Quantum Fusion of Domain Walls with Fluxes

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Abstract

We study how fluxes on the domain wall world volume modify quantum fusion of two distant parallel domain walls into a composite wall. The elementary wall fluxes can be separated into parallel and antiparallel components. The parallel component affects neither the binding energy nor the process of quantum merger. The antiparallel fluxes, instead, increase the binding energy and, against naive expectations, suppress quantum fusion. In the small flux limit we explicitly find the bounce solution and the fusion rate as a function of the flux. We argue that at large (antiparallel) fluxes there exists a critical value of the flux (versus the difference in the wall tensions), which switches off quantum fusion altogether. This phenomenon of flux-related wall stabilization is rather peculiar: it is unrelated to any conserved quantity. Our consideration of the flux-related wall stabilization is based on substantiated arguments that fall short of complete proof.
1 Introduction

In our previous paper [1] we showed how to describe quantum fusion of two parallel elementary domain walls with tension $T_1$ into a composite wall with tension $T_2$, with a binding energy, i.e. $T_2 < 2T_1$. The distance $d$ between the elementary walls is assumed to be much larger than the wall thickness. An illustrative example in which composite domain walls have a binding energy, and our calculation can be applied, is that of the $k$-walls in $\mathcal{N} = 1$ super-Yang–Mills theory [2].

Many microscopic theories supporting domain walls allow one to introduce constant magnetic fluxes inside the walls [3, 4]. In these cases the corresponding wall world-volume theory is (2 + 1)-dimensional QED,

$$L_{2+1} = -\frac{1}{4e_{2+1}^2}F_{\mu\nu}^2. \quad (1)$$

(In what follows we will omit the subscript $2+1$, as the only electric charge that will appear below is that of the (2 + 1)-dimensional theory.) It is well known [5] that in 2 + 1 dimensions the electromagnetic field can be dualized into a compact scalar field $\sigma$, defined mod $2\pi$,

$$F_{\mu\nu} = \frac{e^2}{2\pi} \varepsilon_{\mu\nu\rho} \partial^\rho \sigma. \quad (2)$$

The Lagrangian takes the form

$$L_{2+1} = \frac{e^2}{8\pi^2} (\partial_\mu \sigma)^2. \quad (3)$$

The field $\sigma$ linearly depending on a spatial coordinate $x^i$

$$\sigma = n^\mu x^\mu, \quad n_y \equiv n, \quad (4)$$

with all other components of $n^\mu$ vanishing, describes a constant electric field on the wall world volume, which is in one-to-one correspondence with the magnetic flux trapped inside the wall in the bulk description [4].

In this paper we consider the impact of possible fluxes on the wall quantum fusion. For parallel fluxes, the binding energy and the fusion rate remain unchanged. Therefore, the question we focus on is the impact of antiparallel
fluxes. In this case the flux contribution effectively increases the tension of the elementary walls. Since on the composite wall the flux vanishes, $T_2$ stays intact. As a result, effectively,

$$\Delta T \equiv 2T_1 - T_2$$

increases which, at first sight, should entail an enhancement of the fusion rate.

We will show that, in fact, it is the opposite tendency that prevails: switching on antiparallel fluxes on the elementary walls increases the bounce action and, hence, suppresses the fusion rate.

Our consideration proceeds in two stages. First, we consider the problem in the limit $e^2 n^2 \ll \Delta T$. In this limit the bounce solution can be explicitly determined, and its action analytically calculated. The fusion rate obtained in the no-flux problem [1]

$$\Gamma \propto e^{-S_B}, \quad S_B = \frac{\pi}{3} T_1 d^3 \sqrt{\frac{T_1}{\Delta T}}$$

gets modified in a rather minimal way,

$$\left( S_B \right)_{n \neq 0} = \frac{\pi}{3} T_1 d^3 \sqrt{\frac{T_1}{\Delta T - \frac{e^2 n^2}{4\pi^2}}}.$$  \hspace{1cm} (7)

The suppression of the fusion rate is obvious in Eq. (7). Taking Eq. (7) at its face value, we would conclude that for the fusion to occur $\Delta T$ must exceed a critical value, $\Delta T_* = \frac{e^2 n^2}{4\pi^2}$. At this critical point the fusion rate vanishes; and at $\Delta T < \Delta T_*$ the elementary wall fusion through quantum tunneling becomes impossible since the Euclidean bounce configuration no longer exists.

