Part II: A Practical Approach for Successive Omniscience

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Abstract—In Part I, we studied the communication for omniscience (CO) problem and proposed a parametric (PAR) algorithm to determine the minimum sum-rate at which a set of users indexed by a finite set \( V \) attain omniscience. The omniscience in CO refers to the status that each user in \( V \) recovers the observations of a multiple random source. It is called the global omniscience in this paper in contrast to the study of the successive omniscience (SO), where the local omniscience is attained subsequently in user subsets. We apply the PAR algorithm to search a complimentary subset \( X_\ast \subseteq V \) such that if the local omniscience in \( X_\ast \) is reached first, the global omniscience whereafter can still be attained with the minimum sum-rate. We further utilize the outputs of the PAR algorithm to outline a multi-stage SO approach that is characterized by \( K \leq |V| - 1 \) complimentary subsets \( X(k), \forall k \in \{1, \ldots, K\} \) forming a nesting sequence \( X(k) \subsetneq \ldots \subsetneq X(K) = V \). Starting from stage \( k = 1 \), the local omniscience in \( X(k) \) is attained at each stage \( k \) until the final global omniscience in \( X(K) = V \). A \(|X(k)|\)-dimensional local omniscience achievable rate vector is also derived for each stage \( k \) designating individual users' transmitting rates. The sum-rate of this rate vector in the last stage \( K \) coincides with the minimized sum-rate for the global omniscience.

Index Terms—Communication for omniscience, successive omniscience, Dillworth truncation, submodularity.

I. INTRODUCTION

For the users in a finite set \( V \) observing distinct terminals of a multiple random source in private, the communication for omniscience (CO) problem in [3] studied how to let users exchange data over broadcast channels to share the knowledge of the entire source. The minimum sum-rate problem in [4]–[2] aims at the determination of the least sum-rate for users in \( V \) to attain omniscience, the state that each user recovers the observation sequence of the entire source, and a corresponding optimal rate vector designating the transmission rates for each user. While the CO problem [3] considers the omniscience problem in a one-off manner, the idea of successive omniscience (SO) is proposed in [8]–[10] revealing that the state of omniscience can be reached in a two-stage manner: let a user subset \( X \subseteq V \) exchange the data first to attain omniscience and the rest of the users overhear the communications; then solve the global omniscience problem in \( V \). By recursively applying the two-stage SO approach, the omniscience in \( V \) can be attained in a multi-stage manner.

In [11], the concept of SO has been applied to the coded cooperative data exchange (CCDE), an application of CO in wireless communications. A multi-stage SO process was outlined based on a given user subset sequence specifying which group of users to transmit and attain omniscience in each stage. The problem of determining a local omniscience achievable rate vector for each stage was formulated and solved as a constrained multi-objective optimization problem. But, it is shown in [8, 9] that there is a particular group of complimentary user subsets so that the local omniscience in any of them can be attained first while the overall communication rates for the global omniscience whereafter still remains minimized. By knowing that not all user subsets are complimentary, if a non-complimentary subset reaches local omniscience first, e.g., in the solutions to SO in [11], the users might need to transmit more than the minimum sum-rate to attain the global omniscience finally. Therefore, the essential problem in the two-stage SO is not to determine a local omniscience achievable vector, but how to choose a user subset \( X_\ast \subseteq V \) that is complimentary such that the optimality of the global omniscience at the second stage is still maintained.

