Conditional Linearizability Criteria for Scalar Fourth Order Semi-Linear Ordinary Differential Equations

F M Mahomed\textsuperscript{1} and A Qadir\textsuperscript{2}

\textsuperscript{1}School of Computational and Applied Mathematics, Centre for Differential Equations, Continuum Mechanics and Applications
University of the Witwatersrand, Wits 2050, South Africa
Fazal.Mahomed@wits.ac.za
\textsuperscript{2}Centre for Advanced Mathematics & Physics
National University of Sciences & Technology
Campus of College of Electrical & Mechanical Engineering
Peshawar Road, Rawalpindi, Pakistan
aqadirmath@yahoo.com

Abstract. Using geometric methods for linearizing systems of second order cubically semi-linear ordinary differential equations and third order quintically semi-linear ordinary differential equations, we extend to the fourth order by differentiating the third order equation. This yields criteria for linearizability of a class of fourth order semi-linear ordinary differential equations, which have not been discussed in the literature previously. It is shown that the procedure can be extended to higher order. Though the results for the higher orders are complicated, they are doable by algebraic computing. The standard Lie approach, as developed at present does not seem to be amenable to giving results that can be handled even by algebraic computing.

Keywords. Linearizability, symmetry algebra, geometric approach

1 Introduction

First order ordinary differential equations (ODEs) can always be linearized (i.e. converted to linear form) \cite{1} by point transformations \cite{2}. Lie \cite{3} showed that all second order ODEs that can be converted to linear form must be cubically semi-linear

$$y'' + E_3(x, y)y^3 + E_2(x, y)y^2 + E_1(x, y)y' + E_0(x, y) = 0,$$

the coefficients $E_0$ to $E_3$ satisfying the over-determined integrable system

$$
\begin{align*}
    b_x &= -\frac{1}{3}E_1y + \frac{2}{3}E_{2x} + be - E_0E_3, \\
    b_y &= E_{3x} - b^2 + bE_2 - E_1E_3 + eE_3, \\
    e_x &= E_{0y} + e^2 - eE_1 - bE_0 + E_0E_2, \\
    e_y &= \frac{2}{3}E_1y - \frac{1}{3}E_{2x} - be + E_0E_3,
\end{align*}
$$

(2)
where \( b \) and \( e \) were (for Lie’s purposes) auxiliary variables and the suffices \( x \) and \( y \) refer to partial derivatives. The auxiliary variables arise naturally in the geometric approach mentioned shortly.

The linearization programme was carried forward to the third order by Chern [4, 5] using contact transformations to discuss the linearizability of equations reducible to the linear forms \( u'''(t) = 0 \) and \( u'''(t) + u(t) = 0 \) and Grebot [6, 7] used point transformations that were restricted to the class of transformations \( t = \phi(x) \), \( u = \psi(t, x) \) for the same purpose. It was subsequently shown [8] that there are three classes of third order ODES that are linearizable by point transformations, viz. those that reduce to the above two forms or \( u'''(t) + \alpha(t)u(t) = 0 \). (Note that one cannot choose \( \alpha \) to be a constant, as it would then reduce to the second form by re-scaling the independent variable.) After that, Neut and Petitot [9] and later Ibragimov and Maleshko [10] used the original Lie procedure [3], of postulating some point transformation that yields the desired linearizability, and determining the criteria for it to exist. The resulting criteria are very complicated and barely manageable for algebraic computing. Clearly, higher order attempts would not be feasible for this method. Maleshko [11] also provided a simple algorithm to reduce third order odes of the form \( y''' = F[y, y', y''] \) to second order and used the \( \partial / \partial t \) symmetry to reduce the order to two. He then used the Lie linearizability criteria to determine the linearizability of the third order equation. This could be trivially extended to the fourth order for \( y''' = F[y', y'', y'''] \), but this extension would not work for any non-trivial problem.

Though no progress was made towards providing explicit procedures for linearizing systems of ODEs, it was shown [12] that there will generally be multiple classes, even for second order systems. The number of classes for even dimensions, \( 2m \), is \( 2m^2 + 3 \) and for odd dimensions, \( (2m - 1) \) it is \( 2m^2 - 2m + 4 \). Clearly, there was no question of proceeding to higher order systems with the previous procedures.

