Renormalization of three-quark operators at two loops in the RI’/SMOM scheme

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Abstract

We consider the renormalization of the three-quark operators without derivatives at next-to-next-to-leading order in QCD perturbation theory at the symmetric subtraction point. This allows us to obtain conversion factors between the \textit{MS} scheme and the regularization invariant symmetric MOM (RI/SMOM, RI’/SMOM) schemes. The results are presented both analytically in $R_\xi$ gauge in terms of a set of master integrals and numerically in Landau gauge. They can be used to reduce the errors in determinations of baryonic distribution amplitudes in lattice QCD simulations.

Keywords: Lattice QCD, Baryonic distribution amplitudes, \textit{MS} scheme, Regularization invariant symmetric MOM scheme, Two-loop approximation

1. Introduction

Light cone distribution amplitudes (DAs) play an important rôle in the analysis of hard exclusive reactions involving large momentum transfer from the initial to the final state. The cases of baryon asymptotic states have been considered already long ago\cite{13, 7, 5}.

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The theoretical description of DAs is based on the relation of their moments to matrix elements of local operators. Such matrix elements involve long-distance dynamics and, thus, cannot be accessed via perturbation theory alone.

First estimates of the lower moments of the baryon DAs have been obtained more than 30 years ago using QCD sum rules. An alternative way to access the moments is to calculate them from first principles using lattice QCD. Such studies for the nucleon DAs have a long history. More recently, this analysis has been extended to include the full SU(3) octet of baryons.

To renormalize the matrix elements on the lattice, the RI'/SMOM scheme has been used in Ref. However, in order to embed lattice estimations of hadronic matrix elements into the complex of other studies and to assure comparability, it is necessary to present the result in the widely used MS scheme. Since the RI'/SMOM prescription can be used in both perturbative and non-perturbative calculations, the conversion from the RI'/SMOM to the MS scheme can be evaluated perturbatively as a series in the strong-coupling constant at some typical scale of the order of a few GeV.

In our previous works, we have evaluated the matching constants for the bilinear quark operators with up to two derivatives and up to three loops.

In this paper, we perform the renormalization of the three-quark operators at two loops for RI'/SMOM kinematics, which allows for the conversion between the RI'/SMOM and MS schemes for the lowest moments of the baryonic DAs.

As for baryonic operators, there are additional subtleties due to contributions of evanescent operators that have to be taken into account. In this work, we adopt the calculational scheme proposed in Ref., which allows one to avoid the necessity of additional finite renormalizations and the consideration of evanescent operators. Instead of contracting the operators with different Dirac matrices, we will consider the operators with open spinor indices. The price that one has to pay is that one has a large number of

\[ \epsilon^{ijk}(D_{\mu_1} \ldots D_{\mu_l} \psi_1)^i_{\alpha_1} (D_{\mu_{l+1}} \ldots D_{\mu_{l+m}} \psi_2)^j_{\alpha_2} (D_{\mu_{l+m+1}} \ldots D_{\mu_{l+m+n}} \psi_3)^k_{\alpha_3}, \] (1)

where \(i, j, k\) are color indices, \(\mu_k\) are Lorentz indices, and \(\alpha_l\) are spinor indices.

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1The three-quark operators relevant for the baryonic DAs have the general form
different spinor tensor structures.

This paper is organized as follows. In Section 2, we introduce our notations and definitions. In Section 3, we discuss the tensor decomposition and the renormalization procedure. In Section 4, we present a sample result, while our complete result is provided in ancillary files submitted to the ArXiv along with this paper. In Section 5, we present our conclusions. In Appendix A, we expose the relevant spin tensor structures.

2. Basic setup

The basic object for the three-quark operators without derivatives located at the origin is the amputated four-point function,

$$H_{\beta_1\beta_2\alpha_1\alpha_2\alpha_3}(p_1, p_2, p_3) = -\int d^4 x_1 d^4 x_2 d^4 x_3 e^{i(p_1 x_1 + p_2 x_2 + p_3 x_3)} \epsilon^{b_1 b_2 b_3} \epsilon^{a_1 a_2 a_3} \times \langle u_{b_1}^{\alpha_1}(0) d_{b_2}^{\beta_2}(0) s_{b_3}^{\beta_3}(0) \bar{u}_{a_1}^{\alpha_1}(p_1) d_{a_2}^{\alpha_2}(p_2) s_{a_3}^{\alpha_3}(p_3) \rangle \times G_2^{-1}(p_1) G_2^{-1}(p_2) G_2^{-1}(p_3) G_2^{-1}(p_3),$$

(2)

where all quantities are to be understood as Euclidean. The quark flavors are called $u$, $d$, and $s$, but the only essential feature is that they are all different. All masses are supposed to vanish. $\alpha_i$ and $\beta_j$ are spinor indices, $a_i$ and $b_j$ are color indices in the fundamental representation, and $p_m$ are the external momenta. The matrix element of the three-quark operator is shown schematically in Fig. 1.

