CELESTIAL MECHANICS SOLUTIONS WHERE THE FUTURE IS A PERFECT REFLECTION OF THE PAST

ALI ABDULHUSSEIN\textsuperscript{1} AND HARRY GINGOLD\textsuperscript{2}

Abstract. Newton’s equations of celestial mechanics are shown to possess a continuum of solutions in which the future trajectories of the \( N \) bodies are a perfect reflection of their past. These solutions evolve from zero initial velocities of the \( N \) bodies. Consequently, the future gravitational forces acting on the \( N \) bodies are also a perfect reflection of their past. The proof is carried out via Taylor series expansions. A perturbed system of equations of the \( N \) body problem is also considered. All real valued solutions of this perturbed system have no singularities on the real line. The perturbed system is shown to have a continuum of solutions that possess symmetry where the future velocities of the \( N \) bodies are a perfect reflection of their past. The positions and accelerations of the \( N \) bodies are then odd functions of the time. All \( N \) bodies then evolve from one location in space.

1. Introduction.

Solutions that possess symmetry are of great interest in celestial mechanics. Periodic motion of celestial bodies may be considered a form of symmetry that is central to our understanding of the solar system. Given a periodic function \( y(t) \) with a period \( P > 0, n \in \mathbb{N} \), then the following identity in \( t \) is a manifestation of symmetry

\[
y(t - nP) \equiv y(t + nP). \tag{1.1}
\]

demonstration of the predictive powers of Newtons’ equations of celestial mechanics is the existence of solutions that are conic sections that lie in one plane. This, for the Kepler two body problem. E.g. [16]. Even functions and odd functions respectively, are also a manifestation of symmetry. Namely,

\[
y(t) = y(-t), \quad -y(t) = y(-t). \tag{1.2}
\]

In this article we look for celestial mechanics solutions that possess symmetries other than periodicity. We answer in the affirmative the question whether or not the \( N \) body problem possesses solutions in which the future is a perfect reflection of the past. This is so iff the initial velocities of all \( N \) bodies is zero. Consequently, the future gravitational forces acting on the \( N \) bodies are also a perfect reflection of their past. The derivation of these conclusions point to another interesting aspect of the predictive powers of Newtons’ equations of celestial mechanics. We don’t know for sure what is the past the present and the future of the universe. Are there astronomically remote subsystems of point massess that approximately possess symmetries that Newton’s equations predict? This work is also a contribution to our interest in past present and future of the universe. Compare with [4]. A perturbed system

\[2000 \quad \text{Mathematics Subject Classification.} \quad \text{Primary 37N05; Secondary 70F15.} \]

\[\text{Key words and phrases.} \quad \text{Celestial Mechanics; Gravitation; Even solutions; Odd solutions; Taylor Series; Holomorphic; Analytic.}\]
of equations of the $N$ body problem is also considered. All real valued solutions of this perturbed system have no singularities on the real line. The perturbed system is shown to have a continuum of solutions that possess symmetry where the future velocities of the $N$ bodies are a perfect reflection of their past. The positions and accelerations of the $N$ bodies are then odd functions of the time. This is so iff the $N$ bodies evolve from one location in space. We provide a proof via Taylor series expansions. This method is tangible to scientists. The method of proof guarantees the existence of real analytic even solutions to the $N$ body problem. It also guarantees the existence of even and odd analytic solutions to an approximate model of the equations. For analytic solutions of differential systems compare e.g. with [11, 13, 12].

We prove two theorems that apply to a significant list of second order nonlinear vectorial autonomous differential equations. The list includes central force problems, like the Manev problem, and the Pendulum equation. Compare e.g. with [2, 3, 7, 10, 13, 15, 6, 5]. They could be also useful to numerical approximations of solutions and to phase space analysis. Given a scalar ordinary differential equation $y'' = f(y)$, let $(y, y')$ be the phase plane. Then, the even solutions orbits intercept the $y$ axis and the odd solutions orbits, if any, intercept the $y'$ axis. It is not easy to construct and visualize global phase diagram for a $2n$ dimensions space in. However, analiticity as proven in here could provide guidance to some local phase space analysis for every pair of coordinate $(y_j, y'_j)$. The order of presentations in this article is as follows. In section 2 we discuss preliminary notations and conventions and we motivate an approximation model to the $N$ body problem equations. In section 3 we prove one of the main theorems in this article about the existence of even solutions to the celestial mechanics solutions. In section 4 we provide a discussion about the meaning of $-f(y) = f(-y)$ and we provide a lemma on the symmetry of partial derivatives of $f(y)$. These, are necessary for the proof of our second main theorem on the existence of odd solutions to an approximation model to the $N$ body problem equations. In section 5 we provide a lemma that presents two successive even order derivatives of the components of $y$. In section 6 we provided a proof that the approximation model possesses a continuum of odd solutions.

2. Preliminary notations and conventions and an approximation model

Some of the notation below is motivated by the necessity to formulate initial value problems for the celestial mechanics equations that are complex valued and exists in disks of the complex plane $t$. However, as will be seen in the sequel, the initial point and the initial positions and velocities are required to be real valued. The following notation is used, $m, r, s, j, k, l, \lambda, s_j, z_i, k_0, N \in \mathbb{N}_0$, $\mathbb{N}_0$ is the set of nonnegative integers : $m_1, m_2, \ldots, m_N$ are the masses of the $N$ bodies; $t \in \mathbb{C}$ is the time variable; $y_j \in \mathbb{C}^3$ where $1 \leq j \leq N$, are column position vectors of the $N$ bodies, respectively; $T$ stands for transposition of a vector or a matrix; $y^T = [y_1, y_2, \ldots, y_N]$ and $f(y)^T := [f_1(y), f_2(y), \ldots, f_N(y)]$ are respectively rows of blocks of column vectors; $\overline{y}_j$ is the complex conjugate of $y_j$; $y_k(t_0), y_j(t_0) \in \mathbb{R}^3, y_k(t_0) \neq y_j(t_0), k \neq j, k, j = 1, 2, \ldots, N$. Let

$$D_\epsilon(t_0) := \{t \mid |t - t_0| \leq \epsilon, \epsilon > 0, \ t_0 \in \mathbb{R}, t \in \mathbb{C}\}. \quad (2.1)$$

We also adopt the following definition.
CELESTIAL MECHANICS SOLUTIONS WHERE THE FUTURE IS A PERFECT REFLECTION OF THE PAST

**Definition.** Given an m by n matrix and y an n by 1 column vector with elements in \( \mathbb{C} \). A norm denoted by \( | | \) is called algebraic if it satisifies the following inequality

\[
|Ay| \leq |A||y|.
\]  

(2.2)

The notation \( \|y\| \) that normally stands for \( \|y\| = [y^T y]^{1/2} \) that is the Euclidean norm for \( y \) complex, is replaced in here by the unconventional use

\[
\|y_j\| := [y_j^T y_j]^{1/2}, j = 1, 2, \ldots, N, y_j \in \mathbb{C}^3.
\]  

(2.3)

**Remark.** The unconventional use of notation \( (2.3) \) requires justification. We desire to prove the existence of analytic solutions to the celestial mechanics equations with \((2.4), (2.5)\), and with \((2.7), (2.9)\). The convention \((2.3)\) together with the real values of the initial point and the initial values guarantees that \( f_k(y) \) and \( \hat{f}_k(y) \) are analytic vector functions of \( y \) in some disk in \( \mathbb{C}^{3N} \). More specifically, the denominators in \((2.4)\) and in \((2.7)\) stay analytic in some disk in \( \mathbb{C}^{3N} \) and are bounded away from zero.

