3/2-Institutions: an institution theory for conceptual blending

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Abstract

We develop an extension of institution theory that accommodates implicitly the partiality of the signature morphisms and its syntactic and semantic effects. This is driven primarily by applications to conceptual blending, but other application domains are possible (such as software evolution). The particularity of this extension is a reliance on ordered-enriched categorical structures.

1. Introduction

1.1. Institution theory

The mathematical context of our work is the theory of institutions [12] which is a three-decades-old category-theoretic abstract model theory that traditionally has been playing a crucial foundational role in formal specification (e.g. [22]). It has been introduced in [11] as an answer to the explosion in the number of population of logical systems there, as a very general mathematical study of formal logical systems, with emphasis on semantics (model theory), that is not committed to any particular logical system. Its role has gradually expanded to other areas of logic-based computer science, most notably to declarative programming and ontologies. In parallel, and often in interdependence to its role in computer science, in the past fifteen years it has made important contributions to model theory through the new area called institution-independent model theory [2] – an abstract approach to model theory that is liberated from any commitment to particular logical systems. Institutions thus allowed for a smooth, systematic, and uniform development of model theories for unconventional logical systems, as well as of logic-by-translation techniques and of heterogeneous multi-logic frameworks.

Mathematically, institution theory is based upon a category-theoretic [19] formalization of the concept of logical system that includes the syntax, the semantics, and the satisfaction relation between them. As a form of abstract model theory, it is the only one that treats all these components of a logical system fully abstractly. In a nutshell, the above-mentioned formalization is a category-theoretic structure \((\text{Sign}, \text{Sen}, \text{Mod}, \models)\), called institution, that consists of (a) a category \(\text{Sign}\) of so-called signatures, (b) two functors, \(\text{Sen}: \text{Sign} \to \text{SET}\) for the syntax, given by sets of so-called sentences, and \(\text{Mod}: \text{Sign}^\Sigma \to \text{CAT}\) for the semantics, given by categories of so-called models, and (c) for each signature \(\Sigma\), a binary satisfaction relation \(\models_\Sigma\) between the \(\Sigma\)-models, i.e. objects of \(\text{Mod}(\Sigma)\), and the \(\Sigma\)-sentences, i.e. elements of \(\text{Sen}(\Sigma)\), such that for each morphism \(\varphi: \Sigma \to \Sigma'\) in the category \(\text{Sign}\), each \(\Sigma'\)-model \(M'\), and each \(\Sigma\)-sentence \(\rho\) the following Satisfaction Condition holds:

\[ M' \models_\Sigma \text{Sen}(\varphi)(\rho) \quad \text{if and only if} \quad \text{Mod}(\varphi)(M') \models_\Sigma \rho. \]

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Because of its very high level of abstraction, this definition accommodates not only well established logical systems but also very unconventional ones. Moreover, it has served and it may serve as a template for defining new ones. Institution theory approaches logic and model theory from a relativistic, non-substantialist perspective, quite different from the common reading of formal logic. This does not mean that institution theory is opposed to the established logic tradition, since it rather includes it from a higher abstraction level. In fact, the real difference may occur at the level of the development methodology: top-down in the case of institution theory, versus bottom-up in the case of traditional logic. Consequently, in institution theory, concepts come naturally as presumed features that a logical system might exhibit or not, and are defined at the most appropriate level of abstraction; in developing results, hypotheses are kept as general as possible and introduced on a by-need basis.

1.2. Conceptual blending

Our work constitutes an effort to provide adequate mathematical foundations to conceptual blending, which is an important research problem in the area of computational creativity. This is a relatively recent multidisciplinary science, with contributions from/to artificial intelligence, cognitive sciences, philosophy and arts, going back at least until to the notion of bisociation, presented by Arthur Koestler [18]. Its aims are not only to construct a program that is capable of human-level creativity, but also to achieve a better understanding and to provide better support for it. Conceptual blending was proposed by Fauconnier and Turner [6] as a fundamental cognitive operation of language and common-sense, modelled as a process by which humans subconsciously combine particular elements of two possibly conceptually distinct notions, as well as their relations, into a unified concept in which new elements and relations emerge.

The structural aspects of this cognitive theory have been given rigorous mathematical grounds by Goguen [8, 9], based upon category theory. In this formal model, concepts are represented as logical theories giving their axiomatization. Goguen used the algebraic specification language OBJ [13] to axiomatize the concepts, a language that is based upon a refined version of equational logic; but in fact the approach is independent of the logical formalism used (this is why category theory is involved). This approach is illustrated by the diagram in Figure 1, which has to be read in an order-enriched categorical context: The nodes correspond to logical theories and the arrows to theory morphisms, but the diagram does not commute in a strict sense. There is only a lax form of commutativity, meaning that the compositions in the left- and the right-hand sides of the diagram are both ‘less’ than the arrow at the centre. The ‘less’ comes from the fact that the arrows (to be interpreted as theory morphisms) are subject to an ordering that reflects the fact that they correspond to partial rather than total mappings.

In the above-mentioned work by Goguen there are convincing arguments, supported by examples, for this
partiality aspect, which represents very much a departure to a different mathematical realm than that of logical theories (even when considered in a very general sense, as commonly done in modern computer science). In category-theoretic terms, this means that we need to consider there categories equipped with partial orders on the hom-sets that are preserved by the compositions of arrows/morphisms. These are special instances of 2-categories (a rather notorious concept), somehow half-way between ordinary categories and 2-categories: according to Goguen, this is what motivates the term \( \frac{1}{2} \)-category. To summarise the main mathematical idea underlying theory blending as it stands now:

Theory blending is a cocone in a \( \frac{1}{2} \)-category in which objects represent logical theories and arrows correspond to partial mappings between logical theories.

There is still a great deal of thinking on whether the cocone should actually be a colimit (in other words, a minimal cocone) or not necessarily. An understanding of this issue is that blending should not necessarily be thought as a colimit, but that colimits are related to a kind of optimality principle. Moreover, since \( \frac{1}{2} \)-category theory has several different concepts of colimits, there is still thinking about which of those is most appropriate for modelling the blending operation.

Goguen’s ideas about theory blending benefited from an important boost with the European FP7 project COINVENT [23] that has adopted them as its foundations. Based on this, a creative computational system has been implemented and demonstrated in fields like mathematics [14] and music [5] (although both use the strict rather than the \( \frac{1}{2} \)-version of category theory).

1.3. \( \frac{1}{2} \)-institutions

However, the COINVENT approach still lacks crucial theoretical features, especially a proper semantic dimension. Such a dimension is absolutely necessary when talking about concepts because meaning and interpretation are central to the idea of concept. For example, the idea of consistency of a concept depends on the semantics. If one considers also the abstraction level of Goguen’s approach in its general form, of non-commitment to particular logical systems, then the institution-theoretic dimension appears as inevitable. In fact, Goguen argued for the role of institution theory in [10], and so does the COINVENT project. However, institution theory cannot be used as such in a proper way because, as it stands now, it cannot capture the partiality of theory morphisms (which boils down to the partiality of signature morphisms). Although the treatment of signatures and their morphisms as an abstract category \( \text{Sign} \) seems to do this, the implications of this partiality go beyond the common concept of institution. The the sentence translations \( \text{Sent}(\varphi) \) ought to be allowed to be partial rather than total functions, and that the model reducts \( \text{Mod}(\varphi) \) ought to be allowed to map models to sets of models rather than single models. Therefore we define a \( \frac{1}{2} \)-categorical extension of the concept of institution, called \( \frac{1}{2} \)-institution, that accommodates those aspects and that starts from an abstract \( \frac{1}{2} \)-category of signatures. Moreover, based on this, we unfold a theory of \( \frac{1}{2} \)-institutions aimed as a general institution theoretic foundations for conceptual blending.

1.4. Other applications: the problem of merging software changes

The diagram in Figure 1 that depicts the process of theory blending also has an important interpretation in software engineering: In large software-development projects, it often happens that a part of the system is being modified (deleting of code also allowed) by several different programmers concurrently, after which it is necessary to merge the changes to form a single consistent version. Even cooperative distributed writing of papers or documents may fall under this topic; writing scientific papers in \( \LaTeX \) certainly qualifies, as \( \LaTeX \) is indeed a programming language. Like in the case of theory blending, a \( \frac{1}{2} \)-categorical
approach is necessary (changes being modelled as partial mappings) [7] but this is not enough because of not being able to capture the semantic dimension of software. For example in order to be able to have a notion of consistency for merges we need to enhance the approach with a model theory. This software engineering problem is a second application domain that drives our development of the theory of $\frac{3}{2}$-institutions.

1.5. Contributions and Structure of the Paper
The paper is structured as follows:

1. In a preliminary section we introduce some basic category theoretic notations and terminology, with emphasis on $\frac{3}{2}$-categories.

2. In a section on $\frac{3}{2}$-institutions we start by recalling the basic concepts of (ordinary) institution theory, then we refine this to the concept of $\frac{3}{2}$-institution, provide a collection of relevant examples, and develop basic $\frac{3}{2}$-institution theoretic concepts and results on:
   - $\frac{3}{2}$-institutional seeds, that constitute a simple abstract scheme that underlies the definition of many $\frac{3}{2}$-institutions of interest and that provides a general framework for an easy derivation and understanding of important $\frac{3}{2}$-institutional properties.
   - Theory morphisms, that parallels the corresponding concept from ordinary institution theory but only to a limited extent, since $\frac{3}{2}$-institution theory admits several relevant concepts of theory morphisms.
   - Model amalgamation, that extends the corresponding concept from ordinary institution theory to $\frac{3}{2}$-institutions.

3. We dedicate a special section to the presentation of a scheme for approaching conceptual blending with $\frac{1}{2}$-institutions that essentially replaces the currently prevalent idea of looking for colimits of theories with another idea, of looking for lax cocones with model amalgamation. Our scheme is supported by the mathematical results of the previous sections, and in addition to that it has also a number of parameters that makes it quite flexible in the applications.

2. Category-theoretic and other preliminaries

2.1. Categories, monads
In general we stick to the established category theoretic terminology and notations, such as in [19]. But unlike there we prefer to use the diagrammatic notation for compositions of arrows in categories, i.e. if $f : A \to B$ and $g : B \to C$ are arrows then $f;g$ denotes their composition. The domain of an arrow/morphism $f$ is denoted by $\Box f$ while its codomain is denoted by $f\Box$. SET denotes the category of sets and functions and CAT the “quasi-category” of categories and functors.\footnote{This means it is bigger than a category since the hom-sets are classes rather than sets.}

The dual of a category $C$ (obtained by formally reversing its arrows) is denoted by $C^\oplus$.

Given a category $C$, a triple $(\Delta, \delta, \mu)$ constitutes a monad in $C$ when $\Delta : C \to C$, and $\delta$ and $\mu$ are natural transformations $\Delta^2 \Rightarrow \Delta$ and $1_C \Rightarrow \Delta$, respectively such that following diagrams commute:

\[
\begin{array}{ccc}
\Delta(\Sigma) & \xrightarrow{\Delta(\delta)} & \Delta(\Sigma) \\
\downarrow{\Delta(\Sigma)} & & \downarrow{\Delta(\Sigma)} \\
\Delta(\Sigma) & \xrightarrow{\Delta(\mu)} & \Delta(\Sigma)
\end{array}
\quad
\begin{array}{ccc}
\Delta^2(\Sigma) & \xrightarrow{\Delta^2(\delta)} & \Delta^2(\Sigma) \\
\downarrow{\Delta^2(\Sigma)} & & \downarrow{\Delta^2(\Sigma)} \\
\Delta^2(\Sigma) & \xrightarrow{\Delta^2(\mu)} & \Delta^2(\Sigma)
\end{array}
\]
The Kleisli category \( C_\Delta \) of the monad \( (\Delta, \delta, \mu) \) has the objects of \( C \) but an arrow \( \theta_\Delta : A \to B \) in \( C_\Delta \) is an arrow \( \theta : A \to \Delta(B) \) in \( C \). The composition in \( C_\Delta \) is defined as shown below:

\[
\begin{array}{ccc}
  & A & \downarrow \theta_
  \\
  \theta_\Delta & A & \downarrow \theta
  \\
  B & \Delta(B) & \downarrow \Delta(\theta)
  \\
  \theta_\Delta & C & \Delta^2(C) \xrightarrow{\rho_C} \Delta(C)
\end{array}
\]

The following functor extends the well known power-set functor from sets to categories:

**Definition 2.1.** The power-set functor on categories \( \mathcal{P} : \text{CAT} \to \text{CAT} \) is defined as follows:
- for any category \( C \),
  - \( |\mathcal{P}C| = \{A \mid A \subseteq |C|\} \) and \( \mathcal{P}C(A, B) = \{H \subseteq C \mid \forall h \in A, h\sqsubseteq B \text{ for each } h \in H\}; \) and
  - composition is defined by \( H_1 ; H_2 = \{h_1 ; h_2 \mid h_1 \in H_1, h_2 \in H_2, h_1 \sqsubseteq = \sqsubseteq h_2\}; \) then
    \( 1_A = \{1_a \mid a \in A\} \) are the identities.
- for any functor \( F : C \to C' \), \( \mathcal{P}F(A) = F(A) \subseteq |C'| \) and \( \mathcal{P}F(H) = F(H) \subseteq C' \).

Moreover, like in the case of sets, this construction extends to a monad \( (\mathcal{P}, \sqcup, \sqcap) \) in \( \text{CAT} \). Then \( \text{CAT}_\mathcal{P} \) denotes its associated Kleisli category.

### 2.2. Partial functions

A partial function \( f : A \to B \) is a binary relation \( f \subseteq A \times B \) such that \((a, b), (a, b') \in f \) implies \( b = b' \). The **definition domain** of \( f \), denoted \( \text{dom}(f) \) is the set \( \{a \in A \mid \exists b \ (a, b) \in f\} \). A partial function \( f : A \to B \) is called **total** when \( \text{dom}(f) = A \). We denote by \( f^0 \) the restriction of \( f \) to \( \text{dom}(f) \times B \); this is a total function. Partial functions yield a subcategory of the category of binary relations, denoted \( \text{Pfn} \). If \( A' \subseteq A \) by \( f(A') \) we denote the set \( \{b \mid \exists a \in A', (a, b) \in f\} \). It is easy to check the following (though not as immediate as in the case of the total functions):

**Lemma 2.1.** Given partial functions \( f : A \to B \) and \( g : B \to C \) and \( A' \subseteq A \) we have that \((f; g)(A') = g(f(A'))\).

