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Metrics without isometries are generic

Pierre Mounoud

Abstract

We prove that for any compact manifold of dimension greater than 1, the set of pseudo-Riemannian metrics having a trivial isometry group contains an open and dense subset of the space of metrics.

Keywords: metrics without isometries; space of pseudo-Riemannian metrics

Mathematics Subject Classification (2010): 53C50

Let $V$ be a compact manifold and $\mathcal{M}_{p,q}$ be the set of smooth pseudo-Riemannian metrics of signature $(p,q)$ on $V$ (we suppose that it is not empty). In [2] D’Ambra and Gromov wrote: “everybody knows that $\text{Is}(V, g) = \text{Id}$ for generic pseudo-Riemannian metrics $g$ on $V$, for $\dim(V) \geq 2$.” Nevertheless, as far as we know, no proof of this fact is available. The purpose of this short article is to give a proof of this result in the case where $V$ is compact and to precise the meaning of the word generic. Let us recall that it is known since the work of Ebin [3] (see also [4]) that the set of Riemannian metrics without isometries on a compact manifold is open and dense. We prove:

Theorem 1 If $V$ is a compact manifold such that $\dim(V) \geq 2$ then the set $\mathcal{G} = \{ g \in \mathcal{M}_{p,q} | \text{Is}(g) = \text{Id} \}$ contains a subset that is open and dense in $\mathcal{M}_{p,q}$ for the $C^\infty$-topology.

This result is optimal in the sense that $\mathcal{G}$ is not always open as we showed in [5]. The particularity of the Riemannian case lies in the fact that the natural action of the group of smooth diffeomorphisms of $V$, denoted by $\text{Diff}(V)$, on $\mathcal{M}_{p,0}$ is proper. Furthermore, Theorem 4.2 of [5] says that when this action is proper then $\mathcal{G}$ is an open subset of $\mathcal{M}_{p,q}$. The idea of proof is therefore to find a big enough subset of $\mathcal{M}_{p,q}$ invariant by $\text{Diff}(V)$ on which the action is proper. We have decided to be short rather than self-contained, in particular we are going to use several results from our former work [5].

In the following $\mathcal{M}_{p,q}$ will be endowed with the $C^\infty$-topology without further mention of it and by a perturbation we will always mean an arbitrary small perturbation.

For any $g \in \mathcal{M}_{p,q}$ we denote by $\text{Scal}_g$ its scalar curvature and by $M_g$ the maximum of $\text{Scal}_g$ on $V$. Let $\mathcal{F}_V$ be the set of pseudo-Riemannian metrics $g$ such that $\text{Scal}_g^{-1}(M_g)$ contains a (non trivial) geodesic. The big set we are looking for is actually the complement of $\mathcal{F}_V$.

Proposition 2 The set $\mathcal{O}_V = \mathcal{M}_{p,q} \setminus \mathcal{F}_V$ is an open dense subset of $\mathcal{M}_{p,q}$ invariant by the action of $\text{Diff}(V)$ and the restriction of the action of $\text{Diff}(V)$ to $\mathcal{O}_V$ is proper.

Proof. The set $\mathcal{O}_V$ is clearly invariant. Let $g \in \mathcal{M}_{p,q}$ and $x_0 \in V$ realizing the maximum of $\text{Scal}_g$. It is easy to find a perturbation with arbitrary small support increasing the value of $\text{Scal}_g(x_0)$. Repeating these deformation on smaller and smaller neighborhood of $x_0$ we find a perturbation of $g$ such that the maximum of the scalar curvature is realized by only one point (see [3] p. 35 for a similar construction). Hence $\mathcal{O}_V$ is dense in $\mathcal{M}_{p,q}$.
Let us see now that $\mathcal{F}_V$ is closed. Let $g_n$ be a sequence of metrics of $\mathcal{F}_V$ converging to $g_\infty$. For any $n \in \mathbb{N}$ there exists a $g_\infty$-geodesic $\gamma_n$ such that $\text{Scal}_{g_n}$ is constant and equal to $M_{g_n} = \max_{x \in V} \text{Scal}_{g_n}(x)$ on it. As the sequence of exponential maps converges to the exponential map of $g_\infty$ and as $V$ is compact we see that (up to subsequences) the sequence of geodesics $\gamma_n$ converges to a $g_\infty$-geodesic $\gamma_\infty$. As $\text{Scal}_{g_n} \to \text{Scal}_{g_\infty}$ we know that $\text{Scal}_{g_\infty}$ is constant along $\gamma_\infty$ and its value is necessarily $M_{g_\infty}$. Hence $g_\infty \in \mathcal{F}_V$ and $\mathcal{F}_V$ is closed.