Equation (7), literally speaking, becomes invalid at $\Delta T_* \sim \frac{e^2 n^2}{4\pi^2}$. Its derivation, to be presented below, is based on the small flux assumption. This assumption is relaxed at the second stage. We argue that the conclusion survives at a qualitative level: $(S_B)_{n \neq 0} \to \infty$ at some finite positive value of $\Delta T = \Delta T_*$,

$$\Delta T_* = \text{const} \cdot \frac{e^2 n^2}{4\pi^2},$$

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where the constant appearing on the right-hand side is of order 1. We call this phenomenon flux-induced stabilization of the wall fusion. The flux-induced stabilization is an interesting and rather peculiar phenomenon. Usually, when speaking of flux stabilization, we have in mind something related to conserved quantities. For example, some radius (which can be the size of a soliton or of a cycle on a manifold) can be stabilized by a flux captured inside. Combined application of energy and flux conservation prevents the radius from shrinking to zero, stabilizing the object under consideration. In our problem, instead, conservation laws would be perfectly consistent with, and, moreover, in favor of the fusion of the two separated elementary walls. The fusion is prohibited by the absence of any finite-action configuration (bounce) that could mediate the process. In this sense the situation is similar to that discovered by Coleman and De Luccia [6] who found that gravity-related effects suppress the process of false vacuum decay through bubble creation. It is worth noting that a suppression of the existence of a bounce was also observed in [7] from the string/D-brane theory side. The problem considered in [7] was different (creation of pairs of particles, charged electrically or magnetically, in external fields), but the bounce suppression and, in particular, the existence of a critical electric field, seemingly have a common origin which can be traced back to [8] where the rate of pair production of open bosonic and supersymmetric strings in a constant electric field was discussed. If we continued this parallel, at an intuitive level, we might conjecture that, in our problem, when $n$ reaches its critical value, the bounce solution under consideration breaks into separate pieces.

It should be also mentioned that the process we consider is the fusion of walls due to quantum tunneling. Generally, depending on particular situation, the fusion can proceed classically, e.g. if the walls actually intersect or if they move toward each other. For strictly parallel walls that are exactly at rest a classical fusion is possible due to an exponentially weak attraction between them at large distances. Clearly, the classical fusion crucially depends on the initial conditions. For instance any exponentially weak attraction becomes irrelevant if the walls are slowly moving away from each other, so that the classical fusion never occurs. The behavior of the quantum fusion, considered in the present paper, is different in that the exponential factor in
the probability has a smooth dependence on the initial conditions, such as (a small) relative velocity of the walls.

The microscopic theory supporting domain walls with fluxes, which can be taken as an example, is a straightforward generalization of that of Ref. [3], namely, $\mathcal{N} = 2$ SQED, with three matter hypermultiplets with masses $m_{1,2,3}$, and a Fayet–Iliopoulos term. There are three vacua, according to which hypermultiplet’s scalar field $a$ is locked. Correspondingly, there are three domain walls. We will refer to two walls with smaller tension, say $\langle 12 \rangle$ and $\langle 23 \rangle$ as to elementary walls. Then the wall $\langle 13 \rangle$ is composite. The masses $m_{1,2,3}$ are complex parameters. If they are not aligned in the complex plane, there is a natural binding energy between the elementary walls. If they are aligned, $T_2 = 2T_1$ and $\Delta T = 0$.

We will assume that $\Delta T \ll T_1$ so that, instead of the full Dirac–Born–Infeld action on the wall world volume, we can limit ourselves to quadratic terms. Geometry of the problem, our notation and all constraints and limitations are the same as in Ref. [1].

## 2 Switching on fluxes in the limit $e^2 n^2 \ll \Delta T$

The existence of the flux (4) entails two consequences. First, the effective wall tension changes. If, without the fluxes, it was $T_1$, then, with the fluxes switched on it becomes

$$T_1 \rightarrow T_1 + \delta \equiv T_1 + \frac{e^2}{8\pi^2} n^2 .$$

Without loss of generality we can assume that the fluxes on two elementary walls are the same in the absolute value and antiparallel. Then the flux on the composite wall must vanish. Correspondingly, the tension of the composite wall $T_2$ remains intact. Note that $n$ has dimension of mass, and so does the three-dimensional coupling $e^2$.

