Complex Dynamic Behaviors of A Congestion Control System under A Fractional-order PD Control

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Abstract. This paper proposed an original fractional-order proportional-derivative (PD) feedback controller which is designed to control the Hopf bifurcation caused by the congestion control system. The proposed PD\textsuperscript{\alpha} controller has the different order with the original congestion system. The proposed fractional-order PD controller has the different order with the controlled system and the communication delay is selected as the bifurcation parameter. Then the conditions of the stability and Hopf bifurcation are obtained by analyzing its characteristic equation and the stability domain can be extended under the adjustment of appropriate control gain parameters and the order. Therefore, the congestion system becomes controllable and the desirable behaviors can be realized. Finally, numerical simulations are carried out to testify the validity of the theoretical analysis in the designed fractional-order PD controller.

1. Introduction

With the rapid development of network technology, the number of Internet users are keeping rising, and the concomitant possibility that the resource capacity of computer networks fails to meet the needs of Internet users. The phenomenon of computer network congestion has become increasingly serious. Therefore, there have been a great of achievements in the research of congestion control system [1, 2, 3]. In this paper, we choose the integer-order congestion control system with time delay as our controlled model which was introduced in [4] and it can be expressed as the following differential equation:

\[ \dot{x}(t) = kx(t) \left[ \frac{1}{x(t-\tau)} - c \right], \]  

where the variable $x$ is the price at the link and $\tau$ denotes the communication delay respectively. Meanwhile, $k > 0$ is the gain parameter and the scalar $c$ is the capacity of this model. Obviously, it can be obtained that system (1) has a sole equilibrium $x^* = \frac{1}{c}$. Some control schemes were designed to control the undesirable Hopf bifurcation so that we can adjust the initiation of bifurcation and achieve desirable dynamical characteristic of the controlled system [5, 6, 7].

Fractional-order calculus has a wide range of applications in physics, biology and engineering and it extends the traditional concept of differential and integral calculus. It has been found that fractional-order derivatives has superiority of memory and hereditary effects in mathematical modeling, which guarantees a better global performance of the controlled system [8]. Therefore, abundant significative scientific achievements have been achieved in stability analysis [9], bifurcations [10], and chaos [11].
In recent years, researches have an increasing attention on applying PID controller to control the Hopf bifurcation and a lot of important research results have been achieved [12, 13]. However, few studies have considered that the order of the controller is different from that of the controlled model. Therefore, we propose a fractional-order PD controller to control the integer-order congestion control model in this paper. The proposed controller can be represented in the following form:

\[ u(t) = k_p (x(t) - x^*) + k_d \left[ C_0 D_t^\frac{1}{n} (x(t) - x^*) \right], \]  

(2)

where \( k_p \) and \( k_d \) respectively represent the proportional control parameter and the derivative control parameter and \( n \) is a natural number.

The rest of this paper is organized as follows. Section 2 presents the mathematical model based on fractional-order PD controller and linearizes the controlled model at equilibrium and discuss the impacts of order and gain parameters on stability and Hopf bifurcation and the transversality condition is proved. In Section 3, some numerical simulation results are carried out to verify our results. Finally, Section 4 concludes this paper.

2. Model Description and Linearize Controlled Model at Equilibrium

Applying the fractional-order controller (2) to system (1), we can achieve a controlled fractional system as follows:

\[ \dot{x}(t) = k x(t) \left[ \frac{1}{x(t-\tau)} - c \right] + u(t) \]

\[ = k x(t) \left[ \frac{1}{x(t-\tau)} - c \right] + k_p (x(t) - x^*) + k_d \left[ C_0 D_t^\frac{1}{n} (x(t) - x^*) \right]. \]

(3)

Obviously, the equilibrium of system (3) still locate in \( x^* \) of system (1). Let

\[ x_1(t) = x(t) - x^*, \]

\[ x_2(t) = C_0 D_t^\frac{1}{n} x_1(t), \]

\[ x_3(t) = C_0 D_t^\frac{1}{n} x_2(t), \]

\[ \vdots \]

\[ x_n(t) = C_0 D_t^\frac{1}{n} x_{n-1}(t). \]

(4)

We can derive

\[ C_0 D_t^\frac{1}{n} x_n(t) = C_0 D_t^\frac{1}{n} \left[ C_0 D_t^\frac{1}{n} x_{n-1}(t) \right] \]

\[ = C_0 D_t^\frac{1}{n} x_{n-1}(t) = \cdots = C_0 D_t^\frac{1}{n} x_1(t) = \dot{x}_1(t) = \dot{x}(t) \]

\[ = k [x_1(t) + x^*] \left[ \frac{1}{x_1(t-\tau) + x^*} - c \right] + k_p x_1(t) + k_d x_2(t), \]

(5)

Therefore, system (3) can translate into a \( n \)-dimensional commensurate-order system.

