The Uncertainty of Fluxes

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Abstract. In the ordinary quantum Maxwell theory of a free electromagnetic field, formulated on a curved 3-manifold, we observe that magnetic and electric fluxes cannot be simultaneously measured. This uncertainty principle reflects torsion: fluxes modulo torsion can be simultaneously measured. We also develop the Hamilton theory of self-dual fields, noting that they are quantized by Pontrjagin self-dual cohomology theories and that the quantum Hilbert space is \( \mathbb{Z}/2\mathbb{Z} \)-graded, so typically contains both bosonic and fermionic states. Significantly, these ideas apply to the Ramond-Ramond field in string theory, showing that its \( K \)-theory class cannot be measured.

Fluxes in the classical theory of electromagnetism and its generalizations are real-valued and Poisson-commute. Our main result is a Heisenberg uncertainty principle in the quantum theory: magnetic and electric fluxes cannot be measured simultaneously. This observation applies to any abelian gauge field, including the standard Maxwell field theory in four spacetime dimensions as well as the \( B \)-field and Ramond-Ramond fields in string theories. It is the torsion part of the fluxes which experience uncertainty—the nontrivial commutator of torsion fluxes is computed by the link pairing on the cohomology of space, and there are always nontrivial commutators if torsion is present. We remark that torsion fluxes arise from Dirac charge/flux quantization. This Heisenberg uncertainty relation goes against the conventional wisdom that the quantum Hilbert space is simultaneously graded by the abelian group of magnetic and electric flux; in fact, it is only graded by the free abelian group of fluxes modulo torsion. The most interesting example is the Ramond-Ramond field in 10-dimensional superstring theory. Here the Dirac quantization law

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is expressed in terms of topological $K$-theory, and conventional wisdom holds that the quantum Hilbert space is graded by the integer $K$-theory of space. The main result proves this wrong: the grading is only by $K$-theory modulo torsion. Notice that there are still operators reflecting the quantization by the full $K$-theory group; the assertion is that these operators do not all commute among themselves if there is torsion.

Our exposition in §1 begins with the classical Maxwell equations. We work on a compact\(^1\) 3-dimensional smooth manifold $Y$. We first define the classical fluxes and show that they Poisson commute. The Hamiltonian formulation of Maxwell’s equations has Poisson brackets which are not invertible, so do not derive from a symplectic structure, if the second cohomology of $Y$ is nontrivial: the symplectic leaves of the Poisson structure are parametrized by the fluxes. The most natural quantization of this system is as a family of Hilbert spaces parametrized by the real vector space of fluxes. Dirac charge quantization is implemented in Maxwell theory by writing the electromagnetic field as the curvature of a $T$-connection, where $T = U(1)$ is the circle group. There is now an action principle and the space of classical solutions is a symplectic manifold, the tangent bundle to the space $\mathcal{C}(Y)$ of equivalence classes of $T$-connections on $Y$. Its quantization is a single Hilbert space, defined as the irreducible representation of the Heisenberg group built from the product of $\mathcal{C}(Y)$ and its Pontrjagin dual. (The salient features of Heisenberg groups and their representations are reviewed in Appendix A.) Magnetic and electric fluxes are refined to take values in the abelian group $H^2(Y; \mathbb{Z})$. The Heisenberg uncertainty relation, stated in Theorem 1.20, follows from the commutation relations in the Heisenberg group.

Our second aim in this paper, carried out in §2, is to establish an appropriate Hamiltonian quantization of generalized self-dual fields, such as the Ramond-Ramond field in superstring theory. We highlight the main issues with the simplest self-dual field: the left-moving string on a circle, which we simply call the self-dual scalar field. Its quantization, which does not quite follow from the usual general principles, serves as a model for the general case. The flux quantization condition for other gauge fields, both self-dual and non-self-dual, is expressed in terms of a generalized cohomology theory. The fields themselves live in a generalized differential cohomology theory. We briefly summarize the salient points of the differential theory. The data we give in Definition 2.9 is sufficient for the Hamiltonian theory developed here; the full Lagrangian theory requires a more refined starting point. As in the Maxwell theory one can write classical equations (2.12) and a Heisenberg group (Theorem 2.14), which now is $\mathbb{Z}/2\mathbb{Z}$-graded. The Hilbert space of the self-dual field is defined (up to noncanonical isomorphism) as a $\mathbb{Z}/2\mathbb{Z}$-graded representation of that graded Heisenberg group. The fact that the Hilbert space is $\mathbb{Z}/2\mathbb{Z}$-graded, so in general has fermionic states, is one of the novel points in this paper. The noncommutativity of quantum fluxes (2.16) in the presence of torsion is then a straightforward generalization of Theorem 1.20 in the Maxwell theory. Section 2 concludes by showing how some common examples, including the Ramond-Ramond fields,

\(^1\)There is also an uncertainty principle for fluxes if space is noncompact; we hope to return to that topic in the future.
fit into our framework.

We call attention to one feature which emerged while investigating self-dual fields in general. For non-self-dual generalized abelian gauge fields any generalized cohomology theory may be used to define the Dirac quantization law. However, for a self-dual field the cohomology theory must itself be Pontrjagin self-dual. See Appendix B for an introduction to generalized cohomology theories and duality. Pontrjagin self-duality is a strong restriction on a cohomology theory. Ordinary cohomology, periodic complex $K$-theory, and periodic real $K$-theory\(^2\) are all Pontrjagin self-dual and all occur in physics as quantization laws for self-dual fields. Pontrjagin self-duality is not satisfied by most cohomology theories. For example, if the cohomology of a point in a Pontrjagin self-dual theory contains nonzero elements in positive degrees, then there are nonzero elements in negative degrees as well if the duality is centered about degree zero.

In this paper we confine ourselves to the Hamiltonian point of view. We only construct the quantum Hilbert space and the operators which measure magnetic and electric flux up to non-canonical isomorphism. In future work we plan to develop the entire Euclidean quantum field theory of gauge fields, both self-dual and non-self-dual. We emphasize that the Hamiltonian quantization we use for a self-dual field is a special definition; it does not follow from general principles of quantization—see the discussion at the beginning of §2. Perhaps one should not be surprised that the Hamiltonian theory for self-dual fields requires a separate definition; after all, the same is true for the Lagrangian theory [W]. Our definition is motivated by some preliminary calculations for a full theory of self-dual fields as well as by the special case of the self-dual scalar field in two dimensions.

The role of Heisenberg groups and the noncommutativity of fluxes we find in Maxwell theory was anticipated in [GRW].

Our point of view in this paper is unapologetically mathematical. The fields under discussion are free, their quantum theory is mathematically rigorous, whence our mathematical presentation. In particular, we use the representation theory of Heisenberg groups to define the quantum Hilbert space of a free field. Appendix A reviews these ideas in the generality we need. A companion paper [FMS] presents our ideas and fleshes out the examples in a more physical style.

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\(^2\)KO-theory is Pontrjagin self-dual with a shift; see Proposition B.11.
§1 Maxwell Theory

Classical Maxwell equations

Let $Y$ be a compact oriented Riemannian 3-manifold and $M = \mathbb{R} \times Y$ the associated Lorentzian spacetime with signature $(1, 3)$. The classical electromagnetic field $F \in \Omega^2(M)$ satisfies Maxwell’s equations

\begin{align*}
\text{(1.1)}
\end{align*}

\begin{align*}
dF &= j_B, \\
d \ast F &= j_E,
\end{align*}

where the magnetic and electric currents $j_B, j_E \in \Omega^3(M)$ are closed. The Hodge star operator $\ast$ is defined relative to the Lorentz metric. Let $\Sigma \subset Y$ be a closed oriented surface. The classical magnetic flux $B^{\text{cl}}_{t, \Sigma}$ and classical electric flux $E^{\text{cl}}_{t, \Sigma}$ are defined by

\begin{align*}
B^{\text{cl}}_{t, \Sigma}(F) &= \int_{\{t\} \times \Sigma} F, \\
E^{\text{cl}}_{t, \Sigma}(F) &= \int_{\{t\} \times \Sigma} \ast F.
\end{align*}

It follows from (1.1) that $B^{\text{cl}}_{t, \Sigma}(F)$ and $E^{\text{cl}}_{t, \Sigma}(F)$ are static—that is, independent of time $t$—if we impose $j_B = j_E = 0$. Let $V$ be the vector space of smooth solutions to the vacuum Maxwell equations ($j_B = j_E = 0$) of finite energy. Stokes’ theorem implies that both $B^{\text{cl}}_{\Sigma}(F)$ and $E^{\text{cl}}_{\Sigma}(F)$ depend only on the homology class of $\Sigma$, so define functions

\begin{align*}
\text{(1.2)}
\end{align*}

\begin{align*}
B^\text{cl}, E^\text{cl} : H_2(Y) \times V \rightarrow \mathbb{R}
\end{align*}

which are homomorphisms on $H_2(Y) \times \{F\}$ for all $F \in V$. Put differently, in this vacuum case $B^\text{cl}, E^\text{cl}$ are $H^2(Y; \mathbb{R})$-valued functions on $V$, simply the de Rham cohomology classes of $F, \ast F$.

To express Maxwell’s equations in Hamiltonian form, write

\begin{align*}
\text{(1.3)}
F &= B - dt \wedge E
\end{align*}

for $B = B(t) \in \Omega^2(Y)$ and $E = E(t) \in \Omega^1(Y)$; we set the speed of light to one. Let $\ast$ be the Hodge star operator on $Y$ (relative to its Riemannian metric). Then $B$ and $\ast E$ are closed and their evolution equations are

\begin{align*}
\text{(1.4)}
\frac{\partial B}{\partial t} &= -d_Y E, \\
\frac{\partial \ast E}{\partial t} &= d_Y \ast B.
\end{align*}

\[3\]The orientation assumption is only for convenience of exposition. It is easily removed by working with differential forms twisted by the orientation bundle. The Hodge star operator maps ordinary forms to twisted forms on a non-oriented manifold, so the electric current $j_E$ is a twisted form whereas the magnetic current $j_B$ is untwisted.
Let \( W = \Omega^2(Y)_{\text{closed}} \times \Omega^2(Y)_{\text{closed}} \). For each time \( t \) there is an isomorphism \( V \to W \) obtained by evaluating the restriction of \( (F, *F) \) to \( Y \) at time \( t \). The Hamiltonian of the electromagnetic field, its total energy, is
\[
H = \frac{1}{2} \int_Y B \wedge *B + E \wedge *E.
\]

To put (1.4) in Hamiltonian form we introduce a Poisson structure on \( W \).

**Definition 1.5.** A (translationally-invariant) Poisson structure on a real vector space \( W \) is a skew-symmetric pairing
\[
\{ \cdot, \cdot \} : W^* \times W^* \to \mathbb{R}
\]
on the space of linear functions on \( W \). The symplectic leaves are the affine translates in \( W \) of the annihilator of the kernel of \( \{ \cdot, \cdot \} \).

If \( \{ \cdot, \cdot \} \) is nondegenerate, then \( W \) is symplectic. In general there is a kernel, the subspace
\[
K = \{ \ell \in W^* : \{ \ell, \ell' \} = 0 \text{ for all } \ell' \in W^* \},
\]
and its annihilator is \( \{ w \in W : \ell(w) = 0 \text{ for all } \ell \in K \} \). The Poisson structure induces a Poisson bracket on polynomial functions on \( W \). The Poisson structure for Maxwell is most easily written in terms of the linear functionals
\[
\ell_\eta(B, *E) = \int_Y \eta \wedge B, \quad \ell'_\eta(B, *E) = \int_Y \eta \wedge *E, \quad \eta \in \frac{\Omega^1(Y)}{d\Omega^0(Y)}.
\]
Namely,
\[
\{ \ell_{\eta_1}, \ell_{\eta_2} \} = \{ \ell'_{\eta_1}, \ell'_{\eta_2} \} = 0, \quad \{ \ell_{\eta_1}, \ell'_{\eta_2} \} = \int_Y d\eta_1 \wedge \eta_2.
\]
Then equation (1.4) takes the Hamiltonian form
\[
\frac{\partial B}{\partial t} = \{ H, B \}, \quad \frac{\partial (*E)}{\partial t} = \{ H, *E \}.
\]
The kernel (1.6) of the Poisson bracket consists of \( \ell_\eta, \ell'_\eta \) for closed \( \eta \), and so its annihilator is the subspace
\[
W_0 = d\Omega^1(Y) \times d\Omega^1(Y) \subset W.
\]
Therefore, the symplectic leaves are the fibers of the map \( \pi \) in
\[
0 \to W_0 \to W \xrightarrow{\pi} H^2(Y; \mathbb{R}) \times H^2(Y; \mathbb{R}) \to 0
\]
which assigns to \( (B, *E) \) the pair \( ([B]_{\text{dR}}, [*E]_{\text{dR}}) \) of de Rham cohomology classes.

