Chiral charge transfer along magnetic field lines in a Weyl superconductor

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We identify a signature of chirality in the electrical conduction along magnetic vortices in a Weyl superconductor: The conductance depends on whether the magnetic field is parallel or antiparallel to the vector in the Brillouin zone that separates Weyl points of opposite chirality.

I. INTRODUCTION

Three-dimensional Weyl fermions have a definite chirality, given by the ± sign in the Weyl Hamiltonian $\pm \mathbf{p} \cdot \sigma$. Three spatial dimensions are essential, if $\mathbf{p} \cdot \sigma = p_x \sigma_x + p_y \sigma_y$ contains only two Pauli matrices, then $+\mathbf{p} \cdot \sigma$ and $-\mathbf{p} \cdot \sigma$ can be transformed into each other by a unitary transformation (conjugation with $\sigma_z$). The chirality is therefore a characteristic feature of 3D Weyl semimetals, not shared by 2D graphene.

The search for observable signatures of chirality is a common theme in the study of this new class of materials [1–4]. The basic mechanism used for that purpose is the chirality dependent motion in a magnetic field: Weyl fermions in the zeroth Landau level propagate parallel or antiparallel to the field lines, dependent on their chirality [5]. A population imbalance between the two chiralities of fermions in the zeroth Landau level is a dispersionless flat band in the plane of the layers perpendicular to the layers [6–8]. Right panel: Weyl points of opposite chirality at $K_\mu = \pm K$. For $\mu \neq 0$ the conductance $G = I_2/V_1$ depends on whether the magnetic field points parallel or antiparallel to the vector from $-K$ to $+K$.

The main effect of the superconductor is to renormalize the charge of the quasiparticles [9], by a factor $\kappa = \sqrt{1 - \Delta_0^2/\beta^2}$.

A magnetic field $B$ perpendicular to the layers penetrates in an array of $h/2e$ vortices. The zeroth Landau level is a dispersionless flat band in the plane of the layers — the chirality of the Weyl fermions prevents broadening of the Landau band by vortex scattering [10].

Following Ref. [11] we probe the Landau band by electrical conduction: A voltage $V_1$ applied to contact $N_1$ induces a current $I_2 = GV_1$ in contact $N_2$. This is a three-terminal circuit, the grounded superconductor being the third terminal. The chemical potential $\mu_N$ in the normal-metal contacts is assumed to be large compared to the value $\mu$ in the superconductor. We calculate the dependence of the conductance $G(\pm B)$ on the direction of the magnetic field $B$, relative to the separation of the Weyl points of opposite chirality.

When the chemical potential is at the Weyl point ($\mu = 0$) the conductance is determined by the renormalized charge and $B$ only enters via the Landau band degeneracy [11],

$$G = \kappa^2 G_0, \quad G_0 = (e^2/h)N_\Phi, \quad \text{at } \mu = 0,$$

with $N_\Phi = eBS/h$ the flux through an area $S$ in units of $h/e$. We generalize this result to nonzero $\mu$ and find that

$$\delta G = G(B) - G(-B) = (4\mu/\beta)(\kappa^2 - \kappa)G_0.$$  \hspace{1cm} (1.2)

The conductance thus depends on whether the magnetic field points from + chirality to − chirality, or the other way around.

The outline of the paper is as follows. In the next section we formulate the problem of electrical conduction along the magnetic vortices of a Weyl superconductor. The key quantity to calculate is the charge $e^* \Phi$ transferred by the quasiparticles across the normal-superconductor interface. At $\mu = 0$ this is simply given by the renormalized charge $\kappa e$ of the Weyl fermions [11], but that no longer holds at nonzero $\mu$. In Secs. [11] and [14] we apply a mode matching technique developed in Ref. [12] to calculate $e^*$. The conductance then follows in Sec. [15]. These are all analytical results, we test them on a com-
puter simulation of a tight-binding model in Sec. VI. We conclude in Sec. VII.