Put,

\[
f_k(y) := \sum_{j=1, j \neq k}^N \frac{Gm_j(y_j - y_k)}{\|y_j - y_k\|^3}, 1 \leq k \leq N.
\]  

(2.4)

Denote \( \frac{dy_j}{dt} = \dot{y}_j \), \( \frac{d^2y_j}{dt^2} = \ddot{y}_j \), \( \frac{dy_j}{dt} = \dot{y}_j \), etc, \( t_0 \in \mathbb{R}, y_0, \eta \in \mathbb{R}^{3N} \). Then, the initial value problem for the \( N \) bodies is

\[
y'' = f(y), y(t_0) = y_0, y'(t_0) = \eta, y_k(t_0) \neq y_j(t_0), k \neq j, k, j = 1, 2, \ldots, N. \tag{2.5}
\]

Observe that Newton’s equations of celestial mechanics satisfy

\[
-f_k(-y) = -\sum_{j \neq k} Gm_j(y_j - y_k) \|y_j - y_k\|^{-3} = \sum_{j \neq k} \frac{Gm_j(y_j - y_k)}{\|y_j - y_k\|^3} \implies -f(y) = f(-y). \tag{2.6}
\]

However, we cannot solve for odd solutions the initial value problem \((2.5)\) with a condition \( y(t_0) = \overrightarrow{0} \). This, because \( y(t_0) = \overrightarrow{0} \) means that all of the celestial point masses are in state of mutual collision. Then, the initial value problem \((2.5)\) contains undetermined and unbounded terms which render the equations invalid. Therefore, we consider a modified celestial mechanics system of equations with \( \epsilon(j, k) \) small. Namely,

\[
\hat{f}_k(y) := \sum_{j=1, j \neq k}^N \frac{Gm_j(y_j - y_k)}{\|y_j - y_k\| + \epsilon(j, k)}, 1 \leq j, k \leq N, \epsilon(j, k) = \epsilon(k, j) > 0. \tag{2.7}
\]

It is easily verified that \( -\hat{f}_k(y) = \hat{f}_k(-y) \) as well. Consequently, with

\[
-\hat{f}(y)^T := [\hat{f}_1(y), \hat{f}_2(y), \ldots, \hat{f}_N(y)]^T = \hat{f}(-y)^T, \tag{2.8}
\]

we obtain an initial value problem for an approximated equation

\[
y'' = \hat{f}(y), y(t_0) = y_0, y'(t_0) = \eta, y_0, \eta \in \mathbb{R}^{3N}. \tag{2.9}
\]

Formally, we also have that the limits of all \( \hat{f}_k(y) \) with all \( \epsilon(j, k) \rightarrow 0^+ \) are respectively \( f_k(y) \) of Newton’s equations \((2.4)\). This initial value problem \((2.9)\) can be solved for any \( y_0, \eta \in \mathbb{R}^{3N} \). Notice that

\[
y_0, \eta \in \mathbb{R}^{3N}, t, t_0 \in \mathbb{R} \implies y(t) \in \mathbb{R}^{3N}.
\]
Then, the initial value problem possesses a unique analytic solution \( C_{1, 9, 8} \).

**Proof.** Without loss of generality assume that **Theorem 1.** Assume that: i) for which \( \text{Remark.} \) develop singularities in the complex plane.

These particular cases (3.3) and (3.4) indicate what should be the general form of higher odd order derivatives of \( y \).

Also observe that for \( y(t) \in \mathbb{R}^{3N} \), a solution to the initial value problem 2.9, we have

\[
\|y''(t)\| = \left\| \hat{f}(y) \right\| \leq \left\| \sum_{j=1, j \neq k}^{N} \frac{Gm_j(y_j - y_k)}{\|y_k - y_j\| + \epsilon(j,k)} \right\| \leq NG[\max(m_j)] \max\left( \frac{1}{[\epsilon(j,k)]^2} \right).
\]

(2.10)

\( \max(m_j) \) and \( \max(1/[\epsilon(j,k)]^2) \) are taken over all \( j = 1, 2, \ldots, N \). Consequently, all solutions of 2.9 exist on \((-\infty, \infty)\). Thus, the system of equations \( y'' = \hat{f}(y) \) is (so called) complete. However, solutions of 2.9 as analytic functions of \( t \), could develop singularities in the complex plane.

**Remark.** The formulation of the two main theorems in the sequel assume \( y, f(y) \in \mathbb{C}^n \), \( n \in \mathbb{N} \), that are somewhat more general than the \( f(y) \) and \( \hat{f}(y) \) discussed above for which \( n = 3N \) was restricted.

### 3. Formulation and Proof of Main Theorem 1 by Induction on Odd Derivatives

**Theorem 1.** Assume that: i) \( t \in \mathbb{C}, t_0 \in \mathbb{R}, y_0, \eta \in \mathbb{R}^{3N}, y, f(y) \in \mathbb{C}^n \), \( n \in \mathbb{N} \), where \( f(y) \) is an analytic function in the vector variable \( y \) in a disk such that in \( D \)

\[
D := \{y|\|y - y_0\| \leq b\} \implies \|f(y)\| \leq M.
\]

Then, the initial value problem

\[
y'' = f(y), y(t_0) = y_0, y'(t_0) = 0, \quad 0^T := [0, 0, \ldots, 0],
\]

(3.2)

possesses a unique analytic solution \( y(t) \) for \( |t - t_0| \leq \sqrt{2b/M} \) that satisfies \( y(t - t_0) \equiv y(-t - t_0) \). Namely, \( y^{(m)}(0) = 0^T \) for all odd numbers \( m \).

**Proof.** Without loss of generality assume that \( t_0 = 0 \) since our differential system is autonomous. Below are two successive odd order derivatives. Compare also with 11 91 81.

\[
\frac{d^3y_j(t)}{dt^3} = \sum_{k_1=1}^{n} \frac{\partial f_j(y(t))}{\partial y_{k_1}} \frac{dy_{k_1}(t)}{dt}; j = 1, 2, \ldots, n.
\]

(3.3)

\[
\frac{d^5y_j(t)}{dt^5} = \sum_{k_1=1}^{n} \frac{\partial f_j(y(t))}{\partial y_{k_1}} \frac{d^2y_{k_1}(t)}{dt^2} + \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \frac{\partial^2 f_j(y(t))}{\partial y_{k_1} \partial y_{k_2}} \frac{dy_{k_2}(t)}{dt} \frac{d^2y_{k_1}(t)}{dt^2} + \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} \frac{\partial^3 f_j(y(t))}{\partial y_{k_1} \partial y_{k_2} \partial y_{k_3}} \frac{dy_{k_3}(t)}{dt} \frac{d^3y_{k_2}(t)}{dt^3} \frac{dy_{k_1}(t)}{dt}.
\]

(3.4)

It is easily verified that with \( y^T = [y_1, y_2, \ldots, y_j, \ldots, y_n]^T \) and with \( y_j(t) \) scalars, we get from (3.3) and (3.4)

\[
y'(0) = 0^T \implies y^{(3)}(0) = y^{(5)}(0) = 0^T.
\]

(3.5)

These particular cases 3.3 and 3.4 indicate what should be the general form of higher odd order derivatives of \( y_j(t) \). Moreover, they demonstrate how the property of zero initial velocities \( y_j^{(1)}(0) = 0 \) is inherited by subsequent derivatives of odd order. In what follows we may suppress the notation \( t \) in \( y(t), y_j^{(\lambda)}(t) \) etc,
We view components of the form (3.9) subject to (3.10) and (3.11). It is useful to keep in mind that when clarity is not compromised. Assume that each component \( y_j^{(2+m)}(t) \), \( m \) odd, \( j = 1, 2, \ldots, n \), is a finite sum of products of the form:

\[
T_m := \frac{\partial^m f_j(y(t))}{\partial y_{k_1} \ldots \partial y_{k_m}} J_O J_E, 
\]

(3.6)

i) \( J_O \) is a finite product of an odd number \( r \in \mathbb{N}_0 \) of odd order derivatives of certain components of \( y(t) \). Namely,

\[
J_O = y_{x_1}^{(2e_1+1)}(t) y_{x_2}^{(2e_2+1)}(t) \ldots y_{x_r}^{(2e_r+1)}(t), \quad e_1, e_2, \ldots, e_r \in \mathbb{N}_0, \ r \geq 1. 
\]

(3.7)

ii) \( J_E \) is a finite product of any number \( w \in \mathbb{N}_0 \) of even order derivatives of components of \( y(t) \). Namely,

\[
J_E = y_{z_1}^{(2c_1)}(t) y_{z_2}^{(2c_2)}(t) \ldots y_{z_w}^{(2c_w)}(t), \quad c_1, c_2, \ldots, c_w \in \mathbb{N}_0. 
\]

(3.8)

If \( w = 0 \), we put \( J_E \equiv 1 \). Then, \( y_j^{(4+m)}(t) \) is a finite sum of certain products of the form

\[
\hat{T}_m^{(2)} := \frac{\partial^2 f_j(y(t))}{\partial y_{k_1} \ldots \partial y_{k_2} \partial y_{k_1}} \hat{J}_O \hat{J}_E,
\]

(3.9)

where \( \hat{J}_O \) is a finite product of an odd number \( p \in \mathbb{N} \) of odd order derivatives of components of \( y(t) \). Namely,

\[
\hat{J}_O = y_{x_1}^{(2g_1+1)}(t) y_{x_2}^{(2g_2+1)}(t) \ldots y_{x_p}^{(2g_p+1)}(t), \quad g_1, g_2, \ldots, g_p \in \mathbb{N}_0, \ p \geq 1,
\]

(3.10)

and \( \hat{J}_E \) is a finite product of any number \( q \in \mathbb{N}_0 \) of even order derivatives of components of \( y(t) \). Namely,

\[
\hat{J}_E = y_{z_1}^{(2c_1)}(t) y_{z_2}^{(2c_2)}(t) \ldots y_{z_q}^{(2c_q)}(t), \quad c_1, c_2, \ldots, c_q, q \in \mathbb{N}_0.
\]

(3.11)

