Integration of adaptive control and reinforcement learning approaches for real-time control and learning

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Abstract

This paper considers the problem of real-time control and learning in dynamic systems subjected to parametric uncertainties and proposes a controller that combines Adaptive Control (AC) in the inner loop and a Reinforcement Learning (RL) based policy in the outer loop. Two classes of nonlinear dynamic systems are considered, both of which are control-affine. The first class of dynamic systems utilizes equilibrium points with expansion forms around these points and employs a Lyapunov approach. The second class of nonlinear systems uses contraction theory as the underlying framework. For both classes of systems, the AC-RL controller is shown to lead to online policies that guarantee stability, and leverage accelerated convergence properties using a high-order tuner. Additionally, for the second class of systems, the AC-RL controller is shown to lead to parameter learning with persistent excitation. Numerical validations of all algorithms are carried out using a quadrotor landing task on a moving platform and other academic examples. All results clearly point out the advantage of the proposed integrative AC-RL approach.

1 Introduction

This paper deals with two goals in a dynamic system that is subject to uncertainties, the first of which is real-time control and the second is to learn the uncertainties. The class of uncertainties considered is parametric in nature. Methods for realizing these goals abound in the controls community, examples of which include adaptive control [1–7], robust control [8], model-predictive control [9,10], and sliding-mode control [11]. A more recent entrant into this area is reinforcement learning (RL), a subfield of Machine Learning (ML), which has been proposed for the development of control policies for complex systems and environments [12–16]. The second goal has also been addressed by the control community in the case when the uncertainties are parametric in nature [17,18] with a focus on deriving accurate estimates of parameters, states, and outputs using underlying dynamic models.

The field of adaptive control (AC) has always included as a central element in its design a parametric learning component. With the starting point as the control of a dynamic

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system whose uncertainties are parametric, the adaptive control solution has consisted of
a control input that explicitly includes a parameter estimate which is recursively adjusted
using all available data in real-time so as to converge to its true value. The overall goals
of the adaptive system are to guarantee that the control input results in the closed-loop
adaptive system to have globally bounded solutions and to learn the true parameter. With
adaptation to parametric uncertainties as the key architecture of the controller, robustness
to nonparametric uncertainties such as disturbances and unmodeled dynamics have been
 ensured through regularization and persistent excitation \[20,21\]. The field of model-free RL,
on the other hand, has as its goal the determination of a sequence of inputs that drives a
dynamical system to minimize a suitable objective with minimal knowledge of the system
model. The central structure of the RL solution revolves around a policy that is learned so
as to maximize a desired reward \[12,15\]. This policy is often parameterized in the form of a
neural network, and an offline recursive update of the weights that promotes convergence is
then addressed \[13,16\].

Both AC and RL-based control methods have addressed the problem of control in the
presence of uncertainty, with the component of learning addressed explicitly in both. The
two however have deployed entirely different approaches for accomplishing this objective.
AC methods have been proven to be effective in a “zero-shot” enforcement of objectives for
specific classes of problems, such as control and learning, in real-time \[2,12,22,23\]. These
adaptive techniques are able to accommodate, in real-time, parametric uncertainties and
constraints on the control input magnitude \[24,25\] and rate \[26\]. Despite these abilities
to accommodate the presence of modeling errors over short time-scales and meet tracking
objectives, AC methods are unable to directly guarantee the realization of long-term
optimality-based objectives. RL-trained policies, on the other hand, can handle a broad
range of objectives \[27\], where the control policies are often learned in simulation. Training
in simulation is a powerful technique, allowing for a near infinite number of agent-environment
interactions to allow the policy to become near-optimal. In practice, however, offline policies
trained in simulation often exhibit degenerate performance when used for real-time control
due to modeling errors that can occur online \[28,29\]. It may be difficult to reliably predict the
behavior of a learned policy when it is applied to an environment different from the one seen
during training \[30–32\]. The contribution of the paper is an integration of the AC and RL
approaches so as to bridge this “sim-to-real” gap by realizing a combined set of advantages
of both approaches.

Several RL approaches have been suggested to deal with the sim-to-real gap. In particular,
the approaches used in domain randomization (DR-RL) \[33–37\], and meta-learning based RL
(ME-RL) \[38\] should be noted. In domain randomization, the simulated training environment
is perturbed throughout training. This leads to an RL-trained policy that is robust to
perturbations that lie within the training distribution \[33\]. DR has seen broad success in
applications to RL tasks \[34\], and can be used in conjunction with other methods - such as
iteration-based system identification. One issue with DR is that the test environment is
expected to fall within the test environment distribution. To relax this requirement, in \[35\],
the distribution of training environments is tweaked and improved whenever real world data
is collected. Robust RL has also been utilized to accommodate adversarial perturbations and
disturbances \[36,37\]. In ME-RL \[38,39\], a meta-learning based adaptive algorithm allows
for a refinement of the policy to occur online. This adaptation algorithm is often based on
a neural-network, and thus raises more issues of stability \[40\] and generalizability \[41\]. In
contrast, the approach we propose in this paper combines AC and RL and allows for provable
guarantees on adaptation quality, rates, and bounds, at the cost of being applied to a more constrained class of environments.

The AC-RL approach that we propose in this paper consists of AC-based components in the inner-loop and RL-based ones in the outer-loop. The role of RL is to train through simulation the optimal control needed to minimize a desired objective, where the simulation is assumed to have access to a reference system that is the best plant-model available. The role of AC is to accommodate the effect of parametric uncertainties through a suitably designed control input with nonlinear adaptive laws for adjusting its parameters in real-time. The proposed approach is a combination of these two methods such that in real-time the inner-loop AC contracts the closed-loop dynamics towards the reference system, and as the contraction takes hold, the RL in the outerloop directs the overall system towards optimal performance. These properties are guaranteed formally in the paper, and form one of two main contributions of the paper.

Two classes of nonlinear dynamic systems are considered, both of which are control-affine, and denoted as Problem 1 and Problem 2. The first class of dynamic systems utilizes equilibrium points and expansion forms around these points and employs a Lyapunov approach to establish convergence. The second class of nonlinear systems does not leverage equilibrium points or Lyapunov theory but uses contraction theory as the underlying framework \[42\]. In each class, an AC-RL controller is proposed, and the resulting closed-loop system is proved to be stable when certain parametric uncertainties are present. In all cases, the states of the dynamic system are assumed to be accessible for measurement.

For the first class of systems, the performance of the AC-RL is guaranteed through closed-loop stability and a constant regret \[43, 44\], defined as the difference in an integral control performance between the controller employing a given algorithm and the best controller given full knowledge of the plant. All validations and comparisons are carried out using a numerical experiment of a quadrotor that is required to land on a moving platform, with medium-fidelity models that include realistic mechanisms of nonlinear kinematics, actuator nonlinearities, and measurement noise. The demonstration of real-time control of the AC-RL approach for these classes of dynamic systems through closed-loop stability and constant regret is the first contribution of this paper. Extension to a class of non-affine dynamic systems is also presented.

In the second class of dynamic systems, denoted as Problem 2, we introduce various parametric uncertainties and employ contraction theory to establish convergence which utilizes tools in Riemannian geometry to analyze the convergence of differential dynamics by constructing differential Lyapunov functions \[42, 45\]. In contrast to prior work \[46\], we introduce multiplicative parametric uncertainties in the control input, employ a high-order tuner (HT) rather than a gradient-descent based adjustment for adaptation, and most importantly, an RL-based outer-loop to synthesize the reference system. These three important distinctions enable us to accommodate real-time uncertainties in control effectiveness, fast acceleration, and general unstructured cost functions.

The central component of the AC algorithm is built on a high-order tuner which was first proposed by \[47\] in an effort to develop stable low-order adaptive controllers. Rather than utilizing a gradient-descent algorithm to directly generate a first-order tuner for determining the parameter estimates, the idea here is to use a higher-order tuner so as to allow the controller to implicitly generate a reference model that is strictly positive real. This idea was subsequently explored further in \[48\] for general adaptive control designs and in \[49\] to allow adaptive control of time-delay systems in a stable manner. Independently, higher-order
tuners have also been sought after in the ML community, in an effort to obtain accelerated convergence of an underlying cost function and the associated accelerated learning of the minimizer of this function (see for example, [50–51]). The idea here is to include momentum-based updates so as to get a faster convergence of the performance error and have seen widespread applications in machine learning [55–56]. Elements of the HT algorithms in [47] were utilized to develop several types of HTs in [57] in continuous-time leading to stability and in discrete-time in [58] leading to stability with time-varying regressors and accelerated convergence with constant regressors as in [50]. We leverage these stability and accelerated convergence properties in the AC-RL control design in this paper by fully integrating the properties of RL into the controller.

The second contribution of this paper is the demonstration of parameter learning in uncertain dynamic systems. We focus our attention primarily on the first class of dynamic systems with a scalar input. We show that under conditions of persistent excitation [1], the AC-RL controller with the high-order tuner guarantees that the parameter estimates converge to their true values thereby allowing parameter learning. As the high-order tuner inserts additional filtering actions, demonstrating parameter convergence is a nontrivial result, and represents the second contribution of this paper. Earlier results related to parameter convergence with high-order tuners can be found in [58], which require additional processing and time-scale transformations on the underlying regressors. The results in this paper avoid these transformations and directly leverage properties of persistent excitation of the exogenous signals.

Efforts to combine AC and RL approaches have been highlighted in several recent works, which include [59–62]. Algorithms that combine AC and RL in continuous-time systems can be found in [63–66]. [64] proposes the use of adaptive control for nonlinear systems in a data-driven manner, but requires offline trajectories from a target system and does not address accelerated learning or magnitude saturation. References [65,66] study the linear-quadratic-regulator problem and its adaptive control variants from an optimization and machine learning perspective. In [61] a reinforcement learning approach is used to determine an adaptive controller for an unknown system, while in [62] principles from adaptive control and Lyapunov analysis are used to adjust and train a deep neural network. These approaches, including our earlier work in [63], have not addressed a comprehensive treatment of an integrated AC-RL approach. The results in [63] are restricted to a specific class of nonlinear systems that is a subset of those considered in Problem 1. While the use of an AC in the inner-loop and RL in the outer-loop is a common feature, all other elements including high-order tuner, magnitude saturation, non-affine systems, and contraction-theory based adaptive control design have been considered in this paper for the first time. None of these papers have addressed the problem of parameter convergence using the AC-RL controller and forms a key contribution of this paper.

In summary, the contributions of the paper are the following:

- An integral AC-RL approach that combines AC and RL methods with the AC employing a high-order tuner that has the ability to reduce the performance error to zero quickly. This AC-RL approach is guaranteed to lead to a closed-loop system that is stable, for two different classes of control-affine nonlinear systems. Numerical validation is provided for the first class of systems.

- A demonstration of parameter learning in real-time using the AC-RL controller. We focus on the first class of dynamic systems with a scalar input and show that under
conditions of persistent excitation, the AC-RL controller with the high-order tuner guarantees that the parameter estimates converge to their true values.

In Section 2, we describe the problem statement for the two classes of dynamic systems that we consider in this paper. In each case, the state-of-the-art approaches in AC and ML are described. Section 3 lays out the necessary preliminaries including notation, a brief overview of the RL approach, a summary of the AC approach, a description of the HT algorithm, and definitions of persistent excitation. Sections 4 and 5 lay out the AC-RL controller for Problems 1 and 2 respectively. In section 4 both real-time control and learning solutions, the two central contributions of the paper, are presented for Problem 1. Extensions to the case when multiple equilibrium points are present and for a class of nonaffine dynamic systems are also presented in this section. Section 6 includes a summary and concluding remarks. Proofs of Theorems 1, 4-7 can be found in the Appendix and [67]. Proofs of Theorems 2 and 3 can be found in [45] and [68] respectively.

2 Problem Statement

We consider two classes of problems in this paper, for which we propose control and learning solutions in real-time.

2.1 Problem 1: Real-time control and learning for systems with equilibrium points

Consider a continuous-time, deterministic nonlinear system described by the following dynamics:

\[ \dot{X}(t) = F(X(t), U(t)) \] (1)

where \( X(t) \in \mathbb{R}^n \), and \( U(t) \in \mathbb{R}^m \). We define \( x = X - X_0 \) and \( u = U - U_0 \), where \((X_0, U_0)\) is an equilibrium point\(^1\), i.e., \( F(X_0, U_0) = 0 \). Using a Taylor series expansion on (1) yields

\[ \dot{x} = Ax + Bu + f(x, u) \] (2)

where \( f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \), and \((A, B)\) is controllable. It should be noted that (1) can always be written in the form of (2) for any analytic function \( F \). The following assumption is made about \( f \):

**Assumption 1.** The higher order effects represented by the nonlinearity \( f(x, u) \) in (2) (a) lies in the span of \( B \) and (b) are solely a function of the state \( x \), i.e. the system in (2) is control affine and \( g(x) \) exists such that \( Bg(x) = f(x, u) \):

\[ \dot{x} = Ax + B[u + g(x)] \] (3)

The goal in Problem 1 is the determination of the control input \( u(t) \) in real-time, when parametric uncertainties are present in the system dynamics in (3), so as to minimize the cost function

\[ \min_{u(t) \in \mathcal{U}, \forall t \in [0, T]} \int_0^T c(x(t), u(t), t) dt \] (4)

subject to the dynamics in (3), where \( \mathcal{U} \) represents the action space and \( c \) is a bounded cost function \([16]\).

\(^1\)In most of what follows, we suppress the argument \( t \) for ease of exposition.
2.2 Problem 2: Real-time control for a class of nonlinear systems

Problem 2 corresponds to a class of control-affine nonlinear systems with parametric uncertainties which does not necessarily utilize the presence of an equilibrium point. This class is assumed to be of the form

\[ \dot{X}(t) = f(X, t) + B(X, t)[U(t)] \]  

(5)

where \( X(t) \in \mathbb{R}^n \) and \( U(t) \in \mathbb{R}^m \). We note that (5) differs from the expansion form of (2) and therefore allows both \( f(X, t) \) and \( B(X, t) \) to be nonlinear. The goal once again is to determine \( U(t) \) in real-time which minimizes a cost function of \( X(t) \) and \( U(t) \), in the same form as (4), when parameter uncertainties are present in \( f(X, t) \) and \( U(t) \).

