Semiclassical description of D-branes in $\text{SL}(2)/\text{U}(1)$ gauged WZW model

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Abstract. In this note we examine some semiclassical features of D-branes in the $\text{SL}(2)/\text{U}(1)$ gauged WZW model and determine the small fluctuation spectra for one class of branes. We compare our results with expectations from the CFT side.

1. Introduction

It is well known that WZW models are some of the few known exact string theory backgrounds. They are important elements of the near-horizon geometry of various string theory configurations. For example the background of $Q_5$ NS5 branes and $Q_1$ fundamental strings has a near horizon geometry given by the $\text{SL}(2,\mathbb{R}) \times \text{SU}(2)$ WZW model. On the other hand the near-horizon geometry of a circular distribution of NS5 branes is T-dual to an orbifold of the $\text{SL}(2,\mathbb{R})/\text{U}(1) \times \text{SU}(2)/\text{U}(1)$ gauged WZW model. In addition WZW models and their gauged variations have become increasingly important in studies of the AdS/CFT correspondence and branes-world scenarios.

D-branes are an essential ingredient of string theory since they play crucial role in various dualities and holography. One way to study the properties of D-branes in a given background is to construct the boundary state of the D-branes which describe the couplings of closed string modes on their world-volume. This is in general a difficult question for a general background. Although D-branes in WZW models on compact group manifolds like $\text{SU}(2)$ have been studied extensively, little progress has been achieved towards non-compact cases like $\text{SL}(2,\mathbb{R})$. In fact even the spectrum of these theories is not fully understood. Nevertheless we can get useful information for such D-branes by studying just the semiclassical limit of their boundary states, in other words their low energy effective action. This has been a fruitful approach for the $\text{SL}(2,\mathbb{R})$ case [1 2 3] as well as for the cases of compact group manifolds [4 5]. In this

note we will study the semiclassical features of D-branes in the gauged $SL(2,\mathbb{R})/U(1)$ WZW model along the lines of a similar study for $SU(2)/U(1)$ [6]. Since we are not in position to give the full boundary state for all D-branes in $SL(2,\mathbb{R})/U(1)$, we will restrict ourselves to their geometrical features. Finally, for one set of branes, we use a small fluctuation analysis of the low energy effective action to get the semiclassical limit of the open string spectra. Consequently we compare our results with expectations from a CFT analysis [7].

2. The geometry and the spectrum

The $SL(2,\mathbb{R})$ WZW model is described by a world-sheet sigma model with target space:

$$ds^2 = k(-(\cosh \rho)^2 d\psi^2 + (d\rho)^2 + (\sinh \rho)^2 d\tau^2)$$

and a Kalb-Ramond field strength given by the volume element of the spacetime. The constant $k$ is the level of the affine lie algebra that can take real values with $k \geq 2$, and we used units $\alpha' = 1$.

One can construct the $SL(2,\mathbb{R})/U(1)$ coset from the $SL(2,\mathbb{R})$ WZW model by gauging a U(1) subgroup of $SL(2,\mathbb{R})$ [8,9]. If this subgroup is non compact one ends with a manifold with Lorentzian signature, the 2D black hole. On the contrary if we choose a compact U(1) then we get an Euclidean manifold. There are two possible anomaly free U(1) gauge symmetries: the axial, $g \rightarrow hgh$ and the vector, $g \rightarrow hgh^{-1}$, where $g(z, \bar{z})$ group elements of $SL(2,\mathbb{R})$ and $h \in U(1)$. For a compact U(1) the manifold we get by gauging the axial symmetry has the geometry of a cigar:

$$ds^2 = (d\rho)^2 + (\tanh \rho)^2 d\tau^2 \quad e^{-\Phi} = e^{-\Phi_0} \cosh \rho \quad (cigar)$$

The axial gauging amounts intuitively into gauging the symmetry which corresponds to time translations in $SL(2,\mathbb{R})$. This will be important later in understanding the geometry of branes in the cigar geometry. Notice that unlike AdS$_3$, in $SL(2,\mathbb{R})$ we have closed time-like curves. Therefore the U(1) associated with time translations is a compact subgroup of $SL(2,\mathbb{R})$.

By gauging the vector compact U(1) symmetry we get the geometry of a trumpet:

$$ds^2 = (d\rho)^2 + (\coth \rho)^2 d\psi^2 \quad e^{-\Phi} = e^{-\Phi_0} \sinh \rho \quad (trumpet)$$

This coset amounts to gauging the U(1) symmetry of shifting $\tau$. Notice that unlike the cigar manifold, that of the trumpet is singular due to the fact that the vector gauging action has fixed points. The two backgrounds are actually T-dual to each other. To be exact the cigar is T-dual to the trumpet where $\psi \sim \psi + 2\pi/k$.

