An Estimator for Matching Size in Low Arboricity Graphs with Two Applications

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Abstract

In this paper, we present a new simple degree-based estimator for the size of maximum matching in bounded arboricity graphs. When the arboricity of the graph is bounded by $\alpha$, the estimator gives a $\alpha + 2$ factor approximation of the matching size. For planar graphs, we show the estimator does better and returns a $3.5$ approximation of the matching size.

Using this estimator, we get new results for approximating the matching size of planar graphs in the streaming and distributed models of computation. In particular, in the vertex-arrival streams, we get a randomized $O(\sqrt{\frac{n}{\epsilon^2}} \log n)$ space algorithm for approximating the matching size within $(3.5 + \epsilon)$ factor in a planar graph on $n$ vertices.

Similarly, we get a simultaneous protocol in the vertex-partition model for approximating the matching size within $(3.5 + \epsilon)$ factor using $O(\frac{n^{2/3}}{\epsilon^2} \log n)$ communication from each player.

In comparison with the previous estimators, the estimator in this paper does not need to know the arboricity of the input graph and improves the approximation factor for the case of planar graphs.

1 Introduction

A matching in a graph $G = (V,E)$ is a subset of edges $M \subseteq E$ where no two edges in $M$ share an endpoint. A maximum matching of $G$ has the maximum number of edges among all possible matchings. Here we let $m(G)$ denote the matching size of $G$, i.e. the size of a maximum matching in $G$. In this paper, we present algorithms for approximating $m(G)$ in the sublinear models of computation. In particular, our results fit the vertex-arrival stream model (also known as the adjacency list streams). In the vertex-arrival model, in contrast...
with the edge-arrival version where the input stream is an arbitrary ordering of the edges, here each item in the stream is a vertex of the graph followed by a list of its neighbors.

We also focus on graphs with bounded arboricity. A graph \( G = (V, E) \) has arboricity bounded by \( \alpha \) if the edge set \( E \) can be partitioned into at most \( \alpha \) forests. A well-known fact (known as the Nash-William theorem [NW64]) states that a graph has arboricity \( \alpha \), if and only if every induced subgraph on \( t \) vertices has at most \( \alpha(t - 1) \) number of edges. Graphs with low arboricity cover a wide range of graphs such as constant degree graphs, planar graphs, and graphs with small tree-widths. In particular planar graphs have arboricity bounded by 3.

A simple reduction from counting distinct elements implies that computing \( m(G) \) exactly requires \( \Omega(n) \) space complexity even for trees and randomized algorithms (see [AMS99] for the lower bound on distinct elements problem.) This has initiated the study of finding computationally-light estimators for \( m(G) \) that take small space to compute. With this focus, following the work by Esfandiari et al. [EHL+15], there has been a series of papers [MV16, CJMM17, MV18, BGM+19] that have designed estimators for the matching size based on the degrees of vertices, edges and the arboricity of the input graph. In this paper, we design another degree-based estimator for \( m(G) \) in low arboricity graphs that has certain advantages in comparison with the previous works and leads to new algorithmic results. Before describing our estimator we briefly review some of the previous ideas. In the discussions below, we assume \( G \) has arboricity bounded by \( \alpha \).

**Shallow edges, high degree vertices** Esfandiari et al. [EHL+15] were first to observe that one can approximately characterize the matching size of low arboricity graphs based on the degree information of the vertices and the local neighborhood of the edges. Let \( H \) denote the set of vertices with degree more than \( h = 2\alpha + 3 \) and let \( F \) denote the set of edges with both endpoints having degree at most \( h \). Esfandiari et al. have shown that \( m(G) \leq |H| + |F| \leq (5\alpha + 9)m(G) \). Based on this estimator, the authors in [EHL+15] have designed a \( \tilde{O}(\epsilon^{-2}n^{2/3}) \) space algorithm for approximating \( m(G) \) within \( 5\alpha + 9 + \epsilon \) factor in the edge-arrival model.

