Sobolev $W^1_p$-spaces on closed subsets of $\mathbb{R}^n$

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Abstract

For each $p > n$ we use local oscillations and doubling measures to give intrinsic characterizations of the restriction of the Sobolev space $W^1_p(\mathbb{R}^n)$ to an arbitrary closed subset of $\mathbb{R}^n$.

1. Introduction.

1.1 Main definitions and results. Let $S$ be an arbitrary closed subset of $\mathbb{R}^n$. In this paper we describe the restrictions to $S$ of the functions in the Sobolev space $W^1_p(\mathbb{R}^n)$ whenever $p > n$.

We recall that, for each choice of the open set $\Omega \subset \mathbb{R}^n$ and of the exponent $p \in [1, \infty]$, the Sobolev space $W^1_p(\Omega)$ consists of all (equivalence classes of) real valued functions $f \in L^p(\Omega)$ whose first order distributional partial derivatives on $\Omega$ belong to $L^p(\Omega)$. $W^1_p(\Omega)$ is normed by

$$
\|f\|_{W^1_p(\Omega)} := \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}.
$$

We also recall that $L^1_p(\Omega)$ is the corresponding homogeneous Sobolev space, defined by the finiteness of the seminorm $\|f\|_{L^1_p(\Omega)} := \|\nabla f\|_{L^p(\Omega)}$.

There is an extensive literature devoted to describing the restrictions of Sobolev functions to different classes of subsets of $\mathbb{R}^n$. (We refer the reader to [A, BIN, BK, Bur, FJ, GV1, GV2, Jn, JW, K, M, MP, St, S4, T3] and references therein for numerous results in this direction and techniques for obtaining them.)

In this paper we consider restrictions of Sobolev functions to an arbitrary closed subset $S$ of $\mathbb{R}^n$ and ways of extending such functions back to Sobolev functions on all of $\mathbb{R}^n$. In the case where $p = \infty$, the Sobolev space $W^1_\infty(\mathbb{R}^n)$ can be identified with the space $\text{Lip}(\mathbb{R}^n)$ of Lipschitz functions on $\mathbb{R}^n$ and it is known that the restriction $\text{Lip}(\mathbb{R}^n)|_S$ coincides with the space $\text{Lip}(S)$ of Lipschitz functions on $S$ and that, furthermore, the classical Whitney extension operator linearly and continuously maps the space $\text{Lip}(S)$ into the space $\text{Lip}(\mathbb{R}^n)$ (see e.g., [St], Chapter 6).

Here we will show that, perhaps surprisingly, an analogous result also holds for all $p$ in the range $n < p < \infty$, namely, that very same classical linear Whitney extension operator also provides an extension to $W^1_p(\mathbb{R}^n)$, even an almost optimal extension, of each function on $S$ which is the restriction to $S$ of a function in $W^1_p(\mathbb{R}^n)$. We also provide various intrinsic characterizations of the restriction of $W^1_p(\mathbb{R}^n)$ to the given set $S$. To the best of our knowledge, these are the first results of this type to have been obtained for the range $n < p < \infty$ without the imposition of some extra conditions on the set $S$.

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Recall that, when \( p > n \), it follows from the Sobolev embedding theorem that every function \( f \in L^1_p(\mathbb{R}^n) \) coincides almost everywhere with a function satisfying the Hölder condition of order \( \alpha := 1 - \frac{n}{p} \). I.e., after possibly modifying \( f \) on a set of Lebesgue measure zero, we have

\[
|f(x) - f(y)| \leq C(n, p) \|f\|_{L^1_p(\mathbb{R}^n)} \|x - y\|^{1 - \frac{n}{p}} \text{ for all } x, y \in \mathbb{R}^n. \tag{1.1}
\]

This fact enables us to identify each element \( f \in W^{1}_p(\mathbb{R}^n), p > n \), with its unique continuous representative. This identification, in particular, implies that \( f \) has a well defined restriction to any given subset of \( \mathbb{R}^n \). Furthermore it means that there is no loss of generality in only considering closed subsets of \( \mathbb{R}^n \). Thus, given an arbitrary closed subset \( S \) of \( \mathbb{R}^n \), we define the trace space \( W^{1}_p(\mathbb{R}^n)|_S, p > n, \) of all restrictions of \( W^{1}_p(\mathbb{R}^n) \)-functions to \( S \) by

\[
W^{1}_p(\mathbb{R}^n)|_S := \{ f : S \rightarrow \mathbb{R} : \text{there is a continuous } F \in W^{1}_p(\mathbb{R}^n) \text{ such that } F|_S = f \}.
\]

\( W^{1}_p(\mathbb{R}^n)|_S \) is equipped with the standard quotient space norm

\[
\|f\|_{W^{1}_p(\mathbb{R}^n)|_S} := \inf\{\|F\|_{W^{1}_p(\mathbb{R}^n)} : F \in W^{1}_p(\mathbb{R}^n) \text{ and continuous, } F|_S = f \}.
\]

The trace space \( L^1_p(\mathbb{R}^n)|_S \) is defined in an analogous way.

Our first main result, Theorem 1.1, gives an intrinsic characterization of the space \( L^1_p(\mathbb{R}^n)|_S \). Before formulating this result we need to fix some notation: Throughout this paper, the terminology “cube” will mean a closed cube in \( \mathbb{R}^n \) whose sides are parallel to the coordinate axes. We let \( Q(x,r) \) denote the cube in \( \mathbb{R}^n \) centered at \( x \) with side length \( 2r \). Given \( \lambda > 0 \) and a cube \( Q \) we let \( \lambda Q \) denote the dilation of \( Q \) with respect to its center by a factor of \( \lambda \). (Thus \( \lambda Q(x,r) = Q(x,\lambda r) \).) The Lebesgue measure of a measurable set \( A \subset \mathbb{R}^n \) will be denoted by \( |A| \).

**Theorem 1.1** A function \( f \) defined on a closed set \( S \subset \mathbb{R}^n \) can be extended to a (continuous) function \( F \in L^1_p(\mathbb{R}^n), n < p < \infty \), if and only if there exists a constant \( \lambda > 0 \) such that for every finite family \( \{Q_i : i = 1, \ldots, m\} \) of pairwise disjoint cubes in \( \mathbb{R}^n \) and every choice of points

\[
x_i, y_i \in (11Q_i) \cap S \tag{1.2}
\]

the inequality

\[
\sum_{i=1}^{m} \frac{|f(x_i) - f(y_i)|^p}{(\text{diam } Q_i)^{p-n}} \leq \lambda \tag{1.3}
\]

holds. Moreover,

\[
\|f\|_{L^1_p(\mathbb{R}^n)|_S} \sim \inf \lambda^{\frac{1}{p}} \tag{1.4}
\]

with constants of equivalence depending only on \( n \) and \( p \).
The special case of this result where \( S = \mathbb{R}^n \) has been treated by Yu. Brudnyi [Br1]. He proved that a function \( f \) is in \( L^p_1(\mathbb{R}^n) \), \( p > n \), if and only if the inequality (1.3) holds for every finite family \( \{Q_i\} \) of disjoint cubes and all \( x_i, y_i \in Q_i \).

In particular, Brudnyi’s result shows that, for \( S = \mathbb{R}^n \), the condition (1.2) can be replaced by \( x_i, y_i \in Q_i \). We do not know whether the same is true in the case of an arbitrary subset \( S \). We do know that a more elaborate variant of our proof of Theorem 1.1 enables the factor 11 in (1.2) to be decreased to \( 3 + \varepsilon \) where \( \varepsilon > 0 \) is arbitrary. (Of course, in this case the constants of equivalence in (1.4) will also depend on \( \varepsilon \).

We now formulate the second main result of the paper, Theorem 1.2, which describes the restrictions of \( W^1_p(\mathbb{R}^n) \)-functions to \( S \).

**Theorem 1.2** Let \( S \) be an arbitrary closed subset of \( \mathbb{R}^n \). Let \( \varepsilon \) be an arbitrary positive number and let \( S_\varepsilon \) denote the \( \varepsilon \)-neighborhood of \( S \). Fix a number \( \theta \geq 1 \) and let \( T : S_\varepsilon \to S \) be a measurable mapping such that

\[
\|T(x) - x\| \leq \theta \text{dist}(x, S) \quad \text{for every} \quad x \in S_\varepsilon.
\]

The function \( f : S \to \mathbb{R} \) is an element of \( W^1_p(\mathbb{R}^n)|_S \) if and only if \( f \circ T \in L^p(S_\varepsilon) \) and there exists a constant \( \lambda > 0 \) such that the inequality (1.3) holds for every finite family \( \{Q_i : i = 1,...,m\} \) of disjoint cubes contained in \( S_\varepsilon \), and every choice of points \( x_i, y_i \in (\eta Q_i) \cap S \), where \( \eta := 10 \theta + 1 \). Moreover,

\[
\|f\|_{W^1_p(\mathbb{R}^n)|_S} \sim \|f \circ T\|_{L^p(S_\varepsilon)} + \inf \lambda^{\frac{1}{p}}
\]

with constants of equivalence depending only on \( n, p, \varepsilon \) and \( \theta \).

**Remark 1.3** The mapping \( T \) appearing in this theorem must of course satisfy \( T(x) = x \) for all \( x \in S \).

In fact, a simple example of a mapping \( T \) satisfying (1.5) can be easily constructed with the help of a Whitney decomposition \( W(S) = \{Q_k : k \in I\} \) of the open set \( \mathbb{R}^n \setminus S \). (Recall that \( W(S) \) is a covering of \( \mathbb{R}^n \setminus S \) by non-overlapping cubes such that \( \text{diam } Q_k \sim \text{dist}(Q_k, S), k \in I \).) Let \( a_Q \) be a point in \( S \) which is nearest to \( Q \) and let

\[
T(x) := \begin{cases} 
  x, & x \in S, \\
  a_Q, & x \in \text{int}(Q), \quad Q \in W(S),
\end{cases}
\]

where, for each set \( A \subset \mathbb{R}^n \), \( \text{int}(A) \) denotes the interior of \( A \). Clearly, \( T \) is defined a.e. on \( \mathbb{R}^n \) and satisfies the inequality (1.5) (with a constant \( \theta \) depending only on parameters of the Whitney decomposition \( W(S) \)). If we use this particular mapping \( T \) then (1.6) yields the following formula for the trace norm of a function:

\[
\|f\|_{W^1_p(\mathbb{R}^n)|_S} \sim \|f\|_{L^p(S)} + \left\{ \sum_{Q \in W(S), \text{diam } Q \leq \varepsilon} |f(a_Q)|^p |Q| \right\}^{\frac{1}{p}} + \inf \lambda^{\frac{1}{p}}
\]

(with constants of equivalence depending only on \( n, p, \varepsilon \), and parameters of \( W(S) \)).
Our next result provides a different kind of description of trace spaces. It is expressed in terms of a certain kind of maximal function which is a variant of \( f_p^\# \), the familiar sharp maximal function in \( L_p \). Let us first recall that, for each \( p \in [1, \infty) \) and each 
\( f \in L_{p, \text{loc}}(\mathbb{R}^n) \), one defines \( f_p^\# \) by

\[
f_p^\#(x) := \sup_{r > 0} \frac{1}{r} \left( \frac{1}{|Q(x, r)|} \int_{Q(x, r)} |f(y) - f_Q(x, r)|^p \, dy \right)^{\frac{1}{p}}.
\]

Here \( f_Q := |Q|^{-1} \int_Q f \, dx \) denotes the average of \( f \) on the cube \( Q \). For \( p = \infty \) we put

\[
f_{\infty}^\#(x) := \sup \{|f(y) - f(z)|/r : r > 0, y, z \in Q(x, r)\}.
\]

In [C1] Calderón proved that, for \( 1 < p \leq \infty \), a function \( f \) is in \( W^1_p(\mathbb{R}^n) \) if and only if \( f \) and \( f_1^\# \) are both in \( L_p(\mathbb{R}^n) \), and that, furthermore,

\[
\|f\|_{W^1_p(\mathbb{R}^n)} \sim \|f\|_{L_p(\mathbb{R}^n)} + \|f_1^\#\|_{L_p(\mathbb{R}^n)} \tag{1.8}
\]

with constants of equivalence depending only on \( p \) and \( n \). See also [CS]. We observe that the Sobolev imbedding theorem (for \( p > n \)) allows us to replace \( f_1^\# \) in (1.8)) with the (bigger) sharp maximal function \( f_\infty^\# \). Thus, for \( p > n \) and for every \( f \in W^1_p(\mathbb{R}^n) \), we have

\[
\|f\|_{W^1_p(\mathbb{R}^n)} \sim \|f\|_{L_p(\mathbb{R}^n)} + \|f_\infty^\#\|_{L_p(\mathbb{R}^n)}.
\]

This equivalence motivates us to introduce the new variant of the sharp maximal function \( f_\infty^\# \) that we need here. It is defined with respect to an arbitrary closed set \( S \subset \mathbb{R}^n \) for functions \( f : S \to \mathbb{R} \). It is given by

\[
f_{\infty, S}^\#(x) := \sup \{|f(y) - f(z)|/r : r > 0, y, z \in Q(x, r) \cap S\}, \quad x \in \mathbb{R}^n.
\]

**Theorem 1.4** Let \( S \subset \mathbb{R}^n \) be a closed set and let \( p > n \).

(i) A function \( f \) defined on \( S \) can be extended to a (continuous) function \( F \in L^1_p(\mathbb{R}^n) \)
if and only if \( f_{\infty, S}^\# \in L^p_p(\mathbb{R}^n) \). Moreover,

\[
\|f\|_{L^1_p(\mathbb{R}^n)} \sim \|f_{\infty, S}^\#\|_{L^p_p(\mathbb{R}^n)}. \tag{1.9}
\]

(ii) Given \( \varepsilon > 0 \) and \( \theta \geq 1 \), let \( S_\varepsilon \) be the \( \varepsilon \)-neighborhood of \( S \), and let \( T : S_\varepsilon \to S \) be a measurable mapping satisfying the condition (1.5). A function \( f \) defined on \( S \) is an element of \( W^1_p(\mathbb{R}^n)|_S \) if and only if \( f \circ T \) and \( f_{\infty, S}^\# \) are both in \( L^p_p(S_\varepsilon) \). Furthermore,

\[
\|f\|_{W^1_p(\mathbb{R}^n)}|_S \sim \|f \circ T\|_{L^p_p(S_\varepsilon)} + \|f_{\infty, S}^\#\|_{L^p_p(S_\varepsilon)}. \tag{1.10}
\]

The constants of equivalence in (1.9) depend only on \( n \) and \( p \), and in (1.10) they only depend on \( n \), \( p \), \( \varepsilon \) and \( \theta \).

As already mentioned above, the classical Whitney extension operator provides an almost optimal extension of a function \( f \) defined on \( S \) to a function \( F \) in \( W^1_p(\mathbb{R}^n) \) or in \( L^1_p(\mathbb{R}^n) \), whenever \( p > n \). Since this extension operator is linear, we obtain the following
Theorem 1.5 For every closed subset $S \subset \mathbb{R}^n$ and every $p > n$ there exists a linear extension operator which maps the trace space $W^1_p(\mathbb{R}^n)|_S$ continuously into $W^1_p(\mathbb{R}^n)$ and also maps the trace space $L^1_p(\mathbb{R}^n)|_S$ continuously into $L^1_p(\mathbb{R}^n)$. Its operator norms are bounded by constants depending only on $n$ and $p$.

1.2 Extensions of Sobolev functions and doubling measures. In this subsection we present a series of results which describe the restrictions of $W^1_p(\mathbb{R}^n)$-functions to a closed set $S$ via certain doubling measures supported on $S$. As before, we assume that $p > n$. Our approach here is inspired by a paper of Jonsson [J] devoted to characterization of traces of Besov spaces to arbitrary closed subsets of $\mathbb{R}^n$, see Section 6.

Let $\mu$ be a positive regular Borel measure on $\mathbb{R}^n$ such that $\text{supp} \, \mu = S$.

Following [VK], we say that $\mu$ satisfies the condition $(D_n)$, if there exists a constant $C = C_\mu > 0$ such that, for all $x \in S$, $0 < r \leq 1$ and $1 \leq k \leq 1/r$, the following inequality

$$\mu(Q(x, kr)) \leq C_\mu k^n \mu(Q(x, r))$$

holds. The condition $(D_n)$ is a generalization of the familiar doubling condition:

$$\mu(Q(x, 2r)) \leq c_\mu \mu(Q(x, r)), \quad x \in S, \ 0 < r \leq 1/2.$$  \tag{1.12}

We refer to the constant $c_\mu$ in (1.12) as a doubling constant.

We also assume that the measure $\mu$ satisfies the following condition: there exist constants $C''_\mu, C'''_\mu > 0$ such that

$$C''_\mu \leq \mu(Q(x, 1)) \leq C'''_\mu, \quad \text{for all } x \in S.$$  \tag{1.13}

Dynkin [D1, D2] conjectured that every compact set $S \subset \mathbb{R}^n$ carries a non-trivial doubling measure $\mu$. He constructed a doubling measure on every compact set $S \subset \mathbb{R}^1$ which satisfies a certain “porosity” condition (see the so-called “the ball condition” mentioned below). Dynkin’s conjecture was subsequently proved by Volberg and Konyagin [VK]. (See also Wu [Wu].) Moreover, Volberg and Konyagin showed that every compact set in $\mathbb{R}^n$ carries a non-trivial measure $\mu$ satisfying the $(D_n)$-condition (1.11). Using their argument, Luukkainen and Saksman [LS] extended this result to all closed subsets of $\mathbb{R}^n$. Jonsson [J] showed that it may be also assumed that $\mu$ satisfies (1.13).

Throughout this paper $\sigma$ will denote a measure supported on the boundary $\partial S$ of the set $S$. It will usually be assumed that $\sigma$ satisfies the generalized doubling condition $(D_n)$ and also the condition (1.13). Thus $\text{supp} \, \sigma = \partial S$, and there exist positive constants $C_\sigma, C'_\sigma, C''_\sigma$, such that

$$\sigma(Q(x, kr)) \leq C_\sigma k^n \sigma(Q(x, r)), \quad x \in \partial S, \ 1 \leq k \leq 1/r,$$  \tag{1.14}

and

$$C'_\sigma \leq \sigma(Q(x, 1)) \leq C''_\sigma, \quad x \in \partial S.$$  \tag{1.15}
As preparation for formulating our the next main result, Theorem 1.6, we need to define a special kind of “distance” between points of some subset of $\mathbb{R}^n$. Given a set $A \subset \mathbb{R}^n$ and points $x, y \in A$, let

$$
\rho_A(x, y) := \inf \{ \text{diam} : Q \text{ is a cube, } Q \ni x, y, \frac{1}{15}Q \subset \mathbb{R}^n \setminus A \}. 
$$

(1.16)

Thus we have $\rho_A(x, y) := +\infty$ whenever there does not exist any cube $Q$ for which $x, y \in Q$ and $\frac{1}{15}Q \subset \mathbb{R}^n \setminus A$. Clearly, $\|x - y\| \leq \rho_A(x, y)$ for every $x, y \in A$.

**Theorem 1.6** Let $S$ be an arbitrary closed subset of $\mathbb{R}^n$. Let $\mu$ be a measure with $\text{supp} \mu = S$ satisfying (1.11) and (1.13), and let $\sigma$ be a measure supported on $\partial S$ which satisfies (1.14) and (1.15). Then, for every $p \in (n, \infty)$ and every function $f \in C(S)$, we have

$$
\|f\|^p_{L^p(S)} \sim \|f\|_{L^p(\mu)} + \sup_{0 < t \leq 1} \left( \int \int \frac{|f(x) - f(y)|^p}{t^{p-n}\mu(Q(x, t))^2} \, d\mu(x) \, d\mu(y) \right)^{\frac{1}{p}}
$$

$$
+ \|f\|_{L^p(\sigma)} + \left( \int \int_{\rho(x, y) < 1} \frac{|f(x) - f(y)|^p}{\rho(x, y)^{p-n}\sigma(Q(x, \rho(x, y)))^2} \, d\sigma(x) \, d\sigma(y) \right)^{\frac{1}{p}}
$$

with constants of equivalence depending only on $n, p$, and the constants $C_\mu, C'_\mu, C''_\mu$ and $C_\sigma, C'_\sigma, C''_\sigma$.

**Remark 1.7** Let us make some comments about the expressions

$$
I_1(f; S) := \sup_{0 < t \leq 1} \left( \int \int \frac{|f(x) - f(y)|^p}{t^{p-n}\mu(Q(x, t))^2} \, d\mu(x) \, d\mu(y) \right)^{\frac{1}{p}}
$$

and

$$
I_2(f; S) := \left( \int \int_{\rho(x, y) < 1} \frac{|f(x) - f(y)|^p}{\rho(x, y)^{p-n}\sigma(Q(x, \rho(x, y)))^2} \, d\sigma(x) \, d\sigma(y) \right)^{\frac{1}{p}}
$$

which appear in the statement of the preceding theorem. In particular $I_1(f; S)$ can be rewritten in the form

$$
I_1(f; S) = \sup_{0 < t \leq 1} w_1(f; t)_{L^p(S; \mu)}/t,
$$

(1.17)

where

$$
w_1(f; t)_{L^p(S; \mu)} := \left( \frac{1}{t^n} \int \int \frac{|f(x) - f(y)|^p}{\mu(Q(x, t))^2} \, d\mu(x) \, d\mu(y) \right)^{\frac{1}{p}}, \quad t > 0.
$$
We will refer to the function $w_1(f; \cdot)_{L^p(S; \mu)}$ as the averaged modulus of smoothness of $f$ in $L^p(S; \mu)$. This name is motivated by the following example: If we choose $S = \mathbb{R}^n$ and let $\mu$ be Lebesgue measure, then the function $w_1(f; \cdot)_{L^p(\mathbb{R}^n; \mu)}$ coincides with the function

$$w_1(f; t)_{L^p} := \left( \frac{1}{t^n} \int_{\|h\| < t} \int |f(x + h) - f(x)|^p \, dx \, dh \right)^{\frac{1}{p}},$$

which is known in the literature as the averaged modulus of smoothness in $L^p$. It is also known (see, e.g. [DL]) that $w_1(f; \cdot)_{L^p} \sim \omega(f; \cdot)_{L^p}$ where $\omega(f; \cdot)_{L^p}$ is the classical modulus of smoothness in $L^p$:

$$\omega(f; t)_{L^p} := \sup_{\|h\| < t} \left( \int |f(x + h) - f(x)|^p \, dx \right)^{\frac{1}{p}}. \quad (1.18)$$

It is known that

$$\sup_{t > 0} \frac{\omega(f; t)_{L^p}}{t} \sim \|f\|_{L^2(\mathbb{R}^n)}, \quad (1.19)$$

see, e.g. [Z], p. 45, so that

$$\|f\|_{W^1_{p}(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)} + \sup_{0 < t \leq 1} \omega(f; t)_{L^p} / t. \quad (1.20)$$

Consequently, we have

$$\|f\|_{W^1_{p}(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)} + \sup_{0 < t \leq 1} w_1(f; t)_{L^p} / t. \quad (1.20)$$

In view of (1.20) and (1.17), the quantity $\|f\|_{L^p(\mu)} + I_1(f; S)$ can be interpreted as a certain Sobolev-type norm on $S$. This motivates us to call $I_1(f; S)$ the Sobolev part of the trace norm $\|f\|_{W^1_{p}(\mathbb{R}^n)|_S}$.

We now discuss the second expression $I_2(f; S)$. This quantity depends only on the values of $f$ on $\partial S$, the measure $\sigma$ supported on $\partial S$ and the function $\rho_{\partial S}$ defined on $\partial S$. So we can consider $I_2(f; S)$ as a “characteristic” of the trace $f|_S$ which controls its behavior on the boundary of the set $S$. Let us consider the example when $S = \mathbb{R}^d$ for some $d$, $0 < d < n$. Then, clearly, $S \sim \partial S$ and $\rho_{\partial S}(x, y) \sim \|x - y\|$. Suppose that, for this choice of $S$, we take $\sigma$ to be Lebesgue measure on $\mathbb{R}^d$ so that $\sigma(Q(x, t)) \sim t^d$ for every $x \in S$ and every $t > 0$. It can be readily shown that in this case

$$\|f\|_{L^p(\sigma)} + I_2(f; S) \sim \|f\|_{L^p(\mathbb{R}^d)} + \left( \int \int_{\|x - y\| < 1} \rho_{\partial S}^{p-n} \rho_{\partial S}^d \, d\sigma(x) \, d\sigma(y) \right)^{\frac{1}{p}}$$

$$= \|f\|_{L^p(\mathbb{R}^d)} + \left( \int \int_{\|x - y\| < 1} \rho_{\partial S}^{p-n} \rho_{\partial S}^d \, d\sigma(x) \, d\sigma(y) \right)^{\frac{1}{p}}.$$
In fact this last quantity is the norm in a certain Besov space of functions defined on $\mathbb{R}^d$. We recall that the Besov space $B_{p,p}^s(\mathbb{R}^d)$, for $0 < s < 1$, is defined by the finiteness of the norm

$$\|f\|_{B_{p,p}^s(\mathbb{R}^d)} := \|f\|_{L_p(\mathbb{R}^d)} + \left( \int \int_{\|x-y\| < 1} \frac{|f(x) - f(y)|^p}{\|x-y\|^{d+sp}} \ d\sigma(x) \ d\sigma(y) \right)^{\frac{1}{p}}.$$ 

(For the general theory of Besov spaces we refer the reader to the monographs [BIN, N, T2, T3] and references therein.)

Thus for $S = \mathbb{R}^d$ the quantity $\|f\|_{L_p(\sigma)} + I_2(f; S)$ is an equivalent norm for the Besov space $B_{p,p}^{1-\frac{d}{p}}(\mathbb{R}^d)$. This suggests that we should interpret $\|f\|_{L_p(\sigma)} + I_2(f; S)$ for an arbitrary set $S \subset \mathbb{R}^n$ as a certain "Besov-type" norm on $\partial S$. This example also motivates us to call $\|f\|_{L_p(\sigma)} + I_2(f; S)$ the Besov part of the trace norm $\|f\|_{W_p^1(\mathbb{R}^n)|_S}$.

To summarize, these observations and examples give us some idea of the structure of the trace space $W_p^1(\mathbb{R}^n)|_S$ for $p > n$. In particular, they show that, in general, $W_p^1(\mathbb{R}^n)|_S$ is a certain "Sobolev-type" space of functions on $S$ whose values on the boundary belong to a certain "Besov-type" space of functions defined on $\partial S$.

The next theorem provides additional tools for the calculation of the trace norm $\| \cdot \|_{W_p^1(\mathbb{R}^n)|_S}$. In particular it offers us the options of using either a suitable measure $\sigma$ supported on $\partial S$ or a suitable measure $\mu$ supported on $S$, for this calculation. It is convenient to have this flexibility since, on the one hand, there are many choices of $S$ for which the measure $\sigma$ can be easily determined, whereas it may be a very non-trivial task to specify a suitable measure $\mu$, and, on the other hand, there are sets $S$ for which it is obvious how to choose $\mu$ but very difficult to specify $\sigma$.

**Theorem 1.8** Let $p \in (n, \infty)$.

(i). Let $\sigma$ be a measure supported on $\partial S$ which satisfies conditions (1.14) and (1.15), and let $\Omega := \text{int}(S)$ be the interior of $S$. Then for every $f \in C(S)$ the following equivalence

$$\|f\|_{W_p^1(\mathbb{R}^n)|_S} \sim \|f\|_{\Omega} W_p^1(\Omega) + \|f\|_{L_p(\sigma)} + \sup_{0 < t \leq 1} \left( \int \int_{|x-y| < t} \frac{|f(x) - f(y)|^p}{t^{p-n}\sigma(Q(x,t))^2} \ d\sigma(x) \ d\sigma(y) \right)^{\frac{1}{p}}$$

$$+ \left( \int \int_{\rho_\partial S(x,y) < 1} \frac{|f(x) - f(y)|^p}{\rho_\partial S(x,y)^{p-n}\sigma(Q(x,\rho_\partial S(x,y)))^2} \ d\sigma(x) \ d\sigma(y) \right)^{\frac{1}{p}}$$

holds.

(ii). For every function $f \in C(S)$ and every measure $\mu$ supported on $S$ and satisfying...
Let \( \Omega := \text{int}(S) \), and let \( \sigma \) be a measure with \( \text{supp} \sigma = \partial S \) satisfying conditions (1.14) and (1.15). Suppose that \( \partial S \) satisfies the ball condition then, in the equivalence of Theorem 1.8, (i), one can omit the third term on the right-hand side.

**Theorem 1.10** Let \( p \in (n, \infty) \), \( S = \partial S \), and let \( \sigma \) be a measure with \( \text{supp} \sigma = \partial S \) satisfying conditions (1.14) and (1.15). Suppose that \( \partial S \) satisfies the ball condition. Then, for every \( f \in C(S) \), we have

\[
\|f\|w_p^1(\mathbb{R}^n)_s \sim \|f\|w_p^1(\Omega) + \|f\|L_p(\sigma) + \left( \iint_{|x-y| < 1} \frac{|f(x) - f(y)|^p}{\rho_S(x, y)^{p-n} \mu(Q(x, \rho_S(x, y)))^2} d\mu(x) d\mu(y) \right)^{\frac{1}{p}}.
\]

with constants of equivalence depending only on \( n, p, C_\sigma, C'_\sigma, C''_\sigma \) (for the case (i)) and \( C_\mu, C'_\mu, C''_\mu \) (for (ii)).

The results of Theorems 1.6 and 1.8 can be simplified essentially whenever \( S \) or \( \mathbb{R}^n \setminus S \) possesses certain “plumpness” properties.

Following Triebel [T3], Definition 9.16, we introduce the next

**Definition 1.9** We say that a closed set \( A \) satisfies the ball condition if there exists a constant \( \beta_A > 0 \) such that the following statement is true: For each Euclidean ball \( B \) of diameter at most 1 whose center lies in \( A \), there exists a ball \( B' \subset B \setminus A \) such that \( \text{diam} B \leq \beta_A \text{diam} B' \).

It can be readily seen that \( A \) satisfies the ball condition if and only if \( \rho_A(x, y) \sim \|x - y\| \) for all \( x, y \) such that \( |x - y| \leq 1 \).

As the next theorem shows, if \( \partial S \) satisfies the ball condition then, in the equivalence

\[
\|f\|w_p^1(\mathbb{R}^n)_s \sim \|f\|w_p^1(\Omega) + \|f\|L_p(\mu) + \left( \iint_{|x-y| < 1} \frac{|f(x) - f(y)|^p}{\rho_S(x, y)^{p-n} \mu(Q(x, \rho_S(x, y)))^2} d\mu(x) d\mu(y) \right)^{\frac{1}{p}}.
\]

with \( \text{supp} \sigma = \partial S \) and the ball condition constant \( \beta_{\partial S} \).

In particular, if \( \Omega := \text{int}(S) = \emptyset \), then, clearly, \( S = \partial S \), so that we can put \( \sigma = \mu \). Also, since \( \Omega = \emptyset \), the term \( \|f\|w_p^1(\Omega) \) in (1.21) can be omitted. Thus we have

**Theorem 1.11** Let \( p \in (n, \infty) \) and let \( \mu \) be a measure with \( \text{supp} \mu = \partial S \) satisfying conditions (1.11) and (1.13). Assume that \( \text{int}(S) = \emptyset \) and that \( S \) satisfies the ball condition. Then, for every \( f \in C(S) \),

\[
\|f\|w_p^1(\mathbb{R}^n)_s \sim \|f\|w_p^1(\mu) + \left( \iint_{|x-y| < 1} \frac{|f(x) - f(y)|^p}{\rho_S(x, y)^{p-n} \mu(Q(x, \rho_S(x, y)))^2} d\mu(x) d\mu(y) \right)^{\frac{1}{p}}.
\]

(1.11) and (1.13), we have

\[
\|f\|w_p^1(\mathbb{R}^n)_s \sim \|f\|L_p(\mu) + \sup_{0 < t < 1} \left( \iint_{|x-y| < t} \frac{|f(x) - f(y)|^p}{t^{p-n} \mu(Q(x, t))^2} d\mu(x) d\mu(y) \right)^{\frac{1}{p}}
\]

\[
+ \left( \iint_{\rho_S(x, y) < 1} \frac{|f(x) - f(y)|^p}{\rho_S(x, y)^{p-n} \mu(Q(x, \rho_S(x, y)))^2} d\mu(x) d\mu(y) \right)^{\frac{1}{p}}.
\]
with constants of equivalence depending only on $n, p, C_{\mu}, C_{\mu}', C_{\mu}''$, and $\beta_S$.

Norms similar to the one on the right hand side of (1.22) have been used by Dynkin [D1] to prove results about the trace to the boundary of the unit circle of analytic functions whose derivatives are in the Hardy space $H^p$. Note also that similar norms have also been used by Jonsson [J] to characterize the restrictions of Besov spaces to closed subsets of $\mathbb{R}^n$. See Section 6 for more details about this.

Theorem 1.11 implies several known results about traces of Sobolev space to different classes of “thin” (or “porous”) subsets of $\mathbb{R}^n$. For instance, let $S = \mathbb{R}^d \subset \mathbb{R}^n$ where $0 < d < n$. As we have seen in Remark 1.7, in this case, for $p > n$, the right hand side of (1.22) coincides with one of the norms for the Besov space $B^{1-\frac{n-d}{p}}_{p,p}(\mathbb{R}^d)$. This implies the classical trace description of Sobolev spaces to $\mathbb{R}^d$,

$$W^{1}_{p}(\mathbb{R}^n)|_{\mathbb{R}^d} = B^{1-\frac{n-d}{p}}_{p,p}(\mathbb{R}^d),$$

(1.23)

see, e.g. [BIN]. (Note that this isomorphism holds for all $p \geq 1, 0 < q \leq \infty$.)

Jonsson and Wallin [J1, JW] generalized isomorphism (1.23) to the family of $d$-sets in $\mathbb{R}^n$. We recall that a Borel set $S$ in $\mathbb{R}^n$ is called a $d$-set, $0 < d \leq n$, if the $d$-dimensional Hausdorff measure $\mathcal{H}_d$ on $S$, $\mu := \mathcal{H}_d|_S$, satisfies the following condition: there exist constants $C_1, C_2 > 0$, such that

$$C_1 r^d \leq \mu(Q(x,r)) \leq C_2 r^d \quad \text{for all } x \in S \text{ and } 0 < r \leq 1.$$ 

(1.24)

Given a $d$-set $S \subset \mathbb{R}^n$, the Besov space $B^\alpha_{p,p}(S), 0 < \alpha < 1, 1 \leq p < \infty$, is defined as the family of all functions $f \in L^p(S; \mu)$ such that the following norm

$$\|f\|_{B^\alpha_{p,p}(S)} := \|f\|_{L^p(S; \mu)} + \left( \int \int_{\|x-y\| < 1} \frac{|f(x) - f(y)|^p}{\|x-y\|^{d+\alpha}} d\mu(x) d\mu(y) \right)^{\frac{1}{p}}$$

is finite, see [JW], p. 103. Jonsson and Wallin proved that for every $d$-set $S \subset \mathbb{R}^n$ and every $1 < p < \infty, 0 < n - d < p$,

$$W^{1}_{p}(\mathbb{R}^n)|_S = B^{1-\frac{n-d}{p}}_{p,p}(S).$$

(1.25)

It can be readily seen that every $d$-set with $d < n$ satisfies the ball condition and its interior is empty, so that the isomorphism (1.25) for $p > n$ also follows from Theorem 1.11.

Thus Theorem 1.11 extends both of the isomorphisms (1.23) and (1.25) (in the case $p > n$) to all of the family of sets $S \subset \mathbb{R}^n$ which have empty interior and satisfy the ball condition. In Section 7 we present an example (see Example 7.26) of a constructive description of the trace space $W^{1}_{p}(\mathbb{R}^2)|_S, p > 2$, for such a set $S \subset \mathbb{R}^2$ which is not a $d$-set for any $d > 0$.

1.3 A Whitney-type extension theorem for $W^{k}_{p}(\mathbb{R}^n)$-functions, $p > n$. The proof of the result which we describe in this subsection does not in fact appear in this
As usual we let $L^k_p(\mathbb{R}^n)$ denote the (homogeneous) Sobolev space of all functions defined on $\mathbb{R}^n$ whose distributional partial derivatives of order $k$ belong to $L^p(\mathbb{R}^n)$. $L^k_p(\mathbb{R}^n)$ is normed by

$$
\|f\|_{L^k_p(\mathbb{R}^n)} := \sum \{\|D^\alpha f\|_{L^p(\mathbb{R}^n)} : |\alpha| = k\}.
$$

By the Sobolev imbedding theorem, see e.g., [M], p. 60, every $f \in L^k_p(\mathbb{R}^n), p > n$, can be redefined, if necessary, in a set of Lebesgue measure zero so that it belongs to the space $C^{k-1,\alpha}(\mathbb{R}^n)$ of functions whose partial derivatives of order $k - 1$ satisfy the Hölder condition of order $\alpha := 1 - \frac{n}{p}$:

$$
|D^\beta f(x) - D^\beta f(y)| \leq C(n, p)\|f\|_{L^k_p(\mathbb{R}^n)}\|x - y\|^{1 - \frac{n}{p}}, \quad x, y, \in \mathbb{R}^n, |\beta| = k - 1.
$$

This means that, for $p > n$, we can identify each element $f \in L^k_p(\mathbb{R}^n)$ with its unique $C^{k-1}$-representative, and this enables us to define the trace space $L^k_p(\mathbb{R}^n)|S, p > n$, of all restrictions of $L^k_p(\mathbb{R}^n)$-functions to an arbitrary closed set $S$,

$$
L^k_p(\mathbb{R}^n)|S := \{f : S \rightarrow \mathbb{R} : \text{there is } F \in L^k_p(\mathbb{R}^n) \cap C^{k-1}(\mathbb{R}^n) \text{ such that } F|_S = f\}.
$$

We equip this space with the standard trace norm

$$
\|f\|_{L^k_p(\mathbb{R}^n)|S} := \inf \{\|F\|_{L^k_p(\mathbb{R}^n)} : F \in L^k_p(\mathbb{R}^n) \cap C^{k-1}(\mathbb{R}^n) \text{ and } F|_S = f\}.
$$

Let $P_k$ be the space of polynomials of degree at most $k$ defined on $\mathbb{R}^n$. Given a $k$-times differentiable function $F$ and a point $x \in \mathbb{R}^n$, we let $T^k_x(F)$ denote the Taylor polynomial of $F$ at $x$ of degree at most $k$:

$$
T^k_x(F)(y) := \sum_{|\alpha| \leq k} \frac{1}{\alpha!}(D^\alpha F)(x)(y-x)^\alpha, \quad y, \in \mathbb{R}^n.
$$

Recall that, by Whitney’s theorem [W1], given a family of polynomials $\{P_x \in P_{k-1} : x \in S\}$, there exists a function $F \in L^k_p(\mathbb{R}^n)$ such that $T^k_x(F) = P_x$ for all $x \in S$, if and only if there exists a constant $\lambda > 0$ such that, for every $\alpha, |\alpha| \leq k - 1$, we have

$$
|D^\alpha(P_x - P_y)(x)| \leq \lambda \|x - y\|^{k - |\alpha|}, \quad x, y, \in S.
$$

We have obtained the following generalization of this theorem to the case $p > n$.

