The Foucault pendulum and the additivity of infinitesimal rotations

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Abstract
The purpose of this work is to present the Foucault pendulum precession without making recourse to the Coriolis acceleration or to geometrical methods. The exposition makes use of the fact that, contrary to finite rotations, infinitesimal rotations about different axes in three dimensions are both additive and commutative and that these important properties are readily extended to their time-derivatives, i.e. angular velocities. The angular velocity of the Earth about the polar axis is then decomposed into a pair of orthogonal angular velocity vectors having a common start point at the center of the Earth; the first vector is parallel to the axis that, in the local horizontal plane, points toward the north, while the second vector is directed upward, pointing toward the site where the pendulum is positioned. For the validity of the procedure it is essential to show that an infinitesimal rotation about the horizontal axis can be shown to have a negligible effect on the orientation of the plane of oscillation of the pendulum. At this point, only an infinitesimal rotation about the vertical axis is enough for calculating the change of orientation of the pendulum. Finally, owing to the constancy of the Earth’s spin, the obtained result can be readily extended to any finite rotation of the Earth, for example through the angle $2\pi$ that corresponds to a whole sidereal day.

Keywords: Foucault pendulum, physics education, infinitesimal rotations

(Some figures may appear in colour only in the online journal)

1. Introduction

It is certain that when Umberto Eco set about writing *Foucault’s Pendulum*, his aim was not to publish a new essay on the homonymous problem of theoretical mechanics [1]. Nevertheless, the title of his scholarly novel acted as a scientific stimulus for people attracted by unusual physical effects.
The Foucault pendulum problem (FPP) is historically important because it first allowed one to verify the reality of the Earth’s rotation by means of an experiment performed within a space visually separated from the outside world. After Foucault’s work, simple tall pendulums became popular scientific attractions exhibited in several museums scattered around the world.

This work was mainly motivated by the surprising opinion expressed by a worldwide scientific celebrity, the paleontologist Stephen Jay Gould, in the course of an interview released in 1993 to the New York Times [2], as reported below:

‘At the hanging pendulum, a giant steel ball suspended from a wire in the ceiling, he says: “I’ve never understood why every science museum in the country feels compelled to have one of these. I still don’t understand how they work, and I don’t think most visitors to the museum do either.” [The interviewer objects that] the pendulum is supposed to show how the suspended ball keeps swinging in a straight line while the Earth rotates beneath it, but Dr Gould points out that “the pendulum itself is attached to a building that is rotating with the Earth, so why should the axis of the ball not be rotating as well?”.’

Leaving aside the interviewer’s oversimplified and questionable statement that ‘the Earth rotates beneath the pendulum’, the doubts expressed by an outstanding scientist as Professor Gould are likely to have been unconsciously shared by entire generations of students.

In my life as a teacher, I regularly tried to explain the subtlety of the Foucault precession to university freshmen (and to several friends of mine as well …) using a low-level approach; but, more often than not, my pedagogical efforts were a substantial failure. Now, I hope to have found a description that might work better than my previous attempts and I’d like to submit it to the attention of European Journal of Physics readers.

In the literature, the FPP is presented following two different paths. Most university courses and texts of theoretical mechanics [3–7] adopt a standard approach in which the differential equations of motion of the pendulum are written in a non-inertial local frame rotating with the Earth. This road allows one to calculate both the daily precession of the pendulum and the detailed trajectory of the bob. The didactic problem is that the overwhelming majority of freshmen find this description rather hard because it requires solving a pair of coupled differential equations (after making suitable approximations); this seems a bit too complex for their capabilities. Typically, students admit that when they write the equations of motion in the rotating Earth frame, a Coriolis term automatically appears that, after a long sequence of mathematical steps, outputs the correct precession rate. But they also add that an explanation relying also on intuition would be welcome.

