LOCAL REALISTIC THEORY FOR PDC EXPERIMENTS BASED ON THE WIGNER FORMALISM

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Abstract

In this article we present a local hidden variables model for all experiments involving photon pairs produced in parametric down conversion, based on the Wigner representation of the radiation field. A modification of the standard quantum theory of detection is made in order to give a local realist explanation of the counting rates in photodetectors. This model involves the existence of a real zero-point field, such that the vacuum level of radiation lies below the threshold of the detectors.

\textit{Key words:} Parametric Down Conversion, Wigner Representation, Zero-point Field, Local Realism, Bell’s Inequalities.
1 Introduction

Following Bell, a local hidden variables model (LHV) exists for an Einstein-Podilsky-Rosen (EPR) experiment if it is possible to write the single and joint detection probabilities in the form

\[ p_i = \int \rho(\lambda) P_i(\lambda, \phi_i) d\lambda; \quad p_{12} = \int \rho(\lambda) P_1(\lambda, \phi_1) P_2(\lambda, \phi_2) d\lambda, \quad (1) \]

where \( \lambda \) being the hidden variables with a probability distribution \( \rho(\lambda) \). \( P_i(\lambda, \phi_i) \) \((i = 1, 2)\), are some functions fulfilling the conditions \( 0 \leq P_i(\lambda, \phi_i) \leq 1 \). \( \phi_1 \) and \( \phi_2 \) represent controllable parameters of the experimental setup.

Many experiments have been performed in order to test local realism vs. quantum mechanics via Bell’s inequalities, but none of these experiments have violated Bell’s genuine inequalities, which are based upon the assumptions of realism and locality alone. All inequalities tested up to now involve additional assumptions, like no-enhancement. In the last few years experiments using photon pairs produced in parametric down conversion (PDC) have become very popular. LHV models for EPR experiments with PDC have appeared in the literature, the aim of which has been to show that local realism had not been ruled out by these experiments, and to stress the relevance of the so called “loopholes”, in particular the one due to the low efficiency of optical photon counters. However, those models were mathematical constructs without physical content; they have no predictive power.

The purpose of this paper is to exhibit a physical model which is based on the Wigner formulation of PDC. Our previous work has shown that the Wigner function of the electromagnetic field is positive for all performed experiments, and hence it provides a realistic description for the production and propagation of the radiation in these experiments. In particular, the Wigner function of the vacuum is the gaussian

\[ W(\{\alpha_k, \alpha_k^*\}) = \prod_k \frac{2}{\pi} e^{-2|\alpha_k|^2}, \quad (2) \]

which in our local realistic model is interpreted as the probability distribution of a random radiation filling the whole space (the zeropoint field, ZPF). Thus the amplitudes \( \{\alpha_k\} \) play the role of the hidden variables \( \lambda \) and \( W \) the role of the function \( \rho(\lambda) \) in Eq.\((1)\). The nonlinear crystal, acted by a laser beam, gives an outgoing PDC field...
\[ E^{(+)} = \sum_k \left( \frac{\hbar \omega}{\epsilon_0 L_0^3} \right)^{1/2} \left[ \alpha_k e^{-i(k \cdot r + \omega t)} + g \alpha_k^* e^{-i((k_0 - k) \cdot r + (\omega_0 - \omega) t)} + \frac{1}{2} g^2 \alpha_k e^{-i(k \cdot r + \omega t)} \right], \]

where the first term is the ZPF that crosses the crystal without any modification, and the other terms are produced via the non-linear coupling between the laser and the ZPF (\( g \) is the coupling parameter).

The PDC beam propagates through a number of devices (lenses, beam splitters, etc.) characteristic of every experiment and finally some intensity \( I_j \) arrives at every detector \( j \). The intensity \( I_j \) is a function bilinear of the amplitudes \( \alpha_k \) and \( \alpha_k^* \) of Eq.(1). These functions \( I_j \) are quite involved and have been studied in detail in Refs.[4] for many experiments. The single and joint detection probabilities are given by

\[ p_j = \int W(\{\alpha_k\}, \{\alpha_k^*\}) Q_j(\{\alpha_k\}, \{\alpha_k^*\}, \phi) d^N \alpha_k d^N \alpha_k^* , \]  

\[ p_{12} = \int W(\{\alpha_k\}, \{\alpha_k^*\}) Q_1(\{\alpha_k\}, \{\alpha_k^*\}, \phi_1) Q_2(\{\alpha_k\}, \{\alpha_k^*\}, \phi_2) d^N \alpha_k d^N \alpha_k^* , \]

where

\[ Q = \frac{\eta}{\hbar \nu} \int dt \int d^2 r \left[ I(\{\alpha_k\}, \{\alpha_k^*\}, \phi, r, t) - I_0 \right] , \]

\( I = E^+ E^- \) being the intensity of the field arriving at the corresponding detector and \( I_0 \) is a constant which in our model is interpreted as the mean intensity of the the ZPF. The integration is carried over the time window and the surface aperture of the detector. We have divided \( \frac{\eta}{\hbar \nu} \) by the typical energy of one photon, so that \( Q \) becomes dimensionless, \( \eta \) being the quantum efficiency of the detector.