The second novel element is the necessity of matching of the $\sigma$ fields on the elementary walls at the boundary of the fused domain.

The geometry of the problem is exhibited in Figure 1: we have two elementary parallel walls $\langle 12 \rangle$ and $\langle 23 \rangle$ lying in the $x, y$ plane, at separation $d$
in the z direction. There are antiparallel fluxes on both of them.\footnote{If the fluxes are not antiparallel, say, \( n_{x, (12)} = n_{x, (23)} \) and \( n_{y, (12)} = -n_{y, (23)} \), the tensions of the two elementary walls are \( T_{(12)} = T_{(23)} = T_1 + \frac{e^2}{8\pi^2} (n_x^2 + n_y^2) \) while the tension of the composite wall is \( T_{(13)} = T_2 + \frac{e^2}{8\pi^2} (2n_x)^2 \). Since the \( x \) component matches automatically (and \( n_x = \text{const on the bounce} \), we can include the \( n_x^2 \) terms in the corresponding tensions. Then the problem reduces to that with antiparallel fluxes. Note two factors. The electric coupling \( e^2 \) of the composite wall is half of that of the elementary walls. If the fluxes are parallel (i.e. \( n_y = 0 \)) the energy stored in the flux has no binding effect. Since the \( n_x \) component has no effect, apart from an overall shift of the tensions, we set it to zero.}

In this section we will consider misaligned mass parameters \( m_i \) in the microscopic theory, i.e. \( 2T_1 > T_2 \). Our task is to find the bounce solution satisfying the following conditions: (i) In the linear approximation both \( z \) and \( \sigma \) satisfy the Laplace equation, and, in particular, they do not interact,

\[
\Delta z = 0, \quad \Delta \sigma = 0. \tag{10}
\]

(ii) The boundary conditions are set at infinity in \( \tau, x, y \) where \( \tau \) is Euclidean time, and at the boundary of the fused domain at \( z = 0 \). Both \( z \) and \( \sigma \) must match at \( z = 0 \), while at infinity \( z \to \pm d/2 \) and \( \sigma \to \pm n_\mu x^\mu \). Here

\[
x^\mu \equiv \{ \tau, x, y \}. \tag{11}
\]

The plus-minus signs refer to the upper and lower walls, respectively.

If

\[
e^2 n^2 \ll \Delta T \ll T_1 \tag{12}
\]

the spherical symmetry of the bounce field configuration (i.e. the fact that \( z(x^\mu) = f(\sqrt{x^\mu x^\mu}) \) remains approximately valid, since the \( \sigma \) related contribution can be considered as a small correction and its back reaction ignored. One can use the multipole expansion for the bounce solution keeping only the lowest harmonics, i.e. \( l = 0 \) for \( z \) and \( l = 1 \) for \( \sigma \). Then, to the leading order in this expansion, for the upper brane,

\[
z = \frac{d}{2} \left( 1 - \frac{r_*}{r} \right), \quad \sigma = n_\mu x^\mu \left( 1 - \frac{r_*^3}{r^3} \right), \tag{13}
\]

where

\[
r \equiv \sqrt{x^\mu x^\mu} \tag{14}
\]
and \( r_* \) is the radius of the fused domain with the composite wall at \( z = 0 \). Please, note that the solution (34) satisfies both boundary conditions. At \( r = r_* \), on the boundary of the composite wall, \( z = 0 \) and \( \sigma = 0 \). The value of \( r_* \) is to be determined through extremization (maximization) of the bounce action.

![Figure 1: Geometry of the problem. Two elementary walls at distance \( d \) with anti-parallel fluxes.](image)

Now we have to calculate the bounce action. It consists of a few distinct contributions: (a) at \( r > r_* \) we loose, compared to the two flat walls, due to the fact \( z \) and \( \sigma \) nontrivially depend on \( x^\mu \); on the other hand, at \( r < r_* \) we gain due to the fact \( T_2 < 2T_1 + \frac{e^2}{4\pi^2} n^2 \). The extremal balance between gain and loss is achieved at a critical (extremal) value of \( r_* \).

