EQUALITY CASES IN VITERBO’S CONJECTURE AND ISOPERIMETRIC BILLIARD INEQUALITIES

ALEXEY BALITSKIY

Abstract. In this note we apply the billiard technique to deduce some results on Viterbo’s conjectured inequality between volume of a convex body and its symplectic capacity. We show that the product of a permutohedron and a simplex (properly related to each other) delivers equality in Viterbo’s conjecture. Using this result as long as previously known equality cases, we prove some special cases of Viterbo’s conjecture and interpret them as isoperimetric-like inequalities for billiard trajectories.

1. Introduction

Claude Viterbo [14] conjectured an isoperimetric inequality for any symplectic capacity and the volume of a convex body $X \subset \mathbb{R}^{2n}$:

$$\text{vol}(X) \geq \frac{c(X)^n}{n!}.$$  

The minimum is supposed to be attained (not uniquely) on Euclidean balls. This inequality is proven up to a constant factor to the power of $n$ in [3]. We investigate the case when $c$ is the Hofer–Zehnder symplectic capacity $c_{HZ}$. Among the reasons why the inequality $[14]$ is interesting, we mention that this inequality would imply the longstanding Mahler’s conjecture in convex geometry (as proven in [2]).

The computation of symplectic capacities is considered very difficult, so we restrict to the case of bodies that are Lagrangian products of a component in $V = \mathbb{R}^n$ with $q$-coordinates and a component in $V^* = \mathbb{R}^n$ with $p$-coordinates. For such bodies there is a simple geometric interpretation, established in [4], of the Hofer–Zehnder capacity. Namely,

$$c_{HZ}(K \times T) = \xi_T(K),$$

where $\xi_T(K)$ denotes the length of the shortest closed billiard trajectory in a convex body $K \subset V$ with geometry of lengths determined by another convex body $T \subset V^*$ by the formula

$$\|q\|_T = \max_{p \in T} \langle p, q \rangle, \quad q \in V.$$  

The shortest closed billiard trajectory will be understood in the sense of K. Bezdek and D. Bezdek [6] as the closed polygonal line of minimal $\|\cdot\|_T$-length not fitting into a translate of the interior of $K$. In Section 2 we introduce this billiard technique in detail.

Our first result concerns certain convex bodies for which the equality in [14] holds and for which we do not know whether they are symplectic balls or not. One known (since [2]) family of such examples consists of bodies $X = H \times H^\circ$, where $H$ is a Hanner (or Hanner–Hansen–Lima) polytope. The Hanner bodies are famous for being conjectured minimizers

Key words and phrases. Billiards, Minkowski norm, Viterbo’s conjecture, Permutohedron.

The author is supported by the Russian Foundation for Basic Research grant 15-01-99563 (A), supported in part by the Moebius Contest Foundation for Young Scientists and by the Simons Foundation.
in Mahler’s conjecture. We introduce another family of examples. In Section 3 we prove that
\[ \text{vol}(P_n \times \triangle_n^\circ) = \frac{c_{HZ}(P_n \times \triangle_n^\circ)^n}{n!}. \]
Here \( \triangle_n \) denotes a regular \( n \)-dimensional simplex centered at the origin; \( \triangle_n^\circ \) is its polar simplex. Further, \( P_n \) denotes an \( n \)-dimensional permutahedron, which can be considered as the Minkowski sum of all the edges of \( \triangle_n \). For example, permutahedron \( P_2 \) is a regular hexagon; permutahedron \( P_3 \) is a truncated regular octahedron whose hexagonal facets are regular hexagons (Figure 1).

![Figure 1. Permutahedron \( P_3 \) is a properly truncated octahedron.](https://en.wikipedia.org/wiki/Permutahedron)

The rest of results in this paper are some very special cases of Viterbo’s conjecture. In Section 4 we prove that Viterbo’s conjecture (for \( c_{HZ} \)) holds for the bodies of the form \( K \times (\text{a simplex} \ \triangle_n) \) or \( K \times (\text{a parallelotope} \ \square_n) \), where \( K \subset \mathbb{R}^n \) is any convex body, and products are Lagrangian:
\[ \text{vol}(K \times \triangle_n) \geq \frac{c_{HZ}(K \times \triangle_n)^n}{n!}, \]
\[ \text{vol}(K \times \square_n) \geq \frac{c_{HZ}(K \times \square_n)^n}{n!}. \]
The latter can be interpreted as a sharp isoperimetric-like inequality for billiard trajectories in the \( \ell_1 \)-norm:
\[ \xi_{\square_n}(K) \leq \left( \frac{n!}{2^n \text{vol}(K)} \right)^{1/n}. \]
It seems that that in other norms there are no known sharp isoperimetric inequalities for billiard trajectories. For example, in the Euclidean case the inequality
\[ \xi_{\triangle_n}(K) \leq C_n \text{ vol } K^{1/n} \]
follows from results of [3], even with some absolute factor \( C_n = C \) not depending on \( n \). But the optimal value of the constant factor \( C_n \) seems to be unknown even for \( n = 2 \).

