Constraints from the Detection of Cosmic Topology on the Generalized Chaplygin Gas

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Despite our present-day inability to predict the topology of the universe one may expect that we should be able to detect it in the near future, given the increasing accuracy in the astro-cosmological observations. Motivated by this, we examine to what extent a possible detection of a non-trivial topology of a low curvature (Ω0 ∼ 1) universe, suggested by a diverse set of current observations, may be used to place constraints on the matter content of the universe, focusing our attention on the generalized Chaplygin gas (GCG) model, which unifies dark matter and dark energy in a single matter component. We show that besides constraining the GCG free parameters, the detection of a nontrivial topology also allows to set bounds on the total density parameter Ω0. We also study the combination of the bounds from the topology detection with the limits that arise from current data on 194 SNIa, and show that the determination of a given nontrivial topology sets complementary bounds on the GCG parameters (and on Ω0) to those obtained from the SNIa data.

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I. INTRODUCTION

The isotropic expansion of the universe, the primordial abundance of light elements, and the nearly uniform cosmic microwave background radiation (CMBR) constitute the main observational pillars of the standard Friedmann–Lemaître–Robertson–Walker (FLRW) model, which provides a very successful description of the universe. In this FLRW cosmological context, recent observations of high redshift type Ia supernovae (SNIa) suggest that the universe is undergoing an accelerated expansion. This picture is further strengthened by the combination of recent CMBR observations (which imply that the total density Ω0 is close or equal to unity) and the value Ωm0 ≃ 1/3 for the density of clustered (baryonic plus dark) matter obtained from the x-ray emission in rich clusters of galaxies, and from galaxy redshift surveys. This diverse set of observations has led to an evolving consensus among cosmologists that the universe is smoothly permeated by a negative-pressure dark energy (DE) component 1, which dominates the matter-energy of the universe today (ΩDE ≃ 2Ωm0 ≃ 2/3), although it must have been negligible in the past so as to permit structure formation.

The nature of both dark matter (DM) and DE is still object of intense investigations today. There are some candidates for DM from particle physics, but yet no evidence of these suggested particles has been found in laboratory experiments 2. Regarding DE there seems to exist no natural candidate from particle physics. Thus, the current observational information regarding DM and DE arises only from astro-cosmological observations. In addition to the cosmological constant and a dynamical scalar field (quintessence; see, e.g., refs. 3), the current paradigms for DE include a number of possibilities (see, for example, ref. 4), among which the so-called generalized Chaplygin gas (GCG) 5,6,7,8,9,10, which unifies DM and DE in a single matter-energy component, acting as cold dark matter (CDM) at high redshifts and driving the accelerated expansion today. The behavior of the GCG in the framework of DM and DE unification was extensively discussed in the literature (see, for instance, refs. 6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22 and references therein).1

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1 The GCG model is in agreement with a number of observational data related to the background dynamics, such as SNIa, gravitational lenses, and Fanaroff-Riley Type IIb galaxies (see e.g. refs. 14,15,13), but it fails to reproduce large-scale structure data for adiabatic perturbations 13. However, for a specific type of entropy perturbations the density fluctuation spectrum is consistent with observational data 20. Note that these intrinsic entropy perturbations are not ruled out by other current data (see discussion in ref. 16,21). Thus,
Somewhat parallel to these developments, and owing to the fact that general relativity as a local metrical theory leaves undetermined the space-time topology, a great deal of work has also recently gone into studying the possibility that the universe may possess compact spatial sections with a non-trivial topology. On the one hand, methods and indicators have been devised to search for topological signs of a possible non-trivial topology of our 3-space \( M \) (see, e.g., refs. \[23, 24\], and also the review articles \[25, 26\]). On the other hand, the reported non-Gaussianity in CMBR maps \[27\], the small power of large-angle fluctuations \[28\], and some features in the power spectrum \[27\] are large-scale anomalies which have been suggested as a potential indication of a non-trivial topology of the universe. In this regard the Poincaré dodecahedron space \[29\] has been suggested as an explanation for the weak wide-angle temperature correlation in high precision CMBR data \[30\]. However, preliminary results \[31\], using Wilkinson Microwave Anisotropy Probe (WMAP) data, failed to find the six pairs of matched circles of angular radius of about 35° predicted by the Poincaré space model \[29\]. In this regard it is important to note the results of the recent articles by Roukema et al. \[32\] and Aurich et al. \[33, 34\], and some points made by Luminet \[35\].

The immediate observational consequence of a multiply connected 3-space section \( M \) of the universe is the existence of multiple images of radiating sources: discrete cosmic objects or CMBR from the last scattering surface. However, for these repetitions to be detected the observable horizon radius \( \chi_{hor} \) has to exceed at least the smallest characteristic size \( r_{inj} \) of the 3-space \( M \). In this way, the question of detectability of cosmic (space) topology naturally arises. This question has been recently studied in the light of current astro-cosmological observations, which indicate that our universe is nearly flat (\( \Omega_0 \sim 1 \); see, e.g. ref. \[37\]), and the extent to which a number of non-trivial topologies may or may not be detected for the current bounds on the cosmological density parameters has been determined in a few articles \[35, 38, 39, 40\].

These studies have concentrated in the ACDM framework, where the cold dark (and baryonic) matter plus a cosmological constant (\( \Lambda \)) are the matter-energy constituents. In a recent article \[41\] the detectability of cosmic topology of low curvature universes (\( \Omega_0 \sim 1 \)) has been discussed in the unified DM and DE GCG context.

The main aim of this paper is to address the detectability of cosmic topology inverse problem, i.e. to investigate the extent to which the detection of a non-trivial nearly flat (\( \Omega_0 \sim 1 \)) topology may constrain models for the matter-energy content of the universe. To this end, we shall focus our attention on the GCG model and derive bounds set on its two free parameters from the detection of several possible compact topologies. We also consider the combination of the bounds from the topology detection with the limits on the GCG parameters which arise from current data from 194 SNIa. It is found that detection of a nontrivial topology (through CMBR pattern repetitions) sets complementary limits to those obtained from SNIa data. In particular, the detection of some specific manifolds taken together with current SNIa data may place constraints comparable to those expected from space based experiments (such as the Supernovae Acceleration Probe).

We also show that besides limiting the GCG parameters, the detection of a nontrivial topology allows to set bounds on the total density parameter \( \Omega_0 \), which are shown to be further constrained by the combination of topology detection with the current SNIa data. Such bounds on \( \Omega_0 \) can be confronted with the values which arise from some topology detection methods (such as the circles in-the-sky \[24\]) in order to have further limits on the GCG parameters.