### 2.3. \( \frac{1}{\text{2}} \)-categories

A \( \frac{1}{\text{2}} \)-category is just a category such that its hom-sets are partial orders, and the composition preserve these partial orders. In the literature \( \frac{1}{\text{2}} \)-categories are also called **ordered categories** or **locally ordered categories**. In terms of enriched category theory [16], \( \frac{1}{\text{2}} \)-category are just categories enriched by the monoidal category of partially ordered sets.

Given a \( \frac{1}{\text{2}} \)-category \( C \) by \( C^{\text{op}} \) we denote its ‘vertical’ dual which reverses the partial orders, and by \( C^{\text{op,op}} \) its double dual \( C^{\text{op,op}} \). Given \( \frac{1}{\text{2}} \)-categories \( C \) and \( C' \), a **strict \( \frac{1}{\text{2}} \)-functor** \( F : C \to C' \) is a functor \( C \to C' \) that preserves the partial orders on the hom-sets. **Lax functors** relax the functoriality conditions \( F(h ; h') = F(h ; h') \) to \( F(h ; h') \leq F(h ; h') \) (when \( h\sqsubseteq = \sqsubseteq h' \) and \( F(1_A) = 1_{F(A)} \)) to \( 1_{F(A)} \leq F(1_A) \). If these inequalities are reversed then \( F \) is an **oplax functor**. This terminology complies to [1] and to more recent literature, but in earlier literature [17, 15] this is reversed. Note that oplax + lax = strict. In what
follows whenever we say “\(\frac{3}{2}\)-functor” without the qualification “lax” or “oplax” we mean a functor which is either lax or oplax.

Lax functors can be composed like ordinary functors; we denote by \(\frac{3}{2}\text{CAT}\) the category of \(\frac{3}{2}\)-categories and lax functors.

Most typical examples of a \(\frac{3}{2}\)-category are \(\text{Pfn}\) – the category of partial functions in which the ordering between partial functions \(A \to \to B\) is given by the inclusion relation on the binary relations \(A \to B\), and \(\text{PoSET}\) – the category partial ordered sets (with monotonic mappings as arrows) with orderings between monotonic functions being defined point-wise.\(\leq g\) if and only if \(f(p) \leq g(p)\) for all \(p\).

**Definition 2.2.** Let us consider the power-set monad on categories of \(\text{Dfn}\).

Given the partial order on each \(\mathcal{P}C\) given by category inclusions, the Kleisli category \(\text{CATp}\) admits a two-fold refinement to a \(\frac{3}{2}\)-category:

1. morphisms \(C \to \mathcal{P}C\) are allowed to be lax functors rather than (strict) functors, and
2. we consider the point-wise partial order on the class of the lax functors \(C \to \mathcal{P}C\) that is induced by the partial order on \(\mathcal{P}C\).

Let us denote the \(\frac{3}{2}\)-category thus obtained by \(\frac{3}{2}(\text{CATp})\).

Unlike in the case of ordinary categories, colimits in \(\frac{3}{2}\)-categories come in several different flavours according to the role played by the order on the arrows. Here we recall some of these for the particular emblematic case of pushouts; the extension to other types of colimits being obvious.

Given a span \(\varphi_1,\varphi_2\) of arrows in a \(\frac{3}{2}\)-category, a lax cocone for the span consists of arrows \(\theta_0,\theta_1,\theta_2\) such that there are inequalities as shown in the following diagram:

When the two inequalities are both equalities, this is a strict cocone. In this case \(\theta_0\) is redundant and the data collapses to the equality \(\varphi_1;\theta_1 = \varphi_2;\theta_2\).

A lax cocone like in diagram (1) is:

- **pushout** when it is strict and for any strict cocone \(\theta'_1,\theta'_2\) there exists and unique arrow \(\mu\) that is mediating, i.e. \(\theta_k;\mu = \theta'_k, k = 1, 2;\)
- **lax pushout** when for any lax cocone \(\theta'_0,\theta'_1,\theta'_2\) there exists an unique mediating arrow \(\mu\), i.e. \(\theta_k;\mu = \theta'_k, k = 0, 1, 2;\)
- **weak (lax) pushout** when the uniqueness condition on the mediating arrow is dropped from the above properties;
- **near pushout** when for any lax cocone \(\theta'_0,\theta'_1,\theta'_2\) the set of mediating arrows \(\{\mu | \theta_k;\mu \leq \theta'_k, k = 0, 1, 2\}\) has a maximal element.

Pushouts are not a proper \(\frac{3}{2}\)-categorical concept because they do not involve in any way the orders on the arrows.

Lax pushouts represents the instance of a natural concept of colimit from general enriched category theory [16] to \(\frac{3}{2}\)-categories; however in concrete situations, unlike their cousins from ordinary category theory,
they can be very difficult to grasp and sometimes appearing quite inadequate. For example in \textit{Pfn}, if $\text{dom}\varphi_1 \cap \text{dom}\varphi_2 \neq \emptyset$ then the span $(\varphi_1, \varphi_2)$ does not have a lax pushout. This is caused by the discrepancy between a lot of laxity at the level of diagrams and of the arrows on the one hand (allowing for unbalanced cocones in which low components may coexist with high components), and the strictness required in the universal property on the other hand. A remedy for this would be to restrict the cocones to designated subclasses of arrows as follows.

\textbf{Definition 2.3 (T-colimits).} Given a (1-)subcategory $\mathcal{T} \subseteq C$ of a $\mathcal{T}$-category $C$, a lax $\mathcal{T}$-cocone for a span $(\varphi_1, \varphi_2)$ is a lax cocone $(\theta_0, \theta_1, \theta_2)$ for the span such that $\theta_k \in \mathcal{T}$, $k = 0, 1, 2$. A lax $\mathcal{T}$-pushout is a minimal lax $\mathcal{T}$-cocone, i.e. for any lax $\mathcal{T}$-cocone $(\theta'_0, \theta'_1, \theta'_2)$ there exists an unique mediating arrow $\mu \in \mathcal{T}$ such that $\theta_k; \mu = \theta'_k$, $k = 0, 1, 2$.

This definition extends in the obvious way to general colimits and to the weak case (by dropping off the requirement on the uniqueness of $\mu$).

For example, in \textit{Pfn} by letting $\mathcal{T}$ be the class of total functions, any span of partial functions admits a lax $\mathcal{T}$-pushout. Near pushouts (terminology from \cite{15}) are much easier to grasp than lax pushouts (for example in \textit{Pfn} they are the epimorphic cocones) but nevertheless they have received only little consideration due to their pathology of lacking uniqueness, a property that is considered crucial for any kind of colimits. However in \cite{15} it is argued that they constitute a more proper concept of colimit in a ordered categorical context because it involves only inequalities and moreover Goguen argues \cite{8} that their lack of the uniqueness property is exactly what makes them useful for modelling conceptual blending; there he calls them $\mathcal{T}$-pushouts.

3. $\mathcal{T}$-institutions

The outline of this section is as follows.

1. We recall the concept of institution and provide a couple of emblematic examples. Some basic institution theoretic concepts are also recalled.
2. We introduce the definition of $\mathcal{T}$-institutions.
3. We provide some relevant examples of $\mathcal{T}$-institutions that constitute extensions of well known corresponding institutions that accommodate partiality of the signature morphisms.
4. We introduce the concept of $\mathcal{T}$-institutional seed that serves as a very general way to define $\mathcal{T}$-institutions. This is also mathematically convenient especially within the context of the study of model amalgamation properties.
5. We extend the crucial concept of model amalgamation from common institution theory to $\mathcal{T}$-institution theory, and we give some general and yet pragmatic sufficient conditions for $\mathcal{T}$-institution theoretic model amalgamation.
6. We extend the concept of theory morphism from common institution theory to $\mathcal{T}$-institutions, what happens being an unfolding of the original concept to several concepts of theory morphisms. We establish the relationships between these, and we study their basic compositionality and model amalgamation properties.
7. Finally, we introduce and study theory changes, which represent a different kind of mapping or relationship between theories that is relevant especially in foundational studies for the problem of merging software changes.
3.1. Institutions

An institution $I = (\text{Sign}^I, \text{Sen}^I, \text{Mod}^I, \models^I)$ consists of

- a category $\text{Sign}^I$ whose objects are called signatures,
- a sentence functor $\text{Sen}^I : \text{Sign}^I \to \text{SET}$ defining for each signature a set whose elements are called sentences over that signature and defining for each signature morphism a sentence translation function,
- a model functor $\text{Mod}^I : (\text{Sign}^I)^\otimes \to \text{CAT}$ defining for each signature $\Sigma$ the category $\text{Mod}^I(\Sigma)$ of $\Sigma$-models and $\Sigma$-model homomorphisms, and for each signature morphism $\varphi$ the reduct functor $\text{Mod}^I(\varphi)$,
- for every signature $\Sigma$, a binary $\Sigma$-satisfaction relation $\models^I_\Sigma \subseteq |\text{Mod}^I(\Sigma)| \times \text{Sen}^I(\Sigma)$, such that for each signature $\Sigma$, a binary $\Sigma$-satisfaction relation $\models^I_\Sigma \subseteq |\text{Mod}^I(\Sigma)| \times \text{Sen}^I(\Sigma)$, such that for each morphism $\varphi$, the Satisfaction Condition.

\begin{align*}
(2) & \quad M' \models^I_\Sigma \text{Sen}^I(\varphi) \rho \text{ if and only if } \text{Mod}^I(\varphi)M' \models^I_\Sigma \rho
\end{align*}

holds for each $M' \in |\text{Mod}^I(\varphi)\Sigma|$ and $\rho \in \text{Sen}^I(\varphi)\Sigma$.

We may omit the superscripts or subscripts from the notations of the components of institutions when there is no risk of ambiguity. For example, if the considered institution and signature are clear, we may denote $\models^I_\Sigma$ just by $\models$. For $M = \text{Mod}(\varphi)M'$, we say that $M$ is the $\varphi$-reduct of $M'$.

Example 3.1 (Propositional logic – $\mathcal{P} \mathcal{L}$). This is defined as follows. $\text{Sign}^{\mathcal{P} \mathcal{L}} = \text{SET}$, and for any set $P$, $\text{Sen}(P)$ is generated by the grammar

\begin{align*}
S & \ ::= P \mid S \wedge S \mid \neg S
\end{align*}

and $\text{Mod}^{\mathcal{P} \mathcal{L}}(P) = (2^P, \subseteq)$. For any $M \in |\text{Mod}^{\mathcal{P} \mathcal{L}}(P)|$, depending on convenience, we may consider it either as a subset $M \subseteq P$ or equivalently as a function $M : P \to 2 = \{0, 1\}$.

For any function $\varphi : P \to P'$, $\text{Sen}^{\mathcal{P} \mathcal{L}}(\varphi)$ replaces the each element $p \in P$ that occurs in a sentence $\rho$ by $\varphi(p)$, and $\text{Mod}^{\mathcal{P} \mathcal{L}}(\varphi)(M') = \varphi; M$ for each $M' \in 2^{P'}$. For any $P$-model $M \subseteq P$ and $\rho \in \text{Sen}^{\mathcal{P} \mathcal{L}}(P)$, $M \models \rho$ is defined by induction on the structure of $\rho$ by $\rho = (p \in M)$, $M \models \rho_1 \wedge \rho_2$ = $(M \models \rho_1) \wedge (M \models \rho_2)$ and $(M \models \neg \rho) = \neg(M \models \rho)$.

Example 3.2 (Many-sorted algebra – $\text{MSA}$). The $\text{MSA}$-signatures are pairs $(S, F)$ consisting of a set $S$ of sort symbols and of a family $F = \{F_{w \to s} \mid w \in S^*, s \in S\}$ of sets of function symbols indexed by arities (for the arguments) and sorts (for the results).

Signatures morphisms $\varphi : (S, F) \to (S', F')$ consist of a function $\varphi^\text{M} : S \to S'$ and a family of functions $\varphi^\text{op} = \{\varphi^\text{op}_{w \to s} : F_{w \to s} \to F'_{\varphi^\text{M}(w) \to \varphi^\text{M}(s)} | w \in S^*, s \in S\}$.

The $(S, F)$-models $M$, called algebras, interpret each sort symbol $s$ as a set $M_s$ and each function symbol $\sigma \in F_{w \to s}$ as a function $M_\sigma$ from the product $M_w$ of the interpretations of the argument sorts to the

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2By $S^*$ we denote the set of strings of sort symbols.
interpretation $M_i$ of the result sort. An $(S, F)$-model homomorphism $h : M \rightarrow M'$ is an indexed family of functions $\{h_s : M_i \rightarrow M'_i \mid s \in S\}$ such that $h_s(M_i(m)) = M'_i(h_w(m))$ for each $\sigma \in F_{w \rightarrow s}$ and each $m \in M_i$, where $h_w : M_w \rightarrow M'_w$ is the canonical componentwise extension of $h$, i.e.

$h_w(m_1, \ldots, m_n) = (h_{s_1}(m_1), \ldots, h_{s_n}(m_n))$ for $w = s_1 \ldots s_n$ and $m_i \in M_{s_i}$.

For each signature morphism $\phi : (S, F) \rightarrow (S', F')$, the reduct $\text{Mod}(\phi)(M')$ of an $(S', F')$-model $M'$ is defined by $\text{Mod}(\phi)(M')_s = M'_\phi(s)$ for each sort or function symbol $x$ from the domain signature of $\phi$.

For each signature $(S, F)$, $\text{Th}(S, F) = \{\langle T(S, F) \rangle_\sigma \mid \sigma \in S\}$ is the least family of sets such that $\sigma(t)$ in $\langle T(S, F) \rangle_\sigma$ for all $\sigma \in F_{w \rightarrow s}$ and all tuples $t \in \langle T(S, F) \rangle_w$. The elements of $\langle T(S, F) \rangle_\sigma$ are called $(S, F)$-terms of sort $s$. For each $(S, F)$-algebra $M$, the evaluation of an $(S, F)$-term $\sigma(t)$ in $M$, denoted $M_{\sigma(t)}$, is defined as $M_{\sigma}(M_i)$, where $M_i$ is the componentwise evaluation of the tuple of $(S, F)$-terms $t$ in $M$.

Sentences are the usual first order sentences built from equational atoms $t = t'$, with $t$ and $t'$ (well-formed) terms of the same sort, by iterative application of Boolean connectives ($\land$, $\lor$, $\rightarrow$, $\exists$, $\forall$) and quantifiers ($\forall X$, $\exists X$ — where $X$ is a sorted set of variables). Sentence translations along signature morphisms just rename the sort and function symbols according to the respective signature morphisms. They can be formally defined by recursion on the structure of the sentences. The satisfaction of sentences by models is the usual Tarskian satisfaction defined recursively on the structure of the sentences. (As a special note for the satisfaction of the quantified sentences, defined in this formalisation by means of model reducts, we recall that $M \models \Sigma(\forall X) \rho$ if and only if $M' \models \Sigma + X \rho$ for each expansion $M'$ of $M$ to the signature $\Sigma + X$ that adds the variables $X$ as new constants to $\Sigma$.)