Acknowledgments. The author thanks Roman Karasev for his constant attention to this work.

[1] The figure by w:en:User:Cyp@wikimedia, distributed under CC BY-SA 3.0 license.
2. Billiards in a Minkowski norm

We work in an pair of $n$-dimensional real vector space $V = \mathbb{R}^n$ and $V^* = \mathbb{R}^n$ with a canonical perfect pairing $\langle \cdot , \cdot \rangle$.

A convex body $K \subset V = \mathbb{R}^n$ is a compact convex set with non-empty interior. The polar body to a body $K \subset V$ is defined as $K^o = \{ p \in V^* : \langle q, p \rangle \leq 1 \ \forall q \in K \}$. Let $V$ be endowed with the unit ball $T^n$ (where $T^n \subset V$ is polar to a convex body $T \subset V^*$). We follow the notation of [1] and denote this norm by $\| \cdot \|_T$. By definition, $\|q\|_T = \max_{p \in T} \langle p, q \rangle$, where $\langle \cdot , \cdot \rangle : V^* \times V \to \mathbb{R}$ is the canonical bilinear form of the duality between $V$ and $V^*$. Here we assume that $T$ contains the origin but is not necessarily centrally symmetric. Therefore, our norms might not be symmetric; in general, $\|q\|_T \neq \|-q\|_T$. (Sometimes, such “norms” are called gauges.)

The momentum $p \in \partial T \subset V^*$ of the trajectory fragment $q \to q'$ is defined as a functional reaching its maximum at $q' - q$. If $T$ is not strictly convex, then there is an ambiguity in the definition of $p$.

The cone $N_K(q)$ of outer normals is defined as

$$N_K(q) = \{n \in V^* : \langle n, q' - q \rangle \leq 0 \ \forall q' \in K \}.$$  

The generalized reflection law is the following relation:

$$p' - p \in -N_K(q),$$

where $p$ and $p'$ stand for the momenta of the billiard trajectory before and after the reflection at the point $q$.

We set

$$\mathcal{P}_m(K) = \{(q_1, \ldots , q_m) : q_1, \ldots , q_m \}$$

doesn’t fit into $(\text{int} \ K + t)$ with $t \in V$ =

$$= \{(q_1, \ldots , q_m) : q_1, \ldots , q_m \}$$

doesn’t fit into $(\alpha K + t)$ with $\alpha \in (0, 1)$, $t \in V$

and

$$\xi_T(K) = \min_{Q \in \mathcal{Q}_T(K)} \ell_T(Q),$$

where $Q = (q_1, \ldots , q_m)$, $m \geq 2$, ranges over the set $\mathcal{Q}_T(K)$ of all closed generalized billiard trajectories in $K$ with geometry defined by $T$. (Here we denote the length $\ell_T(q_1, \ldots , q_m) = \sum_{i=1}^m \|q_{i+1} - q_i\|_T$.)

The generalization of the main result of [6], proved in [1], states the following:

**Theorem 2.1.** For any convex bodies $K \subset V$, $T \subset V^*$ ($T$ is smooth) containing the origins of $V$ and $V^*$ in their interiors, the equality holds:

$$\xi_T(K) = \min_{m \geq 2} \min_{Q \in \mathcal{P}_m(K)} \ell_T(Q);$$

and furthermore, the minimum is attained at $m \leq n + 1$.