The mathematical complexity of the standard approach has stimulated many authors to search for alternative procedures based on geometrically-oriented descriptions; these usually link the Foucault precession to the curvature of the Earth and to parallel transport [8–13]. As far as I know, these alternative roads are never reported in textbooks and are variously disseminated in physics journals [8–10, 12] and on the internet [11, 13]. These explanations certainly disclose the brilliant geometric visualization capabilities of the authors, but they also show a weak point: they are often based on the choice of a large closed path on the curved Earth’s surface along which the parallel transport is carried out. This procedure may generate the impression that the Foucault precession depends on some property of the area enclosed by the path, whereas the pendulum responds only to local dynamical inputs as is also shown in standard texts of classical mechanics. In my opinion, another drawback of geometrical explanations is that their pedagogical efficiency is rather low, owing to the non-negligible effort required to follow the details of these methods.

My hope is that, by focusing our attention on the instantaneous motion of the suspension point and on the vector additivity of infinitesimal rotations, the explanation of the FPP will
become less rigorous but more intuitive, so as to be presented with success to freshmen of science and engineering.

2. The oscillations of a plumb line

This work rests on two pillars, the first of which is a home-made experiment. Take a cylindrical can (like an emptied tomato sauce container) and a plumb line (more or less 1 m long); then make a small central hole in the bottom of the can, pass the free end of the wire through the hole and stop it by making a knot just inside the can. The knot becomes the suspension point of the pendulum (figure 1).

Now let us hold the can upright, keeping our arms extended horizontally in front of us and start the bob oscillating sideways; then let us begin to rotate the can around its axis of symmetry, rolling it slowly and smoothly between the palms of our hands. We can easily note that, in the lab’s frame, the orientation of the plane of oscillation of the wire is unaffected by the rotation of the can. As we will see later, this simple instructional experiment is a basic step towards a physical understanding of the Foucault precession. To this end, it is useful to save the above result as it is judged in the frame of the rotating can: the oscillation plane of the pendulum appears to precess through an angle equal (and opposite) to the one we manually imparted to the can itself.

For the benefit of our students we might give an intuitive physical description of the experiment in the lab frame: since the axis of symmetry of the can is aligned with the local gravity field, the rotation of the can does not exert any moment on the pendulum and is not dynamically coupled to it. The pendulum therefore keeps swinging in its original vertical plane. Let us note that this result does not require any calculation: considerations of symmetry are sufficient to explain it. It is also significant to recall that this description is conceptually identical to that given in most texts of classical mechanics to explain the special case of a pendulum located at the Poles; not by chance, these are the only places on the Earth where the direction of the gravity field is parallel to the polar axis. At first sight the only apparent difference between the two cases is the following: in the instructional experiment of figure 1 we are manually rotating the can fast enough to neglect the concurrent diurnal motion of the Earth, so that the precession of the pendulum relative to the can is observed in real time. In the case of the pendulum located at the Pole, instead, the rotation of the point of suspension is
driven by the diurnal motion of the Earth, whose angular speed is so low that several tens of minutes are needed clearly to visualize the precession.

At this point, encouraged by the explanation given for the polar pendulum, we might hope to mandate the diurnal motion of the Earth to rotate the point of suspension of our instructional pendulum; and we try to do this by gluing the upper rim of the can to the ceiling (and then waiting for the Earth to rotate).

Unluckily, our project starts off on the wrong foot. There is an irreducible difference, in fact, between our manual rotation of the can of figure 1 and the rotation of the can fastened to the ceiling driven by the Earth; but this difference is in no way related to the fact that the angular speeds of the two rotations are hugely unbalanced. The key point is that the can of figure 1 is turned by us about the local vertical axis, while the can glued to the ceiling is turned by the Earth about the polar axis. Now, at any point of the Earth, the angle between these two axes is equal to the complement of latitude; only at the two Poles is this angle is zero, confirming the special position of these locations when discussing the Foucault problem.

Our next goal, then, must be to disclose how the motion of the can glued to the ceiling is mathematically related to the diurnal rotation of the Earth. Even though, at first sight, the details of this link may not appear to be immediate, we will be aided by some elementary notions of astronomy, usually well known to college students and amateur astronomers. Let us recapitulate: the can is glued to the ceiling, which is rigidly coupled to the floor, which in turn is part of the horizontal plane. Therefore, the motion of the can is the same as the motion of the local horizontal plane. And it is here that elementary astronomy comes into the foreground to explain the details of how the horizontal plane is moving.

