Large deviation properties of the empirical measure of a stochastic differential equation with small noise

Paul Dupuis∗ and Guo-Jhen Wu†

March 2, 2020

Abstract

The aim of this paper is to develop tractable large deviation approximations for the empirical measure of a small noise diffusion. The starting point is the Freidlin-Wentzell theory, which shows how to approximate via a large deviation principle the invariant distribution of such a diffusion. The rate function of the invariant measure is formulated in terms of quasipotentials, quantities that measure the difficulty of a transition from the neighborhood of one metastable set to another. The theory provides an intuitive and useful approximation for the invariant measure, and along the way many useful related results (e.g., transition rates between metastable states) are also developed.

With the specific goal of design of Monte Carlo schemes in mind, we prove large deviation limits for integrals with respect to the empirical measure, where the process is considered over a time interval whose length grows as the noise decreases to zero. In particular, we show how the first and second moments of these integrals can be expressed in terms of quasipotentials. When the dynamics of the process depend on parameters, these approximations can be used for algorithm design, and applications of this sort will appear elsewhere. The use of a small noise limit is well motivated, since in this limit good sampling of the state space becomes most challenging. The proof exploits a regenerative structure, and a number of new techniques are needed to turn large deviation estimates over a regenerative cycle into estimates for the empirical measure and its moments.

*Division of Applied Mathematics, Brown University, Providence, USA. Research supported in part by the National Science Foundation (DMS-1904992) and the AFOSR (FA-9550-18-1-0214).
†Department of Mathematics, KTH Royal Institute of Technology, Stockholm, Sweden. Research supported in part by the AFOSR (FA-9550-18-1-0214).
Keywords: Large deviations, Freidlin-Wentzell theory, small noise diffusion, empirical measure, quasipotential, Monte Carlo method

1 Introduction

Among the many interesting results proved by Freidlin and Wentzell in the 70’s and 80’s concerning small random perturbations of dynamical systems, one of particular note is the large deviation principle for the invariant measure of such a system. Consider the small noise diffusion

\[ dX^\varepsilon_t = b(X^\varepsilon_t)dt + \varepsilon^{1/2}\sigma(X^\varepsilon_t)dW_t, \quad X^\varepsilon_0 = x, \]

where \( X^\varepsilon_t \in \mathbb{R}^d, b: \mathbb{R}^d \to \mathbb{R}^d, \sigma: \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^k \) (the \( d \times k \) matrices) and \( W_t \in \mathbb{R}^k \) is a standard Brownian motion. Under mild regularity conditions on \( b \) and \( \sigma \), one has that for any \( T \in (0, \infty) \) the processes \( \{X^\varepsilon\}_{\varepsilon > 0} \) satisfy a large deviation principle on \( C([0, T]: \mathbb{R}^d) \) with rate function

\[ I_T(\phi) = \int_0^T \sup_{\alpha \in \mathbb{R}^d} \left( \langle \dot{\phi}_t, \alpha \rangle - \langle b(\phi_t), \alpha \rangle - \frac{1}{2} \| \sigma(\phi_t)\alpha \|^2 \right)dt \]

when \( \phi \) is absolutely continuous and \( \phi(0) = x \), and \( I_T(\phi) = \infty \) otherwise. If \( \sigma(x)\sigma(x)^\prime > 0 \) (in the sense of symmetric square matrices) for all \( x \in \mathbb{R}^d \), then one can evaluate the supremum and find

\[ I_T(\phi) = \int_0^T \frac{1}{2} \left( \langle \dot{\phi}_t - b(\phi_t), \left[ \sigma(\phi_t)\sigma(\phi_t)^\prime \right]^{-1}(\dot{\phi}_t - b(\phi_t)) \right)dt. \]

To simplify the discussion we will assume this non-degeneracy condition. It is also assumed by Freidlin and Wentzell in [8], but can be weakened.

Define the quasipotential \( V(x, y) \) by

\[ V(x, y) = \inf \{ I_T(\phi) : \phi(0) = x, \phi(T) = y, T < \infty \}. \]

Suppose that \( \{X^\varepsilon\} \) is ergodic on a compact manifold \( M \subset \mathbb{R}^d \) with invariant measure \( \mu^\varepsilon \in \mathcal{P}(M) \). Then under a number of additional assumptions, including assumptions on the structure of the dynamical system \( \dot{X}_t^0 = b(X_t^0) \), Freidlin and Wentzell [8] Chapter 6 show how to construct a function \( J : M \to [0, \infty] \) in terms of \( V \), such that \( J \) is the large deviation rate function for \( \{\mu^\varepsilon\}_{\varepsilon > 0} \) : \( J \) has compact level sets, and

\[ \liminf_{\varepsilon \to 0} \varepsilon \log \mu^\varepsilon(G) \geq - \inf_{y \in G} J(y) \text{ for open } G \subset M, \]
\[
\limsup_{\varepsilon \to 0} \varepsilon \log \mu^\varepsilon(F) \leq - \inf_{y \in F} J(y) \text{ for closed } F \subset M.
\]

This gives a very useful approximation to \(\mu^\varepsilon\), and along the way many interesting related results (e.g., transition rates between metastable states) are also developed.

The aim of this paper is to develop large deviation type estimates for a quantity closely related to \(\mu^\varepsilon\), which is the empirical measure over an interval \([0, T^\varepsilon]\). This is defined by

\[
\rho^\varepsilon(A) = \frac{1}{T^\varepsilon} \int_0^{T^\varepsilon} 1_A(X^\varepsilon_s)ds
\]

for \(A \in \mathcal{B}(M)\). For reasons that will be made precise later on, we will assume \(T^\varepsilon \to \infty\) as \(\varepsilon \to 0\), and typically \(T^\varepsilon\) will grow exponentially in the form \(e^{c/\varepsilon}\) for some \(c > 0\).

There is of course a large deviation theory for the empirical measure when \(\varepsilon > 0\) is held fixed and the length of the time interval tends to infinity. However, it can be hard to extract information from the rate function. Our interest in proving large deviations estimates when \(\varepsilon \to 0\) and \(T^\varepsilon \to \infty\) is because one might find it easier to extract information in this double limit, analogous to the simplified approximation to \(\mu^\varepsilon\) just mentioned. These results will be applied in [6] to analyze and optimize a Monte Carlo method known as infinite swapping [5, 10] when the noise is small, which happens to be both common in applications and also the setting in which the Monte Carlo method will have the greatest difficulty. We expect that the general set of results will be useful for other purposes as well.

We note that while developed in the context of small noise diffusions, the collection of results due to Freidlin and Wentzell that are discussed in [8] also hold for other classes of processes, such as scaled stochastic networks, when appropriate conditions are assumed and the finite time sample path large deviation results are available (see, e.g., [14]). We expect that such generalizations are possible for the results we prove as well.

The outline of the paper is as follows. In Section 2 we explain our motivation and the relevance for studying the particular quantities that are the topic of the paper. In Section 3 we provide the definitions and assumptions that are used throughout the paper, and Section 4 states the main asymptotic results of the paper and as well as a related conjecture. In Section 5 we introduce an important tool for our analysis — the regenerative structure, and with this concept, we decompose the original asymptotic problem into two other asymptotic problems that require very different forms of analysis. These two types of asymptotic problems are analyzed in Section 6 and
Section 7 separately. In Section 8 we combine the partial asymptotic results from Section 6 and Section 7 to prove the main large deviation type results that were stated in Section 4. Section 9 gives the proof of a key theorem from Section 7 about the distribution and tail behavior of a return time that arises from the decomposition based on regenerative structure. The last section of the paper, Section 10, presents the proof of an upper bound for the rate of decay of the variance per unit time in the context of a special case, thereby showing for the case that the lower bounds are in a sense tight. To focus on the main discussion, proofs of some lemmas are collected in an Appendix.

2 Quantities of Interest

The quantities we are interested in are the higher order moments, and in particular second moments, of an integral of a risk-sensitive functional with respect to the empirical measure $\rho^\varepsilon$. To be more precise, the integral is of the form

$$\int_M e^{-\frac{1}{\varepsilon}f(x)} 1_A(x) \rho^\varepsilon(dx)$$

with some nice (e.g., bounded and continuous) function $f : M \to \mathbb{R}$ and a closed set $A \in \mathcal{B}(M)$. Note that this integral can also be expressed as

$$\frac{1}{T\varepsilon} \int_0^{T\varepsilon} e^{-\frac{1}{\varepsilon}f(X^\varepsilon_t)} 1_A(X^\varepsilon_t) dt.$$
The second moment is
\[
E \left( \frac{1}{n} \sum_{j=1}^{n} Y_j \right)^2 = \frac{1}{n^2} \sum_{j=1}^{n} E (Y_j)^2 + \frac{1}{n^2} \sum_{i,j:i\neq j} E (Y_i Y_j)
\]
\[
= \frac{1}{n} E (Y_1)^2 + \frac{1}{n^2} (n^2 - n) (E Y_1)^2
\]
\[
= (E Y_1)^2 + \frac{1}{n} \text{Var} (Y_1),
\]
and the second centered moment is
\[
E \left( \frac{1}{n} \sum_{j=1}^{n} (Y_j - E Y_1) \right)^2 = \text{Var} \left( \frac{1}{n} \sum_{j=1}^{n} Y_j \right) = \frac{1}{n} \text{Var} (Y_1).
\]

When analyzing the performance of the Monte Carlo schemes one is concerned of course with both bias and variance, but in certain situations where we would like to apply the results of this paper one can assume $T_\varepsilon$ is large enough that the bias term is unimportant, so that all we are concerned with is the variance. However some care will be needed to determine a suitable measure of quality of the algorithm, since as noted $Y_i$ could scale exponentially with $1/\varepsilon$ with a negative coefficient (exponentially small), while $n$ will be exponentially large. In the analysis of some accelerated Monte Carlo methods for small noise systems over bounded time intervals (e.g., to estimate escape probabilities [2, Chapter 14]), it is standard to use the second moment, which is often easier to analyze, in lieu of the variance. This situation corresponds to $n = 1$, and the substitution makes sense since by Jensen’s inequality one can easily establish a best possible rate of decay of the second moment, and estimators are deemed efficient if they possess that rate of decay. However with $n$ exponentially large this is no longer true. Using the previous calculations, we see that the second moment of $\frac{1}{n} \sum_{j=1}^{n} Y_j$ can be completely dominated by $(E Y_1)^2$, and therefore using this quantity to compare algorithms may be misleading, since our true concern is $\text{Var} (Y_1)$.

This observation suggests that our study of moments of the empirical measure we should consider only centered moments, and in particular quantities like
\[
T_\varepsilon \text{Var} \left( \int_{M} e^{-\frac{1}{2} f(x)} 1_A (x) \rho^\varepsilon (dx) \right) = T_\varepsilon \text{Var} \left( \frac{1}{T_\varepsilon} \int_{0}^{T_\varepsilon} e^{-\frac{1}{2} f(X^\varepsilon_t)} 1_A (X^\varepsilon_t) dt \right),
\]
which is the decay rate of the normalized variance (or variance per unit time). For Monte Carlo one wants to maximize this decay rate, and so we are especially interested in upper bounds.
Thus we need methods that will allow the approximation of at least first and second moments of \((2.2)\). In fact, the methods could be developed further to obtain large deviation estimates of higher moments if desired.

3 Setting of the Problem, Assumptions and Definitions

The process model we would like to consider is an \(\mathbb{R}^d\)-valued solution to an Itô stochastic differential equation, where the drift so strongly returns the process to some compact set that events involving exit of the process from some larger compact set are so rare that when analyzing the empirical measure they can effectively be ignored. However, to simplify the analysis we follow the convention of [8, Chapter 6], and work with small noise diffusion processes that takes values in a compact and connected manifold \(M \subset \mathbb{R}^d\) of dimension \(r\) and with smooth boundary. The precise regularity assumptions for \(M\) are given on [8, page 134]. With this convention in mind, we consider a family of small noise diffusion processes \(\{X_\varepsilon(t)\}_{t \in [0,\infty)}\), \(X_\varepsilon \in C([0,\infty) : M)\), that satisfy the following condition.

**Condition 3.1** Consider continuous \(b : M \to \mathbb{R}^d\) and \(\sigma : M \to \mathbb{R}^d \times \mathbb{R}^d\) (the \(d \times d\) matrices), and assume that \(\sigma\) is uniformly nondegenerate, in that there is \(c > 0\) such that for any \(x\) and any \(v\) in the tangent space of \(M\) at \(x\), \(\langle v, \sigma(x)\sigma(x)'v \rangle \geq c\langle v, v \rangle\). For absolutely continuous \(\phi \in C([0,T] : M)\) define

\[
I_T(\phi) = \int_0^T \frac{1}{2} \left\langle \dot{\phi}_t - b(\phi_t), [\sigma(\phi_t)\sigma(\phi_t)']^{-1}(\dot{\phi}_t - b(\phi_t)) \right\rangle dt,
\]

where the inverse \([\sigma(x)\sigma(x)']^{-1}\) is relative to the tangent space of \(M\) at \(x\). Let \(I_T(\phi) = \infty\) for all other \(\phi \in C([0,T] : M)\). Then we assume that for each \(T < \infty\), \(\{X_\varepsilon(t)\}_{t \in [0,T]}\) satisfies the large deviation principle with rate function \(I_T\), uniformly with respect to the initial condition.

We note that for such diffusion processes nondegeneracy of the diffusion matrix implies there is a unique of invariant measure \(\mu_\varepsilon \in \mathcal{P}(M)\).

**Remark 3.2** There are several ways one can approximate a diffusion of the sort described at the beginning of this section by a diffusion on a smooth compact manifold. One such “compactification” of the state space can be obtained by assuming that for some bounded but large enough rectangle trajectories that exit the rectangle do not affect the large deviation behavior of quantities
of interest, and to then extend the coefficients of the process periodically and smoothly off an even larger rectangle to all of $\mathbb{R}^d$ (a technique sometimes used to bound the state space for purposes of practical numerical implementation). One can then map $\mathbb{R}^d$ to a manifold that is topologically equivalent to a torus, and even arrange that the metric structure on the part of the manifold corresponding to the smaller rectangle coincides with a Euclidean metric.

Define the quasipotential $V(x,y) : M \times M \to [0, \infty)$ by

$$V(x,y) = \inf \{ I_T(\phi) : \phi(0) = x, \phi(T) = y, T < \infty \}.$$  

For a given set $A \subset M$, define $V(x,A) = \inf_{y \in A} V(x,y)$ and $V(A,y) = \inf_{x \in A} V(x,y)$.

**Remark 3.3** For any fixed $y$ and set $A$, $V(x,y)$ and $V(x,A)$ are both continuous functions of $x$. Similarly, for any given $x$ and any set $A$, $V(x,y)$ and $V(A,y)$ are also continuous in $y$.

**Definition 3.4** We say that a set $N \subset M$ is **stable** if for any $x \in N, y \notin N$ we have $V(x,y) > 0$. A set which is not stable is called **unstable**.

**Definition 3.5** We say that $O \in M$ is an **equilibrium point** of the ordinary differential equation (ODE) $\dot{x}_t = b(x_t)$ if $b(O) = 0$. Moreover, we say that this equilibrium point $O$ is **asymptotically stable** if for every neighborhood $E_1$ of $O$ (relative to $M$) there exists a smaller neighborhood $E_2$ such that the trajectories of system $\dot{x}_t = b(x_t)$ starting in $E_2$ converge to $O$ without leaving $E_1$ as $t \to \infty$.

**Remark 3.6** An asymptotically stable equilibrium point is a stable set, but a stable set might contain no asymptotically stable equilibrium point.

Next we give a definition from graph theory which will be used in the statement of the main results.

**Definition 3.7** Given a subset $W \subset L = \{1, \ldots, l\}$, a directed graph consisting of arrows $i \to j$ ($i \in L \setminus W, j \in L, i \neq j$) is called a $W$-graph on $L$ if it satisfies the following conditions.

1. Every point $i \in L \setminus W$ is the initial point of exactly one arrow.
2. There are no closed cycles in the graph.

We note that the second condition can be replaced by the following (i.e., 1 and 2 are equivalent to 1 and 2').

2' For any point \( i \in L \setminus W \), there exists a sequence of arrows leading from \( i \) to some point in \( W \).

We denote by \( G(W) \) the set of \( W \)-graphs; we shall use the letter \( g \) to denote graphs. Moreover, if \( p_{ij} \) \((i, j \in L, j \neq i)\) are numbers, then \( \prod_{i \rightarrow j \in g} p_{ij} \) will be denoted by \( \pi(g) \).

**Remark 3.8** In this paper we mostly consider the set of \( \{i\} \)-graphs, i.e., \( G(\{i\}) \) for some \( i \in L \), and also use \( G(i) \) to denote \( G(\{i\}) \). We occasionally consider the set of \( \{i,j\} \)-graphs, i.e., \( G(\{i,j\}) \) for some \( i, j \in L \) with \( i \neq j \). Again, we also use \( G(i,j) \) to denote \( G(\{i,j\}) \).

The following restrictions on the structure of the dynamical system in \( M \) will be used. These restrictions include the assumption that the equilibrium points are a finite collection. This is a more restrictive framework than that of [8], which allows, e.g., limit cycles. In a remark at the end of this section we comment on possible extension to the general setup of [8].

**Condition 3.9** There exists a finite number of points \( \{O_j\}_{j \in L} \subset M \) with \( L = \{1, 2, \ldots, l\} \) for some \( l \in \mathbb{N} \), such that \( \bigcup_{j \in L} \{O_j\} \) coincides with the \( \omega \)-limit set of the ODE \( \dot{x}_t = b(x_t) \).

Without loss of generality, we may assume that \( O_j \) is stable if and only if \( j \in L_s \) where \( L_s = \{1, \ldots, l_s\} \) for some \( l_s \leq l \).

**Definition 3.10** We use \( G_s(W) \) to denote the collection of all \( W \)-graphs on \( L_s = \{1, \ldots, l_s\} \) with \( W \subset L_s \).

To verify our results, we have to make the following technical assumptions on the structure of the dynamical system. Let \( B_\delta(K) \) denote the \( \delta \)-neighborhood of a set \( K \subset M \). Recall that \( \mu^\varepsilon \) is the unique invariant measure of the diffusion process \( \{X^\varepsilon(t)\}_t \).

1 Parts 2 and 3 of Condition 3.11 relate the depth of the well about \( O_1 \), as measured by the quasipotential, with the depths of other wells, and is used crucially in the proof of Lemma 9.14. We give a complete proof under this condition, but think it likely that parts 2 and 3 can be removed. If so, we will revise this ArXiv version before submission for publication to include this more general result.
Condition 3.11  1. There exists a unique asymptotically stable equilibrium point $O_1$ of the system $\dot{x}_t = b(x_t)$ such that

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} -\varepsilon \log \mu^\varepsilon (B_\delta (O_1)) = 0,$$

and for any $j \in L \setminus \{1\}$,

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} -\varepsilon \log \mu^\varepsilon (B_\delta (O_j)) > 0.$$  

2. $h_1 > \max_{j \in L \setminus \{1\}} h_j$, where for any $k \in L$, $h_k = \min_{j \in L \setminus \{k\}} V(O_k, O_j)$.

3. For all $j \in L \setminus \{1\}$, $h_1 + W(O_1 \cup O_j) > W(O_1)$ with

$$W(O_j) = \min_{g \in G(j)} \left[ \sum_{(m \to n) \in g} V(O_m, O_n) \right]. \quad (3.1)$$

and

$$W(O_1 \cup O_j) = \min_{g \in G(1, j)} \left[ \sum_{(m \to n) \in g} V(O_m, O_n) \right]. \quad (3.2)$$

4. All of the eigenvalues of the matrix of partial derivatives of $b$ at $O_\ell$ relative to $M$ have negative real parts for $\ell \in L_s$.

5. $b : M \to \mathbb{R}^d$ and $\sigma : M \to \mathbb{R}^d \times \mathbb{R}^d$ are $C^1$.

Remark 3.12 Sometimes we use $h$ to denote $h_1$.

Remark 3.13 The existence of the limits appearing in Condition 3.11.1 is ensured by Theorem 4.1 in [8, Chapter 6].

Remark 3.14 According to [8, Theorem 4.3, Chapter 6] and the first and the second parts of Condition 3.11 we know that $W(O_j) > W(O_1)$ for all $j \in L \setminus \{1\}$.

Remark 3.15 We remark on the use of the various parts of the condition. Parts 1, 2 and 3 are used to ensure that the time for a regenerative cycle is basically determined by the travel time from $O_1$ to any other equilibrium point. Parts 4 and 5 are assumed in [4] which gives an explicit exponential bound on the tail probability of the exit time from the domain of attraction. It is largely because of our reliance on the results of [4] that we must assume that equilibrium sets are points, rather than the more general compacta as considered in [8], and our results could be extended to the more general setting if the more general version of the result from [4] were available.
The first part of Condition 3.11 assert that when $\varepsilon$ is small, a small neighborhood of $O_1$ captures most (in fact exponentially more) of the mass of the invariant measure $\mu^\varepsilon$ than a small neighborhood of any other equilibrium point.

**Remark 3.16** The quantities $V(O_i, O_j)$ determine various key transition probabilities and time scales in the analysis of the empirical measure. The more general framework of [8], as well as the one dimensional case in the present setting, require some closely related but slightly more complicated quantities. These are essentially the analogues of $V(O_i, O_j)$ under the assumption that trajectories used in the definition are not allowed to pass through equilibrium compacta (such as a limit cycle) when traveling from $O_i$ to $O_j$. The related quantities, which are designated using notation of the form $\tilde{V}(O_i, O_j)$ in [8], are needed since the probability of a direct transition from $O_i$ to $O_j$ without passing through another equilibrium structure may be zero, which means that transitions from $O_i$ to $O_j$ must be decomposed according to these intermediate transitions. To simplify the presentation we do not provide the details of the one dimensional case in our setup, but simply note that it can be handled by the introduction of these additional quantities.

Consider the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ defined by $\mathcal{F}_t = \sigma(X^\varepsilon_s, s \leq t)$ for any $t \geq 0$. For any $\delta > 0$ smaller than a quarter of the minimum of the distances between $O_i$ and $O_j$ for all $i \neq j$, we consider two types of stopping times with respect to the filtration $\{\mathcal{F}_t\}_t$. The first type are the hitting times of $\{X^\varepsilon(t)\}_t$ at the $\delta$-neighborhood of all equilibrium points $\{O_j\}_{j \in L}$ after traveling a reasonable distance away from those neighborhoods. More precisely, we define stopping times by

$$\tau^\varepsilon_0 = 0,$$

$$\sigma_n = \inf\{t > \tau^\varepsilon_n : X^\varepsilon_t \in \bigcup_{j \in L} \partial B_{2\delta}(O_j)\}$$

and

$$\tau^\varepsilon_n = \inf\{t > \sigma_n : X^\varepsilon_t \in \bigcup_{j \in L} \partial B_{\delta}(O_j)\}.$$

The second type of stopping times are the return times of $\{X^\varepsilon(t)\}_t$ to the $\delta$-neighborhood of $O_1$, where as noted previously $O_1$ is in some sense the most important equilibrium point. The exact definitions are

$$\tau^\varepsilon_0 = 0,$$

$$\sigma^\varepsilon_n = \inf\{t > \tau^\varepsilon_n : X^\varepsilon_t \in \bigcup_{j \in L \setminus \{1\}} \partial B_{\delta}(O_j)\}$$

and

$$\tau^\varepsilon_n = \inf\{t > \sigma^\varepsilon_n : X^\varepsilon_t \in \partial B_{\delta}(O_1)\}.$$
We then define two embedded Markov chains \( \{Z_n\}_{n \in \mathbb{N}_0} = \{X^\varepsilon_t\}_{n \in \mathbb{N}_0} \) with state space \( \bigcup_{j \in L} \partial B_\delta(O_j) \) and \( \{Z_n^\varepsilon\}_{n \in \mathbb{N}_0} = \{X^\varepsilon_t\}_{n \in \mathbb{N}_0} \) with state space \( \partial B_\delta(O_1) \).

Let \( p(x, \partial B_\delta(O_j)) \) denote the one-step transition probabilities of \( \{Z_n\}_{n \in \mathbb{N}_0} \) starting from a point \( x \in \bigcup_{i \in L} \partial B_\delta(O_i) \), namely,

\[
p(x, \partial B_\delta(O_j)) \doteq P_x(Z_1 \in \partial B_\delta(O_j)).
\]

We have the following estimates on \( p(x, \partial B_\delta(O_j)) \) in terms of \( V \). The lemma is a consequence of [8, Lemma 2.1, Chapter 6] and the fact that under our conditions \( V(O_i, O_j) \) and \( \tilde{V}(O_i, O_j) \) as defined in [8] coincide.

**Lemma 3.17** For any \( \eta > 0 \), there exists \( \delta_0 \in (0, 1) \) and \( \varepsilon_0 \in (0, 1) \), such that for any \( \delta \in (0, \delta_0) \) and \( \varepsilon \in (0, \varepsilon_0) \), for all \( x \in \partial B_\delta(O_1) \), the one-step transition probability of the Markov chain \( \{Z_n\}_{n \in \mathbb{N}} \) on \( \partial B_\delta(O_j) \) satisfies

\[
e^{-\frac{1}{\tau_1}V(O_i,O_j)+\eta} \leq p(x,\partial B_\delta(O_j)) \leq e^{-\frac{1}{\tau_1}V(O_i,O_j)-\eta}
\]

for any \( i, j \in L \).

**Remark 3.18** According to Lemma 4.6 in [11], Condition 3.1 guarantees the existence and uniqueness of invariant measures for \( \{Z_n\}_n \) and \( \{Z^\varepsilon_n\}_n \). We use \( \nu^\varepsilon \in \mathcal{P}(\bigcup_{i \in L} \partial B_\delta(O_i)) \) and \( \lambda^\varepsilon \in \mathcal{P}(\partial B_\delta(O_1)) \) to denote the associated invariant measures.

### 4 Results and a Conjecture

The following main results of this paper assume Conditions 3.1, 3.9 and 3.11. Although moments higher than the second moment are not considered in this paper, as noted previously one can use arguments such as those used here to identify and prove the analogous results.

Recall that \( \{O_j\}_{j \in L} \) are the set of all equilibrium points and that they satisfy Condition 3.9 and Condition 3.11. In addition, \( O_j \) is stable if and only if \( j \in L_s \), where \( L_s \doteq \{1, \ldots, l_s\} \) for some \( l_s \leq l = |L| \), and \( \tau^\varepsilon_1 \) is the first return time to the \( \delta \)-neighborhood of \( O_1 \) after having first visited the \( \delta \)-neighborhood of any other equilibrium point.

**Lemma 4.1** For any \( \delta \in (0, 1) \) smaller than a quarter of the minimum of the distances between \( O_i \) and \( O_j \) for all \( i \neq j \), any \( \varepsilon > 0 \) and any bounded measurable function \( g : M \rightarrow \mathbb{R} \)

\[
E_{\lambda^\varepsilon} \left( \int_0^{\tau^\varepsilon_1} g(X^\varepsilon_s) \, ds \right) = E\lambda^\varepsilon \tau^\varepsilon_1 \cdot \int_M g(x) \mu^\varepsilon \mu^\varepsilon(dx),
\]
where \( \lambda^c \in \mathcal{P}(\partial B_\delta(O_1)) \) is the unique invariant measure of \( \{Z^c_n\}_n = \{X^c_\tau\}_n \) and \( \mu^c \in \mathcal{P}(M) \) is the unique invariant measure of \( \{X^c_t\}_t \).

**Proof.** Let \( \mu^c \in \mathcal{P}(M) \) be the unique invariant measure of \( \{X^c(t)\}_t \) and \( \lambda^c \) be the unique invariant measure of We define a measure on \( M \) by

\[
\hat{\mu}^c(B) = E_{\lambda^c} \left( \int_0^{\tau^c_1} 1_B(X^c(t)) \, dt \right)
\]

for \( B \in \mathcal{B}(M) \), so that for any nonnegative measurable function \( g : M \to \mathbb{R} \)

\[
\int_M g(x) \, \hat{\mu}^c(dx) = E_{\lambda^c} \left( \int_0^{\tau^c_1} g(X^c(t)) \, dt \right).
\]

According to the proof of Theorem 4.1 in [11], the measure given by \( \hat{\mu}^c(B) / \hat{\mu}^c(M) \) is an invariant measure of \( \{X^c(t)\}_t \). Since we already know that \( \mu^c \) is the unique invariant measure of \( \{X^c(t)\}_t \), this means that \( \mu^c(B) = \hat{\mu}^c(B) / \hat{\mu}^c(M) \) for any \( B \in \mathcal{B}(M) \). Therefore for any nonnegative measurable function \( g : M \to \mathbb{R} \)

\[
E_{\lambda^c} \left( \int_0^{\tau^c_1} g(X^c(t)) \, dt \right) = \int_M g(x) \mu^c(dx) \cdot \hat{\mu}^c(M)
\]

\[
= \int_M g(x) \mu^c(dx) \cdot E_{\lambda^c} \tau^c_1.
\]

**Theorem 4.2** Let \( T^c = e^{\frac{1}{2}c} \) for some \( c > h = \min_{j \in L \setminus \{1\}} V(O_1, O_j) \). Given a continuous function \( f : M \to \mathbb{R} \) and any compact set \( A \subset M \), there exists \( \delta_0 \in (0, 1) \) such that for any \( \delta \in (0, \delta_0) \)

\[
\liminf_{c \to 0} -\varepsilon \log \left| E_{\lambda^c} \left( \frac{1}{T^c} \int_0^{T^c} e^{-\frac{1}{2}f(X^c)} 1_A(X^c) \, dt \right) - \int e^{-\frac{1}{2}f(x)} 1_A(x) \mu^c(dx) \right|
\]

\[
\geq \inf_{x \in A} \left[ f(x) + W(x) \right] - W(O_1) + c - h,
\]

where \( W(x) = \min_{j \in L}[W(O_j) + V(O_j, x)] \) and

\[
W(O_j) = \min_{g \in G(j)} \left[ \sum_{(m \to n) \in g} V(O_m, O_n) \right]. \tag{4.1}
\]

**Remark 4.3** Since \( W(x) = \min_{j \in L}[W(O_j) + V(O_j, x)] \), the lower bound appearing in Theorem 4.2 is equivalent to

\[
\min_{j \in L} \left( \inf_{x \in A} [f(x) + W(O_j, x)] + W(O_j) - W(O_1) \right) + c - h.
\]
The next result gives an upper bound on the variance per unit time, or equivalently a lower bound on its rate of decay. In the design of a Markov chain Monte Carlo method, one would maximize this rate of decay to improve the method’s performance.

**Theorem 4.4** Let \( T^\varepsilon = e^{c/\varepsilon} \) for some \( c > h \). For any \( \eta > 0 \), and given a continuous function \( f : M \to \mathbb{R} \) and any compact set \( A \subset M \), there exists \( \delta_0 \in (0, 1) \) such that for any \( \delta \in (0, \delta_0) \)

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( T^\varepsilon \cdot \text{Var}_\varepsilon \left( \frac{1}{T^\varepsilon} \int_0^{T^\varepsilon} e^{-\frac{1}{\varepsilon} f(X_t^\varepsilon)} 1_A (X_t^\varepsilon) \, dt \right) \right) \\
\geq \min_{j \in L} \left( R_j^{(1)} \wedge R_j^{(2)} \right) - \eta,
\]

where

\[
R_j^{(1)} = \inf_{x \in A} [2f(x) + V(O_j, x)] + W(O_j) - W(O_1),
\]

\[
R_j^{(2)} = 2 \inf_{x \in A} [f(x) + V(O_j, x)] - h,
\]

and for \( j \in L \setminus \{1\} \)

\[
R_j^{(2)} = 2 \inf_{x \in A} [f(x) + V(O_j, x)] + W(O_j) - 2W(O_1) \\
+ W(O_1 \cup O_j)
\]

with

\[
W(O_1 \cup O_j) = \min_{g \in \mathcal{G}(1,j)} \left[ \sum_{(m \to n) \in g} V(O_m, O_n) \right].
\]

**Remark 4.5** In this remark we interpret the use of the last two results in the context of Monte Carlo, and also explain the role of the time scaling \( T^\varepsilon \).

There is a minimum amount of time that must elapse before the process can visit all stable equilibrium points often enough that good estimation of risk-sensitive integrals is possible. As is well known, this time scales exponentially in the form of \( T^\varepsilon = e^{c/\varepsilon} \), and the issue is the selection of the constant \( c > 0 \), which motives the assumptions on \( T^\varepsilon \) for the two cases. However, when designing a scheme there typically will be parameters available for selection. The growth constant in \( T^\varepsilon \) will then depend on these parameters, which will then be chosen to (either directly or indirectly, depending on the criteria used) reduce the size of \( T^\varepsilon \). For a compelling example we refer to [6], which shows how for a system with fixed well depths a scheme known as infinite swapping can be designed so that given any \( a > 0 \) one can design a scheme so that an interval of length \( e^{a/\varepsilon} \) suffices.
Theorem 4.2 is concerned with bias, and for $T^e$ as above will give a negligible contribution to the total error in comparison to the variance. Thus it is Theorem 4.4, and in particular when combined with a corresponding lower bound, that determines the performance of the scheme and serves as a criteria for optimization.

Finally we note that although the theorems assume the starting distribution $\lambda^e$, they can be extended to general initial distributions by using results from Section 9 which show that the process essentially forgets the initial distribution before leaving the neighborhood of $O_1$.

**Theorem 4.6** The lower bound in Theorem 4.2 can be calculated by only stable equilibrium points. Specifically,

1. $W(x) = \min_{j \in L_s} [W(O_j) + V(O_j, x)]$
2. $W(O_j) = \min_{g \in G_s(j)} \left[ \sum_{(m \to n) \in g} V(O_m, O_n) \right]$
3. $W(O_1 \cup O_j) = \min_{g \in G_s(1,j)} \left[ \sum_{(m \to n) \in g} V(O_m, O_n) \right]$
4. 

$$\min_{j \in L_s} \left( \inf_{x \in A} [f(x) + V(O_j, x)] + W(O_j) - W(O_1) \right)$$

$$= \min_{j \in L_s} \left( \inf_{x \in A} [f(x) + V(O_j, x)] + W(O_j) - W(O_1) \right),$$

**Remark 4.7** Theorem 4.6 says that the lower bound appearing in Theorem 4.2 depends on the set of indices of only stable equilibrium points. This is not surprising since in [8, Chapter 6], it has been shown that the logarithmic asymptotics of the invariant measure of a Markov process in this framework can be characterized by a quantity which can be calculated by considering graphs on the set of indices of only stable equilibrium points.

**Remark 4.8** As we discussed in Remark 4.4, various quantities can be computed by considering graphs on the set of indices of only stable equilibrium points, so it is natural to ask if the same property holds for the lower bound appearing in Theorem 4.4. Notice that part 4 of Theorem 4.6 implies that $\min_{j \in L} R_j^{(1)} = \min_{j \in L_s} R_j^{(1)}$, so if one can prove (possibly under extra conditions, for example, by considering a double-well model as in Section 10) that
min_{j \in L} R^{(2)}_j = \min_{j \in L_s} R^{(2)}_j, then these two equations assert the property we want, namely,

$$\min_{j \in L} \left( R^{(1)}_j \wedge R^{(2)}_j \right) = \min_{j \in L_s} \left( R^{(1)}_j \wedge R^{(2)}_j \right).$$

While Theorem 4.4 gives a lower bound on the rate of decay of variance per unit time, we expect the other direction holds as well.

**Conjecture 4.9** Let $T^\varepsilon = e^{\frac{1}{c} \varepsilon}$ for some $c > h \equiv \min_{j \in L \setminus \{1\}} V(O_1, O_j)$ and $M$ be a compact manifold in $\mathbb{R}^d$. For any $\eta > 0$, there exists $\delta_0 \in (0, 1)$ such that for any $\delta \in (0, \delta_0)$ and any compact set $A \subset M$

$$\liminf_{\varepsilon \to 0} -\varepsilon \log \left( T^\varepsilon \cdot \text{Var}_{\lambda^\varepsilon} \left( \int_0^{T^\varepsilon} e^{-\frac{1}{\varepsilon} f(X_t)} 1_A (X_t) \: dt \right) \right) \leq \min_{j \in L_s} \left( R^{(1)}_j \wedge R^{(2)}_j \right) + \eta.$$

In Section 10 we outline the proof of Conjecture 4.9 for a special case.

5 Wald’s Identities and Regenerative Structure

To prove Theorems 4.2 and 4.4, we will use the regenerative structure to analyze the system over the interval $[0, T^\varepsilon]$. Since the number of regenerative cycles will be random, Wald’s identities will be useful.

Recall that $\tau_n^\varepsilon$ is the $n$-th return time to $\partial B_\delta (O_1)$ after having visited the neighborhood of a different equilibrium point. If we let the process $\{X^\varepsilon(t)\}_t$ start with the invariant measure $\lambda^\varepsilon$ at time 0, that is, assume $X^\varepsilon(0) \overset{d}{=} \lambda^\varepsilon$, then by the strong Markov property of $\{X^\varepsilon(t)\}_t$, we find that $\{X^\varepsilon(t)\}_t$ is a regenerative process and the cycles $C_n \overset{d}{=} \{X^\varepsilon(\tau^\varepsilon_{n-1} + t) : 0 \leq t < \tau^\varepsilon_n - \tau^\varepsilon_{n-1}, \tau^\varepsilon_n - \tau^\varepsilon_{n-1}\}$ are iid objects. Moreover, $\{\tau^\varepsilon_n\}_{n \in \mathbb{N}}$ is a sequence of renewal times under $\lambda^\varepsilon$.

Define the filtration $\{\mathcal{H}_n\}_{n \in \mathbb{N}}$, where $\mathcal{H}_n \overset{d}{=} \mathcal{F}_{\tau^\varepsilon_n}$ and $\mathcal{F}_t \overset{d}{=} \sigma(\{X^\varepsilon(s); s \leq t\})$. With respect to this filtration we consider the stopping times

$$N^\varepsilon (T) \overset{d}{=} \min \{ n \in \mathbb{N} : \tau^\varepsilon_n > T \}.$$

Note that $N^\varepsilon (T) - 1$ is the number of complete renewal intervals contained in $[0, T]$. 