**Remark 2.2.** In fact, Theorem [2.1] is proved in [1] for a smooth body $K$ and classical trajectories. By a classical billiard trajectory we mean a trajectory that, considered as a polygonal line in $\partial(K \times T)$, meets only smooth points of $\partial K$ and smooth points of $\partial T$. Additionally, in what follows we do not allow a classical trajectory to pass the same path multiple times. We obtain the formulation above by approximating a non-smooth $K$ in Hausdorff metrics and passing to the limit. Note that the formula of Theorem [2.1] can be used as the definition of $\xi_T(K)$ for arbitrary $T$ and $K$ without any smoothness assumptions.
3. Equality cases in Viterbo’s conjecture

Consider a regular simplex $\triangle_n = \text{conv}\{v_0, \ldots, v_n\} \subset \mathbb{R}^n$, normalized so that its edges are all of unit length. Choose an orthonormal base $(e_1, \ldots, e_n)$ with $e_n$ pointing to $v_0$. Consider also the permutohedron $P_n \subset \mathbb{R}^n$, defined as the Minkowski sum of the simplex edges (this definition can be found in \cite[Lecture 7.3]{[15]}):

$$P_n = \sum_{0 \leq i < j \leq n} [v_i, v_j].$$

The main result of this section is the following

**Theorem 3.1.** In the configuration $P_n \times \triangle_n^o$ the shortest generalized billiard trajectory has length $\xi_{\triangle_n^o}(P_n) = (n + 1)^2$. Moreover, $X = P_n \times \triangle_n^o$ delivers equality in Viterbo’s conjecture (where $P_n$ and $\triangle_n^o$ lie in Lagrangian subspaces).

This theorem can be viewed as the extension of the first part of the following proposition, proved in \cite{[5]}:

**Proposition 3.2.** (1) In the configuration $P_2 \times \triangle_2^o$ (our regular simplices are always centered at the origin) the shortest generalized billiard trajectory has length

$$\xi_{\triangle_2^o}(P_2) = 9.$$

Hence, $X = P_2 \times \triangle_2^o$ delivers equality in Viterbo’s conjecture.

(2) Any classical billiard trajectory in the configuration $P_2 \times \triangle_2^o$ bounces 4 times and has length 9.

(3) Also, arbitrarily close to any point

$$(q, p) \in \partial(P_2 \times \triangle_2^o)$$

there passes a certain classical billiard trajectory of minimal length.

Another statement of the same spirit, proved in \cite{[5]} is

**Proposition 3.3.** (1) In a Hanner polytope $H \subset \mathbb{R}^n$ with geometry specified by its polar $H^o$, the shortest generalized billiard trajectory has length $\xi_{H^o}(H) = 4$, and $X = H \times H^o$ delivers equality in Viterbo’s conjecture (the latter fact has been known since \cite{[2]}).

(2) Any simple classical billiard trajectory in a Hanner polytope $H \subset \mathbb{R}^n$ with geometry specified by its polar $H^o$ is 2n-bouncing and has length 4.

(3) Moreover, in an arbitrarily small neighborhood of any point $(q, p) \in \partial(H \times H^o)$, there passes a classical billiard (in configuration $H \times H^o$) trajectory of minimal length.

F. Schlenk has established (see \cite{[13]} or \cite[\S 4]{[9]}) that the interior of the Lagrangian product of a crosspolytope and a cube (its polar) equals the interior of a certain ball from the symplectic point of view. In view of results of \cite{[10]}, this fact yields the implications of Proposition 3.3 for the case of a cube and a crosspolytope. A reasonable question is whether such a situation (that the interior of such body is symplectomorphic to an open ball) holds for other configurations delivering equality in Viterbo’s conjecture. In particular, Theorem 3.1 would immediately follow from the affirmative answer to the following

**Question 3.4.** Is $P_n \times \triangle_n^o$ a symplectic ball? More precisely, is the interior of $P_n \times \triangle_n^o$ a symplectic image of a Euclidean ball of the same volume?
We do not know the answer. The negative answer to Question 3.4 would disprove another well-known conjecture. It states that all symplectic capacities coincide on the class of convex bodies. This conjecture is even stronger than 1.1.

To prove Theorem 3.1 we will directly compute \( \xi_\triangle_n(P_n) = c_{HZ}(P_n \times \triangle_n) \) and then check the equality:

\[
\text{vol}(P_n \times \triangle_n^o) = \frac{c_{HZ}(P_n \times \triangle_n^o)^n}{n!}.
\]

This will follow from the following two propositions.

