Second-order accurate finite volume method for well-driven flows

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Abstract
We consider a finite volume method for a well-driven fluid flow in a porous medium. Due to the singularity of the well, modeling in the near-well region with standard numerical schemes results in a completely wrong total well flux and an inaccurate hydraulic head. Local grid refinement can help, but it comes at computational cost. In this article we propose two methods to address well singularity. In the first method the flux through well faces is corrected using a logarithmic function, in a way related to the Peaceman correction. Coupling this correction with a second-order accurate two-point scheme gives a greatly improved total well flux, but the resulting scheme is still not even first order accurate on coarse grids. In the second method fluxes in the near-well region are corrected by representing the hydraulic head as a sum of a logarithmic and a linear function. This scheme is second-order accurate.

Keywords: Finite volume method, Near-well modeling, Flow simulations, Second-order accuracy

1. Introduction

The stationary groundwater flow equation is obtained by substituting the Darcy law
\[ \mathbf{u} = -K \nabla h \quad \text{in} \quad \Omega \]
into the continuity equation
\[ \nabla \cdot \mathbf{u} = g_s, \]
where \( \mathbf{u} \) is the Darcy velocity, \( g_s \) describes sources and sinks, \( K \) is the hydraulic conductivity tensor, \( h \) is the hydraulic head, and \( \Omega \subset \mathbb{R}^3 \) is a bounded domain.

We consider the following boundary conditions:
\[ h = g_D \quad \text{on} \quad \Gamma_D, \]
\[ \mathbf{u} \cdot \mathbf{n} = g_N \quad \text{on} \quad \Gamma_N, \]
where \( \partial \Omega = \Gamma_D \cup \Gamma_N \) is the domain boundary, \( \Gamma_D \cap \Gamma_N = \emptyset, \Gamma_D \neq \emptyset \), and \( \mathbf{n} \) is the unit vector normal to \( \partial \Omega \) pointing outwards.

A colimated layer, also known as the skin effect, is formed along the well wall due to well clogging [1]. This causes additional hydraulic resistance (see Fig. [1]). As a result, flux density through the well filter is
\[ u = \Psi (h_r - h_w), \]
where \( h_w \) is the hydraulic head inside the well, \( h_r \) is the hydraulic head just outside the colimated layer (see Fig. [1]), \( \Psi = K_c/d_c \) is the transfer coefficient, while \( K_c \) and \( d_c \) are the unknown conductivity and thickness of the colimated layer respectively.

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Hydraulic head varies logarithmically and its gradient changes sharply in the well vicinity (Fig. 1). Thus, linear approximation of hydraulic head is inappropriate on coarse grids and numerical methods based on it are inaccurate in the near-well region.

Accurate modeling in the near-well region is important in reservoir engineering. Flow in the entire reservoir is induced mainly by wells, therefore poor near-well modeling results in accuracy loss throughout the model.

Numerous families of second-order accurate numerical methods are applicable to porous media flows. Here we consider non-linear two-point approximations [2, 3, 4, 5, 6, 7, 8]. These schemes do not include special near-well discretization and thus lose accuracy on coarse grids if a well is present.

Local grid refinement can alleviate the problem. However, this comes at a computational cost.

Methods for well modeling have been widely discussed in the literature [9, 10, 11, 12, 13, 14, 15]. A commonly used method is the Peaceman correction [10, 13, 14]. This approach was originally formulated for finite differences, with a well placed in the center of the cell. It has been extended for various other discretization methods [9]. The Peaceman correction yields a correct flow rate but the obtained hydraulic head distribution is not second order accurate on coarse grids.

In finite element methods wells are usually associated with arrays of mesh edges in three dimensional models, or with nodes in two dimensional models. Since there are more tools available to generate unstructured meshes for finite elements then for other methods, in order to be able to use these meshes we construct cylinders (circles in two dimensions) around well edges (nodes). As a result, a well is an array of cylindrical cells in three dimensional models (Fig. 2 right) or a circular cell in two dimensional models (Fig. 2 left).
The well face correction method (WFC) described in Subsection 2.1 is related to the Peaceman method and results in a greatly improved well extraction rate compared to the uncorrected scheme, but the hydraulic head is not second-order accurate on coarse grids near the well.

