A cosmological scenario from the Starobinsky model within the $f(R,T)$ formalism

P.H.R.S. Moraes • P.K. Sahoo • G. Ribeiro • R.A.C. Correa

Abstract In this paper we derive a novel cosmological model from the $f(R,T)$ theory of gravitation, for which $R$ is the Ricci scalar and $T$ is the trace of the energy-momentum tensor. We consider the functional form $f(R,T) = f(R) + f(T)$, with $f(R)$ being the Starobinsky model, named $R + \alpha R^2$, and $f(T) = 2\gamma T$, with $\alpha$ and $\gamma$ being constants. We show that a hybrid expansion law form for the scale factor is a solution for the derived Friedmann-like equations. In this way, the model is able to predict both the decelerated and the accelerated regimes of expansion of the universe, with the transition redshift between these stages being in accordance with recent observations. We also apply the energy conditions to our material content solutions. Such an application makes us able to obtain the range of acceptability for the free parameters of the model, named $\alpha$ and $\gamma$.

Keywords extended theories of gravity; cosmological models; dark energy; cosmic transition

1 Introduction

The $f(R)$ theories of gravity (Sotiriou & Faraoni (2010); Felice & Tsujikawa (2010)) are an optimistic alternative to the shortcomings General Relativity (GR) faces as the underlying gravitational theory, such as those discussed by Padmanabhan (2003); Antoniadis et al. (2013); Demorest et al. (2010); Bull et al. (2016). They can account for the cosmic acceleration (Riess et al. (1998); Perlmutter et al. (1999)) with no need for a cosmological constant, providing good match between theory and cosmological observations (Tsujikawa (2008); Capozziello et al. (2005); Nojiri & Odintsov (2006)).

Particularly, in standard cosmology, derived from GR, the dark energy and dark matter should compose $\sim 95\%$ of the universe (Hinshaw et al. (2013)), but their nature is still dubious (Dil (2017); Behrouz et al. (2017); Germani (2017); Jennen & Pereira (2016); Evslin (2016); Liu et al. (2016); Rinaldi (2017); Zhang (2017); Albert et al. (2017)).

Another crucial trouble surrounding GR is the difficulty in quantizing it. Attempts to do so have been proposed (Fradkin & Tseytlin (1985); Witten (1986); Friedan et al. (1986)) and can, in future, provide us a robust and trustworthy model of gravity - quantum mechanics unification.

Meanwhile it is worthwhile to attempt to consider the presence of quantum effects in gravitational theories. Those effects can rise from the consideration of terms proportional to the trace of the energy-momentum tensor $T$ in the gravitational part of the $f(R)$ action, yielding the $f(R,T)$ gravity theories (Harko et al. (2011)). Those theories were also motivated by the fact that although $f(R)$ gravity is well behaved in cosmological scales, the Solar System regime seems to rule out most of the $f(R)$ models proposed so far (Erickcek et al. (2006); Chiba et al. (2007);
Capozziello et al. (2007); Olmo (2007). Furthermore, rotation curves of spiral galaxies were constructed in $f(R)$ gravity, but the results did not favour the theory, as it can be checked in Chiba (2003); Dolgov & Kawasaki (2003); Olmo (2005). The structure and cosmological properties of the modified gravity starting from $f(R)$ theory to power-counting renormalizable covariant gravity were presented by Nojiri & Odintsov (2011). The review in Nojiri et al. (2017) describes the cosmological developments regarding inflation, bounce and late time evolution in $f(R)$, $f(G)$ and $f(T)$ modified theories of gravity, with $G$ and $T$ being the Gauss-Bonnet and torsion scalars. Despite its recent elaboration, $f(R,T)$ gravity has already been applied to a number of areas, such as Cosmology (Moraes et al. 2015; Moraes et al. 2016; Moraes & Santos 2016; Moraes & Sahoo 2017; Myrzakulov 2012; Reddy et al. 2012; Singh & Singh 2015) and Astrophysics (Moraes et al. 2016; Alves et al. 2016; Moraes & Sahoo 2017; Moraes et al. 2017; Das et al. 2016; Sharif & Yousaf 2014; Yousaf et al. 2017; Baffou et al. 2017). Particularly, solar system tests have been applied to $f(R,T)$ gravity (Sharif & Farhoudi 2014) and the dark matter issue was analysed by Zaregonbadi et al. (2016). The late time behaviour of cosmic fluids consisting of collisional self-interacting dark matter and radiation was discussed by Zubair et al. 2018. Moreover, considering the metric and the affine connection as independent field variables, the Palatini formulation of the $f(R,T)$ gravity can be seen in Correa & Moraes (2016) and Correa et al. (2015) for these theories, respectively.

