The spatial sign covariance operator: Asymptotic results and applications

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Abstract

Due to the increasing recording capability, functional data analysis has become an important research topic. For functional data the study of outlier detection and/or the development of robust statistical procedures has started recently. One robust alternative to the sample covariance operator is the sample spatial sign covariance operator. In this paper, we study the asymptotic behaviour of the sample spatial sign covariance operator when location is unknown. Among other possible applications of the obtained results, we derive the asymptotic distribution of the principal directions obtained from the sample spatial sign covariance operator and we develop test to detect differences between the scatter operators of two populations. In particular, the test performance is illustrated through a Monte Carlo study for small sample sizes.

1 Introduction

Functional data analysis is a field which deals with a sample of curves registered on a continuous period of time. A more general and inclusive framework that can accommodate the situation in which the observations are images or surfaces is to consider realizations of a random element \( X \) on a Hilbert space \( \mathcal{H} \) with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). The area has attracted much interest in the statistics community and has increase its development, since technological advances in data collection and storage require procedures specifically designed for dealing with such data. It has been extensively discussed that simplifying the functional model by discretizing the observations as sequences of numbers can often fail to capture some of its important characteristics, such as the smoothness and continuity of the underlying functions. Statistical methods to analyse such functional data may be found, for instance, in Ramsay and Silverman (2005), Ferraty and Vieu (2006), Ferraty and Romain (2010), Horváth and Kokoszka (2012) and Hsing and Eubank (2015).

For a summary of recent advances in functional statistics see Cuevas (2014) and Goia and Vieu (2016).

In this setting, the analysis of the covariance operator arises in many applied contexts. In particular, functional principal component analysis is a common tool to explore the characteristics of the data within a reduced dimensional space. As it is well known, the principal directions
may be obtained as the eigenfunctions of the covariance operator. By exploring this lower dimensional principal components space, functional principal components analysis allows to detect atypical observations or outliers in the data set, when combined with a robust estimation procedure. Among other procedures to robustly estimate the principal directions, we can mention the spherical principal components of Locantore et al. (1999) and Gervini (2008) that correspond to the eigenfunctions of the spatial sign operator, the projection–pursuit given in Bali et al. (2011), the robust approach given on Sawant et al. (2012), the M-type smoothing spline estimators proposed in Lee et al. (2013) and the S–estimators of Boente and Salibián–Barrera (2015).

One key point when deriving detection rules is that the robust functional principal direction estimators are indeed estimating the target directions. In this sense, the projection–pursuit given in Bali et al. (2011) and the spherical principal components are Fisher consistent for elliptically distributed random elements. The result for the spherical principal components, derived in Boente et al. (2014), extends a previous one obtained in Gervini (2008) for random elements with a finite Karhunen-Loève expansion. Moreover, Boente et al. (2014) proved that the linear space spanned by the q eigenfunctions associated to the q larger eigenvalues of the spatial sign covariance operator, provides the best q dimensional approximation to the centered process, in the sense of having stochastically smallest residuals squared norms among all linear spaces of dimension q. This result does not required second order moment for the process. Up to our knowledge, the asymptotic distribution of the robust principal direction estimators mentioned above is unknown. One of the goals of this paper is to derive the asymptotic distribution of the spherical principal component estimators through that of the sample spatial sign covariance operator.

A more recent statistical problem is that of testing for equality or proportionality between the covariance operators of two populations. For instance, Ferraty et al. (2007) considered tests for comparing groups of curves based on comparison of their covariances. By the Karhunen–Loève expansion, this is equivalent to testing if all the samples have the same set of functional principal components sharing also their size. When considering only two populations, Benko et al. (2009), Panaretos et al. (2010) and Fremdt et al. (2013) used this characterization to develop test statistics. In particular, Benko et al. (2009) proposed two–sample bootstrap tests for specific aspects of the spectrum of functional data, such as the equality of a subset of eigenfunctions. On the other hand, Panaretos et al. (2010) and Fremdt et al. (2013) considered an approach based on the projection of the data over a suitable chosen finite–dimensional space, such as that defined by the functional principal components of each population. The results in Fremdt et al. (2013) generalized those provided in Panaretos et al. (2010) which assume that the processes have a Gaussian distribution. More recently, Pigoli et al. (2014) developed a two–sample test for comparing covariance operators using different distances between covariance operators. Their procedure is based on a permutation test and assumes that the two samples have the same mean, otherwise, an approximate permutation test is considered after the processes are centered using their sample means. Some authors have also consider robust proposals for this problem. Kraus and Panaretos (2012) introduced a class of dispersion operators and proposed a procedure for testing for equality of dispersion operators among two populations. Recently, Boente et al. (2017), extended the classical two populations problem, presenting a test for equality of covariance operators among k populations in which the asymptotic distribution of
the sample covariance operator plays a crucial role in deriving the asymptotic distribution of the proposed statistic. It is well known that the presence of outliers in the sample might drive to invalid conclusions. This motivate the development of robust procedures to deal with these kind of problems. In this paper, we also present as application of our results a test for equality of the spatial sign covariance operators between two populations. The statistic mimics the one presented for the classical setting and, as in the classical setting, its asymptotic distributions depends on that of the empirical spatial sign covariance operator for each population. It is worth noticing that, for functional elliptical distributions, equality of spatial sign covariance operators guarantees that the considered populations have the same principal components.

Unlike the classical case, where the estimation of the mean plays no role in the asymptotic distribution of the covariance operator estimator, the imputation of an estimated location when defining the spatial sign covariance estimator requires some special considerations. One of the goals of this paper is to present a detailed proof of the asymptotic distribution of the sample spatial sign covariance estimator, which extends to the functional setting the results given by Dürr et al. (2014) in the finite–dimensional case.

The paper is organized as follows. In Section 2, we introduce the notation to be used in the paper as well as the spatial sign covariance operator with unknown location, while Section 3 deals with its consistency and asymptotic normality. Section 4 considers the application of the obtained results to two situations: the asymptotic behaviour of the spherical principal component estimators and the proposal of a test to detect differences between the spatial sign covariance operators of two populations, whose performance is also numerically studied for small samples. Proofs are relegated to the Appendix.

2 The spatial covariance operator

Let \( \mathcal{H} \) be a separable Hilbert space, such as \( L^2(\mathcal{I}) \) for some bounded interval \( \mathcal{I} \), with inner product \( \langle \cdot , \cdot \rangle \) and norm \( \| u \| = \langle u, u \rangle^{1/2} \). The functional sign of \( u \in \mathcal{H} \), is defined as \( s(u) = u/\| u \| \), for \( u \neq 0 \), and \( s(0) = 0 \).

Let \( X \) be a random element taking values in \( \mathcal{H} \). For a given \( t \in \mathcal{H} \), the spatial or sign covariance operator of \( X \) centered at \( t \) is defined by

\[
\Gamma^s(t) = \mathbb{E}[s(X - t) \otimes s(X - t)],
\]

where \( \otimes \) denotes the tensor product on \( \mathcal{H} \), e.g., for \( u, v \in \mathcal{H} \), the operator \( u \otimes v : \mathcal{H} \to \mathcal{H} \) is defined as \( (u \otimes v)w = \langle v, w \rangle u \). Notice that \( u \otimes v \) is a compact operator that belongs to \( \mathcal{F} \), the Hilbert space of Hilbert–Schmidt operators over \( \mathcal{H} \). Recall that for \( \Upsilon \in \mathcal{F} \), \( \Upsilon^* \) denotes the adjoint of the operator \( \Upsilon \), while for \( \Upsilon_1, \Upsilon_2 \in \mathcal{F} \), the inner product in \( \mathcal{F} \) is defined as \( \langle \Upsilon_1, \Upsilon_2 \rangle_\mathcal{F} = \text{trace}(\Upsilon_1^* \Upsilon_2) = \sum_{\ell=1}^{\infty} \langle \Upsilon_1 u_\ell, \Upsilon_2 u_\ell \rangle \), and so the norm equals \( \| \Upsilon \|_\mathcal{F} = \langle \Upsilon^*, \Upsilon \rangle_\mathcal{F}^{1/2} = \{ \sum_{\ell=1}^{\infty} \| \Upsilon u_\ell \|^2 \}^{1/2}, \) with \( \{ u_\ell : \ell \geq 1 \} \) any orthonormal basis of \( \mathcal{H} \). These definitions are independent of the basis choice.