Evidently, the induction hypothesis holds for the derivatives \( y^{(1)}(0) = y^{(3)}(0) = y^{(5)}(0) = 0 \). It is easily verified that \( \hat{Q}_3 \) and \( \hat{Q}_4 \) are sums of products of the form \( (3.6) \). The crux of the induction is to show that \( \hat{J}_O \) and \( \hat{J}_E \) respectively, are of the desired form (3.10) and (3.11) for any \( m \) odd. To this end differentiate twice both sides of (3.6). This leads to the relation \( T_m^{(2)} = Q_1 + Q_2 + Q_3 \) where

\[
Q_1 := \sum_{k_{i+2}=1}^{n} \sum_{k_{i+1}=1}^{n} \left\{ \left[ \frac{\partial^{k_{i+2}} f_j(y(t))}{\partial y_{k_{i+2}} \partial y_{k_{i+1}} \partial y_{k_i} \partial y_{k_{i+2}} \partial y_{k_i}} \right] y_{k_{i+2}}^{(1)} y_{k_{i+1}}^{(1)} J_O J_E \right. \\
+ \sum_{k_{i+1}=1}^{n} \left[ \frac{\partial^{k_{i+1}} f_j(y(t))}{\partial y_{k_{i+1}} \partial y_{k_i} \partial y_{k_{i+1}} \partial y_{k_i}} \right] y_{k_{i+1}}^{(2)} J_O J_E, 
\]

(3.12)

\[
Q_2 := 2 \sum_{k_{i+1}=1}^{n} \left[ \frac{\partial^{k_{i+1}} f_j(y(t))}{\partial y_{k_{i+1}} \partial y_{k_i} \partial y_{k_{i+1}} \partial y_{k_i}} \right] y_{k_{i+1}}^{(1)} J_O^{(1)} J_E + J_O J_E^{(1)}, 
\]

(3.13)

\[
Q_3 := \frac{\partial^j f_j(y(t))}{\partial y_{k_j} \partial y_{k_j} \partial y_{k_j}} [J_O J_E]^{(2)} = \frac{\partial^j f_j(y(t))}{\partial y_{k_j} \partial y_{k_j} \partial y_{k_j}} [J_O^{(2)} J_E + J_O J_E^{(1)}], 
\]

(3.14)

We view \( Q_1 \), \( Q_2 \) and \( Q_3 \) as sums of certain products and we show that they are of the form (3.9) subject to (3.10) and (3.11). It is useful to keep in mind that \( J_O \) is
CELESTIAL MECHANICS SOLUTIONS WHERE THE FUTURE IS A PERFECT REFLECTION OF THE PAST

a product of an odd number \( r \geq 1 \) of odd order derivatives. The list of these terms is:

\[
y^{(1)}_{k_{i+2}}y^{(1)}_{k_{i+1}}y^{(2)}_{O}y^{(2)}_{E}, y^{(2)}_{k_{i+1}}y^{(1)}_{O}y^{(1)}_{O}J^{(1)}_{E}, y^{(1)}_{k_{i+1}}y^{(1)}_{O}J^{(1)}_{O}J^{(1)}_{E}, y^{(1)}_{k_{i+1}}y^{(1)}_{O}J^{(2)}_{O}J^{(2)}_{E}.
\]

(3.15)

For the products originating from \( y^{(1)}_{k_{i+2}}y^{(1)}_{k_{i+1}}J^{(1)}_{O}J^{(1)}_{E} \) put \( \tilde{J}_{O} = y^{(1)}_{k_{i+2}}y^{(1)}_{k_{i+1}}J^{(1)}_{O}, \tilde{J} = J^{(1)}_{E} \). For the products originating from \( y^{(2)}_{k_{i+1}}y^{(2)}_{O}J^{(1)}_{O}J^{(1)}_{E} \) put \( \tilde{J}_{O} = J^{(2)}_{O}, \tilde{J} = y^{(2)}_{k_{i+1}}J^{(1)}_{E} \). Consider now \( y^{(1)}_{k_{i+1}}J^{(1)}_{O}J^{(1)}_{E} \). Evidently, \( J^{(1)}_{O} \) is a sum of \( r \) products as follows.

\[
J^{(1)}_{O} = \sum_{j=1}^{r} y^{(2c_{j}+2)}_{s_{j}} \prod_{l \neq j, l=1}^{r} y^{(2c_{l}+1)}_{t_{l}}.
\]

(3.16)

Each product in (3.16) has \((r-1)\) odd order derivatives factors and precisely one factor that is an even order derivative of a certain component of \( y_{s_{j}} \). For a representative product in \( y^{(1)}_{k_{i+1}}J^{(1)}_{O}J^{(1)}_{E} \) put

\[
\tilde{J}_{E} = y^{(2c_{j}+2)}_{s_{j}}J^{(1)}_{E}, r = 1 \Rightarrow \tilde{J}_{O} = y^{(1)}_{k_{i+1}}, r = 2 \Rightarrow \tilde{J}_{O} = y^{(1)}_{k_{i+1}} \prod_{l \neq j, l=1}^{r} y^{(2c_{l}+1)}_{t_{l}}.
\]

(3.17)

Consider now a product originating from \( y^{(1)}_{k_{i+1}}J^{(1)}_{O}J^{(1)}_{E} \). If \( w = 0 \) namely \( J^{(1)}_{E} = 1 \), then \( J^{(1)}_{O} = 0 \) and the product \( y^{(1)}_{k_{i+1}}J^{(1)}_{O}J^{(1)}_{E} = 0 \). If \( w = 1 \), put \( \tilde{J}_{O} = y^{(1)}_{k_{i+1}}y^{(2c_{1}+1)}_{z_{1}}J^{(1)}_{O}, \tilde{J}_{E} = 1 \). Let \( w \geq 2 \). Then, \( J^{(1)}_{E} \) is the following sum of \( w \) products.

\[
J^{(1)}_{E} = \sum_{j=1}^{w} y^{(2c_{j}+1)}_{s_{j}} \prod_{l \neq j, l=1}^{w} y^{(2c_{l})}_{t_{l}}.
\]

(3.18)

Put,

\[
\tilde{J}_{O} = y^{(1)}_{k_{i+1}}y^{(2c_{1}+1)}_{z_{1}}J^{(1)}_{O}, \tilde{J}_{E} = \prod_{l \neq j, l=1}^{w} y^{(2c_{l})}_{t_{l}}.
\]

(3.19)

Consider now the products emanating from \( J^{(1)}_{O}J^{(1)}_{E} \). If \( J^{(1)}_{E} = 1 \) namely \( w = 0 \), then \( J^{(1)}_{O}J^{(1)}_{E} \equiv 0 \). If \( r = 1 \) and \( w = 1 \) put \( \tilde{J}_{O} = y^{(2c_{1}+1)}_{z_{1}}J^{(1)}_{O}, \tilde{J}_{E} = y^{(2c_{1}+2)}_{z_{1}} \). If \( r \geq 2 \) and \( w = 1 \) put

\[
\tilde{J}_{O} = y^{(2c_{1}+1)}_{z_{1}} \prod_{l \neq j, l=1}^{r} y^{(2c_{j}+1)}_{s_{j}}, \tilde{J}_{E} = 1.
\]

(3.20)

If \( r \geq 2 \) and \( w \geq 2 \) we have by (3.16) and (3.18) \( rw \) products in \( J^{(1)}_{O}J^{(1)}_{E} \). Put

\[
\tilde{J}_{O} = y^{(2c_{j}+1)}_{z_{j}} \prod_{l \neq j, l=1}^{r} y^{(2c_{l}+1)}_{s_{l}}, \tilde{J}_{E} = y^{(2c_{j}+2)}_{z_{j}} \prod_{l \neq j, l=1}^{w} y^{(2c_{l})}_{t_{l}}.
\]

It remains to consider the factors originating from \( J^{(2)}_{O}J^{(2)}_{E} \) and from \( J^{(2)}_{O}J^{(1)}_{E} \). To this end we first calculate the sum of products emanating from \( J^{(2)}_{O}J^{(2)}_{E} \) and multiply them...
CELESTIAL MECHANICS SOLUTIONS WHERE THE FUTURE IS A PERFECT REFLECTION OF THE PAST

by \( J_E \). If \( J_O \) has only \( r = 1 \) factors then \( J_O^{(2)} = y_s^{(2e_1+3)} \). Then put \( \hat{J}_O = y_s^{(2e_1+3)} \) and \( \hat{J}_E = J_E \). If \( r \geq 3 \) then the factors in \( J_O^{(2)} \) are of two kinds. The one is

\[
y_{s_j}^{(2r+2)} y_{s_k}^{(2r+2)} \prod_{l \neq j,k,l=1}^r y_{s_l}^{(2e_1+1)}, \quad r \geq 3. \tag{3.21}
\]

