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Model Fractional Chern Insulators

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We devise local lattice models whose ground states are model fractional Chern insulators—Abelian and non-Abelian topologically ordered states characterized by exact ground state degeneracies at any finite size and infinite entanglement gaps. Most saliently, we construct exact parent Hamiltonians for two distinct families of bosonic lattice generalizations of the $Z_2$ parafermion quantum Hall states: (i) color-entangled fractional Chern insulators at band filling fractions $\nu = k/(C + 1)$ and (ii) nematic states at $\nu = k/2$, where $C$ is the Chern number of the lowest band. In spite of a fluctuating Berry curvature, our construction is partially frustration free: the ground states reside entirely within the lowest band and exactly minimize a local $(k+1)$-body repulsion term by term. In addition to providing the first known models hosting intriguing states such as higher Chern number generalizations of the Fibonacci anyon quantum Hall states, the remarkable stability and finite-size properties make our models particularly well-suited for the study of novel phenomena involving e.g. twist defects and proximity induced superconductivity, as well as being a guide for designing experiments.

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Introduction. The prospect of lattice-scale fractional quantum Hall (FQH) phenomena at high temperatures, without the need for a strong magnetic field, has attracted ample recent attention to the theory of fractional Chern insulators (FCIs) [1–3]. While experimental realizations of FCIs are becoming increasingly realistic in the light of the recent realization of integer Chern insulators with unit Chern number in solid state materials [4] and cold atom systems [5], the theoretical frontier has turned towards strongly correlated states featuring an even richer phenomenology than continuum Landau levels.

In the present work, we bridge this divide and provide exact lattice parent Hamiltonians for a large class of Abelian as well as non-Abelian model FCIs in bands carrying any Chern number $C$. We explicitly verify that the ground state multi-plets are exactly degenerate at any finite size, that the gap to excited states remains finite in the thermodynamic limit, and that there is an infinite gap in the particle entanglement spectrum [46, 47].

Flat band model. We begin by constructing a family of multi-orbital models possessing exactly flat lowest bands with arbitrary Chern number $C$. For definitiveness we describe our construction on a square lattice with an effective magnetic flux $\phi = 1/q$ piercing each elementary plaquette [48], although it can be generalized to any Bravais lattice and rational flux.

We assign $M$ internal orbitals to each lattice site $i$ with real-space coordinates $(x_i, y_i)$, where $M$ must be a factor of $q$. The site-dependent orbital index to the lattice site $i$ is $s_i = x_i \mod (\frac{M}{q}) + m \frac{q}{M}$, with $m = 0, 1, \ldots, M − 1$. The single-particle physics is governed by

$$H_0 = \sum_{j,s_j} \sum_{k,s_k} t_{j,k}^{s_j,s_k} a_{j,s_j}^\dagger a_{k,s_k},$$

where $a_{j,s_j}$ $(a_{j,s_j})$ creates (annihilates) a particle on the orbital $s_j$ at site $j$. To achieve an exactly flat lowest Chern band we choose the hopping amplitudes as [49]

$$t_{j,k}^{s_j,s_k} = \delta_{s_j,s_k} \delta_{x_j-x_k, y_j-y_k} e^{-\frac{1}{2}(|1-\phi| z^2 + i\phi(\tilde{x}_j+\tilde{x}_k) y)}$$

where $z_j = x_j + iy_j$, $z = z_j - z_k$, and $\tilde{x}_j = x_j + (s_j - x_j) \mod q$. The hopping amplitudes decay as a function of the distance between site $k$ and site $j$ like a Gaussian. The hopping phase factor depends on both $x_i$ and $s_i$ according to the definition of $\tilde{x}_i$. The unit cell of our model contains
q/M sites in the x direction. The q orbitals in a unit cell lead to q bands, and the lowest thereof is exactly flat and carries Chern number C = M. For M = 1, our construction Eqs. (1) and (2) reduces to the Kapit-Mueller model [30] in Landau gauge. Similar multi-orbital models have also been studied in Refs. [10, 18] albeit with different choices for the hopping amplitudes.

Although it is generally impossible to have an entirely flat band with non-zero Chern number and strictly finite hopping [50], our model is local in the sense of being at least exponentially bounded. Truncating the hopping at a distance of d = 2 lattice constants already gives a high flatness ratio between the band gap and bandwidth: e.g. for φ = 1/6 and C = 2 or 3 one finds f ≈ 85 or 73 respectively. The efficiently quenched kinetic energy amplifies the importance of interaction effects and we will now proceed to show that local interactions indeed generate model FCIs.