Given independent random elements \( X_1, \ldots, X_n \), distributed as \( X \), for each \( t \in \mathcal{H} \) define the
sample version of $\Gamma^S(t)$ as

$$\hat{\Gamma}_n^S(t) = \frac{1}{n} \sum_{i=1}^{n} s(X_i - t) \otimes s(X_i - t).$$

The law of large numbers in $\mathcal{F}$, entails that, for any $t \in \mathcal{H}$, $\hat{\Gamma}_n^S(t)$ converges almost surely to $\Gamma^S(t)$. Moreover, the asymptotic distribution can be obtained from the central limit theorem in $\mathcal{F}$, see, for instance, Dauxois et al. (1982).

Typically, the spatial operator is centered using as location the functional median $\mu$ of the process $X$, that is, the object of interest is the spatial operator $\Gamma^S(\mu)$. However, in most situations $\mu$ is unknown. Hence, when estimating the spatial sign operator, an estimator of $\mu$ must be considered. More precisely, let $\hat{\mu}_n$ be a preliminary consistent estimator of $\mu$, then $\hat{\Gamma}_n^S(\hat{\mu}_n)$ provides an estimator of $\Gamma^S(\mu)$. The asymptotic properties of $\hat{\Gamma}_n^S(\hat{\mu}_n)$ are presented in Section 3.

### 2.1 Some general comments

As mentioned in the Introduction, the sample spatial operator $\hat{\Gamma}_n^S(\hat{\mu}_n)$ has been used as an alternative to the sample covariance operator when considering robust estimation procedures. In particular, it has been considered when one suspects that the underlying distribution may not have finite moments. Elliptical random elements have been introduced in Bali and Boente (2009) and further studied in Boente et al. (2014). For completeness, we recall their definition. Given a random element $X$ in a separable Hilbert space $\mathcal{H}$, we say that $X$ has an elliptical distribution with parameters $\mu \in \mathcal{H}$ and $\Gamma : \mathcal{H} \to \mathcal{H}$, where $\Gamma$ is a self–adjoint, positive semi–definite and compact operator, if and only if for any linear and bounded operator $A : \mathcal{H} \to \mathbb{R}^d$ we have that the vector $AX$ has a $d$–variate elliptical distribution with location parameter $A\mu$, shape matrix $A \Gamma A^*$ and characteristic generator $\varphi$, that is, $AX \sim \mathcal{E}_d(A\mu, A\Gamma A^*, \varphi)$ where $A^* : \mathbb{R}^d \to \mathcal{H}$ denotes the adjoint operator of $A$. We write $X \sim \mathcal{E}(\mu, \Gamma, \varphi)$ and $\Gamma$ is called the scatter operator of $X$. Hence, elliptical families provide a more general setting than considering Gaussian random elements and the sign operator gives a useful tool to obtain Fisher–consistent estimators of the principal directions, that is, estimators consistent to the eigenfunctions of the scatter operator of the elliptical process, even when second moments do not exist (see Boente et al., 2014).

For elliptical random elements, two situations may arise, either the scatter operator $\Gamma$ has a finite rank $q$ or it does not have a finite rank. In the first case, the process $X$ has a finite Karhunen–Loève expansion $X = \mu + \sum_{j=1}^{q} \lambda_j^{1/2} \xi_j \phi_j$, where $\phi_j$ are the eigenfunctions of $\Gamma$ related to the eigenvalues $\lambda_j$ ordered in decreasing order and $\xi = (\xi_1, \ldots, \xi_q)^T \sim \mathcal{N}_q(0, \mathbf{I}_q, \varphi)$, that is, $\xi$ has an spherical distribution. In this setting, the asymptotic behaviour of $\hat{\Gamma}_n^S(\hat{\mu}_n)$ may be derived from the results given in Dürre et al. (2014), since the distribution of $\text{diag}(\lambda_1^{1/2}, \ldots, \lambda_q^{1/2})\xi$ is symmetric around 0. On the other hand, if $\Gamma$ has not a finite rank, Proposition 2.1 in Boente et al. (2014) states that the process is a scale mixture of Gaussian distributions, more precisely there exists a zero mean Gaussian random element $Y$ and a random variable $V > 0$ independent of $Y$ such that $X = \mu + V Y$. Without loss of generality, throughout the paper, we will assume...
that $\Gamma$ is the covariance operator of $Y$. The results given in Section 3 include this case but they also provide a consistency and asymptotic normality results in a framework more general than elliptical families.

3 Asymptotic results

The following results establish the consistency and the asymptotic normality of the spatial sign covariance operator with unknown location. The proofs are relegated to the Appendix. From now on, the notation $u_n \overset{a.s.}{\to} u$ in $\mathcal{H}$ means that $\|u_n - u\| \overset{a.s.}{\to} 0$, while for random operators $\Upsilon_n \in \mathcal{F}$, the convergence $\Upsilon_n \overset{a.s.}{\to} \Upsilon$ in $\mathcal{F}$ stands for $\|\Upsilon_n - \Upsilon\|_{\mathcal{F}} \overset{a.s.}{\to} 0$.

**Theorem 3.1.** Let $\mu \in \mathcal{H}$ be the location parameter of the process. Assume that $\hat{\mu}_n$ is strongly consistent estimator of $\mu$ and that $E\left[\|X - \mu\|^{-1}\right] < \infty$. Then, we have that $\hat{\Gamma}^s(\hat{\mu}_n) \overset{a.s.}{\to} \Gamma^s(\mu)$.

**Remark 3.1.** In a robust context, several estimators of the location parameter $\mu$ have been considered. Among others we can mention the trimmed means proposed by Fraiman and Muñiz (2001), the depth–based estimators of Cuevas et al. (2007) and López–Pintado and Romo (2007), or the functional median defined in Gervini (2008). In particular, as mentioned above, the spatial median is the usual choice to center the data when location is unknown and the spatial covariance operator is considered. The spatial median is defined as

$$\mu = \arg\min_{u \in \mathcal{H}} E[\|X - u\| - \|X\|].$$

and different methods have been proposed to provide estimators, in the functional case. Gervini (2008) shows that the sample spatial median, denoted $\hat{\mu}_n$ and defined as the solution of the empirical version of (2), can be found solving a convex $n$–dimensional minimization problem. Furthermore, $\hat{\mu}_n$ is strongly consistent with respect to the weak topology in $\mathcal{H}$, that is for any $u \in \mathcal{H}$, $\langle \hat{\mu}_n, u \rangle \overset{a.s.}{\to} \langle \mu, u \rangle$. On the other hand, Cardot et al. (2013) propose to estimate the spatial median through an algorithm that can be seen as a stochastic gradient algorithm. Theorem 3.1 in Cardot et al. (2013) shows that this estimator converges to the median almost surely, under mild conditions. This result guarantees the existence of strong consistent estimators of the median in the functional case and hence, that of the estimators of the spatial sign covariance operator.

In order to study the asymptotic distribution of $\hat{\Gamma}^s(\hat{\mu}_n)$, let $\mathcal{B}$ denotes the Banach space of linear and continuous operators from $\mathcal{H}$ to $\mathcal{F}$, that is, $\mathcal{B} = \{T : \mathcal{H} \to \mathcal{F} : T$ linear and continuous$\}$ and denote as $\|T\|_{\mathcal{B}} = \sup_{\|u\| \leq 1} \|T(u)\|_{\mathcal{F}}$. The following assumptions will be required.

