Wegner-type bounds for a two-particle Anderson model in a continuous space

A. Boutet de Monvel, V. Chulaevsky and Y. Suhov

Abstract. We analyse a two-particle quantum system in $\mathbb{R}^d$ with interaction and in presence of a random external potential field with a continuous argument (an Anderson model in a continuous space). Our aim is to establish the so-called Wegner-type estimates for such a model, assessing the probability that random spectra of Hamiltonians in finite volumes intersect with a given set. For the lattice version of the two-particle model, a similar result was obtained in \cite{8}

Keywords. Wegner bound; Anderson model.

1. Introduction. The two-particle Anderson Hamiltonian in a continuous space

This paper is a follow-up of \cite{8} and establishes Wegner-type (more precisely, Wegner-Stollmann-type) bounds for random continuous Schrödinger operators. We focus here on a two-particle interactive Anderson model in a continuous space, subject to a random external field with a continuous argument. The infinite-volume Hamiltonian of the model is a Schrödinger operator $H = H^{(2)}(\omega)$ acting on functions $\phi \in L_2(\mathbb{R}^d \times \mathbb{R}^d)$:

$$H \phi(\mathbf{x}) = H^0 \phi(\mathbf{x}) + W(\mathbf{x}; \omega)\phi(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d. \quad (1.1)$$

Here $H^0$ is the kinetic energy operator:

$$H^0 = -\frac{1}{2} \sum_{j=1,2} \Delta_j, \quad (1.2)$$
where $\Delta_j$ is the Laplacian in variable $x_j = (x_j^{(1)}, \ldots, x_j^{(d)}) \in \mathbb{R}^d$ corresponding to the $j$th particle:

$$\Delta_j = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_j^{(i)}^2}.$$  \hspace{1cm} (1.3)

The randomness in Hamiltonian $H^{(2)}(\omega)$ is concentrated in the potential energy function $W(\underline{x}; \omega)$ which is written in the form

$$W(\underline{x}; \omega) = U(\underline{x}) + \sum_{j=1,2} V(x_j; \omega), \quad \underline{x} = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d.$$  \hspace{1cm} (1.4)

The term $U(\underline{x})$ represents the (nonrandom) interaction potential while the sum $[V(x_1; \omega) + V(x_2; \omega)]$ describes the action of an external potential field on the two-particle system. More precisely, the external potential field is represented by a family of real random variables (RVs) $V = \{V(x; \omega), x \in \mathbb{R}^d\}$, or, in probabilistic terminology, by a real-valued random field with a continuous argument (or simply random field). A sample of such a random field is a (measurable) function $v : x \in \mathbb{R}^d \mapsto \mathbb{R}$, and its graph is a ‘hypersurface’ in $\mathbb{R}^d \times \mathbb{R}$, with a unique point of intersection with any straight line orthogonal to the ‘argument space’ $\mathbb{R}^d$. The argument $\omega$ in the notation $V(x; \omega)$ stresses randomness of the external potential. The distribution of the random field $V$ is denoted by $P$: it is a probability measure on a (suitably chosen) sigma-algebra $\mathcal{M}$ in the space of sample functions $v : \mathbb{R}^d \rightarrow \mathbb{R}$. Formal definitions are given in Section 2; cf. [2]. A classical example is a Gaussian distribution on $M$; see, e.g., [1].

The randomness introduced in Hamiltonian $H$ perplexes its spectral properties. A physically-motivated result expected here is that the spectrum of $H$ near its ‘lower edge’ is pure point with probability one, and the corresponding eigenfunctions decay exponentially in space. (This is called Lifshits-tail-based exponential localisation.) Similar results have been established for a variety of single-particle models where the Hamiltonian acts on a function $\phi \in L^2(\mathbb{R}^d)$ as $-\Delta \phi(x)/2 + V(x; \omega)\phi(x), \; x \in \mathbb{R}^d$. See [17], [15] and references therein. As far as localisation is concerned, the main difference between a single-particle model and its two-particle counterpart is that replacing term $V(x; \omega)$ by the sum $W(\underline{x}; \omega) = U(\underline{x}) + \sum_{j=1,2} V(x_j; \omega)$ leads to ‘strong’ dependencies between RVs $W(\underline{x}; \omega)$ and $W(\underline{x}'; \omega)$, no matter how far points $\underline{x}, \underline{x}' \in \mathbb{R}^d \times \mathbb{R}^d$ are positioned from each other. The ensuing difficulty cannot be relieved no matter how quickly correlations between RVs $V(x; \omega)$ and $V(x'; \omega)$ decay when $x, x' \in \mathbb{R}^d$ are far from each other. Cf. [8, 9] where this discussion has been conducted for the so-called tight-binding two-particle Anderson model on a lattice.

An important ingredient of the existing localisation proofs for single-particle Anderson models or their two-particle counterparts is the so-called Wegner-type (or Wegner–Stollmann type) estimates for the corresponding Hamiltonian. Exceptions are one-dimensional models where the whole physics of localisation is rather
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special.] This explains the title of this paper. The Wegner-type estimates are produced for the eigen-values of a finite-volume approximation of the Hamiltonian; cf. [13]. In our situation, these are operators $H_\Lambda \left( = H^{(2)}_\Lambda (\omega) \right)$ acting on functions $\phi \in L^2(\Lambda)$:

$$H_\Lambda \phi(x) = H^0_\Lambda \phi(x) + W(x; \omega) \phi(x), \quad x = (x_1, x_2) \in \Lambda, \quad (1.5)$$

with

$$H^0_\Lambda = -\frac{1}{2} \sum_{j=1,2} \Delta^j(\Lambda). \quad (1.6)$$

and $W(x; \omega)$ defined, as before, by Equation (1.2.2). Here $\Lambda$ is a bounded domain in $\mathbb{R}^d \times \mathbb{R}^d$ which we choose to be the Cartesian product of two $d$-dimensional cubes: for $u = (u_1, u_2) \in \mathbb{R}^d \times \mathbb{R}^d$ and $L_1, L_2 \in (0, \infty)$,

$$\Lambda = \Lambda_{L_1, L_2}(u) = \Lambda_{L_1}(u_1) \times \Lambda_{L_2}(u_2), \quad (1.7)$$

where, for $u = (u^{(1)}, \ldots, u^{(d)}) \in \mathbb{R}^d$ and $L \in (0, \infty)$,

$$\Lambda_L(u) = \times \left[ -L + u^{(i)}, u^{(i)} + L \right]. \quad (1.8)$$

The shorthand notation $\Lambda$ for $\Lambda_{L_1, L_2}(u)$ (and $\Lambda'$ for $\Lambda'_{L_1', L_2'}(u')$) will be systematically used in this paper. Further, $\Pi_1 \Lambda$ and $\Pi_2 \Lambda$ will stand for cubes $\Lambda_{L_1}(u_1)$ and $\Lambda_{L_2}(u_2)$ forming the projections of ‘two-particle’ parallelepiped $\Lambda = \Lambda_{L_1, L_2}(u)$ onto single-particle configurational spaces (and similarly with $\Pi_1 \Lambda'$ and $\Pi_2 \Lambda'$).

