Radiative generation of realistic neutrino mixing with $A_4$

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Abstract

Radiative generation of realistic mixing in neutrino sector is studied at one-loop level in a sco-
togenic $A_4 \times Z_2$ symmetric framework. A scheme of obtaining non-zero $\theta_{13}$ through small mass
splitting in right-handed neutrino sector is proposed. The model consists of three right-handed neu-
trinos, two of which were required to be degenerate in masses to yield the common structure of the
left-handed neutrino mass matrix that corresponds to $\theta_{13} = 0$, $\theta_{23} = \pi/4$ and any $\theta^0_{12}$ in particular
the choices specific to the Tribimaximal (TBM), Bimaximal (BM) and Golden Ratio (GR) mixings. Non-zero $\theta_{13}$, deviations of $\theta_{23}$ from maximality and small corrections to the solar mixing angle $\theta_{12}$
can be generated in one stroke by shifting from this degeneracy in the right-handed neutrino sector
by a small amount. The lightest among the three $Z_2$ odd inert $SU(2)_L$ doublet scalars present in the
model can be a potential dark matter candidate.

I Introduction

Neutrino oscillation observations have clearly demonstrated the massive nature of neutrinos. The mass
eigenstates are non-degenerate and distinct from the flavour eigenstates and are connected to each other
by the Pontecorvo, Maki, Nakagawa, Sakata – PMNS – matrix usually parametrized as:

$$U = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & -s_{13}e^{-i\delta} \\
    -c_{23}s_{12} + s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} + s_{23}s_{13}s_{12}e^{i\delta} & -s_{23}c_{13} \\
    -s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} + c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13}
\end{pmatrix},$$

(1)

where $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$.

In 2012, the short-baseline reactor anti-neutrino experiments observed non-zero $\theta_{13}$, yet small compared
to the other two mixing angles [1]. Prior to this observation, models leading to several structures like
the Tribimaximal (TBM), Bimaximal (BM) and Golden Ratio (GR) mixings (which we refer henceforth
as popular lepton mixings) were studied all of which were constructed with $\theta_{13} = 0$ and $\theta_{23} = \pi/4$ and
varying $\theta^0_{12}$ yielded the different alternatives.

Putting $\theta_{13} = 0$ and $\theta_{23} = \pi/4$ in Eq. (1) can lead to the common structure for all popular mixings:

$$U^0 = \begin{pmatrix}
    \cos \theta^0_{12} & \sin \theta^0_{12} & 0 \\
    -\sin \theta^0_{12} & \cos \theta^0_{12} & -\frac{1}{\sqrt{2}} \\
    \sin \theta^0_{12} & \cos \theta^0_{12} & \frac{1}{\sqrt{2}}
\end{pmatrix},$$

(2)

where $\theta^0_{12}$ for TBM, BM and GR are listed in Table [1]

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Table 1: $\theta_{12}^{0}$ for the different popular lepton mixings viz. TBM, BM, and GR mixing.

| Model | TBM | BM | GR |
|-------|-----|----|----|
| $\theta_{12}^{0}$ | $35.3^\circ$ | $45.0^\circ$ | $31.7^\circ$ |

The present $3\sigma$ global fits for the three mixing angles [2, 3]:

$$
\begin{align*}
\theta_{12} &= (31.42 - 36.05)^\circ, \\
\theta_{23} &= (40.3 - 51.5)^\circ, \\
\theta_{13} &= (8.09 - 8.98)^\circ.
\end{align*}
$$

The numbers are from NuFIT3.2 of 2018 [2].

Thus the popular mixings are in disagreement with the observed non-zero $\theta_{13}$. A plethora of activities had been taking place since this discovery to incorporate non-zero $\theta_{13}$ in these mixings. Attempts to relate the smallness of solar splitting with that of $\theta_{13}$ can be found in [4]. In [5], $\Delta m^2_{atmos}$ and $\theta_{23} = \pi/4$ were embedded in the dominant component of neutrino masses and mixing and the other oscillation parameters such as $\theta_{13}, \theta_{12}$, the deviation of $\theta_{23}$ from $\pi/4$, and $\Delta m^2_{solar}$ were obtained perturbatively from a smaller see-saw [6] contribution. Vanishing $\theta_{13}$ can be induced by certain symmetries and models generating non-zero $\theta_{13}$ through perturbation to such symmetric structures are also studied [8, 9].