**Proposition 3.5.** In the above notation

\[
\text{vol} \triangle_n^o = \frac{2^{n/2}(n+1)^{n+1/2}}{n!}, \quad \text{vol} P_n = \frac{(n+1)^{n-1/2}}{2^{n/2}}.
\]

**Proposition 3.6.** In the above notation

\[
c_{HZ}(P_n \times \triangle_n^o) = (n+1)^2.
\]

**Proof of Proposition 3.5.** The volume \( \text{vol} \triangle_n = \sqrt{\frac{n+1}{2^{n/2}n!}} \) can be easily computed, and the Mahler volume product \( \text{vol} \triangle_n \cdot \text{vol} \triangle_n^o = \frac{(n+1)^{n+1}}{(n!)^2} \) is well known, so we do not give details here.

To compute \( \text{vol} P_n \), we use another definition of permutohedron (see [7, Chapter 21]): \( \widetilde{P}_n \) is defined as the Voronoi cell of the lattice

\[
A_n^* = \{(x_0, \ldots, x_n) \in \mathbb{Z}^{n+1} : \sum_i x_i = 0, \ x_0 \equiv \ldots \equiv x_n \pmod{n+1}\},
\]

lying in \( \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum_i x_i = 0\} \cong \mathbb{R}^n \). This lattice is generated by the vectors

\[
a_1 = (-1, n, -1, \ldots, -1)^t, \ldots, a_n = (-1, -1, \ldots, n)^t.
\]

Note that \( \widetilde{P}_n \) is \( \sqrt{2} \) times larger than \( P_n \) (since the width of \( P_n \) equals \( n(v_0 - v_i, e_n) = \sqrt{(n^2+n)/2} \), and the width of \( \widetilde{P}_n \) equals \(|a_1| = \sqrt{n^2+n} \)). Then

\[
2^{n/2} \text{vol} P_n = \text{vol} \widetilde{P}_n = \det A_n^* = \det \Gamma(a_1, \ldots, a_n)^{1/2} = \begin{vmatrix} n^2 + n & -n - 1 & -n - 1 \\ -n - 1 & n^2 + n & -n - 1 \\ -n - 1 & -n - 1 & n^2 + n \end{vmatrix}^{1/2}.
\]

The latter \( n \times n \) matrix has an eigenvector \( h = (1, \ldots, 1)^t \) corresponding to the eigenvalue \( n+1 \) without multiplicities. All other eigenvectors, orthogonal to \( h \), have the same \((n-1)\)-fold eigenvalue \((n+1)^2\), as it is easy to check by hand. Therefore, \( \det \Gamma(a_1, \ldots, a_n) = (n+1)^{2n-1} \), and the second result follows.

To prove Proposition 3.6 we use Bezdeks’ characterization of shortest generalized billiard trajectories in \( P_n \), when lengths are measured using the norm with unit body \( \triangle_n \). Note that a 2-periodic trajectory (2-fold bypass of the width of \( P_n \)) gives an example of the trajectory of \( \triangle_{n-1} \)-length \((n+1)^2\) that cannot fit into \( \text{int} P_n \).

Further, we consider an arbitrary closed polygonal line that cannot fit into \( \text{int} P_n \) and show that its length cannot be less than \((n+1)^2\).
First, we replace each segment \([q, q']\) of this line with a certain polygonal line of same length but with edges directed along \(v_0, \ldots, v_n\). This can be done as follows. Consider the convex cone 

\[ C_i = \text{pos}\{\{v_0, \ldots, v_n\} \setminus \{v_i\}\} \]

in which the vector \(q' - q\) lies; then, decompose \(q' - q = \sum_{0 \leq k \leq n, k \neq i} \alpha_k v_k\) with \(\alpha_k \geq 0\). Now the polygonal line with edges congruent to \(\alpha_i v_i\) suits our purpose well. Its length equals the length of \(q' - q\) in the given norm because the norm function is linear on this cone.

Now, we have the closed polygonal line with at most \(n + 1\) directions used. Note that \(\sum_i v_i = 0\) is the only linear dependence among the directions; thus, the total length of the segments of this polygonal line along each direction \(v_i\) does not depend on \(i\). Assume the contrary to the statement of Proposition 3.6: This length along each direction \(v_i\) is less than \(n + 1\). We show that such a line can fit into a smaller homothet of \(P_n\). It remains to establish the following:

**Lemma 3.7.** Suppose a closed polygonal line consists of segments directed only along \(v_0, \ldots, v_n\). Suppose also that the total length of all segments that are directed along \(v_i\) equals \(n + 1\), for each \(i = 0, 1, \ldots, n\). Then this line can be covered by a translate of \(P_n\).

