Enhanced entanglement criterion via symmetric informationally complete measurements

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We show that a special type of measurements, called symmetric informationally complete positive operator-valued measures (SIC POVMs), provide a stronger entanglement detection criterion than the computable cross-norm or realignment criterion based on local orthogonal observables. As an illustration, we demonstrate the enhanced entanglement detection power in simple systems of qubit and qutrit pairs. This observation highlights the significance of SIC POVMs for entanglement detection.

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I. INTRODUCTION

Entanglement is one of the most distinctive features of quantum theory as compared to classical theory, which is also considered to be a useful resource for tasks like quantum communication, quantum cryptography, and quantum metrology. Thus, developing simple and efficient criteria for the detection of entanglement in quantum states is indeed pivotal. However, it has long been proven that entanglement detection is an NP-hard problem as the system size increases [3][4].

A pure state \( |\psi\rangle \) of two particles is called separable, if it is a product state \( |\psi\rangle = |u\rangle \otimes |v\rangle \), otherwise it is entangled. More generally, a bipartite mixed state is separable if it can be written as a convex combination of pure product states,

\[
\rho = \sum_i p_i |u_i\rangle \langle u_i| \otimes |v_i\rangle \langle v_i| ,
\]

where the \( p_i \)'s form a probability distribution, so they are positive and sum up to one. A state that cannot be written in the above form is called entangled.

Many criteria on entanglement detection have been developed. Most of them, however, provide only sufficient conditions for the detection. A well-known example is the positive partial transpose (PPT) criterion [5][6], which is necessary and sufficient for qubit-qubit and qubit-qutrit systems only. For higher dimensions, there exist the so-called bound entangled states which are PPT and nondistillable. It is still an open question whether the PPT criterion completely characterizes bound entanglement, namely, whether all bound entangled states are PPT.

Another popular criterion is the computable cross-norm or realignment criterion [7][8], often acronymed as CCNR. Notably, the CCNR criterion is able to detect the entanglement of many states where the PPT criterion fails. On the other hand, there also exist some states which are detected by the PPT criterion, but cannot be detected by CCNR [10]. Therefore, we should not assess one as either stronger or weaker than the other criterion, but rather complementing each other. There are also nonlinear extensions of the CCNR criterion using, for instance, the local uncertainty relations or covariance matrices [11][12], and other feasible methods [13]. Nevertheless, here we will not consider such extensions.

Empirically, the CCNR criterion can be evaluated by measuring the correlations between local orthogonal observables of two parties. Here, we propose an analogous yet more efficient entanglement detection criterion. Instead of using a set of local orthogonal observables, we use a single generalized measurement, known as the symmetric informationally complete positive operator-valued measure (SIC POVM), for each party. Our criterion is determined by the correlations between POVM elements of the two SIC POVMs for the two parties and shares the simplicity of the CCNR criterion. In addition, it has at least two notable advantages, which are connected to the properties of SIC POVMs. First, this criterion can detect many entangled states (including bound entangled states) that cannot be detected by the CCNR criterion. Second, all correlations featured in the criterion can be

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measured in one go instead of one by one as in the CCNR criterion. These advantages are expected to have both theoretical and experimental interests.

Incidentally, Refs. [15–16] used a different approach, namely, the sum of the correlation entries, to derive separability criteria with SIC POVMs. Very recently, Bae et al. [17] studied entanglement detection via quantum 2-designs, which include SIC POVMs as a special example; however, their entanglement criteria are not so closely related to the properties of 2-designs except for the tomographic completeness. Ref. [16] investigated the entanglement properties of multipartite systems with tight informationally complete measurements, including SIC POVMs. In addition, Ref. [19] considered a nonlinear entanglement criterion based on SIC POVMs, which turns out to be equivalent to the criterion using observables in Refs. [10, 14].

This paper is organized as follows. We will first recall the CCNR criterion in Sec. II and the basic properties of SIC POVMs in Sec. III. In Sec. IV we derive the entanglement criterion based on SIC POVMs. Section V demonstrates the superiority of the criterion with various examples, then we close with a few remarks.