Sooner or later, virtually all people have run into a photograph like that shown in figure 2. It represents a long exposure of the portion of sky centered on the Celestial Pole, i.e. very near to the Pole Star.

In our reference frame the starry sky rotates counter-clockwise at an angular speed \( \Omega_P = 2\pi/T_P \), where \( T_P \) is the duration of the sidereal day (corresponding to a complete rotation of the Earth with respect to the ‘fixed stars’).
Now let us consider figure 3, which shows the essential astronomical variables judged by a hypothetical observer located at the center $C$ of the Earth. The suffixes $P$, $V$, $N$ attached to the vector symbols $\vec{\Omega}(i = P, V, N)$ stand, respectively, for polar, vertical and north.

Physically, the principal ‘vector’ in figure 3 is, of course, $\vec{\Omega}_P$. For now, $\vec{\Omega}_V$ and $\vec{\Omega}_N$ are merely the ‘vector’ components of $\vec{\Omega}_P$ along, respectively, the vertical axis and the north axis; their physical meaning is clarified below. We recall that the decomposition of a vector, in our case $\vec{\Omega}_P$, along two axes leaves a large room for free choices; since, in our problem, the fundamental physical direction is established by the vertical gravity field, a natural choice is to decompose the vector $\vec{\Omega}_P$ in the plane defined by the vertical axis and of course by $\vec{\Omega}_P$ itself, thereby assigning to $\vec{\Omega}_V$ the role of the second orthogonal axis. With this choice the direction of the ‘north axis’ in figure 3 will be parallel to the direction of the north cardinal point in the pendulum lab.

3. A crucial doubt: is it correct to decompose an angular rotation ‘vector’ along two directions?

It is well known that two-dimensional rotations (2DRs) in the $x$-$y$ plane are additive in the sense that the result of two simultaneous or successive rotations $\theta_1$ and $\theta_2$ about the $z$-axis is equivalent to a rotation $(\theta_1 + \theta_2)$ about the same axis. A consequence of this additivity is that two 2DRs also commute, i.e. the order in which they are applied is immaterial. When we shift to three-dimensional rotations (3DRs), commutativity no longer holds, as is illustrated in all courses of classical mechanics. For example, if we rotate a book successively around two different axes, it is manifest that the book’s final orientation depends on the order of the rotations. Now, this absence of commutativity means that 3DRs cannot be handled as true vectors (that is why rotational mechanics in space uses complex tools such as non-commuting matrices, Euler angles or quaternions). Luckily, the good news for our problem is that, neglecting higher order terms, infinitesimal 3DRs about different axes are additive and can be treated as true vectors; this entails that angular velocities are also additive. The fact that infinitesimal rotations can be treated as true vectors is clearly illustrated in [14] (which shows the rotation of a book); to be true, this explanation is certainly intuitive but it does not prove
the essential fact that the neglected terms are infinitesimals of higher order. A complete proof, easily comprehensible because it requires only the basics of vector analysis, is given, for example, in Becker’s text [4], from which we quote these lines: ‘the angular velocity is a vector which has the instantaneous direction of the infinitesimal rotation, has the magnitude of the rotation divided by the scalar $dt$, and obeys the parallelogram law of vector addition. If referred to a set of rectangular coordinate axes, the angular velocity $\Omega$, like any other vector, may be expressed in terms of its components along these axes.’

Now, with the previous statement in mind, let us go back to figure 3 and ask: what are the physical meanings of $\Omega_V$ and $\Omega_N$? First of all, we confine ourselves to the infinitesimal rotations, $d\theta_p, d\theta_V$ and $d\theta_N$, that take place in an infinitesimal time $dt$. Then the quantities

$$
\begin{align*}
d\theta_p &= \Omega_p \cdot dt \\
d\theta_V &= \Omega_V \cdot dt \\
d\theta_N &= \Omega_N \cdot dt
\end{align*}
$$

 denote infinitesimal rotations performed about the three mentioned axes in the interval $dt$, and are represented by vectors along the same axes. If we denote the local latitude by $\varphi$, the additivity quoted above authorizes us to handle these rotations like ordinary vectors, i.e.

