“Nom” Code and Pattern Avoidance in a Catalan Numbered Class of Permutations

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Abstract: We show some new combinatorial interpretations of a code associating each permutation with a “subexceedant function”, that is, a function $f : [n] \rightarrow [n]$ such that $f(i) \leq i, 1 \leq i \leq n$, that we call “nom code”. We show that this code is related to the selection sort algorithm and to the cycle structure of the permutation. We propose a representation of permutations by trees based on this code. In the second half of the paper we study the permutations having a non-decreasing nom code, in particular we study length 3 pattern avoidance in this class of permutations.

Keywords: Permutation; Subexceedant function; Permutation code; Permutation statistics; Sorting; Pattern avoidance; Integer partition.

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1. Introduction

At least since the works of Lehmer (9, 1960) or perhaps even much earlier (8, 1888), permutations over $[n] := \{1, 2, \ldots, n\}$ have been coded with integer $n$-tuples $(a_1, a_2, \ldots, a_n)$ such that $1 \leq a_i \leq i, \forall i \in [n]$, or equivalently with functions $f : [n] \rightarrow [n]$ such that $1 \leq f(i) \leq i, \forall i \in [n]$, called subexceedant functions.

Lehmer, in particular, associates with every permutation the vector of its inversions, which is clearly related to the sorting problem, since the number of inversions of a permutation measures the complexity of the Inversion Sorting algorithm to sort the permutation.

Later, other interesting codes for permutation with subexceedant functions have been introduced, like the Denert code that solved a famous combinatorial problem proposed by Foata and Zeilberger (6).

Permutation codes are interesting because certain algorithms perform better over the codes than they do over the permutations themselves. Codes
allow for instance to implement efficient algorithms for the exhaustive generation of some specific classes of permutations. To do so, one has often to “read” the properties of the permutation in its code and this gives birth to interesting combinatorial problems (see for instance [2,10]).

As shown in [10], Fanja Rakotondrajao and the second author studied a way of coding permutations with subexceedant functions, by defining a natural bijection \( \phi \) from the set of subexceedant functions over \([n]\) to the symmetric group \( S_n \). The function \( \phi^{-1} \) is therefore a permutation code. In particular, they also proved that the image set of a subexceedant function \( f \) is related to the anti-exceedances of the corresponding permutation \( \sigma = \phi(f) \).

This particular code has already found some interesting applications not only in this article, but also in [1] and in two other works of the authors of this article ([3] and [4]).

In this article we investigate further properties of this code. We start by characterizing the positions as well as the letters of the anti-exceedances of a permutation looking at its code. We show that this code is related to the cycle structure of the permutation. More precisely, if \( \sigma \) is a permutation and \( f \) is its code (that is, \( \sigma = \phi(f) \)), then \( \forall i \in [n] \), the value \( f(i) \) is simply the element \( j \leq i \) belonging to the same cycle of \( \sigma \) that contains \( i \) and that is “as near as possible” to \( i \). The name nom code came from “nearest orbital minorant”.

We rewrite the two algorithms that construct the permutation associated with a given function \( f \) and vice-versa. This way, we show that, as the Lehmer code is related to the Insertion Sorting algorithm, the nom code appears to be related to the Selection Sorting algorithm. More precisely, we show that if \( \phi(f) = \sigma \) then the values of \( f \) correspond to the operations that need to be performed to sort the permutation \( \sigma \) using Selection Sorting (selection of the maximum).

We provide two procedures allowing to construct the graph of the permutation \( \sigma \) from the graph of the subexceedant function \( f \) to which \( \sigma \) is associated, and vice-versa. We describe a way to associate a permutation with a forest of increasing ordered trees, which is remindful of an analogous bijection between permutations and increasing trees presented by Stanley [11].

It is well known that non-decreasing subexceedant functions of \([n]\) are enumerated by the Catalan numbers. Some of their properties have been presented in [5,11]. In the second part of the article we include a section studying the permutations whose nom code is a non-decreasing subexceedant function and we provide some characterization results.

We conclude the article with a section studying the avoidance of length 3 patterns in the set of permutations with non-decreasing nom codes. A
complete enumeration is computed for four of the six patterns, while for the remaining two we provide lower bounds showing that they grow exponentially. For two patterns, 132 and 213, the permutations avoiding them are counted by the (cumulative) numbers of integers partitions. An analogous result for permutations having a non-increasing Lehmer code can be found in [12].

2. Notations and Preliminaries

Let \([n]\) denote the interval \(\{1, 2, 3, \ldots, n\}\) and \(\mathfrak{S}_n\) the symmetric group of permutations over the set \([n]\), where \(n\) is a fixed positive integer. A permutation \(\sigma \in \mathfrak{S}_n\) can be given as a word \((\sigma = \sigma (1)\sigma (2)\cdots \sigma (n))\), two-line notation \((\sigma = (1 \ 2 \ \cdots \ n) \ \sigma (1) \ \sigma (2) \ \cdots \ \sigma (n))\), or cycle notation \((\sigma = C_1C_2\cdots C_k\), where \(C_i, 1 \leq i \leq k\), is a cycle of \(\sigma\)). A transposition is a permutation that swaps two integers and fixes the others.

When we multiply two permutations, the leftmost permutation always acts first.

Definition 2.1. We will say that a function \(f : [n] \rightarrow [n]\) is subexceedant if \(1 \leq f(i) \leq i, \forall i \in [n]\).

The set of all subexceedant functions over \([n]\) is denoted by \(F_n\). We will note \(f \in F_n\) as a word \(f_1f_2\cdots f_n\) on the alphabet \([n]\), where \(f_i = f(i)\), then \(\text{Im}(f)\) denotes the set of elements in \(f([n])\) and \(\text{IMA}(f)\) its cardinality. We specify the set \(\text{ImRp}(f)\) of positions of the rightmost occurrences of image values of \(f\). Further we define the statistics \(\text{Plat}\) and \(\text{Fix}\) as follows:

\[
\text{Plat}(f) = \{i : 1 \leq i < n, f_i = f_{i+1}\}, \quad \text{plat}(f) := |\text{Plat}(f)|,
\]

\[
\text{Fix}(f) = \{i : 1 \leq i \leq n, f_i = i\}, \quad \text{fix}(f) := |\text{Fix}(f)|,
\]

For instance, let \(f = 1121345 \in F_7\). Then \(\text{Im}(f) = \{1, 2, 3, 4, 5\}\), \(\text{ImRp}(f) = \{3, 4, 6, 7\}\), \(\text{Plat}(f) = \{1\}\), and \(\text{Fix}(f) = \{1\}\).

As shown in [10], F. Rakotondrajao and the second author studied the bijection \(\phi : F_n \rightarrow \mathfrak{S}_n\) that associates a subexceedant function \(f\) with the permutation \(\sigma = \phi(f) \in \mathfrak{S}_n\), where \(\sigma = (1, f_1)(2, f_2)\cdots(n, f_n)\). In the product of transpositions if \(f(i) = i\) for some \(i\), then the corresponding transposition degenerates into the identity.

Example 2.1. Let \(f = 1121345 \in F_7\). Then

\[
\sigma = \phi(f) = (1, 1)(2, 1)(3, 2)(4, 1)(5, 3)(6, 4)(7, 5) = 7621345.
\]
2.1 The inverse function of $\phi$ (the permutation code)

As shown in [10], the procedure to obtain the subexceedant function $f = f_1 f_2 \cdots f_n$ associated with a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ such that $\phi(f) = \sigma$ is described by an $n$-step loop.

– For $i = n, n-1, \ldots, 1$ :
  – set $f_i = \sigma(i)$
  – set $\sigma = \sigma \cdot (i, \sigma(i))$ (this swaps $i$ and $\sigma(i)$ in the permutation)

For $i = n, n-1, \ldots, 1$ we note $\sigma_i$ is the permutation at the iteration of the “for” loop corresponding to a given $i$, then the procedure sets $f_i = \sigma_i(i)$, where $\sigma_n = \sigma$ and $\sigma_i = \sigma_{i+1} \cdot (i+1, \sigma_{i+1}(i+1))$ for $i = n-1, \ldots, 1$.

Example 2.2. Take $\sigma = 23154$. Then $f_5 = \sigma_5(5) = \sigma(5) = 4$ and $\sigma_4 = \sigma_5 \cdot (5, 4) = 2314$. Now $f_4 = \sigma_4(4) = 4$ and $\sigma_3 = \sigma_4 \cdot (4, 4) = 231$. From $\sigma_3$ we have $f_3 = \sigma_3(3) = 1$ and $\sigma_2 = \sigma_3 \cdot (3, 1) = 21$. So $f_2 = \sigma_2(2) = 1$ and $\sigma_1 = \sigma_2 \cdot (2, 1) = 1$. Finally from $\sigma_1$ we have $f_1 = 1$. Therefore, $f = f_1 f_2 f_3 f_4 f_5 = 11144$. We summarize the steps of the procedure as follows.

\begin{align*}
\sigma_5 &= 23154 \\
\sigma_4 &= 23145 \\
\sigma_3 &= 23145 \\
\sigma_2 &= 21345 \\
\sigma_1 &= 12345
\end{align*}

Remark 2.1. We note that at the end of the iteration for a given $i$, the permutation $\sigma_i$ fixes all points greater than $i$ (it is in fact a permutation on $[i]$). The algorithm therefore sorts the permutation $\sigma$ (by selection of the maximum and swap).

3. Further investigations on $\phi$

3.1 The insertion method to compute $\sigma$ from its code.

By observing the sorting process of the permutation $\sigma$ when we compute its code $f = \phi^{-1}(\sigma)$, we have another way to express and compute $\sigma = \phi(f)$ other than as a product of transpositions. Indeed, a given subexceedant function $f = f_1 f_2 \cdots f_n$ contains the information to easily construct $\sigma$ by “undoing” that sorting process. The algorithm follows. Start with the identity $\sigma = 1$, suppose that the integers 1, 2, \ldots, $i-1$ have already been inserted. Then replace the integer $f_i$ by $i$ and append the integer $f_i$ at the end.
Example 3.1. Take \( f = 11144 \in F_5 \), then we construct \( \sigma = \phi(f) \) as follows:

\[
\begin{align*}
1 \\
21 \\
231 \\
2314 \\
23154 = \sigma
\end{align*}
\]

3.2 \( \phi^{-1} \) code, exceedances and cycles.

