THE IWASAWA MAIN CONJECTURE FOR SEMISTABLE ABELIAN VARIETIES OVER FUNCTION FIELDS

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Abstract. We prove the Iwasawa Main Conjecture over the arithmetic $\mathbb{Z}_p$-extension for semistable abelian varieties over function fields of characteristic $p > 0$.

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1. INTRODUCTION

We prove in this paper an important case of the Iwasawa Main Conjecture for abelian varieties over function fields. We fix a prime number $p$ ($p = 2$ is allowed). Let $K$ be a global field of characteristic $p$. Let $K_{\infty}^{(p)}$ be the unramified $\mathbb{Z}_p$-extension of $K$ (called the arithmetic extension of $K$), with $\Gamma := \text{Gal}(K_{\infty}^{(p)}/K)$, and write $Q(\Lambda)$ for the fraction field of the Iwasawa algebra $\Lambda := \mathbb{Z}_p[[\Gamma]]$. Let $A$ be an abelian variety over $K$ with semistable reduction. Let $X_p(A/K_{\infty}^{(p)})$ denote the Pontryagin dual of the Selmer group $\text{Sel}_p^\infty(A/K_{\infty}^{(p)})$. Then $X_p(A/K_{\infty}^{(p)})$ is finitely generated over $\Lambda$, hence we can define the characteristic ideal $\chi(X_p(A/K_{\infty}^{(p)})).$ It is a principal ideal of $\Lambda$, and we let $c_{A/K_{\infty}^{(p)}} \in \Lambda$ denote a generator, which is unique up to elements in $\Lambda^\times$.

For $\omega$ a continuous character of $\Gamma$ and $T$ a finite set of places of $K$, let $L_T(A, \omega, s)$ denote the twisted Hasse-Weil $L$-function of $A$ with the local factors at $T$ taken away. In practise $T$ will consist of those places where $A$ has bad reduction. Our main theorem is the following:

**Theorem 1.1.** There exists a “$p$-adic $L$-function” $L_{A/K_{\infty}^{(p)}} \in Q(\Lambda)$ such that for any continuous character $\omega: \Gamma \to \mathbb{C}^\times$, $\omega(L_{A/K_{\infty}^{(p)}})$ is defined and

$$\omega(L_{A/K_{\infty}^{(p)}}) = L_T(A, \omega, 1).$$

Furthermore,

$$L_{A/K_{\infty}^{(p)}} \equiv \star_{A/K_{\infty}^{(p)}} \cdot c_{A/K_{\infty}^{(p)}} \mod \Lambda^\times.$$  

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The interpolation formula (1) is proven in Theorems 3.1.5, while (2) is the content of Theorems 1.3. For the precise expression of $\star_{A,K^{(p)}}$ we refer to Proposition 2.2.4.

1.0.1. A closer look at our result. In order to explain our results, we need first to introduce some cohomology groups and operators between them. We write $\mathbb{C}/\mathbb{F}$ for the smooth proper geometrically connected curve which is the model of the function field $K$ over its field of constants $\mathbb{F}$. Let $C_\infty := C \times_p k^{(p)}_\infty$ (where $k^{(p)}_\infty$ denotes the $\mathbb{Z}_p$-extension of $\mathbb{F}$) and $\pi: C_\infty \to C$ be the pro-étale covering with Galois group $\text{Gal}(K^{(p)}/K)$.

Let $\mathcal{A}$ denote the Néron model of $A$ over $C$. Let $\mathbb{Z}$ be the finite set of points where $A$ has bad reduction. Denote by $\text{Lie}(\mathcal{A})$ the Lie algebra of $\mathcal{A}$. Let $L^{j\infty}$ be the $i$th cohomology group of $\mathbb{R}\Gamma(C_\infty, \pi^*\text{Lie}(\mathcal{A})(-Z)) \otimes \mathbb{Q}_p/\mathbb{Z}_p$.

Theorem 1.2. For $i=0,1,2$ and $j=0,1$, $(N^i_\infty)^\vee$ and $(L^j_\infty)^\vee$ are finitely generated torsion $\Lambda$-modules. Here $\vee$ denotes the Pontryagin dual. The proof shall be given in Corollaries 2.1.5 and 2.1.11. Note that, by [KT03], $X_p(A/K^{(p)}_\infty)$ as a submodule of $(N^1_\infty)^\vee$ is also $\Lambda$-torsion.

Formula (17) in section 2.2 will define $f_{A/K^{(p)}_\infty} \in \mathbb{Q}(\Lambda)/\Lambda^\times$ as the alternating product of determinants of the action of “$1$–Frobenius” on the log crystalline cohomology of $D(-Z)$ (see Section 3.1.2 for the precise expression). Theorem 3.1.5 proves that $L_{A/K^{(p)}_\infty}$ satisfies the interpolation formula (1), with $T=Z$.

Finally, we can state our analogue of the Iwasawa Main Conjecture:

Theorem 1.3. Let $A$ be an abelian variety with at worst semistable reduction relative to the arithmetic extension $K^{(p)}_\infty/K$. We have the following equality in $\mathbb{Q}(\Lambda)^\times/\Lambda^\times$:

$$L_{A/K^{(p)}_\infty} = f_{A/K^{(p)}_\infty}$$

The proof is based on a generalization of a lemma of $\sigma$-linear algebra that was used to prove the cohomological formula of the Birch and Swinnerton-Dyer conjecture (see [KT03, Lemma 3.6]).

Section 2.2 investigates the consequences of Theorem 1.3 in the direction of a $p$-adic Birch and Swinnerton-Dyer conjecture. The following result can be seen as a geometric analogue of the conjecture of Mazur-Tate-Teitelbaum ([MTT86]):
Theorem 1.4. Assume that $A/K$ has semistable reduction. Then

$$\text{ord}(L_{A/K}(p)) = \text{ord}_{s=1}(L_{Z}(A, s)) \geq \text{rank}_{\mathbb{Z}} A(K).$$

If moreover $A/K$ verifies the Birch and Swinnerton-Dyer Conjecture, the inequality above becomes an equality and

$$|L(L_{A/K}(p))|^{-1} = c_{BSD} \cdot |(N_{A^2})^{\Gamma}| \mod \mathbb{Z}_p^\times,$$

where $c_{BSD}$ is the leading coefficient at $s = 1$ of $L_{Z}(A, s)$.

Here $\text{ord}$ denotes the “analytic rank” and $L$ the “leading coefficient” of power series in $\Lambda$ (see [3.2.2] for precise definitions). The proof will be provided in Theorem 3.2.6 Proposition 2.2.3 will prove that very often the error term $|(N_{A^2})^{\Gamma}|$ is just 1.

It seems difficult at the present to remove the semistable hypothesis and will require a delicate study of the integral $p$-adic cohomology computed over (wildly) ramified extensions.

Finally we would like to mention to the readers [LLTT1] for a proof of the Iwasawa Main conjecture for constant ordinary abelian varieties over general global fields) and [LLTT2], for a case of non-torsion Selmer group where the Main conjecture nevertheless holds.

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2. Syntomic cohomology of abelian varieties

2.0.2. The arithmetic tower. We fix the notations. For any $n \geq 0$, let $k_n/\mathbb{F}$ be the $\mathbb{Z}/p^n\mathbb{Z}$-extension of $\mathbb{F}$ and let $k^\infty_n := \bigcup_{n \geq 0} k_n$ denote the induced $\mathbb{Z}_p$-extension of $\mathbb{F}$. Thus our tower becomes $K_n := Kk_n$ and $K^\infty_n := Kk^\infty_n$. Since they are canonically isomorphic, we identify $\Gamma$, $\Gamma_n$ and $\Gamma^{(n)}$ with $\text{Gal}(k^\infty_n/\mathbb{F})$, $\text{Gal}(k_n/\mathbb{F})$ and $\text{Gal}(k^\infty_n/k_n)$ respectively. We denote by $Fr_q$ the generator of $\text{Gal}(k^\infty_n/\mathbb{F})$, $x \mapsto x^q$.

Let $C/\mathbb{F}$ denote the smooth proper geometrically connected curve which is the model of $K$ over $\mathbb{F}$. Let $C_{\infty} := C \times_{\mathbb{F}} k^\infty_n$ and $C_n := C \times_{\mathbb{F}} k_n$. Let $\pi: C_{\infty} \rightarrow C$ and $\pi_n: C_n \rightarrow C$ denote the (pro) étale covering with Galois group $\Gamma$ and $\Gamma_n$ respectively. By abuse of notation, we will also denote by $\pi$ and $\pi_n$ the associated morphisms in the log crystalline topos (BBMS2).

2.1. The cohomology.

2.1.1. The Dieudonné crystal. Let $\mathcal{A}$ denote the Néron model of $A$ over $C$. Let $U$ be the dense open subset of $C$ where $A$ has good reduction and $Z := C - U$ the finite (possibly empty) set of points where $A$ has bad (at worst semistable) reduction. We endow $C$ with the log structure induced by the smooth divisor $Z$ and denote $C^\#$ this log-scheme. Let $D$ be the (covariant) log Dieudonné crystal over $C^\#/W(\mathbb{F})$ associated with $A/K$ as constructed in [KT03 IV]. Recall the following theorem of [KT03]:
Theorem 2.1.1. ([KT03]§5.4(b) and §5.5) Let \( i \) be the canonical morphism of topoi of \([BBM82]\) from the topos of sheaves on \( C_{\text{et}} \) to the log crystalline topos \((C^\# \otimes W(F))^\text{crys}\). There exists a surjective map of sheaves \( D \rightarrow i_*(\text{Lie}(A)) \) in \((C^\# \otimes W(F))^\text{crys}\).

2.1.2. A distinguished triangle. We denote by \( D^0 \) the kernel of \( D \rightarrow i_*(\text{Lie}(A)) \) in the topos \((C^\# \otimes W(F))^\text{crys}\). Let \( 1 : D^0 \rightarrow D \) be the natural inclusion. By applying the canonical projection \( u_* \) from the log crystalline topos \((C^\# \otimes W(F))^\text{crys}\) to the topos of sheaves on \( C_{\text{et}} \), we get a distinguished triangle:

\[
Ru_*D^0 \xrightarrow{1} Ru_*D \rightarrow \text{Lie}(A).
\]

We can twist this triangle by the divisor \( Z \) to get a triangle:

\[
Ru_*D^0(-Z) \xrightarrow{1} Ru_*D(-Z) \rightarrow \text{Lie}(A)(-Z).
\]

(3) where \( D(-Z) \) is the twist of the log Dieudonné crystal \( D \) defined in [KT03] §5.11.

2.1.3. The syntomic complex. In [KT03] §5.8, a Frobenius operator \( \varphi : Ru_*D^0(-Z) \rightarrow Ru_*D(-Z) \)

is constructed. We denote by \( S_D \) the mapping fiber of the map

\[
1 - \varphi : Ru_*D^0(-Z) \rightarrow Ru_*D(-Z).
\]

This complex is an object in the derived category of complexes of sheaves over \( C_{\text{et}} \) and we have a distinguished triangle:

\[
S_D \rightarrow Ru_*D^0(-Z) \xrightarrow{1-\varphi} Ru_*D(-Z).
\]

(4)

2.1.4. The cohomology theories. We define the following modules:

1. Let

\[
P_n^i := H_n^i(\text{crys}(C^\#_n \otimes W(k_n), \pi_n^*D(-Z))).
\]

Then for any \( n \), \( P_n^i \) is a finitely generated \( W(k_n) \)-module endowed with a \( \text{Fr}_q \)-linear operator \( F_{i,n} \) induced by the Frobenius operator of the Dieudonné crystal. Using the (log) crystalline base change by the morphism of topoi \( \pi_n : (C^\#_\infty \otimes W(k_n))^\text{crys} \rightarrow (C^\# \otimes W(F))^\text{crys} \) ([Ka94] §2.5.2]) and by flatness of the extensions \( W(k_n)/Z_p \), we have, for \( n \geq 1 \),

\[
P_n^i \simeq P_0^i \otimes W(k_n).
\]

These isomorphisms identify the \( \text{Fr}_q \)-linear operator \( F_{i,n} \) on the left hand side with the \( \text{Fr}_q \)-linear operator \( F_{i,0} \otimes \text{Fr}_q \) on the right hand side.

