On the EO-orientability of vector bundles

P. Bhattacharya | H. Chatham

Abstract
We study the orientability of vector bundles with respect to a family of cohomology theories called EO-theories. The EO-theories are higher height analogues of real K-theory KO. For each EO-theory, we prove that the direct sum of $i$ copies of any vector bundle is EO-orientable for some specific integer $i$. Using a splitting principal, we reduce to the case of the canonical line bundle over $\mathbb{C}P^\infty$. Our method involves understanding the action of an order $p$ subgroup of the Morava stabilizer group on the Morava $E$-theory of $\mathbb{C}P^\infty$. Our calculations have another application: We determine the homotopy type of the $S^1$-Tate spectrum associated to the trivial action of $S^1$ on all EO-theories.

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1 | INTRODUCTION

A real vector bundle $\xi$, is called orientable with respect to a cohomology theory $E$ if it cannot distinguish between $\xi$, and the trivial vector bundle of the same dimension over the same base
space, which we will denote by $e^{\dim \xi}$. In other words, there is an E-Thom isomorphism

$$E \wedge \text{Th}(\xi) \simeq E \wedge \text{Th}(e^{\dim \xi}),$$

where $\text{Th}(\xi)$ denotes the Thom space associated to $\xi$ (see (2.1)). The study of the orientability of vector bundles with respect to cohomology theories has led to foundational results in both geometry and algebraic topology. For example, the $H\mathbb{F}_2$-orientability of all vector bundles leads to the notion of Stiefel–Whitney classes and the $H\mathbb{Z}$-orientability of complex vector bundles leads to Chern classes. It is a standard fact that all complex vector bundles are $\text{KU}$-orientable, however, not all of them are $\text{KO}$-orientable. The following examples motivate some of the work in this paper.

**Example 1.1.** The complex tautological line bundle $\gamma^1$ over $\mathbb{CP}^1 \simeq S^2$ is not $\text{KO}$-orientable because the Thom isomorphism fails:

$$\text{KO} \wedge \text{Th}(\gamma^1) \not\simeq \text{KO} \wedge (\epsilon^2).$$

On one hand, $\text{Th}(\gamma^1) \simeq \mathbb{CP}^2$ (see Example 2.3) and $\text{KO} \wedge \mathbb{CP}^2 \simeq \text{KU}$. On the other hand, $\text{Th}(\epsilon^2) \simeq \Sigma^2 \mathbb{CP}_+^1 \simeq S^2 \vee S^4$ (see Example 2.2) and $\text{KO} \wedge \text{Th}(\epsilon^2) \simeq \Sigma^2 \text{KO} \vee \Sigma^4 \text{KO}$.

**Example 1.2.** While $\gamma^1$ is not $\text{KO}$-orientable, its 2-fold direct sum $\gamma^1 \oplus \gamma^1$ is. In fact, $\xi^\oplus 2$ is $\text{KO}$-orientable for every complex vector bundle $\xi$ (see Lemma 2.18).

From the point of view of chromatic homotopy theory, 2-completed $\text{KO}$ is part of a family of cohomology theories called the $\text{EO}$-theories. In 1985, Jack Morava [18] showed that associated to each finite height formal group $\Gamma$ over a perfect field $F$ of characteristic $p$, there is a cohomology theory called Morava E-theory $E_{\Gamma}$ with an action of $\text{Aut}(\Gamma)$. The group $\text{Aut}(\Gamma)$ is often called the Morava Stabilizer group. When the cyclic group of order $p$ acts faithfully on $\Gamma$, we define the spectrum $\text{EO}_{\Gamma}$ as the homotopy fixed point spectrum

$$\text{EO}_{\Gamma} := E_{\Gamma}^{hC_p}.$$

**Example 1.3.** When $p = 2$ and $\hat{\mathbb{G}}_m$ is the multiplicative formal group over $\mathbb{F}_2$, there is a canonical action of $\mathbb{C}_2$ on $\hat{\mathbb{G}}_m$. In this case, $E_{\hat{\mathbb{G}}_m} \simeq \text{KU}^\wedge$ and $\text{EO}_{\hat{\mathbb{G}}_m} \simeq \text{KO}^\wedge$.

**Remark 1.4.** Let $n$ denote the height of the formal group $\Gamma$. If $p - 1$ divides $n$ and the base field $F$ of $\Gamma$ is algebraically closed, then $\text{Aut}(\Gamma)$ contains a subgroup of order $p$ which is unique up to conjugation. If $F$ is not algebraically closed, then any number of conjugacy classes of subgroups of order $p$ (including zero) is possible depending on $\Gamma$. If $p - 1$ does not divide $n$, then $\text{Aut}(\Gamma)$ does not contain any subgroup of order $p$ (see [23]).

The main goal of this paper is to prove the following result:

**Main Theorem 1.5.** Let $p$ be a prime and $k > 0$. The $p^{k-1}$-fold direct sum of any $\mathbb{C}$-vector bundle is $\text{EO}_{\Gamma}$-orientable if the height of formal group $\Gamma$ is $(p - 1)k$.

For a ring spectrum $R$, the $R$-orientation order of a vector bundle $\xi$, denoted by $\Theta(R, \xi)$, is the smallest positive integer $d$ for which the $d$-fold direct sum of $\xi$ is $R$-orientable (see Definition 2.9 and Remark 2.10).
The proof of Main Theorem 1.5 reduces to the study of a single vector bundle, namely the tautological line bundle $\gamma$ over $\mathbb{C}P^{\infty}$, because of a splitting principal which asserts that $\Theta(\xi, R)$ divides $\Theta(\gamma, R)$ for any $\mathbb{C}$-vector bundle $\xi$ (see Lemma 2.18). Thus, Main Theorem 1.5 follows from the following result.

**Main Theorem 1.6.** $\Theta(\gamma, EO_\Gamma) \text{ divides } p^{p_k-1}$.

Main Theorem 1.6 is a consequence of the study of the action of $C_p \subset \text{Aut}(\Gamma)$ on $E_\Gamma^* \mathbb{C}P^{\infty}$. We first show that $E_\Gamma^* \mathbb{C}P^{\infty}$ admits a 'Free $\oplus$ Finite' decomposition as a $C_p$-module (see Corollary 4.4). Then, using the relative Adams spectral sequence [5], we lift this algebraic splitting to obtain the following $EO_\Gamma$-module splitting:

**Main Theorem 1.7.** In the category of $EO_\Gamma$-modules, there is a splitting

$$EO_\Gamma \wedge \mathbb{C}P^+ \cong \mathcal{M} \lor \mathcal{F},$$

where $\mathcal{M}$ is a finitely presented $EO_\Gamma$-module and $\mathcal{F} \cong \bigvee_{\mathbb{N}} E_\Gamma$ is a wedge of Morava E-theories.

Further, we show that the inclusion map

$$\mathcal{M} \longrightarrow EO_\Gamma \wedge \mathbb{C}P^+_{+}$$

of the compact summand factors through $EO_\Gamma \wedge \mathbb{C}P^+_{+} \wedge_{p^{p_k+1}-p^{k-1}}$ (see Lemma 4.20). Consequently

$$\Theta(\gamma, EO_\Gamma) = \Theta(\gamma^{p^{k+1}-p^{k-1}}, EO_\Gamma),$$

where $\gamma^d$ is the restriction of $\gamma$ to $\mathbb{C}P^d$ (see Corollary 4.26). Atiyah and Todd [4] computed $\Theta(\gamma^d, S)$ for all $d$ which serves as an upper bound for $\Theta(\gamma^d, EO_\Gamma)$ and leads to Main Theorem 1.6 (see Example 2.13). Let $K_\Gamma$ denote the Morava K-theory associated to $\Gamma$. As an application of Main Theorem 1.7, we prove:

**Main Theorem 1.8.** Let $S^1$ act on $EO_\Gamma$ trivially and let $EO_\Gamma^{S^1}$ denote the associated Tate spectrum. Then there is a $K_\Gamma$-local equivalence

$$EO_\Gamma^{S^1} \cong \prod_{-\infty < k < \infty} E_\Gamma.$$

**Remark 1.9.** Main Theorem 1.7 is a generalization of the splitting [10, §15]

$$KO \wedge \mathbb{C}P^+ \cong KO \lor \bigvee_{k \geq 1} \Sigma^{4k-2} KU$$

and Main Theorem 1.8 is a generalization of the fact that

$$KO^{S^1} \cong \prod_{-\infty < k < \infty} KU.$$
Remark 1.11. The second author studied $EO_p$-orientations when $\Gamma$ has height $p - 1$ in his previous work and proved that (see [8, Corollary 1.6])

$$\Theta(\gamma, EO_\Gamma) = p.$$ 

Thus, when $n = p - 1$ our bound is not sharp.

Remark 1.12. At $p = 2$, Kitchloo, Lorman and Wilson [13, 14] studied a similar problem. There is a $C_2$-action on height $n$ Johnson–Wilson theory $E(n)$. The $C_2$ fixed points are commonly called real Johnson–Wilson theory, denoted $ER(n)$. In [13, 14], the authors use genuine $C_2$-equivariant homotopy theory to deduce that $\Theta(\gamma, ER(n)) = 2^n$. Hahn and Shi [11] proved that there is a homotopy ring map $ER(n) \to EO_\Gamma$, where $\Gamma$ is any height $n$ formal group over a perfect field of characteristic 2. Combining these facts, one concludes that $\Theta(\gamma, EO_\Gamma)$ divides $2^n$ when $\Gamma$ has height $n$. Thus, when $p = 2$ our bound on $\Theta(\gamma, EO_\Gamma)$ is not sharp.

Motivated by the $p = 2$ case in Remark 1.12, we conjecture:

Conjecture 1.13. Let $p$ be any prime. If the height of the formal group $\Gamma$ is $n = (p - 1)k$, then $\Theta(\gamma, EO_\Gamma) = p^k$.

A commutative ring spectrum $R$ is called complex orientable if $\gamma$ is $R$-orientable. In our language, $R$ is complex orientable if and only if $\Theta(\gamma, R) = 1$. Quillen proved that an $R$-orientation of $\gamma$ is equivalent data to a homotopy ring map $MU \to R$ (see [1, Part II, Lemma 4.6] and [20, Lemma 4.1.13] for a statement). Let $MU[n]$ be the Thom spectrum of the map

$$\varphi_n : BU \to BU$$

given by multiplication by $n$ in the additive $E_\infty$-structure. If $R$ is a ring spectrum $R$ such that $\Theta(\gamma, R)$ divides $n$, we prove that there is a map $MU[n] \to R$ using a theorem of Segal [22] (see Theorem 2.15). Main Theorem 1.5 and Theorem 2.15 imply the following corollary.

