Orthocompactness and semi-stratifiability in the density topology

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Abstract
The density topology $\mathcal{T}$ is a topology on the real line, finer than the usual topology, having as its open sets the measurable subsets of $\mathbb{R}$, which are of density 1 at each of their points. The aim of this paper is to determine which subsets of the density topology are semi-stratifiable, orthocompact and weakly hereditarily pseudocompact.

1 The density topology: introduction
The density topology on the real line is a finer topology than the usual Euclidean one, serving often as a counterexample in modern Topology. The open sets of the density topology are the measurable subsets of the real line, which are of density 1 at each of their points. The density topology is closely related to the measure structure of the real line and is a bridge between Analysis and Topology. In fact, the density topology is the place where Analysis, Measure theory, Set theory and Topology celebrate their consequent happy reunion, Topology probably being the host.
It was not until 1952, when the density topology was introduced by Haupt and Pauc [7]. For the first time the properties of the density topology were studied in 1961 by Goffman and Waterman [1]. Two very important contributions on the study are due to Tall [15, 16]. In connection with approximate continuity Goffman together with Neugebauer and Nishiura continued the investigation on the density topology.

The effort to explain the striking parallels between the theorems concerning nullsets and first category sets has been for years a major block of research in the field of real analysis. Indeed the density topology on the real line is a topology where these two concepts coincide. As the Lebesgue density plays a central rôle in the study of real functions it has its significant impact on the density topology.

The density topology, with its rich structure, is very often a useful counterexample in the study of general topological spaces. The aim of this paper is to investigate the density topology in the light of the some popular (and less popular) concepts in Modern Topology, namely, we try to determine which subsets of the density topology are semi-stratifiable, orthocompact and weakly hereditarily pseudocompact.

**Definition 1** A measurable set \( E \subseteq \mathbb{R} \) has density \( d \) at \( x \in \mathbb{R} \) if

\[
\lim_{h \to 0} \frac{m(E \cap [x - h, x + h])}{2h}
\]

exists and is equal to \( d \). Set \( \phi(E) = \{x \in \mathbb{R} : d(x, E) = 1\} \). The open sets of the density topology \( T \) are those measurable sets \( E \) that satisfy \( E \subseteq \phi(E) \). Clearly, the density topology \( T \) is finer than the usual topology on the real line.

Tall [15] proved that every subset of the density topology is the union of a CCC-set and a closed discrete set. He also proved that the density topology is hereditarily subparacompact and hence hereditarily submetacompact (= hereditarily \( \theta \)-refinable [7]) and hereditarily countably subparacompact. In the same paper, Tall showed that the \( \aleph_1 \)-compact subsets of the density topology are hereditarily Lindelöf, and that all collectionwise Hausdorff and all \( \sigma \)-metacompact subspaces of the density topology are the union of a hereditarily Lindelöf and a closed discrete set. Moreover, every countably paracompact subspace \( A \) of the density
topology with $|RO(A)| < \kappa$ is $\kappa$-compact \[\] and if $A \subseteq (\mathbb{R}, \mathcal{T})$ is countably paracompact, then $A$ is the union of a $2^{\aleph_0}$-compact set and a closed discrete set.

Since the density topology is not even $\sigma$-metacompact (but subparacompact) \[\], it is natural to ask if the density topology is orthocompact or semi-stratifiable. We also determine the subsets of the density topology which have some properties in the vicinity of pseudocompactness.

## 2 Semi-stratifiability in the density topology

It is well-known that every semi-stratifiable space is subparacompact. Since the density topology is hereditarily subparacompact and since semi-stratifiability is a hereditary property, it is natural to ask if the density topology is (hereditarily) semi-stratifiable.

**Theorem 2.1** For a subset $A$ of the density topology the following conditions are equivalent:

1. $A$ is a nullset.
2. $A$ is semi-stratifiable.

**Proof.** (1) $\Rightarrow$ (2) Clearly every nullset of the density topology is semi-stratifiable, since the nullsets are precisely the closed and discrete subsets.

