Asymptotics for the number of spanning trees in circulant graphs and degenerating $d$-dimensional discrete tori

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Abstract

In this paper we obtain precise asymptotics for certain families of graphs, namely circulant graphs and degenerating discrete tori. The asymptotics contain interesting constants from number theory among which some can be interpreted as corresponding values for continuous limiting objects. We answer one question formulated in a paper from Atajan, Yong and Inaba in [1] and formulate a conjecture in relation to the paper from Zhang, Yong and Golin [21]. A crucial ingredient in the proof is to use the matrix tree theorem and express the combinatorial Laplacian determinant in terms of Bessel functions. A non-standard Poisson summation formula and limiting properties of theta functions are then used to evaluate the asymptotics.

Keywords: spanning trees, circulant graphs, Bessel functions, zeta functions

1 Introduction

The number of spanning trees of a finite graph is an interesting invariant which has many applications in different fields such as network reliability (for example see [7]), statistical physics [16], designing electrical circuits; for more applications see [8]. In 1847 Kirchhoff established the matrix tree theorem [13] which relates the number of spanning trees $\tau(G)$ in a graph $G$ with $|V(G)|$ vertices to the determinant of the combinatorial Laplacian on $G$ by the following relation

$$\tau(G) = \frac{1}{|V(G)|}\det^* \Delta$$

where $\det^* \Delta$ is the product of the non-zero eigenvalues of the Laplacian on $G$.

One type of graphs, so-called circulant graphs, also known as loop networks, has been much studied. Let $1 \leq \gamma_1 < \ldots < \gamma_d \leq \lfloor n/2 \rfloor$ be positive integers. A circulant graph $C^{\gamma_1, \ldots, \gamma_d}_n$ is the $2d$-regular graph with $n$ vertices labelled $0, 1, \ldots, n-1$ such that each vertex $v \in \mathbb{Z}/n\mathbb{Z}$ is connected to $v \pm \gamma_i \mod n$ for all $i \in \{1, \ldots, d\}$. Figure 1 illustrates two examples. The problem of computing the number of spanning trees in these graphs can be approached in several ways. One of the first results, proven by Kleitman and Golden [14], see also [3] and [18], states that $\tau(C^{\gamma_1, \ldots, \gamma_d}_n) = nF_n^2$, where $F_n$ are the Fibonacci numbers. Boesch and Prodinger [4] computed the number of spanning trees for different classes of graphs with algebraic techniques using Chebyshev...
polynomials. Zhang, Yong and Golin [19, 21] used this technique for circulant graphs. The same authors showed in [20] that the number of spanning trees in circulant graphs with fixed generators satisfies a recurrence relation, that is $\tau(C^n_{\gamma_1, \ldots, \gamma_d}) = n a_n^2$, where $a_n$ satisfies a recurrence relation of order $2^{d-1}$. This was also proven combinatorially later by Golin and Leung in [9]. They extended their method to circulant graphs with non-fixed generators in [10]. In [1], Atajan, Yong and Inaba improved the order of the recurrence relation for $a_n$ and found the asymptotic behaviour of $a_n$, i.e. $a_n \sim c\phi^n$, where $c$ and $\phi$ are constants which are obtained from the smallest modulus root of the generating function of $a_n$. They again improved this in [2] by finding an efficient way of solving the recurrence relation of $a_n$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{circulant_graphs.png}
\caption{The circulant graphs $C^1_{1,2}$ and $C^1_{13}$.}
\end{figure}

In this work we are interested in studying the asymptotic behaviour of the number of spanning trees in circulant graphs with fixed generators and in $d$-dimensional discrete tori. This will be done by extending the work of Chinta, Jorgenson and Karlsson in [5] and [6] to these cases. In their papers, the authors developed a technique to compute the asymptotic behaviour of spectral determinants of sequences of discrete tori $\mathbb{Z}^d / \Lambda_n \mathbb{Z}^d$ where $\Lambda_n$ is a $d \times d$ integer matrix such that $\det \Lambda_n \to \infty$ and $\Lambda_n / (\det \Lambda_n)^{1/d} \to A \in SL_d(\mathbb{R})$ as $n \to \infty$. The two families of graphs which will be considered here do not satisfy this condition. An important ingredient is the theta inversion formula (see Proposition 2.1 below) which relates the eigenvalues of the combinatorial Laplacian to the modified $I$-Bessel functions. The method then consists in studying the asymptotics of integrals involving these Bessel functions. In the first part of this work we apply it to the case of circulant graphs with fixed generators. We will prove the following theorem:

**Theorem 1.1.** Let $C^n_\Gamma$ be a circulant graph with $n$ vertices and $d$ generators given by $\Gamma := \{1, \gamma_1, \ldots, \gamma_{d-1}\}$, such that $1 < \gamma_1 < \ldots < \gamma_{d-1} \leq \lfloor \frac{n}{2} \rfloor$, and let $\det^* \Delta C^n_\Gamma$ be the product of the non-zero eigenvalues of the Laplacian on $C^n_\Gamma$. Then as $n \to \infty$

$$
\log \det^* \Delta C^n_\Gamma = n \int_0^\infty (e^{-t} - e^{-2t} I_0^\Gamma(2t, \ldots, 2t)) \frac{dt}{t} + 2 \log n - \log c_\Gamma + o(1)
$$

where $c_\Gamma = 1 + \sum_{i=1}^{d-1} \gamma_i^2$ and

$$
I_0^\Gamma(2t, \ldots, 2t) = \frac{1}{2\pi} \int_{-\pi}^\pi e^{2t(\cos w + \sum_{i=1}^{d-1} \cos \gamma_i w)} dw
$$

is the $d$-dimensional modified $I$-Bessel function of order zero.
The function $I_0^J$ appearing in the lead term is a generalization of the 2-dimensional $J$-Bessel function in [15] and will be defined in section 2.4. Theorem 1.1 can be compared to Lemma 2 of Golin, Yong and Zhang in [11] where they find the lead term of the asymptotic number of spanning trees. With our method we derived also the second term of the asymptotic. These are consistent with numerics given in [20, 1] by these authors. In particular, this answers one of their open problems stated in the conclusion of [1] that asks whether we can find out the exact value of the asymptotic constants. Indeed we show that

$$c^2 = \frac{1}{\zeta(\Gamma)}.$$

Let $\Lambda_n$ be a $d \times d$ invertible diagonal integer matrix. In the second part of this work we extend the method used in [2] to study the asymptotic behaviour of spectral determinants of a sequence of $d$-dimensional degenerating discrete tori, that is, the Cayley graphs of the groups $\mathbb{Z}^d/\Lambda_n\mathbb{Z}^d$ with respect to the generators corresponding to the standard basis vectors of $\mathbb{Z}^d$. It is degenerating in the sense that $d - p$ sides of the torus are tending to infinity at the same rate while $p$ sides tend to infinity sublinearly with respect to the $d - p$ sides. More precisely, let $\alpha_i$, $i = 1, \ldots, p$, and $\beta_i$, $i = 1, \ldots, d - p$, be positive non-zero integers and let $a(n)$ and $a_i(n)$, $i = 1, \ldots, p$, be sequences of positive integers which go to infinity sublinearly with respect to $n$ and such that

$$\frac{a(n)}{n} \to 0, \quad \frac{a_i(n)}{n} \to 0 \quad \text{and} \quad \frac{a_i(n)}{a(n)} \to \alpha_i, \quad \text{as} \ n \to \infty.$$

Let $b_i(n)$, $i = 1, \ldots, d - p$, be a sequence of positive integers such that

$$\frac{b_i(n)}{n} \to \beta_i, \quad \text{as} \ n \to \infty.$$

The $p$ sides tending to infinity sublinearly with respect to the $d - p$ sides means that $a_i(n)/b_j(n) \to 0$ for all $i = 1, \ldots, p$ and $j = 1, \ldots, d - p$. The matrix $\Lambda_n$ considered is then given by $\Lambda_n = \text{diag}(a_1(n), \ldots, a_p(n), b_1(n), \ldots, b_{d-p}(n))$. Figure 2 illustrates an example.

![Figure 2: The discrete torus $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/[\log n]\mathbb{Z}$ with $n = 43$.](image)

We define the spectral or Epstein zeta function associated to the real torus $\mathbb{R}^r/\Lambda\mathbb{Z}^r$, where $\Lambda$ is an $r \times r$ invertible matrix, for $\Re(s) > r/2$ by

$$\zeta_{\mathbb{R}^r/\Lambda\mathbb{Z}^r}(s) = \frac{1}{(2\pi)^r} \sum_{m \in \mathbb{Z}^r \setminus \{0\}} (m^T \Lambda^{-1} m)^{-s}.$$

It has an analytic continuation to the whole complex plane except for a simple pole at $s = r/2$. Let $B$ be an $r \times r$ invertible matrix. The regularized determinant of the Laplacian on the real
torus $\mathbb{R}^d/B\mathbb{Z}^d$ is then defined through the spectral zeta function evaluated at $s = 0$ by
\[
\log \det^* \Delta_{\mathbb{R}^d/B\mathbb{Z}^d} = -\zeta'_{\mathbb{R}^d/B\mathbb{Z}^d}(0).
\]
We will show the following theorem:

**Theorem 1.2.** Let $A = \text{diag}(\alpha_1, \ldots, \alpha_p)$, $B = \text{diag}(\beta_1, \ldots, \beta_{d-p})$, $\Lambda = \text{diag}(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_{d-p})$ and let $\det^* \Delta_{\Lambda_n, \mathbb{Z}^d}$ be the product of the non-zero eigenvalues of the Laplacian on the discrete torus $\mathbb{Z}^d/\Lambda_n \mathbb{Z}^d$. Then as $n \to \infty$
\[
\log \det^* \Delta_{\mathbb{Z}^d/\Lambda_n \mathbb{Z}^d} = n^{d-p}a(n)^p \det(\Lambda)c_d - \frac{n^{d-p}}{a(n)^{d-p}} \det(\Lambda)(4\pi)^{d/2} \Gamma(d/2)/\Lambda_{\mathbb{R}^d/B\mathbb{Z}^d}(d/2) + 2 \log n - \zeta'_{d-p}/B\mathbb{Z}^d(0) + o(1)
\]
where $c_d$ is the following integral
\[
c_d = \int_0^\infty \left( e^{-t} - e^{-2dt}I_0(2t)^d \right) \frac{dt}{t}.
\]

We recall the special values for the gamma function for odd $d$, $\Gamma(d/2) = (d-2)! \sqrt{\pi}/2^{(d-1)/2}$, and for even $d$, $\Gamma(d/2) = (d/2 - 1)!$.