**Proof.** We proceed by induction on \(n\). Base \(n = 1\) is clear.

To begin, we note that the top horizontal (meaning orthogonal to \(e_n\), which is supposed to be directed upward) facet \(F\) of \(P_n\) is congruent to \(P_{n-1}\) placed horizontally in \(\mathbb{R}^n\). Explicitly,

\[ F = n v_0 + \sum_{1 \leq i < j \leq n} [v_i, v_j]. \]

Now let \(Q = q_0 \rightarrow q_1 \rightarrow \ldots \rightarrow q_{m-1} \rightarrow q_m = q_0\) be a polygonal line satisfying the assumptions of the lemma, and let \(q_0\) be the highest (in the sense of largest coordinate along \(e_n\)) vertex of \(Q\). Let us introduce a parametrization \(q : [0, n + 1] \rightarrow Q\) (without loss of generality \(q(0) = q(n + 1) = 0\)), so that the point \(q(t)\) runs along \(Q\) with a constant velocity, and the portion of time when the point travels along any \(v_i\) direction equals 1.

Consider the following transformation of \(Q\). We contract it using the transform

\[
\begin{pmatrix}
\frac{1}{n+1} & 0 & : & 0 & 0 \\
0 & \frac{1}{n+1} & : & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & : & \frac{1}{n+1} & 0 \\
0 & 0 & : & 0 & 1
\end{pmatrix}
\]

and obtain the line \(\tilde{Q}\) with corresponding parametrization \(\tilde{q}(\cdot)\). Also, denote by \(\tilde{v}_i\) the image of \(v_i\) under this transform. (Note that \((n + 1)\tilde{v}_i = v_i - v_0\).

Now we consider the Minkowski sum

\[ R = \tilde{Q} + F = \bigcup_{t \in [0, n + 1]} (F + \tilde{q}(t)) \]

and claim that \(Q\) can be covered by \(R\). We need \(Q \subset R + s\) for some \(s \in \mathbb{R}^n\). Imagine that we track the relative motion of the point \(q(t)\) inside the moving horizontal permutohedron \(F + s + \tilde{q}(t)\) in the same horizontal hyperplane. The relative velocity of \(q(t)\) in \(F + s + \tilde{q}(t)\) equals \(\frac{n}{n+1}\)-fraction of the corresponding horizontal velocity component of \(q(t)\). Therefore, for this relative motion, the inductive assumption holds: In permutohedron \(P_{n-1}\), we have
the trajectory that travels by distance \( n \) along each of the \( n \) distinguished directions (defined as horizontal projections of \( v_1, \ldots, v_n \)). Therefore, such \( s \) can indeed be found.

The next step is to cover the set \( R \) by \( P_n \). To do this, we consider the coordinates \( (\tilde{q}_1(t), \ldots, \tilde{q}_n(t)) \) of the point \( \tilde{q}(t) \) in the base \( \tilde{v}_1, \ldots, \tilde{v}_n \). Shift \( \tilde{Q} \) (and \( R = \tilde{Q} + F \) correspondingly) so that, for any \( i \),

\[
\min_{t \in [0, n+1]} \tilde{q}_i(t) = 0.
\]

We claim that \( R \subset P_n \) after such a shift. Indeed, any point of \( R \) could be represented as

\[
f + \tilde{q}(t) = f + \sum_{i=1}^{n} \tilde{q}_i(t)\tilde{v}_i,
\]

for some \( t \in [0, n+1] \) and \( f \in F \). Note that \( \tilde{q}_i(t) \in [0, n+1] \), and the point \( \sum_{i=1}^{n} \tilde{q}_i(t)\tilde{v}_i \) belongs to the Minkowski sum of the non-horizontal segments

\[
\sum_{i=1}^{n} [0, (n+1)i] = \sum_{i=1}^{n} ([v_0, v_i] - v_0).
\]

Hence,

\[
f + \sum_{i=1}^{n} \tilde{q}_i(t)\tilde{v}_i \in F + \sum_{i=1}^{n} ([v_0, v_i] - v_0) = \sum_{1 \leq i < j \leq n} [v_i, v_j] + \sum_{i=1}^{n} [v_0, v_i] = P_n,
\]

as required. \( \square \)

4. Special cases of Viterbo’s conjecture

Viterbo’s conjecture is proven \([8]\) for ellipsoids, polydiscs and convex Reinhardt domains. However, even for \( X \subset \mathbb{R}^4 \) Viterbo’s conjecture remains widely open. Here we prove the following special cases of the conjecture.

