Planck Stars from Asymptotic Safe Gravity

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The Asymptotic safe gravity program suggests a specific form of the running Newtonian coupling constant which depends on two free parameters, usually denoted with \( \tilde{\omega} \) and \( \gamma \). New metrics can be inferred from a "running" gravitational constant, and in particular new black hole spacetimes. Of course, the minimum requirement satisfied by the new metrics should be the matching with a Schwarzschild field at large radial coordinate. By further imposing the simple request of matching the new metric with the Donoghue quantum corrected potential, we find a negative value of the \( \tilde{\omega} \) parameter, and hence a not yet explored black hole metric, which naturally turns out to describe the so called Planck stars.

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I. INTRODUCTION

In many of the existing approaches to quantum gravity (for an incomplete list see e.g. \[1–7\] and references therein) the fundamental parameters that enter the action, such as Newton’s constant, electromagnetic coupling, the cosmological constant etc, become scale dependent quantities. This looks quite natural, since scale dependence at the level of the effective action is a generic feature of ordinary quantum field theory. In particular, in theories of gravity, the scale dependence is expected to modify the horizon, the thermodynamics, the quasinormal modes spectra of classical black hole backgrounds \[8–14\]. In the group of theories beyond classical GR, among the aforementioned approaches based on scale-dependent gravity, we can find also a particular method, which is usually known as "improved" asymptotically safe (AS) gravity \[15–17\]. For the renormalization group (RG)-improved black hole metrics, cosmologies, and inflationary models from asymptotic safety, the reader can usefully consult e.g. Refs. \[18–30\]. The procedure adopted in that scenario is to integrate the beta function for the gravitational coupling in order to compute the Newton’s constant \( G \) as a function of some energy scale \( k \). The "running" Newton’s constant \( G(k) \) so constructed is then inserted into the classical black hole solution and an "improved" lapse function is obtained, which is thought to automatically include, in this way, the quantum gravity effects. In this approach the gravitational coupling depends on some arbitrary renormalization energy scale \( k \). Therefore, it is pivotal to establish a link between the energy scale \( k \) and the radial coordinate \( r \), and only after this step is done, the improved black hole metric can be considered complete and useful. As said, those extended solutions, inspired by the asymptotic safety program, are expected to modify the classical black hole solutions by incorporating quantum features. Different modified black hole metrics, in particular metrics also not affected by the central singularity, can be found, e.g. in Refs. \[31–33\], although those examples are not directly connected with the ASG program.

On the other hand, by reformulating General Relativity as an effective quantum field theory of gravity at low energies, John Donoghue and other authors \[34–41\] have, along the years, established a solid prediction of the quantum corrections to the Newtonian potential, at least at the first order in \( \hbar \). By comparing the effective Newtonian potential predicted by the ASG approach with the one computed in the framework of GR as an effective QFT, we arrive to establish, for the first time, a negative value for the parameter \( \tilde{\omega} \), unlike previous early predictions (see Refs. \[15, 26\]). This in turn leads directly, without further assumptions, to a specific metric that seems to be able to describe the principal features of the so called Planck stars. It is remarkalbe that, while Planck stars were introduced in Ref. \[42\] on the grounds of plausible quite general physical considerations, here on the contrary they appear as an almost unavoidable consequence of the Asymptotic Safe Gravity approach.

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The paper is organized as follows. In the next Section we shortly recall the main results of the renormalization group approach leading to the ASG-improved classical Newtonian potential, as well to the ASG-modification of the Schwarzschild metric. In Section III we give a precise evaluation of the ASG parameters. Section IV is devoted to discuss the features of the ASG-improved Schwarzschild metric, and then Section V discusses the "prediction" of the Planck star model. In Sections VI, VII, VIII, IX we study, respectively, temperature, specific heat, emission rate equation, and thermodynamic entropy of this ASG-improved Schwarzschild/Planck star metric. The last Section contains some quick hints to the phenomenology of these objects (and related references), and the conclusions.

Here we work in units where \( c = k_B = 1 \), and \( \ell_P \) is the Planck length defined as \( G\hbar = \ell_P^2 \). Then of course the Planck mass \( m_P \) satisfies \( 2Gm_P = \ell_P \) and \( \hbar = 2m_P\ell_P \).

II. BLACK HOLE METRICS FROM RUNNING NEWTONIAN COUPLING

With reference to [15], we can say that the main steps towards the construction of a renormalization group improved Schwarzschild metric are essentially three: first, we integrate the beta function for the gravitational coupling to compute Newton’s constant as a function of some energy scale \( k \), namely \( G(k) \). After that, a link between the energy scale \( k \) and the radial coordinate \( r \) must be established, the so called "identification of the infrared cutoff", namely \( k = k(r) \). Finally, the \( G(r) \) Newton’s constant so obtained is inserted into the classical black hole solution and we arrive to an improved lapse function of the metric. Only after this final step the complete metric of a "renormalization improved" black hole becomes concretely usable for explicit calculations. The above steps are detailed, for example, in Refs. [15, 43].

So, the basic idea of the AS gravity approach in order to obtain the renormalization improved, classical Newtonian or general relativistic, solutions is to replace everywhere the numerical Newton constant \( G \) with the ‘running constant’ \( G(r) \), whose explicit form is given by [15]

\[
G(r) = \frac{Gr^3}{r^3 + \omega G\hbar (r + \gamma GM)},
\]

where, in accordance with our conventions, \( c = k_B = 1 \) and we retained \( \hbar \). Here \( \omega \) and \( \gamma \) are dimensionless numerical parameters, whose concrete value will be discussed later.