Corollary 1.14. There exists a map $MU[n] \to EO_\Gamma$ whenever $p^{p^k - 1}$ divides $n$.

Organization of the paper

In Section 2, we review orientation theory and prove a splitting principle, namely Lemma 2.18, which reduces the proof of Main Theorem 1.5 to Main Theorem 1.6.

In Section 3, we study the action of order $p$ elements in $\text{Aut}(\Gamma)$ on the Morava K-theory of $\mathbb{C}P^\infty$. The main purpose of this section is to establish the algebraic splitting of Corollary 3.26.

We devote Section 4 in proving the main theorems of this paper. In Section 4.1, we lift the algebraic splitting of Corollary 3.26 to a topological splitting of $EO$-modules as in Corollary 4.19, whose special case is Main Theorem 1.7. We then use the $EO$-module splitting to prove Main Theorem 1.6 in Section 4.2, and Main Theorem 1.8 in Section 4.3.
A BRIEF REVIEW OF ORIENTATION THEORY

The goal of this section is to prove a splitting principle (see Lemma 2.18) and generalize a result of Quillen (see Theorem 2.15). We do so after reviewing orientation theory and its connection to chromatic homotopy theory.

Although many of the statements in this section are known to be true more generally for homotopy commutative ring spectra, we restrict to ones with an $E_\infty$ structure because it is a requirement in the proof of Theorem 2.15 and Lemma 2.18. This is acceptable for our application, namely Corollary 1.14, because $EO_\Gamma$ is an $E_\infty$-ring spectrum.

Orientation with respect to cohomology theories

Given an $\mathbb{R}$-vector bundle $\xi$ over a base space $B_\xi$ with total space $E_\xi$, the Thom space of $\xi$ is defined as the cofiber of the projection map

$$\text{Th}(\xi) := \text{Cofib}(\pi_+ \to B_\xi),$$

where $E^\times_\xi$ denotes the complement of the zero section. The Thom spectrum of $\xi$ is the suspension spectrum of $\text{Th}(\xi)$:

$$M(\xi) := \Sigma^\infty \text{Th}(\xi)$$

Example 2.2. When $\xi \cong \mathcal{E}^r$ is a trivial bundle of rank $r$, then

$$\text{Th}(\xi) \cong \Sigma^r B_\xi,$$

where $B_\xi$ is the base space of $\xi$.

Example 2.3. For the complex tautological line bundle $\gamma^n$ over $\mathbb{C}P^n$, $\text{Th}(k \gamma^n)$ is equivalent to

$$\mathbb{C}P^{n+k}_k := \text{Cofib}(\mathbb{C}P^{k-1}_+ \to \mathbb{C}P^{n+k}_+),$$

where $k$ is a positive integer and $1 \leq n \leq \infty$.

The notion of Thom space does not extend to virtual vector bundles, but the notion of Thom spectrum does. We recall in brief the construction of Thom spectrum, the Thom isomorphism, and orientation theory for $E_\infty$ ring spectra following [3]. Many of these ideas originated in [16].

The space of units $\text{GL}_1(R)$ of an $E_\infty$ ring spectrum $R$ is an $E_\infty$ space. Consequently, for any space $B$, the set of homotopy classes of pointed maps $[B_+, \text{BGL}_1(R)]$ forms an abelian group.

Let $\mathbb{S}$ denote the sphere spectrum. For any zero-dimensional virtual vector bundle $\xi$, the composite

$$B_\xi \xrightarrow{f_\xi} BO \xrightarrow{J} \text{BGL}_1(\mathbb{S}) \xrightarrow{B(\theta)} \text{BGL}_1(R),$$
classifies a principal $\text{GL}_1(\mathbb{R})$-bundle $P(\xi, \mathbb{R})$ which is a right $\text{GL}_1(\mathbb{R})$-space where $f_\xi$ is the classifying map, $J$ is the $J$-homomorphism and $i_\mathbb{R}$ is the unit map of $\mathbb{R}$. The $R$-Thom spectrum of $\xi$ is defined as the derived smash product

$$M(\xi, R) := P(\xi, \mathbb{R})_+ \wedge_{\text{GL}_1(\mathbb{R})_+} \mathbb{R}. $$

The above construction of the $R$-Thom spectrum extends to virtual vector bundles of all dimensions.

**Notation 2.4.** For a virtual vector bundle $\xi$ of dimension $d$, let $\xi_0 := \xi - \epsilon^d$ denote the corresponding zero-dimensional bundle. We declare

$$M(\xi, R) := \Sigma^d M(\xi_0, R).$$

We use $M(\xi)$ as a shorthand for $M(\xi, \mathbb{S})$.

There is a natural weak equivalence

$$M(\xi) \wedge R \xrightarrow{\cong} M(\xi, R),$$

which determines the homotopy type of $M(\xi, R)$.

**Notation 2.5.** For all $k \in \mathbb{Z}$, let $\mathbb{C}P^n_{k}$ denote the Thom spectrum

$$\mathbb{C}P^n_{k} := M(\gamma^n).$$

This extends Example 2.3 to all integers, provided that we consider $\mathbb{C}P^b_{\delta}$ to be a spectrum.

An $R$-orientation of $\xi$ is a choice of trivialization of $P(\xi_0, \mathbb{R})$, or equivalently a choice of null-homotopy of the classifying map

$$B(i_\mathbb{R}) \circ J \circ f_{\xi_0} : B_\xi \longrightarrow B\text{GL}_1(\mathbb{R}).$$

Such a null-homotopy leads to an equivalence of $R$-modules

$$M(\xi) \wedge R \simeq \Sigma^d M(\xi_0, R) \simeq \Sigma^d B_{\xi_+} \wedge R,$$

which we refer to as the $R$-Thom isomorphism.

Dually, an $R$-orientation for $\xi$ leads to an equivalence of $R^{B_{\xi_+}}$-modules

$$R^{B_{\xi_+}} \simeq \Sigma^d R^{M(\xi)},$$

which can be regarded as a cohomological version of the $R$-Thom isomorphism. On $\pi_0$, the above isomorphism sends a map $f : \Sigma^\infty B_{\xi_+} \rightarrow R$ to the composite

$$M(\xi) \xrightarrow{f_{M(\xi)}\wedge \mathbb{R}} M(\xi) \wedge R \xrightarrow{\cong} \Sigma^d B_{\xi_+} \wedge R \xrightarrow{f \wedge 1_k} \Sigma^d R \wedge R \xrightarrow{i_\mathbb{R}} \Sigma^d \mathbb{R}. \quad (2.7)$$
When $f : \Sigma^\infty B_+ \to S \to R$ is the composite of the collapse map with the unit of $R$, we refer to the corresponding element of $R^d(M(\xi))$ as an $R$-Thom class. It is classical fact that an $R$-orientation is equivalent to a choice of an $R$-Thom class.

**Remark 2.8.** If $\xi$, is $R$-orientable, then its restriction $\xi^{(k)}$ to the $k$th skeleton $B_{\xi}^{(k)}$ is also $R$-orientable and the Thom spectrum $M(\xi^{(k)})$ is the $(k + \dim \xi)$-th skeleton of $M(\xi)$. Thus, any $R$-Thom isomorphism $\omega : R \wedge M(\xi) \xrightarrow{\sim} R \wedge \Sigma^d B_+$ preserves the Atiyah–Hirzebruch filtration

\[
\omega^{(k)} : R \wedge M(\xi^{(k)}) \xrightarrow{\sim} R \wedge \Sigma^d B^{(k)}.\]

**Definition 2.9.** For an $\mathbb{E}_\infty$ ring spectrum $R$, the $R$-orientation order $\Theta(\xi, R)$ of a virtual vector bundle $\xi$ with base space $B_\xi$ is defined as the order of the class $[B(\iota) \circ J \circ f_{\xi_0}] \in [B_{\xi+}, BGL_1(R)]$.

**Remark 2.10.** We say a vector bundle $\xi$ is $n$-orientable with respect to $R$ if $n \xi$ is $R$-orientable. The number $\Theta(\xi, R)$ is the smallest positive integer $n$ for which $n \xi$ is $R$-orientable. In particular, $\xi$ is $R$-orientable when $\Theta(\xi, R) = 1$.

**Example 2.11.** Since $\mathbb{F}_2$ is the trivial group, every vector bundle is $H\mathbb{F}_2$-orientable. However, for $p > 2$, not every bundle is $H\mathbb{F}_p$-orientable. For example, $H\mathbb{F}_p$-orientation order of the tautological line bundle over $\mathbb{R}\mathbb{P}^\infty$ is 2 (as its $HZ$-orientation order is 2).

If a virtual vector bundle $\xi$, is $R$-orientable and there is a ring map $R \to T$, then $\xi$, is also $T$-orientable. In particular, $\Theta(\xi, T)$ divides $\Theta(\xi, R)$.

**Remark 2.12.** If $R$ is a $p$-complete ring spectrum (in the sense of Bousfield [7]) and $\xi$ is an $HZ$-orientable virtual vector bundle over a compact base, then $\Theta(\xi, R)$ is a power of $p$. This is because any $p$-complete ring spectrum admits a ring map from the $p$-complete sphere spectrum

\[
t_R : \mathbb{S}_p \longrightarrow R,
\]

and $\Theta(\xi, \mathbb{S}_p)$ is a power of $p$ as the higher homotopy groups of $GL_1(\mathbb{S}_p)$ are $p$-torsion.

**Example 2.13.** The $S$-orientation order of the tautological line bundle $\gamma^n$ over $\mathbb{C}\mathbb{P}^n$ is known for all $n \in \mathbb{N}$ due to work of Atiyah and Todd [4] (upper bound) and Adams and Walker [2] (lower bound). It is given by the formula

\[
\nu_p(\Theta(\gamma^n, \mathbb{S})) = \begin{cases} 
\max \left\{ r + \nu_p(r) : 1 \leq r \leq \left\lfloor \frac{n}{p-1} \right\rfloor \right\} & \text{if } p \leq n + 1 \\
0 & \text{if } p > n + 1,
\end{cases}
\]

where $\nu_p$ is the $p$-adic valuation.

