AREA-MINIMIZING CONES IN THE HEISENBERG GROUP $\mathbb{H}^1$

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ABSTRACT. We present a characterization of minimal cones of class $C^2$ and $C^1$ in the first Heisenberg group $\mathbb{H}$, with an additional set of examples of minimal cones that are not of class $C^1$.

1. Introduction

The interest towards Geometric Measure Theory in the Heisenberg group grew drastically in the last decades, see for instance [9, 6, 2, 12] and the references therein. Despite many deep results, fundamental questions still remain open, the main difficulty being that sets of finite perimeter may not be rectifiable sets in the Riemannian sense.

In the effort to understand minimal surfaces in the first Heisenberg group, we are presenting a characterization of minimal cones of class $C^2$ and $C^1$. Furthermore, we also provide a set of examples of minimal cones that are not of class $C^1$.

Complete minimal surfaces of class $C^2$ have been classified in [11]. We provide a self-contained classification of minimal cones of class $C^2$, as it is a simple exercise in our case. Minimal surfaces of class $C^1$ have been studied in [8, 7]. Tentatives to study minimal surfaces with regularity lower than $C^1$ can be found in in [10, 14].
The construction of minimal cones is the following, see Section 3 for details. Given proper disjoint open subarcs $I, J$ of the unit circle $S^1 \subset \mathbb{R}^2$, let $L$ be the bisectrix of $I$. Then consider the family of planar curves made of (see Figure 1):

1. rays emanating from 0 and intersecting $J$;
2. the line $L$ together with half-lines starting from $L$ parallel to the two boundary lines of $0\#I$.

![Figure 1. The configuration of lines in $\mathbb{R}^2$ for given arcs $I, J$.](image)

All these curves in $\mathbb{R}^2$ lift uniquely to horizontal curves in $\mathbb{H}^1$, whose union form a surface $C(I, J) \subset \mathbb{H}^1$ with non-empty boundary in general. The lifted curves are the characteristic curves of $C(I, J)$.

Similarly, we can construct a surface $C(\mathcal{I})$ from a (possibly infinite) family $\mathcal{I}$ of disjoint arcs of $S^1$, see Figure 3. These are minimal cones with different degrees of regularity.

**Theorem A.** Let $\mathcal{I}$ be a family of disjoint arcs of $S^1$.

1. The surface $C(\mathcal{I})$ is a minimal cone.
2. The surface $C(\mathcal{I})$ is of class $C^1$ if and only if $\mathcal{I}$ is finite and the closure of $\bigcup \mathcal{I}$ is $S^1$.

Theorem A is proven in Propositions 3.3 and 3.4. With these examples at hand, we provide a classification of minimal cones of class $C^1$. The classification is based on the study of the singular set of minimal surfaces, that is, the set of points where the tangent plane is horizontal, see [2]. See Section 4.1 for the proof.

**Theorem B.** If $S \subset \mathbb{H}^1$ is a minimal cone of class $C^1$, then one of the following possibilities holds:

1. $S$ is a vertical plane, or
2. $S$ is the horizontal plane $\{t = 0\}$, or
3. $S = C(I_1, \ldots, I_k)$ for some disjoint open arcs $I_1, \ldots, I_k$ in $S^1$ with $S^1 = \bigcup_{j=1}^k \bar{I}_j$.

These cases can be distinguished by their singular set: empty in the first case, a single point in the second case, and a finite family of horizontal half-lines starting from the vertex in the third case.
Not all $C^1$ minimal cones are of class $C^2$. In the third class, the only minimal cones of class $C^2$ are those with $k = 2$.

**Theorem C.** If $S \subset H^1$ is a minimal cone of class $C^2$, then $S$ is a vertical plane, or the horizontal plane $\{t = 0\}$, or rotations about the $t$-axis of the graph of the function $t = -xy$.

Theorem C follows from Theorem 5.1 of [16], where it is proven that the unique entire $C^2$ area-stationary graphs over the plane $H$ in $H^1$ are Euclidean planes and vertical rotations of graphs of the form $t = x y + (a y + b)$, where $a$ and $b$ are real constants. In case the surface is a cone then $a = b = 0$.

**Plan of the paper.** The preliminary Section 2 introducLes the main definitions and properties of the Heisenberg group that we need. The construction of minimal cones that we sketched above is presented in detail in Section 3. Finally, we prove our main results in Section 4.

2. Preliminaries

2.1. The Heisenberg group. We identify the first Heisenberg group $H$ with $\mathbb{R}^3$ with coordinates $(x, y, t)$ where we set the group operation

$$(x, y, t) * (x', y', t') = (x + x', y + y', t + t' + (x'y - xy')).$$

The neutral element is $(0, 0, 0)$ and the inverse of $(x, y, t)$ is $(-x, -y, -t)$. We choose the frame of left-invariant vector fields generated by $\partial_x, \partial_y$ and $\partial_t$ at $0$

$$X = \partial_x + y \partial_t, \quad Y = \partial_y - x \partial_t, \quad T = \partial_t.$$

Notice that $[X, Y] = -2T$. These vector fields form a basis for the Lie algebra $\mathfrak{h}$ of $H$, which is stratified with first layer $\mathcal{H} = \text{span}\{X, Y\}$, the horizontal plane, and second layer $[\mathcal{H}, \mathcal{H}] = \text{span}\{T\}$.

