A DIAGRAMMATIC DEFINITION OF $U_q(\mathfrak{sl}_2)$

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Abstract. We give a diagrammatic definition of $U_q(\mathfrak{sl}_2)$, including its Hopf algebra structure and its relationship with the Temperley-Lieb category.

1. Introduction

This paper is about $U_q(\mathfrak{sl}_2)$, one of the simplest examples of a quantum group. For an account of the early history of quantum groups and some of their applications, see [Jon07].

The Temperley-Lieb category $\mathbf{TL}$ is a category of certain representations of $U_q(\mathfrak{sl}_2)$ and morphisms between them. These morphisms can be usefully represented as formal linear combinations of Temperley-Lieb diagrams. A knot or link diagram represents a morphism between trivial representations, and this leads to a definition of the Jones polynomial.

The aim of this paper is to give a new definition of $U_q(\mathfrak{sl}_2)$ that will include it in this diagrammatic point of view. We will define a category $\mathbf{TL}^*$ that contains both $\mathbf{TL}$ and $U_q(\mathfrak{sl}_2)$. Morphisms in $\mathbf{TL}^*$ will be formal linear combinations of diagrams that may include interior endpoints and orientations.

Orientations appeared in the earliest applications of Temperley-Lieb diagrams to ice-type models in statistical mechanics, such as [Lieb67]. The orientations in $\mathbf{TL}^*$ are very similar, and also satisfy the ice rule, which says that every crossing has two arrows pointing in and two pointing out.

Orientations again appeared in work of Frenkel and Khovanov [FK97]. Their idea is that, whereas a Temperley-Lieb diagram represents a linear map between representations of $U_q(\mathfrak{sl}_2)$, an oriented Temperley-Lieb diagram represents a single matrix entry of that linear map. Such diagrams form a category that is basically the same as our $\mathbf{TL}^*$. This provides the main algebraic motivation behind the definitions in this paper.

One advantage of the diagrammatic approach is that some abstract algebraic facts become “visually obvious”. For example, Theorem 4.1 gives the “intertwining” relationship between $\mathbf{TL}$ and $U_q(\mathfrak{sl}_2)$. The proof that the actions commute involves physically sliding one through the other, in a way reminiscent of Morton’s diagrammatic proofs that certain elements of the Temperley-Lieb algebra commute [Mor02]. I hope this approach makes the algebra more accessible to those, like myself, with a background in knot theory and skein relations.

Throughout the paper, we work over an arbitrary field containing an element $q$ which is neither 0 nor $\pm 1$. We will also need square roots $\sqrt{q}$ and $\sqrt{-q}$.

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2. THE CATEGORY $\text{TL}^*$

In this section, we define a monoidal category $\text{TL}^*$.

We start with a quick review of the Temperley-Lieb category $\text{TL}$. The objects are the non-negative integers. The morphisms from $n$ to $m$ are formal linear combinations of Temperley-Lieb diagrams that have $n$ endpoints at the bottom and $m$ at the top. Composition is by stacking. A closed loop can be deleted in exchange for the scalar $q + q^{-1}$. The tensor product of objects is given by $n \otimes m = n + m$. The tensor product $f \otimes g$ of two diagrams $f$ and $g$ is obtained by placing $f$ to the left of $g$.

We also allow diagrams with crossings, which are defined as follows:

$$\begin{vmatrix}
\hline
\end{vmatrix} = \sqrt{-q}$$

Crossings satisfy Reidemeister moves two and three (as proved in [Kau90]).

We extend $\text{TL}$ to $\text{TL}^*$ by introducing diagrams with univalent vertices. A vertex is the endpoint of a strand, lying in the interior of the diagram. We require that, at every vertex, the strand must have a horizontal tangent vector, and must be given an orientation either into or out of the vertex. Unlike ordinary Temperley-Lieb diagrams, a diagram with vertices is not considered up to planar isotopy. Instead, we only allow planar isotopies that preserve the horizontal tangent vector at each vertex. We also impose the following turning, confetti, and cutting relations.

