Quantized Laplacian growth, I: Statistical theory

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The Laplacian growth problem is regularized by quantization of the area of growing domains, which equals an integer multiple of the area quanta $\hbar$. The nonzero $\hbar$ determines a short distance cutoff preventing the development of cusps at the interface in a finite time. The quantized domain can be considered as an aggregate of tiny particles (area quanta) obeying the Pauli exclusion principle, i.e., as the droplet of incompressible quantum fluid consisting of noninteracting fermions. Then, statistical theory for Laplacian growth is obtained by using Laughlin’s theory of the integer quantum Hall effect. The classical evolution of the aggregate reproduces the deterministic Laplacian growth dynamics. However, the quantization procedure generates inevitable fluctuations at the interface. Remarkably, the statistical behavior of the fluctuations is universal and common to that of quantum chaotic systems, whose properties can be conventionally described by Dyson’s circular orthogonal ensemble on symmetric unitary matrices.

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Laplacian growth is a fundamental problem of pattern formation in nonequilibrium physics [1, 2]. It embraces numerous free boundary dynamics including bidirectional solidification, dendritic formation, electro deposition, dielectric breakdown, bacterial growth, and flows in porous media [3]. These processes can be typically represented by diffusion driven motion of an unstable interface between different phases. Despite its seeming simplicity, Laplacian growth raises enormous interest in physics and mathematics, because it produces a variety of universal patterns with remarkable geometrical and dynamical properties [4, 5].

An intrinsic instability of the interface dynamics is a distinctive feature of Laplacian growth. The relevant example is a viscous fingering in a Hele-Shaw cell, when a less viscous fluid is injected into a more viscous one in a narrow gap between two plates [5, 6]. When an effect of surface tension is negligibly small, the interface dynamics becomes highly unstable. This, the so-called idealized Laplacian growth problem, possesses a reach integrable structure, unusual for most nonlinear physical phenomena [7–9]. The patterns typically observed in experiments are fractals with universal scaling behavior similar to those observed in diffusion-limited aggregation—a discrete stochastic fractal growth process [10].

The idealized problem is known to be ill-defined, because the interfaces typically develop the cusplike singularities in a finite time [7]. Thus, a mechanism stabilizing the growth is necessary. For instance, surface tension restores a stability of the growth process by preventing the formation of singularities with an infinite curvature. However, it is a singular perturbation of the system, which ruins a reach mathematical structure of the idealized Laplacian growth problem. Nevertheless, it is commonly believed that a particular mechanisms regularizing singularities on a microscopic scale (e.g., surface tension, lattice regularization, etc.) should not effect the macroscopic features of the observed patterns.

In this paper, the Laplacian growth problem with zero surface tension is regularized by quantization of the area of growing domains, which equals an integer multiple of the area quanta $\hbar$. Remarkably, this regularization mechanism retains a rich integrable structure of the idealized problem [11]. Besides, the quantized Laplacian growth is known to be closely related to the evolution of an electronic droplet in the quantum Hall regime [12]. Indeed, the quantized domain can be considered as an aggregate of tiny particles with the size $\hbar$ (area quanta) obeying the Pauli exclusion principle, i.e., as the droplet of incompressible quantum fluid consisting of noninteracting fermions. Therefore, there arises a possibility to describe statistical properties of patterns observed in Laplacian growth by means of Laughlin’s theory of the integer quantum Hall effect.

Laughlin’s theory is commonly applied for statistical description of electronic droplets in the quantum Hall regime [13]. In what follows, we will ignore the Coulomb interactions between the electrons, scale a magnetic field by the uniform field $B_0$, so that $l = \sqrt{\hbar c/eB_0}$ is a magnetic length, and the magnetic flux is $\Phi = N\Phi_0$, where $\Phi_0 = 2\pi\hbar$ is a unit of flux quanta [14]. Then, the ground state of the system of $N$ noninteracting electrons can be found exactly even for a nonuniform magnetic field,

$$
\Psi_N(z) = \frac{1}{\sqrt{N!}} \prod_{i<j} (z_i - z_j) e^{-\frac{i}{\hbar} \sum_i |z_i\bar{z}_i - 2V(z_i)|},
$$

where $z_j = x_j + iy_j$ is a position of the $j$th electron in the plane, $z = \{z_1, \ldots, z_N\}$, the overbar denotes complex conjugation, the normalization factor is

$$
\tau_N = \frac{1}{N!} \int \prod_{i<j} |z_i - z_j|^2 \prod_i e^{-\frac{i}{\hbar} [z_i\bar{z}_i - 2V(z_i)]} d^2 z_i,
$$

and $V(z)$ is the potential of a nonuniform part of the magnetic field $\delta B = -\nabla^2 V$. Although $\delta B = 0$ inside
the droplet occupied by electrons, the potential \( V(z) \) is nonzero in this region. It strongly effects the electrons and, therefore, a shape of the domain—this is a manifestation of the Aharonov-Bohm effect. Inside the droplet \( V(z) \) is an analytic function, which possesses the series expansion in terms of the multipole moments \( T_k \), namely,

\[
V(z) = \text{Re} \sum_{k \geq 1} T_k z^k, \quad T_k = \frac{1}{2\pi k} \int_C \delta B(z) z^{-k} d^2 z. \tag{3}
\]