### 2.2 Complex $n$-orientation and the splitting principle

A complex orientation of a $\mathbb{E}_\infty$-ring spectrum $R$ is a choice of an $R$-Thom class for the tautological complex line bundle $\gamma$, that is, a map

\[
u_R : M(\gamma - e^2) \simeq \Sigma^{-2} \mathbb{C}\mathbb{P}^\infty \to R
\]
such that the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\iota_k} & R \\
\downarrow & & \downarrow \\
\Sigma^{-2}C\mathbb{P}_n^\infty & \xrightarrow{u_k} & R \\
\end{array}
\]

commutes up to homotopy. One may generalize this definition to the notion of a complex

\[n\text{-orientation} \text{ as follows.} \]

**Definition 2.14.** A complex \(n\)-orientation of an \(E_\infty\)-ring spectrum \(R\) is a map

\[u_R : M(n(\gamma - \epsilon^2)) \simeq \Sigma^{-2n}C\mathbb{P}_n^\infty \to R\]

such that the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\iota_k} & R \\
\downarrow & & \downarrow \\
\Sigma^{-2n}C\mathbb{P}_n^\infty & \xrightarrow{u_k} & R \\
\end{array}
\]

commutes up to homotopy.

Quillen proved that complex orientations of a ring spectrum \(R\) are in bijection with homotopy ring maps \(u_1 : MU \to R\) (see [20, Lemma 4.1.13] for a proof). Stated differently, Quillen proved that if \(\Theta(\gamma, R) = 1\), then \(\Theta(\xi, R) = 1\) for any complex vector bundle \(\xi\).

Let \(MU[n]\) be the Thom spectrum of the multiplication by \(n\) map \(\varphi_n : BU \to BU\). Because \(\varphi_n\) is an \(E_\infty\)-map (with respect to the additive infinite loop structure on \(BU\)), \(MU[n]\) is an \(E_\infty\)-ring spectrum. We make the following generalization of Quillen’s work.

**Theorem 2.15.** Let \(R\) be an \(E_\infty\)-ring spectrum. There is a one-to-one correspondence between complex \(n\)-orientations of \(R\) and homotopy classes of unital maps \(u_n : MU[n] \to R\).

**Proof.** A complex \(n\)-orientation of \(R\) is a null-homotopy of the composite

\[
\begin{array}{ccc}
\mathbb{C}\mathbb{P}_n^\infty & \xrightarrow{f_\gamma} & BU \\
\downarrow & \xrightarrow{\varphi_n} & \downarrow \\
BU & \xrightarrow{\iota_k} & BGL_1(S) \\
\downarrow & \xrightarrow{BGL_1((\iota_k)_R)} & BGL_1(R) \\
\end{array}
\]

where \(\iota_c\) is the complex J-homomorphism. For a pointed space \(X\), let \(QX\) denote \(\Omega^\infty\Sigma^\infty X\). Using the fact that \(BU\), \(BGL_1(S)\) and \(BGL_1(R)\) are \(E_\infty\)-spaces and \(\varphi_n\), \(J_c\) and \(BGL_1((\iota_k)_R)\) are \(E_\infty\)-maps, we can extend \(f_\gamma\) to \(\tilde{f}_\gamma\)

\[
\begin{array}{ccc}
\mathbb{C}\mathbb{P}_n^\infty & \xrightarrow{f_\gamma} & BU \\
\downarrow & \xrightarrow{\varphi_n} & \downarrow \\
\mathbb{C}\mathbb{P}_n^\infty & \xrightarrow{\tilde{f}_\gamma} & BGL_1(R) \\
\end{array}
\]
and the null-homotopy of \( BGL_1(t_R) \circ J_C \circ f_\gamma \) to a null-homotopy of

\[
BGL_1(t_R) \circ J_C \circ \varphi_n \circ f_\gamma : QCP^\infty \to BGL_1(R).
\]

By [22, Proposition 2.1], there exists a map \( \alpha : BU \to QCP^\infty \) such that \( f_\gamma \circ \alpha \simeq 1_{BU} \). Therefore,

\[
BGL_1(t_R) \circ J_C \circ \varphi_n \simeq BGL_1(t_R) \circ J_C \circ \varphi_n \circ f_\gamma \circ \alpha \simeq \ast .
\]

We let \( u_n : MU[n] \to R \) to be the corresponding \( R \)-Thom class.

\[\square\]

**Remark 2.16.** Theorem 2.15 restricted to \( n = 1 \) is significantly weaker when compared to Quillen’s result (see [20, Lemma 4.1.13]). Quillen only requires that the ring \( R \) to be homotopy commutative and he proves that the \( R \)-Thom class \( u_1 \) is multiplicative. Quillen’s argument when \( n = 1 \) relies on the computation of \( R^*(BU(d)_\gamma) \) in terms of Chern classes to gain control over the multiplicative structure. It is unclear how to generalize such direct methods to \( n > 1 \).

**Remark 2.17.** If \( f_\xi : X \to BU \) is the classifying map of a virtual complex vector bundle \( \xi \), then \( \varphi_n \circ f_\xi \) classifies the bundle \( n\xi \).

**Lemma 2.18** (The splitting principle). If \( R \) is an \( E_\infty \)-ring spectrum, then \( \Theta(R, \xi) \) divides \( \Theta(R, \gamma) \) for any virtual complex vector bundle \( \xi \).

**Proof.** Without loss of generality, we may assume \( \dim \xi = 0 \). By Remark 2.17, the classifying map of \( n\xi \) factors through \( \varphi_n \)

\[
f_{n\xi} : B_{\xi_n} \xrightarrow{f_\xi} BU \xrightarrow{\varphi_n} BU.
\]

In the proof of Theorem 2.15, we show that if \( n\gamma \) is \( R \)-orientable, then the composite

\[
BU \xrightarrow{BGL_1(t_R) \circ J_C \circ f_\gamma} BGL_1(R)
\]

is null-homotopic. Therefore, the composite \( BGL_1(t_R) \circ J_C \circ f_{n\xi} \) is null as well

\[
BGL_1(t_R) \circ J_C \circ f_{n\xi} = (BGL_1(t_R) \circ J_C \circ \varphi_n) \circ f_\xi \simeq (\ast) \circ f_\xi \simeq \ast .
\]

Thus, when \( \Theta(R, \gamma) = n \) then \( n\xi \) is \( R \)-orientable, and therefore, \( \Theta(R, \xi) \) divides \( n \). \[\square\]

### 3 \ THE \( C_p \)-ACTION ON THE MORAVA \( K \)-THEORY OF \( CP^\infty \)

Let \( \Gamma \) be a formal group of height \( n = (p - 1)k \) for some positive \( k \) such that \( C_p \) is a subgroup of \( \text{Aut}(\Gamma) \) (see Remark 1.4). The goal of this section is to study the action of \( C_p \) on \( K^*_T CP^\infty \), where \( K_T = ET/m \) is the associated Morava K-theory. We do so by relating the action of \( C_p \) with the action of an element \( P_k \) of the Steenrod algebra on the homology of \( CP^\infty \) (see (3.18)).
Notation 3.1. We will use the following notations and conventions for the remainder of the paper.

- Fix a prime $p$ and a positive integer $k$. Let $n = k(p - 1)$.
- Fix a perfect field $\mathbb{F}$ of characteristic $p$ and a formal group $\Gamma$ of height $n$ over $\mathbb{F}$ such that $C_p$ is a subgroup of $\text{Aut}(\Gamma)$. Let $E_\Gamma$ denote the associated Lubin–Tate theory. By Lubin–Tate theory $[15]$
  \[ \pi_*E_\Gamma \cong \mathbb{W}(\mathbb{F})[u_1, \ldots, u_{n-1}][u^{\pm}] , \]
  where $\mathbb{W}(\mathbb{F})$ is the ring of Witt vectors over $\mathbb{F}$. The elements $u_1, \ldots, u_{n-1}$ are elements of $\pi_0E_\Gamma$ and $u \in \pi_{-2}E_\Gamma$.
- Let $\mathfrak{m}$ denote the maximal ideal $(p, u_1, \ldots, u_{n-1})$ of $\pi_*E_\Gamma$ and let $K_\Gamma$ denote the corresponding height $n$ Morava K-theory so that $\pi_*K_\Gamma \cong \pi_*(E_\Gamma) / \mathfrak{m}$.
- Fix an embedding $\iota : C_p \to \text{Aut}(\Gamma)$ and let $E_{\Gamma}^\iota := E_{\Gamma}^{hC_p}$.
- Abbreviate $E_{\Gamma}$ by $E$, $K_{\Gamma}$ by $K$, and $E_{\Gamma}^\iota$ by $EO_{\Gamma}$, leaving the dependence on $\Gamma$ and $\iota$ implicit.
- Let $L_K(-)$ denote the Bousfield localization $[7]$ with respect to $K$.
- Let $E_{\Gamma}^*(-) := \pi_*(L_K(E \wedge -))$, $E_{\Gamma}^*EO(-) := \pi_*(E \wedge_{EO} -)$, and $K_{\Gamma}EO(-) := \pi_*(K \wedge_{EO} -)$.
- Fix a $p$-typical coordinate on $\Gamma$ and let $\pi_E : BP \to E$ be the associated map. Let $\pi_K : BP \to K$ denote the composite of $\pi_E$ with the reduction map from $E \to K$.
- Let $\pi_{\mathbb{F}p} : BP \to \text{HF}_p$ be the standard reduction map.

The ring spectrum $EO_{\Gamma} := E_{\Gamma}^{hC_p}$ depends not only on the formal group $\Gamma$ but also on the choice of embedding $\iota : C_p \to \text{Aut}(\Gamma)$ up to a conjugation. We emphasize that our results apply to all choices of $EO_{\Gamma}$ because Lemma 3.7 holds for all pairs $(\Gamma, \iota)$. We begin by recalling some facts from Lubin–Tate theory needed in this paper.

Let $\overline{\mathbb{F}}$ denote the separable closure of $\mathbb{F}$ and let $\overline{\Gamma}$ be the base change of $\Gamma$ to $\overline{\mathbb{F}}$. The endomorphism ring of $\overline{\Gamma}$ is a noncommutative valuation ring which can be explicitly described as

\[ \text{End}(\overline{\Gamma}) \cong \mathbb{W}(\overline{\mathbb{F}})(T)/(Ta - \phi(a)T, T^n - pu) , \]  

where $T$ is a uniformizer, $\phi$ is the Frobenius and $u \in \mathbb{W}(\mathbb{F}_p^n)^\times$ is a unit (see $[15]$). Note that $\nu(T^n) = \nu(pu) = 1$. Any element $e \in \text{End}(\overline{\Gamma})$ can be expressed as

\[ e = \sum_{i=0}^{\infty} a_i T^i , \]

where $a_i$ are Teichmüller lifts of $\overline{\mathbb{F}}$ in $\mathbb{W}(\overline{\mathbb{F}})$ and the valuation is $\nu(e) = j/n$, where $j$ is the smallest integer such that $a_j \neq 0$.

