On Parametric Resonance in Quantum Many-Body System

Collective Motion and Quantum Fluctuation Around It in Coupled Lipkin Model

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Abstract

The dynamics governed by a requantized collective Hamiltonian in the coupled Lipkin model is investigated in the time-dependent variational approach with squeezed state. It is pointed out that there is a possibility of the parametric resonance mechanism which leads to amplifying the amplitude of quantum fluctuation around the collective mode in this model.
§1. Introduction

The time-dependent Hartree-Fock (TDHF) theory and its approximated versions and/or the extensions have played a crucial role in the studies of nuclear collective motions. They are based on the time-dependent variational principle and exhibit two characteristic aspects. The aspect (i) is to give a possible description of time-evolution of the quantal state under investigation in the frame of a chosen form of the trial state of the variation. The aspect (ii) is to give a possible classical counterpart of the original quantal system under a suitable choice of the trial state. Then, it is expected that the original quantal system is reproduced in a disguised form under an appropriate requantization. The aspect (ii) has been deeply concerned in the studies of collective motions.

Since a powerful idea was proposed by Marumori, Maskawa, Sakata and Kuriyama in 1980,1 various studies on the aspect (ii) have been performed until the present. For example, we can find the newest one in Ref.2. At the early stage, the present authors (M. Y. and A. K.) also presented a possible form constructed in terms of canonical variables including the Grassmann variables, which was reviewed in Ref.3. In order to make the significance of this paper understandable, first, we summarize the basic scheme of our form as follows:

(a) Paying attention to the Lie-algebraic structure, we set up a trial state containing parameters. These obey the canonicity condition3,4 and can be regarded as the canonical variables in classical mechanics.

(b) The trial state in (a) leads us to the classical counterpart of the original quantal systems through a certain procedure. It is formulated in the phase space of classical mechanics.

(c) In various cases, requantizations at this stage serve us quantal systems in disguised forms completely equivalent to the original ones, i.e., boson realization of Lie algebra.

(d) We presuppose that classical version for collective motions under investigation can be described on a collective submanifold in the phase space in (b). This is specified by canonical variables which enables us to describe the collective motion, i.e., collective variables. Under the above presupposition, equation of collective submanifold is derived.

(e) Together with the canonicity condition, the equation of collective submanifold enables us to express the variables in (a) as functions of the collective variables. Then, classical Hamiltonian is obtained in terms of the collective variables and by solving the Hamilton’s equation of motion, we get the time-evolution of the original quantal system in the frame of the chosen form for the trial state.

(f) As the result of an appropriate requantization, we obtain a quantal system expressed in terms of the collective variables as the operators. Of course, it is of a disguised form for the original quantal system in the collective subspace.
By diagonalizing the Hamiltonian in (e), the collective motion can be described. Naturally, we expect that the results are in good agreement with the exact one. The above is our basic scheme reviewed in Ref.3). The adiabatic TDHF approach can be formulated in this scheme. However, we must point out the following:

The investigation of the aspect (i) based on the Hamiltonian in (f) has remained untouched. If, as the trial state, we adopt boson coherent state or its equivalent state for the Hamiltonian in (f), the time-dependent variation gives us the same results as those in (e) except the quantum effect coming from the ordering of operators. However, we have a possibility to adopt trial state different from the boson coherent state, for example, such as the squeezed state and we expect the aspect (ii) different from that in (e).

With the aim of making a check on the validity of the above scheme ((a)∼(g)), the present authors (M. Y. and A. K.) with Iida investigated the adiabatic TDHF approximation on the coupled Lipkin model, a kind of the $su(2) \otimes su(2)$-algebraic model, which will be referred to as (A). Pioneering result on this model can be seen in Ref.6). By solving the equation of collective submanifold in the adiabatic TDHF approximation, we can draw the equi-potential curves in two dimensional space. In a certain region of the coupling strength, two bottoms appear and we can determine the collective path passing these two points by one parameter which plays a role of collective coordinate. After the procedure given in the scheme ((a)∼(g)), we derive various results under rather good agreement with the exact one. Therefore, it may be interesting to investigate the aspect (i) in the sense of (h).