The loss due to the wall curvature (i.e. \( z \neq \text{const.} \)) is [1]

\[
(\Delta S_z)_{r>r_*} = \pi T_1 r_* d^2.
\]

(15)

The flux-related loss (due to \( \sigma \neq \sigma_{\text{asymptotic}} \)) is

\[
(\Delta S_\sigma)_{r>r_*} = 2 \times \frac{e^2}{8\pi^2} \int_{r_*}^{\infty} d\Omega \int r^2 \frac{dr}{\Omega} \left[ (\partial \sigma)^2 - n^2 \right] = \frac{2e^2}{3\pi} r_*^3 n^2,
\]

(16)

where \( d\Omega \) presents the angular integration, \( \int d\Omega = 4\pi \), and the overall factor of 2 is due to the fact that both elementary walls are included in (16). (The same is valid for (15).)
The gain from the domain $r < r_*$ is

$$\Delta S_{r<r_*} \equiv (\Delta S_z)_{r<r_*} + (\Delta S_\sigma)_{r<r_*} = -\frac{4\pi r_*^3}{3} \left(2T_1 + \frac{e^2}{4\pi^2} n^2 - T_2\right).$$  \hspace{1cm} (17)$$

The total contributions due to $\sigma$ and $z$ are as follows:

$$\Delta S_\sigma = (\Delta S_\sigma)_{r>r_*} + (\Delta S_\sigma)_{r<r_*} = \frac{e^2}{3\pi} r_*^3 n^2,$$

$$\Delta S_z = (\Delta S_z)_{r>r_*} + (\Delta S_z)_{r<r_*} = \pi T_1 r_* d^2 - \frac{4\pi r_*^3}{3} \Delta T. \hspace{1cm} (18)$$

Note that the $\sigma$-related gain at $r < r_*$ is over-compensated by the loss at $r > r_*$. Equation (18) implies

$$\Delta S = \Delta S_z + \Delta S_\sigma = \pi T_1 r_* d^2 - \frac{4\pi r_*^3}{3} \left(\Delta T - \frac{e^2}{4\pi^2} n^2\right), \hspace{1cm} (19)$$

Next, to find the critical radius, we must extremize $\Delta S$ with respect to $r_*$. The functional form of $\Delta S$ in (19) is the same as in the problem [1] with vanishing fluxes. As a result, the extremal value of the radius of the fused domain takes the form

$$r_* = \frac{d}{2} \sqrt{\frac{T_1}{\Delta T - \frac{e^2 n^2}{4\pi^2}}}, \hspace{1cm} (20)$$

while the bounce action is

$$S_B = \frac{\pi}{3} T_1 d^3 \sqrt{\frac{T_1}{\Delta T - \frac{e^2 n^2}{4\pi^2}}}. \hspace{1cm} (21)$$

As usual, the fusion rate per unit time per unit area of the wall is proportional to

$$\Gamma \sim e^{-S_B}. \hspace{1cm} (22)$$

A few explanatory comments are in order here. Equation (21) presents the maximal value of $\Delta S$ as a function of $r_*$ (as opposed to minimal), in full accordance with the fact the fusion process under consideration is that of quantum instability of two flat elementary walls. The flux-related contribution suppresses the decay rate. If we formally extrapolate the result to
\[ \Delta T = \frac{e^2 n^2}{4\pi}, \] at this point \( S_B \to \infty \) and the suppression becomes absolute. There is no wall fusion below this point. However, Eq. (21) is valid only at small \( e^2 n^2/\Delta T \), to the leading order in this parameter. This is due to the fact that we used the spherically symmetric ansatz (34) for \( z(\tau, x, y) \). When \( e^2 n^2/\Delta T \sim 1 \) this ansatz is no longer justified. Deviations from sphericity will be discussed in the next section.