The classical fluxes (1.2) may be viewed as \( H^2(Y; \mathbb{R}) \)-valued linear functions on \( W \), which together form the map \( \pi \) in (1.7).
Classical Fact 1.8. The classical magnetic and electric fluxes $\mathcal{B}^{cl}$ and $\mathcal{E}^{cl}$ Poisson commute.

To make this precise, for a closed 1-form $\eta$ on $Y$ define the linear functions $\mathcal{B}^{cl}(\eta), \mathcal{E}^{cl}(\eta) : W \to \mathbb{R}$ by

$$
(1.9) \quad \mathcal{B}^{cl}(\eta)(B, \star E) = \int_Y \eta \wedge B, \quad \mathcal{E}^{cl}(\eta)(B, \star E) = \int_Y \eta \wedge \star E.
$$

Integration by parts shows

$$
\{\mathcal{B}^{cl}(\eta_1), \mathcal{E}^{cl}(\eta_2)\} = 0.
$$

We pass to the quantum theory (without Dirac charge quantization) by quantizing the affine symplectic fibers in (1.7), thus obtaining a family of Hilbert spaces parametrized by $H^2(Y; \mathbb{R}) \times H^2(Y; \mathbb{R})$. The parameter is the pair of fluxes, which varies continuously over this real vector space. For simplicity we discuss only the quantization of the fiber at $(0,0)$, the symplectic vector space $W_0$. Briefly, one writes $W_0 \otimes \mathbb{C}$ as a direct sum of lagrangian subspaces and Hilbert space completes the polynomial functions (Fock space) on one of the summands. For finite dimensional symplectic vector spaces the resulting (projective) Hilbert space $\mathcal{H}$ is independent of the lagrangian splitting. For infinite dimensional symplectic vector spaces one needs to fix a polarization to specify the quantization. In Hamiltonian field theory the natural polarization is given by the energy operator $i d/dt$: the complexification of the space of classical solutions is the sum of positive energy solutions and negative energy solutions. The cleanest characterization of the quantum (projective) Hilbert space $\mathcal{H}$ is as the unique irreducible representation of associated Heisenberg group which is compatible with the polarization. We recall the definition of the Heisenberg group and leave further discussion to Appendix A. Let $W_0$ be any symplectic vector space with symplectic form $\Omega$. The Heisenberg group is a central extension of the translation group $W_0$ by the circle group $\mathbb{T}$ of unit complex numbers. It is defined as the set $W_0 \times \mathbb{T}$ with multiplication

$$
(1.10) \quad (w_1, \lambda_1) \cdot (w_2, \lambda_2) = (w_1 + w_2, \lambda_1 \lambda_2 \exp(i \pi \Omega(w_1, w_2))).
$$

The quantum Hilbert space $\mathcal{H}_0$ is heuristically the space of $L^2$ functions on $d\Omega^1(Y)$, but rather than define a measure on this infinite dimensional vector space we appeal to the representation theory of the Heisenberg group.

Semiclassical Maxwell theory: Dirac charge quantization

Our main concern is the modification of this discussion when Dirac charge quantization is taken into account. The quantization of charge leads to the quantization of flux. Thus on the space-time $M = \mathbb{R} \times Y$ the quantum magnetic and electric fluxes are constrained to live in a full lattice
inside the vector space $H^2(Y; \mathbb{R})$, namely the image of integer cohomology $H^2(Y; \mathbb{Z})$ in $H^2(Y; \mathbb{R})$. The map

$$H^2(Y; \mathbb{Z}) \rightarrow H^2(Y; \mathbb{R})$$

has a kernel, the torsion subgroup $\text{Tors} H^2(Y; \mathbb{Z}) \subset H^2(Y; \mathbb{Z})$, so the lattice is naturally identified with $H^2(Y; \mathbb{Z})/\text{torsion}$. A geometric model which implements charge quantization takes the electromagnetic field $F$ to be $-1/2\pi i$ times the curvature of a connection $A$ on a principal circle bundle $P \rightarrow M$. This lifts the magnetic flux from $H^2(M; \mathbb{Z})/\text{torsion}$ to an element of the abelian group $H^2(M; \mathbb{Z})$, the Chern class of $P$. The curvature of a connection depends only its isomorphism class. The space $\mathcal{C}(M)$ of isomorphism classes of smooth connections forms an infinite dimensional abelian Lie group under tensor product of circle bundles with connection. The geometry of $\mathcal{C}(M)$ is important to us, so we pause to elucidate it.

In this paragraph $M$ is any smooth manifold. Let $\Omega^q_Z(M)$, $q > 0$, be the set of closed $q$-forms on $M$ with integral periods. Also, let $\mathbb{T}$ denote the circle group. Then

$$0 \rightarrow H^1(M; \mathbb{T}) \xrightarrow{i} \mathcal{C}(M) \xrightarrow{\text{curvature}} \Omega^2_Z(M) \rightarrow 0$$

is an exact sequence of abelian groups. Thus any closed 2-form with integral periods is realized as the curvature/($-2\pi i$) of a connection and the kernel of the curvature map is the group of isomorphism classes of flat connections. The latter is an abelian Lie group whose identity component is the torus $H^1(M; \mathbb{Z}) \otimes \mathbb{T}$ and whose group of components is the finite group $\text{Tors} H^2(M; \mathbb{Z})$. This is encoded in the exact sequence

$$1 \rightarrow H^1(M; \mathbb{Z}) \otimes \mathbb{T} \rightarrow H^1(M; \mathbb{T}) \xrightarrow{\beta} \text{Tors} H^2(M; \mathbb{Z}) \rightarrow 1,$$

where $\beta$ is the Bockstein homomorphism. Also, $\Omega^2_Z(M)$ is the union of affine spaces $\Omega^q_b$ of closed forms whose de Rham cohomology class is $\tilde{b} \in H^2(M; \mathbb{Z})/\text{torsion}$. Let $C_b \subset \mathcal{C}(M)$ be the preimage of $\Omega^q_b$; then $C_b \rightarrow \Omega^q_b$ is a principal bundle with structure group $H^1(M; \mathbb{T})$. Another view of $\mathcal{C}(M)$ is exhibited by the exact sequence

$$0 \rightarrow \Omega^1(M) / \Omega^1_b(M) \rightarrow \mathcal{C}(M) \xrightarrow{\text{Chern}} H^2(M; \mathbb{Z}) \rightarrow 0;$$

any integral cohomology class of degree two is the Chern class of a principal circle bundle and the kernel of the Chern class map is the set of connections on the trivial bundle up to gauge

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Footnote: In this Hamiltonian setup one postulates that $P$ be the pullback of a bundle on $Y$, i.e., time translation is lifted to $P$. 
equivalence. Let $\mathcal{C}_b$ denote the set of equivalence classes of connections on a bundle with Chern class $b \in H^2(M; \mathbb{Z})$ and $\bar{b}$ the image of $b$ in de Rham cohomology. Then $\mathcal{C}_b \to \Omega_b$ is a principal bundle with structure group the torus $H^1(M; \mathbb{Z}) \otimes \mathbb{T}$. The group $\mathcal{C}(M)$ is naturally a Lie group. The quotient $H^2(M; \mathbb{Z})$ in (1.14) is its group of components and the kernel in that exact sequence is its identity component.

There is an action principle for Maxwell theory on the space\(^5\) of fields $\mathcal{C}(M)$. Namely,

\[ L(A) = -\frac{1}{2} F_A \wedge *F_A, \]

where $-2\pi i F_A$ is the curvature of the connection $A$. Cauchy data at fixed time identifies the space $\mathcal{M}$ of solutions to the corresponding Euler-Lagrange equations as the tangent bundle to $\mathcal{C}(Y)$.

Now, in contrast to the classical theory considered above, the entire space $\mathcal{M}$ has a symplectic structure. The magnetic and electric fluxes are defined as in (1.9): for a closed 1-form $\eta$ on $Y$ we have the functions $\mathcal{B}^{\text{scl}}(\eta), \mathcal{E}^{\text{scl}}(\eta): \mathcal{M} \to \mathbb{R}$ given by

\[ \mathcal{B}^{\text{scl}}(\eta)(A) = \int_Y \eta \wedge F_A, \quad \mathcal{E}^{\text{scl}}(\eta)(A) = \int_Y \eta \wedge *F_A, \]

where the integrals are computed at any fixed time. The semiclassical magnetic flux is quantized—if $\eta$ represents an integral class then $\mathcal{B}^{\text{scl}}$ is integer-valued—whereas $\mathcal{E}^{\text{scl}}$ is not.

**Semiclassical Fact 1.15.** *The semiclassical magnetic and electric fluxes Poisson commute:*

\[ \{ \mathcal{B}^{\text{scl}}(\eta_1), \mathcal{E}^{\text{scl}}(\eta_2) \} = 0. \]

In fact, $\mathcal{B}^{\text{scl}}(\eta_1)$ is locally constant on $\mathcal{M}$, since $[F_A]_{\text{dR}}$ takes discrete values, so $\mathcal{B}^{\text{scl}}(\eta_1)$ Poisson commutes with any function on $\mathcal{M}$.

To motivate our definition of the quantum electric flux below we observe that any function on a symplectic manifold generates an infinitesimal symplectic diffeomorphism. A linear function on a symplectic vector space generates an infinitesimal translation, and for $\mathcal{E}^{\text{scl}}(\eta)$ it is infinitesimal translation by $\eta$, viewed as a static connection on the trivial bundle. Its equivalence class in $\mathcal{C}(Y)$ lies in the torus $H^1(Y; \mathbb{Z}) \otimes \mathbb{T}$. In the quantum version below this torus is augmented to the compact abelian Lie group $H^1(Y; \mathbb{T})$.

**Quantum Maxwell theory**

We quantize the semiclassical Maxwell theory. Recalling that the space $\mathcal{M}$ of classical solutions may be identified with the tangent bundle to $\mathcal{C}(Y)$, we see that the Hilbert space $\mathcal{H}_Y$ is heuristically

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\(^5\)In fact, the fields do not form a space but rather the groupoid of $\mathbb{T}$ connections on $M$. The lagrangian is gauge-invariant, so determines a function on the space $\mathcal{C}(M)$ of equivalence classes.
the space of $L^2$ functions on $C(Y)$. It is naturally graded by the group $H^2(Y;\mathbb{Z})$ of components of $C(Y)$; the homogeneous subspaces consist of $L^2$ functions supported on a single component:

\[(1.16) \quad \mathcal{H}_Y = \bigoplus_{b \in H^2(Y;\mathbb{Z})} \mathcal{H}^b.\]

This is the grading by magnetic flux. As for electric flux, observe that the group of flat connections $H^1(Y;\mathbb{T})$ acts on $C(Y)$ by tensor product. This induces a representation of $H^1(Y;\mathbb{T})$ on $L^2$ functions, and we decompose $\mathcal{H}_Y$ according to the group of characters\(^6\) $\text{Hom}(H^1(Y;\mathbb{T}),\mathbb{T}) \cong H^2(Y;\mathbb{Z})$:

\[(1.17) \quad \mathcal{H}_Y = \bigoplus_{e \in H^2(Y;\mathbb{Z})} \mathcal{H}_e.\]

**Main Observation 1.18.** The gradings (1.16) and (1.17) do not necessarily induce a simultaneous grading of $\mathcal{H}_Y$ by magnetic and electric flux.

In this heuristic picture, where $\mathcal{H}_Y$ is the space of $L^2$ functions on $C(Y)$, the main observation follows immediately from the fact that translation by an element of $H^1(Y;\mathbb{T})$ does not preserve the components of $C(Y)$, which are labeled by $H^2(Y;\mathbb{Z})$. In fact, the action on the group of components is by translation via the Bockstein homomorphism $\beta$ in (1.13). In other words, the identity component of $H^1(Y;\mathbb{T})$ does preserve the group of components, and the noncommutativity is measured strictly by the torsion subgroup $\text{Tors} H^2(Y;\mathbb{Z})$ of the second cohomology.