Notation 3.4. The Morava stabilizer group $\text{Aut}(\Gamma)$ is a profinite group. Therefore, $\text{Aut}(\Gamma)$ and its subgroups are equipped with the profinite topology. Likewise, $E_*$ can be given the $\mathfrak{m}$-adic topology as $E_*$ is a local ring with maximal ideal $\mathfrak{m}$. In (3.5), we use $\text{Map}^c(\text{Aut}(\Gamma), E_*)$ to denote the subset of continuous maps with respect to these topologies. However, $C_p \subset \text{Aut}(\Gamma)$ is discrete, and therefore $\text{Map}^c(C_p, E_*) = \text{Map}(C_p, E_*)$. 

The map $\pi_E : BP \to E$ induces a map $BP \wedge BP \to E \wedge E \to L_K(E \wedge E)$. There is a homotopy coequalizer map $L_K(E \wedge E) \to E \wedge_{E O} E$. These maps fit into a diagram

$$
\begin{array}{ccc}
BP \wedge BP & \xrightarrow{\rho_E} & E^* \wedge E \\
\downarrow & & \downarrow \\
\text{Map}^\Gamma(\text{Aut}(\Gamma), E^*) & \xrightarrow{\text{Res}^C} & \text{Map}(C_p, E^*)
\end{array}
$$

(3.5)

The vertical isomorphisms are by Galois theory [21, Theorem 5.4.4 and Definition 4.1.3]. The map $\text{Res}^C_p$ is restriction along the inclusion map $\iota : C_p \to \text{Aut}(\Gamma)$.

**Notation 3.6.** Given $\theta \in BP \wedge BP$ and $\mathbb{G} \in \text{Aut}(\Gamma)$, the map

$$
\rho_E : BP \wedge BP \longrightarrow E^* \wedge E = \text{Map}^\Gamma(\text{Aut}(\Gamma), E^*)
$$

allows us to interpret the element $\theta(g) := \rho_E(\theta)(g) \in E_*$. Let $\overline{\theta}(g)$ denote the image of $\theta(g)$ under the quotient map

$$
E_* \to E_*/\mathfrak{m} \cong K_*.
$$

There is an isomorphism $BP \wedge BP \cong BP_1$, where $t_1$ is in degree $2(p^i - 1)$ [20, Theorem 4.1.18].

**Lemma 3.7.** Let $\zeta \in \text{Aut}(\Gamma)$ be an element of order $p$. Then $\bar{t}_i(\zeta) = 0$ for $i < k$ and $\bar{t}_k(\zeta)$ is a unit.

**Proof.** Let $\bar{F}$ be the separable closure of $F$, $\bar{\Gamma}$ be the base change of $\Gamma$ to $\bar{F}$ and $\bar{K} := K_{\bar{\Gamma}}$ be the Morava K-theory associated to $\bar{\Gamma}$. The field extension $F \to \bar{F}$ induces the inclusion of groups

$$
\alpha : \text{Aut}(\Gamma) \hookrightarrow \text{Aut}(\bar{\Gamma})
$$

as well as a map

$$
f : K_* \longrightarrow \bar{K}_*
$$

on Morava K-theories. Since $f$ is an injection and

$$
\bar{t}_i(\alpha(\zeta)) = f(\bar{t}_i(\zeta)),
$$

it suffices show that $\bar{t}_i(\zeta) = 0$ for $i < k$ and $\bar{t}_k(\zeta)$ is a unit whenever $\zeta \in \text{Aut}(\bar{\Gamma})$ is an element of order $p$. When expressed as a power series (as in (3.3))

$$
\zeta = 1 + \sum_{i>0} a_i T^i,
$$

and it follows from the properties of $t_i$ that

$$
t_i(\zeta) \equiv a_i u^{1-p^i} \mod \mathfrak{m}
$$

(3.8)
Because $\mathbb{Q}_p(\tilde{\zeta})$ is a totally ramified extension of $\mathbb{Q}_p$ of degree $p - 1$ with $\tilde{\zeta} - 1$ as a uniformizer,

$$v(\tilde{\zeta} - 1) = \frac{1}{p - 1} = \frac{k}{\text{height of } \Gamma} = v(T^k).$$

Therefore, the ideal generated by $\tilde{\zeta} - 1$ is equal to the ideal generated by $T^k$. Thus, $a_i = 0$ if $i < k$ and a unit if $i = k$ in (3.8).

3.1 Filtering the $C_p$-action on $K_\ast X$

For any spectrum $X$, the $E^n_{eo}E$-coaction on $E_\ast X$, given by the composition

$$\Psi : E_\ast X \xrightarrow{\Psi} E^n E \otimes_{E_\ast} E_\ast X \xrightarrow{\text{Res}_E \otimes \text{id}_E} E^n_{eo} E \otimes_{E_\ast} E_\ast X,$$  \hspace{1cm} (3.9)

leads to a dual action of $C_p$ on $E_\ast X$ since $E^n_{eo}E$ is isomorphic $\text{Map}(C_p, E_\ast)$ as a Hopf algebroid. Explicitly, the action of $g \in C_p$ is given by

$$E_\ast X \xrightarrow{\Psi} E^n_{eo} E \otimes_{E_\ast} E_\ast X \xrightarrow{\text{ev}_g \otimes 1} E_\ast X.$$  \hspace{1cm} (3.10)

Note that $E^n_{eo}E \cong \text{Map}(C_p, E_\ast)$ is a quotient Hopf algebra of $E^n E$ and dual to $\pi \ast (\text{EO} - \text{Mod}(E, E)) \cong E_\ast [C_p]^\sigma$ as defined in Remark 3.11.

Remark 3.11. The nontrivial action of $C_p$ on $E_\ast$ means $\pi \ast (\text{EO} - \text{Mod}(E, E))$ is isomorphic as a ring to the twisted group ring

$$E_\ast [C_p]^\sigma := E_\ast \langle \zeta \rangle / (\zeta^p = 1, \zeta \cdot e = \zeta(e) \cdot \zeta),$$

where $e \in E_\ast$. Since the action of $C_p$ on $K_\ast$ is trivial, $E_\ast [C_p]^\sigma / \mathfrak{m}$ is isomorphic to the untwisted group ring $K_\ast [C_p]$.

Definition 3.12. A spectrum $X$ is even if $X$ is bounded below, $H\mathbb{Z}_{2i+1}X = 0$ and $H\mathbb{Z}_{2i}X$ is a finitely generated $\mathbb{Z}$-module for all integers $i$.

When $X$ is an even spectrum, there is an isomorphism

$$K_\ast X \cong E_\ast X / \mathfrak{m},$$

which makes $K_\ast X$ into a $K_\ast [C_p]$-module.

Next, we use the Atiyah–Hirzebruch filtration to relate the $K_\ast [C_p]$-module structure on $K_\ast X$ to the $B(k)_\ast$ comodule structure on $H_\ast X$ for an even spectrum $X$, where $B(k)_\ast$ is a quotient Hopf algebra of the even dual Steenrod algebra. The Atiyah–Hirzebruch filtration on $R_\ast X$ is an
increasing filtration induced by the skeletal filtration of $X$

$$\text{Fil}_d R_* X := R_* X^{(d)} \subseteq R_* X.$$ 

The associated graded

$$\text{gr} R_* X := \bigoplus_{d} \frac{\text{Fil}_d R_* X}{\text{Fil}_{d-1} R_* X}$$

is bigraded as the second grading is induced by the internal grading of $R_* X$.

When $R$ is complex orientable (for example, $\text{BP}$, $K$ and $E$) and $X$ is an even spectrum (for example, $\mathbb{CP}^{n+k}$), the AHSS (Atiyah Hirzebruch spectral sequence) for $R_* X$ collapses on the $E_2$-page. This collapse induces an isomorphism

$$\text{gr} R_* X \cong R_* \otimes H_* X.$$ 

**Lemma 3.13.** Let $X$ be an even spectrum and let $\chi = \zeta - 1 \in K_*[\mathbb{C}_p]$. Then

$$\chi_* \text{Fil}_d K_* X \subseteq \text{Fil}_{d-2p^k+2} K_* X.$$ 

**Proof.** We required that $X$ be even so the map $K_* \otimes_{\text{BP}*} \text{BP}_* X \to K_* X$ is surjective and we can check our claim on the image.

Pick $x^{\text{BP}} \in \text{Fil}_d \text{BP}_* X$. Write $x^K = \pi_{K*}(x^{\text{BP}})$ and $x^{FP} = \pi_{FP*}(x^{\text{BP}})$. Write the BP-coaction on $x^{\text{BP}}$ as

$$\Psi(x^{\text{BP}}) = 1 \otimes x^{\text{BP}} + \sum_i \theta_i \otimes x_i^{\text{BP}},$$

The counit axiom says that

$$(\varepsilon \otimes 1)(\Psi(x^{\text{BP}})) = x^{\text{BP}} = (\varepsilon \otimes 1)(1 \otimes x^{\text{BP}})$$

so we conclude that $\theta_i \in \ker \varepsilon = (t_1, t_2, ...)$.

By definition,

$$\chi_* (x^K) = (\zeta - 1)_*(x^K) = \sum_i \tilde{\theta}_i(\zeta) \cdot x^K_i,$$

where $\tilde{\theta}_i$ is as in Notation 3.6. By Lemma 3.7, $\tilde{\theta}_i(\zeta) = 0$ when

$$\tilde{\theta}_i \in (t_1, ..., t_{k-1}) \subseteq \text{BP}_* \text{BP},$$

therefore $\chi_* (x^K) \in \text{Fil}_{d-|\theta_i|} K_* X = \text{Fil}_{d-2p^k+2} K_* X$ as $t_k$ is the element of lowest degree in $\ker \varepsilon/(t_1, ..., t_{k-1})$. 

We introduce an increasing filtration on $K_*[\mathbb{C}_p]$ by assigning $\chi$ an ‘Atiyah Hirzebruch weight’

$$|\chi|_{AH} = -|t_k| = 2 - 2p^k.$$
We denote the associated graded by $\text{gr} K_\ast[C_p]$ and the representative of $\chi$ in $\text{gr} K_\ast[C_p]$ by $\tilde{\chi}$. It follows from Lemma 3.13 that, for an even spectrum $X$, the $K_\ast[C_p]$-module structure on $K_\ast X$ induces a $\text{gr} K_\ast[C_p]$-module structure on $\text{gr} K_\ast X$ (see Lemma 3.17).