**Theorem 4.1.** Let \( K \subset V = \mathbb{R}^n \) be a convex body. Let \( \triangle_n \subset V^* \) be a simplex. Then

\[
\text{vol}(K \times \triangle_n) \geq \frac{c_{HZ}(K \times \triangle_n)^n}{n!},
\]

where \( K \) and \( \triangle_n \) lie in Lagrangian subspaces.

**Theorem 4.2.** Let \( K \subset V = \mathbb{R}^n \) be a convex body. Let \( \square_n \subset V^* \) be a parallelotope. Then

\[
\text{vol}(K \times \square_n) \geq \frac{c_{HZ}(K \times \square_n)^n}{n!},
\]

where \( K \) and \( \square_n \) lie in Lagrangian subspaces.

The latter can be interpreted as a sharp isoperimetric-like inequality for billiard trajectories in the \( \ell_1 \)-norm, with equality attained on \( K = \square_n^0 \):

\[
\xi_{\square_n}(K) \leq \frac{1}{2} \left( n! \text{vol}(K) \right)^{1/n}.
\]

Similarly, the former is a sharp isoperimetric-like inequality for billiard trajectories in the non-symmetric \( \| \cdot \|_{\triangle_n} \)-norm, with equality attained on permutohedra.

**Proof of Theorem 4.1.** The inequality in question is affine invariant. Any simplex is affinely equivalent to a regular one, so we assume that \( \triangle_n = \text{conv}\{v_0, \ldots, v_n\}^0 \) is a regular simplex centered at the origin, with \( |v_0| = \ldots = |v_n| = 1 \). Further, we scale \( K \) so that \( c_{HZ}(K \times \triangle_n) \) becomes equal to \( (n+1) \).
Consider the lattice $\Lambda$ generated by the vectors $v_0, \ldots, v_n$. The key observation is that $\Lambda$ is just a scaled copy of $A_n^*$, so its Voronoi cell $p_n$ is congruent to $\frac{1}{\sqrt{n^2+n}} \mathbb{P}_n$.

We will prove that $\text{vol} K \geq \text{vol} p_n$. This will suffice:

$$\text{vol}(K \times \triangle_n) \geq \text{vol}(p_n \times \triangle_n) = \frac{c_{HZ}(p_n \times \triangle_n)^n}{n!} = \frac{(n+1)^n}{n!} = \frac{c_{HZ}(K \times \triangle_n)^n}{n!}.$$ 

Here we used the results of Section 3. Note that in our scale, $p_n$ equals $\frac{1}{n+1}$-times the Minkowski sum of the edges of $\triangle_n^0 = \text{conv}\{v_0, \ldots, v_n\}$, hence $c_{HZ}(p_n \times \triangle_n) = n + 1$ and $\text{vol}(p_n \times \triangle_n) = \frac{c_{HZ}(p_n \times \triangle_n)^n}{n!}$.

Now we are proving the estimate $\text{vol} K \geq \text{vol} p_n$. We will do it by showing that

$$\bigcup_{\lambda \in \Lambda} (\lambda + K) = \mathbb{R}^n.$$ 

This will imply that $\text{vol} K$ is not less than the volume of the fundamental domain of $\Lambda$, i.e., $\text{vol} p_n$.

Consider the triangulation $T$ of $\mathbb{R}^n$ that is dual to the Voronoi tessellation of $\mathbb{R}^n$ with respect to $\Lambda$. Every simplex $\sigma$ of $T$ has the following property: There exist a closed oriented polygonal line $Q_\sigma$ such that

- $Q_\sigma$ traverses along edges of $\sigma$ and visits every vertex of $\sigma$ once;
- the segments of $Q_\sigma$ have unit length and the set of their directions coincides with $\{v_0, \ldots, v_n\}$;
- $\text{conv} Q_\sigma = \sigma$.

