Binary Operations for Homotopy Groups with Coefficients

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Abstract

Binary operations are functions $T : \pi_{q_1}(X; G_1) \times \pi_{q_2}(X; G_2) \to \pi_{q_3}(X; G_3)$ compatible with induced homomorphisms. With Moore spaces $A = M(G_1, q_1 - 1)$ and $B = M(G_2, q_2 - 1)$ and a restriction on the degree $q_3$, there exists a unique element $\theta \in \pi_{q_3}(\Sigma(A \wedge B); G_3)$ such that $T(\alpha, \beta) = [\alpha, \beta]_\theta$, where $[\alpha, \beta] \in [\Sigma(A \wedge B), X]$ is the generalized Whitehead product. Properties of the generalized Whitehead product then carry over to these operations. We then study a class of operations which generalizes Hilton’s Whitehead products ($\pi_{q_1}(X; G_1) \times \pi_{q_2}(X; G_2) \to \pi_{q_1 + q_2 - 1}(X; G_1 \oplus G_2)$) and Torsion products ($\pi_{q_1}(X; G_1) \times \pi_{q_2}(X; G_2) \to \pi_{q_1 + q_2}(X; G_1 \ast G_2)$). We introduce a class of operations called Ext operations ($\pi_{q_1}(X) \times \pi_{q_2}(X) \to \pi_{q_1 + q_2 - 2}(X; \mathbb{Z}_k)$) and determine some of its properties. When all coefficient groups equal the same cyclic group of prime power order, we compare the Torsion product to a version of the Whitehead product. We prove, with mild restrictions, that the smash product of two Moore spaces has the homotopy type of a wedge of two Moore spaces.

1 Introduction

Products, such as the cup product for cohomology groups or the Whitehead product for homotopy groups, are important binary operations in Algebraic Topology. The cup product is defined for cohomology groups with coefficients, whereas the Whitehead product is usually defined for integral homotopy groups. In spite of the fact that the concept of homotopy groups with coefficients has been available for some time, there has been very little work on binary operations for homotopy groups with coefficients (two exceptions are [H1] and [N]). Our object in this paper is to study in some detail such binary operations and to provide unifying method. We show that the generalized Whitehead product, a function $[\Sigma A, X] \times [\Sigma B, X] \to [\Sigma(A \wedge B), X]$ of homotopy sets, can be mapped homomorphically onto many different binary operations. In this way, properties of the generalized Whitehead product are carried over to these binary operations.

The following is a brief outline of the paper. We shall refer to homotopy groups with coefficients as homotopy groups and binary operations as operations. After a preliminary section, the generalized Whitehead product and generalized Whitehead product map are recalled in §3. We show that the map can be extended to a map of a cone into a product of suspensions. In §4 we define the general notion of an operation for homotopy groups: $T$ is called a type $\{G_1, G_2, G_3; q_1, q_2, q_3\}$
operation if \( T : \pi_{q_1}(X;G_1) \times \pi_{q_2}(X;G_2) \to \pi_{q_3}(X;G_3) \) is a function for every space \( X \) and integers \( q_1, q_2, q_3 \geq 2 \), which is compatible with induced homomorphisms. A class of operations, called basic operations, is defined and it is proved that, with a restriction on the degree \( q_3 \), these are homomorphic images of the generalized Whitehead product (Theorem 4.2). We prove that the collection of basic operations forms a group isomorphic to a certain homotopy group. Operations called Ext operations are then introduced as a counterexample to extending an earlier result. In §5 we consider operations that have been restricted further (special operations) with a view to studying two particular classes of operations which have been introduced by Hilton, the Whitehead products and the Torsion products. If \( T \) is an operation which is expressed as the image of a generalized Whitehead product, then we give necessary and sufficient conditions for \( T \) to be a special operation. This then applies to Whitehead and Torsion products.

We conclude the section by proving an anti-commutativity statement which also applies to Whitehead and Torsion products. In §6, the final section, we discuss a number of topics which bear upon earlier sections. We first consider operations in which all three coefficient groups are cyclic groups with the same odd prime power order. The Torsion product is then compared to a version of the Whitehead product which was introduced by Neisendorfer. Next we consider the Ext operations and obtain a representation of them. Then we establish a homotopy equivalence between the smash product of two Moore spaces and the wedge of two (different) Moore spaces under very mild restrictions. Finally we briefly discuss the difference between using Moore spaces or Co-Moore spaces to obtain coefficients.

2 Preliminaries

In this section we present our notation and assumptions. All spaces are assumed to be based and of the homotopy type of based CW-complexes and all groups are assumed to be abelian. Maps and homotopies are to preserve base points. The base point is generically denoted by \( \ast \). We let \([f]\) denote the homotopy class of the map \( f \) and \( f \simeq g \) signifies that \( f \) and \( g \) are homotopic, but notationally we often ignore the distinction between maps and homotopy classes. For example, an expression containing a mix of maps and homotopy classes refers to the homotopy class determined by the expression. We write \( \Sigma X \) for the (reduced) suspension of the space \( X \) and \( CX = X \times I/\{\ast\} \times I \cup X \times \{1\} \) for the (reduced) cone. Also \( X \lor Y \) denotes the wedge and \( X \land Y \) the smash product. Furthermore, the join \( X * Y \) is the quotient of \( X \times Y \times I \) with the equivalence relations \((x, y, 0) \sim (x, y', 0)\) and \((x, y, 1) \sim (x', y, 1)\) and base point given by \( \{\ast\} \times \{\ast\} \times I \). We use "\( \approx \)" for isomorphism of groups and "\( \equiv \)" for same homotopy type. We let \([X, Y]\) be the set of homotopy classes of maps from \( X \) to \( Y \). A map \( f \) induces a homomorphism \( f_\ast \) of homotopy groups and a homomorphism \( f_\# \) of homotopy groups. For homomorphisms of groups \( h : G' \to G \) and \( k : H \to H' \), we let \( h^\ast : \text{Hom}(G, H) \to \text{Hom}(G', H) \) and \( k_\ast : \text{Hom}(G, H) \to \text{Hom}(G, H') \) be the induced homomorphisms. We denote by \( \mu : A * B \to \Sigma(A \land B) \) the homotopy equivalence obtained by collapsing the subset determined by \((A \times \{\ast\} \times I) \cup (\{\ast\} \times B \times I)\) to a point. The homotopy inverse of \( \mu \) is denoted \( \mu : \Sigma(A \land B) \to A \ast B \). Let \( G \) be a group and \( n \) an integer \( \geq 2 \). A Moore space \( M(G, n) \) is a simply-connected space with a single non-vanishing reduced homology group \( G \) in degree \( n \). The \( n \text{th homotopy group of } X \text{ with coefficients } G \), denoted \( \pi_n(X;G) \), is defined to be \([M(G, n), X]\). If \((X; A)\) is a pair of spaces then the homotopy group \( \pi_n(X; A; G) \) is defined as the set of homotopy classes of maps \((CM(G, n - 1), M(G, n - 1)) \to (X, A)\). We shall refer several times to the
Universal Coefficient Theorem for homotopy groups:

**Theorem** There is a short exact sequence

$$0 \rightarrow \Ext(G, \pi_{n+1}(X)) \overset{\lambda}{\rightarrow} \pi_n(X; G) \overset{\eta}{\rightarrow} \Hom(G, \pi_n(X)) \rightarrow 0$$

where $\eta[f] = f_\# : G \rightarrow \pi_n(X)$ ([HI], p. 30). When $X$ is replaced by a pair of spaces, the sequence is also exact.