The turning relations let us rotate a vertex at the expense of a power of $\sqrt{q}$.

$$\begin{vmatrix}
\hline
\end{vmatrix} = \begin{vmatrix}
\hline
\end{vmatrix} = \frac{1}{\sqrt{q}} \begin{vmatrix}
\hline
\end{vmatrix}, \quad \frac{1}{\sqrt{q}} \begin{vmatrix}
\hline
\end{vmatrix} = \begin{vmatrix}
\hline
\end{vmatrix} = \sqrt{q} \begin{vmatrix}
\hline
\end{vmatrix}.$$

The confetti relations let us eliminate any straight strand that has univalent vertices at both ends.

$$\begin{vmatrix}
\hline
\end{vmatrix} = \begin{vmatrix}
\hline
\end{vmatrix} = 0, \quad \begin{vmatrix}
\hline
\end{vmatrix} = 1.$$

The cutting relation lets us replace a strand with a sum of “cut” strands with the two possible orientations.

$$\begin{vmatrix}
\hline
\end{vmatrix} = \begin{vmatrix}
\hline
\end{vmatrix} + \begin{vmatrix}
\hline
\end{vmatrix}.$$

Note that univalent vertices do not interact particularly well with crossings. There is no relation to let you pass a strand over or under a vertex, and the orientation on a strand may change when it goes through a crossing.

3. THE HOPF ALGEBRA $H$

In this section, we define a Hopf algebra $H$ consisting of formal linear combinations of certain diagrams. The diagrams in $H$ are similar to those in $\text{TL}^*$, but with a special straight vertical edge called the pole. No other strands are allowed to have endpoints on the top or bottom of the diagram. Strands are allowed to cross over or under the pole.

The turning, confetti, and cutting relations from $\text{TL}^*$ still hold in $H$. We also allow Reidemeister moves involving the pole. That is, we impose the relations

$$\begin{vmatrix}
\hline
\end{vmatrix} = \begin{vmatrix}
\hline
\end{vmatrix} = \begin{vmatrix}
\hline
\end{vmatrix}, \quad \begin{vmatrix}
\hline
\end{vmatrix} = \begin{vmatrix}
\hline
\end{vmatrix}.$$
A DIAGRAMMATIC DEFINITION OF $U_q(sl_2)$ and their horizontal reflections. (Other versions of Reidemeister three follow from Reidemeister two and the definition of a crossing.)

The product $xy$ of two diagrams $x$ and $y$ in $H$ is obtained by stacking $x$ on top of $y$.

The tensor product $x \otimes y$ of two diagrams $x$ and $y$ in $H$ is obtained by placing $x$ to the left of $y$, resulting in a diagram with two poles. In general, any diagram $z$ with two poles represents an element of $H \otimes H$. If $z$ contains strands that go from one pole to the other, then use the cutting relation to write $z$ as a sum of tensor products of diagrams from $H$.

The coproduct $\Delta: H \to H \otimes H$ acts on any diagram by removing the pole and threading two parallel poles in its place. Every crossing where a strand passes over (or under) the pole becomes a pair of crossings where the strand passes over (or under) both poles.

The counit $\epsilon$ acts on any diagram $x$ by deleting the pole. The result is a scalar multiple of the empty diagram in $TL^*$, and $\epsilon(x)$ is defined to be that scalar.

The antipode $S: H \to H$ acts on any diagram by a planar isotopy that rotates the pole clockwise through an angle of 180 degrees. Throughout the isotopy, we must preserve the horizontal tangent vectors at every vertex. The result is the same as rigidly rotating the diagram and then multiplying the result by $\sqrt{q}$ to the power of the number of inward oriented vertices minus the number of outward oriented vertices.

**Lemma 3.1.** $H$ satisfies the axioms of a Hopf algebra.

**Proof.** It is easy to check that $H$ satisfies the axioms of a bialgebra. It remains to check that the antipode satisfies:

$$\nabla \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \epsilon = \nabla \circ (\text{id} \otimes S) \circ \Delta,$$

where, $\eta$ is the unit and $\nabla$ is multiplication.

Here is a schematic representation of the effect of $\nabla \circ (S \otimes \text{id}) \circ \Delta$ on a diagram:

Here, $S \otimes \text{id}$ rotates the left pole clockwise, bringing it above the other pole. As usual, the vertices do not rotate throughout this isotopy. Although the resulting diagram is oddly shaped and has one pole on top of the other, it still represents an element of $H \otimes H$ by the same construction as when the poles are side by side.