In the following we recall some basic concepts from institution theory that will play a role in this work.

For any set $E$ of $\Sigma$-sentences:

- if $M$ is a any $\Sigma$-model, then by $M \models E$ we denote that $M \models e$ for each $e \in E$;
- $E$ is consistent when there exists a $\Sigma$-model $M$ such that $M \models E$;
- if $\rho$ is a $\Sigma$-sentence then $E \models \rho$ denotes the situation when for each $\Sigma$-model $M$ if $M \models E$ then $M \models \rho$ too;
- by $E^*$ we denote $\{\rho \in \text{Sen}(\Sigma) \mid E \models \rho\}$.

In any institution, a theory is a pair $(\Sigma, E)$ consisting of a signature $\Sigma$ and a set $E$ of $\Sigma$-sentences. A theory morphism $\phi : (\Sigma, E) \rightarrow (\Sigma', E')$ is a signature morphism $\phi : \Sigma \rightarrow \Sigma'$ such that $E' = \text{Sen}(\phi)E$. It is easy to check that the theory morphisms are closed under the composition given by the composition of the signature morphisms; this gives the category of the theories of $I$ denoted $\text{Th}(I)$. This fact opens the door for the following general construction, that is quite helpful in several situations, especially in the study of logic encodings.

Let $I = (\text{Sign}, \text{Sen}, \text{Mod}, \models)$ be any institution. The institution of the theories of $I$, denoted by $I' = (\text{Sign}', \text{Sen}', \text{Mod}', \models')$, is defined by

- $\text{Sign}'(\Sigma, E) = \text{Sen}(\Sigma)$,
- $\text{Mod}'(\Sigma, E)$ is the full subcategory of $\text{Mod}(\Sigma)$ determined by those models which satisfy $E$, and
- for each $(\Sigma, E)$-model $M$ and $\Sigma$-sentence $e$, $M \models e$ if and only if $M \models e$.

Model amalgamation properties for institutions formalize the possibility of amalgamating models of different signatures when they are consistent on some kind of generalized ‘intersection’ of signatures. It is one of the most pervasive properties of concrete institutions and it is used in a crucial way in many institution theoretic studies. A few early examples are [21, 24, 20, 4]. For the role played by this property in specification theory and in institutional model theory see [22] and [2], respectively.
A model of a diagram of signature morphisms in an institution consists of a model $M_k$ for each signature $\Sigma_k$ in the diagram such that for each signature morphism $\varphi : \Sigma_i \to \Sigma_j$ in the diagram we have that $M_i = \text{Mod}(\varphi)M_j$.

A commutative square of signature morphisms

\[ \begin{array}{ccc}
\Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\
\downarrow{\varphi_2} & & \downarrow{\theta_1} \\
\Sigma_2 & \xrightarrow{\theta_2} & \Sigma'
\end{array} \]

is an amalgamation square if and only if each model of the span $(\varphi_1, \varphi_2)$ admits an unique completion to a model of the square. When we drop off the uniqueness requirement we call this a weak model amalgamation square.

In most of the institutions formalizing conventional or non-conventional logics, pushout squares of signature morphisms are model amalgamation squares.

In the literature there are several more general concepts of model amalgamation. One of them is model amalgamation for cocones of arbitrary diagrams (rather than just for spans), another one is model amalgamation for model homomorphisms. Both are very easy to define by mimicking the definitions presented above. While the former generalisation is quite relevant for the intended applications of our work, the latter is less so since at this moment model homomorphisms do not seem to play any role in conceptual blending or in merging of software changes. Moreover amalgamation of model homomorphisms is known to play a role only in some developments in institution-independent model theory, but even there most involvements of model amalgamation refers only to amalgamation of models.

### 3.2. $\frac{3}{2}$-institutions: definition

**Definition 3.1 ($\frac{3}{2}$-institution).** A $\frac{3}{2}$-institution $I = (\text{Sign}^I, \text{Sen}^I, \text{Mod}^I, (\models^I)_{\Sigma \in \text{Sign}^I})$ consists of

- a $\frac{3}{2}$-category of signatures $\text{Sign}^I$,
- an $\frac{3}{2}$-functor $\text{Sen}^I : \text{Sign}^I \to \text{Pfn}$, called the sentence functor,
- an lax $\frac{3}{2}$-functor $\text{Mod}^I : (\text{Sign}^I)_{\varnothing} \to \frac{3}{2}(\text{CAT})_{\varnothing}$, called the model functor,
- for each signature $\Sigma \in |\text{Sign}^I|$ a satisfaction relation $\models^I_{\Sigma} \subseteq |\text{Mod}^I(\Sigma)| \times |\text{Sen}^I(\Sigma)|$ such that for each morphism $\varphi \in \text{Sign}^I$, the Satisfaction Condition

\[(3) \quad M' \models^I_{\varphi \sqcup} \text{Sen}^I(\varphi) \rho \quad \text{if and only if} \quad M \models^I_{\varnothing} \rho \]

holds for each $M' \in |\text{Mod}^I(\varphi \sqcup)|$, $M \in |\text{Mod}^I(\varphi)M'|$ and $\rho \in \text{dom}(\text{Sen}^I(\varphi))$.

The difference between $\frac{3}{2}$-institutions and ordinary institutions, from now on called $1$-institutions, is determined by the $\frac{3}{2}$-categorical structure of the signature morphisms which propagates to the sentence and to the model functors. Consequently the Satisfaction Condition (3) takes an appropriate format. Thus, for each signature morphism $\varphi$ its corresponding sentence translation $\text{Sen}(\varphi)$ is a partial function $\text{Sen}(\varphi \sqcup) \to \text{Sen}(\varphi)$ and moreover whenever $\varphi \leq \theta$ we have that $\text{Sen}(\varphi) \subseteq \text{Sen}(\theta)$. The sentence functor $\text{Sen}$ can be either lax or oplax; depending on how is this we may call the respective $\frac{3}{2}$-institution as lax or oplax $\frac{3}{2}$-institution. In many concrete situations it happens that $\text{Sen}$ is strict while some general results require it to be either lax or oplax or both.
The model reduct $\text{Mod}(\varphi)$ is an lax functor $\text{Mod}(\varphi) \to \mathcal{P}\text{Mod}(\varphi)$ meaning that for each $\Sigma'$-model we have a set of reducts rather than a single reduct. In concrete examples this is a direct consequence of the partiality of $\varphi$: in the reducts the interpretation of the symbols on which $\varphi$ is not defined is unconstrained, therefore there may be many possibilities for their interpretations. “Many” here includes also the case when there is no interpretation.

**Definition 3.2.** The model functor $\text{Mod}$ admits emptiness when there exists a signature morphism $\varphi$ and a $\varphi\square$-model $M'$ such that $\text{Mod}(\varphi) = \emptyset$, otherwise it is said that $\text{Mod}$ does not admit emptiness.

In examples most often the model functors $\text{Mod}$ do not admit emptiness, however the general definition does not rule out emptiness and moreover there are significant examples (we will see in Sect. 3.6) when emptiness of $\text{Mod}$ may happen.

– The fact that $\text{Mod}$ is a $\frac{3}{2}$-functor implies also that whenever $\varphi \leq \theta$ we have $\text{Mod}(\theta) \leq \text{Mod}(\varphi)$, i.e. $\text{Mod}(\theta)M' \subseteq \text{Mod}(\varphi)M'$, etc.

– The lax aspect of $\text{Mod}$ means that for signature morphisms $\varphi$ and $\varphi'$ such that $\varphi\square = \square\varphi'$ and for any $\varphi'\square$-model $M''$, we have that

$$\text{Mod}(\varphi)(\text{Mod}(\varphi')M'') \subseteq \text{Mod}(\varphi; \varphi')M''$$

and for each signature $\Sigma$ and for each $\Sigma$-model $M$ that

$$M \in \text{Mod}(1_{\Sigma})M.$$

– The lax aspect of the reduct functors $\text{Mod}(\varphi)$ means that for model homomorphisms $h_1, h_2$ such that $h_1\square = \square h_2$ we have that

$$\text{Mod}(\varphi)(h_1); \text{Mod}(\varphi)(h_2) \subseteq \text{Mod}(\varphi)(h_1; h_2)$$

and for each $M' \in \text{Mod}(\varphi\square)$ and each $M \in \text{Mod}(\varphi)M'$ that

$$1_M \in \text{Mod}(\varphi)1_{M'}.$$  

As already mentioned above model homomorphisms do not play yet any role in conceptual blending or in other envisaged applications of $\frac{3}{2}$-institutions. Hence the lax aspect of model functors is for the moment a purely theoretical feature which is however supported naturally by all examples.

In [25] there is a 2-categorical generalization of the concept of institution, called 2-institution, that consider $\text{Sign}$ to be a 2-category, $\text{Sen} : \text{Sign} \to \text{CAT}$ and $\text{Mod} : \text{Sign}^{\square} \to \text{CAT}$ to be pseudo-functors, and that takes a (quite sophisticated categorically) many-valued approach to the satisfaction relation. From these we can see immediately that 2-institutions of [25] do not cover the concept of $\frac{3}{2}$-institution through the perspective of $\frac{3}{2}$-categories as special cases of 2-categories, the functors $\text{Sen}$ and $\text{Mod}$ in 2-institutions diverging from those in $\frac{3}{2}$-institutions in two ways: they are pseudo-functors (in $\frac{3}{2}$-category theory this means just ordinary functors) and their targets do not match those of $\frac{3}{2}$-institutions. This lack of convergence is due to the two extensions aiming to different application domains.

**Definition 3.3** (Total signature morphisms). A signature morphism $\varphi$ in a $\frac{3}{2}$-institution is

– $\text{Sen}$-maximal when $\text{Sen}(\varphi)$ is total;

– $\text{Mod}$-maximal when for each $\varphi\square$-model $M'$, $\text{Mod}(\varphi)M'$ is a singleton; and

– total when it is both $\text{Sen}$-maximal and $\text{Mod}$-maximal.

**Corollary 3.1.** In each $\frac{3}{2}$-institution the total signature morphisms determine a 1-institution.
3.3. $\frac{1}{2}$-institutions: examples

The following expected example shows that the concept of $\frac{1}{2}$-institution constitute a generalisation of the concept of institution.

Example 3.3 (Institutions). Each $1$-institution can be regarded as a $\frac{1}{2}$-institution that has all its signature morphisms total (cf. Dfn. 3.3 and Cor. 3.1).

Example 3.4 (Propositional logic with partial morphisms of signatures – $\frac{1}{2}\mathcal{PL}$). This example extends the ordinary institution $\mathcal{PL}$ to a $\frac{1}{2}$-institution by considering partial functions rather than total functions as signature morphisms; thus $\text{Sign} = \text{Pfn}.$

SENTENCES. While for each set $P, \text{Sen}(P)$ is like in $\mathcal{PL}$, for any partial function $\varphi : P \to P'$ the sentence translation $\text{Sen}(\varphi)$ translates like in $\mathcal{PL}$ but only the sentences containing only propositional variables $P$ that are translated by $\varphi$, i.e. that belong to $\text{dom}\varphi$; hence the partiality of $\text{Sen}(\varphi)$. More precisely we have that $\text{dom}(\text{Sen}\varphi) = \text{Sen}\mathcal{PL}(\text{dom} \varphi)$ and for each $\rho \in \text{dom}(\text{Sen}\varphi)$ we have that $\text{Sen}(\varphi)\rho = \text{Sen}\mathcal{PL}(\varphi^0)\rho$. The sentence functor is a strict $\frac{1}{2}$-functor; the main main part for the functoriality argument for $\text{Sen}$ goes as follows. Let $\varphi, \varphi'$ be signature morphisms where $\varphi \square = \Box \varphi'$ and let $\rho \in \text{Sen}(\Box \varphi))$.

- First we establish the equality of the definition domains:

$$\text{dom} \text{Sen}(\varphi; \varphi') = \text{Sen}\mathcal{PL}(\text{dom} \varphi; \varphi')$$

$$= \text{Sen}\mathcal{PL}(\{ p \in \text{dom} \varphi | \varphi^0(p) \in \text{dom} \varphi' \})$$

$$= \{ \rho \in \text{Sen}\mathcal{PL}(\text{dom} \varphi) | \text{Sen}\mathcal{PL}(\varphi^0)\rho \in \text{Sen}\mathcal{PL}(\text{dom} \varphi') \}$$

$$= \{ \rho \in \text{dom}(\text{Sen}\mathcal{PL}\varphi) | \text{Sen}\mathcal{PL}(\varphi^0)\rho \in \text{dom}(\text{Sen}\mathcal{PL}\varphi') \}$$

$$= \text{dom}(\text{Sen}\varphi; \text{Sen}\varphi').$$

- The next step is obtained on the basis of the functoriality of $\text{Sen}\mathcal{PL}$. For each $\rho \in \text{dom} \text{Sen}(\varphi; \varphi')$ we have:

$$\text{Sen}(\varphi; \varphi')\rho = \text{Sen}\mathcal{PL}((\varphi^0; \varphi'^0)\rho = \text{Sen}\mathcal{PL}(\varphi^0)(\text{Sen}\mathcal{PL}(\varphi^0)\rho) = \text{Sen}(\varphi')\text{Sen}(\varphi)\rho).$$

MODELS. The $\frac{1}{2}\mathcal{PL}$ models and model homomorphisms are those of $\mathcal{PL}$, but their reducts differ from those in $\mathcal{PL}$. Given a partial function $\varphi : P \to P'$ and a $P'$-model $M' : P' \to 2,$

$$\text{Mod}(\varphi)M' = \{ M : P \to 2 | M_p = M'_{\varphi^0(p)} \text{ for all } p \in \text{dom} \varphi \}.$$

On the model homomorphisms the reduct is defined by

$$\text{Mod}(\varphi)(M' \subseteq N') = \{ M \subseteq N | M \in \text{Mod}(\varphi)M', N \in \text{Mod}(\varphi)N' \}.$$

The main part of the lax functoriality of $\text{Mod}$ is proved as follows. Let $\varphi, \varphi'$ be signature morphisms such that $\varphi \square = \Box \varphi'$ and let $M'' \in \{ \text{Mod}(\varphi') \}$, For any $M \in \text{Mod}(\varphi)(\text{Mod}(\varphi')M'')$ we show that $M \in \text{Mod}(\varphi; \varphi')M''$. Then there exists $M' \in \text{Mod}(\varphi')M''$ such that $M \in \text{Mod}(\varphi)M'$. For any $p \in \text{dom}(\varphi; \varphi') = \{ p \in \text{dom} \varphi | \varphi^0(p) \in \text{dom} \varphi' \}$ we have that

$$M_p = M'_{\varphi^0(p)} \text{ since } p \in \text{dom} \varphi \text{ and } M \in \text{Mod}(\varphi)M'$$

$$= M''_{\varphi'^0(\varphi^0(p))} \text{ since } \varphi^0(p) \in \text{dom} \varphi' \text{ and } M' \in \text{Mod}(\varphi')M''$$

$$= M''_{(\varphi; \varphi')^0(p)}.$$
This shows that $M \in \text{Mod}(\varphi; \varphi')M''$. Note that $\text{Mod}(1_p)M = \{M\}$, hence the second condition of the lax functoriality of $\text{Mod}$ is satisfied in a strict sense.

