Data transmission via erasure type channels protected by linear codes

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Abstract. The paper is concerned with data transmission via channels composed of a memoryless binary symmetric channel and the erasure channel of Peter Elias. Channels of this type play an important role in modelling different types of networks especially wireless networks, and have been investigated using, amongst others, the theory of Markov chains. Channel Capacities and network flows have been determined. The authors focus their interest on some aspects of coding theory. They assume the data transmission to be protected by a linear code, a CRC for example, and determine the probability of undetected error of the code. They then consider redundant transmission via two or more channels with bit inversion, and calculate the probability of undetected error. They prove some inequalities that are useful instruments to estimate the rate of transmission errors and to determine safety integrity levels according to the standards. Finally the authors apply their results to Bluetooth channels suffering from different types of noise.

1. Introduction
Suppose that data are transmitted via a binary symmetric channel without memory (BSC, memoryless noisy channel) suffering from intermittent and temporary erasures, caused by a fading of the channel. Such a channel may be considered as a memoryless noisy channel followed by Peter Elias’ erasure channel (cf. [3], [4], see also [5]). In the present paper this type of channel is called Noisy Elias Channel (see Figure 1).

To begin with, let the binary symmetric channel be characterized by the transition probabilities

\[ p(1|0) = p(0|1) = \varepsilon, \quad p(0|0) = p(1|1) = 1 - \varepsilon, \]  

(1)

where \( \varepsilon \) is the bit error ratio (BER).

If then an erasure happens with probability \( \varphi \), the erasure channel is determined by the transition probabilities

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\begin{align*}
    p(e|0) = p(e|1) = \varphi, \quad p(0|0) = p(1|1) = 1 - \varphi
\end{align*}

where \( e \) is the erasure symbol. The transition probabilities of the Noisy Elias Channel are then given by

\begin{align}
    p(0 | 0) &= (1 - \varepsilon) \cdot (1 - \varphi) \\
    p(1 | 0) &= \varepsilon \cdot (1 - \varphi) \\
    p(0 | 1) &= \varepsilon \cdot (1 - \varphi) \\
    p(1 | 1) &= (1 - \varepsilon) \cdot (1 - \varphi). \\
    p(e | 0) &= \varphi \\
    p(e | 1) &= \varphi
\end{align}

**Erasure Type Channel**

![Figure 1. Noisy Elias Channel.](image)

There are now to problems arising: Firstly, how to become aware of an erasure or a fading. Without additional information it is difficult if not impossible to decide if a single received 0 has been transmitted uncorrupted or has been the result of a fading of the channel. This problem can be solved easily by converting each bit into a doublet of 2 bits, whose first component is the original bit and the second one is an inverted copy, and transmitting the doublet. The receiving device then has to verify that the arriving doublet is antivalent. If it is not, it shall be rejected by the receiver. By this procedure erasures will be detected confidently. The second question is how to describe the probabilistic influence of the erasures onto the overall transition probabilities.

It seemed to the authors that erasure type channels are a good model for data transmission using Bluetooth technology. Table 2 below lists some aspects of this technology.

**Table 1. Bluetooth technology.**

| Frequency band | Max. Data rate | Range     | Bandwidth | Modulation Type       | Data Protection |
|----------------|----------------|-----------|-----------|-----------------------|-----------------|
| 2.4 GHz        | 3-24 Mb/s      | 10-100 m  | 1 MHz     | GFSK, pi/4-DQPSK or 8DPSK | 16-bit CCITT CRC |

2. **The Redundant Antivalent Noisy Elias Channel**

To attack the second problem of the problems mentioned above (the probabilistic influence), let us begin with introducing a Noisy Channel \( \text{NCh}_{\text{anti}} \) describing the process of transmitting the doublets (see Figure 2).

