Relaxing the Differentiability Assumption in Taylor Theorem and Optimization: Applications to Finance

Moawia Alghalith
UWI, St Augustine
malghalith@gmail.com

Abstract. We overcome a major obstacle in mathematical optimization. In so doing, we provide smooth solutions to the HJB PDE without assuming the smoothness of the value function. We apply our method to the portfolio model.

Key words: portfolio, consumption, optimization, HJB PDE, value function, stochastic factor, viscosity solution.
1 Introduction

A major obstacle in dynamic optimization is that the value function may not be smooth (see, for example, Strulovici and Szydlowski (2015) for a discussion). Actually, it is not expected to be smooth. A smooth solution to the Hamilton-Jacobi-Bellman partial differential equation (HJB PDE) may not exist. It is not surprising that a verification result exists only for a few functional forms. In response, weak solutions such as viscosity solutions were introduced (see, for example, Crandall and Lyon (1983), Hata and Sheu (2012) and Grüne and Picarelli (2015), among many others).

In this paper, we overcome this obstacle in dynamic optimization. In doing so, we present a simple method that relaxes the assumption of the differentiability (smoothness) of the value function. That is, we generally establish the existence and the uniqueness of a strong (smooth) solution without the differentiability assumption.

We apply our method to three dominant models (the portfolio model, the consumption-portfolio model, and the stochastic-factor model). However, the extension to other areas is straightforward.
2 Relaxing the differentiability assumption in Taylor theorem

We introduce a pioneering approach that overcomes an obstacle in mathematical sciences. In doing so, we introduce Taylor-like expansions even if the original function is not differentiable. Needless to say, this result is very useful in many applications, such as the areas of regression analysis, optimization, integration, and partial differential equations PDEs.

Consider a continuous, bounded function \( f(x) \) that is not differentiable with respect to \( x \) in a traditional sense; however, with a simple transformation the function can become differentiable in some sense. It can be expressed as \( f(x + \beta) \), where \( \beta \) is a shift parameter with an initial value equal to zero. We define \( i \equiv x + \beta \), so that \( f(i) \equiv f(x + \beta) \), initially \( f(i) = f(x) \). We can show that \( f \) is differentiable (in some sense) w.r.t. \( \beta \) (since every function can be shifted; this can be easily seen graphically as a horizontal shift), and consequently \( f \) is differentiable w.r.t. \( i \) holding \( x \) constant.

The idea is simple and intuitive. However, the mathematical proof is very novel.

Consider the following Taylor expansion around \( c \)

\[
f(i) = f(c) + f_i(c)(i - c) + R,
\]

where \( R \) is the error. We also examine a two-variable function \( f(x, y) \), however, the extension to a multiple-variable function is straightforward. As before, we define \( h \equiv y + \alpha \), where \( \alpha \) is a shift parameter with an initial value equal to zero; so that initially \( f(x + \beta, y + \alpha) \equiv f(i, h) \). As before, \( f \) is differentiable w.r.t. \( h \) and thus \( i \) (holding \( x \) constant).

The Taylor expansion is given by

\[
f(i, h) = f(c_1, c_2) + f_i(c_1, c_2)(i - c_1) + f_h(c_1, c_2)(h - c_2) + R_2.
\]

**Theorem 1:**

i) \( f(i) \) is differentiable w.r.t. \( i \) (holding \( x \) constant).

ii) \( f(i, h) \) is differentiable w.r.t. \( i \) and \( h \) (holding \( x \) and \( y \) constant).

**Proof.**

i) Differentiability with respect to the *shift parameter* (as opposed to a *variable*) stems from the fact that the change in the shift parameter is a constant, since the function is shifted horizontally by a constant amount.
(graphically, this is evidenced by a horizontal shift of the function). Let $\hat{f}$ be the shifted function and $f$ the original function. The impact of the shift at $x$ (holding $x$ constant) is $\hat{f}(x) - f(x) = f(x + \beta) - f(x)$. Let $d\beta = \varphi - 0 = \varphi$ (since the initial values are zero), where $|\varphi|$ is a small non-zero constant. It is non-zero since the function is shifted. Therefore, the change in $V$ as a result of the shift holding $x$ constant ($dV/d\varphi$) can be expressed as this derivative