The following counterexample shows why $\text{Mod}$ is a proper lax functor. Let $\{p_1, p_2, p_3\} \xrightarrow{\varphi} \{p, p\} \xrightarrow{\varphi'} \{p_3\}$ be such that $\varphi(p_1) = \varphi(p_2) = p'$, $\varphi(p_3) = p$ and dom $\varphi' = \{p_3\}$. Then we consider any $\varphi' \square$-model $M''$ and $M \in \text{Mod}(\varphi; \varphi')M''$ such that $M_{p_1} \neq M_{p_2}$. Because of the latter condition there is no $M'$ such that $M \in \text{Mod}(\varphi; \varphi')M'$.

Also in general the reduct functors $\text{Mod}(\varphi)$ are proper lax functors, but this works exactly the other way than in the case of $\text{Mod}$.

- Let $M' \subseteq N' \subseteq T' \in \text{Mod}(\varphi \square)$. Given $M \subseteq T \in \text{Mod}(\varphi \square)$ such that $M \in \text{Mod}(\varphi)M'$ and $T \in \text{Mod}(\varphi \square)T'$, we may define $\{T \} = \mathcal{N}$ by $\mathcal{N} = \mathcal{N} \varphi(p)$ when $p \in \text{dom} \varphi$ and $\mathcal{N} = \mathcal{N} \varphi(p)$ and $\mathcal{N} = \mathcal{N} \varphi(p)$. Otherwise, we may define $\text{dom}(\mathcal{N}) = \mathcal{N} \varphi(p)$ to $\mathcal{N} \varphi(p)$ and $\mathcal{N} \varphi(p)$. Hence $\mathcal{N} \varphi(p)$ is strictly larger than $\mathcal{N} \varphi(p)$.

- Given $M' \in \text{Mod}(\varphi \square)$ and $M \in \text{Mod}(\varphi)M'$ it is obvious that $1_M \in \text{Mod}(\varphi)1_{M'}$.

However $\text{Mod}(\varphi)$ fails to be strict on the identities as shown by the following counterexample. Let $\varphi : \{p, q\} \rightarrow \{p\}$ such that dom $\varphi = \{p\}$. If we take $M' = \{p\}$, $M = \{p\}$ and $N = \{p, q\}$ then we have that $M \subseteq N \in \text{Mod}(\varphi)1_M$, which means that $\text{Mod}(\varphi)1_M$ is strictly larger than $1_{\text{Mod}(\varphi)M'} = \{1_M \mid M \in \text{Mod}(\varphi)M'\}$.

SATISFACTION. The satisfaction relation of $\mathcal{P}_\mathcal{L}$ is inherited from $\mathcal{L}$. The Satisfaction Condition is proved on the basis of that of $\mathcal{L}$ as follows. Let $\varphi : P \rightarrow P'$, $M' : P' \rightarrow 2$ and $M \in \text{Mod}(\varphi)M'$ and $\rho \in \text{dom}(\text{Sen} \varphi)$. Then

$$M' \models \text{Sen}(\varphi) \rho \quad \text{if and only if} \quad M' \models \text{Sen} \mathcal{L}(\varphi) \rho \quad \text{by definition of Sen(\varphi)}$$

$$\text{if and only if} \quad \varphi^0(\rho) \models \varphi^0(\rho) \quad \text{by the Satisfaction Condition in } \mathcal{L} \text{ for } \varphi^0$$

$$\text{if and only if} \quad (\text{dom} \varphi \subseteq P) ; M \models \varphi \quad \text{since (dom} \varphi \subseteq P) ; M = \varphi^0(\varphi) ; M$$

$$\text{if and only if} \quad M \models \varphi \quad \text{by the Satisfaction Condition in } \mathcal{L} \text{ for dom } \varphi \subseteq P.$$

**Example 3.5** (Many sorted algebra with partial morphisms of signatures $\frac{1}{2} \mathcal{M} \mathcal{S} \mathcal{A}$). In this example we extend the $\mathcal{M} \mathcal{S} \mathcal{A}$ institution to its $\frac{1}{2}$ variant in a way that parallels the extension of $\mathcal{L}$ to $\frac{1}{2} \mathcal{L}$. For this reason we will give only the definitions and rather skip the arguments.

Given $\mathcal{M} \mathcal{S} \mathcal{A}$ signatures, a partial $\mathcal{M} \mathcal{S} \mathcal{A}$-signatures morphism $\varphi : (S, F) \rightarrow (S', F')$ consists of

- a partial function $\varphi^\ast : S \rightarrow S'$, and
- for each $w \in (\text{dom} \varphi^\ast)^+$ and $s \in \text{dom} \varphi^\ast$ a partial function $\varphi^\op_{w \rightarrow s} : F_{w \rightarrow s} \rightarrow F'_{w \rightarrow s \ast}$.

Given $\varphi : (S, F) \rightarrow (S', F')$ and $\varphi' : (S', F') \rightarrow (S'', F'')$ their composition $\varphi ; \varphi'$ is defined by

- $((\varphi; \varphi')^\ast = \varphi^\ast; \varphi'^\ast$, and
- for each $w \in (\text{dom}(\varphi; \varphi')^\ast)^+$ and $s \in \text{dom}(\varphi; \varphi')^\ast$: $(\varphi; \varphi')^\op_{w \rightarrow s} = \varphi^\op_{w \rightarrow s}; \varphi'^\op_{w \rightarrow s \ast}$.

Given $\varphi, \theta : (S, F) \rightarrow (S', F')$, then $\varphi \leq \theta$ if and only if

- $\varphi^\ast \subseteq \theta^\ast$, and
- for each $w \in (\text{dom} \varphi^\ast)^+$ and $s \in \text{dom} \varphi^\ast$: $\varphi^\op_{w \rightarrow s} \subseteq \theta^\op_{w \rightarrow s}$.
Under these definitions the partial $\mathcal{MSA}$-signature morphisms form a $\frac{3}{2}$-category, which is the category of the $\frac{3}{2}$-$\mathcal{MSA}$ signatures.

Given a partial $\mathcal{MSA}$-signature morphism $\varphi$ we denote by $\text{dom}\varphi$ the signature $(\text{dom}\varphi^\text{st}, \text{dom}\varphi^\text{op})$ where $(\text{dom}\varphi^\text{op})_{w\rightarrow s} = \text{dom}\varphi^\text{op}_{w\rightarrow s}$ and by $\varphi^0 : \text{dom}\varphi \rightarrow \varphi^\square$ the resulting (total) $\mathcal{MSA}$-signature morphism.

For any signature $\Sigma$, $\text{Sen}^{\frac{3}{2}}\mathcal{MSA}(\Sigma) = \text{Sen}\mathcal{MSA}(\Sigma)$ and for any partial $\mathcal{MSA}$-signature morphism $\varphi$, $\text{Sen}^{\frac{3}{2}}\mathcal{MSA}(\varphi)$ is defined by

- $\text{dom}\text{Sen}^{\frac{3}{2}}\mathcal{MSA}(\varphi) = \text{Sen}\mathcal{MSA}(\text{dom}\varphi)$ and
- for each sentence $\rho \in \text{dom}\text{Sen}^{\frac{3}{2}}\mathcal{MSA}(\varphi)$, $\text{Sen}^{\frac{3}{2}}\mathcal{MSA}(\varphi)\rho = \text{Sen}\mathcal{MSA}(\varphi^0)\rho$.

Like for $\frac{3}{2}\mathcal{PL}$ this yields also a strict $\frac{3}{2}$-functor. For any signature $\Sigma$, $\text{Mod}^{\frac{3}{2}}\mathcal{MSA}(\Sigma) = \text{Mod}\mathcal{MSA}(\Sigma)$ and for any partial $\mathcal{MSA}$-signature morphism $\varphi$, each $\varphi^\square$-model $M'$, $\text{Mod}^{\frac{3}{2}}\mathcal{MSA}(\varphi)M' = M$ is defined by

- for each sort symbol $s$ in $\text{dom}\varphi$, $M_s = M'^{\varphi^\text{st}}_s$, and
- for each operation symbol $\sigma$ in $\text{dom}\varphi$, $M_{\sigma} = M'^{\varphi^\text{op}}_{\sigma}$.

The definition on model homomorphisms is similar, we skip it here. Under these definitions, $\text{Mod}^{\frac{3}{2}}\mathcal{MSA}$ is a lax functor.

The satisfaction relation is inherited from $\mathcal{MSA}$, and the argument for the Satisfaction Condition in $\frac{3}{2}\mathcal{MSA}$ is similar to that in $\frac{3}{2}\mathcal{PL}$.

**Example 3.6.** The $\frac{3}{2}\mathcal{MSA}$ example can be twisted by considering less partiality in the signature morphisms. This can be done in several ways, in each case a different $\frac{3}{2}$-‘sub-institution’ of $\frac{3}{2}\mathcal{MSA}$ emerges.

1. We constrain $\varphi^\text{st}$ to be total functions.
2. We let $\varphi^\text{st}$ to be partial functions but we constrain $\varphi^\text{op}_{w\rightarrow s}$ to be total.

**Example 3.7.** The pattern of Ex. 3.5 can be applied to the extension of $\mathcal{MSA}$ that takes the ‘first order views’ of $[3]$ in the role of signature morphisms. Since first order views are more general the the $\mathcal{MSA}$ signature morphisms, the resulting $\frac{3}{2}$-institution based upon partial first order views can thought as an extension of $\frac{3}{2}\mathcal{MSA}$.

### 3.4. $\frac{3}{2}$-institutional seeds

So far the Examples 3.4, 3.5, 3.6 and 3.7 are based upon a pattern that can be described as follows:

1. Consider a concrete 1-institution (that may be quite common).
2. Consider some form of partiality for its signature morphisms; often this can be done in several different ways (see Ex. 3.6).
3. Keep the sentences and the models of the original institution, but based on the partiality of the signature morphisms extend the concepts of sentence translations and of model reducts to $\frac{3}{2}$-institutional ones. The partiality of the sentence translations amounts to the fact that only the sentences that only involve symbols from the definition domain of the (partial) signature morphism can be translated. The relation-like aspect of the model reducts amounts to the fact that symbols that are outside the definition domain of the (partial) signature morphisms can be interpreted in several different ways in the models.
4. The satisfaction relation of the resulting $\frac{3}{2}$-institution is inherited from the original 1-institution.
This pattern pervades a lot of useful $\frac{1}{2}$-institutions and can be captured as a generic mathematical construction that derives $\frac{3}{2}$-institutions from 1-institutions; this will be the topic of Sect. ?? However there are significant examples of $\frac{3}{2}$-institutions that fall short of this pattern; two of them will appear in Sections 3.6 and 3.7, respectively.

In the following we propose a general scheme for defining $\frac{3}{2}$-institutions that on the one hand serves a technical purpose as it projects a convenient mathematical perspective on situations of interest, and on the other hand constitutes a framework for generating new $\frac{2}{2}$-institutions, some of them not necessarily being partially-based.

**Definition 3.4 ($\frac{3}{2}$-institutional seed).** A $\frac{3}{2}$-institutional seed $(\Sigma, \Omega, \Sigma, T)$ consists of

- a lax $\frac{3}{2}$-functor $\text{Sen} : \text{Sign} \rightarrow \text{Pfn}$ (the ‘sentence functor’), and
- a designated ‘signature’ $\Omega \in |\text{Sign}|$ and a ‘truth’ function $T : \text{Sen}(\Omega) \rightarrow 2$.

**Proposition 3.1.** Any $\frac{3}{2}$-institutional seed $S = (\Sigma, \Omega, T)$ extends canonically to a lax $\frac{3}{2}$-institution $I(S) = (\Sigma, \Omega, \text{Mod}, \models)$ as follows:

- for each signature $\Sigma \in |\text{Sign}|$ we let
  $$\text{Mod}(\Sigma) = \{ M : \Sigma \rightarrow \Omega | \text{Sen}(M) \text{ total} \}$$
- for each signature morphism $\varphi$ and each $\varphi\square$-model $M'$ we let
  $$\text{Mod}(\varphi)M' = \{ M | \varphi; M' \leq M \}$$
- for each $\Sigma$-model $M$ and each $\Sigma$-sentence $\rho$ we let
  $$M \models \rho \text{ if and only if } T(\text{Sen}(M)\rho) = 1.$$

**Proof.** For showing the lax functoriality of $\text{Mod}$ we consider signature morphisms $\varphi, \varphi'$ such that $\varphi\square = \square \varphi'$ and $M'' \in \text{Mod}(\varphi'\square)$. Then

$$\text{Mod}(\varphi')(\text{Mod}(\varphi)M'') = \{ M \in \text{Mod}(\varphi)M' | M' \in \text{Mod}(\varphi')M'' \}$$

(by the definition of composition in $\frac{3}{2}(\text{CAT}_\varphi)$)

$$= \{ M \in \text{Mod}(\square \varphi) | \exists M' \in \text{Mod}(\varphi \square) \text{ such that } \varphi; M' \leq M, \varphi'; M'' \leq M' \}$$

(by the definitions of $\text{Mod}(\varphi), \text{Mod}(\varphi')$)

$$\subseteq \{ M \in \text{Mod}(\square \varphi) | \varphi; \varphi'; M'' \leq M \}$$

(by the monotonicity of the composition in $\text{Sign}$)

$$= \text{Mod}(\varphi; \varphi')M''$$

(by the definition of $\text{Mod}(\varphi; \varphi')$).

