Commuting Jacobi Operators on Real Hypersurfaces of Type B in the Complex Quadric

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Abstract
In this paper, first, we investigate the commuting property between the normal Jacobi operator $\bar{R}_N$ and the structure Jacobi operator $R_\xi$ for Hopf real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$ for $m \geq 3$, which is defined by $\bar{R}_N R_\xi = R_\xi \bar{R}_N$. Moreover, a new characterization of Hopf real hypersurfaces with $\mathfrak{A}$-principal singular normal vector field in the complex quadric $Q^m$ is obtained. By virtue of this result, we can give a remarkable classification of Hopf real hypersurfaces in the complex quadric $Q^m$ with commuting Jacobi operators.

Keywords Commuting Jacobi operator · $\mathfrak{A}$-isotropic · $\mathfrak{A}$-principal · Kähler structure · Complex conjugation · Complex quadric

Mathematics Subject Classification (2010) Primary 53C40 · Secondary 53C55

1 Introduction

In the class of Hermitian symmetric spaces of rank 2, usually we can give the examples of Riemannian symmetric spaces $G_2(C^{m+2}) = SU_{m+2}/S(U_2 U_m)$ and $G_2^*(C^{m+2}) = SU_{2,m}/S(U_2 U_m)$, which are said to be complex two-plane Grassmannians and

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complex hyperbolic two-plane Grassmannians, respectively (see [4, 11, 24, 25] and [27]). These are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with Kähler structure $J$ and quaternionic Kähler structure $J$. There are exactly two types of singular tangent vectors $X$ of complex 2-plane Grassmannians $G_2^m(Cm+2)$ and complex hyperbolic 2-plane Grassmannians $G_2^*(Cm+2)$ which are characterized by the geometric properties $JX \in JX$ and $JX \perp JX$ respectively.

As another kind of Hermitian symmetric space with rank 2 of compact type different from the above ones, we can give the example of complex quadric $Q_m = SO_{m+2}/SO_mSO_2$, which is a complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$ (see [20–23] and [26]). The complex quadric also can be regarded as a kind of real Grassmann manifold of compact type with rank 2 (see [7] and [12]). Accordingly, the complex quadric admits both a complex conjugation structure $A$ and a Kähler structure $J$, which anti-commutes with each other, that is, $AJ = -JA$. Then for $m \geq 3$ the triple $(Q_m, J, g)$ is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see [10] and [20]).

In addition to the Kähler structure $J$ there is another distinguished geometric structure on $Q_m$, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an $S^1$-bundle of real structures, that is, complex conjugations $A$ on the tangent spaces of $Q_m$. The set is denoted by $\mathfrak{A}_z = \{ A_{\lambda z} \mid \lambda \in S^1 \subset \mathbb{C} \}$, $z \in Q_m$, and it is the set of all complex conjugations defined on $Q_m$. Then $\mathfrak{A}_z$ becomes a parallel rank 2-subbundle of $End(TQ_m)$. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $Q$ of the tangent bundle $TM$ of a real hypersurface $M$ in $Q^m$. Here the notion of parallel vector bundle $\mathfrak{A}$ means that $(\bar{\nabla} X A)Y = q(X)JA Y$ for any vector fields $X$ and $Y$ on $Q_m$, where $\bar{\nabla}$ and $q$ denote a connection and a certain 1-form defined on $T_z Q^m$, $z \in Q^m$, respectively (see [23]).

Recall that a nonzero tangent vector $W \in T_z Q^m$ is called singular if it is tangent to more than one maximal flat in $Q^m$. There are two types of singular tangent vectors for the complex quadric $Q^m$:

(a) If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A) := \{ W \mid AW = W \}$, then $W$ is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.

(b) If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then $W$ is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

On the other hand, a typical characterization for real hypersurfaces with $\mathfrak{A}$-principal normal vector field in $Q^m$ was introduced in [2] as follows.

**Theorem A.** Let $M$ be a connected orientable real hypersurface with constant mean curvature in the complex quadric $Q^m$, $m \geq 3$. Then $M$ is a contact hypersurface if and only if $M$ is congruent to an open part of the tube around the $m$-dimensional sphere $S^m$ which is embedded in $Q^m$ as a real form of $Q^m$.

In fact, we say that $M$ is a contact hypersurface of a Kaehler manifold if there exists an everywhere nonzero smooth function $\rho$ such that $d\eta(X, Y) = 2\rho g(\phi X, Y)$ holds on $M$. Here $(\phi, \xi, \eta, g)$ is an almost contact metric structure of $M$. It can
be easily verified that a real hypersurface $M$ is contact if and only if there exists an everywhere nonzero constant function $\rho$ on $M$ such that $S\phi + \phi S = 2\rho\phi$. In particular, this concept of contact real hypersurfaces can be regarded as a typical characterization of model spaces of type $B$ in complex projective space and complex hyperbolic space, respectively (see [13] and [32]). As far as we know, this is the only characterization of the model space of type $B$ in $Q^m$, which is the tube around $m$-dimensional sphere $S^m$ in $Q^m$ (Hereafter we denote this model space $(T_B)$).

In this paper, we investigate some characterization problems for Hopf real hypersurfaces in $Q^m$. The notion of Hopf means that the Reeb vector field $\xi$ of $M$ is principal by the shape operator $S$ of $M$, that is, $S\xi = g(S\xi, \xi)\xi = \alpha\xi$. When the Reeb curvature function $\alpha = g(S\xi, \xi)$ identically vanishes on $M$, we say that $M$ has vanishing geodesic Reeb flow. Otherwise, $M$ has non-vanishing geodesic Reeb flow. Recently, many characterizations of Hopf real hypersurfaces in the complex quadric $Q^m$ have been given by some differential geometers from various geometric points of view (see [1, 2, 8–10, 14, 16, 18, 19] etc).

On the other hand, Jacobi fields along geodesics of a given Riemannian manifold $(\tilde{M}, \tilde{g})$ satisfy a well known differential equation (see [6]). This equation naturally inspires the so-called Jacobi operator. That is, if $\tilde{R}$ denotes the curvature operator of $\tilde{M}$, and $Z$ is a tangent vector field to $\tilde{M}$, defined by $(\tilde{R}_Z Y)(p) = (\tilde{R}(Y, Z)Z)(p)$ for any $Z \in T_p\tilde{M}$, becomes a self-adjoint endomorphism of the tangent bundle $TM$ of $\tilde{M}$. Thus, the normal vector field $N$ of a real hypersurface $M$ in $Q^m$ provides the Jacobi operator $\tilde{R}_N \in \text{End}(TM)$, which is said to be a normal Jacobi operator. Moreover, for the Reeb vector field $\xi := -JN \in TM$ the Jacobi operator $\tilde{R}_\xi \in \text{End}(TM)$ is called a structure Jacobi operator. Here $\tilde{R}$ and $R$ are the Riemannian curvature tensors for $Q^m$ and its real hypersurface $M$, respectively.

By the Kähler structure $J$ of the complex quadric $Q^m$, we can decompose its action on any tangent vector field $X$ on $M$ in $Q^m$ as follows:

$$JX = \phi X + \eta(X)N,$$

where $\phi X$ denotes the tangential component of $JX$ and $\eta$ a 1-form defined by $\eta(X) = g(JX, N) = g(X, \xi)$ for the Reeb vector field $\xi = -JN$ and $N$ a unit normal vector field on $M$ in $Q^m$.

When the Ricci tensor $\text{Ric}$ of $M$ in $Q^m$ commutes with the structure tensor $\phi$, that is, $\text{Ric} \phi = \phi \text{Ric}$, we say that $M$ has Ricci commuting or commuting Ricci tensor. Pérez and Suh [17] proved a non-existence property for Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel and commuting Ricci tensor. In [30] Suh and Hwang gave another classification for real hypersurfaces in $Q^m$ with commuting Ricci tensor. Recently, in [31] the present authors and Woo studied the commuting normal Jacobi operator (resp. the structure Jacobi operator) defined by $\tilde{R}_N \phi = \phi \tilde{R}_N$ (resp. $R_\xi \phi = \phi R_\xi$).

Motivated by these studies, in this paper, we consider the commuting property between the normal Jacobi operator $\tilde{R}_N$ and the structure Jacobi operator $R_\xi$ given by

$$\tilde{R}_N R_\xi = R_\xi \tilde{R}_N.$$  (**)

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Commuting property with Jacobi operators was first initiated by Brozos-Vázquez and Gilkey [5]. They gave two results for a Riemannian manifold \((\tilde{M}^m, \tilde{g}), m \geq 3\), as follows: One is: if \(\tilde{R}_U \tilde{R}_V = \tilde{R}_V \tilde{R}_U\) for all tangent vector fields \(U, V\) on \(\tilde{M}\), then \(\tilde{M}\) is flat. The other is: if the same occurs for any \(U \perp V\), then \(\tilde{M}\) has constant sectional curvature.