The second term in the theorem is new in the asymptotic development which comes from the degeneration. The other terms are the usual terms appearing in the asymptotic behaviour of spectral determinants (see [5] and [6]). As mentioned above the last term is the logarithm of the spectral determinant of the Laplacian on the real torus $\mathbb{R}^{d-p}/B\mathbb{Z}^{d-p}$ where $p$ dimensions are lost because of the degeneration of the sequence of tori. Indeed one can rescale the discrete torus by dividing the number of vertices per dimension by $n$. Therefore the $d$-dimensional sequence of discrete tori converges to the $(d-p)$-dimensional real torus $\mathbb{R}^{d-p}/\text{diag}(\beta_1, \ldots, \beta_{d-p})\mathbb{Z}^{d-p}$.

**Example:** To illustrate the theorem we consider the graphs $\mathbb{Z}^3/\Lambda_n \mathbb{Z}^3$ where
\[
\Lambda_n = \begin{pmatrix} [\log n] & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n \end{pmatrix}.
\]
Then as $n \to \infty$
\[
\log \det^* \Delta_{\Lambda_n, \mathbb{Z}^3} = c_3 n^2 \log n - \frac{n^2}{(\log n)^2} \frac{1}{\pi} \zeta(3) + 2 \log n + 2 \log(\Gamma(1/4)^4/16\pi^3) + o(1)
\]
where the constant in the last logarithm is Dedekind’s eta function evaluated in $i$.

This work is structured as follows. In subsection 2.1 we define the combinatorial Laplacian, and then the spectral zeta function and the theta function in subsection 2.2. In subsection 2.3 we recall some results on modified $I$-Bessel functions and in the next subsection we define the $d$-dimensional modified $I$-Bessel function which will be used in the computation of the asymptotics for the circulant graphs. In the two next subsections we recall some upper bounds on modified $I$-Bessel functions and briefly describe the method used in [5]. In section 4 we explain Theorem 1.3 and compare the results with other papers. In section 5 we treat the case of the degenerating sequence of tori, show Theorem 1.4 and give some examples. In the last section we formulate a conjecture on the number of spanning trees in $C_{5n}^n$, for $n \geq 2$.

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2 Preliminary results

2.1 Laplacians

We define a $d$-dimensional discrete torus to be the quotient $\mathbb{Z}^d/\Lambda\mathbb{Z}^d$ where $\Lambda \in GL_d(\mathbb{Z})$ and a $d$-dimensional real torus by the quotient $\mathbb{R}^d/B\mathbb{Z}^d$ where $B \in GL_d(\mathbb{R})$. Let $B^*$ be the matrix generating the dual lattice of $B\mathbb{Z}^d$ defined by $B^*\mathbb{Z}^d = \{y \in \mathbb{R}^d | \langle x, y \rangle \in \mathbb{Z}, \forall x \in B\mathbb{Z}^d\}$ where $\langle \cdot, \cdot \rangle$ is the usual inner product, which satisfies the two following conditions:

- $\text{span}(B) = \text{span}(B^*)$
- $B^TB^* = 1$.

The eigenfunctions of the Laplace-Beltrami operator $-\sum_{j=1}^d \partial^2 / \partial x_j^2$ on the real torus are given by $\phi(x) = \exp(2\pi i \langle \mu, x \rangle)$, for some $\mu \in \mathbb{R}^d$, with the condition that the opposite sides of the parallelogram generated by $B\mathbb{Z}^d$ are identified. So for all $x \in \mathbb{R}^d$ we have $\phi(x + B\mathbb{Z}^d) = \phi(x)$.

Hence $\exp(2\pi i \langle \mu, B\mathbb{Z}^d \rangle) = 1$ and therefore $\langle \mu, B\mathbb{Z}^d \rangle \in \mathbb{Z}$ if and only if $\mu = B^*m$ for $m \in \mathbb{Z}^d$. It follows that the eigenvalues are given by $\lambda_m = (2\pi)^2 \mu^T \mu = (2\pi)^2 \|B^*m\|^2$ with $m \in \mathbb{Z}^d$. (1)

Let $V(\mathbb{Z}^d/\Lambda\mathbb{Z}^d)$ be the set of vertices of the torus $\mathbb{Z}^d/\Lambda\mathbb{Z}^d$ and $f : V(\mathbb{Z}^d/\Lambda\mathbb{Z}^d) \to \mathbb{C}$. The combinatorial Laplacian on $\mathbb{Z}^d/\Lambda\mathbb{Z}^d$ is defined by

$$\Delta_{\mathbb{Z}^d/\Lambda\mathbb{Z}^d}f(x) = \sum_{y \sim x} (f(x) - f(y))$$

where the sum is over the vertices adjacent to $x$.

Recall Proposition 5 of [6].

**Proposition 2.1.** Let $\lambda_v$, with $v \in \Lambda^*\mathbb{Z}^d/\mathbb{Z}^d$, be the eigenvalues of $\Delta_{\mathbb{Z}^d/\Lambda\mathbb{Z}^d}$. The following formula holds for $t \in \mathbb{R}_{\geq 0}$

$$|\text{det}\Lambda| \sum_{y \in \Lambda^*\mathbb{Z}^d} e^{-2dt} I_{y_1}(2t) \ldots I_{y_d}(2t) = \sum_{v \in \Lambda^*\mathbb{Z}^d/\mathbb{Z}^d} e^{-t\lambda_v}$$

where $I_{y_i}$ is the modified $I$-Bessel function of order $y_i$.

2.2 Spectral zeta function and theta function

In this section we define the spectral zeta function and the theta function and give the relations that will enable us to compute the asymptotics in sections 3 and 4.

Let $\{\lambda_j\}_{j \geq 0}$ be the eigenvalues of the combinatorial Laplacian, respectively the Laplace-Beltrami operator, on a discrete torus, respectively a real torus, $T$, with $\lambda_0 = 0$. The associated theta function on $T$ is defined by

$$\sum_j e^{-\lambda_j t}.$$ (2)
It will be denoted by $\theta_T(t)$ when $T$ denotes a discrete torus and by $\Theta_T(t)$ when $T$ denotes a real torus. The relation in Proposition 2.1 is then called the theta inversion formula on $\mathbb{Z}^d/\Lambda\mathbb{Z}^d$. The associated spectral zeta function on a real torus $T$ is defined for $\text{Re}(s) > d/2$ by

$$
\zeta_T(s) = \sum_{\gamma \neq 0} \frac{1}{\lambda_{\gamma}^s}.
$$

It is related to the theta function through the Mellin transform:

$$
\zeta_T(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} (\Theta_T(t) - 1) t^{s-1} \frac{dt}{t},
$$

where the $-1$ in the integral comes from the fact that the zero eigenvalue is kept in the definition of the theta function, and where $\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt$ is the gamma function.

Let $B \in \text{GL}_d(\mathbb{R})$ be a matrix. By splitting the above integral one can show that the zeta function admits a meromorphic continuation to $s \in \mathbb{C}$ (see section 2.6 in [3]). By differentiating $\zeta_{\mathbb{R}^d/B^{\mathbb{R}}}$ and evaluating at $s = 0$, one has

$$
\zeta'_{\mathbb{R}^d/B^{\mathbb{R}}}(0) = \int_0^1 (\Theta_{\mathbb{R}^d/B^{\mathbb{R}}}(t) - \text{det}(B)(4\pi t)^{-d/2}) \frac{dt}{t} + \Gamma'(1) - \frac{2}{d} \text{det}(B)(4\pi)^{-d/2} + \int_1^{\infty} (\Theta_{\mathbb{R}^d/B^{\mathbb{R}}}(t) - 1) \frac{dt}{t}.
$$

In section 3 a limiting torus will be the circle $S^1 = \mathbb{R}/\mathbb{Z}$. In this case it is convenient to split the integral at $c_T$. The spectral zeta function is defined for $\text{Re}(s) > 1/2$:

$$
\zeta_{S^1}(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} (\Theta_{S^1}(t) - 1) t^{s-1} \frac{dt}{t} = \frac{1}{\Gamma(s)} \int_0^{c_T} (\Theta_{S^1}(t) - \frac{1}{\sqrt{4\pi t}}) t^{s-1} \frac{dt}{t} + \frac{1}{\Gamma(s)} \int_{c_T}^{\infty} \frac{dt}{\sqrt{4\pi t}} 
$$

$$
+ \frac{1}{\Gamma(s)} \int_{c_T}^{\infty} (\Theta_{S^1}(t) - 1) t^{s-1} \frac{dt}{t}.
$$

This defines a meromorphic continuation of $\zeta_{S^1}$ to the whole complex plane, hence the limit of $\zeta_{S^1}(s)$ at $s = 0$ exists. Near $s = 0$ the gamma function behaves as $1/\Gamma(s) = s + O(s^2)$. Therefore

$$
\zeta_{S^1}(0) = \int_0^{c_T} \left( \Theta_{S^1}(t) - \frac{1}{\sqrt{4\pi t}} \right) \frac{dt}{t} - \frac{1}{\sqrt{4\pi c_T}} \log c_T + \Gamma'(1) + \int_{c_T}^{\infty} (\Theta_{S^1}(t) - 1) \frac{dt}{t}.
$$