15
With this notation, we can bound \( \frac{1}{T^\epsilon} \int_0^{T^\epsilon} e^{-\frac{1}{T^\epsilon} f(X_t^\epsilon)} 1_A(X_t^\epsilon) \, dt \) from above and below by

\[
\frac{1}{T^\epsilon} \sum_{n=1}^{N^\epsilon(T^\epsilon)-1} S_n^\epsilon \leq \frac{1}{T^\epsilon} \int_0^{T^\epsilon} e^{-\frac{1}{T^\epsilon} f(X_t^\epsilon)} 1_A(X_t^\epsilon) \, dt \leq \frac{1}{T^\epsilon} \sum_{n=1}^{N^\epsilon(T^\epsilon)} S_n^\epsilon, \quad (5.1)
\]

where

\[
S_n^\epsilon = \int_{t_{n-1}}^{t_n} e^{-\frac{1}{T^\epsilon} f(X_t^\epsilon)} 1_A(X_t^\epsilon) \, dt.
\]

Applying Wald’s first identity shows

\[
E_{\lambda^\epsilon} \left( \frac{1}{T^\epsilon} \sum_{n=1}^{N^\epsilon(T^\epsilon)} S_n^\epsilon \right) = \frac{1}{T^\epsilon} E_{\lambda^\epsilon} (N^\epsilon(T^\epsilon)) E_{\lambda^\epsilon} S_1^\epsilon.
\]

Therefore, the logarithmic asymptotics of \( E_{\lambda^\epsilon} (\int_0^{T^\epsilon} e^{-\frac{1}{T^\epsilon} f(X_t^\epsilon)} 1_A(X_t^\epsilon) \, dt/T^\epsilon) \) are determined by those of \( E_{\lambda^\epsilon} (N^\epsilon(T^\epsilon)) / T^\epsilon \) and \( E_{\lambda^\epsilon} S_1^\epsilon \). Likewise, to understand the logarithmic asymptotics of \( T^\epsilon \cdot \text{Var}_{\lambda^\epsilon} (\int_0^{T^\epsilon} e^{-\frac{1}{T^\epsilon} f(X_t^\epsilon)} 1_A(X_t^\epsilon) \, dt/T^\epsilon) \), it is sufficient to identify the corresponding logarithmic asymptotics of \( \text{Var}_{\lambda^\epsilon} (N^\epsilon(T^\epsilon)) / T^\epsilon \), \( \text{Var}_{\lambda^\epsilon} (S_1^\epsilon) \), \( E_{\lambda^\epsilon} (N^\epsilon(T^\epsilon)) / T^\epsilon \) and \( E_{\lambda^\epsilon} S_1^\epsilon \). This can be done with the help of Wald’s second identity, since

\[
T^\epsilon \cdot \text{Var}_{\lambda^\epsilon} \left( \frac{1}{T^\epsilon} \sum_{n=1}^{N^\epsilon(T^\epsilon)} S_n^\epsilon \right) = T^\epsilon \cdot E_{\lambda^\epsilon} \left( \frac{1}{T^\epsilon} \sum_{n=1}^{N^\epsilon(T^\epsilon)} S_n^\epsilon \right) - E_{\lambda^\epsilon} \left( \frac{1}{T^\epsilon} \sum_{n=1}^{N^\epsilon(T^\epsilon)} S_n^\epsilon \right)^2
\]

\[
= T^\epsilon \cdot E_{\lambda^\epsilon} \left( \frac{1}{T^\epsilon} \sum_{n=1}^{N^\epsilon(T^\epsilon)} S_n^\epsilon - \frac{1}{T^\epsilon} E_{\lambda^\epsilon} (N^\epsilon(T^\epsilon)) E_{\lambda^\epsilon} S_1^\epsilon \right)^2
\]

\[
= T^\epsilon \cdot E_{\lambda^\epsilon} \left( \frac{1}{T^\epsilon} \sum_{n=1}^{N^\epsilon(T^\epsilon)} S_n^\epsilon - \frac{1}{T^\epsilon} N^\epsilon(T^\epsilon) E_{\lambda^\epsilon} S_1^\epsilon \right)^2
\]

\[
\leq 2T^\epsilon \cdot E_{\lambda^\epsilon} \left( \frac{1}{T^\epsilon} \sum_{n=1}^{N^\epsilon(T^\epsilon)} S_n^\epsilon - \frac{1}{T^\epsilon} N^\epsilon(T^\epsilon) E_{\lambda^\epsilon} S_1^\epsilon \right)^2
\]

\[
+ 2T^\epsilon \cdot E_{\lambda^\epsilon} \left( \frac{1}{T^\epsilon} N^\epsilon(T^\epsilon) E_{\lambda^\epsilon} S_1^\epsilon - \frac{1}{T^\epsilon} E_{\lambda^\epsilon} (N^\epsilon(T^\epsilon)) E_{\lambda^\epsilon} S_1^\epsilon \right)^2
\]

\[
= 2 \frac{E_{\lambda^\epsilon} (N^\epsilon(T^\epsilon))}{T^\epsilon} \text{Var}_{\lambda^\epsilon} S_1^\epsilon + 2 \frac{\text{Var}_{\lambda^\epsilon} (N^\epsilon(T^\epsilon))}{T^\epsilon} (E_{\lambda^\epsilon} S_1^\epsilon)^2.
\]
In the next two sections we derive bounds on $E_{\lambda^\varepsilon} S_1^\varepsilon$, $\text{Var}_{\lambda^\varepsilon}(S_1^\varepsilon)$ and $E_{\lambda^\varepsilon}(N^\varepsilon(T^\varepsilon))$, $\text{Var}_{\lambda^\varepsilon}(N^\varepsilon(T^\varepsilon))$, respectively.

6 Asymptotics of Moments of $S_1^\varepsilon$

In this section we will first introduce the elementary theory of an irreducible finite state Markov chain $\{Z_n\}_{n \in \mathbb{N}_0}$ with state space $L$, and then state and prove the bound for the asymptotics of moments of $S_1^\varepsilon$.

For the asymptotic analysis, the following useful facts will be used repeatedly.

**Lemma 6.1** For any nonnegative sequences $\{a_\varepsilon\}_{\varepsilon > 0}$ and $\{b_\varepsilon\}_{\varepsilon > 0}$, we have

$$\liminf_{\varepsilon \to 0} -\varepsilon \log (a_\varepsilon b_\varepsilon) \geq \liminf_{\varepsilon \to 0} -\varepsilon \log a_\varepsilon + \liminf_{\varepsilon \to 0} -\varepsilon \log b_\varepsilon. \quad (6.1)$$

Moreover,

$$\limsup_{\varepsilon \to 0} -\varepsilon \log (a_\varepsilon + b_\varepsilon) \leq \min \left\{ \limsup_{\varepsilon \to 0} -\varepsilon \log a_\varepsilon, \limsup_{\varepsilon \to 0} -\varepsilon \log b_\varepsilon \right\}$$

and

$$\liminf_{\varepsilon \to 0} -\varepsilon \log (a_\varepsilon + b_\varepsilon) = \min \left\{ \liminf_{\varepsilon \to 0} -\varepsilon \log a_\varepsilon, \liminf_{\varepsilon \to 0} -\varepsilon \log b_\varepsilon \right\}. \quad (6.2)$$

6.1 Markov chains and graph theory

In this subsection we state some elementary theory for finite state Markov chains taken from [1, Chapter 2]. For a finite state Markov chain, the invariant measure, the mean exit time, etc., can be expressed explicitly as the ratio of certain determinants, i.e., sums of products consisting of transition probabilities, and these sums only contain terms with a plus sign. Which products should appear in the various sums can be described conveniently by means of graphs on the set of states of the chain. This method of linking graphs and quantities associated with a finite state Markov chain was introduced by Freidlin and Wentzell in [3, Chapter 6].

Consider an irreducible finite state Markov chain $\{Z_n\}_{n \in \mathbb{N}_0}$ with state space $L$. For any $i, j \in L$, let $p_{ij}$ be the one-step transition probability of $\{Z_n\}_n$ from state $i$ to state $j$. Write $P_i(\cdot)$ and $E_i(\cdot)$ for probabilities and expectations of the chain started at state $i$ at time $0$. Recall the notation $\pi(g) = \prod_{(i \to j) \in g} P_{ij}$. 

17
Lemma 6.2  The unique invariant measure of \{Z_n\}_{n \in \mathbb{N}} can be expressed

\[
\lambda(i) = \frac{\sum_{g \in G(i)} \pi(g)}{\sum_{j \in L} \left( \sum_{g \in G(j)} \pi(g) \right)}.
\]

Proof. See Lemma 3.1, Chapter 6 in [8]. 

Remark 6.3  We will use \(\lambda_i\) and \(\lambda(i)\) interchangeably.

\[
T_i = \inf \{n \geq 0 : Z_n = i\}
\]

for the first hitting time of state \(i\), and write

\[
T_i^+ = \inf \{n \geq 1 : Z_n = i\}.
\]

Observe that \(T_i^+ = T_i\) unless \(Z_0 = i\), in which case we call \(T_i^+\) the first return time to state \(i\).

Let \(\hat{N} = \inf \{n \in \mathbb{N}_0 : Z_n \in L \setminus \{1\}\}\) and \(N = \inf \{n \in \mathbb{N} : Z_n = 1, n \geq \hat{N}\}\). \(\hat{N}\) is the first time of visiting a state other than state 1 and \(N\) is the first time of visiting state 1 after \(\hat{N}\). For any \(j \in L\), let \(N_j\) be the number of visits (including time 0) of state \(j\) before \(N\), i.e., \(N_j = |\{n \in \mathbb{N}_0 : n < N \text{ and } Z_n = j\}|\). We would like to understand \(E_{1N_j}\) and \(E_{jN_j}\) for any \(j \in L\). These quantities will appear later on in Subsection 6.2. The next lemma shows how they can be related to the invariant measure of \{Z_n\}_n.

Lemma 6.4  1. For any \(j \in L \setminus \{1\}\)

\[
E_{jN_j} = \frac{\sum_{g \in G(1,j)} \pi(g)}{\sum_{g \in G(1)} \pi(g)} \quad \text{and} \quad E_{jN_j} = \lambda_j (E_j T_1 + E_{1T_j}).
\]

2. For any \(i,j \in L, j \neq i\)

\[
P_i (T_j < T_i^+) = \frac{1}{\lambda_i (E_j T_1 + E_{1T_j})}.
\]

3. For any \(j \in L\)

\[
E_{1N_j} = \frac{1}{1 - p_{11}} \frac{\lambda_j}{\lambda_1}.
\]
Proof. See Lemma 3.4 in [8, Chapter 6] for the first assertion of part 1 and see Lemma 2.7 in [1, Chapter 2] for the second assertion of part 1. For part 2, see Corollary 2.8 in [1, Chapter 2].

For part 3, since
\[ E_1 N_j = \sum_{\ell=1}^{\infty} P_1 (N_j \geq \ell), \]
we need to understand \( P_1 (N_j \geq \ell) \), which means we need to know how to count all the ways to get \( N_j \geq \ell \) before returning to state 1.

We first have to move away from state 1, so the types of sequences are of the form
\[ 1, 1, \ldots, 1, k_1, k_2, \ldots, k_q, 1 \]
for some \( i, q \in \mathbb{N} \) and \( k_1 \neq 1, \ldots, k_q \neq 1 \). When \( j = 1 \), we do not care about \( k_1, k_2, \ldots, k_q \), and therefore

\[ P_1 (N_1 \geq i) = p_{11}^{i-1} \text{ and } E_1 N_1 = \sum_{i=1}^{\infty} P_1 (N_1 \geq i) = \frac{1}{1 - p_{11}}. \]

For \( j \in L \setminus \{1\} \), the event \( \{N_j \geq \ell\} \) requires that within \( k_1, k_2, \ldots, k_q \), we

1. first visit state \( j \) before returning to state 1, which has corresponding probability \( P_1(T_j < T_1^+) \),

2. then start from state \( j \) and again visit state \( j \) before returning to state 1, which has corresponding probability \( P_j(T_j^+ < T_1) \).

Step 2 needs to happen at least \( \ell - 1 \) times in a row, and after that we do not care. Thus,

\[ P_1 (N_j \geq \ell) = \sum_{i=1}^{\infty} (p_{11})^{i-1} P_1 (T_j < T_1^+) \left( P_j \left( T_j^+ < T_1 \right) \right)^{\ell-1} \]

\[ = \frac{1}{1 - p_{11}} P_1 (T_j < T_1^+) \left( P_j \left( T_j^+ < T_1 \right) \right)^{\ell-1} \]
and
\[
\sum_{\ell=1}^{\infty} P_1(N_j \geq \ell) = \frac{1}{1-p_{11}} P_1(T_j < T_1^+) \sum_{\ell=1}^{\infty} \left( P_j(T_j^+ < T_1) \right)^{\ell-1} \\
= \frac{1}{1-p_{11}} P_j(T_1 < T_j^+) \\
= \frac{1}{1-p_{11}} \lambda_j (E_1 T_j + E_j T_1) \\
= \frac{1}{1-p_{11}} \lambda_1 (E_1 T_j + E_j T_1) \\
= \frac{1}{1-p_{11}} \lambda_j
\]

The third equality comes from part 2. □

To apply the preceding results using the machinery developed by Freidlin and Wentzell, one must have analogues that allow for small perturbations of the transition probabilities due to the fact that initial conditions are to be taken in small neighborhoods of the equilibrium points. The addition of a tilde will be used to identify the corresponding objects, such as hitting and return times. Take as given a Markov chain \( \{\tilde{Z}_n\}_{n \in \mathbb{N}_0} \) on a state space \( \mathcal{X} = \bigcup_{i \in L} \mathcal{X}_i \), with \( \mathcal{X}_i \cap \mathcal{X}_j = \emptyset \) \((i \neq j)\), and assume there is \( a \in [1, \infty) \) such that for any \( i, j \in L \) and \( j \neq i \), the transition probability of the chain from \( x \in \mathcal{X}_i \) to \( \mathcal{X}_j \) (denoted by \( p(x, \mathcal{X}_j) \)) satisfies the inequalities
\[
a^{-1} p_{ij} \leq p(x, \mathcal{X}_j) \leq ap_{ij} \quad (6.3)
\]
for any \( x \in \mathcal{X}_i \). Write \( P_x(\cdot) \) and \( E_x(\cdot) \) for probabilities and expectations of the chain started at \( x \in \mathcal{X} \) at time 0. Write
\[
\tilde{T}_i = \inf \left\{ n \geq 0 : \tilde{Z}_n \in \mathcal{X}_i \right\}
\]
for the first hitting time of \( \mathcal{X}_i \), and write
\[
\tilde{T}_i^+ = \inf \left\{ n \geq 1 : \tilde{Z}_n \in \mathcal{X}_i \right\}.
\]
Observe that \( \tilde{T}_i^+ = \tilde{T}_i \) unless \( \tilde{Z}_0 \in \mathcal{X}_i \), in which case we call \( \tilde{T}_i^+ \) the first return time to \( \mathcal{X}_i \). Recall that \( l = |L| \).

**Remark 6.5** Observe that given \( j \in L \) and for any \( x \in \mathcal{X}_j \)
\[
1 - p(x, \mathcal{X}_j) = \sum_{k \in L \setminus \{j\}} p(x, \mathcal{X}_k).
\]
Therefore, we can apply (6.3) to obtain
\[ a^{-1} \sum_{k \in L \setminus \{j\}} p_{jk} \leq 1 - p(x, \mathcal{X}_j) \leq a \sum_{k \in L \setminus \{j\}} p_{jk}. \]

**Lemma 6.6**

1. Consider distinct \(i, j, k \in L\). Then for \(x \in \mathcal{X}_k\),
\[ a^{-4l-2} P_k (T_j < T_i) \leq P_x (\tilde{T}_j < \tilde{T}_i) \leq a^{-4l-2} P_k (T_j < T_i). \]

2. For any \(i \in L\), \(j \in L \setminus \{i\}\) and \(x \in \mathcal{X}_i\),
\[ a^{-4l-2-1} P_i (T_j < T_i^+) \leq P_x (\tilde{T}_j < \tilde{T}_i^+) \leq a^{-4l-2+1} P_i (T_j < T_i^+). \]

**Proof.** For part 1, see Lemma 3.3 in [8, Chapter 6]. We only need to prove part 2. Note that by a first step analysis on \(\{\tilde{Z}_n\}_{n \in \mathbb{N}_0}\), for any \(i \in L\), \(j \in L \setminus \{i\}\) and \(x \in \mathcal{X}_i\),
\[ P_x (\tilde{T}_j < \tilde{T}_i^+) = p(x, \mathcal{X}_j) + \sum_{k \in L \setminus \{i, j\}} \int_{\mathcal{X}_k} P_y (\tilde{T}_j < \tilde{T}_i) p(x, dy) \]
\[ \leq a p_{ij} + \sum_{k \in L \setminus \{i, j\}} \left( a^{-4l-2} P_k (T_j < T_i) \right) (a p_{ik}) \]
\[ \leq a^{-4l-2+1} \left( p_{ij} + \sum_{k \in L \setminus \{i, j\}} P_k (T_j < T_i) p_{ik} \right) \]
\[ = a^{-4l-2+1} P_i (T_j < T_i^+). \]

The first inequality comes from the use of (6.3) and part 1; the last equality holds since we can do a first step analysis on \(\{Z_n\}_n\). Similarly, we can show the lower bound. ■

Let \(\tilde{N} = \inf\{n \in \mathbb{N}_0 : \tilde{Z}_n \in \bigcup_{j \in L \setminus \{1\}} \mathcal{X}_j\}\) and \(\tilde{N} = \inf\{n \in \mathbb{N} : Z_n \in \mathcal{X}_1, n \geq \tilde{N}\}\). For any \(j \in L\), let \(\tilde{N}_j\) be the number of visits (including time 0) of state \(\mathcal{X}_j\) before \(\tilde{N}\), i.e. \(\tilde{N}_j = |\{n \in \mathbb{N}_0 : n < \tilde{N} \text{ and } Z_n \in \mathcal{X}_j\}|\). We would like to understand \(E_x \tilde{N}_j\) for any \(j \in L\) and \(x \in \mathcal{X}_1\) or \(\mathcal{X}_j\).

**Lemma 6.7** For any \(j \in L\) and \(x \in \mathcal{X}_1\)
\[ E_x \tilde{N}_j \leq \frac{a^{4l-1}}{\sum_{t \in L \setminus \{1\}} p_{1t}} \frac{\sum_{g \in G(j)} \pi(g)}{\sum_{g \in G(1)} \pi(g)}. \]
Moreover, for any \( j \in L \setminus \{1\} \)
\[
\sum_{\ell=1}^{\infty} \sup_{x \in \mathcal{X}_j} P_x \left( \tilde{N}_j \geq \ell \right) \leq a^{d^i-1} \sum_{g \in G(1,j)} \pi(g) \sum_{g \in G(1)} \pi(g)
\]
and
\[
\sum_{\ell=1}^{\infty} \sup_{x \in \mathcal{X}_1} P_x \left( \tilde{N}_1 \geq \ell \right) \leq \frac{a}{\sum_{\ell \in L \setminus \{1\}} p_{1}\ell}.
\]

**Proof.** For any \( x \in \mathcal{X}_1 \), note that for any \( \ell \in \mathbb{N} \), by a conditioning argument as in the proof of Lemma 6.4 (3), we find that for \( j \in L \setminus \{1\} \)
\[
P_x \left( \tilde{N}_j \geq \ell \right) \leq \frac{\sup_{y \in \mathcal{X}_1} P_{y} \left( \tilde{T}_j < \tilde{T}_1^+ \right)}{1 - \sup_{y \in \mathcal{X}_1} p(y, \mathcal{X}_1)} \left( \frac{\sup_{y \in \mathcal{X}_j} P_{y} \left( \tilde{T}_j^+ < \tilde{T}_1 \right)}{\sup_{y \in \mathcal{X}_1} p(y, \mathcal{X}_1)} \right)^{\ell-1}
\]
and
\[
P_x \left( \tilde{N}_1 \geq \ell \right) \leq \left( \sup_{y \in \mathcal{X}_1} p(y, \mathcal{X}_1) \right)^{\ell-1}.
\]
Thus, for any \( x \in \mathcal{X}_1 \) and for \( j \in L \setminus \{1\} \)
\[
E_x \tilde{N}_j = \sum_{\ell=1}^{\infty} P_x \left( \tilde{N}_j \geq \ell \right)
\leq \frac{\sup_{y \in \mathcal{X}_1} P_{y} \left( \tilde{T}_j < \tilde{T}_1^+ \right)}{1 - \sup_{y \in \mathcal{X}_1} p(y, \mathcal{X}_1)} \cdot \frac{1}{1 - \sup_{y \in \mathcal{X}_j} p(y, \mathcal{X}_1)} \left( \sup_{y \in \mathcal{X}_j} P_{y} \left( \tilde{T}_j^+ < \tilde{T}_1 \right) \right)^{\ell-1}
\]
\[
= \left( \inf_{y \in \mathcal{X}_j} (1 - p(y, \mathcal{X}_1)) \right) \left( \inf_{y \in \mathcal{X}_j} P_{y} \left( \tilde{T}_1 < \tilde{T}_j^+ \right) \right) \left( P_1 \left( T_j < T_1^+ \right) \right)^{\ell-1}.
\]

The second inequality is from Remark 6.5 and Lemma 6.6 (2); the third equality comes from Lemma 6.4 (2); the last equality holds due to Lemma 6.2.
Moreover,

\[ E_x \tilde{N}_1 = \sum_{\ell=1}^{\infty} P_x \left( \tilde{N}_1 \geq \ell \right) \leq \frac{1}{1 - \sup_{y \in X_1} p(y, X_1)} \leq \frac{1}{\inf_{y \in X_1} (1 - p(y, X_1))} \leq \frac{a}{\sum_{\ell \in L \setminus \{1\}} p_{1\ell}}. \]

The last inequality is from Remark 6.5. This completes the proof of part 1.

Turning to part 2, since for any \( \ell \in \mathbb{N} \)

\[ \sup_{x \in X_1} P_x \left( \tilde{N}_1 \geq \ell \right) \leq \left( \sup_{y \in X_1} p(y, X_1) \right)^{\ell-1}, \]

we have

\[ \sum_{\ell=1}^{\infty} \sup_{x \in X_1} P_x \left( \tilde{N}_1 \geq \ell \right) \leq \frac{1}{1 - \sup_{y \in X_1} p(y, X_1)} \leq \frac{a}{\sum_{\ell \in L \setminus \{1\}} p_{1\ell}}. \]

Furthermore, we use the conditioning argument again to find that for any \( j \in L \setminus \{1\} \) and \( \ell \in \mathbb{N} \)

\[ \sup_{x \in X_j} P_x \left( \tilde{N}_j \geq \ell \right) \leq \left( \sup_{y \in X_j} P_y \left( \tilde{T}_j^+ < \tilde{T}_j^1 \right) \right)^{\ell-1}. \]

This implies that

\[ \sum_{\ell=1}^{\infty} \sup_{x \in X_j} P_x \left( \tilde{N}_j \geq \ell \right) \leq \sum_{\ell=1}^{\infty} \left( \sup_{y \in X_j} P_y \left( \tilde{T}_j^+ < \tilde{T}_j^1 \right) \right)^{\ell-1} \]

\[ = \frac{1}{1 - \sup_{y \in X_j} P_y \left( \tilde{T}_j^+ < \tilde{T}_j^1 \right)} \]

\[ = \frac{1}{\inf_{y \in X_j} \left[ P_y \left( \tilde{T}_j^1 < \tilde{T}_j^+ \right) \right]} \]

\[ \leq \frac{1}{a^{j-2} - 1} \frac{1}{P_j \left( T_1 < T_j^+ \right)} \]

\[ \leq a^{j-1} \frac{1}{P_j \left( T_1 < T_j^+ \right)} \]

\[ = a^{j-1} \lambda_j \left( E_1 T_j + E_j T_1 \right) = a^{j-1} \frac{\sum_{g \in G(j)} \pi(g)}{\sum_{g \in G(1)} \pi(g)}. \]
We use Lemma 6.6 (2) to obtain the second inequality and Lemma 6.4, parts (2) and (1), for the penultimate and last equalities. 

6.2 Asymptotics of Moments of $S_1^\varepsilon$

Recall that $\{X_\varepsilon\}_{\varepsilon \in (0, \infty)} \subset C([0, \infty) : M)$ is a sequence of stochastic processes satisfying Condition 3.3, Condition 3.9 and Condition 3.11. Moreover, recall that $S_1^\varepsilon$ is defined by

$$S_1^\varepsilon = \int_0^{\tau_1^\varepsilon} e^{-\frac{1}{\varepsilon}f(X_\varepsilon^t)} 1_A(X_\varepsilon^t) \, dt.$$ 

As mentioned in Section 5, we are interested in the logarithmic asymptotics of $E_{X_\varepsilon} S_1^\varepsilon$ and $E_{X_\varepsilon}(S_1^\varepsilon)^2$. To find these asymptotics, the main tool we will use is Freidlin-Wentzell theory [8]. In fact, we will generalize the results of Freidlin-Wentzell to the following: For any given continuous function $f : M \to \mathbb{R}$ and any compact set $A \subset M$, we will provide lower bounds for

$$\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_1)} E_z \left( \int_0^{\tau_1^\varepsilon} e^{-\frac{1}{\varepsilon}f(X_\varepsilon^s)} 1_A(X_\varepsilon^s) \, ds \right) \right) \quad (6.4)$$

and

$$\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_1)} E_z \left( \int_0^{\tau_1^\varepsilon} e^{-\frac{1}{\varepsilon}f(X_\varepsilon^s)} 1_A(X_\varepsilon^s) \, ds \right)^2 \right). \quad (6.5)$$

As will be shown, these two bounds can be expressed in terms of the quasipotentials $V(O_i, O_j)$ and $V(O_i, x)$.

Remark 6.8 In the Freidlin-Wentzell theory as presented in [8], they only consider bounds for

$$\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_1)} E_z \tau_1^\varepsilon \right).$$

Thus, their result is a special case of (6.4) with $f \equiv 0$ and $A = M$. Moreover, we generalize their result further by considering the logarithmic asymptotics of higher moment quantities such as (6.5).

Before proceeding, we recall that $L = \{1, \ldots, l\}$ and for any $\delta > 0$, we define $\tau_0 \doteq 0$,

$$\sigma_n \doteq \inf\{t > \tau_n : X_\varepsilon^t \in \bigcup_{j \in L} \partial B_{2\delta}(O_j)\}$$
and
\[ \tau_n \doteq \inf\{ t > \sigma_{n-1} : X_t^\varepsilon \in \bigcup_{j \in L} \partial B_\delta(O_j) \}. \]
Moreover, \( \tau_0^- = 0 \),
\[ \sigma_n^\varepsilon \doteq \inf\{ t > \tau_n^\varepsilon : X_t^\varepsilon \in \bigcup_{j \in L \setminus \{1\}} \partial B_\delta(O_j) \} \]
and
\[ \tau_n^\varepsilon \doteq \inf\{ t > \sigma_{n-1}^\varepsilon : X_t^\varepsilon \in \partial B_\delta(O_1) \}. \]
In addition, \( \{Z_n\}_{n \in \mathbb{N}_0} \doteq \{X_n^\varepsilon\}_{n \in \mathbb{N}_0} \) is a Markov chain on \( \bigcup_{j \in L} \partial B_\delta(O_j) \) and \( \{Z_n^\varepsilon\}_{n \in \mathbb{N}_0} \doteq \{X_n^\varepsilon\}_{n \in \mathbb{N}_0} \) is a Markov chain on \( \partial B_\delta(O_1) \).

Also, following the notation of Section 5 let \( \hat{N} = \inf\{ n \in \mathbb{N}_0 : Z_n \in \bigcup_{j \in L \setminus \{1\}} \partial B_\delta(O_j) \} \), \( N = \inf\{ n \geq \hat{N} : Z_n \in \partial B_\delta(O_1) \} \), and recall \( \mathcal{F}_t = \sigma\{X_s^\varepsilon; s \leq t\} \). Then since \( \{\tau_n\}_{n \in \mathbb{N}_0} \) are stopping times with respect to the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \), \( \mathcal{F}_{\tau_n} \) are well-defined for any \( n \in \mathbb{N}_0 \) and we use \( \mathcal{G}_{\tau_n} \) to denote \( \mathcal{F}_{\tau_n} \). One can prove that \( \hat{N} \) and \( N \) are stopping times with respect to \( \{\mathcal{G}_n\}_{n \in \mathbb{N}} \). For any \( j \in L \), let \( N_j \) be the number of visits of \( \{Z_n\}_{n \in \mathbb{N}_0} \) to \( \partial B_\delta(O_j) \) (including time 0) before \( N \).

**Remark 6.9** We call \( N_j \) the number of visits of the embedded Markov chain to \( \partial B_\delta(O_j) \) within one loop of the regenerative cycle.

The proofs of the following two lemmas are given in the Appendix.

**Lemma 6.10** Given \( \delta > 0 \) sufficiently small, for any \( x \in \partial B_\delta(O_1) \) and any bounded measurable function \( g : M \to \mathbb{R} \),
\[ E_x \left( \int_0^{\tau^\varepsilon_1} g(X_s^\varepsilon) \, ds \right) \leq \sum_{j \in L} \sup_{y \in \partial B_\delta(O_j)} E_y \left( \int_0^{\tau^\varepsilon_1} g(X_s^\varepsilon) \, ds \right) \cdot E_x N_j. \]

**Lemma 6.11** Given \( \delta > 0 \) sufficiently small, for any \( x \in \partial B_\delta(O_1) \) and any bounded measurable function \( g : M \to \mathbb{R} \),
\[ E_x \left( \int_0^{\tau^\varepsilon_1} g(X_s^\varepsilon) \, ds \right)^2 \leq l \sum_{j \in L} \sup_{y \in \partial B_\delta(O_j)} E_y \left( \int_0^{\tau^\varepsilon_1} g(X_s^\varepsilon) \, ds \right)^2 \cdot E_x N_j \]
\[ + 2l \sum_{j \in L} \sup_{y \in \partial B_\delta(O_j)} E_y \left( \int_0^{\tau^\varepsilon_1} g(X_s^\varepsilon) \, ds \right)^2 \cdot E_x N_j \]
\[ \cdot \sum_{k=1}^{\infty} \sup_{y \in \partial B_\delta(O_j)} P_y (k \leq N_j). \]
Although as noted the proofs are given in the Appendix, these results follow in a straightforward way by decomposing the excursion away from $O_1$ during $[0, \tau^1]$, which only stops when returning to a neighborhood of $O_1$, into excursions between any pair of equilibrium points, counting the number of such excursions that start near a particular equilibrium point, and using the strong Markov property.

Remark 6.12 Following an analogous argument as in the proof of Lemma 6.11 and Lemma 6.11, we can prove the following: Given $\delta > 0$ sufficiently small, for any $x \in \partial B_\delta(O_1)$ and any bounded measurable function $g : M \to \mathbb{R}$,

$$E_x \left( \int_{\tau^1_0}^{\tau^1_{\epsilon}} g(X_s^\epsilon) \, ds \right) \leq \sum_{j \in L \setminus \{1\}} \left[ \sup_{y \in \partial B_\delta(O_j)} E_y \left( \int_0^{\tau_1} g(X_s) \, ds \right) \right] \cdot E_x N_j$$

and

$$E_x \left( \int_{\tau^1_0}^{\tau^1_{\epsilon}} g(X_s^\epsilon) \, ds \right)^2 \leq l \sum_{j \in L \setminus \{1\}} \left[ \sup_{y \in \partial B_\delta(O_j)} E_y \left( \int_0^{\tau_1} g(X_s) \, ds \right)^2 \right] \cdot E_x N_j$$

$$+ 2l \sum_{j \in L \setminus \{1\}} \left[ \sup_{y \in \partial B_\delta(O_j)} E_y \left( \int_0^{\tau_1} g(X_s) \, ds \right) \right]^2 \cdot E_x N_j$$

$$+ \sum_{k=1}^{\infty} \sup_{y \in \partial B_\delta(O_j)} P_y \left( k \leq N_j \right).$$

The main difference is that if the integration starts from $\sigma^\epsilon_0$ (the first visiting time of $\bigcup_{j \in L \setminus \{1\}} \partial B_\delta(O_j)$), then any summation appearing in the upper bounds should sum over all indices in $L \setminus \{1\}$ instead of $L$.

Corollary 6.13 Given any measurable set $A \subset M$, a bounded below and measurable function $f : M \to \mathbb{R}$, $j \in L$ and $\delta > 0$, we have

$$\lim_{\epsilon \to 0} \inf -\epsilon \log \left( \sup_{z \in \partial B_\delta(O_1)} E_z(S^\epsilon_1) \right)$$

$$\geq \min_{j \in L} \left\{ \lim_{\epsilon \to 0} \inf -\epsilon \log \left( \sup_{z \in \partial B_\delta(O_1)} E_z N_j \right) \right\}$$

$$+ \lim_{\epsilon \to 0} \inf -\epsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left[ \int_0^{\tau_1} e^{-\frac{1}{\epsilon} f(X_s^\epsilon)} 1_A(X_s^\epsilon) \, ds \right] \right),$$

26
where
\[
S_1^\varepsilon = \int_0^{\tau_1^\varepsilon} e^{-\frac{1}{\varepsilon} f(X_t^\varepsilon) 1_A(X_t^\varepsilon)} dt.
\]

Moreover,
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B(0_1)} E_z \left( S_1^\varepsilon \right)^2 \right) \geq \min_{j \in L} \left( \hat{R}_j^{(1)} \land \hat{R}_j^{(2)} \right),
\]
where
\[
\hat{R}_j^{(1)} = \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B(0_1)} E_z \left( \int_0^{\tau_1^\varepsilon} e^{-\frac{1}{\varepsilon} f(X_t^\varepsilon) 1_A(X_t^\varepsilon)} ds \right) \right)^2
\]
\[+ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B(0_1)} E_z N_j \right)
\]
and
\[
\hat{R}_j^{(2)} = 2 \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B(0_1)} E_z \left[ \int_0^{\tau_1^\varepsilon} e^{-\frac{1}{\varepsilon} f(X_t^\varepsilon) 1_A(X_t^\varepsilon)} ds \right] \right)^2
\]
\[+ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B(0_1)} E_z N_j \right)
\]
\[+ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sum_{\ell=1}^{\infty} \sup_{z \in \partial B(0_1)} P_z (\ell \leq N_j) \right).
\]

**Proof.** For the first part, applying Lemma 6.10 with \(g(x) = e^{-\frac{1}{\varepsilon} f(x) 1_A(x)}\) and using (6.1) and (6.2) completes the proof. For the second part, using Lemma 6.11 with \(g(x) = e^{-\frac{1}{\varepsilon} f(x) 1_A(x)}\) and using (6.1) and (6.2) again completes the proof. \[\]

**Remark 6.14** Owing to Remark 6.12, we can modify the proof of Corollary 6.13 and show that given any set \(A \subset M\), a bounded below and measurable function \(f : M \to \mathbb{R}\), \(j \in L\) and \(\delta > 0\),
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B(0_1)} E_z \left( \int_{\sigma_0^\varepsilon}^{\tau_1^\varepsilon} e^{-\frac{1}{\varepsilon} f(X_t^\varepsilon) 1_A(X_t^\varepsilon)} ds \right) \right)
\]
\[\geq \min_{j \in L \setminus \{1\}} \left\{ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B(0_1)} E_z (N_j) \right) \right. \]
\[+ \left. \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B(0_1)} E_z \left[ \int_0^{\tau_1^\varepsilon} e^{-\frac{1}{\varepsilon} f(X_t^\varepsilon) 1_A(X_t^\varepsilon)} ds \right] \right) \right\}.
\]
Moreover,
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_{2\delta}(O_j)} E_z \left( \int_{\sigma_0}^{\tau_1} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_A(X_s^\varepsilon) \, ds \right)^2 \right)
\geq \min_{j \in L \setminus \{1\}} \left( \hat{R}^{(1)}_j \wedge \hat{R}^{(2)}_j \right),
\]
where the definitions of \( \hat{R}^{(1)}_j \) and \( \hat{R}^{(2)}_j \) can be found in Corollary 6.13.

We next consider lower bounds on
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_{\delta}(O_j)} E_z \left( \int_{\tau_1}^{\tau_1} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_A(X_s^\varepsilon) \, ds \right) \right)
\]
and
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_{\delta}(O_j)} E_z \left( \left( \int_{\tau_1}^{\tau_1} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_A(X_s^\varepsilon) \, ds \right)^2 \right) \right)
\]
for \( j \in L \). We state some useful results before studying the lower bounds. Recall that \( \tau_1 \) is the time to reach the \( \delta \)-neighborhood of any of the equilibrium points after leaving the \( 2\delta \)-neighborhood of one of the equilibrium points.

**Lemma 6.15** For any \( \eta > 0 \), there exists \( \delta_0 \in (0,1) \) and \( \varepsilon_0 \in (0,1) \), such that for all \( \delta \in (0,\delta_0) \) and \( \varepsilon \in (0,\varepsilon_0) \)
\[
\sup_{x \in M} E_x \tau_1 \leq e^{\frac{\eta}{2}} \quad \text{and} \quad \sup_{x \in M} E_x (\tau_1)^2 \leq e^{\frac{\eta}{2}}.
\]

**Proof.** If \( x \) is not in \( \bigcup_{j \in L} B_{2\delta}(O_j) \) then a uniform (in \( x \) and small \( \varepsilon \)) upper bound on these expected values follows from the corollary to [8, Lemma 1.9, Chapter 6].

If \( x \in \bigcup_{j \in L} B_{2\delta}(O_j) \) then we must wait till the process reaches \( \bigcup_{j \in L} \partial B_{2\delta}(O_j) \), after which we can use the uniform bound (and the strong Markov property). Since there exists \( \delta > 0 \) such the lower bound \( P_x(\inf\{t \geq 0 : X_t^\varepsilon \in \bigcup_{j \in L} \partial B_{2\delta}(O_j) \leq 1\} \geq e^{-\eta/2\varepsilon} \) is valid for all \( x \in \bigcup_{j \in L} B_{2\delta}(O_j) \) and small \( \varepsilon > 0 \), upper bounds of the desired form follow from the Markov property and standard calculations.

For any compact set \( A \subset \mathbb{R}^d \), we use \( \vartheta_A \) to denote the first hitting time
\[
\vartheta_A = \inf\{t \geq 0 : X_t^\varepsilon \in A\}.
\]
Note that $\vartheta_A$ is a stopping time with respect to filtration $\{\mathcal{F}_t\}_{t \geq 0}$. The following result is relatively straightforward given the just discussed bound on the distribution of $\tau_1$, and follows by partitioning according to $\tau_1 \geq T$ and $\tau_1 < T$ for large but fixed $T$.

**Lemma 6.16** For any compact set $A \subset M$, $j \in \mathcal{L}$ and any $\eta > 0$, there exists $\delta_0 \in (0, 1)$ and $\epsilon_0 \in (0, 1)$, such that for all $\epsilon \in (0, \epsilon_0)$ and $\delta \in (0, \delta_0)$

$$\sup_{z \in \partial B_\delta(O_j)} P_z (\vartheta_A \leq \tau_1) \leq e^{-\frac{1}{\epsilon} (\inf_{x \in A} [V(O_j, x)] - \eta)}.$$

**Lemma 6.17** Given a compact set $A \subset M$, any $j \in \mathcal{L}$ and $\eta > 0$, there exists $\delta_0 \in (0, 1)$, such that for any $\delta \in (0, \delta_0)$

$$\liminf_{\epsilon \to 0} -\epsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left[ \int_0^{\tau_1} 1_A (X_s^x) \, ds \right] \right) \geq \inf_{x \in A} V(O_j, x) - \eta$$

and

$$\liminf_{\epsilon \to 0} -\epsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left( \int_0^{\tau_1} 1_A (X_s^x) \, ds \right)^2 \right) \geq \inf_{x \in A} V(O_j, x) - \eta.$$

**Proof.** The idea of this proof follows from the proof of Theorem 4.3 in [8, Chapter 4]. For any $x \in \partial B_\delta(O_j)$,

$$E_x \left[ \int_0^{\tau_1} 1_A (X_s^x) \, ds \right] = E_x \left[ \int_0^{\tau_1} 1_A (X_s^x) \, ds \right] 1_{\{\vartheta_A \leq \tau_1\}}$$

$$= E_x \left[ E_x \left[ \left( \int_0^{\tau_1} 1_A (X_s^x) \, ds \right) 1_{\{\vartheta_A \leq \tau_1\}} \mid \mathcal{F}_\vartheta \right] \right]$$

$$= E_x \left[ E_x \left[ \left( \int_0^{\tau_1} 1_A (X_s^x) \, ds \right) \mid \mathcal{F}_\vartheta \right] 1_{\{\vartheta_A \leq \tau_1\}} \right]$$

$$= E_x \left[ \left( E_{X_{\vartheta_A}} \left( \int_0^{\tau_1} 1_A (X_s^x) \, ds \right) \right) 1_{\{\vartheta_A \leq \tau_1\}} \right]$$

$$\leq \left( \sup_{y \in \partial A} E_{y \tau_1} \right) \sup_{z \in \partial B_\delta(O_j)} P_z (\vartheta_A \leq \tau_1).$$

The inequality is due to

$$E_{X_{\vartheta_A}} \left( \int_0^{\tau_1} 1_A (X_s^x) \, ds \right) \leq E_{X_{\vartheta_A}} \tau_1 \leq \sup_{y \in \partial A} E_{y \tau_1}.$$
We then apply Lemma 6.15 and Lemma 6.16 to find that for the given \( \eta > 0 \), there exists \( \delta_0 \in (0, 1) \) and \( \varepsilon_0 \in (0, 1) \), such that for all \( \varepsilon \in (0, \varepsilon_0) \) and \( \delta \in (0, \delta_0) \),

\[
E_x \left( \int_0^{\tau_1} 1_A (X_s^\varepsilon) \, ds \right) \leq \left( \sup_{y \in \partial A} E_y \tau_1 \right) \sup_{z \in \partial B_\delta(O_j)} P_z (\partial A \leq \tau_1) \\
\leq e^{n/2} \cdot e^{-\frac{1}{2} (\inf_{y \in A} V(O_j, y) - \eta)}.
\]

Thus,

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left( \int_0^{\tau_1} 1_A (X_s^\varepsilon) \, ds \right) \right) \\
\geq \liminf_{\varepsilon \to 0} -\varepsilon \log \left( e^{-\frac{1}{2} (\inf_{x \in A} V(O_j, x) - \eta)} \right) = \inf_{x \in A} V(O_j, x) - \eta.
\]

This completes the proof of part 1.

