Typicality of normal numbers with respect to the Cantor series expansion

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Abstract. Fix a sequence of integers $Q = \{q_n\}_{n=1}^{\infty}$ such that $q_n$ is greater than or equal to 2 for all $n$. In this paper, we improve upon results by J. Galambos and F. Schweiger showing that almost every (in the sense of Lebesgue measure) real number in $[0, 1)$ is $Q$-normal with respect to the $Q$-Cantor series expansion for sequences $Q$ that satisfy a certain condition. We also provide asymptotics describing the number of occurrences of blocks of digits in the $Q$-Cantor series expansion of a typical number. The notion of strong $Q$-normality, that satisfies a similar typicality result, is introduced. Both of these notions are equivalent for the $b$-ary expansion, but strong normality is stronger than normality for the Cantor series expansion. In order to show this, we provide an explicit construction of a sequence $Q$ and a real number that is $Q$-normal, but not strongly $Q$-normal. We use the results in this paper to show that under a mild condition on the sequence $Q$, a set satisfying a weaker notion of normality, studied by A. Rényi in [7], will be dense in $[0, 1)$.

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1. Introduction

Definition 1.1. Let $b$ and $k$ be positive integers. A block of length $k$ in base $b$ is an ordered $k$-tuple of integers in $\{0, 1, \ldots, b-1\}$. A block of length

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$k$ is a block of length $k$ in some base $b$. A *block* is a block of length $k$ in base $b$ for some integers $k$ and $b$.

**Definition 1.2.** Given an integer $b \geq 2$, the *$b$-ary expansion* of a real $x$ in $[0, 1)$ is the (unique) expansion of the form

$$x = \sum_{n=1}^{\infty} \frac{E_n}{b^n} = 0.E_1E_2E_3\ldots$$

such that $E_n$ is in $\{0, 1, \ldots, b-1\}$ for all $n$ with $E_n \neq b - 1$ infinitely often.

Denote by $N^b_n(B, x)$ the number of times a block $B$ occurs with its starting position no greater than $n$ in the $b$-ary expansion of $x$.

**Definition 1.3.** A real number $x$ in $[0, 1)$ is *normal in base $b$* if for all $k$ and blocks $B$ in base $b$ of length $k$, one has

$$\lim_{n \to \infty} \frac{N^b_n(B, x)}{n} = b^{-k}.$$  

A number $x$ is *simply normal in base $b$* if (1) holds for $k = 1$.

Borel introduced normal numbers in 1909 and proved that almost all (in the sense of Lebesgue measure) real numbers in $[0, 1)$ are normal in all bases. The best known example of a number that is normal in base 10 is due to Champernowne [3]. The number

$$H_{10} = 0.123456789101112\ldots,$$

formed by concatenating the digits of every natural number written in increasing order in base 10, is normal in base 10. Any $H_b$, formed similarly to $H_{10}$ but in base $b$, is known to be normal in base $b$. Since then, many examples have been given of numbers that are normal in at least one base. One can find a more thorough literature review in [4] and [5].

The $Q$-Cantor series expansion, first studied by Georg Cantor in [9], is a natural generalization of the $b$-ary expansion.

**Definition 1.4.** $Q = \{q_n\}_{n=1}^{\infty}$ is a *basic sequence* if each $q_n$ is an integer greater than or equal to 2.

**Definition 1.5.** Given a basic sequence $Q$, the *$Q$-Cantor series expansion* of a real $x$ in $[0, 1)$ is the (unique) expansion of the form

$$(2) \quad x = \sum_{n=1}^{\infty} \frac{E_n}{q_1q_2\ldots q_n}$$

such that $E_n$ is in $\{0, 1, \ldots, q_n - 1\}$ for all $n$ with $E_n \neq q_n - 1$ infinitely often. We abbreviate (2) with the notation $x = 0.E_1E_2E_3\ldots$ with respect to $Q$. 

Clearly, the $b$-ary expansion is a special case of (2) where $q_n = b$ for all $n$. If one thinks of a $b$-ary expansion as representing an outcome of repeatedly rolling a fair $b$-sided die, then a $Q$-Cantor series expansion may be thought of as representing an outcome of rolling a fair $q_1$ sided die, followed by a fair $q_2$ sided die and so on. For example, if $q_n = n + 1$ for all $n$, then the $Q$-Cantor series expansion of $e - 2$ is

$$e - 2 = \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \ldots$$

If $q_n = 10$ for all $n$, then the $Q$-Cantor series expansion for $1/4$ is

$$\frac{1}{4} = \frac{2}{10} + \frac{5}{10^2} + \frac{0}{10^3} + \frac{0}{10^4} + \ldots$$

For a given basic sequence $Q$, let $N_n^Q(B, x)$ denote the number of times a block $B$ occurs starting at a position no greater than $n$ in the $Q$-Cantor series expansion of $x$. Additionally, define

$$Q_n^{(k)} = \sum_{j=1}^{n} \frac{1}{q_j q_{j+1} \cdots q_{j+k-1}}.$$

A. Rényi [7] defined a real number $x$ to be normal with respect to $Q$ if for all blocks $B$ of length 1,

$$\lim_{n \to \infty} \frac{N_n^Q(B, x)}{Q_n^{(1)}} = 1.$$ 

If $q_n = b$ for all $n$, then (3) is equivalent to simple normality in base $b$, but not equivalent to normality in base $b$. Thus, we want to generalize normality in a way that is equivalent to normality in base $b$ when all $q_n = b$.

**Definition 1.6.** A real number $x$ is $Q$-normal of order $k$ if for all blocks $B$ of length $k$,

$$\lim_{n \to \infty} \frac{N_n^Q(B, x)}{Q_n^{(k)}} = 1.$$ 

We say that $x$ is $Q$-normal if it is $Q$-normal of order $k$ for all $k$. A real number $x$ is $Q$-ratio normal of order $k$ if for all blocks $B$ and $B'$ of length $k$, we have

$$\lim_{n \to \infty} \frac{N_n^Q(B, x)}{N_n^Q(B', x)} = 1.$$ 

$x$ is $Q$-ratio normal if it is $Q$-ratio normal of order $k$ for all positive integers $k$.