### 3 The Generalized Whitehead Product

Let $A$, $B$, and $X$ be spaces and let $\alpha \in [\Sigma A, X]$ and $\beta \in [\Sigma B, X]$. The generalized Whitehead product of $\alpha$ and $\beta$ is an element $[\alpha, \beta] \in [\Sigma(A \wedge B), X]$ and is defined in $[A]$. When $A$ and $B$ are spheres, this is just the ordinary Whitehead product. Let $i_1 \in [\Sigma A, \Sigma A \vee \Sigma B]$ and $i_2 \in [\Sigma B, \Sigma A \vee \Sigma B]$ be the inclusions. Then $[i_1, i_2] \in [\Sigma(A \wedge B), \Sigma A \vee \Sigma B]$ is called the universal element for the generalized Whitehead product. If $f$ represents $\alpha$ and $g$ represents $\beta$, then $[\alpha, \beta] = (f, g)_* [i_1, i_2]$, where $(f, g) : \Sigma A \vee \Sigma B \rightarrow X$ is the map determined by $f$ and $g$ and $(f, g)_*$ is the induced map $[\Sigma(A \wedge B), \Sigma A \vee \Sigma B] \rightarrow [\Sigma(A \wedge B), X]$. Thus any generalized Whitehead product is the image of the universal element. We choose a map $k : (A \wedge B) \rightarrow \Sigma A \vee \Sigma B$ in the homotopy class $[i_1, i_2]$ and call it the generalized Whitehead product map.

**Theorem 3.1** There is a map $\Lambda : C(A \ast B) \rightarrow \Sigma A \times \Sigma B$ such that $\Lambda|A \ast B : A \ast B \rightarrow \Sigma A \vee \Sigma B$. If the latter map is denoted $\lambda$, then

1. $\lambda \mu \simeq k : \Sigma(A \wedge B) \rightarrow \Sigma A \vee \Sigma B$,

2. $\Lambda$ induces $\overline{\Lambda} : \Sigma(A \ast B) \rightarrow \Sigma A \wedge \Sigma B$ such that $\overline{\Lambda} \simeq \sigma(\Sigma \mu') : \Sigma(A \ast B) \rightarrow \Sigma A \wedge \Sigma B$, where $\sigma : \Sigma^2(A \ast B) \rightarrow \Sigma A \wedge \Sigma B$ is the homeomorphism given by $\sigma((a, b), t, u) = ((a, t), (b, u))$, for $a \in A, b \in B$, and $t, u \in I$.

**Proof** The function $\Lambda$ was defined by D. Cohen ([C], Theorem 2.4) as follows:

$$\Lambda((a, b, t), u) = \begin{cases} ((a, u), (b, 1 - 2t(1 - u))) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ ((a, 1 - 2(1 - t)(1 - u)), (b, u)) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

for $a \in A, b \in B$, and $t, u \in I$. The proof of (1) is an immediate consequence of Lemma 4.1 and the proof of Theorem 2.4 of [A]. For the proof of (2) we define a (linear) homotopy between $\overline{\Lambda}$ and $\sigma(\Sigma \mu')$:

$$\Phi_s(x) = \begin{cases} ((a, (1 - s)u + st), (b, (1 - s)(1 - 2t(1 - u)) + su)) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ ((a, (1 - (1 - 2(1 - t)(1 - u)) + st), (b, u)), & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

where $x = ((a, b, t), u) \in \Sigma(A \ast B)$ and $s \in I$.

### 4 Binary Operations

Let $G_1$, $G_2$, and $G_3$ be groups and $q_1$, $q_2$, and $q_3$ be integers. A (binary) operation of type $\{G_1, G_2, G_3; q_1, q_2, q_3\}$ is function $T$ which, for every space $X$ and integers $q_1, q_2 \geq 2$, assigns to each $\alpha \in \pi_{q_1}(X; G_1)$ and $\beta \in \pi_{q_2}(X; G_2)$, an element $T(\alpha, \beta) \in \pi_{q_3}(X; G_3)$ (also written $T_X(\alpha, \beta)$) such that if $f : X \rightarrow Y$ is a map, then $f_\#T_X(\alpha, \beta) = T_Y(f_\#(\alpha), f_\#(\beta))$. In the examples, $q_3$ will be a simple function of $q_1$ and $q_2$ such as $q_1 + q_2 + C$, for some constant $C$. Let $M_i = M(G_i, q_i)$, $i = 1, 2$, or $3$, let $i_1 \in \pi_{q_1}(M_1 \vee M_2; G_1)$ and $i_2 \in \pi_{q_2}(M_1 \vee M_2; G_2)$.
be the inclusions, and let \([f] = \alpha \in \pi_{q_1}(X; G_1)\) and \([g] = \beta \in \pi_{q_2}(X; G_2)\). Then \((f, g)_{\#} T_{(t_1, t_2)} = T(\alpha, \beta)\). We call \(T_{(t_1, t_2)} \in \pi_{q_3}(M_1 \vee M_2; G_3)\) the universal element for \(T\).

Next let the set of all operations of type \(\{G_1, G_2, G_3; q_1, q_2, q_3\}\) be denoted \(\mathcal{O} = \mathcal{O}\{G_1, G_2, G_3; q_1, q_2, q_3\}\). If \(T\) and \(T'\) are two such operations, then \(T + T'\) defined by \((T + T')(\alpha, \beta) = T(\alpha, \beta) + T'(\alpha, \beta)\) is also in \(\mathcal{O}\). Thus \(\mathcal{O}\) is an abelian group. Furthermore, the function from \(\mathcal{O}\) to \(\pi_{q_3}(M_1 \vee M_2; G_3)\) which sends \(T\) to \(T_{(t_1, t_2)}\) is easily seen to be an isomorphism.