**Amelioration 1.19.** The Hilbert space $\mathcal{H}_Y$ is simultaneously graded by magnetic and electric fluxes modulo torsion, that is, by the abelian group $H^2(Y;\mathbb{Z})/\text{Torsion} \times H^2(Y;\mathbb{Z})/\text{Torsion}$.

The main observation is in force whenever $Y$ has torsion in its second cohomology, or equivalently in its first homology. For example, take $Y$ to be a three-dimensional lens space, such as real projective space $\mathbb{RP}^3$.

These observations can be formulated more sharply. For each $\omega \in H^1(Y;\mathbb{T})$ there are (scalar-valued) linear operators $\mathcal{B}^{\text{qtm}}(\omega), \mathcal{E}^{\text{qtm}}(\omega)$. The operator $\mathcal{B}^{\text{qtm}}(\omega)$ is multiplication by $\langle b, \omega \rangle$ on $\mathcal{H}^b$; the operators $\mathcal{E}^{\text{qtm}}(\omega)$ form the representation of $H^1(Y;\mathbb{T})$ on $\mathcal{H}_Y$.

**Theorem 1.20.** For $\omega_1, \omega_2 \in H^1(Y;\mathbb{T})$ we have

\[(1.21) \quad [\mathcal{B}^{\text{qtm}}(\omega_1), \mathcal{E}^{\text{qtm}}(\omega_2)] = (\omega_1 \sim \beta \omega_2)[Y] \quad \text{id}_{\mathcal{H}_Y},\]

where $\beta$ is the Bockstein in (1.13) and $[Y]$ the fundamental class of $Y$ in homology.

\(^{6}\)The identification of the characters with $H^2(Y;\mathbb{Z})$ uses Poincaré duality, which is an important ingredient in the general picture.
The left hand side of (1.21) is the \textit{group commutator} $B_{\text{qtm}}(\omega_1)E_{\text{qtm}}(\omega_2)B_{\text{qtm}}(\omega_1)^{-1}E_{\text{qtm}}(\omega_2)^{-1}$. The pairing on the right-hand side of (1.21) is symmetric and depends only on $\beta\omega_1, \beta\omega_2$, so factors to a symmetric pairing

\[(1.22) \quad \tau: \text{Tors } H^2(Y;\mathbb{Z}) \times \text{Tors } H^2(Y;\mathbb{Z}) \longrightarrow \mathbb{T},\]

the so-called \textit{link pairing} or \textit{torsion pairing} in cohomology. Poincaré duality implies that $\tau$ is a perfect pairing: $\text{Tors } H^2(Y;\mathbb{Z})$ is its own Pontrjagin dual.

These heuristics are made rigorous, and Theorem 1.20 is proved, by defining the quantum Hilbert space $\mathcal{H}_Y$ as a representation of a generalized Heisenberg group. (See Appendix A for more details.)

\textbf{Definition 1.23.} Let $\mathcal{A}$ be an abelian group and $\psi: \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{T}$ a 2-cocycle, that is,

\[(1.24) \quad \psi(a_1, a_2)\psi(a_1 + a_2, a_3) = \psi(a_1, a_2 + a_3)\psi(a_2, a_3), \quad a_1, a_2, a_3 \in \mathcal{A}.\]

The group $\mathcal{G}(\mathcal{A}, \psi)$ attached to $(\mathcal{A}, \psi)$ is the set $\mathcal{A} \times \mathbb{T}$ with multiplication

\[(a_1 + a_2, \lambda_1 \lambda_2 \psi(a_1, a_2)).\]

Commutators in $\mathcal{G}(\mathcal{A}, \psi)$ are measured by a map $s: \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{T}$:

\[(1.25) \quad [(a_1, \lambda_1), (a_2, \lambda_2)] = [0, s(a_1, a_2)] = [0, \psi(a_1, a_2) \psi(a_2, a_1)^{-1}], \quad a_1, a_2 \in \mathcal{A}.\]

The map $s$ is \textit{bimultiplicative}—a homomorphism in each variable separately—and is \textit{alternating}: $s(a, a) = 1$ for all $a \in \mathcal{A}$. We say $s$ is \textit{nondegenerate} if for all $a_1 \in \mathcal{A}$ there exists $a_2 \in \mathcal{A}$ such that $s(a_1, a_2) \neq 1$. If $s$ is nondegenerate, then we call $\mathcal{G}(\mathcal{A}, \psi)$ a \textit{Heisenberg group}.

If $\mathcal{A}$ is a Lie group and $\psi$ is smooth, then $\mathcal{G}(\mathcal{A}, \psi)$ is also a Lie group. The Heisenberg group associated to a symplectic vector space $(V, \Omega)$ is $\mathcal{G}(V, e^{i\pi\Omega})$; see (1.10). The Heisenberg group is a central extension of $\mathcal{A}$ by the circle group:

\[1 \longrightarrow \mathbb{T} \longrightarrow \mathcal{G}(\mathcal{A}, \psi) \longrightarrow \mathcal{A} \longrightarrow 0.\]

For quantum Maxwell theory we take

\[(1.26) \quad \mathcal{A} = \mathcal{C}(Y) \times \mathcal{C}(Y).\]
The pairing $\psi$ is defined in terms of a symmetric pairing

\[
\sigma : \mathcal{C}(Y) \times \mathcal{C}(Y) \rightarrow \mathbb{T}
\]
on connections, namely

\[
(1.27) \quad \psi((A_1, A_2), (A'_1, A'_2)) = \sigma(A_1, A'_2).
\]

Notice that $\psi$ vanishes on $H \times H$ for the subgroups $H = \mathcal{C}(Y) \times \{0\}$ and $H = \{0\} \times \mathcal{C}(Y)$, so the Heisenberg group is canonically split over these subgroups. The simplest way to define $\sigma$ is to use the topological fact that every compact oriented 3-manifold $Y$ with a pair of circle connections $A_1, A_2$ bounds a compact oriented 4-manifold $X$ over which the connections extend. Let $-2\pi i F_j$, $j = 1, 2$, denote the curvatures of the extended connections. Then

\[
(1.28) \quad \sigma(A_1, A_2) = \exp(2\pi i \int_X \tilde{F}_1 \wedge \tilde{F}_2).
\]

We remark that the diagonal value $\sigma(A, A)$ is the Chern-Simons invariant of the connection $A$. For $\alpha_1 \in \Omega^1(Y)/\Omega^1_2(Y)$ (see (1.14)) this formula simplifies to

\[
\sigma(\alpha_1, A_2) = \exp(2\pi i \int_Y \alpha_1 \wedge F_2),
\]

where $-2\pi i F_2$ is the curvature of the connection $A_2$. If $\omega_1$ and $\omega_2$ are flat connections, i.e., live in the subgroup $H^1(Y; \mathbb{T})$ (see (1.12)), then

\[
\sigma(\omega_1, \omega_2) = ([\omega_1 \sim \beta \omega_2])|Y|
\]
is the link pairing $\tau(\beta \omega_1, \beta \omega_2)$ (see (1.22)).

Proposition A.5 in the Appendix A asserts that the Heisenberg group $\mathcal{G}(A, \psi)$ has a unique irreducible unitary representation up to isomorphism which is compatible with positive energy and on which the central circle acts by scalar multiplication.

**Definition 1.29.** The quantum Hilbert space $\mathcal{H}_Y$ of Maxwell theory is this unique irreducible representation of $\mathcal{G}(A, \psi)$.

Recall (1.27) that $\mathcal{G}(A, \psi)$ is canonically split over $\mathcal{C}(Y) \times \{0\}$ and $\{0\} \times \mathcal{C}(Y)$. The operators $\mathcal{B}^{\text{qtm}}(\omega)$ and $\mathcal{E}^{\text{qtm}}(\omega)$ for $\omega \in H^1(Y; \mathbb{T})$ are defined by restricting the representation in Definition 1.29 to the lifts of the subgroups $H^1(Y; \mathbb{T}) \times \{0\}$ and $\{0\} \times H^1(Y; \mathbb{T})$ of $\mathcal{C}(Y) \times \mathcal{C}(Y)$ in $\mathcal{G}(A, \psi)$. Theorem 1.20, and so our main observation, now follows directly from the commutation relations of the Heisenberg group.

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The simplest self-dual field is the self-dual scalar field in 2-dimensional field theory, which with Dirac charge quantization takes values in the circle $\mathbb{T}$. It is also called a left-moving string. Its motion is described by a $\mathbb{T}$-valued function of time $t$ and space $x$ of the form $f(t, x) = \phi(x + t)$, which is a general solution to the first-order wave equation

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}. \quad (2.1)$$

When space is a circle the classical motions are parametrized by the loop group $L\mathbb{T} = \text{Map}(S^1, \mathbb{T})$. Now any $\phi \in L\mathbb{T}$ may be uniquely decomposed as

$$\phi(x) = \lambda e^{iwx} \exp(i \sum_{n \neq 0} q_n e^{inx}),$$

where $w \in \mathbb{Z}$ is the winding number, $\lambda \in \mathbb{T}$, and $q_{-n} = \overline{q_n}$ are complex numbers. This defines a decomposition of the loop group

$$L\mathbb{T} \cong \mathbb{T} \times \mathbb{Z} \times V \quad (2.2)$$

as the product of the circle group, the integers, and a real vector space. In physics $V$ is called the space of “oscillators”. There is a Hamiltonian description of the motion. The Poisson brackets define a nondegenerate pairing on $V$, but the overall structure is degenerate: the symplectic leaves are parametrized by $\mathbb{T} \times \mathbb{Z}$. The quantization of this system does not follow standard rules, which would give a family of Hilbert spaces parametrized by $\mathbb{T} \times \mathbb{Z}$. Rather, we observe that $\mathbb{T}$ and $\mathbb{Z}$ are Pontrjagin dual, and in fact the entire loop group (2.2) is Pontrjagin self-dual. So there is a Heisenberg central extension, the standard central extension of the loop group at level one. Furthermore, we introduce a $\mathbb{Z}/2\mathbb{Z}$-grading.

**Definition 2.3.** A $\mathbb{Z}/2\mathbb{Z}$-grading on a topological group $G$ is a continuous homomorphism $\epsilon : G \to \mathbb{Z}/2\mathbb{Z}$. A graded representation of a $\mathbb{Z}/2\mathbb{Z}$-graded group is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $\mathcal{H}^0 \oplus \mathcal{H}^1$ and a homomorphism $G \to \text{GL}(\mathcal{H}^0 \oplus \mathcal{H}^1)$ such that even elements of $G$ preserve the grading on $\mathcal{H}^0 \oplus \mathcal{H}^1$ and odd elements reverse it.

A $\mathbb{Z}/2\mathbb{Z}$-grading $\epsilon$ is constant on the identity component, so for a Lie group does not induce any structure on the Lie algebra. In particular, there are no sign rules for a grading on a group. For the loop group $L\mathbb{T}$ there is a unique nontrivial grading according to the parity of the winding number, and it lifts to a grading of the Heisenberg central extension. The unique irreducible unitary representation of the Heisenberg central extension is graded—see the discussion at the end.
of Appendix A—and is defined to be the quantum Hilbert space of the self-dual scalar. That the Hilbert space is \( \mathbb{Z}/2\mathbb{Z} \)-graded is expected from the Bose-Fermi correspondence in two dimensions. Another motivation for the grading is \([S,(12.3)]\).

More complicated self-dual fields, such as the Ramond-Ramond fields of superstring theory, lead to a Pontrjagin self-dual abelian group which generalizes (2.2). Its definition requires new ideas. The Dirac quantization law is implemented by a cohomology theory; for Ramond-Ramond fields it is a flavor of \( K \)-theory. The abelian group which plays the role of (2.2) is then a differential cohomology group built by combining the cohomology theory with differential forms. For ordinary cohomology these groups were first introduced by Cheeger-Simons [CS] and are a smooth version of Deligne cohomology \([D]\). The generalization we need is developed by Hopkins-Singer \([HS]\).\(^7\) We summarize what we need and then go on to define self-dual fields and their quantum Hilbert space.

Non self-dual fields quantized by ordinary cohomology, such as the \( B \)-field in Type II superstring theory, fit into our theory by doubling and considering them as self-dual (Example 2.18); there is an equivalent direct treatment as a non-self-dual field.

