The Vainshtein conditions: The Vainshtein mechanism in terms of St¨uckelberg functions

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Abstract

Here I develop the simplest method in order to evaluate whether or not the Vainshtein mechanism can operate for a given set of parameters in a given solution. The method is based on the formulation of the mechanism in terms of the St¨uckelberg functions given in Int.J.Mod.Phys. D24 (2015) 1550022 and arXiv:1305.0475 [gr-qc]. In such a case, the Vainshtein scale appears as an extremal condition of the dynamical metric. If we fix the graviton mass, we can define the parameter-dependent Vainshtein scale. Then for parameters where the Vainshtein scale vanishes or becomes smaller than the gravitational radius, the mechanism is absent. At the other extreme, if the Vainshtein scale is finite or infinite, then the mechanism can operate, although this condition not necessarily guarantees consistent results with respect to General Relativity (GR). Another and equivalent point of view is to fix the Vainshtein scale as an invariant. In such a case we can define a parameter-dependent graviton mass, such that the absence of the Vainshtein mechanism is equivalent to an almost zero (but finite) graviton mass. Then the massive gravitons can propagate everywhere and it becomes difficult to screen its effects. At the other extreme, the validity of the screening mechanism is equivalent to large values of the parameter-dependent graviton mass such that the massive gravitons are not able to propagate and then its effects are screened.

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I. INTRODUCTION

The first massive gravity formulation was done at the linear level by Fierz and Pauli [1]. This first attempt failed after comparing its predictions with the solar system observations. The reason for the failure of the theory at this level is the additional attractive effect due to the coupling between the scalar component and the trace of the energy-momentum tensor. This coupling remains even at the mass-less limit and is known as the vDVZ discontinuity [2]. Then after Vainshtein proposed that it is possible to recover the predictions of GR at scales near the source if we consider a non-linear formulation of massive gravity instead of the standard linear version [3]. Introducing non-linearities however, makes the theory pathological due to the appearance of a ghost [4]. In fact, the recovery of GR at scales of the solar system was due to the presence of this ghost since the negative kinetic energy term reproduced by the ghost exactly cancels the positive kinetic energy contribution coming from the scalar component. The details about this physical fact are explained in [5]. Due to the presence of a ghost, the non-linear theory was considered pathological. Later de-Rham, Gabadadze and Tolley discovered that by tuning in an appropriate way the different parameters of the potential, it was then possible to reproduce a ghost-free formulation of massive gravity. The essential idea is that the pathological terms of the action, after being grouped with higher order terms, are total derivatives and they will never appear in the equations of motion. The theory formulated in this way is called dRGT [6]. Inside this theory, the recovery of GR at scales of the solar system is due to the non-linearities, relevant below the Vainshtein scale which is approximately $r_V \sim (GM/m^2)^{1/3}$. In [7], the author formulated the Vainshtein mechanism in terms of Stückelberg functions. The Vainshtein scale then appeared as an extremal condition of the dynamical metric in unitary gauge. This is equivalent to say that the scale is an extremal condition of the massive action. In this manuscript, I use this formulation in order to explain for which set of parameters it is possible to expect the Vainshtein mechanism to operate and for which set of parameters, the mechanism is absent. The theory has generically three free-parameters, namely, the graviton mass, and the two free-parameters of the potential. By imposing the Schwarzschild de-Sitter background condition, the number of free-parameters is reduced to two. If we fix the graviton mass, then we can define a parameter-dependent Vainshtein scale. For the set of parameters where this scale vanishes or becomes smaller than the gravitational radius $GM$, the mechanism must be absent. On the other hand, for the set of parameters where the mechanism appears, the parameter-dependent Vainshtein scale must be finite. If it becomes infinite, then the screening mechanism should appear at any scale larger than $GM$. An alternative point of view is to set the Vainshtein scale as an invariant. Then we can define a parameter-dependent effective graviton mass. At scales where the effective graviton mass almost vanishes, the mechanism should be expected to be absent and the massive gravitons can propagate freely everywhere. On the other hand, if the effective graviton mass is finite, the mechanism must appear. If the effective graviton mass is infinite, then the massive gravitons cannot propagate so far and then GR can in principle be recovered. It is important to remark that the fact that the Vainshtein mechanism operates, does not guarantee the recovery of GR. This fact will depend on additional physical considerations like the effective Newtonian constant of the theory for example. Then we can say that the conditions able to predict the existence of the Vainshtein mechanism in this manuscript, are necessary but not sufficient in order to recover GR at the solar system scale.
II. THE SCHWARZSCHILD DE-SITTER SOLUTION IN DRGT

