Linear Convergence of Adaptive Stochastic Gradient Descent

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Abstract

We prove that the norm version of the adaptive stochastic gradient method (AdaGrad-Norm) achieves a linear convergence rate for a subset of either strongly convex functions or non-convex functions that satisfy the Polyak-Łojasiewicz (PL) inequality. The paper introduces the notion of Restricted Uniform Inequality of Gradients (RUIG), which describes the uniform lower bound for the norm of the stochastic gradients with respect to the distance to the optimal solution. RUIG plays the key role in proving the robustness of AdaGrad-Norm to its hyper-parameter tuning. On top of RUIG, we develop a novel two-stage framework to prove linear convergence of AdaGrad-Norm without knowing the parameters of the objective functions:

Stage I: the step-size decrease fast such that it reaches to Stage II;
Stage II: the step-size decreases slowly and converges.

This framework can likely be extended to other adaptive stepsize algorithms. The numerical experiments show desirable agreement with our theories.

1 Introduction

Consider the optimization problem of the population risk over the space $\mathbb{R}^d$:

$$\min_{x \in \mathbb{R}^d} F(x) := \mathbb{E}_{z \sim S} [f(x; z)]$$

In practice, for finite dataset with size $n$, we consider a model aiming to minimize the empirical risk $F(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)$, where $f_i(x) = f(x; z_i) : \mathbb{R}^d \to \mathbb{R}$, $i = 1, 2, \ldots$. Here, $F$ satisfies smoothness with either (strong) convexity or by satisfying the Polyak-Łojasiewicz inequality [1, 2]. Functions with these characteristics are fundamental to a variety of machine learning problems [3, 4]. Polynomial convergence results have been established for this class of functions using stochastic gradient descent (SGD) methods or using accelerated SGD [5, 6, 7, 8, 9, 10].

Rather than searching for an algorithm with a faster convergence rate like accelerated SGD, this paper focuses on how hyperparameter choices affect the robustness of the convergence for SGD. For a strongly convex function where each i.i.d. component is chosen uniformly at each iteration, fixed step-size SGD guarantees exponential convergence up to a radius around the optimal solution [11, 12]. Improved algorithms—like SAG [13], SVRG [14] and SAGA [15]—strengthen this exponential convergence to the global minimizer. However, the above convergence requires that fixed stepsizes must meet a certain threshold determined by unknown parameters such as the level of stochastic noise, Lipschitz smoothness constants, and strongly convex parameters. Hence, SGD and variance reduced SGD are highly sensitive to step-size tuning in practice. Thus, seeking an algorithm that is robust to the choice of hyper-parameters is as crucial as designing an algorithm that gives faster convergence.

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Adaptive gradient methods introduced in \[16,17\] update the step-size on the fly: They either adapt a vector of per-coefficient step sizes \[13,19,20,21,22,23\] or a single stepsize depending on the norm of the gradient \[24,25,26\]. The latter one, AdaGrad-Norm \[27\], has the following updates:

\[
\begin{align*}
 b_{j+1}^2 &= b_j^2 + \|\nabla f_{\xi_j}(x_j)\|^2; \\
 x_{j+1} &= x_j - \frac{\eta}{b_{j+1}} \nabla f_{\xi_j}(x_j); \\
 j &= 0, 1, 2, \ldots
\end{align*}
\]

where \( \xi_j \sim \text{Uniform}\{1, 2, \ldots\} \) such that \( \mathbb{E}_{\xi_j}\left[\|\nabla f_{\xi_j}(x)\|\right] = \|\nabla F(x)\| \). AdaGrad-Norm has been shown to be extremely resilient to the functions’ parameters being unknown \[24,28,27\]. In addition to this robustness, AdaGrad-Norm enjoys \( \mathcal{O}(1/\sqrt{T}) \) convergence rate for smooth non-convex functions under the metric \( \min_{j \in [T]} \|\nabla F(x_j)\|^2 \) \[27,29\]. This asymptotic convergence rate has also been proved for general convex functions \[28\]. A linear convergence rate \( \mathcal{O}\left(\exp(-\kappa T)\right) \) is possible for strongly convex smooth functions using variants of AdaGrad-Norm in which the final update uses harmonic sum of the queried gradients \[24\]. Yet, the analysis in \[24,28\] requires a priori information: a convex set with a known diameter in which the global minimizer resides. The analysis in \[27\] considers the smooth function under an assumption of a bounded stochastic gradient norm that rules out the strongly convex cases, while \[29\] only assumes bounded variance but requires prior knowledge of smoothness. Therefore, obtaining a robust convergence guarantee without prior knowledge of a convex set or the smoothness parameters, remains an open question for AdaGrad-Norm with strongly convex objectives.

In this paper, we obtain convergence guarantees for strongly convex functions without requiring knowledge of a convex set, and extend this convergence to non-convex function classes satisfying the Polyak-Łojasiewicz (PL) inequality. Surprisingly, we prove that the AdaGrad-Norm algorithm converges quickly and can achieve a linear rate if there exists no noise in the solution (cf. Assumption (A4)). Our analysis does not follow the standard analysis—which assumes the bounded variance \( \bar{\sigma} \) for \( \mathbb{E}_{\xi_j}\left[\|\nabla f_{\xi_j}(x_j) - \nabla F(x_j)\|^2\right] \leq \sigma^2 \) in \[28,24,27,29\]—and avoids likely sub-linear convergence results. Although Assumption (A4) limits the application of our theories, it is weaker than the (Strong or Weak) Growth Condition in \[35,36\]. We are the first to use this weaker assumption and prove a better convergence rate of AdaGrad-Norm for many optimization problems that satisfy Assumption (A4) (including certain classes of neural networks). Our contributions are not only significant for the two-stage framework for the linear convergence proof, we believe it is easily applicable to other applicable adaptive algorithms such as ADAM \[18\] and AMSGrad \[20\].

1.1 Main Contributions

In this paper, we propose Restricted Uniform Inequality of Gradients (RUG) to measure the uniform lower bound of gradients according to \( \|x - x^\ast\|^2 \) in a restricted region. We show that the evolution of the error can be divided into the following two stages:

- **Stage I** if \( b_0 < \mu \leq L \), \( \|x_t - x^\ast\|^2 \) increases first and decreases after \( b_0 \geq \mu, b_t \) continues growing to exceed \( L \);

- **Stage II** \( b_t > L \), AdaGrad-Norm converges linearly and \( b_t \leq b_{\text{max}} \).

We graphically illustrates in Figure 1.

We prove the non-asymptotic linear convergence of AdaGrad-Norm in the strongly convex setting for stochastic and batch updates; furthermore, we also extend our results for non-convex functions satisfying the PL inequality. Our main results are as follows (informal):

1. In the stochastic setting assuming \( \mu \)-strong convexity and \( L \)-smoothness with \( (\epsilon, \alpha, \gamma) \)-RUG: for fixed \( x \), if \( \|x - x^\ast\|^2 > \epsilon, \forall \xi_j, \exists(\alpha, \gamma) \) s.t. \( \mathbb{P}_{\xi_j}\left[\|\nabla f_{\xi_j}(x)\|^2 \geq \alpha \|x - x^\ast\|^2\right] \geq \gamma \), Theorem 1 shows that AdaGrad-Norm attains \( \min_i \|x_i - x^\ast\|^2 \leq \epsilon \) with high probability after \( T = \mathcal{O}(\log \frac{1}{\epsilon}) \) iterations for \( b_0 > \eta L \); and after \( T = \mathcal{O}(\frac{1}{\epsilon} + \log \frac{1}{\epsilon}) \) iterations for \( b_0 \leq \eta L \).

\( \kappa \) is the condition number

\( ^3 \) We note that our robustness results for AdaGrad-Norm with respect to hyper-parameter choice are for the norm version of AdaGrad. This differs from the convergence guarantees for the diagonal version of AdaGrad and its variants (with momentum) \[30,31,32,33,34\].

![Figure 1: Two-Stage Convergence of AdaGrad-Norm](image)
2. In the batch setting, by using the full gradient, the above probability γ degrades to 1 and α = µ. Theorem 2 shows that \( \min_i \| x_i - x^* \|^2 \leq \epsilon \) after \( T = \mathcal{O}(\log \frac{1}{\epsilon}) \) iterations for \( b_0 > \eta \frac{L}{\Delta} \) and after \( T = \mathcal{O}(\frac{1}{\log(1+\Delta/\epsilon)}) + \log \frac{1}{\epsilon} \) iterations for \( b_0 \leq \eta \frac{L}{\Delta} \).

3. For non-convex functions with PL inequality, we alternatively consider the convergence of \( \min_i F(x_i) - F^* \). Theorem 3 illustrates that \( \min_i F(x_i) - F^* \leq \epsilon \) after \( T = \mathcal{O}(\log \frac{1}{\epsilon}) \) iterations for \( b_0 > \eta L \); and after \( T = \mathcal{O}(\frac{1}{\log(1+\Delta/\epsilon)}) + \log \frac{1}{\epsilon} \) iterations for \( b_0 \leq \eta L \).

