Functional equations of Nekrasov functions proposed by Ito-Maruyoshi-Okuda

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Abstract

We prove functional equations of Nekrasov partition functions for $A_1$-singularity, suggested by Ito-Maruyoshi-Okuda [IMO]. Our proof is given by the computation similar to [O]. This is the method by Nakajima-Yoshioka [NY2] based on the theory of wall-crossing formula developed by Mochizuki [Mo].

1 Introduction

We explain the conjecture by Ito-Maruyoshi-Okuda [IMO].

1.1 Two resolutions

Let $Q$ be an affine plane $\mathbb{C}^2$ and $H$ a finite sub-group of $\text{SL}(Q)$, and consider the left $H$-action on $Q$ induced by the natural $\text{SL}(Q)$-action. We have a unique minimal resolution

$$ g: X_1^\circ = H\text{-Hilb}(Q) \to Q/H = \text{Spec } \mathbb{C}[x_1, x_2]^H $$

of the quotient singularity, and the exceptional curve $C \subset X_1^\circ$ with the dual graph equal to the Dynkin diagram corresponding to $ADE$ classification of $H$ (cf. [N2 Chapter 4], or [K Chapter 12]).

On the other hand, we consider the natural $H$-action on $\mathbb{P}^2 = \mathbb{P}(\mathbb{C} \oplus Q)$, and the quotient stack $X_0 = [\mathbb{P}^2/H]$, and the coarse moduli map $f: X_0 \to \mathbb{P}^2/H$. The only loci with non-trivial stabilizer groups are $O = \{ [x_1 = x_2 = 0] / H \}$ and $\ell_\infty = \{ [x_0 = 0] / H \}$ in $X_0$. Hence we have an isomorphism

$$ X_0 \setminus (O \sqcup \ell_\infty) \cong X_1^\circ \setminus C. \quad (1) $$

We patch $X_1^\circ$ and $X_0 \setminus O$ to get a compactification $X_1$ of $X_1^\circ$ via this isomorphism:

$$ X_1 = X_1^\circ \sqcup \ell_\infty = X_1^\circ \cup (X_0 \setminus O) $$

Framed sheaves on $X_\kappa$ for $\kappa = 0, 1$ are pairs $(E, \Phi)$ of sheaves $E$ on $X_\kappa$ and isomorphisms $\Phi: E|_{\ell_\infty} \cong \mathcal{O}_{\mathbb{P}^1} \otimes \rho$ for some $H$-representation $\rho$, where we identify coherent sheaves on $\ell_\infty$
as $H$-equivariant coherent sheaves on $\mathbb{P}^1 = \mathbb{P}(Q)$. Moduli of framed sheaves are examples of quiver varieties constructed from quivers whose underlying graphs are Dynkin diagrams corresponding to $ADE$ classification of $H$ as shown in [N3]. We define Nekrasov partition functions by integrations on these moduli spaces.

In this paper, we consider the case where $H = \{ \pm \text{id}_Q \}$, and prove relations between Nekrasov partition functions defined from $X_0$ and $X_1$, suggested by [IMO]. In this case, the corresponding graph is of $A^{(1)}_1$-type. We write by $F$ the divisor class defined by $\{ x_i = 0 \}$ for $i = 1, 2$ on $X_0$. Similarly we write by $R_0$ and $R_1$ line bundles $\mathcal{O}_{X_\kappa}$ and $\mathcal{O}_{X_\kappa}(F - \ell_\infty)$ on $X_\kappa$ for $\kappa = 0, 1$, and put $\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1$. When $H = \{ \pm \text{id}_Q \}$ as above, our compactification $X_1$ coincides with the one used in the previous research [BPSS].

![Figure 1: graph of type $A^{(1)}_1$](image)

### 1.2 Derived equivalence

We consider the universal subscheme $U$ over $X_\emptyset^1 = H$-Hilb($Q$).

$$
\begin{array}{ccc}
U & \rightarrow & Q \\
\downarrow & & \downarrow \\
X_\emptyset^1 & \rightarrow & X_\emptyset^1
\end{array}
$$

(2)

Each fibre of $U \rightarrow X_\emptyset^1$ is a $H$-invariant subscheme of $Q$, and we have a natural $H$-action on $U$. We put $W^0 = [U/H]$. Then the above $\Gamma$-equivariant diagram (2) induces

$$
\begin{array}{ccc}
W^0 & \rightarrow & X_0^\emptyset = [Q/H] \\
\downarrow & & \downarrow \\
X_1^\emptyset & \rightarrow & X_1^\emptyset
\end{array}
$$

where $X_0^\emptyset = [Q/H]$ is an open subset $X_0 \setminus \ell_\infty$ of $X_0$. These morphisms are isomorphisms outside exceptional sets. As in the previous section, we patch together to get

$$W = W^0 \cup (X_0 \setminus O) \cong W^0 \cup \ell_\infty$$

and morphisms $p: W \rightarrow X_0, q: W \rightarrow X_1$. We write by $D(X_0), D(X_1)$ bounded derived categories of coherent sheaves on $X_0, X_1$. 

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Proposition 1.1. A functor $F = q_*p^* : D(X_0) \to D(X_1)$ gives an equivalence of categories.

Proof. First we show faithfulness of $F$. By the similar argument to [Bri] Example 2.2, we have a spanning class

$$\Omega = \{O_Z \otimes \rho \mid Z: H\text{-orbit in } \mathbb{P}^2, \rho \in \text{Irrep}(H)\}$$

of $D(X_0)$, and it is enough to check on $\Omega$. Here $\text{Irrep}(H)$ is the set of irreducible $H$-representations. If supports of $L, L' \in \Omega'$ are contained in $X_0 \setminus \ell_\infty$, then the derived McKay correspondence by [KV] between $X_0^o$ and $X_1^o$ implies that $\text{Hom}(L, L') \cong \text{Hom}(F(L), F(L'))$. Otherwise, we also have faithfulness by an isomorphism $X_0 \setminus O \cong W \setminus p^{-1}(O) \cong X_1 \setminus C$. By [Ni], Serre functors of $D(X_0), D(X_1)$ are given by canonical bundles $\omega_{X_0}, \omega_{X_1}$ of $X_0, X_1$. Since we have an isomorphism

$$p^*\omega_{X_0} \cong q^*\omega_{X_1},$$

Serre functors commute with $F$. Hence, $F$ is an equivalence by [BKR] Theorem 2.3.\qed

We also write by $F: K(X_0) \to K(X_1)$ the induced isomorphism of $K$-groups $K(X_\kappa)$.

1.3 Chow ring of inertia stacks

We consider inertia stacks $IX_\kappa \to X_\kappa$, which parametrize pairs $(x, \sigma)$ of objects $x$ of $X_\kappa$ and automorphisms $\sigma$ of $x$. We identify $X_\kappa$ as components of $IX_\kappa$ consisting of objects of $X_\kappa$ together with identities. We have other components $\ell_\kappa^1$ of $IX_0$ and $IX_1$, and $O^1$ of $IX_0$, consisting of objects of $\ell_\infty$ and $O$ together with non-trivial automorphisms. These are isomorphic to $\ell_\infty$ and $O$ as stacks. We have $IX_0 = X_0 \sqcup \ell_\infty \sqcup O^1$ and $IX_1 = X_1 \sqcup \ell_\kappa^1$.

By [B Theorem 2.2], Chow rings of $IX_\kappa$ are described as

$$A(IX_0) = A(X_0) \oplus A(\ell_\infty^1) \oplus A(O^1)$$

$$= (\mathbb{C}[X_0] \oplus \mathbb{C}[\ell_\infty] \oplus \mathbb{C}[O]) \oplus (\mathbb{C}[\ell_\infty^1] \oplus \mathbb{C}[Q^1]) \oplus \mathbb{C}[O^1],$$

$$A(IX_1) = A(X_1) \oplus A(\ell_\kappa^1)$$

$$= (\mathbb{C}[X_1] \oplus \mathbb{C}[\ell_\infty] \oplus \mathbb{C}[F] \oplus \mathbb{C}[P]) \oplus (\mathbb{C}[\ell_\kappa^1] \oplus \mathbb{C}[Q^1]),$$

where $P = F \cap C$ in $X_1$, and $Q = F \cap \ell_\infty$ in $X_\kappa$ for $\kappa = 0, 1$. We have $Q = \frac{1}{2}P$ in $A(X_1)$, and $Q^1$ is a sub-stack of $\ell_\kappa^1$ consisting of a point in $Q$ with the non-trivial stabilizer group.

For $\alpha \in K(X_\kappa)$, we define $\tilde{\text{ch}}(\alpha) \in A(IX_\kappa)$ as follows. Any vector bundle $E$ on $\ell_\kappa^1$ has an eigen decomposition $E_0 \oplus E_1$ for the action of non-trivial automorphisms. We define $\rho(E) = E_0 - E_1$, which gives an operation on $K(\ell_\kappa^1)$. Similarly we define an operator $\rho$ on $K(O^1)$. On $K(X_\kappa)$, we define $\rho = \text{id}_{K(X_\kappa)}$. Then we have operators $\rho$ on $K(IX_\kappa)$ for $\kappa = 0, 1$. We define $\tilde{\text{ch}}(\alpha)$ by $\text{ch}(\rho(\alpha|_{IX_\kappa}))$, where $\alpha|_{IX_\kappa}$ is the pull-back of $\alpha \in K(X_\kappa)$ to $K(IX_\kappa)$.
We write by \( r \) and \( \bar{r} \) the coefficients of \([X_\kappa]\) and \([\ell_\infty]\) in \( \check{c}(\alpha) \) respectively. We put
\[
r_0 = r_0(\alpha) = (r + \bar{r})/2, \quad r_1 = r_1(\alpha) = (r - \bar{r})/2,
\]
and \( r = r(\alpha) = (r_0, r_1) \). For \( \kappa = 1 \), we write by \( k(\alpha) \) and \( n(\alpha) \) the coefficient of \([C] = 2([\ell_\infty] - [F])\) and \(-[P]\) in \( \check{c}(\alpha) \) respectively. For \( \kappa = 0 \), we put \( k(\alpha) = k(F(\alpha)) \) and \( n(\alpha) = n(F(\alpha)) \), where \( F: K(X_0) \to K(X_1) \) is an induced isomorphism from the derived equivalence \( F: D(X_0) \to D(X_1) \) in the previous subsection.

### 1.4 Moduli of framed sheaves

A framed sheaf on \( X_\kappa \) is a pair \((E, \Phi)\) of a torsion free sheaf \( E \) on \( X_\kappa \), and an isomorphism \( \Phi: E|_{\ell_\infty} \cong O_{X_\kappa} \otimes \rho \) called framing, where \( \rho \) is a representation of \( H = \{\pm iQ\} \). Such a representation \( \rho \) is given by a \( \mathbb{Z}_2 \)-graded vector space \( W = W_0 \oplus W_1 \), where \( W_0 \) is trivial, and \( W_1 \) is a sum of non-trivial representations. If we have \( r(E) = (r_0, r_1) \), then we have \( \dim W_0 = r_0, \dim W_1 = r_1 \) by definition.

For \( c \in A(I X_\kappa) \), we write by \( M_{X_\kappa}(c) \) the moduli space of framed sheaves \((E, \Phi)\) on \( X_\kappa \) with \( \check{c}(\alpha) = c \) in \( A(I X_\kappa) \) for \( \kappa = 0, 1 \). This moduli space is constructed, and shown to be smooth and have a universal sheaf \( \check{E} \) on \( X_\kappa \times M_{X_\kappa}(c) \) in [BPSS] at least for \( \kappa = 1 \). For \( \kappa_0 \), see [N2 Remark 2.2]. We also construct moduli in terms of ADHM data in §2.3 and §Appendix A for both \( \kappa = 0 \) and 1. Then smoothness of moduli spaces \( M_{X_\kappa}(c) \) follows from the following lemma (cf. [BPSS 4.2]).

**Lemma 1.2.** For \((E, \Phi) \in M_{X_\kappa}(c)\), we have
\[
\operatorname{Hom}(E, E(-\ell_\infty)) = \operatorname{Ext}^2(E, E(-\ell_\infty)) = 0.
\]

*Proof.* It is similarly proven as in [NYT Proposition 2.1]. \( \square \)

We introduce tautological bundles \( \mathcal{V}_0 = \mathbb{R}^1 p_* \mathcal{E}(-\ell_\infty), \mathcal{V}_1 = \mathbb{R}^1 p_* \mathcal{E}(-F) \), where \( p: X_\kappa \times M_{X_\kappa}(c) \to M_{X_\kappa}(c) \) is the projection.

### 1.5 Torus action on \( M_{X_\kappa}(c) \)

We put \( \check{T} = T^2 \times T^r \times T^2_\tau \), where \( T = \mathbb{C}^* \) is the algebraic torus, and define \( \check{T} \)-action on moduli spaces as follows. To do that, we consider \( X_0 \) and \( X_1 \) as quotient stacks
\[
X_0 = [\mathbb{P}^2/\{\pm 1\}] \quad \quad (3)
\]
\[
X_1 = [\{(y, x_0, x_1, x_2) \in \mathbb{C}^4 \mid (y, x_0) \neq 0, (x_1, x_2) \neq 0\}/(\mathbb{C}_s^* \times \mathbb{C}_t^*)], \quad (4)
\]
where the \( \mathbb{C}_s^* \times \mathbb{C}_t^* \)-action is defined by
\[
(s, t)(y, x_0, x_1, x_2) = \left( s^2 \frac{y}{t^2}, sx_0, sx_1, tx_2 \right).
\]

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Then we have $T^2$-action on $X_0, X_1$ defined by

$$
F_t : X_0 \to X_0, [x_0, x_1, x_2] \mapsto [x_0, t_1x_1, t_2x_2]
$$

$$
F_t : X_1 \to X_1, [y, x_0, x_1, x_2] \mapsto [y, x_0, t_1x_1, t_2x_2]
$$

for $t = (t_1, t_2) \in T^2$.

We define $\tilde{T}$-action on moduli $M_{X_0}(c)$ of framed torsion free sheaves $(E, \Phi)$ by

$$(E, \Phi) \mapsto ((F_t^{-1})^*E, \Phi')$$

for $(t, e^a, e^m) \in \tilde{T}$. Here $\Phi' : (F_t^{-1})^*E|_{\infty} \to O_{\mathbb{P}^1} \otimes \rho$ is the composition of the pull-back

$$(F_t^{-1})^*\Phi : (F_t^{-1})^*E \to (F_t^{-1})^*(O_{\mathbb{P}^1} \otimes \rho) \cong O_{\mathbb{P}^1} \otimes \rho,$$

and the diagonal action of $e^a = (e^{a_1}, \ldots, e^{a_r}) \in T^r$

$$id_{O_{\mathbb{P}^1}} \otimes \text{diag}(e^{a_1}, \ldots, e^{a_r}) : O_{\mathbb{P}^1} \otimes \rho \to O_{\mathbb{P}^1} \otimes \rho.$$

Finally $e^m = (e^{m_1}, \ldots, e^{m_{2r}}) \in T^{2r}$ trivially acts on moduli spaces, but in the next subsection, we consider fiber-wise action of $T^{2r}$ on vector bundles on moduli spaces.

### 1.6 Partition functions

The $\tilde{T}$-equivariant Chow ring $A^*_\tilde{T}(M_{X_0}(\alpha))$ is a module over the $\tilde{T}$-equivariant Chow ring $A^*_\tilde{T}(pt)$ of a point, which is isomorphic to $\mathbb{Z}[\mathbb{e}, a, m]$, where $\mathbb{e} = (e_1, e_2), a = (a_1, \ldots, a_r)$, and $m = (m_1, \ldots, m_{2r})$ correspond to the first Chern classes of characters of $\tilde{T}$ with eigen-values $t \in T^2, e^a \in T^r$, and $e^m \in T^{2r}$ respectively. We write by $S$ the quotient field $\mathbb{Q}(\mathbb{e}, a, m)$ of $\mathbb{Z}[\mathbb{e}, a, m].$

We consider a $\tilde{T}$-equivariant vector bundle

$$\mathcal{F}_r(\mathcal{V}_0) = \left( \mathcal{V}_0 \otimes \frac{e^{m_1}}{\sqrt{t_1t_2}} \right) \oplus \cdots \oplus \left( \mathcal{V}_0 \otimes \frac{e^{m_{2r}}}{\sqrt{t_1t_2}} \right)$$

on $M_{X_0}(\alpha)$, and the $\tilde{T}$-equivariant Euler class $e(\mathcal{F}_r(\mathcal{V}_0))$, where $e^m = (e^{m_1}, \ldots, e^{m_{2r}})$ is an element in the last component $T^{2r}$ of $\tilde{T}$. Here we consider a homomorphism $\tilde{T}' = T' \to \tilde{T}$ defined by

$$(t_1', t_2', e^a, e^m) \mapsto (t_1, t_2, e^a, e^m) = ((t_1')^2, (t_2')^2, e^a, e^m),$$

and use identification $A^*_\tilde{T}'(pt) \otimes S \cong S$ via $t_1' = \sqrt{t_1}, t_2' = \sqrt{t_2}$.

For fixed $r \in \mathbb{Z}^2_{\geq 0} \setminus \{(0, 0)\}$ and $k \in \frac{1}{2}\mathbb{Z}$, we define partition functions for $\kappa = 0, 1$ by

$$Z^k_{X_0}(\mathbb{e}, a, m, q) = \sum_{\alpha \in K(X_0)} q^{n(\alpha)} \int_{M_{X_0}(\mathbb{e}\mathbb{h}(\alpha))} e(\mathcal{F}_r(\mathcal{V}_0)) \in S[[\sqrt{q}]].$$
Precise definitions of integrations are explained in §2.5.

The purpose of this paper is to prove the following statement conjectured by [IMO].

**Theorem 1.3.** We have

\[
Z_k^{X_1}(-\epsilon, a, m, q) = \begin{cases} 
(1 - (-1)^r) w^r Z_k^{X_0}(\epsilon, a, m, q) & \text{for } k \geq 0, \\
Z_k^{X_0}(-\epsilon, a, m, q) & \text{for } k \leq 0,
\end{cases}
\]

where

\[
u_r = (\epsilon_1 + \epsilon_2)(2 \sum_{a=1}^{r} a + \sum_{f=1}^{2^r} m_f) 2^{\epsilon_1 \epsilon_2}.\]

In the rest of the paper, we give a proof of Theorem 1.3 as follows. In §2, we recall ADHM description of framed sheaves, reduce a proof of Theorem 1.3 to wall-crossing formulas, and give a precise definition of integrations over moduli \(M_{\chi_\alpha}(\text{ch}(\alpha))\) of framed sheaves. In §3, we recall Mochizuki method in the manner similar to [O]. In §4, we compute wall-crossing formulas, and complete a proof of Theorem 1.3. In Appendix A, we construct moduli of framed sheaves in terms of ADHM data. In Appendix B, we recall combinatorial description of Nekrasov partition functions similar to [NY1].