We make the following definitions:

**Definition 1.7.** A basic sequence $Q$ is $k$-divergent if $\lim_{n \to \infty} Q_n^{(k)} = \infty$. $Q$ is fully divergent if $Q$ is $k$-divergent for all $k$. $Q$ is $k$-convergent if it is not $k$-divergent.
Definition 1.8. A basic sequence $Q$ is *infinite in limit* if $q_n \to \infty$.

For $Q$ that are infinite in limit, it has been shown that the set of all $x$ in $[0,1)$ that are $Q$-normal of order $k$ has full Lebesgue measure if and only if $Q$ is $k$-divergent [7]. Therefore, if $Q$ is infinite in limit, then the set of all $x$ in $[0,1)$ that are $Q$-normal has full Lebesgue measure if and only if $Q$ is fully divergent. Suppose that $Q$ is infinite in limit, then the set of all $x$ in $[0,1)$ that are $Q$-normal has full Lebesgue measure if and only if $Q$ is fully divergent. Suppose that $Q$ is 1-divergent. Given an arbitrary non-negative integer $a$, F. Schweiger [8] proved that for almost every $x$ with $\epsilon > 0$, one has

$$N_n((a), x) = Q_n^{(1)} + O\left( \sqrt{Q_n^{(1)}} \cdot \log^{3/2+\epsilon} Q_n^{(1)} \right).$$

J. Galambos proved an even stronger result in [10]. He showed that for almost every $x$ in $[0,1)$ and for all non-negative integers $a$,

$$N_Q^n((a), x) = Q_n^{(1)} + O\left( \sqrt{Q_n^{(1)}} \left( \log \log Q_n^{(1)} \right)^{1/2} \right).$$

We provide the following main results:

1. A notion of strong $Q$-normality is provided and we construct an explicit example of a basic sequence $Q$ and a real number that is $Q$-normal, but not strongly $Q$-normal (Theorem 2.15).

2. (Theorem 4.9) If $Q$ is a basic sequence that is infinite in limit and $B$ is a block of length $k$, then for almost every real number $x$ in $[0,1)$, we have

$$N_Q^n(B, x) = Q_n^{(k)} + O\left( \sqrt{Q_n^{(k)}} \left( \log \log Q_n^{(k)} \right)^{1/2} \right).$$

3. If $Q$ is infinite in limit, then almost every real number is $Q$-normal of order $k$ if and only if $Q$ is $k$-divergent (Theorem 4.11).

4. If $Q$ is $k$-convergent for some $k$, then the set of numbers that are $Q$-normal is empty (Proposition 5.1). If $Q$ is infinite in limit, then the set of $Q$-ratio normal numbers is dense in $[0,1)$ (Corollary 5.3).

2. Strongly Normal Numbers

2.1. Basic definitions and results. In this section, we will introduce a notion of normality that is stronger than $Q$-normality. This notion of normality will arise naturally later in this paper and will be useful for studying the typicality of $Q$-normal numbers. We will first need to make definitions similar to those of $N_Q^n(B, x)$ and $Q_n^{(k)}$.

Given a real number $x \in [0,1)$, a basic sequence $Q$, a block $B$ of length $k$, a positive integer $p \in [1,k]$, and a positive integer $n$, we will denote by $N_{n,p}^Q(B, x)$ the number of times that the block $B$ occurs in the $Q$-Cantor series expansion of $x$ with starting position of the form $j \cdot k + p$ for $0 \leq j < \frac{n}{k}$.

If $n$ and $k$ are positive integers, define

$$\rho(n,k) = \lfloor n/k \rfloor - 1 = \max \left\{ i \in \mathbb{Z} : i < \frac{n}{k} \right\}.$$
Suppose that $Q$ is a basic sequence and that $n, p$, and $k$ are positive integers with $p \in [1, k]$. We will write

$$Q^{(k)}_{n, p} = \frac{1}{\sum_{j=0}^{\rho(n,k)} q_{jk+p}\cdots q_{jk+p+k-1}}.$$ 

**Definition 2.1.** Let $k$ be a positive integer. Then a basic sequence $Q$ is strongly $k$-divergent \footnote{It is not true that $k$-divergent basic sequences must be strongly $k$-divergent. The following example of a 2-divergent basic sequence that is not strongly 2-divergent was suggested by C. Altomare (verbal communication): let the basic sequence $Q = \{q_n\}$ be given by

$$q_n = \begin{cases} 
\max(2, \lfloor n^{1/4} \rfloor) & \text{if } n \equiv 0 \pmod{4} \\
\max(2, \lfloor n^{3/4} \log^2 n \rfloor) & \text{if } n \equiv 1 \pmod{4} \\
\max(2, \lfloor n^{3/4} \rfloor) & \text{if } n \equiv 2 \pmod{4} \\
\max(2, \lfloor n^{3/4} \log^2 n \rfloor) & \text{if } n \equiv 3 \pmod{4}
\end{cases}$$}

if for all positive integers $p$ with $p \in [1, k]$, we have $\lim_{n \to \infty} Q^{(k)}_{n, p} = \infty$. A basic sequence $Q$ is strongly fully divergent if it is strongly $k$-divergent for all $k$.

Given a real number $x \in [0, 1)$, a basic sequence $Q$, a block $B$ of length $k$, a positive integer $p \in [1, k]$, and a positive integer $n$, we will denote by $N^Q_{n, p}(B, x)$ the number of times the block $B$ occurs in the $Q$-Cantor series expansion of $x$ with positions of the form $j \cdot k + p$ for $0 \leq j < \frac{n}{k}$.

**Definition 2.2.** Suppose that $Q$ is a basic sequence. A real number $x$ in $[0, 1)$ is strongly $Q$-normal of order $k$ if for all blocks $B$ of length $m \leq k$ and all $p \in [1, m]$, we have

$$\lim_{n \to \infty} \frac{N^Q_{n, p}(B, x)}{Q^{(m)}_{n, p}} = 1.$$ 

A real number $x$ is strongly $Q$-normal if it is strongly $Q$ normal of order $k$ for all $k$.