In the near-well correction scheme (NWC) presented in Sect. 2.2 the flow in the well vicinity is split into a linear part and a part that is due to the influence of the well. A similar idea was used in [10, 11] for a multipoint scheme, but the accuracy of this scheme reduces if a well is much smaller than the grid size.

The paper is organized as follows. The two discretization schemes are presented in Sect. 2 in the two dimensional case. Three dimensional versions of these schemes are presented in Sect. 3. The results of numerical tests are provided in Sect. 4.

2. Discretization in two dimensions

In order to use the same terminology in two-dimensional and three-dimensional cases, the edges of two-dimensional cells are referred to as faces and their lengths are called face areas.

Integrating (2) over cell \( T \) and applying the divergence theorem yields

\[
\sum_{f \in \partial T} \chi_{T,f} u_f = \int_T g_s dT, \quad \text{where} \quad u_f = \int_f \mathbf{u} \cdot \mathbf{n}_f ds.
\]

(6)

Term \( u_f \) denotes the flux through face \( f \), \( \mathbf{n}_f \) is a unit vector normal to face \( f \) fixed once and for all, while \( \chi_{T,f} = 1 \) if \( \mathbf{n}_f \) points outside of \( T \) and \( \chi_{T,f} = -1 \) otherwise. Boundary face normals always point outside.

We associate one hydraulic head value \( h_T \) with each cell centroid \( x_T \). Also, the Dirichlet boundary condition is evaluated at each node belonging to \( \Gamma_D \). These cell centroids and nodes are referred to as primary collocation points.

An auxiliary hydraulic head value is associated with each well face. These auxiliary head values are eliminated and do not enter the discretization matrix. Well face centroids are referred to as auxiliary collocation points.

Either the hydraulic head is set in a well cell or a source/sink term is used to specify the flow rate.

2.1. Well face correction (WFC)

Consider the case of a homogeneous isotropic circular reservoir of radius \( R \) with hydraulic conductivity \( K = K_I \), and with a well of radius \( r \) in its center. The well extraction rate \([16]\) is

\[
Q = AK \frac{h_R - h_r}{r \ln \frac{R}{r}},
\]

(7)

where \( A \) is the total area of the well screen, while \( h_r \) and \( h_R \) are hydraulic head values in the porous medium at distances \( r \) and \( R \), respectively, from the well center.

Based on the flow rate (7) we propose to calculate the flux through well face \( f \) (see Fig. 2) belonging to cell \( T \) as

\[
\mathbf{u}_f = |f| K \frac{h_T - h_f}{r \ln \frac{2|\mathbf{x}_T|}{r}},
\]

(8)

where \( |f| \) is the face area, \( \mathbf{x}_T \) is the centroid of cell \( T \), and \( \rho(\mathbf{x}_T) \) is the distance from \( \mathbf{x}_T \) to the well center. Hydraulic head at cell \( T \) is denoted by \( h_T \) and \( h_f \) is the auxiliary hydraulic head value at face \( f \).

From equation (5) flux through face \( f \) is

\[
\mathbf{u}_f = |f| \Psi(h_f - h_w).\]

(9)

Combining equations (8) and (9) gives a flux approximation that does not include the auxiliary head value at face \( f \):

\[
\mathbf{u}_f = |f| \frac{\Psi K}{r \ln \frac{2|\mathbf{x}_T|}{r} + K}(h_T - h_w).
\]

(10)

This correction leads to an acceptable well extraction rate. However, there is a substantial error in the hydraulic head distribution, which does not decrease quadratically if the mesh is refined. Therefore, the scheme obtained in this way is not second-order accurate on coarse grids, as shown in Sect. 4.
2.2. Near-well correction (NWC)

We assume that the hydraulic conductivity \( K = K_I \) is homogeneous and isotropic in the near-well region. The hydraulic head is represented as

\[
h \approx L + \hat{h},
\]

where \( L \) is a linear function and \( \hat{h} \) is the singular part

\[
\hat{h}(x) = C_0 \ln r(x) + C_1,
\]

\( C_0 \) and \( C_1 \) are arbitrary constants, and

\[
r(x) = \|x - x_w\|
\]

is the distance from the well center \( x_w \).

