POISSON GEOMETRY OF FLAT CONNECTIONS FOR SU(2)-BUNDLES ON SURFACES

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Abstract. In earlier work we have shown that the moduli space $N$ of flat connections for the (trivial) SU(2)-bundle on a closed surface of genus $\ell \geq 2$ is a stratified symplectic space so that, in particular, the data give rise to a Poisson algebra $(C^\infty N, \{\cdot, \cdot\})$ of continuous functions on $N$; furthermore, the strata are Kähler manifolds, and the stratification consists of an open connected and dense submanifold $N_Z$ of real dimension $6(\ell - 1)$, a connected stratum $N_{(T)}$ of real dimension $2\ell$, and $2^{2\ell}$ isolated points. In this paper we show that, close to each point of $N_{(T)}$, the space $N$ and Poisson algebra $(C^\infty N, \{\cdot, \cdot\})$ look like a product of $\mathbb{C}^\ell$ endowed with the standard symplectic Poisson structure with the reduced space and Poisson algebra of the system of $(\ell - 1)$ particles in the plane with total angular momentum zero, while close to one of the isolated points, the Poisson algebra $(C^\infty N, \{\cdot, \cdot\})$ looks like that of the reduced system of $\ell$ particles in $\mathbb{R}^3$ with total angular momentum zero. Moreover, in the genus two case where the space $N$ is known to be smooth we locally describe the Poisson algebra $(C^\infty N, \{\cdot, \cdot\})$ and the various underlying symplectic structures on the strata and their mutual positions explicitly in terms of the Poisson structure.
Introduction

Let $\Sigma$ be a closed surface of genus $\ell \geq 2$, write $\pi$ for its fundamental group, let $G = SU(2)$, let $N$ be the moduli space of flat connections for the trivial $SU(2)$-bundle on $\Sigma$, and suppose the Lie algebra $g$ of $G$ endowed with a $G$-invariant inner product. Let $Z = \{\pm 1\}$ denote the centre of $G$ and $T = S^1 \subseteq G$ the standard circle subgroup inside $G$; it is a maximal torus. The decomposition of $N$ according to orbit types of flat connections looks like

$$N = N_G \cup N_T \cup N_Z$$

where the subscript refers to the conjugacy class of stabilizer. In an earlier paper [8] we proved that this decomposition is a stratification in the strong sense; the stratum $N_Z$ is open, connected, and dense, and referred to as top stratum. While the space $N$ is known to have a structure of a normal projective variety [19], it is unclear whether and how the stratification (0.1) relates to the corresponding complex analytic one; I am indebted to M. S. Narasimhan for this comment. In another earlier paper [11], we proved that the space $N$ inherits a structure of a stratified symplectic space in the sense of Sjamaar-Lerman [21]; in particular, we constructed a Poisson algebra $(C^\infty N, \{\cdot, \cdot\})$ of continuous functions on $N$ which, on each stratum, restricts to a symplectic Poisson algebra of smooth functions in the ordinary sense. In the present paper we describe the strata locally explicitly in terms of certain related classical constrained systems; in particular, for the special case of genus two, we describe the Poisson algebra and resulting Poisson geometry of the space $N$ explicitly. Here is our first result.

**Theorem 1.** Near a point of the top stratum $N_Z$, the space $N$ looks like $C^{3(\ell-1)}$, endowed with the standard symplectic and hence Poisson structure. Furthermore, near a point of $N_T$, the space $N$ and Poisson algebra $(C^\infty N, \{\cdot, \cdot\})$ look like the product of a copy of $C^\ell$, endowed with the standard symplectic Poisson structure, with the reduced reduced space and Poisson algebra of the reduced system of $(\ell-1)$ particles in the plane with total angular momentum zero. Finally, near a point of $N_G$, the space $N$ and Poisson algebra $(C^\infty N, \{\cdot, \cdot\})$ look like the reduced space and Poisson algebra of the reduced system of $\ell$ particles in ordinary space with total angular momentum zero.

Our approach also yields a proof of the following result, established by Narasimhan and Ramanan by other methods [17], cf. also [20].

**Theorem 2.** As a space, $N$ is smooth if and only if $\Sigma$ has genus two.

In fact, for genus two, $N$ is just complex projective 3-space; however, the algebra $C^\infty N$ then does not coincide with the standard one of smooth functions. It follows that, in particular, the reduced space underlying the reduced system of two particles in ordinary space with total angular momentum zero is smooth, in fact, just $\mathbb{R}^6$. This does not seem to have been known before. However its reduced Poisson algebra is considerably more complicated than the standard one on $\mathbb{R}^6$; see Section 7 below for details.

The reduced systems in the genus two case coming into play in Theorem 1 will be described explicitly in Sections 6 and 7 below. In particular, this will show
how the symplectic structures on the strata behave in the genus two case. The reduced system of a single particle in the plane with total angular momentum zero has been understood for a while [4] and is easy to describe: Consider the plane $\mathbb{R}^2$, with coordinates $x_1, x_2$. Consider the algebra of (continuous) functions on the plane generated by $x_1, x_2$ and the radius function $r$; thus these generators are subject to the relation $x_1^2 + x_2^2 = r^2$. The assignment

$$\{x_1, x_2\} = 2r, \quad \{x_1, r\} = 2x_2, \quad \{x_2, r\} = -2x_1$$

(0.2)

endows this algebra with a Poisson structure $\{\cdot, \cdot\}$; notice away from the origin, in suitable coordinates, this Poisson structure amounts to the standard symplectic Poisson structure while it degenerates at the origin. The reduced Poisson algebra of a single particle in the plane with total angular momentum zero is that of continuous functions in the plane that are smooth in the (dependent) variables $x_1, x_2, r$, with the Poisson structure (0.2).

The reduced system of two particles in ordinary space with total angular momentum zero is much more complicated. See Section 7 below for details. Suffice it to mention at this stage that the reduced Poisson algebra is generated by ten algebraic functions on $\mathbb{R}^6$ six thereof being algebraically independent.

Our method is to study the local models obtained in our earlier papers [7] – [11] by means of invariant theory and to exploit a result of Kempf-Ness [14] and Kirwan [15], thereby playing off against each other invariant theory over the reals and over the complex numbers. To get our hands on the local models we calculate various group cohomology spaces as suitable unitary representations explicitly in Section 2 below. Our approach is quite general and applies to arbitrary genus and structure group. Its very simplicity should enable one to study more complicated moduli spaces.

In another guise, the moduli space $N$ is that of semi stable holomorphic vector bundles on $\Sigma$ (with reference to a choice of holomorphic structure) of rank 2, degree 0, and trivial determinant. This space and related ones have been studied extensively in the literature [17], [18], [19]. In particular, for genus $\ell \geq 2$, the space $K = N_G \cup N(T)$ is known to be the Kummer variety of $\Sigma$ associated with its Jacobian $J$ and the canonical involution thereupon. Theorem 1 above implies in particular that, for genus $\geq 3$, the Kummer variety $K$ is precisely the singular locus of $N$, a result due to Narasimhan-Ramanan [17]. Theorem 1 above has the following consequence:

**Corollary.** When $\Sigma$ has genus $\ell \geq 2$, the Poisson algebra $(C^\infty N, \{\cdot, \cdot\})$ detects the Kummer variety $K$ in $N$ together with its $2^{2\ell}$ double points. More precisely, $K$ consists of the points of $N$ where the rank of the Poisson structure is not maximal, the double points being those where the rank is zero.