Then, put

\[
\hat{J}_O = \prod_{l \neq j,k,l=1}^r y_{s_l}^{(2e_1+1)}, \quad \hat{J}_E := y_{s_j}^{(2r+2)} y_{s_k}^{(2r+2)} J_E, \quad r \geq 3.
\]

The other type of product in \( J_O^{(2)} \) is \( y_{s_j}^{(2e_1+3)} \prod_{l \neq j,l=1}^r y_{s_l}^{(2e_1+1)} \). Put

\[
\hat{J}_O := y_{s_j}^{(2e_1+3)} \prod_{l \neq j,l=1}^r y_{s_l}^{(2e_1+1)}, \quad \hat{J}_E := J_E, \quad \text{if} \ r \geq 3.
\]

It remains to analyze the resulting products in \( J_O J_E^{(2)} \). To this end calculate first the resulting products in \( J_E^{(2)} \). If \( w = 0 \) then \( J_E^{(2)} \equiv 0 \) and then the contribution of products from \( J_O J_E^{(2)} \) is 0. If \( w = 1 \) then \( J_E^{(2)} = y_{s_1}^{(2e_1+2)} \) and we put \( \hat{J}_O = J_O, \quad \hat{J}_E = y_{s_1}^{(2e_1+2)} \). Assume that \( w \geq 2 \). Then \( J_E^{(2)} \) is a sum of products that are of two kinds. The first kind is

\[
y_{s_j}^{(2e_1+2)} \prod_{l \neq j,l=1}^w y_{s_l}^{(2e_1)}. \tag{3.22}
\]

Then put

\[
\hat{J}_O = J_O, \quad \hat{J}_E = y_{s_j}^{(2e_1+2)} \prod_{l \neq j,l=1}^w y_{s_l}^{(2e_1)}. \tag{3.23}
\]

The second kind is precisely \( y_{s_j}^{(2e_1+1)} y_{s_k}^{(2e_1+1)} \) if \( w = 2 \). Then put

\[
\hat{J}_O \equiv y_{s_j}^{(2e_1+1)} y_{s_k}^{(2e_1+1)} J_O, \quad \hat{J}_E \equiv 1. \tag{3.24}
\]

If \( w \geq 3 \) then the second kind of a product resulting from the sum of products in \( J_E^{(2)} \) is

\[
y_{s_j}^{(2e_1+1)} y_{s_k}^{(2e_1+1)} \prod_{l \neq j,k,l=1}^w y_{s_l}^{(2e_1)}. \tag{3.25}
\]

Then put

\[
\hat{J}_O := y_{s_j}^{(2e_1+1)} y_{s_k}^{(2e_1+1)} J_O, \quad \hat{J}_E := \prod_{l \neq j,k,l=1}^w y_{s_l}^{(2e_1)}. \tag{3.26}
\]

In sum, all \( T_m(0) = 0 \) imply that all \( \hat{T}_m^{(2)}(0) = 0 \). Consequently, \( y^{(m)}(0) = 0 \) for odd \( m \in \mathbb{N}_0 \). Q.E.D.

Remark. It is evident that the symmetry \( y(t) \equiv y(-t) \) is manifest to the future being a perfect reflection of the past. We show inhere that all odd order derivatives, in the Taylor series expansion of \( y(t) \), vanish at \( t = 0 \). The estimate \( |t| \leq \sqrt{2b/m} \) follows by adaptations of the technicians in e.g. [12], Chapter 1, pages 20. Compare also with [11]. The Taylor series then show that \( y(t) = y(0) + \sum_{l=1}^n [y^{(2l)}(0)/(2l)!]t^{2l} \) satisfy
y(t) \equiv y(-t). Since y'' = f(y) is an autonomous system, then for any t_0 \in \mathbb{C}, y(t) = y(t_0) + \sum_{j=1}^{n} \frac{[f''(t_0)](2t_0)^{2j}}{2j!} (t - t_0)^{2j} are also solutions of y'' = f(y). Furthermore, the velocities are symmetric functions as well. Namely -y'(t) \equiv y'(-t). The future accelerations and forces acting on the bodies, are a perfect reflection of their past. Let t_c \in \mathbb{R} be a real valued collision time, where y_k(t_c) = y_j(t_c) for some k \neq j. Allowing the variable t to be complex valued could make it possible to analytically continue a solution y(t) from the real line into the complex plane, from time t < t_c to t > t_c. Then, y(t) \equiv y(-t) holds for t < t_c as well as for t > t_c. Thus, circumventing a collision at time t = t_c.

4. A LEMMA ON PARTIAL DERIVATIVES OF f(y)

We clarify now what is an even and an odd function of a scalar function u = H(y_1, y_2, \ldots, y_N) of several variables. To this end we denote the transposed column vector y^T = (y_1, y_2, \ldots, y_N) and we put u = H(y_1, y_2, \ldots, y_N) = H(y).

**Definition 2.** Denote by REG an open connected set in \( \mathbb{R}^N \). We say that H(y) is an even function of y in REG if

\[
H(y) = H(-y), \quad y \in \text{REG}.
\]  

(4.1)

We say that H(y) is an odd function of y in REG if

\[
-H(y) = H(-y), \quad y \in \text{REG}.
\]  

(4.2)

This definition is different than requiring that \( u = H(y_1, y_2, \ldots, y_N) = H(y) \) be an even or an odd function in each individual coordinate \( y_j \). In order to bring out the difference we add the following.

**Definition 3.** Denote by REG an open connected set in \( \mathbb{R}^N \). We say that H(y) is an even function of y in REG in the strict sense if

\[
H(y_1, y_2, \ldots, y_j, \ldots, y_N) = H(y_1, y_2, \ldots, -y_j, \ldots, y_N), \quad y \in \text{REG}, j = 1, 2, \ldots, N.
\]  

(4.3)

We say that H(y) is an odd function of y in REG in the stricter sense if

\[
-H(y_1, y_2, \ldots, y_j, \ldots, y_N) = H(y_1, y_2, \ldots, -y_j, \ldots, y_N), \quad y \in \text{REG}, j = 1, 2, \ldots, N.
\]  

(4.4)

Consider the following functions

\[
H(y_1, y_2) := y_1^5 y_2^3, \quad L(y_1, y_2) = y_1^{10} y_2^6.
\]  

(4.5)

Evidently, \( H(y_1, y_2) := y_1^5 y_2^3 \) is an even function in \( \text{REG} := \mathbb{R}^2 \). However, it is an odd function in the strict sense in \( \text{REG} := \mathbb{R}^2 \). Evidently, \( L(y_1, y_2) = y_1^{10} y_2^6 \) is an even function in \( \text{REG} := \mathbb{R}^2 \) and it is also an even function in the strict sense in \( \text{REG} := \mathbb{R}^2 \).

**Remark 4.** The reader may want to consider a multinomial in the \((r + w)\) independent variables \( y_1, y_2, \ldots, y_{r}, y_{r+1}, y_{r+2}, \ldots, y_{r+w} \).

\[
H(y) = \left[ y_{1}^{[2e_{1}+1]} y_{2}^{[2e_{2}+1]} \ldots y_{j}^{[2e_{j}+1]} \ldots y_{r}^{[2e_{r}+1]} \right] \left[ y_{r+1}^{[2e_{r+1}]} y_{r+2}^{[2e_{r+2}]} \ldots y_{r+w}^{[2e_{r+w}]} \right]
\]  

where \( e_1, e_2, \ldots, e_r, c_1, c_2, \ldots, c_w \in \mathbb{N}_0, \quad r, w \in \mathbb{N} \). Formulation of necessary and sufficient conditions on the powers occurring in \( H(y) \) such that a) \( H(y) \) is an even multivariate function b) \( H(y) \) is an even multivariate function in the strict sense.
c) $H(y)$ is an odd multivariate function d) $H(y)$ is an odd multivariate function in the strict sense, could further clarify the difference between these two types of symmetry. Next we formulate an analog to Lemma 1 for multivariate functions.