The lax functoriality of $\text{Mod}$ on identities may be checked as follows:

$$1_{\text{Mod}(\square)(\Sigma)}(M) = \{ M \} \subseteq \{ N : \square M \rightarrow \Omega | M \leq N, \text{Sen}(N) \text{ total} \} = \text{Mod}(1_{\Sigma}M).$$

For showing the Satisfaction Condition we consider a signature morphism $\varphi$, a $\varphi\square$-model $M'$, $M \in \text{Mod}(\varphi)M'$ and $\rho \in \text{dom } \text{Sen}(\square \varphi)$. Since $\varphi; M' \leq M$ by the monotonicity of $\text{Sen}$ we have that $\text{Sen}(\varphi; M') \subseteq \text{Sen}(M)$. By the lax property of $\text{Sen}$ it follows that $\text{Sen}(\varphi); \text{Sen}(M') \subseteq \text{Sen}(M)$. Since $\rho \in \text{dom } \text{Sen}(\varphi)$ and since $\text{Sen}(M')$ is total it follows that $\text{Sen}(M')(\text{Sen}(\varphi)\rho) = \text{Sen}(M)\rho$. Consequently $T(\text{Sen}(M')(\text{Sen}(\varphi)\rho)) = T(\text{Sen}(M)\rho)$ which means $M' \models \text{Sen}(\varphi)\rho = M \models \rho$. \qed
The following two situations show that Prop. 3.1 is a vehicle for obtaining natural \( \frac{3}{2} \)-institutions.

**Example 3.8** (Seeds for \( 2 \mathcal{PL}, \frac{3}{2} \mathcal{MSA} \)).

1. The \( 2 \mathcal{PL} \) variant without model homomorphisms arises easily as an \( I(S) \) by taking \( \Omega = 2 \) and by taking \( T \) to be the function that evaluates Boolean terms (for example \( T(\neg(0 \land 1)) = 1 \), etc.)
2. Even a local variant of \( 2 \mathcal{MSA} \) without model homomorphisms such that all carrier sets of the models are subsets of a fixed set \( U \) arises as a \( I(S) \) by defining \( \Omega = (S^\Omega, F^\Omega) \) by
   - \( S^\Omega = 2^U \), i.e. the sets of the subsets of \( S \), and
   - for any \( s_1, \ldots, s_n, s \subseteq U \), \( F^\Omega_{s_1 \ldots s_n \rightarrow s} \) is the set of all functions \( s_1 \times \cdots \times s_n \rightarrow s \).

The truth function \( T \) is based upon the evaluation of \( \Omega \)-terms by recursion and functional composition as follows:
   - Any term \( t \) of sort \( s \) gets evaluated as an element \( T(t) \in s \) (note here the overloading of \( T \) defined by
     \[ T(\sigma(t_1, \ldots, t_n)) = \sigma(T(t_1), \ldots, T(t_n)). \]
   - For any equation \( t_1 = t_2 \) we set \( T(t_1) = T(t_2) = 1 \) if and only if \( T(t_1) = T(t_2) \).
   - The evaluation function \( T \) extends to composed sentence, in an obvious manner in the case of the Boolean connectives, and as follows in the case of quantifications. Given an \( \Omega \)-sentence \((\forall x)p\) where \( x \) is a variable of sort \( s \), then
     \[ T((\forall x)p) = \bigwedge\{T(p(a)) | a \in s\} \]
   
     where \( \rho(a) \) denotes the \( \Omega \)-sentence obtained by replacing each occurrence of \( x \) in \( \rho \) by \( a \).

Because the definition of \( \frac{3}{2} \)-institutional seeds involves deceptively poor data, there is a significant space for defining relevant \( \frac{3}{2} \)-institutions that do not fall into the pattern of partiality of signature morphisms. The following example, albeit rather artificial, may give an indication about this potential.

**Example 3.9** (A seed beyond partiality). We let
   - \( |\text{Sign}| = \omega \), the set of the natural numbers,
   - arrows \( m \rightarrow n \) are pairs \((a, b)\) of natural numbers such that \( a \leq n - m \),
   - the composition of arrows \((a, b) : m \rightarrow n \) and \((c, d) : n \rightarrow p \) is \((a + c, b \lor d) : m \rightarrow p \)
     (by \( b \lor d \) we denote the maximum of \( b \) and \( d \)); note that the composition is well defined, it is associative and has \((0, 0)\) as identities.

So far this yields a category. Now we make this into a \( \frac{3}{2} \)-category.
   - Given \((a, b), (a', b') : m \rightarrow n \) we let \((a, b) \leq (a', b')\) if and only if \( a = a' \) and \( b' \leq b \). It is easy to check that this yields a partial order which is preserved by the compositions.

The lax \( \frac{3}{2} \)-functor \( \text{Sen} : \text{Sign} \rightarrow \text{Pfn} \) is defined as follows:
   - for each \( m \in \omega \), \( \text{Sen}(m) = \{x \in \omega | x \leq m\} \),
   - for each arrow \((a, b) : m \rightarrow n \) in \( \text{Sign} \), \( \text{dom} \text{Sen}(a, b) = \{x \in \omega | x \leq m, x + a + b \leq n\} \) and \( \text{Sen}(a, b)(x) = x + a \) for each \( x \in \text{dom} \text{Sen}(a, b) \).

The interested reader may check the lax functoriality properties of \( \text{Sen} \); we skip this here.

Now any choice of \( \Omega \) and \( T : \text{Sen}(\Omega) \rightarrow 2 \) completes the definition of a \( \frac{3}{2} \)-institutional seed.
3.5. Model amalgamation in $\frac{3}{2}$-institutions

The following definition extends the crucial notion of model amalgamation concept from 1-institutions to $\frac{3}{2}$-institutions. For the sake of simplicity of presentation, this is presented for lax cocones of spans, the general concept for lax cocones over arbitrary diagrams of signature morphisms being an obvious generalisation. Moreover all the results in this section can be presented in that more general framework without a real additional effort.

**Definition 3.5.** A model for a diagram of signature morphisms in a $\frac{3}{2}$-institution consists of a model $M_k$ for each signature $\Sigma_k$ in the diagram such that for each signature morphism $\varphi : \Sigma_i \rightarrow \Sigma_j$ in the diagram we have that $M_i \in \text{Mod}(\varphi)M_j$.

The diagram is consistent when it has at least one model.

**Definition 3.6** (Model amalgamation in $\frac{3}{2}$-institutions). In any $\frac{3}{2}$-institution, a lax cocone for a span in the $\frac{3}{2}$-category of the signature morphisms

\[
\begin{array}{ccc}
\Sigma_0 & \xrightarrow{\theta_0} & \Sigma_1 \\
\sim & \SEarrow & \sim \\
\varphi_1 & \sim & \varphi_2 \\
\Sigma_1 & \xleftarrow{\theta_1} & \Sigma_2 \\
\end{array}
\]

has model amalgamation when each model of the span admits an unique completion to a model (called the amalgamation) of the lax cocone.

When dropping the uniqueness condition, the property is called weak model amalgamation.

Note that when the signature morphisms involved in Dfn. 3.6 are total (or at least when the model reducts give singletons) we get the ordinary concept of model amalgamation for (1-)institution theory. This also means that $\theta_0$ and $\Sigma_0$-model become redundant. In the proper $\frac{3}{2}$ case their presence is necessary, this being one of the important aspects that distinguishes the $\frac{3}{2}$ case from ordinary (1-)institution theoretic model amalgamation.

**Example 3.10.** In $\mathcal{SP}_L$, for the diagram of Dfn. 3.6 we consider the signatures $\Sigma_0 = \{p, p', p_1, p_2\}$, $\Sigma_1 = \{p, p_1, p'_1\}$, $\Sigma_2 = \{p, p_2, p'_2\}$, $\Sigma = \{p, p', p'_1, p'_2\}$ and let $\varphi_1, \varphi_2, \theta_0, \theta_1, \theta_2$ be the maximal partial inclusions. We prove that this cocone has model amalgamation as follows. We assume $\{M_k \mid k = 0, 1, 2\}$ a model for the span $(\varphi_1, \varphi_2)$ and define the $\Sigma$-model $M$ by $M(p) = M_k(p)$, $M(p'_i) = M_k(p'_i)$, $k \in 1, 2$, and $M(p') = M_0(p')$. It is easy to see that $M$ thus defined is the unique amalgamation of $M_0, M_1, M_2$.

In ordinary institution theory the causal dependency between pushout squares and model amalgamation squares is central and well known (cf. [4, 2, 22], etc.). The following result refines this to $\frac{3}{2}$-institutions in a way intended to maximize its applicability in concrete situations.

**Proposition 3.2.** For any $\frac{3}{2}$-institutional seed $S$ and any 1-subcategory $\mathcal{T} \subseteq \text{Sign}$ such that

- $\text{Sen}$ preserves and reflects maximality ($\varphi$ is maximal if and only if it is $\text{Sen}$-maximal),
- $\mathcal{T}$ contains all maximal signature morphisms, and
- if $\varphi \in \mathcal{T}$ and $\varphi \leq \varphi'$ then $\varphi' \in \mathcal{T}$,

in $I(S)$ each lax $\mathcal{T}$-pushout of signature morphisms has weak model amalgamation.
Proof. We consider a lax $\mathcal{T}$-pushout $(\theta_0, \theta_1, \theta_2)$ for a span $(\varphi_1, \varphi_2)$ of signature morphisms like shown in the diagram below, and a model $\{M_k\}_{k=0,1,2}$ for the span $(\varphi_1, \varphi_2)$. By the first and second assumptions this means that we have a lax $\mathcal{T}$-cocone $(M_0, M_1, M_2)$ for the span $(\varphi_1, \varphi_2)$. By the universal property of $(\theta_0, \theta_1, \theta_2)$ there exists an unique signature morphism $M : \Sigma \rightarrow \Omega$ in $\mathcal{T}$ such that $\theta_k; M = M_k$ for $k = 0, 1, 2$.

In order to establish that $M$ is a model we show that $M$ is maximal; then since $\text{Sen}$ preserves maximality it follows that $\text{Sen}(M)$ is total. Let $M \leq N$. By the third assumption it follows that $N \in \mathcal{T}$. For each $k = 0, 1, 2$, by the monotonicity of the composition, we have that $M_k = \theta_k; M \leq \theta_k; N$. Because $M_k$ is maximal (as a consequence of $\text{Sen}$ reflecting maximality) it follows that $M_k = \theta_k; N$ for each $k = 0, 1, 2$. By the uniqueness of $M$ as a meditating arrow between lax $\mathcal{T}$-cocones it follows that $M = N$. Hence $M$ is maximal.

One quick note on the first condition of Prop. 3.2 which although holds naturally in many $\frac{1}{2}$-institutions of interest (such as those from Ex. 3.4, 3.5, 3.6 and 3.7), it has to be assumed in the abstract setup since there are concrete situations when it does not hold (such as the $\frac{1}{2}$-institution of Ex. 3.9 where $\text{Sen}$ preserves maximality but does not reflect it).

The following result gives the important information that we should in general give up expectations that weak lax cocones may involve ‘non-total’ signature morphisms; this will be also used to strengthen the conclusion of Prop. 3.2.

**Proposition 3.3.** For any $\frac{3}{2}$-institutional seed $\mathcal{S}$ and any 1-subcategory $\mathcal{T} \subseteq \text{Sign}$ such that

- $\text{Sen}$ is strict, and
- $\mathcal{T}$ contains all $\text{Sen}$-maximal signature morphisms,

for any consistent span $(\varphi_1, \varphi_2)$ of signature morphisms in the $\frac{3}{2}$-institution $\mathcal{I}(\mathcal{S})$ any of its each weak lax $\mathcal{T}$-pushout cocones $(\theta_0, \theta_1, \theta_2)$ consists only of $\text{Sen}$-maximal signature morphisms.

Proof. The consistency of the span means that it has a lax cocone $(M_0, M_1, M_2)$ such that each $\text{Sen}(M_k)$ is total for $k = 0, 1, 2$. By the second assumption of the proposition it follows that this is a $\mathcal{T}$-cocone. By the weak lax $\mathcal{T}$-pushout property of $(\theta_0, \theta_1, \theta_2)$ there exists an $M : \Sigma \rightarrow \Omega$ in $\mathcal{T}$ such that $\theta_k; M = M_k$ for $k = 0, 1, 2$ (like in diagram (4)). Since $\text{Sen}$ is strict it follows that $\text{Sen}(\theta_k); \text{Sen}(M) = \text{Sen}(M_k)$, $k = 0, 1, 2$. Because $\text{Sen}(M_k)$ is total, $\text{Sen}(\theta_k)$ must be total too.

The outstanding condition of Prop. 3.3 is that of consistency of the span. Although at the abstract level the consistency of spans has to be assumed axiomatically, in concrete situations, spans of real signature morphisms are very easily consistent. For example in $\frac{1}{2}\text{PCL}$ it is enough to consider $(M_k)p = 1$, $k = 0, 1, 2,
for all propositional symbols $p$, and in $\frac{1}{2}MSA$ to consider $M_k, k = 0, 1, 2$, having a fixed singleton set $\{\ast\}$ as underlying/carrier sets. However the concept gets real substance in $\frac{1}{2}$-institutions where the signature morphisms carry more structure than the common signature morphisms, an important example being given by that of theory morphisms of Sect. 3.6 below.

**Corollary 3.2.** If in addition to the hypotheses of Prop. 3.2 we have that Sen is strict then the conclusion of Prop. 3.2 is that in $I(S)$ each lax $T$-pushout of signature morphisms has model amalgamation.

**Proof.** Let us suppose that a model $\{M_k \mid k = 0, 1, 2\}$ of the span $(\varphi_1, \varphi_2)$ has two amalgamations $M$ and $N$. In other words $\theta_k; M, \theta_k; N \leq M_k$ for $k = 0, 1, 2$.

Note that the second assumption of Prop. 3.2 is a consequence of the assumptions of Prop. 3.2. By the strictness of Sen we have that $\text{Sen}(\theta_k; M) = \text{Sen}(\theta_k; N)$ for $k = 0, 1, 2$ and likewise for $N$. Since $\text{Sen}(\theta_k)$ (by Prop. 3.3), $\text{Sen}(M), \text{Sen}(N)$ (since $M, N$ are models) are total functions, it follows that all $\text{Sen}(\theta_k; M), \text{Sen}(\theta_k; N)$, $k = 0, 1, 2$, are total functions too. By the first assumption of Prop. 3.2 it follows that all $\theta_k; M, \theta_k; N$, $k = 0, 1, 2$, are maximal. Hence $\theta_k; M = \theta_k; N = M_k$, $k = 0, 1, 2$. By the uniqueness part of the universal property of lax $T$-pushouts it follows that $M = N$. □

The following corollary indicates that the result of Cor. 3.2 covers many concrete situations of interest.

**Corollary 3.3.** In both $\frac{1}{2}\mathcal{PL}$ and $\frac{1}{2}MSA$ each lax $T$-pushout of signature morphisms has model amalgamation in any of the following situations for $T$(the latter two apply only for $\frac{1}{2}MSA$):

1. all signature morphisms,
2. the total signature morphisms,
3. the signature morphisms that are total on the sort symbols, i.e. $\varphi_0$ are total functions, and
4. the signature morphisms that are total on the operation symbols, i.e. $\varphi_{\text{op}}$ are total functions.

