Inapproximability of Minimizing a Pair of DNFs or Binary Decision Trees Defining a Partial Boolean Function

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Abstract

The desire to apply machine learning techniques in safety-critical environments has renewed interest in the learning of partial functions for distinguishing between positive, negative and unclear observations. We contribute to the understanding of the hardness of this problem. Specifically, we consider partial Boolean functions defined by a pair of Boolean functions \( f, g : \{0,1\}^J \rightarrow \{0,1\} \) such that \( f \cdot g = 0 \) and such that \( f \) and \( g \) are defined by disjunctive normal forms or binary decision trees. We show: Minimizing the sum of the lengths or depths of these forms while separating disjoint sets \( A \cup B = S \subseteq \{0,1\}^J \) such that \( f(A) = \{1\} \) and \( g(B) = \{1\} \) is inapproximable to within \((1-\epsilon)\ln(|S|-1)\) for any \( \epsilon > 0 \), unless \( P=NP \).

1 Introduction

The desire to apply machine learning techniques in safety-critical environments has renewed interest in the learning of partial functions for distinguishing between positive, negative and unclear observations. For instance, a (total) Boolean function \( f : \{0,1\}^J \rightarrow \{0,1\} \) defines a decision, zero or one, for every assignment \( x \in \{0,1\}^J \) of zeros or ones to the finite set \( J \) of variables. In contrast, a partial Boolean function, i.e. a map from a subset of \( \{0,1\}^J \) to \( \{0,1\} \), distinguishes between positive, negative and undefined observations \( x \) for which \( f(x) \) is 1, 0 and undefined, respectively.

Partial Boolean functions have potential applications in safety-critical environments. In medicine, for instance, partial Boolean functions can potentially distinguish between healthy records (0), pathological records (1), and records prioritized for human inspection (unclear). In the field of autonomous driving, partial Boolean functions can potentially distinguish between an autonomous driving mode (0), emergency breaking (1) and escalation to the driver (unclear).

With this paper, we contribute to the understanding of the hardness of learning partial Boolean functions. This hardness depends on the encoding of partial Boolean functions as well as on the learning problem. We choose to encode any partial Boolean function by a pair \( (f,g) \) of (total) Boolean functions \( f, g : \{0,1\}^J \rightarrow \{0,1\} \) such that \( f \cdot g = 0 \). We define a partial Boolean function w.r.t. \( f \) and \( g \) such that it assumes the value one (resp. zero) iff \( f \) (resp. \( g \)) assumes the value one. More specifically, we choose to encode \( f \) and \( g \) either both by disjunctive normal forms (DNFs) or both by binary decision trees (BDTs).

As a learning problem, we consider the objective of minimizing the sum of the depths or lengths of the DNFs or BDTs defining \( f \) and \( g \), subject to the constraint that disjoint sets \( A \cup B = S \subseteq \{0,1\}^J \) are separated such that \( f(A) = \{1\} \) and \( g(B) = \{1\} \).

We show that this problem is inapproximable to within \((1-\epsilon)\ln(|S|-1)\) for any \( \epsilon > 0 \), unless \( P=NP \). We arrive at this result by generalizing a classic reduction of MIN-SET-COVER by Haussler (1988) and transferring the tight inapproximability bound for MIN-SET-COVER established by Dinur and Steurer (2014). Moreover, we show that a technique known to yield a logarithmic approximation algorithm for minimizing the length of a single DNF defining a (total) Boolean function only yields a \( \frac{1}{2}|S||J| \)-approximation algorithm for the problem of minimizing the sum of the lengths of two DNFs defining a partial Boolean function.

2 Related Work

The problem of extending a partial Boolean function defined by a set of truth points \( A \subseteq \{0,1\}^J \) and a disjoint set of false points \( B \subseteq \{0,1\}^J \) to a total function \( h : \{0,1\}^J \rightarrow \{0,1\} \) such that \( A \subseteq h^{-1}(1) \) and \( B \subseteq h^{-1}(0) \) has been studied comprehensively, for various classes of functions (Crama and Hammer, 2011). In particular, deciding whether a DNF or BDT of bounded depth or length exists which classifies the set of truth points and false points exactly has been shown to be \( \mathsf{NP} \)-complete (Czort, 1999; Hancock et al., 1996; Haussler, 1988). The problem of finding a DNF of bounded length remains \( \mathsf{NP} \)-hard even if the full truth table is given as input (Allender et al., 2008; Czort, 1999).

Toward approximation, the problem of finding a DNF of minimum depth or length, consistent with labeled data, does not admit a polynomial-time \((1-\epsilon)\ln n\)-approximate algorithm for any \( \epsilon > 0 \) unless \( \mathsf{NP} \subseteq \mathsf{P} \) (Czort, 1999; Feige, 1998) where \( \mathsf{P} \) denotes the class of problems solvable in quasi-polynomial time. This bound also holds under the weaker assumption \( \mathsf{P} \neq \mathsf{NP} \) due to the improved bound for MIN-SET-COVER established by Dinur and Steurer (2014). Analogously, the problem of finding a BDT of minimum depth or length does not admit a polynomial-time \((1-\epsilon)\ln n\)-approximate algorithm for any \( \epsilon > 0 \) unless \( \mathsf{NP} \subseteq \mathsf{P} \) (Czort, 1999).
\(\epsilon\) ln \(n\)-approximate algorithm for any \(\epsilon > 0\) unless \(P \neq NP\), by the reduction of Hancock et al. (1996) and the bound of Dinur and Steurer (2014).