The line element for the spherically symmetric, Lorentzian metric preserves the usual form, that is

\[
ds^2 = F(r)dt^2 - F(r)^{-1}dr^2 - r^2 d\Omega^2,
\]

where \( r \) is the radial coordinate, and \( d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \) is the line element of the unit two-sphere. But now, according to the above prescriptions, the lapse function \( F(r) \) of our ASG improved Schwarzschild geometry reads

\[
F(r) = 1 - \frac{2MG(r)}{r} = 1 - \frac{2GMr^2}{r^3 + \omega G\hbar (r + \gamma GM)},
\]

with \( G(r) \) given by (1) and \( M \) the mass of the black hole. Of course, we suppose \( \omega \neq 0 \), otherwise we would go back to the standard Schwarzschild metric. Two very important limiting cases should be considered. The first corresponds to the low energy scales \( (r \to \infty, \text{or} \ k \to 0) \), which implies

\[
F(r \to \infty) \simeq 1 - \frac{2GM}{r},
\]

so the standard Schwarzschild metric at large distances is recovered, and this behavior is independent from the values of \( \omega \) and \( \gamma \).

The second corresponds to the high energy scales \( (r \to 0, \text{or} \ k \to \infty) \). Here we have to distinguish two subcases. If \( \gamma \neq 0 \), then

\[
F(r \to 0) \simeq 1 - \frac{2r^2}{\omega \gamma G\hbar},
\]

and thus the lapse function corresponds to a deSitter (\( \omega \gamma > 0 \)) or an Anti-deSitter (\( \omega \gamma < 0 \)) core of our metric, depending on the sign of \( \omega \gamma \).

If \( \gamma = 0 \), then

\[
F(r) = 1 - \frac{2GMr}{r^2 + \omega G\hbar},
\]
and therefore
\[ F(r \to 0) \simeq 1 - \frac{2Mr}{\omega \hbar}, \tag{7} \]
so in this case we have a conic singularity at the origin. Clearly, the presence of \( \hbar \) signals the quantum character of the correction that the ASG approach gives to the core of the standard Schwarzschild metric. In both cases the central singularity has disappeared.

III. VALUES OF THE PARAMETERS \( \tilde{\omega} \) AND \( \gamma \)

As we said in Sec.II, the ASG-improved Newtonian potential can be obtained from the standard Newton formula
\[ V(r) = -\frac{GMm}{r} \tag{8} \]
by simply replacing the experimentally observed Newton constant \( G \) with the running coupling \( G(r) \) given in Eq.(1). Thus we get
\[ V_{\text{ASG}}(r) = -G(r) \frac{Mm}{r} = -\frac{GMm}{r} \left( 1 + \tilde{\omega} \frac{G}{\hbar} \frac{G}{r} + O \left( \frac{G^2 \hbar^2}{r^4} \right) \right), \tag{9} \]
which can be expanded for large \( r \) as
\[ V_{\text{ASG}}(r) = -\frac{GMm}{r} \left[ 1 - \frac{\tilde{\omega} G}{r^2} - \frac{\gamma \tilde{\omega} G^2 \hbar M}{r^3} + O \left( \frac{G^2 \hbar^2}{r^4} \right) \right]. \tag{10} \]

Clearly, the corrections to the standard Newtonian potential predicted by the ASG approach are all of quantum nature. This is suggested by the presence of \( \hbar \) in each term of correction. In fact, there are no correction terms of classical origin, coming from some kind of post-Newtonian approximation. On the other hand, corrections of quantum origin to the classical Newtonian potential have been elaborated by several researchers [34–41] in the last three decades or so. In particular, it was pointed out by Donoghue [35, 38] that the standard perturbative quantization of Einstein gravity leads to a well defined, finite prediction for the leading large distance quantum correction to Newtonian potential. The numerical coefficients of the quantum expansion have undergone a certain evolution over the years [36, 37], but the result today accepted by the community [38, 39] reads
\[ V_{\text{QGR}}(r) = -\frac{GMm}{r} \left[ 1 + \frac{41}{10\pi} \frac{G}{r^2} \right. + \ldots \right]. \tag{11} \]

This is an expansion at first order in \( \hbar \), where the first correction term represents a genuine quantum correction proportional to \( \hbar \).

The comparison of the two expansions (10) and (11) allows us to fix the parameter \( \tilde{\omega} \), which results to be
\[ \tilde{\omega} = -\frac{41}{10\pi}. \tag{12} \]

The ASG parameter \( \gamma \) cannot be fixed by these considerations. To this aim, we refer the reader to the arguments originally developed in Ref.[15], and then taken up also by other authors (e.g. [26]). Those classical general relativistic arguments have to do with the correct identification of the infrared cutoff, and they fix \( \gamma = 9/2 \). Different kind of considerations, based on the generalized uncertainty principle (see Ref. [13]; see also Refs. [14]), lead to the value \( \gamma = 0 \). In this paper we will assume always \( \gamma \geq 0 \), and in some specific cases we shall comment on the special value \( \gamma = 0 \). However, most of the results will be qualitatively the same for all \( \gamma \geq 0 \).

We should here emphasize that many authors of ASG papers (e.g. [15, 26, 27]), in order to fix the \( \tilde{\omega} \) parameter, use results of the early calculations performed by Donoghue and others [34, 36] in the period 1994-1995. As a direct consequence, they get a positive value of \( \tilde{\omega} \). This of course has nice consequences, as for example, black hole metrics without singularities, where in particular the central singularity is wiped out, in favour of a De Sitter or an Anti de Sitter core, as can be easily inferred from Eqs.(3)(5), and is widely discussed in the above References. However, during the years the analytical techniques used in GR as an effective QFT have been refined, and the results now accepted by the community are those expressed, initially, in Refs.[38, 39], and then confirmed in Refs.[40]. All these results coherently point to a negative value of \( \tilde{\omega} \). This fact has deep consequences on the structure of the black hole metric (5), as we will see in the next Sections.
IV. STUDY OF THE NEW ASG-IMPROVED SCHWARZSCHILD METRIC

The key information obtained in the previous Section is that $\tilde{\omega}$ is negative. This, as we will see, represents a major change in respect to others modified (but regular) Schwarzschild metrics present in literature [10, 14, 15, 32, 33]. Instead, some contact with our results can be found in Ref. [45], although there the authors don’t deal with ASG models. So, according to the previous section, we consider here the case

$$\tilde{\omega} < 0 \Rightarrow \tilde{\omega} = -|\tilde{\omega}|; \quad \gamma > 0. \quad (13)$$