For part 2, following the same conditioning argument as for part 1 with the use of Lemma 6.15 and Lemma 6.16 gives that for the given \( \eta > 0 \), there exists \( \delta_0 \in (0, 1) \) and \( \varepsilon_0 \in (0, 1) \), such that for all \( \varepsilon \in (0, \varepsilon_0) \) and \( \delta \in (0, \delta_0) \),

\[
E_x \left( \int_0^{\tau_1} 1_A (X_s^\varepsilon) \, ds \right)^2 \leq \left( \sup_{y \in \partial A} E_y (\tau_1)^2 \right) \sup_{z \in \partial B_\delta(O_j)} P_z (\partial A \leq \tau_1) \\
\leq e^{n/2} \cdot e^{-\frac{1}{2} (\inf_{x \in A} V(O_j, x) - \eta)}.
\]

Therefore,

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left( \int_0^{\tau_1} 1_A (X_s^\varepsilon) \, ds \right)^2 \right) \\
\geq \liminf_{\varepsilon \to 0} -\varepsilon \log \left( e^{-\frac{1}{2} (\inf_{x \in A} V(O_j, x) - \eta)} \right) = \inf_{x \in A} V(O_j, x) - \eta.
\]

\[\blacksquare\]

Remark 6.18 For the inequalities in Lemma 6.17, we are not able to send \( \eta \) to 0 since \( \delta \) depends on \( \eta \).

Lemma 6.19 Given compact sets \( A_1, A_2 \subset M, j \in L \) and \( \eta > 0 \), there exists \( \delta_0 \in (0, 1) \), such that for any \( \delta \in (0, \delta_0) \)

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left( \left( \int_0^{\tau_1} 1_{A_1} (X_s^\varepsilon) \, ds \right) \left( \int_0^{\tau_1} 1_{A_2} (X_s^\varepsilon) \, ds \right) \right) \right) \\
\geq \max \left\{ \inf_{x \in A_1} V(O_j, x), \inf_{x \in A_2} V(O_j, x) \right\} - \eta.
\]
Proof. We set $\vartheta_{A_i} = \inf \{t \geq 0 : X^x_t \in A_i\}$ for $i = 1, 2$. For any $x \in \partial B_{\delta}(O_j)$, using a conditioning argument as in the proof of Lemma 6.17 we obtain that for any $\eta > 0$, there exists $\delta_0 \in (0, 1)$ and $\varepsilon_0 \in (0, 1)$, such that for all $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0)$,

$$E_x \left[ \left( \int_0^{\tau_1} 1_{A_1} (X^x_s) ds \right) \left( \int_0^{\tau_1} 1_{A_2} (X^x_s) ds \right) \right]$$

(6.6)

$$\leq e^{\frac{\alpha_0}{2}} \cdot \min \left\{ \sup_{y \in \partial A_1} E_y (\tau_1)^2, \sup_{z \in \partial B_{\delta}(O_j)} P_z (\vartheta_{A_1} \leq \tau_1), \sup_{z \in \partial B_{\delta}(O_j)} P_z (\vartheta_{A_2} \leq \tau_1) \right\},$$

The last inequality holds since for $i = 1, 2$

$$\sup_{z \in \partial B_{\delta}(O_j)} P_z (\vartheta_{A_1} \leq \tau_1, \vartheta_{A_2} \leq \tau_1) \leq \sup_{z \in \partial B_{\delta}(O_j)} P_z (\vartheta_{A_i} \leq \tau_1)$$

and owing to Lemma 6.15 for all $\varepsilon \in (0, \varepsilon_0)$

$$\sup_{y \in \partial A_1} E_y (\tau_1)^2 \leq e^{\frac{\alpha_0}{2}} \text{ and } \sup_{y \in \partial A_2} E_y (\tau_1)^2 \leq e^{\frac{\alpha_0}{2}}.$$

Furthermore, for the given $\eta > 0$, by Lemma 6.16 there exists $\delta_i \in (0, 1)$ such that for any $\delta \in (0, \delta_i)$

$$\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_{\delta}(O_j)} P_z (\vartheta_{A_i} \leq \tau_1) \right) \geq \inf_{x \in A_i} \bar{V} (O_j, x) - \eta/2$$

31
for \(i = 1, 2\). Hence, letting \(\delta_0 = \delta_1 \land \delta_2\), for any \(\delta \in (0, \delta_0)\)

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left[ \left( \int_0^{\tau_1} 1_{A_1} (X^\varepsilon_s) \, ds \right) \left( \int_0^{\tau_1} 1_{A_2} (X^\varepsilon_s) \, ds \right) \right] \right)
\]

\[
\geq \liminf_{\varepsilon \to 0} -\varepsilon \log \left( e^{\frac {\eta} {2}} \min \left\{ \sup_{z \in \partial B_\delta(O_j)} P_z (\vartheta_{A_1} \leq \tau_1), \sup_{z \in \partial B_\delta(O_j)} P_z (\vartheta_{A_2} \leq \tau_1) \right\} \right)
\]

\[
= -\eta/2 + \max \left\{ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} P_z (\vartheta_{A_1} \leq \tau_1) \right), \right.
\]

\[
\left. \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} P_z (\vartheta_{A_2} \leq \tau_1) \right) \right\}
\]

\[
\geq \max \left\{ \inf_{x \in A_1} V(O_j, x), \inf_{x \in A_2} V(O_j, x) \right\} - \eta.
\]

The first inequality is from (6.6). □

**Lemma 6.20** Given a compact set \(A \subset M\), a continuous function \(f : M \to \mathbb{R}\), \(j \in L\) and \(\eta > 0\), there exists \(\delta_0 \in (0, 1)\), such that for any \(\delta \in (0, \delta_0)\)

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left[ \int_0^{\tau_1} e^{-\frac {1} {\varepsilon} f(X^\varepsilon_s)} 1_A (X^\varepsilon_s) \, ds \right] \right)
\]

\[
\geq \inf_{x \in A} [f(x) + V(O_j, x)] - \eta
\]

and

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left[ \int_0^{\tau_1} e^{-\frac {1} {\varepsilon} f(X^\varepsilon_s)} 1_A (X^\varepsilon_s) \, ds \right]^2 \right)
\]

\[
\geq \inf_{x \in A} [2f(x) + V(O_j, x)] - \eta.
\]

**Proof.** Since a continuous function is bounded on a compact set, there exists \(m \in (0, \infty)\) such that \(-m \leq f(x) \leq m\) for all \(x \in A\). For \(n \in \mathbb{N}\) and \(k \in \{1, 2, \ldots, n\}\), consider the sets

\[
A_{n,k} = \left\{ x \in A : f(x) \in \left[ -m + \frac {2(k-1)m} {n}, -m + \frac {2km} {n} \right] \right\}.
\]

Note that \(A_{n,k}\) is a compact set for any \(n, k\). In addition, for any \(n\) fixed,
\[ \bigcup_{k=1}^{n} A_{n,k} = A. \] With this expression, for any \( x \in \partial B_\delta(O_j) \) and \( n \in \mathbb{N} \)
\[
E_x \left[ \int_0^{\tau_1} e^{-\frac{1}{2} f(X_s^\varepsilon)} 1_A (X_s^\varepsilon) \, ds \right]
\]
\[
= E_x \left[ \int_0^{\tau_1} e^{-\frac{1}{2} f(X_s^\varepsilon)} 1_{\bigcup_{k=1}^{n} A_{n,k}} (X_s^\varepsilon) \, ds \right]
\]
\[
\leq \sum_{k=1}^{n} E_x \left[ \int_0^{\tau_1} e^{-\frac{1}{2} f(X_s^\varepsilon)} 1_{A_{n,k}} (X_s^\varepsilon) \, ds \right]
\]
\[
\leq \sum_{k=1}^{n} \left( E_x \left[ \int_0^{\tau_1} 1_{A_{n,k}} (X_s^\varepsilon) \, ds \right] e^{-\frac{1}{2} \varepsilon \left( \sup_{y \in A_{n,k}} f(y) - \frac{2m}{n} \right)} \right).
\]

The second inequality holds because by definition of \( A_{n,k} \), for any \( x \in A_{n,k} \)
\[
f(x) \geq -m + \frac{2km}{n} - \frac{2m}{n} \geq \sup_{y \in A_{n,k}} f(y) - \frac{2m}{n}.
\]

We use \( F_{n,k} \) to denote \( \sup_{y \in A_{n,k}} f(y) \). Next we first apply (6.2) and then Lemma 6.17 with compact sets \( A_{n,k} \) for \( k \in \{1, 2, \ldots, n\} \) to get
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left[ \int_0^{\tau_1} e^{-\frac{1}{2} f(X_s^\varepsilon)} 1_A (X_s^\varepsilon) \, ds \right] \right)
\]
\[
\geq \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sum_{k=1}^{n} \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left[ \int_0^{\tau_1} 1_{A_{n,k}} (X_s^\varepsilon) \, ds \right] e^{-\frac{1}{2} \varepsilon \left( F_{n,k} - \frac{2m}{n} \right)} \right) \right)
\]
\[
= \min_{k \in \{1, \ldots, n\}} \left\{ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left[ \int_0^{\tau_1} 1_{A_{n,k}} (X_s^\varepsilon) \, ds \right] e^{-\frac{1}{2} \varepsilon \left( F_{n,k} - \frac{2m}{n} \right)} \right) \right\}
\]
\[
= \min_{k \in \{1, \ldots, n\}} \left\{ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left[ \int_0^{\tau_1} 1_{A_{n,k}} (X_s^\varepsilon) \, ds \right] \right) + F_{n,k} \right\} - \frac{2m}{n}
\]
\[
\geq \min_{k \in \{1, \ldots, n\}} \left\{ \sup_{x \in A_{n,k}} f(x) + \inf_{x \in A_{n,k}} V(O_j, x) \right\} - \eta - \frac{2m}{n}.
\]

Finally, we know that \( V(O_j, x) \) is bounded below by 0, and then we use the fact that for any two functions \( f, g : \mathbb{R}^d \to \mathbb{R} \) with \( g \) being bounded below (to ensure that the right hand side is well defined) and any set \( A \subseteq \mathbb{R}^d \),
\[
\inf_{x \in A} (f(x) + g(x)) \leq \sup_{x \in A} f(x) + \inf_{x \in A} g(x)
\]

33
to find
\[
\min_{k \in \{1, \ldots, n\}} \left\{ \sup_{x \in A_{n,k}} f(x) + \inf_{y \in A_{n,k}} V(O_j, x) \right\}
\geq \min_{k \in \{1, \ldots, n\}} \left\{ \inf_{x \in A_{n,k}} [f(x) + V(O_j, x)] \right\} = \inf_{x \in A} [f(x) + V(O_j, x)].
\]

Therefore,
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left[ \int_0^{\tau_1} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_A (X_s^\varepsilon) \, ds \right] \right)
\geq \inf_{x \in A} [f(x) + V(O_j, x)] - \eta - \frac{2m}{n}.
\]

Since \(n\) is arbitrary, sending \(n \to \infty\) completes the proof for the first part.

Turning to part 2, we follow the same argument as for part 1. For any \(n \in \mathbb{N}\), we use the decomposition of \(A\) into \(\bigcup_{k=1}^n A_{n,k}\) to have that for any \(x \in \partial B_\delta(O_j)\),
\[
E_x \left( \int_0^{\tau_1} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_A (X_s^\varepsilon) \, ds \right)^2
\leq E_x \left( \sum_{k=1}^n \int_0^{\tau_1} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_{A_{n,k}} (X_s^\varepsilon) \, ds \right)^2
= \sum_{k=1}^n \sum_{\ell=1}^n E_x \left[ \left( \int_0^{\tau_1} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_{A_{n,k}} (X_s^\varepsilon) \, ds \right) \left( \int_0^{\tau_1} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_{A_{n,\ell}} (X_s^\varepsilon) \, ds \right) \right].
\]
Recall that \(F_{n,k}\) is used to denote \(\sup_{y \in A_{n,k}} f(y)\). Using the definition of \(A_{n,k}\) gives that for any \(k, \ell \in \{1, \ldots, n\}\)
\[
E_x \left[ \left( \int_0^{\tau_1} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_{A_{n,k}} (X_s^\varepsilon) \, ds \right) \left( \int_0^{\tau_1} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_{A_{n,\ell}} (X_s^\varepsilon) \, ds \right) \right]
\leq \sup_{z \in \partial B_\delta(O_j)} E_z \left[ \left( \int_0^{\tau_1} 1_{A_{n,k}} (X_s^\varepsilon) \, ds \right) \left( \int_0^{\tau_1} 1_{A_{n,\ell}} (X_s^\varepsilon) \, ds \right) \right] e^{-\frac{1}{\varepsilon} (F_{n,k} + F_{n,\ell} - 4m)}.
\]

Applying (6.2) first and then Lemma 6.19 with compact sets \(A_{n,k}\) and \(A_{n,\ell}\)
pairwise for all \( k, \ell \in \{1, 2, \ldots, n\} \) gives that

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_{\delta}(O_j)} E_z \left( \int_0^{\tau_1} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_{A_n,k}(X_s^\varepsilon) \, ds \right)^2 \right)
\]

\[
\geq \min_{k, \ell \in \{1, \ldots, n\}} \left\{ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_{\delta}(O_j)} E_z \left[ \left( \int_0^{\tau_1} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_{A_{n,k}}(X_s^\varepsilon) \, ds \right) \cdot \left( \int_0^{\tau_1} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_{A_{n,\ell}}(X_s^\varepsilon) \, ds \right) \right] \right) \right\}
\]

\[
\geq \min_{k, \ell \in \{1, \ldots, n\}} \left\{ \max \left\{ \inf_{x \in A_{n,k}} V(O_1, x), \inf_{x \in A_{n,\ell}} V(O_j, x) \right\} + F_{n,k} + F_{n,\ell} \right\} - \eta - \frac{4m}{n}
\]

\[
\geq \min_{k \in \{1, \ldots, n\}} \left\{ \sup_{x \in A_{n,k}} [2f(x)] + \inf_{x \in A_{n,k}} V(O_j, x) \right\} - \eta - \frac{4m}{n}
\]

\[
\geq \min_{k \in \{1, \ldots, n\}} \left\{ \inf_{x \in A_{n,k}} [2f(x) + V(O_j, x)] \right\} - \eta - \frac{4m}{n}
\]

\[
= \inf_{x \in A} [2f(x) + V(O_j, x)] - \eta - \frac{4m}{n}.
\]

Sending \( n \to \infty \) completes the proof for the second part.

Our next interest is to find lower bounds for

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_{\delta}(O_1)} E_z N_j \right)
\]

and

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sum_{\ell=1}^{\infty} \sup_{z \in \partial B_{\delta}(O_j)} P_z (\ell \leq N_j) \right).
\]

We first recall that \( N_j \) is the number of visits of the embedded Markov chain \( \{Z_n\}_n = \{X_n^\varepsilon\}_n \) to \( \partial B_{\delta}(O_j) \) within one loop of regenerative cycle. Also, the definitions of \( G(i) \) and \( G(i,j) \) for any \( i, j \in L \) with \( i \neq j \) are given in Definition 3.7 and Remark 3.8.

**Lemma 6.21** For any \( \eta > 0 \), there exists \( \delta_0 \in (0, 1) \), such that for any \( \delta \in (0, \delta_0) \) and for any \( j \in L \)

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_{\delta}(O_1)} E_z N_j \right) \geq - \min_{\ell \in L \setminus \{1\}} V(O_1, O_\ell) + W(O_\ell) - W(O_1) - \eta,
\]

35
where
\[ W(O_j) = \min_{g \in G(j)} \left[ \sum_{(m \to n) \in g} V(O_m, O_n) \right]. \]

**Proof.** According to Lemma 3.17 we know that for any \( \eta > 0 \), there exist \( \delta_0 \in (0, 1) \) and \( \varepsilon_0 \in (0, 1) \), such that for any \( \delta \in (0, \delta_0) \) and \( \varepsilon \in (0, \varepsilon_0) \), for all \( x \in \partial B_\delta(O_i) \), the one-step transition probability of the Markov chain \( \{Z_n\}_n \) on \( \partial B_\delta(O_j) \) satisfies the inequalities
\[ e^{-\frac{1}{\varepsilon} V(O_i, O_j) + \eta/4^{l-1}} \leq p(x, \partial B_\delta(O_j)) \leq e^{-\frac{1}{\varepsilon} V(O_i, O_j) - \eta/4^{l-1}}. \] (6.7)

We can then apply Lemma 6.7 with \( p_{ij} = e^{-\frac{1}{\varepsilon} V(O_i, O_j)} \) and \( a = e^{\frac{1}{\varepsilon} \eta/4^{l-1}} \) to obtain that
\[
\sup_{z \in \partial B_\delta(O_1)} E_{z N_j} \leq \frac{e^{\frac{1}{\varepsilon} \eta} \sum_{g \in G(j)} \pi(g)}{e^{\frac{1}{\varepsilon} \frac{1}{4} V(O_1, O_\ell)} \sum_{g \in G(1)} \pi(g)} \left( e^{\frac{1}{\varepsilon} \sum_{(m \to n) \in g} V(O_m, O_n)} \right).
\]

Thus,
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_1)} E_{z N_j} \right) \geq - \min_{\ell \in L \setminus \{1\}} V(O_1, O_\ell) - \eta + \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \frac{\sum_{g \in G(j)} \pi(g)}{\sum_{g \in G(1)} \pi(g)} \right).
\]

Hence it suffices to show that
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \frac{\sum_{g \in G(j)} \pi(g)}{\sum_{g \in G(1)} \pi(g)} \right) \geq W(O_j) - W(O_1).
\]

Observe that by definition for any \( j \in L \) and \( g \in G(j) \)
\[
\pi(g) = \prod_{(m \to n) \in g} p_{mn} = \prod_{(m \to n) \in g} e^{-\frac{1}{\varepsilon} V(O_m, O_n)}
= \exp \left\{ -\frac{1}{\varepsilon} \sum_{(m \to n) \in g} V(O_m, O_n) \right\},
\]
\]

which implies that

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \frac{\sum_{g \in G(j)} \pi(g)}{\sum_{g \in G(1)} \pi(g)} \right) \\
\geq \min_{g \in G(j)} \left[ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \exp \left\{ -\frac{1}{\varepsilon} \sum_{(m \to n) \in g} V(O_m, O_n) \right\} \right) \right] \\
- \min_{g \in G(1)} \left[ \limsup_{\varepsilon \to 0} -\varepsilon \log \left( \exp \left\{ -\frac{1}{\varepsilon} \sum_{(m \to n) \in g} V(O_m, O_n) \right\} \right) \right] \\
= \min_{g \in G(j)} \left[ \sum_{(m \to n) \in g} V(O_m, O_n) \right] - \min_{g \in G(1)} \left[ \sum_{(m \to n) \in g} V(O_m, O_n) \right] \\
= W(O_j) - W(O_1).
\]

The inequality is from Lemma 6.1; the last equality holds due the definition of \( W(O_j) \).

**Remark 6.22** \( W(O_j) \) has been defined to be \( \min_{g \in G(j)} \left[ \sum_{(m \to n) \in g} V(O_m, O_n) \right] \).

Heuristically, if we interpret \( V(O_m, O_n) \) as the “cost” of moving from \( O_m \) to \( O_n \), then \( W(O_j) \) is the “least total cost” of reaching \( O_j \) from every \( O_i \) with \( i \in L \setminus \{j\} \).

**Remark 6.23** As shown in Theorem 4.1, [8, Chapter 6],

\[
W(O_j) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} -\varepsilon \log \mu^\varepsilon(B_\delta(O_j)),
\]

where \( \mu^\varepsilon \in \mathcal{P}(M) \) is the unique invariant measure of \( \{X^\varepsilon(t)\}_{t \geq 0} \).

Recall the definition of \( W(O_1 \cup O_j) \) in (3.1).

**Lemma 6.24** For any \( \eta > 0 \), there exists \( \delta_0 \in (0, 1) \), such that for any \( \delta \in (0, \delta_0) \)

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sum_{\ell=1}^{\infty} \sup_{z \in \partial B_\delta(O_1)} P_z(\ell \leq N_1) \right) \geq -\min_{\ell \in L \setminus \{1\}} V(O_1, O_\ell) - \eta
\]

and for any \( j \in L \setminus \{1\} \)

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sum_{\ell=1}^{\infty} \sup_{z \in \partial B_\delta(O_j)} P_z(\ell \leq N_j) \right) \geq W(O_1 \cup O_j) - W(O_1) - \eta.
\]
Proof. We again use that by Lemma 3.17 for any $\eta > 0$ there exist $\delta_0 \in (0, 1)$ and $\varepsilon_0 \in (0, 1)$, such that (6.7) holds for any $\delta \in (0, \delta_0)$, $\varepsilon \in (0, \varepsilon_0)$ and all $x \in \partial B_\delta(O_i)$. Then by Lemma 6.7 with $p_{ij} = e^{-\frac{1}{\varepsilon}V(O_i, O_j)}$ and $a = e^{\frac{1}{\varepsilon}V/O_i, O_j}$

$$\sum_{\ell=1}^{\infty} \sup_{x \in \partial B_\delta(O_j)} P_x (N_1 \geq \ell) \leq \frac{e^{\frac{1}{\varepsilon}\eta}}{\sum_{\ell \in L \setminus \{1\}} e^{-\frac{1}{\varepsilon}V(O_1, O_{\ell})}}$$

and for any $j \in L \setminus \{1\}$

$$\sum_{\ell=1}^{\infty} \sup_{x \in \partial B_\delta(O_j)} P_x (N_j \geq \ell) \leq e^{\frac{1}{\varepsilon}\eta} \sum_{g \in G_1(j)} \pi(g) \sum_{g \in G(1)} \pi(g).$$

Thus,

$$\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sum_{\ell=1}^{\infty} \sup_{x \in \partial B_\delta(O_1)} P_x (\ell \leq N_1) \right) \geq - \limsup_{\varepsilon \to 0} -\varepsilon \log \left( \sum_{\ell \in L \setminus \{1\}} e^{-\frac{1}{\varepsilon}V(O_1, O_{\ell})} \right) - \eta$$

and

$$\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sum_{\ell=1}^{\infty} \sup_{x \in \partial B_\delta(O_j)} P_x (\ell \leq N_j) \right) \geq \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sum_{g \in G_1(j)} \pi(g) \sum_{g \in G(1)} \pi(g) \right) - \eta.$$

Following the same argument as for the proof of Lemma 6.21, we can use Lemma 6.1 to obtain that

$$- \limsup_{\varepsilon \to 0} -\varepsilon \log \left( \sum_{\ell \in L \setminus \{1\}} e^{-\frac{1}{\varepsilon}V(O_1, O_{\ell})} \right) \geq - \min_{\ell \in L \setminus \{1\}} V(O_1, O_{\ell})$$

and

$$\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sum_{g \in G_1(j)} \pi(g) \sum_{g \in G(1)} \pi(g) \right) \geq \min_{g \in G_1(j)} \left[ \sum_{(m \to n) \in g} V(O_m, O_n) \right] - \min_{g \in G(1)} \left[ \sum_{(m \to n) \in g} V(O_m, O_n) \right].$$

38
Recalling (3.1), we are done. ■

As mentioned at the beginning of this subsection, our main goal is to provide lower bounds for

$$\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta (O_1)} E_z \left( \int_0^{\tau_\varepsilon} e^{-\frac{1}{2} f(X_s^\varepsilon)} 1_A (X_s^\varepsilon) \, ds \right) \right)$$

and

$$\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta (O_1)} E_z \left( \int_0^{\tau_\varepsilon} e^{-\frac{1}{2} f(X_s^\varepsilon)} 1_A (X_s^\varepsilon) \, ds \right)^2 \right)$$

for a given continuous function \( f : M \to \mathbb{R} \) and compact set \( A \subset M \). We now state the main results of the subsection.

**Lemma 6.25** Given a compact set \( A \subset M \), a continuous function \( f : M \to \mathbb{R} \) and \( \eta > 0 \), there exists \( \delta_0 \in (0, 1) \), such that for any \( \delta \in (0, \delta_0) \)

$$\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta (O_1)} E_z \left( S_{\varepsilon}^1 \right) \right) \geq \inf_{x \in A} \left[ f(x) + V(O_j, x) \right] - \eta,$$

where

$$S_{\varepsilon}^1 \equiv \int_0^{\tau_\varepsilon} e^{-\frac{1}{2} f(X_s^\varepsilon)} 1_A (X_s^\varepsilon) \, ds$$

and

$$W(O_j) \equiv \min_{g \in G(j)} \left[ \sum_{(m \to n) \in g} V(O_m, O_n) \right].$$

**Proof.** Recall that by Lemma 6.20, we have shown that for the given \( \eta \), there exists \( \delta_1 \in (0, 1) \), such that for any \( \delta \in (0, \delta_1) \) and \( j \in L \)

$$\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta (O_1)} E_z \left[ \int_0^{\tau_\varepsilon} e^{-\frac{1}{2} f(X_s^\varepsilon)} 1_A (X_s^\varepsilon) \, ds \right] \right) \geq \inf_{x \in A} \left[ f(x) + V(O_j, x) \right] - \frac{\eta}{2}.$$
In addition, by Lemma 6.21, we know that for the same \( \eta \), there exists \( \delta_2 \in (0,1) \), such that for any \( \delta \in (0,\delta_2) \)

\[
\liminf_{\epsilon \to 0} -\epsilon \log \left( \sup_{z \in \partial B_\delta(O_1)} E_z N_j \right) \\
\geq - \min_{\ell \in L \setminus \{1\}} V(O_1, O_\ell) + W(O_j) - W(O_1) - \frac{\eta}{2}.
\]

Hence for any \( \delta \in (0, \delta_0) \) with \( \delta_0 = \delta_1 \wedge \delta_2 \), we apply Corollary 6.13 to get

\[
\liminf_{\epsilon \to 0} -\epsilon \log \left[ E_z \left( \int_0^{\tau_\delta} e^{-\frac{1}{\epsilon} f(X^z_s)} 1_A(X^z_s) \, ds \right) \right] \\
\geq \min_{j \in L} \left\{ \liminf_{\epsilon \to 0} -\epsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left( \int_0^{\tau_\delta} e^{-\frac{1}{\epsilon} f(X^z_s)} 1_A(X^z_s) \, ds \right) \right) \right. \\
+ \liminf_{\epsilon \to 0} -\epsilon \log \left( \sup_{z \in \partial B_\delta(O_1)} E_z (N_j) \right) \}
\]

\[
\geq \min_{j \in L} \left\{ \inf_{x \in A} \left[ f(x) + V(O_j, x) \right] + W(O_j) \right\} - W(O_1) \\
- \min_{\ell \in L \setminus \{1\}} V(O_1, O_\ell) - \eta,
\]

where \( \tau_\delta \) is the time for a regenerative cycle and \( \tau_1 \) is the first visit time of neighborhoods of equilibrium points after being a certain distance away from them. \( \blacksquare \)

**Remark 6.26** According to Remark 6.14 and using the same argument as in Lemma 6.25, we can find that given a compact set \( A \subset M \), a continuous function \( f : M \to \mathbb{R} \) and \( \eta > 0 \), there exists \( \delta_0 \in (0,1) \), such that for any \( \delta \in (0, \delta_0) \)

\[
\liminf_{\epsilon \to 0} -\epsilon \log \left[ \sup_{z \in \partial B_\delta(O_1)} E_z \left( \int_0^{\tau_\delta} e^{-\frac{1}{\epsilon} f(X^z_s)} 1_A(X^z_s) \, ds \right) \right] \\
\geq \min_{j \in L \setminus \{1\}} \left\{ \inf_{x \in A} \left[ f(x) + V(O_j, x) \right] + W(O_j) \right\} - W(O_1) \\
- \min_{\ell \in L \setminus \{1\}} V(O_1, O_\ell) - \eta.
\]

**Lemma 6.27** Given a compact set \( A \subset M \), a continuous function \( f : M \to \mathbb{R} \) and \( \eta > 0 \), there exists \( \delta_0 \in (0,1) \), such that for any \( \delta \in (0, \delta_0) \)

\[
\liminf_{\epsilon \to 0} -\epsilon \log \left( \sup_{z \in \partial B_\delta(O_1)} E_z (S^z_1)^2 \right) \geq \min_{j \in L} \left( R_j^{(1)} \wedge R_j^{(2)} \right) - h - \eta,
\]

where \( S^z_1 \) is the time for a regenerative cycle and \( \tau_1 \) is the first visit time of neighborhoods of equilibrium points after being a certain distance away from them. \( \blacksquare \)
where
\[ S^\varepsilon_1 \doteq \int_0^{\tau_1^\varepsilon} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_A (X_s^\varepsilon) \, ds \]
and \( h = \min_{\ell \in L \setminus \{1\}} V (O_1, O_\ell) \) and
\[ R_j^{(1)} \doteq \inf_{x \in A} [2f(x) + V(O_j, x)] + W(O_j) - W(O_1) \]
and
\[ R_1^{(2)} \doteq 2 \inf_{x \in A} [f(x) + V(O_1, x)] - h \]
and for \( j \in L \setminus \{1\} \)
\[ R_j^{(2)} \doteq 2 \inf_{x \in A} [f(x) + V(O_j, x)] + W(O_j) - 2W(O_1) + W(O_1 \cup O_j). \]

**Proof.** Following a similar argument as for the proof of Lemma 6.25, given any \( \eta > 0 \), owing to Lemma 6.20, Lemma 6.21 and Lemma 6.24 there exists \( \delta_1, \delta_2, \delta_3 \in (0, 1) \) such that: for any \( \delta \in (0, \delta_1) \) and for any \( j \in L \)
\[ \lim_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left( \int_0^{\tau_1^\varepsilon} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_A (X_s^\varepsilon) \, ds \right) \right) \geq \inf_{x \in A} [f(x) + V(O_j, x)] - \frac{\eta}{4} \]
and
\[ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_1)} E_z \left( \int_0^{\tau_1^\varepsilon} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_A (X_s^\varepsilon) \, ds \right)^2 \right) \geq \inf_{x \in A} [2f(x) + V(O_j, x)] - \frac{\eta}{4} \]
for any \( \delta \in (0, \delta_2) \) and for any \( j \in L \)
\[ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_1)} E_z N_j \right) \geq -h + W(O_j) - W(O_1) - \frac{\eta}{4} ; \]
and for any \( \delta \in (0, \delta_3) \)
\[ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sum_{\ell = 1}^{\infty} \sup_{z \in \partial B_\delta(O_1)} P_z (\ell \leq N_1) \right) \geq -h - \frac{\eta}{4} . \]
and for any \( j \in L \setminus \{1\} \)

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sum_{\ell=1}^{\infty} \sup_{z \in \partial B_{\delta}(O_j)} P_z (\ell \leq N_j) \right) \\
\geq W(O_1 \cup O_j) - W(O_1) - \frac{\eta}{4}.
\]

Hence for any \( \delta \in (0, \delta_0) \) with \( \delta_0 = \delta_1 \land \delta_2 \land \delta_3 \), we apply Corollary 6.13 to get

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_{\delta}(O_1)} E_z (S_1^{\varepsilon})^2 \right) \geq \min_{j \in L} \left( \hat{R}_j^{(1)} \land \hat{R}_j^{(2)} \right),
\]

where

\[
\hat{R}_j^{(1)} = \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_{\delta}(O_j)} E_z \left( \int_0^{T_1} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon) 1_A(X_s^\varepsilon)) ds} \right)^2 \right) \\
+ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_{\delta}(O_1)} E_z N_j \right) \\
\geq \inf_{x \in A} \left[ 2f(x) + V(O_j, x) \right] + W(O_j) - W(O_1) - h - \eta \\
= R_j^{(1)} - h - \eta
\]

and

\[
\hat{R}_1^{(2)} = 2 \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_{\delta}(O_1)} E_z \left[ \int_0^{T_1} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon) 1_A(X_s^\varepsilon)) ds} \right] \right) \\
+ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_{\delta}(O_1)} E_z N_1 \right) \\
+ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sum_{\ell=1}^{\infty} \sup_{z \in \partial B_{\delta}(O_1)} P_z (\ell \leq N_1) \right) \\
\geq 2 \left( \inf_{x \in A} [f(x) + V(O_1, x)] - \frac{\eta}{4} \right) + \left( -h - \frac{\eta}{4} \right) + \left( -h - \frac{\eta}{4} \right) \\
= 2 \inf_{x \in A} [f(x) + V(O_1, x)] - 2h - \eta = R_1^{(2)} - h - \eta
\]
and for \( j \in L \setminus \{1\} \)

\[
\hat{R}_j^{(2)} = 2 \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_j)} E_z \left[ \int_0^{\tau_1} e^{-\frac{\varepsilon}{2} f(X_s^\varepsilon)} 1_A(X_s^\varepsilon) \, ds \right] \right) \\
+ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_\delta(O_1)} E_z N_j \right) \\
+ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sum_{\ell=1}^\infty \sup_{z \in \partial B_\delta(O_j)} P_z (\ell \leq N_j) \right) \\
\geq 2 \left( \inf_{x \in A} \left[ f(x) + V(O_j, x) \right] - \frac{\eta}{4} \right) + \left( -h + W(O_j) - W(O_1) - \frac{\eta}{4} \right) \\
+ \left( W(O_1 \cup O_j) - W(O_1) - \frac{\eta}{4} \right) \\
= 2 \inf_{x \in A} \left[ f(x) + V(O_j, x) \right] + W(O_j) - 2W(O_1) + W(O_1 \cup O_j) - h - \eta \\
= R_j^{(2)} - h - \eta.
\]

### 7 Asymptotics of Moments of \( N^\varepsilon(T^\varepsilon) \)

Recall that the number of renewals in the time interval \([0, T^\varepsilon]\) is defined as

\[
N^\varepsilon(T^\varepsilon) = \min \{ n \in \mathbb{N} : \tau_n^\varepsilon > T^\varepsilon \},
\]

where the \( \tau_n^\varepsilon \) are the return times to \( B_\delta(O_1) \) after ever visiting one of the \( \delta \)-neighborhood of other equilibrium points than \( O_1 \). In addition, \( \lambda^\varepsilon \) is the unique invariant measure of \( \{Z_n^\varepsilon\}_n \).

In this section, we will find the logarithmic asymptotics of the expected value and the variance of \( N^\varepsilon(T^\varepsilon) \) with \( T^\varepsilon = e^{\frac{1}{c} c} \) for some \( c > h_\delta \), \( h_\delta = \min_{x \in \bigcup_{j \in L \setminus \{1\}} \partial B_\delta(O_j)} V(O_1, x) \), in Lemma 7.2 and Lemma 7.3.

**Remark 7.1** For any \( \delta > 0 \), \( h_\delta \leq h = \min_{j \in L \setminus \{1\}} V(O_1, O_j) \).

**Lemma 7.2** For any \( \delta > 0 \) sufficiently small and \( T^\varepsilon = e^{\frac{1}{c} c} \) for some \( c > h_\delta \),

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left| \frac{E_{\lambda^\varepsilon} (N^\varepsilon(T^\varepsilon))}{T^\varepsilon} - \frac{1}{E_{\lambda^\varepsilon} \tau_1^\varepsilon} \right| \geq c.
\]

**Lemma 7.3** For any \( \delta > 0 \) sufficiently small and \( T^\varepsilon = e^{\frac{1}{c} c} \) for some \( c > h_\delta \),

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left| \frac{\text{Var}_{\lambda^\varepsilon} (N^\varepsilon(T^\varepsilon))}{T^\varepsilon} \right| \geq h_\delta.
\]
Before proceeding, we mention a result from [7] and define some notation which will be used in this section. Results in Section 5 and Section 10 from [7, Chapter XI] say that for any \( t > 0 \), the first and second moment of \( N^\varepsilon(t) \) can be represented as

\[
E_{\lambda^\varepsilon}(N^\varepsilon(t)) = \sum_{n=0}^{\infty} P_{\lambda^\varepsilon}(\tau_n^\varepsilon \leq t) \quad \text{and} \quad E_{\lambda^\varepsilon}(N^\varepsilon(t))^2 = \sum_{n=0}^{\infty} (2n + 1) P_{\lambda^\varepsilon}(\tau_n^\varepsilon \leq t).
\]

(7.1)

Let \( \Gamma^\varepsilon \doteq T^\varepsilon/E_{\lambda^\varepsilon} \tau_1^\varepsilon \) and \( \gamma^\varepsilon \doteq (\Gamma^\varepsilon)^{-\ell} \) with some \( \ell \in (0, 1) \) which will be chosen later. Intuitively, \( \Gamma^\varepsilon \) is the typical number of regenerative cycles in \([0, T^\varepsilon]\) since \( E_{\lambda^\varepsilon} \tau_1^\varepsilon \) is the expected length of one regenerative cycle. To simplify notation, we pretend that \((1 + 2\gamma^\varepsilon)\Gamma^\varepsilon \) and \((1 - 2\gamma^\varepsilon)\Gamma^\varepsilon \) are positive integers so that we can divide \( E_{\lambda^\varepsilon}(N^\varepsilon(T^\varepsilon)) \) into three partial sums which are

\[
\mathcal{P}_1 \doteq \sum_{n=(1+2\gamma^\varepsilon)\Gamma^\varepsilon + 1}^{\infty} P_{\lambda^\varepsilon}(\tau_n^\varepsilon \leq T^\varepsilon), \quad \mathcal{P}_2 \doteq \sum_{n=(1-2\gamma^\varepsilon)\Gamma^\varepsilon}^{(1+2\gamma^\varepsilon)\Gamma^\varepsilon} P_{\lambda^\varepsilon}(\tau_n^\varepsilon \leq T^\varepsilon)
\]

and

\[
\mathcal{P}_3 \doteq \sum_{n=0}^{(1-2\gamma^\varepsilon)\Gamma^\varepsilon - 1} P_{\lambda^\varepsilon}(\tau_n^\varepsilon \leq T^\varepsilon).
\]

(7.2)

Similarly, we divide \( E_{\lambda^\varepsilon}(N^\varepsilon(T^\varepsilon))^2 \) into

\[
\mathcal{R}_1 \doteq \sum_{n=(1+2\gamma^\varepsilon)\Gamma^\varepsilon + 1}^{\infty} (2n + 1) P_{\lambda^\varepsilon}(\tau_n^\varepsilon \leq T^\varepsilon),
\]

\[
\mathcal{R}_2 \doteq \sum_{n=(1-2\gamma^\varepsilon)\Gamma^\varepsilon}^{(1+2\gamma^\varepsilon)\Gamma^\varepsilon} (2n + 1) P_{\lambda^\varepsilon}(\tau_n^\varepsilon \leq T^\varepsilon)
\]

and

\[
\mathcal{R}_3 \doteq \sum_{n=0}^{(1-2\gamma^\varepsilon)\Gamma^\varepsilon - 1} (2n + 1) P_{\lambda^\varepsilon}(\tau_n^\varepsilon \leq T^\varepsilon).
\]

(7.3)

The next step is to find upper bounds for these partial sums, and these bounds will help us to find suitable lower bounds for the logarithmic asymptotics of \( E_{\lambda^\varepsilon}(N^\varepsilon(T^\varepsilon)) \) and \( \text{Var}_{\lambda^\varepsilon}(N^\varepsilon(T^\varepsilon)) \). Before looking into the upper bound for partial sums, we establish some properties.
Theorem 7.4 For any $\delta > 0$ sufficiently small,
\[
\lim_{\varepsilon \to 0} \varepsilon \log E_{\lambda^\varepsilon} \tau_1^\varepsilon = h_\delta \quad \text{and} \quad \frac{\tau_1^\varepsilon}{E_{\lambda^\varepsilon} \tau_1^\varepsilon} \xrightarrow{d} \text{Exp}(1).
\]
Moreover, there exists $\varepsilon_0 \in (0, 1)$ and a constant $\tilde{c} > 0$ such that
\[
P_{\lambda^\varepsilon} \left( \frac{\tau_1^\varepsilon}{E_{\lambda^\varepsilon} \tau_1^\varepsilon} > t \right) \leq e^{-\tilde{c}t}
\]
for any $t > 0$ and any $\varepsilon \in (0, \varepsilon_0)$.