We show that the convergence is robust starting from any initialization of \( b_0 \), without knowing the Lipschitz constant or strongly convex parameter a priori. We believe this framework can easily be adapted to other adaptive stepsize algorithms due to its generality.

| Setting          | Algorithm | Initial stepsize | Steps to achieve \( \| x_T - x^* \|^2 \leq \epsilon \) |
|------------------|-----------|------------------|--------------------------------------------------|
| Stochastic GD    | fixed stepsize [12] | \( \eta_0 = \frac{1}{b_0 \sup_i L_i} \) | \( \mathcal{O}(\frac{L}{\mu} \log \frac{\Delta_0}{\epsilon}) \) |
|                  | AdaGrad-Norm | \( \eta_0 = \frac{1}{b_0} < \frac{1}{\sup_i L_i} \) | \( \mathcal{O}(\frac{\sup_i L_i \Delta_0}{\mu} \log \frac{\Delta_0}{\epsilon}) \) |
|                  | AdaGrad-Norm | arbitrary         | \( \mathcal{O}(\frac{1}{\epsilon} + \frac{\sup_i L_i \Delta_0}{\mu} \log \frac{\Delta_0}{\epsilon}) \) |
| Deterministic GD | fixed stepsize [10] | \( \eta_0 = \frac{2}{\mu + L} \) | \( \mathcal{O}(\frac{\mu + L}{4\mu L} \log \frac{\Delta_0}{\epsilon}) \) |
|                  | AdaGrad-Norm | \( \eta_0 = \frac{1}{b_0} < \frac{2}{\mu + L} \) | \( \mathcal{O}(\frac{\Delta_0}{\mu} \log \frac{\Delta_0}{\epsilon}) \) |
|                  | AdaGrad-Norm | arbitrary         | \( \mathcal{O}(\frac{1}{\log(1+\Delta/\epsilon)} + \frac{\Delta_0}{\mu} \log \frac{\Delta_0}{\epsilon}) \) |

\( \Delta_0 = \| x_0 - x^* \|^2 \) is the initial distance to the minimizer \( x^* \).

As it shows in Table 1 when starting at any initial stepsize, the convergence of AdaGrad-Norm is changed with the slope of the linear convergence in Stage II, with only negligible gain from the add-on sublinear part in Stage I. However, changing the initial stepsize for SGD causes the error to blow up. Theorem 1 establishes this robustness for AdaGrad-Norm.

**Paper Organization** This paper is organized as follows. Section ?? presents the setup and assumptions for the problem. Section ?? introduces the interpretation and examples of RUIG, and the convergence under the RUIG. Section 2 shows the main linear convergence results of AdaGrad-Norm in both the stochastic and batch strongly convex settings and the non-convex setting where the function satisfies the PL inequality. Section 3 outlines the two-stage framework of AdaGrad-Norm in the strongly convex setting. Section ?? demonstrates the experimental simulations, and Section ?? concludes the paper. The proofs of all lemmas and theorems are provided in the Appendix.

**Notation** \( \| \cdot \| \) denotes the \( \ell_2 \)-norm. In the batch setting, \( L > 0 \) is the smallest Lipschitz constant of \( \nabla F(x) \); in the stochastic setting, \( L \triangleq \sup_i L_i \), where \( L_i \) is the Lipschitz constant of \( \nabla f_i(x) \). \( P_i(\cdot) \) is the probability with respect to the \( i \)th sample point.

## 2 Problem Setup

### 2.1 AdaGrad-Norm

Consider the empirical risk \( F(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x) \), where \( f_i(x) = f(x, z_i) : \mathbb{R}^d \to \mathbb{R}, i = 1, 2, \ldots, n \) with possibly infinite \( n \). In contrast to (Stochastic) Gradient Descent implemented with fixed stepsize, we use two equivalent update rules that dynamically incorporate the information from previous gradients into the reciprocal of the learning rates:

**Square Form:** \( b_{j+1} = \sqrt{b_j^2 + \| \nabla f_{\xi_j}(x_j) \|^2} \)  
**Solution Form:** \( b_{j+1} = b_j + \frac{\| \nabla f_{\xi_j}(x_j) \|^2}{b_j + b_{j+1}} \)

In the stochastic setting, the update rule is described in Algorithm 1. In the batch setting, we can obtain the full gradient–in which case we use \( \nabla F(x_j) \) of \( F(x_j) \) instead of the unbiased estimator.
The algorithm follows the standard assumptions from [4]: for each \( j \geq 0 \), the random vectors \( \xi_j \), \( j = 0, 1, 2, \ldots \), are mutually independent, independent of \( x_j \), and satisfy \( \mathbb{E}_{\xi_j} [f_{\xi_j}(x_j)] = \nabla F(x_j) \).

Algorithm 1 AdaGrad-Norm

**Input:** Initialize \( \epsilon > 0 \), \( \eta > 0 \), \( T > 0 \), \( x_0 \in \mathbb{R}^d \), \( b_0 \in \mathbb{R} \), \( j \leftarrow 0 \)

while \( j < T \) do

Generate random variable \( \xi_j \) and \( G_j = \nabla f_{\xi_j}(x_j) \) (Batch: \( G_j = \nabla F(x_j) \))

\[
\begin{align*}
    b_{j+1}^2 &\leftarrow b_j^2 + \|G_j\|^2 \\
    x_{j+1} &\leftarrow x_j - \frac{\eta}{b_{j+1}} G_j \\
    j &\leftarrow j + 1
\end{align*}
\]

2.2 Assumptions

Throughout the paper, we use the following assumptions individually or in combination to analyze the convergence rates in both the stochastic and batch settings.

(A1) **convex:** \( F(x) \) is convex and differentiable: \( \langle \nabla F(x) - \nabla F(y), x - y \rangle \geq 0, \forall x, y \).

(A2) \( \mu \)-**strongly convex:** \( \langle \nabla F(x) - \nabla F(y), x - y \rangle \geq \mu \|x - y\|^2, \forall x, y \).

(A3) **L-smooth:** \( f_i(x) \) is \( L_i \)-smooth, \( \forall i \): \( \|\nabla f_i(x) - \nabla f_i(y)\| \leq L_i \|x - y\|, \forall x, y \). Let \( L \triangleq \sup_i L_i, F(x) \) and \( \{ f_i(x), \forall i \} \) are all \( L \)-smooth.

(A4) **almost stationary** [12, 37]: Let \( x^* = \arg\min_x F(x) \), the minimizer of \( F(x) \), then \( \mathbb{P}_i(\|f_i(x^*) = 0\| = 1, \forall i) \). That is, the probability of \( x^* \) being a stationary point is almost surely over all sample data points.

(A5) \( \mu \)-**Polyak-Łojasiewicz (PL):** \( \|\nabla F(x)\|^2 \geq 2\mu(F(x) - F(x^*)) \), \( \forall x \).

(A6) **(\( \epsilon, \alpha, \gamma \))-Restricted Uniform Inequality of Gradients (RIUG):** \( \forall \epsilon > 0, \exists (\alpha, \gamma) \ s.t. \alpha > 0, \gamma > 0, \) and for fixed \( x \in \mathcal{D}_\epsilon \triangleq \{ x \in \mathbb{R}^d : \|x - x^*\|^2 > \epsilon \}, \mathbb{P}_i(\|\nabla f_i(x)\| \geq \alpha\|x - x^*\|^2) \geq \gamma, i = 1, 2, \ldots \)

Assumption (A6) is a sufficient condition to guarantee the linear convergence for AdaGrad-Norm with any initialization of stepsize, but it is not necessary when the initial stepsize is smaller than the unknown critical values, i.e. \( 1/L \) or \( 2/(\mu + L) \).

Assumption (A4) is the key condition for linear convergence of \( \|x - x^*\|^2 \) in the stochastic approximation algorithms [35, 36] as it imposes a very strong condition on each component function at the point \( x^* \). However, this assumption is much weaker than (Strong or Weak) Growth Condition in [35, 36] where it assumes that for all \( x \in \mathbb{R}^d \), \( \max_i \|\nabla f_i(x)\|^2 \leq B\|\nabla F(x)\|^2 \) or \( \mathbb{E}_i(\|\nabla f_i(x)\|^2 < B(\|F(x) - F^*\|) \) (for some constant \( B \)). We are the first to use this weaker assumption and characterize a better convergence rate of AdaGrad-Norm for many optimization problems that satisfy Assumption (A4). One particularly relevant application is the over-parameterized neural network [36, 39, 40, 41, 42]. It is important to note that Assumption (A4) implies that there exists almost no noise at the solution, which may not be appropriate for certain applications.