The outline of the paper is as follows. In the next section, we give a brief account of the detectability of cosmic topology basic context and review a few topological properties of 3-manifolds which will be used in the following sections. Focusing on the GCG as a matter content model, we discuss in section \[IV\] the cosmic topology inverse problem and set bounds on the GCG parameters from the possible detection of several compact topologies. The combination of cosmic topology detection with current SNIa data and its implications for constraining the GCG parameters and \( \Omega_0 \) are discussed in section \[V\]. Finally, in section \[V\] we discuss our main results and present some concluding remarks. A few details of the supernovae analysis are discussed in the appendix.

II. DETECTABILITY PROBLEM IN COSMIC TOPOLOGY

To make the article as clear and self-contained as possible, in this section we shall present the cosmic topology basic context, state the detectability condition, and recall some topological properties of 3-manifolds which will be used in the following sections.

Within the framework of the standard FLRW cosmology, the universe is modeled by a 4-manifold \( \mathcal{M} \) which is decomposed into \( \mathcal{M} = \mathbb{R} \times M \), and is endowed with a locally isotropic and homogeneous Robertson–Walker (RW) metric

\[
ds^2 = -dt^2 + a^2(t) \left[ d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2) \right],
\]

the (non adiabatic) GCG is a viable framework to model the matter-energy content of the universe.
where \( f(\chi) = (\chi, \sin \chi, \text{or } \sinh \chi) \) depending on the sign of the constant spatial curvature \( (k = 0, 1, -1) \), and \( a(t) \) is the scale factor.

The spatial section \( M \) is usually taken to be one of the following simply-connected spaces: Euclidean \( \mathbb{E}^3 \) \( (k = 0) \), spherical \( S^3 \) \( (k = 1) \), or hyperbolic \( H^3 \) \( (k = -1) \) spaces. However, since geometry does not dictate topology, the 3-space \( M \) may equally well be any one of the possible quotient (multiply connected) manifolds \( M = \bar{M}/\Gamma \), where \( \Gamma \) is a discrete and fixed point-free group of isometries of the covering space \( \bar{M} = (\mathbb{E}^3, S^3, H^3) \). In forming the quotient manifolds \( M \) the essential point is that they are obtained from \( \bar{M} \) by identifying points which are equivalent under the action of the discrete group \( \Gamma \). Hence, each point on the quotient manifold \( M \) represents all the equivalent points on the covering manifold \( \bar{M} \). The action of \( \Gamma \) tessellates (tiles) \( \bar{M} \) into identical cells or domains which are copies of what is known as fundamental polyhedron (FP).²

In a multiply connected manifold, any two given points may be joined by more than one geodesic. Since the radiation emitted by cosmic sources follows (space-time) geodesics, the immediate observational consequence of a non-trivial spatial topology of \( M \) is that the sky may (potentially) show multiple images of radiating sources: cosmic objects or specific correlated spots of the CMBR. At large cosmological scales, the existence of these multiple images (or pattern repetitions) is a physical effect often used to examine the detectability of the 3-space topology.

In order to state the conditions for the detectability of cosmic topology in the context of standard cosmology, we note that for non-flat metrics of the form \( \xi \), the scale factor \( a(t) \) can be identified with the curvature radius of the spatial section of the universe at time \( t \). Therefore \( \chi \) is the distance of any point with coordinates \( (\chi, \theta, \phi) \) to the origin (in the covering space) \textit{in units of the curvature radius}, which is a natural unit of length that shall be used throughout this paper.

The study of the detectability of a possible non-trivial topology of the spatial section \( M \) requires a topological typical length which can be put into correspondence with observation survey depths \( \chi_{\text{obs}} \), up to a redshift \( z = z_{\text{obs}} \). A suitable characteristic size of \( M \), which we shall use in this paper, is the so-called injectivity radius \( r_{\text{inj}} \), which is nothing but the radius of the smallest sphere ‘inscribable’ in \( M \), and is defined in terms of the length of the smallest closed geodesics \( \ell_M \) by \( r_{\text{inj}} = \ell_M/2 \).

Now, for a given survey depth \( \chi_{\text{obs}} \), a topology is said to be undetectable if \( \chi_{\text{obs}} < r_{\text{inj}} \). In this case there are no multiple images (or pattern repetitions of CMBR spots) in the survey of depth \( \chi_{\text{obs}} \). On the other hand, when \( \chi_{\text{obs}} > r_{\text{inj}} \), then the topology is detectable in principle or potentially detectable.

In a globally homogeneous manifold the above detectability condition holds regardless of the observer’s position, and so if the topology is potentially detectable (or is undetectable) by an observer at \( x \in M \), it is potentially detectable (or is undetectable) by an observer at any other point in the 3-space \( M \). However, in globally inhomogeneous manifolds the detectability of cosmic topology depends on both the observer’s position \( x \) and the survey depth \( \chi_{\text{obs}} \). Nevertheless, even for globally inhomogeneous manifolds the above defined ‘global’ injectivity radius \( r_{\text{inj}} \) can be used to state an absolute undetectability condition, namely \( r_{\text{inj}} > \chi_{\text{obs}} \). Reciprocally, the condition \( \chi_{\text{obs}} > r_{\text{inj}} \) allows potential detectability (or detectability in principle) in the sense that, if this condition holds, multiple images of topological origin are potentially observable at least for some observers suitably located in \( M \). An important point is that for spherical and hyperbolic manifolds, which we focus on in this work, the ‘global’ injectivity radius \( r_{\text{inj}} \) expressed in terms of the curvature radius, is a constant (topological invariant) for a given topology.

In the remainder of this section we shall recall some relevant results about spherical and hyperbolic 3-manifolds, which will be useful in the following sections. The multiply connected spherical 3-manifolds are of the form \( M = S^3/\Gamma \), where \( \Gamma \) is a finite subgroup of \( SO(4) \) acting freely on the 3-sphere. These manifolds were originally classified by Threlfall and Seifert [42], and are also discussed by Wolf [43] (for a description in the context of cosmic topology see [14]). Such a classification consists essentially in the enumeration of all finite groups \( \Gamma \subset SO(4) \), and then in grouping the possible manifolds in classes. In a recent paper, Gausmann et al. [44] recast the classification in terms of single action, double action, and linked action manifolds. In table [1] we list the single action manifolds together with the symbol often used to refer to them, as well as the order of the covering group \( \Gamma \) and the corresponding injectivity radius. It is known that single action manifolds are globally homogeneous, and thus the detectability conditions for an observer at an arbitrary point \( p \in M \) also hold for an observer at any other point \( q \in M \). Finally we note that the binary icosahedral group \( I^* \) gives the known Poincaré dodecahedral space, whose fundamental polyhedron is a regular spherical dodecahedron, 120 of which tile the 3-sphere into identical cells which are copies of the FP.