We expect that our results and the generalization to \(K\)-theoretic Nekrasov functions have applications to study Pinlvé \(\tau\) functions (cf. [BS1] and [BS2]).

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## 2 ADHM description

We introduce ADHM description of framed moduli.

### 2.1 ADHM data

Let \(W = W_0 \oplus W_1, V = V_0 \oplus V_1\) and \(Q = Q_0 \oplus Q_1\) be \(\mathbb{Z}_2\)-graded vector spaces with \(Q_0 = 0\) and \(Q_1 = \mathbb{C}\). We introduce ADHM data on \((W, V)\).

**Definition 2.1.** ADHM data on \((W, V)\) are collections \((B, z, w)\) of \(\mathbb{Z}_2\)-graded linear maps such that \(B \in \text{Hom}_{\mathbb{Z}_2}(Q^\vee \otimes V, V), z \in \text{Hom}_{\mathbb{Z}_2}(W, V)\) and \(w \in \text{Hom}_{\mathbb{Z}_2}(\wedge^2 Q^\vee \otimes V, W)\), satisfying

\[
[B \wedge B] + zw = 0 \in \text{Hom}_{\mathbb{Z}_2}(\wedge^2 Q^\vee \otimes V, W),
\]

where \([B \wedge B]\) is the restriction of \(B \circ (\text{id}_{Q^\vee} \otimes B): Q^\vee \otimes Q^\vee \otimes V \to V\) to the subspace \(\wedge^2 Q^\vee \otimes V\).

If we write \(Q^\vee = C e_1 \oplus C e_2\) and \(B(e_1 \otimes v_1 + e_2 \otimes v_2) = B_1(v_1) + B_2(v_2)\), then the equation \([B \wedge B] + zw = 0\) is equivalent to \([B_1, B_2] + zw = 0\). We take \(\zeta = (\zeta_0, \zeta_1) \in \mathbb{R}^2\), and put \(\xi(V) = \zeta_0 \dim V_0 + \zeta_1 \dim V_1\) for a \(\mathbb{Z}_2\)-graded vector space \(V\).

**Definition 2.2.** We assume \(W \neq 0\). ADHM data \((B, z, w)\) on \((W, V)\) are said to be \(\zeta\)-semistable if the conditions (i) (ii) hold for any \(\mathbb{Z}_2\)-graded subspace \(S\) of \(V\) with \(B(Q^\vee \otimes S) \subset S\).
(i) If \( S \subset \ker w \), then we have \( \zeta(S) \leq 0 \).

(ii) If \( \text{im} z \subset S \), then we have \( \zeta(V/S) \geq 0 \).

They are said to be \( \zeta \)-stable when the strict inequality always holds for \( S \neq 0 \) in (i), and \( S \neq V \) in (ii).

We put
\[
\begin{align*}
w &= \begin{bmatrix} \dim W_0 \\ \dim W_1 \end{bmatrix}, \\
v &= \begin{bmatrix} \dim V_0 \\ \dim V_1 \end{bmatrix},
\end{align*}
\]
and construct moduli \( M^\zeta(w, v) \) of \( \zeta \)-semistable ADHM data on \( (W, V) \) as follows. We put
\[
\begin{align*}
M &= M(W, V) = \text{Hom}_{\mathbb{Z}^2}(Q^\vee \otimes V, V) \times \text{Hom}_{\mathbb{Z}^2}(W, V) \\
L &= L(W, V) = \text{Hom}_{\mathbb{Z}^2}(\wedge^2 Q^\vee \otimes V, V),
\end{align*}
\]
and define a map \( \mu : M \to L \) by
\[
\mu(B, z, w) = [B \wedge B] + zw.
\] (5)

We take the \( \zeta \)-semistable locus \( \mu^{-1}(0)^\zeta \) and define \( M^\zeta(w, v) = [\mu^{-1}(0)^\zeta/G] \), where \( G = \text{GL}(V_0) \times \text{GL}(V_1) \).

We have a natural \( \text{GL}(Q) \times \text{GL}(W_0) \times \text{GL}(W_1) \)-action on \( M^\zeta(w, v) \). Hence via the diagonal embedding \( T^2 \times T^r \) into \( \text{GL}(Q) \times \text{GL}(W_0) \times \text{GL}(W_1) \) and the projection \( \tilde{T} = T^2 \times T^r \times T^{2r} \to T^2 \times T^r \), we get a \( \tilde{T} \)-action on \( M^\zeta(w, v) \), where \( r = \dim W_0 + \dim W_1 \). Concretely \((t, e^a) \in \tilde{T}\) acts by \((t_1 B_1, t_2 B_2, ze^{-a}, e^{a nat} t_2)\).

2.2 Chamber and wall structure on \( \zeta \)-plane
As in [N3, 1(i)], we have positive roots
\[
R_+ = \{(m, m+1), (m+1, m) \mid m \in \mathbb{Z}_{\geq 0}\} \cup \{(m, m) \mid m \in \mathbb{Z}_{> 0}\},
\]
where \((m, m+1), (m+1, m)\) are real roots, and \((m, m)\) are imaginary roots. These roots \( \alpha \in R^+ \) define the walls
\[
\mathcal{D}_\alpha = \{\zeta = (\zeta_0, \zeta_1) \in \mathbb{R}^2 \mid \zeta(\alpha) = 0, \zeta_0 \leq \zeta_1\}
\]
and \( -\mathcal{D}_\alpha \), where we put \( \zeta(\alpha) = \zeta_0 \alpha_0 + \zeta_1 \alpha_1 \) for \( \alpha = (\alpha_0, \alpha_1) \).

Fix a dimension vector \( v = (v_0, v_1) \), and put \( R_+(v) = \{\alpha \in R_+ \mid \alpha_0 \leq v_0, \alpha_1 \leq v_1\} \).

Chambers are connected components of
\[
\mathbb{R}^2 \setminus \bigcup_{\alpha \in R_+(v)} \mathcal{D}_\alpha \cup (-\mathcal{D}_\alpha).
\]

On these chambers, stability and semi-stability coincides, and all \( \zeta \)-stability conditions are equivalent when \( \zeta \) lies in one fixed chamber \( \mathcal{C} \) by [NT, 2.8]. We put \( M^\zeta(w, v) = M^\zeta(w, v) \).
for $\zeta \in \mathbb{C}$. On $M^\mathcal{C}(w, v)$, we have tautological bundles $\mathcal{V}_i = [\mu^{-1}(0) \times \mathcal{V}_i / G]$ for $i = 0, 1$, and tautological homomorphisms $B : Q^\vee \otimes \mathcal{V} \to \mathcal{V}$, where $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$.

For the imaginary root $\delta = (1, 1) \in R_+$, we put $\mathcal{D}_\infty = \mathcal{D}_\delta$. For $m \in \mathbb{Z}$, we put

$$c_m = \{ \zeta = (\zeta_0, \zeta_1) \in \mathbb{R}^2 \mid m\zeta_0 + (m + 1)\zeta_1 < 0, (m - 1)\zeta_0 + m\zeta_1 > 0 \},$$

and $M^m(w, v) = M^{c_m}(w, v)$. By the above chamber structure, we have chambers $C_+$ containing $\bigcup_{m>\min\{v_0,v_1\}} c_m$, and $C_-$ containing $\bigcup_{m>\min\{v_0-1,v_1\}} c_m$. We put $M^\pm(w, v) = M^{c_{\pm}}(w, v)$.

### 2.3 ADHM description of framed moduli

We recall ADHM description from [N2, Chapter 2] and [N3, Theorem 2.2].

**Theorem 2.3.** We have the following $\bar{T}$-equivariant isomorphisms.

(i) We have $M^0(w, v) \cong M_{X_0}(c)$, where we put

$$c = \bar{c}(w_0 R_0 + w_1 R_1 - v_0 O_P - v_1 O_P \otimes (-1)) \in A(I X_0).$$

(ii) We have $M^+(w, v) \cong M_{X_1}(c)$, where we put

$$c = (w_0 + w_1)[X_1] + (-2v_0 + 2v_1 - w_1) \left[ \frac{C}{2} \right] - \left( v_0 + \frac{v_1}{4} \right) [P] + (w_0 - w_1)[\ell_\infty^1] \in A(I X_1).$$

We note that $[C] = 2[\ell_\infty] - 2[F]$.

Furthermore, via these isomorphisms (i) and (ii), tautological bundles $\mathcal{V}_0, \mathcal{V}_1$ on both sides coincide as $\bar{T}$-equivariant vector bundles.

**Proof.** We recall these isomorphisms as follows.

We put $R_0 = O_{X_0}, R_1 = O_{X_1}(F - \ell_\infty)$ and $R = R_0 \oplus R_1$ as in the introduction. For ADHM data $(B, z, w)$, we consider the following complex

$\begin{array}{cccc}
0 & \to & R_0(-\ell_\infty) \otimes V_0 & \to & (R_1 \otimes V_0) \oplus (R_0 \otimes V_1) \oplus (R_0 \otimes \ell_\infty) \otimes V_0 \\
& & \oplus & \to & \oplus \\
& & R_1(-\ell_\infty) \otimes V_1 & \to & \bigoplus \to 0 \\
\end{array}$

with

$$\sigma = \begin{bmatrix} x_0 B_1 - x_1 I_y \\ x_0 B_2 - x_2 I_y \\ x_0 w \end{bmatrix}, \quad \tau = \begin{bmatrix} x_0 B_2 - x_2 I^y \\ x_0 B_1 - x_1 I^y \\ x_0 z \end{bmatrix},$$

where $I^y = \begin{bmatrix} y^\kappa & 0 \\ 0 & 1 \end{bmatrix}, I_y = \begin{bmatrix} 1 & 0 \\ 0 & y^\kappa \end{bmatrix}$ for $\kappa = 0, 1$. We take its cohomology $E = \ker \tau / \text{im } \sigma$. By restricting to $\ell_\infty$, we get a framing $\Phi$. 8
In the following, we will show that this map \((B,z,w) \mapsto (E,\Phi)\) gives the desired isomorphism. For (ii), this follows from [N3 Theorem 2.2] and Appendix A.

For (i), this follows from [N2 Chapter 2] as follows. For a stability parameter \(\zeta^0 = (\zeta^0_0, \zeta^0_1) \in C_0\) such that \(\zeta^0_0, \zeta^0_1 < 0\), the \(\zeta^0\)-stability condition in Definition 2.2 is equivalent to the condition that for any graded subspace \(S = S_0 \oplus S_1 \subset V\) such that \(B\)-invariant and \(\text{im} z \subset S_k\) for \(k = 0, 1\), we have \(S = V\). This is equivalent to the condition that for any subspace \(S' \subset V\) (without grading) such that \(B\)-invariant and \(\text{im} z \subset S'\), we have \(S' = V\), since we can get a graded subspace \(S = S' \cap V_0 \oplus S' \cap V_1\) as in the former condition. By [N3 Lemma 2.6], this is equivalent to the condition that \(\sigma\) is injective except finitely many points and \(\tau\) is surjective for any point of \(X_0\). This implies that the middle cohomology is a torsion free sheaf on \(X_0\), and we get the desired isomorphism.

The last assertion follows if we compute \(V_0 = R^1 p_* \mathcal{E}(\ell_\infty), V_1 = R^1 p_* \mathcal{E}(F)\) on \(M_{X_1}(c)\) using (8).

2.4 Symmetry

We consider isomorphisms among moduli of semistable ADHM data with various parameters \(\zeta, w\) and \(v\). We consider \(Z_2\)-graded vector spaces \(V[1]\) and \(W[1]\) and put \(\zeta[1] = (\zeta_1, \zeta_0), v[1] = [\dim V_1 \dim V_0], w[1] = [\dim W_1 \dim W_0]\).

Then any \(\zeta\)-semistable ADHM datum on \((W,V)\) is naturally identified with \(\zeta[1]\)-semistable ADHM datum on \((W[1],V[1])\), hence we have an isomorphism

\[ M^\zeta(w, v) \cong M^{\zeta[1]}(w[1], v[1]). \]

(9)

On the other hand, if we take dual vector spaces \(W^\vee, V^\vee\), then \((B^\vee_2, B^\vee_1, w^\vee, z^\vee)\) is a \((-\zeta)\)-semistable ADHM datum on \((W^\vee, V^\vee)\) for any \(\zeta\)-smistable ADHM datum \((B, z, w)\) on \((W, V)\). This gives an isomorphism

\[ M^\zeta(w, v) \cong M^{-\zeta}(w, v), \]

(10)

which is \(\tilde{T}\)-equivariant via \(\tilde{T} \to \tilde{T}, ((t_1, t_2), e^a, e^m) \mapsto ((t_2, t_1), e^{-a}, e^m)\).

Using this we have the following as a corollary of Theorem 2.3.

**Corollary 2.4.** We have an isomorphism

\[ M^{-}(w, v) \cong M_{X_1}(c), \]

where we put

\[ c_- = (w_0 + w_1)[X_1] + (2v_0 - 2v_1 + w_1) \frac{[C]}{2} - \left( v_0 + \frac{w_1}{4} \right) [P] + (w_0 - w_1)[\ell_\infty^2] \in A(I X_1). \]
Proof. We put $R_i^{-} = O_{X_1}, R_1^{-} = O_{X_1}(\ell_{\infty} - F)$ and $R_1^- = R_i^- \oplus R_1^-$. Replacing $R_i$ with $R_i$ in the complex $[3]$ in the proof of Theorem 2.3, we get a $\tilde{T}$-equivariant isomorphism $\pi^{-}: M^{-}(w, v) \cong M_{X_1}(c_-)$. This follows from the following commutative diagram:

$$
\begin{array}{ccc}
M_{X_1}(c_+) & \xrightarrow{\pi} & M_{X_1}(c_-) \\
\downarrow & & \downarrow \\
M\zeta^+(w[1], v[1]) & \xrightarrow{[1]} & M\zeta^-(w, v)
\end{array}
$$

where

$$c_+ = (w_1 + w_0)[X_1] + (2v_0 - 2v_1 - w_0)[C] - \frac{(v_1 + \frac{w_0}{\ell})}{4}[P] + (w_1 - w_0)[f_{\infty}^1] \in A(I X_1).$$

We also note that $[1]$ is the isomorphism in $[9]$, since we can take $\zeta^+$ and $\zeta^-$ such that $M^{\pm}(c_\pm) = M^{\pm}(c_\pm)$, and $\zeta^- = \zeta^+[1]$.

2.5 Integrations

We take $0 \in \mathbb{R}^2$, and put $M_0(w, v) = M^0(w, v)$. Then we have a proper map $\Pi: M^\zeta(w, v) \to M_0(w, v)$ for any $\zeta \in \mathbb{R}^2$.

Proposition 2.5. The $\tilde{T}$-fixed points set $M_0(w, v)^\tilde{T}$ consists of one point.

Proof. If we take a representative $A = (B, z, w) \in \mu^{-1}(0)$ of a point $p \in M_0(w, v)^\tilde{T}$. Then for any $f \in \Gamma(\mu^{-1}(0), O_{\mu^{-1}(0)})^G$, we have $f\{A\} = f\{tA\}$ for any $t \in T^2$. Since $\lim_{t \to 0} tA = (0, z, 0)$, we have $f(\{A\}) = f(0, z, 0)$. Furthermore the closure of $G$-orbit of $(0, z, 0)$ contains $(0, 0, 0)$. Hence we see that $(0, 0, 0) \in \mu^{-1}(0)$ represents the same point $p$.

We write the inclusion by $\iota: M_0(w, v)^\tilde{T} \to M_0(w, v)$. For $\zeta$ in some chambers and $\psi \in A^\zeta_\tilde{T}(M^\zeta(w, v))$, we define the integration over $M^\zeta(w, v)$ by

$$\int_{M^\zeta(w, v)} \psi = (\iota_{\mu})^{-1} \circ \Pi_{\mu}(\psi \cap [M^\zeta(w, v)]) \in S = \mathbb{Q}(e, a, m),$$

where $[M^\zeta(w, v)]$ is the fundamental cycle. Here $S$ is a fractional field of $A^\zeta_\tilde{T}(pt) = \mathbb{Z}[e, a, m]$ as in the introduction.

Integrations over $M_{X_1}(c)$ in $[1.6]$ are defined by these integrations via isomorphisms in Theorem 2.3.
2.6 ADHM descriptions of Partition functions

For \( \zeta \in \mathbb{R}^2 \) on a certain chamber, we consider a \( \tilde{T} \)-equivariant bundle

\[
F_r(V_0) = \left( V_0 \otimes \frac{e^{m_1}}{\sqrt{t_1 t_2}} \right) \oplus \cdots \oplus \left( V_0 \otimes \frac{e^{m_{2r}}}{\sqrt{t_1 t_2}} \right)
\]

on \( M \zeta(w, v) \). For fixed \( k \in \mathbb{Z}^2 \) and \( w \in \mathbb{Z}_{\geq 0}^2 \), we consider \( v = (v_0, v_0 + \frac{w}{2} + k) \), and take sums over \( v_0 \in \mathbb{Z}_{\geq 0} \). Then by Theorem 2.3, we have

\[
Z_{k, X_0}(\epsilon, a, m, q) = \sum q^n \int_{M^0(w, v)} e(F_r(V_0)), \quad Z_{k, X_1}(\epsilon, a, m, q) = \sum q^n \int_{M^+(w, v)} e(F_r(V_0)),
\]

where \( n = v_0 + \frac{w}{2} \), and \( \zeta^\pm \) as in Theorem 2.3.

Our strategy to prove Theorem 1.3 is the following. For \( k \leq 0 \), we show that wall-crossing between \( \zeta^+ \in C_+ \) and \( \zeta^0 \in C_0 \) does not change partition functions in the similar way to [NY2].

For \( k \geq 0 \), we analyze wall-crossing between \( \zeta^+ \in C_+ \) and \( -\zeta^- \in -C_- \) in the similar way to [O]. We use an isomorphism \( M^- \zeta^-(w, v) \cong M \zeta^+(w, v) \) via the homomorphism \( \tilde{T} \to \tilde{T}, (t_1, t_2, e^a, e^m) \to (t_2, t_1, e^{-a}, e^m) \) by (10).

Furthermore, we can show that wall-crossing between \( \zeta^- \) and \( \zeta^0 \) does not change partition functions, since this process is equivalent to the above wall-crossing between \( \zeta^+ \) and \( \zeta^0 \) for \( k \leq 0 \) via the symmetry in §2.4. This completes our proof of Theorem 1.3.

3 Mochizuki method

We apply Mochizuki method [Mo] to ADHM description in the previous section following [NY2]. It is the similar arguments as in [O]. Hence we often omit the detailed description.
3.1 Quiver description

For later purpose, we modify the definition of ADHM data following [3]. As in the previous section, we consider ADHM data on $(W,V)$ for $\mathbb{Z}_2$-graded vector spaces $W = W_0 \oplus W_1$, $V = V_0 \oplus V_1$.