We will use the following lemmas frequently and without mention:

**Lemma 2.3.** Given a real number $x \in [0, 1)$, a basic sequence $Q$, a block $B$ of length $k$, a positive integer $p \in [1, k]$, and a positive integer $n$, we have

$$N^Q_{n, 1}(B, x) + N^Q_{n, 2}(B, x) + \ldots + N^Q_{n, k}(B, x) = N^Q_n(B, x) + O(1)$$

and

$$Q^{(k)}_{n, 1} + Q^{(k)}_{n, 2} + \ldots + Q^{(k)}_{n, k} = Q^{(k)}_n + O(1).$$

**Proof.** This follows directly from the definitions of $N^Q_n(B, x)$ and $Q^{(k)}_n$. □

**Lemma 2.4.** If $g_1, g_2, \ldots, g_n$ are non-negative functions on the natural numbers, then

$$o(g_1) + o(g_2) + \ldots + o(g_n) = o(g_1 + g_2 + \ldots + g_n).$$
Theorem 2.5. If $Q$ is a basic sequence and $x$ is strongly $Q$-normal of order $k$, then $x$ is $Q$-normal of order $k$.

Proof. Let $m \leq k$ be a positive integer and let $B$ be a block of length $k$. Since $x$ is strongly $Q$-normal of $k$, we know that for all $p \in [1, m]$, $N_{n,p}^Q(B,x) = Q_{n,p}^{(k)} + o(Q_{n,p}^{(k)})$. Thus, we see that

$$N_{n}^Q(B,x) = \sum_{p=1}^{m} N_{n,p}^Q(B,x) = \sum_{p=1}^{m} \left( Q_{n,p}^{(k)} + o(Q_{n,p}^{(k)}) \right)$$

$$= \sum_{p=1}^{m} Q_{n,p}^{(k)} + o\left( \sum_{p=1}^{m} Q_{n,p}^{(k)} \right) = Q_{n}^{(k)} + o\left( Q_{n}^{(k)} \right),$$

so $\lim_{n \to \infty} \frac{N_{n}^Q(B,x)}{Q_{n}^{(k)}} = 1$. Therefore, $x$ is $Q$-normal of order $k$. \hfill \Box

Corollary 2.6. Suppose that $Q$ is a basic sequence. If $x$ is strongly $Q$-normal, then $x$ is $Q$-normal.

2.2. Construction of a number that is $Q$-normal, but not strongly $Q$-normal of order 2. In this subsection, we will work towards giving an example of a basic sequence $Q$ and a real number $x$ that is $Q$-normal, but not strongly $Q$-normal of order 2. We will use the conventions found in [6].

Given a block $B$, $|B|$ will represent the length of $B$. Given non-negative integers $l_1, l_2, \ldots, l_n$, at least one of which is positive, and blocks $B_1, B_2, \ldots, B_n$, the block $B = l_1 B_1 l_2 B_2 \ldots l_n B_n$ will be the block of length $l_1 |B_1| + \ldots + l_n |B_n|$ formed by concatenating $l_1$ copies of $B_1$, $l_2$ copies of $B_2$, through $l_n$ copies of $B_n$. For example, if $B_1 = (2,3,5)$ and $B_2 = (0,8)$, then $2 B_1 1 B_2 0 B_2 = (2,3,5,2,3,5,0,8)$. We will need the following definitions:

Definition 2.7. A weighting $\mu$ is a collection of functions $\mu^{(1)}, \mu^{(2)}, \mu^{(3)}, \ldots$ with $\sum_{j=0}^{\infty} \mu^{(1)}(j) = 1$ such that for all $k$, $\mu^{(k)} : \{0,1,2,\ldots\}^k \to [0,1]$ and $\mu^{(k)}(b_1, b_2, \ldots, b_k) = \sum_{j=0}^{\infty} \mu^{(k+1)}(j, b_1, b_2, \ldots, b_k, j)$.

Definition 2.8. The uniform weighting in base $b$ is the collection $\lambda_b$ of functions $\lambda_b^{(1)}, \lambda_b^{(2)}, \lambda_b^{(3)}, \ldots$ such that for all $k$ and blocks $B$ of length $k$ in base $b$

$$\lambda_b^{(k)}(B) = b^{-k}.$$  

Definition 2.9. Let $p$ and $b$ be positive integers such that $1 \leq p \leq b$. A weighting $\mu$ is ($p,b$)-uniform if for all $k$ and blocks $B$ of length $k$ in base $p$, we have

$$\mu^{(k)}(B) = \lambda_b^{(k)}(B) = b^{-k}.$$  

Given blocks $B$ and $y$, let $N(B,y)$ be the number of occurrences of the block $B$ in the block $y$. 
Definition 2.10. Let \( \epsilon \) be a real number such that \( 0 < \epsilon < 1 \) and let \( k \) be a positive integer. Assume that \( \mu \) is a weighting. A block of digits \( y \) is \((\epsilon, k, \mu)\)-normal\footnote{Definition 2.10 is a generalization of the concept of \((\epsilon, k)\)-normality, originally due to Besicovitch \cite{Besicovitch}} if for all blocks \( B \) of length \( m \leq k \), we have
\[
\mu^{(m)}(B)|y|(1 - \epsilon) \leq N(B, y) \leq \mu^{(m)}(B)|y|(1 + \epsilon).
\]

For the rest of this subsection, we use the following conventions. Given sequences of non-negative integers \( \{l_i\}_{i=1}^\infty \) and \( \{b_i\}_{i=1}^\infty \) with each \( b_i \geq 2 \) and a sequence of blocks \( \{x_i\}_{i=1}^\infty \), we set
\[
L_i = |l_1x_1 \ldots l_ix_i| = \sum_{j=1}^i l_j|x_j|,
\]
(7)
(8)
\[q_n = b_i \text{ for } L_{i-1} < n \leq L_i,
\]
and
\[
Q = \{q_n\}_{n=1}^\infty.
\]
Moreover, if \((E_1, E_2, \ldots) = l_1x_1l_2x_2 \ldots \), we set
\[
x = \sum_{n=1}^\infty \frac{E_n}{q_1q_2 \ldots q_n}.
\]

Given \( \{q_n\}_{n=1}^\infty \) and \( \{l_i\}_{i=1}^\infty \), it is assumed that \( x \) and \( Q \) are given by the formulas above.