$$f_i \equiv \frac{df(i)}{di} \bigg|_{\Delta x=0} = \lim_{\Delta i \to 0} \left[ \frac{f(i + \Delta i) - f(i)}{\Delta i} \right]_{\Delta x=0} = \lim_{\Delta i \to 0} \left[ \frac{f(i + \Delta x + \Delta \beta) - f(i)}{\Delta x + \Delta \beta} \right]_{\Delta x=0} = \frac{f(i + \varphi) - f(i)}{\varphi}.$$  

By the continuity of $f$ and the fact that $\varphi \neq 0$, $f_i < \infty$ and thus it is differentiable. □

\textit{ii}) For a two-variable function, the proof is similar. As before, $f(y + \alpha, x + \beta) \equiv f(h, i)$. Let $\hat{f}$ be the shifted function and $f$ the original function. The impact of the shift at $x$ (holding $x$ (and of course $h$) constant) is $\hat{f}(x, y) - f(x, y) = f(x + \beta, y) - f(x, y)$. Thus, consider this derivative

$$f_i \equiv \frac{\partial f(h, i)}{\partial i} \bigg|_{\Delta x=0} = \lim_{\Delta i \to 0} \left[ \frac{f(h + \Delta h, i + \Delta i) - f(h, i)}{\Delta i + \Delta h} \right]_{\Delta x=\Delta h=0} = \lim_{\Delta i \to 0} \left[ \frac{f(h + \Delta h, i + \Delta x + \Delta \beta) - f(h, i)}{\Delta x + \Delta \beta + \Delta h} \right]_{\Delta x=\Delta h=0} = \frac{f(h, i + \varphi) - f(h, i)}{\varphi}.$$  

The same applies to $f_h$. Similarly, the second derivative $f_{ii} \equiv \frac{f(h, i + \varphi) - f(h, i)}{\varphi}$ exists, since $f_i$ is continuous. □

Though we only need local differentiability, this exercise can be repeated at each value of $x$. 

4
We note that if the function is not differentiable locally (at say \(x_0\)), our expansion holds locally in the neighborhood of \(x_0\) (that is, \(f(i) = f(x_0) + f_i(x_0)(\varphi + x_0 - x_0) = f(x_0) + f_i(x_0)\varphi\)). For the other values/intervals of \(x\), clearly the classical Taylor expansion holds. Thus, we obtain expansions at both \(x_0\) and other values of \(x\). Therefore, the expansion holds generally for \(i\).

If the function is no-where differentiable, we repeat the method at each value of \(x\). Thus, once again, the expansion holds generally.

**Verification and Examples:**

We provide examples to verify the correctness of the method.

*Example 1*: The absolute value function at zero

\[
 f_i = \lim_{\Delta t \to 0} \frac{|0 + \varphi| - 0}{\varphi} = \frac{|\varphi|}{\varphi}.
\]

*Example 2*: A no-where differentiable function (the Brownian motion sample path)

\[
 f_i = \lim_{\Delta t \to 0} \left[\frac{\Omega \sqrt{dt + \varphi}}{dt + \varphi} \right]_{\Delta t = 0} = \frac{\Omega}{\sqrt{\varphi}}
\]

for \(\Omega < \infty\).

### 3 The portfolio model

In this paper, we use the standard technical assumptions (except for the smoothness assumption). We first apply our method to the baseline portfolio model (see, for example, Cvitanic and Zapatero (2000)). The risk-free asset price process is given by \(S^0 = e^{rs}\), where \(r\) is the constant risk-free rate of return. The dynamics of the risky asset price are given by

\[
 dS_s = S_s (\mu ds + \sigma dW_s),
\]

where \(\mu\) and \(\sigma\) are the constant rate of return and the volatility, respectively; \(W_s\) is a Brownian motion defined on the probability space \((\Omega, F, F_s, P)\), where \(\{F_s\}_{t \leq s \leq T}\) is the augmentation of filtration.