The input alphabet of \( \text{NCh}_{\text{anti}} \) is be given by \( X = \{(0,1), (1,0)\} \), where \( \{0, 1\} \) is the input resp. output alphabet of the underlying BSC with transition probabilities given by (1). Let further \( Y = \{(0,1), (1,0), (0,0), (1,1)\} \) denote the output alphabet of \( \text{NCh}_{\text{anti}} \). The transition
probabilities are described by table 2:

Table 2. The transition properties of the redundant Noisy Elias Channel.

| Input NChanti | Input BEC | Output BEC | Output Noisy Elias | Erasure Happened? | Transition probability |
|---------------|-----------|------------|--------------------|-------------------|-----------------------|
| 0             | (0,1)     | (0,1)      | (0,1)              | 0                 | no                    | \((1 - \epsilon)^2(1 - \phi)\) |
| 0             | (0,1)     | (0,1)      | (0,1)              | 0                 | yes                   | \((1 - \epsilon)^2 \cdot \phi\) |
| 0             | (0,1)     | (1,0)      | (1,0)              | 1                 | no                    | \(\epsilon^2(1 - \phi)\) |
| 0             | (0,1)     | (1,0)      | (1,0)              | 1                 | yes                   | \(\epsilon^2 \cdot \phi\) |
| 0             | (0,1)     | (0,0)      | (0,0)              | (0,0)             | no                    | \((1 - \epsilon) \cdot \epsilon \cdot (1 - \phi)\) |
| 0             | (0,1)     | (0,0)      | (0,0)              | (0,0)             | yes                   | \((1 - \epsilon) \cdot \epsilon \cdot \phi\) |
| 0             | (0,1)     | (1,1)      | (1,1)              | (1,1)             | no                    | \(\epsilon(1 - \epsilon) \cdot (1 - \phi)\) |
| 0             | (0,1)     | (1,1)      | (1,1)              | (1,1)             | yes                   | \(\epsilon(1 - \epsilon) \cdot \phi\) |
| 1             | (1,0)     | (1,0)      | (1,0)              | 1                 | no                    | \((1 - \epsilon)^2(1 - \phi)\) |
| 1             | (1,0)     | (1,0)      | (1,0)              | 1                 | yes                   | \((1 - \epsilon)^2 \cdot \phi\) |
| 1             | (1,0)     | (0,1)      | (0,1)              | (0,1)             | no                    | \(\epsilon^2(1 - \phi)\) |
| 1             | (1,0)     | (0,1)      | (0,1)              | (0,1)             | yes                   | \(\epsilon^2 \cdot \phi\) |
| 1             | (1,0)     | (0,0)      | (0,0)              | (0,0)             | no                    | \(\epsilon(1 - \epsilon) \cdot (1 - \phi)\) |
| 1             | (1,0)     | (0,0)      | (0,0)              | (0,0)             | yes                   | \(\epsilon(1 - \epsilon) \cdot \phi\) |
| 1             | (1,0)     | (1,1)      | (1,1)              | (1,1)             | no                    | \((1 - \epsilon)^2 \cdot \epsilon \cdot (1 - \phi)\) |
| 1             | (1,0)     | (1,1)      | (1,1)              | (1,1)             | yes                   | \((1 - \epsilon)^2 \cdot \epsilon \cdot \phi\) |

The same erasure symbol \(e\) is used in the non-redundant and in the redundant case. The erasure probabilities must not coincide.
I.e., the result is a memoryless channel with input alphabet $I = \{0, 1\}$, output alphabet $O = \{0, 1, \varepsilon\}$, and transmission probabilities given by

\[
\begin{align*}
    p(0|0) &= (1 - \varepsilon)^2 \cdot (1 - \phi), \\
    p(1|0) &= \varepsilon^2 \cdot (1 - \phi), \\
    p(1|1) &= (1 - \phi)^2 \cdot (1 - \phi), \\
    p(0|1) &= \varepsilon^2 \cdot (1 - \phi), \\
    p(\varepsilon|0) &= \phi + 2 \cdot \varepsilon \cdot (1 - \varepsilon) \cdot (1 - \phi), \\
    p(\varepsilon|1) &= \phi + 2 \cdot \varepsilon \cdot (1 - \varepsilon) \cdot (1 - \phi).
\end{align*}
\] (3)

A channel with transition probabilities of type (2) or (3) is a generalized erasure channel (GEC, see [10]) with transition probabilities

\[
\begin{align*}
    p(0|0) &= \theta, & p(0|1) &= \eta, \\
    p(1|0) &= \eta, & p(1|1) &= \theta, \\
    p(\varepsilon|0) &= \zeta, & p(\varepsilon|1) &= \zeta,
\end{align*}
\]

where the numbers $\theta$, $\eta$ and $\zeta$ are determined by

\[
\begin{align*}
    \eta &= \varepsilon \cdot (1 - \phi), \\
    \theta &= (1 - \varepsilon)^2 \cdot (1 - \phi), \\
    \zeta &= \phi.
\end{align*}
\]

in case of a Noisy Elias Channel, and by

\[
\begin{align*}
    \eta &= \varepsilon^2 \cdot (1 - \phi), \\
    \theta &= (1 - \varepsilon)^3 \cdot (1 - \phi), \\
    \zeta &= \phi + 2 \cdot \varepsilon \cdot (1 - \varepsilon) \cdot (1 - \phi).
\end{align*}
\]

in case of a Redundant Antivalent Noisy Elias Channel (see Figure 3.).