For a user subset \( X_\ast \subseteq V \) to be complimentary, the necessary and sufficient condition was derived in [9, Theorems 4.2 and 5.2] for the asymptotic and non-asymptotic models, where the communication rates are real-valued and integer-valued, respectively. However, [9, Theorems 4.2 and 5.2] are based on the value of the minimum sum-rate for the global omniscience, which is already computationally complex to determine. Meanwhile, the studies on the universal multi-party data exchange problem in [13]–[15] suggest letting users adaptively increase their transmission rates and running an ideal decoder at the same time to keep searching for the user subset that reaches the omniscience state. The recursive application of this process in [14, Protocol 3] results in a multi-stage SO. However, this method requires extra scheduling overheads, e.g., ordering transmission turns based on the information amount (entropy) of individual users’ observations and repetitively checking a so-called constant difference property to determine when a user should transmit. In addition, the ideal decoder needs to be run on line, which also incurs communication overheads between users, e.g., sending ACK/NACK signals. Thus, the current literature is missing an efficient overall scheduling of the multi-stage SO.

The complexity of the omniscience problem is polynomially growing with the dimension, the number of users \(|V|\), of the system [12]. Therefore, attaining omniscience in a user subset is less complex than the global one. This is also a motivation to study the SO problem.
before the transmissions actually take place. More specifically, this
scheduling refers to the design of the $K \leq |V| - 1$ stages,
for each of which, a complimentary user subset $X^{(k)}_\alpha$ that
holds the condition in [9] Theorems 4.2 and 5.2 is selected and a
rate vector $r^{(k)}_\alpha = (r^{(k)}_i : i \in V)$ is determined with its
reduction/projection $r^{(k)}_{X^{(k)}_\alpha}$ on $X^{(k)}_\alpha$ being an achievable local
omniscience vector. In addition, the $X^{(k)}_\alpha$ in the last stage
must equal $V$ and $r_{X^{(k)}_\alpha}$ is an optimal rate vector that attains
global omniscience with the minimum sum-rate.

A. Contributions

In this paper, we efficiently solve the SO problem for both
asymptotic and non-asymptotic source models. In each iteration $i$, the
PAR algorithm [12] Algorithm 2 updates $Q_{\alpha,V_i}$, a partition of
the users in $V_i$, and a rate vector $r_{\alpha,V_i} = (r_{\alpha,i'} : i' \in V_i)$
for all values of the minimum sum-rate estimate $\alpha$ for
the global omniscience problem. Here, $V_i$ for $i \in \{1, \ldots, |V|\}$
contains the first $i$ users based on an ordering of user indices.
Throughout the PAR algorithm, the value of the partition
$Q_{\alpha,V_i}$ is segmented in $\alpha$ and each dimension $r_{\alpha,i}$ of the
rate vector $r_{\alpha,\cdot}$ is piecewise linear in $\alpha$. At the end of the
last iteration $i = |V|$, $Q_{\alpha,V}$ and $r_{\alpha,V}$ are obtained, where
the partition $Q_{\alpha,V}$ is segmented in $\alpha$ by $p < |V|$ critical or turning
points in $\{ \alpha(j) : j \in \{0, \ldots, p-1\} \}$. While Part I [12] applies
the value of $Q_{\alpha,V}$ and $r_{\alpha,V}$ at the first critical point $\alpha = \alpha(1)$
to solve the global omniscience problem, this paper utilizes
$Q_{\alpha,V}$ and $r_{\alpha,V}$ at each iteration $i$ to solve the two-stage SO
problem and $Q_{\alpha,V}$ and $r_{\alpha,V}$ in the last iteration to outline a
multi-stage SO solution.

We first consider the problem of how to efficiently search a
complimentary user subset $X_\alpha \subseteq V$ for the two-stage SO. We
relax the necessary and sufficient condition in [9] Theorems
4.2 and 5.2 to sufficient condition based on a lower bound $\alpha$
on the minimum sum-rate for the global omniscience. This
lower bound can be determined in $O(|V|)$ time. This sufficient condition
is used to prove that, at each iteration $i$ of the
PAR algorithm, any non-singleton user subset contained in the
partition $Q_{\omega,V_i}$ is complimentary. Here, $Q_{\omega,V}$ is the value
of $Q_{\alpha,V_i}$ at $\alpha = \omega$. Once the complimentary subset $X_\alpha$
is chosen as any of the non-singleton subsets in $Q_{\omega,V_i}$ at some
iteration $i$, a local omniscience achievable rate vector $r_{X_\alpha}$
can be determined simultaneously from $r_{\alpha,\cdot}$.

