The Γ-limit of traveling waves in the FitzHugh-Nagumo system

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Abstract: Patterns and waves are basic and important phenomena that govern the dynamics of physical and biological systems. A common theme in investigating such systems is to identify the intrinsic factors responsible for such self-organization. The Γ-convergence is a well-known technique applicable to variational formulation in studying the concentration phenomena of stable patterns. A geometric variational functional associated with the Γ-limit of standing waves of FitzHugh-Nagumo system has recently been built. This article studies the Γ-limit of traveling waves. To the best of our knowledge, this is the first attempt to expand the scope of applicability of Γ-convergence to cover non-stationary problems.

Key words: Γ-convergence, FitzHugh-Nagumo, geometric variational problem, traveling front, traveling pulse.

AMS subject classification: 35K40, 35B08, 49J45

1 Introduction

Patterns and waves are basic and important phenomena that govern the dynamics of physical and biological systems. These can be seen in, for instance, morphological phases in block copolymers, skin pigmentation in cell development and semiconductor gas-discharge systems. In the investigation of such systems, a common theme is to identify the intrinsic factors responsible for such self-organization. For the reaction-diffusion systems, self-organized patterns have not only been found in the neighborhoods of Turing’s instability [32], but recent works [6,7,8,9,14,15,13,17,18,30,29]...
exhibited that some patterns and waves possess localized spatial structures. In fact localized waves in reaction-diffusion systems are commonly observed, referred to as dissipative solitons \cite{4, 20, 24, 27} in physical literature.

The FitzHugh-Nagumo model, which was originally derived as an excitable system for studying nerve impulse propagation, is now of great interest to the scientific community as breeding grounds for patterns, traveling waves, and other localized structures. It has been extensively studied as a paradigmatic activator-inhibitor system for patterns generated from homogeneous media destabilized by a spatial modulation. These patterns are robust in the sense that they are stable and exist for a wide range of parameters.

The $\Gamma$-convergence \cite{5, 23} is a well-known technique applicable to variational formulation in studying the concentration phenomena of stable patterns. When a stationary FitzHugh-Nagumo system is equipped with an appropriate scaling on the parameters, we are led to studying a geometric variational problem \cite{10, 11} with a $\Gamma$-limit energy functional defined by

$$J_D(\Omega) = P_D(\Omega) - \alpha |\Omega| + \frac{\sigma}{2} \int_{\Omega} N_D(\Omega) dx,$$

where $\Omega$ is a measurable subset of the domain $D \subset \mathbb{R}^N$. Denoted by $|\Omega|$ its Lebesgue measure and $P_D(\Omega)$ the perimeter of $\Omega$ in $D$. In case $\Omega$ is of class $C^1$, $P_D(\Omega)$ is the area of the part of the boundary of $\Omega$ that is inside $D$. The integral term represents nonlocal influence with $N_D(\Omega)$ satisfying the linear equation

$$-\Delta N_D(\Omega) + N_D(\Omega) = \chi_{\Omega}; \quad \partial_\nu N_D(\Omega) = 0. \tag{1.2}$$

Here $\partial_\nu$ is the outward normal derivative.

We will build a geometric variational functional associated with traveling wave investigation in later sections. To understand this derivation as opposed to that of the stationary problem, we give a brief review on the connection between (1.1) and the FitzHugh-Nagumo system. Consider

$$u_t = \epsilon^2 \Delta u - u \left( u - \frac{1}{2} \right) (u - 1) + \epsilon \alpha - \epsilon \sigma v, \tag{1.3}$$

$$v_t = \Delta v - v + u. \tag{1.4}$$

With $\alpha > 0$ and $\sigma > 0$ being fixed, a small $\epsilon$ identifies a range where a singular limit will emerge. Recall that $u$ acts as an activator and $v$ is the inhibitor. Physically $\alpha$ measures the driving force towards a non-trivial state while $\sigma$ represents the stabilizing inhibition mechanism. Their competition leads to interesting dynamics and the emergence of patterns. In dealing with stationary solutions of (1.3)-(1.4) both $u_t$ and $v_t$ vanish.
Solving (1.4) for \( v \) in terms of \( u \) and denoted this solution by \( v = L_D u \), we see that (1.3) becomes

\[- \epsilon^2 \Delta u + u \left( u - \frac{1}{2} \right) (u - 1) - \epsilon \alpha + \epsilon \sigma L_D u = 0. \tag{1.5}\]

The solutions of (1.3) are the critical points of

\[ I_{D, \epsilon}(u) = \int_D \left( \frac{\epsilon^2}{2} |\nabla u|^2 + \frac{u^2(u - 1)^2}{4} - \epsilon \alpha u + \frac{\epsilon \sigma}{2} u L_D u \right) dx. \tag{1.6} \]

When \( D \) is bounded, \( \epsilon^{-1} I_{D, \epsilon} \) \( \Gamma \)-converges in \( L^1(D) \) to

\[ \frac{\sqrt{2}}{12} P_D(\Omega) - \alpha |\Omega| + \frac{\sigma}{2} \int_{\Omega} N_D(\Omega) dx, \tag{1.7} \]

a functional equivalent to (1.1). In case \( D = \mathbb{R}^N \), a ball shaped stationary set of \( J_{\mathbb{R}^N} \) is referred as a bubble or an entire solution.

Front and pulse are localized waves, the latter is manifest as a small spot. In the past, \( \Gamma \)-convergence has been employed to establish many interesting results for stable patterns; however to the best of our knowledge, this tool has not been utilized to treat traveling waves. We make attempt toward this goal, starting with the investigation of planar traveling wave solutions of the following FitzHugh-Nagumo system:

\[
\begin{align*}
    u_t &= \Delta u + \frac{1}{4} \left( f_\epsilon(u) - \epsilon \sigma v \right), \\
    v_t &= \Delta v + u - \gamma v,
\end{align*}
\]

where \( f_\epsilon(\xi) = -\xi(\xi - \beta_\epsilon)(\xi - 1) \), \( \beta_\epsilon = 1 - \frac{\alpha \epsilon}{\sqrt{2}} \) and \( \alpha, \gamma, \sigma > 0 \). As a remark, with \( d = \epsilon^2 \) and \( \gamma = 1 \), it can be shown as \( \epsilon \to 0 \), its stationary problem leads to the same geometric variational functional (1.7), if we replace \( \alpha \) by \( \alpha/6 \sqrt{2} \) and work on a bound domain.

A planar traveling wave solution is of the from \((u(x - ct), v(x - ct))\); that is, the wave moves with a speed \( c \) and keeps the same profile along the moving coordinates. To treat traveling wave using \( \Gamma \)-convergence, the ansatz \((u(c(x - ct)), v(c(x - ct)))\), proposed in [21], is more appropriate and will be adopted in the paper. We are thus led to deal with

\[
\begin{align*}
    dc^2 u_{xx} + dc^2 u_x + f_\epsilon(u) - \epsilon \sigma v &= 0, \tag{1.9} \\
    c^2 v_{xx} + c^2 v_x - \gamma v + u &= 0, \tag{1.10}
\end{align*}
\]

with the value of \( c \) to be determined. Let us remark that (1.1) results from setting \( \gamma = 1 \).

Let \( L^2_e = \{ w : \int_{-\infty}^{\infty} e^x w^2 dx < \infty \} \). In studying the \( \Gamma \)-convergence in \( L^2_e \) topology of the traveling wave functional associated with (1.9)-(1.10), the quantity \( \sqrt{dc^2} \) serves the
role of a small parameter $\epsilon$, the conventional notation being used. With $\int_0^1 f_\epsilon(\xi) \, d\xi > 0$ for all small $\epsilon$, we seek traveling wave solutions with $c > 0$. Given a prescribed velocity $c_0 > 0$, we seek traveling wave with speed $c_0 + o(1)$ for small $\epsilon$; to be more precise, $d = \epsilon^2/c^2(\epsilon)$ with $c(\epsilon) \to c_0$ as $\epsilon \to 0$ and $c_0$ indeed is determined by other parameters. For the existence of traveling waves whose $L^2_\epsilon$-limit is a front, it is assumed that the following condition holds:

\[(A1) \quad \alpha > \frac{3\sqrt{2}\sigma}{\gamma} > \alpha - 1 > 0.\]

Let $h_* = 1 - \frac{(\alpha-1)\gamma}{3\sqrt{2}\sigma}$, 
\[c_f = \frac{2h_*\sqrt{\gamma}}{\sqrt{1 - h_*^2}}.\] (1.11)

A translation of a traveling wave solution remains a solution. For concreteness we impose an additional constraint $\|u_\epsilon\|_{L^2_\epsilon} = 1$ when looking for such waves.

**Theorem 1.1.** Assumed that $(A1)$ is satisfied. If $c_f$ satisfies (1.11) for given $\sigma, \gamma$ and $\alpha$, there is an $\epsilon_0 > 0$ such that if $\epsilon \leq \epsilon_0$, there are $c_\epsilon$, $d = \epsilon^2/c_\epsilon^2$ for which a traveling wave solution $(c_\epsilon, u_\epsilon, v_\epsilon)$ of (1.9)-(1.10) exists with $\|u_\epsilon\|_{L^2_\epsilon} = 1$. Moreover $c_\epsilon \to c_f$ and $u_\epsilon \to \chi_{(-\infty,0]}$ in $L^2_\epsilon$, as $\epsilon \to 0$.

In Theorem 1.1, $c_f$ is unique determined when $\sigma, \gamma$ and $\alpha$ are given, and for small $\epsilon$ the speeds $c_\epsilon$ of traveling wave solutions are known to leading order. Since the limit is unique, the convergence takes place along the whole sequence.

Next we investigate the question that the $L^2_\epsilon$-limit is a traveling pulse. In this case, the parameters satisfy the following condition:

\[(A2) \quad \frac{3\sqrt{2}\sigma}{\gamma} > \alpha > 1.\]

**Theorem 1.2.** Assumed that $(A2)$ is satisfied. There is an $\epsilon_0 > 0$ such that if $\epsilon \leq \epsilon_0$, there are $c_\epsilon$ and $d = \epsilon^2/c_\epsilon^2$ for which a traveling wave solution $(c_\epsilon, u_\epsilon, v_\epsilon)$ of (1.9)-(1.10) exists with $\|u_\epsilon\|_{L^2_\epsilon} = 1$. Moreover if $\epsilon \to 0$ then $c_\epsilon \to c_p$ and $u_\epsilon \to \chi_{[a,b]}$ in $L^2_\epsilon$, where $c_p$ and $b - a$ are uniquely determined by the given parameters $\sigma, \gamma$ and $\alpha$.

It was proved [10, 11] that when $N = 1$ there always exists a single bubble; when $N = 2$ there may exist zero, one, two, or even three bubble profiles, depending on the values of $\alpha$ and $\sigma$, while if $N \geq 3$, there can be no more than two bubble profiles. The $\Gamma$-limit of higher dimensional traveling waves of (1.8) is a future work in progress.

We give an outline for the remainder of the paper. In Section 2 we introduce a class of functions which are in the spaces of weighted bounded variation. They fit in with
the \( \Gamma \)-convergence for the traveling wave problem. The variational formulation for the FitzHugh-Nagumo system is given in Section 3 and conditions for the \( \Gamma \)-convergence are verified in Section 4. We then in Sections 5 demonstrate that the limiting functional has a minimizer that corresponds to traveling front or pulse. Section 6 and Section 7 will distinguish front from pulse under different physical parameter regimes. Moreover these front and pulse in the limiting case imply the existence of the traveling waves if \( \epsilon \) is sufficiently small.

2 Weighted BV function

If \( (x, y) \in \Omega = \mathbb{R} \times \tilde{\Omega} \) for a bounded smooth domain \( \tilde{\Omega} \) in \( \mathbb{R}^{n-1} \), we write \( |(x, y)| = \sqrt{x^2 + \sum_{i=1}^{n-1} y_i^2} \) to denote the usual Euclidean norm. Let \( L^p_{e} = \{ w : \int_{\Omega} e^x |w|^p \, dx \, dy < \infty \} \) for \( p \geq 1 \).