**Definition 4.1** Let \(\partial : \pi_{q_3+1}(M_1 \times M_2, M_1 \vee M_2; G_3) \rightarrow \pi_{q_3}(M_1 \vee M_2; G_3)\) be the boundary homomorphism in the homotopy sequence of \((M_1 \times M_2, M_1 \vee M_2)\), let \(T\) be an operation as above, and assume that \(q_1, q_2 \geq 3\). Then \(T\) is called a basic operation if \(T_{(t_1, t_2)} \in \text{Im} \partial\), the image of \(\partial\).

Next let \(A = M(G_1, q_1 - 1)\) and \(B = M(G_2, q_2 - 1)\), so \(\Sigma A = M_1\) and \(\Sigma B = M_2\).

**Theorem 4.2** If \(T\) is a basic operation and \(q_3 < q_1 + q_2 + \min(q_1, q_2) - 3\), then there exists a unique element \(\theta_T \in \pi_{q_3}(\Sigma(A \wedge B); G_3)\) such that \(T(\alpha, \beta) = [\alpha, \beta] \theta_T = h^*[\alpha, \beta]\), for every space \(X\), where \(\alpha \in [\Sigma A, X]\), \(\beta \in [\Sigma B, X]\), \([h] = \theta_T\), and \(h^* : [\Sigma(A \wedge B), X] \rightarrow \pi_{q_3}(X; G_3)\) is induced by \(h\). Furthermore, \(h\) is a suspension and so \(h^*\) is a homomorphism.

**Proof** Consider the diagram

\[
\begin{array}{ccc}
\Sigma(A \wedge B) & \xrightarrow{\lambda_\mu} & C\Sigma(A \wedge B) \\
\downarrow \lambda_\mu & & \downarrow \lambda(C_\mu) \\
\Sigma A \vee \Sigma B & \xrightarrow{p} & \Sigma A \wedge \Sigma B,
\end{array}
\]

where each row is a cofiber sequence, the squares commute, and \(p\) and \(p'\) are projections. By the Blakers-Massey Theorem ([11], p. 49), if \(r < q_1 + q_2 + \min(q_1, q_2) - 3\), then the homomorphisms \(p_{\#} : \pi_{r+1}(C\Sigma(A \wedge B), \Sigma(A \wedge B); G_3) \rightarrow \pi_{r+1}(\Sigma^2(A \wedge B); G_3)\) and \(p_{\#} : \pi_{r+1}(\Sigma A \wedge \Sigma B, \Sigma A \vee \Sigma B; G_3) \rightarrow \pi_{r+1}(\Sigma A \wedge \Sigma B; G_3)\) induced by \(p\) and \(p'\) are isomorphisms. Therefore the exact homotopy group sequences of the pair \((C\Sigma(A \wedge B), \Sigma(A \wedge B))\) and the pair \((\Sigma A \wedge \Sigma B, \Sigma A \vee \Sigma B)\), together with the homomorphism of the first sequence into the second sequence determined by the map \(\Lambda(C_\mu)\), yield the following commutative square

\[
\begin{array}{ccc}
\pi_{q_3+1}(\Sigma^2(A \wedge B); G_3) & \xrightarrow{\delta'} & \pi_{q_3}(\Sigma(A \wedge B); G_3) \\
\downarrow \pi_{q_3+1}(\Sigma A \wedge \Sigma B; G_3) & & \downarrow \pi_{q_3}(\Sigma A \wedge \Sigma B; G_3) \\
(\Sigma \otimes_\mu)_{\#} = \sigma_{\#} & \xrightarrow{\delta} & \pi_{q_3}(\Sigma A \wedge \Sigma B; G_3)
\end{array}
\]

where \(\delta = \partial p_{\#}^{-1}\) for \(p_{\#}^{-1} : \pi_{q_3+1}(\Sigma A \wedge \Sigma B; G_3) \rightarrow \pi_{q_3+1}(\Sigma A \wedge \Sigma B, \Sigma A \vee \Sigma B; G_3)\) and \(\partial : \pi_{q_3+1}(\Sigma A \wedge \Sigma B, \Sigma A \vee \Sigma B; G_3) \rightarrow \pi_{q_3}(\Sigma A \wedge \Sigma B; G_3)\), and \(\delta'\) is similarly defined. Clearly \(\delta'\) and \(\sigma_{\#}\) are isomorphisms. In addition, it follows from the exact sequence of the pair \((\Sigma A \wedge \Sigma B, \Sigma A \vee \Sigma B)\) that \(\delta\) is one-one. Thus \(k_{\#}\) is one-one and \(\text{Im} k_{\#} = \text{Im} \delta = \text{Im} \partial p_{\#}^{-1} = \text{Im} \partial\). But \(T_{(t_1, t_2)} \in \text{Im} \partial\). Therefore there exists a unique \(\theta_T = [h] \in \pi_{q_3}(\Sigma(A \wedge B); G_3)\) with \(T_{(t_1, t_2)} = [t_1, t_2] \theta_T = h^*[t_1, t_2]\). If \(f : \Sigma A \rightarrow X\) and \(g : \Sigma B \rightarrow X\) represent \(\alpha\) and \(\beta\) respectively, then \([\alpha, \beta] \theta_T = (f, g)_{\#}[t_1, t_2] \theta_T = (f, g)_{\#}T_{(t_1, t_2)} = T(\alpha, \beta)\). The second assertion
of the theorem is a consequence of the generalized suspension theorem since the
dimension of \( M(G_3, q_3 - 1) \) is \( \leq q_3 \) and \( q_3 < q_1 + q_2 + \min(q_1, q_2) - 3 \). \( \square \)

**Remark 4.3** If \( G_3 \) is a free-abelian group, then the conclusion of Theorem 4.2
holds when \( q_3 = q_1 + q_2 + \min(q_1, q_2) - 3 \). This is also true for the conclusion
of subsequent results in which this strict inequality appears. This is because the
Blakers-Massey Theorem holds in this case.

**Corollary 4.4** Let \( T \) be an operation of type \( \{ G_1, G_2, G_3; q_1, q_2, q_3 \} \) and let
\( \alpha, \alpha' \in \pi_{q_1}(X; G_1) \) and \( \beta, \beta' \in \pi_{q_2}(X; G_2) \). Consider the following statements:

1. \( T \) is basic;
2. \( j_\# T(t_1, t_2) = 0 \), where \( j : M_1 \vee M_2 \to M_1 \times M_2 \) is the inclusion;
3. \( T \) is bi-additive: \( T(\alpha + \alpha', \beta) = T(\alpha, \beta) + T(\alpha', \beta) \) and \( T(\alpha, \beta + \beta') = T(\alpha, \beta)' \);
4. \( T(\alpha, 0) = 0 \) and \( T(0, \beta) = 0 \).

Then (1) \( \iff \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) \( \Rightarrow \) (2). If in addition \( q_3 < q_1 + q_2 + \min(q_1, q_2) - 3 \), then (2) \( \Rightarrow \) (3), and in this case all four statements are equivalent.