2. Let \( M_{1,\infty}^i \) be the \( i \)th cohomology group of

\[
\Gamma(\text{crys}(C^\#_\infty \otimes W(k^{(p)}), \pi^*D^0(-Z)) \otimes^L \mathbb{Q}_p/Z_p).
\]

3. Let \( M_{2,\infty}^i \) be the \( i \)th cohomology group of

\[
\Gamma(\text{crys}(C^\#_\infty \otimes W(k^{(p)}), \pi^*D(-Z)) \otimes^L \mathbb{Q}_p/Z_p).
\]

4. Let \( M_{1,n}^i \) be the \( i \)th cohomology group of

\[
\Gamma(\text{crys}(C^\#_n \otimes W(k_n), \pi_n^*D^0(-Z)) \otimes^L \mathbb{Q}_p/Z_p).
\]
Let $M_{2,n}^i$ be the $i$th cohomology group of
\[ \mathbb{R}\Gamma_{\text{crys}}(C_n^\# / W(k_n), \pi_n^* D(-Z)) \otimes L_q \mathbb{Q}_p / \mathbb{Z}_p. \]
Again by the base change theorem, we have, for $k = 1, 2$, an isomorphism of torsion $W(k_n)$-modules
\[ M_{k,\infty}^i \simeq M_{k,0}^i \otimes W(k_n) \tag{6} \]
and for any $n \geq 0$ an isomorphism of torsion $W(k_n)$-modules
\[ M_{k,n}^i \simeq M_{k,0}^i \otimes W(k_n) \]
identifying the $\text{Fr}_q$-linear operator $1 - \varphi_{i,n}$ on the left hand side with the $\text{Fr}_q$-linear operator $1 \otimes id - \varphi_{i,0} \otimes \text{Fr}_q$ on the right hand side.

Let $L^i_\infty$ be the $i$th cohomology group of
\[ \mathbb{R}\Gamma(C_\infty, \pi^* \text{Lie}(A(-Z))) \otimes L_q \mathbb{Q}_p / \mathbb{Z}_p = \mathbb{R}\Gamma(C_\infty, \pi^* \text{Lie}(A(-Z))) [1]. \tag{7} \]

Let $L^i_n$ be the $i$th cohomology group of
\[ \mathbb{R}\Gamma(C_n, \pi_n^* \text{Lie}(A(-Z))) \otimes L_q \mathbb{Q}_p / \mathbb{Z}_p = \mathbb{R}\Gamma(C_n, \pi_n^* \text{Lie}(A(-Z))) [1]. \tag{8} \]
By the Zariski base change formula (note that the cohomology of the finite locally free module $\text{Lie}(A)(-Z)$ is the same in the étale or Zariski site), we have isomorphisms
\[ L^i_\infty \simeq L^i_0 \otimes W(k_n^\infty) \]
and, for any $n \geq 0$,
\[ L^i_n \simeq L^i_0 \otimes W(k_n). \]
In particular, since $L^i_0$ is a finite $\mathbb{F}_p$-vector space with rank $d(L^i_0)$, we deduce that $L^i_\infty$ is a finite $k_n^\infty$-vector space while $L^i_n$ is a finite $k_n$-vector space, both with the same rank $d(L^i_0)$.

Let
\[ N_i^\infty := H^i_{\text{syn}}(C_\infty, \pi^* S_D \otimes \mathbb{Q}_p / \mathbb{Z}_p) \]
be the $i$th cohomology group of
\[ \mathbb{R}\Gamma(C_\infty, \pi^* S_D) \otimes L_q \mathbb{Q}_p / \mathbb{Z}_p. \tag{9} \]
Let $N_i^\infty$ be the $i$th cohomology group of
\[ \mathbb{R}\Gamma(C_n, \pi_n^* S_D) \otimes \mathbb{Q}_p / \mathbb{Z}_p. \]
The distinguished triangles $\mathbf{3}$ and $\mathbf{4}$ induce, by passing to the cohomology, the following long exact sequences:
\[ \ldots \longrightarrow N^i_\infty \longrightarrow M^i_{1,\infty} \longrightarrow \quad 1 - \varphi_{i,\infty} \longrightarrow M^i_{2,\infty} \longrightarrow \ldots \tag{7} \]
\[ \ldots \longrightarrow L^i_\infty \longrightarrow M^i_{1,\infty} \longrightarrow \quad 1 \longrightarrow M^i_{2,\infty} \longrightarrow \ldots \tag{8} \]
which are inductive limits of the long exact sequences
\[ \ldots \longrightarrow N^i_n \longrightarrow M^i_{1,n} \longrightarrow \quad 1 - \varphi_{i,n} \longrightarrow M^i_{2,n} \longrightarrow \ldots \tag{9} \]
\[ \ldots \longrightarrow L^i_n \longrightarrow M^i_{1,n} \longrightarrow \quad 1 \longrightarrow M^i_{2,n} \longrightarrow \ldots \tag{10} \]
Note that the cohomology theories $M$ and $N$ are concentrated in degrees 0, 1 and 2 and the cohomology theory $L$ is concentrated in degrees 0 and 1.
**Theorem 2.1.2.** For \( i = 0, 1, 2 \) and \( j = 0, 1 \), the Pontryagin duals of \( N_i^\infty \) and \( L_i^j \) are finitely generated torsion \( \Lambda \)-modules.

The proof shall be given in Corollaries 2.1.3 and 2.1.11. Note that, by [KT03], \( X_p(A/K_\infty^\langle p \rangle) \) is a submodule of the dual of \( N_\infty^\infty \), so Theorem 2.1.2 shows that under our hypothesis \( X_p(A/K_\infty^\langle p \rangle) \) is also a finitely generated torsion \( \Lambda \)-module. This was already known by [OT09], whose argument is simplified in the present paper.

In [KT03], the following was proved (see [KT03] §3.3.3, §3.3.4 and §3.3.5):

**Lemma 2.1.3.** For \( k = 1 \) or \( 2 \), there exists a map

\[
f_k : P_0^i[p^{-1}] \rightarrow M_{k,0}^i
\]

satisfying the following conditions:

1. The kernel of \( f_k \) is a \( \mathbb{Z}_p \)-lattice in \( P_0^i[p^{-1}] \) and the cokernel is a finite group. In particular, \( M_{1,0}^i \) and \( M_{2,0}^i \) are torsion \( \mathbb{Z}_p \)-modules with the same finite corank.
2. The diagrams

\[
\begin{array}{ccc}
P_0^i[p^{-1}] & \xrightarrow{id} & P_0^i[p] \\
\downarrow f_1 & & \downarrow f_2 \\
M_{1,0}^i & \xrightarrow{1} & M_{2,0}^i
\end{array}
\quad
\begin{array}{ccc}
P_0^i[p] & \xrightarrow{p^{-1} f_i} & P_0^i[p] \\
\downarrow f_1 & & \downarrow f_2 \\
M_{1,0}^i & \xrightarrow{\varphi_i} & M_{2,0}^i
\end{array}
\]

commute.

**Lemma 2.1.4.** Let \( M \) be a torsion \( \mathbb{Z}_p \)-module of cofinite type. Then the Pontryagin dual of \( M \otimes \mathbb{Z}_p W(k_\infty^\langle p \rangle) \) is a finitely generated \( \Lambda \)-module. Moreover, we have an isomorphism of \( \Lambda \)-modules

\[
(M \otimes \mathbb{Z}_p W(k_\infty^\langle p \rangle))^\vee \simeq \Lambda^\Gamma \bigoplus_{i=1}^s \Lambda / (p^{n_i})
\]

**Proof.** Let \( X_\infty \) denote the Pontryagin dual of \( M \otimes \mathbb{Z}_p W(k_\infty^\langle p \rangle) \). Then \( X_\infty \) is the limit of the projective system of Pontryagin duals \( X_n := (M \otimes \mathbb{Z}_p W(k_n))^\vee \). Note that the functor \( M \rightsquigarrow (M \otimes \mathbb{Z}_p W(k_\infty^\langle p \rangle))^\vee \) is exact.

In the case \( M = \mathbb{Z}/p\mathbb{Z} \), we have a projective system \( \{X_n = (k_n)^\vee\}_n \) where the transition maps are the Pontryagin dual of the canonical inclusions \( k_n \hookrightarrow k_{n+1} \). Let \( \Omega := F[[\Gamma]] \simeq F[[T]] \). We have

\[
(X_\infty)^\Gamma = X_\infty/TX_\infty = ((k_\infty^\langle p \rangle)^\Gamma)^\vee \simeq F.
\]

Now, by Nakayama’s lemma, this implies that \( X_\infty \) is a cyclic \( \Omega \)-module and since \( X_\infty \) is infinite we have \( X_\infty \simeq \Omega = \Lambda / p \Lambda \).

If \( M = \mathbb{Z}/p^j\mathbb{Z} \), then \( X_\infty = (W_j(k_\infty^\langle p \rangle))^\vee = \lim_{n \rightarrow \infty} (W_j(k_n))^\vee \equiv Y_j \) (where \( W_j \) denotes Witt vectors of length \( j \)) and we prove by the same argument that \( Y_j \) is a cyclic \( W_j(F)[[\Gamma]] \)-module. In particular for each \( j \) we have a surjective map:

\[
W_j(F)[[\Gamma]] \rightarrow Y_j.
\]

We prove by induction on \( j \) that \( Y_j \) is a free \( W_j(F)[[\Gamma]] \)-module. The case \( j = 1 \) was treated above. Now assume the assertion is true for \( j - 1 \) and consider the commutative diagram
of short exact sequences:

\[
\begin{array}{cccccc}
0 & \longrightarrow & Y_{j-1} & \overset{\delta}{\longrightarrow} & Y_j & \overset{\epsilon}{\longrightarrow} & Y_1 & \longrightarrow & 0 \\
\alpha \uparrow & & \beta \uparrow & & \gamma \uparrow & & \\
0 & \longrightarrow & W_{j-1}(F)[[\Gamma]] & \overset{x_p}{\longrightarrow} & W_j(F)[[\Gamma]] & \longrightarrow & \Omega & \longrightarrow & 0.
\end{array}
\]

The upper horizontal line is induced by the exact sequence \( \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p^i\mathbb{Z} \rightarrow \mathbb{Z}/p^{i-1}\mathbb{Z} \).

The vertical maps are constructed as follows: define first \( \beta \) as the map sending 1 to a generator \( x \) of the cyclic \( W_j(F)[[\Gamma]] \)-module \( Y_j \). Then \( px \in \text{Ker}(\epsilon) = \text{Im}(\delta) \). Let \( w \in Y_{j-1} \) be such that \( px = \delta(w) \) and let \( y = \epsilon(x) \). Finally, define \( \alpha \) to be the map sending 1 to \( w \) and \( \gamma \) to be the map sending 1 to \( y \). The map \( \gamma \) is surjective because \( \beta \) and \( \epsilon \) are surjective. So \( \gamma \) is an isomorphism since \( Y_1 \) is infinite and the only proper quotients of \( \Omega \) are finite. We deduce by the snake lemma that \( \alpha \) is surjective and therefore, by the induction hypothesis, an isomorphism. This implies that \( \beta \) is also an isomorphism, by the 5-lemma.