\[\text{Figure 3. The Transition Probabilities of the GEC.}\]
3. **The probability of undetected error of erasure type channels**

Suppose now the communication to be protected by a linear code $C$, i.e. the receiver verifies the checksum of each codeword, and if there is no checksum fault the code word is accepted as correctly transmitted. Otherwise it is rejected. By [2], the probability of undetected error of a GEC can be calculated by the following equation

$$p_{ue}(\zeta, \eta, \theta, C) = \sum_{i=1}^{n} A_i \eta^i \theta^{n-i}.$$  

This fact leads to

**Theorem 1:** Suppose that data transmission via a Noisy Elias Channel or a Redundant Antivalent Noisy Elias Channel is protected by a linear Code $C$. Then the probability of undetected error is given by the equation

$$p_{ue}(\varepsilon, \phi, C) = (1 - \phi)^n \cdot \sum_{i=1}^{n} A_i \varepsilon^i \cdot (1 - \varepsilon)^{n-i}$$

in case of a Noisy Elias Channel, and by the equation

$$p_{ue}(\varepsilon, \phi, C) = (1 - \phi)^n \cdot \sum_{i=1}^{n} A_i \varepsilon^{2i} \cdot (1 - \varepsilon)^{2(n-i)}$$

in case of a Redundant Antivalent Noisy Elias Channel.

This result is somewhat paradox, because $p_{ue}(\varepsilon, \phi, C)$ decreases, if $\phi$ increases. But the paradox resolves if you take into account tow items

1. Each erasure will surely be detected. Therefore a certain fraction of the undetected errors will be detected.
2. The channel capacity decreases, if $\phi$ increases. This means, if $\phi = 1$, each error will be detected, but transmission is impossible (for the channel capacity see [5]).

Suppose now the linear code $C$ to satisfy the $2^r$-bound (see [9])

$$p_{ue}(\varepsilon, C) \leq 2^{-q}.$$  

Then, in exactly the same way as in [9], the next theorem can be proven

**Theorem 2:** Suppose that data transmission via a Noisy Elias Channel or a Redundant Antivalent Noisy Elias Channel is protected by a linear Code $C$ satisfying the $2^r$-bound. Then the probability of undetected error upper bounded by the inequality

$$p_{ue}(\varepsilon, \phi, C) \leq \frac{72}{121} \cdot \sqrt{2 \pi} \cdot (1 - \phi)^n \cdot \sqrt{n} \cdot \frac{1}{2^p} \cdot n^d \cdot \varepsilon^d + R_n(\varepsilon) \quad (4)$$

with

$$R_n(\varepsilon) \leq 2^n \cdot \sqrt{e}^{n-1}, \quad \text{if } n \geq 4.$$
in case of a Noisy Elias Channel, and by the inequality

$$p_{ue}(\varepsilon, \varphi, C^{(2)}) \leq \frac{72}{121} \cdot \sqrt{2 \pi} \cdot (1 - \varphi)^n \cdot \frac{1}{2^n} \cdot \frac{n^d}{d!} \cdot \varepsilon^{2d} + R_a(\varepsilon^2) \quad (5)$$

with

$$R_a(\varepsilon^2) \leq 2^n \cdot \varepsilon^{n-1}, \quad \text{if} \ n \geq 4.$$

in case of a Redundant Antivalent Noisy Elias Channel.