The black-hole solutions inside the non-linear formulation were already found in [8, 9]. The simplest and most generic solution for the spherically symmetric solution, was found in [10]. All the solutions satisfying the stationary condition, are defined as:

\[ ds^2 = G_{tt} dt^2 + G_{rr} S^2 dr^2 + G_{rt} (dr dt + dtdr) + S^2 r^2 d\Omega_2^2, \]  

(1)

where:

\[ G_{tt} = -f(Sr)(\partial_t T_0(r,t))^2, \quad G_{rr} = -f(Sr)(\partial_r T_0(r,t))^2 + \frac{1}{f(Sr)}, \]

\[ G_{tr} = -f(Sr)\partial_t T_0(r,t)\partial_r T_0(r,t), \]

(2)

and \( f(Sr) = 1 - \frac{2GM}{Sr} - \frac{1}{3\zeta} \Lambda (Sr)^2 \), with \( S \) being the scale factor which depends on the free-parameters of the theory. In this previous solution, all the degrees of freedom are inside the dynamical metric. Here we are in unitary gauge and then the fiducial metric is:

\[ f_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dr^2 + r^2( d\theta^2 + r^2 \sin^2 \theta ). \]  

(3)

The solution (2) is equivalent to:

\[ ds^2 = -f(Sr) dT_0(r,t)^2 + \frac{S^2 dr^2}{f(Sr)} + S^2 r^2 d\Omega^2, \]

(4)

where \( T_0(r,t) \) corresponds to the Stückelberg function. The non-triviality of \( T_0(r,t) \) contains the information of the extra-degrees of freedom [7, 10–12]. For the family of solutions with two free-parameters and the Stückelberg function constrained, the relation between the cosmological constant and the graviton mass is given by [10]:

\[ \Lambda = -m^2 \left( 1 - \frac{1}{S} \right) \left( 2 + \alpha - \frac{\alpha}{S} \right). \]  

(5)

Depending on the interpretation of the theory, the cosmological constant can be zero if we fix the graviton mass as an invariant. If however, what we keep as an invariant is the cosmological constant, then the value taken by the graviton mass might change. In particular, the relation \( \beta = (3/4)\alpha^2 \) can give us a zero \( \Lambda \) value or an infinite graviton mass, depending on the point of view. Independent of the interpretation used, the Stückelberg function has to satisfy the constraint [10]:

\[ (T_0(r,t))^2 = \frac{1 - f(Sr)}{f(Sr)} \left( \frac{S^2}{f(Sr)} - T_0^2 \right). \]  

(6)

A global solution of this previous constraint, is given by the Finkelstein-type form [10]:

\[ T_0(r,t) = St \pm \int^{Sr} \left( \frac{1}{f(u)} - 1 \right) du. \]

(7)

For the family of solutions with one free-parameter but with the Stückelberg function arbitrary, in principle there is no clear solution for \( T_0(r,t) \). The arbitrariness of \( T_0(r,t) \) was considered as pathological for the perturbative analysis considered in [10]. An alternative interpretation, was however considered in [12].
III. THE VAINSHTEIN CONDITIONS

