Computing defects associated to bounded domain wall structures: the $\mathbb{Z}/p\mathbb{Z}$ case

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Abstract
We discuss domain walls and defects in topological phases occurring as the Drinfeld center of some fusion category. Domain walls between such phases correspond to bimodules between the fusion categories. Point defects correspond to functors between the bimodules. A domain wall structure consists of a planar graph with faces labeled by fusion categories. Edges are labeled by bimodules. When the vertices are labeled by point defects we get a compound defect. We present an algorithm, called the domain wall structure algorithm, for computing the compound defect. We apply this algorithm to show that the bimodule associator, related to the $O_3$ obstruction of Etingof et al (2010 Quantum Topol. 1 209), is trivial for all domain walls of $\text{Vec}(\mathbb{Z}/p\mathbb{Z})$. In the language of this paper, the ground states of the Levin–Wen model are compound defects. We use this to define a generalization of the Levin–Wen model with domain walls and point defects. The domain wall structure algorithm can be used to compute the ground states of these generalized Levin–Wen type models.

Keywords: Levin–Wen models, fusion categories, topological phases of matter, topological quantum field theory

(Some figures may appear in colour only in the online journal)

Due to their insensitivity to environmental noise, topological phases have promise as materials for encoding quantum information [2–5]. By braiding and fusing the emergent quasiparticle excitations, the encoded information can be manipulated in a robust manner. Such protection from the environment is an important requirement for any large scale quantum device. In many phases, especially those most suited to laboratory realization, the quantum computational power is severely limited. It has become clear that the inclusion of defects can improve the materials from this perspective [5–19]. A complete understanding of defects, both

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invertible and noninvertible, is therefore necessary if we are to utilize topological phases to their fullest. In particular, such an understanding should allow for the computation of fusion of general defects. In this paper, we study non-chiral, two-dimensional, long-range-entangled topological phases with general defect structures.

A defect of a topological phase is a region of positive codimension which differs from the ground state of the underlying bulk phase. Much work has been done on defects in topological phases, for example references [5–19, 20–31]. Although a complete classification for defects exists, it is not computational in nature. In this work, we show how the compound defect associated to a defect network [31] can be computed. Our techniques are not restricted to invertible bimodules or defects.

In reference [32], Levin and Wen (LW) constructed a long range entangled, 2D topological phase of matter associated to any fusion category \( C \). When \( C = \text{Vec}(G) \) for a finite group \( G \), this phase agrees with Kitaev’s quantum double phase defined in reference [2]. For small groups, these Kitaev models are currently of great experimental interest [33, 34]. The excitations of the theory are given by the Drinfeld center \( Z(C) \). In reference [20], Kitaev and Kong demonstrated that \( C - D \) bimodules correspond to domain walls between the corresponding LW phases. The domain walls can equivalently be defined in terms of excitations. In that case, they are described by \( Z(C) - Z(D) \) bimodules which are compatible with the braiding. For our purposes, it is more convenient to work with the input bimodules between the underlying fusion categories. In reference [35], we showed how to compute the tensor product of \( C - D \) bimodules \( M \otimes D N \), corresponding to fusing the domain walls in the LW model. Additionally, we gave an explicit physical interpretation of all bimodules for the case \( C = D = \text{Vec}(\mathbb{Z}/p\mathbb{Z}) \) for prime \( p \). In reference [36], we extended this work to include binary interface defects. We showed how to compute the horizontal fusion (tensor product) and vertical fusion (composition) of these defects. In the case \( C = D = \text{Vec}(\mathbb{Z}/p\mathbb{Z}) \), we provided complete fusion tables and physical interpretations of all binary interface defects.

This paper is a continuation of the work from references [35, 36]. In this paper, we present a new procedure, which we call the domain wall structure algorithm, that computes the compound point defect associated to a domain wall structure once the holes have been filled in with point defects. We use this algorithm to show that all domain wall associators are trivial for \( C = D = \text{Vec}(\mathbb{Z}/p\mathbb{Z}) \). This was not previously known for the noninvertible bimodules over \( \text{Vec}(\mathbb{Z}/p\mathbb{Z}) \). As we discuss below, these associators are related to the \( O_3 \) obstruction [1]. This obstruction plays an important role in gauging as explained in references [37, 38]. From the condensed matter perspective, it would be interesting to find some non-trivial bimodule associators since it would show how a, potentially complicated, defect could be produced from simple domain walls. We now summarize the main result of this work.

**Algorithm 1** (Domain wall structure algorithm). The domain wall structure algorithm decomposes a compound defect into a sum of simple representations. The main steps in the domain wall structure algorithm are as follows:

(a) Construct a compound defect by filling the holes in the domain wall structure with vectors from the corresponding annular category representations, subject to the labels on the internal edges agreeing.

(b) Quotient out the bubble action for each internal cavity.

(c) Compute all relevant idempotent actions on the quotient representation. This lets us decompose the quotient representation into simple annular category representations.

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4 Non-chiral = described by a Drinfeld center.
The remainder of this paper is structured as follows. In section 1 we provide some definitions and preliminaries that are required for the remainder of the manuscript. In section 2, we explain all the data which is required to execute the algorithm and how to execute it by hand. The real virtue of algorithm 1 is that it can be implemented in a computer. The authors have used the algorithm to check the horizontal and vertical fusion tables from reference [36] in the computer. In section 3, we discuss the domain wall associator. We provide an example calculations and physical interpretations. In section 4, we define a lattice model which generalizes the Levin–Wen Lattice model by introducing domain walls and dimension 0 defects. The ground states of this lattice model can be computed using the domain wall structure algorithm. We conclude in section 5.

In appendix A, we recall some important definitions. We also provide the defining data for the $\text{Vec}(\mathbb{Z}/p\mathbb{Z})$ bimodules (reproduced from reference [35]) and their physical interpretations. Appendix B briefly discusses the relationship between the bimodule associator structure and extension theory of a fusion category. In appendix C, we provide tables defining the irreducible representations of the annular categories for $\text{Vec}(\mathbb{Z}/p\mathbb{Z})$. We provide the complete set of bimodule associator defects in appendix D. In the auxiliary material [39], we provide a Mathematica notebook that computes the composition of binary interface defects.