II. WEYL SUPERCONDUCTOR IN A MAGNETIC VORTEX LATTICE

We consider a three-dimensional Weyl superconductor (S, Fermi velocity \( v_F \), chemical potential \( \mu \), s-wave pair potential \( \Delta_0 e^{i\phi} \), sandwiched between metal contacts \( N_1 \) and \( N_2 \) at \( z = \pm L/2 \) (see Fig. 1). A magnetic field \( B > 0 \) in the \( z \)-direction penetrates the superconductor in the form of a vortex lattice. The superconducting phase \( \phi \) winds by \( 2\pi \) around each vortex (at position \( R_n \)),

\[
\nabla \times \nabla \phi = 2\pi^{\frac{\beta}{2}} \sum_{n} \delta(r - R_n). \tag{2.1}
\]

The Bogoliubov-De Gennes Hamiltonian is

\[
\mathcal{H} = v_F \nu_0 \tau_z \left( k \cdot \sigma \right) - ev_F \nu_0 \tau_z (A \cdot \sigma) + \nu_0 \tau_0 \beta \cdot \sigma - \mu \nu_0 \tau_0 \sigma_0 + \Delta_0 (\nu_x \cos \phi - \nu_y \sin \phi) \tau_0 \sigma_0. \tag{2.2}
\]

The Pauli matrices \( \sigma_i \), \( \tau_i \), \( \nu_i \) act respectively on the spin, subband, and electron-hole degree of freedom. We set \( \hbar \) to unity and choose the electron charge as +\( e \). The magnetization \( \beta = \beta n_\beta \) (with \( n_\beta \) a unit vector) may point in an arbitrary direction relative to \( B = \nabla \times A = B\hat{z} \). We choose a gauge in which \( A_z = 0 \) and both \( A \) and \( \phi \) are \( z \)-independent.

The Weyl points in zero magnetic field are at momentum \( k = \pm K = \pm K n_\beta \) with

\[
v_F K = \kappa \beta, \quad \kappa = \sqrt{1 - \Delta_0^2 / \beta^2}. \tag{2.3}
\]

The Weyl cones remain gapless provided that \( \Delta_0 < \beta \). In a magnetic field the states condense into Landau bands, dispersionless in the plane perpendicular to \( B \), but freely moving along \( B \).

A quasiparticle in a Landau band, at energy \( E \), has charge expectation value \( Q = -e \partial E / \partial \mu \). At the Weyl point, \( \mu = 0 = E \), this equals \( 0 \)

\[
Q_0 = \kappa e = e \sqrt{1 - \Delta_0^2 / \beta^2}. \tag{2.4}
\]

We seek the charge \( e^* \) transferred into the normal-metal contact by a quasiparticle in the Landau band. At \( \mu = 0 \) this was calculated in Ref. 11 with the result \( e^* = Q_0 \). We wish to generalize this to nonzero \( \mu \). For that purpose we apply a methodology developed for a different problem in Ref. 12, as described in the next section.

III. FRACTIONAL CHARGE TRANSFER

A. Matching condition

The particle current operator \( \hat{v}_z \) and charge current operator \( \hat{j}_z \), both in the \( z \)-direction, are given by

\[
\hat{v}_z = \partial \mathcal{H} / \partial k_z = ev_F \nu_z \tau_z \sigma_z, \quad \hat{j}_z = -\partial \mathcal{H} / \partial A_z = ev_F \nu_0 \tau_z \sigma_z. \tag{3.1}
\]

In what follows we set \( v_F \) and \( e \) equal to unity, for ease of notation.

The chirality \( \chi = \pm 1 \) of a mode in the superconductor (S) determines whether it propagates in the \( +z \) direction or in the \(-z \) direction. We position the normal-superconductor (NS) interface at \( z = 0 \), so that the mode in S approaches it from \( z < 0 \) for \( \chi = +1 \) and from \( z > 0 \) for \( \chi = -1 \).