With an abuse of language, we denote by $C^k(\Omega; \mathcal{H})$ (and $C^k_c(\Omega; \mathcal{H})$) the space of sections of class $C^k$ (with compact support in $\Omega$) of the left-invariant vector bundle generated by $\mathcal{H}$. These sections are vector fields on $\mathbb{R}^3$.

One can easily see that, if $V = v_1X + v_2Y$ with $v_1$ and $v_2$ smooth functions, then the standard divergence in $\mathbb{R}^3$ applied to $V$ is

$$\text{div}(V) = X v_1 + Y v_2.$$

If we consider the left-invariant Riemannian metric $g$ on $H$ making $X, Y, T$ and orthonormal basis, div$(V)$ is also the divergence with respect to the Riemannian metric $g$.

The left-invariant vector bundle generated by $\mathcal{H}$ is the kernel of the contact form

$$\omega = dt - y dx + x dy.$$

Lipschitz curves in $\mathbb{R}^2$ can be lifted to $H$ in the following way.

**Lemma 2.1.** Let $\gamma : [0, 1] \to \mathbb{R}^2$, $\gamma(s) = (x(s), y(s))$, be a Lipschitz curve with $\gamma(0) = 0$. Define $t : [0, 1] \to \mathbb{R}$ by

$$t(s) = \int_0^s (y(u) - x(u) dy)[\gamma'(u)] du = \int_0^s (y(u)x'(u) - x(u)y'(u)) du.$$

Then, the curve $s \mapsto (x(s), y(s), t(s))$ is the only horizontal Lipschitz curve in $H$ starting from $(0, 0, 0)$ and projecting to $\gamma$. 

Moreover, if \( A(\gamma, s) = \{ v\gamma(u) : u \in [0, s], \ v \in [0, 1] \} \) (with the orientation given by \( \gamma \)), then

\[
t(s) = -2 \int_{A(\gamma, s)} dx \wedge dy,
\]

which is called the balayage area spanned by \( \gamma \).

**Proof.** Notice that a Lipschitz curve \( \eta : s \mapsto (x(s), y(s), t(s)) \) is horizontal if and only if \( \omega|_{\eta(s)}[\eta'(s)] = 0 \) for almost all \( s \), that is, \( t' = yx' - xy' \). Integrating, we get the statement. \( \square \)

### 2.2. Sub-Riemannian perimeter

Given a measurable set \( E \subset \mathbb{H} \) and an open set \( \Omega \subset \mathbb{H} \), the **perimeter** of \( E \) in \( \Omega \) is defined as

\[
P(E; \Omega) := \sup \left\{ \int_E \nabla V \cdot dL^3 : V \in C^1(\Omega; \mathcal{H}), \ |V| \leq 1 \right\},
\]

where \( L^3 \) is the Lebesgue measure in \( \mathbb{R}^3 \) that is, in our chosen coordinate system, a Haar measure of \( \mathbb{H} \).

A measurable set \( E \subset \mathbb{H} \) has **locally finite perimeter** if for every bounded open set \( \Omega \subset \mathbb{H} \) we have \( P(E; \Omega) < \infty \). It turns out (see [6]) that for a locally finite perimeter, the distributional gradient of the characteristic function \( \mathbf{1}_E \) is a vector valued Radon measure, that is, there is a positive Radon measure \( |\partial E| \) and a unit horizontal vector field \( \nu_E : \mathbb{H} \rightarrow H \) such that \( \nabla \mathbf{1}_E = \nu_E |\partial E| \). The measure \( |\partial E| \), and thus \( \nabla \mathbf{1}_E \), is supported on the so-called **reduced boundary** \( \partial^* E \subset \partial E \).

**Proposition 2.2** ([6]). Let \( E \subset \mathbb{H}^1 \) be a set with locally finite perimeter and \( V \in C^\infty(\mathbb{H}; H) \) a smooth horizontal vector field, then

\[
\int_E \nabla(V) \cdot dL^3 = -\int_{\partial^* E} (V, \nu_E) d|\partial E|.
\]

As a corollary, we can easily prove the following formula.

**Corollary 2.3.** Let \( V \in C^1(\Omega; \mathcal{H}) \), \( \phi \in C^1(\mathbb{H}) \) and \( E \subset \mathbb{H} \) a set with locally finite perimeter. Then

\[
\int_E \langle \nabla \phi, V \rangle \cdot dL^3 = -\int_{\partial^* E} \phi \langle V, \nu_E \rangle d|\partial E| - \int_E \phi \nabla(V) \cdot dL^3.
\]

**Proof.** First, by group convolution, the relation (2.1) remains true for \( v \) of class \( C^1 \). Second, notice that \( \nabla(\phi V) = \langle \nabla \phi, V \rangle + \phi \nabla(V) \). Therefore, on the one hand,

\[
\int_E \langle \nabla \phi, V \rangle \cdot dL^3 = \int_E \nabla(V) \cdot dL^3 - \int_E \phi \nabla(V) \cdot dL^3,
\]

on the other hand,

\[
\int_E \nabla(V) \cdot dL^3 = -\int_{\partial^* E} \phi \langle v, \nu_E \rangle d|\partial E|.
\]

by (2.1). Putting these two identities together, we get (2.2).