Finally we would like to summarize the operator spectrum of the theory [9] since it will become useful in our small fluctuation analysis. The operators in the axially gauged theory belong to discrete series with $(k - 1)/2 \geq j \geq -1/2$ and continuous series with $j = -1/2 + iP$, $P \in \mathbb{R}^+$. The operators are charged under the $SL(2,\mathbb{R})$ currents $J_3, \bar{J}_3$ with charges: $\omega_L = 1/2(m + nk)$ and $\omega_R = -1/2(m - nk)$. The classical
angular momentum of an operator is $m$ and the winding $n$. The conformal weights of the operators are:

$$h_{mn}^j = -j(j+1)/k - 2 + (m+nk)/k - 2$$
$$\bar{h}_{mn}^j = -j(j+1)/k - 2 + (m-nk)/k - 2$$

In addition from the harmonic analysis in $SL(2,\mathbb{R})$ we conclude that discrete representations should satisfy: $|nk| > |m|$. We see that we cannot have discrete representations with vanishing winding. For the T-dual vector gauging we need to take $\omega_R \rightarrow -\omega_R$ which results in $m \leftrightarrow n$ in (4). The discrete representations constrain in this case is $|nk| < |m|$, where $m$ is the angular momentum in the trumpet and $n$ the winding. In this case discrete series states with non-zero momentum and zero winding exist. The continuous representations correspond semiclassically to wave like solutions while the discrete ones to bound states. So, in the semiclassical limit $k \rightarrow \infty$, where the winding states decouple, we expect bound states for the trumpet and not for the cigar. These expectations from CFT are nicely confirmed by the computation of the low energy spectrum of the two geometries [9]. The open string spectra of the D-branes satisfy analogous constrains.

3. D-brane descendants from branes of $SL(2)$ and DBI analysis

The geometry of D-branes in $SL(2,\mathbb{R})$ has been studied in [1] (see also i.e. [10, 11, 12] for related work) where it was found that solutions of the DBI action correspond to regular and twined conjugacy classes of $SL(2,\mathbb{R})$. It was found that D1 branes have $AdS_2$ world volumes and D2 branes come in two types with $H_2$ and $dS_2$ world volumes [1]. The first type of D2 branes has a D-instanton density in its world-volume while the second one was found to be unphysical due to a supercritical electric field. As explained in [13, 14, 15, 16] the D-branes in coset models $G/H$ have world-volumes localized on the projection to the coset, of the product of a conjugacy class of $G$ with a conjugacy class of $H$. In geometric words we expect the D-branes we will find on the cigar to be projections on a constant $\psi$ plane of the D-branes in $SL(2,\mathbb{R})$ and those of the trumpet projections on a constant $\tau$ plane. We will verify these expectations by solving the DBI for the cigar and trumpet.

We will assume from now on that the level $k$ is an integer. This needs not to be the general case but in most of the interesting (supersymmetric) backgrounds the $SL(2,\mathbb{R})k/U(1)$ CFT appears with a corresponding $SU(2)k'/U(1)$ with the levels $k$ and $k'$ differing by an integer [17]. Since $k'$ must be an integer then $k$ should be one too.

For the D1 branes in the cigar and with an embedding $\rho(\tau)$ we need to minimize the "energy" of the system $E = \frac{\partial L_{DBI}}{\partial \dot{\rho}} - L_{DBI}$, where $\dot{\rho}$ is derivative with respect to $\tau$. The same method is employed for the D1 branes in the trumpet. For the D2 branes one has to solve the equations of motion for the gauge field $F_{\rho\tau}$. The analysis is straightforward as shown in tables 1 and 2.

In tables 1 and 2 we give the embedding equations for the various branes in trumpet and cigar respectively, as well as the $SL(2,\mathbb{R})$ branes from which they descent. The
D-branes in SL(2)/U(1)

Figure 1. The D-branes in the cigar and trumpet as well as the D-branes of SL(2, R) where they descent from.

Dimensionality of each brane is evident by its name i.e. $DS_{1\text{tr}}$ is an 1-dimensional brane in the trumpet, descending from the $dS_2$ brane in SL(2, R).

