Polynomials in categories with pullbacks

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Abstract. The theory developed in [5], of polynomials over a locally cartesian closed category $E$, is generalised for $E$ just having pullbacks. The 2-categorical analogue of the theory of polynomials and polynomial functors is given. Canonical examples are presented which illustrate the ubiquity of polynomials within 2-categories, and their relevance for the study of internal fibrations [11] and 2-toposes [16].

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1. Introduction

Thanks to unpublished work of André Joyal dating back to the 1980’s, polynomials admit a beautiful categorical interpretation. Given a multivariable polynomial function $p$ with natural number coefficients, like say

$$p(w, x, y, z) = (x^3 y + 2, 3x^2 z + y)$$

one may break down its formation as follows. There is a set $\text{In}$ of input variables and $\text{Out}$ of output variables. In our example $\text{In} = \{w, x, y, z\}$ and $\text{Out}$ has 2-elements. There is a set $\text{MSum}$ of monomial summands, which in this case is the set

$$\text{MSum} = \{x^3 y, 1, 1, x^2 z, x^2 z, x^2 z, y\}$$

of cardinality 7, and a set $\text{UVar}$ of usages of variables which in this case is the set

$$\text{UVar} = \{x, x, x, y, x, x, z, x, x, z, x, z, y\}.$$
of cardinality 14. The task of forming the polynomial $p$ can then be done in 3 steps. First one takes the input variables and duplicates or ignores them according to how often each variable is used. The book-keeping of this step is by means of the evident function $p_1 : \text{UVar} \to \text{In}$, which in our example we have indicated by how we labelled the elements of the set $\text{UVar}$. In the second step one performs all the multiplications, and this is book-kept by taking products over the fibres of the function $p_2 : \text{UVar} \to \text{MSum}$ which sends each usage to the monomial summand in which it occurs. Finally one adds up the summands, and this is book-kept by summing over the fibres of the evident function $p_3 : \text{MSum} \to \text{Out}$. Thus the polynomial $p$ “is” the diagram

$$
\begin{array}{cccc}
\text{In} & \xrightarrow{p_1} & \text{UVar} & \xrightarrow{p_2} & \text{MSum} & \xrightarrow{p_3} & \text{Out} \\
\end{array}
$$

in the category $\text{Set}$. A categorical interpretation of the formula (1) from the diagram (2) begins by regarding an $n$-tuple of variables as (the fibres of) a function into a given set of cardinality $n$. Duplication of variables is then interpreted by the functor $\Delta_{p_1} : \text{Set}/\text{In} \to \text{Set}/\text{UVar}$ given by pulling back along $p_1$, taking products by the functor $\Pi_{p_2} : \text{Set}/\text{UVar} \to \text{Set}/\text{MSum}$ and taking sums by applying the functor $\Sigma_{p_3} : \text{Set}/\text{MSum} \to \text{Set}/\text{Out}$ given by composing with $p_3$. Composing these functors gives

$$P(p) : \text{Set}/\text{In} \to \text{Set}/\text{Out}$$

the polynomial functor corresponding to the polynomial $p$.

Functors of the form $\Delta_{p_1}$, $\Pi_{p_2}$ and $\Sigma_{p_3}$ are part of the bread and butter of category theory. For any map $p_3$ in any category, one may define $\Sigma_{p_3}$ between the appropriate slices, and one requires only pullbacks in the ambient category to interpret $\Delta_{p_1}$ more generally. The functor $\Pi_{p_2}$ is by definition the right adjoint of $\Delta_{p_2}$, and its existence is a condition on the map $p_2$, called exponentiability. Locally cartesian closed categories are by definition categories with finite limits in which all maps are exponentiable. Consequently a reasonable general categorical definition of polynomial is as a diagram

$$
\begin{array}{cccc}
X & \xrightarrow{p_1} & A & \xrightarrow{p_2} & B & \xrightarrow{p_3} & Y \\
\end{array}
$$

in some locally cartesian closed category $\mathcal{E}$. The theory polynomials and polynomial functors was developed at this generality in the beautiful paper [5] of Gambino and Kock. There the question of what structures polynomials in a locally cartesian closed $\mathcal{E}$ form was considered, and it was established in particular that polynomials can be seen as the arrows of certain canonical bicategories, with the process of forming the associated polynomial functor giving homomorphisms of bicategories.

In this paper we shall focus on the bicategory $\text{Poly}_\mathcal{E}$ of polynomials and cartesian maps between them in the sense of [5]. Our desire to generalise the above setting comes from the existence of canonical polynomials and polynomial functors for the case $\mathcal{E} = \text{CAT}$ and the wish that they sit properly within an established framework. While local cartesian closedness is a very natural condition of great importance to categorical logic, enjoyed for example by any elementary topos, it is not satisfied by $\text{CAT}$. The natural remedy of this defect is to define a polynomial $p$ between $X$ and $Y$ in a category $\mathcal{E}$ with pullbacks to be a diagram as in (2) such that $p_2$ is an exponentiable map. Since exponentiable maps are pullback stable and closed under composition, one obtains the bicategory $\text{Poly}_\mathcal{E}$ together
with the “associated-polynomial-functor homomorphism”, as before. We describe this in sections 2 and 3.

The main technical innovation of those sections is to remove any reliance on type theory in the proofs, giving a completely categorical account of the theory. In establishing the bicategory structure on $\text{Poly}_E$ in section 2 of [5], the internal language of $E$ is used in an essential way, especially in the proof of proposition (2.9). Our development makes no use of the internal language nor of the strengths that play such a central role in [5]. Instead we isolate the concept of a distributivity pullback and prove some elementary facts about them. Armed with this technology we then proceed to give an elementary account of the bicategory of polynomials, and the homomorphism which encodes the formation of associated polynomial functors. Our treatment requires only pullbacks in $E$.

Our second extension to the categorical theory of polynomials is motivated by the fact that $\text{CAT}$ is a 2-category, and our canonical examples are very 2-categorical in flavour. Thus in section 7 we develop the theory of polynomials within a 2-category $K$ with pullbacks, and the polynomial 2-functors that they determine. In this context the structure formed by polynomials is a degenerate kind of tricategory, called a 2-bicategory, which roughly speaking is a bicategory whose homs are 2-categories instead of categories. However except for this change, the theory works in the same way as for categories. Though it must be said that our treatment of the 1-categorical version of the theory in sections 2-3 was tailored in order to make the previous sentence true (in addition to giving the desired generalisation).

Just as one may easily consider monads within any bicategory, one may easily consider pseudo-monads within any 2-bicategory. The canonical examples described in section 5 are in fact polynomial pseudo-monads, that is to say, pseudo-monads within the 2-bicategories $\text{Poly}_K$ established in section 7. These examples arise from two sources. One is from the theory of fibrations within a finitely complete 2-category [11]. The other is that a classifying discrete opfibration in the sense of [16] within a nice enough $K$ determines a polynomial pseudo-monad, whose associated pseudo-monad on $K$ amounts, in the paradigmatic example, to the usual $\text{Fun}$-construction.

As explained in [5], polynomial functors arise in diverse mathematical contexts. In particular Tambara [14] studied polynomials over categories of finite $G$-sets motivated by representation theory and group cohomology. Very interesting applications of Tambara’s work were found by Brun in [2] to Witt vectors, and in [3] also to equivariant stable homotopy theory and cobordism. These developments together with our extension of the theory of polynomials to 2-categories in section 7, its connections with certain important aspects of 2-category theory in section 5, and the sense in which the bicategories of polynomials may be iterated to produced higher categorical structures in remark 28, indicates that the further study of polynomials within higher dimensional category theory is likely to be very fruitful.

2. Exponentiable morphisms

Given a morphism $f : X \to Y$ in a category $E$, we denote by $\Sigma f : E/X \to E/Y$ the functor given by composition with $f$. When $E$ has pullbacks $\Sigma f$ has a right adjoint denoted as $\Delta f$, given by pulling back maps along $f$. When $\Delta f$ has a right
adjoint, denoted as $\Pi_f$, $f$ is said to be exponentiable. A commutative square in $\mathcal{E}$ as on the left

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow h & & \downarrow k \\
C & \xrightarrow{g} & D
\end{array}
\quad
\begin{array}{ccc}
\mathcal{E}/A & \xrightarrow{\Sigma_f} & \mathcal{E}/B \\
\downarrow \Delta_h & & \downarrow \Delta_k \\
\mathcal{E}/C & \xrightarrow{\Sigma_g} & \mathcal{E}/D
\end{array}
\quad
\begin{array}{ccc}
\mathcal{E}/A & \xrightarrow{\Delta_f} & \mathcal{E}/B \\
\downarrow \Pi_h & & \downarrow \Pi_k \\
\mathcal{E}/C & \xrightarrow{\Delta_g} & \mathcal{E}/D
\end{array}
\]

determines a natural transformation $\alpha$ as in the middle, as the mate of the identity $\Sigma_h \Sigma_f = \Sigma_g \Sigma_h$ via the adjunctions $\Sigma_h \dashv \Delta_h$ and $\Sigma_k \dashv \Delta_k$. We call $\alpha$ a left Beck-Chevalley cell for the original square. There is another left Beck-Chevalley cell for this square, namely $\Sigma_h \Delta_f \to \Delta_g \Sigma_h$, obtained by mating the identity $\Sigma_h \Sigma_f = \Sigma_g \Sigma_h$ with the adjunctions $\Sigma_f \dashv \Delta_f$ and $\Sigma_g \dashv \Delta_g$. If in addition $h$ and $k$ are exponentiable maps, then taking right adjoints produces the natural transformation $\beta$ from $\alpha$, and we call this a right Beck-Chevalley cell for the original square. There is another right Beck-Chevalley cell $\Delta_k \Pi_g = \Pi_f \Delta_h$ when $f$ and $g$ are exponentiable. It is well-known that the original square is a pullback iff either associated left Beck-Chevalley cell is invertible, and when $h$ and $k$ are exponentiable, these conditions are also equivalent to the right Beck-Chevalley cell $\beta$ being an isomorphism. Under these circumstances we shall speak of the left or right Beck-Chevalley isomorphisms.

Clearly exponentiable maps are closed under composition and any isomorphism is exponentiable. Moreover, exponentiable maps are pullback stable. For given a pullback square as above in which $g$ is exponentiable, one has $\Sigma_h \Delta_f \cong \Delta_g \Sigma_h$, and since $\Sigma_h$ is comonadic, $\Delta_g$ has a right adjoint by the Dubuc adjoint triangle theorem.

When $\mathcal{E}$ has a terminal object $1$ and $f$ is the unique map $X \to 1$, we denote by $\Sigma_X$, $\Delta_X$ and $\Pi_X$ the functors $\Sigma_f$, $\Delta_f$ and $\Pi_f$ (when it exists) respectively. In fact since $\Sigma_X : \mathcal{E}/X \to \mathcal{E}$ takes the domain of a given arrow into $X$, it makes sense to speak of it even when $\mathcal{E}$ doesn’t have a terminal object. An object $X$ of a finitely complete category $\mathcal{E}$ is exponentiable when the unique map $X \to 1$ is exponentiable in the above sense (i.e. when $\Pi_X$ exists). A finitely complete category $\mathcal{E}$ is cartesian closed when all its objects are exponentiable, and locally cartesian closed when all its morphisms are exponentiable.

