Examination of Thai freshmen’s understanding on vectors using a model analysis technique

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Abstract. This study focused on an investigation of Thai students’ comprehension of vectors using the model analysis technique, which is based on the matrix representation of quantum physics. This technique displays students’ knowledge states and the probability of applying each mental model to solve a given vector concept. A well-known vector concept test in physics education research named the Test of Understanding of Vectors (TUV) was translated into Thai language and validated by a group of physics lecturers. It was administered to 651 first-year engineering students at Prince of Songkla University via the online system at the beginning of the first semester. The three common mental states of vectors were identified as correct, popular incorrect, and other models. Model estimation of the model analysis revealed that although 54% of the students correctly answered vector addition, they still had difficulty with vector subtraction. Theoretically, both topics are considered as identical concepts. The students had inconsistent ideas on vector subtraction as reported by eigenvalues < 0.65. Moreover, they demonstrated equal distribution in using the three mental models to solve the dot product of vectors. The most common incorrect idea was that a dot product resultant was a vector quantity. The initial knowledge states of the cross product were in an incorrect model region involving confusion between calculating magnitudes of the cross product and the dot product, and incorrectly applying the right-hand rule to locate a direction of the cross product. Overall, the findings will guide instructors which common vector ideas should be revised and how. The vector concept plays an important role in physics learning since it is embedded in several physics topics as a learning mechanism.

1. Introduction
Physics deals with several quantities that have both magnitude and direction, and it needs a special mathematical language called vectors to describe physical properties. In general, the vector concept is firstly considered in a physics course at both high school and university levels since it is applied to other subsequent physics concepts. However, some studies reported students’ misunderstanding in the qualitative and quantitative aspects of vector analysis [1,2]. These issues could interfere with learning advanced physics concepts.

One of the important things in education research, which is crucial to teachers, is to gain insights about the knowledge students learn after studying in the classroom. Students have different abilities, as
well as their comprehension of physics concepts. Therefore, a detailed analysis is needed to determine their misconceptions. This will be realized using the model analysis technique [3] for the vector concept. This technique differs from conventional methods such as normalized gain [4], which only analyses students correct answers and quantifies the learning gain from total scores before and after instruction. In contrast, the model analysis applies both correct and incorrect answers to identify a single student mental state. An average mental model representation for the whole class is displayed by a class density matrix, in which eigenvalues and eigenvectors can be obtained.

Therefore, this work aims to investigate students’ understanding of vectors using the model analysis technique. The Test of Understanding of Vectors (TUV) [5] is a reliable vector concept test in physics education research and will be used as an instrument in the current study. The findings will suggest what and how teachers should design lessons to facilitate students to learn the vector concept.

2. Data Collection
At the beginning of the first semester in the year 2018, we asked 651 Thai first-year students (65% male) from faculty of engineering at Prince of Songkla University to complete the TUV online via the learning management system. TUV was developed to measure students’ understanding of vector concepts without a physical context for a university level. It consists of 20 questions with five choices for each item [5]. It was translated into Thai language and validated by a group of Thai physics lecturers before it was uploaded online.

3. Model Analysis
Students’ responses to the TUV were analysed and divided into 3 main categories: correct answer (model 1: $e_1$), popular incorrect answer (model 2: $e_2$), and other choices (model 3: $e_3$) in the model analysis technique. These three common models indicate scientific and alternative conceptions of students illustrated in a linear vector space. Each model is represented by a unit vector, namely $e_1 = (1 0 0)^T$, $e_2 = (0 1 0)^T$, and $e_3 = (0 0 1)^T$. These models are orthogonal vectors which denote that one idea is separate from another. A question consisted of different ideas through item options and was linked to a sub-topic. This study covers 4 vector sub-topics of addition and subtraction of vectors, as well as dot and cross product of vectors. The most popular incorrect choice per item and its percentage of responses related to these sub-topics are shown in table 1.

For one sub-topic of vectors with the total number of equivalent concept questions $m$, the model state vector for a single student in a class is described by

$$\bar{u}_k = \frac{1}{\sqrt{m}} \left( \sqrt{n^k_1} \sqrt{n^k_2} \sqrt{n^k_3} \right)^T$$

(1)

where $n^k_1, n^k_2, n^k_3$ are the number of questions which the $k^{th}$ student applied to model 1, model 2, and model 3, respectively [3]. The student vector ($\bar{u}_k$) is used to calculate and obtain a single student density matrix($D_k$), where $D_k = \bar{u}_k \otimes \bar{u}_k^T$. For all students in the class, the class model density matrix ($D$) can be computed from the average of the individual student’s density matrices as [3]

$$D = \frac{1}{N} \sum_{k=1}^{N} D_k = \frac{1}{N \cdot m} \left[ \begin{array}{ccc} n^k_1 & \sqrt{n^k_1 n^k_2} & \sqrt{n^k_1 n^k_3} \\ \sqrt{n^k_1 n^k_2} & n^k_2 & \sqrt{n^k_2 n^k_3} \\ \sqrt{n^k_1 n^k_3} & \sqrt{n^k_2 n^k_3} & n^k_3 \end{array} \right] = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix}$$

(2)

The resulting diagonal elements represent the percentage of responses in applying each common model. The off-diagonal elements reflect the consistency of a single student’s use of each common model. It significantly displays low consistency (large mixing) if $\rho > 50\%$ of its components. The knowledge vector of the class is interpreted from a primary eigenvector. The largest eigenvalue (>0.65) indicates that several single student model vectors are similar to each other, and they can be
adequately represented by the corresponding dominant eigenvector. This class model vector is the weighted average of all single student model vectors.