3 Restricted Uniform Inequality of Gradients

In this section, we concretely explain our assumption (A6) in Section ??, and we restate as follows:

**Assumption.** **(\( \epsilon, \alpha, \gamma \))-Restricted Uniform Inequality of Gradients (RIUG):** \( \forall \epsilon > 0, \exists (\alpha, \gamma) \ s.t. \alpha > 0, \gamma > 0, \) and for fixed \( x \in \mathcal{D}_\epsilon \triangleq \{ x \in \mathbb{R}^d : \|x - x^*\|^2 > \epsilon \}, \mathbb{P}_i(\|\nabla f_i(x)\|^2 \geq \alpha\|x - x^*\|^2) \geq \gamma, i = 1, 2, \ldots \)

The RIUG gives a lower bound of the probability, \( \gamma \), for which the norm of any unbiased gradient estimator \( \|\nabla f_i(x)\| \) is larger than the distance between \( x \) and \( x^* \) by a constant factor \( \alpha \), if \( x \) is in a restricted region \( \mathcal{D}_\epsilon \). This inequality preserves a flat landscape around \( x^* \) for each component loss function \( f_i(x) \) and characterizes the relatively sharper curvature beyond the region.
The constant tuple \((\epsilon, \alpha, \gamma)\) is determined by the distribution of the dataset. In general, \(\alpha\) and \(\gamma\) are negatively correlated, i.e., \(\alpha \to 0, \gamma \to 1\). The error \(\epsilon\) could be independent of \(\alpha\) and \(\gamma\). However, for large \(\epsilon\), the product \(\alpha\gamma\) is more likely far away from zero. If \(\epsilon_2 \geq \epsilon_1 \geq 0\), then \(D_0 = D_{\epsilon_1} \subseteq D_{\epsilon_0} = \mathbb{R}^d\).

We provide some examples where we can directly compute the lower bounds of \(\gamma\) and \(\alpha\) for the restricted region \(D_e\). Note that these bounds depend on the data set \(\{a_i\}_{i=1}^\infty\), hence they are data dependent.

**Example 1. Least Square Problem**

Suppose that \(F(x) = \frac{1}{2n}\|Ax - y\|^2_2 = \frac{1}{n}\sum_{i=1}^n \frac{1}{2}(\langle a_i, x \rangle - y_i)^2\), where \(y = Ax^*\), and any data point \(a_i\) consists of \(d\) features. Consider AdaGrad-Norm following two examples, \(\gamma \to \infty\). As long as \(\epsilon 2 \geq \epsilon 1 \geq 0\), then \(D_e \subseteq D_{\epsilon 1} \subseteq D_{\epsilon 0} = \mathbb{R}^d\).

Let \(x \triangleq \frac{x - x^*}{\|x - x^*\|}\) and \(Y \triangleq \langle a_i, \bar{x} \rangle = \sum_j A_{ij}\bar{x}_j\). Using the fact that a linear combination of independent normal distributions is \(\mathcal{N}(\sum_j c_j\mu_j, \sum_j c_j^2\sigma_j^2)\), then \(Y \sim \mathcal{N}(0, \|\bar{x}\|^2)\), i.e. \(Y \sim \mathcal{N}(0, 1)\) and \(Y^2 \sim \chi^2(d)\). For example, from the distribution table of \(\chi^2(1)\),

\[
P_x\left(\|\nabla f_i(x)\|^2 \geq 0.45 \min_j \|a_j\|^2 \|x - x^*\|^2\right) \geq 0.5, \forall x, \forall i = 1, 2, ..., n.
\]

In the above case, \(\alpha \geq 0.45 \min_j \|a_j\|^2\) and \(\gamma \geq 0.5\) in RUIG, where \(\|a_j\|^2 \sim \chi^2(d)\). Therefore, \(\mathbb{P}_i(\|a_j\|^2 \geq (1 - t)d) \geq 1 - e^{-dt^2}/8\). Generally \(\alpha\) is not small—especially when the data is fairly dense. From the chi-squared distribution, other possible tuples \((\alpha, \gamma)\) in RUIG could be \((0.015, 0.9), (0.1, 0.75), (1.3, 0.25), (2.7, 0.1)\). The inequality \(\forall x_i \in \mathbb{R}^d, D_e\) is extended to \(D_0\).

**Example 2. \(\mu\) – Strongly Convex Function**

(a) Consider \(\{f_i(x)\}\) are \(\mu\) – strongly convex functions and \(x^* = \arg \min_x F(x)\) is the minimizer for each \(f_i(x)\), i.e. \(\nabla f_i(x^*) = 0\). By strong convexity, \(\|f_i(x)\|^2 \geq \mu^2 \|x - x^*\|^2\), \(\forall x, \forall i = 1, 2, ..., n\). In this case, the uniform probability \(\gamma\) degenerates to 1, \(\alpha = \mu^2\), and the data is not restricted to \(D_0\).

(b) A more general function class: \(f_i(x) \in \mathcal{H}_1 \cup \mathcal{H}_2\), where \(\mathcal{H}_1 := \{g(x) : g(x)\) is \(\mu\) – strongly convex\} and \(\mathcal{H}_2 := \{h(x) : h(x)\) is non-strongly convex\}. \(f_i(x)\) draws from \(\mathcal{H}_1\) with probability \(\gamma\) and from \(\mathcal{H}_2\) with probability \(1 - \gamma\), where \(0 < \gamma < 1\).

**Convergence Under RUIG Assumption**

Under RUIG assumption, the recurrence of the step-size \((b_j)\) in AdaGrad-Norm increases quickly with high probability in stage I.

**Lemma 1. (Two-case high-probability lower bound for \(b_N\) in the stochastic setting)** For Algorithm 1, \(\forall \epsilon\), suppose \(F(x)\) satisfies \((\epsilon, \alpha, \gamma)\) – RUIG. Then \(\forall \) fixed \(C\), after \(N = \lceil \frac{C^2 - k^2}{\alpha \gamma \epsilon^2} + \frac{3}{\gamma} \rceil + 1\) steps and with high probability given by \(1 - \exp(-\frac{\epsilon^2}{2N(1 - \gamma) + \delta_1})\), either \(b_N > C\) or \(\min_j \|x_j - x^*\|^2 \leq \epsilon\).

Let \(\delta_1 = \exp(-\frac{\epsilon^2}{2N(1 - \gamma) + \delta_1})\); the high probability \(1 - \delta_1\) is derived from Bernstein Inequality of the Bernoulli distribution. When \(\gamma N/\log N \to \infty\), let \(\delta = \sqrt{4c(1 - \gamma)N \log N}\), then \(\delta_1 \leq N^{-c}\).

On the other hand, if \(\gamma N \sim \log N\), let \(\delta \sim (\log N)^{4.5}\), then \(\delta_1 \sim \exp(-\epsilon^2 \log N \delta_1^{4.5}) \to 0\) as \(N \to \infty\). As long as \(\gamma \gg (\log N)^{4.5}/N\), the number of steps \(\frac{\delta}{\gamma} \ll N\) in Lemma 1. In the following two examples, \(\gamma\) can be chosen to be at least 0.5.  

4 Linear Convergence of AdaGrad-Norm

Throughout this section, we mainly focus on AdaGrad-Norm in the stochastic setting, i.e. Algorithm 1. The linear convergence rate is given in Theorem 1.

**Theorem 1. (AdaGrad-Norm: strongly convex and stochastic setting)** Consider AdaGrad-Norm Algorithm in the stochastic setting, suppose that \(F(x)\) is strongly convex and smooth and \(x^* = \arg \min_x F(x)\) with Assumptions (A1), (A2), (A3), (A4), (A6). Then

**Case 1:** If \(b_0 > \eta L\), \(\|x_T - x^*\|^2 \leq \epsilon\) with high probability \(1 - \delta_h\) for

\[
T = \left\lceil \frac{b_0 + L/\eta \|x_0 - x^*\|^2}{\mu} \log \frac{\|x_0 - x^*\|^2}{\epsilon b_h} \right\rceil + 1;
\]
**Case 2:** If \( b_0 \leq \eta L \), \( \min_i \|x_i - x^*\|^2 \leq \epsilon \) with high probability \( 1 - \delta_h = \exp(-\frac{\delta^2}{2(N \gamma (1 - \gamma) + \delta)}) \) for

\[
T = \left[ \frac{\eta^2 L^2 - b_0^2}{\alpha \gamma} + \frac{\delta}{\gamma} + \frac{L(\eta + D^2/\eta)}{\mu} \log \frac{D^2}{e \delta_h} \right] + 1
\]

where \( D^2 = \|x_0 - x^*\|^2 + \eta^2 (\log(\frac{\eta^2 L^2}{b_0^2}) + 1) \), \( N = \left[ \frac{\eta^2 L^2 - b_0^2}{\alpha \gamma} + \frac{\delta}{\gamma} \right] \).

Our theorem establishes not only the robustness of hyper-parameters of the AdaGrad-Norm algorithm but also, more importantly, the strong linear convergence in the stochastic setting. To put the theorem in context, we compare with the sub-linear convergence rate of AdaGrad-Norm (i.e., \( T = O \left( \frac{1}{\epsilon^2} \right) \)) in \([28, 24, 27, 29]\). The key breakthrough in our theorem is that instead of following the standard analysis on SGD where it often assumes \( E \xi_i [\|\nabla f_j(x) - \nabla F(x)\|^2] \leq \delta^2 \), we use novel methods in high dimensional probability (c.f. RUIG) and utilize the nice landscape property at the solution (c.f. Assumption (A4)). Even though Assumption (A4) limits the application for the class of functions where there exists big noise at the solution, we uncovered the advantage of AdaGrad-Norm and increase its impact over many optimization problems, particularly for those classes of functions with almost no noise at the solution.

