Lefschetz Fixed-Point Theorem and Lattice Points in Convex Polytopes*

Sacha Sardo Infirri †

Mathematical Institute
24-29 St. Giles'
Oxford

Abstract

A simple convex lattice polytope □ defines a torus-equivariant line bundle $L_□$ over a toric variety $X_□$.

Atiyah and Bott's Lefschetz fixed-point theorem is applied to the torus action on the $d''$-complex of $L_□$ and information is obtained about the lattice points of □.

In particular an explicit formula is derived, computing the number of lattice points and the volume of □ in terms of geometric data at its extreme points. The results of Brion [5] are recovered and an elementary convex geometric interpretation is given by performing Laurent expansions similar to those of Ishida [14].

0 Introduction

0.1 The problem

Let □ be a convex polytope all of whose vertices belong to a lattice $M$. The question of calculating the number of points of $M$ contained in □ is a well-known one in convex geometry. The oldest formula appears to be Pick’s classical result [18], valid for arbitrary polygons in 2 dimensions:

$$
\#(□ \cap M) = \text{Area}(□) + \frac{1}{2} \#(\text{boundary}(□) \cap M) + 1.
$$

* M.S.Classification (1991): Primary: 52B20; Secondary: 14M25, 11H06, 11P21.
† email:sacha@maths.oxford.ac.uk
Following Ehrhart’s work on Hilbert polynomials, Macdonald \[14, 15\] subsequently generalised Pick’s formula to arbitrary \( n \). His formula expresses the volume of \( \square \) in terms of the number of lattice points of its’ multiples \( k \square \) for finitely many integers \( k \). Although these formulae are valid for multiples (non-convex) polygons, they do not give any convenient way of calculating either the volume, or the number of lattice points of \( \square \). A review of this and other problems concerning lattice points can be found in \[8, 7\].

From an elementary point of view, for large polytopes one expects the volume to be a good approximation to the number of lattice points, so that one can imagine a general formula of the form

\[
\text{number of points} = \text{volume} + \text{correction terms} \tag{1}
\]

where the corrections terms are negligible in the large limit. The formula we present here is however quite different in nature.

### 0.2 The results

Given a parameter \( \zeta \), to each extreme point \( \alpha \) of a simple convex polytope \( \square \), we associate a rational number depending the local geometry of \( \square \) at \( \alpha \). Their sum is independent of \( \zeta \) and yields the number of lattice points in \( \square \) (Theorem 3). Our main formula (Theorem 6) is more general, since it expresses not just the number, but \textit{which} points of the lattice belong to the polytope, as a finite Laurent polynomial in \( n \) variables (the lattice points corresponding to the monomials via \( m \mapsto x^m = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \)).

I give an initial form of this using the Lefschetz-fixed point theorem for orbifolds. By expanding in Laurent series this is shown to be equivalent to another formulation (Theorem 7) given by Brion \[5\] which doesn’t involve cyclotomic sums. I use this form to calculate the number of lattice points. The volume of \( \square \) is obtained by taking the leading order terms for finer and finer subdivisions of the lattice. The Laurent series expansions extend Ishida’s \[10\] and provide a convex geometric interpretation of the formula (Theorem 8). This in turn suggests a proof of the formula involving \textit{no} toric geometry — only convex geometry and elementary Laurent expansions. This could be considered as a variation of Ishida’s proof \[10\] based on the contractibility of convex sets.

This paper is an amplification of my 1990 transfer dissertation at Oxford university \[20\]. This was originally written whilst I was unaware of Michel Brion’s 1988 paper \[4\], where a toric approach is used to calculate the number of lattice points. There has also been a paper by Ishida \[10\]
where similar Laurent expansions similar to mine are performed. This is
the revised version of my original which takes these works into account. Let
me briefly mention their relationship to this paper.

Brion relies on the Lefschetz-Riemann-Roch theorem for equi-
variant K-theory \cite{3} and obtains theorem \cite{1}. He calculates the number of lattice points
by subdividing the tangent cones into basic cones. The formula that I obtain
using the Lefschetz fixed point theorem involves instead cyclotomic sums
for the action of the finite quotient group. By extending Ishida’s Laurent
series expansions \cite{10} in section 5 of this article, I prove that the two are
equivalent, and provide a combinatorial interpretation of the formula. It is
also not necessary for me to subdivide the tangent cones in order to obtain
a formula for the number of lattice points.

0.3 The method

Our main tool is the theory of toric varieties. This associates a holomor-
phic line bundle $L_\square$ over a complex orbifold $X_\square$ to any $n$-dimensional simple
polytope $\square$ on a lattice $M$. The variety comes equipped with the action of
an algebraic $n$-torus $T_N$ (the character group of $M$) and $L_\square$ is equivariant
with respect to this action. Its cohomology is trivial in positive dimen-
sion, whereas its space of sections is naturally isomorphic to a vector space
generated by the lattice points in $\square$.

In \cite{4,13}, the Rieman-Roch theorem is used to calculate the number of
lattice points in $\square$. This yields a formula similar to equation (1) above. The
problem with this approach, however, is that the correction terms are not
readily computable.

In this paper I follow an idea of Atiyah and exploit the torus action. I
apply Atiyah & Bott’s Lefschetz fixed point theorem \cite{1} — suitably extended
to orbifolds \cite{11} — to the (geometric endomorphism induced by the) action
of $t \in T_N$ on the $d^\prime$-complex of $L_\square$. The $d^\prime$-complex is elliptic \cite{2} and its
cohomology groups are (canonically isomorphic to) those of $(X_\square, L_\square)$. The
fixed points of the torus action on $X_\square$ correspond to the extreme points of
$\square$. The Lefschetz theorem in this case expresses the equality between the
Lefschetz number (an element of $\mathbb{C}[M]$) and the sum of the indexes $\nu_\alpha$ for $\alpha$
in the set of extreme points. The $\nu_\alpha$ define elements of $\mathbb{C}(M)$. The formula I
obtain initially involves sum over the characters of the finite abelian groups
which characterise the singularities at the points $P_\alpha \in X_\square$ corresponding to
$\alpha \in \text{ext } \square$. By studying characteristic series for cones in section 5 I eliminate
the summation over group elements.
If one restricts $t$ to the one-parameter subgroup of the torus determined by an element $\zeta$ of its’ Lie algebra one obtains an equality between a polynomial and a sum of rational functions in one variable. When $t \to 1$ the polynomial tends to the the number of lattice points of $\square$, and this is given by the sum of the constant terms in the one variable Laurent series for the rational functions: this gives theorem 9. By identifying the coefficient of the leading order terms in the asymptotic expansions of the formula for sub-multiples of the lattice — the ‘classical limit’ in quantum terminology — I derive a formula for the volume of $\square$ in Theorem 10.

I review the toric geometry results I shall need in the first part of this paper. The reader who is familiar with the notation in Oda [17] can GOTO PART II, which contains the application proper.

0.4 Acknowledgments

I would like to thank Michael Atiyah and Peter Kronheimer for their stimulating ideas and encouraging support. Thanks also to Frances Kirwan for her suggestions and to Mark Lennsen, Jorgen Andersen and Jorge Ramirez-Alfonsin for interesting discussions. I was supported by a Rhodes Scholarship while I did this research.

0.5 Notation

Throughout this paper, let $N \cong \mathbb{Z}^n$ denote an n-dimensional integral lattice, $M \cong \text{hom}_\mathbb{Z}(N, \mathbb{Z})$ it’s dual and $N_\mathbb{R} = N \otimes \mathbb{R}$ its’ associated real vector space. The complex torus $N \otimes \mathbb{C}^\times \cong \text{hom}_\mathbb{Z}(M, \mathbb{C}^\times)$ is denoted $T_N$ and the compact real sub-torus $N \otimes S^1 \subset N \otimes \mathbb{C}^\times$ is denoted $CT_N$.

If $A$ is any commutative ring with identity and $S$ any additive semi-group, we write $A[S]$ for the group algebra of $S$ with coefficients in $A$; this is generated by elements $e(s)$ for $s \in S$ satisfying the relations $e(s)e(s') = e(s+s')$. We write $A(S)$ for its total quotient ring (i.e., its’ field of fractions if $A = \mathbb{C}$).

Occasionally I choose coordinates $t_i$ for $T_N$. This is equivalent to choosing generators $n_i$ for $N$. I denote the dual generators by $m^j \in M$. Then if $\alpha \in M$ and $z \in T_N$ have coordinates $(\alpha_1, \ldots, \alpha_n)$ and $(z_1, \ldots, z_n)$ with respect to the appropriate bases we have

$$\alpha(z) = z^\alpha = z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_n^{\alpha_n}.$$ 

This identifies $\mathbb{C}[M]$ with the Laurent polynomials in the variables $t_i$. 

4
Part I
Toric Geometry

The theory of toric varieties establishes correspondences between convex geometry in $n$ real dimensions and the geometry of compactifications of $n$-dimensional complex tori. I refer to [12, 17, 6].