Table 1. The most popular incorrect choices, which are named model 2 ($e_2$) in the model analysis, found in this study and their percentage.

| Sub-topics of vectors | Item | Popular incorrect ideas                                                                 | Percentage of responses ($N = 651$) |
|-----------------------|------|----------------------------------------------------------------------------------------|-------------------------------------|
| vector addition       | 1    | Sum two vectors using the head-to-head approach.                                         | 18                                  |
|                       | 7    | A resultant vector of two identical perpendicular vectors has the same magnitude with the initial one. | 12                                  |
|                       | 16   | The addition of two vectors always gives a resultant vector with a greater magnitude than the vectors that are added up. | 17                                  |
| vector subtraction    | 19   | Sum two vectors that have opposite directions.                                           | 46                                  |
| dot product           | 3    | The dot product of two vectors is the magnitude of a vector between both vectors pointing up to the right. | 36                                  |
|                       | 6    | The dot product of two vectors is the sum of $\cos \theta$ and $\sin \theta$.               | 13                                  |
|                       | 8    | The result of the dot product is a vector.                                              | 41                                  |
| cross product         | 12   | The cross product of two vectors is the magnitude of a vector between both vectors pointing up to the right. | 28                                  |
|                       | 18   | Use solution of the dot product to calculate the cross product                           | 26                                  |
|                       | 15   | Confusion in using unit vectors for cross product                                        | 36                                  |

4. Results and Discussion

The study applied the model analysis technique to explore the student mental model states in detail. Average pre-instruction scores of this group of Thai students related to vector addition, vector subtraction, dot product, and cross product were 54%, 36%, 32%, and 19%, respectively, as displayed by $\rho_{11}$ of the class density matrices shown in table 2. Their most popular misconceptions (model 2) as indicated in table 1 were similar to those of Mexican students [5].

The sample had the best understanding of the vector concept of addition. For model 1, students’ responses indicated that the concept of addition of two vectors ($\vec{A} + \vec{B}$) is closely related to the parallelogram law. It can be noted that the parallelogram law is calculated to obtain the magnitude of the diagonal of the parallelogram originating from the common point of two vectors. The direction is drawn from the common point to the opposite vertex of the parallelogram. However, in model 2, the most popular incorrect idea is that the vector sum can be obtained from a line drawn from the head of $\vec{A}$ to the head of $\vec{B}$. It greatly affects the summing error with graphical addition.

Although 54% of the students correctly answered vector addition, they still had difficulty with vector subtraction. This result is similar to a study which reported that most students understand vector addition better than vector subtraction [6]. In theory, vector subtraction is just the inverse operation to vector addition. The class density matrix of vector subtraction displayed somewhat equal distribution among the three common models. Moreover, it revealed that the individual student vectors in the class were widely distributed. There was no appropriate eigenvector acting as a representative of the class model vector, as reported by the eigenvalue < 0.65.

The students also almost equally employed the three common models to solve the dot product of vectors, as shown by the diagonal elements of its class density matrix in table 2. But they had quite similar vectors among students demonstrating their ideas of the dot product, as shown by the eigenvalue = 0.70. Moreover, we found that almost half of the students thought the result of the dot product was a vector. This finding is similar to Barniol and Zavala [5], and Van Deventer and
colleagues [7]. One-third of the students believed that the dot product of two vectors is the magnitude of a vector between the two vectors pointing up to the right.

| Table 2. Class density matrices, eigenvalues, and dominant eigenvectors related to the 4 sub-topics. |
|---------------------------------------------------------------|
| Addition | Subtraction | Dot product | Cross product |
| Class density matrices |
| 0.54 | 0.12 | 0.20 | 0.36 | 0.06 | 0.06 | 0.32 | 0.19 | 0.18 | 0.19 | 0.09 | 0.15 |
| 0.12 | 0.17 | 0.09 | 0.06 | 0.38 | 0.12 | 0.19 | 0.32 | 0.19 | 0.09 | 0.32 | 0.23 |
| 0.20 | 0.09 | 0.28 | 0.06 | 0.12 | 0.26 | 0.18 | 0.19 | 0.35 | 0.15 | 0.23 | 0.49 |
| Eigenvalues |
| 0.69 |
| 0.50 |
| 0.70 |
| 0.71 |
| Dominant eigenvectors |
| (0.84 0.27 0.47)T |
| - |
| (0.56 0.57 0.59)T |
| (0.32 0.54 0.78)T |

The concept of the cross product is usually taught after the dot product in an introductory physics course. It was found that only 19% of the sample applied the correct model to solve the questions. The cross product of two vectors is defined as the vector with the magnitude equal to the parallelogram area of the vector span described by a sine function of the two vectors and direction explained by the right-hand rule. In this study, it was revealed that 26% of the students applied the cosine function for the cross product. They may be confused between the dot and cross product- a more concrete analysis may be needed for further study.

5. Conclusion
This study applied the model analysis, a technique based on the matrix representation of quantum physics, to examine Thai freshmen’s understanding of four sub-concepts of vectors from the TUV test. It was revealed that the cross product was the most difficult concept for the students, while the concept of adding vectors was the easiest one. The technique generated the class model vectors for each vector concept. The trend of using common mental models to solve a set of equivalent-concept questions was presented. It can be applied to detect pre-and-post class model vectors of students in a given topic, as well as comparing class model vectors between two classrooms. In this work, we employed a basic idea of correct, popular incorrect, and other options to design three common mental models of the TUV test. A more concrete analysis may be needed for further study.

Acknowledgment
The authors would like to thank all participants at Prince of Songkla University, Hat Yai, Thailand.

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