In addition, in [15] the authors classified real hypersurfaces in \(G_2(\mathbb{C}^{m+2})\) whose structure Jacobi operator \(R_\xi\) commutes with normal Jacobi operator \(\tilde{R}_N\). Now in this paper, first, we prove that our commuting property (*) is equivalent to the singularity of normal vector field for a Hopf real hypersurface in \(Q^m\) as follows:

**Theorem 1** Let \(M\) be a real hypersurface with non-vanishing geodesic Reeb flow in \(Q^m\) for \(m \geq 3\). Then \(M\) has an A-principal normal vector field if and only if the normal Jacobi operator \(\tilde{R}_N\) commutes with the structure Jacobi operator \(R_\xi\).

Related to Theorem 1, naturally, some characterizations of Hopf hypersurfaces in terms of singularity of the normal vector field are being investigated. Among them we can give one of remarkable results as follows:

**Theorem 2** Let \(M\) be a Hopf real hypersurface in \(Q^m\) for \(m \geq 3\). Then \(M\) has an A-principal normal vector field if and only if \(M\) is locally congruent to the model space of type \((T_B)\), that is, a tube over \(m\)-dimensional sphere \(S^m\) in \(Q^m\).

By virtue of Theorems 1 and 2, we also assert the following: Let \(M\) be a real hypersurface with non-vanishing geodesic Reeb flow in \(Q^m\), \(m \geq 3\). Then \(M\) has the commuting normal Jacobi operator, \(\tilde{R}_N R_\xi = R_\xi \tilde{R}_N\), if and only if \(M\) is locally congruent to the model space of type \((T_B)\). Motivated by this result, we can give another remarkable result as follows:

**Theorem 3** Let \(M\) be a Hopf real hypersurface in \(Q^m\) for \(m \geq 3\). Then \(M\) has the commuting normal Jacobi operator, \(\tilde{R}_N R_X = R_X \tilde{R}_N\) for all tangent vector fields \(X \in \mathcal{C} = \{X \in TM \mid X \perp \xi\}\) if and only if \(M\) is locally congruent to the model space of type \((T_B)\).

### 2 The Complex Quadric

For more background to this section we refer to [3, 10–12] and [20]. The complex quadric \(Q^m\) is the complex hypersurface in \(\mathbb{C}P^{m+1}\) which is defined by the equation \(z_1^2 + \cdots + z_{m+2}^2 = 0\), where \(z_1, \ldots, z_{m+2}\) are homogeneous coordinates on \(\mathbb{C}P^{m+1}\). We equip \(Q^m\) with the Riemannian metric which is induced from the Fubini Study metric on \(\mathbb{C}P^{m+1}\) with constant holomorphic sectional curvature 4. The Kähler structure on \(\mathbb{C}P^{m+1}\) induces canonically a Kähler structure \((J, g)\) on the complex quadric. For a nonzero vector \(z \in \mathbb{C}^{m+1}\) we denote by \([z]\) the complex span of \(z\), that is, \([z] = \mathbb{C}z = \{\lambda z \mid \lambda \in S^1 \subset \mathbb{C}\}\). Note that by definition \([z]\) is a point in \(\mathbb{C}P^{m+1}\). For each \([z]\) \(\in Q^m \subset \mathbb{C}P^{m+1}\) we identify \(T_{[z]} \mathbb{C}P^{m+1}\) with the
orthogonal complement $\mathbb{C}^{m+2} \ominus \mathbb{C}z$ of $\mathbb{C}z$ in $\mathbb{C}^{m+2}$ (see Kobayashi and Nomizu [12]). The tangent space $T[z]Q^m$ can then be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus (\mathbb{C}z \oplus \mathbb{C}\rho)$ of $\mathbb{C}z \oplus \mathbb{C}\rho$ in $\mathbb{C}^{m+2}$, where $\rho \in \nu[z]Q^m$ is a normal vector of $Q^m$ in $\mathbb{C}P^{m+1}$ at the point $[z]$.

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group $SU_{m+2}$, namely $\mathbb{C}P^{m+1} = SU_{m+2}/SU_{m+1}U_1$. We denote by $o = [0, \ldots, 0, 1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $SU_{m+1}U_1$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing $o$ is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_{m}SO_{2}$. This homogeneous space model leads to the geometric interpretation of the complex quadric $Q^m$ as the Grassmann manifold $G^1_2(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of $Q^m$ as a Hermitian symmetric space of rank 2. The complex quadric $Q^1$ is isometric to a sphere $S^2$ with constant curvature, and $Q^2$ is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m \geq 3$ from now on.

For a unit normal vector $\rho$ of $Q^m$ at a point $[z] \in Q^m$ we denote by $A = A_\rho$ the shape operator of $Q^m$ in $\mathbb{C}P^{m+1}$ with respect to $\rho$. The shape operator is an involution on the tangent space $T[z]Q^m$ and

$$T[z]Q^m = V(A_\rho) \oplus JV(A_\rho),$$

where $V(A_\rho)$ is the $(+1)$-eigenspace and $JV(A_\rho)$ is the $(-1)$-eigenspace of $A_\rho$. Geometrically this means that the shape operator $A_\rho$ defines a real structure on the complex vector space $T[z]Q^m$, or equivalently, is a complex conjugation on $T[z]Q^m$. Since the real codimension of $Q^m$ in $\mathbb{C}P^{m+1}$ is 2, this induces an $S^1$-subbundle $\mathcal{A}$ of the endomorphism bundle $End(TQ^m)$ consisting of complex conjugations. There is a geometric interpretation of these conjugations. The complex quadric $Q^m$ can be viewed as the complexification of the $m$-dimensional sphere $S^m$. Through each point $[z] \in Q^m$ there exists a one-parameter family of Lagrangian submanifolds in $Q^m$ which are isometric to the sphere $S^m$. These real forms are congruent to each other under action of the center $SO_2$ of the isotropy subgroup of $SO_{m+2}$ at $[z]$. The isometric reflection of $Q^m$ in such a real form $S^m$ is an isometry, and the differential at $[z]$ of such a reflection is a conjugation on $T[z]Q^m$. In this way the family $\mathcal{A}$ of conjugations on $T[z]Q^m$ corresponds to the family of real forms $S^m$ of $Q^m$ containing $[z]$, and the subspaces $V(A) \subset T[z]Q^m$ correspond to the tangent spaces $T[z]S^m$ of the real forms $S^m$ of $Q^m$.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of $Q^m$ can be described in terms of the complex structure $J$ and the complex conjugation $A \in \mathcal{A}$:

$$\bar{R}(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY$$
$$-2g(JX, Y)JZ + g(AY, Z)AX$$
$$-g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY. \quad (2.1)$$
By using the Gauss and Weingarten formulas, the left-hand side of (2.1) becomes
\[ \bar{R}(X,Y)Z = R(X,Y)Z - g(SY,Z)SX + g(SX,Z)SY + \{g((\nabla_X S)Y, Z) - g((\nabla_Y S)X, Z)\}N, \]
where \( R \) and \( S \) denote the Riemannian curvature tensor and the shape operator of a real hypersurface \( M \) in \( Q^m \), respectively.

From this, taking tangential and normal components of (2.1), we have respectively
\[ g(R(X,Y)Z,W) - g(SY,Z)g(SX,W) + g(SX,Z)g(SY,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(JY,Z)g(JX,W) - g(JX,Z)g(JY,W) - 2g(JX,Y)g(JZ,W) + g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W) + g(JAY,Z)g(JAX,W) - g(JAX,Z)g(JAY,W). \] (2.2)

It is well known that for every unit tangent vector \( W \in T_{[z]}Q^m \) there exist a conjugation \( A \in \mathfrak{A} \) and orthonormal vectors \( Z_1, Z_2 \in V(A) \) such that
\[ W = \cos(t)Z_1 + \sin(t)JZ_2 \]
for some \( t \in [0, \pi/4] \) (see [20]). The singular tangent vectors correspond to the values \( t = 0 \) and \( t = \pi/4 \). If \( 0 < t < \pi/4 \) then the unique maximal flat containing \( W \) is \( \mathbb{R}Z_1 \oplus \mathbb{R}JZ_2 \).