Briefly, there is a functor that associates, to a pair $(N, \Sigma)$ (where $\Sigma$ is a fan in $N$), an irreducible normal Hausdorff complex analytic space $X_{N,\Sigma}$. A convex polytope $\square$ in $M$ determines a unique fan $\Sigma$ in $N$, and we set $X_{\square} = X_{N,\Sigma}$. The polytope contains more information than simply its cone structure, and this determines a piecewise linear function $h = h_{\square}$ on the support $|\Sigma| \subset N_\mathbb{R}$ of $\Sigma$. This corresponds under the functorial construction above to an equivariant line bundle $L_h$ on $X_{N,\Sigma}$, which we denote by $L_{\square}$.

1 Cones and Affine Toric Varieties

1.1 Cones

Let $V$ denote a vector space and $V^*$ its dual.

A cone in a vector space $V$ is a finite intersection of half-spaces in $V$. Cones are always convex and polyhedral. I shall take them to be also strongly convex, namely such that they do not contain any proper subspace of $V$.

For $v_1, \ldots, v_k \in N_\mathbb{R}$, let $\langle v_1, \ldots, v_k \rangle$ denote the smallest cone containing $v_1, \ldots, v_k$. Any cone is generated in this way. A cone is said to be simplicial if it can be generated by linearly independent elements of $N_\mathbb{R}$. If it can be generated by part of a $\mathbb{Z}$-basis of $N$, then the cone is called basic. Finally, a cone is said to be integral with respect to $N$ if it can be generated by elements of $N$. When we speak of a cone in a lattice $N$ we mean a cone in $N_\mathbb{R}$ which is integral with respect to $N$. I only consider such cones henceforth.

The dimension of a cone is the dimension of the subspace it generates. By the interior of a cone we usually mean the relative interior in the subspace it generates.

1.2 Duality

Given a subset $A \subset V$ its dual $A^\vee \subset V^*$ is defined by:

$$A^\vee = \{ \theta \in V^* : \forall v \in V, \langle \theta, v \rangle \geq 0 \}. $$
Proposition 1 The dual of a cone (respectively, a simplicial cone, a basic cone, or an integral cone) is a cone (respectively a simplicial cone, a basic cone, or an integral cone). Moreover, for any cone \( \sigma \) we consider, we have \( \sigma^\vee \vee = \sigma \).

For a proof of all the results regarding cones, see [19]. A summary of the results I require will be found in [17].

1.3 Affine Toric Varieties

Let \( \sigma \) be a cone in \( N \). Recall [17, Prop. 1.1] that the subset of \( M \) given by

\[
S_\sigma = M \cap \sigma^\vee
\]

is finitely generated as an additive semigroup, generates \( M \) as a group, and is saturated. Such semigroups are in one-one correspondance with cones in \( N \).

Denote by \( U_\sigma = U_{N,\sigma} \) the set of semigroup homomorphisms from \( (S_\sigma, +) \) to \( (\mathbb{C}, \cdot) \), namely

\[
U_\sigma = \{ u : S_\sigma \to \mathbb{C} : u(0) = 1, u(m + m') = u(m)u(m'), \forall m, m' \in S_\sigma \}.
\]

This can be given the structure of an \( n \)-dimensional irreducible normal complex analytic space by choosing generators \( m_1, \ldots, m_p \) for \( S_\sigma \) and embedding \( U_\sigma \) in \( \mathbb{C}^p \) via the evaluation maps \( \text{ev}(m_i) : u \mapsto u(m_i) \) on the generators \( m_i \).

The structure is inherited from the usual structure on \( \mathbb{C}^p \) and is independent of the generators chosen.

In other words, \( U_\sigma \) is just equal to the (set of points of the) affine scheme \( \text{Spec}(\mathbb{C}[S_\sigma]) \). Identifying \( U_\sigma \) with its \( \mathbb{C} \)-points corresponds to identifying \( \text{ev}(m) \) with \( e(m) \). I spend little effort making the distinction. The following proposition is easy to show [17, Th. 1.10]:

Proposition 2 The variety \( U_\sigma \) is non-singular if and only if \( \sigma \) is basic.

2 Fans and General Toric Varieties

2.1 Faces, Fans and Gluing

Let \( \sigma \) be a cone in \( N \).

Definition 1 A face of \( \sigma \) is a subset of the form \( \sigma \cap \{m_0\}^\perp \), where \( m_0 \in M = \text{hom}(N, \mathbb{Z}) \) is non-negative on \( \sigma \). A face of a cone is also a cone.
We immediately have:

**Lemma 1** If \( \tau \) is a face of \( \sigma \) then, for some \( m_0 \in M \), we have

\[
U_\tau = \{ u \in U_\sigma : u(m_0) \neq 0 \},
\]
so that \( U_\tau \) is naturally an open subset of \( U_\sigma \).

Given this, one constructs collections of cones (called fans) which have the property that their corresponding varieties fit together in a natural way:

**Definition 2** A fan in \( N \) is a collection \( \Sigma = \{ \sigma : \sigma \) a cone in \( N \} \) satisfying the following conditions:

- if \( \tau \) is a face of \( \sigma \) and \( \sigma \in \Sigma \), then \( \tau \in \Sigma \).
- \( \sigma \cap \sigma' \) is a face of both \( \sigma \) and \( \sigma' \), for all \( \sigma, \sigma' \in \Sigma \).

The set of cones of \( \Sigma \) of dimension \( k \) is called the \( k \)-skeleton of \( \Sigma \) and is denoted \( \Sigma^{(k)} \). The union of all the cones of \( \Sigma \) is called the support of \( \Sigma \) and is denoted \( |\Sigma| \subset N_{\mathbb{R}} \).

**Theorem 1** The toric variety associated to \( (N, \Sigma) \) is the space obtained by gluing together the affine varieties \( U_{N, \sigma} \) for \( \sigma \in \Sigma \), using lemma 4. It is an \( n \)-dimensional Hausdorff complex analytic space \( X_{N, \Sigma} \) which is irreducible and normal [17, Theorem 1.4]. It is compact if and only if \( \Sigma \) is complete, namely if and only if \( |\Sigma| = N_{\mathbb{R}} \).

### 2.2 The torus action

The torus \( T_N \) acts on \( U_\sigma \) by \( (t \cdot u)(m) = t(m)u(m) \), and this gives an action on \( X_{N, \Sigma} \). For \( \sigma = \{0\} \), one has \( U_{\{0\}} = T_N \), and the action coincides with group multiplication on the torus.

The \( T_N \)-orbits on \( X_{N, \Sigma} \) are given by the quotient algebraic tori

\[
\text{orb}(\tau) = \text{hom}_{\mathbb{Z}}(M \cap \tau^\perp, \mathbb{C}^\times),
\]

for each \( \tau \in \Sigma \). The orbit corresponding to \( \tau \) has dimension equal to the codimension of \( \tau \) in \( N_{\mathbb{R}} \). It is also easy to see that \( U_\sigma \) decomposes as the disjoint union of the orbits corresponding to its faces, and that \( \text{orb}(\sigma) \) is the only closed orbit in \( U_\sigma \). I record a special case of this for later use:

**Lemma 2** The fixed points of the \( T_N \) action on \( X_{N, \Sigma} \) are in one- one correspondence with the orbits \( \text{orb}(\sigma) \in U_\sigma \), for the cones \( \sigma \) in the \( n \)-skeleton \( \Sigma^{(n)} \).
2.3 Functoriality

Recall the following characterisation of toric varieties:

\[ X \text{ is a toric variety if and only if it is an irreducible normal variety, locally of finite type over } \mathbb{C}, \text{ with a densely embedded torus whose action on itself extends to the whole variety.} \]

The assignment \( (N, \Sigma) \mapsto X_{N, \Sigma} \) is a functor of categories:

**Definition 3** A map of fans \( \phi : (N', \Sigma') \rightarrow (N, \Sigma) \) is a \( \mathbb{Z} \)-linear homomorphism \( \phi : N' \rightarrow N \) whose scalar extension \( \phi_R : N'_R \rightarrow N_R \) satisfies the following property: for each \( \sigma' \in \Sigma' \), there exists \( \sigma \in \Sigma \) such that \( \phi_R(\sigma') \subset \sigma \).

**Theorem 2** [4, page 19] A map of fans \( \phi : (N', \Sigma') \rightarrow (N, \Sigma) \) gives rise to a holomorphic map

\[ \phi_* : X_{N', \Sigma'} \rightarrow X_{N, \Sigma} \]

whose restriction to the open subset \( T_{N'} \) coincides with the homomorphism of algebraic tori \( \phi_{\mathbb{C}} : N' \otimes_{\mathbb{Z}} \mathbb{C}^\times \rightarrow N \otimes_{\mathbb{Z}} \mathbb{C}^\times \). Through this homomorphism, \( \phi_* \) is \( (T_{N'}, T_N) \)-equivariant. Conversely any holomorphic map \( \psi : X' \rightarrow X \) between toric varieties which restricts to a homomorphism \( \chi : T' \rightarrow T \) on the algebraic tori \( T' \) and \( T \) in such a way that \( \psi \) is \( \chi \)-equivariant corresponds to a unique \( \mathbb{Z} \)-linear homomorphism \( f : N' \rightarrow N \) giving rise to a map of fans \( (N', \Sigma') \rightarrow (N, \Sigma) \) such that \( f_* = \psi \).