As was mentioned above, in (A), it was shown that the form of collective potential in the coupled Lipkin model has two minima in a certain region of coupling strength, that is, the collective potential is similar to the double well potential in the one dimensional problem in classical or quantum mechanics. If the collective variable oscillates around a minimum of the collective potential, it may be expected that the amplitude of this oscillation becomes gradually small because the oscillational energy dissipates to other degree of freedom such as quantum fluctuations or single particle motion. The similar situation has been realized theoretically in the late time of dynamical chiral phase transition in the context of the formation of disoriented chiral condensate. In this case, the chiral condensate oscillates around its vacuum value, where the chiral condensate corresponds to the collective mode in nuclear collective motion. Further, one of the present authors (Y.T.) has pointed out that, when the chiral condensate oscillates around the vacuum value in the late time of chiral phase transition in the context of the relativistic heavy ion collisions, there is a possibility that the amplitudes of quantum pion modes, which correspond to the quantum fluctuations around the mean field, are amplified by the mechanism of the parametric resonance and/or the forced oscillation induced by the oscillation of the chiral condensate in the O(4) linear...
Also, the general investigation for the parametric resonance mechanism in the \(O(N)\) scalar field theory is given by using the \(1/N\) expansion method.\(^9\)

In this paper, as a part of the aspect (i) given in (h), we investigate the possibility of the parametric resonance mechanism to amplify the quantum fluctuation around the collective variable in the coupled Lipkin model developed in (A). It is important to investigate whether the amplification of fluctuation mode occurs or not. The reason is as follows: The amplification of the fluctuation mode may possibly lead to the damping of collective mode because the energy flow from collective motion to fluctuations should exist. In \(\S2\), the specification of collective submanifold in the coupled Lipkin model is recapitulated following (A). The derived collective Hamiltonian is requantized and we treat this system governed by the collective Hamiltonian as a quantum mechanical system. In \(\S3\), the time-dependent variational approach with squeezed state is applied to the dynamical problem of collective motion including quantum fluctuation on the collective submanifold. In \(\S4\), it is shown that the time-dependent part of collective mode induces the parametric resonance for the quantum fluctuation mode, in which it is demonstrated that the Mathieu equation with additional term is derived. In \(\S5\), discussion and concluding remarks are presented.

\(\S2\). Recapitulation of specification of collective submanifold in the coupled Lipkin model

In this section, we present a brief review about a specification of collective submanifold of the coupled Lipkin model by the adiabatic time-dependent Hartree-Fock method given in (A).

The Hamiltonian of the coupled Lipkin model is given by

\[
\hat{H} = \sum_{\sigma=1}^{2} \hat{H}_\sigma - V_3(\hat{S}_1+\hat{S}_2^- + \hat{S}_2+\hat{S}_1^-) ,
\]

\[
\hat{H}_\sigma = 2\epsilon_\sigma \hat{S}_\sigma^0 - \frac{1}{2}V_\sigma (\hat{S}_\sigma^2 + \hat{S}_{\sigma-}^2) ,
\]

where the quasi-spin operators are defined in terms of the particle and hole creation and annihilation operators, \(\hat{a}_{\sigma j m}^*, \hat{b}_{\sigma j m}^*\) and \(\hat{a}_{\sigma jm}, \hat{b}_{\sigma jm}\):

\[
\hat{S}_\sigma^+ = \sum_{m=-j}^{j} \hat{a}_{\sigma jm}^*(-)^{j-m}\hat{b}_{\sigma jm}^* , \quad \hat{S}_\sigma^- = \hat{S}_\sigma^+ ,
\]

\[
\hat{S}_\sigma^0 = \frac{1}{2} \sum_{m=-j}^{j} (\hat{a}_{\sigma jm}^*\hat{a}_{\sigma jm} + \hat{b}_{\sigma jm}^*\hat{b}_{\sigma jm}) - \Omega ,
\]

\[
\Omega = j + \frac{1}{2} .
\]
In this paper, we consider only the case of zero seniority number. Thus, the classical image of this system described by (2.1) can be obtained by the TDHF method where the Slater determinantal state is used. As a result, a classical correspondence of the above quasi-spin operators is obtained in the following forms in terms of a certain set of canonical variables \((q^\sigma, p^\sigma)\) \((\sigma = 1, 2)\):

\[
S_{\sigma x} = \frac{1}{2}(S_{\sigma +} + S_{\sigma -}) = \sqrt{\Omega^2 - p^2_\sigma} \sin q^\sigma, \\
S_{\sigma y} = \frac{1}{2i}(S_{\sigma +} - S_{\sigma -}) = -p_\sigma, \\
S_{\sigma z} = S_{\sigma 0} = -\sqrt{\Omega^2 - p^2_\sigma} \cos q^\sigma. 
\]

