A Relation Between N=8 Gauge Theories in Three Dimensions

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We show that three-dimensional N=8 $Sp(N)$ and $SO(2N + 1)$ gauge theories flow to the same strong coupling fixed point. As a consequence, the corresponding orientifold two-planes in type IIA string theory are described at strong coupling and low-energies by the same M theory background. In the large $N$ limit, these assertions are confirmed by studying discrete torsion in the supergravity theory corresponding to membranes on $\mathbb{R}^8/\mathbb{Z}_2$. 

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1. N=8 Gauge Theories in Three Dimensions

In four dimensions, N=4 Yang-Mills theories provide some of the simplest examples of interacting conformal field theories. The existence of S duality implies that there are at least two distinct realizations of the same conformal field theory \[1\]. The purpose of this letter is to explain a similar identification between conformal field theories in three dimensions. Our starting point is N=8 Yang-Mills theory in three dimensions with gauge group \(G\). The coupling constant, \(g^2\), has dimension one in three dimensions. For simplicity, we will assume there is only one coupling constant. To obtain an interacting conformal field theory, we must therefore consider the infra-red limit of the theory where \(g^2 \to \infty\) so the theory is strongly coupled.

Let the gauge group \(G\) have rank \(N\). The gauge theory contains seven scalar fields, \(\phi^i\), in the adjoint representation of \(G\). The moduli space of the theory is,

\[
\mathcal{M} = \frac{\mathbb{R}^{7N} \times \hat{T}^N}{\mathcal{W}},
\]

(1.1)

where \(\mathcal{W}\) is the Weyl group of \(G\) and \(\hat{T}^N\) is the Cartan torus for the dual group \(\hat{G}\) \[2\]. The manifest R-symmetry of the theory is \(Spin(7)\). The compact directions in the moduli space \(\mathcal{M}\) correspond to the expectation values of the scalar fields, \(\sigma\), dual to the \(N\) massless photons present at generic points in \(\mathcal{M}\). The size of the compact directions in \(\mathcal{M}\) is proportional to \(g^2\) and in the strong coupling limit, \(\hat{T}^N \to \mathbb{R}^N\).

A non-trivial conformal field theory can only appear at a singularity of the moduli space \(\mathcal{M}\) in the strong coupling limit. For the case where \(G = U(N)\), arguments given in \[3,4,2\] showed that the theory at the origin of the moduli space flows to an interacting \(Spin(8)\) invariant fixed point. Note that there are other singularities in the moduli space but we will focus on the theory at the origin.

There are two ways to show \(Spin(8)\) invariance. One argument only involves supersymmetry and the conformal group in three dimensions, which is isomorphic to \(Spin(3, 2)\). Closure of the superconformal algebra with sixteen supersymmetry generators simply requires a \(Spin(8)\) symmetry \[3,4\]. This argument is true for any gauge group \(G\).

The second argument for \(Spin(8)\) invariance will provide additional information. This argument uses the strong-weak coupling duality of four-dimensional gauge theory. Let us

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1 Both four-dimensional N=4 Yang-Mills and three-dimensional N=8 Yang-Mills have sixteen real supersymmetries.
start with N=4 Yang-Mills in four dimensions with gauge group $G$ and coupling constant $\tau$. This theory has a dual realization in terms of a theory with gauge group $\hat{G}$ and coupling $-1/\tau$. We can set the theta angle to zero for our purposes. We can then restrict to the $\mathbb{Z}_2$ subgroup of the $SL(2,\mathbb{Z})$ strong-weak coupling duality which relates the dual coupling $\tilde{\lambda}$ to the original four-dimensional coupling constant $\lambda$ in the following way:

$$\tilde{\lambda} = \frac{2\pi}{\lambda}. \quad (1.2)$$

Let us compactify the gauge theory on $\mathbb{R}^3 \times S^1$. The effective three-dimensional coupling constant is given by,

$$g^2 = \frac{\lambda^2}{2\pi R}, \quad (1.3)$$

where $R$ is the radius of the circle. The component $A$ of the four-dimensional gauge field along $S^1$ gives an extra adjoint-valued scalar,

$$\phi_e = \frac{1}{2\pi R} \int_{S^1} A. \quad (1.4)$$

This scalar field has a compact moduli space with a size proportional to $1/R$. In the $R \to 0$ limit with $g^2$ held fixed, this scalar field becomes $Spin(7)$ symmetric with the six scalars of the four-dimensional theory.