**Theorem 1.12 ([S5])** Given positive integers $k$ and $n$, a number $p \in (n, \infty)$ and a family of polynomials $\{P_x \in P_{k-1} : x \in S\}$, there exists a $(k - 1)$-continuously differentiable function $F \in L^k_p(\mathbb{R}^n)$ such that $T^k_x(F) = P_x$ for every $x \in S$ if and only if there exists a constant $\lambda > 0$ such that for every finite family $\{Q_i : i = 1, \ldots, m\}$ of disjoint cubes in $\mathbb{R}^n$, every $x, y_i \in (11Q_i) \cap S$, and every multi-index $\alpha, |\alpha| \leq k - 1$, the following inequality

$$
\sum_{i=1}^m \frac{|D^\alpha(P_{x_i} - P_{y_i})(x_i)|^p}{(\text{diam } Q_i)^{(k - |\alpha|)p - n}} \leq \lambda
$$

holds. Moreover, $\|f\|_{L^k_p(\mathbb{R}^n)|S} \sim \inf \lambda^{\frac{1}{p}}$ with constants of equivalence depending only on $k, n$ and $p$.  

11
Analogously to the role played by the Whitney extension theorem in the study of the case \( p = \infty \), we expect that Theorem 1.12 will turn out to be useful for the study of the following variant of the classical Whitney extension problem [W1, W2] for the space \( L_p^k(\mathbb{R}^n) \), \( p > n \):

Given \( p \in (n, \infty] \), a positive integer \( k \), and an arbitrary function \( f : S \to \mathbb{R} \), what is a necessary and sufficient condition for \( f \) to be the restriction to \( S \) of a \((k - 1)\)-times continuously differentiable function \( F \in L_p^k(\mathbb{R}^n) \)?

We observe that the Whitney problem for the spaces \( L_\infty^k, C^{k,\alpha}, C^k \), and their generalizations has attracted a lot of attention in recent years. We refer the reader to [BS1]-[BS3], [F1]-[F4], [BM, BMP, S6, Zob1] and references therein for numerous results in this direction, and for a variety of techniques for obtaining them.

We also note that, as in the case \( k = 1 \) (see Theorem 1.1), the classical Whitney extension method provides an almost optimal extension operator for all the scale of \( L_p^k(\mathbb{R}^n) \)-spaces, \( p > n \).

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2. Traces of Sobolev and Besov spaces and local oscillations.

2.1 Local oscillations and \( A_p \)-functionals.

The proofs of the results formulated in subsections 1.1 and 1.2 are based on estimates of local oscillations and corresponding oscillation functionals (\( A_p \)-functionals) of functions and their extensions. The \( A_p \)-functionals, see Definitions 2.1 and 2.3, play a role of certain generalized moduli of smoothness on a closed set providing one more approach to description of traces of Sobolev spaces. We present these results in Theorems 2.4 and 2.5 below.

Let us fix additional notation. Given a bounded function \( f \) defined on a set \( U \subset \mathbb{R}^n \) we let \( E(f; U) \) denote its oscillation on \( U \), i.e., the quantity

\[
E(f; U) := \sup_{x,y \in U} |f(x) - f(y)|. \tag{2.1}
\]

We put \( E(f; U) := 0 \) whenever \( U = \emptyset \).

We call a finite family of disjoint cubes a packing.

The point of departure for our approach is a description of Sobolev spaces in terms of local oscillations due to Brudnyi [Br1]: a continuous function \( f \in L_1^p(\mathbb{R}^n), p > n \), if and only if there is a constant \( \lambda > 0 \) such that for every \( t > 0 \) we have

\[
\sup_{\pi} \left\{ \sum_{Q \in \pi} |Q|E(f; Q)^p \right\}^{\frac{1}{p}} \leq \lambda t. \tag{2.2}
\]

Here the supremum is taken over all packings \( \pi = \{Q\} \) of equal cubes with \( \text{diam} \, Q \leq t \). Moreover, \( \|f\|_{L_1^p(\mathbb{R}^n)} \sim \inf \lambda \) with constants of equivalence depending only on \( n \) and \( p \).

Inequality (2.2) motivates us to introduce the following definition: given a locally bounded function \( f : \mathbb{R}^n \to \mathbb{R} \) we let \( A_p(\cdot; f) \) denote the function

\[
A_p(f; t) := \sup_{\pi} \left\{ \sum_{Q \in \pi} |Q|E(f; Q)^p \right\}^{\frac{1}{p}} \tag{2.3}
\]
where \( \pi \) runs over all packings of equal cubes with diameter at most \( t \). Now, by (2.2), for every \( p > n \) we have
\[
\|f\|_{L^p_R(\mathbb{R}^n)} \sim \sup_{t>0} A_p(f; t)/t. \tag{2.4}
\]

Observe that, an approximation functional, similar to \( A_p \), has been introduced by Yu. Brudnyi in [Br1]. Let us recall its definition.

Given a cube \( Q \) and a function \( f \in L^q(Q)\), \( q > 0 \), we let \( E_1(f; Q)_{L^q} \) denote the so-called normalized local best approximation of \( f \) on \( Q \) in \( L^q \)-norm:
\[
E_1(f; Q)_{L^q} := \left| Q \right|^{-\frac{1}{q}} \inf_{c \in \mathbb{R}} \|f - c\|_{L^q(Q)} = \inf_{c \in \mathbb{R}} \left( \frac{1}{|Q|} \int_Q |f - c|^q \, dx \right)^{\frac{1}{q}}. \tag{2.5}
\]

Following [Br1], given a function \( f \in L^q_{loc}(\mathbb{R}^n) \), \( q, p \in (0, \infty) \), we define its \((1, p)\)-modulus of continuity in \( L^q \) by letting
\[
\omega_{1,p}(f; t)_{L^q} := \sup_{\pi} \left\{ \sum_{Q \in \pi} |Q| E_1(f; Q)_{L^q}^p \right\}^{\frac{1}{p}}, \quad t > 0. \tag{2.6}
\]
Here \( \pi \) runs over all packings of equal cubes in \( \mathbb{R}^n \) with diameter at most \( t \).

There is an integral form of the \((1, p)\)-modulus of continuity,
\[
\omega_{1,p}(f; t)_{L^q} \sim \|E_1(f; Q(\cdot; t))_{L^q}\|_{L^p(\mathbb{R}^n)}, \tag{2.7}
\]
where constants of equivalence depend only on \( n, p \), and \( q \), see [Br2], Chapter 3, and also [S4]. It is also clear that, \( E_1(f; Q)_{L^\infty} \sim E(f; Q) \) so that
\[
\omega_{1,p}(f; \cdot)_{L^\infty} \sim A_p(f; \cdot) \tag{2.8}
\]
(with absolute constants). Therefore, by (2.7), for every locally bounded function \( f \) we have
\[
A_p(f; t) \sim \|E(f; Q(\cdot; t))\|_{L^p(\mathbb{R}^n)}, \quad t > 0, \tag{2.9}
\]
with constants of equivalence depending only on \( n \) and \( p \).

Let us also note an important property of the \((1, p)\)-modulus of continuity, see, e.g. [Br1] or [DL, DS1]: for every \( f \in L^p_{loc}(\mathbb{R}^n) \) and \( 1 \leq p \leq \infty \)
\[
\omega(f; t)_{L^p} \sim \omega_{1,p}(f; t)_{L^p}, \quad t > 0, \tag{2.10}
\]
with absolute constants of equivalence. (Recall that \( \omega(f; \cdot)_{L^p} \), see (1.18), denotes the modulus of smoothness of \( f \) in \( L^p_{loc} \).

Let us extend definition (2.3) of the \( A_p \)-functional to an arbitrary closed set \( S \subset \mathbb{R}^n \).
Definition 2.1  Let $f : S \to \mathbb{R}$ be a locally bounded function. We define the functional $A_p(f ; \cdot : S)$ by the formula

$$A_p(f ; t : S) := \sup_{\pi} \left\{ \sum_{Q \in \pi} |Q| \mathcal{E}(f ; Q \cap S)^p \right\} \frac{1}{p}, \quad t > 0,$$

(2.11)

where the supremum is taken over all packings $\pi = \{Q\}$ of equal cubes centered in $S$ with diameter at most $t$.

Thus, $A_p(F ; t) = A_p(F ; t : \mathbb{R}^n), t > 0$.

Observe that equivalence (2.4) provides a necessary condition for a function $f : S \to \mathbb{R}$ to belong to the trace space $L^1_p(\mathbb{R}^n)|_S, p > n$. In fact, since

$$A_p(f ; t : S) \leq A_p(F ; t : \mathbb{R}^n) = A_p(F ; t)$$

for every extension $F \in L^1_p(\mathbb{R}^n)$ of $f$, by (2.4),

$$\sup_{t > 0} A_p(f ; t : S) / t \leq C \|f\|_{L^1_p(\mathbb{R}^n)|_S},$$

(2.12)

so that

$$f \in L^1_p(\mathbb{R}^n)|_S \implies \sup_{t > 0} A_p(f ; t : S) / t < \infty.$$

However, simple examples (for instance, a straight line in $\mathbb{R}^n$) show that, in general, this condition is not sufficient. Below we present an additional condition which will enable us to describe the restrictions of Sobolev functions via $A_p$-functionals.

To formulate this new condition we need the following definitions and notation.

Definition 2.2  Let $0 \leq \alpha \leq 1$. A cube $Q \subset \mathbb{R}^n$ is said to be $\alpha$-porous with respect to $S$ if there exists a cube $Q'$ such that

$$Q' \subset Q \setminus S \quad \text{and} \quad \text{diam } Q' \geq \alpha \text{ diam } Q.$$

We say that a packing $\pi$ is $\alpha$-porous with respect to $S$ if every $Q \in \pi$ is $\alpha$-porous (with respect to $S$).

Let us introduce a corresponding $A_p$-functional over the family of all $\alpha$-porous packings.

Definition 2.3  Given $0 \leq \alpha \leq 1$ and a locally bounded function $f : S \to \mathbb{R}$ we define the functional $A_{p,\alpha}(f ; t : S)$ by letting

$$A_{p,\alpha}(f ; t : S) := \sup_{\pi} \left\{ \sum_{Q \in \pi} |Q| \mathcal{E}(f ; Q \cap \partial S)^p \right\} \frac{1}{p}. $$

(2.13)

Here $\pi = \{Q\}$ runs over all $\alpha$-porous (with respect to $S$) packings of equal cubes with centers in $\partial S$ and diameter at most $t$. 

Note that
\[ A_{p,\alpha}(f; t : S) \leq A_p(f; t : S), \quad 0 < \alpha < 1, \quad t > 0, \]  
and
\[ A_{p,\alpha_2}(f; t : S) \leq A_{p,\alpha_1}(f; t : S), \quad 0 < \alpha_1 < \alpha_2 < 1, \quad t > 0. \]

**Theorem 2.4** Let \( \alpha \in (0, 3/20] \) and let \( p \in (n, \infty) \). Then for every function \( f \in C(S) \) the following equivalence

\[
\| f \|_{L_p^1(\mathbb{R}^n)} |_S \sim \sup_{0 < t \leq 2 \text{diam } S} \frac{A_p(f; t : S)}{t} + \left\{ \int_0^{\text{diam } S} A_{p,\alpha}(f; t : S)^p \frac{dt}{tp+1} \right\}^{\frac{1}{p}}
\]

holds with constants depending only on \( n, p \) and \( \alpha \).

We turn to characterization of the traces of \( W^1_p \)-functions via \( A_p \) and \( A_{p,\alpha} \)-functionals. Let \( \varepsilon > 0 \) and let \( T : S_\varepsilon \rightarrow S \) be a measurable mapping satisfying inequality (1.5). In addition, we assume that

\[ T(x) \in \partial S, \quad x \in S_\varepsilon \setminus S. \]  

(In particular, formula (1.7) provides an example of such a mapping).

**Theorem 2.5** Let \( \alpha \in (0, 3/(10 + 10\theta)] \) and let \( p \in (n, \infty) \). Then for every \( f \in C(S) \) we have

\[
\| f \|_{W^1_p(\mathbb{R}^n)} |_S \sim \| f \circ T \|_{L_p(S_\varepsilon)} + \sup_{0 < t \leq \varepsilon} \frac{A_p(f; t : S)}{t} + \left\{ \int_0^{\varepsilon} A_{p,\alpha}(f; t : S)^p \frac{dt}{tp+1} \right\}^{\frac{1}{p}}
\]

with constants of equivalence depending only on \( n, p, \alpha, \theta \) and \( \varepsilon \).

Observe that, by choosing a more delicate Whitney covering of \( \mathbb{R}^n \setminus S \), see Remark 5.1, the intervals \( (0, 3/20] \) and \( (0, 3/(10 + 10\theta)] \) in Theorems 2.4 and 2.5 can be replaced with slightly smaller intervals \( (0, 1/4) \) and \( (0, 1/(2 + 2\theta)) \) respectively.

### 2.1 Besov spaces on closed subsets of \( \mathbb{R}^n \).

A. Jonsson [J] characterized the trace of the Besov space \( B_{p,q}^s(\mathbb{R}^n) \), \( s \in (n/p, 1) \), to an arbitrary closed subset \( S \subset \mathbb{R}^n \) via doubling measures supported on \( S \), see Theorem 6.3. In Section 6 we present another intrinsic description of the space \( B_{p,q}^s(\mathbb{R}^n)|_S \) which involves only the values of a function defined on \( S \) and does not use doubling measures.

First we observe that every function \( f \in B_{p,q}^s(\mathbb{R}^n) \), \( s \in (n/p, 1) \), can be redefined in a set of measure zero to satisfy the Hölder condition of order \( s - n/p \), see, e.g., [BIN] or [N], Chapter 6. Therefore, similar to the case of Sobolev spaces, in what follows we will identify \( f \) with its continuous refinement. As usual, given a closed subset \( S \subset \mathbb{R}^n \), we define the trace space

\[ B_{p,q}^s(\mathbb{R}^n)|_S := \{ f : S \rightarrow \mathbb{R} : \text{there is a continuous } F \in B_{p,q}^s(\mathbb{R}^n) \text{ such that } F|_S = f \}, \]
and equip this space with the standard quotient space norm.

The next theorem provides an intrinsic characterization of the restrictions of the Besov space $B^s_{p,q}(\mathbb{R}^n)$, $n/p < s < 1$, to an arbitrary closed subset $S \subset \mathbb{R}^n$. In its formulation given $\varepsilon > 0$ and $\theta \geq 1$, we again let $T$ denote a measurable mapping from $S_\varepsilon$ into $S$ satisfying on $S_\varepsilon$ inequality (1.5).

**Theorem 2.6** A function $f \in C(S)$ can be extended to a (continuous) function $F \in B^s_{p,q}(\mathbb{R}^n)$, $n/p < s < 1$, if and only if $f \circ T \in L_p(S_\varepsilon)$ and

$$
\varepsilon \int_0^\varepsilon A_p(f; t : S)_t^q \frac{dt}{t^{sq+1}} < \infty.
$$

Moreover,

$$
\|f\|_{B^s_{p,q}(\mathbb{R}^n)|_S} \sim \|f \circ T\|_{L_p(S_\varepsilon)} + \left\{ \varepsilon \int_0^\varepsilon A_p(f; t : S)_t^q \frac{dt}{t^{sq+1}} \right\}^{\frac{1}{q}}, \tag{2.16}
$$

with constants of equivalence depending only on $n, s, p, q, \theta$ and $\varepsilon$.

Similar to the case of the Sobolev space, Whitney’s method works for all the scale of the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, $n/p < s < 1$, providing an almost optimal extension operator. For Whitney-type extension theorems for the Besov spaces we refer the reader to [JW, Gud].

3. The Whitney covering and local approximation properties of $W^1_p(\mathbb{R}^n)$-functions.

Throughout the paper $C, C_1, C_2, ...$ will be generic positive constants which depend only on parameters determining sets (say, $n$, doubling or “porosity” constants, etc.) or function spaces ($p, q, s$, etc). These constants can change even in a single string of estimates. The dependence of a constant on certain parameters is expressed, for example, by the notation $C = C(n, p)$. We write $A \sim B$ if there is a constant $C \geq 1$ such that $A/C \leq B \leq CA$.

Recall that $|A|$ denotes the Lebesgue measure of $A$. Given a Borel measure $\mu$ on $\mathbb{R}^n$, a $\mu$-measurable set $A \subset \mathbb{R}^n$ and $p \in (0, \infty]$ we let $L_p(A : \mu)$ denote the $L_p$-space (with respect to the measure $\mu$) of functions on $A$. We write $L_p(\mu)$ whenever $A = \mathbb{R}^n$, and $L_p(A)$ whenever $\mu$ is the Lebesgue measure.

If $f \in L_1(A : \mu)$ and $\mu(A) < \infty$, we let $f_{A,\mu} := \mu(A)^{-1} \int_A f \, d\mu$ denote the $\mu$-average of $f$ on $A$. By $f_A$ we denote the average of $f$ on $A$ with respect to the Lebesgue measure.

Recall that $Q = Q(x, r)$ denotes the closed cube in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with side length $2r$ whose sides are parallel to the coordinate axes. We write $x_Q = x$ and $r_Q = r$ so that $Q = Q(x_Q, r_Q)$. Also recall that $\lambda > 0$ and $Q = Q(x, r)$ we let $\lambda Q$ denote the cube $Q(x, \lambda r)$. By $Q^*$ we denote the cube $Q^* := \frac{9}{8}Q$. 

16
It will be convenient for us to measure distances in $\mathbb{R}^n$ in the uniform norm

$$\|x\| := \max\{|x_i| : i = 1, \ldots, n\}, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$  \hfill (3.1)

Thus every cube

$$Q = Q(x, r) := \{y \in \mathbb{R}^n : \|y - x\| \leq r\}$$

is a “ball” in $\| \cdot \|$-norm of “radius” $r$ centered at $x$.

Given subsets $A, B \subset \mathbb{R}^n$, we put $\text{diam } A := \sup\{\|a - a'\| : a, a' \in A\}$ and

$$\text{dist}(A, B) := \inf\{\|a - b\| : a \in A, b \in B\}.$$  

For $x \in \mathbb{R}^n$ we also set $\text{dist}(x, A) := \text{dist}([x], A)$. By $\text{int}(A)$ we denote the interior of $A$ and by $\chi_A$ the characteristic function of $A$; we put $\chi_A \equiv 0$ whenever $A = \emptyset$.

As usual, $\nabla F(x) := \left(\frac{\partial F}{\partial x_1}(x), \ldots, \frac{\partial F}{\partial x_n}(x)\right)$ stands for the gradient of a function $F$ at $x$; thus, by (3.1), $\|\nabla F(x)\| = \max\left\{\left|\frac{\partial F}{\partial x_i}(x)\right| : i = 1, \ldots, n\right\}$.

Let $\pi = \{Q\}$ be a family of cubes in $\mathbb{R}^n$. By $M_\pi$ we denote its covering multiplicity, i.e., the minimal positive integer $M$ such that every point $x \in \mathbb{R}^n$ is covered by at most $M$ cubes from $\pi$. Thus

$$M_\pi := \sup_{x \in \mathbb{R}^n} \sum_{Q \in \pi} \chi_Q(x).$$

Everywhere in this paper $S$ denotes a closed subset of $\mathbb{R}^n$. Since the set $\mathbb{R}^n \setminus S$ is open, it admits a Whitney covering $\mathbb{W}(S)$ with the following properties.

**Theorem 3.1** (see, e.g. [St], or [G]) $\mathbb{W}(S) = \{Q_k\}$ is a countable family of cubes such that

(i). $\mathbb{R}^n \setminus S = \bigcup\{Q : Q \in \mathbb{W}(S)\}$;

(ii). For every cube $Q \in \mathbb{W}(S)$

$$\text{diam } Q \leq \text{dist}(Q, S) \leq 4 \text{ diam } Q;$$  \hfill (3.2)

(iii). The covering multiplicity $M_{\mathbb{W}(S)}$ of $\mathbb{W}(S)$ is bounded by a constant $N = N(n)$. Thus every point of $\mathbb{R}^n \setminus S$ is covered by at most $N$ cubes from $\mathbb{W}(S)$.

We are also needed certain additional properties of Whitney’s cubes which we present in the next lemma. These properties easily follow from Theorem 3.1.

**Lemma 3.2** (1). If $Q, K \in \mathbb{W}(S)$ and $Q^* \cap K^* \neq \emptyset$, then

$$\frac{1}{4} \text{ diam } Q \leq \text{ diam } K \leq 4 \text{ diam } Q.$$  

(Recall that $Q^* := \frac{1}{8}Q$.)

(2). For every cube $K \in \mathbb{W}(S)$ there are at most $N = N(n)$ cubes from the family $\mathbb{W}^*(S) := \{Q^* : Q \in \mathbb{W}(S)\}$ which intersect $K^*$.

(3). If $Q, K \in \mathbb{W}(S)$, then $Q^* \cap K^* \neq \emptyset$ if and only if $Q \cap K \neq \emptyset$. 

17
Given $\varepsilon > 0$ we put
\[ W(S : \varepsilon) := \{ Q \in W(S) : \text{diam } Q \leq \varepsilon \}. \] (3.3)

The proofs of the necessity part of several results stated in Sections 1 and 2 are based on a series of auxiliary statements which we present in this section. First of them is the following combinatorial

**Theorem 3.3** ([BrK, Dol]) Let $\mathcal{A} = \{Q\}$ be a collection of cubes in $\mathbb{R}^n$ with the covering multiplicity $M_A < \infty$. Then $\mathcal{A}$ can be partitioned into at most $N = 2^{n-1}(M_A - 1) + 1$ families of disjoint cubes.

**Lemma 3.4** Let $0 < \alpha \leq 1 \leq \beta \leq \lambda$ and let $t > 0$. Let $\pi = \{Q\}$ be a family of cubes in $\mathbb{R}^n$ with the finite covering multiplicity such that $\alpha t \leq \text{diam } Q \leq \beta t$ for every $Q \in \pi$.

Let us assign every $Q \in \pi$ a cube $K_Q$ of diameter at most $\lambda t$ and such that $K_Q \supseteq Q$. Then the family of cubes $\mathcal{A} = \{K_Q : Q \in \pi\}$ can be partitioned into at most $N = N(n, \alpha, \beta, \lambda, M_\pi)$ families of disjoint cubes.

**Proof.** One can easily show that the covering multiplicity of the family $\mathcal{A}$ is bounded by a constant depending only on $n, \alpha, \beta, \lambda$, and $M_\pi$. It remains to make use of Theorem 3.3 and the lemma follows. $\square$

The following proposition presents a classical Sobolev embedding inequality for the case $p > n$, see, e.g. [M], p. 61, or [MP], p. 55. This inequality is also known in the literature as Sobolev-Poincaré inequality (for $p > n$).

**Proposition 3.5** Let $F \in L^1_p(\Omega)$ be a continuous function defined on a domain $\Omega \subset \mathbb{R}^n$ and let $n < q \leq p < \infty$. Then for every cube $Q \subset \Omega$ and every $x, y \in Q$ the following inequality
\[ |F(x) - F(y)| \leq C(n, q) r_Q \left( \frac{1}{|Q|} \int_Q \|\nabla F(z)\|^q \, dz \right)^{\frac{1}{q}} \]
holds.

This inequality enables us to prove the following useful property of the $A_p$-functional, see Definition 2.1.

**Proposition 3.6** Let $p > n$ and let $\Omega := \text{int}(S)$. Assume that $f \in C(S)$ and $f|_\Omega \in L^1_p(\Omega)$. Then
\[ A_p(f; t : S) \leq C(n, p)\{A_p(f; 4t : \partial S) + t\|f|_\Omega\|_{L^1_p(\Omega)}\}, \quad t > 0. \]

**Proof.** Let $t > 0$ and let $\pi = \{Q\}$ be a packing of equal cubes of diameter $\text{diam } Q = 2\tau \leq t$ centered in $S$. If $Q \subset \Omega$ for every $Q \in \pi$, then, by Proposition 3.5,
\[ J(f; Q) := \sum_{Q \in \pi} |Q| \mathcal{E}(f; Q \cap S)^p = \sum_{Q \in \pi} |Q| \mathcal{E}(f; Q)^p \leq C \sum_{Q \in \pi} r_Q^p \int_Q \|\nabla f(y)\|^p \, dy \leq C t^p \int_\Omega \|\nabla f(y)\|^p \, dy \]
so that

\[ J(f; Q) \leq C\|f\|_{L^p_0(\Omega)}^p. \quad (3.4) \]

Now assume that \( Q \cap \partial S \neq \emptyset \) for each \( Q \in \pi \); thus there exists a point \( b_Q \in Q \cap \partial S \), \( Q \in \pi \). Put \( Q' := Q(b_Q, 2\tau) = Q(b_Q, 2r_Q) \). Then, clearly, \( Q \subset Q' \subset 3Q \).

We have

\[ \mathcal{E}(f; Q \cap S) = \sup_{x,y \in Q \cap S} |f(x) - f(y)| \leq 2 \sup_{x \in Q \cap S} |f(x) - f(b_Q)| \]

so that

\[ \mathcal{E}(f; Q \cap S) \leq 2 \max \{ \sup_{x \in Q \cap S} |f(x) - f(b_Q)|, \sup_{x \in Q \cap \Omega} |f(x) - f(b_Q)| \}. \]

Hence,

\[ \mathcal{E}(f; Q \cap S) \leq 2 \{ \mathcal{E}(f; Q' \cap \partial S) + \sup_{x \in Q \cap \Omega} |f(x) - f(b_Q)| \}. \quad (3.5) \]

Given \( x \in Q \cap \Omega \) we let \( a_x \) denote a point nearest to \( x \) on \( \partial S \). Thus

\[ \|x - a_x\| = \text{dist}(x, \partial S) \leq \|x - b_Q\| \leq r_Q = \tau. \]

Put \( Q^{(x)} := Q(x, \text{dist}(x, \partial S)) \). Then, clearly, \( \text{int}(Q^{(x)}) \subset \Omega \) and \( a_x \in Q^{(x)} \cap \partial S \) so that for every \( y \in Q^{(x)} \) we have

\[ \|y - x_Q\| \leq \|y - x\| + \|x - x_Q\| \leq \text{dist}(x, \partial S) + r_Q \leq 2r_Q \]

proving that \( Q^{(x)} \subset 2Q \subset 2Q' \).

By Lemma 3.4, we can assume that the family of cubes \( 2\pi' := \{ 2Q' : Q \in \pi \} \) is a packing. We have

\[ |f(x) - f(b_Q)| \leq |f(x) - f(a_x)| + |f(a_x) - f(b_Q)| \]

so that

\[ |f(x) - f(b_Q)| \leq \mathcal{E}(f, Q^{(x)}) + \mathcal{E}(f, (2Q') \cap \partial S). \quad (3.6) \]

Since \( f \) is continuous on \( S \) and \( \text{int}(Q^{(x)}) \subset \Omega \), there exists a cube \( \tilde{Q}^{(x)} \subset \text{int}(Q^{(x)}) \) such that \( \mathcal{E}(f, Q^{(x)}) \leq 2\mathcal{E}(f, \tilde{Q}^{(x)}) \). Also, since \( \tilde{Q}^{(x)} \subset Q^{(x)} \subset 2Q \), we have \( r_{\tilde{Q}^{(x)}} \leq 2r_Q = 2\tau \).

Then, by Proposition 3.5, for any choice of \( x = x(Q) \in Q \cap \Omega \), we have

\[ \sum_{Q \in \pi} |Q| \mathcal{E}(f; Q^{(x)})^p \leq C \sum_{Q \in \pi} |Q| r_Q^{p-n} \int_{\tilde{Q}^{(x)}} \|\nabla f(y)\|^p dy \]

\[ \leq C \tau^n (2\tau)^{p-n} \sum_{Q \in \pi} \int_{\tilde{Q}^{(x)}} \|\nabla f(y)\|^p dy \leq C \tau^p \int_{\Omega} \|\nabla f(y)\|^p dy. \]
Hence, by (3.6) and Definition 2.1,
\[
\sum_{Q \in \pi} |Q| \sup_{x \in Q \cap \Omega} |f(x) - f(b_Q)|^p \leq C \{ \sum_{Q \in \pi} |Q| \sup_{x \in Q \cap \Omega} \mathcal{E}(f; Q(x))^p \\
+ \sum_{Q \in \pi} |2Q'| \mathcal{E}(f; (2Q') \cap \partial S)^p \} \\
\leq C \{ t^p \int_{\Omega} \| \nabla f(y) \|^p dy + A_p(f; 2t : \partial S)^p \}.
\]

Finally, by (3.5) and Definition 2.1,
\[
J(f; \pi) \leq C \{ \sum_{Q \in \pi} |Q| \mathcal{E}(f; Q \cap \partial S)^p + \sum_{Q \in \pi} |Q| \sup_{x \in Q \cap \Omega} |f(x) - f(b_Q)|^p \} \\
\leq C \{ A_p(f; 2t : \partial S)^p + t^p \| f \|_{L^p_{\mathcal{L}^1}(\Omega)} + A_p(f; 4t : \partial S)^p \}.
\]

Combining this inequality with (3.4), we obtain
\[
J(f; \pi) \leq C \{ A_p(f; 4t : \partial S)^p + t^p \| f \|_{L^p_{\mathcal{L}^1}(\Omega)} \}
\]
provided \( \pi \) is an arbitrary packing of equal cubes of diameter at most \( t \) centered in \( S \). It remains to take the supremum over all such packings \( \pi \), and the proposition follows.

**Lemma 3.7** Let \( \beta > 1, \gamma \geq 1 \), and let \( \pi \) be a family of cubes with the covering multiplicity \( M_\pi < \infty \). Then for every function \( G \in L^\beta(\mathbb{R}^n) \) we have
\[
\{ \sum_{Q \in \pi} |Q| |G_{\gamma Q}|^\beta \} \frac{1}{\beta} \leq C \| G \|_{L^\beta(\mathbb{R}^n)}.
\]
Here \( C \) is a constant depending only on \( n, \beta \), and \( M_\pi \).

**Proof.** Let \( Q \in \pi \) and let \( x \in Q \). Then \( \gamma Q \subset Q_x \) where \( Q_x := Q(x, (\gamma + 1)r_Q) \). It is also clear that \( Q_x \subset 3\gamma Q \). Hence
\[
|G_{\gamma Q}| \leq \frac{1}{|\gamma Q|} \int_{\gamma Q} |G| dx \leq C(n) \frac{1}{|Q_x|} \int_{Q_x} |G| dx \leq \mathcal{M}G(x), \quad x \in Q.
\]
Here \( \mathcal{M} \) denotes the Hardy-Littlewood maximal function. Integrating this inequality over \( Q \) we obtain
\[
|Q| |G_{\gamma Q}|^\beta \leq C \int_Q \mathcal{M}G(x)^\beta dx
\]
with \( C = C(n, \beta) \). Hence
\[
I := \sum_{Q \in \pi} |Q| |G_{\gamma Q}|^\beta \leq C \sum_{Q \in \pi} \int_Q \mathcal{M}G(x)^\beta dx = C \int_{\mathbb{R}^n} \left( \sum_{Q \in \pi} \chi_Q(x) \right) \mathcal{M}G(x)^\beta dx.
\]
This inequality and the Hardy-Littlewood maximal theorem yield

\[ I \leq C M_\pi \int |MG(x)^\beta| dx \leq C M_\pi \int |G(x)|^\beta dx, \]

proving the required inequality \( I \leq C \|G\|_{L_\beta(\mathbb{R}^n)}^{\beta}. \) □

This lemma and Theorem 3.1 imply the following

Lemma 3.8 Let \( \beta > 1, \gamma \geq 1, \) and let \( A \) be a closed subset of \( \mathbb{R}^n. \) Then for every \( G \in L_\beta(\mathbb{R}^n) \) the following inequality

\[
\{ \sum_{Q \in \mathcal{W}(A)} |Q| |G_{\gamma Q}|^\beta \}^{\frac{1}{\beta}} \leq C \|G\|_{L_\beta(\mathbb{R}^n)}.
\]

holds. Here \( C \) is a constant depending only on \( n \) and \( \beta. \)

Lemma 3.9 Let \( n \in (p, \infty), \gamma \geq 1, \) and let \( \pi = \{Q\} \) be a family of cubes with the covering multiplicity \( M_\pi < \infty. \) Then for every continuous function \( F \in L_1^p(\mathbb{R}^n) \) the following inequality

\[
\sum_{Q \in \pi} |Q| \left( \frac{E(F; \gamma Q)}{\text{diam } Q} \right)^p \leq C \int_{\mathbb{R}^n} \|\nabla F\|^p dx
\]

holds. Here \( C \) is a constant depending only on \( n, p, \gamma, \) and \( M_\pi. \)

**Proof.** We put \( q := (p + n)/2 \) so that \( n < q < p. \) By Proposition 3.5,

\[
E(F; \gamma Q) = \sup_{x, y \in \gamma Q} |F(x) - F(y)| \leq C|Q| \left( \frac{1}{|Q|} \int_{\gamma Q} \|\nabla F\|^q dx \right)^\frac{1}{q}
\]

with \( C = C(n, p). \) Put \( G := \|\nabla F\|^q \) and \( \beta := p/q. \) Then

\[
I := \sum_{Q \in \pi} |Q| \left( \frac{E(F; \gamma Q)}{\text{diam } Q} \right)^p \leq C \sum_{Q \in \pi} |Q| \left( \frac{1}{|Q|} \int_{\gamma Q} \|\nabla F\|^q dx \right)^\frac{p}{q} = C \sum_{Q \in \pi} |Q| |G_{\gamma Q}|^\beta.
\]

Since \( \beta > 1 \) and \( M_\pi < \infty, \) by Lemma 3.7,

\[
I \leq C \|G\|_{L_\beta(\mathbb{R}^n)}^{\beta} = C \int_{\mathbb{R}^n} (\|\nabla F\|^q)^{\frac{p}{q}} dx = C \int_{\mathbb{R}^n} \|\nabla F\|^p dx,
\]

proving the lemma. □

Now, let us replace in Definition 2.3 the boundary \( \partial S \) of \( S \) with the set \( S \) itself. We denote a functional obtained by \( A_{p, \alpha}(f; \cdot : S). \)

21
**Definition 3.10** Let \( f : S \to \mathbb{R} \) be a locally bounded function and let \( \alpha \in (0, 1] \). We define the functional \( A_{p, \alpha}(f; \cdot : S) \) by the formula

\[
A_{p, \alpha}(f; t : S) := \sup_{\pi} \left\{ \sum_{Q \in \pi} |Q| E(f; Q \cap S)^p \right\}^{\frac{1}{p}}, \quad t > 0.
\] (3.7)

Here the supremum is taken over all \( \alpha \)-porous (with respect to \( S \)) packings \( \pi = \{Q\} \) of equal cubes with centers in \( S \) and diameter at most \( t \).

Clearly, for all \( t > 0 \) and \( \alpha \in (0, 1] \) we have

\[
A_{p, \alpha}(f; t : S) \leq A_{p, \alpha}(f; t : S). \quad (3.8)
\]

**Proposition 3.11** Let \( \alpha \in (0, 1] \) and let \( p \in (n, \infty) \). Then for every continuous function \( F \in L^1_{p}(\mathbb{R}^n) \) we have

\[
\int_0^\infty A_{p, \alpha}(F|_S; t : S)^p \frac{dt}{t^{p+1}} \leq C(n, p, \alpha)\|F\|_{L^1_{p}(\mathbb{R}^n)}^p.
\]

**Proof.** Put \( f := F|_S \). Since \( A_{p, \alpha}(f; \cdot : S) \) is a non-decreasing function, we have

\[
I := \int_0^\infty A_{p, \alpha}(f; t : S)^p \frac{dt}{t^{p+1}} = \sum_{m=\infty}^\infty \int_{2m-1}^{2m} A_{p, \alpha}(f; t : S)^p \frac{dt}{t^{p+1}} \\
\leq C \sum_{m=\infty}^\infty 2^{-mp} A_{p, \alpha}(f; 2^m : S)^p.
\]

By (3.7), for every integer \( m \) such that \( A_{p, \alpha}(f; 2^m : S) > 0 \) there exists an \( \alpha \)-porous packing \( \pi_m = \{Q\} \) which consists of equal cubes centered in \( S \) with \( r_Q = t_m \leq 2^m \) and satisfies the following inequality:

\[
A_{p, \alpha}(f; 2^m : S)^p \leq 2 \sum_{Q \in \pi_m} |Q| E(f; Q \cap S)^p. \quad (3.9)
\]

Now fix an integer \( m \). Since \( \pi_m \) is \( \alpha \)-porous, for each \( Q \in \pi \) there exists a cube \( Q' \subset Q \setminus S \) such that \( r_{Q'} \geq \alpha r_Q = \alpha t_m \). Since \( x_{Q'} \in Q \setminus S \), there is a Whitney’s cube \( K_Q \in \mathcal{W}(S) \) which contains \( x_{Q'} \).