2.4 Finite Quotients

I will be interested in the case when \( N' \) is a \( \mathbb{Z} \)-submodule of \( N \) of finite index and \( \Sigma' = \Sigma \). I write \( X' \) and \( X \) for the corresponding varieties:

**Proposition 3** With the data as above, \( X' \rightarrow X \) coincides with the projection of \( X' \) with respect to natural action of the finite group

\[ K = N/N' \cong \text{hom}_\mathbb{Z}(M'/M, \mathbb{C}^\times) = \ker[T_{N'} \rightarrow T_N]. \]

**Proof:** [17, Cor. 1.16, p.22]
3 Toric Varieties, Equivariant Line Bundles and Convex Polytopes

3.1 Polytopes

Recall first some basic notions of convex geometry.

A convex polytope \( \Box \) in a vector space \( V \) is a bounded intersection of a finite number of affine half-spaces of \( V \). The set of extreme points of \( \Box \) is denoted \( \text{ext} \ Box \). Since \( \Box \) is bounded, it is equal to the convex hull of \( \text{ext} \ Box \).

By a polytope on the lattice \( M \) we mean a polytope in \( M_\mathbb{R} \) such that \( \text{ext} \ Box \subset M \). Suppose \( \Box \) is such a polytope, and let \( \alpha \) be an extreme point. I define the (tangent) cone of \( \Box \) at \( \alpha \) to be the cone \( C_\alpha \) in \( M \) given by:

\[
C_\alpha = \mathbb{R}_{\geq 0}(\Box - \alpha) = \{ r(v - \alpha) : r \geq 0, v \in \Box \}.
\]

Let \( \lambda^i_\alpha, i = 1, \ldots, k \) be the shortest generators for \( C_\alpha \) which belong to the lattice \( M \). I call these the edges of \( \Box \) emanating from \( \alpha \), or simply the edge vectors for \( \Box \) at \( \alpha \). If \( C_\alpha \) is simplicial (respectively, basic), then \( \Box \) is called simple (respectively, basic) at \( \alpha \). Henceforth, all the polytopes we consider are convex, integral and simple at all extreme points. They may be non-basic.

3.2 Toric Varieties Defined by Polytopes

3.2.1 The Fan Defined by a Polytope

The construction of \( C_\alpha \) described in the previous section can be generalised to show that a polytope \( \Box \) in \( M \) defines a complete fan in \( N \). To each face \( \Gamma \) of \( \Box \) we associate the cone \( C_\Gamma \) in \( M \) defined by

\[
C_\Gamma = \mathbb{R}_{\geq 0}(\Box - m_\Gamma),
\]

where \( m_\Gamma \) is any element of \( M \) strictly in the interior of the face \( \Gamma \). If \( F = \{ \alpha \} \) we set \( C_{\{\alpha\}} = C_\alpha \), as defined previously in equation (3). Taking duals one obtains a collection of cones in \( N \)

\[
\Sigma_\Box = \{ \sigma_\Gamma = C_\Gamma^\vee : \Gamma \text{ a face of } \Box \}.
\]

One has the following easy lemma:

**Lemma 3** \( \Sigma_\Box \) is equal to the fan consisting of the cones \( \sigma_\alpha = C_\alpha^\vee \), for \( \alpha \in \text{ext} \ Box \) and all their faces. It is complete, and its n-skeleton is \( \Sigma^{(n)} = \{ \sigma_\alpha : \alpha \in \text{ext} \ Box \} \).
3.2.2 The Variety Defined by a Polytope

I define \( X_{\Box} \) to be \( X_{\Sigma} \), for \( \Sigma = \Sigma_{\Box} \). By [17, Theorem 2.22], \( X_{\Box} \) is an orbifold (i.e., it has at worst quotient singularities) if \( \Box \) is simple.

**Proposition 4** The variety \( X_{\Box} \) is compact, and is covered by affine pieces

\[
U_\alpha = U_{\sigma_\alpha} = \text{Spec}(\mathbb{C}[M \cap C_\alpha]),
\]
for \( \alpha \in \text{ext} \Box \), each containing a unique \( T_N \)-fixed point \( P_\alpha = \text{orb} (\sigma_\alpha) \) (see equation (2)). Furthermore, when \( U_\alpha \) is non-singular, the weights of the \( T_N \) action on the tangent space \( T_{P_\alpha} U_\alpha \) are given by the edges vectors for \( \Box \) at \( \alpha \).

**Proof:** The first claim follows directly from theorem 1 and lemmas 2 and 3. For the second part, observe (prop. 2 and 1) that \( U_\alpha \) is non-singular if and only if the edge vectors at \( \alpha \) generate \( M \) as a group. The semigroup \( C_\alpha \) is then free on these generators. They correspond to the weights of \( T_N \) on \( U_\alpha \), and hence, by linearity, to the weights on \( T_{P_\alpha} U_\alpha \).

3.3 Equivariant Line Bundles

The polytope \( \Box \) contains more information than the fan \( \Sigma_{\Box} \). This extra information turns out to be exactly what one needs to specify a \( T_N \)-equivariant line bundle \( L_{\Box} \) over \( X_{\Box} \).

3.3.1 Line Bundles and Piecewise Linear Functions

In general (equivalence classes of) equivariant line bundles over \( X_{N,\Sigma} \) are in one-one correspondence with the space \( PL(N, \Sigma) \) of **piecewise linear functions** on \( (N, \Sigma) \), namely functions \( h : |\Sigma| \to \mathbb{R} \) that are linear on each \( \sigma \in \Sigma \) and which take integer values on the integer points of \( |\Sigma| \).

Defining an element \( h \in PL(N, \Sigma) \) involves, by definition, specifying an element \( l_\sigma \in M \) for each \( \sigma \in \Sigma \) such that \( h(n) = (l_\sigma, n) \) for all \( n \in \sigma \). These elements determine a line bundle \( L_h \) equipped with a \( T_N \)-action and whose projection \( L_h \to X_\Sigma \) is equivariant with respect to that action. Note that in general, the elements \( l_\sigma \) are not uniquely determined by \( h \), but different choices give rise to equivariantly equivalent bundles.

The bundle \( L_h \) is defined to be trivial over the varieties \( U_\sigma \), with transition functions given by

\[
g_{\tau\sigma}(x) = e(l_\sigma - l_\tau)(x).
\]
The action of $T_N$ on the piece $U_\sigma \times \mathbb{C} \subset L_h$ is defined by

$$t(x,c) = (tx, e(-l_\sigma)(t)c).$$

(5)

3.3.2 Cohomology

The cohomology groups for equivariant line bundles decompose under the action of $T_N$ into weight spaces, and can be expressed as a direct sum (see [17, Th. 2.6]):

$$H^q(X_\Sigma, \mathcal{O}_{X_\Sigma}(L_h)) = \bigoplus_{m \in M} H^q_{Z(h,m)}(N_R, \mathbb{C})e(m),$$

where $Z(h,m) = \{n \in N_R : \langle m, n \rangle \geq h(n)\}$, and $H^q_{Z(h,m)}(N_R, \mathbb{C})$ denotes the $q$-th cohomology group of $N_R$ with support in $Z(h,m)$ and coefficients in $\mathbb{C}$.

The Line Bundle $L_\square$ The polytope $\square$ defines a piecewise linear function $h_\square$ on $\Sigma_\square$ by putting $l_{\sigma_\alpha} = \alpha$ (and $l_\sigma = \alpha$ for the faces $\sigma$ of $\sigma_\alpha$). The corresponding bundle is denoted $L_\square$. Its cohomology is given by [17, Cor. 2.9]

$$H^q(X_\square, \mathcal{O}_{X_\square}(L_\square)) = \begin{cases} \mathbb{C}[M]_\square = \bigoplus_{m \in M \cap \square} \mathbb{C}e(m) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

(6)
Part II
The Polytope Formula

4 The Lefschetz Fixed-Point Theorem

Recall [2, Theorem 4.12] the following application of the Lefschetz fixed point theorem to the case of holomorphic vector bundles:

**Theorem 3** Let $X$ be a compact complex manifold, $F$ a holomorphic vector bundle over $X$, $f : X \to X$ a holomorphic map with simple fixed points and $\phi : f^*F \to F$ a holomorphic bundle homomorphism. Let $L(T)$ be the Lefschetz number of the endomorphism $T$ of the $d''$-complex of $F$:

\[ L(T) = \sum (-1)^q \text{trace} H^q T|_{H^q(X,F)}. \]

Then $L(T) = \sum_{P=f(P)} \nu_P$, where

\[ \nu_P = \frac{\text{trace}_C \phi_P}{\det_C(1 - df_P)}. \]

(Recall that since $P$ is a fixed point, $\phi_P$ and $df_P$ are endomorphisms of $F_P$ and $T_P X$ respectively.)