**Generalized differential cohomology**

Let \( E \) be a multiplicative cohomology theory, and \( E_\mathbb{R}, E_\mathbb{T} \) the associated theories with real and circle coefficients. (See Appendix B for a brief introduction.) The real \( E \)-cohomology of a point \( V_E^\bullet = E^\bullet(\text{pt}; \mathbb{R}) \) is a \( \mathbb{Z} \)-graded real vector space. For any space \( X \) there is a natural map

\[
E^\bullet(X) \to H(X; V_E)^\bullet
\]

whose image is a full lattice and whose kernel is the torsion subgroup. The codomain—the space on the right hand side of (2.4)—is a direct sum of ordinary real cohomology groups. The total degree is the sum of the cohomological degree and the degree in \( V_E \). In case \( E \) is ordinary cohomology, \( V_H^\bullet \) equals \( \mathbb{R} \) in degree zero and vanishes otherwise, and (2.4) is the map (1.11). In case \( E \) is complex \( K \)-theory, \( V_K^\bullet \cong \mathbb{R}[u, u^{-1}] \) is a Laurent series ring with the inverse Bott element \( u \) of degree 2. Now suppose \( M \) is a smooth manifold. The generalized differential cohomology \( \tilde{E}^\bullet(M) \) is a \( \mathbb{Z} \)-graded abelian Lie group. We content ourselves here with a description of its geometry, analogous to the discussion of the geometry of \( \tilde{C}(M) = \tilde{H}^2(M) \) in §1. First, the \( E \)-cohomology with circle coefficients is a compact abelian group whose identity component is a torus, as indicated in the sequence

\[
1 \to E^{d-1}(M) \otimes \mathbb{T} \to E^{d-1}_\mathbb{T}(M) \xrightarrow{\beta} \text{Tors } E^d(M) \to 1,
\]

valid for any \( d \in \mathbb{Z} \). (The \( E \)-cohomology is graded by the integers and can be nonzero in negative degrees, as for example for \( K \)-theory.) Let \( \Omega_E(M; V_E)^9 \) denote the space of closed \( V_E \)-valued
differential forms of total degree $q$ whose de Rham cohomology class lies in the image of (2.4). There are exact sequences

\[(2.5) \quad 0 \to E_{d-1}^d(M) \xrightarrow{i} \tilde{E}^d(M) \xrightarrow{\text{field strength}} \Omega_E(M; V_E)^d \to 0\]

and

\[(2.6) \quad 0 \to \Omega(M; V_E)^{d-1} / \Omega_E(M; V_E)^{d-1} \to \tilde{E}^d(M) \xrightarrow{\text{characteristic class}} E^d(M) \to 0.\]

In the physics language, a gauge field of degree $d$ up to gauge equivalence is an element of $\tilde{E}^d(M)$. Its field strength is computed using the labeled map in (2.5); the isomorphism classes of flat fields form the compact abelian group which is its kernel. A gauge field also has a magnetic flux, the characteristic class in (2.6), and the space of equivalence classes of gauge fields with trivial magnetic flux is a quotient space of differential forms.

Because $E$ is a multiplicative cohomology theory, the associated differential theory has a graded multiplication

$$\tilde{E}^d(M) \otimes \tilde{E}^{d'}(M) \to \tilde{E}^{d+d'}(M).$$

It combines the wedge product on differential forms and the multiplication in $E$: the field strength of the product is the wedge product of the field strengths and the characteristic class of the product is the $E$-product of the characteristic classes. Also, if $\omega \in E_{d-1}^d(M)$ and $\tilde{F} \in \tilde{E}^d(M)$, then the product $i(\omega) \cdot \tilde{F}$ only depends on the characteristic class of $\tilde{F}$ and is the usual cohomological product

$$E_{d-1}^d(M) \otimes E^{d'}(M) \to E_{d-1}^{d+d'}(M) \to \tilde{E}^{d+d'}(M).$$

In particular,

$$i(\omega) \cdot i(\omega') = \omega \cdot \beta \omega' = \tau(\beta \omega, \beta \omega'), \quad \omega \in E_{d-1}^d(M), \quad \omega' \in E_{d-1}^{d'}(M),$$

where $\tau$ is the “link (torsion) pairing”

\[(2.7) \quad \tau: \text{Tors } E^d(M) \otimes \text{Tors } E^{d'}(M) \to \text{Tors } E_{d-1}^{d+d'-1}(M).\]

Similarly, if $\alpha$ lies in the kernel of (2.6) then the product $\alpha \cdot \tilde{F}$ only depends on the field strength of $\tilde{F}$ and is given by wedge product.

Integration (also called pushforward or direct image) is defined in generalized differential cohomology. Suppose $M$ is a compact $n$-manifold which is oriented for $E$-cohomology. An orientation for
differential $\tilde{E}$-cohomology includes an orientation for $E$-cohomology, but might involve more data. For ordinary cohomology an $\tilde{H}$-orientation is an $H$- orientation is the usual notion of orientation. For real $KO$-theory a $KO$-orientation is a spin structure on $M$. A differential $KO^\chi$-orientation is a spin structure together with a Riemannian metric. Similarly, for complex $K$-theory a $K$-orientation is a spin$^c$ structure, whereas a differential $\tilde{K}$-orientation is a spin$^c$-structure together with a Riemannian metric and compatible covariant derivative on the spin$^c$ structure. For a compact $\tilde{E}$-oriented $n$-manifold $M$ we have the integration map

$$\int_{M} : \tilde{E}^\bullet(M) \rightarrow \tilde{E}^\bullet(-n)(pt).$$

There is an extension to integration in fiber bundles and for arbitrary maps as well. These integrations satisfy the usual Stokes’ theorem and are compatible with the corresponding integration $E$-cohomology. However, in general it does not commute with the field strength: there is an invertible differential form $\hat{A}_E(M) \in \Omega(M; V_E)^0$ such that for $\tilde{F} \in \tilde{E}^\bullet(M)$ the field strength of $\int_M \tilde{F}$ is

\begin{equation}
\int_M \hat{A}_E(M) \wedge F,
\end{equation}

where $F$ is the field strength of $\tilde{F}$. This form is derived from the Riemann-Roch theorem, which relates the pushforwards in $E$ and ordinary cohomology. Thus when $E$ is ordinary cohomology this form is 1; its value for real $K$-theory explains our choice of notation. There is an extension to integration in fiber bundles. Finally, if $\tilde{F} = i(\omega)$ in (2.5), then $\int_M \tilde{F} = i(\omega')$ for $\omega' = \int_M \hat{A}_E(M) \wedge \omega$.

Notice that $\tilde{E}^\bullet(pt)$ is not concentrated in degree zero. For example, even in ordinary cohomology the exact sequences (2.5) and (2.6) imply

$$\tilde{H}^\bullet(pt) = \begin{cases} 
\mathbb{Z}, \bullet = 0, \\
\mathbb{T}, \bullet = 1, \\
0, \text{ otherwise.}
\end{cases}$$

A cohomology theory determines a Pontrjagin dual cohomology theory. The cohomology theories which express the quantization law for self-dual fields are quite special: they are Pontrjagin self-dual. See Appendix B, especially Definition B.2, for a discussion.

**Self-dual abelian gauge fields**

We now describe a generalized self-dual field. Just as a $\sigma$-model depends on a target Riemannian manifold as well as a spacetime dimension, so too does a generalized self-dual field depend on external data. We begin by specifying enough of this data to define the Hamiltonian theory, then
write the classical theory in a Lorentzian spacetime of the Hamiltonian form \( \mathbb{R} \times Y \), and finally define the quantum Hilbert space of a self-dual field. We emphasize that our definition of the quantum Hilbert space and certain operators on it is only up to noncanonical isomorphism. This is enough to demonstrate the Heisenberg uncertainty principle for flux. Thus the following definition is only good enough for our noncanonical construction here; we need more precise data for the full quantum theory.

**Definition 2.9.** The data which define a *Hamiltonian self-dual generalized abelian gauge field* are:

(i) a Pontrjagin self-dual multiplicative cohomology theory \((E^\bullet, i)\) with shift \(s \in \mathbb{Z}\);

(ii) a dimension \(m\) and a multi-degree \(d\), i.e., an ordered collection \(d = (d_1, d_2, \ldots, d_k)\) of integer degrees;

and (iii) a natural isomorphism

\[
\phi: E^d \rightarrow E^{m-s+1-d}
\]

such that for any compact \(\tilde{E}\)-oriented manifold \(Y\) of dimension \(m\) the pairing

\[
s_Y: \tilde{E}^d(Y) \times \tilde{E}^d(Y) \rightarrow \mathbb{T}
\]

\[
(\tilde{F}_1, \tilde{F}_2) \mapsto i \int_Y \tilde{\phi}(\tilde{F}_1) \cdot \tilde{F}_2
\]

is skew-symmetric.

In (i) \(s \in \mathbb{Z}\) is a shift in degree—part of Pontrjagin self-duality—and \(i\) determines a homomorphism \(E^{−s}(pt) \rightarrow \mathbb{Z}\). (In all cases we know \(i\) is specified by that homomorphism.) In (ii) \(m\) is the dimension of space, so we are working in a theory with \((m + 1)\)-dimensional spacetimes. The field strength of the self-dual field has degree \(d\), which is a multi-degree in case there are several gauge fields. The notation for multi-degrees is \(d + 1 = (d_1 + 1, d_2 + 1, \ldots, d_k + 1)\), etc. The physical meaning of (iii) is that \(\phi\) induces a map of a non-self-dual gauge field to its dual; it appears in the classical equation of motion (2.12) satisfied by a self-dual gauge field. See Proposition B.6 for a description of the pairing in (2.11). The self-dual scalar is the case when \(E\) is ordinary cohomology, \(m = 1\), \(d = 1\), and \(\phi\) is the identity map. Other examples are given in the next subsection. We remark that whereas the pairing (2.11) is skew-symmetric, it is not necessarily alternating—that is, its values on the diagonal may be nontrivial. It is always a perfect pairing by Proposition B.6.

The Maxwell story of §1 generalizes to self-dual gauge fields. (Example 2.18 below explains how to write Maxwell theory as a special case.) Let \(Y\) be an \(E\)-oriented Riemannian manifold of dimension \(m\) and \(M = \mathbb{R} \times Y\) the associated Lorentzian spacetime of signature \((1, n-1)\). The role of the electromagnetic field is played by a differential form of total degree \(d\) on spacetime:

\[
F \in \Omega(M; V_E)^d = \bigoplus_{q \in \mathbb{Z}} \Omega^q(M; V_E^{d-q}).
\]
The analog of the classical Maxwell equations (1.1) with zero current are the self-duality equations

\[ dF = 0 \]
\[ \phi(F) = i(\ast F) \]

The second equation, the self-duality condition, takes values in \( \Omega(M; V_E^{m-s+1-d}) \). For the notation observe first that (2.10) applied to \( \mathbb{R}^q \) with compact supports gives, after tensoring with \( \mathbb{R} \), a map \( V_E^{d-q} \to V_E^{m-s+1-d-q} \). Thus on the component of \( F \) which is a differential form of degree \( q \) we have

\[ \phi: \Omega^q(M; V_E^{d-q}) \to \Omega^q(M; V_E^{m-s+1-d-q}). \]

On the right-hand side appears the Lorentzian Hodge star operator

\[ \ast: \Omega^{m+1-q}(M; V_E^{d+q-m-1}) \to \Omega^q(M; (V_E^{d+q-m-1})^\ast) \]

followed by the duality map (B.10)

\[ i: \Omega^q(M; (V_E^{d+q-m-1})^\ast) \to \Omega^q(M; V_E^{m-s+1-d-q}). \]

We remark that if there is a twisting in the definition of the gauge field, then the de Rham differential in the first equation is also twisted; see Example 2.27. Write \( F = B - dt \wedge E \) as in (1.3); then (2.12) implies

\[ \phi(B) = i(\ast E), \]

where \( \ast \) is the Riemannian Hodge star operator on \( Y \). In other words, up to some invertible algebraic maps the self-duality equates the electric field and the magnetic field. The classical flux, now both electric and magnetic combined, is an \( H(Y; V_E)^d \)-valued function on the space of solutions to (2.12), defined simply as the de Rham cohomology class of \( F \). There is a Poisson structure with symplectic leaves parametrized by \( H(Y; V_E)^d \) and the classical fluxes Poisson commute. The self-duality equation (2.12) is a first-order linear hyperbolic equation, and so a solution is determined by the value at any fixed time, as for the special case of the self-dual scalar field in two dimensions—see (2.1). Specifically, the space of solutions is isomorphic to the real vector space \( \Omega(Y; V_E)^d \) closed.