We now make a few comments about the special case where $\Sigma$ has genus two. The space $N$ then equals complex projective 3-space and $K$ is the Kummer surface associated with the Jacobian of $\Sigma$, cf. Narasimhan-Ramanan [17]. In the literature, this case has been considered somewhat special since as a space $N$ is then actually smooth. However, from our point of view, there is no exception. As a stratified symplectic space, $N$ still has singularities, our algebra $C^\infty N$ is not that of smooth functions in the ordinary sense, and the Kummer surface $K$ is the complement of the
top stratum $N_Z$ and hence still precisely the singular locus in the sense of stratified symplectic space; in particular, the symplectic structure on the top stratum does not extend to the whole space. It is interesting to observe that the stratification (0.1) is finer than the standard complex analytic one on complex projective 3-space.

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1. The stratification

Consider the standard presentation

\[
\mathcal{P} = \langle x_1, y_1, \ldots, x_\ell, y_\ell; r \rangle, \quad r = \prod_{j=1}^\ell [x_j, y_j],
\]

of $\pi$. The choice of generators induces an embedding of $\text{Hom}(\pi, G)$ into $G^{2\ell}$, and in this way the former will henceforth be viewed as a subspace of the latter. Moreover the holonomies with reference to the chosen generators induce a diffeomorphism in the appropriate sense from $N$ onto the representation space $\text{Rep}(\pi, G) = \text{Hom}(\pi, G)/G$; see [10] for details. Henceforth we do not distinguish in notation between $N$ and $\text{Rep}(\pi, G)$.

We now reproduce briefly the stratification of the moduli space $N$ of flat $\text{SU}(2)$-connections: The choice of $T$ induces an embedding of $\text{Hom}(\pi, T)$ into $\text{Hom}(\pi, G)$, and the choice of generators identifies $\text{Hom}(\pi, T)$ with $T^{2\ell}$. Let $Y$ be the $G$-orbit of $\text{Hom}(\pi, T)$ in $\text{Hom}(\pi, G)$ under the adjoint action. Then $\text{Hom}(\pi, \text{SU}(2))$ decomposes into $\text{Hom}(\pi, \text{SU}(2)) \setminus Y$ and $Y$. Each point in $\text{Hom}(\pi, \text{SU}(2)) \setminus Y$ has stabilizer the centre $Z = \{\pm 1\}$ of $G$, in fact, these are the irreducible representations of $\pi$ in $G$, while each point in $\text{Hom}(\pi, T) \cong T^{2\ell}$ has stabilizer $T$. Furthermore the inclusion of $\text{Hom}(\pi, T)$ into $Y$ induces a bijection of $\text{Hom}(\pi, T)/W$ onto $Y/G$ where $W = \mathbb{Z}/2$ refers to the Weyl group of $\text{SU}(2)$. Now $\text{Hom}(\pi, T)$ looks like $T^{2\ell}$, and the non-trivial element $w$ of $W$ acts on $T^{2\ell}$ by the assignment of $(\zeta_1, \ldots, \zeta_{2\ell})$ to $(\zeta_1, \ldots, \zeta_{2\ell})$ where as usual “$\zeta \mapsto \overline{\zeta}$” refers to complex conjugation. Hence the fixed point set of the action of $W$ on $\text{Hom}(\pi, T)$ is

\[
N_G = \{ \phi; \phi(x_j) = \pm 1, \ \phi(y_j) = \pm 1, \ 1 \leq j \leq \ell \},
\]

and the $W$-action is free on $\text{Hom}(\pi, T) \setminus N_G$. Thus, cf. (0.1), the resulting decomposition of $N$ looks like $N = N_G \cup N(T) \cup N_Z$; here $N_Z = (\text{Hom}(\pi, G) \setminus Y)/G$ and $N(T) = (Y \setminus N_G)/G$. In [8] this decomposition has been proved to be a stratification in the strong sense. Notice the projection map from $\text{Hom}(\pi, G) \setminus Y$ onto $N_Z$ is actually a principal $\text{SO}(3, \mathbb{R})$-bundle map, whence $N_Z$ is a smooth manifold of real dimension $6(\ell - 1)$. Furthermore $N_G$ consists of $2^{2\ell}$ isolated points while $N(T)$ is manifestly a smooth connected manifold of real dimension $2\ell$; in fact, the projection map $\text{Hom}(\pi, T) \to N(T)$ is a branched 2-fold covering, branched at the points of $N_G$.

We conclude with the remark that $\text{Hom}(\pi, T)$ is the group of characters of $\pi$ and hence may be canonically identified with the Jacobien of $\Sigma$. The quotient $\text{Hom}(\pi, T)/W$ is then the Kummer variety mentioned in the Introduction.
2. The local model

Let $\phi: \pi \to G$ be a homomorphism, and consider the group cohomology $H^*(\pi, g_\phi)$. The Lie bracket on $g$ induces a graded Lie bracket

\[ [\cdot, \cdot]: H^*(\pi, g_\phi) \otimes H^*(\pi, g_\phi) \to H^*(\pi, g_\phi) \]  

on $H^*(\pi, g_\phi)$. Further, after a choice of fundamental class in $H_2(\pi, \mathbb{Z})$ has been made, the orthogonal structure on $g$ induces a non-degenerate bilinear pairing

\[ (\cdot, \cdot)_\phi: H^j(\pi, g_\phi) \otimes H^{2-j}(\pi, g_\phi) \to \mathbb{R} \]  

which, for $j = 1$, amounts to a symplectic structure

\[ \sigma_\phi: H^1(\pi, g_\phi) \otimes H^1(\pi, g_\phi) \to \mathbb{R} \]  

on $H^1(\pi, g_\phi)$. Moreover, $H^0(\pi, g_\phi)$ equals the Lie algebra $z_\phi$ of the stabilizer $Z_\phi \subset G$ of $\phi$, the pairing (2.2) identifies $H^2(\pi, g_\phi)$ with the dual $z_\phi^*$, and the assignment

\[ \Theta_\phi: H^1(\pi, g_\phi) \to H^2(\pi, g_\phi), \quad \Theta_\phi(\eta) = \frac{1}{2}[\eta, \eta]_\phi, \quad \eta \in H^1(\pi, g_\phi), \]  

is a momentum mapping for the action of the stabilizer $Z_\phi$ on $H^1(\pi, g_\phi)$. See [7] for details.