Let us suppose that the action of $\text{Diff}(V)$ on $\mathcal{M}_{p,q}$ is not proper (otherwise there is nothing to prove). It means (see [5]) that there exists a sequence of metrics $(g_n)_{n \in \mathbb{N}}$ converging to $g_\infty$ and a non equicontinuous sequence of diffeomorphisms $(\phi_n)_{n \in \mathbb{N}}$ such that the sequence of metrics $(\phi_n^* g_n)$ converges to $g'$. The proposition will follow from the fact that $g_\infty$ or $g'_\infty$ have to belong to $\mathcal{F}_V$.

We first remark that the sequence of linear maps $(D\phi_n(x_n))_{n \in \mathbb{N}}$ lies in $O(p, q)$ up to conjugacy by a converging sequence. As the sequence $(\phi_n)_{n \in \mathbb{N}}$ is non equicontinuous, we know by [5, Proposition 2.3] that there exists a subsequence such that $\|D\phi_n(x_n)\| \to \infty$ when $k \to \infty$. We deduce from the $KAK$ decomposition of $O(p, q)$ the existence of what are called in [6], see subsection 4.1 therein for details, strongly approximately stable vectors, more explicitly we have:

**Fact 3** For any sequence $(x_n)_{n \in \mathbb{N}}$ of points of $V$, there exist a sequence $(v_n)_{n \in \mathbb{N}}$ such that (up to subsequences):

- $\forall n \in \mathbb{N}$, $v_n \in T_{x_n} V$,
- $D\phi_n(x_n)v_n \to 0,$
- $v_n \to v_\infty \neq 0$

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of points of $V$ realizing the maximum of the function $\text{Scal}_{\phi_n^* g_n}$. The manifold being compact, we can assume that this sequence is convergent to a point $x_\infty$. We can also assume that the sequence $(\phi_n(x_n))_{n \in \mathbb{N}}$ converges to $y_\infty$. Of course $x_\infty$ (resp. $y_\infty$) realizes the maximum of $\text{Scal}_{g_\infty}$ (resp. $\text{Scal}_{g'_\infty}$).

Let $(v_n)_{n \in \mathbb{N}}$ be a sequence given by Fact 3. Reproducing the computation p. 471 of [5], we see that the scalar curvature of $g'_\infty$ is constant along the $g'_\infty$-geodesic starting from $x_\infty$ with speed $v_\infty$ (by symmetry the scalar curvature of $g_\infty$ is constant along a geodesic containing $y_\infty$):

$$\text{Scal}_{g'_\infty}(\exp_{g'_\infty}(x_\infty, v_\infty)) - \text{Scal}_{g'_\infty}(x_\infty) = \lim_{n \to \infty} \text{Scal}_{\phi_n^* g_n}(\exp_{\phi_n^* g_n}(x_n, v_n)) - \text{Scal}_{\phi_n^* g_n}(x_n)$$

$$= \lim_{n \to \infty} \text{Scal}_{g_n}(\exp_{g_n}(D\phi_n(x_n, v_n)) - \text{Scal}_{g_n}(\phi_n(x_n))$$

$$= \text{Scal}_{g_\infty}(y_\infty) - \text{Scal}_{g_\infty}(y_\infty) = 0.$$

Hence $g_\infty$ and $g'_\infty$ do not belong to $O_V$. □

It follows from Theorem 4.2 of [5] and Proposition 2 that $G \cap O_V$ is open. As $O_V$ is dense we just have to show that $G$ is dense in $O_V$ in order to prove Theorem 1. Let $g$ be a metric in $O_V$, as we saw earlier we can perturb it in such a way that the maximum of $\text{Scal}_g$ is realized by only one point $p$. This point is now fixed by isometry. We choose now $U$ an open subset of $V$ that do not contain $p$ in its closure but whose closure is contained in some normal neighborhood $O$ of $p$. According to [1, Theorem 3.1] by Beig et al., we can perturb again $g$ in such a way that there are no local Killing fields on $U$. We choose the perturbation in order that $\text{Scal}^{-1}(M_p) = \{p\}$. The new metric has now a finite isometry group (it is 0-dimensional and compact by Proposition 2 as the metric still lies in $O_V$). Actually, the proof of Proposition 2 implies also that the set of germs of local isometries fixing $p$ is itself compact. It means that any isometry of a perturbation of $g$ with support not containing $O$ can send a geodesic $\gamma_1$ starting from $p$ only on a finite number of
geodesics that do not depend on the perturbation. Therefore, it is easy to find a perturbation of the metric along $\gamma_1$ (with support away from $O$) in order to destroy these possibilities. Now, any isometry has to fix pointwise $\gamma_1$ (we chose a non symmetric perturbation). Repeating this operation for $n = \dim V$ geodesics $\gamma_1, \ldots, \gamma_n$ such that the vectors $\gamma'_i(0)$ span $T_p V$, we obtain a perturbation of $g$ such that any of its isometries has to be the identity i.e. the perturbed metric is in $G$.

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