It can be readily seen that inequality (3.2) implies the following two properties of the cube \( K_Q \): \( \text{diam} K_Q \leq 2^m \) and \( \gamma K_Q \supset Q \) for some \( \gamma = \gamma(\alpha) > 0 \).

Hence \( E(f; Q \cap S) \leq E(F; \gamma K_Q) \) so that by (3.9)

\[
I \leq C \sum_{m=\infty}^\infty \sum_{Q \in \pi_m} 2^{-mp} |Q| E(F; \gamma K_Q)^p.
\]
Since \( \text{diam } K_Q \leq 2^m \), we obtain

\[
I \leq C \sum_{m=-\infty}^{\infty} \sum_{Q \in \pi_m} 2^{-m(p-n)} \mathcal{E}(F; \gamma K)^p \\
\leq C \sum_{K \in \mathcal{W}(S)} \mathcal{E}(F; \gamma K)^p \sum_{\{m: \text{diam } K \leq 2^m\}} 2^{-m(p-n)} \\
\leq C \sum_{K \in \mathcal{W}(S)} \frac{\mathcal{E}(F; \gamma K)^p}{(\text{diam } K)^{p-n}} = 2^n C \sum_{K \in \mathcal{W}(S)} |K| \left( \frac{\mathcal{E}(F; \gamma K)}{\text{diam } K} \right)^p.
\]

Since \( M_{\mathcal{W}(S)} \leq N(n) \), by Lemma 3.9, \( I \leq C \|F\|_{L^p(S^a)}^p \), and the proof is finished. \( \square \)

Given \( \varepsilon > 0 \) and \( \theta \geq 1 \) we let \( T : S_\varepsilon \to S \) denote a measurable mapping satisfying inequality (1.5). Also recall that \( \mathcal{W}(S : \varepsilon) \) is a subfamily of \( \mathcal{W}(S) \) of all cubes with diameter at most \( \varepsilon \), see (3.3).

**Lemma 3.12** For every continuous function \( F \in L^p(S^a) \), \( p \in (1, \infty) \), we have

\[
\| F \circ T \|_{L^p(S_\varepsilon)}^p \leq C(\|F\|_{L^p(S^a)}^p + \sum_{Q \in \mathcal{W}(S : \varepsilon)} |Q| \mathcal{E}(F; \eta Q)^p)
\]

where \( \eta := 10\theta + 1 \) and \( C \) is a constant depending only on \( n, p \) and \( \theta \).

**Proof.** Put

\[
A_\varepsilon := \{Q \in \mathcal{W}(S): Q \cap S_\varepsilon \neq \emptyset\}.
\]

Then \( \text{dist}(Q, S) \leq \varepsilon \) for every \( Q \in A_\varepsilon \) so that, by part (ii) of Theorem 3.1, \( \text{diam } Q = 2r_Q \leq \text{dist}(Q, S) \leq \varepsilon \), proving that \( A_\varepsilon \subset \mathcal{W}(S : \varepsilon) \).

Prove that

\[ T(Q) \subset \eta Q. \tag{3.10} \]

In fact, by (1.5), for each \( x \in Q \) we have

\[
\|T(x) - x_Q\| \leq \|T(x) - x\| + \|x - x_Q\| \leq \theta \text{dist}(x, S) + r_Q.
\]

But \( \text{dist}(x, S) \leq \text{dist}(Q, S) + \text{diam } Q \), so that, by (ii), Theorem 3.1, we have \( \text{dist}(x, S) \leq 5 \text{diam } Q = 10r_Q \). Hence

\[
\|T(x) - x_Q\| \leq (10\theta + 1)r_Q = \eta r_Q
\]

proving (3.10).

Put \( \tilde{Q} := \eta Q \). We have

\[
\int_{Q \cap S_\varepsilon} |F(T(x))|^p dx \leq 2^p \int_{Q \cap S_\varepsilon} (|F(T(x)) - F_{\tilde{Q}}|^p + |F_{\tilde{Q}}|^p) dx \\
\leq 2^p \int_{Q \cap S_\varepsilon} |F(T(x)) - F_{\tilde{Q}}|^p dx + 2^p |Q| \|F_{\tilde{Q}}|^p.
\]

23
Since $T(x) \in \tilde{Q}$, we obtain
\[
\int_{Q \cap S} |F(T(x)) - F_Q|^p \, dx \leq |Q| \sup_{x \in Q} |F(x) - F_Q|^p \leq |Q| \mathcal{E}(F; \tilde{Q})^p,
\]
so that
\[
\int_{Q \cap S} |F(T(x))|^p \, dx \leq 2^p (|Q| \mathcal{E}(F; \tilde{Q})^p + |Q| \|F_Q\|^p).
\]
This inequality and Lemma 3.8 yield
\[
\int_{S \setminus S} |F(T(x))|^p \, dx \leq \sum_{Q \in A_{\varepsilon}} \int_{Q \cap S} |F(T(x))|^p \, dx
\]
\[
\leq 2^p \left( \sum_{Q \in \mathcal{W}(S; \varepsilon)} |Q| \mathcal{E}(F; \tilde{Q})^p + \sum_{Q \in \mathcal{W}(S)} |Q| \|F_Q\|^p \right)
\]
\[
\leq C \left( \sum_{Q \in \mathcal{W}(S; \varepsilon)} |Q| \mathcal{E}(F; \tilde{Q})^p + \|F\|^p_{L_p(\mathbb{R}^n)} \right).
\]
It remains to note that $T(x) = x, x \in S$, so that
\[
\int_{S \setminus S} |F(T(x))|^p \, dx = \int_{S} |F(T(x))|^p \, dx + \int_{S \setminus S} |F(T(x))|^p \, dx
\]
\[
= \int_{S} |F(x)|^p \, dx + \int_{S \setminus S} |F(T(x))|^p \, dx
\]
and the lemma follows. \(\square\)

**Proposition 3.13** For every continuous function $F \in W^1_p(\mathbb{R}^n), p > n$, we have
\[
\|F \circ T\|_{L_p(S; \varepsilon)} \leq C \|F\|_{W^1_p(\mathbb{R}^n)}
\]
where $C$ is a constant depending only on $n, p$ and $\theta$.

**Proof.** Recall that $\text{diam} Q \leq \varepsilon$ for every $Q \in \mathcal{W}(S; \varepsilon)$. Hence, by Lemma 3.9, we have
\[
\sum_{Q \in \mathcal{W}(S; \varepsilon)} |Q| \mathcal{E}(F; \eta Q)^p \leq \varepsilon^p \sum_{Q \in \mathcal{W}(S; \varepsilon)} |Q| \left( \frac{\mathcal{E}(F; \eta Q)}{\text{diam} Q} \right)^p \leq C \|F\|_{L_p(\mathbb{R}^n)},
\]
so that by Lemma 3.12
\[
\|F \circ T\|_{L_p(S; \varepsilon)}^p \leq C(\|F\|_{L_p(\mathbb{R}^n)}^p + \sum_{Q \in \mathcal{W}(S; \varepsilon)} |Q| \mathcal{E}(F; \eta Q)^p) \leq C(\|F\|_{L_p(\mathbb{R}^n)}^p + \|F\|_{L_p(\mathbb{R}^n)}),
\]
proving the proposition. \(\square\)
4. Local approximation properties of the extension operator.

Let \( \delta > 0, \bar{c} \in \mathbb{R} \), and let \( f \) be a function defined on a closed set \( S \subset \mathbb{R}^n \). We assign every Whitney’s cube \( Q \in \mathcal{W}(S) \) a point \( a_Q \in S \) which is nearest to \( Q \) on \( S \) in the metric \( \| \cdot \| \). We put

\[
c_Q := \begin{cases} 
  f(a_Q), & \text{diam } Q \leq 2\delta, \\
  \bar{c}, & \text{diam } Q > 2\delta,
\end{cases}
\]

(4.1)

and define a linear extension operator \( \text{Ext}_{\delta,\bar{c},S} \) by letting

\[
\text{Ext}_{\delta,\bar{c},S} f(x) := \begin{cases} 
  f(x), & x \in S, \\
  \sum_{Q \in \mathcal{W}(S)} c_Q \varphi_Q(x), & x \in \mathbb{R}^n \setminus S.
\end{cases}
\]

(4.2)

Here \( \{ \varphi_Q : Q \in \mathcal{W}(S) \} \) is a smooth partition of unity subordinated to the Whitney decomposition \( \mathcal{W}(S) \), see, e.g. [St]. Recall main properties of this partition:

(a). \( \varphi_Q \in C^\infty(\mathbb{R}^n) \) and \( 0 \leq \varphi_Q \leq 1 \) for every \( Q \in \mathcal{W}(S) \);

(b). \( \text{supp } \varphi_Q \subset Q^*(:= \frac{2}{3} Q), Q \in \mathcal{W}(S) \);

(c). \( \sum \{ \varphi_Q(x) : Q \in \mathcal{W}(S) \} = 1 \) for every \( x \in \mathbb{R}^n \setminus S \);

(d). for every cube \( Q \in \mathcal{W}(S) \) we have

\[
\sup_{\mathbb{R}^n} \| \nabla \varphi_Q \| \leq C/ \text{diam } Q,
\]

where \( C \) is a constant depending only on \( n \).

We fix \( \delta \) and \( \bar{c} \) and put

\[
\tilde{f} := \text{Ext}_{\delta,\bar{c},S} f.
\]

In this section we present estimates of local oscillations of \( \tilde{f} \) via those of the function \( f \). These estimates are based on a series of auxiliary lemmas. To formulate the first of them, given a cube \( K \subset \mathbb{R}^n \) we define two families of Whitney’s cubes:

\[
Q_1(K) := \{ Q \in \mathcal{W}(S) : Q \cap K \neq \emptyset \}
\]

and

\[
Q_2(K) := \{ Q \in \mathcal{W}(S) : \exists Q' \in Q_1(K) \text{ such that } Q' \cap Q^* \neq \emptyset \}.
\]

(4.3)

**Lemma 4.1** Let \( K \) be a cube centered in \( S \). Then for every \( Q \in Q_2(K) \) we have

\[
\text{diam } Q \leq 2 \text{ diam } K \text{ and } \| x_K - x_Q \| \leq \frac{5}{2} \text{ diam } K.
\]

For the proof see, e.g. [S4].

**Lemma 4.2** For every cube \( K \) in \( \mathbb{R}^n \) and every \( c_0 \in \mathbb{R} \) we have

\[
\sup_{K \setminus S} | \tilde{f} - c_0 | \leq \sup_{Q \in Q_2(K)} | c_Q - c_0 |.
\]
Proof. Clearly, \( K \setminus S \subset \bigcup \{ Q : Q \in \mathcal{Q}_1(K) \} \) so that
\[
\sup_{K \setminus S} |\tilde{f} - c_0| \leq \sup_{Q \in \mathcal{Q}_1(K)} \sup_{Q} |\tilde{f} - c_0|.
\]

Let \( Q \in \mathcal{Q}_1(K) \) and let \( V(Q) := \{ Q' \in \mathcal{W}(S) : (Q')^* \cap Q \neq \emptyset \} \). By (4.3), \( V(Q) \subset \mathcal{Q}_2(K) \).

Properties (a)-(c) of the partition of unity and formula (4.2) imply the following inequality:
\[
\sup_{Q} |\tilde{f} - c_0| = \sup_{x \in Q} |\sum_{Q' \in \mathcal{W}(S)} (c_{Q'} - c_0) \varphi_{Q'}(x)| = \sup_{x \in Q} |\sum_{Q' \in V(Q)} (c_{Q'} - c_0) \varphi_{Q'}(x)| \leq \left( \max_{Q' \in V(Q)} |c_{Q'} - c_0| \right) \left( \sup_{x \in Q} \sum_{Q' \in V(Q)} \varphi_{Q'}(x) \right) \leq \max_{Q' \in V(Q)} |c_{Q'} - c_0|.
\]

Hence,
\[
\sup_{K \setminus S} |\tilde{f} - c_0| \leq \sup_{Q \in \mathcal{Q}_1(K)} \max_{Q' \in V(Q)} |c_{Q'} - c_0| \leq \sup_{Q \in \mathcal{Q}_2(K)} |c_{Q'} - c_0|.
\]

\( \blacksquare \)

Lemma 4.3 We have
\[
\|\tilde{f}\|^p_{L^p(\mathbb{R}^n \setminus S)} \leq C \sum_{Q \in \mathcal{W}(S; 2\delta)} |f(a_Q)|^p |Q|.
\]

Proof. Let \( Q \in \mathcal{W}(S) \). By Lemma 4.2 with \( c_0 = 0 \),
\[
\sup_{Q} |\tilde{f}| \leq \max\{|c_{Q'}| : Q' \in \mathcal{Q}_2(Q)\}
\]
so that
\[
\|\tilde{f}\|^p_{L^p(\mathbb{R}^n \setminus S)} \leq \sum_{Q \in \mathcal{W}(S)} \int_Q |\tilde{f}|^p dx \leq \sum_{Q \in \mathcal{W}(S)} |Q| \sup_{Q} |\tilde{f}|^p \leq \sum_{Q \in \mathcal{W}(S)} |Q| \max_{Q' \in \mathcal{Q}_2(Q)} |c_{Q'}|^p.
\]

By Lemma 3.2 \( |Q| \sim |Q'|, Q' \in \mathcal{Q}_2(Q) \), so that
\[
\|\tilde{f}\|^p_{L^p(\mathbb{R}^n \setminus S)} \leq C \sum_{Q \in \mathcal{W}(S)} \max_{Q' \in \mathcal{Q}_2(Q)} |c_{Q'}|^p |Q'| \leq C \sum_{Q \in \mathcal{W}(S)} |c_{Q}|^p |Q| \text{card } J_Q
\]
where
\[
J_Q := \{ Q' \in \mathcal{W}(S) : \mathcal{Q}_2(Q') \ni Q \}.
\]
By (4.3), \( Q' \in \mathcal{Q}_2(Q) \Leftrightarrow Q \in \mathcal{Q}_2(Q') \) so that \( J_Q = \mathcal{Q}_2(Q) \). Moreover, by (4.3) and Lemma 3.2,
\[
\text{card } J_Q = \text{card } \mathcal{Q}_2(Q) \leq (\text{card } \mathcal{Q}_1(Q))^2 \leq N(n)^2.
\]
Hence,
\[
\|\tilde{f}\|^p_{L^p(\mathbb{R}^n \setminus S)} \leq C N(n)^2 \sum_{Q \in \mathcal{W}(S)} |c_{Q}|^p |Q|.
\]
But, by (4.1), \( c_Q = \bar{c} = 0 \) whenever \( \text{diam} \, Q = 2r_Q > 2\delta \), so that
\[
\| \tilde{f} \|_{L_p(\mathbb{R}^n \setminus S)}^p \leq CN(n)^2 \sum_{Q \in \mathcal{W}(S), \text{diam} \, Q \leq 2\delta} |c_Q|^p |Q| = CN(n)^2 \sum_{Q \in \mathcal{W}(S:2\delta)} |f(a_Q)|^p |Q|.
\]

The lemma is proved. \( \square \)

**Lemma 4.4** (a). For every cube \( K \) centered in \( S \) we have:
\[
\sup_K |\tilde{f} - \bar{c}| \leq \sup_{(14K) \cap S} |f - \bar{c}|.
\]

(b). If \( K \) is a cube centered in \( S \) and \( \text{diam} \, K \leq \delta \), then for every \( c_0 \in \mathbb{R} \)
\[
\sup_K |\tilde{f} - c_0| \leq \sup_{(14K) \cap S} |f - c_0|.
\]

**Proof.** It can be readily seen that Theorem 3.1 and Lemma 4.1 imply the following:
\[
a_Q \in (14K) \cap S, \quad Q \in Q_2(K). \tag{4.4}
\]

Prove (a). By Lemma 4.2 with \( c_0 = \bar{c} \),
\[
\sup_{K \setminus S} |\tilde{f} - \bar{c}| \leq \sup_{Q \in Q_2(K)} |c_Q - \bar{c}|.
\]

By (4.1), \( c_Q \) takes the value \( f(a_Q) \) or \( \bar{c} \) so that by (4.4)
\[
\sup_{K \setminus S} |\tilde{f} - \bar{c}| \leq \sup_{Q \in Q_2(K)} |f(a_Q) - \bar{c}| \leq \sup_{(14K) \cap S} |f - \bar{c}|.
\]

Finally,
\[
\sup_K |\tilde{f} - \bar{c}| \leq \max\{\sup_{K \cap S} |f - \bar{c}|, \sup_{K \setminus S} |\tilde{f} - \bar{c}|\} \leq \sup_{(14K) \cap S} |f - \bar{c}|,
\]
proving part (a) of the lemma.

Prove (b). Again, applying Lemma 4.2, we obtain
\[
\sup_K |\tilde{f} - c_0| \leq \max\{\sup_{K \cap S} |f - c_0|, \sup_{K \setminus S} |\tilde{f} - c_0|\} \leq \max\{\sup_{K \cap S} |f - c_0|, \sup_{Q \in Q_2(K)} |c_Q - c_0|\}.
\]

But, by Lemma 4.1, \( \text{diam} \, Q \leq 2 \text{diam} \, K \leq 2\delta \) for every \( Q \in Q_2(K) \) so that by (4.1)
\( c_Q = f(a_Q) \) for each \( Q \in Q_2(K) \). Hence, by (4.4),
\[
\sup_{Q \in Q_2(K)} |c_Q - c_0| = \sup_{Q \in Q_2(K)} |f(a_Q) - c_0| \leq \sup_{(14K) \cap S} |f - c_0|
\]
proving part (b) of the lemma. \( \square \)

We turn to estimates of local oscillations of \( \tilde{f} \) on cubes which are located rather far from the set \( S \). Let \( K \) be a cube satisfying the following inequality
\[
\text{diam} \, K \leq \text{dist}(K, S)/40. \tag{4.5}
\]

We let \( Q_K \in \mathcal{W}(S) \) denote a Whitney's cube which contains center of \( K \), the point \( x_K \).

The proof of the next lemma easily follows from condition (4.5) and Theorem 3.1.
Lemma 4.5 $K \subset Q^*_K$ and

$$\left(\frac{8}{9}\right) \text{diam } Q_K \leq \text{dist}(K, S) \leq 5 \text{ diam } Q_K.$$ 

We present one more property of cubes satisfying condition (4.5).

Lemma 4.6 For every cube $K$ satisfying (4.5) the following inequality

$$\mathcal{E}(\tilde{f}; K) \leq C(n) \frac{r_K}{r_{Q_K}} \max \{|c_Q - c_{Q_K}| : Q \in \mathbb{W}(S), Q^* \cap K \neq \emptyset\}$$

holds.

Proof. Since $K \subset \mathbb{R}^n \setminus S$, the function $\tilde{f}_K \in C^\infty(\mathbb{R}^n)$ so that for every $x, y \in K$ we have

$$|\tilde{f}(x) - \tilde{f}(y)| \leq C(\sup_K \|
abla \tilde{f}\|) \|x - y\| \leq C r_K \sup_K \|
abla \tilde{f}\|.$$ 

Since $K \subset Q^*_K$, see Lemma 4.5, by properties (b)-(d) of partition of unity, for every $x \in K$ we have

$$\|
abla \tilde{f}(x)\| = \|
abla( \sum_{Q \in \mathbb{W}(S)} (c_Q - c_K) \varphi_Q(x))\| = \| \sum_{Q^* \cap K \neq \emptyset, Q \in \mathbb{W}(S)} (c_Q - c_K) \nabla \varphi_Q(x) \| \leq \sum_{Q^* \cap K \neq \emptyset, Q \in \mathbb{W}(S)} |c_Q - c_K| \|
abla \varphi_Q(x)\| \leq C \sum_{Q^* \cap K \neq \emptyset, Q \in \mathbb{W}(S)} \frac{|c_Q - c_K|}{\text{diam } Q}.$$ 

Clearly, if $Q^* \cap K \neq \emptyset$, then $Q^* \cap Q_K \neq \emptyset$ as well, so that by Lemma 3.2 the number of such cubes $Q \in \mathbb{W}(S)$ is bounded by $N(n)$ and $\text{diam } Q \sim \text{diam } Q_K$. Hence

$$\sup_K \|
abla \tilde{f}\| \leq C r_{Q_K}^{-1} \max \{|c_Q - c_K| : Q \in \mathbb{W}(S), Q^* \cap K \neq \emptyset\}$$

proving the lemma. □

Let $\pi$ be a packing of cubes in $\mathbb{R}^n$. We put

$$I(\tilde{f}; \pi) := \left\{ \sum_{K \in \pi} r_K^{n-p} \mathcal{E}(\tilde{f}; K)^p \right\}^{\frac{1}{p}}.$$ 

We also define a family of cubes $Q(\pi)$ by letting

$$Q(\pi) := \{ Q \in \mathbb{W}(S) : \text{ there is } K \in \pi \text{ such that } x_K \in Q \}.$$ 

(4.6)

Lemma 4.7 For every packing $\pi$ of cubes satisfying inequality (4.5) we have

$$I(\tilde{f}; \pi)^p \leq C(n, p) \sum_{Q \in Q(\pi)} r_Q^{n-p} \max \{|c_Q - c_{Q'}|^p : Q' \in \mathbb{W}(S), Q \cap Q' \neq \emptyset\}.$$ 

28
Proof. Fix a cube $\tilde{Q} \in \mathbb{W}(S)$ and suppose that all cubes $K \in \pi$ are centered in $\tilde{Q}$. By Lemma 4.6 for each $K \in \pi$ we have
\[
\mathcal{E}(\tilde{f}; K) \leq C_{\tilde{Q}}^{r_{\tilde{Q}}} \max \{|c_Q - c_{\tilde{Q}}| : Q \in \mathbb{W}(S), Q^* \cap K \neq \emptyset\}.
\]
Since $x_K \in \tilde{Q}$, by Lemma 4.5, $K \subset \tilde{Q}^*$ so that $Q^* \cap \tilde{Q}^* \neq \emptyset$ for every $Q \in \mathbb{W}(S)$ such that $Q^* \cap K \neq \emptyset$. Then, by part (3) of Lemma 3.2, $Q \cap \tilde{Q} \neq \emptyset$ as well so that
\[
\mathcal{E}(\tilde{f}; K) \leq C_{\tilde{Q}} r_{\tilde{Q}} J(\tilde{f}; \tilde{Q})/r_{\tilde{Q}}
\]
where
\[
J(\tilde{f}; \tilde{Q}) := \max \{|c_Q - c_{\tilde{Q}}| : Q \in \mathbb{W}(S), Q \cap \tilde{Q} \neq \emptyset\}.
\]
Hence
\[
I(\tilde{f}; \pi)^p \leq C \sum_{K \in \pi} r_{\tilde{Q}}^{-p} J(\tilde{f}; \tilde{Q})^p r_{K} = C r_{\tilde{Q}}^{-p} J(\tilde{f}; \tilde{Q})^p \sum_{K \in \pi} |K|.
\]
Since $\pi$ consists of disjoint cubes lying in $\tilde{Q}^*$, we obtain
\[
\sum_{K \in \pi} |K| = |\bigcup_{K \in \pi} K| \leq |\tilde{Q}^*| \leq C |\tilde{Q}| = C r_{\tilde{Q}}^n
\]
proving that in the case under consideration $I(\tilde{f}; \pi)^p \leq C r_{\tilde{Q}}^{-p} J(\tilde{f}; \tilde{Q})$.

Now let $\pi$ be an arbitrary packing of cubes satisfying inequality (4.5). Then
\[
I(\tilde{f}; \pi)^p \leq C \sum_{Q \in \mathbb{W}(S)} \sum_{K \in \pi, x_K \in Q} r_{\tilde{Q}}^{-p} \mathcal{E}(\tilde{f}; K)^p
\]
\[
\leq C \sum_{Q \in \mathbb{W}(S)} r_{\tilde{Q}}^{-p} \max \{|c_Q - c_{Q'}|^p : Q' \in \mathbb{W}(S), Q \cap Q' \neq \emptyset\}.
\]

We let $\mathcal{K}(\mathbb{R}^n)$ denote the family of all cubes in $\mathbb{R}^n$. We put
\[
\mathcal{K}_1 := \{K \in \mathcal{K}(\mathbb{R}^n) : \text{dist}(K, S) < 40 \text{ diam } K, \text{ diam } K \leq \delta/82\}, \quad (4.7)
\]
\[
\mathcal{K}_2 := \{K \in \mathcal{K}(\mathbb{R}^n) : \text{dist}(K, S) < 40 \text{ diam } K, \text{ diam } K > \delta/82\}, \quad (4.8)
\]
\[
\mathcal{K}_3 := \{K \in \mathcal{K}(\mathbb{R}^n) : 40 \text{ diam } K \leq \text{dist}(K, S) \leq (2/5)\delta\}, \quad (4.9)
\]
\[
\mathcal{K}_4 := \{K \in \mathcal{K}(\mathbb{R}^n) : 40 \text{ diam } K \leq \text{dist}(K, S), (2/5)\delta < \text{dist}(K, S) \leq 40\delta\}, \quad (4.10)
\]
and, finally,
\[
\mathcal{K}_5 := \{K \in \mathcal{K}(\mathbb{R}^n) : 40 \text{ diam } K \leq \text{dist}(K, S), \text{dist}(K, S) > 40\delta\}. \quad (4.11)
\]
Clearly, subfamilies $\{\mathcal{K}_i, i = 1, ..., 5\}$ provide a partition of the family $\mathcal{K}(\mathbb{R}^n)$. Let us estimate $I(\tilde{f}; \pi)$ for $\pi \subset \mathcal{K}_i, i = 1, ..., 5$.

We begin with the estimates of the quantity $I(\tilde{f}; \pi)$ for $\pi \subset \mathcal{K}_1$. 

29
Lemma 4.8 For every packing $\pi \subset K_1$ of equal cubes of diameter $t$ there exists a packing $\tilde{\pi} = \{Q\}$ of equal cubes centered in $S$ of diameter $\gamma t$ such that

$$I(\tilde{f}; \pi)^p \leq C(n, p) \sum_{Q \in \tilde{\pi}} r_Q^{n-p} E(f; Q \cap S)^p.$$ 

Here $\gamma (= 574)$ is an absolute constant.

Proof. For each $K \in \pi$ we let $K' := Q(a_K, \text{diam } K + \text{dist}(K, S))$. (Recall that $a_K$ is a point nearest to $K$ on $S$.) It can be easily seen that $K' \supset K$. Observe that for every cube $K \in K_1$ we have

$$r_{K'} := \text{diam } K + \text{dist}(K, S) \leq \text{diam } K + 40 \text{ diam } K = 41 \text{ diam } K \leq \delta/2. \quad (4.12)$$

Let us apply part (b) of Lemma 4.4 to $K'$ (with $c_0 := \tilde{f}(a_K) = f(a_K)$). We obtain

$$E(\tilde{f}; K') := \sup_{x, y \in K'} |\tilde{f}(x) - \tilde{f}(y)| \leq 2 \sup_{K'} |\tilde{f} - \tilde{f}(a_K)|$$

$$\leq 2 \sup_{(14K') \cap S} |f - f(a_K)| \leq 2 E(f; (14K') \cap S).$$

Since $K \subset K'$, we have

$$I(\tilde{f}; \pi)^p \leq \sum_{K \in \pi} r_{K'}^{n-p} E(\tilde{f}; K')^p \leq C^p \sum_{K \in \pi} r_{K'}^{n-p} E(f; (14K') \cap S)^p. \quad (4.13)$$

Given $K \in \pi$ we put $\tilde{K} := Q(a_K, \gamma t)$ where $\gamma = 14 \cdot 41 = 574$. Clearly, by (4.12), $\tilde{K} \supset 14 K' \supset K$.

We also put $\tilde{\pi} := \{\tilde{K} : K \in \pi\}$. Since $r_{\tilde{K}} \sim r_K$ and $\tilde{K} \supset K$, $K \in \pi$, by Lemma 3.4 the packing $\tilde{\pi}$ can be represented as union of at most $C(n)$ packings. Therefore, without loss of generality, we may suppose that $\tilde{\pi}$ is a packing.

Finally, (4.13) and inclusion $\tilde{K} \supset K$ imply the following:

$$I(\tilde{f}; \pi)^p \leq C^p \sum_{K \in \pi} r_{K'}^{n-p} E(f; (14K') \cap S)^p \leq C^p \sum_{K \in \pi} r_{K}^{n-p} E(f; \tilde{K} \cap S)^p.$$ 

The lemma is proved.

This lemma and Definition 2.1 imply the following

Lemma 4.9 For every packing $\pi \subset K_1$ of equal cubes of diameter $t$ we have

$$I(\tilde{f}; \pi) \leq C(n, p) A_p(f; \gamma t : S)/t.$$ 

Here $\gamma (= 574)$ is an absolute constant.

We turn to estimates of oscillations of $\tilde{f}$ over cubes from the family $K_3$. (We do not present here such estimates for the family $K_2$. As we shall see below, these estimates are either elementary or can be easily avoided.)
Lemma 4.10 For every packing $\pi \subset \mathcal{K}_3$ of equal cubes of diameter $t$ there exists a finite family $\pi' = \{Q\} \subset \mathcal{W}(S)$ of Whitney’s cubes of diameter $2t \leq \text{diam } Q \leq 2\delta$ such that

$$I(\tilde{f}; \pi)^p \leq C(n, p) \sum_{Q \in \pi'} \{r_Q^{n-p}|f(a_Q) - f(a_Q')|^p : Q, Q' \in \pi', Q \cap Q' \neq \emptyset, r_{Q'} \leq r_Q\}.$$  

Proof. By Lemma 4.7

$$I(\tilde{f}; \pi)^p \leq C \sum_{Q \in \pi'} r_Q^{n-p} \max\{|c_Q - c_{Q'}|^p : Q' \in \mathcal{W}(S), Q \cap Q' \neq \emptyset\}, \quad (4.14)$$

where $Q(\pi)$ is defined by (4.6). Thus for every $Q \in Q(\pi)$ there is a cube $K \in \pi$ such that $x_K \in Q$. Since $K \in \mathcal{K}_3$, we have $40 \text{diam } K \leq \text{dist}(K, S)$ and $\text{dist}(K, S) \leq (2/5)\delta$. Clearly,

$$\text{dist}(K, S) \leq \text{diam } Q + \text{dist}(Q, S) \leq 5 \text{diam } Q,$$

so that

$$40t = 40 \text{diam } K \leq \text{dist}(K, S) \leq 5 \text{diam } Q.$$  

Hence $8t \leq \text{diam } Q$. If $Q' \cap Q \neq \emptyset$, $Q' \in \mathcal{W}(S)$, then, by Lemma 3.2, $\text{diam } Q \leq 4 \text{diam } Q'$ so that $\text{diam } Q' \geq 2t$.

On the other hand, by Lemma 4.5, $\text{diam } Q \leq (9/8)\text{dist}(K, S)$ so that

$$\text{diam } Q \leq (9/8)(2/5)\delta = (9/20)\delta.$$  

Then, by Lemma 3.2, for every $Q' \in \mathcal{W}(S)$ such that $Q' \cap Q \neq \emptyset$ we have

$$\text{diam } Q' \leq 4 \text{diam } Q \leq (9/5)\delta < 2\delta,$$

or, equivalently, $r_{Q'} \leq \delta$. Therefore, by (4.1), $c_Q = f(a_Q), \; c_{Q'} = f(a_{Q'})$. (Recall that $a_Q$ stands for a point nearest to $Q$ on $S$.)

Finally, we put

$$\pi' := \{Q \in \mathcal{W}(S) : \text{ there is } Q' \in Q(\pi) \text{ such that } Q \cap Q' \neq \emptyset\}.$$  

As we have proved, $2t \leq \text{diam } Q \leq 2\delta$ for every $Q \in \pi'$.

Now, by (4.14),

$$I(\tilde{f}; \pi)^p \leq C \sum_{Q \in \pi'} r_Q^{n-p} \max\{|f(a_Q) - f(a_{Q'})|^p : Q' \in \mathcal{W}(S), Q \cap Q' \neq \emptyset\}$$

$$\leq C \sum_{Q \in \pi'} \{r_Q^{n-p}|f(a_Q) - f(a_{Q'})|^p : Q, Q' \in \pi', Q \cap Q' \neq \emptyset\}$$

$$\leq 2C \sum_{Q \in \pi'} \{r_Q^{n-p}|f(a_Q) - f(a_{Q'})|^p : Q, Q' \in \pi', Q \cap Q' \neq \emptyset, r_{Q'} \leq r_Q\}$$

proving the lemma. \hfill \Box

Lemma 4.11 For every packing $\pi \subset \mathcal{K}_3$ there exists a packing $\pi^\perp \subset \mathcal{W}(S)$ of Whitney’s cubes of diameter at most $2\delta$ such that

$$I(\tilde{f}; \pi)^p \leq C(n, p) \sum_{Q \in \pi^\perp} r_Q^{n-p} \mathcal{E}(f; (11Q) \cap S)^p. \quad (4.15)$$
Proof. By Lemma 4.10 there exists a finite family \( \pi' \subset \mathbb{W}(S) \) of Whitney’s cubes of diameter at most \( 2\delta \) such that
\[
I(\tilde{f}; \pi)^p \leq C \sum_{Q \in \pi'} \{r_Q^n - p|f(a_Q) - f(a_{Q'})|^p : Q, Q' \in \pi', Q \cap Q' \neq \emptyset, r_{Q'} \leq r_Q\}.
\]
Observe that for every two cubes \( Q, Q' \in \mathbb{W}(S) \) such that \( Q \cap Q' \neq \emptyset, r_{Q'} \leq r_Q \), we have \( a_Q, a_{Q'} \in (11Q) \cap S \). This property easily follows from Theorem 3.1 and Lemma 3.2. Hence,
\[
|f(a_Q) - f(a_{Q'})| \leq \mathcal{E}(f; (11Q) \cap S)
\]
so that
\[
I(\tilde{f}; \pi)^p \leq C \sum_{Q \in \pi'} r_Q^n \mathcal{E}(f; (11Q) \cap S)^p \text{card}\{Q' \in \mathbb{W}(S) : Q' \cap Q \neq \emptyset\}.
\]
By Lemma 3.2, (2), \( \text{card}\{Q' \in \mathbb{W}(S) : Q' \cap Q \neq \emptyset\} \leq N(n) \) so that
\[
I(\tilde{f}; \pi)^p \leq C \sum_{Q \in \pi'} r_Q^n \mathcal{E}(f; (11Q) \cap S)^p. \tag{4.16}
\]
Since \( \pi' \subset \mathbb{W}(S) \), its covering multiplicity is bounded by a constant \( N(n) \), see Theorem 3.1, so that, by Theorem 3.3, \( \pi' \) can be partitioned into \( C(n) \) packings \( \{\pi'_i : i = 1, \ldots, C(n)\} \). We let \( \pi^\wedge \) denote a packing \( \pi'_i \) for which the sum
\[
\sum_{Q \in \pi'_i} r_Q^n \mathcal{E}(f; (11Q) \cap S)^p
\]
is maximal. Clearly, by inequality (4.16), \( \pi^\wedge \) satisfies inequality (4.15), and the lemma follows. \( \square \)

In the next three lemmas we present estimates of local oscillation of \( \tilde{f} \) over cubes from \( \mathcal{K}_3 \) via the functional \( \mathcal{A}_{p,\alpha}(f; \cdot : S) \), see (2.13).

**Lemma 4.12** Let \( 0 < \beta < 1, 0 < t < T, \varsigma \geq 1, 0 \leq s \leq 1, \) and let \( \pi = \{Q\} \subset \mathbb{W}(S) \) be a collection of Whitney’s cubes of diameter \( t \leq \text{diam} \, Q \leq T \). Given \( Q \in \pi \) let \( \tilde{Q} \) be a \( \beta \)-porous cube centered in \( \partial S \) such that
\[
\|x_Q - x_{\tilde{Q}}\| \leq \varsigma \text{diam} \, Q \quad \text{and} \quad \text{diam} \, Q \leq \text{diam} \, \tilde{Q} \leq \varsigma \text{diam} \, Q. \tag{4.17}
\]

Then for every \( \alpha \in (0, \beta) \) and every \( f \in C(S) \) we have
\[
\sum_{Q \in \pi} r_Q^{n-sp} \mathcal{E}(f; \tilde{Q} \cap \partial S)^p \leq C \int_t^{4\varsigma T} A_{p,\alpha}(f; u : S)^p \frac{du}{u^{sp+1}}.
\]
Here \( C \) is a constant depending only on \( n, p, \varsigma, \beta \) and \( \alpha \).

**Proof.** Since \( \mathcal{A}_{p,\alpha} \) is a non-increasing function of \( \alpha \), it suffices to prove the statement of the lemma for \( \alpha \in (\beta/2, \beta) \). We put \( \tau := \beta/\alpha \). Since \( \alpha < \beta, \tau > 1 \).
Given integer \( m \) consider a family of cubes \( \tilde{\pi}_m := \{ Q \in \pi : \tau^{m-1} < r_Q \leq \tau^m \} \). Since for each \( Q \in \pi \)
\[
\text{diam} \tilde{Q} \leq \varsigma \text{diam} Q \quad \text{and} \quad t \leq \text{diam} Q \leq T,
\]
we have
\[
t \leq \text{diam} \tilde{Q} = 2r_{\tilde{Q}} \leq \varsigma \text{diam} Q \leq \varsigma T, \quad Q \in \pi.
\]
Therefore we may assume that \( t \leq 2\tau^m \) and \( 2\tau^{m-1} \leq \varsigma T \). Thus \( M_0 \leq m \leq M_1 \) where \( M_0, M_1 \) are positive integers such that \( 2\tau^{M_1-1} \leq \varsigma T < 2\tau^{M_1} \) and \( 2\tau^{M_0-1} \leq t \leq 2\tau^{M_0} \).