Definition: Given a function \( u \in L^1_{loc}(\Omega) \) the total variation of \( u \) in \( \Omega \) with respect to the measure \( e^x \, dx \, dy \) is defined as

\[
\|Du\|_e(\Omega) = \sup \left\{ \int_{\Omega} u \, \text{div} (e^x \varphi) \, dx \, dy : \varphi \in C_0^1(\Omega; \mathbb{R}^n), |\varphi| \leq 1 \right\}. \tag{2.1}
\]

If \( \|Du\|_e(\Omega) < \infty \), it is easily checked \( \|Du\|_e(\Omega) \) that for any open set \( \Omega' \subset \subset \Omega \) the total variation of \( u \) in \( \Omega' \)

\[
\|Du\|(\Omega') = \sup \left\{ \int_{\Omega'} u \, \text{div} \varphi \, dx \, dy : \varphi \in C_0^1(\Omega'; \mathbb{R}^n), |\varphi| \leq 1 \right\}
\]

is also finite. Therefore if \( u \in L^1_{loc}(\Omega) \) and \( \|Du\|_e(\Omega) < \infty \), then the function \( u \) has bounded variation on any \( \Omega' \subset \subset \Omega \). By the Riesz representation theorem there exist a Radon measure \( \mu \) and a \( \mu \)-measurable function \( \sigma \) with \( |\sigma| = 1 \) such that for any \( \varphi \in C_0^1(\Omega; \mathbb{R}^n) \)

\[
\int_{\Omega} u \, \text{div} \varphi \, dx \, dy = -\int_{\Omega} \varphi \cdot \sigma \, d\mu. \tag{2.2}
\]

It follows from (2.2) that the vector valued measure \( \sigma d\mu \) coincides with the distributional gradient \( Du \) of \( u \). Hence, if \( u \) is smooth then \( \sigma d\mu = \nabla u \, dx \, dy \) and \( d\mu = |\nabla u| \, dx \, dy \).

Note that (2.2) in particular implies that for any \( \varphi \in C_0^1(\Omega; \mathbb{R}^n) \)

\[
\int_{\Omega} u \, \text{div} (e^x \varphi) \, dx \, dy = -\int_{\Omega} e^x \varphi \cdot \sigma \, d\mu. \tag{2.3}
\]

For \( \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \), it is clear that \( e^x \, \text{div} \varphi = \text{div} (e^x \varphi) - e^x \varphi_1 \) and thus

\[
\int_{\Omega} e^x u \, \text{div} \varphi \, dx \, dy = -\int_{\Omega} (e^x u \varphi_1 \, dx \, dy + e^x \varphi \cdot \sigma \, d\mu), \tag{2.4}
\]
a formula that we will use later.

We shall denote by $BV_e(\Omega)$ the set of functions $u \in L^1_e(\Omega)$ such that $\|Du\|_e(\Omega) < \infty$. This function space is equipped with the norm

$$\|u\|_{BV_e} = \|Du\|_e(\Omega) + \|u\|_{L^1_e}.$$ 

Also, if $u \in BV_e(\Omega)$ then

$$\|Du\|_e(\Omega) = \int_{\Omega} e^x \, d\mu.$$  

(2.5)

The first of the next two lemmas is a straightforward consequence of (2.1), while the second can be proved exactly the same as the standard $BV$ functions; see [19, p.172, Theorem 2].

**Lemma 2.1. (lower semicontinuity of weighted variation measure)**

Suppose $\{u_k\} \subset BV_e$ and $u_k \to u_0$ in $L^1_{e,loc}$, then $\|Du_0\|_e(\Omega) \leq \liminf_{k \to \infty} \|Du_k\|_e(\Omega)$.

**Lemma 2.2. (local approximation by smooth functions)**

Suppose $u \in BV_e$. Then there exists function $\{u_k\} \subset BV_e \cap C^\infty$ such that

(i) $u_k \to u$ in $L^1_e$ and
(ii) $\|Du_k\|_e(\Omega) \to \|Du\|_e(\Omega)$ as $k \to \infty$.

Our attention will be focused on the one dimensional case from now on; that is, $\Omega = (-\infty, \infty)$. Suppose $u = \chi_{[a,b]}$ with $-\infty \leq a < b < \infty$, an easy calculation gives

$$\|D\chi_{[a,b]}\|_e(\Omega) = \sup_{|\varphi| \leq 1} \int_a^b (e^x \varphi)' \, dx = \sup_{|\varphi| \leq 1} \{e^b \varphi|_{x=a} - e^a \varphi|_{x=b}\} = e^b + e^a.$$  

(2.6)

The following lemma will be useful in the sequel.

**Lemma 2.3.** If $E = \cup_i [a_i, b_i]$ is a union of countably many disjoint intervals, then

$$\|D\chi_E\|_e(\Omega) = \sum_i (e^{a_i} + e^{b_i}).$$  

(2.7)

Moreover $\chi_E \in BV_e$ if and only if $E$ is the union of countably many disjoint intervals $[a_i, b_i]$ and the right hand side of (2.7) is finite.

**Proof.** It is clear that (2.7) is an immediate consequence of (2.6). Furthermore, if $E = \cup_i [a_i, b_i]$ is the union of countably many disjoint intervals and the right hand side of (2.7) is finite, then

$$\int_{-\infty}^{+\infty} e^x \chi_E(x) \, dx \leq \sum_i e^{b_i} < \infty.$$ 

This implies $\chi_E \in L^1_e$ and thus $\chi_E \in BV_e$.

Conversely, if $\chi_E \in BV_e$, then, as observed before, $\chi_E$ has finite total variation in any bounded interval. It follows that, (see, e.g. [2, Proposition 3.52]), $E$ is the union of countably many disjoint intervals $[a_i, b_i]$ and the conclusion is immediate from (2.7).  

$\square$
Lemma 2.4. Let $u \in BV_e$ and $h \in \mathbb{R}$. Then

(i) $\|u(\cdot + h)\|_{L^1_e} = e^{-h}\|u\|_{L^1_e}$;
(ii) $\|D(u(\cdot + h))\|_{e(\Omega)} = e^{-h}\|Du\|_{e(\Omega)}$;
(iii) $u^+$ and $u^-$ are in $BV_e$ with $\|Du^+\|_e \leq \|Du\|_e$ and $\|Du^-\|_e \leq \|Du\|_e$;
(iv) $\|u\|_{L^1_e} \leq \|Du\|_{e(\Omega)}$;
(v) $|u(x)| e^x \leq 2\|Du\|_{e(\Omega)}$ a.e.

Proof. (i) to (ii) are directly follow from the definitions.

If $u \in BV_e$ then $u$ has locally finite variation in $\mathbb{R}$. This implies that $u^+$ has locally finite variation in $\mathbb{R}$ and $d(Du^+) = \chi_{\{u>0\}} d(Du) = \chi_{\{u>0\}} \sigma d\mu$ (see [2, Example 3.100]). It follows from (2.5) that $\|Du^+\|_{e(\Omega)} = \int_{\{u>0\}} e^x d\mu \leq \|Du\|_{e(\Omega)}$.

Similar calculations work for $u^-$, so (iii) follows.

By Lemma 2.2 it suffices to verify (iv) and (v) for smooth $u$ only. Consider the case $u \geq 0$ first. For any $\delta > 0$, there exists a large $R > 0$ such that $0 \leq \int_{|x|>R} e^x u dx \leq \delta$. Take a $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi = 1$ on $[-R,R]$, $\varphi = 0$ for $|x| \geq R + 2$, and $0 \leq \varphi \leq 1$ on $[R,R + 2]$ with $|\varphi'| \leq 1$. Then

$$
\|u\|_{L^1_e} - \delta \leq \int_{-\infty}^{\infty} e^x u \varphi dx = -\int_{-\infty}^{\infty} e^x \varphi \cdot \sigma d\mu - \int_{-\infty}^{\infty} e^x u \varphi' dx \\
\leq \|Du\|_{e(\Omega)} + \int_{R \leq |x| \leq R+2} e^x |u| dx \\
\leq \|Du\|_{e(\Omega)} + \delta .
$$

This gives $\|u\|_{L^1_e} \leq \|Du\|_{e(\Omega)}$ if $u \geq 0$. In the general case, setting $u = u^+ - u^-$ yields

$$
\|u\|_{L^1_e} = \|u^+\|_{L^1_e} + \|u^-\|_{L^1_e} \\
\leq \|Du^+\|_{L^1_e} + \|Du^-\|_{L^1_e} \\
= \|Du\|_{L^1_e} \\
\leq \|Du\|_{e(\Omega)},
$$

which completes the proof of (iv).

Suppose $u$ is smooth and with compact support. Then $-u e^x = \int_x^{\infty} (ue^t)' dt = \int_x^{\infty} e^t (u + u') dt$. Therefore

$$
|u(x)| e^x \leq \int_x^{\infty} e^t (|u| + |u'|) dt \leq \|u\|_{L^1_e} + \|Du\|_{e(\Omega)} \leq 2\|Du\|_{e(\Omega)},
$$
which yields (v) in this special case. This inequality can now be extended to $u \in C^\infty$ by using $u \varphi$ as an approximation, where $\varphi$ is the cut-off function that we employed earlier.

3 Variational formulation

We now turn to the variational formulation for studying the traveling waves of (1.8). Let $H^1_\varepsilon = \{ w : \int_{-\infty}^{\infty} e^x (w'^2 + w^2) \, dx < \infty \}$ and $F_\varepsilon(w) \equiv -\int_0^w f_\varepsilon(\xi) \, d\xi$. For a given $u$, we designate the unique solution of (1.10) by $v = L_c u$; here $L_c : L^2_\varepsilon \to H^1_\varepsilon$ is a self-adjoint operator with respect to the inner product on $L^2_\varepsilon$. For given $c, d > 0$, let $I_{c,d} : H^1_\varepsilon \to \mathbb{R}$ defined by

$$I_{c,d}(w) = \int_{-\infty}^{\infty} e^x \left( \frac{dc^2}{2} w'^2 + F_\varepsilon(w) + \frac{\epsilon \sigma}{2} w L_c w \right) \, dx.$$ (3.1)

The standard variational argument shows that $(u, v, c)$ solves (1.9)-(1.10) provided $v = L_c u$ and $u$ is a critical point of $I_{c,d}$. For easy referral, the terms on the right of (3.1) are called the gradient energy, the $F$-integral (or potential) and the nonlocal energy, respectively. The nonlocal energy is always non-negative since $\int_{-\infty}^{\infty} e^x L_c w \, dx \geq 0$ for any $w \in L^2_\varepsilon$.

Given an $\epsilon_1 > 0$ such that $\max\{\frac{1}{\sigma}, \frac{1}{2\alpha}\} \geq \epsilon_1 > 0$. For all $\epsilon \in (0, \epsilon_1)$, there exist $\beta_2 > 1$ with $\beta_2 - 1$ being small and $\tilde{M}_1 = M_1(\gamma) > 0$ satisfying $f_\varepsilon(-\tilde{M}_1) \geq \frac{\beta_2}{\gamma}$. We consider $I_{c,d} : Y \to \mathbb{R}$ with $Y$ being a admissible set defined by

$$Y \equiv \{ w \in H^1_\varepsilon : \int_{-\infty}^{\infty} e^x w^2 \, dx = 1, -\tilde{M}_1 - 1 \leq w \leq \beta_2 \}.$$

The constraint $||w||_{L^2_\varepsilon} = 1$ imposed in $Y$ is to eliminate a continuum of critical points due to translation. Suppose $u \in Y$ is a constrained minimizer of $I_{c,d}$, it is the sought-after traveling wave solution provided $I_{c,d}(u) = 0$ and $-\tilde{M}_1 - 1 < u < \beta_2$. We refer to [8] for the detailed argument.

The constrained variational approach has been employed [6, 8, 9] to establish the existence of traveling wave solutions of FitzHugh-Nagumo system. There all the parameters are fixed and of order $O(1)$, except that $d$ can be sufficiently small. In the situation as $d \to 0$ the wave speed $c$ tends to infinity and $dc^2$ approaches to a positive number, which depends only on $\beta$ if $f(\xi) = \xi(\xi - \beta)(1 - \xi)$ with $\beta \in (0, 1/2)$. This is a case that the $\Gamma$-limit of $I_{c,d}$ does not exist when $d \to 0$. It is interesting to investigate if the tool of $\Gamma$-convergence can be utilized to study traveling waves; this will, for the first time, expand its scope of applicability to non-stationary problems. In so doing we require other parameters to change in some coordinate fashion with $d$. 

Remark 3.1. The proofs given in \([6, 8, 9]\) used a different constraint \(\|w\|_{L^2} = \sqrt{2}\); it will be seen that imposing \(\|w\|_{L^2} = 1\) in \(Y\) facilitates easier \(\Gamma\)-convergence analysis.

The \(\Gamma\)-limit of \(I_{c,d}\) and its minimizer will be studied in the later sections. The next two lemmas enable us to recover the traveling wave solutions from the minimizer of the limiting functional of \(I_{c,d}\).

