ALMOST ALL $k$-COP-WIN GRAPHS CONTAIN A DOMINATING SET OF CARDINALITY $k$

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Abstract. We consider $k$-cop-win graphs in the binomial random graph $G(n, 1/2)$. It is known that almost all cop-win graphs contain a universal vertex. We generalize this result and prove that for every $k \in \mathbb{N}$, almost all $k$-cop-win graphs contain a dominating set of cardinality $k$. From this it follows that the asymptotic number of labelled $k$-cop-win graphs of order $n$ is equal to $(1+o(1))(1-2^{-k})^{-k}2^n/2-(1/2-\log_2(1-2^{-k}))/n$.

1. Introduction

Cops and Robbers is vertex-pursuit game played on a reflexive graph. There are two players consisting of a set of cops and a single robber. The game is played over a sequence of discrete time-steps or rounds, with the cops going first in the first round and then playing on alternate time-steps. The cops and robber occupy vertices. When a player is ready to move in a round, they must move to a neighbouring vertex. Because of the loops, players can pass, or remain on their own vertex. Observe that any subset of cops may move in a given round. The cops win if after some finite number of rounds, one of them can occupy the same vertex as the robber. This is called a capture. The robber wins if he can evade capture indefinitely. A winning strategy for the cops is a set of rules that, if followed, result in a win for the cops. A winning strategy for the robber is defined analogously.

If we place a cop at each vertex, then the cops are guaranteed to win. Therefore, the minimum number of cops required to win in a graph $G$ is a well-defined positive integer, named the cop number (or copnumber) of the graph $G$. We write $c(G)$ for the cop number of a graph $G$. If $c(G) \leq k$, then we say $G$ is $k$-cop-win. In the special case $k = 1$, we say $G$ is cop-win (or copwin). Nowakowski and Winkler [9], and independently Quilliot [13], considered the game with one cop only; the introduction of the cop number came in [1]. Many papers have now been written on cop number since these three early works; see the monograph [5].

Since their introduction, the structure of cop-win graphs has been relatively well-understood. In [9 13 14] a kind of ordering of the vertex set—now called a cop-win or elimination ordering—was introduced which completely characterizes such graphs. If $u$ is a vertex, then the closed neighbour set of $u$, written $N[u]$, consists of $u$ along with the neighbours of $u$. A vertex $u$ is a corner if there is some vertex $v$, $v \neq u$, such that $N[u] \subseteq N[v]$. A graph is dismantlable if some sequence of deleting corners results

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in the graph with a single vertex. For example, each tree is dismantlable, and more generally, so are chordal graphs (that is, graphs with no induced cycles of length more than three). To prove the latter fact, note that every chordal graph contains a vertex whose neighbour set is a clique; see, for example, [15].

The following theorem gives the main results characterizing cop-win graphs.

**Theorem 1.1.** [9, 13, 14]

1. If $u$ is a corner of a graph $G$, then $G$ is cop-win if and only if $G - u$ is cop-win.
2. A graph is cop-win if and only if it is dismantlable.

We say that an event holds *asymptotically almost surely* (a.a.s.), if it holds with probability tending to one as $n$ tends to infinity. The probability of an event $A$ is denoted by $\mathbb{P}(A)$.

Our goal is to investigate the structure of random cop-win graphs. The random graph model we use is the familiar $G(n, 1/2)$ probability space of all labelled graphs on $n$ vertices where each pair of vertices is joined with probability $1/2$, independent from the events for other pairs of vertices. Note that a given graph $G$ occurs with probability $\mathbb{P}(G \in G(n, 1/2)) \approx \left(\frac{1}{2}\right)^{|E(G)|}(1 - \frac{1}{2})^{\binom{n}{2} - |E(G)|} = \left(\frac{1}{2}\right)^{\binom{n}{2}}$, which does not depend on $G$. Thus, $G(n, 1/2)$ is in fact a uniform probability space over all labelled graphs on $n$ vertices. We heavily use this interpretation of $G(n, 1/2)$ in the proof of our main result, Theorem 2.1 stated below. We expect results analogous to Theorem 2.1 (that is, with 2 replaced by $1/p$) for other constants $p \in (0, 1)$ and $p = p(n)$ tending to zero with $n$. (The argument for $p = p(n)$ tending to one needs to be modified when the expected number of dominating sets of cardinality $k$ is $\Omega(1)$; see [10].) However, this seems not to be an interesting research direction in the theory of random graphs, where we usually focus on investigating typical properties that hold a.a.s. in $G(n, p)$. Therefore, studying bounds for the cop number that hold a.a.s. are of interest, and a number of papers have been published on this topic (see, for example [2, 4, 8, 11, 12]). We focus on $G(n, 1/2)$ in this paper since it gives the typical structure of a $k$-cop-win graph. Therefore, from now on our probability space is always taken to be $G(n, 1/2)$.

