Possible solutions of inverse dynamical problems with regards for nonlinear constraint stabilization function

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Abstract. Baumgarte’s method of constraint stabilization can be applied to find a stable numerical solution in relation for constraint equations during the numerical integration. According to this method full time derivatives of constraints are equated to their arbitrary linear form. Choosing values of the coefficients of this form allow us to obtain a stable solution. In this paper a method of constructing Lagrange equations of the second kind is considered with regard for a nonlinear homogeneous stabilization function. A necessity if introducing dissipative function to find a complete stabilization problem’s solution can be the main conclusion of the research.

1. Introduction
Numerical integration of differential equations with constraints can not always provide a stable solution. Accumulation of deviations on each step of summation cause an increase of constraint deviation values. J. Baumgarte in his paper [1] suggested to modify constraint equations. During the procedure of determining Lagrange multipliers derivatives of constraint equations equate to linear form of constraints themselves. Appropriate choice of coefficients of such form allows us to achieve a stable solution. Some estimations of safe range of coefficient values were established in papers [2], [3] for Euler’s approach and in [4] for Runge-Kutta approach.

The problem of constructing a system of motion equations with regard for constraint equations were considered in [5]. To establish whether an obtained system is reduced to the form of Lagrange equations and whether it allows us to solve a stabilization problem we consider generalized Helmholtz conditions [6] as a necessary criteria to our system.

2. Problem statement
Let a mechanical system be described by a set of \( n \) generalized coordinates \( q = \{q^1, ..., q^n\} \) and velocities \( \dot{q} = dq/dt = \{q^1, ..., q^n\} \). Let us also assume that a mechanical system is a restricted by a set of mechanical constraints

\[ g_\alpha(q) = 0; \alpha = 1, ..., s_1; \]
\[ h_{\mu}^{\alpha}(q)q^a = h^{0}_{\mu}, \mu = s_1 + 1, ..., s_2, a = 1, ..., n, s_2 < n. \]


Here and in the future, repeating indexes will undermine summation.

This set of equations can be represented in matrix form $\mathbf{H} \mathbf{q}' = \mathbf{H}^0$ where

$$h_{\rho b} = \begin{cases} \frac{\partial g_b}{\partial q^\rho} & \text{if } \rho = 1, \ldots, s_1, \\ h_{\rho b} & \text{if } \rho = s_1 + 1, \ldots, s_2, \end{cases} \quad H^0_{b\rho} = \begin{cases} 0 & \text{if } \rho = 1, \ldots, s_1, \\ h^0_{b\rho} & \text{if } \rho = s_1 + 1, \ldots, s_2. \end{cases}$$

We can express the generalized velocities as a function of coordinates. Let’s assume that $\mathbf{q}' = \mathbf{v}(\mathbf{q})$ where vector function $\mathbf{v}(\mathbf{q})$ is chosen such that $\mathbf{H} \mathbf{v} = \mathbf{H}^0$. Because rank of a matrix $\mathbf{H}$ is less than a number of independent coordinates, functions $\mathbf{v}(\mathbf{q})$ can be selected ambiguously. If we consider a specific problem, then by choosing the functions $\mathbf{v}(\mathbf{q})$ we determine the system’s trajectory. In other words, a system of curves is reduced to one true trajectory.

So we can rewrite our constraint equations in the following form

$$h_a = q^a - v^a(\mathbf{q}) = 0, \quad a = 1, \ldots, n. \quad (1)$$

This form of equations is much easier to apply to Baumgarte’s constraint stabilization method

$$h_a = F_a(h_1, \ldots, h_n, \mathbf{q}, \mathbf{q}', t), \quad a = 1, \ldots, n. \quad (2)$$

Functions $F_a$ must satisfy the following conditions:

- $F(0, \ldots, 0, \mathbf{q}, \mathbf{q}', t) = 0$;
- the trivial solution must be Lyapunov stable.