**Proof**

(1) \( \iff \) (2): This is an immediate consequence of the exactness of the homotopy
sequence of the pair \( (M_1 \times M_2, M_1 \vee M_2) \).

(3) \( \Rightarrow \) (4): \( T(\alpha, 0) = T(\alpha, 0 + 0) = T(\alpha, 0) + T(\alpha, 0) \) and so \( T(\alpha, 0) = 0 \).

\( T(0, \beta) = 0 \) is similar.

(4) \( \Rightarrow \) (2): Let \( j_k : M_k \to M_1 \times M_2 \) be the inclusions and \( p_k : M_1 \times M_2 \to M_k \)
be the projections, \( k = 1, 2 \). Then

\[
\text{\( j_\# T(t_1, t_2) = T(j_1, j_2) \in \pi_{q_3}(M_1 \times M_2; G_3). \)}
\]

But \( T(j_1, j_2) = 0 \iff p_1 T(j_1, j_2) = 0 \) and \( p_2 T(j_1, j_2) = 0 \). However by (4),
\( p_1 T(j_1, j_2) = T(p_1 j_1, 0) = 0 \) and similarly \( p_2 T(j_1, j_2) = 0 \). This proves (2).

(2) \( \Rightarrow \) (3): Since \( q_3 < q_1 + q_2 + \min(q_1, q_2) - 3 \) and since \( T \) is basic, (3) follows
from Theorem 4.2 and the bi-additivity of the generalized Whitehead product
(\[A\], p. 14).

Next let \( \text{BO} = \text{BO}\{G_1, G_2, G_3; q_1, q_2, q_3\} \) be the set of all basic operations
of type \( \{ G_1, G_2, G_3; q_1, q_2, q_3 \} \). Clearly \( \text{BO} \subseteq \text{O} \) is a subgroup.

**Corollary 4.5** If \( q_3 < q_1 + q_2 + \min(q_1, q_2) - 3 \), then there is an isomorphism
from \( \text{BO} \) to \( \pi_{q_3}(\Sigma(A \wedge B); G_3) \).

**Proof** For \( T \in \text{BO} \), we have \( T(t_1, t_2) = \tilde{k}_\#(\theta) \), for \( \theta \in \pi_{q_3}(\Sigma(A \wedge B)); G_3 \).

Conversely given \( \theta \), we define \( T \) by \( T(t_1, t_2) = \tilde{k}_\#(\theta) \). It suffices to prove that \( T \)

is basic, that is, \( j\tilde{k}\theta \simeq 0 \), by Corollary 4.4. But \( j\tilde{k} \simeq j\lambda \mu \simeq 0 \) since \( j\lambda \mu \) factors
through \( C\Sigma(A \wedge B) \) (see the diagram in the proof of Theorem 4.2).

We have seen that if \( T \) is basic and \( q_3 < q_1 + q_2 + \min(q_1, q_2) - 3 \), then \( T \) is bi-additive.
The following corollary gives additional properties with this hypothesis.

**Corollary 4.6** If \( T \in \text{BO} \) and \( q_3 < q_1 + q_2 + \min(q_1, q_2) - 3 \), then

1. \( T(\alpha, \beta) = 0 \) if \( X \) is an H-space;
2. \( ET(\alpha, \beta) = 0 \), where \( E : \pi_{q_3}(X; G_3) \to \pi_{q_3+1}(\Sigma X; G_3) \) is the suspension
   homomorphism;
3. If \( q_3 \leq q_1 + q_2 - 3 \), then \( T(\alpha, \beta) = 0 \), for all \( \alpha \) and \( \beta \).
Proof The generalized Whitehead product satisfies the first two properties (see [A], p. 13), and so (1) and (2) follow from Theorem 4.2. Property (3) follows since the dimension of $M(G_3, q_3)$ is $\leq q_3 + 1$ and $\Sigma(A \wedge B)$ is $(q_1 + q_2 - 2)$-connected, and so $\theta_T$ is nullhomotopic. □

The inequality in property (3) of Corollary 4.6 cannot be improved. That is, there are non-trivial operations with $q_3 = q_1 + q_2 - 2$. To see this, let $q = q_1 + q_2$, let $G_1 = G_2 = \mathbb{Z}$ and $G_3 = \mathbb{Z}_k$, for some integer $k > 1$ (so that $M_1 = S^{q_1}$ and $M_2 = S^{q_2}$), and let $(Y, X) = (M_1 \times M_2, M_1 \vee M_2)$. Consider the homomorphisms

$$\operatorname{Ext}(\mathbb{Z}_k, \pi_q(Y, X)) \xrightarrow{\lambda} \pi_{q-1}(Y, X; \mathbb{Z}_k) \xrightarrow{\partial} \pi_{q-2}(M_1 \vee M_2; \mathbb{Z}_k),$$

where $\lambda$ is the homomorphism of the Universal Coefficient Theorem and $\partial$ is the boundary homomorphism. Both homomorphisms are monomorphisms. Moreover, $\pi_q(Y, X)$ is isomorphic to $\mathbb{Z}$ and so the Ext term is $\operatorname{Ext}(\mathbb{Z}_k, \mathbb{Z}) = \mathbb{Z}_k$. Then the monomorphism $\partial \lambda$ maps these $k$ elements into $\pi_{q-2}(M_1 \vee M_2; \mathbb{Z}_k)$ and hence determines $k$ basic homotopy operations. Since $q - 2 < q_1 + q_2 + \min(q_1, q_2) - 3$, all of these operations are bi-additive and have the properties listed in Corollary 4.6. We shall refer to these operations of type $\{\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_k; q_1, q_2, q_1 + q_2 - 2\}$ as Ext operations. For more about them, see §6.

5 Whitehead and Torsion Products

We next define a class of basic operations. The purpose is to unify Hilton’s treatment of Whitehead products and Torsion products in ([H] pp. 110-120), Let $T$ be an operation of type $\{G_1, G_2, G_3; q_1, q_2, q_3\}$ and let $\omega \in \pi_{q_3}(M_1 \vee M_2; G_3)$ be the universal element for $T$. Furthermore, let $(Y, X) = (M_1 \times M_2, M_1 \vee M_2)$ and $Z = M_1 \wedge M_2$. We give three conditions:

1. $H_{q_3+1}(Z) \cong G_3$, and we let $\phi : G_3 \rightarrow H_{q_3+1}(Z) = H_{q_3+1}(M_1 \wedge M_2)$ be the isomorphism of the Künneth Theorem.

2. $T$ is basic, and so there is a unique $\xi \in \pi_{q_3+1}(Y, X; G_3)$ such that $\partial(\xi) = \omega$.