We introduce a quiver with relations $\Gamma = (\overline{\Gamma}, I)$ as follows, where $\overline{\Gamma}$ is a quiver, and $I$ is an ideal of the path algebra of $\overline{\Gamma}$. The set $\overline{\Gamma}_0$ of vertex consists of 0, 1 and $\infty$. The set $\overline{\Gamma}_1$ of arrows consists of $\alpha: 0 \to 1, \beta: 1 \to 0, \gamma_1, \ldots, \gamma_n : \infty \to 0, \delta_1, \ldots, \delta_m : \infty \to 1$, and their converse $\alpha^*: 1 \to 0, \beta^*: 0 \to 1, \gamma_1^*, \ldots, \gamma_n^* : 0 \to \infty, \delta_1^*, \ldots, \delta_m^* : 1 \to \infty$. The ideal $I$ is generated by $\beta \beta^* - \alpha^* \alpha + \sum_{i=1}^r \gamma_i \gamma_i^*$ and $\alpha \alpha^* - \beta \beta^* + \sum_{i=1}^s \delta_i \delta_i^*$.

A $\Gamma$-representation $A$ consists of finite dimensional vector spaces $A_0, A_1$ and $A_\infty$ corresponding to each vertices in $\overline{\Gamma}_0$, and linear maps $A_a : A_{s(a)} \to A_{\ell(a)}$ for each arrow $a \in \overline{\Gamma}_1$, where $s(a)$ and $\ell(a)$ are source and target of an arrow $a$.

We identify ADHM data and $\Gamma$-representations as follows. We put

$$A_0 = V_0, A_1 = V_1, A_\infty = \begin{cases} \mathbb{C} & \text{if } W \neq 0, \\ 0 & \text{if } W = 0. \end{cases} \quad (11)$$

We consider the graded vector space $V[1]$ with $V[1]_0 = V_1$ and $V[1]_1 = V_0$, and take a basis $e_1, \ldots, e_n$ and $e_{r_0+1}, \ldots, e_{r_0+r_1}$ of $W_0$ and $W_1$, and their dual basis $e_1^*, \ldots, e_n^*$ and $e_{r_0+1}^*, \ldots, e_{r_0+r_1}^*$. Then from $\Gamma$-representations $A$, we can assign ADHM data $(B_1, B_2, z, w)$ by

$$B_1 = \begin{bmatrix} 0 & A_b & A_\gamma \\ A_\alpha & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & A_\gamma^* \\ A_{\alpha^*} & 0 \end{bmatrix} \in \text{Hom}_{\mathbb{Z}_2}(V_1, V[1]),$$

$$z = \sum_{i=1}^r A_{\gamma_i} e_i^* + \sum_{j=1}^s A_{\delta_j} e_{r_0+j}^* \in \text{Hom}_{\mathbb{Z}_2}(W, V),$$

and

$$w = \sum_{i=1}^r e_i A_{\gamma_i}^* + \sum_{j=1}^s e_{r_0+j} A_{\delta_j}^* \in \text{Hom}_{\mathbb{Z}_2}(V, W).$$

Conversely, we can assign a $\Gamma$-representation from ADHM data on $(W,V)$ by the above equations.

For $\zeta = (\zeta_0, \zeta_1) \in \mathbb{R}^2$, we define stability of $\Gamma$-representations as follows. For any sub-representation $S = S_0 \oplus S_1 \oplus S_\infty$ of $\Gamma$-representations $A = V_0 \oplus V_1 \oplus \mathbb{C}$, we put

$$\zeta(S) = \zeta_0 \dim S_0 + \zeta_1 \dim S_1 - (\zeta_0 v_0 + \zeta_1 v_1) \dim S_\infty,$$

where $v_0 = \dim V_0, v_1 = \dim V_1$.

**Definition 3.1.** A $\Gamma$-representation $A$ is said to be $\zeta$-semistable if for any sub-representation $S$ of $A$, we have $\zeta(S) \leq 0$. In addition, if we have $\zeta(S) < 0$ for any $S \neq 0, A$, we say $A$ is $\zeta$-stable.

This coincides with Definition 2.2 of stability for ADHM data via the above identification between ADHM data and $\Gamma$-representations.
3.2 ADHM data with full flags

For \( \mathbb{Z}_2 \)-graded vector spaces \( W = W_0 \oplus W_1, V = V_0 \oplus V_1 \), we consider pairs \((\mathcal{A}, F^*)\) of ADHM data \( \mathcal{A} = (B, z, w) \) on \((W, V)\) and full flags \( F^* \) of \( V_i \), where \( i = 0, 1 \) are chosen as follows. In the rest of this section, we fix one of the walls \( \mathcal{D} \) defined in \( \S 2.2 \). Then there exists a root \( \alpha = (\alpha_0, \alpha_1) \in R^+ \) such that \( \mathcal{D} = \pm \mathcal{D}_\alpha \). We choose \( i \in \{0, 1\} \) such that \( \alpha_i \neq 0 \). When \( \alpha \neq t(1, 0), t(0, 1) \), we can choose both \( i = 0 \) and 1. However, for simplicity, when \( \alpha \) is an imaginary root \( p\delta \), we always choose \( i = 1 \), and set \( p = 1 \), that is, \( \alpha = \delta \).

We have two chambers adjacent to the wall \( \mathcal{D} \), and write by \( \mathcal{C} \) the one whose element \( \zeta \) satisfy \( \zeta(\alpha) < 0 \), and by \( \mathcal{C}' \) the other one. We take \( \zeta^0 \) on the wall \( \mathcal{D} \). We consider ADHM data \( \mathcal{A} \) as \( \Gamma \)-representations as in \( \S 3.1 \).

**Definition 3.2.** For \( \ell \geq 0 \), a pair \((\mathcal{A}, F^*)\) is said to be \((\mathcal{C}, \ell)\)-stable if \( \mathcal{A} \) is \( \zeta^0 \)-semistable and any sub-representation \( S = S_0 \oplus S_1 \oplus S_\infty \) of \( \mathcal{A} \) with \( \zeta^0(S) = 0 \) satisfies the following two conditions:

1. If \( S_\infty = 0 \) and \( S \neq 0 \), we have \( S_i \cap F^\ell = 0 \).
2. If \( S_\infty = \mathcal{C} \) and \( S \neq \mathcal{A} \), we have \( F^\ell \not\subset S_i \).

We write by \( \widetilde{M}^{(\mathcal{C}, \ell)}(w, v) \) moduli of \((\mathcal{C}, \ell)\)-stable ADHM data on \((W, V)\) with full flags of \( V_i \), which will be constructed in the next subsection. For \( \mathcal{C} = \mathcal{C}_m \) and \( \mathcal{C}_+ \) in \( \S 2.2 \) we put \( \widetilde{M}^{m, \ell}(w, v) = \widetilde{M}^{(\mathcal{C}_m, \ell)}(w, v) \) and \( \widetilde{M}^{+, \ell}(w, v) = \widetilde{M}^{(\mathcal{C}_+, \ell)}(w, v) \). We remark that \( \zeta^0(S) = 0 \) implies that \((\dim S_0, \dim S_1)\) is proportional to the root \( \alpha \) defining the wall \( D \), and when \( \ell = 0 \) (resp. \( \ell = v_i \)), an object \((\mathcal{A}, F^*)\) is \((\mathcal{C}, \ell)\)-stable if and only if \( \mathcal{A} = (B, z, w) \) is \( \mathcal{C} \)-stable (resp. \( \mathcal{C}' \)-stable). Hence we see that \( \widetilde{M}^{(\mathcal{C}, 0)}(w, v) \) and \( \widetilde{M}^{(\mathcal{C}, v_i)}(w, v) \) are full flag bundles of tautological bundles \( \mathcal{V}_i \) on \( M^{\mathcal{C}}(w, v) \) and \( M^{\mathcal{C}'}(w, v) \) respectively.

We also interpret \((\mathcal{C}, \ell)\)-stability in terms of \( \Gamma \)-representations as follows. We take \( \eta = (\eta_1, \ldots, \eta_{v_i}) \in (\mathbb{Q}_{>0})^{v_i} \), and for any sub-representation \( S \) of \( \mathcal{A} \), we put

\[
\mu_{\zeta, \eta}(S) = \frac{\zeta(S) + \sum_{j=1}^{v_i} \eta_j \dim(S_i \cap F^j)}{\text{rk} S},
\]

where \( \text{rk} S = \dim S_0 + \dim S_1 + \dim S_\infty \). We say that \((\mathcal{A}, F^*)\) is \((\zeta, \eta)\)-semistable if for any non-zero proper sub-representation \( S \), we have

\[
\mu_{\zeta, \eta}(S) \leq \mu_{\zeta, \eta}(\mathcal{A}) = \frac{\sum_{j=1}^{v_i} j \eta_j}{\text{rk} \mathcal{A}}.
\]  

(12)

If inequality is always strict, we say that \((\mathcal{A}, F^*)\) is \((\zeta, \eta)\)-stable.
We consider the following condition

\[ \sum_{j=1}^{v_i} j\eta_j < \min_{\mathcal{C}(S) \neq 0} \frac{|\zeta(S)|}{\text{rk} A}, \quad (13) \]

\[ \text{rk} A \sum_{j=\ell+1}^{n} j\eta_j < \min \left( \sum_{j=1}^{\ell} j\eta_j - \frac{\text{rk} A}{\text{rk} \alpha} \zeta(\alpha), \; - \sum_{j=1}^{\ell} j\eta_j + \frac{\text{rk} A}{\text{rk} \alpha} \zeta(\alpha) + \eta_\ell \right), \quad (14) \]

\[ \eta_k > \text{rk} A \sum_{i=k+1}^{n} i\eta_i \text{ for } k = 1, \ldots, v_i, \quad (15) \]

\[ \sum_{i=1}^{n} k_i \eta_i \neq 0 \text{ for any } (k_1, \ldots, k_n) \in \mathbb{Z}^n \setminus \{0\} \text{ with } |k_i| \leq n^2, \quad (16) \]

where in (13) minimum is taken over the set of all graded subspaces \( S \) of \( V \) with \( \zeta^0(S) \neq 0 \).

By the similar arguments as in [O, §4.1], we have the following

**Proposition 3.3** ([Mo, Proposition 4.2.4], [NY2, Lemma 5.6]). Assume that \((\zeta, \eta)\) satisfies (13) and (14). Then for \((A, F^*)\), the \((C, \ell)\)-stability is equivalent to the \((\zeta, \eta)\)-stability. Furthermore the \((\zeta, \eta)\)-semistability automatically implies that the \((\zeta, \eta)\)-stability.

We can choose \(\zeta, \eta\) satisfying (13), (14), (15) and (16) as follows. First, we determine the neighborhood of \(\zeta^0\) in which we take \(\zeta\) so that \(|\zeta(S)|\) is large enough for any graded sub-space \( S \subset V \) with \(\zeta^0(S) \neq 0 \). We choose \(\eta\) in the region satisfying (13) and (16), such that they satisfy (15) when we put \(\eta_{\ell+1} = \cdots = \eta_{v_i} = 0\). Then we take \(\zeta\) so that it satisfies

\[ \sum_{i=1}^{\ell} i\eta_i - \eta_\ell < \frac{\text{rk} A}{\text{rk} \alpha} \zeta(\alpha) < \sum_{i=1}^{\ell} i\eta_i. \]

Finally we take \(\eta_{\ell+1}, \ldots, \eta_n\) satisfying (14) and (15).

### 3.3 Enhanced master space

We introduce an enhanced master space using ADHM description. For \(\mathbb{Z}_2\)-graded vector spaces \( W = W_0 \oplus W_1, V = V_0 \oplus V_1 \), we consider pairs \((A, F^*)\) of ADHM data \( A = (B, z, w) \) on \((W, V)\) and full flags \( F^* \) of \( V \).

Let \([v_i]\) denote the set \(\{1, \ldots, v_i\}\) of integers, and \(Fl = Fl(V_i, [v_i])\) denote the full flag variety of \( V_i \), where \( v_i = \dim V_i \). We consider natural projections \( \rho_j: Fl \to G_j = Gr(V_i, j) \) to Grassmanian manifolds \( G_j \) of \( j \)-dimensional subspace of \( V_i \) and pull-backs \( \rho_j^* O_{G_j}(1) \) of polarizations \( O_{G_j}(1) \) by Plucker embeddings.

In the following, we fix \( \ell \in [v_i] \), and choose \( \zeta^- \in \mathcal{C}, \zeta \in \mathcal{C}', \) and \( \eta \in (\mathbb{Q}_{\geq 0})^{v_i} \) as follows. \(|\zeta|, |\eta|\) are enough smaller than \(|\zeta^-|\) so that any \((A, F^*)\) is \((\zeta^-, \eta)\)-stable if and only if \( A \) is \( \mathcal{C} \)-stable,
and \((\zeta, \eta)\) satisfy the conditions \([\text{13}], [\text{14}], [\text{15}]\) and \([\text{16}]\). We take a positive integer \(k\) enough divisible such that \(k\zeta, k\zeta^-\) and \(k\eta\) are all integer valued, and consider ample \(G\)-linearizations

\[
L_+ = \big( O_M \otimes (\det V)^{\otimes k\zeta} \big) \boxtimes \big( \bigotimes_{j=1}^n \rho_j^* O_{G_j}(k\eta_j) \big),
\]

\[
L_- = \big( O_M \otimes (\det V)^{\otimes k\zeta^-} \big) \boxtimes \big( \bigotimes_{j=1}^n \rho_j^* O_{G_j}(k\eta_j) \big)
\]
on \(\tilde{M} = \tilde{M}(W,V) = M(W,V) \times Fl\). We consider the composition \(\tilde{\mu}: \tilde{M} \to L\) of the projection \(\tilde{M} \to M\) and \(\mu: M \to L\) in \([\text{15}]\), and semistable loci \(\tilde{\mu}^{-1}(0)^+\) and \(\tilde{\mu}^{-1}(0)^-\) with respect \(L_+\) and \(L_-\) respectively.

We put \(\tilde{M} = \tilde{M}(W,V) = \mathbb{P}(L_- \oplus L_+)\) and consider a composition \(\hat{\mu}: \tilde{M} \to L\) of the projection \(\tilde{M} \to M\) and \(\mu: M \to L\). Then we have a natural \(G = \mathrm{GL}(V) \times \mathrm{GL}(V)\)-action on \(\tilde{M}\) compatible with \(\hat{\mu}\). We also write by \(O(1)\) the restriction of the tautological bundle \(O(1)\) to \(\hat{\mu}^{-1}(0)\), which defines semistable locus \(\hat{\mu}^{-1}(0)^{ss}\). We define an enhanced master space by \(\mathcal{M} = [\hat{\mu}^{-1}(0)^{ss}/G]\). The projection \(\hat{\mu}^{-1}(0) \to \mu^{-1}(0)\) induces a proper morphism \(\Pi: \mathcal{M} \to M_0(w, v)\).

We have a \(\mathbb{C}_h^*\)-action on \(\mathcal{M}\) defined by

\[
(A, F^*, [x_-, x_+]) \mapsto (A, F^*, [e^h x_-, x_+]),
\]

where \([x_-, x_+]\) is the homogeneous coordinate of \(\mathbb{P}(L_- \oplus L_+)\). Finally, we get \(\mathbb{C}_h^*\)-fixed points sets \(\mathcal{M}^{\mathbb{C}_h^*}\).

### 3.4 Direct sum decompositions of fixed points sets

In this section, we follow the similar argument as in \([\text{10}] \S 4.3\).

For ADHM data with full flags \((A, F^*)\), if we have a direct sum decomposition \((A, F^*) = (A_b, F_b^*) \oplus (A_t, F_t^*)\), then we put \(I_{\alpha} = \{ j \in [v_i] | F_{\alpha}^i/F_{\alpha}^{i-1} \neq 0 \}\) for \(\alpha = b, t\) so that \([v_i] = I_b \sqcup I_t\). We choose \(A_q\) such that \((A_q)_{\infty} = 0\). The data \(J = (I_b, I_t)\) are called the decomposition type.

By \(2\)-stability condition \([\text{16}]\), we see that \(x \in \hat{\mu}^{-1}(0)^{ss} \setminus (\mathbb{P}(L_-) \sqcup \mathbb{P}(L_+))\) over \((A, F^*) \in \tilde{M}(r, n)\) represents a \(\mathbb{C}_h^*\)-fixed point in \(\mathcal{M}\) if and only if we have a decomposition \((A, F^*) = (A_b, F_b^*) \oplus (A_t, F_t^*)\) satisfying the following conditions (cf \([\text{NY2}]\) Lemma 5.16)). The decomposition type \(J = (I_b, I_t)\) satisfies \(\min(I_b) \leq \ell\), and there exists a \(\zeta'\) on the segment connecting \(\zeta^-\) and \(\zeta\) such that

\[
\mu_{\zeta', \eta}(A_b) = \mu_{\zeta', \eta}(A_t) = \mu_{\zeta', \eta}(A),
\]

and \((A_b, F_b^*)\) and \((A_t, F_t^*)\) are \((\zeta', \eta)\)-stable.

Since \(\eta\) is smaller enough than \(|\zeta^-|, |\zeta'|\), there exists such a \(\zeta'\) such that the last equation holds if and only if \(\zeta^0(A_t) = 0\). Hence the dimension vector of \(A_t\) is uniquely determined up
to scalar multiplication. For a fixed wall $D = \pm \alpha$ for $\alpha = (\alpha_0, \alpha_1) \in \mathbb{R}_+$ and $\ell \in [v_i]$, we introduce the set

$$D^{\alpha, \ell}(v_i) = \{ \overline{3} = (I_3, I_2) \mid [v_i] = I_3 \cup I_2, I_2 \neq \emptyset, \min(I_2) \leq \ell, \text{ and } |I_1| = p\alpha_i \text{ for some } p > 0 \}.$$  \hspace{1cm} (19)

We also put

$$w_2 = \begin{cases} 0 & \text{if } \alpha \text{ is a real root,} \\ e_i & \text{if } \alpha = \delta, \end{cases} \hspace{1cm} (20)$$

$$v_2 = p\alpha, \text{ and } v_3 = v - v_2,$$

where

$${e_0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad {e_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$ 

**Definition 3.4.** $(A_3, F^\bullet_3)$ is said to be $(D, +)$-stable if $(A_3) = 0$, $\zeta^0(A_3) = 0$, $A_2$ is $\zeta^0$-semistable, and for any proper sub-representation $S = S_0 \oplus S_1$ of $A_2$ with $\zeta^0(S) = 0$, we have $F_1^s \cap S = 0$.