We linearize the controlled system at the equilibrium point and can achieve the associated characteristic equation of system as follows:

\[ s - k_d s^\frac{1}{n} - (k_p - cke^{-s\tau}) = 0. \]

(6)
2.1. The case of a fractional-order PD controller \((n \geq 2)\) without delays \((\tau = 0)\)

When \(n \geq 2\) and \(\tau = 0\), (6) can be switched to

\[
s - k_d s^n - k_p + c k = 0. \tag{7}
\]

**Lemma 1** If \(k_d < 0\) and \(k_p < c k\), the roots of (7) are all in the left half of the complex plane.

**Proof 1** Let \(s = A e^{\theta i} = A (\cos \theta + i \sin \theta)\), \(A\) is the modulus and \(\theta\) argument of \(s\) on the complex plane. Therefore, (7) can be separated the real and imaginary parts. Obviously, if \(k_d < 0\), \(k_p < c k\) and \(\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\), (7) has no real positive roots. It means that all roots of (7) are on the left part of the complex plane. The proof is complete.

**Theorem 1** Assume that \(n \geq 2\), \(k_p < c k\) and \(k_d < 0\) hold, system (3) without delay is asymptotically stable at the equilibrium point \(x^* = \frac{1}{c}\).

2.2. The case of a fractional-order PD controller \((n \geq 2)\) with delays \((\tau > 0)\)

When \(n > 2\) and \(\tau > 0\), letting \(s = i \omega\) be a purely imaginary root of (6). Separating the real and imaginary parts and adding the squares of two parts, we can obtain

\[
(ck)^2 = k_p^2 + 2k_p k_d \omega^n \cos \frac{\pi}{2n} + \omega^2 - 2k_d \omega^{n+1} \sin \frac{\pi}{2n} + (k_d \omega^n)^2. \tag{8}
\]

Let \(\omega^n = z\), then \(\omega = z^n\). Therefore, (8) can be formulated as

\[
z^{2n} - 2k_d \sin \frac{\pi}{2n} z^{n+1} + k_d^2 z^2 + 2k_p k_d \cos \frac{\pi}{2n} z + k_p^2 - (ck)^2 = 0. \tag{9}
\]

**Lemma 2** When \(-c k < k_p < c k\), \(k_d < 0\) hold, (9) has real positive roots. When \(k_p < -c k\), \(k_d < 0\), (9) has not a purely imaginary root for any \(\tau\).

**Proof 2** We define a function \(H(z)\) as follows

\[
H(z) = z^{2n} - 2k_d \sin \frac{\pi}{2n} z^{n+1} + k_d^2 z^2 + 2k_p k_d \cos \frac{\pi}{2n} z + k_p^2 - (ck)^2. \tag{10}
\]

When \(k_d < 0\) and \(k_p < 0\), then \(H'(z) > 0\) is permanent for \(z \in [0, +\infty)\), which means that \(H(z)\) is a monotone increasing function.

(1) When \(k_p < -c k\), \(k_d < 0\), then \(k_p^2 - (ck)^2 > 0\). Therefore, \(H(0) > 0\). Moreover, (9) has no real positive roots.

(2) When \(-c k < k_p < 0\), \(k_d < 0\), then \(k_p^2 - (ck)^2 < 0\). Therefore, \(H(0) < 0\) and the \(H(z)\) has a unique zero \(z_0\). Moreover, (9) has real positive roots.

When \(k_d < 0\) and \(0 \leq k_p < c k\), then \(H(0) < 0\) and the \(H(z)\) has also a unique zero \(z_0\). This ends the proof.