Using the connection between the isometry algebra and the symmetries of the system of geodesic equations [13, 14], linearizability criteria were stated for a system of second order quadratically semi-linear ODEs, of a class that could be regarded as a system of geodesic equations [15], that we will call of geodesic type. The criteria came from requiring that the coefficients in the equations, regarded as Christoffel symbols, yield a zero curvature tensor. The criteria also provide the required compatibility conditions. Further, even for larger dimensional systems, it is possible to associate a metric with the system of geodesics when these criteria are met [16]. The flatness of the metric allows a coordinate transformation to be defined from the given metric to a Cartesian (Euclidean or pseudo-Euclidean) form. These are the linearizing transformations. They can be determined using complex variables [15] and used to write down the solution in terms of the original variables.

Utilizing the projection procedure of Aminova and Aminov [13], which appeals to the fact that the geodesic equations do not depend on the geodetic parameter, the system of \( n \) second order ODEs of geodesic type can be reduced to system of \( (n - 1) \) second order
cubically semi-linear ODEs [17]. When applied to a system of two dimensions we get a scalar cubically semi-linear ode that yields the Lie criteria! Applied to a system of three dimensions, one obtains a system of two cubically semi-linear odes with extended Lie criteria. The system of odes so obtained is in the class of maximally symmetric equations (out of the total of five classes mentioned earlier).

Differentiating the quadratically and cubically semi-linear system of ODEs relative to the independent variable gives third order ODEs. Taking the general class of the scalar third order ODE one gets linearizability criteria for scalar third order ODEs [18]. This class is is not included in the Neut and Petitot [9] and Ibragimov and Maleshko classes [10]. Though there can be an overlap with the Maleshko class [11] it is obviously not contained in that either. It may be wondered how it can be outside the three classes allowed by [8]. The reason is that it is not obtained by point transformations but is linearized conditionally to a second order ODE being linearizable. In effect its first integral is a linearizable second order semi-linear ODE. This is, thus, a new type of linearizability.

The quadratically semi-linear system of geodesic type, on differentiation yields a cubically semi-linear third order system of the same dimension. This is obtained by replacing the second order term in the differentiated equation by a quadratic expression using the original equation. It could also be written as a third order ODE with a second order term multiplied by a first order term and then followed by a term quadratic in the first derivative. The linearizability criteria for this system for two dimensions have also been derived [19].

It is clear that the geometrical approach has far-reaching consequences for the linearizability of ODEs. In this paper it will be used for stating criteria for the conditional linearizability of fourth order ODEs in some detail and its utilization for higher orders will also be discussed. First a brief summary of the notation used, relevant for our present purposes, will be provided.

\section{Notation and review}

Equations of geodesic type for two dimensions are of the form

\begin{align*}
x'' &= a(x, y)x'^2 + 2b(x, y)x'y' + c(x, y)y'^2,
y'' &= d(x, y)x'^2 + 2e(x, y)x'y' + f(x, y)y'^2. \tag{3}
\end{align*}

There are six coefficients and they may be identified with the six Christoffel symbols for a 2-dimensional space,

\begin{align*}
\Gamma^1_{11} &= -a, \quad \Gamma^1_{12} = -b, \quad \Gamma^1_{22} = -c, \\
\Gamma^2_{11} &= -d, \quad \Gamma^2_{12} = -e, \quad \Gamma^2_{22} = -f. \tag{4}
\end{align*}
Requiring that the space corresponding to these Christoffel symbols is flat

\[ R^i_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{mk} \Gamma^m_{jl} - \Gamma^i_{ml} \Gamma^m_{jk} = 0, \]  

yields the linearizability conditions

\[ a_y - b_x + be - cd = 0, b_y - c_x + (ac - b^2) + (bf - ce) = 0, \]
\[ d_y - e_x - (ae - bd) - (df - e^2) = 0, (b + f)_x = (a + e)_y, \]  

which provide the metric coefficients through

\[ p_x = -2(ap + dq), q_x = -bp - (a + e)q - dr, r_x = -2(bq + er), \]
\[ p_y = -2(bp + eq), q_y = -cp - (b + f)q - er, r_y = -2(cq + fr). \]  

The compatibility of these equations is guaranteed by the curvature tensor being zero. The metric coefficients being obtained the linearizing transformation is available as a coordinate transformation and its inverse can then be computed.