Note that as right adjoints the functors $\Delta_f$ and $\Pi_f$ preserve terminal objects. An object $h : A \to X$ of the slice category $\mathcal{E}/X$ is terminal iff $h$ is an isomorphism in $\mathcal{E}$, but there is also a canonical choice of terminal object for $\mathcal{E}/X$ – the identity $1_X$. So for the sake of convenience we shall often assume below that $\Delta_f$ and $\Pi_f$ are chosen so that $\Delta_f(1_Y) = 1_X$ and $\Pi_f(1_X) = 1_Y$.

## 3. Distributivity pullbacks

For $f : A \to B$ in $\mathcal{E}$ a category with pullbacks, $\Delta_f : \mathcal{E}/B \to \mathcal{E}/A$ expresses the process of pulling back along $f$ as a functor. One may then ask: what basic categorical process is expressed by the functor $\Pi_f : \mathcal{E}/A \to \mathcal{E}/B$, when $f$ is an exponentiable map?

Let us denote by $\varepsilon_f^{(1)}$ the counit of $\Sigma_f \dashv \Delta_f$, and when $f$ is exponentiable, by $\varepsilon_f^{(2)}$ the counit of $\Delta_f \dashv \Pi_f$. The components of these counits fit into the following
pullbacks:

\[
\begin{array}{c}
\Delta_f b \\
\downarrow \\
A \\
\rightarrow \\
B
\end{array}
\quad
\begin{array}{c}
\varepsilon^{(1)}_{f,b} \\
\downarrow \\
Y \\
\rightarrow \\
X
\end{array}
\quad
\begin{array}{c}
\Pi_f a \\
\downarrow \\
R \\
\rightarrow \\
B
\end{array}
\quad
\begin{array}{c}
P a \\
\downarrow \\
Q \\
\rightarrow \\
A
\end{array}
\]

Now the universal property of \(\varepsilon^{(1)}_f\), as the counit of the adjunction \(\Sigma_f \dashv \Delta_f\), is equivalent to the square on the left being a pullback as indicated. An answer to the above question is obtained by identifying what is special about the diagram on the right in (4), that corresponds to the universal property of \(\varepsilon^{(2)}_f\) as the counit of \(\Delta_f \dashv \Pi_f\). To this end we make

**Definition 1.** Let \(g : Z \to A\) and \(f : A \to B\) be a composable pair of morphisms in a category \(E\). Then a pullback around \((f, g)\) is a diagram

\[
\begin{array}{c}
X \\
\downarrow q \\
Y \\
\rightarrow \\
Z \\
\rightarrow \\
A \\
\rightarrow \\
B
\end{array}
\]

in which the square with boundary \((gp, f, r, q)\) is, as indicated, a pullback. A morphism \((p, q, r) \to (p', q', r')\) of pullbacks around \((f, g)\) consists of \(s : X \to X'\) and \(t : Y \to Y'\) such that \(p's = p\), \(qs = tq'\) and \(r = r's\). The category of pullbacks around \((f, g)\) is denoted \(\text{PB}(f, g)\).

For example the pullback on the right in (4) exhibits \((\varepsilon^{(2)}_{f,a}, \varepsilon^{(1)}_{f,\Pi_f a}, \Pi_f a)\) as a pullback around \((f, a)\). One may easily observe directly that the universal property of \(\varepsilon^{(2)}_{f,a}\) is equivalent to \((\varepsilon^{(2)}_{f,a}, \varepsilon^{(1)}_{f,\Pi_f a}, \Pi_f a)\) being a terminal object of \(\text{PB}(f, a)\). Thus we make

**Definition 2.** Let \(g : Z \to A\) and \(f : A \to B\) be a composable pair of morphisms in a category \(E\). Then a distributivity pullback around \((f, g)\) is a terminal object of \(\text{PB}(f, g)\). When \((p, q, r)\) is a distributivity pullback, we denote this diagramatically as follows:

\[
\begin{array}{c}
X \\
\downarrow q \\
Y \\
\rightarrow \\
Z \\
\rightarrow \\
A \\
\rightarrow \\
B
\end{array}
\]

and we say that this diagram exhibits \(r\) as a distributivity pullback of \(g\) along \(f\).

Thus the answer to the question posed at the beginning of this section is: when \(f : A \to B\) is an exponentiable map in \(E\) a category with pullbacks, the functor \(\Pi_f : E/A \to E/B\) encodes the process of taking distributivity pullbacks along \(f\).

For any \((p, q, r) \in \text{PB}(f, g)\) one has a Beck-Chevalley isomorphism \(\Pi_q \Delta_p \Delta_g \cong \Delta_p \Pi_f\), which when you mate it by \(\Sigma_r \dashv \Delta_r\) and \(\Sigma_g \dashv \Delta_g\) gives a natural transformation \(\delta_{p,q,r} : \Sigma_r \Pi_q \Delta_p \to \Pi_f \Sigma_g\). When this is an isomorphism, it expresses a type of distributivity of “sums” over “products”, and so the following proposition explains why we use the terminology distributivity pullback.
**Proposition 3.** Let $f$ be an exponentiable map in a category $\mathcal{E}$ with pullbacks. Then $(p, q, r)$ is a distributivity pullback around $(f, g)$ iff $\delta_{p, q, r}$ is an isomorphism.

**Proof.** Since $(\varepsilon_{f,g}^{(2)}, \varepsilon_{f,\Pi fg}^{(1)}, \Pi fg)$ is terminal in $\text{PB}(f, g)$, one has unique morphisms $d$ and $e$ fitting into a commutative diagram

\[
\begin{array}{c}
\varepsilon_{f,g}^{(2)} \quad X \quad Y \\
\downarrow d \quad \downarrow pb \quad \downarrow e \\
\varepsilon_{f,\Pi fg}^{(1)} \quad E \quad B \\
\end{array}
\]

in which the middle square is a pullback by the elementary properties of pullbacks. Thus $(p, q, r)$ is a distributivity pullback iff $e$ is an isomorphism. Since the adjunctions $\Sigma_r \dashv \Delta_r$ and $\Sigma_g \dashv \Delta_g$ are cartesian, $\delta_{p, q, r}$ is cartesian, and so it’s an isomorphism iff its component at $1_Z \in \mathcal{E}/Z$ is an isomorphism. Since $\Delta_p(1_Z) = 1_X$ and $\Pi_q(1_X) = 1_Y$ one may easily witness directly that $(\delta_{p, q, r})_{1_X} = e$. \hfill \Box

When manipulating pullbacks in a general category, one uses the “elementary fact” that given a commutative diagram of the form

\[
\begin{array}{c}
A \quad B \quad C \\
\downarrow \quad \downarrow pb \\
D \quad E \quad F \\
\end{array}
\]

then the front square is a pullback iff the composite square is. In the remainder of this section we identify three elementary facts about distributivity pullbacks.

**Lemma 4.** (Composition/cancellation) Given a diagram of the form

\[
\begin{array}{c}
B_0 \quad B_1 \quad B_2 \\
\downarrow h_1 \quad \downarrow \quad \downarrow \downarrow pb \\
B_1 \quad B_3 \quad B_4 \\
\downarrow h_3 \quad \downarrow pb \quad \downarrow h_5 \\
B_2 \quad B_5 \\
\downarrow h_2 \quad \downarrow h_6 \quad \downarrow \downarrow h_7 \\
B \\
\end{array}
\]

in any category with pullbacks, then the right-most pullback is a distributivity pullback around $(g, h_4)$ iff the composite diagram is a distributivity pullback around $(gf, h)$. 
are given such that the square with boundary \((hk_1, gf, k_3, k_2)\) is a pullback. Then we must exhibit \(r : C_1 \to B_6\) and \(s : C_2 \to B_5\) unique such that \(h_2h_8r = k_1\), \(h_8h_9r = sk_2\) and \(h_7s = k_3\). Form \(C_3, k_4\) and \(k_5\) by taking the pullback of \(k_3\) along \(g\), and then \(k_6\) is unique such that \(k_5k_6 = k_2\) and \(k_4k_6 = fhk_3\). Clearly the square with boundary \((hk_1, f, k_4, k_6)\) is a pullback around \((f, h)\). From the universal property of the left-most distributivity pullback, one has \(k_7\) and \(k_8\) as shown unique such that \(k_1 = h_2k_7\), \(h_3k_7 = ks_6\) and \(h_4k_8 = k_4\). From the universal property of the right-most distributivity pullback, one has \(k_9\) and \(k_{10}\) as shown unique such that \(k_8 = h_5k_9\), \(h_6k_9 = k_{10}k_5\) and \(h_7k_{10} = k_3\). Clearly \(h_5k_9k_6 = h_3k_7\) and so by the universal property of the top-left pullback square one has \(k_{11}\) as shown unique such that \(h_8k_{11} = k_7\) and \(h_9k_{11} = k_9k_6\). Clearly \(h_2h_8k_{11} = k_1\), \(h_6h_9k_{11} = k_{10}k_2\) and \(h_7k_{10} = k_3\) and so we have established the existence of maps \(r\) and \(s\) with the required properties.

As for uniqueness, let us suppose now that \(r : C_1 \to B_6\) and \(s : C_2 \to B_5\) are given such that \(h_2h_8r = k_1\), \(h_6h_9r = sk_2\) and \(h_7s = k_3\). We must verify that \(r = k_{11}\) and \(s = k_{10}\). Since the right-most distributivity pullback is in particular a pullback, one has \(k_6' : C_3 \to B_4\) unique such that \(h_4h_5k_6' = k_4\) and \(h_6k_6' = sk_5\). Since \((h_4h_5, h_6)\) are jointly monic, and clearly \(h_4h_5h_9r = h_4h_5k_6'k_6\) and \(h_6h_9r = h_6k_6'k_6\), we have \(h_9r = k_6'k_6\). By the universal property of the left-most distributivity pullback, it follows that \(h_5k_6' = k_8\) and \(h_8r = k_7\). Thus by the universal property of the left-most distributivity pullback, it follows that \(k_9 = k_9'\) and \(k_{10} = s\). Since \((h_8, h_9)\) are jointly monic, \(h_8k_{11} = k_7\) and \(h_9k_{11} = k_9k_6 = h_9r\), we have \(r = k_{11}\).
Conversely, suppose that the composite diagram is a distributivity pullback around \((gf, h)\), and that \(C_1, C_2, k_1, k_2\) and \(k_3\) as in

are given such that the square with boundary \((h_4k_1, g, k_3, k_2)\) is a pullback. We must give \(r : C_1 \to B_4\) and \(s : C_2 \to B_5\) unique such that \(k_1 = h_5r, h_6r = sk_2\) and \(h_7s = k_3\). Pullback \(k_1\) along \(h_3\) to produce \(C_3, k_4\) and \(k_5\). This makes the square with boundary \((hh_2k_4, gf, k_3, h_2k_3)\) a pullback around \((gf, h)\). Thus one has \(k_6\) and \(k_7\) as shown unique such that \(h_8k_6 = k_4, h_6h_9k_6 = k_7k_2k_1\) and \(k_7h_7 = k_3\). By universal property of the right pullback and since \(gh_4k_1 = h_7k_7k_2\), one has \(k_8\) as shown unique such that \(h_5k_8 = k_1\) and \(h_6k_8 = k_7k_2\). By the uniqueness part of the universal property of the left distributivity pullback, it follows that \(h_5k_8 = k_1\), and so we have established the existence of maps \(r\) and \(s\) with the required properties.