Lemma 3.2. Let \(\gamma, \sigma, \alpha\) and \(\epsilon_1\) be given. If \(c, d > 0\) and \(\epsilon \in (0, \epsilon_1)\) then \(\inf_{w \in Y} I_{c,d}(w)\) is, uniformly in \(\epsilon\), bounded from below. Suppose, in addition, \(\inf_{w \in Y} I_{c,d}(w) \leq 0\), then there is a minimizer \(u \in Y\).

Proof. There exists a constant \(M_2 > 0\) such that \(-M_2 \xi^2 \leq F_\epsilon(\xi)\) for all \(\xi \in \mathbb{R}\). For \(w \in Y\),

\[
I_{c,d}(w) \geq \int_{-\infty}^{\infty} e^x F_\epsilon(w) \, dx \geq -M_2 \int_{-\infty}^{\infty} e^x w^2 \, dx = -M_2,
\]

which shows \(\inf_{w \in Y} I_{c,d}(w)\) is, uniformly in \(\epsilon\), bounded from below. Taking a minimizing sequence \(\{w_n\} \subset Y\) with \(I_{c,d}(w_n) \leq \inf_{w \in Y} I_{c,d}(w) + 1\), then

\[
\frac{dc^2}{2} \int_{-\infty}^{\infty} e^x w'_n \, dx \leq I_{c,d}(w_n) - \int_{-\infty}^{\infty} e^x F_\epsilon(w_n) \, dx \leq \inf_{w \in Y} I_{c,d}(w) + 1 + M_2.
\]

Recall a Poincare type inequality for \(w \in H^1_e\):

\[
\int_{\mathbb{R}} e^x w^2 \, dx \geq \frac{1}{4} \int_{\mathbb{R}} e^x w^2 \, dx. \quad (3.2)
\]

This gives a uniform bound for \(\|w_n\|_{H^1_e}\) for all \(n\). Along a subsequence there is a \(W \in H^1_e\) such that \(w_n \rightharpoonup W\) weakly in \(H^1_e\) and strongly in \(L^{\infty}_{\text{loc}}(\mathbb{R}) \cap L^2_{\text{loc}}(\mathbb{R})\). As in the proof of Lemma 4.2 in \([8]\), we obtain \(-M_1 - 1 \leq W \leq \beta_2\),

\[
\begin{align*}
\int_{\mathbb{R}} e^x W^2 \, dx &\leq \liminf_{n \to \infty} \int_{\mathbb{R}} e^x w'_n \, dx, \\
\int_{\mathbb{R}} e^x F_\epsilon(W) \, dx &\leq \liminf_{n \to \infty} \int_{\mathbb{R}} e^x F_\epsilon(w_n) \, dx, \\
\int_{\mathbb{R}} e^x W \mathcal{L}_c W \, dx &\leq \liminf_{n \to \infty} \int_{\mathbb{R}} e^x w_n \mathcal{L}_c w_n \, dx,
\end{align*}
\]

and thus \(I_{c,d}(W) \leq \liminf_{n \to \infty} I_{c,d}(w_n)\). Moreover \(\int_{\mathbb{R}} e^x W^2 \, dx \leq 1\), since \(\int_{-l}^{l} e^x W^2 \, dx = \lim_{n \to \infty} \int_{-l}^{l} e^x w^2_n \, dx \leq 1\) for any \(l > 0\).
Suppose \( \inf_{w \in Y} I_{c,d}(w) \leq 0 \) then \( I_{c,d}(W) \leq 0 \). We claim \( W \not\equiv 0 \); for otherwise

\[
0 \geq \liminf_{n \to \infty} I_{c,d}(w_n)
\geq \frac{dc^2}{8} \int_{\mathbb{R}} e^x w_n^2 \, dx + \liminf_{n \to \infty} \int_{\mathbb{R}} e^x \{ F_\epsilon(w_n) + \frac{1}{2} w_n \mathcal{L}_c w_n \} \, dx
\geq \frac{dc^2}{8} + \int_{\mathbb{R}} e^x \{ F_\epsilon(W) + \frac{1}{2} W \mathcal{L}_c W \} \, dx
= \frac{dc^2}{8},
\]

which is absurd. Consequently \( 1 \geq \int_{\mathbb{R}} e^x W^2 \, dx > 0 \).

Take \( a \geq 0 \) such that \( e^a \int_{\mathbb{R}} e^x W^2 \, dx = 1 \). Letting \( u(x) \equiv W(x - a) \) gives \( u \in Y \) and

\[
I_{c,d}(u) = e^a I_{c,d}(W) \leq I_{c,d}(W) \leq \inf_{w \in Y} I_{c,d}(w) \leq I_{c,d}(u).
\]

Hence \( u \) is a minimizer of \( I_{c,d} \). In case \( \inf_{w \in Y} I_{c,d}(w) < 0 \) then \( a = 0 \) and \( u = W \).

**Lemma 3.3.** Suppose \( u \in Y \) is a minimizer of \( I_{c,d} \) and \( I_{c,d}(u) = 0 \), then \( (u, \mathcal{L}_c u) \) is a traveling wave solution of (1.9)-(1.10) with \( c \) as its wave speed.

**Proof.** The first step is to show \( -\tilde{M}_1 - 1 < u < \beta_2 \), using the arguments of \cite{6, 8}. Since \( I_{c,d}(u) = 0 \), we may slightly modify the proofs given in \cite{8, 6} to get rid of the Lagrange multiplier associated with the constraint \( \int_{\mathbb{R}} e^x w^2 \, dx = 1 \). This ensures that \( (u, \mathcal{L}_c u) \) is a traveling wave solution, because \( u \) acts like an unconstrained critical point of \( I_{c,d} \).

\[ \square \]

### 4 \( \Gamma \)-convergence

To investigate the \( \Gamma \)-convergence for the traveling wave functional, we rewrite (1.9)-(1.10) in the following form:

\[
e^2 u_{xx} + e^2 u_x + f_\epsilon(u) - \epsilon \sigma v = 0, \quad (4.1)
\]
\[
c^2 v_{xx} + c^2 v_x - \gamma v + u = 0; \quad (4.2)
\]

that is, set \( \epsilon = \sqrt{dc^2} \). It is required that \( d \) be related to \( \epsilon \) in some suitable fashion to induce an interesting geometric variational problem to be the \( \Gamma \)-limit. Thus at least one of \( c, d \) is depending on \( \epsilon \) when we consider a sequence along \( \epsilon \to 0 \).

Note that \( F_\epsilon = F_0 + \alpha \epsilon G \), where \( F_0(u) \equiv 4u^2(u - 1)^2 \) and \( G(u) \equiv \frac{1}{\sqrt{2}}(\frac{u^3}{3} - \frac{u^2}{2}) \). In the decomposition of \( F_\epsilon \), \( F_0 \) is a balanced bistable nonlinearity and \( F_0(0) = F_0(1) = \min F_0 = 0 \). \( G \) has a local maximum at 0 and a local minimum at 1 with \( G(0) = 0 \).
and \( G(1) = -1/6\sqrt{2} \). Their combined \( F_\epsilon \) has a local maximum at \( \beta_\epsilon \) and \( F_\epsilon(\beta_\epsilon) > 0 \). \( F_\epsilon(0) = 0 \) is a local minimum and \( F_\epsilon(1) = -\frac{1-2\beta_\epsilon}{12} = -\frac{1}{6\sqrt{2}}a_\epsilon \) is the global minimum.

To evaluate the \( \Gamma \)-limit of the functional \( \mathcal{I}_{c,d}/\epsilon \) in the topology \( L^2_\epsilon \) as \( \epsilon \to 0 \), we set \( J_{c(\epsilon)} \equiv \mathcal{I}_{c,d}/\epsilon \); that is,

\[
J_{c(\epsilon)}(w) = \int_{-\infty}^{\infty} e^x \left\{ \frac{\epsilon w'^2}{2} + \frac{F_0(w)}{\epsilon} + \alpha G(w) + \frac{\sigma}{2} w L_{c(\epsilon)} w \right\} dx,
\]

for \( w \in Y \). In (4.3), \( c \) is not necessarily a constant but a function of \( \epsilon \) with the property that \( c(\epsilon) \to c_0 \) for some positive constant \( c_0 \) as \( \epsilon \to 0 \). If \( u \) is a minimizer of \( J_{c(\epsilon)} \) in \( Y \) and

\[
J_{c(\epsilon)}(u) = 0
\]

then by Lemma 3.3, we obtain a solution of (4.1)-(4.2) by setting \( v = L_{c(\epsilon)} u \). Our goal aims for traveling wave with speed close to \( c \).

Next we examine the \( \Gamma \)-convergence of \( J_{c(\epsilon)} \) as \( \epsilon \to 0 \). As an application of the limiting functional later on, we prove the existence of traveling wave solutions of the original problems by seeking suitable conditions on the parameters \( \alpha, \gamma \) and \( \sigma \). Let \( \phi(\xi) = \int_0^\xi \sqrt{2F_0(\eta)} d\eta \). This function has been frequently used in the calculation of phase transition problems. Although these latter problems bear certain similarity to our study, additional complexities arise due to the treatment for unbounded domains, the presence of a nonlocal term, and the appearance of the weight \( e^x \) in dealing with traveling waves instead of stationary solutions. As a remark, \( \phi \) is a strictly increasing function, \( \phi(0) = 0 \) and \( \phi(1) = \frac{\sqrt{2}}{12} \).

In what follows, when we say a sequence converges, it might be passing through a subsequence without further comment. Denoted by \( C_k, k = 0, 1, 2, \ldots \), a positive constant not depending on \( \epsilon \). For instance, let \( C_1 \equiv \frac{1}{\sqrt{2}}(\frac{1+\sqrt{17}}{3} + \frac{1}{2}) \) in the following compactness lemma.

**Lemma 4.1. (compactness)** Let \( \{w_\epsilon\} \subset Y \) such that \( \liminf_{\epsilon \to 0} J_{c(\epsilon)}(w_\epsilon) \leq C_0 \). Then

(i) \( \phi(w_\epsilon) \in BV_\epsilon \);

(ii) there exists a subsequence, still denoted by \( \{w_\epsilon\} \), and a characteristic function \( \chi_E \in BV_\epsilon \cap L^2_\epsilon \) such that \( w_\epsilon \to \chi_E \) in \( L^2_\epsilon \) and \( \|D\chi_E\|_\epsilon(\Omega) \leq 6\sqrt{2}(C_0 + C_1 \alpha) \);

(iii) \( \int_{-\infty}^{\infty} e^2 \chi_E = 1 \) with \( E \subset (-\infty, \log 6\sqrt{2}(C_0 + C_1 \alpha)] \).

**Proof.** For \( \{w_\epsilon\} \subset Y \), we know \( \|w_\epsilon\|_{L^\infty} \leq 1 + \tilde{M}_1 \) and \( \|w_\epsilon\|_{L^p_\epsilon} \leq \|w_\epsilon\|_{L^2_\epsilon}^{2/p} \|w_\epsilon\|_{L^{p-2}_\epsilon}^{(p-2)/p} \leq \)
which shows \( \phi(w_{\epsilon}) \in BV_{e} \) with a uniform bound in norm. On each finite interval \([-k, k]\), applying the compactness theorem for bounded variation functions on finite domains, we obtain a function \( \Phi_{0} \) such that along a subsequence \( \phi(w_{\epsilon}) \to \Phi_{0} \) in \( L_{e}^{1}(-k, k) \) and pointwise a.e.. Then using a diagonal process, we conclude that \( \Phi_{0} \) is a strictly increasing function, setting \( w_{0} = \phi^{-1}(\Phi_{0}) \) yields \( w_{\epsilon} \to w_{0} \) pointwise a.e. and thus \( -\tilde{M}_{1} - 1 \leq w_{0} \leq \beta_{2} \) a.e.. Observe that

\[
\int_{-\infty}^{\infty} e^{x} F_{0}(w_{\epsilon}) \, dx \leq \epsilon \{ J_{c(\epsilon)}(w_{\epsilon}) + \alpha \int_{-\infty}^{\infty} e^{x}|G(w_{\epsilon})| \, dx \} \leq \epsilon(C_{0} + C_{1}\alpha + o(1)).
\]  