We assume that the chemical potential \( \mu_N \) in N is large compared to the value \( \mu \) in S. The potential step at the NS interface boosts the momentum component \( k_z \) perpendicular to the interface, without affecting the parallel component \( k_x, k_y \), so in N only modes are excited with \( |k_z| \gg |k_x|, |k_y| \). These are eigenstates of \( \nu_z \tau_z \sigma_z \) with eigenvalue \( \chi \), moving away from the interface in the \( +z \) direction if \( \chi = +1 \) and in the \(-z \) direction for \( \chi = -1 \). Continuity of the wave function \( \Psi \) at the interface then gives the matching condition

\[
\nu_z \tau_z \sigma_z \Psi = \chi \Psi \quad \text{at} \quad z = 0. \tag{3.2}
\]

B. Projection

Because the Hamiltonian (2.2) commutes with \( \tau_z \) we can replace this Pauli matrix by the subband index \( \tau \) and rewrite the matching condition (3.2) as \( \chi \nu \tau \nu_z \sigma_z \Psi = \Psi \). We define the projection operator

\[
\mathcal{P} = \frac{1}{2} \left( 1 + \chi \nu \tau \sigma_z \right), \quad \text{such that} \quad \mathcal{P} \Psi = \Psi \quad \text{at} \quad z = 0, \tag{3.3}
\]

and project the Hamiltonian (2.2),

\[
\mathcal{P} \mathcal{H} \mathcal{P} = (\tau \beta_z - \chi \mu) \mathcal{P} \mathcal{J}_z \mathcal{P} + \mathcal{P} \hat{k}_z \hat{v}_z \mathcal{P}. \tag{3.4}
\]

We have used that \( A \) only has components in the \( x-y \) plane. The hat on \( \hat{k}_z = -i \partial / \partial z \) is there to remind us it is an operator.

We take the \( z \)-dependent inner product

\[
\langle \Psi_1 | \Psi_2 \rangle_z = \int dx \int dy \Psi_2^*(x, y, z) \Psi_2(x, y, z) \tag{3.5}
\]

of Eq. (3.4),

\[
(\tau \beta_z - \chi \mu) \langle \Psi | \mathcal{P} \mathcal{J}_z \mathcal{P} | \Psi \rangle_z = \langle \Psi | \mathcal{P} \delta \mathcal{H} \mathcal{P} | \Psi \rangle_z, \tag{3.6}
\]

with \( \delta \mathcal{H} = \mathcal{H} - \hat{k}_z \hat{v}_z \).
At the NS interface $z = 0$ the projector may be removed,

$$\langle \tau \beta_z - \chi \mu \rangle \langle \hat{\psi} \mid \hat{\psi} \rangle_0 = \langle \hat{\psi} \mid \delta \mathcal{H} \mid \hat{\psi} \rangle_0, \quad (3.7)$$

since neither $\hat{\psi}$ nor $\delta \mathcal{H}$ contain a $z$-derivative, so that these operators commute with the limit $z \to 0$ and we may replace $\mathcal{P} \hat{\psi}$ by $\hat{\psi}$ in view of the matching condition \[3.3\]. Eq. (3.7) is the key identity that allows us to calculate the transferred charge.

### C. Transferred charge

Let $\hat{\psi}$ be an eigenstate of $\mathcal{H}$ at energy $E$. The transferred charge $e^*$ through the NS interface is given by the ratio

$$e^* = \frac{\langle \hat{\psi} \mid \hat{\psi} \rangle_0}{\langle \hat{\psi} \mid \hat{\psi} \rangle_0}. \quad (3.8)$$

Substitution of Eq. (3.7) equates this to

$$e^* = (\tau \beta_z - \chi \mu)^{-1} \frac{\langle \hat{\psi} \mid \hat{\psi} \rangle_0}{\langle \hat{\psi} \mid \hat{\psi} \rangle_0}, \quad (3.9a)$$

$$= (\tau \beta_z - \chi \mu)^{-1} \frac{\chi E \langle \hat{\psi} \mid \hat{\psi} \rangle_0}{\langle \hat{\psi} \mid \hat{\psi} \rangle_0}. \quad (3.9b)$$

The term $\chi E$ appears because

$$\langle \hat{\psi} \mid \hat{\psi} \rangle_0 = E \langle \hat{\psi} \mid \hat{\psi} \rangle_0 = \chi E \langle \hat{\psi} \mid \hat{\psi} \rangle_0, \quad (3.10)$$

where in the last equality we used the matching condition \[3.2\].

