A simple unconditionally secure quantum bit commitment protocol via quantum teleportation*

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By using local quantum teleportation of a fixed state to one qubit of an entangled pair sent from the other party, it is shown how one party can commit a bit with only classical information as evidence that results in an unconditionally secure protocol. The well-known “impossibility proof” does not cover such protocols due to its different commitment and opening prescriptions, which necessitate actual quantum measurements among different possible systems that cannot be entangled as a consequence.

It is nearly universally accepted that unconditionally secure quantum bit commitment is impossible due to entanglement cheating. For a brief summary of the problem and further references to the literature, see Ref. 1. In this paper, a concise self-contained description of a protocol belonging to what we call Type 5 QBC protocols, is presented together with a security proof. Such protocols are based on two-way quantum and classical communications within the framework of ordinary quantum mechanics, without the use of additional constraints such as relativity or super-selection rules, and in which only classical information is committed as evidence. The preliminary protocol QBC5p is a two-stage protocol in which Babe sends Adam many maximally entangled pairs of qubits $H_{k1} \otimes H_{k2}$, $\ell \in \{1, \ldots, n\}$, on each of which a unitary transformation $U_{k\ell}$ unknown to Adam has been applied to $H_{k1}$. Adam picks one pair $H_{k1} \otimes H_{k2}$ and teleports to $H_{k1}$ one of two known orthogonal states in $H_3$ corresponding to the bit value $b = 0$ or 1, and commits the Bell-basis measurement result on $H_{k2} \otimes H_{k3}$ to Babe. He opens by sending $H_{k1}$ and the remaining $n - 1$ qubit pairs to Babe, who verifies by making the corresponding projection measurements. This protocol is $\epsilon$-concealing and can be extended to an $\epsilon$-binding one, QBC5, in a sequence. The reasons underlying the success of this protocol will be explained in this paper. But the drastic difference between this protocol and the ones covered by the “impossibility proof” (IP) should be evident. Aside from details, the main step of QBC5 outlined above is quite simple, as is the idea behind it which is the reverse of the usual formulation: measurement result committed and quantum state given at opening instead of state committed and measurement result given at opening. The teleportation prevents Adam from entangling the opening possibilities.

Consider first the case of a single qubit pair. Starting from a fixed openly known, say, $\Psi_{12} = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2)$, Babe applies a unitary $U_k$ known only to herself to $H_1$, and sends the resulting $H_1 \otimes H_2$ in state $\Psi_k \equiv (U_k \otimes I)\Psi_{12}$ to Adam. If Adam measures the Bell basis

$$\Psi^\pm \equiv \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle),$$

$$\Phi^\pm \equiv \frac{1}{\sqrt{2}}(|\uparrow\rangle |\uparrow\rangle \pm |\downarrow\rangle |\downarrow\rangle)$$

on $H_2 \otimes H_3$ for any state $|\phi\rangle$ on $H_3$, he would obtain each result $i \in \{1, 2, 3, 4\}$, with probability $\frac{1}{4}$ corresponding to $\Psi^\pm$ and $\Phi^\pm$, with resulting state in $H_1$ given by:

$$U_k \sigma_i |\phi\rangle$$

where $\sigma_i$ are given by $\sigma_1 = -\sigma_z, \sigma_2 = -\sigma_0 = -I, \sigma_3 = -i\sigma_y, \sigma_4 = \sigma_x$ in terms of the Pauli spin operators. Let $\{|0\rangle, |1\rangle\}$ be an openly known orthonormal basis of a qubit. To commit $b = 0$ or 1, Adam may use $|\phi\rangle = |b\rangle$ on $H_3$ and obtains $U_k \sigma_i |\phi\rangle$ on $H_1$ from 1, announcing $b$ to Babe as the committed evidence. He opens by claiming $b$ and sending $H_1$ to Babe, who verifies by measuring the one-dimensional projection of the state $U_k \sigma_i |\phi\rangle$. It is evident that the protocol is perfectly concealing if she does not entangle the possible $U_k$: with orthogonal states $|i\rangle$ representing the classical information committed by Adam, $p_B^R = |\langle i | i\rangle|^2$. Thus, her conditional probabilities for optimally estimating the bit $b$ on the basis of the committed evidence $i$ are $p(b|i) = \frac{1}{2}$ for both $b = 0, 1$.