The lapse function (3) can therefore be written as

$$F(r) = 1 - \frac{2GMr^2}{r^3 - |\tilde{\omega}|G\hbar(r + \gamma GM)}. \quad (14)$$

While for $\tilde{\omega} > 0$ the lapse (3) is regular everywhere when $r > 0$ (see Refs. [15], etc.), here, with $\tilde{\omega} < 0$, the situation is very different. First, we notice that the behavior of $F(r)$ at $r \to \infty$ remains that described by Eq.(4), namely standard Schwarzschild for large $r$. At $r \to 0$ we have an Anti-DeSitter core, namely $F(r) \approx 1 + 2r^2/(|\tilde{\omega}|\gamma G\hbar)$. But now the denominator $D(r)$ appearing in (14) can develop zeros. Luckily, a simple graphical analysis is sufficient to clarify the situation. In Fig.1 we compare the two lines

$$y_1 = r^3 \quad y_2 = |\tilde{\omega}|G\hbar(r + \gamma GM) \quad (15)$$

for various values of $M > 0$, the mass of the central body. As we see, since $D = y_1 - y_2$, there is always only one single zero for $D(r)$ when $r > 0$, let’s call it $r_0$. So

$$D(r) > 0 \quad \text{for} \quad r > r_0$$

$$D(r) = 0 \quad \text{for} \quad r = r_0$$

$$D(r) < 0 \quad \text{for} \quad 0 < r < r_0$$

In the unphysical region $r < 0$, $D(r)$ can develop two distinct zeros, or two coincident zeros, or no real zeros at all. On the ground of the above situation, we can develop a straightforward analysis of the function $F(r)$, valid for any $M > 0$:

$$\lim_{r \to \pm \infty} F(r) = 1^-; \quad \lim_{r \to r_0^+} F(r) = 1 - \frac{2GMr_0^2}{0^+} = -\infty;$$

$$\lim_{r \to r_0^-} F(r) = 1 - \frac{2GMr_0^2}{0^-} = +\infty; \quad \lim_{r \to 0^+} F(r) = 1^+;$$

$$F(r) \sim 1 + \frac{2r^2}{|\tilde{\omega}|\gamma G\hbar} \quad \text{for} \quad r \sim 0; \quad F'(r) = \frac{2GM(r^4 + |\tilde{\omega}|Ghr^2 + 2|\tilde{\omega}|\gamma G^2 Mhr)}{(r^3 - |\tilde{\omega}|Ghr - |\tilde{\omega}|\gamma G^2 hM)^2} > 0 \quad \text{for} \quad r > 0.$$
The information collected above allows us to draw a general graph (Fig. 2) of the lapse function \( F(r) \) in the region \( r > 0 \), valid for any \( \gamma > 0 \) and of course \( M > 0 \). As we see, there is one single positive zero \( r = r_h \) of \( F(r) \), which is the horizon of the ASG-improved black hole metric \(^\dagger\). There is also an essential singularity at \( r = r_0 > 0 \). The most relevant difference with the standard Schwarzschild metric is the fact that the essential (ineliminable) singularity is at \( r_0 > 0 \), instead of being at \( r_0 = 0 \). It is also clear that it is always \( r_0 < r_h \), for any \( M > 0 \). So the singularity is always protected by the event horizon. The singularity is never naked, in full accordance with the Cosmic Censorship Conjecture.

We now examine the asymptotic behavior of the horizon \( r_h \) and of the singularity \( r_0 \) in the physical relevant limits of large \( M \) and small \( M \).

### A. Horizon \( r_h \)

The only real positive zero of \( F(r) \), the horizon, must be a solution \( r = r_h \) of the equation

\[
 r^3 - 2GMr^2 - |\varpi|Gh r - \gamma|\varpi|G^2 hM = 0. \tag{16}
\]

The classical limit, \( \hbar \to 0 \), of the above equation reads: \( r^3 - 2GMr^2 = 0 \), whose positive solution is \( r = 2GM \). Therefore, to get an approximate solution of (16) for large \( M \), we pose

\[
 r = 2GM + \varepsilon \tag{17}
\]

and we perturb around the classical solution \( 2GM \) by keeping only the first order terms in \( \varepsilon \).

\(^1\) In so doing, we get a linear equation for \( \varepsilon \), from which results

\[
 \varepsilon = \frac{(2 + \gamma)|\varpi|\hbar}{4M}, \tag{18}
\]

and finally the behavior of the horizon for large \( M \), \( M \to \infty \), is

\[
 r_h \simeq 2GM + \frac{(2 + \gamma)|\varpi|\hbar}{4M}, \tag{19}
\]

**FIG. 2:** Lapse function \( F(r) \): physical region for \( r > 0 \), singularity at \( r = r_0 \), horizon at \( r = r_h \), where \( F(r_h) = 0 \).