In the $f(R)$ gravity, a reliable and reputed functional form was proposed by A.A. Starobinsky as (Starobinsky 1980, 2007) can satisfy cosmological observational tests (Starobinsky 2007). On the other hand, the model seems to predict an overproduction of scalarons in the very early universe. In an astrophysical context, SM is also of great importance. In Sharif & Siddiqua (2017), the authors have explored the source of a gravitational radiation in SM by considering axially symmetric dissipative dust under geodesic condition. In Resco et al. (2016), it has been showed that in SM it is possible to find neutron stars with $2M_{\odot}$, which raises as an important alternative to some of the GR shortcomings mentioned above (Antoniadis et al. 2013; Demorest et al. 2010). In Astashenok et al. (2015), the macroscopical features of quark stars were obtained.

Our proposal in this paper is to construct a cosmological scenario from an $f(R,T)$ functional form whose $R$-dependence is the same as in the SM, i.e., with a quadratic extra contribution of $R$, as in Eq. (1). The $T$-dependence will be considered to be linear, as $2\gamma T$, with $\gamma$ a constant. Therefore, we will take

$$f(R,T) = R + \alpha R^2 + 2\gamma T.$$  \hspace{1cm} (2)

As far as the present authors know, the SM has not been considered for the $R$-dependence within $f(R,T)$ models for cosmological purposes so far, only in the study of astrophysical compact objects (Zubair & Noureen 2015; Noureen & Zubair 2015; Noureen et al. 2015) and wormholes (Zubair et al. 2016; Yousaf et al. 2017). We believe this is due to the expected high non-linearity of the resulting differential equation for the scale factor. Anyhow, the consideration of linear material corrections together with quadratic geometrical terms can imply interesting outcomes in a cosmological perspective as it does in the astrophysical level.

2 The $f(R,T) = R + \alpha R^2 + 2\gamma T$ gravity

Following the steps in Harko et al. (2011), we can write the $f(R,T) = R + \alpha R^2 + 2\gamma T$ gravity total action as

$$S = \frac{1}{16\pi} \int (R + \alpha R^2 + 2\gamma T)\sqrt{-g}d^4x + \int L_m\sqrt{-g}d^4x,$$ \hspace{1cm} (3)

in which $g$ is the determinant of the metric $g_{\mu\nu}$, $L_m$ is the matter lagrangian and we are working with natural units.

By taking $L_m = -p$, with $p$ being the pressure of the universe, the variational principle applied in Eq. (3) yields the following field equations:
\[ \frac{1}{2} G_{\mu \nu} + \alpha \left( R_{\mu \nu} - \frac{1}{4} R g_{\mu \nu} + g_{\mu \nu} \Box - \nabla_\mu \nabla_\nu \right) R \]
\[ = 4\pi T_{\mu \nu} + \gamma \left[ T_{\mu \nu} + \left( p + \frac{1}{2} \right) g_{\mu \nu} \right]. \tag{4} \]

In \( [1] \), \( G_{\mu \nu} \) is the Einstein tensor, \( R_{\mu \nu} \) is the Ricci tensor, \( T_{\mu \nu} = \text{diag}(\rho, -p, -p, -p) \), \( \rho \) is the matter-energy density of the universe and \( T = \rho - 3p \). Still in \( [1] \), it can be straightforwardly seen that the limit \( \alpha = \gamma = 0 \) recovers the GR field equations.

Also, the above choice for the matter lagrangian is usually assumed in the literature as it can be checked in \cite{Harko et al. (2011), Moraes et al. (2016, 2017)}, among many others.

### 3 The \( f(R, T) = R + \alpha R^2 + 2\gamma T \) cosmology

Let us assume a flat Friedmann-Robertson-Walker metric in the field equations above. Such a substitution yields the following Friedmann-like equations

\[ \left( \frac{\dot{a}}{a} \right)^2 + 6\alpha G(a, \dot{a}, \ddot{a}) = \frac{8\pi}{3} \rho + \gamma \left( \rho - \frac{p}{3} \right), \tag{5} \]

\[ \frac{\ddot{a}}{a} + \frac{1}{2} \left( \frac{\dot{a}}{a} \right)^2 + 6\alpha S(a, \dot{a}, \ddot{a}) = -4\pi p + \frac{1}{2} \gamma (\rho - p), \tag{6} \]