1. Preliminaries

For definitions of fusion categories and bimodules we refer to reference [35] and appendix A. Throughout, we work with a skeletonized category. This means that we assume (without loss of generality) that all isomorphism classes contain a single object. Additionally, all categories are enriched over $\text{Vec}$, the category of $\mathbb{C}$-vector spaces. This means that the Hom spaces (written $C(A,B)$) are vector spaces and composition is bilinear. In particular, the endomorphisms of any object form an algebra. Throughout, $c$ denotes a simple object in $C$, written $c \in C$. Non-simple objects are represented as direct sums of our chosen simple objects.

We primarily use a graphical approach for clarity. For simplicity, we specialize the diagrams to $\text{Vec}(\mathbb{Z}/p\mathbb{Z})$. As discussed in appendix A, the diagrams must be refined if they are applied to a more general setting, for example to categories with nontrivial Frobenius–Schur indicators.

**Definition 2** (Category representation). Let $\mathcal{A}$ be a category. A representation of $\mathcal{A}$ is a functor $V : \mathcal{A} \to \text{Vec}$. This functor is comprised of a vector space $V_a$ for each object $a \in \mathcal{A}$ and a linear map $M_f : V_a \to V_b$ for each morphism $f : a \to b$. The linear maps must satisfy the equations $M_{fg} = M_f \cdot M_g$ and $M_{id_a} = 1_{V_a}$. A basis for a representation consists of a basis for each vector space $V_a$.

**Definition 3** (Annular category). Consider a sectioned annulus where the faces are labeled by fusion categories and the section dividers are labeled by bimodules, for example
Associated to this, we define an annular category $\text{Ann}$ (also known as a sphere category in reference [40]) whose objects are tuples of simple objects from the bimodules and morphisms are string diagrams which can be drawn in the annulus modulo isotopy and local relations.

\[
\begin{array}{c}
\includegraphics{diagram} \\
\text{(2)}
\end{array}
\]

We denote by $\text{Ann}(a, b)$ the space of annular morphisms $a \to b$, where $a$ is the inner labels. Composition $g \circ f$, with $f \in \text{Ann}(a, b)$ and $g \in \text{Ann}(b, c)$ is defined by stacking the diagram $g$ on the outside of the diagram $f$ and reducing using local relations. In our setting, it is important to not isotope the bimodule strings or rotate the annulus.\(^5\)

**Definition 4** (Point defect). Physically, representations of annular categories parameterize point defects at the domain wall junction. The leads to the following definition. Given a decorated sectioned annulus, such as in equation (1), a point defect is a representation of the associated annular category.\(^6\) A simple point defect is an irreducible representation of the annular category.

**Definition 5** (Domain wall structure). A domain wall structure consists of a graph embedded into a disc where the edges don’t have critical points. We label the faces of the graph with fusion categories, and the edges of the graph with bimodules between the corresponding fusion categories. For example,

\[
\begin{array}{c}
\includegraphics{diagram2} \\
\text{(3)}
\end{array}
\]

\(^5\)In order to do this, we would need to specify a consistent set of rigid structures for all the bimodules involved. This can be done [45], but is beyond the scope of this work.

\(^6\)In reference [40], representations of the annular category are called sphere modules.
**Definition 6** (Compound defect). A **compound defect** consists of a domain wall structure, along with an assignment of a point defect (annular category representation) to each vertex. For example,

\[
\text{(Diagram 4 defines an \textit{annular category representation}. The vectors in this representation are constructed by choosing vectors from the representations } \alpha_1, \ldots, \alpha_4 \text{ subject to consistent labeling of the edges with bimodule objects.)}
\]

The annular category action on a vector in a compound defect is
There is also a bubble action for each internal cavity where $d_a$ is the dimension of the object labeling the inserted loop. We must quotient away the bubble actions for every internal cavity because bubbles internal to a cavity should evaluate to the dimension of their labeling object. That is, within the cavity we can create a loop labeled by an object $a$ at a cost of dividing by the dimension of $a$. This loop can then be pushed into the domain walls, as shown in equation (7). Physically, the result of such a transformation should be trivial, since when viewed from far away (i.e. the compound defect is regarded as a representation of the annular category equation (6)) there is no internal action. For $\text{Vec}(\mathbb{Z}/p\mathbb{Z})$ every simple object has dimension 1. This bubble action appears in the definition of the Levin–Wen Hamiltonian from reference [32], and the version presented in section 4.

2. The domain wall structure algorithm

The goal of this section is to explain how to compute the compound defect. We shall demonstrate how the computation works using $\text{Vec}(\mathbb{Z}/p\mathbb{Z})$ as our central example, but everything we describe works in much more generality.

All the annular categories $\text{Ann}$ of interest in this paper are semi-simple, so we can describe their representations as functors $\text{Ann} \to \text{Vec}$ or as primitive idempotent endomorphisms in $\text{Ann}$. In reference [36], we exclusively used the idempotent description. In this paper, we shall use both ways of presenting a representation of $\text{Ann}$. The translation between these notions can be found in lemma 20.

All of the functors corresponding to binary interface defects described in reference [36] have been tabulated in appendix C. The vector space in which the tabulated vectors live can
be read off from the string labels. We refer to reference [35] for the bimodule definitions for \( \text{Vec}(\mathbb{Z}/p\mathbb{Z}) \). Definitions of idempotents corresponding to all 2-string annular categories can be found in reference [36]. We will use the notation (introduced in reference [36]) \( \text{Vec}(\mathbb{Z}/p\mathbb{Z}) \) for a defect called ‘\( x \)’ occurring at the interface of a bimodule \( M \) and a bimodule \( N \).

We shall now demonstrate how an entry of the representation tables (appendix C) is computed.