The case \( M = \mathbb{Q}_p/\mathbb{Z}_p \) is deduced from the case \( M = \mathbb{Z}/p^i\mathbb{Z} \) by passing to the inductive limit in \( j \) and the general case follows from the cases \( M = \mathbb{Q}_p/\mathbb{Z}_p \) and \( M = \mathbb{Z}/p^i\mathbb{Z} \). \( \square \)

We deduce from Lemmas 2.1.4 and 2.1.3

**Corollary 2.1.5.**

1. For any \( i \geq 0 \), \( (M_i^1)\mathcal{V} \) and \( (M_i^2_\infty)\mathcal{V} \) are two \( \Lambda \)-modules of finite type with the same rank \( r_i \) (equal to the \( \mathbb{Z}_p \)-rank of \( P_i^0 ) \) and torsion parts isomorphic to \( \bigoplus_{j=1}^\infty \Lambda /p^{r_j} \Lambda \).

2. For any \( i \), \( (L_i^1)\mathcal{V} \) is a finitely generated torsion \( \Lambda \)-module isomorphic to \( \bigoplus_{j=1}^\infty \Lambda /p \Lambda \).

3. Passing to the Pontryagin dual, the long exact sequences 7 and 8 induce long exact sequences of \( \Lambda \)-modules with \( \Lambda \)-linear operators.

**Proof.** The first assertion is a consequence of Lemmas 2.1.4 and 2.1.3. Assertion (2) is proved similarly to the case of \( M_i^1 \). For the third assertion, note that \( \Gamma \) is an abelian group and therefore any \( \tau \in \Gamma \) commutes with \( F_{q_i} \). Hence, the operator \( 1 - \varphi \) is \( \Lambda \)-linear. \( \square \)

Next, we are going to prove that \( (N_i^0)\mathcal{V} \) is a finitely generated torsion \( \Lambda \)-module for \( i = 0, \ldots, 2 \).

**Proposition 2.1.6.** The \( \Lambda \)-modules \( (N_\infty^0)\mathcal{V} \) and \( (\text{Coker}(1 - \varphi_{0,\infty}))\mathcal{V} \) are \( \Lambda \)-torsion.

**Proof.** Reasoning as in [KT03] §2.5.2 (see (10) below), one obtains that the module \( (N_i^0)\mathcal{V} \) is a quotient of \( A(K_{\infty}^0)[p^\infty]\mathcal{V} \). The latter is a finitely generated \( \mathbb{Z}_p \)-module (and so a torsion \( \Lambda \)-module), hence the claim for \( (N_i^0)\mathcal{V} \) is proven. As for \( (\text{Coker}(1 - \varphi_{0,\infty}))\mathcal{V} \), note that we have an exact sequence of \( \Lambda \)-modules

\[
0 \longrightarrow (\text{Coker}(1 - \varphi_{0,\infty}))\mathcal{V} \longrightarrow (M_i^0)\mathcal{V} \overset{(1-\varphi_{0,\infty})\mathcal{V}}{\longrightarrow} (M_i^1)\mathcal{V} \longrightarrow (N_i^0)\mathcal{V} \longrightarrow 0.
\]

Since \( (M_i^0)\mathcal{V} \) and \( (M_i^2_\infty)\mathcal{V} \) have the same \( \Lambda \)-rank, it implies that the \( \Lambda \)-rank of \( (\text{Coker}(1 - \varphi_{0,\infty}))\mathcal{V} \) is equal to that of \( (N_i^0)\mathcal{V} \), which is zero. \( \square \)

We now prove that \( (N_i^0)\mathcal{V} \) is \( \Lambda \)-torsion.

The exact sequence:

\[
\cdots \longrightarrow H_1^1(\text{syn}(C_\infty, \pi^*S_D) \otimes \mathbb{Q}_p \overset{\alpha}{\longrightarrow} H_1^1(\text{syn}(C_\infty, \pi^*S_D \otimes \mathbb{Q}_p/\mathbb{Z}_p)) \overset{\beta}{\longrightarrow} H_2^1(\text{syn}(C_\infty, \pi^*S_D) \otimes \mathbb{Q}_p) \longrightarrow \cdots
\]
induces a short exact sequence

\[ 0 \rightarrow \text{Im}(\alpha) \rightarrow N_1^1 \rightarrow \text{Im}(\beta) \rightarrow 0. \]  

(12)

By taking the Pontryagin dual of \([12]\), we have

\[ 0 \rightarrow \text{Im}(\beta)^\vee \rightarrow (N_1^1)^\vee \rightarrow \text{Im}(\alpha)^\vee \rightarrow 0, \]  

(13)

where the modules and the morphisms are naturally defined over \(\Lambda\).

**Lemma 2.1.7.** \(H^1_{\text{syn}}(C_\infty, \pi^*S_D) \otimes \mathbb{Q}_p\) is a finite dimensional \(\mathbb{Q}_p\)-vector space.

**Proof.** The long exact sequence

\[ \cdots \rightarrow H^i_{\text{syn}}(C_\infty, \pi^*S_D) \otimes \mathbb{Q}_p \rightarrow H^i_{\text{crys}}(C_\infty^#/W(k_\infty^{(p)}), \pi^*D^0(-Z)) \otimes \mathbb{Q}_p \]

\[ 1-\varphi_1 H^i_{\text{crys}}(C_\infty^#/W(k_\infty^{(p)}), \pi^*D(-Z)) \otimes \mathbb{Q}_p \rightarrow \cdots \]

can be rewritten

\[ \cdots \rightarrow H^i_{\text{syn}}(C_\infty, \pi^*S_D) \otimes \mathbb{Q}_p \rightarrow H^i_{\text{crys}}(C_\infty^#/W(k_\infty^{(p)}), \pi^*D(-Z)) \otimes \mathbb{Q}_p \]

\[ 1-\varphi_2 H^i_{\text{crys}}(C_\infty^#/W(k_\infty^{(p)}), \pi^*D(-Z)) \otimes \mathbb{Q}_p \rightarrow \cdots \]

(because the difference between the middle terms in these sequences is \(L_\infty^1 \otimes \mathbb{Q}_p\), which, thanks to Lemma 2.1.5 (2) is trivial).

We deduce from this long exact sequence the following short exact sequence:

\[ 0 \rightarrow \text{Coker}(1 - \varphi_0) \rightarrow H^1_{\text{syn}}(C_\infty, \pi^*S_D) \otimes \mathbb{Q}_p \rightarrow \text{Ker}(1 - \varphi_1) \rightarrow 0. \]

Since \(H^i_{\text{crys}}(C_\infty^#/W(k_\infty^{(p)}), \pi^*D(-Z)) \otimes \mathbb{Q}_p\) is a finite dimensional \(W(k_\infty^{(p)})[\frac{1}{\ell}]\)-vector space and \([\mathbb{Q}_p]\) holds, the assertion is implied by the following:

**Lemma 2.1.8.** Let \(V\) be a finite dimensional \(\mathbb{Q}_p\)-vector space endowed with a linear operator \(\varphi: V \rightarrow V\). Then \(1 - \varphi \otimes \text{Fr}_q: V \otimes_{\mathbb{Z}_p} W(k_\infty^{(p)}) \rightarrow V \otimes_{\mathbb{Z}_p} W(k_\infty^{(p)})\) is a surjective map whose kernel is a finite dimensional \(\mathbb{Q}_p\)-vector space.

**Proof.** One can easily check that the proof of [14, Lemme 6.2] remains true if we replace \(\mathbb{F}_p\) by \(k_\infty^{(p)}\).

\[ \square \]

**Corollary 2.1.9.** The group \(\text{Im}(\alpha)^\vee\) of \([13]\) is a free \(\mathbb{Z}_p\)-module of finite rank.

**Proof.** By \([14]\), \(N_1^1\) is a torsion \(\mathbb{Z}_p\)-module. Thus, \(\text{Im}(\alpha)\) is a torsion \(\mathbb{Z}_p\)-module which is a quotient of \(H^1_{\text{syn}}(C_\infty, \pi^*S_D) \otimes \mathbb{Q}_p\). We deduce from Lemma 2.1.7 that \(\text{Im}(\alpha)\) is cofree of finite corank \(n \leq \dim_{\mathbb{Q}_p}(H^1_{\text{syn}}(C_\infty, \pi^*S_D) \otimes \mathbb{Q}_p)\).

\[ \square \]

We now study the term \(\text{Im}(\beta)\):

**Lemma 2.1.10.** The group \(\text{Im}(\beta)^\vee\) of \([13]\) is a finitely generated torsion \(\Lambda\)-module.

**Proof.** Note that \([11]\) yields an isomorphism \(\text{Im}(\beta) \simeq \text{Ker}(\gamma)\). The kernel of the map

\( \gamma: H^2_{\text{syn}}(C_\infty, \pi^*S_D) \rightarrow H^2_{\text{syn}}(C_\infty, \pi^*S_D) \otimes \mathbb{Q}_p\)

is \(H^2_{\text{syn}}(C_\infty, \pi^*S_D)[\frac{1}{\ell}]\). Recall that we have a short exact sequence:

\[ 0 \rightarrow \text{Coker}(1 - \varphi_{1,\infty}) \rightarrow H^2_{\text{syn}}(C_\infty, \pi^*S_D) \rightarrow \text{Ker}(1 - \varphi_{2,\infty}) \rightarrow 0. \]
By taking the \( p \)-power torsion part of this sequence, we have
\[
0 \to \text{Coker}(1 - \varphi_{1,\infty})[p^\infty] \to \text{Im}(\beta) \to \text{Ker}(1 - \varphi_{2,\infty})[p^\infty].
\] (14)

Taking Pontryagin duals, we have the following:
\[
\text{Ker}(1 - \varphi_{2,\infty})[p^\infty]^\vee \to \text{Im}(\beta)^\vee \to \text{Coker}(1 - \varphi_{1,\infty})[p^\infty]^\vee \to 0,
\] (15)
where the modules and the morphisms are defined over \( \Lambda \). By the sequence (15), it is enough to show that \( \text{Coker}(1 - \varphi_{1,\infty})[p^\infty]^\vee \) and \( \text{Ker}(1 - \varphi_{2,\infty})[p^\infty]^\vee \) are finitely generated torsion \( \Lambda \)-modules. But these two groups are both \( p \)-torsion \( \Lambda \)-modules, so their Pontryagin duals are \( \Lambda \)-torsion by Lemma 2.1.4.

**Corollary 2.1.11.** The groups
\[
(N^1_\infty)^\vee, \ (\text{Ker}(1 - \varphi_{1,\infty}))^\vee, \ (\text{Coker}(1 - \varphi_{1,\infty}))^\vee, \ (\text{Ker}(1 - \varphi_{2,\infty}))^\vee \text{ and } (N^2_\infty)^\vee
\]
are \( \Lambda \)-torsion. In particular, \( X_p(A/K^{(p)}_\infty) \) is \( \Lambda \)-torsion.