Definition 3.1. A permutation \( \sigma \) is said to have an exceedance at \( i \in [n] \) if \( \sigma(i) > i \) and an anti-exceedance if \( \sigma(i) \leq i \). If \( \sigma \) has an exceedance (respectively, anti-exceedance) at \( i \) then \( i \) is called the position of the exceedance (respectively, anti-exceedance) while \( \sigma(i) \) is called an exceedance (respectively, anti-exceedance) letter.

Given a permutation \( \sigma = \sigma(1) \sigma(2) \cdots \sigma(n) \in S_n \), we note:

\[
\begin{align*}
\text{Exc}(\sigma) &= \{i \in [n] : \sigma(i) > i\}, \\
\text{AX}(\sigma) &= \{i \in [n] : \sigma(i) \leq i\}.
\end{align*}
\]

Furthermore, \( \text{exc}(\sigma) := |\text{Exc}(\sigma)| \) and \( \text{ax}(\sigma) := |\text{AX}(\sigma)| \).

As shown in [10], the set \( \text{Im}(f) \) coincides with the set of anti-exceedance letters of \( \sigma = \phi(f) \), that is, \( i \in \text{Im}(f) \) if and only if \( \sigma^{-1}(i) \in \text{AX}(\sigma) \).

In this subsection we are giving more precise results relating the subexceedant function \( f \) to the exceedances and anti-exceedances of \( \sigma \). In particular, the following result characterizes the positions of these anti-exceedances.

Proposition 3.1. Let \( f = f_1 f_2 \cdots f_n \in F_n, \phi(f) = \sigma(1) \sigma(2) \cdots \sigma(n) \), and \( i \in [n] \).

1. \( i \in \text{ImRp}(f) \) iff \( i \in \text{AX}(\sigma) \).

2. \( f_i = \sigma(i) \) iff \( i \in \text{AX}(\sigma) \).

Proof. Point 1. In the product \((1, f_1)(2, f_2) \cdots (n, f_n)\) the transpositions \((j, f_j)\) with \( j < i \) fix the integer \( i \); the transposition \((i, f_i)\) transforms \( i \) into an integer \( f_i \leq i \). If \( \sigma(i) > i \), then the integer \( f_i \) must appear in one of the terms \((j, f_j)\) for \( j > i \). Therefore, there exists \( j > i \) such that \( f_j = f_i \).

Point 2. Suppose \( \sigma \) has an anti-exceedance at \( i \). Then \( \sigma(i) \leq i \). Observe that the procedure to compute \( \phi^{-1}(\sigma) \) does not move \( \sigma(i) \) during its first
$n-i$ iterations. Indeed each of the steps for $k=n,n-1,\ldots,i+1$ swaps the integers $k$ and $\sigma(k)$. None of the integers involved in the swaps can be $\sigma(i)$, because on one hand $k$ only take values larger than $i$ and $\sigma(i) \leq i$ and on the other hand none of the $\sigma(k)$ can be equal to $\sigma(i)$ because $\sigma$ is a bijection. Thus $\sigma_i(i) = \sigma(i)$, and hence $f_i = \sigma(i)$.

**Example 3.2.** In $f = 1121345637 \in F_{10}$, the values 2, 4, 5, 6 and 7 appear only once, therefore their positions are in $\text{ImRp}(f)$ and are positions of anti-exceedances. The repeated values are 1 (in positions $\{1,2,4\}$) and 3 (in positions $\{5,9\}$). Among these positions only 4 and 9 are in $\text{ImRp}(f)$ and correspond to anti-exceedances while 1, 2, 5 correspond to exceedances.

Observe indeed that $\phi = \sigma(f) = 10 \ 8 \ 2 \ 1 \ 9 \ 4 \ 5 \ 6 \ 3 \ 7$ and it has exceedances at 1, 2, 5.

Now, we can provide an alternative definition of the permutation code $\phi^{-1}(\sigma)$, that is, related to the cycle structure of the permutation. It suffices to observe the effect of multiplying the initial permutation by the transpositions $(i, \sigma(i))$.

**Theorem 3.1.** For every $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) \in S_n$ we have $\phi^{-1}(\sigma) = f = f_1f_2\cdots f_n$, where $f_i = \sigma^t(i)$, and $t$ is the smallest positive integer such that $\sigma^t(i) \leq i, \forall i \in [n]$.

**Proof.** We use descending recursion. In other terms, to prove that the claim is true for all $i \in [n]$, we show that:

1. it is true for $n$ and

2. if it is true for all integers $j$ with $i < j \leq n$ then it is true for $i$.

The claim is trivially true for $i = n$ since $f_n = \sigma(n)$ and $\sigma(n) \leq n$. Let then be $i < n$. By Proposition 3.1 (2) the claim is trivially true if $i$ is an anti-exceedance (with $t = 1$). Let us suppose then that $\sigma$ has an exceedance at $i$, say $\sigma(i) = j_1$ with $j_1 > i$. Observe that during the iteration step (of the procedure to construct $f$ from $\sigma$) corresponding to $j_1$, the multiplication of the permutation by the transposition $(j_1, \sigma_{j_1}(j_1))$ changes the image of $i$ from $j_1$ to $\sigma_{j_1}(j_1) = f_{j_1}$. But by recurrence hypotheses, $f_{j_1} = \sigma^{t_1}(j_1)$, where $t_1$ is the smallest integer such that $j_2 := \sigma^{t_1}(j_1) < j_1$. We can then define a sequence of integers $j_1 > j_2 > \cdots > j_p > j_{p+1}$ such that:

- $j_p > i \geq j_{p+1}$;
- $j_1 = \sigma(i)$ and $j_{p+1} = \sigma_i(i) = f_i$;
• \( j_{k+1} = \sigma_{jk}(j_k) = \sigma^t_k(j_k) \) for \( 1 \leq k \leq p \) and \( t_k \) is the smallest positive integer such that \( \sigma^t_k(j_k) < j_k \). (Thus \( j_{k+1} = \sigma^{1+f_1+\cdots+f_k}(i) \).)

Therefore, if \( t = 1 + \sum_{k=1}^{p} t_k \) we have \( f_i = j_p = \sigma^t(i) \), we only have to prove that \( t \) is the smallest integer such that \( \sigma^t(i) \leq i \). Suppose there exists an integer \( s < t \) such that \( \sigma^s(i) \leq i \). Hence there exists an integer \( p_0 \leq p \) such that

\[
1 + \sum_{k=1}^{p_0-1} t_k < s < 1 + \sum_{k=1}^{p_0} t_k. \]

Let \( r = s - (1 + \sum_{k=1}^{p_0-1} t_k) \), then

\[
\sigma^s(i) = \sigma^{(1+\sum_{k=1}^{p_0-1} t_k)+r}(i) = \sigma^r(\sigma^{1+\sum_{k=1}^{p_0-1} t_k}(i)) = \sigma^r(j_{p_0}),
\]

this integer is smaller than \( i \), therefore it is smaller than \( j_{p_0} \), but \( r \) is smaller than \( t_{p_0} \) and this is a contradiction because, by recurrence hypothesis \( t_{p_0} \) is the smallest integer such that \( \sigma^{t_{p_0}}(j_{p_0}) \leq j_{p_0} \).

We introduce the following definition.

**Definition 3.2.** Let \( \sigma \in S_n \). For any integer \( i \in [n] \) we define the nearest orbital minorant of \( i \) (under \( \sigma \)) as the integer \( j \leq i \) such that \( j = \sigma^t(i) \) with \( t \geq 1 \) chosen as small as possible. We denote this integer by \( \text{nom}_\sigma(i) \) or simply \( \text{nom}(i) \) when there is no ambiguity regarding the permutation \( \sigma \).

**Remark 3.1.** For a permutation \( \sigma \) we refer to the code \( \phi^{-1}(\sigma) \) as its nom code.

Therefore, Theorem [3.1] can be simply re-stated as follows.

**Theorem 3.1.** (New form) For every \( \sigma = \sigma(1)\sigma(2)\cdots \sigma(n) \in S_n \), we have \( \phi^{-1}(\sigma) = f = f_1f_2 \cdots f_n \), where \( f_i = \text{nom}(i), \forall i \in [n] \).

When \( \sigma \) is represented graphically as the reunion of cyclic graphs (each corresponding to a cycle), to find \( \text{nom}(i) \) it suffices to start from \( i \) and follow the arcs of the cycle (connecting \( k \) and \( \sigma(k) \)) until one meets an integer \( j \leq i \). Obviously, when (and only when) \( i \) is the minimum of its own cycle, one has \( \text{nom}(i) = i \).

**Example 3.3.** Take \( \sigma = 10 \ 6 \ 8 \ 5 \ 1 \ 4 \ 9 \ 3 \ 2 \ 7 = (1, 10, 7, 9, 2, 6, 4, 5)(3, 8) \).

By Theorem [3.1], \( f_i = \sigma^t(i) \), where \( t \) is the smallest positive integer such that \( \sigma^t(i) \leq i \). Now we compute the values of \( f \) only at the exceedances of \( \sigma \): \( f_1 = \sigma^8(1) = 1, f_2 = \sigma^4(2) = 1, f_3 = \sigma^2(3) = 3, f_4 = \sigma^2(4) = 1 \) and \( f_7 = \sigma^2(7) = 2 \). Thus, \( f = 1131142327 \). In the following graph the blue solid edges represent the graph of \( \sigma \) and the red dashed ones represent the graph of \( f \).
Figure 1: Representation of a permutation by cycles

The main Theorem 3.1 suggests also a new interpretation of the integer $f_i$ and a new algorithm to go from $f$ to $\sigma$ (as a product of disjoint cycles and not as a word).