3. There is a homomorphism $\widehat{\eta}$ defined by commutativity of the following diagram

$$\begin{array}{ccc}
\pi_{q_3+1}(Y, X; G_3) & \xrightarrow{p^*} & \pi_{q_3+1}(Z; G_3) \\
& \xrightarrow{\eta} & \operatorname{Hom}(G_3, \pi_{q_3+1}(Z)),
\end{array}$$

where $p : Y = M_1 \times M_2 \rightarrow Z = M_1 \wedge M_2$ is the projection and $\eta$ is the epimorphism in the Universal Coefficient Theorem. Then the third condition is that the following composition is equal to the identity map

$$G_3 \xrightarrow{\widehat{\eta}(\xi)} \pi_{q_3+1}(M_1 \wedge M_2) \xrightarrow{h} H_{q_3+1}(M_1 \wedge M_2) \xrightarrow{\phi^{-1}} G_3,$$

where $h$ is the Hurewicz homomorphism.

Definition 5.1 Any operation which satisfies these three conditions will be called a special operation.

Remark 5.2 1. From the Künneth Theorem we have that there are two possibilities for a special operation:

(a) $q_3 = q_1 + q_2 - 1$ and $G_3 = G_1 \otimes G_2$

(b) $q_3 = q_1 + q_2$ and $G_3 = G_1 \ast G_2 = \operatorname{Tor}(G_1, G_2)$. 

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2. We comment on the third condition. The set of special operations of a given type may be empty since the third condition may not be satisfied (assuming that the first two are). Since \( q_3 < q_1 + q_2 + \min(q_1, q_2) - 3 \), \( p_\# \) is an epimorphism and so \( \tilde{\eta} \) is an epimorphism. If, in addition, \( h \) is an isomorphism, then the set \( \tilde{\eta}^{-1}(h^{-1}\phi) \) is non-empty and so the set of special operations equals \( \tilde{\eta}^{-1}(h^{-1}\phi) \). In the general case \( (h \) not necessarily an isomorphism), \( \tilde{\eta}(\xi) = \eta(p_\# \xi) = \eta(p\xi) \). There is a commutative diagram

\[
\begin{array}{ccc}
p_{q_3+1}(M(G_3, q_3 + 1)) & \longrightarrow & p_{q_3+1}(M_1 \wedge M_2) \\
\downarrow h' & & \downarrow h \\
G_3 & \rightarrow & H_{q_3+1}(M(G_3, q_3 + 1))
\end{array}
\]

and \( \eta(p\xi) = (p\xi)_\# h'^{-1} \). Therefore

\[
\phi = h\tilde{\eta}(\xi) = h\eta(p\xi) = h(p\xi)_\# h'^{-1} = (p\xi)_\# h'^{-1} = (p\xi)_*. \]

Thus the third condition is

\( p_{\xi_\#} = (p\xi)_* : G_3 = H_{q_3+1}(M(G_3, q_3 + 1)) \rightarrow H_{q_3+1}(M_1 \wedge M_2) \).

In \([H1]\), Hilton defined two classes of binary homotopy operations, the Whitehead products and the Torsion products. We give slightly different definitions which are equivalent to Hilton’s definition. One difference is that we apply the Universal Coefficient Theorem to \( M_1 \wedge M_2 \) instead of to the pair \( (M_1 \times M_2, M_1 \vee M_2) \). In the relevant degrees the homotopy groups of the two are isomorphic. A second difference concerns the existence of the isomorphism \( \phi \). In \((H1, pp. 110, 115)\), \( H_{q_3+1}(M_1 \wedge M_2) \) is identified with \( G_3 \), whereas we make the isomorphism explicit.

Whitehead products are defined as special operations \( T \) satisfying 1(a) in Remark \([5,2]\) with \( G_3 = G_1 \otimes G_2 \) and \( q_3 = q_1 + q_2 - 1 \). They are called Whitehead products of type \( \{G_1, G_2; q_1, q_2\} \). Then \( \tilde{\eta} : \eta_{q_3+1}(M_1 \times M_2, M_1 \vee M_2; G_3) \rightarrow \text{Hom}(G_3, \eta_{q_3+1}(M_1 \wedge M_2)) \) is onto and \( h : \eta_{q_3+1}(M_1 \wedge M_2) \rightarrow H_{q_3+1}(M_1 \wedge M_2) \) is an isomorphism by the Hurewicz Theorem. Therefore if \( \omega \) is an element of the non-empty set \( \tilde{\eta}^{-1}(h^{-1}\phi) \), then by definition \( \omega \) is the universal element of a Whitehead product \( T \). There can be several different Whitehead products, in fact, since \( \partial \) is one-one, the number of Whitehead products is just the cardinality of the set \( \text{Ker} \tilde{\eta} = \text{Ker} \eta = \text{Ext}(G_3, \eta_{q_3+2}(M_1 \wedge M_2)) \) by the Universal Coefficient Theorem. Moreover, if \( T \) is a Whitehead product, then there exists a unique \( \theta_T \in \eta_{q_3}(\Sigma(A \wedge B); G) \) such that \( T(\alpha, \beta) = [\alpha, \beta] \theta_T \) by Theorem \([4,2]\). Thus each Whitehead product \( T \) satisfies bi-additivity and the properties listed in Corollary \([4,6]\).

Torsion products are defined as special operations \( T \) which satisfy 1(b) in Remark \([5,2]\) with \( G_3 = G_1 \ast G_2 = \text{Tor}(G_1, G_2) \), \( q_3 = q_1 + q_2 \), and, in addition, \( q_1, q_2 \geq 4 \). They will be called Torsion products of type \( \{G_1, G_2; q_1, q_2\} \). A Torsion product is determined by an element \( \tau \in \eta_{q_3}(M_1 \vee M_2; G_3) \) such that \( \tau = \partial(\zeta) \), for some \( \zeta \in \eta_{q_3+1}(M_1 \times M_2, M_1 \vee M_2; G_3) \), and such that \( \phi^{-1} h\tilde{\eta}(\zeta) \) is the identity homomorphism of \( G_3 \) (see \([H1], p. 115)\). Note that the set of Torsion products may be empty, though not if \( h \) is an isomorphism. In this case the number of Torsion products equals the cardinality of the set \( \text{Ker} \tilde{\eta} = \text{Ker} \eta = \text{Ext}(G_3, \eta_{q_3+2}(M_1 \wedge M_2)) \). Furthermore, the hypotheses of Theorem \([4,2]\) are satisfied, and so there exists a unique \( \theta_T \in \eta_{q_3}(\Sigma(A \wedge B); G_3) \)
such that \( T(\alpha, \beta) = [\alpha, \beta] \theta_T \). As in the previous case, each Torsion product \( T \) is bi-additive and satisfies the properties listed in Corollary 4.6.