In the semiclassical picture with Dirac charge quantization the field is a geometric representative of a class in \( \tilde{E}^d(M) \). The space of classical solutions on \( M \) is the differential cohomology group \( \tilde{E}^d(Y) \). Our definition of its quantization is motivated by the discussion at the beginning of \( \S 2 \) of the self-dual scalar in two dimensions, as well as computations in other examples. The pairing (2.11) is skew bimultiplicative, but not necessarily alternating. We apply Proposition A.3 of Appendix A to construct the corresponding graded Heisenberg group.
Theorem 2.14. There exists a central extension
\[ 1 \rightarrow \mathbb{T} \rightarrow \mathcal{G}_Y \rightarrow \tilde{E}^d(Y) \rightarrow 0 \]

of \( A = \tilde{E}^d(Y) \), unique up to noncanonical isomorphism, with graded commutator \( (2.11) \).

Then Proposition A.5 and the remarks which follow imply

Theorem 2.15. There exists an irreducible \( \mathbb{Z}/2\mathbb{Z} \)-graded unitary representation \( \mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1 \) of \( \mathcal{G}_Y \) on which the central circle acts by scalar multiplication, and it is unique up to noncanonical isomorphism.

The quantum Hilbert space of the self-dual field on \( Y \) is the irreducible representation \( \mathcal{H}_Y \) in Theorem 2.15. The generalization of Theorem 1.20 to self-dual gauge fields is now immediate. The quantum fluxes are defined using the kernel torus in \( E^d(Y) \), lifted to the Heisenberg group \( \mathcal{G}_Y \). Namely, for \( \omega \in E^d_{\mathbb{T}}(Y) \subset \tilde{E}^d(Y) \) and \( \tilde{\omega} \) a lift to \( \mathcal{G}_Y \), define \( \mathcal{F}^{\text{qtm}}(\tilde{\omega}) \) to be the corresponding unitary operator on \( \mathcal{H}_Y \). Then the commutation relation in \( \mathcal{G}_Y \) implies

\[ [\mathcal{F}^{\text{qtm}}(\tilde{\omega}_1), \mathcal{F}^{\text{qtm}}(\tilde{\omega}_2)] = s_Y(\omega_1, \omega_2) \text{id}_{\mathcal{H}_Y}, \quad \omega_i \in E^d_{\mathbb{T}}(Y), \]

where \( s_Y \) is the pairing \( (2.11) \) and the left hand side is the graded group commutator \( (A.2) \). The nondegeneracy of \( s_Y \) implies the following.

Generalized Main Observation 2.17. If \( \text{Tors} E^d(Y) \neq 0 \), then not all fluxes commute.

This is the generalization of Main Observation 1.18. Analogous to Amelioration 1.19 there is a grading of \( \mathcal{H}_Y \) by \( E^d(Y)/\text{torsion} \).

Examples

The lagrangian versions of the following examples are discussed in [F, §3]. The first example demonstrates how the framework of self-dual fields encompasses ordinary gauge fields.

Example 2.18 (Dual gauge fields). Fix a dimension \( m \) for space and a degree \( 1 \leq d \leq m \) for a gauge field. Set \( d = (d, m + 1 - d) \) and quantize using ordinary cohomology \( E = H \). The natural isomorphism \( \phi \) in \( (2.10) \) is

\[ \phi(\tilde{F}, \tilde{F}') = ((-1)^{d(m-d)} \tilde{F}' , (-1)^{d-1} \tilde{F}'), \quad \tilde{F} \in \tilde{H}^d(Y), \quad \tilde{F}' \in \tilde{H}^{m+1-d}(Y), \]

for any compact oriented manifold \( Y \). The homomorphism \( i: H^0(\text{pt}) \rightarrow \mathbb{Z} \) is the natural isomorphism. (The signs in equation \( (2.19) \) are derived from [F, Example 3.17].)
The Maxwell theory of §1 is the case $m = 3, d = 2$. Then up to an overall sign (2.11) is equal to the commutator in that theory—see (1.25), (1.27), and (1.28). (Note that (1.28) is the product in $\tilde{H}^2$.) In the discussion of §1 the alternating form $s_Y$ is written as the skew-symmetrization of a 2-cocycle $\psi$. For a general self-dual field the 2-cocycle $\psi$ is not determined by the data, but rather its existence is guaranteed by Theorem 2.14. The “lagrangian splitting” (1.26) gives rise to $\psi$ as well as to a partitioning of fluxes into magnetic and electric.

Example 2.20 (Standard self-dual gauge field [HS], [W]). Here $m = 4k + 1$ for an integer $k \geq 0$, we use ordinary cohomology $E = H$, and there is a single degree $d = 2k + 1$. The automorphism $\phi$ is trivial in this case. The self-dual scalar in two dimensions is the case $k = 0$. Let $L^3$ be a three-dimensional lens space. For $k \geq 1$ the manifold $Y = L^3 \times S^{2k-1} \times S^{2k-1}$ exhibits noncommuting fluxes.

Example 2.21 (Type II Ramond-Ramond field ($\tilde{B} = 0$)). Here $m = 9$ and we quantize using complex $K$-theory $E = K$. We assume the $B$-field vanishes. The Ramond-Ramond field on a compact Riemannian spin manifold $Y$ has an equivalence class in $\tilde{K}^d(Y)$, where $d = 0$ in Type IIA and $d = -1$ in Type IIB. Recall that $K^*(pt) \cong \mathbb{Z}[[u, u^{-1}]]$ is a Laurent series ring with deg $u = 2$. The automorphism $\phi$ is essentially complex conjugation:

\begin{equation}
\phi(\tilde{F}) = u^\ell \overline{F}, \quad \ell = \begin{cases} 5, & \text{Type IIA;} \\ 6, & \text{Type IIB.} \end{cases}
\end{equation}

The homomorphism $i : K^0(pt) \to \mathbb{Z}$ is the augmentation, an isomorphism. To check that (2.11) is skew-symmetric in Type IIA we note that for any $\tilde{F}_1, \tilde{F}_2 \in \tilde{K}^0(Y)$, the element $\tilde{F}_1 \overline{\tilde{F}_2} + \tilde{F}_2 \overline{\tilde{F}_1}$ is the complexification of the realification of $\tilde{F}_1 \overline{\tilde{F}_2}$. Using the periodicity of $K$-theory we can shift the integral in (2.11) to the right-hand vertical map in the commutative diagram

\[
\begin{array}{ccc}
\tilde{K}^0(Y) & \longrightarrow & \tilde{K}^0(Y) \\
\downarrow & & \downarrow \\
\tilde{K}^{-9}(pt) & \longrightarrow & \tilde{K}^{-9}(pt)
\end{array}
\]

Starting with $\tilde{F}_1 \overline{\tilde{F}_2}$ in the upper left-hand corner we deduce the skew-symmetry, since the composition $\mathbb{T} \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{T}$ on the bottom line is zero. A similar argument works for Type IIB.

We make the classical self-dual equations (2.12) explicit for Type IIA. The field strength $F \in \Omega(\mathbb{R} \times Y[[u, u^{-1}]])^0$ has an expansion

\begin{equation}
F = F_0 + F_2 u^{-1} + F_4 u^{-2} + F_6 u^{-3} + F_8 u^{-4} + F_{10} u^{-5},
\end{equation}

\footnote{Complex conjugation maps the inverse Bott element $u$ to $-u$.}
where the subscript \( q \) indicate the degree of the scalar-valued differential form \( F_q \), and the forms \( F_q \) are closed. Then

\[
\phi(F) = F_0 u^5 - F_2 u^4 + F_4 u^3 - F_6 u^2 + F_8 u - F_{10}
\]

and

\[
i(F) = *F_0 + *F_2 u + *F_4 u^2 + *F_6 u^3 + *F_8 u^4 + *F_{10} u^5.
\]

The self-duality equation is then

\[
F_6 = -*F_4, \quad F_8 = *F_2, \quad F_{10} = -*F_0.
\]

In the quantum theory the Generalized Main Observation 2.17 asserts that any manifold with torsion in its \( K \)-theory exhibits noncommuting fluxes. For \( K^0 \), hence for Type IIA, there is torsion in \( K \)-theory on any manifold with finite abelian fundamental group. (Calabi-Yau manifolds with this property have been considered in the physics literature.) For \( K^1 \), hence for Type IIB, one can take the product of a circle and a manifold with finite abelian fundamental group to obtain examples.

**Example 2.27 (Type II Ramond-Ramond field \((\tilde{B} \neq 0)\)).** Now we allow nonzero \( B \)-field, which is a cocycle \( \tilde{B} \) representing an element of \( \tilde{H}^3(Y) \). Then the Ramond-Ramond field has an equivalence class in differential \( K \)-theory twisted by \( \tilde{B} \), denoted \( \tilde{K}^{d+B} \). As before \( d = 0 \) in Type IIA and \( d = -1 \) in Type IIB. The automorphism (2.22) maps to differential \( K \)-theory twisted by \( -\tilde{B} \). The de Rham differential in (2.12) is replaced by the “twisted” differential \( d + u^{-1}H \), where \( H \in \Omega^3(Y)_{\text{closed}} \) is the field strength of \( \tilde{B} \). Equations (2.23)–(2.26) hold as in the untwisted case, but now the condition that \( F \) be closed is replaced by the equation \((d + u^{-1}H)F = 0\). In components this reads

\[
dF_0 = 0, \quad dF_2 + H \wedge F_0 = 0, \quad dF_4 + H \wedge F_2 = 0, \quad \text{etc.}
\]

**Example 2.28 (Type I Ramond-Ramond field).** The Type I Ramond-Ramond field, or ‘\( B \)-field’, is quantized by periodic real \( K \)-theory, whose Pontrjagin self-duality is proved in Proposition B.11. The degree of the field strength is \( d = -1 \), as in Type IIB. Note the nonzero shift \( s = 4 \). The ring \( KO^\bullet(\text{pt}) \) has torsion; after tensoring over the reals we obtain \( V_{KO} \cong \mathbb{R}[u^2, u^{-2}] \), where \( \deg u = 2 \) as in \( K \)-theory. The automorphism (2.10) is

\[
\phi(\hat{F}) = \lambda^{-1} \hat{F},
\]
where $\lambda^{-1} \in KO^8(\text{pt})$ is the generator whose complexification is $u^4 \in K^8(\text{pt})$. In fact, there is a background self-dual current—part of the Green-Schwarz mechanism—and the $B$-field is a ‘cochain’ of degree $-1$ which trivializes the current. See [F] for details on the lagrangian theory.\footnote{The Pontrjagin self-duality was expressed in that paper in terms of a hybrid of real and quaternionic $K$-theory, but should have been stated as it is here.}

**APPENDIX A: CENTRAL EXTENSIONS OF ABELIAN GROUPS**

We shall consider the class of abelian Lie groups $\mathcal{A}$ which fit into an exact sequence

$$0 \to \pi_1(\mathcal{A}) \to V \to \mathcal{A} \to \pi_0(\mathcal{A}) \to 0,$$

where $V$ — the Lie algebra of $\mathcal{A}$ — is a locally convex and complete topological vector space. We shall assume that the exponential map $V \to \mathcal{A}$ is a local diffeomorphism which makes $V$ a covering space of the connected component of $\mathcal{A}$, and that the group $\pi_0(\mathcal{A})$ of connected components, and the fundamental group $\pi_1(\mathcal{A})$, are finitely generated discrete abelian groups.

Requiring $\pi_0$ and $\pi_1$ to be finitely generated may seem unduly restrictive, but it includes the examples we are interested in, and going beyond it makes things decidedly more complicated. In particular, if $F$ is a discrete closed subgroup of a topological vector space $V$ it need not be true that a homomorphism $F \to \mathbb{R}$ extends to a continuous homomorphism $V \to \mathbb{R}$ if $F$ is not finitely generated; for example, if $F$ is the subgroup of a Hilbert space $V$ generated by the elements of an orthonormal basis $\{e_n\}$ then the map which takes each $e_n$ to 1 does not extend continuously to $V$.

We shall now classify the central extensions $\mathcal{G}$ of $\mathcal{A}$ by the circle group $\mathbb{T}$ — we shall call these groups *generalized Heisenberg groups*. In fact we shall describe the category $\mathcal{E}_\mathcal{A}$ of extensions, in which a morphism from $\mathcal{G}$ to $\mathcal{G}'$ is an isomorphism $\mathcal{G} \to \mathcal{G}'$ which makes the diagram

\[
\begin{array}{ccc}
\mathbb{T} & \longrightarrow & \mathcal{G} \\
\downarrow & & \downarrow \\
\mathbb{T} & \longrightarrow & \mathcal{G}'
\end{array}
\]

commute. The class of extensions we consider are those which as manifolds are smooth locally trivial circle bundles over $\mathcal{A}$. It follows from Proposition A.1(ii) that these circle bundles are in fact globally trivial.