![Figure 1: Matrix element of a three-quark baryonic operator in momentum space. The momentum $p_4 = -(p_1 + p_2 + p_3)$ is the momentum coming into the operator.](image)

The two-point function $G_2(p)$ required for the amputation of the external legs is defined by

$$\delta^{a' a} G_2(p)_{a' a} = \int d^4 x e^{i p x} \langle u_{a'}^{a'}(0) \bar{u}_a^{a}(x) \rangle.$$

(3)
To compute the conversion factor for a particular multiplet of operators, the amputated four-point function (2) has to be contracted with coefficients

\[ C^{\beta_1 \beta_2 \beta_3, \alpha_1 \alpha_2 \alpha_3}, \tag{4} \]

which can be easily provided. Notice, however, that these coefficients refer to a particular representation of the Dirac matrices.

In the following, we use the kinematics that was adopted in the analysis of Ref. [1],

\[ p_1^2 = p_2^2 = p_3^2 = (p_1 + p_2 + p_3)^2 = \mu^2, \tag{5} \]
\[ p_1 \cdot p_2 = p_3 \cdot p_1 = -\frac{1}{2} \mu^2, \tag{6} \]
\[ p_2 \cdot p_3 = 0, \tag{7} \]

where \( \mu \) is some euclidean point that fixes the SMOM subtraction point.

3. Tensor decomposition and projection

As was already mentioned in the Introduction, we renormalize Eq. (2) without contracting the spinor indices and projecting on some particular baryonic currents. For this purpose, let us decompose the tensor in Eq. (2) as

\[ H_{\beta_1 \beta_2 \beta_3, \alpha_1 \alpha_2 \alpha_3}(p_1, p_2, p_3) = \sum_{n=1}^{N} T_{n, \beta_1 \beta_2 \beta_3, \alpha_1 \alpha_2 \alpha_3}(p_1, p_2, p_3) f_n \left( \{p_i p_j\} \right), \tag{8} \]

where \( T_n \) are spin tensor structures and \( f_n \) are scalar form factors. The explicit construction of these structures is discussed in Appendix A. The form factors \( f_n \) generally depend on six kinematic invariants, \( p_1^2, p_2^2, p_3^2, p_1 \cdot p_2, p_2 \cdot p_3, \) and \( p_3 \cdot p_1. \) In the following discussion, we omit spinor indices and arguments and simply write

\[ H = \sum_{n=1}^{N} T_n f_n. \tag{9} \]

The upper limit \( N \) of summation in Eqs. (8) and (9) is the number of the linearly independent spin tensor structures. It depends on the number of loops. We also have to distinguish between the decompositions in \( d \) and four
Table 1: Number of form factors for different numbers of loops in $d$ and four dimensions.

| # of loops | $N$ (in $d$ dimensions) | $N$ (in 4 dimensions) |
|------------|-------------------------|----------------------|
| 0          | 1                       | 67                   |
| 1          | 67                      | 247                  |
| 2          | 581                     | 247                  |

dimensions. In $d$ dimensions, the number of independent structures is larger, owing to the presence of evanescent operators. The values $N$ of independent form factors through two loops are given in Table 1.

Let us introduce the following notation. If $X_{\beta_1 \beta_2 \alpha_1 \alpha_2}$ is an object with six spinor indices, we denote by $\text{tr}_3(X)$ the trace over three pairs of indices, i.e.,

$$\text{tr}_3(X) = \sum_{\alpha_1, \alpha_2, \alpha_3=1}^4 X_{\alpha_1 \alpha_2 \alpha_3, \alpha_1 \alpha_2 \alpha_3}. \quad (10)$$

Using this definition, we can introduce the $N \times N$ symmetric matrix

$$M_{kn} = \text{tr}_3(T_k T_n), \quad (11)$$

where $T_j$ are the spin tensor structures from Eqs. (8) and (9). Then, the projectors on the form factors $f_j$ take the form

$$P_l = \sum_{k=1}^N M^{-1}_{lk} T_k, \quad (12)$$

where $M^{-1}$ is the inverse matrix, and we obviously have

$$f_l = \text{tr}_3(P_l H). \quad (13)$$

The use of Eqs. (12) and (13) for unrenormalized amplitudes is delicate within dimensional regularization, since the projectors $P_l$ depend nontrivially on the dimension $d$. A better way is to first renormalize the amplitude $H$ and then use the projectors in four dimensions. In order to achieve this, we construct $N$ scalar amplitudes $A_k$ as