As for uniqueness let us suppose that we are given \(r : C_1 \to B_4\) and \(s : C_2 \to B_5\) such that \(k_1 = h_5r, h_6r = sk_2\) and \(h_7s = k_3\). We must verify that \(r = k_8\) and \(s = k_7\). By the universal property of the top-left pullback one has \(k_6'\) unique such that \(h_8k_6' = k_4\) and \(h_9k_6' = rk_3\). By the uniqueness part of the universal property of the left distributivity pullback, it follows that \(h_8k_6 = h_8k_6'\) and \(h_5k_8 = h_5r\). Thus by the uniqueness part of the universal property of the composite distributivity pullback, it follows that \(k_6 = k_6'\) and \(s = k_7\). Since \((h_4h_5, h_6)\) are jointlymonic, it follows that \(r = k_8\).

\[\square\]

**Lemma 5.** (The cube lemma). Given a diagram of the form

in any category with pullbacks, in which regions \((1)\) and \((2)\) commute, region \((3)\) is a pullback around \((f_2, d_2)\), the square with boundary \((f_1, k_1, g_1, h_1)\) is a pullback
and the bottom distributivity pullback is around \((g_2, d_4)\). Then regions (1) and (2) are pullbacks iff region (3) is a distributivity pullback around \((f_2, d_2)\).

**Proof.** Let us suppose that (1) and (2) are pullbacks and \(p, q \text{ and } r\) are given as in

![Diagram](image)

such that the square with boundary \((q, r, f_2, d_2 p)\) is a pullback. Then one can use the bottom distributivity pullback to induce \(s_2 \text{ and } t_2\) as shown, and then the pullbacks (1) and (2) to induce \(s\) and \(t\), and these clearly satisfy \(d_1 s = p, f_1 s = t q\) and \(r = d_3 t\). On the other hand given \(s' : X \to A_1\) and \(t' : Y \to B_1\) satisfying these equations, define \(s_2' = h_1 s'\) and \(t_2' = k_1 t'\). But then by the uniqueness part of the universal property of the bottom distributivity pullback it follows that \(s_2' = s_2\) and \(t_2' = t_2\), and from the uniqueness parts of the universal properties of the pullbacks (1) and (2), it follows that \(s = s'\) and \(t = t'\), thereby verifying that \(s\) and \(t\) are unique satisfying the aforementioned equations.

For the converse suppose that (3) is a distributivity pullback. Note that (2) being a pullback implies that (1) is by elementary properties of pullbacks, so we must show that (2) is a pullback. To that end consider \(s\) and \(t\) as in

![Diagram](image)

such that \(k_2 s = d_0 t\), and then pullback \(s\) and \(f_2\) to produce \(P, u\) and \(v\). Using the fact that the bottom distributivity pullback is a mere pullback, one has \(w\) unique such that \(d_4 d_3 w = h_2 u\) and \(q_1 w = t v\). Using the inner left pullback, one has \(x\) unique such that \(h_3 x = d_3 w\) and \(d_2 x = u\). Using the distributivity pullback (3), one has \(y\) and \(z\) unique such that \(d_1 y = x, f_1 y = z v\) and \(s = d_5 z\). By the uniqueness part of the universal property of the bottom distributivity pullback, it
follows that \( t = k_1z \). Thus we have constructed \( z \) satisfying \( s = d_5z \) and \( t = k_1z \). On the other hand given \( z' : Z \to B_1 \) such that \( s = d_5z' \) and \( t = k_1z' \), one has \( y' : P \to A_1 \) unique such that \( d_2d_1y = u \) and \( f_1y = vz \), using the fact that the top distributivity pullback is a mere pullback. Then from the uniqueness part of the universal property of that distributivity pullback, it follows that \( y = y' \) and \( z = z' \). Thus as required \( z \) is unique satisfying \( s = d_5z \) and \( t = k_1z \).

**Lemma 6.** (Sections of distributivity pullbacks). Let

\[
\begin{array}{ccc}
D & \xrightarrow{p} & A \\
q & \downarrow & \downarrow \\
E & \xrightarrow{dpb} & C \\
& \downarrow f & \\
& B & \xrightarrow{g} & B \\
& \downarrow & & \downarrow \\
& C & \xrightarrow{r} & C \\
\end{array}
\]

be a distributivity pullback around \((f, g)\) in any category with pullbacks. Three maps

\[
s_1 : B \to A \quad s_2 : B \to D \quad s_3 : C \to E
\]

which are sections of \( g \), \( gp \) and \( r \) respectively, and are natural in the sense that \( s_1 = ps_2 \) and \( qs_2 = s_3f \), are determined uniquely by the either of the following: (1) the section \( s_1 \); or (2) the section \( s_3 \).

**Proof.** Given \( s_1 \) a section of \( g \), induce \( s_2 \) and \( s_3 \) uniquely as shown:

\[
\begin{array}{ccc}
B & \xrightarrow{s_2} & D \\
f & \downarrow & \downarrow \\
C & \xrightarrow{s_3} & E \\
\end{array}
\]

using the universal property of the distributivity pullback. On the other hand given the section \( s_3 \), one induces \( s_2 \) using the fact that the distributivity pullback is a mere pullback, and then put \( s_1 = ps_2 \). □

We often assume that in a given category \( E \) with pullbacks, some choice of all pullbacks, and of all existing distributivity pullbacks, has been fixed. Moreover we make the following harmless assumptions, for the sake of convenience, on these choices once they have been made. First we assume that the chosen pullback of an identity along any map is an identity. This ensures that \( \Delta_{1X} = 1_{E/X} \) and that \( \Delta_f(1_B) = 1_A \) for any \( f : A \to B \). Similarly we assume that all diagrams of the form

\[
\begin{array}{ccc}
1 & \xrightarrow{f} & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & \xrightarrow{dpb} & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & \xrightarrow{g} & 1 \\
\end{array}
\]

are among our chosen distributivity pullbacks. This has the effect of ensuring that \( \Pi_f(1_A) = 1_B \) for any exponentiable \( f : A \to B \), and that \( \Pi_{1X} = 1_{E/X} \).
4. Bicategories of polynomials

Let $\mathcal{E}$ be a category with pullbacks. In this section we give a direct description of a bicategory $\text{Poly}_\mathcal{E}$, whose objects are those of $\mathcal{E}$, and whose one cells are polynomials in $\mathcal{E}$ in the following sense. For $X, Y$ in $\mathcal{E}$, a polynomial $p$ from $X$ to $Y$ consists of three maps

$$X \xrightarrow{p_1} A \xrightarrow{p_2} B \xrightarrow{p_3} Y$$

such that $p_2$ is exponentiable. Let $p$ and $q$ be polynomials in $\mathcal{E}$ from $X$ to $Y$. A cartesian morphism $f : p \to q$ is a pair of maps $(f_0, f_1)$ fitting into a commutative diagram

We call $f_0$ the 0-component of $f$, and $f_1$ the 1-component of $f$. With composition inherited in the evident way from $\mathcal{E}$, one has a category $\text{Poly}_\mathcal{E}(X, Y)$ of polynomials from $X$ to $Y$ and cartesian morphisms between them. These are the homs of our bicategory $\text{Poly}_\mathcal{E}$.

In order to describe the bicategorical composition of polynomials, we introduce the concept of a subdivided composite of a given composable sequence of polynomials. Consider a composable sequence of polynomials in $\mathcal{E}$ of length $n$, that is to say, polynomials

$$X_{i-1} \xrightarrow{p_{i-1}} A_i \xrightarrow{p_i} B_i \xrightarrow{p_i} X_i$$

in $\mathcal{E}$, where $0 < i \leq n$. We denote such a sequence as $(p_i)_{1 \leq i \leq n}$, or more briefly as $(p_i)_i$.

**Definition 7.** A subdivided composite over $(p_i)_i$ consists of objects $(Y_0, ..., Y_n)$, morphisms

$$q_1 : Y_0 \to X_0 \quad q_{2,i} : Y_{i-1} \to Y_i \quad q_3 : Y_n \to X_n$$

for $0 < i \leq n$, and morphisms

$$r_i : Y_{i-1} \to A_i \quad s_i : Y_i \to B_i$$

for $0 < i \leq n$, such that $p_{1,i}r_1 = q_1$, $p_{n,n}8_n = q_3$ and

$$\begin{align*}
Y_i & \xrightarrow{r_{i+1}} A_{i+1} \\
B_i & \xrightarrow{p_{i,3}} X_i
\end{align*} \quad \begin{align*}
Y_{i-1} & \xrightarrow{q_{2,i}} Y_i \\
A_i & \xrightarrow{p_{i,2}} B_i
\end{align*}$$
For example a subdivided composite over \((p_1, p_2, p_3)\), that is when \(n = 3\), assembles into a commutative diagram like this:

We denote a general subdivided composite over \((p_i)_i\) simply as \((Y, q, r, s)\).

**Definition 8.** A morphism \((Y, q, r, s) \to (Y', q', r', s')\) of subdivided composites consists of morphisms \(t_i : Y_i \to Y'_i\) for \(0 \leq i \leq n\), such that \(q_1 = q'_1 t_0, q_2 t_1 = q'_2, q_3 = q'_3 t_n, r_i = r'_i t_{i-1}\) and \(s_i = s'_i t_i\). With compositions inherited from \(\mathcal{E}\), one has a category \(\text{SdC}(p_i)_i\) of subdivided composites over \((p_i)_i\) and morphisms between them.

Given a subdivided composite \((Y, q, r, s)\) over \((p_i)_i\), note that the morphisms \(q_2\) are exponentiable since exponentiable maps are pullback stable, and that the composite \(q_2 : Y_0 \to Y_n\) defined as \(q_2 = q_{2n} \cdots q_{21}\) is also exponentiable, since exponentiable maps are closed under composition. Thus we make

**Definition 9.** The associated polynomial of a given subdivided composite \((Y, q, r, s)\) over \((p_i)_i\) is defined to be

\[
X_0 \xrightarrow{q_1} Y_0 \xrightarrow{q_2} Y_n \xrightarrow{q_3} X_n
\]

The process of taking associated polynomials is the object map of a functor

\[
\text{ass} : \text{SdC}(p_i)_i \to \text{Poly}_\mathcal{E}(X_0, X_n).
\]

Having made the necessary definitions, we now describe the canonical operations on subdivided composites which give rise to the bicategorical composition of polynomials. Let \(n > 0\) and \((p_i)_{1 \leq i \leq n}\) be a composable sequence of polynomials in \(\mathcal{E}\). One has evident forgetful functors

\[
\text{res}_n : \text{SdC}(p_i)_{1 \leq i \leq n} \to \text{SdC}(p_i)_{1 \leq i \leq n-1}
\]

\[
\text{res}_0 : \text{SdC}(p_i)_{1 \leq i \leq n} \to \text{SdC}(p_i)_{1 \leq i \leq n-1}
\]

and we now give a description of the right adjoints, respectively \(p_n \cdot (-)\) and \((-) \cdot p_1\), of these forgetful functors.