**A.1** $\sqrt{n}(\hat{\mu}_n - \mu) = O_p(1)$

**A.2** $E[\|X - \mu\|^{-3/2}] < \infty$.

**Theorem 3.2.** Under assumptions A.1 and A.2, we have that

$$\sqrt{n}(\hat{\Gamma}^s(\hat{\mu}_n) - \hat{\Gamma}^s(\mu)) = \sqrt{n}G_X(\hat{\mu}_n - \mu) + o_p(1),$$
where \( G_X = 2F_X - 2S_X \in \mathcal{B} \), with \( F_X \) and \( S_X \) defined as follows

\[
F_X(u) = \mathbb{E} \left[ \frac{\langle (X - \mu), u \rangle}{\|X - \mu\|^2} \langle X - \mu \rangle \otimes \langle X - \mu \rangle \right] \\
S_X(u) = \frac{1}{2} \left\{ u \otimes \mathbb{E} \left[ \frac{X - \mu}{\|X - \mu\|^2} \right] + \mathbb{E} \left[ \frac{X - \mu}{\|X - \mu\|^2} \right] \otimes u \right\}.
\]

(3)

(4)

**Remark 3.2.** Assumption A.1 is satisfied for the spatial median \( \hat{\mu}_n \), taking \( \widehat{\mu}_n \) as the averaged of the stochastic gradient algorithm estimator, presented in Cardot et al. (2013), where the asymptotic distribution of this estimator is obtained (Theorem 3.4). Regarding the assumptions \( \mathbb{E} \left[ \|X - \mu\|^{-1} \right] < \infty \) and A.2, as noted in Dürre et al. (2014) in the multivariate case, they require that the process has a finite Karhunen–Loève expansion, \( X = \mu + \sum_{k=1}^{q} y_k \phi_k \) where \( \phi_k \in \mathcal{H} \) are orthonormal and \( y_k \) are random variables, then \( E_\nu = \mathbb{E} \left[ \|X - \mu\|^{-\nu} \right] = \mathbb{E} \left[ \|y\|^{-\nu} \right] \), with \( y = (y_1, \ldots, y_q)^T \). Hence \( E_\nu < \infty \) for \( \nu = 1, 3/2 \) when \( y \) has a bounded density at 0 while a weaker requirement may be given when \( y \) has an elliptical distribution (see Remark V in Dürre et al., 2014). For properly infinite–dimensional processes, \( E_\nu < \infty \) if there exists an orthonormal basis \( \{\Psi_k\}_{k \geq 1} \) in \( \mathcal{H} \) such that, for some \( q \geq 1 \), the random vector \( y = (\langle X, \Psi_1 \rangle, \ldots, \langle X, \Psi_q \rangle)^T \) is such that \( \mathbb{E} \left[ \|y\|^{-\nu} \right] < \infty \). For elliptical distributed random elements, one may take as basis \( \{\Psi_k\}_{k \geq 1} \) the eigenfunctions of the scatter operator defining the distribution. When the scatter operator of the elliptical distribution has not a finite rank, we have that \( X = \mu + SY \), where \( Y \) is a zero mean Gaussian random element with covariance operator \( \Gamma \) and the random variable \( V > 0 \) is independent of \( Y \). Hence, \( E_\nu = \mathbb{E} V^{-\nu} \mathbb{E} \|Y\|^{-\nu} \) and \( E_\nu < \infty \) if and only if \( \mathbb{E} V^{-\nu} < \infty \). In particular, when \( V \) is such that \( k/V^2 \sim \chi^2_k \), which corresponds to the functional version of a multivariate \( \mathcal{T} \)–distribution with \( k \) degrees of freedom, we have that \( E_\nu < \infty \).

**Remark 3.3.** It is worth noticing that when \( F_X \equiv 0 \) and \( \mathbb{E} \left[ (X - \mu) \|X - \mu\|^2 \right] = 0 \), Theorem 3.2 provides an extension to the functional data setting of the result given in Theorem 2 of Dürre et al. (2014). More precisely, in this case \( \sqrt{n}(\hat{\Gamma}_n(\mu) - \Gamma(\mu)) = o_p(1) \) meaning that the asymptotic behaviour of the spatial sign covariance operator is not affected by the imputation of a location estimator. In particular, if \( X \) has a symmetric distribution around its spatial median, meaning that \( X - \mu = \mu - X \) have the same distribution and \( \hat{\mu}_n \) stands for the estimator defined in Cardot et al. (2013), then \( \sqrt{n}(\hat{\Gamma}_n(\mu) - \Gamma(\mu)) = o_p(1) \), so that the asymptotic distribution of \( \hat{\Gamma}_n(\mu) \) can be obtained from that of \( \Gamma(\mu) \) using the Central Limit Theorem.

In particular, for elliptical families this assertion holds. Furthermore, if \( X \sim \mathcal{E}(\mu, \Gamma, \varphi) \) and \( \Gamma \) has not a finite rank, using that \( X \) can be written as \( X = \mu + Y \), where \( Y \) is a zero mean Gaussian random element \( Y \) with covariance operator \( \Gamma \) and \( V > 0 \) is a random variable independent of \( Y \), we get that \( \Gamma(\mu) = \mathbb{E}[s(Y) \otimes s(Y)] \) the sign operator of the process \( Y \). Furthermore, noticing that

\[
\hat{\Gamma}_n(\mu) = \frac{1}{n} \sum_{i=1}^{n} s(Y_i) \otimes s(Y_i),
\]

we have that \( \sqrt{n}(\hat{\Gamma}_n(\mu) - \Gamma(\mu)) \) converges in \( \mathcal{F} \) to a zero mean Gaussian element with covariance
operator equal to the covariance operator of \( s(Y) \otimes s(Y) \) if the estimator of the location \( \mu \) is the functional median \( \hat{\mu}_n \) defined in Cardot et al. (2013).

For multivariate data, Theorem 2 of Dürr et al. (2014) gives the asymptotic distribution of the spatial sign operator. Corollary 3.2 below extends this result to the functional setting. In the general situation in which one cannot guarantee that \( F_X \equiv 0 \) and \( \mathbb{E} \left[ \left( X - \mu \right) \| X - \mu \|^{-2} \right] = 0 \), a joint asymptotic distribution between the location parameter estimator and \( \hat{\Gamma}_s(\hat{\mu}) \) is needed.

**Corollary 3.1.** Assume that **A.2** holds and that

\[
\sqrt{n} \left( \hat{\mu}_n - \mu, \sqrt{n} \left( \hat{\Gamma}_s(\mu) - \Gamma_s(\mu) \right) \right) \overset{D}{\to} Z,
\]

where \( Z \) is a zero mean Gaussian random object in \( \mathcal{H} \times \mathcal{F} \), with covariance operator \( \Upsilon : \mathcal{H} \times \mathcal{F} \to \mathcal{H} \times \mathcal{F} \). Then, \( \sqrt{n} (\hat{\Gamma}_s(\hat{\mu}_n) - \Gamma_s(\mu)) \) converges in \( \mathcal{F} \) to a zero mean Gaussian element with covariance operator given by \( (G_X \Pi_1 + \Pi_2) \Upsilon (G_X \Pi_1 + \Pi_2)^* \), where \( \Pi_i \), for \( i = 1, 2 \), are the projection operators from \( \mathcal{H} \times \mathcal{F} \) to \( \mathcal{H} \) and \( \mathcal{F} \), respectively. Moreover, \( G_X^* = 2F_X^* - 2S_X^* \) with \( F_X^* \) and \( S_X^* \) the adjoint operators of \( F_X \) and \( S_X \), respectively given by

\[
S_X^*(\Upsilon) = \frac{1}{2} \left\{ \Upsilon \left( \mathbb{E} \left[ \frac{X - \mu}{\| X - \mu \|^2} \right] \right) + \Upsilon^* \left( \mathbb{E} \left[ \frac{X - \mu}{\| X - \mu \|^2} \right] \right) \right\},
\]

\[
F_X^*(\Upsilon) = \mathbb{E} \left[ \frac{\langle (X - \mu) \otimes (X - \mu), \Upsilon \rangle_{\mathcal{F}}}{\| (X - \mu) \|^4} (X - \mu) \right].
\]

### 4 Applications

In this section, we consider two applications of the results obtained in Section 3. The first one is a result allowing to derive the asymptotic behaviour of the principal direction estimators obtained as the eigenfunctions of \( \hat{\Gamma}_s(\hat{\mu}_n) \). The second one uses the asymptotic distribution of the sample spatial sign operator to obtain a test for equality among sign covariance operators.

#### 4.1 On the asymptotic behaviour of the spherical principal direction estimators

Robust estimators of the principal directions for functional data have been extensively studied since the spherical principal components proposed in Locantore et al. (1999) and studied in Gervini (2008). As mentioned in the Introduction, Fisher–consistency of several proposals including the spherical principal directions has been studied in a framework more general than Gaussian random elements, without requiring finite moments, such as that given by elliptically distributed random elements.

