Langevin description of critical phenomena with two symmetric absorbing states

Omar Al Hammal,1 Hugues Chaté,2 Ivan Dornic,2 and Miguel A. Muñoz1

1Instituto de Física Teórica y Computacional Carlos I,
Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

2CEA – Service de Physique de l’État Condensé, CEN Saclay, 91191 Gif-sur-Yvette, France

(Dated: November 6, 2018)

On the basis of general considerations, we propose a Langevin equation accounting for critical phenomena occurring in the presence of two symmetric absorbing states. We study its phase diagram by mean-field arguments and direct numerical integration in physical dimensions. Our findings fully account for and clarify the intricate picture known so far from the aggregation of partial results obtained with microscopic models. We argue that the direct transition from disorder to one of two absorbing states is best described as a (generalized) voter critical point and show that it can be split into an Ising and a directed percolation transitions in dimensions larger than one.

PACS numbers: 05.70.Ln, 05.50.+q, 02.50.-r, 64.60.Ht

The classification of equilibrium phase transitions into universality classes by just identifying their relevant ingredients (i.e. symmetries, conservation laws, and dimensionalities) constitutes one of the most remarkable achievements of modern statistical mechanics. Ginzburg-Landau-Wilson free-energy functionals, either in their static [1] or dynamic versions (usually written in terms of Langevin equations, as the Model A of Hohenberg and Halperin describing the kinetic Ising class [2]) provide a compact and systematic theoretical framework to represent universality classes: Being continuous (coarse-grained) theories, they are thus susceptible to analytical studies by using the tools of statistical field theory and the renormalization group.

Out of equilibrium, the situation is far from being as satisfactory. In spite of evidence of universality, the relevant ingredients for classification are often not known, and continuous descriptions in terms of Langevin equations or dynamical generating functionals are mostly lacking. For instance, within the prototypical case of absorbing phase transitions, where the ordered “absorbing” states are devoid of fluctuations allowing the return to the disordered “active” ones, the directed percolation (DP) class is prominent and very robust [3, 4]. Loosely defined as the class of all phase transitions into a single effective absorbing state without extra symmetries or conservation laws, it is represented by a Langevin equation which can be renormalized satisfactorily [3, 4]. But such a continuous description or even a controlled renormalisation procedure is lacking for the also rather well-established class of phase transitions into one of two symmetric absorbing states, despite some thoughtful attempts [5].

In this Letter, we propose, on the basis of general considerations, a Langevin equation accounting for critical phenomena with two ($Z_2$)-symmetric absorbing states. Pending a renormalisation group approach, we study the phase diagram of this equation by mean-field arguments and direct numerical integration and show that it fully accounts for the rather intricate picture known so far from the aggregation of partial results obtained with microscopic models, a situation which we briefly recall now before proceeding with our findings.

The lack of consensus about the characterization of phase transitions into two symmetric absorbing states is reflected by the different names given to this class in the literature [3, 5, 6, 7, 8, 9, 10]. Sometimes simply called DP2 (marking the existence of two absorbing states), or, more accurately, “directed Ising” (referring to both the Ising $Z_2$-symmetry and the presence of absorbing states), it is most often called “parity-conserving” because of its usual interface representation where, in one space dimension, diffusing particles $A$ undergo the reactions $A → 3A$, $2A → ∅$. (These branching and annihilating random walks with an even number of offsprings stand for interfaces between domains of the + and − absorbing states, see Fig. 1a.). However, there now exists ample evidence that the conservation of the parity (of the number of interfaces or particles) is not the relevant ingredient [3, 5]. Moreover, this particle representation just gives rise to trivial phase transitions (at zero branching rate and with mean-field exponents) in higher space dimensions $d ≥ 2$. Rather, as was briefly hinted at in [11] and suggested in [12], we endorse the viewpoint that this type of critical phenomenon, where interfaces between the two symmetric absorbing states branch and annihilate, is best described as the (generalized) voter class in the sense of $[13]$. Recall that in the usual voter model $[14]$, randomly chosen Ising spins take the value of one of their randomly chosen neighbors: then only interfaces (+− pairs) evolve, with $+− → + +$ or $−−$ with equal probability $\frac{1}{2}$. In dimension $d = 2$, this model is critical and at its upper critical dimension. It is characterized by a marginal ordering process during which the density of interfaces (+− pairs) decays like $1/\ln t$ [14]. In contrast, this simple rule is not critical in other dimensions: for $d = 1$, it coincides with the annihilation process $2A → ∅$, while in $d = 3$ it leads to a disordered phase. One possible generalisation of this “classical” voter rule preserving the $Z_2$ symmetry is to al-
low for spin swaps $+\rightarrow -+$ [which in $d = 1$ amounts to the branching reaction $A \rightarrow 3A$, see Fig. 1(a)]. It is simple to realize that this generalisation (and others) allows for tuning the model at criticality in any dimension, that in $d = 2$ the critical properties of the original model are preserved, and that in $d = 1$ the critical point is nothing but of the DP2/directed Ising/parity-conserving class. It is thus both meaningful and useful to denote this class, in any dimension, as the generalised voter (GV) class.