**Proposition 3.2.** If $f \in F_n$, then $\sigma = \phi(f)$ can be constructed as follows: For $i = 1, 2, \ldots, n$

- if $f_i = i$, then add a new singleton cycle: $(i)$
- if $f_i < i$, then insert $i$ before $f_i$ in its cycle.

**Example 3.4.** Take $f = 1132532 \in F_7$. Then $\phi(f) = \sigma$ can be obtained as follows:

\[
(1) \\
(2, 1) \\
(2, 1)(3) \\
(4, 2, 1)(3) \\
(4, 2, 1)(3)(5) \\
(4, 2, 1)(6, 3)(5) \\
(4, 7, 2, 1)(6, 3)(5) = \phi(f) = \sigma.
\]

The above construction shows that the number of fixed points of $f$ and the number of cycles of $\sigma = \phi(f)$ are equal.

Theorem 3.1 implies in particular that the integers $i$ and $f_i$ are always in the same cycle of the permutation for all $i, 1 \leq i \leq n$.

**Corollary 3.1.** Let $f = f_1 f_2 \cdots f_n \in F_n$ and $\phi(f) = \sigma$.

1. $i \in \text{Fix}(f)$ iff $i$ is a minimum of a cycle of $\sigma$.
2. If $f_i = f_j$, then $i$ and $j$ are in the same cycle of $\sigma$. 

3.3 The graphical representation.

An alternative way to define and visualize the bijection $\phi$ is to use the graphical representation of permutations and subexceedant functions on the grid $[n] \times [n]$, each graph being made by the dots of coordinates $(i, \sigma(i))$ and $(i, f_i)$, respectively.

**Proposition 3.3.** Consider the graph of a permutation $\sigma \in S_n$, then the graph of the corresponding nom code $f = \phi^{-1}(\sigma)$ can be evaluated as follows. For all $i = n, n-1, \ldots, 1$: if the leftmost point on line $y = i$ is higher than the diagonal $y = x$, then reduce its $y$ coordinate by moving it down on the same level as the point on the line $x = i$.

**Proof.** The construction in the statement defines $f_i \leq i, 1 \leq i \leq n$ so $f = f_1f_2 \cdots f_n$ is a subexceedant function. Now we prove that $f$ satisfies $\phi(f) = \sigma$. The operation described in the statement modifies the value of $\sigma_i(\sigma^{-1}_i(i))$ and changes it from $i$ to $\sigma_i(i)$. This coincides with the value $\sigma_i(\sigma^{-1}_i(i))$ after the exchange of $i$ and $\sigma_i(i)$ in the permutation $\sigma_i$, which is the operation performed at step $i$ in the procedure to build $f$ starting from $\sigma$. \hfill $\Box$

**Example 3.5.** Let $\sigma = 54136287 \in S_8$. The following figure shows how to construct the graphic representation of $f = \phi^{-1}(\sigma) = 11132277$ from the graphic representation of $\sigma$. Here is the description of the first few steps of the procedure:

For $i = 8$ the leftmost point of $y = 8$ is $(7, 8)$, which is above the diagonal, we move it down to the same level of the point of the line $x = 8$, that is, at level 7.

For $i = 7$ the leftmost point of $y = 7$ is right on the diagonal, nothing happens.

For $i = 6$ the leftmost point of $y = 6$ is $(5, 6)$, we move it down to the same level of the point of the line $x = 6$, that is, at level 2.

$\ldots$

The full procedure is illustrated in Figure 2.
In fact, at every step $i$, the current graph coincides with the graph of $\sigma_i$ for $j < i$ and with the graph of $f$ for $j \geq i$, as shown in the following table:

| $\sigma$  | $f$  |
|----------|------|
| 5 4 1 3 6 2 8 7 | 7 7 |
| 5 4 1 3 6 2 7 8 | 2 7 7 |
| 5 4 1 3 2 6 7 8 | 2 2 7 7 |
| 2 4 1 3 5 6 7 8 | 3 2 2 7 7 |
| 2 3 1 4 5 6 7 8 | 1 3 2 2 7 7 |
| 2 1 3 4 5 6 7 8 | 1 1 3 2 2 7 7 |
| 1 2 3 4 5 6 7 8 | 1 1 1 3 2 2 7 7 |

Remark 3.2. From the above construction we have that $f$ and $\sigma = \phi(f)$ only differ in the exceedances of $\sigma$.

Conversely, consider the subexceedant function $f = f_1 f_2 \cdots f_n$ and its corresponding graphical representation, then we can construct a permutation $\sigma = \sigma(1) \cdots \sigma(n)$ whose nom code is $f$ as follows: For all $i = 1, 2, \ldots, n$: move the leftmost point of the line $y = f_i$ different from the point $(i, f_i)$ up to the level $y = i$.

### 3.4 Increasing Tree

Whenever a permutation is coded with a subexceedant function $f$, it is possible to represent the permutation as a forest of (increasing) trees in which the parenthood relationship is defined by father($i$) = $f(i)$ for all $i \in [n]$. In the case of the code $\phi^{-1}$, each tree of the forest corresponds to a cycle of the permutation.

Without loss of generality, we consider then a cyclic permutation $\sigma = (w_1, w_2, \ldots, w_n)$ (written as a cycle) and we associate to $\sigma$ an (increasing, ordered) tree $T(\sigma)$ with nodes $w_1, w_2, \ldots, w_n$, constructed by defining the
node $i$ to be a child of the node $\text{nom}(i)$, $\forall i \in \{w_1, w_2, \ldots, w_n\}$. If $\text{nom}(i) = i$, then $i$ is the root of its tree. The trees obtained this way are obviously increasing in the sense that the labels of the nodes increase as one moves along any path from the root to a leaf. Furthermore, we impose an order on sibling nodes (that is, nodes having the same parent) by arranging them in increasing order from left to right.

**Example 3.6.** Let $\sigma$ be the cycle $(1, 7, 3, 4, 6, 2, 9, 5, 8)$. Then $T(\sigma)$ is given in the following figure.

![Figure 3: The ordered increasing tree $T(\sigma)$.](image)

**Remark 3.3.** With the order we defined on siblings, the postorder transversal of the tree $T(\sigma)$ yields the permutation $\sigma$ itself.

**Remark 3.4.** Let $\sigma = (w_1, w_2, \ldots, w_{n-1})$ be a cyclic permutation on $[n-1]$ and $(i, n)$ a transposition with $i \in [n-1]$, then the tree associated to the (cyclic) permutation $\tau = \sigma \cdot (i, n)$ is obtained by adding to the tree $T(\sigma)$ a node labeled $n$ as new last successor of $i$.

Now we present alternative proofs of some results related to Eulerian numbers that can be found in the literature.

**Proposition 3.4.** The number of unordered increasing trees on $n + 1$ nodes and $k$ internal nodes is the Eulerian number $A(n,k-1)$.

**Proof.** The number of unordered increasing trees on $n + 1$ nodes and $k$ internal nodes is equal to the number of cyclic permutations over $[n+1]$ whose subexceedant function $f$ has $IMA$ equal to $k$. The set of the subexceedant functions of these permutations is in bijection with the set of all subexceedant functions over $[n]$ having $IMA$ equal to $k$. The bijection consist simply in erasing the $1$ at the beginning of $f$. This operation yields a subexceedant function because the cyclic nature of the permutations guarantees that $f(i) \leq i - 1$
for all $i$, furthermore it preserves the number of images because necessarily $f(2) = 1$. We know that IMA is an Eulerian statistic over the set of subexceedant functions, in the sense that $IMA(n, k) = A(n, k - 1)$.

**Proposition 3.5.** The number of forests of increasing ordered trees having $n$ nodes and $k$ leaves is the Eulerian number $A(n, k)$.

**Proof.** Observe that every leaf $i$ of the tree corresponds to an exceedance letter of the associated permutation.

**Proposition 3.6.** The number of forests of increasing ordered trees having $n$ nodes and $k$ nodes that are either internal or roots of singleton trees is the Eulerian number $A(n, k - 1)$.

**Proof.** While this result can be deduced from the previous one using the symmetry property of Eulerian numbers and some classical involution of the symmetric group, a quicker proof can be given using subexceedant functions. Let $\sigma$ be a permutation and $\sigma = C_1 \cdot C_2 \cdots \cdot C_k$ its decomposition into disjoint cycles. The number of nodes that are either internal or roots of singleton trees in the forest $T(C_1) \cup T(C_2) \cup \cdots \cup T(C_k)$, is equal to $IMA(f)$, where $f = \phi^{-1}(\sigma)$, and IMA is an Eulerian statistic over the set of subexceedant functions (in the sense that $IMA(n, k) = A(n, k - 1)$).

R. Stanley in [11] also presented a way to represent a permutation by an (increasing, unordered) tree.

Given $w = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$ (written as a word), construct an (unordered) tree $T(\sigma)$ with nodes $0, 1, \ldots, n$ and with 0 at the root by defining node $i$ to be the successor of the rightmost element $j$ of $w$ that precedes $i$ in the word $\sigma$ and that is less than $i$. If there is no such element $j$, then let $i$ be the successor of the root 0.

Stanley’s trees are defined as “unordered”, however they appear to be drawn ordering siblings in increasing order.

Regardless, since the rightmost element $j$ of $w = w_1 w_2 \cdots w_n$ which precedes $i$ and which is less than $i$ is the nom of $i$ in $\sigma^{-1}$, where $\sigma = (0, w_1, w_2, \ldots, w_n)$, we deduce the following.

**Proposition 3.7.** The tree associated to the permutation $w = w_1 w_2 \cdots w_n$ by the Stanley bijection is the same as the (unordered version of the) tree associated to the cycle $(0, w_n, w_{n-1}, \ldots, w_1)$ via the nom code.