(2) $\Rightarrow$ (1) Assume that $A$ is semi-stratifiable and that $A$ is not a nullset. Let $B$ be a subset of $A$ with the smallest possible cardinality $\kappa$ such that $B$ is not a nullset and accordingly well-order $B$. Note that due to heredity, $B$ is also semi-stratifiable. For each $x \in B$ let $W\{x\} = B \setminus B_x$, where $B_x = \{y: y < x\}$. Note that each $B_x$ is a nullset and hence closed in the density topology. Moreover, if $x \in W\{y\}$, then $y \notin W\{x\}$. Thus $W$ is an antisymmetric neighboret of $B$ (see \[\] for the definition of a neighboret). By \[\] Corollary 4.9, $B$ is $\sigma$-discrete. Since in the density topology every discrete set is a nullset, then $B$ is a nullset. By contradiction, $A$ is a nullset. \[\]

**Corollary 2.2** The density topology is not semi-stratifiable and the stratifiable subsets are precisely the closed and discrete ones.
3 $N$-pseudocompactness in the density topology

It is well-known that closed subspaces of pseudocompact spaces are not necessarily pseudocompact. If every closed (resp. nowhere dense) subspace of a pseudocompact space $(X, \tau)$ is pseudocompact, then $X$ is called weakly hereditarily pseudocompact (resp. $N$-pseudocompact).

Clearly, every countably compact space is weakly hereditarily pseudocompact and within the class of nodec spaces (= nowhere dense subsets are closed) weakly hereditarily pseudocompactness implies $N$-pseudocompactness. The following result relates to Theorem 3.6 from [15].

**Theorem 3.1** For a subset $A$ of the density topology the following conditions are equivalent:

1. $A$ is (weakly) hereditarily pseudocompact.
2. $A$ is $N$-pseudocompact.
3. $A$ is finite.

**Proof.** (1) $\Rightarrow$ (2) is obvious, since the density topology is nodec.

(2) $\Rightarrow$ (3) In the notion of [15, Theorem 3.6], we need to prove that $A$ is countably compact. Since $A$ is Tychonoff and pseudocompact, then $A$ is lightly compact, i.e. every countable open cover has a finite dense subsystem. Let $\mathcal{U} = \{U_i : i \in I\}$ be a countable open cover of the subspace $A$ and let $\{U_i : i \in F\}$ be a finite dense subsystem of $\mathcal{U}$ in $A$. If $W = A \setminus \bigcup_{i \in F} U_i$ is nonempty, then $W$ is nowhere dense in $A$ and hence closed and discrete [15]. Since $A$ is $N$-pseudocompact, then $W$ is pseudocompact and hence lightly compact. Thus the discrete subspace $W$ must be finite. Hence, $A$ is countably compact and so finite.

(3) $\Rightarrow$ (1) is obvious. $\square$

**Remark 3.2** (i) By Theorem 3.1 all weakly hereditarily lightly compact subspaces of the density topology are finite.

(ii) In Theorem 3.1, ‘$N$-pseudocompact’ can not be replace by ‘$Z$-pseudocompact’.

Topological spaces in which every locally finite collection of open sets is countable are called *pseudo $\aleph_1$-compact*. 


Proposition 3.3  (i) The density topology is pseudo $\aleph_1$-compact;
   (ii) The density topology is not $\aleph_1$-compact.

Proof. In order to observe (i), note that even point-finite collections of open sets are countable provided the space is both Baire and CCC as such is the case with the density topology \[15\]. Since submetacompact $\aleph_1$-compact spaces must be Lindelöf \[17\], then (ii) is also clear. \qed

Remark 3.4  (i) The density topology is even strongly pseudo $\aleph_1$-compact, i.e. every point-finite collection of open sets is countable;
   (ii) The density topology is not a stable quasi-pseudo metric space, since as proved in \[10\] in stable quasi-pseudo metric spaces pseudo $\aleph_1$-compactness implies separability;
   (iii) Recall that a topological space is called mildly countably compact \[14\] if every disjoint open cover of $X$ has only a finite number of non-empty members or equivalently if there is no continuous function from $X$ onto the integers. Clearly, the density topology is mildly countably compact.

4 In the vicinity of orthocompactness

A topological property strictly weaker than metacompactness is orthocompactness. Recall first that a family $\mathcal{U}$ of open subsets is called interior preserving if for every $\mathcal{V} \subseteq \mathcal{U}$, $\cap \mathcal{V}$ is open. A topological space $(X, \tau)$ is called (countably) orthocompact \[5\] if every (countable) open cover of $X$ has an interior-preserving open refinement. Orthocompactness is a strictly weaker property than metacompactness, since all linearly ordered topological spaces are orthocompact. Also, all principal spaces (= union of closed sets is always a closed set) are orthocompact and all P-spaces are countably orthocompact.