Despite the enormous advances made in the last few decades, there is at present no complete classification of hyperbolic 3-manifolds. However, a number of important results have been obtained, including the two important

² A simple example of quotient manifold in two dimensions is the 2-torus \( T^2 = S^1 \times S^1 = \mathbb{E}^2/\Gamma \). The covering space clearly is \( \mathbb{E}^2 \), and a FP is a rectangle with opposite sides identified. This FP tiles the covering space \( \mathbb{E}^2 \). The group \( \Gamma = \mathbb{Z} \times \mathbb{Z} \) consists of discrete translations associated with the side identifications.
TABLE I: Single action spherical manifolds together with the order of the covering group and the injectivity radius.

| Name & Symbol | Order of $\Gamma$ | Injectivity Radius |
|---------------|-------------------|--------------------|
| Cyclic $Z_n$  | $n$               | $\pi/n$            |
| Binary dihedral $D_4^m$ | $4m$ | $\pi/2m$ |
| Binary tetrahedral $T^*$ | 24 | $\pi/6$ |
| Binary octahedral $O^*$ | 48 | $\pi/8$ |
| Binary icosahedral $I^*$ | 120 | $\pi/10$ |

Theorems of Mostow [46] and Thurston [47]. According to the former, geometrical quantities of orientable hyperbolic manifolds, such as their volumes and the lengths of their closed geodesics, are topological invariants. Therefore quantities such as the ‘global’ injectivity radius $r_{inj}$ (expressed in units of the curvature radius) are fixed for each manifold. Clearly this property also holds for spherical manifolds.

According to Thurston’s theorem, there is a countable infinity of sequences of compact orientable hyperbolic manifolds, with the manifolds of each sequence being ordered in terms of their volumes. Moreover, each sequence has as an accumulation point a cusped manifold, which has finite volume, is non-compact, and has infinitely long cusped corners [47].

TABLE II: First seven manifolds in the Hodgson-Weeks census of closed hyperbolic manifolds, ordered by the injectivity radius $r_{inj}$, together with their corresponding volume.

| Manifold     | Volume | Injectivity Radius |
|--------------|--------|--------------------|
| m004(1,2)    | 1.398  | 0.183              |
| m004(6,1)    | 1.284  | 0.240              |
| m003(-4,3)   | 1.264  | 0.287              |
| m003(-2,3)   | 0.981  | 0.289              |
| m003(-3,1)   | 0.943  | 0.292              |
| m009(4,1)    | 1.414  | 0.397              |
| m007(3,1)    | 1.015  | 0.416              |

Closed orientable hyperbolic 3-manifolds can be constructed from these cusped manifolds. The compact manifolds are obtained through a so-called Dehn surgery which is a formal procedure identified by two coprime integers, i.e. winding numbers $(n_1, n_2)$. These manifolds can be constructed and studied with the publicly available software package SnapPea [48]. SnapPea names manifolds according to the seed cusped manifold and the winding numbers. So, for example, the smallest volume hyperbolic manifold known to date (Weeks’ manifold) is named as m003(−3, 1), where m003 corresponds to a seed cusped manifold, and (−3, 1) is a pair of winding numbers. Hodgson and Weeks [48, 49] have compiled a census containing 11031 orientable closed hyperbolic 3-manifolds ordered by increasing volumes. In Table II we collect the first seven manifolds from this census with the lowest volumes, ordered by increasing injectivity radius $r_{inj}$, together with their volumes.

III. DETECTABILITY INVERSE PROBLEM

The detectability of cosmic topology as well as its inverse problem can be said to have two main ingredients, namely one of mixed geometric-and-topological nature, and another which comes from the fact the current astro-cosmological observations allow a set of possible models for the matter-energy content of the universe. As we have discussed in the previous section, the former arises from the fact that RW geometry does not dictate the topology of the 3-space $M$, giving rise to a multiplicity of non-trivial topologies for $M$ and the possible existence of multiple images of radiating sources.

Regarding the second important ingredient, we shall focus on the GCG unification of DM and DE paradigm to concretely illustrate the cosmic topology inverse problem. In other words, we assume the current matter content of the universe to be given by the ordinary baryonic matter of density $\rho_b$ plus a generalized Chaplygin gas (with density $\rho_{ch}$ and pressure $p_{ch}$) whose equation of state is given by:

$$p_{ch} = -\frac{M^{4(\alpha+1)}}{\rho_{ch}^\alpha},$$

(2)
where the constant $M > 0$ has dimension of mass and $\alpha$ is a real dimensionless constant. The Friedmann equation is then given by

$$H^2 = \frac{8\pi G}{3} (\rho_b + \rho_{ch}) - \frac{k}{a^2},$$

(3)

where $H = \dot{a}/a$ is the Hubble parameter, overdot stands for derivative with respect to time $t$, and $G$ is Newton’s constant. Introducing the critical density $\rho_{cr} = 3H^2/(8\pi G)$, and the corresponding densities $\Omega_b = \rho_b/\rho_{cr}$ and $\Omega_{ch} = \rho_{ch}/\rho_{cr}$, equation (3) can be rewritten as

$$a^2H^2(\Omega - 1) = k,$$

(4)

where $\Omega = \Omega_b + \Omega_{ch}$.

If one further assumes that these two matter components do not interact, then the energy conservation equation $\dot{\rho}_{tot} + 3H(\rho_{tot} + p_{tot}) = 0$ can be integrated separately for the baryonic matter and Chaplygin gas, giving the well known result $\rho_b = \rho_{b0}(a_0/a)^3$, and

$$\rho_{ch} = \rho_{ch0} \left[ (1 - A) \left( \frac{a_0}{a} \right)^{3(1+\alpha)} + A \right]^{1/(1+\alpha)},$$

(5)

where $A = (M^4/\rho_{ch0})^{(1+\alpha)}$, and the index 0 denotes evaluation at present time $t_0$. We note that at high redshifts $(a_0/a \gg 1)$ one has $\rho_{ch} \propto a^{-3}$, whereas at late times $(a_0/a \ll 1)$ one has $\rho_{ch} = -\rho_{ch} = -M^4 = const.$, making clear that the GCG interpolates between a dust dominated (CDM) phase in the past, and a cosmological constant phase in the future.