Extrapolating from their numerical results in $d = 2$, Dornic et al. [13] conjectured that transitions with $Z_2$ symmetry and no bulk fluctuations (i.e. with two symmetric absorbing states) should all display GV critical points. In a recent paper, though, Droz, Ferreira, and Lipowski [14] somewhat challenged this picture by showing that such a transition, in some versions of two-dimensional generalised voter models, may not be direct, but split into a first Ising-class, symmetry-breaking transition followed later by a DP-class transition to the absorbing state chosen after the previous Ising critical point. One can thus legitimately wonder whether the direct GV class transitions observed by Dornic et al. —as well as Droz et al.— exist at all (at codimension-1 manifolds of parameter space) or whether they are just the artifact of close-by Ising and DP transitions, coinciding only at special points like the classical voter model.

To summarize, in two space dimensions, the question of the possibility of the merging of an Ising and a DP line into a full GV line remains open, while in one space-dimension the relevance of parity conservation is still debated. Below, we address both of these points and clarify the nature of all phase transitions in the presence of two symmetric absorbing states via the introduction of a unique, well-behaved, Langevin equation for this general problem.

Our proposal is by no means unique, but it is constrained by general guidelines: The equation has to be symmetric under reversal of the field ($\phi \rightarrow -\phi$), which takes values between two absorbing barriers, set, without loss of generality, at $\pm 1$ ($\phi \in [-1,1]$). Because, in two dimensions, the transition can be split into an Ising and a DP point, each of the absorbing barriers must be similar to those of the Langevin equation for DP, i.e. the square root of the distance to each barrier must appear as a multiplicative factor of the noise. This is also corroborated by the fact that the Langevin equation proposed once for the classical (integrable) voter model [10]:

$$\partial_t \phi = D \nabla^2 \phi + \sigma \sqrt{1 - \phi^2} \eta$$  

where $\eta$ is a Gaussian noise delta-correlated in space and time, was recently shown to behave as expected (i.e. logarithmic decay of the density of interfaces) [17, 18]. In order to represent the possibility of Ising-like spontaneous symmetry breaking, we need to add a minimal number of polynomial terms with odd powers of $\phi$. (At least two free parameters are needed to describe for the splitting scenario uncovered by Droz et al.). We are then almost ineluctably led to the following equation:

$$\partial_t \phi = (a\phi - b\phi^3)(1 - \phi^2) + D \nabla^2 \phi + \sigma \sqrt{1 - \phi^2} \eta$$  

Note that removing the $1 - \phi^2$ factors, both in the deterministic force and in the noise amplitude, leads to the Model A for the Ising class [2]. Let us now describe the different possible regimes of Eq. (2) in the $(a, b)$ parameter plane, a natural choice since for $a = b = 0$ one recovers the voter equation (1).

We start with a discussion at the mean-field level, i.e. reducing Eq. (2) to its first term, rewritten as $-V'(\phi)$, with the “potential” $V(\phi) = -\frac{a}{2}\phi^2 + \frac{ab}{4}\phi^4 - \frac{b}{6}\phi^6$.

$b > 0$: separate Ising and DP transitions. For $a < 0$, $\phi = 0$ is locally stable, while it is unstable for $a > 0$ (Fig. 1b). The $b\phi^3$ term enforces stability as in Model A (even if the absorbing barriers are removed). At $a = -1$, where the local stability around $\phi = 0$ changes, the symmetry is broken, and we expect an Ising transition in the full problem [19]. Increasing $a > 0$, the minima of the potential move progressively closer to the absorbing barriers and, for $a = b$, a collapse onto the absorbing barrier selected by the previous spontaneous symmetry breaking takes place. This second transition should be in the DP class once fluctuations are incorporated.