For \((Y, q, r, s)\) a subdivided composite over \((p_i)_{1 \leq i \leq n}\), we construct the subdivided composite \(p_n \cdot (Y, q, r, s) := (p_n \cdot Y, p_n \cdot q, p_n \cdot r, p_n \cdot s)\) over \((p_i)_{1 \leq i \leq n}\) as follows. First we form the diagram on the left...
and then for \(1 \leq k < n\) we form pullbacks as on the right in the previous display. Finally we define

\[
(p_n \cdot q)_i = q_1 \varepsilon_0 \quad (p_n \cdot r)_i = r_i \varepsilon_{i-1} \quad (p_n \cdot s)_i = s_i \varepsilon_i.
\]

The \(\varepsilon_i\) are the components of a morphism

\[
\varepsilon_{(Y, q, r, s)}: \text{res}_n(p_n \cdot (Y, q, r, s)) \to (Y, q, r, s)
\]

of subdivided composites. The \(n = 4\) case of this construction is depicted in the diagram:

![Diagram](attachment:image.png)

**Lemma 10.** The morphisms \(\varepsilon_{(Y, q, r, s)}\) just described are the components of the counit of an adjunction \(\text{res}_n \vdash p_n \cdot (-)\).

**Proof.** Let \((Y', q', r', s')\) be a subdivided composite over \((p_i)_{1 \leq i \leq n}\), then for \(t\) as in

\[
\begin{align*}
\text{res}_n(p_n \cdot (Y, q, r, s)) & \xrightarrow{\varepsilon_{(Y, q, r, s)}} (Y, q, r, s) \\
& \xrightarrow{\text{res}_n(t')} (Y', q', r', s')
\end{align*}
\]

we must give \(t'\) unique so that the above triangle commutes. The following commutative diagram assembles this given data in the case \(n = 4\).

![Diagram](attachment:image.png)

Since \(q_{n-1} t_{n-1} = p_{n-1} r'_{n-1}\), one induces \(u : Y'_{n-1} \to C\) using the defining pullback of \(C\), and then one induces \(t'_{n-1} : Y'_{n-1} \to Y_{n-1}\) and \(t'_{n} : Y'_{n} \to Y_{n}\) from the maps \(u\) and \(s'_{n}\) using the distributivity pullback. The rest of the \(t'_{i}\) are induced inductively as follows. For \(0 < i < n\) given \(t'_{i} : Y'_{i} \to Y_{i}\), one induces \(t'_{i-1}\) using the maps \(t_{i-1}\) and \(t'_{i}\) and the pullback which defines \((p_n \cdot Y)_{i-1}\). By construction the \(t'_{i}\) are the components of the required unique map \(t'\). \(\Box\)

For \((Y, q, r, s)\) a subdivided composite over \((p_i)_{1 \leq i \leq n}\), we construct the subdivided composite

\[
(Y, q, r, s) \cdot p_1 := (Y \cdot p_1, q \cdot p_1, r \cdot p_1, s \cdot p_1)
\]
In the case $\varepsilon$ The equations of an adjunction $\text{res}$ $0$ $n$ $\leq i \leq n$ as follows. First one takes the pullback on the left, then for $0 < i < n$ the distributivity pullbacks as in the middle,

$$
\begin{array}{c}
C_{02} \xrightarrow{f_0} Y_0 \\
\downarrow g_0 \quad \downarrow \quad \downarrow q_1 \\
B_1 \xrightarrow{p_{13}} X_1
\end{array}
\quad
\begin{array}{c}
C_{i-1,2} \xrightarrow{g_{i-1}} \quad \text{dpb} \\
\downarrow f_{i-1} \quad \downarrow Y_{i-1} \quad \downarrow q_{2,i}
\end{array}
\quad
\begin{array}{c}
(Y \cdot p_1)_{n-i-1} \quad (q \cdot p_1)_{2,n-i} \quad (q \cdot p_1)_{n-i} \\
\downarrow g'_{i-1} \quad \downarrow \quad \downarrow q'_{2,n-i-1}
\end{array}
\quad
\begin{array}{c}
C_{n-i-1,1} \quad C_{n-i-1,2}
\end{array}

and then for $0 < i < n$ one takes the pullbacks as on the right in the previous display, setting $(q \cdot p_1)_{2,n} = g'_{2,n-1}, g''_i = g_{n-1}, g''_{i+1} = g_{n-i+1}g'_{i+1}$ for $i + 1 < n$, $g''_{n-1} = g_0g_1g''_{n-1}$ and $q''_{02} = p_{12}$. Finally one defines

$$(q \cdot p_1)_1 = p_{11}g''_n \quad (q \cdot p_1)_3 = q_3f_{n-1}.$$ 

In the case $n = 4$ one obtains a diagram like this:

The equations $\varepsilon'_0 = f_0g_1g'_0, \varepsilon'_{n-1} = f_{n-1}$ and $\varepsilon'_i = f_iq''_{n-i-1}$ for $0 < i < n - 1$, define the components of the morphism

$$\varepsilon'_{Y,q,r,s} : \text{res}_0((Y, q, r, s) \cdot p_1) \to (Y, q, r, s).$$

**Lemma 11.** The morphisms $\varepsilon'_{Y,q,r,s}$ just described are the components of the counit of an adjunction $\text{res}_0 \dashv (-) \cdot p_1$.

**Proof.** Given $(Y, q, r, s)$ in $\text{SdC}(p_i)_{1 \leq i \leq n}$ and $t$ as in

$$
\begin{array}{c}
\text{res}_0((Y, q, r, s) \cdot p_1) \xrightarrow{\varepsilon'_{Y,q,r,s}} (Y, q, r, s) \\
\downarrow \text{res}_0(t') \quad \downarrow \quad \downarrow \text{res}_0(Y', q', r', s')
\end{array}
$$

This completes the proof.
we must exhibit $t'$ as shown unique so that the above diagram commutes. In the case $n = 4$ the data $(Y', q', r', s')$ and $t$ fit into the following diagram:

Using the pullback that defines $C_{02}$ and the maps $s'_1$ and $t_1$, one induces $Y'_1 \to C_{02}$. Using the distributivity pullbacks one induces successively the morphisms $Y'_i \to C_{i1}$ and $Y'_{i+1} \to C_{i2}$ for $0 < i < n$. In the case $i = n - 1$ we denote these maps as $t_{n-1}'$ and $t'_n$ respectively. The components $t'_i$ for $0 \leq i < n - 1$ are then induced from this data and the pullbacks that define the objects $(Y \cdot p_1)_i$. By construction the $t'_i$ are the components of the required unique map $t'$.

**Proposition 12.** For any composable sequence $(p_i)_{1 \leq i \leq n}$ of polynomials in a category $\mathcal{E}$ with pullbacks, the category $\text{SdC}(p_i)_i$ has a terminal object.

**Proof.** We proceed by induction on $n$. In the case $n = 0$, observe that a subdivided composite consists just of the data $Y_0, q_1 : Y_0 \to X_0$ and $q_3 : Y_0 \to X_0$, and that $\text{SdC}()$ is the category $\text{span}_\mathcal{E}(X_0, X_0)$ of endospans of $X_0$. The identity endospan is terminal. For the inductive step apply either of the functors $p_n \cdot (-)$ or $(-) \cdot p_1$ which as right adjoints, preserve terminal objects.

**Definition 13.** Let $\mathcal{E}$ be a category with pullbacks. A *composite* of a composable sequence $(p_i)_{1 \leq i \leq n}$ of polynomials in $\mathcal{E}$, is defined to be the associated polynomial of a terminal object in the category $\text{SdC}(p_i)_i$. When such a composite has been chosen, it is denoted as $p_n \circ \ldots \circ p_1$.

Let us consider now some degenerate cases of definition 13.

- **$n = 0$:** Choosing identity spans as terminal nullary subdivided composites (see the proof of proposition 12), nullary composition of polynomials gives polynomials whose constituent maps are all identities. That is,

$$
X \xrightarrow{1_X} X \xrightarrow{1_X} X \xrightarrow{1_X} X
$$

is the “identity polynomial on $X$” as one would hope.

- **$n = 1$:** One may identify $\text{SdC}(p)$ as the slice $\text{Poly}_\mathcal{E}(X_0, X_1)/p$, and thus choose $1_p$ as the terminal unary subdivided composite over $(p)$. Thus the unary composite of a given polynomial $p$ is just $p$. 
• $n = 2$: applying $p_2 \cdot (\cdot)$ to $p_1$, or $(\cdot) \cdot p_1$ to $p_2$, gives the same subdivided composite, namely

which is terminal by the case $n = 1$ and since the functors $p_2 \cdot (\cdot)$ and $(\cdot) \cdot p_1$, as right adjoints, preserve terminal objects. Thus the associated composite of the above is the binary composite $p_2 \circ p_1$, and this agrees with the binary composition of polynomials given in [5].

Lemma 14. Let $n > 0$ and $(p_i)_{1 \leq i \leq n}$ be a composable sequence of polynomials in a category $\mathcal{E}$ with pullbacks. Then one has canonical isomorphisms

\[
\begin{align*}
\text{SdC}(p_i)_{1 \leq i \leq n} & \xrightarrow{\text{ass}} \text{Poly}_\mathcal{E}(X_0, X_{n-1}) & \text{SdC}(p_i)_{1 \leq i \leq n} & \xrightarrow{\text{ass}} \text{Poly}_\mathcal{E}(X_1, X_n) \\
\text{SdC}(p_i)_{1 \leq i \leq n} & \xrightarrow{\text{ass}} \text{Poly}_\mathcal{E}(X_0, X_n) & \text{SdC}(p_i)_{1 \leq i \leq n} & \xrightarrow{\text{ass}} \text{Poly}_\mathcal{E}(X_0, X_n)
\end{align*}
\]

PROOF. The canonical isomorphism on the left follows from the definitions and the elementary properties of pullbacks. The canonical isomorphism on the right follows from the definitions, and iterated application of lemma [4].

Theorem 15. Let $\mathcal{E}$ be a category with pullbacks. One has a bicategory $\text{Poly}_\mathcal{E}$, whose objects are those of $\mathcal{E}$, whose hom from $X$ to $Y$ is $\text{Poly}_\mathcal{E}(X, Y)$, and whose compositions are given by definition [13].

PROOF. By induction on $n$, using the fact that the functors $p_n \cdot (\cdot)$ and $(\cdot) \cdot p_1$ preserve terminal objects, and lemma [4], it follows that any iterated binary composite of polynomials of length $n$, is a composite in the sense of definition [13]. That is, such an iterated composite is the associated polynomial of a terminal subdivided polynomial of the composable sequence of polynomials that participates in the given iterated binary composite. Hence between any two alternative bracketings of a given composite, there is a unique isomorphism of their underlying subdivided composites, giving rise to a “coherence” isomorphism of the composites themselves upon application of “ass”. Any diagram of such coherence isomorphisms must commute, since it is the image by the appropriate “ass” functor, of a diagram whose vertices are all terminal subdivided composites.