Definition 2.11. A block friendly family is a 6-tuple \( W = \{(l_i, b_i, p_i, \epsilon_i, k_i, \mu_i)\}_{i=1}^\infty \) with non-decreasing sequences of non-negative integers \( \{l_i\}_{i=1}^\infty, \{b_i\}_{i=1}^\infty, \{p_i\}_{i=1}^\infty \) and \( \{k_i\}_{i=1}^\infty \), for which \( b_i \geq 2, b_i \to \infty \) and \( p_i \to \infty \), such that \( \{\mu_i\}_{i=1}^\infty \) is a sequence of \((p_i, b_i)\)-uniform weightings and \( \{\epsilon_i\}_{i=1}^\infty \) strictly decreases to 0.

Definition 2.12. Let \( W = \{(l_i, b_i, p_i, \epsilon_i, k_i, \mu_i)\}_{i=1}^\infty \) be a block friendly family. If \( \lim k_i = K < \infty \), then let \( R(W) = \{0, 1, 2, \ldots, K\} \). Otherwise, let \( R(W) = \{0, 1, 2, \ldots\} \). A sequence \( \{x_i\}_{i=1}^\infty \) of \((\epsilon_i, k_i, \mu_i)\)-normal blocks of non-decreasing length is said to be \( W \)-good if for all \( k \) in \( R \), the following three conditions hold:
\[
\frac{b_i^k}{\epsilon_{i-1} - \epsilon_i} = o(|x_i|);
\]
(11)
\[
\frac{l_{i-1}}{l_i} \frac{|x_{i-1}|}{|x_i|} = o(i^{-1}b_i^{-k});
\]
(12)
\[
\frac{1}{l_i} \frac{|x_{i+1}|}{|x_i|} = o(b_i^{-k}).
\]
(13)

We now state a key theorem of \cite{6}.
Theorem 2.13. Let $W$ be a block friendly family and $\{x_i\}_{i=1}^\infty$ a $W$-good sequence. If $k \in R(W)$, then $x$ is $Q$-normal of order $k$. If $k_i \to \infty$, then $x$ is $Q$-normal.

If $b$ and $w$ are positive integers where $b$ is greater than or equal to 2 and $w \geq 3$ is odd, then we let $C_{b,w}$ be one of the blocks formed by concatenating all the blocks of length $w$ in base $b$ in such a way that there are at least twice as many copies of the block (0) at odd positions as the block (1). For example, we could pick

$$C_{2,3} = 1(0,0,0)1(1,0,1)1(0,1,0)1(0,0,1)1(0,1,1)1(1,0,0)1(1,1,0)1(1,1,1),$$

which has 9 copies of (0) at the odd positions and 3 copies of (1) at the odd positions. Note that $|C_{b,w}| = wb^w$. The next lemma is proven identically to Lemma 4.2 in [6]:

Lemma 2.14. If $K < w$ and $\epsilon = \frac{K}{w}$, then $C_{b,w}$ is $(\epsilon, K, \lambda_b)$-normal.

Theorem 2.15. There exists a basic sequence $Q$ and a real number $x$ such that $x$ is $Q$-normal, but not strongly $Q$-normal of order 2.

Proof. Let $x_1 = (0,1)$, $b_1 = 2$, and $l_1 = 0$. For $i \geq 2$, let $x_i = C_{2i,(2i+1)^2}$, $b_i = 2i$, and $l_i = (2i)^{9i+8}$. Set $\epsilon_1 = 1/2$, $k_1 = 1$, $p_1 = 2$ and $\mu_1 = \lambda_2$. For $i \geq 2$, put $\epsilon_i = 1/(2i + 1)$, $k_i = 2i + 1$, $p_i = b_i$, $\mu_i = \lambda_2$, and $W = \{(l_i, b_i, \epsilon_i, k_i, \mu_i)\}_{i=1}^\infty$. Thus, since $x_i = C_{b,w}$ where $b = 2i$ and $w = (2i + 1)^2$, $x_i$ is $(\epsilon_i, k_i, \lambda_b)$-normal by Lemma 2.14.

In order to show that $\{x_i\}$ is a $W$-good sequence we need to verify (11), (12), and (13). Since $k_i \to \infty$, we let $k$ be an arbitrary positive integer. We will make repeated use of the fact that $|x_i| = (2i + 1)^2 \cdot (2i)^{(2i+1)^2}$. We first verify (11):

$$\lim_{i \to \infty} |x_i|/\left(\frac{(2i)^k}{2(2i+1)^2 \cdot (2i)^{(2i+1)^2}}\right) = \lim_{i \to \infty} \frac{2(2i + 1)^2 \cdot (2i)^{(2i+1)^2}}{(2i)^k \cdot (4i^2 - 1)} = \infty.$$  

We next verify (12). Since $l_{i-1}/l_i < 1$, $(2i - 1)^2/(2i + 1)^2 < 1$ and

$$\left(1 - \frac{1}{i}\right)^{(2i+1)^2} < e^{-2(2i+1)},$$

we have

$$\lim_{i \to \infty} \frac{l_{i-1} \cdot x_{i-1}}{x_i} \leq \lim_{i \to \infty} \frac{i \cdot (2i)^k \cdot (2i - 1)^2 \cdot (2i - 2)^{(2i-1)^2}}{(2i + 1)^2 \cdot (2i)^{(2i+1)^2}}$$

Theorem 2.13 may be used to construct other explicit examples of $Q$-normal numbers that satisfy some unusual conditions. Given a basic sequence $Q$, we say that $x$ is $Q$-distribution normal if the sequence $\{q_1q_2 \cdots q_nx\}$ is uniformly distributed mod 1. [3] uses Theorem 2.13 to give an example of a basic sequence $Q$ and a real number $x$ such that $x$ is $Q$-normal, but $q_1q_2 \cdots q_nx$ (mod 1) $\to 0$, so $x$ is not $Q$-distribution normal.
3. Random Variables Associated With Normality

For this section, we must recall a few basic notions from probability theory. Given a random variable $X$, we will denote the expected value of $X$ as $E[X]$. We will denote the variance of $X$ as $\text{Var}[X]$. Lastly, $P(X = j)$ will represent the probability that $X = j$.