II. THE CCNR CRITERION

The CCNR criterion can be formulated in different forms. One approach makes use of the Schmidt decomposition of the quantum state in operator space. According to the Schmidt theorem, any bipartite density matrix \( \rho \) on \( \mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B \) with dimension \( d = d_A \times d_B \) (assuming \( d_A \leq d_B \)) can be written as

\[
\rho = \sum_{k=1}^{d^2} \lambda_k G^A_k \otimes G^B_k, \quad \lambda_k = \langle G^A_k \otimes G^B_k \rangle. \tag{2}
\]

Here \( \lambda_k \geq 0 \) are the Schmidt coefficients, \( \{G^A_k\} \) is an orthonormal basis of Hermitian operators on \( \mathcal{H}^A \), and \( \{G^B_k\} \) is a set of orthonormal Hermitian operators on \( \mathcal{H}^B \), that is, \( \text{tr}(G^A_k G^A_l) = \text{tr}(G^B_k G^B_l) = \delta_{kl} \). In terms of the Schmidt decomposition, the CCNR criterion can be stated as follows.

**Proposition 1 (CCNR).** If a state \( \rho \) is separable and has Schmidt decomposition as in Eq. (2), then

\[
\sum_k \lambda_k \leq 1 \tag{3}
\]

has to hold; otherwise, it is entangled.

**Proof.** See, for instance, the proof in Refs. [17, 18]. \( \square \)

Incidentally, the CCNR criterion is closely related to the linear entanglement witness \( W = \mathbb{1} - \sum_k G^A_k \otimes G^B_k \); see Ref. [20] for another form.

Besides the Schmidt form, the state \( \rho \) can also be decomposed with arbitrary local orthonormal operator bases, i.e., \( \rho = \sum_{ij} c_{ij} A_i \otimes B_j \), where \( \{A_i\}, \{B_j\} \) forms an orthonormal basis for the space of linear operators on \( \mathcal{H}^A \) (\( \mathcal{H}^B \)). It turns out the trace norm of the correlation matrix \( [C]_{ij} = \langle A_i \otimes B_j \rangle = c_{ij} \) is equal to the sum of the Schmidt coefficients,

\[
\|C\|_{tr} = \sum_k \lambda_k, \tag{4}
\]

where \( \|C\|_{tr} \equiv \text{tr}\left(\sqrt{CC^\dagger}\right) \) denotes the trace norm. This equality follows from the fact that the Schmidt coefficients \( \lambda_k \) happen to be the singular values of \( C \).

In terms of the correlation matrix, the CCNR criterion can be formulated as follows: the trace norm of the correlation matrix of any separable state is upper bounded by one, that is,

\[
\|C\|_{tr} \leq 1. \tag{5}
\]

This conclusion can be proved directly as follows. For any given product state \( \rho = \rho_A \otimes \rho_B \), the matrix elements of the correlation matrix have the form \( [C]_{ij} = \langle A_i \rangle \langle B_j \rangle = a_i b_j \), where \( a_i = \langle A_i \rangle \) and \( b_j = \langle B_j \rangle \). So the correlation matrix can be expressed as an outer product \( C \equiv [a](b) \), where the curly ket \( [a] \equiv (a_1, a_2, \ldots)^T \) denotes the column vector composed of \( a \)'s, while the curly bra \( (b) \equiv (b_1, b_2, \ldots) \) denotes the row vector composed of \( b \)'s. In addition, we have

\[
\|C\|_{tr} = \langle a | a \rangle^{1/2} \langle b | b \rangle^{1/2} = \sqrt{\sum_i a_i^2 \sum_j b_j^2} = \sqrt{\text{tr}(\rho_A^2) \text{tr}(\rho_B^2)} \leq 1. \tag{6}
\]

By employing the convexity property of the trace norm under mixing, we conclude that \( \|C\|_{tr} \leq 1 \) for any separable state.