\(^1\) An alternative derivation is the following. Since \( G\hbar = \ell_P^2 \), equation (16) reads: \( r^3 - 2GMr^2 - |\varpi|\ell_P^2 r - \gamma|\varpi|\ell_P^2 GM = 0 \), which, in the limit of large \( M \), approximately becomes: \( r^3 - 2GMr^2 - \gamma|\varpi|\ell_P^2 GM \simeq 0 \). The last term is suppressed by the factor \( \ell_P^2 \), so finally the large \( M \) limit of the horizon equation is: \( r^3 - 2GMr^2 \simeq 0 \), whose positive solution is \( r = 2GM \). Therefore we look for an approximate solution of Eq. (16) by perturbing around the classical Schwarzschild solution, namely we pose \( r = 2GM + \varepsilon \), and we keep only the first order terms in \( \varepsilon \).
so the usual Schwarzschild expression for the horizon is recovered in the large \( M \) limit. We can also investigate the behavior of the horizon in the small \( M \) limit. For \( M \to 0 \), Eq.(16) reads
\[
  r^3 - |\tilde{\omega}|Gh r \simeq 0 \quad \Rightarrow \quad r(r^2 - |\tilde{\omega}|Gh) \simeq 0.\tag{20}
\]
The last equation has solutions \( r_{1/2} = \mp \sqrt{|\tilde{\omega}|Gh} \) and \( r_3 = 0 \). We can perturb around these solutions by posing \( r_{1/2} = \mp \sqrt{|\tilde{\omega}|Gh} + \varepsilon \), \( r_3 = 0 + \varepsilon \) and setting them back into Eq.(16). Under the hypothesis \( |\varepsilon| \ll \sqrt{|\tilde{\omega}|Gh} \)
\[
  \text{we can retain the first order in } \varepsilon \text{ only, then we arrive at a linear equation in } \varepsilon, \text{ and from that finally we get}
\]
\[
r_{1/2} \simeq \mp \sqrt{|\tilde{\omega}|Gh} + \left(1 + \frac{\gamma}{2}\right)GM; \quad r_3 \simeq -\gamma GM. \tag{22}
\]
Clearly, the only acceptable solution is \( r_2 \), since \( r_2 > 0 \) always. So we have the behavior of the horizon for small \( M \) as
\[
r_h \simeq \sqrt{|\tilde{\omega}|Gh} + \left(1 + \frac{\gamma}{2}\right)GM. \tag{23}
\]
However from the physical point of view, this solution is practically meaningless, since the condition \( |\varepsilon| \ll \sqrt{|\tilde{\omega}|Gh} \) implies
\[
  GM \ll \sqrt{|\tilde{\omega}|Gh} \sim \ell_P, \tag{24}
\]
which means that this solution is valid when \( M \ll m_p \), i.e. when the collapsing mass is much smaller than the Planck mass. Equivalently, the correction \( \varepsilon \) should be smaller than the Planck length, which again would mean that \( \varepsilon \) has no definite physical meaning.

\[\text{B. Singularity } r_0\]

As we have seen, the singularity of the metric (14) is located at the only positive root \( r = r_0 \) of the equation
\[
r^3 - |\tilde{\omega}|Ghr - \gamma |\tilde{\omega}|G^2hM = 0 \tag{25}
\]
Here also we have a look at the behavior of the solution for \( M \) small and for \( M \) large. For \( M \to 0 \), the equation is again of the form
\[
r^3 - |\tilde{\omega}|Ghr = 0, \tag{26}
\]
which has the exact solutions \( r_{1/2} = \mp \sqrt{|\tilde{\omega}|Gh} \) and \( r_3 = 0 \). Perturbing around them, namely posing
\[
r_{1/2} = \mp \sqrt{|\tilde{\omega}|Gh} + \varepsilon, \quad r_3 = 0 + \varepsilon, \tag{27}
\]
and substituting them back into (25), we get, under the usual condition \( |\varepsilon| \ll \sqrt{|\tilde{\omega}|Gh} \),
\[
r_{1/2} \simeq \mp \sqrt{|\tilde{\omega}|Gh} + \frac{\gamma}{2}GM; \quad r_3 \simeq -\gamma GM. \tag{28}
\]
Repeating similar considerations as before we can say that the only acceptable solution is \( r_2 \) (because \( r_2 > 0 \) always), so the behavior of the radial coordinate of the singularity for small \( M \) is
\[
r_0 \simeq \sqrt{|\tilde{\omega}|Gh} + \frac{\gamma}{2}GM. \tag{29}
\]
However, the above condition on \( \varepsilon \) once again implies \( |\varepsilon| \ll \ell_P \), or \( M \ll m_p \), which makes such mathematical solutions of little, if any, physical interest.

Instead, for \( M \to \infty \) the situation is much more interesting. The large \( M \) limit of the singularity equation (25) is
\[
r^3 - \gamma |\tilde{\omega}|G^2hM \simeq 0, \tag{30}
\]
so the real positive solution is \( r = (\gamma |\tilde{\omega}|G^2hM)^{1/3} \). We perturb around it by posing
\[
r = (\gamma |\tilde{\omega}|G^2hM)^{1/3} + \varepsilon \tag{31}
\]
and under the usual condition $|\varepsilon| \ll (\gamma|\tilde{\omega}|G^2\hbar M)^{1/3}$, we get for large $M$

$$r_0 \simeq (\gamma|\tilde{\omega}|G^2\hbar M)^{1/3} + \frac{|\tilde{\omega}|G\hbar}{3(\gamma|\tilde{\omega}|G^2\hbar M)^{1/3}}.$$ \hspace{1cm} (32)

An important physical consideration can now be stated. As it was already suggested by the initial analysis, the singularity results to be always protected by the horizon. Namely, by comparing Eqs. (19), (32) for large $M$, or instead by comparing Eqs. (23), (29) in the small $M$ limit, we always have

$$r_0 < r_h,$$ \hspace{1cm} (33)

so there are no naked singularities.

For sake of clarity, in Fig.3 the reader can find the plots of the mass function $M(r_h)$ for the horizon (red dashed line)

$$GM(r_h) = \frac{r^3 - |\tilde{\omega}|\ell_p^2 r}{2r^2 + |\tilde{\omega}|\ell_p^2 \gamma} \bigg|_{r=r_h},$$ \hspace{1cm} (34)

the mass function $M(r_0)$ for the singularity (blue dot-dashed line)

$$GM(r_0) = \frac{r^3 - |\tilde{\omega}|\ell_p^2 r}{|\tilde{\omega}|\ell_p^2 \gamma} \bigg|_{r=r_0},$$ \hspace{1cm} (35)

and the standard Schwarzschild horizon $GM = r_{SCH}^2/2$ (green solid line). Any horizontal line (black dashed) representing an arbitrary $M > 0$ intersects first the blue line and then the red line, namely $r_0(M) < r_h(M)$ for any $M > 0$. Notice that both the horizon and the singularity mass functions have a simple zero at

$$r = r_c = \sqrt{|\tilde{\omega}|\ell_p}$$ \hspace{1cm} (36)

with $r_c^2 = |\tilde{\omega}|\ell_p^2$.