### 4. Non-Decreasing nom codes

We say that a subexceedant function $f = f_1 f_2 \cdots f_n$ is **non-decreasing** if $1 = f_1 \leq f_2 \leq \cdots \leq f_n$. For every $n \geq 0$, the number of non-decreasing
subexceedant functions over \([n]\) is the Catalan number \(C_n = \frac{1}{n+1} \binom{2n}{n}\). A proof of this result is based on a natural bijection between the set of non-decreasing subexceedant functions over \([n]\) and the set of *N-E lattice paths* from \((0,0)\) to \((n,n)\). A *N-E lattice path* is a path on the integer grid going from the point \((0,0)\) to the point \((n,n)\), made only by North = \((0,1)\) and East = \((1,0)\) steps and never going above the diagonal \(y = x\). If \(|w|_x\) denotes the number of occurrences of a letter \(x\) in the N-E path, then the N-E path is a word \(w\) of length \(2n\) over the alphabet \(\{E,N\}\) in which:

- \(|w|_E = |w|_N\),
- for every prefix \(w'\) of the word \(w\), one has \(|w'|_E \geq |w'|_N\).

The bijection between the set of non-decreasing subexceedant functions and the set of N-E lattice paths simply looks at each N-E lattice path as the profile of the graph of a non-decreasing subexceedant function (whose values have all been decreased by 1).

**Example 4.1.** Figure 4 shows the 14 lattice paths over \([4]\).

![Lattice paths on [4]](image)

The corresponding non-decreasing subexceedant functions over \([4]\) are, respectively:

1111, 1112, 1113, 1122,
1222, 1123, 1114, 1223, 1133, 1124,
1134, 1224, 1233, 1234.

Each subexceedant function is obtained by increasing by 1 the column heights of the area below the corresponding N-E lattice path. We include a direct proof of the fact that the cardinality of this set of non-decreasing functions over \([n]\) is equal to the catalan number \(C_n\).
Proposition 4.1. The set of non-decreasing subexceedant functions over \([n]\) satisfies the Catalan recurrence relation:

\[
C_n = \sum_{i=2}^{n+1} C_{i-2}C_{n-i+1}
\]

with initial conditions \(C_0 = C_1 = 1\).

Proof. Let \(f = f_1 f_2 \cdots f_n\) be a non-decreasing subexceedant function and \(i \neq 1\) be the first integer such that \(f_i = i\), then we split \(f\) into two parts: the first part is \(f_2 \cdots f_{i-1}\) which is a non-decreasing subexceedant function of length \(i - 2\) and there are \(C_{i-2}\) of these, and the second part is \(f_i \cdots f_n\) which is also a non-decreasing subexceedant function over \([n-i+1]\) (just reduce every element by \(i-1\)), so it is counted by \(C_{n-i+1}\). Thus there are a total of \(C_{i-2}C_{n-i+1}\) non-decreasing subexceedant functions with the first \(i \neq 1\) such that \(f_i = i\), and summing these terms up gives \(C_n\), which is the recurrence relation. \(\square\)

Remark 4.1. From two subexceedant functions \(f_1 \cdots f_i\) over \([i]\), and \(g_1 \cdots g_{n-i-1}\) over \([n-i-1]\) we can obtain a unique subexceedant function over \([n]\) as follows: Define \(f = 1 f_1 f_2 \cdots f_n\) by \(f_t = g_j + t\), where \(i < t \leq n, j = 1, \ldots, n-i-1\). Clearly \(f\) is a subexceedant function over \([n]\).

Let \(F_n^\rightarrow\) denote the set of all non-decreasing subexceedant functions over \([n]\) and \(\mathfrak{S}_n^\rightarrow\) the set of permutations in \(\mathfrak{S}_n\) whose nom code is in \(F_n^\rightarrow\).

Example 4.2. The 14 permutations in \(\mathfrak{S}_4^\rightarrow\) that correspond to the above non-decreasing subexceedant functions (and hence to the lattice paths in Figure 4) are:

\[
2341, 4312, 2413, 3142, 1342, 4123, 2314, 1423, 2143, 3124, 2134, 1324, 1243, 1234
\]

In the case of other permutation codes representing permutations by subexceedant functions (such as the Lehmer or the Denert code), the Catalan-sized classes of permutations corresponding to non-decreasing codes have been combinatorially characterized, often in terms of pattern avoidance. For instance, in the case of the Lehmer inversion code, the class of permutations corresponding to non decreasing codes is the class of permutations avoiding the pattern 213. Here we present a characterization for the permutations having non-decreasing nom codes and describe a certain number of their properties.
If \( f = f_1 f_2 \cdots f_n \) is a non-decreasing subexceedant function, then we can encode \( f \) by the vector \( r = (r_1, r_2, \ldots, r_n) \), where \( r_i \) is the number of occurrences of the letter \( i \) in \( f \). These vectors have the following characterization:

\[
0 \leq r_i \leq n - i + 1, \forall i \in [n], r_1 \neq 0 \quad \text{and} \quad \sum_{i=1}^{n} r_i = n.
\]

**Example 4.3.** Let \( f = 111334555 \in F_9 \). Then \( r = (3, 0, 2, 1, 3, 0, 0, 0) \).

**Proposition 4.2.** If \( \sigma = \sigma(1) \cdots \sigma(n-1) \in S_{n-1} \) is associated to a non-decreasing nom code, and \( \sigma' = \sigma \cdot (n,j) \), where \( \sigma(n-1) \leq j \leq n \), then so is \( \sigma' \).

**Proof.** Only it suffices to show that \( \sigma'(n) \geq \sigma(n-1) \) for \( \sigma(n-1) \leq j \leq n \). This is obviously true since \( \sigma'(n) = (\sigma \cdot (n,j))(n) = j \).

**Proposition 4.3.** Let \( \sigma \in S_n \) and let \( f \) be its nom code. Then, \( f \) is non-decreasing if and only if:

1. the sub-word of anti-exceedance letters of \( \sigma \) is increasing, in other terms, for any two anti-exceedances \( i \) and \( j \) of \( \sigma \) with \( i < j \), one has \( \sigma(i) < \sigma(j) \), and

2. elements of \([n]\) having the same nearest orbital minorant under \( \sigma \) form an integer interval.

**Proof.** Let \( r = (r_1, r_2, \ldots, r_n) \) be the vector code of \( f \). The values of \( \text{Im}(f) \) are the anti-exceedances of \( \sigma \). These values correspond to the integers \( k \) such that \( r_k \neq 0 \). Furthermore, by Proposition 3.1, only the rightmost occurrences of each letter of \( \text{Im}(f) \) in \( f \) correspond to the (positions of the) anti-exceedances. For each \( k \) such that \( r_k \neq 0 \) this position is exactly \( \sum_{\ell=1}^{k} r_{\ell} \).

Further, observe that the nearest orbital minorants of the elements of \([n]\) under \( \sigma \) are elements of \( \text{Im}(f) \), where \( \phi(f) = \sigma \). If \( i < k < j \), where \( i, j, k \in [n] \) with \( \text{nom}(i) = \text{nom}(j) \) and \( \text{nom}(i) \neq \text{nom}(k) \), then the corresponding subexceedant function is not non-decreasing.

Conversely, it can easily be seen that the subexceedant function \( f = \phi^{-1}(\sigma) \) with \( \sigma \) of the given conditions is non-decreasing.

**Proposition 4.4.** If \( \sigma \in S_n \) is a permutation associated by \( \phi \) with a non-decreasing nom code \( f \), then the set of the anti-exceedances of \( \sigma \) and their values entirely characterize \( \sigma \).
Proof. Let \( \{ i_1 < i_2 < \cdots < i_k = n \} \) be the anti-exceedances of \( \sigma \) and \( \{ j_1 < j_2 < \cdots < j_k \} \) their respective images (with \( i_t \geq j_t \) for all \( t = 1,2,\ldots,k \)). Then the graph of the function \( f \) passes through the points \((i_1,j_1),(i_2,j_2),\ldots,(n,j_k)\). Furthermore, by Proposition \( 3.1 \), \((i_t,j_t)\) is the rightmost point among those of height \( j_t \) and the number of columns of height \( j_t \) is \( i_t - i_{t-1} \) (if one poses \( i_0 = 0 \)). Therefore, (the graph of) \( f \) is entirely determined and consequently \( \sigma \) is too.

Example 4.4. Let \( \sigma \in \mathfrak{S}_9 \) with \( \AX(\sigma) = \{3,4,6,9\} \) and anti-exceedance values \( \{1,2,3,5\} \). If \( f = \phi^{-1}(\sigma) \), then \( \{3,4,6,9\} \) are the positions of the rightmost occurrences of \( \{1,2,3,5\} \), respectively. Then the graph of \( f \) is the one shown in the following picture and we have \( f = 111233555 \in \mathcal{F}_9 \). Using one of the methods to compute \( \phi(f) \), we obtain \( \sigma = 471263895 \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{graph.png}
\caption{Graph of \( f \)}
\end{figure}

Let the set of \( t \)-uples of integer pairs \( \langle (i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k) \rangle \) satisfy the following :

\begin{enumerate}
  \item \( 1 \leq k \leq n \),
  \item \( 1 \leq i_1 < i_2 < \cdots < i_k = n \),
  \item \( 1 = j_1 < j_2 < \cdots < j_k \), and
  \item \( j_s \leq i_{s-1} + 1 \), \( \forall s, 2 \leq s \leq k \).
\end{enumerate}

We can deduce the following corollary as a new proof of a result already known as an incarnation of Catalan numbers whose proof can be found in Stanley \cite{Stanley} and Hu \cite{Hu}.

Corollary 4.1. The cardinality of the set of \( t \)-uples of integer pairs \( \langle (i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k) \rangle \) satisfying the conditions in \( \) is the \( n \)-th Catalan number \( C_n \).