Thus, the Hamiltonian is reduced to the classical Hamilton function as

\[
H = \sum_{\sigma=1}^{2} H_\sigma - V_3(S_{1+}S_{2-} + S_{2+}S_{1-}) , \\
H_\sigma = - \left[ 2\epsilon_\sigma \sqrt{\Omega^2 - p^2_\sigma} \cos q^\sigma + \frac{V_\sigma}{\Omega} \left( \Omega - \frac{1}{2} \right) \{(\Omega^2 - p^2_\sigma)\sin^2 q^\sigma - p^2_\sigma \} \right] . 
\]

For the collective variables \((Q, P)\), we can derive the equation of collective submanifold and the canonicity condition following Ref.4:

\[
\partial_{q^\sigma} H = \lambda \partial_{P^\sigma} - \mu \partial_{Q^\sigma} P^\sigma , \\
\partial_{P^\sigma} H = -\lambda \partial_{P} q^\sigma + \mu \partial_{Q} q^\sigma , \\
\sum_{\sigma=1}^{2} p^\sigma \partial_{Q} q^\sigma = P , \quad \sum_{\sigma=1}^{2} p^\sigma \partial_{P} q^\sigma = 0 . 
\]

Here, \(\partial_z = \partial/\partial z\). Thus, the basic equations consist of Eqs.(2.5) and (2.6).

To proceed the calculation concretely, we introduce the adiabatic approximation which corresponds to the lowest approximation in terms of a power series of \(P\). Under this approximation, \(q^\sigma\) and \(p^\sigma\) can be expressed as

\[
q^\sigma = q^\sigma(Q) , \quad p^\sigma = p^\sigma(Q)P . 
\]

In this approximation, the Hamiltonian (2.4) takes the following form :

\[
H = \frac{1}{2} \sum_{\sigma,\sigma'=1}^{2} M^{\sigma\sigma'} p^\sigma p^{\sigma'} + V(q^\sigma) , \\
M^{\sigma\sigma} = \frac{1}{\Omega} \left[ 2\epsilon_\sigma \cos q^\sigma + 2V_\sigma \left( \Omega - \frac{1}{2} \right) \left( 1 + \sin^2 q^\sigma \right) \right] + 2V_3 \sin q^1 \sin q^2 , \\
M^{12} = M^{21} = -2V_3 , \\
V(q^\sigma) = - \sum_{\sigma=1}^{2} \Omega \left[ 2\epsilon_\sigma \cos q^\sigma + V_\sigma \left( \Omega - \frac{1}{2} \right) \sin q^\sigma \right] - 2V_3 \Omega^2 \sin q^1 \sin q^2 . 
\]
Then, the basic equations (2.5) and (2.6) are reduced to the following equations:

\[ C(Q)Q \cdot p_\sigma(Q) = \partial_q V , \]
\[ \frac{dq^\sigma(Q)}{dQ} = \sum_{\sigma' = 1}^2 M^{\sigma\sigma'} p_{\sigma'}(Q) , \]
\[ \sum_{\sigma = 1}^2 p_\sigma(Q) \frac{dq^\sigma(Q)}{dQ} = 1 . \]

(2.9a)

The second of Eq.(2.6) is automatically satisfied for (2.7). The first of Eq.(2.6) is nothing but Eq.(2.9b). The equations of collective submanifold (2.5) are reduced to Eq.(2.9a) with the explicit expressions of the Lagrange multipliers. Here, we do not need to know the explicit form of \( C(Q) \).

It should be here noted that the basic equations (2.5) and (2.6) do not give a unique collective coordinate system because these equations are still invariant with respect to the point canonical transformation \( Q' = Q'(Q) \) and \( P' = (dQ/dQ') \cdot P \). In (A), a possible method to avoid this ambiguity has been proposed. According to (A), the mass of collective motion has to be taken as unit in order to fix the collective coordinate system. In our case, (2.9a) and (2.9b), the mass of collective motion, \( M = (\sum_{\sigma,\sigma' = 1}^2 p_\sigma(Q)M^{\sigma\sigma'} p_{\sigma'}(Q))^{-1} \), is automatically one from the second equation of (2.9a) and (2.9b). Further, by multiplying \( dq^\sigma(Q)/dQ \) on both sides of the first equation of (2.9a) and summing up \( \sigma = 1,2 \), it is shown that the following relation should be satisfied:

\[ C(Q)Q = \partial_q V . \]

(2.10)

Thus, the collective Hamilton function is given by

\[ H_C = \frac{1}{2} P^2 + V_C(Q) , \]
\[ V_C(Q) = \int Q' C(Q')Q'dQ' . \]

(2.11)