**Proof.** Recall from Sect. 3 how $\frac{1}{2}\mathcal{PL}$ arises as an $I(S)$. In the case of $\frac{1}{2}MSA$, although due to cardinality issues it cannot be presented as a whole as an $I(S)$, we may consider ‘localised’ versions that have all carriers of models included in a fixed set $U$. Thus, given a span of signature morphisms an a model $\{M_k \mid k = 0, 1, 2\}$ of it, we may take $U$ to be the union of all the carrier sets in $M_0, M_1, M_2$. Then the hypotheses of Prop. 3.2 and Cor. 3.2 can be checked quite easily in each of the cases for $T$ listed in the statement of the corollary. □

So far we have established model amalgamation for classes of lax cocones that enjoy a universal property of a colimit. In the following we develop some results that may be used to extend model amalgamation to other classes of lax cocones. First we need a couple of new concepts.

**Definition 3.7** (Model conservativeness). In a $\frac{1}{2}$-institution a signature morphism $\varphi$ is model conservative when for each $\square\varphi$-model $M$ there exists a $\varphi\square$-model $M'$ such that $M \in \text{Mod}(\varphi)M'$.

In general, in many concrete situations of interest $\frac{1}{2}\mathcal{PL}$ and $\frac{1}{2}MSA$ included – a signature morphism is model conservative if and only if it is injective (this does not exclude the possibility of partiality).

**Definition 3.8** (Model strictness). In a $\frac{1}{2}$-institution a signature morphism $\varphi$ is model Mod-strict when for each signature morphism $\theta$ such that $\theta\square = \square\varphi$ we have that $\text{Mod}(\varphi); \text{Mod}(\theta) = \text{Mod}(\theta; \varphi)$.

In general, in many concrete situations of interest $\frac{1}{2}\mathcal{PL}$ and $\frac{1}{2}MSA$ included – a signature morphism is $\text{Mod}$-strict whenever it is total. One way to see this is through the following general result.
**Proposition 3.4.** For any \( \frac{1}{2} \)-institutional seed \( S \), any \( \text{Sen} \)-maximal signature morphism is \( \text{Mod} \)-strict in the associated \( \frac{1}{2} \)-institution \( I(S) \).

**Proof.** Since the other inclusion holds by the lax functoriality of \( \text{Mod} \), we need only to prove that for each \( \varphi \square \)-model \( M'' \) we have that

\[
\text{Mod}(\theta; \varphi)M'' \subseteq \text{Mod}(\theta)(\text{Mod}(\varphi)M'').
\]

Any \( M \in \text{Mod}(\theta; \varphi)M'' \) is characterised by the properties that \( \text{Sen}(M) \) is total and that

\[
\theta; \varphi; M'' \leq M.
\]

Now since \( \text{Sen}(\varphi) \) and \( \text{Sen}(M'') \) are total functions it follows that their composition is a total function too, hence by the lax functoriality of \( \text{Sen} \) is follows that \( \text{Sen}(\varphi; M'') \) is a total function too. This means that \( \varphi; M'' \) is a model in \( \text{Mod}(\varphi)M'' \). This and (5) imply that \( M \in \text{Mod}(\theta)(\text{Mod}(\varphi)M'') \).

**Proposition 3.5.** In any \( \frac{1}{2} \)-institution, consider a lax cocone \((\theta_0, \theta_1, \theta_2)\) of a span of signature morphisms \((\varphi_1, \varphi_2)\) and a signature morphism \( \mu \) such that \( \theta \square = \square \mu \). Then

1. if the lax cocone \( \theta \) has weak model amalgamation and \( \mu \) is model conservative then the lax cocone \( \theta; \mu \) has it too, and
2. if there exists a lax cocone \( \theta' \) that has weak model amalgamation and such that \( \theta; \mu \leq \theta' \), and \( \mu \) is \( \text{Mod} \)-maximal and model \( \text{Mod} \)-strict then the lax cocone \( \theta \) has weak model amalgamation too.

**Proof.**

1. Consider a model \( \{ M_k \mid k = 0, 1, 2 \} \) for the span \((\varphi_1, \varphi_2)\). There exists a \( \theta \square \)-model \( M \) such that \( M_k \in \text{Mod}(\theta_k)M, k = 0, 1, 2 \). Since \( \mu \) is model conservative there exists a model \( M' \) such that \( M \in \text{Mod}(\mu)M' \). Then for each \( k = 0, 1, 2 \), \( M_k \in \text{Mod}(\theta_k)(\text{Mod}(\mu)M') \subseteq \text{Mod}(\theta_k; \mu)M' \) (by the lax property of \( \text{Mod} \)). Hence \( M' \) is an amalgamation of \( \{ M_k \mid k = 0, 1, 2 \} \).

2. Consider a model \( \{ M_k \mid k = 0, 1, 2 \} \) for the span \((\varphi_1, \varphi_2)\). There exists a \( \mu \square \)-model \( M' \) such that \( M_k \in \text{Mod}(\theta_k')M', k = 0, 1, 2 \). Since \( \theta_k; \mu \leq \theta_k', k = 0, 1, 2 \), and since \( \text{Mod} \) preserves orders, we have that \( \text{Mod}(\theta_k')M' \subseteq \text{Mod}(\theta_k; \mu)M', k = 0, 1, 2 \). Hence \( M_k \in \text{Mod}(\theta_k; \mu)M', k = 0, 1, 2 \).

By the \( \text{Mod} \)-maximality assumption we have that \( \text{Mod}(\mu)M' = \{ M \} \). By the \( \text{Mod} \)-strictness assumption it follows that for each \( k = 0, 1, 2 \), \( M_k \in \text{Mod}(\theta_k)(\text{Mod}(\mu)M') = \text{Mod}(\theta_k)M \). Hence \( M \) is an amalgamation of \( \{ M_k \mid k = 0, 1, 2 \} \).

We can combine Prop. 3.2 and 3.5 for getting a larger class of lax cocones enjoying weak model amalgamation.

**Corollary 3.4.** Under the hypotheses of Prop. 3.2 we consider a lax cocone \((\theta_0, \theta_1, \theta_2)\) for a span of signature morphisms \((\varphi_1, \varphi_2)\) and a signature morphism \( \mu \) such that \( \theta \square = \square \mu \). Then

1. if \( \theta \) is a lax \( \mathcal{T} \)-pushout and \( \mu \) is model conservative then the lax cocone \( \theta; \mu \) has weak model amalgamation, and
2. if there exists a lax \( \mathcal{T} \)-pushout \( \theta' \) such that \( \theta; \mu \leq \theta' \) and \( \mu \) is \( \text{Sen} \)-maximal then the lax cocone \( \theta \) has weak model amalgamation.

**Proof.** While 1. is a direct consequence of Prop. 3.2 and 3.5, the argument for 2. needs a bit of elaboration. By Prop. 3.4 we get that \( \mu \) is \( \text{Mod} \)-strict.

Now let \( M \) be any \( \mu \square \)-model. Because \( \text{Sen}(\mu) \) and \( \text{Sen}(M) \) are total functions, by the lax functoriality of \( \text{Sen} \) it follows that \( \text{Sen}(\mu; M) \) is a total function too. Since \( \text{Sen} \) reflects maximality (one of the hypothesis of Prop. 3.2) it follows that \( \mu; M \) is maximal, hence \( \text{Mod}(\mu)M = \{ \mu; M \} \). This shows that \( \mu \) is \( \text{Mod} \)-maximal.
Now all conditions of Prop. 3.2 and 3.5 are fulfilled, therefore the conclusion 2. follows.

Example 3.11. The (weakened version of the) model amalgamation situation of Ex. 3.10 can be obtained from Cor. 3.4 (2.) as follows.

- We set $T$ to be the class of the total functions.
- For each $k = 0, 1, 2$ we let $\theta_k'$ to be the inclusion of $\Sigma_k$ into $\{p, p', p_1, p'_1, p_2, p'_2\}$. This is a $T$-pushout.
- We let $\mu$ be the inclusion $\{p, p', p_1, p'_1, p_2, p'_2\} \subseteq \{p, p', p_1, p'_1, p_2, p'_2\}$.

3.6. Theory morphisms in $\frac{3}{2}$-institutions

In 1-institution theory, the concept of theory morphism plays an important role in connection to foundational works in computer science. It was one of the central institution theoretic concepts introduced and studied in the seminal publication [12]. The mathematical foundations of conceptual blending are based on theory morphisms since concepts are modelled as logical theories and their translations as theory morphisms [8, 10]. While theories in $\frac{3}{2}$-institutions are the same as theories in 1-institutions, the $\frac{3}{2}$-institution theoretic concept of theory morphism is much more subtle because of the partiality of the sentence translations. In fact there are at least four ways to extend the 1-institution concept of theory morphism to $\frac{3}{2}$-institutions.

Definition 3.9. In a $\frac{3}{2}$-institution a theory $(\Sigma, E)$ consists of a signature $\Sigma$ and a set $E$ of $\Sigma$-sentences ($E \subseteq \text{Sen} (\Sigma)$).

Given two theories $(\Sigma, E)$ and $(\Sigma', E')$ in a $\frac{3}{2}$-institution, a signature morphism $\varphi : \Sigma \to \Sigma'$ is

- a pseudo-morphism of theories when $\text{Sen}(\varphi)E \subseteq E'^*$,
- a weak morphism of theories when $\text{Sen}(\varphi)E* \subseteq E'^*$,
- a strong morphism of theories when for each $\Sigma'$-model $M'$ such that $M' \models E'$ there exists $M \in \text{Mod}(\varphi)M'$ such that $M \models E$, and
- a ultra-strong morphism of theories when for all $\Sigma'$-models $M'$ and $\Sigma$-models $M$ such that $M' \models E'$ and $M \in \text{Mod}(\varphi)M'$ we have that $M \models E$.

Fact 3.1. Any weak morphism is pseudo-morphism, any strong morphism is weak. If $\text{Mod}$ does not admit emptiness then any ultra-strong morphism is strong.

In 1-institution theory the four concepts of theory morphisms of Dfn. 3.9 collapse to the single established 1-institution concept of theory morphism (cf. [12, 2], etc.). But in the realm of $\frac{3}{2}$-institutions they are in general different concepts as shown by the following very simple counterexamples:

- In $\frac{3}{2}\mathcal{P}L$ consider $\Sigma = \{p, q\}, \Sigma' = \{q\}, E = \{p \land q\}, E' = \emptyset$. Then $\varphi$, the maximal partial inclusion of $\Sigma$ into $\Sigma'$ (dom$\varphi = \{q\}$), is a pseudo-morphism $(\Sigma, E) \rightarrow (\Sigma', E')$ but it is not a weak one since $q \in \text{Sen}(\varphi)E* \setminus E'^*$.

- In the quantifier-free variant of $\frac{3}{2}\mathcal{MSA}$ (which means sentences without quantifiers) consider $\Sigma$ consisting of one sort symbol $s$ and two constants $c, c'$, $\Sigma'$ consisting only of the sort symbol $s$ and a constant $c', E = \{\neg(c = c')\}$, and $E' = \emptyset$. Then $\varphi$, the maximal partial inclusion of $\Sigma$ into $\Sigma'$, is a (trivially) weak morphism $(\Sigma, E) \rightarrow (\Sigma', E')$ but it is not a strong one since any singleton set does not admit a $\varphi$-reduce that satisfies $E'$.

\cite{Counterexample communicated by Daniel Găină.
In \( \mathcal{CL} \) consider \( \Sigma = \{ p, q \}, \Sigma' = \{ q \}, E = \{ p \land q \}, E' = \{ q \} \). Then \( \varphi \), the maximal partial inclusion of \( \Sigma \) into \( \Sigma' \) (domain = \( \{ q \} \)), is a strong morphism \( (\Sigma, E) \rightarrow (\Sigma', E') \) but it is not an ultra-strong one. There exists only one model \( M' \models E' \), namely \( \varphi M(q) = 1 \). Then \( M' \) has a \( \varphi \)-reduct \( M \) such that \( M \models E \) defined by \( M(p) = M(q) = 1 \). However not any \( \varphi \)-reduct of \( M' \) enjoys this property, for example \( N \) such that \( N(p) = 0 \) and \( N(q) = 1 \).

In general pseudo-morphisms and do not compose and the ultra-strong ones compose under the condition that \( \text{Mod} \) is strict rather than (properly) lax. The strictness condition on \( \text{Mod} \) is a very heavy and unrealistic one in the applications (actually unlike the strictness condition on \( \text{Sen} \) which holds in a lot of \( \frac{3}{2} \)-institutions of interest). This makes both extremes, the pseudo-morphisms and the ultra-strong morphisms, unsuitable as a \( \frac{3}{2} \)-institutional replacement for the 1-institution theory morphisms and leaves us only with the middle options. But it is not only the failure in compositionality that makes them unsuitable, their very nature also feel inadequate as can be for example seen by inspecting the very simple examples above. Pseudo-morphisms are too weak and the ultra-strong morphisms seem to require too much. The strong theory morphisms compose unconditionally, while the weak ones compose under a certain condition that holds often in concrete situations.

**Proposition 3.6.** In any \( \frac{3}{2} \)-institution \( I \), by inheriting the \( \frac{3}{2} \)-categorical structure of \( \text{Sign}^I \)

- strong morphisms of theories yield a \( \frac{3}{2} \)-category – denoted \( \text{Th}^I \), and
- when \( \text{Sen} \) is oplax, the weak theory morphisms yield a \( \frac{3}{2} \)-category – denoted \( \text{Th}^I_w \).

*Proof.* The proof is based on the fact that the composition of theory morphisms yields a theory morphism; the rest being straightforward. Let us consider theory morphisms \( \varphi : (\Sigma, E) \rightarrow (\Sigma', E') \) and \( \varphi' : (\Sigma', E') \rightarrow (\Sigma'', E'') \).

For the ‘strong’ case we consider \( M'' \in |\text{Mod}(\Sigma'')| \) such that \( M'' \models E'' \). Then there exists \( M' \in \text{Mod}(\varphi')M'' \) such that \( M' \models E' \). It follows that there exists \( M \in \text{Mod}(\varphi)M' \) such that \( M \models E \). Then by the lax property of \( \text{Mod} \) it follows that \( M \in \text{Mod}(\varphi; \varphi') \).

For the ‘weak’ case we have:

\[
\text{Sen}(\varphi; \varphi')E^* \subseteq (\text{Sen}(\varphi); \text{Sen}(\varphi'))E^* \quad \text{by the oplax functoriality of} \ \text{Sen}
\]

\[
= \text{Sen}(\varphi')(\text{Sen}(\varphi)E^*) \quad \text{by Lemma 2.1}
\]

\[
\subseteq \text{Sen}(\varphi'E')E^* \quad \text{since} \ \varphi \text{ is theory morphism}
\]

\[
\subseteq E''\quad \text{since} \ \varphi' \text{ is theory morphism}.
\]

\[\square\]

From now on whenever we encounter weak theory morphisms we tacitly assume that \( \text{Sen} \) is oplax.