3 Underlying Tools

3.1 An Offline Approach Based on Reinforcement Learning

Control policies are determined using RL through offline training in simulation, which implies that the system in (2) needs to be fully known and accessible by the simulator during the offline training. We rewrite (2) as

\[ \dot{x}_r = Ax_r + Bu_r + f_r(x_r, u_r) \]  

(6)

where the subscript \( r \) is used to denote signals and parameters belonging to the reference system. We briefly describe the offline training procedure: First, it is assumed that the continuous-time dynamics in (6) are sampled with sufficient accuracy, resulting in the discrete time dynamics:

\[ x_{r,k+1} = h(x_{r,k}, u_{r,k}) \]  

(7)

An appropriate numerical integration scheme ensures that this discrete-time formulation closely approximates the dynamics. When training in simulation the RL process begins with the construction of a simulation environment defined by (7). The reference system in (7) is used to collect data with which RL learns a feedback policy \( u_{r,k} = \pi(x_{r,k}) \) with repeated training of \( \pi(\cdot) \) so as to achieve the control objective of (4) as follows.

At each timestep, an observation \( x_{r,k} \) is received, a control \( u_{r,k} \) is chosen, and the resulting cost \( c_k = c(x_{r,k}, u_{r,k}) \) is received. Repeating this process, a set of input-state-cost tuples \( \mathcal{D} = [(x_{r,1}, u_{r,1}, c_1), \ldots, (x_{r,N}, u_{r,N}, c_N)] \) is formed. This data is used to train and update the policy \( \pi \) [69]. In many RL algorithms the policy is parametric, often using neural networks, so that \( u_k = \pi_\theta(x_k) \). The learning algorithm then seeks to adjust \( \theta \) so that the expected accumulated cost is minimized [15, 70], i.e

\[ \min_\theta J(\theta) = \mathbb{E}_{\pi_\theta} \left[ \sum_{k=0}^{T} c_k \right] \]  

(8)

In this paper, we use the Proximal Policy Optimization (PPO) RL algorithm [71] to train the policy \( \pi \) which constitutes the RL portion of the proposed AC-RL controller.
3.1.1 DR-RL

When using domain randomization the discrete time dynamics (7) may be rewritten as:

\[ x_{r,k+1} \sim h(x_{r,k}, u_{r,k}, \omega_k) \quad \omega_k \sim \mathcal{P} \]

where \( \omega \) represents environmental parameters (e.g., pendulum weight in an inverted pendulum environment), and \( \mathcal{P} \) is the distribution over these parameters \[33\]-\[37\]. Depending on application, \( \omega \) may be redrawn at every timestep, or once per episode. Standard RL methods may be directly applied to the altered environment in order to determine a robust policy.

3.1.2 ME-RL

A general meta-learning RL policy can be split into two sub-policies \[38\]-\[39\]: a base learner and an adaptive learner. The base learner constructs a policy \( \pi_\theta \) that generalizes robustly over the distribution of environments. A separate adaptive learner, parametrized as \( \pi_\phi \) is then trained to update the outputs of \( \pi_\theta \) in an online fashion. Colloquially, the adaptive meta-learner is “learning to adapt”. The base learner may first be trained to find \( \theta^* \) (dropping the \( r \) subscripts for brevity):

\[
\theta^* = \arg \min_\theta \mathbb{E} \left[ \sum_{k=0}^{T} c(x_k, u_k) \right] \\
\text{s.t} \quad x_{k+1} = h(x_k, u_k) \\
\quad u_k = \pi_\theta(x_k)
\]

A typical adaptive meta learner is then trained to use state-control histories to determine an adaptive policy:

\[
\phi^* = \arg \min_\phi \mathbb{E} \left[ \sum_{k=0}^{T} c(x_k, u_k) \right] \\
\text{s.t} \quad x_{k+1} = f(x_k, u_k, \omega_k) \\
\quad \omega_k \sim \mathcal{P} \\
\quad \hat{u}_k = \pi_{\theta^*}(x_k) \\
\quad u_k = \pi_{\phi}(\hat{u}_k, x_k, x_{k-1}, u_{k-1}, \ldots, x_0, u_0)
\]

Such a method is somewhat similar to classical adaptive control techniques, with the main distinction that that a deep network parameterizes the adaptive law.

It should be noted that though RL has led to compelling successes, analytical guarantees of convergence to optimal policies have been shown only in simple settings \[16\]-\[72\]-\[74\]. No analytical guarantees are provided when general deep networks are used, except for papers such as \[10\] which use several other correction terms to guarantee stability.

3.2 The Classical Online Approach Based on Adaptive Methods

We start with an error model description of the form

\[
\dot{e}(t) = A_m e(t) + BA \left[ \tilde{\Theta}(t) \Phi(t) \right]
\]
where \( e(t) \in \mathbb{R}^n \) is a performance error that is required to be brought to zero. Tracking error, identification error, or state estimation error are a few examples. \( \tilde{\Theta}(t) \in \mathbb{R}^{m \times l} \) is a matrix of parameter errors that quantify the learning error. If the true parameter in the system dynamics is \( \Theta^* \in \mathbb{R}^{m \times l} \), and it is estimated as \( \hat{\Theta}(t) \) at time \( t \), then \( \tilde{\Theta} = \hat{\Theta} - \Theta^* \). \( A_m \) is a Hurwitz matrix, \( A_m \) and \( B \) are known with \((A_m, B)\) controllable, and \( \Lambda \in \mathbb{R}^{m \times m} \) is an unknown parameter matrix that is positive definite. Finally \( \Phi(t) \in \mathbb{R}^l \) is a vector of regressors that correspond to all real-time information measured or computed from the system dynamics and controllers at time \( t \). Such an error model is ubiquitous in adaptive control of a large class of nonlinear dynamic systems with parametric uncertainties [22] including that in (3). The following theorem summarizes the standard adaptive control result in the literature:

**Theorem 1.** Let \( \Gamma \in \mathbb{R}^{m \times m} \) and \( Q \) be symmetric positive-definite matrices with \( P \) corresponding to the solution to the Lyapunov equation

\[
A_m^T P + PA_m = -Q.
\]

An adaptive law that adjusts the parameter error as

\[
\dot{\tilde{\Theta}} = -\Gamma B^T Pe\Phi^T
\]

guarantees that \( e(t) \) and \( \tilde{\Theta}(t) \) are bounded for any initial conditions \( e(0) \) and \( \tilde{\Theta}(0) \). If in addition \( \Phi(t) \) is bounded for all \( t \), then \( \lim_{t \to \infty} e(t) = 0 \).

In addition to closed-loop stability and tracking performance, we also use Regret to quantify the performance of proposed controllers, defined as in [59], as

\[
\mathcal{R} = \int_0^T e^T(\tau)Qe(\tau) d\tau.
\]

where \( Q \) is a positive definite matrix. It is clear that for the classical AC approach, \( \mathcal{R} = O(1) \), i.e., independent of \( T \). Decaying exponential signals can also be included in [12, 59].

### 3.3 High-order Tuners

The core of the adaptive components proposed in this paper is based on high-order tuners (HT) [17, 49]. The idea of HT is to use a high-order filter in order to generate the parameter estimate rather than the first-order gradient method employed in the classical adaptive controller as in [11]. The motivation for its use in adaptive control has come from the need to develop robust adaptive controllers that are low-order and can accommodate large lags in the system dynamics. These controllers require parameter estimates that are differentiable to an arbitrary order. The contribution in [17] lies in the development of such a HT that guarantees that the closed-loop adaptive system is globally stable. A completely independent direction of research that has also led to HT is due to a particular body of work in ML which has focused on accelerated convergence of an underlying cost function [50–54]. The idea behind HT is briefly summarized below.

Parameter identification in a linear regression problem of the form \( \gamma^*(t) = \theta^T \phi(t) \) where \( \theta^* \in \mathbb{R}^n \) represents an unknown parameter, and \( \phi(t) \in \mathbb{R}^n \) is an underlying regressor that can be measured at each \( t \) can be formulated as a minimization problem \( L_t(\theta) = e^2(t) \) where
\[ e = y - y^* \text{ and } y(t) = \theta^T \phi(t). \] A first-order tuner that can be used to solve the minimization problem is of the form \[ \dot{\theta} = -\gamma \nabla_\theta L_t(\theta). \] In [57] [58], a HT of the form

\[ \ddot{\theta} + \beta \dot{\theta} = -\frac{\gamma \beta}{N_t} \nabla_\theta L_t(\theta) \tag{13} \]

was proposed and shown to correspond to the Lagrangian

\[ L(\theta, \dot{\theta}, t) = e^{\beta(t-t_0)} \left( \frac{1}{2} \| \dot{\theta} \|^2 - \frac{\gamma \beta}{N_t} L_t(\theta) \right). \tag{14} \]

The benefits of the HT in [13] are that (a) it can be guaranteed to be stable even with time-varying regressors for a dynamic error model with a scalar control input [57], and (b) a particular discretization was shown in [58] to lead to an accelerated algorithm which reaches an \(\epsilon\) sub-optimal point in \(O(1/\sqrt{\epsilon})\) iterations for a linear regression-type convex loss function with constant regressors, as compared to the \(O(1/\epsilon)\) guaranteed rate for the standard gradient descent algorithm. As our goal here is to achieve fast real-time control, we employ elements of HT proposed in [57] in the AC part of the control design.

### 3.4 Contraction Theory

Contraction theory provides tools to analyze nonlinear system’s stability beyond attraction to an equilibrium point. Based on results in Riemannian geometry, control contraction metrics [15] have been developed to design a feedback controller that stabilizes a class of control-affine systems to a reference trajectory. A brief review of the salient features of this approach follows.

For an arbitrary pair of points \(X_1, X_2 \in \mathbb{R}^n\), let \(\Gamma(X_1, X_2)\) denote the set of all smooth curves connecting \(X_1\) and \(X_2\), in which each path \(c \in \Gamma(X_1, X_2)\) is parameterized by \(s \in [0, 1]\), i.e. \(c(s) : [0, 1] \to \mathbb{R}^n, c(0) = X_1, c(1) = X_2\). Let \(S_+^n\) be the set of \(n \times n\) positive-definite matrices. A Riemannian metric \(M(X, t) : \mathbb{R}^n \times \mathbb{R} \to S_+^n\) is a function that defines a smoothly varying inner product \(<\cdot, \cdot>\) on the tangent space of \(\mathbb{R}^n\).

A Riemannian metric \(M(X, t)\) is uniformly bounded if there exists \(\alpha_2 \geq \alpha_1 > 0\) such that \[ \alpha_1 I \leq M(X, t) \leq \alpha_2 I. \] For a smooth path \(c(s, t)\), which is a time-varying smooth curve parameterized by \(s\) at any given \(t \geq 0\), its Riemannian length and energy function are defined as follows,

\[ L(c, t) := \int_0^1 \| c_s \|_{c, t} ds, \quad E(c, t) := \int_0^1 \| c_s \|^2_{c, t} ds \]

where \(c_s := \frac{\partial c(s)}{\partial s}\) and \(\| c_s \|_{c, t} = \sqrt{c_s^TM(c(s), t)c_s}\). A smooth curve is regular if \(\frac{\partial c(s)}{\partial s} \neq 0\) for all \(s \in [0, 1]\). By Hopf-Rinow theorem, a smooth regular minimum-length curve (a geodesic) \(\gamma(s)\) exists and connects any pair \((X_1, X_2)\) [15]. Let the minimum Riemannian length of all paths connecting \(X_1\) and \(X_2\) be a Riemannian distance function, i.e. \(d(X_1, X_2, t) := \inf_{c \in \Gamma(X_1, X_2)} L(c, t)\). The Riemannian energy of the geodesic \(\gamma(s)\) satisfies \(d(X_1, X_2, t)^2 = E(\gamma, t) = L(\gamma, t)^2 \leq E(c, t)\) for all \(c \in \Gamma(X_1, X_2)\) [15], where the second equality follows from the fact that geodesic \(\gamma(s)\) has constant speed \(\gamma_\gamma\).

We now consider the dynamic system in [5], let \(b_i\) be the \(i\)th row of \(B(X, t)\) and \(U_i\) be the \(i\)th element of control input vector \(U(t)\). We can write the differential dynamics of [5] as

\[ \dot{\delta}_{x} = A(X, U, t)\delta_{x} + B(X, t)\delta_{U}, \tag{15} \]
where \(A(X,U,t) = \frac{\partial f}{\partial x} + \sum_{i=1}^{m} \frac{\partial g_i}{\partial x} U_i\), and \(\delta_x = \gamma_s\) is a differential element of the geodesic connecting a pair of points on two trajectories \(X_1(t)\) and \(X_2(t)\) in the state space \(\mathbb{R}^n\). This allows us to introduce the control contraction metric.

**Theorem 2.** [4] If there exists a uniformly bounded metric \(M(X,t)\) and \(\lambda > 0\) such that

\[
\delta_X^T M B = 0 \implies \delta_X^T (A^T M + MA + \dot{M} + 2\lambda M) \delta_X \leq 0
\]

holds for all \(\delta_X \neq 0\), then system \((5)\) is universally exponentially stabilizable with rate \(\lambda\), and \(M(X,t)\) is called a control contraction metric.

**Remark 1.** The above theorem states that under the metric \(M(X,t)\), every tangent vector \(\delta_X\) orthogonal to the span of \(B(X,t)\) is naturally contracting with rate \(\lambda\) \([4,3]\). Note that the differential dynamics in \((15)\) is linear time-varying, which reduces the complexity in finding a suitable \(\delta_U\) that stabilizes \((15)\) when \(\delta_X^T M B \neq 0\). By computing path integrals of such a \(\delta_U\), control contraction metrics \(M(X,t)\) allows construction of stabilizing controllers \(U(t)\).

### 3.5 Persistent Excitation and Parameter Learning

**Definition 1** (Persistent Excitation). \([22]\) A bounded function \(\Phi : [t_0, \infty) \to \mathbb{R}^N\) is persistently exciting (PE) if there exists \(T > 0\) and \(\alpha > 0\) such that

\[
\int_{t}^{t+T} \Phi(\tau) \Phi^T(\tau) d\tau \geq \alpha I, \quad \forall t \geq t_0.
\]

The definition of PE in \((16)\) is equivalent to the following:

\[
\frac{1}{T} \int_{t}^{t+T} \left| \Phi(\tau)^T w \right| d\tau \geq \epsilon_0, \forall \text{unit vectors } w \in \mathbb{R}^N, \forall t \geq t_0
\]

for some \(\epsilon_0 > 0\). In what follows, we will utilize an alternate definition, which is equivalent to \((16)\) and \((17)\) if \(\|\dot{\Phi}(t)\|\) is bounded for all \(t\) \([75]\).