The proof for Theorem 7.4 will be given in Section 9. In that section, we will first prove an analogous result for the exit time (or first visiting time to other stable equilibrium points to be more precise), and then show how one can extend those results to the return time.

Lemma 7.5 Suppose $T_1^\varepsilon = e^{\frac{1}{\varepsilon}c}$ for some $c > h$. For any $\delta > 0$ sufficiently small, the limit of $-\varepsilon \log \Gamma^\varepsilon$ as $\varepsilon \to 0$ exists and
\[
\lim_{\varepsilon \to 0} -\varepsilon \log \Gamma^\varepsilon = h_\delta - c.
\]

Proof. For any $\delta > 0$, since $\Gamma^\varepsilon = T_1^\varepsilon / E_{\lambda^\varepsilon} \tau_1^\varepsilon$ and $T_1^\varepsilon = e^{\frac{1}{\varepsilon}c}$,
\[
-\varepsilon \log \Gamma^\varepsilon = -\varepsilon \log \left( \frac{1}{E_{\lambda^\varepsilon} \tau_1^\varepsilon} \right) - c.
\]
We complete the proof by applying Theorem 7.4.

Lemma 7.6 Define $Z_1^\varepsilon = \tau_1^\varepsilon / E_{\lambda^\varepsilon} \tau_1^\varepsilon$. Then
\begin{itemize}
    \item there exists some $\varepsilon_0 \in (0, 1)$ such that $\sup_{\varepsilon \in (0, \varepsilon_0)} E_{\lambda^\varepsilon} (Z_1^\varepsilon)^3 < \infty$,
    \item there exists some $\varepsilon_0 \in (0, 1)$ such that $\inf_{\varepsilon \in (0, \varepsilon_0)} \text{Var}_{\lambda^\varepsilon}(Z_1^\varepsilon) > 0$ and $E_{\lambda^\varepsilon} (Z_1^\varepsilon)^2 = E_{\lambda^\varepsilon} (\tau_1^\varepsilon)^2 / (E_{\lambda^\varepsilon} \tau_1^\varepsilon)^2 \to 2$ as $\varepsilon \to 0$.
\end{itemize}

Proof. For the first part, we use Theorem 7.4 to find that there exists $\varepsilon_0 \in (0, 1)$ and a constant $\tilde{c} > 0$ such that
\[
P_{\lambda^\varepsilon} (Z_1^\varepsilon > t) = P_{\lambda^\varepsilon} \left( \frac{\tau_1^\varepsilon}{E_{\lambda^\varepsilon} \tau_1^\varepsilon} > t \right) \leq e^{-\tilde{c}t}
\]
for any $t > 0$ and any $\varepsilon \in (0, \varepsilon_0)$. Therefore, for $\varepsilon \in (0, \varepsilon_0)$
\[
E_{\lambda^\varepsilon} (Z_1^\varepsilon)^3 = 3 \int_0^\infty t^2 P_{\lambda^\varepsilon} (Z_1^\varepsilon > t) dt \leq 3 \int_0^\infty t^2 e^{-\tilde{c}t} dt < \infty.
\]

For the second assertion, since $\sup_{0<\varepsilon<\varepsilon_0} E_{\lambda^\varepsilon} (Z_1^\varepsilon)^3 < \infty$, it implies that 
\{
(Z_1^\varepsilon)^2\}_0<\varepsilon<\varepsilon_0 \text{ and } \{Z_1^\varepsilon\}_0<\varepsilon<\varepsilon_0 \text{ are both uniformly integrable. Moreover,}
\[
E_{\lambda^\varepsilon} \left( \frac{\tau_1^\varepsilon}{E_{\lambda^\varepsilon} \tau_1^\varepsilon} \right)^2 = E_{\lambda^\varepsilon} (Z_1^\varepsilon)^2 \to 2 \text{ and } E_{\lambda^\varepsilon} Z_1^\varepsilon \to 1.
\]
as $\varepsilon \to 0$. This implies $\text{Var}_{\lambda^\varepsilon} (Z_1^\varepsilon) \to 1$ as $\varepsilon \to 0$. Obviously, there exists some $\varepsilon_0 \in (0, 1)$ such that
\[
\inf_{\varepsilon \in (0, \varepsilon_0)} \text{Var}_{\lambda^\varepsilon} (Z_1^\varepsilon) \geq \frac{1}{2} > 0.
\]
This completes the proof. ■

**Remark 7.7** Throughout the rest of this section, we will use $C$ to denote a constant in $(0, \infty)$ which is independent of $\varepsilon$ but whose value may change from use to use.

### 7.1 Chernoff bound

In this subsection we will provide upper bounds for
\[
\mathfrak{P}_1 \overset{\Delta}{=} \sum_{n=(1+2\gamma^\varepsilon)^\Gamma^\varepsilon+1}^{\infty} P_{\lambda^\varepsilon} (\tau_n^\varepsilon \leq T^\varepsilon)
\]
and
\[
\mathfrak{R}_1 \overset{\Delta}{=} \sum_{n=(1+2\gamma^\varepsilon)^\Gamma^\varepsilon+1}^{\infty} (2n+1) P_{\lambda^\varepsilon} (\tau_n^\varepsilon \leq T^\varepsilon)
\]
via a Chernoff bound. The following result is well known.

**Lemma 7.8 (Chernoff bound)** Let $X_1, \ldots, X_n$ be an iid sequence of random variables. For any $a \in \mathbb{R}$ and for any $t > 0$
\[
P (X_1 + \cdots + X_n \leq a) \leq \left( E e^{-tX_1} \right)^n e^{ta}.
\]
Proof. For any $t > 0$

\[ P(X_1 + \cdots + X_n \leq a) = P\left( e^{-t(X_1 + \cdots + X_n)} \geq e^{-ta} \right) \leq \frac{Ee^{-t(X_1 + \cdots + X_n)}}{e^{-ta}} = (Ee^{-tX_1})^n e^{ta}. \]

Recall that $\Gamma^\varepsilon = T^\varepsilon / E\lambda^\varepsilon \tau_1^\varepsilon$ and $\gamma^\varepsilon = (\Gamma^\varepsilon)^{-\ell}$ with some $\ell \in (0, 1)$ which will be chosen later.

Lemma 7.9 Given any $\delta > 0$ and any $\ell > 0$, there exists $\varepsilon_0 \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon_0)$

\[ P_{\lambda^\varepsilon} (\tau_n^\varepsilon \leq T^\varepsilon) \leq e^{-n\frac{1}{2}(\Gamma^\varepsilon)^{-2\ell}} \]

for any $n \geq (1 + 2\gamma^\varepsilon) \Gamma^\varepsilon$. Consequently,

\[ \mathfrak{R}_1 = \sum_{n=(1+2\gamma^\varepsilon)\Gamma^\varepsilon+1}^{\infty} P_{\lambda^\varepsilon} (\tau_n^\varepsilon \leq T^\varepsilon) \leq C (\Gamma^\varepsilon)^{2\ell} e^{-\frac{1}{2}(\Gamma^\varepsilon)^{1-2\ell}} \]

and

\[ \mathfrak{R}_1 = \sum_{n=(1+2\gamma^\varepsilon)\Gamma^\varepsilon+1}^{\infty} (2n + 1) P_{\lambda^\varepsilon} (\tau_n^\varepsilon \leq T^\varepsilon) \leq C (\Gamma^\varepsilon)^{1+2\ell} e^{-\frac{1}{2}(\Gamma^\varepsilon)^{1-2\ell}} + C (\Gamma^\varepsilon)^{4\ell} e^{-\frac{1}{2}(\Gamma^\varepsilon)^{1-2\ell}}. \]

Proof. Given $\delta > 0$ and $\varepsilon \in (0, 1)$, we find that for $n \geq (1 + 2\gamma^\varepsilon) \Gamma^\varepsilon$

\[ P_{\lambda^\varepsilon} (\tau_n^\varepsilon \leq T^\varepsilon) = P_{\lambda^\varepsilon} (\tau_n^\varepsilon \leq \Gamma^\varepsilon \cdot E\lambda^\varepsilon \tau_1^\varepsilon) \]

\[ = P_{\lambda^\varepsilon} \left( \frac{\tau_1^\varepsilon + (\tau_2^\varepsilon - \tau_1^\varepsilon) + \cdots + (\tau_n^\varepsilon - \tau_{n-1}^\varepsilon)}{E\lambda^\varepsilon \tau_1^\varepsilon} \leq \Gamma^\varepsilon \right) \leq P_{\lambda^\varepsilon} \left( \frac{\tau_1^\varepsilon + (\tau_2^\varepsilon - \tau_1^\varepsilon) + \cdots + (\tau_n^\varepsilon - \tau_{n-1}^\varepsilon)}{E\lambda^\varepsilon \tau_1^\varepsilon} \leq \frac{n}{1 + 2\gamma^\varepsilon} \right) \]

\[ \leq \left( (E\lambda^\varepsilon e^{-\gamma^\varepsilon \varepsilon_1^\varepsilon} e^{\frac{\gamma^\varepsilon}{1 + 2\gamma^\varepsilon}})^n, \right. \]

where $\varepsilon_1^\varepsilon = \tau_1^\varepsilon / E\lambda^\varepsilon \tau_1^\varepsilon$. We use the fact that $\{\tau_n^\varepsilon - \tau_{n-1}^\varepsilon\}_{n \in \mathbb{N}}$ are iid and apply Lemma 7.8 (Chernoff bound) with $a = n / (1 + 2\gamma^\varepsilon)$ and $t = \gamma^\varepsilon$ for the last inequality.
Therefore, in order to verify the first claim, it suffices to show that
\[
(E_\lambda e^{-\gamma e Z_1^\varepsilon}) e^{-\frac{1}{2}(\gamma e)^2} \leq e^{-\frac{1}{2}(\Gamma e)^2 - 2\varepsilon}.
\]
We observe that for any \( x \geq 0 \), \( e^{-x} \leq 1 - x + \frac{1}{2}x^2 \), and this gives
\[
E_\lambda e^{-\gamma e Z_1^\varepsilon} \leq 1 - E_\lambda (\gamma e Z_1^\varepsilon) + \frac{1}{2}E_\lambda (\gamma e Z_1^\varepsilon)^2
= 1 - \gamma e + \frac{1}{2}(\gamma e)^2 E_\lambda (Z_1^\varepsilon)^2.
\]
Moreover, since we can apply Lemma 7.6 to find \( E_\lambda (Z_1^\varepsilon)^2 \to 2 \) as \( \varepsilon \to 0 \), there exists \( \varepsilon_0 \in (0, 1) \) such that for any \( \varepsilon \in (0, \varepsilon_0) \), \( E_\lambda (Z_1^\varepsilon)^2 \leq 9/4 \). Thus, for any \( \varepsilon \in (0, \varepsilon_0) \)
\[
E_\lambda e^{-\gamma e Z_1^\varepsilon} \leq 1 - \gamma e + \frac{9}{8}(\gamma e)^2
\]
and
\[
(E_\lambda e^{-\gamma e Z_1^\varepsilon}) e^{-\frac{1}{2}(\gamma e)^2} \leq \exp \left\{ \frac{\gamma e}{1 + 2\gamma e} + \log \left( 1 - \gamma e + \frac{9}{8}(\gamma e)^2 \right) \right\}.
\]
Using a Taylor series expansion we find that for all \( |x| < 1 \)
\[
\frac{1}{1 + x} = 1 - x + x^2 + O(x^3) \quad \text{and} \quad \log(1 + x) = x - \frac{x^2}{2} + O(x^3),
\]
which gives
\[
\frac{\gamma e}{1 + 2\gamma e} + \log \left( 1 - \gamma e + \frac{9}{8}(\gamma e)^2 \right)
= \frac{1}{2} \left( 1 - \frac{1}{1 + 2\gamma e} \right) + \log \left( 1 - \gamma e + \frac{9}{8}(\gamma e)^2 \right)
= \gamma e - 2(\gamma e)^2 + \left( -\gamma e + \frac{9}{8}(\gamma e)^2 \right) - \frac{1}{2} \left( -\gamma e + \frac{9}{8}(\gamma e)^2 \right)^2 + O((\gamma e)^3)
= -2(\gamma e)^2 + \frac{9}{8}(\gamma e)^2 - \frac{1}{2}(\gamma e)^2 + O((\gamma e)^3)
= -\frac{11}{8}(\gamma e)^2 + O((\gamma e)^3) \leq -\frac{1}{2}(\gamma e)^2,
\]
for all \( \varepsilon \in (0, \varepsilon_0) \). We are done for part 1.

For part 2, we use the estimate from part 1 and find
\[
\sum_{n=(1+2\gamma e)\Gamma e + 1}^{\infty} P_{\lambda e}(\tau_n^\varepsilon \leq T_n^\varepsilon) \leq \sum_{n=(1+2\gamma e)\Gamma e + 1}^{\infty} e^{-n\frac{1}{2}(\gamma e)^2} \leq \frac{e^{-(1+2\gamma e)\Gamma e \frac{1}{2}(\gamma e)^2}}{1 - e^{-\frac{1}{2}(\gamma e)^2}}.
\]
Since $e^{-x} \leq 1 - x + x^2/2$ for any $x \in \mathbb{R}$, we have $1 - e^{-x} \geq x - x^2/2 \geq x - x/2 = x/2$ for all $x \in (0, 1)$, and thus $1/(1 - e^{-x}) \leq 2/x$ for all $x \in (0, 1)$.

As a result

$$
\sum_{n=(1+2\gamma^e)\Gamma^e+1}^{\infty} P_{\lambda^e} (\tau_n^e \leq T^e) \leq \frac{e^{-(1+2\gamma^e)\Gamma^e \frac{1}{2}(\gamma^e)^2}}{1 - e^{-\frac{1}{2}(\gamma^e)^2}} \leq \frac{4}{(\gamma^e)^2} e^{-(1+2\gamma^e)\Gamma^e \frac{1}{2}(\gamma^e)^2} \leq 4 (\Gamma^e)^{2\ell} e^{-\frac{1}{2}(\Gamma^e)^{1-2\ell}}.
$$

This completes the proof of part 2.

Finally, for part 3, we use the fact that for $x \in (0, 1)$, and for any $k \in \mathbb{N}$,

$$
\sum_{n=k}^{\infty} nx^n = k \frac{x^k}{1-x} + \frac{x^{k+1}}{(1-x)^2} \leq \left( \frac{k}{1-x} + \frac{1}{(1-x)^2} \right) x^k.
$$

Using the estimate from part 1 once again, we have

$$
\sum_{n=(1+2\gamma^e)\Gamma^e+1}^{\infty} nP_{\lambda^e} (\tau_n^e \leq T^e)
\leq \sum_{n=(1+2\gamma^e)\Gamma^e}^{\infty} n e^{-n \frac{2}{(\gamma^e)^2}}
\leq \left( \frac{1 + 2\gamma^e}{1 - e^{-\frac{1}{2}(\gamma^e)^2}} + \frac{1}{(1-e^{-\frac{1}{2}(\gamma^e)^2})^2} \right) e^{-(1+2\gamma^e)\Gamma^e \frac{1}{2}(\gamma^e)^2}
\leq \left( \frac{2}{(\gamma^e)^2} 2\Gamma^e + \left( \frac{2}{(\gamma^e)^2} \right)^2 \right) e^{-(1+2\gamma^e)\Gamma^e \frac{1}{2}(\gamma^e)^2}
\leq \left( 8 (\Gamma^e)^{1+2\ell} + 16 (\Gamma^e)^{4\ell} \right) e^{-\frac{1}{2}(\Gamma^e)^{1-2\ell}}.
$$

We are done. ■

**Remark 7.10** If $0 < \ell < 1/2$, then $\mathfrak{P}_1$ and $\mathfrak{R}_1$ converge to 0 doubly exponentially fast as $\varepsilon \to 0$. To be more precise, for any $k > 0$

$$
\liminf_{\varepsilon \to 0} -\varepsilon \log \left[ (\Gamma^e)^k e^{-\frac{1}{2}(\Gamma^e)^{1-2\ell}} \right] = \infty.
$$
This is true since
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left((\Gamma^{\varepsilon})^k e^{-\frac{1}{2}(\Gamma^{\varepsilon})^{1-2\ell}}\right) \\
\geq k \liminf_{\varepsilon \to 0} -\varepsilon \log \Gamma^{\varepsilon} + \liminf_{\varepsilon \to 0} -\varepsilon \log e^{-\frac{1}{2}(\Gamma^{\varepsilon})^{1-2\ell}} \\
= k \liminf_{\varepsilon \to 0} -\varepsilon \log \Gamma^{\varepsilon} + \frac{1}{2} \liminf_{\varepsilon \to 0} \varepsilon (\Gamma^{\varepsilon})^{1-2\ell},
\]
and using Lemma 7.5 \(\Gamma^{\varepsilon} \geq e^{(c-h_\delta-\delta_0)/\varepsilon}\) for all \(\varepsilon\) sufficiently small, where \(\delta_0 = (c - h_\delta)/2 > 0\). Therefore
\[
\liminf_{\varepsilon \to 0} \varepsilon (\Gamma^{\varepsilon})^{1-2\ell} \geq \liminf_{\varepsilon \to 0} \varepsilon e^{(1-2\ell)(c-h_\delta-\delta_0)/\varepsilon} = \infty.
\]

### 7.2 Berry-Essen bound

In this subsection we will provide upper bounds for
\[
\mathcal{R}_2 := \sum_{n=(1-2\gamma^{\varepsilon})\Gamma^{\varepsilon}} (1+2\gamma^{\varepsilon})\Gamma^{\varepsilon} \sum_{n=(1-2\gamma^{\varepsilon})\Gamma^{\varepsilon}} P^{\varepsilon}_{\lambda^{\varepsilon}} (\tau^{\varepsilon}_n \leq T^{\varepsilon})
\]
and
\[
\mathcal{R}_2 := \sum_{n=(1-2\gamma^{\varepsilon})\Gamma^{\varepsilon}} (1+2\gamma^{\varepsilon})\Gamma^{\varepsilon} \sum_{n=(1-2\gamma^{\varepsilon})\Gamma^{\varepsilon}} (2n+1) P^{\varepsilon}_{\lambda^{\varepsilon}} (\tau^{\varepsilon}_n \leq T^{\varepsilon})
\]
via the Berry-Essen bound.

We first recall that \(\Gamma^{\varepsilon} = T^{\varepsilon}/E^{\varepsilon}_{\lambda^{\varepsilon}} \tau^{\varepsilon}_1\). The following is Theorem 1 in [7, Chapter XVI.5].

**Theorem 7.11 (Berry-Essen)** Let \(\{X_n\}_{n \in \mathbb{N}}\) be independent real-valued random variables with a common distribution such that
\[E(X_1) = 0, \quad \sigma^2 := E(X_1)^2 > 0, \quad \rho := E(|X_1|^3) < \infty.\]
Then for all \(x \in \mathbb{R}\) and \(n \in \mathbb{N}\),
\[
\left| P \left( \frac{X_1 + \cdots + X_n}{\sigma \sqrt{n}} \leq x \right) - \Phi(x) \right| \leq \frac{3\rho}{\sigma^3 \sqrt{n}},
\]
where \(\Phi(\cdot)\) is the distribution function of \(N(0,1)\).
Corollary 7.12 For any \( \varepsilon > 0 \), let \( \{X_n^{\varepsilon}\}_{n \in \mathbb{N}} \) be independent real-valued random variables with a common distribution such that

\[
E(X_1^{\varepsilon}) = 0, \quad (\sigma^{\varepsilon})^2 = E(X_1^{\varepsilon})^2 > 0, \quad \rho^{\varepsilon} = E(|X_1^{\varepsilon}|^3) < \infty.
\]

Assume that there exists \( \varepsilon_0 \in (0, 1) \) such that

\[
\hat{\rho} = \sup_{\varepsilon \in (0, \varepsilon_0)} \rho^{\varepsilon} < \infty \quad \text{and} \quad \hat{\sigma}^2 = \inf_{\varepsilon \in (0, \varepsilon_0)} (\sigma^{\varepsilon})^2 > 0,
\]

then for all \( x \in \mathbb{R} \), \( n \in \mathbb{N} \) and \( \varepsilon \in (0, \varepsilon_0) \),

\[
\left| P \left( \frac{X_1^{\varepsilon} + \cdots + X_n^{\varepsilon}}{\sigma^{\varepsilon} \sqrt{n}} \leq x \right) - \Phi(x) \right| \leq \frac{3\hat{\rho}}{(\sigma^{\varepsilon})^2 \sqrt{n}} \leq \frac{3\hat{\rho}}{\hat{\sigma}^2 \sqrt{n}},
\]

where \( \Phi(\cdot) \) is the distribution function of \( N(0, 1) \).

Lemma 7.13 Given any \( \delta > 0 \) and any \( \ell > 0 \), there exists \( \varepsilon_0 \in (0, 1) \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) and \( k \in \mathbb{N}_0 \), \( 0 \leq k \leq 2\gamma^{\varepsilon} T^{\varepsilon} \)

\[
P_{\varepsilon}(\tau_{k+1}^{\varepsilon} \leq T^{\varepsilon}) \leq 1 - \Phi \left( \frac{k}{\sigma^{\varepsilon} \sqrt{T^{\varepsilon} + k}} \right) + \frac{3\hat{\rho}}{\hat{\sigma}^2 \sqrt{T^{\varepsilon} + k}}
\]

and

\[
P_{\varepsilon}(\tau_k^{\varepsilon} \leq T^{\varepsilon}) \leq \Phi \left( \frac{k}{\sigma^{\varepsilon} \sqrt{T^{\varepsilon} - k}} \right) + \frac{3\hat{\rho}}{\hat{\sigma}^2 \sqrt{T^{\varepsilon} - k}}
\]

where \( (\sigma^{\varepsilon})^2 = E_{\varepsilon}(X_1^{\varepsilon})^2 \), \( \hat{\rho} = \sup_{\varepsilon \in (0, \varepsilon_0)} E_{\varepsilon} \left( |X_1^{\varepsilon}|^3 \right) < \infty \) and \( \hat{\sigma}^2 = \inf_{\varepsilon \in (0, \varepsilon_0)} (\sigma^{\varepsilon})^2 > 0 \) with \( X_1^{\varepsilon} \sim \tau_1^{\varepsilon} / E_{\varepsilon} \tau_1^{\varepsilon} - 1 \).

Proof. For any \( n \in \mathbb{N} \), we define \( X_n^{\varepsilon} \sim Z_n^{\varepsilon} - E_{\varepsilon} Z_1^{\varepsilon} \) with \( Z_n^{\varepsilon} = (\tau_n^{\varepsilon} - \tau_{n-1}^{\varepsilon}) / E_{\varepsilon} \tau_1^{\varepsilon} \). Obviously, \( E_{\varepsilon} Z_n^{\varepsilon} = 1 \) and \( E_{\varepsilon} X_n^{\varepsilon} = 0 \) and if we apply Lemma 7.6 then we find that there exists some \( \varepsilon_0 \in (0, 1) \) such that

\[
\sup_{\varepsilon \in (0, \varepsilon_0)} E_{\varepsilon}(Z_1^{\varepsilon})^3 < \infty \quad \text{and} \quad \inf_{\varepsilon \in (0, \varepsilon_0)} \operatorname{Var}_{\varepsilon}(Z_1^{\varepsilon}) > 0.
\]

Since \( Z_1^{\varepsilon} \geq 0 \), Jensen’s inequality implies \( E_{\varepsilon}(Z_1^{\varepsilon})^3 \leq E_{\varepsilon}(Z_1^{\varepsilon})^3 \), and therefore

\[
\hat{\rho} = \sup_{\varepsilon \in (0, \varepsilon_0)} E_{\varepsilon} \left( |Z_1^{\varepsilon} - E_{\varepsilon} Z_1^{\varepsilon}|^3 \right)
\]

\[
\leq 4 \sup_{\varepsilon \in (0, \varepsilon_0)} \left( E_{\varepsilon}(Z_1^{\varepsilon})^3 + (E_{\varepsilon} Z_1^{\varepsilon})^3 \right)
\]

\[
\leq 8 \sup_{\varepsilon \in (0, \varepsilon_0)} E_{\varepsilon}(Z_1^{\varepsilon})^3 < \infty,
\]

51
and
\[ \hat{\sigma}^2 = \inf_{\varepsilon \in (0, \varepsilon_0)} E_{\lambda^\varepsilon} \left( X_1^\varepsilon \right)^2 = \inf_{\varepsilon \in (0, \varepsilon_0)} \text{Var}_{\lambda^\varepsilon} \left( Z_1^\varepsilon \right) > 0. \]

Therefore we can use Corollary 7.12 with the iid sequence \( \{ X_n^\varepsilon \}_{n \in \mathbb{N}} \) to find that for any \( k \in \mathbb{N}_0 \) and \( 0 \leq k \leq 2\gamma \Gamma^\varepsilon \)

\[ P_{\lambda^\varepsilon} \left( \tau_{\Gamma^\varepsilon + k}^\varepsilon \leq T^\varepsilon \right) = P_{\lambda^\varepsilon} \left( Z_1^\varepsilon + \cdots + Z_{\Gamma^\varepsilon + k}^\varepsilon \leq \Gamma^\varepsilon \right) = P_{\lambda^\varepsilon} \left( \frac{Z_1^\varepsilon + \cdots + Z_{\Gamma^\varepsilon + k}^\varepsilon - (\Gamma^\varepsilon + k) E_{\lambda^\varepsilon} Z_1^\varepsilon}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon + k}} \right) \leq \frac{\Gamma^\varepsilon - (\Gamma^\varepsilon + k) E_{\lambda^\varepsilon} Z_1^\varepsilon}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon + k}} \]

\[ = P_{\lambda^\varepsilon} \left( \frac{X_1^\varepsilon + \cdots + X_{\Gamma^\varepsilon + k}^\varepsilon - k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon + k}} \right) \leq \Phi \left( \frac{-k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon + k}} \right) + \frac{3\hat{\rho}}{\sigma^3 \sqrt{\Gamma^\varepsilon + k}}, \]

and

\[ P_{\lambda^\varepsilon} \left( \tau_{\Gamma^\varepsilon - k}^\varepsilon \leq T^\varepsilon \right) = P_{\lambda^\varepsilon} \left( Z_1^\varepsilon + \cdots + Z_{\Gamma^\varepsilon - k}^\varepsilon \leq \Gamma^\varepsilon \right) = P_{\lambda^\varepsilon} \left( \frac{Z_1^\varepsilon + \cdots + Z_{\Gamma^\varepsilon - k}^\varepsilon - (\Gamma^\varepsilon - k) E_{\lambda^\varepsilon} Z_1^\varepsilon}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon - k}} \right) \leq \frac{\Gamma^\varepsilon - (\Gamma^\varepsilon - k) E_{\lambda^\varepsilon} Z_1^\varepsilon}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon - k}} \]

\[ = P_{\lambda^\varepsilon} \left( \frac{X_1^\varepsilon + \cdots + X_{\Gamma^\varepsilon - k}^\varepsilon - k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon - k}} \right) \leq \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon - k}} \right) + \frac{3\hat{\rho}}{\sigma^3 \sqrt{\Gamma^\varepsilon - k}}. \]

\[ \blacksquare \]

**Lemma 7.14** Given any \( \delta > 0 \) and any \( \ell \in (0, 1/2) \), there exists \( \varepsilon_0 \in (0, 1) \) such that for any \( \varepsilon \in (0, \varepsilon_0) \)

\[ Q_2 = \sum_{n=(1-2\gamma^\varepsilon)\Gamma^\varepsilon}^{(1+2\gamma^\varepsilon)\Gamma^\varepsilon} P_{\lambda^\varepsilon} \left( \tau_n^\varepsilon \leq T^\varepsilon \right) \leq C \left( \Gamma^\varepsilon \right)^{\frac{1}{2}-\ell} + 2 \left( \Gamma^\varepsilon \right)^{1-\ell} \]
Proof. We rewrite $\mathcal{P}_2$ as
\[
\mathcal{P}_2 = \sum_{n=(1-2^\gamma)\Gamma^\varepsilon}^{(1+2^\gamma)\Gamma^\varepsilon} P_{\lambda^\varepsilon} (\tau_n^\varepsilon \leq T^\varepsilon)
\]
\[
= \sum_{k=1}^{2^\gamma \Gamma^\varepsilon} P_{\lambda^\varepsilon} (\tau_{k-1}^\varepsilon \leq T^\varepsilon) + P_{\lambda^\varepsilon} (\tau_{\Gamma^\varepsilon}^\varepsilon \leq T^\varepsilon) + \sum_{k=1}^{2^\gamma \Gamma^\varepsilon} P_{\lambda^\varepsilon} (\tau_{k+1}^\varepsilon \leq T^\varepsilon).
\]

Then we use the upper bounds from Lemma 7.13 to get
\[
\mathcal{P}_2 \leq \sum_{k=1}^{2^\gamma \Gamma^\varepsilon} \left[ \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon} - k} \right) + \frac{3\tilde{\rho}}{\sigma^3 \sqrt{\Gamma^\varepsilon} - k} \right] + 1
\]
\[
+ \sum_{k=1}^{2^\gamma \Gamma^\varepsilon} \left[ 1 - \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon} + k} \right) + \frac{3\tilde{\rho}}{\sigma^3 \sqrt{\Gamma^\varepsilon} + k} \right]
\]
\[
\leq \gamma^\varepsilon \Gamma^\varepsilon \frac{12\tilde{\rho}}{\sigma^3 \sqrt{(1 - \gamma^\varepsilon) \Gamma^\varepsilon}} + 1 + 2^\gamma \Gamma^\varepsilon
\]
\[
+ \sum_{k=1}^{2^\gamma \Gamma^\varepsilon} \left[ \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon} - k} \right) - \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon} + k} \right) \right]
\]
\[
\leq \frac{24\tilde{\rho}}{\sigma^3} \gamma^\varepsilon \sqrt{\Gamma^\varepsilon} + 1 + 2^\gamma \Gamma^\varepsilon + \sum_{k=1}^{2^\gamma \Gamma^\varepsilon} \left[ \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon} - k} \right) - \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon} + k} \right) \right].
\]

The sum of the first three terms is easily bounded according to
\[
\frac{24\tilde{\rho}}{\sigma^3} \gamma^\varepsilon \sqrt{\Gamma^\varepsilon} + 1 + 2^\gamma \Gamma^\varepsilon \leq C (\Gamma^\varepsilon)^{\frac{1}{2} - \ell} + 2 (\Gamma^\varepsilon)^{1 - \ell}.
\]

We will show that the last term
\[
\sum_{k=1}^{2^\gamma \Gamma^\varepsilon} \left[ \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon} - k} \right) - \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon} + k} \right) \right]
\]
is bounded above by a constant to complete the proof.

To prove this, we observe that for any $k \leq 2^\gamma \Gamma^\varepsilon$, we may assume $k \leq \Gamma^\varepsilon/2$ by taking $\varepsilon$ sufficiently small. Then we apply the Mean Value Theorem and find
\[
\left| \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon} - k} \right) - \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon} + k} \right) \right|
\]
\[
\leq \frac{\sup_{x \in \left[ \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon} - k}, \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon} + k} \right]} \phi (x) \cdot \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon} - k} - \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon} + k} \right),
\]

53
where \( \phi(x) = e^{-\frac{x^2}{2}}/\sqrt{2\pi} \). Since \( 0 \leq k \leq \Gamma / 2 \), we have

\[
\left[ \frac{k}{\sigma \sqrt{\Gamma \varepsilon}}, \frac{k}{\sigma \sqrt{\Gamma \varepsilon} + k} \right] \subset \left[ \frac{k}{\sigma \sqrt{\Gamma \varepsilon}}, \frac{\sqrt{2}k}{\sigma \sqrt{\Gamma \varepsilon}} \right].
\]

Additionally, because \( \phi(x) = e^{-\frac{x^2}{2}}/\sqrt{2\pi} \) is a monotone decreasing function on \([0, \infty)\), we find that for any \( x \in \left[ k/(\sigma \sqrt{\Gamma \varepsilon} - k), k/(\sigma \sqrt{\Gamma \varepsilon} + k) \right] \),

\[
\phi(x) \leq \phi \left( \frac{k}{\sigma \sqrt{\Gamma \varepsilon}} \right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2(\sigma \varepsilon)^2 \Gamma \varepsilon}}.
\]

On the other hand,

\[
\frac{k}{\sqrt{\Gamma \varepsilon} - k} - \frac{k}{\sqrt{\Gamma \varepsilon} + k} = \frac{k}{\sqrt{\Gamma \varepsilon}} \left( \frac{1}{\sqrt{1 - \frac{k}{\Gamma \varepsilon}}} - \frac{1}{\sqrt{1 + \frac{k}{\Gamma \varepsilon}}} \right) = \frac{k}{\sqrt{\Gamma \varepsilon}} \frac{\sqrt{1 + \frac{k}{\Gamma \varepsilon}} - \sqrt{1 - \frac{k}{\Gamma \varepsilon}}}{\sqrt{1 - \left( \frac{k}{\Gamma \varepsilon} \right)^2}},
\]

then we use the fact that \( \sqrt{1+x} - \sqrt{1-x} \leq 2x \) for all \( x \in [0, 1] \) and that \( k \leq \Gamma / 2 \) implies

\[
\frac{1}{\sqrt{1 - \left( \frac{k}{\Gamma \varepsilon} \right)^2}} \leq \frac{1}{\sqrt{1 - \frac{1}{4}}} < 2
\]

to get

\[
\frac{k}{\sqrt{\Gamma \varepsilon} - k} - \frac{k}{\sqrt{\Gamma \varepsilon} + k} \leq \frac{2k}{\sqrt{\Gamma \varepsilon}} \left( \frac{k}{\Gamma \varepsilon} \right) = \frac{4k^2}{\Gamma \varepsilon \sqrt{\Gamma \varepsilon}}.
\]

54
Therefore we find
\[
\sum_{k=1}^{2\gamma \Gamma \varepsilon} \left[ \Phi \left( \frac{k}{\sigma \sqrt{\Gamma \varepsilon} - k} \right) - \Phi \left( \frac{k}{\sigma \sqrt{\Gamma \varepsilon} + k} \right) \right] 
\leq \sum_{k=1}^{2\gamma \Gamma \varepsilon} \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2(\sigma \varepsilon)^2}} \frac{4k^2}{\sigma \varepsilon \sqrt{\Gamma \varepsilon}} 
= \frac{4}{\sigma \varepsilon \Gamma \varepsilon} \sum_{k=1}^{2\gamma \Gamma \varepsilon} \frac{k^2}{\sqrt{2\pi \varepsilon}} e^{-\frac{k^2}{2(\sigma \varepsilon)^2}} 
\leq \frac{4}{\sigma \varepsilon \Gamma \varepsilon} \sum_{k=1}^{2\gamma \Gamma \varepsilon} \int_{k-1}^{k} \frac{(1+x)^2}{\sqrt{2\pi \varepsilon}} e^{-\frac{x^2}{2(\sigma \varepsilon)^2}} \, dx 
\leq \frac{4}{\Gamma \varepsilon} \int_{0}^{\infty} \frac{(1+x)^2}{\sqrt{2\pi (\sigma \varepsilon)^2 \Gamma \varepsilon}} e^{-\frac{x^2}{2(\sigma \varepsilon)^2}} \, dx 
= \frac{4}{\Gamma \varepsilon} E \left( 1 + X^+ \right)^2,
\]
where \( X \sim N(0, (\sigma \varepsilon)^2 \Gamma \varepsilon) \). Finally, since \( E (1 + X^+)^2 \leq 2 + 2E(X^2) = 2 + 2(\sigma \varepsilon)^2 \Gamma \varepsilon \), this implies that
\[
\sum_{k=1}^{2\gamma \Gamma \varepsilon} \left[ \Phi \left( \frac{k}{\sigma \sqrt{\Gamma \varepsilon} - k} \right) - \Phi \left( \frac{k}{\sigma \sqrt{\Gamma \varepsilon} + k} \right) \right] \leq \frac{4}{\Gamma \varepsilon} \left( 2 + 2(\sigma \varepsilon)^2 \Gamma \varepsilon \right) \quad (7.4)
\]
\[
\leq 8 + 8\hat{\rho}^{2/3},
\]
where the last inequality is from
\[
\sup_{\varepsilon \in (0, \varepsilon_0)} \sup_{\varepsilon \in (0, \varepsilon_0)} \left( E_{\lambda \varepsilon} (X_1^2) \right)^{1/2} \leq \sup_{\varepsilon \in (0, \varepsilon_0)} \left( E_{\lambda \varepsilon} |X_1^3| \right)^{1/3} = \hat{\rho}^{1/3}.
\]
Since according to Lemma 7.13 \( \hat{\rho}^{1/3} \) is finite, we are done. ■

**Lemma 7.15** Given any \( \delta > 0 \) and any \( \ell \in (0, 1/2) \), there exists \( \varepsilon_0 \in (0, 1) \) and a constant \( C < \infty \) such that for any \( \varepsilon \in (0, \varepsilon_0) \)
\[
\mathfrak{R}_2 = \sum_{n=(1-2\gamma \varepsilon) \Gamma \varepsilon}^{(1+2\gamma \varepsilon) \Gamma \varepsilon} (2n + 1) P_{\lambda \varepsilon} \left( \tau_n \leq T \varepsilon \right) \leq 4 (\Gamma \varepsilon)^2 - \ell + C (\Gamma \varepsilon)^2 (1-\ell).
\]
The proof of this lemma is very similar to the proof of Lemma \[ \text{Lemma 7.14} \]

We rewrite \( R_2 \) as

\[
R_2 = \sum_{k=1}^{2\gamma \varepsilon \Gamma^\varepsilon} (2\Gamma^\varepsilon - 2k + 1) P_{\lambda^\varepsilon} \left( Z_1^\varepsilon + \cdots + Z_{\Gamma^\varepsilon - k}^\varepsilon \leq \Gamma^\varepsilon \right)
\]

\[
+ (2\Gamma^\varepsilon + 1) P_{\lambda^\varepsilon} \left( Z_1^\varepsilon + \cdots + Z_{\Gamma^\varepsilon}^\varepsilon \leq \Gamma^\varepsilon \right)
\]

\[
+ \sum_{k=1}^{2\gamma \varepsilon \Gamma^\varepsilon} (2\Gamma^\varepsilon + 2k + 1) P_{\lambda^\varepsilon} \left( Z_1^\varepsilon + \cdots + Z_{\Gamma^\varepsilon + k}^\varepsilon \leq \Gamma^\varepsilon \right).
\]