The above system of two equations can be projected to the scalar cubically semi-linear ODE

\[ y'' + cy^3 - (f - 2b)y'^2 + (a - 2e)y' - d = 0, \]  

which is clearly of the form (1) for suitable values of the coefficients in the former equation. It is here that one sees the “auxilliary variables” arising naturally. Note that there are only four coefficients here as against the six Christoffel symbols and for the original system of two equations. This degeneracy allows for extra classes when this procedure is applied to larger systems of ODEs but does not interfere with the uniqueness for the scalar equation [16]. We shall write \( g = f - 2b \) and \( h = a - 2e \) for convenience. Again, the coordinate transformations can be obtained, but there is freedom of choice of \( g \) and \( h \) that can lead to more or less convenient coordinate transformations. The linearizability criteria can then be given in the form [20]

\[ 3(ch)_x + 3dc_y - 2gg_x - gh_y - 3c_{xx} - 2g_{xy} - h_{yy} = 0, \]
\[ 3(dg)_y + 3cd_x - 2hh_y - hg_x - 3d_{yy} - 2h_{xy} - g_{xx} = 0. \]  

We shall be using this form.

This equation can be differentiated to yield a third order equation of the general form

\[ y''' + (A_2y'^2 - A_1y' + A_0)y'' + B_4y'^4 - B_3y'^3 + B_2y'^2 - B_1y' + B_0 = 0, \]  

which is linear in the second derivative, but with a coefficient quadratic in the first derivative, where the coefficients are of this equation are identified with the original equation by,

\[ c = A_2 / 3, g = A_1 / 2, h = A_0, \]
\[ d = - \int B_2dx + k(y) = \int (B_1 - A_0x)dy + l(x), \]  

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\[ y''' + (A_2y'^2 - A_1y' + A_0)y'' + B_4y'^4 - B_3y'^3 + B_2y'^2 - B_1y' + B_0 = 0, \]  

which is linear in the second derivative, but with a coefficient quadratic in the first derivative, where the coefficients are of this equation are identified with the original equation by,
the constant of integration being fixed by compatibility with the metric tensor, provided
the compatibility conditions

\[ B_4 = A_{2y}/3, B_3 = A_{1y}/2 - A_{1x}/3, B_2 = A_{0y} - A_{1x}/2, \]

(13)
hold and the linearizability conditions (6) are satisfied. Once again, the coordinate trans-
formations are obtainable and provide the solution.

The above third order ODE is a total derivative. Though it may not be immediately
obvious that a given ODE is an exact derivative, the equation in this form may seem
trivial in some sense. One can use (8) in (10) to replace the second derivative term and
obtain a third order ODE quintically semi-linear in the first derivative,

\[ y''' - \alpha y^5 + \beta y^4 - \gamma y^3 + \delta y^2 - \epsilon y' + \phi = 0, \]

(14)
where

\[ \alpha = 3c^2, \beta = 5cg + c_y, \gamma = 4ch + 2g^2 + g_y - c_x, \delta = 3cd + 3gh + h_y - g_x, \]

(15)
and the compatibility conditions are

\[ \epsilon = 2dg + h^2 + d_y - h_x, \phi = dh - d_x. \]

(16)

This equation is not an exact derivative, and hence provides a non-trivial utilization of
the procedure given. The coefficients can be inverted to obtain the original coefficients
and hence the metric coefficients, yielding the solution.

We shall use the above formulation to obtain the linearizability criteria for higher order
equations in the next section.

3 Conditional linearizability criteria for fourth order ODEs

On differentiating (14) for the scalar third order ODE, writing the independent variable
as \( x \) and the dependent variable as \( y \), we get

\[ y''' - (5\alpha y^4 - 4\beta y^3 + 3\gamma y^2 - 2\delta y' + \epsilon)y'' - \alpha y^6 + (\beta y - \alpha_x)y^5 - (\gamma y - \beta_x)y^4 + \\
(\delta y - \gamma_x)y^3 - (\epsilon y - \delta_x)y^2 + (\phi_x - \epsilon_x)y' + \phi_x = 0. \]

(17)
The general form of this equation is

\[ y''' - (A_4y^4 - A_3y^3 + A_2y^2 - A_1y' + A_0)y'' - B_6y^6 + B_5y^5 - B_4y^4 + B_3y^3 - B_2y^2 + B_1y' - B_0 = 0, \]