**Fractional matchings** By establishing an interesting connection with fractional matchings and the Edmonds Polytope theorem, McGregor and Vorotnikova [MV16] have shown the following quantity approximates \( m(G) \) within \( (\alpha + 2) \) factor.

\[
(\alpha + 1) \sum_{(u,v) \in E} \min\left\{ \frac{1}{\deg(u)}, \frac{1}{\deg(v)}, \frac{1}{\alpha + 1} \right\}.
\]

Based on this estimator, the authors in [MV16] have given a \( \tilde{O}(\epsilon^{-2}n^{2/3}) \) space streaming algorithm (in the edge-arrival model) that approximate \( m(G) \) within \( \alpha + 2 + \epsilon \) factor. Also in the same work, another degree-based estimator is given that returns a \( \frac{(\alpha + 2)^2}{2} \) factor approximation of \( m(G) \). A notable property of this estimator is that it can be implemented in the vertex-arrival stream model in \( O(\log n) \) bits of space.

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\textbf{α-Last edges} Cormode et al. \cite{CJMM17} (later revised by McGregor and Vorotnikova \cite{MV18}) have designed an estimator that depends on a given ordering of the edges. Given a stream of edges \( S = e_1, \ldots, e_m \), let \( E_\alpha(S) \) denote a subset of edges where \((u, v) \in E_\alpha(S)\) iff the vertices \( u \) and \( v \) both appear at most \( \alpha \) times in \( S \) after the edge \((u, v)\). It is shown that \( m(G) \leq |E_\alpha(S)| \leq (\alpha + 2)m(G) \). Moreover a \( O(\frac{1}{\varepsilon^2} \log n) \) space streaming algorithm is given that approximates \( |E_\alpha(S)| \) within \( 1 + \varepsilon \) factor in the edge-arrival model.

1.1 The estimator in this paper

The new estimator is purely based on the degree of the vertices in the graph without any dependence on \( \alpha \). To estimate the matching size, we count the number of what we call \emph{locally superior} vertices in the graph. Namely,

\textbf{Definition 1} In graph \( G = (V, E) \), we call \( u \in V \) a locally superior vertex if \( u \) has a neighbor \( v \) such that \( \deg(u) \geq \deg(v) \). We let \( \ell(G) \) denote the number of locally superior vertices in \( G \).

We show if the arboricity of \( G \) is bounded by \( \alpha \), then \( \ell(G) \) approximates \( m(G) \) within \( (\alpha + 2) \) factor (Lemma\footnote{We do not have tight examples for our analysis. In fact, we conjecture that \( \ell(G) \) approximates \( m(G) \) within 3 factor when \( G \) is planar.}). This repeats the same bound obtained by the estimators in \cite{MV16} and \cite{CJMM17}, however for planar graphs, we show that the approximation factor is at most 3.5 which beats the previous bounds (Lemma\footnote{We do not have tight examples for our analysis. In fact, we conjecture that \( \ell(G) \) approximates \( m(G) \) within 3 factor when \( G \) is planar.}). As an evidence, consider the 4-regular planar graph on 9 vertices. Both of the estimators in \cite{MV16} and \cite{CJMM17}, report 18 as the estimation for \( m(G) \) while the exact answer is 4. It follows their approximation factor is at least 4.5.

Unfortunately, the new estimator, in spite of its simplicity, does not seem to be applicable in the edge-arrival model without an extra pass over the stream. To decide if a vertex is locally superior, we need to know its neighbors and learn their degrees which becomes burdensome in one pass. However in the vertex-partition model, we can obtain this information in one pass and consequently can achieve sublinear space bounds. More formally, we get a randomized \( O(\sqrt{n} \varepsilon^2 \log n) \) space algorithm for approximating \( m(G) \) within \( (3.5 + \varepsilon) \) factor in this model. In terms of approximation factor, this improves over existing sublinear algorithms \cite{MV16, MV18}.

As another application of our estimator, we get a sublinear simultaneous protocol in the \emph{vertex-partition} model for approximating \( m(G) \) when \( G \) is planar. In this model, vertex set \( V \) is partitioned into \( t \) subsets \( V_1, \ldots, V_t \) where each subset is given to a player. The \( i \)-th player knows the edges on \( V_i \). The players do not communicate with each other. They only send one message to a referee whom at the end computes an approximation of the matching size. (The referee does not get any part of the input.) We assume the referee and the players have a shared source of randomness. Within this setting, we design a protocol that approximates \( m(G) \) within \( 3.5 + \varepsilon \) factor using \( O(\frac{n^{2/3}}{\varepsilon^2} \log n) \) communication from each player. Note that for \( t > 3 \) and \( t = o(n^{1/3}) \), this result is non-trivial. The best previous result implicit in the works...
of \cite{CCE+16, MV16} computes a $5 + \varepsilon$ factor approximation using $\tilde{O}(n^{4/5})$ communication from each player. We should also mention that, based on the estimator in \cite{MV16}, there is a simultaneous protocol that reports a 12.5 factor approximation of $m(G)$ using $O(\log n)$ communication.