In addition, we have another $N$ scalars, $\phi_m$, coming from dualizing the three-dimensional photons. The moduli space for these scalars, which correspond to the choice of 't Hooft lines or magnetic Wilson lines along $S^1$, is also compact with a size again proportional to $1/R$. In the $R \to 0$ limit with $\lambda$ held fixed, these scalars become rotationally symmetric with the remaining non-compact scalars. Electric-magnetic duality then exchanges $G$ with $\hat{G}$ and exchanges the $\phi_e$ and $\phi_m$ directions.

When $G$ and $\hat{G}$ are different, this duality action is not a symmetry of the theory for any finite value of the three-dimensional coupling constant. Rather, the $\mathbb{Z}_2$ duality identifies one theory with a different theory. In the infra-red limit where $g^2 \to \infty$, we can combine this duality with the known $Spin(8)$ invariance. The $Spin(8)$ invariance includes rotation of $\phi_e$ into $\phi_m$ but does not exchange $G$ with $\hat{G}$. We can therefore conclude that gauge theories in three dimensions with either gauge group $G$ or gauge group $\hat{G}$ flow to the same $Spin(8)$ invariant superconformal field theory.

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2 We are only interested in differences at the level of the Lie algebra rather than the global structure of the group. For example when $G = U(N)$, this $\mathbb{Z}_2$ action is a symmetry if we choose the self-dual value for the coupling $[3]$. 

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2. Orientifolds and Supergravity

The argument given in section one is primarily interesting for the non-simply-laced gauge groups $Sp(N)$ and $SO(2N + 1)$. For any finite coupling constant, these two theories are quite different but they flow in the infra-red to the same conformal field theory at the origin of the moduli space. These gauge theories appear naturally in type IIA string theory as the low-energy excitations of $N$ D2-branes coincident with various orientifold two-planes. Our starting point is then type IIA string theory on the orientifold $\mathbb{R}^7/\mathbb{Z}_2$. The gauge theory coupling for D2-branes on top of the orientifold plane is related to the string coupling $g_s$ and string scale $M_s$ by,

$$g^2 = g_s M_s.$$  \hfill (2.1)

The $\mathbb{Z}_2$ action must act on the circle on which we reduce from M theory to type IIA if the moduli space seen by a D2-brane probing the orientifold is to agree with (1.1) for the gauge theory [6,7]. When compactified on, $\mathbb{R}^7 \times S^1/\mathbb{Z}_2$, the Chern-Simons interaction in eleven-dimensional supergravity,

$$-\frac{1}{6} \int C \wedge G \wedge G,$$  \hfill (2.3)

is only invariant if $C$ is invariant under the $\mathbb{Z}_2$ action. The M theory lift of the orientifold is therefore the $\mathbb{Z}_2$ orbifold (2.2). The gauge theory coupling can be expressed in terms of the eleven-dimensional scale, $M_{pl}$, and the size $R_{11}$ of the $S^1$ in (2.2):

$$g^2 = M_{pl}^3 R_{11}^2.$$  

As usual, to restrict to the field theory modes on the branes, we take the limit $M_{pl} \to \infty$ holding fixed $g^2$. The strong coupling limit for the gauge theory then corresponds to decompactifying the circle on which we reduce from M theory to type IIA.

We will distinguish between three kinds of orientifold two-planes. The gauge symmetry of $N$ D2-branes on top of the orientifold plane $O2^-$ is $SO(2N)$. We will count branes and charges on the quotient space. This O2-plane carries $-1/8$ units of membrane charge. We can also have a stuck $1/2$ membrane at the orientifold fixed point. We will call this case $\tilde{O2}^+$. In this case, the gauge symmetry is $SO(2N + 1)$ and the charge is $3/8$. Lastly, we
will also consider $O2^+$ which corresponds to the choice of Chan-Paton factors giving gauge group $Sp(N)$. This O2-plane has charge $1/8$.\footnote{We will not distinguish cases where the Ramond gauge-field has a non-trivial holonomy around the circle in (2.2). This holonomy was important in the case of $O4$-planes studied in \cite{8}. In those cases, the limit where $R_{11} \rightarrow \infty$, which we will primarily study, was not necessarily smooth. For example, $O4^0$ only exists with a finite size circle. In this respect, the case of $O2$-planes is nicer since the gauge theory has a smooth strong coupling limit so we can take the limit where $R_{11} \rightarrow \infty$. In this limit, any holonomy involving the circle goes away.}