where we are using the following definitions

\[ G(a, \dot{a}, \ddot{a}) \equiv 2 \left( \frac{\dot{a}}{a} \right)^2 \left[ \frac{\ddot{a}}{a} + \frac{\dot{a}}{a} - 3 \left( \frac{\dot{a}}{a} \right)^2 \right] - \left( \frac{\dot{a}}{a} \right)^2, \tag{7} \]

\[ S(a, \dot{a}, \ddot{a}) \]
\[ = \frac{3}{2} \left[ \left( \frac{\dot{a}}{a} \right)^4 + \frac{\dot{a}}{a} \right] + 2 \left[ \frac{\dot{a}^2}{a^2} - 3 \left( \frac{\dot{a}}{a} \right)^2 \frac{\dot{a}}{a} \right] + \frac{\dot{a}}{a}. \tag{8} \]

In the equations above, \( a = a(t) \) is the scale factor and dots represent time derivatives. Once again, the limit \( \alpha = \gamma = 0 \) recovers the standard formalism.

In terms of the Hubble parameter \( H = \frac{\dot{a}}{a} \), the values of \( \rho \) and \( p \) from Equations \( (5) \)-\( (6) \) are

\[ \rho = \frac{F(\gamma)}{2} \left[ \gamma \dot{F}(H) - (8\pi + \gamma) \dot{G}(H) \right], \tag{9} \]

\[ p = F(\gamma) \left[ 4\pi \dot{F}(H) + 3\gamma \dot{G}(H) \right], \tag{10} \]

with the following definitions

\[ \dot{F}(H) \equiv -F_3(H) - 6\alpha \left[ -3F_2(H) + G_3(H) + 3F_1(H) \right], \tag{11} \]

\[ \dot{G}(H) \equiv \frac{3}{2} \left[ H^2 + 6\dot{G}_1(H) \right], \tag{12} \]

\[ \dot{F}(H) \equiv F_4(H) + 18\alpha F_2(H) - F_1(H) + \dot{H} + 2\dot{H}^2, \tag{13} \]

\[ \dot{G}(H) \equiv -H^2 + \dot{H} + 3\alpha \left[ G_1(H) + G_2(H) + 2H^5 + 6H^3 \dot{H} \right]. \tag{14} \]

For Equations \( (9)-(10) \), we considered

\[ F(\gamma) = \frac{1}{12\pi^2 + 16\gamma \pi + \gamma^2}. \] Moreover, \( F_i \) and \( G_i \), with \( i = 1, 2, 3, 4 \), are functions of \( H \) and its time derivatives, expressed by the following:

\[ F_1(H) = H^2 \left( 1 + 4\dot{H} \right), \tag{15} \]

\[ F_2(H) = H^4 - 4H\dot{H}, \tag{16} \]

\[ F_3(H) = 2 \left( H^5 + 3H^3 \dot{H} \right), \tag{17} \]

\[ F_4(H) = 3H^2 + 2\dot{H}, \tag{18} \]

\[ G_1(H) = H^4 + \dot{H} \left( 3 + 7\dot{H} \right), \tag{19} \]

\[ G_2(H) = -12H\dot{H} + H^2 \left( -3 - 12\dot{H} + 2\dot{H} \right) - 2\dot{H}, \tag{20} \]

\[ G_3(H) = 3\dot{H} \left( 1 + 2\dot{H} \right) + 2\dot{H}, \tag{21} \]

\[ G_4(H) = -2H^4 - \dot{H}^2 + 2H^2 \dot{H}. \tag{22} \]

We can consider, as a solution for Eqs. \( (9)-(10) \), the scale factor in the hybrid expansion law form \cite{Ozgur et al. (2014)}:

\[ a(t) = e^{mt^n}, \tag{23} \]
with \( m \) and \( n \) being constants. It can be seen that such a form for the scale factor consists of a product of power law and exponential law functions. Eq. (23) mimics the power law and de-Sitter cosmologies as particular cases and can predict the transition from a decelerated to an accelerated regime of the universe expansion. It has been applied to Brans-Dicke models in Ozgur et al. \((2014)\), yielding observational constraints to \( m \) and \( n \).

From (23), the Hubble and deceleration parameters are

\[
H = m + \frac{n}{t}, \quad (24)
\]

\[
q = -\frac{\ddot{a}}{aH^2} = -1 + \frac{n}{(mt + n)^2}, \quad (25)
\]

such that the deceleration parameter is defined in such a way that its negative values describe an accelerated expansion of the universe.