Suppose that we are given a pair $(A_3, F^\bullet_3), (A_1, F^\bullet_1)$ with the decomposition type $\overline{3} = (I_3, I_2)$ satisfying $\min(I_2) \leq \ell, (A_3) = 0, (A_2) = 0$ and $\zeta^0(A_3) = 0$. We take $\zeta'$ on the segment connecting $\zeta^-$ and $\zeta$ satisfying [13].

**Lemma 3.5 ([Mo, Proposition 4.4.4], [NY2, Lemma 5.26]).** We have the following.

1. $(A_3, F^\bullet_3)$ is $(\zeta', \eta)$-stable if and only if it is $(C, (\min(I_2) - 1))$-stable.

2. $(A_3, F^\bullet_3)$ is $(\zeta', \eta)$-stable if and only if it is $(D, +)$-stable.

**Proof.** It is similarly proven as in [NY2, Lemma 5.26].

When $\alpha = \delta$, for ADHM data $(A_3, F^\bullet_3)$ on $(0, V')$ with full flags of $V'$, we take a generator of $F^1_2$, that is, a non-zero element $f^1_2$ in $F^1_2$. We consider a new ADHM data $A^+_2 = (B, z, w)$ on $(W', V')$ as follows, where $W_1 = W^0 \oplus W^1$ is a $\mathbb{Z}_2$-graded vector space such that

$$w_2 = \begin{bmatrix} \dim W^0 \\ \dim W^1 \end{bmatrix}.$$ 

We define $B \in \text{Hom}_{\mathbb{Z}_2}(Q^\vee \otimes V, V)$ by the same data as in $A_2$, and $z \in \text{Hom}_{\mathbb{Z}_2}(W, V)$ by $z(1) = f^1_2$, where 1 is a generator of $W^1$, and $w = 0 \in \text{Hom}_{\mathbb{Z}_2}(\wedge^2 Q^\vee \otimes V, W)$.

**Lemma 3.6.** ADHM data $(A_3, F^\bullet_3)$ are $(D, +)$-stable if and only if ADHM data $A^+_2$ are $C$-stable.

From this lemma, $(D, +)$-stable objects $(A_3, F^\bullet_3)$ are parametrized by the full flag bundle of a quotient of a tautological homomorphism $W \otimes \mathcal{O}_{M^C(W_2, v_2)} \rightarrow V_2$ over $M^C(W_2, v_2)$.
3.5 Moduli stacks parametrizing destabilizing objects

For $\mathcal{J} = (I_1, I_2) \in \mathcal{D}^{\alpha_f}(v_1)$, we further decompose $\mathcal{M}_3$ as follows. We fix a direct sum decomposition $V = V_1 \oplus V_2$ such that $V_1 = \mathbb{C}^{\nu - p\alpha_0} \oplus \mathbb{C}^{\nu - p\alpha_1}, V_2 = \mathbb{C}^{p\alpha_0} \oplus \mathbb{C}^{p\alpha_1}$. We consider a moduli stack

$$\mathcal{T}_p = \left[\left((\tilde{\mu}^{-1}(0)_{\mathcal{D},+}) \times \mathbb{C}^*_\mu\right) / \text{GL}(V_2)\right]$$

parametrizing tuples $(X_2, F^*_2, \rho_2)$ of $(\mathcal{D}, +)$-stable pairs $(X_2, F^*_2)$ and orientations

$$\rho_2 : \text{det} V^\otimes k_{(\zeta_0 - \zeta_0)} \otimes \text{det} V^\otimes k_{(\zeta_1 - \zeta_1)} \cong \mathbb{C}$$

Here $\tilde{\mu} : \tilde{M}(0, V_2) \rightarrow L(V_2)$ is defined in \[\tilde{\mu}^{-1}(0)(\mathcal{D},+)\] is the $(\mathcal{D}, +)$-stable locus of $\tilde{\mu}^{-1}(0)$, and $g_2 \in \text{GL}(V_2)$ acts naturally. In the following, we put

$$D = k(\zeta - \zeta)(\alpha) = k((\zeta_0 - \zeta_0)\alpha_0 + (\zeta_1 - \zeta_1)\alpha_1).$$

We consider $\zeta^0$-semistable ADHM data on $(W, V)$ with $W = 0$. It depends on $\alpha \in R_+$ defining the fixed wall $\mathcal{D} = \mathcal{D}_\alpha$. By the symmetry \[\zeta_0 < 0, \text{ or } \zeta_1 > 0,\]

If $\alpha$ is a real root, we have $\alpha = (m, m + 1)$, or $(m + 1, m)$. Then $\zeta^0$-stable ADHM data on $(W, V) = (0, \mathbb{C}^\nu \oplus \mathbb{C}^\nu)$ is unique object, written by $P(m)$, up to isomorphisms. Hence vector spaces $P_0^{(m)}$ and $P_1^{(m)}$ have $\tilde{T}$-module structures. Every $\zeta^0$-semistable ADHM data on $(W, V) = (0, V_2)$ is a direct sum of $P(m)$. For a $(\mathcal{D}, +)$-stable object $((P(m))^{\oplus p}, F^*_1)$, we see that $F^1$ are parametrized by the Grassmannian $\text{Gr}((P_1^{(m)})^\lor, p)$ of surjections $u \in \text{Hom}_\mathbb{C}((P_1^{(m)})^\lor, \mathbb{C}^p) = \mathbb{V}_2$, such that $F^1$ is generated by $u$. We write by $\mathcal{Q}$ the universal quotient on $\text{Gr}((P_1^{(m)})^\lor, p)$.

On the other hand, if $\alpha$ is an imaginary root $\delta$, then $\zeta^0$-stable ADHM data on $(0, \mathbb{C}^p \oplus \mathbb{C}^p)$ is not unique. We put

$$M^p_\alpha = \begin{cases} 
\text{Gr}((P_1^{(m)})^\lor, p) & \text{if } \alpha = \alpha_m, \\
\mathcal{M}^+(w^\perp_2, v_2) & \text{if } \alpha = \delta,
\end{cases}$$

where $w_2$ is defined in \[\text{20}.\] We also write by $\mathcal{V}_2$ the vector bundle $P_1^{(m)} \otimes \mathcal{Q}$ on $M^p_\alpha$ when $\alpha = \alpha_m$, and the tautological bundle when $\alpha = \delta$.

By Lemma \[\text{5.6}.\] we have the following proposition.

**Proposition 3.7.** $\mathcal{T}_p$ is the full flag bundle of $\mathcal{V}_1 / \mathcal{O}_{M^p_\alpha}$ over the quotient stack

$$M^p_\alpha = \begin{cases} 
\left[ \left( (\Lambda^2 \mathcal{Q})^\otimes D \right)^\times / \mathbb{C}^*_u \right] & \text{when } \alpha \text{ is a real root}, \\
\left[ \left( (\text{det } \mathcal{V}_2) \otimes D \right)^\times / \mathbb{C}^*_u \right] & \text{when } \alpha = \delta.
\end{cases}$$
where \( D = k(\zeta - \zeta^-)(\alpha) \), and \( C^*_u \) acts by fiber-wise multiplication of \( u^{pD} \).

Proof. It is proven similarly as in [NY2, Proposition 5.9] for a real root \( \alpha \), and [O, Proposition 6.1] for \( \alpha = \delta \). □

The homomorphism \( C^*_u \to C^*_s \) given by \( s = u^{pD} \) induces an étale and finite morphism \( \tilde{M}_u \to M_u \) of degree \( 1/pD \).

### 3.6 Moduli stacks of fixed points sets

By the observation in §3.4, we have a decomposition,

\[
\mathcal{M}^C = \mathcal{M}_+ \sqcup \mathcal{M}_- \sqcup \bigsqcup_{\mathfrak{J} \in \mathcal{D}} \mathcal{M}_{\mathfrak{J}},
\]

where \( \mathcal{M}_+ = \{ x = 0 \} \), and \( \mathcal{M}_3 \) is described as follows. We fix a direct sum decomposition \( V = V_\flat \oplus V_\sharp \) of a \( \mathbb{Z}_2 \)-graded vector space corresponding to decomposition data \( \mathfrak{J} \), where \( V_\flat = V_{0\flat} \oplus V_{1\flat} \) and \( V_\sharp = V_{0\sharp} \oplus V_{1\sharp} \) are also \( \mathbb{Z}_2 \)-graded.

We write by \( V_\mathfrak{J}^\flat, V_\mathfrak{J}^\sharp \) \( \mathbb{Z}_2 \)-graded tautological bundles on \( \mathcal{M}_3 \) such that we have \( V|_{\mathcal{M}_3} = V_\mathfrak{J}^\flat \oplus V_\mathfrak{J}^\sharp \) for the \( \mathbb{Z}_2 \)-graded tautological bundle \( V \) on \( \mathcal{M}_3 \). We also write by \( \mathcal{V}_\mathfrak{J}^\flat, \mathcal{V}_\mathfrak{J}^\sharp \) \( \mathbb{Z}_2 \)-graded tautological bundles on \( M^{C,\min(I_\flat)}(w, v_\flat), T_p \) corresponding to \( V_\flat, V_\sharp \), and write by the same letters their pull-backs to \( M^{C,\min(I_\flat)}(w, v_\flat) \times T_p \) by projections.

**Theorem 3.8 ([NY2, Theorem 5.18]).** For \( C^*_h \)-action on \( \mathcal{M} \) defined by (17), we have a decomposition

\[
\mathcal{M}^C = \mathcal{M}_+ \sqcup \mathcal{M}_- \sqcup \bigsqcup_{\mathfrak{J} \in \mathcal{D}} \mathcal{M}_{\mathfrak{J}}
\]

such that the followings hold.

(i) We have \( \mathcal{M}_+ \cong M^{C}(w, v) \) and \( \mathcal{M}_- \cong M^{C,0}(w, v) \), that is, the full flag bundle \( FL(V_i, [v_i]) \) of the tautological bundle \( V_i \) over \( M^{C}(w, v) \).

(ii) For each \( \mathfrak{J} = (I_\flat, I_\sharp) \in \mathcal{D}_{\alpha}(v_\flat) \), we have finite étale morphisms

\[
F: S_\mathfrak{J} \to \mathcal{M}_3, G: S_\mathfrak{J} \to M^{C,\min(I_\flat)}(r, n - p) \times T_p
\]

of degree \( \frac{1}{pD} \), where \( p = |I_\sharp|/\alpha \).

(iii) There exists a line bundle \( L_{S_\mathfrak{J}} \) on \( S_\mathfrak{J} \) such that we have isomorphisms

\[
L_{S_\mathfrak{J}}^{pD} \cong \bigotimes_{i=0,1} G^*(\det \mathcal{V}_i)^{k(\zeta_i - \zeta^-)}
\]

\[
F^* \mathcal{V}_\mathfrak{J}^\flat \cong G^* \mathcal{V}_\mathfrak{J}, \quad F^* \mathcal{V}_\mathfrak{J}^\sharp \cong G^* \mathcal{V}_\mathfrak{J} \oplus e^{\frac{1}{pD}} \otimes L_{S_\mathfrak{J}}^{\vee} \quad \text{as } C^*_h\text{-equivariant vector bundles on } S_\mathfrak{J}.
\]

Proof. This is similarly proven as in [NY2, Theorem 5.18]. □
4 Proof of Theorem 1.3

In this section, we deduce functional equations of Nekrasov partition functions as an application of the previous section. These are the similar calculations to NY2 [6], hence we omit detail explanation.

In the following, we use wall-crossing formula deduced from analysis in the previous section. For simplicity, we always assume that $\alpha$ is equal to $\alpha_m$, or $\delta$, and $i=1$, that is, use full flags of $V_1$.

4.1 Iterated cohomology classes

We prepare notation for iteration of wall-crossing formula. For a $\tilde{T}$-manifold $M$ and a $\tilde{T}$-equivariant coherent sheaf $V$ on $M$, we put $\Phi(V) = e(\mathcal{F}_r(V)) \in A^*_h(M)$, where

$$\mathcal{F}_r(V) = \bigoplus_{j=1}^{2r} V \otimes e^{m_j} / t_1 t_2.$$ 

We write by $\Theta^{rel}$ the pull backs to various moduli stacks of the relative tangent bundle of $[\mu^{-1}(0)/G] \to [\mu^{-1}(0)/G]$. We put

$$\tilde{\phi} = \Phi(V_0) \frac{\Phi(V_0) \cup e(\Theta^{rel})}{\nu^1} \in A^*_h(G \times \tilde{T} E_m) \otimes \mathbb{Q},$$

where $E_m \to E_m / \tilde{T}$ is any approximation of the universal bundle $E \tilde{T} \to B \tilde{T}$ over the classifying space.

For $j > 0$ and $\tilde{p} = (p_1, \ldots, p_j) \in \mathbb{Z}^j_{>0}$, we consider a product $M_{\tilde{p}} = \prod_{i=1}^j M^{p_i}_{\tilde{p}}$, and $M \times M_{\tilde{p}}$ for a $\tilde{T}$-equivariant manifold $M$, and write by $\int_{M_{\tilde{p}}}$ the push-forward by the projection $M \times M_{\tilde{p}} \to M$. Here $M^{p_i}_{\tilde{p}}$ is defined in [21] and appears in wall-crossing formula by Proposition 1.7. We write by $V^{(i)}_{\tilde{p}0}, V^{(i)}_{\tilde{p}1}$ the pull-backs to $M \times M_{\tilde{p}}$ of $V_{\tilde{p}0}, V_{\tilde{p}1}$ on $i$-th component of $M_{\tilde{p}}$.

For $\mathbb{Z}_2$-graded $\tilde{T}$-equivariant vector bundles $\mathcal{W}, \mathcal{V}$ on $M$, we write by the same letters the pull-backs to the product $M \times M_{\tilde{p}}$, and put

$$\mathcal{W}^{(i)} = \int_{M_{\tilde{p}}} \mathcal{W}_{\tilde{p}} \text{ Res}_{h^1=\infty} \cdots \text{ Res}_{h_j=\infty} \prod_{i=1}^j \Phi(V^{(i)}_{\tilde{p}0} \otimes e^{h_i}) e((V^{(i)}_{\tilde{p}1}/O_{M \times M_{\tilde{p}}})^\vee)$$

in $A^*_h(M)$, where $V^{(k)}_{\tilde{p}} = P_m \otimes Q^{(k)}$ for $\alpha = \alpha_m$ and $\mathbb{Z}_2$-graded tautological bundle for $\alpha = \delta$, $e^h$ is a trivial bundle with $e^h$-weight, and $\mathfrak{R}(\mathcal{W}', \mathcal{V}', \mathcal{V}'')$ is defined by

$$\mathfrak{R}(\mathcal{W}', \mathcal{V}', \mathcal{V}'') = \text{Hom}_{\mathbb{Z}_2}(Q^\vee \otimes \mathcal{V}', \mathcal{V}'') + \text{Hom}_{\mathbb{Z}_2}(\mathcal{W}', \mathcal{V}'')$$

$$- \text{Hom}_{\mathbb{Z}_2}(\mathcal{V}', \mathcal{V}'') - \text{Hom}_{\mathbb{Z}_2}(\land^2 Q^\vee \otimes \mathcal{V}', \mathcal{V}'')$$

$$+ \text{Hom}_{\mathbb{Z}_2}(Q^\vee \otimes \mathcal{V}'', \mathcal{V}') + \text{Hom}_{\mathbb{Z}_2}(\land^2 Q^\vee \otimes \mathcal{V}'', \mathcal{W}')$$

$$- \text{Hom}_{\mathbb{Z}_2}(\mathcal{V}'', \mathcal{V}') - \text{Hom}_{\mathbb{Z}_2}(\land^2 Q^\vee \otimes \mathcal{V}'', \mathcal{V}')$$ (22)
for $\mathbb{Z}_2$-graded vector bundles $\mathcal{W}', \mathcal{V}', \mathcal{V}''$.

### 4.2 Localizations

By the main result [GP(1)] and Theorem 3.8, we have the following diagram

$$
\lim_{\nu \to \infty} A^*_{\nu}(M \times \bar{\nu}) E_\nu \otimes_{\mathbb{C}[h]} C[h^\pm 1] \\
\downarrow \Pi_*([M]^{vir}) \\
\lim_{\nu \to \infty} A_*(M_0(w, v) \times \bar{\nu}) E_\nu \otimes_{\mathbb{C}[h, h^{-1}]} C[h, h^{-1}]
$$

where the upper horizontal arrow is given by

$$
\frac{\ell_*}{e(\mathcal{R}(M_+))} + \frac{\ell_*}{e(\mathcal{R}(M_-))} + \sum_{\lambda \in D_{\alpha, i}(v_i)} \frac{\ell_3}{e(\mathcal{R}(M_3))}.
$$

Here $h$ corresponds to the first Chern class in $A^*_{C^h}(pt)$ of the weight $e^h \in C^*_h$, and $\epsilon_\pm$ and $\epsilon_3$ are embeddings of $M_\pm$ and $M_3$ into $M$.

For $j_0 > 0$ and $\bar{p}_0 = (p_0, \ldots, p_{j_0}) \in \mathbb{Z}_{>0}^{j_0}$, we take equivariant classes $\varphi = \tilde{\Phi} \cup \Psi^{\bar{p}_0}$ on $M$. For the convenience, we also put $\Psi^{(1)} = 1$ for $j_0 = 0$. By the above diagram, we have

$$
\Pi_*([M]^{vir} \cap \varphi) = \Pi_* \left( \frac{[M_+]^{vir} \cap \ell_3 \varphi}{e(\mathcal{R}(M_+))} + \frac{[M_-]^{vir} \cap \ell_3 \varphi}{e(\mathcal{R}(M_-))} + \sum_{\lambda \in D_{\alpha, i}(v_i)} \frac{[M_3]^{vir} \cap \ell_3 \varphi}{e(\mathcal{R}(M_3))} \right).
$$

The left hand side is a limit of polynomials in $h$, hence taking coefficients of $h^{-1}$ we have

$$
\int_{M^c} \tilde{\Phi}(\nu_0) \cup \Psi^{\bar{p}_0} - \int_{M^c} \Phi(\nu_0) \cup \Psi^{\bar{p}_0} = \text{Res}_{h=\infty} \sum_{\lambda \in D_{\alpha, i}(v_i)} \int_{[M_3]^{vir}} \frac{\ell_3 \varphi}{e(\mathcal{R}(M_3))} \tag{23}
$$

by Theorem 3.8 (i) and $e(\mathcal{R}(M_{\pm})) = \pm (h - c_1(L_+ \otimes L_-))$. Here $\text{Res}_{h=\infty}$ denotes the operator taking the minus of coefficients of $h^{-1}$.