Proof. Two such sequences \( \langle i_1,i_2,\ldots,i_k \rangle \) and \( \langle j_1,j_2,\ldots,j_k \rangle \) constitute respectively the set of anti-exceedances and their images of a permutation associated with a non-decreasing code, namely \( 1^{i_1}j_2^{i_2-i_1}\cdots j_k^{n-i_k-1} \). \( \square \)
Note that Proposition 4.4 suggests the existence of a procedure to re-
construct directly the unique permutation associated with a non-decreasing
subexceedant function only knowing its anti-exceedances and their values,
without passing through the associated nom code. We do this in the follow-
ing lemma.

**Lemma 4.1.** Let $(i_1, j_1 = 1), (i_2, j_2), ..., (i_k = n, j_k)$ be pairs of integers sats-
ifying the conditions in [1]. Then it is possible to construct $\sigma$ as follows.

- For all $s \in \{1, 2, \ldots, k\}$, insert $j_s$ at position $i_s$ (that is, set $\sigma(i_s) = j_s$);
- For all $i \not\in \{i_1, i_2, \ldots, i_k\}$ taken in decreasing order, if $i_{s-1} < i < i_s$
  for a certain $s$, then insert the value $\sigma^{-1}(j_s)$, where $t$ is the smallest
  positive integer such that $\sigma^{-t}(j_s)$ has not yet been inserted as the image
  of another integer, at the position $i$, that is, define $\sigma(i) = \sigma^{-t}(j_s)$.

**Proof.** We prove that the procedure is well defined, namely that a $t$
such that $\sigma^{-t}(j_s)$ has not yet been inserted as the image of another integer can
always be found. We prove this by backward recursion. If $i = n$, then
$\sigma(n) = \sigma^{-1}(j_k)$ and we have $t = 1$. The two line representation of part of the
permutation $\sigma$ (the integers 1, 2, $\ldots$, $n$ on the first row as usual, their images
in the second) can be useful to follow the rest of the proof.

$$\begin{pmatrix}
\cdots & i & i+1 & \cdots & i_s & y & y+1 & \cdots & i_r & i_{p-1} & i_{p-1}+1 & \cdots & i_p & \cdots \\
\sigma^{-t_1}(j_s) & j_s & i_{p-1}+1 & j_r & j_{p-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}$$

Assume now that the assertion is true for all integers in $[i + 1, n]$, therefore
we have $\sigma(i + 1) = \sigma^{-t_1}(j_s)$. If $i + 1 = \sigma^{-t_1-1}(j_s)$ has not yet been inserted
then the procedure will insert it as the image of $i$ because $t_1 + 1$ is the smallest
positive integer such that this happens. Otherwise, if $i + 1$ has already been
inserted, it cannot have been inserted as an image of an integer on its left,
because the only integers already inserted on its left are the images of anti-
exceedances at positions smaller than $i + 1$, then it has been inserted as the
image of integer on its right (larger than $i + 1$), and therefore it is the image
of some anti-exceedance. Since

$$j_s \leq i_{s-1} + 1 < i + 1$$

then $i + 1 = j_p$, where $p \geq s + 1$, that is, $i + 1 \in \{j_{s+1}, j_{s+2}, \ldots, j_k\}$, thus
we have $j_p = \sigma^{-t_1-1}(j_s)$. By induction hypothesis, there exists an integer $t_2$
such that $\sigma(i_{p-1} + 1) = \sigma^{-t_2}(j_p)$. If $i_{p-1} + 1$ has not yet been inserted, then
we insert it as image of $i$, in other terms we set :

$$\sigma(i) = i_{p-1} + 1 = \sigma^{-t_2-1}(j_p) = \sigma^{-t_1-t_2-2}(j_s).$$

Otherwise, we have two cases.
1. $i_{p-1} + 1$ is an image of some integer on its right and therefore it is the image of an anti-exceedance to the right of $i_p$.

2. $i_{p-1} + 1$ is an image of some integer on its left and therefore it is the image of an exceedance to the left of $i_{p-1}$, say $y$, with $i_{r-1} < y < i_r$ for a certain $r$. This means that the integer $y + 1 = \sigma^{-1}(j_r)$ for a certain $t$ had already been inserted when we tried to define $\sigma(y)$, but the only integers already inserted at that point were the images of all integers larger than $y$ and the images of the anti-exceedances smaller than $y$, but these anti-exceedances cannot have $y+1$ as image since $y+1 > i_{r-1}$, therefore $y+1$ needs to be the image of an anti-exceedance on its right. Furthermore, by applying repeatedly $\sigma^{-1}$ to $y + 1$ we must eventually find $i_{p-1} + 1$ otherwise we would not have $\sigma(y) = i_{p-1} + 1$, consequently there must exist a $m$ such that $\sigma^{-m}(y + 1)$ is an anti-exceedance larger than or equal to $i_p$, but the image of $i_p$ has already been defined as $i + 1$ and $i + 1 > y + 1$ (because the image of $i$ has yet to be defined while the image of $y$ is already defined), consequently we can say that there exists a $m$ such that $\sigma^{-m}(y + 1)$ is an anti-exceedance strictly larger than $i_p$.

In both cases, where $i_{p-1} + 1$ has already been inserted, we can identify an integer in the orbit of $j_s$ that is the image of an anti-exceedance larger than $i_p$. We can then construct a sequence of integers such that each of them is the image of an anti-exceedance larger than the precedent elements of the sequence and each of these integers is of the form $\sigma^{-t}(j_s)$ for a certain $t$. Since there is a finite number of anti-exceedances, we repeat the procedure until we find an anti-exceedance, say $i_q$ such that $i_{q-1} + 1$ has not yet been inserted and $i_{q-1} + 1$ is obtained by applying $\sigma^{-1}$ to $j_s$ a certain number of times. Then we set $\sigma(i) = i_{q-1} + 1$. \hfill \Box

**Theorem 4.1.** Let $(i_1, j_1 = 1), (i_2, j_2), \ldots, (i_k = n, j_k)$ be pairs of integers satisfying the conditions in \[ . \] Then $\sigma$ constructed as in the previous lemma has anti-exceedances exactly at $i_1, i_2, \ldots, i_k$ with values $j_1, j_2, \ldots, j_k$ and its nom code is $f = 1^{i_1} j_2^{i_2-i_1} \cdots j_k^{n-i_k-1}$.

**Proof.** Trivially, if $i \in \{i_1, i_2, \ldots, i_k\}$, say $i = i_s$, then $\sigma$ has an anti-exceedance at $i$, because $\sigma(i_s) = j_s \leq i_{s-1} + 1 \leq i_s$, therefore $(\phi^{-1}(\sigma))(i_s) = \sigma(i_s) = j_s$. Assume that $i \notin \{i_1, i_2, \ldots, i_k\}$, say $i_{s-1} < i < i_s$ for a certain $s$. Now we prove that $\sigma$ has an exceedance at $i$ and that $(\phi^{-1}(\sigma))(i) = j_s$. Let $i_{s-1} < i < i_s$ for a certain $s$, and let $\sigma(i) = \sigma^{-t}(j_s)$ with $t$ the integer such that $\sigma^{-t}(j_s)$
has not yet been inserted as the image of another integer. We start by proving that for all $p \leq t$ we have $\sigma^{-p}(j_s) > i$. This is trivial for $p = 1$ since $\sigma^{-1}(j_s) = i_s > i$. Suppose that for some $1 < p_0 \leq t$ one has $\sigma^{-p_0}(j_s) \leq i$, that is, $\sigma^{-1}(\sigma^{-\langle p_0 \rangle}(j_s)) \leq i$ and let us choose $p_0$ as small as possible with this property. This implies that there exists an integer $\ell \leq i$ such that the image $\sigma(\ell)$ is already defined as $\sigma^{-\langle p_0 \rangle}(j_s)$, which is larger than $i$ for the minimality hypotheses on $p_0$. But the only integers smaller than or equal to $i$ whose images are already defined are the anti-exceedances at positions on the left of $i$, while

$$\sigma(\ell) = \sigma^{-\langle p_0 \rangle}(j_s) > i \geq \ell,$$

so $\ell$ would be an exceedance, a contradiction. This implies in particular that $\sigma(i) = \sigma^{-\langle j \rangle}(j_s) > i$ and hence that $i$ is an exceedance for $\sigma$.

Let us prove now that $(\phi^{-1}(\sigma))(i) = j_s$. Since $\sigma(i) = \sigma^{-\langle j \rangle}(j_s)$, we have $\sigma^{t+1}(i) = j_s$. By hypotheses $j_s \leq i_{s-1} + 1 \leq i$, according to Theorem 3.1, $j_s$ is equal to $(\phi^{-1}(\sigma))(i)$ if $t+1$ is the smallest integer such that $\sigma^{t+1}(i) \leq i$, so we only need to prove that for all $q < t + 1$ one has $\sigma^q(i) > i$. Suppose there exists $q_0 < t + 1$ such that $\sigma^{q_0}(i) \leq i$. We have $q_0 > 1$, because $i$ is an exceedance for $\sigma$. Now,

$$\sigma^{-\langle t+1-q_0 \rangle}(j_s) = \sigma^{-\langle t+1-q_0 \rangle}(\sigma^{t+1}(i)) = \sigma^{t+1-t-1+q_0}(i) = \sigma^{q_0}(i) \leq i.$$

This is a contradiction with the previous part of the proof because $t+1-q_0 < t$. Thus, for all $p \leq t$, we have $\sigma^{-p}(j_s) > i$. \qed

5. Avoidance of length 3 patterns in $\mathfrak{S}_n^\rightarrow$

In this section we study length 3 pattern avoidance in $\mathfrak{S}_n^\rightarrow$. The set of permutations in $\mathfrak{S}_n^\rightarrow$ avoiding a pattern $\pi$ is denoted by $\mathfrak{S}_n^\rightarrow(\pi)$.

The case 123.