**Proof.** The module \( (N^1_\infty)^\vee \) is \( \Lambda \)-torsion by Lemma 2.1.10 and Corollary 2.1.9. By (7), using the short exact sequence
\[
0 \to \text{Coker}(1 - \varphi_{i-1,\infty}) \to N^i_\infty \to \text{Ker}(1 - \varphi_{i,\infty}) \to 0
\] (16)
we find that \( (\text{Ker}(1 - \varphi_{1,\infty}))^\vee \) is \( \Lambda \)-torsion. Then by using the exact sequence
\[
0 \to \text{Ker}(1 - \varphi_{1,\infty}) \to M^1_{1,\infty} \to M^2_{1,\infty} \to \text{Coker}(1 - \varphi_{1,\infty}) \to 0
\]
we deduce that \( (\text{Coker}(1 - \varphi_{1,\infty}))^\vee \) is \( \Lambda \)-torsion. Finally, by the exact sequence
\[
0 \to \text{Ker}(1 - \varphi_{2,\infty}) \to M^2_{1,\infty} \to M^2_{2,\infty} \to 0
\]
we know that \( (\text{Ker}(1 - \varphi_{2,\infty}))^\vee \) is \( \Lambda \)-torsion. Hence, the short exact sequence
\[
0 \to \text{Coker}(1 - \varphi_{1,\infty}) \to N^2_\infty \to \text{Ker}(1 - \varphi_{2,\infty}) \to 0
\]
tells us that \( (N^2_\infty)^\vee \) is also \( \Lambda \)-torsion. For the last assertion, note that the surjections \( N^1_n \to \text{Sel}_{p^n}(A/K_n) \) proved in [KT03 §2.5.2] induce a surjection \( N^1_\infty \to \text{Sel}_{p}(A/K^{(p)}_\infty) \) by passing to the inductive limit in \( n \). But since \( (N^1_\infty)^\vee \) is \( \Lambda \)-torsion, the same assertion holds for \( X_p(A/K^{(p)}_\infty) \).

This completes the proof of Theorem 2.1.2.

### 2.2. The characteristic element.
For any finitely generated \( \Lambda \)-torsion module \( M \), we let \( f_M \in \Lambda \) be a characteristic element associated with \( M \); \( f_M \) is defined uniquely up to \( \Lambda^\times \).

We set
\[
f_{A/K^{(p)}_\infty} := \frac{f(N^1_\infty)^\vee f(L^2_\infty)^\vee}{f(N^2_\infty)^\vee f(L^2_\infty)^\vee} \in \mathbb{Q}(\Lambda)^\times / \Lambda^\times
\] (17)
and call it the characteristic element associated with \( A/K \) relative to the arithmetic \( \mathbb{Z}_p \)-extension of \( K \).
2.2.1. Arithmetic interpretation. The characteristic element $f_{A/K_n^{(p)}}$ is related to the arithmetic invariants of $A$ as follows. By considering the $p$-torsion part of the exact sequence in [KT03] §2.5.2 and using the fact that the functor “take the $p$-primary part” is exact in the category of finite abelian groups, we deduce an exact sequence:

$$0 \to N_0^0 \to A(K)[p^\infty] \to (\oplus_{v \in Z \cup M_v})[p^\infty] \to N_0^1 \to Sel_{p^\infty}(A/K) \to 0,$$

with $M_v := A(K_v)/A(m_v)$, where $O_v$ and $m_v$ are ring of integers and maximal ideal of $K_v$ and

$$A(m_v) := \ker(A(K_v) = A(O_v) \to A(k(v))).$$

Observe that, since we assumed that $A/K$ has semistable reduction, we can take the group $V_v$ defined in [KT03] Proposition 5.13 to be equal to $A(m_v)$. Since $A/O_v$ is smooth we have in fact $M_v = A(k(v))$. The finite group $M_v$ is controlled by the short exact sequence:

$$0 \to Q_v \to M_v \to \Phi_v \to 0,$$

where $\Phi_v$ is the (finite) group of components, $\Phi_v := (A/A^0(k(v)))$, and

$$Q_v = A^0(O_v)/(A(m_v) \cap A^0(O_v)) = A^0(k(v))$$

by Hensel’s lemma. Moreover, since $A/K$ has semistable reduction at $v$, we have a short exact sequence:

$$0 \to T_v \to A_v^0 \to B_v \to 0,$$

where $T_v$ is a torus and $B_v$ an abelian variety over $k(v)$.

Since the Néron model functor is stable by étale base change, we also have for any $n \geq 0$ an exact sequence:

$$0 \to N_n^0 \to A(K_n)[p^\infty] \to (\oplus_{w \in C_n, w \mid \infty} M_w)[p^\infty] \to N_n^1 \to Sel_{p^\infty}(A/K_n) \to 0,$$

which induces, by passing to the inductive limit in $n$, an exact sequence:

$$0 \to N_\infty^0 \to A(K_\infty^{(p)})[p^\infty] \to (\oplus_{w \in C_\infty, w \mid \infty} M_w)[p^\infty] \to N_\infty^1 \to Sel_{p^\infty}(A/K_\infty^{(p)}) \to 0$$

and then, by passing to the Pontryagin dual, an exact sequence of finitely generated torsion $\Lambda$-modules. We set

$$M_n := (\oplus_{w \in C_n, w \mid \infty} M_w)[p^\infty]$$

and $M_\infty := \lim M_n$. By multiplicativity of characteristic elements associated with torsion $\Lambda$-modules, we have:

$$f_{(N_n^0)^\vee} = \frac{f_{N_n^0}(A(K_n^{(p)})) f_{M_\infty^{N_n}}}{f_{A(K_n^{(p)}))[p^\infty]^\vee}}.$$  

Let $\zeta_1, \ldots, \zeta_l$ be the eigenvalues of the Galois actions of $Fr_q$ on $T_p A(K_\infty^{(p)})[p^\infty]$. Then by [Tan12 Proposition 2.3.5] we can write

$$f_{A(K_\infty^{(p)}))[p^\infty]^\vee} = \prod_{i=1}^l (1 - \zeta_i^{-1} Fr_q^{-1}).$$

Lemma 2.2.1. Denote by $g(v)$ the dimension of $B_v$ and let $\beta_1^{(v)}, \ldots, \beta_2g(v)^{(v)} \in \mathbb{Q}_p$ be the eigenvalues of the Frobenius endomorphism $F^deg(v)_{B_v}$ (Here we denote $F_B$ the absolute Frobenius of $B_v$ and $deg(v) := [k(v) : \mathbb{F}_p]$). Then

$$f_{M_\infty^{N_n}} = \prod_{v \in Z} \prod_{i=1}^{2g(v)} (\beta_i^{(v)} - Fr_v^{-1}).$$
Proof. For $v$ a place of $K$, let $\Gamma_v \subset \Gamma$ denote the decomposition group at $v$ and put $\Lambda_v := \mathbb{Z}_p[\mathbb{I}^\vee_v]$. Thus we get $\Lambda = \bigoplus_{\sigma \in \Gamma / \Gamma_v} \sigma \Lambda_v$. For each $v \in \mathbb{Z}$, choose a $w_0 \in C_{\infty}$ sitting over $v$. Then

$$\mathcal{M}_{\infty,v} := \bigoplus_{w \in C_{\infty}, w \mid v} \mathcal{M}_w[p^\infty] = \bigoplus_{\sigma \in \Gamma / \Gamma_v} \sigma \mathcal{M}_{w_0}[p^\infty]$$

and hence $\mathcal{M}_{\infty,v} \mid \Lambda \otimes_{\Lambda_v} \mathcal{M}_{w_0}[p^\infty]$. In particular, the characteristic element $f_{\mathcal{M}_{\infty,v}}$ can be chosen to be that of $\mathcal{M}_{w_0}[p^\infty] \mid \Lambda_v$. Also, since $\Phi_v$ is finite and $T_v$ is a torus, $\mathcal{M}_{w_0}[p^\infty] \mid \Lambda$ is pseudo-isomorphic to $B_v[p^\infty](\mathbb{k}_v)$. For each $v$, we order the eigenvalues $\beta_i(v)$ so that $\beta_i(v)$ is a $p$-adic unit if and only if $i \leq f(v) \leq g(v)$.

Let $Fr_v : x \mapsto x^{q^v}$ denote the Frobenius substitution as an element of $\text{Gal}(\mathbb{k}(v)/k(v))$ (and also, by abuse of notation, the corresponding element of $\text{Gal}(\mathbb{k}(p)/k(v))$ and of $\Gamma_v$). The product $\beta_1(v) \cdots \beta_f(v)$ is a $p$-adic unit and $\prod_{i > f(v)}(\beta_i(v) - Fr_v^{-1})$ is a unit in $\Lambda$. We claim that $\beta_1(v), \ldots, \beta_f(v)$ are the eigenvalues of the action of $Fr_v$ on the Tate module $T_p B_v$. Then by [Tan12 Proposition 2.3.6] we can write

$$f_{\mathcal{M}_{\infty,v}} = \prod_{i=1}^f (\beta_i(v) - Fr_v^{-1}) = \prod_{i=1}^{2g(v)} (\beta_i(v) - Fr_v^{-1})$$

and follows from $f_{\mathcal{M}_{\infty,v}} = \prod_{v \in \mathbb{Z}} f_{\mathcal{M}_{\infty,v}}$.

We have to prove the claim. Let $\rho : \text{End}_{k(v)}(B_v) \to \text{End}(T_p B_v)$ denote the $p$-adic representation. Then for every $f \in \text{End}_{k(v)}(B_v)$ the eigenvalues of $\rho(f)$, counting multiplicities, are a portion of those of $f$ (this can be seen e.g. mimicking the argument in the proof of [GM Theorem 12.18], with $\text{Ker}(f)$ replaced by its maximal étale subgroup). In particular, we can rearrange the order of the $\beta_i(v)$s so that, for every positive integer $N$, the eigenvalues of $\rho(F_{B_v,k_v}) = Fr_v^N$ equal $(\beta_1(v))^N, \ldots, (\beta_N(v))^N$ for some $h(v) \leq f(v)$. Let $k(v)_N$ denote the degree $N$ extension of $k(v)$. Letting $\equiv_p$ denote congruence modulo $p$-adic units, we have

$$\prod_{i=1}^{h(v)} (1 - (\beta_i(v))^N) \equiv_p |B_v[p^\infty]|(k(v)_N) \equiv_p \prod_{i=1}^{2g(v)} (1 - (\beta_i(v))_N) \equiv_p \prod_{i=1}^{f(v)} (1 - (\beta_i(v))^N),$$

where the second equality is from Weil’s formula and the third is from the fact that $1 - (\beta_i(v))^N$ is a $p$-adic unit if $i > f(v)$. Taking $N$ such that $(\beta_i(v))^N \equiv 1 \mod p$ for all $i < f(v)$, we deduce $h(v) = f(v)$. \hfill \Box

Write $\mathbb{A}_K$ for the adelic ring. Let $\mu = (\mu_v)_v$ be the Haar measure on $\text{Lie}(\mathcal{A})(\mathbb{A}_K)$ such that

$$\mu_v(\text{Lie}(\mathcal{A})(O_v)) := 1$$

for every $v$ and let $\alpha_v$ denote the Haar measure on $A(K_v) = A(O_v)$ such that for $n \geq 1$

$$\alpha_v(A(m_v^n)) := \mu_v(\text{Lie}(\mathcal{A})(m_v^n)).$$

Then since $|A(O_v)/A(m_v)| = |M_v|$ and $\text{Lie}(\mathcal{A})(O_v)/\text{Lie}(\mathcal{A})(m_v) \simeq (O_v/m_v)^g$, we have

$$\alpha_v(A(O_v)) = |M_v| \cdot q_v^{-g}. \quad (23)$$

By [KT03 p.552] we have the relation:

$$|M_0| \cdot |L_0^1| \cdot |L_0|^{-1} = \mu(\text{Lie}(\mathcal{A})(\mathbb{A}_K)/\text{Lie}(\mathcal{A})(K))^{-1} \cdot \prod_{v \in \mathbb{Z}} \alpha_v(A(O_v)). \quad (24)$$
Thus, by (23) and (24)
\[ |L_0^0| \cdot |L_0^1|^{-1} = q^{-g \deg(Z)} \cdot \mu(\text{Lie}(A)(\mathbb{A}_K)/\text{Lie}(A)(K))^{-1}. \tag{25} \]

Next, we choose a basis \(e_1, \ldots, e_g\) of the \(K\)-vector space \(\text{Lie}(A)(K) = \text{Lie}(A)(K)\).