In Stage I, the high probability \( \delta_1 \equiv \exp(-\frac{\delta^2}{2(N \gamma (1 - \gamma) + \delta)}) \) is guaranteed in the interpretation of high probability in Lemma[1]. In Stage II, \( \delta_h \) is derived from removing expectation in \( E\|x - x^*\|^2 \) with high probability \( 1 - \delta_h \) by Markov Inequality. In general, \( \sigma_h \) is appropriately chosen as a small term.

**Remark 1.** As it shows in Table[1] the classical result [12] for SGD in the strongly convex setting with \( \sigma^2 = 0 \), where \( \sigma^2 \equiv E\|\nabla f_i(x^*)\|^2 \), after \( T = \frac{2 \sup \mu}{\eta} \log \frac{2\|x_0 - x^*\|^2}{\epsilon} \) steps, \( E\|x_t - x^*\|^2 \leq \epsilon \) by setting constant stepsize \( \eta_t = \frac{1}{2 \sup \mu \eta} \). Compared with SGD, our high probability result recovers the convergence rate up to a factor \( \|x_0 - x^*\|^2 \), if \( b_0 > \sup \eta_i \), see Table[1] as well.

In the batch setting, the full gradient at each step is available. Now, the moving direction \( G_t = \nabla F(x_t) \) and the uniform probability \( \gamma \) in \( P_i(\|\nabla f_i(x)\|^2 \geq \alpha \|x - x^*\|^2) \geq \gamma \) degenerates to 1. Therefore, the linear convergence rate is guaranteed in Stage II without high probability.

**Theorem 2.** *(AdaGrad-Norm: strongly convex and batch setting)* Consider AdaGrad-Norm Algorithm in the batch setting. Suppose that \( F \) is \( L \)-smooth, \( \mu \)-strongly convex with Assumptions (A1), (A2), (A3), (A4), and \( x^* = \min_{x} F(x) \), then \( \min_{0 \leq T \leq T-1} \|x_t - x^*\|^2 \leq \epsilon \) after

**Case 1:** If \( b_0 > \eta \frac{\mu^2 L}{2} \),

\[
T = 1 + \left[ \frac{L(1 + \|x_0 - x^*\|^2/\eta^2)}{\mu} + \frac{L(1 + D^2/\eta^2)}{2 \mu} \right] \log \frac{\|x_0 - x^*\|^2}{\epsilon} + 1
\]

**Case 2:** If \( b_0 \leq \eta \frac{\mu^2 L}{2} \),

\[
T = 1 + \left[ \frac{\log(\eta^2 (\mu + L)^2/4b_0^2)}{\log(1 + 4 \mu^2 \epsilon/(\mu + L)^2)} + \max \left\{ \frac{L(1 + D^2/\eta^2)}{\mu}, \frac{L(1 + D^2/\eta^2)}{2 \mu} \right\} \log \frac{D^2}{\epsilon} \right] + 1
\]

where \( D^2 = \|x_0 - x^*\|^2 + \eta^2 (\log(\frac{\mu^2 L^2}{4b_0^2}) + 1) \).

**Remark 2.** Let \( b_0 > (\mu + L)/2 \) and \( \eta = O(\|x_0 - x^*\|) \). Then theorem[2] recovers the classic result of GD with constant stepsize, i.e., \( (\mu + L)^2 \log \frac{\|x_0 - x^*\|^2}{\epsilon} \), with only a factor related to \( L \) and \( \mu \).

For non-convex functions that satisfies \( \mu \)-PL inequality, we prove the linear convergence rate by bounding \( F(x_t) - F^* \) at each step in Theorem[3]

**Theorem 3.** *(AdaGrad-Norm: convergence in non-convex batch setting)* Consider AdaGrad-Norm Algorithm in the batch setting with Assumptions (A3), (A4), (A5), suppose that \( F \) is \( L \)-smooth and satisfies \( \mu \)-PL inequality and \( F^* = \inf_{x} F(x) \).

**Case 1:** If \( b_0 > \eta L \), then \( \min_{0 \leq T \leq T-1} F(x_t) - F^* \leq \epsilon \) for

\[
T = \left[ \frac{b_0 + \frac{2}{\eta} (F(x_0) - F^*)}{\mu \eta} \log \frac{F(x_0) - F^*}{\epsilon} \right] + 1
\]
We develop the following two-stage proof framework to analyze the convergence rates starting at any AdaGrad-Norm in over-parameterization problem, [26] proved the same convergence rate as ours. Well-known condition satisfied by a wide range of non-convex optimization problems including recent popular over-parameterized neural networks [43, 44, 45, 36, 26]. In particular, for the convergence of AdaGrad-Norm in over-parameterization problem, [26] proved the same convergence rate as ours. However, the convergence rate in [26] was tailored for a multilayer network with two fully connected layers, while our theorems are more general.

5 Two-Stage Framework of AdaGrad-Norm

We develop the following two-stage proof framework to analyze the convergence rates starting at any point $x_0$ and any initial stepsize parameter $b_0$ in both the stochastic and batch settings.

Stage I: If we initialize with small $b_0$, i.e. our initial step size is large—we can get a better convergence in Stage I than SGD with constant stepsize. In Stage I, $b_0$ grows to some given parameters, such as $L$ and $\mu + L$, which depend on different settings, with deterministic steps unless the function achieves a global minimal with tolerance $\epsilon$, i.e. $\|x - x^*\|^2 \leq \epsilon$. Details can be found in two-case lemmas: Lemma 1 and 2. By Lemma 3, $\|x - x^*\|$ is bound by radius $D = R(b_0, \|x_0 - x^*\|, C)$ before $b$ grows up to $C$, instead of blowing up.

Stage II: After Stage I, $b_j$ in AdaGrad-Norm exceeds certain thresholds deterministically in the batch setting and with high probability in the stochastic setting. Conditioned on this, the optimization process is a contraction in the strongly convex setting, i.e. $\|x_{j+1} - x^*\|^2 \leq (1 - \mathcal{P}(\beta_{\text{max}}, \mu, L))\|x_{j} - x^*\|^2$, where $\mathcal{P}$ is a function s.t. $0 < \mathcal{P}(\beta_{\text{max}}, \mu, L) < 1$. $\beta_{\text{max}}$ is bounded by Lemma 4.

Note that in the stochastic setting, RUG is a necessary condition for $b_j$’s growth in Stage I to achieve a certain threshold with high probability. The two-case growth of $b_j$ is provided in Lemma 1.

5.1 Growth of $b_j$ in Stage I

First, we introduce some lemmas in addition to Lemma 1 that are critical in the proof of the growth of $b_j$ in Stage I. Detailed proofs are provided in the Appendix C.

Lemma 2. (Two-case lower bound for $b_N$ in the batch setting) For fixed $\epsilon \in (0, 1)$ and $C$, implement AdaGrad-Norm in the batch setting, convex function $F(x)$ with $\mu$ strongly convex assumption, after $N = \lceil \log(\epsilon^2 / b_j^2) \rceil + 1$ steps, either $b_N > C$ or $\min_{0 \leq t \leq N-1} \|x_t - x^*\|^2 \leq \epsilon$, where $x^* = \arg \min F(x)$; non-convex function $F(x)$ with $\mu - PL$ inequality, after $N = \lceil \log(\epsilon^2 / b_j^2) \rceil$ steps, either $b_N > C$ or $\min_{0 \leq t \leq N-1} F(x_t) - F(x^*) \leq \epsilon$.

Remark 3. In Lemma 2 and 3, we provide the worst cases for the growth of $b_j$. However, $b_j$ actually grows very quickly in practice, especially in the stochastic setting. For Lemma 3, since $\log(1 + x) \sim x$, for $x = \mu^2 / C^2$, $\log \left( \frac{c^2}{\mu^2} \right)$ small, $N \sim \frac{c^2}{\mu^2} \log \left( \frac{c^2}{\mu^2} \right)$.

Lemma 3. (Upper bound for $\|x_{j+1} - x^*\|^2$) For any fixed $C$ and $\eta$, in Algorithm 7, suppose that $J$ is the first index s.t. $b_J > C$, then $\|x_{j+1} - x^*\|^2 \leq \|x_{j+1} - x^*\|^2 + \eta^2 (\log(\epsilon^2 / b_j^2) + 1)$.

5.2 Upper Bounds of $b_j$ in Stage II

In Stage II, we focus on the maximum value that $b_j$ can obtain during the optimization process.

Lemma 4. (Upper bound for $\beta_{\text{max}}$) If implementing the update rules in Algorithm 7, given fixed $C \geq \eta L$, if $J$ is the first index s.t. $b_J > C$, then $\beta_{\text{max}} \triangleq \max_{t \geq 0} b_{j+t}$ can be bounded above by

$$b_{\text{max}} \leq C + \frac{L}{\eta} (\|x_0 - x^*\|^2 + \eta^2 (\log \left( \frac{C^2}{b_0^2} \right) + 1))$$
Lemma 5. (Descent lemma for \(\|x - x^*\|^2\)) Once \(b_j > \eta L/2\), Algorithm 1 is a descent algorithm for the error \(\|x - x^*\|^2\). Furthermore, if \(\|x_{j-1} - x^*\| \leq D\), then \(\forall t \geq 0\), \(x_{j-1+t}\) will stay in the ball centering at \(x^*\) with radius \(D\), i.e. \(\|x_{j-1+t} - x^*\| \leq D\).