**Lemma 3.14.** The bigraded Hopf algebra $\text{gr} K_\ast[C_p]$ is isomorphic to

$$K_\ast[\tilde{\chi}]/(\tilde{\chi}^p),$$

where $\tilde{\chi}$ is a primitive in the Atiyah Hirzebruch filtration $2 - 2p^k$.

**Proof.** Since $\Delta(\zeta) = \zeta \otimes \zeta$ and $\chi = \zeta - 1$, we see $\Delta(\chi) = \chi \otimes 1 + 1 \otimes \chi + \chi \otimes \chi$. Thus, in the associated graded $\Delta(\tilde{\chi}) = \tilde{\chi} \otimes 1 + 1 \otimes \tilde{\chi}$.

Let $\mathcal{P}$ be the quotient Hopf algebra of the Steenrod algebra $\mathcal{P} = A/(Q_0, Q_1, \ldots)$, where $Q_i$ are the Milnor primitives. Its dual is the sub Hopf algebra of $A_\ast$ generated by the even degree generators

$$\mathcal{P}_\ast \cong \begin{cases} F_p[\xi_1, \xi_2, \ldots] \subset A_\ast & \text{if } p \text{ is odd} \\ F_p[\xi_1^2, \xi_2^2, \ldots] \subset A_\ast & \text{if } p = 2. \end{cases}$$

Under the map $(\pi_F \wedge \pi_F)_\ast : BP_*BP \to A_\ast$, the image of $t_k$ is

$$(\pi_F \wedge \pi_F)_\ast(t_k) = \begin{cases} c(\xi_k) & \text{if } p \text{ is odd} \\ c(\xi_k^2) & \text{if } p = 2, \end{cases}$$

where $c$ denotes the antipode map of the dual Steenrod algebra. It follows from the definition of $\xi_k \in \mathcal{P}_\ast$ that its linear dual $P_k \in \mathcal{P}$ satisfies the formula

$$P_k(x) = x^{p^k},$$

(3.15)

where $x \in H^2\mathbb{C}P^\infty_+$ is a generator (see [17]).

**Definition 3.16.** Let $B(k) \subset \mathcal{P}$ denote the sub Hopf algebra generated by $P_k$. As a Hopf algebra,

$$B(k) \cong F_p[P_k]/(P_k^p),$$

where $P_k$ is a primitive.

Note that $c(\xi_k) \equiv -\xi_k \mod (\xi_1, \ldots, \xi_{k-1})$, which leads to the negative sign in Lemma 3.17. Let

$$\iota_X : H_\ast X \longrightarrow \text{gr} K_\ast X \cong K_\ast \otimes H_\ast X$$

be the map that sends $x \mapsto 1 \otimes x$. 
Lemma 3.17. For an even spectrum $X$,

$$\zeta(t_X(x)) = -\tilde{t}_k(\zeta)\zeta(P_k(x)) \quad (3.18)$$

for all $x \in H_*X$.

Proof. Let $p$ be an odd prime. Let $x^p = x$ and $y^p = y = P_k(x)$, $I^p_r = P_\ast \otimes \Fil_r H_*X$ and $I^p_{BP} = BP_\ast BP \otimes_{BP_\ast} \Fil_r BP_*X$. Suppose $x$ is in filtration $d$, then the $A_\ast$ coaction on $x^p$ is

$$\Psi(x^p) \equiv 1 \otimes x^p + \xi_k \otimes y^p \pmod{(\xi_1, \ldots, \xi_{k-1}, \zeta^p)}.$$

Let $x_{BP}$ be a lift of $x$ to $BP_dX$. It follows that the $BP$ coaction on $x_{BP}$ is

$$\Psi(x_{BP}) \equiv 1 \otimes x^p - t_k \otimes y_{BP} \pmod{(p, \ldots, v_{n-1}, t_1, \ldots, t_{k-1}, \tilde{I}^{p}_{d-2p^k})},$$

where $y_{BP}$ is a lift of $y^p$. Since $0(\zeta) \equiv 0 \pmod{m}$ for $0 \in (t_1, \ldots, t_{k-1})$, we get

$$\zeta \cdot x^K \equiv x^K - \tilde{t}_k(\zeta)y^K \pmod{\Fil_{d-2p^k} K_*X}.$$

Since $t_X(x) = [\pi_{K_*}(x_{BP})]$ and $t_X(P_k(x)) = [\pi_{K_*}(y_{BP})]$, the result follows.

The same argument works for $p = 2$ after replacing $\xi_i$ with $\xi_i^2$ above. □

Remark 3.19. Following Lemmas 3.14 and 3.17, we see that there is a Hopf algebra isomorphism

$$K_* \otimes B(k) \cong \gr K_*[C_p]$$

obtained by sending $P_k \mapsto -\tilde{t}_k(\zeta)\zeta$. This isomorphism relates the $K_* \otimes (k)$-module structure on the left to the $\gr K_*[C_p]$-module structure on the right, of the isomorphism

$$K_* \otimes H_*X \cong \gr K_*X$$

for any even spectrum $X$.

3.2 The action of $C_p$ on $K_*\mathbb{C}P^\infty_+$

We now explicitly compute the coaction of $B(k)$ on $H_*\mathbb{C}P^\infty_+$ and deduce that $K_*\mathbb{C}P^\infty_+$ has a large $K_*[C_p]$-free summand.

Notation 3.20. Let $x$ be the generator of $H^0\mathbb{C}P^\infty_+ \cong F_p[\llbracket x \rrbracket]$ and let $b_i \in H_{2i}\mathbb{C}P^\infty_+$ be the linear dual of $x^i$. The $HF_\ast$-Thom isomorphism for $c\gamma$ implies $H^0\mathbb{C}P^\infty_+$ is a free module of rank one over $H^0\mathbb{C}P^\infty_+$. Fix an $HF_\ast$-Thom class $u_c \in H^2\mathbb{C}P^\infty_+$. Let

$$b_i \in H_{2i}\mathbb{C}P^\infty_+$$
denote the linear dual to $x^{i-c} \cdot u_c$. We also use $b_i \in H_2[\mathbb{C}P^{a+c}_c]$ to denote the element which maps to $b_i$ under the skeletal inclusion $\mathbb{C}P^{a+c}_c \to \mathbb{C}P^\infty_c$.

**Notation 3.21.** Let $\beta_k := \frac{(p-1)(p^k - 1)}{2}$.

**Proposition 3.22.** Let $c$ be an integer. The action of $\tilde{\chi}$ on

$$\text{gr} \ K_s \mathbb{C}P^\infty_c \cong K_s\{b_c, b_{c+1}, \ldots\}$$

is given by

$$\tilde{\chi}_*(b_i) = \begin{cases} (i - p^k + 1)b_{i-p^k+1} & \text{if } i \geq p^k - 1 + c \\ 0 & \text{otherwise} \end{cases}$$

Moreover, there is a $\text{gr} \ K_s[C_p]$-module isomorphism

$$\text{gr} \ K_s \mathbb{C}P^\infty_c \cong F^{\text{gr} K}_c \bigoplus M^{\text{gr} K}_c,$$

where $F^{\text{gr} K}_c$ is a free $\text{gr} \ K_s[C_p]$-module of infinite rank and $M^{\text{gr} K}_c$ is a finite dimensional $K_s$-module.

**Remark 3.23.** The Hopf algebras $K_s[C_p]$ and $\text{gr} \ K_s[C_p]$ are self-injective. In other words, if $M$ is a $K_s[C_p]$-module and

$$\iota : K_s[C_p] \longrightarrow M$$

is an injective map of $K_s[C_p]$-modules, then $\iota$ must split (and similarly for $\text{gr} \ K_s[C_p]$-modules).

**Proof.** By Lemma 3.17, it suffices to compute the action of $B(k)$ on $H_* \mathbb{C}P^\infty_c$. Since $P_k \in \mathcal{P}$ is primitive (see Remark 3.24), its action on $H^* \mathbb{C}P^\infty_c \cong F_p[\mathbb{Z}]$ follows the Leibniz rule

$$P_k(x^i) = ix^{i-1}P_k(x).$$

The action of $P_k$ on $H^* \mathbb{C}P^\infty_c \cong F_p[\mathbb{Z}] \cdot u_c$ can be calculated using the formula

$$P_k(x^i \cdot u_c) = P_k(x^i) \cdot u_c + x^i \cdot P_k(u_c).$$

By (3.15), $P_k(x) = x^{p^k}$ and $P_k(u_c) = cx^{p^k-1} \cdot u_c$. Therefore,

$$P_k(x^i \cdot u_c) = (i + c)x^{i+p^k-1} \cdot u_c.$$

Dually

$$P_k(b_i) = (i - p^k + 1)b_{i-p^k+1},$$
where it is understood that $P_k(b_i) = 0$ if $i - p^k + 1 < c$. Thus, $\text{gr } K_\bullet \mathbb{C}P^\infty_c$ contains an infinite rank free $\text{gr } K_\bullet \mathbb{C}P^\infty_c$ submodule $F^\text{gr } K_c$ generated by the set

$$\{b_{pi} : i > (\beta_k - c)/p\}.$$ 

By Remark 3.23, $F^\text{gr } K_c$ is a summand of $\text{gr } K_\bullet \mathbb{C}P^\infty_c$. We denote its complement by $M^\text{gr } K_c$. □

Remark 3.24. If $p = 2$, then $\Delta(P_k) = P_k \otimes 1 + Q_k \otimes Q_k + 1 \otimes P_k$ in $\mathcal{A}$ but the quotient map $\mathcal{A} \to \mathcal{P}$ sends $Q_k \mapsto 0$ and so we see that $P_k$ is primitive as an element of $\mathcal{P}$ even though it is not a primitive element of $\mathcal{A}$. If $p$ is odd, $P_k$ is primitive even in $\mathcal{A}$.

Lemma 3.25. Let $M$ be a $K_\bullet \mathbb{C}P^\infty$-module which admits an increasing filtration

$$\{0\} = \text{Fil}_{-c} M \subset \cdots \subset \text{Fil}_0 M \subset \text{Fil}_1 M \subset \cdots \subset M,$$

so that $M = \text{colim}_i \text{Fil}_i M$. Suppose that the action of $K_\bullet \mathbb{C}P^\infty$ on $M$ satisfies

$$\chi \text{Fil}_k M \subset \text{Fil}_{k-2p^k+2} M$$

so that $\text{gr } M$ is a $\text{gr } K_\bullet \mathbb{C}P^\infty$-module. Also, suppose there is an injective $\text{gr } K_\bullet \mathbb{C}P^\infty$-module map

$$\tilde{\alpha} : \tilde{F} \longrightarrow \text{gr } M,$$

where $\tilde{F}$ is a free $\text{gr } K_\bullet \mathbb{C}P^\infty$-module. Then there exists a lift of $\tilde{\alpha}$ to an injective $K_\bullet \mathbb{C}P^\infty$-module map

$$\alpha : F \longrightarrow M,$$

where $F$ is a free $K_\bullet \mathbb{C}P^\infty$-module. In fact, $\alpha$ is the inclusion of a summand.