An similarity between the quantized domains and incompressible electronic droplets allows one to introduce statistical theory for Laplacian growth. Laughlin’s wave functions (1) are the main tools to study statistical properties of quantized domains, whose shapes can be reconstructed from the potential (3) by means of the inverse potential problem. Thus, we will refer to electrons as to particles (area quanta) when a statistical theory of quantized Laplacian growth is discussed.

If the number of particles \( N \) constituting the droplet is large, whereas their size \( \hbar \) is small, and the area of the droplet \( \pi t_0 = \hbar N \) is kept finite, the aggregate can be described semiclassically. In this case, the density of particles steeply drops down at the edge of the droplet [15]. The integral (2) is dominated by a uniform distribution of particles, which fill the domain \( D \) with a constant density. This domain is characterized by the area \( \pi t_0 = \hbar N \), and the set of harmonic moments,

\[
t_k = \frac{1}{\pi k} \int_{\mathbb{C} \setminus D} z^{-k} d^2 z, \quad k = 1, 2, \ldots, \tag{4}
\]

which are equal to the multipole moments of the potential (3). The semiclassical distribution of particles also determines the leading contribution to the free energy, \( F = \lim_{\hbar \to 0} \hbar^2 \log \tau N \) [16], thus establishing a relation to the tau function of analytic curves [17],

\[
F(t_0, t_1, \bar{t}_1, \ldots) = -\frac{1}{\pi^2} \int_D \int_D \log \left| \frac{1}{z} - \frac{1}{z'} \right| d^2 z d^2 z'. \tag{5}
\]

The equality of the harmonic moments \( t_k \) and the multipole moments \( T_k \) indicates a state of static mechanical equilibrium, when all particles are “frozen” in their equilibrium positions. An adiabatic variation of the potential (3) (equivalently, the magnetic field) disturbs the system away from the equilibrium, so that \( t_k \neq T_k \). If the system is connected to a large capacitor that maintains a certain chemical potential, the domain will change due to newly arrived particles tending to compensate the variation of \( V(z) \) [18].

By increasing the total magnetic flux through the system, \( \delta \Phi = K \Phi_0 \), one adds \( K \gg 1 \) particles to the aggregate, so that its edge advances. The growth rate of the area of the droplet is \( Q = \hbar K / \delta t \), where \( \delta t \) is a time unit. We will show, that the semiclassical evolution of the boundary reproduces the deterministic Laplacian growth dynamics [19], when the velocity of the advance is proportional to the harmonic measure of the boundary [20],

\[
u(\zeta, t) = Q|w'(\zeta, t)|, \quad \zeta \in \partial D(t), \tag{6}
\]

where \( w(z, t) \) is the time dependent conformal from the exterior of the droplet \( D(t) \) in the physical \( z \) plane to the complement of the unit disk, \( |w| \geq 1 \), in the auxiliary \( w \) plane. The dashed line in the \( z \) plane represents an advance of the domain, \( D(t) \to D(t + \delta t) \), per time unit, which is determined by density of eigenvalues (the dashed line in the \( w \)-plane) of Dyson’s circular ensemble (15).

**FIG. 1.** The time dependent conformal map \( w(z, t) \) from \( \mathbb{C} \setminus D(t) \) in the physical \( z \) plane to the complement of the unit disk, \( |w| \geq 1 \), in the auxiliary \( w \) plane. The dashed line in the \( z \) plane represents an advance of the domain, \( D(t) \to D(t + \delta t) \), per time unit, which is determined by density of eigenvalues (the dashed line in the \( w \)-plane) of Dyson’s circular ensemble (15).

The translational probability is determined by the absolute square \( |\psi_{N,K}(z)|^2 \) of the overlap function [21],

\[
\psi_{N,K}(z) = \int_C \cdots \int_C \overline{\psi_N(z')} \psi_{N+K}(z', z) d^2 z' . \tag{7}
\]

Let us study a semiclassical limit of the overlap function, when \( K \) is large (1 \( < K \ll N \)), whereas \( \hbar \) is small and \( K \hbar \) is kept finite. In this case, the newly arrived particles form a thin layer \( l \) with the area \( K \hbar \) in the immediate vicinity of the boundary of the aggregate. Below, we will show that the Laplacian growth equation (6) describes the most probable growth scenario, when the newly attached particles do not change the harmonic moments (4), causing the area growth \( \pi \delta t_0 = Q \delta t \) only. In the other exponentially suppressed growth scenarios the newly arrived particles at the boundary will slightly deviate from the most probable distribution.