We provide two ways for determining a solution to multi-
stage SO. The first method is to recursively apply the PAR
algorithm to choose $X_\alpha$, let users in $X_\alpha$ transmit at the rates
$r_{X_\alpha}$ and merge them to a super-user after the local omniscience
is reached. Without incurring any transmissions from the users,
the second method outlines a multi-stage SO process based
on the values of $Q_{\alpha,V}$ and $r_{\alpha,V}$ at the end of the PAR
algorithm. For the asymptotic source model, a $p$-stage SO is
determined from the critical points as follows. For each stage
$k \in \{1, \ldots, p\}$, a complimentary $X^{(k)}_\alpha$ is extracted as a non-
singleton user subset of the partition $Q_{\alpha(p-k),V}$. By doing so,
all $X^{(k)}_\alpha$ form a nesting sequence $X^{(1)}_\alpha \subseteq \ldots \subseteq X^{(p)}_\alpha$
such that $X^{(p)}_\alpha = V$. For the complimentary subset $X^{(k)}_\alpha$ at
stage $k$, a rate vector $r^{(k)}_{X_\alpha}$ is also determined from $r_{\alpha,V}$ with
$r^{(k)}_{X^{(k)}_\alpha}$ designating transmission rates for all users in $X^{(k)}_\alpha$
that are sufficient to attain local omniscience in $X^{(k)}_\alpha$ and $r^{(k)}_i = 0$
for all $i \in V \setminus X^{(k)}_\alpha$ indicating that the rest of users are
merely overhearing, i.e., not transmitting. All $r^{(k)}_V$'s form a
nondecreasing sequence $r^{(1)}_V \leq \ldots \leq r^{(p)}_V$ such that $r^{(p)}_V$
is an optimal rate vector for attaining the global omniscience.
Here, the nesting property of $X^{(k)}_\alpha$ and the monotonicity
of $r^{(k)}_V$ guarantee the achievability of this multi-stage SO in
practice: start from stage $k = 1$; at each stage $k \in \{2, \ldots, p\}$,
a larger user subset $X^{(k)}_\alpha$ attains local omniscience by the
nonnegative transmission rates $r^{(k)}_V - r^{(k-1)}_V$. The number of
users reaching omniscience $|X^{(k)}_\alpha|$ increases with $k$ until the
global omniscience in $V$ is attained in the last stage $k = p$.

Similarly, for the non-asymptotic model, a $K$-stage SO with
$K \leq p$ can be determined by the PAR algorithm: $X^{(k)}_\alpha$
is extracted as a non-singleton user subset of the partition $Q_{\alpha,V}$
at the integer-valued $\alpha \in \{\alpha(p-k) : k \in \{1, \ldots, p\}\}$. The transmission rate vector $r^{(k)}_V$ for each $k$ can be determined
by one more call of the PAR algorithm. The study in this
paper shows that the SO problem in both asymptotic and
non-asymptotic models can be solved by the PAR algorithm
in $O(|V| \cdot SFM(|V|))$ time. Here, $SFM(|V|)$ denotes the
complexity of solving a submodular function minimization
(SFM) problem and is a polynomial function of $|V|$.

B. Organization

The rest of paper is organized as follows. The system
model and the SO problem are described in Section III
Section [13] derives the sufficient condition for a user subset
to be complimentary and Section IV shows how to implement
the two-stage SO in both asymptotic and non-asymptotic
models. In Section [V] we show how to extract the multi-stage SO
procedures from $Q_{\alpha,V}$ and $r_{\alpha,V}$ for asymptotic and non-
asymptotic models.