**Definition 2.** \(\Phi\) is PE if there exists an \(\epsilon > 0\) a \(t_2\) and a sub-interval \([t_2, t_2 + \delta_0] \subset [t, t + T]\) with

\[
\frac{1}{T} \int_{t_2}^{t_2 + \delta_0} \Phi(\tau)^T w d\tau \geq \epsilon_0, \forall \text{unit vectors } w \in \mathbb{R}^N, \forall t \geq t_0.
\]

The goal of this paper is to find solutions not only for real-time control but also for learning the unknown parameters. Adaptive control systems enable parameter learning by imposing properties of persistent excitation defined in Section 3.5. We briefly summarize the classical result related to this topic, which was first established for continuous-time systems in 1977 in \([68,75]\). The starting point is the same error model as in \((11)\), which contains two errors, \(e(t) \in \mathbb{R}^n\) is a performance error that can be measured, and \(\tilde{\Theta}(t) = \tilde{\Theta}(t) \in \mathbb{R}^{1 \times l}\) is a parameter learning error. As before we assume that \(A_m\) is known and Hurwitz, and \(B\) is known. The following theorem summarizes the result in \([68]\).

**Theorem 3.** The solutions of the error dynamics in \((9)\) together with the adaptive law in \((11)\) lead to \(\lim_{t \to \infty} \tilde{\Theta}(t) = 0\) if the regressor \(\Phi\) satisfies the PE condition in Definition 2.

It should be noted that the PE condition that leads to parameter convergence is stronger than that required for error models that have an algebraic relation between the performance error and the parameter error.
4 Integrated AC-RL solutions for real-time control and learning

We address both Problems 1 and 2 in this section, and propose integrated AC-RL control solutions based on adaptive control (AC) and reinforcement learning (RL) approaches.

4.1 A Restatement of Problem 1

As the problem we will address in this paper considers the effect of parametric uncertainties in (3), we shall denote (3) when there are no parametric uncertainties as the reference system, and express it as

\[ \dot{x}_r = Ax_r + B[u_r + g(x_r)] \]  \hspace{1cm} (19)

We assume that \( u_r \) is designed using RL:

\[ u_r = \pi(x_r, t) \]  \hspace{1cm} (20)

so that for all bounded exogenous inputs, \( u_r \) ensures that \( x_r \) is bounded, and is such that (4) is accomplished. It should be noted that the dependence of the policy \( \pi \) on \( t \) represents the effect of all exogenous inputs needed to accomplish the control objective (e.g., a reference trajectory). It should also be noted that \( \pi(x_r, t) \) in (20) does not necessarily cancel \( g(x_r) \), but accommodates it so that the closed-loop system behaves in a satisfactory manner for the requisite control task. This is quantified in Assumption 2, a desirable property of RL controllers.

**Assumption 2.** RL is used to train a feedback policy \( \pi : x \rightarrow u \) such that for a given positive constant \( R_2 \), a positive constant \( R_1 \) exists such that \( ||x_r(0)|| \leq R_1 \) implies \( ||x_r(t)|| \leq R_2 \) for all \( t \).

4.1.1 Parametric uncertainties

We now introduce the following assumption to address parametric uncertainties.

**Assumption 3.** The higher order term in (3) is parameterized linearly, i.e., \( g(x) = \Theta_{n,r} \Phi_n(x) \) where \( \Theta_{n,r} \in \mathbb{R}^{m \times l} \) and \( \Phi_n(x) \in \mathbb{R}^l \).

Assumption 3 implies that the nonlinearities in (3) can be approximated using \( l \) basis functions. In what follows, we consider two dominant sources of uncertainties, one in the form of control effectiveness, i.e., \( u \) gets perturbed as \( \Lambda u \), and the second as a perturbation in \( g(x) \) from \( \Theta_{n,r} \Phi_n(x) \) to \( \Theta'_{n} \Phi_n(x) \). The plant equation in (3) then becomes

\[ \dot{x} = Ax + B\Lambda \left[ u + \Lambda^{-1}\Theta'_{n} \Phi_n(x) \right] \]  \hspace{1cm} (21)

where

\[ B \in \mathbb{R}^{n \times m}, \Lambda \in \mathbb{R}^{m \times m}, \Theta'_{n} \in \mathbb{R}^{m \times l} \]  \hspace{1cm} (22)

Loss of control effectiveness is ubiquitous in practical problems, due to unforeseen anomalies that may occur in real-time, such as accidents or aging in system components, especially in actuators. Parametric uncertainties in the nonlinearity \( g(x) \) may be due to modeling errors. Together we note that Problem 1 is restated as the control of (21) where \( A, B, \) and \( \Phi_n(x) \) are known, but \( \Lambda \) and \( \Theta'_{n} \) are unknown parameters.
4.2 The AC-RL controller

Suppose that an AC-RL control input is designed as

$$ u = u_r + u_{ad} $$  \hspace{1cm} (23)

whose goal is to ensure that in the presence of the aforementioned perturbations in $\Lambda$ and $\Theta'_n$, the true system state $x$ converges to the reference system state $x_r$. Subtracting (19) from (21), we obtain the error dynamics:

$$ \dot{e} = A_H e + B\Lambda [u_{ad} + (I - \Lambda^{-1})u_r + \Lambda^{-1}\Theta'_n \Phi_n(x) \\
- \Lambda^{-1}(g(x_r) + \Theta_{l,r}e)] $$  \hspace{1cm} (24)

where $e := x - x_r$ and $\Theta_{l,r}$ is such that $A_H := A + B\Theta_{l,r}$ is Hurwitz. The bracketed terms in (24) can be rearranged into:

$$ u_{ad} + u_r - \Lambda^{-1}(u_r + g(x_r) + \Theta_{l,r}e) + \Lambda^{-1}\Theta'_n \Phi_n(x) $$

leading to the compact error dynamics

$$ \dot{e} = A_H e + B\Lambda [u - \Theta \Phi] $$  \hspace{1cm} (25)

where

$$ \Theta := \begin{bmatrix} \Lambda^{-1}, & -\Lambda^{-1}\Theta'_n \end{bmatrix} \quad \Phi := \begin{bmatrix} \Phi_r(u_r, x_r, x) \\
\Phi_n(x) \end{bmatrix} $$  \hspace{1cm} (26)

and

$$ \Phi_r(u_r, x_r, x) = u_r + g(x_r) + \Theta_{l,r}e $$  \hspace{1cm} (27)

In (26), $\Theta \in \mathbb{R}^{m \times (m+l)}$ corresponds to an unknown parameter matrix and $\Phi(t) \in \mathbb{R}^{m+l}$ is the regressor vector used for adaptation. The error equation (25) is central to the development of the AC-RL algorithms derived in the following subsections.

The AC-RL approach is exemplified in the choice of two different regressors in the adaptive control input: $\Phi_r$ in (27) and $\Phi_n$, which are utilized to address the two different sources of parametric uncertainties $\Lambda$ and $\Theta_n$:

(i) The first regressor component, $\Phi_r$, comes predominantly from the closed RL system: $u_r$ and $g(x_r)$ come from the RL policy and reference environment, respectively. Note from (19) that $B[u_r + g(x_r)]$ can be viewed as a variable determined by an oracle; this could potentially be accomplished by monitoring the entire vector field $\dot{x}_r$ and subtracting the linear part $Ax_r$.

(ii) The second regressor accommodates the uncertainty $\Theta'_n$ in the nonlinear component $g(x)$ (and employs assumption 3).

(iii) The additional feedback from $e$ in (27) is essential in guaranteeing global stability.

(iv) In contrast to the standard AC formulation, the AC-RL controller allows the use of a nonlinear reference model as in (19) and a nonlinear controller based on RL as in (20).

We now propose the AC-RL controller.

$$ u = \hat{\Theta}(t)\Phi(t) $$  \hspace{1cm} (28)

$$ \dot{\Xi} = -\gamma B^TPe\Phi^T $$  \hspace{1cm} (29)

$$ \dot{\hat{\Theta}} = -\beta(\hat{\Theta} - \Xi)N_t $$  \hspace{1cm} (30)
where

$$N_t = 1 + \mu \Phi^T \Phi$$

(31)

$$\mu \geq \frac{2\gamma}{\beta \|PB\|_F^2}$$

(32)

$\| \cdot \|_F$ in (32) denotes the Frobenius matrix norm, and $P = P^T \in \mathbb{R}^{n \times n}$ is a positive definite matrix that solves the equation $A^T_H P + PA_H = -Q$, where $Q$ is a positive-definite matrix and $Q \geq 2I$. It should be noted that (29) - (30) is a second-order tuner, with a slightly different Lagrangian when compared to (14), and an extension of our earlier results in [57] to the multivariable case. The AC-RL controller in (28)-(30) can be implemented as in Algorithm 1.

The following theorem presents the stability property of the AC-RL as well as Regret (defined in (12)), which requires the following additional assumption:

**Assumption 4.** $\Lambda$ is symmetric and positive definite, with $\|\Lambda\| \leq 1$.

**Theorem 4.** Under Assumptions 1-4, the closed-loop adaptive system specified by the plant in (21), the reference system in (19), and the adaptive controller in (28) - (30) leads to globally bounded solutions with $\lim_{t \to \infty} e(t) = 0$ with $\mathcal{R} = O(1)$.

It is clear from (28) - (30) that if there are no parametric uncertainties, and if the initial conditions of (21) are identical to those of (19), then the choices of $\hat{\Theta}(0) = [I, \Theta^T_{n,r}]^T$ and $\Xi(0) = 0$ ensures that the AC-RL control $u(t)$ coincides with the RL-input, thereby accomplishing the objective in (14). When there are parametric uncertainties, the control input needs to be modified from (20) as in (28). Theorem 1 guarantees that with such a modification, the closed-loop system state $x$ converges to $x_r$. Additional smoothness conditions on $\Phi$ have to be imposed to show that $\hat{\Theta}$ goes to a constant. Denoting this constant as $\Theta_{ss}$, it follows that the parametric uncertainties are accommodated by the AC-RL policy by replacing $u_r$, which may be in the form of $\Theta_{n,r} \Phi(\cdot)$, by $u_r + u_{ad}$ as in (28), which is of the form $\Theta_{ss} \Phi(\cdot)$.

### 4.3 Validity of the AC-RL controller

The RL and AC components of the AC-RL controller imposed distinctly different requirements on the system (1). The AC controller in (28)- (30) required that the system be expanded as in (2), and imposed Assumptions 1 and 3 to bring (2) into the form (21). The RL controller in (20) required none of these assumptions, but that the reference system be amenable to a simulation experiment offline, with $u_r$ determined through training, as quantified in Assumption 2. We explore the benefits of combining these two methods by evaluating the implications of these assumptions and requirements on the original system (2).

We note that the effect of the approximations due to Assumptions 1 and 3 can be expressed by considering (2) in the presence of uncertainties $\Lambda$ and $\Theta'_n$ given by

$$\dot{x} = Ax + B\Lambda u + f'(x,u)$$

(33)

and expressing the nonlinearity $f'(x,u)$ as:

$$f'(x,u) = B\Theta'_n \Phi(x) + \epsilon(x)$$

(34)
where $\epsilon(x)$ represents the approximation error due to assumptions 1 and 2.

We assume that for a given non-negative bound $M$, basis functions $\Phi(x)$ can be chosen such that

$$||\epsilon(x)|| \leq M \quad \forall x \in S(M)$$

where $S(M)$ is a compact set in $\mathbb{R}^n$. It can be shown that a compact set $S_0$ exists such that the closed-loop system with the AC-RL controller guarantees that for all $x(0) \in S_0$, $x(t) \in S(M)$. This is established via the following: We start with a bound

$$||\epsilon(0)|| \leq X_0 + R_1$$

where $X_0 = \max_{x \in S_0} ||x||$ with $R_1, R_2$ given by Assumption 2. From the Lyapunov function, we have

$$||\epsilon(t)|| \leq (k_0 + X_0 + R_1)\frac{\sigma_{\text{max}}(P)}{\sigma_{\text{min}}(P)}$$

where $k_0$ is the maximum bound on the parameter estimates. Finally, denoting $E_0 = (k_0 + X_0 + R_1)\frac{\sigma_{\text{max}}(P)}{\sigma_{\text{min}}(P)}$, we have

$$||x(t)|| \leq E_0 + R_2 \forall t \geq t_0$$

The bound $k_0$ can be shown to exist with a projection operator added to the HT laws using tools in [58, 76]. Let $R_M = \max_{x \in S(M)} ||x||$. If $E_0 + R_2 \leq R_M$, there exists a compact set $S_0$ such that the closed-loop AC-RL system solutions $x(t)$ for the original system in (2) are guaranteed to lie inside $S(M)$. Although Assumption 2 guarantees that $||x_r(t)|| \leq R_2$ it provides no such statement about the states $x(t)$ in (34). Therefore, if RL is applied without AC one cannot make any claim on the boundedness of $x(t)$. In comparison, AC-RL guarantees that $||x(t)|| \leq E_0 + R_2$.

We note that the condition that $E_0 + R_2 \leq R_M$ trivially holds when there is no approximation error, i.e $||\epsilon(x)|| = 0$ which in turn implies that $R_M = \infty$. Therefore AC-RL guarantees the boundedness of $x(t) \forall t \geq t_0$. In comparison, direct application of the RL policy $\pi$ to the target system cannot provide any such guarantee.

In summary, the actual solutions of the closed-loop system with the AC-RL can have a bound that is as large as $E_0 + R_2$, whereas the RL-based controller leads to a solution that is less than $R_2$. Therefore it is clear that the addition of the AC in the inner-loop allows an online policy that accommodates a larger compact set.

