On $k$-measures and Durfee squares of partitions

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Abstract

Recently, Andrews, Bhattacharjee and Dastidar introduced the concept of $k$-measure of an integer partition, and proved a surprising identity that the number of partitions of $n$ which have 2-measure $m$ is equal to the number of partitions of $n$ with a Durfee square of side $m$. The authors asked for a bijective proof of this result and also suggested a further exploration of the properties of the number of partitions of $n$ which have $k$-measure $m$ for $k \geq 3$. In this note, we perform these tasks. That is, we obtain a short combinatorial proof of the result of Andrews, Bhattacharjee and Dastidar, and using this proof, we obtain a natural generalization for $k$-measures.

1 Introduction

Andrews, Bhattacharjee and Dastidar \cite{1} introduced a new statistic of integer partitions, which they named as the $k$-measure.

**Definition 1.** The $k$-measure of a partition is the length of the largest subsequence of parts in the partition in which the difference between any two consecutive parts of the subsequence is at least $k$.

Recall that the Durfee square of a partition is the largest square that can be constructed in the Ferrers diagram of the partition beginning from the top left corner. Andrews, Bhattacharjee and Dastidar \cite{1} found a deep connection between these two statistics of a partition as described in the theorem below.

**Theorem 2** (Andrews, Bhattacharjee and Dastidar (2022)). The number of partitions of $n$ with 2-measure $m$ equals the number of partitions of $n$ with Durfee square of side $m$.

The authors proved Theorem 2 using $q$-series analysis. They concluded the paper by noting that a theorem as simply stated as Theorem 2 should definitely have a bijective proof. They also suggested a further study of the properties of $k$-measures of partitions for $k > 2$. In a subsequent work, Andrews, Chern and Li \cite{2} used generalized Heine transformations to establish some trivariate generating function identities counting both the length and the $k$-measure for partitions and distinct partitions, respectively. As a corollary of their result, they obtained the following refinement of Theorem 2.
Theorem 3 (Andrews, Chern and Li). The number of partitions of \( n \) with \( l \) parts and 2-measure \( m \) equals the number of partitions of \( n \) with \( l \) parts and Durfee square of side \( m \).

As another corollary of their result, they got the following result for partitions with distinct odd parts. They defined \( l(\lambda) \) to be the number of parts of \( \lambda \) and \( \mu_2(\lambda) \) to be the 2-measure of \( \lambda \).

Theorem 4 (Andrews, Chern and Li). The excess of the number of partitions \( \lambda \) of \( n \) with \( l(\lambda) + \mu_2(\lambda) \) even over those with \( l(\lambda) + \mu_2(\lambda) \) odd equals the number of partitions of \( n \) into distinct odd parts.

Also refer to [3] for some recent refinements of Theorem 2 and an alternate \( q \)-series proof of Theorem 3. In Section 2 we obtain a short combinatorial proof of Theorem 2. In fact, our proof also proves the more general result of Theorem 3. In Section 3 we use the ideas in this proof to generalize Theorem 2 for \( k \)-measures.

2 Proof of Theorem 2

Let

- \( a_m(n) \) denotes the number of partitions of \( n \) with 2-measure \( m \).
- \( b_m(n) \) denotes the number of partitions of \( n \) with Durfee square of side \( m \) by \( b_m(n) \).

Thus, Theorem 2 asserts that \( a_m(n) = b_m(n) \) for all \( m \) and \( n \). We begin by noting that

\[
a_m(n) = c_m(n) - c_{m+1}(n),
\]

and

\[
b_m(n) = d_m(n) - d_{m+1}(n),
\]

where \( c_m(n) \) and \( d_m(n) \) are defined as follows.

- \( C_m(n) \) denotes the set of partitions of \( n \) in which there exists a subsequence of length \( m \) of parts in the partition in which the difference between any two consecutive parts of the subsequence is at least 2.
- \( D_m(n) \) denotes the set of partitions of \( n \) which have at least \( m \) parts greater than or equal to \( m \).
- \( c_m(n) = |C_m(n)| \).
- \( d_m(n) = |D_m(n)| \).

From here, it immediately follows that \( a_m(n) = b_m(n) \) for all \( m \) and \( n \) if and only if \( c_m(n) = d_m(n) \) for all \( m \) and \( n \). That is Theorem 2 is equivalent to Theorem 5 described below, for which we provide a short bijective proof.
Theorem 5. The number of partitions of $n$ in which there exists a subsequence of length $m$ of parts in the partition in which the difference between any two consecutive parts of the subsequence is at least 2 is equal to the number of partitions of $n$ which have at least $m$ parts greater than or equal to $m$.