From (11) and (12) the hydraulic head gradient is

\[
\nabla h \approx \nabla L + C_0 \nabla \ln (r(x)).
\]

Thus flux (6) can be written as

\[
-u_f = -\int_f (K \nabla h) \cdot n_f ds \approx -\int_f K \nabla L \cdot n_f ds - \int_f C_0 K \nabla \ln (r(x)) \cdot n_f ds,
\]

Since \( \nabla L \) is constant, the first integral can be expressed as

\[
-\int_f K \nabla L \cdot n_f ds = -|f| K \nabla L \cdot n_f.
\]

Let \( Pr(f) \) denote the radial projection of face \( f \) onto the well wall from the well center (see Fig. 3). The flow component described by the second integral is directed toward the well center. Therefore, this flux component through faces \( f \) and \( f_1 \) is the same as through \( Pr(f) \):

\[
-\int_f C_0 K \nabla \ln (r(x)) \cdot n_f ds = -\int_{Pr(f)} C_0 K \nabla \ln (Pr(x)) \cdot n_f ds = -\sigma_f \frac{|Pr(f)| C_0 K}{r},
\]

because

\[
\nabla \ln (Pr(x)) \cdot n_f = \frac{1}{r},
\]

where \( n_f \) is the outer unit normal on the circle in \( Pr(x) \), and \( \sigma_f = -1 \) if \( n_f \) points inside the triangle defined by face \( f \) and the well center, or \( \sigma_f = 1 \) otherwise. Substituting (16) and (17) in (15) gives

\[
u_f \approx -|f| (K \nabla L) \cdot n_f - \sigma_f \frac{|Pr(f)| C_0 K}{r}.
\]

Let cells \( T_+ \) and \( T_- \) share the face \( f \), and let \( n_f \) point from \( T_+ \) to \( T_- \). By \( x_i \) and \( x_- \) we denote the collocation points in cells \( T_+ \) and \( T_- \). Let \( x_i \) be another collocation point. The difference of hydraulic head values at \( x_i \) and \( x_+ \) is

\[
h_i - h_+ \approx \nabla L \cdot (x_i - x_+) + C_0 \ln \frac{r(x_i)}{r(x_+)}
\]

from (11) and (12). Equations of this form for \( i \in \{1, 2, 3\} \) make up a linear system

\[
AC \approx b,
\]

where

\[
A = \begin{bmatrix}
x_1 - x_+ & y_1 - y_+ & \ln \frac{r(x_1)}{r(x_+)} \\
x_2 - x_+ & y_2 - y_+ & \ln \frac{r(x_2)}{r(x_+)} \\
x_3 - x_+ & y_3 - y_+ & \ln \frac{r(x_3)}{r(x_+)} 
\end{bmatrix},
\]

\[
b = \begin{bmatrix}
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
h_1 \\
h_2 \\
h_3
\end{bmatrix},
\]

\[
b = \begin{bmatrix}
h_1 \\
h_2 \\
h_3
\end{bmatrix}.
\]
\[ \mathbf{x}_i = [x_i, y_i]^T, \quad \mathbf{C} = \left[ \frac{\partial L}{\partial x} \frac{\partial L}{\partial y} C_0 \right]^T, \quad \mathbf{b} = [h_1 - h_+ \quad h_2 - h_+ \quad h_3 - h_+]^T. \] (23)

One or more equations in system (21) may be replaced with a condition that the flux computed using formula (15) matches the flux imposed at a Neumann boundary face: integrating (4) over some Neumann face \( \bar{f} \) and using (19) results in

\[ \left| \bar{f} \right| (K \nabla L) \cdot \mathbf{n}_{\bar{f}} + \sigma_f \left| \text{Pr}(\bar{f}) \right| \frac{C_0 K}{r} = -g_N(x_{\bar{f}}) |\bar{f}|. \] (24)

Let us assume that matrix \( A \) is invertible and let the elements of matrix \( A^{-1} \) be denoted by \( a_{ij} \). Let index \( k \) correspond to equations of form (20), while index \( \bar{k} \) corresponds to equations of form (24). From (21), (23), and (24) the coordinates of unknown vector \( \mathbf{C} \) are:

\[
\begin{align*}
\frac{\partial L}{\partial x} & \approx \sum_k a_{1k}(h_k - h_+) - \sum_k a_{1k} g_N(x_{f_k}) |f_k|, \\
\frac{\partial L}{\partial y} & \approx \sum_k a_{2k}(h_k - h_+) - \sum_k a_{2k} g_N(x_{f_k}) |f_k|, \\
C_0 & \approx \sum_k a_{3k}(h_k - h_+) - \sum_k a_{3k} g_N(x_{f_k}) |f_k|.
\end{align*}
\] (25)