III. SIC POVM

In a \( d \)-dimensional Hilbert space, a SIC POVM \( \mathcal{M}_4 = \{\Pi_k\}_{k=1}^{d^2} \) is composed of \( d^2 \) outcomes \( \Pi_k = |\psi_k\rangle \langle \psi_k|/d \) which are subnormalized rank-1 projectors onto pure states, with equal pairwise fidelity, that is,

\[
|\langle \psi_i | \psi_j \rangle|^2 = \frac{d \delta_{ij} + 1}{d + 1}, \quad i, j = 1, 2, \ldots, d^2. \tag{7}
\]

It is not difficult to verify the completeness condition, that is, \( \sum_{k=1}^{d^2} \Pi_k = \mathbb{1} \). It has been conjectured that SIC POVMs exist in all finite dimensions [21], although a general proof is still missing. So far analytical solutions have been found in dimensions \( d = 2, 4, 8, 24, 28, 30, 31, 35, 37, 39, 43, 48, 124 \), and numerical solutions with high precision have been found up to dimension \( d = 151 \); see Ref. [22] for a recent review. Experimentally, SIC POVMs in low dimensions have been realized in various quantum systems [23–25].
FIG. 1. (a) Tomographically complete single measurement setting $\mathcal{M}_\text{s}$ with $d^2$ outcomes. (b) Employing this type of measurement setting for each party of a bipartite system, we have a new paradigm for entanglement detection. Unlike that for CCNR, only a single measurement setting for each party is involved, which may help to reduce the experimental complexity.

Given a SIC POVM $\mathcal{M}_\text{s} = \{\Pi_k\}_{k=1}^{d^2}$ and a quantum state $\rho$, the probability of obtaining outcome $k$ is given by the Born rule, $p_k = (\Pi_k) = \text{tr}(\rho \Pi_k)$. Conversely, the quantum state $\rho$ can be reconstructed from these probabilities as follows,

$$\rho = \sum_{k=1}^{d^2} [d(d+1)p_k - 1] \Pi_k$$

$$= d(d+1) \sum_{k=1}^{d^2} p_k \Pi_k - \mathbb{I},$$

(8)

see Fig. 1(a). Calculation shows that

$$\sum_{k=1}^{d^2} p_k^2 = \frac{1 + \text{tr}(\rho^2)}{d(d+1)} \leq \frac{2}{d(d+1)},$$

(9)

where the upper bound is saturated iff $\rho$ is pure.

For the convenience of later applications, we need to renormalize the elements in the SIC POVM,

$$E_k \equiv \sqrt{\frac{d(d+1)}{2}} \Pi_k = \sqrt{\frac{d+1}{2d}} |\psi_k\rangle\langle\psi_k|.$$  

(10)

Let

$$e_k = \text{tr}(\rho E_k) = \sqrt{\frac{d(d+1)}{2}} p_k;$$

(11)

then the constraint in Eq. (9) can be formulated as follows,

$$(e|e) = \sum_{k=1}^{d^2} e_k^2 = \frac{1 + \text{tr}(\rho^2)}{2} \leq 1.$$  

(12)

Let $\{G_k\}$ be an arbitrary Hermitian operator basis and $a_k = \text{tr}(\rho G_k)$. Then $\langle a|a \rangle = \text{tr}(\rho^2)$, so we have

$$(e|e) = \frac{1 + \langle a|a \rangle}{2}.$$  

(13)

This equality is instructive to understanding the relation between our entanglement criterion introduced in the next section and the CCNR criterion.

IV. ENTANGLEMENT DETECTION VIA SIC POVMS

Consider a bipartite state $\rho$ acting on the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with dimension $d = d_A \times d_B$, and denote by $\{E_k^A\}_{k=1}^{d_A^2}$ and $\{E_k^B\}_{k=1}^{d_B^2}$ the normalized SIC POVMs for the two respective subsystems; see the schematic setup in Fig. 1(b). The linear correlations between $E_A$ and $E_B$ read

$$[P_s]_{ij} = \langle E_i^A \otimes E_j^B \rangle,$$

(14)

from which we can construct a simple, but useful entanglement criterion via SIC POVMs (ESIC).

Proposition 2 (ESIC). If a state $\rho$ is separable, then

$$\|P_s\|_{\text{tr}} \leq 1$$

(15)

has to hold; otherwise, it is entangled.