Consider the last of the above sequence of four diagrams. The curved part of the rectangle represents a collection of parallel strands that can be moved off the pole, one by one, using Reidemeister two. Thus the entire collection of strands can be slid off the pole to the right. We can then use an isotopy to straighten out the rectangle again. The overall effect is to delete the pole from the original diagram and insert a new pole some distance to the left. But this exactly describes the action of $\eta \circ \epsilon$. Thus

$$\nabla \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \epsilon.$$

An upside-down version of this argument works for $\nabla \circ (\text{id} \otimes S) \circ \Delta$. \qed
4. REPRESENTATIONS OF $H$

We define a morphism $\rho$ from $H$ to the algebra of automorphisms of the object 1 in $\mathbf{TL}^\bullet$. Let $\rho$ act on any diagram by replacing the pole with an ordinary strand. Extend this by linearity to an algebra morphism of all of $H$. Using the coproduct on $H$, we see that $\rho^\otimes n$ acts on a diagram by threading $n$ parallel strands in place of the pole.

The most important relationship between $H$ and $\mathbf{TL}$ is that their actions "intertwine", as follows.

**Theorem 4.1.** Suppose $h \in H$ and $f$ is a morphism in $\mathbf{TL}$ from $n$ to $m$. Then

$$\rho^\otimes m(h) \circ f = f \circ \rho^\otimes n(h)$$

in the category $\mathbf{TL}^\bullet$.

**Proof.** We can assume $h$ and $f$ are diagrams. To obtain $\rho^\otimes m(h) \circ f$, replace the pole in $h$ with $m$ parallel strands and attach $f$ to the bottom. To obtain $f \circ \rho^\otimes n(h)$, replace the pole in $h$ with $n$ parallel strands and attach $f$ to the top. The resulting diagrams represent the same element of $\mathbf{TL}^\bullet$, since we can use Reidemeister moves to slide $f$ through $h$. □

The above theorem expresses the fact that $\mathbf{TL}$ is a category of tensor powers of the fundamental two-dimensional representation of $U_q(\mathfrak{sl}_2)$, and $U_q(\mathfrak{sl}_2)$-module morphisms between them. I do not know a diagrammatic proof that $\mathbf{TL}$ includes all such morphisms.

5. GENERATORS AND RELATIONS IN $H$

We define the following elements of $H$.

$$e = \quad , \quad e_0 = \quad , \quad k = \quad , \quad k' = \quad ,$$

$$f = \quad , \quad f_0 = \quad , \quad \ell = \quad , \quad \ell' = \quad ,$$

**Lemma 5.1.** $H$ is generated by the above eight elements.

**Proof.** Start with an arbitrary diagram in $H$. Apply the definition of a crossing to eliminate any crossings that do not involve the pole. Use the cutting relation to cut all strands into segments that cross the pole at most once. Use the turning relations to straighten out all of the strands. Finally, use the confetti relations to eliminate any strands that do not cross the pole. We are left with only horizontal segments that cross the pole exactly once. There are eight possibilities, depending on the two orientations and the nature of the crossing. These are the eight generators above. □

We now give the Hopf algebra structure of $H$. To save space, we only list the four generators in which the strand passes under the pole. These calculations remain the same if we switch the crossing.
Lemma 5.2. In $H$, the coproduct satisfies:
\[ \Delta(e) = e \otimes k + k' \otimes e, \quad \Delta(e_0) = e_0 \otimes k' + k \otimes e_0, \]
\[ \Delta(k) = k \otimes k + e_0 \otimes e, \quad \Delta(k') = k' \otimes k' + e \otimes e_0, \]
the counit satisfies:
\[ \epsilon(e) = \epsilon(k) = \epsilon(k') = 0, \quad \epsilon(e_0) = 1, \]
and the antipode satisfies:
\[ S(e) = qe, \quad S(e_0) = q^{-1}e_0, \quad S(k) = k', \quad S(k') = k. \]

Proof. These follow immediately from the definitions. \(\square\)

We list some relations satisfied by the generators of $H$. We do not attempt a complete presentation of $H$, since we will soon be taking a quotient anyway.