The grainy structure of the interface, arising because of the microscopic cutoff \( \hbar \), is a primary cause of fluctuations in the quantized Laplacian growth problem. The
probability distribution functions of the fluctuations is given by the absolute square of the overlap function (7). Using the properties of the Vandermonde determinant, it is convenient to recast the overlap function in the form

\[
\psi_{N,K}(z) = \sqrt{\frac{N!}{(N+K)!}} e^{-\frac{\pi}{\hbar} \sum_{i<j} |z_i - z_j|^2} \left( \prod_{i<j} (z_i - z_j) \right)^{N} \cdot \frac{1}{\sqrt{T_N T_{N+K}}},
\]

where the operator \( D(z) = \sum_{k>0} (z^{-k}/k) \partial / \partial t_k \) generates the infinitesimal deformations of the magnetic field. On expanding (8) in \( \hbar \) the probability \( |\psi_{N,K}(z)|^2 \) can be recast in terms of the variations of the free energy (5) at the boundary of the droplet,

\[
|\psi_{N,K}(z)|^2 \simeq \frac{1}{\sqrt{2\pi \hbar}} \prod_{i<j} |z_i - z_j|^2 \exp \left[ -\frac{2}{\hbar} \sum_i A(z_i) \right] \exp \left[ -\sum_{i,j} \left( \frac{1}{2} \partial^2_{t_0} - \text{Re} D(z_i) D(z_j) \right) F \right],
\]

where \( A(z) \equiv A(z, z) = |z|^2/2 - \text{Re} \Omega(z) \) is the modified Schwarz potential written in terms of the following generating function \([22, 23]\).

\[
\Omega(z) = \sum_{k>0} T_k z^k + t_0 \log z - \left( \frac{1}{2} \partial^2_{t_0} + D(z) \right) F.
\]

The modified Schwarz potential characterizes the curve, which bounds a planar domain. As a function of \( z \) it reaches a maximum on the boundary of the domain \( D \) with the harmonic moments \( T_k \) and the area \( \pi t_0 \). The extremum condition yields \( z = \partial_z \Omega(z) \), so that \( S(z) = \partial_z \Omega(z) \) is the Schwarz function of the contour \([23]\). Besides, everywhere on the boundary the potential vanishes, \( A(z) = 0, z \in \partial D \), and, therefore, \( A(z + \delta z) = -\partial_z S(z)(\delta z)^2 + \ldots \) remains positive in the vicinity of the contour.

The free energy \( F \), as a real function of the harmonic moments (4), satisfies certain symmetry relations reflecting integrability of the problem \([24]\). These relations allow one to express the conformal map \( w(z, t) \) in terms of the variation of \( F \), in particular, \([17]\).

\[
\log \frac{w(z_i) - w(z_j)}{z_i - z_j} = \left( -\frac{1}{2} \partial^2_{t_0} + D(z_i) D(z_j) \right) F.
\]

Here and below, we do not indicate the time dependence of \( w(z) \) explicitly. Then, the absolute square of the overlap function can be recast in the form

\[
|\psi_{N,K}(z)|^2 \simeq \frac{1}{\sqrt{2\pi \hbar}} e^{\kappa} \int F \chi(z) \log |z - z'| \chi(z') dz^2 dz' + \frac{\kappa}{\hbar} \int F \chi(z) \log \left| \frac{w(z) - w(z')}{|z - z'|} \right| \chi(z') dz^2 dz'.
\]

where \( \chi(z) = \sum_{i=1}^K \delta(z - z_i) \) is a density (a sum of two-dimensional delta functions) of the newly arrived particles constituting the layer \( l \). The density is normalized so that \( \int \chi(z) d^2 z = K \hbar \) is the area of the layer. In the semiclassical limit \( \chi(z) \) is a characteristic function of the layer, i.e., \( \chi(z) = 1 \) if \( z \in l \), and \( \chi(z) = 0 \) otherwise.