A span in $\mathcal{E}$ may be identified as a polynomial in which the middle map is an identity. Polynomial composition of spans coincides exactly with span composition, giving us a strict inclusion

$$\text{Span}_\mathcal{E} \hookrightarrow \text{Poly}_\mathcal{E}$$

of bicategories which is the identity on objects and locally fully faithful. For a given map $f : X \to Y$ in $\mathcal{E}$, we denote by $f^* : X \to Y$ and $f_* : Y \to X$ the polynomials

$$X \xleftarrow{1} X \xleftarrow{1} X \xrightarrow{f} Y \quad Y \xrightarrow{f} X \xleftarrow{1} X \xleftarrow{1} X$$
respectively. These are spans, it’s well known that one has \( f^\bullet + f_* \) and that this is part of the basic data of the proarrow equipment \((\mathcal{E}, \text{Span}_{\mathcal{E}})\) [17, 18]. By the above strict inclusion, this extends to another proarrow equipment \((\mathcal{E}, \text{Poly}_{\mathcal{E}})\) \([17, 18]\). By the above strict inclusion, this extends to another proarrow equipment \((\mathcal{E}, \text{Poly}_{\mathcal{E}})\), and all this at the generality of a category \(\mathcal{E}\) with pullbacks. It’s worth noting that polynomial composites of the form \( f^\bullet \circ p \) and \( q \circ g_* \) are particularly easy, these being

\[
\begin{array}{ccc}
p_1 & \bullet & p_2 & \bullet & \bullet & f p_3 & \bullet & g q_1 & \bullet & q_2 & \bullet & q_3 & \bullet
\end{array}
\]

respectively.

The homs of \(\text{Poly}_{\mathcal{E}}\) interact well with the slices of \(\mathcal{E}\). For all \(X\) and \(Y\) one has obvious forgetful functors

\[
\begin{array}{ccc}
\mathcal{E}/X & \xrightarrow{\Sigma g l_{X,Y}} & \text{Poly}_{\mathcal{E}}(X, Y) & \xrightarrow{\Sigma f r_{X,Y}} & \mathcal{E}/Y
\end{array}
\]

and we refer to these as the left and right projections of the homs of \(\text{Poly}_{\mathcal{E}}\). From the above descriptions of composites of the form \( f^\bullet \circ p \) and \( q \circ g_* \), one obtains immediately the sense in which these forgetful functors are natural.

**Lemma 16.** For all \(f : Y \rightarrow Z\) and \(g : X \rightarrow W\) one has

\[
\begin{align*}
\Sigma g l_{X,Y} &= l_{W,Y}((-) \circ g_*) \\
l_{X,Y} &= l_{X,Z}(f^\bullet \circ (-))
\end{align*}
\]

\[
\begin{align*}
\Sigma f r_{X,Y} &= r_{X,Z}((f^\bullet \circ (-))) \\
r_{X,Y} &= r_{W,Y}((-) \circ g_*)
\end{align*}
\]

5. Polynomial functors

Let \(\mathcal{E}\) be a category with pullbacks. In this section we define a pseudo functor

\[
\text{P}_\mathcal{E} : \text{Poly}_{\mathcal{E}} \rightarrow \text{CAT} \quad X \mapsto \mathcal{E}/X
\]

with object map as indicated. Given a polynomial \(p : X \rightarrow Y\) in \(\mathcal{E}\), the functor \(\text{P}_\mathcal{E}(p) : \mathcal{E}/X \rightarrow \mathcal{E}/Y\) is defined to be the composite \(\Sigma p_3 \Pi p_2 \Delta p_1\), which for the sake of brevity, will also be denoted as \(p(-) : \mathcal{E}/X \rightarrow \mathcal{E}/Y\). In more elementary terms the effect of \(p(-)\) on an object \(x : C \rightarrow X\) of \(\mathcal{E}/X\) is described by the following commutative diagram:

\[
\begin{array}{ccc}
C_2 & \xrightarrow{d_{pb}} & C_3 \\
X & \xrightarrow{A} & B & \xrightarrow{p(x)} & Y
\end{array}
\]

Similarly one may, by exploiting the universal property of the pullback and distributivity pullback in this description, induce the maps which provide the arrow map of \(p(-)\). These explicit descriptions together with lemma(16) enable us to catalogue all the ways one can use the composition of \(\text{Poly}_{\mathcal{E}}\) to describe the functor \(p(-)\), and we record this in

**Lemma 17.** Let \(p : X \rightarrow Y\) be a polynomial in \(\mathcal{E}\).

1. Given \(x : C \rightarrow X\) in \(\mathcal{E}/X\), one has

\[
p(x) = l_{Z,Y}(p \circ x^\bullet \circ g_*)
\]

for all \(Z\) and \(g : C \rightarrow Z\).
(2) Given \( x_1 : C_1 \to X \), \( x_2 : C_2 \to X \) and \( h : C_1 \to C_2 \) over \( X \), one has
\[
p(h) = l_{Z,Y} (p \circ h' \circ g_*)
\]
for all \( Z \) and \( g : C_2 \to Z \), where \( h' : x_2^* \to x_1^* h_* \) is the mate of the identity \( x_2^* h_* = x_1^* \) via \( h_* \sqcup h_* \).

Similarly there are two ways of describing the 2-cell map of \( \mathbf{P}_\mathcal{E} \). The first, which was described in [5], is to associate to a given cartesian morphism \( f : p \to q \) between polynomials from \( X \) to \( Y \), the following natural transformation

\[
\begin{array}{c}
\mathcal{E}/A \\
\downarrow \Delta_{p_1} \\
\mathcal{E}/X \\
\downarrow \Delta_{q_1} \\
\mathcal{E}/A' \\
\downarrow \Pi_{p_2} \\
\mathcal{E}/A \\
\end{array}
\begin{array}{c}
\mathcal{E}/B \\
\downarrow \Sigma_{p_3} \\
\mathcal{E}/Y \\
\downarrow \Sigma_{q_3} \\
\mathcal{E}/B' \\
\end{array}
\]

\( \Delta_{f_0} \cong \Delta_{f_1} \cong \Delta_f \)

in which the isomorphism in the middle is a Beck-Chevalley isomorphism, and \( \Sigma_{p_3} \Delta_{f_1} \to \Sigma_{q_3} \) is the mate of the identity via \( \Sigma f_1 \sqcup \Delta f_1 \). The advantage of this description is that the functoriality of the resulting hom functor

\( (\mathbf{P}_\mathcal{E})_{X,Y} : \text{Poly}\mathcal{E}(X,Y) \to \text{CAT}(\mathcal{E}/X, \mathcal{E}/Y) \)

is evident. On the other hand it is also worthwhile to have a direct elementary description of the components of this natural transformation. We will show that the component at \( x : C \to X \) is given by the map \( f_{4,x} \) constructed in

\[
\begin{array}{c}
C_2 \\
pb \\
f_{2,x} \\
p \\
pb \\
C \\
\end{array}
\begin{array}{c}
A \\
p_1 \\
f_0 \\
p_2 \\
B \\
\end{array}
\begin{array}{c}
C_3 \\
pb \\
f_{3,x} \\
q_1 \\
pb \\
C_3' \\
\end{array}
\begin{array}{c}
X \\
\downarrow f_{1,x} \\
A' \\
\downarrow f_{4,x} \\
B \\
\end{array}
\begin{array}{c}
C_4 \\
p_3 \\
pb \\
C_4' \\
\end{array}
\]

To construct this diagram one induces \( f_{2,x} \) using \( f_0 \) and the bottom pullback, and then it follows that the square \((C_2, A, A', C_3')\) is a pullback. Thus one can then induce \( f_{3,x} \) and \( f_{4,x} \) using the bottom distributivity pullback. The cube lemma ensures that \((C_4, B, B', C_4')\) is a pullback, and elementary properties of pullbacks ensure that \((C_2, C_3, C_3', C_2')\) and \((C_3, C_4, C_4', C_3')\) are also pullbacks.

Our next task is to explain why [6] really does describe the components of [5]. This verification begins by unpacking, for a given commuting square as shown on
the right

\[ \begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow{k} \\
C & \xrightarrow{g} & D
\end{array} \quad \begin{array}{ccc}
\mathcal{E}/A & \xrightarrow{\Sigma_f} & \mathcal{E}/B \\
\Delta_k & \xrightarrow{\alpha} & \Delta_k \\
\mathcal{E}/C & \xrightarrow{\Sigma_g} & \mathcal{E}/D
\end{array} \quad \begin{array}{ccc}
\mathcal{E}/A & \xrightarrow{\Delta f} & \mathcal{E}/B \\
\Pi_h & \xrightarrow{\beta} & \Pi_h \\
\mathcal{E}/C & \xrightarrow{\Delta g} & \mathcal{E}/D
\end{array} \]

the left Beck-Chevalley cell \( \alpha \), and when \( h \) and \( k \) are exponentiable, the right Beck-Chevalley cell \( \beta \), in elementary terms. Since \( \alpha \) is obtained by taking the mate of the identity \( \Sigma_g \Sigma_h = \Sigma_k \Sigma_f \) via the adjunctions \( \Sigma_h \dashv \Delta_h \) and \( \Sigma_k \dashv \Delta_k \), it follows that \( \alpha \) is uniquely determined by the equation \( \Sigma_k \varepsilon_f^{(1)} = (\varepsilon_g^{(1)} \Sigma_k)(\Sigma_f \alpha) \). On the other hand one has the commutative diagram

\[ (7) \]

\[ \begin{array}{c}
A_2 \\
\downarrow{\varepsilon_{k,gx}^{(1)}} \\
A_3 \\
\downarrow{\varepsilon_{k,gx}^{(2)}} \\
C_2 \\
\downarrow{x} \\
C \\
\downarrow{g} \\
D
\end{array} \quad \begin{array}{c}
A \\
\downarrow{h} \\
B \\
\downarrow{k} \\
\mathcal{E}/A \\
\Delta f \\
\mathcal{E}/B \\
\Delta_k \\
\mathcal{E}/C \\
\Delta g \\
\mathcal{E}/D
\end{array} \]

and the above equation, for the component \( x \), is witnessed by the commutativity of the bottom triangle. Thus

**Lemma 18.** The components of \( \alpha \) are induced as in (7).

Moreover we can see directly from (7) that if the original square is a pullback, then \( \alpha \) is invertible, and the converse follows by considering the case \( x = 1_C \). The right Beck-Chevalley cell \( \beta \) may be obtained by taking the mate of \( \Delta f \Delta_k \cong \Delta_h \Delta g \) via \( \Delta_h \dashv \Pi_h \) and \( \Delta_k \dashv \Pi_k \). Thus it is uniquely determined by the commutativity of

\[ (8) \]

\[ \begin{array}{ccc}
\Delta h \Delta f \Pi k & \xrightarrow{\Delta h \beta} & \Delta h \Pi h \Delta f \\
\downarrow{\text{coh} \Pi k} & & \downarrow{\varepsilon_{k}^{(2)}} \\
\Delta f \Delta_k \Pi k & \xrightarrow{\Delta f \varepsilon_k^{(2)}} & \Delta f
\end{array} \]

whereas inside \( \mathcal{E} \) for all \( B_2 \) and \( x : B_2 \to B \) we have

\[ (9) \]

\[ \begin{array}{ccc}
A_2 & \xrightarrow{pb} & B_2 \\
\downarrow{dpb} & & \downarrow{x} \\
A_3 & \xrightarrow{dpb} & B_3 \\
\downarrow{h} & & \downarrow{k} \\
C_3 & \xrightarrow{dpb} & D_2 \\
\downarrow{pb} & & \downarrow{pb} \\
C_2 & \xrightarrow{pb} & D \\
\downarrow{g} & & \downarrow{dpb}
\end{array} \]
constructed as follows. Take the distributivity pullback of \( x \) along \( k \) and then pullback the result along \( g \). Pullback \( x \) along \( f \) and then take the distributivity pullback of the result along \( h \). Then form the top pullback and induce the morphism \( A_4 \to C_2 \). By the elementary properties of pullbacks, it follows that the squares \((A_4, B_3, D_2, C_2)\) and \((A_4, A, C, C_2)\) are pullbacks. From this last we induce the dotted arrows using the left distributivity pullback.