For the corresponding binding proof of the situation in which Adam opens one bit value perfectly, say opening $b = 0$ with probability 1, we will first show that Adam cannot cheat perfectly, i.e., his optimal cheating probability $P^A_c$ of opening $b = 1$ instead is bounded away from 1. This can be seen by exhausting all of his possible actions. First note that there is no entanglement possibility for him either over the single state $|b\rangle$, or over the classical information $i$ even if it is represented by $|i\rangle$, as only a specific $i$ is accepted as legitimate. In order to cheat perfectly, Adam has several courses of action left. The first one is for him to teleport $|0\rangle$ to $U_k \sigma_i |0\rangle$ and then apply

*NOTE: This paper and the following two quant-ph/0305143 and quant-ph/0305144 together provide a detailed description of various gaps in the QBC “impossibility proof,” as well as security proofs for four different protocols, QBC1, QBC2, QBC4, and QBC5. They also explain and correct some of the claims on my previous QBC protocols. This v3 on QBC5 is a minor improvement over v2.

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some $V^A$ to try changing it to $U_k\sigma_i|1\rangle$. Another is for him to announce some $i$ without a Bell measurement and perform the teleportation on some $|\phi\rangle \in H_4$ with result $j$ when he opens $b$, applying some $V^A$ to $H_1$ to obtain the proper opening state. The third is just sending some $|\phi\rangle$ back to Babe and try to open correctly. Finally, he may try to determine $U_k$. In the first case, his probability of successful cheating is given by, for a given $i$,

$$P_c^A(1) = \sum_k \lambda_k |\langle 1|\sigma_i^A U_k^\dagger V^A U_k \sigma_i|0\rangle|^2$$

(2)

where $\lambda_k$ is probability that $U_k$ was employed by Babe. To obtain $P_c^A = 1$, one would need each of the terms in the sum of (2) to be one, which is impossible if $\{U_k\}$ is properly chosen, in particular if it is $\{I, R_x, R_y, R_z\}$ where $R_x$ is the rotation about the $x$-axis of the qubit by an angle $\pi/2$ in its Bloch-sphere representation, etc.,

$$\{U_k\} = \{I, R_x, R_y, R_z\}.$$  

(3)

This is due to the fact that for each $U_k$, there are only one possible rotations $V^A$ that makes $|\langle 1|\sigma_i^A U_k^\dagger V^A U_k \sigma_i|0\rangle|^2 = 1$ and there is no common rotation that works for all $k$. In the second situation the successful cheating probability is, for a given pair $\{i, j\}$ and a given $b$,

$$P_c^A(2) = \sum_k \lambda_k |\langle b|\sigma_i^A U_k^\dagger V^A U_k \sigma_j|\phi\rangle|^2.$$  

(4)

Again, equation (4) cannot be made equal to 1 if

$$P_c^A(3) = \sum_k \lambda_k |\langle 1|\sigma_i^A U_k^\dagger V^A |\phi\rangle|^2,$$  

(5)

which clearly cannot be 1 as $V^A|\phi\rangle$ is independent of $k$. Note that the last two courses of action actually do not allow him to open $b = 0$ perfectly already. Also, other $\{U_k\}$ can be used in lieu of (3).

Note the role of the classical evidence $i$ and the role of Babe’s application of $U_k$ in preventing Adam’s perfect cheating. Note also that Adam has no entanglement cheating, in contrast to the IP formulation. To summarize, Adam’s optimum cheating probability is some fixed number $P_c^A = p_A < 1$ for perfect $b=0$ opening. The exact value of $p_A$ depends on the set $\{U_k\}$ and its probability distribution, and can be determined by solving the optimization problems corresponding to his possible actions described above. But there is no need to determine this value for the purpose of proving the possibility of obtaining unconditionally secure protocols.