We consider $x$ as a random variable which has uniform distribution on the interval $[0, 1)$. If $x = 0.E_1(x)E_2(x)E_3(x)\ldots$ with respect to $Q$, then we consider $E_1(x), E_2(x), E_3(x), \ldots$ to be random variables. So for all $n$, we have

$$P(E_n(x) = j) = \begin{cases} \frac{1}{q_n} & \text{if } 0 \leq j \leq q_n - 1 \\ 0 & \text{if } j \geq q_n \end{cases}.$$ 

**Lemma 3.1.** If $Q$ is a basic sequence, then the random variables $E_1(x), E_2(x), E_3(x), \ldots$ are independent.

**Proof.** Suppose that $n_1$ and $n_2$ are distinct positive integers and that $0 \leq F_j < q_j - 1$ for all $j$. Then

$$P(E_{n_1}(x) = F_{n_1}, E_{n_2}(x) = F_{n_2}) = \lambda \{ x \in [0, 1) : E_{n_1}(x) = F_{n_1} \text{ and } E_{n_2}(x) = F_{n_2} \}$$

$$= \frac{1}{q_{n_1}q_{n_2}} = \frac{1}{q_{n_1}} \cdot \frac{1}{q_{n_2}} = P(E_{n_1}(x) = F_{n_1}) \cdot P(E_{n_2}(x) = F_{n_2}).$$
Suppose that \( Q \) is a basic sequence, \( b \) is a natural number, \( B \) is a block of length \( k \), and \( m = ik + p \) is an integer with \( p \in [0, k - 1] \). We set
\[
\zeta_{b,n}^Q(x) = \begin{cases} 1 & \text{if } E_n(x) = b \\ 0 & \text{if } E_n(x) \neq n \end{cases},
\]
\[
\zeta_{B,i,p}^Q(x) = \begin{cases} 1 & \text{if } E_{ik+p,k}(x) = B \\ 0 & \text{if } E_{ik+p,k}(x) \neq B \end{cases},
\]
\[
F_m^{(k)} = \mathbb{E} \left[ \zeta_{B,i,p}^Q(x) \right],
\]
\[
V_m^{(k)} = \text{Var} \left[ \zeta_{B,i,p}^Q(x) \right],
\]
and \( t_{m,p}^{(k)} = \sum_{j=0}^{\rho(n,k)} V_j^{(k)} \).

**Lemma 3.2.** For all non-negative integers \( b \), the random variables \( \zeta_{b,1}^Q(x), \zeta_{b,2}^Q(x), \zeta_{b,3}^Q(x), \ldots \)
are independent.

**Proof.** This follows directly from Lemma 3.1 as the random variables \( E_1(x), E_2(x), E_3(x), \ldots \)
are independent.

**Lemma 3.3.** If \( B = (b_1, b_2, \ldots, b_k) \) is a block of length \( k \), then
\[
\zeta_{B,i,p}^Q(x) = \zeta_{b_1,ik+p}(x) \cdot \zeta_{b_2,ik+p+1}(x) \cdots \zeta_{b_k,ik+p+k-1}(x).
\]

**Proof.** By definition,
\[
\zeta_{B,i,p}^Q(x) = \begin{cases} 1 & \text{if } E_{ik+p,k} = B \\ 0 & \text{if } E_{ik+p,k} \neq B \end{cases},
\]
or in other words, \( \zeta_{B,i,p}^Q(x) = 1 \) if
\[
\zeta_{b_1,ik+p}(x) = \zeta_{b_2,ik+p+1}(x) = \cdots = \zeta_{b_k,ik+p+k-1}(x) = 1
\]
and \( \zeta_{B,i,p}^Q(x) = 0 \) otherwise.

**Corollary 3.4.** For all blocks \( B = (b_1, b_2, \ldots, b_k) \) of length \( k \) and non-negative integers \( p_1, p_2 \in [1, k] \), \( i_1 \), and \( i_2 \) with \( (i_1, p_1) \neq (i_2, p_2) \), the random variables \( \zeta_{B,i_1,p_1}^Q(x) \) and \( \zeta_{B,i_2,p_2}^Q(x) \) are independent.

**Proof.** Using Lemma 3.2 and Lemma 3.3 we see that
\[
\mathbb{E} \left[ \zeta_{B,i_1,p_1}^Q(x) \cdot \zeta_{B,i_2,p_2}^Q(x) \right] = \mathbb{E} \left[ \left( \prod_{j=0}^{k-1} \zeta_{b_j,i_1+k+p_1+j}(x) \right) \cdot \left( \prod_{j=0}^{k-1} \zeta_{b_j,i_2+k+p_2+j}(x) \right) \right]
\]
\[
= \left( \prod_{j=0}^{k-1} \mathbb{E} \left[ \zeta_{b_j,i_1+k+p_1+j}(x) \right] \right) \cdot \left( \prod_{j=0}^{k-1} \mathbb{E} \left[ \zeta_{b_j,i_2+k+p_2+j}(x) \right] \right)
\]
\[
= \mathbb{E} \left[ \prod_{j=0}^{k-1} \zeta_{b_j,i_1+k+p_1+j}(x) \right] \cdot \mathbb{E} \left[ \prod_{j=0}^{k-1} \zeta_{b_j,i_2+k+p_2+j}(x) \right]
\]
\[
= \mathbb{E} \left[ \zeta_{B,i_1,p_1}^Q(x) \right] \cdot \mathbb{E} \left[ \zeta_{B,i_2,p_2}^Q(x) \right].
\]