II. SYSTEM MODEL

Let the ground set $V$ with $|V| > 1$ contain all users in the
system. The multiple random source is denoted by $Z_V = (Z_i : \ i \in V)$
with each $Z_i$ being a discrete random variable. User
$i$ privately observes an $n$-sequence $Z^n_i$ of $Z_i$. The users are
allowed to exchange their data to help each other recover the
observation of the source. For a user subset $X \subseteq V$, the state
that each user in $X$ recovers the observation sequence $Z^n_X$
of $Z_X$ is called the local omniscience in $X$. In the case when
$X = V$, we say that the global omniscience is attained.

For the local omniscience in the user subset $X \subseteq V$, we
briefly summarize the results on the CO problem in [12]
Section [22]). For a vector $r_X = (r_i : i \in X)$ with each $r_i$
denoting the rate at which user $i$ broadcasts/transmits, $r_X$ is
called a local omniscience achievable rate vector if all the
users in $X$ are able to recover $Z^n_X$ after transmitting at the
rates $r_X$. The corresponding local omniscience achievable rate
region is [3]. [16], [17]:

$$\mathcal{R}_{CO}(X) = \{ r_X \in \mathbb{R}^{|X|} : r(C) \geq H(C|V, C), \forall C \subseteq X \}.$$
where \( r(C) = \sum_{i \in C} r_i \) is the sum-rate in the subset \( C \) of \( X \). In an asymptotic model, we consider the asymptotic limits as the block length \( n \) goes to infinity so that the communication rates could be real or fractional; In a non-asymptotic model, the block length \( n \) is restricted to be finite and the communication rates are required to be integral.

The minimum sum-rates for attaining the local omniscience in \( X \) in the asymptotic and non-asymptotic models are 

\[
R_{ACO}(X) = \min \{ r(X) : r_X \in \mathcal{R}_{ACO}(X) \} \quad \text{and} \quad R_{NCO}(X) = \min \{ r(X) : r_X \in \mathcal{R}_{NCO}(X) \}.
\]

The corresponding optimal rate sets are \( \mathcal{R}^*_ {ACO}(X) = \{ r_X \in \mathcal{R}_{ACO}(X) : r(X) = R_{ACO}(X) \} \) and \( \mathcal{R}^*_ {NCO}(X) = \{ r_X \in \mathcal{R}_{NCO}(X) : r(X) = R_{NCO}(X) \} \) for the asymptotic and non-asymptotic models, respectively. Let \( \Pi(X) \) be the set of all partitions of \( X \). It is shown in [3, Example 4] [18, 17, Corollary 6] that 

\[
R_{ACO}(X) = \max_{\mathcal{P} \in \Pi(X) : |\mathcal{P}| > 1} \frac{H(X) - H(C)}{|\mathcal{P}| - 1} \tag{1}
\]

and \( R_{NCO}(X) = \lfloor R_{ACO}(X) \rfloor \).

A. Successive Omniscience

The concept of SO is outlined in [9, 10]. It is shown that, instead of the one-off approach in [4, 5, 7, 19–21], the communications between the users in \( V \) can be organized in a way such that global omniscience in \( X \) is attained in a two-stage manner. First, let the users in a subset \( X \) broadcast to attain the local omniscience and the remaining users \( i \in V \setminus X \) overhear these transmissions; Then, solve the global omniscience problem in \( V \). This two-stage approach can be implemented in a way without losing the optimality of the global omniscience: it is shown in [9] that there is a particular group of nonsingleton subsets \( X \) such that the local omniscience in \( X \) can be attained first so that the overall sum-rate for attaining the global omniscience in \( V \) still remains minimized. We call \( X \) a complimentary subset [9].