A smooth bimultiplicative map

$$\psi : \mathcal{A} \times \mathcal{A} \to \mathbb{T}$$
defines an extension $G(A, \psi)$: as a manifold $G(A, \psi)$ is the product $A \times T$, and its multiplication is given by

$$(a_1, \lambda_1) (a_2, \lambda_2) = (a_1 + a_2, \lambda_1 \lambda_2 \psi(a_1, a_2)).$$

(Notice that a bimultiplicative map automatically satisfies the cocycle condition (1.24).)

Let us recall that a quadratic map $f : A \to T$ is a smooth map such that

$$(a_1, a_2) \mapsto \psi_f(a_1, a_2) = f(a_1 + a_2) f(a_1)^{-1} f(a_2)^{-1}$$

is bimultiplicative. We introduce the category $B_A$ whose objects are the smooth bimultiplicative maps, and whose morphisms from $\psi$ to $\psi'$ are the smooth quadratic maps $f$ such that $\psi_f = \psi' / \psi$. There is a functor $B_A \to E_A$, as a quadratic map $f$ defines a homomorphism $G(A, \psi) \to G(A, \psi_f)$ by

$$(a, \lambda) \mapsto (a, \lambda f(a)).$$

**Proposition A.1.** (i) An object of $E_A$ is determined up to isomorphism by its commutator map

$$s : A \times A \to T,$$

which is a bimultiplicative alternating map, and every bimultiplicative alternating map can arise. (ii) The functor $B_A \to E_A$ is an equivalence of categories.

**Proof.** To prove that an extension is determined by its commutator map we observe that the category $E_A$ has a composition-law. For the fibre-product $G_1 \times_A G_2$ over $A$ of two extensions $G_1$ and $G_2$ is an extension of $A$ by $T \times T$, and we can define the desired composite $G_1 \ast G_2$ as the quotient of the fibre-product by the antidiagonal subgroup of $T \times T$ — the image of $T$ by $\lambda \mapsto (\lambda, \lambda^{-1})$. The isomorphism-classes of $E_A$ form a group $\text{Ext}(A)$ under this composition-law, for the inverse of $G$ is the quotient of the product extension $G \times T$ of $A$ by $T \times T$ by the diagonal subgroup of $T \times T$. Using the composition-law, it is enough to show that an extension which is itself commutative is trivial. But any of the abelian groups of the type we are considering can be written non-canonically as a product $K \times V \times \pi$, where $K$ is a compact torus, $V$ is a vector space, and $\pi$ is discrete. (To see this, we first split off $\pi$, using the fact that the identity component of the group is divisible; then we split the Lie algebra of the identity-component as $V$ times the Lie algebra of the finite-dimensional maximal compact subgroup $K$, using the version of the Hahn-Banach theorem which asserts that any continuous linear map $V_0 \to \mathbb{R}$ defined on a closed subspace of a locally convex topological vector space $V$ can be extended to a continuous linear map $V \to \mathbb{R}$.) This reduces us to showing that any inclusion of a circle in a torus is split, which follows from Pontrjagin duality.
To complete the proof of Proposition A.1, let $\text{Bim}(A_1, A_2)$ denote the group of bimultiplicative maps $A_1 \times A_2 \rightarrow \mathbb{T}$, and let $\text{Alt}(A)$ denote the group of alternating maps $A \times A \rightarrow \mathbb{T}$. There is an obvious map $\text{Bim}(A, A) \rightarrow \text{Alt}(A)$ which takes $\psi$ to the commutator

$$(a_1, a_2) \mapsto s(a_1, a_2) = \psi(a_1, a_2)/\psi(a_2, a_1)$$

of the extension $G(A, \psi)$. We need only show that $\text{Bim}(A, A) \rightarrow \text{Alt}(A)$ is surjective for all abelian groups $A$ in our class. Using the fact that $\text{Alt}(A_1 \oplus A_2) \cong \text{Alt}(A_1) \oplus \text{Alt}(A_2) \oplus \text{Bim}(A_1, A_2)$, together with the more obvious decomposition of $\text{Bim}(A_1 \oplus A_2, A_1 \oplus A_2)$ as a sum of four groups, we see that if the desired surjectivity holds for $A_1$ and $A_2$ then it holds for $A_1 \oplus A_2$. It is therefore enough to consider separately the cases when $A$ is a torus, a topological vector space, and a discrete cyclic group, and each of these is trivial.

In fact the abelian groups $A$ which are important in the present work come equipped with skew bimultiplicative maps $s : A \times A \rightarrow \mathbb{T}$ rather than alternating ones — i.e. $s(b, a) = s(a, b)^{-1}$, and so $s(a, a)$ has order 2, but we need not have $s(a, a) = 1$. Such a skew map $s$ defines a mod 2 grading $A = A^{\text{even}} \cup A^{\text{odd}}$ of $A$, as $a \mapsto s(a, a)$ is a continuous homomorphism $A \rightarrow \mathbb{Z}/2$. In fact the group $\text{Skew}(A)$ of these forms can be identified with the group $\text{Ext}_{gr}(A)$ of isomorphism classes of graded central extensions, as we shall now explain.

A central extension of a — not necessarily abelian — group $A$ by $\mathbb{T}$ can be regarded as a rule that associates a hermitian line $L_a$ to each $a \in A$, and associative unitary isomorphisms $m_{a,b} : L_a \otimes L_b \rightarrow L_{a+b}$ to each $a,b \in A$. (The elements of the extended group are then pairs $(a, \lambda)$, with $a \in A$ and $\lambda \in L_a$ of unit length.) To define a graded central extension we simply replace the lines $L_a$ by graded lines (with a mod 2 grading), and require the isomorphisms $m_{a,b}$ to preserve the grading. A graded line is either even or odd, and evidently the group $A$ acquires a grading from the degree of $L_a$. In fact a graded central extension is simply a central extension $G$ of $A$ by $\mathbb{T}$ which is at the same time a mod 2 graded group — i.e. is equipped with a homomorphism $G \rightarrow \mathbb{Z}/2$.

The analogue of the commutator map $s : A \times A \rightarrow \mathbb{T}$ in the graded case is the graded commutator, which maps $(a, b) \in A \times A$ to the composite

$$s(a, b) = m_{b,a} \circ T_{a,b} \circ m_{a,b}^{-1} : L_{a+b} \rightarrow L_{a+b},$$

where $T_{a,b} : L_a \otimes L_b \rightarrow L_b \otimes L_a$ expresses the symmetry of the graded tensor product — i.e. it multiplies by $-1$ if both $L_a$ and $L_b$ are odd. We think of $s(a, b)$ as an element of $\mathbb{T}$: concretely, it is

$$(-1)^{\deg(L_a) \deg(L_b)}$$

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times the naive commutator in the extension.

Graded central extensions of $A$ possess a composition law (which takes $\{L_a\}$ and $\{L'_a\}$ to $\{L_a \otimes L'_a\}$), and there is an obvious short exact sequence

$$0 \longrightarrow \text{Ext}(A) \longrightarrow \text{Ext}_{gr}(A) \longrightarrow \text{Hom}(A; \mathbb{Z}/2) \longrightarrow 0.$$ 

Comparing this with the other obvious exact sequence

$$0 \longrightarrow \text{Alt}(A) \longrightarrow \text{Skew}(A) \longrightarrow \text{Hom}(A; \mathbb{Z}/2) \longrightarrow 0,$$

we have

**Proposition A.3.** An object of $E_{A}^{gr}$ is determined up to isomorphism by its graded commutator map

$$s : A \times A \longrightarrow T,$$

which is a skew bimultiplicative map, and every skew bimultiplicative map can arise.

We can also make an assertion analogous to Proposition A.1(ii) in the graded case: if we consider the category of graded extensions with a given grading on $A$ the assertion is verbally the same as in the ungraded case.

**Representations**

Our next task is to describe the irreducible unitary representations of a generalized Heisenberg group $G(A, \psi)$. It will be enough to consider representations in which the central subgroup $T$ acts by multiplication: the general case reduces easily to this one.

The centre $Z$ of the group $G = G(A, \psi)$ is an extension of $Z$ by $T$, where $Z$ is the kernel $\{a \in A : s(a, b) = 1 \text{ for all } b \in A\}$ of the commutator map. Being abelian, the extension $Z$ is split, but not canonically, for the cocycle $\psi$ need not vanish on $Z$. By Schur’s lemma in any irreducible representation $\rho$ of $G$ the subgroup $Z$ acts by scalar multiplication, i.e. by a homomorphism $\chi : Z \rightarrow T$ which is a splitting of the extension.

**Proposition A.4.** For a finite-dimensional generalized Heisenberg group $G$ an irreducible unitary representation $\rho$ is determined up to isomorphism by the splitting homomorphism $\chi : Z \rightarrow T$, and any such homomorphism can arise.

**Proof.** If the Heisenberg group $G$ is non-degenerate, i.e. its centre is exactly $T$, the assertion is that there is only one possible representation. This is essentially the classical theorem of Stone and von Neumann. Let us assume that it is known in the case when $A$ is a finite-dimensional vector space. In general, let $T$ be the kernel of the commutator map restricted to the identity
component of $\mathcal{A}$; this is necessarily a torus over which the extension $\mathcal{G}$ is canonically split (for the bimultiplicative cocycle $\psi$ which defines $\mathcal{G}$ must vanish when restricted to a torus). Furthermore, $T$ is Pontrjagin-dual to $\mathcal{G}/\mathcal{G}_1$, where $\mathcal{G}_1$ is the centralizer of $T$ in $\mathcal{G}$. Any Hilbert space $\mathcal{H}$ on which $\mathcal{G}$ acts unitarily can be decomposed according to the action of $T$ as $\mathcal{H} = \bigoplus \mathcal{H}_\alpha$, where $\alpha$ runs through the characters of $T$. Now $\mathcal{H}_0$ is a representation of $\mathcal{G}_1$, and evidently $\mathcal{H}$ is the representation of $\mathcal{G}$ induced from the representation $\mathcal{H}_0$ of $\mathcal{G}_1$, which must therefore be irreducible. In fact $\mathcal{H}_0$ is a representation of the non-degenerate Heisenberg group $\mathcal{G}_1/T$, whose group of components is finite, and whose identity-component is a vector Heisenberg group. We have therefore reduced ourselves to treating the case of a Heisenberg group $\mathcal{G}$ arising from an abelian group $\mathcal{A}$ of the form $V \times \pi$, where $V$ is a vector space and $\pi$ is a finite abelian group. In this case, let $\Gamma$ be a maximal abelian subgroup of the restriction of the extension to $\pi$. We can write $\Gamma = \mathbb{T} \times \pi'$, though non-canonically. In any representation of $\mathcal{G}$ the action of the subgroup $\pi'$ decomposes the Hilbert space into pieces according to the characters of $\pi'$, which are permuted transitively by the conjugation-action of $\pi$. By the argument we have already used we see that the representation is induced from an irreducible representation of $\mathcal{G}_0 \times \pi'$ in which $\pi'$ acts trivially. (Here $\mathcal{G}_0$ is the identity-component of $\mathcal{G}$.) But $\mathcal{G}_0$ is a vector Heisenberg group, and so we have reduced ourselves to the case we have assumed to be known.

When the group $\mathcal{G}$ is degenerate, the representation $\rho$ is actually a representation of $\mathcal{G}/Z'$, where $Z'$ is the kernel of the homomorphism $\chi : \mathbb{Z} \to \mathbb{T}$. But $\mathcal{G}/Z'$ is a non-degenerate Heisenberg group, and so we are back to the previous case.

Turning now to infinite-dimensional groups, to have an analogue of the Stone-von Neumann theorem we must introduce the concept of a positive-energy representation, which is defined when the abelian group $\mathcal{A}$ is polarized. There are many versions of this concept. The following is a rather narrow one, but seems simplest for our purposes.

We shall say $\mathcal{A}$ is polarized if there is a continuous action of the group $\mathbb{R}$ on the Lie algebra $V$ of $\mathcal{A}$ by operators $\{u_t\}_{t \in \mathbb{R}}$ which preserve the skew form coming from the commutator and decompose the complexification $V_C$ into a countable sum of finite-dimensional subspaces $V_\lambda$ in which $u_t$ acts by multiplication by $e^{i\lambda t}$. Here each $\lambda$ is real, and we assume that the algebraic sum of the $V_\lambda$ is dense in $V$. In our applications $\{u_t\}$ will be the Hamiltonian flow on the phase space $V$ induced by a positive quadratic energy-function — but we allow the symplectic structure on $V$ to be degenerate.