Furthermore by Theorem 3.8 (ii), Proposition 3.7, the right hand side is equal to

$$
\text{Res}_{h=\infty} \sum_{\lambda \in D_{\alpha, i}(v_i)} \frac{(v_1 - p_{01})(p_{01} - 1)!}{v_1!} \int_{M^c, \epsilon_{(v_i)}} \tilde{\Phi}(\nu_0) \cup \Psi^{\bar{p}_0} \cup \Psi^p, \tag{24}
$$

where $p = |\lambda|/\alpha_1$ is determined from $\lambda = (\lambda_0, \lambda_1)$. We also note that $\Psi^{\bar{p}_0} \cup \Psi^p = \Psi^{(\bar{p}_0, p)}$ by the definition in the last subsection, where $(\bar{p}_0, p) = (p_0, \ldots, p_{j_0}, p) \in \mathbb{Z}_{\geq 0}^{j_0+1}$.

In this expression, we deleted some line bundles and a parameter $pD$, since we have $\text{Res}_{h=\infty} f(h) = pD \text{Res}_{h=\infty} f(pDh + a)$ and $T_p$ are full flag bundles of $1/pD$-degree étale covering of $M_\alpha$ (cf. [O] §8.2).
4.3 Iterations

For $j > 0$, we put

$$S_j^\alpha(v_1) = \left\{ \vec{p} = (p_1, \ldots, p_j) \in \mathbb{Z}_{\geq 0}^j \mid \sum_{i=1}^j p_i \alpha_1 \leq v_1 \right\}.$$ 

Theorem 4.1. We have

$$\int_{M^c(w,v)} \Phi(V_0) - \int_{M^c(w,v)} \Phi(V_0)$$

$$= \sum_{j=1}^{\left\lceil \frac{v_1}{\alpha_1} \right\rceil} \frac{1}{\alpha_1} \sum_{p \in S_j^\alpha(v_1)} \prod_{i=1}^j \sum_{1 \leq k \leq p_i} \int_{M^c(w,v-|p_i|\vec{p})} \Phi(V_0) \cup \Psi_{\vec{p}}. \quad (25)$$

Proof. Let $\text{Dec}^{(j)}(v_1)$ be the set of collections $\mathcal{Y}^{(j)} = \langle I_0^{(j)}, I_1^{(j)}, \ldots, I_j^{(j)} \rangle$ such that

- $v_1 = I_j^{(j)} \sqcup I_{j-1}^{(j)} \sqcup \cdots \sqcup I_0^{(j)},$
- $|I_i^{(j)}| = p_i \alpha_1$ for $p_i > 0$ ($i = 1, \ldots, j$), and
- $\min(I_i^{(j)}) > \cdots > \min(I_j^{(j)}).$

We note that $\text{Dec}^{(1)}(v_1) = D^{\alpha,v_1}(v_1).$ We consider maps $\sigma_j: \text{Dec}^{(j+1)}(v_1) \to \text{Dec}^{(j)}(v_1),$

$$\mathcal{Y}^{(j+1)} = \langle I_0^{(j+1)}, I_1^{(j+1)}, \ldots, I_j^{(j+1)} \rangle \to \sigma_j(\mathcal{Y}^{(j+1)}) = \langle I_0^{(j+1)} \sqcup I_1^{(j+1)}, I_2^{(j+1)}, \ldots, I_j^{(j+1)} \rangle,$$

and $\rho_j: \text{Dec}^{(j)}(v_1) \to S_j^\alpha(v_1), \mathcal{Y}^{(j)} \mapsto \vec{p}_{\mathcal{Y}^{(j)}} = \left( \frac{|I_0|}{\alpha_1}, \ldots, \frac{|I_j|}{\alpha_1} \right).$

Lemma 4.2. We have

$$\int_{M^c(w,v)} \Phi(V_0) - \int_{M^c(w,v)} \Phi(V_0)$$

$$= \sum_{i=1}^{\left\lceil \frac{v_1}{\alpha_1} \right\rceil} \frac{|I_i^{(j)}|! \prod_{k=1}^i \left| I_k^{(j)} \right| - 1!}{v_1!} \int_{M^c(w,v-|I_i^{(j)}|\vec{p}_{\mathcal{Y}^{(j)}})} \Phi(V_0) \cup \Phi_{\vec{p}_{\mathcal{Y}^{(j)}}}$$

$$+ \sum_{\mathcal{Y} \in \text{Dec}^{(j)}(v_1)} \frac{|I_j^{(j)}|! \prod_{k=1}^j \left| I_k^{(j)} \right| - 1!}{v_1!} \int_{M^c(w,v-|I_j^{(j)}|\vec{p}_{\mathcal{Y}^{(j)}})} \Phi(V_0) \cup \Phi_{\vec{p}_{\mathcal{Y}^{(j)}}}. \quad (26)$$

Proof. We prove by induction on $j$. In fact, for $j = 1$, (26) is nothing but (23) and (24) for $j_0 = 0$ and $\ell = v_1$. For $j \geq 1$, we assume the formulas (24), then again by (23) and (24), the
By Theorem 4.1, it is enough to show that all summands in the right hand of (25) vanish. For $j > 1$, we put $\Phi(\mathcal{V}_0) = \Psi_{\tilde{f}_j}^{(j)}(\alpha)$. From Theorem 4.1, we get the following vanishing theorem, which gives a proof of Theorem 4.4.

\[
\int_{M^c(w,v)} \Phi(\mathcal{V}_0) = \int_{M^c(w,v)} \Phi(\mathcal{V}_0) = \sum_{j=1}^{\left\lfloor \frac{\alpha_1}{\alpha} \right\rfloor} \sum_{\mathcal{Y}^{(j)} \in \mathcal{Y}^{(j)}(v_1)} |I_{2k}^{(j)}|! \prod_{k=1}^{(j)} |I_{2k+1}^{(j)}| - 1|! \int_{M^c(w,v-|\tilde{f}_j|,\alpha)} \Phi(\mathcal{V}_0) \Psi_{\tilde{f}_j}^{(j)}(\alpha),
\]

where $p_{j+1} = |I_{2j+1}^{(j+1)}|/\alpha_1$, and $(\tilde{p}_{j+1}, p_{j+1}) = (\tilde{f}_j^{(j)}, j+1)$. Hence we have (26) for general $j \geq 1$.

For $j > \frac{\alpha_1}{\alpha_1}$, the set $\mathcal{Y}^{(j)}(v_1)$ is empty. Thus we have

\[
\int_{M^c(w,v)} \Phi(\mathcal{V}_0) = \int_{M^c(w,v)} \Phi(\mathcal{V}_0) = \sum_{j=1}^{\left\lfloor \frac{\alpha_1}{\alpha} \right\rfloor} \sum_{\mathcal{Y}^{(j)} \in \mathcal{Y}^{(j)}(v_1)} |I_{2k}^{(j)}|! \prod_{k=1}^{(j)} |I_{2k+1}^{(j)}| - 1|! \int_{M^c(w,v-|\tilde{f}_j|,\alpha)} \Phi(\mathcal{V}_0) \Psi_{\tilde{f}_j}^{(j)}(\alpha). \tag{27}
\]

We note that each summand in the last sum depends only on $\tilde{p}_{j}^{(j)}$ at this stage.

**Lemma 4.3.** For $\tilde{\rho} \in S^a(v_1)$ and $\mathcal{Y}^{(j)} = (I_{2q_1}^{(j)}, I_{2q_2}^{(j)}, \ldots, I_{2q_{j+1}}^{(j)}) \in \rho_j^{-1}(\tilde{\rho})$, we have

\[
|\rho_j^{-1}(\tilde{\rho})| = \frac{1}{\alpha_1^{\frac{\alpha_1}{\alpha}} \prod_{k=1}^{j+1} p_k |I_{2k}^{(j)}|! \prod_{k=1}^{j+1} (|I_{2k}^{(j)}| - 1)!}.
\]

**Proof.** This follows from [NY2, Lemma 6.8] since $|I_{2k}^{(j)}| = p_k \alpha_1$. \hfill \Box

By this lemma and (26), we get (25) and complete the proof of Theorem 4.1. \hfill \Box

### 4.4 Wall-crossing across $\mathcal{D}_m$

We take a real root $\alpha_m = (m, m + 1) \in R_+$, and consider the case where $\mathcal{D} = \mathcal{D}_{\alpha_m}$ when $k \leq 0$.

**Theorem 4.4.** We put $k = -2v_0 + 2v_1 - w_1$. Then we have

\[
\int_{M^m(w,v)} e(F_r(\mathcal{V}_0)) = \int_{M^o(w,v)} e(F_r(\mathcal{V}_0)) \text{ for } \begin{cases} m \leq 0 & \text{if } k \geq 0, \\ m \geq 0 & \text{if } k \leq 0. \end{cases}
\]

**Proof.** By Theorem 4.1, it is enough to show that all summands in the right hand of (25) vanish. For $k \leq 0$, it follows from the similar arguments in [NY2, Theorem 2.1]. In fact, we
have cohomological degrees
\[
\deg \int_{M^{m+1}(w, v)} \Phi(\mathcal{E}) = \deg \int_{M^m(w, v)} \Phi(\mathcal{E}) = \dim M^m(w, v) - 2r v_0 = 2(w_0 v_0 + w_1 v_1) - 2(v_0 - v_1)^2 - 2r v_0.
\]

On the other hand, we can write summands in the right hand of (23) as \( \int_{M^m(w, v) - j \alpha_m} e(\mathcal{F}_r(\mathcal{V})) \cup \)? for some cohomology class \( ? \). Therefore its degree is at most
\[
\dim M^m(w, v - j \alpha_m) - 2r(v_0 - jm) = 2w_0(v_0 - jm) + 2w_1(v_1 - jm - j) - 2(v_0 - v_1 + j)^2 - 2r v_0 + 2rjm = 2(w_0 v_0 + w_1 v_1) - 2(v_0 - v_1)^2 - 2r v_0 + 2j(-2v_0 + v_1 - w_1 - j) < \dim M^m(w, v) - 2r v_0
\]
if \( k = -2v_0 + 2v_1 - w_1 \leq 0 \) and \( j > 0 \). Hence it is zero.

For the case where \( k \geq 0 \), we can use isomorphisms (3) to reduce the case where \( k \leq 0 \).

\[ \square \]

This gives a proof of Theorem 1.3 for \( k \leq 0 \).

### 4.5 Wall-crossing across \( \mathcal{D}_\infty \)

We assume \( k \geq 0 \), and apply (23) in the case where \( \mathcal{D} = \mathcal{D}_\infty, \mathcal{C} = \mathcal{C}_+ \) and \( \mathcal{C}' = -\mathcal{C}_- \) (cf. §2.6). Here \( \mathcal{D}_\infty \) and \( \mathcal{C}_\pm \) are defined in (22).

By Theorem 1.4, the integrations over \( M^0(w, v) \) and \( M^-(w, v) \) are same. Furthermore, since we have a \( \tilde{T} \)-equivariant isomorphism \( M^-(w, v) \cong M^{-\mathcal{C}_-}(w, v) \) by (10) modulo the automorphism of \( G \times \tilde{T} \), we can show that
\[
Z^k_{X_0}(\epsilon, -\alpha, -m, q) = \sum_{v_0 \in \mathbb{Z}} q^{v_0} \int_{M^{-\mathcal{C}_-}(w, v)} e(\mathcal{F}(\mathcal{V}))
\]
similarly as in [3] §3.2.

For \( n = v_0 + \frac{m}{2} \), we put
\[
\alpha_n = \int_{M^+(w, v)} e(\mathcal{F}(\mathcal{V})), \beta_n = \int_{M^{-\mathcal{C}_-}(w, v)} e(\mathcal{F}(\mathcal{V}))
\]
so that \( Z^k_{X_1}(\epsilon, a, m, q) = \sum \alpha_n q^n \), and \( Z^k_{X_0}(\epsilon, -a, -m, q) = \sum \beta_n q^n \). From the combinatorial description in (13) we can show \( Z^k_{X_0}(-\epsilon, -a, -m, q) = Z^k_{X_0}(\epsilon, a, m, q) \) as in [NY1] Lemma 6.3 (3). Hence to prove Theorem 1.3 for \( k \geq 0 \), we must show
\[
\alpha_n = \sum_{k=0}^n \frac{(-1)^k r (u_r + 1) \cdots (u_r + k - 1)}{k!} \beta_{n-k}, \quad (28)
\]
Proposition 4.5. When $\alpha = \delta$, we have

$$\mathcal{F}(\mathcal{G}) = \mathcal{F}(\mathcal{D}) + \mathcal{F}(\mathcal{E})$$

where

$$\mathcal{F}(\mathcal{G}) = \mathcal{F}(\mathcal{D}) + \mathcal{F}(\mathcal{E})$$

Proof. We have

$$\mathcal{F}(\mathcal{G}) = \sum_{|\mathcal{G}|=p} \text{Res}_{h=\infty} \frac{\mathcal{F}(\mathcal{G})}{\mathcal{D}(\mathcal{H})} + \frac{\mathcal{F}(\mathcal{D})}{\mathcal{E}(\mathcal{G})} + \frac{\mathcal{F}(\mathcal{E})}{\mathcal{G}(\mathcal{D})}$$

Here $\mathcal{G}(\mathcal{D}) = T_{\mathcal{D}}M^+(\mathcal{W}, \mathcal{V})/\text{Hom}_{\mathcal{D}}(\mathcal{D}, \mathcal{V})$, and $T_{\mathcal{D}}M^+(\mathcal{W}, \mathcal{V})$ is the tangent space at $\mathcal{D}$ in $M^+(\mathcal{W}, \mathcal{V})$. This is the normal bundle induced from obstruction theories on $\mathcal{M}$ defined similarly as in [O] §5. Since $\mathcal{D} = t_1t_2$ as a $\mathcal{T}$-module, we have

$$\mathcal{F}(\mathcal{G}) = \mathcal{F}(\mathcal{D}) + \mathcal{F}(\mathcal{E})$$

Here we may only take pairs of Young diagrams $\mathcal{D} = (Y^1, Y^2)$ such that either one of $Y^1$ or $Y^2$ is the empty set. For if neither one is not empty, we see by Proposition [3.5] that there is a two dimensional trivial $\mathcal{T}$-submodule in the fiber of $\mathcal{V}$ over the fixed point corresponding to $\mathcal{D}$, and the Euler class $e(\mathcal{V})$ vanishes there.

On the other hand, we have

$$\text{Res}_{h=\infty} \frac{\mathcal{F}(\mathcal{G})}{\mathcal{D}(\mathcal{H})} =$$

$$\left(2 \sum_{a=1}^{r} a_k + 2 \sum_{f=1}^{m_f} m_f \right) + 4(v_{11} - v_{00} - w_1)\mathcal{F}(\mathcal{V} - \mathcal{V}_{01} - \mathcal{V}_{11}) - c_1$$

$$c_1 = \text{Res}_{h=\infty} \frac{\mathcal{F}(\mathcal{G})}{\mathcal{D}(\mathcal{H})} =$$

$$\left(2 \sum_{a=1}^{r} a_k + 2 \sum_{f=1}^{m_f} m_f \right) + 4(v_{11} - v_{00} - w_1)\mathcal{F}(\mathcal{V} - \mathcal{V}_{01} - \mathcal{V}_{11}) - c_1$$

$$c_1 = \text{Res}_{h=\infty} \frac{\mathcal{F}(\mathcal{G})}{\mathcal{D}(\mathcal{H})} =$$

$$\left(2 \sum_{a=1}^{r} a_k + 2 \sum_{f=1}^{m_f} m_f \right) + 4(v_{11} - v_{00} - w_1)\mathcal{F}(\mathcal{V} - \mathcal{V}_{01} - \mathcal{V}_{11}) - c_1$$
since \( v_{21} - v_{10} = p - p = 0 \). By Proposition [B.3] we have

\[
\iota_{Y}^*(c_1(\det \mathcal{V}_{21}) - c_1(\det \mathcal{V}_{10})) = \begin{cases} 
\epsilon & \text{if } Y = (Y, \emptyset) \\
\epsilon' & \text{if } Y = (\emptyset, Y).
\end{cases}
\]

Furthermore by [O, Proposition 8.1] and Proposition [B.5] and [B.7] we have

\[
\sum_{|Y^1| + |Y^2| = p, Y^1 \neq \emptyset} \iota_{Y}^* e(\mathcal{V}_1/\mathcal{O}_{M^+(w_1, v_1)} \oplus \mathcal{V}_2^\vee \otimes \iota_{1} \iota_{2}) = \begin{cases} 
\epsilon & \text{if } j = 2 \\
\epsilon' & \text{if } j = 1.
\end{cases}
\]

Combining these together, we get the assertion.

By [23], [24] and Proposition [B.5] we have the following equation

\[
\int_{\bar{\mathcal{M}}^+(\tau(w, v))} \tilde{\Phi}(\mathcal{V}_0) - \int_{\mathcal{M}^+(\tau(w, v))} \Phi(\mathcal{V}_0) = (-1)^{\tau p+1} \binom{v_1 - p}{(p-1)!} u_r \sum_{\delta \in \Delta^+(\tau)} \int_{\bar{\mathcal{M}}^+(\tau \cup \delta)^{-1}(\tau(w, v) - \delta)} \tilde{\Phi}(\mathcal{V}_0),
\]

where \( u_r = \frac{\epsilon(2 \sum_{a_i = 1}^{r} a_i + 2 \sum_{j=1}^{r} m_j)}{2\epsilon (\epsilon')}. \)

Then the similar arguments as in [13] or [O, §8.3] imply (28), and we complete the proof of Theorem [13].

### A Construction of framed moduli on \( X_K \)

We show that moduli of ADHM data gives framed moduli on \( X_K \) for \( k = 0, 1 \). This is just a slight modification of the proof of [N3, Theorem 2.2] to the relative setting.