The related problem of isolating points by BDTs is \(NP\)-hard (Hyafil and Rivest, 1976). It does not admit a polynomial-time \(o(\log n)\)-algorithm unless \(NP \subseteq \tilde{P}\) (Laber and Nogueira, 2004). Moreover, it does not admit a polynomial-time \((1 - \epsilon)\ln n\)-approximate algorithm for any \(\epsilon > 0\) unless \(P \neq NP\), by the reduction of Laber and Nogueira (2004) and the bound of Dinur and Steurer (2014).

The problem of deciding, for any BDT given as input, whether an equivalent BDT of size at most \(D\) is \(\{1,0\}\)-hard (Hyafil and Rivest, 1976). It does not admit a polynomial-time \((1 - \epsilon)\ln n\) approximation algorithm for any \(\epsilon > 0\) unless \(P \neq NP\), by the reduction of Hancock et al. (1996) and the bound of Dinur and Steurer (2014).

The related problem of isolating points by BDTs is \(NP\)-complete (Zantema and Bodlaender, 2000). The corresponding optimization problem of finding an equivalent BDT of minimal size does not admit a polynomial-time \(r\)-approximation algorithm for any constant \(r > 1\), unless \(P = NP\) (Sieling, 2008).

### 3 Problem Statement

**Definition 1.** A tuple \((J, X, A, B)\) is called Boolean labeled data with the feature space \(X\) iff \(X = \{0, 1\}^J\) and \(A \cup B \subseteq X\) and \(A \neq \emptyset\) and \(B \neq \emptyset\) and \(\emptyset \cap B = \emptyset\).

**Definition 2.** For any Boolean labeled data \((J, X, A, B)\) = \(D\), any non-empty set \(\Theta\) and family \(f: \Theta \to \{0, 1\}^X\) of Boolean functions, and for any function \(R: \Theta \to \mathbb{N}_0\), called a regularizer, the instance of the problem separation w.r.t. \(D, \Theta, f\) and \(R\) has the form

\[
\min_{(\theta, \theta')} R(\theta) + R(\theta') \\
\text{subj. to} \\
\forall x \in A: f_\theta(x) = 1 \quad \text{(exact)} \quad (2) \\
\forall x \in B: f_{\theta'}(x) = 1 \quad \text{(exact)} \quad (3) \\
f_\theta \cdot f_{\theta'} = 0 \quad \text{(non-contradictory)} \quad (4)
\]

For any \(m \in \mathbb{N}_0\), the instance of the problem separability w.r.t. \(D, \Theta, f, R, m\) and \(R\) is to decide whether there exist \(\theta, \theta' \in \Theta\) such that

\[
\forall x \in A: f_\theta(x) = 1 \quad \text{(exact)} \quad (5) \\
\forall x \in B: f_{\theta'}(x) = 1 \quad \text{(exact)} \quad (6) \\
f_\theta \cdot f_{\theta'} = 0 \quad \text{(non-contradictory)} \quad (7) \\
R(\theta) + R(\theta') \leq m \quad \text{(bounded)} \quad (8)
\]

Four remarks are in order.

- Any feasible solution \((\theta, \theta') \in \Theta^2\) defines a partial Boolean function \(h: X' \to \{0, 1\}\) with the domain \(X' = f_\theta^{-1}(1) \cup f_{\theta'}^{-1}(1)\) such that for all \(x' \in X'\), we have \(h(x') = 1\) iff \(f_\theta(x') = 1\) and, equivalently, \(h(x') = 0\) iff \(f_{\theta'}(x') = 1\).
- Exactness means the labeled data is classified totally and without errors, i.e., for all \(x \in A\) we have \(x \in X'\) and \(h(x) = 1\), and for all \(x \in B\) we have \(x \in X'\) and \(h(x) = 0\).
- The problems are symmetric in the sense that \((\theta, \theta') \in \Theta^2\) is a (feasible) solution to an instance w.r.t. Boolean labeled data \((J, X, A, B)\) iff \((\theta', \theta)\) is a (feasible) solution to the same instance but with Boolean labeled data labeled data \((J, X, B, A)\).
- Totality could be enforced by the additional constraint

\[
1 \leq f_\theta + f_{\theta'} . \quad (9)
\]

In this sense, the problem of learning partial Boolean functions is a relaxation of the problem of learning (total) Boolean functions.

Our work is motivated by the following research question: Are the problems separability and separation w.r.t. the family \(f\) of DNFs or BDTs easier to solve or approximate than the related problems from the classic works of Haussler (1988); Hancock et al. (1996); Feige (1998) and Czort (1999) for learning a single DNF or BDT defining a (total) Boolean function? In this paper, we answer this question in the negative by establishing \(NP\)-completeness of separability and an analogous inapproximability bound for separation.

### 4 Preliminaries

#### 4.1 Disjunctive Normal Forms

**Definition 3.** For any finite set \(X = \{0, 1\}^J\), \(\Gamma = \{(V, \bar{V}) \in 2^J \times 2^J | V \cap \bar{V} = \emptyset\}\) and \(\Theta = 2^\Gamma\), the family \(f: \Theta \to \{0, 1\}^X\) such that, for any \(\theta \in \Theta\) and any \(x \in X\),

\[
f_\theta(x) = \sum_{(J_0, J_1) \in \theta} \prod_{j \in J_0} x_j \prod_{j \in J_1} (1 - x_j) \quad (10)
\]

is called the family of \(J\)-variante disjunctive normal forms (DNFs).