**Example 7** (Constructing irreducible representations). Consider the defect \( \frac{F_r}{R} \) which was defined in reference [36] by the idempotent

\[
\frac{F_r}{R} = \frac{1}{p} \sum_k \omega^{kx}.
\]  

This idempotent serves two purposes. Firstly, it labels an irreducible representation of \( \text{Ann}_{\mathcal{E},F_r} \). Secondly, the idempotent projects onto the representation it labels. If \( f : (0; +) \rightarrow (m; +) \), then \( f \mapsto f \circ \frac{R}{F_r} \) is an endomorphism of \( \text{Ann}_{\mathcal{E},F_r}((0; +), (m; +)) \). We choose the following basis for the image of this endomorphism

\[
\frac{F_r}{R} = \frac{1}{p} \sum_k \omega^{k(x+rm)}.
\]  

Acting by a general morphism

\[
\omega^{h(x+rm)}
\]  

on the basis vectors gives

\[
\omega^{h(x+rg+vm)}
\]  

as tabulated in appendix C.
2.1. Vertical defect fusion

The simplest case of the domain wall structure algorithm is vertical defect fusion, corresponding to the domain wall structure

\[
\begin{align*}
\text{(12)}
\end{align*}
\]

In reference [36], we computed these vertical defect fusions for all compatible pairs of defects in the \textbf{Vec}(\mathbb{Z}/p\mathbb{Z}) model. These vertical fusions can also be computed using the domain wall structure algorithm. Given a pair of point defects \(\alpha_1, \alpha_2\) (equivalently representations of 2-string annular categories), the compound defect is formed by filling the holes in equation (12) with these defects

\[
\begin{align*}
\text{(13)}
\end{align*}
\]

This forms a (possibly reducible) annular category representation. A vector in this representation looks like

\[
\begin{align*}
\text{(14)}
\end{align*}
\]

If \(\alpha\) is a binary interface defect and \(i_\alpha : (m_{\alpha}, n_{\alpha}) \rightarrow (m, n)\) is the corresponding idempotent from [36], then we have

\[
\begin{align*}
&V \cong \bigoplus_{\alpha} \dim(i_\alpha V(m_{\alpha}, n_{\alpha})) \cdot \alpha. \\
&\text{(15)}
\end{align*}
\]

In representation theory, this is called an isotypic decomposition. The general theory of isotypic decompositions is explained in chapter 4 of reference [41]. We use equation (15) to decompose \(V\) into irreducible representations.

**Example 8.** Consider the vertical defect fusion

\[
\begin{align*}
\text{Example 8.}
\end{align*}
\]

We begin by building the compound representation of the annular category \textbf{Ann}_{R,R} from our chosen defects. It has basis vectors of the form
In reference [36], we define the idempotent

\[ R_{(\alpha, \zeta)} = \frac{1}{p} \sum_{\gamma} \omega^{\gamma \zeta} \]

Recall that this idempotent projects onto the irreducible representation labeled by \((\alpha, \zeta)\). Applying this idempotent to the basis vector

\[ (18) \]

\[ \frac{1}{p} \sum_{\gamma} \omega^{\gamma (\zeta - x - z + r \alpha)} \]

which is zero unless \(\zeta = x + z - r \alpha\). Therefore we have

\[ F_r \left| \begin{array}{c} 0 \\ R_x \end{array} \right. \overset{\otimes_{\alpha}}{\cong} \left. \begin{array}{c} 0 \\ R_{(\alpha, x + z - r \alpha)} \end{array} \right. \]

2.2. Horizontal defect fusion

If we only use annular categories with two bimodule strings, the domain wall structure algorithm only computes vertical composition of defects. To compute more interesting compound defects, we need to include annular categories with three or more bimodule strings. In reference [35], we computed the Brauer–Picard ring for the fusion category \(\text{Vec}(\mathbb{Z}/p\mathbb{Z})\). More precisely, for all pairs of \(\text{Vec}(\mathbb{Z}/p\mathbb{Z})\) bimodules \(M, N\), we computed an explicit isomorphism \(M \otimes_{\mathbb{Z}/p\mathbb{Z}} N \cong \oplus P_i\). These explicit isomorphisms are recorded in the inflation tables in reference [36]. If we take the identity (under vertical fusion) defect on \(M\) and inflate the top or bottom part, we get an idempotent in a three string annular category. The corresponding representations play the role of bimodule trivalent vertices. These representations have been tabulated in appendix C. Now we demonstrate how to compute an entry of this table.
Example 9. Consider the trivial defect on the $X_x$ domain wall

$$X_x|_{(0,0)} = \frac{1}{p} \sum_g x_g^*$$  \hspace{1cm} (21)

If we inflate the top half of this idempotent along the isomorphism $X_x \cong F_q \otimes_{\mathbb{Z}/p\mathbb{Z}} F_r$ where $x = q^{-1} r$, then we get the idempotent

$$\frac{1}{p^2} \sum_{g,k} x_g$$  \hspace{1cm} (22)

Composing morphisms $(0; *, *) \rightarrow (m; *, *)$ on the outside gives us a linear endomorphism of $\text{Ann}_{F_q, F_r, X_x}((0; *, *), (m; *, *))$. We choose the following basis for the image of this endomorphism:

$$\vec{y}_{\vec{m}} := \frac{1}{p^2} \sum_{g,k} \omega^{qmk}$$  \hspace{1cm} (23)

This forms the basis for our representation. Applying

$$\omega^{-c(q(a + m + zb))} \vec{y}_{\vec{m}}$$  \hspace{1cm} (24)

and making the substitutions $g \rightarrow g + b, k \rightarrow k - c$ gives

$$\omega^{-c(q(a + m + zb))} \vec{y}_{\vec{m}}$$  \hspace{1cm} (25)

as recorded in table C3.

Now that we have a collection of 2 and 3 bimodule string annular category representations at our disposal, we can discuss some more complicated domain wall structures and
compute the corresponding compound defects. Of particular interest is the domain wall structure

\[
\text{(26)}
\]

This domain wall structure corresponds to horizontal defect fusion. In the Vec \((\mathbb{Z}/p\mathbb{Z})\) case, we computed all possible horizontal defect fusions in reference [36]. In the following example, we demonstrate how to compute horizontal defect fusion using the domain wall structure algorithm. This example is the first time we encounter the internal cavity bubble action of equation (7), which we need to trivialize to get the correct answer.