The sequence of the number of 123-avoiding permutations over $[n], n \geq 1$ having non-decreasing nom codes is $1, 2, 4, 4, 3, 0, 0, 0, \ldots$

Proposition 5.1. We have $|\mathfrak{S}_n^\rightarrow(123)| = 0$ for $n \geq 6$.

Proof. If $\sigma \in \mathfrak{S}_n^\rightarrow$ such that $ax(\sigma) \geq 3$, then $\sigma$ must contain the pattern 123 since the sub-word of anti-exceedance letters of $\sigma$ is increasing. Let $ax(\sigma) \leq 2$. If $ax(\sigma) = 1$, then $\sigma = 234 \cdots n1$ and it contains the pattern 123. If $ax(\sigma) = 2$, then let $AX(\sigma) = \{i, n\}$, where $i < n$ and $\sigma(i) = 1 < \sigma(n)$. In this case $\sigma$ avoids the pattern 123 if and only if the sub-words of exceedance
letters \( \sigma(1)\sigma(2) \cdots \sigma(i-1) \) and \( \sigma(i+1)\sigma(i+2) \cdots \sigma(n-1) \) are decreasing. But the integers in \( [n] \setminus \{ \sigma(i) = 1, \sigma(n) \} \) can be arranged in decreasing order so that they are sub-words of exceedance letters of \( \sigma \) if and only if \( n \leq 5 \). Therefore, every \( \sigma \in \mathcal{G}_n^\varphi \) for \( n \geq 6 \) contains the pattern 123.

\[ \square \]

The case 132.

In a subexceedant function \( f \), a plateau at \( i \) of \( f \) is said to be of height \( h \) if \( f_i = f_{i+1} = h \). Further, \( f \) has a stair at \( i \) if \( f_{i+1} = f_i + 1, 1 \leq i \leq n-1 \).

Let \( V_n = \{ f \in F_n^\varphi : f \text{ may only have plateaus of height 1} \} \), that is, \( f \in V_n \) iff \( f_1 = f_2 = \cdots = f_k = 1 < f_{k+1} < f_{k+2} < \cdots < f_n \), where \( 1 \leq k \leq n \). For instance, \( V_3 = \{111, 112, 113, 123\} \).

**Proposition 5.2.** If \( \sigma \in \mathcal{G}_n^\varphi(132) \), then \( f = \phi^{-1}(\sigma) \in V_n \).

**Proof.** Let \( \sigma \in \mathcal{G}_n^\varphi(132) \) and suppose \( f = \phi^{-1}(\sigma) \notin V_n \), that is, \( f \) has plateaus of height \( h \) with \( h \geq 2 \). Let \( k \) be the rightmost position such that \( f_k = 1 \) and \( p > k + 1 \) be the rightmost position such that \( f_p = h \). By proposition 3.1 \( \sigma \) has an anti-exceedance in \( k \) with value 1 and an anti-exceedance in \( p \) with value \( h \leq p \). In the position \( p - 1 \), we have \( f_{p-1} = h \) and an exceedance for \( \sigma \), that is, a letter larger than \( p - 1 \), therefore \( \sigma \) would have a pattern 132 in the positions \( k, p - 1 \) and \( p \), a contradiction. \( \square \)

**Remark 5.1.** Let \( f \in V_n \) having \( k \) plateaus of height 1 and let \( \sigma = \phi(f) \). Then

1. for all \( i \) with \( k + 1 \leq i \leq n \) we have \( \sigma(i) = f_i \), because all these are anti-exceedance positions, then \( \text{Exc}(\sigma) = [k] \) and the \( n-k \) remaining letters are a sequence of increasing anti-exceedances letters.

2. if there exists \( i \) such that \( f_i = i \), then \( f_j = j \) for all \( j > i \), in particular \( f_n = n \).

Let \( \mathcal{D}_n \) denotes the set of all nom codes \( f \) with no fixed points greater than 1 in \( F_n^\varphi \) such that \( \sigma = \phi(f) \in \mathcal{G}_n^\varphi(132) \), and \( \mathcal{D}_{n,k} \) the set of \( f \in \mathcal{D}_n \) with \( k \) plateaus of height 1. Further, \( d_n = |\mathcal{D}_n| \) and \( d_{n,k} = |\mathcal{D}_{n,k}| \). The following proposition characterizes the elements of \( \mathcal{D}_n \).

**Proposition 5.3.** Let \( f \in \mathcal{D}_{n,k} \). Decompose \( f = 1 \cdot H_0 \cdot L_1 \cdots \cdot L_p \), where \( H_0 = 1^{k_0} \), where \( k_0 = k_i \), and \( L_i, i \geq 1 \) is a maximal sequence of stairs. Let \( l_i = |L_i|, i \geq 1 \). We cut \( L_i \) into blocks of size \( k_{i-1} \) (with a possible remainder of size \( l_{i-1} \mod k_{i-1} \) and define \( k_i = ((l_{i-1} - 1) \mod k_{i-1}) + 1 \). Let the resulting decomposition be \( f = 1 |H_0| H_1 |H_2| \cdots |H_s \). Then, \( \sigma = \)
\(\phi(f) \in \mathcal{S}_n(132)\) if and only if \(f_{k_0+k_1+\ldots+k_i+2} = k_0 + k_1 + \cdots + k_{i-1} + 2, 1 \leq i \leq s\), that is, each block \(H_i\) with \(i > 0\) begins with the integer equal to the initial position of the block \(H_{i-1}\).

**Proof.** Let \(f_{k_0+2} > 2\). In this case we have \(\sigma(1) = 2\) since \(\text{Exc}(\sigma) = [k_0]\) and 2 is not the image of any anti-exceedance of \(\sigma\). Since \(f_n < n\) we have some \(j\) such that \(2 \leq j \in [k_0]\) and \(\sigma(j) = n\). This implies that \(\sigma\) has a pattern 132 in positions 1, \(j, k_0 + 2\) and this is a contradiction. Thus, \(f_{k_0+2} = 2\). Proceeding by induction let us suppose that \(f_{k_0+k_1+\ldots+k_i+2} \neq k_0 + k_1 + \cdots + k_{i-1} + 2\). Let \(t = k_0 + k_1 + \cdots + k_i + 2\). Then \(k_0 + k_1 + \cdots + k_{i-1} + 2 = t - k_{i-1}\). At the \(t\)-th step of the insertion method of the construction of \(\sigma\) we have the largest \(k_{i-1}\) integers already inserted in to the first \(k_{i-1}\) positions of \(\sigma\) in increasing order. Since \(k_{i-1} \geq h_i\) the next \(k_i\) steps move the largest \(k_i\) integers in \(\sigma\). So \(t\) certainly moves the integer in the first position. If \(f_t < t - k_{i-1}\), then \(f_t > f_{t-1} + 1\) and there is some \(j \in [k_0]\), \(j > k_{i-1} + 1\) such that \(\sigma(j) = t\). So, we have \(\sigma(k_{i-1} + 1) < f_t = \sigma(t) < \sigma(j)\), where \(k_{i-1} + 1 < j < t\). Since in the successive steps \(\sigma(k_{i-1} + 1)\) and \(\sigma(j)\) remain unmoved, and \(f_t\) can only be moved by a larger integer we have a pattern 132 in \(\sigma\) and this is a contradiction. If \(f_t > t - k_{i-1}\), then \(f_t\) is one of the largest \(k_{i-1} - 1\) already inserted in positions \([2, k_{i-1}]\) and the positions \(1, \sigma^{-1}(f_t)\) and \(t\) would create the pattern 132. Therefore, \(f_t = t - k_{i-1}\).

Conversely, suppose that \(f\) satisfies the given condition. We can further decompose the block of exceedance letters of \(\sigma\) into blocks \(P_1|P_2|\cdots|P_u\) made of increasing numbers and such that for every \(a \in P_j, b \in P_{j+1}\) we have \(a > b\), where \(j = 1, 2, \ldots, u - 1\). Indeed for all \(r\) with \(1 \leq r \leq s\), it can be verified that if \(\tau = \phi(1|H_0|H_1|H_2|\cdots|H_r)\) then \(\tau\) is obtained from \(\tau' = \phi(1|H_0|H_1|H_2|\cdots|H_{r-1})\) by replacing the integers \(\tau'(1), \ldots, \tau'(h_r)\) with the \(h_r\) largest integers in increasing order and appending \(\tau'(1), \ldots, \tau'(h_r)\) at the end. It is straightforward to verify that such a permutation cannot contain the pattern 132. \(\square\)

We now present one of the main results of this section.

**Theorem 5.1.** The number \(a_n\) of nom codes \(f\) over \([n]\) such that \(\sigma = \phi(f) \in \mathcal{S}_n(132)\) satisfies the recurrence relation:

\[
a_n = a_{n-1} + p_{n-1}, n \geq 1, \quad a_0 = 1,
\]

where \(p_n\) is the number of integer partitions of \(n\).

**Proof.** The term \(a_{n-1}\) in the right-hand side of the recurrence relation can be explained easily. Let \(f = f_1f_2\cdots f_{n-1}\) such that \(\sigma = \phi(f) \in \mathcal{S}_{n-1}(132)\).
Then the nom code \( f' \) obtained from \( f \) by appending \( n \) at its end corresponds to a permutation \( \sigma' = \phi(f') \in \mathcal{S}_n^\varphi(132) \). This term counts the non-decreasing nom codes with fixed points and such that \( \phi(f') \in \mathcal{S}_n^\varphi(132) \) and it contributes \( a_{n-1} \) to \( a_n \). For the term \( p_{n-1} \), we will show in the following lemmas and propositions that the number of nom codes \( f \) with no fixed points, with \( \sigma = \phi(f) \in \mathcal{S}_n^\varphi(132) \), equals the number \( p_{n-1} \) of all integer partitions of \( n - 1 \) and more precisely that the number \( d_{n,k} \) equals the number of integer partitions over \( [n-1] \) with the largest part equal to \( k \) \((\square)\). Thus, this contributes \( p_{n-1} \) to the number \( a_n \).

\[ \square \]

**Lemma 5.1.** Let \( f \in \mathcal{D}_{n,k} \) and \( f' \) is obtained from \( f \) by increasing each integer greater than or equal to \( k + 2 \) by 1 in \( f \) and inserting a 1 at the beginning. Then \( f' \in \mathcal{D}_{n+1,k+1} \).