4.1 Application

I apply this to the case where $X = X_{\square}$, $L = L_{\square}$ and $f : X \to X$ is given by the action of a non-trivial element of $t \in T_N$. The fixed points are simple and are given by $P_\alpha = \text{orb}(\sigma_\alpha) \in U_{\sigma_\alpha}$, for $\alpha \in \text{ext } \square$. The bundle homomorphism $\phi_t : t^*L \to L$ is given by the action of $-t$ (recall that $T_N$ acts on line bundles). The cohomology groups are all zero, except $H^0(X_{\square}, L_{\square})$ which is isomorphic to the subspace $\mathbb{C}[M]_{\square}$ of $\mathbb{C}[M]$ determined by $\square$.

In this context, the Lefschetz number is an element of $\mathbb{C}[M]$ and the indexes $\nu_\alpha = \nu_{P_\alpha}$ are elements of $\mathbb{C}(M)$. (As we shall see in section 5 they are characteristic functions for the tangent cones to $\square$.)

**Lemma 4** We have

\[ \text{trace} (\phi_t)_{P_\alpha} = \alpha(t). \]

**Proof:** Recall (equation (3)), that $t$ acts on the fibres of $L$ over $U_{\sigma}$ by multiplication by $e(-l_{\sigma})(t)$, where $l_{\sigma}$ are the elements of $M$ corresponding to $L$ as in (3.3.1). In the present case, at a fixed point $P_\alpha \in U_{\alpha}$ we have $l_{\sigma_\alpha} = \alpha$, so $\phi_t$ acts by $e(-\alpha)(-t) = \alpha(t)$. ■
In the case of a basic polytope $\Box$ in $M$, applying Theorem 3 directly one obtains:

**Theorem 4** For a basic simple convex polytope $\Box$ in $M$, we have

$$\sum_{m \in \Box} m(t) = \sum_{\alpha \in \text{ext} \; \Box} \nu_\alpha(t) \quad (7)$$

where

$$\nu_\alpha(t) = \sum_{\alpha \in \text{ext} \; \Box} \frac{\alpha(t)}{(1 - \lambda^1_\alpha(t)) \cdots (1 - \lambda^n_\alpha(t))}. \quad (8)$$

the vectors $\lambda^1_\alpha, \ldots, \lambda^n_\alpha$ are the edge vectors of $\Box$ at $\alpha$.

**Proof:** The decomposition of $H^0(X_\Box; \mathcal{O}_{X_\Box}(L_\Box))$ given by equation (6) shows that the left-hand side of equation (7) is equal to the Lefschetz number of the endomorphism induced by the action of $t$. Lemma 4 and Proposition 4 yield equation (8). \(\blacksquare\)

### 4.2 The Lefschetz Fixed-Point Theorem for Orbifolds

In [11] the Lefschetz formula is generalised to orbifolds (also known as V-manifolds), using zeta-function techniques. As I do not need the full power of this approach, I present an alternative more elementary argument.

The Lefschetz fixed-point formula is essentially local in nature, the formula for the multiplicities $\nu_\alpha$ only involving the properties of $f$ and $\phi$ at the point $P_\alpha$. This fact is clearly apparent in Atiyah and Bott’s proof in [11] (see their remarks at the beginning of section 5, and Proposition 5.3). To extend the formula to orbifolds, it is sufficient therefore to extend it to global quotient spaces, of the form $X = X'/K$.

**Proposition 5** Suppose that a finite abelian group $K$ acts on a smooth $X'$ and equivariantly on a holomorphic bundle $F'$ over $X'$. Let $f' : X' \to X'$ and $\phi' : f'^* F' \to F'$ be as in Theorem 3, and suppose they are $K$-equivariant. Denote by $L'(T')$ the Lefschetz number of the corresponding endomorphism $T'$ of the $d'^*$-complex of $F'$. Because of the $K$-equivariance, we can define $X = X'/K$, $f : X \to X$, $F = (F')^K$, $\phi : f^* F \to F$ and the corresponding Lefschetz number

$$L(T) = \sum (-1)^q \text{trace } H^q(T)|_{H^q(X; F)}. $$

13
Then we have

$$L(T) = \frac{1}{|K|} \sum_{k \in K} L'(k \circ T).$$

(9)

Proof: Note that since $T$ determines an endomorphism of the primed complex, it makes sense to write $L'(T)$. The claim then follows by applying the following easy lemma of linear algebra, recalling that $H^q(X; F)$ is just the $K$-invariant part of $H^q(X'; F')$.

Lemma 5 Suppose we have a linear action of a finite abelian group $K$ on a finite dimensional vectorspace $V$, commuting with an endomorphism $T$ of $V$. Denote by $V^K$ the $K$-invariant subspace of $V$. Then $T$ is an endomorphism of $V^K$ and we have

$$\text{trace } T|_{V^K} = \frac{1}{|K|} \sum_{k \in K} \text{trace } (k \circ T)|_V.$$

Proof: Define $P$ to be the following endomorphism of $V$:

$$Pv = \frac{1}{|K|} \sum_{k \in K} k \cdot v.$$

Then $P^2 = P$, so $P$ is the projection $V \to V^K$. Since $T$ commutes with $P$, it follows that $T$ respects the decomposition $V = V^K \oplus \ker P$. Furthermore we have

$$\text{trace } T|_{V^K} = \text{trace } TP|_V = \text{trace } PT|_V,$$

so the result follows.

Now, given a general orbifold $X$, at each point $P \in X$, choose a local model $(U'_P, f'_P, K_P, L'_P)$ as follows:

Let $U_P$ be an $f$-invariant neighbourhood of $P$ in $X$ and $U'_P$ be a smooth cover with an action of a finite group $K_P$, free away from $P$, such that $U_P = U'_P/K_P$. Thus $X$ has a quotient singularity of type $K_P$ at $P$. Let $f'_P : U'_P \to U'_P$ be a $K_P$-equivariant lifting of $f|_{U_P}$. A line bundle $L$ over $X$ is understood to be an invertible sheaf $L$ over $X$ such that for any $P \in X$ with local model $(U'_P, f'_P, K_P)$, there exists a line bundle $L'_P \to U'_P$ such that $L|_{U_P} = (L'_P)^{K_P}$.

With these definitions, our remarks at the beginning of the section and Proposition 5 imply the following:
Theorem 5 Let $X$ be a compact complex orbifold, $F$ a holomorphic vector bundle over $X$, $f : X \to X$ a holomorphic map with simple fixed points and $\phi : f^*F \to F$ a holomorphic bundle homomorphism. Let $L(T)$ be the Lefschetz number of the endomorphism $T$ of the $d''$-complex of $F$:

$$L(T) = \sum (-1)^q \text{trace} H^q T|_{H^q(X; F)}.$$

Then $L(T) = \sum_{P=f(P)} \nu_P$, where

$$\nu_P = \frac{1}{|K_P|} \sum_{k \in K_P} \frac{\text{trace}(k \circ \phi'_P)}{\det C_k(1 - (k \circ df')_P)},$$

and $\phi', f'$ are lifts for $\phi, f$ respectively, in the same spirit as that of the local models above.

### 4.3 Singular Case

Suppose that $\square$ is not basic relative to $M$ at $\alpha$. Then $X = X_{\square}$ has a singularity at the point $P_\alpha$. Let $C_{\alpha}$ be the cone of $\square$ at $\alpha$ and let $\sigma_{\alpha}$ be the dual cone.

**Definition 4** I define the dual edge vectors for $\square$ at $\alpha$ to be the primitive generators of the cone $\sigma_{\alpha}$ in $N$. When $\sigma_{\alpha}$ is not basic, the dual edge vectors do not generate $N$ as a group, but instead a sublattice $N'_{\alpha}$ of $N$ of finite index, which I call the dual edge lattice for $\square$ at $\alpha$.