## 2 Graph properties

In the following proofs, we let $M \subseteq E$ denote a maximum matching in graph $G$. When the underlying graph is clear from the context, for the vertex set $S$, we use $N(S)$ to denote the neighbors of the vertices in $S$ excluding $S$ itself. For vertex $u$, we simply use $N(u)$ to denote the neighbors of $u$. The vertex $v$ is a neighbor of the edge $(x, y)$ if $v$ is adjacent with $x$ or $y$. When $x$ is paired with $y$ in the matching $M$, abusing the notation, we define $M(x) = y$.

**Lemma 2** Let $G = (V, E)$ be a graph with arboricity $\alpha$. We have

$$m(G) \leq \ell(G) \leq (\alpha + 2)m(G).$$

**Proof:** The left hand side of the inequality is easy to show. For every edge in $E$, at least one of the endpoints is locally superior. Since edges in $M$ are disjoint, at least $|M|$ number of endpoints must be locally superior. This proves $m(G) \leq \ell(G)$.

To show the right hand side, we use a charging argument. Let $L$ denote the locally superior vertices in $G$. Our goal is to show an upper bound on $|L|$ in terms of $|M|$ and $\alpha$. Let $X \subseteq L$ be the set of locally superior vertices that are NOT endpoints of a matching edge. The challenge is to prove an upper bound on $|X|$.

The vertices in $X$ do not contribute to the maximum matching. However all the vertices in $N(X)$ must be endpoints of matching edges (otherwise $M$ would not be a maximal matching.) For the same reason, there cannot be an edge between the vertices in $X$. To prove an upper bound on $|X|$, in the first step, using an assignment procedure, we assign a subset of vertices in $X$ to edges in $M$ in a way any target edge gets at most $\alpha - 1$ locally superior vertices. We do the assignments in the following way.

**The Assignment Procedure** If we find a $y \in N(X)$ with at most $\alpha - 1$ neighbors in $X$, we assign all the neighbors of $y$ in $X$ to the matching edge $(y, M(y))$. We repeat this process, every time picking a vertex in $N(X)$ with less than $\alpha$ neighbors in $X$ and do the assignment that we just described, until we cannot find such a vertex in $N(X)$. Note that when we assign a locally superior vertex $x$, we remove the edges on $x$ before continuing the procedure.

Here we emphasize the fact that if $y$ has a neighbor $x \in X$, then $M(y)$ cannot have neighbors in $X \setminus \{x\}$ (otherwise it would create an augmenting path and contradict the optimality of $M$.)
Let $X_1 \subseteq X$ be the assigned locally superior vertices and $M_1 \subseteq M$ be the used matching edges in the assignment procedure. We have

$$|X_1| \leq (\alpha - 1)|M_1|. \quad (1)$$

Let $X_2 = X \setminus X_1$ be the unassigned vertices in $X$. Now we try to prove an upper bound on $|X_2|$. For this, we need to make a few observations.

**Observation 3** Let $Y_2 = N(X_2)$. The pair $y$ and $M(y)$ cannot be both in $Y_2$.

**Proof:** Suppose $y$ and $M(y)$ are both in $N(X_2)$. Let $B$ and $C$ be the neighbors of $y$ and $M(y)$ in $X_2$ respectively. If $|B \cup C| > 1$, then one can find an augmenting path of length 3 (with respect to $M$.) A contradiction.