Given \( Q \in \tilde{\pi}_m \) we put
\[
Q^\sharp := (\tau^m / r_Q) \tilde{Q} = Q(x_Q, \tau^m).
\]
Then \( \tilde{Q} \subset Q^\sharp \subset \tau \tilde{Q} = (\beta/\alpha) \tilde{Q} \) so that
\[
\tilde{Q} \subset Q^\sharp \subset 2\tilde{Q}. \tag{4.18}
\]

Since \( \tilde{Q} \) has porosity \( \beta \), it contains a cube of diameter at least \( \beta \text{diam} \tilde{Q} \) which lies in \( \mathbb{R}^n \setminus S \). Put \( Q^\sharp := (\tau^m / r_Q) \tilde{Q} \). Then the set \( Q^\sharp \setminus S \) contains a cube of diameter at least
\[
(\tau^m / r_Q) \beta \text{diam} Q^\sharp > (\tau^{m-1} / \tau^m) \beta \text{diam} Q^\sharp = \beta / \tau \text{diam} Q^\sharp = \alpha \text{diam} Q^\sharp,
\]
so that \( Q^\sharp \) is an \( \alpha \)-porous cube.

Thus the family of cubes
\[
\pi^\sharp_m := \{ Q^\sharp : Q \in \tilde{\pi}_m \}
\]
is a family of equal cubes of porosity \( \alpha \). By (4.17) and by Lemma 3.4, the family \( \pi^\sharp_m \) can be partitioned into at most \( C(n, \tau) = C(n, \alpha, \beta) \) packings. Therefore, without loss of generality, later on we can assume that \( \pi^\sharp_m \) is a packing.

Thus \( \pi^\sharp_m \) is a packing of cubes centered in \( \partial S \) with diameter \( 2\tau^m \) and porosity \( \alpha \). Since \( \tilde{Q} \subset Q^\sharp \), we have
\[
E(f; \tilde{Q} \cap \partial S) \leq E(f; Q^\sharp \cap \partial S), \tag{4.19}
\]
so that
\[
I := \sum_{Q \in \pi} r_Q^{n-sp} E(f; \tilde{Q} \cap \partial S)^p \leq C \sum_{Q \in \pi} r_Q^{n-sp} E(f; Q^\sharp \cap \partial S)^p
\]
\[
= C \sum_{m=M_0}^{M_1} \tau^{-spm} \sum_{Q \in \pi^\sharp_m} |Q^\sharp| E(f; Q^\sharp \cap \partial S)^p.
\]

Hence, by (2.13), we have
\[
I \leq C \sum_{m=M_0}^{M_1} \tau^{-spm} A_{p,\alpha}(f; 2\tau^m : S)^p.
\]
Since \( A_{p,\alpha}(f; \cdot : S) \) is a non-decreasing function, it can be readily shown that

\[
I \leq C \int_{2\tau^{M_0}}^{2\tau^{M_1+1}} A_{p,\alpha}(f; u : S)^p \frac{du}{u^{sp+1}}.
\]

It remains to note that \( t \leq 2\tau^{M_0} \), and

\[
2\tau^{M_1+1} \leq \tau^2 T = (\beta/\alpha)^2 T \leq 4\varsigma T,
\]

and the lemma follows. \( \square \)

The following lemma easily follows from Theorem 3.1 and Lemma 3.2 and we leave the details of its proof to the reader.

**Lemma 4.13** Let \( Q, Q' \in \mathcal{W}(S) \), \( Q \cap Q' \neq \emptyset \), and let \( r_{Q'} \leq r_Q \). Then

(i). \( a_{Q'} \in \bar{Q} := Q(a_Q, 10 \text{diam } Q) \);

(ii). For every \( \beta \in (0, 3/20) \) and every \( \eta \in (0, 1] \) the cube \( \eta \bar{Q} \) is \( \beta \)-porous;

(iii). \( 3Q \subset \bar{Q} \setminus S. \)

**Lemma 4.14** Let \( 0 < \alpha < 3/20 \). Then for every packing \( \pi \subset K_3 \) of equal cubes of diameter \( t \) the following inequality

\[
I \left( \tilde{f}; \pi \right)^p \leq C \int t \ A_{p,\alpha}(f; u : S)^p \frac{du}{u^{sp+1}}
\]

holds. Here \( C = C(n, p, \alpha) \) and \( \gamma = 160 \).

**Proof.** By Lemma 4.10, there exists a finite family \( \pi' = \{Q\} \subset \mathcal{W}(S) \) of cubes of diameter \( 2t \leq \text{diam } Q \leq 2\delta \) such that

\[
I(\tilde{f}; \pi)^p \leq C^p \sum \{r_Q^n p |f(a_Q) - f(a_{Q'})|^p : Q, Q' \in \pi', Q \cap Q' \neq \emptyset, r_{Q'} \leq r_Q\}.
\]

Fix \( Q \in \pi' \). Then, by Lemma 4.13, \( a_{Q'} \in \bar{Q} := Q(a_Q, 10 \text{diam } Q) \) for every cube \( Q' \in \pi' \) such that \( Q' \cap Q \neq \emptyset \) and \( r_{Q'} \leq r_Q \). Clearly, \( a_Q, a_{Q'} \in \partial S \) so that

\[
|f(a_Q) - f(a_{Q'})| \leq \mathcal{E}(f; \bar{Q} \cap \partial S).
\]

Hence

\[
I(\tilde{f}; \pi)^p \leq C^p \sum \{r_Q^n p \mathcal{E}(f; \bar{Q} \cap \partial S)^p : Q, Q' \in \pi', Q \cap Q' \neq \emptyset, r_{Q'} \leq r_Q\}
\]

\[
\leq C^p \sum_{Q \in \pi'} r_Q^n p \mathcal{E}(f; \bar{Q} \cap \partial S)^p \text{card}\{Q' : Q' \in \mathcal{W}(S), Q' \cap Q \neq \emptyset\},
\]

so that by Lemma 3.2

\[
I(\tilde{f}; \pi)^p \leq C^p \sum_{Q \in \pi'} r_Q^n p \mathcal{E}(f; \bar{Q} \cap \partial S)^p.
\]
Let $\beta := \min\{2\alpha, (\alpha + 3/20)/2\}$. Then $\alpha < \beta < 3/20$. Since $20\beta < 3$, by Lemma 4.13, $(20\beta)Q \subset \tilde{Q} \setminus S$, so that the cube $\tilde{Q} = Q(aQ, 20rQ)$ is $\beta$-porous.

Now, applying Lemma 4.12 with $s = 1$, $T = 2\delta$ and $\varsigma = 20$, we have

$$I(\tilde{f}; \pi)^p \leq C \int_t^{\gamma \delta} A_{p, \alpha}(f; u : S)^p \frac{du}{u^{p+1}}$$

with $\gamma = 2\varsigma/\alpha^2$. But $\beta/\alpha \leq 2$ so that we can put $\gamma = 160$, and the proof of the lemma is finished. $\square$

We turn to the family $\mathcal{K}_4$.

**Lemma 4.15** For every packing $\pi \subset \mathcal{K}_4$ there exists a packing $\pi' = \{Q\}$ of Whitney’s cubes with $0.01\delta \leq r_Q \leq 90\delta$ such that

$$I(\tilde{f}; \pi)^p \leq C(n, p) \sum_{Q \in \pi'} r_Q^{-p} |f(a_Q) - \bar{c}|^p.$$  \hspace{1cm} (4.20)

**Proof.** Let $Q \in \mathcal{Q}(\pi)$, i.e., there exists a cube $K \in \pi$ centered in $Q$. Since $K \in \mathcal{K}_4$, by (4.10), $40 \text{diam} K \leq \text{dist}(K, S)$ and $(2/5)\delta < \text{dist}(K, S) \leq 40\delta$. Also, by Lemma 4.5,

$$\frac{1}{5} \text{dist}(K, S) \leq \text{diam} Q \leq \frac{9}{8} \text{dist}(K, S)$$

so that $(2/25)\delta \leq \text{diam} Q \leq 45\delta$. Therefore, by Lemma 3.2, for each $Q' \in \mathcal{W}(S)$, $Q' \cap Q \neq \emptyset$, we have $0.02\delta \leq \text{diam} Q' \leq 180\delta$, or, equivalently, $0.01\delta \leq r_{Q'} \leq 90\delta$.

Now, by Lemma 4.7, we obtain

$$I(\tilde{f}; \pi)^p \leq C \sum_{Q \in \mathcal{Q}(\pi)} r_Q^{-p} \max\{|c_Q - c_{Q'}|^p : Q' \in \mathcal{W}(S), Q \cap Q' \neq \emptyset\}.$$ 

We put

$$\pi' := \{Q \in \mathcal{W}(S) : \text{there is } Q' \in \mathcal{Q}(\pi) \text{ such that } Q \cap Q' \neq \emptyset\}.$$ 

Then $\pi' = \{Q\}$ is a finite family of cubes such that $0.01\delta \leq r_Q \leq 90\delta$ for every $Q \in \pi'$. Moreover, since $\pi' \subset \mathcal{W}(S)$, its covering multiplicity is bounded by a constant $N(n)$. Therefore, by Theorem 3.3, $\pi'$ can be partitioned into at most $C(n)$ packings so that, without loss of generality, we may assume that $\pi'$ itself is a packing.

Hence,

$$I(\tilde{f}; \pi)^p \leq C \sum\{(r_Q + r_{Q'})^{-p} |c_Q - c_{Q'}|^p : Q, Q' \in \pi', Q \cap Q' \neq \emptyset\}.$$ 

Since $r_Q \sim r_{Q'}$ for every $Q, Q' \in \mathcal{W}(S), Q \cap Q' \neq \emptyset$ (see Lemma 3.2), we have

$$(r_Q + r_{Q'})^{-p} |c_Q - c_{Q'}|^p \leq C(r_Q^{-p} |c_Q - \bar{c}|^p + r_{Q'}^{-p} |c_{Q'} - \bar{c}|^p)$$

so that

$$I(\tilde{f}; \pi)^p \leq C \sum\{r_Q^{-p} |c_Q - \bar{c}|^p + r_{Q'}^{-p} |c_{Q'} - \bar{c}|^p : Q, Q' \in \pi', Q \cap Q' \neq \emptyset\} \leq C \sum_{Q \in \pi'} (r_Q^{-p} |c_Q - \bar{c}|^p) \text{card}\{Q' \in \mathcal{W}(S) : Q' \cap Q \neq \emptyset\}.$$
Hence, by Lemma 3.2, we obtain
\[ I(\tilde{f}; \pi)^p \leq C \sum_{Q \in \pi'} r_Q^{n-p} |c_Q - \bar{c}|^p. \]

By (4.1), \( c_Q \in \{ f(a_Q), \bar{c} \} \) for every \( Q \in \mathbb{W}(S) \) so that
\[ |c_Q - \bar{c}| \leq |f(a_Q) - \bar{c}|, \quad Q \in \mathbb{W}(S) \]
proving that \( \pi' \) satisfies inequality (4.20). \( \square \)

This lemma implies the following

**Lemma 4.16** Let \( \bar{c} = 0 \). Then for every packing \( \pi \subset K_4 \) there exists a packing \( \pi' = \{ Q \} \subset \mathbb{W}(S) \) with \( 0.01\delta \leq \text{diam} Q \leq 90\delta \) such that the following inequality
\[ I(\tilde{f}; \pi)^p \leq C(n, p) \sum_{Q \in \pi'} |f(a_Q)|^p |Q| \]
holds.

It remains to consider the case \( \pi \subset K_5 \).

**Lemma 4.17** For every packing \( \pi \subset K_5 \) we have \( I(\tilde{f}; \pi) = 0 \).

**Proof.** Let \( Q \in Q(\pi) \), i.e., there exists a cube \( K \in \pi \) centered in \( Q \). Since \( K \in K_5 \), by (4.11), \( 40 \text{diam} K \leq \text{dist}(K, S) \) and \( \text{dist}(K, S) > 40\delta \). On the other hand, by Lemma 4.5, \( \text{diam} Q \geq (1/5) \text{dist}(K, S) \) so that \( \text{diam} Q > (40/5)\delta = 8\delta \).

Therefore, by Lemma 3.2, for each \( Q' \in \mathbb{W}(S) \), \( Q' \cap Q \neq \emptyset \), we have
\[ \text{diam} Q' \geq (1/4) \text{diam} Q > 2\delta. \]
Hence, by (4.1), \( c_{Q'} = c_Q = \bar{c} \) so that, by Lemma 4.7, \( I(\tilde{f}; \pi)^p = 0 \). \( \square \)

In the last lemma of this section we estimate the \( L_p \)-norm of the extension \( \tilde{f} \). For its formulation we put
\[ W^\sharp = \mathbb{W}(S : 180\delta) := \{ Q \in \mathbb{W}(S) : r_Q \leq 90\delta \}. \] (4.21)

Let \( \varepsilon > 0 \) and let \( T : S_\varepsilon \to S \) be a mapping such that \( T(x) = x \) for every \( x \in S \). Let \( f \) be a function defined on \( S \) such that \( f \circ T \in L_p(S_\varepsilon) \).

**Lemma 4.18** For every family \( \pi \subset W^\sharp \) the following is true:

(i). If \( \delta \leq 0.001\varepsilon \), then \( Q \subset S_\varepsilon \) for each \( Q \in \pi \);

(ii). There exists a packing \( \pi' \subset \pi \) such that
\[ \sum_{Q \in \pi} |f(a_Q)|^p |Q| \leq C \left\{ \|f \circ T\|_{L_p(S_\varepsilon)}^p + \sum_{Q \in \pi'} |Q| \sup_Q |f \circ T - f(a_Q)|^p \right\}. \]
Proof. For every $Q \in \pi$ we have

$$|f(a_Q) - (f \circ T)_{a_Q}| \leq \frac{1}{|Q|} \int_{Q} |f(T(x)) - f(a_Q)| \, dx$$

so that

$$I := \sum_{Q \in \pi} |f(a_Q)|^p |Q| \leq 2^p \{ \sum_{Q \in \pi} (|f(a_Q) - (f \circ T)_{a_Q}|^p |Q| + |(f \circ T)_{a_Q}|^p |Q|) \}$$

$$\leq 2^p \{ \sum_{Q \in \pi} J(f; Q)^p |Q| + \sum_{Q \in \pi} |(f \circ T)_{a_Q}|^p |Q| \} = 2^p \{ I_1 + I_2 \}.$$ 

By Lemma 4.13, $Q \subset Q(a_Q, 10 r_Q)$ for every $Q \in \mathbb{W}(S)$. Since $a_Q \in S$, we have $Q \subset S_{10 r_Q}$. But $r_Q \leq 90 \delta$, so that

$$10 r_Q \leq 900 \delta \leq 900 \cdot (0.001) \varepsilon < \varepsilon.$$ 

Hence, $Q \subset S_{\varepsilon}$ for every $Q \in \pi$ proving the part (i) of the lemma.

Prove (ii). We can assume that $I_1 < \infty$, otherwise part (ii) of the lemma is trivial. In this case there exists a finite subfamily $\pi' \subset \pi$ such that

$$I_1 := \sum_{Q \in \pi} |Q| J(f; Q)^p \leq 2 \sum_{Q \in \pi'} |Q| J(f; Q)^p.$$ 

By Theorem 3.3, the family $\pi'$ can be partitioned into at most $C(n)$ packings so that without loss of generality we may assume that $\pi'$ is a packing.

Let us estimate the quantity $I_2$. We have

$$I_2 := \sum_{Q \in \pi} |(f \circ T)_{a_Q}|^p |Q| = \sum_{Q \in \pi} |Q| \left| \frac{1}{|Q|} \int_{Q} f(T(x)) \, dx \right|^p \leq \sum_{Q \in \pi} \int_{Q} |f(T(x))|^p \, dx.$$ 

Put $U := \bigcup\{ Q : Q \in \pi \}$. Since $Q \subset S_{\varepsilon}$, $Q \in \pi$, we have $U \subset S_{\varepsilon}$. Then

$$I_2 \leq \sum_{Q \in \pi} \int_{Q} |f(T(x))|^p \, dx = \int_{U} |f(T(x))|^p \sum_{Q \in \pi} \chi_Q(x) \, dx$$

so that, by Theorem 3.1,

$$I_2 \leq N(n) \int_{U} |f(T(x))|^p \, dx \leq N(n) \int_{S_{\varepsilon}} |f(T(x))|^p \, dx = N(n) \| f \circ T \|_{L_p(S_{\varepsilon})}^p.$$ 

Finally, we have

$$I \leq 2^p \{ I_2 + I_1 \} \leq C \{ \| f \circ T \|_{L_p(S_{\varepsilon})}^p + \sum_{Q \in \pi'} |Q| J(f; Q)^p \}. \quad \square$$
5. Proofs of the main theorems.

Proof of Theorem 1.1. The necessity part of the theorem follows from Lemma 3.9 so that we turn to the proof of

Sufficiency. Let $f$ be a function defined on $S$. Theorem’s hypothesis is equivalent to the next statement: there exists a constant $\lambda > 0$ such that for every packing $\pi$ the following inequality

$$\sum_{Q \in \pi} r_Q^{n,p} \mathcal{E}(f; (11Q) \cap S)^p \leq \lambda$$

(5.1)

holds.

We have to prove that there exists a continuous function $F \in L^1_p(\mathbb{R}^n)$ such that $F|_S = f$ and

$$\|F\|_{L^1_p(\mathbb{R}^n)} \leq C\lambda^{\frac{1}{p}}.$$  

(5.2)

We define $F$ as follows. We fix a point $x_0 \in S$ and set

$$\delta := \text{diam } S \text{ and } \bar{c} := f(x_0).$$

Then we put

$$F := \text{Ext}_{\delta, \bar{c}, S} f$$

where $\text{Ext}_{\delta, \bar{c}, S}$ is the extension operator defined by formulas (4.2).

Let us prove that inequality (5.2) is satisfied. By criterion (2.2) it suffices to show that for every packing $\pi = \{K\}$ of equal cubes the following inequality

$$I(F; \pi) := \left\{ \sum_{K \in \pi} r_K^{n,p} \mathcal{E}(F; K)^p \right\}^{\frac{1}{p}} \leq C\lambda^{\frac{1}{p}}$$

(5.3)

holds.

We prove this inequality in 5 steps.

Put $t := \text{diam } K$, $K \in \pi$.

The first step: $\pi \subset \mathcal{K}_1$, see (4.7). By Lemma (4.8), there exist an absolute constant $\gamma > 1$ and a packing $\tilde{\pi} = \{Q\}$ of equal cubes centered in $S$ of diameter $\gamma t$ such that

$$I(F; \pi) \leq C \sum_{Q \in \tilde{\pi}} r_Q^{n,p} \mathcal{E}(f; Q \cap S)^p$$

so that, by (5.1), $I(F; \pi) \leq C\lambda^{\frac{1}{p}}$ proving (5.3).

The second step: $\pi \subset \mathcal{K}_2$, see (4.8). In this case

$$t = \text{diam } K > \delta/82 = \text{diam } S/82, \quad K \in \pi$$

so that we may assume that $\delta = \text{diam } S < \infty$.  

38
We put \( \tilde{K} := Q(x_0, \text{diam } S) \). The reader can easily prove that in this case

\[
I(F; \pi)^p \leq C \sum_{Q \in \pi} r_Q^{n-p} \mathcal{E}(f; \tilde{K} \cap S)^p
\]  

(5.4)

so that applying (5.1) with \( \pi = \{ \tilde{K} \} \) we obtain the required inequality (5.3).

*The third step:* \( \pi \subset \mathcal{K}_3 \), see (4.9). In this case, by Lemma 4.11, there exists a packing \( \pi^L \subset \mathbb{W}(S) \) of Whitney’s cubes of diameter at most \( 2\delta \) such that

\[
I(F; \pi)^p \leq C \sum_{Q \in \pi^L} r_Q^{n-p} \mathcal{E}(f; (11Q) \cap S)^p,
\]

so that, by (5.1), \( I(F; \pi)^p \leq C\lambda \) proving (5.3).

*The forth step.* \( \pi \subset \mathcal{K}_4 \), see (4.10). In this case

\[
40 \text{diam } K \leq \text{dist}(K, S) \quad \text{and} \quad (2/5)\delta < \text{dist}(K, S) \leq 40\delta, \quad K \in \pi,
\]

so that we may suppose that \( \delta = \text{diam } S < \infty \).

By Lemma 4.15, there exists a packing \( \pi' = \{ Q \} \) with \( r_Q \sim \delta = \text{diam } S \) such that

\[
I(F; \pi)^p \leq C \sum_{Q \in \pi'} r_Q^{n-p} |f(a_Q) - \bar{c}|^p.
\]

Recall that \( \bar{c} = f(x_0) \) so that

\[
I(F; \pi)^p \leq C \sum_{Q \in \pi'} r_Q^{n-p} |f(a_Q) - f(x_0)|^p \leq C \sum_{Q \in \pi'} r_Q^{n-p} \mathcal{E}(f; S)^p
\]

\[
\leq C(\text{diam } S)^{n-p} \mathcal{E}(f; S)^p.
\]

Hence

\[
I(F; \pi)^p \leq C r_{\tilde{K}}^{n-p} \mathcal{E}(f; \tilde{K} \cap S)^p
\]  

(5.5)

where \( \tilde{K} := Q(x_0, \text{diam } S) \). It remains to apply (5.1) with \( \pi = \{ \tilde{K} \} \) and the required inequality \( I(F; \pi)^p \leq C\lambda \) follows.

*The fifth step.* \( \pi \subset \mathcal{K}_5 \). In this case, by Lemma 4.17, \( I(F; \pi) = 0 \), so that (5.3) trivially holds.

Theorem 1.1 is completely proved. \( \square \)

**Remark 5.1** Given \( \varepsilon \in (0, 1] \) the factor \( \gamma = 11 \) in the formulation of Theorem 1.1 can be decreased to \( \gamma = 3 + \varepsilon \). It can be done by a slight modification of the Whitney extension formula (4.2) where the Whitney covering \( \mathbb{W}(S) \) is replaced with the covering \( \mathbb{W}_\varepsilon(S) = \{ Q \} \) of \( \mathbb{R}^n \setminus S \) satisfying the following conditions:

(a). \( \varepsilon \text{diam } Q \leq \text{dist}(Q, S) \leq 4\varepsilon \text{diam } Q, \quad Q \in \mathbb{W}_\varepsilon(S); \)

(b). Every point of \( \mathbb{R}^n \setminus S \) is covered by at most \( N = N(n, \varepsilon) \) cubes from \( \mathbb{W}_\varepsilon(S) \).

Also we put \( Q^* := (1 + \varepsilon/8)Q \). The existence of the family \( \mathbb{W}_\varepsilon(S) \) readily follows from Besicovitch covering theorem, see, e.g. [G]).

We observe that the factor \( \gamma = 11 \) first appears in the proof of Lemma 4.11. Elementary changes in this proof show that the corresponding factor for the modified Whitney’s method does not exceed the quantity \( \gamma = 3 + 9\varepsilon \).

Observe that in this case the constants of the equivalence \( \|f\|_{L^p_{\lambda}(\mathbb{R}^n)} \sim \inf \lambda^\frac{1}{p} \) depend on \( n, p \) and \( \varepsilon \).
Proof of Theorem 2.4. The necessity part of the theorem follows from inequality (2.12), Proposition 3.11 and inequality (3.8).

Let us prove the sufficiency. Let \( \lambda > 0 \), \( 0 < \alpha < 3/20 \), and let \( f : S \rightarrow \mathbb{R} \) be a function satisfying the following inequality:

\[
\sup_{0 < t \leq 2 \text{ diam } S} \left( \frac{A_p(f; t : S)}{t} \right)^p + \int_0^{\text{diam } S} \left( \frac{A_{p,\alpha}(f; t : S)}{t} \right)^p \frac{dt}{t} \leq \lambda.
\] (5.6)

We have to prove the existence of a continuous function \( F \in L^1_p(\mathbb{R}^n) \) such that \( F|_S = f \) and \( \|F\|_{L^1_p(\mathbb{R}^n)} \leq C\lambda^{\frac{1}{p}} \) where \( C \) is a constant depending only on \( n, p \) and \( \alpha \).

We use the same extension operator as in the proof of Theorem 1.1, i.e., we fix a point \( x_0 \in S \) and put

\( F := \text{Ext}_{\delta, \bar{c}, S} f \)

where \( \delta := \text{diam } S \), \( \bar{c} := f(x_0) \), and \( \text{Ext}_{\delta, \bar{c}, S} \) is the extension operator defined by formulas (4.2).

We are needed the following lemma which easily follows from definition (2.11) of the \( A_p \)-functional.

Lemma 5.2 Let \( p > n \). Then

\[
\frac{1}{u} A_p(f; u : S) \leq (2 \text{ diam } S)^{-\frac{1}{p}} A_p(f; 2 \text{ diam } S : S), \quad u \geq 2 \text{ diam } S.
\]

Our aim is to show that inequality (5.3) is satisfied. We prove this inequality following the same scheme as in the proof of Theorem 1.1, i.e., in 5 steps.

The first step: \( \pi \subset K_1 \). Let \( \pi \subset K_1 \) be a packing of equal cubes of diameter \( t \). Then, by Lemma 4.9, \( I(F; \pi) \leq Ct^{-1}A_p(f; \gamma t : S) \). On the other hand, by Lemma 5.2,

\[
\frac{A_p(f; u : S)}{u} \leq u^{p-1} \frac{A_p(f; 2 \text{ diam } S : S)}{(2 \text{ diam } S)^{\frac{p}{2}}} \leq \frac{A_p(f; 2 \text{ diam } S : S)}{2 \text{ diam } S}, \quad u \geq 2 \text{ diam } S,
\]

so that, by (5.6),

\[
I(F; \pi) \leq C \sup_{0 < u \leq 2 \text{ diam } S} \frac{A_p(f; u : S)}{u} \leq C\lambda^{\frac{1}{p}}
\]

proving (5.3).

The second (\( \pi \subset K_2 \)), forth (\( \pi \subset K_4 \)) and fifth (\( \pi \subset K_5 \)) steps. Suppose that \( \pi \subset K_2 \) or \( \pi \subset K_4 \). In these cases inequality (5.3) easy follows from inequalities (5.4) and (5.5). In fact, by these inequalities in both cases

\[
I(F; \pi)^p \leq C^p r^{n-p}_{\tilde{K}} \mathcal{E}(f, \tilde{K} \cap S)^p
\]

where \( \tilde{K} \) is a cube centered in \( S \) with diam \( \tilde{K} = 2 \text{ diam } S \). Hence, by (5.6)

\[
I(F; \pi) \leq C A_p(f, 2 \text{ diam } S : S)/(2 \text{ diam } S) \leq C\lambda^{\frac{1}{p}}
\]

so that inequality (5.3) is satisfied.
The fifth step \((\pi \subset \mathcal{K}_5)\) is trivial because in this case \(I(F; \pi) = 0\).

The third step: \(\pi \subset \mathcal{K}_3\). By Lemma 4.14, for every \(0 < \alpha < 3/20\) we have

\[
I(F; \pi)^p \leq C^n \int_0^\infty \left( \frac{A_{p,\alpha}(f; u : \partial S, \alpha)}{u} \right)^p \frac{du}{u}.
\]

Elementary calculations show that, this inequality, Lemma 5.2 and (5.6) imply the required inequality \(I(F; \pi) \leq C\lambda^{\frac{1}{p}}\). We leave the technical details of the proof to the reader. 

Proof of Theorem 1.2. The necessity part of the theorem follows from Lemma 3.9 and Proposition 3.13.

Let us prove the sufficiency. We will be needed the following

**Lemma 5.3** Let \(\varsigma > 0\) be a positive number and let \(F\) be a continuous function from \(L_p(\mathbb{R}^n), p > 1\). Suppose that there exists a constant \(\lambda > 0\) such that for every packing \(\pi\) of equal cubes of diameter at most \(\varsigma\) the following inequality

\[
\sum_{Q \in \pi} r_Q^{-p} E(F; Q)^p \leq \lambda
\]

holds. Then \(F \in L_p^1(\mathbb{R}^n)\) and

\[
\|F\|_{L_p^1(\mathbb{R}^n)} \leq C(\lambda^{\frac{1}{p}} + \varsigma^{-1}\|F\|_{L_p(\mathbb{R}^n)}).
\]

**Proof.** Let \(\pi\) be a packing of equal cubes. Put

\[
J(F; \pi) := \left\{ \sum_{Q \in \pi} r_Q^{-p} E_1(F; Q)_{L_p}^p \right\}^{\frac{1}{p}}
\]

where \(E_1(F; Q)_{L_p}\) is the normalized best approximation in \(L_p\) defined by (2.5).

If \(\text{diam } Q \leq \varsigma\) for every \(Q \in \pi\), then by (5.7)

\[
J(F; \pi)^p \leq \sum_{Q \in \pi} r_Q^{-p} E_1(F; Q)_{L_p}^p \leq \sum_{Q \in \pi} r_Q^{-p} E(F; Q)^p \leq \lambda.
\]

Suppose that for every \(Q \in \pi\) we have \(\text{diam } Q = 2r_Q > \varsigma\). Then

\[
J(F; \pi)^p \leq \sum_{Q \in \pi} r_Q^{-p} \left( \frac{1}{|Q|} \int_Q |F|^p \, dx \right) = \sum_{Q \in \pi} r_Q^{-p} \int_Q |F|^p \, dx,
\]

so that

\[
J(F; \pi)^p \leq 2^p \varsigma^{-p} \sum_{Q \in \pi} \int_Q |F|^p \, dx \leq 2^p \varsigma^{-p} \|F\|_{L_p(\mathbb{R}^n)}^p.
\]
Thus
\[ \sup_{\pi} J(F; \pi)^p \leq \lambda + 2^p \varsigma^{-p} \|F\|_{L_p(R^n)}^p < \infty, \]
where \( \pi \) runs over all packings of equal cubes in \( R^n \). Hence, by a result of Brudnyi [Br1], \( F \in L^1_p(R^n) \) and
\[ \|F\|_{L^1_p(R^n)} < \infty, \]
where \( \pi \) runs over all packings of equal cubes in \( R^n \). Hence, by a result of Brudnyi [Br1], \( F \in L^1_p(R^n) \) and
\[ \|F\|_{L^1_p(R^n)} \leq C \sup_{\pi} J(F; \pi) \leq C(\lambda^{1\over p} + \varsigma^{-1} \|F\|_{L_p(R^n)}). \]
The lemma is proved. \( \square \)

Theorem's hypothesis can be formulated in the following equivalent way, c.f., (5.1).

Let \( \theta \geq 1 \) and let \( T : S_\varepsilon \rightarrow S \) be a measurable mapping satisfying the following condition:
\[ \|T(x) - x\| \leq \theta \text{dist}(x, S), \quad x \in S_\varepsilon. \] 
(5.8)

Suppose that there exists is a constant \( \lambda > 0 \) such that:
(a).
\[ \|f \circ T\|_{L_p(S_\varepsilon)} \leq \lambda^{1\over p}. \] 
(5.9)
(b). For every packing \( \pi \) of cubes lying in \( S_\varepsilon \), we have
\[ \sum_{Q \in \pi} r_Q^{n-p} E(f; (\eta Q) \cap S)^p \leq \lambda, \] 
(5.10)
where \( \eta := 10^{\theta + 1} \).

We have to prove that there exists a continuous function \( F \in W^1_p(R^n) \) such that \( F|_S = f \) and \( \|F\|_{W^1_p(R^n)} \leq C\lambda^{1\over p} \) where the constant \( C \) depends only on \( n, p, \varepsilon \) and \( \theta \).

We define the function \( F := \text{Ext}_{\delta,\varepsilon,S} f \) by the formula (4.2) where the parameter \( \delta := 0 \) and the parameter \( \bar{c} := 0.001\varepsilon \).

First we prove that
\[ \|F\|_{L_p(R^n)} \leq C\lambda^{1\over p}. \] 
(5.11)

**Lemma 5.4** Under conditions (5.10) and (5.9) the following inequality
\[ \sum_{Q \in W^\sharp} |f(a_Q)|^p|Q| \leq C\lambda. \]
holds. (Recall that \( W^\sharp = \mathbb{W}(S : 180\delta) \), see (4.21).)

**Proof.** We put \( \pi = W^\sharp \) in Lemma 4.18. Then, by this lemma, there exists a packing \( W' \subset W^\sharp \) of cubes lying in \( S_\varepsilon \) such that
\[ I := \sum_{Q \in W^\sharp} |f(a_Q)|^p|Q| \leq C\{\|f \circ T\|_{L_p(S_\varepsilon)}^p + \sum_{Q \in W'} |Q| \sup_Q |f \circ T - f(a_Q)|^p\}. \]

42
Prove that
\[ \sup_{x \in Q} |f(T(x)) - f(a_Q)| \leq \mathcal{E}(f; (\eta Q) \cap S). \quad (5.12) \]

In fact, by (3.10), \( T(x) \in \eta Q \). Also, it can be readily seen that \( a_Q \) (a point nearest to \( Q \) on \( S \)) lies in the cube \( 9Q \subset \eta Q \). Thus \( a_Q, T(x) \in \eta Q \) so that
\[ |f(T(x)) - f(a_Q)| \leq \mathcal{E}(f; (\eta Q) \cap S), \]
proving (5.12).

Since \( Q \in W^\sharp \), \( r_Q \leq 90 \delta \) so that
\[
I' := \sum_{Q \in W'} |Q| \mathcal{E}(f; (\eta Q) \cap S)^p \leq (90 \delta)^p \sum_{Q \in W'} r_Q^{n-p} \mathcal{E}(f; (\eta Q) \cap S)^p. \quad (5.13)
\]

Since \( W' \) is a packing of cubes lying in \( S_\varepsilon \), by (5.10), \( I' \leq C \lambda \).

Combining this inequality with (5.9), we finally obtain
\[
I \leq C \{\|f \circ T\|_{L^p(S_\varepsilon)}^p + I'\} \leq C \lambda. \quad \square
\]

We finish the proof of inequality (5.11) as follows. By Lemma 5.4 and Lemma 4.3 we have \( \|F\|_{L^p(\mathbb{R}^n \setminus S)}^p \leq C \lambda \). Since \( T(x) = x \) on \( S \), by inequality (5.9), \( \|F\|_{L^p(S)}^p = \|f\|_{L^p(S)}^p \leq \lambda \). Hence,
\[
\|F\|_{L^p(\mathbb{R}^n)}^p = \|F\|_{L^p(S)}^p + \|F\|_{L^p(\mathbb{R}^n \setminus S)}^p \leq \lambda + C \lambda, \]
proving (5.11).

Let us prove that \( \|F\|^p_{L^p_{\mathbb{H}}(\mathbb{R}^n)} \leq C \lambda \frac{1}{2} \). We will follow the same scheme as in the proof of Theorem 1.1.

We put \( \varsigma := \delta/90 \). By Lemma 5.3 it suffices to show that for every packing \( \pi = \{K\} \) of equal cubes of diameter
\[
\text{diam } K = t \leq \varsigma \quad (5.14)
\]
the following inequality
\[
I(F; \pi) := \left\{ \sum_{K \in \pi} r_K^{n-p} \mathcal{E}(F; K)^p \right\}^{\frac{1}{p}} \leq C \lambda \frac{1}{2} \quad (5.15)
\]
holds. Similarly to the proof of inequality (5.3), we prove (5.15) in 5 steps.

The first step: \( \pi \subset K_1 \). Recall that by (4.7), \( \text{diam } K \leq \delta/82 \) for every cube \( K \in \pi \). By Lemma (4.8), there exist a packing \( \tilde{\pi} = \{Q\} \) of equal cubes centered in \( S \) of diameter \( \gamma t \) such that
\[
I(F; \pi) \leq C \sum_{Q \in \tilde{\pi}} r_Q^{n-p} \mathcal{E}(f; Q \cap S)^p.
\]
Here $\gamma = 574$. Since $t = \text{diam } K \leq \delta/82$, $K \in \pi$, and $\text{diam } Q = \gamma t, Q \in \tilde{\pi}$, we have

$$r_Q = \text{diam } Q/2 = \gamma t/2 \leq \gamma \delta/164 = 574\delta/164 < 4\delta.$$ 

Since $\delta = 0.001\varepsilon$, we have $r_Q < \varepsilon$. In addition, every cube $Q \in \tilde{\pi}$ is centered in $S$ so that $Q \subset S_\varepsilon$.

Hence, by assumption (5.10), $I(F; \pi)^p \leq C\lambda$ proving (5.15).

There is no necessity to consider the case of packings $\pi \subset K$. In fact, in this case $\text{diam } K > \delta/82$ for every $K \in K_2$, see (4.8), so that

$$\text{diam } K > \delta/82 > \delta/90 = \zeta$$

which contradicts (5.14).

Therefore we turn to

The third step: $\pi \subset K_3$. This case can be treated in the same way as in the proof of Theorem 1.1.

By Lemma 4.11 there exists a packing $\pi^\wedge \subset \mathcal{W}(S)$ of Whitney’s cubes of diameter at most $2\delta$ satisfying inequality (4.15). Since $\eta := 10\theta + 1 \geq 11$, we obtain

$$I(F; \pi)^p \leq C \sum_{Q \in \pi^\wedge} r_Q^{n-p}\mathcal{E}(f; (\eta Q) \cap S)^p. \tag{5.16}$$

By Lemma 4.13, every cube $Q \in \pi^\wedge$ lies in $5\text{diam } Q$-neighborhood of $S$ so that $Q \subset S_{10\delta}$. Since $\delta = 0.001\varepsilon$, the cube $Q \subset S_\varepsilon$.

Combining (5.16) with (5.10), we finally obtain the required inequality $I(F; \pi)^p \leq C\lambda$.

The forth step: $\pi \subset K_4$. In this case the inequality $I(F; \pi)^p \leq C\lambda$ follows from Lemma 4.16 and Lemma 5.4.

The remaining case $\pi \subset K_5$ is trivial because in this case $I(\tilde{f}; \pi) = 0$. Thus (5.15) holds for every packing $\pi$ of equal cubes proving the theorem.  