Therefore, we can derive

\[
\omega_{\theta_0} = z_0^n, \tau_{\theta_0} = \frac{1}{\omega_{\theta_0}} \arccos \frac{k_p + k_d \omega_{\theta_0}^{n+1} \cos \frac{\pi}{2n}}{ck}. \tag{11}
\]

**Lemma 3** Let \(s(\tau) = \kappa(\tau) + i \omega(\tau)\) be the root of (6) near \(\tau = \tau_{\theta_0}\) satisfying \(\kappa(\tau_{\theta_0}) = 0\), \(\omega(\tau_{\theta_0}) = \omega_{\theta_0}\). Then the following transversality condition of (6) holds:

\[
\text{Re} \left[ \frac{ds}{d\tau} \right]_{\tau = \tau_{\theta_0}} > 0. \tag{12}
\]
Proof 3 Differentiating (6) on both sides with respect of \( \tau \), we can deduce
\[
Re \left[ \frac{dr}{ds} \right] = Re \left[ \frac{1 - \frac{k_d}{n} \frac{n}{\tau} - 1}{cke^{-\alpha \tau} s} \right] = \frac{g(\omega_0)}{(ck)^\omega_0},
\]
where
\[
g(\omega) = \omega - k_d \left( \frac{1}{n} + 1 \right) \sin \frac{\pi}{2n} \omega_n + k_p \frac{k_d}{n} \cos \frac{\pi}{2n} \omega_n - \frac{n}{\omega_n} - 1 + \frac{k_d^2}{n} \omega_n^2 - 1.
\]
According to (8), we can suppose
\[
G(\omega) = k_p^2 + 2k_pk_d \omega_n \cos \frac{\pi}{2n} \omega_n + \omega_n^2 - 2k_d \omega_n^{\frac{1}{n} + 1} \sin \frac{\pi}{2n} \omega_n + (k_d \omega_n^{\frac{1}{n}})^2 - (ck)^2.
\]
Differentiating (15) on both sides with respect of \( \omega \), we can achieve
\[
G'(\omega) = 2g(\omega).
\]
\( G'(\omega) > 0 \) is permanent based on the condition of \(-ck < k_p < ck \) and \( k_d < 0 \). Therefore,
\[
Re \left[ \frac{dr}{ds} \right]_{s=\omega_0}^{\tau=\tau_0} > 0. \text{ Hence, } Re \left[ \frac{ds}{dr} \right]_{s=\omega_0}^{\tau=\tau_0} > 0. \text{ This ends the proof.}
\]

Theorem 2 Assume that \( n \geq 2 \), \(-ck < k_p < ck \) and \( k_d < 0 \) hold, system (3) is asymptotically stable at the equilibrium point \( x^* = \frac{1}{c} \) for all \( \tau \in [0, \tau_0] \) and undergoes a Hopf bifurcation at the equilibrium \( x^* = \frac{1}{c} \) when \( \tau = \tau_0 \). Suppose that \( n \geq 2 \), \( k_p < -ck \) and \( k_d < 0 \) are met, system (3) is asymptotically stable at the equilibrium point \( x^* = \frac{1}{c} \) for all \( \tau \geq 0 \).

3. Numerical Simulation and Analysis
In this section, we illustrate the validity and practicability of theoretical analyses which are obtained in the previous section by numerical simulations and discuss the influence of the control gain parameters of PD controller on system stability in this paper. For the sake of comparison, we consider system (1) with the same parameters \( k = 0.01 \) and \( c = 50 \) which are used in [4]. Therefore, system (1) has the unique equilibrium \( x^* = \frac{1}{c} = 0.02 \).

3.1. The case of without control
We first consider the original congestion control system (1) and the time delay \( \tau \) is chosen as the bifurcation parameter. By (6), it can be obtained \( \tau_{00} = \frac{\pi}{2c} = \pi \) without control \( (k_p = k_d = 0) \). System (1) is asymptotically stable at the equilibrium point \( x^* \) when \( \tau \in [0, \tau_{00}) \) and undergoes a Hopf bifurcation when \( \tau > \tau_{00} \). These properties are shown in Figures. 1 and 2.

3.2. The case of a fractional-order PD controller \((n \geq 2)\)
For advancing the bifurcation of system (3), we take the \( k_p = 0.25 \), \( k_d = -0.01 \), \( n = 3 \) and obtain \( \tau_{00} = 3.08 \) by Lemma 2.2. It can be seen from Theorem 2.2 that system (3) is asymptotically stable at the equilibrium point \( x^* \) for \( \tau \in [0, \tau_{00}) \) and undergoes a Hopf bifurcation at the equilibrium \( x^* \) when \( \tau > \tau_{00} \). Figure 3 displays the stable \( x^* \) of system (3) when \( \tau = 2.45 < \tau_{00} = 3.08 \) and a periodic oscillation occurs through a Hopf bifurcation when \( \tau = 3.10 > \tau_{00} = 3.08 \) as shown in Figure 4. Note that \( \tau_{00} = 3.08 < \tau_{00} = \pi \).