When considering the spherical principal directions two possible situations may arise: either (a) the distribution is concentrated on a finite-dimensional subspace or (b) the rank of \( \Gamma_s(\mu) \) is infinite, where \( \mu \) stands for the location parameter of \( X \) which is typically the functional median. Gervini (2008) showed that the spherical principal direction estimators are Fisher–consistent for the principal directions when the process admits a Karhunen–Loève expansion with only finitely
many terms, while Boente et al. (2014) derived that the spherical principal components are in fact Fisher–consistent for any elliptical distribution. More precisely, assume that either:

\[ X = \mu + \sum_{k=1}^{q} \lambda_k^{1/2} \xi_k \phi_k, \quad \text{where} \quad \lambda_1 \geq \cdots \geq \lambda_k > 0, \quad \phi_k \in \mathcal{H} \quad \text{are orthonormal and} \quad \xi_k \quad \text{are random variables such that} \quad (\xi_1, \ldots, \xi_q)^T \quad \text{has symmetric and exchangeable marginals,} \]

\[ X \sim \mathcal{E}(\mu, \Gamma, \varphi) \quad \text{and denote} \quad \lambda_1 \geq \lambda_2 \geq \cdots \quad \text{the eigenvalues of the scatter operator} \quad \Gamma \quad \text{with associated eigenfunctions} \quad \phi_j, \]

hold. Note that in the situation a), the scatter operator \( \Gamma = \sum_{i=1}^{q} \lambda_i \phi_i \) has finite rank. As shown in Gervini (2008) and Boente et al. (2014), the eigenfunctions of \( \Gamma^s(\mu) \) are those of \( \Gamma \) and in the same order. More precisely, if \( \lambda_1^s \geq \lambda_2^s \geq \cdots \) stand for the ordered eigenvalues of \( \Gamma^s(\mu) \), under a) or b), we have that \( \phi_k \) is the eigenfunction of \( \Gamma^s(\mu) \) related to the eigenvalue \( \lambda_k^s \), meaning that the spatial principal directions are Fisher–consistent. Moreover, we also have that \( \lambda_1^s > \lambda_2^s + 1 \) if \( \lambda_j > \lambda_{j+1} \).

Beyond Fisher–consistency, consistency and order of consistency are also desirable properties for any robust procedure. However, for most of the proposed methods only consistency results were obtained. In this section, we derive the asymptotic distribution of the spherical principal direction estimators, which correspond to the eigenfunctions of the spatial sign operator estimator. In this sense, our result provides the first asymptotic normality result for robust principal direction estimators in a general setting.

Even though the asymptotic behaviour of the eigenfunctions of \( \widehat{\Gamma}^s(\mu) \) can easily be obtained from the central limit theorem and the results in Dauxois et al. (1982), Theorem 3 states that this asymptotic behaviour may not be the same when location is unknown and estimated. However, it should be noticed that for elliptical distributed random elements or under the symmetry assumptions required in Gervini (2008) to ensure Fisher consistency, we have that the asymptotic behaviour of the eigenfunctions of \( \widehat{\Gamma}^s(\mu_n) \) is that of the eigenfunctions of \( \Gamma^s(\mu) \), since as mentioned in Remark 3 \( \sqrt{n}(\widehat{\Gamma}^s(\mu_n) - \Gamma^s(\mu)) = o_p(1) \).

Similar arguments to those considered in Dauxois et al. (1982) and Corollary 1 allow to obtain the asymptotic distribution of the spatial principal direction estimators not only for elliptical families. For that purpose, denote \( \{\lambda_j^s\}_{j \geq 1} \) the sequence of eigenvalues of \( \Gamma^s(\mu) \) ordered in decreasing order and as \( \{\phi_j\}_{j \geq 1} \) their related eigenfunctions. Let \( \widehat{\phi}_1^s, \widehat{\phi}_2^s, \ldots \) be the eigenfunctions of \( \widehat{\Gamma}^s(\mu_n) \) related to the ordered eigenvalues \( \widehat{\lambda}_1^s \geq \widehat{\lambda}_2^s \geq \ldots \).

Recall that if the process has an elliptical distribution with scatter operator \( \Gamma \), \( \phi_j^s = \phi_j \) the \( j \)-th eigenfunction of \( \Gamma \).

Define \( \Lambda_i = \{j \in \mathbb{N} : \lambda_j^s = \lambda_i^s\} \), \( \Lambda = \{i \in \mathbb{N} : \text{card}(\Lambda_i) = 1\} \) and the projection operators \( \Pi_i = \sum_{j \in \Lambda_i} \phi_j^s \otimes \phi_j^s \) and \( \Pi_i^s = \sum_{j \in \Lambda_i} \widehat{\phi}_j^s \otimes \widehat{\phi}_j^s \). The following result is a direct consequence of Propositions 3, 4, 6 and 10 in Dauxois et al. (1982) and Corollary 1. Taking into account that the \( i \)-th principal direction is defined up to a sign change when the eigenvalue \( \lambda_i^s \) has multiplicity one, in the sequel, we choose the direction of the eigenfunction estimator so that \( \langle \phi_i^s, \phi_i^s \rangle > 0 \).

**Proposition 1** Assume that A.2 holds and that \( \left( \sqrt{n} (\hat{\mu}_n - \mu), \sqrt{n} (\widehat{\Gamma}^s(\mu) - \Gamma^s(\mu)) \right) \to^D Z \),
where $Z$ is a zero mean Gaussian random object in $\mathcal{H} \times \mathcal{F}$, with covariance operator $\Upsilon : \mathcal{H} \times \mathcal{F} \rightarrow \mathcal{H} \times \mathcal{F}$. Denote as $\Pi_i$, for $i = 1, 2$, the projection operators from $\mathcal{H} \times \mathcal{F}$ to $\mathcal{H}$ and $\mathcal{F}$, respectively and as $U$ a zero mean Gaussian random object in $\mathcal{F}$ with covariance operator $\Upsilon^* = (G_X \Pi_1 + \Pi_2) \Upsilon (G_X \Pi_1 + \Pi_2)^*$. Then, we have that

\[ \hat{\Pi}_i^S \xrightarrow{a.s.} \Pi_i^S \text{ in } \mathcal{F}. \]

Moreover, for any $i \in \Lambda$, $\hat{\phi}_i^S \xrightarrow{a.s.} \phi_i^S$ in $\mathcal{H}$.

\[ \sqrt{n}\left(\hat{\Pi}_i^S - \Pi_i^S\right) \xrightarrow{D} \left(\Delta_i \cup \Pi_i^S + \Pi_i^S \Delta_i\right) \]

Furthermore, when $i \in \Lambda$, we have that $\sqrt{n}\left(\hat{\phi}_i^S - \phi_i^S\right) \xrightarrow{D} (\Delta_i \cup \phi_i^S)$, which is a zero mean Gaussian process in $\mathcal{H}$.

Note that when $i \in \Lambda$, $\Delta_i = \sum_{\ell \neq i} \{1/(\lambda_i^S - \lambda_\ell^S)\} \phi_\ell^S \otimes \phi_i^S$.

### 4.2 Tests for equality of the sign covariance operators

The asymptotic distribution of the spatial covariance operator given in Corollary 3.1 allows to construct a test for equality between spatial covariance operators between two different populations. More precisely, assume that we have independent observations $X_{i,1}, \cdots, X_{i,n_i}, i = 1, 2$ such that $X_{i,j} \sim X_i, 1 \leq j \leq n_i$ with location parameter $\mu_i$. For the sake of simplicity, let us denote $\Gamma_i^S = \mathbb{E}[s(X_i - \mu_i) \otimes s(X_i - \mu_i)]$ the spatial sign covariance operator of the $i$-th population. We are interested in testing the null hypothesis

\[ H_0 : \Gamma_1^S = \Gamma_2^S \text{ against } H_1 : \Gamma_1^S \neq \Gamma_2^S. \]  

As in Boente et al. (2017), we will reject the null hypothesis when the difference between the estimated spatial sign covariance operators is large. Namely, if $\hat{\Gamma}_i^S$ stands for a consistent estimator of $\Gamma_i^S$ based on $X_{i,1}, \cdots, X_{i,n_i}, i = 1, 2$, we define

\[ T_n^S = n\|\hat{\Gamma}_2^S - \hat{\Gamma}_1^S\|_F^2, \]

where $n = n_1 + n_2$. The asymptotic distribution of $T_n^S$ can be obtained from the asymptotic distribution of $\sqrt{n}(\hat{\Gamma}_i^S - \Gamma_i^S)$, as stated in the following proposition, which can be considered as a robust version of Corollary 1 in Boente et al. (2017). Its proof can be obtained using Theorem 1 from the above-mentioned paper.