$$
\begin{align*}
d\theta_V + d\theta_N &= d\theta_p \\
d\theta_V &= d\theta_p \cdot \sin (\varphi) \\
d\theta_N &= d\theta_p \cdot \cos (\varphi).
\end{align*}
$$

The importance of these equations for the problem at hand should be clear. If we perform a rotation $d\theta_V$ about the vertical axis and another one $d\theta_N$ about the north axis, related by the previous equations, they are equivalent to a rotation $d\theta_p$ around the polar axis. And the reverse operation is also valid. Finally, let us investigate more closely how $d\theta_N$ and $d\theta_V$ would act separately on the pendulum.

Let us start (see figure 3) with the rotation around the horizontal axis $\Omega_N$ stating beforehand that, in my opinion, most readers will appreciate more easily an intuitive description given in the lab frame. Then, if $\Omega_N$ acts alone, at each instant the can glued to the ceiling is subjected to a centrifugal acceleration $R^2 \Omega_N$ where $R$ is the Earth’s radius [15]. This acceleration acts along the local vertical axis (i.e. along the gravity field) and therefore it can only change the period of oscillation of the pendulum without altering the orientation of the oscillation plane. This also explains at once why no Foucault precession is detectable at the Equator: simply because in this case the horizontal north axis of figure 3 coincides with the polar axis.

We are now at the finish line. Since the horizontal component $\Omega_N$ does not affect the orientation plane of the pendulum, we are left only with the contribution of the vertical component, i.e. the one that constrains the can fastened to the ceiling to rotate at angular velocity $\Omega_V$ along its axis of symmetry. Recalling the comments following figure 1, we conclude that in the interval $dt$ the orientation of the pendulum plane as seen in the lab’s frame lags behind the rotation of the suspension point by the infinitesimal angle $d\theta_V$, whose magnitude is given by

$$
\begin{align*}
d\theta_V &= \Omega_V dt = \Omega_p \cdot \sin (\varphi) dt.
\end{align*}
$$

Now, since the angular speed of the Earth is constant, this lag $d\theta_V$ is the same in any interval of length $dt$ during a whole day and depends only on the latitude of our lab, i.e. on the angle that the gravity field makes with the equatorial plane. This confirms that the nature of the Foucault precession is local and, therefore, need not be derived using large closed paths on the Earth’s surface. The total lag $\Delta \theta_V$ of the plane of oscillation in one sidereal day is immediately obtained by setting $dt = TP$ in the previous equation:

$$
\Delta \theta_V = \Omega_p \sin (\varphi) TP = 2\pi \sin (\varphi).
$$
It may be interesting to recall that the Earth is approximately an ellipsoid distorted by local geophysical anomalies so that, in geodesy, three different kinds of latitude are defined at each point $P$: geocentric, geographic and astronomic [16, 17]. These indicate the angles that three different lines passing through $P$ form, respectively, with the equatorial plane. The first line (geocentric latitude) goes through the center of the Earth; the second line (geographic latitude) is normal to the surface; the third line (astronomic latitude) coincides with the direction of gravity, i.e. with the equilibrium position of the pendulum. Hence, our discussion shows that the natural variable appearing in the Foucault equation is, strictly speaking, astronomic latitude.

An interdisciplinary observation may usefully complete this presentation. The vector composition of two very small rotations about different axes is not a mathematical curiosity: it is actually practised, for example, by amateur astronomers who employ an altazimuth telescope mount. As depicted by the sketch in figure 4, the telescope can be independently rotated about the vertical axis to vary the horizontal direction (azimuth) and about the horizontal axis to vary the vertical direction (altitude).

Suppose that, at some instant, a certain star is exactly centered in the field of view of the telescope. After a small interval, $dt$, the star, owing to the Earth’s rotation, appears to have migrated (by a very small amount) in a direction that, in general, is neither purely horizontal nor purely vertical. In order to bring the star back to the center of the field, the operator performs two very small rotations: one, $d\theta_V$, about the vertical axis and one, $d\theta_N$, about the horizontal axis. In modern telescopes, while the Earth rotates, a computerized system automatically performs this operation in real time, but the conceptual basis is the same as discussed before: an infinitesimal rotation about the polar axis can be obtained by composing two infinitesimal rotations, one about the vertical axis and one about the horizontal axis.