### 3 Some General Equations

Let \( M \) be a real hypersurface in \( Q^m \) and denote by \( (\phi, \xi, \eta, g) \) the induced almost contact metric structure. Note that \( JX = \phi X + \eta(X)N \) and \( JN = -\xi \), where \( \phi X \) is the tangential component of \( JX \) and \( N \) is a (local) unit normal vector field of \( M \). The tangent bundle \( TM \) of \( M \) splits orthogonally into \( TM = \mathcal{C} \oplus \mathbb{R}\xi \), where \( \mathcal{C} = \ker \eta \) is the maximal complex subbundle of \( TM \). The structure tensor field \( \phi \) restricted to \( \mathcal{C} \) coincides with the complex structure \( J \) restricted to \( \mathcal{C} \), and \( \phi \xi = 0 \). Moreover, since \( Q^m \) has also a real structure \( A \), we decompose \( AX \) into its tangential and normal components for a fixed \( A \in \mathfrak{A}_{[z]} \) and \( X \in T_{[z]}M \):
\[ AX = BX + \rho(X)N \] (3.1)
where \( BX \) is the tangential component of \( AX \) and
\[ \rho(X) = g(AX,N) = g(X,AN) = g(X,AJ\xi) = g(JX,AX). \]

At each point \([z] \in M \) we can choose \( A \in \mathfrak{A}_{[z]} \) such that
\[ N = \cos(t)Z_1 + \sin(t)JZ_2 \]
for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$ (see Proposition 3 in [20]). Note that $t$ is a function on $M$. From this and $\xi = -JN$, we have

$$
\begin{aligned}
\begin{cases}
\xi = \sin(t)Z_2 - \cos(t)JZ_1, \\
AN = \cos(t)Z_1 - \sin(t)JZ_2, \\
A\xi = \sin(t)Z_2 + \cos(t)JZ_1.
\end{cases}
\end{aligned}
$$

(3.2)

These formulas lead to $g(\xi, AN) = 0$ and $g(A\xi, \xi) = -g(AN, N) = -\cos(2t)$ on $M$.

We now assume that $M$ is a Hopf real hypersurface in $Q^m$. Then the shape operator $S$ of $M$ satisfies $S\xi = \alpha\xi$ with Reeb curvature function $\alpha = g(S\xi, \xi)$ on $M$. By virtue of the equation of Codazzi (2.3), we obtain the following lemma.

**Lemma 3.1** ([1, 28]) Let $M$ be a Hopf real hypersurface in $Q^m, m \geq 3$. Then we obtain

$$(X\alpha) = (\xi\alpha)\eta(X) + 2g(A\xi, \xi)g(X, AN)$$

(i.e. $\text{grad} \alpha = (\xi\alpha)\xi - 2g(A\xi, \xi)\phi A\xi$) \hspace{1cm} (3.3)

and

$$
\begin{aligned}
2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\
+ 2g(X, AN)g(Y, A\xi) - 2g(Y, AN)g(X, A\xi) \\
- 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X) = 0
\end{aligned}
$$

(3.4)

for any tangent vector fields $X$ and $Y$ on $M$.

In addition, if $M$ has a singular normal vector field $N$, then the gradient of $\alpha$ should be $\text{grad} \alpha = (\xi\alpha)\xi$. From the property of $g(\nabla_X\text{grad} \alpha, Y) = g(\nabla_Y\text{grad} \alpha, X)$ we obtain

$$(X(\xi\alpha))\eta(Y) + (\xi\alpha)g(\phi SX, Y) = (Y(\xi\alpha))\eta(X) + (\xi\alpha)g(\phi SY, X)$$

(3.5)

for all $X, Y \in TM$. Putting $Y = \xi$ in (3.5) it follows $(X(\xi\alpha)) = (\xi(\xi\alpha))\eta(X)$. From this, the (3.5) becomes

$$(\xi\alpha)g((\phi S + S\phi)X, Y) = 0.$$ 

On the other hand, in [14] the authors gave that there does not exist a real hypersurface with anti-commuting property, $S\phi + \phi S = 0$, in $Q^m, m \geq 3$. By virtue of this result, we get $(\xi\alpha) = 0$. Then from this and (3.3), we assert:

**Lemma 3.2** Let $M$ be a Hopf real hypersurface in $Q^m, m \geq 3$. If $M$ has a singular normal vector field, then the Reeb curvature function $\alpha$ should be constant.

Specially, it has been known for a Hopf real hypersurface with $\mathfrak{A}$-principal normal vector field as follows:

**Lemma 3.3** [26] Let $M$ be a Hopf real hypersurface in $Q^m$ with $\mathfrak{A}$-principal normal vector field $N$. Then the Reeb curvature function $\alpha$ is constant. Moreover, if $X \in C$
is a principal curvature vector of $M$ with principal curvature $\lambda$, then $2\lambda \neq \alpha$ and its corresponding vector $\phi X$ is a principal curvature vector of $M$ with principal curvature $\frac{\alpha \lambda + 2}{2\lambda - \alpha}$.

When the normal vector $N$ is $\mathfrak{A}$-isotropic, the tangent vector space $T_{[z]}M$ at $[z] \in M$ is decomposed by

$$T_{[z]}M = [\xi] \oplus [A\xi, AN] \oplus Q_{[z]},$$

where $C_{[z]} \oplus Q_{[z]} = Q_{[z]} = \text{Span}[A\xi, AN]$. From this decomposition we obtain:

**Lemma 3.4** [14] Let $M$ be a Hopf hypersurface in $Q^m$ with $\mathfrak{A}$-isotropic normal vector field $N$. Then $SA\xi = 0$ and $SAN = 0$. Moreover, if $X \in Q$ is a principal curvature vector of $M$ with principal curvature $\lambda$, then $2\lambda \neq \alpha$ and its corresponding vector $\phi X$ is a principal curvature vector of $M$ with principal curvature $\frac{\alpha \lambda + 2}{2\lambda - \alpha}$.

On the other hand, from the property of $g(A\xi, N) = 0$ on a real hypersurface $M$ in $Q^m$ we see that the non-zero vector field $A\xi$ is tangent to $M$. Hence by the Gauss formula, $\bar{\nabla}_U V = \nabla_U V + \sigma(U, V)$ for $U, V \in TM$, it follows

$$\nabla_X(A\xi) = \bar{\nabla}_X(A\xi) - \sigma(X, A\xi)$$

for any $X \in TM$. Taking the inner product of (3.6) with unit normal vector field $N$, we obtain

$$q(X)g(A\xi, \xi) = -g(AN, \nabla_X\xi) + g(SX, \xi)g(A\xi, \xi) + g(SX, A\xi)$$

by using $g(AN, N) = -g(A\xi, \xi)$. In particular, if $M$ is Hopf, then this equation becomes

$$q(\xi)g(A\xi, \xi) = 2\alpha g(A\xi, \xi).$$

### 4 Commuting Jacobi Operator

Now, we consider the commuting condition with respect to the Jacobi operator $\bar{R}_N$ and the structure Jacobi operator $R_\xi$ on a Hopf real hypersurface $M$ in the complex quadric $Q^m$, $m \geq 3$. The Jacobi operator $\bar{R}_N \in \text{End}(TQ^m)$ with respect to the unit tangent vector $N \in T_{[z]}Q^m$, $[z] \in Q^m$, is induced from the curvature tensor $R$ of $Q^m$ given in Section 2 as follows:

$$\bar{R}_N U = \bar{R}(U, N)N$$

$$= U - g(U, N)N + 3g(U, \xi)\xi + g(AN, N)AU - g(AN, U)AN - g(A\xi, U)A\xi$$

for all vector field $U \in TQ^m$. Since $TQ^m = TM \oplus \text{span}[N]$, we obtain

$$\bar{R}_N Y = (\bar{R}_N Y)^\top + (\bar{R}_N Y)^\perp$$

and

$$(\bar{R}_N Y)^\perp = g(\bar{R}_N Y, N)N = g(\bar{R}(Y, N)N, N) = 0$$
for any vector field $Y \in TM \subset TQ^m$. Hence $\bar{R}_N \in \text{End}(TM)$ is defined by
\[
\bar{R}_N Y = \bar{R}(Y, N) N = Y + 3\eta(Y)\xi + g(AN, N)BY + g(AN, Y)\phi A\xi - g(A\xi, Y)A\xi \quad (4.1)
\]
for all vector field $Y \in TM$. Here we have used (3.4) and $AN = AJ\xi = -JA\xi = -\phi A\xi - g(A\xi, \xi)N$.