**Proposition 4.2** Let $X_{i,1}, \cdots, X_{i,n_i} \in \mathcal{H}, i = 1, 2$, be independent observations from two independent populations with location parameter $\mu_i$ and spatial sign covariance operator $\Gamma_i^S$. Assume that $n_i/n \rightarrow \tau_i$ with $\tau_i \in (0, 1)$ where $n = n_1 + n_2$. Let $\hat{\Gamma}_i^S$ be independent estimators of the $i$-th population spatial sign covariance operator such that $\sqrt{n_i}(\hat{\Gamma}_i^S - \Gamma_i^S) \xrightarrow{D} U_i$, with $U_i$
a zero mean Gaussian random element with covariance operator $\Upsilon_i$. Denote $\Upsilon_i : \mathcal{F} \to \mathcal{F}$ the linear operator defined as $\Upsilon_i = (1/n_i) \sum_{j=1}^{n_i} s(X_{i,j} - \hat{\mu}_n) \otimes s(X_{i,j} - \hat{\mu}_n)$, with $\hat{\mu}_n$, any consistent estimators of the functional median $\hat{\mu}_i$ of the process $X_i$ satisfying $A.1$, for instance, the spatial median are given in Cardot et al. (2013) (see Remark $A.2$). In such a case, as noted in Boente et al. (2017), equation (7) motivates the use of the bootstrap methods, to decide whether to reject the null hypotheses, as follows:

**Step 1.** For $i = 1, 2$ and given the sample $X_{i,1}, \ldots, X_{i,n_i}$, let $\hat{\Upsilon}_i$ be consistent estimators of $\Upsilon_i$. Define $\hat{\Upsilon}_n = \hat{\Upsilon}_1^{-1} \hat{\Upsilon}_1 + \hat{\Upsilon}_2^{-1} \hat{\Upsilon}_2$ with $\hat{\Upsilon}_n = n_i/(n_1 + n_2)$.

**Step 2.** For $1 \leq q \leq n_i$ denote by $\hat{\theta}_q$ the positive eigenvalues of $\hat{\Upsilon}_n$.

**Step 3.** Generate $Z_{1}^*, \ldots, Z_{q_i}^*$ i.i.d. such that $Z_{i}^* \sim N(0, 1)$ and let $U_{n}^r = \sum_{q=1}^{n_i} \hat{\theta}_q Z_{q_i}^2$.

**Step 4.** Repeat Step 3 $N_B$ times, to get $N_B$ values of $U_{n}^r$ for $1 \leq r \leq N_B$.

The $(1 - \alpha)$–quantile of the asymptotic null distribution of $T_{n}^*\Upsilon$ can be approximated by the $(1 - \alpha)$–quantile of the empirical distribution of $U_{n}^r$ for $1 \leq r \leq N_B$. Besides, the $p$–value can be estimated by $\hat{p} = s/N_B$ where $s$ equals the number of $U_{n}^r$ which are larger or equal than the observed value of the statistic $T_{n}^*\Upsilon$. The validity of the bootstrap procedure can be derived from Theorem 3 in Boente et al. (2017) if the estimators of estimators of $\Upsilon_n$ are such that $\sqrt{n_{i}}|\hat{\Upsilon}_W - \Upsilon_n\|_F = O_p(1)$ ensuring that the asymptotic significance level of the test based on the bootstrap critical value is indeed $\alpha$. A possible choice for $\hat{\Upsilon}_i$, $i = 1, 2$ is the sample covariance operator of $Y_i = s(X_i - \mu_i) \otimes s(X_i - \mu_i)$.

**Remark $A.1$.** Proposition $A.2$ ensures that, under mild assumptions, it is possible to provide a test to decide if $\Gamma_{1}^s = \Gamma_{2}^s$. An important point to highlight is what this null hypothesis represents, for instance, in terms of the covariance operators of the two populations, when they exist. Let us consider the situation in which the two populations have an elliptical distribution, that is, $X_i \sim \mathcal{E}(\mu_i, \Gamma_i, \varphi_i)$, for $i = 1, 2$. Recall that the eigenfunctions of $\Gamma_{i}^s$ are those of $\Gamma_i$ and in the same order, while the eigenvalues of the sign covariance operator $\Gamma_{i}^s$, denoted $\lambda_{i,\ell}^s$, are shrunken with respect to those of $\Gamma_i$ (that are denoted as $\lambda_{i,\ell}$) as follows

$$
\lambda_{i,\ell}^s = \lambda_{i,\ell} \left( \frac{\epsilon_{i,\ell}^2}{\sum_{\ell \geq 1} \lambda_{i,\ell}^s \epsilon_{i,j}^2} \right), \quad (8)
$$
where $\xi_{i,j} = \lambda_{i,\ell}^{-1/2} \langle X_i - \mu_i, \phi_{i,\ell} \rangle$ with $\phi_{i,\ell}$ the eigenfunction of $\Gamma_i$.

Assume that $\varphi_1 = \varphi_2$, that is, if the two populations have the same underlying distribution up to location and scatter. Note that, if the scatter operators are proportional, i.e., if $\Gamma_2 = \rho \Gamma_1$ for some positive constant $\rho$, then $\Gamma_i^s = \Gamma_2^s$. Thus, when the two populations have the same elliptical distribution up to changes in location and scatter, the test based on $T_n^s$ provides a way for testing proportionality of the scatter operators, even when second moments do not exist.

It is worth noticing that, when second moment exists the covariance operator of $X_i$ is up to a constant equal to $\Gamma_i$, hence the statistic $T_n^s$ allows to test proportionality between the two covariance operators. Note that when $\Gamma_i^s = \Gamma_2^s$, both scatter operators have the same rank and share the same eigenfunctions. Furthermore, if the scatter operators have finite rank, from Proposition 1 in Dürre et al. (2016), we get that $\Gamma_i^s = \Gamma_2^s$ if and only if $\Gamma_2 = \rho \Gamma_1$ for some positive constant $\rho$. Hence, for finite rank scatter operators, testing proportionality of the scatter operators is equivalent to testing equality of the spatial sign operators.

### 4.2.1 Monte Carlo study

This section contains the results of a simulation study devoted to illustrate the finite–sample performance of the test procedure described in Section 4.2 under the null hypothesis and different alternatives, when atypical data are introduced in the samples. The numerical study also aim to compare the performance of the sign operator testing procedure with that based on the sample covariance operator introduced in Boente et al. (2017).

We have performed $N = 1000$ replications taking samples of size $n_i = 100$, $i = 1, 2$. The generated samples $X_{i,1}, \ldots, X_{i,n_i}$, $i = 1, 2$ are such that $X_{i,j} \sim X_i \in L^2(0,1)$. In all cases, each trajectory was observed at $m = 100$ equidistant points in the interval $[0,1]$ and we performed $N_B = 5000$ bootstrap replications. To summarize the tests performance, we compute the observed frequency of rejections over replications with nominal level $\alpha = 0.05$.

#### Simulation settings

The distribution of the two populations correspond, under the null hypothesis, to independent centred Brownian motion processes, denoted from now on as $BM(0,1)$. Hence, both processes have the same spatial sign operators and also the same covariance operators. On the other hand, to check the test power performance, we consider the same alternatives as in Boente et al. (2017) and also Gaussian alternatives. More precisely, we generate independent observations $X_{1,j} \sim X_1, 1 \leq j \leq n_1$, and $X_{2,j} \sim X_2, 1 \leq j \leq n_2$, such that $X_1$ has the distribution of a centred Brownian motion denoted $BM(0,1)$ while the second population has a distribution according to the one of the following models

- **Model 1:** $X_2 \sim Y_1 + \delta_n Y_2^2$, where $Y_1$ and $Y_2$ are independent $Y_i \sim BM(0,1), i = 1, 2$ and $\delta_n = \Delta n^{-1/4}$ with $n = n_1 + n_2$ and $\Delta$ takes values from 0 to 8 with step 1 and from 10 to 20 with step 2. The situation $\Delta = 0$ corresponds to the null hypothesis in which both processes have a Gaussian distribution.
• Model 2: $X_2 \sim Y_1 + \delta_n Y_2$, where $Y_1$ and $Y_2$ are independent $Y_1 \sim BM(0,1)$, $Y_2$ is a Gaussian process with covariance kernel $\text{Cov}(Y_2(t), Y_2(s)) = \exp(-|s - t|/0.2)$ and $\delta_n = \Delta n^{-1/4}$ with $n = n_1 + n_2$ and $\Delta \in \{0, 0.5, 1, 1.5, 2, 2.5, 3, 4, 5\}$. In this case both processes have a Gaussian distribution under the null and under the alternative which implies that, for each population, the spatial sign operator has the same eigenfunctions as the covariance operator. Moreover, the eigenvalues of the spatial operator and of the covariance operator of the $i$–th population are related through $\xi_s \in \mathbb{S}$ with $\xi_{i,t} \sim N(0,1)$ independent of each other.