\textbf{3 The general case:} \( e^2 n^2 \lesssim \Delta T \)

In supersymmetric theories with critical (BPS saturated) walls the degenerate situation

\[ T_2 = 2T_1 \] (23)

is not uncommon. Then, if there are no fluxes the two elementary domain walls at rest at separation \( d \) present an absolutely stable configuration. Quantum fusion is impossible. If we switch on antiparallel fluxes on these walls, according to (9), the tension of the elementary walls increases while that of the composite wall stays intact, i.e. effectively \( 2T_1 \) becomes larger than \( T_2 \). Therefore, one might suspect that switching on fluxes induces quantum fusion in the degenerate case. In fact, as was shown above, the tendency is just opposite. Equation (21) suggests that there may exist a critical value of \( \Delta T/e^2 n^2 \) below which the wall fusion through tunneling becomes impossible.

To verify this hypothesis we have to move away from the small flux limit, i.e. relax the first condition in (12). The spherical approximation valid for small fluxes is not expected to be applicable at \( e^2 n^2/4\pi \sim \Delta T \). Indeed, at \( e^2 n^2/4\pi \sim \Delta T \) the dipole contribution due to \( \sigma \) becomes important in the determination of the bounce shape. It feeds back into the solution for \( z(\tau, x, y) \), generating angular momentum \( l = 2 \) in the \( z \) profile. This, in turn, triggers \( l = 3 \) harmonics in the \( \sigma \) solution, and so on. All terms in the multipole expansion enter the game. We will not be able to find the exact answer for the bounce configuration in this case. However, certain predictions are still possible.

Let us begin by discussing general lessons we can abstract from Sect. 2. The bounce solution has the following features. There is a certain domain \( \mathcal{M} \) (which includes the origin) inside which the two elementary walls are
merged; on the boundary of this domain $\partial M$ and inside it $z = 0$ and $\sigma = 0$. Outside $\mathcal{M}$ both $z$ and $\sigma$ satisfy the Laplace equations (10) with the boundary conditions at $\infty$

$$z \to \pm d/2, \quad \sigma \to \pm n^\mu x^\mu, \quad r \to \infty,$$

where $n^\mu = \{0, 0, n\}$. A typical linear dimension of $\mathcal{M}$

$$\ell_\mathcal{M} \gg d.$$

In addition to these general lessons from Sect. 2 we should add a particular lesson: the absolute value of the flux-related loss $|\Delta S_{\sigma}|_{r>r_*}$ is larger than that of the flux related gain $|\Delta S_{\sigma}|_{r<r_*}$. Below we will argue that this crucial feature bears a more general nature than sphericity.

As was mentioned, beyond the small flux limit, spherical symmetry is lost (although the axial symmetry survives); in particular, $\partial M$ is no longer $S_2$. It is worth emphasizing that the equations are still linear (see Eq. (10)); the coupling between $z$ and $\sigma$ is realized through the shape of $\partial M$. The condition of the tension balance at the boundary $\partial M$ is

$$\left(2T_1 + \frac{e^2}{4\pi^2} (\partial \sigma)^2\right) \frac{1}{\sqrt{1 + (\partial z)^2}} = T_2,$$

where on the left-hand side we have the tension of the two external branes multiplied by the cosine of the angle at which they merge, while on the right-hand side the tension of the composite brane. This information seems to be sufficient to determine, at least qualitatively, the shape of the domain $\mathcal{M}$.

The presence of the flux $\partial \sigma$ is what causes deviation from sphericity. The domain $\mathcal{M}$ will be elongated (Fig. 2). A crucial question is whether this elongation is in the $\vec{n}$ direction or is perpendicular to $\vec{n}$? There are a few arguments one can give supporting the first option: that the exact solution of (25) will be elongated along the direction of $\vec{n}$, see Fig. 2. Consider the spherical solution as a starting point. Then the value of $\partial \sigma$ in the direction orthogonal to the sphere is $n \cos \theta$ where the angle $\theta$ is measured from the vertical axis. This means that the north and south poles feel an outward force, while at the equator the extra force vanishes.
Figure 2: A slice of the auxiliary hyperplane $Z_0$ at $\tau = x =$const. (straight line). The curved hypersurface $Z(\tau, x, y)$, which consists of free domains, including a horizontal domain $AB$ near the origin, approaches the hyperplane $Z_0$ at large distances.