In [10, 11], discrete flavour symmetries like $A_4$, $S_3$ were used to devise a two-component Lagrangian formalism at tree-level to ameliorate all the popular mixing patterns in single stroke. The dominant contribution to the Lagrangian was obtained from Type II see-saw mechanism characterized by popular mixing patterns, to which corrections were obtained from a sub-dominant Type I see-saw contribution. In [12], the same scheme was performed for the no solar mixing (NSM) case i.e., $\theta_{12}^{0} = 0$ case with $A_4$ symmetry. The basic difference between [11, 12] and earlier works with $A_4$ [13, 14, 15] is that in the earlier works Type II see-saw was used to generate the mass matrices and obtaining TBM was the prime goal. More realistic mixings can be found in recent works [16, 17].

In this paper, we intend to generate:

1. the popular mixing structure in Eq. (2) with $\theta_{13} = 0$, $\theta_{23} = \pi/4$ and $\theta_{12}^{0}$ as listed in Table 1
2. non-zero $\theta_{13}$, deviations of $\theta_{23}$ from $\pi/4$ and small corrections to $\theta_{12}$

radiatively with $A_4$ flavour symmetry [4]. Precisely, this $A_4 \times Z_2$ symmetric model will produce neutrino masses at one-loop level using three right-handed neutrinos that transform as a triplet under $A_4$. To get Eq. (2) it is necessary that two of these right-handed neutrinos are degenerate. A little shift from that degeneracy will yield non-zero $\theta_{13}$, deviations of atmospheric mixing from maximality and tinker the solar mixing by a small amount in one go.

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1 Some such earlier models can be found in [7].
2 The dominant Type II see-saw structure was kept devoid of solar splitting and degenerate perturbation theory was used to obtain large $\theta_{12}$.
3 For review of radiative neutrino mass models see [18].
4 For a brief discussion on $A_4$ see Appendix of the paper.
In order to accomplish this we also need to introduce a $Z_2$ odd $A_4$ triplet scalar field $\eta$, the lightest of which could be a potential dark matter candidate.

II The Model

The neutrino mass matrix in the mass basis is given by $M_{\nu L}^{\text{mass}} = \text{diag} \ (m_1, m_2, m_3)$. This when expressed in flavour basis using the common structure of $U_0$ in Eq. (2) for the popular lepton mixings, leads to:

$$M_{\nu L}^{\text{flavour}} = U_0 M_{\nu L}^{\text{mass}} U_0^T = \begin{pmatrix} a & c & c \\ c & b & d \\ c & d & b \end{pmatrix}$$

where,

$$a = m_1 \cos^2 \theta_{12}^0 + m_2 \sin^2 \theta_{12}^0$$
$$b = \frac{1}{2} \left( m_1 \sin^2 \theta_{12}^0 + m_2 \cos^2 \theta_{12}^0 + m_3 \right)$$
$$c = \frac{1}{2\sqrt{2}} \sin 2\theta_{12}^0 (m_2 - m_1)$$
$$d = \frac{1}{2} \left( m_1 \sin^2 \theta_{12}^0 + m_2 \cos^2 \theta_{12}^0 - m_3 \right)$$

Equivalently,

$$\tan 2\theta_{12}^0 = \frac{2\sqrt{2}c}{b + d - a}$$

For non-degenerate realistic neutrino masses $a, b, c$ and $d$ are non-zero.

