b t_{ab} = 2 m^a l^b t_{ab} = -2 \theta^a \theta^b t_{ab} = -2 t_{1-} \text{ and inverting: } t_{1-} = -\frac{1}{2} \bar{\partial} t_{11}. \]

Since the components of the trace-free part of the stress tensor constitute the right side of the trace-free Einstein equations, then by the same operations suggested above for \( t_{ij} \) we can obtain the nine trace-free Einstein equations from \( \partial^2 \nu = t_{11} \).

**APPENDIX D: THE TRACE OF THE EINSTEIN SYSTEM AND THE METRICITY CONDITIONS**

Here we show that

\[ \bar{\partial} \left( 3 \Box \nu + \partial_1^2 \bar{\partial} \bar{\partial} \nu \right) = 0 \]

by virtue of the metricity conditions. Though the calculation is fairly lengthy, we will outline the argument. We need Eq.\((*)\), (A) and Eq.(46):

\[ \Box \Lambda = 4 \partial_+ \partial \nu - 2 \partial_1 \partial^2 \nu = -2 (\partial_1 \bar{\partial} - 2 \partial_-) \partial \nu \tag{D1} \]

(obtained from (*) and (A)).

First we turn \( \bar{\partial} \partial_1^2 \bar{\partial} \nu \) into a more suitable expression:

\[ \bar{\partial} \partial_1^2 \bar{\partial} \nu = \partial_1 \bar{\partial}(\partial_1 \bar{\partial} - 2 \partial_-) \partial \nu + \partial_1(\partial_+ \bar{\partial} + 2 \partial_1 - 2 \partial_0) \partial \nu. \]

Now we use (D1) to put the first term in terms of \( \Box \Lambda \) and use (*) to eliminate \( \nu \) in favor of \( \Lambda \) (also commute \( \partial_1 \bar{\partial} \) in the first term):

\[ \bar{\partial} \partial_1^2 \bar{\partial} \nu = -\frac{1}{2} \Box \Lambda_1 - \frac{1}{2} \Box \Lambda_+ + \frac{1}{2} \partial_1(\partial_+ \bar{\partial} + 2 \partial_1 - 2 \partial_0)(\Lambda_+ + \frac{1}{2} \bar{\partial} \Lambda_1). \]

If we put it together and after some work we obtain:
\[ \partial \left( 3 \Box \nu + \partial^2 \partial \bar{\nu} \right) = \frac{1}{2} \partial_+ \left( \Box \Lambda + (\partial_t \bar{\nu} - 2\partial_-) (\Lambda_+ + \frac{1}{2} \bar{\partial} \Lambda_1) \right). \]

If we commute \( \bar{\nu} \) and \( \partial_- \) in the term \( \partial_- \bar{\partial} \Lambda_1 \) and use (*) to eliminate \( \Lambda_- \) in favor of \( \bar{\partial} \Lambda_1 \) we obtain:

\[ \partial \left( 3 \Box \nu + \partial^2 \partial \bar{\nu} \right) = \frac{1}{2} \partial_+ \left( \Box \Lambda + (\partial_t \bar{\nu} - 2\partial_-) \Lambda_+ + \frac{1}{2} \partial_1 \bar{\partial} \Lambda_1 \right) - \frac{1}{2} \partial_1 \bar{\partial} \Lambda_1 + \frac{1}{2} \partial_+ \partial \Lambda_1 - \partial_0 \partial_1 \Lambda_1 + 2 \partial_1 \Lambda_1 \).

With some more work using the commutation relations only we obtain:

\[ \partial \left( 3 \Box \nu + \partial^2 \partial \bar{\nu} \right) = \frac{1}{2} \partial_+ \left( -3 \partial_+ (\Lambda_- - \frac{1}{2} \bar{\partial} \Lambda_1) + \frac{1}{2} \partial_1 (\bar{\partial} \partial - \bar{\partial} \bar{\partial} + 4) \Lambda_1 \right). \]

This last result is clearly vanishing by virtue of the metricity condition (A) and the commutation relations for \( \bar{\partial} \) and \( \partial \) on quantities of spin weight equal to 2.

[1] C. N. Kozameh and E. T. Newman, J. Math. Phys. \textbf{24}, 2481 (1983).
[2] C. N. Kozameh and E. T. Newman, in \textit{Asymptotic Behavior of Mass and Spacetime Geometry}, edited by F. J. Flaherty (Springer-Verlag, New York, 1984).
[3] S. V. Iyer, Ph. d. dissertation, University of Pittsburgh, 1993.
[4] R. M. Wald, \textit{General Relativity} (The University of Chicago Press, Chicago, 1984).
[5] E. T. Newman and R. Penrose, J. Math. Phys. \textbf{5}, 863 (1966).
[6] S. Frittelli, C. N. Kozameh, and E. T. Newman, in preparation.
[7] J. Ivancovich, C. N. Kozameh, and E. T. Newman, J. Math. Phys. \textbf{30}, 45 (1989).
[8] C. N. Kozameh, E. T. Newman, and J. Porter, Foundations of Physics \textbf{14}, 1061 (1984).