Many of these properties for Whitehead and Torsion products have been proved in ([11], pp. 111–113 and 116–119) directly from the definitions, though some of our results (such as bi-additivity) are more general and the proofs are shorter (see [11], Theorems 12.3 and 12.6).

Since \( \pi_i(X; G \oplus G') \approx \pi_i(X; G) \oplus \pi_i(X; G') \), for any \( i \geq 2 \) and groups \( G \) and \( G' \), and since \( G_1 \ast G_2 = 0 \) if \( G_1 \) or \( G_2 \) is free-abelian, for the Torsion product we may restrict attention to the case when \( G_1 = \mathbb{Z}_m \) and \( G_2 = \mathbb{Z}_n \) are cyclic groups. Then \( G_1 \ast G_2 = \mathbb{Z}_d \), where \( d \) is the greatest common divisor of \( m \) and \( n \). In ([11], pp. 115–116) the following was proved: A Torsion product of type \( \{ \mathbb{Z}_m, \mathbb{Z}_n; q_1, q_2 \} \) exists if and only if (1) \( d \) is odd or (2) \( m \) and \( n \) are even and either \( m \) or \( n \) is a multiple of 4. In particular, a Torsion product exists if \( m = n = p^k \), where \( p \) is an odd prime and \( k \geq 1 \).

Another approach to Whitehead and Torsion products is suggested by Theorem 4.2. In that theorem it is proved that many basic operations \( T \) can be written as \( T(\alpha, \beta) = [\alpha, \beta] \theta_T \), for a unique \( \theta_T \in \pi_{q_3}(\Sigma(A \wedge B); G_3) \), where \( A = M(G_1, q_1-1) \) and \( B = M(G_2, q_2-1) \). This suggests that we define an operation by \( T(\alpha, \beta) = [\alpha, \beta] \theta \), for some \( \theta \in \pi_{q_3}(\Sigma(A \wedge B); G_3) \).

**Proposition 5.3** Let the operation \( T \) of type \( \{ G_1, G_2, G_3; q_1, q_2, q_3 \} \) be defined by \( T(\alpha, \beta) = [\alpha, \beta] \theta \) and let \( \phi : G_3 \rightarrow H_{q_3+1}(M_1 \wedge M_2) = H_{q_3+1}(\Sigma A \wedge \Sigma B) \) be the Künneth isomorphism. Then \( T \) is a special operation if and only if \( (\Sigma \phi)_* = \sigma^{-1} \phi \), where \( \sigma : \Sigma(A \wedge B) \rightarrow \Sigma A \wedge \Sigma B \) is defined in Theorem 5.7.

**Proof** First we show that \( T \) is basic. Let \( \omega \) be the universal element of \( T \) and consider the homotopy-commutative diagram obtained from Theorem 5.1 (see also the diagram in the proof of Theorem 5.2).

\[
\begin{array}{cccccc}
M(G_3, q_3) & \xrightarrow{\theta} & \Sigma(A \wedge B) & \xrightarrow{\tilde{k}} & \Sigma A \vee \Sigma B \\
CM(G_3, q_3) & \xrightarrow{C\phi} & C\Sigma(A \wedge B) & \xrightarrow{\Lambda'} & \Sigma A \times \Sigma B \\
\Sigma M(G_3, q_3) & \xrightarrow{\Sigma \phi} & \Sigma^2(A \wedge B) & \xrightarrow{\sigma} & \Sigma A \wedge \Sigma B,
\end{array}
\]

where \( \Lambda' = \Lambda(C\mu) \). If \( \xi = \Lambda'(C\theta) \in \pi_{q_3+2}(\Sigma A \times \Sigma B; \Sigma A \vee \Sigma B; G_3) \), then \( \partial(\xi) = \tilde{k} \theta = \omega \), and so \( T \) is basic. Furthermore, \( p(\xi) = \sigma(\Sigma \theta) \), and so the third condition is

\[\sigma_*(\Sigma \theta)_* = p_* \xi_* = \phi,\]

which is equivalent to \( (\Sigma \theta)_* = \sigma^{-1} \phi \). This completes the proof. \(\square\)

Note that if \( T \), given by the universal element \( \tilde{k} \theta \), is a special operation, then \( \theta_* : H_{q_3}(M(G_3, q_3)) \rightarrow H_{q_3}(\Sigma(A \wedge B)) \) is an isomorphism.

By Remark 5.2, there are only two possibilities for special operations. The following result is then a consequence of Proposition 5.3.

**Corollary 5.4** Let \( T \) be an operation of type \( \{ G_1, G_2, G_3; q_1, q_2, q_3 \} \) with universal element \( \tilde{k} \theta \) for some \( \theta \in \pi_{q_3}(\Sigma(A \wedge B); G_3) \).

1. Let \( q_3 = q_1 + q_2 - 1 \) and \( G_3 = G_1 \otimes G_2 \). Then \( T \) is a Whitehead product if and only if \( (\Sigma \theta)_* = \sigma^{-1} \phi \).
2. Let \( q_3 = q_1 + q_2 \) and \( G_3 = G_1 \ast G_2 \). Then \( T \) is a Torsion product if and only if \( (\Sigma \theta)_* = \sigma^{-1}\phi \).

**Remark 5.5** For Whitehead products \( (q = q_1 + q_2 = q_3 + 1 \text{ and } G_3 = G_1 \otimes G_2) \), we claim that this corollary can identify those \( \theta \in \pi_{q-1}(\Sigma(A \wedge B); G_3) \) such that \( \overline{k}\theta \) are all the Whitehead universal elements. We choose any universal element \( \omega \in \pi_{q-1}(M_1 \times M_2; G_3) \). Then \( \omega = \partial(\xi) \) for \( \xi \in \pi_q(M_1 \times M_2, M_1 \vee M_2; G_3) \). Thus \( p\xi \in \pi_q(M_1 \wedge M_2; G_3) \) and \( \sigma^{-1}(p\xi) \in \pi_q(\Sigma^2(A \wedge B); G_3) \). Because the suspension homomorphism \( E : \pi_{q-1}(\Sigma(A \wedge B); G_3) \to \pi_q(\Sigma^2(A \wedge B); G_3) \) is an isomorphism, there exists a unique \( \theta \in \pi_{q-1}(\Sigma(A \wedge B); G_3) \) such that \( \Sigma\theta = \sigma^{-1}(p\xi) \). We will show that \( \overline{k}\theta = \omega \). From the diagram in the proof of Proposition 5.3 we see that

\[
\partial(\Lambda'(C\theta)) = \overline{k}\theta \quad \text{and, with } p\#: \pi_q(M_1 \times M_2, M_1 \vee M_2; G_3) \to \pi_q(M_1 \wedge M_2; G_3),
\]

\[
p\#(\Lambda'(C\theta)) = p(\Lambda'(C\theta)) = \sigma(\Sigma \theta) = \sigma\sigma^{-1}(p\xi) = p\#(\xi).
\]

Since \( p\# \) is an isomorphism, \( \Lambda'(C\theta) = \xi \), and so

\[
\overline{k}\theta = \partial(\Lambda'(C\theta)) = \partial(\xi) = \omega.
\]

This establishes the claim.