Equations (2.9) can be solved in the following power series expansion technique:

\[ q^\sigma(Q) = \sum_{n=0} q^n_{\sigma} Q^{2n+1} , \quad p_\sigma(Q) = \sum_{n=0} p^n_{\sigma} Q^{2n} , \quad C(Q) = \sum_{n=0} c_n Q^{2n} . \]

(2.12)

It is interesting to the collective potential energy \( V_C(Q) \) because we investigate the dynamics of collective motion and of the quantum fluctuation around the collective variable in this paper. In the above expansion, the collective potential \( V_C(Q) \) can be represented as

\[ V_C(Q) = \frac{1}{2} c_0 Q^2 + \frac{1}{4} c_1 Q^4 + \frac{1}{6} c_2 Q^6 + \cdots + \text{const.} . \]

(2.13)
Table I. The values of coefficients, $c_0$, $c_1$ and $c_2$, of the collective potential $V_C = (1/2)c_0Q^2 + (1/4)c_1Q^4 + (1/6)c_2Q^6$ are listed.

| $\chi$ | $c_0$ | $c_1$ | $c_2$ | $(1/6)c_2/|(1/4)c_1|$ |
|--------|-------|-------|-------|---------------------|
| 1.5    | 4.69  | 0.328 | -0.161| 0.3278              |
| 2.0    | 2.77  | 0.894 | -0.175| 0.130               |
| 2.5    | 0.942 | 1.24  | -0.106| 0.0568              |
| 3.0    | -0.747| 1.40  | -0.002| 9.5x10^{-4}         |
| 3.5    | -2.26 | 1.38  | 0.125 | 0.0602              |
| 4.0    | -3.56 | 1.20  | 0.265 | 0.147               |
| 4.5    | -4.62 | 0.881 | 0.403 | 0.305               |
| 5.0    | -5.40 | 0.441 | 0.522 | 0.789               |

Here, we numerically estimate the coefficients of power expansion in (2.12). In the numerical evaluation, we adopt the following set of values for the model parameters as

$$
\epsilon_1 = 1.5, \quad \epsilon_2 = 2.0, \\
V_1 = 0.05\chi, \quad V_2 = 0.1\chi, \quad V_3 = 0.075\chi, \\
\Omega = 5,
$$

(2.14)

which are those used in Ref.[3] and in (A). Here, $\chi$ remains as a parameter which controls the force strength. The numerical values for $c_0$, $c_1$ and $c_2$ in the collective potential are listed up in Table I for the various $\chi$. It is found that, in the neighborhood of the phase transition point, that is around $c_0 = 0$, the collective potential can be approximated up to the order of $Q^4$ safely because the relation between the coefficients of $Q^4$ and $Q^6$ satisfies $|c_2/6|/|c_1/4|\approx 0.1$.

We quantize the collective Hamiltonian in (2.11) by the canonical quantization: $[\hat{Q}, \hat{P}] = i\hbar$. In the next section, we will investigate the dynamics of the collective motion and the quantum fluctuation around it based on the above derived collective Hamiltonian up to the order of $\hat{Q}^4$.

§3. Time-dependent variational approach with squeezed state to the coupled Lipkin model on the collective submanifold

We apply the time-dependent variational approach with a squeezed state to the coupled Lipkin model on the collective submanifold. Our task is reduced to solving the dynamical
problem in one-dimensional quantum mechanical system. The time-dependent variational method with the squeezed state gives a useful approximation including the quantum effects in various quantal systems.\textsuperscript{12, 13, 14}

3.1. Squeezed state approach

The squeezed state is defined as

\[ |\psi(\alpha, \beta)\rangle = (1 - \beta^* \beta)^{\frac{1}{4}} \exp \left( \frac{\beta \hat{b}^2}{2} \right) \exp \left( -\frac{1}{2} \alpha^* \alpha \right) \exp(\alpha \hat{a}^*) |0\rangle . \tag{3.1} \]

Here, \( \hat{a}^* \) is a boson creation operator and \( |0\rangle \) is a vacuum state for the boson annihilation operator \( \hat{a} : \hat{a}|0\rangle = 0 \). The operators \( \hat{b}^* \) and \( \hat{b} \) are defined as

\[ \hat{b}^* = \hat{a}^* - \alpha^* , \quad \hat{b} = \hat{a} - \alpha . \tag{3.2} \]