Applying Fatou’s lemma gives

\[
0 \leq \int_{-\infty}^{\infty} e^{x} F_{0}(w_{0}) \, dx \leq \lim inf_{\epsilon \to 0} \int_{-\infty}^{\infty} e^{x} F_{0}(w_{\epsilon}) \, dx \\
\leq \lim inf_{\epsilon \to 0} \epsilon(C_{0} + C_{1}\alpha + o(1)) \\
= 0,
\]

which implies \( F_{0}(w_{0}) = 0 \) a.e.. This can be valid only if \( w_{0}(x) \in \{0, 1\} \) a.e.; in other words, \( w_{0} = \chi_{E} \) for some set \( E \subset \mathbb{R} \). As a consequence, \( \phi(w_{\epsilon}) \to \Phi_{0} = \phi(w_{0}) = \phi(1)\chi_{E} \) in \( L_{e, loc}^{1} \) and pointwise a.e.. Using Lemma 2.1 and Lemma 2.4 again, we get

\[
\phi(1)\|\chi_{E}\|_{L_{e}^{1}} \leq \phi(1)\|D\chi_{E}\|_{e}(\Omega) = \|D\phi(w_{0})\|_{e}(\Omega) \leq \lim inf \|D\phi(w_{\epsilon})\|_{e}(\Omega) \leq C_{0} + C_{1}\alpha.
\]  

\[(4.7)\]
Hence $\chi_E \in BV_\varepsilon$ and $\|D\chi_E\|_e(\Omega) \leq \frac{C_0+C_1\alpha}{\phi(1)}$, which shows $E \subset (-\infty, \log \frac{C_0+C_1\alpha}{\phi(1)}]$. Invoking Lemma 2.4 yields $e^{x} |\phi(w_\varepsilon(x))| \leq 2 \|D\phi(w_\varepsilon)\|_e(\Omega) \leq 2(C_0 + C_1\alpha + o(1))$ and thus $|\phi(w_\varepsilon(x))| \leq 2(C_0 + C_1\alpha + 1)e^{-x}$. Since $\phi^{-1}$ is continuous and $\phi^{-1}(0) = 0$, there exists a $y_0$ such that $|w_\varepsilon(x)| \leq 1/2$ if $x \geq y_0$. This together with (4.6) gives

$$\int_{y_0}^{\infty} e^{x} w_\varepsilon^2 \, dx \leq 4 \int_{y_0}^{\infty} e^{x} w_\varepsilon^2(w_\varepsilon - 1)^2 \, dx \leq 16 \int_{-\infty}^{\infty} e^{x} F_0(w_\varepsilon) \, dx \leq 16\epsilon(C_0 + C_1\alpha + o(1)).$$

Next we extract a subsequence of $\{w_\varepsilon\}$ to form a Cauchy sequence in $L^2_\varepsilon$. For any $\delta > 0$, in view of

$$\int_{-\infty}^{\infty} e^{x}|w_\varepsilon - w_\eta|^2 \, dx \leq \int_{-\infty}^{-y_5} e^{x}|w_\varepsilon - w_\eta|^2 \, dx + \int_{y_5}^{y_0} e^{x}|w_\varepsilon - w_\eta|^2 \, dx + \int_{y_0}^{\infty} e^{x}|w_\varepsilon - w_\eta|^2 \, dx \leq 4(1 + \widetilde{M}_1)^2 \int_{-\infty}^{-y_5} e^{x} \, dx + \int_{y_5}^{y_0} e^{x}|w_\varepsilon - w_\eta|^2 \, dx + 2 \int_{y_0}^{\infty} e^{x}(w_\varepsilon^2 + w_\eta^2) \, dx \leq 4(1 + \widetilde{M}_1)^2 e^{-y_5} + \int_{-y_5}^{y_0} e^{x}|w_\varepsilon - w_\eta|^2 \, dx + 64(C_0 + C_1\alpha + o(1)) \max\{\epsilon, \eta\},$$

in the last line the first term and the third term are smaller than $\delta$ if we pick $y_5$ large and $\max\{\epsilon, \eta\}$ small enough. With $w_\varepsilon$ being converging to $w_0$ pointwise a.e. and $\|w_\varepsilon\|_{L^\infty} \leq 1 + \widetilde{M}_1$, the second term is also smaller than $\delta$ since the dominated convergence theorem implies that $w_\varepsilon \rightharpoonup w_0$ in $L^2_\varepsilon(-y_5, y_0)$. Along this Cauchy subsequence, $\{w_\varepsilon\}$ converges to a function in $L^2_\varepsilon$ and pointwise a.e. and consequently this limit function has to be $\chi_E$. Hence

$$\int_{-\infty}^{\infty} e^{x}\chi_E \, dx = \int_{-\infty}^{\infty} e^{x}\chi_E^2 \, dx = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} e^{x}w_\varepsilon^2 \, dx = 1.$$ The proof is complete.

To employ $\Gamma$-convergence for studying the existence and qualitative behavior of traveling wave solutions as $\epsilon \to 0$, we extend the domain of $J_{\epsilon(\cdot)}$ to $L^2_\epsilon$ by setting

$$J_{\epsilon(\cdot)}(w) = \begin{cases} \text{as } (4.3), & \text{if } w \in \mathcal{Y}, \\ \infty, & \text{if } w \in L^2_\epsilon \setminus \mathcal{Y}. \end{cases} \quad (4.8)$$

Next we propose a possible candidate for the $\Gamma$-limit of $J_{\epsilon(\cdot)}$ Let $E = \bigcup_i[a_i, b_i]$, a union of countably many disjoint intervals; here $b_1 > a_1 > b_2 > a_2 > \ldots$ and $a_i \to -\infty$, $b_i \to -\infty$ as $i \to \infty$. We introduce a functional $J^*_\epsilon: L^2_\epsilon \to \mathbb{R}$ defined by

$$J^*_\epsilon(w) = \begin{cases} \sqrt{2} \sum_i(e^{a_i} + e^{b_i}) - \sqrt{2a_i} \sum_i(e^{b_i} - e^{a_i}) + \frac{\sigma}{2} \int_{-\infty}^{\infty} e^{x}\chi_E \mathcal{L}_e \chi_E \, dx, & \text{if } w = \chi_E \in BV_\varepsilon \text{ and } \int_{-\infty}^{\infty} e^{x}\chi_E \, dx = 1, \\ \infty, & \text{otherwise}. \end{cases} \quad (4.9)$$
Remark 4.2. Since $\sum(e^{b_i} - e^{a_i}) \leq e^{b_1} < \infty$, if $J^*_c(\chi_E) < \infty$ then both the first and the third terms of $J^*_c(\chi_E)$ are positive and bounded from above.

From (4.2), it follows from integration by parts that
\[
c^2(\epsilon)\|L_c(\epsilon)w\|_{L^2}^2 + \gamma\|L_c(\epsilon)w\|_{L^2}^2 = \int_{-\infty}^{\infty} e^x L_c(\epsilon)w\,dx.
\]
Since $c(\epsilon) \to c_0$, it follows that $\|L_c(\epsilon)w\|_{L^2} \leq C_3\|w\|_{L^2}$ for some positive constant $C_3$, depending on $c_0$ and $\gamma$ only. Suppose $w_\epsilon \to w_0$ in $L^2_\epsilon$; it is easily verified that $L_c(\epsilon)w_\epsilon \to L_{c_0}w_0$ in $H^1_\epsilon$ and $\int_{-\infty}^{\infty} e^x w_\epsilon L_c(\epsilon)w_\epsilon\,dx \to \int_{-\infty}^{\infty} e^x w_0 L_{c_0}w_0\,dx$ as $\epsilon \to 0$. Using this fact, we apply a well-known stability theorem \[5, proposition 2.3\] to establish the liminf inequality as follows.

Lemma 4.3. (liminf inequality) If $w_0 \in L^2_\epsilon$ then
\[
J^*_c(w_0) \leq \liminf_{\epsilon \to 0} J_{c(\epsilon)}(w_\epsilon) \tag{4.10}
\]
for any sequence $w_\epsilon \to w_0$ in $L^2_\epsilon$.

Proof. It suffices to treat the case that $\liminf J_{c(\epsilon)}(w_\epsilon) \leq \infty$ and $\{w_\epsilon\} \subset Y$; otherwise there is nothing to prove. Then $\|w_\epsilon\|_{L^2_\epsilon} = 1$ and $\alpha_1 - 1 \leq w_\epsilon \leq \beta_2$. It follows from Lemma 4.1 and Lemma 2.3 that $w_0 = \chi_E \in BV_\epsilon \cap L^2_\epsilon$, $\int_{-\infty}^{\infty} e^x \chi_E\,dx = 1$, and $E = \cup_{i=1}^{\infty} [a_i, b_i]$, a union of countably many disjoint intervals.

As mentioned earlier, the nonlocal term can be ignored in checking the $\Gamma$-convergence; the same is true for the term $\alpha \int_{-\infty}^{\infty} e^x G(u)\,dx$ by the stability theorem \[5, proposition 2.3\], since $w_\epsilon \to \chi_E$ in $L^2_\epsilon$ and a uniform $L^\infty$ norm bound on $w_\epsilon$ imply $w_\epsilon \to \chi_E$ in $L^2_\epsilon$ as well. As a consequence,
\[
\alpha \int_{-\infty}^{\infty} e^x G(w_\epsilon)\,dx \to \alpha \int_{-\infty}^{\infty} e^x G(\chi_E)\,dx = \alpha G(1) \sum_i (e^{b_i} - e^{a_i}) = -\frac{\sqrt{2} \alpha}{12} \sum_i (e^{b_i} - e^{a_i}).
\]

Thus it remains to verify the liminf inequality for the gradient term and the $F_0$-integral only, and this is well known except for the weight function $e^x$ appeared in the formulation.

Extracting useful calculations from (4.3) and (4.7), we arrive at
\[
\frac{\sqrt{2}}{12} \sum_i (e^{a_i} + e^{b_i}) = \phi(1)\|D\chi_E\|_\epsilon(\Omega)
\leq \liminf \|D\phi(w_\epsilon)\|_\epsilon
\leq \liminf \int_{-\infty}^{\infty} e^x \left(\frac{e^2 w_\epsilon^2}{2} + \frac{F_0(w_\epsilon)}{\epsilon}\right)\,dx,
\]
from which the proof of (4.10) can be completed. \qed
In studying the limsup inequality, the first step in the proof is to construct auxiliary functions as in dealing with phase transition problems [? ]. Let \( f_0(\xi) = -\xi(\xi-1/2)(\xi-1) \).

Integrating the equation
\[
e^2U''_\varepsilon + f_0(U_\varepsilon) = 0 \tag{4.11}
\]
once yields
\[
e^2U'_\varepsilon^2/2 - F_0(U_\varepsilon) = \text{constant}.
\]
Assigning this constant to be \( \varepsilon/2 \) gives
\[
U'_\varepsilon = \frac{\sqrt{\varepsilon + 2F_0(U_\varepsilon)}}{\varepsilon} \tag{4.12}
\]
with initial condition \( U_\varepsilon(0) = 0 \). This function \( U_\varepsilon(x) \) is strictly increasing and
\[
\int_0^U \frac{\varepsilon}{\sqrt{\varepsilon + 2F_0(s)}} \, ds = x.
\]
It is now immediate that \( U_\varepsilon(\rho_\varepsilon) = 1 \) and
\[
\rho_\varepsilon = \int_0^1 \frac{\varepsilon}{\sqrt{\varepsilon + 2F_0(s)}} \, ds \leq \sqrt{\varepsilon} \tag{4.13}
\]
By the same token, a strictly decreasing function \( \tilde{U}_\varepsilon \) satisfies
\[
\tilde{U}_\varepsilon' = -\frac{\sqrt{\varepsilon + 2F_0(\tilde{U}_\varepsilon)}}{\varepsilon}, \tag{4.14}
\]
\( \tilde{U}_\varepsilon(0) = 1 \) and \( \tilde{U}_\varepsilon(\rho_\varepsilon) = 0 \).