**Example 10.** Consider the horizontal fusion \(R \overset{F_q}{\otimes} L\). Using the trivalent vertices corresponding to the isomorphisms \(R \otimes_{\mathbb{Z}/p\mathbb{Z}} L \cong p \cdot T\) and \(F_q \otimes_{\mathbb{Z}/p\mathbb{Z}} L \cong T\), we can construct a (reducible) representation of the category \(\text{Ann}_{T,T}\). It has the basis

\[
\text{(27)}
\]

This representation is too large. It has a \(\mathbb{Z}/p\mathbb{Z}\) action by introducing a bubble into the middle cavity. In order to get a physically relevant representation, we need to quotient away this action to construct the representation of interest. Acting by a \(g\) bubble multiplies the above vector by

\[
\text{(28)}
\]
Therefore, unless \( t = q^{-1}(x + z - \nu) + m \), the vector projects onto zero in the quotient. After taking the quotient, the idempotent \( T |_{(\alpha, \beta)} \) acts as zero unless \( \alpha = q^{-1}(x + z - \nu) \) and \( \beta = c \).

This is exactly the horizontal fusion outcome \( R | F_q \otimes L |_{(x, z)} \approx T |_{(q^{-1}(x + z - \nu), c)} \) which was computed in reference [36].

**Example 11.** Consider the horizontal fusion \( X_l \otimes R | F_0 \). As in example 10, we construct a representation of the category \( \text{Ann}_{F_0, R} \). It has the basis

\[
m + (l - k)s + lr
\]

Acting by a bubble labeled with \( r \) in the internal cavity sends this vector to

\[
m + (l - k)s + lr
\]

Therefore, if we relabel \( t' = m + (l - k)s + lr \) and \( m' = m + kr \) we have the following basis when we quotient away the bubble action

\[
m \otimes 0
\]
If we want to act by the idempotent $R_{F_0} |_{\zeta}$, we must have $t = 0$. The result of applying the projection is nonzero if and only if $\zeta = z$. Since $m$ is arbitrary, we have $X \otimes R_{F_0} |_{\zeta} \cong p$. 

### Example 12.

Consider the horizontal fusion $F_r F_0 \otimes F_t T$. First we construct the compound representation of $\text{Ann}_{L, X_r^{-1}}$, 

![Diagram](image)

Acting by an $s$ bubble sends this vector to 

![Diagram](image)

which forms a basis for the quotient. To apply $X_r^{-1} L$, we must have $m = n = 0$. Since $\alpha \in \mathbb{Z}/p\mathbb{Z}$ (the representation corresponding to $F_r F_0$) is $p$-dimensional, we have $F_r F_0 \otimes F_t T \cong p \cdot X_r^{-1} L$. 

### 3. Bimodule associators

Now that we have seen that we must quotient away the bubble actions corresponding to internal cavities, we have seen everything needed to compute the compound defects corresponding to arbitrarily complex domain wall structures. Another interesting example is the following compound defect.
We shall call this domain wall structure the bimodule associator for the triple \(M, N, P\). If this defect projects onto nontrivial point defects, it indicates an obstruction to defining an extension (as described in definition 22). For \(M, N, P\) invertible, this is closely related to the \(O_3\) obstruction of reference [1] being nontrivial (see appendix B). From a physics viewpoint, this obstruction means we cannot gauge the defects [26, 37, 38]. We note that this obstruction does not compute a generalized \(F\)-symbol (generalizing equation (A2)) for defects. An algorithm for this computation can be found in our later work reference [42].

We now provide an example calculation of a bimodule associator. The full set of associators (all trivial) can be found in table D1.

**Example 13.** Let us compute the bimodule associator \([F_q, L, X_l]\). First, we can construct the following representation out of our trivalent vertices

\[
(t, n) \quad \text{lg} \quad (t, (n+1)) \quad (n, n) \quad (t, l_n) \quad (t, ln) \quad (m, n)
\]

We need to quotient out the bubble actions from both of the cavities. Acting by an \(lg\) bubble in the top cavity sends the vector to

\[
(t, n) \quad 0 \quad (t, ln) \quad (m, n)
\]
Acting by an $h$ bubble in the bottom cavity multiplies this vector by $\omega^{m(m-1)}$. So the vector is projected to zero in the quotient unless $m = t$. Therefore we have the following basis for the quotient

$$\{(m, n)\}$$

To apply $T\bigg|_{(\alpha, \beta)}\bigg|_{(0,0)}$, we must have $m = n = \alpha = \beta = 0$. Therefore the compound defect is $T\bigg|_{(0,0)}$.

**Example 14.** We can also compute bimodule associators using the physical interpretations of the bimodules from reference [35] (table A2). The parameters $\mu$ and $\nu$ in our 3-string annular category representations physically correspond to the presence of a non-condensable anyon at the corner. Recall that the rough boundary condenses the $e$ anyons and the smooth boundary condenses the $m$ anyons.
since the internal disc must contain 0 anyons that cannot be fused into the boundary. Therefore, this associator is
\[
\delta^{\nu}_{\mu} \left[ \frac{R}{R} \Psi^{\nu}_{\mu} \right]_{(0,0)}.
\]

4. A defect Levin–Wen Lattice model

A particularly important application of the domain wall structure algorithm is finding the ground states of Levin–Wen Hamiltonians [32], which are compound defects. This requires a slight modification of the usual definition of these models. In this section, we show how a Levin–Wen type model [32], complete with defects, can be constructed using annular category representations. For invertible bimodules, a related model was previously studied in reference [43], however the work there did not allow noninvertible bimodules or point defects to be included.

For simplicity, we restrict to the following lattice

\begin{equation}
(40)
\end{equation}

with either periodic boundary conditions, or an infinite lattice. The model described here has degrees of freedom both on edges and vertices, indicated by gray and black dots respectively.

Each face of the lattice is labeled with a fusion category \( C_f \), edges are labeled with bimodules \( M_e \) between the appropriate \( C_f \). Each vertex is labeled with an (irreducible) representation \( V^z \) of the corresponding annular category (equivalently a (primitive) idempotent of the annular category). Recall that this representation consists of a set of vector spaces \( V_{m_1,m_2,m_3} \), where the \( m_i \) are the edge labels. A partial list of irreducible 3-bimodule annular category representations are provided in tables \( C2 \) and \( C3 \). All other irreducible representations can be obtained by composing with the 2-bimodule vertex representations from table \( C1 \) in the following way

\begin{equation}
(41)
\end{equation}

A basis for the edge Hilbert space is given by the set of objects \( m \in M_e \). A basis for the vertex degree of freedom is given by a basis for the assigned representation.