**Color-entangled FCIs.** We begin by considering N particles with the (k + 1)-body on-site repulsion on a finite lattice of Nσ × Nσ unit cells with periodic boundary conditions. The interaction Hamiltonian reads

\[
H_{\text{int}} = \sum_i^{N} \sum_{\sigma_k \leq \sigma_1 \leq \cdots \leq \sigma_k} \cdots n_i, \sigma_k n_i, \sigma_k \cdots n_i, \sigma_k \cdots , \tag{3}
\]

where \(n_i, \sigma\) is the occupation operator on the orbital \(\sigma\) at lattice site \(i\), and \(\cdots\) enforces the normal ordering. For \(C = M = 1\), the single-particle wave functions of the lowest band of Eq. (1) have the structure of a discretized lowest Landau level, and lattice analogs of the Zk Read-Rezayi states are unique zero-energy ground states of Eq. (3) at \(\nu = N/(N_{x}N_{y}) = k/2\) up to an exact \((k+1)\)-fold degeneracy, when the number of particles is a multiple of \(k\), because the exact clustering properties of these wave functions [44] carry over directly to the present lattice setting. This is astonishing given that many other properties such as the fluctuating Berry curvature in reciprocal space and the excitation spectrum already deviate from that in the continuum since the discretized Landau level orbitals are non-orthogonal. Furthermore, if the wave functions are written in a properly orthogonalized Wannier basis [31–33], they differ from the continuum model states [19, 38]. Nevertheless, we find that these states are characterized by an infinite gap in the particle entanglement spectrum (PES) which probes the quasihole excitations of the system [46, 47]. Remarkably, we find that this scenario generalizes to any \(C = M > 1\): at filling fractions \(\nu = k/(C + 1)\), there are \((C+k)\)-fold exactly degenerate zero-energy ground states when the number of particles is a multiple of \(k\), and their PES has an infinite gap.

To establish this, we project the interaction Hamiltonian Eq. (3) for large number of samples onto the lowest band, and compute the many-body eigenvalues and eigenstates by exact diagonalization. Indeed, we always observe the expected number of zero-energy modes in the energy spectrum [48, 51], which in turn implies that the band projection leaves the ground states unchanged. We also find the number of zero-energy modes is robust against the flux insertion (twisted boundary conditions). To demonstrate the infinite entanglement gap in the particle entanglement spectrum (PES), we truncate the hopping range in Eq. (1) at distance d, then track the evolution of the PES with increasing d. While the lowest band is dispersive for finite d, we study the band projected version thus ignoring the band dispersion.

Figure 1. (Color online) Typical particle entanglement spectra (PES), here displayed for the \(k = 2, \phi = 2/3\) non-Abelian state in a \(C = M = 3\) band with \(N = 8, N_x \times N_y = 4 \times 4, \phi = 1/3\) and hoppings truncated at (a) \(d = \sqrt{2}\), (b) \(d = 3\), (c) \(d = \sqrt{10}\), and (d) \(d = \infty\) lattice constants. Generally a PES obtained by numerical diagonalization includes three parts: the low-lying levels with quasihole excitation information (blue), the high non-universal levels (red), and numerical noise (gray) set by the double precision above \(\xi_c \approx \ln(2^{−53}) \approx 36.7\). The high, non-universal levels merge into numerical noise for large d, preventing us from further numerically tracking the growth of the entanglement gap, which we argue to increase without bound with increasing d.
results in the literature [6, 12, 53]. Increasing \( d \) further elevates the non-universal part of the PES and quickly enlarges the entanglement gap [Figs. 1(b) and 1(c)]. Our capability of tracking the growth of the entanglement gap is only limited by the machine precision, which determines that the PES levels can be computed reliably at most up to \( \xi \approx 36.7 \), which corresponds to an exponentially small amplitude, of order \( O(e^{-\xi/2}) \), in the ground state wave function. When the non-universal levels merge into the numerical noise, it is impossible to identify the entanglement gap [Fig. 1(d)] accurately. This happens when the machine error dominates the high-energy part in the PES, the entanglement gap has already \( \Delta \xi \) grown to a purely determined by the interaction and the projection, except for the many-body gap as long as the band gap is much larger than the interaction strength, since low-lying excitations are essentially indistinguishable from model FCIs with infinite entanglement gaps. A short-range truncation of Eq. (1) is enough to get FCIs which clearly suggest infinite entanglement gaps of model FCIs. A finite entanglement gaps.