Thus, the problem of SO boils down to searching the complimentary user subset \( X \) and determining the local omniscience achievable rate vectors \( r_X \in \mathcal{R}_{CO}(X) \) and \( r_X \in \mathcal{R}_{CO}(X) \cap \mathbb{Z}^{X \setminus X} \) for the asymptotic and non-asymptotic models, respectively. In a multi-stage SO, the problem is to determine the complimentary user subset \( X \) and the local omniscience achievable rate vectors \( r_X \in \mathcal{R}_{CO}(X) \) and \( r_X \in \mathcal{R}_{CO}(X) \cap \mathbb{Z}^{X \setminus X} \) for the asymptotic and non-asymptotic models, respectively, for each stage \( k \in \{1, \ldots, K-1\} \) and ensure \( X^{(K)} = V \) and \( r^{(K)} \in \mathcal{R}_{ACO}(V) \) and \( r^{(K)} \in \mathcal{R}_{NCO}(V) \) for the asymptotic and non-asymptotic models, respectively, in last stage \( K \). This paper shows how to solve the two-stage and multi-stage SO by the PAR algorithm proposed in Part I [12]. Before presenting the solutions in Sections III to V, we briefly review the exiting results on the global omniscience problem in [6, 7, 22] and the PAR algorithm below.

B. The PAR Algorithm

For a given minimum sum-rate estimate \( \alpha \in [0, H(V)] \), the residual entropy function is 

\[
f_{\alpha}(X) = \alpha - H(V) + H(X), \forall X \subseteq V.
\]

The solution to the global omniscience problem in \( V \) is closely related to the Dilworth truncation of \( f_{\alpha}(V) = \min_{P \in \Pi(V)} f_{\alpha}(P) \), where \( f_{\alpha}(P) = \sum_{C \in P} f_{\alpha}(C) \). In Part I [12] Algorithm ??, we proposed a parametric (PAR) algorithm (see Algorithm 1), where the iteration are run in the order of an arbitrary linear ordering/permutation \( \Phi = (\phi_1, \ldots, \phi_{|V|}) \) of user indices.

It is shown in [12, Proposition ??] that, for \( V = \{\phi_1, \ldots, \phi_n\} \) that contains the first \( i \) users in \( \Phi \), the finest minimizer \( Q_{\alpha, V} = \arg \min_{P \in \Pi(V)} f_{\alpha}(P) \) for all \( \alpha \) segmented as

\[
Q_{\alpha, V} = \left\{ \begin{array}{ll}
P^{(p)} & \alpha \in [0, \alpha^{(p)}], \\
P^{(p-1)} & \alpha \in (\alpha^{(p)}, \alpha^{(p-1)}], \\
\vdots & \\
P^{(0)} & \alpha \in (\alpha^{(1)}, \alpha^{(0)}] 
\end{array} \right.
\]

and is obtained at end of each iteration \( i \) of the PAR algorithm, where \( 0 \leq \alpha^{(p)} < \ldots < \alpha^{(0)} = H(V) \) and \( \{m : m \in V_1 \} = P^{(p)} < \ldots < P^{(1)} < P^{(0)} = \{V_1\} \). Here, \( P < P' \) denotes \( P \) is strictly finer than \( P' \). The critical points \( \alpha^{(i)} \)'s and partitions \( P^{(i)} \)'s constitute the principal sequence of partitions (PSP) of \( V_i \). The PAR algorithm finally outputs the PSP of \( V \) that segments \( Q_{\alpha, V} = \arg \min_{P \in \Pi(V)} f_{\alpha}(P) \), where the first critical/turning point refers to \( \alpha^{(1)} \) and equals the minimum sum-rate \( R_{ACO}(V) \) for attaining the global omniscience. The corresponding value of \( Q_{\alpha^{(1)}, V} = P^{(1)} \) is the finest maximizer of (1) for \( X = V \) and is called the fundamental partition.