Particle current conservation requires that

$$\frac{d}{dz} \langle \hat{\psi} \mid \hat{\psi} \rangle_0 = 0. \quad (3.11)$$

More generally, for our case of a $z$-independent Hamiltonian it holds that

$$\frac{d}{dz} \langle \hat{\psi} \mid \hat{\psi} \rangle_0 = 0 \quad (3.12)$$

for any function of $f$ of $\hat{\psi}$ (see App. A for a proof). Each of the two expectation values $\langle \cdots \rangle_0$ on the right-hand-side of Eq. (3.9b) can thus be replaced by $\langle \cdots \rangle_z$. This ratio can then be evaluated for large $|z|$, far from the NS interface, where evanescent waves have decayed and $\hat{\psi} \propto e^{ikz}$ is an eigenstate of $\hat{\psi}$.

We finally obtain the transferred charge

$$e^* = e \frac{\chi E - v_F k_z}{\tau \beta_z - \chi \mu}. \quad (3.13)$$

reinstating units of $e$ and $v_F$. For $\mu = 0 = E$, $\beta = \beta_z$, $k_z = K = \kappa \beta / v_F$ we recover the result $e^* = \pm \kappa e = \pm Q_0$ of Ref. [11].

It remains to relate the momentum $k_z$ of a propagating mode at the Fermi level to the parameters of the Weyl superconductor. For that we need the dispersion relation $E(k_z)$ of the Landau band, which we calculate in the next section.

### IV. Dispersion relation of the Landau band

#### A. Block diagonalization

We calculate the dispersion relation of the Landau band by means of the block diagonalization approach of Ref. [10]. Starting from the BdG Hamiltonian \[2.2\] we first make the Anderson gauge transformation \[4.5\]

$$H \to \Omega \hat{\mathcal{H}} \Omega, \quad \text{with} \quad \Omega = \left( \begin{array}{cc} e^{i\phi} & 0 \\ 0 & 1 \end{array} \right). \quad (4.1)$$

The subblocks of $\Omega$ refer to the electron-hole $(\nu_e, \nu_h)$ degree of freedom. The resulting Hamiltonian is

$$H = \nu_z \tau_z (k + a) \cdot \sigma + \nu_0 \tau_z q \cdot \sigma + \nu_0 \tau_0 \beta \cdot \sigma - \mu \nu_z \tau_0 \sigma_0 + \Delta_0 \nu_0 \tau_0 \sigma_0, \quad (4.2)$$

$$a = \frac{i}{2} \nabla \phi, \quad q = \frac{i}{2} \nabla \phi - A. \quad (4.3)$$

Both fields $a$ and $q$ have only components in the $x$–$y$ plane and are $z$-independent.

To focus on states near $K$ we set $k = \kappa \beta + \delta k$ and consider $\delta k$ small. The component parallel to $\beta$ of a vector $v$ is denoted by $v_\parallel = v \cdot n_\beta$.

One more unitary transformation $H \to U^\dagger H U$ with

$$U = \sigma_\parallel \exp \left( \frac{i}{2} \mu \nu_0 \tau_\parallel \sigma_\parallel \right),$$

$$\tan \alpha = -\frac{\Delta_0}{K}, \quad \cos \alpha = -(1 + \Delta_0^2 / K^2)^{-1/2} = -\kappa, \quad (4.4)$$

followed by a projection onto the $\nu = \tau = \pm 1$ blocks, gives a pair of $2 \times 2$ low-energy Hamiltonians,

$$H_\tau = \tau \kappa \mu \sigma_0 - (\delta k + a - \tau \kappa q) \cdot \sigma$$

$$+ (1 - \kappa) (\delta k_\parallel + a_\parallel + \tau q_\parallel) \sigma_\parallel. \quad (4.5)$$

Eq. (4.5) is an anisotropic Dirac Hamiltonian, the velocity parallel to the magnetization is reduced by a factor $\kappa$. The same factor renormalizes the quasiparticle charge,

$$Q = -e \frac{\partial H_\tau}{\partial \mu} = -e \tau \kappa. \quad (4.6)$$

The two Hamiltonians $H_\tau = H_\perp$ near $k = K$ thus describe quasiparticles of opposite charge. Another pair of oppositely charged Weyl cones exists near $k = -K$.