We now state limsup inequality:

For every \( w_0 \in L^2_e \), there exists a sequence \( \{w_\varepsilon\} \subset L^2_e \) such that \( w_\varepsilon \to w_0 \) in \( L^2_e \) and
\[
\limsup_{\varepsilon \to 0} J_{\varepsilon(\cdot)}(w_\varepsilon) \leq J^{\ast}_{\varepsilon_0}(w_0).
\]

By (4.9) it suffices to consider \( J^{\ast}_{\varepsilon_0}(w_0) < \infty \). Then \( w_0 = \chi_E \in BV_\varepsilon \) for some \( E \subset \mathbb{R} \),
\[
\int_{-\infty}^{\infty} e^x \chi_E \, dx = 1 \text{ and it is clear that } \{w_\varepsilon\} \subset Y.
\]

**Lemma 4.4.** (limsup inequality) For every \( \chi_E \in BV_\varepsilon \) with \( \int_{-\infty}^{\infty} e^x \chi_E \, dx = 1 \), there exists a sequence \( \{w_\varepsilon\} \subset Y \) such that \( w_\varepsilon \to \chi_E \) in \( L^2_e \) and
\[
\limsup_{\varepsilon \to 0} J_{\varepsilon(\cdot)}(w_\varepsilon) \leq J^{\ast}_{\varepsilon_0}(\chi_E).
\]

**Proof.** Lemma [2.3] indicates that \( \chi_E \) can jump at countably many points \( x_1 > x_2 > x_3 > \ldots \) with \( x_n \to -\infty \). (If \( E = \cup_j [a_j, b_j] \), then we set \( x_1 = b_1, x_2 = a_1 \) and \( x_{2j-1} = b_j, x_{2j} = a_j \)). For any \( \delta > 0 \), choose a large \( i_\delta \) such that \( e^{-i_\delta} \leq \delta \) and thus \( \int_{-\infty}^{-i_\delta} e^x \chi_E \, dx \leq \delta. \)
Without loss of generality, we may assume that $x_j \neq -i_\delta$ for all $j$.

Let $x_j \in (-i_\delta, \infty)$ if $j = 1, 2, \ldots, k_\delta$ and be outside of the interval $[-i_\delta, \infty)$ otherwise. There is a $\Delta_\delta > 0$ such that $\min_{1 \leq j \leq k_\delta} (x_j - x_{j+1}) \geq \Delta_\delta$. Let $\varepsilon$ satisfy $\rho_\varepsilon \leq \sqrt{\varepsilon} < \Delta_\delta/2$.

It is clear that as $x$ increases, $\chi_E(x)$ jumps from 0 to 1 at $x_{2j}$ and from 1 to 0 at $x_{2j-1}$.

Take $0 \leq \theta_i \leq 1$ for $i = 1, 2, \ldots, k_\delta$ and define

$$w_\varepsilon(x) = \begin{cases} 
U_\varepsilon(x - q_{2j}), & x \in [q_{2j}, q_{2j} + \rho_\varepsilon], q_{2j} = x_{2j} - \theta_{2j}\rho_\varepsilon, \text{ when } 2j \leq k_\delta, \\
\tilde{U}_\varepsilon(x - q_{2j-1}), & x \in [q_{2j-1}, q_{2j-1} + \rho_\varepsilon], q_{2j-1} = x_{2j-1} - \theta_{2j-1}\rho_\varepsilon, \text{ when } 2j - 1 \leq k_\delta, \\
0, & \text{if } x \leq -i_\delta - \rho_\varepsilon, \\
\chi_E(x), & \text{otherwise.}
\end{cases}$$

At a point $x_{2j}$,

$$\begin{align*}
\text{if } \theta_{2j} = 0, \text{ then } \int_{x_{2j} - \rho_\varepsilon}^{x_{2j} + \rho_\varepsilon} e^x U_\varepsilon^2 \, dx &\leq \int_{x_{2j} - \rho_\varepsilon}^{x_{2j} + \rho_\varepsilon} e^x \chi_E \, dx, \\
\text{if } \theta_{2j} = 1, \text{ then } \int_{x_{2j} - \rho_\varepsilon}^{x_{2j} + \rho_\varepsilon} e^x U_\varepsilon^2 \, dx &\geq \int_{x_{2j} - \rho_\varepsilon}^{x_{2j} + \rho_\varepsilon} e^x \chi_E \, dx.
\end{align*}$$

Similar inequalities hold at $x_{2j-1}$. By the immediate value theorem we can select each $\theta_i \in [0, 1]$ to ensure that $\int_{-\infty}^{\infty} e^x w_\varepsilon^2 \, dx = 1$. Observe that $w_\varepsilon \in Y$ and $w_\varepsilon(x) = 0$ when $x \geq x_1 + 1$.

Since $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$, it follows that $w_\varepsilon \to \chi_E$ pointwise a.e. on $[-i_\delta - 1, x_1 + 1]$. The dominated convergence theorem then gives $w_\varepsilon \to \chi_E$ in $L^2_e(-i_\delta, x_1 + 1)$. Now $\int_{-\infty}^{-i_\delta} e^x |w_\varepsilon - \chi_E|^2 \, dx \leq \int_{-\infty}^{-i_\delta} e^x \, dx \leq \delta$. Since $\delta$ is arbitrary, we conclude that $w_\varepsilon \to \chi_E$ in $L^2_e$. Together with $0 \leq w_\varepsilon \leq 1$, the stability theorem is applicable to the nonlocal term and the $G$-integral. Thus it suffices to check the limsup inequality for only the gradient term and the $F_0$-integral.
\[
\int_{-\infty}^{\infty} e^{x} \left( \frac{\epsilon w_{\epsilon}'^2}{2} + \frac{F_{0}(w_{\epsilon})}{\epsilon} \right) \, dx \\
= \sum_{j=1}^{k} \int_{q_{j}}^{q_{j} + \rho_{\epsilon}} e^{x} \left( \frac{\epsilon w_{\epsilon}'^2}{2} + \frac{F_{0}(w_{\epsilon})}{\epsilon} \right) \, dx \\
= \sum_{\text{j is even}}^{k} e^{q_{j}} \int_{0}^{\rho_{\epsilon}} e^{x} \left( \frac{\epsilon U_{\epsilon}'^2}{2} + \frac{F_{0}(U_{\epsilon})}{\epsilon} \right) \, dx + \sum_{\text{j is odd}}^{k} e^{q_{j}} \int_{0}^{\rho_{\epsilon}} e^{x} \left( \frac{\epsilon \bar{U}_{\epsilon}'^2}{2} + \frac{F_{0}(\bar{U}_{\epsilon})}{\epsilon} \right) \, dx \\
\leq \sum_{\text{j is even}}^{k} e^{q_{j}} e^{\rho_{\epsilon}} \int_{0}^{\rho_{\epsilon}} \epsilon + 2 \frac{F_{0}(U_{\epsilon})}{\epsilon} \, dx + \sum_{\text{j is odd}}^{k} e^{q_{j}} e^{\rho_{\epsilon}} \int_{0}^{\rho_{\epsilon}} \epsilon + 2 \frac{F_{0}(\bar{U}_{\epsilon})}{\epsilon} \, dx \\
= \sum_{j}^{k} e^{q_{j}} e^{\rho_{\epsilon}} \int_{0}^{\epsilon} \epsilon + 2 \frac{F_{0}(U_{\epsilon})}{\epsilon} \, dx \quad \text{by using symmetry,} \\
\leq \sum_{j}^{k} e^{q_{j}} e^{\rho_{\epsilon}} \int_{0}^{\epsilon} \epsilon + 2 \frac{F_{0}(U_{\epsilon})}{\epsilon} \, dx \quad \text{by using (4.12),} \\
= \sum_{j}^{k} e^{q_{j}} e^{\rho_{\epsilon}} \int_{0}^{\epsilon} \sqrt{\epsilon + 2 F_{0}(U_{\epsilon})} \, U_{\epsilon}' \, dx \\
= \sum_{j}^{k} e^{q_{j}} e^{\rho_{\epsilon}} \int_{0}^{1} \sqrt{\epsilon + 2 F_{0}(s)} \, ds .
\]

Taking limit as \( \epsilon \to 0 \) yields

\[
\lim \sup \int_{-\infty}^{\infty} e^{x} \left( \frac{\epsilon w_{\epsilon}'^2}{2} + \frac{F_{0}(w_{\epsilon})}{\epsilon} \right) \, dx \leq \phi(1) \sum_{j}^{k} e^{q_{j}} \\
\leq \frac{\sqrt{2}}{12} \sum_{j}^{\infty} (e^{q_{j}} + e^{b_{j}}) ,
\]

from which the proof can be completed. \( \square \)

By virtue of Lemma 4.3 and Lemma 4.4, the \( \Gamma \)-limit of \( J_{c}(\epsilon) \) in \( L_{\epsilon}^{2} \) has been established;

\[
\Gamma \lim J_{c}(\epsilon) = J_{c_{0}}^{\ast} . \quad (4.15)
\]

We claim that \( \liminf_{\epsilon \to 0} (\inf_{w \in Y} J_{c}(\epsilon)(w)) < \infty \). Indeed taking

\[
w_{\epsilon}(x) = \begin{cases} 
1, & \text{if } x \leq 0, \\
1 - \frac{x}{\epsilon}, & \text{if } 0 \leq x \leq \epsilon, \\
0, & \text{if } x \geq \epsilon,
\end{cases}
\]

we see that \(w_{\epsilon}(\cdot - \theta_{\epsilon}) \in Y\) for some \(\theta_{\epsilon} \to 0\). A direct calculation yields \(\liminf_{\epsilon \to 0} J_{c(\epsilon)}(w_{\epsilon}) < \infty\), which verifies the claim.

Since there exists \(C_{4} > 0\) such that \(G(\xi) \geq -C_{4}\xi^{2}\) for \(\xi \in [-\tilde{M}_{1}, \beta_{2}]\), if \(w \in Y\) then

\[
J_{c(\epsilon)}(w) \geq \alpha \int_{-\infty}^{\infty} e^{2}G(w)\, dx \\
\geq -\alpha C_{4} \int_{-\infty}^{\infty} e^{2}w^{2}\, dx \\
= -\alpha C_{4}.
\]

Hence

\[
\infty > \liminf_{\epsilon \to 0} (\inf_{w \in Y} J_{c(\epsilon)}(w)) \geq -\alpha C_{4}
\]

and there is a sequence \(\{\zeta_{\epsilon}\} \subset Y\) such that

\[
\lim_{\epsilon \to 0} J_{c(\epsilon)}(\zeta_{\epsilon}) = \liminf_{\epsilon \to 0} (\inf_{w \in Y} J_{c(\epsilon)}(w)) .
\]

Such a minimizing sequence will be utilized to establish the existence of minimizers for \(J_{c_{0}}\) in the next section.

5 Minimizer for \(J_{c}^{*}\)

In the investigation of the minimizers of \(J_{c_{0}}^{*}\), we assume \(c(\epsilon) \to c\) as \(\epsilon \to 0\) throughout the section; it allows us to use a simpler notation \(J_{c}^{*}\) instead of \(J_{c_{0}}^{*}\). Also, if \(\chi_{E} \in BV_{\epsilon}\) we will follow the formula stated in (4.9) to evaluate \(J_{c}^{*}(\chi_{E})\) even if \(\|\chi_{E}\|_{L_{1}^{\epsilon}} \neq 1\).

Lemma 5.1. If \(c(\epsilon) \to c\) as \(\epsilon \to 0\), then \(J_{c}^{*}\) has a minimizer, denoted by \(\chi_{E_{c}}\). Moreover

(i) \(\int_{-\infty}^{\infty} e^{2}\chi_{E_{c}}\, dx = 1\);  
(ii) \(E_{c} = [a, b]\) for some \(a < b\); the scenario \(a = -\infty\) is allowed.

Proof. Since \(\liminf_{\epsilon \to 0} (\inf_{w \in Y} J_{c(\epsilon)}(w)) \leq C_{0}\) for some \(C_{0} > 0\), it follows from (4.16) and Lemma 4.1 that \(\zeta_{\epsilon} \to \chi_{E_{c}}\) in \(L_{2}^{\epsilon}\) with \(\|D\chi_{E_{c}}\|_{e} \leq 6\sqrt{2}(C_{0} + C_{1}\alpha)\). By a fundamental theorem of \(\Gamma\)-convergence \([3, Theorem 2.1]\), \(\chi_{E_{c}}\) is a minimizer of \(J_{c}^{*}\) and

\[
J_{c}^{*}(\chi_{E_{c}}) = \lim_{\epsilon \to 0} J_{c(\epsilon)}(\zeta_{\epsilon}).
\]

Moreover \(\|\chi_{E_{c}}\|_{L_{1}^{\epsilon}} = \|\chi_{E_{c}}\|_{L_{2}^{\epsilon}} = \lim_{\epsilon \to 0} \|\zeta_{\epsilon}\|_{L_{2}^{\epsilon}} = 1\). Thus (i) is proved.