V. A METRIC FOR THE PLANCK STARS

It is of the greatest interest to examine the core of the black hole metric we obtained. For masses $M$ larger than the Planck mass $m_p$, $M \gg m_p$, the central hard singularity is located at $r = r_0$, and it has, in fact, a finite positive size $r_0 \simeq (\gamma|\tilde{\omega}|G^2\hbar M)^{1/3} > 0$, contrary to what happens in the standard Schwarzschild black hole, where the singularity is point-like. Observe that this finite size is completely of quantum origin: in fact, $r_0 \rightarrow 0$ if we take the classical limit $\hbar \rightarrow 0$.

![FIG. 3: Horizon mass function $M(r_h)$ (red dashed line), singularity mass function $M(r_0)$ (blue dot-dashed line), and Schwarzschild mass function (green solid line). The horizontal black dashed line represents an arbitrary $M > 0$, and always intersects blue and red lines at $r_0(M) < r_h(M)$, namely there are no naked singularities.](image-url)
But most importantly, if we presume that the whole collapsing mass $M$ is concentrated into the central hard sphere of radius $r_0$, then we can compute the (non covariant) volume of this sphere, and hence the density of this matter (as seen by an observer at infinity), which will result to be finite, and precisely

$$r_0 = (\gamma|\tilde{\omega}|\ell_p^2GM)^{1/3} \, \left(\frac{M}{m_p}\right)^{\frac{1}{3}} \ell_p \quad \Rightarrow \quad \varrho = \frac{M}{V_{\text{core}}} = \frac{3}{2\pi \gamma |\tilde{\omega}|\ell_p^4} \approx \frac{m_p}{2\gamma |\tilde{\omega}|\ell_p^4} = \frac{\varrho_{\text{Planck}}}{2\gamma |\tilde{\omega}|},$$

where we used the definitions $Gh = \ell_p^2$, $2Gm_p = \ell_p$, and

$$\varrho_{\text{Planck}} = \frac{m_p}{\ell_p^4}.$$  

So the central hard core of our black hole results to have a finite size, and a density of the order of the Planck density. These are exactly the characteristics of the so called Planck stars, first proposed in Ref. [42], on the ground of general qualitative considerations. Many of the general properties described in [42] can now be repeated for our black hole. The finite positive size of the central core, being of pure quantum origin, is presumably due to the action of the Heisenberg uncertainty principle, which prevents matter to be arbitrarily concentrated into a geometrical point of size zero. The central kernel can presumably keep trace of all the information swallowed by the black hole; we see here a possible way out of the information paradox. Of course, all the above considerations make sense only for $\gamma > 0$ strictly.

The original proposal [42] contains a certain amount of qualitative considerations, including an educated guess on the form of the metric able to describe a Planck star. Such metric was initially chosen to be the Hayward metric [32]

$$F(r) = 1 - \frac{2GMr^2}{r^3 + 2GML^2},$$

where $L$ is a parameter with dimensions of a length. No particularly compelling argument, from the physical point of view, was exhibited for that choice, with the exception, perhaps, that the Hayward metric is a well known example of singularity-free metric. For large $M$ the metric [39] develops two horizons, one inner

$$r_- \approx L + \frac{L^2}{4GM},$$

and one outer

$$r_+ \approx 2GM - \frac{L^2}{2GM}.$$  

However, no specific indication is contained in the metric [39] about the size of the hard kernel of a Planck star. And certainly not of a hard kernel with a size increasing with $M$, as instead Eq.(37) suggests (to be compared with (40)). Even worse, the Hayward metric per se is unable to mimic the well established quantum correction to the Newtonian potential [38] that occurs at low energies. This is due to the lack of a term $1/r^3$ in the expansion of the metric [39]. The authors of Ref. [40] found a smart way to cure this shortcoming, but at the price of introducing a further metric function $H(r)$, determined through a bunch of additional constraints, so that their "modified Hayward" metric now reads

$$ds^2 = -H(r)F(r)dt^2 + \frac{1}{F(r)}dr^2 + r^2d\Omega^2, \quad \text{with} \quad H(r) = 1 - \frac{\beta GM\alpha}{ar^3 + \beta GM};$$

where $\beta$ is a parameter that in Ref. [40] plays the rôle of our $|\tilde{\omega}|$. The above metric 2 finally contains the $1/r^3$ term necessary to mimic the Donoghue modified Newtonian potential [38] for large $r$.

Although smart and working, the above solution is undeniably contorted and intricate. On the contrary, within the formalism of the Renormalization Group, the mathematical structure of the metric is dictated by the general properties of the AS Gravity, and its lapse function [14] results clearly simpler than the above product $H(r)F(r)$. The ASG metric already contains the right terms to match, at large distances, the quantum corrected Newtonian potential. Moreover, and this is quite astonishing, by simply imposing that match, the final form of the metric is uniquely fixed, and it automatically displays the size of the central hard kernel of the Planck star.

In the following, we shall therefore study the thermodynamic properties of the metric [14].