Using the recipe exploited several times in our earlier papers, we endow the Marsden-Weinstein reduced space $H_\phi = \Theta_\phi^{-1}(0)/Z_\phi$ with a smooth structure, cf. e. g. [10]. Let $V_\phi$ denote the zero locus $\Theta_\phi^{-1}(0)$ of $\Theta_\phi$ in $H^1(\pi, g_\phi)$, write $I_\phi \subseteq C^\infty(H^1(\pi, g_\phi))$ for the ideal of smooth functions on $H^1(\pi, g_\phi)$ that vanish on $V_\phi$, and let

\[ C^\infty(H_\phi) = (C^\infty(H^1(\pi, g_\phi)))^{Z_\phi} / (I_\phi)^{Z_\phi}, \]  

the algebra of smooth $Z_\phi$-invariant functions on $H^1(\pi, g_\phi)$, modulo the ideal $(I_\phi)^{Z_\phi}$ of smooth $Z_\phi$-invariant functions that vanish on $V_\phi$. With respect to the decomposition into connected components of orbit types, (2.5) endows $H_\phi$ with a smooth structure. Moreover, cf. [1], the symplectic Poisson structure on $H^1(\pi, g_\phi)$ passes to a Poisson structure $\{\cdot, \cdot\}_\phi$ on $C^\infty(H_\phi)$. The space $H_\phi$ together with the Poisson algebra $(C^\infty(H_\phi), \{\cdot, \cdot\}_\phi)$ is our local model for the moduli space $N$ near the point represented by $\phi$, as a stratified symplectic space. See [10] and [11] for details.

For a homomorphism $\phi$ representing a point in the top stratum $N_Z$, that is, having the property that $H^0(\pi, g_\phi)$ is zero, the cohomology space $H^1(\pi, g_\phi)$ is a finite dimensional symplectic vector space of real dimension $6(\ell - 1)$ and, near a point represented by $\phi$, as a symplectic manifold, the moduli space $N$ looks like $H^1(\pi, g_\phi)$, viewed as a symplectic vector space. In particular the Poisson structure then amounts to the usual symplectic one.

To understand the other strata we must at first calculate the cohomology groups $H^j(\pi, g_\phi)$, for $j = 0, 1, 2$, as $Z_\phi$-representation spaces. It seems wise to proceed in somewhat greater generality:
Let $K$ be an arbitrary compact Lie group, let the genus $\ell$ of $\Sigma$ be arbitrary $\geq 2$, and let $M$ be a finite dimensional (real) orthogonal representation of $K$ together with a compatible $\pi$-module structure in the sense that for every $x \in K$, $y \in \pi$, $m \in M$, 
\[ x(ym) = y(xm). \]

The cohomology groups $H_j(\pi, M)$, $j = 0, 1, 2$, are calculated by means of the standard chain complex arising from the presentation (1.1) in the usual way. It looks like

\[ (2.6) \quad M \overset{d^0}{\longrightarrow} M^{2\ell} \overset{d^1}{\longrightarrow} M. \]

Now $H_1(\pi, M)$ inherits a Hermitian structure in the following way: The inner product on $M$ extends to one on $M^{2\ell}$ in the obvious way and the operation of taking orthogonal complements yields a canonical decomposition

\[ (2.7) \quad M^{2\ell} \cong d^0(M) \oplus H^1(\pi, M) \oplus d^1(M^{2\ell}) \]

whence in particular $H^1(\pi, M)$ inherits an inner product from that on $M^{2\ell}$. The symplectic structure arises in the same way as (2.3) above, that is, it now looks like

\[ (2.8) \quad \sigma: H^1(\pi, M) \otimes H^1(\pi, M) \rightarrow \mathbb{R}; \]

it involves a choice of fundamental class in $H_2(\pi, \mathbb{Z})$. The two structures induce a complex structure $\ast$ and then combine to a Hermitian structure on $H^1(\pi, M)$. Hence the orthogonal $K$-representation on $M$ induces a structure of unitary $K$-representation on $H^1(\pi, M)$.

Until the end of this Section we now suppose that $G$ is an arbitrary compact Lie group, with Lie algebra $g$ endowed with a $G$-invariant inner product. Let $\phi: \pi \rightarrow G$ be a homomorphism. The Lie algebra $g_\phi$, endowed with the resulting $\pi$-module structure, decomposes as a $\pi$-module into the direct sum of the Lie algebra $z_\phi = H^0(\pi, g_\phi)$ of the stabilizer $Z_\phi \subseteq G$ of $\phi$ and its orthogonal complement $z_\phi^\perp$ in $g$ with reference to the inner product on $g$. The above remarks apply with $K = Z_\phi$ and $M$ any one of $g_\phi, z_\phi, z_\phi^\perp$. Hence the cohomology spaces $H^1(\pi, z_\phi), H^1(\pi, g_\phi), H^1(\pi, z_\phi^\perp)$, inherit structures of a unitary $Z_\phi$-representation.

**Theorem 2.9.** For $\ell \geq 2$, as a unitary $Z_\phi$-representation, the space $H^1(\pi, g_\phi)$ decomposes into a direct sum of $\ell$ copies of $z_\phi \otimes \mathbb{C}$ and $\ell - 1$ copies of $z_\phi^\perp \otimes \mathbb{C}$, both $z_\phi \otimes \mathbb{C}$ and $z_\phi^\perp \otimes \mathbb{C}$ being viewed as unitary $Z_\phi$-representations in the obvious way.

**Corollary 2.10.** Under the circumstances of the Theorem, as far as the resulting symplectic structure is concerned, viewed as an affine symplectic manifold with a Hamiltonian $Z_\phi$-action, the space $H^1(\pi, g_\phi)$ decomposes into a product

\[ (z_\phi^\ell) \times (z_\phi^\perp)^{\ell-1} \times i ((z_\phi^\ell) \times (z_\phi^\perp)^{\ell-1}) \]

of two Lagrangian subspaces, and hence looks like a cotangent bundle.

To prove the theorem we observe first that $z_\phi$ is a trivial $\pi$-module in such a way that $z_\phi = H^0(\pi, z_\phi)$ whence the canonical map from $H^2(\pi, z_\phi)$ to $H^2(\pi, g_\phi)$ is an
isomorphism and, furthermore, $H^0(\pi, z_\phi^\bot)$ and $H^0(\pi, z_\phi^\bot)$ are both zero. Consequently the standard chain complex computing $H^*(\pi, g_\phi)$ decomposes into the two chain complexes
\[
z_\phi \xrightarrow{d^0} z_\phi^2 \xrightarrow{d^1} z_\phi, \quad z_\phi^\bot \xrightarrow{d^0} (z_\phi^\bot)^{2\ell} \xrightarrow{d^1} z_\phi^\bot,
\]
the former having both $d^0$ and $d^1$ zero while the latter having $d^0$ injective and $d^1$ surjective. In particular, as a symplectic, in fact Hermitian vector space, $H^1(\pi, g_\phi)$ decomposes into the direct sum of $H^1(\pi, z_\phi)$ and $H^1(\pi, z_\phi^\bot)$ in such a way that (i) $H^1(\pi, z_\phi)$ amounts to a direct sum of $2\ell$ copies of $z_\phi$ and (ii) the real dimension of $H^1(\pi, z_\phi^\bot)$ equals $2(\ell - 1)$ times the real dimension of $z_\phi^\bot$. To complete the argument for the theorem it will therefore suffice to prove the following.