If \( \psi \) is a basis state on the lattice, in a neighborhood of the vertex \( z \) it looks like

16
where $\psi \in \mathcal{V}_{m_1, m_2, m_3}$, $\psi_1, m_i \in \mathcal{M}_i$. The Hamiltonian for this model consists of two parts. The vertex operator $H_z$ projects onto states where $(\psi_1, \psi_2, \psi_3) = (m_1, m_2, m_3)$, thereby ensuring the fusion rules are obeyed. In a neighborhood of the face $f$, the state $\psi$ looks like

Fix $g \in \mathbb{Z}/p\mathbb{Z}$. The operator acts $H_{f, g}$ maps this to:
An example of the computation of an explicit face term is shown in equation (28). Since the local degrees of freedom which live on the vertices are vectors in annular category representations, they can absorb the blue strings (as shown in tables C2 and C3). We define
\[ H_f = \frac{1}{p} \sum_{g} H_{f,g}. \] (45)

The Hamiltonian for the model is
\[ \sum_{z} (1 - H_z) + \sum_{f} (1 - H_f). \] (46)
where \( z \) ranges over vertices and \( f \) ranges over faces.

4.1. Ground state configurations

For clarity, we now show how the ‘string-net’ ground state arises from this construction. Restrict to the bimodule \( X_1 \) for \( \mathbb{Z}/p\mathbb{Z} \). From tables C2 and C3, all vector spaces arising in the annular category representation at a trivertex are one dimensional. We can therefore label the vertex degrees of freedom by the edge labels. A basis for the vertex Hilbert space is given by
\[ \left\{ \begin{array}{c}
  a + b \\
  a \\
  b
\end{array} \right\}, \quad a, b \in \mathbb{Z}/p\mathbb{Z}. \] (47)

The edge terms in the Hamiltonian force the edge labels to be consistent. The face terms fluctuate between configurations of edge labelings. For example, in the \( \mathbb{Z}/2\mathbb{Z} \) case
Excitations in this model are created by mismatches along the edges, for example the configuration

\begin{equation}
\text{(49)}
\end{equation}

violates two vertex terms, corresponding to a pair of particles (anyons). By inserting phases along the string, plaquette terms can also be violated.

Ground states with defect lines and points can also be described in this model. Consider the following arrangement of defect lines

\begin{equation}
\text{(50)}
\end{equation}

where the blue, red, green lines are the bimodules $F_0$, $T$ and $L$ respectively. The remaining edges are labeled by the $X_1$ bimodule, corresponding to the bulk theory. Recall from reference [35] (reproduced in table A2) that these domain walls correspond to boundaries in the theory. At each vertex, we assign the representation from tables C2 and C3.

In addition to the vertex basis from equation (47), we now require bases for the new vertices (see tables C2 and C3). As before, excitations can be created by violating either vertex or face terms, both in the bulk and in the vicinity of the boundary.

5. Remarks

In this work, we have described a framework for computing the compound defect associated to a domain wall structure. The algorithm described is agnostic to the invertibility of the bimodules and point defects forming the structure. Using this algorithm, we have shown how the
fusion (both vertical and horizontal) of defects are expressed as domain wall structures, and how the results of reference [35] can be replicated in this new, computer-friendly manner. Additionally, we have applied our algorithm to show that the domain wall associators for all bimodules over $\text{Vec}(\mathbb{Z}/p\mathbb{Z})$ are trivial.

Although we have specialized to $\text{Vec}(\mathbb{Z}/p\mathbb{Z})$ for this work, the ideas described here are not restricted to this class of fusion categories. Of particular interest is the category $\text{Vec}(\mathbb{Z}/2\mathbb{Z}) \times \text{Vec}(\mathbb{Z}/2\mathbb{Z})$. The Levin–Wen model associated to this is called the ‘color code’ in quantum computing, and is important due to the group of fault tolerant gates that can be implemented [11, 44]. The large number of bimodules of this model (270) make a computer-implementable method, such as that outlined here, necessary to study the defects. We also expect these techniques to be useful for $\text{Vec}(G)$ when $G$ is not abelian, and other non-abelian fusion categories.

We have shown how the domain wall associators can be computed in this framework. These associators are closely related to the $O_3$ obstruction of reference [1]. When this obstruction vanishes (as is the case for $\text{Vec}(\mathbb{Z}/p\mathbb{Z})$), a further obstruction, called $O_4$ in reference [1], can arise. This obstruction is related to natural isomorphisms of defects. It would be extremely useful if the techniques developed in this work can be extended to include this data.

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**Appendix A. A gentle review of bimodule categories over a fusion category**

To aid the unfamiliar reader, in this appendix we provide a sketch of the definitions of some of the mathematical structures used in this paper. While these definitions are sufficient for studying the $\text{Vec}(\mathbb{Z}/p\mathbb{Z})$ case, we note that they must be refined if the algorithms described here are to be used for more general fusion categories. In particular, care must be taken when the fusion categories have nontrivial Frobenius–Schur indicators. In the general case, all the strings must be oriented and we need to be more careful with rigid structures (see definition 15). More complete definitions can be found in reference [46].

The particle types in a long range entangled (2 + 1)D topological phase are labeled by isomorphism classes of simple objects in a modular tensor category [47]. The Drinfeld center $Z(C)$ of a fusion category $C$ is a modular tensor category. The fusion category $C$ encodes a local Hamiltonian which can be used to construct a gapped commuting projector Hamiltonian with anyonic excitations $Z(C)$ as described by Levin and Wen in reference [32]. It is important to not think of the objects in $C$ as anyons. The objects in $C$ are local degrees of freedom in the Levin–Wen lattice model.