**Lemma 19.** The components of \( \beta \) are induced as in (9).

**Proof.** Note that the square \((A_4, A_3, C_3, C_2)\) is also a pullback, and so one can identify the commuting triangle \((A_4, A_3, A_2)\) with (8), once one has understood that \( A_4 \to A_2 \) is the (appropriate component of) the lower composite in (8). \( \Box \)

From this construction and the cube lemma one may witness directly that if the original square is a pullback, then \( \beta \) is invertible, and the converse is easily witnessed by considering the case \( x = 1_B \). With these details sorted out we can now proceed to the proof of

**Lemma 20.** The component at \( x : C \to X \) of the natural transformation described in (5) is the morphism \( f_{4,x} \) described in (8).

**Proof.** The proof consists of unpacking the definition of \((P \xi)_X, Y (f)_x\) with reference to the diagram (8), keeping track of the canonical isomorphisms that participate in the definition. All of this may be witnessed in

![Diagram](image-url)

in which the solid arrows appeared already in (8), and the dotted arrows are constructed as follows. Form \( D_1 \) by pulling back \( f_0 \) and \( C'_2 \to A' \), then \( D_1 \to C \) is the composite for the triangle \((D_1, C'_2, C)\). The form \( D_2, D_3 \) and \( D_4 \) by taking the distributivity pullback of \( D_1 \to A \) along \( p_2 \). Form \( D_4 \) by pulling back \( f_1 \) and \( C'_4 \to B' \). The construction of the rest of the data proceeds in the same way as for
Thus the arrow labelled as $\beta$ is by lemma(16) the right Beck-Chevalley isomorphism, and $\gamma$ is also invertible. Clearly $\phi$ is an isomorphism witnessing the pseudo-functoriality of $\Delta(-)$, and considering (7) for the square

\[
\begin{array}{ccc}
B & \xrightarrow{p_3} & Y \\
\downarrow f_0 & & \downarrow 1_Y \\
B' & \xrightarrow{q_3} & Y
\end{array}
\]

the arrow labelled $\alpha$ is evidently the appropriate component of the left Beck-Chevalley cell by lemma(18). Thus $(P_{E})_{X,Y}(f)_x$ is by definition the composite

\[
C_4 \xrightarrow{\beta^{-1}} D_3 \xrightarrow{\gamma^{-1}} D_2 \xrightarrow{\alpha} C_4'
\]

and to finish the proof we must show that this composite is $f_{4,x}$. Provisionally let us denote by $\xi$ this composite, and by $\zeta$ the composite

\[
C_3 \xrightarrow{\gamma^{-1}} D_2 \xrightarrow{\alpha} C_4'
\]

Observe that the squares

\[
\begin{array}{ccc}
C_2 & \xleftarrow{\zeta} & C_3 \\
\downarrow \zeta & & \downarrow \zeta \\
C'_2 & \xleftarrow{\zeta} & C'_3
\end{array}
\]

\[
\begin{array}{ccc}
C_3 & \xrightarrow{\zeta} & C_4 \\
\downarrow \zeta & & \downarrow \zeta \\
C'_3 & \xrightarrow{\zeta} & C'_4
\end{array}
\]

\[
\begin{array}{ccc}
C_4 & \xrightarrow{\zeta} & B \\
\downarrow \zeta & & \downarrow f_x \\
C'_4 & \xrightarrow{\zeta} & B'
\end{array}
\]

are commutative, and so by the uniqueness aspect of the universal property of the bottom distributivity pullback, it follows that $\zeta = f_{3,x}$ and $\xi = f_{4,x}$.

The importance of this alternative description is that it can, in various ways, be written in terms of composition in the bicategory $\text{Poly}_E$. These ways are described in the following result, which follows immediately from lemma(16) and lemma(20).

**Lemma 21.** Let $p$ and $q : X \to Y$ be polynomials in $E$ and $f : p \to q$ a cartesian morphism between them. Then given $x : C \to X$ in $E/X$, one has

\[
P_{E}(f)_x = f_{4,x} = l_{Z,Y}(p \circ x^* \circ g_*)
\]

for all $Z$ and $g : C \to Z$. 

\[\square\]
The fact that the one and 2-cell maps of \( P_\mathcal{E} \) have, by lemmas \( \text{(17)} \) and \( \text{(21)} \), been described in terms of the bicategory structure of \( \text{Poly}_\mathcal{E} \), is the conceptual reason why they give a homomorphism of bicategories. We expand on this further in the proof of

**Theorem 22.** Let \( \mathcal{E} \) be a category with pullbacks. With the object map \( X \to \mathcal{E}/X \), arrow map \( p \mapsto \Sigma_p, \Pi_p, \Delta_p \), and 2-cell map depicted in \( \text{(5)} \), one has a homomorphism

\[
P_\mathcal{E} : \text{Poly}_\mathcal{E} \to \text{CAT}
\]

of bicategories.

**Proof.** It remains to exhibit the coherence isomorphisms and verify the coherence axioms. We assume a canonical choice of all pullbacks and existing distributivity pullbacks as explained at the end of section \( \text{(3)} \). In particular this implies that identities in \( \text{Poly}_\mathcal{E} \) are strict, making \( P_\mathcal{E}(1_X) = 1_{\mathcal{E}/X} \) for all \( X \in \mathcal{E} \) by lemma \( \text{(17)} \). Let \( p : X \to Y \) and \( q : Y \to Z \) be polynomials. For \( x : C \to X \) in \( \mathcal{E}/X \) one has the associativity isomorphism

\[
\alpha_{p,q,x}^{-1} : q \circ (p \circ x^\bullet) \cong (q \circ p) \circ x^\bullet
\]

and so the component of the coherence isomorphism

\[
\pi_{p,q,x} : P_\mathcal{E}(q)P_\mathcal{E}(p)(x) \cong P_\mathcal{E}(q \circ p)(x)
\]

is defined to be \( l_{C,Z}(\alpha_{p,q,x}^{-1}) \). Naturality in \( q, p \) and \( x \) is clear by definition. Note that by lemma \( \text{(16)} \) there are many other descriptions of this same component, namely

\[
\pi_{p,q,x} = l_{D,Z}(\alpha_{q,p,x}^{-1} \circ g^\bullet)
\]

for any \( D \) and \( g : C \to D \). Using this and lemmas \( \text{(21)} \) and \( \text{(17)} \), one can exhibit any component of any bicategorical homomorphism coherence diagram, as the image of a diagram of coherence isomorphisms in \( \text{Poly}_\mathcal{E} \), by a right projection of one of \( \text{Poly}_\mathcal{E} \)'s homs. By theorem \( \text{(15)} \) all such diagrams commute. \( \square \)

A **polynomial functor** over \( \mathcal{E} \) is by definition a composite of functors of the form \( \Sigma_f, \Delta_g \) and \( \Pi_h \), where \( f \) and \( g \) can be arbitrary morphisms of \( \mathcal{E} \), and \( h \) can be an exponentiable morphism of \( \mathcal{E} \). It follows from theorem \( \text{(22)} \) that a functor between slices of \( \mathcal{E} \) is polynomial if it is in the image of \( P_\mathcal{E} \).

We conclude this section by observing that the hom functors of \( P_\mathcal{E} \) are faithful and conservative.

**Proposition 23.** For any category \( \mathcal{E} \) with pullbacks and objects \( X, Y \in \mathcal{E} \), the hom functor \( (P_\mathcal{E})_{X,Y} \) is faithful and conservative.

**Proof.** Considering the instance of \( \text{(6)} \) in which \( x = 1_X \), it is clear that \( f_{4,1_X} = f_1 \), and so by lemma \( \text{(20)} \) \( (P_\mathcal{E})_{X,Y}(f) \) uniquely determines \( f_1 \). Let us now consider the case \( x = q_1 \). In that case \( C = A' \) and the morphisms \( C'_2 \to C \) and \( C'_2 \to A' \), namely the projections of the pullback defining \( C'_2 \), have a common section \( s_1 : A' \to C'_2 \). Applying lemma \( \text{(18)} \) to the bottom distributivity pullback, one obtains the sections \( s_2 : A' \to C'_3 \) and \( s_3 : B' \to C'_4 \) satisfying the naturality conditions of that lemma. Since the square \( (C_4, B, B', C'_4) \) is a pullback, one induces unique \( s_4 : B \to C'_4 \) which is a section of the given map \( C_4 \to B \) and satisfies \( f_{4,x}s_4 = s_1f_1 \). Applying lemma \( \text{(6)} \), this time to the top distributivity pullback, one induces the natural sections \( s_5 : A \to C_3 \) and \( s_6 : A \to C_2 \). From the naturality
conditions of the sections so constructed and the commutativities in the original
general diagram (5), it follows easily that \( f_0 \) is equal to the composite

\[
A \xrightarrow{\eta_b} C_2 \longrightarrow C = A',
\]

which by construction and lemma (20), is determined uniquely by \((P_\mathcal{E})_{X,Y}(f)\),
and so \((P_\mathcal{E})_{X,Y}\) is faithful. If \((P_\mathcal{E})_{X,Y}(f)\) is invertible, then \(f_1\),
which we saw is the component at \(1_X\) of this natural transformation, must also be invertible.
Since \(f_0\) is a pullback along \(q_2\) of \(f_1\), \(f_0\) is also invertible, and so \((P_\mathcal{E})_{X,Y}\) is conservative. □

6. Enrichment over \(\text{CAT}_{pb}\)

Recall from [1] that the category \(\text{CAT}_{pb}\) of categories with pullbacks and
pullback preserving functors is cartesian closed. The product in \(\text{CAT}_{pb}\) is as in
\(\text{CAT}\), and the internal hom \([X, Y]\) is the category of pullback preserving factors
\(X \to Y\) and cartesian transformations between them. A \(\text{CAT}_{pb}\)-bicategory
is a bicategory \(B\) whose homs have pullbacks and whose compositions

\[
\text{comp}_{X,Y,Z} : B(Y, Z) \times B(X, Y) \to B(X, Z)
\]
preserve them. The basic example is \(\text{CAT}_{pb}\) itself. A homomorphism \(F : B \to C\)
of \(\text{CAT}_{pb}\)-bicategories is a homomorphism of their underlying bicategories whose
hom functors preserve pullbacks. The point of this section is to show that for any
category \(\mathcal{E}\) with pullbacks, the homomorphism \(P_\mathcal{E}\) is in fact a homomorphism of
\(\text{CAT}_{pb}\)-bicategories.