After substituting (25) in (19), the flux approximation becomes

\[ u_f \approx -\sum_k \alpha_k (h_k - h_+) + \sum_k \alpha_k g_N(x_{f_k}) |f_k|, \] (26)

where

\[
\begin{align*}
\alpha_k &= K \left( |f| \left( a_{1k} n_{f}^1 + a_{2k} n_{f}^2 \right) + a_{3k} \frac{|\text{Pr}(f)|}{r} \right), \\
\alpha_{\bar{k}} &= K \left( |f| \left( a_{1\bar{k}} n_{f}^1 + a_{2\bar{k}} n_{f}^2 \right) + a_{3\bar{k}} \frac{|\text{Pr}(f)|}{r} \right).
\end{align*}
\] (27) (28)

Term \( n_{f}^l \) denotes the \( l \)-th coordinate of vector \( n_f \). In the same way we obtain the one-sided approximation from cell \( T_- \):

\[ u_f \approx \sum_l \alpha_l (h_l - h_-) - \sum_l \alpha_l g_N(x_{f_l}) |f_l|, \] (29)
The derivation is carried out further like in [2, 6]. One-sided approximations of form (26) and (29) are linearly combined using non-negative weights $\mu_+$ and $\mu_-:

$$u_f \approx -\mu_+ \sum_k \alpha_k^+ (h_k - h_+) + \mu_- \sum_l \alpha_l^- (h_l - h_-) + \mu_+ \sum_k \alpha_k^+ g_N(x_{f_k}) |f_k| - \mu_- \sum_l \alpha_l^- g_N(x_{f_l}) |f_l|. \quad (30)$$

For this approximation to be valid it is required that:

$$\mu_+ + \mu_- = 1. \quad (31)$$

We choose $\mu_+$ and $\mu_-$ so that in (30) contributions of hydraulic head values other than $h_-$ and $h_+$ as well as contributions of inflow Neumann boundary conditions, cancel out:

$$-\mu_+ d_+ + \mu_- d_- = 0, \quad d_{\pm} = \sum_{x_k \neq x_{\mp}} \alpha_k^\pm h_k - \sum_k \alpha_k^\pm g_N(x_{f_k}) |f_k|. \quad (32)$$

If $d_+ + d_- \neq 0$, $\mu_{\pm}$ is computed from (31) and (32)

$$\mu_+ = \frac{d_-}{d_+ + d_-}, \quad \mu_- = \frac{d_+}{d_+ + d_-}, \quad (33)$$

otherwise we set $\mu_{\pm} = 0.5$.

In this way, a two-point flux approximation is obtained:

$$u_f \approx M_f^+ h_+ - M_f^- h_- + r_f, \quad (34)$$

where

$$M_f^+ = \mu_+ \sum_k \alpha_k^+ + \mu_- \sum_{x_k = x_+} \alpha_l^+, \quad (35)$$

$$M_f^- = \mu_- \sum_l \alpha_l^- + \mu_+ \sum_{x_k = x_-} \alpha_k^+, \quad (36)$$

$$r_f = \mu_+ \sum_{g_N(x_{f_k}) > 0} \alpha_k^+ g_N(x_{f_k}) |f_k| - \mu_- \sum_{g_N(x_{f_l}) > 0} \alpha_l^- g_N(x_{f_l}) |f_l|. \quad (37)$$

Following the same logic as in [2], it is required that $\alpha_k^+, \alpha_l^- \geq 0$, for all $k, l$, which implies that $M_f^+ \geq 0$, so the resulting computational matrix is an M-matrix. If this is not the case, then another set of collocation points and Neumann faces is chosen to form (21).

The proposed scheme is used within the near-well region, which can be of any shape, as long as it includes at least the cells nearest to the well. Near-well regions belonging to different wells must not overlap. Scheme [6] is used outside of these regions. Fluxes through the faces at the edge of the near-well region are obtained by coupling one-sided flux approximation (26) with an uncorrected one-sided flux approximation [6] in the same way as in (30).

The auxiliary hydraulic head values at the well faces are eliminated using the requirement that approximation (34) is equal to (5), as done in [6] for a material discontinuity.