6 Numerical Experiments

In this section, we present numerical results to compare AdaGrad-Norm and vanilla (Stochastic) Gradient Descent methods with fixed stepsize \(\eta_j = \frac{1}{b_0}\) or square-root decaying stepsize \(\eta_j = \frac{1}{b_0 + 0.2 \sqrt{t}}\), in the stochastic and batch settings, respectively.

Consider the least square problem \(F(x) = \frac{1}{2n} \|Ax - y\|^2 = \frac{1}{n} \sum_{i=1}^n \frac{1}{2} (a_i, x) - y_i)^2\). In this case, \(x^* = \arg\min \frac{1}{2n} \|Ax - y\|^2\). Since \(\forall x, z, \|\nabla f_i(x) - \nabla f_i(z)\| = \|a_i ((a_i, x) - y_i) - a_i ((a_i, z) - y_i)\| = \|a_i (a_i, x - z)\| \leq \|a_i\|^2 \|x - z\|\), the Lipschitz constants are \(L_i = \|a_i\|^2\) and \(L = \sum_{i=1}^n \frac{1}{2} \|a_i\|^2 = \frac{1}{2} \|A\|_F^2\), respectively. In the experiments, we use a 1000 \(\times\) 20 random matrix A, a randomly generated vector \(x^*\) for the noiseless case. The figure on the right shows the loss \(F(x_i)\) in the noisy case.

We first show noiseless cases to illustrate the linear convergence and robustness we proved in our theorems. Then, in the noisy case, our experiment shows that AdaGrad-Norm converges to the \(\epsilon\)-error solution in the stochastic setting, while SGD with constant stepsize fluctuates significantly.

Figure 2 verifies the expected linear convergence of AdaGrad-Norm in the stochastic and batch settings. In order to compare the convergence rates of AdaGrad-Norm with vanilla SGD, we choose a \(b_0 > \sup_{L_i} L_i = L\) to prevent SGD from blowing up. AdaGrad-Norm with \(b_0 = 1\) and \(b_0 > L\) have similar linear convergence to SGD with \(\eta_j = \frac{1}{b_0}\), up to a constant difference, while SGD with \(\eta_j = \frac{1}{b_0 + 0.2 \sqrt{t}}\) converges more slowly.

Figure 2 shows that \(b_0 = 1\) has a better convergence rate in this case than that of the other two methods, since it takes big steps when \(x\) is far away from \(x^*\), very small steps around \(x^*\) when \(b_j\) grows to a value up to \(b_{\text{max}}\) in corresponding theorems and keeps stable since \(\|G_j\| \|\nabla F(x_j)\| \to 0\) as \(x_j \to x^*\). In the noisy case (Figure 2 right), AdaGrad-Norm has a similar convergence rate up to a constant factor and achieves a better approximation of \(x^*\), without vibrations compared with SGD with constant or square-root decaying stepsize.

Figure 3 shows the growth of \(b_t\) using different methods. The growth of \(b_t\) of AdaGrad-Norm is similar to the square root decay at first, but after exceeding the threshold and approximating to \(b_{\text{max}}\), \(b_t\)‘s growth is analogous to SGD with constant. Figure 4 and 5 show that the linear convergence rates of AdaGrad-Norm are more robust to the choice of initial stepsize \(1/b_0\) compared to SGD with constant stepsizes. The error \(\|x_T - x^*\|^2\) of Stochastic AdaGrad-Norm after \(T\) steps keeps stable automatically for a relatively arbitrary range of \(b_0\) while the error of SGD blows up at first and then decreases significantly when \(b_0\) approaches to \(L\) since SGD is sensitive to the choice of stepsize.

8
7 Discussions

In this work, we propose the notion of RUIG to measure the uniform lower bound of gradients with respect to $\|x - x^*\|^2$ in a restricted region. We propose a two-stage framework and use it to prove the non-asymptotic convergence rates for AdaGrad-Norm starting from any initialization and without knowing the smooth or strongly convex parameter as a priori. In the stochastic setting, we prove linear convergence with high probability under strongly convex and RUIG assumptions, without uniform bound on $E\|G_t\|^2$. In the batch setting, we prove deterministic linear convergence for strongly convex functions and non-convex functions with PL inequality. Our results validate the robustness of AdaGrad-Norm starting at different initial stepsizes.

There are still some open problems to be solved: First, drawing on [12], we may improve $L = \sup_i L_i$ in convergence rates to $\bar{L} = \frac{1}{n} \sum_i L_i$ with importance sampling. Second, we note that AdaGrad-Norm has an aggressively decaying stepsize in online situations—which reduces the contributions from subsequent samples over time. The ADAM algorithm [18] solves this problem by incorporating an exponential moving average, and AMSGrad [20] solves this problem by utilizing the maximum of $v_t$. However, providing rigorous theoretical guarantees for these methods remains an open problem of research. Since AdaGrad-Norm is fundamentally related to both of these algorithms, extending our theoretical guarantees to ADAM and AMSGrad is an exciting direction for future research.

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Supplementary Material

Supplementary material for the paper: "Linear Convergence of Adaptive Stochastic Gradient Descent".

This appendix is organized as follows:

- Appendix A: Proof of Theorem 1 in the Stochastic Strongly Convex Setting
- Appendix B: Proof of Theorem 2 and 3 in the batch Setting
- Appendix C: Proof of Lemmas in Stage I
- Appendix D: Proof of Lemmas in Stage II
- Appendix E: Numerical Experiments of AdaGrad-Norm with Different Initialization

A Proof of Theorem 1 in the Stochastic Strongly Convex Setting

From Lemma 1, let $C = \eta L$, after $N \geq \frac{\eta^2 L^2 - \frac{b_0^2}{\alpha \gamma}}{\alpha \gamma} + \frac{\delta}{\gamma}$ steps, if $\min_{0 \leq i < N} \|x_i - x^*\|^2 > \epsilon$, then with high probability $1 - \exp\left(\frac{-\delta^2}{2N(\gamma(1+\eta))}\right)$, $b_N > \eta L$. Then, there exists a first index $k_0 < N$, s.t. $b_{k_0} > \eta L$ but $b_{k_0-1} < \eta L$.

If $k_0 \geq 1$, then

$$
\|x_{k_0+l} - x^*\|^2 = \|x_{k_0+1} - x^*\|^2 + \frac{\eta^2}{b_{k_0+l}^2} \|G_{k_0+1}\|^2 - \frac{2\eta}{b_{k_0+l}} \langle x_{k_0+1} - x^*, G_{k_0+1} \rangle
$$

$$
\leq \|x_{k_0+1} - x^*\|^2 + (\frac{\eta^2 L}{b_{k_0+l}^2} - \frac{2\eta}{b_{k_0+l}}) \langle x_{k_0+1} - x^*, G_{k_0+1} \rangle
$$

$$
\leq \|x_{k_0+1} - x^*\|^2 - \frac{\eta}{b_{k_0+l}} \langle x_{k_0+1} - x^*, G_{k_0+1} \rangle
$$

$$
\leq \|x_{k_0+1} - x^*\|^2 - \frac{\eta}{b_{\max}} \langle x_{k_0+1} - x^*, G_{k_0+1} \rangle
$$

(2)

where the last second inequality is from the condition $b_{k_0} > \eta L$. The last inequality holds since $b_{k_0+l} \leq b_{\max}$ and $f_{k_0+1}(x)$ is convex, which implies $\langle x_{k_0+1} - x^*, G_{k_0+1} - f_{k_0+1}(x) \rangle \geq 0$, $P(f_{k_0+1}(x^*) = 0) = 1$ by Assumption (A4).

Take expectation regarding to $\xi_{k_0+1}$, and use the fact that when $j > k_0$, $b_j > L$, when $l \geq 1$ and $0 < \frac{\mu \eta}{b_{\max}} < \frac{\delta}{\gamma} < 1$, then we can get

$$
E_{\xi_{k_0+1}} \|x_{k_0+l} - x^*\|^2 \leq \|x_{k_0+1} - x^*\|^2 - \frac{\eta}{b_{\max}} \langle x_{k_0+1} - x^*, \nabla F(x_{k_0+1}) \rangle
$$

$$
\leq (1 - \frac{\mu \eta}{b_{\max}}) \|x_{k_0+1} - x^*\|^2
$$

$$
\leq \prod_{j=0}^{l-1} (1 - \frac{\mu \eta}{b_{\max}}) \|x_{k_0+1} - x^*\|^2
$$

(3)

$$
\leq \prod_{j=0}^{l-1} (1 - \frac{\mu \eta}{b_{\max}}) (\|x_0 - x^*\|^2 + \eta^2 (\log(\frac{C^2}{b_0^2}) + 1))
$$

$$
\leq (\|x_0 - x^*\|^2 + \eta^2 (\log(\frac{\eta^2 L^2}{b_0^2}) + 1)) \exp(-\frac{\mu l}{b_{\max}})
$$

where the second inequality is from the strong convexity of $F(x)$, i.e. $(x - y, \nabla f(x) - \nabla f(y)) \geq \mu \|x - y\|^2$ and $\nabla^2 F(x^*) = 0$. From Lemma 2, we can give an upper bound for $b_{\max} = \max_{t \geq 0} b_{k_0+t} = C + \frac{\ell}{\eta} (\|x_0 - x^*\|^2 + \eta^2 (\log(\frac{C^2}{b_0^2}) + 1)) = \eta L + \frac{L}{\eta} (\|x_0 - x^*\|^2 + \eta^2 (\log(\frac{\eta^2 L^2}{b_0^2}) + 1))$.