**Lemma 2.11.** Suppose that $H^0(\pi, M)$ is zero. Then, as a real unitary $K$-representation, $H^1(\pi, M)$ is isomorphic to a direct sum of $\ell - 1$ copies of $M \otimes C$ with the obvious structure of unitary $K$-representation.

**Proof.** Since $H^0(\pi, M)$ is zero the canonical decomposition (2.7) of real orthogonal $K$-representations implies at once that, as real orthogonal $K$-representations, $H^1(\pi, M)$ and $M^2(\ell - 1)$ are isomorphic. However the complex, euclidean and symplectic structures respectively $\ast$, $\cdot$, and $\sigma$, on $H^1(\pi, M)$ are related by
\[
u \cdot v = \sigma(u, \ast v), \quad u, v \in H^1(\pi, M).
\]
Hence as a hermitian space, $H^1(\pi, M)$ is isomorphic to a direct sum of $\ell - 1$ copies of $M \oplus iM = M \otimes C$. $\square$

3. The symplectic links of the points in the middle stratum

The purpose of this Section is to prove the following.

**Lemma 3.1.** Topologically, the points of $N(T)$ are non-singular in $N$ if and only if $\Sigma$ has genus $\ell = 2$. Consequently the moduli space $N$ is smooth only if $\Sigma$ has genus $\ell = 2$.

As a $T$-module, $g = t \oplus t^\bot$, the direct sum of the Lie algebra $t \cong R$ of $T$ and its orthogonal complement $t^\bot \cong R^2$, and, by construction, $T$ acts on $t^\bot \cong R^2$ through the 2-fold covering map $T \to SO(2, R)$, the group $SO(2, R)$ being identified with that of rotations of $t^\bot \cong R^2$. In view of what was said in the previous Section, as a unitary $S^1$-representation, the space $H^1(\pi, g_\phi)$ decomposes into a direct sum
\[
H^1(\pi, g_\phi) \cong (t \otimes C)^{\ell} \oplus (t^\bot \otimes C)^{\ell - 1}, \tag{3.2}
\]
that on $(t \otimes C)^{\ell}$ is manifestly trivial, and we are left with the unitary $S^1$-representation on $(t^\bot \otimes C)^{\ell - 1}$ and its momentum mapping. By standard principles the momentum mapping is determined by the representation. Clearly it will suffice to examine the corresponding $SO(2, R)$-representation.

On a single copy of
\[
C^2 = t^\bot \oplus i t^\bot = R^2 \oplus iR^2, \tag{3.3}
\]
the requisite unitary SO(2, \mathbb{R})-representation amounts to rotation on the real and imaginary summands in the decomposition (3.3). However, under the canonical identification of SO(2, \mathbb{R}) with the group U(1), in a suitable basis of \mathbb{C}^2, the representation is given by the assignment of \[
abla \begin{bmatrix} \zeta & 0 \\ 0 & \zeta \end{bmatrix}
abla \text{ to } \zeta \in \text{U}(1). \]
Hence, with the notation \( n = \ell - 1 \),
the resulting unitary representation of SO(2, \mathbb{R}) on \( H^1(\pi, \ell) \cong (\mathbb{C}^2)^n \) looks like
\[
\zeta \mapsto \left( \begin{bmatrix} \zeta & 0 \\ 0 & \zeta \end{bmatrix}, \ldots, \begin{bmatrix} \zeta & 0 \\ 0 & \zeta \end{bmatrix} \right), \quad \text{for } \zeta \in \text{U}(1).
\]
Its momentum mapping \( \mu : (\mathbb{C}^2)^n \to \mathbb{R} \) is given by
\[
\mu(w_1, z_1, \ldots, w_n, z_n) = \frac{1}{2} \left( |z_1|^2 + \cdots + |z_n|^2 - (|w_1|^2 + \cdots + |w_n|^2) \right)
\]
whence the zero locus \( \mu^{-1}(0) \) consists of the \( 2n \)-tuples \((w_1, z_1, \ldots, w_n, z_n) \in (\mathbb{C}^2)^n\) satisfying the equation
\[
|z_1|^2 + \cdots + |z_n|^2 = |w_1|^2 + \cdots + |w_n|^2.
\]
Consequently the intersection of the zero locus \( \mu^{-1}(0) \) with the unit sphere \( S^{4n-1} \subseteq (\mathbb{C}^2)^n \) is the product \( S^{2n-1} \times S^{2n-1} \) of two spheres of radius \( \frac{1}{2} \), and the circle group \( S^1 \) acts diagonally thereupon. Thus the symplectic link \( L \) looks like \( L = S^{2n-1} \times \text{S}^1 \times S^{2n-1} \) and, moreover, fits into a fibre bundle over \( P_{n-1}\mathbb{C} \times P_{n-1}\mathbb{C} \), with fibre a circle. In particular we see that \( L \) is a sphere, in fact a circle, if and only if \( n = 1 \), that is, \( \ell = 2 \). Thus, topologically, the points of \( N(T) \) are non-singular in \( N \) if and only if the genus of \( \Sigma \) equals two. This proves the Lemma.

4. Symplectic and categorical quotients

We reproduce a special case of a result due to Kempf-Ness and Kirwan.

Let \( E \) be a finite dimensional unitary representation of a compact Lie group \( K \). The \( K \)-action on \( E \) extends to a complex representation of its complexification \( K^\mathbb{C} \). Associated with it is the well known unique momentum mapping \( \mu : E \to k^* \) vanishing at the origin which, after a choice of basis so that \( E \cong \mathbb{C}^m \), looks like
\[
(4.1) \quad \mu^*(z) = \frac{i}{2} \sum_{j,k}^n x_{j,k} z_j \bar{z}_k, \quad \text{where } z = (z_1, \ldots, z_m) \in E, \quad x = [x_{j,k}] \in k.
\]

Lemma 4.2. The canonical map \( E//K \to E//K^\mathbb{C} \) from the symplectic quotient \( E//K = \mu^{-1}(0)/K \) to the (affine) categorical quotient \( E//K^\mathbb{C} \) induced by the inclusion of \( \mu^{-1}(0) \) into \( E \) is a homeomorphism.

Proof. Injectivity is established in [15] (7.2) while the surjectivity may be found in [14]. \( \square \)

5. Some classical invariant theory

We shall determine the requisite categorical quotients by means of classical invariant theory. We briefly reproduce the necessary results from Weyl [22], simultaneously over \( \mathbb{R} \) and \( \mathbb{C} \), and we write \( K \) for either ground field, to have a neutral notation:
5.1. Let $n \geq 2$ and $\ell \geq 1$. The following is a complete list of invariants for the $\text{SO}(n, K)$-action on $(K^n)^\ell$:

- (5.1.1) the $\frac{\ell(\ell+1)}{2}$ inner products $u_j u_k$, where $u_j, u_k \in K^n, 1 \leq j \leq k \leq \ell$;
- (5.1.2) the $\binom{\ell}{n}$ determinants $|u_j u_{j_2} \ldots u_{j_n}|$ where $u_{j_1}, u_{j_2}, \ldots, u_{j_n} \in K^n, 1 \leq j_1 < j_2 < \cdots < j_n \leq \ell$; these, of course, may be non-zero only if $\ell \geq n$.