Suppose \(E_{c} = \bigcup_{i=1}^{\infty} [a_{i}, b_{i}]\) with \(b_{1} > a_{1} > b_{2} > a_{2} > \ldots\). Let \(E^{1} = [a_{1}, b_{1}]\) and \(E^{2} = E_{c} \setminus E^{1}\) with both \(E^{1}\) and \(E^{2}\) being non-empty. For \(i = 1, 2\), let \(\|\chi_{E^{i}}\|_{L_{1}^{h_{i}}} = e^{-h_{i}}\) for some \(h_{i} > 0\). Thus

\[
1 = \|\chi_{E_{c}}\|_{L_{1}^{\epsilon}} = e^{-h_{1}} + e^{-h_{2}}
\]
and \( \| \chi_{E^1} (\cdot - h_i) \|_{L^1} = 1 \). Since \( \chi_{E_c} \) is a minimizer of \( J^*_c \), it is clear that \( J^*_c (\chi_{E_c}) \leq J^*_c (\chi_{E^1} (\cdot - h_i)) = e^{h_i} J^*_c (\chi_{E^1}) \). A simple calculation gives

\[
J^*_c (\chi_{E_c}) = J^*_c (\chi_{E^1}) + J^*_c (\chi_{E^2}) + \sigma_0 \int_{-\infty}^{\infty} e^x \chi_{E^1} \mathcal{L}_c \chi_{E^2} \, dx .
\]  

(5.3)

Then

\[
J^*_c (\chi_{E_c}) \geq e^{-h_1} J^*_c (\chi_{E_c}) + e^{-h_2} J^*_c (\chi_{E_c}) + \sigma_0 \int_{-\infty}^{\infty} e^x \chi_{E^1} \mathcal{L}_c \chi_{E^2} \, dx ,
\]

together with (5.2) leads to

\[
0 \geq \sigma_0 \int_{-\infty}^{\infty} e^x \chi_{E^1} \mathcal{L}_c \chi_{E^2} \, dx .
\]

This is absurd, since \( \mathcal{L}_c \chi_{E^2} (x) > 0 \) for all \( x \), which implies \( \int_{-\infty}^{\infty} e^x \chi_{E^1} \mathcal{L}_c \chi_{E^2} \, dx > 0 \). As a conclusion, \( E_c = [a, b] \) or \((-\infty, 0] \); in the latter case \( \| \chi_{E_c} \|_{L^1} = 1 \). \( \square \)

Note that \( E_c \) can be a finite or semi-infinite interval. We need preliminary analysis in order to distinguish such two cases in later sections. Define \( \ell = b - a \). With \( \| \chi_{[a,b]} \|_{L^1} = e^b - e^a = 1 \), a direct calculation from (1.9) gives

\[
e^{-b} J^*_c (\chi_{[a,b]}) = J^*_c (\chi_{[\ell - \ell, 0]}) = \frac{\sqrt{2}}{12} (e^{-\ell} + 1) - \frac{\sqrt{2} \alpha}{12} (1 - e^{-\ell}) + \frac{\sigma}{2} \int_{-\ell}^{0} e^x \mathcal{L}_c \chi_{[\ell - \ell, 0]} \, dx .
\]

(5.4)

We next calculate the nonlocal term. The solutions of the characteristic equation \( c^2 r^2 + c^2 r - \gamma = 0 \) are

\[
r = \frac{1}{2c} (-c \pm \sqrt{c^2 + 4\gamma}) ,
\]

(5.5)

which will be denoted by \( r_1, r_2 \); here \( r_1 < -1 < 0 < r_2 \) and note that \( r_1 + r_2 = -1 \). The general solution of \( c^2 u'' + c^2 u' - \gamma u = 0 \) is an element of \( \text{span} \{ e^{r_1 x}, e^{r_2 x} \} \). Solving (1.2) with \( u = \chi_{[-\ell, 0]} \) and \( \ell \) being finite, we obtain

\[
\mathcal{L}_c \chi_{[-\ell, 0]} = \begin{cases} 
A_1 e^{r_2 x}, & \text{for } x \leq -\ell, \\
\frac{1}{\gamma} + A_3 e^{r_2 x} + A_2 e^{r_2 x}, & \text{for } -\ell \leq x \leq 0, \\
A_1 e^{r_1 x}, & \text{for } x \geq 0,
\end{cases}
\]

(5.6)
Evaluating the nonlocal term in (5.4), we get
\[
\int_{-\ell}^{0} e^{x} L_{c} \chi_{[-\ell,0]}(x) \, dx = \int_{-\ell}^{0} e^{x}(\frac{1}{\gamma} + A_{3} e^{r_{1}x} + A_{2} e^{r_{2}x}) \, dx = \frac{2}{\gamma(r_{2} - r_{1})}(r_{2} + r_{1} e^{-\ell} + e^{r_{1}\ell}) .
\]

Substituting into (5.4) gives
\[
J_{c}^{*}(\chi_{[a,b]}) = e^{b} \mathcal{J}(\ell, c)
\]
if we define
\[
\mathcal{J}(\ell, c) \equiv \sqrt{\frac{2}{12}} (1 - \alpha) + \sqrt{\frac{2}{12}} (1 + \alpha)e^{-\ell} + \frac{\sigma}{\gamma(r_{2} - r_{1})}(r_{2} + r_{1} e^{-\ell} + e^{r_{1}\ell}) . \quad (5.7)
\]

As a reminder, the last term on the right hand side of (5.7) is positive, since it is generated from \(\int_{-\ell}^{0} e^{x} L_{c} \chi_{[-\ell,0]}(x) \, dx\).

Recall that \(E_{c} = [a, b]\) and \(1 = \|\chi_{[a,b]}\|_{L_{1}} = e^{b}(1 - e^{-\ell})\). Hence \(b = -\log(1 - e^{-\ell})\) and
\[
J_{c}^{*}(\chi_{E_{c}}) = \frac{1}{1 - e^{-\ell}} \mathcal{J}(\ell, c). \quad (5.8)
\]

It has been shown that \(E_{c}\) is a minimizer of \(J_{c}^{*}\). To see how such a minimizer depends on \(c\), we introduce an auxiliary function \(H(c) = \frac{\sigma}{\sqrt{c^{2} + 4\gamma}}\). Clearly \(H\) is strictly increasing, \(H(0) = 0\) and \(H(c) \to 1\) as \(c \to \infty\). Straightforward calculation gives
\[
\frac{r_{2}}{r_{2} - r_{1}} = \frac{1}{2} (1 - H(c)), \quad \frac{r_{1}}{r_{2} - r_{1}} = -\frac{1}{2} (1 + H(c)), \quad dr_{1} \quad dc > 0, \quad dr_{2} \quad dc < 0, \quad (5.9)
\]
\[
r_{2} - r_{1} = \frac{1}{H(c)}, \quad \frac{d(r_{2} - r_{1})}{dc} = -H'(c). \quad (5.10)
\]

For a given \(c\), we intend to solve a \(\ell_{c} \in (0, \infty]\) such that
\[
\frac{1}{1 - e^{-\ell_{c}}} \mathcal{J}(\ell_{c}, c) \leq \frac{1}{1 - e^{-\ell}} \mathcal{J}(\ell, c) \quad \text{holds for all } \ell > 0. \quad (5.12)
\]

When \(c\) is the correct wave speed, the case \(\ell_{c} = \infty\) corresponds to a traveling front for the limiting problem, while a finite \(\ell_{c}\) indicates a traveling pulse.
6 Traveling front

In this section we examine the case $c = \infty$. Define

$$F(c) \equiv \lim_{\ell \to \infty} J(\ell, c) = \frac{\sqrt{2}}{12} (1 - \alpha) + \frac{\sigma}{2\gamma} (1 - H(c)).$$

(6.1)

It is clear that

$$F'(c) = -\frac{\sigma}{2\gamma} H'(c) < 0.$$

(6.2)

**Lemma 6.1.** For $c > 0$, a necessary and sufficient condition for $\ell_c = \infty$ is

$$\frac{\sqrt{2}}{6\sigma} \gamma \geq H(c).$$

(6.3)

Under the hypothesis (6.3), $\chi_{(-\infty,0]}$ is the unique minimizer of $J^*_c$.

**Proof.** A direct calculation gives

$$\frac{1}{1 - e^{-\ell}} \left( \frac{\sqrt{2}}{6} e^{-\ell} - \frac{\sigma H(c)}{\gamma} e^{-\ell} + \frac{\sigma}{\gamma (r_2 - r_1)} e^{r_1 \ell} \right).$$

(6.4)

If $\ell_c = \infty$ then $\frac{1}{1 - e^{-\ell}} J(\ell, c) - F(c) \geq 0$ for all $\ell > 0$. In particular for large $\ell$, it follows from (6.4) and $r_1 < -1$ that (6.3) holds.

As to show that (6.3) is a sufficient condition, it is clear from (6.4) that $\frac{1}{1 - e^{-\ell}} J(\ell, c) - F(c) > 0$ for all $\ell > 0$ Since the last inequality is strict, $\chi_{(-\infty,0]}$ is the unique minimizer of $J^*_c$.

Besides $\ell_c = \infty$, we next investigate the additional constraints on the physical parameters such that $F(c) = 0$ is satisfied as well. We now aim at the following assumption:

(A1)* $\alpha \geq \frac{3\sqrt{2}\sigma}{\gamma} > \alpha - 1 > 0.$

**Lemma 6.2.** (A1)* is a necessary and sufficient condition for both (i) and (ii) to be held:

(i) there is a unique $c_f > 0$ such that $F(c_f) = 0$.

(ii) $\chi_{(-\infty,0]}$ is the unique global minimizer of $J^*_c$ and $J^*_c(\chi_{(-\infty,0]}) = 0$.

**Proof.** First we prove the sufficiency. If $\alpha > 1$ and $\frac{3\sqrt{2}\sigma}{\gamma} > (\alpha - 1)$, then by direct calculation $F(0) = \frac{\sqrt{2}}{12} (1 - \alpha) + \frac{\sigma}{2\gamma} > 0$ and $F(\infty) = \frac{\sqrt{2}}{12} (1 - \alpha) < 0$. Since $F$ is strictly decreasing in $c$, there is a unique $c_f$ such that $F(c_f) = 0$. This together with (6.1) yields

$$H(c_f) = 1 - \frac{2\phi(1)(\alpha - 1)\gamma}{\sigma}.$$

(6.5)
Recall that \( h_* = 1 - \frac{(\alpha - 1)\gamma}{3\sqrt{2}\sigma} \). Solving (6.5) yields (1.11). Moreover, to ensure \( \ell_c = \infty \) at the same time, we need to check that (6.3) holds with \( c = c_f \). This is true if \( \alpha \geq \frac{3\sqrt{2}\sigma}{\gamma} \).

Conversely if (i) holds, this unique \( c_f \) has to be determined by (1.11). Since \( \mathcal{F}(c_f) = 0 \) and \( \mathcal{F} \) is strictly decreasing, it follows that \( \mathcal{F}(0) > 0 \) and \( \mathcal{F}(\infty) < 0 \). Together with (6.3) lead to \((A1)^*\).

**Remark 6.3.** (a) As a consequence of (1.11), \( c_f \) is an increasing function of \( h_* \). Thus \( c_f \) is a decreasing function of \( \alpha \) but an increasing function of \( \sigma \). In fact \( c_f \) can get close to 0 or tend to \( \infty \) with appropriate values on the parameters.

(b) Note that (1.11) gives a formula to calculate the \( \Gamma \)-limit speed for the traveling front solutions.

In Theorem 1.1 a stronger condition \((A1)\) is imposed; that is, requiring \((A1)^*\) be satisfied with strict inequalities. It enables us to prove the existence of traveling wave solutions of (1.1)-(1.2) when \( \epsilon \) is small, as stated in Theorem 1.1

**Proof of Theorem 1.1.** Since \( c_f \) satisfies (6.3) with a strict inequality, there is an \( \eta > 0 \) such that if \( c \in [c_f - \eta, c_f + \eta] \) then (6.3) continue to hold and thus \( \inf_{12} J_c^* = \mathcal{F}(c) \). Furthermore, \( \mathcal{F} \) is a strictly decreasing function of \( c \), hence \( \mathcal{F}(c_f - \eta) > 0 = \mathcal{F}(c_f) > \mathcal{F}(c_f + \eta) \).