Then take the iterated expectation, and use Markov in inequality, with high probability $1 - \delta_h$,

$$
\|x_{k_0+l} - x^*\|^2 \leq \frac{1}{\delta_h} (\|x_0 - x^*\|^2 + \eta^2 (\log(\frac{\eta^2 L^2}{b_0^2}) + 1)) \exp(-\frac{\mu l}{b_{\max}})
$$

(4)
Then after $M \geq \frac{\eta L + \frac{\mu}{L} \eta^2 (2 + \eta^2 \log(\frac{\eta^2}{\delta^2} \log(\frac{\eta^2}{\delta^2} \log(\frac{\eta^2}{\delta^2})))}{\mu \delta^2 \log(\frac{\eta^2}{\delta^2})}}{\mu \delta^2 \log(\frac{\eta^2}{\delta^2})}$, with high probability more than $1 - \delta_h$, we have

$$\|x_{k_0} - x^*\|^2 \leq \epsilon$$

Otherwise, if $k_0 = 0$, i.e. $b_0 > \eta L$, then use the same inequality as above,

$$E_{\xi_{M-1}} \|x_M - x^*\|^2 \leq \left(1 - \frac{\mu}{b_{\max}'}\right) \|x_{M-1} - x^*\|^2 \leq \|x_0 - x^*\|^2 \exp\left(-\frac{\mu M}{b_{\max}'}\right)$$

Then after $M \geq \frac{b_{\max}'}{\mu} \log\left(\frac{\|x_0 - x^*\|^2}{\varepsilon \delta^2}\right)$, by Markov’s inequality,

$$P\left(\|x_M - x^*\|^2 \geq \varepsilon\right) \leq \frac{E\|x_M - x^*\|^2}{\varepsilon} \leq \delta_h$$

where $b_{\max}' = b_0 + \frac{L}{\eta} \|x_0 - x^*\|^2$ can be estimated as follows:

$$\|x_j - x^*\|^2 \leq \|x_{j-1} - x^*\|^2 - \frac{\eta}{L} \frac{\|G_j\|^2}{b_j + 1}$$

Then for any $j + 1$:

$$b_{j+1} = b_0 + \sum_{i=0}^{j} \frac{\|G_i\|^2}{b_i + b_{i+1}} \leq b_0 + \frac{L}{\eta} (\|x_0 - x^*\|^2 - \|x_{j+1} - x^*\|^2)$$

Plug in the value, we can get $M \geq \frac{b_0 + \frac{L}{\mu} \|x_0 - x^*\|^2}{\mu} \log\left(\frac{\|x_0 - x^*\|^2}{\varepsilon \delta^2}\right)$.

**B Proof of Theorems in the batch Setting**

**B.1 Proof of Theorem 2**

**Lemma B1. (Co-coercivity with Strong Convexity)** If $F(x)$ is $\mu$–strongly convex and $L$–smooth, then

$$\langle \nabla F(x) - \nabla F(y), x - y \rangle \geq \frac{\mu L}{\mu + L} \|x - y\|^2 + \frac{1}{\mu + L} \|\nabla F(x) - \nabla F(y)\|^2$$

**Proof.** Let $\phi(x) = F(x) - \frac{\mu}{L} \|x\|^2$, then $\phi(x)$ is convex and $(L - \mu)$–smooth. By Lemma C4

$$\langle \nabla \phi(x) - \nabla \phi(y), x - y \rangle \geq \frac{1}{L - \mu} \|\nabla \phi(x) - \nabla \phi(y)\|^2$$

Plugging in $\nabla \phi(x) = \nabla F(x) - \mu x$, we have

$$\langle \nabla F(x) - \nabla F(y), x - y \rangle - \mu \|x - y\|^2 \geq \frac{1}{L - \mu} (\|\nabla F(x) - \nabla F(y)\|^2 + \mu^2 \|x - y\|^2 - 2\mu \langle \nabla F(x) - \nabla F(y), x - y \rangle)$$

With simple algebra, we can get the result. \hfill \Box

By Lemma 2, after $N = \lceil \frac{\log(\eta^2 (\mu + L)^2 / (4\delta^2)))}{\log(\eta^2 (\mu + L)^2 / (4\delta^2)))} \rceil + 1$ steps, if $\min_{0 \leq i \leq N-1} \|x_i - x^*\|^2 > \epsilon$, then $\exists k_0 \leq N$, such that it is the first index s.t. $b_{k_0} > \frac{\eta \mu + L}{2 \epsilon}$.

If $k_0 > 1$, since $F$ is $\mu$–strongly convex and $L$–smooth, we have:

$$\langle \nabla F(x) - \nabla F(y), x - y \rangle \geq \frac{\mu L}{\mu + L} \|x - y\|^2 + \frac{1}{\mu + L} \|\nabla F(x) - \nabla F(y)\|^2.$$
For $j \geq 0$, we have $0 < \eta \frac{2\mu L}{\mu + L} \leq \frac{4\mu L}{(\mu + L)^2} < 1$, since $2\mu L < \mu^2 + L^2$.

We divide the analysis into two situations to get a better bound instead of using $b_{\text{max}}$ for all the following steps. First, assume $\eta \frac{\mu + L}{2} < b_0 < \eta L$ and after $l$ steps, $b_{k+1}$ is still less than $\eta L$.

$$\|x_{k+1} - x^*\|^2 = \|x_{k+1} - x^*\|^2 + \frac{\eta^2}{b_{k+1}^2} \|\nabla F(x_{k+1})\|^2$$

$$- \frac{2\eta}{b_{k+1}^2} (x_{k+1} - x^*, \nabla F(x_{k+1}))$$

$$\leq (1 - \frac{2\mu L}{(\mu + L)b_{k+1}})\|x_{k+1} - x^*\|^2$$

$$+ \frac{\eta}{b_{k+1}^2} (\frac{\eta}{L} - \frac{2}{\mu + L})\|\nabla F(x_{k+1})\|^2$$

$$\leq \prod_{j=0}^{l} (1 - \frac{2\mu L}{(\mu + L)b_{k+1} + j})\|x_{k+1} - x^*\|^2$$

$$\leq \exp(-\frac{2\mu(l + 1)}{\mu + L})\|x_{k+1} - x^*\|^2$$

where $\|x_{k+1} - x^*\|^2$ can be upper bounded similar to Lemma \(3\), setting $C = \frac{\mu + L}{2}$

$$\|x_{k+1} - x^*\|^2 \leq \|x_0 - x^*\|^2 + \eta^2 \log(\frac{(\mu + L)^2}{4b_0^2} + 1)$$

Second, if $b_{k+1} > \eta L$, $M_0$ can be 0, 1, 2, ..., then for $l \geq 0$,

$$\|x_{k+1} + M_0 + l - x^*\|^2 = \|x_{k+1} + M_0 + l - x^*\|^2 + \frac{\eta^2}{b_{k+1}^2} \|\nabla F(x_{k+1} + M_0 + l)\|^2$$

$$- \frac{2\eta}{b_{k+1}^2} (x_{k+1} + M_0 + l - x^*, \nabla F(x_{k+1} + M_0 + l))$$

$$\leq \|x_{k+1} + M_0 + l - x^*\|^2 + \frac{\eta^2 L}{b_{k+1}^2} + \frac{2\eta}{b_{k+1}^2} (\frac{\eta}{L} - \frac{2}{\mu + L}) (x_{k+1} + M_0 + l - x^*, \nabla F(x_{k+1} + M_0 + l))$$

$$\leq (1 - \frac{\eta}{b_{\text{max}}})\|x_{k+1} + M_0 + l - x^*\|^2$$

$$\leq \exp(-\frac{\eta}{b_{\text{max}}})\|x_{k+1} + M_0 - x^*\|^2$$

$$\leq \exp(-\frac{\eta}{b_{\text{max}}}) \exp(-\frac{2\mu(M_0 + 1)}{\mu + L})\|x_{k+1} - x^*\|^2$$

$b_{\text{max}}$ can be upper bounded similar to Lemma \(4\), $b_{\text{max}} \leq \eta L + \frac{L}{\eta}\|x_{k+1} + M_0 - x^*\|^2$ Once $b_l > \frac{\mu + L}{2} > \frac{\mu}{2}$, by Lemma \(5\), the update result of $\|x_j - x^*\|^2$ in AdaGrad-Norm is a contraction, so $\|x_{k+1} + M_0 - x^*\|^2 \leq \|x_{k+1} - x^*\|^2$. Hence, $b_{\text{max}} \leq \eta L + \frac{L}{\eta}\|x_{k+1} - x^*\|^2$

Combining the two situations above, we have

$$\|x_{k+1} + M_0 - x^*\|^2 \leq \exp(-M \min\{\frac{\mu}{b_{\text{max}}}, \frac{2\mu}{\mu + L}\})\|x_{k+1} - x^*\|^2$$

$$\leq \exp(-M \min\{\frac{\mu}{L(1 + D^2/\eta^2)}, \frac{2\mu}{\mu + L}\})\|x_{k+1} - x^*\|^2$$
where \( D^2 = \|x_{k0-1} - x^*\|^2 \).