#### A.1 Beilinson complex

We consider tautological bundles \( \mathcal{R}_0 = \mathcal{O}_{X_0} \) and \( \mathcal{R}_1 = \mathcal{O}_{X_0}(F - \ell_{\infty}) \), and put \( \mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1 \) as in [14]. For \( i \in \mathbb{Z}/2\mathbb{Z} \), we define tautological homomorphisms \( \xi : \mathcal{R}_i \rightarrow \mathcal{R}_{i+1}(\ell_{\infty}) \) and \( \tilde{\xi} : \mathcal{R}_{i+1} \rightarrow \mathcal{R}_i(\ell_{\infty}) \) so that \( \xi_0, \tilde{\xi}_0, \xi_1, \tilde{\xi}_1, \xi_{-1}, \tilde{\xi}_{-1} \) on both \( X_0 \) and \( X_1 \) as follows. On \( X_0 \), we define \( \xi \) and \( \tilde{\xi} \) by the multiplication of \( x_1 \) and \( x_2 \) respectively, where \( [x_0, x_1, x_2] \) is the homogeneous coordinate of \( \mathbb{P}^2 \). On \( X_1 \), we define \( \xi_0, \xi_1, \tilde{\xi}_0, \tilde{\xi}_1 \) and \( \xi_{-1}, \tilde{\xi}_{-1} \) by the multiplication of \( x_1, yx_1, yx_2 \) and \( x_2 \) respectively, where we use description of \( X_1 \) in [15]. By the construction [3] and [1], these define homomorphisms \( \xi, \tilde{\xi} \) on both \( X_0 \) and \( X_1 \), and satisfy \( \tilde{\xi}_i \xi_{i-1} = \xi_{i-1} \tilde{\xi}_{i-1} \). We can also check easily that these homomorphisms coincide on \( X_0 \setminus O \cong X_1 \setminus C \). We put

\[
\Xi = e \oplus 0 \begin{bmatrix} 0 & \xi_1 \\ \xi_0 & 0 \end{bmatrix} + \bar{e} \oplus 0 \begin{bmatrix} 0 & \tilde{\xi}_0 \\ \tilde{\xi}_1 & 0 \end{bmatrix} \in \text{Hom}_{\mathbb{Z}_2}(Q^\vee \otimes \mathcal{R}, \mathcal{R}(\ell_{\infty})),
\]

where \( Q \) is the degree 2 line bundle on \( \mathbb{P}^2 \) corresponding to the second projection of \( \mathcal{R} \).
where $e, \bar{e}$ is a basis of $Q$.

We take $\zeta = (\zeta_0, \zeta_1) \in \mathbb{R}^2$ such that

$$
\zeta_0 + \zeta_1 < 0, \ z_0 < 0
$$

and does not lie on any wall. For a moduli $M^\zeta(\mathbf{w}, \mathbf{v})$ of ADHM data on $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces $W, V$, we also have tautological bundle $W_i, V_i$ for $i \in \mathbb{Z}/2\mathbb{Z}$ corresponding to $W_i, V_i$, and put $V = V_0 \oplus V_1, W = W_0 \oplus W_1$. Write by $B: Q^\vee \otimes V \to V, z: W \to V$ and $w: \wedge^2 Q \otimes V \to W$ the tautological homomorphism corresponding to $B, z$ and $w$ in Definition 2.1

We consider the following complex on $X_\kappa \times M^\zeta(\mathbf{w}, \mathbf{v})$:

$$
\text{Hom}_{\mathbb{Z}_2}(p_1^* R^\vee(\ell_\infty), p_2^* V) \xrightarrow{\text{Hom}_{\mathbb{Z}_2}(Q^\vee \otimes p_1^* R^\vee, p_2^* V)} \wedge^2 R^\vee(-\ell_\infty), p_2^* V), \ (30)
$$

where $p_1$ and $p_2$ are projections to the first and second components respectively. We call this Beilinson complex. The differentials $\sigma$ and $\tau$ are defined by

$$
\sigma(\eta) = \left[ B \eta x_0 - \eta \Xi^\vee \right], \ \tau(\eta, \gamma) = \left[ B \eta x_0 - \eta \Xi^\vee \right].
$$

Here $\eta \in \text{Hom}_{\mathbb{Z}_2}(p_1^* R^\vee(\ell_\infty), p_2^* V), \eta' \in \text{Hom}_{\mathbb{Z}_2}(Q^\vee \otimes p_1^* R^\vee, p_2^* V)$ and $\gamma \in \text{Hom}_{\mathbb{Z}_2}(p_1^* R^\vee, p_2^* V)$. $\Xi^\vee$ is the dual of $\Xi$, and we regard $x_0$ as a homomorphism among suitable line bundles. We identify $B: Q^\vee \otimes V \to V$ with the adjoint $V \to Q \otimes V$, and also write by the same symbol $B$ the composition $\text{id}_Q \otimes B: Q \otimes V \to Q \otimes Q \otimes V$ and the natural surjection $Q \otimes Q \otimes V \to \wedge^2 Q \otimes V$.

We use same notation for $\Xi^\vee$. Then this is just a relative version of [3], since $Q = Q_0 \oplus Q_1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space with $Q_0 = 0$ and $Q_1 = \mathbb{C}^2$.

By the condition [20], we can show that the restriction of $\sigma$ and $\tau$ to $X_1 \times \{m\}$ for any closed point $m \in M^\zeta(\mathbf{w}, \mathbf{v})$ is injective and surjective respectively as a sheaf homomorphism by the similar argument as in [NY1]. Hence, using [MR] Theorem 22.5, we see that $E = \ker \tau/\text{im} \sigma$ is flat over $M^\zeta(\mathbf{w}, \mathbf{v})$. This gives a family of framed sheaves $(E, \Phi)$ on $X_\kappa \times M^\zeta(\mathbf{w}, \mathbf{v})$, where $\Phi$ is naturally given by restricting the Beilinson complex [30] to $\ell_\infty \times M^\zeta(\mathbf{w}, \mathbf{v})$. We call $(E, \Phi)$ a universal framed sheaf, and will check that this is true in the rest of this section.

We take $\zeta^0 \in \mathbb{C}$ and $\zeta^1 \in \mathbb{C}$. Then we have a morphism from $M^\zeta(\mathbf{w}, \mathbf{v})$ to framed moduli $M_{X_\kappa}(c)$ on $X_\kappa$ for $\kappa = 0, 1$, where $c \in A(I X_\kappa)$ is defined by [4] and [7]. That is, if we have a family of ADHM data on a scheme $S$, we have a morphism $S \to M^\zeta(\mathbf{w}, \mathbf{v})$, and the pull-back $(E_S, \Phi_S)$ of universal framed sheaf $(E, \Phi)$ to $X_\kappa \times S$ is a flat family over $S$ of framed sheaves on $X_\kappa$. This gives a morphism $M^\zeta(\mathbf{w}, \mathbf{v}) \to M_{X_\kappa}(c)$, and by [N3], this is bijection at least set theoretically.
A.2 Resolution of diagonal on $X_\kappa \times X_\kappa$

To construct the converse $M_{X_\kappa}(c) \to M^C(w,v)$, we consider resolutions of diagonals $\Delta, \mathcal{O}_{X_\kappa}$ on $X_\kappa \times X_\kappa$, where $\Delta: X_\kappa \to X_\kappa \times X_\kappa$ is the diagonal embedding.

First we construct resolutions on $X^0_\kappa \times X^\circ_\kappa$. We regard $X^0_\kappa \times X^0_\kappa$ as $[Q \times Q/H_1 \times H_2]$, where $H_1$ and $H_2$ are copies of $H = \{ \pm \text{id}_Q \}$ and acts on the first and second components respectively. Then $\Delta, \mathcal{O}_{X_0}$ can be identified with $\mathcal{O}_Q \oplus \mathcal{O}_{-Q}$, where

$$-Q = \{(x_1, x_2), -(x_1, x_2)\} \in Q \times Q \mid (x_1, x_2) \in Q = \mathbb{C}^2\}.$$ 

We consider the following complex on $X^0_\kappa \times X^\circ_\kappa$:

$$\mathcal{H}om_{\mathbb{Z}_2}(p_1^*\mathcal{R}'', p_2^*\mathcal{R}'')|_{X^0_\kappa \times X^0_\kappa} \to \mathcal{H}om_{\mathbb{Z}_2}(Q' \otimes p_1^*\mathcal{R}', p_2^*\mathcal{R}')|_{X^0_\kappa \times X^\circ_\kappa} \to \mathcal{H}om_{\mathbb{Z}_2}(Q' \otimes p_1^*\mathcal{R}', p_2^*\mathcal{R})|_{X^0_\kappa \times X^\circ_\kappa} \to \Delta, \mathcal{O}_{X^\circ_\kappa}. \quad (31)$$

Here $d^{-2}, d^{-1}$ is defined by replacing $B$ in first components of $\sigma, \tau$ in $[50]$ with $\Xi'$. The last differential $d_0$ is defined by the restriction to the diagonal and taking the contraction, where $p_2\mathcal{R}_{-Q}$ on $-Q \cong Q$ is identified with $\mathcal{R}$ by multiplication of $-1$. This gives a resolution of the diagonal $\Delta, \mathcal{O}_{X_\kappa}$ by $[N2]$ Lemma 4.10] and $[N3]$ 3(iii)], and they are identified on $X_0 \setminus O \cong X_1 \setminus C$. This complex without the last component can be viewed as a special case of (30) where $W = 0$.

But when $W = 0$, we need to change definition of stability. We take $\zeta$ such that $(\zeta, v) = 0$.

**Definition A.1.** We say that ADHM data $(B, 0, 0)$ on $W = 0, V$ are $\zeta$-semistable if the following conditions hold: For any $\mathbb{Z}_2$-graded subspace $S$ of $V$, if $B(S) \subset S$, then we have $\zeta(S) \leq 0$. They are said to be $\zeta$-stable when the strict inequality always holds for non-trivial proper subspace $S$.

For $\zeta = \zeta(1, -1)$ with $\zeta > 0$ and $\delta = (1, 1)$, we have an isomorphism $M^C(0, \delta) \cong X^\circ_\kappa = X_1 \setminus \ell_\infty$ such that tautological bundles $\mathcal{V}_0, \mathcal{V}_1$ coincide with $\mathcal{R}_0, \mathcal{R}_1$, and $B = \xi$. Hence $d_{-2}$ and $d_{-3}$ in $[51]$ are naturally obtained from $[50]$.

To extend the complex $[51]$ on $X^\circ_\kappa$ to the compactification $X_\kappa$, we identify $X_0 \times X_0$ with $[\mathbb{P}^2 \times \mathbb{P}^2/H_1 \times H_2]$ as above, where $H_1$ and $H_2$ are copies of $H$ and acts on the first and second components respectively. We construct a resolution of the diagonal on $X_0 \times X_0$ as complexes of $H_1 \times H_2$-equivariant vector bundles on $\mathbb{P}^2 \times \mathbb{P}^2$. We construct vector bundles $\mathcal{Q}$ on $X_0$ and $X_1$ by

$$Q = \text{coker}(x_0 \oplus \xi_0 \oplus \xi_1: \mathcal{O}_{X_\kappa}(-\ell_\infty) \to \mathcal{O}_{X_\kappa} \oplus \mathcal{R}_1 \oplus \mathcal{R}_1).$$

On $X_0 \times X_0$, we consider a map $p_2^*\mathcal{O}(-\ell_\infty) \to \mathcal{C}[H] \otimes p_1^*\mathcal{Q}$, defined by $x_2 \mapsto \sum_{\gamma \in H} \gamma \otimes \varphi(\gamma x_2)$, where $\varphi$ is defined by the compositions

$$p_2^*\mathcal{O}_{\mathbb{P}^2}(-1) \to p_1^*\mathcal{O}_{\mathbb{P}^2} \otimes (\mathcal{C} \oplus \mathcal{Q}) \to p_1^*\mathcal{O}_{\mathbb{P}^2} \otimes (\mathcal{C} \oplus \mathcal{Q})/p_*^2\mathcal{O}_{\mathbb{P}^2}(-1)$$

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on $\mathbb{P}^2 \times \mathbb{P}^2$. Here $(\gamma_1, \gamma_2) \in H_1 \times H_2$ acts on $\mathbb{C}[H]$ by $\gamma_1 \gamma_2^{-1}$. Then this map is $H_1 \times H_2$-equivariant.

Hence we have a section $s \in H(X_0 \times X_0, \mathbb{C}[H] \otimes \mathcal{O}_{X_0}(\ell_{\infty}) \otimes \mathcal{Q})$, whose zero locus is equal to $\Delta_* \mathcal{O}_{X_0} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{-\mathbb{P}^2}$, and the Koszul resolution

$$0 \to C_0^{-2} \to C_0^{-1} \to C_0^0 \to \Delta_* \mathcal{O}_{X_0} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{-\mathbb{P}^2} \to 0. \quad (32)$$

Here $-\mathbb{P}^2 = \{(\xi x_0, x_1, x_2), [x_0, -x_1, -x_2]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid [x_0, x_1, x_2] \in \mathbb{P}^2\}$, and

$$C_0 = \mathbb{C}[H] \otimes \mathcal{O}_{X_0 \times X_0}, C_0^{-1} = \mathbb{C}[H] \otimes \mathcal{Q} \otimes \mathcal{O}_{X_0}(\ell_{\infty}), C_0^{-2} = \mathbb{C}[H] \otimes \mathcal{Q} \otimes \mathcal{O}_{X_0}(-2\ell_{\infty}).$$

We have $\mathcal{Q}|_{X_0^2} \cong \mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{Q}$, and $\mathbb{C}[H] \cong \bigoplus_{\ell \in \mathbb{Z}/2\mathbb{Z}} \text{Hom}(R^\vee, R_i^\vee)$, where $R_0$ and $R_1$ are trivial and non-trivial irreducible $H$-representations. We can identify $R_0$ and $R_1$ with $H$-equivariant line bundles $\mathcal{O}_{\mathbb{P}^2} \otimes R_0$ and $\mathcal{O}_{\mathbb{P}^2} \otimes R_1$. Then we have isomorphisms

$$C_0^{-1} |_{X_0^2 \times X_0^2} \cong \mathcal{H}om_{\mathbb{Z}_2}(p_1^\vee \mathcal{R}^\vee, p_2^\vee \mathcal{R}^\vee)|_{X_0^2 \times X_0^2}, \quad C_0^{-1} |_{X_0^2 \times X_0^2} \cong \mathcal{H}om_{\mathbb{Z}_2}(\mathcal{Q} \otimes p_1^\vee \mathcal{R}^\vee, p_2^\vee \mathcal{R}^\vee)|_{X_0^2 \times X_0^2},$$

and $C_0^0 |_{X_0^2 \times X_0^2} \cong \mathcal{H}om_{\mathbb{Z}_2}(\mathcal{Q} \otimes p_1^\vee \mathcal{R}^\vee, p_2^\vee \mathcal{R}^\vee)|_{X_0^2 \times X_0^2}$. Via these isomorphisms, we can check that the complex $^{43}$ coincides with $^{41}$.

Hence we can patch the restriction of $^{42}$ to $(X_0 \setminus \mathcal{O}) \times (X_0 \setminus \mathcal{O})$ and $^{43}$ on $X_1^0 \times X_1^0$ to get the resolution of the diagonal on $X_1 \times X_1$:

$$0 \to C_1^{-2} \to C_1^{-1} \to C_1^0 \to \Delta_* \mathcal{O}_{X_1} \to 0. \quad (33)$$

Here we can write $C_1^{-2} = \bigoplus_{\ell \in \mathbb{Z}/2\mathbb{Z}} \mathcal{R}^\vee(-2\ell_{\infty}) \boxtimes \mathcal{R}_i \otimes \wedge^2 Q^\vee, C_1^{-1} = \bigoplus_{\ell \in \mathbb{Z}/2\mathbb{Z}} \mathcal{R}_i^\vee(-\ell_{\infty}) \boxtimes \mathcal{R}_i \otimes Q^\vee$, and $C_1^0 = \bigoplus_{\ell \in \mathbb{Z}/2\mathbb{Z}} \mathcal{R}_i \boxtimes \mathcal{R}_i$ for $\kappa = 0, 1$.

### A.3 Beilinson spectral sequence

For a finitely generated $\mathbb{C}$-algebra $A$, we put $S = \text{Spec} A$. Here we consider families of framed sheaves $(\mathcal{E}_S, \Phi_S)$ on $X_\kappa \times S$, where $\mathcal{E}_S$ is a torsion free sheaf on $X_\kappa \times S$ flat over $S$, and $\Phi_S : \mathcal{E}_S|_{X_\kappa \times S} \cong W \otimes \mathcal{O}_{X_\kappa \times S}$ is an isomorphism. Pulling back resolutions $^{42}$ and $^{43}$, we get resolutions of diagonals on $X_\kappa \times X_\kappa \times S$.

We construct Beilinson complexes from framed sheaves $(\mathcal{E}_S, \Phi_S)$. We consider

$$\mathbb{R}p_{1*}(p_2^* \mathcal{E}_S(-\ell_{\infty}) \otimes \Delta_* \mathcal{O}_{X_\kappa}) = \mathbb{R}p_{1*}(p_2^* \mathcal{E}_S(-\ell_{\infty}) \otimes C^\bullet_\kappa)$$

as a double complex, where $p_1, p_2 : X_\kappa \times X_\kappa \times S \to X_\kappa \times S$ are the first and second projections, and $C^\bullet_\kappa$ is the complex in $^{42}$ and $^{43}$. We have a spectral sequence associated to this double complex, whose $E_2$-term is given by

$$E^{pq}_2 = \mathbb{R}q_{1*}(p_2^* \mathcal{E}_S(-\ell_{\infty}) \otimes C^p_\kappa).$$
Explicitly, we have

\[
E'^p = \begin{cases} 
\bigoplus_{i \in \mathbb{Z}/2\mathbb{Z}} \mathcal{R}^i_s(-2\ell) \boxtimes \mathbb{R}^q \mathcal{P}_s(\mathcal{E}_s(-2\ell) \otimes \mathcal{R}_i) & \text{for } p = -2, \\
\bigoplus_{i \in \mathbb{Z}/2\mathbb{Z}} \mathcal{R}^i_s(-\ell) \boxtimes \mathbb{R}^q \mathcal{P}_s(\mathcal{E}_s(-\ell) \otimes \mathcal{R}_i \otimes \mathcal{Q}^\vee) & \text{for } p = -1, \\
\bigoplus_{i \in \mathbb{Z}/2\mathbb{Z}} \mathcal{R}^i_s(-\ell) \boxtimes \mathbb{R}^q \mathcal{P}_s(\mathcal{E}_s(-\ell) \otimes \mathcal{R}_i) & \text{for } p = 0,
\end{cases}
\]

where \( p_s: X_\kappa \times S \to S \) is the projection.

We need the following vanishing lemma.

**Lemma A.2.** For \( i = 0, 1 \), we have

\[
\begin{align*}
\mathbb{R}^q \mathcal{P}_s(\mathcal{E}_s(-k\ell) \otimes \mathcal{R}_i) &= 0 & \text{for } k = 1, 2, q = 0, 2, \\
\mathbb{R}^q \mathcal{P}_s(\mathcal{E}_s(-\ell) \otimes \mathcal{R}_i \otimes \mathcal{Q}^\vee) &= 0 & \text{for } q = 0, 2.
\end{align*}
\]

**Proof.** This follows from [N2] Lemma 2.4. \( \square \)

From this lemma, \( \mathcal{V}_i = \mathbb{R}^1 \mathcal{P}_s(\mathcal{E}_s(-2\ell) \otimes \mathcal{R}_i) \), \( \mathcal{V}^\prime_i = \mathbb{R}^1 \mathcal{P}_s(\mathcal{E}_s(-\ell) \otimes \mathcal{R}_i \otimes \mathcal{Q}^\vee) \), and \( \mathcal{W} = \mathbb{R}^1 \mathcal{P}_s(\mathcal{E}_s(-\ell) \otimes \mathcal{R}_i \otimes \mathcal{Q}^\vee) \) are vector bundles on \( S \). Furthermore, the complex \( E^q_{\ell}(-\ell) \) on \( X_\kappa \times S \)

\[
0 \to \bigoplus_{i \in \mathbb{Z}/2\mathbb{Z}} \mathcal{R}^i_s(-\ell) \boxtimes \mathcal{V}_i \xrightarrow{d} \bigoplus_{i \in \mathbb{Z}/2\mathbb{Z}} \mathcal{R}^i_s \boxtimes \mathcal{W} \xrightarrow{\delta} \bigoplus_{i \in \mathbb{Z}/2\mathbb{Z}} \mathcal{R}^i_s(\ell) \boxtimes \mathcal{V}_i' \to 0 \tag{34}
\]

satisfies \( \ker a = 0, \text{coker } b = 0, \) and \( \mathcal{E}_s \cong \ker b/\text{im } a \).