## 2. Main results

A vertex set $S \subseteq V$ is a dominating set for a graph $G = (V, E)$ if every vertex not in $S$ is adjacent to at least one member of $S$. A vertex $v \in V$ is universal if it is joined to all others (that is, if $S = \{v\}$ is a dominating set). Let $k$-cop-win be the event that the graph is $k$-cop-win, let $k$-dom be the event that there is a dominating set of cardinality $k$, and let universal be the event that there is a universal vertex. If a graph has a dominating set of cardinality $k$, then it is clearly $k$-cop-win. Hence, in a certain sense, graphs with dominating vertices of cardinality $k$ are the simplest $k$-cop-win graphs. The probability that a random graph is $k$-cop-win can be estimated
as follows:

\[ \mathbb{P}(k\text{-cop-win}) \geq \mathbb{P}(k\text{-dom}) = (1 + o(1)) \left(1 - 2^{-k}\right)^{-k} \binom{n}{k} (1 - 2^{-k})^n. \]  

(2.1)

Indeed, for any \( S \subseteq V = [n] \) of cardinality \( k \), let \( A_S \) denote the event that \( S \) is a dominating set. By the union bound,

\[ \mathbb{P}(k\text{-dom}) = \mathbb{P} \left( \bigcup_S A_S \right) \leq \sum_S \mathbb{P}(A_S) = \binom{n}{k} (1 - 2^{-k})^{n-k}. \]

On the other hand, it follows from Bonferroni inequality that

\[ \mathbb{P}(k\text{-dom}) = \mathbb{P} \left( \bigcup_S A_S \right) \geq \sum_S \mathbb{P}(A_S) - \sum_{S,T:S \neq T} \mathbb{P}(A_S \cap A_T). \]

Let \( S, T \subseteq V \) be such that \( |S| = |T| = k \) and \( |S \cap T| = \ell \) for some \( 0 \leq \ell < k \). The probability that a vertex \( v \in V \setminus (S \cup T) \) is dominated by both \( S \) and \( T \) is equal to

\[
(1 - 2^{-\ell}) + (1 - 2^{-(k-\ell)})^2 2^{-\ell} = 1 - 2^{-k} - 2^{-k} (1 - 2^{-(k-\ell)}) \leq 1 - \frac{3}{2} \cdot 2^{-k} < 1 - 2^{-k}.
\]

Hence,

\[
\sum_{S,T:S \neq T} \mathbb{P}(A_S \cap A_T) \leq n^{2k} \left(1 - \frac{3}{2} \cdot 2^{-k}\right)^{n-2k} = o \left( \sum_S \mathbb{P}(A_S) \right)
\]

and so the lower bound (2.1) holds.

Surprisingly, this lower bound is in fact the correct asymptotic value for \( \mathbb{P}(k\text{-cop-win}) \).

Our main result is the following theorem.

**Theorem 2.1.** In \( G(n, 1/2) \), we have that

\[ \mathbb{P}(k\text{-cop-win}) = (1 + o(1)) \left(1 - 2^{-k}\right)^{-k} \binom{n}{k} (1 - 2^{-k})^n = \Theta \left( n^k \left(1 - 2^{-k}\right)^n \right). \]

We prove Theorem 2.1 in the next section. Using it, we derive the asymptotic number of labelled \( k \)-cop-win graphs.

**Corollary 2.2.** The number of \( k \)-cop-win graphs on \( n \) labelled vertices is

\[ \mathbb{P}(k\text{-cop-win}) 2^{\binom{n}{2}} = (1 + o(1))(1 - 2^{-k})^{-k} \binom{n}{k} 2^{n^2/2 - \left(1/2\log_2(1 - 2^{-k})\right)n}. \]

It also follows that almost all \( k \)-cop-win graphs contain a dominating set of cardinality \( k \), a fact not obvious a priori.