For \(R_1, R_2: \Theta \to \mathbb{N}_0\) such that for all \(\theta \in \Theta\),

\[
\begin{align*}
R_1(\theta) &= \sum_{(J_0, J_1) \in \theta} (|J_0| + |J_1|) \\
R_2(\theta) &= \max_{(J_0, J_1) \in \theta} (|J_0| + |J_1|)
\end{align*}
\]

\(R_1(\theta)\) and \(R_2(\theta)\) are called the length and depth, respectively, of the DNF defined by \(\theta\).

In the context of DNFs, we refer to separability w.r.t. \(R_1\) and \(R_2\) as bounded-length-DNF and bounded-depth-DNF. And we refer to separation w.r.t. \(R_1\) and \(R_2\) as min-length-DNF and min-depth-DNF.

#### 4.2 Binary Decision Trees

**Definition 4.** A tuple \(\theta = (J, Y, D, D', d^*, E, \delta, v, y)\) is called a \(J\)-variante \(Y\)-valued binary decision tree iff the following conditions hold:

- \(J \neq \emptyset\) is the finite set of observed variables,
- \(Y\) is the finite set of labels,
- \((D \cup D', E)\) is a finite, non-empty, directed binary tree,
\( d^* \in D \cup D' \) is the unique root of the tree,
\( \delta : E \to \{0, 1\} \),
\( \delta(e) = 0 \) and \( \delta(e') = 1 \),
\( v : D \to J \) assigns a variable to each interior node, and
\( y : D' \to Y \) assigns a label to each leaf.

For any BDT \( \theta = (J, Y, D, D', d^*, E, \delta, v, y) \), any \( d \in D \) and any \( j \in \{0, 1\} \), we denote by \( d_{ij} \in D \cup D' \) the unique node such that \( e = (d, d_i) \in E \) and \( \delta(e) = j \).

**Definition 5.** For any BDT \( \theta = (J, Y, D, D', d^*, E, \delta, v, y) \) and any \( d \in D \subseteq D' \), the tuple \( \theta[d] = (J, Y, D_2, D_2', d, E', \delta', v', y') \) is called the binary decision subtree of \( \theta \) rooted at \( d \) iff \((D_2 \cup D_2', E')\) is the subtree of \((D \cup D', E)\) rooted at \( d \), and \( \delta', v', y' \) are the restrictions of \( \delta, v, y \) to the subtrees \( D_2, D_2', E' \).

**Definition 6.** For any BDT \( \theta = (J, Y, D, D', d^*, E, \delta, v, y) \) defined by the BDT \( \theta \) that is for all \( x \in \{0, 1\}^J \):
\[
\begin{align*}
    f_\theta(x) &= \begin{cases} 
        y(d^*) & \text{if } D = \emptyset \\
        (1 - x_{v(d^*)}) f_\theta[x_{v(d^*)}] + x_{v(d^*)} f_\theta[x_{v(d^*)}] & \text{otherwise}
    \end{cases} 
\end{align*}
\] (13)

For \( R_l, R_d : \Theta \to \mathbb{N}_0 \) such that for all \( \theta \in \Theta \),
\[
R_l(\theta) = |D \cup D'| \\
R_d(\theta) = \begin{cases} 
0 & \text{if } D = \emptyset \\
1 + \max \{R_l(\theta[d]), R_l(\theta[d^*])\} & \text{otherwise}
\end{cases}
\] (14) (15)

\( R_l(\theta) \) and \( R_d(\theta) \) are called length and depth, respectively, of the BDT defined by \( \theta \).

In the context of BDTs, we refer to separability w.r.t. \( R_l \) and \( R_d \) as BOUNDED-LONG-BDT and BOUNDED-DEPTH-BDT. And we refer to separation w.r.t. \( R_l \) and \( R_d \) as MIN-LENGTH-BDT and MIN-DEPTH-BDT.

### 4.3 Set Cover Problem

**Definition 7.** For any finite set \( U \neq \emptyset \), any collection \( \Sigma \subseteq 2^U \) and any \( m \in \mathbb{N}_0 \) the instance of the problem set-cover w.r.t. \( U, \Sigma \) and \( m \) is to decide whether there is a \( \Sigma' \subseteq \Sigma \) such that \( \bigcup_{\sigma \in \Sigma'} \sigma = U \) and \( |\Sigma'| \leq m \).

The problem MIN-SET-COVER w.r.t. \( U \) and \( \Sigma \) is to solve
\[
\min_{\Sigma' \subseteq \Sigma} |\Sigma'| \\
\text{subject to } U = \bigcup_{\sigma \in \Sigma'} \sigma .
\] (16) (17)

**Theorem 1** (Dimur and Steurer (2014)). For every \( \epsilon > 0 \), it is NP-hard to approximate MIN-SET-COVER to within \((1 - \epsilon) \ln n\), where \( n \) is the size of the instance.

In this section, we establish NP-completeness of separability for the family of DNFs or BDTs, w.r.t. \( R_l \) or \( R_d \), by reduction of set-cover. Beyond this reduction, we already relate feasible solutions to any instance of MIN-SET-COVER to feasible solutions to the induced instance of separation, as shown in Fig. 1, in a way that will allow us in Section 6 to transfer an inapproximability bound from set-cover to separability.

**Definition 8** (Haussler (1988)). For any finite set \( U \neq \emptyset \) and any collection \( \Sigma \subseteq 2^U \) of subsets of \( U \), the Haussler data is the Boolean labeled data \( D_{U, \Sigma} = (\Sigma, X, A, B) \) such that \( B = \{0^\Sigma\} \) and \( A = \{x^u \in \{0, 1\}^\Sigma \mid u \in U\} \) such that for any \( u \in U \) and any \( \sigma \in \Sigma \), we have \( x_u^u = 1 \) iff \( u \in \sigma \).