Table 1.

| $SL(2, \mathbb{R})$ | Trumpet | Embedding equations | Moduli |
|---------------------|---------|---------------------|--------|
| $H_2$               | $D_{1\text{tr}}$ | $\cosh \rho \sin(\psi - \psi_0) = C$ | $C = \sin \frac{2\pi}{N} k$ |
| $dS_2$              | $DS_{1\text{tr}}$ | $\cosh \rho \sin(\psi - \psi_0) = C$ | $C \geq 1$, $\psi_0$ |
| $AdS_2$             | $D_{2\text{tr}}$ | $2\pi F_{\rho \psi} = \frac{N \cosh \rho \sinh \rho_{\min}}{\sqrt{\sinh^2 \rho - \sinh^2 \rho_{\min}}} \rho_{\min}$, $A_\psi$ |
| $?$                 | $D_{1\text{tr}}'$ | $\rho = 0$ | none |

Table 2.

| $SL(2, \mathbb{R})$ | Cigar | Embedding equations | Moduli |
|---------------------|-------|---------------------|--------|
| $H_2$               | $D_{2\text{cig}}$ | $2\pi F_{\rho \tau} = N \frac{C \tanh \rho}{\sqrt{\cosh^2 \rho - C^2}}$ | $C = \sin \frac{2\pi}{N} N_{D0}$ |
| $dS_2$              | $DS_{2\text{cig}}$ | $2\pi F_{\rho \tau} = N \frac{C \tanh \rho}{\sqrt{\cosh^2 \rho - C^2}}$ | $C = \cosh \rho_{\min} \geq 1$, $A_\tau$ |
| $AdS_2$             | $D_{1\text{cig}}$ | $\sin(\tau - \tau_0) = \frac{\sinh \rho_{\min}}{\sinh \rho}$ | $\rho_{\min}$, $\tau_0$ |
| $?$                 | $D_{0\text{cig}}$ | $\rho = 0$ | none |

At this point a few comments are in order. First looking at the table of the cigar and the trumpet we notice that the branes which descent from the $AdS_2$ and $dS_2$ branes have two classical moduli. Unlike these branes, the DBI analysis analogous to [6] for D2 branes in the cigar, implies that they carry a D0 brane charge which leads to the quantization of the modulus $C$. Also since they have trivial topology they do not admit a Wilson line. By T-duality these branes are related to D1 branes in the trumpet which hit the infinite radius circle at $\rho = 0$. In principal one expects no quantization of
their incidence angle given by the modulus $C$ and in addition that they carry an extra modulus, the angle $\psi_0$ where they hit the circle $\rho = 0$. T-duality with the cigar suggests that these D1 branes can hit only special points on the $\rho = 0$ circle of the trumpet and it is suggested also (see [13]) that they correspond to bound states of D1 branes orthogonal to the circle with a number of $D1'_{tr}$ branes.

We should also emphasize that the trumpet has a strong coupling singularity at $\rho = 0$ and therefore it is plausible that our analysis might be valid only for branes which extend far enough from the singularity. Nevertheless the considerable success of our semiclassical analysis and the similar results of [9] for the spectrum of the theory suggest that there should be a grain of truth in our results.

Notice that we have not been able to identify the brane of $SL(2, \mathbb{R})$, if any, where the $D0_{cig}$ branes on the cigar and their T-dual $D1'_{tr}$ descent from. One might think that the D0 brane of the cigar comes from the point brane of $SL(2, \mathbb{R})$ but there seems to be a contradiction when one tries to construct the $D1'_{tr}$ by descending from this brane to the trumpet. The point in $SL(2, \mathbb{R})$ will still be a point in cigar or trumpet. These shortcomings of our semiclassical analysis should be eliminated once we are able to have the complete boundary state for these branes.

Finally notice that since the cigar is actually T-dual to an orbifold of the trumpet, one needs to rescale the string coupling by $1/\sqrt{N}$ when comparing quantities under T-duality.

4. Small fluctuation analysis for $AdS_2$ descendants and the open string spectrum

The boundary states for branes in $SL(2, \mathbb{R})$ has been a long and rather subtle problem. The non-compactness of the group results in infinities for most physically relevant quantities which must be dealt with caution. On top of that only recently [19] was a significant aspect of the spectrum, that of spectral flow, understood. Nevertheless one can work in the Euclidean analog of $AdS_3$ and try to construct branes in the $H^+_{3}$ hyperbolic space. This method was employed in [20] † and the boundary state of a brane with $AdS_2$ world-volume was constructed among other branes of the background. It was claimed that due to the time independence of this particular set of branes on can descent trivially to the corresponding branes in the coset $H^+_3/\mathbb{R}_\psi$ which describes the cigar geometry. These branes couple only to closed string momentum modes as expected by their geometry. This in turn implies, due to the discrete representation constrains, that only closed strings in the continuous representations couple to the brane.