Proof. We construct a bijection $\phi$ between the sets $C_m(n)$ and $D_m(n)$. Prior to that, we describe the motivation behind this bijection with the help of an example. Suppose $m = 5$. Then, we are given a subsequence $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ with

$$\lambda_1 \geq \lambda_2 + 2, \lambda_2 \geq \lambda_3 + 2, \lambda_3 \geq \lambda_4 + 2, \lambda_4 \geq \lambda_5 + 2.$$  

Thus, in particular, we have

$$\lambda_1 \geq 9, \lambda_2 \geq 7, \lambda_3 \geq 5, \lambda_4 \geq 3, \lambda_5 \geq 1.$$  

We need to map this to a partition with all parts greater than or equal to 5. Note that the average of these lower bounds $9, 7, 5, 3, 1$ is equal to 5. Therefore, we modify all the members of the subsequence $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ in such a way that their lower bound sequence $9, 7, 5, 3, 1$ becomes the average sequence $5, 5, 5, 5, 5$. To do this, we basically reduce the first number by 4, reduce the second number by 2, keep the third number unchanged, increase the fourth number by 2 and increase the last number by 4. That is, we map the subsequence $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ to $(\lambda_1 - 4, \lambda_2 - 2, \lambda_3, \lambda_4 + 2, \lambda_5 + 4)$. Clearly, each component of this vector is a number greater than or equal to 5. To make the pattern even more clear, we can also express the above vector as $(\lambda_1 - 4, \lambda_2 - 2, \lambda_3 - 0, \lambda_4 - (-2), \lambda_5 - (-4))$. Now, we describe the bijection $\phi$ for a general $m$.

Suppose we have a partition $\lambda \in C_m(n)$. Further, suppose $(\lambda_1, \lambda_2, \cdots, \lambda_m)$ be a subsequence of parts of $\lambda$ in which the difference between any two consecutive parts of the subsequence is at least 2. That is, $\lambda_i \geq \lambda_{i+1} + 2$ for all $i$. In particular, we have $\lambda_i \geq 1 + 2(m - i)$ for all $1 \leq i \leq m$. We define the map $\phi$ to be such that the elements of $\lambda$ not in our chosen subsequence $(\lambda_1, \lambda_2, \cdots, \lambda_m)$ are left unchanged, and $(\lambda_1, \lambda_2, \cdots, \lambda_m)$ is mapped under $\phi$ to

$$\left(\lambda_1 - (m - 1), \lambda_2 - (m - 3), \cdots, \lambda_i - (m - (2i - 1)), \cdots, \lambda_{m-1} + (m - 3), \lambda_m + (m - 1)\right). \quad (1)$$

We verify that the resultant partition is indeed a member of $D_m(n)$. For that, we note two things. First, we have

$$\sum_{i=1}^{m} \left(\lambda_i - (m - (2i - 1))\right) = \sum_{i=1}^{m} \lambda_i.$$  

That is, the other terms with the positive and negative signs cancel out. Secondly, using $\lambda_i \geq 1 + 2(m - i)$ for all $1 \leq i \leq m$, one easily observes that

$$\lambda_i - (m - (2i - 1)) \geq m$$  

for all $1 \leq i \leq m$. Thus, all the members of the vector in $(1)$ are greater than or equal to $m$, and therefore the resultant partition is indeed a member of $D_m(n)$. Next, we show that
the map $\phi$ is invertible by constructing the inverse map $\psi$. Though $\psi$ is easy to predict, we describe it below in some detail.

Suppose we have a partition $\pi \in D_m(n)$. There exist some parts $\pi_1, \pi_2, \cdots, \pi_m$ of $\pi$ such that $$\pi_1 \geq \pi_2 \cdots \geq \pi_m \geq m.$$ We define the map $\psi$ to be such that the elements of $\pi$ other than $(\pi_1, \pi_2, \cdots, \pi_m)$ are left unchanged, and $(\pi_1, \pi_2, \cdots, \pi_m)$ is mapped under $\psi$ to $$\left(\pi_1 + (m - 1), \pi_2 + (m - 3), \cdots, \pi_i + (m - (2i - 1)), \cdots, \pi_{m-1} - (m - 3), \pi_m - (m - 1)\right).$$

It is straightforward to verify that consecutive members of the above vector differ by at least 2, and thus the resultant partition is indeed a member of $C_m(n)$. Finally, it is easy to check that the maps $\phi$ and $\psi$ are indeed inverses of each other, completing the proof of Theorem 5, and thus also of Theorem 2.