**Lemma 3.5.** If \( B = (b_1, b_2, \ldots, b_k) \) is a block of length \( k \), then
\[
\frac{F_m^{(k)}}{V_m^{(k)}} = \frac{1}{q_{ik+p,q_{ik+p+1}} \cdots q_{ik+p+k-1}}
\]
and
\[
\frac{V_m^{(k)}}{V_m^{(k)}} = \frac{1}{q_{ik+p,q_{ik+p+1}} \cdots q_{ik+p+k-1}} - \left( \frac{1}{q_{ik+p,q_{ik+p+1}} \cdots q_{ik+p+k-1}} \right)^2.
\]
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Proof. We first compute the expected value of \( \zeta_{B,i,p}^Q(x) \). By Lemma 3.2 and Lemma 3.3 we see that

\[
E\left[\zeta_{B,i,p}^Q(x)\right] = E\left[\zeta_{b_1,ik+p}^Q(x) \cdot \zeta_{b_2,ik+p+1}^Q(x) \cdots \zeta_{b_{k},ik+p+k-1}^Q(x)\right]
\]

\[
= \frac{1}{q_{ik+p}} \cdot \frac{1}{q_{ik+p+1}} \cdots \frac{1}{q_{ik+p+k-1}} = \frac{1}{q_{ik+p} q_{ik+p+1} \cdots q_{ik+p+k-1}}.
\]

Next, we recall that \( \text{Var}[\zeta_{B,i,p}^Q(x)] = E\left[\zeta_{B,i,p}^Q(x)^2\right] - E[\zeta_{B,i,p}^Q(x)]^2 \). Since \( \zeta_{B,i,p}^Q(x) \) may only be 0 or 1, we see that \( \left( \zeta_{B,i,p}^Q(x) \right)^2 = \zeta_{B,i,p}^Q(x) \), so

\[
\text{Var}[\zeta_{B,i,p}^Q(x)] = \frac{1}{q_{ik+p} q_{ik+p+1} \cdots q_{ik+p+k-1}} - \left( \frac{1}{q_{ik+p} q_{ik+p+1} \cdots q_{ik+p+k-1}} \right)^2.
\]

Lastly, we remark that \( Q_{n,p}^{(k)} = \sum_{i=0}^{\rho(n,k)} f_{ik+p}^{(k)} \), by Lemma 3.5 and will use this fact frequently and without mention.

4. Typicality of Normal Numbers

We will need the following:

**Theorem 4.1.** \(^4\) Let \( X_1, X_2, \ldots, X_n \) be independent random variables. Assume that there exists a constant \( c > 0 \) such that \( |X_j| < c \) for all \( j \). Let \( G_j = E[X_j], U_j = \text{Var}[X_j], \) and \( t_n = \sum_{j=1}^n U_j \). If \( t_n \rightarrow \infty \), then, with probability one,

\[
\limsup_{n \rightarrow \infty} \frac{X_1 + X_2 + \ldots + X_n - G_1 - G_2 - \ldots - G_n}{\sqrt{2t_n \log \log t_n}} = 1.
\]

**Corollary 4.2.** Under the same assumptions of Theorem 4.1 with probability one,

\[
X_1 + X_2 + \ldots + X_n = G_1 + G_2 + \ldots + G_n + O(t_n^{1/2} \log \log t_n)^{1/2}.
\]

We will also need the Borel-Cantelli Lemma:

**Theorem 4.3.** (The Borel Cantelli Lemma) If \( \sum_{n=1}^{\infty} P(A_n) < \infty \), then \( P(A_n \ i.o.) = 0 \).

Given a basic sequence \( Q \), we will define \( t_{n,p}^{(k)} = \sum_{i=0}^{\rho(n,k)} v_{jk+p}^{(k)} \).

**Lemma 4.4.** If \( Q \) is a basic sequence and \( n, k, \) and \( p \) are positive integers with \( p \in [1, k] \), then

\[
\frac{1}{2} Q_{n,p}^{(k)} \leq t_{n,p}^{(k)} < Q_{n,p}^{(k)}.
\]

\(^4\)See, for example, [11]
Proof.

\[ t_{n,p}^{(k)} = \sum_{i=0}^{\rho(n,k)} \left( \frac{1}{q_{ik+p}q_{ik+p+1} \cdots q_{ik+p+k-1}} - \left( \frac{1}{q_{ik+p}q_{ik+p+1} \cdots q_{ik+p+k-1}} \right)^2 \right) \]

\[ < \sum_{i=0}^{\rho(n,k)} \frac{1}{q_{ik+p}q_{ik+p+1} \cdots q_{ik+p+k-1}} = \sum_{i=0}^{\rho(n,k)} F_{ik}^{(k)} = Q_{n,p}^{(k)}. \]

To show the other direction of the inequality, we recall that since \( Q \) is a basic sequence, \( q_m \geq 2 \) for all \( m \), so for all \( i \)

\[ \sum_{i=0}^{\rho(n,k)} \frac{1}{q_{ik+p}q_{ik+p+1} \cdots q_{ik+p+k-1}} \geq \sum_{i=0}^{\rho(n,k)} \frac{1}{2} \cdot \frac{1}{q_{ik+p}q_{ik+p+1} \cdots q_{ik+p+k-1}} = \frac{1}{2} Q_{n,p}^{(k)}. \]

\[ \square \]

Lemma 4.5. If \( Q \) is infinite in limit and \( B \) is a block of length \( k \), then for almost every real number \( x \) in \([0, 1)\), we have

\[ N_{n,p}^{Q}(B, x) = Q_{n,p}^{(k)} + O \left( \sqrt{Q_{n,p}^{(k)}} \left( \log \log Q_{n,p}^{(k)} \right)^{1/2} \right). \]

Proof. We consider two cases. The first case is when \( \lim_{n \to \infty} Q_{n,p}^{(k)} < \infty \).

We see that

\[ \lim_{n \to \infty} Q_{n,p}^{(k)} = \lim_{n \to \infty} \sum_{i=0}^{\rho(n,k)} \mathbb{P} \left( \zeta_{B,i,p}^{Q} = 1 \right) < \infty, \]

so by Theorem 4.3 we have \( \mathbb{P} \left( \zeta_{B,i,p}^{Q} = 1 \text{ i.o} \right) = 0 \). Thus, for almost every \( x \in [0, 1) \), \( \lim_{n \to \infty} N_{n,p}^{Q}(B, x) < \infty \) and (14) holds.