On the other hand, the structure Jacobi operator $R_\xi$ from (2.3) can be rewritten as follows:
\[
g(R_\xi Y, W) = g(R(Y, \xi)\xi, W)
\]
\[
= g(Y, W) - \eta(Y)\eta(W) + \beta g(AY, W) - g(AY, \xi)g(A\xi, W)
\]
\[
- g(AY, N)g(AN, W) + \alpha g(SY, W) - \alpha^2 \eta(Y)\eta(W),
\]
where we have put $\alpha = g(S\xi, \xi)$ and $\beta = g(A\xi, \xi)$, because we assume that $M$ is Hopf. The Reeb vector field $\xi = -JN$ and the anti-commuting property $AJ = -JA$ gives $\beta = -g(AN, N)$. When the function $\beta = g(A\xi, \xi)$ identically vanishes, we say that a real hypersurface $M$ in $Q^m$ is $\mathfrak{A}$-isotropic as in Section 1. From this equation, we get the structure operator $R_\xi \in \text{End}(TM)$ as follows:
\[
R_\xi Y = Y - \eta(Y)\xi + \beta B(Y) - g(A\xi, Y)A\xi
\]
\[
= -g(\phi A\xi, Y)\phi A\xi + \alpha SY - \alpha^2 \eta(Y)\xi. \quad (4.2)
\]
By the linearity of $\bar{R}_N$ and $R_\xi$, the commuting condition $(\bar{R}_N R_\xi)Y = (R_\xi \bar{R}_N)Y$ for any $Y \in TM$ becomes
\[
R_\xi Y - \beta B(R_\xi Y) - g(\phi A\xi, R_\xi Y)\phi A\xi - (A\xi, R_\xi Y)A\xi
\]
\[
= \bar{R}_N Y - \eta(\bar{R}_N Y)\xi + \beta B(\bar{R}_N Y) - g(A\xi, \bar{R}_N Y)A\xi
\]
\[
- g(\phi A\xi, \bar{R}_N Y)\phi A\xi + \alpha S(\bar{R}_N Y) - \alpha^2 \eta(\bar{R}_N Y)\xi,
\]
where we have used (4.1) and (4.2).

Using this condition, now let us prove that the unit normal vector field $N$ of $M$ is singular, that is, $N$ is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.

Substituting $Y = \xi$ in (4.3), then it yields
\[
0 = \bar{R}_N \xi - \eta(\bar{R}_N \xi)\xi + \beta B(\bar{R}_N \xi) - g(A\xi, \bar{R}_N \xi)A\xi
\]
\[
- g(\phi A\xi, \bar{R}_N \xi)\phi A\xi + \alpha S(\bar{R}_N \xi) - \alpha^2 \eta(\bar{R}_N \xi)\xi
\]
\[
= 2\alpha\beta(\alpha\beta \xi - SA\xi), \quad (4.4)
\]
where we have used $R_\xi \xi = 0$, $\bar{R}_N \xi = 4\xi - 2\beta A\xi$ and $A^2 = I$.

By virtue of Remark 3.3 in [14] we see that if the Reeb curvature function $\alpha$ is vanishing, then the normal vector field $N$ of $M$ is singular. Hence, from now on, we only investigate the case of $\alpha \neq 0$. Then (4.4) gives us the following two cases:

**Case 1** $\beta = g(A\xi, \xi) = 0$

From the result of Reckziegel [20], we obtain that $g(\xi, AN) = 0$ and $\beta = g(A\xi, \xi) = -g(AN, N) = -\cos(2t), t \in [0, \pi/4]$, on $M$ (see (3.2) in Section 3). It leads that $t = \frac{\pi}{4}$, which means that the normal vector field $N$ is $\mathfrak{A}$-isotropic.
Case I \[ \beta = g(A\xi, \xi) \neq 0 \] (that is, \( SA\xi = \alpha \beta \xi, \alpha \beta \neq 0 \))

From (3.4), the assumption \( SA\xi = \alpha \beta \xi \) leads to

\[ \alpha S\phi A\xi = -2g^2(A\xi, \xi)\phi A\xi = -2\beta^2\phi A\xi. \tag{4.5} \]

Taking the covariant derivative for our assumption \( SA\xi = \alpha \beta \xi \) along any tangent vector \( X \) of \( M \), we have

\[ (\nabla_X S)A\xi + S(\nabla_X (A\xi)) = (X\alpha)\beta\xi + \alpha g(\nabla_X (A\xi), \xi) + \alpha g(A\xi, \nabla_X \xi) + \alpha \beta \nabla_X \xi. \tag{4.6} \]

In addition, taking the inner product of (4.6) with \( Y \in T_{[z]}M, [z] \in M \), it yields

\[ \begin{align*}
&g((\nabla_Y S)X, A\xi) - 2\beta g(\phi X, Y) + g(X)g(\phi A\xi, SY) \\
&+ g(\phi SX, ASY) - \alpha \eta(X)g(\phi A\xi, SY) \\
&= (X\alpha)\beta\eta(Y) + \alpha g(A\xi, \phi SX)\eta(Y),
\end{align*} \tag{4.7} \]

for any \( X \in C = \{X \in TM | X \perp \xi\} \) and \( Y \in TM \).

Substituting \( X = \phi A\xi \in C \) and \( Y = \xi \) and using (4.5) this equation becomes

\[ \alpha g((\nabla_\xi S)\phi A\xi, A\xi) + 2\alpha \beta^2(1 - \beta^2) = 0, \tag{4.8} \]

where we have used \( g(\phi A\xi, \phi A\xi) = 1 - g^2(A\xi, \xi) = 1 - \beta^2. \) On the other hand, from our assumption \( SA\xi = \alpha \beta \xi \) we obtain \( \alpha g((\nabla_\xi S)\phi A\xi, A\xi) = \alpha g((\nabla_\xi S)A\xi, \phi A\xi) = -2\beta^2(q(\xi) - \alpha)(1 - \beta^2). \) Hence (4.8) yields

\[ 0 = -2\beta^2q(\xi)(1 - \beta^2) = 4\alpha \beta^2(1 - \beta^2), \]

where we have used (3.8) in the second equality. Since \( \alpha \beta \neq 0 \), it yields \( \beta = -\cos 2t = \pm 1, t \in [0, \frac{\pi}{2}) \). So, we get \( t = 0 \), which means that the normal vector field \( N \) of \( M \) should be \( \mathfrak{A} \)-principal. Thus the proof of the following lemma is completed.

**Lemma 4.1** Let \( M \) be a Hopf real hypersurface with commuting Jacobi operators such that \( \overline{R}_N R_{\xi} = R_{\xi} \overline{R}_N \) in the complex quadric \( Q^m, m \geq 3 \). Then the unit normal vector field \( N \) should be singular. It means that \( N \) becomes either \( \mathfrak{A} \)-isotropic or \( \mathfrak{A} \)-principal.

### 5 Proof of Theorem 1

In this section, we will give a proof of Theorem 1. Let \( M \) be a real hypersurface with non-vanishing geodesic Reeb flow, \( \alpha = g(S\xi, \xi) \neq 0 \), in \( Q^m, m \geq 3 \). As mentioned in Section 4, if \( M \) satisfies the commuting condition (*) , the normal vector field \( N \) of \( M \) should be singular. Then from the definition of singular tangent vector field of \( Q^m \), we can divided the following two cases.
5.1 Commuting Jacobi operator with $\mathfrak{A}$-isotropic unit normal vector field

Let us consider the case that the unit normal vector field $N$ of $M$ is $\mathfrak{A}$-isotropic singular. It means that the normal vector field $N$ can be expressed by

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for $Z_1, Z_2 \in V(A)$, where $V(A)$ denotes a $(+1)$-eigenspace of the complex conjugation $A \in \mathfrak{A}$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2), \quad AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2), \quad and \quad JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

Thus, we obtain that

$$g(\xi, A\xi) = g(JN, AJN) = 0, \quad g(\xi, AN) = 0 \quad and \quad g(AN, N) = 0,$$

which means that two vector fields $AN$ and $A\xi$ are tangent to $M$. By virtue of these formulas with respect to $\mathfrak{A}$-isotropic unit normal vector field and $g(JAY, \xi) = -g(AY, J\xi) = -g(AY, N)$, the normal Jacobi operator $\tilde{R}_N$ and the structure Jacobi operator $R_\xi$ can be rearranged as follows:

$$\tilde{R}_NY = Y + 3\eta(Y)\xi - g(AN, Y)AN - g(A\xi, Y)A\xi$$

and

$$R_\xi Y = Y - \eta(Y)\xi - g(A\xi, Y)A\xi - g(AN, Y)AN + \alpha SY - \alpha^2 \eta(Y)\xi$$

respectively. So, the property for the commuting Jacobi operator $R_\xi \tilde{R}_N = \tilde{R}_N R_\xi$ on $M$ is equivalent to

$$\alpha SY = -6\alpha^2 \eta(Y)\xi,$$

where we have used

$$\begin{cases}
\tilde{R}_N(A\xi) = \tilde{R}_N(AN) = 0, \quad \tilde{R}_N\xi = 4\xi, \\
R_\xi\xi = R_\xi(AN) = R_\xi(A\xi) = 0.
\end{cases}$$

It gives us

$$SY = -6\alpha \eta(Y)\xi \quad (5.1)$$

for all $Y \in TM$, since $M$ has a non-vanishing geodesic Reeb flow, that is, $\alpha = g(S\xi, \xi) \neq 0$ on $M$. It makes a contradiction. In fact, if we substitute $Y = \xi$ in (5.1), then $M$ should have a vanishing geodesic Reeb flow.