Proof. For a product state $\rho = \rho_A \otimes \rho_B$, we have

$$P_s = \begin{pmatrix} e_{A,1} & e_{A,2} & \cdots \\ e_{B,1} & e_{B,2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \equiv |e_A\rangle\langle e_B|,$$

(16)

where $e_{A,j} = \text{tr}(\rho E_j^A)$ for $j = 1, 2, \ldots, d_A^2$ and $e_{B,k} = \text{tr}(\rho E_k^B)$ for $k = 1, 2, \ldots, d_B^2$; cf. Eq. (11). Then,

$$\|P_s\|_{\text{tr}} = (e_A|e_A)^{1/2}(e_B|e_B)^{1/2}$$

$$= \sqrt{\sum_j e_{A,j}^2} \sqrt{\sum_k e_{B,k}^2} \leq 1$$

(17)

according to Eq. (12). Again, by employing the convexity property of the trace norm under mixing, we have $\|P_s\|_{\text{tr}} \leq 1$ for separable states in general. \qed

Proposition 3. The ESIC criterion is independent of the specific SIC POVMs for systems A and B. In other words, any pair of SIC POVMs leads to the same criterion.

Proof. Let $\{E_k^A\}_{k=1}^{d_A^2}$ and $\{\tilde{E}_k^A\}_{k=1}^{d_A^2}$ be two arbitrary normalized SIC POVMs for system A, while $\{E_k^B\}_{k=1}^{d_B^2}$ and $\{\tilde{E}_k^B\}_{k=1}^{d_B^2}$ are two arbitrary normalized SIC POVMs for system B. Let $P_s$ be the correlation matrix as defined in Eq. (14) and

$$[\tilde{P}_s]_{ij} = \langle \tilde{E}_i^A \otimes \tilde{E}_j^B \rangle.$$

(18)

To prove the proposition it suffices to prove that $\|\tilde{P}_s\|_{\text{tr}} = \|P_s\|_{\text{tr}}$. To this end, note that $\tilde{E}_i^A$ can be expressed as a linear combination of $E_j^A$, that is,
\[ E^A_i = \sum_j O^A_{ij} E^A_j, \] where \( O^A_{ij} \) are uniquely determined by 
\( (E^A_k)_{k=1}^{d_A^2} \) and \( (E^A_k)_{k=1}^{d_A^2} \), and form an orthogonal matrix. Similarly, \( E^B_i = \sum_j O^B_{ij} E^B_j \) with \( O^B_{ij} \) forming an orthogonal matrix. Therefore,

\[ \tilde{P}_{ij} = \sum_{k,l} O^A_{ij} O^B_{kl} (E^A_k \otimes E^B_l) = \sum_{k,l} O^A_{ij} O^B_{kl} |P_{kl}|, \tag{19} \]

that is, \( \tilde{P} = O^A P_{\omega} (O^B)^T \). Since both \( O^A \) and \( O^B \) are orthogonal matrices, it follows that \( \| \tilde{P} \|_{\text{tr}} = \| P \|_{\text{tr}} \), so the entanglement criterion in Proposition 2 does not depend on the specific SIC POVMs for systems A and B.

It is not easy to analytically compare the ESIC criterion with CCNR. Nevertheless, through extensive numerical evidence, we find that the ESIC criterion is stronger than the CCNR criterion. Here we present several observations and a conjecture. For a product state, we have

\[ \| C \|_{\text{tr}} \leq \| P \|_{\text{tr}}. \tag{20} \]

This inequality follows from Eqs. (9), (13), and (17). Numerical calculations suggest that this inequality also holds for separable states.

**Conjecture 1.** If \( \| C \|_{\text{tr}} > 1 \), then \( \| P \|_{\text{tr}} > 1 \).

Remark. The converse of the above relation does not hold in general. However, for the special case of bipartite pure states, our numerical calculations suggest that

\[ \| C \|_{\text{tr}} - 1 = 2(\| P \|_{\text{tr}} - 1). \tag{21} \]

The same is true if the bipartite state is a convex combination of a pure state and the completely mixed state. So the ESIC criterion and the CCNR criterion become equivalent in this special case. Incidentally, in this case, they are also equivalent to the nonlinear criteria presented in Refs. [10][13]. Note that Eq. (21) can be verified analytically in the case of two-qubit pure states.