As mentioned in the introduction, we notice that for a real torus $T$ the regularized determinant of the Laplacian, $\det^* \Delta_T$, is defined by the following identity (for more details see [17]):

$$
\log \det^* \Delta_T = -\zeta'_T(0).
$$

Let $s \in \mathbb{C}$ with $\text{Re}(s) > d/2$, and $B = \text{diag}(\beta_1, \ldots, \beta_d)$ be a positive diagonal matrix. Using (1), the zeta function can be rewritten as

$$
\zeta_{\mathbb{R}^d/B^{\mathbb{R}}}(s) = \frac{1}{(4\pi)^{d/2}} \sum_{(m_1, \ldots, m_d) \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{(\sum_{i=1}^d m_i^2/\beta_i^2)^s}.
$$
Let $\zeta$ be the Riemann zeta function. In the case of the circle $\mathbb{R}/\beta\mathbb{Z}$ the eigenvalues of the Laplacian are given by $\lambda_j = (2\pi)^2 (j/\beta)^2$ for $j \in \mathbb{Z}$, so the spectral zeta function is related to the Riemann zeta function by

$$\zeta_{\mathbb{R}/\beta\mathbb{Z}}(s) = 2(\beta/2\pi)^{2s} \zeta(2s).$$

Using the special values of the Riemann zeta function $\zeta(0) = -1/2$ and $\zeta'(0) = -(1/2) \log(2\pi)$, the derivative evaluated at zero is given by

$$\zeta'_{\mathbb{R}/\beta\mathbb{Z}}(0) = 4 \log(\beta/2\pi) \zeta(0) + 4 \zeta'(0) = -2 \log \beta. \quad (6)$$

In particular for the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$, one has

$$\zeta'_{S^1}(0) = 0. \quad (7)$$

### 2.3 Modified $I$-Bessel functions

Let $I_x$ be the modified $I$-Bessel function of the first kind of index $x$. For positive integer values of $x$, $I_x(t)$ has the following series representation

$$I_x(t) = \sum_{n=0}^{\infty} \frac{(t/2)^{2n+x}}{n! \Gamma(n+1+x)} \quad (8)$$

and the integral representation

$$I_x(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{t \cos \theta} \cos(x \theta) d\theta. \quad (9)$$

For negative values of $x$ we have that $I_x(t) = I_{-x}(t)$ for all $t$.

From Theorem 9 in [12] which is a special case of Proposition 2.1, we have the theta inversion formula on $\mathbb{Z}/m\mathbb{Z}$, that is, for every integer $m > 0$ and all $t$,

$$e^{-t} \sum_{k \in \mathbb{Z}} I_{km}(t) = \frac{1}{m} \sum_{j=0}^{m-1} e^{-(1-\cos(2\pi j/m))t}. \quad (9)$$

The two following propositions give some results on the asymptotics of the $I$-Bessel function. The first result has been proved in [5].

**Proposition 2.2.** Let $b(n)$ be a sequence of positive integers parametrized by $n \in \mathbb{N}$ such that $b(n)/n \to \beta > 0$ as $n \to \infty$. Then for any $t > 0$ and non-negative integer $k \geq 0$, we have

$$\lim_{n \to \infty} b(n)e^{-2n^2t} (b(n)k)(2n^2t) = \frac{\beta}{\sqrt{4\pi t}} e^{-\beta k^2/4t}. \quad (10)$$

**Proposition 2.3.** Let $a(n)$ be a sequence of positive integers tending to infinity sublinearly with respect to $n$. Then we have that

$$\lim_{n \to \infty} a(n)e^{-2n^2t} \sum_{k \in \mathbb{Z}} I_{a(n)k}(2n^2t) = 1. \quad (11)$$

**Proof.** From the theta inversion formula on $\mathbb{Z}$,

$$a(n)e^{-2n^2t} \sum_{k \in \mathbb{Z}} I_{a(n)k}(2n^2t) = 1 + \sum_{j=1}^{a(n)-1} e^{-4 \sin^2(\pi j/a(n))n^2}. \quad (12)$$
If $a(n)$ is even,
\[
\sum_{j=1}^{a(n)-1} e^{-4\sin^2(\pi j/a(n))}n^2t = e^{-4n^2t} + 2 \sum_{j=1}^{a(n)/2-1} e^{-4\sin^2(\pi j/a(n))}n^2t.
\]
If $a(n)$ is odd,
\[
\sum_{j=1}^{a(n)-1} e^{-4\sin^2(\pi j/a(n))}n^2t = 2 \sum_{j=1}^{a(n)-1/2} e^{-\pi^2n^2/a(n)^2}j^2t
\]
\[
\leq \sum_{j=1}^{\infty} e^{-\pi^2n^2/a(n)^2}j^2t = \frac{1}{e^{\pi^2tn^2/a(n)^2} - 1} \to 0
\]
since $n/a(n) \to \infty$ as $n \to \infty$.

**Proposition 2.4.** For all $x \geq 2$,
\[
\int_0^\infty \left( e^{-t} - e^{-xt}I_0(2t) \right) \frac{dt}{t} = \text{argcosh}(x/2).
\]

**Proof.** Setting $x = 0$ in (8), we have
\[
I_0(2t) = \sum_{n \geq 0} \frac{t^{2n}}{(n!)^2}.
\]
It follows
\[
\int_0^\infty e^{-xt}(I_0(2t) - 1)\frac{dt}{t} = \int_0^\infty e^{-xt} \sum_{n \geq 1} \frac{t^{2n}}{(n!)^2} \frac{dt}{t}
\]
\[
= \sum_{n \geq 1} \frac{(2n-1)!}{(n!)^2} \frac{1}{x^{2n}}.
\]
Let $y = 1/x^2$ with $y \leq 1/4$, so the above is equivalent to the following sum $\sum_{n \geq 1} y^n (2n-1)!/(n!)^2$.
Let $C_n = C_{2n}/(n+1) = (2n)!/(n+1)!n!$ be the Catalan numbers, $n \geq 0$, where $C_m = m!/n!(m-n)!$ is the binomial coefficient. The generating function of the Catalan numbers is given by
\[
\sum_{n \geq 0} C_n y^n = \frac{2}{1 + \sqrt{1 - 4y}}
\]
(10)
The integration over $y$ of the above leads to
\[
\sum_{n \geq 0} \frac{C_n}{n+1} y^{n+1} = \log(1 + \sqrt{1 - 4y}) - \sqrt{1 - 4y} + \text{constant}.
\]
Taking the limit $y \to 0$ on both sides gives the constant $= 1 - \log 2$. Hence,

$$\sum_{n \geq 0} C_n \frac{y^{n+1}}{n+1} = y + \sum_{n \geq 2} \frac{(2n-2)!}{(n!)^2} y^n$$

$$= \log(1 + \sqrt{1-4y}) - \sqrt{1-4y} + 1 - \log 2.$$  

Let $\alpha_n = C_{n-1}/n = (2n-2)!/(n!)^2$, $n \geq 2$, and $\alpha_1 = 1$, and let $g(y) = \log(1 + \sqrt{1-4y}) - \sqrt{1-4y} + 1 - \log 2$. So the previous equation can be written as

$$\sum_{n \geq 1} \alpha_n y^n = g(y).$$

So (10) is equivalent to

$$\sum_{n \geq 1} n\alpha_n y^{n-1} = g'(y).$$

Finally,

$$\sum_{n \geq 1} \frac{(2n-1)!}{(n!)^2} y^n = \sum_{n \geq 1} (2n-1)\alpha_n y^n$$

$$= 2y \sum_{n \geq 1} n\alpha_n y^{n-1} - \sum_{n \geq 1} \alpha_n y^n$$

$$= 2yg(y) - g(y)$$

$$= \log \left( \frac{2}{1 + \sqrt{1-4y}} \right).$$

Writing the above in terms of $x$ gives for all $x \geq 2$,

$$\int_0^\infty e^{-xt}(I_0(2t) - 1) \frac{dt}{t} = \log \frac{x}{2} + \log(x - \sqrt{x^2 - 4}).$$

Notice that the above is the generating function of the Catalan numbers, and therefore is equal to $\log(\sum_{n=0}^\infty C_n x^{-2n})$.

Using the following integral identity for all $x \in \mathbb{C}$ with $\text{Re}(x) > 0$

$$\int_0^\infty (e^{-t} - e^{-xt}) \frac{dt}{t} = \log x$$

one has

$$\int_0^\infty (e^{-t} - e^{-xt}I_0(2t)) \frac{dt}{t} = \log \left( \frac{x + \sqrt{x^2 - 4}}{2} \right) = \text{argcosh}(x/2).$$

\[ \square \]

### 2.4 $d$-dimensional modified I-Bessel function

Let $m, p_1, \ldots, p_d$ be positive integers. By analogy with the two-dimensional $J$-Bessel function defined in [13] we define the $d$-dimensional modified $I$-Bessel function of order $m, I_{m,p_1\ldots,p_d}(u_1, \ldots, u_d)$, as the generating function of $e^{\sum_{i=1}^d u_i \cos p_i t}$, that is

$$e^{\sum_{i=1}^d u_i \cos p_i t} = \sum_{m=-\infty}^\infty I_{m,p_1\ldots,p_d}(u_1, \ldots, u_d)e^{imt}.$$
In our computation we will only need \( u_1 = \ldots = u_d = 2n^2t \) so we set \( u_1 = \ldots = u_d = u \). We have
\[
\mathcal{P}^{m \ldots p_d}(u, \ldots, u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{i=1}^{d} \mu_i(u) e^{i \sum_{i=1}^{d} (\mu_i - 1) t} dt.
\]
The integral is non-zero only for \( \sum_{i=1}^{d} \mu_i p_i = m \). Let \( (\mu_1, \ldots, \mu_d) = (M_1, \ldots, M_d) \) be a particular solution, then the set of solutions is given by
\[
\mu_1 = M_1 - \sum_{k=2}^{d} p_{d+1-k}, \quad \mu_i = M_i + p_1 k_i, \quad i = 2, \ldots, d, \quad k_2, \ldots, k_d \in \mathbb{Z}.
\]
So we have
\[
\mathcal{P}^{m \ldots p_d}(u, \ldots, u) = \sum_{(k_2, \ldots, k_d) \in \mathbb{Z}^{d-1}} I^{M_1 - \sum_{i=2}^{d} p_i k_i}(u) \prod_{i=2}^{d} I_{M_i + p_i k_i}(u).
\]
Let \( \Gamma := \{1, \gamma_1, \ldots, \gamma_{d-1}\} \) be a set of integral parameters, and \( k_1 \in \mathbb{N} \). We set \( M_1 = nk_1 \), \( M_2 = \ldots = M_d = 0 \), \( p_1 = 1 \), \( p_i = \gamma_i - 1 \), \( i = 2, \ldots, d \), then the \( d \)-dimensional modified \( I \)-Bessel function of order \( nk_1 \) and parameters set \( \Gamma \) is given by
\[
I_{nk_1}^{\Gamma}(u, \ldots, u) := I_{nk_1}^{1, \gamma_1, \ldots, \gamma_{d-1}}(u, \ldots, u) = \sum_{(k_2, \ldots, k_d) \in \mathbb{Z}^{d-1}} I_{nk_1 - \sum_{i=2}^{d-1} \gamma_i k_i + 1}(u) \prod_{i=2}^{d} I_{k_i}(u)
\]
which has the integral representation
\[
I_{nk_1}^{\Gamma}(u, \ldots, u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iu \cos w + \sum_{i=1}^{d-1} \gamma_i w} e^{-ink_1 w} dw.
\]
Since \( I_n(u) = I_n(u) \), notice that
\[
I_{-nk_1}^{\Gamma}(u, \ldots, u) = I_{nk_1}^{\Gamma}(u, \ldots, u).
\]

### 2.5 Upper bounds for \( I \)-Bessel functions

Recall Remark 4.2 in [5]: For all \( t > 0 \) we have the bounds:
\[
0 \leq ne^{-n^2t} I_0(n^2t) \leq C t^{-1/2}
\]
for some positive constant \( C \).