On the other hand, if $|B \cup C| = 1$, then $y$ and $M(y)$ have only a shared neighbor $x \in X_2$ which means the edge $e = (y, M(y))$ should have been used by the assignment procedure and as result $x \in X_1$. Another contradiction. $\square$  

**Observation 4** Every vertex $x \in X_2$ has degree at least $\alpha + 1$.

**Proof:** Consider $x \in X_2$. Suppose, for the sake of contradiction, $\deg(x)$ is $k$ where $k \leq \alpha$. Since $x$ is a locally superior vertex, there must be a $y \in N(x)$ with degree at most $k$ in $G$. We know that $y$ is an endpoint of a matching edge. In the assignments procedure, whenever we used an edge $e \in M$ all the neighbors of its endpoints (in $X$) were assigned. Since $x$ is not assigned yet, it means the edge $(y, M(y))$ has not been used. Consequently $y$ must have at least $\alpha$ neighbors in $X_2$. Counting the edge $(y, M(y))$, we should have $\deg(y) \geq \alpha + 1$. A contradiction. $\square$

Let $G' = (X_2 \cup Y_2, E')$ be a bipartite graph where $E'$ is the set of edges between $X_2$ and $Y_2$. From Observation 4 we have

$$(\alpha + 1)|X_2| \leq |E'|. \quad (2)$$

Since $G'$ is a subgraph of $G$, its arboricity is bounded by $\alpha$. As result,

$$|E'| \leq \alpha(|X_2| + |Y_2|). \quad (3)$$

Recall that $Y_2$ are endpoints of matching edges. Let $M_2$ be those matching edges. Observation 3 implies that $|Y_2| = |M_2|$. As result, combining (2) and (3), we get the following.

$$|X_2| \leq \alpha|Y_2| = \alpha|M_2|. \quad (4)$$
To prove an upper bound on $|L|$, we also need to count the locally superior vertices that are endpoints of matching edges. Let $Z = L \setminus X$. We have $|Z| \leq 2|M|$. Summing up, we get

$$|L| = |X_1| + |X_2| + |Z|$$
$$\leq (\alpha - 1)|M_1| + \alpha|M_2| + 2|M|$$
$$= \alpha(|M_1| + |M_2|) + 2|M| - |M_1|$$
$$\leq (\alpha + 2)|M|$$

This proves the lemma. \qed

**Lemma 5** Let $G = (V, E)$ be a planar graph. We have $\ell(G) \leq 3.5m(G)$.

**Proof:** For planar graphs, similar to what we did in the proof of Lemma 2, we first try to assign some of the vertices in $X$ to the matching edges using a simple assignment procedure. (Recall that $X$ is the set of vertices in $L$ that are not endpoints of edges in $M$.)

**The Assignment Procedure** Let $Y_1 = \emptyset$. If we find a $y \in N(X)$ with only 1 neighbor $x \in X$, we assign $x$ to the matching edge $(y, M(y))$. Also we add $y$ to $Y_1$. We continue the procedure until we cannot find such a vertex in $N(X)$. Note that when we assign a locally superior vertex $x$, we remove the edges on $x$.

Let $X_1 \subseteq X$ be the assigned locally superior vertices and $M_1 \subseteq M$ be the used matching edges in the assignment procedure. Note that $|Y_1| = |M_1|$. We have

$$|X_1| \leq |M_1|. \tag{5}$$

Let $X_2 = X \setminus X_1$. Using a similar argument that we used for proving Observation 4, we can show every vertex in $X_2$ has degree at least 3. Also letting $Y_2 = N(X_2)$, we observe that $y \in Y_2$ and $M(y)$ cannot be both in $Y_2$ as we noticed in the Observation 4. Let $M_2 \subseteq M$ be the matching edges with one endpoint in $Y_2$. We have $|Y_2| = |M_2|$.

Now consider the bipartite graph $G' = (X_2 \cup Y_2, E')$ where $E'$ is the set of edges between $X_2$ and $Y_2$. Every planar bipartite graph with $n$ vertices has at most $2n - 4$ edges. Since $G'$ is a bipartite planar graph, it follows,

$$3|X_2| \leq |E'| < 2(|X_2| + |Y_2|) = 2(|X_2| + |M_2|). \tag{6}$$

This shows $|X_2| < 2|M_2|$. Letting $Z = L \setminus X$ and $M_3 = M \setminus (M_1 \cup M_2)$, we get

$$|L| = |X_1| + |X_2| + |Z| \leq |M_1| + 2|M_2| + 2|M| \leq 3|M| + |M_2| - |M_3|. \tag{7}$$

\footnote{For a short proof of this, combine the Euler’s formula $|V| - |E| + |F| = 2$ with the inequality $2|E| \geq 4|F|$ caused by each face having at least 4 sides (since there are no odd cycles) and we get $|E| \leq 2|V| - 4$.}
This already proves $|L|$ is bounded by $4|M|$. To prove the bound claimed in the lemma, we also show that $|L| \leq 3|M| + |M_1| + |M_3|$. Combined with the inequality (1), this proves the lemma.