Remark 6. Since the maps $\phi$ and $\psi$ preserve the number of parts, our proof also gives a combinatorial proof of Theorem 3.

3 Generalization of Theorem 2 for $k$-measures

It turns out that Theorem 5 is easy to generalize by appropriately modifying the maps $\phi$ and $\psi$. We provide all the details for the sake of completeness. We denote the floor and ceiling functions of $x$ by $\lfloor x \rfloor$ and $\lceil x \rceil$ respectively. Note that for any natural number $m$, $$\left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil = m.$$ This fact will be used frequently in the proof of the next theorem.

**Theorem 7.** The number of partitions of $n$ in which there exists a subsequence of length $m$ of parts in the partition in which the difference between any two consecutive parts of the subsequence is at least $k$ is equal to the number of partitions of $n$ which have at least $\left\lfloor \frac{m}{2} \right\rfloor$ parts greater than or equal to $1 + \left\lceil \frac{k(m-1)}{2} \right\rceil$, and an additional at least $\left\lfloor \frac{m}{2} \right\rfloor$ parts greater than or equal to $1 + \left\lfloor \frac{k(m-1)}{2} \right\rfloor$.

**Proof.** Let

- $C_{k,m}(n)$ denotes the set of partitions of $n$ in which there exists a subsequence of length $m$ of parts in the partition in which the difference between any two consecutive parts of the subsequence is at least $k$.

- $D_{k,m}(n)$ denotes the set of partitions of $n$ which have at least $\left\lfloor \frac{m}{2} \right\rfloor$ parts greater than or equal to $1 + \left\lceil \frac{k(m-1)}{2} \right\rceil$, and an additional at least $\left\lfloor \frac{m}{2} \right\rfloor$ parts greater than or equal to $1 + \left\lfloor \frac{k(m-1)}{2} \right\rfloor$.

We construct a bijection $\phi'$ between $C_{k,m}(n)$ and $D_{k,m}(n)$. Prior to that, we describe the motivation behind this bijection.
Suppose we have a partition \( \lambda \in C_{k,m}(n) \). Further, suppose \((\lambda_1, \lambda_2, \cdots, \lambda_m)\) be a subsequence of parts of \( \lambda \) in which the difference between any two consecutive parts of the subsequence is at least \( k \). That is, \( \lambda_i \geq \lambda_{i+1} + k \) for all \( i \). In particular, we have \( \lambda_i \geq 1 + k(m-i) \) for all \( 1 \leq i \leq m \). As suggested by the proof of Theorem 5, we calculate the average of the lower bounds on the \( \lambda_i \)'s, which comes out to be \( 1 + \frac{k(m-1)}{2} \). If \( k \) is even or \( m \) is odd, then \( \frac{k(m-1)}{2} \) is an integer and our idea in the proof of Theorem 5 can be easily generalized to construct the required bijection.

However if \( k \) is odd and \( m \) is even, then the average \( 1 + \frac{k(m-1)}{2} \) of the lower bounds on the \( \lambda_i \)'s is not an integer. This problem makes the analysis of this case a little harder. We explain this difficulty by considering an example. Suppose \( k = 3 \) and \( m = 4 \). Then, we are given a subsequence \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) with

\[
\lambda_1 \geq \lambda_2 + 3, \lambda_2 \geq \lambda_3 + 3, \lambda_3 \geq \lambda_4 + 3.
\]

Thus, in particular, we have

\[
\lambda_1 \geq 10, \lambda_2 \geq 7, \lambda_3 \geq 4, \lambda_4 \geq 1.
\]

The average of these lower bounds on the \( \lambda_i \)'s comes out to be 5.5. Now if we just try to use the previous approach, we would map this subsequence to \((\lambda_1 - 4.5, \lambda_2 - 1.5, \lambda_3 + 1.5, \lambda_4 + 4.5)\). However, since the parts of a partition must be integers, we cannot map it like this. Thus, the appropriate modification we make in this case is to map the subsequence to \((\lambda_1 - 4, \lambda_2 - 1, \lambda_3 + 1, \lambda_4 + 4)\) instead. Note that in this case, the resultant partition has the property that at least two parts are greater than or equal to 6 and an additional two parts are greater than or equal to 5, explaining the strange condition in the definition of \( D_{k,m}(n) \). Next, we describe the map \( \phi' \) for any \( k \) and \( m \). Using floor and ceiling functions, we will be able to handle together the two cases when \( k(m-1) \) is odd or even.