The operator \( \hat{b} \) is identical with the annihilation operator for the usual coherent state: \( \hat{b} \exp \left( -\frac{1}{2} \alpha^* \alpha \right) \exp(\alpha \hat{a}^*) |0\rangle = 0 \). Introducing the operators which correspond to the coordinate and momentum operators, we have another expression of the squeezed state (3.1):

\[ |\psi(\alpha, \beta)\rangle = e^{i\varphi}(2G)^{-1/4} \exp \left( \frac{i}{\hbar} \left( P \hat{Q} - Q \hat{P} \right) \right) \exp \left\{ \frac{1}{2\hbar} \left( 1 - \frac{1}{2G} + i2\Pi \right) \hat{Q}^2 \right\} |0\rangle , \tag{3.3} \]

\[ \hat{Q} = \sqrt{\frac{\hbar}{2}} (\hat{a}^* + \hat{a}) , \quad \hat{P} = i \sqrt{\frac{\hbar}{2}} (\hat{a}^* - \hat{a}) , \tag{3.4} \]

\[ Q = \sqrt{\frac{\hbar}{2}} (\alpha^* + \alpha) , \quad P = i \sqrt{\frac{\hbar}{2}} (\alpha^* - \alpha) , \]

\[ G = \sqrt{\frac{1}{2} + |y|^2 + y^2} , \quad \Pi = \frac{i}{2} (y^* - y) \sqrt{\frac{1}{2} + |y|^2} G^{-1} , \tag{3.5} \]

\[ e^{-i2\varphi} = \frac{1}{\sqrt{G}} \left( \sqrt{\frac{1}{2} + |y|^2 + y} \right) , \]

where \( y \) is related to \( \beta \) as

\[ y = \beta / \sqrt{2(1 - |\beta|^2)} . \tag{3.6} \]

The reason why we have introduced the new variables \( y \) and \( y^* \) is that these variables correspond to the boson-type canonical variables. The expectation values for the coordinate and the momentum operators are derived easily as

\[ \langle \psi(\alpha, \beta)|\hat{Q}|\psi(\alpha, \beta)\rangle = Q , \]
\[ \langle \psi(\alpha, \beta) | \hat{P} | \psi(\alpha, \beta) \rangle = P, \]
\[ \langle \psi(\alpha, \beta) | \hat{Q}^2 | \psi(\alpha, \beta) \rangle = Q^2 + \hbar G, \]
\[ \langle \psi(\alpha, \beta) | \hat{P}^2 | \psi(\alpha, \beta) \rangle = P^2 + \hbar \left( \frac{1}{4G} + 4G^2 \right), \]  
\[ \langle \psi(\alpha, \beta) | V(\hat{Q}) | \psi(\alpha, \beta) \rangle = \exp \left\{ \frac{1}{2} \hbar G \left( \frac{\partial}{\partial Q} \right)^2 \right\} V(Q), \]  
\[ \langle \psi(\alpha, \beta) | \partial_z | \psi(\alpha, \beta) \rangle = \frac{i}{2\hbar} (Q \partial_z P - P \partial_z Q) + iG \partial_z \Pi + i\varphi, \]  

where \( \partial_z = \partial / \partial z \).

The squares of the standard deviations for \( \hat{Q} \) and \( \hat{P} \) are then expressed as \( \langle \psi(\alpha, \beta) | (\hat{Q} - Q)^2 | \psi(\alpha, \beta) \rangle = \hbar G \) and \( \langle \psi(\alpha, \beta) | (\hat{P} - P)^2 | \psi(\alpha, \beta) \rangle = \hbar (1/(4G) + 4G^2) \). Thus, the uncertainty relation is expressed in terms of \( G \) and \( \Pi \) as \( \langle \psi(\alpha, \beta) | (\hat{Q} - Q)^2 | \psi(\alpha, \beta) \rangle \langle \psi(\alpha, \beta) | (\hat{P} - P)^2 | \psi(\alpha, \beta) \rangle = \hbar^2 (1/4 + 4G^2) \). It can be seen from this uncertainty relation that, if one direction of the uncertainty is relaxed, the other can be squeezed. Also, one can see from (3.8) that the quantum effects beyond the order of \( \hbar \) are included in this formalism. This novel feature is originated from the degree of freedom of the squeezing, namely \( \beta \) and \( \beta^* \) in (3.1) or \( G \) and \( \Pi \) in (3.3).