Then we use the upper bounds from Lemma \[ \text{Lemma 7.13} \] to get

\[
R_2 \leq \sum_{k=1}^{2\gamma \varepsilon \Gamma^\varepsilon} (2\Gamma^\varepsilon - 2k + 1) \left[ \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon - k}} \right) + \frac{3\hat{\rho}^\varepsilon}{\sigma^3 \sqrt{\Gamma^\varepsilon - k}} \right] + (2\Gamma^\varepsilon + 1)
\]

\[
+ \sum_{k=1}^{2\gamma \varepsilon \Gamma^\varepsilon} (2\Gamma^\varepsilon + 2k + 1) \left[ 1 - \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon + k}} \right) + \frac{3\hat{\rho}^\varepsilon}{\sigma^3 \sqrt{\Gamma^\varepsilon + k}} \right].
\]

The next thing is to pair all the terms carefully and bound these pairs separately. We start with

\[
\sum_{k=1}^{2\gamma \varepsilon \Gamma^\varepsilon} (2\Gamma^\varepsilon - 2k + 1) \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon - k}} \right) - \sum_{k=1}^{2\gamma \varepsilon \Gamma^\varepsilon} (2\Gamma^\varepsilon + 2k + 1) \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon + k}} \right)
\]

\[
= (2\Gamma^\varepsilon + 1) \sum_{k=1}^{2\gamma \varepsilon \Gamma^\varepsilon} \left[ \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon - k}} \right) - \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon + k}} \right) \right]
\]

\[
- 2 \sum_{k=1}^{2\gamma \varepsilon \Gamma^\varepsilon} k \left[ \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon - k}} \right) + \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon + k}} \right) \right]
\]

\[
\leq (2\Gamma^\varepsilon + 1) \sum_{k=1}^{2\gamma \varepsilon \Gamma^\varepsilon} \left[ \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon - k}} \right) - \Phi \left( \frac{k}{\sigma^\varepsilon \sqrt{\Gamma^\varepsilon + k}} \right) \right]
\]

\[
\leq C \Gamma^\varepsilon.
\]

We use \[ \text{Lemma 7.14} \] for the last inequality.
The second pair is

\[
\sum_{k=1}^{2\gamma^e\Gamma^e} (2\Gamma^e - 2k + 1) \frac{3\hat{\rho}}{\sigma^3 \sqrt{\Gamma^e - k}} + \sum_{k=1}^{2\gamma^e\Gamma^e} (2\Gamma^e + 2k + 1) \frac{3\hat{\rho}}{\sigma^3 \sqrt{\Gamma^e + k}} \\
= \sum_{k=1}^{2\gamma^e\Gamma^e} (\Gamma^e - k) \frac{6\hat{\rho}}{\sigma^3 \sqrt{\Gamma^e - k}} + \sum_{k=1}^{2\gamma^e\Gamma^e} (\Gamma^e + k) \frac{6\hat{\rho}}{\sigma^3 \sqrt{\Gamma^e + k}} \\
+ \sum_{k=1}^{2\gamma^e\Gamma^e} 3\hat{\rho} \frac{3\hat{\rho}}{\sigma^3 \sqrt{\Gamma^e - k}} + \sum_{k=1}^{2\gamma^e\Gamma^e} 3\hat{\rho} \frac{3\hat{\rho}}{\sigma^3 \sqrt{\Gamma^e + k}} \\
= \frac{6\hat{\rho}}{\sigma^3} \sum_{k=1}^{2\gamma^e\Gamma^e} \left( \sqrt{\Gamma^e - k} + \sqrt{\Gamma^e + k} \right) + \frac{3\hat{\rho}}{\sigma^3} \sum_{k=1}^{2\gamma^e\Gamma^e} \left( \frac{1}{\sqrt{\Gamma^e - k}} + \frac{1}{\sqrt{\Gamma^e + k}} \right) .
\]

Using \( k \leq \Gamma^e / 2 \)

\[
\frac{6\hat{\rho}}{\sigma^3} \sum_{k=1}^{2\gamma^e\Gamma^e} \left( \sqrt{\Gamma^e - k} + \sqrt{\Gamma^e + k} \right) \leq \frac{6\hat{\rho}}{\sigma^3} \sum_{k=1}^{2\gamma^e\Gamma^e} 2\sqrt{2\Gamma^e} \leq C\gamma^e\Gamma^e \sqrt{\Gamma^e} ,
\]

Moreover,

\[
\frac{3\hat{\rho}}{\sigma^3} \sum_{k=1}^{2\gamma^e\Gamma^e} \left( \frac{1}{\sqrt{\Gamma^e - k}} + \frac{1}{\sqrt{\Gamma^e + k}} \right) \leq \frac{3\hat{\rho}}{\sigma^3} \sum_{k=1}^{2\gamma^e\Gamma^e} 2 \leq C\gamma^e\Gamma^e
\]

which implies

\[
\sum_{k=1}^{2\gamma^e\Gamma^e} (2\Gamma^e - 2k + 1) \frac{3\hat{\rho}}{\sigma^3 \sqrt{\Gamma^e - k}} + \sum_{k=1}^{2\gamma^e\Gamma^e} (2\Gamma^e + 2k + 1) \frac{3\hat{\rho}}{\sigma^3 \sqrt{\Gamma^e + k}} \\
\leq C\gamma^e\Gamma^e \sqrt{\Gamma^e} + C\gamma^e\Gamma^e \\
\leq C (\Gamma^e)^{\frac{3}{2}} .
\]
The third term is
\[
\sum_{k=1}^{2\gamma^\varepsilon \Gamma^\varepsilon} (2\Gamma^\varepsilon + 2k + 1) + (2\Gamma^\varepsilon + 1) \\
= (2\Gamma^\varepsilon + 1) 2\gamma^\varepsilon \Gamma^\varepsilon + 2 \sum_{k=1}^{2\gamma^\varepsilon \Gamma^\varepsilon} k + (2\Gamma^\varepsilon + 1) \\
= (2\Gamma^\varepsilon + 1) 2\gamma^\varepsilon \Gamma^\varepsilon + 2 \left( \frac{2\gamma^\varepsilon \Gamma^\varepsilon + 1}{2} \right) 2\gamma^\varepsilon \Gamma^\varepsilon + (2\Gamma^\varepsilon + 1) \\
= 4\gamma^\varepsilon (\Gamma^\varepsilon)^2 + 2\gamma^\varepsilon \Gamma^\varepsilon + 4 (\gamma^\varepsilon \Gamma^\varepsilon)^2 + 2\gamma^\varepsilon \Gamma^\varepsilon + (2\Gamma^\varepsilon + 1) \\
\leq 4\gamma^\varepsilon (\Gamma^\varepsilon)^2 + C (\gamma^\varepsilon \Gamma^\varepsilon)^2 \\
= 4 (\Gamma^\varepsilon)^{2-\ell} + C (\Gamma^\varepsilon)^{2(1-\ell)},
\]
where the inequality holds since for \(\ell \in (0, 1/2)\), \(2 - 2\ell \geq 1\) and this implies that
\[(2\Gamma^\varepsilon + 1) \leq C (\gamma^\varepsilon \Gamma^\varepsilon)^2.\]

Lastly, combining all the pairs and the corresponding upper bounds, we find that for any \(\ell \in (0, 1/2)\),
\[
\Omega_2 \leq \left[ 4 (\Gamma^\varepsilon)^{2-\ell} + C (\Gamma^\varepsilon)^{2(1-\ell)} \right] + C\Gamma^\varepsilon + C (\Gamma^\varepsilon)^{3/2 - \ell} \\
\leq 4 (\Gamma^\varepsilon)^{2-\ell} + C (\Gamma^\varepsilon)^{2(1-\ell)},
\]
where \(C\) is a constant which depends on \(\ell\) only (and in particular is independent of \(\varepsilon\)). □

### 7.3 Asymptotics of moments of \(N^\varepsilon(T^\varepsilon)\)

In this subsection, we prove Lemma 7.2 and Lemma 7.3.

**Proof of Lemma 7.2** First, recall that
\[
E_{\lambda^\varepsilon} (N^\varepsilon(T^\varepsilon)) = \sum_{n=0}^{\infty} P_{\lambda^\varepsilon} (\tau_n^\varepsilon \leq T^\varepsilon) = \Psi_1 + \Psi_2 + \Psi_3,
\]
where the \(\Psi_i\) are defined in (7.2). We can simply bound \(\Psi_3\) from above by \((1 - 2\gamma^\varepsilon) \Gamma^\varepsilon\). Applying Lemma 7.1 and Lemma 7.14 for the other terms, we
have for any \( \ell \in (0, 1/2) \) that
\[
E_{\lambda^\varepsilon} (N^{\varepsilon} (T^{\varepsilon})) \\
\leq C (\Gamma^{\varepsilon})^{2\ell} e^{-\frac{1}{2} (\Gamma^{\varepsilon})^{1-2\ell}} + (C (\Gamma^{\varepsilon})^{\frac{1}{2} - \ell} e^{\frac{1}{2} (\Gamma^{\varepsilon})^{1-2\ell}}) + (1 - 2\gamma) \Gamma^{\varepsilon} \\
= \Gamma^{\varepsilon} + C (\Gamma^{\varepsilon})^{\frac{1}{2} - \ell} + C (\Gamma^{\varepsilon})^{2\ell} e^{-\frac{1}{2} (\Gamma^{\varepsilon})^{1-2\ell}} \\
= \frac{T^{\varepsilon}}{E_{\lambda^\varepsilon} \tau_{1}^{\varepsilon}} + C (\Gamma^{\varepsilon})^{\frac{1}{2} - \ell} + C (\Gamma^{\varepsilon})^{2\ell} e^{-\frac{1}{2} (\Gamma^{\varepsilon})^{1-2\ell}}.
\]

On the other hand, from the definition of \( N^{\varepsilon} (T^{\varepsilon}) \), \( E_{\lambda^\varepsilon} \tau_{N^{\varepsilon} (T^{\varepsilon})} \geq T^{\varepsilon} \).

Using Wald’s first identity, we find
\[
E_{\lambda^\varepsilon} \tau_{N^{\varepsilon} (T^{\varepsilon})} = E_{\lambda^\varepsilon} \left[ \sum_{n=1}^{N^{\varepsilon} (T^{\varepsilon})} (\tau_n^\varepsilon - \tau_{n-1}^\varepsilon) \right] = E_{\lambda^\varepsilon} (N^{\varepsilon} (T^{\varepsilon})) \cdot E_{\lambda^\varepsilon} \tau_{1}^\varepsilon.
\]

Hence
\[
\frac{E_{\lambda^\varepsilon} (N^{\varepsilon} (T^{\varepsilon}))}{T^{\varepsilon}} \geq \frac{1}{E_{\lambda^\varepsilon} \tau_{1}^{\varepsilon}}
\]
and
\[
0 \leq \frac{E_{\lambda^\varepsilon} (N^{\varepsilon} (T^{\varepsilon}))}{T^{\varepsilon}} - \frac{1}{E_{\lambda^\varepsilon} \tau_{1}^{\varepsilon}} \leq \frac{1}{T^{\varepsilon}} \left[ C (\Gamma^{\varepsilon})^{\frac{1}{2} - \ell} + C (\Gamma^{\varepsilon})^{2\ell} e^{-\frac{1}{2} (\Gamma^{\varepsilon})^{1-2\ell}} \right].
\]

Therefore,
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left| \frac{E_{\lambda^\varepsilon} (N^{\varepsilon} (T^{\varepsilon}))}{T^{\varepsilon}} - \frac{1}{E_{\lambda^\varepsilon} \tau_{1}^{\varepsilon}} \right| \geq \liminf_{\varepsilon \to 0} -\varepsilon \log \left[ \frac{1}{T^{\varepsilon}} \left( C (\Gamma^{\varepsilon})^{\frac{1}{2} - \ell} + (\Gamma^{\varepsilon})^{2\ell} e^{-\frac{1}{2} (\Gamma^{\varepsilon})^{1-2\ell}} \right) \right].
\]

It remains to find an appropriate lower bound for the liminf.

We use (6.2), Lemma 7.5 and Remark 7.10 to find that for any \( \ell \in (0, 1/2) \)
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left[ \frac{1}{T^{\varepsilon}} \left( C (\Gamma^{\varepsilon})^{\frac{1}{2} - \ell} + (\Gamma^{\varepsilon})^{2\ell} e^{-\frac{1}{2} (\Gamma^{\varepsilon})^{1-2\ell}} \right) \right] \geq \min \left\{ \liminf_{\varepsilon \to 0} -\varepsilon \log \left( \frac{1}{T^{\varepsilon}} \right), \liminf_{\varepsilon \to 0} -\varepsilon \log \left( (\Gamma^{\varepsilon})^{\frac{1}{2} - \ell} \right), \liminf_{\varepsilon \to 0} -\varepsilon \log \left( (\Gamma^{\varepsilon})^{2\ell} e^{-\frac{1}{2} (\Gamma^{\varepsilon})^{1-2\ell}} \right) \right\}
= c + \min \left\{ \left( \frac{1}{2} - \ell \right) (h_{\delta} - c), \infty \right\}
= c + \left( \frac{1}{2} - \ell \right) (h_{\delta} - c).
\]
March 2, 2020

Sending \( \ell \) to 1/2,

\[
\liminf_{\epsilon \to 0} -\epsilon \log \left| \frac{E_{\lambda^\epsilon} (N^\epsilon (T^\epsilon))}{T^\epsilon} - \frac{1}{E_{\lambda^\epsilon} \tau_1^\epsilon} \right| \geq c.
\]

\[\Box\]

**Proof of Lemma 7.3.** Recall that

\[
E_{\lambda^\epsilon} (N^\epsilon (T^\epsilon))^2 = \sum_{n=0}^{\infty} (2n + 1) P_{\lambda^\epsilon} (\tau_n^\epsilon \leq t) = R_1 + R_2 + R_3
\]

where the \( R_i \) are defined in (7.3). We can bound \( R_3 \) from above by

\[
(1 - 2\gamma^\epsilon) \Gamma^\epsilon - \sum_{n=0}^{(1-2\gamma^\epsilon) \Gamma^\epsilon - 1} (2n + 1)
\]

\[
= 2 \left( \frac{(1 - 2\gamma^\epsilon) \Gamma^\epsilon - 1)(1 - 2\gamma^\epsilon) \Gamma^\epsilon}{2} + (1 - 2\gamma^\epsilon) \Gamma^\epsilon \right)
\]

\[
= \left( 1 - 4\gamma^\epsilon + 4(\gamma^\epsilon)^2 \right) (\Gamma^\epsilon)^2.
\]

Applying Lemma 7.9 and Lemma 7.15, we have for any \( \ell \in (0, 1/2) \) that

\[
E_{\lambda^\epsilon} (N^\epsilon (T^\epsilon))^2 \leq C (\Gamma^\epsilon)^{1+2\ell} e^{-\frac{1}{2}(\Gamma^\epsilon)^{1-2\ell}} + \left[ 4 (\Gamma^\epsilon)^{2-\ell} + C (\Gamma^\epsilon)^{2(1-\ell)} \right]
\]

\[
+ \left( 1 - 4\gamma^\epsilon + 4(\gamma^\epsilon)^2 \right) (\Gamma^\epsilon)^2
\]

\[
\leq (\Gamma^\epsilon)^2 + C (\Gamma^\epsilon)^{2(1-\ell)} + C (\Gamma^\epsilon)^{1+2\ell} e^{-\frac{1}{2}(\Gamma^\epsilon)^{1-2\ell}}.
\]

As in the proof of Lemma 7.2,

\[
E_{\lambda^\epsilon} (N^\epsilon (T^\epsilon)) \geq \frac{T^\epsilon}{E_{\lambda^\epsilon} \tau_1^\epsilon} = \Gamma^\epsilon.
\]

Thus for any \( \ell \in (0, 1/2) \)

\[
\text{Var}_{\lambda^\epsilon} (N^\epsilon (T^\epsilon))
\]

\[
= E_{\lambda^\epsilon} (N^\epsilon (T^\epsilon))^2 - (E_{\lambda^\epsilon} (N^\epsilon (T^\epsilon)))^2
\]

\[
\leq E_{\lambda^\epsilon} (N^\epsilon (T^\epsilon))^2 - (\Gamma^\epsilon)^2
\]

\[
\leq \left[ (\Gamma^\epsilon)^2 + C (\Gamma^\epsilon)^{2(1-\ell)} + C (\Gamma^\epsilon)^{1+2\ell} e^{-\frac{1}{2}(\Gamma^\epsilon)^{1-2\ell}} \right] - (\Gamma^\epsilon)^2
\]

\[
= C (\Gamma^\epsilon)^{2(1-\ell)} + C (\Gamma^\epsilon)^{1+2\ell} e^{-\frac{1}{2}(\Gamma^\epsilon)^{1-2\ell}}.
\]

60
Again we use (6.2), Lemma 7.5 and Remark 7.10 to find that for any \( \ell \in (0, 1/2) \),

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \frac{\text{Var}_{\lambda \varepsilon} \left( N_{\varepsilon} (T) \right)}{T_{\varepsilon}} \geq \liminf_{\varepsilon \to 0} -\varepsilon \log \frac{1}{T_{\varepsilon}} + \min \left\{ \liminf_{\varepsilon \to 0} -\varepsilon \log (\Gamma_{\varepsilon})^{2(1-\ell)}, \liminf_{\varepsilon \to 0} -\varepsilon \log \left( (\Gamma_{\varepsilon})^{1+2\ell} e^{-\frac{1}{2}(\Gamma_{\varepsilon}^{1-2\ell})} \right) \right\} = c + \min \{ 2 (1 - \ell) (h_{\delta} - c), \infty \} = c + 2 (1 - \ell) (h_{\delta} - c) = 2 (1 - \ell) h_{\delta} + (2\ell - 1) c.
\]

Sending \( \ell \) to 1/2 gives

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \frac{\text{Var}_{\lambda \varepsilon} \left( N_{\varepsilon} (T) \right)}{T_{\varepsilon}} \geq h_{\delta},
\]

and we are done. ■

8 Large Deviation Type Lower Bounds

In this section we collect results from the previous sections to prove the main results of the paper, Theorems 4.2 and 4.4, which give large deviation upper bounds on the bias under the empirical measure and the variance per unit time. We also give the proof of Theorem 4.6 which shows how to simplify some expressions appearing in the large deviation bounds. Before giving the proof of the first result we establish Lemma 8.1 and Lemma 8.2 which are needed in the proof of Theorems 4.2. Recall that for any \( n \in \mathbb{N} \)

\[
S_{n}^{\varepsilon} = \int_{T_{n-1}}^{T_{n}} e^{-\frac{1}{2}f(x)} 1_{A}(X_{t}^{\varepsilon}) dt.
\]

**Lemma 8.1** Given a compact set \( A \subset M \), for any \( \delta > 0 \) sufficiently small

\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left| \frac{E_{\lambda \varepsilon} N_{\varepsilon} (T_{\varepsilon})}{T_{\varepsilon}} - E_{\lambda \varepsilon} S_{1}^{\varepsilon} - \int e^{-\frac{1}{2}f(x)} 1_{A}(x) \mu_{\varepsilon} (dx) \right| \geq \inf_{x \in A} [f(x) + W(x)] - W(O_{1}) + c - h_{\delta}.
\]
Proof. To begin, by Lemma 4.1 with $g(x) = e^{-\frac{1}{\varepsilon} f(x)} 1_A(x)$, we know that for any $\delta$ sufficiently small and $\varepsilon > 0,$

$$E_{\lambda^e} S_1^\varepsilon = E_{\lambda^e} \left( \int_0^{T_1^\varepsilon} e^{-\frac{1}{\varepsilon} f(X_s^\varepsilon)} 1_A(X_s^\varepsilon) ds \right) = E_{\lambda^e} \tau_1^\varepsilon \cdot \int e^{-\frac{1}{\varepsilon} f(x)} 1_A(x) \mu^\varepsilon(dx).$$

This implies that

$$\left| \frac{E_{\lambda^e} (N^\varepsilon (T^\varepsilon))}{T^\varepsilon} E_{\lambda^e} S_1^\varepsilon - \int e^{-\frac{1}{\varepsilon} f(x)} 1_A(x) \mu^\varepsilon(dx) \right| = \int e^{-\frac{1}{\varepsilon} f(x)} 1_A(x) \mu^e(dx) \cdot E_{\lambda^e} \tau_1^\varepsilon \cdot \left| \frac{E_{\lambda^e} (N^\varepsilon (T^\varepsilon))}{T^\varepsilon} - \frac{1}{E_{\lambda^e} \tau_1^\varepsilon} \right|.$$ 

Hence, by (6.1), Lemma 7.2 and Theorem 7.4, we find that there exists $\varepsilon > 0$ such that for any $\delta \in (0, \delta_0)$

$$\lim_{\varepsilon \to 0} -\varepsilon \log \left| \frac{E_{\lambda^e} (N^\varepsilon (T^\varepsilon))}{T^\varepsilon} E_{\lambda^e} S_1^\varepsilon - \int e^{-\frac{1}{\varepsilon} f(x)} 1_A(x) \mu^\varepsilon(dx) \right| \geq \lim_{\varepsilon \to 0} -\varepsilon \log \left( \int e^{-\frac{1}{\varepsilon} f(x)} 1_A(x) \mu^\varepsilon(dx) \right) + \lim_{\varepsilon \to 0} -\varepsilon \log E_{\lambda^e} \tau_1^\varepsilon$$

$$+ \lim_{\varepsilon \to 0} -\varepsilon \log \left| \frac{E_{\lambda^e} (N^\varepsilon (T^\varepsilon))}{T^\varepsilon} - \frac{1}{E_{\lambda^e} \tau_1^\varepsilon} \right|$$

$$\geq \lim_{\varepsilon \to 0} -\varepsilon \log \left( \int e^{-\frac{1}{\varepsilon} f(x)} 1_A(x) \mu^\varepsilon(dx) \right) + c - h_\delta.$$ 

It remains to show that

$$\lim_{\varepsilon \to 0} -\varepsilon \log \left( \int e^{-\frac{1}{\varepsilon} f(x)} 1_A(x) \mu^\varepsilon(dx) \right) \geq \inf_{x \in A} [f(x) + W(x)] - \min_{j \in L} W(O_j).$$

Since $A$ is compact, for any $\delta > 0$ we can cover it by a finitely many open balls $B_\delta(y_m)$ with $y_m \in A$ for all $m \in \{1, \ldots, n_\delta\}$ for some $n_\delta < \infty.$ For this given $\delta,$ we apply Theorem 4.3 in [8, Chapter 6] to find that for any $\xi > 0$ and any $m$

$$\mu^e(B_\delta(y_m)) \leq e^{-\frac{1}{\varepsilon}(W(y_m) - \min_{j \in L} W(O_j) - \xi)}$$

for all $\varepsilon$ sufficiently small. Since

$$\int e^{-\frac{1}{\varepsilon} f(x)} 1_A(x) \mu^\varepsilon(dx) \leq n_\delta \max_{m \in \{1, \ldots, n_\delta\}} \left[ e^{-\frac{1}{\varepsilon}(\inf_{x \in B_\delta(y_m)} f(x))} \mu^\varepsilon(B_\delta(y_m)) \right],$$

March 2, 2020
the previous display gives
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \int e^{-\frac{1}{\varepsilon} f(x) \mathbf{1}_A (x) \mu (dx)} \right) \geq \min_{m \in \{1, \ldots, n\}} \left[ \inf_{x \in \mathcal{B}_\delta(y_m)} f(x) + W(y_m) \right] - \min_{j \in \mathcal{L}} W(O_j) - \xi.
\]

We complete the proof by sending $\xi$ to 0 and then $\delta$ to 0, and noting that according to Remark 3.14 $\min_{j \in \mathcal{L}} W(O_j) = W(O_1)$. 

In the application of Wald’s identity a difficulty arises in that, owing to the randomness of $N^\varepsilon (T^\varepsilon)$, $S_{N^\varepsilon (T^\varepsilon)}^\varepsilon$ need not have the same distribution as $S_1^\varepsilon$. Since as we will see the expected value of this quantity is needed to bound an error in the use of Wald’s identity, we will need to identify a related decay rate.

**Lemma 8.2** Given a compact set $A \subset M$, for all $\delta > 0$ sufficiently small
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \frac{E_{\lambda^\varepsilon} S_{N^\varepsilon (T^\varepsilon)}^\varepsilon}{T^\varepsilon} \geq \inf_{x \in A} [f(x) + W(x)] - W(O_1) + c - h_\delta.
\]

**Proof.** The main idea of the proof comes from [13, Theorem 3.16].

Given any $\varepsilon > 0$, we define $g^\varepsilon (t) = E_{\lambda^\varepsilon} S_{N^\varepsilon (t)}^\varepsilon$ for any $t \geq 0$. Conditioning on $\tau_1^\varepsilon$ yields
\[
g^\varepsilon (t) = \int_0^\infty E_{\lambda^\varepsilon} \left[ S_{N^\varepsilon (t)}^\varepsilon | \tau_1^\varepsilon = x \right] dF^\varepsilon (x),
\]
where $F^\varepsilon (\cdot)$ is the distribution function of $\tau_1^\varepsilon$. Note that
\[
E_{\lambda^\varepsilon} \left[ S_{N^\varepsilon (t)}^\varepsilon | \tau_1^\varepsilon = x \right] = \left\{ \begin{array}{ll} g^\varepsilon (t - x) & \text{if } x \leq t \\ E_{\lambda^\varepsilon} [S_1^\varepsilon | \tau_1^\varepsilon = x] & \text{if } x > t \end{array} \right.,
\]
which implies
\[
g^\varepsilon (t) = \int_0^t g^\varepsilon (t - x) dF^\varepsilon (x) + h^\varepsilon (t),
\]
with
\[
h^\varepsilon (t) = \int_t^\infty E_{\lambda^\varepsilon} [S_1^\varepsilon | \tau_1^\varepsilon = x] dF^\varepsilon (x).
\]
March 2, 2020

Since $E_{\lambda^e} S_1^e = \int_0^\infty E_{\lambda^e} [S_1^e | \tau_1^e = x] \, dF^e (x) < \infty$, we have $h^e \left( t \right) \leq E_{\lambda^e} S_1^e$ for all $t \geq 0$. Moreover, if we apply Hölder’s inequality first and then the conditional Jensen’s inequality, we find that for all $t \geq 0$,

$$h^e \left( t \right) = \int_t^\infty E_{\lambda^e} [S_1^e | \tau_1^e = x] \, dF^e (x)$$

$$\leq \left( \int_t^\infty \left( E_{\lambda^e} [S_1^e | \tau_1^e = x] \right)^2 \, dF^e (x) \right)^{\frac{1}{2}} \left( \int_t^\infty 1^2 \, dF^e (x) \right)^{\frac{1}{2}}$$

$$\leq (1 - F^e \left( t \right))^\frac{1}{2} \left( \int_t^\infty E_{\lambda^e} \left[ (S_1^e)^2 | \tau_1^e = x \right] \, dF^e (x) \right)^{\frac{1}{2}}$$

$$\leq (1 - F^e \left( t \right))^\frac{1}{2} \left( E_{\lambda^e} (S_1^e)^2 \right)^{\frac{1}{2}}.$$ 

For $\ell \in (0, c - h)$ let $U^e \triangleq e^{\frac{1}{2} \ell} E_{\lambda^e} \tau_1^e$. According to Theorem 7.4, there exists $\varepsilon_0 \in (0, 1)$ and a constant $\tilde{c} > 0$ such that

$$1 - F^e \left( U^e \right) = P_{\lambda^e} \left( \frac{\tau_1^e}{E_{\lambda^e} \tau_1^e} > e^{\frac{1}{1 + \ell}} \right) \leq e^{-\tilde{c} e^{\frac{\ell}{1 + \ell}}}$$

for any $\varepsilon \in (0, \varepsilon_0)$. Also by Theorem 7.4, $U^e < T^e$ for all $\varepsilon$ small enough. Hence for any $t \geq U^e$,

$$1 - F^e \left( t \right) \leq 1 - F^e \left( U^e \right) \leq e^{-\tilde{c} e^{\frac{\ell}{1 + \ell}}} \text{ and } h^e \left( t \right) \leq e^{-\tilde{c} e^{\frac{\ell}{1 + \ell}}} \left( E_{\lambda^e} (S_1^e)^2 \right)^{\frac{1}{2}}.$$ 

By Proposition 3.4 in [13], we know that for any $\varepsilon > 0$, for $t \in [0, \infty)$

$$g^e \left( t \right) = h^e \left( t \right) + \int_0^t h^e \left( t - x \right) \, dm^e \left( x \right),$$

where

$$m^e \left( t \right) \equiv \int_0^\infty E_{\lambda^e} \left[ N^e \left( t \right) | \tau_1^e = x \right] \, dF^e (x) = E_{\lambda^e} \left( N^e \left( t \right) \right).$$
This implies
\[
\frac{E_\lambda \epsilon S_\epsilon^\ell (T_\epsilon)}{T_\epsilon} = \frac{g^\epsilon (T_\epsilon)}{T_\epsilon} = \frac{h^\epsilon (T_\epsilon)}{T_\epsilon} + \frac{1}{T_\epsilon} \int_0^{T_\epsilon - U_\epsilon} h^\epsilon (T_\epsilon - x) \, dm^\epsilon (x)
\]
\[+ \frac{1}{T_\epsilon} \int_{T_\epsilon - U_\epsilon}^{T_\epsilon} h^\epsilon (T_\epsilon - x) \, dm^\epsilon (x),
\]
\[\leq \frac{E_\lambda \epsilon S_\epsilon^\ell}{T_\epsilon} + (1 - F^\epsilon (U_\epsilon)) \left( \left( E_\lambda \epsilon (S_1^\epsilon)^2 \right)^{\frac{1}{2}} \frac{m^\epsilon (T_\epsilon - U_\epsilon)}{T_\epsilon} \right)
\]
\[+ E_\lambda \epsilon S_\epsilon \frac{m^\epsilon (T_\epsilon) - m^\epsilon (T_\epsilon - U_\epsilon)}{T_\epsilon},
\]
where we use \( h^\epsilon (t) \leq E_\lambda \epsilon S_1^\epsilon \) to bound the first term and the third term, and
\( h^\epsilon (t) \leq e^{-c_\epsilon \ell^4/2} (E_\lambda \epsilon (S_1^\epsilon)^2)^{1/2} \) for any \( t \geq U_\epsilon \) for the second term.

To calculate the decay rate of the first term, we apply Lemma 4.1 to find that
\[
\liminf_{\epsilon \to 0} -\epsilon \log \frac{E_\lambda \epsilon S_\epsilon^\ell}{T_\epsilon} = \liminf_{\epsilon \to 0} -\epsilon \log \left( \frac{1}{T_\epsilon} E_\lambda \epsilon \tau_1^\epsilon \cdot \int e^{-\frac{1}{2} f(x)} 1_A (x) \, \mu^\epsilon (dx) \right)
\]
\[\geq \liminf_{\epsilon \to 0} -\epsilon \log \left( \int e^{-\frac{1}{2} f(x)} 1_A (x) \, \mu^\epsilon (dx) \right) + \liminf_{\epsilon \to 0} -\epsilon \log \frac{E_\lambda \epsilon \tau_1^\epsilon}{T_\epsilon}
\]
\[= \inf_{x \in A} [f (x) + W (x)] - W (O_1) + c - h_\delta.
\]

For the decay rate of the second term, given any \( \delta > 0 \)
\[
\liminf_{\epsilon \to 0} -\epsilon \log \left( e^{-c_\epsilon \ell^4/2} \left( E_\lambda \epsilon (S_1^\epsilon)^2 \right)^{\frac{1}{2}} \frac{m^\epsilon (T_\epsilon - U_\epsilon)}{T_\epsilon} \right)
\]
\[= \hat{c} \liminf_{\epsilon \to 0} \epsilon \epsilon \ell^4 + \liminf_{\epsilon \to 0} -\epsilon \log \left( \left( E_\lambda \epsilon (S_1^\epsilon)^2 \right)^{\frac{1}{2}} \frac{m^\epsilon (T_\epsilon - U_\epsilon)}{T_\epsilon} \right) = \infty,
\]
where the last equality holds since \( \ell > 0 \) implies \( \liminf_{\epsilon \to 0} \epsilon \epsilon \ell^4 = \infty \) and also because Lemma 6.2 and Lemma 7.2 ensure the finiteness of \( \liminf_{\epsilon \to 0} -\epsilon \log \left( (E_\lambda \epsilon (S_1^\epsilon)^2)^{1/2} m^\epsilon (T_\epsilon - U_\epsilon)/T_\epsilon \right) \).

For the last term, note that for any \( \epsilon \) fixed, the renewal function \( m^\epsilon (t) \) is subadditive in \( \ell \) (see for example Lemma 1.2 in [12]), so we have \( m^\epsilon (T_\epsilon) - m^\epsilon (T_\epsilon - U_\epsilon) \leq m^\epsilon (U_\epsilon) \). Thus we apply by Lemma 7.2 and Theorem 4.3 to
find that for the $\delta > 0$,
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( E_{\lambda^\varepsilon} S_1^{m^\varepsilon (T^\varepsilon) - m^\varepsilon (T^\varepsilon - U^\varepsilon)} \right) = 
\geq \liminf_{\varepsilon \to 0} -\varepsilon \log \left( E_{\lambda^\varepsilon} S_1^{m^\varepsilon (U^\varepsilon)} \frac{U^\varepsilon}{T^\varepsilon} \right)
\geq \liminf_{\varepsilon \to 0} -\varepsilon \log E_{\lambda^\varepsilon} S_1^{\varepsilon} + \liminf_{\varepsilon \to 0} -\varepsilon \log \frac{m^\varepsilon (U^\varepsilon)}{U^\varepsilon} + \liminf_{\varepsilon \to 0} -\varepsilon \log \frac{U^\varepsilon}{T^\varepsilon}
= \liminf_{\varepsilon \to 0} -\varepsilon \log E_{\lambda^\varepsilon} S_1^{\varepsilon} + \liminf_{\varepsilon \to 0} -\varepsilon \log \frac{E_{\lambda^\varepsilon} (N^\varepsilon (U^\varepsilon))}{U^\varepsilon}
\geq \liminf_{\varepsilon \to 0} -\varepsilon \log \frac{e^{\int_{F} \lambda^\varepsilon \tau_t^\varepsilon} T^\varepsilon}{T^\varepsilon}
\geq \inf_{x \in A} [f (x) + W (x)] - W (O_1) - h_\delta + (-\ell - h_\delta + c)
= \inf_{x \in A} [f (x) + W (x)] - W (O_1) + (c - h_\delta - \ell).
\]

Sending $\ell$ to 0, we have
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( E_{\lambda^\varepsilon} S_1^{m^\varepsilon (T^\varepsilon) - m^\varepsilon (T^\varepsilon - U^\varepsilon)} \right) \geq \inf_{x \in A} [f (x) + W (x)] - W (O_1) + c - h_\delta.
\]

Putting the bounds (8.1), (8.2) and (8.3) together gives that for any $\delta > 0$
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \frac{E_{\lambda^\varepsilon} S_1^{N^\varepsilon (T^\varepsilon)}}{T^\varepsilon} \geq \inf_{x \in A} [f (x) + W (x)] - W (O_1) + c - h_\delta.
\]

**Proof of Theorem 4.2.** Recall that
\[
\frac{1}{T^\varepsilon} \sum_{n=1}^{N^\varepsilon (T^\varepsilon) - 1} S_n^\varepsilon \leq \frac{1}{T^\varepsilon} \int_{0}^{T^\varepsilon} e^{-\frac{t}{2} f (X_t^\varepsilon)} 1_A (X_t^\varepsilon) dt \leq \frac{1}{T^\varepsilon} \sum_{n=1}^{N^\varepsilon (T^\varepsilon)} S_n^\varepsilon,
\]
where
\[
S_n^\varepsilon = \int_{\tau_{n-1}^\varepsilon}^{\tau_n^\varepsilon} e^{-\frac{t}{2} f (X_t^\varepsilon)} 1_A (X_t^\varepsilon) dt.
\]

Then we apply Wald’s first identity to obtain
\[
E_{\lambda^\varepsilon} \left( \sum_{n=1}^{N^\varepsilon (T^\varepsilon) - 1} S_n^\varepsilon \right) = E_{\lambda^\varepsilon} \left( \sum_{n=1}^{N^\varepsilon (T^\varepsilon)} S_n^\varepsilon \right) - E_{\lambda^\varepsilon} S_1^{\varepsilon} = E_{\lambda^\varepsilon} (N^\varepsilon (T^\varepsilon)) E_{\lambda^\varepsilon} S_1^{\varepsilon} - E_{\lambda^\varepsilon} S_1^{\varepsilon}.
\]
Thus
\[
\left| E^{\lambda} \left( \frac{1}{T^\varepsilon} \int_0^{T^\varepsilon} e^{-\frac{1}{2} f(X_i^\varepsilon)}_A (X_i^\varepsilon) \, dt \right) - \int e^{-\frac{1}{2} f(x)}_A (x) \, \mu^\varepsilon (dx) \right| \\
\leq \left| E^{\lambda} \left( \frac{N^\varepsilon (T^\varepsilon)}{T^\varepsilon} \right) E^{\lambda} S^\varepsilon_1 - \int e^{-\frac{1}{2} f(x)}_A (x) \, \mu^\varepsilon (dx) \right| + \frac{E^{\lambda} S^\varepsilon_{N^\varepsilon (T^\varepsilon)}}{T^\varepsilon}.
\]

Therefore, by Lemma 8.1 and Lemma 8.2 we have
\[
\liminf_{\varepsilon \to 0} \varepsilon \log \left| E^{\lambda} \left( \frac{1}{T^\varepsilon} \int_0^{T^\varepsilon} e^{-\frac{1}{2} f(X_i^\varepsilon)}_A (X_i^\varepsilon) \, dt \right) - \int e^{-\frac{1}{2} f(x)}_A (x) \, \mu^\varepsilon (dx) \right| \\
\geq \inf_{x \in A} [f(x) + W(x)] - W(O_1) + c - h_\delta.
\]

The following lemma bounds quantities that will arise in the proof of Theorem 4.4.