The relevant question is whether (3) and (4) may be considered particular cases of (1) with \( Q_j \) playing the role of \( P_j \). The answer is not affirmative because we cannot guarantee that \( 0 \leq Q \leq 1 \). Consequently we conclude that it is not possible to interpret directly the Wigner-function formalism as an LHV model for the PDC experiments.

2 The Detection Model

We shall devote the rest of the article to the description of the model of a detector that works in a strictly local way and provides the LHV model we
are searching for. We will first show the basic points of our model:

1. The detector consists of a set of individual photodetector elements, $D_l$, each characterized by a frequency $\omega_l$, and a wave vector $k_l (\omega_l = |k_l|/c)$, to which $D_l$ responds. We shall consider the direction of $k_l$ to be normal to the surface of the detector, which is taken as a cylinder of area $\pi R^2$ and length $L$.

2. The relevant quantity for the detection is a filtered field corresponding to a detector element $D_j$:

$$
\mathbf{E}_l^{(+)} = \frac{1}{\pi R^2 LT} \int_V dV \int_0^T E^{(+)}(r,t)e^{-i(k_l \cdot r + i\omega_l t)} dt.
$$

Hence it follows that the photodetector element $D_l$ is sensitive to radiation with frequencies in the interval $(\omega_l - \frac{\Delta \omega}{2}, \omega_l + \frac{\Delta \omega}{2})$ with $\Delta \omega \approx \frac{2\pi}{T}$, $T$ being the detection time window. If we assume that the incoming light beam has frequencies in the interval $(\omega_{\text{min}}, \omega_{\text{max}})$ with an average frequency $\bar{\omega} = (\omega_{\text{max}} + \omega_{\text{min}})/2$, and $\tau$ is the coherence time of the beam, we shall have $\delta \omega \equiv \omega_{\text{max}} - \omega_{\text{min}} \approx 2\pi/\tau$, so that the minimum number of detecting elements is $N \approx \delta \omega/\Delta \omega \approx T/\tau$. By putting typical values, $T = 10^{-8}$ s and $\tau = 10^{-12}$ s, we have the condition $N > 10^4$.

3. We now define the effective intensity obtained from the filtered fields in the form

$$
\mathbf{T} = c\epsilon_0 \sum_{l=1}^N \mathbf{E}_l^{(+)}\mathbf{E}_l^{(-)}.
$$

After that we replace the standard quantum Eq.(5) by the expression

$$
Q(\mathbf{T}) = (1 - e^{-\zeta(\mathbf{T} - \mathbf{T}_0)})\Theta(\mathbf{T} - \mathbf{I}_m) \quad ; \quad \zeta = \eta(h\nu)^{-1},
$$

which completes the definition of our model. $\mathbf{T}_0$ is the average of $\mathbf{T}$ for the ZPF. $\mathbf{I}_m$ is some threshold intensity fulfilling the condition $\mathbf{I}_m > \mathbf{T}_0$, and $\Theta(x)$ is the Heaviside function.
4. Now we rewrite the detection probabilities (3) and (4) in the following equivalent form

\[ p = \int \rho(T) Q(T) dT \quad ; \quad p_{12} = \int \rho_{12}(T_1, T_2) Q_1(T_1) Q_2(T_2) dT_1 dT_2 \] (9)

The probability distribution of the \textit{effective intensity} is a gaussian, which is determined by the mean and the standard deviation (for details see [5]). For instance, the probability distribution for the zeropoint field is

\[ \rho_0(T) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{(T - T_0)^2 / 2\sigma_0^2}; \quad T_0 = \frac{\omega_0 \delta \omega}{8\pi cL} ; \quad \sigma_0 = T_0 \frac{\sqrt{T}}{T}. \] (10)