We assume that the wealth process satisfies this equation

\[
 dX_s = \{rX_s + (\mu - r)\pi_s\} ds + \pi_s \sigma dW_s,
\]

where \(\pi_s\) is the optimal portfolio at time \(s\).
or

\[ X_T^\pi = x + \int_t^T \{ rX_s^\pi + (\mu - r) \pi_s \} \, ds + \int_t^T \pi_s \sigma dW_s, \]

where \( x \) is the initial wealth, \( \{ \pi_s, \mathcal{F}_s \}_{t \leq s \leq T} \) is the portfolio process, and

\[ E \int_t^T \pi_s^2 \, ds < \infty. \]

The trading strategy \( \pi_s \in \mathcal{A}(x) \) is admissible.

The investor maximizes the expected utility of the terminal wealth

\[ V(t, x) = \sup_\pi E \left[ U(X_T^\pi) \mid \mathcal{F}_t \right], \]

where \( V(.) \) is the value function, \( U(.) \) is a continuous and bounded utility function. It is well known that if \( V(t, x) \in C^{1,2}([0, T], R) \), it satisfies (in the classical sense) the HJB PDE (suppressing the notations)

\[ V_t + rxV_x + \sup_\pi \left\{ \pi_t (\mu - r) V_x + \frac{1}{2} \pi_t^2 \sigma^2 V_{xx} \right\} = 0; \quad V(T, x) = U(x), \]

where the subscripts of \( V \) denote partial derivatives. Therefore, the optimal portfolio is given by

\[ \pi_t^* = -\frac{(\mu - r) V_x}{\sigma^2 V_{xx}}. \]

We define \( h \equiv t + \alpha, \ i \equiv x + \beta, \) and \( d\beta = \varphi = 0 = \varphi, \) where \( \alpha \) and \( \beta \) are deterministic shift parameters, each with an initial value equal to zero (see, for example, Dalal (1990) and Alghalith (2008)); so that initially \( V(t, x) = V(t + \alpha, x + \beta) \equiv V(h, i), \) and \( \varphi \) is a (small) non-zero constant. Evidently, by construction, \( V \) is continuously differentiable w.r.t. each shift parameter, since any function can be shifted (graphically the derivative is depicted as a small horizontal shift of the graph of the function; thus the derivative exists); and hence it is continuously differentiable w.r.t. \( h \) and \( i \), even if it is non-differentiable w.r.t. \( x \) or \( t \) (see Appendix 1 for the proof).

In the following proof (and the extensions), we use the standard technical assumptions (see, for example, Touzi (2002), Touzi (2010) and Strulovici and Szydlowski (2015) for a discussion), except for the smoothness of value function.
**Theorem:** The value function \( V(h, i) \) satisfies (in the classical sense) this HJB PDE

\[
V_h + (rx + \varphi)V_i + \sup_{\pi_t} \left\{ \pi_t (\mu - r) V_i + \frac{1}{2} \pi_t^2 \sigma^2 V_{ii} \right\} = 0, \quad V(T, x) = U(x).
\]

**PROOF\[1\]** Define the function \( \bar{V}(h, i) \) as

\[
\bar{V}(h, i) \equiv \bar{V}(t, x) = \mathbb{E}\left[ \frac{U(X_{\pi(T)})}{F_t} \right].
\]

Applying Ito’s rule to \( \bar{V}(h, i) \), we obtain (suppressing the notations)

\[
d\bar{V} = \bar{V}_h dh + \bar{V}_i di + \frac{1}{2} (di)^2 \bar{V}_{ii} = \bar{V}_h dh + \bar{V}_i [dX_s + d\beta] + \frac{1}{2} (dX_s)^2 \bar{V}_{ii} =
\]

\[
\left[ \bar{V}_h + \bar{V}_i (\pi_s (\mu - r) + rX_s^\pi + \varphi) + \frac{1}{2} \bar{V}_{ii} \pi_s^2 \sigma^2 \right] ds + \bar{V}_i \pi_s \sigma dW_s,
\]

where \( h \) and \( i \) are defined the same as before (and more generally for each time \( s \), \( h \equiv s + \alpha \), \( i \equiv X_s + \beta \), \( d\beta = \varphi \)). Integrating the previous equation yields

\[
\bar{V}(T, X_{\pi(T)}) = U(X_{\pi(T)}) = \bar{V}(t, x) + \int_t^T \left( \bar{V}_h + \bar{V}_i (\pi_s (\mu - r) + rX_s^\pi + \varphi) + \frac{1}{2} \bar{V}_{ii} \pi_s^2 \sigma^2 \right) ds + \int_t^T \bar{V}_i \pi_s \sigma dW_s.
\]