Writing the above expressions for the densities $\rho_b$ and $\rho_{ch}$ in terms of the redshift ($z = a_0/a - 1$), the Friedmann equation (2) can be rearranged to give the Hubble function

$$H(z) = H_0 \left\{ \Omega_{ch0} \left[ A + (1 - A)(1 + z)^{3(1+\alpha)} \right]^{1/(1+\alpha)} + \Omega_{b0}(1 + z)^3 + (1 - \Omega_0)(1 + z)^2 \right\}^{1/2},$$

(6)

where $\Omega_0 = \Omega_{ch0} + \Omega_{b0}$. It is clear from this equation that, regardless of the value of $\alpha$, for $A = 0$ the GCG component behaves as CDM with density $\Omega_{ch0}$, while for $A = 1$ the GCG plays the role of a cosmological constant term, whose density is again $\Omega_{ch0}$. On the other hand, for $\alpha = 0$ the GCG behaves as $\Lambda$CDM, whose matter and cosmological constant components are, respectively, given by $\Omega_{ch0}(1 - A)$ and $\Omega_{ch0}A$.

From (6) we find that the redshift-distance relation for non-flat cases, in units of curvature radius $a_0$, is given by

$$\chi = \sqrt{|1 - \Omega_0|} \int_1^{1+z} \left\{ \Omega_{ch0} \left[ A + (1 - A) x^{3(1+\alpha)} \right]^{1/(1+\alpha)} + \Omega_{b0} x^3 + \left( 1 - \Omega_0 \right) x^2 \right\}^{-1/2} dx,$$

(7)

where $x = 1 + z$ is an integration variable and we have used that, for non-flat models ($k \neq 0$), the curvature radius is identified with the scale factor, which from (3) is given by $a_0 = (H_0 \sqrt{|1 - \Omega_0|})^{-1}$ today. For simplicity, on the left hand side of (7) and in many places in the remainder of this article, we have left implicit the dependence of the function $\chi$ on its variables.

For a given survey with redshift cut-off $z_{obs}$, the redshift distance function $\chi_{obs}$ clearly depends on the way one models the matter-energy content of the universe. So, for example, in the $\Lambda$CDM context $\chi_{obs} = \chi(\Omega_m, \Omega_\Lambda)$, and therefore the potential detectability (or the undetectability) of a given topology depends on these density parameters. Similarly, in dealing with the detectability of cosmic topology in the GCG unified framework it is clear from (7) that besides the Chaplygin density $\Omega_{ch0}$ one has to consider the GCG model parameters $\alpha$ and $A$, assuming we know $\Omega_{b0}$. Unless otherwise stated, we shall assume in what follows that $\Omega_{b0} = 0.04$, which is the value that arises from recent observations.3

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3 The determination of light element abundances together with primordial Big-Bang nucleosynthesis furnishes $\Omega_b h^2 = 0.0214 \pm 0.0018$, which combined with $h = 0.72 \pm 0.08$ ($H_0 = 100 h \text{ km/s/Mpc}$), from the Hubble Space Telescope Key Project, gives the central value $\Omega_b \approx 0.04$. Although these results were derived in the $\Lambda$CDM context, the values remain unaltered in the GCG unified framework, since the GCG behaves as CDM at the relevant (high) redshifts and the observational determination of the $H_0$ (made through the linear Hubble law) is also unaffected by the nature of the DM and DE components.
To examine how a possible detection of cosmic topology may constrain the GCG parameters, consider a manifold \( M \) with \( r_{inj} = r_{inj}^{M} \) (say), and a given survey depth \( z = z_{obs} \). It is clear that, for a fixed value of a variable of the redshift-distance function (7), equation

\[
\chi_{obs}(A, \alpha, \Omega_0) = r_{inj}^{M}
\]

defines an implicit function of the remaining two parameters, whose graph is a curve in the corresponding parameter plane. In the parameter planes \( \Omega_0-A \) and \( \Omega_0-\alpha \) for all points above (below) a curve solution of (8), the topology of the corresponding spherical (or respectively hyperbolic) manifold is potentially detectable.\(^4\) Obviously, for all points below (or above in the hyperbolic case) these curves, the corresponding spherical (hyperbolic) topologies are undetectable \((\chi_{obs} < r_{inj})\). In this way, the detection of a non-trivial cosmic topology \( M \) with injectivity radius \( r_{inj}^{M} \) may be used to set constraints on the GCG parameters \( A \) and \( \alpha \), for \( \Omega_0 \) given by recent observations.

Before we proceed further we note that, for \( \rho_{ch} \) to be well defined for any value of the scale factor, the parameter \( A \) has to be limited to the interval \( 0 \leq A \leq 1 \). Regarding \( \alpha \), we must have \( \alpha > -1 \), so as to ensure that the GCG behaves as CDM at early times and as a positive cosmological constant at late times. As we shall see in section IV, recent SNIa data give \( \alpha \lesssim 12 \) (see also refs. 13 22). Thus, in what follows, we shall take \(-1 < \alpha < 12\).

As a first concrete example of bounds set by the detection of cosmic topology on the GCG model parameters, we consider the behavior of these curves solution for fixed values of \( A \) for some single action spherical and small volume hyperbolic manifolds of Tables II and III\(^5\) taking into account the redshift \( z = 1089 \) of the last scattering surface 23. As an illustration, we shall consider the interval \( 0.99 < \Omega_0 < 1.03 \), which arises from a combined analysis of CMBR, SNIa, and large-scale structure data (at 68% confidence level) in the \( \Lambda \)CDM framework.

Figure 4 gives plots of the curves solution of \( \chi_{obs}(\Omega_0, A) = r_{inj} \), for \( \alpha = -1, 0, 1, \) and 12 and \( \Omega_0 = 0.04 \). Given that each of these curves separates the parameter plane into (potentially) detectable and undetectable regions, it is clear that, for example, the detection of a binary tetrahedral \( T^* \) topology (or binary dihedral \( D^*_m \), or cyclic \( Z_n \), which have the same \( r_{inj} \)) would set the bounds \( A \gtrsim 0.2 \), \( A \gtrsim 0.7 \), and \( A \gtrsim 0.9 \), for \( \alpha = -1, \alpha = 0 \) (\( \Lambda \)CDM), and \( \alpha = 1 \), respectively. On the other hand, the detection of the Poincaré dodecahedron space topology would not set any constraint on \( A \) for the considered upper bound on \( \Omega_0 \).