Our objective is to obtain the structure of the matrix shown in Eq. (4) at one-loop level. For that we assign specific $A_4 \times Z_2$ charges to the scalars and fermions in our model. This model has three right-handed neutrino fields. As we will see in course of the discussion that in order to obtain the structure in Eq. (4), two of these right-handed neutrino states will require to be degenerate in masses. Once the structure in Eq. (4) is produced, we will exploit small relaxation of this degenerate feature in right-handed neutrino sector to yield the realistic neutrino mixings, namely non-zero $\theta_{13}$.

In this model, apart from the three $SU(2)_L$ lepton doublets we have three right-handed neutrinos, $N_{\alpha R}$, ($\alpha = 1, 2, 3$) invariant under the standard model (SM) gauge group. Under $A_4$ these three right-handed neutrinos transform as a triplet and so does the three $SU(2)_L$ lepton doublets. In the scalar sector we have two $A_4$ symmetric triplet fields $\Phi$ and $\eta$ each of which comprises of three $SU(2)_L$ doublet fields $\Phi_i \equiv (\phi_i^+, \phi_i^0)^T$ and $\eta_j \equiv (\eta_j^+, \eta_j^0)^T$, ($i, j = 1, 2, 3$). In addition to $A_4$ we have an unbroken $Z_2$ under which all the fields are even except the scalar field $\eta$ and the right-handed neutrinos. Thus the scalars $\eta_j$ do not acquire vacuum expectation value (vev) after spontaneous symmetry breaking (SSB), whereas the fields $\phi_i$ do. All the fields along with their quantum numbers are listed in Table 2. Here we restrict ourselves to the neutrino sector only.

We work in a basis in which the charged lepton mass matrix is diagonal and the mixing is entirely from the neutrino sector.

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This model differs from [19] in terms of the particle content. Unlike [19], here we consider all the popular mixings viz. TBM, BM and GR and also generate non-zero $\theta_{13}$, deviations of $\theta_{23}$ from maximality and small corrections to $\theta_{12}$ simultaneously through small mass splitting in the right-handed neutrino sector.
Leptons $SU(2)_L$ $A_4$ $Z_2$

$L_\beta \equiv \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$

|  | 2 | 3 | 1 |

$N_{\alpha R} \equiv \begin{pmatrix} N_{1R} \\ N_{2R} \\ N_{3R} \end{pmatrix}$

|  | 1 | 3 | −1 |

Scalars $SU(2)_L$ $A_4$ $Z_2$

$\Phi \equiv \begin{pmatrix} \phi_1^+ \\ \phi_2^+ \\ \phi_3^+ \\ \phi_1^0 \\ \phi_2^0 \\ \phi_3^0 \end{pmatrix}$

|  | 2 | 3 | 1 |

$\eta \equiv \begin{pmatrix} \eta_1^+ \\ \eta_2^+ \\ \eta_3^+ \\ \eta_1^0 \\ \eta_2^0 \\ \eta_3^0 \end{pmatrix}$

|  | 2 | 3 | −1 |

Table 2: Fields and their quantum numbers. Here we are concerned with the neutrino sector only.

With these fields one can generate neutrino mass at one-loop level as shown in Fig. (1). The relevant part of the scalar potential from the four-point scalar vertex contributing to the neutrino mass matrix is given by:

$$V_{\text{relevant}} \supset \lambda_1 \left[ \{ \eta_1^\dagger \phi_1 + \eta_2^\dagger \phi_2 + \eta_3^\dagger \phi_3 \}^2 + \text{h.c.} \right]$$

$$+ \lambda_2 \left[ \{ (\eta_1^\dagger \phi_1 + \omega \eta_3^\dagger \phi_2 + \omega^2 \eta_3^\dagger \phi_3) (\eta_1^\dagger \phi_1 + \omega^2 \eta_2^\dagger \phi_2 + \omega \eta_3^\dagger \phi_3) \} + \text{h.c.} \right]$$

$$+ \lambda_3 \left[ \{ (\eta_2^\dagger \phi_2)^2 + (\eta_3^\dagger \phi_2)^2 + (\eta_3^\dagger \phi_1)^2 + (\eta_1^\dagger \phi_2)^2 + (\eta_2^\dagger \phi_1)^2 \} + \text{h.c.} \right]$$