The system of 2nd order differential equations that satisfy the constraint equations (1) with regard for stabilization function (2) can be written as follows

$$f_a = q^a - \frac{\partial v^a}{\partial q^c} q^c - F_a(h_1, \ldots, h_n, \mathbf{q}, \mathbf{q}', t) = 0. \quad (3)$$

If we want this system to be reduced to the form of Lagrange equations of the second kind, then we must demand the fulfillment of Helmholtz’s conditions:

$$\frac{\partial f_a}{\partial q^b} - \frac{\partial f_b}{\partial q^a} = 0; \quad (4)$$

$$\frac{\partial f_a}{\partial q^b} + \frac{\partial f_b}{\partial q^a} = \frac{d}{dt} \left( \frac{\partial f_a}{\partial q^b} \right); \quad (5)$$

$$\frac{\partial f_a}{\partial q^c} - \frac{\partial f_a}{\partial q^c} = \frac{d}{dt} \left( \frac{\partial f_a}{\partial q^c} \right); \quad (6)$$

$$\frac{\partial f_a}{\partial q^b} = 0, \quad a, b, c = 1, \ldots, n. \quad (7)$$

Conditions (4) and (7) are obviously satisfied. Condition (5) leads to the following expression

$$\frac{\partial v^a}{\partial q^b} + \frac{\partial F_a}{\partial h_b} + \frac{\partial v^b}{\partial q^a} + \frac{\partial F_b}{\partial h_a} + \frac{\partial v^a}{\partial q^a} = 0. \quad (8)$$

Let’s just assume that the stabilization function is chosen such as

$$\frac{\partial F_a}{\partial q^b} + \frac{\partial F_b}{\partial q^a} = 0. \quad (9)$$

To achieve this equality, one of the following relationships must take place.

$$\frac{\partial F_a}{\partial h_b} = -\frac{\partial v^a}{\partial q^b}. \quad (10)$$
These two expressions can be considered as equations to find a nonlinear stabilization function that satisfies all necessary conditions. But solution (10) does not satisfy the condition (6). And solution (11) does satisfy (6) only if the following equations take place

\[ \frac{\partial F_a}{\partial q^b} - \frac{\partial F_b}{\partial q^a} = 0; \]  

(12)

So, conditions (9), (11) and (1) allow us to set stabilization function such that the system of equations (3) will be reduced to the form of Lagrange equations of the second kind. One of possible solutions of (11) is a linear function

\[ F_a = k_a^b h_b, \quad k_a^b = -\frac{\partial v^b}{\partial q^a}. \]  

(13)

However, this approach cannot always be helpful. As we have discussed earlier functions \( v^b(q) \) are unambiguously set for the specific trajectory and they cannot always provide us with a Lyapunov stable solution.

### 3. Dissipative function and generalized Helmholtz conditions

In this chapter we consider conditions of reducing equations (3) to the form of Lagrange equations of the second kind with dissipative function.

\[ \frac{\partial f_a}{\partial q^b} - \frac{\partial f_b}{\partial q^a} = 0; \]  

(14)

\[ \frac{1}{2} \left( \frac{\partial f_a}{\partial q^a} + \frac{\partial f_b}{\partial q^b} \right) - \frac{d}{dt} \left( \frac{\partial f_a}{\partial q^a} \right) = s_{ab}; \]  

(15)

\[ \frac{\partial f_a}{\partial q^b} - \frac{\partial f_b}{\partial q^a} - \frac{d}{dt} \left( \frac{\partial f_a}{\partial q^a} - \frac{\partial f_b}{\partial q^b} \right) = r_{ab}; \]  

(16)

\[ \frac{\partial^2 f_a}{\partial q^a \partial q^c} = 0; \]  

(17)

\[ \frac{\partial s_{ab}}{\partial q^c} - \frac{\partial s_{ac}}{\partial q^b} = 0; \]  

(18)

\[ \frac{\partial r_{ab}}{\partial q^c} = \frac{\partial s_{ac}}{\partial q^b} - \frac{\partial s_{bc}}{\partial q^a}; \]  

(19)

\[ \frac{\partial r_{ab}}{\partial q^c} + \frac{\partial r_{ac}}{\partial q^a} + \frac{\partial r_{ca}}{\partial q^b} = 0, \quad a, b, c = 1, \ldots n. \]  