**Corollary 2.3.** In \( G(n, 1/2) \), we have that

\[ \mathbb{P}(k\text{-dom} \mid k\text{-cop-win}) = 1 - o(1). \]

Let us mention that these observations generalize the result obtained in [3].
Theorem 2.4 ([3]). In $G(n,1/2)$, we have that
\[
P(\text{cop-win}) = (1 + o(1))n2^{-n+1}.
\]

In [3], it was conjectured that the theorem can be generalized, but this was left as an open problem. The proof of Theorem 2.4 used the characterization of cop-win graphs (see Theorem 1.1). The limited current understanding of graphs with cop number two or higher was the main stumbling block in extending the result. For example, there are no elementary analogues of cop-win orderings for higher $k$. An elimination ordering characterization of $k$-cop-win graphs for $k > 1$ was given in [6], although it becomes more complex as $k$ increases (in particular, a vertex ordering is provided but in the $(k + 1)$th strong power of the graph). The proof of Theorem 2.1 avoids these complications and analyzes the winning strategy for the robber instead.

3. Proofs of main results

We start this section with some notation that will be used in the proof. The neighbourhood of a vertex $v$ is the set containing all neighbours of $v$ and is denoted by $N(v)$. We use $N^c(v) = V(G) \setminus (N(v) \cup \{v\})$ for the set of non-neighbours of $v$. For a given $k \in \mathbb{N}$, let
\[
\delta_k = \min_{S \subseteq V, |S| = k} \left| \bigcap_{v \in S} N^c(v) \right|.
\]

Note that $\delta_k = 0$ if and only if there exists a dominating set of cardinality $k$.

To prove Theorem 2.1, we bound the probability of $k$-cop-win for graphs with $\delta_k \geq 1$. Since the proof for small values of $\delta_k$ has a different flavour than that for large ones, we prove it independently.

Theorem 3.1. There exist $\xi > 0$ and $\varepsilon > 0$ such that the following hold. In $G(n,1/2)$, we have that

(a) $P(\text{k-cop-win and } 1 \leq \delta_k \leq \xi n) \leq 2^{-(\log_2(1-2^{-k})+\varepsilon)n}$, and
(b) $P(\text{k-cop-win and } \delta_k > \xi n) \leq 2^{-(\log_2(1-2^{-k})+\varepsilon)n}$.

Theorem 2.1 follows immediately from Theorem 3.1 and (2.1).

Let $G$ be a random graph drawn from the $G(n,1/2)$ distribution. Our goal is to investigate the probability that $\delta_k \geq 1$ and that $c(G) \leq k$. We show that this event holds with extremely small probability (wesp), which means that the probability it holds is at most $2^{-(\log_2(1-2^{-k})+\varepsilon)n}$ for some $\varepsilon > 0$. Observe that if we can show that each of a polynomial number of events holds wesp, then the same is true for the union of these events.

3.1. Proof of Theorem 3.1(a). Let $S \subseteq V(G)$ be a set of vertices that dominates all but $\delta_k \geq 1$ vertices; let $v \in \bigcap_{v \in S} N^c(v)$ be some vertex not dominated by $S$. To estimate the probability (from above) that $c(G) \leq k$ we introduce a strategy for the robber and show that wesp $k$ cops can win (against this given strategy). The strategy for the robber is as follows:

(a) if cops occupy the whole set $S$, then the robber goes to $v$;
(b) if cops occupy a proper subset $T$ of $S$ and no cop occupies $v$, then the robber goes to a vertex of $N_c(T) \cap N(v)$ that is adjacent to no cop;
(b') in particular, if no cop occupies $S \cup \{v\}$, then the robber goes to a vertex of $N(v)$ that is adjacent to no cop;
(c) if cops occupy a proper subset $T$ of $S$ and some cop occupies $v$, then the robber goes to a vertex of $(N_c(T) \cap N_c(v)) \setminus S$ that is adjacent to no cop;
(c') in particular, if no cop occupies $S$ but some cop occupies $v$, then the robber goes to a vertex of $N_c(v) \setminus S$ that is adjacent to no cop.

If the robber has more than one vertex to choose from, then she can make an arbitrary choice. This strategy is a greedy one that guarantees that the robber stays alive for at least one more round but does not “think about the future”. If all the cops “pause”, then we may assume that the robber “pauses” too.

The following technical lemma is the key observation. In order to simplify the notation, we consider the probability space $G(n + k + 1, 1/2)$ instead of $G(n, 1/2)$.