4.4 MSAC-RL Controller

As control inputs are often subject to magnitude saturation, we propose another AC-RL controller, MSAC-RL, which builds on the ideas introduced in [24]. The saturated control input into the true plant is calculated as:

$$u_i(t) = u_{i,\text{max}} \text{sat} \left( \frac{u_{i,c}(t)}{u_{i,\text{max}}} \right)$$

where $u_c(t)$ denotes the output of the controller and $u_{i,\text{max}}$ is the allowable magnitude limit on $u_i$. This induces a saturation-triggered disturbance $\Delta u$ vector defined by

$$\Delta u(t) = u_c(t) - u(t)$$
It is easy to see that $\Delta u(t) = 0$ when the desired control $u_c(t)$ does not saturate. The output of the controller, $u_c$, is given by:

$$u_c = \hat{\Theta}(t)\Phi(t) \tag{41}$$

The presence of the disturbance $\Delta u$ causes the error equation to vary from (25) to

$$\dot{e} = A_H e + B\Lambda[u_c - \Delta u - \Theta\Phi] \tag{42}$$

We introduce a new performance error $e_a$ in order to accommodate the disturbance $\Delta u$ as follows:

$$\dot{e}_a = A_H e_a + BK_a(t)\Delta u \tag{43}$$

which leads to a new augmented error $e_u = e - e_a$:

$$\dot{e}_u = A_H e_u + B\Lambda\hat{\Theta}(t)\Phi(t) + B(\Lambda - K_a^T)\Delta u \tag{44}$$

This suggests a different set of adaptive laws,

$$\dot{\Xi} = -\gamma B^T P e_u \Phi^T \tag{45}$$

$$\dot{\Theta} = -\beta(\Theta - \Xi)\Lambda^T, \tag{46}$$

where $\Theta = [\hat{\Theta}^T, -K_a^T]^T$ and $\Phi = [\Phi^T, \Delta u^T]^T$, $\gamma$ and $\beta$ are positive constants, and $\Lambda^T$ defined as in (31) with $\Phi$ replaced by $\Phi$ and $\mu$ as in (32). The following theorem provides the analytical guarantees for this MSAC-RL controller.

**Theorem 5.** Under Assumptions 1-4, the closed-loop adaptive system specified by the plant in (21), the reference system in (19), the magnitude constraint in (39) and the MSAC-RL controller given by (41) and (45) leads to

(i) globally bounded solutions, with $\lim_{t \to \infty} \|e_u(t)\| = 0$ if the target system in (21) is open-loop stable.

(ii) bounded solutions for all initial conditions $x(0), \Xi(0), \hat{\Theta}(0)$ and $K_a(0)$ in a bounded domain, with $\|e(t)\| = O[\int_0^t \|\Delta u(\tau)\|d\tau]$ if the target system in (19) is not open-loop stable.

It is clear that the performance guarantees of the closed-loop system in Theorems 4-5 approximate the online control goal stated in (1) for the case when the cost function does not depend on the control input, as $x(t)$ approaches $x_r(t)$ and the choice of $u_r(t)$ optimizes the behavior of the reference system. Any dependence on the control input is implicitly addressed in MSAC-RL by accommodating magnitude saturation. More remains to be done in bridging this optimization gap, a topic for future research.

### 4.5 Extension to multiple equilibrium points

Suppose the system in (11) has $p$ equilibrium points $(X_1, U_1), \ldots, (X_p, U_p)$, so that $F(X_i, U_i) = 0$ for $i = 1, \ldots, p$. Define $x_i = x - X_i$, $u_i = u - U_i$. Denoting the state and action spaces as $X$ and $U$, respectively, partition the composite domain $\mathcal{D} = X \times U$ into $p$ disjoint subsets $S_1, \ldots, S_p$, such that:

$$\mathcal{D} = S_1 \cup S_2 \cup \ldots S_p$$

$$S_i \cap S_j = \emptyset, i \neq j$$
One can then express (1) in the presence of parametric uncertainties as

\[
\dot{x} = \begin{cases} 
A_1 x_1 + B_1 [\Lambda u_1 + g(x_1)], & x_1, u_1 \in S_1 \\
\vdots \\
A_p x_p + B_p [\Lambda u_p + g(x_p)], & x_p, u_p \in S_p
\end{cases}
\]

(47)

If we now consider the effect of parametric uncertainties in \(u_i\) and \(g(x_i)\) as in section III.A, it is easy to have a switching set of controllers as in (28)-(30) that are invoked when the trajectories enter the set \(S_i\). A corresponding result to Theorem 4 can be derived with stability guaranteed provided the command signal is such that the dwell time in each set \(S_i\) exceeded a certain threshold, which is not discussed here due to lack of space. We summarize the overall AC-RL controller in Algorithm 1 which specifies the discrete time implementation of the overall AC-RL Controller. \(\Delta t\) denotes the integration timestep and the AdaptiveControl function corresponds to the adaptive control input; in the single equilibrium case, this function corresponds to equations (28)-(32).

**Algorithm 1** Multiple Equilibrium AC-RL

Require: \(F, F_r, \pi, x_r, x\)

while running do

- \(u_r = \pi(x_r)\)
- \(x_i \leftarrow x - X_i\)
- \(u_i \leftarrow u - U_i\)
- \(e \leftarrow x_i - (x_r - X_i)\)
- \(\Phi \leftarrow [x_i, u_i]\)
- \(u \leftarrow \text{AdaptiveControl}(\Phi, e, \Theta_i, P_i, B_i) + U_i\)
- \(x_r \leftarrow x_r + F_r(x_r, u_r) \Delta t\)
- \(x \leftarrow x + F(x, u) \Delta t\)
- \(i \leftarrow j : x, u \in S_j\)

end while

4.6 Learning in AC-RL controllers with persistent excitation

Sections IV-A through IV-E focused on the control solution, i.e. the solution to the problem in (1). The AC-RL controller in (28)-(30) guaranteed that the resulting solution \(x(t)\) of the closed loop system converged to the nominal reference solution \(x_r(t)\) with the AC-RL policy \(u(t)\) converging to the RL policy \(u_r(t)\). In this section, we return to the learning problem, i.e. the conditions under which the parameter estimate \(\hat{\Theta}\) converges to the true parameter \(\Theta\) in Problem 1 with the AC-RL algorithm. We limit our discussion to the case when \(u(t)\) is a scalar. A few remarks follow regarding its extension to the multi-input case, which falls outside the scope of this paper.

The starting point is the error equation in (25) and the AC-RL control input in (28). We note that with a scalar input, \(m = 1\), which leads to \(\Lambda \in \mathbb{R}^+\) (using Assumption 1), \(\Theta \in \mathbb{R}^{1 \times (l+1)}\), \(\Phi_r(t) \in \mathbb{R}\), and \(\Phi_n(t) \in \mathbb{R}^l\) in (26). Similar to Assumption 1 we assume that \(0 < \Lambda < 1\). This in turn leads to the error equation

\[
\dot{e} = A_H e + B \Lambda [(\hat{\Theta} - \Theta) \Phi] 
\]

(48)
and the adaptive laws (29)-(30). The following theorem establishes the conditions under which \( \lim_{t \to \infty} \tilde{\Theta}(t) = 0 \).

Defining \( \tilde{\theta} = (\tilde{\Theta} - \Theta)^T, \tilde{\vartheta} = (\Xi - \Theta)^T \) and \( B_0 = BA \), we rewrite (48), (22), and (30) as

\[
\dot{e} = A_H e + B_0 \tilde{\theta}^T \varphi
\]

and

\[
\dot{\vartheta} = -\gamma \varphi e^T P B
\]

\[
\dot{\theta} = -\beta(\tilde{\theta} - \vartheta) \mathcal{N}_i
\]

where \( \mathcal{N}_i = 1 + \mu \Phi^T \Phi, \mu \geq 2\gamma \|PB\|^2 / \beta \). As before, the matrix \( P \) solves \( A_H^T P + PA_H = -Q \) and \( Q \geq 2I \) is a positive-definite matrix. The constants \( \gamma \) and \( \beta \) are positive. Let \( x_1(t) = [e(t)^T, (\vartheta(t) - \tilde{\vartheta}(t))^T]^T \) and \( z(t) = [x_1(t)^T, \tilde{\vartheta}(t)^T]^T \).

We note that the stability result related to the AC-RL controller stated in Theorem 4 guarantees that \( z(t) \) is uniformly bounded, which follows from a Lyapunov function of the form

\[
V = \frac{\Lambda}{\gamma} \tilde{\vartheta}^T \tilde{\vartheta} + \frac{\Lambda}{\gamma} [(\tilde{\vartheta} - \vartheta)^T(\tilde{\vartheta} - \vartheta)] + e^T P e
\]

whose derivative is given by

\[
\dot{V} \leq -\frac{2\beta \Lambda}{\gamma} \|\tilde{\vartheta} - \vartheta\|^2 - \Lambda \|e\|^2
\]

Noting that our goal is to show that \( \lim_{t \to \infty} \tilde{\theta}(t) = 0 \), it follows that our goal is to show that \( V(t) \to 0 \) as \( t \to \infty \). As the time-derivative \( \dot{V} \) in (53) is negative-definite only in \( x_1 \), the goal cannot be accomplished in a straightforward manner. We show below that this is indeed possible under conditions of persistent excitation, and corresponds to the second contribution of this paper, which is parameter learning using the AC-RL controller.

The following lemmas are useful in proving the main result, which is stated in Theorem 6. We note that Theorem 4 guarantees that \( z(t) \) and therefore \( \Phi(x, e, u_r) \) is bounded, making the properties of persistent excitation applicable for what follows.

**Lemma 1.** Let \( \epsilon_1 > \epsilon_2 > 0 \), then there is an \( n = n(\epsilon_1, \epsilon_2) \) such that if \( z(t) = [x_1(t)^T, \tilde{\vartheta}(t)^T]^T \) is a solution with \( \|z(t_1)\| \leq \epsilon_1 \) and \( S = \{ t \in [t_1, \infty) | \|x_1(t)\| > \epsilon_2 \} \), then \( \mu(S) \leq n \) where \( \mu \) denotes Lebesgue measure.

**Lemma 2.** Let \( \delta > 0 \) and \( \epsilon_1 > 0 \) be given. Then there exist positive numbers \( \epsilon \) and \( T \) such that if \( z(t) \) is a solution with \( \|z(t_1)\| \leq \epsilon_1 \) and if \( \|\tilde{\vartheta}(t)\| \geq \delta \) for \( t \in [t_1, t_1 + T] \), then there exists a \( t_2 \in [t_1, t_1 + T] \) such that \( \|x_1(t_2)\| \geq \epsilon \).

**Lemma 3.** Let \( \epsilon_1 \) and \( \delta \) be given positive numbers. Then there is a \( T = T(\epsilon_1, \delta) \) such that if \( z(t) \) is a solution and \( \|z(t_1)\| \leq \epsilon_1 \), then there exist some \( t_2 \in [t_1, t_1 + T] \) such that \( \|\tilde{\vartheta}(t_2)\| \geq \delta \).

**Theorem 6.** If \( \Phi(t) \) satisfies the persistent excitation property in (18), then the origin \( (e = 0, \tilde{\vartheta} = 0, \tilde{\theta} = 0) \) in (49)-(51) is uniformly asymptotically stable.
The proof of Theorem 6 stems from the three lemmas listed above, which represent the three main steps. Lemmas 1 and 2 establish that $x_1(t)$ cannot remain small over the entire period of persistent excitation. Lemma 3 then leverages this fact to show that this leads to the parameter error $\hat{\theta}(t)$ to decrease. Together this allows the conclusion of u.a.s. of the origin (49)-(51) and therefore that $\lim_{t \to \infty} \hat{\theta}(t) = 0$. As is apparent from the details of the proof provided in the appendix, first principles based arguments had to be employed in order to derive this result. No standard observability properties or time-scale transformations as in [48] have been employed; these are inadequate as the error model structure in (49) includes system dynamics and no filtering techniques are used to convert the error model to a static linear regression model.

### 4.7 Numerical Validation

We validate the AC-RL controller with the MSAC-RL algorithm described in Section IV-D using a simulated task. The task requires a quadrotor moving in 3-D to land on a moving platform, assuming full-state feedback. A sparse reward function is chosen in which positive reward (negative cost) is attained only when the quadrotor enters a relatively small compact set in the state-space. Negative reward (positive cost) is attained if the quadrotor altitude falls below the platform altitude, or if a termination time of 15 seconds is reached. A reinforcement learning algorithm based on PPO [71] is used to train a control policy using (19). The policy is then applied to a target environment which contains parametric uncertainties (see [67] for details).

We test two types of parametric uncertainties: 1) the mass, length and inertia properties of the quadrotor are varied between $\pm 25\%$ of their nominal (reference) values (Table 1) and 2) an abrupt loss of effectiveness (LOE) in the fourth propeller occurs (Table 2). The LOE diminishes the total thrust producible by the quadrotor, and induces an additional moment on the quadrotor if the LOE is not accounted for. Such a LOE may occur if the propeller blades are broken midflight, as demonstrated in [77].

Both types of parametric uncertainties correspond to the target-system structure as in (21), with the symmetric part of $\Lambda$ being positive definite. Two success criteria, success rate (SR) and success time (ST), are measured. These methods are compared to a straightforward RL approach in which the learned policy $\pi$ is applied directly to the target system so as to maximize the success criteria. The AC-RL method (which uses the PPO-trained policy as RL) is compared to a different PPO policy trained using domain randomization (DR-RL), and a meta-learning RL policy (ME-RL). For further details on the quadrotor model and adaptive control implementation, refer to [63, 67, 77].

| Algorithm | Results |
|-----------|---------|
| **SR**    | **ST**  |
| **RL**    | 48%     | 7.5s    |
| **AC-RL** | 82%     | 3.5s    |
| **DR-RL** | 75%     | 7.1s    |
| **ME-RL** | 88%     | 3.4s    |

Table 1: $\pm 25\%$ parametric uncertainty results
From Table 1, AC-RL performs favorably when compared to pure RL or DR-RL. We note that ME-RL outperforms AC-RL on both metrics. This comes with two qualifiers: 1) the meta-learner in ME-RL is trained using the same distributional shift on which it was tested (the ±25% parameter perturbations) and 2) the meta-learner is a DNN - hence the ME-RL policy does not provide the guarantees of convergence within a compact set that are afforded by AC-RL. The relevance of point 1) can be in the ME-RL results within Table 2 in which ME-RL greatly underperforms AC-RL. This is because the specific type of perturbation (asymmetric LOE) studied in Table 2 was not incorporated into the meta-learner’s training regimen.