The Vainshtein conditions were derived originally by the author in [7]. They express the Vainshtein scale as an extremal condition of the dynamical metric. This result helps us to find a relation between the Stückelberg function $T_0(r, t)$ and the function $f(Sr)$ contained inside the components of the dynamical metric. The key relations are:

$$\partial_r T_0(r, t) = 0 \rightarrow r << r_V, \quad \partial_r T_0(r, t) \neq 0 \rightarrow r >> r_V,$$

$$T''_0(r, t) = 0 \rightarrow r = r_V,$$  \hspace{1cm} (8)

with $r_V$ being the Vainshtein scale. The vanishing condition $T''_0(r, t) = 0$, guarantees the recovery of GR in the usual Schwarzschild-like coordinates. The same set of relations, could also be derived from the most direct condition $dU(g, \phi) = 0$ [7].

IV. THE VAINSHTEIN CONDITIONS: EXTREMAL CONDITIONS ON THE MASSIVE ACTION

The Vainshtein scale, corresponding to an extremal conditions of the dynamical metric, can also be derived directly from the potential $U(g, \phi)$ as has been explained in the previous section. The key point is to understand that in unitary gauge, namely, when all the degrees of freedom (Nambu-Goldstone bosons) are inside the dynamical metric (eaten up by the dynamical metric), then any variation of the potential, is directly related to the variations on the dynamical metric. The relevant mathematical condition is [7]:

$$dU(g, \phi) = \left( \frac{\partial U(g, \phi)}{\partial g} \right)_\phi dg + \left( \frac{\partial U(g, \phi)}{\partial \phi} \right)_g d\phi.$$  \hspace{1cm} (9)

The importance of the Vainshtein conditions is that the details of the potential are not relevant at all in order to derive the Vainshtein scale. All the information about the Vainshtein scale is contained inside the dynamical metric if it contains all the degrees of freedom. For stationary cases, where the metric components are time-independent, then eq. (9) becomes:

$$dU(G) = \left( \frac{\partial U(G)}{\partial g} \right) dg = 0,$$  \hspace{1cm} (10)

and this is just equivalent to:

$$dg = 0,$$  \hspace{1cm} (11)

if the matrix $\left( \frac{\partial U(G)}{\partial g} \right)$ is non-singular. The condition (11), has to be satisfied by each component of the metric containing the information of the extra-degrees of freedom. The conditions are valid at the Vainshtein scale. Then whenever we want to calculate the Vainshtein radius of the theory, all what we have to do is to translate all the degrees of freedom to the dynamical metric and then apply the conditions (11) and finally solve algebraically for $r = r_V$ [7].
FIG. 1: The parameter-dependent Vainshtein scale for a fixed graviton mass. Note that The Vainshtein scale becomes asymptotically infinite when $\alpha \to 0$ and $\alpha \to \infty$. The Vainshtein scale has a minimum for $\alpha = 2$. The factor $\left( \frac{3GM}{m^2} \right)^{1/3}$ has been normalized to be one. This figure also shows the qualitative behavior of the Vainshtein scale for one of the branches of the solution with $\beta = 0$ and $\alpha$ arbitrary.

V. THE PARAMETER-DEPENDENT VAINSHTEIN SCALE: THE CASE WITH ONE FREE-PARAMETER WITH A FIXED GRAVITON MASS

Here I derive the parameter-dependent Vainshtein scale for the two type of solutions obtained in [10]. The first solution to be considered is the one with one free-parameter but with the St"{u}ckelberg function arbitrary. In this case, in principle there is no correspondence between $T_0(r, t)$ and the function $f(Sr)$. The dynamical metric in unitary gauge takes the explicit form given by eq. (2) but taking into account that $T_0(r, t)$ is in principle arbitrary. However, the Vainshtein conditions can still be applied. For the stationary case, in agreement with [7], the Vainshtein scale can be obtained as:

$$f'(Sr) = 0, \quad \Rightarrow \quad r = r_V,$$

$$r_V = \frac{1}{S} \left( \frac{3GM}{\Lambda} \right)^{1/3} = \frac{\alpha + 1}{\alpha^{2/3}} \left( \frac{3GM}{m^2} \right)^{1/3}. \quad (12)$$