We are interested in perimeter minimizers. A measurable set \( E \subset \mathbb{H} \) is a **perimeter minimizer** in an open set \( \Omega \subset \mathbb{H} \) if, for every \( F \subset \mathbb{H} \) of locally finite perimeter with \( E \Delta F \in \Omega \), we have

\[
P(E; \Omega) \leq P(F; \Omega).
\]

A set is **local perimeter minimizer** if it is perimeter minimizer in every bounded open set. A surface \( S \) in \( \mathbb{H} \) is an **area-minimizing surface**, or just a **minimal surface**, if

\[\begin{align*}
\int_S \nabla u \cdot d\nu &\leq \int_S \nabla u \cdot d\nu \\
\int_S \kappa \cdot n &\leq \int_S \kappa \cdot n \\
\int_S \kappa &\leq \int_S \kappa
\end{align*}\]

where \( u \) is the local parameterization of \( S \), \( \kappa \) is the mean curvature, and \( n \) is the unit normal.
if it coincides with the reduced boundary of a perimeter minimizer. The following proposition yields a method via calibrations to prove that a given set is perimeter minimizer.

**Proposition 2.4** ([13, Theorem 2.1]). Let \( E \subset \mathbb{H} \) be a measurable set, \( \Omega \subset \mathbb{H} \) an open set and \( v : \Omega \to H \) a Borel map. Assume that

(i) \( E \) has locally finite perimeter in \( \Omega \);
(ii) \( v = \nu_E \) \( |\partial E| \)-almost everywhere in \( \Omega \);
(iii) there exists an open set \( \tilde{\Omega} \subset \Omega \) such that \( |\partial E|(\Omega \setminus \tilde{\Omega}) = 0 \) and \( v \) is continuous on \( \tilde{\Omega} \);
(iv) \( \text{div}(v) = 0 \) in distributional sense in \( \Omega \).

Then \( E \) is a perimeter minimizer in \( \Omega \).

The vector field \( v \) above is called a *calibration* for \( \partial^* E \). In applications of Proposition 2.4, we will give the calibration \( v \) by putting together smooth vector fields in different domains. The following proposition gives a way to check that the resulting vector field has zero distributional divergence. Notice that condition (2.3) below is automatically satisfied if \( v \) is continuous.

**Proposition 2.5.** Let \( \{ \Omega_j \}_j \) be a family of open disjoint sets with locally finite perimeter in \( \mathbb{H} \) such that \( \{ \Omega_j \}_j \) is a locally finite cover of \( \mathbb{H} \) with \( \mathcal{L}^3(\mathbb{H} \setminus \bigcup \Omega_j) = 0 \). For each \( j \), let \( V_j \in C^1(\Omega_j; H) \) be a horizontal vector field of class \( C^1 \) on \( \Omega_j \) (extendible to a \( C^1 \) horizontal vector field on a neighborhood of \( \Omega_j \)).

The distributional divergence of \( V := \sum_j V_j \mathbb{1}_{\Omega_j} \) is zero if and only if \( \text{div}(V_j | \Omega_j) = 0 \) for every \( j \) and

\[
\sum_j \langle V_j(p), \nu_{\Omega_j}(p) \rangle = 0 \quad \text{for } \sum_j |\partial \Omega_j|-\text{a.e. } p \in \mathbb{H},
\]

where we set \( \nu_{\Omega_j}(p) = 0 \) if \( p \notin \partial \Omega_j \).

**Proof.** Let \( \phi \in C^\infty_c(\mathbb{H}) \). Using (2.2), we have

\[
\int_{\mathbb{H}} \langle \nabla \phi, V \rangle \, d\mathcal{L}^3 = \sum_j \int_{\Omega_j} \langle \nabla \phi, V_j \rangle \, d\mathcal{L}^3
\]

\[
= -\sum_j \left( \int_{\partial^* \Omega_j} \phi(V_j, \nu_{\Omega_j}) \, d|\partial \Omega_j| + \int_{\Omega_j} \phi \text{div}(V_j) \, d\mathcal{L}^3 \right).
\]

The latter expression is zero for every \( \phi \in C^\infty_c(\mathbb{H}) \) if and only if \( \text{div}(V_j | \Omega_j) = 0 \) for every \( j \) and (2.3) holds.

Finally, the following stability of perimeter minimizers is well known.

**Proposition 2.6.** Let \( \{ E_k \}_{k \in \mathbb{N}} \) be a sequence of locally perimeter minimizers and \( E \) a set of locally finite perimeter such that \( \mathbb{1}_{E_k} \) converge locally in \( L^1 \) to \( \mathbb{1}_E \). Then \( E \) is also locally perimeter minimizer.