Table 2: Results from the simulated quadrotor experiments. The LOE column represents the degree of propeller thrust lost (with 0% being no loss). Success time is measured as the mean time required to complete a task, averaged over all successful tests. For a 75% LOE there is no data on the RL or ME-RL success time because there were no successful tests. No measurement noise was introduced in these experiments. See [67] for results with noise.

| LOE | AC-RL | ME-RL | RL |
|-----|-------|-------|----|
| 0%  | 97%   | 98%   | 97%|
| 10% | 74%   | 41%   | 34%|
| 25% | 14%   | 8%    | 50%|
| 75% | 0%    | 0%    | 75%|

| LOE | AC-RL | ME-RL | RL |
|-----|-------|-------|----|
| 0%  | 2.6s  | 2.8s  | 2.6s|
| 10% | 3.1s  | 4.7s  | 5.4s|
| 25% | 5.4s  | 5.4s  | 5.4s|

4.8 Extensions to a class of nonaffine systems

We once again start with (1), expand the dynamics around \((X_0, U_0)\), which together with Assumption 1 leads to a class of nonlinear systems

\[
\dot{x} = Ax + B[u + h(x, u)]
\]

(54)

Noting that \(u\) is required to minimize the cost functional subject to the constraints specified in (4), we assume that \(u\) can be expressed as an analytic function of the state \(x\), which in turn leads to the assumption that

\[ h(x, u) = g(x, t) \]

(55)

The nonautonomous nature in (55) is due to the cost functional dependent on time which in turn may be due to an objective that may vary with \(t\). The goal is to determine \(u\) in real
time when parametric uncertainties are present in (54)-(55) so that (4) is accomplished for initial condition \(x_0\). Denoting the case when there are no parametric uncertainties as the reference system, we obtain its dynamics to be

\[
\dot{x}_r = Ax_r + B[u_r + g(x_r, t)]
\]

(56)

We proceed to make a similar assumption as in Section 4.1 regarding the higher order term:

**Assumption 5.** The higher order term in (56) is parameterized linearly, i.e., \(g(x) = \Theta_{nl,r} \Phi_{nl}(x, t)\) where \(\Phi_{nl}(x, t) \in \mathbb{R}^l\).

Similar to Assumption 3, Assumption 5 implies that the nonlinearities of the affine nonlinear system (3) are confined to the first \(l\) higher-order terms, with additional nonautonomous components as well due to the nonaffine nature of the dynamics.

As in Section 4.1.1, we assume that the plant dynamics in (54)-(55) are subject to two parametric uncertainties, with \(u\) perturbed as \(\Lambda u\), and \(g(x, t)\) from \(\Theta_{nl,r} \Phi_{nl}(x, t)\) to \(\Theta_{nl} \Phi_{nl}(x, t)\), which causes the plant equation to become

\[
\dot{x} = Ax + B\Lambda[u + \Lambda^{-1}\Theta_{nl}' \Phi_{nl}(x)]
\]

(57)

As the structure of the plant dynamics in (57) is almost identical to that in (21), the development of the AC-RL controller and its control solution is identical to the descriptions above and are summarized in the following section.

### 4.8.1 The AC-RL Controller

The control input \(u\) in (57) is chosen as

\[
u = \hat{\Theta}(t)\Phi(t)
\]

(58)

\[
\dot{\xi} = -\gamma B^T Pe \Phi^T
\]

(59)

\[
\dot{\hat{\Theta}} = -\beta(\hat{\Theta} - \xi)N_t,
\]

(60)

where

\[
\Phi = \begin{bmatrix}
\Phi_r(u_r, x_r, x) \\
\Phi_{nl}(x, t)
\end{bmatrix}
\]

(61)

\[
\Phi_r = u_r + g(x_r, t) + \Theta_{l,r} e
\]

(62)

\[
N_t = 1 + \mu \Phi^T \Phi
\]

(63)

\[
\mu \geq \frac{2\gamma}{\beta} ||PB||_F^2
\]

(64)

The following theorem presents the stability property of the AC-RL controller for a class of nonaffine systems:

**Theorem 7.** Under Assumptions 3, 4, and 5, the closed-loop adaptive system specified by the plant in (57), the reference system in (56), and the adaptive controller in (58)-(60) leads to globally bounded solutions with \(\lim_{t \to \infty} e(t) = 0\) with \(\mathcal{R} = O(1)\).
5 Integrated AC-RL Solutions: Problem 2

We now consider Problem 2, which pertains to the class of dynamic system (4), and introduce parametric uncertainties \( \theta \in \mathbb{R}^{n_\theta} \) and \( \nu \in \mathbb{R}^{n_\nu} \) in \( f(X, t) \) which may be due to changes in the system dynamics, and \( \Lambda \in S^+_m \) in \( U \) which may be due to loss of effectiveness, leading to the dynamic system

\[
\dot{X}(t) = f(X, t) - \rho(X, t)\nu + B(X, t) \left[ \Lambda U(t) - \phi(X, t)\theta \right],
\]

where \( \rho(X, t) \in \mathbb{R}^{n \times n_\nu} \) and \( \phi(X, t) \in \mathbb{R}^{n \times n_\theta} \) are known nonlinear dynamics of the parametric uncertainties \( \theta \) and \( \nu \), respectively. It is clear that \( \phi(X, t)\theta \) denotes a matched uncertainty similar to that introduced by Assumptions 3 and 5, while \( \rho(X, t)\nu \) denotes an unmatched uncertainty. We make the following assumption regarding the latter:

**Assumption 6.** The unmatched uncertainty satisfies \( \rho(X, t)\nu \in \text{span}\{ad_f B\} \), where \( ad_f B = [f, B] = \frac{\partial f}{\partial X} B - \frac{\partial B}{\partial X} f \) denotes the Lie bracket of \( f, B \). \( f(X, t) \) and \( B(X, t) \) are nonlinear functions such that \( B \) and \( ad_f B \) are linearly independent.

For the system in (65) with Assumption 6, we now propose an AC-RL controller. We first consider a control-affine reference system, similar to (5), of the form,

\[
\dot{X}_r(t) = f(X_r, t) + B(X_r, t)U_r(t).
\]

Given an initial condition \( X_0 \), \( U_r(t) \) minimizes the cost function given in (4) where \( x(t) \) and \( u(t) \) are replaced by \( X(t) \) and \( U(t) \), respectively. The reference control input \( U_r(t) \) can be constructed using the offline RL-based approach proposed in Section 3.1. Rather than requiring an equilibrium point \((X_0, U_0)\), or Assumption 4, 3, 5 we introduce the following assumption:

**Assumption 7.** A uniformly bounded parameter-dependent contraction metric \( M(X, \hat{\nu}, t) \) can be computed for each parameter estimate \( \hat{\nu} \in \mathbb{R}^{n_\nu} \).

Under Assumption 7 let \( \gamma(t, s) \) be the geodesic that connects \( X_r(t) \) to \( X(t) \), i.e., \( \gamma(t, 0) = X_r(t) \) and \( \gamma(t, 1) = X(t) \). The Riemannian energy of the geodesic is defined as \( E = \int_0^1 \dot{\gamma}_s^T M(X, \hat{\nu}, t) \gamma_s ds \).

We now describe the AC-RL controller in the equations (67a)–(69) and the stability result. The AC-RL controller is

\[
U = \Psi \left( U_c + U_\nu + \phi(X, t)\hat{\theta} \right),
\]

\[
U_\nu = \sum_{i=1}^{n_\nu} \Omega_i \hat{\nu}_i \int_0^1 \left[ \frac{\partial \rho_i^T(\gamma(s))}{\partial X_1} \ldots \frac{\partial \rho_i^T(\gamma(s))}{\partial X_{n-1}} \right] \gamma_s(s) ds,
\]

\[
\dot{\hat{\nu}} = \beta_\nu (\nu_a - \hat{\nu}) N_\nu,
\]

\[
\dot{\nu}_a = -\gamma_\nu(\rho(X, t)^T M \gamma_s(1)) E,
\]

\[
\dot{\hat{\theta}} = \beta_\theta (\theta_a - \hat{\theta}) N_\theta,
\]

\[
\dot{\theta}_a = -\gamma_\theta(\phi(X, t)^T B(X, t)^T M \gamma_s(1)) E,
\]

\[
\dot{\Psi} = \beta_\Psi (\Psi_a - \Psi) N_\Psi,
\]

\[
\dot{\Psi}_a = -\gamma_\Psi B(X, t)^T M \gamma_s(1) EU^T,
\]
where

\[ N_\nu = 1 + \omega_\nu \omega_\nu, \quad \omega_\nu = \rho(X, t)^T M \gamma_s(1), \quad (68a) \]

\[ N_\theta = 1 + \omega_\theta \omega_\theta, \quad \omega_\theta = \phi(X, t)^T B(X, t)^T M \gamma_s(1), \quad (68b) \]

\[ N_\Psi = 1 + \omega_\Psi \omega_\Psi, \quad \omega_\Psi = U, \quad (68c) \]

\( \rho_i \) is the \( i \)th column of \( \rho(X, t) \), \( \Omega_i \in \mathbb{R}^m \) extracts the \( j \)th column vector of \( B(x, t) \) such that \( \rho_i(x, t) \nu_i \in \text{span}\{ad_j b\} \), \( U_c \) is a nominal controller that satisfies the following condition stated with a positive contraction rate \( \lambda > 0 \),

\[
\begin{align*}
\frac{\partial E}{\partial t} + 2\gamma_s(1)^T M(X, \hat{\nu}, t)(f(X, t) - \rho(X, t)^T \hat{\nu} + B(X, t)U_c) \\
- 2\gamma_s(0)^T M(X_r, \hat{\nu}, t) [f(X_r, t) + B(X_r, t)U_r] \leq -2\lambda E.
\end{align*}
\] (69)

**Theorem 8.** Under Assumption 4, 6, and 7, the closed-loop system specified by the plant in (65), and the adaptive controller in (67)-(69) has bounded solutions and ensures that \( \lim_{t \to \infty} \|X(t) - X_r(t)\| = 0. \)

### 6 Summary and Conclusions

This paper proposes solutions for real-time control and learning in dynamic systems using a combination of adaptive control and reinforcement learning. Two classes of nonlinear dynamic systems are considered, both of which are control-affine. The first class of dynamic systems utilizes equilibrium points and expansion forms around these points and employs a Lyapunov approach. The second class of nonlinear systems uses contraction theory as the underlying framework. For both classes of systems, the AC-RL controller is shown to lead to online policies that guarantee stability, and leverage accelerated convergence properties using a high-order tuner. Additionally, for the first class of systems, the AC-RL controller is shown to lead to parameter learning with persistent excitation. Together, this paper takes a first step towards online machine learning with provable guarantees by drawing upon key insights, tools, and approaches developed in these two disparate and powerful methodologies of adaptive control and reinforcement learning. Several more steps need to be carried out in this direction and address extensions to more complex nonlinear systems with generic cost functions.

### 7 Appendix

#### 7.1 Proof of Theorem 1

A Lyapunov function candidate is chosen as

\[ V(e, \tilde{\Theta}) = e^T Pe + \text{Tr} \left( \tilde{\Theta}^T \left( \Lambda^T S \right) \tilde{\Theta} \right) \] (70)

where \( S = \Gamma^{-1} \) in (11) with the symmetric part of \( \Lambda S \) positive-definite. Using arguments as in [78], we can show that \( \tilde{V} = -e^T Q e \) by using (9) and (11). Thus, \( e(t) \) and the parameter estimates \( \tilde{\Theta} \) are uniformly bounded. If in addition \( \Phi(t) \) is bounded, we note that \( \dot{e}(t) \) is bounded. It follows from Barbalat’s lemma that \( \lim_{t \to \infty} \|e(t)\| = 0. \)
7.2 Proof of Theorem 4

The closed-loop system dynamics with the AC-RL controller is represented by the error equation (25), the adaptive control input (28), and the adaptive laws in (29)-(32). A Lyapunov function candidate

\[ V = \frac{1}{\gamma} \text{Tr} \left[ (\Xi - \Theta)^T \Lambda^T (\Xi - \Theta) \right] + \frac{1}{\gamma} \text{Tr} \left[ (\hat{\Theta} - \Xi)^T \Lambda^T (\hat{\Theta} - \Xi) \right] + e^T Pe \]  

(71)

yields a time-derivative

\[ \dot{V} = -\text{Tr} \left[ (\Xi - \Theta)^T (\Lambda + \Lambda^T) B^T Pe \Phi^T \right] \\
- \frac{\beta}{\gamma} \text{Tr} \left[ (\hat{\Theta} - \Xi)^T (\Lambda + \Lambda^T) (\hat{\Theta} - \Xi) \right] N_i \\
+ \text{Tr} \left[ (\hat{\Theta} - \Xi)^T (\Lambda + \Lambda^T) B^T Pe \Phi^T \right] \\
+ e^T \left( A^T H P + PA_H \right) e + 2e^T PB \Lambda \tilde{\Theta} \Phi \\
= -\frac{\beta}{\gamma} \text{Tr} \left[ (\hat{\Theta} - \Xi)^T (\Lambda + \Lambda^T) (\hat{\Theta} - \Xi) \right] \\
- \frac{\mu \beta}{\gamma} \text{Tr} \left[ (\hat{\Theta} - \Xi)^T (\Lambda + \Lambda^T) (\hat{\Theta} - \Xi) \right] \| \Phi \|^2 \\
- e^T Qe + 4e^T PB \Lambda (\hat{\Theta} - \Xi) \Phi \\
\]

Through algebraic manipulations, it can be shown that

\[ \dot{V} \leq -\frac{2\beta}{\gamma} \text{Tr} \left[ (\hat{\Theta} - \Xi)^T \Omega^T \Omega (\hat{\Theta} - \Xi) \right] - 2\| e \|^2 \\
- 4 \| PB \|^2 \text{Tr} \left[ (\hat{\Theta} - \Xi)^T \Omega^T \Omega (\hat{\Theta} - \Xi) \right] \| \Phi \|^2 \\
+ 4\| e \| \| PB \Lambda (\hat{\Theta} - \Xi) \|_F \| \Phi \| \\
\leq -\| e \|^2 - \frac{2\beta}{\gamma} \left[ (\hat{\Theta} - \Xi)^T \Omega \right]_F^2 \\
- \left[ \| e \| - 2\| PB \|_2 \| (\hat{\Theta} - \Xi)^T \Omega \|_F \| \Phi \| \right]^2 \leq 0 \\
\]

where we have expressed the positive definite matrix \( \Lambda \) using \( \Omega \) with \( 2\Omega^T \Omega = \Lambda + \Lambda^T \), \( \| \cdot \|_F \) is the Frobenius matrix norm and \( \| \cdot \|_2 \) is the matrix 2-norm. In the above derivation we have used (31), (32), Assumption 4, and the inequality \( \| AB \|_F \leq \| A \|_2 \| B \|_F \). It can be concluded that \( e, \hat{\Theta}, \Xi \in L_\infty \). Using Barbalat’s lemma, as before, we conclude that \( \lim_{t \to \infty} e(t) = 0 \) and that \( R = O(1) \).