Let $Y = Y_1 \cup Y_2$. Note that $Y$ are one side of the matching edges in $M_1 \cup M_2$. Let $Y' = \{M(y) \mid y \in Y\}$. We use a special subset of $Y'$, named $Y''$ which is defined as follows. We let $Y''$ denote the locally superior vertices in $Y'$ that have degree 2 or they are adjacent with both endpoints of an edge in $M_3$. We make the following observation regarding the vertices in $Y''$.

**Observation 6** We can assign each vertex $y' \in Y''$ to a distinct $e \in Y_1 \cup Y_3$ where $e$ has no neighbor in $Y' \setminus \{y'\}$.

**Proof:** Consider $y' \in Y''$. If $y'$ is adjacent with both endpoints of an edge $e = (z, z') \in M_3$, we assign $y'$ to $e$ (when there are multiple edges with this condition we pick one of them arbitrarily.) Note that $z$ and $z'$ cannot have neighbors in $Y''$ other than $y'$ because otherwise it would create an augmenting path.

Now suppose $y'$ has degree 2. Since $y'$ is a locally superior vertex, it must have a neighbor $z$ of degree at most 2. The neighbor $z$ cannot be in $Y_2 \cup X_2$ because the vertices in $Y_2 \cup X_2$ have degree at least 3. We distinguish between two cases.

- $M(y') \subseteq Y_2$. In this case, $z$ cannot be in $Y_1$ either because the vertices in $Y_1$ are already of degree 2 without $y'$. Also $z \notin X_1$ because otherwise it would create an augmenting path. The only possibility is that $z$ is an endpoint of a matching edge in $M_3$. We assign $y'$ to the matching edge $(z, z') \in M_3$. Note that $z'$ cannot have a neighbor in $Y' \setminus \{y'\}$ because it would create an augmenting path.

- $M(y') \subseteq Y_1$. Here $z$ could be in $X_1$. If this is the case, then $M(y')$ cannot have a neighbor in $Y' \setminus \{y'\}$ because it would create an augmenting path. In this case, we assign $y'$ to $M(y')$. If $z = M(y')$, then again we assign $y'$ to $M(y')$. The only remaining possibility is that $z$ an endpoint of a matching edge in $M_3$ which we handle it similar to the previous case.

Now, assume we assign the vertices in $Y''$ to the elements in $Y_1 \cup Y_3$ according to the above observation. Let $Y'_1 \subseteq Y_1$ and $M'_3 \subseteq M_3$ be the vertices and edges that were used in the assignment. Let $Y'''$ be the remaining locally superior vertices in $Y'$. Namely, $Y''' = (L \cap Y') \setminus Y''$. Before making the final point, we observe that only one endpoint of the edges in $M_3$ are adjacent with vertices in $Y'''$. Let $Y_3$ be the endpoint of edges in $M_3 \setminus M'_3$ that have neighbors in $Y'''$. Consider the bipartite graph $G''(V'', E'')$ where

$$V'' = (X_2 \cup Y''') \cup (Y_2 \cup (Y_1 \setminus Y'_1) \cup Y_3)$$

and $E''$ is the set of edges between $X_2$ and $Y_2$, and the edges between $Y'''$ and $Y_2 \cup (Y_1 \setminus Y'_1) \cup Y_3$. 


Figure 1: A demonstration of the construction in the proof of lemmas 2 and 3. Thick edges represent matching edges. The unfilled vertices belong to the set $Y''$.

Relying on the facts that $G''$ is a planar bipartite graph, $Y'''$ is composed of vertices with degree at least 3, and the edges on $Y'''$ are all in $E''$, we have

$$3|X_2| + 3|Y'''| \leq |E''| \leq 2(|X_2| + |Y_2| + |Y_1 \setminus Y_1'| + |Y'''| + |Y_3|).$$