Summing up these observations, we assert that

Lemma 5.1 There does not exist a real hypersurface in the complex quadric $Q^m$, $m \geqslant 3$, with the following three conditions:

(C1) the non-vanishing geodesic Reeb flow,
(C2) the $\mathfrak{A}$-isotropic normal unit vector, and
(C3) the commuting Jacobi operator, that is, $R_\xi \tilde{R}_N = \tilde{R}_N R_\xi$.

Remark 5.2 Let $M$ be a Hopf real hypersurface with $\mathfrak{A}$-isotropic normal vector field $N$ in $Q^m$, $m \geqslant 3$. By virtue of the proof given in Lemma 5.1, we assert that if
$M$ has the vanishing geodesic Reeb flow, then $M$ naturally satisfies the commuting Jacobi operator, $(R_\xi \tilde{R}_N) = (\tilde{R}_N R_\xi)$.

### 5.2 Commuting Jacobi Operator with $\mathfrak{A}$-Principal Unit Normal Vector Field

Assume that $M$ is a Hopf real hypersurface with $\mathfrak{A}$-principal unit normal vector field $N$ in the complex quadric $Q^m$, $m \geq 3$.

The assumption that $N$ is $\mathfrak{A}$-principal implies that $N$ satisfies $AN = N$ for a complex conjugation $A \in \mathfrak{A}$. It yields that a vector field $AY$ should be tangent to $M$ for all $Y \in TM$, because

$$AY = BY + g(AY, N)N = BY \in TM$$

(in particular, $A\xi = -AJN = JAN = JN = -\xi \in TM$). From this, the anti-commuting property, $JA = -AJ$, with respect to the complex structure $J$ and the real structure $A$ tells us that

$$\phi AY = -A\phi Y$$  \hspace{1cm} (5.2)

for all $Y \in TM$. By virtue of these properties and (4.1) and (4.2), the normal Jacobi operator $\tilde{R}_N$ and the structure Jacobi operator $R_\xi$ are given by respectively

$$\tilde{R}_N Y = Y + 2\eta(Y)\xi + AY$$

and

$$R_\xi Y = Y - 2\eta(Y)\xi - AY + \alpha SY - \alpha^2 \eta(Y)\xi.$$  

So, the commuting property defined by $(R_\xi \tilde{R}_N) = (\tilde{R}_N R_\xi)$ with respect to $\tilde{R}_N$ and $R_\xi$ is equivalent to

$$\alpha SY = \alpha ASY,$$  \hspace{1cm} (5.3)  

because the real structure $A$ is an anti-linear involution on $TQ^m$, that is, $A^2 = I$.

On the other hand, taking the covariant derivative $\tilde{V}$ of $Q^m$ to $AN = N$ along the direction of $Y \in TM$ and using the formula of Weingarten, we have:

$$-SY = \tilde{V}_Y N = (\tilde{V}_Y A)N + A(\tilde{V}_Y N)$$

$$= q(Y)JAN - ASY$$

$$= -2\alpha \eta(Y)\xi - ASY,$$

that is,

$$ASY = SY - 2\alpha \eta(Y)\xi,$$  \hspace{1cm} (5.4)

where we have used (3.7) and $(\tilde{V}_U V) = q(U)JAV$ for $U, V \in T[w]Q^m$, $[w] \in Q^m$. Taking the symmetric part of (5.4), we see that the shape operator $S$ commutes with the real structure $A$ on $TM$, that is, $ASY = SAY$ for any $Y \in TM$. From this fact and (5.3) we can assert the following: Let $M$ be a Hopf real hypersurface in $Q^m$, $m \geq 3$. If $M$ has an $\mathfrak{A}$-principal normal vector field, then the normal Jacobi operator $\tilde{R}_N$ commutes with the structure Jacobi operator $R_\xi$.

From this, together with Lemmas 4.1 and 5.1, we can give a complete proof of Theorem 1 in the introduction.
6 Proof of Theorem 2

Now, we try to classify a Hopf real hypersurface with $\mathfrak{A}$-principal normal vector field in $Q^m$, $m \geq 3$.

**Lemma 6.1** Let $M$ be a Hopf real hypersurface in $Q^m$, $m \geq 3$. If the normal vector $N$ of $M$ is $\mathfrak{A}$-principal, then the Reeb curvature function $\alpha$ is non-vanishing constant on $M$. Moreover, $M$ is a contact real hypersurface with constant mean curvature in $Q^m$.

**Proof** For any $X, Y, Z \in \mathcal{C}$, the Codazzi equation yields

$$g((\nabla_X S)Y - (\nabla_Y S)X, Z) = 0,$$

so we obtain

$$(\nabla_X S)Y - (\nabla_Y S)X = g((\nabla_X S)Y - (\nabla_Y S)X, \xi)\xi. \quad (6.1)$$

Taking an inner product with $\xi$ of (6.1), it becomes

$$g(\alpha \phi SX + \alpha S\phi X - 2S\phi SX, Y) = -2g(\phi X, Y), \quad (6.2)$$

where the right side (resp. the left side) is induced from the equation of Codazzi (resp. the assumption of $M$ being Hopf).

On the other hand, since $g(AY, N) = 0$ and $g(AY, \xi) = 0$ for any $Y \in \mathcal{C}$, we see that $AY \in \mathcal{C}$. Then (6.2) yields

$$-2g(\phi X, AY) = g(\alpha \phi SX + \alpha S\phi X - 2S\phi SX, AY) \iff -2g(A\phi X, Y) = g(\alpha A\phi SX + \alpha AS\phi X - 2AS\phi SX, Y) \iff 2g(\phi AX, Y) = -g(\alpha \phi ASX + \alpha \phi SX + 2\phi X, Y) \iff g(\alpha \phi ASX + \alpha \phi SX + 2\phi X + 2\phi AX, Y) = 0$$

for any $Y \in \mathcal{C}$. In fact, in the above second line we have used (5.2) and (5.4), because $\phi X, \phi SX \in \mathcal{C}$. Moreover, the third line follows from (3.4). Together with $\phi \xi = 0$, consequently, it leads to:

$$\alpha \phi ASX + \alpha \phi SX + 2\phi X + 2\phi AX = 0, \quad \forall X \in \mathcal{C}. \quad (6.3)$$

Suppose $\alpha = 0$. Then it becomes

$$\phi AX = -\phi X, \quad \forall X \perp \xi. \quad (6.4)$$

Applying the structure tensor $\phi$, it yields that

$$AX = -X, \quad \forall X \in \mathcal{C}. \quad (6.5)$$

In addition, by (5.2), the (6.4) becomes $A\phi X = \phi X$ for all $X \in \mathcal{C}$. Since $\phi X \in \mathcal{C}$, it follows that $AX = X$ for any $X \in \mathcal{C}$. From this, together with (6.5), we get $X = 0$ for all $X \in \mathcal{C}$, which means that $\dim_{\mathbb{R}} \mathcal{C} = 0$. It makes a contradiction for the dimension of $\mathcal{C}$. In fact, from the geometric structure of the tangent vector space $T_{[z]}M$ of $M$ at
\([z] \in M \subset Q^m\) we can take one basis for \(T_{[z]} M\) as \(\{\xi, e_1, \cdots, e_{2m-2}\} = \{\xi\} \oplus \mathcal{C}\). From this, we get
\[
\dim \mathcal{C} = 2m - 2 = 0.
\]
that is, \(m = 1\). So, we assert that the Reeb curvature function \(\alpha\) is non-vanishing on \(M\). Moreover, by Lemmas 3.2 and 3.3, the Reeb curvature function \(\alpha\) is constant on \(M\).

Applying the structure tensor \(\phi\) to (6.3) and using (5.4), it yields
\[
\alpha SX = -X - AX, \quad \forall X \in \mathcal{C}.
\]
(6.6)

Let \(X \in \mathcal{C}\) be an eigenvector with its eigenvalue \(\lambda\), that is, \(SX = \lambda X\). Then the (6.6) leads to
\[
AX = -(\alpha \lambda + 1)X,
\]
taking the real structure \(A\) of this equation and using the previous equation again, it gives us
\[
\alpha \lambda (\alpha \lambda + 2) = 0.
\]

- **Case 1.** \(\alpha \lambda = 0\).

Since \(\alpha \neq 0\) on \(M\), we obtain \(\lambda = 0\). From Lemma 3.3, \(\phi X\) is also an eigenvector with eigenvalue \(\mu = -\frac{2}{\alpha}\). Thus, the expression of the shape operator \(S_1\) of \(M\) is given by
\[
S_1 = \text{diag}(\alpha, 0, 0, \cdots, 0, \underbrace{-\frac{2}{\alpha}, -\frac{2}{\alpha}, \cdots, -\frac{2}{\alpha}}_{(m-1)}).
\]

- **Case 2.** \(\alpha \lambda + 2 = 0\).