Babe may entangle over the possible $U_k$ so that she keeps $H^C$ and sends $H_1 \otimes H_2$ for $|\Psi\rangle \in H^C \otimes H_1 \otimes H_2$ with $|f_k\rangle$ orthonormal in $H^C$,

$$|\Psi\rangle = \sum_k \sqrt{\lambda_k} |f_k\rangle (U_k \otimes I)|\Psi^-_{12}\rangle.$$  

(6)

In this situation, one can cast the protocol in the IP formulation for a fixed Bell-measurement result $i$ known to both parties by combining (1) and (6), and the protocol is not perfectly concealing. If Adam commits only a partial Bell-measurement result, the protocol can still be cast in the IP formulation if he only makes the partial (degenerate) Bell-measurement, and not the full one, and then randomizes. To obtain $\epsilon$-cheating, one can employ the following strategy — Babe sends Adam an ordered sequence of $n$ qubit pairs $\{H_1 \otimes H_2\}$, each entangled in the form (4) with $\lambda_k = \frac{1}{2}$, $|f_k\rangle \in H^C$, and $(U_k)\}$ given by (3). Adam randomly picks one of these pairs, $H_{12} \otimes H_{12}$, and performs the quantum teleportation. He commits the result $i$ of the Bell-measurement as evidence, but not the name of the pair or $H_{12}$ itself. He opens by sending in $H_{12}$ and all the other $n-1$ qubit pairs. Babe verifies by measuring the projections onto $U_k \sigma_i |b\rangle$ on $H_{12}$ (or the projection onto (6) in $H^C_{12} \otimes H_{12}$), and $(U_k \otimes I)|\Psi_{12}\rangle$ on the other pairs.

If Babe entangles pair by pair, it is easily seen that this preliminary protocol QBC5p is $\epsilon$-concealing, as Babe has probability $\frac{1}{n}$ of guessing the correct $H^C_{12}$ for entanglement cheating to obtain a cheating probability $P_{cB} < \frac{1}{2}$. If she does not guess correctly, her cheating probability is $\frac{1}{2}$ as shown above. Thus, her optimum $P_{cB} \leq \frac{1}{2} + \frac{1}{2\sqrt{n}}$. If Babe entangles over all the pairs, a calculation similar to that of Section IV in Ref. 3 shows that, conditioned on any of Babe’s measurement results on her ancilla, the same bound on $\bar{P}_{cB}$ applies. This also follows from the fact that pair-by-pair entanglement is entirely general in the present case. See also Appendix A of this paper. At the same time, Adam cannot cheat perfectly by operation on $H_{12}$ as above, and he cannot entangle the different possibilities of using any of the $n$ qubit pairs to obtain $i$ — due to the verification procedure he could use only one qubit pair for teleportation. One cannot entangle the different possibilities involving measurements and no-measurements on different state spaces. A classical measurement result is to be obtained, by Babe and thus on Adam’s qubits too, even if Adam does not perform the measurement himself and just sends her a quantum state representing the result. The impossibility proof assumes that all opening possibilities can be purified in an entanglement without a proof that it is true in all possible situations. The above protocol provides an example for which such an assumption is not valid.

Thus far the “impossibility proof” has already been contradicted, since it asserts that whenever the protocol is $\epsilon$-concealing, i.e., Babe’s optimum cheating probability is $P_{c} \sim \frac{1}{2}$, then $P_{c} \sim 1$ even in the situation when he opens $b=0$ perfectly. If the $b=0$ perfect opening condition is relaxed, Adam has more possible actions open to him. Exhaustively, they can be described by the following and their combinations: approximate cloning of $|\Psi_k\rangle$, using different states than $|b\rangle$ on $H_2$, announcing a different $i$ from his Bell measurement, opening by sending a different qubit from $H_1$ to Babe, and simply announcing a different bit value at opening. Two of these are already included in the above analysis. By continuity of all relevant functions on a finite-dimensional space, it
is readily seen that if Adam opens on \( b = 0 \) with probability \( P_A(0) = 1 - \delta_1 \), then he can cheat no better than \( P^A_e = p_A + \delta_2 \) with \( \lim_{\delta_1 \to 0} \delta_2 = 0 \). While the optimum \( P^A_e \) for arbitrary \( P_A(0) \) remains to be determined, there is no need to include it here as there does not seem to be much interest in protocols where both \( P_A(0) \) and \( P_A(1) \) are not close to 1. In any event, the optimum tradeoff between \( P_A(0) \) and \( P_A(1) \) may be determined and the average probability of opening correctly to Adam’s choice can be brought down to arbitrarily small level similar to the following \( P_A(0) = 1 \) case.