**Lemma 8.3** Let \( h = \min_{\ell \in L \setminus \{1\}} V(O_1, O_\ell) \),
\[
R^{(2)}_1 = 2 \inf_{x \in A} [f(x) + V(O_1, x)] - h,
\]
and for \( j \in L \setminus \{1\} \)
\[
R^{(2)}_j = 2 \inf_{x \in A} [f(x) + V(O_j, x)] + W(O_j) - 2W(O_1) \\
+ W(O_1 \cup O_j).
\]

Then
\[
2 \inf_{x \in A} [f(x) + W(x)] - 2W(O_1) - h \geq \min_{j \in L} R^{(2)}_j.
\]

**Proof.** By the definition of \( W(x) \),
\[
\begin{align*}
2 \inf_{x \in A} [f(x) + W(x)] - 2W(O_1) - h \\
&= 2 \inf_{x \in A} \left[ f(x) + \min_{j \in L} (V(O_j, x) + W(O_j)) \right] - 2W(O_1) - h \\
&= \min_{j \in L} \left( 2 \inf_{x \in A} \left[ f(x) + V(O_j, x) \right] + 2W(O_j) - 2W(O_1) - h \right).
\end{align*}
\]

Define \( Q_j = 2 \inf_{x \in A} [f(x) + V(O_j, x)] + 2W(O_j) - 2W(O_1) - h \). Then it suffices to show that \( Q_j \geq R^{(2)}_j \) for all \( j \in L \).
For $j = 1$, $Q_1 = 2 \inf_{x \in A} [f(x) + V(O_1, x)] - h = R_1^{(2)}$. For $j \in L \setminus \{1\}$, $Q_j \geq R_j^{(2)}$ if and only if $W(O_j) - h \geq W(O_1 \cup O_j)$. Recall that

$$W(O_j) = \min_{g \in G(j)} \left[ \sum_{(m \to n) \in g} V(O_m, O_n) \right]$$

and

$$W(O_1 \cup O_j) = \min_{g \in G(1,j)} \left[ \sum_{(m \to n) \in g} V(O_m, O_n) \right].$$

Therefore, for any $\hat{g} \in G(j)$ such that

\[
W(O_j) = \sum_{(m \to n) \in \hat{g}} V(O_m, O_n),
\]

if we remove the arrow starting from 1, and assume that it goes to $i$, then it is easy to see that $\hat{g} = \hat{g} \setminus \{(1,i)\} \in G(1,j)$. Since $V(O_1, O_j) \geq h$, we find that

\[
W(O_j) - h = \sum_{(m \to n) \in \hat{g}} V(O_m, O_n) - h
= \sum_{(m \to n) \in \hat{g}} V(O_m, O_n) + V(O_1, O_j) - h
\geq \min_{g \in G(1,j)} \left[ \sum_{(m \to n) \in g} V(O_m, O_n) \right]
= W(O_1 \cup O_j).
\]

Proof of Theorem 4.4. We begin with the observation that for any random variables $X, Y$ and $Z$ satisfying $0 \leq Y - Z \leq X \leq Y$,

\[
\text{Var}(X) = EX^2 - (EX)^2 \leq EY^2 - (E(Y - Z))^2
= \text{Var}(Y) + 2EY \cdot EZ - (EZ)^2 \leq \text{Var}(Y) + 2EY \cdot EZ.
\]

Since

\[
0 \leq \frac{1}{T\varepsilon} \sum_{n=1}^{N^\varepsilon(T\varepsilon)} S_n \leq \frac{1}{T\varepsilon} \int_0^{T\varepsilon} e^{-\frac{1}{2T\varepsilon} f(X_t^\varepsilon)} 1_A(X_t^\varepsilon) dt \leq \frac{1}{T\varepsilon} \sum_{n=1}^{N^\varepsilon(T\varepsilon)} S_n,
\]

68
we have
\[
\text{Var}_{\lambda^\varepsilon} \left( \frac{1}{T^\varepsilon} \int_0^{T^\varepsilon} e^{-\frac{1}{\varepsilon} f(X_t^\varepsilon)} 1_A(X_t^\varepsilon) \, dt \right)
\]
\[
\leq \text{Var}_{\lambda^\varepsilon} \left( \frac{1}{T^\varepsilon} \sum_{n=1}^{N^\varepsilon(T^\varepsilon)} S_n^\varepsilon \right) + 2 E_{\lambda^\varepsilon} \left( \frac{1}{T^\varepsilon} \sum_{n=1}^{N^\varepsilon(T^\varepsilon)} S_n^\varepsilon \right) \frac{E_{\lambda^\varepsilon} S_{N^\varepsilon(T^\varepsilon)}^{\varepsilon}}{T^\varepsilon},
\]
and with the help of (6.2)
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \text{Var}_{\lambda^\varepsilon} \left( \frac{1}{T^\varepsilon} \int_0^{T^\varepsilon} e^{-\frac{1}{\varepsilon} f(X_t^\varepsilon)} 1_A(X_t^\varepsilon) \, dt \right) T^\varepsilon \right)
\]
\[
\geq \min \left\{ \liminf_{\varepsilon \to 0} -\varepsilon \log \left[ \text{Var}_{\lambda^\varepsilon} \left( \frac{1}{T^\varepsilon} \sum_{n=1}^{N^\varepsilon(T^\varepsilon)} S_n^\varepsilon \right) T^\varepsilon \right] , \right.
\]
\[
\left. \liminf_{\varepsilon \to 0} -\varepsilon \log \left[ E_{\lambda^\varepsilon} \left( \frac{1}{T^\varepsilon} \sum_{n=1}^{N^\varepsilon(T^\varepsilon)} S_n^\varepsilon \right) \frac{E_{\lambda^\varepsilon} S_{N^\varepsilon(T^\varepsilon)}^{\varepsilon}}{T^\varepsilon} \right] \right\} .
\]

We complete the proof by showing
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left[ \text{Var}_{\lambda^\varepsilon} \left( \frac{1}{T^\varepsilon} \sum_{n=1}^{N^\varepsilon(T^\varepsilon)} S_n^\varepsilon \right) T^\varepsilon \right] \geq \min_{j \in L} \left( R_j^{(1)} \wedge R_j^{(2)} \right) - \eta
\]
and
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left[ E_{\lambda^\varepsilon} \left( \frac{1}{T^\varepsilon} \sum_{n=1}^{N^\varepsilon(T^\varepsilon)} S_n^\varepsilon \right) \frac{E_{\lambda^\varepsilon} S_{N^\varepsilon(T^\varepsilon)}^{\varepsilon}}{T^\varepsilon} \right]
\]
\[
\geq \min_{j \in L} \left( R_j^{(1)} \wedge R_j^{(2)} \right),
\]
where we recall
\[
R_j^{(1)} \doteq \inf_{x \in A} \left[ 2 f(x) + V(O_j, x) \right] + W(O_j) - W(O_1),
\]
\[
R_1^{(2)} \doteq 2 \inf_{x \in A} \left[ f(x) + V(O_1, x) \right] - h,
\]
and for \( j \in L \setminus \{1\} \)
\[
R_j^{(2)} \doteq 2 \inf_{x \in A} \left[ f(x) + V(O_j, x) \right] + W(O_j) - 2W(O_1)
\]
\[
+ W(O_1 \cup O_j).
\]
For the second term, we apply Wald’s first identity, Lemma 8.1 and Lemma 8.2 to find
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left[ E_{\lambda \varepsilon} \left( \frac{N(\varepsilon(T))}{T} \sum_{n=1}^{N(\varepsilon(T))} S_n \right) \right] \\
\geq \liminf_{\varepsilon \to 0} -\varepsilon \log \frac{E_{\lambda \varepsilon} S_{N(\varepsilon(T))}}{T} T \\
+ \liminf_{\varepsilon \to 0} -\varepsilon \log \frac{E_{\lambda \varepsilon} S_{N(\varepsilon(T))}}{T} \\
\geq -c + \inf_{x \in A} [f(x) + W(x)] - W(O_1) \\
+ \left( \inf_{x \in A} [f(x) + W(x)] - W(O_1) + (c - h) \right) \\
= 2 \inf_{x \in A} [f(x) + W(x)] - 2W(O_1) - h \\
\geq \min_{j \in L} R_j(2) \geq \min_{j \in L} \left( R_j(1) \land R_j(2) \right).
\]
The second to last inequality is from Lemma 8.3 and $h_\delta < h$.

Turning to the first term, we can bound the variance by (5.2):
\[
\text{Var}_{\lambda \varepsilon} \left( \frac{1}{T} \sum_{n=1}^{N(\varepsilon(T))} S_n \right) \\
\leq 2 \frac{E_{\lambda \varepsilon} (N(\varepsilon(T)))}{T} \text{Var}_{\lambda \varepsilon} S_1^2 + 2 \frac{E_{\lambda \varepsilon} (N(\varepsilon(T)))}{T} (E_{\lambda \varepsilon} S_1^2)^2 \\
\leq 2 \frac{E_{\lambda \varepsilon} (N(\varepsilon(T)))}{T} E_{\lambda \varepsilon} (S_1^2)^2 + 2 \frac{E_{\lambda \varepsilon} (N(\varepsilon(T)))}{T} (E_{\lambda \varepsilon} S_1^2)^2.
\]
Moreover, if we use Lemma 7.2 and Lemma 6.27 then we know that given $\eta > 0$, there exists $\delta_0 \in (0, 1)$, such that for any $\delta \in (0, \delta_0)$
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left[ \frac{E_{\lambda \varepsilon} (N(\varepsilon(T)))}{T} E_{\lambda \varepsilon} (S_1^2)^2 \right] \\
\geq \liminf_{\varepsilon \to 0} -\varepsilon \log \frac{E_{\lambda \varepsilon} (N(\varepsilon(T)))}{T} + \liminf_{\varepsilon \to 0} -\varepsilon \log E_{\lambda \varepsilon} (S_1^2)^2 \\
\geq \min_{j \in L} \left( R_j(1) \land R_j(2) \right) + h_\delta - h - \eta \geq \min_{j \in L} \left( R_j(1) \land R_j(2) \right) - \eta.
\]
In addition, we can apply Lemma 7.3 and Lemma 4.1 as in (8.1) to show
that for any $\delta > 0$ sufficiently small
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left[ \frac{\text{Var}_\lambda (N^\varepsilon (T^\varepsilon))}{T^\varepsilon} (E_{S_1} S_1^\varepsilon)^2 \right] \\
\geq \liminf_{\varepsilon \to 0} -\varepsilon \log \frac{\text{Var}_\lambda (N^\varepsilon (T^\varepsilon))}{T^\varepsilon} + 2 \liminf_{\varepsilon \to 0} -\varepsilon \log E_{S_1} S_1^\varepsilon \\
= 2 \inf_{x \in A} [f(x) + W(x)] - 2W(O_1) - h_\delta \\
\geq \min_{j \in L} R_j^{(2)} + h - h_\delta \geq \min_{j \in L} (R_j^{(1)} \land R_j^{(2)}) .
\]

The second last inequality comes from Lemma 8.3 and $h > h_\delta$.

Hence, we find that for any $\delta \in (0, \delta_0)$
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \text{Var}_\lambda \left( \frac{1}{T^\varepsilon} \sum_{n=1}^{N^\varepsilon (T^\varepsilon)} S_n^\varepsilon \right) \right) T^\varepsilon \geq \min_{j \in L} (R_j^{(1)} \land R_j^{(2)}) - \eta ,
\]
and we are done. ■

**Proof of Theorem 4.6** Parts 1, 2 and 3 are from Theorem 4.3, Lemma 4.3 (b) and Theorem 6.1 in [8, Chapter 6], respectively.

We now turn to part 4. Before giving the proof, we state a result from [8]. The result is Lemma 4.3 (c) in [8, Chapter 6], which says that for any unstable equilibrium point $O_j$, there exists a stable equilibrium point $O_i$ such that
\[
W(O_j) = W(O_i) + V(O_i, O_j) .
\]

Now suppose that
\[
\min_{j \in L} \left( \inf_{x \in A} [f(x) + V(O_j, x)] + W(O_j) \right)
\]
is attained at some $\ell \in L$ such that $O_\ell$ is unstable (i.e., $\ell \in L \setminus L_s$). Then since there exists a stable equilibrium point $O_i$ (i.e., $i \in L_s$) such that $W(O_\ell) =
\(W(O_t) + V(O_i, O_\ell)\) we find

\[
\min_{j \in L} \left( \inf_{x \in A} \left[ f(x) + V(O_j, x) + W(O_j) \right] \right) \\
= \inf_{x \in A} \left[ f(x) + V(O_\ell, x) + W(O_\ell) \right] \\
= \inf_{x \in A} \left[ f(x) + V(O_i, x) + V(O_i, O_\ell) + W(O_i) \right] \\
\geq \inf_{x \in A} \left[ f(x) + V(O_i, x) + W(O_i) \right] \\
\geq \min_{j \in L_s} \left( \inf_{x \in A} \left[ f(x) + V(O_j, x) + W(O_j) \right] \right) \\
\geq \min_{j \in L} \left( \inf_{x \in A} \left[ f(x) + V(O_j, x) + W(O_j) \right] \right).
\]

The first inequality is from dynamic programming inequality. Therefore, the minimum is also attained at \(i \in L_s\) and \(\min_{j \in L} R_j^{(1)} = \min_{j \in L_s} R_j^{(1)}\).

\section{Exponential Return Law and Tail Behavior}

In this section we give the proof of Theorem 7.4, which was the key fact needed to obtain bounds on the distribution of \(N^\varepsilon(T^\varepsilon)\). A result of this type first appears in [4], which asserts that the time needed to escape from an open subset of the domain of attraction of a stable equilibrium point that contains the equilibrium point has an asymptotically exponential distribution. [4] also proves a nonasymptotic bound on the tail of the probability of escape before a certain time that is also of exponential form. Theorem 7.4 is a more complicated statement, in that it asserts the asymptotically exponential form for the return time to the neighborhood of \(O_1\). To prove this we build on the results of [4], and decompose the return time into times of transitions between equilibrium points. This in turn will require the proof of a number of related results, such as establishing the independence of certain estimates with respect to initial distributions.

The existence of an exponentially distributed first hitting time is a central topic in the theory of quasistationary distributions. For a recent book length treatment of the topic we refer to [3]. However, so far as we can tell the types of situations we encounter are not covered by existing results, and so as noted we will develop what we need by using the results of [4] as the starting point.

For any \(j \in L_s\), define \(\nu_j^\varepsilon\) as the hitting time of \(B_\delta(O_k)\) for some \(k \in \)
\( L \setminus \{j\} \), i.e., \( v_j^\varepsilon = \inf \{ t > 0 : X_t^\varepsilon \in \bigcup_{k \in L \setminus \{j\}} \partial B_\delta(O_k) \} \).

We will prove the following result for first hitting times, and later extend to return times.

**Lemma 9.1** For any \( \delta > 0 \) sufficiently small and \( x \in B_\delta(O_j) \) with \( j \in L_s \)

\[
\lim_{\varepsilon \to 0} \varepsilon \log E_x v_j^\varepsilon = \min_{y \in \bigcup_{k \in L \setminus \{j\}} \partial B_\delta(O_k)} V(O_j, y) \quad \text{and} \quad \frac{v_j^\varepsilon}{E_x v_j^\varepsilon} \xrightarrow{d} \text{Exp}(1).
\]

Moreover, there exists \( \varepsilon_0 \in (0, 1) \) and a constant \( \tilde{c} > 0 \) such that

\[
P_x \left( \frac{v_j^\varepsilon}{E_x v_j^\varepsilon} > t \right) \leq e^{-\tilde{c}t}
\]

for any \( t > 0 \) and any \( \varepsilon \in (0, \varepsilon_0) \).

The organization of this section is as follows.

- Prove the first part of Lemma 9.1 that is concerned with mean first hitting times in subsection 9.1.

- Prove the second part of Lemma 9.1 that is concerned with an asymptotically exponential distribution but when starting with a special distribution in Subsection 9.2.

- Prove the third part of Lemma 9.1 that is concerned with bounds on the tail of the escape time but when starting with a special distribution in Subsection 9.3.

- Extend the second and third parts of Lemma 9.1 to general initial distributions in Subsection 9.4.

- Extend Lemma 9.1 to return times in Subsection 9.5.

### 9.1 Mean first hitting time

**Lemma 9.2** For any \( \delta > 0 \) sufficiently small and \( x \in B_\delta(O_j) \) with \( j \in L_s \)

\[
\lim_{\varepsilon \to 0} \varepsilon \log E_x v_j^\varepsilon = \min_{y \in \bigcup_{k \in L \setminus \{j\}} \partial B_\delta(O_k)} V(O_j, y).
\]
Proof. For the given \( j \in L_s \) let \( D_j \) denote the corresponding domain of attraction. We claim there is \( k \in L \setminus L_s \) such that

\[
h_j = \inf_{y \in \partial D_j} V(O_j, y) = V(O_j, O_k).
\]

Since \( V(O_j, \cdot) \) is continuous and \( \partial D_j \) is compact, there is a point \( y^* \in \partial D_j \) such that \( h_j = V(O_j, y^*) \). If \( y^* \in \cup_{k \in L \setminus L_s} O_k \), then we are done. If this is not true, then since \( y^* \notin (\cup_{k \in L_s} D_k) \cup (\cup_{k \in L_s} O_k) \), and since the solution to \( \dot{\phi} = b(\phi), \phi(0) = y^* \) must converge to \( \cup_{k \in L} O_k \) as \( t \to \infty \), it must in fact converge to a point in \( \cup_{k \in L \setminus L_s} O_k \), say \( O_k \). Since such trajectories have zero cost, by a standard argument for any \( \epsilon > 0 \) we can construct by concatenation a trajectory that connects \( O_j \) to \( O_k \) in finite time and with cost less than \( h_j + \epsilon \). Since \( \epsilon > 0 \) is arbitrary we have \( h_j = V(O_j, O_k) \).

There may be more than one \( l \in L \setminus L_s \) such that \( O_l \in \partial D_j \) and \( h_j = V(O_j, O_l) \), but we can assume that for some \( k \in L \setminus L_s \) and \( y \in \partial B_\delta(O_k) \) we attain the min in (9.1). Then

\[
\lim \inf_{\epsilon \to 0} \epsilon \log E_x v_j^\epsilon = \bar{h}_j.
\]

Given \( s < \bar{h}_j \), let \( D_j(s) = \{ x : V(O_j, x) \leq s \} \) and assume \( s \) is large enough that \( B_\delta(O_j) \subset D_j(s) \). Then \( D_j(s) \subset D_j^0 \) is closed and contained in the open set \( D_j \setminus \cup_{l \in L \setminus \{j\}} B_\delta(O_l) \), and thus the time to reach \( \partial D_j(s) \) is never greater than \( v_j^\epsilon \). Given \( \eta > 0 \) we can find a set \( D_j^\eta(s) \) that is contained in \( D_j(s) \) and satisfies the conditions of [8, Theorem 4.1, Chapter 4], and also \( \inf_{z \in \partial D_j^\eta(s)} V(O_j, z) \geq s - \eta \). Thus

\[
\lim \inf_{\epsilon \to 0} \epsilon \log E_x v_j^\epsilon \geq \lim \inf_{\epsilon \to 0} \epsilon \log E_x \inf \{ t \geq 0 : X_t^\epsilon \in \partial D_j^\eta(s) \} \geq s - \eta.
\]

Letting \( \eta \downarrow 0 \) and then \( s \uparrow \bar{h}_j \) gives \( \lim \inf_{\epsilon \to 0} \epsilon \log E_x v_j^\epsilon \geq \bar{h}_j \).

For the reverse inequality we also adapt an argument from the proof of [8, Theorem 4.1, Chapter 4]. One can find \( T_1 < \infty \) such that the probability for \( X_t^\epsilon \) to reach \( \cup_{l \in L} B_\delta(O_l) \) by time \( T_1 \) from any \( x \in M \setminus \cup_{l \in L} B_\delta(O_l) \) is bounded below by \( 1/2 \). (This follows easily from the law of large numbers and that all trajectories of the noiseless system reach \( \cup_{l \in L} B_\delta/2(O_l) \) in some finite time that is bounded uniformly in \( x \in M \setminus \cup_{l \in L} B_\delta(O_l) \).) Also, given \( \eta > 0 \) there is \( T_2 < \infty \) and \( \varepsilon_0 > 0 \) such that \( P_x \{ X_t^\epsilon \text{ reaches } \partial B_\delta(O_k) \text{ before } T_2 \} \geq \exp - (\bar{h}_j + \eta) / \epsilon \) for all \( x \in \partial B_\delta(O_j) \). It then follows from the strong Markov property that for any \( x \in M \setminus \cup_{l \in L} B_\delta(O_l) \)

\[
P_x \{ v_j^\epsilon \leq T_1 + T_2 \} \geq \frac{1}{2} e^{-\frac{1}{\epsilon}(\bar{h}_j+\eta)}.
\]
Using the ordinary Markov property we have
\[
E_x v_j^\varepsilon \leq \sum_{n=0}^{\infty} (n+1)(T_1 + T_2)P_x\{n(T_1 + T_2) < v_j^\varepsilon \leq (n+1)(T_1 + T_2)\}
\]
\[
= (T_1 + T_2) \sum_{n=0}^{\infty} P_x\{v_j^\varepsilon > n(T_1 + T_2)\}
\]
\[
\leq (T_1 + T_2) \sum_{n=0}^{\infty} \left[1 - \inf_{x \in M \cup \bigcup_{l \in L} \partial B_\delta(O_l)} P_x\{v_j^\varepsilon \leq T_1 + T_2\}\right]^n
\]
\[
= (T_1 + T_2) \left(\inf_{x \in M \cup \bigcup_{l \in L} \partial B_\delta(O_l)} P_x\{v_j^\varepsilon \leq T_1 + T_2\}\right)^{-1}
\]
\[
\leq 2(T_1 + T_2)e^{\frac{1}{2}(\bar{h} + \eta)}.
\]
Thus \(\lim_{\varepsilon \to 0} \varepsilon \log E_x v_j^\varepsilon \leq \bar{h}_j + \eta\), and letting \(\eta \downarrow 0\) completes the proof.

### 9.2 Asymptotically exponential distribution

**Lemma 9.3** For each \(j \in L_s\) there is a distribution \(u^\varepsilon\) on \(\partial B_\delta(O_j)\) such that
\[
\frac{v_j^\varepsilon}{E_{u^\varepsilon}v_j^\varepsilon} \xrightarrow{d} \text{Exp}(1).
\]

**Proof.** To simplify notation and since it plays no role, we write \(j = 1\) throughout the proof. We call \(\partial B_\delta(O_1)\) and \(\partial B_\delta(O_1)\) the inner and outer rings of \(O_1\) and denote them by \(B_1\) and \(B_2\). We can then decompose the hitting time as
\[
v_1^\varepsilon = \sum_{k=1}^{N^\varepsilon - 1} \theta_k^\varepsilon + \zeta^\varepsilon, \quad (9.2)
\]
where \(\theta_k^\varepsilon\) is the \(k\)-th amount of time that the process travels from the outer ring to the inner ring and back without visiting \(\bigcup_{j \in L \setminus \{1\}} \partial B_\delta(O_j)\), \(\zeta^\varepsilon\) is the amount of time that the process travels from the outer ring directly to \(\bigcup_{j \in L \setminus \{1\}} \partial B_\delta(O_j)\) without visiting the inner ring, and \(N^\varepsilon - 1\) is the number of times that the process goes back and forth between the inner ring and outer ring. (It is assumed that \(\delta > 0\) is small enough that \(B_\delta(O_1) \subset M \setminus \bigcup_{j \in L \setminus \{1\}} \partial B_\delta(O_j)\).) Note that \(\theta_k^\varepsilon\) grows exponentially of the order \(\delta\), due to the time taken to travel from the inner ring to the outer ring, and \(\zeta^\varepsilon\) is uniformly bounded in expected value.
For any set $A$, define the first hitting time by
\[\tau(A) = \inf \{ t > 0 : X_t^\epsilon \in A \} .\]

Consider the conditional transition probability from $x \in B_2$ to $y \in B_1$ given by
\[\psi_1^\epsilon(dy|x) = P \left( X^\epsilon_{\tau(B_1)} \in dy | X_0^\epsilon = x, X^\epsilon_t \notin \cup_{j \in L \setminus \{1\}} \partial B_\delta(O_j), t \in [0, \tau(B_1)] \right) ,\]
and the transition probability from $y \in B_1$ to $x \in B_2$ given by
\[\psi_2^\epsilon(dx|y) = P \left( X^\epsilon_{\tau(B_2)} \in dx | X_0^\epsilon = y \right) .\]

Then we can create a transition probability from $x \in B_2$ to $y \in B_2$ by
\[\psi^\epsilon(dy|x) = \int_{B_1} \psi_2^\epsilon(dx|z) \psi_1^\epsilon(dz|x) .\]

Since $B_2$ is compact and $\{X^\epsilon_t\}_t$ is non-degenerate and Feller, there exists an invariant measure $u^\epsilon \in \mathcal{P}(B_2)$ with respect to the transition probability $\psi^\epsilon(dy|x)$. If we start with the distribution $u^\epsilon$ on $B_2$, then it follows from the definition of $u^\epsilon$ and the strong Markov property that the $\{\theta_k^\epsilon\}_{k < N^\epsilon}$ are iid. Moreover, the indicators of escape (i.e., $1\{\tau(\cup_{j \in L \setminus \{1\}} \partial B_\delta(O_j)) = \tau(\cup_{j \in L \setminus \{1\}} \partial B_\delta(O_j))\}$) are iid Bernoulli, and we write them as $Y_k^\epsilon$, where
\[P_{u^\epsilon}(Y_k^\epsilon = 1) = e^{-h_1^\epsilon(\delta)/\epsilon},\]
where $\delta > 0$ is from the construction, $h_1^\epsilon(\delta) \to h_1(\delta)$ as $\epsilon \to 0$ and $h_1(\delta) \uparrow h_1$ as $\delta \downarrow 0$ with $h_1 = \min_{j \in L \setminus \{1\}} V(O_1, O_j)$. Note that
\[N^\epsilon = \min \{ k \in \mathbb{N} : Y_k^\epsilon = 1 \} .\]

We therefore have
\[P_{u^\epsilon}(N^\epsilon = k) = (1 - e^{-h_1^\epsilon(\delta)/\epsilon})^{k-1} e^{-h_1^\epsilon(\delta)/\epsilon},\]
and thus
\[E_{u^\epsilon}v_{1}^\epsilon = E_{u^\epsilon} \left[ \sum_{j=1}^{N^\epsilon-1} \theta_j^\epsilon \right] + E_{u^\epsilon} \zeta^\epsilon = E_{u^\epsilon}(N^\epsilon - 1)E_{u^\epsilon} \theta_1^\epsilon + E_{u^\epsilon} \zeta^\epsilon ,\]

76
where the second equality comes from Wald’s identity. Using \( \sum_{k=1}^{\infty} k a^{k-1} = 1/(1-a)^2 \) for \( a \in [0,1) \), we also have
\[
E_{u^\varepsilon N^\varepsilon} = \sum_{k=1}^{\infty} k \left( 1 - e^{-h_1^\varepsilon(\delta)/\varepsilon} \right) k^{-1} e^{-h_1^\varepsilon(\delta)/\varepsilon} \\
= e^{-h_1^\varepsilon(\delta)/\varepsilon} \sum_{k=1}^{\infty} k \left( 1 - e^{-h_1^\varepsilon(\delta)/\varepsilon} \right) k^{-1} \\
= e^{-h_1^\varepsilon(\delta)/\varepsilon} e^{2h_1^\varepsilon(\delta)/\varepsilon} = e^{h_1^\varepsilon(\delta)/\varepsilon},
\]
and therefore
\[
E_{u^\varepsilon \upsilon_1^\varepsilon} = e^{h_1^\varepsilon(\delta)/\varepsilon} E_{u^\varepsilon \theta_1^\varepsilon} + (E_{u^\varepsilon \zeta_1^\varepsilon} - E_{u^\varepsilon \theta_1^\varepsilon}). \tag{9.3}
\]

Next consider the characteristic function
\[
\phi_{\upsilon_1^\varepsilon}(t) = E_{u^\varepsilon} e^{it \upsilon_1^\varepsilon} = \phi_{\upsilon_1^\varepsilon}(t/E_{u^\varepsilon} \upsilon_1^\varepsilon),
\]
where \( \phi_{\upsilon_1^\varepsilon} \) is the characteristic function of \( \upsilon_1^\varepsilon \). By (9.2), we have
\[
\phi_{\zeta_1^\varepsilon}(s) = E_{u^\varepsilon} e^{is \zeta_1^\varepsilon} E_{u^\varepsilon} e^{is \theta_1^\varepsilon} = \phi_{\upsilon_1^\varepsilon}(s/E_{u^\varepsilon} \zeta_1^\varepsilon).
\]

where \( \phi_{\theta_1^\varepsilon} \) and \( \phi_{\zeta_1^\varepsilon} \) are the characteristic functions of \( \theta_1^\varepsilon \) and \( \zeta_1^\varepsilon \), respectively.

We want to show that for any \( t \in \mathbb{R} \)
\[
\phi_{\theta_1^\varepsilon}(t/E_{u^\varepsilon} \upsilon_1^\varepsilon) \to \frac{1}{1 - it} \text{ as } \varepsilon \to 0.
\]

We first show that \( \phi_{\zeta_1^\varepsilon}(t/E_{u^\varepsilon} \upsilon_1^\varepsilon) \to 1 \). By definition,
\[
\phi_{\zeta_1^\varepsilon}(t/E_{u^\varepsilon} \upsilon_1^\varepsilon) = E_{u^\varepsilon} \cos \left( \frac{t \zeta_1^\varepsilon}{E_{u^\varepsilon} \upsilon_1^\varepsilon} \right) + i E_{u^\varepsilon} \sin \left( \frac{t \zeta_1^\varepsilon}{E_{u^\varepsilon} \upsilon_1^\varepsilon} \right).
\]
We know that there exist $T_0 < \infty$ and $c > 0$ such that for any $T \in (0, \infty)$ and for all $\varepsilon$ sufficiently small

$$P_\varepsilon (\zeta \varepsilon > T) \leq e^{-\frac{1}{\varepsilon}(T-T_0)}, \quad (9.4)$$

and therefore for any bounded and continuous function $f : \mathbb{R} \to \mathbb{R}$

$$\left| E_\varepsilon f \left( \frac{t\zeta \varepsilon}{E_\varepsilon v_1^\varepsilon} \right) - f(0) \right|$$

$$\leq E_\varepsilon \left[ f \left( \frac{t\zeta \varepsilon}{E_\varepsilon v_1^\varepsilon} \right) - f(0) \right] \left( 1_{\{\zeta \varepsilon > T\}} + 1_{\{\zeta \varepsilon \leq T\}} \right)$$

$$\leq 2 \|f\|_\infty P_\varepsilon (\zeta \varepsilon > T) + E_\varepsilon \left[ f \left( \frac{t\zeta \varepsilon}{E_\varepsilon v_1^\varepsilon} \right) - f(0) \right] 1_{\{\zeta \varepsilon \leq T\}}.$$

The first term in the last display goes to 0 as $\varepsilon \to 0$. For any fixed $t$, $t/E_\varepsilon v_1^\varepsilon \to 0$ as $\varepsilon \to 0$. Since $f$ is continuous, the second term in the last display also converges to 0 as $\varepsilon \to 0$. $\phi_\varepsilon (t/E_\varepsilon v_1^\varepsilon) \to 1$ follows by taking $f$ to be $\sin x$ and $\cos x$.

It remains to show that for any $t \in \mathbb{R}$

$$\frac{e^{-h_1^\varepsilon(\delta)/\varepsilon}}{1 - \left[ (1 - e^{-h_1^\varepsilon(\delta)/\varepsilon}) \phi_\delta(t/E_\varepsilon v_1^\varepsilon) \right]} \to \frac{1}{1 - it}$$

as $\varepsilon \to 0$. Observe that

$$\frac{e^{-h_1^\varepsilon(\delta)/\varepsilon}}{1 - \left[ (1 - e^{-h_1^\varepsilon(\delta)/\varepsilon}) \phi_\delta(t/E_\varepsilon v_1^\varepsilon) \right]} = \frac{1}{1 - \phi_\delta(t/E_\varepsilon v_1^\varepsilon) + \phi_\delta(t/E_\varepsilon v_1^\varepsilon)},$$

so it suffices to show that $\phi_\delta(t/E_\varepsilon v_1^\varepsilon) \to 1$ and $[1 - \phi_\delta(t/E_\varepsilon v_1^\varepsilon)]/e^{-h_1^\varepsilon(\delta)/\varepsilon} \to -it$ as $\varepsilon \to 0$.

For the former, note that by (9.3)

$$0 \leq E_\varepsilon (t\theta_1^\varepsilon/E_\varepsilon v_1^\varepsilon) \leq \frac{tE_\varepsilon \theta_1^\varepsilon}{e^{h_1^\varepsilon(\delta)/\varepsilon} - 1} E_\varepsilon \theta_1^\varepsilon \to 0$$

as $\varepsilon \to 0$, and so $t\theta_1^\varepsilon/E_\varepsilon v_1^\varepsilon$ converges to 0 in distribution. Since $\sin x$ and $\cos x$ are bounded functions, by the convergence in distribution version of the Dominated Convergence Theorem

$$\phi_\delta(t/E_\varepsilon v_1^\varepsilon) = E_\varepsilon \cos (t\theta_1^\varepsilon/E_\varepsilon v_1^\varepsilon) + iE_\varepsilon \sin (t\theta_1^\varepsilon/E_\varepsilon v_1^\varepsilon) \to 1.$$
For the second part, using
\[ x - \frac{x^3}{3!} \leq \sin x \leq x \quad \text{and} \quad 1 - \frac{x^2}{2} \leq \cos x \leq 1 \]
for \( x \in \mathbb{R} \) we find that
\[ 0 \leq \frac{1 - E_{w^\varepsilon} \cos \left( \frac{t\theta_1^\varepsilon}{E_{w^\varepsilon} v_1^\varepsilon} \right)}{e^{-h_1^\varepsilon(\delta)/\varepsilon}} \leq \frac{E_{w^\varepsilon} \left( \frac{t\theta_1^\varepsilon}{E_{w^\varepsilon} v_1^\varepsilon} \right)^2}{2e^{-h_1^\varepsilon(\delta)/\varepsilon}} \]
and
\[ \frac{E_{w^\varepsilon} \left( \frac{t\theta_1^\varepsilon}{E_{w^\varepsilon} v_1^\varepsilon} \right) - E_{w^\varepsilon} \left( \frac{t\theta_1^\varepsilon}{E_{w^\varepsilon} v_1^\varepsilon} \right)^3}{3!e^{-h_1^\varepsilon(\delta)/\varepsilon}} \leq \frac{E_{w^\varepsilon} \sin \left( \frac{t\theta_1^\varepsilon}{E_{w^\varepsilon} v_1^\varepsilon} \right)}{e^{-h_1^\varepsilon(\delta)/\varepsilon}} \leq \frac{E_{w^\varepsilon} \left( \frac{t\theta_1^\varepsilon}{E_{w^\varepsilon} v_1^\varepsilon} \right)}{e^{-h_1^\varepsilon(\delta)/\varepsilon}}. \]
From our previous observation regarding the distribution of \( \zeta^\varepsilon \) and \([9.3]\)
\[ \frac{E_{w^\varepsilon} \left( \frac{t\theta_1^\varepsilon}{E_{w^\varepsilon} v_1^\varepsilon} \right)}{e^{-h_1^\varepsilon(\delta)/\varepsilon}} \to t \quad \text{as} \quad \varepsilon \to 0. \]
In addition, since \( \theta_1^\varepsilon \) can be viewed as the time from the outer ring to the inner ring without visiting \( \cup_{j \in L \setminus \{1\}} \partial B_\delta(O_j) \) plus the time from the inner ring to the outer ring, by applying \([9.4]\) to the former and using \([4, \text{Theorem 4 and Corollary 1}]\) under Condition \([3.11]\) to the later, we find that
\[ P_{w^\varepsilon} \left( \frac{\theta_1^\varepsilon}{E_{w^\varepsilon} \theta_1^\varepsilon} > t \right) \leq 2e^{-t} \]
for all \( t \in [0, \infty) \) and \( \varepsilon \) sufficiently small. This implies that
\[ E_{w^\varepsilon} \left( \frac{\theta_1^\varepsilon}{E_{w^\varepsilon} \theta_1^\varepsilon} \right)^2 = 2 \int_0^\infty \tilde{t}^2 P_{w^\varepsilon} \left( \frac{\theta_1^\varepsilon}{E_{w^\varepsilon} \theta_1^\varepsilon} > t \right) dt \leq 4 \int_0^\infty \tilde{t}^2 e^{-t} dt = 8 \]
and
\[ E_{w^\varepsilon} \left( \frac{\theta_1^\varepsilon}{E_{w^\varepsilon} \theta_1^\varepsilon} \right)^3 = 3 \int_0^\infty \tilde{t}^3 P_{w^\varepsilon} \left( \frac{\theta_1^\varepsilon}{E_{w^\varepsilon} \theta_1^\varepsilon} > t \right) dt \leq 6 \int_0^\infty \tilde{t}^3 e^{-t} dt = 36. \]
Then combined with \([9.3]\), we have
\[ 0 \leq \frac{E_{w^\varepsilon} \left( \frac{t\theta_1^\varepsilon}{E_{w^\varepsilon} v_1^\varepsilon} \right)^2}{2e^{-h_1^\varepsilon(\delta)/\varepsilon}} \leq \frac{\tilde{t}^2 E_{w^\varepsilon} \left( \frac{\theta_1^\varepsilon}{E_{w^\varepsilon} \theta_1^\varepsilon} \right)^2}{2e^{-h_1^\varepsilon(\delta)/\varepsilon} \left( e^{h_1^\varepsilon(\delta)/\varepsilon} - 1 \right)^2} \to 0 \]
and
\[ 0 \leq \frac{E_{u^\varepsilon} (t\theta^\varepsilon/E_{u^\varepsilon}v^\varepsilon_j)^3}{3!e^{-h_1^1(\delta)/\varepsilon}} \leq \frac{t^3 E_{u^\varepsilon} (\theta^\varepsilon_j/E_{u^\varepsilon} \theta^\varepsilon_j)^3}{3!e^{-h_1^1(\delta)/\varepsilon}(e^{h_1^1(\delta)/\varepsilon} - 1)^3} \to 0. \]

Therefore, we have shown that for any \( t \in \mathbb{R} \)
\[ \frac{1 - \phi^\varepsilon(t/E_{u^\varepsilon}v^\varepsilon_j)}{e^{-h_1^1(\delta)/\varepsilon}} = \frac{1 - E_{u^\varepsilon} \cos (t\theta^\varepsilon_j/E_{u^\varepsilon}v^\varepsilon_j)}{e^{-h_1^1(\delta)/\varepsilon}} - i \frac{E_{u^\varepsilon} \sin (t\theta^\varepsilon_j/E_{u^\varepsilon}v^\varepsilon_j)}{e^{-h_1^1(\delta)/\varepsilon}} \to -i t. \]

In general, one can adapt the argument to prove that if the process starts from an appropriate distribution around an equilibrium point, then the first hitting time of other equilibrium points is exponential distributed as \( \varepsilon \to 0 \).

### 9.3 Tail probability

The goal of this subsection is to prove the following.

**Lemma 9.4** For each \( j \in L_s \) there is a distribution \( u^\varepsilon \) on \( \partial B_{2\delta}(O_j) \) and \( \bar{c} > 0 \) such that for any \( t \in [0, \infty) \)
\[ P_{u^\varepsilon} \left( \frac{v^\varepsilon_j}{E_{u^\varepsilon}v^\varepsilon_j} > t \right) \leq e^{-\bar{c}t} \]
(here \( v^\varepsilon_j \) and \( u^\varepsilon \) are defined as in the last subsection).

**Proof.** As in the last subsection we give the proof for the case \( j = 1 \). To begin we note that for any \( c > 0 \) Chebyshev’s inequality implies \[ P_{u^\varepsilon} \left( \frac{v^\varepsilon_j}{E_{u^\varepsilon}v^\varepsilon_j} > t \right) = P_{u^\varepsilon} \left( \frac{e^{ct}v_j}{E_{u^\varepsilon}v^\varepsilon_j} > e^{ct} \right) \leq e^{-ct} \cdot E_{u^\varepsilon} e^{ct}v^\varepsilon_j. \]

By picking \( c = c^* \doteq 1/8 \), it suffices to show that \( E_{u^\varepsilon} e^{ct}v^\varepsilon_j/E_{u^\varepsilon}v^\varepsilon_j \) is bounded by a constant.

Before proving the bound, we first show the finiteness of \( E_{u^\varepsilon} e^{ct}v^\varepsilon_j/E_{u^\varepsilon}v^\varepsilon_j \) which is implied by the finiteness of \( E_{u^\varepsilon} e^{ct\theta^\varepsilon_j}/E_{u^\varepsilon}v^\varepsilon_j \) and \( E_{u^\varepsilon} e^{ct\theta^\varepsilon_j}/E_{u^\varepsilon}v^\varepsilon_j \).