Let us look at the same question from a slightly different perspective. Let us call $r_1$ the “radius” of $\mathcal{M}$ perpendicular to the $\vec{n}$ direction, and $r_2$ in the parallel direction. Then the volume of $\mathcal{M}$ scales as $V(\mathcal{M}) \sim r_1^2 r_2$. If $r_1 < r_2$ then $\mathcal{M}$ is elongated in the $y$ direction, otherwise the elongation is in the perpendicular direction. In the search of solution we may first find the shape for the $z$ field. The problem is analogous to that from electrostatics, of a conducting object $\mathcal{M}$ at a certain potential. The charges in general are denser near the domain of strong curvature, implying that $\partial z$ is larger at the poles in the case of vertical orientation of $\mathcal{M}$ (Fig. 2) or at the equator in the case of perpendicular orientation. The larger the deviation from sphericity, the larger is the agglomeration of charges near the curved areas. Now let us consider the sigma-field solution. This case is different: $\partial \sigma$ always vanishes at the equator and is always maximal at the poles, independently of the shape. The balance of tensions on the surface $\partial \mathcal{M}$ requires that, where $\partial \sigma$ is larger, the slope $\partial z$ must also be larger to achieve compensation, implying $r_1 < r_2$.

Finally, we can try various particular ansätze for $\sigma(\tau, x, y)$ compatible
with the asymptotic behavior (24). For instance, if we take\(^2\)

\[
\sigma_0 = n^\mu x^\mu + c \left( \frac{1}{|x + a|} - \frac{1}{|x - a|} \right),
\]

(26)

where \(c\) is a numerical coefficient and \(a\) is a vector \(a^\mu = \{a_0, a_1, a_2\}\), then \(\partial \mathcal{M}\) on which \(\sigma_0\) vanishes exists only if the vectors \(n\) and \(a\) are parallel, and then \(\mathcal{M}\) is elongated along \(n\) with necessity.

The direction of elongation of \(\mathcal{M}\) will be crucial in what follows. Summarizing, we think it is fair to say that the arguments presented above are compelling, although stop short of proving the statement. To be cautious, for the time being, we will accept it as a motivated assumption.

What can be said of the bounce action under the above conditions? Let us first consider the contribution coming from \(x^\mu \in \mathcal{M}\). Given that the composite brane is flat inside \(\mathcal{M}\), and \(\sigma = 0\) on the composite brane, we get

\[
\Delta S_\prec = -V(\mathcal{M}) \left( 2T_1 + \frac{e^2}{4\pi^2} n^2 - T_2 \right),
\]

\[
\equiv -V(\mathcal{M}) \Delta T + (\Delta S_\sigma)_\prec
\]

(27)

where \(V(\mathcal{M})\) is the volume of \(\mathcal{M}\), and the subscript \(\prec\) indicates integration over \(x^\mu \in \mathcal{M}\).

Now, we have to calculate the loss \(\Delta S_\succ\) coming from integration over \(x^\mu \notin \mathcal{M}\). At first let us deal with \((\Delta S_\sigma)_\succ\),

\[
(\Delta S_\sigma)_\succ = \frac{e^2}{4\pi^2} \int_\succ d^3x \left[ (\partial \sigma)^2 - n^2 \right].
\]

(28)

Our task is to prove that \((\Delta S_\sigma)_\succ\) is larger than the absolute value of the flux-related gain,

\[
-(\Delta S_\sigma)_\prec = \frac{e^2}{4\pi^2} \int_\prec d^3x n^2.
\]

(29)

This requirement is identical to the condition

\[
\frac{e^2}{4\pi^2} \int d^3x \left[ (\partial \sigma)^2 - n^2 \right] > 0
\]

(30)