The probability distribution of the effective intensity when there is a PDC signal present, \( \rho(T) \), is similar to that of the ZPF, but the following remarks are in order: (a) We shall define the signal mean effective intensity by \( I_s = \langle T \rangle - T_0 \). Here the word \textit{mean} refers to both, time average over the detection window and ensemble average with the distribution function Eq.(2). If we assume that the signal enters parallel to the axis of the detector, it can be shown that the effective intensity, \( T_s \), is equal to the actual intensity, i.e. \( T_s \approx I_s \). In practical situations the relation \( T_s \ll T_0 \) is fulfilled. (b) The relation between the mean and the standard deviation is completely analogous to that of the ZPF alone (see Eq. (10)), and the distribution of the effective intensity \( T \) may be written

\[ \rho(T) = \frac{\sqrt{T/\tau}}{\sqrt{2\pi\langle T \rangle}} e^{-\frac{(T - T_0 - T_s)^2 T}{2\sigma_0^2 \tau}}. \] (11)

On the other hand, \( \rho(T_1, T_2) \) is a double gaussian function which is defined by the mean values of its marginals, their standard deviations and the correlation function \( \langle (T_1 - T_{1s} - T_{10})(T_2 - T_{2s} - T_{20}) \rangle \).

3 The Detection Probabilities

In this section we shall compare the predicted detection probabilities of our model with those of quantum optics. In the case of single detection probability, let us consider the three following possible situations:
i) $I_s = 0$. In sharp contrast with quantum optics, our model predicts the existence of some counts in any detector even in the absence of signal, the probability being very small if $I_m - \overline{T}_0 \gg \sigma_0$.

ii) $\zeta \overline{I}_s \ll \sigma_0$. This should be the normal situation in experimental practice. In this case we may choose $I_m$ so that $\overline{T}_0 + \overline{I}_s - I_m \gg \sigma_0$, but $I_m > \overline{T}_0$ in order to preserve the positivity of $Q$ in Eq. (8). With these two constraints it can be shown that $p \approx \zeta I_s$, a result that agrees with the quantum one if we put $\zeta = \eta i \hbar \nu i$.

iii) $\zeta \overline{I}_s \gg \sigma_0$. In this case the detector saturates and gives a count in every time window.

Finally, in the case of the joint detection probability the predictions of our model coincide with the quantum result in the limits $\frac{\nu_t}{\nu_i} (\overline{T}_i - \overline{T}_i) \ll 1$, which requires $I_{is} + I_{i0} - I_{im} \gg \sigma_0$.

4 Constraints of the Model

In our model there is a trade-off between the constraint $\overline{T}_0 + \overline{I}_s - I_m \gg \sigma_0$, required for the linearity of the response at low efficiency, and $I_m - \overline{T}_0 \gg \sigma_0$, needed for the smallness of the dark counting probability. The need to satisfy both conditions implies that $\overline{T}_s \gg \sigma_0$. This means that there is a minimal intensity of the signal which may be reliably detected, a constraint absent in the quantum theory of detection, but certainly existing in experimental practice.

Let us analyze the consequences of the constraint. In experimental practice a lens is placed in front of the detector in such a way that the signal field has spatial coherence on the surface of the lens. The condition for having spatial coherence is $d \lambda \geq R_t R_C$, $d$ being the typical distance between the nonlinear medium (with radius $R_C$) and the detector; $R_t$ is the radius of the lens. On the other hand, the zeropoint field is not modified by the lens, which is evident because of the fact that energy cannot be extracted from the vacuum. As a consequence, the intensity of the incident signal is amplified by a factor $b^2 \equiv \pi^2 R_t^4 / \lambda^2 f^2$, $f$ being the focal distance. On the other hand, 84% (91%) of the total intensity is concentrated within the first (second) ring of the diffraction pattern with a radius $R = a \times (f \lambda / 2 R_t) = a \lambda / A_r$, where $A_r = 2 R_t / f$ is the relative aperture of the lens and $a = 1.22 (2.23)$ for the first (second) ring. Consequently, the optimum radius of the detector is
given by $R$.

By taking into account the above considerations, and using Eq. (10) and the condition $d\lambda \geq R_l R_C$, the constraint $T_s \gg \sigma_0$ gives

$$\text{Rate} \gg \frac{\eta f^2 R_C^2}{2Ld^2 \lambda \sqrt{\tau T}}$$

which puts a lower bound on the single rates which may be used in reliable experiments. This result cannot be derived from (conventional) quantum theory. By putting typical parameters, that is $\eta \approx 0.1$, $R_l$, $L$ and $f$ of the order of fractions of a centimeter, 10 nm (which gives a coherence time $\tau \approx 1$ ps) we get a minimal counting rate of the order of $10^5 - 10^6$ counts per second. This figure agrees fairly well with the actual experiments and should be considered a success of our model.

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