Orthocompact spaces were first considered in 1966 by Sion and Willmott \[13\] as the spaces having the property Q. The name orthocompact was given by Arens.

Proposition 4.1  The density topology is (hereditarily) countably orthocompact.
Proof. As mentioned in [2], every countably metacompact (= countably submetacompact [3]) space is countably orthocompact. Note that the density topology is even countably subparacompact. □

Recall that a space \((X, \tau)\) is called \(\sigma\)-orthocompact [4] if \(C\) is an open cover of \(X\), then there exists an open refinement \(R = \bigcup_{n=1}^{\infty} R_n\) of \(C\) such that for each \(n \in \omega\), \(R_n\) is interior preserving.

**Theorem 4.2** (CH) *The density topology has a dense hereditarily orthocompact subspace of power continuum.*

*Proof.* First, we establish that the density topology has a dense \(\sigma\)-orthocompact subspace of power continuum. It is proved in [3] that every CCC, Baire, dense-in-itself space with \(\pi\)-weight \(\leq 2^{\aleph_0}\) contains a dense generalized Lusin subspace (= every nowhere dense subset has cardinality \(\leq 2^{\aleph_0}\)) of power continuum. Since the density topology is cometrizable [16], then its \(\pi\)-weight \(\leq 2^{\aleph_0}\). Let \(S\) be the dense subspace (of the density topology) in question. By [13, Theorem 3.1], \(S = A \cup B\), where \(A\) is CCC and \(B\) is nowhere dense. Clearly, \(B\) is countable. Thus in order to show that \(S\) is \(\sigma\)-orthocompact, it suffices to verify that \(A\) is \(\sigma\)-orthocompact. Let \(U\) be an open cover of \(A\). Let \(V = \{V_n; n \in \omega\}\) be a maximal disjoint collection of open sets such that each one is included in a member of \(U\). Since \(A \setminus \cup V\) is nowhere dense in \(A\), then it is nowhere dense in \(S\) and hence countable. Since \(V\) is interior preserving, then \(A\) is \(\sigma\)-orthocompact. Thus \(S\) is \(\sigma\)-orthocompact and since \(S\) is also perfect due to heredity, then \(S\) is hereditarily orthocompact. □

Recall that a topological space \((X, \tau)\) is called *weakly orthocompact* (in the sense of Peregudov) [11] if every open cover \(U\) of \(X\) has an open refinement \(V\) such that for each \(x \in X\) the set \(\cap V_x\) has nonempty interior. In [12], Scott defined another form of orthocompactness, which he also called *weakly orthocompact*. Scott’s definition requires that directed open covers have interior preserving open refinements. A topological space \((X, \tau)\) is called *semi-metacompact* [4, 11] if every open cover of \(X\) has an open-finite refinement, where an open-finite cover means that no nonempty open set is a subset of infinitely many members of the cover [11].
Proposition 4.3 (CH) (i) The density topology is a weakly orthocompact space in the sense of Peregudov.

(ii) The density topology is not semi-metacompact.

Proof. (i) Follows from [11, Theorem 3] and the fact that under CH, the density topology is meta-Lindelöf.

(ii) Follows from the fact that all semi-metacompact weakly orthocompact spaces are metacompact. □

A slightly stronger form of orthocompactness was recently considered by Junnila and Künzi. A topological space $(X, \tau)$ is called ortho-refinable [9] provided that for each open cover $C$ of $X$ there are an ordinal $\delta$ and a decreasing chain $(T_\alpha)_{\alpha<\delta}$ of transitive partial neiborheads on $X$ so that for each $x \in X$ there exists $\alpha < \delta$ such that $x \in \text{St}(x, T_\alpha) \subseteq C$ for some $C \in C$. Here $T_\alpha = \{T_\alpha(x) : x \in T_\alpha(X)\}$. Concerning neiborheads the reader may refer to [8]. The class of ortho-refinable spaces is placed between the classes of spaces having ortho-bases and orthocompactness [9].

Lemma 4.4 [9] A submetacompact space is ortho-refinable if and only if it is orthocompact.

Theorem 4.5 (CH) The density topology has a dense hereditarily ortho-refinable subspace of power continuum.

Proof. Follows from Theorem 4.2 and Lemma 4.4. □

In the notion of Lemma 4.4, the orthocompactness of the density topology would imply its ortho-refinability. On the other hand, if the density topology is a $\sigma$-orthocompact space, then it clearly would be hereditarily orthocompact.

Question. Is the density topology ($\sigma$-)orthocompact?
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