If $\beta = (\beta \sin \theta, 0, \beta \cos \theta)$ makes an angle $\theta$ with the magnetic field we have...
\[ H_\tau = \tau \kappa \mu \sigma_0 - \sum_{\alpha=x,y} (\delta k_\alpha + a_\alpha - \tau \kappa q_\alpha) \sigma_\alpha - \delta k_z \sigma_z \]
\[ + (1 - \kappa) (\delta k_x \sin \theta + \delta k_z \cos \theta + a_x \sin \theta + \tau q_x \sin \theta) (\sigma_x \sin \theta + \sigma_z \cos \theta), \] (4.7)

where we used that \( a_z = 0 = q_z \).

**B. Zeroth Landau band**

A major simplification appears if the magnetization \( \beta \) and the magnetic field \( B \) are either parallel or perpendicular, so \( \cos \theta \equiv \gamma \in \{0, \pm 1\} \). In these cases the Hamiltonian \( \text{Eq. (4.7)} \) anticommutes with \( \sigma_z \) when \( \mu = 0 = \delta k_z \). This so-called chiral symmetry implies that the zeroth Landau band is an eigenstate of \( \sigma_z \), with eigenvalue \( -\tau \) \( \text{Eq. (10)} \). The dispersion relation then follows immediately,

\[ E(k_z) = \tau \kappa \mu + \tau \delta k_z [1 - (1 - \kappa) \gamma^2] \]
\[ = \chi \kappa \mu + \chi (k_z - \kappa \beta \gamma) [1 - (1 - \kappa) \gamma^2]. \] (4.8)

In the second equation we have identified the chirality index \( \chi \equiv \text{sign}(dE/dk_z) = \tau \). Equating \( E(k_z) = E \) and solving for \( k_z \) gives

\[ k_z = \kappa \beta \gamma - \frac{\kappa \mu - \chi E}{1 - (1 - \kappa) \gamma^2}, \] (4.9)

to first order in \( E \) and \( \mu \). (Higher order terms are not captured by the linearization around the Weyl point.) We substitute Eq. \( \text{Eq. (4.9)} \) in the expression \( \text{Eq. (3.13)} \) for the transferred charge,

\[ e^* = \frac{\chi e}{\mu - \beta \gamma} \left( \kappa \beta \gamma - \frac{\kappa \mu - \chi E}{1 - (1 - \kappa) \gamma^2} - \chi E \right). \] (4.10)

For \( \beta \parallel B \) this gives

\[ e^* = -\chi e \frac{\pm \kappa \beta - \mu + \chi E (1/\kappa - 1)}{\pm \beta - \mu}, \quad \gamma = \pm 1. \] (4.11)

In contrast, for \( \beta \perp B \) the \( \mu \) and \( E \) dependence drops out,

\[ e^* = -\chi e \nu, \quad \gamma = 0. \] (4.12)

These are the results for the charge transferred by a mode with \( k_z \) near \(+K\). The mode with \( k_z \) near \(-K\) is its charge-conjugate, the transferred charge is given by \( e^*(E) \rightarrow -e^*(-E) \).