Next, $\Delta^j(\Lambda)$ is the Laplacian in variable $x_j = (x_j^{(1)}, \ldots, x_j^{(d)}) \in \Pi_2 \Lambda$ confined to cube $\Pi_j \Lambda$, $j = 1, 2$, with a specified boundary condition which we choose to be Dirichlet’s:

$$\Delta^\Lambda \left( = \Delta^j(\Pi_j \Lambda) \right) = \sum_{i=1}^d \frac{\partial^2}{\partial^2 x_j^{(i)}} + \text{Dirichlet’s boundary condition.} \quad (1.9)$$

In fact, a possible choice of boundary conditions is rather broad and includes periodic and ‘elastic’, in particular, Neumann’s. (What we need for our method to work is that the self-adjoint extension specified by a boundary condition is half-bounded and has a compact resolvent $(H_{\Lambda} - \lambda I)^{-1}$ for $\lambda \in \mathbb{C}$ away from the real line.) See [6], [17] for details.

Thus, operator $H^{(2)}_\Lambda$ in Eqn (1.6) is understood in terms of the sesquilinear form $\langle \phi, H^{(2)}_\Lambda \psi \rangle$ in $L^2(\Lambda)$, where $\langle \cdot, \cdot \rangle$ stand for the standard inner product. A similar meaning is attributed to the multiplication operator by $W(x; \omega)$ in $L^2(\Lambda)$; consequently, we treat the function $W(x; \omega)$, $x \in \Lambda$, as an element of $L^2(\Lambda)$ defined for $\mathbb{P}$-a.a. $\omega$. Additional conditions imposed below mean that this function is bounded on $\Lambda$, and the supremum $\overline{W}_\Lambda(\omega)$ of its absolute value has certain moments; see below.

Moreover, under the assumptions introduced in this paper, the (self-adjoint) operator $H_{\Lambda}$ is bounded from below and has, with probability one, a discrete
spectrum of a finite multiplicity. It is convenient to write its eigenvalues $E^{(\Lambda)}$ in an increasing order:

$$E^{(\Lambda)}_0 \leq E^{(\Lambda)}_1 \leq E^{(\Lambda)}_2 \leq \ldots$$

(1.10)

In the ‘one-volume’ Wegner estimate one assesses the probability that at least one eigenvalue $E^{(\Lambda)}_k$ of operator $H_{\Lambda}$ falls in a (narrow) interval around a given point $E$ on the spectral axis:

$$\mathbb{P} \left( \exists \ k \text{ with } \left| E - E^{(\Lambda)}_k \right| \leq \epsilon \right), \quad (1.11)$$

Next, the ‘two-volume’ Wegner estimate addresses the probability that the eigenvalues $E^{(\Lambda)}_k$ and $E^{(\Lambda')}$ of operators $H_{\Lambda}$ and $H_{\Lambda'}$ come near to each other in a given interval $J \subseteq \mathbb{R}$, for two (distant) parallelepipeds $\Lambda = \Lambda_{L_1, L_2}(\omega)$ and $\Lambda' = \Lambda'_{L'_1, L'_2}(\omega')$. That is,

$$\mathbb{P} \left( \exists \ k \text{ and } l \text{ with } E^{(\Lambda)}_k, E^{(\Lambda')}_l \in J \text{ and } \left| E^{(\Lambda)}_k - E^{(\Lambda')}_l \right| \leq \epsilon \right), \quad (1.12)$$

Here and below, $\mathbb{P}$ stands for the corresponding probability measure on the underlying probability space (see Section 2).

From the probabilistic point of view, the estimates for probabilities (1.11) and (1.12) are examples of concentration inequalities, albeit for rather implicit RVs $E^{(\Lambda)}_k$ carrying a considerable amount of dependence. For single-particle Anderson models, assuming natural properties of the random term $V(x; \omega)$, the Wegner estimates are rather straightforward. For the two-particle models, the study of these estimates, with a view of localisation, started in [14], [8], [4] and is continued in this paper.

In the main body of this work, the interaction potential $U$ satisfies the following property:

(I) $U$ is a (measurable) bounded real function $\mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ obeying

$$U(x) = U(\sigma x), \quad \forall x \in \mathbb{R}^d \times \mathbb{R}^d, \quad U(x) = 0, \quad \text{if } \|x_1 - x_2\|_{\text{max}} > r_1. \quad (1.13)$$

Here $r_1 \in (0, \infty)$ is the interaction radius, $\sigma$ stands for the permutation of the (vector) variables ($x_1 = (x_2, x_1)$ for $x = (x_1, x_2)$) and $\|x_1 - x_2\|_{\text{max}}$ denotes the max-norm in $\mathbb{R}^d$: for $x_j = (x_j^{(1)}, \ldots, x_j^{(d)}) \in \mathbb{R}^d, j = 1, 2$:

$$\|x_1 - x_2\|_{\text{max}} = \max \left[ |x_j^{(k)} - x_j^{(k)}| : k = 1, \ldots, d \right] \quad (1.14)$$

We can also allow the case where $U$ has a hard core, i.e.,

$$U(x) = +\infty \text{ if } \|x_1 - x_2\|_{\text{max}} < r_0, \quad (1.15)$$

with $r_0 \in (0, r_1)$ being the diameter of the hard core. However, we have to assume that $|U(x)|$ remains uniformly bounded for $x$ with $r_0 < \|x_1 - x_2\|_{\text{max}} < r_1$. 

\[ \]
In Section 2, we give formal conditions upon the structure of the potential energy terms in Hamiltonian (1.5). A useful notion is the ‘shadow’ $\Pi \Lambda$ of a parallelepiped $\Lambda = \Lambda_{L_1, L_2}$:

$$\Pi \Lambda = \Pi_1 \Lambda \cup \Pi_2 \Lambda.$$  \hfill (1.16)

It can be a cube or a union of two cubes in $\mathbb{R}^d$, possibly disjoint. In what follows we call sets of this kind ‘cellular’.

Throughout the paper, $|\Lambda|$ stands for the (Euclidean) volume of parallelepiped $\Lambda \subset \mathbb{R}^d \times \mathbb{R}^d$ and $|\Pi_j \Lambda|$ for that of a projection cube $\Pi_j \Lambda \subset \mathbb{R}^d$. In addition, we use a similar notation $|A|$ for a cellular set $A \subset \mathbb{R}^d$.

Finally, note that an extension of the results of the present paper to the general case with $N \geq 1$ particles is also possible. To this end, one needs to apply the technique proposed recently in [10]. We plan to publish such an extension in a forthcoming paper.

2. The external potential field

A common model of a random field is a Gaussian random field on $\mathbb{R}^d$, with its characteristic ‘linearity structure’; it also serves as a ‘base’ for producing wider families of random fields. Our presentation will take this fact into account: comments on properties of the external potential field will be made from a ‘Gaussian perspective’. However, the class of the external fields under consideration is much larger and includes various ‘perturbations’ of and ‘operations’ with Gaussian random fields. We will not venture in this direction in the current paper but plan to address this issue elsewhere.