When $\ell < n$ these invariants are independent. When $\ell \geq n$, there are
- $\binom{\ell}{n}$ relations of the kind

$$|u_{j_1} u_{j_2} \ldots u_{j_n}| |v_{j_1} v_{j_2} \ldots v_{j_n}| = \begin{vmatrix} u_{j_1} v_{j_1} & u_{j_1} v_{j_2} & \cdots & u_{j_1} v_{j_n} \\ u_{j_2} v_{j_1} & u_{j_2} v_{j_2} & \cdots & u_{j_2} v_{j_n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{j_n} v_{j_1} & u_{j_n} v_{j_2} & \cdots & u_{j_n} v_{j_n} \end{vmatrix}$$

and, when $\ell > n$, there are
- $\binom{\ell}{n+1}^2$ additional relations of the kind

$$\begin{vmatrix} u_{j_0} v_{j_0} & u_{j_0} v_{j_1} & \cdots & u_{j_0} v_{j_n} \\ u_{j_1} v_{j_0} & u_{j_1} v_{j_1} & \cdots & u_{j_1} v_{j_n} \\ u_{j_2} v_{j_0} & u_{j_2} v_{j_1} & \cdots & u_{j_2} v_{j_n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{j_n} v_{j_0} & u_{j_n} v_{j_1} & \cdots & u_{j_n} v_{j_n} \end{vmatrix} = 0.$$

These relations constitute a complete set.

To spell out a crucial consequence thereof for our purposes, write $S^2(K^\ell)$ for the second symmetric power of $K^\ell$ and $\Lambda^n(K^\ell)$ for the $n'$th exterior power of $K^\ell$. Moreover, we denote by

(5.2) $J: (C^n)^\ell \to \text{so}(n, R)^*$

the momentum mapping for the diagonal action of $\text{SO}(n, R)$ on $(C^n)^\ell$, with its canonical symplectic structure.

Theorem 5.3. For $n \geq 2$ and $\ell \geq 1$, the invariants (5.1.1) and (5.1.2) induce an $\text{SO}(n, C)$-invariant map

(5.3.1) $(C^n)^\ell \to S^2(C^\ell) \times \Lambda^n(C^\ell)$

which induces an embedding

(5.3.2) $J^{-1}(0)/\text{SO}(n, R) \to S^2(C^\ell) \times \Lambda^n(C^\ell)$

of the reduced space $J^{-1}(0)/\text{SO}(n, R)$ into $S^2(C^\ell) \times \Lambda^n(C^\ell)$ as the complex affine algebraic set described by the equations (5.1.3) and (5.1.4).

Proof. In fact, (5.3.1) induces an embedding of the categorical quotient $(C^n)^\ell/\text{SO}(n, C)$ into $S^2(C^\ell) \times \Lambda^n(C^\ell)$ as the complex affine algebraic set described
by the equations (5.1.3) and (5.1.4). By (4.3), the canonical map identifies the symplectic quotient with the categorical one. □

6. The middle stratum

Consider a homomorphism $\phi: \pi \to G$ representing a point of $N(T)$. From (3.2) we know already that, as a $T$-representation, $H^1(\pi, g_{\phi})$ decomposes into a direct sum of $(t \otimes C)^{\ell}$ and $(t^\perp \otimes C)^{\ell-1}$ in such a way that the $T$-representation on $(t \otimes C)^{\ell}$ is trivial. Moreover $t^\perp$ is just the ordinary plane $\mathbb{R}^2$, and the circle group $T$ acts on it through a 2-fold covering map onto the group $SO(2, \mathbb{R})$. The $SO(2, \mathbb{R})$-representation on $(t^\perp \otimes C)^{\ell-1}$, in turn, amounts to the classical constrained system of $\ell - 1$ particles moving in the plane with total angular momentum zero. This proves the corresponding assertion in Theorem 1. With the notation $C^2 = \mathbb{R}^2 \oplus i \mathbb{R}^2$, the momentum mapping of this system looks like

$$\mu: (C^2)^{\ell-1} \to \mathbb{R}, \quad \mu(q_1 + ip_1, \ldots, q_{\ell-1} + ip_{\ell-1}) = |q_1 p_1| + \cdots + |q_{\ell-1} p_{\ell-1}|.$$

Furthermore, the extension of the $SO(2, \mathbb{R})$-representation to the complexification $SO(2, \mathbb{C})$ of $SO(2, \mathbb{R})$ is just the standard diagonal action of $SO(2, \mathbb{C})$ on $\ell - 1$ copies of $C^2$, and hence the embedding (5.3.2) identifies the reduced space as a complex affine algebraic subset of $S^2(C^{\ell-1}) \times \Lambda^2(C^{\ell-1})$.

We now examine the special case of genus $\ell = 2$. In this case, there is a single $SO(2, \mathbb{C})$-invariant of the kind (5.1.1); with the notation $w = q + ip \in \mathbb{R}^2 \oplus i \mathbb{R}^2 = C^2$,

it looks like

$$ww = (q + ip)(q + ip) = qq - pp + 2ipq.$$

The holomorphic map from $C^2$ to $\mathbb{C}$ induced by this invariant, cf. (5.3.1), assigns $ww \in \mathbb{C}$ to $w \in C^2$ and identifies the reduced space $\mu^{-1}(0)/SO(2, \mathbb{R})$ with a copy of one-dimensional complex affine space $C$, that is, with the usual real affine plane $\mathbb{R}^2$. In particular, we see once more that, when the genus equals two, near a point of $N(T)$, the moduli space $N$ is smooth; however we cannot here conclude the converse also, that is, that $N$ is smooth near a point of $N(T)$ only if the genus equals two.

We now determine the reduced Poisson algebra near a point of $N(T)$ and make a few comments about the resulting Poisson geometry. Working with polynomial functions instead of smooth functions, following the construction (2.5), we must take the $SO(2, \mathbb{R})$-invariants on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ and thereafter divide out the appropriate ideal. We continue to write $w = (q, p) \in \mathbb{R}^2 \times \mathbb{R}^2$. By (5.1), the four $SO(2, \mathbb{R})$-invariants

$$qq, \ pp, \ qp, \ |qp|,$$

constitute a complete set; however since the invariant $|qp|$ vanishes on the zero locus, the three invariants $qq, \ pp, \ qp$ already generate the reduced Poisson algebra. Let