Then for a polarized group $\mathcal{A}$ we say a unitary representation of the Heisenberg group $\mathcal{G}$ on a Hilbert space $\mathcal{H}$ is of positive energy if there is a unitary action of $\mathbb{R}$ on $\mathcal{H}$ by operators $U_t = e^{iHt}$ whose generator $H$ has discrete non-negative spectrum, and which intertwines with the action of $\mathcal{G}$ on $\mathcal{H}$ in the sense that the action of $\exp(u_t(v))$ for any $v \in V$ is the conjugate by $U_t$ of the action of $\exp(v)$. In our applications $\{U_t\}$ will be the time-evolution of a quantum system, which we require to have positive energy in the usual quantum-mechanical sense.

Now we have the following version of the Stone-von Neumann theorem.
Proposition A.5. For a polarized generalized Heisenberg group \( \mathcal{G} \) an irreducible unitary representation of positive energy is completely determined by the splitting homomorphism \( \chi : \mathbb{Z} \to \mathbb{T} \), and any such homomorphism can occur.

Proof. By exactly the same arguments as in the finite-dimensional case we reduce first to the case of a non-degenerate Heisenberg group, and then to a vector Heisenberg group. We then find, just as in [PS,§9.5], that the unique positive energy irreducible unitary representation of the Heisenberg group formed from \( V \) is realized on the completion of the symmetric algebra \( S(W) \), where \( V_{\mathbb{C}} = W \oplus \bar{W} \) is the decomposition into positive- and negative-energy pieces.

To conclude, we return to graded central extensions. For a mod 2 graded group \( \mathcal{G} \) it is natural to consider unitary representations on mod 2 graded Hilbert spaces, and to require that the action of the even elements of \( \mathcal{G} \) preserves the grading, while that of the odd elements reverses it. But if \( \mathcal{G} \) is a graded generalized Heisenberg group every representation automatically has such a grading. To see this, we may as well assume that \( \mathcal{G} \) is non-degenerate, for in any irreducible representation \( \mathcal{G} \) acts through a non-degenerate quotient. If \( \mathcal{G} \) is non-degenerate, the grading homomorphism \( \mathcal{A} \to \mathbb{Z}/2 \) is necessarily of the form \( a \mapsto s(\varepsilon, a) \) for some \( \varepsilon \in \mathcal{A} \) which has order 2. If \( \hat{\varepsilon} \) is a lift of \( \varepsilon \) in \( \mathcal{G} \) then \( \hat{\varepsilon} \) commutes with the even elements of \( \mathcal{G} \) and anticommutes with the odd elements. Furthermore, \( \hat{\varepsilon}^2 \) acts as a scalar, so its eigenspaces define the desired mod 2 grading on the Hilbert space. (We do not have a preferred way of naming the eigenspaces odd and even; but that does not matter, as reversing the choice gives us an isomorphic graded representation.)

Appendix B: Self-Dual Cohomology Theories

Any cohomology theory \( E^\bullet \) arises from a loop-spectrum, i.e. a sequence \( \mathcal{E} = \{ \mathcal{E}_q \}_{q \in \mathbb{Z}} \) of spaces with base-point such that \( E^q(X) \) is the set of homotopy classes of maps \( X \to \mathcal{E}_q \). For a space \( X \) with base-point, the reduced cohomology \( \hat{E}^q(X) \) is the set of homotopy classes of base-point-preserving maps \( X \to \mathcal{E}_q \). The term “loop”-spectrum reflects the existence of canonical homotopy equivalences \( \mathcal{E}_q \to \Omega \mathcal{E}_{q+1} \) — where \( \Omega \) denotes the based loop-space — which express the behaviour of cohomology groups under suspension. The spectrum is determined up to homotopy equivalence by the cohomology theory, and algebraic topologists usually mean the spectrum when they refer to a cohomology theory. The spectrum also defines a homology theory \( E_\bullet \) by

\[
E_q(X) = \lim_i \pi_{q+i}(X_+ \wedge \mathcal{E}_i),
\]

where \( X_+ \) denotes the space \( X \) — which is not assumed to have a given base-point — with a disjoint base-point adjoined.\(^\text{10}\) In particular, \( E_q(\text{pt}) = E^{-q}(\text{pt}) \).

\(^\text{10}\)For two spaces \( Y \) and \( Z \) with base-points \( y_0 \) and \( z_0 \) the wedge product \( Y \wedge Z \) denotes the spaces obtained from
It may be helpful to notice that the spaces $X_+ \land \mathcal{E}_q$ form a *spectrum*, in the sense that there are natural maps

$$X_+ \land \mathcal{E}_q \longrightarrow \Omega(X_+ \land \mathcal{E}_{q+1}).$$

This is not a loop-spectrum, but one can make it into a loop-spectrum $X_+ \otimes \mathcal{E}$ whose $q$-th space is

$$\lim_i \Omega^i(X_+ \land \mathcal{E}_{q+i}),$$

and then $E_q(X)$ is the $q$-th homotopy group of $X_+ \otimes \mathcal{E}$.

In general there is no way to calculate the groups $E^\bullet(X)$ algebraically from the groups $E^\bullet(X)$, although if $X$ is compact and can be embedded as a neighbourhood-deformation-retract in Euclidean space $\mathbb{R}^N$ we have

$$E_q(X) \cong \tilde{E}^{N-q}(D^N X),$$

where $D^N X$ is the $N$-*dual* of $X$, defined as the one-point compactification of an open neighbourhood of $X$ in $\mathbb{R}^N$ of which $X$ is a deformation-retract, and $\tilde{E}^\bullet$ denotes *reduced* cohomology. This notion of duality — usually called *S-duality* — is explained by the fact that the category of spectra forms — when the morphisms are defined appropriately — a tensor category in which the tensor product is induced by the wedge-product of spaces and the sphere-spectrum $\{S^q\}$ is the neutral object. In this category the natural dual of a compact space $X$ is the spectrum formed by the mapping-spaces $\{\text{Map}(X; S^q)\}$. The sequence of spaces $\{D^q X\}$ forms a spectrum, and there is a natural map — the “scanning” map\footnote{To define the scanning map, choose $\varepsilon > 0$ so that the $\varepsilon$-neighbourhood of $X$ in $\mathbb{R}^q$ in contained in the open neighbourhood $U_X$ of $X$ of which $D^q(X)$ is the compactification. Then the scanning map is the extension to the compactification of the map which takes $x \in U_X$ to the composite $U_X \rightarrow \mathbb{R}^q \rightarrow \mathbb{R}^q/(\mathbb{R}^q - U_x) \cong S^q$, where $U_x$ is the $\varepsilon$-neighbourhood of $x$ in $\mathbb{R}^q$.} $D^q X \rightarrow \text{Map}(X; S^q)$ which induces an equivalence of the associated loop-spectra.

If $E^\bullet$ is a multiplicative theory, in the sense that $X \mapsto E^\bullet(X)$ is a contravariant functor to anticommutative graded rings, then the multiplication is induced by maps $\mathcal{E}_p \land \mathcal{E}_q \rightarrow \mathcal{E}_{p+q}$. These define maps

$$E^q(X) \times E_q(X) \longrightarrow E_{q-p}(X)$$

taking $f : X_+ \rightarrow \mathcal{E}_p$ and $g : S^{q+i} \rightarrow X_+ \land \mathcal{E}_i$ to the composite

$$S^{q+i} \longrightarrow X_+ \land \mathcal{E}_i \longrightarrow X_+ \land \mathcal{E}_i \land \mathcal{E}_p \longrightarrow X_+ \land \mathcal{E}_{p+i},$$

which make $E^\bullet(X)$ a graded module over $E^\bullet(X)$.
There is another way — essentially purely algebraic — to pass between cohomology theories and homology theories which from the point of view of spectra does not look at all natural. Let us recall that if \( I \) is a divisible abelian group such as \( \mathbb{R} \) or \( \mathbb{T} \) then the contravariant functor \( A \mapsto \text{Hom}(A; I) \) from the category of abelian groups to itself takes exact sequences to exact sequences. So if \( E_\bullet \) is a homology theory we can define a cohomology theory \( e_\bullet \) by

\[
e^q(X) = \text{Hom}(E_q(X); I).
\]

Unfortunately this does not work when \( I = \mathbb{Z} \), as \( \mathbb{Z} \) is not a divisible group. It is reasonable, however, to define the theory \( e^\bullet = e^\bullet_\mathbb{Z} \) as the theory that fits into a long exact sequence

\[
\ldots \rightarrow e^q(X) \rightarrow e^q_\mathbb{R}(X) \rightarrow e^q_\mathbb{T}(X) \rightarrow e^{q+1}(X) \rightarrow \ldots,
\]

where the transformation of theories \( e^\bullet_\mathbb{R} \rightarrow e^\bullet_\mathbb{T} \) is induced by the obvious homomorphism \( \mathbb{R} \rightarrow \mathbb{T} \). (This does entail representing \( e^\bullet_\mathbb{R} \) and \( e^\bullet_\mathbb{T} \) by spectra, and defining \( e^\bullet \) as the theory represented by the fibre of the map of spectra; but it is a fairly anodyne operation as we shall in the end have a short exact sequence

\[
0 \rightarrow \text{Ext}(E_{q-1}(X); \mathbb{Z}) \rightarrow e^q(X) \rightarrow \text{Hom}(E_q(X); \mathbb{Z}) \rightarrow 0.
\]

Because \( E_\bullet(X) \) is a module over \( E^\bullet(X) \) it is easy to see that the cohomology theories \( e^\bullet_\mathbb{R} \) and \( e^\bullet \) are module-theories over \( E^\bullet \). Let \( s \) be an integer termed a “shift”. If we choose once and for all an element of \( i \in e^{-s}(pt) \) — which is simply \( \text{Hom}(E^{-s}(pt); \mathbb{Z}) \) if \( E^{-s}(pt) = 0 \), as is the case for all the theories we shall be concerned with — then we get a natural element in \( i_X \in e^s(X) \) for all spaces \( X \), and hence a natural transformation

\[
(B.1) \quad E^\bullet(X) \rightarrow e^{\bullet+s}(X)
\]

which takes \( \xi \) to \( \xi \cdot i_X \).

**Definition B.2.** The theory \((E^\bullet, i)\) is called Pontrjagin self-dual if (B.1) defines an isomorphism of cohomology theories.

This is a very strong constraint on a theory, as it implies, for instance, that the groups \( E^q(pt) \) and \( E^{-s-q}(pt) \) have the same rank for every \( q \). Known examples include classical homology and periodic complex and real \( K \)-theory — the shift \( s \) is zero for the former two and we take \( s = -4 \) for the latter; see below.
In treating the examples it is more convenient to use an equivalent version of the self-duality condition which is stated in terms of the theory $E^\bullet$ called “$E^\bullet$ with coefficients in $T$”. This is a module-theory over $E^\bullet$ which fits into a long exact sequence

(B.3) \[ \ldots \rightarrow E^q(X) \rightarrow E^q_R(X) \rightarrow E^q_T(X) \rightarrow E^{q+1}(X) \rightarrow \ldots , \]

where — at least if certain finiteness conditions are satisfied which hold in the cases at hand — the theory $E^\bullet_R$ with real coefficients is defined simply by $E^q_R(X) = E^q(X) \otimes \mathbb{R}$.

The transformation $E^\bullet \rightarrow e^{s+}^\bullet$ induced by a choice of $i$ also leads to a transformation $E^\bullet_T \rightarrow e^{s+}^\bullet_T$, and the latter is an isomorphism if and only if the former is. To check self-duality, therefore, it is enough to prove that the map $E^\bullet_T(pt) \rightarrow e^{s+}^\bullet_T(pt)$, which is a map of modules over $E^\bullet(pt)$, is an isomorphism, i.e. that the module-action

(B.4) \[ E^q-s(pt) \times E^{-q}_T(pt) \rightarrow E^{-s}_T(pt) \rightarrow \mathbb{T} \]

is a perfect pairing.