3. Discretization in three dimensions

In the three dimensional case, the well is represented as an array of cylindrical cells. In the well cell where the pump is located, either the hydraulic head or a source term is specified. The flow through the

\[ \text{\textbf{rf}} \]
well is modeled using the Hagen-Poiseuille law \[17\], meaning that the hydraulic conductivity along the well is computed as

\[
K = \frac{r^2 \rho g}{8 \mu},
\]

(38)

where \(\rho\) is the density, \(g\) is the gravity and \(\mu\) is the dynamic viscosity.

The well face correction scheme derived in Sect. 2.1 is directly applicable to the three-dimensional case. The near-well correction scheme is derived in a manner analogous to the two dimensional case, with \(\rho(x_T)\) representing the distance between \(x_T\) and the well axis, and \(Pr(f)\) denoting a projection of face \(f\) onto the well cylinder. This projection, defined in Appendix A, is known in cartography as Lambert cylindrical equal-area projection.

System (21) is formed with four instead of three equations of type (20) and (24). The vector of unknowns is

\[
C = \begin{bmatrix}
\frac{\partial L}{\partial x} & \frac{\partial L}{\partial y} & \frac{\partial L}{\partial z} & C_0
\end{bmatrix}^T.
\]

(39)

Thus, instead of (25) we have

\[
\frac{\partial L}{\partial x} \approx \sum_k a_{1k}(h_k - h_0) - \sum_k a_{1k}g(x_{f_k})|f_k|,
\]

\[
\frac{\partial L}{\partial y} \approx \sum_k a_{2k}(h_k - h_0) - \sum_k a_{2k}g(x_{f_k})|f_k|,
\]

\[
\frac{\partial L}{\partial z} \approx \sum_k a_{3k}(h_k - h_0) - \sum_k a_{3k}g(x_{f_k})|f_k|,
\]

\[
C_0 \approx \sum_k a_{4k}(h_k - h_0) - \sum_k a_{4k}g(x_{f_k})|f_k|.
\]

(40)

Therefore instead of (27) and (28) we have

\[
\alpha_k = K \left( |f| \sum_{i=1}^{3} a_{1k}n_f^i + a_{3k}f \frac{|Pr(f)|}{r} \right),
\]

(41)

and

\[
\alpha_k = K \left( |f| \sum_{i=1}^{3} a_{1k}n_f^i + a_{3k}f \frac{|Pr(f)|}{r} \right).
\]

(42)

4. Numerical tests

To verify the schemes we solve several problems whose analytical solutions are available. In each example we compare the results obtained with uncorrected, WFC and NWC schemes. Near-well regions are taken to be circular or cylindrical.

The meshes used in the examples were independently generated and are not hierarchically related. Mesh parameter \(P\) is the square root of the largest cell area in the two dimensional cases (Examples 1 and 2). In the three dimensional case (Example 3) the mesh parameter is the cubic root of the largest cell volume. Unstructured triangular meshes are used in the first two examples and unstructured triangular prismatic meshes are used in the third example.

Discrete \(L_2\) and maximum norms were used to evaluate relative hydraulic head errors:

\[
\epsilon^h_2 = \left[ \frac{\sum_T (h(x_T) - h_T)^2 |T|}{\sum_T (h(x_T))^2 |T|} \right]^{1/2},
\]

(43)
\( h_{\text{max}} = \max_{T} \left| h(x_T) - h_T \right| \)

\[
\epsilon^h_{\text{max}} = \frac{\max_T |h(x_T) - h_T|}{\sum_T (h(x_T))^2 |T| / \sum_T |T|^{1/2}}.
\]

where \( |T| \) stands for volume (area in 2D) of cell \( T \) and \( |f| \) stands for area (length in 2D) of face \( f \). The analytical hydraulic head evaluated at the centroid of cell \( T \) is denoted by \( h(x_T) \), while the head value numerically obtained in this cell is denoted by \( h_T \).

The relative error of the total well flux is also presented:

\[
\epsilon_Q = \frac{Q - Q_A}{Q_A},
\]

where \( Q \) is the numerical well flux and \( Q_A \) is the analytical flux.