### 2.3. Regularity of \( C^1 \) area-minimizing surfaces in \( \mathbb{H}^1 \)

Given a \( C^1 \) surface \( S \), the set \( S_0 \subset S \) is composed of the points \( p \) where \( T_p S \) is horizontal. It is referred to as the *singular set* of \( S \). Points in \( S \setminus S_0 \) are called *regular points*. A *horizontal line segment* is the image in \( \mathbb{H} \) of an interval in \( \mathbb{R} \) through a curve of the form \( s \mapsto p \exp(sv) \), for some \( p \in \mathbb{H} \) and \( v \in H \).
**Proposition 2.7** ([2, 7, 8]). If $S$ is a minimal $C^1$ surface, then $S \setminus S_0$ is ruled by horizontal line segments whose endpoints lie in $S_0$. If $S$ is a $t$-graph then at most one endpoint lies in $S_0$.

Given a function $u : A \to \mathbb{R}$ defined on a domain $A \subset \mathbb{R}^2$, its $t$-graph is the surface \[ \{(x, y, u(x, y)) : (x, y) \in A\} \]. We always consider a $t$-graph as boundary of the subgraph $E := \{(x, y, t) : t \leq u(x, y), \ (x, y) \in A\}$. The following lemma characterize minimal $t$-graphs of continuous functions.

**Lemma 2.8.** The $t$-graph $S$ of a continuous function $u : \mathbb{R}^2 \to \mathbb{R}$ is a minimal surface if and only if the unit normal of $S$, extended to $\mathbb{H}$ as a Borel vector field independent of $t$, has zero distributional divergence.

### 3. Construction of minimal cones

Consider a finite family $I_1, \ldots, I_k$ of disjoint open arcs in the unit circle $S^1$, and let $J_1, \ldots, J_r$ the open connected arcs in $S^1 \setminus \bigcup_{i=1}^k I_i$. This set could be eventually empty if $S^1 = \bigcup_{i=1}^k I_i$. Let $2\alpha_i$ be the length (opening angle) of $I_i$ and $L_i$ be the bisectrix of the arc $I_i$.

![Figure 2. An initial configuration with three open arcs $I_1, I_2, I_3$.](image_url)

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**ii** The conical sector $0#J_i$ in $\mathbb{R}^2$ (the cone of vertex 0 over $J_i$) is filled with half-lines leaving the origin. The conical sectors $0#I_i$ are filled with pairs of half-lines making angle $\alpha_i$ with the half-line $L_i$. This way, every point of $\mathbb{R}^2$ can be joined to some $L_i$ or 0 by a unique shortest path that follows these lines. We lift these paths to $\mathbb{H}$ as in Lemma 2.1. So, we first lift as horizontal curves the half-lines $L_1, \ldots, L_k$ and those in the sectors $0#J_i$, $i = 1, \ldots, r$, which remain in the plane \{ $t = 0$ \}. Then, for $i = 1, \ldots, k$, we lift the half-lines making angle $\alpha_i$ with $L_i$ to horizontal half-lines starting from the corresponding Lifted line $L_i$.

We obtain a surface, which we call $C(I_1, \ldots, I_k)$, that is the $t$-graph of a function $u : \mathbb{R}^2 \to \mathbb{R}$. The following lemma gives an explicit formula in a specific case. Notice that, up to a rotation of $\mathbb{R}^2$, the restriction of $u$ to $0#I_i$ is equal to the function $u_{\alpha_i}$ on $0#I$ described below.
Lemma 3.1 ([15]). Let \( \alpha \in (0, \pi) \) and define the open arc \( I = \{(\cos(\theta), \sin(\theta)) : |\theta| < \alpha\} \subset \mathbb{S}^1 \). Then \( C(I) \) is the t-graph of the function

\[
(3.1) \quad u_\alpha(x, y) = \begin{cases} 
\frac{y(|y| \cot \alpha - x)}{\sin(\alpha)} & \text{if } (x, y) \in 0\#I, \\
0 & \text{otherwise}.
\end{cases}
\]

The function \( u_\alpha \) is continuous, but not \( C^1 \), and has derivatives

\[
\partial_x u_\alpha(x, y) = \begin{cases} 
-y & \text{if } (x, y) \in 0\#I, \\
0 & \text{if } (x, y) \in \mathbb{R}^2 \setminus 0\#I;
\end{cases}
\]

\[
\partial_y u_\alpha(x, y) = \begin{cases} 
2|y| \cot \alpha - x & \text{if } (x, y) \in 0\#I, \\
0 & \text{if } (x, y) \in \mathbb{R}^2 \setminus 0\#I.
\end{cases}
\]