**Lemma 3.2.** Let $S \subseteq V = \{1, 2, \ldots, n + k + 1\}$ with $|S| = k$, $v \notin S$. Let $T \subseteq S$ with $0 \leq |T| = \ell < k$. Let $x = x(n)$, $y = y(n)$, $z = z(n)$ be any deterministic functions such that $x_n, y_n, z_n \in \mathbb{Z}$, $0 \leq y, z \leq 1$, and $0 \leq x \leq y$. Consider the following properties:

- (p1) $v \notin N(S)$,
- (p2) $y_n$ vertices of $W := V \setminus (S \cup \{v\})$ are adjacent to $v$,
- (p3) $S$ dominates all but $z_n$ vertices of $W$,
- (p4) $x_n$ vertices of $N(v)$ that are dominated by $S$ are not adjacent to $T$,
- (p5) the following is not true: for every $U \subseteq W$ with $|U| = k - \ell$ and $w \in W \setminus U$, there exists a vertex $x \in N(v)$ adjacent to $w$ but not adjacent to any vertex of $T \cup U$.

Let $G = (V, E) \in G(n + k + 1, 1/2)$. (Note that all the variables are chosen in advance before a random graph $G$ is generated.) Then, there exists $\xi > 0$ such that if $z \leq \xi$, then properties (p1)-(p5) hold simultaneously wesp. Moreover, the statement holds when in both (p4) and (p5), $N(v)$ is replaced by $N_c(v)$.

Note the unusual statement of property (p5). This could be avoided but, since we are going to use the negation of (p5) later on, this is a convenient way of thinking about this property. Moreover, note that property (p1) holds with probability $(1/2)^k = \Theta(1)$ and is independent of the remaining properties. Hence, property (p1) could be clearly omitted, since it cannot help with showing that properties (p1)-(p5) hold simultaneously wesp. Again, we keep it for convenience, as these are all the properties used later on.

Before we prove the lemma, let us show how the lemma implies Theorem 3.1(a). Let $G \in G(n, 1/2)$. We want to estimate $\mathbb{P}(k$-cop-win and $\delta_k = \delta)$ for some $\delta \geq 1$. Let $S \subseteq V(G)$ be a set of vertices that dominates all but $\delta_k$ vertices; let $v \in \bigcap_{v \in S} N_c(v)$ be the vertex not dominated by $S$. The robber follows the strategy we introduced above.

If cops move to $S$, the robber wants to move to $v$ (see rule (a) of the robber’s strategy). Since there is no edge between $v$ and $S$, no cop occupied $v$ in the previous round. Hence, the robber is currently in $N(v)$ (see rule (b)) and can easily move to $v$.

Suppose then that cops move to a proper subset $T$ of $S$ and but not to $v$ (that is, some cops perhaps go to $U$ that is outside of $S \cup \{v\}$). The robber is at $v$ (if cops were
in $S$ in the previous round) or some vertex of $V(G) \setminus (S \cup \{v\})$, and she wants to move to a vertex of $N^c(T) \cap N(v)$ that is adjacent to no cop (see rule (b)). If for every $T \subseteq S$, $U \subseteq V(G) \setminus (S \cup \{v\})$ such that $|T \cup U| = k$, and $w \in V(G) \setminus (S \cup \{v\} \cup U)$, there exists a vertex $x \in N(v)$ adjacent to $w$ but not adjacent to any vertex of $T \cup U$, the robber can move from $w$ to $x$ and survive for at least one more round. In other words, the robber’s strategy fails for (b) if $G$ satisfies properties (p1)-(p5) in the Lemma 3.2 for some specific choice of $S, v, T, x(n), y(n), z(n)$. A symmetric argument can be used to analyze the case when cops move to $T \subseteq S$ and to $v$ (rule (c)) to get a conclusion that if the robber’s strategy fails for (c), then $G$ must have properties (p1)-(p5) where $N(v)$ is replaced by $N^c(v)$.

Since the number of possible choices for $S, v, T, x(n), y(n), z(n)$ to consider is at most $O(n^{k+1+3}) = n^O(1)$ (recall that $k = O(1)$) and for each choice we get a statement that holds wesp, we get the desired upper bound for $\mathbb{P}(k$-cop-win and $1 \leq \delta_k \leq \xi n)$, where $\xi > 0$ is a constant guaranteed by Lemma 3.2.

It remains to prove Lemma 3.2.