A convenient way to describe our assumptions on random field $V$ is as follows. Fix a (measurable) function

$$C : \begin{array}{c}
  x, y \in \mathbb{R}^d \mapsto C(x, y) \in \mathbb{R},
\end{array}$$  \hfill (2.1)

which is (strictly) positive-definite: $\forall$ (measurable) function $\zeta : \mathbb{R}^d \to \mathbb{C}$, the Lebesgue integral

$$\langle \zeta, \zeta \rangle_C := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} C(x, y) \zeta(x) \overline{\zeta(y)} \, dx \, dy \geq 0,$$  \hfill (2.2)

and $\langle \zeta, \zeta \rangle_C > 0$ unless $\zeta = 0$ a.e.. In these inequalities we allow the integral in the LHS of (2.1) to equal $+\infty$. However, given a cellular set $A \subset \mathbb{R}^d$, we denote by $\mathcal{L}_C^2(A)$ the set of functions $\zeta$ with support in $A$ and finite $\langle \zeta, \zeta \rangle_C$:

$$\mathcal{L}_C^2(A) = \{ \zeta : \begin{array}{c}
  \zeta(x) = 0 \text{ for } x \notin A \text{ and } \langle \zeta, \zeta \rangle_C < +\infty \},$$  \hfill (2.3)

equipped with the inner product

$$\langle \zeta, \eta \rangle_C := \int_A \int_A C(x, y) \zeta(x) \overline{\eta(y)} \, dx \, dy$$  \hfill (2.4)
and the norm \( \| \zeta \|_C := \langle \zeta, \zeta \rangle_C^{1/2} \). We will often suppose that an orthonormal basis in \( L^2_C(A) \) has been given, \( \{ \eta_i^A, i = 0, 1, \ldots \} \), where
\[
\eta_0^A = \mathbf{1}_A \quad \text{where} \quad \mathbf{1}_A = \frac{1}{Z_A} A.
\] (2.5)

Here and below, \( \mathbf{1}_A \) stands for the indicator function of set \( A \) and the normalising constant is given by
\[
Z_A = \| \mathbf{1}_A \|_C^{1/2}.
\]

In the case where \( C(x, y) \) admits a bound \( |C(x, y)| \leq a(x - y) \) where \( \int_a(z) \, dz < +\infty \), we have that \( Z_A \sim |A| \).

Our first condition on random field \( \mathbf{V} \) is:

(S) Summability: \( \forall \) cellular set \( A \subset \mathbb{R}^d \) and \( \zeta \in L^2_C(A) \):
\[
\int_A |\zeta(y)V(y; \omega)| \, dy < +\infty, \quad \mathbb{P} \text{-a.s.; (2.6)}
\]
consequently, we set
\[
[\zeta](\omega) = \int_A \zeta(y)V(y; \omega) \, dy,
\] (2.7)
which yields a correctly defined (and a.s. finite) RV \([\zeta]\). In particular, given a basis \( \{ \eta_i^A \} \) in \( L^2_C(A) \), we obtain a sequence of RVs \([\eta_i^A]\) \( \omega \).

In view of Condition (S), we can represent a random realisation \( V(x; \omega) \), \( x \in A \), of external potential random field \( \mathbf{V} \) in \( A \) as the sum
\[
V(x; \omega) = \sum_i [\eta_i^A](\omega) \eta_i^A(x), \quad \text{(2.8.1)}
\]
where the series is considered in space \( L^2_C(A) \). For that reason, we will employ an alternative notation \( \Gamma_i^A(\omega) = [\eta_i^A](\omega) \) and the decomposition
\[
V(x; \omega) = \sum_i \Gamma_i^A(\omega) \eta_i^A(x), \quad \text{(2.8.2)}
\]
and call RVs \( \Gamma_i^A(\omega) \) coefficient random variables in \( L^2_C(A) \). The whole sequence \( \{ \Gamma_i^A \} \) of the coefficient RVs in \( L^2_C(A) \) is denoted by \( \mathbf{\Gamma}_A \) (it is, of course, basis-dependent). Furthermore, if \( \eta_0^A \) is a member of the basis \( \{ \eta_i^A \} \) in \( L^2_C(A) \), then we denote by \( \mathbf{\Gamma}_{A[\eta_0^A]} \) the sequence of the remaining RVs \( \{ \Gamma_i^A, i \neq 0 \} \).

To illustrate the role of space \( L^2_C(A) \), take the example of a Gaussian random field \( \mathbf{V} \) in \( \mathbb{R}^d \) with the covariance kernel \( C(x, y) \). In this case, RVs \([\zeta_1]\) and \([\zeta_2]\) defined in (2.7) for functions \( \zeta_1, \zeta_2 \in L^2_C(A) \) with \( \langle \zeta, \zeta' \rangle_C = 0 \) are independent normal \( \mathcal{N}(0, 1) \).

When \( A = \Pi A \), with \( A = A_{L_1, L_2}(\mathbf{u}) \), the series (2.8.1) and (2.8.2) also determine the map
\[
\phi(\mathbf{x}) \mapsto W(\mathbf{x}; \omega) \phi(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2) \in A,
\]
as a (random) multiplication operator in $L_2(\Lambda)$. This in turn allows us to define the (random) Hamiltonian $H_\Lambda$ in $L_2(\Lambda)$; cf. Eqn (1.4). Given a basis in $L_2^c(\Pi \Lambda)$, the randomness in $H_\Lambda$ is represented by a sequence of coefficient RVs $\Gamma_{\Pi \Lambda} = \{\Gamma_{\Pi \Lambda}^i\}$.

We denote by $\mathcal{M}_\Lambda$ the sigma-algebra (in the space of locally square-integrable functions $\mathbb{R}^d \to \mathbb{R}$) generated by the RVs $[\zeta]$, $\zeta \in L_2^c(\Lambda)$. Then the sigma-algebra $\mathcal{M}$ is the smallest one containing $\mathcal{M}_\Lambda$, $\forall$ cellular sets $\Lambda$. The underlying probability distribution $\mathbb{P}$ is defined on $\mathcal{M}$; the restriction of $\mathbb{P}$ to $\mathcal{M}_\Lambda$ is denoted by $\mathbb{P}_\Lambda$ and the expectation relative to $\mathbb{P}_\Lambda$ by $E_\Lambda$.

Our second condition on $V$ is:

(T) **Temperedness**: $\forall$ cellular set $\Lambda \subset \mathbb{R}^d$, the RV $\overline{V}_\Lambda(\omega) = \sup \{ |V(y; \omega)| : y \in \Lambda \}$, (2.9)

is $\mathbb{P}_\Lambda$-a.s. finite, and has a finite moment $E_\Lambda (\overline{V}_\Lambda)^d < +\infty$. (2.10)

In this condition, we employ the representation $|V(x; \omega)| = \left| \sum _i \Gamma^\Lambda_i(\omega) \eta^\Lambda_i(x) \right|$, (2.11)

following Eqn (2.8.2). Although the choice of coefficient RVs $\Gamma^\Lambda_i$ depends on the basis in $L_2^c(\Lambda)$, condition T is basis-independent: when it holds for a particular choice of the orthonormal basis in $L_2^c(\Lambda)$, it also holds for all bases.

**Remark 2.1.** Condition (2.10) holds for a Gaussian random field under a mild assumption that its samples are ‘regular’. (In fact, a much stronger property takes place, guaranteeing (2.10).) See, e.g., [3] and references therein.