The cone $\sigma_{\alpha}$ is basic with respect to $N'_{\alpha}$, and the corresponding variety $X'_{\alpha} = X_{\sigma_{\alpha},N'_{\alpha}}$ is smooth at $P_\alpha$. By Corollary 3, the map $X'_{\alpha} \to X_{\alpha} = X_{\sigma_{\alpha},N}$ is the quotient map by the action of the finite abelian group $K_{\alpha} = N/N'_{\alpha} \cong \text{hom}_\mathbb{Z}(M'_{\alpha}/M, \mathbb{Q}/\mathbb{Z})$. Here $M'_{\alpha}$ is the dual of $N'_{\alpha}$ and is naturally a superlattice of $M$. There is a unique pairing $M' \times N \to \mathbb{Q}/\mathbb{Z}$ which extends the pairings $M \times N \to \mathbb{Z}$ and $M' \times N' \to \mathbb{Z}$. We then use the morphism $\mathbb{Q}/\mathbb{Z} \to \mathbb{C}^\times$ given by the exponential map to identify $K_{\alpha}$ with $\text{hom}_\mathbb{Z}(M'_{\alpha}/M, \mathbb{C}^\times)$. If we identify $k \in K$ with the morphism $k : M'_{\alpha} \to \mathbb{Q}/\mathbb{Z}$ such that $k(M) = 0$, the action is given by

$$k \cdot u'(m') = \exp(2\pi i (k, m')) u'(m'),$$

for $u' \in U'_{\alpha}$. Since the invariant part of $M'_{\alpha}$ under $K_{\alpha}$ is $M$, the line bundles $L_{\alpha}$ and $L'_{\alpha}$ over $X_{\alpha}$ and $X'_{\alpha}$ defined by the polytope $\square$ are related by $L_{\alpha} = \ldots$
$L^p_{\alpha}$. Equation (8) shows that the cohomology of $L_\alpha$ can be identified with the $K_\alpha$-invariant part of that of $L'_\alpha$.

In summary, $(U'_\alpha, t, K_\alpha, L'_\alpha)$ is a local model for $X$ at $P_\alpha$. Applying the Lefschetz formula for orbifolds, one deduces:

**Theorem 6** For a simple convex polytope $\square$ in $M$, we have

$$\sum_{m \in \square} m(t) = \sum_{\alpha \in \text{ext} \square} \nu_\alpha(t)$$

(11)

where

$$\nu_\alpha(t) = \frac{1}{|K_\alpha|} \sum_{k \in K_\alpha} \frac{\alpha(t)}{(1 - e_k(\lambda'_1\alpha))\lambda'_1\alpha(t) \cdots (1 - e_k(\lambda'_n\alpha))\lambda'_n\alpha(t))}.$$  

(12)

and we write $e_k(\lambda)$ for $\exp(2\pi i \langle k, \lambda \rangle)$. Here, the vectors $\lambda'_1\alpha, \ldots, \lambda'_n\alpha$ are the edge vectors of $\square$ at $\alpha$ in the dual $M'_\alpha$ of the dual edge lattice $N'_\alpha$ of definition 4, and $K_\alpha$ is the finite abelian group $N/N'_\alpha$ acting according to equation (10).

5 Laurent Expansions

In this section I expand the rational functions $\nu_\alpha$ away from their poles, i.e., in the domains where $|\lambda'_i\alpha(t)|$ is not 1, for $i = 1, \ldots, n$. This has two benefits.

Firstly, it produces another formula which does not involve sums over roots of unity. We shall use this in calculating the number of lattice points and the volume.

Secondly it leads us to interpret the formula as a combinatorial statement, decomposing the (characteristic polynomial for the) polytope $\square$ as an algebraic sum of the (characteristic series for the) cones $C_\alpha$ for each extreme point. Ultimately this could be used to prove the formula using elementary convex geometric reasoning. We don’t attempt this here, as Ishida has already reduced the proof to the contractibility of convex sets [10].

We begin by some general remarks about characteristic series for convex cones.

5.1 Characteristic functions and series for convex cones

We recall some notation, following [10]. Let $A$ be a commutative ring with identity. Recall that $A[M]$ denotes the group algebra of $M$ generated by
elements $e(m)$ for $m \in M$ satisfying relations $e(m)e(m') = e(m + m')$ and $e(0) = 1$. We denote by $A(M)$ the total quotient ring of $A[M]$.

We define $A[[M]] = \text{Map}(M, A)$. Elements $f \in A[[M]]$ can also be expressed as formal Laurent series $f = \sum_{m \in M} f(m)e(m)$ and this defines a $A[M]$-module structure on $A[[M]]$ by:

$$e(x)(\sum f(m)e(m)) = \sum f(m - x)e(m).$$

The relationship of $A[[M]]$ to $A(M)$ is as follows. To a given element $\nu \in A(M)$ correspond (possibly) several elements of $A[[M]]$ called the Laurent expansions of $\nu$. As we see below a convex cone $C$ in $M$ gives rise to elements $\nu^M_C \in A(M)$ and $\chi_{C \cap M} \in A[[M]]$ and the latter is a Laurent expansion of the former.

**Definition 5** For $S$ a subset of $M$, we define the characteristic series of $S$ to be the element $\chi[S] = \chi_S$ of $A[[M]]$ corresponding to the set-theoretic characteristic function of $S$ (the function which takes values 1 on $S$ and 0 elsewhere), namely to the series

$$\chi_S = \sum_{m \in S} e(m).$$

Let $C$ be a (strongly convex rational simplicial) cone in $M_\mathbb{R}$. We write $\text{gen}_C^M = \{\lambda_1, \ldots, \lambda_n\}$ for the primitive generators in $M$ of $C$. The unit parallelepiped $Q_C^M = \{\sum a_i \lambda_i : 0 \leq a_i < 1\}$ defined by $C$ in $M$ intersects $M$ in $\{c_1, \ldots, c_k\}$. Here $k = |K|$, the order of the finite abelian group which is the quotient of the dual lattice $N$ to $M$ by the lattice generated by the primitive generators $\text{gen}_N^C = \{\sigma_1, \ldots, \sigma_n\}$ of $C^\vee$ in $N$.

**Definition 6** For $C$ strictly convex, we define the characteristic function for $C$ with respect to $M$ is the following element of $A(M)$:

$$\nu^M_C = \sum_{c \in Q_C^M \cap M} e(c) \prod_{\lambda \in \text{gen}_C^M} (1 - e(\lambda))^{-1}.$$

$$= \sum_{j=1}^{|K|} e(c_j) \prod_{i=1}^n (1 - e(\lambda_i))^{-1}.$$

For the translate of a cone $C$ by $\alpha \in M$, we set $\nu^M_{\alpha + C} = e(\alpha)\nu^M_C$.  

Denote by $\text{PL}_A(M)$ the $A[M]$-submodule of $A[[M]]$ generated by the set of polyhedral Laurent series:

$$\{\chi_{C \cap M} : C \text{ a basic cone in } M_\mathbb{R}\}.$$ 

Ishida proves that the following

**Proposition 6** There exists a unique $A[M]$-homomorphism

$$\varphi : \text{PL}_A(M) \to A(M)$$

such that $\varphi(\chi_{C \cap M}) = \nu^M_C$, for all basic cones $C$ in $M_\mathbb{R}$.

Actually, we have:

**Proposition 7** For any cone $C$, $\chi_{C \cap M} \in \text{PL}_A(M)$ and $\varphi(\chi_{C \cap M}) = \nu^M_C$ for $\varphi$ defined above.

**Proof:** This follows from the remark that any element of $m \in M$ can be expressed uniquely as $q + \sum x_i \lambda_i$ with $q \in Q^M_C \cap M$ and $x_i \in \mathbb{N}$.

The existence of $\varphi$ says essentially that we lose no information by passing from the characteristic function of a cone to its Laurent series, even though the latter might not always have a well defined convergence on all of $T_N$ (in the case $A = \mathbb{C}$).

**Remark** Whereas Ishida uses open cones, we find it more convenient to use closed ones. The correspondence between the two is of course that $C \cap M = \cup_{F < C} \text{int}(F) \cap M$, where the union runs over the faces of $C$.

**5.1.1 Action of $K$**

The group $K$ acts on $M'$ and hence on $A[[M']]$ by

$$k \cdot f = \sum_{m \in M} e_k(m)f(m)e(m),$$

and we have $A[[M]] = A[[M']]^K$. The following elementary remark gives the relationship between the characteristic series for $C$ with respect to the two lattices $M$ and $M'$. 

18
Proposition 8 For any cone $C$, we have

$$\chi_{C \cap M} = \frac{1}{|K|} \sum_{k \in K} k \cdot \chi_{C \cap M'}.$$ 

Proof: Note that $k \cdot \chi(m') = e_k(m')\chi(m')$. Since $e_k$, for $k \in K$, are nothing but the characters of the finite abelian group $M'/M$, we have $e_k(M) = |K|$ and $e_k(m' + M) = 0$, for all $m' \not\in M$. Hence the formula follows. ■

By the uniqueness of $\varphi$ we deduce that the same equality holds between the characteristic functions of $C$:

Corollary 1 For any cone $C$, we have

$$\nu^M_C = \frac{1}{|K|} \sum_{k \in K} k \cdot \nu^M_{C'}.$$ 

5.2 Recovery of Brion’s result

We apply the results of the previous section with $A = C$. Then $\mathbb{C}[M]$ is the affine coordinate ring for the algebraic torus $T_N$ and its’ field of fractions $\mathbb{C}(M)$ is the ring of rational functions on $T_N$.