**Proof of Lemma 3.2.** Since edges of a random graph are generated independently, the probability that properties (p2) and (p3) hold simultaneously is equal to

$$\left(\frac{n}{yn}\right)^2 2^{-\kappa n} \left(\frac{2^{-k} \kappa n}{zn}\right) (1 - 2^{-k})^{(1-\kappa)n} \leq \left(\frac{n}{yn}\right) 2^{-\kappa n} \left(\frac{n}{\xi n}\right) (1 - 2^{-k})^n.$$

Using Stirling’s formula $(n! = (1 + o(1)) \sqrt{2\pi n}(n/e)^n)$ and taking the exponential part we obtain an upper bound of

$$\exp\left((-y \ln y - (1 - y) \ln(1 - y) - \xi \ln \xi - (1 - \xi) \ln(1 - \xi) - \ln 2)n\right) (1 - 2^{-k})^n.$$

It is straightforward to see that $f(x) := -x \ln x - (1 - x) \ln(1 - x)$ tends to zero as $x \to 0$, and that $f(x)$ is maximized at $x = 1/2$ giving $f(1/2) = \ln 2$. Moreover, if $x \leq 1/2 - \varepsilon_1$ or $x \geq 1/2 + \varepsilon_1$ for some $\varepsilon_1 > 0$, we get that $f(x) \leq \ln 2 - \varepsilon_2$ for some $\varepsilon_2 = \varepsilon_2(\varepsilon_1) > 0$, and so the desired property holds wesp after taking $\xi$ sufficiently small so that, for example, $f(\xi) \leq \varepsilon_2/2$.

We may assume that $1/2 - \varepsilon_1 \leq y \leq 1/2 + \varepsilon_1$ for some $\varepsilon_1 > 0$. (The constant $\varepsilon_1$ can be made arbitrarily small by assuming that $\xi$ is small enough.) The probability that properties (p2) and (p3) hold simultaneously is equal to

$$\left(\frac{n}{yn}\right)^2 2^{-\kappa n} \left(\frac{n}{\xi n}\right) (1 - 2^{-k})^{(1-\kappa)n} \leq \left(\frac{n}{\xi n}\right) (1 - 2^{-k})^n,$$

and we need to consider property (p4) to obtain the desired bound. We first expose edges from $v$ to the vertices of $W$. For a vertex $u \in N(v)$, let $A(u)$ be the event that $u$ is dominated by $S$ and let $B(u)$ be the event that $u$ is nonadjacent to $T$. We can perform a “double exposure” for each vertex. First, we determine whether $u$ is dominated by $S$; $\mathbb{P}(A(u)) = (1 - 2^{-k})$. If this is the case, then we determine whether $u$ is also nonadjacent to $T$. This time,

$$\eta := \mathbb{P}(B(u) | A(u)) = \frac{\mathbb{P}(B(u) \wedge A(u))}{\mathbb{P}(A(u))} = \frac{2^{-\ell}(1 - 2^{-(k-\ell)})}{1 - 2^{-k}}.$$
(Note that, as expected, \( \eta = 1 \) if \( T = \emptyset \); in any case, \( \eta > 0 \).) Since the number of vertices not dominated by \( S \) is \( zn \leq \xi n \) and \( |N(v)| \geq (1/2 - \varepsilon_1)n \), the expected number of neighbours of \( v \) that are dominated by \( S \) but not adjacent to \( T \) is at least

\[
\left( \frac{1}{2} - \varepsilon_1 - \xi \right) n \cdot \eta > \frac{\eta n}{3},
\]

provided \( \xi \) (and so \( \varepsilon_1 \) as well) are small enough. The events associated with two distinct vertices \( u_1 \) and \( u_2 \) are independent. We next use a consequence of Chernoff’s bound (see e.g. [7, p. 27, Corollary 2.3]), that

\[
\Pr \left( |X - \mathbb{E}X| \geq \varepsilon \mathbb{E}X \right) \leq 2 \exp \left( -\frac{\varepsilon^2 \mathbb{E}X}{3} \right)
\]

for \( 0 < \varepsilon < 3/2 \). We get that the probability that (p4) holds for \( x \leq \eta/6 \) (conditioned on (p2) and (p3) holding) is at most

\[
\Pr (\text{Bin}(n/3, \eta) \leq \eta n/6) \leq \exp \left( -\frac{(1/2)^2(\eta n/3)}{3} \right) = \exp \left( -\frac{\eta}{36} n \right).
\]

Hence, if \( x \leq \eta/6 \), then properties (p2)-(p4) hold simultaneously wesp, after taking \( \xi \) sufficiently small so that, for example, \( f(\xi) \leq \eta/40 \).