We know that the universe not always has accelerated its expansion (Riess et al. \((1998)\); Perlmutter et al. \((1999)\)). The accelerated regime of expansion is considered to be a late-time phenomenon.

In this way, one can choose the constants \( m \) and \( n \) in such a way that the power-law dominates over exponential law in the early universe and the exponential law dominates over power-law at late times. Therefore, the decelerated and accelerated regimes of the universe expansion can be respectively well described, as well as the transition between these regimes.

From (25), it is clear that there is a transition from deceleration to acceleration phases of the universe expansion at 

\[
t = \frac{1}{m}(n \pm \sqrt{n}), \quad 0 < n < 1.
\]

Since the negativity of the second term leads to a negative time, which indicates an non-physical situation, we conclude that the cosmic transition may have occurred at

\[
t = \frac{\sqrt{n} - n}{m}.
\]

From \( \frac{a(t)}{a_0} = \frac{1}{1+z} \), with \( a_0 = 1 \) being the present value of the scale factor and \( z \) being the redshift, we obtain the following time-redshift relation:

\[
t = \frac{n}{m} W \left[ \frac{m}{n} \left( \frac{1}{1+z} \right)^{1/n} \right], \quad (26)
\]

where \( W \) denotes the Lambert function (also known as “product logarithm”).

Using Equation (26), we can plot the deceleration parameter with respect to the redshift, which can be appreciated in Fig.[1] below, in which the values chosen for \( m \) and \( n \) are in agreement with the observational constraints found in Ozgur et al. \((2014)\).

Plotting \( q \) as a redshift function has the advantage of checking the reliability of the model, through the redshift value in which the decelerated-accelerated expansion of the universe transition occurs. We will call the transition redshift as \( z_{tr} \) and in our model it can be seen that it depends directly on the parameter \( m \). From Fig.1, the transition occurs at \( z_{tr} = 0.5836, 0.6777 \), corresponding to \( m = 0.25, 0.27 \), respectively. The values of the transition redshift \( z_{tr} \) for our model model are in accordance with the observational data, as one can check in Capozziello et al. \((2014, 2015)\); Farooq et al. \((2017)\).

Now, let us write the solutions for the material content of our model, named \( \rho \) and \( p \). Using (24) in Eqs. (9)-(10), we have

\[
\rho = \frac{F(\gamma)}{2}(-\gamma(P_1 + P_2) + P_3 + 3(8\pi + \gamma)(P_4 + P_5 + P_6)), \quad (27)
\]

\[
p = F(\gamma)[-(G_1 + G_2 + G_3 + G_4) + G_5], \quad (28)
\]

where we are using the definitions

\[
P_1 \equiv 32\alpha n F_3(t)^3 \times 4F_1(t) \left[ 1 - \frac{F_1(t)}{2} \right] + n \left[ 1 - \frac{F_3(t)^2}{2} \right] - 2, \quad (29)
\]

\[
P_2 \equiv 18\alpha F_2(t)^4 \left[ F_2(t)^2 - 1 \right], \quad (30)
\]

\[
P_3 \equiv 3F_2(t)^2 - 2nF_3(t)^2, \quad (31)
\]
\[ \mathcal{P}_4 \equiv 6\alpha n F_3(t)^5 \left\{ 2F_1(t)^2 \left[ 2 - 3F_1(t) \right] - nF_3(t)^{-1} \right\}, \]  
\[ \mathcal{P}_5 \equiv 12\alpha F_2(t)^5 \left[ 1 - F_2(t)^{-1} \right], \]  
\[ G_1 \equiv \alpha n F_3(t)^4 F_1(t)^2 \left\{ [9\gamma F_3(t)] \left[ 6F_1(t) - 4 \right] - 2F_1^{-1}(t)G(\gamma) + G(\gamma) \right\}, \]  
\[ G_2 \equiv n F_3(t)^2 \left[ \frac{\alpha}{4} G(\gamma) - 3\gamma - 8\pi \right], \]  
\[ G_3 \equiv \alpha^2 F_3(t)^2 \left[ G(\gamma) + 18\gamma \right] + \alpha G(\gamma), \]  
\[ G_4 \equiv \alpha n F_3(t)^4 G(\gamma), \]  
\[ G_5 \equiv F_2(t)^2 \times \]  
\[ \left\{ 9\alpha F_2(t)^2 \left[ -2\gamma F_2(t) - \gamma - 8\pi \right] - \frac{\alpha}{4} G(\gamma) + 3\gamma + 12\pi \right\}, \]  

with \( G(\gamma) = -108\gamma - 288\pi \) and \( F_j \), with \( j = 1, 2, 3 \), being functions of \( t \) only, defined as:

\[ F_1(t) = mt + n, \]  
\[ F_2(t) = m + \frac{n}{t}, \]  
\[ F_3(t) = \frac{1}{t}. \]  