A similar remark holds for the Torsion product.

We next consider commutativity of special operations. Our proof is an adaptation of the argument in \((\text{Hi}), \text{pp. 113-114 and 117-118})\.

Let \( T \) be a special operation of type \( \{G_1, G_2, G_3; q_1, q_2, q_3\} \). Then \( G_3 = G_1 \otimes G_2 \) or \( G_3 = G_1 \ast G_2 \), and we set \( G'_3 = G_2 \otimes G_1 \) or \( G'_3 = G_2 \ast G_1 \), accordingly. Furthermore, let \( t : G'_3 \to G_3 \) be the switching isomorphism \( (G_2 \otimes G_1) \to G_1 \otimes G_2 \) or \( G_2 \ast G_1 \to G_1 \ast G_2 \). Then there is a map \( \tau : M(G'_3, q_3) \to M(G_3, q_3) \) such that \( \tau_* = t \).

**Proposition 5.6** With \( T \) a special operation as above, we define an operation \( S \) by

\[
S(\beta, \alpha) = (-1)^{\delta}T(\alpha, \beta)\tau,
\]

for \( \alpha \in \pi_q(W; G_3) \) and \( \beta \in \pi_{q_2}(W; G_2) \) and for any space \( W \), where \( \delta = q_1q_2 \) when \( G_3 = G_1 \otimes G_2 \) and \( \delta = q_1q_2 + 1 \) when \( G_3 = G_1 \ast G_2 \). Then \( S \) is a special operation of type \( \{G_2, G_1, G'_3; q_2, q_1, q_3\} \).

**Proof** Let \( (Y, X) = (M_1 \times M_2, M_1 \vee M_2) \), \( (Y', X') = (M_2 \times M_1, M_2 \vee M_1) \), \( Z = M_1 \wedge M_2 \), and \( Z' = M_2 \wedge M_1 \). If \( \rho : (Y, X) \to (Y', X') \) be the switching map, then \( \rho \) determines maps \( \rho' : X \to X' \) and \( \rho'' : Z \to Z' \). There is a commutative diagram

\[
\begin{array}{ccc}
\pi_{q_3+1}(Y, X; G_3) & \xrightarrow{\partial} & \text{Hom}(G_3, \pi_{q_3+1}(Z)) \\
\rho_* \tau^* & \downarrow & h_* \downarrow \rho'\tau''_* \\
\pi_{q_3+1}(Y', X'; G'_3) & \xrightarrow{\partial'} & \text{Hom}(G'_3, \pi_{q_3+1}(Z')).
\end{array}
\]

If \( \omega \in \pi_{q_3}(X; G_3) \) is the universal element with \( \omega = \partial(\xi) \) for \( \xi \in \pi_{q_3+1}(Y, X; G_3) \), then \( h_* \partial(\xi) = \phi \in \text{Hom}(G_3, H_{q_3+1}(Z)) \). We show that if \( \xi' = \rho'\tau''(\xi) \), then \( h'_* \partial'(\xi') = (\rho''\tau')^* \phi \), where \( \phi' : G'_3 \to H_{q_3+1}(Z') \) is the K"unneth isomorphism. We have

\[
h'_* \partial'(\xi') = (\rho''\tau')^* h_* \partial(\xi) = (\rho''\tau')^* (\rho^* \phi t) = \rho''\phi t.
\]
It follows immediately from results of Hilton (Hilton, pp. 114 and 118) that the following diagram is commutative

\[
\begin{array}{ccc}
G_3 & \xrightarrow{(-1)^s t^{-1}} & G'_3 \\
\phi & & \phi' \\
H_{q_3+1}(Z) & \xrightarrow{\rho''} & H_{q_3+1}(Z')
\end{array}
\]

Therefore

\[
h^*_q(\xi') = \rho''(\xi) = (-1)^s \phi' t^{-1} t = (-1)^s \phi'.
\]

We set \( \omega' = \partial'(\xi') \) where \( \partial' : \pi_{q_3+1}(Y', X'; G'_3) \rightarrow \pi_{q_3}(X'; G'_3) \). Thus \((-1)^s \omega'\) is the universal element of an operation \( S \) of type \( \{ G_2, G_1, G_3; q_2, q_1, q_3 \} \). Note that

\[
\rho^t \omega = \rho^t \psi^t (\partial \xi) = \partial' (\rho^t \psi^t (\xi)) = \partial' (\xi') = \omega'.
\]

Let \( j_1 : M_2 \rightarrow M_2 \vee M_1 \) and \( j_2 : M_1 \rightarrow M_2 \vee M_1 \) be inclusions. Then

\[
S(j_1, j_2) = (-1)^s \rho^t (t_1, t_2) \tau = (-1)^s T(j_2, j_1),
\]

and so

\[
S(\beta, \alpha) = (-1)^s T(\alpha, \beta) \tau,
\]

for \( \alpha \in \pi_{q_3}(W; G_1) \) and \( \beta \in \pi_{q_3}(W; G_2) \). The conclusion of the proposition now follows. \( \square \)

Note that if the operation \( T \) is unique, then there is the following anti-commutative rule

\[
T(\beta, \alpha) = (-1)^s T(\alpha, \beta) \tau.
\]

**Corollary 5.7**

1. If \( T \) is a Whitehead product of type \( \{ G_1, G_2; q_1, q_2 \} \), then the special operation \( S \) defined by \( S(\beta, \alpha) = (-1)^{q_1+q_2} T(\alpha, \beta) \tau \) is a Whitehead product of type \( \{ q_2, q_1; G_2, G_1 \} \).

2. If \( T \) is a Torsion product of type \( \{ G_1, G_2; q_1, q_2 \} \), then the special operation \( S \) defined by \( S(\beta, \alpha) = (-1)^{q_1+q_2+1} T(\alpha, \beta) \tau \) is a Torsion product of type \( \{ q_2, q_1; G_2, G_1 \} \).

6 Concluding Remarks and Results

1. Neisendorfer’s Approach

   We consider operations with coefficients \( \mathbb{Z}_p^k \), \( p \) an odd prime and \( k \geq 1 \). As mentioned earlier, any operation with finite groups of coefficients of odd order can be expressed in terms of operations with these coefficients.