Recall Lemma 4.6 in [5]:

**Lemma 2.5.** Fix \( t \geq 0 \) and non-negative integers \( x \) and \( n_0 \). Then for all \( n \geq n_0 \), we have the uniform bound
\[
0 \leq \sqrt{n^2t} e^{-n^2t} I_n x (n^2t) \leq \left( \frac{n_0 t}{x + n_0 t} \right)^{n_0 x/2} = \left( 1 + \frac{x}{n_0 t} \right)^{-n_0 x/2} \leq 1.
\]
2.6 Method

The method developed in [5] consists in studying the asymptotic behaviour of the Gauss transform of the theta function evaluated at zero in order to obtain the product of the Laplacian eigenvalues. This leads to the two following theorems which are adapted from Theorem 3 in [5]. They express the logarithm of the determinant of the combinatorial Laplacian on the corresponding discrete torus in terms of integrals of theta and I-Bessel functions. The study of the asymptotics of these integrals will therefore lead to the asymptotic behaviour of the number of spanning trees.

In the case of the circulant graph we have:

**Theorem 2.6.** We have the identity

\[
\log \left( \prod_{\lambda_j \neq 0} \lambda_j \right) = n I_d^\Gamma + \mathcal{H}_{C_n}^\Gamma
\]

where

\[
I_d^\Gamma = \int_0^\infty \left( e^{-t} - e^{-2dt I_0^\Gamma(2t, \ldots, 2t)} \right) \frac{dt}{t}
\]

and

\[
\mathcal{H}_{C_n}^\Gamma = -\int_0^\infty \left( \theta_{C_n}^\Gamma(t) - ne^{-2dt I_0^\Gamma(2t, \ldots, 2t)} - 1 + e^{-t} \right) \frac{dt}{t}.
\]

And in the case of the diagonal discrete torus we have:

**Theorem 2.7.** We have the identity

\[
\log \left( \prod_{\lambda_j \neq 0} \lambda_j \right) = \det(\Lambda_n) x_d^{(n)}_{\Gamma} + \mathcal{H}_{\Lambda_n}
\]

where

\[
x_d^{(n)}_{\Gamma} = \int_0^\infty \left( e^{-t} - e^{-2dt I_0^\Gamma(2t)} \sum_{(k_1, \ldots, k_p) \in \mathbb{Z}^p} \prod_{i=1}^p I_{k_i, a_i(n)}(2t) \right) \frac{dt}{t}
\]

and

\[
\mathcal{H}_{\Lambda_n} = -\int_0^\infty \left( \theta_{\Lambda_n}(t) - e^{-2dt I_0^\Gamma(2t)} \sum_{(k_1, \ldots, k_p) \in \mathbb{Z}^p} \prod_{i=1}^p I_{k_i, a_i(n)}(2t) - 1 + e^{-t} \right) \frac{dt}{t}.
\]

3 Asymptotic behaviour of spectral determinant on circulant graphs

3.1 Computation of the asymptotics

Let \( 1 < \gamma_1 < \ldots < \gamma_{d-1} \leq \lfloor n/2 \rfloor \) be positive integers and \( C_n^\Gamma \) denote the circulant graph where \( \Gamma := \{1, \gamma_1, \ldots, \gamma_{d-1}\} \) is the set of generators. In this work we only consider circulant graphs with first generator equals to 1. In this case one can verify that \( C_n^\Gamma \) is isomorphic to the \( d \)-dimensional discrete torus \( \mathbb{Z}^d/\Lambda_\Gamma \mathbb{Z}^d \) where \( \Lambda_\Gamma \) is the following matrix

\[
\Lambda_\Gamma = \begin{pmatrix}
\frac{n}{d} & -\gamma_1 & \cdots & -\gamma_{d-1} \\
0 & \ddots & & \vdots \\
& & 0 & \ddots \\
& & & \ddots & I_{d-1}
\end{pmatrix}
\]
where $I_{d-1}$ is the identity matrix of order $d-1$. Indeed, all the points on the lattice $\Lambda_{\Gamma}Z^d$ are identified according to the numbers, where the nearest neighbours are connected to each other. Denote by $e_i$, $i=1,\ldots,d$, the canonical basis of $Z^d$. Then 0 is connected to $e_i$ and $-e_i$ for $i=1,\ldots,d$. For $v \in Z/nZ$, all the points $ve_1 + \Lambda_{\Gamma}Z^d$ are identified to $v$. Hence 0 is connected to 1. Since $-e_1 = (n-1)e_1 - \Lambda_{\Gamma}e_1$, 0 is connected to $-1$. Using that $e_{i+1} = -\gamma_i e_1 - \Lambda_{\Gamma}e_{i+1}$, $i=1,\ldots,d$, 0 is connected to $\gamma_i$ for all $i=1,\ldots,d-1$. Similarly, all $v \in Z/nZ$ are connected to $v \pm \gamma_i$ mod $n$ for all $i=1,\ldots,d$. Therefore the quotient $Z^d/\Lambda_{\Gamma}Z^d$ with nearest neighbours connected to each other is isomorphic to the circulant graph $C^\Gamma_n$. Figure 3 illustrates the lattice corresponding to the circulant graph $C^\Gamma_{1,2}$ represented in Figure 1. The fact that the matrix is almost diagonal simplifies the expression of the theta function. Indeed from Proposition 2.1 the theta function on $C^\Gamma_n$ is given by

$$\theta_{C^\Gamma_n}(n^2t) = ne^{-2dn^2t} \sum_{(k_1,\ldots,k_d) \in Z^d} I_{nk_1} - \sum_{i=1}^{d-1} \gamma_i k_{i+1} (2n^2t) \prod_{i=2}^d I_{k_i}(2n^2t).$$

Rewriting it in terms of the $d$-dimensional modified $I$-Bessel function defined in section 2.4 we get

$$\theta_{C^\Gamma_n}(n^2t) = ne^{-2dn^2t} \sum_{k_1 \in Z} I^\Gamma_{nk_1}(2n^2t,\ldots,2n^2t).$$

A circulant graph is the Cayley graph of a finite abelian group, so the eigenvectors of the Laplacian on $C^\Gamma_n$ are the characters

$$\chi_j(x) = e^{2\pi ijx/n}, \quad j = 0,1,\ldots,n-1.$$

By applying the Laplacian on the characters, we obtain the eigenvalues

$$\lambda_j = 2d - 2\cos(2\pi j/n) - 2\sum_{i=1}^{d-1} \cos(2\pi \gamma_i j/n), \quad j = 0,1,\ldots,n-1.$$
Therefore, by definition of the theta function (12) it can also be written as

\[
\theta_{C^n}(n^2 t) = \sum_{j=0}^{n-1} e^{-(2d-2)\cos(2\pi j/n) - 2\sum_{i=1}^{d-1} \cos(2\pi \gamma_{ij}/n))}n^2 t
\]

\[
= \sum_{j=0}^{n-1} e^{-4\sin^2(\pi j/n) + \sum_{i=1}^{d-1} \sin^2(\pi \gamma_{ij}/n))}n^2 t.
\]

(15)

**Proposition 3.1.** With the above notation we have for all \( t \geq 0 \),

\[
\lim_{n \to \infty} \theta_{C^n}(n^2 t) = \Theta_S(t)
\]

where \( \Theta_S \) is the theta function on the circle \( S^1 = \mathbb{R}/\mathbb{Z} \) given by

\[
\Theta_S(t) = \frac{1}{\sqrt{4\pi t}} \sum_{k=-\infty}^{\infty} e^{-k^2/(4t)}.
\]

**Proof.** From the theta inversion formula on \( \mathbb{Z}/m\mathbb{Z} \) (Theorem 10 in [12]) we have for any \( z \in \mathbb{C} \), and integers \( x \) and \( m > 0 \),

\[
\sum_{k=-\infty}^{\infty} I_{x+km}(z) = \frac{1}{m} \sum_{j=0}^{m-1} e^{\cos(2\pi j/m)z + 2\pi ijx/m}.
\]

Using the expression of the theta function in terms of \( I \)-Bessel functions, it follows that for all \( n \gg 1 \) and \( t > 0 \),

\[
|\theta_{C^n}(n^2 t)| = |ne^{-2dn^2 t} \sum_{(k_2, \ldots, k_d) \in \mathbb{Z}^{d-1}} \frac{1}{n} \sum_{j=0}^{n-1} e^{2n^2 t \cos(2\pi j/n) - 2\pi ij \sum_{i=1}^{d-1} \gamma_{ik_{i+1}/n}} \prod_{i=2}^{d} I_{k_i}(2n^2 t)|
\]

\[
\leq \prod_{i=2}^{d} \sum_{k_i \in \mathbb{Z}} e^{-2n^2 t} I_{k_i}(2n^2 t) \sum_{j=0}^{n-1} e^{-2n^2 t(1-\cos(2\pi j/n))}
\]

\[
\leq \sum_{j=0}^{n-1} e^{-8\pi^2 ctj^2} \leq \sum_{j=0}^{n-1} e^{-ctj} \leq \frac{1}{1 - e^{-ct}}
\]

where \( c' > 0 \). In the second inequality we used the fact that for all \( v \in [0, \pi] \), \( (1 - \cos v)/v^2 \geq c \), with \( c = 1/2 - \pi^2/24 > 0 \), and \( e^{-t} \sum_{x \in \mathbb{Z}} I_x(t) = 1 \).