We sketch the proof of this statement. A simplex $\sigma$ of $T$ corresponds to a vertex $u$ of the permutohedral tessellation. A vertex $\lambda$ of $\sigma$ corresponds to a permutohedron $P \ni u$. We would like to show that among the edges of $\sigma$ adjacent to $\lambda$, there precisely two of length 1, and one of them is pointed along a vector $v_i$ while the other is pointed against another $v_j$. This follows from the well-known description of the combinatorial structure of $P$ (see, for example, [14 Proposition 2.6]). The $d$-dimensional faces of $P$ are in one-to-one correspondence with the ordered partitions of $\{0, 1, \ldots, n\}$ into $n - d$ subsets. The vertex $u$ corresponds to a partition into singletons, that is, to a permutation $\pi$ of $\{0, 1, \ldots, n\}$. The facets of $P$ correspond to the pairs of sets $(S, \{0, 1, \ldots, n\} \setminus S)$. Furthermore, there are exactly two facets of $P$ containing $u$ and congruent to a scaled copy of $P_{n-1}$. They correspond to the pairs $\{\{\pi(0), \pi(1), \ldots, \pi(n)\}\}$ and $\{\{\pi(0), \ldots, \pi(n-1)\}, \{\pi(n)\}\}$. These two facets correspond to shortest edges of $\sigma$ pointed along, say, $v_i$ and $-v_j$. Note that vertices $u - v_i$ and $u + v_j$ of $P$ correspond to the cyclic shifts of $P$. Let us mark all those (oriented along the $v_i$’s) edges of $\sigma$ we found, over all vertices $\lambda$ of $\sigma$. They form a family of oriented cycles passing through each vertex $\lambda \in \sigma$ once. We claim that this family is in fact a single cycle. To see that, we keep track of the permutations $\pi$ corresponding to $u$ in all adjacent permutohedra. As we move along a cycle from the family, these permutations shift cyclically, as described above. Thus, the cycle consists of $(n + 1)$ segments, and we found $Q_\sigma$ as desired. The statement is proven.

Since $c_{HZ}(K \times \triangle_n) = (n + 1)$, any such line $Q_\sigma$ can be covered by a translate of $K$, hence any simplex of $T$ can be covered by a translate of $K$.

Assume that $\bigcup_{\lambda \in \Lambda} (\lambda + K) \neq \mathbb{R}^n$, that is, there exists $x \in \mathbb{R}^n$ such that $(x + \Lambda) \cap K = \emptyset$. So we found a translate $K - x$ of $K$ that avoids $\Lambda$, while every simplex of $T$ can be covered by a translate of $K$. This contradicts Lemma 4.3 below. \hfill \Box

**Lemma 4.3.** Let $T$ be a triangulation of $\mathbb{R}^n$ with the diameters of simplices bounded from above by a number $d \in \mathbb{R}$. Let $\Lambda$ be the vertex set of $T$. Assume that every simplex of $T$
can be covered by a translate of a convex body \( K \subset \mathbb{R}^n \). Then any translate of \( K \) meets \( \Lambda \).

**Proof.** Suppose there is a translate \( K + t \) of \( K \) avoiding \( \Lambda \). We define a continuous vector field \( \nu : \mathbb{R}^n \rightarrow \mathbb{R}^n \) as follows. For every vertex \( \lambda \in \Lambda \), set \( \nu(\lambda) \) equal to any unit vector pointing away from \( K + t \). More precisely, take any hyperplane \( H \ni \lambda \) avoiding \( K + t \), and set \( \nu(\lambda) \) to be the normal vector of \( H \) pointing to the halfspace missing \( K + t \) (see Figure 2). Then extend \( \nu \) affinely to the entire \( \mathbb{R}^n \).

![Figure 2. Construction of vector field \( \nu \)](image)

Now take a ball \( B \) centered at any point of \( K + t \). We take it large enough, say, of radius \( 100(d + \text{diam } K) \). Then \( \nu \big|_{\partial B} : \partial B \rightarrow S^1 \) has degree 1, hence there is \( x \in \text{int } B \) such that \( \nu(x) = 0 \). If \( x \) lies in a simplex \( \sigma \) of \( T \) with vertices \( v_0, \ldots, v_n \), then there are non-negative multipliers \( \alpha_0, \ldots, \alpha_n \) such that \( \sum \alpha_i \nu(v_i) = 0 \). Informally, this means that \( K + t \) is “blocked” inside the “cage” \( \sigma \) with the “cage bars” \( v_i \). It is pretty easy to check that in this situation \( \sigma \) cannot be covered by a translate of \( K \). The details of this argument can be found in [6, Lemma 2.2]. \( \square \)