Proof. Let $\{i_i : i \in I\}$ and $\{i_i : i \in I\}$ be a basis of $\tilde{F}$ and $F$, respectively, for a fixed indexing set $I$. Then we may write

$$\tilde{F} = \bigoplus_{i \in I} \text{gr } K_\bullet \mathbb{C}P^\infty_c(i_i) \text{ and } F = \bigoplus_{i \in I} K_\bullet \mathbb{C}P^\infty_c(i_i).$$

Clearly $\text{gr } F \cong \tilde{F}$ as a $\text{gr } K_\bullet \mathbb{C}P^\infty_c$-module. By assumption, any element $m_i \in M$ such that $[m_i] = \tilde{\alpha}(i_i)$ satisfies

$$\tilde{\chi}(m_i) = [\chi^i(m_i)].$$

Fix such an $m_i$ for each $i \in I$. Now define the map $\alpha$ by setting $\alpha(i_i) = m_i$ and extending it linearly to a $K_\bullet \mathbb{C}P^\infty$-module map. Clearly, $\alpha$ is injective as $\tilde{\alpha}$ is injective, and it splits as $K_\bullet \mathbb{C}P^\infty$ is self-injective (see Remark 3.23). □

Proposition 3.22 and Lemma 3.25 imply:
Corollary 3.26. For any \( c \in \mathbb{Z} \), there exists a \( K_*[C_p] \)-module isomorphism

\[ K_*\mathbb{C}P^\infty_c \cong F^K_c \oplus M^K_c, \]

where \( F^K_c \) is free and \( M^K_c \) is finitely generated as \( K_*[C_p] \)-modules.

4. AN UPPER BOUND ON THE EO-ORIENTATION ORDER OF \( \gamma \)

In Section 4.1, we lift the \( K_*[C_p] \)-module splitting of Corollary 3.26 to an EO-module splitting (see Corollary 4.19). In Section 4.2, we show that the inclusion of the compact summand of \( EO \wedge \mathbb{C}P^\infty_c \) factors through \( EO \wedge \mathbb{C}P^{c+d}_c \), where \( d = p^k(p-1) - 1 \). This leads to the proof of Main Theorem 1.5. In Section 4.3, we prove Main Theorem 1.8.

4.1. A splitting of \( EO \wedge \mathbb{C}P^\infty_c \)

Now we extend the ‘free \( \oplus \) finite’ decomposition of \( K_*[C_p] \)-modules in Corollary 3.26 to a ‘free \( \oplus \) finite’ decomposition of \( E_*[C_p]^{\sigma} \)-modules (see Corollary 4.4) using the fact that a free \( E_*[C_p]^{\sigma} \)-module is injective relative to \( E_*^{\sigma} \) (see Proposition 4.2).

Definition 4.1 [12]. Let \( S \) be a subring of a commutative ring \( R \). An \( R \)-module \( M \) is \((R,S)\)-injective (injective relative to \( S \)) if every \( R \)-module injection \( M \hookrightarrow N \) that splits in the category of \( S \)-modules also splits in the category of \( R \)-modules.

Proposition 4.2. A free \( E_*[C_p]^{\sigma} \)-module \( F \) is \((E_*[C_p]^{\sigma}, E_*)\)-injective.

Proof. Hochschild [12, Lemma 1] showed that \( \text{Hom}_S(R, A) \) is \((R,S)\)-injective when \( S \) is a subring of \( R \) and \( A \) is an \( S \)-module. Since

\[ E_*[C_p]^{\sigma} \cong \text{Hom}_{E_*}(E_*[C_p]^{\sigma}, E_*) \]

as an \( E_*[C_p]^{\sigma} \)-module, the result follows. \( \square \)

Lemma 4.3. Suppose \( Q^E \) is a free \( E_*[C_p]^{\sigma} \)-module whose underlying \( E_* \)-module is free and there is a \( K_*[C_p] \)-module splitting

\[ Q^E/m \cong F^K \oplus M^K, \]

where \( F^K \) is free and \( M^K \) is finitely generated. Then the \( K_*[C_p] \)-module splitting of \( Q^E/m \) lifts to an \( E_*[C_p]^{\sigma} \)-module splitting

\[ Q^E \cong F^E \oplus M^E, \]

where \( F^E \) is free and \( M^E \) is finitely presented as \( E_*[C_p]^{\sigma} \)-modules.
Proof. Let $F^E$ be a free $E_+[C_p]^\sigma$-module such that $F^E / \mathfrak{m} \cong F^K$. Fix an $E_+[C_p]^\sigma$-basis $B = \{b_i : i \in \mathbb{N}\}$ of $F^E$. Let

$$t_K : F^E \xrightleftharpoons{\pi_K} Q^E / \mathfrak{m} : \pi_K$$

denote the $K_+[C_p]$-maps that split $F^K$ from $Q^E / \mathfrak{m}$. We will now show that the maps $t_K$ and $\pi_K$ can be lifted to $E_+[C_p]^\sigma$-module maps $t_E$ and $\pi_E$

$$\begin{array}{ccc}
F^E & \xrightarrow{t_E} & Q^E \\
\pi_1 & \downarrow & \downarrow \pi_2 \\
F^K & \xleftarrow{\pi_K} & Q^E / \mathfrak{m}
\end{array}$$

such that $\pi_E \circ t_E$ is the identity.

The $E_+[C_p]^\sigma$-linear map $t_E$ can be defined by sending $b_i$ to an arbitrary lift of $t_K(\pi_1(b_i))$. Since $E_+$ is Noetherian, $F^E$ and $Q^E$ are both $E_+$-free and $t_K$ is injective, we conclude that $t_E$ is also injective (essentially from the Krull intersection theorem). The freeness of $F^E$ and $Q^E$ as $E_+$-modules also implies the existence of an $E_+$-linear map $\pi_E$ such that $\pi_E \circ t_E$ is the identity. By Proposition 4.2, $\pi_E$ can be chosen so that it is $E_+[C_p]^\sigma$-linear. □

Corollary 4.4. For any $c \in \mathbb{Z}$, there exists an $E_+[C_p]^\sigma$-module isomorphism

$$E_+\mathbb{C}P^\infty_c \cong F^E_c \oplus M^E_c,$$

where $F^E_c$ is free and $M^E_c$ is finitely presented as $E_+[C_p]^\sigma$-modules.

Next, we use the relative Adams spectral sequence for the map $\text{EO} \to \text{E}$ to show that the algebraic splitting of Corollary 4.4 originates from an EO-module splitting of $\text{EO} \wedge \mathbb{C}P^\infty_c$.

Definition 4.5. An EO-module $\mathcal{X}$ is relatively projective (respectively, relatively free) if $E_+^E \mathcal{X}$ is a projective (respectively, free) $E_+$-module.

Example 4.6. The spectrum $E$ is a relatively free EO-module.

Example 4.7. When $X$ is an even spectrum $\text{EO} \wedge X$ is a relatively free EO-module.

Theorem 4.8 [9, Corollary 3.4]. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are EO-modules such that $\mathcal{X}$ is finite and relatively projective. The Adams spectral sequence relative to the map $\text{EO} \to \text{E}$

$$E_2^{s,t} : = \text{Ext}^{s,t}_{E_+^E} (E_+^{EO} \mathcal{X}, E_+^{EO} \mathcal{Y}) \Rightarrow \pi_{t-s} \text{EO} - \text{Mod}(\mathcal{X}, \mathcal{Y}),$$

(4.9)

is strongly convergent.

The relative Adams spectral sequence was developed by Baker and Lazarev [5]. Devinatz [9] defined the homotopy fixed points $E^hG$ for $G$ an arbitrary closed subgroup of $\text{Aut}(\Gamma)$. He identi-
fied the $E_2$-page of the relative Adams spectral sequence for $E^{hG} \to E$ with the group cohomology of $G$ and showed that for every $E^{hG}$-module the relative Adams spectral sequence is strongly convergent and that there is a uniform horizontal vanishing line independent of the $E^{hG}$-module. Rognes [21] developed the theory of Galois extensions of ring spectra and reinterpreted the results of Devinatz by concluding that the map $EO \to E$ is Galois (see [21, Theorem 5.4.4]). Let

$$D_{EO}(-) := EO - \text{Mod}(-, EO) : EO\text{-module} \to EO\text{-module}$$

denote the relative Spanier–Whitehead dual functor.

**Proposition 4.10** [21, Proposition 6.4.7]. $D_{EO}(E) \simeq E$.

**Lemma 4.11.** The edge homomorphism for the relative Adams spectral sequence

$$\pi_* \text{EO-Mod}(\mathcal{X}, \mathcal{Y}) \longrightarrow \text{Hom}_{E^E}(E_\ast^E \mathcal{X}, E_\ast^E \mathcal{Y})$$  \hspace{1cm} (4.12)

is an isomorphism if:

(I) $\mathcal{Y} = E$ and $\mathcal{X}$ is a finite relatively-projective $EO$-module; or

(II) $\mathcal{X} = E$ and $\mathcal{Y}$ is an arbitrary $EO$-module.

**Proof.** When $\mathcal{Y} = E$ and $\mathcal{X}$ is finite, the edge map (4.12) is an isomorphism, as $E_\ast^E E$ is a cofree $E_\ast^E E$-comodule and the spectral sequence (4.9) is concentrated in the zero line.

When $\mathcal{X} = E$ is a finite self-dual $EO$-module (see Proposition 4.10) and therefore,

$$\mathcal{X} \wedge_{EO} \mathcal{Y} \simeq D_{EO}(\mathcal{X}) \wedge_{EO} \mathcal{Y} \simeq EO - \text{Mod}(\mathcal{X}, \mathcal{Y}).$$

Since $E_\ast^EO E_\ast^E$ is $E_\ast^E$-free, we have a Kunneth isomorphism

$$E_\ast^EO (D_{EO}(E) \wedge_{EO} \mathcal{Y}) \cong E_\ast^EO (E \wedge_{EO} \mathcal{Y}) \cong E_\ast^EO (E) \otimes_{E_\ast^E} E_\ast^EO \mathcal{Y}$$

and consequently $E_\ast^EO (D_{EO}(E) \wedge_{EO} \mathcal{Y})$ is cofree as an $E_\ast^EO E$-comodule. Therefore, the spectral sequence (4.9) is concentrated on the zero line and the edge homomorphism (4.12) is an equivalence as desired.