It follows that

\[
\lim_{n \to \infty} \theta_{C^n}(n^2 t) = \sum_{k \in \mathbb{Z}} \lim_{n \to \infty} ne^{-2dn^2 t} I_{nk_1}(2n^2 t, \ldots, 2n^2 t)
\]

(16)

Let \( k_1 > 0 \). From the integral representation of the \( d \)-dimensional \( I \)-Bessel function we have

\[
ne^{-2dn^2 t} I_{nk_1}(2n^2 t, \ldots, 2n^2 t) = \frac{1}{2\pi k_1} \int_{-\pi k_1}^{\pi k_1} e^{iw} e^{-2n^2 t(d - \cos(w/(nk_1)) - \sum_{i=1}^{d-1} \cos(\gamma_i w/(nk_1)))} dw.
\]

Since \( (1 - \cos v)/v^2 \geq c > 0 \) for all \( v \in [0, \pi] \), we have that

\[
n^2(d - \cos(w/(nk_1)) - \sum_{i=1}^{d-1} \cos(\gamma_i w/(nk_1))) \geq c (w/k_1)^2
\]
for all \( w \in [0, \pi n k_1] \). Hence for all \( n \geq 1 \),

\[
|n e^{-2dn^2t} I_{nk_1} (2n^2 t, \ldots, 2n^2 t)| \leq \frac{1}{2\pi k_1} \int_{-\pi n k_1}^{\pi n k_1} e^{-2tw^2/k_1^2} dw \leq \frac{1}{2\pi k_1} \int_{-\infty}^{\infty} e^{-2tw^2/k_1^2} dw = \frac{2}{\pi ct}.
\]

We also have that

\[
\lim_{n \to \infty} n^2 (d - \cos(w/(nk_1)) - \sum_{i=1}^{d-1} \cos(\gamma_i w/(nk_1))) = \frac{c_t}{2} (w/k_1)^2.
\]

So by the Lebesgue dominated convergence Theorem, we have for all \( k_1 > 0 \)

\[
\lim_{n \to \infty} n e^{-2dn^2t} I_{nk_1} (2n^2 t, \ldots, 2n^2 t) = \frac{1}{2\pi k_1} \int_{-\infty}^{\infty} e^{-ctw^2/k_1^2} e^{iw} dw = \frac{1}{\sqrt{4\pi ct}} e^{-k_1^2/(4ct)}. \tag{17}
\]

Let \( k_1 = 0 \). From the integral representation of the \( d \)-dimensional \( I \)-Bessel function we have

\[
n e^{-2dn^2t} I_{0} (2n^2 t, \ldots, 2n^2 t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-2nt(d - \cos(w/n) - \sum_{i=1}^{d-1} \cos(\gamma_i w/n))} dw.
\]

With the same argument as in the case \( k_1 > 0 \) we can apply the Lebesgue dominated convergence Theorem and we get

\[
\lim_{n \to \infty} n e^{-2dn^2t} I_{0} (2n^2 t, \ldots, 2n^2 t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ctw^2} dw = \frac{1}{\sqrt{4\pi ct}}. \tag{18}
\]

Putting \((17)\) and \((18)\) in \((16)\) and using \((12)\), the result follows. \( \Box \)

**Proposition 3.2.** With the above notation we have

\[
\lim_{n \to \infty} \int_{0}^{1} \left( \theta_{Ct_x} (n^2 t) - n e^{-2dn^2t} I_{0} (2n^2 t, \ldots, 2n^2 t) \right) \frac{dt}{t} = \int_{0}^{1} \left( \theta_{S^1} (ct) - \frac{1}{\sqrt{4\pi ct}} \right) \frac{dt}{t}.
\]

**Proof.** From the integral representation of the \( d \)-dimensional modified \( I \)-Bessel function \((11)\) and Cauchy-Schwarz inequality we have

\[
\theta_{Ct_x} (n^2 t) - n e^{-2dn^2t} I_{0} (2n^2 t, \ldots, 2n^2 t) = \frac{n}{2\pi} \sum_{k_1 \in \mathbb{Z}^d} \int_{-\pi}^{\pi} e^{-2nt(d - \cos w - \sum_{i=1}^{d-1} \cos \gamma_i w)} e^{-ink_1 w} dw
\]

\[
\leq \frac{n}{2\pi} \sum_{k_1 \in \mathbb{Z}^d} \left( \int_{-\pi}^{\pi} e^{-4nt(1 - \cos w)} e^{-ink_1 w} dw \right)^{1/2} \left( \int_{-\pi}^{\pi} e^{-4nt(1 - \sum_{i=1}^{d-1} \cos \gamma_i w)} e^{-ink_1 w} dw \right)^{1/2}
\]

\[
= \frac{1}{\sqrt{2\pi}} \sum_{k_1 \in \mathbb{Z}^d} \sqrt{n} e^{-2nt} I_{nk_1} (4n^2 t)^{1/2} \left( n \int_{-\pi}^{\pi} e^{-4nt(1 - \sum_{i=1}^{d-1} \cos \gamma_i w)} e^{-ink_1 w} dw \right)^{1/2}.
\]

\[14\]
Let $k_1 > 0$. We have

$$\left| n \int_{-\pi}^{\pi} e^{-4n^2 t (d-1-\sum_{i=1}^{d-1} \cos \gamma_i w)} e^{-i n k_1 w} dw \right| \leq \frac{1}{k_1} \prod_{i=1}^{d-2} \left( \int_{-\pi}^{\pi} e^{-8 n^2 t \sin^2 (\gamma_i w/(2k_1))} dw \right)^{1/2} \times \left( \int_{-\pi}^{\pi} e^{-8 n^2 t \sin^2 (\gamma_{d-1} w/(2k_1))} dw \right)^{1/2 - 1}$$

where Cauchy-Schwarz inequality is used $d - 2$ times in the second step. Using that $\sin x \geq x/2$, for all $x \in [0, \pi/2]$, the above is less than or equal to

$$\frac{1}{k_1} \prod_{i=1}^{d-2} \left( \int_{-\infty}^{\infty} e^{-2^{i+1} \gamma_i w^2/k_i^2} dw \right)^{1/2} \left( \int_{-\infty}^{\infty} e^{-2^{d-1} \gamma_{d-1} w^2/k_d^2} dw \right)^{1/2 - 1}$$

Let $n_0$ be a positive integer and $k_1 \geq 1$. From Lemma 2.5, we have for all $n \geq n_0$ and $t > 0$

$$\sqrt{n e^{-2n^2 t} I(nk_1(4n^2 t)^{1/2})} \leq (4t)^{-1/4} \left( 1 + \frac{k_1}{4n_0 t} \right)^{-n_0 k_1/4}.$$ 

Let $c = 2^{1/2 - 3/2} \pi^{-1/4}$. Hence for all $n \geq n_0$ and $t > 0$

$$\theta_c(n^2 t) - n e^{-2d n^2 t} I_0(2n^2 t, \ldots, 2n^2 t) \leq ct^{-1/2} \sum_{k_1=1}^{\infty} \left( 1 + \frac{k_1}{4n_0 t} \right)^{-n_0 k_1/4}$$

$$= ct^{-1/2} \left( \frac{1}{(1 + 1/4n_0 t)^n + 1} \right) \leq ct^{n_0/4 - 1/2}.$$ 

Let $n_0 = 3$, then the above is integrable on $(0, 1)$ with respect to the measure $dt/t$. The proposition then follows from the Lebesgue dominated convergence Theorem and from the pointwise convergence. 

Recall the following lemma from [5]:

**Lemma 3.3.** For $n \in \mathbb{R}$, we have the asymptotic formula

$$\int_0^t (e^{-n^2 t} - 1) \frac{dt}{t} = \Gamma'(1) - 2 \log n + o(1) \quad \text{as } n \to \infty.$$
Proposition 3.4. With the above notation we have that
\[ \lim_{n \to \infty} \int_{1}^{\infty} (\theta_{C_n}(n^2t) - 1) \frac{dt}{t} = \int_{1}^{\infty} (\Theta_{S^1}(c_T t) - 1) \frac{dt}{t}. \]

Proof. From Proposition 2.5 we have for all \( t > 0 \), the pointwise limit
\[ \lim_{n \to \infty} \theta_{C_n}(n^2t) - 1 = \Theta_{S^1}(c_T t) - 1. \]
From Proposition 3.1 we have
\[ \lim_{n \to \infty} \theta_{C_n}(n^2t) - 1 = \Theta_{S^1}(c_T t) - 1. \]

From Lemma 2.5, we have
\[ \theta_{C_n}(n^2t) = 1 + \sum_{j=1}^{n-1} e^{-4\sin^2(\pi j/n)n^2t} \prod_{i=1}^{d-1} e^{-4\sin^2(\pi \gamma_{i,j}/n)n^2t}. \]
Since the product on \( i \) is smaller than 1, we have
\[ \theta_{C_n}(n^2t) \leq 1 + \sum_{j=1}^{n-1} e^{-4\sin^2(\pi j/n)n^2t} = 1 + 2 \sum_{j=1}^{[n/2]} e^{-4\sin^2(\pi j/n)n^2t}. \]
Using the elementary bound
\[ \sin(x) \geq x \left(1 - \pi^2 x^2 / 6\right) \geq c \pi x \]
for all \( x \in [0,1/2] \), where \( c = 1 - \pi^2/24 > 0 \), we have
\[ \theta_{C_n}(n^2t) - 1 \leq 2 \sum_{j=1}^{[n/2]} e^{-4c^2 \pi^2 j^2 n^2 t} \leq 2 \sum_{j=1}^{\infty} e^{-d_j t} = \frac{2}{e^{dt} - 1} \leq \frac{2}{1 - e^{-d_T t}}, \]
for all \( t \geq 1 \), where \( d = 4c^2\pi^2 > 0 \). Since it is integrable on \((1, \infty)\) with respect to the measure \( dt/t \), the proposition follows from the Lebesgue dominated convergence Theorem.