$$+ \lambda_4 \left[ \{ (\eta_2^\dagger \phi_3)(\eta_3^\dagger \phi_2) + (\eta_3^\dagger \phi_1)(\eta_1^\dagger \phi_3) + (\eta_1^\dagger \phi_2)(\eta_2^\dagger \phi_1) \} + \text{h.c.} \right],$$

where all the quartic couplings $\lambda_i$ ($i = 1, 2, 3, 4$) are considered to be real.

As discussed earlier, after SSB, $\phi_i^0$ will get vevs whereas the $\eta_i^0$ will not owing to the $Z_2$ assignments. Let $\langle \Phi_i \rangle = v_i$ where $i = 1, 2, 3$. In [21, 22], it has been shown that for $A_4$ symmetric three-Higgs-doublets

Note at the four-point scalar vertex, both the $\phi$ fields are annihilated and both the $\eta$ fields are created. So terms of the scalar potential of $(\eta^\dagger \phi)(\eta^\dagger \phi)$ kind will contribute to the neutrino mass matrix and are therefore relevant.
the four vev configurations for which the scalar potential acquires the global minima are:

\[
\langle \Phi \rangle_{\text{case 1}} = v \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \langle \Phi \rangle_{\text{case 2}} = v \begin{pmatrix} 0 & 1 \\ 0 & e^{i\alpha} \\ 0 & 0 \end{pmatrix}, \quad \langle \Phi \rangle_{\text{case 3}} = v \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \langle \Phi \rangle_{\text{case 4}} = v \begin{pmatrix} 0 & 1 \\ 0 & \omega \\ 0 & \omega^2 \end{pmatrix}.
\]

Let \( \eta_{Rj} \) and \( \eta_{Ij} \) be the real and imaginary parts of \( \eta_j^0 \) respectively. Splitting among the masses of \( \eta_{Rj} \) and \( \eta_{Ij} \) is proportional to \( \lambda v_j \) and is expected to be small. Here \( \lambda \) stands for the quartic couplings \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) in Eq. (7). Also the mass splittings between \( \eta_j \) (\( j = 1, 2, 3 \)) constituting the A4 triplet are neglected and \( m_0 \) is their common mass. Our model has three right-handed neutrinos, \( N_{\alpha R} \) (\( \alpha = 1, 2, 3 \)), \( M_\alpha \) being their masses. In the limit \( M_\alpha^2 >> m_0^2 \), the diagram in Fig. (1) leads to neutrino mass of the kind [20]:

\[
(M_{\text{flavour}}_{\nu L})_{ij} = \lambda \frac{v_m v_n}{8\pi^2} \sum_{\alpha,k,l \neq (i,j)} \frac{h_{iak} h_{jal}}{M_\alpha} \left[ \ln z_\alpha - 1 \right].
\]

where \( z_\alpha \equiv \frac{M_\alpha^2}{m_0^2} \). The vevs of \( \phi_0^m \) and \( \phi_0^n \) are given by \( v_m \) and \( v_n \) respectively. The Yukawa couplings at the two vertices where the left-handed and right-handed neutrinos couple to the inert \( SU(2)_L \) doublet fields \( \eta \) are given by \( h_{iak} \) and \( h_{jal} \). No two of the three indices appearing in each of \( h_{iak} \) and \( h_{jal} \) individually can be same owing to A4 symmetry. Thus \( h_{iak} \) and \( h_{jal} \) are determined by the A4 invariance which in its turn governs the structure of the neutrino mass matrix. Since the logarithm is a slowly varying function and the heavy right handed neutrino masses \( M_\alpha \) (\( \alpha = 1, 2, 3 \)) are expected to be close to each other, the RHS of Eq. (9) can be approximated as proportional to \( \frac{1}{M_\alpha} \). Leaving the vevs \( v_m, v_n \) and the quartic couplings \( \lambda \), let us denote the contribution to left-handed neutrino mass matrix \( (M_{\text{flavour}}_{\nu L})_{ij} \) from everything else in Eq. (9) by loop contributing factors \( r_\alpha \propto \frac{1}{M_\alpha} \).