**C. Comparison of transferred charge and charge expectation value**

For the case \( \chi = 1 \) that \( \beta \) is parallel to \( B \) we can use the more accurate dispersion relation from Ref. \( \text{10} \) without making the linearization around the Weyl point:

\[ E(k_z) = -\chi M(k_z) - \chi M'(k_z) \mu, \quad M(k_z) = \beta - \sqrt{\Delta_0^2 + k_z^2}. \] (4.13)

![FIG. 2. Comparison of the transferred charge \( e^* \) across the NS interface and the charge expectation value \( Q \) of the Weyl fermions. The curves are computed from Eqs. \( \text{4.14} \) and \( \text{4.17a} \) using \( k_0 \) from the full nonlinear dispersion \( \text{4.13} \).](image)

\( Q = \chi e M'(k_0) = -\frac{\chi e k_0}{\sqrt{\Delta_0^2 + k_0^2}} \) (4.17a)
\[ = -\chi e \kappa [1 + (\mu/\beta)(\kappa - 1/\kappa) + O(\mu^2)], \] (4.17b)

see Fig. \( \text{2} \). We conclude that the \( \mu \)-dependence of the transferred charge \( e^* \) is not simply accounted for by the \( \mu \)-dependence of the charge expectation value \( Q \).

**V. CONDUCTANCE**

**A. Transmission matrix**

The Landau band contains \( N_\Phi = eB \Sigma/h \) modes propagating along the magnetic field through a cross-sectional
area $S$. For each of these modes the transmission matrix $t(E)$ at energy $E$ from contact $N_1$ to $N_2$ is a rank-two matrix of the form

$$t(E) = e^{ik_z L} \langle \Psi_2^+ | \langle \Psi_1^+ | + e^{-ik_z L} \langle \Psi_2^- | \langle \Psi_1^- |. \ (5.1)$$

The incoming mode $| \Psi_1^\pm \rangle$ from contact $N_1$ is matched in $S$ to a Landau band mode at $\pm k_z$. This chiral mode propagates over a distance $L$ to contact $N_2$, picking up a phase $e^{\pm ik_z L}$, and is then matched to an outgoing mode $| \Psi_2^\pm \rangle$. The matching condition gives a charge $\pm e^* (\pm E)$ to $\Psi_n^\pm$,

$$\langle \Psi_n^\pm | \nu_z | \Psi_n^\pm \rangle = \pm e^* (\pm E). \ (5.2)$$

The transmission matrix $t(E)$ has electron and hole submatrices $t_{ee}$ and $t_{he}$ (transmission of an electron as an electron or as a hole). These determine the differential conductance

$$\frac{dI_2}{dV_1} = \lim_{E \to eV_1} \text{Tr} \left( t_{ee}^e t_{ee}^e - t_{he}^e t_{he}^e \right) = \frac{1}{2} G_0 \text{Tr} (1 + \nu_z) t_l^e (eV_1) \nu_z t_e (eV_1), \ (5.3)$$

with $G_0 = N_\Phi e^2 / h$.

### B. Linear response

The linear response conductance $G = \lim_{V_1 \to 0} dI_2 / dV_1$ simplifies because at the Fermi level we can use the particle-hole symmetry relations

$$\nu_x \sigma_y \nu_y \sigma_y = t^* \left| \Psi_1^+ \right\rangle = \nu_y \sigma_y \left| \Psi_1^- \right\rangle^* \ \text{at} \ E = 0. \ (5.4)$$

These two relations imply that

$$\text{Tr} t^e \nu_z t = 0 \ \text{at} \ E = 0. \ (5.5)$$

The equation (5.3) for the differential conductance thus reduces in linear response to

$$G = \frac{1}{2} G_0 \nu_z t_l^e \nu_z t_e = \frac{1}{2} G_0 \sum_{\alpha=x,y,z} \langle \Psi_2^\pm | \nu_z | \Psi_2^\pm \rangle \langle \Psi_1^\pm | \nu_z | \Psi_1^\pm \rangle = N_\Phi \frac{(e^*)^2}{h}. \ (5.6)$$

The charge $e \mapsto e^*$ quadratically renormalizes the conductance $[11]$. Application of Eq. (4.10) at $E = 0$ then gives the result

$$G / G_0 = \begin{cases} \kappa^2 \pm (2 \mu / \beta) (\kappa^2 - \kappa) & \text{if} \ \beta \parallel B, \\ \kappa^2 & \text{if} \ \beta \perp B, \end{cases} \ (5.7)$$

to first order in $\mu$. The $\pm$ sign refers to $\beta$ parallel $(\pm)$ or antiparallel $(-)$ to $B$. The difference $\delta G = G(B) - G(-B)$ is thus given by the formula (1.2) announced in the introduction.