This previous result is the parameter-dependent Vainshtein scale. It is an effective value of the Vainshtein radius depending on the free-parameters of the theory. Fig. 1 shows the behavior of this scale as a function of $\alpha$. Note that the Vainshtein scale is non-zero for any positive value taken by the parameter $\alpha$. This means that the Vainshtein mechanism will always appear for this solution. When $r_V \to \infty$, GR should be recovered everywhere. However, if we analyze term by term in the function $f(Sr)$, given in terms of $\alpha$ by:

$$f(Sr) = 1 - \frac{2GM(\alpha + 1)}{sr} - \frac{1}{3} \frac{\alpha^2 r^2}{(\alpha + 1)^2}, \quad (13)$$

then it is clear that GR is recovered everywhere for $\alpha \to \infty$. However, for $\alpha \to 0$, the Newtonian term $\frac{2GM(\alpha + 1)}{sr}$ becomes dominant, rescaling the effective Newtonian constant and making gravity very strong and attractive. Then the effective Schwarzschild radius (gravitational radius) becomes very large. This means that even if the Vainshtein mechanism operates everywhere outside the gravitational radius, this part of the solution is unphysical from the experimental point of view. However, the main point here is that as far as the
FIG. 2: The behavior of the scale factor $S$ as a function of the free-parameters of the theory. This plot corresponds to the negative root square branch in agreement with eq. (15). This picture is the mirror image of the positive root square branch of $S$.

Vainshtein scale is non-zero, the mechanism is able to operate outside the gravitational radius. The case explored here is equivalent to the one with the condition $\beta = \alpha^2$.

A. The case with two free-parameters and the St"uckelberg function constrained and a fixed graviton mass

For this case, there is a direct correspondence between $T_0(r,t)$ and $f(Sr)$. However, still we can define the Vainshtein scale in the same way as in the previous case. It is given by the result (13), but with the cosmological constant now defined as in eq. (5). The scale factor $S$ is given by (10):

$$S = \frac{\alpha + \beta \pm \sqrt{\alpha^2 - \beta}}{1 + 2\alpha + \beta}. \quad (15)$$

The solution can then be classified in terms of the different values taken by the parameters $\alpha$ and $\beta$. Figure 2 shows the behavior of $S$ as a function of the two free-parameters of the theory for the negative root square branch. The same factor but for the positive branch is just the mirror image of this figure. In what follows, I will analyze two cases.

B. Case i). $\beta = 0$, $\alpha$ arbitrary and a fixed graviton mass

For this case, there are two possible values for the scale factor $S$. They are:

$$S_1 = \frac{2\alpha}{1 + 2\alpha}, \quad S_2 = 0. \quad (16)$$

The cosmological constant as a function of the graviton mass is in agreement with eq. (5) the following:

$$\Lambda_1 = -\frac{m^2}{S^2}(S - 1)(2S + \alpha S - \alpha), \quad \Lambda_2 = -\frac{m^2\alpha}{S^2}. \quad (17)$$
FIG. 3: The parameter-dependent Vainshtein scale for a fixed graviton mass and arbitrary parameter combination. The peaks of the figure represent the largest possible Vainshtein scales. There is a region of discontinuity around the peaks representing the absence of the mechanism for the parameter combination representing the region. The factor \( \left( \frac{3GM}{m^2} \right)^{1/3} \) has been normalized to one. This plot corresponds to the positive root square branch in agreement with eq. (15).