It follows,

$$|X_2| + |Y'''| \leq 2(|Y_2| + |Y_1 \setminus Y_1'| + |Y_3|)$$

$$\leq 2(|M_2| + |M_1| - |Y_1'| + |M_3| - |M_3'|)$$

$$= 2(|M| - |Y_1'| - |M_3'|)$$

Since $|Y''| = |Y_1'| + |M_3'|$, we get

$$|X_2| + |Y'''| \leq 2|M| - 2|Y''| \quad (8)$$

Let $Z_1$, $Z_2$ and $Z_3$ denote the locally superior vertices that are endpoints of matching edges in $M_1$, $M_2$ and $M_3$ respectively. From the definition of $Y''$ and $Y'''$, we have

$$|Z_1| + |Z_2| \leq |M_1| + |M_2| + |Y''| + |Y'''| \quad (9)$$

From (8) and (9), we get

$$|L| = |X_1| + |X_2| + |Z_1| + |Z_2| + |Z_3|$$

$$\leq |M_1| + |X_2| + (|M_1| + |M_2| + |Y''| + |Y'''|) + 2|M_3|$$

$$= 2|M_1| + (|X_2| + |Y'''|) + |M_2| + |Y''| + 2|M_3|$$

$$\leq 2|M_1| + |M_2| + 2|M| - |Y''| + 2|M_3|$$

$$= 3|M| + |M_1| + |M_3| - |Y''|$$

$$\leq 3|M| + |M_1| + |M_3|$$

This finishes the proof of the lemma. □
3 Algorithms

We first present a high-level sampling-based estimator for $\ell(G)$. Then we show how this estimator can be implemented in the streaming and distributed settings using small space and communication. For our streaming result, we use a combination of the estimator for $\ell(G)$ and the greedy maximal matching algorithm. For the simultaneous protocol, we use the estimator for $\ell(G)$ in combination with the edge-sampling primitive in [CCE+16] and an estimator in [MV16].

The high-level estimator (described in Algorithm 1) samples a subset of vertices $S \subseteq V$ and computes the locally superior vertices in $S$. The quantity $\ell(G)$ is estimated from the scaled ratio of the locally superior vertices in the sample set.

**Algorithm 1**: The high-level description of the estimator for $\ell(G)$

Run the following estimator $r = \lceil \frac{8}{\epsilon^2} \rceil$ number of times in parallel. In the end, report the average of the outcomes.

1. Sample $s$ vertices (uniformly at random) from $V$ without replacement.
2. Let $S$ be the set of sampled vertices.
3. Compute $S'$ where $S'$ is the set of locally superior vertices in $S$.
4. Return $\frac{n}{s} |S'|$ as an estimation for $\ell(G)$.

**Lemma 7** Assuming $s \geq \frac{n}{\ell(G)}$, the high-level estimator in Algorithm 1 returns a $1 + \frac{\epsilon}{2}$ factor approximation of $\ell(G)$ with probability at least $\frac{7}{8}$.

**Proof**: Fix a parallel repetition of the algorithm and let $X$ denote the outcome of the associated estimator. Assuming an arbitrary ordering on the locally superior vertices, let $X_i$ denote the random variable associated with $i$-th locally superior vertex. We define $X_i = 1$ if the $i$-th locally superior vertex has been sampled, otherwise $X_i = 0$. We have $X = \frac{n}{s} \sum_{i=1}^{\ell(G)} X_i$. Since $Pr(X_i = 1) = \frac{s}{n}$, we get $E[X] = \ell(G)$. Further we have

$$E[X^2] = \frac{n^2}{s^2} E \left[ \sum_{i,j} X_i X_j \right] = \frac{n^2}{s^2} \left[ \sum_i E[X_i^2] + \sum_{i \neq j} E[X_i X_j] \right]$$

$$= \frac{n^2}{s^2} \frac{s}{n} \ell(G) + \frac{\ell(G)}{2} \frac{s(s-1)}{n(n-1)}$$

$$= \frac{n}{s} \ell(G) + \frac{\ell(G)}{2} \frac{n(s-1)}{s(n-1)}$$

$$< \frac{n}{s} \ell(G) + \ell^2(G)$$

Consequently, $Var[X] = E[X^2] - E^2[X] < \frac{n}{s} \ell(G)$. 
Let $Y$ be the average of the outcomes of $r$ parallel and independent repetitions of the basic estimator. We have $E[Y] = \ell(G)$ and $\text{Var}[Y] < \frac{n}{sr} \ell(G)$. Using the Chebyshev’s inequality,

$$Pr(\{|Y - E[Y]| \geq \varepsilon E[Y]\} \leq \frac{\text{Var}[Y]}{\varepsilon^2 E^2[X]} < \frac{n/s}{r\varepsilon^2 \ell(G)}.$$