From the assumption of \(\alpha \lambda + 2 = 0\), we see that \(\lambda = -\frac{2}{\alpha}\). Moreover, by virtue of Lemma 3.3, \(\phi X\) is also an eigenvector with eigenvalue \(\mu = 0\). Thus, the expression of the shape operator \(S_2\) of \(M\) is given by
\[
S_2 = \text{diag}(\alpha, -\frac{2}{\alpha}, -\frac{2}{\alpha}, \cdots, -\frac{2}{\alpha}, 0, 0, \cdots, 0).
\]

Summing up the above two cases, we obtain that the shape operator \(S\) of \(M\) satisfies \(S\phi + \phi S = \delta \phi\), where \(\delta = -\frac{2}{\alpha} \neq 0\). That is, \(M\) should be a contact real hypersurface in \(Q^m\). In addition, we obtain that \(M\) is a Hopf real hypersurface with constant mean curvature, because the trace of the shape operator \(S\) is given by
\[
\text{Tr} S = \alpha - (m - 1)(\frac{2}{\alpha}).
\]

It completes the proof.

By virtue of Theorem A and Lemma 6.1, we assert that if \(M\) is a Hopf real hypersurface with \(\mathfrak{A}\)-principal singular normal vector field in \(Q^m\), \(m \geq 3\), then \(M\) should be congruent to an open part of the tube around \(S^m\) in \(Q^m\). We call this tube a model space of \((T_B)\). Conversely, the model space \((T_B)\) becomes a Hopf real hypersurface with \(\mathfrak{A}\)-principal singular normal vector field in \(Q^m\) (see [29]). Thus we obtain Theorem 2 given in the introduction.
Remark 6.2 By virtue of Theorem 1, the above result can be rewritten as follows: Let $M$ be a real hypersurface with non-vanishing geodesic Reeb flow in $Q^m$, $m \geq 3$. Then $M$ has commuting normal Jacobi operator, $\bar{R}_N R_\xi = R_\xi \bar{R}_N$, if and only if $M$ is locally congruent to the model space of $(T_B)$.

7 Proof of Theorem 3

In this section we assume that $M$ is a Hopf real hypersurface in $Q^m$, $m \geq 3$, with the commuting normal Jacobi operator which is different from the formulas in Sections 4 and 5.

As mentioned in the introduction, the Jacobi operator $R_X$ with respect to a tangent vector field $X \in TM$ is defined by $R_X Y := R(Y, X)X$ for any $Y \in TM$, and it becomes a self-adjoint endomorphism of the tangent bundle $TM$ of $M$. That is, the Jacobi operator satisfies $R_X \in \text{End}(TM)$ and is symmetric in the sense of $g(R_X Y, Z) = g(Y, R_X Z)$ for any tangent vector fields $Y$ and $Z$ on $M$.

From now on, assume that the normal Jacobi operator $\bar{R}_N$ of $M$ satisfies the new commuting condition given by

$$\bar{R}_N R_X = R_X \bar{R}_N \quad (**),$$

where $R_X$ is the Jacobi operator with respect to $X \in \mathcal{C} = \{X \in TM \mid X \perp \xi\}$. From the equation of Gauss (2.2) of $M$, the Jacobi operator $R_X$ for $X \in \mathcal{C}$ is defined by

$$R_X Y := R(Y, X)X = g(X, X)Y - g(X, Y)X - 3g(X, \phi Y)\phi X + g(BX, X)BY - g(BX, Y)BX + g(\phi BX, X)\phi BY - g(\phi BX, X)g(AN, Y)\xi - g(\phi BX, X)g(AN, X)\xi + g(SX, X)SY - g(SY, X)SX$$ (7.1)

for any $X \in \mathcal{C}$ and $Y \in TM$. By using this equation, we obtain:

Lemma 7.1 Let $M$ be a Hopf real hypersurface in $Q^m$, $m \geq 3$. If the normal Jacobi operator $\bar{R}_N$ of $M$ commutes with the Jacobi operator $R_X$ for any vector field $X \in \mathcal{C}$, then the normal vector field $N$ of $M$ is singular.

Proof Since $\bar{R}_N$ and $R_X$ are symmetric, the commuting condition (**)) yields that

$$g(R_X(A\xi), \bar{R}_N\xi) = g(\bar{R}_N(A\xi), R_X\xi)$$ (7.2)

for any $X \in \mathcal{C}$. On the other hand, from (4.1) we get $\bar{R}_N\xi = 4\xi - 2\beta A\xi$ and $\bar{R}_N(A\xi) = 2\beta \xi$. In addition, by using (7.1) the condition (7.2) leads to

$$2g(BX, X) + \alpha\beta g(SX, X) + 2\beta g(A\xi, X)g(A\xi, X) - 2\beta g(AN, X)g(AN, X) - 2\beta^2 g(BX, X) - \beta g(SX, X)g(SA\xi, A\xi) + \beta g(SA\xi, X)g(SA\xi, X) = 0$$
for any $X \in C$. This equality also holds for $X + Y$ where $Y \in C$. Then it gives

$$2g(BX, Y) + \alpha \beta g(SX, Y) + 2\beta g(A \xi, X)g(A \xi, Y) - 2\beta g(AN, X)g(AN, Y) - 2\beta^2 g(BX, Y) - \beta g(SX, Y)g(SA \xi, A \xi) + \beta g(SA \xi, X)g(SA \xi, Y) = 0$$

(7.3)

for any $X, Y \in C$. Now we may put

$$W := 2BX + \alpha \beta SX + 2\beta g(A \xi, X)A \xi + 2\beta g(AN, X)\phi A \xi - 2\beta^2 BX - \beta g(SA \xi, A \xi)SX + \beta g(SA \xi, X)SA \xi.$$  

Then by means of (7.3), the vector field $W \in TM$ is expressed by

$$W = \sum_{i=1}^{2m-1} g(W, e_i) e_i + g(W, \xi) \xi = 2g(A \xi, X) + \alpha \beta^2 g(SA \xi, X)$$

for a basis $\{e_1, e_2, \cdots, e_{2m-2}, e_{2m-1} = \xi\}$ of $TM$. Taking the inner product with $A \xi$ to this equation, we get

$$\alpha \beta (1 - \beta^2) g(SA \xi, X) = 0$$

(7.4)

for any $X \in C$.

As mentioned before, if the Reeb curvature function $\alpha = g(S \xi, \xi)$ vanishes, then the normal vector field $N$ is singular. Moreover, since $\beta = g(A \xi, \xi) = -\cot 2t$ where $t \in [0, \pi/4]$, the normal vector field $N$ is singular if $\beta = 0$ or $\beta = -1$. That is, when $\beta = 0$ (resp. $\beta = -1$), the normal vector field $N$ is $A$-isotropic (resp. $A$-principal). Other case, $\beta = 1$, does not occur.

Finally, let us consider the remained case, $g(SA \xi, X) = 0$ for any $X \in C$. In other words, it implies that

$$SA \xi = \sum_{i=1}^{2m-2} g(SA \xi, e_i) e_i + g(SA \xi, \xi) \xi = g(SA \xi, \xi) \xi = \alpha \beta \xi$$

(7.5)

with $\alpha \beta (1 - \beta^2) \neq 0$ on $M$. From this and (3.4), we get

$$S \phi A \xi = \sigma \phi A \xi \quad \text{where } \sigma = -2\beta^2/\alpha.$$  

(7.6)