The Lefschetz formula is expressing the characteristic series $\chi_{\Box}$ of $\Box$ as a sum of elements of $\mathbb{C}(M)$. The theorem below says that these are simply the characteristic functions for the tangent cones of $\Box$ at its’ extreme points. See [5, Théorème 2.2]

Theorem 7 Let $\Box$ be a simple convex polytope $\Box$ in $M$. Denote by $C_\alpha$ its’ tangent cone at $\alpha \in \text{ext } \Box$. Then we have

$$\chi_{\Box \cap M} = \sum_{\alpha \in \text{ext } \Box} \nu^M_{C_\alpha}. \quad (13)$$

Proof: By theorem 6 we have $\nu_\alpha = \frac{1}{|K|} \sum_{k \in K} k \cdot \nu^M_{C_\alpha}$, which by the corollary of the previous section is nothing but $\nu^M_{C_\alpha}$. ■

5.3 Laurent expansions of $\nu_C$ and their domains of validity

We take $A = \mathbb{C}$ and give all the different possible Laurent expansions of $\nu^M_C$ for a cone $C$. When we attempt to evaluate these on elements of $T_N$ these series only converge on certain open subsets which we specify here.
5.3.1 The expansions

We adopt the same notation as in section 5.1. The primitive generators of $C_\alpha$ are $\lambda_\alpha^i$ in $M$ and $\lambda_\alpha^i$ in $M'_\alpha$.

Proposition 9 (Basic Expansion) For $|\lambda_\alpha^n(t)| < 1$, for $i = 1, \ldots, n$, we have

$$\nu_\alpha(t) = \chi_\alpha + C_\alpha \cap M(t).$$

(14)

Proof: Applying the elementary expansion (valid for $|z| < 1$)

$$(1 - z)^{-1} = 1 + z + z^2 + z^3 + \cdots$$

to the individual factors $(1 - e_k(\lambda_\alpha^i)\lambda_\alpha^n(t))^{-1}$ gives:

$$\nu_\alpha(t) = \alpha(t) \frac{1}{|K_\alpha|} \sum_k \sum_{c_1, \ldots, c_n = 0} e_k(c \cdot \lambda_\alpha^i)(c \cdot \lambda_\alpha^n(t)),$$

where I have written $c \cdot \lambda_\alpha^i$ for $\sum_{i=1}^n c_i \lambda_\alpha^i$. Since the series is convergent, one has

$$\nu_\alpha(t) = \sum_{c_1, \ldots, c_n = 0} (\alpha + c \cdot \lambda_\alpha^n(t)) \frac{1}{|K_\alpha|} \sum_k e_k(c \cdot \lambda_\alpha^n),$$

and the result follows from the proof of proposition 8. \[\blacksquare\]

There are in fact $2^n$ different possible expansions for $\nu_\alpha(t)$ depending on whether we expand about $\lambda_\alpha^n(t) = 0$ or $\infty$, each expansion being valid for $|\lambda_\alpha^n(t)| < 1$ or $> 1$ respectively.

**Notation:** Let $s$ be an $n$-tuple $s \in \{\pm 1\}^n$. As a shorthand, I will write:

$$\lambda_\alpha^i \overset{\text{def}}{=} (\lambda_\alpha^1, \ldots, \lambda_\alpha^n)$$

$$s \lambda_\alpha^i \overset{\text{def}}{=} (s_1 \lambda_\alpha^1, \ldots, s_n \lambda_\alpha^n).$$

I also write $\langle \lambda_\alpha^i \rangle$ for the cone $\langle \lambda_\alpha^1, \ldots, \lambda_\alpha^n \rangle$. I define the quantity $s_- \cdot \lambda_\alpha^i$ by:

$$s_- \cdot \lambda_\alpha^i = \sum_{s_i = -1} s_i \lambda_\alpha^i.$$

An element $m \in M'$ defines a region $T_m$ of $T_{N'}$ by:

$$T_m = \{ t \in T_{N'} : |m(t)| < 1 \}.$$
I also write, for a cone $C$ in $M$,

$$T_C = \{ t \in T_N : |m(t)| < 1, \forall m \in C \cap M \}.$$ 

Thus, for example,

$$T_{(\lambda_\alpha')} = T_{\lambda_\alpha'} \cap \cdots \cap T_{\lambda_\alpha'}.$$ 

**Proposition 10 (General Expansion)** Given $s \in \{\pm 1\}^n$, we have, for $t \in T_{(s\lambda_\alpha')}$,

$$\nu_\alpha(t) = \left( \prod_{i=1}^n s_i \right) \chi[\alpha + s_\cdot \lambda_\alpha' + (s\lambda_\alpha') \cap M](t).$$ (15)

**Proof:** In order to expand $\nu_\alpha$ when, for some $i$, we have $|\lambda_\alpha^i(t)| > 1$, I use the other expansion of $(1 - z)^{-1}$, valid for $|z| > 1$:

$$(1 - z)^{-1} = -z - z^2 - z^3 - z^4 - \cdots.$$ 

The result follows in the same way as the basic expansion. Note that compared to the basic expansion, the cone whose characteristic series we end up with undergoes a reflection plus a translation: $(\lambda_\alpha') \cap M$ becomes $s_\cdot \lambda_\alpha' + (s\lambda_\alpha') \cap M$. This is due to the shift from $1 + z + z^2 + \cdots$ to $-z^1 - z^2 - z^3 - \cdots$. \[\square\]

### 5.3.2 Consistency of expansions

It doesn’t make sense to expand all the $\nu_\alpha$ according to (14) because the variable $t$ can’t satisfy the condition $|\lambda_\alpha^i(t)| < 1$ for all $i$ and $\alpha$. For one thing, if $\alpha$ and $\beta$ are two extreme vertices of $\square$ connected by an edge, we will have $\lambda_\alpha^i = -\lambda_\beta^j$ for some $i$ and $j$, so that $|\lambda_\alpha^i(t)| < 1 \iff |\lambda_\beta^j(t)| > 1$.

I can find a domain for $t \in T_N$ such that all the expansions we perform are valid at the same time, then when we sum up all the $\nu_\alpha(t)$, all but a finite number of terms in the infinite series cancel, and we get the characteristic polynomial $\chi_{\square}$ evaluated on $t$.

For each $\beta \in \text{ext } \square$, we choose an element $s_\beta \in \{\pm 1\}^n$, and expand according to (15). We require that the set

$$\bigcap_{\beta \in \text{ext } \square} T_{(s_\beta\lambda_\beta')} = T_{\cup \{ (s_\beta\lambda_\beta') : \beta \in \text{ext } \square \}}$$ (16)

be non-empty. I turn next to the necessary conditions for this to be so.
5.3.3 Necessary conditions for a consistent expansion

The above requirement implies, for instance, that if \( \lambda'_{i\alpha} = -\lambda'_{j\beta} \), as it happens for adjacent vertices, then \( s^\alpha_i = -s^\beta_j \). This can be thought of graphically as choosing a direction for each edge of the polytope \( \square \) and sticking to it throughout the expansion. For each vertex \( \alpha \) if the \( i \)-th edge is pointing into \( \alpha \) then we set \( s^\alpha_i = -1 \), if it is pointing out, we set \( s^\alpha_i = +1 \).

Another necessary condition is that we choose \( s^\alpha = (1, 1, \ldots, 1) \) for some \( \alpha \in \text{ext} \, \square \). This can be seen easily, if one thinks for a moment of decomposing \( \chi \, \square \) as a sum of characteristic series for cones:

\[
\chi \, \square = \sum_{\beta \in \text{ext} \, \square} \pm \chi_{C'_\beta \cap M} \tag{17}
\]

where the cones \( C'_\beta \) are obtained from the tangent cones \( C_\beta \) eventually by the ‘reflection + translation’ process prescribed in the general expansion in proposition 10 and the sign is determined by the number of reflections specified by \( s^\beta \). One of the cones involved must be \( C_\alpha \), for some \( \alpha \in \text{ext} \, \square \). It will have all of its' edges pointing outwards in the above orientation and will correspond to the characteristic series \( +\chi_{C_\alpha \cap M} \). I will call this the base vertex for the expansion.

The non-emptiness requirement above then implies that the following condition on the orientations be satisfied:

**Orientation condition** Let \( \lambda'_{i\alpha} \) for \( i = 1, \ldots, p \) be any set of edges emanating from \( \alpha \) that have been oriented so that they are all outgoing with respect to \( \alpha \). Then we require that for all \( \beta \neq \alpha \),

\[
\text{if } (\lambda'_{i\beta})^j \in \pm(\lambda'_{n1}, \ldots, \lambda'_{np}) \text{ then } (s^\beta)^j = \pm 1. \tag{18}
\]

In words, this says that if an edge \( \lambda'_{i\beta} \) is a linear combination, all of whose coefficients are of the same sign or zero, of oriented edges \( \lambda'_{i\alpha} \) all going outwards from a given vertex \( \alpha \), then it should be oriented in the direction which includes it in the cone spanned by these outgoing edges. This is because, if it were oriented oppositely, it would mean that \( T_{(\lambda'_{n1}, \ldots, \lambda'_{np})} \cap T_{s^\beta, \lambda'_{i\beta}} = \emptyset \), since one cannot have both \( |\lambda'_{i\beta}(t)| < 1 \) and \( | -\lambda'_{i\beta}(t)| < 1 \).