Taking expectation expectations on both sides yields

\[
\bar{V}(t, x) = \mathbb{E}\left[ \frac{U(X_{\pi(T)})}{F_t} \right] - \mathbb{E}\left[ \int_t^T \left( \bar{V}_h + \bar{V}_i (\pi_s (\mu - r) + rX_s^\pi + \varphi) + \frac{1}{2} \bar{V}_{ii} \pi_s^2 \sigma^2 \right) ds/F_t \right].
\]

The above equation implies that for any \( \pi_t \)

\[
\bar{V}_h + (\pi_t (\mu - r) + rx + \varphi) \bar{V}_i + \frac{1}{2} \bar{V}_{ii} \pi_t^2 \sigma^2 = 0. \tag{1}
\]

\[1\]The proof also relies on Appendix 1.
Now, by definition
\[ V(h, i) \equiv V(t, x) = \sup_{\pi} \tilde{V}(t, x; \pi), \]
and thus (I) holds for the optimal portfolio $\pi^*_t$
\[ V_h + (r x + \varphi) V_i + \sup_{\pi_t} \left\{ \pi_t (\mu - r) V_i + \frac{1}{2} \pi^2_t \sigma^2 \right\} = 0, V(T, x) = U(x). \square \]
We also note that integrating over $[0, x]$ and $[0, t]$ will yield the original value function $V(t, x)$ as the solution. Also, the optimal portfolio is given by
\[ \pi^*_t = -\frac{(\mu - r) V_i(t, x)}{\sigma^2 V_{ii}(t, x)}, \]
since the derivatives are taken at the initial values $h = t$ and $i = x$.

4 Extensions

4.1 The portfolio and consumption

If a part of the wealth can be consumed by the investor (see Hata and sheu (2012) and Trybola (2015), among others), the wealth process is given by
\[ X^{\pi, c}_T = x + \int_t^T \{ r X^{\pi, c}_s + (\mu - r) \pi_s - c_s \} ds + \int_t^T \pi_s \sigma dW_s, \]
where $\{c_s, \mathcal{F}_s\}_{t \leq s \leq T}$ is the consumption rate process, with $E \int_t^T \pi^2_s ds < \infty$, $E \int_t^T c_s ds < \infty$ and $c_s \geq 0$. The strategy $(\pi_s, c_s) \in \mathcal{A}$ is admissible. The investor maximizes the expected utility of the terminal wealth and consumption
\[ V(t, x) = \sup_{\pi, c} E \left[ U_1(X^{\pi, c}_T) + \int_t^T U_2(c_s) ds \mid \mathcal{F}_t \right]. \]
If it is smooth, the value function satisfies this HJB PDE
\[ V_t + r x V_x + \]
\[
\begin{align*}
&\sup_{\pi_t, c_t} \left\{ \frac{1}{2} \pi_t^2 \sigma^2 V_{xx} + [\pi_t (\mu - r) - c_t] V_x + U_2 (c_t) \right\} = 0, \\
&V (T, x) = U (x).
\end{align*}
\]

The optimal solutions are
\[
\pi_t^* = -\frac{(\mu - r) V_x}{\sigma^2 V_{xx}},
\]
\[
U_2' (c_t^*) = V_x (t, x).
\]