For simplicity let us consider the right-handed neutrino mass matrix to be already diagonal i.e., \( M_{N_R} \equiv diag(M_1, M_2, M_3) \). In terms of the right-handed neutrino loop contributing factors \( r_\alpha \) we have the contribution coming from right-handed neutrino sector as \( diag(r_1, r_2, r_3) \). Using Eqs. (9) and (7), the left-handed neutrino mass matrix that arises from Fig. (1) is given by [19]:

\[\text{Figure 1: Neutrino mass generation at one-loop level.}\]
where,

\[ M_{\nu L}^{\text{Flavour}} = \begin{pmatrix} \chi_1 & \chi_4 & \chi_5 \\ \chi_4 & \chi_2 & \chi_6 \\ \chi_5 & \chi_6 & \chi_3 \end{pmatrix} \]  

where,

\[ \chi_1 \equiv (\lambda_1 + \lambda_2)(r_3v_2^2 + r_2v_3^2) + \lambda_3[r_2(v_1^2 + v_2^2) + r_3(v_1^2 + v_3^2)] \]

\[ \chi_2 \equiv (\lambda_1 + \lambda_2)(r_1v_3^2 + r_3v_1^2) + \lambda_3[r_1(v_2^2 + v_3^2) + r_3(v_2^2 + v_3^2)] \]

\[ \chi_3 \equiv (\lambda_1 + \lambda_2)(r_1v_2^2 + r_2v_1^2) + \lambda_3[r_2(v_2^2 + v_3^2) + r_1(v_2^2 + v_3^2)] \]

\[ \chi_4 \equiv r_3[\lambda_4 + 2\lambda_1 - \lambda_2]v_1v_2 \]

\[ \chi_5 \equiv r_2[\lambda_4 + 2\lambda_1 - \lambda_2]v_1v_3 \]

\[ \chi_6 \equiv r_1[\lambda_4 + 2\lambda_1 - \lambda_2]v_2v_3. \]  

(11)

In order to obtain the neutrino mass matrix of the form of Eq. (4) from Eq. (10), one will simultaneously require \( \chi_1 \neq \chi_2 = \chi_3 \) and \( \chi_4 = \chi_5 \). Let us now try each of the vev configurations in Eq. (8) and find out the one suitable to obtain this feature along with the constraints put on to \( r_1, r_2 \) and \( r_3 \).

1. Choice A: For \( (v_1, v_2, v_3) = v(1,0,0) \), irrespective of the choices for \( r_1, r_2 \) and \( r_3 \), the off-diagonal entries in Eq. (10) will vanish and one cannot obtain mixing in the neutrino sector.

2. Choice B: For \( (v_1, v_2, v_3) = v(1,e^{i\alpha},0) \), two of the three off-diagonal entries in Eq. (10) will vanish for any \( r_1, r_2 \) and \( r_3 \), and one cannot obtain the structure in Eq. (4).

3. Choice C: For \( (v_1, v_2, v_3) = v(1,\omega,\omega^2) \), one cannot achieve \( \chi_2 = \chi_3 \) as required to obtain the structure of the mass matrix in Eq. (11) from Eq. (10), whatever may be the choices for \( r_1, r_2, r_3 \).

4. Choice D: For \( (v_1, v_2, v_3) = v(1,1,1) \), note first that \( r_1 = r_2 = r_3 \) implies all the diagonal terms to be equal to each other and the off-diagonal entries are equal among themselves. This leads to two left-handed degenerate states and only TBM is admissible. We will not consider that choice. However the form in Eq. (11) starting from Eq. (10) is achieved for \( r_1 \neq r_2 = r_3 = r \) when \( (v_1, v_2, v_3) = v(1,1,1) \), which we refer to as choice D from now onwards. This choice allows all three mixings viz. TBM, BM, GR and all three left-handed neutrinos to be non-degenerate. Hence we will consider this case for further analysis. Such choice of \( r_2 = r_3 \) is achieved when the right-handed neutrinos \( N_{2R} \) and \( N_{3R} \) are degenerate in masses.