**Proof of Theorem 1.4.** *(Necessity.)*

(i). Let $F \in L^1_p(\mathbb{R}^n)$ be a continuous function such that $F|_S = f$ and

$$\|F\|_{L^1_p(\mathbb{R}^n)} \leq 2\|f\|_{L^1_p(\mathbb{R}^n)|_S}. \tag{5.17}$$

Fix $q \in (n, p)$. Then by Sobolev-Poincaré inequality (3.5), for every cube $Q = Q(x, r)$ and every $y, z \in Q \cap S$ we have

$$|f(y) - f(z)|/r = |F(y) - F(z)|/r \leq C \left(\frac{1}{|Q|} \int_Q \|\nabla F\|^q(u) \, du \right)^{\frac{1}{q}} \leq CM(\|\nabla F\|^q)^{\frac{1}{q}}(x).$$

Hence,

$$f_{\infty, S}(x) := \sup\{|f(y) - f(z)|/r : \ r > 0, y, z \in Q \cap S\} \leq CM(\|\nabla F\|^q)^{\frac{1}{q}}(x), \ \ x \in \mathbb{R}^n,$$
so that, by the Hardy-Littlewood maximal theorem,

$$\|f_{\infty,S}^z\|_{L_p(\mathbb{R}^n)} \leq C\|M(\|\nabla F\|^q)^{\frac{1}{q'}}\|_{L_p(\mathbb{R}^n)} \leq C\|\nabla F\|_{L_p(\mathbb{R}^n)},$$

proving that $$\|f_{\infty,S}^z\|_{L_p(\mathbb{R}^n)} \leq C\|F\|_{L_p^1(\mathbb{R}^n)}.$$

Combining this inequality with (5.17), we obtain

$$\|f_{\infty,S}^z\|_{L_p(\mathbb{R}^n)} \leq C\|f\|_{L_p^1(\mathbb{R}^n)}.$$  \hspace{1cm} (5.18)

(ii). The required inequality

$$\|f \circ T\|_{L_p(S_z)} + \|f_{\infty,S}^z\|_{L_p(S_z)} \leq C\|f\|_{W_p^1(\mathbb{R}^n)}$$

immediately follows from equivalence (1.6) and inequality (5.18).

(Sufficiency).

(i). Given $$\gamma \geq 1$$ consider a packing $$\{Q_i : i = 1, \ldots, m\}$$ and points $$x_i, y_i \in (\gamma Q_i) \cap S.$$ Let $$z \in Q_i$$ and let $$Q_z := Q(z, (\gamma + 1)r_{Q_i}).$$ Then, clearly, $$Q_z \supseteq Q_i$$ so that $$x_i, y_i \in Q_z \cap S.$$ Since $$r_{Q_z} \sim \text{diam} Q_i,$$ we have

$$\frac{|f(x_i) - f(y_i)|^p}{(\text{diam} Q_i)^{p-n}} \leq C |Q_i| (|f(x_i) - f(y_i)|/r_{Q_z})^p \leq C |Q_i| (f_{\infty,S}^z(z))^p, \quad z \in Q_i.$$

Integrating this inequality (with respect to $$z$$) on cube $$Q_i$$ we obtain

$$\frac{|f(x_i) - f(y_i)|^p}{(\text{diam} Q_i)^{p-n}} \leq C \int_{Q_i} (f_{\infty,S}^z)^p(z) \, dz$$

so that

$$I := \sum_{i=1}^m \frac{|f(x_i) - f(y_i)|^p}{(\text{diam} Q_i)^{p-n}} \leq C \int_{U_x} (f_{\infty,S}^z)^p(z) \, dz \hspace{1cm} (5.19)$$

where $$U_x := \bigcup_{i=1}^m Q_i.$$ Hence $$I \leq C\|f_{\infty,S}^z\|_{L_p(\mathbb{R}^n)}^p.$$ 

We put $$\gamma = 11$$ and obtain inequality (1.3) with $$\lambda := C(n, p)\|f_{\infty,S}^z\|_{L_p(\mathbb{R}^n)}^p.$$ Then, by (1.4),

$$\|f\|_{L_p^1(\mathbb{R}^n)} \leq C\lambda^\frac{1}{p} \leq C\|f_{\infty,S}^z\|_{L_p(\mathbb{R}^n)}$$

proving the sufficiency part of the theorem for the case (i).

(ii). By inequality (5.19) with $$\gamma = \eta,$$ for every packing $$\{Q_i : i = 1, \ldots, m\}$$ lying in $$S_z$$ and every two points $$x_i, y_i \in (\eta Q_i) \cap S$$ we have

$$I := \sum_{i=1}^m \frac{|f(x_i) - f(y_i)|^p}{(\text{diam} Q_i)^{p-n}} \leq C \int_{U_x} (f_{\infty,S}^z)^p(z) \, dz \leq C\|f_{\infty,S}^z\|_{L_p(S_z)}^p =: \lambda.$$

Hence, by Theorem 1.2,

$$\|f\|_{W_p^1(\mathbb{R}^n)} \leq C(\|f \circ T\|_{L_p(S_z)} + \lambda^\frac{1}{p}) \leq C(\|f \circ T\|_{L_p(S_z)} + \|f_{\infty,S}^z\|_{L_p(S_z)}).$$
Theorem 1.4 is completely proved. □

**Proof of Theorem 2.5.** The *necessity* part of the theorem follows from inequality (2.12), Proposition 3.11, inequality (3.8) and Proposition 3.13.

Prove the *sufficiency*. Let \( \theta \geq 1 \) and let \( T : S_\varepsilon \to S \) be a measurable mapping such that

\[
T(x) \in \partial S, \quad x \in S_\varepsilon \setminus S, \tag{5.20}
\]

and inequality (5.8) is satisfied.

We put

\[
\bar{c} := 0 \quad \text{and} \quad \delta := \frac{\varepsilon}{7200(1 + \theta)}, \tag{5.21}
\]

and define the function

\[
F := \text{Ext}_{\delta, \bar{c}, S} f \tag{5.22}
\]

by the formula (4.2).

In the remaining part of the section \( C \) will denote a positive constant depending only on \( n, p, \varepsilon, \theta \) and \( \alpha \).

**Lemma 5.5** Let \( t > 0 \) and let \( \pi \subset W^2 \) be a collection of Whitney’s cubes of diameter at least \( t \). Then for every continuous function \( f \) on \( S \) and every \( 0 < \alpha < 0.3/(1 + \theta) \) the following inequality holds.

\[
\sum_{Q \in \pi} |f(a_Q)|^p |Q| \leq C \left\{ \| f \circ T \|_{L_p(S_\varepsilon)}^p + \int_{t}^{\varepsilon/2} A_{p, \alpha}(f; t : S)^p \frac{dt}{t} \right\}
\]

**Proof.** By Lemma 4.18 each cube \( Q \in \pi \) lies in \( S_\varepsilon \) and

\[
I := \sum_{Q \in \pi} |f(a_Q)|^p |Q| \leq C \left\{ \| f \circ T \|_{L_p(S_\varepsilon)}^p + \sum_{Q \in \pi} |Q| \sup_{Q} |f \circ T - f(a_Q)|^p \right\}. \tag{5.23}
\]

Given \( Q \in \pi \), we put \( \tilde{Q} := Q(a_Q, 10(1 + \theta)r_Q) \). The reader can readily see that inequality (5.8) and property (3.2) of Whitney’s cubes imply the following:

\[
T(x) \in \tilde{Q} \quad \text{for every} \quad x \in Q.
\]

Since \( a_Q \in \partial S \) and \( T(x) \in \partial S, x \in Q \), see (5.20), we have

\[
|f(a_Q) - f(T(x))| \leq \mathcal{E}(f; \tilde{Q} \cap \partial S), \quad x \in Q,
\]

so that

\[
I' := \sum_{Q \in \pi} |Q| \sup_{Q} |f \circ T - f(a_Q)|^p \leq \sum_{Q \in \pi} \tilde{Q} \mathcal{E}(f; \tilde{Q} \cap \partial S)^p.
\]
Let $0 < \beta < 3/(10 + 10\theta)$. Prove that $\tilde{Q}$ has porosity $\beta$. We put $\varsigma := 10(1 + \theta)\beta$ so that $0 < \varsigma < 3$. Therefore by Lemma 4.13

$$Q_\varsigma := \varsigma Q \subset Q(a_Q, 10 \text{ diam } Q) \setminus S.$$ 

Since $\theta \geq 1$,

$$Q(a_Q, 10 \text{ diam } Q) \subset \tilde{Q} := Q(a_Q, 5(1 + \theta) \text{ diam } Q),$$

so that $Q_\varsigma \subset \tilde{Q} \setminus S$. But

$$\beta \text{ diam } \tilde{Q} = 10\beta(1 + \theta) \text{ diam } Q = \varsigma \text{ diam } Q = \text{ diam } Q_\varsigma,$$

proving that $\tilde{Q}$ is a $\beta$-porous cube.

Now put

$$\beta := \min\{2\alpha, (\alpha + 3/(10 + 10\theta))/2\}. \quad (5.24)$$

Then $0 < \alpha < \beta < 3/(10 + 10\theta)$. In addition, the collection $\{Q : Q \in \pi\}$ is a family of $\beta$-porous cubes centered in $S$. We apply Lemma 4.12 with $s = 0$, $T := 90\delta$, $\varsigma := 10(1 + \theta)$ and parameter $\beta$ defined by (5.24), and obtain the following inequality:

$$I' \leq \sum_{Q \in \pi} r^a_Q \mathcal{E}(f; \tilde{Q} \cap \partial S)^p \leq C \int_0^\gamma T A_{p,\alpha}(f; u : S)^p \frac{du}{u}. \quad (5.25)$$

Here $\gamma = \beta^2 \varsigma/\alpha^2$. Since $\beta/\alpha \leq 2$, we have

$$\gamma T \leq 4 \cdot 10(1 + \theta) \cdot 90\delta = 3600(1 + \theta),$$

so that by (5.21) $\gamma T \leq \epsilon/2$. Combining this inequality, (5.23) and (5.25), we obtain the statement of the lemma. \hfill \Box

**Proposition 5.6** Let $f \in C(S)$ and let $\alpha \in (0, 0.3/(1 + \theta))$. Then

$$\|F\|_{L^p(\mathbb{R}^n)}^p \leq C \left\{ \|f \circ T\|_{L^p(S_\varsigma)}^p + \int_0^{\epsilon/2} A_{p,\alpha}(f; t : S)^p \frac{dt}{t} \right\}. \quad (5.26)$$

**Proof.** By Lemma 4.3 and Lemma 5.5,

$$\|F\|_{L^p(\mathbb{R}^n \setminus S)}^p \leq \sum_{Q \in \mathcal{W}(S), r_Q \leq \delta} |f(a_Q)|^p|Q| \leq \sum_{Q \in \mathcal{W}^2} |f(a_Q)|^p|Q| \leq C \left\{ \|f \circ T\|_{L^p(S_\varsigma)}^p + \int_0^{\epsilon/2} A_{p,\alpha}(f; t : S)^p \frac{dt}{t} \right\}. \quad (5.27)$$

Since $F|_S = f$ and $T(x) = x$ on $S$, we have

$$\|F\|_{L^p(S)} = \|f\|_{L^p(S)} = \|f \circ T\|_{L^p(S)} \leq \|f \circ T\|_{L^p(S_\varsigma)}^p.$$
Finally, these estimates and equality $\|F\|^p_{L_p(\mathbb{R}^n)} = \|F\|^p_{L_p(S)} + \|F\|^p_{L_p(\mathbb{R}^n \setminus S)}$ imply the statement of the proposition. \(\square\)

The next proposition provides an estimate of the $A_p$-functional of the extension $F$ of $f$ via corresponding $A_p$- and $A_{p,\alpha}$-functionals of $f$. To formulate the result we put $\varepsilon_0 := \delta/100 = \varepsilon/(7200 \cdot 100(1 + \theta))$, see (5.21).

**Proposition 5.7** Let $n < p < \infty$ and let $0 < \alpha < 0.3/(1 + \theta)$. Then for every $f \in C(S)$ and every $0 < t \leq \varepsilon_0$ we have

$$A_p(F; t) \leq C \left\{ A_p(f; \gamma t : S) + t \left( \int_0^t A_{p,\alpha}(f; u : S)^p \frac{du}{u^{p+1}} \right)^{\frac{1}{p}} + t\|f \circ T\|_{L_p(S)} \right\}. $$

Here $\gamma$ is an absolute constant ($\gamma \leq 574$).

**Proof.** Let $\pi$ be a packing of equal cubes of diameter $s > 0$. Put

$$J(F; \pi) := \left\{ \sum_{Q \in \pi} |Q| \mathcal{E}(F; Q)^p \right\}^{\frac{1}{p}}$$

and

$$H(f; t) := A_p(f; \gamma t : S) + t \left( \int_0^t A_{p,\alpha}(f; u : S)^p \frac{du}{u^{p+1}} \right)^{\frac{1}{p}} + t\|f \circ T\|_{L_p(S)}$$

(with $\gamma = 574$). By (2.3), we have to proof that

$$J(F; \pi) \leq C H(f; t), \quad 0 < s \leq t \leq \varepsilon_0. \quad (5.26)$$

Recall that

$$I(F; \pi) := \left\{ \sum_{Q \in \pi} \gamma_Q^{n-p} \mathcal{E}(F; Q)^p \right\}^{\frac{1}{p}}$$

so that $J(F; \pi) = sI(F; \pi)/2$.

We prove (5.26) in 5 steps.

*The first step: $\pi \subset \mathcal{K}_1$. By Lemma 4.9 in this case*

$$J(F; \pi) = sI(F; \pi)/2 \leq Cs(Cs^{-1}A_p(f; \gamma s : S) \leq CA_p(f; \gamma s : S)$$

with $\gamma \leq 574$, so that (5.26) is satisfied.

*The second step: $\pi \subset \mathcal{K}_2$. Recall that $\text{diam}Q > \delta/82$ for every $Q \in \mathcal{K}_2$. But $\text{diam}Q = s \leq \delta/100$ for each $Q \in \pi$ so that this case can be omitted.*
The third step: $\pi \subset K_3$. Since $\theta \geq 1$, the parameter $\alpha \in (0, 3/20)$ so that by Lemma 4.14 and inequality $\gamma \delta \leq \varepsilon$, see (5.21),

\[
J(F; \pi)^p = (sI(F; \pi)/2)^p \leq CS^p \int_{s}^{\gamma \delta} A_{p, \alpha}(f; u : S)^p \frac{du}{u^{p+1}} \\
\leq CS^p \int_{s}^{\delta} A_{p, \alpha}(f; u : S)^p \frac{du}{u^{p+1}} \\
\leq C \left( s^p \int_{s}^{t} A_{p, \alpha}(f; u : S)^p \frac{du}{u^{p+1}} + t^p \int_{t}^{\varepsilon} A_{p, \alpha}(f; u : S)^p \frac{du}{u^{p+1}} \right) = C(I_1 + I_2).
\]

Since $A_{p, \alpha}(f; \cdot : S)$ is a non-decreasing function and $A_{p, \alpha}(f; \cdot : S) \leq A_p(f; \cdot : S)$, see (2.14), the item $I_1$ is bounded by $(1/p)A_p(f; t : S)^p$ proving (5.26) in the case under consideration.

The forth step: $\pi \subset K_4$. By Lemma 4.16 there exists a packing of Whitney’s cubes $\pi' = \{Q\}$ with $0.01\delta \leq \text{diam } Q \leq 90\delta$ such that

\[
I(F; \pi)^p \leq C \sum_{Q \in \pi'} |f(a_Q)|^p |Q|.
\]

Since $\pi'$ is a subfamily of $W' := \{Q \in \mathbb{W}(S) : \text{diam } Q \leq 90\delta\}$ of cubes of diameter at least $\delta/100 = \varepsilon_0$, and $0 < t \leq \varepsilon_0$, by Lemma 5.5

\[
I(F; \pi)^p \leq C \left( \|f \circ T\|_{L_p(S_0)}^p + \int_{t}^{\varepsilon} A_{p, \alpha}(f; u : S)^p \frac{du}{u^{p+1}} \right).
\]

Clearly, $J(F; \pi) = 0$ whenever $\pi \subset K_5$, see Lemma 4.17, and the proof is finished.

Let us finish the proof of the sufficiency part of Theorem 2.5. We let $\lambda$ denote the right-hand side of the theorem’s equivalence.

Then by Proposition 5.6 $\|F\|_{L_p(R^n)} \leq C\lambda$. On the other hand, by Proposition 5.7 $A_p(F; t) \leq C\lambda t$ for every $0 < t \leq \varepsilon_0$. Therefore, by (2.3), for every packing of equal cubes of diameter at most $\varepsilon_0$ the inequality (5.7) of Lemma 5.3 holds (with constant $C\lambda^p$ instead of $\lambda$). Then by this lemma $F \in L^1_p(R^n)$ and

\[
\|F\|_{L^1_p(R^n)} \leq C(\lambda + \varepsilon_0^{-1}\|F\|_{L_p(R^n)}) \leq C(\lambda + C\lambda) \leq C\lambda.
\]

Theorem 2.5 is completely proved.

This theorem enables us to prove the following

**Proposition 5.8** Let $p > n$ and let $\Omega := \text{int}(S)$. Then for every function $f \in C(S)$ the following equivalence

\[
\|f\|_{W^1_p(R^n)_{\partial \Omega}} \sim \|f|_\Omega\|_{W^1_p(\Omega)} + \|f|_{\partial \Omega S}\|_{W^1_p(R^n)_{\partial \Omega S}}
\]

holds with constants depending only on $p$ and $n$. 49
Proof. We put
\[ \lambda := \|f|_w^p(\Omega) + \|f|_{\partial S}^p|_w^p(\mathbb{R}^n) \] (5.27)

Observe that inequality \( \lambda \leq 2\|f|_w^p|_w^p(\mathbb{R}^n) \) is trivial. Prove the converse inequality.

Fix \( \varepsilon > 0 \) and a mapping \( T : S_\varepsilon \rightarrow S \) satisfying conditions (1.5) and (2.15). By Theorem 2.5, it suffices to show that for some \( \alpha \in (0, 3/(10 + 10\theta)) \) the following inequalities
\[ \|f \circ T\|_{L_w^p(S_\varepsilon)} \leq C\lambda, \] (5.28)
\[ \sup_{0 < t \leq \varepsilon} \frac{A_p(f; t : S)}{t} \leq C\lambda \] (5.29)
and
\[ \left\{ \int_{0}^{\varepsilon} A_{p, \alpha}(f; t : S)^p \frac{dt}{t^{p+1}} \right\}^{\frac{1}{p}} \leq C\lambda \] (5.30)
hold with a constant \( C \) depending only on \( n \) and \( p \). (Recall that \( \theta \) is a constant from inequality (1.5).)

Let \( T_S : \mathbb{R}^n \rightarrow S \) is a mapping defined by the formula (1.7). Also, let us consider the corresponding mapping \( T_{\partial S} : \mathbb{R}^n \rightarrow S \) defined by the same formula (1.7) but for \( \partial S \) rather than \( S \). Clearly, \( T_{\partial S}|_{\mathbb{R}^n \setminus \Omega} = T_S \).

Put
\[ I := \|f \circ T_{\partial S}\|_{L_w^p((\partial S)_\varepsilon)} + \sup_{0 < t \leq 4 \varepsilon} \frac{A_p(f; t : \partial S)}{t} + \left\{ \int_{0}^{4 \varepsilon} A_{p, \alpha}(f; t : \partial S)^p \frac{dt}{t^{p+1}} \right\}^{\frac{1}{p}} \] (5.33)
and apply Theorem 2.5 to the set \( \partial S \). By this theorem,
\[ I \leq C\|f|_{\partial S}^p|_w^p(\mathbb{R}^n) \leq C\lambda \] (5.31)
provided
\[ \alpha \in (0, 3/(10 + 10\theta)) \] (5.32)
where \( \theta = \theta(n) \) is a constant from inequality (1.5) corresponding to the mapping \( T_{\partial S} \).

Prove that inequalities (5.28)-(5.30) hold for \( T = T_S \) and every \( \alpha \) satisfying (5.32).

We begin with inequality (5.28). Since \( T_{\partial S}|_{\mathbb{R}^n \setminus \Omega} = T_S \), we have
\[ \|f \circ T_S\|_{L_w^p(S_\varepsilon)}^p = \|f\|_{L_w^p(\Omega)}^p + \|f \circ T_S\|_{L_w^p((\partial S)_\varepsilon \setminus \Omega)}^p = \|f\|_{L_w^p(\Omega)}^p + \|f \circ T_{\partial S}\|_{L_w^p((\partial S)_\varepsilon \setminus \Omega)}^p, \]
so that
\[ \|f \circ T_S\|_{L_w^p(S_\varepsilon)}^p \leq \|f|_w^p|_w^p(\Omega) + \|f \circ T_{\partial S}\|_{L_w^p((\partial S)_\varepsilon \setminus \Omega)}^p. \] (5.33)
Hence, by (5.31) and (5.27), we have \( \| f \circ T_S \|_{L^p(S, \varepsilon)} \leq \lambda^p + (C\lambda)^p \) proving (5.28).

Prove (5.29). By Proposition 3.6,

\[
A_p(f; t : S) \leq C \{ A_p(f; 4t : \partial S) + t \| f \|_{L^p(\partial S)} \}, \quad t > 0
\]

so that, by (5.31) and (5.27), \( A_p(f; t : S) \leq C \lambda t, t \in (0, \varepsilon) \), proving (5.29).

The remaining inequality (5.30) is trivial. In fact, by Definition 2.3,

\[
A_{p, \alpha}(f; t : S) \leq A_{p, \alpha}(f; t : \partial S), \quad t > 0,
\]

so that, by (5.31),

\[
\int_0^\varepsilon \frac{A_{p, \alpha}(f; t : S)^p dt}{t^{p+1}} \leq \int_0^\varepsilon \frac{A_{p, \alpha}(f; t : \partial S)^p dt}{t^{p+1}} \leq C \lambda^p.
\]

The proposition is completely proved. □

**Remark 5.9** Theorem 2.5 and estimates (5.33) and (5.34) imply the following equivalence

\[
\| f \|_{W^1_p(\mathbb{R}^n)} \sim \| f \|_{W^3_p(\mathbb{R}^n)} + \| f \circ T_S \|_{L^p(\partial S, \varepsilon)} + \sup_{0 < t \leq \varepsilon} \frac{A_p(f; t : \partial S)}{t}
\]

\[
+ \left\{ \int_0^\varepsilon A_{p, \alpha}(f; t : S)^p \frac{dt}{t^{p+1}} \right\}^{\frac{1}{p}}.
\]

Here \( f, p, \alpha, \varepsilon \) and constants of equivalence are the same as in Theorem 2.5.

We finish the section with an important consequence of Proposition 5.7.

Recall that by \( \omega(F; t)_{L^p} \) we denote the modulus of continuity of a function \( F \) in \( L^p \), see (1.18).

**Theorem 5.10** Let \( \varepsilon > 0, \theta \geq 1 \), and let \( T : S_\varepsilon \to S \) be a measurable mapping satisfying inequality (5.8). There exists a linear continuous extension operator \( E : C(S) \to C(\mathbb{R}^n) \) such that for every \( f \in C(S) \), \( n < p < \infty \), and every \( t \in (0, \varepsilon_0) \) we have:

\[
\omega(Ef; t)_{L^p} \leq C t \left\{ \left( \int_t^\varepsilon A_p(f; u : S)^p \frac{du}{u^{p+1}} \right)^{\frac{1}{p}} + \| f \circ T \|_{L^p(S, \varepsilon)} \right\}.
\]

In addition, if the mapping \( T \) satisfies condition (5.20), then for every \( t \in (0, \varepsilon_0) \) and every \( \alpha \in (0, 0.3/(1 + \theta)) \) the following inequality

\[
\omega(Ef; t)_{L^p} \leq C \left\{ A_p(f; \gamma t : S) + t \left( \int_t^\varepsilon A_{p, \alpha}(f; u : S) \frac{du}{u^{p+1}} \right)^{\frac{1}{p}} + t \| f \circ T \|_{L^p(S, \varepsilon)} \right\}
\]

holds. Here \( \gamma \) is an absolute constant, and \( \varepsilon_0 \) is a positive constant depending only on \( \varepsilon \) and \( \theta \).
Proof. We let $E := \text{Ext}_{\delta, \varepsilon, S}$ denote the same extension operator as in the proof of Theorem 2.5, i.e., we put $\bar{c} := 0$ and define $\delta$ by formula (5.21).

Clearly, by (2.6), $\omega_{1, p}(Ef; t)_{L_p} \leq \omega_{1, p}(Ef; t)_{L_\infty}$, $t > 0$, so that by (2.8) and (2.10) $\omega(Ef; t)_{L_p} \leq CA_p(Ef; t)$, $t > 0$. This inequality and Proposition 5.7 imply the second inequality of the theorem.

Prove the first inequality. Observe that $A_{p, \alpha}(f; \cdot : S) \leq A_p(f; \cdot : S)$, see (2.14), so that in the second inequality the quantity $A_{p, \alpha}(f; \cdot : S)$ can be replaced with $A_p(f; \cdot : S)$. Moreover, a trivial modification of the proof (see, in particular, Lemma 5.5) allows us to omit condition (5.20) in this inequality. Moreover, we can assume that $2\gamma \varepsilon_0 \leq \varepsilon$ so that $2\gamma t \leq \varepsilon$ for every $t \in (0, \varepsilon_0)$. Since $A_p(f; \cdot : S)$ is non-decreasing, we have

$$t^p \frac{\omega(t)}{u^{p+1}} \geq (2\gamma)^{-p}(2^p - 1)A_p(f; \gamma t : S),$$

proving the theorem. \(\square\)

6. Restrictions of Besov $B^s_{p,q}(\mathbb{R}^n)$-spaces, $n/p < s < 1$, to closed sets.

In this section we present an intrinsic characterization of the restrictions of Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, $s \in (n/p, 1)$, to closed subsets of $\mathbb{R}^n$ via local oscillations.

We recall that the space $B^s_{p,q}(\mathbb{R}^n)$, $0 < s < 1$, $1 \leq p \leq \infty$, $0 < q \leq \infty$, is defined by the finiteness of the norm $\|f\|_{B^s_{p,q}(\mathbb{R}^n)} := \|f\|_{L_p(\mathbb{R}^n)} + \left(\int_0^T \omega(f; t)_{L_q}^{q} \frac{dt}{t^{p+1}}\right)^{\frac{1}{q}}$ (with standard modification if $q = \infty$), see, e.g. [BIN, T2, T3].

Here $\Delta_h f(x) = f(x+h) - f(x)$. In particular, for $p = q$,

$$\|f\|_{B^s_{p,p}(\mathbb{R}^n)} = \|f\|_{L_p(\mathbb{R}^n)} + \left(\int_0^T \omega(f; t)_{L_p}^{q} \frac{dt}{t^{p+1}}\right)^{\frac{1}{q}}.$$

There are many different expressions for equivalent norms in the Besov space. We will be interested in two of them. The first equivalence is one of the classical definition of the Besov space norm: Let $p \geq 1$, $q > 0$, $s \in (0, 1)$ and $T > 0$. Then for every $f \in L_p(\mathbb{R}^n)$

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \sim \|f\|_{L_p(\mathbb{R}^n)} + \left(\int_0^T \omega(f; t)_{L_q}^{q} \frac{dt}{t^{p+1}}\right)^{\frac{1}{q}}. \quad (6.1)$$
The second equivalence involves the notion of local oscillation $E(f, Q(x, t))$, see (2.1). This result, see [Br1] and [T2], p. 186, states that for every $f \in L_p(\mathbb{R}^n)$

$$
\|f\|_{B^p,q(\mathbb{R}^n)} \sim \|f\|_{L_p(\mathbb{R}^n)} + \left( \int_0^T \|E(f; Q(\cdot; t))\|^q_{L_p(\mathbb{R}^n)} \frac{dt}{t^{sp+1}} \right)^{\frac{1}{q}}
$$

provided $p \geq 1, q > 0, s \in (0, 1)$ and $sp > n$.

This equivalence and (2.9) imply the following description of Besov spaces via $A_p$-functionals: Let $p \geq 1, q > 0, s \in (0, 1)$ and $sp > n$. Then for every $f \in L_p(\mathbb{R}^n)$ we have

$$
\|f\|_{B^p,q(\mathbb{R}^n)} \sim \|f\|_{L_p(\mathbb{R}^n)} + \left( \int_0^T A_p(f; t)^q \frac{dt}{t^{sp+1}} \right)^{\frac{1}{q}}.
$$

(6.2)

In all cases the constants of equivalence depend only on $n, s, p, q$ and $T$.

We turn to

**Proof of Theorem 2.6.**

We are needed two technical lemmas. First of them is a particular case of the well known Hardy inequality. Let $0 < \varsigma, \eta \leq 1, 0 < p, q \leq \infty, a > 0$. Given $g \in L_{1,loc}(\mathbb{R}^n)$ put

$$
U_\varsigma g(t) := t^\varsigma \left( \int_0^t \left( \frac{|g(u)|}{u^{\varsigma}} \right)^p \frac{du}{u} \right)^{\frac{1}{p}}, \quad V_\varsigma g(t) := t^\varsigma \left( \int_{t}^a \left( \frac{|g(u)|}{u^{\varsigma}} \right)^p \frac{du}{u} \right)^{\frac{1}{p}},
$$

and

$$
\|g\|_{\eta,q} := \left( \int_0^a \left( \frac{|g(t)|}{t^{\eta}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}
$$

(with the corresponding modification for $p$ or $q = \infty$).

**Lemma 6.1** For every $g \in L_{1,loc}(\mathbb{R}^n)$ the following two inequalities

$$
\|U_\varsigma g\|_{\eta,q} \leq C\|g\|_{\eta,q}, \quad \text{if } \varsigma < \eta,
$$

(6.3)

and

$$
\|V_\varsigma g\|_{\eta,q} \leq C\|g\|_{\eta,q}, \quad \text{if } \varsigma > \eta,
$$

(6.4)

hold with constant $C = C(\varsigma, \eta, p, q)$.

Let us formulate the second lemma (its proof is elementary and can be left to the reader as an easy exercise).
Lemma 6.2 Let \( g = g(t), t > 0 \), be a non-decreasing function. Then for every \( 0 < p, q \leq \infty \) and every \( 0 < s, a < \infty \) the following inequality
\[
\left( \int_0^a g(t)^p \frac{dt}{t} \right)^{\frac{1}{p}} \leq C \left( \int_0^{2a} \left( \frac{g(u)}{u^s} \right)^q \frac{du}{u} \right)^{\frac{1}{q}}
\]
holds (with the standard modification for \( p = \infty \) or \( q = \infty \)). Here \( C \) is a constant depending only on \( p, q, s, \) and \( a \).

We turn to the proof of Theorem 2.6.
(Necessity.) We put \( \tilde{Q} := 11\theta Q \) and
\[
I := \sum_{Q \in \mathbb{W}(S; \varepsilon)} |Q| \mathcal{E}(F; \tilde{Q})^p.
\]

By \( \pi_m \) we denote the collections of cubes
\[
\pi_m := \{ Q \in \mathbb{W}(S; \varepsilon) : 2^m < r_Q \leq 2^{m+1} \}.
\]
Since \( r_Q = (1/2) \text{diam} Q \leq \varepsilon/2 \) for each \( Q \in \mathbb{W}(S; \varepsilon) \), we may suppose that \( m \leq M \) where \( M \) satisfies the following inequality: \( 2^M \leq \varepsilon/2 \leq 2^{M+1} \). (Otherwise \( \pi = \emptyset \).

Given \( Q \in \pi_m \) we put \( Q^\Lambda := Q(x_Q, 11\theta 2^{m+1}) \). Then \( Q \subset \tilde{Q} \subset Q^\Lambda \). By Lemma 3.4, the collection \( \pi_m^\Lambda := \{ Q^\Lambda : Q \in \pi_m \} \) of equal cubes (of diameter \( 22\theta 2^{m+1} \)) can be partitioned into at most \( N = N(n, \theta) \) packings. Therefore without loss of generality we may assume that \( \pi^\Lambda \) is a packing.

Put \( \tau_m := 11\theta 2^{m+1} \). Then, by (2.3),
\[
\sum_{Q \in \pi_m} |Q| \mathcal{E}(F; \tilde{Q})^p \leq \sum_{Q \in \pi_m} |Q| \mathcal{E}(F; Q^\Lambda)^p \leq C \sum_{K \in \pi_m} |K| \mathcal{E}(F; K)^p \leq CA_p(F; \tau_{m+1})^p.
\]
Hence,
\[
I = \sum_{m=-\infty}^M \sum_{Q \in \pi_m} |Q| \mathcal{E}(F; \tilde{Q})^p \leq C \sum_{m=-\infty}^M A_p(F; \tau_{m+1})^p.
\]
Since \( A_p(\cdot; t) \) is a non-decreasing function, it can be readily seen that
\[
I \leq C \sum_{m=-\infty}^M \int_{\tau_{m+1}}^{\tau_{m+2}} A_p(F; t)^p \frac{dt}{t} = C \int_{\tau_{M+2}}^\infty A_p(F; t)^p \frac{dt}{t}.
\]
Since \( 2^{M+1} \leq \varepsilon \), we have \( \tau_{M+2} = 11\theta 2^{M+2} \leq 22\theta \varepsilon =: \tilde{\varepsilon} \). Hence, by Lemma 6.2, we obtain
\[
I^{\frac{1}{p}} \leq C \left( \int_0^{2\tilde{\varepsilon}} \left( \frac{A_p(F; t)}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.
\]
On the other hand, by Lemma 3.12, \( \|F \circ T\|_{L_p(S_\epsilon)} \leq C \left\{ \|F\|_{L_p(\mathbb{R}^n)} + I_p^\frac{1}{p} \right\} \). Combining this inequality, equivalence (6.2) and obvious inequality

\[
A_p(F|_S; t : S) \leq A_p(F; t), \quad F \in C(\mathbb{R}^n),
\]

we obtain the statement of the necessity part of the theorem.

(Sufficiency.) The proof of the sufficiency part follows the same scheme as the proof of Theorem 2.5. Given a function \( f \in C(S) \) we let \( \lambda \) denote the right-hand side of the equivalence (2.16).

As in Theorem 2.5, we define an extension \( F \) of \( f \) by formulas (5.21) and (5.22). Then we prove that \( F \in B_{p,q}^s(\mathbb{R}^n) \) and \( \|F\|_{B_{p,q}^s(\mathbb{R}^n)} \leq C\lambda \).

First we show that \( \|F\|_{L_p(\mathbb{R}^n)} \leq C\lambda \). Note that \( A_{p,\alpha}(f; \cdot : S) \leq A_p(f; \cdot : S) \), so that by Lemma 5.6

\[
\|F\|_{L_p(\mathbb{R}^n)} \leq C \left\{ \|f \circ T\|_{L_p(S_\epsilon)} + \left( \int_0^\frac{\epsilon}{2} A_{p,\alpha}(f; t : S)^p \frac{dt}{t} \right)^\frac{1}{p} \right\}
\]

\[
\leq C \left\{ \|f \circ T\|_{L_p(S_\epsilon)} + \left( \int_0^\frac{\epsilon}{2} A_p(f; t : S)^p \frac{dt}{t} \right)^\frac{1}{p} \right\}.
\]

This inequality and Lemma 6.2 yield

\[
\|F\|_{L_p(\mathbb{R}^n)} \leq C \left\{ \|f \circ T\|_{L_p(S_\epsilon)} + \left( \int_0^\frac{\epsilon}{2} \left( \frac{A_p(f; t : S)}{t^s} \right)^q \frac{dt}{t} \right)^\frac{1}{q} \right\} \leq C\lambda.
\]

On the other hand, inequality (5.35) and the Hardy inequality (6.4) imply the following estimate

\[
\left( \int_0^\frac{\epsilon}{2} \left( \frac{\omega(F; t)_L}{t^s} \right)^q \frac{dt}{t} \right)^\frac{1}{q} \leq C\|f \circ T\|_{L_p(S_\epsilon)} + C \left( \int_0^\frac{\epsilon}{2} \left( \frac{A_p(f; t : S)}{t^s} \right)^q \frac{dt}{t} \right)^\frac{1}{q} \leq C\lambda.
\]

Finally, by (6.1), \( \|F\|_{B_{p,q}^s(\mathbb{R}^n)} \leq C\lambda \).

Theorem 2.6 is proved. \( \Box \)

In the aforementioned paper [J] A. Jonsson describes the trace space \( B_{p,q}^s(\mathbb{R}^n)|_S \), \( n/p < s < 1 \), via certain \( L_p \)-oscillations of functions with respect to a doubling measure supported on \( S \). Let us recall this result.

Let \( \mu \) be a doubling measure with support \( S \) satisfying the condition \((D_n)\) (1.11) and the condition (1.13). Observe that for every cube \( Q = Q(x, r) \) such that \( x \in S \) and \( r \leq 1 \), by the \((D_n)\)-condition,

\[
\mu(Q(x, 1)) \leq C_\mu (1/r)^n \mu(Q(x, r)) = 2^n C_\mu \mu(Q(x, r))/|Q(x, r)|.
\]
By (1.13), $C'_\mu \leq \mu(Q(x,1))$ so that
\[ |Q(x,r)| \leq C\mu(Q(x,r)), \text{ for every } x \in S, \ r \leq 1, \] (6.5)
with $C = 2^nC'_\mu/C''_\mu$.

**Theorem 6.3** ([J]) Let $n/p < s < 1$, $1 \leq p, q \leq \infty$, and let $\mu$ be a measure with support $S$ satisfying conditions $(D_n)$ and (1.13). Then for every $f \in C(S)$
\[
\|f\|_{B_{p,q}^{s}(\mathbb{R}^n)|_S} \sim \|f\|_{L_p(\mu)} + \left\{ \sum_{\nu=0}^{\infty} \left( 2^{\nu(s-n/p)} \left( \iint_{||x-y||<2^{-\nu}} \frac{|f(x)-f(y)|^p}{\mu(Q(x,2^{-\nu}))\mu(Q(y,2^{-\nu}))} \, d\mu(x) \, d\mu(y) \right) \right)^{\frac{1}{p}} \right\}^{\frac{q}{p}}
\]
(with the standard modification of the right-hand side whenever $p = \infty$ or $q = \infty$). The constants of this equivalence depend only on $n, p, q, s$, and parameters $C_\mu, C'_\mu, C''_\mu$.

It is shown in [J] that for $p = q$
\[
\|f\|_{B_{p,q}^{s}(\mathbb{R}^n)|_S} \sim \|f\|_{L_p(\mu)} + \left( \iint_{||x-y||<1} \frac{|f(x)-f(y)|^p}{\mu(Q(x,|x-y|))} \, d\mu(x) \, d\mu(y) \right)^{\frac{1}{p}}.
\]

Let us show that the second item on the right-hand side of these equivalences may be expressed in terms of certain $L_p(\mu)$-version of the functional $A_p$, see Definition 6.4 below.