Following the previous procedure in Section 2, we can show that the value function \( V (h, i) \) satisfies (in a classical sense)
\[
V_h + (r x + \varphi_1) V_i + \sup_{\pi_t, c_t} \left\{ \frac{1}{2} \pi_t^2 \sigma^2 V_{ii} + [\pi_t (\mu - r) - c_t] V_i + U_2 (g) \right\} = 0,
\]
where \( g \equiv c_t^* + \gamma \), and \( \gamma \) is a shift parameter with an initial value equal to zero (defined the same as before) and \( d\gamma = \varphi_1 \) is a non-zero constant; also, \( i \) and \( h \) are defined the same as before. Thus, the optimal solutions are
\[
\pi_t^* = -\frac{(\mu - r) V_i (t, x)}{\sigma^2 V_{ii} (t, x)},
\]
\[
U_2' (c_t^*) = V_i (t, x).
\]

4.2 The portfolio with a (stochastic) economic factor

The stochastic factor model assumes that the rate of return and volatility are functions of a stochastic (economic) factor (see, for example, Alghalith (2009) and Trybola (2015)). This implies a two-dimensional standard Brownian motion \( \{ (W_{1s}, W_{2s}) , \mathcal{F}_s \}_{t \leq s \leq T} \) defined on the probability space \( (\Omega, \mathcal{F}, \mathcal{F}_s, P) \). The risk-free asset price process is \( S^0 = e^{rs} \), where \( r \) is the rate of return and \( Y_s \) is the economic factor.

The risky asset price process is given by
\[
\frac{dS_s}{S_s} = \mu (Y_s) ds + \sigma (Y_s) dW_s^1,
\]
where \( \mu(Y_s) \) and \( \sigma(Y_s) \in C^2_b(R) \) are the rate of return and the volatility, respectively. The economic factor process satisfies

\[
dY_s = b(Y_s) \, ds + \rho dW_1^s + \sqrt{1 - \rho^2} dW_2^s, Y_t = y,
\]

where \(|\rho| < 1\) is the correlation factor between the two Brownian motions and \( b(Y_s) \in C^1(R) \).

The wealth process satisfies

\[
X_{\pi}^T = x + \int_t^T \{ rX_{\pi}^s + [\mu(Y_s) - r] \pi_s \} \, ds + \int_t^T \pi_s \sigma(Y_s) \, dW_1^s.
\]

The investor maximizes the expected utility of the terminal wealth

\[
V(t, x, y) = \sup_{\pi} E \left[ U(X_{\pi}^T) \mid \mathcal{F}_t \right].
\]

If it is smooth, the value function satisfies this Hamilton-Jacobi-Bellman PDE

\[
V_t + rxV_x + b(y)V_y + \frac{1}{2}V_{yy} + \\
\sup_{\pi_t} \left\{ \frac{1}{2} \pi_t^2 \sigma^2(y) V_{xx} + [\pi_t (\mu(y) - r)] V_x + \rho \sigma(y) \pi_t V_{xy} \right\} = 0,
\]

\[
V(T, x, y) = U(x).
\]

Hence, the optimal portfolio is

\[
\pi^*_t = -\frac{[\mu(y) - r] V_x}{\sigma^2(y) V_{xx}} - \frac{\rho V_{xy}}{\sigma(y) V_{xx}}.
\]

Using the previous procedure, we can show that the value function \( V(h, i, j) \) satisfies (in a classical sense) this HJB PDE

\[
V_h + (rx + \varphi_2) V_i + (b(y) + \iota) V_j + \frac{1}{2} V_{jj} + \\
\sup_{\pi_t} \left\{ \frac{1}{2} \pi_t^2 \sigma^2(y) V_{ii} + [\pi_t (\mu(y) - r)] V_i + \rho \sigma(y) \pi_t V_{ij} \right\} = 0,
\]

\[
V(T, x, y) = U(x),
\]
where \( j \equiv y + \zeta \), and \( \zeta \) is a shift parameter with an initial value equal to zero (defined the same as before) and \( d\zeta = \varphi_2 \) is a non-zero constant; \( h \) and \( i \) are defined the same as before. Therefore, the optimal solution is

\[
\pi^* = -\frac{[\mu(y) - r] V_i(t, x, y)}{\sigma^2(y) V_{ii}(t, x, y)} - \frac{\rho V_{ij}(t, x, y)}{\sigma(y) V_{ii}(t, x, y)}.
\]