Proof. The value of the function \( u_\alpha : \mathbb{R}^2 \to \mathbb{R} \) at a point \((x, y)\) is the balayage area of the curve from \((0, 0)\) to \((x, y)\) that follows the half-lines singled out in the above construction. So, if \((x, y) \notin 0\#I\), then \( u_\alpha(x, y) = 0 \). If \((x, y) \in 0\#I\) and \(y \geq 0\), then there are \(x_0 \geq 0\) and \(s \geq 0\) such that

\[
\begin{cases}
x = x_0 + s \cos(\alpha) \\
y = s \sin(\alpha)
\end{cases}
\]

that is

\[
\begin{cases}
x_0 = x - y \cot(\alpha) \\
s = \frac{y}{\sin(\alpha)}
\end{cases}
\]

So, define \( u_\alpha(x, y) \) as the Balayage area of the curve from \((0, 0)\) to \((x, y)\) that follows the \(x\)-axis until \((x_0, 0)\) and then follows the line parallel to \((\cos \alpha, \sin \alpha)\), that is,

\[
u_\alpha(x, y) = -2\frac{x_0 s \sin(\alpha)}{2} = y(y \cot \alpha - x).
\]

Similarly, if \((x, y) \in 0\#I\) and \(y \leq 0\), one finds that \( u_\alpha(x, y) = y(-y \cot \alpha - x) \) and so (3.1) is proven. \( \square \)

Proposition 3.2. Let \( I_1, \ldots, I_k \) be a finite set of disjoint open arcs in \( \mathbb{S}^1 \) and \( C(I_1, \ldots, I_k) \) the associated surface. Then

1. \( C(I_1, \ldots, I_k) \) is a conical continuous t-graph with vertex at 0;
2. \( C(I_1, \ldots, I_k) \) is a \( C^{1,1} \) surface outside the lines \( 0\#\partial I_i \), with singular set \( \bigcup_{i=1}^k L_i \). It is not \( C^2 \) at points of the singular set unless \( \alpha = \pi/2 \).
3. \( C(I_1, \ldots, I_k) \) is area-minimizing.
4. The horizontal unit normal of \( C(I_1, \ldots, C_k) \) is continuous.

Proof. The surface \( C(I_1, \ldots, I_k) \) is the graph of version of the function (3.1) in each sector \( 0\#I_k \), up to a pre-composition with a rotation of the plane. Therefore, the first two statements are clear.

We prove that \( C(I_1, \ldots, I_k) \) is area-minimizing by presenting a calibration and thus applying Proposition 2.4. Figure 3 helps the understanding. Let \( v \) be the horizontal vector field that is invariant along \( t \) and that is equal to the upward unit normal to \( C(I_1, \ldots, I_k) \) outside the half-lines \( \bigcup_{i=1}^k L_i \) and \( \bigcup_{i=1}^k 0\#\partial I_j \). We claim that the distributional divergence of \( v \) is zero.

In fact, the unit normal of \( C(I_1, \ldots, I_k) \) is the upward unit horizontal vector that is orthogonal to the horizontal characteristic lines we lifted. Above the sectors \( 0\#J_i \) is simply \( \frac{\sqrt{x^2 + y^2}}{y^2} \), which is actually the calibration of the plane \( \{ t = 0 \} \); in particular, it is smooth and with zero divergence. Above the other sectors, \( v \) has
constant coefficients in the basis \((X, Y)\) above the regions between the half-lines \(L_j\) and the boundaries \(#\partial I_j\), where it has thus zero divergence.

Finally, one easily sees that \(v\) satisfies (2.3) above the half-lines \(#\partial I_j\) and the lines \(L_j\).

We conclude that \(\text{div}(v) = 0\) by Proposition 2.5. \(\square\)

**Proposition 3.3.** Given a finite set of disjoint open arcs \(I_1, \ldots, I_k\) in \(S^1\), the associated surface \(C(I_1, \ldots, I_k)\) is of class \(C^1\) if and only if \(S^1 = \bigcup_{j=1}^k \bar{I}_j\).

**Proof.** Let \(u : \mathbb{R}^2 \to \mathbb{R}\) be the function whose \(t\)-graph is \(C(I_1, \ldots, I_k)\). In each sector \(#I_j\), the function \(u\) is a version of \(u_{\alpha_j}\) as in (3.1), up to a rotation of the plane.

Since \(\partial_x u_{\alpha}\) is not continuous along the half-lines \(#\partial I\), then we conclude that, if \(C(I_1, \ldots, I_k)\) is of class \(C^1\), then \(S^1 = \bigcup_{j=1}^k \bar{I}_j\).

Next, notice that the derivative of \(u_{\alpha}\) along the vector \((\cos \alpha, \sin \alpha)\) (or the vector \((\cos \alpha, -\sin \alpha))\) is continuous in the half-plane \(\{y > 0\}\) (in the half-plane \(\{y < 0\}\), respectively), and zero along the half-line \(#(\cos \alpha, \sin \alpha)\) (or \(#(\cos \alpha, -\sin \alpha)\), respectively).

So, if \(#I_1\) and \(#I_2\) share a half-line \(#\hat{v}\), where \(|\hat{v}| = 1\), then the derivative of \(u\) along \(\hat{v}\) is continuous across \(#\hat{v}\).