Then for every \(v\) the exterior product \(e := e_1 \wedge \cdots \wedge e_g\) determines the Haar measure \(\mu^{(e)}_v\) on \(\text{Lie}(A)(K_v)\) that has measure 1 on the compact subset \(\mathcal{L}(O_v) := O_ve_1 + \cdots + O_ve_g\).

Similarly, if we choose a basis \(f_1, \ldots, f_g\) of \(\text{Lie}(A)(O_v)\) over \(O_v\), then the exterior product \(f_v := f_1 \wedge \cdots \wedge f_g\) actually determines the Haar measure \(\mu_v\). Define the number \(\delta\) by
\[ q^{-\delta} := \prod_{\text{all } v} \frac{\mu^{(e)}_v(\text{Lie}(A)(O_v))}{\mu_v(\text{Lie}(A)(O_v))} = \prod_{\text{all } v} \mu^{(e)}_v(\text{Lie}(A)(O_v)), \tag{26} \]
so that the Haar measures \(\mu\) and \(\mu^{(e)}\) are related by \(\mu^{(e)} = q^{-\delta} \mu\). By a well-known computation (see e.g. [We74, VI, Corollary 1 of Theorem 1]) one finds
\[ \mu^{(e)}(\text{Lie}(A)(\mathbb{A}_K)/\text{Lie}(A)(K)) = q^{g(\kappa-1)}, \]
with \(\kappa\) the genus of \(C/F\), whence, by (25) and (26),
\[ |L_0^0| \cdot |L_0^1|^{-1} = q^{-g(\deg(Z)+\kappa-1)-\delta}. \tag{27} \]

**Lemma 2.2.2.** Under the above notation we can write
\[ \frac{f(L_0^0)^{\vee}}{f(L_0^1)^{\vee}} = q^{-g(\deg(Z)+\kappa-1)-\delta}. \]

**Proof.** Since \(L_0^1 \simeq L_0^1 \otimes_{\mathbb{Z}_p} W(k^{(p)}_\infty)\), the lemma follows from (27) and Lemma 2.1.4 (as well as its proof). \(\square\)

Finally, by [KT03, 2.5.3], we have for any \(n \geq 0\) an isomorphism:
\[ N^2_n \simeq \text{Sel}_{\mathbb{Z}_p}(A^f/K_n)^{\vee}, \tag{28} \]
where \(\text{Sel}_{\mathbb{Z}_p}(\cdot) := \varprojlim \text{Sel}_{p^n}(\cdot)\) denotes the compact Selmer group as in [KT03, §2.3]. These isomorphisms induce, when passing to the inductive limit, an isomorphism
\[ (N^2_\infty)^{\vee} \simeq \text{Sel}_{\mathbb{Z}_p}(A^f/K^{(p)}_{\infty}) := \varinjlim_{n} \text{Sel}_{\mathbb{Z}_p}(A^f/K_n). \tag{29} \]

Let \(T_p(A(K^{(p)}_{\infty})) := \varprojlim \text{A}(K^{(p)}_{\infty})[p^n]\) denote the Tate-module of \(A(K^{(p)}_{\infty})\).

**Proposition 2.2.3.** If \(A_{\infty} = K^{(p)}_{\infty}\) is a finite group, then \(N^2_\infty = 0\). In general, we can write
\[ f(N^2_\infty)^{\vee} = f(T_p(A^{(p)}(K^{(p)}_{\infty})))^{\vee}. \]

**Proof.** For simplicity denote \(D_n := A^f(K_n)[p^{\infty}]\). Also, write \(V_n := \varprojlim_m \text{Sel}_{p^n}(A^f/K_n)[p^m]\).

Since \(\text{Sel}_{p^n}(A^f/K_n)\) is cotimately generated over \(\mathbb{Z}_p\), actually
\[ V_n = \varprojlim_m \text{Sel}_{p^n}(A^f/K_n)_{\text{der}}[p^m]. \]

For each \(m\) we have the exact sequence
\[ 0 \rightarrow D_n/p^mD_n \rightarrow \text{Sel}_{p^m}(A^f/K_n) \rightarrow \text{Sel}_{p^n}(A^f/K_n)[p^m] \rightarrow 0, \]
which induces, by taking \(m \rightarrow \infty\), the exact sequence
\[ 0 \rightarrow D_n \rightarrow \text{Sel}_{\mathbb{Z}_p}(A^f/K_n) \rightarrow V_n \rightarrow 0. \]

\(^1\)If \(g = 1\) and \(\Delta\) denote the global discriminant, then \(\delta = \frac{\deg(\Delta)}{12}\) (see e.g. [Tan95, eq. (9)]).
For each $n$, let $\text{Sel}_{p^n}(A/K_n)_{\text{div}}$ denote the $p$-divisible part of $\text{Sel}_{p^n}(A/K_n)$ and let $Y_p(A/K_n)$ be its Pontryagin dual. Also, let $Y_p(A/K(p))$ denote the projective limit of $\{Y_p(A/K_n)\}_n$, so that

$$Y_p(A/K(p)) = \text{Sel}_{\text{div}}(A/K(p))$$

where

$$\text{Sel}_{\text{div}}(A/K(p)) := \varprojlim_n \text{Sel}_{p^n}(A/K_n)_{\text{div}}.$$ 

Since $\text{Sel}_{p^n}(A'/K(p))$ is cotorsion over $\Lambda$, the divisible subgroup $\text{Sel}_{\text{div}}(A'/K(p))$ must be cofinitely generated over $\mathbb{Z}_p$. Thus, for $n$ sufficiently large the restriction map $\text{Sel}_{p^n}(A'/K_n)_{\text{div}} \to \text{Sel}_{\text{div}}(A'/K(p))$ is surjective. The kernel of this map is finite (note that we can show as in [Gr03] that the groups $H^1(\Gamma(n), A'_{p^n}(K(p)))$ are finite). It follows that if $n$ is sufficiently large then the restriction map $\text{Sel}_{p^n}(A'/K_n)_{\text{div}} \to \text{Sel}_{p^n}(A'/K(r))_{\text{div}}$ is an isomorphism for every $r > n$, and hence $V_n$ can be identified with $V_r$. This implies $\lim_{r \to n} V_r = 0$ as the map $V_r \to V_n$ becomes multiplication by $p^{r-n}$ on $V_r$ for sufficiently large $r, n$. Hence, (29) and the above exact sequence yield

$$(N_n^2)^{\vee} = \varprojlim_n \text{Sel}_{p^n}(A'/K_n) = \varprojlim_n A'(K_n)[p^\infty].$$

This proves the first assertion, while the second follows from [Tan12] Proposition 2.3.5. □

Since $A$ and $A'$ are isogenous, the Galois actions of $\text{Fr}_q$ on both $T_p(A(K(p)))$ and $T_p(A'(K(p)))$ have the same eigenvalues. Then, by [Tan12] Proposition 2.3.5] again, we can write

$$f_{T_p(A'(K(p)))} = \prod_{i=1}^l (1 - \zeta_i^{-1} \text{Fr}_q). \quad (30)$$

2.2.2. The value of $\star_{A,K(p)}$. Summarizing the above and observing that both (22) and (30) are units, we can make more precise the statement of Theorem 1.1. Recall that, in the notation of our Introduction, $c_{A,K(p)} = f_{X_p(A/K(p))}$. Proposition 2.2.4. Let notation be as above. Then we can write

$$f_{A/K(p)} \equiv \Lambda^{\star_{A,K(p)} \cdot c_{A/K(p)}}.$$ 

with

$$\star_{A,K(p)} = \frac{q^{-g(\deg(Z)+\kappa-1)-\delta} \prod_{v \in Z} \prod_{i=1}^{2g(v)} (\beta_i^{(v)} - \text{Fr}_v^{-1})}{\prod_{i=1}^l (1 - \zeta_i^{-1} \text{Fr}_q)(1 - \zeta_i^{-1} \text{Fr}_q^{-1})}.$$ 

3. The Main Conjecture

In this section we give a geometric analogue of the Iwasawa Main Conjecture of abelian varieties with semistable reduction. We keep the notations and the hypotheses of Section 2.

3.1. Interpolation and the Main Conjecture.
3.1.1. The modules $P_i^i$. Let $\mathbb{Q}_{p,n}$ denote the fraction field of $W(k_n)$ and $\mathbb{Q}_{p,\infty} := \cup_n \mathbb{Q}_{p,n}$.

Lemma 3.1.1. For any $i$, the map

$$(1 \otimes W(k^{(p)}_{\infty}))^\vee \left[ \frac{1}{p} \right] : (M^i_{2,\infty})^\vee \left[ \frac{1}{p} \right] \rightarrow (M^i_{1,\infty})^\vee \left[ \frac{1}{p} \right]$$

is an isomorphism induced by the identity on $(P^i_{0}[\frac{1}{p}] \otimes Q_p \mathbb{Q}_{p,\infty})^\vee$.

Proof. The first assertion follows from the exact sequence $[3]$ and from Corollary 2.1.3 (2). To show that the map is induced by the identity on $(P^i_{0}[\frac{1}{p}] \otimes Q_p \mathbb{Q}_{p,\infty})^\vee$, note that, by Lemma 2.1.3 (1), for all $n$ and for $k = 1, 2$ there exist some exact sequences:

$$0 \rightarrow B^i_0 \otimes_{\mathbb{Z}_p} W(k_n) \rightarrow P^i_{0}[p^{-1}] \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p,n} \rightarrow M^i_{1,0} \otimes_{\mathbb{Z}_p} W(k_n) \rightarrow C^i_0 \otimes_{\mathbb{Z}_p} W(k_n) \rightarrow 0$$

where $B^i_0$ is a lattice of $P^i_{0}[\frac{1}{p}]$ and $C^i_0$ a finite abelian $p$-group.

Passing to the inductive limit in $n$, then to the Pontryagin dual, and finally by inverting $p$, we obtain an exact sequence:

$$0 \rightarrow (M^i_{1,\infty})^\vee[p^{-1}] \rightarrow (P^i_{0}[p^{-1}] \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p,\infty})^\vee \rightarrow (\lim_n B^i_n)^\vee[p^{-1}] \rightarrow 0,$$

since $(\lim_n C^i_0 \otimes_{\mathbb{Z}_p} W(k_n))^\vee$ is $p$-torsion (actually, by Lemma 2.1.3 it is a finite direct sum of $((\mathbb{Z}/p^\infty\mathbb{Z})[[\Gamma]])$. In particular, $(M^i_{k,\infty})^\vee[\frac{1}{p}]$ is a submodule of $(P^i_{0}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p,\infty})^\vee$ for $k = 1, 2$ and the map $(1 \otimes W(k^{(p)}_{\infty}))^\vee[\frac{1}{p}]$ is compatible with the identity on $(P^i_{0}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p,\infty})^\vee$ by Lemma 2.1.3 (2). \[\square\]

As a consequence of the previous lemma, we denote $P^i_{\infty}$ the module $(M^i_{2,\infty})^\vee[\frac{1}{p}]$, endowed with the operator $\Phi_i := (1 \otimes W(k^{(p)}_{\infty}))^\vee[\frac{1}{p}]^{-1} \circ ((\varphi_i,0 \otimes \text{Fr}_q)^\vee[\frac{1}{p}])$. First, we define the $p$-adic $L$-function.