Putting choice D i.e., \( v_1 = v_2 = v_3 = v \) and \( r_1 \neq r_2 = r_3 = r \) in Eq. (10) one gets the following form of the left-handed neutrino mass matrix in the flavour basis:

\[ M_{\nu L}^{\text{Flavour}} = \begin{pmatrix} \lambda_{123}(2rv_2^2) & \lambda_{124}rv^2 & \lambda_{124}rv^2 \\ \lambda_{124}rv^2 & \lambda_{123}(r + r_1)v^2 & \lambda_{124}r^2v^2 \\ \lambda_{124}rv^2 & \lambda_{123}(r + r_1)v^2 & \lambda_{123}(r + r_1)v^2 \end{pmatrix} \]  

(12)

where, \( \lambda_{123} = \lambda_1 + \lambda_2 + 2\lambda_3 \) and \( \lambda_{124} = \lambda_4 + 2\lambda_1 - \lambda_2 \). Thus the neutrino mass matrix generated at one-loop level as shown in Fig. (11) can produce the form of \( M_{\nu L}^{\text{Flavour}} \) as in Eq. (11) that corresponds to \( \theta_{13} = 0, \theta_{23} = \pi/4 \) and \( \theta_{12}^{0} \) of the popular mixing alternatives, with the vevs and right-handed neutrino
masses as specified in choice D. This follows from the identifications:

\[
a \equiv \lambda_{123}(2r_2v^2) = (\lambda_1 + \lambda_2 + 2\lambda_3)(2rv^2)
\]

\[
b \equiv \lambda_{123}(r + r_1)v^2 = (\lambda_1 + \lambda_2 + 2\lambda_3)(r + r_1)v^2
\]

\[
c \equiv \lambda_{124}r_2v^2 = (\lambda_4 + 2\lambda_1 - \lambda_2)r_2v^2
\]

\[
d \equiv \lambda_{124}r_1v^2 = (\lambda_4 + 2\lambda_1 - \lambda_2)r_1v^2
\]

Having achieved this, next we concentrate on generation of realistic neutrino mixing i.e., non-zero \(\theta_{13}\), deviations of \(\theta_{23}\) from maximality and small corrections in the solar mixing \(\theta_{12}\). For that one has to deviate from the \(r_\alpha (\alpha = 1, 2, 3)\) of choice D. Let us now split the degeneracy in the right handed neutrino sector by a small amount \(\epsilon\) i.e., consider \(r_3 = r_2 + \epsilon\) and \(r_1 \neq r_2 \neq r_3 \neq r_1\), keeping the vevs still to be \(v_1 = v_2 = v_3 = v\). With such a choice one is expected to get a dominant contribution of the form of \(M_{\nu}^{\text{flavour}}\) as was achieved in Eq. (12), say \(M^0\), together with small shift from it, \(M'\), proportional to \(\epsilon\). Thus,

\[
M_{\nu}^{\text{flavour}} = M^0 + M'
\]

where,

\[
M^0 = \begin{pmatrix}
\lambda_{123}(2rv^2) & \lambda_{124}r_2v^2 & \lambda_{124}r_2v^2 \\
\lambda_{124}r_2v^2 & \lambda_{123}(r_1 + r_2)v^2 & \lambda_{123}(r_1 + r_2)v^2 \\
\lambda_{124}r_2v^2 & \lambda_{123}(r_1 + r_2)v^2 & \lambda_{123}(r_1 + r_2)v^2
\end{pmatrix}
\quad \text{and} \quad
M' = \epsilon \begin{pmatrix}
x & y & 0 \\
y & x & 0 \\
0 & 0 & 0
\end{pmatrix}
\]  