$$x_1 = qq - pp, \quad x_2 = 2qp, \quad r = qq + pp.$$
It is clear that these three functions also constitute a set of generators. Its advantage is that \(x_1\) and \(x_2\) are the obvious coordinate functions on the reduced space since, with the above notation,
\[
ww = x_1 + ix_2.
\]
Moreover, the zero locus \(\mu^{-1}(0)\) is defined by the equation \(|qp| = 0\) whence on the reduced space we have
\[
0 = |qp|^2 = (qq)(pp) - (qp)^2
\]
that is,
\[
x_2^2 = 4(qp)^2 = 4(qq)(pp) = (qq + pp)^2 - (qq - pp)^2 = r^2 - x_1^2;
\]
in other words, the third generator \(r\) amounts to the usual radius function in the plane with coordinates \(x_1, x_2\). Thus the reduced Poisson algebra has generators \(x_1, x_2, r\), subject to the relation \(x_1^2 + x_2^2 = r^2\); moreover, still by construction, the Poisson brackets among the generators are calculated in the unreduced Poisson algebra, that is, in the algebra of polynomials in the indeterminates \(q_1, q_2, p_1, p_2\), with the canonical symplectic Poisson brackets \(\{q_j, p_k\} = \delta_{j,k}\) etc. This yields
\[
\{x_1, x_2\} = 2r, \quad \{x_1, r\} = 2x_2, \quad \{x_2, r\} = -2x_1.
\]
Thus the reduced Poisson algebra is symplectic everywhere except at the origin and we see that, symplectically, or, more appropriately, in the Poisson sense, near a point \(Q\) of \(N(T)\), the space \(N\) looks like the product of a copy of \(\mathbb{R}^4\), endowed with the standard symplectic or Poisson structure, with a copy of \(\mathbb{R}^2\), endowed with the above Poisson structure. The copy \(\mathbb{R}^4\) corresponds to the stratum \(N(T)\). This shows in particular how the symplectic structure on the top stratum degenerates at the points of \(N(T)\). Notice however that, at a point of \(N(T)\), the function \(r\) is genuinely no longer smooth in the usual sense. To describe the mutual positions of the symplectic structures on the strata, we are thus forced to go beyond the usual smooth setting and have to admit continuous functions which are no longer smooth everywhere.

The complexified reduced Poisson algebra admits the following somewhat simpler description: It is generated by the three functions \(ww, \overline{ww}, w\overline{w}\), with Poisson brackets
\[
\{ww, \overline{ww}\} = -8iww, \quad \{ww, w\overline{w}\} = -4iww, \quad \{w\overline{w}, w\overline{w}\} = -4iww.
\]
Notice \(w\overline{w}\) coincides with the above radius function \(r\).

For genus \(\ell > 2\), we can still take the coordinate functions prescribed by the corresponding embedding given by the real \(\text{SO}(2, \mathbb{R})\)-invariants and calculate their Poisson brackets. This yields the reduced Poisson algebra in the general case. We refrain from spelling out the details.

7. The isolated points

To begin with, consider a surface \(\Sigma\) of arbitrary genus \(\ell \geq 2\). Let \(\phi: \pi \rightarrow G\) be a homomorphism whose values lie in the centre \(Z \subseteq G\), so that \(\phi\) represents one of the isolated points of \(N_G\). By (2.9), \(H^1(\pi, g)\) and \((g \otimes \mathbb{C})^\ell\) are isomorphic as real unitary \(G\)-representations. Hence the \(G\)-action on \(H^1(\pi, g)\) amounts to the standard diagonal action on \(\ell\) copies of \(\mathbb{R}^3 \oplus i\mathbb{R}^3 = \mathbb{C}^3\), with \(G\) acting on each copy of \(\mathbb{R}^3\) through
the 2-fold covering \( G \to \text{SO}(3, \mathbb{R}) \). This is just the classical constrained system of \( \ell \) particles moving in space with total angular momentum zero, except that its group is \( \text{SU}(2) \) and acts through the 2-fold covering \( \text{SU}(2) \to \text{SO}(3, \mathbb{R}) \). This establishes the corresponding assertion in Theorem 1. In particular, with the notation

\[
q + ip \in \mathbb{C}^3 = \mathbb{R}^3 \oplus i\mathbb{R}^3,
\]

the momentum mapping \( \Theta \) and reduced space \( H_\phi \) look like

\[
\begin{align*}
\Theta(q_1 + ip_1, \ldots, q_\ell + ip_\ell) &= q_1 \wedge p_1 + \cdots + q_\ell \wedge p_\ell, \\
H_\phi &= \Theta^{-1}(0)/\text{SU}(2) = \Theta^{-1}(0)/\text{SO}(3, \mathbb{R}).
\end{align*}
\]

Furthermore, the extension of the \( \text{SO}(3, \mathbb{R}) \)-representation to the complexification \( \text{SO}(3, \mathbb{C}) \) of \( \text{SO}(3, \mathbb{R}) \) is just the standard diagonal action of \( \text{SO}(3, \mathbb{C}) \) on \( \ell \) copies of \( \mathbb{C}^3 \), and (5.3.2) identifies \( H_\phi \) as a complex affine algebraic subset of \( S^2(\mathbb{C}^\ell) \times \Lambda^3(\mathbb{C}^\ell) \).

How does the middle stratum \( N(T) \) look like in this model of \( H_\phi \)?

**Proposition 7.3.** The middle stratum arises from the subspace of the zero locus \( \Theta^{-1}(0) \) consisting of \( \ell \)-tuples

\[
(\lambda_1 w, \ldots, \lambda_\ell w) \in \Theta^{-1}(0) \subseteq (\mathbb{C}^3)^\ell, \quad w \in \mathbb{C}^3, \quad \lambda_1, \ldots, \lambda_\ell \in \mathbb{C},
\]

and hence corresponds to the image thereof in \( S^2(\mathbb{C}^\ell) \times \Lambda^3(\mathbb{C}^\ell) \) under (5.3.2). Thus in the constrained system picture, the middle stratum corresponds to states different from the origin where each one of the \( \ell \) particles individually has angular momentum zero.

**Proof.** By construction, the elements of \( N(T) \) are represented by points in \((\mathbb{C}^3)^\ell\) having stabilizer conjugate to \( T \) and hence by \( \ell \)-tuples of the kind

\[
(w_1, \ldots, w_\ell) = (\psi_1 v + i\rho_1 v, \ldots, \psi_\ell v + i\rho_\ell v)
\]

\[
= ((\psi_1 + i\rho_1)v, \ldots, (\psi_\ell + i\rho_\ell)v) \in \Theta^{-1}(0) \subseteq (\mathbb{C}^3)^\ell,
\]

where \( v \in \mathbb{R}^3, \psi_j, \rho_j \in \mathbb{R} \). It is clear that these \( \ell \)-tuples \((w_1, \ldots, w_\ell)\) are of the asserted kind. Conversely, let

\[
(w_1, \ldots, w_\ell) = (\lambda_1 w, \ldots, \lambda_\ell w) \in \Theta^{-1}(0) \setminus \{0\}
\]

so that, for \( 1 \leq j \leq \ell \), with the notation \( \lambda_j = u_j + iv_j \),

\[
w_j = \lambda_j w = (u_j + iv_j)(q + ip) = u_j q - v_j p + i(v_j q + u_j p) = q_j + ip_j
\]