For a bipartite pure state \( \rho = |\psi\rangle\langle\psi| \), the Schmidt decomposition in Eq. (2) is closely related to the Schmidt decomposition of \( |\psi\rangle \). Suppose \( |\psi\rangle \) has the Schmidt decomposition

\[ |\psi\rangle = \sum_{k=1}^{d_A} \tilde{\lambda}_k |u^A_k\rangle |v^B_k\rangle, \tag{22} \]

where \( \tilde{\lambda}_k \) are the Schmidt coefficients and satisfy \( \sum_k \tilde{\lambda}_k^2 = 1 \). Then the Schmidt coefficients in Eq. (2) are given by \( \tilde{\lambda}_i \tilde{\lambda}_j \) for \( i, j = 1, 2, \ldots, d_A \). Therefore,

\[ \| C \|_{\text{tr}} = \sum_i \sum_j \tilde{\lambda}_i \tilde{\lambda}_j = \left( \sum_j \tilde{\lambda}_j \right)^2, \tag{23} \]

In Fig. 2 we plot the value of \( \| C \|_{\text{tr}} - 1 \) [that is, \( 2(\| P \|_{\text{tr}} - 1) \) according to Eq. (21)] against the two independent squared Schmidt coefficients for two-qutrit pure states. All states are entangled, except for the ones corresponding to the minimum value (the three fulcrums) of the surface plot, which are product states. Top of the surface (or center of the contours) represents the maximally entangled states. Incidentally, for two-qubit pure states, the corresponding plot is an arc lying on the plane formed by one of the horizontal axes and the vertical axis.

In the following section, we use various examples to demonstrate that Conjecture 1 holds true.

**V. Examples**

For the first example, let us consider two-qubit states. In this case, the PPT criterion [5][6] is necessary and sufficient for detecting entanglement. For a randomly generated two-qubit state \( \rho_{2q} \), if it is entangled, then we add white noise to it and get the state

\[ \rho(q) = q \rho_{2q} + (1-q) \frac{1}{4}, \quad 0 \leq q \leq 1. \tag{24} \]

The threshold value of \( q \) at which \( \rho(q) \) becomes separable can be determined by the PPT criterion \( \| \rho^{\text{tr}} \|_{\text{tr}} \). Then we compare it with the values determined by the ESIC \( \| P \|_{\text{tr}} \) and CCNR \( \| C \|_{\text{tr}} \) criteria, respectively. The smaller the difference is as compared to the PPT criterion, the better the criterion is. Figure 3 illustrates the results on 10 000 entangled two-qubit states which are generated randomly according to the Hilbert-Schmidt measure. As can be seen, the ESIC criterion is better than CCNR in most of the cases. Notably, all states that are detected by CCNR can also be detected by ESIC.

Moreover, the advantage of ESIC over CCNR is not only tied to the Hilbert-Schmidt measure. To corroborate this point, we have considered random mixed states.
generated according to various different measures. In particular, we have studied induced measures on mixed states obtained by taking partial trace of the Haar random pure states of bipartite systems [27]. For example, if \(|\Psi\rangle\) is a Haar random pure state in dimension \(N \times K\), then taking partial trace over the second system yields a random mixed state \(\rho = \text{tr}_K(|\Psi\rangle\langle\Psi|)\) on the Hilbert space of dimension \(N\). The resulting induced measure will be denoted by \((N,K)\). Note that this measure is equivalent to the Hilbert-Schmidt measure when \(N = K\).

In general, as \(K\) increases, the measure gets more biased towards more mixed states. Here we are interested in the two-qubit random mixed states, so \(N = 2 \times 2 = 4\). Table I shows the distinction between ESIC and CCNR for three different choices of \(K\), namely \(K = 3, 4, 6\). In all three cases, ESIC can detect more entangled states than CCNR. In addition, we have considered random states generated according to the Jeffreys prior over the probability space [28]. Again, the ESIC criterion is better than CCNR.