**Example 1.** We consider a circular reservoir with a well in the center. The exact flow rate is given by (7), and the exact hydraulic head at distance \( \rho \) from the center is

\[
h(\rho) = h_w \ln \frac{R}{\rho} + h_R \ln \frac{\rho}{r}.
\]

In this example we specify the hydraulic head in the well \( h_w \) and at \( \rho = R \). We take \( r = 0.05, R = 200, h_w = 55, h_R = 100 \) and \( K = 0.0001 \). Transfer coefficient \( \Psi \) is set so that the hydraulic head at the well wall is \( h_r = 60 \).

**Table 1: Errors in Example 1**

| P | 32√2 | 16√2 | 8√2 | 4√2 | 2√2 | √2 |
|---|---|---|---|---|---|---|
| uncorrected scheme | | | | | | |
| \( \epsilon_2 \) | 3.3349e-02 | 3.2881e-02 | 3.2454e-02 | 3.0796e-02 | 2.8715e-02 | 2.3497e-02 |
| \( \epsilon_{\text{max}} \) | 1.1237e-01 | 1.2507e-01 | 1.7871e-01 | 1.8020e-01 | 2.1814e-01 | 1.8954e-01 |
| \( \epsilon_Q \) | -9.8285e-01 | -9.7334e-01 | -9.5137e-01 | -9.0151e-01 | -8.368e-01 | -6.8381e-01 |
| WFC scheme | | | | | | |
| \( \epsilon_2 \) | 3.4011e-03 | 6.1547e-04 | 7.4125e-04 | 4.1751e-04 | 3.7675e-04 | 3.104e-04 |
| \( \epsilon_{\text{max}} \) | 1.6428e-02 | 5.1832e-03 | 1.853e-02 | 7.5021e-03 | 6.8333e-03 | 5.5382e-01 |
| \( \epsilon_Q \) | 1.1732e-02 | 9.8658e-03 | 4.4707e-03 | 1.0318e-02 | 8.9214e-03 | 8.7904e-03 |
| NWC scheme | | | | | | |
| \( \epsilon_2 \) | 6.5656e-04 | 1.6437e-04 | 4.1625e-04 | 9.1401e-04 | 2.5938e-06 | 6.2480e-07 |
| \( \epsilon_{\text{max}} \) | 2.7643e-03 | 9.8072e-04 | 3.317e-04 | 7.8684e-05 | 3.3677e-05 | 7.8565e-06 |
| \( \epsilon_Q \) | 1.4182e-02 | -1.2922e-04 | 1.5902e-04 | 2.9709e-05 | 1.7036e-06 | 2.4278e-06 |

The errors are presented in Table 1. The uncorrected scheme is not second-order accurate for the considered meshes and the flow rate through the well is completely wrong. The hydraulic head error is larger near the well, as shown in Fig. 4 (left). This is as expected because the flow velocity changes quickly in this region. If even finer meshes could be used, this scheme would likely approach second-order accuracy at some point.

If the WFC scheme is used, errors are smaller than those obtained without any correction. The largest errors are still located near the well (Fig. 4 middle). The well flow rate error is around one percent on the coarsest mesh and it decreases very slowly as the mesh is refined. However, the scheme is still not even first-order accurate for the considered meshes.

The results for the NWC scheme were obtained using a near-well region with radius \( \rho = 40 \). The absolute hydraulic head error distribution is shown in Fig. 4 (right). Results obtained in this way are second-order accurate. If we took \( R \) for the radius of the near-well region, then this scheme would be exact.
The reduction of the well flow rate error with the mesh parameter is less predictable because it depends on the particular geometry of the few cells around the well, which changes in a random fashion. Nevertheless, a comparison of flow rate errors on the finest and coarsest meshes shows that this flow rate is at least first order accurate.

**Figure 4:** Absolute hydraulic head error using mesh \( P = 4\sqrt{2} \) in Example 1 with uncorrected scheme (left), WFC scheme (middle), NWC scheme (right).

**Example 2.** Here we consider a rectangular reservoir with corners \((\pm 300, \pm 150)\). Two wells with radii \( r_l \) and \( r_r \) are specified at \((-150,0)\) and \((150,0)\), respectively.