Given a pair of disjoint cellular sets $\Lambda, \Lambda' \subset \mathbb{R}^d$, we will be working with the conditional distribution functions $F_\Lambda \left( y; \Gamma_{\overline{A} \cup A'} \right)$ defined as follows. Let us fix a basis $\{ \eta^\Lambda_{\overline{A} \cup A'}, i = 0, 1, \ldots \}$ in $L_2^c(\Lambda \cup \Lambda')$ with $\eta^\Lambda_{\overline{A} \cup A'} = \tilde{1}_A$. Then set:

$F_\Lambda \left( y; \Gamma_{\overline{A} \cup A'} \right) := \mathbb{P} \left( \left[ \tilde{1}_A > y \right] \right)$, (2.12)

for the coefficient RV $\tilde{1}_A = \{ \eta^\Lambda_i \}$, given $\Gamma_{\overline{A} \cup A'} = \{ \eta^\Lambda_i : i \geq 1 \}$, a collection of other basis-related coefficient RVs in $L_2^c(\Lambda \cup \Lambda')$. In fact, in Eqn (2.12) we are interested in sets of the form $A = \Pi \Lambda$ and $A' = \Pi \Lambda'$ where parallelepipeds $\Lambda = \Lambda_{L_1, L_2}$ and $\Lambda' = \Lambda'_{L'_1, L'_2}$ are distant apart, namely with $\| \underline{u} - \underline{u}' \|_{\max} > 8 \max \{ L_1, L_2, L_1', L_2' \}$. (2.13)

More precisely, we exploit a simple geometric fact stated in Lemma 2.1.
Lemma 2.1. Consider two parallelepipeds $\Lambda = \Lambda_{L_1, L_2}(u)$ and $\Lambda' = \Lambda_{L_1', L_2'}(u')$ and suppose that condition (2.13) holds true. Then there are two possibilities (which in general do not exclude each other):

(i) $\Lambda$ and $\Lambda'$ are 'completely separated', when

$$\text{dist} \left[ \Pi \Lambda, \Pi \Lambda' \right] > 0.$$  \hfill (2.14)

(ii) $\Lambda$ and $\Lambda'$ are 'partially separated'. In this case one (or more) of the four possibilities can occur:

(A) $\text{dist} \left[ \Pi_1 \Lambda, \Pi_1 \Lambda' \right] > 0,$
(B) $\text{dist} \left[ \Pi_2 \Lambda, \Pi_2 \Lambda' \right] > 0,$
(C) $\text{dist} \left[ \Pi_1 \Lambda', \Pi_1 \Lambda' \right] > 0,$
(D) $\text{dist} \left[ \Pi_2 \Lambda', \Pi_2 \Lambda' \right] > 0.$  \hfill (2.15)

The proof of Lemma 2.1 is straightforward; it has been given in our earlier paper [8] and applied to the lattice case, but geometrical arguments in [8] actually refer to parallelepipeds in Euclidean space $\mathbb{R}^d$.

Pictorially, case (ii) is where one of the cubes $\Pi_j \Lambda$, $\Pi_j \Lambda'$, $j = 1, 2$, is disjoint from the union of the rest of the projections of $\Lambda$ and $\Lambda'$. We note that the use of the max-norm $\| \cdot \|_{\text{max}}$ is convenient here as it leads to the constant 8 (equal to 2 times 4, the number of projections $\Pi_j \Lambda$ and $\Pi_j \Lambda'$, $j = 1, 2$) which does not depend on dimension $d$.

We make use of Lemma 2.1 as follows. First, for a given pair of bounded cellular sets $\Lambda, \Lambda' \subset \mathbb{R}^d$, with $\Lambda \cap \Lambda' = \emptyset$, we set:

$$\nu_{\Lambda, \Lambda'}(b) := \sup_{y \in \mathbb{R}} \sup \text{ess} \left[ F_\Lambda \left( y + b; \Gamma_{\Lambda \cup \Lambda'}^\perp \right) - F_\Lambda \left( y; \Gamma_{\Lambda \cup \Lambda'}^\perp \right) \right].$$  \hfill (2.16)

Here, it is assumed that we have been given a basis $\left\{ \eta_{\Lambda \cup \Lambda'}^i, i = 0, 1, \ldots \right\}$ in $L^2_\mu (\Lambda \cup \Lambda')$, with $\eta_{\emptyset}^{\Lambda \cup \Lambda'} = \hat{1}_\Lambda$.

In particular, with $\Lambda' = \emptyset$, we have:

$$\nu_{\Pi \Lambda, \emptyset}(b) := \sup_{y \in \mathbb{R}} \sup \text{ess} \left[ F_\Lambda \left( y + b; \Gamma_A^{\perp} \right) - F_\Lambda \left( y; \Gamma_A^{\perp} \right) \right].$$  \hfill (2.17)

We then set:

$$\mu_{\Pi \Lambda}(b) = \nu_{\Pi \Lambda, \emptyset}(b), \quad b > 0.$$  \hfill (2.18)

Next, given $b > 0$, we denote:

$$\mu_{\Lambda}^{(0)}(b) := \sup \left\{ \nu_{\Pi \Lambda, \Pi \Lambda'}(b) : \ L_1', L_2' > 0, \ u' \in \mathbb{R}^d \times \mathbb{R}^d, \ \|u - u'\|_{\text{max}} > 8 \max \left[ L_1, L_2, L_1', L_2' \right] \text{ and } \Pi \Lambda \cap \Pi \Lambda' = \emptyset \right\},$$  \hfill (2.19)
\[
\mu^{(1)}_\Lambda(b) := \sup \left\{ \nu_{\Pi_1 \Lambda \cup \Pi_2 \Lambda}(b) : \begin{array}{l}
L_1', L_2' > 0, \\
\mathbf{u}' \in \mathbb{R}^d \times \mathbb{R}^d, \|\mathbf{u} - \mathbf{u}'\|_{\text{max}} > 8 \max [L_1, L_2, L_1', L_2']
\end{array}
\right\}
\tag{2.20}
\]
and
\[
\mu^{(2)}_\Lambda(b) := \sup \left\{ \nu_{\Pi_2 \Lambda \cup \Pi_1 \Lambda}(b) : \begin{array}{l}
L_1', L_2' > 0, \\
\mathbf{u}' \in \mathbb{R}^d \times \mathbb{R}^d, \|\mathbf{u} - \mathbf{u}'\|_{\text{max}} > 8 \max [L_1, L_2, L_1', L_2']
\end{array}
\right\}
\tag{2.21}
\]

(We suppose here that we have bases \(\eta_{\Pi_1 \Lambda \cup \Pi_2 \Lambda}'', i = 0, 1, \ldots\) in \(\mathcal{L}_2^G(\Pi_1 \Lambda \cup \Pi_2 \Lambda)\), with \(\eta_{\Pi_1 \Lambda \cup \Pi_2 \Lambda}' = \mathbf{1}_{\Pi_1 \Lambda}\) in Eqn (2.19), \(\eta_{\Pi_1 \Lambda \cup \Pi_2 \Lambda}' = \mathbf{1}_{\Pi_2 \Lambda}\) in Eqn (2.20) and \(\eta_{\Pi_1 \Lambda \cup \Pi_2 \Lambda}' = \mathbf{1}_{\Pi_1 \Lambda}\) in Eqn (2.21).