**Lemma 4.4.** Let \( L \subset \mathbb{R}^{n-1} \) be a convex body. Suppose we established the inequality

\[
\text{vol}(M \times L) \geq \frac{c_{HZ}(M \times L)^{n-1}}{(n-1)!}
\]

for all convex bodies \( M \subset \mathbb{R}^{n-1} \). Then for any convex body \( K \subset \mathbb{R}^n \) the following holds:

\[
\text{vol}(K \times L \times [-1,1]) \geq \frac{c_{HZ}(K \times L \times [-1,1])^n}{n!}.
\]

**Proof.** Without loss of generality assume that \( L \) contains the origin in its interior. Denote by \( H \) the hyperplane in \( \mathbb{R}^n \) where \( L \) sits. Note that \( (L \times [-1,1])^c = \text{conv}(L^o \times \mathbb{R} \cup \mathbb{R}^{n-1} \times [-1,1]) \). Sometimes this body is called the free sum or the \( \ell_1 \)-sum of \( L^o \) and \([-1,1]\).

Scale \( K \) so that \( c_{HZ}(K \times L \times [-1,1]) = \xi_{L \times [-1,1]}(K) = 4 \). This implies that any polygonal line of \( \| \cdot \|_{L \times [-1,1]} \)-length 4 fits into \( K \). In particular, any such polygonal line that is parallel to \( H \) fits into \( \pi_H(K) \), the orthogonal projection of \( K \) onto \( H \). Thus, \( c_{HZ}(\pi_H(K) \times L) \geq 4 \). By the assumption,

\[
\text{vol}(\pi_H(K) \times L) \geq \frac{c_{HZ}(\pi_H(K) \times L)^{n-1}}{(n-1)!} \geq \frac{4^{n-1}}{(n-1)!}.
\]
Now we apply the following inequality (proven by Rogers and Shepard [12] in greater generality):

$$\text{vol}_n(K) \geq \left( \frac{n}{n-1} \right)^{-1} \text{vol}_{n-1}(\pi_H(K)) \cdot \max_{x \in \mathbb{R}^n} \text{vol}_1((x + H^\perp) \cap K).$$

Since $\xi_{L \times [-1,1]}(K) = 4$, $K$ contains a segment orthogonal to $H$ of $\| \cdot \|_{L \times [-1,1]}$-length 2, and we conclude that $\max_{x \in \mathbb{R}^n} \text{vol}_1((x + H^\perp) \cap K) \geq 2$. Finally, we combine all inequalities above:

$$\text{vol}(K \times L \times [-1,1]) \geq 2 \cdot \text{vol}(\pi_H(K)) \cdot \text{vol}(L \times [-1,1]) = \frac{4}{n} \cdot \text{vol}(\pi_H(K) \times L) \geq \frac{4}{n} \cdot \frac{4^{n-1}}{(n-1)!} = c_{HZ}(K \times L \times [-1,1])^n.$$

□

Proof of Theorem 4.2. The inequality in question is affine invariant, so we assume that $\square_n = [-1,1]^n$ is a hypercube centered at the origin. Now the claim follows from 4.4 by induction. The base case

$$\text{vol}([a,b] \times [-1,1]) \geq c_{HZ}([a,b] \times [-1,1])$$

is trivial.

□

Remark 4.5. Starting from [4.1] as the induction base, we see that the same induction proves the $c_{HZ}$-version of Viterbo’s conjecture for a product of any convex body $K \subset \mathbb{R}^{k+m}$ with a simplex in $\mathbb{R}^k$ and a parallelotope in $\mathbb{R}^m$:

$$\text{vol}(K \times \Delta_k \times \square_m) \geq \frac{c_{HZ}(K \times \Delta_k \times \square_m)^{k+m}}{(k+m)!}.$$
[13] F. Schlenk. *Embedding problems in symplectic geometry*, volume 40. Walter de Gruyter, 2005.
[14] C. Viterbo. Metric and isoperimetric problems in symplectic geometry. *Journal of the American Mathematical Society*, 13(2):411–431, 2000.
[15] G. M. Ziegler. *Lectures on polytopes*, volume 152. Springer Science & Business Media, 1995.

E-mail address: balitski@mit.edu

DEPT. OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, 182 MEMORIAL DR., CAMBRIDGE, MA 02142

DEPT. OF MATHEMATICS, MOSCOW INSTITUTE OF PHYSICS AND TECHNOLOGY, INSTITUTSKIY PER. 9, DOLGOPRUDNY, RUSSIA 141700

INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS RAS, BOLSHOY KARETNY PER. 19, MOSCOW, RUSSIA 127994