### VI. NUMERICAL RESULTS

To test these analytical results, we have calculated the conductance numerically from a tight-binding model obtained by discretizing the Hamiltonian (2.2) of the Weyl superconductor on a cubic lattice (lattice constant $a_0$):

$$H_S = (v_F / a_0) \tau_z \sum_{\alpha=x,y,z} \sigma_\alpha \sin (a_0 \nu_z k_\alpha - e a_0 \nu_0 A_n) + \nu_0 \gamma_0 \beta \cdot \sigma - \mu \nu_z \gamma_0 \sigma_0 + \Delta_0 (\nu_z \cos \phi - \nu_y \sin \phi) \gamma_0 \sigma_0 + (v_F / a_0) \nu_z \tau_z \sigma_0 \sum_{\alpha=x,y,z} (1 - \cos a_0 k_\alpha). \ (6.1)$$

The term on the last line is added to avoid fermion doubling.

The vortex lattice (a square array with lattice constant $d_0$ and two $h / 2 e$ vortices per unit cell) is introduced as described in Ref. [10]. The scattering matrix is calculated using the Kwant code [14], and then the linear-response conductance follows from

$$G = \frac{I_2}{V_1} = \frac{e^2}{h} \text{Tr} (t_l^e \nu_z t_l^e - t_{he}^e t_{he}^e), \ (6.2)$$

where the trace is taken over all the $N_\Phi$ modes in the magnetic Brillouin zone and the transmission matrices are evaluated at the Fermi level ($E = 0$).

In Fig. 4 we compare the conductance with and without a potential step at the NS interfaces. In the absence of a potential step, when the Hamiltonian $H_N$ in $N$ equals $H_S$ with $\Delta_0 = 0$, the conductance has the bare value of $G_0 = N_\Phi e^2 / h$, as long as $\Delta_0$ remains well below $\beta$. When $\Delta_0$ exceeds $\beta$ a gap opens up at the Weyl point and the three-terminal conductance $G$ vanishes: All the carriers injected into the superconductor by contact $N_1$ are then drained to ground before they reach contact $N_2$.

The theory developed here does not apply to this case $\mu_N = \mu$, but instead addresses the more realistic case $\mu_N \gg \mu$ of a large potential step at the NS interfaces. In the numerics we implement the large-$\mu_N$ limit by removing the transverse hoppings from the tight-binding Hamiltonian in the normal-metal leads, which is then given by

$$H_N = (v_F / a_0) \nu_z \tau_z \sigma_2 \sin a_0 k_z + \nu_0 \gamma_0 \beta \cdot \sigma + (v_F / a_0) \nu_z \tau_z \sigma_0 (1 - \cos a_0 k_z). \ (6.3)$$

As shown in the same Fig. 3 in that case the conductance at $\mu = 0$ follows the predicted $\kappa^2 = 1 - \Delta_0^2 / \beta^2$ parabolic profile [11]. The agreement is better for $B$ perpendicular to $\beta$ than it is for $B$ parallel to $\beta$.

Fig. 4 is the test of our key result, the difference (1.2) of the conductance for $B$ parallel or antiparallel to $\beta$. The linear $\mu$-dependence has the predicted slope, without any adjustable parameter. Backscattering from the NS interfaces produces Fabry-Perot-type oscillations around this linear dependence, more rapidly oscillating when the separation $L$ of the NS interfaces is larger (compare dashed and solid curves).
FIG. 3. Dependence of the conductance $G$ on the pair potential $\Delta_0$, computed from the tight-binding model for $B$ parallel to $\beta$ (left panel) and for $B$ perpendicular to $\beta$ (right panel). The parameters are $d_0 = 18 a_0$, $L = 30 a_0$, and $\mu = 0$ (so there is no difference between parallel or antiparallel orientation of $B$). The red and blue curves show the results with and without a large potential step at the NS interfaces. The black curve is the $G = \kappa^2 G_0$ from Ref. 11.