(18)
subject to the identification of coefficients
\[ c = \sqrt{A_1/15}, g = (A_3 - 4c_y)/20c, h = (A_2 - 6g^2 - 3g_y + 3c_x)/12c, \]
\[ d = (A_1 - 6gh - 2h_y + 2g_x)/6c, \]  
with the constraints
\[ B_0 = A_4/5, B_5 = A_{3y}/4 - A_{4x}/5, B_4 = A_{2y}/3 - A_{3x}/4, B_3 = A_{1y}/2 - A_{2x}/3, \]
\[ B_2 = A_{0y} - A_{1x}/2, B_1 = dh_y + hd_y - d_{xy} - A_{0x}/2, B_0 = dh_x + hd_x - d_{xx}, \]
\[ A_0 = 2gd + h^2 + d_y - h_x. \]  

In this form it is a total derivative and may be regarded as trivial in some sense.

Replacing the second derivative by the first derivative expressions from (8) we get the fourth order ODE semi-linear in the first derivative in the seventh degree
\[ y''' + P_2 y'^7 - P_6 y^6 + P_5 y^6 + P_3 y^4 + P_2 y^2 + P_1 y' - P_0 = 0, \]  
which is not a total derivative. For consistency of the identification of coefficients
\[ c = (P_7/15)^{1/3}, g = P_6/35c^2 - 2c_y/7c, \]
\[ h = P_5/27c^2 - 26g^2/27c + c_x/3c - gc_y/3c^2 - 8g_y/27c - c_yy/27c^2, \]
\[ d = P_4/21c^2 - 38gh/21c - 2g^3/7c^2 + 8gc_x/21c^2 - 8hc_y/21c^2 \]
\[ + g_x/3c - gg_y/3c^2 - 2h_y/7c + 2c_{xy}/21c^2 - g_{yy}/21c^2, \]  
so that all four coefficients are explicitly given. We now have four differential constraints
\[ P_3 = 28cdg + 13ch^2 + 12g^2h - 3(h + d)cx + 4dcy - (2g + 3h)g_x + (3h + 2d)g_y \]
\[-(c + 3g)h_x + 2(g + h)h_y - 3cd_x + (c + 2g)d_y + g_{xx} - 2h_{xy} + d_{yy}, \]
\[ P_2 = 18chd + 8g^2d + 7gh^2 - 6dc_x - 5hgx + 4dg_y - 4gh_x + 4hh_y - 3cd_x + 2gd_y \]
\[ + g_{xx} - 2h_{xy} + d_{yy}, \]
\[ P_1 = 6cd^2 - 8ghd + h^3 - 4dg_x - 3hh_x + 3dh_y - 2gd_x + 2hd_y + h_{xx} - 2d_{xy}, \]
\[ P_0 = 2gd^2 + h^2d - 2dh_x - hd_x + dd_y + d_{xx}, \]  
apart from the earlier stated linearizability conditions (9).