**Lemma 5.** Let $H(y) \in C^1(\text{REG})$. i) Assume that $H(y) = H(-y), y \in \text{REG}$. Then the partial derivatives

$$
\Psi_j(y) := \frac{\partial H(y)}{\partial y_j}, \; j = 1, 2, \ldots, N,
$$

are odd function in REG. ii) Assume that $-H(y) = H(-y), y \in \text{REG}$. Then the partial derivatives

$$
\Psi_j(y) := \frac{\partial H(y)}{\partial y_j}, \; j = 1, 2, \ldots, N,
$$

are even functions in REG. iii) Assume $f(y)$ to be a column vector function, $f^T(y) := [f_1(y), f_2(y), \ldots, f_n(y)]$, where $f_j(y), j = 1, 2, \ldots, n$ are the scalar component of $f(y)$ such that $f_j(y) \in C^1(\text{REG})$. Then,

$$
f(y) = f(-y), y \in \text{REG} \implies \Psi(y) := \frac{\partial f(y)}{\partial y_j} = -\Psi(-y) := -\frac{\partial f(-y)}{\partial y_j}, j = 1, 2, \ldots, N.
$$

Moreover,

$$
-f(y) = f(-y), y \in \text{REG} \implies \Psi(y) := \frac{\partial f(y)}{\partial y_j} = \Psi(-y) := \frac{\partial f(-y)}{\partial y_j}, j = 1, 2, \ldots, N.
$$

We first prove i) and we focus on the quotient below: with $y \in \text{REG}$; with $h \neq 0$; and with $h$ arbitrarily small.

$$
Q_j(y, h) := \frac{H(y_1, y_2, \ldots, y_{j-1}, y_j + h, y_{j+1}, \ldots, y_N) - H(y_1, y_2, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_N)}{h}.
$$

Put as short hand notation

$$
\hat{H}(y_j + h) := H(y_1, y_2, \ldots, y_{j-1}, y_j + h, y_{j+1}, \ldots, y_N),
$$

$$
\hat{H}(-y_j - h) := H(-y_1, -y_2, \ldots, -y_{j-1}, -y_j - h, -y_{j+1}, \ldots, -y_N).
$$

**Proof.** Since $H(y)$ is an even function then

$$
H(y_1, y_2, \ldots, y_{j-1}, y_j + h, y_{j+1}, \ldots, y_N) = H(-y_1, -y_2, \ldots, -y_{j-1}, -y_j, -y_{j+1}, \ldots, -y_N).
$$

$$
H(y_1, y_2, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_N) = H(-y_1, -y_2, \ldots, -y_{j-1}, -y_j, -y_{j+1}, \ldots, -y_N).
$$

Substitute from (4.12) and (4.10) and (4.11) into the right hand side of (4.9) to obtain

$$
Q_j(y, h) = -Q_j(-y, -h) = -\frac{\hat{H}(-y_j - h) - \hat{H}(-y_j)}{-h}.
$$

Take the limit as $h \to 0$ in (4.14) and obtain

$$
\Psi(y) := \frac{\partial H(y)}{\partial y_j} = \lim_{h \to 0} Q_j(y, h) = -\lim_{h \to 0} Q_j(-y, -h) = -\Psi(-y).
$$
Next we prove ii). We focus again on the quotient
\[
Q_j(y, h) := \frac{H(y_1, y_2, \ldots, y_{j-1}, y_j + h, y_{j+1}, \ldots, y_N) - H(y_1, y_2, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_N)}{h}.
\] (4.16)

Since \( H(y) \) is an odd function then
\[
H(y_1, y_2, \ldots, y_{j-1}, y_j + h, y_{j+1}, \ldots, y_N)
= -H(-y_1, -y_2, \ldots, -y_{j-1}, -y_j - h, -y_{j+1}, \ldots, -y_N).
\] (4.17)
\[
H(y_1, y_2, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_N) = -H(-y_1, -y_2, \ldots, -y_{j-1}, -y_j, -y_{j+1}, \ldots, -y_N).
\] (4.18)

Substitute from (4.17) into the right hand side of (4.16) to obtain
\[
Q_j(y, h) = Q_j(-y, -h).
\] (4.19)

Take the limit as \( h \to 0 \) in (4.19) and obtain
\[
\Psi(y) := \frac{\partial H(y)}{\partial y_j} = \lim_{h \to 0} Q_j(y, h) = \lim_{h \to 0} Q_j(-y, -h) = \Psi(-y). \quad (4.20)
\]

The proof of iii) follows from the proofs of i) and ii) and the definition of \( f(y) \). □

5. SAMPLE OF TWO SUCCESSIVE EVEN DERIVATIVES LEMMA

A straight forward calculation reveals that
\[
\frac{d^4 y_j(t)}{dt^4} = \sum_{k_1=1}^{n} \frac{\partial f_j(y(t))}{\partial y_{k_1}} \frac{d^2 y_{k_1}(t)}{dt^2} + \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \frac{\partial^2 f_j(y(t))}{\partial y_{k_2} \partial y_{k_1}} \frac{dy_{k_2}(t)}{dt} \frac{dy_{k_1}(t)}{dt}. \quad (5.1)
\]
\[
\frac{d^6 y_j(t)}{dt^6} = \sum_{k_1=1}^{n} \frac{\partial f_j(y(t))}{\partial y_{k_1}} \frac{d^4 y_{k_1}(t)}{dt^4} + \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} \frac{\partial^2 f_j(y(t))}{\partial y_{k_2} \partial y_{k_1}} \left[ 3 \frac{d^3 y_{k_1}(t)}{dt^3} \frac{dy_{k_2}(t)}{dt} + \frac{dy_{k_2}(t)}{dt} \frac{d^3 y_{k_1}(t)}{dt^3} + \frac{d^3 y_{k_1}(t)}{dt^3} \frac{dy_{k_2}(t)}{dt} \right] \frac{dy_{k_2}(t)}{dt} \frac{dy_{k_1}(t)}{dt}
\]
\[
+ \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} \sum_{k_4=1}^{n} \frac{\partial^3 f_j(y(t))}{\partial y_{k_4} \partial y_{k_2} \partial y_{k_1}} \left[ 3 \frac{d^2 y_{k_1}(t)}{dt^2} \frac{dy_{k_2}(t)}{dt} \frac{dy_{k_4}(t)}{dt} + 2 \frac{dy_{k_1}(t)}{dt} \frac{d^2 y_{k_2}(t)}{dt^2} \frac{dy_{k_4}(t)}{dt} + \frac{dy_{k_1}(t)}{dt} \frac{dy_{k_2}(t)}{dt} \frac{d^2 y_{k_4}(t)}{dt^2} \right] \frac{dy_{k_2}(t)}{dt} \frac{dy_{k_1}(t)}{dt} \frac{dy_{k_4}(t)}{dt}
\]
\[
+ \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} \sum_{k_4=1}^{n} \sum_{k_5=1}^{n} \frac{\partial^4 f_j(y(t))}{\partial y_{k_5} \partial y_{k_4} \partial y_{k_2} \partial y_{k_1}} \frac{dy_{k_5}(t)}{dt} \frac{dy_{k_4}(t)}{dt} \frac{dy_{k_2}(t)}{dt} \frac{dy_{k_1}(t)}{dt} \frac{dy_{k_2}(t)}{dt}. \quad (5.2)
\]

These aid us in formulating the elements of induction i) , ii) in theorem 6 below.

6. INDUCTION ON EVEN DERIVATIVES

We prove

**Theorem 6.** Assume that: i) \( t \in \mathbb{C}, t_0 \in \mathbb{R}, y_0, \eta \in \mathbb{R}^{3N}, y, f(y) \in \mathbb{C}^n, n \in \mathbb{N}, \) where \( f(y) \) is an analytic function in the vector variable \( y \) in a disk such that in \( D \)
\[
D := \{ y | \| y - y_0 \| \leq b \} \implies \| f(y) \| \leq M. \quad (6.1)
\]

Then, the initial value problem
\[
y'' = f(y), y(t_0) = \eta_0, y'(t_0) = \eta_1, \quad \eta := [0, 0, \ldots, 0], \quad (6.2)
\]
possesses a unique analytic solution \( y(t) \) for \( |t - t_0| \leq \sqrt{2b/M} \) that satisfies \( -y(t - t_0) \equiv y(-(t - t_0)) \). Namely, \( y^{(m)}(t_0) = \overline{0} \) for all even numbers \( m \).

**Proof.** We proceed to prove this theorem by induction. The proof uses some calculations and representations analogous to those in section 3. However, these come with different interpretations. This is necessitated by conditions i), ii) of theorem 6, that are different than the analogous conditions i) and ii) in theorem 1. In theorem 6, the assumption \( -f(y) = f(-y) \) is the source of the differences. The following claim clarifies the role of i) in this theorem 6. If \( f(y) \) is an odd function of \( y \) then \( f(0) = \overline{0} \). Moreover, by lemma 5, the even order partial derivatives of \( f_j(y) \) with respect to the variables \( y_k \), (like \( f_j^{(0)}(y) := f(y) \) ) , are odd functions of \( y \). Namely,

\[
- \frac{\partial^l f_j(y(t))}{\partial y_{k_1} \ldots \partial y_{k_l}} = \frac{\partial^l f_j(-y(t))}{\partial y_{k_1} \ldots \partial y_{k_l}}, \quad l = 0, 2, 4, \ldots, \Rightarrow \frac{\partial^l f_j(0)}{\partial y_{k_1} \ldots \partial y_{k_l}} = 0. \tag{6.3}
\]

Per lemma 5, the odd order partial derivatives of \( f_j(y) \) with respect to \( y_j \) (unlike \( f_j^{(0)}(y) = f(y) \) ) are even functions of \( y \). Namely,

\[
\frac{\partial^l f_j(y(t))}{\partial y_{k_1} \ldots \partial y_{k_l}} = \frac{\partial^l f_j(-y(t))}{\partial y_{k_1} \ldots \partial y_{k_l}}, \quad l \text{ is odd.}
\]