Let \( c^+ = c_f + \eta \) and \( c^- = c_f - \eta \). For \( w \in Y \), define

\[
I_{\epsilon,c}(w) = \int_{-\infty}^{\infty} e^{x} \left\{ \frac{w'^2}{2} + \frac{F_0(w)}{\epsilon} + \alpha G(w) + \frac{\sigma}{2} w L_c w \right\} dx. \tag{6.6}
\]

In other words, we take \( I_{\epsilon,c} = J_{c(\epsilon)} \) with \( c(\epsilon) = c \) for all \( \epsilon \). Consequently

\[
\Gamma \text{-lim } I_{\epsilon,c} = J_c^*. \tag{6.7}
\]

Hence there is an \( \epsilon_0 > 0 \) such that \( \inf_Y I_{\epsilon,c^-} > 0 \) and \( \inf_Y I_{\epsilon,c^+} < 0 \) if \( \epsilon < \epsilon_0 \). With only slight modification, the argument of Lemma 3.5 of [1] shows that \( I_{\epsilon,c} \) is a continuous function of \( c \). By the intermediate value theorem, there exists a \( c_\epsilon \in (c_f - \eta, c_f + \eta) \) such that \( \inf_Y I_{\epsilon,c_\epsilon} = 0 \). Furthermore by Lemma 3.2 and Lemma 3.3 \( I_{\epsilon,c_\epsilon} \) has a minimizer \( u_\epsilon \in Y \). Together with \( I_{\epsilon,c_\epsilon}(u_\epsilon) = 0 \), we set \( v_\epsilon = L_{c_\epsilon} u_\epsilon \) to obtain a traveling wave solution \( (u_\epsilon, v_\epsilon) \) of (1.1)-(1.2) with a speed \( c_\epsilon \). With \( c_\epsilon \in (c_f - \eta, c_f + \eta) \) and Lemma 4.1 as \( \epsilon \to 0 \) we see that along a subsequence \( c_\epsilon \to c_f \) and \( u_\epsilon \to \chi_E \) in \( L^2_c \) for some \( c_f \in \mathbb{R} \) and \( \chi_E \in BV_\epsilon \). As a consequence of \( \Gamma \)-convergence, \( \chi_E \) is a minimizer of \( J_{c_f}^* \) and

\[
0 = \lim_{\epsilon} I_{\epsilon,c_\epsilon}(u_\epsilon) = J_{c_f}^*(\chi_E). \]

By Lemma 6.2 \( \hat{c}_f = c_f \) and \( \chi_E = \chi_{(-\infty,0)} \). Finally it follows from the uniqueness that the limit of each subsequence is the same. Hence the convergence \( c_\epsilon \to c_f \) and \( u_\epsilon \to \chi_{(-\infty,0)} \) is along the whole sequence.
To give a physical interpretation for the assumption (A1), we take a look of the nullclines. Denoted by $\gamma_*=\gamma_*(\epsilon)$ such that the two regions enclosed by the line $v = u/\gamma_*$ and the curve $v = f_\epsilon(u)/\sigma$ are equal in area with opposing signs.

**Lemma 6.4.** If $\epsilon$ is sufficiently small then $\gamma_* = \frac{3\sqrt{2}\sigma}{\alpha}(1 + O(\epsilon))$.

*Proof.* The nullclines $v = u/\gamma_*$ and $v = f_\epsilon(u)/\sigma$ intersect at $u = 0, \mu_2^*, \mu_3^*$, where $\mu_2^*, \mu_3^*$ are the roots of the quadratic equation $u^2 - (1 + \beta_\epsilon)u + \beta_\epsilon + \frac{\alpha}{\gamma_*} = 0$. As both nullclines are anti-symmetric about $(u, v) = (\mu_2^*, \mu_2^*/\gamma_*)$, it can be easily checked $\mu_2^* = \frac{1 + \beta_\epsilon}{3}$ and $\mu_3^* = \frac{2(1 + \beta_\epsilon)}{3}$. Using the fact that $\mu_2^*\mu_3^* = \beta_\epsilon + \frac{\alpha}{\gamma_*}$, we obtain

$$\frac{\epsilon\sigma}{\gamma_*} = \frac{1}{9}(1 - 2\beta_\epsilon)(2 - \beta_\epsilon).$$

(6.8)

The lemma follows by substituting $\beta_\epsilon = \frac{1}{2} - \frac{\epsilon\alpha}{\sqrt{2}}$ into (6.8).

For sufficiently small $\epsilon$, Lemma 6.4 shows that (A1) is equivalent to

$$\alpha > 1 \quad \text{and} \quad \frac{\alpha}{\alpha - 1} \gamma_* > \gamma > \gamma_*,$$

which can be viewed as the constraints on $\gamma$ to generate a wave front.

### 7 Traveling pulse

In this section our attention turns to traveling pulse. Let $c > 0$ and suppose that there is an $\ell_c \in (0, \infty)$ such that

$$\frac{1}{1 - e^{-\ell_c}} J(\ell_c, c) \leq \frac{1}{1 - e^{-\ell}} J(\ell, c) \quad \text{holds for all } \ell > 0.$$  
(7.1)

It follows that

$$\frac{\partial}{\partial \ell} \left( \frac{J(\ell, c)}{1 - e^{-\ell}} \right) \bigg|_{\ell = \ell_c} = 0.$$  
(7.2)

For the existence of traveling pulses, in addition to (7.2) we look for suitable ranges for the parameters under which a value of $c$ can be found to satisfy

$$J(\ell_c, c) = 0.$$  
(7.3)

Then $\chi_{[a,b]}$ is a minimizer of $J^*_c$ if we take $b = -\log(1 - e^{-\ell_c})$ and $a = b - \ell_c$. A lower bound of $\ell_c$ is given in the next lemma.

**Lemma 7.1.** If $\chi_{[a,b]}$ is a minimizer of $J^*_c$ satisfying $J^*_c(\chi_{[a,b]}) = 0$ then $\alpha > 1$ and $\ell_c = b - a > \log \frac{\alpha}{\alpha - 1}$. 

Proof. It simply follows from \( J(\ell, c) = 0 \) and the last term in (5.7) is positive. □

**Lemma 7.2.** Let \( c > 0 \) and assume that \( J^*_c \) has a minimizer \( \chi_{[a,b]} \) with \( \|\chi_{[a,b]}\|_{L^1} = 1 \) and \( J^*_c(\chi_{[a,b]}) = 0 \). If \( \ell_c = b - a \) then \((\ell_c, c)\) satisfies

\[
\frac{\sigma}{2\gamma} (1 + H(c)) \left(1 - e^{-r_2\ell_c}\right) = \frac{\sqrt{2}}{12} (\alpha + 1) \quad (7.4)
\]

and

\[
\frac{\sigma}{2\gamma} (1 - H(c)) \left(1 - e^{r_1\ell_c}\right) = \frac{\sqrt{2}}{12} (\alpha - 1) . \quad (7.5)
\]

**Proof.** By direct calculation

\[
\frac{\partial}{\partial \ell} J(\ell, c) = -\frac{\sqrt{2}}{12} \left((1 + \alpha)e^{-\ell} - (1 + H(c)) \frac{\sigma}{2\gamma} (-e^{-\ell} + e^{r_1\ell}) \right). \quad (7.6)
\]

Since both (7.2) and (7.3) must hold, we obtain

\[
\frac{\partial J}{\partial \ell}(\ell_c, c) = 0 . \quad (7.7)
\]

This gives (7.4). Using it together with (7.3) yields

\[
\frac{\sqrt{2}}{12} (1 - \alpha) + \frac{\sigma}{\gamma(r_2 - r_1)} (r_2 + e^{r_1\ell}) + \frac{\sigma r_1}{\gamma(r_2 - r_1)} e^{r_1\ell} = 0 ,
\]

from which (7.5) follows. □

**Lemma 7.3.** Let \( \sigma \) and \( \gamma \) be given. If \( J^*_c \) has a minimizer \( \chi_{[a,b]} \) with \( \|\chi_{[a,b]}\|_{L^1} = 1 \) and \( J^*_c(\chi_{[a,b]}) = 0 \), then

\[
\frac{3\sqrt{2}\sigma}{\gamma} > \alpha > 1 . \quad (7.8)
\]

**Proof.** By Lemma 7.2, \((\ell_c, c)\) satisfies (7.4) and (7.5). Clearly the left hand side of (7.5) is positive, which implies \( \alpha > 1 \); a fact which is already indicated in Lemma 7.1. Define

\[
Q(\ell, c) \equiv \frac{\sqrt{2}\gamma}{12\sigma} \left\{ \frac{1 + \alpha}{1 - e^{-r_2\ell}} + \frac{\alpha - 1}{1 - e^{r_1\ell}} \right\} - 1 . \quad (7.9)
\]

From (7.4) and (7.5), eliminating \( H(c) \) yields \( Q(\ell_c, c) = 0 \).

By direct calculation

\[
\frac{\partial Q}{\partial \ell} = \frac{\sqrt{2}\gamma}{12\sigma} \left\{ - \frac{(1 + \alpha)r_2}{(1 - e^{-r_2\ell})^2} e^{-r_2\ell} + \frac{(\alpha - 1)r_1}{(1 - e^{r_1\ell})^2} e^{r_1\ell} \right\} < 0 . \quad (7.10)
\]

Since \( \frac{\partial Q}{\partial \ell} < 0 \) for all \( \ell \) and \( Q(\ell_c, c) = 0 \), it follows that \( 0 > \lim_{\ell \to \infty} Q(\ell, c) = \frac{\sqrt{2}\sigma\gamma}{6\sigma} - 1 \), which completes the proof. □
Remark 7.4. If we get a solution from $J(\ell, c) = \partial J / \partial \ell(\ell, c) = 0$, it satisfies $J(\ell, c) = 0$, and is also a solution of $\partial J / \partial \ell(\ell, c) = Q(\ell, c) = 0$. As $Q$ is a linear combination of $J$ and $\partial J / \partial \ell$, we expect that this solution satisfies $Q(\ell, c) = J(\ell, c) = 0$.

Lemma 7.5. If $J(\ell, c) = 0$ and $\partial J / \partial \ell(\ell, c) = 0$, then $\partial^2 J / \partial \ell^2(\ell, c) > 0$.

Proof. Direction calculation gives

$$\frac{\partial^2}{\partial \ell^2} J(\ell, c) = \frac{\sqrt{2}}{12} e^{-\ell}[1 + \alpha] - (1 + H(c)) \frac{3\sqrt{2} \sigma}{\gamma} (1 + r_1 e^{-r_2 \ell})]. \quad (7.11)$$

Invoking (7.1) yields

$$\frac{\partial^2}{\partial \ell^2} J(\ell, c) = -\frac{\sqrt{2}}{12} e^{-\ell} \left[ \frac{3\sqrt{2} \sigma}{\gamma} (1 + H(c))(1 + r_1 e^{-r_2 \ell}) \right] > 0. \quad (7.12)$$

Lemma 7.3 shows that (7.8) is a necessary condition for the existence of the specified minimizer, we now intend to establish that it is a sufficient condition as well. For given $\sigma$ and $\gamma$, if $\alpha \in (1, \frac{3\sqrt{2} \sigma}{\gamma})$, it follows from straightforward calculation that

$$\frac{\partial Q}{\partial c} = \frac{\sqrt{2} \gamma}{12 \sigma} \left\{ -\frac{(1 + \alpha) \ell}{(1 - e^{-r_2 \ell})^2} e^{-r_2 \ell} \frac{dr_2}{dc} + \frac{(\alpha - 1) \ell}{(1 - e^{-r_1 \ell})^2} e^{-r_1 \ell} \frac{dr_1}{dc} \right\} > 0. \quad (7.13)$$

For $c > 0$, it is clear that $\frac{\partial Q}{\partial c} < 0$ and $\lim_{\ell \to 0^+} Q(\ell, c) = \infty$. Moreover by (7.8), $\lim_{\ell \to \infty} Q(\ell, c) = \frac{\sqrt{2} \alpha}{6 \sigma} - 1 < 0$. Hence there is a unique $L = L(c)$ satisfying $Q(L(c), c) = 0$. Moreover

$$L'(c) = -\frac{\partial Q / \partial c}{\partial Q / \partial \ell} > 0. \quad (7.14)$$

Next we derive asymptotic properties of $L(c)$. As $c \to 0^+$, it can be easily checked that $r_1 \sim -\frac{\sqrt{\gamma}}{c}(1 + O(c))$ and $r_2 \sim \frac{\sqrt{\gamma}}{c}(1 + O(c))$. Plugging into $Q(L(c), c) = 0$ yields

$$\frac{\sqrt{2} \gamma}{12 \sigma} \left( \frac{1 + \alpha}{1 - e^{-\sqrt{2} \gamma L/c}} + \frac{\alpha - 1}{1 - e^{-\sqrt{2} \gamma L/c}} \right) = 1 + o(1), \quad (7.15)$$

which can be simplified as

$$1 - e^{-\sqrt{2} \gamma L/c} = \frac{\sqrt{2} \gamma \alpha}{6 \sigma} (1 + o(1)). \quad (7.16)$$