$$A_k = \text{tr}_3(T_k H), \quad k = 1, \ldots, N. \quad (14)$$
After renormalization of all $A_k$ amplitudes in the $\overline{\text{MS}}$ scheme, the form factors can be obtained as\footnote{Indeed, we have $\sum_k M_{lk}^{-1} A_k = \sum_{k,n} M_{lk}^{-1} M_{kn} f_n = \sum_n \delta_{ln} f_n = f_l$.}

$$f_l = \sum_{k=1}^{N} M_{lk}^{-1} A_k,$$  \hspace{1cm} (15)

where $M^{-1}$ is now taken in four dimensions. In this limit, all elements of $M^{-1}$ are just rational numbers.

However, in four dimensions, we cannot apply the formula in Eq. (15) directly, since the determinant of the matrix $M_{lk}$ is then zero. This may be understood from Table 1 by observing that the number of independent structures in four dimensions is less than in $d$ dimensions. In this case, we need to solve the system (in matrix notation)

$$M \vec{A} = \vec{f},$$  \hspace{1cm} (16)

where $\vec{f} = (f_1, \ldots, f_N)^T$ etc.

The system (16) is over-determined, but consistent by construction. We find the solution for $\vec{f}$ in the form

$$\vec{f} = \vec{f}_0 + \sum_{j=0}^{N_d - N_4} C_j \vec{y}_j,$$  \hspace{1cm} (17)

where $\vec{f}_0$ is some particular solution of the system (16), the vectors $\vec{y}_j$ form a basis of the $N_d - N_4 = 334$ dimensional null space of the matrix $M$, and $C_j$ are arbitrary constants.

After renormalization, we have 581 two-loop form factors $f_n$ in four dimensions, 247 of which are linearly independent. We have calculated all of them analytically in $R_\xi$ gauge in terms of a set of complicated master integrals, which we have evaluated numerically.

4. Results

Because of their large number, we refrain from listing the renormalized two-loop form factors $f_n$ here, but supply them in ancillary files submitted to
the ArXiv along with this manuscript. Specifically, we present our analytic results in \( R_\xi \) gauge in the form of Eq. (17) including explicit expressions for the constants \( C_i \), and our numerical results in Landau gauge (\( \xi = 0 \)) for \( C_i = 0 \).

To illustrate the structure and typical size of the corrections, we present here, in numerical form, the two-loop form factor \( f_1 \), corresponding to the structure \( \Gamma_0 \otimes \Gamma_0 \otimes \Gamma_0 \), in \( R_\xi \) gauge:

\[
\frac{f_1}{f_{1,\text{Born}}} = 1 + a(0.6204053 + 0.595702 \xi) + a^2[10.45 + 3.59 \xi + 1.42 \xi^2 \pm 0.03 - (0.689 \pm 0.001)n_f],
\]

where \( f_{1,\text{Born}} = \epsilon^{ijk}\epsilon^{ijk} = 6 \) is the Born result, \( a = \alpha_s/\pi \), \( n_f \) is the number of light quark flavors, and \( \xi \) is the gauge parameter.

5. Conclusion

In this work, we have established a framework for the evaluation of the corrections to the baryonic current without derivatives through the two-loop order. The main difficulty in the study of the baryonic operators is the presence of evanescent operators that mix under renormalization with the physical operators. This leads to a large mixing matrix and the necessity for finite renormalizations. On the other hand, if we use the open-indices approach, there is no ambiguity in the interpretation in the \( \overline{\text{MS}} \) scheme. Exploiting this observation, we have evaluated all the form factors appearing through two loops and presented them in a numerical form that is ready for use in lattice QCD simulations.

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Appendix A. Spin tensor structures

In this Section, we explicitly enumerate all linearly independent spin tensor structures \( T_n \) through two loops in \( d \) dimensions. All tensors \( T_n \) are
represented as a tensor products of three Dirac structures, as

\[ T_{\alpha_1 \alpha_2 \alpha_3, \beta_1 \beta_2 \beta_3} = \Gamma_{\alpha_1 \beta_1} \otimes \Gamma_{\alpha_2 \beta_2} \otimes \Gamma_{\alpha_3 \beta_3}. \]  (A.1)

The building blocks \( \Gamma \) are anti-symmetric products of Dirac \( \gamma \) matrices,

\[
\Gamma_0 = \mathbb{1},
\]

\[
\Gamma_{\mu_1 \mu_2} = \frac{1}{2!} \gamma_{[\mu_1 \gamma_{\mu_2]}},
\]

\[
\Gamma_{\mu_1 \mu_2 \mu_3 \mu_4} = \frac{1}{4!} \gamma_{[\mu_1 \gamma_{\mu_2 \gamma_{\mu_3} \gamma_{\mu_4}}]},
\]

where \( \mathbb{1} \) is the unit Dirac matrix and square brackets \([...]\) denote antisymmetrisation. Notice that Dirac structures with odd numbers of Dirac matrices do not appear in our calculation.