We note that, according to equation (7), the smaller the value of \(|\Omega_0 - 1|\) the greater is the number of undetectable topologies. However, for any value of \(|\Omega_0 - 1|\), no matter how small, there will always remain an infinite number of topologies in the binary dihedral \( D^*_m \) and cyclic \( Z_n \) families, which are in principle detectable for large order covering groups \( \Gamma \) (see Table II). Nevertheless, figure 4 makes clear that for \( n > 10 \) and \( m > 5 \) the detection of topology (using CMBR) puts no constraints on \( A \), for the values of \( \alpha \) consistent with recent SNIa observations, with \( \Omega_0 \) within the interval given by CMBR and large-scale structure data.

The picture is rather similar for hyperbolic manifolds. For example, the detection of the Week’s manifold m003(−3, 1) would give the bounds \( A \gtrsim 0.2 \), \( A \gtrsim 0.7 \) and \( A \gtrsim 0.9 \) for, respectively, \( \alpha = -1, \alpha = 0, \) and \( \alpha = 1 \). The smaller the injectivity radius the weaker is the restriction on \( A \) that arises from the detection of the topology, for any fixed \( \alpha \). So, for example, for \( \alpha = 1 \), the detection of the m004(1, 2) topology imposes no bounds on \( A \), while for the manifold m004(6, 1) it gives \( A \gtrsim 0.7 \).

Finally, from figure 4 one also has that in general the greater is \( A \) the smaller is the interval allowed for \( A \) as consequence of a possible detection of spherical and hyperbolic topologies.

An important point to be noticed here is that since \( \chi_{obs} \) is independent of \( \alpha \) for \( A = 1 \), according to (7), there is an absolute minimum (maximum) of \( \Omega_0 \) which arises from the detection of a given spherical (hyperbolic) manifold. This clearly illustrates how bounds on a local physical quantity can be imposed by the global topology of the universe. The fact that a possible detection of a nontrivial topology places constraints on \( \Omega_0 \) was first mentioned in the cosmological context by Bernshetein and Shvartsman 54, and has been recently dealt with in the \( \Lambda \)CDM framework by Roukema and Luminet 51. We shall show in the next section that the combination of cosmic topology detection with bounds from other experiments, such as SNIa data, allows to place even stronger limits on \( \Omega_0 \).

The role played by the baryonic matter component is not significant for positive values of \( \alpha \) (and \( A \) not too close to 1), but is very important for \(-1 \leq \alpha < 0 \), as figure 2 demonstrates for \( \alpha = -1 \). The comparison of this figure with figure 4 clearly shows that the detection of any spherical or hyperbolic topology would set less stringent bounds on \( A \) in the absence of the baryonic matter.

We shall now focus our attention on the parameter plane \( A-\alpha \). Figure 4 gives plots of the curves solution of \( \chi_{obs}(A, \alpha) = r_{inj} \), for the limiting values of the total density parameter \( \Omega_0 \) and \( \Omega_{\text{m003}} \) in the remainder of this work.

\(^4\) In the \( A-\alpha \) plane, for points above the curves solution of (5), the topology is detectable for both spherical and hyperbolic manifolds.

\(^5\) As the three manifolds in table II obtained from the cusped seed manifold m003 have very close injectivity radii, for the sake of simplicity we shall present only the results corresponding to Weeks manifold m003(−3, 1) in the remainder of this work.
FIG. 1: The solution curves of (8) as plots of $\Omega_0$ versus $A$ for fixed values of $\alpha$ and $r_{\text{inj}}$ of some spherical and hyperbolic manifolds of tables I and II. A survey depth $z_{\text{obs}} = 1089$ (CMBR) was used in all cases.

The figure shows the curves of some single action spherical manifolds (table I), while on the right panel we have the curves of some small hyperbolic manifolds (table II).

From the region above the curves of single action manifolds one reads that, for example, the detection of a $D^*_3$ topology (or equivalently $T^*$ and $Z_6$) gives the bounds $\alpha \lesssim 2$ and $A \gtrsim 0.2$ (for $\Omega_0 = 1.03$). The Poincaré dodecahedron (as well as $D^*_5$ and $Z_{10}$) is detectable for most of the range of $A$ and $\alpha$. Its detection only places the limits $A \gtrsim 0.05$ and $\alpha \gtrsim -0.5$. For the binary dihedral $D^*_m$ and cyclic $Z_n$ families, figure 3 makes clear that the detection of topology puts no constraints on both $\alpha$ and $A$, using CMBR ($z = 1089$), for $n > 10$ and $m > 5$, for the limiting value $\Omega_0 = 1.03$.

For $\Omega_0 = 0.99$, the detection of the smallest (known) hyperbolic manifold $m003(-3,1)$ would set the constraints $\alpha \lesssim 2.5$ and $A \gtrsim 0.2$, while the detection of $m004(1,2)$ gives almost no restriction on both GCG parameters.

Hitherto we have examined the detectability of cosmic topology inverse problem in terms of the parameters $A$ and $\alpha$, which are directly constrained by some experiments, such as SNIa observations. However, these experiments probe the redshift-distance function (7), and do not give rise to strong constraints on that parameters using current data (see, e.g., [11]). On the other hand, observables related to the clustering of matter allow to place tighter limits on the GCG parameters [11], and are sensitive to the part of the GCG that clumps. In this way, to proceed further with our
FIG. 2: The solution curves of $\chi_{\text{obs}} = r_{\text{inj}}$ for $\alpha = -1$ and $r_{\text{inj}}$ of some spherical and hyperbolic manifolds of tables II and III. A survey depth $z_{\text{obs}} = 1089$ (CMBR) was used. This figure illustrates the role played by the baryonic matter in the detectability of cosmic topology and its inverse problem.

study of the inverse problem we identify the clumping matter density parameter with

$$\Omega_{m0} = \Omega_{ch0} (1 - A)^{1/(1+\alpha)}, \quad (9)$$

such that in the Hubble function [9] both the baryonic matter and the GCG scale as $(1 + z)^3$ at high redshifts. In this way, the density $\Omega_{m0}$ corresponds to the fraction of the GCG that behaves as CDM, and therefore an effective mass