Let us start with the M theory lift of $O2^-$. There are two singularities in (2.2) at 0 and $\pi$ on the circle. At strong coupling, the geometry around either singularity becomes $\mathbb{R}^8/\mathbb{Z}_2$ and the D2-branes become membranes which we can place at the fixed point. To compute the membrane charge of this configuration, we can consider M theory on $T^8/\mathbb{Z}_2$. There is a net membrane charge in this theory coming from the interaction $\int_C \wedge X_8(R)$,\footnote{We will not distinguish cases where the Ramond gauge-field has a non-trivial holonomy around the circle in (2.2). This holonomy was important in the case of $O4$-planes studied in \cite{8}. In those cases, the limit where $R_{11} \rightarrow \infty$, which we will primarily study, was not necessarily smooth. For example, $O4^0$ only exists with a finite size circle. In this respect, the case of $O2$-planes is nicer since the gauge theory has a smooth strong coupling limit so we can take the limit where $R_{11} \rightarrow \infty$. In this limit, any holonomy involving the circle goes away.}

\begin{equation}
- \int C \wedge X_8(R),
\end{equation}

in M theory. The total membrane charge is given by $-\chi/24$ and must be cancelled to avoid a tadpole anomaly \cite{12}. Although the space $T^8/\mathbb{Z}_2$ is singular, standard string theory technology for orbifolds can be used to show that $\chi/24 = 16$. Each fixed point, and consequently the $\mathbb{R}^8/\mathbb{Z}_2$ fixed point, is then a source of $-1/16$ units of membrane charge \cite{13,6}.

To count the distinct M theory configurations corresponding to the remaining O2-planes, we turn to the supergravity solutions dual to these conformal field theories \cite{14}. The supergravity solutions for the various O3-planes were analyzed in \cite{15}, while membranes on the orbifold space $\mathbb{R}^8/\mathbb{Z}_2$ were mentioned in \cite{16}. The $SO(2N)$ $(2,0)$ theory was considered from a supergravity perspective in \cite{17}. M theory compactified on $AdS_4 \times \mathbb{R}P^7$ is dual to the world-volume theory of membranes on $\mathbb{R}^8/\mathbb{Z}_2$ in the limit where $M_{pl} \rightarrow \infty$. We wish to count the number of distinct fluxes for the four-form $G$ on $\mathbb{R}P^7$. This counts the number of distinct strong coupling limits for O2-planes. Note that since $w_4 = 0$ for $\mathbb{R}^8/\mathbb{Z}_2$, no half-integral $G$ flux is possible \cite{18}.

The counting of fluxes is then rather simple. We need to compute $H^4(\mathbb{R}P^7, \mathbb{Z})$ which, using the results of \cite{13}, is $\mathbb{Z}_2$. Therefore, there is only one possible choice of discrete torsion. This is a supergravity confirmation of the argument in section one. The low-energy theories on the three orientifold two-planes, $O2^-, \tilde{O}2^+$ and $O2^+$ must flow to two
distinct strong coupling conformal field theories. We know that $O2^-$ flows to the case without torsion so $O2^+$ and $O2^-$ must flow to the case with torsion.

In the case where we turn on discrete torsion, we can ask: by how much does the membrane charge change? The charge shift is generated by the term (2.3) and is determined by computing,

$$-\frac{1}{2} \int_{\mathbb{R}P^7} \frac{G}{2\pi} \wedge \frac{C}{2\pi},$$

where $G$ is the torsion class. To compute this integral, we will view $\mathbb{R}P^7$ as the boundary of a smooth eight-dimensional space $\mathcal{M}$. We can then evaluate,

$$-\frac{1}{2} \int_{\mathcal{M}} \frac{G}{2\pi} \wedge \frac{G}{2\pi},$$

rather than (2.5). Let us begin by recalling the construction of the Hopf fibration of $S^7$ over $\mathbb{C}P^3$. The sphere $S^7$ is the locus of points in $\mathbb{C}^4$ obeying,

$$|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1,$$

where $\vec{z} = (z_1, z_2, z_3, z_4)$ coordinatize $\mathbb{C}^4$. To obtain $\mathbb{C}P^3$ from $S^7$, we quotient by the $U(1)$ action,

$$\vec{z} \sim e^{i\alpha} \vec{z}.$$  \hspace{1cm} (2.6)

The fibers over $\mathbb{C}P^3$ are then circles. We can describe this construction in a slightly different way. Let us start with the total space of the bundle $O(-1)$ over $\mathbb{C}P^3$. Let $w$ be a coordinate for the fiber, which is a copy of $\mathbb{C}$. We obtain a smooth eight manifold by taking a disk $|w| < 1$ in each fiber over $\mathbb{C}P^3$. The boundary of this space is $S^7$. We can mimick this construction for the case of $\mathbb{R}P^7$ by taking the bundle $O(-2)$ rather than $O(-1)$. This gives us the eight manifold $\mathcal{M}$.