Note that \( \Lambda_2 \) diverges if \( \alpha \neq 0 \). If \( \alpha = 0 \), then we recover the special case of the previous section, namely, with one free-parameter but with the Stückelberg function arbitrary. This case is also divergent for \( \alpha \to 0 \). Here we can define the parameter-dependent Vainshtein scale as the extremal condition of the dynamical metric in unitary gauge [7]. The result is:

\[
\begin{align*}
    r_{V1} &= \frac{2\alpha + 1}{(6\alpha^2)^{1/3}} \left( \frac{3GM}{m^2} \right)^{1/3}, \\
    r_{V2} &= -\left( \frac{3GM}{Sm^2} \right)^{1/3}.
\end{align*}
\]

(18)

It is evident from the result (16) that \( r_{V2} \) will diverge for any value of \( \alpha \). This means that the screening effects are expected to appear everywhere up to the gravitational radius scale. However, it is easy to find that the effective gravitational radius is also divergent, then the Vainshtein mechanism and any physically relevant solution related to this branch is absent. For the case of \( r_{V1} \), the behavior is just the same as in the case studied previously when \( \beta = \alpha^2 \). Fig. (1) still represents the behavior for this case.

C. Case ii). Both, \( \beta \) and \( \alpha \) arbitrary and a fixed graviton mass

In this case it is evident that two branches of solutions will appear in agreement with eq. (15). The parameter-dependent Vainshtein scale for this case becomes:

\[
\begin{align*}
    r_V &= \left( \frac{1}{\alpha \pm \sqrt{\alpha^2 - \beta}} \right) \left( \frac{3GM(1 + 2\alpha + \beta)^3 \beta^2}{m^2(-2\alpha^3 \pm 2\alpha^2 \sqrt{\alpha^2 - \beta} + 3\alpha\beta \mp 2\beta \sqrt{\alpha^2 - \beta})} \right)^{1/3}.
\end{align*}
\]

(19)

The two branches of solutions can be observed in Figs. (3), (4) and (5). For the positive root square branch, note that the Vainshtein scale has a discontinuity region. In agreement with the plots, the Vainshtein mechanism operates whenever we have a finite value for \( r_V \). Regarding the branch of solutions corresponding to the negative root square in eq. (15), the result can be observed in Fig. (5). From the figure, we can see that the parameter-dependent Vainshtein scale is absent in most of the parameter ranges. There is only a narrow region where the mechanism can in principle works.
FIG. 4: The same Figure (3), but from a different angle. From this image it is clear to distinguish the regions where the parameter-dependent Vainshtein scale becomes large. The mechanism is almost absent for negative values of the parameter $\alpha$.

FIG. 5: The parameter-dependent Vainshtein scale for a fixed graviton mass and arbitrary parameter combinations. This plot corresponds to the negative root square branch in agreement with eq. (15). The mechanism is absent for most of the parameter ranges. There is only a small region where the mechanism can operate.

VI. THE PARAMETER-DEPENDENT GRAVITON MASS: THE VAINSHTEIN RADIUS AS AN INVARIANT

If we fix as an invariant the Vainshtein scale, then we can construct a parameter-dependent the graviton mass. In this situation, we can interpret any of the two free-parameters, namely $\alpha$ or $\beta$ as the graviton mass. Here I will analyze the cases exposed in the previous sections but by keeping the Vainshtein scale as an invariant and then observing the behavior of $m^2$. For the case with one free-parameter corresponding to the Vainshtein scale given in eq. (13), the effective graviton mass should behave as:

$$m^2 \sim \frac{(\alpha + 1)^3}{\alpha^2}. \quad (20)$$

Fig. (6) shows the behavior of $m^2$ for the different values taken by the free-parameter of the theory. The graviton mass has two branches. The negative values correspond to tachyons and for the moment we will consider this range of parameters as unphysical. Note that the graviton mass goes to infinite near $\alpha = 0$. In the neighborhood of this value, the massive gravitons cannot propagate and for moment we will consider this range of parameters as unphysical. Note that the graviton mass goes to infinite near $\alpha = 0$. In the neighborhood of this value, the massive gravitons cannot propagate and then GR can in principle be recovered. This is consistent with the ideas expressed in the previous section suggesting that for a fixed graviton mass, infinite values of the parameter-dependent Vainshtein scale correspond the range of parameters where GR can in principle be recovered.
FIG. 6: The parameter-dependent graviton mass as a function of the free-parameter of the theory. In this case, the Vainshtein scale is kept as an invariant. This case corresponds to the condition $\alpha = \beta^2$.