Throughout this paper, all categories are enriched over $\text{Vec}$, the category of $C$-vector spaces. This means that the Hom spaces are vector spaces and composition is bilinear. In particular,
Table A1. Data for all \( \text{Vec}(\mathbb{Z}/p\mathbb{Z}) \to \text{Vec}(\mathbb{Z}/p\mathbb{Z}) \) bimodules. \( q \in H^2(\mathbb{Z}/p\mathbb{Z}, U(1)) \cong \mathbb{Z}/p\mathbb{Z} \). Bimodules below the thick line are invertible. Reprinted by permission from Springer Nature Customer Service Centre GmbH: Springer Nature, *Commun. Math. Phys.* [35], Copyright 2019.

| Bimodule label | Left action | Right action | Associator |
|----------------|-------------|--------------|------------|
| \( T \)       | \( (a + g, b) \) | \( (a, b + g) \) | \( (a + g, b + h) \) |
|                | \( g(a, b) \)   | \( g(a, b) \)   | \( g(a, b) h \)   |
| \( L \)       | \( a + h \)     | \( a + h \)     | \( a + h \)     |
| \( R \)       | \( g(a, h) \)   | \( g(a, h) \)   | \( g(a, h) \)   |
| \( F_0 \)     | \( * \)        | \( * \)        | \( * \)        |
| \( X_k \)     | \( a + g \)     | \( a + kg \)    | \( g + a + kh \) |
| \( F_q \)     | \( g \)        | \( g \)        | \( e^{2\pi i q g h} \) |

the endomorphisms of any object form an algebra. Throughout, \( c \) denotes a simple object in \( \mathcal{C} \) and \( \mathcal{C}(A, B) \) the space of morphisms \( A \to B \).

**Definition 15** (Fusion category). A *tensor category* \( \mathcal{C} \) is a category \( \mathcal{C} \) equipped with a functor \( - \otimes - : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \), a natural isomorphism \( (- \otimes -) \otimes - \cong - \otimes (- \otimes -) \) called the associator and a special object \( 1 \in \mathcal{C} \) which satisfy the pentagon equation and unit equations respectively. These can be found on page 22 of reference [48]. If \( \mathcal{C} \) is semi-simple, using the string diagram notation as explained in reference [46], a vector in \( \mathcal{C}(a \otimes b, c) \) can be represented by a trivalent vertex:

\[
\begin{align*}
\begin{array}{c}
\cdot \\
\downarrow \\
\cdot
\end{array}
\end{align*}
\]

(A1)
Table A2. Domain walls on the lattice corresponding to bimodules. Reprinted by permission from Springer Nature Customer Service Centre GmbH: Springer Nature, Commun. Math. Phys. [35], Copyright 2019.

| Bimodule label | Domain wall | Action on particles |
|----------------|-------------|---------------------|
| \( T \)       |             | Condenses \( e \) on both sides |
| \( L \)       |             | Condenses \( m \) on left and \( e \) on right |
| \( R \)       |             | Condenses \( e \) on left and \( m \) on right |
| \( F_0 \)     |             | Condenses \( m \) on both sides |
| \( X_k \)     |             | \( X_k : e^a m^b \mapsto e^{k a} m^{k^{-1} b} \) |
| \( F_q = F_1 X_q \) |       | \( F_q : e^a m^b \mapsto e^q m^a \) |

If we choose bases for all the vector spaces \( \mathcal{C}(a \otimes b, c) \), then the associator can be represented as a tensor \( F \), where

\[
\alpha \beta \varepsilon \quad \gamma \quad \delta = \sum_{(f, \mu, \nu)} \left[ F_{\delta}^{\varepsilon \mu \nu} \right]_{(\gamma, \alpha, \beta)} \left[ F_{\delta}^{\varepsilon \mu \nu} \right]_{(\gamma, \alpha, \beta)}
\]

\( (A2) \)

where \( \alpha \in \mathcal{C}(a \otimes b, c), \quad \beta \in \mathcal{C}(c \otimes d), \quad \mu \in \mathcal{C}(b \otimes c, f) \) and \( \nu \in \mathcal{C}(a \otimes f, d) \) are basis vectors.

An object is simple if there are no non-trivial sub-objects. Semi-simple means that every object in \( \mathcal{C} \) is a direct sum of simple objects. Rigid is a technical condition defined on page 40 of reference [48] which implies that objects in \( \mathcal{C} \) have duals and they behave like duals in the category of vector spaces. A fusion category is a semi-simple rigid tensor category \( \mathcal{C} \) with a finite number of isomorphism classes of simple objects and a simple unit. When doing fusion category computations, it is convenient to skeletonize the category. This amounts to choosing a single object to represent each isomorphism class. We refer the reader to exercise 2.8.8 of reference [48] for more details.

A subtle but important issue is the interaction between rigid structures and the graphical calculus used to describe morphisms in a fusion category. In order for the graphical calculus to fully capture the rigid structure, all strings need to be oriented. If the Frobenius–Schur indicators associated to the rigid structure are trivial, then the orientations can be ignored. This is case for the fusion category \( \text{Vec}(G) \), which is why we can ignore the string orientations.

**Example 16** (\( \text{Vec}(G) \)). Let \( G \) be a finite group. The fusion category \( \text{Vec}(G) \) has simple objects the elements \( g \in G \). The tensor product is \( g \otimes h \cong gh \). Trivalent vertices

\[
\begin{align*}
\alpha & \quad \beta \\
\gamma & \quad \delta
\end{align*}
\]

\( (A3) \)
Table A3. Primitive idempotents for 2-string annuli of all domain walls, corresponding to defects. For \( p = 2 \), \( \Theta_{x}(g) = (-1)^{w_1 \cdot p^0} \), while for odd \( p \), \( \Theta_{x}(g) = \omega^{x_1 \cdot p^0 + 2^{-1}} \), where \( 2^{-1} \) is the modular inverse of 2. Reprinted from [36], with the permission of AIP Publishing.

| \( p \) | \( x \) | \( \Theta_{x}(g) \) |
|-------|-------|------------------|
| 2     | 0     | \( -1^{x_1} \)   |
| 2     | 1     | \( \omega^{x_1 + 2^{-1}} \) |

where \( 2^{-1} \) is the modular inverse of 2.
can be chosen so that

\[
\begin{aligned}
g h k \\
h k
\end{aligned}
= \begin{aligned}
g h \\
k
\end{aligned}
\tag{A4}
\]

**Definition 17** (Bimodule category). Let \( C, D \) be fusion categories. A *bimodule category* \( C \rtimes M \rtimes D \) is a semi-simple category equipped with functors \(- \lhd - : C \otimes M \to M\) and \(- \rhd - : M \otimes D \to M\) and three natural isomorphisms

\[
\begin{aligned}
- \lhd (- \lhd -) & \cong (- \otimes -) \lhd - \\
(- \lhd -) \rhd - & \cong - \lhd (- \otimes -) \\
- \rhd (- \rhd -) & \cong (- \lhd -) \rhd -
\end{aligned}
\tag{A5}
\]