The evolution of the energy density, pressure and corresponding equation of state (EoS) parameter \( \omega = p/\rho \) with \( m = 0.27, n = 0.75 \) are shown in Figures 2-4 in which the time units are Gyr.

4 Energy conditions

Energy condition (ECs), in the context of a wide class of covariant theories including GR, are relations one demands for the energy-momentum tensor of matter to satisfy in order to try to capture the idea that “energy should be positive”. By imposing so, one can obtain constraints to the free parameters of the concerned model.

5 Discussion and conclusions

In this article we have proposed an \( f(R, T) \) cosmological model. For the functional form of the function \( f(R, T) \), we investigated a quadratic correction to \( R \), as in the well-known SM, together with a linear term on \( T \). The substitution of such an \( f(R, T) \) function in the gravitational action generates extra terms in the field
\[ \alpha = -0.02, \quad \gamma = 2 \]

\[ \alpha = -0.02, \quad \gamma = 3 \]

\[ -1.0 \quad -0.8 \quad -0.6 \quad -0.4 \quad -0.2 \quad 0.0 \]

\[ 0 1 2 3 4 5 6 \]

\( \omega \)

Fig. 4 Evolution of \( \omega \) with time.

Fig. 5 Evolution of \( \rho \).

Fig. 6 Validation of NEC.

Specifically, for the deceleration parameter, we could separate it in two phases: one describing the decelerated and the other describing the accelerated regime of the universe expansion. This can be well-checked in Figure 1 in which the transition redshift between these two stages agrees with observational data. Moreover, one can also note that, remarkably, the present \((z = 0)\) values for \( q \), named \(-0.178\) for \( m = 0.27 \) and \(-0.157\) for \( m = 0.25 \), are also in agreement with observational data (Hinshaw et al. (2013)).

We have also obtained solutions for the material content of the universe, named \( \rho \) and \( p \) (Eqs. (27)-(41)). In Figs. 2-4 we plot the evolution of \( \rho \), \( p \) and \( \omega \), the EoS parameter. The values chosen for \( \alpha \) and \( \gamma \) respect the energy conditions outcomes presented in Section 4.

In Fig. 3 we see that the pressure of the universe starts in positive values and then assumes negative values. In standard model of cosmology, a negative pressure fluid is exactly the mechanism responsible for accelerating the universe expansion. In the present model, such a behavior for the pressure was naturally obtained.

It is also worth stressing that the EoS parameter shows a transition from a decelerated to an accelerated regime of the expansion of the universe. This can well
be seen in Fig. 4, by recalling that from standard cosmology, the latter regime may happen if \( \omega < -1/3 \)\(^\text{(Ryden (2003))}\). Moreover, it can be seen from such a figure that as time passes by, \( \omega \to -1 \), in accordance with recent observational data on the cosmic microwave background temperature fluctuations (Hinshaw et al. (2013)).

Furthermore, in Figs. 5-8 we plotted the ECs from the material solutions of our cosmological model. Those figures were plotted in terms of \( \gamma \), for fixed \( \alpha = -0.02 \). They show the validation of WEC, NEC and DEC, with a wide range of acceptable values for \( \gamma \).

On the other hand, Fig. 7 shows violation of SEC. However, as it has been deeply discussed by Barcelo & Visser (2002), an early and late-time accelerating universe must violate SEC.

As a further work, we can look for the \( f(R,T) = R + \alpha R^2 + 2\gamma T \) gravity universe at very early times, particularly investigating the production of scalarons in this model. Moraes & Santos (2016) have shown that the trace of the energy-momentum tensor contribution in the theory is higher for the early universe when compared to the late-time contribution. According to Starobinsky (2007), some mechanism should work in the early universe to prohibit the scalaron over-production within SM. The high contribution of the terms proportional to \( T \) in the early universe may well be this mechanism.

**Data Availability** The data used to support the findings of this study are included within the article.

**Conflicts of Interest** The author declares that they have no conflicts of interest.

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