We make \( k_p = -0.15 \), \( k_d = -0.05 \) and \( n = 3 \) in order to delay the Hopf bifurcation. By Lemma 2.2, we have \( \tau_{00} = 4.0 \). According to Theorem 2.2, system (3) is asymptotically stable at the equilibrium point \( x^* \) for all \( \tau \in [0, \tau_{00}) \) and undergoes a Hopf bifurcation at the equilibrium \( x^* \) when \( \tau \geq \tau_{00} \). The numerical simulations for different values of the delay are graphically displayed in Figures 5 and 6.
Figure 1. Waveform plot of system (1) with \( k = 0.01, c = 50, k_p = 0 \) and \( k_d = 0 \). System (1) is asymptotically stable when \( \tau = 2.80 < \tau_0 = \pi \).

Figure 2. Waveform plot of system (1) with \( k = 0.01, c = 50, k_p = 0 \) and \( k_d = 0 \). A periodic oscillation bifurcates when \( \tau = 3.20 > \tau_0 = \pi \).

Figure 3. Waveform plot of system (3) with \( n = 3, k = 0.01, c = 50, k_p = 0.25 \) and \( k_d = -0.01 \). System is asymptotically stable for when \( \tau = 2.45 < \tau_0 = 3.08 \).

Figure 4. Waveform plot of system (3) with \( n = 3, k = 0.01, c = 50, k_p = 0.25 \) and \( k_d = -0.01 \). A periodic oscillation bifurcates when \( \tau = 3.10 > \tau_0 = 3.02 \).

Figure 5. Waveform plot of system (3) with \( n = 3, k = 0.01, c = 50, k_p = -0.15 \) and \( k_d = -0.05 \). System is asymptotically stable for when \( \tau = 3.80 < \tau_0 = 4.0 \).

Figure 6. Waveform plot of system (3) with \( n = 3, k = 0.01, c = 50, k_p = -0.15 \) and \( k_d = -0.05 \). A periodic oscillation bifurcates when \( \tau = 4.20 > \tau_0 = 4.0 \).

3.3. The influence of \( k_p, k_d \) and \( n \) on the critical value \( \tau_0 \)

Figure 7 describes the relationship of \( k_p \) and \( \tau_0 \) with \( k_d = -0.1 \). We conclude that the values of \( \tau_0 \) is declining with the rise of \( k_p \). \( k_d \) and \( \tau_0 \) have a similar relationship shown in Figure 8. According to the above analysis, we discover that \( n \) also have a great influence on the change of the bifurcation point \( \tau_0 \). Table 1 displays the influence of \( n \) on the bifurcation point \( \tau_0 \) when \( k_p = 0.25, k_d = -0.1, k = 0.01, \) and \( c = 50 \). Regardless of the case of integer-order \( (n = 1) \), we draw a conclusion that the bifurcation point \( \tau_0 \) monotonically declines with the rise of \( n \) \((n \geq 2)\).

We can draw a conclusion from Table 1 that the controlled system is difficult to maintain stable in a larger total delay, and are more likely to have a Hopf bifurcation when the total delay \( \tau \) varies.
Figure 7. The relationship between $k_p$ and $\tau_0$ with $k_d = -0.1$.

Figure 8. The relationship between $k_d$ and $\tau_0$ with $k_p = -0.2$.

Table 1. Influence of $n$ on the value of the bifurcation point $\tau_0$ when $k_p = 0.25$, $k_d = -0.1$.

| Fractional-order parameter $n$ | Bifurcation point $\tau_0$ |
|-------------------------------|-----------------------------|
| 1                             | 3.3484                      |
| 2                             | 3.4337                      |
| 3                             | 3.3983                      |
| 4                             | 3.3635                      |
| 5                             | 3.3369                      |
| 6                             | 3.3167                      |
| 7                             | 3.3011                      |

4. Conclusion
In the beginning, in order to better control the Hopf bifurcation, we consider the advantages of the fractional-order PD controller. Furthermore, through a series of mathematical substitutions, we have expanded the one-dimensional controlled network with different orders to $n$-dimensional one with the same order. Meanwhile, the dynamic behavior of the controlled network is obtained by analyzing the stability problem of the $n$-dimensional network with same order. Finally, it is found that the Hopf bifurcation point of the controlled system can be effectively delayed and advanced. Despite the limitations of Caputo derivatives, recent studies have shown that it still plays a vital role in many physical systems.

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