$b \leq 0$: unique GV transition. If $b = 0$ the potential is Gaussian around the origin, which is a stable extremum if $a < 0$, and unstable otherwise. The transition is at $a = 0$, but there is no $\phi^3$ term in the potential forcing it to be continuous: the location of the potential minimum changes abruptly from $\phi = 0$ to $\phi = \pm 1$. This time the
symmetry breaking occurs simultaneously with a fall into
one of the absorbing states. The critical point should
be that of the voter model once fluctuations are taken
into account. For \( b < 0 \), the \( \phi^4 \) term is present in the
potential, but it is not stabilizing, and it does not lead
to a continuous transition. For \( a < 0 \) the origin is locally
stable, and there are also extrema at the barrier, and
additional maxima at \( \pm \sqrt{a/b} \) that may or may not lie
in the interval \([-1, 1]\). As \( a \) approaches 0 the extrema
move closer to the origin, and the minima at the barriers
deepen [Fig. 1(c)]. At some point, the stability is globally
changed and we expect to have a situation similar to the
one for \( b = 0 \), i.e. a unique GV transition.

The naive mean-field parameter diagram of Eq. (2)
thus consists of a symmetry-breaking line along the
\( b \)-axis, joined by a DP line \( a = b \) at the origin. For \( b \leq 0 \),
the two lines merge, and a unique GV transition occurs,
while for \( b > 0 \) the Ising-like symmetry breaking does
not lead directly to one of the absorbing states, a sit-
tuation similar to that uncovered by Droz et al. More
elaborated, self-consistent mean-field approaches lead to
similar results, albeit with the symmetry-breaking line
not being along the \( b \)-axis anymore.

In order to go beyond mean-field and elucidate the in-
fluence of fluctuations in the phenomenology of equation
(2), we integrate it numerically using the approach de-
tailed in [18, 20]. This method is designed to circumvent
the numerical difficulties associated with the presence
a singular square-root noise near an absorbing state in
Langevin equations. It consists in separating the integra-
tion of the deterministic terms from the stochastic piece,
the latter being performed by sampling exactly the con-
tional probability distribution function (p.d.f.) solution
of the associated (forward) Fokker-Planck equation. This
calculated value is then used to evolve the remaining de-
terministic part. Here, the Fokker-Planck equation for
\( \frac{d\phi}{dt} = \sigma \sqrt{1 - \phi^2} \eta(t) \) can be solved through an eigenfunc-
tion expansion, leading to a rather complicated p.d.f.,
with a continuous part and two delta peaks at the barri-
ers \( \phi = \pm 1 \). However, as remarked in [18], discarding the
(exponentially suppressed) influence of the nearest
absorbing state, one can treat the noise term as two in-
dependent DP barriers, and apply the existing efficient
procedure for the DP noise. Thereby, using time-mesh
\( \Delta t = 0.1 \), space-mesh \( \Delta \sigma = 1 \), and the parameter values
\( D = 0.5 \) and \( \sigma^2 = 0.8 \), Eq. (2) can be faithfully integrated.

We first present our results obtained in two dimen-
sions, which agree qualitatively with the phase diagram
predicted from mean-field arguments:

For \( b \) larger than some \( b^* \) \( (b^* \approx 0.50 \) with our choice
of parameters), two distinct transitions are encountered
upon increasing \( a \) and they merge linearly as \( b \to b^* \)
[Fig. 2(a) and 2(b)]. At low \( a \), any initial condition leads to
a disordered state \( \langle \phi \rangle = 0 \). For \( a_{\text{Ising}} < a < a_{\text{DP}} \),
the steady state, reached after some phase ordering transient,
has a non-zero magnetization \( m = \langle \phi(r, t) \rangle_r \) but is still
fluctuating \( (0 < |m| < 1) \) and the density of interfaces
\( \rho = 1 - \langle \phi(r, t)\phi(r + e_\ell, t) \rangle \) (where \( e_\ell \) represents any of
the unit vectors of the underlying square lattice) is finite.
For \( a > a_{\text{DP}} \), ordering is complete \( (m = \pm 1, \rho = 0) \). We
have checked that the symmetry-breaking transition oc-
curring at \( a_{\text{Ising}} \) is in the Ising universality class, both by
steady-state finite-size scaling analysis, and by measur-
ing the decay of the time auto-correlation function from
disordered initial conditions. For instance, the curves for
the so-called Binder cumulant at different system sizes
all cross each other around the universal value \( U^* \approx 0.61 \)
(not shown). We have also checked that the fall into one
of the absorbing states is a DP-class phase transition, as
e.g. testified by the algebraic decay in time of the activity
in a large system after a critical quench [Fig. 2(d)].