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2 A further requirement imposed by the authors on the function $H(r)$ is that $H(r)$ should allow for a time delay between a clock sitting at the center of the collapsed object ($r = 0$), and a clock at infinity (put a clock at $r = 0$ is in principle conceivable, just because the Hayward metric is regular at $r = 0$). To get this, authors demand that $H(r = 0) = 1 - \alpha$. They justify this further request by saying that "is a physically unmotivated restriction" to leave $H(0) = 1$. In any case, we do not have this kind of problem with the ASG metric [14], since the center $r = 0$ cannot be reached, being protected by the singularity at $r = r_0 > 0$. 
VI. HAWKING TEMPERATURE

According to the celebrated works of Hawking (see e.g. Ref. [47]), a generic Lorentzian black hole metric as of a lapse function $F(r)$ bearing an horizon (namely a simple zero at some $r = r_h$, with $F(r_h) = 0$, $F'(r_h) \neq 0$), displays on the horizon a Hawking-Bekenstein temperature given by

$$T_{BH} = \frac{\hbar}{4\pi} F'(r_h).$$

(43)

Therefore, reminding $G\hbar = \ell_p^2$ and $|\hat{\omega}|^2 = r_c^2$, from [14] we can compute,

$$\frac{4\pi}{\hbar} T = F'(r) = 2GMr \left( r^3 + r_c^2r + 2\gamma GMr_c^2 \right) \left( r^3 - r_c^2r - \gamma GMr_c^2 \right)^2$$

(44)

where $r = r_h$, namely $r$ should satisfy the identity $F(r) = r^3 - 2GMr^2 - r_c^2r - \gamma GMr_c^2 = 0$. We can use this to simplify $r^3 - r_c^2r - \gamma GMr_c^2 = 2GMr_c^2$, and get

$$F'(r) = \frac{1}{2GM} \left( 1 + \frac{r_c^2}{r^2} + 2\gamma GMr_c^2 \right).$$

(45)

The above expression is exact. Now we study its limits, namely the limits of $T(M)$ for $M$ large and for $M$ small.

Repeating twice the procedure which led us to (19), for $M \to \infty$, we can arrive to write the expansion of $r_h(M)$ to the third order in $1/M$

$$r = r_h \approx 2GM + \frac{(\gamma + 2)r_c^2}{4GM} + \frac{(\gamma + 2)r_c^4}{16GM^3M^3} + \ldots$$

(46)

Inserting this back into (45) we get finally

$$\frac{4\pi}{\hbar} T(M) = F'(r) = \frac{1}{2GM} \left[ 1 + \frac{\gamma + 1}{4} \left( \frac{r_c}{GM} \right)^2 - \frac{(\gamma + 2)(3\gamma + 2)}{32} \left( \frac{r_c}{GM} \right)^4 + \ldots \right].$$

(47)

We recover here the standard behavior $T_{BH} = \hbar/8\pi GM$ of the BH temperature for large $M$, when our black hole looks even more like a Schwarzschild one. Moreover, notice also that the horizon mass function $M(r_h)$ is an odd function and its expansion can hence contain odd powers of $M$ only, as in fact results in Eq. (46). Moreover, the expression in round brackets in (45) must be an even function of $M$, since $r_h(M)$ is odd, and therefore we find in its expansion (47), square brackets, only even powers of $1/M$.

For small $M$, first we note that, clearly, the horizon mass function $M(r_h)$, diagram Fig.3, does not have a positive minimum, $M_{min} > 0$. So the Hawking evaporation can in principle proceed until $M \to 0$. Reminding Eq. (23), we can write for small $M$

$$r_h(M) = r_c + \left( 1 + \frac{\gamma}{2} \right) GM + \ldots$$

(48)

where we suppose $GM \ll r_c$. Inserting this $r_h(M)$ back into (45) we get

$$\frac{4\pi}{\hbar} T(M) = F'(r) = \frac{1}{2GM} \left[ 2 + (\gamma - 2) \left( \frac{GM}{r_c} \right) + \frac{3(\gamma + 2)(2 - 3\gamma)}{4} \left( \frac{GM}{r_c} \right)^2 + \ldots \right].$$

(49)

We see here again a behavior similar to that of standard Hawking temperature for the standard Schwarzschild black hole, namely $T(M) \to \infty$ when $M \to 0$. However, for small $M$ we have that our ASG improved black hole has a temperature double of that of the standard Hawking, i.e. $T(M) \simeq 2T_{BH}(M)$ for small $M$.

At this point, for the forthcoming developments, it is also extremely useful to observe that for a metric with a lapse function $F(r, M)$, the Hawking temperature can be expressed in two different ways, either $T$ as a function of $M$, $T = T(M)$, or instead $T$ as a function of $r$, $T = T(r)$. In fact, usually we write

$$\frac{4\pi}{\hbar} T(M) = \frac{\partial F(r, M)}{\partial r} \bigg|_{r=r(M)},$$

(50)
where \( r(M) \equiv r_h \) is a solution of the equation \( F(r(M), M) = 0 \). However, the equation \( F(r, M) = 0 \) in respect to \( r \) can be complicated, containing radicals, etc., as actually it is in our present case, where we have the third degree equation (16). It is much easier to consider \( T \) as a function of \( r \), as

\[
\frac{4\pi}{\hbar}T(r) = \left. \frac{\partial F(r, M)}{\partial r} \right|_{M=M(r)},
\]

where \( M(r) \) is a solution of \( F(r, M(r)) = 0 \), namely \( M(r) \) is the mass function, \( M(r) \equiv M(r_h) \). Mathematically, \( r = r(M) \) and \( M = M(r) \) are just the same implicit function, locally defined by the equation \( F(r, M) = 0 \). Of course, the equation \( F(r, M) = 0 \) is usually much easier to be solved in respect to \( M \) than in \( r \), being usually an equation of first degree in \( M \).

Following the above considerations, we insert the mass function expression (34) into Eq.(45), and we get

\[
\frac{8\pi}{\hbar}T(r) = \left( \frac{2r^2 + \gamma r_c^2}{r^3 - r_c^2 r} \right) \left( 1 + r_c^2 \right) + \frac{2\gamma r_c^2}{r^3},
\]

where as usual \( r_c^2 = |\tilde{\omega}|l_P^2 \), and \( r \equiv r_h \) is the only real positive solution of the horizon equation (16). As expected, we obtain here an exact, simple, rational expression of \( M \) as function of \( r \), with no radicals displayed. Notice that we can compute the behavior of \( T(r) \) for large \( r \) (corresponding to \( M \to \infty \)), or for \( r \to r_c \) (corresponding to \( M \to 0 \)), and we get respectively

\[
T(r) \approx \frac{\hbar}{4\pi r} \quad \text{for} \quad r \to \infty; \quad T(r) \to \infty \quad \text{for} \quad r \to r_c.
\]

The above confirms the behavior of \( T(M) \) showed, respectively, in the Eqs. (17), (49).