The constructions in the Corollaries 3.5 and 3.6 constitute natural examples of \( \frac{3}{2} \)-institutions that are not based on an explicit form of partiality of signature morphisms.

**Corollary 3.5.** For any \( \frac{3}{2} \)-institution \( I = (\text{Sign}, \text{Sen}, \text{Mod}, \models) \) its \( \frac{3}{2} \)-category of weak/strong theory morphisms determines a \( \frac{3}{2} \)-institution \( I_w/I_s \) as follows (i is w or s):

- the \( \frac{3}{2} \)-category of signatures \( \text{Sign}_i \) is \( \text{Th}^I_i \),
- \( \text{Sen}_i \) is a trivial lifting of \( \text{Sen} \) to theories, i.e. \( \text{Sen}_i(\Sigma, E) = \text{Sen}_i(\Sigma) \), etc.,
We consider $\text{Th}(\Sigma, E)$ the full subcategory of $\text{Mod}(\Sigma)$ of the $\Sigma$-models satisfying $E$, and for each theory morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ and each $(\Sigma', E')$-model $M'$

$$\text{Mod}_i(\varphi)M' = \{ M \in \text{Mod}(\varphi)M' | M \models E \}$$

and the satisfaction relation is inherited from $I$.

**Proof.** The only interesting part of the proof is the lax functoriality of $\text{Mod}_i$, the rest being straightforward. We consider $\varphi : (\Sigma, E) \to (\Sigma', E')$ and $\varphi' : (\Sigma', E') \to (\Sigma'', E'')$ theory morphisms. For any $(\Sigma'', E'')$-model $M''$ we have that

$$\text{Mod}_i(\varphi)(\text{Mod}_i(\varphi')M'') = \text{Mod}_i(\varphi)[M' \in \text{Mod}(\varphi')M'' | M' \models E']$$

by definition of $\text{Mod}_i$

$$\subseteq \{ M \in \text{Mod}(\varphi)M' | M' \models \text{Mod}(\varphi')M'' | M \models E \}$$

by definition of $\text{Mod}_i$

$$\subseteq \{ M \in \text{Mod}(\varphi; \varphi')M'' | M \models E \}$$

since $\text{Mod}$ is lax

$$\text{Mod}_i(\varphi; \varphi')M''$$

by definition of $\text{Mod}_i$.

$I_w/I_s$ generalise the concept of the “institution of theories” from 1-institution theory [2] to $\frac{1}{2}$-institutions. Note that both of them constitute examples of $\frac{1}{2}$-institutions where the model functor may naturally admit emptiness, and this without being inherited from the base institution.

There is also an alternative way to complete the definition of $\text{Th}_s^I/\text{Th}_s^I$ to that of a $\frac{1}{2}$-institution by shifting the weight of the construction from the models side to the sentences side. However this construction is conditioned by $I$ being a lax $\frac{1}{2}$-institution.

**Corollary 3.6.** For any lax $\frac{1}{2}$-institution $I = (\text{Sign}, \text{Sen}, \text{Mod}, \models)$ its $\frac{1}{2}$-category of weak/strong theory morphisms determines a lax $\frac{1}{2}$-institution $I_w/I_s$ as follows (i is w or s):

- the $\frac{1}{2}$-category of signatures $\text{Sign}_i$ is $\text{Th}_i^I$, 
- $\text{Sen}_i(\Sigma, E) = E^*$ and for each theory morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ we let
  - $\text{dom} \text{Sen}_i(\varphi) = E^* \cap \text{dom} \text{Sen}(\varphi)$, and
  - $\text{Sen}_i(\varphi)\rho = \text{Sen}(\varphi)\rho$ for all $\rho \in \text{dom} \text{Sen}_i(\varphi)$.
- $\text{Mod}_i$ is the trivial lifting of $\text{Mod}$, i.e. $\text{Mod}_i(\Sigma, E) = \text{Mod}(\Sigma)$, etc.,
- and the satisfaction relation is inherited from $I$.

**Proof.** The only interesting part of the proof is the lax functoriality of $\text{Sen}_i$, the rest being straightforward. We consider $\varphi : (\Sigma, E) \to (\Sigma', E')$ and $\varphi' : (\Sigma', E') \to (\Sigma'', E'')$ theory morphisms. On the one hand we have that

$$\text{dom} \text{Sen}_i(\varphi); \text{Sen}_i(\varphi') = \text{dom} \text{Sen}_i(\varphi) \cap \text{Sen}_i^{-1}(\varphi)(\text{dom} \text{Sen}_i(\varphi'))$$

by definition of $\text{Sen}_i$

$$\subseteq E^* \cap \text{dom} \text{Sen}(\varphi) \cap \text{Sen}(\varphi)^{-1}(E^* \cap \text{dom} \text{Sen}(\varphi'))$$

by definition of $\text{Sen}_i$

$$\subseteq E^* \cap \text{dom}(\text{Sen}(\varphi); \text{Sen}(\varphi')) \cap \text{Sen}(\varphi)^{-1}(E^*)$$

since $\text{Sen}$ is lax

$$\subseteq E^* \cap \text{dom} \text{Sen}(\varphi; \varphi')$$

by definition of $\text{Sen}_i$. 

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On the other hand for each \( \rho \in \text{dom} \ Sen_i(\varphi) \):
\[
Sen_i(\varphi')(Sen_i(\varphi)\rho) = Sen(\varphi')(Sen(\varphi)\rho) = Sen(\varphi; \varphi')\rho = Sen_i(\varphi; \varphi')\rho.
\]

One of the starting motivations in 1-institution theory was the development of a general logic-independent method for the aggregation of software modules, modelled as institutional theories [12]. The process of “putting together” – just to use a favourite phrase of Goguen and Burstall – institutional theories relies on colimits in the category of theory morphisms, an important result being the automatic lifting of colimits from the category of signature morphisms to that of theory morphisms (see [12, 2, 22]). The following results replicate this in the context of \( \frac{1}{1} \)-institutions in support of conceptual blending theory. The more complicated situation of colimits and theory morphisms in \( \frac{1}{2} \)-institutions leads to a significantly more complex situation with respect to the lifting of colimits from signatures to theories.

**Proposition 3.7** (Lifting lax cocones from signatures to theories). Consider a span of weak/strong theory morphisms \( \varphi_k: (\Sigma_0, E_0) \to (\Sigma_k, E_k), k = 1, 2 \), and a lax cocone for the underlying span of signature morphisms like shown in the following diagram.

\[
\begin{array}{ccc}
\Sigma_1 & \leq & \Sigma_2 \\
\varphi_1 & & \varphi_2 \\
\Sigma_0
\end{array}
\]

Then for any \( E \subseteq \text{Sen}(\Sigma) \) such that \( \bigcup_{k=0,1,2} \text{Sen}(\gamma_k)E_k^* \subseteq E^* \) the following diagram displays a lax cocone of theory morphisms

\[
\begin{array}{ccc}
(\Sigma, E) & \leq & (\Sigma, E) \\
(\Sigma_1, E_1) & & (\Sigma_2, E_2) \\
(\Sigma_0, E_0)
\end{array}
\]

where

1. in the ‘weak’ case, \( \gamma_k = \theta_k, k = 0, 1, 2 \), and
2. in the ‘strong’ case, \( \gamma_k, k = 0, 1, 2 \) are any signature morphisms such that \( \theta_k \leq \gamma_k \) and \( \text{Sen}(\gamma_k) \) are total functions.

**Proof.** We have to only to show that \( \theta_k: (\Sigma_k, E_k) \to (\Sigma, E), k = 0, 1, 2 \) are theory morphisms. The ‘weak’ case is straightforward. For the ‘strong’ case we consider any \( M \in |\text{Mod}(\Sigma)| \) such that \( M \models E \). Because \( \text{Sen}(\gamma_k) \) are total, by the Satisfaction Condition it follows that for any \( M_k \in \text{Mod}(\gamma_k)M, M_k \models E_k \). Since \( \theta_k \leq \gamma_k \) by the monotonicity of \( \text{Mod} \) it follows that \( M_k \in \text{Mod}(\theta_k)M \). Hence \( \theta_k \) are strong theory morphisms. \( \square \)
Corollary 3.7 (Lifting lax pushouts from signature to theories). In the context of Prop. 3.7, given a \( \mathcal{T} \subseteq \mathcal{S} \), let \( \mathcal{T}_w/\mathcal{T}_s \) denotes the class of weak theory morphisms \( \varphi \) such that \( \varphi \in \mathcal{T} \). We further assume that

- \( \mathcal{M} \) does not admit emptiness,
- the lax cocone of signature morphisms is a lax \( \mathcal{T} \)-pushout,
- \( E^* = (\bigcup_{k=0,1,2} \text{Sen}(\gamma_k)E^*_k) \).

Then the lax cocone of theory morphisms obtained by Prop. 3.7

- is a lax \( \mathcal{T}_w \)-pushout when \( \text{Sen} \) is lax (therefore it is strict) and each signature morphism in \( \mathcal{T} \) is \( \text{Sen} \)-maximal,
- is a lax \( \mathcal{T}_s \)-pushout when each signature morphism in \( \mathcal{T} \) is \( \text{Mod} \)-maximal.

Proof. We consider a lax \( \mathcal{T}_w/\mathcal{T}_s \)-cocone \( \theta' \) for the span of weak/strong theory morphisms. By the lax \( \mathcal{T} \)-pushout property in \( \mathcal{S} \) (the category of signature morphisms) there exists an unique \( \mu \in \mathcal{T} \) such that \( \theta_k; \mu = \theta'_k \), \( k = 0, 1, 2 \). It only remains to show that \( \mu \) is a weak/strong theory morphism \( (\Sigma, E) \to (\Sigma', E') \), where \( (\Sigma', E') \) is the vertex of \( \theta' \).

We first solve the weak case. Let us recall that in this case \( \gamma_k = \theta_k \). For that we need the following lemma (we skip its proof):

Lemma 3.1. In any \( \frac{3}{2} \)-institution such that \( \mathcal{M} \) does not admit emptiness, for any signature morphism \( \varphi \) that is \( \text{Sen} \)-maximal and for any set \( E \) of \( \Box \varphi \)-sentences, we have that

\[
\text{Sen}(\mu)E^* \subseteq (\text{Sen}(\mu)E)^*.
\]

Then

\[
\text{Sen}(\mu)E^* = \text{Sen}(\mu)(\bigcup_{k=0,1,2} \text{Sen}(\theta_k)E^*_k)^* \\
\subseteq (\text{Sen}(\mu) \bigcup_{k=0,1,2} \text{Sen}(\theta_k)E^*_k)^* \quad \text{by the second and third assumptions and by Lemma 3.1} \\
= (\bigcup_{k=0,1,2} \text{Sen}(\mu)(\text{Sen}(\theta_k)E^*_k))^* \\
= (\bigcup_{k=0,1,2} \text{Sen}(\theta_k; \mu)E^*_k)^* \quad \text{by the strictness assumption on Sen} \\
\subseteq (E^*)^* \quad \text{since } \theta'_k \text{ are weak theory morphisms} \\
= E^*.
\]

Now comes the strong case. We consider a \( \Sigma' \)-model \( M' \) such that \( M' \models E' \). Since \( \mu, \theta_k \in \mathcal{T} \) are \( \text{Mod} \)-maximal, let \( M \) be the unique model in \( \text{Mod}(\mu)M' \) and for each \( k = 0, 1, 2 \) let \( M_k \) be the unique model in \( \text{Mod}(\theta_k)M \). Since \( \theta_k \leq \gamma_k \), the monotonicity of \( \text{Mod} \) it follows that \( \text{Mod}(\gamma_k)M \subseteq \text{Mod}(\theta_k)M \). Since \( \text{Mod} \) does not admit emptiness this means that \( M_k \) is the unique member of \( \text{Mod}(\gamma_k)M \) too.

By the lax property of \( \text{Mod} \) and by the equalities \( \theta'_k = \theta_k; \mu \) it follows that

\[
\text{Mod}(\theta_k)(\text{Mod}(\mu)M') \subseteq \text{Mod}(\theta'_k)M'
\]

which means

\[
\text{Mod}(\theta_k)M \subseteq \text{Mod}(\theta'_k)M'.
\]

By the \( \text{Mod} \)-maximality assumption it follows that \( \text{Mod}(\theta'_k)M' = \{M_k\} \). Since \( \theta'_k \) is a strong theory morphism \( (\Sigma_k, E_k) \to (\Sigma', E') \) we have that \( M_k \models E_k \). By the Satisfaction Condition for \( \gamma_k \) (and by keeping in mind that \( \text{Sen}(\gamma_k) \) is total) we obtain that \( M \models \text{Sen}(\gamma_k)E^*_k \), \( k = 0, 1, 2 \). This shows that \( M \models E \).
The only apparently restrictive assumption in the applications is the \(Sen/Mod\)-maximality condition on the signature morphisms in \(T\). Very often \(Sen\) and \(Mod\)-maximality say the same thing, namely that the corresponding signature morphisms are total. However Prop. 3.3 tells us that in many situations of interest, anyway one cannot get beyond that with lax \(T\)-pushouts. Although this does not constitute a real restriction in the applications, we may also note that the weak case adds a supplementary technical condition to the strong case, namely that \(Sen\) is lax.

**Proposition 3.8** (Lifting model amalgamation from signatures to theories). *Under the framework of Prop. 3.7, if*

- the lax cocone of signature morphisms has (weak) model amalgamation, and
- \(E^* = (\bigcup_{k=0,1,2} Sen(\gamma_k) E_k^*)^*\)

*then the lax cocone of theory morphisms has (weak) model amalgamation too.*

**Proof.** We treat both the ‘weak’ and the ‘strong’ case in one shot because there is no essential difference between them.

Let \(i \in \{w, s\}\). We consider \((M_0, M_1, M_2)\) a model for the span of theory morphisms. According to the definition of \(Mod_i\) we have that \(M_0 \in Mod(\varphi_i) M_k\) for \(k = 1, 2\). We show that if \(M\) is an amalgamation of \(M_0, M_1,\) and \(M_2\) with respect to the lax cocone of signature morphisms then it is an amalgamation with respect to the lax cocone of theory morphisms too.