We can write

\[
a = \begin{bmatrix} x_0 \text{id}_{\mathcal{R}_0} \boxtimes a_0^0 & \xi_1 \boxtimes a_1^0 + \xi_1 \boxtimes a_1^1 \\ \xi_0 \boxtimes a_0^0 + \xi_0 \boxtimes a_0^1 & x_0 \text{id}_{\mathcal{R}_0} \boxtimes a_0^1 \end{bmatrix},
\]

\[
b = \begin{bmatrix} x_0 \text{id}_{\mathcal{R}_0} \boxtimes b_0^0 & \xi_1 \boxtimes b_1^0 + \xi_1 \boxtimes b_1^1 \\ \xi_0 \boxtimes b_0^0 + \xi_0 \boxtimes b_0^1 & x_0 \text{id}_{\mathcal{R}_0} \boxtimes b_0^1 \end{bmatrix}.
\]

We put \( \mathcal{V} = \bigoplus_{i=0}^1 \mathcal{V}_i, \mathcal{W} = \bigoplus_{i=0}^1 \mathcal{W}_i \) and \( \mathcal{V}' = \bigoplus_{i=0}^1 \mathcal{V}'_i \), and

\[
a_0 = \begin{bmatrix} a_0^0 & 0 \\ 0 & a_0^1 \end{bmatrix}, a_k = \begin{bmatrix} 0 & a_k^1 \\ a_k^0 & 0 \end{bmatrix}, b_0 = \begin{bmatrix} b_0^0 & 0 \\ 0 & b_0^1 \end{bmatrix}, b_k = \begin{bmatrix} 0 & b_k^1 \\ b_k^0 & 0 \end{bmatrix}
\]

for \( k = 1, 2 \). Then we have \( a_0, a_1, a_2 \in \text{Hom}_S(\mathcal{V}, \mathcal{W}) \) and \( b_0, b_1, b_2 \in \text{Hom}_S(\mathcal{W}, \mathcal{V}') \).

Over \( X_0 \setminus O \cong X_1 \setminus C \), we can write

\[
a = a_0 x_0 + a_1 x_1 + a_2 x_2, b = b_0 x_0 + b_1 x_1 + b_2 x_2.
\]

Since \( ba = 0 \), we have \( b_0 a_1 = 0, b_1 a_{i+1} + b_{i+1} a_i = 0 \) for \( i \in \mathbb{Z}/3\mathbb{Z} \). Restricting the complex \( E^q_{\ell}(-\ell) \) to \( \ell \), we have

\[
0 \to \mathcal{O}_\mathbb{P}^{(-1)}(1) \boxtimes \mathcal{V}^{\ell} \xrightarrow{a_{\ell}} \mathcal{O}_\mathbb{P}^{(1)}(1) \boxtimes \mathcal{V}' \to 0,
\]

where \( a_{\ell} = x_1 \boxtimes a_1 + x_2 \boxtimes a_2, b_{\ell} = x_1 \boxtimes b_1 + x_2 \boxtimes b_2 \), and we regard \( \mathcal{V}, \mathcal{W} \) and \( \mathcal{V}' \) as \( H \)-equivariant vector bundles on \( S \) by \( \mathbb{Z}/2\mathbb{Z} \)-grading.
By the similar arguments as in [N2] §2.1 using framing $\Phi$: $E|_{\ell_\infty \times S} \cong W \otimes O_{\ell_\infty \times S}$, we see that $b_1 a_2 = -b_2 a_1$ gives an isomorphism $\mathcal{V} \cong \mathcal{V}'$, and $\widetilde{W} = \text{im } a_1 \oplus \text{im } a_2 \oplus \mathcal{W} \cong \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{W}$, where $\mathcal{W} = \ker b_1 \cap \ker b_2$. Via these identifications, we have

$$b_1 = \begin{bmatrix} 0 & -\text{id}_{\mathcal{V}} & 0 \\ -\text{id}_{\mathcal{V}} & 0 & 0 \end{bmatrix}, b_2 = \begin{bmatrix} \text{id}_{\mathcal{V}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

From the condition $ba = 0$, we can write

$$a_0 = \begin{bmatrix} B_1 \\ B_2 \\ w \end{bmatrix}, b_1 = \begin{bmatrix} -B_2 & B_1 & z \end{bmatrix}$$

with $[B_1, B_2] + zw = 0$. By [N3] Proposition 4.1] and [N2] Lemma 2.7, these are family of $\zeta^0$-stable ADHM data with $\zeta^0 \in \mathcal{C}_0$ and $\zeta^1 \in \mathcal{C}_1$.

### A.4 Isomorphisms of moduli spaces

We summarize results in the previous subsection. If we have a family of framed sheaves $(E_S, \Phi_S)$ on $X_\kappa \times S$, then we have a family of $\zeta^\kappa$-stable ADHM data $(B_1, B_2, z, w)$ on $S$. This defines a morphism to $S \rightarrow M_{\zeta^\kappa}(w, v)$ such that the pull back of the complex (30) coincides with the complex (34). This means that the pull-back of the universal framed sheaf $(E, \Phi)$ on $X_\kappa \times M_{\zeta^\kappa}(w, v)$ is isomorphic to $(E_S, \Phi_S)$. Furthermore such a morphism is unique, since isomorphisms of framed sheaves induces isomorphisms of ADHM data. Hence we get a morphism $M_{X_\kappa}(c) \rightarrow M_{\zeta^\kappa}(w, v)$.

Together with §A.1, we have two morphisms between $M_{\zeta^\kappa}(w, v)$ and $M_{X_\kappa}(c)$. To check whether these are converse to each other, it is enough to see set theoretically that the compositions are identities, and this is proven in [N3] Theorem 2.2]. Hence $M_{\zeta^\kappa}(w, v)$ is a moduli of framed sheaves on $X_\kappa$, and we complete the proof of Theorem 2.3.

To extend Theorem 2.3, we introduce $m$-stability for framed sheaves on $X_1$. For the following definition, we put $\tilde{C} = \ell_\infty - F = \frac{1}{2} C$.

**Definition A.3.** For $m \in \mathbb{Z}_{\geq 0}$, a framed sheaf $(E, \Phi)$ on $X_1$ is said to be $m$-stable if $E(-m\tilde{C})$ is perverse coherent, i.e.,

(i) $\text{Hom}_{X_1}(E(-m\tilde{C}), O_C(-1)) = 0$,

(ii) $\text{Hom}_{X_1}(O_C, E(-m\tilde{C})) = 0$,

(iii) $E(-m\tilde{C})$ is torsion free outside $C$.

For $c \in A(I X_1)$, we write by $M^m_{X_\kappa}(c)$ the moduli space of $m$-stable framed sheaves $(E, \Phi)$ on $X_1$ with $\hat{ch}(E) = c$ in $A(I X_1)$. We can show the following theorem similarly as in [NY2], but we will give a proof elsewhere.
**Theorem A.4.** We have an isomorphism between $M^m(w, v)$ and $M^m_{X_1}(c)$, where we put
\[ c = (w_0 + w_1)[X_1] + (-2v_0 + 2v_1 - w_1) \frac{C}{2} - \left( v_0 + \frac{w_1}{4} \right) P + (w_0 - w_1)\ell_1 \in A(I X_1). \]

By this theorem and Theorem 2.3, we have $M^m_{X_1}(c) \cong M_{X_1}(c)$ for $m \gg 0$, and $M^0_{X_1}(c) \cong M_{X_0}(c)$.

**B Combinatorial description of partition functions**

Following the same arguments as in [NY1], we give a combinatorial description of Nekrasov partition functions $Z_{X_0}$ for $\kappa = 0, 1$, and compare with the original Nekrasov partition function $Z$ defined from framed moduli $M(r, n)$ of torsion free sheaves on the plane $\mathbb{P}^2$ with the rank $r$ and $c_2 = n$.

In the following, we consider $M_{X_0}(c)$ for $\kappa = 0, 1$, where
\[
 c = r[X_1] + k[C] - n[P] + r[\ell_1] \in A(I X_1),
\]
and for $\kappa = 0$, $c \in A(I X_0)$ is defined via the equivalence $F: D(X_0) \cong D(X_1)$ in Proposition 1.1. We put $r_0 = (r + \bar{r})/2$ and $r_1 = (r - \bar{r})/2$.

**B.1 Fixed point sets of framed moduli**

For $\tilde{T}$-action on $M_{X_0}(c)$ defined in §1.3, we consider fixed point sets $M_{X_0}(c)^\tilde{T}$ in $M_{X_0}(c)$.

**Proposition B.1.** For $c \in A(I X_0)$ as in Theorem 2.3, the set of fixed points of $M_{X_0}(c)$ consists of pairs
\[
\left( \bigoplus_{\alpha=1}^{r_0} I_\alpha \oplus \bigoplus_{\alpha=r_0+1}^{r} I_\alpha (F - \ell_\infty), \Phi \right),
\]
where $I_\alpha$ are ideal sheaves supported on $P$, and $\Phi$ is a direct sum of natural isomorphisms $I_\alpha|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}$ for $\alpha = 1, \ldots, r_0$ and $I_\alpha(F - \ell_\infty)|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty} \otimes (-1)$ for $\alpha = r_0 + 1, \ldots, r_0 + r_1$.

Furthermore each $I_\alpha$ corresponds to a Young diagram $Y_\alpha$ for $\alpha = 1, \ldots, r$ such that
\[
v_s = \sum_{\alpha=1}^{r} \sharp\{ (i, j) \in Y_\alpha \mid l_\alpha + (i - 1) + (j - 1) \equiv s \ mod \ 2 \},
\]
for $s = 0, 1$, where $l_1 = \cdots = l_{r_0} = 0$ and $l_{r_0+1} = \cdots = l_{r_0+r_1} = 1$.

**Proof.** For a $\tilde{T}$-fixed point $(E, \Phi) \in M_{X_0}(c)$, we consider $E \in \text{Coh} X_0$ as a $\mathbb{Z}_2$-equivariant torsion free sheaf on $\mathbb{P}^2$. As in [NY1 Proposition 2.9], we have an eigen-vector decomposition of $E$ for $\tilde{T}$-action. Since $\tilde{T}$-action is compatible with $\mathbb{Z}_2$-action, this gives a decomposition of sheaves on $X_0$. \[\square\]
Proposition B.2. For $c \in A(I X_1)$ as in Theorem 2.3, the set of fixed point of $M_{X_1}(c)$ consists of pairs

$$\left( \bigoplus_{\alpha=1}^r I_{\alpha}(k_\alpha C), \Phi \right),$$

where $I_\alpha$ are ideal sheaves supported on $\{P_1, P_2\}$, and $\Phi$ is a direct sum of natural isomorphisms $I_{\alpha}(k_\alpha C)|_{\ell_{\infty}} \cong O_{\mathbb{P}^1} \otimes (-1)^{2k_\alpha}$. Vectors $\tilde{k} = (k_1, \ldots, k_r) \in \frac{1}{2} \mathbb{Z}^r$ satisfy $k_1, \ldots, k_{r_0} \in \mathbb{Z}$, $k_{r_0+1}, \ldots, k_r \in \frac{1}{2} + \mathbb{Z}$, and $\sum_{\alpha=1}^r k_\alpha = k$. Furthermore each $I_\alpha$ corresponds to a pair of Young diagrams $(Y_\alpha^1, Y_\alpha^2)$ for $\alpha = 1, \ldots, r$ such that $\sum_{\alpha=1}^r (k_\alpha^1 + |Y_\alpha^1| + |Y_\alpha^2|) = n$.

Proof. $(E, \Phi) \in M_{X_1}(c)$ is fixed by $T^*$-action if and only if it has eigenvector decomposition $E = I_1 \oplus \cdots \oplus I_r$ and $\Phi$ is direct sum of isomorphisms $I_i \cong O_{\ell_{\infty}}$ of the $i$-th factor for each $i = 1, \ldots, r$.

In the following, we compute $\tilde{T}$-module structures of fibers of $\tilde{T}$-equivariant vector bundles on framed moduli. These are considered as elements of the representation ring $R(\tilde{T})$ of the torus $\tilde{T}$. We identify it with the Laurent polynomial ring $\mathbb{Z}[t_1^\pm, t_2^\pm, e_1^\pm, \ldots, e_r^\pm, m_1^\pm, \ldots, m_2^\pm]$. We also consider $\mathbb{Z}_2$-grading of $R(\tilde{T})$ as $S$ in [2.3] that is, defined by $\deg t_1 = \deg t_2 = \deg e_{r_0+1} = \cdots = \deg e_r = 1$, and $\deg e_1 = \deg e_{r_0} = \deg \mu_1 = \cdots = \deg \mu_2 = 0$. For an element $F \in R(\tilde{T})$, the degree $s$ part is denoted by $[F]_s$ for $s \in \mathbb{Z}_2$.

B.2 $\tilde{T}$-module structures of tautological bundles on framed moduli

We compute $\tilde{T}$-module structures of tautological bundles $V_0 = \mathbb{R}^1 p_\ast \mathcal{E}(\ell_{\infty}), V_1 = \mathbb{R}^1 p_\ast \mathcal{E}(F)$ over $\tilde{T}$-fixed points of framed moduli, where $\mathcal{E}$ are universal sheaves on $X_\kappa \times M_{X_1}(\alpha)$, and $p: X_\kappa \times M_{X_1}(\alpha) \to M_{X_1}(\alpha)$ is the projection.

Proposition B.3. For a fixed point $(E, \Phi) \in M_{X_0}(c)$ corresponding to a datum $Y = (Y_1, \ldots, Y_r)$, we have isomorphisms $V_1|_{(E, \Phi)} \cong \bigoplus_{\alpha=1}^r \bigoplus_{(i,j) \in Y_\alpha} [e_\alpha t_1^{i+1} t_2^{j+1}]_s$ of $\tilde{T}$-modules, where $[e_\alpha t_1^{i+1} t_2^{j+1}]_s$ is the degree $s$ part of $e_\alpha t_1^{i+1} t_2^{j+1}$ in $R(\tilde{T})$ for $s = 0, 1$.

Proof. We compute $V_1|_{(E, \Phi)} = H^1(X_0, E(s(\ell_{\infty} - F) - \ell_{\infty}))$ as follows. We have a decomposition $E = \bigoplus_{\alpha=1}^r e_\alpha I_{Y_\alpha}$. For each $\alpha$, we consider an exact sequence

$$0 \to O_{Z_\alpha}(s(\ell_{\infty} - F)) \to I_{Y_\alpha}(s(\ell_{\infty} - F) - \ell_{\infty}) \to O_{X_0}(s(\ell_{\infty} - F) - \ell_{\infty}) \to 0,$$

where $Z_\alpha$ is a 0-dimensional sub-scheme of $\mathbb{P}^2$ defined by $I_{Y_\alpha}$. Hence we have $H^1(X_0, I_{Y_\alpha}) \cong H^0(X_0, O_{Z_\alpha}(s(\ell_{\infty} - F) - \ell_{\infty}))$. This is the space of $\mathbb{Z}_2$-invariant sections of $O_{Z_\alpha} \otimes (-1)^s$, where $(-1)^s$ denote representations $H = \{\pm 1\} \to \mathbb{C}^*, m \mapsto m^s$ for $s = 0, 1$. This gives the assertion. □
For tautological bundles on the minimal resolution $X_1$, we need the following computations. For $k \in \frac{1}{2}\mathbb{Z}$, we consider a $T^2$-equivariant sheaf $\mathcal{O}_{X_1}(kC - \ell_\infty)$ and a $T^2$-modules $L_k(t_1, t_2) = H^1(X_1, \mathcal{O}_{X_1}(kC - \ell_\infty))$.

**Lemma B.4.** We have the following isomorphisms of $T^2$-modules.

$$L_k(t_1, t_2) \cong \begin{cases} \bigoplus_{i,j \geq 0, i+j \equiv 2k \text{ mod } 2} t_1^{i+1}t_2^{-j} & \text{if } k < -\frac{1}{2} \\ \bigoplus_{i,j \geq 0, i+j \equiv 2k \text{ mod } 2} t_1^{i-1}t_2^{-j} & \text{if } k > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

**Proof.** For $k = 0, \pm \frac{1}{2}$, since $\chi(\mathcal{O}_{X_1}(kC - \ell_\infty)) = 0$ we have $H^1(X_1, \mathcal{O}_{X_1}(kC - \ell_\infty)) = 0$ by Lemma A.2. For $k > \frac{1}{2}$ we consider the exact sequence

$$0 \to \mathcal{O}_{X_1}((k - 1)C - \ell_\infty) \to \mathcal{O}_{X_1}(kC - \ell_\infty) \to \mathcal{O}_C(kC) = \mathcal{O}_{\mathbb{P}^1}(-2k) \to 0.$$  
From the cohomology long exact sequence

$$0 \to H^1(X_1, \mathcal{O}_{X_1}((k - 1)C - \ell_\infty)) \to H^1(X_1, \mathcal{O}_{X_1}(kC - \ell_\infty)) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2k)) \to 0$$

we get a decomposition $H^1(X_1, \mathcal{O}_{X_1}(kC - \ell_\infty)) = H^1(X_1, \mathcal{O}_{X_1}((k - 1)C - \ell_\infty)) \oplus H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2k))$. Since we have the $T^2$-equivariant dualizing sheaf $t_1^{-1}t_2^{-1}\mathcal{O}_{\mathbb{P}^1}(-2)$, by the Serre duality we have

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2k)) = (t_1^{-1}t_2^{-1}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2k - 2))^*) = \bigoplus_{i,j \geq 0, i+j = 2k-2} t_1^{i+1}t_2^{j+1}.$$  
Repeating this we get the assertion.