7.3 Proof of Theorem 5

As in Theorem 4, we consider a candidate

\[ V = \frac{1}{\gamma} \text{Tr} \left[ (\Xi - \Theta)^T \Lambda^T (\Xi - \Theta) \right] + \frac{1}{\gamma} \text{Tr} \left[ (\hat{\Xi} - \Xi)^T \Lambda^T (\hat{\Xi} - \Xi) \right] + e_u^T Pe_u \]  

(72)
Using (44) and (45), we obtain that $\dot{V} \leq -e^T u$ since $Q \geq 2I$. This in turn allows us to conclude that $e_u, \Xi, \Theta \in L^\infty$. For case (i), since all parameters are bounded, together with magnitude saturation, we have that the input $u$ is bounded. For this case, it implies that the state $x$ is bounded. Therefore $e_u, \Xi, \Theta \in L^\infty$. For case (ii), as the structure of the error model in (44) is identical to that considered in Theorem 1 in [79], the same arguments can be used to establish boundedness of the state.

7.4 Proof of Lemma 1

Note: In the proofs of Lemma 1 - 3 and Theorem 6, we assume that $\|B_0 \Phi(t)^T\| \leq c_1$. Such a $c_1$ exists as $\Phi$ is bounded following Theorem 4.

Consider the following candidate Lyapunov function

$$V = \frac{\Lambda}{\gamma} \|\tilde{\vartheta}\|^2 + \frac{\Lambda}{\gamma} \|\tilde{\theta} - \tilde{\vartheta}\|^2 + e^T P e$$

Similar to what has been shown in the proof of Theorem 4, the time derivative of $V$ may be bounded by

$$\dot{V} \leq -\frac{2\beta\Lambda}{\gamma}\|\tilde{\theta} - \tilde{\vartheta}\|^2 - \Lambda\|e\|^2 - \Lambda \left(\|e\| - 2\|PB\|\|\tilde{\theta} - \tilde{\vartheta}\|\Phi\right)^2$$

$$\leq -c_2 \|x_1\|^2$$

where $c_2 = \Lambda \min \left\{1, \frac{2\beta}{\gamma}\right\}$. Noting that $P$ is a positive definite matrix, it follows that for any vector $v \in \mathbb{R}^n$, $\alpha, \rho > 0$ exist such that

$$\alpha v^T v \leq v^T P v \leq \rho v^T v \quad (73)$$

We prove Lemma 1 by using contradiction. Assume $\mu(S) > n$ and denote $\bar{S} = \{t \in [t_1, \infty)\}$. Integrating $\dot{V}$, we have

$$\int_{t_1}^{\infty} \dot{V}(\tau) d\tau = \int_{S} \dot{V}(\tau) d\tau + \int_{S-S} \dot{V}(\tau) d\tau$$

$$\leq \int_{S} -c_2 \|x_1(\tau)\|^2 d\tau + \int_{S-S} \dot{V}(\tau) d\tau$$

$$\leq -c_2 n \epsilon_2^2$$

Choose $n(\epsilon_1, \epsilon_2) = c_3 \epsilon_1^2/(c_2 \epsilon_2^2)$, then we have a contradiction since $\|V(t_1)\| \leq c_3 \epsilon_1^2$, where $c_3 = \max \left\{\frac{\Lambda}{\gamma}, \rho\right\}$.

7.5 Proof of Lemma 2

Let $z(t)$ be a solution with initial condition $\|z(t_1)\| \leq \epsilon_1$. Suppose that $\left\|\tilde{\vartheta}(t)\right\| \geq \delta$ for all $t \in [t_1, t_1 + T]$, where $T = T_0 + \delta_0$. 

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From the error model in (49), for any $t \geq t_1$,

$$e(t + \delta_0) = e(t) + \int_t^{t + \delta_0} A_H e(\tau) + B_0 \Phi(\tau)^T \tilde{\theta}(\tau) d\tau$$

(74)

from which we have

$$\|e(t + \delta_0)\| \geq \left\| \int_t^{t + \delta_0} B_0 \Phi(\tau)^T \tilde{\theta}(\tau) d\tau \right\| - \left\| e(t) + \int_t^{t + \delta_0} A_H e(\tau) d\tau \right\|$$

(75)

Given $\epsilon' = \epsilon_0 \delta / (2c_2 \delta_0)$ and $T = T_0 + \delta_0$, from the adaptive law in (51), there is an $\epsilon_2 > 0$ such that if $z(\tau)$ is a solution to (50)-(51) with $\|x_1(\tau)\| \leq \epsilon_2$ for all $\tau \in [t_1, t_1 + T]$, then $\|\tilde{\theta}(\tau) - \tilde{\theta}(t_1)\| \leq \epsilon'$. Define $\epsilon = \min \left\{ \frac{\delta_0}{8}, \frac{\delta_0}{8 \epsilon_1 \delta_0}, \epsilon_2 \right\}$. Now we show the lemma holds for this choice of $T$ and $\epsilon$.

If $\|x_1(t_2)\| \geq \epsilon$ for some $t_2 \in [t_1, t_1 + T]$, then we are done. Assume $\|x_1(t)\| \leq \epsilon$ for all $t \in [t_1, t_1 + T]$, then

$$\left\| e(t) + \int_t^{t + \delta_0} A_H e(\tau) d\tau \right\| \leq \epsilon + c_1 \epsilon \delta_0 \leq \frac{\epsilon_0 \delta}{4}$$

for all $t \in [t_1, t_1 + T]$. By hypothesis, there exist a $t' \in [t_1, t_1 + T_0]$ such that

$$\left\| \int_{t'}^{t' + \delta_0} B_0 \Phi(\tau)^T w d\tau \right\| \geq \epsilon_0$$

where $w = \tilde{\theta}(t_1)/\|\tilde{\theta}(t_1)\|$ is a unit vector.

We have

$$\left\| \int_{t'}^{t' + \delta_0} B_0 \Phi(\tau)^T \left[ w \| \tilde{\theta}(t_1) \| - \tilde{\theta}(\tau) \right] d\tau \right\| \leq c_2 \int_{t'}^{t' + \delta_0} \| \tilde{\theta}(t_1) - \tilde{\theta}(\tau) \| d\tau$$

$$\leq c_2 \delta_0 \epsilon' = \frac{\epsilon_0 \delta}{2}$$

Since $\|e(\tau)\| \leq \|x_1(\tau)\| \leq \epsilon \leq \epsilon_2$ for $\tau \in [t_1, t_1 + T]$,

$$\left\| \tilde{\theta}(t_1) \right\| \left\| \int_{t'}^{t' + \delta_0} B_0 \Phi(\tau)^T w d\tau \right\| - \left\| \int_{t'}^{t' + \delta_0} B_0 \Phi(\tau)^T \tilde{\theta}(\tau) d\tau \right\| \leq \frac{\epsilon_0 \delta}{2}$$

which implies

$$\left\| \int_{t'}^{t' + \delta_0} B_0 \Phi(\tau)^T \tilde{\theta}(\tau) d\tau \right\| \geq \epsilon_0 \delta - \frac{\epsilon_0 \delta}{2} = \frac{\epsilon_0 \delta}{2}$$

Thus

$$\|x_1(t' + \delta_0)\| \geq \|e(t' + \delta_0)\| \geq \frac{\epsilon_0 \delta}{2} - \frac{\epsilon_0 \delta}{4} = \frac{\epsilon_0 \delta}{4} > \epsilon$$

which is a contradiction.
Theorem 6

Consider the candidate

\[ V = \frac{\Lambda}{\gamma} \| \tilde{\theta} \|^2 + \frac{\Lambda}{\gamma} \| \tilde{\varphi} - \tilde{\varphi} \|^2 + e^T P e \]  

(76)

With \( \mu \geq 2\gamma \| P B \|^2 / \beta \) and \( Q \geq 2I \) which solves \( A_H^T P + PA_H = -Q \), the time derivative of (76) may be bounded by

\[ \dot{V} \leq -\frac{2\beta \Lambda}{\gamma} \| \tilde{\varphi} - \tilde{\varphi} \|^2 - \Lambda \| e \|^2 - \Lambda \| e \| - 2 \| P B \| \| \tilde{\varphi} - \tilde{\varphi} \| \| \Phi \|^2 \leq 0 \]  

(77)

Since \( P \) is positive-definite, (73) holds for some \( \alpha, \rho > 0 \). Now we show that given \( \epsilon_1 > \epsilon_2 > 0 \), there is a \( \eta \) with \( 0 < \eta < 1 \) and \( \Delta T_1 > 0 \) such that if \( z(t) \) is a solution with

\[ \epsilon_2 \leq V(t) \leq \epsilon_1, \quad \text{for} \ t \in [t_1, t_1 + \Delta T_1], \]

then there is a \( t_2 \in [t_1, t_1 + \Delta T_1] \) such that \( V(t_2) \leq \eta V(t_1) \). Choose \( 0 < \nu < 1, \nu < \sigma < 1 \) and \( \Delta T_2 > 0 \) so that \( \rho \sqrt{1 - \sigma - \Delta T_2} \left( \frac{\epsilon}{\gamma} + 2c_2 \sqrt{\gamma} \right) > 0, \sqrt{\gamma(1 - \nu) - \Delta T_2} \left( \beta \sqrt{\gamma} + \frac{c_2 \rho}{\gamma} \right) > 0, 0 < \Delta T_2 \left( \rho \sqrt{1 - \sigma - \Delta T_2} \left( \frac{\epsilon}{\gamma} + 2c_2 \sqrt{\gamma} \right) \right)^2 < 1 \) and \( 0 < \frac{2\beta \Delta T_2}{\gamma} \left[ \sqrt{\gamma(1 - \nu) - \Delta T_2} \left( \beta \sqrt{\gamma} + \frac{c_2 \rho}{\gamma} \right) \right]^2 < 1 \). From Lemma 3 we can obtain a \( T \) when \( \epsilon = \epsilon_1 \) and \( \delta = \sqrt{\epsilon_2 \nu} \). Define \( \eta = 1 - \min \left\{ \Delta T_2 \left( \rho \sqrt{1 - \sigma - \Delta T_2} \left( \frac{\epsilon}{\gamma} + 2c_2 \sqrt{\gamma} \right) \right)^2, \frac{2\beta \Delta T_2}{\gamma} \left[ \sqrt{\gamma(1 - \nu) - \Delta T_2} \left( \beta \sqrt{\gamma} + \frac{c_2 \rho}{\gamma} \right) \right]^2 \right\} \) and \( \Delta T_1 = T + \Delta T_2 \). Next we show that for this \( \eta \) and \( \Delta T_1 \) the results hold.

Let \( t_2' \in [t_1, t_1 + T] \) be such that \( \| \tilde{\varphi}(t_2') \| \leq \delta \sqrt{\gamma} \). If \( V(t_2') \leq \epsilon_2 \), we are done. If \( V(t_2') \geq \epsilon_2 \), then

\[ V(t_2') = \frac{\Lambda}{\gamma} \| \tilde{\varphi}(t_2') \|^2 + \frac{\Lambda}{\gamma} \| \tilde{\theta}(t_2') - \tilde{\theta}(t_2') \|^2 + e(t_2')^T P e(t_2') \]  

(78)

implies

\[ (1 - \nu) V(t_2') \leq V(t_2') - \delta^2 \]  

(79)

\[ \leq \frac{\Lambda}{\gamma} \| \tilde{\theta}(t_2') - \tilde{\theta}(t_2') \|^2 + e(t_2')^T P e(t_2') \]  

(80)

\[ \leq \frac{\Lambda}{\gamma} \| \tilde{\theta}(t_2') - \tilde{\theta}(t_2') \|^2 + \rho e(t_2')^2 \]  

(81)

Case 1: \( \frac{\Lambda}{\gamma} \| \tilde{\theta}(t_2') - \tilde{\theta}(t_2') \|^2 < (1 - \nu) V(t_2') \). From (79),

\[ e(t_2')^2 \geq \frac{1}{\rho} (1 - \sigma) V(t_2'), \]  

(82)
where $0 < \nu \leq \sigma < 1$. From (49), for any $t \geq t'_2$,
\[
\|e(t'_2)\| - \|e(t)\| \leq \int_{t'_2}^{t} \left\| A_H e(\tau) + B_0 \Phi(\tau)^T \tilde{\theta}(\tau) \right\| d\tau \\
\leq \left( \frac{c_1}{\sqrt{\alpha}} + 2c_2\sqrt{\gamma} \right) (t - t'_2) \sqrt{V(t'_2)}
\]
where the last inequality is due to the assumption that $\|A_H\| \leq c_1$ and $\|B_0 \Phi(\tau)^T\| \leq c_2$ for all $\tau$. If $t_2 = t'_2 + \Delta T_2$, we obtain
\[
\|e(t)\| \geq \|e(t'_2)\| - \left( \frac{c_1}{\sqrt{\alpha}} + 2c_2\sqrt{\gamma} \right) (t_2 - t'_2) \|\dot{z}(t'_2)\|
\geq \rho \sqrt{1 - \sigma} \sqrt{V(t'_2)} - \left( \frac{c_1}{\sqrt{\alpha}} + 2c_2\sqrt{\gamma} \right) \Delta T_2 \sqrt{V(t'_2)}
= \left[ \rho \sqrt{1 - \sigma} - \Delta T_2 \left( \frac{c_1}{\sqrt{\alpha}} + 2c_2\sqrt{\gamma} \right) \right] \sqrt{V(t'_2)}
\]
Integrating the derivative of the Lyapunov candidate function, we have
\[
V(t'_2) - V(t_2) = \int_{t'_2}^{t_2} -\mathbb{\dot{V}}(\tau) d\tau \\
\geq \int_{t'_2}^{t_2} \left( \|e(\tau)\|^2 + \frac{2\beta}{\gamma} \|\dot{\varphi}(\tau) - \tilde{\varphi}(\tau)\|^2 \right) d\tau \\
\geq \Delta T_2 \left[ \rho \sqrt{1 - \sigma} - \Delta T_2 \left( \frac{c_1}{\sqrt{\alpha}} + 2c_2\sqrt{\gamma} \right) \right]^2 V(t'_2)
\]
Therefore, $V(t_2) \leq \eta V(t'_2)$ and uniform asymptotic stability holds.