Let $\mu$ be a doubling measure on $\mathbb{R}^n$ with support supp $\mu = S$ and let $1 \leq q \leq \infty$. Given a function $f \in L_{q,loc}(\mu)$ and a cube $Q \subset \mathbb{R}^n$, we define $L_q$-oscillation of $f$ on $Q$ (with respect to $\mu$) by letting
\[
\mathcal{E}(f; Q)_{L_q(\mu)} := \left( \frac{1}{\mu(Q)} \iint_{Q \times Q} |f(x) - f(y)|^q \, d\mu(x) \, d\mu(y) \right)^{\frac{1}{q}},
\] (6.6)
whenever $1 \leq q < \infty$, and
\[
\mathcal{E}(f; Q)_{L_\infty(\mu)} := \mathcal{E}(f; Q \cap S) = \sup_{x,y \in Q \cap S} |f(x) - f(y)|.
\]

It can be readily seen that
\[
\mathcal{E}(f; Q)_{L_q(\mu)} \sim \mathcal{E}_1(f; Q)_{L_q(\mu)} \quad (6.7)
\]
where
\[
\mathcal{E}_1(f; Q)_{L_q(\mu)} := \inf_{c \in \mathbb{R}} \mu(Q)^{-\frac{1}{q}} \|f - c\|_{L_q(Q:\mu)}.
\]
In addition,
\[
\mathcal{E}(f;Q)_{L_q(\mu)} \sim \left( \frac{1}{\mu(Q)} \int_Q |f(x) - f_{Q,\mu}|^q d\mu(x) \right)^{\frac{1}{q}}.
\]  
(6.8)

Recall that \( f_{Q,\mu} \) denotes the \( \mu \)-average of \( f \) on \( Q \).

Since \( Q \subset Q(x, 2r_Q) \) and \( Q(x, r_Q) \subset 2Q \), whenever \( x \in Q \), by the doubling condition (1.12),
\[
\mathcal{E}(f;Q)_{L_q(\mu)} \sim \left( \int_{Q \times Q} \frac{|f(x) - f(y)|^q}{\mu(Q(x, r_Q))^2} d\mu(x)d\mu(y) \right)^{\frac{1}{q}}
\]  
(6.9)

provided \( r_Q \in (0, 1/2] \). Here the constants of equivalence depend only on \( q \) and the doubling constant \( c_\mu \).

**Definition 6.4** Given \( 1 \leq p < \infty \), \( 1 \leq q \leq \infty \) and a function \( f \in L_{q, \text{loc}}(\mu) \) we define the functional \( A_p(f, \cdot : S)_{L_q(\mu)} \) by letting
\[
A_p(f; t : S)_{L_q(\mu)} := \sup_{\pi} \left\{ \sum_{Q \in \pi} |Q|^p \mathcal{E}(f; Q)_{L_q(\mu)}^p \right\}^{\frac{1}{p}}
\]
where \( \pi \) runs over all packings of equal cubes with diameter at most \( t \) whose centers belong to \( S \).

Clearly, if \( S = \text{supp} \mu \), then
\[
A_p(f; t : S)_{L_{\infty}(\mu)} = A_p(f; t : S), \quad t > 0,
\]
see Definition 2.1, so that for all \( 1 \leq q \leq \infty \) we have
\[
A_p(f; t : S)_{L_q(\mu)} \leq A_p(f; t : S), \quad t > 0.
\]  
(6.10)

We present two important properties of the functional \( A_p(\cdot; \cdot : S)_{L_p(\mu)} \).

**Lemma 6.5** For every function \( f \in L_p(\mu) \), we have
\[
A_p(f; t : S)_{L_p(\mu)} \leq C(1 + t^p) \| f \|_{L_p(\mu)}, \quad t > 0,
\]
where \( C \) is a constant depending only on \( p, n, C_\mu \) and \( C'_\mu \).

*Proof.* Let \( \pi = \{Q\} \) be a packing of equal cubes of diameter \( \text{diam} Q = s \leq t \) centered in \( S \). By (6.7) for each \( Q \in \pi \) we have
\[
\mathcal{E}(f; Q)_{L_p(\mu)}^p \leq C \inf_{c \in \mathbb{R}} \frac{1}{\mu(Q)} \int_Q |f - c|^p d\mu \leq C \frac{1}{\mu(Q)} \int_Q |f|^p d\mu.
\]
By (6.5), \(|Q| \leq C \mu(Q)\) provided \(r_Q \leq 1\), so that whenever \(s \leq 1\) we obtain

\[
|Q| \mathcal{E}(f; Q)_{L_p(\mu)}^p \leq C \frac{|Q|}{\mu(Q)} \int_Q |f|^p \, d\mu \leq C \int_Q |f|^p \, d\mu.
\]

Let \(1 < s = \text{diam } Q \leq t\). Then \(\mu(Q) \geq \mu(Q(x_Q, 1)) \geq C'_\mu\) so that

\[
|Q| \mathcal{E}(f; Q)_{L_p(\mu)}^p \leq C \frac{|Q|}{\mu(Q)} \int_Q |f|^p \, d\mu \leq C \frac{|Q|}{C'_\mu} \int_Q |f|^p \, d\mu \leq C \frac{t^n}{C'_\mu} \int_Q |f|^p \, d\mu.
\]

Hence, for every \(Q \in \pi\) of diameter \(\text{diam } Q = s \in (0, t]\) we have

\[
|Q| \mathcal{E}(f; Q)_{L_p(\mu)}^p \leq C (1 + t^n) \int_Q |f|^p \, d\mu,
\]

so that

\[
\sum_{Q \in \pi} |Q| \mathcal{E}(f; Q)_{L_p(\mu)}^p \leq C (1 + t^n) \sum_{Q \in \pi} \int_Q |f|^p \, d\mu \leq C (1 + t^n) \int_{\mathbb{R}^n} |f|^p \, d\mu.
\]

It remains to take the supremum over all packings \(\pi\) of equal cubes with diameter at most \(t\) centered in \(S\), and the lemma follows. \(\Box\)

**Proposition 6.6** Let \(\mu\) be a doubling measure and let \(f \in L_{p, \text{loc}}(\mu), 1 \leq p < \infty\). Then for every \(t \in (0, 1/4]\) we have

\[
\frac{1}{C} \mathcal{A}_p(f; t/4 : S)_{L_p(\mu)}^p \leq \int \int_{|x-y|<t} \frac{t^n |f(x) - f(y)|^p}{\mu(Q(x,t))^2} \, d\mu(x) \, d\mu(y) \leq C \mathcal{A}_p(f; 4t : S)_{L_p(\mu)}^p.
\]

Here \(C\) is a positive constant depending only on \(n, p\) and \(c_\mu\).

**Proof.** By (6.7), for every two cubes \(Q, \tilde{Q}, Q \subset \tilde{Q}\), we have

\[
\mathcal{E}(f; Q)_{L_p(\mu)} \sim \mu(Q)^{-\frac{1}{p}} \inf_{c \in \mathbb{R}} \|f - c\|_{L_p(Q; \mu)} \leq \mu(Q)^{-\frac{1}{p}} \inf_{c \in \mathbb{R}} \|f - c\|_{L_p(\tilde{Q}; \mu)},
\]

so that

\[
\mathcal{E}(f; Q)_{L_p(\mu)} \leq C \frac{\mu(\tilde{Q})^{1/p}}{\mu(Q)^{1/p}} \mathcal{E}(f; \tilde{Q})_{L_p(\mu)}.
\]

First we prove the second inequality of the proposition. We have

\[
I := \int \int_{|x-y|<t} \frac{t^n |f(x) - f(y)|^p}{\mu(Q(x,t))^2} \, d\mu(y) \, d\mu(x) = t^n \int S_{Q(x,t)} \int_{Q(x,t)} \frac{|f(x) - f(y)|^p}{\mu(Q(x,t))^2} \, d\mu(y) \, d\mu(x).
\]

Let

\[
\tilde{\pi} := \{Q_i = Q(x_i, t/2) : i = 1, 2, \ldots\} \quad (6.11)
\]

58
be a covering of $S$ by cubes centered in $S$ with the covering multiplicity bounded by a constant $C = C(n)$. Then,

$$I \leq C t^n \sum_{i=1}^{\infty} \int_{Q_i, Q(x,t)} \int \frac{|f(x) - f(y)|^p}{\mu(Q(x,t))^2} d\mu(y) d\mu(x).$$

Clearly, for each $x \in Q_i$ we have $Q(x,t) \subset 3Q_i = Q(x_i, 3t) \subset 3Q(x,t)$ so that, by the doubling condition, $\mu(Q(x,t)) \sim \mu(3Q_i) \sim \mu(3Q(x,t))$ provided $t \leq 1/3$. Hence, by (6.6),

$$I \leq C t^n \sum_{i=1}^{\infty} \int_{3Q_i, 3Q_i} \int \frac{|f(x) - f(y)|^p}{\mu(Q(x,t))^2} d\mu(y) d\mu(x)$$

$$\leq C t^n \sum_{i=1}^{\infty} \frac{1}{\mu(3Q_i)^2} \int_{3Q_i, 3Q_i} \int |f(x) - f(y)|^p d\mu(y) d\mu(x)$$

$$\leq C \sum_{i=1}^{\infty} |3Q_i| E(f; 3Q_i)^p_{L_p(\mu)}.$$

By Lemma 3.4, the collection of cubes $\{2Q_i : i = 1, 2, \ldots\}$ can be partitioned into at most $N(n)$ packings so that, by Definition 6.4,

$$I \leq CN(n) A_p(f; 4t : S)^p_{L_p(\mu)}$$

proving the second inequality of the proposition.

Prove the first inequality. Let $0 < \tau \leq t$ and let $\pi$ be a packing of equal cubes of diameter $\tau$ centered in $S$. Put

$$J := \sum_{Q \in \pi} |Q| E(f; Q)^p_{L_p(\mu)}.$$ Then by (6.7) and (1.11)

$$J \leq C \sum_{Q \in \pi} \frac{\tau^n}{\mu(Q)} \inf_{c \in R} \int_{Q} |f - c|^p d\mu \leq C \sum_{Q \in \pi} \frac{t^n}{\mu(Q(x_Q, t))} \inf_{c \in R} \int_{Q} |f - c|^p d\mu.$$

Given a cube $Q_i \in \tilde{\pi}$, see (6.11), we put $\pi_i := \{Q \in \pi : x_Q \in Q_i\}$. Since $\tau \leq t$, each cube $Q \in \pi_i$ lies in the cube $3Q_i$ so that

$$\sum_{Q \in \pi_i} \inf_{c \in R} \int_{Q} |f - c|^p d\mu \leq \sum_{Q \in \pi_i} \inf_{c \in R} \int_{3Q_i} |f - c|^p d\mu \leq \int_{3Q_i} |f - c|^p d\mu.$$

On the other hand, since $x_Q \in Q_i$, the cube $3Q_i \subset Q(x_Q, 2t)$, so that, by the doubling
condition, \( \mu(3Q_i) \sim \mu(Q(x, t)) \) provided \( t \leq 1/2 \). Hence,

\[
J \leq C t^n \sum_{i=1}^{\infty} \sum_{Q \in \pi_i} \mu(Q(x, t)) \inf_{c \in \mathbb{R}} \int_{Q} |f - c|^p \, d\mu
\]

\[
\leq C t^n \sum_{i=1}^{\infty} \frac{1}{\mu(3Q_i)} \sum_{Q \in \pi_i} \inf_{c \in \mathbb{R}} \int_{3Q_i} |f - c|^p \, d\mu,
\]

so that, by (6.7),

\[
J \leq C t^n \sum_{i=1}^{\infty} E(f; 3Q_i)_{L_p(\mu)}.
\]

Hence, by (6.9),

\[
J \leq C t^n \sum_{i=1}^{\infty} \int_{S} \int_{Q(x, 3t/2)} \frac{|f(x) - f(y)|^p}{\mu(Q(x, 3t/2))^2} \, d\mu(y) \, d\mu(x).
\]

Note that \( 3Q_i \subset Q(x, 3t) \) for each \( x \in 3Q_i \). In addition, by the doubling condition, \( \mu(Q(x, 4t)) \sim \mu(Q(x, 3t/2)) \), \( x \in 3Q_i \), provided \( t \leq 1/4 \). Hence,

\[
J \leq C t^n \sum_{i=1}^{\infty} \int_{3Q_i} \int_{Q(x, 3t)} \frac{|f(x) - f(y)|^p}{\mu(Q(x, 4t))^2} \, d\mu(y) \, d\mu(x).
\]

By Lemma 3.4, without loss of generality one can assume that the collection of cubes \( \{3Q_i : i = 1, 2, \ldots\} \) is a packing. Hence

\[
J \leq C t^n \sum_{i=1}^{\infty} \int_{3Q_i, Q(x, 3t)} \frac{|f(x) - f(y)|^p}{\mu(Q(x, 4t))^2} \, d\mu(y) \, d\mu(x).
\]

Taking the supremum over all packings \( \pi \) of equal cubes centered in \( S \) with diameter \( \tau \leq t \), we finally obtain

\[
A_p(f; t : S)^p_{L_p(\mu)} \leq C t^n \int_{\|x-y\|<4t} \int_{\|x-y\|<4t} \frac{|f(x) - f(y)|^p}{\mu(Q(x, 4t))^2} \, d\mu(y) \, d\mu(x).
\]

The proposition is proved. \( \square \)

Now, combining this proposition with Theorem 6.3, we obtain

\[
\|f\|_{B_p,q(\mathbb{R}^n)} \sim \|f\|_{L_p(\mu)} + \left\{ \sum_{\nu=0}^{\infty} 2^{\nu sq} A_p(f; 2^{-\nu} : S)^q_{L_p(\mu)} \right\}^{\frac{1}{q}}.
\]
Finally, this equivalence and Lemma 6.5 imply the following formula for the trace norm in the Besov space:

Let $\mu$ be a measure with support $S$ satisfying conditions $(D_n)$ and (1.13). Let $1 \leq p, q \leq \infty, \varepsilon > 0$, and let $n/p < s < 1$. Then for every $f \in C(S)$ the following equivalence holds. The constants of this equivalence depend on $\varepsilon$ and the same parameters as the constants in Theorem 6.3.

7. Restrictions of Sobolev spaces and doubling measures.

In this section we prove the results stated in subsection 1.2.

Let $\mu$ be a positive Borel regular measure supported on the closed set $S$. We assume that $\mu$ satisfies condition $(D_n)$ (1.11) and condition (1.13).

**Definition 7.1** Let $\alpha \in (0, 1]$, $p \in [1, \infty)$ and $q \in [1, \infty]$. Given $f \in L_q, \text{loc}(\mu)$ we define the functional $A_{p,\alpha}(f; t : S)_{L_q(\mu)}$ by letting

$$A_{p,\alpha}(f; t : S)_{L_q(\mu)} := \sup_\pi \left\{ \sum_{Q \in \pi} |Q| E(f; Q)_{L_q(\mu)}^{p/2} \right\}, \quad t > 0,$$

where $\pi = \{Q\}$ runs over all $\alpha$-porous (with respect to $S$) packings of equal cubes with centers in $\partial S$ and diameter at most $t$.

Clearly, by Definition 3.10, for every $1 \leq q \leq \infty$ and $\alpha \in (0, 1]$

$$A_{p,\alpha}(f; t : S)_{L_q(\mu)} \leq A_{p,\alpha}(f; t : S), \quad t > 0.$$  \hspace{1cm} (7.2)

It is also clear that

$$A_{p,\alpha}(f; t : S)_{L_q(\mu)} \leq A_{p}(f; t : S)_{L_q(\mu)}, \quad t > 0,$$  \hspace{1cm} (7.3)

see Definition 6.4. Also observe that the functional $A_{p,\alpha}(f; \cdot : S)$, see Definition 2.3, equals the functional $A_{p,\alpha}(f; \cdot : S)_{L_{\infty}(\sigma)}$ where $\sigma$ is a measure whose support coincides with $\partial S$.

The first main result of the section is the following

**Theorem 7.2** Let $p \in (n, \infty)$, $\alpha \in (0, 1/7)$, and $\varepsilon > 0$. Then for every function $f \in C(S)$ we have

$$\|f\|_{W_p^1(\mathbb{R}^n)_S} \sim \|f\|_{L_p(\mu)} + \sup_{0 < t \leq \varepsilon} \frac{A_{p}(f; t : S)_{L_p(\mu)}}{t} \left\{ \int_0^\varepsilon A_{p,\alpha}(f; t : S)_{L_p(\mu)}^{p/2} \frac{dt}{t^{p/2+1}} \right\}^{1/p}$$

with constants of equivalence depending only on $n, p, \alpha, \varepsilon$ and the parameters $C_\mu, C'_\mu, C''_\mu$.  

61
The proof is based on a series of auxiliary lemmas and propositions. In all of these results we will assume that \( f \in C(S), \ p \in (n, \infty), \epsilon > 0, \) and \( C \) is a constant depending only on \( n, p, \alpha, \epsilon \) and \( C_\mu, C'_\mu, C''_\mu. \)

**Lemma 7.3** We have \( \|f\|_{L_p(\mu)} \leq C \|f\|_{W^1_p(\mathbb{R}^n)} \).

**Proof.** Let \( F \in W^1_p(\mathbb{R}^n) \) be a continuous function such that \( F|_S = f \) and

\[
\|F\|_{W^1_p(\mathbb{R}^n)} \leq 2 \|f\|_{W^1_p(\mathbb{R}^n)}.
\]

Let \( \pi = \{Q_i = Q(x_i, 1/2) : i = 1, 2, \ldots \} \) be a covering of \( S \) by equal cubes of diameter 1 centered in \( S \) whose covering multiplicity \( M_\pi \leq N(n). \) By condition (1.13), \( \mu(Q_i) \leq C_\mu'' \) so that

\[
\|f\|_{L_p(\mu)}^p = \|f\|_{L_p(\mu)}^p \leq \sum_{i=1}^{\infty} \int Q_i |F|^p d\mu \leq \sum_{i=1}^{\infty} (\sup_{Q_i} |F|)^p \mu(Q_i) \leq C_\mu'' \sum_{i=1}^{\infty} \sup_{Q_i} |F|^p.
\]

Hence

\[
\|f\|_{L_p(\mu)}^p \leq C \{ \sum_{i=1}^{\infty} \sup_{Q_i} |F - F_{Q_i}|^p + \sum_{i=1}^{\infty} |F_{Q_i}|^p \} \leq C \{ \sum_{i=1}^{\infty} \mathcal{E}(F; Q_i)^p + \sum_{i=1}^{\infty} |Q_i|^{-1} \int Q_i |F|^p \}.
\]

(Recall that \( F_Q \) stands for the average of \( F \) on \( Q_i). \) Since \( |Q_i| = 1 \), we obtain

\[
\|f\|_{L_p(\mu)}^p \leq C \{ \sum_{i=1}^{\infty} |Q_i| \mathcal{E}(F; Q_i)^p + \sum_{i=1}^{\infty} \int Q_i |F|^p \}.
\]

Since \( M_\pi \leq N(n) \), the second term on the right-hand side of this inequality is bounded by \( N(n)\|F\|_{L_p(\mathbb{R}^n)}^p \). Let us estimate the first term. By Theorem 3.3, the collection \( \pi \) can be partitioned into at most \( C(n) \) packings so that without loss of generality we may assume that \( \pi \) itself is a packing. Then, by (2.3) and (2.4),

\[
\sum_{i=1}^{\infty} |Q_i| \mathcal{E}(F; Q_i)^p \leq A_p(F; 1)^p \leq C \|f\|_{L_p(\mathbb{R}^n)}^p.
\]

Hence,

\[
\|f\|_{L_p(\mu)}^p \leq C(\|f\|_{L_p(\mathbb{R}^n)}^p + \|f\|_{L_p(\mathbb{R}^n)}^p) \leq C \|f\|_{W^1_p(\mathbb{R}^n)}^p \leq C \|f\|_{W^1_p(\mathbb{R}^n)}^p.
\]

proving the lemma. \( \Box \)

We put

\[
\|f\|_S = \|f\|_{L_p(\mu)} + \sup_{0 < t \leq \epsilon} \frac{A_p(f; t : S)_{L_p(\mu)}}{t} + \left\{ \int_0^\epsilon A_p(t; f; t : S)_{L_p(\mu)} \frac{dt}{t^{p+1}} \right\}^{\frac{1}{p}}.
\]

62
Combining Lemma 7.3 with inequalities (6.10), 2.12, (7.2), and Proposition 3.11, we obtain the required inequality $\|f\|_{S}^{p} \leq C\|f\|_{W_{p}(\mathbb{R}^{n})S}$ provided $n < p < \infty$, $\varepsilon > 0$, and $0 < \alpha \leq 1$.

Let us prove that for every $p > n$, $\varepsilon > 0$, and $0 < \alpha < 1/7$ the opposite inequality

$$\|f\|_{P_{\mu}(S)} \leq C\|f\|_{L_{p}(\mu)} + A_{p}(f; \varepsilon : S)$$

holds as well. We begin the proof with the next auxiliary

**Lemma 7.4** For every $\varepsilon > 0$ we have

$$\|f\|_{L_{p}(S)} \leq C\{\|f\|_{L_{p}(\mu)} + A_{p}(f; \varepsilon : S)\}.$$

**Proof.** It suffices to prove the lemma for $\varepsilon \in (0, 1]$. We let $\pi := \{Q_{i} = Q(x_{i}, \varepsilon/2) : x_{i} \in S, i \in I\}$ denote a covering of $S$ with the covering multiplicity $M_{\pi} \leq C(n)$. We have

$$\|f\|_{L_{p}(S)} \leq \sum_{Q_{i} \cap S} |Q_{i}| \int_{Q_{i} \cap S} |f|^p \, dx \leq \sum_{Q_{i} \cap S} |Q_{i}| \sup_{Q_{i} \cap S} |f|^p.$$

Hence,

$$\|f\|_{L_{p}(S)} \leq 2^{p}\left\{ \sum_{Q_{i} \cap S} |Q_{i}| \sup_{Q_{i} \cap S} |f - f_{Q_{i}, \mu}|^p + \sum_{Q_{i} \cap S} |Q_{i}| |f_{Q_{i}, \mu}|^p \right\}$$

$$\leq 2^{p}\left\{ \sum_{Q_{i} \cap S} |Q_{i}| \mathcal{E}(f; Q_{i} \cap S)^p + \sum_{Q_{i} \cap S} |Q_{i}| |f_{Q_{i}, \mu}|^p \right\}.$$

By (6.5), $|Q_{i}| \leq C\mu(Q_{i})$ so that

$$\|f\|_{L_{p}(S)} \leq C\sum_{Q_{i} \cap S} |Q_{i}| \mathcal{E}(f; Q_{i} \cap S)^p + \sum_{Q_{i} \cap S} \int_{Q_{i}} |f|^p \, d\mu.$$

Again, by Theorem 3.3, we may suppose that $\pi$ is a packing, so that, by (2.3),

$$\|f\|_{L_{p}(S)} \leq C\left\{ \sum_{Q_{i} \cap S} |Q_{i}| \mathcal{E}(f; Q_{i} \cap S)^p + \int_{\mathbb{R}^{n}} |f|^p \, d\mu \right\}$$

proving the lemma. \[\square\]

Recall that given $\varepsilon > 0$ by $\mathcal{W}(S; \varepsilon)$ we denote the family of all Whitney’s cubes of diameter at most $\varepsilon$, see (3.3). Given $Q \in \mathcal{W}(S)$ we put

$$\overline{Q} := Q(a_{Q}, r_{Q}).$$

**Lemma 7.5** For every $0 < \varepsilon \leq 1/50$ we have

$$\sum_{Q \in \mathcal{W}(S; \varepsilon)} |Q| |f_{Q, \mu}|^p \leq C\{\|f\|_{L_{p}(\mu)}^p + \|f\|_{L_{p}(S)}^p\}.$$
Proof. The reader can easily see that, by Theorem 3.1, for \( Q \in \mathcal{W}(S : \varepsilon) \) and every \( y \in Q = Q(a_Q, r_Q) \) we have the following:

\[
\overline{Q} \subset Q(x_Q, 10r_Q) = 10Q,
\]

\[
\overline{Q} \subset 10Q \subset Q(x, 11Q), \quad x \in Q,
\]

and

\[
Q(x, 11r_Q) \subset Q(a_Q, 22r_Q) = 22\overline{Q}, \quad x \in Q.
\]  

This and (7.5) yield

\[
\mu(\overline{Q}) \leq \mu(Q(x, 11Q)) \leq \mu(22\overline{Q}), \quad x \in Q.
\]  

Since \( Q \in \mathcal{W}(S : \varepsilon) \), its diameter \( \text{diam} Q = 2r_Q \leq \varepsilon \) so that

\[
\text{diam} 22\overline{Q} = 44r_Q = 44r_Q \leq 44\varepsilon \leq 1.
\]

Therefore, by the doubling condition and (7.6),

\[
\mu(Q(x, 11Q)) \sim \mu(\overline{Q}), \quad x \in Q.
\]  

In addition, by (6.5),

\[
|Q| = |\overline{Q}| \leq C\mu(\overline{Q}).
\]  

We put \( \tilde{\mu} := \mu + \lambda \) where \( \lambda \) denotes the Lebesgue measure on \( \mathbb{R}^n \). By \( f^\wedge \) we denote the extension of \( f \) by 0 from \( S \) to all of \( \mathbb{R}^n \). Thus \( f^\wedge(x) = f(x), x \in S \), and \( f^\wedge(x) = 0 \) on \( \mathbb{R}^n \setminus S \). Given \( g \in L_{1, \text{loc}}(\tilde{\mu}) \) we let \( \mathcal{M}_{\tilde{\mu}}g \) denote the Hardy–Littlewood maximal function of \( g \) (with respect to the measure \( \tilde{\mu} \)):

\[
\mathcal{M}_{\tilde{\mu}}g(x) := \sup_{r > 0} \frac{1}{\tilde{\mu}(Q(x, r))} \int_{Q(x,r)} |g| \, d\tilde{\mu}.
\]

Now, by (7.5), for every \( x \in Q \) we have

\[
|f_{\overline{Q},\mu}| \leq \frac{1}{\mu(\overline{Q})} \int_{Q} |f| \, d\mu \leq \frac{1}{\mu(\overline{Q})} \int_{Q(x,11r_Q)} |f| \, d\mu \leq \frac{1}{\mu(\overline{Q})} \int_{Q(x,11r_Q)} |f^\wedge| \, d\tilde{\mu},
\]

so that

\[
|f_{\overline{Q},\mu}| \leq \frac{\tilde{\mu}(Q(x, 11r_Q))}{\mu(\overline{Q})} \frac{1}{\tilde{\mu}(Q(x, 11r_Q))} \int_{Q(x,11r_Q)} |f^\wedge| \, d\tilde{\mu} \leq \frac{\tilde{\mu}(Q(x, 11r_Q))}{\mu(\overline{Q})} [\mathcal{M}_{\tilde{\mu}}(f^\wedge)](x).
\]

But, by (7.7) and (7.8),

\[
\frac{\tilde{\mu}(Q(x, 11r_Q))}{\mu(\overline{Q})} = \frac{\mu(Q(x, 11r_Q)) + |Q(x, 11r_Q)|}{\mu(\overline{Q})} \leq C \frac{\mu(\overline{Q}) + |Q|}{\mu(\overline{Q})} \leq C,
\]
so that 

$$|f|_{\mathbb{Q}, \mu} \leq C [M_\mu(f^\wedge)](x), \quad x \in Q.$$ 

Since $|Q| \leq |Q| + \mu(Q) = \tilde{\mu}(Q)$, we obtain 

$$|Q| |f|_{\mathbb{Q}, \mu}^p \leq \tilde{\mu}(Q) |f|_{\mathbb{Q}, \mu}^p \leq C \int_Q [M_\mu(f^\wedge)]^p(x) d\tilde{\mu}(x).$$ 

Hence, 

$$I := \sum_{Q \in \mathcal{W}(S; \varepsilon)} |Q| |f|_{\mathbb{Q}, \mu}^p \leq C \sum_{Q \in \mathcal{W}(S; \varepsilon)} \int_Q [M_\mu(f^\wedge)]^p(x) d\tilde{\mu}(x) \leq C\cdot\int_{\mathbb{R}^n} [M_\mu(f^\wedge)]^p(x) d\tilde{\mu}(x).$$ 

(Recall that $M_\mathcal{W}(S)$ denotes the covering multiplicity of the family $\mathcal{W}(S)$ of Whitney’s cubes.) By Theorem 3.1, $M_\mathcal{W}(S) \leq N(n)$ so that 

$$I \leq C\cdot\int_{\mathbb{R}^n} [M_\mu(f^\wedge)]^p(x) d\tilde{\mu}(x) = C\|M_\mu(f^\wedge)\|_{L^p(\tilde{\mu})}^p.$$ 

Observe that $\tilde{\mu} = \mu + \lambda$ is a positive Borel regular measure on $\mathbb{R}^n$. In this case the Hardy–Littlewood maximal operator is a bounded sublinear operator from $L_p(\tilde{\mu})$ into $L_p(\tilde{\mu}), p > 1$, whose operator norm does not exceed a constant depending only on $p$ and $n$. See, e.g., [GK], Corollary 3.3. Thus, 

$$\|M_\mu(f^\wedge)\|_{L^p(\tilde{\mu})} \leq C\|f^\wedge\|_{L^p(\tilde{\mu})}^p,$$ 

so that 

$$I \leq C\|f^\wedge\|_{L^p(\tilde{\mu})}^p = C \int_{\mathbb{R}^n} |f^\wedge|^p d\tilde{\mu} = C \int_{\mathbb{R}^n} |f|^p d\mu + \int_{\mathbb{R}^n} |f|^p dx \leq C\{\|f\|_{L^p(\mu)}^p + \|f\|_{L^p(S)}^p\},$$ 

proving the lemma.

**Definition 7.6** Let $0 \leq \alpha \leq 1$ and let $Q$ be a cube centered in $S$. We say that $Q$ is **strongly $\alpha$-porous** if the cube $\eta Q$ is $\alpha$-porous (with respect to $S$) for every $0 < \eta \leq 1$.

A packing $\pi$ is said to be **strongly $\alpha$-porous** if every cube $Q \in \pi$ is strongly $\alpha$-porous.

Given a function $f \in C(S)$ and a cube $Q \subset \mathbb{R}^n$, we put 

$$\tilde{E}(f; Q) := \frac{1}{\mu(Q)} \int_Q |f - f(x_Q)| d\mu. \quad (7.9)$$ 

Let us introduce a corresponding analog of the functional (7.1) over families of strongly porous cubes.
Definition 7.7 Let \( p \in [1, \infty), \alpha \in [0, 1], \) and let \( f \in C(S) \). We define the functional \( J_{p,\alpha}(f, \cdot : S, \mu) \) by the formula
\[
J_{p,\alpha}(f; t : S, \mu) := \sup_{\pi} \left\{ \sum_{Q \in \pi} |Q| \tilde{E}(f; Q)^{p} \right\}^{\frac{1}{p}}, \quad t > 0.
\]
Here \( \pi = \{ Q \} \) runs over all strongly \( \alpha \)-porous (with respect to \( S \)) packings of equal cubes of diameter at most \( t \) centered in \( \partial S \).

Proposition 7.8 For every \( \alpha \in (0, 1/7) \) we have
\[
\| f \|_{W_1^p(\mathbb{R}^n) | S} \leq C \left( \| f \|_{L_p(\mu)} + \sup_{0 < t \leq \varepsilon} \frac{A_p(f; t : S)}{t} + \left\{ \int_0^\varepsilon J_{p,\alpha}(f; t : S, \mu)^p \frac{dt}{t^{p+1}} \right\}^{\frac{1}{p}} \right).
\]

Proof. Since the righthand side of the proposition’s inequality is a non-decreasing function of \( \varepsilon \), it suffices to prove the proposition for \( \varepsilon \in (0, 0.02) \). (7.10)

Put
\[
\bar{\varepsilon} := 0 \quad \text{and} \quad \delta := \frac{\varepsilon}{2000}, \quad (7.11)
\]
and define the function
\[
F := \text{Ext}_{\delta, \bar{\varepsilon}, S} f
\]
by the formula (4.2). Since \( \| f \|_{W_1^p(\mathbb{R}^n) | S} \leq \| F \|_{W_1^p(\mathbb{R}^n)} \), our aim is to show that
\[
\| F \|_{W_1^p(\mathbb{R}^n)} := \| F \|_{L_p(\mathbb{R}^n)} + \| F \|_{L_1^p(\mathbb{R}^n)}
\]
is bounded by the righthand side of the proposition’s inequality.

To estimate the norm \( \| F \|_{L_p(\mathbb{R}^n)} \) we will be needed the following auxiliary lemmas.

Lemma 7.9 Let \( 0 < \beta < 1, \varsigma \geq 1 \) and \( 0 < T \leq 1/\varsigma \). Let \( \pi = \{ Q \} \subset \mathcal{W}(S) \) be a collection of Whitney’s cubes of diameter at most \( T \). Given \( Q \in \pi \) let \( \tilde{Q} \) be a strongly \( \beta \)-porous cube centered in \( \partial S \) such that
\[
\| x_Q - x_{\tilde{Q}} \| \leq \varsigma \text{diam } Q \quad \text{and} \quad \text{diam } Q \leq \text{diam } \tilde{Q} \leq \varsigma \text{diam } Q.
\]
Then for every \( f \in C(S) \) and every \( \alpha \in (0, \beta) \) we have
\[
\sum_{Q \in \pi} r_Q^{n-p} \tilde{E}(f; \tilde{Q})^p \leq C \int_0^{4kT} J_{p,\alpha}(f; t : S, \mu)^p \frac{dt}{t^{p+1}}
\]
where \( C \) is a constant depending on \( n, p, \varsigma \) and \( c_\mu \).
Proof. The lemma can be proven precisely in the same way as Lemma 4.12 (with \( t = 0 \) in its formulation). We need only to replace in this proof (4.19) with the inequality
\[
\tilde{E}(f; \tilde{Q} \cap S) \leq C \tilde{E}(f; Q^2 \cap S).
\] (7.12)
To prove this let us recall that, by (4.18), \( \tilde{Q} \subset Q^2 \subset 2\tilde{Q} \) and \( x_{\tilde{Q}} = x_{Q^2} \) so that
\[
\tilde{E}(f; \tilde{Q} \cap S) = \frac{1}{\mu(\tilde{Q})} \int_{\tilde{Q}} |f - f(x_{\tilde{Q}})| \, d\mu \leq \frac{1}{\mu(Q^2)} \int_{Q^2} |f - f(x_{Q^2})| \, d\mu.
\]
Since \( \text{diam} \tilde{Q} \leq \xi \text{diam} Q \leq \xi T \leq 1 \), by the doubling condition (1.12) \( \mu(Q^2) \leq \mu(2\tilde{Q}) \leq c_\mu \mu(\tilde{Q}) \), proving (7.12) with \( C = c_\mu \). \qed

**Lemma 7.10** For every \( \alpha \in (0, 3/20) \)
\[
\sum_{Q \in \mathcal{W}(S; \delta_0)} |Q||f(a_Q)|^p \leq C \left\{ \|f\|_{L_p(\mu)} + A_p(f; \varepsilon : S) + \left( \int_0^\varepsilon J_p,\alpha(f; t : S, \mu)^p \frac{dt}{t^{p+1}} \right)^{\frac{1}{p}} \right\}.
\]

Proof. Put
\[
I := \sum_{Q \in \mathcal{W}(S; \delta_0)} |Q||f(a_Q)|^p.
\]
Then
\[
I \leq 2^p \left\{ \sum_{Q \in \mathcal{W}(S; \delta_0)} |Q||f(a_Q) - f_{\overline{Q}, \mu}|^p + \sum_{Q \in \mathcal{W}(S; \delta_0)} |Q||f_{\overline{Q}, \mu}|^p \right\} = 2^p \{ I_1 + I_2 \}. \tag{7.13}
\]
Recall that \( \overline{Q} := Q(a_Q, r_Q) \). By (7.10) and (7.11), we have \( 90\delta \leq \varepsilon < 1/50 \) so that by Lemma 7.5
\[
I_2 := \sum_{Q \in \mathcal{W}(S; \delta_0)} |Q||f_{\overline{Q}, \mu}|^p \leq C \{ \|f\|_{L_p(\mu)} + \|f\|_{L_p(S)} \}.
\]
Applying again Lemma 7.4, we obtain
\[
I_2 \leq C \{ \|f\|_{L_p(\mu)} + A_p(f; \varepsilon : S) \}. \tag{7.14}
\]
Let us estimate the quantity \( I_1 := \sum_{Q \in \mathcal{W}(S; \delta_0)} |Q||f(a_Q) - f_{\overline{Q}, \mu}|^p \). We have
\[
I_1 \leq \sum_{Q \in \mathcal{W}(S; \delta_0)} |Q| \left( \frac{1}{\mu(\overline{Q})} \int_{\overline{Q}} |f - f(a_{\overline{Q}})| \, d\mu \right)^p \leq C \sum_{Q \in \mathcal{W}(S; \delta_0)} r_{\overline{Q}}^{n-p} \tilde{E}(f; \overline{Q})^p.
\]
67
Observe that, by Definition 7.6 and by Lemma 4.13, the cube $\bar{Q} := Q(a_Q, 20r_Q)$ is strongly $\beta$-porous for every $\beta \in (0, 3/20)$. Since $\bar{Q} = Q(a_Q, r_Q) = (1/20) \bar{Q}$, the cube $\bar{Q}$ is strongly $\beta$-porous for every $\beta \in (0, 3/20)$ as well.