#### 5.1 Disjunctive Normal Forms

We now consider the family \( f : \Theta \to \{0, 1\}^X \) of DNFs according to Def. 3.

**Lemma 1.** For any instance \((U, \Sigma)\) of MIN-SET-COVER and the instance of MIN-LENGTH-DNF (resp. MIN-DEPTH-DNF) w.r.t. the Haussler data \( D_{U, \Sigma} \), the function \( h : 2^\Sigma \to \Theta^2 \) such that for any \( \Sigma' \subseteq \Sigma \), \( h(\Sigma') = (\theta, \theta') \) with
\[
\begin{align*}
\theta &= \{\{\sigma, \emptyset \mid \sigma \in \Sigma'\} \} \\
\theta' &= \{\emptyset, \Sigma'\}
\end{align*}
\] (18) (19)

has the following properties: For any feasible solution \( \Sigma' \) to the instance of MIN-SET-COVER w.r.t. \((U, \Sigma)\):

1. \( h(\Sigma') \) is computable in linear time, \( O(|\Sigma'|) \).
2. \( h(\Sigma') = (\theta, \theta') \) is such that \( R_l(\theta) + R_l(\theta') = 2|\Sigma'| \) and \( R_d(\theta) + R_d(\theta') = |\Sigma'| + 1 \).
3. \( h(\Sigma') \) is a feasible solution to the instance of MIN-LENGTH-DNF (resp. MIN-DEPTH-DNF) w.r.t. \( D_{U, \Sigma} \).

**Proof.** (1) and (2) hold by construction of \( h \). (3) Firstly, \( f_\theta \) and \( f_{\theta'} \) are such that for any \( x \in \{0, 1\}^\Sigma \):
\[
\begin{align*}
f_\theta(x) &= \sum_{\sigma \in \Sigma} x_{\sigma} \\
f_{\theta'}(x) &= \prod_{\sigma \in \Sigma} (1 - x_{\sigma})
\end{align*}
\] (20) (21)
Secondly, these functions are non-contradicting, i.e. $f_\theta \cdot f_{\theta'} = 0$, since $f_\theta = 1 - f_{\theta'}$. Thirdly, $f_{\theta'}(0^2) = 1$, by (21). Moreover, since $\Sigma'$ is a cover of $U$:

\begin{align}
\forall u \in U \quad \exists \sigma \in \Sigma' : u \in \sigma & \quad \text{(Def. of } D_{U,\Sigma}) \tag{22} \\
\Rightarrow \forall u \in U \quad \exists \sigma \in \Sigma' : x_\sigma^u = 1 & \quad \text{(Def. of } D_{U,\Sigma}) \tag{23} \\
\Rightarrow \forall u \in U : f_\theta(x^u) = 1 & \quad \text{(Def. of } \theta) \tag{24}
\end{align}

Thus, $h(\Sigma') = (\theta, \theta')$ is a feasible solution to the instance of MIN-LENGTH-DNF (resp. MIN-DEPTH-DNF) w.r.t. $D_{U,\Sigma}$.

**Corollary 1.** For any instance $(U, \Sigma, m)$ of SET-COVER and any solution $\Sigma' \subseteq \Sigma$ to this instance, there is a solution $(\theta, \theta')$ to the instance of BOUNDED-LENGTH-DNF w.r.t. $D_{U,\Sigma}$ and $2m$ (resp. BOUNDED-DEPTH-DNF w.r.t. $D_{U,\Sigma}$ and $m + 1$). Moreover, this solution $(\theta, \theta')$ can be computed efficiently from $\Sigma'$.

**Lemma 2.** For any instance $(U, \Sigma)$ of MIN-SET-COVER and the instance of MIN-LENGTH-DNF (resp. MIN-DEPTH-DNF) w.r.t. the Haussler data $D_{U,\Sigma}$, the function $g : \Theta^2 \to 2^\Sigma$ such that for any $(\theta, \theta') \in \Theta^2$,

\[ g(\theta, \theta') = \begin{cases} 
\sum_0 \text{ if } |\Sigma_0| \leq |\Sigma_1| \\
\Sigma_1 \quad \text{otherwise} 
\end{cases} \quad \text{(25)} \]

with

\[ \Sigma_0' = \bigcup_{(\Sigma_0, \Sigma_1) \in \theta} \Sigma_0 \quad \text{(26)} \]

and

\[ \Sigma_1' \in \begin{cases} 
\{ \Sigma_1 \subseteq \Sigma : (\theta, \Sigma_1) \in \theta' \} & \text{if non-empty} \\
\{ \emptyset \} & \text{otherwise} \end{cases} \quad \text{(27)} \]

has the following properties: For any feasible solution $(\theta, \theta') \in \Theta^2$ to the instance of MIN-LENGTH-DNF (resp. MIN-DEPTH-DNF) w.r.t. $D_{U,\Sigma}$:

1. $g(\theta, \theta')$ is computable in linear time, $O(R_l(\theta) + R_l(\theta'))$.
2. $g(\theta, \theta')$ is such that $|g(\theta, \theta')| \leq (R_l(\theta) + R_l(\theta'))/2$ and $|g(\theta, \theta')| \leq R_d(\theta) + R_d(\theta') - 1$.
3. $g(\theta, \theta')$ is a feasible solution to the instance of MIN-SET-COVER w.r.t. $(U, \Sigma)$.