To analyse the behaviour when atypical data are introduced in the sample, for each generated sample, we also consider the following contamination. We first generate two independent samples $V_{i,j} \in \mathbb{R}$, $1 \leq i \leq n_j$ and $j = 1, 2$, such that $V_{i,j} \sim |T_1|$, where $|T_1|$ corresponds to the absolute value of an univariate $T$-Student distribution with 1 degree of freedom. We then generated the contaminated samples, denoted $X_{i,j}^{(c)}$, as follows $X_{i,j}^{(c)} = (1 - B_i) X_{i,j} + B_i V_{i,j}$, where $B_i \sim B(1, 0.1)$ are independent and independent of $(X_{i,j}, V_{i,j})$. Note that under the null hypothesis, both populations have the same elliptical distribution since they can be written as $W_{i,j} X_{i,j}$ with $W_{i,j} = (1 - B_i) + B_i V_{i,j}$ a positive random variable independent of $X_{i,j}$ and $W_{1,j} \sim W_{2,j}$.

The test statistics

We computed two test statistics, the statistic based on the spatial sign operator defined above and the procedure defined in Boente et al. (2017). The test statistic given in this last paper is defined as $T_n = n \|\hat{\Gamma}_1 - \hat{\Gamma}_2\|^2$ where $\hat{\Gamma}_i = (1/n_i) \sum_{j=1}^{n_i} (X_{i,j} - \overline{X}_i) \otimes (X_{i,j} - \overline{X}_i)$ is the sample covariance operator. This testing method is designed to test equality of the two populations covariance operators, which is fulfilled when $\Delta = 0$. On the other hand, the statistic $T_n^S$ defined in [3] is designed to test equality of the spatial operators, that is, $\Gamma_1^S = \Gamma_2^S$. As mentioned in Remark 4.1, this null hypothesis is fulfilled when the scatter operators related to the elliptical distribution are proportional which holds when $\Delta = 0$, both for clean and contaminated samples. When computing the spatial sign operators $\hat{\Gamma}_i^S$, we center the data with the functional median computed through the function $\text{llmedian}$ from the R package pcaPP.

The testing procedure requires bootstrap calibration. For that purpose, following the procedure described in Boente et al. (2017), we project the centred data onto the $M$ largest principal components of the pooled operators $\hat{\Gamma}_\text{POOL}$, where the pooled operator was adapted to the testing procedure used. More precisely, $\hat{\Gamma}_\text{POOL} = n^{-1} \sum n_i \hat{\Gamma}_i$, when the test statistic is based on the sample covariance matrices, while $\hat{\Gamma}_\text{POOL} = n^{-1} \sum n_i \hat{\Gamma}_i^S$, when the test statistic corresponds to the sample sign operator. The covariance operator of each estimator, denoted $\hat{\Gamma}_W$ in Proposition 4.2 for the spatial operator, is then estimated through a finite dimensional matrix. We choose different values of the number of principal directions $M = 3, 10, 20$ and 30 to study the dependence on the finite–dimensional approximation considered. As noted in Boente et al. (2017), the value $q_n$ used in Step 2 equals $q_n = M(M + 1)/2$. With the selected number of principal directions, we explained more than 80% of the total variability (see Table 1).
Table 1: Percentage of the total variance explained by the first $M$ principal components when using the test $T_{B,M}$ or $T_{S,B,M}$.

| $\Delta$ | $M$ | Clean samples | Contaminated samples |
|---------|-----|---------------|---------------------|
|         |     | 3 10 20 30    | 3 10 20 30          |
| $T_{B,M}$ | 0  | 0.934 0.981 0.991 0.995 | 0.962 0.992 0.996 0.998 |
| $T_{S,B,M}$ | 0  | 0.828 0.950 0.979 0.989 | 0.828 0.950 0.979 0.989 |

When the populations have a Gaussian distribution, the asymptotic covariance operator of the sample covariance operator $\hat{\Gamma}_i = (1/n_i) \sum_{j=1}^{n_i} (X_{i,j} - \bar{X}_i) \otimes (X_{i,j} - \bar{X}_i)$ can be estimated using the eigenvalues and eigenfunctions of $\hat{\Gamma}_i$. Taking into account that, under the null hypothesis, the processes are Gaussian, we have also used this approximation when considering the sample covariance operator.

From now on we denote as $T_{B,M}$ and $T_{S,B,M}$, for $M = 3, 10, 20$ and $30$ the bootstrap calibration of the statistics $T_n$ and $T_{S,n}$, respectively, computed using $M$ principal components. Finally, $T_{B,G}$ stands for the bootstrap calibration of $T_n$ computed using the Gaussian approximation.

**Simulation results**

For the alternatives given through Models 1 and 2, Tables 2 and 4 summarize, respectively, the frequency of rejection for the procedure based on the sample covariance operator for the uncontaminated samples and for the contaminated samples, while those corresponding to the test based on the sample spatial sign operator are reported in Table 3 and 5.

Table 2: Frequency of rejection for the bootstrap test $T_{B,M}$ based on the sample covariance operators, under Model 1, when $M = 3, 10, 20$ and $30$ principal components are used. The row labelled $T_{B,G}$ reports the frequencies obtained when the eigenvalues $\theta_\ell$ are estimated using that the processes are Gaussian.

| $\Delta$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 10 | 12 | 14 | 16 | 18 | 20 |
|---------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Clean samples | $T_{B,3}$ | 0.066 | 0.083 | 0.315 | 0.694 | 0.895 | 0.948 | 0.959 | 0.967 | 0.972 | 0.973 | 0.974 | 0.974 | 0.974 | 0.975 |
|          | $T_{B,10}$ | 0.065 | 0.082 | 0.299 | 0.681 | 0.890 | 0.942 | 0.957 | 0.962 | 0.969 | 0.971 | 0.972 | 0.973 | 0.973 | 0.973 |
|          | $T_{B,20}$ | 0.061 | 0.081 | 0.296 | 0.671 | 0.885 | 0.941 | 0.956 | 0.961 | 0.965 | 0.968 | 0.971 | 0.973 | 0.973 | 0.971 |
|          | $T_{B,30}$ | 0.060 | 0.079 | 0.290 | 0.666 | 0.882 | 0.940 | 0.956 | 0.961 | 0.964 | 0.967 | 0.970 | 0.973 | 0.973 | 0.971 |
|          | $T_{B,G}$ | 0.050 | 0.064 | 0.333 | 0.801 | 0.975 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

| $T_{B,3}$ | 0.011 | 0.016 | 0.037 | 0.082 | 0.152 | 0.217 | 0.268 | 0.291 | 0.315 | 0.364 | 0.394 | 0.412 | 0.436 | 0.451 | 0.461 |
| $T_{B,10}$ | 0.010 | 0.013 | 0.033 | 0.076 | 0.148 | 0.214 | 0.260 | 0.283 | 0.306 | 0.355 | 0.389 | 0.402 | 0.423 | 0.441 | 0.451 |
| $T_{B,20}$ | 0.009 | 0.011 | 0.032 | 0.074 | 0.145 | 0.212 | 0.256 | 0.280 | 0.302 | 0.348 | 0.381 | 0.396 | 0.420 | 0.436 | 0.446 |
| $T_{B,30}$ | 0.009 | 0.011 | 0.031 | 0.074 | 0.144 | 0.209 | 0.253 | 0.279 | 0.300 | 0.345 | 0.379 | 0.395 | 0.417 | 0.434 | 0.445 |
| $T_{B,G}$ | 0.843 | 0.836 | 0.856 | 0.923 | 0.961 | 0.974 | 0.979 | 0.984 | 0.989 | 0.994 | 0.997 | 0.996 | 0.998 | 0.999 | 0.999 |

As noted in Boente et al. (2017) when using the Gaussian approximation the test based on the sample covariance operators shows an improvement in size for uncontaminated samples.
Table 3: Frequency of rejection for the bootstrap test $T_{B,M}^S$ based on the spatial sign operator, under Model 1, when $M = 3, 10, 20$ and $30$ principal directions are used.