We define the map \( \phi' \) to be such that the elements of \( \lambda \) not in our chosen subsequence \((\lambda_1, \lambda_2, \cdots, \lambda_m)\) are left unchanged, and \((\lambda_1, \lambda_2, \cdots, \lambda_m)\) is mapped under \( \phi' \) to

\[
\left( \lambda_1 - \left\lfloor \frac{k(m-1)}{2} \right\rfloor, \lambda_2 - \left\lfloor \frac{k(m-3)}{2} \right\rfloor, \cdots, \lambda_{\left\lceil \frac{m}{2} \right\rceil} - \left\lfloor \frac{k}{2} \left( m + 1 - 2 \left\lceil \frac{m}{2} \right\rceil \right) \right\rfloor, \right.
\]

\[
\left. \lambda_{\left\lceil \frac{m}{2} \right\rceil + 1} + \left\lfloor \frac{k}{2} \left( m + 1 - 2 \left\lceil \frac{m}{2} \right\rceil \right) \right\rfloor, \lambda_{\left\lceil \frac{m}{2} \right\rceil + 2} + \left\lfloor \frac{k}{2} \left( m + 3 - 2 \left\lceil \frac{m}{2} \right\rceil \right) \right\rfloor, \cdots, \lambda_{m-1} + \left\lfloor \frac{k(m-3)}{2} \right\rfloor, \lambda_m + \left\lfloor \frac{k(m-1)}{2} \right\rfloor \right) \quad \text{(2)}
\]

The members of the vector in (2) can be described compactly as follows. The first \( \left\lfloor \frac{m}{2} \right\rfloor \) members can be written as

\[
\left\{ \lambda_i - \left\lfloor \frac{k(m-(2i-1))}{2} \right\rfloor : 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor \right\}.
\]

while the remaining \( \left\lceil \frac{m}{2} \right\rceil \) members can be written (beginning from right to left) as

\[
\left\{ \lambda_{m-i} + \left\lfloor \frac{k(m-(2i+1))}{2} \right\rfloor : 0 \leq i < \left\lceil \frac{m}{2} \right\rceil \right\}.
\]
We prove that the resultant partition is indeed a member of $D_{k,m}(n)$. First, considering two cases based on the parity of $m$, it is an easy exercise to confirm that the sum of the members of the vector in (2) is equal to the sum of $\lambda_i$'s. That is, the other terms with the positive and negative signs cancel out. Secondly, we show that the first $\left\lfloor \frac{m}{2} \right\rfloor$ members of the vector in (2) are greater than or equal to $1 + \left\lceil \frac{k(m-1)}{2} \right\rceil$, and the last $\left\lceil \frac{m}{2} \right\rceil$ members are greater than or equal to $1 + \left\lceil \frac{k(m-1)}{2} \right\rceil$. To prove these facts, we crucially use $\lambda_i \geq 1 + k(m-i)$ for all $1 \leq i \leq m$. Therefore, for $1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor$, we have

$$\lambda_i - \left\lfloor \frac{k(m-(2i-1))}{2} \right\rfloor \geq 1 + k(m-i) - \left\lfloor \frac{k(m-(2i-1))}{2} \right\rfloor$$

$$= 1 + km - \left\lfloor \frac{k(m+1)}{2} \right\rfloor$$

$$= 1 + k(m-1) - \left\lfloor \frac{k(m-1)}{2} \right\rfloor$$

$$= 1 + \left\lceil \frac{k(m-1)}{2} \right\rceil.$$

Similarly, for $0 \leq i < \left\lceil \frac{m}{2} \right\rceil$, we have

$$\lambda_{m-i} + \left\lceil \frac{k(m-(2i+1))}{2} \right\rceil \geq 1 + ki + \left\lceil \frac{k(m-(2i+1))}{2} \right\rceil$$

$$= 1 + \left\lfloor \frac{k(m+1)}{2} \right\rfloor,$$

as required. Thus, the resultant partition is indeed a member of $D_{k,m}(n)$. Next, we show that the map $\phi'$ is invertible by constructing the inverse map $\psi'$. The map $\psi'$ is again easy to guess but we describe it in some detail below.