On the other hand, from $\bar{R}_N(\phi A \xi) = 0$ and (7.1), the commuting condition $g(R_X (\phi A \xi), \bar{R}_N(\phi A \xi)) = g(R_X \xi, \bar{R}_N(\phi A \xi))$ becomes

$$4\beta g(X, AN)g(A \xi, X) - 4\beta^2 g(\phi BX, X) + 4g(\phi BX, X) - 2\beta^3 g(A \xi, X)g(AN, X) - 2\beta g(SX, X)g(S \phi A \xi, A \xi) + 2\beta g(S \phi A \xi, X)g(SX, A \xi) = 0$$


for $X \in \mathcal{C}$. By using the linearity of the inner product, we have
\begin{align*}
4\beta g(X, AN)g(A\xi, Y) &+ 4\beta g(Y, AN)g(A\xi, X) - 4\beta^2 g(\phi BX, Y) \\
-4\beta^2 g(\phi BY, X) &+ 4g(\phi BX, Y) + 4g(\phi BY, X) \\
-2\beta^2 g(A\xi, Y)g(BX, AN) &- 2\beta^2 g(A\xi, X)g(BY, AN) \\
-2\beta^3 g(A\xi, Y)g(AN, X) &- 2\beta^3 g(A\xi, X)g(AN, Y) \\
+2\beta g(S\phi A\xi, X)g(SY, A\xi) &+ 2\beta g(S\phi A\xi, Y)g(SX, A\xi) = 0
\end{align*}
(7.7)
for $X, Y \in \mathcal{C}$. By the way, from the property of $JA = -AJ$ it follows
\[
\phi BZ - g(AN, Z)\xi = -B\phi Z + \eta(Z)\phi A\xi
\]
for any $Z \in TM$. Hence $B\phi A\xi = \beta\phi A\xi$. From this and putting $Y = A\xi \in \mathcal{C}$ in (7.7) we get
\[
\beta(2 - 2\beta^2 + \beta^4)g(AN, X) = 0 \quad \text{for } X \in \mathcal{C},
\]
together with (7.5) and (7.6). Since $\beta(2 - 2\beta^2 + \beta^4) \neq 0$, it implies that
\[
g(AN, X) = -g(\phi A\xi, X) = 0 \quad (7.8)
\]
for all $X \in \mathcal{C}$, which gives a contradiction for $(1 - \beta^2) \neq 0$. In fact, from (7.8), we obtain
\[
\phi A\xi = \sum_{i=1}^{2m-2} g(\phi A\xi, e_i)e_i + g(\phi A\xi, \xi)\xi = 0.
\]
It leads to $A\xi = \beta\xi$, which means $\beta^2 = 1$.

Making use of these facts, we can conclude that the normal vector field $N$ of $M$ is singular. So we give a complete proof of Lemma 7.1. \hfill \Box

Furthermore, we obtain:

**Lemma 7.2** There does not exist a Hopf real hypersurface with $\mathfrak{A}$-isotropic normal vector field $N$ and the commuting condition (***) in $Q^m$, $m \geq 3$.

**Proof** Suppose that the normal vector field $N$ of $M$ is $\mathfrak{A}$-isotropic, that is, $\beta = 0$. Let $Q$ is a distribution of $TM$ defined by
\[
Q[z] = C[z] - [A\xi, AN][z] = \{Z \in T[z]M \mid Z \perp \xi, A\xi, AN\} \quad \text{at } [z] \in M.
\]
Since $\tilde{R}_N\xi = 4\xi$ and $\tilde{R}_NZ = Z$ for any $Z \in Q$, the commuting condition (**), $g(\tilde{R}_N RX_\xi, Z) = g(R_X\tilde{R}_N\xi, Z)$, gives us $g(R_X\xi, Z) = 0$ for all $Z \in Q$. Thus, by using (7.1) we get
\[
-g(A\xi, X)g(BX, Z) + g(AN, X)g(\phi BX, Z) = 0.
\]
By virtue of the linearity in terms of the inner product, it is equal to
\begin{align*}
-g(A\xi, Y)g(BX, Z) &- g(A\xi, X)g(BY, Z) \\
+g(AN, Y)g(\phi BX, Z) &+ g(AN, X)g(\phi BY, Z) = 0
\end{align*}
(7.9)
for any $X, Y \in \mathcal{C}$ and $Z \in \mathcal{Q}$. Then for any basis $\{e_1, e_2, \cdots, e_{2m-4}, e_{2m-3} = A\xi, e_{2m-2} = AN, e_{2m-1} = \xi\}$ of $TM$, the (7.9) yields that

$$\tilde{W} = \sum_{i=1}^{2m-1} g(W, e_i)e_i$$

$$= \sum_{i=1}^{2m-2} g(\tilde{W}, e_i)e_i + g(\tilde{W}, A\xi)A\xi + g(\tilde{W}, AN)AN + g(W, \xi)\xi$$

$$= g(\tilde{W}, A\xi)A\xi + g(\tilde{W}, AN)AN + g(W, \xi)\xi$$

where $\tilde{W} = -g(A\xi, Y)BX - g(A\xi, X)BY + g(AN, Y)\phi BX + g(AN, X)\phi BY$. Since $BAN = -B\phi A\xi = 0$ and $BA\xi = \xi$, it follows

$$2g(A\xi, X)g(A\xi, Y)\xi = g(A\xi, Y)BX + g(A\xi, X)BY$$

$$-g(AN, Y)\phi BX - g(AN, X)\phi BY$$

(7.10)

for any $X, Y \in \mathcal{C}$. Since $\beta = g(A\xi, \xi) = 0$, we know that $A\xi \in \mathcal{C}$. Hence substituting $Y = A\xi$ in (7.10), it leads to $BX = g(A\xi, X)\xi$ for $X \in \mathcal{C}$. Therefore, we obtain

$$B^2X = g(A\xi, X)A\xi$$

(7.11)

because of $A\xi = B\xi$. In general, from the properties of $AJ = -JA$ and $A^2 = I$, we get

$$AN = AJ\xi = -JA\xi = -\phi A\xi - g(A\xi, \xi)N,$$

and

$$B^2Y = Y + g(AN, Y)\phi A\xi$$

for any $Y \in TM$. From this, (7.11) becomes

$$X = g(A\xi, X)A\xi + g(AN, X)AN$$

for any $X \in \mathcal{C} = [A\xi, AN] \oplus \mathcal{Q}$, which implies $\dim_{\mathbb{R}} \mathcal{C} = 2$. It gives a contradiction for $m \geq 3$. From this we get a complete proof of Lemma 7.2.

From Lemmas 7.1 and 7.2, we can assert that the normal vector field $N$ of $M$ satisfying the condition of (**) in $Q^m$ must be $\mathfrak{A}$-principal. Accordingly, by virtue of Theorem 2, we arrive at the conclusion that

Let $M$ be a Hopf real hypersurface in $Q^m$, $m \geq 3$. If the normal Jacobi operator $R_N$ commutes with the Jacobi operator $R_X$ with respect to $X \in \mathcal{C}$, then $M$ is locally congruent to the model space of type $(T_B)$.

Now, we want to check whether the model space of type $(T_B)$ satisfy the commuting condition (**). According to Remark 5.1. in [29] we obtain:

**Proposition A** Let $(T_B)$ be the tube of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around the $m$-dimensional sphere $S^m$ in $Q^m$. Then the following holds:

(i) $(T_B)$ is a Hopf hypersurface.

(ii) The normal bundle of $(T_B)$ consists of $\mathfrak{A}$-principal singular.

(iii) $(T_B)$ has three distinct constant principal curvatures.
(iv) \[ S\phi + \phi S = 2\delta\phi, \delta = -\frac{1}{\alpha} \neq 0 \] (contact hypersurface).

| Principal curvature | Eigenspace | Multiplicity |
|----------------------|------------|--------------|
| \( \alpha = -\sqrt{2} \cot(\sqrt{2}r) \) | \( T_\alpha = \text{Span}\{\xi\} \) | 1 |
| \( \lambda = \sqrt{2} \tan(\sqrt{2}r) \) | \( T_\lambda = V(A) \cap C = \{ X \in C \mid AX = X \} \) | \( m - 1 \) |
| \( \mu = 0 \) | \( T_\mu = JV(A) \cap C = \{ X \in C \mid AX = -X \} \) | \( m - 1 \) |

By virtue of (ii) in Proposition A, we know that \( AN = N \). So it follows that \( AY \) is a tangent vector field of \((T_B)\) for any \( Y \in T(T_B)\). Thus from (2.2) and (4.1) the Jacobi operators with respect to \( N \) and \( X \), respectively, are given by

\[
\tilde{R}_N Y = Y + 2\eta(Y)\xi + AY
\]

and

\[
R_X Y = g(X, X)Y - g(X, Y)X + 3g(\phi X, Y)\phi X + g(AX, X)AY
\]

\[
- g(AX, Y)AX + g(\phi AX, X)\phi AY + g(A\phi X, Y)\phi AX
\]

\[
+ g(SX, X)SY - g(SX, Y)SX
\]

for any \( X, Y \in T(T_B) \). Then we see that all of tangent vector fields are principal by \( \tilde{R}_N \), that is,

\[
\tilde{R}_N Y = \begin{cases} 
2Y & \text{if } Y \in T_\alpha \\
2Y & \text{if } Y \in T_\lambda \\
0 & \text{if } Y \in T_\mu 
\end{cases} \tag{7.12}
\]

where \( T(T_B) = T_\alpha \oplus T_\lambda \oplus T_\mu \). On the other hand, the Jacobi operator with respect to \( X \in T(T_B) \) can be expressed by the following three cases.