The time-evolution of this quantum state is governed by the time-dependent variational principle:
\[ \delta \int_{t_0}^{t_1} dt \langle \psi(\alpha, \beta) | i\hbar \partial_t - \hat{H} | \psi(\alpha, \beta) \rangle = 0. \]  
In order to formulate the time-dependent variational approach in the canonical form, we impose the canonicity conditions as
\[ \langle \psi(\alpha, \beta) | i\hbar \partial_Q | \psi(\alpha, \beta) \rangle = P_i + \partial_Q s(Q_i, P_i), \]
\[ \langle \psi(\alpha, \beta) | i\hbar \partial_P | \psi(\alpha, \beta) \rangle = \partial_P s(Q_i, P_i), \quad (i = 1, 2). \]  

A set of possible solutions are obtained as
\[ Q_1 = Q, \quad P_1 = P, \]
\[ Q_2 = \hbar G, \quad P_2 = \Pi, \quad s = -PQ/2 + \hbar G \Pi + \hbar \varphi. \]  

Thus, the equations of motion are formulated as canonical equations of motion with the same form in the classical mechanics:
\[ \dot{Q} = \frac{\partial \langle \hat{H} \rangle}{\partial P} = P, \quad \dot{P} = -\frac{\partial \langle \hat{H} \rangle}{\partial Q}, \]
\[ \hbar \dot{G} = \frac{\partial \langle \hat{H} \rangle}{\partial \Pi} = \hbar \cdot 4G \Pi, \quad \hbar \dot{\Pi} = -\frac{\partial \langle \hat{H} \rangle}{\partial G}, \]  

where the dot denotes the time-derivative and \( \langle \hat{H} \rangle \) is the expectation value of the Hamiltonian with respect to the state (3.1) or (3.3).
3.2. Squeezed state approach to the coupled Lipkin model with the double well potential on the collective submanifold

Let us consider the case in which the collective Hamiltonian of the coupled Lipkin model derived in §2 has a form of the double well potential, for example, in the region around \( \chi \sim 3.5 \). In this case, the Hamiltonian is simply written as

\[
\hat{H} = \frac{1}{2} \hat{P}^2 + \frac{1}{2} c_0 \hat{Q}^2 + \frac{1}{4} c_1 \hat{Q}^4 \tag{3.15}
\]

with \( c_0 < 0 \) and \( c_1 > 0 \), where the constant term in the collective potential is omitted because of no influence of the dynamics discussed later. The expectation value of this Hamiltonian with respect to the squeezed state is calculated as

\[
\langle \hat{H} \rangle = \frac{1}{2} \left[ \hat{P}^2 + \hbar \left( \frac{1}{4G} + 4G \Pi^2 \right) \right] + \frac{1}{2} c_0 (\hat{Q}^2 + \hbar G) + \frac{1}{4} c_1 (\hat{Q}^4 + 6 \hbar \hat{Q}^2 G + 3 \hbar^2 G^2) . \tag{3.16}
\]

Then, the equations of motion in (3.13) and (3.14) are summarized as

\[
\dot{Q} = P ,
\]
\[
\dot{P} = - (c_0 + c_1 Q^2 + 3 \hbar c_1 G) Q ,
\]
\[
\hbar \dot{G} = \hbar \cdot 4G \Pi ,
\]
\[
\hbar \dot{\Pi} = - \hbar \left( - \frac{1}{8G^2} + 2 \Pi^2 + \frac{1}{2} c_0 + \frac{3}{2} c_1 Q^2 + \frac{3}{2} \hbar c_1 G \right) . \tag{3.17}
\]

Eliminating \( p \) and \( \Pi \), the above equations of motion are rewritten as

\[
\ddot{Q} + \left( c_0 + c_1 Q^2 + 3 \hbar c_1 G \right) Q = 0 , \tag{3.18}
\]
\[
\frac{\dot{G}}{4G} - \frac{\dot{G}^2}{8G^2} - \frac{1}{8G^2} + \frac{1}{2} c_0 + \frac{3}{2} c_1 Q^2 + \hbar \cdot \frac{3}{2} G = 0 . \tag{3.19}
\]

Further, since \( G \) is positive definite, we introduce new variables \( \eta \) instead of \( G \) as

\[
G = \eta^2 . \tag{3.20}
\]

Then, the equations of motion (3.18) and (3.19) are further recast into

\[
\ddot{Q} + \left( c_0 + c_1 Q^2 + \hbar \cdot 3 c_1 \eta^2 \right) Q = 0 , \tag{3.21}
\]
\[
\ddot{\eta} + \left( c_0 + 3 c_1 Q^2 + \hbar \cdot 3 c_1 \eta^2 \right) \eta - \frac{1}{4 \eta^3} = 0 . \tag{3.22}
\]

Here, \( Q \) represents the collective coordinate which presents a classical image of the collective motion. Also, \( \eta \) represents quantum fluctuations around the collective variables found in Eq.(3.7) with (3.20).
§4. Generalized Mathieu’s equation and amplification of quantal fluctuation mode by the parametric resonance mechanism