After \( M \geq \max \{ \frac{L(1+D^2/\eta^2)}{\mu}, \frac{\mu+L}{2\mu} \} \log \frac{D^2}{\epsilon} - 1 \) steps,
\[
\|x_{k0+M} - x^*\|^2 \leq \epsilon
\]

Otherwise, if \( k_0 = 1 \), then
\[
\|x_M - x^*\|^2 \leq \exp(-\sum_{j=1}^{M} \min\{ \frac{\mu \eta}{b^*_\max}, \frac{2\mu}{\mu+L} \})\|x_0 - x^*\|^2
\]

where \( b^*_\max = \eta L + \frac{L}{\mu} \|x_0 - x^*\|^2 \).

Then after \( M \geq \max \{ \frac{L(1+\|x_0-x^*\|^2/\eta^2)}{\mu}, \frac{\mu+L}{2\mu} \} \log \|x_0-x^*\|^2 \) steps, we can assure that
\[
\|x_M - x^*\|^2 \leq \epsilon
\]

### B.2 Proof of Theorem 3

By Lemma [2] after \( N \geq \frac{\log(\eta^2 L^2/b_0^2)}{\log(1+2\mu)/(\eta L^2)} \) steps, if \( \min_{0 \leq i \leq N-1} F(x_i) - F^* > \epsilon \), then \( \exists k_0 \leq N \), such that it is the first index s.t. \( b_{k_0} > \eta L \).

If \( k_0 > 1 \), then for \( j \geq 0 \), from Assumption (A3), we have
\[
F(x_{k0+j}) \leq F(x_{k0+j-1}) - \frac{\eta}{b_{k0+j}^l} (1 - \frac{\eta L}{2b_{k0+j}}) \|\nabla F(x_{k0+j-1})\|^2
\]
\[
\leq F(x_{k0+j-1}) - \frac{\eta}{2b_{k0+j}} \|\nabla F(x_{k0+j-1})\|^2
\]
\[
\leq F(x_{k0+j-1}) + \frac{\mu \eta}{b_{k0+j}} (F^* - F(x_{k0+j-1}))
\]

Then add \(-F^*\) on both sides, we can get
\[
F(x_{k0+j}) - F^* \leq (1 - \frac{\mu \eta}{b_{k0+j}})(F(x_{k0+j-1}) - F^*)
\]

The last inequality is from \( \mu - \text{PL} \) inequality, i.e., \( \|F(x)\|^2 \leq 2\mu(F^* - F(x)) \), \( \forall x \). Since \( b_{k0+j} > \eta L \geq \eta \mu, 1 - \frac{\mu \eta}{b_{k0+j}} \in (0, 1) \) holds for all \( j \geq 0 \), it is a contraction at every step. Then,
\[
F(x_{k0+j}) - F^* \leq \left( \prod_{l=0}^{j} (1 - \frac{\mu \eta}{b_{k0+l}}) \right)(F(x_{k0}) - F^*)
\]
\[
\leq \exp(- \sum_{l=0}^{j} \frac{\mu \eta}{b_{k0+l}})(F(x_{k0}) - F^*)
\]
\[
\leq \exp(- \sum_{l=0}^{j} \frac{\mu \eta}{b_{k0+l}})(F(x_{0}) - F^*) + \frac{\eta^2 L}{2} (1 + \log(\frac{b_{k0-1}^2}{b_0^2}))
\]

where we use the fact that \( 1 - x \leq e^{-x}, \forall x \in (0, 1) \) and the the fact from lemma in [27]: \( F(x_{k0}) \leq F(x_0) + \frac{\eta^2 L}{2} (1 + \log(\frac{b_{k0-1}^2}{b_0^2})) \).

The upper bound of \( b_j \) is [27]:
\[
b_{\max} = b_{k0-1} + \frac{2}{\eta} (F_{k0-1} - F^*) \leq \eta L + \frac{2}{\eta} (F(x_0) - F^*) + \frac{\eta^2 L}{2} (1 + \log(\frac{\eta^2 L^2}{b_0^2}))
\]

Then
\[
F(x_{k0+M-1}) - F^* \leq \exp(- \frac{\mu \eta M}{b_{\max}})(F(x_0) - F^*) + \frac{\eta^2 L}{2} (1 + 2 \log(\frac{\eta L}{b_0}))
\]

Hence, we need
\[
M \geq \frac{b_{\max} \mu \eta}{\log(\frac{F(x_0) - F^*}{\frac{\eta^2 L}{2} (1 + 2 \log(\frac{\eta L}{b_0}))})} \frac{\log \frac{D^2}{\epsilon}}{\epsilon}
\]

15
It is sufficient that it
\[ M \geq \frac{\eta L + \frac{2}{\eta}(F(x_0) - F^*) + \frac{\eta^2 L^2}{2}(1 + \log(\frac{2L^2}{b_0^2})))}{\mu \eta} \log \frac{F(x_0) - F^* + \frac{\eta^2 L^2}{2}(1 + 2 \log \frac{n_k}{b_0})}{\epsilon} \]

Then,
\[ \min_{0 \leq i \leq N + M - 1} F(x_i) - F^* \leq \epsilon \]
where \( N = \left\lceil \frac{\log(b_0^2 L^2)}{\log(1 + 2 \mu \eta /(\eta L^2))} \right\rceil + 1. \)

Otherwise, if \( k_0 = 1 \), the upper bound of \( b_j \) degenerates to
\[ b'_{max} = b_0 + \frac{2}{\eta}(F(x_0) - F^*) \]

Then, use the same procedure,
\[ F(x_M) - F^* \leq \exp(-\sum_{k=0}^{M-1} \frac{\mu \eta b_{k+1}}{b_{max}})(F(x_0) - F^*) \]
\[ \leq \exp(-\frac{\mu \eta M}{b_{max}})(F(x_0) - F^*) \]

Once
\[ M \geq \frac{b'_{max}}{\mu \eta} \log \frac{F(x_0) - F^*}{\epsilon} = \frac{b_0 + \frac{2}{\eta}(F(x_0) - F^*)}{\mu \eta} \log \frac{F(x_0) - F^*}{\epsilon} \]
we can get the expected result: \( F(x_M) - F^* \leq \epsilon. \)

C Proof of Lemmas in Stage I

C.1 Proof of Lemma 1

Lemma C2. (Bernstein’s Inequality)[46] Let \( X \) be a random variable, \( \mathbb{E}[X] = \mu, \text{Var}(X) = \sigma^2 \), if it satisfies the Bernstein condition with parameter \( b > 0 \), i.e. if \( |\mathbb{E}(X - \mu)|^k \leq \frac{1}{2} k! \sigma^2 b^{k-2}, \forall k \geq 2 \), then
\[ \mathbb{P}(|X - \mu| \geq t) \leq 2 \exp(-\frac{t^2}{2(\sigma^2 + bt)}) \]

Lemma C3. Let \( X_i \sim \text{Bernoulli}(p) \), i.i.d. \( \forall i = 1, 2, \ldots, n \), and \( X = \sum_{i=1}^{n} X_i \). Since \( X_i \in [0,1] \), \( \{X_i\} \) satisfy Bernstein condition, then[46]
\[ \mathbb{P}(|X - np| > t) \leq 2 \exp(-\frac{t^2}{2(np(1-p) + t)}) \]

Proof of Lemma 1 If \( \min_j \|x_j - x^*\|^2 \leq \epsilon \), we are done. Otherwise, we have \( \|x_j - x^*\|^2 > \epsilon, \forall j = 0, 1, 2, ..., N \). Assume that \( F(x) \) satisfies \((\epsilon, \alpha, \gamma)\)-RUIG (Assumption (A6)), we can use independent identical Bernoulli random variables \( \{Z_j\} \) to represent them with the following distribution:
\[ Z_j = \begin{cases} 1 & \text{if } \|\nabla f_{x_j}(x_j)\|^2 \geq \alpha \|x_j - x^*\|^2 \\ 0 & \text{else} \end{cases} \]
(15)
where \( \mathbb{P}(Z_j = 1) = \gamma, \forall j \). Then from Lemma C3 and let \( Z = \sum_{j} Z_j \), with high probability bigger than \( 1 - \exp(-\frac{\delta^2}{2N(1-\gamma+\delta)}) \), \( Z \geq \gamma N - \delta, \forall N \). Thus, after \( N \geq \frac{\epsilon^2 - b_0^2 + \delta}{\alpha \gamma \epsilon} \) steps, with \( 1 - \exp(-\frac{\delta^2}{2N(1-\gamma+\delta)}) \), we have
\[ b_N^2 = b_0^2 + \sum_{i=0}^{N-1} \|\nabla f_{x_i}(x_i)\|^2 > b_0^2 + (\gamma N - \delta) \alpha \epsilon \geq C^2 \]
C.2 Proof of Lemma 2

If $b_0 > C$, we are done. Otherwise if $b_0 < C$, and after $N \geq \frac{\log(C^2/b_0^2)}{\log(1 + 2\mu\epsilon/C^2)}$ steps, $b_N < C$ and $\min_{0 \leq i \leq N-1} \|x_i - x^*\|^2 > \epsilon$. Since $F(x)$ is $\mu$-strongly convex, $\epsilon < \|x_i - x^*\|^2 \leq \frac{1}{\mu^2} \|\nabla F(x_i) - \nabla F(x^*)\|^2, \forall x_i$ and $\nabla F(x^*) = 0$. Then,

$$b_N^2 = b_{N-1}^2 + \|\nabla F(X_{N-1})\|^2$$

$$= b_{N-1}^2(1 + \frac{\|\nabla F(x_{N-1})\|^2}{b_{N-1}^2})$$

$$\geq b_0^2 \prod_{j=0}^{N-1}(1 + \frac{\|\nabla F(x_j)\|^2}{b_j^2})$$

(16)

Contradiction! Hence, at least one of $\min_{0 \leq i \leq N-1} \|x_i - x^*\|^2 \leq \epsilon$ or $b_N > C$ holds. When $\mu$ is small and $C$ is big, we have $\log(1 + \frac{\mu^2\epsilon}{C^2}) \approx \frac{\mu^2\epsilon}{C^2}$.