Suppose we have a partition $\pi \in D_m(n)$. There exist some parts $\pi_1, \pi_2, \ldots, \pi_m$ of $\pi$ such that

$$\pi_1 \geq \pi_2 \geq \cdots \geq \pi_m \geq m.$$ 

We define the map $\psi'$ to be such that the elements of $\pi$ other than $(\pi_1, \pi_2, \ldots, \pi_m)$ are left unchanged, and $(\pi_1, \pi_2, \ldots, \pi_m)$ is mapped under $\psi'$ to

$$\left( \pi_1 + \left\lfloor \frac{k(m-1)}{2} \right\rfloor, \pi_2 + \left\lfloor \frac{k(m-3)}{2} \right\rfloor, \ldots, \pi_{\left\lfloor \frac{m}{2} \right\rfloor} + \left\lfloor \frac{k}{2} \left( m + 1 - 2 \left\lceil \frac{m}{2} \right\rceil \right) \right\rfloor, \right.$$ 

$$\pi_{\left\lfloor \frac{m}{2} \right\rfloor + 1} - \left\lceil \frac{k}{2} \left( m + 1 - 2 \left\lceil \frac{m}{2} \right\rceil \right) \right\rceil, \pi_{\left\lfloor \frac{m}{2} \right\rfloor + 2} - \left\lfloor \frac{k}{2} \left( m + 3 - 2 \left\lceil \frac{m}{2} \right\rceil \right) \right\rfloor, \ldots$$

$$\pi_{m-1} - \left\lceil \frac{k(m-3)}{2} \right\rceil, \pi_m - \left\lfloor \frac{k(m-1)}{2} \right\rfloor \right).$$

It is easy to verify that consecutive members of the above vector differ by at least $k$, and thus the resultant partition is indeed a member of $C_{k,m}(n)$. Finally, it is also straightforward
to verify that the maps $\phi'$ and $\psi'$ are indeed inverses of each other, completing the proof of Theorem 7.

Next, we use Theorem 7 to generalize Theorems 2 and 3. First, we deduce two immediate corollaries of Theorem 7 that will be helpful to obtain these generalizations.

Corollary 8. Suppose either $k$ is even or $m$ is odd. Then the number of partitions of $n$ in which there exists a subsequence of length $m$ of parts in the partition in which the difference between any two consecutive parts of the subsequence is at least $k$ is equal to the number of partitions of $n$ which have at least $m$ parts of $1 + \frac{k(m-1)}{2}$.

Corollary 9. Suppose $k$ is odd and $m$ is even. Then the number of partitions of $n$ in which there exists a subsequence of length $m$ of parts in the partition in which the difference between any two consecutive parts of the subsequence is at least $k$ is equal to the number of partitions of $n$ which have at least $m$ parts of $1 + \frac{k(m-1)+1}{2}$, and an additional at least $m$ parts of $1 + \frac{k(m-1)+3}{2}$.

Based on the results in these corollaries, we define a $(k,m)$-polygon associated to an integer partition.

- Suppose $k$ is even or $m$ is odd. Then define the $(k,m)$-polygon of a partition $\pi$ to be the rectangle containing $m$ rows of $1 + \frac{k(m-1)}{2}$ nodes (In other words, the rectangle with vertical side $m$ and horizontal side $1 + \frac{k(m-1)}{2}$) beginning from the top left corner of the Ferrers diagram of $\pi$. For example, the partition $9 + 9 + 8 + 7 + 4 + 3 + 1$ of 41 has the following $(4,3)$-polygon.

- Suppose $k$ is odd and $m$ is even. Then define the $(k,m)$-polygon of a partition $\pi$ to be the polygon containing $\frac{m}{2}$ rows of $1 + \frac{k(m-1)+3}{2}$ nodes and $\frac{m}{2}$ rows of $1 + \frac{k(m-1)+1}{2}$ nodes beginning from the top left corner of the Ferrers diagram of $\pi$. For example, the partition $9 + 9 + 8 + 7 + 4 + 3 + 1$ of 41 has the following $(3,4)$-polygon.
Based on this definition, we can rewrite Corollaries 8 and 9 together as follows.

**Corollary 10.** Suppose $k$ is odd and $m$ is even. Then the number of partitions of $n$ in which there exists a subsequence of length $m$ of parts in the partition in which the difference between any two consecutive parts of the subsequence is at least $k$ is equal to the number of partitions of $n$ whose Ferrers diagram contains its $(k, m)$-polygon.