If we choose bases for the Hom spaces \( M(c \rhd m, n) \) and \( M(m \lhd d, n) \), then these natural isomorphisms can be represented as tensors:

\[
\begin{aligned}
\rho = \sum_{(q, \mu, \nu)} \left[ L_{n}^{abm} \right]_{(p, \alpha, \beta)} \left( q, \mu, \nu \right) \quad \tag{A8}
\end{aligned}
\]

\[
\begin{aligned}
\alpha = \sum_{(q, \mu, \nu)} \left[ F_{n}^{abm} \right]_{(p, \alpha, \beta)} \left( q, \mu, \nu \right) \quad \tag{A9}
\end{aligned}
\]

\[
\begin{aligned}
\beta = \sum_{(q, \mu, \nu)} \left[ R_{n}^{abm} \right]_{(p, \alpha, \beta)} \left( q, \mu, \nu \right) \quad \tag{A10}
\end{aligned}
\]

These natural isomorphisms must be chosen to be compatible with the \( F \) tensors of the fusion categories \( C \) and \( D \). The compatibility equations can be found in definition 7.1.7 of [48].

**Example 18** (\( \text{Vec}(\mathbb{Z}/p\mathbb{Z}) \)-bimodules). In reference [35], following reference [48], we gave a complete list of the simple \( \text{Vec}(\mathbb{Z}/p\mathbb{Z}) \)-bimodules (reproduced in table A1). In this example, we can always gauge away the left and right associators (by choosing an appropriate basis for the Hom spaces, we can ensure \( L \) and \( R \) are +1), so we only tabulate the center associator in table A1. In reference [35], we computed the physical interpretations for each of these bimodules. This data is reproduced in table A2.

**Definition 19** (Bimodule functors). Let \( C, D \) be fusion categories, and let \( M, N \) be \((C, D)\)-bimodules. A \((C, D)\)-bimodule functor \( F : M \to M \) is a functor \( M \to N \), together with two natural isomorphisms

\[
A : F(- \rhd -) \cong - \lhd F(-) \quad \tag{A11}
\]
If we choose bases for all the Hom spaces, then these natural isomorphisms can be represented as tensors:

$$B : F(- \langle - - \rangle) \cong F(- \langle - - \rangle)$$

(A12)

These natural isomorphisms must be chosen to be compatible with the $F$ tensors of the fusion categories $C$ and $D$. The compatibility equations can be found in definition 7.2.1 of [48].

**Lemma 20.** As explained by Morrison and Walker in chapter 6 of reference [40], bimodule functors correspond to representations of annular categories $\text{Ann}$ from definition 3.

Given a primitive idempotent $i : a \to a$ in $\text{Ann}$, the corresponding functor $V^i : \text{Ann} \to \text{Vec}$ is defined as follows. Denote by $\text{Ann}(a, x)$, the space of annular morphisms $a \to x$. The vector space $V^i_x$ associated to $x$ is the image of the projection defined by $i$

$$V^i_x = \text{Span} \{ f \circ i \mid f \in \text{Ann}(a, x) \}$$

(A15)

The vector space $V^i_x$ is nontrivial exactly when there is a nontrivial morphism $a \to x$ of the form $f \circ i$. Where it does not cause confusion, we suppress the upper label $i$.

**Example 21** (Vec $(\mathbb{Z}/p\mathbb{Z})$-bimodule functors). We use the notation $\frac{M}{N}$ to refer to primitive idempotents in Vec $(\mathbb{Z}/p\mathbb{Z})$-bimodule annular categories. All the primitive idempotents are reproduced in table A3.

**Appendix B. Relationship to extension theory**

The computations described in this paper are closely related to extension theory as described in reference [1].

**Definition 22.** We define the Brauer–Picard 3-category $\text{BPR}$ as follows: objects are fusion categories, 1-morphisms are bimodules, 2-morphisms are bimodule functors and 3-morphisms are natural transformations.

Let $M$ be a finite monoid and $\mathcal{M}$ the tensor category $\text{Vec}(M)$ of $M$-graded vector spaces considered as a 3-category with a single object $*$ and only identity 3-morphisms. Then extension data is exactly a 3-functor $\mathcal{M} \to \text{BPR}$. Such a 3-functor contains the following data:

- A fusion category $* \mapsto \mathcal{C}$.
- A $\mathcal{C} - \mathcal{C}$ bimodule $g \mapsto M_g$ for each element $g \in M$. 

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• Annular category representations

\[ \begin{array}{c}
g \quad \downarrow \quad h \\
g \quad \downarrow \\
h \\
gh \end{array} \rightarrow \Lambda_{g,h} \in \text{Rep} \begin{pmatrix} M_{gh} \\ M_g \\ M_h \end{pmatrix}, \quad (B1) \]

\[ \begin{array}{c}
g \quad \downarrow \\
g \quad \downarrow \quad h \quad \downarrow \\
gh \end{array} \rightarrow V_{g,h} \in \text{Rep} \begin{pmatrix} M_g \\ M_{gh} \\ M_h \end{pmatrix}. \quad (B2) \]

In order for the 3-functor to be defined at this level, the following diagrams must map to the identity defect

\[ \begin{array}{c}
g \quad \downarrow \\
g \quad \downarrow \quad h \quad \downarrow \\
gh \end{array} \rightarrow \Lambda_{g,h,k} \quad \Lambda_{gh,k} \]

\[ \begin{array}{c}
g \quad \downarrow \\
gh \end{array} \rightarrow M_g \quad \Lambda_{g,h} \]

This is closely related to the vanishing of the \( O_3 \) obstruction from reference [1]. For invertible bimodules \( g, h, k \), the annular category representations equation B define bimodule equivalences \( M_g \otimes_c M_h \cong M_{gh} \) (In section 8 of reference [1], these equivalences are called \( M_{g,h} \)). The domain wall structure equation (B3) then corresponds to \( T_{g,h,k} \) in reference [1].