Protocol QBC5’ is obtained when the above QBC5p is repeated in a sequence of \( N \) such \( n \)-pairs, each consisting of the above protocol step with the same \( b \) opening. Adam’s cheating probability \( P^A_e = P^A_n \) can be made arbitrarily small for large \( N \). Babe’s entanglement of the \( Nn \) pairs clearly does not help her cheat as she is left with \( N \) repetitions, but it can be made arbitrarily small for arbitrarily large \( N \) of Ref. [4]. The key point is that Adam’s optimal cheating probability of successfully sneaking in a non-allowed state can be made arbitrarily small for any fixed \( N \) by making \( n \) large. The quantitative treatment of a similar situation has been given in Section VI of Ref. [4]. The key point is that Adam’s optimal cheating probability does not keep increasing with \( N \), so that arbitrarily large \( N \) can be used to reduce \( P^B_e \) to any level without corresponding increase in \( P^A_e \). Thus, the protocol QBC5 is \( \epsilon \)-concealing and \( \epsilon \)-binding for any desired \( \epsilon > 0 \).

So far we have allowed Adam to cheat during both commitment and opening but assumed that Babe is honest in sending Adam the “legal” states. Honesty during commitment in a multi-stage protocol is an assumption of the “impossibility proof,” which has to be relaxed for a truly secure protocol. In QBC5, Babe can cheat by, e.g., sending \( |\psi_k\rangle|\psi_k\rangle \), an unentangled state in \( H_1 \otimes H_2 \), so that the Bell measurement probability on \( |\psi_k\rangle|b\rangle \) in \( H_2 \otimes H_3 \) depends on \( b \). This kind of cheating, sending in a different state other than one allowed in a multi-stage protocol, can be handled in two different ways in general that also apply effectively to the “impossibility proof” formulation for certain protocols. The first way is to let one party send in a large number of the allowed states so that the other party can check for honesty on most of them and then use the rest. In the present case, Babe would send Adam a large number \( M \) of pairs, so Adam can set aside \( m \ll M \) of them, and asks Babe for the exact state in the other \( M - m \) pairs. With or without her entanglement over such pairs, Babe has to tell exactly what each state is, which Adam can verify by a projection measurement. With \( M \) sufficiently large, it may be shown that the probability that any of the \( m \) remaining states is not a legal one allowed by the protocol, up to entanglement by Babe, given all \( M - m \) are, can be made arbitrarily small for any \( M \). The detailed analysis of this approach for the present problem is given in Appendix A of this paper.

What if Babe is found cheating during such testing? Clearly, one party can always refuse to cooperate in any protocol, which is what repeated cheating and getting caught amounts to. It does not alter the fact that the protocol allows arbitrarily small cheating probability. One meaningful way to deal with repeated cheating applicable to realistic environments is to allow a fixed number \( n_c \) of cheating detection, beyond which the cheating party is taken to be the loser in an essentially classical game-theoretic formulation. The above \( N \)-ensemble can be generalized to deal with such a formulation, but it is conceptually simpler to have the following explicit game-theoretic formulation, our second way to handle the use of non-allowed states during commitment. In such a formulation, there is no \( N \)-ensemble sent simultaneously. Only one pair is sent, but the other party has a large probability of choosing to check the validity of that state instead of going further with the protocol. During the successive trials, the other party may decide at any point to stop checking and accept the state. It is clear that with potentially an unlimited number of trials, the probability of successfully sneaking in a non-allowed state can be made arbitrarily small for any fixed \( n_c \). With \( n_c \) set to be zero, essentially no cheating can be attempted at all without undue risk of losing the game. A quantitative description of this approach is given in Appendix B of this paper. Note that neither of these approaches is needed if one adopts the IP assumption that Babe is honest in submitting only legal states of the protocol to Adam.