Theorem 2 may be used to determine the maximum $\varepsilon$, allowed for $p_{uc}(\varepsilon, \varphi, C)$ to match a certain bound $\sigma$. Apart from $R_a(\varepsilon)$, which is negligibly small compared with the first term on the upper side of (4) or (5), solve (4) and (5) by $\varepsilon$, determine $\varepsilon_{\text{max}}$ such that

$$\varepsilon_{\text{max}} < n^{-1/2} \left( (1 - \varphi)^n \cdot (121 \cdot \varphi \cdot d \cdot 2^n) \right) \left( \frac{72}{\sqrt{2 \pi \cdot n^{0.5}}} \right)^{1/d}, \quad (6)$$

in case of a Noisy Elias Channel, and by

$$\varepsilon_{\text{max}} < n^{-1/2} \left( (1 - \varphi)^n \cdot (121 \cdot \varphi \cdot d \cdot 2^n) \right) \left( \frac{72}{\sqrt{2 \pi \cdot n^{0.5}}} \right)^{1/2d}, \quad (7)$$

in case of a Redundant Antivalent Noisy Elias Channel, and assure that $R_a(\varepsilon_{\text{max}})$ is small enough.

4. A procedure for condition monitoring a Bluetooth channel

Bluetooth is protected by the 16-bit CCITT-CRC which satisfies the $2^e$-bound for block lengths $n = 260, \ldots, 1024$ (see [1]), i.e. Theorem 2 may be applied. Assume further the Bluetooth channel to be an AWGN-channel $\pi/4$ DQPSK modulation scheme, then the BER-fraction due to electrical noise is given by

$$\varepsilon_E = Q\left( \sqrt{E_b/N_0} \cdot 2 \cdot (2 - \sqrt{2}) \right) \quad (8)$$

with signal to noise ratio $E_b/N_0$. Here $Q$ is the complemented normal distribution

$$Q(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, dt.$$

Assuming however a Rayleigh fading channel with uncorrelated, stationary Gaussian noise and the $\pi/4$ DQPSK modulation scheme, the BER-fraction due to electrical noise is given by

$$\varepsilon_E = 0.5 \cdot \frac{E_b}{N_0} \cdot \left( 1 - \frac{E_b}{N_0} \right) \cdot \left( \frac{1}{\sqrt{(E_b/N_0)^2 + 0.5 \cdot (E_b/N_0)}} \right). \quad (9)$$

(See [6], [7], [8], or [11] for (8) and (9).) Both equations can be used to determine the BER in dependence of the signal to noise ratio $E_b/N_0$.

Hence, in view of these results, it is suggested to use the following procedure for condition monitoring of a Noisy Elias Channel or Redundant Antivalent Noisy Elias Channel protected by a linear code satisfying the $2^e$-bound. For details see [9].

1. Fix a bound $\sigma$ or the probability of undetected error.
2. Determine BER-fractions due to transmitter and receiver errors $\varepsilon_T$ resp. $\varepsilon_R$. 
3. Determine the erasure probability $\phi$.
4. Determine $\epsilon_{\text{max}}$ according to (6) and (7).
5. Determine the maximal allowed signal to noise ratio $E_b/N_0$ in order that

$$\epsilon = \epsilon_R + \epsilon_T + \epsilon_E < \epsilon_{\text{max}}$$

The graph of BER vs. SNR for different Bluetooth modulation schemes is shown in Figure 4.

![Graph of BER vs SNR](image)

**Figure 4.** Bluetooth modulation scheme based BER vs SNR.

**Remark:** If there is redundant data transmission via $\mu$ Noisy Elias Channels or a $\mu$ Redundant Antivalent Noisy Elias Channels, it can be shown by the same proof as in the case of Theorem 1 that the respective probabilities of undetected error are given by

$$p_{\text{ud}}(\epsilon, \phi, C) = (1 - \phi)^n \cdot \sum_{i=1}^{n} A_i \epsilon^{n-i} \cdot (1 - \epsilon)^{\mu(n-i)}$$

in case of a Noisy Elias Channel, and by

$$p_{\text{ud}}(\epsilon, \phi, C) = (1 - \phi)^n \cdot \sum_{i=1}^{n} A_i \epsilon^{\mu(n-i)} \cdot (1 - \epsilon)^{\mu(2(n-i))}$$

in case of a Redundant Antivalent Noisy Elias Channel.
The aim of the authors was to investigate erasure type channels protected by a linear code, and to establish condition monitoring of safety related communication via Bluetooth. To achieve this aim, they introduced the concept of redundant transmission with antivalent doublets of bits. They determined the probability of undetected error, and proved some inequalities useful for condition monitoring. The concept can be generalized to an arbitrary number of channels. Finally the authors presented a procedure for condition monitoring Bluetooth channel.

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