and \( q_j \wedge p_j = \lambda_j \overline{\lambda}_j q \wedge p \). However the constraint equation

\[
\Theta(q_1 + ip_1, \ldots, q_\ell + ip_\ell) = 0
\]

yields

\[
0 = q_1 \wedge p_1 + \cdots + q_\ell \wedge p_\ell = (\lambda_1 \overline{\lambda}_1 + \cdots + \lambda_\ell \overline{\lambda}_\ell) q \wedge p,
\]
that is \( p \) is proportional to \( q \) and hence each \( p_j \) to \( q_j \); in view of (7.4), each \( q_j \) is also proportional to \( q \) and each \( p_j \) to \( p \), that is, the \( 2\ell \)-tuple \((q_1, p_1, \ldots q_\ell, p_\ell)\) looks like
\[
(q_1, p_1, \ldots q_\ell, p_\ell) = (\psi_1 v, \rho_1 v, \ldots, \psi_\ell v, \rho_\ell v), \quad v \in \mathbb{R}^3, \quad \psi_j, \rho_j \in \mathbb{R},
\]
and hence has stabilizer conjugate to \( T \). □

We now examine the special case of genus \( \ell = 2 \) in detail. By (5.1), the three holomorphic invariants
\[
(7.5) \quad w_1 w_1, \ w_1 w_2, \ w_2 w_2
\]
constitute a complete set of invariants for the \( \text{SO}(3, \mathbb{C}) \)-action on \((\mathbb{C}^3)^2\), and these are independent. Hence the categorical quotient is just a three-dimensional complex affine space and, by virtue of (5.3), the \( \text{SO}(3, \mathbb{C}) \)-invariant holomorphic map
\[
(7.6) \quad (\mathbb{C}^3)^2 \to \mathbb{C}^3, \quad (w_1, w_2) \mapsto (w_1 w_1, w_1 w_2, w_2 w_2),
\]
cf. (5.3.1), induces a homeomorphism
\[
(7.7) \quad H_\phi = \Theta^{-1}(0)/\text{SO}(3, \mathbb{R}) \to \mathbb{C}^3.
\]
In particular, this shows that, in genus two, near any of the 16 isolated points of \( N_G \), the moduli space \( N \) is smooth, that is, the proof of Theorem 2 is now complete.

**Proposition 7.8.** For genus two, in the model \( \mathbb{C}^3 \) for the reduced space \( H_\phi \), the middle stratum \( N(\Theta(T)) \) corresponds to the complex quadric
\[
(7.8.1) \quad Q = \{ (x, y, z) \in \mathbb{C}^3; \ y^2 = xz \}
\]
with the origin removed.

*Proof.* This is immediate from (7.4). In fact, under (7.6), the subspace consisting of pairs \((\lambda_1 w, \lambda_2 w)\) is mapped onto the asserted quadric. □

We now determine the corresponding reduced Poisson algebra. Working with polynomial functions instead of smooth functions, following the construction (2.5), we must take the \( \text{SO}(3, \mathbb{R}) \)-invariants on \((\mathbb{R}^3)^4\) and thereafter divide out the appropriate ideal. We continue to write \((q_1, p_1, q_2, p_2) \in (\mathbb{R}^3)^4\). In view of (5.1) above, the ten distinct invariants
\[
(7.9) \quad q_i q_j, \ q_i p_j, \ p_i p_j, \quad 1 \leq i, j \leq 2,
\]
among the scalar products, together with the four determinants
\[
(7.10) \quad |q_1 p_1 q_2|, \ |q_1 p_1 p_2|, \ |q_1 q_2 p_2|, \ |p_1 q_2 p_2|,
\]
constitute a complete set of invariants for the \( \text{SO}(3, \mathbb{R}) \)-action on \((\mathbb{R}^3)^4\). However, for \((q_1 + ip_1, q_2 + ip_2) \in \Theta^{-1}(0)\), that is, when
\[
\Theta(q_1 + ip_1, q_2 + ip_2) = q_1 \wedge p_1 + q_2 \wedge p_2 = 0,
\]
any three of \((q_1, p_1, q_2, p_2)\) are linearly dependent, whence the four determinants (7.10) vanish on the zero locus \(\Theta^{-1}(0)\), and the ten scalar products (7.9) induce an embedding

\[
H_\phi \to S^2(\mathbb{R}^4)
\]

of the reduced space \(H_\phi\) into the 10-dimensional real vector space \(S^2(\mathbb{R}^4)\) as a real semi-algebraic set. More details are given in Section 12 of our paper [10]. Hence our reduced Poisson algebra of continuous functions will be generated by the ten scalar product invariants (7.9). It is straightforward to calculate the Poisson brackets between these functions; in fact, cf. (2.5), this is done in the unreduced symplectic Poisson algebra generated by the coordinate functions on \((\mathbb{R}^3)^4\), and we have

\[
\{q_1q_1, q_1p_1\} = 2q_1q_1, \quad \{q_1p_2, q_1p_1\} = q_1p_2,
\]

e tc.

As a real semi-algebraic subset of \(S^2(\mathbb{R}^4)\), the reduced space looks more complicated than as an affine complex algebraic set. On the other hand, the complex algebraic structure ignores the reduced Poisson algebra since this algebra involves \textit{additional} functions which in the complex algebraic picture are \textit{not} visible. In particular, not all of the ten generators of the reduced Poisson algebra will be smooth functions on \(C^3\) in the ordinary sense.

Remark 7.14. In (5.4) of [16] it is asserted that, for arbitrary \(\ell\), the \(\text{SO}(3, \mathbb{R})\)-reduced space is a double branched cover of the \(\text{O}(3, \mathbb{R})\)-reduced space. Under the present circumstances, \(\ell = 2\) and the two spaces actually coincide, that is, the space in fact coincides with its branching locus. The reason is that the determinant invariants (7.10) which distinguish between the \(\text{O}(3, \mathbb{R})\)- and \(\text{SO}(3, \mathbb{R})\)-reduced spaces vanish on the zero locus of the momentum mapping which is the same for the two. Moreover the results of [16] imply that, as a stratified symplectic space, \(H_\phi\) may be identified with the closure of a certain a nilpotent orbit in \(\text{sp}(2, \mathbb{R})^*\). This orbit is made precise in Section 12 of [10]. Moreover, the ten invariants (7.9) may be viewed as a basis of \(\text{sp}(2, \mathbb{R})\), and the Poisson brackets (7.12) are given by the Lie bracket of \(\text{sp}(2, \mathbb{R})\).

We summarize the resulting \textit{Poisson geometry} of the moduli space \(\mathcal{N}\) near any of the isolated points in the following.

\textbf{Theorem 7.15.} In the model \(\mathbb{C}^3\) for the reduced space \(H_\phi\), away from the quadric (7.8.1), the Poisson structure has maximal rank equal to six, while at a point on the quadric (7.8.1), the Poisson structure has rank four except at the origin where it has rank zero. Moreover the ten Hamiltonian vector fields of the generators of the Poisson algebra, restricted to the quadric (7.8.1), are tangential to the quadric except at the origin where they all vanish.