Finally,
\[
\overline{\mu}_\Lambda(b) := \max \left\{ \mu^{(0)}_\Lambda(b), \mu^{(1)}_\Lambda(b), \mu^{(2)}_\Lambda(b) \right\}.
\tag{2.22}
\]

**Remark 2.2.** Quantity \(\nu_{\Lambda\Lambda'}(b)\) in Eqn (2.16) describes a ‘conditioned continuity modulus’ of (the distribution function of) RV \(\mathbf{1}_\Lambda\), given a sample \(\Gamma_{\Lambda\Lambda'}\). It is important to have a grasp of the magnitude of this RV and therefore of quantity \(\nu_{\Lambda\Lambda'}(b)\). As an example, we can think of variable \(\mathbf{1}_\Lambda\) as normal \(\mathcal{N}(0, 1)\): this is the case where random field \(\mathbf{V}\) is Gaussian with covariance kernel \(C(x, y)\). In this case, \(\nu_{\Lambda\Lambda'}(b) \leq (2\pi)^{-1/2}b\) and consequently, quantities \(\mu_{\Pi_1 \Lambda}(b)\) and \(\overline{\mu}_{\Pi_1 \Lambda}(b)\), \(j = 0, 1, 2\), from Eqns (2.18) and (2.19)–(2.21) satisfy a similar bound.

Concluding this section, we would like to comment on our approach to analysis of random fields with a continuous argument, particularly, on the meaning and use of quantities introduced in Eqns (2.16)–(2.22). Our main intention is to provide a benefit to a readers with a physical background.

We start by referring back to Eqn (1.3) specifying the structure of a two-particle potential energy under consideration. In a single-particle model, the Hamiltonian \(H^{(1)}\) is simply \(-\frac{1}{2}\Delta + V(x, \omega)\). In this case, Wegner-type bounds were proved in [13], for a certain class of external potential random fields (including Gaussian fields). The proof given in [13] is based on a special kind of decomposition of an random field in a cube \(\Lambda_L(u)\) (see (1.8)), of the form
\[
V(x; \omega) = \varphi_{\Lambda_L(u)}(x) + \Theta(x; \omega), \quad x \in \Lambda_L(u),
\tag{2.23}
\]
where the (scalar) random variable \(\varphi_{\Lambda_L(u)}(\omega)\) is independent of the ‘fluctuation field’ \(\Theta(x; \omega)\). This suggests conditioning upon values of \(\Theta(x; \omega)\), so that the finite-volume Hamiltonian \(H^{(1)}_{\Lambda_L(u)}\) (again with Dirichlet boundary conditions) becomes a function of the scalar random parameter \(\varphi_{\Lambda_L(u)}(\omega)\). Naturally, its eigenvalues are also viewed as functions of \(\varphi_{\Lambda_L(u)}(\omega)\); their analysis, based on the so-called Dirichlet–Neumann bracketing and other techniques, leads to Wegner-type bounds.
These bounds are helpful for proving the existence of the limiting density of states for single-particle model in question. Conditions on the random field $V(x; \omega)$ in [13] cover some non-Gaussian random fields, but they are somewhat restrictive even in the Gaussian case.

In the present paper, we make use of a similar decomposition (2.8.1)–(2.8.2). For $A = \Lambda$ where $\Lambda = \Pi_j \Lambda(j)$, it looks like

$$V(x; \omega) = \left[ \eta^\Lambda_0(\omega) \right] \tilde{1}_\Lambda(x) + \Xi_\Lambda(x; \omega), \quad x \in \Lambda,$$

where $\Xi_\Lambda(x; \omega) = \sum_{\eta^\Lambda \in \Gamma \perp \Lambda} \eta^\Lambda_{\Lambda_0}(\omega)$, with a (normalised) constant ‘ground level’ $\tilde{1}_\Lambda(x)$ and a coefficient $[\eta^\Lambda_0(\omega)]$ ‘moderately dependent’ on the residual ‘fluctuation field’ $\Xi(x; \omega)$. Such an idea is well-known in probability theory: in elementary courses of statistics, it is usually proved that "the empiric mean of a Gaussian sample is independent of the empiric variance". See, e.g., [18].

In the context of this paper, decomposition (2.24) implies a similar decomposition for two-particle finite-volume eigenvalues $E_\Lambda(k)$ (see (1.10)). In particular, conditional on a value of $\Xi_\Lambda(x; \omega)$, the eigenvalues $E_\Lambda(k)$ are behaving, as functions of $\eta_\Lambda(\omega)$, in a controllable fashion, which leads to our Wegner-type bounds. In the special case when random field $V$ is Gaussian, the random variables $E_\Lambda(k)$ are (conditionally) normal, with the means and variances determined by the conditions but behaving in a controllable fashion.

### 3. Wegner-type bounds

A one-volume Wegner-type bound for finite-volume Hamiltonians is given in Theorem 3.1 below. Let $\Sigma(H_\Lambda)$ denote the (random) spectrum of operator $H_\Lambda$ from Eqn (1.4) (i.e., the collection of its eigenvalues $E_\Lambda$, without their multiplicities). Next, set

$$W_\Lambda(\omega) := \max \left| W(x; \omega) \right| : x \in \Lambda \leq U_\Lambda + \Omega_{\Pi_1 \Lambda}(\omega) + \Omega_{\Pi_2 \Lambda}(\omega) \leq U_\Lambda + 2\Omega_{\Pi \Lambda}(\omega).$$

where

$$U_\Lambda = \sup_{x \in \Lambda} |U(x)|,$$

and $\Omega_{\Pi \Lambda}$ is the supremum from Eqn (2.9).

**Theorem 3.1.** Assume the above conditions (I), $(E0)$ and $(E1)$ on function $U$ and external potential random field $V$. Then, $\forall E \in \mathbb{R}$, $L_1, L_2 \geq 1$, $u \in \mathbb{R}^d \times \mathbb{R}^d$ and
\( \epsilon \in (0, 1) \), with \( \Lambda = \Lambda_{L_1, L_2}(\underline{u}) \):

\[
P\left( [E - \epsilon, E + \epsilon] \cap \Sigma (H_\Lambda) \neq \emptyset \right) 
\leq c_1 \cdot |\Lambda| \cdot E \left( E + 2 + W_\Lambda \right)^d \cdot \mu_\Lambda \left( 4Z_\Pi \Lambda \cdot \epsilon \right). \tag{3.2}
\]

Here \( c_1 \in (0, +\infty) \) is a constant independent of \( E, L_1, L_2, \underline{u} \) and \( \epsilon \).