Stokes’ theorem allows one to reduce the integral over the layer to the contour integrals over its boundaries, which further can be contracted to a single contour integral. By focusing attention on the logarithmic singularities in the integrands in (12), one obtains \([25]\)

\[
\int \int \log |z - z'| d^2 z d^2 z' - 2 \int \mathcal{A}(z) d^2 z = \frac{1}{\hbar} (\delta t)^2 \int_{\partial D} \int_{\partial D} v(\zeta) \log |\zeta - \zeta'| |v(\zeta')| d\zeta ||d\zeta'|.
\]

where \( \partial D \) denotes the internal boundary of the layer coinciding with the boundary of the domain, and \( v(\zeta) \delta t \) is the advance of the interface per time unit \( \delta t \). These relations allow one to recast the probability \( |\psi_{N,K}(z)|^2 \simeq P[v(\zeta)] \) in the functional on the velocity of the boundary,

\[
P[v(\zeta)] \simeq e^{\frac{(\delta t)^2}{\hbar} \int_{\partial D} \int_{\partial D} v(\zeta) \log |w(\zeta) - w(\zeta')| v(\zeta') |d\zeta||d\zeta'|}.
\]

The classical point for the action, \( -\log P[v(\zeta)] \), is the most probable growth scenario, which reproduces the deterministic Laplacian growth dynamics (6). This maximum is exponentially sharp in the vanishing particle size limit \( \hbar \to 0 \). In this case, all fluctuations are suppressed, so that only the deterministic Laplacian growth survives. However, in computer simulations with diffusion-limited aggregation the initially featureless interfaces quickly develop into a branched graph with a characteristic width \( \hbar \) \([10]\). Thus, this naive limit \( \hbar \to 0 \) is impossible, because it yields the development of cusps at the interface.

An arbitrary small nonzero \( \hbar \) makes fluctuations of the interface inevitable, thus modifying the growth process. For instance, experimental observations in the Hele-Shaw cell usually show a chaotic dynamics of the interfaces. The mechanism leading to this behavior is missing in the deterministic Laplacian growth problem, where the complex irregular shapes are commonly explained by tiny uncontrollable initial deviations of the interface from the perfect shape. In our case, the fluctuations are inevitable because of the grainy structure of the boundary.

The analytic treatment of the functional (14) can be considerably simplified by mapping it to the auxiliary \( w \) plane. Let \( \rho(\theta) \) be a macroscopic density function on the unit circle, \( |w| = 1 \), related to the interface velocity in
the $z$ plane by the conformal factor (see Fig. 1),

$$ v(\zeta) = \nu \nu'(\zeta) |\rho(\theta)|, \quad \nu = \hbar / \delta t. \quad (15) $$

where $\nu$ is the rate quanta, so that the total growth rate $Q = K \nu$ is quantized. Then, the functional $P$ becomes

$$ P[\rho(\theta)] \sim \exp \left[ \int_0^{2\pi} \int_0^{2\pi} \rho(\theta) \log |e^{i\theta} - e^{i\theta'}| |\rho(\theta')| d\theta d\theta' \right], \quad (16) $$

and the exponent of $P$ is a negative Coulomb energy of the electric charge distributed on the unit circle with the density $\rho(\theta)$. The uniform density $\rho = K / 2\pi$ maximizes the functional, and corresponds to the most probable deterministic Laplacian growth dynamics (6).

The nonuniform densities describe tiny fluctuations at the interface arising during the evolution. Remarkably, the statistical behavior of these fluctuations (16) is common with Dyson’s circular ensemble on symmetric unitary $K \times K$ matrices in the limit $K \to \infty$ [26],

$$ P(\theta_1, \ldots, \theta_K) \sim \prod_{i<j} \sin \left( \frac{\theta_i - \theta_j}{2} \right), \quad (17) $$

so that the density $\rho(\theta) = \sum_{i=1}^{K} \delta(\theta - \theta_i)$ is a sum of one-dimensional delta functions. The important characteristic of Dyson’s ensemble is the mean square deviation $\Delta$ of eigenvalues from their equilibrium positions. Denoting the mean level spacing by $D = (2\pi / K)$, one obtains [27]

$$ (\Delta / D)^2 \sim \log(Q / \nu), \quad (18) $$

where the growth rate $Q$ is an integer multiple of rate quanta $\nu$. Therefore, when the growth rate $Q$ is large the fluctuations become significant. We will show elsewhere, that fluctuations seriously effect the interface dynamics, yielding the development of chaotic patterns in a long time asymptotic.

In summary, a short distance regularization of the Laplacian growth problem is achieved by quantization of the area of growing domains. This regularization mechanism prevents the naive limit $\hbar \to 0$, and makes the fluctuations of the interface during the evolution inevitable. The effect of jumpy motion of the grainy boundary is equivalent to noise on a microscale, which drastically effects the dynamics for large growth rates. The noise is commonly believed to be the main mechanism responsible for the tip splitting, instability, and chaotic dynamics resulting in the universal fractal patterns observed in experiments.

Remarkably, the statistical behavior of the fluctuations is universal and common to that of quantum chaotic systems, whose properties can be conventionally described by Dyson’s circular orthogonal ensemble on symmetric unitary matrices. Then, the analysis of the functional (14) promises to elucidate statistical properties of Laplacian growth (in particular, the selection problem) and its relation to diffusion-limited aggregation.

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