Without loss of generality assume that \( t_0 = 0 \) since our differential system is autonomous. Assume that each component \( y_{j}^{(m)}, j = 1, 2, \ldots, n \), is a finite sum of terms of the form

\[
T_m := \frac{\partial^l f_j(y(t))}{\partial y_{k_1} \ldots \partial y_{k_l}} J_O J_E, \tag{6.4}
\]

where: i) \( J_E \) is a finite number \( w \) of factors of even order derivatives of the form \( y^{(2e)}_j \). Namely,

\[
J_E = y_{z_1}^{(2e_1)}(t)y_{z_2}^{(2e_2)}(t) \ldots y_{z_w}^{(2e_w)}(t), \quad c_1, c_2, \ldots, c_w \in \mathbb{N}_0. \tag{6.5}
\]

If \( l \) is an odd number then \( w \geq 1 \) is an odd number. If \( l \) is an even number then \( w \geq 0 \) is an even number. Thus, making \( T_m(0) = 0 \) in \( (6.4) \) and consequently \( y^{(m)}(0) = 0 \). ii) \( J_O \) has an even number \( r \) of factors of the form \( y^{(2g+1)}_j \). Namely,

\[
J_O = y_{s_1}^{(2e_1+1)}(t)y_{s_2}^{(2e_2+1)}(t) \ldots y_{s_r}^{(2e_r+1)}(t), \quad e_1, e_2, \ldots, e_r \in \mathbb{N}_0. \tag{6.6}
\]

Then, \( y_j^{(m+2)}(t) \) is a finite sum of terms of the form

\[
\overline{T_m^{(2)}} := \frac{\partial^l f_j(y(t))}{\partial y_{k_1} \ldots \partial y_{k_l}} \overline{J_O} \overline{J_E}, \tag{6.7}
\]

where: i) \( \overline{J_E} \) is a finite product of a number \( q \) of even order derivatives of components of \( y(t) \). Namely,

\[
\overline{J_E} = y_{z_1}^{(2e_1)}(t)y_{z_2}^{(2e_2)}(t) \ldots y_{z_q}^{(2e_q)}(t), \quad c_1, c_2, \ldots, c_q, q \in \mathbb{N}_0. \tag{6.8}
\]

If \( s \) is an odd number then \( w \geq 1 \) is an odd number. If \( s \) is an even number then \( w \geq 0 \) is an even number. Thus, making \( T_m^{(2)}(0) = 0 \) and consequently \( y^{(m+2)}(0) = 0 \). ii) \( \overline{J_O} \) is a finite product of an even number \( p \in \mathbb{N} \) of odd order derivatives of components of \( y(t) \). Namely,

\[
\overline{J_O} = y_{s_1}^{(2g_1+1)}(t)y_{s_2}^{(2g_2+1)}(t) \ldots y_{s_p}^{(2g_p+1)}(t) \quad g_1, g_2, \ldots, g_p \in \mathbb{N}_0. \tag{6.9}
\]
A calculation of the second derivative $T_m^{(2)}(t)$ shows that

$$T_m^{(2)}(t) = Q_1 + Q_2 + Q_3,$$  \hspace{1cm} (6.10)

where

$$Q_1 := \sum_{k_{l+1}=1}^{n} \sum_{k_{l+2}=1}^{n} \left\{ \frac{\partial^l f_j(y(t))}{\partial y_{k_{l+1}} \partial y_{k_{l+1}} \partial y_{k_l} \ldots \partial y_{k_2} \partial y_{k_1}} \right\} y_{k_{l+1}}^{(1)} (k_{l+1} J_{O} J_{E})
+ \sum_{k_{l+1}=1}^{n} \left\{ \frac{\partial^{l+1} f_j(y(t))}{\partial y_{k_{l+1}} \partial y_{k_l} \ldots \partial y_{k_2} \partial y_{k_1}} \right\} y_{k_{l+1}}^{(2)} (k_{l+1} J_{O} J_{E}),$$  \hspace{1cm} (6.11)

$$Q_2 := 2 \sum_{k_{l+1}=1}^{n} \left\{ \frac{\partial^{l+1} f_j(y(t))}{\partial y_{k_{l+1}} \partial y_{k_l} \ldots \partial y_{k_2} \partial y_{k_1}} \right\} y_{k_{l+1}}^{(1)} [J_{O}^{(1)} J_{E} + J_{O} J_{E}^{(1)}],$$  \hspace{1cm} (6.12)

$$Q_3 := \frac{\partial^l f_j(y(t))}{\partial y_{k_l} \ldots \partial y_{k_2} \partial y_{k_1}} [J_{O} J_{E}]^{(2)} = \frac{\partial^{l+1} f_j(y(t))}{\partial y_{k_l} \ldots \partial y_{k_2} \partial y_{k_1}} [J_{O}^{(2)} J_{E} + 2J_{O}^{(1)} J_{E}^{(1)} + J_{O} J_{E}^{(2)}].$$  \hspace{1cm} (6.13)

Below is the list of the different types of products of the form $T_m^{(2)}$ that occur in (6.10). The products are:

\begin{align*}
\left\{ \frac{\partial^l f_j(y(t))}{\partial y_{k_{l+1}} \partial y_{k_{l+1}} \partial y_{k_l} \ldots \partial y_{k_2} \partial y_{k_1}} \right\} y_{k_{l+1}}^{(1)} (k_{l+1} J_{O} J_{E}) &; \left\{ \frac{\partial^{l+1} f_j(y(t))}{\partial y_{k_{l+1}} \partial y_{k_l} \ldots \partial y_{k_2} \partial y_{k_1}} \right\} y_{k_{l+1}}^{(2)} (k_{l+1} J_{O} J_{E}); \\
\left\{ \frac{\partial^{l+1} f_j(y(t))}{\partial y_{k_{l+1}} \partial y_{k_l} \ldots \partial y_{k_2} \partial y_{k_1}} \right\} y_{k_{l+1}}^{(1)} [J_{O}^{(1)} J_{E}] &; \left\{ \frac{\partial^{l+1} f_j(y(t))}{\partial y_{k_{l+1}} \partial y_{k_l} \ldots \partial y_{k_2} \partial y_{k_1}} \right\} y_{k_{l+1}}^{(1)} [J_{O} J_{E}^{(1)}]; \\
\left\{ \frac{\partial^{l+1} f_j(y(t))}{\partial y_{k_{l+1}} \partial y_{k_l} \ldots \partial y_{k_2} \partial y_{k_1}} \right\} [J_{O}^{(2)} J_{E}] &; \left\{ \frac{\partial^{l+1} f_j(y(t))}{\partial y_{k_{l+1}} \partial y_{k_l} \ldots \partial y_{k_2} \partial y_{k_1}} \right\} [J_{O} J_{E}^{(2)}].
\end{align*}

(6.14) (6.15) (6.16)

Now we proceed to show that each product in (6.10) is of the desired form. Consider first each term in (6.14) and start with

$$Q_{11} := \left\{ \frac{\partial^{l+2} f_j(y(t))}{\partial y_{k_{l+2}} \partial y_{k_{l+1}} \partial y_{k_l} \ldots \partial y_{k_2} \partial y_{k_1}} \right\} y_{k_{l+2}}^{(1)} (k_{l+2} J_{O} J_{E}).$$  \hspace{1cm} (6.17)

Put,

$$\tilde{J}_{O} := y_{k_{l+2}}^{(1)} (k_{l+2} J_{O}), \quad \tilde{J}_{E} := J_{E}.$$  \hspace{1cm} (6.18)

Observe that $J_{O}$ has an even number of factors of odd order derivatives that is $r \geq 0$. Consequently, $\tilde{J}_{O}$ has an even number of odd order derivatives of components of $y(t)$ that is $(r + 2)$ as required by ii). Assume that $l$ is an odd number then $s = l + 2$ is also an odd number. Assume that $l$ is an even number then $s = l + 2$ is also an even number. Since $\tilde{J}_{E} := J_{E}$ then all conditions of i) are satisfied. And hence all conditions in (6.7) holds. Then we have as desired

$$\frac{\partial^l f_j(y(0))}{\partial y_{k_{l+2}} \partial y_{k_{l+1}} \partial y_{k_l} \ldots \partial y_{k_2} \partial y_{k_1}} J_{O}(0) J_{E}(0) = 0 \implies \frac{\partial^{l+2} f_j(y(0))}{\partial y_{k_{l+2}} \partial y_{k_{l+1}} \partial y_{k_l} \ldots \partial y_{k_2} \partial y_{k_1}} \tilde{J}_{O}(0) \tilde{J}_{E}(0) = 0.$$  \hspace{1cm} (6.19)
Next we focus on the second representative product in (6.14) that is
\[ Q_{21} := \frac{\partial^{l+1} f_{z}(y(t))}{\partial y_{k_{l+1}} \partial y_{k_{l}} \ldots \partial y_{k_{2}} \partial y_{k_{1}}} y^{(2)}_{k_{l+1}} J_{O} J_{E}. \] (6.20)