3.1.2. The $p$-adic $L$-function. By Corollary 2.1.3 (1), we know that the $P_i^i$’s are free $\Lambda[\frac{1}{p}]$-modules of finite rank. We set

$$L_{A/K[\frac{1}{p}]} := \prod_{i=0}^{2} \det_{\Lambda[\frac{1}{p}]}(\text{id} - \Phi_i, P^i_{\infty})^{-1}$$

and call it the $p$-adic $L$-function associated with $A/K$ relative to the arithmetic $\mathbb{Z}_p$-extension of $K$.

Lemma 3.1.2. Let $E$ be a finite extension of $\mathbb{Q}_p$ and $\omega: \Gamma \rightarrow E^\times$ an Artin character factoring through $\Gamma_N$. Then we have

$$\omega(L_{A/K[\frac{1}{p}]}) = \prod_{i=0}^{2} \det_E \left( \text{id} - p^{-1} F_{i,0} \otimes m_{\omega(F_{i,q})}, P^i_{0}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} E \right)^{-1}$$

where $m_{\omega(F_{i,q})}: E \rightarrow E$ is the multiplication by $\omega(F_{i,q})$.

Proof. We have for any $i$,

$$\omega(\det_{\Lambda[\frac{1}{p}]}(\text{id} - \Phi_i, P^i_{\infty})) = \det_E(\text{id} - \Phi_i \otimes \text{id}_E, P^i_{\infty} \otimes_{\Lambda[\frac{1}{p}]} E).$$

(where, as usual, $E$ is a $\Lambda[\frac{1}{p}]$-module via $\omega$).
Since $\omega$ factors through $\Gamma_N$, we have
\[
P^\infty_\infty \otimes_{\mathcal{A}(\frac{1}{p})} E = (P^\infty_\infty)_{\Gamma(N)} \otimes_{\mathcal{Q}_p} \mathcal{Q}_p[\Gamma_N] E
\]
\[
= (P^\infty_\infty)_{\Gamma(N)} \otimes_{\mathcal{Q}_p} \mathcal{Q}_p[\Gamma_N] E
\]
\[
= \left( (M_k^I \otimes W(k^\infty))^{r(N)} \right)^{p-1} \otimes_{\mathcal{Q}_p} \mathcal{Q}_p[\Gamma_N] E
\]
\[
= (M_k^I \otimes W(k^N))^{p-1} \otimes_{\mathcal{Q}_p} \mathcal{Q}_p[\Gamma_N] E
\]
\[
= (P^I_0[p^{-1}] \otimes_{\mathcal{Q}_p} \mathcal{Q}_p[\Gamma_N])^{p-1} \otimes_{\mathcal{Q}_p} \mathcal{Q}_p[\Gamma_N] E,
\]
where the last equality is a consequence of Lemma 2.1.3. An operator and its dual share the same determinant: hence we are reduced to compute $\det_E((id - p^{-1}F_{i,0} \otimes Fr_q) \otimes id_E)$ on $P^I_0[p^{-1}] \otimes_{\mathcal{Q}_p} \mathcal{Q}_p[\Gamma_N] E$. By the normal basis theorem $\mathcal{Q}_p[\Gamma_N] \otimes_{\mathcal{Q}_p[\Gamma_N]} E$ endowed with its $E$-endomorphism $Fr_q \otimes id_E$ is isomorphic to $E$ endowed with the endomorphism $m_\omega(Fr_q)$ and so the assertion is clear.

3.1.3. Twisted Hasse-Weil $L$-function. Let $\omega: \Gamma \to E^\times_0$ be any character factoring through $\Gamma_N$, with $E_0$ some totally ramified finite extension of $\mathbb{Q}_p,0$ endowed with a Frobenius operator $\sigma$ which acts trivially on $\mathbb{Q}_p,0$ and on $\omega(\Gamma)$. Then we can see $\omega$ as a one-dimensional $E_0$-representation of the fundamental group of $U$, having finite local monodromy. By [Ts98 Theorem 7.2.3] this representation corresponds to a unique constant unit-root overconvergent isocrystal $U(\omega)^\dagger$ over $\mathbb{F}/E_0$ endowed with $m_\omega(Fr_q)$ as Frobenius operator. Let
\[
pr^\dagger_1: F^n\text{-}\text{iso}^\dagger(U_{/\mathbb{F}/\mathbb{Q}_p,0}) \to F^n\text{-}\text{iso}^\dagger(U_{/\mathbb{F}/E_0})
\]
and
\[
pr^\dagger_2: F^n\text{-}\text{iso}^\dagger(\mathbb{F}/E_0) \to F^n\text{-}\text{iso}^\dagger(U_{/\mathbb{F}/E_0}),
\]
denote the two restriction functors in the categories of overconvergent isocrystals endowed with Frobenius. From $I^\dagger \in \text{F}^n\text{-}\text{iso}^\dagger(U_{/\mathbb{F}/\mathbb{Q}_p,0})$ we obtain the $F^n$-isocrystal $pr^\dagger_1 I^\dagger \otimes pr^\dagger_2 U(\omega)^\dagger$ endowed with the natural Frobenius induced by the Frobenius on $I^\dagger$ and the one on $U(\omega)^\dagger$.

Definition 3.1.3. Let $I^\dagger \in \text{F}^n\text{-}\text{iso}^\dagger(U_{/\mathbb{F}/\mathbb{Q}_p,0})$. Then we set
\[
L(U, I^\dagger, \omega, t) := L(U, pr^\dagger_1 I^\dagger \otimes pr^\dagger_2 U(\omega)^\dagger, t)
\]
where the right hand term is the classical $L$-function associated with the $F^n$-isocrystal $pr^\dagger_1 I^\dagger \otimes pr^\dagger_2 U(\omega)^\dagger$, as defined in [EL93]. We call the function $L(U, I^\dagger, \omega, t)$ the $\omega$-twisted $L$-function of $I^\dagger$.

Recall ([EL93 Théorème 6.3]) that this is a rational function in the variable $t$ and we have
\[
L(U, I^\dagger, \omega, t) = \prod_{i=0}^{2} \det(1 - t\varphi_i, H^{i}_{rig,c}(U_{/E_0}, pr^\dagger_1 I^\dagger \otimes pr^\dagger_2 U(\omega)^\dagger))(-1)^{i+1}.
\]

3.1.4. Interpolation. Recall ([KT03 IV]) that $D$, the log Dieudonné crystal associated with our semistable abelian variety $A/K$, induces an overconvergent $F^n$-isocrystal $D^\dagger$ over $U_{/\mathbb{Q}_p,0}$ and that we have a canonical isomorphism:
\[
P^I_0[p^{-1}] \simeq H^{i}_{rig,c}(U_{/\mathbb{Q}_p,0}, D^\dagger)
\]
(32)
compatible with the Frobenius operators.

Moreover, we have, by [KT03 3.2.2],
\[
L(U, D^\dagger, q^{-s}) = L_Z(A, s),
\]
where $L_Z(A, s)$ is the Hasse-Weil $L$-function of $A$ without Euler factors outside $U$. In fact, more generally, one can show that for any character $\omega : \Gamma \to \mathbb{C}^\times$ we have

$$L(U, D^1, \omega, q^{-s}) = L_Z(A, \omega, s),$$

(33)

since the Euler factors on both sides can be written as $\prod_{v \in U} (1 - \omega([v])\varepsilon_{i,v}q^{-s})^{-1}$, where the $\varepsilon_{i,v}$'s are the eigenvalues of the arithmetic Frobenius at $v$ acting on $T_{1,v}(A)$ (or, equivalently, of the geometric Frobenius acting on the fibre at $v$ of $D^1$). These eigenvalues don’t depend on $\ell$, as results of [KM] (for the details see e.g. the proof of [Tr02, Corollaire 1.4], where this independence is used to deduce the equality of different definitions of $L$-functions).

The Kähler formula for rigid cohomology (formula (1.2.4.1) in [Ke06a]) implies:

**Lemma 3.1.4.** Let $\omega : \Gamma \to E_0^\times$ be as in (7.1.5). There is an isomorphism of $E_0$-vector spaces compatible with Frobenius operators:

$$H^i_{rig,c}(U/E_0, pr_1^*D^1 \otimes pr_2^*U(\omega)^1) \simeq H^i_{rig,c}(U/\mathbb{Q}_p, D^1) \otimes E_0.$$

Together with Lemma 3.1.2 (52) and (53), this immediately yields the following:

**Theorem 3.1.5.** For any Artin character $\omega : \Gamma \to \mathbb{Q}_p^\times$, we have

$$\omega(L_A/K(p)) = L_Z(A, \omega, 1).$$

(34)

**Remark 3.1.6.** In order to discuss $\omega(L_A/K(p))$, one needs to know that the denominator of $L_A/K(p)$ is not killed by $\omega$. Actually, we are going to see (formula (43) below) that $L_A/K(p)$ is an alternating product of terms $1 - \alpha_{ij} \text{Fr}_q$, By [Ke06b, Theorem 5.4.1], the coefficients $\alpha_{ij}$ are Weil numbers of weight respectively $-1$ (for $i = 0$) and $1$ (for $i = 2$): in particular their complex absolute values do not include $1$. Hence the left-hand side of (44) is well defined.

3.1.5. The Main Conjecture. Finally, we prove the Iwasawa Main Conjecture in this setting.

**Theorem 3.1.7.** Let $A$ be an abelian variety with at worst semistable reduction relative to the arithmetic extension $K(p)/K$. We have the following equality in $Q(\Lambda)^\times/\Lambda^\times$:

$$L_A/K(p) = f_A/K(p).$$

**Proof.** For any morphism $g : M \to N$ of $\Lambda$-modules whose kernel and cokernel are both torsion $\Lambda$-modules, we denote by $\text{char}(g)$ the element

$$f_{\text{Coker}(g)} \cdot f^{-1}_{\text{Ker}(g)} \in Q(\Lambda)^\times/\Lambda^\times.$$

Dualizing the exact sequence (16) we get

$$0 \to (\text{Ker}(1 - \varphi_{i,1}))^\vee \to (N^i_{\infty})^\vee \to (\text{Coker}(1 - \varphi_{i-1,0}))^\vee \to 0$$

which, remembering that $\text{Ker}(h^\vee) = (\text{Coker}(h))^\vee$, implies

$$f_{(N^i_{\infty})^\vee} = f_{\text{Coker}(1 - \varphi_{i,1})}f_{\text{Ker}(1 - \varphi_{i-1,0})}.$$ Similarly, (9) yields $f_{(L^i_{\infty})^\vee} = f_{\text{Coker}(1)}f_{\text{Ker}(1)}$. Replacing in (17), we obtain

$$f_{A/K(p)} = \frac{f_{\text{Coker}(1 - \varphi_{i,1})}f_{\text{Ker}(1 - \varphi_{i-1,0})}f_{\text{Coker}(1 - \varphi_{i-1,\infty})}}{f_{\text{Coker}(1 - \varphi_{i-1,\infty})}f_{\text{Coker}(1 - \varphi_{i-1,\infty})}f_{\text{Coker}(1 - \varphi_{i-1,\infty})}} \cdot \frac{f_{\text{Coker}(1)}f_{\text{Ker}(1)}}{f_{\text{Coker}(1)}f_{\text{Ker}(1)}}.$$