While the ground states and quasi-hole excitations have identically zero interaction energy in our model, the gap \( \Delta E \)—measured at fixed particle number corresponding to a particle-hole excitation pair—is in principle size-dependent. The projection of the interaction to the lowest band will not affect the many-body gap as long as the band gap is much larger than the interaction strength, since low-lying excitations are purely determined by the interaction and the projection, excluding excitations caused by hopping from lower to higher bands. In Fig. 3, we plot \( \Delta E \) versus the inverse particle number \( 1/N \) for various model FCIs. In each case we find that the gap clearly extrapolates to a finite value in the thermodynamic limit, and, compared to other FCI models, the gap is remarkably insensitive to the system size (cf. e.g. Ref. [28]).

Having established the ideal nature of FCIs in our model, we now turn to its color-entangled nature. If we interpret \( m = 0, 1, \ldots, M - 1 \) in the orbital indices \( \sigma_i \) as “layers” or “colors”, Eq. (1) on an infinite lattice is equivalent to a shifted stacking of \( M \) layers of the infinite \( M = 1 \) model. However, for a finite lattice of \( N_x \times N_y \) unit cells with periodic boundary conditions, the corresponding stacking has color-entangled boundary conditions [7, 14, 18] in the \( x \)-direction in the sense that the hopping across the boundary may occur between orbitals belonging to different layers (usual periodic boundary conditions apply in the \( y \)-direction) [48]. Crucially, each layer is not necessarily a complete \( M = 1 \) model with integer number of unit cells and periodic boundary conditions. Instead, one finds that Eq. (1) can be mapped to \( \text{gcd}(N_x, M) \) copies of complete \( M = 1 \) model with usual periodic boundary conditions. Ref. [14] provided a color-entangled basis built from continuum Landau levels and showed that provides a promising approach to the \( \nu = k/(C + 1) \) FCIs by providing numerical evidence for a few states with small \( C \) and \( k \) [6, 12, 14]. When \( N_x \) is divisible by \( M \), the FCIs correspond to color-dependent magnetic-flux inserted versions of the Halperin [54] or non-Abelian spin singlet states [55, 56]. Our construction extends this list of color-entangled FCIs and, by contrast, gives an exact construction directly in the real-space lattice.

Nematic states. The Hamiltonian Eq. (3) includes interactions within the same orbital and between different orbitals. Now we consider the zero-energy states in the presence of only on-site \( (k + 1) \)-body intra-orbital repulsion, i.e.

\[
H_{\text{int}} = \sum_i \sum_{\sigma_0 = \sigma_1 = \cdots = \sigma_k \in \{s_i\}}:n_i,\sigma_0 n_i,\sigma_1 \cdots n_i,\sigma_k: \quad (4)
\]

As discussed above, the single-particle problem can be mapped to \( \text{gcd}(N_x, M) \) copies of the \( M = 1 \) model. Because the interaction Hamiltonian Eq. (4) does not couple different copies, the many-body physics in this case is equivalent to distributing \( N \) on-site interacting particles in \( \text{gcd}(N_x, M) \) decoupled copies of \( M = 1 \) models, each with
promising basis for investigating lattice dislocations, which
nematic higher Chern number states provides a particularly
of the Fibonacci anyon quantum Hall states. Moreover, the
curvature at small
tice effects, e.g. reflected in a strongly non-uniform Berry
energy gap remains finite in the thermodynamic limit. These
finite gap in the particle entanglement spectrum and that the
$C$ in the existing literature. Notably the
$k$-Halperin state which is a
provides
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Discussion. Following the constructions detailed above,
it is straightforward to construct parent Hamiltonians for an
entire zoo of new model FCIs. For instance, for $M = 2$ and
even $N_x$, Eq. (1) has a bilayer FQH system as the continuum
counterpart. Thus with an on-site inter-orbital three-body repul-
sion in combination with a two-body intra-orbital repulsion is
expected to mimic the parent Hamiltonian for the coupled
Moore-Read state [57, 58] in the continuum. The degener-
acy of this state on the torus is $2N + 3$ for even number of
particles [59] which is consistent with our numerics [48].

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[52] Each eigenstate can be labeled by a two-dimensional total quasi-momentum $(K_{x}, K_{y})$. The ground state degeneracy is
not always possible to resolve for $d = 1$. In that case, we select the same number of lowest states in each $(K_x, K_y)$ sector as the $d = \infty$ case to construct the ground state manifold $\{ | \Psi_\alpha \rangle \}$.

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