Table 4: Frequency of rejection for the bootstrap test $T_{B,M}^S$ based on the sample covariance operators, under Model 2, when $M = 3, 10, 20$ and $30$ principal components are used. The row labelled $T_{B,G}$ reports the frequencies obtained when the eigenvalues $\theta_\ell$ are estimated using that the processes are Gaussian.

Table 5: Frequency of rejection for the bootstrap test $T_{B,M}^S$ based on the spatial sign operator, under Model 2, when $M = 3, 10, 20$ and $30$ principal directions are used.

However, when contaminating the data the level breaks-down and the test becomes uninformative.

On the other hand, when projecting the data on the first $M$ principal components, the empirical size of the tests based on the bootstrap calibration either using the sample covariance or the spatial sign operators is quite close to the nominal one, for uncontaminated samples. To analyse the significance of the empirical size, we study if the empirical size is significantly different from the nominal level $\alpha = 0.05$ by testing $H_0,\pi : \pi = \alpha$ with nominal level $\gamma$, where $\pi$ stands for the value such that $\pi_n \xrightarrow{p} \pi$ with $\pi_n$ the empirical size of the considered test. This null hypothesis is rejected at level $\gamma$ versus $H_1,\pi : \pi \neq \alpha$ if $\pi_n \notin [a_1(\alpha), a_2(\alpha)]$ where
Throughout this section, we will assume that $\mu = 0$, without loss of generality. Furthermore, we will denote as $\Gamma_0^S = \Gamma_0^S(0)$, $\Gamma_i(\mu_n) = s(X_i - \hat{\mu}_n) \otimes s(X_i - \hat{\mu}_n)$ and $\Gamma_i = s(X_i) \otimes s(X_i)$.

**Proof of Theorem 3.1.** Note that the strong law of large numbers entails that it is enough to prove that $\hat{\Gamma}_n^S(\mu_n) - \Gamma_0^S \xrightarrow{a.s.} 0$. Consider the following random set

$$A_n = \{ x \in \mathcal{H} : \| x - \hat{\mu}_n \| \geq \frac{1}{2} \| x \| \}.$$

Therefore, we have that

$$\| \hat{\Gamma}_n^S(\mu_n) - \Gamma_0^S \|_F \leq \frac{1}{n} \sum_{X_i \in A_n} \| \Gamma_i(\mu_n) - \Gamma_i \|_F + \frac{1}{n} \sum_{X_i \notin A_n} \| \Gamma_i(\mu_n) - \Gamma_i \|_F = A_{n,1} + A_{n,2} \quad (A.1)$$

To show that $A_{n,1} \xrightarrow{a.s.} 0$, note that straightforward calculations lead to the bound

$$\| \Gamma_i(\mu_n) - \Gamma_i \|_F^2 = \frac{2}{\| X_i \|^2 \| X_i - \mu_n \|^2} \left( \| X_i \|^2 \| \hat{\mu}_n \|^2 - \langle \hat{\mu}_n, X_i \rangle \right)^2 \leq \frac{4 \| X_i \|^2 \| \hat{\mu}_n \|^2}{\| X_i \|^2 \| X_i - \mu_n \|^2}.$$  

On the other hand, if $X_i \in A_n$, we have that $\| \Gamma_i(\mu_n) - \Gamma_i \|_F^2 \leq 16 \| \hat{\mu}_n \|^2 / \| X_i \|^2$ which implies that

$$\frac{1}{n} \sum_{X_i \in A_n} \| \Gamma_i(\mu_n) - \Gamma_i \|_F \leq 4 \| \hat{\mu}_n \| \frac{1}{n} \sum_{i=1}^n \frac{1}{\| X_i \|}.$$  

**Acknowledgements.** This research was partially supported by Grants PIP 112-201101-00742 from CONICET, PICT 2014-0351 and 201-0377 from ANPCyT and 20020130100279BA and 20020150200110BA from the Universidad de Buenos Aires at Buenos Aires, Argentina.

**Appendix**

Throughout this section, we will assume that $\mu = 0$, without loss of generality. Furthermore, we will denote as $\Gamma_0^S = \Gamma_0^S(0)$, $\Gamma_i(\mu_n) = s(X_i - \hat{\mu}_n) \otimes s(X_i - \hat{\mu}_n)$ and $\Gamma_i = s(X_i) \otimes s(X_i)$.

**Proof of Theorem 3.1.** Note that the strong law of large numbers entails that it is enough to prove that $\hat{\Gamma}_n^S(\mu_n) - \Gamma_0^S \xrightarrow{a.s.} 0$. Consider the following random set

$$A_n = \{ x \in \mathcal{H} : \| x - \hat{\mu}_n \| \geq \frac{1}{2} \| x \| \}.$$  

Therefore, we have that

$$\| \hat{\Gamma}_n^S(\mu_n) - \Gamma_0^S \|_F \leq \frac{1}{n} \sum_{X_i \in A_n} \| \Gamma_i(\mu_n) - \Gamma_i \|_F + \frac{1}{n} \sum_{X_i \notin A_n} \| \Gamma_i(\mu_n) - \Gamma_i \|_F = A_{n,1} + A_{n,2} \quad (A.1)$$

To show that $A_{n,1} \xrightarrow{a.s.} 0$, note that straightforward calculations lead to the bound

$$\| \Gamma_i(\mu_n) - \Gamma_i \|_F^2 = \frac{2}{\| X_i \|^2 \| X_i - \mu_n \|^2} \left( \| X_i \|^2 \| \hat{\mu}_n \|^2 - \langle \hat{\mu}_n, X_i \rangle \right)^2 \leq \frac{4 \| X_i \|^2 \| \hat{\mu}_n \|^2}{\| X_i \|^2 \| X_i - \mu_n \|^2}.$$  

On the other hand, if $X_i \in A_n$, we have that $\| \Gamma_i(\mu_n) - \Gamma_i \|_F^2 \leq 16 \| \hat{\mu}_n \|^2 / \| X_i \|^2$ which implies that

$$\frac{1}{n} \sum_{X_i \in A_n} \| \Gamma_i(\mu_n) - \Gamma_i \|_F \leq 4 \| \hat{\mu}_n \| \frac{1}{n} \sum_{i=1}^n \frac{1}{\| X_i \|}.$$  

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Therefore, using that $\hat{\mu}_n \xrightarrow{a.s.} 0$, $\mathbb{E} \left[ \|X\|^{-1} \right] < \infty$ and the strong law of large numbers we conclude that $A_{n,1} \xrightarrow{a.s.} 0$.

In remains to show that the second term $A_{n,2}$ in the right hand side of (A.1) converges almost surely to zero. The fact that $\|\Gamma_i(\hat{\mu}_n)\| = \|\Gamma_i\|_F = 1$ implies that

$$A_{n,2} = \frac{1}{n} \sum_{X_i \in A_n} \|\Gamma_i(\hat{\mu}_n) - \Gamma_i\|_F \leq \frac{2}{n} \sum_{i=1}^{n} Z_{n,i},$$

where $Z_{n,i} = I_{A_n^c}(X_i)$.