We also introduce the following notation for the contraction of a vector and a tensor (Schoonship notation)

\[
p^\mu \Gamma_{...\mu...} = \Gamma_{...p...}.
\]

Furthermore, we introduce the following wild-cards: \( p \) can take one of \( p_1, p_2, p_3 \), \( pp \) can take one of \( p_1 p_2, p_2 p_3, p_3 p_1 \), and \( ppp \) stands for \( p_1 p_2 p_3 \).

For the sake of systematics, we assign to each tensor structure a signature, which is an ordered triplet of the numbers 0, 2, and 4 of \( \gamma \) matrices appearing in each \( \Gamma \) factor, and a number \([p]\) counting the overall appearances of momenta. Furthermore, we distinguish between symmetric and non-symmetric structures. The symmetric structures do not have co-partners arising under the change of order of the \( \Gamma \) factors in the tensor products, while the non-symmetric ones do. So, the numbers of non-symmetric structures should be multiplied by 3. In Tables \( \text{A.2} \) and \( \text{A.3} \), we systematically list the symmetric and non-symmetric tensor structures, respectively, and specify the number \((\#)\) of entities for each signature and each value of \([p]\). We also give the total number \((\#\#)\) of entities for each signature.
| signature | $[p]$ | tensor structure | # | ## |
|-----------|------|-----------------|---|----|
| 000       | 0    | $\Gamma_0 \otimes \Gamma_0 \otimes \Gamma_0$ | 1 | 1  |
| 222       | 0    | $\Gamma_{\mu_1 \mu_2} \otimes \Gamma_{\mu_2 \mu_3} \otimes \Gamma_{\mu_3 \mu_1}$ | 1 |    |
| 222       | 6    | $1/(\mu^2)^3 \Gamma_{pp} \otimes \Gamma_{pp} \otimes \Gamma_{pp}$ | 27 | 28 |

Table A.2: Symmetric structures ordered according to their signatures and values of $[p]$, numbers # of entities for given signature and value of $[p]$, and total numbers ## of entities for given signature.
\[
\begin{array}{lll}
422 & 0 & \Gamma_{\mu_1\mu_2\mu_3\mu_4} \otimes \Gamma_{\mu_1\mu_2} \otimes \Gamma_{\mu_3\mu_4} \\
422 & 2 & 1/(\mu^2)^1 \Gamma_{\mu_1\mu_2\mu_3} \otimes \Gamma_{\mu_1} \otimes \Gamma_{\mu_2\mu_3} \\
422 & 2 & 1/(\mu^2)^1 \Gamma_{\mu_1\mu_2\mu_3} \otimes \Gamma_{\mu_2\mu_3} \otimes \Gamma_{\mu_1} \\
422 & 2 & 1/(\mu^2)^1 \Gamma_{pp\mu_1\mu_2} \otimes \Gamma_{\mu_2\mu_3} \otimes \Gamma_{\mu_3\mu_1} \\
422 & 4 & 1/(\mu^2)^2 \Gamma_{pp\mu_1\mu_2} \otimes \Gamma_{\mu_1} \otimes \Gamma_{pp\mu_2} \\
422 & 4 & 1/(\mu^2)^2 \Gamma_{pp\mu_1\mu_2} \otimes \Gamma_{pp} \otimes \Gamma_{\mu_1\mu_2} \\
422 & 4 & 1/(\mu^2)^2 \Gamma_{pp\mu_1\mu_2} \otimes \Gamma_{\mu_1\mu_2} \otimes \Gamma_{pp} \\
422 & 4 & 1/(\mu^2)^2 \Gamma_{ppp\mu_1} \otimes \Gamma_{pp} \otimes \Gamma_{\mu_1\mu_2} \\
422 & 4 & 1/(\mu^2)^2 \Gamma_{ppp\mu_1} \otimes \Gamma_{ppp\mu_2} \otimes \Gamma_{\mu_1\mu_2} \\
422 & 6 & 1/(\mu^2)^3 \Gamma_{ppp\mu_1} \otimes \Gamma_{ppp} \otimes \Gamma_{\mu_1} \\
422 & 6 & 1/(\mu^2)^3 \Gamma_{ppp\mu_1} \otimes \Gamma_{pp} \otimes \Gamma_{ppp\mu_2} \\
\end{array}
\]

Table A.3: Non-symmetric structures. The meaning of the columns is the same as in Table A.2.

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