We note that a Hamiltonian vector field is not necessarily smooth but under the present circumstances can still be written in terms of suitable frames with continuous coefficients which are smooth on each stratum.

The Theorem is, of course, an immediate consequence of properties of our local model \(H_\phi\). However it is instructive to verify its statement directly as a \textit{formal consequence} of properties of the reduced Poisson algebra of functions on the model \(\mathbb{C}^3\) of \(H_\phi\). We now explain this.
With the ten scalar product invariant generators (7.9) for the reduced Poisson algebra it seems hard to visualize the resulting Poisson geometry directly. However, after complexification, the reduced Poisson algebra admits the following much more perspicuous description: We may take the six SO(3, \mathbb{R})-invariants (7.16)

\[ w_1w_1, \ w_1w_2, \ w_2w_2, \ w_1\overline{w}_1, \ w_1\overline{w}_2, \ w_2\overline{w}_2 \]

as independent (complex) coordinate functions, and these six invariants, together with the four SO(3, \mathbb{R})-invariants (7.17)

\[ w_1w_1, \ w_1\overline{w}_2, \ w_1\overline{w}_2, \ w_2\overline{w}_2 \]

generate the same algebra of continuous complex valued functions on the reduced space as the ten scalar product invariants (7.9), viewed as complex valued functions. We note that the functions (7.17) are not smooth on the reduced space in the ordinary sense.

A straightforward calculation in the unreduced Poisson algebra yields the following Poisson brackets:

\[
\begin{align*}
\{w_1w_1, w_1w_1\} &= -8iw_1\overline{w}_1 \\
\{w_1w_1, w_1w_2\} &= -4iw_1\overline{w}_2 \\
\{w_1w_2, w_1w_1\} &= -4i\overline{w}_1w_2 \\
\{w_1w_2, w_1w_2\} &= -2i(w_1\overline{w}_1 + w_2\overline{w}_2) \\
\{w_1w_2, w_1w_2\} &= -4iw_1\overline{w}_2 \\
\{w_2w_2, w_1w_2\} &= -4i\overline{w}_1w_2 \\
\{w_2w_2, w_2w_2\} &= -8iw_2\overline{w}_2 \\
\{\overline{w}_1w_1, w_1w_2\} &= 4i\overline{w}_1w_2 \\
\{\overline{w}_1w_2, w_1w_1\} &= 4iw_1\overline{w}_2 \\
\{\overline{w}_1w_2, w_1w_2\} &= 2i(w_1\overline{w}_1 + w_2\overline{w}_2) \\
\{\overline{w}_1w_2, w_2w_2\} &= 4i\overline{w}_1w_2 \\
\{w_2\overline{w}_2, w_1w_2\} &= 4iw_1\overline{w}_2,
\end{align*}
\]

still with the convention that Poisson brackets between coordinate functions not spelled out explicitly are zero. From these we derive at first the six Hamiltonian vector fields of the six chosen coordinate functions (7.16). With the notation

\[ \alpha = w_1\overline{w}_1, \ \beta = 2w_1\overline{w}_2, \ \gamma = w_2\overline{w}_2, \]

in the above chosen order of coordinate functions, the system of Hamiltonian vector fields of the functions (7.16) is given by the matrix

\[
A = 2i \begin{bmatrix}
0 & 0 & 0 & 4\alpha & \beta & 0 \\
0 & 0 & 0 & \overline{\beta} & \alpha + \gamma & \beta \\
0 & 0 & 0 & \overline{\beta} & \beta & 4\gamma \\
-4\alpha & -\beta & 0 & 0 & 0 & 0 \\
-\beta & -(\alpha + \gamma) & -\overline{\beta} & 0 & 0 & 0 \\
0 & -\beta & -4\gamma & 0 & 0 & 0
\end{bmatrix}.
\]
Thus the matrix has rank six if and only if
\[
\begin{vmatrix}
4\alpha & \beta & 0 \\
\beta & \alpha + \gamma & \beta \\
0 & \beta & 4\gamma \\
\end{vmatrix} = 4(\alpha + \gamma)(4\alpha\gamma - \beta^2) \neq 0.
\]

However
\[
\alpha + \gamma = w_1\overline{w}_1 + w_2\overline{w}_2 = q_1q_1 + p_1p_1 + q_2q_2 + p_2p_2,
\]
\[
4\alpha\gamma - \beta^2 = 4((w_1\overline{w}_1)(w_2\overline{w}_2) - (w_1\overline{w}_2)(\overline{w}_1w_2)) = 4(w_1 \wedge w_2)(\overline{w}_1 \wedge \overline{w}_2).
\]

Hence the matrix is non-singular if and only if
\[
q_1q_1 + p_1p_1 + q_2q_2 + p_2p_2 \neq 0
\]
and if \(w_1\) and \(w_2\) are linearly independent in \(\mathbb{C}^3\). In other words, the matrix has full rank on every point of the top stratum. Furthermore, the matrix is zero if and only if
\[
q_1q_1 + p_1p_1 + q_2q_2 + p_2p_2 = 0;
\]
this corresponds to the image of the origin in the reduced space and hence to a point in the stratum \(N_G\). Finally a little thought reveals that the matrix has rank equal to 4 if and only of \(w_1\) and \(w_2\) are linearly dependent in \(\mathbb{C}^3\). In view of (7.4), the points having this property correspond exactly to the remaining stratum, given by orbit type \((T)\) examined earlier. However, cf. (7.8), this corresponds to the complex quadric \(y^2 = zx\).

To get a complete picture, we must examine the Poisson brackets with the four remaining \(SO(3,\mathbb{R})\)-invariants (7.17) and their Hamiltonian vector fields. This does not lead to any new difficulty, and we leave the details to the reader. This completes our discussion of the local Poisson geometry of the moduli space of flat \(SU(2)\)-connections.

For genus \(\ell > 2\), we can proceed in much the same way and describe the complexified reduced Poisson algebra and corresponding Poisson geometry. We refrain from spelling out the details here.

We conclude with a comment about the geometric significance of the Poisson structure: In the above model of the genus two moduli space \(N\) near any of the isolated points, the non-singular part of the complex quadric \(Q \subseteq \mathbb{C}^3\) given by the equation \(y^2 = xz\) comes of course with its standard Kähler and hence symplectic structure, and the latter gives rise to the corresponding symplectic Poisson structure. However this Poisson structure does not arise from restriction to \(Q\) of the corresponding standard symplectic Poisson structure on the ambient space \(\mathbb{C}^3\). In fact, in the language of theoretical physics, the quadric \(Q\) is a second class constraint, and to get the correct Poisson brackets on \(Q\) one has to introduce Dirac brackets on the ambient space. On the other hand, in our approach, the Poisson algebra on \(Q\) does arise from restriction of our Poisson algebra on the ambient space \(\mathbb{C}^3\). In other words, the ideal of functions in this algebra that vanish on \(Q\) is a Poisson ideal. Thus we see explicitly how the Poisson structure encapsulates the mutual positions of the symplectic structures on the strata.
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