Note that the assumption $\mathbb{E} \left[ \|X\|^{-1} \right] < \infty$ implies that $\mathbb{P}(\|X\| = 0) = 0$. Hence, for any $\epsilon > 0$, let $\delta > 0$ be such that $\mathbb{P}(\|X\| \leq \delta) \leq \epsilon$ and denote $Z_{\delta,i} = I_{B_\delta}(X_i)$, where $B_\delta = \{\|x\| \leq \delta\}$. Then,

$$\frac{1}{n} \sum_{i=1}^{n} Z_{n,i} \leq \frac{1}{n} \sum_{i=1}^{n} Z_{\delta,i} + \frac{1}{n} \sum_{i=1}^{n} (Z_{n,i} - Z_{\delta,i})_+ = B_{n,1} + B_{n,2},$$

where $a_+ = \max(a,0)$. The strong law of large numbers entails that $B_{n,1} \xrightarrow{a.s.} 0$, $\mathbb{P}(\|X\| \leq \delta) \leq \epsilon$. To show that $B_{n,2} \xrightarrow{a.s.} 0$, note that $\{\|\hat{\mu}_n\| \leq \delta/2\} \subset \{(Z_{n,i} - Z_{\delta,i})_+ = 0\}$. Hence, using that $\hat{\mu}_n \xrightarrow{a.s.} 0$, we get that there exists a null probability set $N$ such that for $\omega \notin N$, there exists $n_0$ such that, for all $n > n_0$, $\|\hat{\mu}_n\| \leq \delta/2$ implying that $B_{n,2} = (1/n)\sum_{i=1}^{n} (Z_{n,i} - Z_{\delta,i})_+ = 0$ and concluding the proof. $\square$

**Proof of Theorem 3.2.** Note that $\Gamma_i(\hat{\mu}_n) - \Gamma_i$ can be written as follows

$$\Gamma_i(\hat{\mu}_n) - \Gamma_i = \|X_i\|^{-2} \left\{ \|X_i\|^2 \Gamma_i(\hat{\mu}_n) - X_i \otimes X_i \right\}$$

(A.2)

$$= \|X_i\|^{-2} \left\{ \|X_i - \hat{\mu}_n\|^2 + \|\hat{\mu}_n\|^2 + 2\langle X_i - \hat{\mu}_n, \hat{\mu}_n \rangle \right\} \Gamma_i(\hat{\mu}_n) - X_i \otimes X_i$$

$$= \|X_i\|^{-2} \left\{ \hat{\mu}_n \otimes \hat{\mu}_n - \hat{\mu}_n \otimes X_i - X_i \otimes \hat{\mu}_n + 2 \langle X_i, \hat{\mu}_n \rangle - \|\hat{\mu}_n\|^2 \right\} \Gamma_i(\hat{\mu}_n).$$

Therefore, $\sqrt{n} \left( \hat{\Gamma}_s(\hat{\mu}_n) - \hat{\Gamma}_0 \right) = (1/\sqrt{n}) \sum_{i=1}^{n} (\Gamma_i(\hat{\mu}_n) - \Gamma_i) = S_{n,1} - S_{n,2} - S_{n,3} + 2 S_{n,4} - S_{n,5}$, where

$$S_{n,1} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\hat{\mu}_n \otimes \hat{\mu}_n}{\|X_i\|^2} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i}{\|X_i\|^2}$$

$$S_{n,2} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\hat{\mu}_n \otimes X_i}{\|X_i\|^2} = \sqrt{n} \hat{\mu}_n \otimes \left( \frac{1}{n} \sum_{i=1}^{n} \frac{X_i}{\|X_i\|^2} \right)$$

$$S_{n,3} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i \otimes \hat{\mu}_n}{\|X_i\|^2} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i}{\|X_i\|^2} \otimes \hat{\mu}_n$$

$$S_{n,4} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\langle X_i, \hat{\mu}_n \rangle \Gamma_i(\hat{\mu}_n)}{\|X_i\|^2}$$

$$S_{n,5} = \|\hat{\mu}_n\|^2 \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\Gamma_i(\hat{\mu}_n)}{\|X_i\|^2} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\Gamma_i(\hat{\mu}_n)}{\|X_i\|^2}.$$
Note that A.2 entails that $\mathbb{E}V_i^{2/3} < \infty$ where $V_i = 1/\|X_i\|^2$, so the Marcinkiewicz’s strong law of large numbers implies that $n^{-3/2}\sum_{i=1}^n 1/\|X_i\|^2 \overset{a.s.}{\rightarrow} 0$. Hence, Assumptions A.1 and A.2 together with the strong law of large numbers and the fact that $\|\Gamma_i(\hat{\mu}_n)\|_F = 1$ entail that $S_{n,j} \overset{p}{\rightarrow} 0$ for $j = 1, 5$.

The decomposition of $\Gamma_i(\hat{\mu}_n) - \Gamma_i$ obtained in (A.2) entails that $S_{n,4}$ can be written as $S_{n,4} = S_{n,41} + S_{n,42} - S_{n,43} - S_{n,44} + S_{n,45} - S_{n,46}$, where

$$S_{n,41} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\langle X_i, \hat{\mu}_n \rangle}{\|X_i\|^4} X_i \otimes X_i$$

$$S_{n,43} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\langle X_i, \mu_n \rangle}{\|X_i\|^4} \hat{\mu}_n \otimes \hat{\mu}_n$$

$$S_{n,45} = \frac{2}{\sqrt{n}} \sum_{i=1}^n \frac{\langle X_i, \hat{\mu}_n \rangle}{\|X_i\|^4} \langle X_i, \hat{\mu}_n \rangle \Gamma_i(\hat{\mu}_n)$$

Using again the Marcinkiewicz’s strong law of large numbers, we get that $n^{-2}\sum_{i=1}^n 1/\|X_i\|^3 \overset{a.s.}{\rightarrow} 0$, since $\mathbb{E}\|X_i\|^{-3/2} < \infty$ by A.2. Hence, using A.1 and that $\|\Gamma_i(\hat{\mu}_n)\|_F = 1$, we get that $S_{n,j} \overset{p}{\rightarrow} 0$, for $j = 2, 6$. On the other hand, using again that $n^{-3/2}\sum_{i=1}^n 1/\|X_i\|^2 \overset{a.s.}{\rightarrow} 0$, we obtain that $S_{n,j} \overset{p}{\rightarrow} 0$ for $j = 3, 4, 5$.

It remains to study the asymptotic behaviour of $S_{n,2}, S_{n,3}$ and $S_{n,41}$. We will show that

$$S_{n,41} - \sqrt{n}SF(X(\hat{\mu}_n) - \mu) = o_p(1) \quad (A.3)$$

$$S_{n,2} + S_{n,3} - 2\sqrt{n}SF(\hat{\mu}_n - \mu) = o_p(1) \quad (A.4)$$

Let us begin by showing (A.3). Denote as $W_i : \mathcal{H} \to \mathcal{F}$, the random objects in $\mathcal{B}$, defined as $W_i(u) = (\langle X_i, u \rangle/\|X_i\|^4) X_i \otimes X_i$, for $u \in \mathcal{H}$. It is easy to see that $\|W_i\|_{\mathcal{B}} \leq \|X_i\|^{-1}$ and assumption A.2 guarantee that $\mathbb{E}\|X_i\|^{-1} < \infty$, hence the strong law of large number on $\mathcal{B}$ allows to conclude that

$$\frac{1}{n} \sum_{i=1}^n \frac{\langle X_i, u \rangle}{\|X_i\|^4} X_i \otimes X_i \overset{a.s.}{\rightarrow} F_X,$$

where $F_X$ is defined in (3), which together with A.1 concludes the proof of (A.3).

To obtain (A.4), note that the strong law of large number on $\mathcal{H}$ and the fact that $\mathbb{E}\|X_i\|^{-1} < \infty$ imply that $(1/n)\sum_{i=1}^n X_i/\|X_i\|^2 \overset{a.s.}{\rightarrow} \mathbb{E}[X^2/\|X\|^2]$. Thus, if we define a sequence $\{T_n\}_{n \geq 1}$ of random objects in $\mathcal{B}$ as

$$T_n(u) = u \otimes \frac{1}{n} \sum_{i=1}^n \frac{X_i}{\|X_i\|^2} + \frac{1}{n} \sum_{i=1}^n \frac{X_i}{\|X_i\|^2} \otimes u$$

for any $u \in \mathcal{H}$

we obtain that $T_n \overset{a.s.}{\rightarrow} 2S_X$, where $S_X$ is defined in (4). Hence, using A.1, we obtain (A.3) concluding the proof. \qed

**Proof of Corollary 3.1.** Note that from Theorem 3.2 we get that

$$\sqrt{n}\left(\hat{\Gamma}^S(\hat{\mu}_n) - \Gamma^S(\mu)\right) = \sqrt{n}\left(\hat{\Gamma}^S(\mu) - \Gamma^S(\mu)\right) + \sqrt{n}G_X(\hat{\mu}_n - \mu) + o_P(1).$$
Now, the results follows immediately defining, for any fixed $v \in \mathcal{H}$, the operators $R_v : \mathcal{H} \to \mathcal{F}$ and $L_v : \mathcal{H} \to \mathcal{F}$ as $R_v(u) = u \otimes v$ and $L_v(u) = v \otimes u$ and using that $R_v^*(\Upsilon) = \Upsilon(v)$ and $L_v^*(\Upsilon) = \Upsilon^*(v)$.

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