This shows $L / c \to k_1 \equiv -\frac{1}{\sqrt{\gamma}} \log(1 - \frac{\sqrt{2} \gamma \alpha}{6 \sigma})$ as $c \to 0^+$. Then from (5.7),

$$\lim_{c \to 0^+} J(L(c), c) = \frac{\sqrt{2}}{6}.$$
Next if \( c \to \infty \) then \( r_2 \sim \frac{c}{\gamma^2} \) and \( r_1 \sim -1 \). As \( L(c) \) is an increasing function, \( \lim_{c \to \infty} L(c) \) exists; however it cannot be a finite number, from the fact of \( Q(L(c), c) = 0 \) which then gives

\[
\frac{1 + \alpha}{1 - \lim_{c \to \infty} e^{-r_2L}} + \frac{\alpha - 1}{1 - \lim_{c \to \infty} e^{r_1L}} = \frac{6\sqrt{2} \sigma}{\gamma}.
\]

(7.17)

This indeed also eliminates the scenario that \( \lim_{c \to \infty} r_2L = 0 \). Furthermore, \( \limsup_{c \to \infty} r_2L = \infty \) would lead to \( \frac{3\sqrt{2} \sigma}{\gamma} = \alpha \), which violates (7.8). Thus the only possibility left is \( \limsup_{c \to \infty} r_2L \equiv R_L \), a positive constant that can be solved from

\[
\frac{1 + \alpha}{1 - \lim_{c \to \infty} e^{-r_2L}} + \alpha - 1 = \frac{6\sqrt{2} \sigma}{\gamma}.
\]

Consequently along this subsequence,

\[
L \sim \frac{R_L c^2}{\gamma} \text{ as } c \to \infty
\]

(7.18)

and then

\[
\lim_{c \to \infty} J(L(c), c) = \frac{\sqrt{2}}{12} (1 - \alpha) < 0.
\]

By the intermediate value theorem there is a \( c > 0 \) such that \( J(L(c), c) = 0 \). As indicated in Remark 7.4 if \( (\ell, c) \) satisfies \( J(\ell, c) = Q(\ell, c) = 0 \), then \( \frac{\partial J}{\partial \ell}(\ell, c) = 0 \) should be true as well. Indeed,

\[
\frac{\partial J}{\partial \ell}(L(c), c) = -\frac{\sqrt{2}}{12} (1 + \alpha) e^{-L} + \frac{\sigma r_1}{\gamma (r_2 - r_1)} (-e^{-L} + e^{r_1L})
\]

(7.19)

\[
= -\frac{\sqrt{2}}{12} (1 + \alpha) e^{-L} + \frac{\sigma r_1}{\gamma (r_2 - r_1)} (-e^{-L} + e^{r_1L}) + J(L(c), c)
\]

\[
= \frac{\sqrt{2}}{12} (1 - \alpha) + \frac{\sigma}{\gamma (r_2 - r_1)} (r_2 + e^{r_1L}) + \frac{\sigma r_1}{\gamma (r_2 - r_1)} e^{r_1L}
\]

\[
= \frac{\sigma}{2\gamma} (1 - H(c)) (1 - e^{r_1L}) - \frac{\sqrt{2}}{12} (\alpha - 1).
\]

(7.20)

Also, we see from (7.19) that

\[
e^L \frac{\partial J}{\partial \ell}(L(c), c) = \frac{\sigma}{2\gamma} (1 + H(c)) (1 - e^{-r_2L}) - \frac{\sqrt{2}}{12} (\alpha + 1).
\]

(7.21)
Eliminating $H(c)$ from (7.20)-(7.21) yields
\[
\frac{\partial J}{\partial \ell} (L(c), c) = -\frac{\sigma}{\gamma} [(1 - e^{r_1 L})^{-1} + e^L(1 - e^{-r_2 L})^{-1}]^{-1} Q(L(c), c) = 0.
\] (7.22)

By (5.8), a direct calculation gives
\[
\left. \frac{\partial^2 J^*_c}{\partial \ell^2} \right|_{(L(c), c)} = \left. \frac{1}{1 - e^{-\ell}} \frac{\partial^2 J}{\partial \ell^2} \right|_{(L(c), c)}.
\]

As a consequence of Lemma (7.3), this solution $(L(c), c)$ has to be a local minimizer of $J$ and we use $c_p$ to designate the speed $c$ obtained from the above calculation. In the remainder of the proof, let $c = c_p$. Then it follows from (7.21) and (7.6) that
\[
\frac{\partial J}{\partial \ell} = e^{-\ell} \left( -\frac{\sqrt{2}}{12} (1 + \alpha) + (1 + H(c_p)) \frac{\sigma}{2\gamma} (1 - e^{-r_2 \ell}) \right) \begin{cases} > 0, & \text{if } \ell > L(c_p), \\ < 0, & \text{if } \ell < L(c_p), \end{cases}
\] (7.23)
from which we know $L(c_p)$ is a global minimizer of $J(\cdot, c_p)$; thus denote $L(c_p)$ by $\ell_p$, the notion corresponding to $\ell_c$ when $c = c_p$. If we take $c = c_p$ and $b = -\log(1 - e^{-\ell_p})$ then $J^*_c$ has a minimizer $\chi|_{[b-\ell_p,b]}$ with $J^*_c(\chi|_{[b-\ell_p,b]}) = 0$ and $\|\chi|_{[b-\ell_p,b]}\|_{L^1_\ell} = 1$. It follows from Lemma (7.1) that $\ell_p > \log \frac{\alpha+1}{\alpha-1}$.

In summary, we have proved the following lemma under the assumption $(A2)$.

**Lemma 7.6.** Let $\sigma$ and $\gamma$ be given. If $(A2)$ is satisfied, there exists a $c_p > 0$ such that, when $c = c_p$, $J^*_c$ has a minimizer $\chi|_{[a,b]}$ with $\|\chi|_{[a,b]}\|_{L^1_\ell} = 1$ and $J^*_c(\chi|_{[a,b]}) = 0$. This $c_p$ is referred to as a $\Gamma$-limit speed for the traveling pulse solutions and $b - a > \log \frac{\alpha+1}{\alpha-1}$.

We next study the uniqueness of $c_p$, as $\Gamma$-limit speed. This will in turn implies the uniqueness of both $L(c_p)$ and the minimizer of $J^*_c$. In the first step, we prove the following lemma.

**Lemma 7.7.** Let $\ell, c > 0$. Then
\[
\frac{\partial J}{\partial c} (\ell, c) < 0.
\]
Proof. If \( \ell, c > 0 \), it follows from (5.7), the first equation in (5.11) and (5.5) that

\[
\gamma \sigma \frac{\partial J}{\partial c} (\ell, c) = \frac{\partial}{\partial c} \left[ \frac{1}{r_2 - r_1} (r_2 + r_1 e^{-\ell} + e^{r_1 \ell}) \right] \\
= \frac{\partial}{\partial c} \left[ \frac{1}{r_2 - r_1} (r_1 (1 + e^{-\ell}) + e^{r_1 \ell}) \right] \\
= \frac{\partial}{\partial c} \left[ H(c) (r_1 (1 + e^{-\ell}) + e^{r_1 \ell}) \right] \\
= H'(c) (r_1 (1 + e^{-\ell}) + e^{r_1 \ell}) + H(c) \frac{\partial r_1}{\partial c} (1 + e^{-\ell} + \ell e^{r_1 \ell}) \\
= \frac{4\gamma}{(c^2 + 4\gamma)^{3/2}} \left[ r_1 (1 + e^{-\ell}) + e^{r_1 \ell} + 1 + e^{-\ell} + \ell e^{r_1 \ell} \right] \\
= \frac{4\gamma}{(c^2 + 4\gamma)^{3/2}} [r_1 (1 + e^{-\ell}) + e^{r_1 \ell} + 1 + e^{-\ell} + \ell e^{r_1 \ell}] \\
=\]

It is easy to verify that \( 2r_1 H(c) = -1 - H(c) \). Using such an identity we obtain

\[
\frac{\gamma \sigma \partial J}{\partial c} = \frac{2\gamma}{H(c)(c^2 + 4\gamma)^{3/2}} \left[ -(1 + H(c))(1 + e^{-\ell}) + 2H(c)e^{r_1 \ell} + 1 + e^{-\ell} + \ell e^{r_1 \ell} \right] \\
= \frac{2\gamma}{(c^2 + 4\gamma)^{3/2}} \mathcal{K},
\]

where

\[
\mathcal{K}(\ell, c) \equiv -1 - e^{-\ell} + 2e^{r_1 \ell} + \frac{\ell}{H(c)} e^{r_1 \ell}. \tag{7.24}
\]

A direct calculation yields

\[
\frac{\partial \mathcal{K}}{\partial \ell} = e^{-\ell} + 2r_1 e^{r_1 \ell} + \frac{1}{H(c)} (e^{r_1 \ell} + r_1 \ell e^{r_1 \ell}) \\
= e^{-\ell} - e^{r_1 \ell} + \frac{r_1 \ell}{H(c)} e^{r_1 \ell}. \tag{7.25}
\]

It suffices to show \( \mathcal{K} < 0 \) for all \( \ell, c > 0 \). Note that \( \mathcal{K}(0, c) = 0 \). Then for small \( \ell \), a Taylor’s expansion gives

\[
\mathcal{K} = -1 - (1 - \ell + \frac{\ell^2}{2}) + 2(1 + r_1 \ell + \frac{r_1^2 \ell^2}{2}) + \frac{\ell}{H(c)} (1 + r_1 \ell) + O(\ell^3) \\
= \ell^2 (-\frac{1}{2} + r_1^2 + \frac{r_1}{H(c)}) + O(\ell^3) \\
= \ell^2 (-\frac{1}{2} + r_1 r_2) + O(\ell^3) \\
= \ell^2 (-\frac{1}{2} - \frac{\gamma}{c^2}) + O(\ell^3) \\
< 0.
\]
Therefore $\mathcal{K} = \frac{\partial K}{\partial \ell} = 0$ and $\frac{\partial^2 K}{\partial \ell^2} < 0$ at $\ell = 0$, which implies $\frac{\partial K}{\partial \ell} < 0$ for small positive $\ell$. Besides $\ell = 0$, we claim that there exists another unique non-negative root of $\frac{\partial K}{\partial \ell}$. Indeed such a root satisfies

$$e^{r_2 \ell} = 1 - \frac{r_1}{H(c)} \ell,$$

which corresponds to an intersection point of the exponential function $y = e^{r_2 \ell}$ and the straight line $y = 1 - \frac{r_1}{H(c)} \ell$ in the $(\ell, y)$ plane.

As $\frac{\partial K}{\partial \ell} < 0$ for small positive $\ell$, the straight line lies above the graph of exponential function near $\ell = 0$. On the other hand the exponential function will dominate the straight line when $\ell$ is large, hence there exists another unique intersection point, which is called $\ell_1$. Clearly $\frac{\partial K}{\partial \ell} < 0$ for $\ell < \ell_1$ and $\frac{\partial K}{\partial \ell} > 0$ for $\ell > \ell_1$. As $\ell \to \infty$, it is easily checked that $\mathcal{K} \to -1$. We can now conclude that $\mathcal{K}$ dips below 0 near $\ell = 0$, reaching a negative minimum at $\ell = \ell_1$, then increases again to reach $\mathcal{K} = -1$ at $\ell = \infty$. Hence for any given $c > 0$, we know $\mathcal{K} < 0$ for $\ell > 0$. This concludes the proof of the lemma.

Lemma 7.8. Under the assumption of Lemma 7.6, the $\Gamma$-limit speed $c_p$ is unique.

Proof. We argue indirectly. From Lemma 7.6, suppose that $\chi_{[a_i, b_i]}$ is a minimizer of $J_{c_i}$ for $i = 1, 2$ and $\ell_i = b_i - a_i$ with $\ell_2 > \ell_1$. By (7.14), it is necessary that $c_2 > c_1$. Then $J(\ell, c_i) > J(\ell, c_i) = 0$ for all $\ell \neq \ell_i$. Together with $\frac{\partial J}{\partial c} < 0$, we obtain $0 = J(\ell_1, c_1) > J(\ell_1, c_2) > J(\ell_2, c_2) = 0$, which is absurd.

Remark 7.9. As a consequence of Lemma 7.6 and Lemma 7.8, when consider a positive speed for the $\Gamma$-limit, $c_p$ is the unique speed and $J_{c_p}$ has a unique minimizer $\chi_{[a,b]}$.

The proof of Theorem 1.2 is analogous to that of Theorem 1.1, we omit it. Finally recall $\gamma_*$ from Lemma 6.4 and note that when $\epsilon$ is sufficiently small, (A2) is equivalent to

$$\alpha > 1 \quad \text{and} \quad \gamma_* > \gamma. \quad (7.26)$$

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