Instead of starting with the quintically semi-linear third order ODE, we can start with the form involving the second derivative (10). In this case we get
\[ y''' + (A_2 y'^2 - A_1 y' + A_0) y'' + (B_1 y' - B_0) y'^2 + (C_3 y'^3 - C_2 y'^2 + C_1 y' - C_0) y'' \]
\[-(D_5 y'^5 - D_4 y'^4 + D_3 y'^3 - D_2 y'^2 + D_1 y' - D_0) = 0, \]  
which is semilinear involving the third third derivative with a coefficient quadratic in the first derivative, second derivative squared with a coefficient linear in the first derivative, and otherwise quintic in the first derivative. It is a total derivative with the identification
\[ c = A_2/3 = B_1/6, g = A_1/2 = B_0/2, h = A_0, \]  
\[ d = (A_1 - 6gh - 2h_y + 2g_x)/6c. \]
the differential constraints
\[
C_3 = 7A_{2y}/3, C_2 = 5A_{1y}/2 - 2A_{2x}, C_1 = 3A_{0y} - 2A_{1x},
\]
\[
D_5 = A_{2yy}/3, D_4 = A_{1yy}/2 - 2A_{2xy}/3, D_3 = A_{0yy}/2 - A_{1xy} + A_{2xx}/3,
\]
and the additional constraints that yield \( d \):
\[
C_0 = d_y - 2A_{0x}, D_0 = d_{xx},
\]
which gives the value as a single and a double integral to be solved simultaneously
\[
d = \int (C_0 + 2A_{0x})dy + k(x) = \int (\int D_0 dx) dx + l(y)x + m(y),
\]
where \( k, l, m \) are arbitrary functions of the respective variables. We have the additional requirements for \( D_1 \) and \( D_2 \),
\[
D_1 + 2C_{0x} + 3A_{0xx} = 0, D_2 - C_{0y} - A_{1xx}/2 = 0.
\]
One can now use the third order equation \((10)\) to replace the third derivative term in \((24)\) and obtain the equation in a form quadratically semilinear in the second derivative that has a coefficient quadratically semilinear in the first derivative for the higher order, quartic in the first derivative for the one linear in the second derivative and otherwise of sixth order in the first derivative,
\[
y''' + (Q_1y' - Q_0)y'' - (R_4y'^4 - R_3y'^3 + R_2y'^2 - R_1y' + R_0)y''
\]
\[-(S_6y'^6 - S_5y'^5 + S_4y'^4 - S_3y'^3 + S_2y'^2 - S_1y' + S_0) = 0,
\]
with the identification of the first three coefficients given by
\[
c = Q_1/6, g = Q_0/2, h = (R_2 - Q_0^2 + Q_{1x} - 5Q_{0y}/2)/Q_1.
\]
Since the coefficient \( h \) is very complicated we shall use it as a shorthand for the above expression. The fourth coefficient is given by the solution of
\[
d = \int (R_0 - h^2 + 2h_x) dy + k(x).
\]
In the subsequent equations replacing \( d_y \) by the integrand above and writing \( d_x \) (as there is no convenient expression for it), the following constraint equations must hold
\[
R_4 = Q_1^2/4, R_3 = Q_1Q_0 + 7Q_{1y}/6, R_1 = 2Q_0h + h^2 + 3h_y - 2Q_{0x},
\]
\[
S_6 = Q_1Q_{1y}/12, S_5 = -Q_1Q_{1x}/36 + Q_0Q_{0y}/6 + Q_1Q_{0y}/4 + Q_{1yy}/6,
\]
\[
S_4 = -Q_0Q_{1x}/6 + hQ_{1y}/6 - Q_1Q_{0x}/4 + Q_0Q_{0y}/2 + Q_1h_y/2 - Q_{1xy}/3 + Q_{0yy}/2,
\]
\[
S_3 = Q_1(R_0 - h^2 - h_x)/2 - hQ_1/6 - Q_0Q_{0x}/2 + hQ_{0y}/2 + Q_0h_y + Q_{1xx}/6 - Q_{0xy} + h_{yy},
\]
\[
S_2 = Q_1d_x/2 + Q_0(R_0 - h^2 + h_x) - hQ_0 - hQ_{0x}/2 + R_{0y} + Q_{0xx}/2,
\]
\[
S_1 = h(R_0 - h^2 + 5h_x) - 2R_{0x} - Q_0d_x - 3h_{xx}, S_0 = hd_x - d_{xx}.\]
along with the linearization criteria. This is not a total derivative. Though looking more messy, it is perfectly usable.

Yet another form that is not a total derivative can be obtained from (24) by using (8) to replace the second derivative term. In this case we get

\[y''' + (A_2 y'^2 - A_1 y' + A_0)y''' + B_7 y'^7 - B_6 y'^6 + B_5 y'^5 - B_4 y'^4 + B_3 y'^3 - B_2 y'^2 + B_1 y' - B_0 = 0,\]  

(34)

with the identification of all four coefficients given by

\[c = \sqrt{A_2/3}, g = A_1/2, h = A_0,\]
\[d = (B_4 - 8cA_1A_0 - \frac{1}{4} A_1^3 + 3A_1c_x - 7A_0c_y + 2cA_1x - 5A_1A_1y/4 - 3cA_0x + 2c_{xy} - \frac{1}{2} A_1y))/4A_2,\]  