**Proof.** (1) holds because $\Sigma_0'$ and $\Sigma_1'$ can be constructed in time $O(R_l(\theta) + R_l(\theta'))$. (2) Firstly:

\[ |g(\theta, \theta')| = \min\{|\Sigma_0'|, |\Sigma_1'|\} \quad \text{(by (25))} \tag{28} \]

\[ \leq \frac{|\Sigma_0'| + |\Sigma_1'|}{2} \quad \text{(by (26), (27))} \tag{29} \]

\[ \leq \frac{R_l(\theta) + R_l(\theta')}{2} \quad \text{(by (26), (27))} \tag{30} \]

Secondly:

\[ |g(\theta, \theta')| = \min\{|\Sigma_0'|, |\Sigma_1'|\} \quad \text{(by (25))} \tag{31} \]

\[ \leq |\Sigma_1'| \quad \text{(by (27))} \tag{32} \]

\[ \leq R_d(\theta') \quad \text{(by (27))} \tag{33} \]

\[ \leq R_d(\theta) + R_d(\theta') - 1 \quad \text{(as } 1 \leq R_d(\theta)) \tag{34} \]

(3) We recall from Def. 3 that for any DNF $\theta \in \Theta$, the function $f_\theta$ is such that for any $x \in \{0, 1\}^\Sigma$:

\[ f_\theta(x) = \sum_{(\Sigma_0, \Sigma_1) \in \theta} \prod_{\sigma \in \Sigma_0} x_\sigma \prod_{\sigma \in \Sigma_1} (1 - x_\sigma) \quad \text{(35)} \]

Firstly, we show that $\Sigma_0'$ is a feasible solution to the instance of MIN-SET-COVER w.r.t. $(U, \Sigma)$. On the one hand:

\[ f_{\theta'}(0^2) = 1 \quad \text{(by Def. of } D_{U,\Sigma}) \tag{36} \]

\[ \implies f_{\theta'}(0^2) = 0 \quad \text{(as } f_\theta \cdot f_{\theta'} = 0) \tag{37} \]

On the other hand:

\[ \forall u \in U : f_\theta(x^u) = 1 \quad \text{(Def. of } D_{U,\Sigma}) \tag{38} \]

\[ \forall u \in U \exists (\Sigma_0, \Sigma_1) \in \theta : \forall \sigma \in \Sigma_0 : x_{\sigma}^u = 1 \tag{39} \]

\[ \forall u \in U \exists \sigma \in \Sigma_0' : x_{\sigma}^u = 1 \quad \text{(Def. of } \Sigma_0') \tag{40} \]

\[ \forall u \in U \exists \sigma \in \Sigma_1' : u \in \sigma \quad \text{(Def. of } D_{U,\Sigma}) \tag{41} \]

\[ \forall \sigma \in \Sigma_0' \quad \exists u \in U : f_\theta(x^u) = 1 \quad \text{(Def. of } D_{U,\Sigma}) \tag{42} \]

Secondly, we show that $\Sigma_1'$ is a feasible solution to the instance of MIN-SET-COVER w.r.t. $(U, \Sigma)$. On the one hand:

\[ f_{\theta'}(0^2) = 1 \quad \text{(Def. of } D_{U,\Sigma}) \tag{43} \]

\[ \exists (\Sigma_0, \Sigma_1) \in \theta : \Sigma_0 = \emptyset \tag{44} \]

On the other hand:

\[ \forall u \in U : f_\theta(x^u) = 1 \quad \text{(Def. of } D_{U,\Sigma}) \tag{45} \]

\[ \forall u \in U \forall (\Sigma_0, \Sigma_1) \in \theta' : \exists (\Sigma_0, \Sigma_1) : x_{\sigma}^u = 0 \tag{46} \]

\[ \forall u \in U \exists \sigma \in \Sigma_1' : x_{\sigma}^u = 1 \quad \text{(Def. of } \Sigma_1' \text{ and } (44)) \tag{47} \]

\[ \forall u \in U \exists \sigma \in \Sigma_1' : u \in \sigma \quad \text{(Def. of } D_{U,\Sigma}) \tag{48} \]

\[ \forall \sigma \in \Sigma_1' \quad \exists u \in U : f_\theta(x^u) = 1 \quad \text{(Def. of } D_{U,\Sigma}) \tag{49} \]

Thus $g(\theta, \theta')$ is a feasible solution to the instance of MIN-SET-COVER w.r.t. $(U, \Sigma)$.

**Corollary 2.** For any instance $(U, \Sigma, m)$ of SET-COVER and any solution $(\theta, \theta')$ to the instance of BOUNDED-LENGTH-DNF w.r.t. $D_{U,\Sigma}$ and $2m$ (resp. BOUNDED-DEPTH-DNF w.r.t. $D_{U,\Sigma}$ and $m + 1$), there is a solution $\Sigma' \subseteq \Sigma$ to the instance of SET-COVER w.r.t. $(U, \Sigma, m)$. Moreover, this solution $\Sigma'$ can be computed efficiently from $(\theta, \theta')$.

**Theorem 2.** BOUNDED-LENGTH-DNF and BOUNDED-DEPTH-DNF are NP-complete.
Then, for any feasible solution $\Sigma$ that the following conditions hold: $X_{\Sigma}$

Thus, $w.r.t.$

Lemma 3. For any instance $(U, \Sigma)$ of MIN-SET-COVER and the instance of MIN-DEPTH-BDT (resp. MIN-LENGTH-BDT) w.r.t. the Haussler data $D_{U, \Sigma}$, let $h : 2^\Sigma \to \Theta^2$ such that the following conditions hold:

- $h(0) = (\theta, \theta')$ where $\theta$ and $\theta'$ are BDTs consisting of only a root node labeled 0 and 1, respectively.
- For any $\Sigma' \subseteq \Sigma$ with $|\Sigma'| > 0$, there is a bijection $\sigma' : \{1, \ldots, |\Sigma'|\} \to \Sigma'$ such that $h(\Sigma') = (\theta, \theta')$ where $\theta$ and $\theta'$ are the BDTs depicted in Fig. 2.