What remains to be checked is the continuity across \(#\hat{v}\) of the derivative of \(u\) along the orthogonal direction to \(\hat{v}\). Going back to \(u_{\alpha}\), a computation shows that

\[-\sin(\alpha)\partial_x u_{\alpha} + \cos(\alpha)\partial_y u_{\alpha}]_{(s \cos \alpha, s \sin \alpha)} = s\]

and

\[(\sin(\alpha)\partial_x u_{\alpha} + \cos(\alpha)\partial_y u_{\alpha})_{(s \cos \alpha, -s \sin \alpha)} = s,\]

where the derivatives are the continuous limit from inside \(#I\). Therefore, the derivative of \(u\) along the orthogonal direction to \(\hat{v}\) are continuous across \(#\hat{v}\). \(\square\)
In the special case of two disjoint open intervals $I_1, I_2$ such that $S^1 = \overline{I_1} \cup \overline{I_2}$, the singular line is a horizontal straight line $L$ which complement is foliated by two families of parallel lines making a constant angle with $L$. This is merely $C^{1,1}$ except in the case $\alpha = \pi/2$ when we get the cone $t \leq -xy$ with $C^\infty$ boundary.

Via approximation, we can consider also the above cones constructed using infinitely many arcs. More precisely, let $\mathcal{I}$ be a family of disjoint open arcs of $S^1$, possibly countable. For each $I \in \mathcal{I}$, let $u_I$ be the function whose $t$-graph is $C(I)$. Define

$$(3.2) \quad u_\mathcal{I} = \sum_{I \in \mathcal{I}} u_I,$$

where the sum is well defined, because for every $v \in \mathbb{R}^2$ there is at most one $I \in \mathcal{I}$ with $u_I(v) \neq 0$.

**Proposition 3.4.** Given a family $\mathcal{I}$ of disjoint open arcs of $S^1$, the function $u_\mathcal{I}$ is continuous and its $t$-graph $C(\mathcal{I})$ is a minimal cone. Moreover, if $\mathcal{I}$ is infinite, then $C(\mathcal{I})$ is not a $C^1$ surface.

**Proof.** From (3.1), one easily sees that $|u_\alpha(v)| \leq |v|^2 \tan(\alpha)$. We deduce that the sum in (3.2) converges uniformly on compact sets. So, $u_\mathcal{I}$ is continuous and its $t$-graph is a cone. By Proposition 2.6, $C(\mathcal{I})$ is a minimal surface.

Finally, if $\mathcal{I}$ is infinite, then there are $\hat{v} \in S^1$ and a sequence $\{I_k\}_k \subset \mathcal{I}$ so that $\text{dist}(\hat{v}, I_k) \to 0$ and the amplitude of $I_k$ also goes to zero. Now, if we consider the
function $u_\alpha$ in (3.1), we see that its $y$-derivative is
\[
\partial_y u_\alpha(x, y) = \begin{cases} 
2|y| \cot \alpha - x & \text{if } (x, y) \in 0 \# I, \\
0 & \text{if } (x, y) \in \mathbb{R}^2 \setminus 0 \# I.
\end{cases}
\]
In particular, if $(x, y) \in 0 \# I$ is close enough to $(1, \tan(\alpha))$, then $\partial_y u_\alpha(x, y)$ is arbitrary close to 1, while $\partial_y u_\alpha(1, 0) = -1$. We conclude that for every $k$ there are points in $0 \# I_k$ where some derivative of $u_{\mathbb{T}}$ oscillates between 1 and $-1$, so $\nabla u_{\mathbb{T}}$ is not continuous at $\hat{v}$. Since $\nabla u_{\mathbb{T}}$ remains bounded, $C(\mathbb{T})$ is not a $C^1$ surface.

\section{4. Classification results}

\subsection{4.1. Characterization of $C^1$ minimal cones.}
This section is devoted to the proof of our main classification result in the $C^1$ case, Theorem B.

\begin{lemma}
A conical $C^1$ surface $S \subset \mathbb{H}^1$ without singular points is a vertical plane.
\end{lemma}

\begin{proof}
For any $p = (p_1, p_2, p_3)$ in $S$ out of the vertical axis $V$ we consider the curve $\gamma(s) = (sp_1, sp_2, s^2 p_3)$, whose tangent vector at $s = 0$ is the horizontal vector $\gamma'(0) = p_1 X_0 + p_2 Y_0 \neq 0$. Since $0$ is not a singular point, $S \setminus V$ must be contained in the vertical plane $p_2 x - p_1 y = 0$ and so is a vertical plane.
\end{proof}

\begin{lemma}
Let $S \subset \mathbb{H}^1$ be a conical $C^1$ surface, and let $p \in S_0 \setminus \{0\}$. Then $0$ and $p$ belong to a horizontal half-line contained in $S_0$.
\end{lemma}

\begin{proof}
We let $p = (p_1, p_2, p_3)$ and consider the curve $\gamma(s) := (sp_1, sp_2, s^2 p_3)$, whose image is contained in $S$. We trivially have $\gamma'(s) = p_1 X_{\gamma(s)} + p_2 Y_{\gamma(s)} + 2tp_3 T_{\gamma(s)}$. Since $\gamma(1) = p$ and $p$ is a singular point, the vector $\gamma'(1)$ is horizontal and so $p_3 = 0$. This implies that $\gamma(s)$ is a parameterization of a horizontal half-line starting from $0$. Since dilations preserve the horizontal distribution, $\gamma(s) \in S_0$ for all $s \geq 0$.
\end{proof}