We mentioned early that the CCNR criterion is able to detect certain bound entangled states, where the PPT criterion fails completely. In the second example, we consider the family of \(3 \times 3\) bound entangled states introduced by P. Horodecki [29],

\[
\rho_{PH}^x = \frac{1}{8x+1} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i|x| & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

(25)

Although these states cannot be detected by the PPT criterion and are not distillable, they are nevertheless entangled for \(0 < x < 1\). Consider the mixtures of \(\rho_{PH}^x\) with the white noise,

\[
\rho(x,q) = q\rho_{PH}^x + (1-q)\frac{1}{9}, \quad 0 \leq q \leq 1.
\]

(26)

Figure 3 illustrates the parameter range for which the state \(\rho(x,q)\) is entangled and can be detected by the ESIC (CCNR) criterion. It is clear from the figure that the ESIC criterion can detect strictly more entangled states than the CCNR criterion.

For the next example, we consider the \(3 \times 3\) bound entangled states called chessboard states [30]. They are defined as

\[
\rho_{cb} = \mathcal{N} \sum_{j=1}^{4} |V_j\rangle\langle V_j|,
\]

(27)

with \(\mathcal{N}\) being the normalization constant, where the un-normalized vectors are

\[
|V_1\rangle = |m_5,0,m_1m_3/m_6;0,m_6;0,0,0\rangle, \\
|V_2\rangle = |0,m_1,0;m_2,0,m_3;0,0,0\rangle, \\
|V_3\rangle = |m_6,0,0;0,-m_5,0;m_1m_4/m_6,0,0\rangle, \\
|V_4\rangle = |0,m_2,0;0,-m_1,0,0;0,m_4,0\rangle.
\]

(28)

These chessboard states are characterized by the six real parameters \(m_k\)s for \(k = 1,2,\ldots,6\). We compare the ESIC criterion with the CCNR criterion on randomly generated chessboard states, where the six parameters are drawn independently from the normal distribution with zero mean value and standard deviation of 2. Altogether 50000 random states are generated, and the re-
FIG. 4. Entanglement detection of $\rho(x, q)$, mixtures of Horodecki $3 \times 3$ bound entangled states with the white noise. States above the red solid (blue dashed) curve are detected by the ESIC (CCNR) criterion. The ESIC criterion can detect all states detected by the CCNR criterion and some additional states.

FIG. 5. Entanglement detection of $3 \times 3$ chessboard states. Altogether 50,000 states are generated randomly, and the plot shows the fractions of states that are detected by three criteria, respectively. Here the PPT criterion fails completely, the ESIC criterion can detect roughly 2% more states than the CCNR criterion.

random states are generated according to the Hilbert-Schmidt measure in each dimension ranging from $2 \times 2$ up to $7 \times 7$, respectively, see the results in Fig. [6]. In all these cases, the ESIC criterion is shown to be better than CCNR, although the differences vary with the dimension. It is quite surprising that an unusually large percentage of entangled states is detected in the $3 \times 3$ case compared to other dimensions. Dimension 3 has been known to be very special in the study of SIC POVMs [21]. The reason behind such anomaly deserves further study.

VI. CONCLUSION

A SIC POVM represents a special single measurement setting, which is tomographically complete. In this paper we show that by using SIC POVMs we can construct a stronger and more efficient entanglement detection criterion than the CCNR criterion based on local orthogonal observables. The superiority of our criterion is illustrated with various examples. The reason behind this superiority is worthy of further study. In passing, we note that an equivalent criterion can be constructed by replacing SIC POVMs with the complete sets of mutually unbiased bases.

For the CCNR criterion, the local uncertainty relation provides a straightforward construction of nonlinear entanglement witnesses. Then, an interesting open question is how to achieve a nonlinear improvement for the ESIC criterion with SIC POVMs. Note that the nonlinear criterion with SIC POVMs introduced recently in Ref. [19] is identical to the criterion using observables in Refs. [10, 14].
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