Then, for any feasible solution $\Sigma'$ to the instance of MIN-SET-COVER w.r.t. $(U, \Sigma)$:

1. $h(\Sigma')$ is computable in linear time $O(|\Sigma'|)$.
2. $h(\Sigma') = (\theta, \theta')$ is such that $R_t(\theta) + R_t(\theta') = 4|\Sigma'| + 2$ and $R_d(\theta) + R_d(\theta') = 2|\Sigma'|$.
3. $h(\Sigma')$ is a feasible solution to the instance of MIN-DEPTH-BDT (resp. MIN-LENGTH-BDT) w.r.t. the Haussler data $D_{U, \Sigma}$.

Proof. (1) and (2) hold by construction of $h$. (3) Firstly, $f_\theta \cdot f_{\theta'} = 0$, since $f_\theta = 1 - f_{\theta'}$. Secondly, $f_{\theta'}(0^{\Sigma'}) = 1$ by construction of $h$. Thirdly, since $\Sigma'$ is a cover of $U$:

Thus, $h(\Sigma')$ is a feasible solution to the instance of MIN-DEPTH-BDT (resp. MIN-LENGTH-BDT) w.r.t. $D_{U, \Sigma}$.

Corollary 3. For any instance $(U, \Sigma, m)$ of SET-COVER and any solution $\Sigma' \subseteq \Sigma$ to this instance, there is a solution $(\theta, \theta')$ to the instance of BOUNDED-DEPTH-BDT w.r.t. $D_{U, \Sigma}$ and $2m$ (resp. BOUNDED-LENGTH-BDT w.r.t. $D_{U, \Sigma}$ and $4m + 2$). Moreover, this solution $(\theta, \theta')$ can be computed efficiently from $\Sigma'$.

Definition 9. For any BDT $\theta = (J, Y, D, D', d^*, E, \delta, \epsilon, v, y)$ and any input $x \in \{0, 1\}^J$, let $J_\theta(x) \subseteq J$ the set of variables visited on the path that decides $x$, i.e.

$L_\theta(x) = \begin{cases} \emptyset & \text{if } D = \emptyset \\ \{v(d^*)\} \cup J_\theta(\xi_{\epsilon(v)(\delta)}) & \text{otherwise} \end{cases}, \quad (53)$

Lemma 4. For any instance $(U, \Sigma)$ of MIN-SET-COVER and the instance of MIN-DEPTH-BDT (resp. MIN-LENGTH-BDT) w.r.t. the Haussler data $D_{U, \Sigma}$, the function $g : \Theta^2 \to 2^\Sigma$ such that for any $(\theta, \theta') \in \Theta^2$,

$g(\theta, \theta') = \begin{cases} J_\theta(0^\Sigma) & \text{if } |J_\theta(0^\Sigma)| \leq |J_{\theta'}(0^\Sigma)| \\ J_{\theta'}(0^\Sigma) & \text{otherwise} \end{cases} \quad (54)$

has the following properties: For any feasible solution $(\theta, \theta') \in \Theta^2$ to the instance of MIN-DEPTH-BDT (resp. MIN-LENGTH-BDT) w.r.t. $D_{U, \Sigma}$:

1. $g(\theta, \theta')$ is computable in linear time $O(R_d(\theta) + R_d(\theta'))$.
2. $g(\theta, \theta')$ is such that $|g(\theta, \theta')| \leq (R_d(\theta) + R_d(\theta'))/2$ and $|g(\theta, \theta')| \leq (R_d(\theta) + R_d(\theta') - 2)/4$.
3. $g(\theta, \theta')$ is a feasible solution to the instance of MIN-SET-COVER w.r.t. $(U, \Sigma)$.

Proof. (1) holds since $J_\theta(0^\Sigma)$ and $J_{\theta'}(0^\Sigma)$ can be constructed in $O(R_d(\theta) + R_d(\theta'))$ time.

(2) holds by the fact that $R_d(\theta) \geq 2R_d(\theta) + 1$ for any $\theta$ and the following argument:

$|g(\theta, \theta')| = \min \{|J_\theta(0^\Sigma)|, |J_{\theta'}(0^\Sigma)|\} \quad (55)$

$\leq \frac{|J_\theta(0^\Sigma)| + |J_{\theta'}(0^\Sigma)|}{2} \quad (56)$

$\leq \frac{R_d(\theta) + R_d(\theta')}{2} \quad (57)$

$\leq \frac{R_d(\theta) + R_d(\theta') - 2}{4} \quad (58)$

(3) W.l.o.g., the BDTs $\theta$ and $\theta'$ have the form depicted in Fig. 3 with $N_1 = |J_\theta(0^\Sigma)|$, $N_2 = |J_{\theta'}(0^\Sigma)|$ and bijections $\sigma' : \{1, \ldots, N_1\} \to J_\theta(0^\Sigma)$ and $\sigma'' : \{1, \ldots, N_2\} \to J_{\theta'}(0^\Sigma)$.

Firstly, we show that $J_\theta(0^\Sigma)$ is a feasible solution to the instance of MIN-SET-COVER w.r.t. $(U, \Sigma)$:

Thus, $h(\Sigma')$ is a feasible solution to the instance of MIN-DEPTH-BDT (resp. MIN-LENGTH-BDT) w.r.t. $D_{U, \Sigma}$.