FIG. 7: The parameter-dependent graviton mass as a function of the two free-parameters of the theory. In this case, the Vainshtein scale is kept as an invariant. In this case both parameters are arbitrary. The plot corresponds to the positive root square branch of the scale parameter $S$.

A. The parameter-dependent graviton mass for $\beta = 0$, arbitrary $\alpha$ and a fixed Vainshtein scale

If we fix the Vainshtein scale as an invariant, the parameter-dependent graviton masses corresponding to the case with $\beta = 0$ and $\alpha$ arbitrary are in agreement with eqns. (18):

$$m_1^2 \sim \frac{(2\alpha + 1)^3}{\alpha^2}, \quad m_2^2 \sim -\frac{1}{S\alpha}.$$ (21)

FIG. 8: The parameter-dependent graviton mass as a function of the two free-parameters of the theory. This plot corresponds to the Fig. (7), but observed from a different perspective.
FIG. 9: The parameter-dependent graviton mass as a function of the two free-parameters of the theory. This plot corresponds to the case of two free-parameters for the negative root square branch of the scale factor $S$.

For $m_2$, $S \to 0$, which indicates that the parameter-dependent graviton mass diverges in that case for any value of $\alpha$. This again means that the massive gravitons cannot propagate for this branch of solutions and then GR could in principle be recovered everywhere. However, as has been discussed before, the gravitational radius for this solution also diverges, making it unphysical. The behavior of $m_1$ is exactly the same as in the case indicated in eq. (20) and then Fig. (6) is also appropriate for representing this branch of solutions.

B. The parameter-dependent graviton mass for arbitrary $\beta$ and $\alpha$ parameters with a fixed Vainshtein scale

In this case, the behavior of the parameter-dependent graviton mass is more complicated, but it can be visualized in Figs. (7) and (8) for the positive branch of the root square of $S$ (See Fig. (2)). The regions where the parameter-dependent graviton mass takes large values, correspond to regions where the massive gravitons can travel shorter distances. This is equivalent to an screening effect. Whenever the graviton mass is almost zero (but still finite), then the massive gravitons are able to travel larger scales and the screening effects are absent. Fig. (9) illustrates the same situation but for the branch where the root square factor in $S$ has a negative sign. The graviton mass takes finite values only for a small region among the possible values of $\alpha$ and $\beta$.

VII. SUMMARY

Here I have derived the simplest method in order to predict whether the Vainshtein screening mechanism might appear for some specific parameter combinations of the theory. The method is based on the Vainshtein conditions derived by the author in [7]. The details of the potential are not important for the present analysis. The regions where the Vainshtein mechanism operates are not necessarily relevant from the physical point of view. This means that the conditions derived in this manuscript are necessary but not sufficient in order to guarantee the recovery of GR. The recovery of GR not only implies the predictions at the solar system scale but also the predictions related to the physics of black-holes.
In [12] for example, it was discovered that some parameter combinations can reproduce branch point effects when we analyze the periodicity structure of the propagators. This manuscript is developed by using two equivalent points of view. The first one is worked by fixing the graviton mass and then deriving an effective Vainshtein scale. The second point of view is developed by fixing the Vainshtein scale and then deriving an effective version of the graviton mass. Here I consider that both points of view are physically equivalent and considering one or the other depends on the physical situation analyzed. It would be interesting to extend these ideas to other solutions like the ones proposed in [13] among others, where other gravitational contributions might appear. The method could also be extended for the analysis of more complicated solutions like the Kerr one, discovered recently inside the massive gravity formulations [14]. For the case of solutions containing the time variable inside the metric components, the Vainshtein conditions [9] cannot ignore the time variation in such a case. Then the parameter combinations where the Vainshtein (screening) mechanism appears might change with time.

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