\(^2\)We assume \(c\) and \(a\) to be nonvanishing. The ansatz (26) contains \(l = 1\) and all higher odd waves in a certain combination.
where the integral runs over the entire three-dimensional space. The easiest way to see that Eq. (30) is satisfied is through an auxiliary geometrical picture. Indeed, let us introduce an auxiliary coordinate $Z \perp \tau, x, y$ such that

$$Z = \frac{\sigma}{\mu}, \quad Z_0 = \frac{n y}{\mu},$$

(31)

where $\mu$ is an auxiliary (large) parameter of dimension of mass. Then $Z_0$ represents a slightly tilted three-dimensional hyperplane, see Fig. 3. The integral in Eq. (30) represents $\frac{e^2 \mu^2}{4\pi^2} \times$ the difference between the hyperareas (volumes) of the curved and flat three-dimensional hypersurfaces (their $\tau = x =$ const. slice is shown in Fig. 3). Needless to say, considering $AB$ alone we would get a negative contribution. However, overall, the curved hypersurface has a larger area than the flat one given that both hypersurfaces touch each other at $r = \infty$. This completes the proof of the most crucial statement – the total flux-related contribution in $S_B$ is always positive, as was the case in the small-flux limit (Section 2).

Various estimates we have carried out for reasonably shaped probe $M$’s show that

$$\Delta S_\sigma = \kappa_M V(M) \frac{e^2 n^2}{4\pi^2}$$

(32)

where $\kappa_M$ is a positive number depending in geometry of $M$. Moreover, if $M$ is elongated in the $n$ direction, as was argued above, $\kappa_M > 1$. Although we were unable to obtain a general proof of this inequality, it is quite transparent. The spherical shape has $\kappa_{S^2} = 1$. The previous argument showed that the actual solution is elongated along the $n^\mu$ direction. Given the same volume
of $\mathcal{M}$, elongation in the direction of $n$ will create a larger disturbance (see Fig. 3), increasing the coefficient $\kappa$.

It might be instructive to consider a particular example. Consider $\sigma$ comprised of $l = 1$ and $l = 3$ harmonics,

$$\sigma = \cos \theta \left( r - \frac{1}{r^2} \right) + \alpha \frac{5 \cos^3 \theta - 3 \cos \theta}{r^4}$$

where $\alpha$ is a coefficient that deviates the surface $\sigma = 0$ from sphericity. Positive or negative $\alpha$ implies elongation of $\mathcal{M}$ in the $vecn$ direction or perpendicular. The flux-related contribution to the bounce action is independent of $\alpha$. On the other hand, the volume of $\mathcal{M}$ does depend on it. The volume $V(\mathcal{M})$ becomes smaller in the case of parallel orientation (Fig. 2) and larger in the perpendicular case.

The fact that $\kappa_\mathcal{M}$ is positive can be seen from an alternative argument. Indeed, let us use the multipole expansion for the $\sigma$ field at large $r$. The expansion then takes the form

$$\sigma = \frac{n^\mu x^\mu}{r^3} + O \left( \frac{1}{r^4} \right)$$

where $p_\mathcal{M}$ presents the dipole term, while all higher multipoles are hidden in $O(1/r^4)$ terms. It is natural to expect $p_\mathcal{M}$ to be positive. The flux-related contribution to the bounce action is

$$\Delta S_\sigma = \frac{e^2}{4\pi^2} \int d^3x \left\{ (\partial^\mu \sigma)^2 - [\partial^\mu (n^\alpha x^\alpha)]^2 \right\}$$

where we performed integration by parts and used the fact that $\Delta \sigma = 0$. The second integral in Eq. (35) runs over the surface of the large sphere, $S_2(R \to \infty)$. From Eq. (35) it is clearly seen that only the dipole term in $\sigma$ contributes to $\Delta S_\sigma$, all other multipoles fall off at infinity too fast to contribute. The result is

$$\Delta S_\sigma = p_\mathcal{M} \frac{e^2 n^2}{3\pi}$$

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3Strictly speaking, $\partial \sigma$ is not continuous on $\partial \mathcal{M}$. The discontinuity should be smoothed out. This produces no impact in Eq. (35).
Comparing with Eq. (32) we conclude that
\[ p_{\mathcal{M}} = \frac{3}{4\pi} \kappa_{\mathcal{M}} V(\mathcal{M}). \]  
(37)

To compute exactly the critical value of flux we need the relation between \( p_{\mathcal{M}} \) and \( V \) (i.e. the coefficient \( \kappa_{\mathcal{M}} \)). As was discussed above, \( \kappa_{\mathcal{M}} \geq 1 \) is to be expected.