The expression in the RHS of (3.2) includes a ‘volume’ factor \(|\Lambda_{L_1, L_2}(\underline{u})|\) and a factor \( E \left( E + 2 + W_\Lambda \right)^d \) reflecting the growth of the (value of the) external potential random field \( V \). Next, we have a factor \( \mu_\Lambda \left( 4Z_\Pi \Lambda \cdot \epsilon \right) \) controlling ‘singularity’ of the conditional distribution functions

\[
F^\Lambda_{\Pi \Lambda} \left( y; \Gamma^\Lambda_{\Pi \Lambda j} \right) \text{ and } F^\Lambda_{\Pi \Lambda} \left( y; \Gamma^\Lambda_{\Pi \Lambda j} \right), \quad j = 1, 2, y \in \mathbb{R},
\]

with a controlled distance between projections \( \Pi \Lambda \) and \( \Pi \Lambda' \). Here \( j = 3 - j \). In a ‘smooth’ situation where this distribution function has a density that is bounded uniformly in \( y \) and in \( L_1, L_2, L'_1, L'_2, \underline{u}, \underline{u}' \) satisfying (2.13), (see Remark above), we have that \( \mu_\Lambda \left( 4Z_\Pi \Lambda \epsilon \right) \approx c'_1 Z_\Pi \Lambda \epsilon \) where \( c'_1 > 0 \) is a constant. Again, such a form of the bound is important for the proof of Anderson’s localisation, the main area of application of Wegner-type inequalities.

Our conditions on the random external potential field are straightforward, as well as the core argument of the proof given below. However, they require from the reader sufficient familiarity with standard constructions used in the analysis of random fields with a continuous argument. It may be of some help to the reader to know that these conditions are written down by postulating, in general terms, well-known qualitative properties of Gaussian random fields. On the other hand, the obtained eigenvalue concentration bounds are not optimal, and they are not supposed to be, since our main motivation here is to lay ground for the multi-particle Multi-Scale Analysis in a Euclidean space \( \mathbb{R}^d \), using ideas and techniques similar to those used in \( [9], [10] \).

**Proof.** Given a two-particle parallelepiped \( \Lambda = \Lambda_{L_1, L_2}(\underline{u}) \), the randomness in operator \( H_\Lambda(\omega) \) is represented by the sample of the random field \( \mathbf{V}^\Lambda_{\Pi \Lambda} = \{V(x; \omega), x \in \Pi \Lambda\} \). More conveniently, we can refer to a sequence \( \Gamma^\Lambda_{\Pi \Lambda} = \{\Gamma^\Lambda_{\Pi \Lambda i}, i = 0, 1, \ldots\} \) of random variables \( \Gamma^\Lambda_{\Pi \Lambda i} \) representing sample \( \mathbf{V}^\Lambda_{\Pi \Lambda} \) in \( L^2_2(\Pi \Lambda) \):

\[
V(x; \omega) = \sum_{i \geq 0} \Gamma^\Lambda_{\Pi \Lambda i}(\omega) \eta^\Lambda_{\Pi \Lambda i}(x), \quad x \in \Pi \Lambda.
\]
It is convenient to think of $H_{\Lambda}(\omega)$ as a family of operators $H_{\Lambda}(\gamma)$ in $L_2(\Lambda)$ parametrised by vectors $\gamma = \{\gamma_i : i = 0, 1, \ldots\}$ (i.e., sample vectors of $\Gamma_{\Pi \Lambda}$):

$$[H_{\Lambda}(\gamma)\phi](\underline{x}) = [H^{(0)}_{\Lambda} \phi](\underline{x}) + W_{\Lambda}(\underline{x}; \gamma) \phi(\underline{x})$$

$$= -\frac{1}{2} \sum_{j=1,2} \Delta_j^{(\Lambda_{L_j}(\underline{u}_j))} \phi(\underline{x}) + \left( U(\underline{x}) + \sum_{j=1,2} \sum_{i=0,1,\ldots} \gamma_i \eta_i^{(\Pi \Lambda)}(x_j) \right) \phi(\underline{x}), \quad (3.3)$$

Vectors $\gamma$ are then made random, subject to the distribution $\mathbb{P}_{\Pi \Lambda}$ of random field $V_{\Pi \Lambda}$. Consequently, the eigenvalues $E_k(\Lambda)$ of $H_{\Lambda}(\gamma)$ (written in the non-decreasing order) are parametrised by $\gamma$, i.e., are random variables on $(\mathbb{R}^d, \mathbb{P}_{\Pi \Lambda})$.

We can denote them by $E_k(\Lambda, \gamma)$ but in fact, to stress the dependence on function $W(= W(\cdot ; \omega))$, we will use an alternative notation $E_k(\Lambda, W)$, $k = 0, 1, \ldots$.

In this setting, it is convenient to choose a basis $\{\eta_i^{(\Pi \Lambda)} : i = 0, 1, \ldots\}$ in $L_2^d(\Pi \Lambda)$ with $\eta_0^{(\Pi \Lambda)} = \tilde{1}_{\Pi \Lambda}$ and work with $\mathbb{P}_\Lambda(\cdot | \gamma \geq 1)$, the probability distribution for RV $\Gamma_{\Pi \Lambda}^{(0)} = \tilde{1}_{\Pi \Lambda}$ conditional on a given vector of values $\gamma \geq 1 = \{\gamma_j : j \geq 1\}$ of RVs $\Gamma_{\Pi \Lambda}^{(1)} = \{\Gamma_{\Pi \Lambda}^{(1)}, i \geq 1\}$.

Our first remark is that the additive change $v_{\Pi \Lambda} \mapsto v_{\Pi \Lambda} + t\tilde{1}_{\Pi \Lambda}$ in the sample of the external potential field $V_{\Pi \Lambda}$ generates the change $W(\underline{x}) \mapsto W(\underline{x}) + 2t$, $\underline{x} \in \Lambda$,

in the value of the potential energy function $W(\underline{x})$, $\underline{x} \in \Lambda$. Correspondingly, the eigenvalues $E_0(\Lambda, W) \leq E_1(\Lambda, W) \leq \ldots$ and $E_0(\Lambda, W + 2t) \leq E_1(\Lambda, W + 2t) \leq \ldots$ of the operator $H_{\Lambda}$ before and after the change are related with

$$E_j(\Lambda, W + 2t) = E_j(\Lambda, W) + 2t, \quad j = 0, 1, \ldots$$

Recalling that $\eta_0^{(\Pi \Lambda)} = \tilde{1}_{\Pi \Lambda}/Z_{\Pi \Lambda}$, we obtain that $\forall j = 0, 1,\ldots$:

$$\mathbb{P}_\Lambda \left( E_j(\Lambda, W) \in [E - \epsilon, E + \epsilon] | \gamma \geq 1 \right) \leq \mu_{\Pi \Lambda} \left( 4Z_{\Pi \Lambda} \epsilon \right), \quad (3.4)$$

The second remark is that, when we pass from (3.4) to (3.2), we only have to examine a finite number of eigenvalues $E_j(\Lambda, W)$ of Hamiltonian $H_{\Lambda}$. More precisely, denote by $E_0(\Lambda, 0) \leq E_1(\Lambda, 0) \leq \ldots$ the eigenvalues of the kinetic energy operator $H_0^\Lambda$. Then the bound

$$\left| E_j(\Lambda, 0) - E(\Lambda, W) \right| \leq 2V_{\Pi \Lambda} \quad (3.5)$$
implies that, \( \forall \ l = 1, 2, \ldots \), the conditional probability
\[
\mathbb{P}\left( [E - \epsilon, E + \epsilon] \cap \Sigma (H_\Lambda) \neq \emptyset \mid \nabla_{\mathfrak{m}_\Lambda} \in [l - 1, l] \right) 
\leq \# \left\{ \text{eigenvalues of } H_\Lambda^\alpha \text{ in } [E - \epsilon - l, E + \epsilon + l] \right\} \cdot \mu_{\mathfrak{m}_\Lambda} \left( 4Z\mathfrak{m}_\Lambda \epsilon \right). 
\] (3.6)