where \(x = \lambda_{123}v^2\) and \(y = \lambda_{124}v^2\). Here \(M^0\) is the form of the \(M_{\nu}^{\text{flavour}}\) required for \(\theta_{13} = 0, \theta_{23} = \pi/4\) and \(\theta_{12}^0\) of the popular mixings. Thus in analogy to Eq. (13), one can identify the following:

\[
a' \equiv \lambda_{123}(2rv^2) = (\lambda_1 + \lambda_2 + 2\lambda_3)(2rv^2)
\]

\[
b' \equiv \lambda_{123}(r_1 + r_2)v^2 = (\lambda_1 + \lambda_2 + 2\lambda_3)(r_1 + r_2)v^2
\]

\[
c' \equiv \lambda_{124}r_2v^2 = (\lambda_4 + 2\lambda_1 - \lambda_2)r_2v^2
\]

\[
d' \equiv \lambda_{124}r_1v^2 = (\lambda_4 + 2\lambda_1 - \lambda_2)r_1v^2
\]

It is straightforward to incorporate the corrections offered by \(M'\) to \(M^0\) using the non-degenerate perturbation theory. Columns of \(U^0\) in Eq. (2) is the unperturbed flavour basis. From Eq. (16) one can define:

\[
\gamma \equiv (b' - 3d' - a') \quad \text{and} \quad \rho \equiv \sqrt{a'^2 + b'^2 + 8c'^2 + d'^2 - 2a'b' - 2a'd' + 2b'd'}
\]  

The first order corrected third ket is then given by:

\[
|\psi_3\rangle = \begin{pmatrix}
\frac{x}{\sqrt{\gamma^2 - \rho^2}} \left[ \gamma(x \sin 2\theta_{12}^0 - \sqrt{2}y \cos 2\theta_{12}^0) + \rho \sqrt{2}y \right] \\
-\frac{1}{\sqrt{2}} [1 + \xi \epsilon] \\
\frac{1}{\sqrt{2}} [1 - \xi \epsilon]
\end{pmatrix},
\]

where,

\[
\xi \equiv \frac{[\gamma x + \rho (x \cos 2\theta_{12}^0 + \sqrt{2}y \sin 2\theta_{12}^0)]}{(\gamma^2 - \rho^2)}.
\]

Thus one can write:

\[
\sin \theta_{13} = \frac{\epsilon}{\gamma^2 - \rho^2} \left[ \gamma(x \sin 2\theta_{12}^0 - \sqrt{2}y \cos 2\theta_{12}^0) + \rho \sqrt{2}y \right].
\]

\textsuperscript{9}To distinguish from \(r_2 = r_3\) case, let us use a primed notation.

\textsuperscript{10}Here we restrict ourselves to no CP-violation.
Using Eqs. (16), (17) and (20), one can easily read off non-zero \( \theta_{13} \) in terms of the model parameters, namely, \( \epsilon \), the quartic couplings and the vevs. Throughout our discussion we have assumed \( r_\alpha (\alpha = 1, 2, 3) \) are real and restricted ourselves to a CP-conserving scenario. In principle, the right-handed neutrino masses can have Majorana phases causing these \( r_\alpha \) to be complex. Then one can have a complex \( \epsilon \), from which one can generate CP-violation in the lepton sector.

From Eq. (18) the deviation of atmospheric mixing from maximality is given by:

\[
\tan \varphi \equiv \tan(\theta_{23} - \pi/4) = \xi \epsilon. \tag{21}
\]

Similarly, one can obtain small corrections to \( \theta_{12} \) from the corrections of the first and second kets. The solar mixing angle after receiving first order corrections is given by:

\[
\tan \theta_{12} = \frac{\sin \theta_{12}^0 + \epsilon \beta \cos \theta_{12}^0}{\cos \theta_{12}^0 - \epsilon \beta \sin \theta_{12}^0} \tag{22}
\]

where,

\[
\beta \equiv \frac{\left[ \frac{\sqrt{2}}{2} \cos 2\theta_{12}^0 + \frac{\sqrt{4}}{\rho} \sin 2\theta_{12}^0 \right]}{\rho} \tag{23}
\]

The corrections to the solar mixing and deviations of atmospheric mixing from \( \pi/4 \) in Eq. (22) and (21) respectively can be expressed in terms of the model parameters using Eqs. (16), (17), (19) and (23).