VII. SPECIFIC HEAT

We can now proceed swiftly to the computation of the specific heat capacity. In general, it is defined as \( C_s = dE/dT \), where \( E \) is the total energy of the system under consideration. As usual we identify the total energy of our ASG black hole with its total mass \( M \) \((c=1)\) (see Ref.15). Once again, as above, the "best" analytical parameter through which express the mass and the temperature of our black hole is not its mass \( M \), but rather its gravitational radius \( r_h = r \) (positive solution of Eq.(16)). So we have

\[
C_s = \frac{dM(r)}{dT(r)} = \frac{M'(r)}{T'(r)}.
\]

For agility of calculation, let’s rename the horizon mass function (34) as

\[
GM(r) = \frac{r^3 - r_c^2 r}{2r^2 + \gamma r_c^2} =: N(r),
\]

with \( r_c^2 = |\tilde{\omega}|l_P^2 \), then Eq. (52) reads

\[
\frac{8\pi}{\hbar}T(r) = \frac{1}{N(r)} \left( 1 + \frac{r_c^2}{r^2} \right) + \frac{2\gamma r_c^2}{r^3}.
\]

Therefore we have

\[
C_s = -\frac{8\pi}{G\hbar} \frac{N'(r)N(r)^2}{[N'(r)(1 + r_c^2/r^2) + 2N(r)r_c^2/r^3 + 6N(r)^2\gamma r_c^2/r^4]}.
\]

We can easily verify that \( N(r) > 0 \), and \( N'(r) > 0 \), for \( r > r_c \) (being \( \gamma \geq 0 \)). Therefore

\[
C_s(r) < 0 \quad \text{for any} \quad r > r_c,
\]

namely the specific heat is negative for any \( r > r_c \), i.e. when \( M > 0 \). This behavior is analogous to the Schwarzschild black hole. More specifically, since \( N(r_c) = 0 \) and \( N'(r_c) = 2/(\gamma + 2) > 0 \), we have \( C_s(r_c) = 0 \), or, more precisely,

\[
C_s(r) \approx -\frac{4\pi}{G\hbar} \left( \frac{2}{2 + \gamma} \right)^2 (r - r_c)^2 + \ldots.
\]
Summarizing, when \( r \to r_c \) then

\[
M(r) \to 0, \quad T(r) \to \infty, \quad C_s(r) \to 0, \tag{60}
\]

in perfect analogy with the Schwarzschild black hole.

When instead \( r \to \infty \), we have \( r \approx 2GM \) and therefore

\[
C_s(r) \approx -\frac{2\pi}{G} \frac{r^2}{\hbar} \approx \frac{8\pi}{\hbar} GM^2 \tag{61}
\]

which coincides with the behavior of \( C_s \) for large \( M \) for the Schwarzschild black hole.

### VIII. EMISSION RATE EQUATION

We can investigate how long it takes a black hole with an initial mass \( M \) to reduce to a final mass \( M_f \) via the Hawking radiation. In our case we have seen that the black hole can evaporate until \( M_f = 0 \). The Stefan-Boltzmann law allows us to write an emission rate differential equation, which, once integrated, yields the above life-time for Hawking radiation. In our case we have seen that the black hole can evaporate until \( M \rightarrow \infty \).

The mass/energy loss per unit proper time of an infinitely far away static observer is approximately given by

\[
-\frac{dM}{dt} = \sigma A T^4, \tag{62}
\]

where \( \sigma \) is a constant (related to the Stefan-Boltzmann constant) and \( A \) is the area of the event horizon. In the standard Hawking calculation for a Schwarzschild black hole we have \( A \sim r_h^2 \sim M^2 \), and \( T \sim 1/M \). Therefore we get \( t_{\text{Haw}} \sim M^3 \). As before, for the description of our system the variable \( r \equiv r_h \) (positive solution of the horizon equation \( \sigma \)) appears analytically more viable than the mass \( M \). So we consider everything as a function of \( r \), namely \( M(r), \quad A(r), \quad T(r), \) and the above equation becomes an evolution equation for the gravitational radius

\[
-\frac{dA}{dt} = \sigma A(r) T(r)^4 \frac{r}{M'(r)} \tag{63}
\]

Inserting \( A(r) = 4\pi r^2 \), and \( M(r), \quad T(r) \) from Eqs. (55), (56), respectively, we arrive at

\[
-\frac{dA}{dt} = 4\pi G \sigma \left( \frac{\hbar}{8\pi} \right)^4 \frac{r^2}{N'(r)} \left[ \frac{1}{N(r)} \left( 1 + \frac{r_c^2}{r^2} \right) + \frac{2\gamma r_c^2}{r^3} \right]^4 \tag{64}
\]

This equation doesn’t look very expressive, although it can be integrated, in principle, without a particular effort (a rational function, quite tedious calculation). However, it can be used to check the two important limits of our physical system, namely \( r \to \infty \) and \( r \to r_c \).

When \( r \to \infty \), then \( N(r) \approx r/2 \) and \( N'(r) \approx 1/2 \). Hence

\[
-\frac{dA}{dt} \approx \sigma G \hbar \left( \frac{1}{32\pi} \right) \frac{r}{r^2} \quad \text{for} \quad r \to \infty. \tag{65}
\]

Since in this approximation \( r \sim M \), then we recover here the standard Hawking result, namely \( -dM/dt \sim 1/M^2 \). Therefore, in this approximation, for the life-time of black hole of initial mass \( M \), the above equation once integrated gives \( t_{\text{Haw}} \sim M^3 \), as expected.