Since $\text{Ker}(1 - \varphi_{i,1})$ and $\text{Coker}(1 - \varphi_{i,1})$ are $\Lambda$-torsion modules by Proposition 2.1.6 and Corollary 2.1.11 this can be rewritten as

$$f_{A/K(p)} = \frac{\text{char}(1 - \varphi_{i,1}) \cdot \text{char}(1)}{\text{char}(1 - \varphi_{i-1,0}) \cdot \text{char}(1) \cdot \text{char}(1 - \varphi_{i-1,\infty})} \cdot \frac{\text{char}(1)}{\text{char}(1 - \varphi_{i-1,\infty}) \cdot \text{char}(1)}.$$
On the other hand, \( \mathcal{L}_{A/K_{\infty}^0} \) is defined as an alternating product of determinants of
\[
\text{id} - \Phi_i = (1^\vee_i)^{-1} \circ (1^\vee - \varphi_{i,\infty}).
\]
Thus Theorem \[3.1.7\] becomes an immediate consequence of the following lemma (whose proof is an easy exercise which we omit):

**Lemma 3.1.8.** Let \( g, h: M \to N \) be two homomorphisms of finitely generated \( \Lambda \)-modules with torsion kernel and cokernel: then
\[
\det_{Q(\Lambda)}(g_{Q(\Lambda)}h_{Q(\Lambda)}^{-1}) = \text{char}(g)\text{char}(h)^{-1}.
\]

\( \square \)

### 3.2. Euler characteristic

#### 3.2.1. Generalized Euler characteristic

We recall the definition of the generalized \( \Gamma \)-Euler characteristic. Let \( M \) be a finitely generated torsion \( \Lambda \)-module and let \( g_M: M^\Gamma \to M \) denote the composed map \( M^\Gamma \hookrightarrow M \to M \), where the first map is the canonical inclusion and the second map the canonical projection. Then we say that \( M \) has finite generalized \( \Gamma \)-Euler characteristic, denoted \( \text{char}(\Gamma, M) \), if \( \text{Ker}(g_M) \) and \( \text{Coker}(g_M) \) are finite groups and in this case we set
\[
\text{char}(\Gamma, M) := \left| \frac{\text{Coker}(g_M)}{\text{Ker}(g_M)} \right|.
\]
By the identifications \((M^\vee)^\Gamma = (M^\Gamma)^\vee\) and \((M^\vee)^\Gamma = (M^\Gamma)^\vee\), we see that \( g_{M^\vee} \) is the dual of \( g_M \) and hence
\[
\text{char}(\Gamma, M^\vee) = \text{char}(\Gamma, M)^{-1} \tag{35}
\]
if one of them is defined.

#### 3.2.2. Twisted Euler characteristic

Let \( \omega: \Gamma \to \mathcal{O}^\times \) be an Artin character, with \( \mathcal{O} \) the ring of integers of some finite extension of \( \mathbb{Q}_p \), and \( M \) be a finitely generated torsion \( \Lambda \)-module. Let \( \omega^*: \Lambda_{\mathcal{O}} \to \Lambda_{\mathcal{O}} \) be the automorphism \( \gamma \mapsto \omega(\gamma)^{-1} \gamma \) and denote \( M_{\mathcal{O}}(\omega) := \Lambda_{\mathcal{O}} \otimes_{\Lambda_{\mathcal{O}}} M_{\mathcal{O}} \), where we see \( \Lambda_{\mathcal{O}} \) as \( \Lambda_{\mathcal{O}} \)-module via \( \omega^* \). Then \( M_{\mathcal{O}}(\omega) \) has again a structure of finitely generated torsion \( \Lambda \)-module.

Assuming that \( M_{\mathcal{O}}(\omega) \) has finite generalized \( \Gamma \)-Euler characteristic, we denote \( \mathcal{L}_\omega(f_M) \) the leading term of \( f_{M_{\mathcal{O}}(\omega)} \) and \( \text{ord}_\omega(f_M) \) the order of \( f_{M_{\mathcal{O}}(\omega)} \). We have the following result (compare \[Zer09\] Lemma 2.11 and also \[BV06\] Prop. 3.19):

**Lemma 3.2.1.** Let \( M \) be a finitely generated torsion \( \Lambda \)-module with characteristic element \( f_M \in \Lambda / \Lambda^\times \) and let \( \omega: \Gamma \to \mathcal{O}^\times \) be a character. Let \( d_{\mathcal{O}} := [\mathcal{O} : \mathbb{Z}_p] \). Then
\[
\text{rank}_{\mathbb{Z}_p}(M_{\mathcal{O}}(\omega)^\Gamma) = \text{rank}_{\mathbb{Z}_p}(M_{\mathcal{O}}(\omega)^\Gamma) \leq \text{ord}_\omega(f_M) = \text{ord}(\omega^*(f_M)),
\]
with equality if and only if \( M_{\mathcal{O}}(\omega) \) has finite generalized \( \Gamma \)-characteristic and in this case we have
\[
\text{char}(\Gamma, M_{\mathcal{O}}(\omega)) = [\mathcal{L}_\omega(f_M)]_{p^{-d_{\mathcal{O}}}} = [\omega^*(f_M)(0)]_{p^{-d_{\mathcal{O}}}}.
\]
Proof. First, note that if $M \sim \Lambda_\mathcal{O}/f \Lambda_\mathcal{O}$ then $M_\mathcal{O}(\omega) \sim \Lambda_\mathcal{O}/\omega^*(f) \Lambda_\mathcal{O}$. It is an easy exercise to check that if $M$ and $N$ are pseudo-isomorphic $\Lambda_\mathcal{O}$-modules, then they have the same Euler characteristic (for a hint, see [CSS03 Lemma 3.5]). Besides, the Euler characteristic is multiplicative: hence we are reduced to compute it for the case $M = \Lambda_\mathcal{O}/f \Lambda_\mathcal{O}$, with $f$ a power of some prime $\xi \in \Lambda_\mathcal{O} \simeq \mathcal{O}[[T]]$, and in the rest of the proof we will assume we are in this situation. Then if $f = T^i$ the map $g_M$ is the identity, while if $f = T^i$ for some $i > 1$ we have $g_M = 0$ and $M^\Gamma \simeq \mathcal{O} \simeq M_\Gamma$. Finally, if $f$ is coprime with $T$ we get $M^\Gamma = 0$ and $M_\Gamma \simeq \Lambda_\mathcal{O}/(f, T) \simeq \mathcal{O}/f(0)\mathcal{O}$.

Now just remember that, by basic number theory, $|\mathcal{O}/x\mathcal{O}| = |x|^{-d_\mathcal{O}}$ for any $x \in \mathcal{O}$. □

**Lemma 3.2.2.** Let $\omega: \Gamma \to \mathcal{O}^\times$ and $d_\mathcal{O}$ be as in the previous lemma. Then, for $j = 0, 1,$ we have

$$\text{char} \left(\Gamma, (L^j_\infty)^\vee(\omega)\right) = p^{d_\mathcal{O}d(L^j_0)}$$

and

$$\text{rank}_{\mathbb{Z}_p} \left((L^j_\infty)^\vee(\omega)\right)^\Gamma = 0.$$

Proof. By Corollary (2.1.5) (2), for $j = 0, 1,$ $(L^j_\infty)^\vee$ is a finite direct sum of copies of $\Lambda/p\Lambda$ and therefore $(L^j_\infty)^\vee(\omega)$ is a finite direct sum of copies of $\Lambda_\mathcal{O}/p\Lambda_\mathcal{O}$ and the assertion is clear. □

We deduce now from Lemma 3.2.1, Lemma 3.2.2, Theorem 3.1.7 and (27) the following result:

**Theorem 3.2.3.** Let $\omega: \Gamma \to \mathcal{O}^\times$ be a character. Assume that, for $i = 0, 1, 2,$ the $\Lambda$-modules $(N^i_\infty)^\vee(\omega)$ have finite generalized $\Gamma$-Euler characteristic. Then

$$\text{ord}_\omega \left(L_{A/K^\gamma(\mathcal{O})} \right) = \sum_{i = 0}^{2} (-1)^{i+1} \text{rank}_{\mathbb{Z}_p} \left((N^i_\infty)^\vee(\omega)\right)^\Gamma$$

and

$$|L_\omega \left(L_{A/K^\gamma(\mathcal{O})} \right)|_{p-d_\mathcal{O}}^{-d_\mathcal{O}} = q^{-d_\mathcal{O}(\delta + \gamma(\deg(Z) + \kappa - 1))} \prod_{i = 0}^{2} \text{char} \left(\Gamma, (N^i_\infty)^\vee(\omega)\right)^{(-1)^{i+1}}.$$

If $\omega$ is the trivial character, we can obtain more precise results. We consider first the problem of finiteness of the generalized $\Gamma$-Euler characteristic.

3.2.3. **Hochschild-Serre spectral sequence.** Since $\Gamma$ has cohomological dimension one, the natural Hochschild-Serre spectral sequence

$$H^j(\Gamma, N^i_\infty) \Rightarrow N^{i+j}_\infty$$

induces ([Mil80 Appendix B]) the following two exact sequences,

$$0 \to (N^0_\infty)_\Gamma \to N^1_0 \xrightarrow{\beta} (N^1_\infty)^\Gamma \to 0$$

and

$$0 \to (N^1_\infty)_\Gamma \to N^2_0 \xrightarrow{\gamma} (N^2_\infty)^\Gamma \to 0.$$

**Lemma 3.2.4.** The groups $(N^i_\infty)_\Gamma$ and $(N^i_\infty)^\Gamma$ are finite for $i = 0$ and $i = 2$. In particular, we have

$$\text{rank}_{\mathbb{Z}_p} \left((N^0_\infty)^\vee\right) = \text{rank}_{\mathbb{Z}_p} \left((N^1_\infty)^\vee\right)_\Gamma = \text{rank}_{\mathbb{Z}_p} \left(X_p(A/K)\right).$$
It is known that \(\text{Disc}\) is independent of the choices of basis and we call it the discriminant of the height pairing.

On the other hand, for the group \((\mathbb{N}_0^1)^{\nu}\) where the last equality is a consequence of the control theorem [Tan10, Theorem 4]. This yields (39). On the other hand, we have \(\text{trivial}\). Therefore we have

\[
\text{rank}_{\mathbb{Z}_p}((\mathbb{M}_\infty^{\nu})^\Gamma) = \text{rank}_{\mathbb{Z}_p}((\oplus_{w|p}\mathbb{B}_w(k(w))[p^\infty])^\Gamma) = 0
\]

where the first equality results from the facts that \(\Phi_w\) is a finite group and \((T_w)[p^\infty]\) is trivial. Therefore we have

\[
\text{rank}_{\mathbb{Z}_p}((\mathbb{N}_0^1)^{\nu})^\Gamma = \text{rank}_{\mathbb{Z}_p}X_p(A/K(p)^\Gamma) = \text{rank}_{\mathbb{Z}_p}X_p(A/K),
\]

where the last equality is a consequence of the control theorem [Tan10, Theorem 4]. This yields (39). On the other hand, for the group \((\mathbb{N}_0^2)^\Gamma\), we have

\[
\text{rank}_{\mathbb{Z}_p}(\mathbb{N}_0^2)^{\nu} = \text{rank}_{\mathbb{Z}_p}\text{Sel}_{\mathbb{Z}_p}(A^1/K),
\]

by (29). On the other hand, we have

\[
\text{rank}_{\mathbb{Z}_p}\text{Sel}_{\mathbb{Z}_p}(A^1/K) = \text{rank}\, A^1(K) + \text{rank}_{\mathbb{Z}_p}\mathbb{II}_{p^\infty}(A^1/K)^{\nu} = \text{rank}_{\mathbb{Z}_p}X_p(A^1/K),
\]

In particular,

\[
\text{rank}_{\mathbb{Z}_p}(\mathbb{N}_0^1)^{\nu} = \text{rank}_{\mathbb{Z}_p}X_p(A^1/K) = \text{rank}_{\mathbb{Z}_p}X_p(A/K),
\]

where the second equality results from the existence of an isogeny between \(A\) and \(A^1\). Hence, from the short exact sequence (38) we deduce \(\text{rank}_{\mathbb{Z}_p}((\mathbb{N}_0^2)^{\nu}) = 0\).