Using \([9.5]\) we find that for any \( \alpha > 0 \)
\[ P_{u^\varepsilon} \left( \frac{e^{ct\theta^\varepsilon_j}/E_{u^\varepsilon}\theta^\varepsilon_j}{E_{u^\varepsilon}v^\varepsilon_j} > t \right) \leq 2e^{-1/\alpha \log t} = 2t^{-1/\alpha} \]
for all \( t \in [1, \infty) \) and \( \varepsilon \) sufficiently small.
Then (9.3) implies $E_{u^\varepsilon} v_1^\varepsilon \geq (e^{h_1^\varepsilon(\delta)/\varepsilon}) - 1) E_{u^\varepsilon} \theta_1^\varepsilon$ and therefore

\[
E_{u^\varepsilon} e^{c^* \theta_1^\varepsilon} \leq E_{u^\varepsilon} \exp \left( \frac{c^* \theta_1^\varepsilon}{(e^{h_1^\varepsilon(\delta)/\varepsilon}) - 1) E_{u^\varepsilon} \theta_1^\varepsilon} \right) = \int_0^1 P_{u^\varepsilon} \left( \exp \left( \frac{c^* \theta_1^\varepsilon}{(e^{h_1^\varepsilon(\delta)/\varepsilon}) - 1) E_{u^\varepsilon} \theta_1^\varepsilon} \right) > t \right) dt \\
+ \int_1^\infty P_{u^\varepsilon} \left( \exp \left( \frac{c^* \theta_1^\varepsilon}{(e^{h_1^\varepsilon(\delta)/\varepsilon}) - 1) E_{u^\varepsilon} \theta_1^\varepsilon} \right) > t \right) dt \\
\leq 1 + 2 \int_1^\infty t^{-\left(e^{h_1^\varepsilon(\delta)/\varepsilon} - 1) / c^* \right)} dt \\
= 1 + \frac{2}{(e^{h_1^\varepsilon(\delta)/\varepsilon} - 1) / c^* - 1} = 1 + \frac{2c^*}{e^{h_1^\varepsilon(\delta)/\varepsilon} - c^* - 1}.
\]

To estimate $\zeta^\varepsilon$, we use that by (9.4) there are $T_0$ and $c > 0$ such that for any $t > 0$ and for all $\varepsilon$ sufficiently small

\[
P_{u^\varepsilon} (\zeta^\varepsilon > t) \leq e^{-\frac{1}{c}(t - T_0)},
\]
so that for any $\alpha > 0$

\[
P_{u^\varepsilon} (e^{\alpha \zeta^\varepsilon} > t) \leq e^{-\frac{1}{c}(\frac{1}{\alpha} \log t - T_0)}
\]
for any $t \geq e^{\alpha T_0}$. Given $n \in \mathbb{N}$, for all sufficiently small $\varepsilon$ we have $c^* / E_{u^\varepsilon} v_1^\varepsilon \leq 1/n$, and thus

\[
P_{u^\varepsilon} \left( e^{c^* \varepsilon v_1^\varepsilon} \zeta^\varepsilon > t \right) \leq P_{u^\varepsilon} \left( e^{\zeta^\varepsilon / n} > t \right) \leq e^{-\frac{1}{c}(n \log t - T_0)}.
\]
Hence for any $n$ such that $e^{T_0/n} \leq 3/2$ and $(-cn + 1) \log (3/2) + cT_0 < 0$,
and for $\varepsilon$ small enough that $c^*/E_u\varepsilon_1^\varepsilon \leq 1/n$, we have

$$E_u\varepsilon e^{c^*/E_u\varepsilon_1^\varepsilon \zeta \varepsilon} \leq \int_0^\infty P_{u\varepsilon} \left( e^{c^*/E_u\varepsilon_1^\varepsilon \zeta \varepsilon} > t \right) dt$$

$$\leq \frac{3}{2} + \int_0^\infty P_{u\varepsilon} \left( e^{c^*/E_u\varepsilon_1^\varepsilon \zeta \varepsilon} > t \right) dt$$

$$\leq \frac{3}{2} + \int_0^\infty e^{-\frac{1}{2}c^*(n \log t - T_0)} dt$$

$$= \frac{3}{2} + \frac{1}{\varepsilon c^* n - 1} \left( -c^* n + 1 \right) \frac{1}{2} \frac{3}{2} \left( 1 - e^{-h(\delta)/\varepsilon} \right)$$

$$\leq \frac{3}{2} + \frac{1}{\varepsilon c^* n - 1}$$

$$\leq 2.$$

We have shown that for such $c^*$, $E_u\varepsilon e^{c^*/E_u\varepsilon_1^\varepsilon \zeta \varepsilon}$ and $E_u\varepsilon e^{c^*/E_u\varepsilon_1^\varepsilon \theta \varepsilon_1^\varepsilon}$ are uniformly bounded for all $\varepsilon$ sufficiently small. Lastly, using the same calculation as used for the characteristic function

$$E_u\varepsilon e^{c^*/E_u\varepsilon_1^\varepsilon \zeta \varepsilon}$$

$$= \frac{1}{1 - \left( 1 - e^{-h(\delta)/\varepsilon} \right) E_u\varepsilon e^{c^*/E_u\varepsilon_1^\varepsilon \theta \varepsilon_1^\varepsilon}}$$

$$\leq 2e^{-h(\delta)/\varepsilon}$$

$$\frac{1}{1 - \left( 1 - e^{-h(\delta)/\varepsilon} \right) \left( 1 + \frac{2c^*}{e^{h(\delta)/\varepsilon} - c^* - 1} \right)}$$

$$= 2e^{-h(\delta)/\varepsilon}$$

$$\frac{1}{e^{-h(\delta)/\varepsilon} - \frac{2c^*}{e^{h(\delta)/\varepsilon} - c^* - 1} + \frac{2c^*}{e^{h(\delta)/\varepsilon} - c^* - 1} e^{-h(\delta)/\varepsilon}}$$

$$= \frac{2}{1 - 2c^* e^{h(\delta)/\varepsilon - 1}}$$

$$\leq \frac{2}{1 - 4c^*} = 4.$$
9.4 General initial condition

This section presents results that will allow us to extend the results in the previous two subsections to arbitrary deterministic initial conditions \( x \in B_1 = \partial B_\delta(O_1) \). Under our assumptions, for any \( j \in L_s \) we observe that the process model

\[
dX_t^\varepsilon = b(X_t^\varepsilon) \, dt + \sqrt{\varepsilon} \sigma(X_t^\varepsilon) \, dW_t
\]

has the property that \( b(x) = A(x - O_j)[1 + o(1)] \) and \( \sigma(x) = \bar{\sigma}[1 + o(1)] \), where \( o(1) \to 0 \) as \( \|x - O_j\| \to 0 \), \( A \) is stable and \( \bar{\sigma} \) is invertible. By an invertible change of variable we can arrange so that \( O_j = 0 \) and \( \bar{\sigma} = I \), and to simplify we assume this in the rest of the section.

Since \( A \) is stable there exists a positive definite and symmetric solution \( M \) to the matrix equation

\[
AM + MA^T = -I
\]

(we can in fact exhibit the solution in the form \( M = \int_0^\infty e^{At} e^{A^T t} \, dt \)). To prove the ergodicity we introduce some additional sets:

\[
S_1(\varepsilon) = \{x : \langle x, Mx \rangle \leq a_1 \sqrt{\varepsilon} \}
\]

and

\[
S_2(\varepsilon) = \{x : \langle x, Mx \rangle \leq a_2 \sqrt{\varepsilon} \},
\]

where \( 0 < a_1 < a_2 \) and \( \varepsilon_0 > 0 \) is small enough that

\[
S_2(\varepsilon_0) \subset B_0,
\]

where \( B_0 = \{x : \langle x, Mx \rangle \leq b_0^2 \}, b_0 \in (0, b_1) \), and assume \( \varepsilon \in (0, \varepsilon_0) \) henceforth.

Remark 9.5 Although elsewhere in this paper as well as in the reference \[8\], the sets \( B_i \), \( i = 1, 2 \) are taken to be balls with respect to the Euclidean norm, in this section we take them to be level sets of \( V(x) = \langle x, Mx \rangle \), so that \( B_i = \{x : V(x) \leq b_i^2 \} \). The shape of these sets plays no role in the analysis of \[8\] or in our prior use in this paper, but as we will see it is notationally convenient in the present setting for them to be level sets of \( V \), since \( V \) is a Lyapunov function for the noiseless dynamics near 0.

In addition to the restrictions \( a_1 < a_2 \) and \( a_2^2 \varepsilon_0 \leq b_0 \), we also assume that \( a_1, a_2 \) and \( \varepsilon_0 > 0 \) are such that if \( \phi^x \) is the solution to the noiseless dynamics \( \dot{\phi} = b(\phi) \) with initial condition \( x \), then: (i) for all \( x \in \partial S_2(\varepsilon) \), \( \phi^x \) never enters \( B_1 \); (ii) for all \( x \in \partial S_1(\varepsilon) \), \( \phi^x \) never exits \( S_2(\varepsilon) \).

The idea that will be used to establish asymptotic independence from the starting distribution is the following. We start the process on \( B_1 \). With some
small probability it will hit $B_2$ before hitting $S_2(\varepsilon)$. This gives a contribution to $\psi_2^\varepsilon(dz|x)$ that will be relatively unimportant. If instead it hits $S_2(\varepsilon)$ first, then we do a Freidlin-Wentzell type analysis, and decompose the trajectory into excursions between $\partial S_2(\varepsilon)$ and $\partial S_1(\varepsilon)$, before a final excursion from $\partial S_2(\varepsilon)$ to $B_2$.

To exhibit the asymptotic independence from $\varepsilon$, we introduce the scaled process $Y^\varepsilon(t) = X^\varepsilon(t)/\sqrt{\varepsilon}$, which solves the SDE

$$dY^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} b(\sqrt{\varepsilon}Y^\varepsilon(t))dt + \sigma(\sqrt{\varepsilon}Y^\varepsilon(t))dW(t).$$

Let

$$S_1 = \partial S_1(1) \text{ and } S_2 = \partial S_2(1).$$

Let $\omega^\varepsilon(w|x)$ denote the density of the hitting location on $S_2$ by the process $Y^\varepsilon$, given $Y^\varepsilon(0) = x \in S_1$. The following estimate is essential. The density function can be identified with the normal derivative of a related Green’s function, which is bounded from above by the boundary gradient estimate and bounded below by using the Hopf lemma.

**Lemma 9.6** Given $\varepsilon_0 > 0$, there are $0 < c_1 < c_2 < \infty$ such that

$$c_1 \leq \omega^\varepsilon(w|x) \leq c_2$$

for all $x \in S_1$, $w \in S_2$ and $\varepsilon \in (0, \varepsilon_0)$.

Next let $p^\varepsilon(u|w)$ denote the density of the return location for $Y^\varepsilon$ on $S_2$, conditioned on visiting $S_1$ before $B_2/\sqrt{\varepsilon}$, and starting at $w \in S_2$. The last lemma then directly gives the following.

**Lemma 9.7** For $\varepsilon_0 > 0$ and $c_1, c_2$ as in the last lemma

$$c_1 \leq p^\varepsilon(u|w) \leq c_2$$

for all $u, w \in S_2$ and $\varepsilon \in (0, \varepsilon_0)$.

Let $r^\varepsilon(w)$ denote the unique stationary distribution of $p^\varepsilon(u|w)$, and let $p^\varepsilon,n(u|w)$ denote the $n$-step transition density. The preceding lemma, [9 Theorem 10.1 Chapter 3], and the existence of a uniform strictly positive lower bound on $r^\varepsilon(u)$ for all sufficiently small $\varepsilon > 0$ imply the following.

**Lemma 9.8** There is $K < \infty$ and $\alpha \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\sup_{w \in S_2} |p^\varepsilon,n(u|w) - r^\varepsilon(u)|/r^\varepsilon(u) \leq K \alpha^n.$$
Let $\tilde{\eta}^\varepsilon(dx|w)$ denote the distribution of $X^\varepsilon$ upon first hitting $B_2$ given that $X^\varepsilon$ reaches $\partial S_1(\varepsilon)$ before $B_2$ and starts at $w \in \partial S_2(\varepsilon)$.

**Lemma 9.9** There is $\kappa > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$
\[
\sup_{x \in B_1} P_x \{ X^\varepsilon \text{ reaches } B_2 \text{ before } S_2(\varepsilon) \} \leq e^{-\kappa/\varepsilon} \text{ as } \varepsilon \to 0.
\]

**Lemma 9.10** There are $\tilde{\eta}^\varepsilon(dz) \in \mathcal{P}(B_2)$ and $s^\varepsilon$ that tends to 0 as $\varepsilon \to 0$, such that for all $A \in \mathcal{B}(B_2), w \in \partial S_2(\varepsilon)$
\[
\tilde{\eta}^\varepsilon(A)[1 - s^\varepsilon K/(1 - \alpha)] \leq \tilde{\eta}^\varepsilon(A|w) \leq \tilde{\eta}^\varepsilon(A)[1 - s^\varepsilon K/(1 - \alpha)],
\]

where $K$ and $\alpha$ are from Lemma 9.8.

**Proof of Lemma 9.9** We choose $a_1$ according to $a_1^2 = 2 \sup_{x \in B_1} \text{tr} |\sigma(x)\sigma(x)^T M|$. We then use that $AM + MA^T = -I$ and the relation of $a_2$ and $B_2$ to get that with $V(x) = \langle x, Mx \rangle / 2$,
\[
\langle DV(x), b(x) \rangle \leq -\varepsilon a_1^2 \tag{9.7}
\]
for $x \in B_2 \setminus S_2(\varepsilon)$, and
\[
\langle DV(x), b(x) \rangle \leq -\frac{1}{8} b_0^2 \tag{9.8}
\]
for $B_2 \setminus (B_0/2)$. By Itô’s formula
\[
dV(X^\varepsilon(t)) = \langle DV(X^\varepsilon(t)), b(X^\varepsilon(t)) \rangle dt + \frac{\varepsilon^2}{2} \text{tr} [\sigma(X^\varepsilon(t))\sigma(X^\varepsilon(t))^T M] dt + \varepsilon^{1/2} \langle DV(X^\varepsilon(t)), \sigma(X^\varepsilon(t))dW(t) \rangle. \tag{9.9}
\]
Starting at $x \in B_1$, we are concerned with the probability
\[
P_x \{ V(X^\varepsilon(t)) \text{ reaches } b_2^2 \text{ before } a_2^2 \varepsilon \},
\]
where $V(x) = b_1^2$. However, according to (9.9) and (9.8), reaching $b_2^2$ before $b_0^2/4$ is a rare event, and its probability decays exponentially in the form $e^{-\kappa/\varepsilon}$ for some $\kappa > 0$ and uniformly in $x \in B_1$. Once the process reaches $B_0/2$, (9.9) and (9.7) imply $V(X^\varepsilon(t))$ is a supermartingale as long as it is in the interval $[0, b_0^2]$, and therefore after reaching $B_0/2$ the probability to reach $a_2^2 \varepsilon$ before $b_0^2$ is greater than $1/2$.

**Proof of Lemma 9.10** Consider a starting position $w \in \partial S_2(\varepsilon)$, and recall that $\tilde{\eta}^\varepsilon(dz|w)$ denotes the hitting distribution on $B_2$ after starting at $w$. Let
\( \theta_k^\varepsilon \) denote the return times to \( \partial S_2(\varepsilon) \) after visiting \( \partial S_1(\varepsilon) \), and let \( q_n^\varepsilon(w) \) denote the probability that the first \( k \) for which \( X^\varepsilon \) visits \( B_2 \) before visiting \( \partial S_1(\varepsilon) \) during \([\theta_k^\varepsilon, \theta_{k+1}^\varepsilon]\) is \( n \). Then by the strong Markov property and using the rescaled process

\[
\int_{B_2} g(z) \eta^\varepsilon(dz|w) = \sum_{n=0}^\infty \int_{B_2} g(z) q_n^\varepsilon(w) \int_{\partial S_2(\varepsilon)} \eta^\varepsilon(dz|u) J^\varepsilon(u) p^{\varepsilon,n}(\varepsilon^{1/2}u|\varepsilon^{1/2}w) du,
\]

where \( J^\varepsilon(u) \) is the Jacobian that accounts for the mapping between \( \partial S_2(\varepsilon) \) and \( \partial S_2(1) \) given by \( u/\varepsilon^{1/2} \). We next use that uniformly in \( w \in \partial S_2(\varepsilon) \)

\[
p^{\varepsilon,n}(\varepsilon^{1/2}u|\varepsilon^{1/2}w) \leq r^\varepsilon(\varepsilon^{1/2}u)[1 + K \alpha^n]
\]

to get

\[
\sum_{n=0}^\infty \int_{B_2} g(z) q_n^\varepsilon(w) \int_{\partial S_2(\varepsilon)} \eta^\varepsilon(dz|u) J^\varepsilon(u) p^{\varepsilon,n}(\varepsilon^{1/2}u|\varepsilon^{1/2}w) du
\]

\[
\leq \left( \sum_{n=0}^\infty \int_{B_2} g(z) q_n^\varepsilon(w) \int_{\partial S_2(\varepsilon)} \eta^\varepsilon(dz|u) J^\varepsilon(u) r^\varepsilon(\varepsilon^{1/2}u) du \right) [1 + K \alpha^n]
\]

\[
= \int_{B_2} g(z) \int_{\partial S_2(\varepsilon)} \eta^\varepsilon(dz|u) J^\varepsilon(u) r^\varepsilon(\varepsilon^{1/2}u) du \left[ 1 + K \sum_{n=0}^\infty q_n^\varepsilon(w) \alpha^n \right].
\]

Now use that \( K \sum_{n=0}^\infty \alpha^n = K/(1 - \alpha) \leq \infty \) and

\[
s^\varepsilon = \sup_{w \in \partial S_2(\varepsilon)} \sup_{n \in \mathbb{N}_0} q_n^\varepsilon(w) \to 0
\]
as \( \varepsilon \to 0 \) to get the upper bound with

\[
\bar{\eta}^\varepsilon(dz) = \int_{\partial S_2(\varepsilon)} \eta^\varepsilon(dz|u) J^\varepsilon(u) r^\varepsilon(\varepsilon^{1/2}u) du.
\]

When combined with the lower bound which has an analogous proof, Lemma 9.10 follows. \( \blacksquare \)

### 9.5 Leaving times with general initial distribution

**Lemma 9.11** There exist \( \tilde{c} > 0 \) and \( \varepsilon_0 \in (0, 1) \) such that for any distribution \( \lambda^\varepsilon \) on \( B_1 \),

\[
P_{\lambda^\varepsilon}(v_1^\varepsilon/E_{\lambda^\varepsilon} v_1^\varepsilon > t) \leq e^{-\tilde{c}t}
\]

for all \( t > 0 \) and \( \varepsilon \in (0, \varepsilon_0) \).
Proof. We first show for any $c \in (0, 1)$ there is $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\lambda^\varepsilon, \theta^\varepsilon \in P(B_1)$

$$\frac{E_{\lambda^\varepsilon} v_1^\varepsilon}{E_{\theta^\varepsilon} v_1^\varepsilon} \geq c.$$  

We use that $v_1^\varepsilon$ can be decomposed into $\bar{v}^\varepsilon + \hat{v}^\varepsilon$, where $\bar{v}^\varepsilon$ is the first hitting time to $B_2 = B_{2\delta}(O_1)$. Since by standard large deviation theory the exponential growth rate of the expected value of $v_1^\varepsilon$ is strictly greater than that of $\bar{v}^\varepsilon$ (uniformly in the initial distribution) $E_{\lambda^\varepsilon} \bar{v}^\varepsilon_1$ (respectively $E_{\theta^\varepsilon} \bar{v}^\varepsilon_1$) is negligible compared to $E_{\lambda^\varepsilon} v_1^\varepsilon$ (respectively $E_{\theta^\varepsilon} v_1^\varepsilon$), and so it is enough to show

$$\frac{E_{\lambda^\varepsilon} \hat{v}^\varepsilon_1}{E_{\theta^\varepsilon} \hat{v}^\varepsilon_1} \geq c.$$  

Owing to Lemma 9.9 (and in particular because $\kappa > 0$) the contribution to either $E_{\lambda^\varepsilon} \hat{v}^\varepsilon_1$ or $E_{\theta^\varepsilon} \hat{v}^\varepsilon_1$ from trajectories that reach $B_2$ before $\partial S_2(\varepsilon)$ can be neglected. Using Lemma 9.10 and the strong Markov property gives

$$\inf_{u_1, u_2 \in \partial S_2(\varepsilon)} \frac{E_{u_1} \hat{v}^\varepsilon_1}{E_{u_2} \hat{v}^\varepsilon_1} \geq \frac{[1 - s^\varepsilon K/(1 - \alpha)]}{[1 + s^\varepsilon K/(1 - \alpha)]},$$

and the lower bound follows since $s^\varepsilon \to 0$.

Hence, there exist $\varepsilon_1 \in (0, 1)$ such that

$$P_{X^\varepsilon}(\bar{v}_1^\varepsilon/E_{X^\varepsilon} v_1^\varepsilon > t) = E_{X^\varepsilon}(P_{X^\varepsilon(\bar{v}_1^\varepsilon)}(\bar{v}_1^\varepsilon/E_{X^\varepsilon} v_1^\varepsilon > t))$$

$$\leq 2P_{u^\varepsilon}(v_1^\varepsilon/E_{X^\varepsilon} v_1^\varepsilon > t)$$

$$\leq 2P_{u^\varepsilon}(v_1^\varepsilon/(E_{u^\varepsilon} v_1^\varepsilon/2) > t)$$

$$\leq 2P_{u^\varepsilon}(v_1^\varepsilon/E_{u^\varepsilon} v_1^\varepsilon > t/2)$$

$$\leq 2e^{-\frac{1}{2}t}$$

for any $t > 0$ and $\varepsilon \in (0, \varepsilon_1)$. Moreover, since as noted previously

$$\lim_{\varepsilon \to 0} \varepsilon \log \left( \frac{E_{X^\varepsilon} v_1^\varepsilon}{E_{X^\varepsilon} \hat{v}^\varepsilon_1} \right) > 0$$

and since by [3] Theorem 4 and Corollary 1 there exists $\varepsilon_2 \in (0, 1)$ such that

$$P_{X^\varepsilon}(\hat{v}_1^\varepsilon/E_{X^\varepsilon} \hat{v}^\varepsilon_1 > t) \leq 2e^{-\frac{1}{2}t}$$

for any $t > 0$ and $\varepsilon \in (0, \varepsilon_2)$, we conclude that for any $t > 0$

$$P_{X^\varepsilon}(\bar{v}_1^\varepsilon/E_{X^\varepsilon} v_1^\varepsilon > t/2) = P_{X^\varepsilon}(\bar{v}_1^\varepsilon/E_{X^\varepsilon} \bar{v}_1^\varepsilon > (t/2) \cdot (E_{X^\varepsilon} v_1^\varepsilon/E_{X^\varepsilon} \bar{v}_1^\varepsilon))$$

$$\leq 2e^{-\frac{1}{2}t}$$

for any $t > 0$ and $\varepsilon \in (0, \varepsilon_2)$.
We conclude that
\[
P_{\lambda^e}(v_1^e/E_{\lambda^e}v_1^e > t) \leq P_{\lambda^e}(\bar{v}_1^e/E_{\lambda^e}v_1^e > t/2) + P_{\lambda^e}(\hat{\nu}_1^e/E_{\lambda^e}v_1^e > t/2)
\]
\[
\leq 4e^{-\frac{t}{4}} \leq e^{-\frac{\varepsilon t}{16}}
\]
for any \( t > 0 \) and \( \varepsilon \in (0, \varepsilon_0) \) with \( \varepsilon_0 = \varepsilon_1 \land \varepsilon_2. \)

**Lemma 9.12** For any distribution \( \lambda^e \) on \( B_1 \), \( v_1^e/E_{\lambda^e}v_1^e \) converges in distribution to an \( \text{Exp}(1) \) random variable under \( P_{\lambda^e} \).

**Proof.** Recall that \( E_{u^e} e^{itv_1^e/E_{u^e}v_1^e} \rightarrow 1/(1 - it) \) as \( \varepsilon \rightarrow 0 \). We would like to show that \( E_{\lambda^e} e^{itv_1^e/E_{\lambda^e}v_1^e} \rightarrow 1/(1 - it) \). Since \( v_1^e = \bar{v}_1^e + \hat{\nu}_1^e \) with \( \bar{v}_1^e \) the first hitting time to \( B_2 = \partial B_{2\delta}(O_1) \), we know that \( E_{\lambda^e} \bar{v}_1^e/E_{\lambda^e}v_1^e \rightarrow 0 \) and thus \( E_{\lambda^e} \hat{\nu}_1^e/E_{\lambda^e}v_1^e \rightarrow 1 \). Observe that
\[
E_{\lambda^e} e^{itv_1^e/E_{\lambda^e}v_1^e} = E_{\lambda^e} \left[ e^{it\bar{v}_1^e/E_{\lambda^e}v_1^e} \cdot E_{X^e(\bar{v}_1^e)} \left( e^{it\hat{\nu}_1^e/E_{\lambda^e}v_1^e} \right) \right],
\]
\[
E_{\lambda^e} \left[ E_{X^e(\bar{v}_1^e)} \left( e^{it\hat{\nu}_1^e/E_{\lambda^e}v_1^e} \right) \right] \leq \frac{1 + s^K(1 - \alpha)}{1 - s^K(1 - \alpha)} E_{u^e} e^{itv_1^e/E_{\lambda^e}v_1^e} \rightarrow 1/(1 - it)
\]
and
\[
E_{\lambda^e} \left[ E_{X^e(\bar{v}_1^e)} \left( e^{it\hat{\nu}_1^e/E_{\lambda^e}v_1^e} \right) \right] \geq \frac{1 - s^K(1 - \alpha)}{1 + s^K(1 - \alpha)} E_{u^e} e^{itv_1^e/E_{\lambda^e}v_1^e} \rightarrow 1/(1 - it).
\]

Since \( E_{\lambda^e} \bar{v}_1^e/E_{\lambda^e}v_1^e \rightarrow 0 \) and \( e^{ix} \) is a bounded and continuous function, a conditioning argument gives
\[
\left| E_{\lambda^e} e^{itv_1^e/E_{\lambda^e}v_1^e} - E_{\lambda^e} \left[ E_{X^e(\bar{v}_1^e)} \left( e^{it\hat{\nu}_1^e/E_{\lambda^e}v_1^e} \right) \right] \right| \leq E_{\lambda^e} \left| e^{it\hat{\nu}_1^e/E_{\lambda^e}v_1^e} - 1 \right| \rightarrow 0.
\]
We conclude that \( E_{\lambda^e} e^{itv_1^e/E_{\lambda^e}v_1^e} \rightarrow 1/(1 - it) \). \hspace{1cm} \Box

### 9.6 Return times

In this subsection, we will extend all the three results to return times.

**Lemma 9.13** There exists \( \delta_0 \in (0, 1) \) such that for any \( \delta \in (0, \delta_0) \) and any distribution \( \lambda^e \) on \( \partial B_{\delta}(O_1) \),

\[
\lim_{\varepsilon \to 0} \varepsilon \log E_{\lambda^e} \tau_1^e = \min_{y \in \bigcup_{k \in \mathbb{N}} \partial B_{\delta}(O_k)} V(O_1, y).
\]
Proof. Since $E_{\lambda^e} \tau_1^e = E_{\lambda^e} v_1^e + E_{\lambda^e} (\tau_1^e - v_1^e)$ and from Lemma 9.2 we know that
\[
\lim_{\varepsilon \to 0} \varepsilon \log E_{\lambda^e} v_1^e = \min_{y \in \cup_{k \in L \setminus \{1\}} \partial B_3(O_k)} V(O_1, y).
\]
Moreover, observe that $W(O_j) > W(O_1)$ for any $j \in L \setminus \{1\}$ due to Condition 3.1 and Remark 6.23. If we apply Remark 6.26 with $W$ for any $\varepsilon \in (0, 1)$ such that for any $\eta = [\min_{j \in L \setminus \{1\}} W(O_j) - W(O_1)]/3$, we find that there exists $\delta_1 \in (0, 1)$ such that for any $\delta \in (0, \delta_1)$
\[
\liminf_{\varepsilon \to 0} -\varepsilon \log \left( \sup_{z \in \partial B_3(O_1)} E_z (\tau_1^e - v_1^e) \right) \\
\geq \min_{j \in L \setminus \{1\}} W(O_j) - W(O_1) - \min_{j \in L \setminus \{1\}} V(O_1, O_j) - \eta \\
= - \min_{j \in L \setminus \{1\}} V(O_1, O_j) + 2\eta.
\]
On the other hand, by continuity of $V(O_1, \cdot)$, for this given $\eta$, there exists $\delta_2 \in (0, 1)$ such that for any $\delta \in (0, \delta_2)$
\[
\min_{y \in \cup_{k \in L \setminus \{1\}} \partial B_3(O_k)} V(O_1, y) > \min_{j \in L \setminus \{1\}} V(O_1, O_j) - \eta.
\]
Thus, for any $\delta \in (0, \delta_0)$ with $\delta_0 = \delta_1 \wedge \delta_2$
\[
\limsup_{\varepsilon \to 0} \varepsilon \log E_{\lambda^e} (\tau_1^e - v_1^e) \leq \limsup_{\varepsilon \to 0} \varepsilon \log \left( \sup_{z \in \partial B_3(O_1)} E_z (\tau_1^e - v_1^e) \right) \\
\leq \min_{j \in L \setminus \{1\}} V(O_1, O_j) - 2\eta \\
< \min_{y \in \cup_{k \in L \setminus \{1\}} \partial B_3(O_k)} V(O_1, y) - \eta \\
= \lim_{\varepsilon \to 0} \varepsilon \log E_{\lambda^e} v_1^e - \eta
\]
and
\[
\lim_{\varepsilon \to 0} \varepsilon \log E_{\lambda^e} \tau_1^e = \lim_{\varepsilon \to 0} \varepsilon \log E_{\lambda^e} v_1^e = \min_{y \in \cup_{k \in L \setminus \{1\}} \partial B_3(O_k)} V(O_1, y).
\]

\[\square\]

Lemma 9.14 Given $\delta > 0$, for any distribution $\lambda^e$ on $\partial B_3(O_1)$, there exist $\tilde{c} > 0$ and $\varepsilon_0 \in (0, 1)$ such that
\[
P_{\lambda^e} (\tau_1^e / E_{\lambda^e} \tau_1^e > t) \leq e^{-\tilde{c} t}
\]
for all $t \geq 1$ and $\varepsilon \in (0, \varepsilon_0)$. 89
Proof. For any $t > 0$, $P_{\lambda^\varepsilon}(\tau_1^\varepsilon/E_{\lambda^\varepsilon}\tau_1^\varepsilon > t) \leq P_{\lambda^\varepsilon}(v_1^\varepsilon/E_{\lambda^\varepsilon}\tau_1^\varepsilon > t/2) + P_{\lambda^\varepsilon}((\tau_1^\varepsilon - v_1^\varepsilon)/E_{\lambda^\varepsilon}\tau_1^\varepsilon > t/2)$. It is easy to see that the first term has this sort of bound due to Lemma 9.11 and $E_{\lambda^\varepsilon}\tau_1^\varepsilon \geq E_{\lambda^\varepsilon}v_1^\varepsilon$.

It suffices to show that this sort of bound holds for the second term, namely, there exists a constant $c > 0$ such that

$$P_{\lambda^\varepsilon} \left( \frac{\tau_1^\varepsilon - v_1^\varepsilon}{E_{\lambda^\varepsilon}\tau_1^\varepsilon} > t \right) \leq e^{-ct}$$

for all $t \in [0, \infty)$ and $\varepsilon$ sufficiently small.

By Chebyshev’s inequality,

$$P_{\lambda^\varepsilon} \left( \frac{\tau_1^\varepsilon - v_1^\varepsilon}{E_{\lambda^\varepsilon}\tau_1^\varepsilon} > t \right) = P_{\lambda^\varepsilon} \left( e^{(\tau_1^\varepsilon - v_1^\varepsilon)/E_{\lambda^\varepsilon}\tau_1^\varepsilon} > e^t \right) \leq e^{-t} E_{\lambda^\varepsilon} e^{(\tau_1^\varepsilon - v_1^\varepsilon)/E_{\lambda^\varepsilon}\tau_1^\varepsilon},$$

and it therefore suffices to prove that $E_{\lambda^\varepsilon} e^{(\tau_1^\varepsilon - v_1^\varepsilon)/E_{\lambda^\varepsilon}\tau_1^\varepsilon}$ is less than a constant for all $\varepsilon$ sufficiently small. Observe that

$$\tau_1^\varepsilon - v_1^\varepsilon = \sum_{j \in L \setminus \{1\}} \sum_{k=1}^{N_j} v_j^\varepsilon(k),$$

where $N_j$ is the number of visits of $j$-th well, and $v_j^\varepsilon(k)$ is the $k$-th copy of the first visit time to $\bigcup_{k \in L \setminus \{j\}} \partial B_\delta(O_k)$ after starting from $\partial B_\delta(O_j)$.

If we consider $\partial B_\delta(O_j)$ as the starting location of a regenerative cycle, as was done previously in the paper for $\partial B_\delta(O_1)$, then there will be a unique stationary distribution, and if the process starts with that as the initial distribution then the times $v_j^\varepsilon(k)$ are independent from each other and from the number of returns to $\partial B_\delta(O_j)$ before first visiting $\partial B_\delta(O_1)$. While these random times as used here do not arise from starting with such a distribution, we can use Lemma 9.10 to bound the error in terms of a multiplicative factor that is independent of $\varepsilon$ for small $\varepsilon > 0$, and thereby justify treating $N_j$ as though it is independent of the $v_j^\varepsilon(k)$.

Recalling that $l = |L|$,

$$E_{\lambda^\varepsilon} e^{(\tau_1^\varepsilon - v_1^\varepsilon)/E_{\lambda^\varepsilon}\tau_1^\varepsilon} = E_{\lambda^\varepsilon} e^{\left( \sum_{j \in L \setminus \{1\}} \sum_{k=1}^{N_j} v_j^\varepsilon(k) \right)/E_{\lambda^\varepsilon}\tau_1^\varepsilon}$$

$$= E_{\lambda^\varepsilon} \left[ \prod_{j \in L \setminus \{1\}} e^{\left( \sum_{k=1}^{N_j} v_j^\varepsilon(k) \right)/E_{\lambda^\varepsilon}\tau_1^\varepsilon} \right]$$

$$\leq \prod_{j \in L \setminus \{1\}} \left( E_{\lambda^\varepsilon} e^{\left( \sum_{k=1}^{N_j} v_j^\varepsilon(k) \right)(l-1)/E_{\lambda^\varepsilon}\tau_1^\varepsilon} \right)^{1/(l-1)},$$

90
where we use the generalized Hölder’s inequality for the last line. Thus, if we can show for each $j \in L \setminus \{1\}$ that $E_{\lambda^c} \exp((\sum_{k=1}^{N_j} v^c_j(k)) (l-1)/E_{\lambda^c} \tau^c_1)$ is less than a constant for all $\varepsilon$ sufficiently small, then we are done.

Such an estimate is straightforward for the case of an unstable equilibrium, i.e., for $j \in L \setminus L_\delta$, and so we focus on the case $j \in L \setminus \{1\}$. For this case, we apply Lemma 9.11 to find that there exists $\tilde{c} > 0$ and $\varepsilon_0 \in (0,1)$ such that for any $j \in L$ and any distribution $\lambda^c$ on $\partial B(O_j)$,

$$P_{\lambda^c}(v_j^c/E_{\lambda^c} v_j^c > t) \leq e^{-\tilde{c} t} \quad (9.10)$$

for any $t > 0$ and $\varepsilon \in (0,\varepsilon_0)$. Hence, given any $\eta > 0$, there is $\bar{\varepsilon}_0 \in (0,\varepsilon_0)$ such that for all $\varepsilon \in (0,\bar{\varepsilon}_0)$ and any $j \in L \setminus \{1\}$

$$E_{\lambda^c} \left[ e^{v_j^c(l-1)/E_{\lambda^c} \tau^c_1} \right] = \int_0^\infty P_{\lambda^c}(e^{(l-1)v_j^c/E_{\lambda^c} \tau^c_1} > t) dt$$

$$\leq 1 + \int_1^\infty P_{\lambda^c}(e^{(l-1)v_j^c/E_{\lambda^c} \tau^c_1} > t) dt$$

$$\leq 1 + \int_1^\infty P_{\lambda^c}((l-1)v_j^c/E_{\lambda^c} \tau^c_1 > \log t) dt$$

$$= 1 + \int_1^\infty P_{\lambda^c} \left( v_j^c/E_{\lambda^c} v_j^c > \frac{1}{(l-1)} \frac{E_{\lambda^c} \tau^c_1}{E_{\lambda^c} v_j^c} \log t \right) dt$$

$$\leq 1 + \int_1^\infty t^{-\frac{E_{\lambda^c} \tau^c_1}{(l-1)E_{\lambda^c} v_j^c}} dt$$

$$= 1 + \frac{\tilde{c} E_{\lambda^c} \tau^c_1}{l-1} \frac{1}{E_{\lambda^c} v_j^c - 1} \leq 1 + \frac{1}{\tilde{c} \left( \frac{E_{\lambda^c} \tau^c_1}{E_{\lambda^c} v_j^c} - \frac{1}{l-1} \right) }$$

$$\leq 1 + \frac{2(l-1)E_{\lambda^c} v_j^c}{\tilde{c} E_{\lambda^c} \tau^c_1} \leq 1 + e^{-\frac{1}{\tilde{c}}(h_1-h_2-\eta)},$$

where the last inequality comes from Lemma 9.12 and Lemma 9.13.

By using induction and a conditioning argument, it follows that for any $\eta > 0$, for any $j \in L \setminus \{1\}$ and for any $n \in \mathbb{N}$,

$$E_{\lambda^c} \left[ e^{\sum_{k=1}^{N_j} v_j^c(k) (l-1)/E_{\lambda^c} \tau^c_1} \right] \leq \left( 1 + e^{-\frac{1}{\tilde{c}}(h_1-h_2-\eta)} \right)^n.$$
The next thing we need to know is the distribution of $N_j$, i.e., $P_{\lambda^\varepsilon}(N_j = n)$ for $n \in \mathbb{N}$. Following a similar argument as in the proof of Lemma 6.4 and the proof of Lemma 6.7 for sufficiently small $\varepsilon > 0$ we find

$$P_{\lambda^\varepsilon}(N_j = n) \leq (1 - q_j)^{n-1} q_j,$$

where

$$q_j = \frac{1 - \sup_{x \in \partial B_\delta(O_j)} P_x(\tilde{T}_j^+ < \tilde{T}_1)}{1 - \sup_{y \in \partial B_\delta(O_j)} p(y, \partial B_\delta(O_j))} = \frac{\inf_{x \in \partial B_\delta(O_j)} P_x(\tilde{T}_1 < \tilde{T}_j^+)}{1 - \sup_{y \in \partial B_\delta(O_j)} p(y, \partial B_\delta(O_j))} \leq e^{-\frac{1}{\varepsilon} (W(O_1) - W(O_1 \cup O_j) - h_j - \eta)}.$$

Therefore,

$$E_{\lambda^\varepsilon} \left[ e^{\left( \sum_{k=1}^{N_j} v_j^\varepsilon(k) \right) (1-1/E_{\lambda^\varepsilon} \tau^\varepsilon)} \right]$$

$$\leq E_{\lambda^\varepsilon} \left[ \left( 1 + e^{-\frac{1}{\varepsilon} (h_1 - h_j - \eta)} \right)^{N_j} \right]$$

$$\leq \sum_{n=1}^{\infty} \left( 1 + e^{-\frac{1}{\varepsilon} (h_1 - h_j - \eta)} \right)^n (1 - q_j)^{n-1} q_j$$

$$= q_j \left( 1 + e^{-\frac{1}{\varepsilon} (h_1 - h_j - \eta)} \right) \sum_{n=1}^{\infty} \left( 1 + e^{-\frac{1}{\varepsilon} (h_1 - h_j - \eta)} \right)^{n-1} (1 - q_j)^{n-1}$$

$$= \frac{q_j \left( 1 + e^{-\frac{1}{\varepsilon} (h_1 - h_j - \eta)} \right)}{1 - \left( 1 + e^{-\frac{1}{\varepsilon} (h_1 - h_j - \eta)} \right) (1 - q_j)}$$

$$= \frac{q_j \left( 1 + e^{-\frac{1}{\varepsilon} (h_1 - h_j - \eta)} \right)}{-e^{-\frac{1}{\varepsilon} (h_1 - h_j - \eta)} + q_j + q_j e^{-\frac{1}{\varepsilon} (h_1 - h_j - \eta)}}$$

$$\leq \frac{q_j \left( 1 + e^{-\frac{1}{\varepsilon} (h_1 - h_j - \eta)} \right)}{q_j / 2} \leq 3,$$

where we use part 3 of Condition 3.11 for the third equality and the third inequality, and the last inequality is due to part 2 of Condition 3.11. This completes the proof. 