With either of these two ways, one can safely be assured that the parties are honest during the exchanges of states or quantum communications in the commitment phase of a multi-stage protocol, as was assumed in the “impossibility proof” formulation. Under this condition, we have completed the concealing and binding proof of the following simplification of the above protocol QBC5’.

| PROTOCOL QBC5 |
|----------------|
| (i) Babe sends Adam \( nN \) modified singlet pairs \( \{U_{ik}\otimes I\}\Psi_{12}^i, \Psi_{12}^i \in H_{i1} \otimes H_{i2}, i \in \{1, \ldots, nN\} \) by their positions, with each \( U_{ik} \), \( k \in \{1, 2, 3, 4\} \), randomly drawn from \( \{U_{ik}\} = \{I, R_x, R_y, R_z\} \), the \( R \)'s being \( \pi/2 \) rotations about the qubit axes. |
| (ii) Adam teleports the state \(|b\rangle\) for the bit \( b \) he wants to commit to \( n \) randomly selected pairs among the \( nN \) ones, committing to Babe the Bell-measurement results \( i_m, m \in \{1, \ldots, n\} \) without the corresponding names of the pairs. |
| (iii) Adam opens by sending \( \{H_m\} \) and all the other pairs together with their names to Babe, who verifies by corresponding measurements. |

To recapitulate the main reason for the security of QBC5: one cannot purify the different possibilities of measuring different components of a tensor product space.
while leaving the remaining part of the space untouched. In QBC5, the tensor product space is $\otimes_{\ell} (\mathcal{H}_{\ell_1} \otimes \mathcal{H}_{\ell_2})$ and the different components are $\mathcal{H}_{\ell_1} \otimes \mathcal{H}_{\ell_2}$ indexed by $\ell$. Adam cannot entangle the different possible actual measurements, and if he does not actually measure, he could not open $b = 0$ perfectly corresponding to the committed $i$, which Babe could measure for him anyway in case he just sends her the register containing the information $i$.

In addition to teleportation, it is shown in Ref. 5 that the use of a split entangled pair can yield an unconditionally secure protocol. A general discussion on the scope of the “impossibility proof” is given in Ref. [2] together with the security proofs of several other protocols.

APPENDIX A: HONESTY GUARANTEE FROM AN ENSEMBLE

One general approach to guarantee with probability arbitrarily close to 1 that a party $B$ is sending a “legal” state from $\{\psi_k\}$ allowed by the protocol, or at least sending the entangled superposition

$$|\Psi\rangle = \sum_k \lambda_k |\psi_k\rangle_B |f_k\rangle_C$$

(A1)

for orthonormal $|f_k\rangle_C \in \mathcal{H}_C$, is the following. She sends in $N$ states, each randomly drawn independently from the given allowed set $\{\psi_k\}$ and named, say, by its temporal position. The other party $A$ randomly picks $N - n$ of such states and asks $B$ to reveal them. After verifying that they are correct, the probability that all $n$ remaining states are at least of the form $|\psi_k\rangle$ can be made arbitrarily close to 1 by proper choice of $n$, $N$ as follows.