Put
\[ \hat{J}_{O} := J_{O}, \quad \hat{J}_{E} := y^{(2)}_{k_{l+1}} J_{E}. \] (6.21)

If \( w \) is the number of factors of even order derivatives in \( J_{E} \) then the number of even order derivatives in \( \hat{J}_{E} \) is \( w + 1 \). Observe that \( s = l + 1 \). Thus condition i) in (6.7) holds. Next we put
\[ Q_{12} := \frac{\partial^{l+1} f_{z}(y(t))}{\partial y_{k_{l+1}} \partial y_{k_{l}} \ldots \partial y_{k_{2}} \partial y_{k_{1}}} y^{(1)}_{k_{l+1}} J_{O}^{(1)} J_{E}. \] (6.22)

There is an even number \( r \) of factors of odd derivatives in \( J_{O} \) that make \( J_{O}^{(1)} \) a sum of \( r \) products as follows.
\[ J_{O}^{(1)} = y^{(2e_{1}+2)}_{s_{1}} y^{(2e_{2}+1)}_{s_{2}} \ldots y^{(2e_{r}+1)}_{s_{r}} + y^{(2e_{1}+1)}_{s_{1}} y^{(2e_{2}+2)} \ldots y^{(2e_{r}+1)}_{s_{r}} + \ldots + y^{(2e_{1}+1)}_{s_{1}} y^{(2e_{2}+2)} \ldots y^{(2e_{r}+2)}_{s_{r}}. \] (6.23)

Each summand in (6.23) is a product of \( (r - 1) \) odd order derivatives and precisely one factor is an even order derivative of a certain component of \( y \). Without loss of generality we relabel each product in (6.23) as
\[ J_{O}^{(1)} = y^{(2u_{1}+1)}_{s_{1}} y^{(2u_{2}+1)}_{s_{2}} \ldots y^{(2u_{r-1}+1)}_{s_{r-1}} y^{(2u_{r}+2)}_{s_{r}}, \] (6.24)

and put
\[ \hat{J}_{O} = y^{(1)}_{k_{l+1}} y^{(2u_{1}+1)}_{s_{1}} y^{(2u_{2}+1)}_{s_{2}} \ldots y^{(2u_{r-1}+1)}_{s_{r-1}} y^{(2u_{r}+2)}_{s_{r}} J_{E} = y^{(2u_{r}+2)}_{s_{r}} J_{E}. \] (6.25)

Evidently, \( \hat{J}_{O} \), like \( J_{O} \), has the same even number \( r \) of factors of odd order derivatives of components of \( y \). Observe that \( \hat{J}_{E} \) has \( (w + 1) \) number of factors of even order derivatives that is one more than \( J_{E} \). However, \( s = l + 1 \). Therefore, \( Q_{12} \) is of the desired form (6.7) implying \( Q_{12}(0) = 0 \). Consider now the second term in (6.15). Put
\[ Q_{22} := \frac{\partial^{l+1} f_{z}(y(t))}{\partial y_{k_{l+1}} \partial y_{k_{l}} \ldots \partial y_{k_{2}} \partial y_{k_{1}}} y^{(1)}_{k_{l+1}} J_{O} J_{E}^{(1)}. \] (6.26)

First we scrutinize the expression \( J_{E}^{(1)} \). If \( J_{E} \equiv 1 \) then \( Q_{22} \equiv 0 \) and trivially \( Q_{22}(0) = 0 \) as desired. If \( w \geq 1 \) then \( J_{E}^{(1)} \) is the sum of \( w \) products as follows
\[ J_{E}^{(1)} = y^{(2e_{1}+1)}_{z_{1}}(t) y^{(2e_{2})}_{z_{2}}(t) \ldots y^{(2e_{w})}_{z_{w}}(t) + y^{(2e_{1})}_{z_{1}}(t) y^{(2e_{2}+1)}_{z_{2}}(t) \ldots y^{(2e_{w})}_{z_{w}}(t) \]
\[ + \ldots + y^{(2e_{1})}_{z_{1}}(t) y^{(2e_{2})}_{z_{2}}(t) \ldots y^{(2e_{w}+1)}_{z_{w}}(t), \quad c_{1}, c_{2}, \ldots, c_{w}, w \in \mathbb{N}. \] (6.27)

Without loss of generality assume that a representative product in (6.27) has the form
\[ F_{22} := y^{(2e_{1}+1)}_{z_{1}}(t) y^{(2e_{2})}_{z_{2}}(t) \ldots y^{(2e_{w})}_{z_{w}}(t). \] (6.28)

Combine \( y^{(1)}_{k_{l+1}} \) from (6.26) with the factor \( y^{(2e_{1}+1)}_{z_{1}} \) in (6.28) and put
\[ \hat{J}_{O} := y^{(1)}_{k_{l+1}} y^{(2e_{1}+1)}_{z_{1}} J_{O}, \quad \hat{J}_{E} := y^{(2e_{2})}_{z_{2}}(t) \ldots y^{(2e_{w})}_{z_{w}}(t). \] (6.29)

Evidently, \( \hat{J}_{O} \) has an even number \( (r + 2) \) of odd order derivatives. However, the number of factors in \( \hat{J}_{E} \) is now \( q = w - 1 \). Recall, that if \( l \) is an even number
They are left to analyze the representative products in the first and third term in (6.16). Moreover, we have again that \( \hat{J}_O \) in (6.23) and \( \hat{J}_E \) in (6.27) respectively. There are \( rw \) terms in \( J_O \) \( J_E \). Each term is a product of \( rw \) factors. We focus on each product. Assume that \( l \) is even. If \( J_E \equiv 1 \) or \( w = 0 \) then

\[
 J_O^{(1)} J_E^{(1)} \equiv 0 \implies Q_{23} \equiv 0 \implies \frac{\partial f_j(y(0))}{\partial y_{k_1} \ldots \partial y_{k_3}} J_O^{(1)}(0) J_E^{(1)}(0) = 0, \tag{6.31}
\]

Notice that \( s = l \). Therefore, we may assume without loss of generality that a representative of one of these \( rw \) products has the form

\[
 J_{OS} J_{ES} := y_{s_1}^{(2u_1+1)} y_{s_2}^{(2u_2+1)} \ldots y_{s_r}^{(2u_r-1+1)} y_{z_1}^{(2c_1+1)} y_{z_2}^{(2c_2)} \ldots y_{z_w}^{(2c_w)}(t). \tag{6.32}
\]

Put

\[
 \hat{J}_O := y_{s_1}^{(2u_1+1)} y_{s_2}^{(2u_2+1)} \ldots y_{s_r}^{(2u_r-1+1)} y_{z_1}^{(2c_1+1)}(t), \quad \hat{J}_E := y_{z_2}^{(2c_2)}(t) \ldots y_{z_w}^{(2c_w)}(t). \tag{6.33}
\]

One can easily verify that \( \hat{J}_O \) and \( \hat{J}_E \) have the same number of factors as \( J_O \) and \( J_E \) respectively. Since \( s = l \), \( \hat{J}_O \) and \( \hat{J}_E \) in (6.33) are of the desired form (6.7). We are left to analyze the representative products in the first and third term in (6.16). They are

\[
 Q_{13} := \frac{\partial f_j(y(t))}{\partial y_{k_1} \ldots \partial y_{k_2} \partial y_{k_3}} J_O^{(2)} J_E, \quad Q_{33} := \frac{\partial f_j(y(t))}{\partial y_{k_1} \ldots \partial y_{k_2} \partial y_{k_3}} J_O J_E^{(2)}, \tag{6.34}
\]

To this end we scrutinize the factors that make up the products in \( J_O J_E^{(2)} \) and in \( J_O^{(2)} J_E \). We start with \( J_O J_E^{(2)} \). If \( w = 0 \) we have \( Q_{33} \equiv 0 \). If \( w = 1 \) then \( s = l \) where \( s \) and \( l \) are both odd numbers and we put

\[
 \hat{J}_O = J_O, \quad \hat{J}_E = J_E^{(2)} = y_{z_1}^{(2c_1+2)}. \tag{6.35}
\]

Evidently, conditions i) and ii) are satisfied in (6.7). Assume that \( w \geq 2 \). Then \( J_E^{(2)} \) is a sum of products that are of two kinds. The first kind is

\[
 y_{z_j}^{(2c_j+2)} \prod_{l \neq j, l = 1}^w y_{z_l}^{(2c_l)}. \tag{6.36}
\]