The final remark is that
\[
\# \left\{ \text{eigenvalues of } H_\Lambda^\alpha \text{ in } [E - \epsilon - l, E + \epsilon + l] \right\} 
\leq c_1 |\Lambda| (E + \epsilon + l)^d \cdot \mu_{\mathfrak{m}_\Lambda} \left( 4Z\mathfrak{m}_\Lambda \epsilon \right), 
\] (3.7)

where \( c_1 \) is the the constant from Weyl’s formula. Hence, by taking expectation, we obtain from (3.6) that the LHS in (3.2) is
\[
\leq c_1 \cdot |\Lambda| \cdot \mathbb{E} \left( E + 2 + \nabla_{\mathfrak{m}_\Lambda} \right)^d \cdot \mu_{\mathfrak{m}_\Lambda} \left( 4Z\mathfrak{m}_\Lambda \epsilon \right),
\] i.e., the bound (3.2) holds true. This completes the proof of Theorem 3.1. \( \square \)

We now turn to a two-volume Wegner-type bound, dealing with a pair of parallelepipeds \( \Lambda = \Lambda_{L_1, L_2, u} \) and \( \Lambda' = \Lambda_{L_1', L_2', u'} \) (more precisely, with the corresponding Hamiltonians \( H_\Lambda \) and \( H_{\Lambda'} \)), under an assumption that the distance between \( \Lambda \) and \( \Lambda' \) is of the same order of magnitude as the size of these parallelepipeds. Given an interval \( J \subset \mathbb{R} \), set:
\[
\text{dist} \left[ \Sigma (H_\Lambda) \cap I, \Sigma (H_{\Lambda'} \cap J) \right] 
= \inf \left\{ |E_k(\Lambda) - E_{k'}(\Lambda')| : E_k(\Lambda), E_{k'}(\Lambda') \in J \right\},
\] (3.8)

where \( E_k(\Lambda) \) are the eigenvalues of \( H_\Lambda \) and \( E_{k'}(\Lambda') \) those of \( H_{\Lambda'} \), \( k, k' = 0, 1, \ldots \).

Because the potential
\[
W(\xi) = U(\xi) + g[V(x_1; \omega) + V(x_2; \omega)]
\] (3.9)
is a symmetric function of the pair \( \xi = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d \), with
\[
W(S\xi) = W(\xi), \text{ where } S : (x_1, x_2) \mapsto (x_2, x_1).
\]
Consequently, the spectra \( \Sigma (H_\Lambda) \) and \( \Sigma (H_{S(\Lambda)}) \) are identical, and the same is true for \( \Sigma (H_{\Lambda'}) \) and \( \Sigma (H_{S(\Lambda')}) \).

**Theorem 3.2.** \( \forall \ L_1, L_2, L_1', L_2' > 1, \ u, u' \in \mathbb{R}^d \times \mathbb{R}^d \) with
\[
\min \left\{ \|u - u'\|_{\text{max}}, \|S(u) - u'\|_{\text{max}} \right\} > 8 \max[L_1, L_2, L_1', L_2']
\] (3.10)
and $\forall \epsilon \in (0, 1)$ and interval $J = [b - \delta, b + \delta] \subset \mathbb{R}$, where $b \in \mathbb{R}$ and $\delta > 0$:

$$\mathbb{P}\left( \text{dist} \left[ \Sigma (H_{\Lambda}) \cap J, \Sigma (H_{\Lambda'}) \cap J \right] \leq \epsilon \right) \leq c_2 \cdot |\Lambda| \cdot |\Lambda'| \cdot \mathbb{E}\left[ (b + \delta + 1 + \overline{W}_{\Lambda'})^d(b + \delta + 1 + \overline{W}_{\Lambda'})^d \right] \times \max_{j=1,2} \left[ \mathbb{P}_{H_\Lambda} (4\epsilon \cdot Z_{\Lambda}) , \mathbb{P}_{H_\Lambda'} (4\epsilon \cdot Z_{\Lambda'}) \right].$$  \hfill (3.11)

**Remark 3.1.** As in the case of the one-volume Wegner-type estimate (3.2), the RHS in (3.11) is composed by ‘volume’ factors $|\Lambda|$ and $|\Lambda'|$ and factors

$$\mathbb{E}\left[ (b + \delta + 1 + \overline{W}_{\Lambda'})^d(b + \delta + 1 + \overline{W}_{\Lambda'})^d \right] \hfill \hfill (3.12)$$

and

$$\max_{j=1,2} \left[ \mathbb{P}_{H_\Lambda} (4\epsilon Z_{\Lambda}) , \mathbb{P}_{H_\Lambda'} (4\epsilon Z_{\Lambda'}) \right]. \hfill \hfill (3.13)$$

These factors assess various aspects of randomness introduced in Hamiltonians $H_{\Lambda}$ and $H_{\Lambda'}$. For us, an immediate use of Theorems 3.1 and 3.2 is in proving Anderson’s localisation in a two-particle model (with interval $J = [b - \delta, b + \delta]$ at the ‘edge of the spectrum, specified by further assumptions on RF $\mathbf{V}$).

**Proof.** Owing to Lemma 2.1 parallelepipeds $\Lambda$ and $\Lambda'$ obeying (3.10) satisfy either (i) or (ii), i.e. they are either completely or partially separated. Consider first case (i) (complete separation). Write

$$\mathbb{P}\left( \text{dist} \left[ \Sigma (H_{\Lambda}) \cap J, \Sigma (H_{\Lambda'}) \cap J \right] \leq \epsilon \right) = \mathbb{E}\left[ \mathbb{P}\left( \text{dist} \left[ \Sigma (H_{\Lambda}) \cap J, \Sigma (H_{\Lambda'}) \cap J \right] \leq \epsilon \big| \mathbf{V}_{\Lambda'} \right) \right]. \hfill \hfill (3.14)$$

Note first that, under conditioning in Eqn (3.14), the eigen-values $E_{k'}(\Lambda')$, $k' = 0, 1, \ldots$, forming the set $\Sigma (H_{\Lambda'})$ are non-random. Therefore, it makes sense to use the following inequality:

$$\mathbb{P}\left( \text{dist} \left[ \Sigma (H_{\Lambda}) , \Sigma (H_{\Lambda'}) \right] \leq \epsilon \big| \mathbf{V}_{\Lambda'} \right) \leq N_{\Lambda'} (J; \overline{W}_{\Lambda'}) \sup_{E \in J} \mathbb{P}\left( \text{dist} \left[ \Sigma (H_{\Lambda}) \cap J, E \right] \leq \epsilon \big| \mathbf{V}_{\Lambda'} \right), \hfill \hfill (3.15)$$

where $N_{\Lambda'} (J; \overline{W}_{\Lambda'})$ is the number of the eigenvalues $E_{k'}(\Lambda')$ that can ‘eventually’ fall in $J$. As in (3.11), for $J = [a, b]$ and with $E_{k'}^0$ standing for the eigenvalues of $H_{\Lambda'}^0$, we have:

$$N_{\Lambda'} (J; \overline{W}_{\Lambda'}) \leq \# \left\{ \text{eigenvalues } E_{k'}^0 \in \left[ (a - \overline{W}_{\Lambda'})^+, b + \overline{W}_{\Lambda'} \right] \right\} \leq b_{2d}(L_1 L_2)^d(b + \overline{W}_{\Lambda'})^d.$$