**Proof.** If \( f \) is decomposed as in the Proposition \[5.3\] we have \( f_{k_0+k_1+2} = k_0 + 2 \).

If \( f' = f'_1 f'_2 \cdots f'_{n+1} \), then

\[
f'_{i} = \begin{cases} 
1, & \text{if } 1 \leq i \leq k_0 + 2, \\
f_{i-1}, & \text{if } k_0 + 3 \leq i \leq k_0 + k_1 + 2, \\
f_{i-1} + 1, & \text{if } k_0 + k_1 + 3 \leq i \leq n + 1. 
\end{cases}
\]

Let us show that \( f' \in \mathcal{D}_{n+1,k+1} \), that is, \( f' \) satisfies the condition in Proposition \[5.3\]. The number of blocks of \( f \) and \( f' \) are the same but the size of \( H'_0 \) of \( f' \) is increased by 1. So \( f'_{k_0+k_1+\cdots+k'_i+2} = f_{k_0+k_1+\cdots+k_i+1} + 1 = k_0 + k_1 + \cdots + k_{i-1} + 2, i \geq k_0 + k_1 + 2 \). Thus, \( f' \in \mathcal{D}_{n+1,k+1} \). \( \square \)

**Lemma 5.2.** Let \( f \in \mathcal{D}_{n,k} \) and \( f' \) is obtained from \( f \) by increasing each integer greater than 1 by \( k \) and then inserting the sub-word \( 23 \cdots k+1 \) after the rightmost occurrence of 1. Then \( f' \in \mathcal{D}_{n+k,k} \).

**Proof.** Since \((l_1 - 1) \mod k + 1 = (l_1 + k - 1) \mod k + 1 \) the insertion of the sub-word \( 23 \cdots k+1 \) in the operation creates a new block of size \( k \) in \( f' \) after \( H_0 \). That is, \( f' = 1 \cdot H_0 \cdot H'_1 \cdot H'_2 \cdots H'_{s+1} \), where \( H'_1 = 234 \cdots k+1 \) and \( a + k \in H'_i \) if \( a \in H_{i-1}, i \geq 2 \). So, \( f'_{k_0+2} = 2 \) and

\[
f'_{k_0+k_1+\cdots+k'_i+2} = f'_{k_0+k_0+k_1+\cdots+k_{i-1}+2} \\
= f_{k_0+k_0+k_1+\cdots+k_{i-1}+2-k_0+k_0} \\
= (k_0 + k_1 + \cdots + k_{i-2} + 2) + k_0 \\
= k'_0 + k'_1 + \cdots + k'_{i-1} + 2, \quad i \geq k_0 + k_0 + 2.
\]

Thus, \( f' \in \mathcal{D}_{n+k,k} \). \( \square \)
From Lemma 5.1 and Lemma 5.2 we deduce that all nom codes in \( f \in \mathcal{D}_{n,k} \) can uniquely be obtained either from a nom code in \( \mathcal{D}_{n-1,k-1} \) by the construction in Lemma 5.1 or from a nom code in \( \mathcal{D}_{n-k,k} \) by the construction in Lemma 5.2. Furthermore, the two sets of nom codes obtained this way are disjoint because those obtained by the construction in 5.1 do not have \( k + 2 \) in their image, while those those obtained by the construction in 5.2 do. As a result, we have the following proposition.

**Proposition 5.4.** The number \( d_{n,k} \) of nom codes \( f \) with \( k \) plateaus of 1’s and no fixed points other than 1 such that \( \sigma = \phi(f) \in \mathcal{S}_n(132) \) satisfies the recurrence relation:

\[
d_{n,k} = d_{n-1,k-1} + d_{n-k,k}, n \geq 2 \quad d_{1,1} = 0. \tag{3}
\]

This recurrence is the same as the one satisfied by the number of integer partitions of \( n \) whose largest part is \( k \), we then deduce the following:

**Corollary 5.1.** The number \( d_n \) of nom codes \( f \) with no fixed points other than 1 such that \( \sigma = \phi(f) \in \mathcal{S}_n(132) \) equals the number of integer partitions of \( n - 1 \).

**Remark 5.2.** The number \( d_{n,k} \) equals the number of permutations \( \sigma \in \mathcal{S}_n(132) \), where \( \sigma(n) < n \) and having \( k \) exceedances.

It is possible to define a direct bijection between the set \( \mathcal{D}_n \) and the set \( P_{n-1} \) of all integer partitions of \( n - 1 \). Let \( f \in \mathcal{D}_n \) and let \( \rho : \mathcal{D}_n \mapsto P_{n-1} \) be a map which associates \( f \) with an integer partition \( \rho(f) = \lambda \vdash n - 1 \) defined as follows:

Decompose \( f \) as in indicated Proposition 5.3: \( f = 1|H_0|H_1|\cdots|H_s \), and call \( k_i \) the size of \( H_i \). Then we set \( \rho(f) = \lambda = k_0k_1\cdots k_s \), this is clearly a partition of \( n - 1 \) with the largest part equal to \( k_0 \).

**Proposition 5.5.** The map \( \rho : \mathcal{D}_n \mapsto P_{n-1} \) is a bijection.

**Proof.** We will show that \( \rho \) can be inverted. Let \( \lambda = \lambda_1\lambda_2\cdots\lambda_l \) be a partition of \( n - 1 \). Let us define \( \rho^{-1}(\lambda) = f \) over \([n]\) as follows: First start with \( k = \lambda_1 \) plateaus of height 1, that is, set \( f_2 = \cdots = f_{k+1} = 1 \). Then for \( i \geq 2 \) insert \( \lambda_i \) consecutive integers starting from the integer \( 2 + \sum_{j=1}^{i-2} \lambda_j \), where \( \lambda_0 = 0 \).

**Example 5.1.** Let \( \lambda = 55533221 \). We have \( \lambda \vdash 26 \). Since \( k = \lambda_1 = 5 \) we have

\[
f = 1|1 1 1 1 1|2 3 4 5 6|7 8 9 10 11|12 13 14|15 16 17 18 19|20 21|22 23 24 25.
\]
The case \(213\).

We will prove that \(|\mathfrak{S}_n^\tau(213)| = |\mathfrak{S}_n^\tau(132)|, n \geq 1.\n
In order to do so we introduce an involution in the set \(F_n\), called “Flip”. This involution associates each \(f = f_1f_2\cdots f_n \in F_n\) with the nom code \(\text{Flip}(f) = f' = f'_1f'_2\cdots f'_n\) given as follows: \(f'_i = n + 1 - \sum_{k=1}^{n-i+1} r_k\), where \((r_1, r_2, \ldots, r_n)\) is the \(r\)-vector of \(f\).

Although non-decreasing nom codes can be seen as partitions of the integer \(\sum_{i=1}^n f_i\) and although this operation reminds the conjugation in the set of integer partitions, the two operations are different.

**Remark 5.3.** If \(f \in F_n\), then \(\text{Flip}(f)\) starts with \(n-f_n\) plateaus of 1.

**Lemma 5.3.** If the nom code \(f\) is obtained from \(f' \in F_{n-1}\) by concatenating \(j \geq f_{n-1}\), then \(\text{Flip}(f)\) is obtained from \(\text{Flip}(f')\) by adding 1 to the integers in the \(j-1\) largest positions and inserting a 1 at the beginning.

**Proof.** Let \(f = f_1f_2\cdots f_{n-1} \in F_{n-1}\) has \(r\)-vector \((r_1, r_2, \ldots, r_n)\), and \(j \geq f_{n-1}\). Then the \(r\)-vector of \(f'\) is obtained by replacing \(r_j\) by \(r_j + 1\) and appending a 0 at the end. So, we have \(\text{Flip}(f') = g'_1g'_2\cdots g'_n\), where \(g'_i = n+1-r_1-r_2-\cdots-r_{n-i+1}, i = n, n-1, \ldots, 1\) which is also the same as a nom code obtained from \(\text{Flip}(f)\).

**Theorem 5.2.** If \(f \in F_n\) is the nom code of the permutation \(\sigma\), then \(\text{Flip}(f)\) is the nom code of the permutation \(\tau = ((\sigma^{-1})^r)^c\), where \(r\) and \(c\) denote the operations of reverse and complement of a permutation respectively.