For all \(f : A \to B\) in a category \(\mathcal{E}\) with pullbacks, it is easy to witness directly
that the adjunction \(\Sigma_f \dashv \Delta_f\) lives in \(\text{CAT}_{pb}\). So polynomial functors preserve
pullbacks, and the diagram (5) may be regarded as living in \(\text{CAT}_{pb}\). Thus the
hom maps of \(P_\mathcal{E}\) may be regarded as landing in the homs of \(\text{CAT}_{pb}\), that is, one
can write

\[
(P_\mathcal{E})_{X,Y} : \text{Poly}_\mathcal{E}(X, Y) \to \text{CAT}_{pb}(\mathcal{E}/X, \mathcal{E}/Y).
\]

In fact these functors themselves live in \(\text{CAT}_{pb}\). To see this we first we note that

Lemma 24. For any category \(\mathcal{E}\) with pullbacks and objects \(X, Y \in \mathcal{E}\), the category
\(\text{Poly}_\mathcal{E}(X, Y)\) has pullbacks, and a commutative square in \(\text{Poly}_\mathcal{E}(X, Y)\) is a pullback
iff its 1-component is a pullback in \(\mathcal{E}\).

Proof. One has a canonical inclusion

\[
\text{Poly}_\mathcal{E}(X, Y) \to \mathcal{E}^{\to \to}
\]
of \(\text{Poly}_\mathcal{E}(X, Y)\) into the functor category. In general, given a category \(C\), an arrow
\(\alpha\) in \(C\), and a square

\[
\begin{array}{ccc}
P & \xrightarrow{q} & B \\
\downarrow p & & \downarrow g \\
A & \xrightarrow{f} & C
\end{array}
\]
in \([C, \mathcal{E}]\), then if the naturality squares of \(f\) and \(g\) at \(\alpha\) are pullbacks, then so
are those for \(p\) and \(q\), by the elementary properties of pullback squares. Since
exponentiable maps are pullback stable, and one may choose pullbacks in \(\mathcal{E}\) so that
identity arrows are pullback stable, it follows that pullbacks in \(\text{Poly}_\mathcal{E}(X, Y)\) exist
and are formed as in \(\mathcal{E}^{\to \to}\). Thus it follows in particular that a commutative
square in $\text{Poly}_E(X,Y)$ as in the statement is a pullback iff its 0 and 1-components are pullbacks in $\mathcal{E}$. But from the elementary properties of pullbacks, if the 1-component square is a pullback then so is the 0-component. □

and so one has

**Proposition 25.** For any category $\mathcal{E}$ with pullbacks and objects $X, Y \in \mathcal{E}$, the functor $(P_\mathcal{E})_{X,Y}$ preserves and reflects pullbacks.

**Proof.** Since by proposition 23 $(P_\mathcal{E})_{X,Y}$ is conservative, it suffices to show that it preserves pullbacks. But by the elementary properties of pullbacks, a square in $\text{CAT}_{\text{pb}}(\mathcal{E}/X, \mathcal{E}/Y)$ is a pullback iff its component at 1 is a pullback. Since by lemma 20 the component at 1 of $(P_\mathcal{E})_{X,Y}(f)$ is just $f_1$, the result follows from lemma 24. □

**Theorem 26.** Let $\mathcal{E}$ be a category with pullbacks. Then $\text{Poly}_E$ is a $\text{CAT}_{\text{pb}}$-bicategory and

$$P_\mathcal{E} : \text{Poly}_\mathcal{E} \to \text{CAT}_{\text{pb}}$$

is a homomorphism of $\text{CAT}_{\text{pb}}$-bicategories.

**Proof.** By proposition 25 it suffices to show that the composition functors of $\text{Poly}_\mathcal{E}$ preserve pullbacks. One has for each $X, Y, Z \in \mathcal{E}$, an isomorphism

$$\text{Poly}_\mathcal{E}(Y, Z) \times \text{Poly}_\mathcal{E}(X, Y) \cong \text{Poly}_\mathcal{E}(X, Z)$$

The vertical functors preserve and reflect pullbacks by proposition 25, the bottom one preserves pullbacks since $\text{CAT}_{\text{pb}}$ is a $\text{CAT}_{\text{pb}}$-bicategory by cartesian closed-ness, and so the composition functor for $\text{Poly}_\mathcal{E}$ preserves pullbacks as required. □

**Remark 27.** In the case where $\mathcal{E}$ is locally cartesian closed, the work of Gambino and Kock [5] tells us more. In that case $\mathcal{E}$ is in particular a monoidal category via its cartesian product, and it acts as a monoidal category on its slices. Moreover polynomial functors over such $\mathcal{E}$ acquire a canonical strength. Then by proposition (2.9) of [5], the image of $P_\mathcal{E}$ consists of the slices of $\mathcal{E}$, polynomial functors over $\mathcal{E}$ and *strong* cartesian transformations between them.

**Remark 28.** Since for any category $\mathcal{E}$ with pullbacks the homs of $\text{Poly}_\mathcal{E}$ also have pullbacks, the above result can be applied to any of those homs in place of $\mathcal{E}$, giving a sense in which the theory of polynomials may be iterated.

### 7. Polynomial 2-functors

In this section we extend the above to the setting of 2-categories. Let $\mathcal{K}$ be a 2-category with pullbacks. Recall that when one speaks of pullbacks, or more generally any weighted limit in a 2-category, the universal property has a 2-dimensional aspect. That is, a square $S$ in $\mathcal{K}$ is by definition a pullback in $\mathcal{K}$ iff for all $X \in \mathcal{K}$, the square $\mathcal{K}(X,S)$ in $\text{CAT}$ is a pullback in $\text{CAT}$. On objects this is the usual universal property of a pullback as in ordinary category theory, and on arrows this is the “2-dimensional aspect”. Recall also [7] that if $\mathcal{K}$ admits tensors with [1], then the usual universal property implies this 2-dimensional aspect, but in the absence of tensors, one must verify the 2-dimensional aspect separately.
Similarly when we speak of distributivity pullbacks in \( K \) we will also demand that these satisfy a 2-dimensional universal property. Let \( g : Z \to A \) and \( f : A \to B \) be in \( K \). We describe first the 2-category \( \text{PB}(f,g) \) of pullbacks around \((f,g)\). The underlying category of \( \text{PB}(f,g) \) is described as in definition\([1]\). Let \((s,t)\) and \((s',t') : (p,q,r) \to (p',q',r')\) be morphisms in \( \text{PB}(f,g) \). Then a 2-cell between them consists of 2-cells \( \sigma : s \to s' \) and \( \tau : t \to t' \) of \( K \), such that \( p' \sigma = 1_p, \quad q \tau = r' \tau \) and \( 1_r = r' \tau \). Compositions for \( \text{PB}(f,g) \) are inherited from \( K \). One thus defines a distributivity pullback around \((f,g)\) in \( K \) to be a terminal object of the 2-category \( \text{PB}(f,g) \).

The meaning of distributivity pullbacks in this 2-categorical environment is the same as in the discussion of section\([3]\). First note that \( \Sigma_f : \mathcal{K}/A \to \mathcal{K}/B \) is a 2-functor and that by virtue of the 2-dimensional universal property of pullbacks in \( K \), \( \Delta_f : \mathcal{K}/B \to \mathcal{K}/A \) is a 2-functor and \( \Sigma_f \dashv \Delta_f \) is a 2-adjunction. To say that all distributivity pullbacks along \( f \) exist in \( K \) is to say that \( \Delta_f \) has a right 2-adjoint, denoted \( \Pi_f \) as before, and this right adjoint encodes the process of taking distributivity pullbacks along \( f \). Such morphisms \( f \) in \( K \) are said to be exponentiable, and as in the 1-dimensional case, exponentiable maps are closed under composition and are stable by pullback along arbitrary maps. Moreover lemmas\([11], [5] \) and \([10] \) remain valid in our 2-categorical environment. The verification of this is just a matter of using the 2-dimensional aspects of pullbacks and distributivity pullbacks to induce the necessary 2-cells, in exact imitation of how one induced the arrows during these proofs in section\([3] \).

Polynomials in \( K \) and cartesian morphisms between them are defined as in section\([1] \). Given polynomials \( p \) and \( q : X \to Y \), and cartesian morphisms \( f \) and \( g : p \to q \), a 2-cell \( \phi : p \to q \) consists of 2-cells \( \phi_0 : f \to g_0 \) and \( \phi_1 : f_1 \to g_1 \) such that \( p_1 = q_1 \phi_0, \quad q_2 \phi_0 = \phi_1 p_2 \) and \( q_3 \phi_1 = p_3 \). With compositions inherited from \( K \) one has a 2-category \( \text{Poly}_K(X,Y) \) together with left and right projections

\[
\begin{array}{ccc}
\mathcal{K}/X & \xrightarrow{\iota_{X,Y}} & \text{Poly}_K(X,Y) \xrightarrow{r_{X,Y}} \mathcal{K}/Y.
\end{array}
\]

For a composable sequence \( (p_i)_i \) of polynomials as in

\[
X_{i-1} \xrightarrow{p_{i-1}} A_i \xrightarrow{p_i} B_i \xrightarrow{p_{i+1}} X_i
\]

one defines the 2-category \( \text{SdC}(p_i)_i \) of subdivided composites over \((p_i)_i\) as follows. The objects and arrows are defined as in definitions\([7] \) and \([8] \). Given morphisms \( t \) and \( t' : (Y,q,r,s) \to (Y',q',r',s') \) of subdivided composites, a 2-cell \( \tau : t \to t' \) consists of 2-cells \( \tau_i : t_i \to t'_i \) in \( K \) for \( 0 \leq i \leq n \), such that \( q_1 = q'_1 \tau_0, \quad q_2 \tau_1 = \tau_2 q_2, \quad q_3 = q'_3 \tau_n, \quad r_i = r'_i \tau_1 \) and \( s_i = s'_i \tau_i \). Compositions in \( \text{SdC}(p_i)_i \) are inherited from \( K \). The process of taking the associated polynomial of a subdivided composite, as described in definition\([13] \), is 2-functorial. The forgetful functors \( \text{res}_n \) and \( \text{res}_0 \) become 2-functors. The fact that lemmas\([11], [9] \) and \([10] \) remain valid in our 2-categorical environment, is once again a matter of using the 2-dimensional aspects of pullbacks and distributivity pullbacks to induce the necessary 2-cells in the same way that the arrows during these proofs were induced in the 1-dimensional case. Thus these 2-categories of subdivided composites admit terminal objects, and so one may define the composition of polynomials as in definition\([13] \). Moreover composition is 2-functorial.