Figure 4. A rough sketch of an altazimuth amateur telescope.
4. Summary

A short summary in five steps may illustrate the conceptual simplicity of the method:

1. Referring to figure 1 we note that, in the lab frame, the plane of oscillation of the pendulum does not change when the can is rotated about its vertical axis.

2. Three-dimensional finite rotations about different axes cannot be handled as vectors, but infinitesimal rotations can (and so do angular velocities).

3. The angular velocity $\Omega_P$ of the Earth about its polar axis can be decomposed into two angular velocity components: the first, $\Omega_N$, of magnitude $\Omega_N = \Omega_P \cdot \cos(\varphi)$, parallel to the horizontal north axis and the second, $\Omega_V$, of magnitude $\Omega_V = \Omega_P \cdot \sin(\varphi)$, aligned with the local vertical axis.

4. The component $\Omega_N$ generates a small vertical acceleration that alters only the magnitude (but not the direction) of the local gravity and cannot produce a rotation of the oscillation plane of the pendulum. (A supplementary and useful comment about this statement is described after point 5.)

5. As a consequence of the previous point, and in accordance with the description related to figure 1, only the component $\Omega_V$ is responsible for the Foucault precession rate.

A supplementary comment

A careful reviewer has called attention to a possible critique that could be posed to the mathematical consistency of the method. The objection runs as follows. During each oscillation cycle the vertical coordinate of the bob varies from a minimum at the point of equilibrium up to a maximum at the moment of reversing its motion (as in all pendulums, there is a continuous trade between kinetic and gravitational energy). This means that the bob velocity has a non-null variable vertical component, $V_N$, with the consequence that the Coriolis force $V_N \times \vec{V}$ associated with the infinitesimal rotation $\Omega_N$ is horizontal and directed in the east–west direction.

For completeness of exposition and for the readers’ clarity I present some answers to this objection. The first is that all university texts that discuss the Foucault precession (see e.g. [3] and [4]) assume at the outset the small angle approximation according to which (up to infinitesimals of higher order) the motion of the bob is restricted to the horizontal plane (and it would be probably excessive for our intuitive and pedagogical approach to place conditions stricter than those used in the specialized literature). At this point, if we accept that the motion is practically horizontal, the vector $\Omega_N \times \vec{V}$ is practically vertical and it cannot alter the orientation of the oscillation plane.

But I think it is possible to understand why the effect of the vertical component of the bob’s velocity can be neglected, using a reasoning in line with the pedagogical approach of the paper.

Let us make recourse to the logic of gedanken, or thought experiments, by progressively shifting the motion of the pendulum toward a small angle condition. For this purpose:

(1) We first imagine to increase the length of the pendulum by a factor $K$; and in so doing (2) we arrange things so that the maximum amplitude of the bob motion is not allowed to increase (in other words we keep confining the motion of the bob within the same circular space in the lab or museum); oscillations become slower and all speeds decrease.

Energy conservation shows that the height of the bob at the turning points of the oscillations is inversely proportional to $K$; and that the maximum speed of the bob is inversely proportional to $\sqrt{K}$. Thus, by increasing the value of $K$, the magnitude of the term $\Omega_N \times \vec{V}$
can be made smaller than any preselected value, while the precession rate due to the Foucault term, \( \Omega_p \cdot \sin(\varphi) \), is unaffected by an increase of the pendulum length.

A different comment might also be added. When the bob swings towards the turning point (i.e. maximum height), its vertical component velocity is directed upwards; when the bob swings away from the turning point, the vertical component of the velocity is directed downwards. Owing to the fact that \( N_W \) never reverses its direction, it is easy to see that at each point of a half cycle the horizontal force due to the Coriolis term \( V_N \times \dot{V}_W \) imparts two small, equal and opposite pushes (this process may only produce imperceptible rosetta-like leaves).

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