It can be easily seen that the cubes $\{Q : Q \in \mathcal{W}(S : 90\delta)\}$ satisfy conditions of Lemma 7.9 with $\varsigma = 5$. Since $90\delta \leq 1/\varsigma = 1/5$, see (7.10) and (7.11), in the lemma’s hypothesis one can put $T := 90\delta$. Observe also that $4\varsigma T = 72\delta \leq \varepsilon$ so that, by Lemma 7.9, for every $\alpha \in (0, 3/20)$ we have

$$I_1 \leq C \int_0^{4\varsigma T} J_{p,\alpha}(f; t : S, \mu)^p \frac{dt}{t^{p+1}} \leq C \int_0^\varepsilon J_{p,\alpha}(f; t : S, \mu)^p \frac{dt}{t^{p+1}}.$$

Combining (7.13) with this estimate and inequality (7.14), we finally obtain the statement of the lemma.

**Lemma 7.11** For every $\alpha \in (0, 3/20)$ we have

$$\|F\|_{L_p(\mathbb{R}^n)} \leq C \left\{ \|f\|_{L_p(\mu)} + A_p(f; \varepsilon : S) + \left( \int_0^\varepsilon J_{p,\alpha}(f; t : S, \mu)^p \frac{dt}{t^{p+1}} \right)^{\frac{1}{p}} \right\}.$$  

**Proof.** By Lemma 7.4

$$\|F\|_{L_p(\mu)} = \|f\|_{L_p(\mu)} \leq C \{ \|f\|_{L_p(\mu)} + A_p(f; \varepsilon : S) \} \quad (7.15)$$

provided $\varepsilon \in (0, 1]$. In turn, by Lemma 4.3,

$$\|F\|_{L_p(\mathbb{R}^n \setminus S)} \leq C \sum_{Q \in \mathcal{W}(S : 2\delta)} |Q|\|f(a_Q)|^p,$$

so that, by Lemma 7.10,

$$\|F\|_{L_p(\mathbb{R}^n \setminus S)} \leq C \left\{ \|f\|_{L_p(\mu)} + A_p(f; \varepsilon : S) + \left( \int_0^\varepsilon J_{p,\alpha}(f; t : S, \mu)^p \frac{dt}{t^{p+1}} \right)^{\frac{1}{p}} \right\}.$$

This inequality and (7.15) imply the result of the lemma. □

We turn to estimates of the norm $\|F\|_{L_p^b(\mathbb{R}^n)}$. We will mainly follow the scheme used in the proofs of the sufficiency part of Theorems 1.2 and 2.5.

We put

$$\lambda(f) := \|f\|_{L_p(\mu)} + \sup_{0 < t \leq \varepsilon} \frac{A_p(f; t : S)}{t} + \left\{ \int_0^\varepsilon J_{p,\alpha}(f; t : S, \mu)^p \frac{dt}{t^{p+1}} \right\}^{\frac{1}{p}}.$$

Our goal is to show that

$$\|F\|_{L_p^b(\mathbb{R}^n) | S} \leq C \lambda(f), \quad (7.16)$$
provided $p > n$, $0 < \varepsilon \leq 1/50$, and $0 < \alpha < 1/7$.

To prove this inequality given a packing $\pi = \{K\}$ of equal cubes in $\mathbb{R}^n$ we again put

$$I(F; \pi) := \left\{ \sum_{K \in \pi} r_K^{n-p} \mathcal{E}(F; K)^p \right\}^{1/p}.$$

We also put $\varsigma := \delta/90$. Let us see that it suffices to show that for every packing $\pi = \{K\}$ of equal cubes of diameter $t \in (0, \varsigma)$ the following inequality

$$I(F; \pi) \leq C \lambda(f) \quad (7.17)$$

holds.

In fact, in this case, by Lemma 5.3, $F \in L^1_p(\mathbb{R}^n)$ and

$$\|F\|_{L^1_p(\mathbb{R}^n)} \leq C(\lambda(f) + \varsigma^{-1}\|F\|_{L^p(\mathbb{R}^n)}).$$

In turn, by Lemma 7.11, $\|F\|_{L^p(\mathbb{R}^n)} \leq C \lambda(f)$ so that $\|F\|_{L^1_p(\mathbb{R}^n)} \leq C\lambda(f)$ as well. Hence,

$$\|F\|_{W^1_p(\mathbb{R}^n)} = \|F\|_{L^p(\mathbb{R}^n)} + \|F\|_{L^1_p(\mathbb{R}^n)} \leq C\lambda(f),$$

proving (7.16).

As above, we prove inequality (7.17) in five steps.

The first step: $\pi \subset \mathcal{K}_1$, see (4.7). By Lemma 4.9,

$$I(F; \pi) \leq Ct^{-1} A_p(f; \gamma t : S)$$

with $\gamma \leq 574$. Since $t \leq \varsigma = \delta/90$, we have $\gamma t \leq \gamma\delta/90 \leq 7\delta \leq \varepsilon$ (recall that $\delta = \varepsilon/2000$). Hence,

$$I(F; \pi) \leq C \sup_{0 < t \leq \varepsilon} \frac{A_p(f; t : S)}{t} \leq C\lambda(f)$$

so that (7.17) is satisfied.

There is no necessity to consider the case $\pi \subset \mathcal{K}_2$. In fact, by (4.8), every cube $K \in \mathcal{K}_2$ has the diameter $\operatorname{diam} K > \delta/82$ which contradicts our assumption $\operatorname{diam} K \leq \delta/90, K \in \pi$.

The third step: $\pi \subset \mathcal{K}_3$. The proof of inequality (7.17) in this case is based on the following

**Lemma 7.12** Let $\delta \in (0,1]$. Then for every packing $\pi \subset \mathcal{K}_3$ of equal cubes and every $0 < \alpha < 1/7$ we have

$$I(F; \pi)^p \leq C \int_0^\delta J_{p,\alpha}(f; t : S, \mu)^p \frac{dt}{tp+1}.$$
Proof. By Lemma 4.10, there exists a finite family \( \pi' = \{Q\} \subset \mathcal{W}(S) \) of cubes of diameter \( \text{diam} \, Q \leq 2\delta \) such that

\[
I(F; \pi)^p \leq C \sum \{t^p \, |f(a_Q) - f(a_{Q'})|^p : Q, Q' \in \pi', Q \cap Q' \neq \emptyset, r_{Q'} \leq r_Q \}.
\]

Given \( Q \in \pi' \) we put

\[
\overline{Q} := Q(a_Q, r_Q), \quad \hat{Q} := Q(a_Q, 21r_Q).
\]

By Lemma 4.13, for every \( Q' \in \pi' \) such that \( Q' \cap Q \neq \emptyset \) and \( r_{Q'} \leq r_Q \) we have

\[
\|a_Q - a_{Q'}\| \leq 20r_Q. \tag{7.18}
\]

In addition, the lemma states the following: Let \( \widetilde{Q} := Q(a_Q, 20r_Q) \). Then the cube \( \eta \widetilde{Q} \) is \( \beta \)-porous for every \( \eta \in (0, 1] \) and \( \beta \in (0, 3/20) \). Since \( \hat{Q} = (21/20)\widetilde{Q} \), the cube \( \eta \hat{Q} \) is \( \beta \)-porous for every \( \eta \in (0, 1] \) but with \( \beta < (20/21)3/20 = 1/7 \).

Thus, by Definition 7.6, the cube \( \hat{Q} \) is strongly \( \beta \)-porous for every \( \beta \in (0, 1/7) \). Now we have

\[
|f(a_Q) - f(a_{Q'})| \leq |f(a_Q) - f_{\eta\hat{Q};\mu}| + |f_{\eta\hat{Q};\mu} - f(a_{Q'})| \\
\leq \frac{1}{\mu(\widetilde{Q})} \int_{\widetilde{Q}} |f - f(a_Q)| \, d\mu + \frac{1}{\mu(\hat{Q})} \int_{\hat{Q}} |f - f(a_{Q'})| \, d\mu.
\]

Observe that \( \overline{Q} \subset \hat{Q} \), see (7.18), so that, by (7.9),

\[
|f(a_Q) - f(a_{Q'})| \leq \frac{\mu(\hat{Q})}{\mu(\overline{Q})} \frac{1}{\mu(\overline{Q})} \int_{\overline{Q}} |f - f(x_{\overline{Q}})| \, d\mu + \frac{\mu(\hat{Q}')}{\mu(\hat{Q})} \frac{1}{\mu(\hat{Q})} \int_{\hat{Q}} |f - f(x_{\hat{Q}})| \, d\mu
\]

\[
= \frac{\mu(\hat{Q})}{\mu(\overline{Q})} \tilde{E}(f; \hat{Q}) + \frac{\mu(\hat{Q}')}{\mu(\hat{Q})} \tilde{E}(f; \hat{Q}')
\]

Recall that for every cube \( Q \in \mathcal{K}_3 \) we have \( \text{diam} \, Q \leq 0.01\delta \) so that

\[
\text{diam} \, \hat{Q} = 21 \text{diam} \, Q \leq 0.21\delta \leq 1.
\]

Therefore, by the doubling condition, \( \mu(\hat{Q})/\mu(\overline{Q}) \leq C \). Also, by (7.18),

\[
\overline{Q}' = Q(a_{Q'}, 21r_{Q'}) \subset Q(a_{Q'}, 21r_Q) \subset Q(a_Q, 41r_Q) = 41\overline{Q}.
\]

Since

\[
\text{diam} \, 41\overline{Q} = 41 \text{diam} \, Q \leq 0.41\delta \leq 1,
\]

by the doubling condition

\[
\frac{\mu(\overline{Q}')}{\mu(\overline{Q})} \leq \frac{\mu(41\overline{Q})}{\mu(\overline{Q})} \leq C.
\]

Hence

\[
|f(a_Q) - f(a_{Q'})| \leq C(\tilde{E}(f; \hat{Q}) + \tilde{E}(f; \hat{Q}'))
\]
so that
\[ I(F; \pi)^p \leq C \sum_{Q \in \pi'} \{ r_Q^{n-p}(\tilde{E}(f; \hat{Q})^p + \tilde{E}(f; \hat{Q})^p) : Q, Q' \in \pi', Q \cap Q' \neq \emptyset, r_{Q'} \leq r_Q \}. \]
Since \( r_Q \sim r_{Q'} \) for every \( Q, Q' \in \mathcal{W}(S) \) such that \( Q \cap Q' \neq \emptyset \), we obtain
\[ I(F; \pi)^p \leq C \sum_{Q \in \pi'} \{ r_Q^{n-p} \tilde{E}(f; \hat{Q})^p + r_{Q'}^{n-p} \tilde{E}(f; \hat{Q})^p : Q, Q' \in \pi' \} = 2C \sum_{Q \in \pi'} r_Q^{n-p} \tilde{E}(f; \hat{Q})^p. \]

We also have
\[ \text{diam } Q \leq \text{diam } \hat{Q} = 21 \text{ diam } Q. \]
Put \( \varsigma := 21, T := 0.01\delta \). Since \( \delta \leq 1 \), we have \( T \leq 1/\varsigma \). We also put
\[ \beta := \min \{ 2\alpha, (\alpha + 1/7)/2 \}. \]
Since \( \alpha \in (0, 1/7) \), we have \( 0 < \alpha < \beta \leq 2\alpha \) and \( \beta \in (0, 1/7) \). We have also proved above that for every \( Q \in \pi' \) the cube \( \hat{Q} \) is strongly \( \beta \)-porous.

Thus all conditions of Lemma 7.9 are satisfied. By this lemma,
\[ I(F; \pi)^p \leq C^p \sum_{Q \in \pi'} r_Q^{n-p} \tilde{E}(f; \hat{Q})^p \leq \int_0^{4\varsigma T} J_{p,\alpha}(f; t : S, \mu)^p \frac{dt}{t^{p+1}}. \]
But \( 4\varsigma T = 0.84\delta \leq \delta \), and the proof of the lemma is finished. \( \square \)

Since \( \delta \leq \varepsilon \), by this lemma,
\[ I(F; \pi)^p \leq C \int_0^\varepsilon J_{p,\alpha}(f; t : S, \mu)^p \frac{dt}{t^{p+1}} \leq C \lambda(f)^p, \]
proving inequality (7.17) for \( \pi \subset \mathcal{K}_3 \).

The forth step: \( \pi \subset \mathcal{K}_4 \). By Lemma 4.16, there exists a finite collection \( \pi' = \{ Q \} \subset \mathcal{W}(S) \) of Whitney’s cubes of diameter \( \text{diam } Q \leq 90\delta \) such that
\[ I(F; \pi)^p \leq C \sum_{Q \in \pi'} |Q||f(a_Q)|^p. \]
Therefore, by Lemma 7.10,
\[ I(F; \pi)^p \leq C \sum_{Q \in \mathcal{W}(S, 90\delta)} |Q||f(a_Q)|^p \leq C \lambda(f)^p, \]
proving (7.17) for \( \pi \subset \mathcal{K}_4 \).

The fifth step: \( \pi \subset \mathcal{K}_5 \). This case is trivial because, by Lemma 4.17, \( I(F; \pi) = 0 \) for every packing \( \pi \subset \mathcal{K}_5 \).

Proposition 7.8 is completely proved. \( \square \)

We are in a position to finish the proof of Theorem 7.2. We observe that, by Proposition 7.8, it remains to replace the functional \( A_p(f; \cdot : S) \) and \( J_{p,\alpha}(f; \cdot : S, \mu) \) in the righthand side of the proposition’s inequality with the corresponding functionals \( A_p(f; \cdot ; S)_{L_p(\mu)} \), see Definition 2.1, and \( A_{p,\alpha}(f; \cdot : S)_{L_p(\mu)} \), see Definition 7.7.

We will make this with the help of the two following lemmas.
Lemma 7.13 Let \( s \in \left( \frac{n}{p}, 1 \right) \) and let \( \alpha \in (0, 1) \). Then for every \( t \in (0, 1] \) we have

\[
\mathcal{J}_{p, \alpha}(f; t : S, \mu) \leq C t^s \left( \int_0^{2t} \left( \frac{A_{p, \alpha}(f; u : S)_{L_p(\mu)}}{u^s} \right)^p \frac{du}{u} \right)^{\frac{1}{p}}.
\]

Here \( C \) is a constant depending only on \( n, p, s \) and \( c_\mu \).

Proof. Let \( Q \) be a strongly \( \alpha \)-porous cube of diameter at most \( t \) centered in \( \partial S \), see Definition 7.6. Put

\[
Q_i := 2^{-i}Q = Q(x_Q, 2^{-i}r_Q), \quad i = 0, 1, ...
\]

Then, by (6.8),

\[
\tilde{E}(f; Q) := \frac{1}{\mu(Q)} \int_Q |f - f(x_Q)| \, d\mu \leq \frac{1}{\mu(Q)} \int_Q |f - f_{Q, \mu}| \, d\mu + |f_{Q, \mu} - f(x_Q)|
\]

\[
\leq \left( \frac{1}{\mu(Q)} \int_Q |f - Q, \mu|^p \, d\mu \right)^{\frac{1}{p}} + |f_{Q, \mu} - f(x_Q)|
\]

\[
\leq C \mathcal{E}(f; Q)_{L_p(\mu)} + |f_{Q, \mu} - f(x_Q)|.
\]

Since \( f \) is continuous on \( S \),

\[
\lim_{r \to 0} \frac{1}{\mu(rQ)} \int_{rQ} f \, d\mu = f(x_Q),
\]

so that

\[
|f_{Q, \mu} - f(x_Q)| = \left| \sum_{i=0}^{\infty} (f_{Q_i, \mu} - f_{Q_{i+1}, \mu}) \right| \leq \sum_{i=0}^{\infty} |f_{Q_i, \mu} - f_{Q_{i+1}, \mu}|.
\]

Since \( \text{diam} Q_i = 2^{-i} \text{diam} Q \leq t \leq 1 \), by the doubling condition,

\[
|f_{Q_i, \mu} - f_{Q_{i+1}, \mu}| \leq \frac{1}{\mu(Q_{i+1})} \int_{Q_{i+1}} |f - f_{Q_i, \mu}| \, d\mu
\]

\[
\leq \frac{\mu(Q_i)}{\mu(Q_{i+1})} \frac{1}{\mu(Q_i)} \int_{Q_i} |f - f_{Q_i, \mu}| \, d\mu \leq c_\mu \frac{1}{\mu(Q_i)} \int_{Q_i} |f - f_{Q_i, \mu}| \, d\mu,
\]

so that, by (6.8),

\[
|f_{Q_i, \mu} - f_{Q_{i+1}, \mu}| \leq c_\mu \left( \frac{1}{\mu(Q_i)} \int_{Q_i} |f - f_{Q_i, \mu}|^p \, d\mu \right)^{\frac{1}{p}} \leq C \mathcal{E}(f; Q_i)_{L_p(\mu)}.
\]

We obtain

\[
|f_{Q, \mu} - f(x_Q)| \leq C \sum_{i=0}^{\infty} \mathcal{E}(f; Q_i)_{L_p(\mu)}.
\]

72
Hence, by the Hölder inequality,
\[
|f_{Q,\mu} - f(x_Q)| \leq C \left( \sum_{i=0}^{\infty} r_{Q_i}^{(s-n/p)p^*} \right)^{\frac{1}{p^*}} \left( \sum_{i=0}^{\infty} \left( \frac{\mathcal{E}(f; Q_i)_{L_p(\mu)}}{r_{Q_i}^{s-n/p}} \right)^p \right)^{\frac{1}{p}}, \quad \frac{1}{p} + \frac{1}{p^*} = 1.
\]
Since \( r_{Q_i} = 2^{-i} r_Q \),
\[
\left( \sum_{i=0}^{\infty} r_{Q_i}^{(s-n/p)p^*} \right)^{\frac{1}{p^*}} \leq C r_Q^{s-n/p}.
\]
Hence,
\[
|f_{Q,\mu} - f(x_Q)| \leq C r_Q^{s-n/p} \left( \sum_{i=0}^{\infty} \left( \frac{\mathcal{E}(f; Q_i)_{L_p(\mu)}}{r_{Q_i}^{s-n/p}} \right)^p \right)^{\frac{1}{p}},
\]
so that
\[
\tilde{\mathcal{E}}(f; Q)^p \leq C \{ \mathcal{E}(f; Q)^p \}_{L_p(\mu)}^{\{Q\}} + r_Q^{sp-n} \sum_{i=0}^{\infty} |Q_i| \left( \frac{\mathcal{E}(f; Q_i)_{L_p(\mu)}}{r_{Q_i}^s} \right)^p \leq C r_Q^{sp-n} \sum_{i=0}^{\infty} |Q_i| \left( \frac{\mathcal{E}(f; Q_i)_{L_p(\mu)}}{r_{Q_i}^s} \right)^p.
\]
We obtain
\[
|Q| \tilde{\mathcal{E}}(f; Q)^p \leq C r_Q^{sp} \sum_{i=0}^{\infty} |Q_i| \left( \frac{\mathcal{E}(f; Q_i)_{L_p(\mu)}}{r_{Q_i}^s} \right)^p. \tag{7.19}
\]
Let \( \pi = \{Q\} \) be a packing of strongly \( \alpha \)-porous equal cubes of diameter \( \text{diam} Q = \tau \leq t \) centered in \( \partial S \). Then, by Definition 7.9, for every \( Q \in \pi \) and every \( i = 0, 1, \ldots \) the cube \( Q_i = 2^{-i} Q \) is \( \alpha \)-porous. Now, by (7.19),
\[
I := \sum_{Q \in \pi} |Q| \tilde{\mathcal{E}}(f; Q)^p \leq C \sum_{Q \in \pi} (\text{diam} Q)^{sp} \sum_{i=0}^{\infty} |Q_i| \left( \frac{\mathcal{E}(f; Q_i)_{L_p(\mu)}}{r_{Q_i}^s} \right)^p
= C \tau^{sp} \sum_{Q \in \pi} \sum_{i=0}^{\infty} |Q_i| \left( \frac{\mathcal{E}(f; Q_i)_{L_p(\mu)}}{r_{Q_i}^s} \right)^p
= C \tau^{sp} \sum_{i=0}^{\infty} (2^{-i}\tau)^{-sp} \sum_{Q \in \pi} |2^{-i} Q| \mathcal{E}(f; 2^{-i} Q)^p,\]
so that, by Definition 7.1,
\[
I \leq C \tau^{sp} \sum_{i=0}^{\infty} (2^{-i}\tau)^{-sp} \mathcal{A}_{p,\alpha}(f; 2^{-i}\tau : S)^p_{L_p(\mu)}.
\]
Since \( \mathcal{A}_{p,\alpha}(f; \cdot : S)_{L_p(\mu)} \) is a non-decreasing function, we obtain
\[
I \leq C \tau^{sp} \int_0^{2\tau} \left( \frac{\mathcal{A}_{p,\alpha}(f; u : S)_{L_p(\mu)}}{u^s} \right)^p du \leq C \tau^{sp} \int_0^{2\tau} \left( \frac{\mathcal{A}_{p,\alpha}(f; u : S)_{L_p(\mu)}}{u^s} \right)^p du,
\]
It remains to take the supremum over all packings \( \pi \) satisfying conditions of Definition 7.7, and the result of the lemma follows. \( \Box \)

In a similar way one can prove the following estimate of the functional \( A_p(f; \cdot : S) \) via the corresponding functional \( A_p(f; \cdot : S)_{L_p(\mu)} \), see Definitions 2.1 and 6.4. (Recall that \( A_p(f; t : S) = A_p(f; t : S)_{L_\infty(\mu)} \) provided \( S = \text{supp} \mu \).)

**Lemma 7.14** For every \( s \in (\frac{n}{p}, 1) \) the following inequality

\[
A_p(f; t : S)^p \leq C t^{e_p} \left( \int_0^t \left( \frac{A_p(f, u : S)_{L_p(\mu)}}{u^{s}} \right)^p \frac{du}{u} \right)^{\frac{1}{p}}, \quad 0 < t \leq 1,
\]

holds. Here \( C \) is a constant depending only on \( n, p, s \) and \( c_\mu \).

Now, we put \( s := (\frac{n}{p} + 1)/2 \) in Lemmas 7.13 and 7.14 and then apply the Hardy inequality (6.3). We obtain:

\[
\int_0^\varepsilon J_{p, \alpha}(f; t : S, \mu)^p \frac{dt}{t^{p+1}} \leq C \int_0^\varepsilon A_{p, \alpha}(f; t : S)^p_{L_p(\mu)} \frac{dt}{t^{p+1}}
\]

and

\[
\sup_{0 < t \leq \varepsilon} \frac{A_p(f; t : S)}{t} \leq C \sup_{0 < t \leq \varepsilon} \frac{A_p(f; t : S)_{L_p(\mu)}}{t}
\]

with \( C \) depending only on \( n, p \) and \( c_\mu \). Combining these inequalities with the result of Proposition 7.8, we obtain the required inequality (7.4).

Theorem 7.2 is proved. \( \Box \)

This theorem leads us to a constructive description of the trace space \( W^1_p(\mathbb{R}^n)|_S \). To formulate this result given \( x, y \in S \) and \( \alpha \in (0, 1] \) we put

\[
\rho_{\alpha, S}(x, y) := \inf \{ \text{diam} Q : Q \ni x, y, \alpha Q \subset \mathbb{R}^n \setminus S \}.
\]

(7.20)

We put \( \rho_{\alpha, S}(x, y) := +\infty \) whenever there no exists a cube \( Q \) such that \( x, y \in Q \) and \( \alpha Q \subset \mathbb{R}^n \setminus S \).

Clearly, the function \( \rho_S \) defined by (1.16) coincides with \( \rho_{\alpha, S} \) where \( \alpha = \frac{1}{15} \).

We prove the following generalization of part (ii) of Theorem 1.8.

**Theorem 7.15** Let \( p \in (n, \infty) \), \( \varepsilon > 0 \), and let \( \alpha \in (0, 1/14) \). Then for every function \( f \in C(S) \) the following equivalence

\[
\|f\|_{W^1_p(\mathbb{R}^n)|_S} \sim \|f\|_{L_p(\mu)} + \sup_{0 < t \leq \varepsilon} \left( \int \int \frac{|f(x) - f(y)|^p}{t^{p-n} \mu(Q(x, t)) \mu(Q(y, t))} \, d\mu(x) \, d\mu(y) \right)^{\frac{1}{p}}
\]

\[
+ \left( \int \int \frac{|f(x) - f(y)|^p}{\rho_{\alpha, S}(x, y)^{p-n} \mu(Q(x, \rho_{\alpha, S}(x, y)))^2} \, d\mu(x) \, d\mu(y) \right)^{\frac{1}{p}}
\]

holds with constants depending only on \( n, p, \varepsilon, \alpha \), and parameters \( C_\mu, C_\mu', C_\mu'' \).
The proof of the theorem relies on results of Sections 6 and 7 and the following two lemmas.

**Lemma 7.16** Let \( p \in (n, \infty), \alpha \in (0, 1) \) and \( \varepsilon \in (0, 1/4] \). Then for every function \( f \in C(S) \) we have

\[
\frac{1}{C} \int_0^{\alpha \varepsilon} \mathcal{A}_{p,2\alpha}(f; t : S)_{L_p(\mu)}^p \frac{dt}{t^{p+1}} \leq \int \int \frac{|f(x) - f(y)|^p}{\rho_{\alpha,S}(x,y)^{p-n} \mu(Q(x, \rho_{\alpha,S}(x,y)))^2} d\mu(x) d\mu(y) \leq C \int_0^{4\varepsilon} \mathcal{A}_{p,\alpha/8}(f; t : S)_{L_p(\mu)}^p \frac{dt}{t^{p+1}}.
\]

Here \( C \) is a positive constant depending only on \( n, p, \varepsilon, \alpha, \) and the doubling constant \( c_\mu \).

**Proof.** Let us prove the first inequality of the lemma. We put \( t_i := 2^{-i} \alpha \varepsilon, i = 0, 1, \ldots \).

We have

\[
I_1 := \int_0^{\alpha \varepsilon} \left( \mathcal{A}_{p,2\alpha}(f; t : S)_{L_p(\mu)} \right)^p \frac{dt}{t} = \sum_{i=0}^{\infty} \int_{t_{i+1}}^{t_i} \left( \mathcal{A}_{p,2\alpha}(f; t : S)_{L_p(\mu)} \right)^p \frac{dt}{t} \leq C \sum_{i=0}^{\infty} t_i^{-p} \mathcal{A}_{p,2\alpha}(f; t_i : S)_{L_p(\mu)}^p.
\]

By Definition 7.1, for every \( i = 0, 1, \ldots \) there exists a packing \( \pi_i = \{Q\} \) of \( 2\alpha \)-porous equal cubes centered in \( \partial S \) with diameter

\[
diam Q = \tau_i \leq t_i = 2^{-i} \alpha \varepsilon \quad (7.21)
\]

such that

\[
\mathcal{A}_{p,2\alpha}(f; t_i : S)_{L_p(\mu)} \leq 2 \sum_{Q \in \pi_i} |Q| \mathcal{E}(f; Q)_{L_p(\mu)}^p.
\]

We may assume that \( \{\tau_i : i = 0, 1, \ldots\} \) is non-decreasing sequence. By (6.6),

\[
I_1 \leq C \sum_{i=0}^{\infty} \sum_{Q \in \pi_i} t_i^{-p} t_{i+1} \mathcal{E}(f; Q)_{L_p(\mu)}^p = C \sum_{i=0}^{\infty} \sum_{Q \in \pi_i} t_i^{-p} t_{i+1} \mu(Q)^{\frac{p}{2}} \int_{Q \times Q} |f(x) - f(y)|^p d\mu(y) d\mu(x).
\]

We let \( H \) denote a subset of \( \mathbb{R}^n \times \mathbb{R}^n \) defined by the formula

\[
H := \cup \{Q \times Q : Q \in \pi_i, \ i = 0, 1, 2, \ldots\}.
\]

Also, given \((x, y) \in H, x \neq y\), we define a collection of cubes \( Q_{xy} \) by letting

\[
Q_{xy} := \{Q : \exists i \in \{0, 1, 2, \ldots\} \text{ such that } Q \in \pi_i \text{ and } Q \ni x, y\}.
\]
Then
\[ I_1 \leq C \int_{H} \int |f(x) - f(y)|^{p} g(x, y) d\mu(y) d\mu(x). \]

where
\[ g(x, y) := \sum \{ t_i^{-p} r_{Q_i}^{n} \mu(Q_i)^{-2} : Q_i \in \pi_i, Q_i \ni x, y, \ i = 0, 1, \ldots \}. \]

Fix \((x, y) \in H \) and \( Q \in \mathcal{Q}_{xy}; \) thus \( Q \in \pi_i \) for some \( i = 0, 1, \ldots \), and \( x, y \in Q \). Recall that \( x_{Q} \in S \), \( \text{diam} \ Q \leq 2^{-1} \varepsilon \), and \( Q \) is \( 2\alpha \)-porous. The latter implies the existence of a cube \( \tilde{Q} \subset Q \setminus S \) such that \( \text{diam} \ Q \leq (2\alpha)^{-1} \text{diam} \tilde{Q} \). Then
\[ Q \subset 2(2\alpha)^{-1} \tilde{Q} = \alpha^{-1} \tilde{Q}, \]
so that \( x, y \in \alpha^{-1} \tilde{Q} \). Therefore, by (7.20), \( \rho_{\alpha, S}(x, y) \leq \alpha^{-1} \text{diam} \tilde{Q} \). Note also that \( x_{Q} \in S \) and \( \tilde{Q} \subset Q \setminus S \) so that \( \text{diam} \tilde{Q} \leq \frac{1}{2} \text{diam} \ Q \). Hence,
\[ \rho_{\alpha, S}(x, y) \leq \alpha^{-1} \text{diam} \tilde{Q} \leq \frac{1}{2} \alpha^{-1} \text{diam} Q \leq 2^{-i-1} \varepsilon. \]...

In particular, \( \rho_{\alpha, S}(x, y) < \varepsilon \) so that
\[ I_1 \leq C \int_{\rho_{\alpha, S}(x, y) < \varepsilon} \int |f(x) - f(y)|^{p} g(x, y) d\mu(y) d\mu(x). \]

Let us estimate \( g(x, y) \). Since \( \text{diam} \ Q_i \geq \|x - y\| > 0 \) for each \( Q_i \ni x, y \),
\[ i_0 := \sup \{ i : \exists Q_i \in \pi_i, Q_i \ni x, y \} < \infty. \]

In addition, for each \( Q_i \in \pi_i, Q_i \ni x, y \), we have \( \text{diam} Q_i \geq \text{diam} Q_{i_0} \) and \( Q_i \cap Q_{i_0} \neq \emptyset \) so that \( 2Q_i \ni Q_{i_0} \). On the other hand, since \( \varepsilon \leq 1/4 \),
\[ r_{Q_i} \leq t_i/2 = 2^{-i-1} \alpha \varepsilon \leq \alpha \varepsilon /2 \leq 1/2, \ i = 1, 2, \ldots, \]
so that, by the doubling condition, \( \mu(Q_{i_0}) \leq \mu(2Q_i) \leq C \mu(Q_i) \). Hence, by (7.21),
\[ g(x, y) := \sum \{ t_i^{-p} r_{Q_i}^{n} \mu(Q_i)^{-2} : Q_i \in \pi_i, Q_i \ni x, y, \ i = 0, 1, \ldots \} \]
\[ \leq C \mu(Q_{i_0})^{-2} \sum \{ t_i^{-p} r_{Q_i}^{n} : Q_i \in \pi_i, Q_i \ni x, y, \ i = 0, 1, \ldots \} \]
\[ \leq C \mu(Q_{i_0})^{-2} \sum \{ t_i^{-p} t_i^{n} : Q_i \in \pi_i, Q_i \ni x, y, \ i = 0, 1, \ldots \} \]
\[ \leq C \mu(Q_{i_0})^{-2} \sum \{ t_i^{n-p} : i \leq i_0 \} \]
\[ \leq C \mu(Q_{i_0})^{-2} \sum \mu(Q_i)^{n-p} \leq C \mu(Q_{i_0})^{-2} r_{Q_{i_0}}^{n-p} \]

Observe that, by (7.22),
\[ \rho_{\alpha, S}(x, y) \leq \frac{1}{2} \alpha^{-1} \text{diam} Q_{i_0} = r_{Q_{i_0}} / \alpha. \]

Since \( Q_{i_0} \ni x \), we obtain
\[ Q(x, \rho_{\alpha, S}(x, y)) \subset (1 + \alpha^{-1})Q_{i_0} \subset 2\alpha^{-1}Q_{i_0}. \]
But
\[ \text{diam}(2\alpha^{-1}Q_{i_0}) = 2\alpha^{-1} \text{diam} Q_{i_0} \leq 2\alpha^{-1}t_{i_0} \leq 2\varepsilon \leq 1, \]
so that by the doubling condition
\[ \mu(Q(x, \rho, S(x, y))) \leq \mu(2\alpha^{-1}Q_{i_0}) \leq C\mu(Q_{i_0}). \]
Combining this estimate with (7.23), we obtain
\[ g(x, y) \leq Cr^{n-p} \mu(Q_{i_0})^{-2} \leq C\rho \mu(Q(x, \rho, S(x, y)))^{-2} \]
proving that
\[ I_1 \leq C \int \int \frac{|f(x) - f(y)|^p}{\rho_{\alpha,S}(x, y)^{p-n} \mu(Q(x, \rho, S(x, y)))^2} d\mu(x) d\mu(y). \]

Prove the second inequality of the lemma. We have
\[ I_2 := \int \int \frac{|f(x) - f(y)|^p}{\rho_{\alpha,S}(x, y)^{p-n} \mu(Q(x, \rho, S(x, y)))^2} d\mu(x) d\mu(y) \]
\[ = \sum_{i=0}^{\infty} \int \int_{t_{i+1} \leq \rho_{\alpha,S}(x, y) < t_i} \frac{|f(x) - f(y)|^p}{\rho_{\alpha,S}(x, y)^{p-n} \mu(Q(x, \rho, S(x, y)))^2} d\mu(x) d\mu(y), \]
where \( t_i := 2^{-i} \varepsilon, i = 0, 1, \ldots \). Put
\[ A_i := \int \int_{t_{i+1} \leq \rho_{\alpha,S}(x, y) < t_i} \frac{|f(x) - f(y)|^p}{\rho_{\alpha,S}(x, y)^{p-n} \mu(Q(x, \rho, S(x, y)))^2} d\mu(x) d\mu(y) \]
and prove that
\[ A_i \leq Ct_i^{p-1} A_{p,\alpha/4}(f; 2t_i : S)^p_{L_p(\mu)}. \quad (7.24) \]

To this end we let \( S_i \) denote the projection of the set
\[ \{(x, y) \in S \times S : t_{i+1} \leq \rho_{\alpha,S}(x, y) < t_i\} \]
ono onto \( S \). Thus
\[ S_i := \{x \in S : \exists x' \in S \text{ such that } t_{i+1} \leq \rho_{\alpha,S}(x, x') < t_i\}, \quad i = 0, 1, \ldots \]
Then, by (7.20), for each \( x \in S \) there exists \( x' \in S \) and a cube \( Q(x) \) with
\[ t_{i+1} \leq \text{diam} Q(x) < t_i \]
such that \( \alpha Q(x) \subset \mathbb{R}^n \setminus S \) and
\[ x, x' \in Q(x). \]
Put $K(x) := Q(x, t_i)$. Since $x \in Q(x)$ and $\text{diam} Q(x) < t_i = \text{diam} K(x)$, we have $K(x) \supset Q(x)$. In addition, since $\alpha Q(x) \subset Q(x) \setminus S \subset K(x) \setminus S$ and
\[
\text{diam}(\alpha Q(x)) \geq \alpha t_{i+1} = \frac{\alpha}{4} \text{diam} K(x),
\]
by Definition 2.2, $K(x)$ is an $\alpha/4$-porous cube (with respect to $S$).

By the Besicovitch covering theorem, see, e.g. [G], there exists a countable subfamily $V_i := \{K_j = Q(x_j, t_i) : j \in J\}$ of the family of cubes $\{K(x) : x \in S\}$ which covers $S$ and has the covering multiplicity $M_{V_i} \leq C(n)$. Consider a cube $K_j \in V_i$. Prove that if $x \in K_j$ and $\rho_{\alpha, S}(x, y) < t_i$, then
\[
y \in 2K_j.
\]
(7.25)

In fact, by (7.20), there exists a cube $Q$ of $\text{diam} Q \leq t_i$ such that $x, y \in Q$ and $\alpha Q \subset \mathbb{R}^n \setminus S$. Hence $\|x - y\| \leq \text{diam} Q \leq t_i$ so that
\[
\|x_j - y\| \leq \|x_j - x\| + \|x - y\| \leq t_i + t_i = 2t_i,
\]
proving (7.25).

Also note that for each $x \in K_j$ we have
\[
2K_j \subset Q(x, 2 \text{diam} K_j) = Q(x, 4t_i) = Q(x, 8t_{i+1}).
\]
But $8t_{i+1} \leq 4\varepsilon \leq 1$ provided $\varepsilon \leq 1/4$ so that, by the doubling condition, $\mu(2K_j) \leq C\mu(Q(x, t_{i+1}))$.

Now we have
\[
A_i \leq \sum_{j \in J} \int_{K_j} \int_{t_{i+1} \leq \rho_{\alpha, S}(x,y) < t_i} \frac{|f(x) - f(y)|^p}{\rho_{\alpha, S}(x, y)^{p-n} \mu(Q(x, \rho_{\alpha, S}(x, y)))^2} \, d\mu(y) \, d\mu(x)
\]
\[
\leq \sum_{j \in J} \int_{K_j} \int_{t_{i+1} \leq \rho_{\alpha, S}(x,y) < t_i} \frac{|f(x) - f(y)|^p}{\rho_{i+1}(Q(x, t_{i+1}))^2} \, d\mu(y) \, d\mu(x)
\]
\[
\leq C \sum_{j \in J} t_i^{n-p} \mu(2K_j)^{-2} \int_{K_j} \int_{t_{i+1} \leq \rho_{\alpha, S}(x,y) < t_i} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x).
\]

Hence, by (7.25), we have
\[
A_i \leq C \sum_{j \in J} t_i^{n-p} \mu(2K_j)^{-2} \int_{2K_j} \int_{2K_j} |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y).
\]

This and (6.6) yield
\[
A_i \leq C \sum_{j \in J} t_i^{n-p} |2K_j| \frac{1}{\mu(2K_j)^{-2}} \int_{2K_j \times 2K_j} |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y)
\]
\[
\leq C t_i^{n-p} \sum_{j \in J} |2K_j| \mathcal{E}(f; 2K_j)^p \mu_p(\mu).
\]

78
Recall that $V_i$ is a collection of equal $\alpha/4$-porous cubes of diameter $t_i$ centered in $S$ with the covering multiplicity $M_{V_i} \leq C(n)$. Then the family $2V_i := \{2K_j : K_j \in V_i\}$ is a collection of equal $\alpha/8$-porous cubes of diameter $2t_i$ with centers in $S$.