Lemma 5.3. $H$ satisfies the relations
\[ \bullet \quad k'k + q^{-1}ee_0 = 1, \]
\[ \bullet \quad kk' + qee_0 = 1, \]
\[ \bullet \quad ek' + qk'e = 0, \]
\[ \bullet \quad ef - fe = (q - q^{-1})(\ell k - k' \ell'). \]

Proof. The first three relations follow from Reidemeister two:
\[ \begin{array}{c}
\text{Diagram 1} = \sqrt{q}, \quad \text{Diagram 2} = \frac{1}{\sqrt{q}}, \quad \text{Diagram 3} = 0.
\end{array} \]

The fourth relation follows from Reidemeister three:
\[ \begin{array}{c}
\text{Diagram 4} = \text{Diagram 5}.
\end{array} \]

In each case, we can express the diagrammatic relation in terms of the generators of $H$, using the method described in the proof of Lemma 5.1. After some algebraic manipulation, we obtain the desired relations. \(\square\)

6. Connection to $U_q(\mathfrak{sl}_2)$

Definition 6.1. Let $H'$ be the quotient of $H$ by the intersection of the kernels of all $\rho \otimes \rho$.

Theorem 6.2. $H'$ has generators $e$, $f$, $k$ and $k^{-1}$, which satisfy the relations:
\[ \bullet \quad kk^{-1} = k^{-1}k = 1, \]
\[ \bullet \quad ek = -q^{-1}ke, \]
\[ \bullet \quad f k = -q k f, \]
\[ \bullet \quad ef - fe = (q - q^{-1})(k^2 - k^{-2}). \]

The Hopf algebra structure on $H'$ is given by the coproduct:
\[ \bullet \quad \Delta(e) = e \otimes k + k^{-1} \otimes e, \]
\[ \bullet \quad \Delta(f) = f \otimes k + k^{-1} \otimes f, \]
\[ \bullet \quad \Delta(k^\pm) = k^\pm \otimes k^\pm, \]
the counit:
\[ \bullet \quad \epsilon(e) = \epsilon(f) = 0, \]
\[ \text{Lemma 6.4.} \] The Cartan involution is a bialgebra automorphism of \( H \), preserves the kernel of \( \rho \otimes n \) for all \( n \), and satisfies \( \theta \circ S = S^{-1} \circ \theta \).

**Proof.** The bialgebra operations on \( H \) have diagrammatic descriptions that are easily seen to commute with \( \theta \). We can also say \( \rho \otimes n \) commutes with \( H \), if we interpret \( \theta \) as acting on \( TL^\bullet \) in the obvious way. Finally, \( \theta \) does not commute with \( S \), but instead reverses the direction of rotation of the pole in the definition of \( S \). \( \square \)

**Lemma 6.5.** For all \( n \geq 0 \), we have:
\[ \begin{align*}
\rho^{\otimes n}(e_0) &= \rho^{\otimes n}(f_0) = 0, \\
\rho^{\otimes n}(k) &= \rho^{\otimes n}(\ell), \\
\rho^{\otimes n}(k') &= \rho^{\otimes n}(\ell').
\end{align*} \]

**Proof.** The proof is by induction on \( n \). The case \( n = 0 \) is easy. The case \( n = 1 \) is a simple computation involving diagrams with a single crossing. For \( n > 1 \), use the formulae for the coproduct taken from Lemma 5.2 and Lemma 6.4. \( \square \)

**Proof of Theorem 6.2.** Let \( k^{-1} = k' \) and combine Lemmas 5.2, 5.3, 6.4 and 6.5. \( \square \)

**Corollary 6.6.** \( U_q(\mathfrak{sl}_2) \), as defined in Kassel’s textbook [Kas95], maps onto the algebra of words of even length in the generators \( e, f \) and \( k^{\pm 1} \) of \( H' \).

**Proof.** The homomorphism is:
\[ \begin{align*}
K^{\pm 1} &\mapsto k^{\pm 1}, \\
e &\mapsto \frac{1}{q-q^{-1}}ek, \\
f &\mapsto \frac{1}{q-q^{-1}}k^{-1}f.
\end{align*} \]

The most interesting relations given in [Kas95] are:
\[ \begin{align*}
EK &= q^{-2}KE, & FK &= q^2KF, & EF - FE &= (K - K^{-1})/(q - q^{-1}), \\
\Delta(E) &= 1 \otimes E + E \otimes K, & \Delta(F) &= K^{-1} \otimes F + F \otimes 1, \\
S(E) &= -EK^{-1}, & S(F) &= -KF.
\end{align*} \]

The other relations are the definition of \( \epsilon \) and some obvious relations involving only \( K^{\pm 1} \). All of these are easy to check by hand. \( \square \)
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