Since $\mathbb{R}P^7$ is orientable, we can associate a homology class in $H_3(\mathbb{R}P^7, \mathbb{Z})$ to $G$ using Poincaré duality. This class can be represented by an $\mathbb{R}P^3$ subspace of $\mathbb{R}P^7$ which is the boundary of a four-cycle $\mathcal{W}$ in $\mathcal{M}$. Let $\pi : \mathcal{M} \rightarrow \mathbb{C}P^3$ denote the projection map. The Hopf bundle $\mathbb{R}P^3 \rightarrow \mathbb{C}P^1$ is compatible with the projection map $\pi$ in the sense that $\pi(\mathcal{W})$ is a $\mathbb{C}P^1$ in $\mathbb{C}P^3$.

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4 I am especially grateful to E. Witten for a detailed explanation about how to compute this integral.
The connection $C$ for the torsion class $G$ obeys,

$$\int_{\partial W} \frac{C}{2\pi} = \frac{1}{2},$$

(2.7)

and therefore,

$$\int_{W} \frac{G}{2\pi} = \frac{1}{2}.$$  

(2.8)

We need to identify a class $G$ in $\mathcal{M}$ obeying (2.8). The Poincaré dual of the base $\mathbb{C}P^3$ of the bundle $\mathcal{M}$ is a two-form $X$ satisfying,

$$\int_X X \wedge X = -2.$$  

We can then take $G/2\pi = -X^2/4$. The charge shift is then given by,

$$-\frac{1}{2(2\pi)^2} \int_{\mathcal{M}} G \wedge G = \frac{1}{16} \int_{\mathcal{M}} X^4,$$

(2.9)

The final integral is evaluated by integrating over one $X$ which restricts the remaining integral to the base $\mathbb{C}P^3$, which is then standard.

Turning on the discrete torsion therefore shifts the membrane charge from $-1/16$ to $3/16$. There is an amusing interpretation for this $1/4$ membrane charge. $\tilde{O}2^+$ differs from $O2^-$ by a stuck $1/2$ membrane. In studying the strong coupling description of $\tilde{O}2^+$, we see that the orientifold plane splits into two singularities from (2.2). Apparently the $1/2$ stuck membrane also splits into two fluxes, each carrying $1/4$ unit of membrane charge. The case of $O2^+$ is a little different. The singularity at the origin of (2.2) has charge $3/16$ but the other singularity at $\pi$ should have charge $-1/16$ since the total membrane charge for $O2^+$ is $1/8$.

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References

[1] C. Montonen and D. Olive, Phys. Lett. B72 (1977) 117; A. Sen, hep-th/9402032, Phys. Lett. B329 (1994) 217; hep-th/9402002, Int. J. Mod. Phys. A9 (1994) 3707.
[2] N. Seiberg, hep-th/9705117, Nucl. Phys. Proc. Suppl. 67 (1998) 158.
[3] S. Sethi and L. Susskind, hep-th/9702101, Phys. Lett. B400 (1997) 265.
[4] T. Banks and N. Seiberg, hep-th/9702187, Nucl. Phys. B497 (1997) 41.
[5] W. Nahm, Nucl. Phys. B135 (1978) 149.
[6] A. Sen, hep-th/9603113, Mod. Phys. Lett. A11 (1996) 1339.
[7] N. Seiberg, hep-th/9606017, Phys. Lett. B384 (1996) 81.
[8] K. Hori, hep-th/9805141.
[9] E. G. Gimon, hep-th/9806226.
[10] C. Vafa and E. Witten, hep-th/9505053, Nucl. Phys. B447 (1995) 261.
[11] M. Duff, J. Liu and R. Minasian, hep-th/9506126, Nucl. Phys. B452 (1995) 261.
[12] S. Sethi, C. Vafa and E. Witten, hep-th/9606122, Nucl. Phys. B480 (1996) 213.
[13] K. Dasgupta, D. P. Jatkar and S. Mukhi, hep-th/9707224, Nucl. Phys. B523 (1998) 465.
[14] J. Maldacena, hep-th/9711200.
[15] E. Witten, hep-th/9805112, J.H.E.P. 7 (1998) 6.
[16] O. Aharony, Y. Oz and Z. Yin, hep-th/9803051, Phys. Lett. B430 (1998) 87.
[17] C. Ahn, H. Kim and H. Yang, hep-th/9808182.
[18] E. Witten, hep-th/9609122, J. Geom. Phys. 22 (1997) 1.