As a consequence of the convergence of the relative Adams spectral sequence, we deduce that $K_{ER}(-)$ detects equivalences of $EO$-modules:

**Corollary 4.13.** Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are relatively projective $EO$-modules. An $EO$-module map

$$f : \mathcal{X} \longrightarrow \mathcal{Y}$$

is a weak equivalence if and only if

$$f_* : K^BO_{E_\ast^E} \mathcal{X} \longrightarrow K^BO_{E_\ast^E} \mathcal{Y}$$

is a $K_{E_\ast^E}$-isomorphism.
Proof. By the Nakayama lemma, if \( f^\ast : K^\ast \to K^\ast \) is an isomorphism, then so is \( f^\ast : E^\ast \to E^\ast \). Thus \( f \) induces an isomorphism of relative Adams \( E_2 \)-pages. It follows that \( f \) induces an isomorphism on \( \pi^\ast \). The converse is obvious. \(\square\)

**Definition 4.14.** We say an EO-module \( \mathcal{X} \) is **freely filtered** if there is a filtration

\[
\mathcal{X}^{(0)} \to \mathcal{X}^{(1)} \to \cdots \to \mathcal{X}
\]

such that:

1. \( \mathcal{X}^{(0)} \) is contractible;
2. each \( \mathcal{X}^{(d)} \) is a compact relatively free EO-module;
3. the map \( \mathcal{X}^{(d)} \to \mathcal{X}^{(d+1)} \) is an EO-module map; and
4. \( \operatorname{colim}_d \mathcal{X}^{(d)} \simeq \mathcal{X} \).

**Example 4.15.** For any even spectrum \( X \) of finite type, \( EO \wedge X \) is a freely filtered EO-module where the filtration is induced by the skeletal filtration of \( X \).

**Lemma 4.16.** Suppose that \( \mathcal{X} \) is a freely filtered EO-module with a \( K_\ast [C_p] \)-module splitting

\[
K^\ast (\mathcal{X}) / \mathfrak{m} \cong F^K \oplus M^K,
\]

where \( F \) is free with countable basis. Then there exists an EO-module splitting

\[
\mathcal{X} \simeq F \vee M,
\]

where \( F \) is a free \( E \)-module such that \( K^\ast F \cong F \) and \( M \) is an EO-module such that \( K^\ast M \cong M \).

Proof. Let us present the free \( K_\ast [C_p] \)-module \( F \) as

\[
F^K := \bigoplus_B K_\ast [C_p] \cdot b_i
\]

using a basis \( B := \{ b_i : 0 \leq i < n \leq \infty \} \). Let \( F^0_0 := K_\ast [C_p] \cdot b_i \subset F^K \). Let \( t^K \) and \( \pi^K \) denote the inclusion and the projection map for the split-summand corresponding to \( F^K_0 \). Because \( F^K_0 \) is a finite \( K_\ast [C_p] \)-module, there exists a solution to the diagram of \( K_\ast [C_p] \)-modules

\[
\begin{array}{cccccccc}
F^K & \longrightarrow & F^K & \longrightarrow & \cdots & \longrightarrow & F^K \\
\hat{\pi}^K & \uparrow & \hat{\pi}^K & \uparrow & \hat{\pi}^K & \uparrow & \hat{\pi}^K & \uparrow \\
K^\ast (\mathcal{X})^{(\ell)} & \longrightarrow & K^\ast (\mathcal{X})^{(\ell+1)} & \longrightarrow & \cdots & \longrightarrow & K^\ast (\mathcal{X}) \\
\end{array}
\]

for some \( \ell \gg 0 \). Using Lemma 4.3, we can extend (4.17) to a diagram of \( E_\ast [C_p]^p \)-modules (or equivalently \( E_\ast EO \)-comodules)

\[
\begin{array}{cccccccc}
E^\ast EO & \longrightarrow & E^\ast EO & \longrightarrow & \cdots & \longrightarrow & E^\ast EO \\
\hat{\pi}^K & \uparrow & \hat{\pi}^K & \uparrow & \hat{\pi}^K & \uparrow & \hat{\pi}^K & \uparrow \\
E^\ast (\mathcal{X})^{(\ell)} & \longrightarrow & E^\ast (\mathcal{X})^{(\ell+1)} & \longrightarrow & \cdots & \longrightarrow & E^\ast (\mathcal{X}) \\
\end{array}
\]
By Lemma 4.11, the maps $i_j^E$ and $\pi_j^E$ can be realized in the homotopy category of EO-modules

$$i_j : E \xleftarrow{} \mathcal{X}^{(j)} : \pi_i$$

for all $i \geq \ell$. Since $\text{colim}_i \mathcal{X}^{(i)} \simeq \mathcal{X}$, we conclude that there exists an EO-module $\mathcal{M}$ such that

$$K^E_{\ast} \mathcal{M} \cong \bigoplus_{i=1}^{n-1} K_{\ast} [C_p] \cdot b_i \oplus M^K$$

and an EO-module equivalence

$$\mathcal{X} \simeq E \vee \mathcal{M}.$$  

Note that $\mathcal{M}$ can be filtered as

$$\mathcal{M}^{(0)} \to \mathcal{M}^{(1)} \to \cdots \to \mathcal{M},$$

where $\mathcal{M}^{(i)} = \text{Cofib}(i_{\lambda+i})$ and $E^E_{\ast} \mathcal{M}^{(i)}$ is $E_\ast$-free. Thus our results follow from an inductive argument. \hfill \Box

Combining Corollary 3.26 and Lemma 4.16, we get:

**Corollary 4.19.** For any $c \in \mathbb{Z}$, there exists an EO-module splitting

$$\text{EO} \wedge \mathbb{C}P^\infty_c \simeq \mathcal{F}_c \vee \mathcal{M}_c,$$

where $\mathcal{F}_c \simeq \bigvee_n E$ and $\mathcal{M}_c$ is a finitely presented EO-module.

Main Theorem 1.7 is the case $c = 0$ in Corollary 4.19.

### 4.2 An EO-Thom isomorphism

Recall from the proof of Proposition 3.22 that the summand $M_0^{\text{gr} K}$ of $\text{gr} K_\ast \mathbb{C}P^\infty_+$ factors through $\text{gr} K_\ast \mathbb{C}P_{+}^{\hat{\beta}_k}$, where

$$\hat{\beta}_k := p^k(p-1)$$

is the smallest multiple of $p$ which is greater than $\beta_k = |p^{k-1}|/2 = (p-1)(p^k-1)$. This algebraic factorization can be leveraged to obtain the following result.

**Lemma 4.20.** For all $c \in \mathbb{Z}$, the inclusion map $s'': \mathcal{M}_{pc} \hookrightarrow \text{EO} \wedge \mathbb{C}P^\infty_{pc}$ of the splitting in Corollary 4.19 can be chosen so that the inclusion of $\mathcal{M}_{pc}$ factors through $\text{EO} \wedge \mathbb{C}P_{pc}^{\hat{\beta}_k + pc}$.
in the homotopy category of $EO$-modules.

Proof. We present the argument for $c = 0$. The general argument follows from the $B(k)$-module isomorphism of Lemma 4.21, where $B(k)$ is the subalgebra of the even Steenrod algebra defined in Definition 3.16.

We start with an arbitrary splitting of $EO \wedge \mathbb{C}\mathbb{P}_\infty^+$, where

$$F_0 \xrightarrow{s} EO \wedge \mathbb{C}\mathbb{P}_\infty^+ \xrightarrow{f} F_0$$

are the inclusion and projection maps of the components. We will modify this splitting to produce a new one which satisfies our conclusion.

The free summand $F_0$ of the $B(k)$-module $H_\ast \mathbb{C}\mathbb{P}_\infty^+$ surjects onto $H_\ast \mathbb{C}\mathbb{P}_\infty^+ \beta_k$ under the coskeletal map. Let $R$ be the fiber of $\cosk \circ s \circ f$. By Lemma 3.17, the composite

$$EO \wedge \mathbb{C}\mathbb{P}_\infty^+ \xrightarrow{f} F_0 \xrightarrow{s} EO \wedge \mathbb{C}\mathbb{P}_\infty^+ \xrightarrow{\cosk} EO \wedge \mathbb{C}\mathbb{P}_\infty^+ \beta_k$$

induces a surjection on $K^EO_\ast(-)$. Hence, the map

$$R \longrightarrow EO \wedge \mathbb{C}\mathbb{P}_\infty^+,$$

induces an injection on $K^EO_\ast(-)$. Because $EO \wedge \mathbb{C}\mathbb{P}_\infty^+ \beta_k^{-1}$ is the fiber of $\cosk$, there is a map

$$\alpha : R \longrightarrow EO \wedge \mathbb{C}\mathbb{P}_\infty^+ \beta_k^{-1}.$$

Since $K^EO_\ast(\cosk \circ s \circ f) = K^EO_\ast(\cosk)$, the induced map

$$K^EO_\ast \alpha : K^EO_\ast R \longrightarrow K^EO_\ast(EO \wedge \mathbb{C}\mathbb{P}_\infty^+ \beta_k^{-1}) \cong K_\ast \mathbb{C}\mathbb{P}_\infty^+ \beta_k^{-1}$$

is an isomorphism of $K_\ast$-modules. By Corollary 4.13, $\alpha$ is a weak equivalence of $EO$-modules.

Because $f \circ s' \circ f$ is null, the composite $(\cosk \circ s \circ f) \circ s'$ is also null and therefore there is a map $\mathcal{M}_0 \to R$. Now let $s' : \mathcal{M}_0 \to R \xrightarrow{\cong} EO \wedge \mathbb{C}\mathbb{P}_\infty^+ \beta_k^{-1}$ and define $s''$ to be the composite

$$\mathcal{M}_0 \xrightarrow{s} EO \wedge \mathbb{C}\mathbb{P}_\infty^+ \beta_k^{-1} \longrightarrow EO \wedge \mathbb{C}\mathbb{P}_\infty^+.$$
It is easy to see that $K^*_s(\text{Cofib}(s'')) \cong K^*_s F_0$. Thus, by Lemma 4.16,
\[
\text{Cofib}(s'') \simeq F_0 \simeq \bigvee \mathbb{N} E
\]
so Cofib(s'') is a split summand of $EO \wedge \mathbb{C}P^\infty_+$. \qed

**Lemma 4.21.** The $H\mathbb{F}_p$-Thom isomorphism $H_{*+c} CP^\infty_+ \cong H_{*} CP^\infty_+$ is a map of $B(k)$-modules if $c$ is divisible by $p$.