An analytical solution is obtained by superposing two solutions of form (46):

\[
h(x) = \frac{h_{r_l} \ln \frac{R_l}{\rho_l} + h_{R_l} \ln \frac{R_l}{r_l}}{\ln \frac{R_l}{r_l}} + \frac{h_{r_r} \ln \frac{R_r}{\rho_r} + h_{R_r} \ln \frac{R_r}{r_r}}{\ln \frac{R_r}{r_r}}, \tag{47}
\]

where the distances from the left and the right well are denoted with \( \rho_l \) and \( \rho_r \) respectively. We take \( h_l = 5 \), \( h_r = 10 \), \( h_{R_l} = h_{R_r} = 20 \), \( r_l = 0.5 \), \( r_r = 0.6 \) and \( R_l = R_r = 1200 \).

Transfer coefficient \( \Psi \) is set for each well face separately so that (5) and (47) give level 23 in the left well and 27 in the right well.

As in the previous example, uncorrected scheme is not second order accurate for the considered meshes and the well flow rates are very inaccurate (Table 2). The total well flux error is much smaller with WFC scheme, but the scheme is still not second-order accurate. The errors for NWC scheme are obtained using a circular near-well region with radius \( \rho_l = \rho_r = 100 \). The results show that the NWC scheme is second-order accurate.

**Example 3.** The domain is a rectangular prism with corners \((\pm 100, \pm 50, \pm 50)\). It contains two straight wells, one horizontal from \((-50, -50, 0)\) to \((-50, 50, 0)\) and one vertical from \((50, 0, -50)\) to \((50, 0, 50)\).

An analytical solution is again obtained by superposition and it is given by (47). Distances \( \rho_l \) and \( \rho_r \) are calculated as

\[
\rho_l = \sqrt{(x - x_l)^2 + (z - z_l)^2}, \quad \rho_r = \sqrt{(x - x_r)^2 + (y - y_r)^2}, \tag{48}
\]

where \( x_l = -50, z_l = 0, x_r = 50 \) and \( y_r = 0 \).

In this example we take \( R_l = R_r = 1000, h_{R_l} = 50, h_{R_r} = 53, r_l = 0.1, r_r = 0.15, h_l = 40 \) and \( h_r = 45 \).

The transfer coefficient in each well face is chosen according to the Hagen-Poiseuille law so that the head in the horizontal well pump is 90 and that in the vertical well pump is 92. The pumps are located at \((-50, -50, 0)\) and \((50, 0, -50)\) for the horizontal and vertical wells, respectively. Hydraulic head isosurfaces are shown in Fig. 6 on the left and the mesh (for \( P = 8 \)) is shown on the right.
Table 2: Errors in Example 2

| $P$  | 64  | 32  | 16  | 8   | 4   | 2    |
|------|-----|-----|-----|-----|-----|------|
| uncorrected scheme | | | | | | |
| $\epsilon_2$ | 3.3782e-02 | 3.2982e-02 | 2.9440e-02 | 2.4404e-02 | 1.3192e-02 | 6.1115e-03 |
| $\epsilon_{\text{max}}$ | 8.1375e-02 | 1.1471e-01 | 1.3808e-01 | 1.4784e-01 | 1.0101e-01 | 5.9049e-02 |
| $\epsilon_{Q_l}$ | -9.4180e-01 | -8.9760e-01 | -7.9625e-01 | -6.4903e-01 | -3.7113e-01 | -1.7455e-01 |
| $\epsilon_{Q_r}$ | -9.3404e-01 | -8.6878e-01 | -7.4095e-01 | -6.1714e-01 | -2.7756e-01 | -1.2008e-01 |
| WFC scheme | | | | | | |
| $\epsilon_2$ | 2.6163e-03 | 2.7284e-04 | 9.1249e-04 | 6.9715e-04 | 6.4924e-04 | 6.2680e-04 |
| $\epsilon_{\text{max}}$ | 1.1375e-02 | 1.8067e-02 | 1.1118e-02 | 1.3259e-02 | 8.2544e-02 | 8.1610e-03 |
| $\epsilon_{Q_l}$ | 1.8027e-02 | 1.1693e-03 | 1.2771e-02 | 1.0014e-02 | 1.5029e-02 | 1.3153e-02 |
| $\epsilon_{Q_r}$ | 1.6572e-02 | 7.7413e-03 | 1.3536e-02 | 1.2957e-02 | 1.9314e-02 | 2.1323e-02 |
| NWC scheme | | | | | | |
| $\epsilon_2$ | 1.0757e-03 | 1.8726e-04 | 3.7874e-04 | 7.8463e-06 | 2.1300e-06 | 5.7798e-07 |
| $\epsilon_{\text{max}}$ | 2.4906e-03 | 7.0931e-04 | 1.9740e-04 | 4.3807e-05 | 1.2458e-05 | 1.2968e-05 |
| $\epsilon_{Q_l}$ | 7.5779e-03 | 4.2305e-04 | -3.1607e-05 | 2.0544e-05 | -4.5723e-06 | 1.8546e-06 |
| $\epsilon_{Q_r}$ | 1.1794e-02 | 6.6805e-04 | 1.6003e-04 | -2.3436e-05 | 1.4109e-06 | -7.2901e-07 |