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6 Another possible choice of the matter density parameter, which has been made in ref. [11], is $\Omega_{m0} = \Omega_{ch0} (1 - A)$. In this case only for $\alpha = 0$ this density scales as $a^{-3}$ for $a \ll 1$. 
density parameter $\Omega_m^{eff} = \Omega_m + \Omega_b$ can be identified as the fraction of the total density that evolves as pressureless matter. In terms of $\Omega_m$ the redshift-distance relation \( \chi \) takes the form

$$\chi (\Omega_0, \Omega_m, \alpha) = \sqrt{1 - \Omega_0} \int_1^{1+z} \left\{ \Omega_{ch0} \left[ 1 - \left( \frac{\Omega_m^{0}}{\Omega_{ch0}} \right)^{1+\alpha} \right] + \left( \frac{\Omega_m^{0}}{\Omega_{ch0}} \right)^{1+\alpha} \right\}^{-1} dx \quad (10)$$

FIG. 4: The behavior of $\chi_{obs}$ as function of $\alpha$ for different values of total density $\Omega_0$, for $\Omega_m^{eff} = 0.3$ and $\Omega_b = 0.04$. The redshift $z_{obs} = 1089$ of the last scattering surface was used.

Assuming that $\Omega_m^{eff}$ is fixed by some independent measurement, say the x-ray gas mass fraction in clusters, for a given survey depth $z_{obs}$ we have $\chi_{obs} = \chi_{obs}(\Omega_0, \alpha)$, which can be shown to be a monotonically increasing function of $\alpha$, for $\alpha \in (-1, \infty)$, and any fixed total density. Figure 3 illustrates this behavior for different values of $\Omega_0$, for $\Omega_m^{eff} = 0.3$, $\Omega_b = 0.04$, and for the redshift corresponding to the last scattering surface ($z_{obs} = 1089$). At first sight, the monotonically increasing behavior of $\chi_{obs}$ indicates that the detectability of a given topology becomes more likely as $\alpha$ increases. However, for $\alpha > 1$ the behavior of $\chi_{obs}$ is not very sensitive to $\alpha$ as can be inferred from the asymptotic values of $\chi_{obs}$, whose values are indicated in the figure by small line segments. On the other hand, $\chi_{obs}$ changes substantially for $\alpha \lesssim -0.5$, and for universes that are not so nearly flat, i.e. for $0.01 \lesssim |\Omega_0 - 1| \lesssim 0.1$. As a consequence, for $\alpha \lesssim -0.5$ the detectability of cosmic topology, for fixed $\Omega_m^{eff}$, is much less likely than in the $\Lambda$CDM-dominated universe. We note that the relevant features of this figure do not change significantly for $\Omega_0 = 0$, keeping $\Omega_m^{eff} = 0.3$, of course.

To close this section we consider some specific manifolds of tables II and III and focus our attention on the parameter plane $\Omega_0 - \alpha$. Taking again $\Omega_m^{eff} = 0.3$, with $\Omega_b = 0.04$, for $z_{obs} = 1089$, and $\Omega_0$ within the previously discussed bounds, we have plotted the curves solution of (5) for some spherical and hyperbolic manifolds, as shown in figure 5. From this figure we have that the greater the order of the group for the cyclic and binary dihedral families the weaker is the restriction on $\alpha$. So, for example, for $n > 10$ and $m > 5$ the detectability of a given topology of these families would set no restriction on the GCG parameter $\alpha$. An important point that can be inferred from this figure is that if no sign is found of a given topology in the $\Lambda$CDM framework this does not necessarily mean that such topology is ruled out by observation. For example, if no sign of either a binary tetrahedron $T^*$ or the Weeks manifold $m003(-3,1)$ is found in CMBR maps, it can be that $\alpha \lesssim -0.4$ such that these topologies are unobservable using pattern repetitions. Finally, if future observations such as the Planck mission tighten the range of $\Omega_0$ even closer to 1, as for example $\Omega_0 = 1 \pm 0.003$, none of the topologies considered in this figure would be detectable, for any value of $\alpha$.

The identification \( \Omega_m^{eff} \) observationally well motivated as the effective matter density of the GCG. Indeed, using $\Omega_m^{eff}$ as the cold dark matter density parameter in the initial fluctuation spectrum (and assuming entropy perturbations) leads to a large-scale matter power spectrum in agreement with observational data [\cite{22}]. Furthermore, the values of $\Omega_m^{eff}$ obtained from x-ray gas mass fraction in clusters [\cite{21}] are consistent with those derived from the power spectrum.
FIG. 5: The solution curves of \( \chi_{\text{obs}}(\Omega_0, \alpha) = r_{\text{inj}} \) for some spherical and small hyperbolic manifolds of tables III and IV. An effective matter density \( \Omega_{\text{eff}} = 0.3 \), with \( \Omega_0 = 0.04 \), and a redshift \( z_{\text{obs}} = 1089 \) were used.

**IV. COMBINATION WITH SUPERNOVAE DATA**

In the preceding section, we have investigated the constraints on the matter content of the universe placed by the detection of a nontrivial topology. Very often in cosmology the measurement of a single observable does not allow to place strong limits on the parameters of a model. It is therefore worthwhile to consider the constraints that could arise from the detection of the topology together with other sets of observational data. We shall consider in this section the constraints on the GCG parameters which arise from the detection of the cosmic topology combined with the limits imposed by recent supernovae data (194 SNIa from refs. [57] and [58]). We shall follow a similar procedure as in section III, superimposing the \( \chi_{\text{obs}} = r_{\text{inj}} \) curves corresponding to some spherical and hyperbolic manifolds to the regions in the parameter space allowed by supernovae data.