Summing up, a scotogenic \( A_4 \times Z_2 \) symmetric model of radiatively obtaining realistic neutrino mixing is proposed. Among others, the model comprises of three gauge singlet right-handed neutrino fields \( N_{\alpha R}, (\alpha = 1, 2, 3) \). If \( N_{2R} \) and \( N_{3R} \) are degenerate in masses, one can obtain the common structure of the left-handed neutrino mass matrix required by \( \theta_{13} = 0, \theta_{23} = \pi/4 \) and \( \theta_{12}^0 \) of the particular choices leading to popular lepton mixing scenarios viz. TBM, BM, GR at one-loop level. A slight shift from this degeneracy of right-handed neutrino masses could generate realistic mixing viz. non-zero \( \theta_{13} \), deviations of \( \theta_{23} \) from \( \pi/4 \) and also tweak \( \theta_{12} \) by a small amount. The model has three inert \( SU(2)_L \) doublet scalars \( \eta \), odd under the unbroken \( Z_2 \), the lightest of which can be a dark matter candidate.

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## A Appendix: The discrete group \( A_4 \)

\( A_4 \) being the group of even permutations of four objects has 12 elements. The group \( A_4 \) has two generators \( S \) and \( T \). These generators satisfy \( S^2 = T^3 = (ST)^3 = I \). The inequivalent irreducible representations for \( A_4 \) are four in number out of which three are 1-dimensional viz. \( 1, 1' \) and \( 1'' \) and one is 3-dimensional. The 1-dimensional representations transform as 1, \( \omega \), and \( \omega^2 \) under \( T \) but are invariant under \( S \). Thus, \( 1' \times 1'' = 1 \). The generators are represented by,

\[
S = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}. \tag{A.1}
\]

\[1^1 \text{Here } \omega \text{ is a cube root of 1.}\]
Below is the combination rule for two $A_4$ triplets:

$$3 \otimes 3 = 1 \oplus 1' \oplus 1'' \oplus 3 \oplus 3.$$  \hfill (A.2)

Let us have two $A_4$ triplet fields, $3_a \equiv a_i$ and $3_b \equiv b_i$, where $i = 1, 2, 3$, and combine them according to Eq. (A.2). The triplets that we get can be written as $3_c \equiv c_i$ and $3_d \equiv d_i$ where,

$$c_i = \left( \frac{a_2 b_3 + a_3 b_2}{2}, \frac{a_3 b_1 + a_1 b_3}{2}, \frac{a_1 b_2 + a_2 b_1}{2} \right), \quad \text{or,} \quad c_i \equiv \alpha_{ijk} a_j b_k,$$

$$d_i = \left( \frac{a_2 b_3 - a_3 b_2}{2}, \frac{a_3 b_1 - a_1 b_3}{2}, \frac{a_1 b_2 - a_2 b_1}{2} \right), \quad \text{or,} \quad d_i \equiv \beta_{ijk} a_j b_k, \quad (i, j, k, \text{are cyclic}) \hfill (A.3)$$

The $1$, $1'$ and $1''$ in this case are:

$$1 = a_1 b_1 + a_2 b_2 + a_3 b_3 \equiv \rho_{1ij} a_i b_j,$$

$$1' = a_1 b_1 + \omega^2 a_2 b_2 + a_3 b_3 \equiv \rho_{3ij} a_i b_j,$$

$$1'' = a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3 \equiv \rho_{2ij} a_i b_j. \hfill (A.4)$$

The group was studied in context of neutrino mass and mixings in the pioneering works [13][14].

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