**Case 1.** \( X \in T_\alpha \) (that is, \( X = \xi \))

\[
R_\xi Y = Y - 2\eta(Y)\xi - AY + \alpha SY - \alpha^2 \eta(Y)\xi
\]

\[
= \begin{cases} 
0 & \text{if } Y \in T_\alpha \\
\alpha\lambda Y & \text{if } Y \in T_\lambda \\
2Y & \text{if } Y \in T_\mu 
\end{cases} \tag{7.13}
\]

**Case 2.** \( X \in T_\lambda = \{ X \in C \mid AX = X \} \)

\[
R_X Y = g(X, X)Y - 2g(X, Y)X + 3g(\phi X, Y)\phi X + g(X, X)AY
\]

\[
+ g(A\phi X, Y)\phi X + \lambda g(X, X)SY - \lambda^2 g(X, Y)X
\]

\[
= \begin{cases} 
\alpha\lambda g(X, X)Y & \text{if } Y \in T_\alpha \\
(\lambda^2 + 2)g(X, X)Y - (\lambda^2 + 2)g(X, Y)X & \text{if } Y \in T_\lambda \\
2g(\phi X, Y)\phi X & \text{if } Y \in T_\mu 
\end{cases} \tag{7.14}
\]
**Case 3.** \( X \in T_{\mu} = \{ X \in \mathcal{C} \mid AX = -X \} \)

\[
R_X Y = g(X, X)Y - 2g(X, Y)X + 3g(\phi X, Y)\phi X - g(X, X)AY
- g(A\phi X, Y)\phi X
= \begin{cases} 
2g(X, X)Y & \text{if } Y \in T_{\alpha} \\
2g(\phi X, Y)\phi X & \text{if } Y \in T_{\lambda} \\
2g(X, X)Y - 2g(X, Y)X & \text{if } Y \in T_{\mu}
\end{cases}
\text{(7.15)}
\]

where we have used that \( \phi Z \in T_{\mu} \) (resp. \( \phi Z \in T_{\lambda} \)), provided that \( Z \in T_{\lambda} \) (resp. \( Z \in T_{\mu} \)). From these equations, consequently, we obtain:

\[
\tilde{R}_N R_X Y = \begin{cases} 
0 & \text{if } X \in T_{\alpha}, Y \in T_{\alpha} \\
2\alpha \lambda Y & \text{if } X \in T_{\alpha}, Y \in T_{\lambda} \\
0 & \text{if } X \in T_{\alpha}, Y \in T_{\mu} \\
2\alpha \lambda g(X, X)Y & \text{if } X \in T_{\lambda}, Y \in T_{\alpha} \\
2(\lambda^2 + 2)g(X, X)Y - 2(\lambda^2 + 2)g(X, Y)X & \text{if } X \in T_{\lambda}, Y \in T_{\lambda} \\
0 & \text{if } X \in T_{\lambda}, Y \in T_{\mu} \\
4g(X, X)Y & \text{if } X \in T_{\mu}, Y \in T_{\alpha} \\
4g(\phi X, Y)\phi X & \text{if } X \in T_{\mu}, Y \in T_{\lambda} \\
0 & \text{if } X \in T_{\mu}, Y \in T_{\mu}
\end{cases}
\]

\[
= R_X \tilde{R}_N Y
\]

It implies that the model space of \((TB)\) satisfies the commuting condition (***) between the normal Jacobi operator \( \tilde{R}_N \) and the Jacobi operator \( R_X \) for \( X \in \mathcal{C} \). Moreover, in the above calculations it can be easily checked that \( \tilde{R}_N R_\xi = R_\xi \tilde{R}_N \) for the tube of type \((TB)\).

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**References**

1. Berndt, J., Suh, Y.J.: Real hypersurfaces with isometric Reeb flow in complex quadrics. Internat. J. Math. 24, no. 7(1350050), 18 (2013)
2. Berndt, J., Suh, Y.J.: Real hypersurfaces in Hermitian Symmetric Spaces, Advances in Analysis and Geometry, Editor in Chief, Jie Xiao, ©2021 Copyright-Text. Walter de Gruyter GmbH, Berlin/Boston (in Press)
3. Berndt, J., Suh, Y.J.: Contact hypersurfaces in Kaehler manifolds. Proc. Amer. Math. Soc. 143(6), 2637–2649 (2015)
4. Berndt, J., Lee, H., Suh, Y.J.: Contact hypersurfaces in noncompact complex Grassmannians of rank two. Internat. J. Math. 24, no. 11(1350089), 11 (2013)
5. Brozos-Vázquez, M., Gilkey, P.: Manifolds with commuting Jacobi operator. J. Geom. 86, 21–30 (2006)
6. do Carmo, M.P.: Riemannian Geometry. Birkhäuser (1992)
7. Helgason, S.: Differential Geometry, Lie Groups and Symmetric Spaces. Grad. Stud. Math., Amer. Math. Soc., 34 (2001)
8. Jeong, I., Suh, Y.J.: Real hypersurfaces in the complex quadric with Killing structure Jacobi operator. J. Geom. Phys. 139, 88–102 (2019)
9. Kim, G.J., Suh, Y.J.: Real hypersurfaces in the complex quadric with Lie invariant normal Jacobi operator. Adv. Appl. Math. 104, 117–134 (2019)
10. Klein, S.: Totally geodesic submanifolds in the complex quadric. Differ. Geom. Appl. 26, 79–96 (2008)
11. Klein, S.: Totally geodesic submanifolds in the complex and the quaternionic 2-Grassmannians. Trans. Amer. Math. Soc. 361(9), 4927–4967 (2009)
12. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, Vol. II. A Wiley-Interscience Publication, Wiley, New York (1996)
13. Kon, M.: Pseudo-einstein real hypersurfaces in complex space forms. J. Diff. Geom. 14(3), 339–354 (1979)
14. Lee, H., Suh, Y.J.: Real hypersurfaces with recurrent normal Jacobi operator in the complex quadric. J. Geom. Phys. 123, 463–474 (2018)
15. Machado, C.J.G., Pérez, J.D., Suh, Y.J.: Commuting structure Jacobi operator for real hypersurfaces in complex two-plane Grassmannians. Acta Math. Sin. 31(1), 111–122 (2015)
16. Pérez, J.D.: On the structure vector field of a real hypersurface in complex quadric. Open Math. 16(1), 185–189 (2018)
17. Pérez, J.D., Suh, Y.J.: The Ricci tensor of real hypersurfaces in complex two-plane Grassmannians. J. Korean Math. Soc. 44, 211–235 (2007)
18. Pérez, J.D., Suh, Y.J.: Derivatives of the shape operator of real hypersurfaces in the complex quadric. Results Math. 73(3), Art 126 (10) (2018)
19. Pérez, J.D.; Jeong, I., Ko, J., Suh, Y.J.: Real hypersurfaces with killing shape operator in the complex quadric. Mediterr. J. Math. 15(1), Art 6 (15) (2018)
20. Reckziegel, H.: On the Geometry of the Complex Quadric. In: Dillen, F., Komrakov, B., Simon, U., Van de Woestyne, I., Verstraelen, L. (eds.) Geometry and Topology of Submanifolds, VIII, pp. 302–315. World Sci. Publ., River Edge (1996)
21. Romero, A.: Some examples of indefinite complete complex Einstein hypersurfaces not locally symmetric. Proc. Amer. Math. Soc. 98, 283–286 (1986)
22. Romero, A.: On a certain class of complex Einstein hypersurfaces in indefinite complex space forms. Math. Z. 192, 627–635 (1986)
23. Smyth, B.: Differential geometry of complex hypersurfaces. Ann. Math. 85, 246–266 (1967)
24. Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with harmonic curvature. J. Math. Pures Appl. 100, 16–33 (2013)
25. Suh, Y.J.: Hypersurfaces with isometric Reeb flow in complex hyperbolic two-plane Grassmannians. Adv. Appl. Math. 50, 645–659 (2013)
26. Suh, Y.J.: Real hypersurfaces in the complex quadric with Reeb parallel shape operator. Internat. J. Math. 25(6), 1450059 (17) (2014)
27. Suh, Y.J.: Real hypersurfaces in complex hyperbolic two-plane Grassmannians with commuting Ricci tensor. Internat. J. Math. 26(1), 155008 (26) (2015)
28. Suh, Y.J.: Real hypersurfaces in the complex quadric with parallel structure Jacobi operator. Diff. Geom. Appl. 51, 33–48 (2017)
29. Suh, Y.J.: Pseudo-einstein real hypersurfaces in the complex quadric. Math. Nachr. 290(11-12), 1884–1904 (2017)
30. Suh, Y.J., Hwang, D.H.: Real hypersurfaces in the complex quadric with commuting Ricci tensor. Sci. China Math. 59(11), 2185–2198 (2016)
31. Suh, Y.J., Lee, H., Woo, C.: Real hypersurfaces with commuting Jacobi operator in the complex quadric. Publ. Math. Debrecen 93(3-4), 425–443 (2018)
32. Vernon, M.H.: Contact hypersurfaces of a complex hyperbolic space. Tōhoku Math. J. 39, 215–222 (1987)

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