When \( r \to r_c \), from Eq. (55) we get the behavior of \( N(r) \) as

\[
GM(r) = N(r) \sim \frac{2}{2 + \gamma} (r - r_c) + \ldots \tag{66}
\]

Therefore the RHS of the emission rate equation (64) diverges badly as,

\[
-\frac{dA}{dt} \approx 2\pi G \sigma \left( \frac{\hbar}{8\pi} \right)^4 (2 + \gamma)^5 r_c^2 \frac{1}{(r - r_c)^4} \to +\infty, \tag{67}
\]

which can be expressed in terms of the \( M \to 0 \) as

\[
-\frac{dM}{dt} \approx \frac{\sigma \hbar r_c^2}{G} \left( \frac{\hbar}{4\pi G} \right)^3 \frac{1}{M^4}. \tag{68}
\]

Here the divergence is worse than in the standard Hawking calculation, where \( -dM/dt \sim 1/M^2 \to +\infty \) when \( M \to 0 \). Although trusting in Eq. (68) to its very end appears risky and perhaps incorrect, nevertheless it’s behavior signals as well an explosive character of the last instants of life of a black hole.
IX. ENTROPY OF THE PLANCK STAR

We consider here the computation of the thermodynamic entropy of our system, while we leave to future work any possible statistical mechanics interpretation of such an entropy through the counting of "microscopic" states (maybe inaccessible to our observation). Therefore, by identifying, as usual, the energy of our system with the mass $M$ of the black hole ($c = 1$), we can write from general thermodynamics

$$dS = \frac{dM}{T},$$

where of course $T$ is the Hawking temperature of the hole.

The standard calculation proceeds by considering $T$ as a function of $M$, and then computing $T(M), S(M)$. On the contrary in our case, as widely illustrated above, the positive solution $r$ of Eq.(16), namely the radius of the event horizon, is an analytical variable better viable than $M$. Therefore we shall write

$$dS = \frac{M'(r)}{T(r)} dr.$$  

Hence, using Eqs.(55), (56), we can write

$$\frac{dS}{dr} = \frac{8\pi GM''(r)N(r)}{G\hbar} \left[ 1 + \frac{r^2}{r^2} + \frac{2\gamma r^2}{r^3} \right]^{-1}.$$  

In principle, as before, we can integrate exactly the above equation, but this step doesn’t seem to bring particularly useful information. It appears more clever to expand the above expression around the two significant limits $r \to \infty$ and $r \to r_c$, and then integrate.

For $r \to \infty$, then $N(r) \approx r/2$ and $N'(r) \approx 1/2$. Hence

$$\frac{dS}{dr} = \frac{2\pi}{G\hbar} r,$$

which can be integrated to give

$$S(r) - S(\Lambda) = \int_\Lambda^r \frac{2\pi}{G\hbar} \rho d\rho = \frac{4\pi r^2}{4\ell_p^2} - \frac{4\pi \Lambda^2}{4\ell_p^2},$$

where we inserted a large (infrared) cutoff $\Lambda$ to take into account the fact that we are integrating Eq.(71) in a region of large $r$ (where (71) takes the form (72)). Once again, we recover for large $r$ the well known behavior of the entropy of a Schwarzschild black hole

$$S(r) \sim \frac{\text{Area}}{4\ell_p^2}.$$  

When on the contrary $r \to r_c$, then Eq.(66) holds, and $N'(r) \approx 2/(2 + \gamma)$, hence

$$\frac{dS}{dr} \approx \frac{4\pi}{G\hbar} \left( \frac{2}{2 + \gamma} \right)^2 (r - r_c),$$

which integrated yields

$$S(r) - S(r_c) \approx \frac{2\pi}{G\hbar} \left( \frac{2}{2 + \gamma} \right)^2 (r - r_c)^2.$$

$S(r_c)$ could be perhaps interpreted as the entropy of the central core (after complete evaporation), which can in the end account for the information swallowed by the black hole, and therefore represent a way out of the information paradox (although a zero mass remnant remains anyway puzzling).
X. CONCLUSIONS

In this paper we have derived an exact value of the parameter $\tilde{\omega}$ characterizing the spherically symmetric metric suggested by the Asymptotic Safe Gravity approach. The result has been obtained by comparing the corrected Newtonian potential computed through ASG, with the analog correction suggested by the Donoghue approach to GR as a low energy effective QFT. The decisive novelty in respect to the previously computed values of $\tilde{\omega}$ is that we get a negative value of $\tilde{\omega}$, and this because we used the more recent results of Donoghue, Khrilpovich, and collaborators.

The fact that $\tilde{\omega} < 0$ completely changes the geometry of the ASG "improved" black hole metric. Previously unexplored aspects of this metric have been studied, the most relevant one being the presence of a finite-size singularity at the core of the black hole. Surprisingly, the size of this "black core" results to be exactly what needed to describe the so called Planck stars. These objects were introduced years ago on the basis of semi-qualitative arguments, while in our context they appear as a natural mathematical consequence of the ASG metric with a negative $\tilde{\omega}$ parameter (see also e.g. [48]).

Hawking temperature, specific heat, emission rate equation, and thermodynamic entropy have been studied for our Planck star metric, and they yield illuminating insights. It is worth mentioning that the phenomenology of these objects could be quite rich, and presents both astrophysical and cosmological signatures, in particular in the realm of (primordial) black hole evaporation [49]. As a Planck star evaporates, with a hard core of finite positive size, then the final explosion may occur at macroscopic scale. So, Planck star explosions could be naturally associated with some of the measured short gamma-ray bursts (SGRBs) [50]. In Ref. [51] authors estimated that several short gamma-ray bursts per day, around 10 MeV, with isotropic distribution, can be expected coming from a region of a few hundred light years around us. On the other hand, also fast radio bursts, strong signals with millisecond duration, which are probably of extragalactic origin, have been shown in Ref. [52] to have wavelengths not far from the expected size of the exploding hole.

On the theoretical side, further investigations, aimed to better understand Penrose diagrams, energy conditions, singularity theorems, quasi-normal modes, as well as a statistical interpretation of entropy and information paradox, related to this kind of metrics are being carried out, and will appear in forthcoming works.

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