In the following we denote by $H$ the plane $t = 0$.

\begin{lemma}
Let $S \subset \mathbb{H}^1$ be a conical $C^1$ minimal $t$-graph. If $p \in S \setminus H$, then $p$ is a regular point and there are a singular point $q \in S \cap H$ and a horizontal half-line $L$ starting from $q$ and containing $p$.
\end{lemma}

\begin{proof}
Let $p = (x, y, t) \in S$ with $t \neq 0$. We know that the point $p$ is regular by Lemma 4.2. By Proposition 2.7, there are $\hat{v} \in H$, with $|\hat{v}| = 1$, and $s_0 < 0$ (possibly $s_0 = -\infty$) such that $\gamma(s_0, +\infty) \subset S$, where $\gamma(s) = p \exp(s \hat{v})$, and $s_0$ is minimal with this property. We have two cases.
First, if $s_0 = -\infty$, then there is $s_1 \in \mathbb{R}$ such that $\gamma(s_1) \in H \cap S$. Indeed, if this were not the case, the horizontal line $\gamma(\mathbb{R})$ would meet the $t$-axis in a non-zero point contradicting the hypotheses that $S$ is a $t$-graph and $0 \in S$. Now, notice that $\gamma'(s) = \hat{v}$ is not parallel to $\frac{1}{\partial_y} \alpha \delta \gamma(s_1)$, but these two vectors are both horizontal and tangent to $S$. Therefore $\gamma'(s_1)$ is a singular point of $S$ and thus, the lemma is proven if we take $L = \gamma([s_1, +\infty))$ if $s_0 > s_1$ or $L = \gamma((-\infty, s_1])$ if $s_0 < s_1$.
Second, if $s_0 > -\infty$, then $\gamma(s_0)$ is a singular point and thus it belongs to $H$. The lemma is proven if we take $L = \gamma([s_0, +\infty))$.
\end{proof}

\begin{lemma}
Let $S \subset \mathbb{H}^1$ be a $C^1$ minimal surface invariant by dilations centered at 0. If $S_0 = \{0\}$, then $S$ is the horizontal plane $\{t = 0\}$.
\end{lemma}
Proof. Let \( p_0 = (x_0, y_0, t_0) \in S \) and suppose that \( t_0 \neq 0 \). Since \( p_0 \) is a regular point but no horizontal line passing through \( p_0 \) contains 0, then, by Proposition 2.7, there exists \( \hat{v} \in H \) such that the entire line \( s \mapsto p_0 \exp(s\hat{v}) \) is contained in \( S \). Since \( S \) is a cone, for every \( s \in \mathbb{R} \) and \( \lambda > 0 \), we have

\[
\delta_\lambda(p_0 \exp(s\hat{v}/\lambda)) = (\lambda x_0, \lambda y_0, \lambda^2 t_0) \exp(s\hat{v}) \in S.
\]

However, direct computations show that in such a surface 0 is not the only singular point, in contradiction with the assumption \( S_0 = \{0\} \). Therefore, \( t_0 = 0 \) and so \( S \subset H \). Since \( S \) is a cone, \( S = H \). \( \square \)

Proof of Theorem B. In case \( S \) has no singular points, Lemma 4.1 implies that \( S \) is a vertical plane. If \( S \) has only one as a singular point, then \( S = H \) by Lemma 4.4.

Finally let us assume that \( S_0 \) contains at least two points and that \( S \) is invariant by dilations centered at 0. Then 0 is a singular point, and Lemma 4.2 implies that \( S_0 \) is a union of horizontal half-lines leaving the origin.

Since 0 is a singular point, \( S \) can be represented near 0 as the \( t \)-graph of a \( C^1 \) function and thus, since \( S \) is a cone, the whole \( S \) is the \( t \)-graph of a function \( u : \mathbb{R}^2 \to \mathbb{R} \).

If \( L \) is one of the singular half-lines leaving the origin and \( p \in L \setminus \{0\} \), then there is a neighborhood \( U \) of \( p \) such that \( U \cap S_0 = U \cap L \), because \( T_p S = pH \neq H \) and \( S_0 \subset H \). Therefore, these singular half-lines cannot accumulate and so we have a finite number of them \( L_1, \ldots, L_k \), with \( k \geq 1 \) (see also Theorem C(b) in [2]).