$\forall u \in U : f_\sigma(u) = 1 \quad (\text{Def. } D_{U, \Sigma})$

$\Rightarrow \forall u \in U : f_\theta(x_u) = 1 \quad (\text{Def. } D_{U, \Sigma})$

Thus, $h(\Sigma')$ is a feasible solution to the instance of MIN-DEPTH-BDT (resp. MIN-LENGTH-BDT) w.r.t. $D_{U, \Sigma}$. –

5
Secondly, we show that $J_{\nu}(\hat{\Sigma})$ is a feasible solution to the instance of MIN-SET-COVER w.r.t. $(U, \Sigma)$.

\[
\forall u \in U: f_{\nu}(x_u) = 0 \quad \text{(Def. } D_{U, \Sigma}) \tag{62}
\]

\[
\Rightarrow \forall u \in U \exists j \in \{1, \ldots, N_2\}: x^u_{\sigma^j} = 1 \quad \text{(Fig. 3)} \tag{63}
\]

\[
\Rightarrow \forall u \in U \exists j \in \{1, \ldots, N_2\}: u \in \sigma^j \quad \text{(Def. } D_{U, \Sigma}) \tag{64}
\]

\[
\Rightarrow \bigcup_{j \in [N_2]} \sigma^j = \bigcup_{\sigma \in J_{\nu}(\hat{\Sigma})} \sigma = U \quad \text{(65)}
\]

Thus, $g(\theta, \theta')$ is a feasible solution to the instance of MIN-SET-COVER w.r.t. $(U, \Sigma)$. □

**Corollary 4.** For any instance $(U, \Sigma, m)$ of SET-COVER and any solution $(\theta, \theta')$ to the instance of BOUNDED-DEPTH-BDT w.r.t. $D_{U, \Sigma}$ and $2m$ (resp. BOUNDED-LENGTH-BDT w.r.t. $D_{U, \Sigma}$ and $4m + 2$), there is a solution $\Sigma' \subseteq \Sigma$ to the instance of SET-COVER w.r.t. $(U, \Sigma, m)$. Moreover, this solution $\Sigma'$ can be computed efficiently from $(\theta, \theta')$.

**Theorem 3.** BOUNDED-DEPTH-BDT and BOUNDED-LENGTH-BDT are NP-complete.

**Proof.** Firstly, BOUNDED-DEPTH-BDT and BOUNDED-LENGTH-BDT are in the complexity class NP, since solutions $(\theta, \theta')$ can be verified in polynomial time. Secondly, both problems are NP-hard, as SET-COVER $\leq_p$ BOUNDED-DEPTH-BDT and SET-COVER $\leq_p$ BOUNDED-LENGTH-BDT, by Corollaries 3 and 4. □

### 6 Hardness of Approximation

Next, we show for any $\epsilon \in (0, 1)$ that no polynomial-time $(1 - \epsilon) \ln(|A| + |B| - 1)$-approximation algorithm exists for SEPARATION w.r.t. the family of DNFs or BDTs and w.r.t. $R_l$ or $R_d$, unless $P = \text{NP}$. In order to arrive at this result, we construct an approximation-preserving reduction of MIN-SET-COVER that allows us to transfer the tight approximation bound for MIN-SET-COVER established by Dinur and Steurer (2014).

To begin with, we observe that SEPARATION is not in the complexity class NP, neither w.r.t. the family of DNFs nor w.r.t. the family of BDTs, and neither w.r.t. $R_l$ nor $R_d$. The reason is that feasible solutions can be of a size exponential in the size of the problem. However, it is sufficient to rule out redundant solutions in order to arrive at a problem in NP: For DNFs, we call a feasible solution $(\theta, \theta')$ irreducible if removing any single term from $\theta$ or $\theta'$ would violate exactness or non-contradictoriness. For BDTs, we call a feasible solution $(\theta, \theta')$ irreducible iff pruning any subtree from $\theta$ or $\theta'$ would violate exactness or non-contradictoriness. The restrictions of SEPARATION to irreducible feasible solutions are in the complexity class NP.

**Lemma 5.** Consider any instance $(U, \Sigma)$ of MIN-SET-COVER and any solution $\Sigma$ to this instance. Consider the instance of SEPARATION w.r.t. the Haussler data $D_{U, \Sigma}$, w.r.t. the family of DNFs (resp. BDTs), and with $R \in \{R_l, R_d\}$. Moreover, consider any solution $(\hat{\theta}, \hat{\theta'})$ and feasible solution $(\theta, \theta')$ to this instance. The function $g : \Theta^2 \rightarrow 2^\Sigma$ from Lemma 2 (resp. Lemma 4) is such that for any $r > 1$:

\[
\frac{R(\theta) + R(\theta')}{R(\hat{\theta}) + R(\hat{\theta'})} \leq r \Rightarrow \frac{|g(\theta, \theta')|}{|\Sigma|} \leq r \quad \text{if} \quad r = 2 \tag{66}
\]

**Proof.** We have

\[
|g(\theta, \theta')| \leq \frac{R(\theta) + R(\theta')}{2} \quad \text{by Lemma 2 (resp. Lemma 4)} \tag{67}
\]

by Lemma 2 (resp. Lemma 4).