We now may assume that \( 1/2 - \varepsilon_1 \leq y \leq 1/2 + \varepsilon_1 \) for some \( \varepsilon_1 > 0 \) and \( x \geq \eta/6 \). (As before, the constant \( \varepsilon_1 \) can be made arbitrarily small by assuming that \( \xi \) is small enough but recall that \( \eta \) is not a function of \( \xi \) and depends only on \( k \) and \( \ell \).) As before, we note that the probability that properties (p2) and (p3) hold is at most \( \left( \frac{n}{\xi n} \right) (1 - 2^{-k})^n \), but this time we need to consider property (p5) to obtain the desired bound. In order to do it, we first expose edges from \( S \cup \{v\} \) to \( W \) (to estimate the probability that (p2) and (p3) hold) but do not yet expose any edge between vertices of \( W \). Hence, we may estimate the probability that (p5) holds by exposing the edges of the subgraph induced by \( W \), and all events are independent. The number of choices of \( U \) and \( w \) is \( n^{O(1)} \). For a particular choice of \( U \) and \( w \), the probability that no suitable \( x \) can be found can be estimated by

\[
\left( 1 - 2^{-1}2^{-k-\ell} \right)^{xn - (k-\ell) - 1} \leq \left( 1 - 2^{-(k-\ell)-1} \right)^{\eta n/7} = \exp (-\varepsilon_3 n),
\]

where \( \varepsilon_3 = \varepsilon_3(k, \ell) > 0 \) and does not depend on \( \xi \). Indeed, with probability \( 2^{-1} \) a given candidate vertex \( x \) (neighbour of \( v \), dominated by \( S \) but not adjacent to \( T \)) is adjacent to \( w \), and with probability \( 2^{-(k-\ell)} \) it is not adjacent to any vertex of \( U \). The bound holds, since we have at least \( xn - (k-\ell) - 1 \geq \eta n/7 \) candidates to test and the corresponding events are independent. The properties (p2), (p3), and (p5) hold wesp after taking \( \xi \) sufficiently small so that, for example, \( f(\xi) \leq \varepsilon_3/2 \). The proof of the lemma is finished. \( \square \)

3.2. Proof of Theorem 3.1(b). Let \( \xi > 0 \) be the constant yielded by Lemma 3.2. Let \( G \subseteq G(n, 1/2) \). We want to estimate \( \Pr (k\text{-}\text{cop-win} \text{ and } \delta_k = \delta) \) for some \( \delta \geq \xi n \). A vertex \( b \) of \( G \) is called dangerous if there exists a threatening set of \( k \) vertices \( A = \{a_1, a_2, \ldots, a_k\}, b \notin A \), such that

\[
|N^c(a_1) \cap N^c(a_2) \cap \ldots \cap N^c(a_k) \cap N(b)| \leq 2q,
\]

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where $q := 3(k+1)/\xi = O(1)$. The robber might be afraid of going to dangerous vertices because if she does so, the cop can move to the threatening set and the robber has at most $2q$ vertices to escape to (possibly zero!). However, if the number of dangerous vertices is, say, smaller than $q$, the robber can easily win. If the robber occupies a vertex that is not dangerous, no matter where the cops are, there are always more than $2q$ vertices to go to that guarantees that she is safe for another round, and at least one of them is not dangerous. The robber will be able to stay away from dangerous vertices forever.

It remains to show that wesp there are at least $q$ dangerous vertices. Suppose that $b_i$ is dangerous because of a threatening set $A_i = \{a_{i1}^1, a_{i2}^1, \ldots, a_{ik}^1\}$ with
\[
|N^c(a_{i1}^1) \cap N^c(a_{i2}^1) \cap \ldots \cap N^c(a_{ik}^1)| = t_i \geq \delta_k \geq \xi n,
\]
but only at most $2q$ neighbours of $b_i$ are in $N^c(a_{i1}^1) \cap N^c(a_{i2}^1) \cap \ldots \cap N^c(a_{ik}^1)$. Since there are only $O(n^{k+q/2}) = n^{O(1)}$ configurations to consider, we may focus on one. It may happen that $b_i = a_{i\ell}^j$ for some $\ell$ and $j$. However, we can always select a subset of $b_i$’s of cardinality $q/(k+1) = 3/\xi$ without this property. This is a desired situation, since we want to be able to expose a lot of potential edges and insist of not generating any. We get that the probability that there are at least $q$ dangerous vertices is at most
\[
\left((1/2)^{t_i - O(1)}\right)^{\frac{\xi}{3}} \leq \left((1/2)^{2xi/3}\right)^{\frac{3}{\xi}} = 2^{-2n},
\]
and so the property holds wesp and the proof is finished.

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