Setting $r = \frac{8}{\varepsilon^2}$ and $s \geq \frac{n}{\ell(G)}$, the above probability will be less than $1/8$. □

### 3.1 The streaming algorithm

We first note that we can implement the high-level estimator of Algorithm 1 in the vertex-arrival stream model using $O(s \varepsilon^2 \log n)$ space. Consider a single repetition of the estimator. The sampled set $S$ is selected in the beginning of the algorithm (before the stream.) This can be done using a reservoir sampling strategy [Vit85] in $O(|S| \log n)$ space. To decide if $u \in S$ is locally superior or not, we just need to store $\text{deg}(u)$ and the minimum degree of the neighbors that are visited so far. This takes $O(\log n)$ bits of space. As result, the whole space needed to implement a single repetition is $O(s \log n)$ bits.

The streaming algorithm runs two threads in parallel. In one thread it runs the streaming implementation of Algorithm 1 after setting $s = \lceil \sqrt{n} \rceil$. In the other thread, it runs a greedy algorithm to find a maximal matching in the input graph. We stop the greedy algorithm whenever the size of the discovered matching $F$ exceeds $\sqrt{n}$. In the end, if $|F| < \sqrt{n}$, we output $|F|$ as an approximation for $m(G)$, otherwise we report the outcome of the first thread.

Note that if $|F| < \sqrt{n}$, $F$ is a maximal matching in $G$. Hence $|F| \geq \frac{1}{2} m(G)$. Assume $|F| \geq \sqrt{n}$. In this case the algorithm outputs the result of first thread. In this case, by Lemma 2 we know $\ell(G) \geq \sqrt{n}$. Consequently, it follows from Lemma 7, the first thread returns a $1 + O\varepsilon$ approximation of $\ell(G)$ and consequently it returns a $3.5 + O(\varepsilon)$ approximation of $m(G)$. Since the greedy algorithm takes at most $O(\sqrt{n})$ space, the space complexity of the algorithm is dominated by the space usage of the first thread. We get the following result.

**Theorem 8** Let $G$ be a planar graph. There is a streaming algorithm (in the vertex-arrival model) that returns a $3.5 + \varepsilon$ factor approximation of $m(G)$ using $O(\frac{\sqrt{n}}{\varepsilon^2})$ space.

### 3.2 A simultaneous communication protocol

To describe the simultaneous protocol, we consider two cases separately: (a) when the matching size is low; to be precise, when it is smaller than some fixed value $k = n^{1/3}$, and (b) when the matching size is high, i.e. at least $\Omega(k)$. For each case, we describe a separate solution. The overall protocol will be these solutions (run in parallel) combined with a sub-protocol (in parallel) to distinguish between the cases.
Graphs with large matching size  In the case when matching size is large, similar to what was done in the streaming model, we run an implementation of Algorithm 1 in the simultaneous model. To see the implementation, in the simultaneous model all the players (including the referee) know the sampled set $S$. This results from access to the shared randomness. For each $u \in S$, the players send the minimum degree of the neighbors of $u$ in his input to the referee. The player that owns $u$, also sends $\deg(u)$ to the referee. Having received this information, the referee can decide if $u$ is a locally superior vertex or not. As result, we can implement Algorithm 1 in the simultaneous model using a protocol with $O(\frac{\log n}{\varepsilon})$ message size.

Graphs with small matching size  In the case when the matching size is small, we use the edge-sampling method of \cite{CCE+16}. We review their basic sampling primitive in its general form. Given a graph $G(V, E)$, let $c : V \to [b]$ be a totally random function that assigns each vertex in $V$ a random number (color) in $[b] = \{1, \ldots, b\}$. The set $\text{Sample}_{b,d,1}$ is a random subset of $E$ picked in the following way. Given a subset $K \subseteq [b]$ of size $d \in \{1, 2\}$, let $E_K$ be the edges of $G$ where the color of their endpoints matches $K$. For example when $K = \{3, 4\}$, the set $E_{(3,4)}$ contains all edges $(u, v)$ such that $\{c(u), c(v)\} = \{3, 4\}$. For all $K \subseteq [b]$ of size $d$, the set $\text{Sample}_{b,d,1}$ picks a random edge from $E_K$. Finally, the random set $\text{Sample}_{b,d,r}$ is the union of $r$ independent instances of $\text{Sample}_{b,d,1}$. We have the following lemma from \cite{CCE+16} (see Theorems 4 in the reference.)