Lemma\([20] \) gives a direct description of the arrow map of the hom functors of \( \mathbf{P}_\mathcal{E} \) as being induced by the universal properties of pullbacks and distributivity
pullbacks. Thus in our 2-categorical setting, with the 2-dimensional aspects of these universal properties available, we can do the same one dimension higher and induce directly the components of the modification induced by a 2-cell between maps of polynomials. Thus we have 2-functors

$$(P_K)_{X,Y} : \text{Poly}_K(X,Y) \to \mathbf{2-CAT}(\mathcal{K}/X, \mathcal{K}/Y)$$

for all objects $X$ and $Y$ of a 2-category $\mathcal{K}$ with pullbacks.

We now describe the structure that polynomials in a 2-category form. A 2-bicategory consists of a bicategory $\mathcal{B}$ whose hom categories are endowed with 2-cells making them 2-categories and the composition functors

$$\text{comp}_{X,Y,Z} : \mathcal{B}(Y,Z) \times \mathcal{B}(X,Y) \to \mathcal{B}(X,Z)$$

are endowed with 2-cell maps making them into 2-functors. The 2-cells in the homs of $\mathcal{B}$ are called 3-cells. In addition we ask that the coherence isomorphisms of $\mathcal{B}$ be natural with respect to the 3-cells. Similarly a homomorphism $F : \mathcal{B} \to \mathcal{C}$ of 2-bicategories is a homomorphism of their underlying bicategories whose hom functors are endowed with 2-cell maps making them into 2-functors, and whose coherence data is natural with respect to 3-cells.

A basic example of a 2-bicategory is obtained from any 2-category $\mathcal{K}$ with pullbacks, by adapting the bicategory of spans in $\mathcal{K}$ to take $\mathcal{K}$’s 2-cells into account using the 2-dimensional aspect of $\mathcal{K}$’s pullbacks. We denote this 2-bicategory as $\text{Span}_\mathcal{K}$.

Any (strict) 3-category is a 2-bicategory, and so $\mathbf{2-CAT}$ is another example.

**Theorem 29.** Let $\mathcal{K}$ be a 2-category with pullbacks. One has a 2-bicategory $\text{Poly}_\mathcal{K}$ whose objects are those of $\mathcal{K}$, whose hom between $X$ and $Y \in \mathcal{K}$ is $\text{Poly}_\mathcal{K}(X,Y)$, and whose compositions are defined as above. Moreover with object, arrow and 2-cell maps defined as in the categorical case, and 3-cell map as defined above, one has a homomorphism

$$P_\mathcal{K} : \text{Poly}_\mathcal{K} \to \mathbf{2-CAT}$$

of 2-bicategories.

**Proof.** By the way that things have been set up, lemma(14) and theorem(15) lift to our 2-categorical setting, with the extra naturality of the coherences coming from the 2-dimensional aspect of all the universal properties being used. Thus $\text{Poly}_\mathcal{K}$ is a 2-bicategory. Since in the proof of theorem(22) the coherences of $P_\mathcal{E}$ were obtained from associativity coherences in $\text{Poly}_\mathcal{E}$, the extra naturality enjoyed by the associativities in $\text{Poly}_\mathcal{K}$ gives the extra naturality required for the coherences of $P_\mathcal{K}$. Thus $P_\mathcal{K}$ is a homomorphism of 2-bicategories. □

8. Canonical examples

In this section we give some canonical examples of polynomial 2-functors. Recall [6, 4] that the functors $f : A \to B$ which are exponentiable as arrows of $\text{CAT}$ have been characterised combinatorially – for any arrow $\alpha$ in $A$ and factorisation of $f\alpha$ in $B$, one can consider the category of factorisations of $\alpha$ in $A$ over the given factorisation of $f(\alpha)$, and the necessary and sufficient condition for exponentiability is that these categories of factorisations be connected. In particular it follows that both Grothendieck fibrations and Grothendieck opfibrations are exponentiable, since the factorisation lifting condition may easily be shown to follow from the lifting conditions one has in these special cases. Thus one has a very explicit understanding of what a polynomial is in $\text{CAT}$. 
We shall assume in this section that the 2-categories $K$ under consideration are finitely complete and satisfy the condition that fibrations and opfibrations are exponentiable. This is true in $\mathbf{CAT}$, in any 2-category of the form $[\mathcal{C}^{\text{op}}, \mathbf{CAT}]$ and more generally in any 2-category of categories internal to a locally cartesian closed category, by corollary(2.17) of [13].

The first class of canonical examples comes from considering a morphism $p : E \to B$ which is a classifying discrete opfibration, in the sense of [16], in some such 2-category $\mathcal{K}$. A discrete opfibration in $\mathcal{K}$ is then defined to be $p$-small when it arises by pulling back $p$. By definition $p$-small discrete opfibrations are pullback stable in $\mathcal{K}$. Often they are also closed under composition, and when this is the case, one can consider the full sub-2-bicategory $\mathcal{S}_p$ of $\text{Poly}_{\mathcal{K}}$ consisting of those polynomials

$$
\begin{array}{ccc}
X & \xleftarrow{q_1} & A \\
& \downarrow & \downarrow \\
& B & \xrightarrow{q_3} Y
\end{array}
$$

such that $q_2$ is a $p$-small discrete opfibration. The condition of being a classifying discrete opfibration then implies that the polynomial

$$
\begin{array}{ccc}
1 & \xleftarrow{1} & E \\
& \downarrow \scalebox{0.8}{$p$} & \downarrow \\
& B & \xrightarrow{1} 1
\end{array}
$$

is a biterminal object in $\mathcal{S}_p(1, 1)$.

Just as monads can be considered within any 2-category, there is a nice notion of pseudo-monad internal to any 2-bicategory. A pseudo-monad on an object $b$ in a 2-bicategory $\mathcal{B}$ is a pseudo-monoid in the hom $\mathcal{B}(b, b)$. Clearly homomorphisms of 2-bicategories send pseudo-monads to pseudo-monads. In particular if $t : b \to b$ is biterminal in the hom $\mathcal{B}(b, b)$, then it admits a pseudo-monad structure. Thus in particular (10) is the underlying endomap of a pseudo-monad in $\text{Poly}_{\mathcal{K}}$. That is to say, such classifying discrete opfibrations give rise in a canonical way to polynomial pseudo-monads.

**Example 30.** From [15] corollary(5.12) the Fam-construction is the result of applying $\mathbf{P}_{\mathbf{CAT}}$ to (10) when $p$ is the forgetful functor $\text{Set}_* \to \text{Set}$. To be $p$-small in this case amounts to having small fibres, and thus $p$-small fibrations are evidently closed under composition. The induced pseudo-monad structure on (10) is, after application of $\mathbf{P}_{\mathbf{CAT}}$, identifiable with the usual pseudo-monad structure on $\text{Fam}$.

Recall [11] that taking comma objects along identity arrows give 2-monads on the slices of $\mathcal{K}$, and that using these 2-monads one may give a definition of split fibration (resp. fibration) in $\mathcal{K}$ as the algebras (resp. pseudo-algebras) of this 2-monad. We now describe how these arise from polynomials in $\mathcal{K}$. For $X \in \mathcal{K}$ the endofunctor part of the monad on $\mathcal{K}/X$ for fibrations over $X$ sends $f : Y \to X$ to $\Phi(f)$ defined via the comma object

$$
\begin{array}{ccc}
X/f & \xrightarrow{\Phi(f)} & Y \\
\downarrow & \Downarrow \Rightarrow & \downarrow \\
\downarrow & f & \downarrow \\
X & \xrightarrow{1_X} & X
\end{array}
$$

in $\mathcal{K}$. However this can be rewritten in terms of pullbacks and cotensors. Writing $[n]$ for the ordinal $\{0 < \ldots < n\}$ one has a cospan

$$
\begin{array}{ccc}
[0] & \xleftarrow{\delta_1} & [1] \\
& \downarrow & \downarrow \\
& [0] & \xleftarrow{\delta_0}
\end{array}
$$
and cotensoring this with $X$ gives the span

$$X = X[0] \xrightarrow{d_1} X[1] \xrightarrow{d_0} X[0] = X$$

in $\mathcal{K}$, where $Z^A$ denotes the cotensor of $Z \in \mathcal{K}$ with the category $A$. One then has $\Phi(f) = \Sigma d_i \Delta d_i$, which expresses on objects that $\Phi$ is the result of applying $P_{\mathcal{K}}$ to the polynomial

$$X \xrightarrow{d_0} X[1] \xrightarrow{1} X[1] \xrightarrow{d_1} X.$$ 

That the monad structure of $\Phi$ is polynomial follows from well-known facts about the simplicial category $\Delta$ and the 2-functoriality of taking cotensors, as we now proceed to explain.

Recall [10, 12] that the standard presentation of (topologists') $\Delta$ as a subcategory of $\text{CAT}$ has the 2-categorical feature that the generating coface and codegeneracy maps are adjoint in the manner depicted in

$$
\cdots \xrightarrow{\delta_0} [0] \xrightarrow{\delta_1} [1] \xrightarrow{\delta_2} [2] \xrightarrow{\delta_3} [3] \xrightarrow{\delta_0} \cdots
$$

and recall also that the above is a cocategory object in $\text{CAT}$. These observations enable a useful interpretation of the diagram

$$
(11) \quad [0] \xleftarrow{\sigma_0} [1] \xleftarrow{\sigma_1} [2] \xleftarrow{\sigma_2} [3] \xleftarrow{\sigma_0} \cdots
$$

in which each functor labelled as "$t$" picks out the top element of its codomain, and each functor labelled as "$b$" picks out a bottom element. Since the horizontal arrows of $\text{(11)}$ are top and bottom preserving maps, each of them underlies a map of cospans. Since $\Delta$ is a cocategory object, the $n$-th cospan in $\text{(11)}$, reading from left to right, is an $(n-1)$-fold composite of the second one.

In a 2-category $\mathcal{K}$ with pushouts, one has the 2-bicategory $\text{Cospans}_{\mathcal{K}}$ whose arrows are cospans and compositions given by pushout of cospans. Moreover the notion of lax idempotent pseudo-monad $1$ on a 2-category admits an evident internalisation to any 2-bicategory. Thus $\text{(11)}$ is a lax idempotent pseudo comonad in $1$.

---

1Here we use the terminology of [8]. In terms of older terminology [9], these are called Kock-Zöberlein doctrines.
Cospan\textsubscript{CAT}. Cotensoring it with an object \(X\) in any finitely complete 2-category \(\mathcal{K}\), gives a lax idempotent 2-monad in \(\text{Span}_{\mathcal{K}}\) and by means of the inclusion \(\text{Span}_{\mathcal{K}} \hookrightarrow \text{Poly}_{\mathcal{K}}\), one has a polynomial lax idempotent 2-monad on \(X\). The associated 2-monad on \(\mathcal{K}/X\), obtained via application of \(\mathbf{P}_{\mathcal{K}}\), is exactly 2-monad on \(\mathcal{K}/X\) for fibrations.

Almost all the details of our account of the fibrations monads as polynomial monads, are contained already in section(2) of \([12]\). All that we have done here beyond mere recollection is to explain how these constructions of \([12]\) factor through the world of polynomials.

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