Proposition 3.5. With the above notation we have
\[ \lim_{n \to \infty} \int_{1}^{\infty} ne^{-2dn^2 t} I_0^\Gamma(2n^2t, \ldots, 2n^2t) \frac{dt}{t} = \frac{1}{\sqrt{\pi c_T}}. \]

Proof. By definition, we have
\[ I_0^\Gamma(2n^2t, \ldots, 2n^2t) = \sum_{(k_2, \ldots, k_d) \in \mathbb{Z}^{d-1}} I - \sum_{i=1}^{d-1} \gamma_{i,k_{i+1}}(2n^2t) \prod_{i=2}^{d} I_{k_i}(2n^2t). \]
From Lemma 2.5, we have the uniform upper bound
\[ ne^{-2n^2 t} I - \sum_{i=1}^{d-1} \gamma_{i,k_{i+1}}(2n^2t) \leq \frac{1}{\sqrt{2t}} \]
Hence
\[ ne^{-2dn^2 t} I_0^\Gamma(2n^2t, \ldots, 2n^2t) \leq \frac{1}{\sqrt{2t}} \left( e^{-2n^2 t} \sum_{k \in \mathbb{Z}} f_k(2n^2t) \right)^{d-1} = \frac{1}{\sqrt{2t}} \]
which is integrable on \((1, \infty)\) with respect to the measure \( dt/t \). By the Lebesgue dominated convergence Theorem it follows that
\[ \lim_{n \to \infty} \int_{1}^{\infty} ne^{-2dn^2 t} I_0^\Gamma(2n^2t, \ldots, 2n^2t) \frac{dt}{t} = \int_{1}^{\infty} \frac{1}{\sqrt{4\pi c_T t}} \frac{dt}{t} = \frac{1}{\sqrt{\pi c_T}}. \]
Since \( \int_1^\infty e^{-n^2t} dt/t \) converges to zero as \( n \to \infty \), putting Lemma 3.3 and Propositions 3.2, 3.4 and 3.5 together in Theorem 2.6 leads to the asymptotic of the \( H_{C_n}^\Gamma \) term as \( n \to \infty \):
\[
H_{C_n}^\Gamma = 2 \log n - \int_0^{c_r} \left( \Theta_{S^1}(t) - \frac{1}{\sqrt{4\pi t}} \right) \frac{dt}{t} - \Gamma'(1) - \int_{c_r}^\infty \left( \Theta_{S^1}(t) - 1 \right) \frac{dt}{t} + \frac{1}{\sqrt{\pi c}} + o(1).
\]

Using equation (4) we can then rewrite:
\[
H_{C_n}^\Gamma = 2 \log n - \zeta'_{S^1}(0) - \log c + o(1) \quad \text{as } n \to \infty.
\]

Since \( \zeta'_{S^1}(0) = 0 \) (7) we get
\[
H_{C_n}^\Gamma = 2 \log n - \log c + o(1) \quad \text{as } n \to \infty
\]
and so
\[
\log \det^* \Delta_{C_n}^\Gamma = n \int_0^\infty \left( e^{-t} - e^{-2dt} I_0^\Gamma(2t, \ldots, 2t) \right) \frac{dt}{t} = 2 \log n - \log c + o(1) \quad \text{as } n \to \infty.
\]

which proves Theorem 1.1.

3.2 Asymptotic number of spanning trees and comparison of the results

Notice that in the trivial case \( d = 1 \), the cycle has \( n \) spanning trees so \( \log \det^* \Delta_{C_n} = \log n^2 \). On the other hand, from Proposition 2.4
\[
\int_0^\infty \left( e^{-t} - e^{-2dt} I_0^\Gamma(2t) \right) \frac{dt}{t} = 0
\]
and so the right hand side of the asymptotic development is \( 2 \log n \). Therefore the theorem is verified in this particular case.

From Kirchhoff’s matrix tree theorem and Theorem 1.1, the number of spanning trees in the circulant graph \( C_n^\Gamma \) with \( \Gamma = \{1, \gamma_1, \ldots, \gamma_{d-1}\} \) is asymptotically given by
\[
\tau(C_n^\Gamma) = \frac{n}{c} e^{n I_d^\Gamma + o(1)} \quad \text{as } n \to \infty.
\]

(19)
The lead term can be rewritten as
\[
I_d^\Gamma = \int_0^\infty \left( e^{-t} - e^{-2dt} I_0^\Gamma(2t, \ldots, 2t) \right) \frac{dt}{t} = \log(2d) + \int_0^\infty e^{-2dt} (1 - I_0^\Gamma(2t, \ldots, 2t)) \frac{dt}{t}.
\]

From the integral representation of \( I_0^\Gamma \) and writing the exponential as a series one has
\[
\int_0^\infty e^{-2dt} (1 - I_0^\Gamma(2t, \ldots, 2t)) \frac{dt}{t} = -\frac{1}{2\pi} \int_0^\infty e^{-2dt} \sum_{n=1}^{\infty} \frac{2n}{n!} \int_{-\pi}^\pi (\cos w + \sum_{i=1}^{d-1} \cos(\gamma_i w))^n dw \frac{dt}{t}
\]
\[
= -\frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{d^n n!} \int_{-\pi}^\pi (\cos w + \sum_{i=1}^{d-1} \cos(\gamma_i w))^n dw
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^\pi \log \left( 1 - \frac{\cos w + \sum_{i=1}^{d-1} \cos(\gamma_i w)}{d} \right) dw
\]
\[
= \int_0^1 \log(\sin^2(\pi w) + \sum_{i=1}^{d-1} \sin^2(\pi \gamma_i w)) dw + \log \frac{2}{d}.
\]
Hence the lead term is given by

\[ T_d^\gamma = \log 4 + \int_0^1 \log(\sin^2(\pi w)) + \sum_{i=1}^{d-1} \sin^2(\pi \gamma_i w) \, dw \]

which corresponds to Lemma 2 of [11].

As mentioned in the introduction, the authors show in [20] that the number of spanning trees in a circulant graph is given by

\[ \tau(C_{\gamma_1, \ldots, \gamma_d}^n) = n a_n^2 \]

where \( a_n \) satisfies a recurrence relation which behaves asymptotically as \( c \phi^n \) for some constants \( c \) and \( \phi \) which can be determined numerically. Comparing with (19) it follows that

\[ c^2 = \frac{1}{c^\gamma} \]

which is numerically verified with the values in Table 1 in [20]. This answers to one of the questions asked in the conclusion of [1].

4 Asymptotic behaviour of spectral determinant on degenerating tori

We consider the sequence of \( d \)-dimensional discrete tori described in the introduction. For simplicity, we denote by \( \theta_{\Lambda_n} \) the theta function associated to \( \mathbb{Z}^d / \Lambda_n \mathbb{Z}^d \). It is given by

\[ \theta_{\Lambda_n}(t) = \sum_{\lambda_j} e^{-\lambda_j t} \]

where

\[ \{\lambda_j\}_{j=0,1,\ldots,\det(\Lambda_n) - 1} = \left\{ 2d - 2 \sum_{i=1}^{p} \cos(2\pi m_i / a_i(n)) - 2 \sum_{i=1}^{d-p} \cos(2\pi m_i / b_i(n)) : 
\right. \\
\left. 0 \leq m_i < a_i(n), i = 1, \ldots, p \text{ and } 0 \leq m_i < b_i(n), i = 1, \ldots, d-p \right\} \]

are the eigenvalues of the combinatorial Laplacian on \( \mathbb{Z}^d / \Lambda_n \mathbb{Z}^d \). From the theta inversion formula on \( \mathbb{Z}^d / \Lambda_n \mathbb{Z}^d \) (Proposition 2.1) we have for all \( t \geq 0 \)

\[ \theta_{\Lambda_n}(t) = \left( \prod_{i=1}^{p} a_i(n) e^{-2t} \sum_{k \in \mathbb{Z}} I_{ka_i(n)}(2t) \right) \left( \prod_{i=1}^{d-p} b_i(n) e^{-2t} \sum_{k \in \mathbb{Z}} I_{kb_i(n)}(2t) \right). \] (20)

4.1 Computation of the lead term

Let \( c_d \) be the integral below. A numerical estimation of it is discussed in section 7.2 of [5].

\[ c_d = \int_0^\infty \left( e^{-t} - e^{-2d t} I_0(2t) \right) \frac{dt}{t}. \]
The lead term of \( \log \det^* \Delta_{\mathbb{Z}^d / \Lambda_n \mathbb{Z}^d} \) in Theorem 2.4 is given by

\[
\det(\Lambda_n)Z_{d}^{(a)}_{\alpha=1} = \det(\Lambda_n) \int_0^\infty \left(e^{-t} - e^{-2dt} I_0(2t)\right)^{d-p} \sum_{(k_1, \ldots, k_p) \in \mathbb{Z}^p} \prod_{i=1}^p I_{k_i a_i n}(2t) \frac{dt}{t}
\]

\[
= \det(\Lambda_n) c_d - \det(\Lambda_n) \int_0^\infty e^{-2dt} I_0(2t)^{d-p} \sum_{(k_1, \ldots, k_p) \in \mathbb{Z}^p \setminus \{0\}} \prod_{i=1}^p I_{k_i a_i n}(2t) \frac{dt}{t}
\]

\[
= n^{d-p} a(n)^p \prod_{i=1}^p \frac{a_i(n)}{a(n)} \prod_{i=1}^p \frac{b_i(n)}{n} c_d
\]

\[
- \frac{n^{d-p}}{a(n)^{d-p}} \prod_{i=1}^p \frac{b_i(n)}{n} \int_0^\infty \left[(a(n) e^{-2a(n)^2 t} I_0(2a(n)^2 t)\right)^{d-p}
\]

\[
\times \sum_{(k_1, \ldots, k_p) \in \mathbb{Z}^p \setminus \{0\}} \prod_{i=1}^p a_i(n) e^{-2a(n)^2 t} I_{k_i a_i n}(2a(n)^2 t) \frac{dt}{t}
\]

where in the last equality the integration variable \( t \) is changed into \( a(n)^2 t \).