If \( p \in S \setminus H \), then \( p \) is a regular point and, by Lemma 4.3, there is a half-line \( L \subset S \) starting from a singular point \( q \in S \setminus H \), say \( q \in L_j \). Then \( \bigcup_{\lambda > 0} \delta_\lambda L \) describes \( S \) on one side of \( L_j \). In other words, there is an arc \( I_j^1 \) so that \( L_j \) is on the boundary of \( 0 \# I_j^1 \), so that \( u \) is a version of the function \( u_\alpha |_{\{y \geq 0\}} \) or \( u_\alpha |_{\{y \leq 0\}} \) in (3.1) on \( 0 \# I_j^1 \).

We conclude that for every \( j \in \{1, \ldots, k\} \) there are two arcs \( I_j^1 \) and \( I_j^2 \), possibly empty, such that \( S \setminus H \) is the graph of \( u \) above the sectors \( 0 \# I_j^{1,2} \).

Notice that \( u \) does not have other singular points on \( 0 \# I_j^{1,2} \) other than \( L_j \). Therefore, if \( i \neq j \), then \( L_j \cap 0 \# I_i^{1,2} = \emptyset \). It follows that, in fact, \( I_j^{1,2} \) are never empty. Indeed, on each side of every \( L_j \) there are regular points, as we noticed above. However, they cannot belong to a sector of a singular half-line other than \( L_j \).

Finally, the unit normal to \( S \) is constant over each \( 0 \# I_j^{1,2} \). By Lemma 2.8, it must have zero distributional divergence, while, by Proposition 2.4, this happens exactly when it reflects across \( L_j \), that is, the characteristic lines in \( I_j^1 \) and in \( I_j^2 \) meet \( L_j \) with the same angle, exactly as it happens with the function \( u_\alpha \) in (3.1).

Set \( I_j = I_j^1 \cup I_j^2 \). It is clear that \( L_j \) is the bisectrix of \( I_j \) and that \( I_j \cap I_i = \emptyset \) if \( i \neq j \). \( \square \)

References

1. Jih-Hsin Cheng, Jenn-Fang Hwang, Andrea Malchiodi, and Paul Yang, Minimal surfaces in pseudohermitian geometry, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 4 (2005), no. 1, 129–177. MR 2165405
2. ________, A Codazzi-like equation and the singular set for \( C^1 \) smooth surfaces in the Heisenberg group, J. Reine Angew. Math. 671 (2012), 131–198. MR 2983199
3. Jih-Hsin Cheng, Jenn-Fang Hwang, and Paul Yang, Existence and uniqueness for p-area minimizers in the Heisenberg group, Math. Ann. 337 (2007), no. 2, 253–293. MR 2262784
4. __________, Regularity of $C^1$ smooth surfaces with prescribed $p$-mean curvature in the Heisenberg group, Math. Ann. 344 (2009), no. 1, 1–35. MR 2481053
5. G. B. Folland and Elias M. Stein, Hardy spaces on homogeneous groups, Mathematical Notes, vol. 28, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982. MR 657581
6. Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano, Rectifiability and perimeter in the Heisenberg group, Math. Ann. 321 (2001), no. 3, 479–531. MR 1871966 (2003g:49062)
7. Matteo Galli and Manuel Ritoré, Area-stationary and stable surfaces of class $C^4$ in the sub-Riemannian Heisenberg group $\mathbb{H}^1$, Adv. Math. 285 (2015), 737–765. MR 3406514
8. __________, Regularity of $C^1$ surfaces with prescribed mean curvature in three-dimensional contact sub-Riemannian manifolds, Calc. Var. Partial Differential Equations 54 (2015), no. 3, 2503–2516. MR 3412382
9. Nicola Garofalo and Duy-Minh Nhieu, Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces, Comm. Pure Appl. Math. 49 (1996), no. 10, 1081–1144. MR 1404326
10. Sebastiano Golo, Some remarks on contact variations in the first Heisenberg group, Ann. Acad. Sci. Fenn. Math. 43 (2018), no. 1, 311–335. MR 3753176
11. Ana Hurtado, Manuel Ritoré, and César Rosales, The classification of complete stable area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$, Adv. Math. 224 (2010), no. 2, 561–600. MR 2609016
12. Antoine Julia, Sebastiano Nicolussi Golo, and Davide Vittone, Area of intrinsic graphs and coarea formula in carnott groups, arXiv e-prints (2020), arXiv:2004.02520.
13. Roberto Monti, Francesco Serra Cassano, and Davide Vittone, A negative answer to the Bernstein problem for intrinsic graphs in the Heisenberg group, Boll. Unione Mat. Ital. (9) 1 (2008), no. 3, 709–727. MR 2455341 (2009m:35158)
14. Sebastiano Nicolussi and Francesco Serra Cassano, The Bernstein problem for Lipschitz intrinsic graphs in the Heisenberg group, Calc. Var. Partial Differential Equations 58 (2019), no. 4, Art. 141, 28. MR 3984100
15. Manuel Ritoré, Examples of area-minimizing surfaces in the sub-Riemannian Heisenberg group $\mathbb{H}^2$ with low regularity, Calc. Var. Partial Differential Equations 34 (2009), no. 2, 179–192. MR 2448649
16. Manuel Ritoré and César Rosales, Area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$, Adv. Math. 219 (2008), no. 2, 633–671. MR 2435652

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