5.3.4 Domain of validity of simultaneous expansions

It is always possible to choose at least one orientation of the edges of \( \square \) which satisfies the orientation condition (18). Suppose we have chosen such
an orientation. For what values of \( t \in T_N \) is it valid? In order to answer this, let us first make some remarks about the regions \( T_C \subset T_N \), for \( C \) a cone in \( M \).

It is helpful, to describe \( T_C \), to decompose \( T_N \) as \( CT_N \times H \), corresponding to the Lie algebra decomposition \( t_C = t \oplus i t \). By identifying the second factor in the Lie algebra decomposition with \( N_{\mathbb{R}} \), we have the exponential map

\[
N_{\mathbb{R}} \xrightarrow{\exp} H.
\]

**Lemma 6** If \( C \) is a cone in \( M \), then \( T_C \) is given by

\[
T_C = CT_N \times \exp(-\text{int}(C^\vee)) \subset CT_N \times H.
\]

**Proof:** The interior of \( C^\vee \) is the set of \( n \in N_{\mathbb{R}} \) such that \( \langle n, c \rangle > 0, \forall c \in C \). Under the exponential map, the orbit \( CT_N \times \{-n\} \) corresponds to an orbit of constant modulus strictly less than 1.

From this, we see that

\[
\bigcap_{\beta \in \text{ext } \Box} T_{\langle s^\beta \lambda_\beta \rangle} = CT_N \times \exp(-\text{int}(\sigma)),
\]

where

\[
\sigma = \left( \bigcup_{\beta \in \text{ext } \Box} \langle s^\beta \lambda_\beta' \rangle \right)^\vee.
\]

If we respect condition \((18)\), we see that \( \bigcup_{\beta \in \text{ext } \Box} \langle s^\beta \lambda_\beta' \rangle \) never contains a whole subspace, so that \( \sigma \) is non-zero. The expansion determined by \( s^\beta \) for \( \beta \in \Box \) is thus valid in the region \( T_\sigma \subset T_N \) given by equation \((19)\).

### 5.4 Elementary convex geometric interpretation

According to the work we have done in the previous sections, one can prove the extreme point formula as follows:

Begin by orienting the edges of \( \Box \) such as to respect condition \((18)\). This defines a cone (with a sign) for each extreme vertex, according to proposition \([10]\), and the algebraic sum of their characteristic series should yield the characteristic polynomial for the polytope \( \Box \). If one can prove this for one admissible orientation of the edges of \( \Box \), then the formula for the characteristic functions follows by the existence of Ishida’s \( \mathbb{C}[M] \)-homomorphism in
the previous section. This gives a proof of the formula involving only elementary convex geometry. We won’t bother with this, as Ishida [10] already gives a proof which reduces the problem to the contractibility of convex sets.

Instead we can deduce the following result in convex geometry:

**Theorem 8** For all orientations \( \{ s^{\alpha} \} \) of the edges of \( \square \) satisfying the orientation condition (18) we have

\[
\chi_{\square \cap M} = \sum_{\alpha \in \text{ext } \square} \pm \chi_{C^{s}_{\alpha} \cap M}
\]

where \( \pm = \prod_{i} (s^{\alpha})_{i} \) and

\[
C^{s}_{\alpha} = \alpha + s_{-} \cdot \lambda'_{\alpha} + \langle s^{\lambda'}_{\alpha} \rangle.
\]

### 6 Number of Lattice Points and Volume

In this section I expand the functions \( \nu_{\alpha}(t) \) around \( t = 1 \) and derive formulae for the number of lattice points and volume of \( \square \).

#### 6.1 The Number of Lattice Points

Equation (13) expresses an equality between the finite Laurent polynomial determined by \( \square \) and a sum of rational functions. When evaluated on \( t \in \mathbb{T}^{N} \) with \( t \to 1 \) the left-hand side tends to the number of lattice points of \( \square \) whereas on the right-hand side the rational functions may have poles.

I choose a one-parameter subgroup \( \{ \exp(s\zeta) : s \in \mathbb{R} \} \) determined by some element \( \zeta \) of the Lie algebra \( t \) of \( CT_{N} \). Substituting \( \exp(s\zeta) \) for \( t \), the formula reduces to an equality between rational functions of \( s \) — provided I choose a one-parameter subgroup that does not coincide with the singular loci of the \( \nu_{\alpha} \).

**Definition 7** For short, I call \( \zeta \in t \) generic if \( \langle \zeta, \lambda_{\alpha} \rangle \neq 0 \), for all \( i \) and \( \alpha \). (This is indeed the case generically).

For generic \( \zeta \), the functions \( \nu^{\square}_{\alpha,\zeta} : s \mapsto \nu^{\square}_{\alpha}(e^{s\zeta}) \) can expanded in Laurent series:

\[
\nu^{\square}_{\alpha,\zeta}(s) = \sum_{i=-\infty}^{\infty} \nu^{\square}_{\alpha,\zeta,i}s^{i},
\]
and their sum as $s \to 0$ is obviously given by the sum of the constant terms $\nu_{\alpha, \zeta, 0}$ in each expansion.

Denote by $C_\alpha$ the tangent cone of $\Box$ at $\alpha \in \text{ext } \Box$, and by $\lambda^i_\alpha$ for $i = 1, \ldots, n$, its' primitive generators in $M$. The semi-open unit parallelepiped determined by the generators of $C_\alpha$ in $M$ is denoted

$$Q_\alpha = Q^M_{C_\alpha} = \{ \sum a_i \lambda^i_\alpha : 0 \leq a_i < 1 \}. \quad (20)$$

We have

$$\nu_{\alpha, \zeta}(s) = \frac{\sum_{q \in Q_\alpha \cap M} e^{s\langle \zeta, \alpha + q \rangle}}{(1 - e^{s\langle \zeta, \lambda^1_\alpha \rangle}) \cdots (1 - e^{s\langle \zeta, \lambda^n_\alpha \rangle})}$$

provided $\langle \zeta, \lambda^i_\alpha \rangle \neq 0$. The zero-th order term in the expansion of $\nu_{\alpha, \zeta}(s)$ is a homogeneous function of $\zeta$, which is equal to:

$$\sum_{q \in Q_\alpha \cap M} e^{s\langle \zeta, \alpha + q \rangle} \prod_i (-\langle \zeta, \lambda^i_\alpha \rangle)^{j_i} \prod_i (1 - \exp(s\langle \zeta, \lambda^i_\alpha \rangle))$$

which gives

$$\frac{1}{\prod_i \langle \zeta, \lambda^i_\alpha \rangle} \sum_{j=0}^n \frac{(-1)^j}{j!} \sum_{q \in Q_\alpha \cap M} \langle \zeta, \alpha + q \rangle^j T_{n-j}(\langle \zeta, \lambda_\alpha \rangle),$$

where $T_k$ are the Todd polynomials, homogeneous polynomials of degree $k$ whose coefficients can be expressed in terms of the Bernoulli numbers \[9\]. They are defined by the formal series

$$\sum_{k=0}^{\infty} s^k T_k(x_1, x_2, \ldots) = \prod_{i \geq 1} \frac{s x_i}{1 - \exp(-sx_i)}.$$

By $T_k(\langle \zeta, \lambda_\alpha \rangle)$ I mean $T_k(\langle \zeta, \lambda^1_\alpha \rangle, \ldots, \langle \zeta, \lambda^n_\alpha \rangle)$.

**Theorem 9** Let $\Box$ be a simple convex lattice polytope. Denote by $C_\alpha$ the tangent cone of $\Box$ at $\alpha \in \text{ext } \Box$, and by $\lambda^i_\alpha$, for $i = 1, \ldots, n$, the primitive generators of $C_\alpha$ in $M$. The semi-open unit parallelepiped determined by the generators of $C_\alpha$ in $M$ as in equation 27 is denoted $Q_\alpha$. Then, for generic $\zeta \in \mathfrak{t}$, the number of lattice points in $\Box$ is given by

$$\sum_{\alpha \in \text{ext } \Box} \frac{1}{\prod_i \langle \zeta, \lambda^i_\alpha \rangle} \sum_{j=0}^n \frac{(-1)^j}{j!} \sum_{q_\alpha \in Q_\alpha \cap M} \langle \zeta, \alpha + q_\alpha \rangle^j T_{n-j}(\langle \zeta, \lambda_\alpha \rangle).$$
**Remark 1** It might be more convenient in some cases to subdivide the tangent cone into non-singular cones. One obtains a similar formula (see [3, Théorème 3.1]).