Second, we consider the case where \( \lim_{n \to \infty} Q_{n,p}^{(k)} = \infty \). By Lemma 4.4 we have \( \lim_{n \to \infty} t_{n,p}^{(k)} \geq \lim_{n \to \infty} Q_{n,p}^{(k)} = \infty \). Note that

\[ N_{n,p}^{Q}(B, x) = \sum_{i=0}^{\rho(n,k)} \zeta_{B,i,p}(x). \]

By Corollary 4.2

\[ N_{n,p}^{Q}(B, x) = \sum_{i=0}^{\rho(n,k)} F_{ik+p}^{(k)} + O \left( \sqrt{t_{n,p}^{(k)}} \left( \log \log t_{n,p}^{(k)} \right)^{1/2} \right) \]
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for almost every \( x \in [0,1) \). By Lemma 4.4, \( \ell^{(k)}_{n,p} < Q^{(k)}_{n,p} \), so the lemma follows. □

Lemma 4.5 allows us to prove the following results on strongly normal numbers:

**Theorem 4.6.** Suppose that \( Q \) is strongly \( k \)-divergent and infinite in limit. Then almost every \( x \in [0,1) \) is strongly \( Q \)-normal of order \( k \).

**Proof.** Let \( B \) be a block of length \( m \leq k \) and \( p \in [1,m] \). Then by Lemma 4.5, for almost every \( x \in [0,1) \), we have that

\[
N_{n,p}^{(m)}(B, x) = Q_{n,p}^{(m)} + O \left( \sqrt{Q_{n,p}^{(m)}} \left( \log \log Q_{n,p}^{(m)} \right)^{1/2} \right),
\]

so

\[
\frac{N_{n,p}^{(m)}(B, x)}{Q_{n,p}^{(m)}} = 1 + O \left( \sqrt{Q_{n,p}^{(m)}} \left( \log \log Q_{n,p}^{(m)} \right)^{1/2} \right).
\]

However, \( Q \) is strongly \( k \)-divergent, so \( Q_{n,p}^{(m)} \to \infty \) and

\[
\lim_{n \to \infty} \frac{N_{n,p}^{(m)}(B, x)}{Q_{n,p}^{(m)}} = \lim_{n \to \infty} \left( 1 + O \left( \sqrt{Q_{n,p}^{(m)}} \left( \log \log Q_{n,p}^{(m)} \right)^{1/2} \right) \right) = 1.
\]

Since there are finitely many choices of \( m \) and \( p \) and only countably many choices of \( B \), the result follows. □

**Corollary 4.7.** If \( Q \) is strongly fully divergent and infinite in limit, then almost every real \( x \in [0,1) \) is strongly \( Q \)-normal.

We now work towards proving a result much stronger than Corollary 4.7 on the typicality of \( Q \)-normal numbers. We will need the following lemma in addition to Lemma 4.5:

**Lemma 4.8.** If \( Q \) is a basic sequence and \( k \) and \( p \) are positive integers with \( p \in [1,k] \), then

\[
\sum_{p=1}^{k} \left( Q_{n,p}^{(k)} + O \left( \sqrt{Q_{n,p}^{(k)}} \left( \log \log Q_{n,p}^{(k)} \right)^{1/2} \right) \right) = Q_{n}^{(k)} + O \left( \sqrt{Q_{n}^{(k)}} \left( \log \log Q_{n}^{(k)} \right)^{1/2} \right).
\]

**Proof.** We first note that \( \sum_{p=1}^{k} Q_{n,p}^{(k)} \leq Q_{n}^{(k)} + \left( Q_{n}^{(k)} - Q_{n-k}^{(k)} \right) \). Since \( Q_{n}^{(k)} - Q_{n-k}^{(k)} \leq (k+1)2^{-k} \to 0 \), we see that

\[
\sum_{p=1}^{k} Q_{n,p}^{(k)} = Q_{n}^{(k)} + o(1).
\]
Next, note that
\[
\sum_{p=1}^{k} \sqrt{Q_{n,p}^{(k)}} \left( \log \log Q_{n,p}^{(k)} \right)^{1/2} \leq k \sqrt{\sum_{p=1}^{k} Q_{n,p}^{(k)} \left( \log \log \sum_{p=1}^{k} Q_{n,p}^{(k)} \right)^{1/2}}.
\]

By (15) and (16),
\[
\sum_{p=1}^{k} O \left( \sqrt{Q_{n,p}^{(k)}} \left( \log \log Q_{n,p}^{(k)} \right)^{1/2} \right) = O \left( \sqrt{Q_{n}^{(k)}} \left( \log \log Q_{n}^{(k)} \right)^{1/2} \right).
\]

Thus, the lemma follows by combining (15) and (17). □

**Theorem 4.9.** If \( Q \) is a basic sequence that is infinite in limit and \( B \) is a block of length \( k \), then for almost every real number \( x \) in \([0, 1)\), we have
\[
N_{Q}^{n}(B, x) = Q_{n}^{(k)} + O \left( \sqrt{Q_{n}^{(k)}} \left( \log \log Q_{n}^{(k)} \right)^{1/2} \right).
\]

**Proof.** We first note that
\[
N_{Q}^{n}(B, x) = \sum_{p=1}^{k} N_{n,p}(B, x) + O(1).
\]

Thus, by (18) and Lemma 4.5 for almost every \( x \in [0, 1) \), we have
\[
N_{Q}^{n}(B, x) = \sum_{p=1}^{k} \left( Q_{n,p}^{(k)} + O \left( \sqrt{Q_{n,p}^{(k)}} \left( \log \log Q_{n,p}^{(k)} \right)^{1/2} \right) \right) + O(1).
\]

Thus, the theorem follows by applying Lemma 4.8 to (19). □

We recall the following standard result on infinite products:

**Lemma 4.10.** If \( \{a_{n}\}_{n=1}^{\infty} \) is a sequence of real numbers such that \( 0 \leq a_{n} < 1 \) for all \( n \), then the infinite product \( \prod_{n=1}^{\infty} (1 - a_{n}) \) converges if and only if the sum \( \sum_{n=1}^{\infty} a_{n} \) is convergent.

**Theorem 4.11.** Suppose that \( Q \) is a basic sequence that is infinite in limit. Then almost every real number in \([0, 1)\) is \( Q \)-normal of order \( k \) if and only if \( Q \) is \( k \)-divergent.