From Proposition 2.2 we have that

\[
\lim_{n \to \infty} a(n) e^{-2a(n)^2 t} I_0(2a(n)^2 t) = \frac{1}{\sqrt{4\pi t}}
\]

and

\[
\lim_{n \to \infty} a_i(n) e^{-2a(n)^2 t} I_{k_i a_i n}(2a(n)^2 t) = \frac{\alpha_i}{\sqrt{4\pi t}} e^{-\alpha_i^2 k_i^2 / 4t}.
\]

To compute the behaviour of the lead term we use that

\[
\lim_{n \to \infty} \prod_{i=1}^p \frac{a_i(n)}{a(n)} \prod_{i=1}^p \frac{b_i(n)}{n} = \det(\Lambda)
\]

and

\[
\int_0^\infty \frac{1}{(4\pi t)^{d/2}} \sum_{(k_1, \ldots, k_p) \in \mathbb{Z}^p \setminus \{0\}} e^{-\sum_{i=1}^p \alpha_i^2 k_i^2 / 4t} \frac{dt}{t} = \frac{\pi^{d/2}}{\Gamma(d/2)} \sum_{(k_1, \ldots, k_p) \in \mathbb{Z}^p \setminus \{0\}} \frac{1}{\prod_{i=1}^p \alpha_i^2 k_i^2 / 4t^{d/2}}
\]

\[
= (4\pi)^{d/2} \Gamma(d/2) \zeta_{R^d / A - 1 \mathbb{Z}^p}(d/2)
\]

where the second equality comes from [5]. Hence as \( n \to \infty \) the lead term behaves as

\[
\det(\Lambda_n)Z_{d}^{(a)}_{\alpha=1} = n^{d-p} a(n)^p \det(\Lambda) c_d - \frac{n^{d-p}}{a(n)^{d-p}} \det(\Lambda) (4\pi)^{d/2} \Gamma(d/2) \zeta_{R^d / A - 1 \mathbb{Z}^p}(d/2) + o(1).
\]

### 4.2 Asymptotic behaviour of the second term

In this section we compute the asymptotics of the \( \mathcal{H}_{\Lambda_n} \) term. To do this we change the integration variable \( t \) into \( n^2 t \) in [13].

\[
\mathcal{H}_{\Lambda_n} = - \int_0^\infty \left( \theta_{\Lambda_n} (n^2 t) - \det(\Lambda_n) e^{-2dn^2 t} I_0(2n^2 t) \left( \sum_{(k_1, \ldots, k_p) \in \mathbb{Z}^p} \prod_{i=1}^p I_{k_i a_i n}(2n^2 t) - 1 - e^{-n^2 t} \right) \right) \frac{dt}{t}
\]

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Proposition 4.1. With the above notation, we have for all \( t \geq 0 \),
\[
\lim_{n \to \infty} \theta_{\Lambda_n}(n^2 t) = \Theta_{\mathbb{R}^{d-p}/BZ^{d-p}}(t).
\]

Proof. The theta function \((20)\) with the change of variable is given by
\[
\theta_{\Lambda_n}(n^2 t) = \left( \prod_{i=1}^{p} a_i(n) e^{-2n^2 t} \sum_{k \in \mathbb{Z}} I_{k_0,\Lambda_n}(2n^2 t) \right) \left( \prod_{i=1}^{d-p} b_i(n) e^{-2n^2 t} \sum_{k \in \mathbb{Z}} I_{k_{-},\Lambda_n}(2n^2 t) \right).
\]

From Proposition 2.3 we have that
\[
\lim_{n \to \infty} \prod_{i=1}^{p} a_i(n) e^{-2n^2 t} \sum_{k \in \mathbb{Z}} I_{k_0,\Lambda_n}(2n^2 t) = 1
\]
and from Proposition 2.2 we have
\[
\lim_{n \to \infty} \prod_{i=1}^{d-p} b_i(n) e^{-2n^2 t} I_{k_{-},\Lambda_n}(2n^2 t) = \prod_{i=1}^{d-p} \frac{\beta_i}{\sqrt{4\pi t}} e^{-(\beta_i k)^2/4t}.
\]

The proposition follows if we can exchange the limit with the sum. This can be justified in the same way as the proof of Proposition 5.2 in [5].

Proposition 4.2. With the above notation, we have that
\[
\lim_{n \to \infty} \int_{0}^{1} \left( \theta_{\Lambda_n}(n^2 t) - \det(\Lambda_n) e^{-2dn^2 t} I_0(2n^2 t)^{d-p} \sum_{(k_1, \ldots, k_p) \in \mathbb{Z}^p} \prod_{i=1}^{p} I_{k_{i},\Lambda_n}(2n^2 t) \right) \frac{dt}{t}
\]
\[
= \int_{0}^{1} \left( \Theta_{\mathbb{R}^{d-p}/BZ^{d-p}}(t) - \prod_{i=1}^{d-p} \frac{\beta_i}{(4\pi t)^{(d-p)/2}} \right) \frac{dt}{t}.
\]

Proof. From Propositions 4.1, 2.2 and 2.3 we have the pointwise convergence:
\[
\lim_{n \to \infty} \theta_{\Lambda_n}(n^2 t) - \det(\Lambda_n) e^{-2dn^2 t} I_0(2n^2 t)^{d-p} \sum_{(k_1, \ldots, k_p) \in \mathbb{Z}^p} \prod_{i=1}^{p} I_{k_{i},\Lambda_n}(2n^2 t)
\]
\[
= \Theta_{\mathbb{R}^{d-p}/BZ^{d-p}}(t) - \prod_{i=1}^{d-p} \frac{\beta_i}{(4\pi t)^{(d-p)/2}}.
\]

We have
\[
\theta_{\Lambda_n}(n^2 t) - \det(\Lambda_n) e^{-2dn^2 t} I_0(2n^2 t)^{d-p} \sum_{(k_1, \ldots, k_p) \in \mathbb{Z}^p} \prod_{i=1}^{p} I_{k_{i},\Lambda_n}(2n^2 t)
\]
\[
= \left( \sum_{(k_1, \ldots, k_p) \in \mathbb{Z}^p} \prod_{i=1}^{p} a_i(n) e^{-2n^2 t} I_{k_{i},\Lambda_n}(2n^2 t) \right) \left( \sum_{(k_1, \ldots, k_{d-p}) \in \mathbb{Z}^{d-p} \setminus \{0\}} \prod_{i=1}^{d-p} b_i(n) e^{-2n^2 t} I_{k_{-},\Lambda_n}(2n^2 t) \right).
\]

The first product of the above can be bounded using Proposition 2.6. Indeed we have that for all \( i = 1, \ldots, p \) there exists an \( n_{i,0} \) such that for all \( n \geq n_{i,0} \)
\[
a_i(n) e^{-2n^2 t} \sum_{k_i \in \mathbb{Z}} I_{k_{i},\Lambda_n}(2n^2 t) < \frac{3}{2}.
\]
Lemma 4.3. Let $n$ and $c$ with proposition: convergence. Theorem we can exchange the limit and integral. Hence we proved the following

Therefore it follows that $\theta$

Lemma 2.5 we have that for $t > d$

Hence if we choose $n_0 = (2(d-p)+4)/\min_{1 \leq i \leq d-p} \beta_i$ the above is integrable on $(0, 1)$ with respect to the measure $dt/t$. The proposition then follows from the Lebesgue dominated convergence Theorem.

We now study the convergence of the integral over $(1, \infty)$. The theta function can be written as the product of two theta functions, that is

$\theta_{\Lambda_n}(n^2 t) = \theta_{\text{diag}(b_1(n), \ldots, b_{d-p}(n))}(n^2 t) \theta_{\text{diag}(a_1(n), \ldots, a_p(n))}(n^2 t)$.

The first theta function can be bounded using Lemma 5.3 in [5] that we recall below.

Lemma 4.3. Let

$\theta_{\text{abs}}(t) = 2 \sum_{j=1}^{\infty} e^{-c j^2 t}$

with $c = 4\pi^2(1-\pi^2/24)^2 > 0$. Assume $n_0$ is such that $\beta_i/2 \leq b_i(n)/n \leq 2\beta_i$ for all $i = 1, \ldots, d-p$ and $n \geq n_0$. Then for any $t > 0$ and $n \geq n_0$ we have the bound

$\theta_{\text{diag}(b_1(n), \ldots, b_{d-p}(n))}(n^2 t) \leq \prod_{i=1}^{d-p} \left( 1 + e^{-4n^2 t} + \theta_{\text{abs}}(t/(4\beta_i^2)) \right)$.

It is easy to verify that similarly the second theta function can be bounded by the following

$\theta_{\text{diag}(a_1(n), \ldots, a_p(n))}(n^2 t) \leq (1 + e^{-4t} + \theta_{\text{abs}}(t))^p$.

Therefore it follows that $\theta_{\Lambda_n}(n^2 t) - 1$ is $dt/t$-integrable on $(1, \infty)$. So by the Lebesgue dominated convergence Theorem we can exchange the limit and integral. Hence we proved the following proposition:

Proposition 4.4. With the above notation we have that

$$\lim_{n \to \infty} \int_1^{\infty} \left( \theta_{\Lambda_n}(n^2 t) - 1 \right) \frac{dt}{t} = \int_1^{\infty} \left( \Theta_{B_2^d} - 1 \right) \frac{dt}{t}.$$ 

Proposition 4.5. With the above notation we have that

$$\lim_{n \to \infty} \int_1^{\infty} \det(\Lambda_n)e^{-2dn^2 t} I_0(2n^2 t)^{d-p} \sum_{(k_1, \ldots, k_p) \in \mathbb{Z}^p} \prod_{i=1}^{p} t_{k_i, a_i(n)}(2n^2 t) \frac{dt}{t} = \frac{2 \det(B)}{d-p (4\pi)^{(d-p)/2}}.$$
4.3.1 Example with constants appearing in some particular cases.