**Proof.** Since the action of $B(k)$ on $H_{*-}(\mathbb{C}P^\infty_+)$ is obtained from the action of $B(k)$ on $H^*(\mathbb{C}P^\infty_+)$ via the isomorphism

\[
H^*_*(\mathbb{C}P^\infty_+) \cong \text{Hom}_{\mathbb{F}_p}(H^*_{*-}(\mathbb{C}P^\infty_+), \mathbb{F}_p),
\]
it is enough to show the $H\mathbb{F}_p$-Thom isomorphism in cohomology

\[
\tau: H^* \mathbb{C}P^\infty_+ \xrightarrow{\cong} H^{*+c} \mathbb{C}P^\infty_+
\]
is a map of $B(k)$-modules when $p$ divides $c$. Note that $P_k(u_c) = cx^{p^{k-1}} \cdot u_c$, where $u_c \in H^* \mathbb{C}P^\infty_+$ is the Thom class. Therefore, $P_k(u_c) = 0$ when $c$ is divisible by $p$ and

\[
\tau(P_k(x^i)) = P_k(x^i) \cdot u_c = P_k(x^i \cdot u_c) = P_k(\tau(x^i)),
\]
as desired. \qed

Our next goal is to prove the following EO-Thom isomorphism.

**Theorem 4.22.** There exists an equivalence of $EO$-modules

\[
EO \wedge \mathbb{C}P^\infty_+ \cong EO \wedge \Sigma^{2r} \mathbb{C}P^\infty_+,
\]
where $r = \Theta(\gamma^{k-1}, EO)$.

**Proof.** Since $r = \Theta(\gamma^{k-1}, EO)$, we have an EO-Thom isomorphism

\[
EO \wedge \mathbb{C}P^{\hat{k}-1+r}_r \cong EO \wedge \Sigma^{2r} \mathbb{C}P^{\hat{k}-1}_+.
\]

Using Corollary 4.19 and Lemma 4.20, we construct the composite map

\[
c: M_r \xrightarrow{\delta} EO \wedge \mathbb{C}P^{\hat{k}-1+r}_r \xrightarrow{\alpha} EO \wedge \Sigma^{2r} \mathbb{C}P^{\hat{k}-1}_+ \xrightarrow{\delta} EO \wedge \Sigma^{2r} \mathbb{C}P^\infty_+.
\]

By Remark 2.8, (4.24) respects the Atiyah–Hirzebruch filtration, therefore we analyze the image of the map induced by $c$ on AHSS calculating $K^*_s(\mathbb{C}P^\infty_+)$-groups.

Since $M_r$ does not have a natural Atiyah–Hirzebruch filtration, we induce one

\[
M_r^{(0)} \xrightarrow{\delta} \ldots \xrightarrow{\delta} M_r^{(\hat{k}-1)} = M_r^{(\hat{k})} = \ldots = M_r
\]
by pulling back the Atiyah–Hirzebruch filtration on $EO \wedge \Sigma^{2r} \mathbb{C}P_{\infty}^r$. The associated graded of the corresponding filtration on $K^*_{*} \mathcal{M}_r$, denote it by $gr K^*_{*} \mathcal{M}_r$, is a gr $K_*[C_p]$-module, and the induced map

$$gr K^*_{*} (c) : gr K^*_{*} \mathcal{M}_r \longrightarrow gr K_* \Sigma^{2r} \mathbb{C}P_{\infty}^r,$$

(4.25)
is an injection with image $\Sigma^{2r} M^0_{gr K}$. This is because of the explicit description of $\Sigma^{2r} M^0_{gr K}$ as in the proof of Proposition 3.22 and Lemma 4.21. The cofiber of $c$ also admits a filtration

$$\text{Cofib}(c) := \text{colim}_n \{ \text{Cofib}(c)^{(0)} \longrightarrow \text{Cofib}(c)^{(1)} \longrightarrow \ldots \},$$

where $\text{Cofib}(c)^{(i)} = \text{Cofiber}(\mathcal{M}_r^{(i)} \rightarrow EO \wedge \Sigma^{2r} \mathbb{C}P_{\infty}^r)$, such that the associated graded of the induced filtration on $K^*_{*} \text{Cofib}(c)$ is a free gr $K_*[C_p]$-module isomorphic to $\Sigma^{2r} F_{0}^{gr K}$. Thus, by Lemma 3.25 and Lemma 4.16

$$\text{Cofib}(c) \simeq \Sigma^{2r} F_{0}$$
is a split summand of $EO \wedge \Sigma^{2r} \mathbb{C}P_{\infty}^r$. Therefore, we have an equivalence

$$EO \wedge \Sigma^{2r} \mathbb{C}P_{\infty}^r \simeq \mathcal{M}_r \vee \Sigma^{2r} F_{0} \simeq \mathcal{M}_r \vee \mathcal{F}_r \simeq EO \wedge \mathbb{C}P_{\infty}^r$$
as desired. □

Corollary 4.26. $\Theta(\gamma, EO) = \Theta(\gamma^\beta k^{-1}, EO)$.

Proof. Immediate from Theorem 4.22 and the fact that $M(\gamma) \cong \mathbb{C}P_{\infty}^r$. □

Main Theorem 1.6 follows from Corollary 4.26 because $\Theta(\gamma^\beta k^{-1}, EO)$ divides $\Theta(\gamma^\beta k^{-1}, S_p) = p^{\beta k - 1}$ (see Example 2.13).

4.3 The $S^1$-Tate fixed points of $EO$

We shift our focus to identifying the $S^1$-Tate spectrum of $EO$. If a group $G$ acts on a spectrum $X$, the Tate spectrum $X^G$ of $X$ is the cofiber of the norm map from the homotopy orbits spectrum $X_{hG}$ to the homotopy fixed points spectrum $X^{hG}$

$$X^tG := \text{Cofib}(Nm : X_{hG} \rightarrow X^{hG}).$$

When $G = S^1$ acts on $X$ trivially, [10] describes the Tate fixed points as

$$X^{tS^1} \simeq \lim_{c \rightarrow \infty} \Sigma^{2c} X \wedge \mathbb{C}P^{\infty}_{-c}.$$  (4.27)

Inverse limits do not commute with $\pi_* (-)$. Instead, there is a short exact sequence

$$0 \rightarrow \lim_{c} (\pi_*(X_* \mathbb{C}P^{\infty}_{-c}) \rightarrow \pi_* X^{tS^1} \rightarrow \lim_\geq (\pi_*(X_* \mathbb{C}P^{\infty}_{-c}) \rightarrow 0$$
called the ‘Milnor sequence’.
Proof of Main Theorem 1.8. By Lemma 4.20, we can choose a splitting of $EO \wedge \mathbb{C}P^\infty_{i\hat{\beta}_k}$ for all $i \in \mathbb{Z}$ such that we have the commutative diagram

$$
\begin{array}{cccccc}
\cdots & \rightarrow & M_{(i-1)\hat{\beta}_k} & 0 & M_{i\hat{\beta}_k} & 0 & M_{(i+1)\hat{\beta}_k} & 0 & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & EO \wedge \mathbb{C}P^\infty_{(i-1)\hat{\beta}_k} & \rightarrow & EO \wedge \mathbb{C}P^\infty_{i\hat{\beta}_k} & \rightarrow & EO \wedge \mathbb{C}P^\infty_{(i+1)\hat{\beta}_k} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & F_{(i-1)\hat{\beta}_k} & \rightarrow & F_{i\hat{\beta}_k} & \rightarrow & F_{(i+1)\hat{\beta}_k} & \cdots \\
\end{array}
$$

(4.28)

such that the horizontal maps in the top row are null maps, in the middle row are coskeletal collapse maps.

The horizontal maps in the bottom row induce surjections in homotopy. To see this, we filter $F_{i\hat{\beta}_k}$ using the Atiyah–Hirzebruch filtration of $\mathbb{C}P^\infty_{i\hat{\beta}_k}$. Let $\text{gr}KEO^{i\hat{\beta}_k}$ denote the associated graded of the induced filtration on $KEO^{i\hat{\beta}_k}$. From Proposition 3.22, we observe that the composite

$$f_i : F_{i\hat{\beta}_k} \longrightarrow EO \wedge \mathbb{C}P^\infty_{i\hat{\beta}_k} \longrightarrow EO \wedge \mathbb{C}P^\infty_{(i+1)\hat{\beta}_k} \longrightarrow F_{(i+1)\hat{\beta}_k}$$

induces a surjection in $\text{gr}KEO^{i\hat{\beta}_k}$, and therefore a surjection in $KEO^{i\hat{\beta}_k}$. By the Nakayama Lemma, $f_i$ induces surjection on $E^{i\hat{\beta}_k}_*$. By studying the induced map of relative Adams spectral sequence, we conclude that $f_i$ is a surjection.

Consequently, we get a six-term exact sequence

$$0 \longrightarrow \lim_i \pi_* M_{-i\hat{\beta}_k} \longrightarrow \lim_i EO_* C\mathbb{P}^\infty_{-i\hat{\beta}_k} \longrightarrow \lim_i \pi_* F_{-i\hat{\beta}_k} \longrightarrow \lim_1 \pi_* M_{-i\hat{\beta}_k} \longrightarrow \lim_1 EO_* C\mathbb{P}^\infty_{-i\hat{\beta}_k} \longrightarrow \lim_1 \pi_* F_{-i\hat{\beta}_k} \longrightarrow 0$$

in which

$$\lim_i \pi_* M_{-i\hat{\beta}_k} = 0$$

$$\lim_1 \pi_* M_{-i\hat{\beta}_k} = 0$$

$$\lim_1 \pi_* F_{-i\hat{\beta}_k} = 0$$

and

$$\lim_1 F_{-i\hat{\beta}_k} \simeq \prod_{-\infty < k < \infty} E.$$

Thus, $\lim_1 EO_* C\mathbb{P}^\infty_{-i\hat{\beta}_k} = 0$ and

$$EO^{S^1} \simeq \lim_i EO \wedge C\mathbb{P}^\infty_{-i\hat{\beta}_k} \simeq \lim_i EO \wedge C\mathbb{P}^\infty_{-i\hat{\beta}_k} \simeq \lim_i F_{-i\hat{\beta}_k} \simeq \prod_{-\infty < k < \infty} E$$

as desired. □
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