First, we wish to derive the static solutions of the equations of motion (3.21) and (3.22). We denote the static solutions of $Q$ and $\eta$ as $Q_0$ and $\eta_0$, respectively. Then, $Q_0$ for $Q_0 \neq 0$ and $\eta_0$ satisfy

$$c_0 + c_1 Q_0^2 + \hbar \cdot 3 c_1 \eta_0^2 = 0 ,$$

$$\left( c_0 + 3 c_1 Q_0^2 + \hbar \cdot 3 c_1 \eta_0^2 \right) \eta_0^4 = \frac{1}{4} .$$

(4.1)

If $\hbar \to 0$ at this stage, the static solutions are easily derived as

$$Q_0^2 = -\frac{c_0}{c_1} (> 0) ,$$

$$\eta_0^4 = -\frac{1}{8c_0} (> 0) .$$

(4.2)

Next, let us investigate the time-dependent solutions $Q$ and $\eta$. We are restricted ourselves to seek the time-dependent solutions around the static configuration. Thus, the variables $Q$ and $\eta$ can be expanded as

$$Q = Q_0 + \delta Q , \quad \eta = \eta_0 + \delta \eta .$$

(4.3)

Here, $\delta Q$ and $\delta \eta$ have time-dependence and we assume that these are small deviation. Substituting (4.3) into (3.21) and (3.22) and using the relation (4.1), we can obtain the following equations up to the order of $\delta Q$ and $\delta \eta$ :

$$\ddot{\delta Q} + \left( 2 c_1 Q_0^2 + \hbar \cdot 6 c_1 \eta_0 \delta \eta \right) \delta Q = -\hbar \cdot 6 c_1 Q_0 \eta_0 \delta \eta ,$$

(4.4)

$$\ddot{\delta \eta} + \left( \frac{1}{\eta_0^4} + 6 c_1 Q_0 \delta Q + \hbar \cdot 6 c_1 \eta_0^2 \right) \delta \eta = -6 c_1 Q_0 \eta_0 \delta Q ,$$

(4.5)

where we have neglected the terms with $\delta Q^2$ and $\delta \eta^2$ and their higher order terms. If the semi-classical limit is adopted at this stage, namely $\hbar \to 0$, then we obtain

$$\ddot{\delta Q} - 2 c_0 \delta Q \approx 0 ,$$

(4.6)

$$\ddot{\delta \eta} + \left( -8 c_0 \pm 6 \sqrt{-c_0 c_1} \delta Q \right) \delta \eta \approx \mp 6 \sqrt{-c_0 c_1^2} \delta Q ,$$

(4.7)

where we used the static solutions in (4.2). The solution of (4.6) gives the small oscillation around the static configuration :

$$\delta Q = \sigma \cos(\sqrt{-2c_0} \ t + \theta_0) .$$

(4.8)
If we adopt the static solutions as $Q_0 = +\sqrt{-c_0/c_1}$ and $\eta_0 = +\sqrt{-1/8c_0}$, the equation for the quantum fluctuation $\eta$ are reduced to

$$\delta \ddot{\eta} + \left(-8c_0 + 6\sigma\sqrt{-c_0c_1}\cos\left(\sqrt{-2c_0}t + \theta_0\right)\right)\delta \eta = -6\sqrt{-\frac{c_0c_1^2}{8}}\sigma \cos\left(\sqrt{-2c_0}t + \theta_0\right).$$

(4.9)

Adopting $\theta_0 = \pi$ without the loss of generality, the above equation can be expressed as

$$\delta \ddot{\eta} + \omega_0^2 (1 - h \cos \gamma t) \delta \eta = f \cos \gamma' t,$$

(4.10a)

where

$$\omega_0^2 = -8c_0, \quad h = \frac{3}{4}\sigma\sqrt{-\frac{c_1}{c_0}}, \quad f = 6\sqrt{-\frac{c_0c_1^2}{8}}\sigma, \quad \gamma = \gamma' = \sqrt{-2c_0}.$$

(4.10b)

If the right-hand side of (4.10a) or (4.9) can be neglected, this equation of motion for quantum fluctuation is reduced to the well-known equation of forced oscillation in the classical mechanics. However, $\omega_0 = 2\gamma'$ is satisfied as is seen in (4.10b), it is not expected to realize the resonance phenomena proportional to $t$ originated by the forced oscillation, while the beat may occur.