In figure 6 we display the 95% (2\( \sigma \)) confidence regions in the \( A - \alpha \) parametric plane, for \( \Omega_0 = 1.03 \) (left) and \( \Omega_0 = 0.99 \) (right), from SNIa data. The contours of constant confidence are nearly identical in both cases, since supernovae data are not very sensitive to the total density. We refer the readers to the appendix for a discussion on how these contours are derived. The curves solution of \( \chi_{\text{obs}} = r_{\text{inj}} \) are shown for some spherical (left) and hyperbolic (right) manifolds. It is clear from the figure that the higher the value of \( r_{\text{inj}} \), the tighter are the combined constraints on both \( A \) and \( \alpha \) (recall that only in models whose parameters are above these curves the corresponding topology is detectable). For instance, a detection of the dihedral \( D^*_2 \) (or \( Z_4 \)) topology, together with the SNIa constraints, would imply \(-1 \leq \alpha \leq -0.5 \) and \( 0.55 \leq A \leq 0.65 \) at 95% confidence level. This constraint on \( \alpha \) is of the same order of what would be obtained with the Supernovae Acceleration Probe [11]. The detection of the \( D^*_2 \) topology would allow to discard the \( \Lambda \)CDM model taking into account current supernovae bounds. Incidentally, this makes clear that a given topology that is undetectable in the \( \Lambda \)CDM model (\( \alpha = 0 \)) may be detectable for other matter content models, as in the GCG case [41]. The detection of a tetrahedral \( T^* \) (as well as \( D^*_3 \) or \( Z_6 \)) would set the constraints \(-1 \leq \alpha \leq 0 \) and \( 0.55 \leq A \leq 0.75 \) at 95% confidence level. The determination of the octahedral topology (as well as \( D^*_4 \) or \( Z_8 \)) would rule out only high values of \( \alpha \) (setting \( \alpha \leq 7 \)). As expected from the discussion of the previous section, for \( \Omega_0 = 1.03 \), the Poincaré dodecahedral does not add limits to the supernovae bounds, nor do the dihedral and cyclic families for \( n > 4 \) and \( m > 8 \).

Similarly, from the left panel of figure 6 the detection of the higher \( r_{\text{inj}} \) manifolds of table IV namely m007(3,1), m009(4,1), and m003(-3,1), would set rather strong constraints on \( \alpha \). The detection of m004(6,1) topology gives rise to the bound \( \alpha \leq 5.5 \), restricting therefore the limits set by supernovae data on this parameter. As for the topology with the smallest \( r_{\text{inj}} \) of table IV m004(1,2), its detection does not add new bounds to those which arise from supernovae.

Let us now investigate what happens when \( \Omega_0 \) is allowed to vary within the bound \( 0.99 < \Omega_0 < 1.03 \) for some noteworthy values of \( \alpha \). We shall take the following values for \( \alpha \). First \( \alpha = -1/2 \), which is a lower limit set by the
FIG. 6: Superposition of the 95% confidence regions from SNIa data (light gray) and the curves $\chi_{obs} = r_{inj}$ for some spherical (left) and hyperbolic (right) manifolds, and for a survey depth of $z_{obs} = 1089$, as in figure 3.

combination of several observables assuming a flat universe.$^8$ Second, $\alpha = 0$, which corresponds to the $\Lambda$CDM case. Third, $\alpha = 1$, which is the standard Chaplygin gas. Finally, we shall take $\alpha = 2$, which is approximately the best fit from the supernovae data for fixed $\Omega_0$ in the interval $0.99 < \Omega_0 < 1.03$.

In figure 7 the 95% confidence regions from supernovae data for these values of $\alpha$ are shown superimposed with the solutions of $\chi_{obs} = r_{inj}$ for several manifolds. As expected, the supernovae constraints on $A$ are almost independent of $\Omega_0$ within the narrow interval of $\Omega_0$ that we are considering. Therefore, the determination of a nontrivial topology sets no further constraints on $A$ than those arising from the supernovae data. However, knowing the topology, in combination with the supernovae limits, allows to constrain the total density $\Omega_0$. Indeed, for $\alpha = -1/2$, for example, if the tetrahedral $T^*$ (or $D^*_2$, or $Z_6$) is found to be the cosmic topology, this sets the constraint $\Omega_0 \gtrsim 1.02$. It should be noticed that, according to SNIa data, a few topologies of tables I and II are already unobservable at $2\sigma$ (95% confidence) for $\alpha = -1/2$ and $0.99 < \Omega_0 < 1.03$, such as $D^*_2$, $Z_4$, m009(4,1), and m007(3,1). The $T^*$ topology is undetectable for $\alpha > 0$, while the Poincaré dodecahedron $I^*$ is detectable for any $\alpha$ in the considered range.

For a given spherical topology, the greater is $\alpha$ the higher is the lower bound on $\Omega_0$ which arises from the detection of the corresponding topology combined with SNIa data. Thus, e.g., while for $\alpha = -1/2$ the detection of the tetrahedral $T^*$ sets the bound $\Omega_0 \gtrsim 1.02$, for $\alpha = 0$ it gives rise to the lower bound $\Omega_0 \gtrsim 1.029$.

As far as hyperbolic topologies are concerned, the greater is $\alpha$ the smaller is the value for lower bound on $\Omega_0$ which arises from the combination of the topology detection with SNIa data. Thus, for example, for the manifold m004(6,1) and for $\alpha = -1/2$ and $\alpha = 1$ the lower bounds are $\Omega_0 \gtrsim 0.995$ and $\Omega_0 \gtrsim 0.990$, respectively. For smaller $r_{inj}$ manifolds, however, such as $D^*_5$, $Z_{18}$, and m004(1,2), the change of the lower bound on $\Omega_0$ with $\alpha$ is not very significative, as can be seen from the figures. This allows to set constraints which are rather independent of $\alpha$, for these manifolds. For instance if the topology of the universe is found to be the m004(1,2) manifold, then we must have $\Omega_0 \lesssim 0.998$ at 95% confidence level, for any value of $\alpha \gtrsim -1/2$.

V. FINAL REMARKS

Regardless of our present-day inability to predict the topology of the universe, its detection and determination is ultimately expected to be an observational problem. The search for topological signs of a nontrivial topology of the universe has become particularly topical. A finite universe is not only consistent with the main pillars of the FLRW

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$^8$ This corresponds to the $2\sigma$ lower limit derived in ref. 11 and at $3\sigma$ in 13.
FIG. 7: Superposition of the 95% confidence regions from SNIa data (light gray) and the curves $\chi_{\text{obs}} = r_{\text{inj}}$ for several spherical and hyperbolic manifolds, and for a survey depth of $z_{\text{obs}} = 1089$, as in figure 1.

The detection of a nontrivial cosmic topology would certainly be a major scientific discovery. Besides its importance per se, we have shown in this article that knowledge of the cosmic topology can be used to set bounds on the matter content of the universe. To concretely illustrate this fact, we have considered the GCG model for dark matter and dark energy unification and determined the constraints on its parameters that would be obtained from the detection of a number of topologies. Specifically, we have considered the seven smaller volume hyperbolic manifolds and the globally homogeneous spherical manifolds.