Let \(k \in \{0, 1, 2\}\). Since \(M_k \in Mod(\gamma_k) M\), since \(M_k \models E_k^*\) by the Satisfaction Condition it follows that \(M \models Sen(\gamma_k) E_k^*\). Hence \(M \models \bigcup_{k=0,1,2} Sen(\gamma_k) E_k^*\). Therefore \(M \models (\bigcup_{k=0,1,2} Sen(\gamma_k) E_k^*)^* = E^*\). This completes the proof for the weak model amalgamation case.

The conclusion can be extended to the proper (non-weak) model amalgamation case by noting (by a simple *reductio ad absurdum* argument) that the uniqueness of amalgamation at the level of signature morphisms implies the uniqueness at the level of theory morphisms.

\[\square\]

### 3.7. Theory changes

In this section we develop an alternative concept of mapping between theories in \(\downarrow\)-institutions that does not resemble or generalise the theory morphisms from 1-institution theory, but which models software changes. Theory changes formalise the process of modifications in specification or declarative programs.

In this modelling a flat (unstructured) specification or program is modelled by a theory. Modifications or changes operate at two different levels, at the signature and the sentences level. The changes at the signature level are encapsulated in the respective concept of signature morphism, while those at the sentences level are made explicit and modelled by the partial inclusion component of the concept of theory changes. This represents a marking of the part of the translated sentences that is not touched by the change, which may consist both of deletions or of adding sentences. The fact that the partial inclusion is not necessarily maximal accounts for the possibility that sentences may be deleted and later added back, or vice versa. Also we assume that the programmer is not committed to the parts that he leaves unchanged.

First we develop a theory of partial inclusions. A partial function \(f: A \rightarrow B\) is an *inclusion* when \(f\) consists only of pairs of elements of the form \((a, a)\). It follows that \(f \subseteq (A \cap B)^2\) and that \(f = \{(a, a) \mid a \in \text{dom}(f)\}\). Note that, unlike in the case of total inclusions, given two sets \(A\) and \(B\) they may admit more than one partial inclusion between them and in any case at least one (the empty one).

Given \(A_1, A_2 \subseteq A\), a partial function \(f: A \rightarrow B\) and a partial inclusion \(i: A_1 \rightarrow A_2\) we let \(f(i) = \{(f^0(a), f^0(a)) \mid a \in \text{dom}(f), (a, a) \in i\}\).
Lemma 3.2. \( f(i) \) is a partial inclusion \( f(A_1) \hookrightarrow f(A_2) \).

Another fact gives a functorial property for the above notation:

**Lemma 3.3.** Given \( A_1, A_2, A_3 \subseteq A \), a partial function \( f : A \rightarrow B \) and partial inclusions \( i_1 : A_1 \twoheadrightarrow A_2, i_2 : A_2 \twoheadrightarrow A_3 \), we have that \( f(i_1; i_2) = f(i_1); f(i_2) \).

Based on Lemmas 2.1 and 3.2 we get another property:

**Lemma 3.4.** Given partial functions \( f : A \rightarrow B \) and \( g : B \rightarrow C \), sets \( A_1, A_2 \subseteq A \) and partial inclusion \( i : A_1 \rightarrow A_2 \) we have \( (f; g)(i) = g(f(i)) \).

**Definition 3.10** (Theory changes). In any \( \frac{I}{2} \)-institution a theory change \( (\varphi, i) : (\Sigma, E) \rightarrow (\Sigma', E') \) consists of:

- theories \( (\Sigma, E) \) and \( (\Sigma', E') \);
- a signature morphism \( \varphi : \Sigma \rightarrow \Sigma' \); and
- a partial inclusion \( i : \text{Sen}(\varphi)E \rightarrow E' \).

**Proposition 3.9.** For any \( \frac{I}{2} \)-institution \( I \) with a strict sentence functor theory changes form a \( \frac{I}{2} \)-category as follows:

- the composition of theory changes is as shown by the following diagram:

\[
(\Sigma, E) \xrightarrow{(\varphi, i)} (\Sigma', E') \xrightarrow{(\varphi', i')} (\Sigma'', E'')
\]

- the partial order on theory changes \( (\Sigma, E) \rightarrow (\Sigma', E') \) is given by:

\( (\varphi, i) \leq (\varphi', i') \) if and only if \( \varphi \leq \varphi' \) and \( i \subseteq i' \).

**Proof.** The composition of theory changes is correctly defined because

- by lemma 3.2 \( \text{Sen}(\theta)(i) \) is a partial inclusion \( \text{Sen}(\theta)(\text{Sen}(\varphi)E) \twoheadrightarrow \text{Sen}(\theta)E' \),
- the composition of partial inclusions is a partial inclusion, hence \( \text{Sen}(\theta)(i); i' \) is a partial inclusion \( \text{Sen}(\theta)(\text{Sen}(\varphi)E) \twoheadrightarrow E'' \), and
- by Lemma 2.1 and by the strict functoriality of \( \text{Sen} \) we have that \( \text{Sen}(\theta)(\text{Sen}(\varphi)E) = \text{Sen}(\varphi; \theta)E \).

The partial order on theory changes is also correctly defined because whenever \( \varphi \leq \theta \) this implies \( \text{Sen}(\varphi) \subseteq \text{Sen}(\theta) \) which implies \( \text{Sen}(\varphi)E \subseteq \text{Sen}(\theta)E \). Then \( i \subseteq j \) parses as a subset relationship between subsets of \( \text{Sen}(\theta)E \times E' \).

The understanding of the proof of the associativity of the composition of theory changes is helped by inspecting the following diagram:

\[
(\Sigma, E) \xrightarrow{(\varphi, i)} (\Sigma', E') \xrightarrow{(\varphi', i')} (\Sigma'', E'') \xrightarrow{(\varphi'', i'')} (\Sigma''', E''')
\]
Thus all we have to show is that $\text{Sen}(\varphi'')(\text{Sen}(\varphi')(i); i'') = \text{Sen}(\varphi''; \varphi')(i); \text{Sen}(\varphi'')(i'); i''$, its proof being:

\[
\text{Sen}(\varphi'')(\text{Sen}(\varphi')(i); i'') = \text{Sen}(\varphi''; \varphi')(i); \text{Sen}(\varphi'')(i'); i'' \\
= (\text{Sen}(\varphi''; \varphi')(i); \text{Sen}(\varphi'')(i') \text{by Lemma 3.3} \\
= \text{Sen}(\varphi''; \varphi''; i); \text{Sen}(\varphi'')(i') \text{by Lemma 3.4} \\
= \text{by the strict functoriality of Sen.}
\]

For showing the preservation of partial orders by compositions we consider only the case when $(\varphi; i) \leq (\varphi', i')$ and $\varphi \square = \varphi' \square = \square \theta$, the other situation getting a similar proof. By the definition of composition we have that

- $(\varphi, i); (\theta, j) = (\varphi; \theta, \text{Sen}(\theta)(i); j)$, and
- $(\varphi', i'); (\theta, j) = (\varphi'; \theta, \text{Sen}(\theta)(i'); j)$.

From the monotonicity of composition in $\text{Sign}$ it follows that $(\varphi, i) \leq (\varphi', i')$. From $i \leq i'$ it follows that $\text{Sen}(\theta)(i) \leq \text{Sen}(\theta)(i')$ and further that $\text{Sen}(\theta)(i); j \leq \text{Sen}(\theta)(i'); j$. 

The following is another example of a $\frac{1}{2}$-institution that does not fall into the partiality pattern characteristic to $\mathcal{T}_{\mathcal{P}}, \mathcal{T}_{\mathcal{MS}}$, etc.

**Corollary 3.8.** For any $\frac{1}{2}$-institution $\mathcal{I}$ with a strict sentence functor, the $\frac{1}{2}$-category of theory changes determines a $\frac{1}{2}$-institution $\mathcal{I}'$ as follows:

- the $\frac{1}{2}$-category of signatures $\text{Sign}'$ is the $\frac{1}{2}$-category of theory changes,
- $\text{Sen}'$ is a trivial lifting of $\text{Sen}$ to theories, i.e. $\text{Sen}'(\Sigma, E) = \text{Sen}(\Sigma)$ and $\text{Sen}'(\varphi, i) = \text{Sen}(\varphi)$,
- $\text{Mod}'(\Sigma, E)$ is the full subcategory of $\text{Mod}(\Sigma)$ of the $\Sigma$-models satisfying $E$, and for each theory change $(\varphi, i); (\Sigma, E) \to (\Sigma', E')$ and each $(\Sigma', E')$-model $M'$

\[
\text{Mod}'(\varphi, i)M' = \{ M \in \text{Mod}(\varphi)M' \mid M \models E \}
\]

- and the satisfaction relation is inherited from $\mathcal{I}$.

In what follows we investigate the possibility of modelling merges of theory changes by pushout constructions. In principle, this should be based upon lifting pushouts from the category of signatures to that of theory changes.

**Proposition 3.10.** In general, lax $\mathcal{T}$-pushouts do not lift from the category of signatures to that of theory changes.

**Proof.** Consider a trivial (lax) $\mathcal{T}$-pushout of signature morphisms consisting only of identities; let the span be $\varphi_1 = \varphi_2 = 1_\Sigma$ and the cocone be $\theta_0 = \theta_1 = \theta_2 = 1_\Sigma$. Let $\rho$ be a $\Sigma$-sentence and let $E_0 = E_1 = E_2 = \{ \rho \}$ and $i_1 = i_2 = 1_{E_0}$.

Let us suppose that there exists a lax $\mathcal{T}$-pushout $(1_\Sigma, j_k), k = 0, 1, 2$ for the span given by $(1_\Sigma, i_1)$ and $(1_\Sigma, i_2)$.

- By considering the lax cocone given by $(1_\Sigma, 1_{E_0})$ everywhere we infer that all $j_k, k = 0, 1, 2$ are total.
- By considering the lax cocone given by $(1_\Sigma, 0), (1_\Sigma, 1_{E_0}), (1_\Sigma, 0)$, let $(1_\Sigma, u)$ be the unique mediating theory change. From $(1_\Sigma, j_k); (1_\Sigma, u) = (1_\Sigma, 0), k = 1, 2$ we infer that $\rho \notin \text{dom} u$. It follows that $j_0; u \neq 1_{E_0}$ which is a contradiction. 

By contrast to lax pushout, near pushouts lift trivially from signatures to theory changes.
Proposition 3.11. Given a span of theory changes \((\varphi_k, i_k) : (\Sigma_0, E_0) \rightarrow (\Sigma_k, E_k), k = 1, 2,\) and a near pushout for the underlying span of signature morphisms like shown in the following diagram

\[
\begin{array}{ccc}
\Sigma_1 & \xrightarrow{\theta_1} & \Sigma_0 \\
\downarrow \theta_0 \geq & \downarrow & \downarrow \theta_0 \geq \\
\Sigma_2 & \xleftarrow{\theta_2} & \Sigma_1 \\
\varphi_1 \downarrow & & \downarrow \varphi_2 \\
\Sigma_0 & & \\
\end{array}
\]

for any \(E \subseteq \text{Sen}(\Sigma), (\theta_k, \emptyset) : (\Sigma_k, E_k) \rightarrow (\Sigma, E), k = 0, 1, 2\) constitutes a near pushout cocone for the given span of theory changes.

Proof. First, it is to establish that we have a lax cocone as \((\varphi_k, i_k); (\theta_k, \emptyset) \leq (\theta_0, \emptyset)\) for \(k = 1, 2.\) Let \((\theta_k', j_k') : (\Sigma_k, E_k) \rightarrow (\Sigma', E'), k = 0, 1, 2\) be a lax cocone for the given span of theory changes. Then let \(\mu\) be the maximal signature morphism such that \(\theta_k; \mu \leq \theta_0, k = 0, 1, 2.\) We define the partial inclusion \(u : \text{Sen}(\mu)E \rightarrow E'\) by \(\text{dom } u = E' \cap \text{Sen}(\mu)E.\) Then \((\theta_k, \emptyset); (\mu, u) = (\theta_k; \mu, \emptyset) \leq (\theta_0', j_0'), k = 0, 1, 2.\)

Now, for any \((\mu', u')\) such that \((\theta_k, \emptyset); (\mu', u') \leq (\theta_0', j_0'), k = 0, 1, 2\) we have that \(\theta_k; \mu' \leq \theta_k', k = 0, 1, 2.\) By the maximality assumption on \(\mu\) it follows that \(\mu' \leq \mu.\) Since \(\text{dom } u' \subseteq \text{Sen}(\mu')E \cap E',\) since \(\mu' \leq \mu,\) by the monotonicity of \(\text{Sen}\) it follows that \(\text{dom } u' \subseteq E' \cap \text{Sen}(\mu)E = \text{dom } u,\) hence \(u' \subseteq u.\)

The results of Propositions 3.10 and 3.11 tell that the established concepts of pushouts in \(\mathfrak{C}_2\)-categories cannot be used for modelling merges of software changes. A new concept is needed for that.

4. Theory blending in \(\mathfrak{C}_2\)-institutions

Now we are in the position to be able to refine Goguen’s approach to conceptual blending within the context of \(\mathfrak{C}_2\)-institutions. This appears as a stepwise process as follows:

1. The input is a consistent span of theory morphisms \(\varphi_1, \varphi_2\) in a \(\mathfrak{C}_2\)-institution \(I,\) which means a consistent span in \(I'.\)
2. Then we consider an appropriate lax cocone for the underlying span of signature morphisms that has weak model amalgamation:

\[
\begin{array}{ccc}
\Sigma_1 & \xrightarrow{\theta_1} & \Sigma_0 \\
\downarrow \theta_0 \geq & \downarrow & \downarrow \theta_0 \geq \\
\Sigma_2 & \xleftarrow{\theta_2} & \Sigma_1 \\
\varphi_1 \downarrow & & \downarrow \varphi_2 \\
\Sigma_0 & & \\
\end{array}
\]

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3. Next we lift it as in Prop. 3.7 to a lax cocone of theory morphisms:

\[ \begin{array}{c}
\Sigma_0 \leq \Sigma_1 \geq \Sigma_2 \\
E_0 \leq E_1 \geq E_2
\end{array} \]

By virtue of Prop. 3.8 it follows that we obtain an oplax cocone of theory morphisms also enjoying weak model amalgamation. Since we started from a consistent span of theory morphisms, it follows that the vertex of the blending cocone – the new theory \((\Sigma, E)\) – is consistent.

This is a very general scheme that has a number of parameters.

- A choice of an appropriate \(\frac{1}{2}\)-institution for modelling the respective concepts as theories, and their translations by theory morphisms.
- What is an ‘appropriate’ lax cocone for the underlying span of signature morphisms is a challenging issue that seems to be difficult to answer at the general level; perhaps seeking for a precise answer at a general level does not even make sense. Some consider that the near pushout solution proposed by Goguen [8] may be too permissive. Though what should be indisputable is the weak amalgamation property for the lax cocone.

References

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