(20)

where functions \( s_{ab} \) and \( r_{ab} \) are connected to the dissipative function \( D(q, q, t) \) through the following expressions

\[ s_{ab} = \frac{\partial^2 D}{\partial q_a \partial q_b}, \quad r_{ab} = \frac{\partial^2 D}{\partial q_a \partial q_b} - \frac{\partial^2 D}{\partial q_b \partial q_a}. \]  

(21)

Conditions (14) and (17) are also obviously satisfied. Let's represent the tensor \( s_{ab} \) as a sum of two nonsymmetrical tensors \( s_{ab} = 1/2(d_{ab} + d_{ba}) \). Then condition (15) leads us to the following expressions

\[ -\frac{\partial v^a}{\partial q^b} + \frac{\partial f_a}{\partial h_b} + \frac{\partial f_a}{\partial q^b} - \left( \frac{\partial v^b}{\partial q^a} + \frac{\partial f_b}{\partial h_a} + \frac{\partial f_b}{\partial q^a} \right) = d_{ab} + d_{ba}. \]  

(22)

Obtained expression can be treated as a system of equations in relation to the constraint stabilization function. Let us consider two possible solutions.
Let us combine the terms so the derivatives of the stabilization functions in relation to the constraint equations would compensate the terms with constraints and dissipative function

\[
\begin{align*}
\frac{\partial F_a}{\partial h_b} &= -d_{ab} - \frac{\partial \nu^b}{\partial q^a}; \\
\frac{\partial F_a}{\partial h_b} &= \frac{\partial F_b}{\partial q^b} + \frac{\partial F_b}{\partial q^a} = 0.
\end{align*}
\]

(23)

Condition (16) for this solution leads to the following equations

\[
\begin{align*}
\frac{\partial \nu^c}{\partial q^a} - d_{bc} \frac{\partial \nu^c}{\partial q^b} - \frac{1}{2} \frac{d}{dt} (d_{ab} - d_{ba}); \\
\frac{\partial F_a}{\partial h_b} - \frac{\partial F_b}{\partial q^b} &= d \left( \frac{\partial F_a}{\partial h_b} \right). 
\end{align*}
\]

(24)

(25)

Let the dissipative function be a quadratic form of the generalized velocities. Then condition (18) is obviously satisfied and condition (19) leads to the equation

\[
\frac{\partial d_{ac}}{\partial q^b} + \frac{\partial d_{cb}}{\partial q^a} - \frac{\partial d_{bc}}{\partial q^a} - \frac{\partial d_{ab}}{\partial q^b} = 0.
\]

(26)

Substituting the relations (24)-(26) in the condition (20), we get the identity. It means that the solution satisfies all generalized Helmholtz conditions. As functions $d_{ab}$ are quadratic coefficients of dissipative function $D$, then by choosing their values we can manipulate the values of stabilization function so the numerical solution would be stable. For instance, if $F$ is linear then

\[
k_a^b = -\frac{\partial \nu^b}{\partial q^a} - d_{ab}(q).
\]

As we can see extra dissipative term $d_{ab}$ generates some sort of ambiguity with the help of which we can solve the stabilization problem.

Let’s combine terms in (22) as follows

\[
\begin{align*}
\frac{\partial F_a}{\partial h_b} &= -\frac{\partial \nu^b}{\partial q^a}; \\
\frac{\partial F_a}{\partial h_b} &= -d_{ab}(q).
\end{align*}
\]

(27)

By substituting it into (16) we obtain

\[
r_{ab} = \frac{\partial F_a}{\partial q^b} - \frac{\partial F_b}{\partial q^a} - \frac{1}{2} \frac{d}{dt} (d_{ab} - d_{ba}).
\]

If we choose tensor $d_{ab}$ symmetric then conditions (18) – (20) will be satisfied. In this case we can also solve the stabilization problem by choosing the values of tensor $d_{ab}$. However, this solution does not provide a linear solution for $F$ function. Because linear function that is a conclusion of (27) can be homogeneous by constraint equations $h$.

4. Conclusion

In this paper it was shown that introduction of dissipative function is necessary to solve the constraint stabilization problem. In the case of nonlinear stabilization several solutions for the stabilization function exist. Some of them are more preferable then others according to the problem statement.

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