\hfill \hfill (3.16)
Next, as in Theorem 3.1
\[
P \left( \text{dist} \left[ \Sigma (H_{\Lambda}), E \right] \leq c_1 \cdot |\Lambda| \cdot E \left[ (E + 2 + \overline{W_{\Lambda}})^d \left| |\Lambda| \right| E \left[ 4 \epsilon Z_{\Pi \Lambda} \right] \right] \right)
\]

This yields
\[
P \left( \text{dist} \left[ \Sigma (H_{\Lambda}), E \right] \leq c_1 \cdot |\Lambda| \cdot E \left[ (E + 2 + \overline{W_{\Lambda}})^d \left| |\Lambda| \right| E \left[ 4 \epsilon Z_{\Pi \Lambda} \right] \right] \right)
\]

implying that
\[
P \left( \text{dist} \left[ \Sigma (H_{\Lambda}), E \right] \leq c_1 \cdot |\Lambda| \cdot E \left[ (E + 2 + \overline{W_{\Lambda}})^d \left| |\Lambda| \right| E \left[ 4 \epsilon Z_{\Pi \Lambda} \right] \right] \right)
\]

where
\[
c_1 \in (0, +\infty)
\]

(ii) Now consider the case of partial separation. For example, assume case (A) from Lemma 2.1 (see Eqn (3.11)), when projection \( \Pi_{1,\Lambda} \) is disjoint from the union of the rest of the projections of \( \Lambda \) and \( \Lambda' \):
\[
\Pi_{1,\Lambda} \cap [\Pi_{2,\Lambda} \cup \Pi_{1,\Lambda}'] = \emptyset.
\]

We then estimate the probability in the LHS of (3.16) with the help of the conditional expectation
\[
P \left( \text{dist} \left[ \Sigma (H_{\Lambda}), \Sigma (H_{\Lambda'}) \right] \leq \epsilon \right)
= E \left[ P \left( \text{dist} \left[ \Sigma (H_{\Lambda}), \Sigma (H_{\Lambda'}) \right] \leq \epsilon \right| \Gamma_{\Pi_{2,\Lambda} \cup \Pi_{1,\Lambda}'} \right] \Gamma_{\Pi_{1,\Lambda}}.
\]

Note that, owing to (3.20), the sigma-algebra generated by \( \Gamma_{\Pi_{2,\Lambda} \cup \Pi_{1,\Lambda}'} \) does not include any of the RVs \( \Gamma_{\Pi_{1,\Lambda}} \) forming \( \Gamma_{\Pi_{1,\Lambda}} \). Thus, the argument used in the proof of Theorem 3.1 is still applicable if, instead of \( P \), we work with the probability distribution \( P_{\Pi_{1,\Lambda}}(\cdot | \Gamma_{\Pi_{2,\Lambda} \cup \Pi_{1,\Lambda}'} \) conditional on \( V_{\Gamma_{2,\Lambda'}} \)) and restricted to the the sigma-algebra generated by \( \Gamma_{\Pi_{1,\Lambda}} \).

This allows us to write
\[
P \left( \text{dist} \left[ \Sigma (H_{\Lambda}), \Sigma (H_{\Lambda'}) \right] \leq \epsilon \right)
\leq c_2 \cdot |\Lambda| \cdot E \left[ (b + 2 + \overline{W_{\Lambda}})^d \left| |\Lambda| \right| E \left[ 4 \epsilon Z_{\Pi \Lambda} \right] \right]
\]
\[
x E \left[ (b + 2 + \overline{W_{\Lambda}})^d \left| \Gamma_{\Pi_{1,\Lambda}} \right| \Gamma_{\Pi_{2,\Lambda} \cup \Pi_{1,\Lambda}} \right] \Gamma_{\Pi_{2,\Lambda} \cup \Pi_{1,\Lambda}'} \times \Gamma_{\Pi_{1,\Lambda}} (4 \epsilon Z_{\Pi \Lambda})
\]

(3.22)
and
\[
\mathbb{P}\left( \text{dist } \left[ \Sigma (H\Lambda), \Sigma (H\Lambda') \right] \leq \epsilon \right) \\
\leq c_2^{(2)} |\Lambda| |\Lambda'| \mathbb{E}\left[ (b + 2 + W_\Lambda)^d (b + 2 + W_{\Lambda'})^d \right] \\
\times \mathbb{P}_{\Pi\Lambda} \left( 4\epsilon Z_{\Pi\Lambda} \right).
\]

We are then in position to deduce the required bound for the the conditional probability in the LHS of (3.21).

If, instead of (3.20), we have one of the other disjointedness relations (B)-(D) in Eqn (3.11), then the argument is conducted in a similar fashion. Specifically, in case (B) we swap projections \( \Pi_1\Lambda \) and \( \Pi_2\Lambda \) in the above argument. Furthermore, in cases (C) and (D), we should exchange \( u \) and \( u' \) as compared to arguments in cases (A) and (B).

We now briefly comment on the case where the interaction potential has a hard core (cf. (1.15)). In this case Hamiltonians \( H \) and \( H\Lambda \) in Eqns (1.1) and (1.5) act in subspaces
\[
L_2 \left( (\mathbb{R}^d \times \mathbb{R}^d) \setminus \mathbb{D}_{r_0} \right) = \left\{ \phi \in L_2(\mathbb{R}^d \times \mathbb{R}^d) : \phi(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in \mathbb{D}_{r_0} \right\}
\]
and
\[
L_2 \left( \Lambda \setminus \mathbb{D}_{r_0} \right) = \left\{ \phi \in L_2(\mathbb{R}^d \times \mathbb{R}^d) : \phi(\mathbf{z}) = 0 \text{ for } \mathbf{z} \in \mathbb{D}_{r_0} \right\}
\]
where
\[
\mathbb{D}_{r_0} = \{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d : \|x_1 - x_2\|_{\max} < r_0 \}.
\]
More precisely, \( H \) and \( H\Lambda \) carry additional Dirichlet’s boundary conditions on \( \partial \mathbb{D}_{r_0} \). However, the scheme of the proof of Theorems 3.1 and 3.2 remains unchanged, and all arguments carry through.

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A. Boutet de Monvel
Institut de Mathématiques de Jussieu
Université Paris Diderot Paris 7
175 rue du Chevaleret, 75013 Paris, France
e-mail: aboutet@math.jussieu.fr

V. Chulaevsky
Département de Mathématiques et Informatique,
Université de Reims, Moulin de la Housse, B.P. 1039,
51687 Reims Cedex 2, France
e-mail: victor.tchoulaevski@univ-reims.fr
Y. Suhov
Statistical Laboratory, DPMMS
University of Cambridge, Wilberforce Road,
Cambridge CB3 0WB, UK
e-mail: Y.M.Suhov@statslab.cam.ac.uk