**Appendix 1. Proof of the differentiability**

Differentiability with respect to the shift parameter (as opposed to a variable) stems from the fact that the change in the shift parameter is a constant (graphically, this is evidenced by a horizontal shift of the function). As before, \( V(t + \alpha, x + \beta) \equiv V(h, i) \), and let \( d\alpha = \epsilon - 0 = \epsilon, d\beta = \varphi - 0 = \varphi \) (since the initial values are zero), where \( \epsilon \) and \( \varphi \) are small non-zero constants. Consider this derivative

\[
\left. \frac{\partial V(h, i)}{\partial i} \right|_{\Delta x=0} = \lim_{\Delta i \to 0} \frac{V(h, i + \Delta i) - V(h, i)}{\Delta i}.
\]

\[
= \lim_{\Delta i \to 0} \frac{V(h, i + \Delta x + \Delta \beta) - V(h, i)}{\Delta x + \Delta \beta} = \lim_{\Delta i \to 0} \frac{V(h, i + \Delta x + \varphi) - V(h, i)}{\Delta x + \varphi}.
\]

\[
= \frac{V(h, i + \varphi) - V(h, i)}{\varphi}.
\]

By the continuity and boundedness of \( V \) and the fact that \( \varphi \neq 0 \), the derivative exists. Since \( x \) and \( \beta \) are independent (\( \frac{dx}{d\beta} = 0 \)), \( \frac{\partial V(h, i)}{\partial h} \left|_{\Delta t=0} = \frac{\partial V(h, i)}{\partial h} \right|_{\Delta t=0} \equiv V_i \). Similarly, \( \frac{\partial V(h, i)}{\partial h} \left|_{\Delta t=0} = \frac{\partial V(h, i)}{\partial h} \right|_{\Delta t=0} \equiv V_i \), since \( \frac{dt}{dx} = 0 \). Similarly, the second derivative \( V_{ii} = \frac{\partial V_i(h, i + \varphi) - V_i(h, i)}{\varphi} \) exists, since \( V_i \) exists and \( \varphi \neq 0 \). \( \square \)

Under the assumption of a non-constant marginal utility, \( V_{ii} \neq 0 \).

**References**

[1] Alghalith, M. (2008). Recent applications of theory of the firm under uncertainty. European Journal of Operational Research, 186, 443-450.

[2] Alghalith, M. (2009). A new stochastic factor model: general explicit solutions. Applied Mathematics Letters, 22, 1852-1854.
[3] Crandall, M.G. and Lions, P.L. (1983). Viscosity solutions of Hamilton-Jacobi equations. Transactions of the American Mathematical Society, 277, 1–42.

[4] Cvitanic, J. and Zapatero, F. (1994). Introduction to the economics and mathematics of financial markets. MIT Press, Cambridge, MA.

[5] Dalal, A. (1990). Symmetry Restrictions in the Analysis of the Competitive Firm Under Price Uncertainty. International Economic Review, 31, 207-211.

[6] Grüne, L., and Picarelli, A. (2015). Zubov’s method for controlled diffusions with state constraints. University of Bayreuth working paper. https://epub.uni-bayreuth.de/id/eprint/1854.

[7] Hata, H. and Sheu, S. (2012). On the Hamilton-Jacobi-Bellman equation for an optimal consumption problem: I. Existence of solution, SIAM J. Control Optim., 50, 2373–2400.

[8] Strulovici, B. and Szydlowski, M. (2015). On the smoothness of value functions and the existence of optimal strategies in diffusion models. Journal of Economic Theory, 159, 1016–1055.

[9] Touzi, N. (2002). Stochastic control problems and viscosity solutions, and application to finance. Special Research Semester on Financial Markets: Mathematical, Statistical and Economic Analysis Pisa. http://www.cmap.polytechnique.fr/~touzi/pise02.pdf

[10] Touzi, N. (2010). Deterministic and stochastic control, application to finance. Lecture Notes. Ecole Polytechnique Paris Departement de Mathematiques Appliquees.

[11] Trybu/suppress la, J. (2015). Optimal consumption problem in the Vasicek model. Opuscula Math. 35, 547-560.