Let \(\tau \in H^1_{\text{fl}}(\mathbb{F}_p, \mathbb{Z}_p) = \text{Hom}_{\text{cont}}(\text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p), \mathbb{Z}_p)\) be the element which sends the arithmetic Frobenius to 1. By [KT03, Lemma 6.9] the composed map

\[
\cup': N_0^1 \longrightarrow M^1_{1,0} \xrightarrow{1} M^1_{2,0} \longrightarrow N_0^2
\]

coinsides up to sign with \(\tau \cup\), the cup product by the image of \(\tau\) in \(H^1_{\text{fl}}(X, \mathbb{Z}_p)\). Using [Mil86, Proposition 6.5], we can prove the following result.

**Lemma 3.2.5.** The composed map

\[
\cup: N_0^1 \xrightarrow{\beta} (N_0^1)^\Gamma \xrightarrow{g_{N_0^1}} (N_0^1)^\Gamma \xrightarrow{\gamma} N_0^2
\]

coinsides (up to the sign) with the map

\[
\tau \cup = \cup': N_0^1 \longrightarrow N_0^2.
\]

**3.2.4. The height pairing.** We denote

\[
\tilde{h}_{A/K}: A(K) \times A^1(K) \rightarrow \mathbb{R},
\]

the Néron-Tate height pairing (see for example [Lan83]). Let \(e_1, \ldots, e_r\) be elements of \(A(K)\) which form a \(\mathbb{Z}\)-basis of \(A(K)/A(K)_{\text{tor}}\) and let \(e_1^*, \ldots, e_r^*\) be elements of \(A^1(K)\) which form a \(\mathbb{Z}\)-basis of \(A^1(K)/A^1(K)_{\text{tor}}\). Then

\[
\text{Disc}(\tilde{h}_{A/K}) := |\det(\tilde{h}_{A/K}(e_i, e_j^*))| \in \mathbb{R}
\]

is independent of the choices of basis and we call it the discriminant of the height pairing. It is known that \(\text{Disc}(\tilde{h}_{A/K}) \neq 0\). We write

\[
\text{Disc}(h_{A/K}) = \log(p)^{-r}\text{Disc}(\tilde{h}_{A/K}),
\]
with \( r = \text{rank}(A(K)) \).

Consider the quotient category \((ab)/(fab)\), where \((ab)\) is the category of abelian groups and \((fab)\) the category of finite abelian groups. Let \( \theta \) be the composed map in \((ab)/(fab)\) defined by:

\[
A(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\theta} N^\Gamma_0 \xrightarrow{\beta} (N^\Gamma_0) \xrightarrow{g_{N^\Gamma_0}} (N^\Gamma_0) \xrightarrow{\gamma} N^2_0
\]

where:

1. the map \( \alpha: A(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow N^\Gamma_0 \) is the canonical morphism in \((ab)/(fab)\) coming from \( A(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \text{Sel}_{p}\infty (A/K) \), by \([18]\);
2. the map \( u: N^2_0 \rightarrow \text{Hom}(A^t(K), \mathbb{Q}_p/\mathbb{Z}_p) \) is the map constructed by using the isomorphism \([28]\) and the natural map \( A^t(K) \otimes \mathbb{Z}_p \rightarrow \text{Sel}_{p}\infty (A^t/K) \);
3. the map \( v \) is induced by the quotient map (which is an isomorphism, since we are in the quotient category \((ab)/(fab)\)).

Thanks to Lemma \([3,2,5]\) and \([1,4,0,3]\) 3.3.6.2 and \([6,8]\), \( \theta \) coincides (up to sign) with the map induced by \( h_{A/K} \) in \((ab)/(fab)\). In particular, since the Néron-Tate height pairing is non-degenerate, \( \theta \) is a quasi-isomorphism (i.e., an isomorphism in the quotient category).

### 3.2.5. Computation of the Euler characteristic.

If \( f \) is a quasi-isomorphism of abelian groups, we denote \( \text{char}(f) := |\text{Ker}(f)|/|\text{Coker}(f)| \). Since this characteristic is multiplicative, \([10]\) gives

\[
\text{char}(\alpha) \cdot \text{char}(\beta) \cdot \text{char}(g_{N^\Gamma_0}) \cdot \text{char}(\gamma) \cdot \text{char}(u) \cdot \text{char}(v)^{-1} = \text{char}(\theta) \equiv_p \text{Disc}(h_{A/K})
\]

where \( \equiv_p \) means “\( \equiv \mod \mathbb{Z}_p^\times \)”. If we assume that \( A/K \) has semistable reduction and the Tate-Shafarevich group of \( A/K \) is finite, then we have:

\[
\text{char}(\alpha) = \frac{|A_{p}\infty (K)|}{|\prod_{p \neq \infty} (A/K)| \cdot |\mathcal{M}_0[p]^{\infty}||} \cdot |N^\Gamma_0|
\]

by \([18]\):

\[
\text{char}(v) \equiv_p |A_{p}\infty (K)|^{-1}
\]

because \( v: \text{Hom}(A^t(K)/A_{p}\infty (K), \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow \text{Hom}(A^t(K), \mathbb{Q}_p/\mathbb{Z}_p) \) is injective with cokernel \( A_{p}\infty (K)^\vee \);

\[
\text{char}(\beta) = |(N^\Gamma_0)|
\]

and

\[
\text{char}(\gamma) = |(N^\Gamma_0)|^{-1}
\]

by \([37]\) and \([38]\), using Lemma \([3,2,4,2]\) and \( \text{char}(u) = 1 \) since the Tate-Shafarevich group is assumed to be finite.

The following result can be seen as a geometric analogue of the conjecture of Mazur-Tate-Teitelbaum \([M,T,T,86]\):

**Theorem 3.2.6.** Assume that \( A/K \) has semistable reduction. Then

\[
\text{ord}(L_{A/K}(s)) = \text{ord}_{s=1}(L_s(A, s)) \geq \text{rank}_\mathbb{Z} A(K).
\]

(41)
If moreover $A/K$ verifies the Birch and Swinnerton-Dyer Conjecture, the inequality above becomes an equality and

$$|L(L_{A/K}(\infty)_{\infty})_{p}^{-1} = c_{BSD} \cdot |(N_{\infty}^0)_{\Gamma} | \mod Z_{p}^{\times},$$

where $c_{BSD}$ is the leading coefficient at $s = 1$ of $L_{Z}(A, s)$.

**Proof.** The last inequality in (41) has been proved in [KT03]§3.5. We show that the analytic rank is equal to the rank of our $p$-adic $L$ function $L_{A/K}(\infty)$ First, note that the operator $(\varphi_{\infty, 0} \otimes \operatorname{Fr}_{q})^{\vee}$ is induced by the operator $(p^{-1}F_{i, 0} \otimes \operatorname{Fr}_{q})^{\vee}$ on $(P_{0}^{1}[p] \otimes \mathbb{Q}_{p, \infty})^{\vee}$. Moreover, using (31), we observe that we have an injection of $\mathbb{Q}_{p}[\Gamma]$-modules

$$P_{\infty}^{i} \otimes_{\Lambda^{[p]}} \mathbb{Q}_{p}[\Gamma] \hookrightarrow (P_{0}^{i}[p^{-1}] \otimes \mathbb{Q}_{p, \infty})^{\vee} \simeq \mathbb{Q}_{p}[\Gamma]^{r_{i}}$$

with $r_{i} := \dim_{\mathbb{Q}_{p}}(P_{0}^{[p]})$ and where the operator $(p^{-1}F_{i, 0} \otimes \operatorname{Fr}_{q})^{\vee}$ on the left-hand side corresponds to the operator $\operatorname{Fr}_{q} p^{-1}F_{i, 0}$ on the right-hand side (as shown in Lemma 3.1.1). Also, Lemma 2.1.3 shows that $P_{\infty}^{i}$ is a free $\Lambda^{[p]}$-module of rank equal to the $Z_{p}$-corank of $M_{2, 0}$, which, by Lemma 2.1.3 is precisely $r_{i}$. Hence, the $p$-adic $L$-function $L_{A/K}(p)$ can be written

$$\prod_{i=0}^{2} \det_{\mathbb{Q}_{p}[\Gamma]} (id - Fr_{q}, F_{i, 0}, P^{i} \otimes \mathbb{Q}_{p}[\Gamma])^{(1-1)(1-1)} = \prod_{i=0}^{2} \prod_{j=1}^{r_{i}} (1 - \alpha_{ij} Fr_{q})^{(1-1)(1-1)},$$

where the $\alpha_{ij}$'s are the eigenvalues (in $\mathbb{Q}_{p}$) of $p^{-1}F_{i, 0}$. In particular, $\operatorname{ord}_{1}((\prod_{j=1}^{r_{i}} (1 - \alpha_{ij} Fr_{q}))$ is the number of $\alpha_{ij}$ equal to 1 (note that $1 - \lambda Fr_{q}$ has order 0 if $\lambda \neq 1$ and order 1 else), that is, the multiplicity of the eigenvalue 1 of the operator $p^{-1}F_{i, 0}$ and the assertion follows from [KT03] 3.5.2.

We proceed to the proof of the second assertion of the theorem. By Lemma 3.2.3 $(N_{\infty}^{0})^{\vee}$ and $(N_{\infty}^{2})^{\vee}$ have finite generalized $\Gamma$-characteristic. For $(N_{\infty}^{2})^{\vee}$, remark that under our assumption $\alpha$, $\beta$ and $\gamma$ are quasi-isomorphisms while $u$ and $v$ are isomorphisms. Therefore, the map $g_{N_{\infty}^{0}}$ is a quasi-isomorphism and so is its Pontryagin dual, $g_{(N_{\infty}^{2})^{\vee}}$. By Theorem 3.2.3 and Lemma 3.2.4, we have

$$\operatorname{ord}_{1}(L_{A/K}(\infty)) = \operatorname{rank}_{\mathbb{Z}_{p}} X_{p}(A/K) = \operatorname{rank}_{\mathbb{Z}} A(K),$$

since we have assumed that the Tate-Shafarevich group of $A/K$ is finite. The last equality $\operatorname{rank}_{\mathbb{Z}} A(K) = \operatorname{ord}_{k=1} L_{Z}(A, s)$ follows from the main theorem of [KT03]. The same theorem also proves that if $\alpha$ and $\mu_{v}$ are the Haar measure defined in (22) and (27) one gets

$$c_{BSD} = \frac{|\prod(A/K)| \cdot \operatorname{Disc}(h_{A/K})}{|A(K)_{tor}| \cdot |A'(K)_{tor}|} \cdot \mu(Lie(A)(\Lambda_{K})/Lie(A)(K))^{-1} \cdot \prod_{v \in \mathbb{Z}} \alpha_{v}(A(K_{v})).$$

Replacing the values above and applying (21) and (27) one gets

$$c_{BSD} \equiv_{p} q^{-g(\deg(Z) + \kappa - 1) - \delta} \cdot \frac{|(N_{\infty}^{0})_{\Gamma}| \cdot \operatorname{char}(g_{N_{\infty}^{0}})}{|N_{\infty}^{0}| \cdot |(N_{\infty}^{2})^{\vee}|}.$$
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