There are further obstructions called \( O_4 \) in reference [1], which appear when scrutinizing the 3-morphisms. It is not clear if these obstructions can be easily expressed in our framework.

A good introduction to extension theory is reference [49] by Edie-Michell.

**Appendix C. Representation tables**

This appendix records the irreducible representations for each annular category. In the following tables, we record the following data for each irreducible representation:

• A chosen basis for the representation.
• The action of a generating set of annular morphisms.
Table C1. Bivalent representation tables. Upper table shows the chosen basis vectors for each bimodule pair. Rows correspond to lower bimod., columns label upper bimod. Lower table records the action of the annulus equation (C1) on the basis.

|      | $T$   | $L$   | $R$   | $R_0$ | $X_1$ | $F_2$ |
|------|-------|-------|-------|-------|-------|-------|
| $T$  | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ |
| $L$  | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ |
| $R$  | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ |
| $R_0$| $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ |
| $X_1$| $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ |
| $F_2$| $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ | $\omega^{m,n}$ |
Table C2. 2:1 trivalent basis vectors and annular action. Gray column denotes the top bimodule, left column indicates the lower left $\otimes_{Vec(Z/pZ)}$ right bimodules, central column records the chosen basis and right column records the action of equation (C2).

| Decomposition | Basis vectors | Action |
|---------------|--------------|--------|
| $T \otimes_{Vec(Z/pZ)} T$ | $\omega$ | $\omega$ |
| $T \otimes_{Vec(Z/pZ)} L$ | $\omega$ | $\omega$ |
| $R \otimes_{Vec(Z/pZ)} X_l$ | $\omega^{-1}$ | $\omega^{-1}$ |
| $R \otimes_{Vec(Z/pZ)} T$ | $\omega^{-1}$ | $\omega^{-1}$ |
| $R \otimes_{Vec(Z/pZ)} L$ | $\omega^{-1}$ | $\omega^{-1}$ |
| $R \otimes_{Vec(Z/pZ)} F_0$ | $\omega^{-1}$ | $\omega^{-1}$ |
| $X_0 \otimes_{Vec(Z/pZ)} T$ | $\omega^{-1}$ | $\omega^{-1}$ |
| $F_0 \otimes_{Vec(Z/pZ)} T$ | $\omega^{-1}$ | $\omega^{-1}$ |
| $L \otimes_{Vec(Z/pZ)} T$ | $\omega^{-1}$ | $\omega^{-1}$ |
| $L \otimes_{Vec(Z/pZ)} L$ | $\omega^{-1}$ | $\omega^{-1}$ |
| $L \otimes_{Vec(Z/pZ)} X_l$ | $\omega^{-1}$ | $\omega^{-1}$ |
| $F_l \otimes_{Vec(Z/pZ)} T$ | $\omega^{-1}$ | $\omega^{-1}$ |
| $F_l \otimes_{Vec(Z/pZ)} L$ | $\omega^{-1}$ | $\omega^{-1}$ |
| $F_l \otimes_{Vec(Z/pZ)} F_0$ | $\omega^{-1}$ | $\omega^{-1}$ |
| $X_0 \otimes_{Vec(Z/pZ)} X_l, x = kl$ | $\omega^{-1}$ | $\omega^{-1}$ |
| $X_0 \otimes_{Vec(Z/pZ)} F_0, \ z = q^{-1}$ | $\omega^{-1}$ | $\omega^{-1}$ |
| $F_0 \otimes_{Vec(Z/pZ)} X_l, y = k^{-1}$ | $\omega^{-1}$ | $\omega^{-1}$ |
| $F_0 \otimes_{Vec(Z/pZ)} F_0, \ y = q^{-1}$ | $\omega^{-1}$ | $\omega^{-1}$ |
Table C3. 1:2 trivalent basis vectors and annular action. Gray column denotes the bottom bimodule, left column indicates the upper left $\otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})}$ right bimodules, central column records the chosen basis and right column records the action of equation (C3).

| Decomposition | Basis vectors | Action |
|---------------|---------------|--------|
| $T \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} T$ | $T$ | $T \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} T$ |
| $T \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} L$ | $L$ | $T \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} L$ |
| $T \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l$ | $X_l$ | $T \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l$ |
| $R \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} T$ | $T$ | $R \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} T$ |
| $R \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} L$ | $L$ | $R \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} L$ |
| $R \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l$ | $X_l$ | $R \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l$ |
| $X_l \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} R$ | $R$ | $X_l \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} R$ |
| $F_0 \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$ | $F_0$ | $F_0 \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$ |
| $L \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} L$ | $L$ | $L \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} L$ |
| $L \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l$ | $X_l$ | $L \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l$ |
| $F_0 \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$ | $F_0$ | $F_0 \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$ |
| $F_0 \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l$ | $X_l$ | $F_0 \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l$ |
| $X_l \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$ | $F_0$ | $X_l \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$ |
| $X_l \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l, x = k_l$ | $X_l$ | $X_l \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l, x = k_l$ |
| $F_0 \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0, x = q^{-1}$ | $F_0$ | $F_0 \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0, x = q^{-1}$ |
| $F_0 \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l, y = q_l$ | $X_l$ | $F_0 \otimes_{\text{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l, y = q_l$ |

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For the bivalent vertices, we have tabulated the action by

\[\text{(C1)}\]

For the two-down-one-up trivalent vertices, we have tabulated the action by

\[\text{(C2)}\]

For the one-down-two-up trivalent vertices, we have tabulated the action by

\[\text{(C3)}\]

Appendix D. Bimodule associator tables

\[\text{(D1)}\]
Table D1. Bimodule associator tables. All associators are trivial. Rows, columns, table label (top left) label M, N, P respectively in equation (D1). Highlighted cells correspond to associators of invertible bimodules, the only previously known associators for this model.

| M   | N   | P   | T   | L   | R   | K   | S   |
|-----|-----|-----|-----|-----|-----|-----|-----|
| H_m | H_n | H_p | 1   | 1   | 1   | 1   | 1   |
| H_m | H_n | H_p | 1   | 1   | 1   | 1   | 1   |
| H_m | H_n | H_p | 1   | 1   | 1   | 1   | 1   |
| H_m | H_n | H_p | 1   | 1   | 1   | 1   | 1   |

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