(35)

where we have largely used the symbol \(c\) rather than \(A_2\) to avoid the extra complications due to a square root and its differentiation. Further, though \(d\) is given algebraically, it is too complicated to use conveniently. The constraint equations are

\[B_7 = 6c^3, B_6 = 7c(A_1 + c_y), B_5 = 3A_2A_0 + 5cA_1^2/2 - 6cc_x + 7A_1c_y/2 + 5cA_1y/2 + c_{yy},\]
\[B_3 = 8cA_1d + 6cA_0^2 + A_1^2A_0 - 6A_0c_x + 7dc_y - A_1A_1x + 5A_0A_1y/2 - 2cA_0x + 3A_1A_0y/2 + cd_y + c_{xx} - A_1y + A_0y,\]
\[B_2 = 12cA_0d + A_1^2d + A_1A_0^2 - 6dc_x - 2A_0A_1x + 5dA_1y/2 - A_1A_0x,\]
\[+3A_0A_0y + A_1dy/2 + A_1xx/2 - 2A_0xy + d_{yy},\]
\[B_1 = 6cd^2 + 2A_1A_0d - 4dA_1x - 2A_0A_0x + 3dA_0y + A_0dy + A_0xx - 2d_{xy},\]
\[B_0 = d(A_1d + d_y - A_0x) + d_{xx},\]  

(36)

along with the linearization conditions. Again, this seems very messy but it has the advantage that all four coefficients are identified.

We could have one other procedure, to use both equations (8) and (10) to replace the second and third derivatives. Since this only involves the first derivative to the same power as before, this will not yield a new class. Thus we have the following theorems.

**Theorem 1:** Equation (21) is linearizable with the identifications (22) if the constraints (23) and the linearizability criteria (9) are satisfied, where \(c \neq 0\), with \(h = a - 2c, g = f - 2b\), after requiring consistency of the Christoffel symbols with the deduced metric coefficients.

It is to be noted that the four linearizability conditions (6) are stated in terms of the 6 coefficients \(a, ..., f\) and not the four coefficients \(c, d, g, h\). Thus there is degeneracy in the choices available. Any choices of \(a\) and \(e\) for a given \(h\), or \(f\) and \(b\) for a given \(g\), compatible with the metric coefficient relations (7) are permissible. For each such choice we would
get corresponding linearizability conditions. Instead, we use the conditions (9) to check linearizability and exploit the freedom of choice to construct a convenient metric.

Theorem 2: Equation (30) is linearizable with the identifications (31) and (32) if the constraints (33) along with the linearizability criteria (9) are satisfied, after requiring consistency of the Christoffel symbols with the deduced metric coefficients.

Theorem 3: Equation (34) is linearizable with the identifications (35) if the constraints (36) and the linearizability criteria (9) are satisfied, after requiring consistency of the Christoffel symbols with the deduced metric coefficients.

Note that there are two other equations, (18) and (24), that are total derivatives of linearizable equations with appropriate identifications, that are also linearizable subject to the corresponding constraints and conditions. We do not stress on these because they may be regarded, in some sense, as trivial.

4 Examples

In this section we present some examples of fourth order equations that can be linearized by our procedure.

1. The equation
   \[ y''' - (18y^2/y^2 - 16y'/y + k^2 - 5l)y'' + 12y^4/y^3 - 8ky^3/y^2 + kly' = 0, \]  
   (37)
   is a fourth order total derivative of the form of (18). However, it is clearly not trivial to spot this fact by looking at the equation. It satisfies the required constraints (20) and linearizability conditions (9).

2. The equation
   \[ y''' - 24y'^4/y^3 + 33ky'^3/y^2 + (28l - 10k^2)y'^2/y + (k^2 - 35l)ky'/2 - (k^2 - 5l)ly = 0, \]
   (38)
   which is a fourth order equation that is not a total derivative, of the form (21) and is linearizable as it satisfies the constraints (23) and the linearizability conditions (9).

3. The equation
   \[ y''' - (4y'/y - k)y'' - 4y'^2/y + (10y^2/y^2 - l)y'' - 4y^4/y^3 = 0, \]
   (39)
   which is a total derivative of the form of (24) and may be more easily identified as a total derivative. It is linearizable as it satisfies the constraints (26) and (29) and the linearizability conditions (9).
4. The equation
\[ y'''' - 4y''^2/y^4 - (6y''/y^2 - 8ky'/y + k^2 + l)y'' + 4y'/y^3 - 2ky''/y^2 - 4kly'^2/y + kly' = 0, \quad (40) \]
which is of the form (30) and satisfies the constraints (33) and the linearizability conditions (9). It is not a total derivative.

5. The equation
\[ y'''' - (4y'/y - k)y''' + 4y'^3/y^2 + (4l - k^2)y'^2/y - 7kly'/2 - 3l^2y = 0, \quad (41) \]
which is not a total derivative and is of the form (34) satisfying the constraints (36) and the linearizability criteria (9). As such it is linearizable.