4.3 Examples

This finishes the proof of Theorem 1.2.

\[ G_n \]

Let \( \Lambda_n \) be a sequence of diagonal matrices with \( a(n) \rightarrow \alpha \) and \( b(n)/n \rightarrow \beta \) as \( n \rightarrow \infty \). From [5], \( \zeta_G(0) = 2 \log \beta \). In [4] the authors showed that \( c_G = 4G/\pi \) where \( G \) is the Catalan constant. Then as \( n \rightarrow \infty \)

\[ \log \det \Delta_{G^2/\Lambda_n} = na(n)\alpha \beta \frac{4G}{\pi} - \frac{n}{a(n)} \frac{\beta}{3} \pi + 2 \log n + 2 \log \beta + o(1) \]

4.3.2 Example with \( p = 1 \) and \( d = 3 \)

Let \( \Lambda_n = \text{diag}(a_1(n), b_1(n)) \) be a sequence of diagonal matrices with \( a_1(n)/a(n) \rightarrow \alpha \) and \( b_1(n)/n \rightarrow \beta_1 \) for \( i = 1, 2 \) as \( n \rightarrow \infty \). From section 6.3 in [4], we have that

\[ -\zeta_G^{\Lambda_n}(\beta_1, \beta_2)_G(0) = 2 \log(\beta_2 \eta(i\beta_2/\beta_1)^2) \]

where \( \eta \) is the Dedekind eta function defined for \( z \in \mathbb{C} \) with \( \text{Im}(z) > 0 \) by

\[ \eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\piinz}) \]

Hence as \( n \rightarrow \infty \)

\[ \log \det \Delta_{G^2/\Lambda_n} = n^2 a(n) \alpha \beta_1 \beta_2 \zeta(3) + 2 \log n + 2 \log(\beta_2 \eta(i\beta_2/\beta_1)^2) + o(1) \]
Using the special value of \( \eta \) at \( z = i, \eta(i) = \Gamma(1/4)/2\pi^{3/4} \), one has for the special case \( \beta_1 = \beta_2 =: \beta \) the asymptotic behaviour as \( n \to \infty \)

\[
\log \det^{*} \Delta_{Z^n/\Lambda_n Z^{d-1}} = n^{d}a(n)\alpha^2 c_3 - \frac{n^2}{(a(n))^2} \frac{\beta^2}{\alpha^2} \frac{1}{\pi} \zeta(3) + 2 \log n + \log(\Gamma(1/4)^4/16\pi^3\beta^2) + o(1).
\]

### 4.3.3 Example with \( p = 1 \) and any \( d \)

Let \( \Lambda_n = \text{diag}(a_1(n), b_1(n), \ldots, b_{d-1}(n)) \) be a sequence of diagonal matrices with \( a_1(n)/a(n) \to \alpha \) and \( b_i(n)/n \to \beta_i \), for \( i = 1, \ldots, d - 1 \) as \( n \to \infty \). Then as \( n \to \infty \)

\[
\log \det^{*} \Delta_{Z^n/\Lambda_n Z^{d-1}} = n^{d-1}a(n)\det(\Lambda)c_d - \frac{n^{d-1}}{a(n)^{d-1}} \frac{\beta_1 \ldots \beta_{d-1}}{\alpha} \frac{2}{\pi d/2} \Gamma(d/2)\zeta(d)
+ 2 \log n - \zeta_{\mathbb{R}^d-1/\text{diag}(\beta_1, \ldots, \beta_{d-1})Z^{d-1}}(0) + o(1).
\]

### 4.3.4 Example with \( p = d - 1 \)

Let \( \Lambda_n = \text{diag}(a_1(n), \ldots, a_{d-1}(n), b(n)) \) be a sequence of diagonal matrices with \( a_i(n)/a(n) \to \alpha_i \) for \( i = 1, \ldots, d - 1 \), and \( b(n)/n \to \beta \) as \( n \to \infty \). Then as \( n \to \infty \)

\[
\log \det^{*} \Delta_{Z^n/\Lambda_n Z^{d-1}} = na(n)^{d-1}\det(\Lambda)c_d - \frac{n}{a(n)} \det(\Lambda)(4\pi)^{d/2} \Gamma(d/2)\zeta_{\mathbb{R}^d-1/A^{-1}Z^{d-1}}(d/2)
+ 2 \log n + 2 \log \beta + o(1).
\]

### 4.3.5 Example with \( a_i(n) \) constant for \( i = 1, \ldots, p \)

Let \( \Lambda_n = \text{diag}(c_1, \ldots, c_p, b_1(n), \ldots, b_{d-p}(n)) \) be a sequence of diagonal matrices where \( c_i, i = 1, \ldots, p, \) are constants and \( b_i(n)/n \to \beta_i \) for \( i = 1, \ldots, d - p \) as \( n \to \infty \). The leading term is given by

\[
\det\Lambda_n \mathcal{I}_d^{(c_i)}_{i=1} = \det\Lambda_n \int_0^{\infty} \left( e^{-t} - e^{-2dt} I_0(2t)^{d-p} \sum_{(k_1, \ldots, k_p) \in \mathbb{Z}^p} \prod_{i=1}^p I_{k_i, c_i}(2t) \right) \frac{dt}{t}.
\]

From the theta inversion formula \(^{[2]}\) we have for \( i = 1, \ldots, p \)

\[
c_i e^{-2t} \sum_{k_i \in \mathbb{Z}} I_{k_i, c_i}(2t) = \sum_{j=0}^{\alpha-1} e^{-2(1-\cos(2\pi j/c_i))t}.
\]

Let

\[
\{ \lambda_j \}_j = \{ 2p - 2 \sum_{i=1}^p \cos(2\pi j/c_i) : j = 0, 1, \ldots, c_i - 1, \text{ for } i = 1, \ldots, p \},
\]

with \( j = 0, 1, \ldots, \prod_{i=1}^p c_i - 1 \), be the eigenvalues of the Laplacian on \( \mathbb{Z}^p/\text{diag}(c_1, \ldots, c_p)\mathbb{Z}^p \).

Hence

\[
\det\Lambda_n \mathcal{I}_d^{(c_i)}_{i=1} = \prod_{i=1}^{d-p} b_i(n) \prod_{j=0}^{\prod_{i=1}^p c_i - 1} \int_0^{\infty} \left( e^{-t} - I_0(2t)^{d-p} e^{-2(\lambda_j)t} \right) \frac{dt}{t}.
\]

It follows that as \( n \to \infty \)

\[
\log \det^{*} \Delta_{Z^n/\Lambda_n Z^{d-1}} = n^{d-p} \prod_{i=1}^{d-p} \beta_i \prod_{j=0}^{\prod_{i=1}^p c_i - 1} \int_0^{\infty} \left( e^{-t} - I_0(2t)^{d-p} e^{-2(\lambda_j)t} \right) \frac{dt}{t}
+ 2 \log n - \zeta_{\mathbb{R}^{d-p}/BZ^{d-p}}(0) + o(1).
\]
4.3.6 Example with $p = d - 1$ and $a_i(n)$ constant for $i = 1, \ldots, d - 1$

Let $p = d - 1$ in the above example, then using Proposition 2.4 one has as $n \to \infty$

$$\log \det^* \Delta_{\mathbb{Z}^d/\Lambda_n, \mathbb{Z}^d} = n\beta \prod_{j=0}^{d-1} \arg \cosh \left( 1 + \frac{\lambda_j}{2} \right) + 2 \log n + 2 \log \beta + o(1).$$

5 A comment on circulant graphs with non-fixed generators

In [11, 21] the authors considered circulant graphs with non-fixed generators. In [11] they computed the lead term of the asymptotic number of spanning trees. It is conceivable that the techniques used here could be extended to improve their result and compute the second term. In [21] they computed the exact number of spanning trees in $C_{\beta n}^1$ for $\beta \in \{2, 3, 4, 6, 12\}$ via Chebyshev polynomials, but were not able to generalize to other values of $\beta$. We propose a conjecture for the case $\beta = 5$:

For all $n \geq 2$,

$$\tau(C_{\beta n}^1) = \frac{n}{5} \left( \frac{9 - \sqrt{5} + \sqrt{70 - 18\sqrt{5}}}{4} \right)^n + \left( \frac{9 - \sqrt{5} + \sqrt{70 - 18\sqrt{5}}}{4} \right)^{n-1} + \frac{1 - \sqrt{5}}{2} \right)^2$$

$$\times \left( \frac{9 + \sqrt{5} + \sqrt{70 + 18\sqrt{5}}}{4} \right)^n + \frac{1 + \sqrt{5}}{2} \right)^2.\right.$$  

Notice that the coefficients in the formula can be expressed in terms of integrals involving modified $I$-Bessel function. Indeed, let

$$J^\beta_k = \int_0^\infty \left( e^{-t} - e^{-2t(2-\cos(2\pi k/\beta))} I_0(2t) \right) \frac{dt}{t}, \quad k = 1, \ldots, \beta - 1.$$  

Then from Proposition 2.4 the above can be rewritten as

$$\tau(C_{\beta n}^1) = \frac{n}{5} \left( e^{nJ^\beta_1} + e^{-nJ^\beta_1} + \frac{1}{2} (1 - \sqrt{5}) \right) \left( e^{nJ^\beta_2} + e^{-nJ^\beta_2} + \frac{1}{2} (1 + \sqrt{5}) \right)$$

$$\times \left( e^{nJ^\beta_3} + e^{-nJ^\beta_3} + \frac{1}{2} (1 + \sqrt{5}) \right) \left( e^{nJ^\beta_4} + e^{-nJ^\beta_4} + \frac{1}{2} (1 - \sqrt{5}) \right).$$

Therefore for other values of $\beta$ the general formula might have the form

$$\tau(C_{\beta n}^1) = \frac{n}{\beta} \prod_{k=1}^{\beta-1} \left( e^{nJ^\beta_k} + e^{-nJ^\beta_k} + \alpha_k^\beta \right), \text{ for all } n \geq 1,$$

where $\alpha_k^\beta$ are coefficients which are not known for $\beta \geq 7$.

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