The PAR algorithm also returns a rate vector \( r_{\alpha, V} = (r_{\alpha, i} : i \in V) \) that is piecewise linear in \( \alpha \) such that \( r_{\alpha, V} \in B(f_{\alpha}) \) for all \( \alpha \), where \( B(f_{\alpha}) = \{r_V \in P(f_{\alpha}) : r(V) = f_{\alpha}(V) \} \) and \( P(f_{\alpha}) = \{r_V \in \mathbb{R}^n : r(X) \leq f_{\alpha}(X), X \subseteq V \} \) is the base polyhedron and polyhedron of \( f_{\alpha} \), respectively. Due to the equivalence, \( B(f_{\alpha}) = \{r_V \in \mathcal{R}_{ACO}(V) : r(V) = \alpha \} \) for all \( \alpha \geq R_{ACO}(V) \) [12, Section III-B and Theorem 4], \( r_{\alpha, V} \in \mathcal{R}_{ACO}(V) \) and \( r_{\alpha, V} \in \mathcal{R}_{NCO}(V) \) are the optimal rate vectors for asymptotic and non-asymptotic

As in Part I [12], ‘for all \( \alpha \) means ‘for all \( \alpha \in [0, H(V)] \)’ in this paper.

\begin{algorithm}[!h]
\caption{Parametric (PAR) Algorithm [12] Algorithm ??}
\begin{algorithmic}[1]
\State \textbf{input :} \( H, V \) and \( \Phi \)
\State \textbf{output:} segmented variables \( r_{\alpha, V} \in B(f_{\alpha}) \) and \( Q_{\alpha, V} = \arg \min_{P \in \Pi(V)} f_{\alpha}(P) \) for all \( \alpha \)
\State \( \alpha = (H(V)) \) for all \( \alpha \)
\State \( r_{\alpha, \emptyset} := f_\alpha(\{\phi_1\}) \) and \( Q_{\alpha, V} := \{\{\phi_1\}\} \) for all \( \alpha \)
\For {\( i = 2 \) to \( |V| \)}
\State \( Q_{\alpha, V_i} := Q_{\alpha, V_{i-1}} \cup \{\{\phi_i\}\} \) for all \( \alpha \)
\State Obtain the minimal minimizer \( U_{\alpha, V_i} \) of \( \min\{g_\alpha(X) : \{\phi_i\} \in X \subseteq Q_{\alpha, V_i} \} \) for all \( \alpha \)
\State For all \( \alpha \), update \( r_{\alpha, V} \) and \( Q_{\alpha, V} \) by
\State \( r_{\alpha, V_i} := r_{\alpha, V_i} + g_\alpha(U_{\alpha, V_i}) \) for all \( \alpha \)
\State \( Q_{\alpha, V_i} := (Q_{\alpha, V_i} \setminus U_{\alpha, V_i}) \cup \{U_{\alpha, V_i}\} \) for all \( \alpha \)
\EndFor
\State \textbf{return} \( r_{\alpha, V} \) and \( Q_{\alpha, V} \) for all \( \alpha \)
\end{algorithmic}
\end{algorithm}
models, respectively. The PAR algorithm completes in $O(|V| \cdot SFM(|V|))$ time, where $SFM(|V|)$ denotes the complexity of minimizing a submodular function $\min \{ f(X) : X \subseteq V \}$.

### III. Complimentary Subset

In this section, we show that the existence and non-existence of a complimentary subset can be determined by a lower bound on the minimum sum-rate, $R_{ACO}(V)$ and $R_{NCO}(V)$ for asymptotic and non-asymptotic models, respectively. When this lower bound is applied to $\mathcal{Q}_{\alpha,V}$ and $r_{\alpha,V}$, obtained at the end of each iteration $i$ of the PAR algorithm, a complimentary subset $X_*$ and a local omniscience achievable rate vector $r_X$, are both determined for SO.

In [8, 9], the authors derived the necessary and sufficient condition for a user subset to be complimentary for both asymptotic and non-asymptotic models.

**Theorem III.1** (necessary and sufficient condition) For asymptotic and non-asymptotic models, respectively, when $H(V)|P| > 1$, if $H(V) - H(X_*) + R_{ACO}(X_*) \leq R_{NCO}(V)$, then $f_{\alpha}(X_*) = \tilde{f}_{\alpha}(X_*)$. If $f_{\alpha}(X_*) = \tilde{f}_{\alpha}(X_*)$, then $f_{\alpha}(X_*) = \tilde{f}_{\alpha}(X_*)$.