For $k < -\frac{1}{2}$, we consider the exact sequence

$$0 \to \mathcal{O}_{X_1}(kC - \ell_\infty) \to \mathcal{O}_{X_1}((k + 1)C - \ell_\infty) \to \mathcal{O}_C((k + 1)C) = \mathcal{O}_{\mathbb{P}^1}(-2k - 2) \to 0.$$  
We get a decomposition

$$H^1(X_1, \mathcal{O}_{X_1}(kC - \ell_\infty)) = H^1(X_1, \mathcal{O}_{X_1}((k + 1)C - \ell_\infty)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2k - 2)).$$  
Repeating this procedure we get the assertion.  

**Proposition B.5.** For a $\tilde{T}$-fixed point $(E, \Phi) \in M_{X_1}(c)$ corresponding to a datum $(\vec{k}, \vec{Y}^1, \vec{Y}^2)$, the $\tilde{T}$-module $\mathcal{V}_s|_{(E, \Phi)}$ is isomorphic to

$$\bigoplus_{\alpha=1}^r e_\alpha \left( L_{k_\alpha+\sigma}(t_1, t_2) \oplus \bigoplus_{(i,j) \in Y^2_1} t_1^{2(k_\alpha - i + 1 + \sigma)} \left( \frac{t_2}{t_1} \right)^{-j+1} \oplus \bigoplus_{(i,j) \in Y^2_2} \left( \frac{t_1}{t_2} \right)^{-i+1} t_2^{2(k_\alpha - j + 1 + \sigma)} \right)$$

for $s = 0, 1$.  

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Proof. The torsion free sheaf $E$ is decomposed into $\bigoplus_{\alpha=1}^{r} I_{\alpha}(k_{\alpha} C)$. The $T^{r}$-action on each component $H^{1}(X_{1}, I_{\alpha} \left(\left(k_{\alpha} + \frac{s}{2}\right) C - \ell_{\infty}\right))$ is given by a multiplication of $e_{\alpha}$ for $\alpha = 1, \ldots, r$. Hence it is enough to compute a $T^{2}$-module structure on $H^{1}(X_{1}, I_{\alpha} \left(\left(k_{\alpha} + \frac{s}{2}\right) C - \ell_{\infty}\right))$ induced by the natural $T^{2}$-equivariant structure on $I_{\alpha} \left(\left(k_{\alpha} + \frac{s}{2}\right) C - \ell_{\infty}\right)$. By the exact sequence

$$0 \to I_{\alpha} \left(\left(k_{\alpha} + \frac{s}{2}\right) C - \ell_{\infty}\right) \to \mathcal{O}_{X_{1}} \left(\left(k_{\alpha} + \frac{s}{2}\right) C - \ell_{\infty}\right) \to \mathcal{O}_{Z_{\alpha}} \left(\left(k_{\alpha} + \frac{s}{2}\right) C - \ell_{\infty}\right) \to 0,$$

we get a decomposition $H^{1}(X_{1}, I_{\alpha} \left(\left(k_{\alpha} + \frac{s}{2}\right) C - \ell_{\infty}\right))$.

We have $Z = Z^{1}_{\alpha} \amalg Z^{2}_{\alpha}$, where $Z_{\alpha}^{i}$ is the sub-scheme supported at $P_{i}$ corresponding to $Y_{\alpha}^{i}$. The multiplication of $yx^{2k}/x^{2k-1}$ gives an equivariant isomorphism $\mathcal{O}_{Z_{\alpha}}(k_{\alpha} C - \ell_{\infty}) \cong t_{1}^{k_{\alpha}} \mathcal{O}_{Z_{\alpha}}$ for $i = 1, 2$. Hence we have the desired isomorphism

$$H^{0}(X_{1}, \mathcal{O}_{Z_{\alpha}} \left(\left(k_{\alpha} - \frac{s}{2}\right) C - \ell_{\infty}\right)) \cong \bigoplus_{(i,j) \in Y_{\alpha}^{i}} t_{1}^{2(k_{\alpha} - i + 1 + \frac{s}{2})} t_{2}^{-j+1} + \bigoplus_{(i,j) \in Y_{\alpha}^{2}} t_{1}^{-i+1} t_{2}^{2(k_{\alpha} - j + 1 + \frac{s}{2})}.$$ 

\[\square\]

### B.3 $\bar{T}$-module structures of tangent bundles on framed moduli

We also compute the $\bar{T}$-module structure of the tangent bundle of framed moduli $M_{X_{a}}(c)$. Let $Y_{a} = \left\{ \lambda_{a,1}, \lambda_{a,2}, \cdots, \right\}$ be a Young diagram where $\lambda_{a,i}$ is the height of the $i$-th column. We set $\lambda_{a,i} = 0$ when $i$ is larger than the width of the diagram $Y_{a}$. Let $Y_{a}^{T} = \left\{ \lambda_{a,1}^{T}, \lambda_{a,2}^{T}, \cdots \right\}$ be its transpose. For a box $s = (i,j)$ in the $i$-th column and the $j$-th row, we define its arm-length $a_{Y_{a}}(s)$ and leg-length $l_{Y_{a}}(s)$ with respect to the diagram $Y_{a}$ by $a_{Y_{a}}(s) = \lambda_{a,i} - j$ and $l_{Y_{a}}(s) = \lambda_{a,j} - i$.

We consider framed moduli $M(r, n)$ of torsion free sheaves on the plane $\mathbb{P}^{2}$ with the rank $r$ and $c_{2} = n$. We recall from \[\text{\cite{NY11}}\] Theorem 2.11 that the fibre of $TM(r, n)$ over a fixed point corresponding to a datum $Y = (Y_{1}, \ldots, Y_{r})$ consisting of Young diagrams is isomorphic to $\bigoplus_{\alpha, \beta=1}^{r} N_{\alpha, \beta}(t_{1}, t_{2})$ as $\bar{T}$-modules, where

$$N_{\alpha, \beta}(t_{1}, t_{2}) = e_{\beta} e_{\alpha}^{-1} \times \left( \bigoplus_{s \in Y_{a}} \left( t_{1}^{-l_{Y_{a}}(s)} t_{2} a_{Y_{a}}(s)+1 \right) \bigoplus_{t \in Y_{a}} \left( t_{1}^{l_{Y_{a}}(t)+1} t_{2}^{-a_{Y_{a}}(t)} \right) \right).$$

**Proposition B.6.** The fibre of $TM_{X_{a}}(c)$ over a fixed point corresponding to a datum $\bar{Y} = (Y_{1}, \ldots, Y_{r})$ is isomorphic to $\bar{T}$-modules $\bigoplus_{\alpha, \beta=1}^{r} [N_{\alpha, \beta}(t_{1}, t_{2})]_{0}$, where $[N_{\alpha, \beta}(t_{1}, t_{2})]_{0}$ is the degree 0 parts of $N_{\alpha, \beta}(t_{1}, t_{2})$ in $R(\bar{T})$. 

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Proof. Let \((E, \Phi)\) be a \(\tilde{T}\)-fixed point corresponding to \(\tilde{Y}\). Then the \(\tilde{T}\)-module structure of \(T_{(E, \Phi)}M_X(c) = \text{Ext}_E(E, E(-\ell_\infty))\) is computed similarly as in [NY1] Theorem 2.11. But in addition we must take \(\mathbb{Z}_2\)-invariant sections. In this way we get the assertion.

We also compute the \(\tilde{T}\)-module structure of the tangent bundle of \(M_{X_1}(c)\).

**Proposition B.7.** The fibre of the tangent bundle \(TM_{X_1}(c)\) over a fixed point corresponding to a datum \((\tilde{k}, \tilde{Y}^1, \tilde{Y}^2)\) is isomorphic to

\[
\bigoplus_{\alpha, \beta = 1}^{r} \left( e_\beta c_\alpha^{-1} L_{k_\beta - k_\alpha} (t_1, t_2) \oplus t_1^{2k_\beta - 2k_\alpha} M_{\alpha, \beta}^1 (t_1, t_2) \oplus t_2^{2k_\beta - 2k_\alpha} M_{\alpha, \beta}^2 (t_1, t_2) \right)
\]

as a \(\tilde{T}\)-module, where \(M_{\alpha, \beta}^1 (t_1, t_2)\) (resp. \(M_{\alpha, \beta}^2 (t_1, t_2)\)) is equal to \(N_{\alpha, \beta} (t_1^2, t_2 / t_1)\) (resp. \(N_{\alpha, \beta} (t_1 / t_2, t_2^2)\)), with \((Y_\alpha, Y_\beta)\) replaced by \((Y_\alpha^1, Y_\beta^1)\) (resp. \((Y_\alpha^2, Y_\beta^2)\)).

**Proof.** Let \(\text{Ext}^*_X\) be the alternating sum \(\sum (-1)^i \text{Ext}^i_X\) of \(\tilde{T}\)-modules. By Lemma \(A.2\) we have

\[
T_{(E, \Phi)}M_X(c) = - \text{Ext}^*_X (E, E(-\ell_\infty)) = \bigoplus_{\alpha, \beta = 1}^{r} - \text{Ext}^*_X (I_\alpha (k_\alpha C), I_\beta (k_\beta C - \ell_\infty)).
\]

Each summand is multiplied by \(e_\beta c_\alpha^{-1}\) for \(T^r\)-action. In the rest of proof we compute the \(T^2\)-action on each summand.

By the exact sequence \(0 \to I_\alpha \to \mathcal{O}_{X_1} \to \mathcal{O}_{Z_\alpha} \to 0\), we get the following decomposition of \(T_{(E, \Phi)}M_X(c)\):

\[
\bigoplus_{\alpha, \beta = 1}^{r} \left( - \text{Ext}^*_X (\mathcal{O}_{X_1} (k_\alpha C), \mathcal{O}_{X_1} (k_\beta C - \ell_\infty)) + \text{Ext}^*_X (\mathcal{O}_{X_1} (k_\alpha C), \mathcal{O}_{Z_\beta} (k_\beta C - \ell_\infty)) 
\]

\[
+ \text{Ext}^*_X (\mathcal{O}_{Z_\alpha} (k_\alpha C), \mathcal{O}_{X_1} (k_\beta C - \ell_\infty)) - \text{Ext}^*_X (\mathcal{O}_{Z_\alpha} (k_\alpha C), \mathcal{O}_{Z_\beta} (k_\beta C - \ell_\infty)) \right).
\]

The first component in \(35\) is isomorphic to \(\bigoplus_{\alpha, \beta = 1}^{r} L_{k_\beta - k_\alpha} (t_1, t_2)\).

For \(\alpha = 1, \ldots, r\), we have \(Z_\alpha = Z_\alpha^1 \sqcup Z_\alpha^2\), where \(Z_\alpha^i\) are closed sub-schemes supported at \(P_i\) corresponding to \(Y^i_\alpha\) for \(i = 1, 2\). By an equivariant isomorphism \(\mathcal{O}_{Z_\alpha^i} (kC) \cong t_i^{2k} \mathcal{O}_{Z_\alpha}\), the last three terms are isomorphic to

\[
\bigoplus_{i=1,2} \bigoplus_{\alpha, \beta = 1}^{r} t_i^{2k_\beta - 2k_\alpha} \left( \text{Ext}^*_X (\mathcal{O}_{X_1}, \mathcal{O}_{Z_\beta} (-\ell_\infty)) + \text{Ext}^*_X (\mathcal{O}_{Z_\alpha}, \mathcal{O}_{X_1} (-\ell_\infty)) 
\]

\[
- \text{Ext}^*_X (\mathcal{O}_{Z_\alpha}, \mathcal{O}_{Z_\beta} (-\ell_\infty)) \right).
\]

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We consider these components as derived functors from the category of coherent sheaves supported at the origin of $\mathbb{C}^2$ via the coordinate $(y_1, \frac{a}{x_2})$ and $(\frac{x_1}{x_2}, y_2)$ around $P_1$ and $P_2$ respectively. Then we have $\text{Ext}^n_{X_1} = \text{Ext}^n_{\mathbb{P}^2}$. If we write by $I_\alpha^1$ the ideal sheaf of $Z_\alpha^i$ in $\mathbb{P}^2$, then it is isomorphic to

$$\bigoplus_{i=1,2} \bigoplus_{\alpha,\beta=1,\ldots} t_i^{2k_\beta - 2k_\alpha} (\text{Ext}^i_{\mathbb{P}^2} (O_{\mathbb{P}^2}, O_{\mathbb{P}^2}(-\ell_\infty)) - \text{Ext}^i_{\mathbb{P}^2} (I_\alpha, I_\beta(-\ell_\infty))).$$

Since $\text{Ext}^i_{\mathbb{P}^2} (O_{\mathbb{P}^2}, O_{\mathbb{P}^2}(-\ell_\infty)) = 0$ and $\text{Ext}^i_{\mathbb{P}^2} (I_\alpha, I_\beta(-\ell_\infty)) = 0$ for $i = 0, 2$, it is isomorphic to

$$\bigoplus_{i=1,2} \bigoplus_{\alpha,\beta=1,\ldots} t_i^{2k_\beta - 2k_\alpha} \text{Ext}^i_{\mathbb{P}^2} (I_\alpha, I_\beta(-\ell_\infty)) = \bigoplus_{i=1,2} \bigoplus_{\alpha,\beta=1,\ldots} t_i^{2k_\beta - 2k_\alpha} M^i_{\alpha,\beta}(t_1, t_2)$$

as desired. $\square$

### B.4 Comparison to $Z_{\mathbb{P}^2}(\varepsilon, a, m, q)$

We consider a $\bar{T}$-equivariant bundle

$$F_r(V_\emptyset) = \left( V_0 \otimes \frac{e^{m_1}}{\sqrt{t_1 t_2}} \right) \oplus \cdots \oplus \left( V_0 \otimes \frac{e^{m_2r}}{\sqrt{t_1 t_2}} \right)$$

on $M_{X_\alpha}(\alpha)$, where $(e^{m_1}, \ldots, e^{m_2r})$ is an element in the last component $T^{2r}$ of $\bar{T}$. Here we consider a homomorphism $\bar{T}' = \bar{T} \rightarrow \bar{T}$ defined by

$$(t'_1, t'_2, e^{a'}, e^{m'}) \mapsto (t_1, t_2, e^{a}, e^{m}) = ((t'_1)^2, (t'_2)^2, e^{a'}, e^{m'}),$$

and use identification $t'_1 = \sqrt{t_1}, t'_2 = \sqrt{t_2}$ and $A_{\bar{T}'}^* (\text{pt}) \otimes S \cong S$. Nekrasov partition functions are defined by

$$Z^k_{X_\alpha}(\varepsilon, a, m, q) = \sum_{\alpha \in K(X_\alpha)} q^{n(\alpha)} \int_{M_{X_\alpha}(\alpha)} e(F_r(V_\emptyset)).$$

as in the introduction.

We consider the other Nekrasov partition function

$$Z_{\mathbb{P}^2}(\varepsilon, a, m, q) = \sum_{n=0}^{\infty} q^n \int_{M(r,n)} e(F_r(V)) = \sum_q q^{\ell(V)} \frac{t_1^* e(F_r(V))}{e(TM(r,n))} \in \mathcal{S}[q]$$

where $M(r,n)$ is framed moduli of torsion free sheaves on the plane $\mathbb{P}^2$ with the rank $r$ and $c_2 = n$. We consider the equivariant Euler class $e(F_r(V))$ in $\mathcal{S}$ of a $\bar{T}$-equivariant vector bundle

$$F_r(V) = \left( V \otimes \frac{e^{m_1}}{\sqrt{t_1 t_2}} \right) \oplus \cdots \oplus \left( V \otimes \frac{e^{m_2r}}{\sqrt{t_1 t_2}} \right), \quad (37)$$
where $V$ is the tautological bundle on $M(r,n)$.

We write $Z_{P^2}(\varepsilon, a, m, q) = \sum_{n=0}^{\infty} \alpha_n q^n$, where we also have description of $\alpha_n$ in terms of Young diagrams [NY1]. Then by Proposition B.5 and Proposition B.7, we have

\[ Z_{X_1}(\varepsilon, a, m, q) = \sum_{\vec{k} \in \mathcal{K}(w, k)} q^{\sum \vec{k}^2} \ell_k(\varepsilon, a, m) Z_{P^2}(\varepsilon, a^0, m, q) Z_{P^2}(\varepsilon^1, a^1, m, q) \]

for $k \in \frac{1}{4} \mathbb{Z}$ as in [IMO (3.5)]. Here $\mathcal{K}(w, k) = \left\{ \vec{k} \in \mathbb{Z}^{w_0} \oplus (1/2 + \mathbb{Z})^{w_1} \mid \sum_{\alpha=1}^{r} k_\alpha = k \right\}$.

\[ \varepsilon^0 = (2\varepsilon_1, -\varepsilon_1 + \varepsilon_2), \varepsilon^1 = (\varepsilon_1 - \varepsilon_2, 2\varepsilon_1), a^0 = a + 2\varepsilon_1 \vec{k}, a^1 = a + 2\varepsilon_2 \vec{k}, \]

and

\[ \ell_k(\varepsilon, a, m) = \prod_{\alpha=1}^{r} e \left( \sum_{t=1}^{\frac{\varepsilon_1 + \varepsilon_2}{2}} L_{k_\alpha}(t_1, t_2) \frac{e^{\alpha L_{k_\alpha}}}{e^{\alpha L_{k_\alpha}} t_1 t_2} \right). \]

### B.5 Comparison to Ito-Maruyoshi-Okuda

Ito-Maruyoshi-Okuda [IMO] introduced a similar partition functions

\[ Z_{C^2/Z_2}^{N_f=2N, \text{inst}, c}(\vec{a}, \vec{I}; \mu; q; \varepsilon_1, \varepsilon_2), Z_{A^1, \text{resolved}}^{N_f=2N}(\vec{a}, \vec{I}; \mu; q; \varepsilon_1, \varepsilon_2). \]

We substitute $\varepsilon_1 = -\varepsilon_1, \varepsilon_2 = -\varepsilon_2, \vec{a} = a, \mu_i = m_i - \frac{\varepsilon_1 + \varepsilon_2}{2}, \mu_{r+i} = -\mu_{r+i} + \frac{\varepsilon_1 + \varepsilon_2}{2}$ for $i = 1, \ldots, r, c = -k, N = r$, and

\[ \vec{I} = (0, \ldots, 0, 1, \ldots, 1). \]

Then we have

\[ Z_{X_0}^k(\varepsilon, a, m, q) = Z_{C^2/Z_2}^{N_f=2N, \text{inst}, c}(\vec{a}, \vec{I}; \mu; q; \varepsilon_1, \varepsilon_2), Z_{X_1}^k(\varepsilon, a, m, q) = Z_{A^1, \text{resolved}}^{N_f=2N}(\vec{a}, \vec{I}; \mu; q; \varepsilon_1, \varepsilon_2). \]

Furthermore, after this substitution their proposed relations [IMO (4.1)] coincides with Theorem [B3]

**Remark B.8.** These computations in Appendix B can be justified by [N3] and earlier results without using framed moduli spaces constructed in [BPSS] (cf. [BPSS line 4 - 6 in p.1179]).

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