Though it is not apparent from looking at the equations they come from differentiating the linearizable third order equation
\[ y''' - 6y'/y^2 + 8ky'^2/y - (k^2 - 5l)y'/2 + kly = 0, \quad (42) \]
and the total derivative linearizable equation
\[ y''' - (4y'/y - k)y'' + 2y'^3/y^2 - ly' = 0, \quad (43) \]
(in some cases subject to conditions of some equation holding) which both come from differentiating the second order linearizable equation
\[ y'' - 2y'^2/y + ky'/2 + ly = 0, \quad (44) \]
and in one case applying a condition. As such, their solutions necessarily have two arbitrary constants but may not have more. It may be noted that these equations do not have any explicit dependence on \( x \). That symmetry being guaranteed only one more needs to be looked for to find the solution by symmetry analysis. Using the procedure of writing the equation as of geodesic type, we can write down the solution directly.

The fact that all of these fourth order equations have a common root becomes obvious from our analysis right in the beginning, as the identification of the four coefficients in each case gives
\[ c = 0, g = 2, h = k/2, d = -l. \quad (45) \]

6. The fourth order ODE
\[ y''' + (3xy'^2 + 2/x)y''' + 6xy'y'' + (6y'^2 - 4/x^2)y'' + 4y'/x^3 = 0, \quad (46) \]
is of the form (24) and is hence a total derivative (though this fact is not obvious by inspection), with the coefficients given by (25), (27) and (28). It satisfies the constraints (26) and (29) and the linearizability criteria (9). As such it is linearizable.
7. The fourth order ODE
\[ y'''' + 6xy'y'' - (9x^2y^4 + 6y^2 + 8/x^2)y'' + 4y'/x^3 = 0, \] (47)
is of the form of (30), and hence is not a total derivative, with the coefficients given by (31) and (32) and satisfies the constraints (33) and the linearizability criteria (9). As such it is linearizable.

8. The fourth order ODE
\[ y'''' + (3xy^2 + 2/x)y'' + 6x^3y'^7 + 18y'^5 + 16y'^3/x + 12y'/x^3 = 0, \] (48)
is of the form of (34), and hence is not a total derivative, with the coefficients given by (35) and satisfying the constraints (36) and the linearizability criteria (9). As such it is linearizable.

9. The fourth order ODE
\[ y'''' - (15x^2y^4 + 21y^2 + 6/x^2)y'' - 6xy'^5 + 12y'/x^3 = 0, \] (49)
is of the form of (18), and hence is a total derivative, with the coefficients given by (19), satisfying the constraints (20) and the linearizability criteria (9). As such it is linearizable.

10. The fourth order ODE
\[ y'''' + (3xy^2 + 2/x)y'' + 6x^3y'^7 + 18y'^5 + 16y'^3/x + 12y'/x^3 = 0, \] (50)
is of the form of (21), and hence is not a total derivative, with the coefficients given by (22) and satisfying the constraints (23) and the linearizability criteria (9). As such it is linearizable.

It is again apparent that even the total derivative equation is not obviously so, and that the other equations being linearizable would not be clear by inspection. The coefficients of examples (6) to (10) are all \( c = x, h = 2/x, g = d = 0 \) and the root equation is the geodesic equation for flat space in polar coordinates. Thus the linearizing transformation is simply the conversion from polar to Cartesian coordinates.

It will be noticed that in the above examples either \( x \) or \( y \) are missing from the coefficients. This makes the application of the identification and constraints relatively trivial. We end with a couple of non-trivial examples, in which both arise.

11. The fourth order ODE
\[ y''' - 15x^3y'/y^6 - 15xyy^5/y^6 + 39y^5/y^4 + 39y'^4/xy^3 - 36y'^3/xy^2 - 36y'^2/x^2y + 24y'/x^3 = 0, \] (51)
is of the form of (21), and hence is not a total derivative. Its coefficients are given by (22) and come out to be \( c = -x/y^2, g = 1/y, h = 2/x, d = 0 \). They satisfy the constraints...
and linearizability criteria (9). As such this equation is linearizable. In fact it can be derived by differentiating the linearizable third order equation [19]

\[ y''' - 3x^2 y^5/y^4 - 3xy^4/y^3 + 6y^3/y^2 + 6y^2/xy - 6y'/x^2 = 0, \]  

writing the root equation from the coefficients, and using it to substitute the \( y'' \) arising from the differentiation. The solution is

\[ Ax + Bx/y = 1, \]  

where \( A \) and \( B \) are arbitrary constant real numbers.