We will work along the lines of [20] to study the semiclassical approximation to the open string spectrum of the branes. It is easy to understand by the shape of the $D1_{cig}$ branes (see also [3]) that open string winding states should appear in the spectrum

† See [21] for a different approach to the problem and some objections to the construction employed in [20].
and not momentum ones ‡. Since open string states satisfy the same constrains for the allowed quantum numbers as closed ones, continuous states as well as discrete ones with winding \( w > 0 \) are allowed. In the semiclassical limit only the winding zero state survives since all other ones become heavy and decouple. Then computing the open string spectrum of these branes using the DBI in the cigar geometry we expect to see only continuous representations since the discrete ones are forbidden for \( w = 0 \). The purpose of the following analysis is to compute the small fluctuation spectra on these branes and confirm the expected spectrum. In addition we will compute the semiclassical reflection coefficient and determine the density of open string states.

We use the DBI lagrangian for D1 strings with embedding \( \tau = \cos^{-1} \frac{C}{r} + \delta \tau(r,t) \) where \( r = \sinh \rho, C = \sinh \rho_{\min} \) and we assume a fluctuation \( \delta \tau(r,t) \) around the classical background. The time is \( t \) not to be confused with \( \tau \). Expanding the DBI action we get:

\[
L_{DBI} \simeq T_1 e^{-\Phi_0} \left[ \frac{1}{\sqrt{1 - \frac{C^2}{r^2}}} + C(\partial_r \delta \tau) \right] \]

The first term is the energy density of our D-brane. The first order variation vanishes when we integrate it on the world-volume of the brane since to have a small fluctuation \( \delta \tau \to \tau_0 \) for \( x \to \pm \infty \) in cartesian coordinates.

The equations of motion for the quadratic fluctuations give a hypergeometric equation. The solution shows no poles which would signify bound states and it turns out that for the wave like solutions with the correct asymptotic behavior the solutions is:

\[
u = \frac{\Gamma(-iP)}{\Gamma(1 - iP/2)^2} r^{-iP} (1 + C^2)^{iP/2} (-1)^{-|j|/2} + \frac{\Gamma(iP)}{\Gamma(1 + iP/2)^2} r^{iP} (1 + C^2)^{-iP/2} (-1)^{+|j|/2}
\]

where we have assumed (semiclassical approximation) \(|j| >> 1\), for \( j = -1/2 + iP \). The term \( r^{-iP} \) describes an outgoing wave and \( r^{iP} \) an incoming one. The ratio of the coefficients of these two terms is the semiclassical approximation to the reflection coefficient of a state coming from infinity. Now, we can compute the semiclassical limit of the relative density of states between a brane with modulus \( \rho_{\min} \) and a reference brane \( \rho_{\min}^* \):

\[
N_{\text{rel}}(P|\rho_{\min}, \rho_{\min}^*) \simeq \frac{1}{2\pi i} \frac{\partial}{\partial P} \log \frac{R_c(\rho_{\min}; P)}{R_c(\rho_{\min}^*; P)} = \frac{1}{\pi} \log \frac{\cosh \rho_{\min}}{\cosh \rho_{\min}^*}
\]

where the function \( R_c(\rho_{\min}; P) \) is the semiclassical limit of the reflection coefficient for an open string state. We compute a relative density of states since the density of states ‡ This is also a consequence of closed/open string duality in the annulus string amplitude. Momentum in the closed string channel becomes winding in the open string channel.
for a single brane is infinite due to the non-compactness of its world-volume. Our results agree with [7].

5. Conclusions

In this note we have determined solutions of the DBI for D0, D1 and D2 branes in the cigar and trumpet geometries. Some aspects of those branes in realistic string backgrounds will be further discussed in [18]. We have also computed the small fluctuations spectrum for one kind of them (the $D_{1cig}$) and found agreement with geometrical and CFT expectations.

One might wonder if we can get useful information for the winding states on $D_{1cig}$, by studying the spectrum of momentum modes in its T-dual, the $D_{2tr}$. We have calculated the small fluctuation spectra of $D_{2tr}$ and found both discrete series states as well as continuous representations. The discrete series states appear with degeneracy one. For the continuous part of the spectrum the density of states turns out to be the same as [7].

As a final remark, in [3] the spectral flow of semiclassical solution gives strings which wind half a time around the center of $AdS_3$. These in the cigar should descent to half-winding open string states which is intuitively obvious from the shape of the $D_{1cig}$. This implies that on the trumpet the $D_{2tr}$ should allow states which have half-angular momentum and are periodic only under a $4\pi$ rotation. This deserves further investigation and we hope to return to this point in the future.

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