**Proof.** For any permutation \(\pi \in \mathfrak{S}_n\) one has \((\pi^c)^r = (\pi^r)^c = \psi_n \pi \psi_n\), where \(\psi_n\) is the permutation \(\begin{array}{cccc} 1 & 2 & \cdots & n \\
 & n & n-1 & \cdots & 1 \end{array}\), therefore the equality to prove becomes: \(\tau = \psi_n \sigma^{-1} \psi_n\). We prove the result by induction on \(n\). The result is trivially true for \(n = 1\). Suppose \(n > 1\) and that the result is true for \(n-1\). Let \(f = f' \cdot j\), where \(f'\) is the prefix of length \(n-1\) of \(f\) and \(j = f_n \geq f_{n-1} = f'_{n-1}\). Let \(\sigma'\) be the permutation whose nom code is \(f'\) and let \(\tau'\) be the permutation whose nom code is \(\text{Flip}(f')\). By induction hypotheses \(\tau' = \psi_{n-1} (\sigma')^{-1} \psi_{n-1}\) or equivalently (since \(\psi_{n-1}\) is an involution) \(\psi_{n-1} \tau' \psi_{n-1} = (\sigma')^{-1}\). On the other hand, by the insertion method, \(\sigma = \sigma' \cdot (n, j)\) and hence \(\sigma^{-1} = (n, j) \cdot (\sigma')^{-1}\), therefore the relation to be proven becomes:

\[
\tau = \psi_n (n, j) \psi_{n-1} \tau' \psi_{n-1} \psi_n.
\] (4)

By the definition of the bijection \(\phi\) we have: \(\tau' = \prod_{i=1}^{n-1}(i, f'_i)\), then by Lemma 5.3, \(\tau = \prod_{i=1}^{n-1}(i+1, f'_i) \cdot \prod_{i=n-j+1}^{n-1}(i+1, f'_i+1)\). By plugging these
values in Equation 4 we obtain:
\[
\prod_{i=1}^{n-j}(i+1,f'_i) \cdot \prod_{i=n-j+1}^{n-1}(i+1,f'_i + 1) = \psi_n(n,j) \psi_{n-1}\left(\prod_{i=1}^{n-1}(i,f'_i)\right) \psi_{n-1} \psi_n
\]

In the symmetric group, if a permutation \( \alpha \) is written as product of cycles and \( \beta \) is another permutations, the conjugate \( \beta \alpha \beta^{-1} \) can be computed by replacing every integer \( i \) with \( \beta(i) \) in the product of cycles giving \( \alpha \).

\[
\prod_{i=1}^{n-j}(i+1,f'_i) \cdot \prod_{i=n-j+1}^{n-1}(i+1,f'_i + 1) = \psi_n(n,j) \left(\prod_{i=1}^{n-1}(n-i,n-f'_i)\right) \psi_n = (1,n+1-j) \prod_{i=1}^{n-1}(i+1,f'_i + 1)
\]

by simplifying:

\[
\prod_{i=1}^{n-j}(i+1,f'_i) = (1,n+1-j) \prod_{i=1}^{n-j}(i+1,f'_i + 1)
\]

because of Remark 5.3 all the \( f'_i \) for \( i = 1, \ldots, n-j-1 \) are equal to 1.

\[
(2,1)(3,1) \cdots (n-j+1,1) = (1,n+1-j)(2,2)(3,2) \cdots (n-j+1,2)
\]

It is straightforward to check that these two permutations are equal (and equal to the cycle \( (1,2,\ldots,n-j+1) \)).

\[\square\]

**Corollary 5.2.** We have \(|\mathcal{S}_n^{(213)}| = |\mathcal{S}_n^{(132)}|, n \geq 1|.

**Proof.** A permutation \( \sigma \) contains the pattern 132 if and only if \(((\sigma^{-1})^c)^c\) contains the pattern 213, therefore \((\phi \circ \text{Flip} \circ \phi^{-1})\) is a bijection between \(\mathcal{S}_n^{(213)}\) and \(\mathcal{S}_n^{(132)}\).

\[\square\]

**The case 231.**

We do not have an explicit expression for the number of permutation of this class but we can provide a lower bound showing that it grows exponentially with \( n \).

Let us recursively define the set \( X_n \) of non-decreasing nom codes over \([n]\), where \( f \in X_n \) is obtained from \( f' \in X_{n-1} \) by appending \( n-1 \) or \( n \) at its
Proposition 5.6. If $f \in X_n$ then $\sigma = \phi(f) \in \mathcal{S}_n^\sigma(231)$.

Proof. Let $f \in X_n$ and $\sigma = \phi(f)$. Then $f$ is obtained from $f' \in X_{n-1}$ or $f' \in X_{n-3}$ by the operations defined above. Let $f \in X_n$ be obtained from $f' \in X_{n-1}$ by appending $n-1$ or $n$ at its end. Then $\sigma = \sigma' \cdot (n, n-1)$ or $\sigma = \sigma' \cdot (n, n)$, respectively, where $\sigma' = \phi(f') \in \mathcal{S}_{n-1}^\sigma(231)$. We now show that $\sigma \in \mathcal{S}_n^\sigma(231)$. The case where $\sigma = \sigma' \cdot (n, n)$ is obvious. Let $\sigma = \sigma' \cdot (n, n-1)$ and assume that $\sigma$ contains a pattern 231. That is, there are integers $i < j < k$ such that $\sigma(j) > \sigma(i) > \sigma(k)$. If $\sigma(j) \neq n$, then since $\sigma(n) = n - 1$ we have that $\sigma(i) = \sigma'(i), \sigma(j) = \sigma'(j), \sigma(k) = \sigma'(k)$, and the sub-word $\sigma'(i)\sigma'(j)\sigma'(k)$ forms a pattern 231 in $\sigma'$. But this is a contradiction.

If $\sigma(j) = n$, then since $\sigma'$ is obtained from $\sigma$ by replacing the integer $n$ by $n-1$, $\sigma'(j) = n - 1$ in $\sigma'$ which is the largest integer in $\sigma'$. This implies that $\sigma'$ contains a pattern 231, but this is also a contradiction. Therefore, $\sigma$ avoids the pattern 231. Now consider that $f$ is obtained from $f' \in X_{n-3}$ by appending the sub-word $(n-3)(n-3)(n-2)$ at its end. Then we have $\sigma = \sigma' \cdot (n-2, n-3)(n-1, n-3)(n, n-2)$ and $\sigma(n-2) = n-1, \sigma(n-1) = n-3$ and $\sigma(n) = n - 2$. If $\sigma$ has a 231 pattern, then we have the integers $i, j, k$ with $i < j < k \leq n - 3$. The operation of obtaining $\sigma'$ from $\sigma$ affects the pattern 231 only if $\sigma(j) = n$. In this case we have $\sigma'(j) = n - 3$ which is the largest integer in $\sigma'$. Thus, $\sigma'$ contains the 231 pattern and this is a contradiction. \qed

The case 312.

Let $Y_n = \{f = f_1f_2 \cdots f_n \in F_n^\sigma : f_i = i, \forall i \in \text{Im}(f)\}$. For instance, 111, 113, 122, 123 $\in Y_3$.

Proposition 5.7. A nom code $f \in Y_n$ if and only if $\sigma = \phi(f) \in \mathcal{S}_n^\sigma(312)$.

Proof. Assume that $f = f_1f_2 \cdots f_n \in Y_n$ and $\sigma = \phi(f)$. Let $\text{Im}(f) = \{l_1, l_2, \ldots, l_k\}$. Then $f_{l_i} = l_i$, $i \in \{1, 2, \ldots, k\}$. We know that $l_i$ is a fixed point of $f$ if and only if $l_i$ is the minimum of its cycle in $\sigma$. So all integers in the cycle of $l_i$ have nom equal to $l_i$, hence they have to appear in increasing order in the cycle (when we write the cycle with its minimum $l_i$ at the beginning). Furthermore, since $f$ is a non-decreasing nom code, all the integers in the
cycle of \( l_i \) form an integer interval, therefore this cycle is \((l_i, l_i+1, \ldots, l_{i+1}-1)\). Thus, if \( \sigma \) contains a 312-pattern, that is, there are \( i < j < k \) such that \( \sigma(i) > \sigma(k) > \sigma(j) \), then \( i, j, k, \sigma(i), \sigma(j), \sigma(k) \) must be in the same cycle, say \( C_r = (l_r, \sigma(l_r), \ldots) \), where \( l_r < \sigma(l_r) < \sigma^2(l_r) < \cdots \). But this a contradiction. Conversely, let \( \sigma \in S_n^\rightarrow(312) \) and \( f = \phi^{-1}(\sigma) = f_1 f_2 \cdots f_n \). Suppose that \( \text{Im}(f) = \{l_1, l_2, \ldots, l_k\} \) and \( l_j \) is the smallest image value of \( f \) such that \( f_{l_j} < l_j \). Let \( t \) and \( s \) be the positions of the leftmost and the rightmost occurrences of the value \( l_j \) in \( f \). Since \( l_{j-1} \in \text{Im}(f) \) is a fixed point it begins a cycle in \( \sigma \), say \( C \). Note that \( \text{nom}(l_j) = l_j \). Thus, \( C = (l_{j-1}, l_{j-1} + 1, \ldots, l_j - 1, \ldots, t, \ldots, s, l_j, \ldots, t - 1) \), where the integers between \( l_j - 1 \) and \( t \) (if any) are greater than \( t \). Thus, there is a 312 pattern in the positions \( l_j - 1, t - 1 \) and \( s \), but this is a contradiction.

**Corollary 5.3.** The number of 312-avoiding permutations in \( S_n^\rightarrow \) having exactly \( k \) cycles is equal to

\[
|\{f \in Y_n : \text{IMA}(f) = k\}| = \binom{n-1}{k-1}, 1 \leq k \leq n - 1, 1 \leq k \leq n - 1.
\]

**Proof.** For every nom code \( f = f_1 f_2 \cdots f_n \) we have \( f_1 = 1 \). Thus we choose \( k - 1 \) positions from the remaining \( n - 1 \) spaces in \( \binom{n-1}{k-1} \) so that we have \( f_i = i \) and we fill in the remaining positions in one way to get the required function \( f \). The number of cycles of the permutation coincides with \( \text{IMA}(f) \) because \( f_i = i \) if and only if \( i = \text{min}(C) \), where \( C \) is a cycle of \( \sigma = \phi(f) \).

**The case 321.**

In this case we can provide a lower bound that shows that the size of this class grows exponentially with respect to \( n \).

We note that a permutation \( \sigma \in S_n \) is 321-avoiding if and only if both the subword of exceedance letters and the subword of anti-exceedance letters are increasing. Therefore, a permutation \( \sigma \in S_n^\rightarrow \) is 321-avoiding if and only if the subword of exceedance letters is increasing. That is, if \( i_1 < i_2 \cdots < i_k \) is the increasing sequence of exceedances of \( \sigma \), then \( \sigma(i_1) < \sigma(i_2) < \cdots < \sigma(i_k) \). Obviously we obtain nom codes of permutations satisfying this property when we append \( n \) or \( n - 1 \) to the nom code of a permutation in \( S_n^{\rightarrow}(321) \), therefore \( |S_n^{\rightarrow}(321)| \geq 2^n \).

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