Suppose $B$ mixes in states $\psi'$ that allows her to cheat with probability $P_c - \frac{1}{2} \geq \epsilon$ for a given $\epsilon$. Then the cheating detection probability $\delta = 1 - |\langle \psi'|\psi_k\rangle|^2 > 0$ minimized over the choice of $k$ and $\psi'$ is a fixed number dependent only on $\epsilon$ and the protocol, independent of $n$ and $N$. Suppose $B$ mixes in $m$ such states $\psi'$ out of the $N$ states, $A$ sends to $B$. We grant that $B$’s cheating is successful if there is just one copy of $\psi'$ in the $n$-group untested by $A$ and the measurements by him reveals no different states from $\{\psi_k\}$. In order that there is a nonvanishing probability found in the random $n$-group that $A$ sets aside, $m/N$ must be non-vanishing with $m/N \to p$ in the limit $N \to \infty$. Let $\alpha = m/N$ so that asymptotically large $N, \alpha$ is the fraction of states among the $N$ set aside. In order for $B$ to be able to cheat, one has $m\alpha \geq 1$ for large $N$ because $m\alpha$ is the average number of $\psi'$ in the $n$-group. The probability that the cheating detection fails in the $N - n$ group is then

$$(1 - \delta)^{m(1 - \alpha)} \leq (1 - \delta)^{\frac{1 - \alpha}{\alpha}} \to 0, \quad (A2)$$

which can be made arbitrarily small by having $\alpha$ arbitrarily small that obtains with $N \to \infty$ for fixed $n$. This argument can be completely quantified via the hypergeometric distribution and the Chernov bound without passing to the limit $N \to \infty$, although the limiting argument suffices for the present purpose.

In QBC4, a simple modification is needed, in which Adam would need Babe to send him $\mathcal{H}_B^i$ together with the exact $\{|f_k\rangle\}$ for checking that the states are indeed $|\psi_k\rangle/|\psi_k\rangle$ of the form $[\rangle$.

When the number of states $\{\psi_k\}$ is equal to or larger than the dimension of the state space $\mathcal{H}_B$ sent to Adam, an entanglement of the form $[\rangle$ for each member of the ensemble is entirely general. Any other entanglement can be obtained by local transformation on $\mathcal{H}_C$. This situation covers all our applications.

The use of this method suffers from two disadvantages compared to the next one in Appendix B via game-theoretic checking, but has the advantage that no game payoffs need to be imposed. The first disadvantage is that one needs to make sure that the evaluation of $P_c$ from Babe’s ensemble takes into account the possibility, depending on the $|\psi_k\rangle$’s, that Adam may be able to determine it exactly with a nonzero probability. Of course, since he usually has to return the unused one for Babe’s verification, such a strategy may not improve his $P_c$. In our qubit formulation such possibility indeed does not arise.

The second disadvantage is that this approach gives Adam an ensemble of choice for each individual $|\psi_k\rangle$ he is going to operate upon, thus the opportunity of entanglement. Thus, one has to deal with such entanglement possibility explicitly. This possibility does not exist in QBC5, which is indeed the reason why it is secure as elaborated in this paper. It does not exist in QBC4 because it is perfectly concealing, but it does in our QBC1 and QBC2 which are dealt with in Ref. [2].

APPENDIX B: HONESTY GUARANTEE IN A GAME-THEORETIC FORMULATION

During the commitment phase of a multi-stage QBC protocol involving exchanges of quantum states, a party can try to cheat by using “illegal” states not allowed in the protocol. In the “impossibility proof”, this problem is not tackled by assuming each party is honest, which is an unreasonable assumption in a protocol that precisely does not place trust in either party. It turns out, however, that a simple classical game formulation can take care of this problem, up to entanglement of legal states, in the following way.

Suppose $B$ is giving $A$ a quantum system in possible allowable states $\{\psi_k\}$, to be further processed by $A$. To make sure that the state is legal with a probability arbitrarily close to 1, $A$ can ask $B$ to reveal the state and check it by corresponding projection measurement. The cheating detection probability is denoted by $p_d$, which is determined by the illegal states $\psi'$ used by Babe that would allow her to cheat beyond $\epsilon$, $P_c^B - \frac{1}{2} \geq \epsilon$. After checking, the whole protocol would need to be repeated up to that point, but there is no question of resource or
efficiency in the present context.