Figure 5: Absolute hydraulic head error using mesh $P = 8$ in Example 2 with uncorrected scheme (top left), WFC scheme (top right), NWC scheme (bottom).
The errors of the uncorrected, WFC and NWC schemes are shown in Table 3. Near-well region radius \( \rho_l = \rho_r = 30 \) is used.

![Figure 6: Hydraulic head isosurfaces (left) and the mesh (right) in Example 3.](image)

| \( P \) | 16 | 8 | 4 | 2 |
|--------|----|----|----|----|
| uncorrected scheme | | | | |
| \( \epsilon_2 \) | 4.2876e-03 | 3.9696e-03 | 3.4098e-03 | 2.6704e-03 |
| \( \epsilon_{max} \) | 1.2832e-02 | 1.9503e-02 | 2.3277e-02 | 2.3188e-02 |
| \( \epsilon_{Q_l} \) | -9.5013e-01 | -8.9142e-01 | -7.9405e-01 | 6.3175e-01 |
| \( \epsilon_{Q_r} \) | -9.2382e-01 | -8.7037e-01 | -6.6288e-01 | 5.2980e-01 |
| WFC scheme | | | | |
| \( \epsilon_2 \) | 9.3401e-04 | 3.0169e-04 | 1.5315e-04 | 5.7184e-05 |
| \( \epsilon_{max} \) | 5.3188e-03 | 4.0249e-03 | 3.3704e-03 | 2.6835e-03 |
| \( \epsilon_{Q_l} \) | -1.1654e-02 | -2.0510e-02 | -1.3227e-02 | -9.5725e-03 |
| \( \epsilon_{Q_r} \) | -1.4505e-02 | 6.9811e-03 | 1.7615e-02 | 6.3175e-01 |
| NWC scheme | | | | |
| \( \epsilon_2 \) | 9.7085e-05 | 1.8482e-05 | 4.5855e-06 | 1.1298e-06 |
| \( \epsilon_{max} \) | 3.2256e-04 | 6.2802e-05 | 2.1404e-05 | 4.6046e-06 |
| \( \epsilon_{Q_l} \) | 3.5085e-03 | 1.6127e-04 | 3.7865e-05 | 7.0735e-06 |
| \( \epsilon_{Q_r} \) | 5.1169e-04 | 2.3796e-04 | -9.8912e-05 | -3.5768e-05 |

5. Conclusion

Discretization schemes based on linear approximations produce very inaccurate results if a well is present. The commonly used Peaceman correction improves the computed well flow rate but does not give a second-order accurate hydraulic head.

We have developed two schemes for the discretization of near-well fluxes. The first scheme (WFC scheme, Sect. 2.1), is similar to the Peaceman correction. It reduces the well flux error, but it is not second order accurate on coarse grids.
Numerical examples show that the second scheme (NWC scheme, Sect. 2.2) gives at least a first-order accurate total well flux and a second-order accurate hydraulic head without near-well mesh refinement.

Both schemes are developed for the case of isotropic hydraulic conductivity. An extension to the anisotropic case is planned.

Appendix A. Lambert cylindrical equal-area projection

![Figure A.7: Projection of triangular face onto a cylinder.](image)

Projection of \( x \) onto a cylinder is defined by

\[
\text{Pr}(x) = x_p + r \frac{(x_i - x_p)}{\|x_i - x_p\|},
\]

where \( x_p \) is the orthogonal projection of \( x \) onto the cylinder axis and \( r \) is the cylinder radius.

The projection of a straight line is generally not a second-order curve (Fig. A.7). Numerical integration is used to calculate the area of \( \text{Pr}(f) \) in Sect 3. The results presented in this paper were obtained using the 6th order Gauss-Legendre integration formula. In our case this formula was accurate enough to calculate the integrals with machine precision.

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