On the other hand, with PL inequality $\frac{1}{2\mu} \|\nabla F(x)\|^2 \geq F(x) - F(x^*)$ instead of $\mu$-strongly convex assumption, if $\min_{0 \leq i \leq N-1} F(x_i) - F^* > \epsilon$ and $b_N < C$, then if $N \geq \frac{\log(C^2/b_0^2)}{\log(1 + 2\mu\epsilon/C^2)}$ steps, $b_N^2 \geq b_0^2(1 + \frac{2\mu\epsilon}{C^2})^N \geq C^2$, contradiction! Hence, either $\min_{0 \leq i \leq N-1} F(x_i) - F^* \leq \epsilon$ or $b_N > C$.

C.3 Proof of Lemma 3

Lemma C4. (Co-coercivity)[27] For a $L$-smooth convex function $f(x)$, $\forall x, y$

$$\|\nabla f(x) - \nabla f(y)\|^2 \leq L\langle x - y, \nabla f(x) - \nabla f(y) \rangle$$

Lemma C5. (Integral lemma)[27] For any non-negative sequence $a_1, ..., a_T$, such that $a_1 \geq 1$,

$$\sum_{i=1}^{T} \frac{a_i}{\sum_{i=1}^{T} a_i} \leq \log(\sum_{i=1}^{T} a_i) + 1$$

(17)

$$\sum_{i=1}^{T} \frac{a_i}{\sqrt{\sum_{i=1}^{T} a_i}} \leq 2 \sqrt{\sum_{i=1}^{T} a_i}$$

(18)

Proof. The lemma can be proved by induction. Besides, we can take above sums as Riemman sums, then the sums should be proportional to integrals, $\log(x)$ and $2\sqrt{x}$, respectively.

Proof of Lemma 3 With above two lemmas, we can bound $\|x_{J-1} - x^*\|^2$ as follows:

$$\|x_{J-1} - x^*\|^2 = \|x_{J-2} - \frac{\eta G_{J-2}}{b_{J-1}} - x^*\|^2$$

$$= \|x_{J-2} - x^*\|^2 + \|\frac{\eta G_{J-2}}{b_{J-1}}\|^2 - 2\eta \langle \frac{G_{J-2}}{b_{J-1}}, x_{J-2} - x^* \rangle$$

$$\leq \|x_{J-2} - x^*\|^2 + \|\frac{\eta G_{J-2}}{b_{J-1}}\|^2 - \frac{2\eta}{b_{J-1}} \|G_{J-2} - f_{J-2}(x^*)\|^2$$

$$\leq \|x_{J-2} - x^*\|^2 + \frac{\eta^2 \|G_{J-2}\|^2}{b_{J-1}^2}$$

$$\leq \|x_0 - x^*\|^2 + \eta^2 \sum_{j=0}^{J-2} \frac{\|G_j\|^2}{b_{j+1}^2}$$

$$\leq \|x_0 - x^*\|^2 + \eta^2 \sum_{j=0}^{J-2} \frac{\|G_j\|^2}{\sum_{l=0}^{j} \|G_l\|^2 / b_0^2}$$
where the first inequality is from the co-coercivity (Lemma C4) and Assumption (A4) $P(f_{\xi_{i-1}}(x^*) = 0) = 1$; last second inequality is from lemma C5 and the last inequality is from the assumption that $J$ is the first index s.t. $b_J > C$.

D Proof of Lemmas in Stage II

D.1 Proof of Lemma 4

Since $b_J > \eta L$, we have the following bound for $\|x_{J+1} - x^*\|^2$:

\[
\|x_{J+1} - x^*\|^2 = \|x_{J+1-1} - x^*\|^2 + \frac{\eta^2 \|G_{J+1-1}\|^2}{b_{J+1}} - \frac{2\eta}{b_{J+1}} (G_{J+1-1} - \nabla f_{\xi_{J+1-1}}(x^*), x_{J+1-1} - x^*)
\]

\[
\leq \|x_{J+1-1} - x^*\|^2 + \frac{\eta \|G_{J+1-1}\|^2}{b_{J+1}} - \frac{2\eta}{b_{J+1}} L \|G_{J+1-1}\| + \frac{2\eta}{L} \|G_{J+1-1}\|^2
\]

\[
\leq \|x_{J+1-1} - x^*\|^2 - \frac{\eta \|G_{J+1-1}\|^2}{b_{J+1}}
\]

\[
\leq \|x_{J+1-1} - x^*\|^2 - \frac{\eta \sum_{j=0}^{l} \|G_{J+1-j}\|^2}{b_{J+1}}
\]

inequalities are from $f_{\xi_{J+1-1}}(x)$ is $L$-smooth and co-coercivity. Then, we have the bound of the sum:

\[
\sum_{j=0}^{l} \frac{\|G_{J+1-j}\|^2}{b_{J+1}} \leq \frac{L}{\eta} (\|x_{J+1-1} - x^*\|^2 - \|x_{J+1} - x^*\|^2)
\]

(20)

Therefore, $b_{\max}$ is bounded as follows:

\[
b_{J+1} = b_{J+1-1} + \frac{\|G_{J+1-1}\|^2}{b_{J+1} + b_{J+1-1}}
\]

\[
\leq b_{J+1-1} + \sum_{j=1}^{l} \frac{\|G_{J+1-j}\|^2}{b_{J+1-j}}
\]

\[
\leq C + \frac{L}{\eta} \|x_{J+1-1} - x^*\|^2
\]

\[
= C + \frac{L}{\eta} (\|x_0 - x^*\|^2 + \eta^2 (\log(\frac{C^2}{b_0^2}) + 1))
\]

where $\|x_{J+1-1} - x^*\|^2 \leq \|x_0 - x^*\|^2 + \eta^2 (\log(\frac{C^2}{b_0^2}) + 1)$ is from Lemma 3.

D.2 Proof of Lemma 5

Use similar technique as above,

\[
\|x_J - x^*\|^2 \leq \|x_{J-1} - x^*\|^2 + \frac{\eta^2}{b_J} \|G_J\|^2
\]

\[
= \|x_{J-1} - x^*\|^2 - \frac{\eta}{b_J} (\frac{2}{L} - \frac{\eta}{b_J}) \|G_J\|^2
\]

\[
\leq \|x_{J-1} - x^*\|^2
\]
where the first inequality if from $f_j(x)$ is $L$-smooth, $\nabla f_{j-1}(x^*) = 0$ (Assumption (A4)) and Lemma C4. Therefore, once $b_j > \eta L/2$, the algorithm is a descent one.

E Numerical Experiments of AdaGrad-Norm with Different Initialization

In this section, we demonstrate the numerical experiments of AdaGrad-Norm with $x_0$ (Figure 6) and the extreme case (Figure 7), $x_0$ is far away from $x^*$ and $\|x_0\|$ is large. Then, we tune the hyperparameter $\eta$ in the extreme case $\eta = O(\|x_0 - x^*\|^2)$ (Figure 6). In these figures, the x-axis represents iteration $t$ while y-axis is the approximation error $\|x_t - x^*\|^2$ for the first and third columns and it is $b_t$ for the second and fourth columns.

We show that when starting from $x_0 = 0$, the result is close to the experiment we show in Figure 2. When initialize $x_0$ with extremely bad one, $x_0 = 100*y_0$, where $y_0$ is a randomly generated vector $y_0$ and $y_0 \sim N(0, I)$, AdaGrad-Norm takes much more iterations than before. However, after tuning $\eta = 10000$, the convergence rate of AdaGrad-Norm is better again. In this case, $b_0$ plays a small role.

Figure 6: Error and growth of $b_t$ with $x_0 = 0$

Figure 7: Error and growth of $b_t$ with extremely bad initialization

Figure 8: Error and growth of $b_t$ with extremely bad initialization and tuning $\eta = O(\|x_0 - x^*\|^2)$