Assuming first that the allowable number of cheats is \( n_c = 0 \), i.e., if \( B \) is found cheating the game is over and \( B \) will be declared the loser. Such a situation occurs, e.g., when the penalty of being found cheating is arbitrarily large. We let \( A \) employ a randomized strategy with a probability \( p_a \) of accepting \( B \)'s state without checking, independently from trial to trial without loss of generality in this case. Thus \( B \)'s strategy can also be represented by a probability \( p_c \) of cheating at each trial. In an indefinitely long sequence of trials, \( B \)'s successful cheating probability \( P_C \) can be found as follows. The probability \( B \) will succeed at the first trial is \( p_c p_a \), at the second is \( p_c p_a(1 - p_a - p_c p_d) \), and at the \( n \)th is \( p_c p_a(1 - p_a - p_c p_d)^{n-1} \). Thus, the total successful cheating probability after \( n \) trials is

\[
P_C(n) = \frac{p_a}{p_a + p_d} [1 - (1 - p_c p_a - p_c p_d)^n]. \tag{B1}
\]

Similarly, the probability that a legal state is accepted after \( n \) trials is

\[
P_A(n) = \frac{p_a(1 - p_c)}{p_a(1 - p_c) + p_c p_d} \times [1 - (1 - p_a + p_a p_c - p_c p_d)^n], \tag{B2}
\]

and the probability a cheating would be detected is

\[
P_D(n) = \frac{p_c p_d}{p_a + p_c p_d} [1 - (1 - p_a - p_c p_d)^n]. \tag{B3}
\]

Observing that for fixed \( p_d > 0 \), \( P_D(n) \to 0 \) is equivalent to \( p_c \to 0 \) from (B3). We have

\[
\lim_{n \to \infty} P_C(n) = 0, \tag{B4}
\]

\[
\lim_{n \to \infty} P_A(n) = 1. \tag{B5}
\]

With a large penalty for cheating, \( P_D \) would be driven to zero from minimizing the average penalty. Hence \( p_c \) and \( P_C \) are also driven to zero, as expected, with \( P_A \to 1 \). Even if no penalty is imposed, \( P_C(n) \) may be made arbitrarily small for any \( p_c > 0 \) by making \( n_l \) large and \( p_a/p_d \) small from (B1), while the corresponding \( P_A(n) \) needs not be close to 1. Since a party has to cooperate and accept a protocol if his/her security is guaranteed, as formalized by the “Intent Principle” of ref. [1] on protocol agreement, \( B \) must pick a small \( p_c \) to accept a viable protocol in this situation.

With \( n_c > 0 \) but finite, the same situation as above arises after \( n_c - 1 \) detections of cheating, which would occur with probability arbitrarily close to 1 for a long enough number \( n' \) of trials. It is also possible for \( A \) to termi- nate the game at any point by accepting the state after a fixed number of trials \( n_a \). Even though a more complicated optimal strategy that is trial stage-dependent for either party would then emerge, it would go over to the above for large \( n \).

If \( \{\psi_k\} \) is the set of allowable states, this procedure only assures that the state \( \psi \in \mathcal{H}^B \otimes \mathcal{H}^C \) that \( B \) sends in is of the form

\[
|\psi\rangle = \sum_k \lambda_k |\psi_k\rangle_B |f_k\rangle_C \tag{B6}
\]

for orthonormal \( |f_k\rangle \in \mathcal{H}^C \). Often \( B \) cannot cheat already in such a case as the Type 5 ones in this paper, although in general \( B \) may still be able to cheat using (B6). In some situations, as in QBC4 of Ref. [5], one party may be asked to entangle as in (B6) and later send in \( \mathcal{H}^C \) for the other party to verify the total entangled state.

In the full unconditionally secure protocol that requiring a sequence, the above game can be repeated successively one by one. The levels of each probability (B1)-(B5) can be adjusted to accommodate the desired level on the overall probabilities for the \( n \)-sequence.

The generality of the entanglement (B6) for our protocols, as well as advantages and disadvantages of this game-theoretic approach versus the ensemble approach, are briefly discussed in the above Appendix A.

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