For \( b < b^* \), a unique transition is observed at \( a = a_{\text{GV}} \),
across which the steady-state magnetization jumps from
zero to \( \pm 1 \). On the other hand, the density of interfaces
goes continuously to zero as \( a \to a_{\text{GV}} \) from below (not shown).
At \( a = a_{\text{GV}} \), the logarithmic time decay of \( \rho \) is
one of the hallmarks of the GV class [Fig. 2(c)].

In one space dimension, where general arguments ex-
clude the existence of an Ising transition, we expect a
unique, continuous direct transition from a disordered
phase \( (m = 0, \rho > 0) \) to one of the two absorbing states
\( (m = \pm 1, \rho = 0) \). It is therefore near at hand to sur-
mise that the ensuing critical point should be character-
It is not clear to us whether Eq. (2) can be derived from first-principles, especially given the importance, in microscopic models exhibiting the GV transition, of the annihilation reaction $2A \rightarrow \emptyset$, for which this is reportedly impossible [21]. Nevertheless, both the generating functional associated to Eq. (2) and the one rigorously obtainable from the master equation for the corresponding reaction-diffusion processes enjoy the same symmetry and feature similar characteristic invariants. Thus analytical studies of our proposal appear promising, be it either within renormalisation-group perturbative calculations (maybe akin to those of [22]), or via the nonperturbative approach put forward in [3]. First results under the latter auspices are very encouraging [23].

We warmly thank P. Grassberger for stimulating discussions. M. A. M. acknowledges financial support from the Spanish MCyT (FEDER) under project BFM2001-2841. I.D. is also indebted to the Service de Physique Théorique (CEA Saclay) for generous financial support.

[1] D. Amit, *Field theory, the renormalization group and critical phenomena*, (World Scientific, Singapore, 1984).
[2] P.C. Hohenberg and B.I. Halperin, Rev. Mod. Phys. 49, 435, (1977).
[3] H. Hinrichsen, Adv. Phys. 49, 815 (2000).
[4] H.K. Janssen, Z. Phys. B 42, 151 (1981). P. Grassberger, Z. Phys. B 47, 365 (1982).
[5] L. Canet et al., Phys. Rev. Lett. 92, 195703 (2004).
[6] J. L. Cardy and U. C. Täuber, J. Stat. Phys. 90, 1 (1998).
[7] P. Grassberger, F. Krause, and T. von der Twer, J. Phys. A 17, L105 (1984). I. Jensen, J. Phys. A 26, 3921 (1993).
[8] N. Menyhárd and G. Ódor, J. Phys. A 29, 7739 (1996).
[9] W. Hwang et al., Phys. Rev. E 57, 6438 (1998).
[10] G. Ódor, Rev. Mod. Phys. 76, 663 (2004).
[11] H. Hinrichsen, Phys. Rev. E 55, 219 (1997).
[12] J. Kockelkoren and H. Chaté, Phys. Rev. Lett. 90, 125701 (2003).
[13] I. Dornic et al., Phys. Rev. Lett. 87, 045701 (2001).
[14] T. M. Liggett, *Interacting Particle Systems*, (Springer, New York, 1985).
[15] M. Droz, A. L. Ferreira, and A. Lipowski, Phys. Rev. E 67, 056108 (2003).
[16] R. Dickmann and A. Yu. Tretyakov, Phys. Rev. E 52, 3218 (1995). See also: M. A. Muñoz, G. Grinstein, and Y. Tu, Phys. Rev. E 56, 5101 (1997).
[17] H. K. Janssen, Phys. Cond. Matter 17, S1973 (2005).
[18] I. Dornic, H. Chaté, and M. A. Muñoz, Phys. Rev. Lett 94, 100601 (2005).
[19] G. Grinstein, C. Jayaprakash and Y. He, Phys. Rev. Lett 55, 2527 (1985).
[20] L. Pechenik and H. Levine, Phys. Rev. E 59, 3893 (1999).
[21] See, e.g., M. A. Muñoz, Phys. Rev. E 57, 1377 (1998).
[22] J. L. Cardy, Nucl. Phys. B 565, 506 (2000).
[23] L. Canet et al., cond-mat/0505170.