For a compact $E$-oriented $m$-manifold $Y$ Poincaré duality there is a perfect pairing

(B.5) \[ E^{m-s-q}(Y) \otimes E^q_T(Y) \rightarrow \mathbb{T} \]

for each integer $q$. For Pontrjagin self-duality identifies $E^q_T(Y) \cong \text{Hom}(E_{q+s}(Y); T)$ under which (B.5) becomes the composition

\[ E^{m-s-q}(Y) \otimes \text{Hom}(E_{q+s}(Y); T) \rightarrow \text{Hom}(E_m(Y); T) \rightarrow \mathbb{T} \]

of the module-action of $E^\bullet$ on $e^s_T^\bullet$ and evaluation on the fundamental class, and this is a perfect pairing by Poincaré duality.

Now let us turn to the differential theory $\dot{E}^\bullet$ associated to $E^\bullet$. If $E^\bullet$ is Pontrjagin self-dual then we have the following very attractive differential version of Poincaré duality. We make the finiteness assumption alluded to after (B.3).

**Proposition B.6.** If $Y$ is a compact $m$-dimensional manifold which is oriented for $\dot{E}^\bullet$, and the theory $(E^\bullet, i)$ is Pontrjagin self-dual, then we have an integration operation

(B.7) \[ i \int_Y : \dot{E}^{m-s+1}(Y) \rightarrow \mathbb{T} \]

which, together with the natural multiplication, gives us a perfect pairing

(B.8) \[ \dot{E}^q(Y) \times \dot{E}^{m-s-q+1}(Y) \rightarrow \mathbb{T}. \]

The proof relies on the following, which is also used in (2.13).
Lemma B.9. For each \( q \in \mathbb{Z} \) the natural pairing

\[
i: V^{q-s}_E \otimes V^{-q}_E \to \mathbb{R}
\]

is nondegenerate.

Proof. By the finiteness assumption \( V^q_E = E^q(\text{pt}) \otimes \mathbb{R} \). The element \( i \in e^s(\text{pt}) \) induces a homomorphism \( E^{-s}(\text{pt}) \to \mathbb{Z} \), and so a linear map \( i: V^{-s}_E \to \mathbb{R} \) and the pairing \( B.10 \). If there is a nonzero \( v \in V^{q-s}_E \) in the kernel of the pairing, we can assume \( v \) is in the image of \( E^{q-s}(\text{pt}) \to E^{q-s}(\text{pt}) \otimes \mathbb{R} \), so defines an element \( e \in E^{q-s}(\text{pt})/\text{torsion} \). Now \( E^{-q}(\text{pt}) \to E_T^{-q}(\text{pt}) \) is onto the identity component, and the perfection of the pairing \( B.4 \) implies that \( E^{q-s}(\text{pt})/\text{torsion} \times E_T^{-q}(\text{pt})_{\text{id}} \to \mathbb{T} \) is also perfect, from which \( e = 0 \).

Proof of Proposition B.6. First, \( B.7 \) is defined as the composition

\[
E^{m-s+1}(Y) \xrightarrow{\int_Y} \bar{E}^{-s+1}(\text{pt}) \xrightarrow{(2.5)} E_T^{-s}(\text{pt}) \xrightarrow{i} \mathbb{T}
\]

Suppose \( \bar{F} \in \bar{E}^q(Y) \) is in the kernel of \( B.8 \) and let \( F \in \Omega_E(M; V_E)^q \) be its field strength. Let \( \hat{G} \in \check{E}^{m-s-q+1}(Y) \) have trivial characteristic class, so it is the image of a differential form \( \alpha \) in the exact sequence \( 2.6 \). Then by the remark following \( 2.7 \) and \( 2.8 \) et seq. we see that \( \int_Y \bar{F} \cdot \hat{G} \) is image of \( \int_Y \hat{A}_E(Y) \wedge F \wedge \alpha \) under the natural map \( \mathbb{R} \to \mathbb{T} \). Decompose \( F \) and \( \alpha \) as a sum of differential forms of fixed degree:

\[
F = \sum_j F_j, \quad F_j \in \Omega^j(V^{q-j}_E),
\]

\[
\alpha = \sum_k \alpha_k, \quad \alpha_k \in \Omega^k(V^{m-s-q-k}_E).
\]

Then since \( \bar{F} \) is assumed in the kernel of \( B.8 \), and since \( \hat{A}_E(Y) \) is invertible, it follows that \( i \) applied to

\[
\int_Y F \wedge \alpha = \sum_j \int_Y F_j \wedge \alpha_{m-j} \in \bigoplus_j V^{q-j}_E \otimes V^{q-s-j}_E
\]

vanishes for all \( \alpha \). Then from Lemma B.9 we conclude \( F = 0 \). Thus \( \bar{F} \) is the image in \( 2.5 \) of \( \omega \in E_T^{-1}(Y) \), and furthermore \( \omega \) lies in the kernel of the pairing

\[
E^{m-s-q+1}(Y) \otimes E^{-1}_T(Y) \to \mathbb{T}.
\]

But this is the perfect Poincaré-Pontrjagin pairing \( B.5 \), whence \( \omega = 0 \).
The Pontrjagin self-duality of ordinary cohomology and complex $K$-theory are relatively easy, but for real $K$-theory it is more subtle. In fact, the Pontrjagin dual of $KO$-theory — the associated theory denoted $\mathfrak{e}^\mathfrak{e}$ above — is again $KO$-theory but with a shift of degree $s = 4$. Note $KO^{-3}(pt) = 0$ and $KO^{-4}(pt)$ is generated by an element $\mu$ whose complexification is $2a^{-2} \in K^{-4}(pt)$. We choose the element $i \in \text{Hom}(KO^{-4}(pt); \mathbb{Z})$ to map $\mu \mapsto 1$.

**Proposition B.11.** $(KO, i)$ is Pontrjagin self-dual.

| $q$ | $KO_q(pt) \cong KO^{-q}(pt)$ | $KO_T^q(pt)$ |
|-----|-------------------------------|---------------|
| 4   | $\mathbb{Z}$ ($\mu$)         | $\mathbb{T}$  |
| 3   | 0                             | 0             |
| 2   | $\mathbb{Z}/2\mathbb{Z}$ ($\eta^2$) | 0             |
| 1   | $\mathbb{Z}/2\mathbb{Z}$ ($\eta$) | 0             |
| 0   | $\mathbb{Z}$                 | $\mathbb{T}$  |
| −1  | 0                             | 0             |
| −2  | 0                             | $\mathbb{Z}/2\mathbb{Z}$ |
| −3  | 0                             | $\mathbb{Z}/2\mathbb{Z}$ |
| −4  | $\mathbb{Z}$                 | $\mathbb{T}$  |

**Proof.** According to (B.4) it suffices to verify that the pairing

$$KO_q^{-4}(pt) \otimes KO_T^{-q}(pt) \rightarrow KO_T^{-4}(pt) \xrightarrow{i} \mathbb{T}$$

is an isomorphism for all $q$. The chart above, together with Bott periodicity $KO^{q+8} \cong KO^q$, reduces our task to the following four statements:

(B.12) $KO_T^0(pt) \otimes KO_T^{-4}(pt) \rightarrow KO_T^{-4}(pt)$ is an isomorphism;
(B.13) $KO^{-4}(pt) \otimes KO_T^0(pt) \rightarrow KO_T^{-4}(pt)$ is an isomorphism;
(B.14) $KO^{-1}(pt) \otimes KO_T^{-3}(pt) \rightarrow KO_T^{-4}(pt)$ is injective;
(B.15) $KO^{-2}(pt) \otimes KO_T^{-2}(pt) \rightarrow KO_T^{-4}(pt)$ is injective.

To verify (B.12) and (B.13) we introduce quaternionic $K$-theory $KSp$ and use Bott periodicity $KSp^{q} \cong KO^{q+4}$.

Since $KSp_T^0(pt) \cong KSp_0^0(pt) \otimes \mathbb{T}$ and $KO_T^0(pt) \cong KO^0(pt) \otimes \mathbb{T}$, it suffices to know that the natural pairing $KO_0^0(pt) \otimes KSp_0^0(pt) \rightarrow KSp_0^0(pt)$ is an isomorphism, which is clear: the generators of $KO_0^0(pt)$ and $KSp_0^0(pt)$ are the trivial real and quaternionic lines, and the map is the tensor product over the reals.
Now the generator of $KO^{-1}(pt) \cong \mathbb{Z}/2\mathbb{Z}$ is a class $\eta$, and $\eta^2$ generates $KO^{-2}(pt)$, whence (B.14) follows from (B.15). Let $KO_{\mathbb{Z}/2\mathbb{Z}}^q(X)$ denote the $KO$-group with $\mathbb{Z}/2\mathbb{Z}$-coefficients. The diagram of short exact sequences

\[
\begin{array}{ccccccc}
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z}/2\mathbb{Z} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{R} & \rightarrow & \mathbb{T} & \rightarrow & 0
\end{array}
\]

induces a diagram of long exact sequences

\[
\begin{array}{ccccccc}
0 & \rightarrow & KO^{-2}(pt) & \rightarrow & KO_{\mathbb{Z}/2\mathbb{Z}}^{-2}(pt) & \rightarrow & KO^{-1}(pt) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & KO_{\mathbb{Z}/2\mathbb{Z}}^{-2}(pt) & \rightarrow & KO_T^{-2}(pt) & \rightarrow & KO^{-1}(pt) & \rightarrow & 0
\end{array}
\]  

Each of $KO^{-1}(pt)$ and $KO^{-2}(pt)$ is cyclic of order two, from which $KO_{\mathbb{Z}/2\mathbb{Z}}^{-2}(pt)$ is either isomorphic to $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; we will see shortly that it is the former. Then the sequence shows that the generator of this group maps to the generator of $KO_T^{-2}(pt) \cong \mathbb{Z}/2\mathbb{Z}$. A similar argument with a different stretch of (B.16) shows that $KO_{\mathbb{Z}/2\mathbb{Z}}^{-3}(pt)$ is cyclic of order two, and with yet another stretch of (B.16) we find $KO_{\mathbb{Z}/2\mathbb{Z}}^{-4}(pt)$ is cyclic of order two and $KO_{\mathbb{Z}/2\mathbb{Z}}^{-4}(pt) \rightarrow KO_T^{-4}(pt)$ is injective. Hence we are reduced to the statement that $\eta^2: KO_{\mathbb{Z}/2\mathbb{Z}}^{-2}(pt) \rightarrow KO_{\mathbb{Z}/2\mathbb{Z}}^{-4}(pt)$ is nonzero.

Cohomology groups with $\mathbb{Z}/2\mathbb{Z}$ coefficients may be computed by smashing with $\mathbb{R}P^2$ and shifting degree by two,\(^{12}\) so we must show that

\[
\eta^2: KO^0(\mathbb{R}P^2) \rightarrow KO^{-2}(\mathbb{R}P^2)
\]

is nonzero. Note that the former group is generated by $H - 1$, where $H \rightarrow \mathbb{R}P^2$ is the nontrivial real line bundle, and since $w_2(H \oplus H) \neq 0$ it follows that $KO^0(\mathbb{R}P^2)$ is cyclic of order four, verifying the claim above. Finally, we deduce that (B.17) is nonzero by applying the long exact “Bott sequence”\(^{13}\)

\[
\begin{array}{ccccccc}
& \rightarrow & \tilde{K}^{-2}(\mathbb{R}P^2) & \rightarrow & \tilde{KO}^q(\mathbb{R}P^2) & \eta & \rightarrow & \tilde{KO}^{q-1}(\mathbb{R}P^2) & \rightarrow & \tilde{K}^{q-1}(\mathbb{R}P^2) & \rightarrow & \\
& & & & & & & & & & &
\end{array}
\]

\(^{12}\)For any cohomology theory $E$ and any pointed space $X$, $E^q_{\mathbb{Z}/2\mathbb{Z}}(X) \cong E^{q+2}(X_+ \wedge \mathbb{R}P^2)$.

\(^{13}\)One derivation begins with the fibration $U/O \rightarrow BO \rightarrow BU$ and the Bott periodicity $\Omega(U/O) \sim \mathbb{Z} \times BO$.  

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twice. First, set \( q = 0 \) and use \( K^{-1}(\mathbb{RP}^2) = 0 \) to deduce that \( \eta: \tilde{KO}^0(\mathbb{RP}^2) \to \tilde{KO}^{-1}(\mathbb{RP}^2) \) is surjective. Then set \( q = -1 \) and use \( K^{-3}(\mathbb{RP}^2) = 0 \) to deduce that \( \eta: \tilde{KO}^{-1}(\mathbb{RP}^2) \to \tilde{KO}^{-2}(\mathbb{RP}^2) \) is an isomorphism. (Both groups are cyclic of order two.)

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