Case 2: $\frac{1}{\gamma} \left\| \dot{\theta}(t'_2) - \tilde{\theta}(t'_2) \right\|^2 \geq (1 - \nu) V(t'_2)$. For any $t \geq t'_2$, following the process in case 1, we can show that
\[
\left\| \dot{\theta}(t'_2) - \tilde{\theta}(t'_2) \right\| - \|\dot{\theta}(t) - \tilde{\theta}(t)\| \leq \left\| \dot{\theta}(t) - \tilde{\theta}(t) - \dot{\theta}(t'_2) + \tilde{\theta}(t'_2) \right\|
\leq \int_{t'_2}^{t} \left\| \dot{\theta}(\tau) - \tilde{\theta}(\tau) \right\| d\tau \\
= \int_{t'_2}^{t} \left\| -\beta \left[ \dot{\theta}(\tau) - \tilde{\theta}(\tau) \right] + \frac{\gamma}{N_t} \Phi e^T P B \right\| d\tau \\
\leq (t - t'_2) \left( \beta \sqrt{\gamma} + \frac{c_2\gamma \rho}{\sqrt{\alpha}} \right) \sqrt{V(t'_2)}
\]
If we let $t_2 = t'_2 + \Delta T_2$, then
\[
\left\| \dot{\theta}(t) - \tilde{\theta}(t) \right\| \geq \left\| \dot{\theta}(t'_2) - \tilde{\theta}(t'_2) \right\| - (t_2 - t'_2) \left( \beta \sqrt{\gamma} + \frac{c_2\gamma \rho}{\sqrt{\alpha}} \right) \sqrt{V(t'_2)}
\geq \sqrt{\gamma(1 - \nu)} \sqrt{V(t'_2)} - \Delta T_2 \left( \beta \sqrt{\gamma} + \frac{c_2\gamma \rho}{\sqrt{\alpha}} \right) \sqrt{V(t'_2)}
\geq \left[ \sqrt{\gamma(1 - \nu)} - \Delta T_2 \left( \beta \sqrt{\gamma} + \frac{c_2\gamma \rho}{\sqrt{\alpha}} \right) \right] \sqrt{V(t'_2)}
\]
Integrating \( \dot{V} \), we have
\[
V(t'_2) - V(t_2) = \int_{t_2}^{t'_2} -\dot{V}(\tau) d\tau \\
\geq \int_{t_2}^{t'_2} \left( \|e(\tau)\|^2 + \frac{2\beta}{\gamma} \|\ddot{\theta}(\tau) - \ddot{\vartheta}(\tau)\|^2 \right) d\tau \\
\geq 2\beta \Delta T_2 \left[ \sqrt{\gamma(1 - \nu)} - \Delta T_2 \left( \beta \sqrt{\gamma} + \frac{c_2 \gamma \rho}{\sqrt{\alpha}} \right) \right]^2 V(t'_2)
\]

Therefore, \( V(t_2) \leq \eta V(t'_2) \) which proves the theorem.

7.8 Proof of Theorem 7:
This is omitted as it is identical to the proof of Theorem 4.

7.9 Proof of Theorem 8:
The first variation of the Riemannien energy \([80]\) is written as
\[
\dot{E} = \frac{\partial E}{\partial t} + \frac{\partial E}{\partial \nu} \dot{\nu} + 2(\gamma_s(s), \dot{\gamma}(s))|_{s=0}^{s=1} - 2 \int_0^1 \langle \frac{D}{ds} \gamma_s, \dot{\gamma} \rangle ds
\]
where \( \frac{D}{ds} \) is the covariant derivative. Since \( \gamma(s) \) is the geodesic, \( \frac{D\gamma}{ds} = 0 \).
Under Assumption \([\text{B.}6]\) Lemma 2 in \([46]\) implies
\[
\frac{\partial E}{\partial \nu} \dot{\nu} + 2\gamma_s(1)^T M(X, \dot{\nu}, t) B(X, t) U_\nu = 0
\]
Therefore, using \([67]\) and \([69]\), we have
\[
\dot{E} = \frac{\partial E}{\partial t} + \frac{\partial E}{\partial \nu} \dot{\nu} + 2\gamma_s(1)^T M(X, \dot{\nu}, t) \dot{X} - 2\gamma_s(0)^T M(X_r, \dot{\nu}, t) \dot{X}_r \\
= \frac{\partial E}{\partial t} + \left[ \frac{\partial E}{\partial \nu} \dot{\nu} + 2\gamma_s(1)^T M(X, \dot{\nu}, t) B(X, t) U_\nu \right] \\
+ 2\gamma_s(1)^T M(X, \dot{\nu}, t) B(X, t)(\Lambda \Psi - I) U_\nu \\
+ 2\gamma_s(1)^T M(X, \dot{\nu}, t) \left[ f(X, t) - \rho(X, t) \nu \right. \\
+ B(X, t)(\Lambda \Psi - I)(U_c + \phi(X, t) \dot{\theta}) \bigg] \\
+ 2\gamma_s(1)^T M(X, \dot{\nu}, t) B(X, t)(U_c + \phi(X, t) \dot{\theta}) \\
- 2\gamma_s(0)^T M(X_r, \dot{\nu}, t) \dot{X}_r \\
\leq -2\lambda E + 2\gamma_s(1)^T M(X, \dot{\nu}, t) \left[ \rho(X, t) \nu \\
+ B(X, t)((\Lambda \Psi - I) U + \phi(X, t) \dot{\theta}) \right]
\]
Consider a Lyapunov-like function,

\[
V = \frac{1}{\gamma_\nu}(\nu_a - \dot{\nu})^T(\nu_a - \dot{\nu}) + \frac{1}{\gamma_\nu}(\nu_a - \nu)^T(\nu_a - \nu)
\]

\[
+ \frac{1}{\gamma_\theta}(\theta_a - \hat{\theta})^T(\theta_a - \hat{\theta}) + \frac{1}{\gamma_\theta}(\theta_a - \theta)^T(\theta_a - \theta)
\]

\[
+ \frac{1}{\gamma_\Psi} \operatorname{Tr} \left[ (\Lambda \Psi_a - \Lambda \Psi)^T \Lambda^{-1} (\Lambda \Psi_a - \Lambda \Psi) \right]
\]

\[
+ \frac{1}{\gamma_\Psi} \operatorname{Tr} \left[ (\Lambda \Psi_a - I)^T \Lambda^{-1} (\Lambda \Psi_a - I) \right] + \frac{1}{2} E(\gamma, \nu, t)^2
\]

Define \( e = \gamma_s(1)ME \), and use short-hand notation \( \rho = \rho(X, t), \phi = \phi(X, t), \) and \( B = B(X, t) \). Since the Riemannian energy is always non-negative, i.e. \( E \geq 0 \), we have

\[
\frac{1}{2} \dot{V} \leq -\lambda E^2 + e^T \rho(\nu_a - \nu) + e^T B[(\Lambda \Psi - I)U
\]

\[
+ \phi(\theta_a - \theta)] - (\nu_a - \dot{\nu})^T \rho^T e - \frac{\beta_\nu}{\gamma_\nu}(\nu_a - \dot{\nu})^T(\nu_a - \dot{\nu})N_\nu
\]

\[
- (\nu_a - \nu)^T \rho^T e - (\theta_a - \hat{\theta})^T \phi^T B^T e - \frac{\beta_\theta}{\gamma_\theta}(\theta_a - \hat{\theta})^T(\theta_a - \hat{\theta})N_\theta
\]

\[
- (\theta_a - \theta)^T \phi^T B^T e + \operatorname{Tr} \left[ -ue^T B(\Lambda \Psi_a - I) \right]
\]

\[
+ \operatorname{Tr} \left[ -Ue^T B\Lambda(\Psi_a - \Psi) - \frac{\beta_\Psi}{\gamma_\Psi} N_\Psi^T(\Psi_a - \Psi)^T \Lambda(\Psi_a - \Psi) \right]
\]

Let \( \lambda_1, \lambda_2, \lambda_3 \) be positive constants such that \( \lambda_1 + \lambda_2 + \lambda_3 = \lambda \), we have

\[
\frac{1}{2} \dot{V} \leq -\lambda E^2 + e^T \rho(\dot{\nu} - \nu) - (\nu_a - \dot{\nu})^T \rho^T e
\]

\[
- \frac{\beta_\nu}{\gamma_\nu}(\nu_a - \dot{\nu})^T(\nu_a - \dot{\nu})N_\nu - (\nu_a - \nu)^T \rho^T e
\]

\[
+ e^T B\phi(\theta - \hat{\theta}) - (\theta_a - \hat{\theta})^T \phi^T B^T e
\]

\[
- \frac{\beta_\theta}{\gamma_\theta}(\theta_a - \hat{\theta})^T(\theta_a - \hat{\theta})N_\theta - (\theta_a - \theta)^T \phi^T B^T e
\]

\[
+ e^T B(\Lambda \Psi - I)U - e^T B(\Lambda \Psi_a - I)U
\]

\[
+ \operatorname{Tr} \left[ -Ue^T B\Lambda(\Psi_a - \Psi) - \frac{\beta_\Psi}{\gamma_\Psi} N_\Psi^T(\Psi_a - \Psi)^T \Lambda(\Psi_a - \Psi) \right]
\]

\[
= V_1 + V_2 + V_3
\]

where

\[
V_1 = -\lambda_1 E^2 + 2e^T \rho(\dot{\nu} - \nu_a) - \frac{\beta_\nu}{\gamma_\nu}(\nu_a - \dot{\nu})^T(\nu_a - \dot{\nu})N_\nu
\]

\[
V_2 = -\lambda_2 E^2 + 2e^T B\phi(\theta - \theta_a) - \frac{\beta_\theta}{\gamma_\theta}(\theta_a - \hat{\theta})^T(\theta_a - \hat{\theta})N_\theta
\]

\[
V_3 = -\lambda_3 E^2 + 2e^T B\Lambda(\Psi - \Psi_a)U - \frac{\beta_\Psi}{\gamma_\Psi} \operatorname{Tr} \left[ (\Psi_a - \Psi)^T \Lambda(\Psi_a - \Psi)N_\Psi \right]
\]

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Choose the adaptive gain parameters $\beta_\nu, \gamma_\nu, \beta_\theta, \gamma_\theta, \beta_\Psi, \gamma_\Psi$ as positive numbers that satisfy

$$\frac{\beta_\nu}{\gamma_\nu} \geq \frac{1}{\lambda_1}, \quad \frac{\beta_\theta}{\gamma_\theta} \geq \frac{1}{\lambda_2}, \quad \frac{\beta_\Psi}{\gamma_\Psi} \geq \frac{\|\gamma_s(1)\| \|M\| \|B\|}{\lambda_3}$$

It follows from (68) that

$$V_1 \leq -\frac{\beta_\nu}{\gamma_\nu} \|\nu_a - \hat{\nu}\|^2 - \lambda_1 E^2 + 2E \|\omega_\nu\| \|\nu_a - \hat{\nu}\| - \frac{\beta_\nu}{\gamma_\nu} \|\nu_a - \hat{\nu}\|^2 \|\omega_\nu\|^2_2$$

$$= -\frac{\beta_\nu}{\gamma_\nu} \|\nu_a - \hat{\nu}\|^2 - \lambda_1 (E - 2\|\omega_\nu\| \|\nu_a - \hat{\nu}\|)^2 \leq 0$$

$$V_2 \leq -\frac{\beta_\theta}{\gamma_\theta} \|\theta_a - \hat{\theta}\|^2 - \lambda_2 E^2 + 2E \|\omega_\theta\| \|\theta_a - \hat{\theta}\| - \frac{\beta_\theta}{\gamma_\theta} \|\theta_a - \hat{\theta}\|^2 \|\omega_\theta\|^2_2$$

$$= -\frac{\beta_\theta}{\gamma_\theta} \|\theta_a - \hat{\theta}\|^2 - \lambda_2 (E - \|\omega_\theta\| \|\theta_a - \hat{\theta}\|)^2 \leq 0$$

Under Assumption 4, $\Lambda$ is a positive definite matrix. Using Cholesky decomposition, we can find a lower triangular matrix with real and positive diagonal entries, $\Omega$, such that $\Omega \Omega^T = \Lambda$. For simplicity, we use $\|\cdot\|$ to denote $\|\cdot\|_2$. Therefore,

$$\text{Tr} \left[ (\Psi_a - \Psi)^T \Lambda (\Psi_a - \Psi) \omega_\Psi^T \omega_\Psi \right]$$

$$= \text{Tr} \left[ (\Psi_a - \Psi)^T \Omega^T (\Psi_a - \Psi) \right] \|U\|^2$$

$$= \|(\Psi_a - \Psi)^T \Omega^T\|_F \|U\|^2$$

Using the inequality $\|AB\|_F \leq \|A\|_2 \|B\|_F$ and the fact that for scalars and vectors, $\|\cdot\|_F = \|\cdot\|_2$,

$$\|2e^T BA(\Psi - \Psi_a)U\| = \|2e^T BA(\Psi - \Psi_a)U\|_F$$

$$= \|2\gamma_s(1)^T MEB\Omega^T (\Psi - \Psi_a)U\|_F$$

$$\leq 2 \|\gamma_s(1)^T MEB\| \|\Omega^T (\Psi - \Psi_a)\|_F \|U\|_F$$

$$\leq 2 \|\gamma_s(1)^T MB\| \|\Omega\| \|(\Psi - \Psi_a)^T \Omega\|_F \|U\|_F$$

Therefore,

$$V_3 \leq -\frac{\beta_\Psi}{\gamma_\Psi} \|\Psi_a - \Psi\|_F^2 - \lambda_3 E^2 + 2e^T BA(\Psi - \Psi_a)U - \frac{\beta_\Psi}{\gamma_\Psi} \|\Psi_a - \Psi\|_F^2 \|\Omega\|_F \omega_\Psi^2$$

$$= -\frac{\beta_\Psi}{\gamma_\Psi} \|\Psi_a - \Psi\|_F^2 - \lambda_3 (E - \|\gamma_s(1)^T MB\| \|\Psi_a - \Psi\|_F \|U\|_F^2)^2 \leq 0$$

Thus we have $\dot{V} \leq 2(V_1 + V_2 + V_3) \leq 0$, which implies that $E, \nu, \hat{\nu}, \hat{\Psi} \in L_\infty$, and that $E \in L_2$. Since $\theta$ is a constant vector and $\bar{\theta}$ is bounded, $\hat{\theta}$ is bounded. Similarly, $\bar{\nu}$ and $(\Lambda \Psi - I)$ are also bounded. By Assumption 7, $M(x, \nu, t)$ is bounded. Since geodesics have constant speed and $E = \int_0^1 \gamma_s(s)^T M(x, \nu, t) \gamma_s(s) ds = \langle \gamma_s, \gamma_s \rangle$ is bounded, $\gamma_s$ is bounded. Therefore, $\dot{E} \in L_\infty$. By Barbalat’s Lemma, $\lim_{t \to \infty} E(t) \to 0$, thus $x(t) \to x_d(t)$ as $t \to \infty$. 