We have shown that for $\Omega_0$ within the current observational bounds, the determination of cosmic topology generally sets upper limits on $\alpha$ and lower limits on $A$ (see, for example, figures 1 and 3). Furthermore, the constraints on $\alpha$ improve as the allowed interval of $\Omega_0$ narrows, as expected from the forthcoming experiments (see, e.g., figure 5). Also, the higher the injectivity radius $r_{\text{inj}}$, the stronger are the constraints from the detection of the corresponding cosmological model, but offers a possible explanation for several CMBR anisotropy features which are unexpected in the context of an infinite flat $\Lambda$CDM model [27, 29, 33, 34]. Although it is not clear whether evidences of topological pattern repetitions have been found in CMBR anisotropies maps [31, 32, 33, 35], it is expected that we should be able to detect the cosmic topology in the future, given the wealth of increasingly accurate cosmological observations, especially the recent results from the WMAP, and the development of new methods and strategies for such a detection.
manifolds through pattern repetitions.

We have also shown that the cosmic topology detection sets additional constraints on the matter content parameters to the bounds that arise from the SNIa data analysis, by using the GCG parameters as a concrete example. To this end we have determined the confidence regions in the $A - \alpha$ parameter plane from 194 SNIa for fixed $\Omega_0$ at the extreme values of the current observational bounds, obtained the 95% limits on the $\Omega_0 - A$ plane for some physically motivated values of $\alpha$, and combined these limits with the bounds from cosmic topology detection for several spherical and hyperbolic topologies. As a result we found that this combination allows to set stringent constraints on the GCG parameters. Thus, for example, the detection of $D_L^2$ implies $-1 \lesssim \alpha \lesssim -0.5$ and $0.55 \lesssim A \lesssim 0.65$ at 95% confidence level, for $\Omega_0 = 1.03$. These constraints are stronger than those obtained from any current data based on the background dynamics [18]. Clearly, if limits from other sets of observational data are combined with constraints from the determination of the topology, the latter would greatly improve the former bounds.

We found that, besides the GCG parameters, the detection of the topology provides constraints on the total density $\Omega_0$. First, the detection of a spherical (hyperbolic) topology obviously implies $\Omega_0 > 1$ ($\Omega_0 < 1$). Second, for each manifold the topology detection implies a lower (upper) bound for spherical (hyperbolic) $\Omega_0$. Finally, the combination of cosmic topology detection with supernovae data further constrains these bounds on $\Omega_0$, by restricting the allowed interval of $A$ for different values of $\alpha$.

The determination of a possible nontrivial cosmic topology is a problem that can be addressed with the current observational data. Several searches have been conducted and more will be carried on in the future. The fact that the determination of the topology allows to constrain the matter-energy content of the universe provides an extra motivation for this search.

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APPENDIX: CONSTRAINTS ON THE GCG FROM RECENT SNIA DATA

Empirical studies show that SNIa can be used as standard candles after light curve calibration (see, e.g., refs. [55, 56]). Therefore, they offer a direct probe of the luminosity distance-redshift relation, which can be used to constrain theoretical models.

In section IV we displayed confidence levels on the parameters $A$ and $\alpha$ of the GCG model obtained from SNIa data. To produce such contours, a sample of 253 SNIa was compiled from references [57] and [58] which provide tables with the redshift $z$, $\log D_L$, and its variance $\sigma^2_{\log D_L}$ for each supernovae. Here $D_L$ is the luminosity distance times the Hubble constant, which is given by

$$D_L = d_L H_0 = \frac{1+z}{\sqrt{1-\Omega_0}} S_k \left( \sqrt{1-\Omega_0} H_0 \int_0^z \frac{dz'}{H(z')} \right) = \frac{1+z}{\sqrt{1-\Omega_0}} S_k(\chi),$$

where $S_k(x) = \sin(x)$, $\sinh(x)$, or $x$, for $\Omega_0$ greater, smaller or equal to unity, respectively. The Hubble function $H(z)$ is given by eq. (6) and $\chi$ is given by (7). As mentioned in the text, we fix $\Omega_{b0} = 0.04$, in agreement with the observed abundances of light elements and measurements of the Hubble constant.

Following [57] and [58], we discard local supernovae with $z < 0.01$, because the peculiar motion contribution to $z$ is too high, and those with high host extinction, $A_V > 0.5$, which could cause a strong bias in the determination of $D_L$. After these cuts, we end up with a sample of 194 SNIa extending up to $z = 1.75$. In computing the Chi-squared, we take into account the scatter in $z$ caused by peculiar velocities. Assuming a velocity dispersion of $\sigma_v = 500 km/s$, we propagate the uncertainty in $z$ into the luminosity distance, adding the result in quadrature with the observational uncertainty in $D_L$. Therefore, the Chi-squared is given by:

$$\chi^2 = \sum_{i=1}^{194} \left[ \frac{\log (D_L^{\text{Obs}}(z_i)) - \log (D_L^{\text{Th}}(z_i))}{\sigma^2_{\log (D_L(z_i))} + \left( \frac{\partial \log D_L^{\text{Th}}}{\partial z} \right)_{z_i} \sigma_z} \right]^2,$$

where the theoretical prediction is computed from (6) together with (7) and the observational values are given in the tables of [58] and [57].
To generate the contours displayed in section XV we assume that the 95% confidence levels are well approximated by the same value of $\Delta \chi^2$ as for a two-dimensional normal distribution. That is, we obtain the best fit (minimum of $\chi^2$) and plot the contour levels of $\Delta \chi^2 = 6.17$.

To produce the confidence regions of figure $6$ we fixed $\Omega_0 = 1.03$ and $\Omega_0 = 0.99$ for the spherical and hyperbolic universes. The parameters $A$ and $\alpha$ are allowed to vary, with no priors. On the other hand, for the graphs of figure $7$ we fix $\alpha$ at several physically motivated values and leave $\Omega_0$ and $A$ totally free. The results are shown only for a narrow interval of $\Omega_0$ allowed by a combination of several observables in the context of the $\Lambda$CDM model, namely $0.99 < \Omega_0 < 1.03$.

As far as we know, this is the first time that such analysis is done for the GCG model. By fixing $\Omega_0$ motivated by other non SNIa data, one is able to obtain stronger limits on $A$ and $\alpha$ when $\Omega_0$ is left completely free. As expected, these limits are almost insensitive to $\Omega_0$ in the range $0.99 < \Omega_0 < 1.03$, i.e., they are robust for nearly flat geometries. For fixed $\alpha$, the constraints on $A$ are almost independent of $\Omega_0$ in the range above, as can be seen in figure $7$ [section XV]. It is clear that SNIa data alone cannot place significative constraints on $\Omega_0$ within that narrow range (see, e.g., ref. [14]).