**Remark 2** Putting \( t = \exp(s\zeta) \) corresponds to considering the Lefschetz number for the action of the one-parameter subgroup \( G_\zeta \) of \( CT_N \) generated by \( \zeta \in t = \text{Lie} CT_N \). Generically this has a dense orbit, and therefore the same fixed points on \( X \) as the whole real torus \( CT_N \), and so the Lefschetz formula for \( G_\zeta \) is the same as that obtained by substituting \( \exp(s\zeta) \) for \( t \). This is not true of course when \( \langle \zeta, \lambda, \alpha_i \rangle = 0 \), for some \( i \) and \( \alpha \). Indeed in that case the group \( G_\zeta \) has whole circles of fixed points. Restricting to \( G_\zeta \) corresponds to projecting the vertices and edges of \( \Box \) onto the hyperplane in \( M_\mathbb{R} \) defined by the form \( \zeta \in N_\mathbb{R} \).

### 6.2 The Volume

#### 6.2.1 The “Classical Limit”

In the introduction I mentioned the fact that for larger and larger polytopes (or finer and finer lattices) the number of points is asymptotically equal to their volume — I call this “the classical limit” by analogy with the limit \( h \to 0 \) in quantum mechanics. More precisely, for any \( n \)-dimensional polytope \( \Box \), the volume of \( \Box \) is given by

\[
\text{vol}_n(\Box) = \lim_{k \to \infty} \frac{\#(k^{-1}M \cap \Box)}{k^n} = \lim_{k \to \infty} \frac{\#(M \cap k\Box)}{k^n}.
\]

Indeed [15], the function

\[
H_\Box(k) = \#(k^{-1}M \cap \Box) = \#(M \cap (k\Box))
\]

is a polynomial of degree \( n \), for \( k \in \mathbb{N} \), with leading coefficient \( \text{vol}_n(\Box) \), and is called the Hilbert polynomial for \( \Box \). The polynomial \( H_\Box \) is in fact equal to the Hilbert polynomial \( H_{(X_\Box, L_\Box)} \) for the pair \( (X_\Box, L_\Box) \), namely

\[
H_{(X_\Box, L_\Box)}(k) = \chi(X_\Box, \mathcal{O}_{X_\Box}(kL_\Box)) = \sum (-1)^i \dim H^i(X_\Box, \mathcal{O}_{X_\Box}(kL_\Box)).
\]

This follows from equation [18] and because taking tensor powers \( L_\Box^\otimes k \) of \( L_\Box \) corresponds to taking multiples of \( kN \) of \( N \), and hence submultiples \( k^{-1}M \) of \( M \).
Theorem 10 Let $\square$ be a simple convex lattice polytope and adopt the same notation as theorem 9. Let $|K_\alpha|$ denote the order of the singularity of $\square$ at $\alpha$. Then for generic $\zeta$ the volume of $\square$ is given by

$$\text{vol}_n(\square) = \frac{(-1)^n}{n!} \sum_{\alpha \in \text{ext} \square} \frac{\langle \zeta, \alpha \rangle^n |K_\alpha|}{\langle \zeta, \lambda_\alpha^1 \rangle \cdots \langle \zeta, \lambda_\alpha^n \rangle}.$$  

Proof: The proposition follows from taking the coefficients of the $k^n$ terms in theorem 9 applied to the polytope $k \square$. Note that $\text{ext} k \square = k(\text{ext} \square)$ and that $C_{k \alpha} = C_{\alpha}$. Note that the order $|K_\alpha|$ of the singularity at $\alpha$ is equal to the cardinality of $Q_\alpha \cap M$. See [5, Corollaire 2]. \[\Box\]

6.2.2 The Riemann-Roch approach

The volume of $\square$ appears if one uses the same geometric approach based on the $d''$-complex but directly applies the Riemann-Roch theorem, instead of computing the Lefschetz number for the action of $t \in T$ and then letting $t \to 1$.

The Riemann-Roch theorem expresses the Euler characteristic of a holomorphic vector bundle $E$ over a complex manifold $X$ in terms of characteristic classes of $E$ and (tangent bundle to the) $X$:

$$\chi(X, E) = \{\text{ch}(E) \cdot T(X)\}[X],$$  \hspace{1cm} (22)

where $\text{ch}(E)$ and $T(X)$ are the Chern character of $E$ and the Todd class of $X$, respectively. If $E$ has rank $n$ and $c_1, \ldots, c_n$ denote the characteristic classes of $E$ then the Chern character can be defined by the power series

$$\sum_{i=1}^n e^{x_i} = n + x_1 + \frac{x_1^2}{2!} + \cdots,$$

where the $c_i$ are to be thought of formally as the elementary symmetric functions in the $x_i$.

Since we are in a one-dimensional situation and $c_1(L_{\square})$ is represented by the Kähler form $\omega$, the Chern character is given by

$$\text{ch}(L_{\square}) = 1 + \omega + \frac{\omega^2}{2!} + \frac{\omega^3}{3!} + \cdots + \frac{\omega^n}{n!}.$$  

The Todd class is a polynomial in the characteristic classes $c_i^j$ of the tangent bundle of $X$. If the $c_i^j$ are regarded formally as the elementary symmetric
functions of the $x'_i$ (as in the case above), the Todd class can be expressed as

$$\mathcal{T}(X) = \prod_i \frac{x'_i}{1 - e^{-x'_i}}.$$ 

(Presumably, there is some relationship between these and the Todd polynomials of theorem \[\text{[3]}\] which in this case exhibits the Riemann-Roch formula as the “classical limit” of the Lefschetz fixed point formula.) By multiplying the two series selecting the terms of order $n$ and evaluating them on $[X]$, we get

$$\chi(X, L) = \text{vol}_n(X) + \text{lower order terms},$$

where the “lower order terms” are terms involving powers of $\omega$ of order less than $n$. Again, because refining the lattice $M$ corresponds to multiplying $\omega$, we see that $\chi(X, tL)$ is given asymptotically by $\text{vol}_n(X)t^n$.

References

[1] M.F. Atiyah & R. Bott. “A Lefschetz fixed point formula for elliptic complexes: I,” Ann. of Math. 86 (1967), 374-407.

[2] M.F. Atiyah & R. Bott. “A Lefschetz fixed point formula for elliptic complexes: II. Applications,” Ann. of Math. 88 (1968), 451-491.

[3] P. Baum, W. Fulton, G. Quart. “Lefschetz-Rieman-Roch for singular varieties,” Acta Mathematica 143 (1979) 193-211.

[4] D.I. Bernshtein. “The number of integral points in integral polyhedra,” Functional Anal. Appl. 10 (1976), 223-224.

[5] M. Brion. “Points entiers dans les polyhèdres convexes”, Ann. scient. Ec. Norm. Sup. 4e série, 21 (1988) 653-663.

[6] V.I. Danilov. The geometry of toric varieties, Russ. Math. Surveys 33:2 (1978), 97-154.

[7] P. Erdős, P.M. Grüber, J. Hammer. Lattice Points, Pitman Monograph Series in Pure & Applied Maths n.39, Longman, 1989.

[8] J. Hammer. Unsolved Problems concerning Lattice Points, Research Notes in Mathematics 15, Pitman, London, 1977.
[9] F. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 1962.

[10] M.-N. Ishida. “Polyhedral Laurent series and Brion’s equalities,” International Journal of Mathematics, 1 (1990), n.3, 251-265.

[11] T. Kawasaki. “The signature theorem for V-manifolds,” Topology 17, 75-83.

[12] G. Kempf, F. Knudsen, D. Mumford, B. Saint-Donat. *Toroidal Embeddings I*, Lecture Notes in Mathematics 339, Springer, Berlin, 1973.

[13] A.G. Khovanski. “Newton polyhedra and toric varieties,” *Functional Anal. Appl.* 11 (1977), 289-296.

[14] I.G. Macdonald. “The volume of a lattice polyhedron,” Proc. Camb. Phil. Soc. 59 (1963), 719.

[15] I.G. Macdonald. “Polynomials associated with finite cell complexes,” J. London Math. Soc. (2), 4 (1971), 181-192.

[16] Milnor, J., *Characteristic Classes*, Princeton Univ. Press., 1974

[17] T. Oda. *Convex Bodies and Algebraic Geometry : An introduction to the theory of toric varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3.Folge–Band 15, Springer-Verlag, Berlin, Heidelberg, New York, 1988

[18] G. Pick. *Geometrisches zur Zahlenlehre*, Sitzungsber, Lotos Prag. (2) 19 (1870) 311-319.

[19] R.T. Rockafellar. *Convex Analysis*, Princeton Univ. Press, 1970.

[20] S. Sardo Infirri. “An application of the Lefschetz fixed point theorem to convex lattice polyhedra,” Dissertation for transfer to D.Phil student status, Mathematical Institute, University of Oxford (1990), unpublished.