   Let \( G_1 = G_2 = G_3 = \mathbb{Z}_p^k \) so that \( G_1 \circ G_2 = G_1 * G_2 = \mathbb{Z}_p^k \) and let \( M(i) = M(\mathbb{Z}_p^k, i) \). Then Neisendorfer proved that there is a homotopy equivalence \( \delta : M(q-2) \vee M(q-1) \rightarrow M(q_1-1) \wedge M(q_2-1) = A \wedge B \), where \( q = q_1 + q_2 \) (Neisendorfer, p. 167). We suspend and obtain (after the identification of \( \Sigma(M(q-2) \vee M(q-1)) \) with \( M(q-1) \vee M(q) \)) a homotopy equivalence \( \delta' : M(q-1) \vee M(q) \rightarrow \Sigma(A \wedge B) \). If \( j_2 : M(q) \rightarrow M(q-1) \vee M(q) \) is the inclusion, then we set \( \theta = \delta' j_2 : M(q) \rightarrow \Sigma(A \wedge B) \) and define an operation \( T \) of type \( \{ \mathbb{Z}_p^k, Z_p^k, Z_p^k; q_1, q_2, q_1 + q_2 \} \) by \( T(\alpha, \beta) = [\alpha, \beta] \theta \).

   This operation was originally defined in (Neisendorfer, Section 6.3) where many of its properties were studied in detail. It was referred to as a Whitehead product. This may seem puzzling at first since the degrees of \( T \) are not those of a Whitehead product. But Neisendorfer used a definition of homotopy groups.
3. The Smash Product of Two Moore Spaces

By applying $-p$ of degree $k$. A projection $\pi M \to \pi$ that is homologically trivial. We shall show that $\pi$ operation and $
abla q_1 + q_2 = 1$ is basic as in the proof of Proposition 5.3 by taking

3. The Smash Product of Two Moore Spaces

Let $G_1$ and $G_2$ be finitely-generated abelian groups such that neither $G_1$ nor $G_2$ has $2$-torsion. Then there is a homotopy equivalence

$$M(G_1, q_1) \wedge M(G_2, q_2) \equiv M(G_1 \otimes G_2, q_1 + q_2) \vee M(G_1 \ast G_2, q_1 + q_2 + 1).$$

**Proof** Let $M_i = M(G_i, q_i)$ for $i = 1, 2$, let $q = q_1 + q_2$, and let $G_3 = G_1 \otimes G_2$ and $G_3 = G_1 \ast G_2$. It is easily seen (e.g., by a homology decomposition) that there is a map $l : M(G_3, q) \to M(G_3, q)$ such that $M(G_1, q_1) \wedge M(G_2, q_2)$ has the homotopy type of the mapping cone $M(G_3, q) \cup_l CM(G_3, q)$ and that $l$ is homologically trivial. We shall show that $l = 0$. Now $l \in \pi_q(M(G_3, q); G_3)$ and, with $M = M(G_3, q)$, we consider

$$\pi_q(M; G_3) \xrightarrow{\eta} \operatorname{Hom}(G_3, \pi_q(M)) \xrightarrow{h^*} \operatorname{Hom}(G_3, H_q(M)),$$

where $\eta$ is the Universal Coefficient homomorphism and $h^*$ is induced by the Hurewicz isomorphism $h$. Then

$$h^*(\eta(l)) = hl\# = l_* = 0,$$

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and so \( \eta(l) = 0 \). Therefore by exactness of the Universal Coefficient Theorem, \( l = \lambda(l) \), for \( l \in \text{Ext}(G_3, \pi_{q+1}(M(G_3, q))) \). It suffices to show that \( \tilde{l} = 0 \). We set \( E = \text{Ext}(G_3, \pi_{q+1}(M(G_3, q))) \) and show that \( E = 0 \). We write \( G_i = F_i \oplus T_i, i = 1, 2 \), where \( F_i \) is a free-abelian group and \( T_i \) is a finite torsion group. Then \( E = \text{Ext}(T_1 \ast T_2, A \oplus B \oplus C \oplus D) \), where \( A = \pi_{q+1}(M(F_1 \otimes F_2, q)), B = \pi_{q+1}(M(F_1 \otimes T_2, q)), C = \pi_{q+1}(M(T_1 \otimes F_2, q)), \) and \( D = \pi_{q+1}(M(T_1 \otimes T_2, q)) \). Then each of \( F_1 \otimes T_2, T_1 \otimes F_2, \) and \( T_1 \otimes T_2 \) is a finite direct sum of cyclic groups of order a power of an odd prime. But \( \pi_{m+1}(M(\mathbb{Z}_n, m)) = 0 \) if \( n \) is odd \( [\text{B}], \) p. 268). Therefore \( B = C = D = 0 \). Thus \( E = \text{Ext}(T_1 \ast T_2, A) = \text{Ext}(T_1 \ast T_2, \pi_{q+1}(M(F_1 \otimes F_2, q))) \). Now \( F_1 \otimes F_2 \) is a direct sum of finitely many copies of \( \mathbb{Z} \) and so \( M(F_1 \otimes F_2, q) \) is a wedge of finitely many \( q \)-spheres \( S^q \). Hence \( A = \pi_{q+1}(M(F_1 \otimes F_2, q)) \) is a direct sum of terms \( \pi_{q+1}(S^q) \), that is, a direct sum of copies of \( \mathbb{Z}_2 \). Therefore \( E = \text{Ext}(T_1 \ast T_2, A) = 0 \) and so \( l = 0 \). It follows that the mapping cone is a wedge of \( M(G_1 \otimes G_2, q_1 + q_2) \) and \( M(G_1 \ast G_2, q_1 + q_2 + 1) \). This completes the proof.

\[ \square \]

**Remark 6.3** Theorem \([12] \) holds if either \( G_1 \) or \( G_2 \) has 2-torsion (but not both). For definiteness suppose that \( G_1 \) has 2-torsion and \( G_2 \) does not. Then \( T_1 \otimes F_2 \) is a finite direct sum of cyclic groups of order a power of a prime including the prime 2. Thus \( C = \pi_{q+1}(M(T_1 \otimes F_2, q)) \) is a finite direct sum of copies of \( \mathbb{Z}_2 \) \([\text{B}], \) p. 268) and it follows that \( \text{Ext}(T_1 \ast T_2, C) = 0 \).

4. Moore vs. Co-Moore Spaces

As a final comment we observe that there are advantages and disadvantages to using either Co-Moore spaces or Moore spaces for coefficients. Co-Moore spaces are the dual of Eilenberg-MacLane spaces within the context of Eckmann-Hilton duality, where homotopy groups and cohomology groups are considered dual to each other, but Co-Moore spaces do not exist for every group \( G \) \([\text{Ha}], \) p. 318). Moore spaces exist for every group, but they are not dual to Eilenberg-MacLane spaces.

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