**Proof.** First, we suppose that \( Q \) is \( k \)-divergent. Then by Theorem 4.9 for almost every \( x \in [0, 1) \), we have
\[
\lim_{n \to \infty} \frac{N_{Q}^{n}(B, x)}{Q_{n}^{(k)}} = \lim_{n \to \infty} \frac{Q_{n}^{(k)} + O \left( \sqrt{Q_{n}^{(k)}} \left( \log \log Q_{n}^{(k)} \right)^{1/2} \right)}{Q_{n}^{(k)}} = 1.
\]

We now suppose that \( Q \) is \( k \)-convergent. We will now use similar reasoning to that found in [7]. Set \( B = (0, 0, \ldots, 0) \) (\( k \) zeros). We will show that the set
of real numbers in \([0, 1]\) whose \(Q\)-Cantor series expansion does not contain
the block \(B\) has positive measure. Call this set \(V\). We see that
\[
\lambda(V) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{q_n q_{n+1} \cdots q_{n+k-1}}\right).
\]
Set \(a_n = q_n q_{n+1} \cdots q_{n+k-1}\). Since \(Q\) is \(k\)-convergent, we have \(\sum a_n < \infty\).
Thus, \(\lambda(V) > 0\) by Lemma \([4, 10]\).

**Corollary 4.12.** Suppose that \(Q\) is a basic sequence that is infinite in limit.
Then almost every real number in \([0, 1]\) is \(Q\)-normal if and only if \(Q\) is fully
divergent.

5. Ratio normal numbers

We are now in a position to compare the prevalence of \(Q\)-normal numbers
to \(Q\)-ratio normal numbers, depending on properties of the basic sequence \(Q\). In particular, we will show that if \(Q\) is infinite in limit, then the set of
\(Q\)-ratio normal numbers is dense in \([0, 1]\) even though the set of \(Q\)-normal numbers may be empty. Suppose that \(Q\) is a \(k\)-convergent basic sequence
and define
\[(20) \quad Q_n^{(k)} = \lim_{n \to \infty} Q_n^{(k)} < \infty.\]

**Proposition 5.1.** If \(Q\) is a basic sequence that is \(k\)-convergent for some \(k\),
then the set of \(Q\)-normal numbers is empty.

**Proof.** We make the observation that since \(q_n \geq 2\) for all \(n\), \(Q_n^{(k)} \leq \frac{1}{2} Q_n^{(k-1)}\)
for all \(k\). Thus, there exists a \(K > 0\) such that for all \(k > K\), we have
\(Q_n^{(k)} < 1\). Thus, no blocks of length \(k > K\) can occur in any \(Q\)-normal number and the set of \(Q\)-normal numbers is empty. \(\square\)

If \(B = (b_1, b_2, \cdots, b_k)\) is a block of length \(k\), we write
\[
\max(B) = \max(b_1, b_2, \cdots, b_k).
\]
If \(E = (E_1, E_2, \cdots)\), then set \(E_{n,k} = (E_n, E_{n+1}, \cdots, E_{n+k-1})\).

**Proposition 5.2.** If \(Q = \{q_n\}_{n=1}^{\infty}\) is infinite in limit, then there exists a
real number that is \(Q\)-ratio normal.

**Proof.** Let \(Q' = \{q'_n\}_{n=1}^{\infty}\) be any fully divergent basic sequence that is
infinite in limit. Then we know that there exists a \(Q'\)-normal number by
Corollary \([4, 12]\). Let \(x = 0.E_1'E_2'E_3'\cdots\) with respect to \(Q'\) be \(Q'\)-normal
and let \(E' = (E_1', E_2', \cdots)\). Set \(M_k = \min\{m : q_n > k \ \forall n \geq m\}, E_n = \min(E_n', q_n - 1)\), and \(E = (E_1, E_2, \cdots)\). Suppose that \(B\) and \(B'\) are two
blocks of length \(k\) and let \(l = \max(\max(B), \max(B')) + 2\).

Thus, if \(n > M_l\), then \(E'_{n,k} = B\) is equivalent to \(E_{n,k} = B\) and \(E'_{n,k} = B'\)
is equivalent to \(E_{n,k} = B'\). Since \(x\) is \(Q'\)-normal, there are infinitely
many occurrences of every block. Additionally, \(E_n \leq q_n - 1\) for all \(n\), so
\[
\sum_{n=1}^{\infty} \frac{E_n}{q_{m+1}q_{m+2}\cdots q_n} \text{ is } Q\text{-ratio normal.} \]
Corollary 5.3. If \( Q \) is infinite in limit, then the set of numbers that are \( Q \)-ratio normal is dense in \([0, 1)\).

References

[1] Altomare, C., Mance, B.: Cantor Series Constructions Contrasting Two Notions of Normality. Monatsh. Math. 164, 1–22 (2011)
[2] Besicovitch, A. S.: The asymptotic distribution of the numerals in the decimal representation of the squares of the natural numbers. Math. Zeit. 39, 146–156 (1934)
[3] Champernowne, D. G.: The construction of decimals normal in the scale of ten. Journal of the London Mathematical Society 8, 254–260 (1933)
[4] Drmota, M., Tichy, R. F.: Sequences, Discrepancies and Applications. Springer-Verlag, Berlin Heidelberg (1997)
[5] Kuipers, L., Niederreiter, H.: Uniform Distribution of Sequences. Dover, Mineola, NY (2006)
[6] Mance, B.: Construction of normal numbers with respect to the \( Q \)-Cantor series representation for certain \( Q \). Acta Arith. 148, 135–152 (2011)
[7] Rényi, A.: On the distribution of the digits in Cantor’s series. Mat. Lapok 7, 77–100 (1956)
[8] Schweiger, F.: Über den Satz von Borel-Rényi in der Theorie der Cantorschen Reihen. Monats. Math. 74, 150–153 (1969)
[9] G. Cantor, Über die einfachen Zahlensysteme, Zeitschrift für Math. und Physik 14, pp. 121–128 (1869)
[10] J. Galambos, Representations of real numbers by infinite series, Lecture Notes in Math. 502, Springer-Verlag, Berlin, Hiedelberg, New York, 1976.
[11] Vervaat, W.: Success epochs in Bernoulli trials with applications in number theory. Math. Centre Tracts, Amsterdam, 1972. Vol 42