**Lemma 9**  Let $G = (V, E)$ be a graph. When $m(G) \leq k$, with probability $1 - 1/poly(k)$, the random set $\text{Sample}_{100k,2, O(\log k)}$ contains a matching of size $m(G)$.

Note that, in the simultaneous vertex-partition model, the referee can obtain an instance of $\text{Sample}_{b,d,1}$ via a protocol with $O(b^d \log n)$ message size. To see this, using the shared randomness, the players pick the random function $c : V \to [b]$. Let $E^{(i)}$ be the subset of edges owned by the $i$-th player. We have $E = \bigcup_{i=1}^{b} E^{(i)}$. To pick a random edge from $E_K$ for a given $K \subseteq [b]$, the $i$-th player randomly picks an edge $e \in E_K \cap E^{(i)}$ and sends it along with $|E_K \cap E^{(i)}|$ to the referee. After receiving this information from all the players, the referee can generate a random element of $E_K$. Since there are $O(b^d)$ different $d$-subsets of $[b]$, the size of the message from a player to the referee is bounded by $O(b^d \log n)$ bits. Consequently, the referee can produce a rightful instance of $\text{Sample}_{b,d,r}$ using $O(rb^d \log n)$ communication each from player.

How to distinguish between the cases?  To accomplish this, here we use a degree-based estimator by McGregor and Vorotnikova \cite{MV16} described in the following lemma.

**Lemma 10**  Let $G$ be a planar graph. We have

$$m(G) \leq A'(G) = \sum_{u \in V} \min\{\deg(u)/2, 4 - \deg(u)/2\} \leq 12.5 \cdot m(G).$$

It is easy to see that, in the simultaneous vertex-partition model, we can implement this estimator with $O(\log n)$ bits communication from each player.
The final protocol Let \( k = \lfloor n^{1/3} \rfloor \). We run the following threads in parallel.

1. A protocol that implements the high-level estimator (Algorithm 1) with \( s = 12.5n/k \) as its input parameter according to the discussions above. Let \( z_1 \) be the output of this protocol.

2. A protocol to compute an instance of \( \text{Sample}_{b,d,r} \) for \( b = 100k \) and \( d = 2 \) and \( r = O(\log k) \). Let \( z_2 \) be the size of maximum matching in the sampled set.

3. A protocol to compute \( A'(G) \). Let \( z_3 \) be the output of this thread.

In the end, if \( z_3 \geq k^{12.5} \), the referee outputs \( z_1 \) as an approximation for \( m(G) \), otherwise the referee reports \( z_2 \) as the final answer.

Theorem 11 Let \( G \) be a planar graph on \( n \) vertices. The above simultaneous protocol with probability \( 3/4 \) returns a \( 3.5 + O(\varepsilon^2) \) approximation of \( m(G) \) where each player sends \( O(n^{2/3}/\varepsilon^2) \) bits to the referee.

Proof: First we note that by choosing the constants large enough, we can assume the thread (2) errs with probability at most \( 1/8 \). If \( z_3 \geq k^{12.5} \), then we know \( m(G) \geq k^{12.5} \). This follows from Lemma 10. Consequently by Lemma 2 we have \( \ell(G) \geq \frac{k}{k^{12.5}} \). Therefore from Lemma 7, we have \( |z_1 - \ell(G)| \leq \varepsilon \ell(G) \) with probability at least \( 7/8 \). It follows from Lemma 3 that \((1 - \varepsilon)m(G) \leq z_1 \leq (3.5 + 3.5 \varepsilon)m(G)\).

On the other hand, if \( z_3 < k^{12.5} \), by Lemma 10 we know that \( m(G) \) must be less than \( k \). Having this, from Lemma 9 with probability at least \( 7/8 \), we get \( z_2 = m(G) \). In this case the protocol computes the exact matching size of the graph.

The communication complexity each player is dominated by the cost of the first thread which is \( O(n^{2/3}/\varepsilon^2 \log n) \). The total error probability is bounded by \( 1/4 \). This finishes the proof. \( \square \)

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