12. The fourth order ODE

\[
y'''' - (6xy'/y^2 + 2/y)y^2 - (9x^2 y^4/y^4 - 2xy^3/y^3 - 7y^2/y^2 - 8y'/xy + 8/x^2)y''
+ 6x^2 y^6/y^5 - 2xy^5/y^4 - 2y^4/y^3 - 6y^3/xy^2 - 4y^2/x^2 y + 8y'/x^3 = 0, \]

is of the form of (30). Its coefficients are given by (31) and (32) and are the same as of the above example. Hence they share a common root equation. This equation arises by differentiating the root equation twice and using its first derivative to replace the \( y''' \). It satisfies the constraints (33) and the linearizability criteria (9) and is linearizable, yielding the same linear equation and possessing the same solution as of the previous example.

5 Concluding remarks

We have provided a procedure to determine the linearizability of some fourth order scalar semi-linear odes that are linearizable. We have written them as five classes. There could have been other ways of getting to the five classes, e.g. by first differentiating and then replacing or first replacing and then differentiating. Since, the general form of both procedures would be the same they are not different. Again, we could have used a lower order equation to partially replace terms in the higher order equation. For example, we could have used the second order equation to replace the quadratic term in the second derivative in (52) but retain the linear term as it is. These are not really independent in some sense. In this sense there are five independent classes. Two of these may be regarded as trivial as they are total derivatives. However, a glance at the equations will show that even these are not so easy to identify as total derivatives. The other three classes are not total derivatives. All can be thought of as arising from the same second order linearizable differential equation that satisfies the Lie conditions, by double differentiation, but conditional to the original differential equation and in some cases the differentiated differential equation. As such, the linearizability classes are non-classical and would not lie in the three classical classes reducible to the fourth order odes \( y''(x) = 0 \), \( y''(x) + y(x) = 0 \) and \( y'' + \alpha(x)y(x) = 0 \). These are guaranteed four arbitrary constants while our solutions are only guaranteed two.
Let us call the underlying second order equation the root equation, and the specific fourth order equations forms of the similar fourth order equations. It is clear that all forms arising from a common second order equation will form an equivalence class. As such, all similar fourth order equations will have two solutions in common but may have other, different solutions. The question arises whether these equivalence classes of linearizable fourth order semi-linear odes are disjoint and can be used to decompose the space of those odes that are linearizable of this type. In this context, it seems unlikely that this be the case because there would be some of them that are linearizable and have two solutions in common with others. As such, we conjecture that the set of linearizable fourth order odes with common root equations will not be decomposable into disjoint classes.

It is worth noting that there is only one class of linearizable second order semi-linear ode (as proved by Lie) and two classes of third order odes, one of which is a total derivative. For the fourth order we find five of which two are total derivatives and three are not. Can this procedure be carried further? The answer is “obviously it can”. The fifth order will have one form that only involves the first derivatives apart from the fifth derivative and is of ninth order in them. It is obvious that here all four coefficients will be easily identified, but now instead of the four constraints (22) we will have six constraints. Similarly, for the sixth order the corresponding class will have the first derivative to the eleventh power and will have eight constraints and so on. Of course the usual linearizability conditions (6) would also have to be met.

The question arises as to the number of classes that we can now have. Clearly, for the fifth order there will be five classes of total derivative forms obtained by differentiating the five classes of the fourth order. How many other classes will there be? The number can be obtained by counting the different forms that one could obtain and turns out to be six. Thus the total number of classes is eleven. It is obvious that there will then be eleven classes of total derivative sixth order equations obtained. One could continue the procedure of counting but it would be nice to have a general formula for the total number of classes for any order.

It is worth stressing that as can be seen from the examples, our procedure provides the solutions of the linearizable semi-linear equations of our classes for any order that is identified. It should not be too difficult to prepare an algebraic code for identification of coefficients, the constraints and the linearizability conditions for the various classes. Certainly, the form involving only the first order could be relatively trivially provided. It would be worth while to get the algebraic codes prepared for at least some of the higher order classes.

Acknowledgements
AQ is most grateful to DECMA and the School of Computational and Applied Mathematics, University of the Witwatersrand.
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