It can be readily seen that the covering multiplicity of the family $2V_i$ is also bounded by a constant $C = C(n)$. Then, by Theorem 3.3, $2V_i$ can be partitioned into at most $N(n)$ families of pairwise disjoint cubes which allows us to assume that the cubes of the collection $2V_i$ are pairwise disjoint. Hence,

$$A_i \leq C t_i^{-p} \sup_{\pi} \sum_{K \in \pi} |K| \mathcal{E}(f; K)^p_{L_p(\mu)}$$

where $\pi$ runs over all finite subfamilies of $2V_i$. Since every such $\pi$ is a packing, by Definition 7.1, we have

$$A_i \leq C t_i^{-p} A_{p,\alpha/8}(f; 2t_i : S)^p_{L_p(\mu)}$$

proving (7.24).

Finally,

$$I_2 = \sum_{i=0}^{\infty} A_i \leq C \sum_{i=0}^{\infty} \left( \frac{A_{p,\alpha/8}(f; u_i : S)^{p}_{L_p(\mu)}}{u_i} \right)^p$$

where $u_i := 2t_i$, $i = 0, 1, \ldots$.

Since $A_{p,\alpha/8}(f; u : S)^p_{L_p(\mu)}$ is a non-decreasing function, this implies the required inequality

$$I_2 \leq C \int_{0}^{4\varepsilon} A_{p,\alpha/8}(f; u : S)^p_{L_p(\mu)} \frac{du}{u^{p+1}},$$

proving the second inequality of the lemma. \qed

Given a function $f \in C(S)$, $\alpha \in (0, 1)$ and $\varepsilon > 0$ we put

$$J_1(f; t) := \left( \int \int \frac{|f(x) - f(y)|^p}{\rho^{p-n}(Q(x, t))^p} \, d\mu(x) \, d\mu(y) \right)^{\frac{1}{p}}, \quad t > 0, \quad (7.26)$$

and

$$J_2(f; \alpha, \varepsilon) := \left( \int \int \frac{|f(x) - f(y)|^p}{\rho_{\alpha,s}(x, y)^{p-n}\mu(Q(x, \rho_{\alpha,s}(x, y)))} \, d\mu(x) \, d\mu(y) \right)^{\frac{1}{p}}. \quad (7.27)$$

**Lemma 7.17** Let $p \in (n, \infty)$, $\alpha \in (0, 1)$, and let $0 < \varepsilon' < \varepsilon$. Then for every function $f \in C(S)$ we have

$$J_1(f; t) \leq C \|f\|_{L_p(\mu)}, \quad t \in (\varepsilon', \varepsilon), \quad (7.28)$$

and

$$J_2(f; \alpha, \varepsilon') \leq J_2(f; \alpha, \varepsilon) \leq C \{J_2(f; \alpha, \varepsilon') + \|f\|_{L_p(\mu)}\}. \quad (7.29)$$

Here $C$ is a positive constant depending only on $n, p, \varepsilon, \varepsilon'$, and the constants $C_\mu, C_\mu', C_\mu''$. 

79
Proof. Put \( \varepsilon := \min \{1, \varepsilon' \} \). Then, by the doubling condition,
\[
\mu(Q(x, 1)) \leq C(\varepsilon', n)\mu(Q(x, \varepsilon)) \leq C\mu(Q(x, \varepsilon')), \quad x \in S.
\]
In turn, by (1.13), \( \mu(Q(x, 1)) \geq C' \mu \) so that
\[
\mu(Q(x, \varepsilon'))^{-1} \leq C/C'.
\]
Hence, for every \( t > \varepsilon \) we have
\[
J_1(f; t)^p := t^{n-p} \iint_{\|x-y\|<t} \frac{|f(x) - f(y)|^p}{\mu(Q(x, t))^2} \, d\mu(x) \, d\mu(y)
\leq (\varepsilon')^{n-p} \iint_{\|x-y\|<\varepsilon} \frac{|f(x) - f(y)|^p}{\mu(Q(x, \varepsilon'))^2} \, d\mu(x) \, d\mu(y)
\leq C \iint_{\|x-y\|<\varepsilon} |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y)
\leq C \iint_{\|x-y\|<\varepsilon} (|f(x)|^p + |f(y)|^p) \, d\mu(x) \, d\mu(y).
\]
But
\[
I_1 := \iint_{\|x-y\|<\varepsilon} (|f(x)|^p + |f(y)|^p) \, d\mu(x) \, d\mu(y) = 2 \iint_{\|x-y\|<\varepsilon} |f(x)|^p \, d\mu(y) \, d\mu(x)
\leq 2 \int_S \int_{y \in Q(x, \varepsilon)} |f(x)|^p \, d\mu(y) \, d\mu(x) = 2 \int_S \mu(Q(x, \varepsilon)) |f(x)|^p \, d\mu(x).
\]
If \( \varepsilon \leq 1 \), then, by (1.13), we have \( \mu(Q(x, \varepsilon)) \leq \mu(Q(x, 1)) \leq C'' \mu \). If \( \varepsilon > 1 \), then we can cover the set \( Q(x, \varepsilon) \cap S \) by a finite family of cubes \( \pi = \{Q(x_i, 1) : x_i \in Q(x, \varepsilon) \cap S, i = 1, ..., N(n, \varepsilon)\} \), so that
\[
\mu(Q(x, \varepsilon)) \leq \sum_{i=1}^{N(n, \varepsilon)} \mu(Q(x_i, 1)) \leq N(n, \varepsilon)C''.
\]
Hence,
\[
I_1 \leq 2 \int_S \mu(Q(x, \varepsilon)) |f(x)|^p \, d\mu(x) \leq C\|f\|_{L^p(\mu)}^p,
\]
so that
\[
J_1(f; t)^p \leq C I_1 \leq C\|f\|_{L^p(\mu)}^p, \quad t \in (\varepsilon', \varepsilon),
\]
proving inequality (7.28).
Prove inequality (7.29). The first inequality of (7.29) is trivial. Let us prove the second one.

We have \( J_2(f; \alpha, \varepsilon)^p = J_2(f; \alpha, \varepsilon')^p + I_2 \), where

\[
I_2 := \int_0^\varepsilon \int_{\rho \cdot \varepsilon} \frac{|f(x) - f(y)|^p}{\rho \cdot \varepsilon} \, d\mu(x) \, d\mu(y).
\]

By (7.30),

\[
I_2 \leq \int_0^\varepsilon \int_{\rho \cdot \varepsilon} \frac{|f(x) - f(y)|^p}{|\varepsilon'|^{p-n} \mu(Q(x, \varepsilon'))^2} \, d\mu(x) \, d\mu(y)
\leq C \int_0^\varepsilon \int_{\rho \cdot \varepsilon} |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y) \leq C \int_\rho^{\varepsilon} |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y).
\]

By (7.29), \( \rho \cdot \varepsilon \geq \|x - y\| \), so that

\[
I_2 \leq C \int_0^\varepsilon \int_{\|x - y\|} |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y) \leq C \int_0^\varepsilon \int_{\|x - y\|} (|f(x)|^p + |f(y)|^p) \, d\mu(x) \, d\mu(y) = CI_1.
\]

Hence, by (7.31), \( I_2 \leq CI_1 \leq C \|f\|_{L_p(\mu)}^p \), proving that

\[
J_2(f; \alpha, \varepsilon)^p \leq J_2(f; \alpha, \varepsilon') + C \|f\|_{L_p(\mu)}^p.
\]

The lemma is proved. \( \Box \)

**Proof of Theorem 7.15.** The proof easily follows from Theorem 7.2, Proposition 6.6 and Lemma 7.16. In fact, given a function \( f \in C(S) \), \( \alpha \in (0, 1) \) and \( \varepsilon > 0 \) we put

\[
J_3(\alpha, \varepsilon) := \left\{ \int_0^\varepsilon A_p(\varepsilon, t: S)^p \, dt \right\}^{\frac{1}{p+1}}.
\]

Then, by Theorem 7.2,

\[
\|f\|_{W_p^1(R^n)} \sim \|f\|_{L_p(\mu)} + \sup_{0 < t < \varepsilon} \frac{A_p(f; t: S)^{L_p(\mu)}}{t} + J_3(\beta, \varepsilon)
\]

provided \( p \in (n, \infty) \), \( \beta \in (0, 1/7) \) and \( \varepsilon > 0 \). In turn, by Proposition 6.6, for every \( t \in (0, 1/4] \) we have

\[
\frac{1}{C} A_p(f; t/4: S)^{L_p(\mu)} / t \leq J_1(f; t) \leq CA_p(f; 4t: S)^{L_p(\mu)} / t,
\]

where \( J_1(f; t) \) is defined by (7.26). Taking the supremum in this inequality over all \( t \in (0, \varepsilon) \) where \( 0 < \varepsilon \leq 1/4 \), we obtain

\[
C^{-1} \sup_{0 < t < \varepsilon} A_p(f; t/4: S)^{L_p(\mu)} / t \leq \sup_{0 < t < \varepsilon} J_1(f; t) \leq C \sup_{0 < t < \varepsilon} A_p(f; 4t: S)^{L_p(\mu)} / t,
\]
so that
\[C^{-1} \sup_{0 < t < \varepsilon/4} A_p(f; t : S)_{L_p(\mu)} / t \leq \sup_{0 < t < \varepsilon} J_1(f; t) \leq C \sup_{0 < t < 4\varepsilon} A_p(f; t : S)_{L_p(\mu)} / t.\]
Applying Lemma 6.5, we obtain
\[\|f\|_{L_p(\mu)} + \sup_{0 < t < \varepsilon} A_p(f; t : S)_{L_p(\mu)} / t \sim \|f\|_{L_p(\mu)} + \sup_{0 < t < \varepsilon} J_1(f; t), \quad \varepsilon \in (0, 1/4]. \quad (7.33)\]
In turn, by Lemma 7.16, we have
\[C^{-1} J_3(2\alpha, \alpha\varepsilon) \leq J_2(f; \alpha, \varepsilon) \leq C J_3(\alpha/8, 4\varepsilon). \quad (7.34)\]
(Recall that \(J_2(f; \alpha, \varepsilon)\) is defined by (7.27).) On the other hand, by (7.3) and Lemma 6.5,
\[A_{p, \alpha}(f; t : S)_{L_p(\mu)} \leq A_p(f; t : S)_{L_p(\mu)} \leq C(1 + t^{\beta_p})\|f\|_{L_p(\mu)},\]
so that for every \(\alpha \in (0, 1)\) we have
\[J_3(\alpha/8, 4\varepsilon)^p = J_3(\alpha/8, \varepsilon)^p + \int_{\varepsilon}^{4\varepsilon} A_{p, \alpha/8}(f; t : S)_{L_p(\mu)} \frac{dt}{t^{p+1}} \leq J_3(\alpha/8, \varepsilon)^p + C\|f\|_{L_p(\mu)},\]
and
\[J_3(\alpha, \varepsilon)^p = J_3(\alpha, \alpha\varepsilon)^p + \int_{\alpha\varepsilon}^{\varepsilon} A_{p, \alpha}(f; t : S)_{L_p(\mu)} \frac{dt}{t^{p+1}} \leq J_3(\alpha, \alpha\varepsilon)^p + C\|f\|_{L_p(\mu)}.\]
These inequalities and (7.34) imply
\[C^{-1}(\|f\|_{L_p(\mu)} + J_3(2\alpha, \varepsilon)) \leq \|f\|_{L_p(\mu)} + J_2(f; \alpha, \varepsilon) \leq C(\|f\|_{L_p(\mu)} + J_3(\alpha/8, \varepsilon)). \quad (7.35)\]
Observe that \(\alpha/8, 2\alpha \in (0, 1/7)\), so that, by (7.32), we have
\[\|f\|_{W^2_p(\mathbb{R}^n)} \sim \|f\|_{L_p(\mu)} + \sup_{0 < t < \varepsilon} A_p(f; t : S)_{L_p(\mu)} / t + J_3(2\alpha, \varepsilon) \sim \|f\|_{L_p(\mu)} + \sup_{0 < t < \varepsilon} A_p(f; t : S)_{L_p(\mu)} / t + J_3(\alpha/8, \varepsilon).\]
These equivalences, (7.33) and inequalities (7.35) yield
\[\|f\|_{W^2_p(\mathbb{R}^n)} \sim \|f\|_{L_p(\mu)} + \sup_{0 < t < \varepsilon} J_1(f; t) + J_2(f; \alpha, \varepsilon) \quad (7.36)\]
proving Theorem 7.15 for \(\varepsilon \in (0, 1/4].\)
Consider the case \(\varepsilon > \varepsilon' := 1/4.\) By Lemma 7.17,
\[\sup_{0 < t \leq \varepsilon} J_1(f; t) \leq C \{ \sup_{0 < t \leq 1/4} J_1(f; t) + \|f\|_{L_p(\mu)} \},\]
and
\[ J_2(f; \alpha, 1/4) \leq J_2(f; \alpha, \varepsilon) \leq C\{J_2(f; \alpha, 1/4) + \|f\|_{L_p(\mu)}\}, \]
proving that equivalence (7.36) holds for \( \varepsilon > 1/4 \) as well.

Theorem 7.15 is completely proved.

This theorem and Proposition 5.8 imply the following generalization of part (i) of Theorem 1.8.

**Theorem 7.18** Let \( p \in (n, \infty) \), \( \varepsilon > 0 \) and let \( \alpha \in (0, 1/14) \). Let \( \sigma \) be a measure supported on \( \partial S \) which satisfies conditions (1.14) and (1.15), and let \( \Omega := \text{int}(S) \) be the interior of \( S \). Then for every \( f \in C(S) \) the following equivalence

\[
\|f\|_{W^1_p(R^n), S} \sim \|f\|_{\Omega} \|W^1_p(\Omega) + \sup_{0 < t \leq \varepsilon} \left( \int \int_{|x-y| < t} \frac{|f(x) - f(y)|^p}{t^{p-n}\sigma(Q(x, t))^2} \, d\sigma(x) \, d\sigma(y) \right) \frac{1}{p}
\]

holds with constants depending only on \( n, p, \) and \( C_\sigma, C'_\sigma, C''_\sigma \).

**Proof.** Let us apply Theorem 7.15 to the set \( \partial S \) and the measure \( \sigma \) supported on \( \partial S \). By this theorem, the quantity \( \|f\|_{\partial S} \|W^1_p(R^n)_{\partial S} \) is equivalent to the sum of the last three terms on the right-hand side of the equivalence of Theorem 7.18. On the other hand, by Proposition 5.8,

\[
\|f\|_{W^1_p(R^n), S} \sim \|f\|_{\Omega} \|W^1_p(\Omega) + \|f\|_{\partial S} \|W^1_p(R^n)_{\partial S},
\]

and the proof is finished. \( \Box \)

**Remark 7.19** The theorem remains true whenever the function \( \rho_{\alpha, \partial S} \) in its formulation is replaced with the (bigger) function \( \rho_{\alpha, S} \). In fact, Remark 5.9 shows that we can prove Theorem 7.18 basing on the remark’s equivalence rather than on Proposition 5.8. This allows us to accomplish the proof using the functional \( A_{p, \alpha}(f; \cdot : S) \) rather than \( A_{p, \alpha}(f; \cdot : \partial S) \) so that, by Lemma 7.16, the proof can be continued with the function \( \rho_{\alpha, S} \) rather than \( \rho_{\alpha, \partial S} \).

The next result presents a generalization of Theorem 1.6.

**Theorem 7.20** Let \( p \in (n, \infty) \), \( \alpha \in (0, 1/14) \), \( \varepsilon > 0 \), and let \( \sigma \) be a measure supported on \( \partial S \) which satisfies conditions (1.14) and (1.15). Then for every \( f \in C(S) \) we have

\[
\|f\|_{W^1_p(R^n), S} \sim \|f\|_{L_p(\mu)} + \sup_{0 < t \leq \varepsilon} \left( \int \int_{|x-y| < t} \frac{|f(x) - f(y)|^p}{t^{p-n}\mu(Q(x, t))^2} \, d\mu(x) \, d\mu(y) \right) \frac{1}{p} \]

\[
+ \left( \int \int_{\rho_{\alpha, \partial S}(x,y) < \varepsilon} \frac{|f(x) - f(y)|^p}{\rho_{\alpha, \partial S}(x,y)^{p-n}\sigma(Q(x, \rho_{\alpha, \partial S}(x,y)))^2} \, d\sigma(x) \, d\sigma(y) \right) \frac{1}{p}. \]
The constants of this equivalence depend only on $n, p, \varepsilon, \alpha, C_{\mu}, C'_{\mu}, C''_{\mu}$ and $C_{\sigma}, C'_{\sigma}, C''_{\sigma}$.

We will be needed the following generalization of the equivalence (2.2).

**Proposition 7.21** Let $\Omega$ be an open subset of $\mathbb{R}^n$ and let $1 < p < \infty$. Then for every function $f \in L^1(p)(\Omega)$ we have

$$
\| f \|_{L^1(p)(\Omega)} \leq C \sup_\pi \left\{ \sum_{Q \in \pi} r_Q^{n-p} E_1(f; Q)^p_{L^p} \right\}^{\frac{1}{p}}
$$

(7.37)

where the supremum is taken over all packings $\pi$ of equal cubes lying in $\Omega$. Here the constant $C$ depends only on $n$ and $p$.

**Proof.** Recall that the quantity $E_1(f; Q)_{L^p}$ is defined by (2.5).

It suffices to show that, for any packing $\pi' = \{K\}$ of cubes $K \subset \Omega$, there exists a packing $\pi = \{Q\}$ of equal cubes $Q \subset \Omega$ such that

$$
\sum_{K \in \pi'} \int_K \| \nabla f(x) \|^p dx \leq C \sum_{Q \in \pi} r_Q^{n-p} E_1(f; Q)^p_{L^p}.
$$

(7.38)

Note that, the proof of equivalence (1.19) given in [Z] can be easily extended to the case of an arbitrary cube $K \subset \mathbb{R}^n$. In other words, for every $f \in L^1(p)(\text{int}(K)), 1 < p < \infty$, we have

$$
\left( \int_K \| \nabla f(x) \|^p dx \right)^{\frac{1}{p}} \sim \sup_{0 < t \leq \text{diam } K} \omega(f; t)_{L^p(K)}/t
$$

(7.39)

where $\omega(f; \cdot)_{L^p(K)}$ is the modulus of smoothness in $L^p(K)$:

$$
\omega(f; t)_{L^p(K)} := \sup_{\|h\|<t} \left( \int_{\{x \in K : x+h \in K\}} |f(x+h) - f(x)|^p dx \right)^{\frac{1}{p}}.
$$

It is also known that the function $\omega(f; t)_{L^p(K)}/t$ is quasi-monotone; the reader can easily prove that

$$
\omega(f; T)_{L^p(K)}/T \leq 2\omega(f; t)_{L^p(K)}/t, \quad 0 < t < T.
$$

(7.40)

Finally, by Brudnyi’s theorem [Br1, Br3],

$$
\omega(f; t)^p_{L^p(K)} \sim \sup_\pi \sum_{Q \in \pi} |Q| E_1(f; Q)_{L^p}^p, \quad 0 < t \leq \text{diam } K,
$$

(7.41)

where $\pi = \{Q\}$ runs over all packings $\pi$ of equal cubes $Q \subset K$ of diameter $\text{diam } Q = t$. 

84
The results cited above easily imply inequality (7.38). In fact, in view of (7.39), for every cube $K \in \pi'$ there exists $t_K \in (0, \text{diam } K]$ such that
\[
\left( \int_K \| \nabla f(x) \|^p \, dx \right)^{\frac{1}{p}} \leq C \omega(f; t_K) L_p(K)/t_K
\]
with $C = C(n, p)$. Let $\tilde{t} := \min \{ t_K : K \in \pi' \}$. Then, by (7.40),
\[
\left( \int_K \| \nabla f(x) \|^p \, dx \right)^{\frac{1}{p}} \leq C \omega(f; \tilde{t}) L_p(K)/\tilde{t}.
\] (7.42)

On the other hand, by (7.41), there exists a packing $\pi_K = \{Q\}$ of equal cubes lying in $K$ of diameter $\text{diam } Q = 2r_Q = \tilde{t}$ such that
\[
\omega(f; \tilde{t}) L_p(K) \leq C \sum_{Q \in \pi_K} |Q| E_1(f; Q) L_p.
\]
This inequality and (7.42) yield
\[
\int_K \| \nabla f(x) \|^p \, dx \leq \left( \frac{C}{\tilde{t}^p} \right) \sum_{Q \in \pi_K} |Q| E_1(f; Q) L_p \leq C \sum_{Q \in \pi_K} r_Q^{n-p} E_1(f; Q) L_p.
\]
(Recall that $\tilde{t} = 2r_Q$, $Q \in \pi_K$.)

It remains to put $\pi := \cup \{ \pi_K : K \in \pi' \}$ and the required inequality (7.38) follows.

□

**Proof Theorem 7.20.** We let $\| f \|_S$ denote the right-hand side of theorem’s equivalence. Then, by Theorem 7.18 and Theorem 7.15, $\| f \|_S \leq C \| f \|_{W^1_p(\mathbb{R}^n)}$.

Prove the converse inequality. Given a function $f \in C(S)$ and a Borel measure $\nu$ on $\mathbb{R}^n$ we put
\[
J(f; t : \nu) := \left( \int \int_{\| x-y \| < t} \frac{|f(x) - f(y)|^p}{t^{p-n} \nu(Q(x, t))^2} \, d\nu(x) \, d\nu(y) \right)^{\frac{1}{p}}, \quad t > 0.
\]

Then, by Proposition 6.6,
\[
J(f; t : \sigma) \leq C^{-1} A_p(f; 4t : \partial S) L_p(\sigma) \leq C t^{-1} A_p(f; 4t : S), \quad t \in (0, 1/4].
\]
Again, by Proposition 6.6,
\[
A_p(f; t : S) \leq C t J(f; 4t : \mu), \quad t \in (0, 1/16],
\] (7.43)
so that
\[
J(f; t : \sigma) \leq C t J(f; 16t : \mu), \quad t \in (0, 1/64].
\] (7.44)

85
Let $\Omega := int(S)$. Let us estimate $\|f\|_{W^1_p(\Omega)}$ via $\|f\|_{L_p(\mu)}$ and the $A_p$-functional of $f$. By Lemma 7.4,

$$\|f\|_{L_p(\Omega)} \leq \|f\|_{L_p(\sigma)} \leq C\{\|f\|_{L_p(\mu)} + A_p(f; \varepsilon : S)\}.$$  \hfill (7.45)

By Proposition 7.21, we have

$$\|f\|_{L_p(\Omega)} \leq C\left\{\sup_{0 < t \leq \varepsilon} A_p(f; t : S)/t + \sup_\pi \left(\sum_{Q \in \pi} r_Q^{n-p} \mathcal{E}_1(f; Q)_{L_p}^p\right)^{\frac{1}{p}}\right\}$$  \hfill (7.46)

where the supremum is taken over all packings $\pi = \{Q\}$ of equal cubes of diameter $\text{diam} \ Q = 2r_Q \geq \varepsilon$ lying in $\Omega$. Since

$$|Q|\mathcal{E}_1(f; Q)_{L_p}^p \leq |Q| \left(\frac{1}{Q} \int_Q |f|^p \, dx\right) = \int_Q |f|^p \, dx,$$

see (2.5), for every such a packing $\pi$ we have

$$\sum_{Q \in \pi} r_Q^{n-p} \mathcal{E}_1(f; Q)_{L_p}^p \leq 2^p \varepsilon^{-p} \sum_{Q \in \pi} |Q|\mathcal{E}_1(f; Q)_{L_p}^p \leq C \sum_{Q \in \pi} \int_Q |f|^p \, dx \leq C\|f\|_{L_p(\mu)}^p.$$

Combining this inequality with (7.46) and (7.45), we obtain

$$\|f\|_{L_p(\Omega)} \leq C\left\{\sup_{0 < t \leq \varepsilon} A_p(f; t : S)/t + \|f\|_{L_p(\Omega)}\right\}$$

$$\leq C\left\{\sup_{0 < t \leq \varepsilon} A_p(f; t : S)/t + \|f\|_{L_p(\mu)} + A_p(f; \varepsilon : S)\right\}$$

$$\leq C\left\{\sup_{0 < t \leq \varepsilon} A_p(f; t : S)/t + \|f\|_{L_p(\mu)}\right\}.$$

Hence

$$\|f\|_{W^1_p(\Omega)} = \|f\|_{L_p(\Omega)} + \|f\|_{L^1_p(\Omega)} \leq C\{\|f\|_{L_p(\mu)} + \sup_{0 < t \leq \varepsilon} A_p(f; t : S)/t\},$$

so that, by (7.43),

$$\|f\|_{W^1_p(\Omega)} \leq C\{\|f\|_{L_p(\mu)} + \sup_{0 < t \leq \varepsilon} J(f; t : \mu)\},$$

provided $\varepsilon \in (0, 1/64)$. This and inequality (7.44) yield

$$\|f\|_{W^1_p(\Omega)} + \sup_{0 < t \leq \varepsilon} J(f; t : \sigma) \leq C\{\|f\|_{L_p(\mu)} + \sup_{0 < t \leq \varepsilon} J(f; t : \mu)\} \leq C\|f\|_{S}^u.$$

Finally, by Theorem 7.18,

$$\|f\|_{W^1_p(\mathbb{R}^n)_S} \leq C\{\|f\|_{W^1_p(\Omega)} + \sup_{0 < t \leq \varepsilon} J(f; t : \sigma) + \|f\|_{S}^u\} \leq C\|f\|_{S}^u,$$

proving the theorem. \hfill $\square$

86
Remark 7.22 The function $\rho_{\partial S}$ in the equivalence of Theorem 7.20 can be replaced with the (bigger) function $\rho_S(x, y)$. In fact, such a replacement us possible in Theorem 7.18, see Remark 7.19. Since the proof of Theorem 7.20 relies on the result of Theorem 7.18, the same replacement can be made in Theorem 7.20 as well.

Proof of Theorem 1.10 and Theorem 1.11.
We let $\mathcal{B}(\mathbb{R}^n)$ denote the class of all closed subsets of $\mathbb{R}^n$ satisfying the ball condition, see Definition 1.9. It will be convenient for us to use the following equivalent definition of the class $\mathcal{B}(\mathbb{R}^n)$:

We say that a closed set $A \in \mathcal{B}(\mathbb{R}^n)$ if there exists a constant $\beta_A > 0$ such that the following is true: For every cube $Q$ of diameter at most 1 centered in $A$ there exists a cube $Q' \subset Q \setminus A$ such that $\text{diam} \, Q \leq \beta_A \text{diam} \, Q'$.

The notion of a set satisfying “the ball condition” is well known in the literature and has been used in many papers (in equivalent forms and with different names), see, e.g. [HM, JK, MV]. Observe that a set $A \in \mathcal{B}(\mathbb{R}^n)$ satisfies the ball condition if and only if every cube $Q$ of $\text{diam} \, Q \leq 1$ centered in $A$ is $1/\beta_A$-porous (with respect to $A$), see Definition 2.2. Also, it can be readily seen that for every set $A \in \mathcal{B}(\mathbb{R}^n)$ and every $\alpha \in (0, 1/2\beta_A]$ the following inequality

$$\|x - y\| \leq \rho_{\alpha, A}(x, y) \leq 4\|x - y\|, \quad x, y \in A, \quad \|x - y\| \leq 1/4,$$

holds. (See (7.20) for the definition of $\rho_{\alpha, A}$.)

The next theorem presents a slight generalization of Theorem 1.11.

Theorem 7.23 Let $p \in (n, \infty)$, $\varepsilon > 0$, and let $S$ be a closed subset of $\mathbb{R}^n$. Assume that $\text{int}(S) = \emptyset$ and the ball condition is satisfied. Then for every $f \in C(S)$

$$\|f\|_{W^1_p(\mathbb{R}^n)_{|S}} \sim \|f\|_{L_p(\mu)} + \left( \int_{R^+} \int \frac{|f(x) - f(y)|^p}{\|x - y\|^{p-n} \mu(Q(x, \|x - y\|))} d\mu(x) d\mu(y) \right)^{1/p}$$

with constants of equivalence depending only on $n, p, \varepsilon, C, C', C'',$ and $\beta_S$.

Proof. Since the interior of $S$ is empty, $S = \partial S$ so that one can put $\mu = \sigma$. Since $S$ satisfies the ball condition, every cube $Q$ of $\text{diam} \, Q \leq 1$ centered in $S$ is $\alpha := 1/\beta_S$-porous so that, by Definition 2.1 and Definition 2.3,

$$A_p(f; t : S) = A_{p, \alpha}(f; t : S), \quad t \in (0, 1].$$

Thus the $A_p$-functional is majorized by the $A_{p, \alpha}$-functional which allows us in the proof of Theorem 7.2 to omit all estimates related to the $A_p$-functional. This leads us to a variant of Theorem 7.2 where the expression related to $A_p$ is omitted as well; thus for every $f \in C(S)$ we have

$$\|f\|_{W^1_p(\mathbb{R}^n)_{|S}} \sim \|f\|_{L_p(\mu)} + \left\{ \int_0^\varepsilon A_{p, \alpha}(f; t : S)_{L_p(\mu)} \frac{dt}{t^{p+1}} \right\}^{1/p}$$

87
provided \( p \in (n, \infty) \), \( 0 < \alpha < \min\{1/7, 1/\beta_S\} \), and \( \varepsilon > 0 \).

We proof Theorem 7.18 basing on this version of Theorem 7.2. This leads us to a
variant of Theorem 7.18 (with \( \sigma = \mu \)) where the third term in the theorem’s equivalence
\( \|\cdot\|_{W^p_1(\mathbb{R}^n)} \) can be omitted. Recall that \( \text{int}(S) = \emptyset \) so that the first term can be omitted as well. We obtain that for every \( p \in (n, \infty) \), \( \varepsilon > 0 \), \( 0 < \alpha < \min\{1/14, 1/\beta_S\} \), and every \( f \in C(S) \)

\[
\|f\|_{W^p_1(\mathbb{R}^n)|_S} \sim \|f\|_{L_p(\mu)} + \left( \int \int \frac{|f(x) - f(y)|^p}{\rho_{\alpha,S}(x,y)^{p-n} \mu(Q(x,\rho_{\alpha,S}(x,y)))^2} d\mu(x) d\mu(y) \right)^{\frac{1}{p}}.
\]

Finally, by (7.47), \( \rho_{\alpha,S}(x,y) \sim \|x - y\| \) which allows us to replace in this equivalence \( \rho_{\alpha,S}(x,y) \) with \( \|x - y\| \). The theorem is proved. \( \square \)

This Theorem and Proposition 5.8 imply the following generalization of Theorem 1.10.

**Theorem 7.24** Let \( p \in (n, \infty) \), \( \varepsilon > 0 \), \( \Omega := \text{int}(S) \), and let \( \sigma \) be a measure with \( \text{supp} \sigma = \partial S \) satisfying conditions (1.14) and (1.15). Suppose that \( \partial S \) satisfies the ball condition. Then for every \( f \in C(S) \) the following equivalence

\[
\|f\|_{W^p_1(\mathbb{R}^n)|_S} \sim \|f\|_{W^p_1(\Omega)} + \|f\|_{L_p(\sigma)} + \left( \int \int \frac{|f(x) - f(y)|^p}{\|x - y\|^{p-n} \sigma(Q(x,\|x - y\|))^2} d\sigma(x) d\sigma(y) \right)^{\frac{1}{p}}
\]

holds with constants depending only on \( n, p, \varepsilon, \beta_S \), and \( C_\sigma, C'_\sigma, C''_\sigma \).

**Proof.** By Proposition 5.8,

\[
\|f\|_{W^p_1(\mathbb{R}^n)|_S} \sim \|f\|_{W^p_1(\Omega)} + \|f\|_{\partial S} \|f\|_{W^p_1(\mathbb{R}^n)|_{\partial S}}
\]

provided \( p > n \) and \( f \in C(S) \). Since \( \partial S \) satisfies the ball condition and \( \text{int}(\partial S) \) is empty, one can apply Theorem 7.23 to \( \partial S \) with \( \mu = \sigma \). By this theorem, the quantity \( \|f\|_{\partial S} \|f\|_{W^p_1(\mathbb{R}^n)|_{\partial S}} \) is equivalent to the sum of the last two terms in the equivalence of Theorem 7.24. The proof is finished. \( \square \)

Let us present two examples of description of the traces of Sobolev functions.

**Example 7.25** Let us apply the results of Theorems 7.15 and 7.24 to \( n \)-sets in \( \mathbb{R}^n \), see (1.24). In this case the measure \( \mu \) in (1.24) is the restriction to \( S \) of the Lebesgue measure. Thus \( S \) is an \( n \)-set if there is a constant \( \theta = \theta_S \geq 1 \) such that for every cube \( Q \) centered in \( S \) of diameter at most 1 we have \( |Q| \leq \theta_S |Q \cap S| \). We call \( n \)-sets regular subsets of \( \mathbb{R}^n \).

In [S3, S4] we described the traces of Sobolev functions to regular sets via the sharp
maximal function

\[
f^g_{1,S}(x) := \sup_{r > 0} \frac{1}{r^{n+1}} \int_{Q(x,r) \cap S} |f - f_{Q(x,r) \cap S}| dx, \quad x \in S.
\]

88
We have proved that, for every regular set $S$ and every function $f \in L_p(S)$, $p > 1$,
\[
\|f\|_{W^1_p(R^n)|_S} \sim \|f\|_{L_p(S)} + \|f^\sharp_1\|_{L_p(S)}
\]
with constants of equivalence depending only on $n, p$ and $\theta_S$.

Theorem 7.15 provides another description of the trace space $W^1_p(R^n)|_S$ which does not use the notion of the sharp maximal function: Let $p \in (n, \infty)$, $\alpha \in (0, 1/14)$ and $f \in C(S)$. Then
\[
\|f\|_{W^1_p(R^n)|_S} \sim \|f\|_{L_p(S)} + \sup_{0 < t \leq 1} \left( \int \int_{\|x-y\| < t} \frac{|f(x) - f(y)|^p}{t^{p+n}} \ dx \ dy \right)^{\frac{1}{p}}
\]
\[\quad + \left( \int \int_{\rho_\alpha, S(x,y) < 1} \frac{|f(x) - f(y)|^p}{\rho_\alpha, S(x,y)^{p+n}} \ dx \ dy \right)^{\frac{1}{p}},\]
with constants of equivalence depending only on $n, p$, and $\theta_S$.

**Example 7.26** Let us present an example of a set in $S \subset R^2$ satisfying the ball condition such that the one dimensional Hausdorff measure $\mu := H_1|S$ is doubling, but $S$ is not a $d$-set for any $0 < d < 2$.

By $\ell_i$, $i = 0, 1, \ldots$, we denote a line segment in $R^2$,
\[
\ell_i := \{x = (x_1, x_2) : x_1 = 2^{-i}, 0 \leq x_2 \leq 4^{-i}\},
\]
and put $S := \{0\} \cup (\cup_{i=0}^\infty \ell_i)$.

Simple calculations show that for every square $Q = Q(x, r)$, $x \in S$, $0 < r \leq 1$, we have
\[
\mu(Q(x, r)) := H_1(Q(x, r) \cap S) \sim \begin{cases} r^2, & \|x\| \leq r, \\ \|x\|^2, & \|x\|^2 \leq r < \|x\|, \\ r, & 0 < r < \|x\|^2. \end{cases}
\]

Using this equivalence, one can easily see that $\mu$ is doubling.

Clearly, $S$ satisfies the ball condition. It is also obvious that $S$ is not a $d$-set for any $d$. Let us calculate the trace norm $\|f\|_{W^1_p(R^2)|_S}$.

It can be easily seen that for every $x = (x_1, x_2)$ and $y = (y_1, y_2) \in S$ such that $\|x - y\| \leq 1$ we have
\[
\mu(Q(x, \|x - y\|)) \sim \begin{cases} \|x - y\|, & x_1 = y_1, \\ \|x - y\|^2, & x_1 \neq y_1. \end{cases}
\]
Hence, by Theorem 7.24,

\[
\|f\|_{W^1_p(\mathbb{R}^n), S} \sim \|f\|_{L^p(\mu)} + \left( \int \int_{\{x, y \in S: x_1 = y_1\}} \frac{|f(x) - f(y)|^p}{\|x - y\|^p} d\mu(x) d\mu(y) \right)^{\frac{1}{p}}
+ \left( \int \int_{\{x, y \in S: x_1 \neq y_1\}} \frac{|f(x) - f(y)|^p}{\|x - y\|^{p+2}} d\mu(x) d\mu(y) \right)^{\frac{1}{p}}.
\]

Observe that, \(d\mu(x) = dx_2\) (because the segments \(\ell_i\) are parallel to the axis \(Ox_2\) and \(\mu\) is the one dimensional Hausdorff measure), \(\|x - y\| = |x_2 - y_2|\) whenever \(x_1 = y_1\), and \(\|x - y\| = |x_1 - y_1|\) whenever \(x_1 \neq y_1\). This enables us to present the above equivalence in the following form:

\[
\|f\|_{W^1_p(\mathbb{R}^n), S} \sim \|f\|_{L^p(\mu)} + \left( \int \int_{\{x, y \in S: x_1 = y_1\}} \frac{|f(x_1, x_2) - f(x_1, y_2)|^p}{|x_2 - y_2|^p} dx_2 dy_2 \right)^{\frac{1}{p}}
+ \left( \int \int_{\{x, y \in S: x_1 \neq y_1\}} \frac{|f(x_1, x_2) - f(x_1, y_2)|^p}{|x_1 - y_1|^{p+2}} dx_2 dy_2 \right)^{\frac{1}{p}}.
\]

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93
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