FIG. 4. Dependence of the conductance on the orientation of $B$, when it is perpendicular to $\beta$ and there is a large potential step at the NS interfaces ($\mu N \to \infty$). The colored curves show $\delta G = G(B) - G(-B)$ as a function of $\mu$, computed from the tight-binding model ($d_0 = 18 a_0$, three values of $\Delta_0/\beta$, two values of $L$). The black dotted line is the linear $\mu$-dependence following from Eq. (5.7).

VII. CONCLUSION

In summary, we have calculated the charge $e^* = \kappa Q_0^*$ that Weyl fermions in a superconducting vortex lattice transport into a normal-metal contact. When the chemical potential $\mu$ in the superconductor is at the Weyl point, the transferred charge equals the charge expectation value $Q_0$ of the Weyl fermions [11] (in the limit of a large chemical potential $\mu_N$ in the metal contacts). There is then no dependence on the relative orientation of the magnetic field $B$ and the separation vector $\beta$ of the Weyl points of opposite chirality. But when $\mu \neq 0$ a dependence on $B \cdot \beta$ appears.

This signature of chirality shows up in the conductance, which differs if $B$ is parallel or antiparallel to $\beta$.

It is not a large effect, a few percent (see Fig. 4), but since it is specifically tied to the sign of the magnetic field it should stand out from other confounding effects.

We have taken a simple layered model for a Weyl superconductor [8], to have a definite form for the pair potential. We expect the effect to be generic for Weyl semimetals in which superconductivity is intrinsic rather than induced [15, 16]. We also expect the effect to be robust to long-range disorder scattering, in view of the chirality of the motion along the magnetic field lines (backscattering needs to couple states at $\pm K$).

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Appendix A: Derivation of Eq. (3.12)

We wish to show that the derivative

$$
\frac{d}{dz}(\Psi|f(\hat{k}_z)\hat{v}_z|\Psi) = \langle \Psi|f(\hat{k}_z)\hat{v}_z\partial_z\Psi|\Psi\rangle_z + \langle \partial_z\Psi|f(\hat{k}_z)\hat{v}_z\Psi|\Psi\rangle_z = i\langle \Psi|f(\hat{k}_z)\hat{v}_z\Psi|\Psi\rangle_z - i\langle \hat{v}_z\Psi|f(\hat{k}_z)\Psi\rangle_z
$$

(A1)

vanishes for any function $f(\hat{k}_z)$ of $\hat{k}_z = -i\partial/\partial z$.

We rewrite

$$
\hat{k}_z\hat{v}_z\Psi = (\mathcal{H} - \delta\mathcal{H})\Psi = (E - \delta\mathcal{H})\Psi
$$

(A2)

and use firstly that

$$
\langle \Psi_1|\delta\mathcal{H}\Psi_2\rangle_z = \langle \delta\mathcal{H}\Psi_1|\Psi_2\rangle_z,
$$

(A3)
because $\delta H$ does not contain any $z$-derivatives, and secondly that

$$\left[ f(\hat{k}_z), \delta H \right] = 0,$$

(A4)

because $\delta H$ does not depend on $z$. This gives the sequence of identities

$$\langle \Psi | f(\hat{k}_z) \hat{k}_z \hat{v}_z \Psi \rangle_z = \langle \Psi | f(\hat{k}_z)(H - \delta H) \Psi \rangle_z = \langle \Psi | f(\hat{k}_z)(E - \delta H) \Psi \rangle_z = \langle (E - \delta H) \Psi | f(\hat{k}_z) \Psi \rangle_z = \langle (H - \delta H) \Psi | f(\hat{k}_z) \Psi \rangle_z = \langle \hat{k}_z \hat{v}_z | f(\hat{k}_z) \Psi \rangle_z.$$ (A5)

Substitution into Eq. (A1) then proves Eq. (3.12) from the main text.

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