Then we put

\[
 \hat{J}_O = J_O, \quad \hat{J}_E = y_{z_1}^{(2c_1+2)} \prod_{l \neq j, l = 1}^w y_{z_l}^{(2c_l)}. \tag{6.37}
\]

We have again that \( \hat{J}_O \) and \( \hat{J}_E \) have the same number of factors as \( J_O \) and \( J_E \) respectively. Moreover, \( s = l \). Thus, conditions i) and ii) are satisfied in (6.7). The second kind of products is

\[
 y_{z_i}^{(2c_i+2)} \prod_{l \neq j, l = 1}^w y_{z_l}^{(2c_l)}. \tag{6.38}
\]
CELESTIAL MECHANICS SOLUTIONS WHERE THE FUTURE IS A PERFECT REFLECTION OF THE PAST

\[ J^{(2)}_E = y_{s_j}^{(2c_j+1)} y_{s_k}^{(2c_k+1)} \prod_{l \neq j, k, l=1}^w y_{s_l}^{(2c_l)}, \quad \prod_{l \neq j, k, l=1}^w y_{s_l}^{(2c_l)} : = 1 \quad \text{if} \quad w = 2. \quad (6.38) \]

Put

\[ \widehat{J}_O := y_{s_j}^{(2c_j+1)} y_{s_k}^{(2c_k+1)} J_O, \quad \widehat{J}_E := \prod_{l \neq j, k, l=1}^w y_{s_l}^{(2c_l)}. \quad (6.39) \]

Evidently, with \( s = l \) the parity of \( s, l \), and the number of factors in \( \widehat{J}_E \) that is \((w - 2)\) is the same. If \( l, w \) are both even numbers so are \( s, w - 2 \). If \( l, w \) are both odd numbers so are \( s, w - 2 \). The number of factors in \( J_O \) is \( r \) and the number of factors in \( J_E \) is \((r + 2)\) as needed. Thus, conditions i) and ii) are satisfied in (6.7). We are left to conclude the inductive process by focusing on the first term \( Q_{13} \) in (6.33). To this end we focus on the two kinds of products that emanate in the second derivative \( J^{(2)}_O \). \( J_O \) has the form

\[ J_O = y_{s_1}^{(2e_1+1)}(t)y_{s_2}^{(2e_2+1)}(t) \ldots y_{s_r}^{(2e_r+1)}(t), \quad e_1, e_2, \ldots, e_r \in \mathbb{N}_0. \quad (6.40) \]

The first kind of second derivatives \( J^{(2)}_O \) will contain the following \( r \) products

\[ [y_{s_1}^{(2e_1+1)}(t)y_{s_2}^{(2e_2+1)}(t) \ldots y_{s_r}^{(2e_r+1)}(t)], \quad [y_{s_1}^{(2e_1+1)}(t)y_{s_2}^{(2e_2+3)}(t) \ldots y_{s_r}^{(2e_r+1)}(t)] \]

\[ \ldots, [y_{s_1}^{(2e_1+1)}y_{s_2}^{(2e_2+1)}(t) \ldots y_{s_r}^{(2e_r+3)}(t)] \ldots [y_{s_1}^{(2e_1+1)}(t)y_{s_2}^{(2e_2+1)}(t) \ldots y_{s_r}^{(2e_r+3)}(t)]. \quad (6.41) \]

Consequently, a representative product in \( J^{(2)}_O J_E \) will have the form

\[ [y_{s_1}^{(2e_1+1)}(t)y_{s_2}^{(2e_2+1)}(t) \ldots y_{s_r}^{(2e_r+1)}(t)] y_{k_1}^{(2c_1)}(t)y_{k_2}^{(2c_2)}(t) \ldots y_{k_m}^{(2c_m)}(t). \]

Choose

\[ \widehat{J}_O = [y_{s_1}^{(2e_1+1)}y_{s_2}^{(2e_2+1)}(t) \ldots y_{s_r}^{(2e_r+1)}(t)], \quad \widehat{J}_E = J_E. \quad (6.42) \]

Evidently, \( \widehat{J}_O \) and \( \widehat{J}_E \) have the same number of factors as \( J_O \) and \( J_E \) respectively. Since \( s = l \), \( J_O \) and \( J_E \) are of the desired form (6.7). One may assume without loss of generality that the second kind of products in \( J^{(2)}_O \) is

\[ y_{s_1}^{(2e_1+2)}(t)y_{s_2}^{(2e_2+2)}(t)y_{s_3}^{(2e_1+1)} \ldots y_{s_r}^{(2e_r+1)}(t). \quad (6.43) \]

In general we will have a product of the form

\[ y_{s_j}^{(2c_j+2)} y_{s_k}^{(2c_k+2)} \quad \text{if} \quad r = 2, \quad y_{s_j}^{(2c_j+2)} y_{s_k}^{(2c_k+2)} \prod_{l \neq j, k}^r y_{s_l}^{(2c_l+1)}, \quad l = 1, 2, \ldots, r, \ r \geq 3. \quad (6.44) \]

Then put:

\[ \widehat{J}_O \equiv 1, \quad \widehat{J}_E = y_{s_j}^{(2c_j+2)} y_{s_k}^{(2c_k+2)} J_E, \quad \text{if} \quad r = 2, \quad (6.45) \]

\[ \widehat{J}_O := \prod_{l \neq j, k, l=1}^r y_{s_l}^{(2c_l+1)}, \quad \widehat{J}_E := y_{s_j}^{(2c_j+2)} y_{s_k}^{(2c_k+2)} J_E, \quad l = 1, 2, \ldots, r, \ r \geq 3. \quad (6.46) \]
CELESTIAL MECHANICS SOLUTIONS WHERE THE FUTURE IS A PERFECT REFLECTION OF THE PAST

It is readily observed that $\hat{J}_O, \hat{J}_E$, are of the desired form \(6.7\). Hence, all $T_m(0) = 0$ imply that all $\hat{T}_m^{(2)}(0) = 0$. Consequently, $y^{(m)}(0) = 0$ for all even numbers $m \in \mathbb{N}_0$. Q.E.D.

References

[1] Arbogast, L. F. A., “Du calcul des derivations,” [On the calculus of derivatives] (in French) Strasbourg: Lervault, pp. xxiii+404 (1800). (Entirely freely available from Google books).
[2] Boyce, E. W., DiPrima, R. C., “Elementary Differential Equations and Boundary Value Problems,” John Wiley & Sons N.Y. Last Edition, (1992).
[3] Brauer, F., Nohel, J. A., “The qualitative theory of ordinary differential equations: An introduction,” Dover N.Y., (1989).
[4] Brumberg, V. A., “Celestial Mechanics: Past, Present, Future,” Solar System Research 47, 376–389 (2013).
[5] Delligado, J., Diacu, F., Lacomba, E. A., Mingarelli, A., Mioc, V., Pérez-Chavela, E., and Stoica, C., “The Global Flow of the Manev Problem,” J. Math. Phys 37, 2748–2761 (1996).
[6] Diacu, F., Mioc, V., Stoica, C., “Phase-space structure and regularization of Manev-type problems,” Nonlinear Analysis 41, 1029–1055 (2000).
[7] Diacu, F., Pérez-Chavelab, E., and Santoprete, M., “The Kepler problem with anisotropic perturbations,” Journal of Mathematical Physics 46, 072701.1-072701.21 (2005).
[8] Encinas, L. H., Masque, J. M., “A Short Proof of the Generalized Faà di Bruno’s Formula,” Applied Mathematics Letters 16, 975-979 (2003).
[9] Faà di Bruno, F., "Sullo sviluppo delle funzioni," [On the development of the functions] Annali di Scienze Matematiche e Fisiche (in Italian) 6, 479–480 LCCN 06036680. Entirely freely available from Google books (1855).
[10] Guckenheimer, J., Holmes, P., “Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector fields,” Springer-Verlag New York (1983).
[11] Hille, E., “Lectures on Ordinary Differential Equations,” Addison-Wesley Reading Massachusetts (1969).
[12] Hsieh, Po-Fang., Sibuya, Y., “Basic Theory of Ordinary Differential Equations,” Universitext Springer-Verlag New York (1999).
[13] Jordan, D. W., Smith, P., “Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers,” 4th edition Oxford Texts in Applied and Engineering Mathematics (2007).
[14] Krantz, S. G., Parks, H. R., “A Primer of Real Analytic Functions,” Birkhäuser Advanced Texts - Basler Lehrbücher (Second ed.) Boston: Birkhäuser Verlag, pp. xiv+205 (2002).
[15] Perko, L., “Differential Equations and Dynamical Systems,” Texts in Applied Mathematics 7 Springer-Verlag New York 3rd edition, (2001).
[16] Pollard, H., “Celestial mechanics,” Mathematical Association of America, (1976).

Department of Mathematics, WVU, Morgantown WV 26506
Email address: aaabdulhussein@mix.wvu.edu, gingold@math.wvu.edu

1, gingold@math.wvu.edu