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References

[1] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*. NJ: Dover Publications, 2005, (original publication by Prentice-Hall Inc., 1989).

[2] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. PTR Prentice-Hall, 1996.

[3] S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence and Robustness*. Prentice-Hall, 1989.

[4] J.-J. E. Slotine and W. Li, *Applied nonlinear control*. Prentice hall Englewood Cliffs, NJ, 1991.

[5] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos, *Nonlinear and adaptive control design*. John Wiley & Sons, Inc., 1995.

[6] A. Astolfi, D. Karagiannis, and R. Ortega, *Nonlinear and adaptive control with applications*. Springer Science & Business Media, 2007.

[7] M. Guay and T. Zhang, “Adaptive extremum seeking control of nonlinear dynamic systems with parametric uncertainties,” *Automatica*, vol. 39, no. 7, pp. 1283–1293, 2003.

[8] K. Zhou, J. C. Doyle, K. Glover et al., *Robust and optimal control*. Prentice hall New Jersey, 1996, vol. 40.

[9] L. Hewing, K. P. Wabersich, M. Menner, and M. N. Zeilinger, “Learning-based model predictive control: Toward safe learning in control,” *Annual Review of Control, Robotics, and Autonomous Systems*, vol. 3, pp. 269–296, 2020.

[10] J. B. Rawlings, D. Q. Mayne, and M. Diehl, *Model predictive control: theory, computation, and design*. Nob Hill Publishing Madison, WI, 2017, vol. 2.

[11] V. I. Utkin, *Sliding modes in control and optimization*. Springer Science & Business Media, 2013.

[12] B. Recht, “A tour of reinforcement learning: The view from continuous control,” *Annual Review of Control, Robotics, and Autonomous Systems*, vol. 2, pp. 253–279, 2019.

[13] D. P. Bertsekas and J. N. Tsitsiklis, *Neuro-dynamic programming*. Athena Scientific, 1996.

[14] R. S. Sutton and A. G. Barto, *Reinforcement learning: An introduction*. MIT press, 2018.

[15] M. Vidyasagar, “Recent advances in reinforcement learning,” in *2020 American Control Conference (ACC)*. IEEE, 2020, pp. 4751–4756.

[16] C. J. Watkins and P. Dayan, “Q-learning,” *Machine learning*, vol. 8, no. 3-4, pp. 279–292, 1992.

[17] L. Ljung, *System Identification: Theory for the User*. Prentice-Hall, 1987.
[33] J. Tobin, R. Fong, A. Ray, J. Schneider, W. Zaremba, and P. Abbeel, “Domain randomization for transferring deep neural networks from simulation to the real world,” in 2017 IEEE/RSJ international conference on intelligent robots and systems (IROS). IEEE, 2017, pp. 23–30.

[34] R. Polvara, M. Patacchiola, M. Hanheide, and G. Neumann, “Sim-to-real quadrotor landing via sequential deep q-networks and domain randomization,” Robotics, vol. 9, no. 1, p. 8, 2020.

[35] Y. Chebotar, A. Handa, V. Makoviychuk, M. Macklin, J. Isaac, N. Ratliff, and D. Fox, “Closing the sim-to-real loop: Adapting simulation randomization with real world experience,” in 2019 International Conference on Robotics and Automation (ICRA). IEEE, 2019, pp. 8973–8979.

[36] L. Pinto, J. Davidson, R. Sukthankar, and A. Gupta, “Robust adversarial reinforcement learning,” in International Conference on Machine Learning. PMLR, 2017, pp. 2817–2826.

[37] D. J. Mankowitz, N. Levine, R. Jeong, Y. Shi, J. Kay, A. Abdolmaleki, J. T. Springenberg, T. Mann, T. Hester, and M. Riedmiller, “Robust reinforcement learning for continuous control with model misspecification,” International Conference on Learning Representations, 2020.

[38] K. Arndt, M. Hazara, A. Ghadirzadeh, and V. Kyrki, “Meta reinforcement learning for sim-to-real domain adaptation,” in 2020 IEEE International Conference on Robotics and Automation (ICRA). IEEE, 2020, pp. 2725–2731.

[39] W. Zhao, J. P. Queralta, and T. Westerlund, “Sim-to-real transfer in deep reinforcement learning for robotics: a survey,” in 2020 IEEE Symposium Series on Computational Intelligence (SSCI). IEEE, 2020, pp. 737–744.

[40] F. L. Lewis, A. Yesildirek, and K. Liu, “Multilayer neural-net robot controller with guaranteed tracking performance,” IEEE Transactions on neural networks, vol. 7, no. 2, pp. 388–399, 1996.

[41] A. Patkar and A. M. Annaswamy, “An adaptive controller for a class of nonlinear plants based on neural networks and convex parameterization,” in 2020 59th IEEE Conference on Decision and Control (CDC). IEEE, 2020, pp. 126–131.

[42] W. Lohmiller and J.-J. E. Slotine, “On contraction analysis for non-linear systems,” Automatica, vol. 34, no. 6, pp. 683–696, 1998.

[43] A. Rantzer, “Concentration bounds for single parameter adaptive control,” in 2018 Annual American Control Conference (ACC). IEEE, 2018, pp. 1862–1866.

[44] N. M. Boffi and J.-J. E. Slotine, “Implicit regularization and momentum algorithms in nonlinerly parameterized adaptive control and prediction,” Neural Computation, vol. 33, no. 3, pp. 590–673, 2021.

[45] I. R. Manchester and J.-J. E. Slotine, “Control contraction metrics: Convex and intrinsic criteria for nonlinear feedback design,” IEEE Transactions on Automatic Control, vol. 62, no. 6, pp. 3046–3053, 2017.
[46] B. T. Lopez and J.-J. E. Slotine, “Adaptive nonlinear control with contraction metrics,” *IEEE Control Systems Letters*, vol. 5, no. 1, pp. 205–210, 2020.

[47] A. S. Morse, “High-order parameter tuners for the adaptive control of linear and nonlinear systems,” in *Systems, Models and Feedback: Theory and Applications*. Birkhauser Boston, 1992, pp. 339–364.

[48] R. Ortega, “On morse’s new adaptive controller: parameter convergence and transient performance,” *IEEE transactions on Automatic Control*, vol. 38, no. 8, pp. 1191–1202, 1993.

[49] S. Evesque, A. M. Annaswamy, S. Niculescu, and A. P. Dowling, “Adaptive control of a class of time-delay systems,” *Journal of Dynamic Systems, Measurement, and Control*, vol. 125, no. 2, p. 186, 2003.

[50] Y. Nesterov, “A method of solving a convex programming problem with convergence rate $O(1/k^2)$,” *Soviet Mathematics Doklady*, vol. 27, pp. 372–376, 1983.

[51] ——, *Lectures on Convex Optimization*. Springer International Publishing, 2018.

[52] W. Su, S. Boyd, and E. J. Candès, “A differential equation for modeling nesterov’s accelerated gradient method: Theory and insights,” *Journal of Machine Learning Research*, vol. 17, no. 153, pp. 1–43, 2016.

[53] A. Wibisono, A. C. Wilson, and M. I. Jordan, “A variational perspective on accelerated methods in optimization,” *Proceedings of the National Academy of Sciences*, vol. 113, no. 47, pp. E7351–E7358, 2016.

[54] J. Zhang, A. Mokhtari, S. Sra, and A. Jadbabaie, “Direct runge-kutta discretization achieves acceleration,” in *Advances in Neural Information Processing Systems*, vol. 31, 2018.

[55] A. Krizhevsky, I. Sutskever, and G. E. Hinton, “Imagenet classification with deep convolutional neural networks,” in *Advances in Neural Information Processing Systems 25*. Curran Associates, Inc., 2012, pp. 1097–1105.

[56] I. Sutskever, J. Martens, G. Dahl, and G. Hinton, “On the importance of initialization and momentum in deep learning,” in *Proceedings of the 30th International Conference on Machine Learning*, vol. 28, no. 3. PMLR, 2013, pp. 1139–1147.

[57] J. E. Gaudio, A. M. Annaswamy, M. A. Bolender, E. Lavretsky, and T. E. Gibson, “A class of high order tuners for adaptive systems,” *IEEE Control Systems Letters*, vol. 5, no. 2, pp. 391–396, apr 2021.

[58] J. E. Gaudio, A. M. Annaswamy, J. M. Moreu, M. A. Bolender, and T. E. Gibson, “Accelerated learning with robustness to adversarial regressors,” *Proceedings of the 3rd Conference on Learning for Dynamics and Control*, PMLR 144:636-650, 2020.

[59] J. E. Gaudio, T. E. Gibson, A. M. Annaswamy, M. A. Bolender, and E. Lavretsky, “Connections between adaptive control and optimization in machine learning,” *58th IEEE Conference on Decision and Control (CDC)*, 2019.
[60] N. Matni, A. Proutiere, A. Rantzer, and S. Tu, “From self-tuning regulators to reinforcement learning and back again,” in *IEEE 58th Conference on Decision and Control (CDC)*, 2019.

[61] T. Westenbroek, E. Mazumdar, D. Fridovich-Keil, V. Prabhu, C. J. Tomlin, and S. S. Sastry, “Adaptive control for linearizable systems using on-policy reinforcement learning,” in *2020 59th IEEE Conference on Decision and Control (CDC)*. IEEE, 2020, pp. 118–125.

[62] R. Sun, M. L. Greene, D. M. Le, Z. I. Bell, G. Chowdhary, and W. E. Dixon, “Lyapunov-based real-time and iterative adjustment of deep neural networks,” *IEEE Control Systems Letters*, 2021.

[63] A. Guha and A. Annaswamy, “Online policies for real-time control using mrae-rl,” in *60th IEEE Conference on Decision and Control (CDC)*. IEEE, 2021, pp. 1806–1811.

[64] S. M. Richards, N. Azizan, J.-J. Slotine, and M. Pavone, “Adaptive-Control-Oriented Meta-Learning for Nonlinear Systems,” in *Proceedings of Robotics: Science and Systems*, Virtual, July 2021.

[65] S. Dean, H. Mania, N. Matni, B. Recht, and S. Tu, “Regret bounds for robust adaptive control of the linear quadratic regulator,” in *Advances in Neural Information Processing Systems 31*. Curran Associates, Inc., 2018, pp. 4192–4201.

[66] M. Fazel, R. Ge, S. Kakade, and M. Mesbahi, “Global convergence of policy gradient methods for the linear quadratic regulator,” in *International Conference on Machine Learning*. PMLR, 2018, pp. 1467–1476.

[67] A. M. Annaswamy, A. Guha, Y. Cui, S. Tang, and J. E. Gaudio, “Online algorithms and policies using adaptive and machine learning approaches,” *arXiv preprint arXiv:2105.06577*, 2021.

[68] A. Morgan and K. Narendra, “On the stability of nonautonomous differential equations $\dot{x} = [A + B(t)]x$, with skew symmetric matrix $B(t)$,” *SIAM Journal on Control and Optimization*, vol. 15, no. 1, pp. 163–176, 1977.

[69] L. P. Kaelbling, M. L. Littman, and A. W. Moore, “Reinforcement learning: A survey,” *Journal of artificial intelligence research*, vol. 4, pp. 237–285, 1996.

[70] J. Schulman, P. Moritz, S. Levine, M. Jordan, and P. Abbeel, “High-dimensional continuous control using generalized advantage estimation,” in *Proceedings of the International Conference on Learning Representations (ICLR)*, 2016.

[71] J. Schulman, F. Wolski, P. Dhariwal, A. Radford, and O. Klimov, “Proximal policy optimization algorithms,” *arXiv preprint arXiv:1707.06347*, 2017.

[72] A. M. Devraj and S. P. Meyn, “Zap q-learning,” in *Proceedings of the 31st International Conference on Neural Information Processing Systems*, 2017, pp. 2232–2241.

[73] F. Berkenkamp, M. Turchetta, A. Schoellig, and A. Krause, “Safe model-based reinforcement learning with stability guarantees,” in *Advances in neural information processing systems*, 2017, pp. 908–918.
[74] A. Loquercio, E. Kaufmann, R. Ranftl, A. Dosovitskiy, V. Koltun, and D. Scaramuzza, “Deep drone racing: From simulation to reality with domain randomization,” *IEEE Transactions on Robotics*, vol. 36, no. 1, pp. 1–14, 2019.

[75] A. P. Morgan and K. S. Narendra, “On the uniform asymptotic stability of certain linear nonautonomous differential equations,” *SIAM Journal on Control and Optimization*, vol. 15, no. 1, pp. 5–24, 1977.

[76] J. E. Gaudio, A. M. Annaswamy, E. Lavretsky, and M. A. Bolender, “Parameter estimation in adaptive control of time-varying systems under a range of excitation conditions,” *IEEE Transactions on Automatic Control (to appear)*, 2022.

[77] Z. T. Dydek, A. M. Annaswamy, and E. Lavretsky, “Adaptive control of quadrotor UAVs: a design trade study with flight evaluations,” *IEEE Transactions on control systems technology*, vol. 21, no. 4, pp. 1400–1406, 2012.

[78] A. Somanath and A. Annaswamy, “Adaptive control of hypersonic vehicles in presence of aerodynamic and center of gravity uncertainties,” in *49th IEEE Conference on Decision and Control (CDC)*. IEEE, 2010, pp. 4661–4666.

[79] M. Schwager and A. M. Annaswamy, “Direct adaptive control of multi-input plants with magnitude saturation constraints,” in *Proceedings of the 44th IEEE Conference on Decision and Control*. IEEE, 2005, pp. 783–788.

[80] M. P. Do Carmo, *Riemannian geometry*. Springer Science & Business Media, 2013.

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