On the other hand, if $f$ in (4.10a) is neglected or the right-hand side in (4.10a) has no effect, Eq.(4.10a) can be reduced to the famous Mathieu equation. This can be realized even

§5. Discussion and concluding remarks

The equation (4.9), or equivalently (4.10a), describes the time-evolution of quantum fluctuation around the collective motion. Here, $\delta \eta$ represents the time-dependent part of quantum fluctuation around the static configuration which is denoted as $\eta_0$. In this coupled Lipkin model, the amplification of the amplitude of quantum fluctuation, $\delta \eta$, may be induced by the collective oscillation, which is represented in terms of $\delta Q$, around the static configuration $Q_0$. Equation (4.10a) reveals two possibility for the amplification of quantum fluctuation around the collective motion: One is by the resonance mechanism in the forced oscillation and the other is by the parametric resonance mechanism governed by the Mathieu equation. The amplification of the amplitude of quantum fluctuation inversely means the damping of collective mode.

If $h$ in Eq.(4.10a) can be neglected, this equation of motion for quantum fluctuation is reduced to the well-known equation of forced oscillation in the classical mechanics. However, $\omega_0 = 2\gamma'$ is satisfied as is seen in (4.10b), it is not expected to realize the resonance phenomena proportional to $t$ originated by the forced oscillation, while the beat may occur.

On the other hand, if $f$ in (4.10a) is neglected or the right-hand side in (4.10a) has no effect, Eq.(4.10a) can be reduced to the famous Mathieu equation. This can be realized even
when the amplitude of collective oscillation, $\sigma$, is small. The Mathieu equation

$$\delta \ddot{\eta} + \omega_0^2 (1 - h \cos \gamma t) \delta \eta = 0$$

(5.1)
describes a parametric resonance phenomena. The parametric resonance occurs in the region around $\gamma = 2\omega_0/n$ where $n$ is natural integer. When we denote $\gamma = 2\omega_0/n + \varepsilon$ where $\varepsilon \ll 1$ and $h \ll 1$, the parametric resonance occurs in the region $-O(h^n) < \varepsilon < O(h^n)$. In this coupled Lipkin model in Eq.(4.10a), $\omega_0 = 2\gamma$ is realized which corresponds to the case $n = 4$ and $\varepsilon = 0$ in the Mathieu equation (5.1). Thus, the parametric resonance occurs inevitably, which may lead to the damping of the collective motion because of the growing of fluctuation energy induced by the amplification of the amplitude of quantum fluctuation around the collective variable. However, the damping time may be rather long because the parametric resonance works weakly because of $n = 4$.

It should be noted here the possible scenario for the amplification of quantum fluctuation modes obtained from the results in this paper and the previous work in Ref.9). In the coupled Lipkin model, it has been indicated that the parametric resonance occurs for the quantum fluctuation mode around the collective motion. However, the instability of this fluctuation mode is weak because of $n = 4$ in terms of the usual Mathieu equation in (5.1). If the other fluctuation modes exist, the unstable modes with the lower $n$ such as $n = 1$ may appear. This situation has actually been seen in the O(4) linear sigma model in the quantum field theory with sigma meson and pions. In that case, the chiral condensate corresponds to the collective variable and the sigma meson modes correspond to the quantum fluctuation mode around the collective motion. In the O(4) linear sigma model, amplification of quantum meson mode with $n = 4$ in the Mathieu equation has been also realized in the lowest quantum sigma meson mode around the chiral condensate. This phenomenon is identical with that seen in the coupled Lipkin model investigated in this paper. In addition to the sigma meson modes, in the O(4) linear sigma model, the quantum pion modes exist. Then, it has been shown that the low momentum pion modes become unstable modes with $n = 1$ in terms of the Mathieu equation. The amplification is strong compared with $n = 4$ case in the sigma direction around the chiral condensate. Thus, if the other modes, such as the intrinsic modes, except for the collective mode and the quantum fluctuation mode around it exist and they are coupled with collective mode, the strong unstable modes may be realized and the dissipation of collective motion may occur. It is interesting to investigate the possibility of the dissipation of collective motion. In addition to the parametric resonance mechanism, the resonance by the forced oscillation may occur, as was seen in the O(4) linear sigma model. These are further problems to study.

In summary, the collective oscillation in the coupled Lipkin model inevitably leads to
the amplification of the amplitude of quantum fluctuation mode around the collective mode. When the amplitude of the collective oscillation is small, the parametric resonance mechanism works to amplify the amplitude of quantum fluctuation mode, which may lead to damping of the collective oscillation.

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