92
Lemma 9.15  Given $\delta > 0$, for any distribution $\lambda^\varepsilon$ on $\partial B_\delta(O_1)$, $\tau^\varepsilon_1/E_{\lambda^\varepsilon} \tau^\varepsilon_1$ converges in distribution to an Exp(1) random variable under $P_{\lambda^\varepsilon}$.

Proof. Note that

$$E_{\lambda^\varepsilon}\left(e^{it(\tau^\varepsilon_1/E_{\lambda^\varepsilon} \tau^\varepsilon_1)}\right) = E_{\lambda^\varepsilon}\left(e^{it(v^\varepsilon_1/E_{\lambda^\varepsilon} \tau^\varepsilon_1)}E_{X^\varepsilon}(v^\varepsilon_1)\left(e^{it((\tau^\varepsilon_1-v^\varepsilon_1)/E_{\lambda^\varepsilon} \tau^\varepsilon_1)}\right)\right).$$

Since

$$E_{\lambda^\varepsilon}\left(e^{it(v^\varepsilon_1/E_{\lambda^\varepsilon} \tau^\varepsilon_1)}\right) = E_{\lambda^\varepsilon}\left(e^{it(E_{\lambda^\varepsilon} v^\varepsilon_1/E_{\lambda^\varepsilon} \tau^\varepsilon_1)(v^\varepsilon_1/E_{\lambda^\varepsilon} v^\varepsilon_1)}\right)$$

and we know that $E_{\lambda^\varepsilon}v^\varepsilon_1/E_{\lambda^\varepsilon} \tau^\varepsilon_1 \to 1$ from the proof of Lemma 9.13 by applying Lemma 9.12 we have $E_{\lambda^\varepsilon}(e^{it(v^\varepsilon_1/E_{\lambda^\varepsilon} \tau^\varepsilon_1)}) \to 1/(1-it)$ for any $t \in \mathbb{R}$. Also

$$\left|E_{\lambda^\varepsilon}\left(e^{it(\tau^\varepsilon_1/E_{\lambda^\varepsilon} \tau^\varepsilon_1)}\right) - E_{\lambda^\varepsilon}\left(e^{it(v^\varepsilon_1/E_{\lambda^\varepsilon} \tau^\varepsilon_1)}\right)\right| \leq E_{\lambda^\varepsilon}\left|E_{X^\varepsilon}(v^\varepsilon_1)\left(e^{it((\tau^\varepsilon_1-v^\varepsilon_1)/E_{\lambda^\varepsilon} \tau^\varepsilon_1)}\right) - 1\right|,$$

where the right hand side converges to 0 using $E_{\lambda^\varepsilon}(\tau^\varepsilon_1-v^\varepsilon_1)/E_{\lambda^\varepsilon} \tau^\varepsilon_1 \to 0$ and the dominated convergence theorem. The convergence of $\tau^\varepsilon_1/E_{\lambda^\varepsilon} \tau^\varepsilon_1$ to an Exp(1) random variable under $P_{\lambda^\varepsilon}$ follows.

10 Sketch of the Proof of Conjecture 4.9 for a Special Case

In this section we outline the proof of the upper bound on the decay rate (giving a lower bound on the variance per unit time) that complements Theorem 4.4 for a special case. Consider $V : \mathbb{R} \to \mathbb{R}$ shown as in the graph below.

In particular, assume $V$ is a bounded $C^2$ function satisfying the following conditions:
Condition 10.1  

• $V$ is defined on a compact interval $D = [\bar{x}_L, \bar{x}_R] \subset \mathbb{R}$ and extends periodically as a $C^2$ function.

• $V$ has two local minima at $x_L$ and $x_R$ with values $V(x_L) < V(x_R)$ and $(x_L, x_R) \subset D$

• $V$ has one local maximum at $0 \in (x_L, x_R)$.

• $V(x_L) = 0$, $V(0) = h_L$ and $V(x_R) = h_L - h_R > 0$.

• $\inf_{x \in \partial D} V(x) > h_L$.

Consider the diffusion process $\{X^\varepsilon(t)\}_{t \geq 0}$ satisfying the stochastic differential equation

$$dX^\varepsilon(t) = -\nabla V(X^\varepsilon(t)) \, dt + \sqrt{2\varepsilon} \, dW(t),$$

where $W$ is a 1-dimensional standard Wiener process. Then there are just two stable equilibrium points $O_1 = x_L$ and $O_2 = x_R$, and one unstable equilibrium point $O_3 = 0$. Moreover, one easily finds that $V(O_1, O_2) = h_L$ and $V(O_2, O_1) = h_R$, and these give that $W(O_1) = V(O_2, O_1)$ and $W(O_2) = V(O_1, O_2)$ (since $\mathcal{M}_s = \{1, 2\}$), this implies that $G_s(1) = \{2 \rightarrow 1\}$ and $G_s(2) = \{1 \rightarrow 2\}$. Another observation is that $h = \min_{x \in \partial D_1} V(O_1, x) = \min_{t \in \mathcal{M}_s \setminus \{1\}} V(O_1, O_t) = h_L$ in this model.

If $f \equiv 0$, then one obtains

$$R_1^{(1)} = \inf_{y \in A} V(O_1, y) + W_s(O_1) - W_s(O_1) = \inf_{y \in A} V(O_1, y);$$

$$R_1^{(2)} = 2 \inf_{y \in A} V(O_1, y) - \min_{t \in \mathcal{M}_s \setminus \{1\}} V(O_1, O_t)$$

$$= 2 \inf_{y \in A} V(O_1, y) - h_L;$$

$$R_2^{(1)} = \inf_{y \in A} V(O_2, y) + W_s(O_2) - W_s(O_1)$$

$$= \inf_{y \in A} V(O_2, y) + h_L - h_R;$$

$$R_2^{(2)} = 2 \inf_{y \in A} V(O_2, y) + W_s(O_2) - W_s(O_1) + 0 - W_s(O_1)$$

$$= 2 \inf_{y \in A} V(O_2, y) + h_L - 2h_R.$$
the stationary distribution (the case of bounded and continuous \( f \) can be
dealt with by approximation, as in the case of the lower bound).

**Case I.** If \( x_R \in A \), then \( \inf_{y \in A} V(O_1, y) = h_L \) and \( \inf_{y \in A} V(O_2, y) = 0 \). Thus the decay rate of variance per unit time is bounded below by

\[
\min_{j=1,2} \left[ R_j^{(1)} \land R_j^{(2)} \right] = \min \left\{ R_{1}^{(1)} \land R_{1}^{(2)}, R_{2}^{(1)} \land R_{2}^{(2)} \right\} \\
= \min \left\{ R_{1}^{(1)}, R_{2}^{(2)} \right\} \\
= \min \{ h_L, h_L - 2h_R \} \\
= h_L - 2h_R.
\]

**Case II.** If \( A \subset [0, x_R - \delta] \) for some \( \delta > 0 \) and \( \delta < x_R \), then \( \inf_{y \in A} V(O_1, y) = h_L \) and \( \inf_{y \in A} V(O_2, y) > 0 \) (we denote it by \( b \in (0, h_R) \)). Thus the decay rate of variance per unit time is bounded below by

\[
\min_{j=1,2} \left[ R_j^{(1)} \land R_j^{(2)} \right] = \min \left\{ R_{1}^{(1)} \land R_{1}^{(2)}, R_{2}^{(1)} \land R_{2}^{(2)} \right\} \\
= \min \left\{ R_{1}^{(1)}, R_{2}^{(2)} \right\} \\
= \min \{ h_L, h_L + 2(b - h_R) \} \\
= h_L + 2(b - h_R).
\]

**Case III.** If \( A \subset [x_R + \delta, x^*] \) with \( V(x^*) = h_L \) for some \( \delta > 0 \) and \( \delta < x^* - x_R \), then \( \inf_{y \in A} V(O_1, y) = h_L + \inf_{y \in A} V(O_2, y) \) and \( \inf_{y \in A} V(O_2, y) > 0 \) (we denote it by \( b \in (0, h_R) \)). Thus the decay rate of variance per unit time is bounded below by

\[
\min_{j=1,2} \left[ R_j^{(1)} \land R_j^{(2)} \right] = \min \left\{ R_{1}^{(1)} \land R_{1}^{(2)}, R_{2}^{(1)} \land R_{2}^{(2)} \right\} \\
= \min \left\{ R_{1}^{(1)}, R_{2}^{(2)} \right\} \\
= \min \{ h_L + b, h_L + 2(b - h_R) \} \\
= h_L + 2(b - h_R).
\]

**Case IV.** If \( A \subset [x^*, \infty) \) with \( V(x^*) = h_L \) and \( x^* > x_R \), then \( \inf_{y \in A} V(O_1, y) = h_L + \inf_{y \in A} V(O_2, y) \) and \( \inf_{y \in A} V(O_2, y) > 0 \) (we denote it by \( b > h_R \)).
Thus the decay rate of variance per unit time is bounded below by
\[
\min_{j=1,2} \left[ R_j^{(1)} \land R_j^{(2)} \right] = \min \left\{ R_1^{(1)} \land R_1^{(2)}, R_2^{(1)} \land R_2^{(2)} \right\} \\
= \min \left\{ R_1^{(1)}, R_1^{(2)} \right\} \\
= \min \{ h_L + \bar{b}, h_L + (\bar{b} - h_R) \} \\
= h_L + (\bar{b} - h_R).
\]

To find an upper bound for the decay rate of variance per unit time, we recall that
\[
\frac{1}{T^\varepsilon} \int_0^{T^\varepsilon} 1_A(X^\varepsilon_t) \, dt \leq \frac{1}{T^\varepsilon} \sum_{j=1}^{N^\varepsilon(T^\varepsilon)-1} \int_{\tau_j^\varepsilon}^{\tau_{j-1}^\varepsilon} 1_A(X^\varepsilon_t) \, dt \leq \frac{1}{T^\varepsilon} \int_0^{T^\varepsilon} 1_A(X^\varepsilon_t) \, dt
\]
with \( \tau_j^\varepsilon \) being the \( j \)-th regenerative cycle. In Case I, one might guess that
\[
\int_{\tau_{j-1}^\varepsilon}^{\tau_j^\varepsilon} 1_A(X^\varepsilon_t) \, dt \quad (10.2)
\]
has approximately the same distribution as the exit time from the shallow well, which has been shown to asymptotically have an exponential distribution with parameter \( \exp(-h_R/\varepsilon) \). Additionally, since the exit time from the shallower well is quantity is exponentially smaller than \( \tau_j^\varepsilon \), it suggests that the random variables \([10.2]\) can be taken as independent of \( N^\varepsilon(T^\varepsilon) \) when \( \varepsilon \) is small. We also know that
\[
\frac{E N^\varepsilon(T^\varepsilon)}{T^\varepsilon} \approx \frac{1}{E \tau_1^\varepsilon} \approx \exp \left( -h_L(\delta)/\varepsilon \right),
\]
where \( h_L(\delta) \uparrow h_L \) as \( \delta \downarrow 0 \) and \( \approx \) means that quantities on either side have the same exponential decay rate. Using Wald’s identity and Jensen’s inequality
to argue that $E[N^\varepsilon(T^\varepsilon)]^2 \geq [EN^\varepsilon(T^\varepsilon)]^2$, we obtain

\[
T^\varepsilon \text{Var} \left( \frac{1}{T^\varepsilon} \int_0^{T^\varepsilon} 1_A(X^\varepsilon_t) \, dt \right) \\
\approx \frac{1}{T^\varepsilon} E \left[ \sum_{j=1}^{N^\varepsilon(T^\varepsilon)} \int_{r^j_{j-1}}^{r^j_j} 1_A(X^\varepsilon_t) \, dt - EN^\varepsilon(T^\varepsilon) E \left( \int_{r^j_{j-1}}^{r^j_j} 1_A(X^\varepsilon_t) \, dt \right) \right]^2 \\
= \frac{1}{T^\varepsilon} E \left[ \sum_{j=1}^{N^\varepsilon(T^\varepsilon)} \left( \int_{r^j_{j-1}}^{r^j_j} 1_A(X^\varepsilon_t) \, dt \right)^2 \right] - \frac{1}{T^\varepsilon} (E(N^\varepsilon(T^\varepsilon)))^2 \left( E \left( \int_{r^j_{j-1}}^{r^j_j} 1_A(X^\varepsilon_t) \, dt \right) \right)^2 \\
= \frac{1}{T^\varepsilon} EN^\varepsilon(T^\varepsilon) E \left( \int_{r^j_{j-1}}^{r^j_j} 1_A(X^\varepsilon_t) \, dt \right)^2 \\
+ \frac{1}{T^\varepsilon} \left( E[N^\varepsilon(T^\varepsilon)]^2 - EN^\varepsilon(T^\varepsilon) \right) \left( E \left( \int_{r^j_{j-1}}^{r^j_j} 1_A(X^\varepsilon_t) \, dt \right) \right)^2 \\
- \frac{1}{T^\varepsilon} [EN^\varepsilon(T^\varepsilon)]^2 \left( E \left( \int_{r^j_{j-1}}^{r^j_j} 1_A(X^\varepsilon_t) \, dt \right) \right)^2 \\
\geq \frac{EN^\varepsilon(T^\varepsilon)}{T^\varepsilon} \text{Var} \left( \int_{r^j_{j-1}}^{r^j_j} 1_A(X^\varepsilon_t) \, dt \right) \\
\approx \exp \left( -h_L(\delta)/\varepsilon \right) \cdot \exp(2h_R/\varepsilon) \\
= \exp((2h_R - h_L(\delta))/\varepsilon).
\]

Letting $\delta \to 0$, we see that the decay rate of variance per unit time is bounded above by $h_L - 2h_R$, which is the same as lower bound found for Case I.

For the other three Cases II, III and IV, the process spends only a very small fraction of the time while in the shallower well in the set $A$. In fact, using the stopping time arguments of the sort that appear in [8, Chapter 4], the event that the process enters $A$ during an excursion away from the neighborhood of $x_R$ can be accurately approximated (as far as large deviation behavior goes) using independent Bernoulli random variables $\{B^\varepsilon_i\}$ with success parameter $e^{-b/\varepsilon}$, and when this occurs the process spends an order one amount of time in $A$ before returning to the neighborhood of $x_R$. There is however another sequence of independent Bernoulli random variables with success parameter $h_R$, and the process accumulates time in $A$ only up till the time of first success of this sequence.
Then
\[ \text{Var} \left( \int_{t_{j-1}}^{t_j} 1_A (X_t^\varepsilon) \, dt \right) \]
has the same logarithmic asymptotics as
\[ \text{Var} \left( \sum_{i=1}^{R^\varepsilon} 1_{B_i^\varepsilon = 1} \right), \]
where \( R^\varepsilon \) is geometric with success parameter \( e^{-h_R/\varepsilon} \) and independent of the \( \{B_i^\varepsilon\} \). Straightforward calculation using Wald’s identity then gives the exponential rate of decay \( 2h_R - 2b \), and so we obtain
\[
T^\varepsilon \text{Var} \left( \frac{1}{T^\varepsilon} \int_0^{T^\varepsilon} 1_A (X_t^\varepsilon) \, dt \right) \geq \frac{E N^\varepsilon (T^\varepsilon)}{T^\varepsilon} \text{Var} \left( \int_{t_{j-1}}^{t_j} 1_A (X_t^\varepsilon) \, dt \right) \\
\approx \exp (-h_L/\varepsilon) \cdot \exp(2h_R/\varepsilon) \cdot \exp (-2b/\varepsilon) \\
= \exp \left( (2 (h_R - b) - h_L) / \varepsilon \right).
\]
This means that the decay rate of variance per unit time is bounded above by \( h_L + 2(b - h_R) \) which is again the same as lower bound.

11 Appendix

Proof of Lemma 6.10. By definition, \( \tau_j^\varepsilon = \tau_N^\varepsilon \) and observe that
\[
\int_0^{\tau_N^\varepsilon} g (X_s^\varepsilon) \, ds = \sum_{\ell=1}^N \int_{\tau_{\ell-1}}^{\tau_\ell} g (X_s^\varepsilon) \, ds = \sum_{\ell=1}^\infty \int_{\tau_{\ell-1}}^{\tau_\ell} g (X_s^\varepsilon) \, ds \cdot 1_{\{\ell \leq N\}} \\
= \sum_{\ell=1}^\infty \int_{\tau_{\ell-1}}^{\tau_\ell} g (X_s^\varepsilon) \, ds \cdot 1_{\{\ell \leq N\}} + \sum_{\ell=1}^\infty \int_{\tau_{\ell-1}}^{\tau_\ell} g (X_s^\varepsilon) \, ds \cdot 1_{\{\tilde{N} + 1 \leq \ell \leq N\}} \\
= \sum_{\ell=1}^\infty \int_{\tau_{\ell-1}}^{\tau_\ell} g (X_s^\varepsilon) \, ds \cdot 1_{\{\ell \leq \tilde{N}\}} \\
+ \sum_{j \in \mathcal{L} \setminus \{1\}} \sum_{\ell=1}^\infty \left( \int_{\tau_{\ell-1}}^{\tau_\ell} g (X_s^\varepsilon) \, ds \cdot 1_{\{\tilde{N} + 1 \leq \ell \leq N, Z_{\ell-1} \in \partial B_{\delta}(O_j)\}} \right).
\]
Since \( \tilde{N} \) and \( N \) are stopping times with respect to filtration \( \{\mathcal{G}_n\}_n \), it implies that
\[
\{ \ell \leq \tilde{N} \} = \left( \tilde{N} \leq \ell - 1 \right)^c \in \mathcal{G}_{\ell-1}
\]
and
and 
\[ \left\{ \hat{N} + 1 \leq \ell \leq N, Z_{\ell-1} \in \partial B_\delta(O_j) \right\} \in \mathcal{G}_{\ell-1}. \]

Let 
\[ \mathcal{S}_1 = \sum_{\ell=1}^{\infty} \int_{\tau_{\ell-1}}^{\tau_\ell} g(X_\varepsilon^s) \, ds \cdot 1_{\{\ell \leq \hat{N}\}} \]

and 
\[ \mathcal{S}_j = \sum_{\ell=1}^{\infty} \left( \int_{\tau_{\ell-1}}^{\tau_\ell} g(X_\varepsilon^s) \, ds \cdot 1_{\{\hat{N}+1 \leq \ell \leq N, Z_{\ell-1} \in \partial B_\delta(O_j)\}} \right) \]

for all \( j \in L \setminus \{1\} \). We find

\[
E_x(\mathcal{S}_1) = E_x \left( \sum_{\ell=1}^{\infty} \int_{\tau_{\ell-1}}^{\tau_\ell} g(X_\varepsilon^s) \, ds \cdot 1_{\{\ell \leq \hat{N}\}} \right)
\]

\[
= \sum_{\ell=1}^{\infty} E_x \left[ E_x \left( \int_{\tau_{\ell-1}}^{\tau_\ell} g(X_\varepsilon^s) \, ds \cdot 1_{\{\ell \leq \hat{N}\}} \right) \left| \mathcal{G}_{\ell-1} \right\} \right]
\]

\[
= \sum_{\ell=1}^{\infty} E_x \left( 1_{\{\ell \leq \hat{N}\}} E_{Z_{\ell-1}} \left[ \int_{\tau_{\ell-1}}^{\tau_\ell} g(X_\varepsilon^s) \, ds \right] \right)
\]

\[
\leq \sup_{y \in \partial B_\delta(O_1)} E_y \left[ \int_{\tau_{\ell-1}}^{\tau_\ell} g(X_\varepsilon^s) \, ds \right] \cdot \left( \sum_{\ell=1}^{\infty} P_x \left( \hat{N} \geq \ell \right) \right).
\]

In addition, for \( j \in L \setminus \{1\} \),

\[
E_x(\mathcal{S}_j) = \sum_{\ell=1}^{\infty} E_x \left( \int_{\tau_{\ell-1}}^{\tau_\ell} g(X_\varepsilon^s) \, ds \cdot 1_{\{\hat{N}+1 \leq \ell \leq N, Z_{\ell-1} \in \partial B_\delta(O_j)\}} \right)
\]

\[
= \sum_{\ell=1}^{\infty} E_x \left[ E_x \left( \int_{\tau_{\ell-1}}^{\tau_\ell} g(X_\varepsilon^s) \, ds \cdot 1_{\{\hat{N}+1 \leq \ell \leq N, Z_{\ell-1} \in \partial B_\delta(O_j)\}} \right) \left| \mathcal{G}_{\ell-1} \right\} \right]
\]

\[
= \sum_{\ell=1}^{\infty} E_x \left( 1_{\{\hat{N}+1 \leq \ell \leq N, Z_{\ell-1} \in \partial B_\delta(O_j)\}} E_{Z_{\ell-1}} \left[ \int_{\tau_{\ell-1}}^{\tau_\ell} g(X_\varepsilon^s) \, ds \right] \right)
\]

\[
\leq \sup_{y \in \partial B_\delta(O_j)} E_y \left[ \int_{\tau_{\ell-1}}^{\tau_\ell} g(X_\varepsilon^s) \, ds \right] \cdot \left( \sum_{\ell=1}^{\infty} E_x \left( 1_{\{\hat{N}+1 \leq \ell \leq N, Z_{\ell-1} \in \partial B_\delta(O_j)\}} \right) \right).
\]

It is straightforward to see that \( \hat{N} = N_1 \). This implies that

\[
\sum_{\ell=1}^{\infty} P_x \left( \hat{N} \geq \ell \right) = E_x \hat{N} = E_x N_1.
\]
Moreover, observe that for any \( j \in L \setminus \{1\} \),
\[
\sum_{\ell = 1}^{\infty} E_x \left( 1_{\{N + 1 \leq \ell \leq N, Z_{\ell - 1} \in \partial B_\delta(O_j)\}} \right) = E_x N_j.
\]
Hence,
\[
E_x \left( \int_0^{\tau_N} g(X_s^x) \, ds \right) = \sum_{j \in L} E_x (\mathcal{G}_j)
\leq \sum_{j \in L} \left[ \sup_{y \in \partial B_\delta(O_j)} E_y \left( \int_0^{\tau_1} g(X_s^x) \, ds \right) \right] E_x N_j.
\]

**Proof of Lemma 6.11** For any \( j \in L \) and \( n \in \mathbb{N} \), we define \( \xi_n^{(j)} = \inf\{k \in \mathbb{N}_0 : Z_k \in \partial B_\delta(O_j)\} \), \( \xi_n^{(j)} = \inf\{k \in \mathbb{N} : k > \xi_n^{(j)} \text{ and } Z_k \in \partial B_\delta(O_j)\} \) i.e. \( \xi_n^{(j)} \) is the \( n \)th time of hitting \( \partial B_\delta(O_j) \). Moreover, we define \( N^{(j)} = \inf\{n \in \mathbb{N} : \xi_n^{(j)} \geq N\} \), recalling that \( N = \inf\{n \geq \hat{N} : Z_n \in \partial B_\delta(O_1)\} \) and \( \hat{N} = \inf\{n \in \mathbb{N} : Z_n \in \bigcup_{j \in L \setminus \{1\}} \partial B_\delta(O_j)\} \). Since \( \xi_n^{(j)} \) is a stopping time with respect to \( \{\mathcal{G}_n\}_n \), we can define the filtration \( \{\mathcal{G}_{\xi_n^{(j)}}\}_n \), and one can verify that \( N^{(j)} \) is a stopping time with respect to \( \{\mathcal{G}_{\xi_n^{(j)}}\}_n \). With these notations, we can write
\[
\int_0^{\tau_N} g(X_s) \, ds = \sum_{j \in L} \sum_{\ell = 1}^{\infty} \int_{\tau_{\xi_{\ell}^{(j)}}}^{\tau_{\xi_{\ell}^{(j)} + 1}} g(X_s) \, ds \cdot 1_{\{\ell \leq N^{(j)} - 1\}}.
\]
Since \((x_1 + \cdots + x_l)^2 \leq l(x_1^2 + \cdots + x_l^2)\) for any \((x_1, \ldots, x_l) \in \mathbb{R}^l\) and \(l \in \mathbb{N}\),
\[
\left( \int_0^{\tau_N} g(X_s) \, ds \right)^2 = \left( \sum_{j \in L} \sum_{\ell = 1}^{\infty} \int_{\tau_{\xi_{\ell}^{(j)}}}^{\tau_{\xi_{\ell}^{(j)} + 1}} g(X_s) \, ds \cdot 1_{\{\ell \leq N^{(j)} - 1\}} \right)^2
\leq l \sum_{j \in L} \left( \sum_{\ell = 1}^{\infty} \int_{\tau_{\xi_{\ell}^{(j)}}}^{\tau_{\xi_{\ell}^{(j)} + 1}} g(X_s) \, ds \cdot 1_{\{\ell \leq N^{(j)} - 1\}} \right)^2.
\]
Thus, \[
\sum_{\ell=1}^{\infty} \int_{\tau_{\xi_{\ell}}^{(j)}}^{\tau_{\xi_{\ell}}^{(j)}+1} g(X_s) \, ds \cdot 1_{\{\ell \leq N^{(j)}-1\}}^2
\]
\[
\begin{aligned}
&= \sum_{\ell=1}^{\infty} \left( \int_{\tau_{\xi_{\ell}}^{(j)}}^{\tau_{\xi_{\ell}}^{(j)}+1} g(X_s) \, ds \right)^2 1_{\{\ell \leq N^{(j)}-1\}} \\
&= 2 \sum_{\ell=2}^{\infty} \sum_{k=1}^{\ell-1} \int_{\tau_{\xi_{\ell}}^{(j)}}^{\tau_{\xi_{\ell}}^{(j)}+1} g(X_s) \, ds \cdot 1_{\{\ell \leq N^{(j)}-1\}} \int_{\tau_{\xi_{\ell}}^{(j)}}^{\tau_{\xi_{\ell}}^{(j)}+1} g(X_s) \, ds \cdot 1_{\{k \leq N^{(j)}-1\}}.
\end{aligned}
\]

For the expected value of the first sum, note that \(\{\ell \leq N^{(j)}-1\} = \{N^{(j)} \leq \ell\}^c \in \mathcal{G}_{\xi_{\ell}^{(j)}}\), we have
\[
\sum_{\ell=1}^{\infty} E_x \left[ \left( \int_{\tau_{\xi_{\ell}}^{(j)}}^{\tau_{\xi_{\ell}}^{(j)}+1} g(X_s) \, ds \right)^2 1_{\{\ell \leq N^{(j)}-1\}} \right]
\]
\[
= \sum_{\ell=1}^{\infty} E_x \left[ 1_{\{\ell \leq N^{(j)}-1\}} E_x \left[ \left( \int_{\tau_{\xi_{\ell}}^{(j)}}^{\tau_{\xi_{\ell}}^{(j)}+1} g(X_s) \, ds \right)^2 \mathcal{G}_{\xi_{\ell}^{(j)}} \right] \right]
\]
\[
= \sum_{\ell=1}^{\infty} E_x \left[ 1_{\{\ell \leq N^{(j)}-1\}} E_{\mathcal{G}_{\xi_{\ell}^{(j)}}} \left( \int_{\tau_{\xi_{\ell}}^{(j)}}^{\tau_{\xi_{\ell}}^{(j)}+1} g(X_s) \, ds \right)^2 \right]
\]
\[
\leq \sup_{y \in \partial B_3(O_j)} E_y \left( \int_{0}^{\tau_1} g(X_s) \, ds \right)^2 \sum_{\ell=1}^{\infty} P_x \left( N^{(j)} - 1 \geq \ell \right).
\]

In addition, since \(N^{(j)} - 1 = N_j\) (recall that \(N_j\) is the number of visits of \(\{Z_n\}_{n \in \mathbb{N}_0}\) to \(\partial B_3(O_j)\) before \(N\) including the initial position) this implies that
\[
\sum_{\ell=1}^{\infty} P_x \left( N^{(j)} - 1 \geq \ell \right) = \sum_{\ell=1}^{\infty} P_x \left( N_j \geq \ell \right) = E_x \left( N_j \right).
\]

Thus,
\[
\sum_{\ell=1}^{\infty} E_x \left[ \left( \int_{\tau_{\xi_{\ell}}^{(j)}}^{\tau_{\xi_{\ell}}^{(j)}+1} g(X_s) \, ds \right)^2 1_{\{\ell \leq N^{(j)}-1\}} \right] \leq \sup_{y \in \partial B_3(O_j)} E_y \left( \int_{0}^{\tau_1} g(X_s) \, ds \right)^2 E_x \left( N_j \right).
\]
Turning to the expected value of the second sum,

\[
\sum_{\ell=2}^{\infty} \sum_{k=1}^{\ell-1} E_x \left[ \int_{T^{(j)+1}_{\xi_{\ell}}} g(X_s) \, ds \cdot 1_{\{\ell \leq N^{(j)}-1\}} \int_{T^{(j)+1}_{\xi_{k}}} g(X_s) \, ds \cdot 1_{\{k \leq N^{(j)}-1\}} \right]
\]

\[
= \sum_{\ell=2}^{\infty} \sum_{k=1}^{\ell-1} E_x \left[ \int_{T^{(j)+1}_{\xi_{\ell}}} g(X_s) \, ds \cdot G_{\xi_{\ell}} \right] 1_{\{\ell \leq N^{(j)}-1\}} \int_{T^{(j)+1}_{\xi_{k}}} g(X_s) \, ds \cdot 1_{\{k \leq N^{(j)}-1\}}
\]

\[
\leq \sup_{y \in \partial B_j(O_j)} E_y \left( \int_{0}^{T^{(j)+1}} g(X_s) \, ds \right) \sum_{\ell=2}^{\infty} \sum_{k=1}^{\ell-1} E_x \left[ 1_{\{\ell \leq N^{(j)}-1\}} \int_{T^{(j)+1}_{\xi_{k}}} g(X_s) \, ds \cdot 1_{\{k \leq N^{(j)}-1\}} \right].
\]

Now since for any \( k \leq \ell - 1 \), i.e. \( k + 1 \leq \ell \)

\[
\int_{T^{(j)+1}_{\xi_{k}}} g(X_s) \, ds \in G_{\xi_{k}^{(j)+1}} \text{ and } 1_{\{\ell \leq N^{(j)}-1\}} \in G_{\xi_{\ell}^{(j)}},
\]
we have

\[
E_x \left[ 1\{ \ell \leq N(j) - 1 \} \int_{\tau_{\xi_k}^{(j)}}^{\tau_{\xi_k}^{(j)} + 1} g(X_s) \, ds \cdot 1\{ k \leq N(j) - 1 \} \right] \\
= E_x \left[ \int_{\tau_{\xi_k}^{(j)}}^{\tau_{\xi_k}^{(j)} + 1} g(X_s) \, ds \cdot 1\{ \tau_{\xi_k}^{(j)} < N(j), \ldots, \tau_{\xi_k}^{(j)} < N(j) \} \right] \\
= E_x \left[ E_G \left[ 1\{ \tau_{\xi_k}^{(j)} < N(j), \ldots, \tau_{\xi_k}^{(j)} < N(j) \} \int_{\tau_{\xi_k}^{(j)}}^{\tau_{\xi_k}^{(j)} + 1} g(X_s) \, ds \cdot 1\{ \tau_{\xi_k}^{(j)} < N(j), \ldots, \tau_{\xi_k}^{(j)} < N(j) \} \right] ight] \\
= E_x \left[ E_G \left[ 1\{ \tau_{\xi_k}^{(j)} < N(j), \ldots, \tau_{\xi_k}^{(j)} < N(j) \} \int_{\tau_{\xi_k}^{(j)}}^{\tau_{\xi_k}^{(j)} + 1} g(X_s) \, ds \cdot 1\{ \tau_{\xi_k}^{(j)} < N(j), \ldots, \tau_{\xi_k}^{(j)} < N(j) \} \right] ight] \\
\leq \sup_{y \in \partial B_1(O_j)} P_y ( \ell - k \leq N(j) - 1 ) E_x \left[ \int_{\tau_{\xi_k}^{(j)}}^{\tau_{\xi_k}^{(j)} + 1} g(X_s) \, ds \cdot 1\{ k \leq N(j) - 1 \} \right] \\
= \sup_{y \in \partial B_1(O_j)} P_y ( \ell - k \leq N_j ) E_x \left[ E_G \left[ \int_{\tau_{\xi_k}^{(j)}}^{\tau_{\xi_k}^{(j)} + 1} g(X_s) \, ds \cdot 1\{ \tau_{\xi_k}^{(j)} < N(j), \ldots, \tau_{\xi_k}^{(j)} < N(j) \} \right] \right] \\
\leq \sup_{y \in \partial B_1(O_j)} E_y \left( \int_0^{\tau_j} g(X_s) \, ds \right) \cdot \sup_{y \in \partial B_1(O_j)} P_y ( \ell - k \leq N_j ) \cdot P_x ( k \leq N_j ).
Therefore, putting the estimates together gives

\[ \sum_{\ell=2}^{\infty} \sum_{k=1}^{\ell-1} E_x \left[ \int_{\tau_{\ell_k}^{(j)}}^{\tau_{\ell_k}^{(j)+1}} g(X_s) \, ds \cdot 1_{\{\ell \leq N^{(j)}-1\}} \int_{\tau_{\ell_k}^{(j)}}^{\tau_{\ell_k}^{(j)+1}} g(X_s) \, ds \cdot 1_{\{k \leq N^{(j)}-1\}} \right] \]

\[ \leq \sup_{y \in \partial B \setminus (O_j)} E_y \left( \int_0^{\tau_1} g(X_s) \, ds \right)^2 \sum_{\ell=2}^{\infty} \sum_{k=1}^{\ell-1} E_x \left[ 1_{\{\ell \leq N^{(j)}-1\}} \int_{\tau_{\ell_k}^{(j)}}^{\tau_{\ell_k}^{(j)+1}} g(X_s) \, ds \cdot 1_{\{k \leq N^{(j)}-1\}} \right] \]

\[ \leq \left( \sup_{y \in \partial B \setminus (O_j)} E_y \left( \int_0^{\tau_1} g(X_s) \, ds \right) \right)^2 \sum_{\ell=2}^{\infty} \sum_{k=1}^{\ell-1} \sup_{y \in \partial B \setminus (O_j)} P_y (\ell - k \leq N_j) \cdot P_x (k \leq N_j) \]

\[ = \left( \sup_{y \in \partial B \setminus (O_j)} E_y \left( \int_0^{\tau_1} g(X_s) \, ds \right) \right)^2 \sum_{\ell=2}^{\infty} \sum_{k=1}^{\ell-1} \sup_{y \in \partial B \setminus (O_j)} P_y (\ell - k \leq N_j) \cdot \sum_{k=1}^{\infty} P_x (k \leq N_j) \]

\[ = \left( \sup_{y \in \partial B \setminus (O_j)} E_y \left( \int_0^{\tau_1} g(X_s) \, ds \right) \right)^2 \sum_{\ell=1}^{\infty} \sup_{y \in \partial B \setminus (O_j)} P_y (\ell \leq N_j) \cdot E_x N_j. \]

Therefore, putting the estimates together gives

\[ E_x \left( \int_0^{\tau_1} g(X_s) \, ds \right)^2 \]

\[ \leq \sum_{j \in L} E_x \left( \sum_{\ell=1}^{\infty} \int_{\tau_{\ell}^{(j)}}^{\tau_{\ell}^{(j)+1}} g(X_s) \, ds \cdot 1_{\{\ell \leq N^{(j)}-1\}} \right)^2 \]

\[ \leq \sum_{\ell=1}^{\infty} E_x \left[ \left( \int_{\tau_{\ell}^{(j)}}^{\tau_{\ell}^{(j)+1}} g(X_s) \, ds \right)^2 1_{\{\ell \leq N^{(j)}-1\}} \right] \]

\[ + 2l \sum_{\ell=2}^{\ell-1} \sum_{k=1}^{k-1} E_x \left[ \int_{\tau_{\ell_k}^{(j)}}^{\tau_{\ell_k}^{(j)+1}} g(X_s) \, ds \cdot 1_{\{\ell \leq N^{(j)}-1\}} \int_{\tau_{\ell_k}^{(j)}}^{\tau_{\ell_k}^{(j)+1}} g(X_s) \, ds \cdot 1_{\{k \leq N^{(j)}-1\}} \right] \]

\[ \leq \sum_{j \in L} \sup_{y \in \partial B \setminus (O_j)} E_y \left( \int_0^{\tau_1} g(X_s) \, ds \right)^2 \cdot E_x N_j \]

\[ + 2l \sum_{j \in L} \sup_{y \in \partial B \setminus (O_j)} E_y \left( \int_0^{\tau_1} g(X_s) \, ds \right)^2 \cdot E_x N_j \cdot \sum_{\ell=1}^{\infty} \sup_{y \in \partial B \setminus (O_j)} P_y (\ell \leq N_j). \]
References

[1] D. Aldous and J. Fill. Reversible markov chains and random walks on graphs, 2002. Unfinished monograph, recompiled 2014.

[2] A. Budhiraja and P. Dupuis. Analysis and Approximation of Rare Events: Representations and Weak Convergence Methods. Number 94 in Probability Theory and Stochastic Modelling. Springer-Verlag, New York, 2019.

[3] P. Collet, S. Martínez, and J. San Martín. Quasi-Stationary Distributions. Springer-Verlag, Berlin, 2013.

[4] M.V. Day. On the exponential exit law in the small parameter exit problem. Stochastics, 8(4):297–323, 1983.

[5] P. Dupuis, Y. Liu, N. Plattner, and J.D. Doll. On the infinite swapping limit for parallel tempering. SIAM J. Multiscale Model. Simul., 10:986–1022, 2012.

[6] P. Dupuis and G.-J. Wu. Analysis and optimization of certain parallel Monte Carlo methods in the low temperature limit. page working paper, 2020.

[7] W. Feller. An Introduction to Probability Theory and Its Applications, Vol. 2. John Wiley, New York, 1971.

[8] M. I. Freidlin and A. D. Wentzell. Random Perturbations of Dynamical Systems. Springer Berlin Heidelberg, second edition, 2012.

[9] T.E. Harris. The Theory of Branching Processes. Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen. Springer-Verlag, 1963.

[10] P. Dupuis J. Doll and P. Nyquist. A large deviations analysis of certain qualitative properties of parallel tempering and infinite swapping algorithms. Applied Math. And Opt., 78:103–144, 2018.

[11] R. Khasminskii. Stochastic Stability of Differential Equations. Springer Berlin Heidelberg, second edition, 2012.
[12] N. Limnios and G. Oprisan. *Semi-Markov Processes and Reliability*. Statistics for Industry and Technology. Birkhäuser Boston, 2001.

[13] S. Ross. *Applied Probability Models with Optimization Applications*. Dover Publications, 1992.

[14] A. Shwartz and A. Weiss. *Large Deviations for Performance Analysis: Queues, Communication and Computing*. Chapman and Hall, New York, 1995.