Our next task is to analyze \( \Delta S_z \). The value of \( (\Delta S_z)_< \) can be read off from Eq. (27).

\[ (\Delta S_z)_< = -V(\mathcal{M}) \Delta T. \]  
(38)

This is the only negative contribution (gain) in \( S_B \). The value of \( (\Delta S_z)_> \) can be determined as follows. At \( x^\mu \notin \mathcal{M} \) the solution for \( z(\tau, x, y) \) can be expanded in spherical harmonics, starting from \( l = 0 \). Given the boundary conditions for \( z \) this multipole expansion at large \( r \) takes the form

\[ z = \frac{d}{2} - \frac{q}{r} + \text{higher harmonics}. \]  
(39)

All angular dependence resides in higher harmonics. The \( l = 2k \) harmonics are suppressed by \( 1/r^{1+2k} \) where \( k = 1, 2, ... \). The coefficient \( q \) depends on details of the bounce solution, and in particular, on geometry of \( \mathcal{M} \), i.e. \( q = q_{\mathcal{M}} \). What is important for us in what follows is that

\[ q_{\mathcal{M}} > 0. \]  
(40)

Moreover, \( q_{\mathcal{M}} \) has dimension \([m]^{-2}\) and is of the order of \( d \ell_{\mathcal{M}} \) (cf. the small flux limit in which \( q_{\mathcal{M}} = r_s d/2 \)). Then \( (\Delta S_z)_> \) can be expressed uniquely as a function of this coefficient. Indeed, integrating by parts and using the fact that \( z \) is a harmonic function at \( x^\mu \notin \mathcal{M} \) which vanishes at \( \partial \mathcal{M} \) we get

\[ (\Delta S_z)_> = T_1 \int_> d^3 x (\partial z)^2 = T_1 \left[ \int_{\partial \mathcal{M}} \partial^\mu (z\partial^\mu z) + \int_\infty \partial^\mu (z\partial^\mu z) \right] \]

\[ - T_1 \int_> d^3 x z \Delta z = 2\pi T_1 q_{\mathcal{M}} d. \]  
(41)

The only nonvanishing contribution to \( (\Delta S_z)_> \) comes from the surface integral \( \int_\infty \partial^\mu (z\partial^\mu z) \) and only from the \( l = 0 \) harmonics in Eq. (39). All higher
harmonics fall off too fast at infinity and do not affect the above surface integral. As a result,

$$\Delta S_z = 2\pi T_1 q_M d - V(M) \Delta T.$$  \hfill (42)

The bounce action is determined by one extra positive parameter $\kappa_M$ (see Eq. (32)) which at the moment is not yet firmly established,

$$\Delta S_\sigma + \Delta S_z = \kappa_M V(M) \frac{e^2 n^2}{4\pi^2} - V(M) \Delta T + 2\pi T_1 q_M. \quad (43)$$

The quantum fusion occurs only if $\Delta T > \kappa_M/e^2 n^2 4\pi^2$. This determines a critical value of $\Delta T$ below which the domain wall fusion is impossible,

$$\Delta T_* = \frac{\kappa_M e^2 n^2}{4\pi^2}. \quad (44)$$

To find the critical bounce action we have to extremize (43) with respect to the bounce size. What we can do is to find the extremum with regards to dilatations of $x_\mu$. To this end we take into account that $V(M)$ scales as $\ell_M^3$ while $q_M \sim \ell_M d$. From this we conclude that

$$S_B \geq \frac{4\pi T_1 q_M}{3}. \quad (45)$$

At $\Delta T < \Delta T_*$ no balance between gain and loss in $S_B$ is achievable. At $\Delta T = \Delta T_*$ the coefficient $q_M$ must tend to infinity.

4 Conclusions

In the small flux limit we explicitly find the bounce solution and the fusion rate as a function of the flux. We argue that at large (antiparallel) fluxes there exists a critical value of the flux (versus $2T_1 - T_2$), which switches off quantum fusion altogether. However, a reservation is in order here: our consideration of the flux-related wall stabilization is based on substantiated arguments that, nevertheless, stop short of unquestionable proof. We used the same framework as in Ref. [1]. In particular, the binding energy is assumed to be much less than the wall tension, so that the DBI action can be expanded up to quadratic in derivative terms, while higher terms neglected.
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