It is easy to observe the following properties of the $(k, m)$-polygons of partitions.

- For a given $k$ and a partition $\pi$, the $(k, m)$-polygon of $\pi$ is strictly contained in the $(k, m + 1)$-polygon of $\pi$ for any $m$, irrespective of the parities of $k$ and $m$.
- For a given $k$ and a partition $\pi$, there are only finitely many values of $m$ such that the Ferrers diagram of $\pi$ contains the $(k, m)$-polygon of $\pi$.

From these observations, it is obvious that for a given $k$ and a partition $\pi$, there exists a largest value of $m$ such that the Ferrers diagram of $\pi$ contains the $(k, m)$-polygon of $\pi$. We say that $\pi$ has a $(k, m)$-Durfee polygon. Then from Corollary 10, the following result immediately follows.

**Theorem 11.** The number of partitions of $n$ with $k$-measure $m$ is equal to the number of partitions of $n$ with $(k, m)$-Durfee polygon.

**Remark 12.** Note that for $k = 2$, the $(2, m)$-polygonal of $\pi$ is a square of side $m$, and thus for a partition $\pi$ to have $(2, m)$-Durfee polygon is the same thing as $\pi$ having a Durfee square of side $m$. Thus, Theorem 11 is a generalization of Theorem 2.

**Remark 13.** Since the maps $\phi'$ and $\psi'$ preserve the number of parts, our proof also gives a generalization of Theorem 3. That is, the number of partitions of $n$ with $l$ parts and $k$-measure $m$ is equal to the number of partitions of $n$ with $l$ parts and $(k, m)$-Durfee polygon.

**Remark 14.** For even $k$, the $(k, m)$-polygons are rectangles irrespective of the value of $m$. Thus, for an even $k$, Theorem 11 provides a very simple looking generalization of Theorem 2. For example, the number of partitions of $n$ with 4-measure $m$ is equal to the number of partitions for which the largest value of $j$ such that the Ferrers diagram of the partition contains a $j \times (2j - 1)$ rectangle equals $m$. Similarly, the number of partitions of $n$ with 6-measure $m$ is equal to the number of partitions for which the largest value of $j$ such that the Ferrers diagram of the partition contains a $j \times (3j - 2)$ rectangle equals $m$. For the case when $k$ is odd, the situation is relatively complex as $(k, m)$-polygons can be rectangles or not depending on the parity of $m$. 

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These results seem to yield interesting properties even in the case $k = 1$. For example, substituting $k = 1$ in Theorem 7, we get the following result.

**Corollary 15.** The number of partitions of $n$ which have at least $m$ distinct parts is equal to the number of partitions of $n$ which have at least $\left\lfloor \frac{m}{2} \right\rfloor$ parts greater than or equal to $\left\lfloor \frac{m+1}{2} \right\rfloor$, and an additional at least $\left\lfloor \frac{m}{2} \right\rfloor$ parts greater than or equal to $\left\lceil \frac{m+1}{2} \right\rceil$.

That is for an odd number $m$, the number of partitions of $n$ which have at least $m$ distinct parts is equal to the number of partitions of $n$ which have at least $m$ parts greater than or equal to $\frac{m+1}{2}$. For example, the number of partitions of $n$ which have at least 7 distinct parts is equal to the number of partitions of $n$ which have at least 7 parts greater than or equal to 4.

Similarly, for an even number $m$, the number of partitions of $n$ which have at least $m$ distinct parts is equal to the number of partitions of $n$ which have at least $\frac{m}{2}$ parts greater than or equal to $\frac{m}{2} + 1$, and an additional at least $\frac{m}{2}$ parts greater than or equal to $\frac{m}{2}$. For example, the number of partitions of $n$ which have at least 6 distinct parts is equal to the number of partitions of $n$ which have at least 3 parts greater than or equal to 4, and an additional at least 3 parts greater than or equal to 3.

Finally, substituting $k = 1$ in Corollary 10 gives the following result.

**Corollary 16.** The number of partitions of $n$ with $m$ distinct parts is equal to the number of partitions of $n$ with $(1, m)$-Durfee polygon.

## 4 Concluding Remarks

It will be very interesting to see if one could utilize the ideas in this paper to obtain a combinatorial proof and possibly generalizations of Theorem 4.

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