Consider the function $h : 2^\Sigma \rightarrow \Theta^2$ from Lemma 1 (resp. Lemma 3). For $(\hat{\theta}, \hat{\theta'}) = h(\Sigma)$ we have, by optimality of $(\theta, \theta')$:

\[
R(\hat{\theta}) + R(\hat{\theta'}) \leq R(\theta) + R(\theta') \leq 2|\Sigma| \quad \text{(by Lemma 1 and 3)} \tag{68}
\]

\[
|g(\theta, \theta')| \leq \frac{R(\theta) + R(\theta')}{R(\hat{\theta}) + R(\hat{\theta'})} \quad \text{(by (67))} \tag{69}
\]

Analogously, for MIN-DEPTH-DNF, we have

\[
\frac{|g(\theta, \theta')|}{|\Sigma|} \leq \frac{R_d(\theta) + R_d(\theta') - 1}{R_d(\hat{\theta}) + R_d(\hat{\theta'}) - 1} \leq \frac{R_d(\theta) + R_d(\theta')}{R_d(\hat{\theta}) + R_d(\hat{\theta'})} - 1 \tag{70}
\]

And for MIN-LENGTH-BDT, we have

\[
|g(\theta, \theta')| \leq \frac{R_l(\theta) + R_l(\theta') - 2}{R_l(\hat{\theta}) + R_l(\hat{\theta'}) - 2} \leq \frac{R_l(\theta) + R_l(\theta')}{R_l(\hat{\theta}) + R_l(\hat{\theta'})} - 2 \tag{71}
\]
Theorem 4. For any number \( n \geq 2 \) of data points and any \( \epsilon \in (0,1) \), there is no polynomial-time \((1-\epsilon)\ln(n-1)\)-approximate algorithm for \textsc{min-length-dnf}, \textsc{min-depth-dnf}, \textsc{min-length-bdt} or \textsc{min-depth-bdt}, unless \( P=NP \).

Proof. Suppose there is a polynomial-time algorithm \( A_1 \) that outputs for every instance of \textsc{separation} w.r.t. the family of DNFs (resp. BDTs), \( R \in \{R_1, R_2\} \) and any Boolean labeled data \( D \), a solution \( A_1(D) = (\theta, \theta') \) with

\[
\frac{R(\theta) + R(\theta')}{R(\theta)} \leq (1-\epsilon) \ln(|A|+|B|-1) .
\]

Let \( g \) be the function defined in Lemma 5. Define the algorithm \( A_2 \) such that for any instance \((U, \Sigma)\) of \textsc{min-set-cover}, \( A_2(U, \Sigma) = g(A_1(D_U, \Sigma)) \). Let \( n = |U| = |A| + |B| - 1 \). By Lemma 5, the algorithm \( A_2 \) is a polynomial-time \((1-\epsilon)\ln n\)-approximate algorithm for \textsc{min-set-cover}. This contradicts Theorem 1.

Next, we show that the analysis of the approximation algorithm is essentially tight, by constructing an example which matches the approximation ratio.

Lemma 7. For any finite, non-empty set \( J \), the set \( X = \{0,1\}^J \), the \( x^i \in X \) such that

\[
\forall i, j \in J: \quad x^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}
\]

any \( k \in J \), the set \( A = \{x^i \mid i \in J \setminus \{k\}\} \) and the set \( B = \{x^k\} \), the algorithm \( A' \) outputs a solution \((\hat{\theta}, \hat{\theta}')\) to the instance of \textsc{min-length-dnf} w.r.t. \((J, X, A, B)\) with the approximation ratio \( \frac{1}{2}|J||A \cup B| \).

Proof. For the instance of \textsc{min-length-dnf-total} w.r.t. \( D_1 = (J, X, A) \), the algorithm \( A \) outputs the DNF

\[
\theta = \{(\{i\}, J \setminus \{i\}) \mid i \in J \setminus \{k\}\}
\]

as this is the only feasible solution.

For the instance of \textsc{min-length-dnf-total} w.r.t. \( D_0 = (J, X, B) \), the algorithm \( A \) outputs the DNF

\[
\theta' = \{(\{k\}, J \setminus \{k\}\)
\]

as this is the only feasible solution.

Therefore, the total length of the DNFs output by \( A' \) is

\[
R_l(\theta) + R_l(\theta') = |J||A \cup B| .
\]

The optimal solution to the instance of \textsc{min-length-dnf} w.r.t. \( D \) is \((\hat{\theta}, \hat{\theta}')\) such that

\[
\hat{\theta} = \{(\emptyset, \{k\})\}
\]

\[
\hat{\theta}' = \{\emptyset, \{k\}\}
\]

with length \( R_l(\hat{\theta}) + R_l(\hat{\theta}') = 2 \). Therefore, the approximation ratio is

\[
\frac{R_l(A(D_1)) + R_l(A(D_0))}{R_l(\hat{\theta}) + R_l(\hat{\theta}')} = \frac{1}{2}|J||A \cup B| .
\]
8 Conclusion

We have studied the problem of separating disjoint, non-empty sets \( A, B \subseteq \{0, 1\}^J \) by a pair of DNFs or BDTs defining (total) Boolean functions \( f, g : \{0, 1\}^J \to \{0, 1\} \) such that \( f(A) = \{1\} \) and \( g(B) = \{1\} \) and \( f \cdot g = 0 \). We have shown: Deciding whether such pairs of DNFs or BDTs exist, for which the sum of their lengths or depths is bounded by a constant, is \( \text{NP} \)-complete. Moreover, minimizing the sum of their lengths or depths is inapproximable to within \((1 - \epsilon) \ln(|A| + |B| - 1)\) for any \( \epsilon > 0 \), unless \( \text{P} = \text{NP} \). In light of analogous inapproximability bounds known from related work for the problem of learning a single DNF or BDT of a (total) Boolean function, we conclude that a transition to partial Boolean functions does not overcome the hardness of the learning problem that becomes manifests in a fundamental relation to \textsc{min-set-cover}.

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