**A. A Sufficient Condition**

We rewrite Theorem III.1 in terms of the Dilworth truncation $f_\alpha$ below and relax it to a sufficient condition that only requires a lower bound on the minimum sum-rate $R_{ACO}(V)$ or $R_{NCO}(V)$. The proof of Corollary III.2 is in Appendix A.

**Corollary III.2.** In an asymptotic model, a user subset $X_* \subseteq V$ such that $|X_*| > 1$ is complimentary if and only if $f_{\alpha}(X_*) = \tilde{f}_{\alpha}(X_*)$. In a non-asymptotic model, a user subset $X_* \subseteq V$ such that $|X_*| > 1$ is complimentary if and only if $f_{\alpha}(X_*) = \tilde{f}_{\alpha}(X_*)$.

**Lemma III.3** (sufficient condition). In an asymptotic model, a user subset $X_* \subseteq V$ such that $|X_*| > 1$ is complimentary if $f_{\alpha}(X_*) = \tilde{f}_{\alpha}(X_*)$ for $\alpha \leq R_{ACO}(V)$; In a non-asymptotic model, a user subset $X_* \subseteq V$ such that $|X_*| > 1$ is complimentary if $f_{\alpha}(X_*) = \tilde{f}_{\alpha}(X_*)$ for an integer-valued $\alpha \leq R_{NCO}(V)$.

**Proof:** For any $\alpha, \alpha' \leq R_{ACO}(V)$ such that $\alpha < \alpha'$, if $f_{\alpha}(X_*) = \tilde{f}_{\alpha}(X_*)$, then

$$f_{\alpha}(|P|) - f_{\alpha}(X_*) = H(|P|) - H(X_*) - (|P| - 1)(H(V) - \alpha') > H(|P|) - H(X_*) - (|P| - 1)(H(V) - \alpha) = f_{\alpha}(|P|) - f_{\alpha}(X_*) \geq 0 \tag{2}$$

for all $\mathcal{P} \in \Pi(|V|)$ such that $|\mathcal{P}| > 1$, where $H(|\mathcal{P}|) = \sum_{C \in \mathcal{P}} H(C)$. So, $f_{\alpha}(X_*) = \tilde{f}_{\alpha}(X_*) \geq 0, \forall \mathcal{P} \in \Pi(|V|) : |\mathcal{P}| > 1$. The condition $f_{\alpha}(X_*) = \tilde{f}_{\alpha}(X_*)$ in Corollary III.2 holds. Therefore, $X_*$ is complimentary in the asymptotic model. In the same way, one can prove the sufficient condition $f_{\alpha}(X_*) = \tilde{f}_{\alpha}(X_*)$ for the non-asymptotic model.

It is not difficult to obtain a lower bound on the minimum sum-rate for the global omniscience problem. According to [7, Proposition 14], a possible value for $\alpha$ in Corollary III.2 can be determined as the value of objective function in (1) over only the singleton partition and the bi-partitions in $\Pi_i(V) = \{ \{m\} : m \in V\}, \{\{i\}, V \setminus \{i\} : i \in V\}$:

$$\alpha = \max_{\mathcal{P} \in \Pi_i(V)} \sum_{C \in \mathcal{P}} \frac{H((V) - H(C))}{|\mathcal{P}| - 1} \leq R_{ACO}(V), \tag{3a}$$

$$\alpha = \left[ \max_{\mathcal{P} \in \Pi_i(V)} \sum_{C \in \mathcal{P}} \frac{H((V) - H(C))}{|\mathcal{P}| - 1} \right] \leq R_{NCO}(V) \tag{3b}$$

for the asymptotic and non-asymptotic models, respectively. The lower bound in (3a) can be determined by $O(|V|)$ calls of the entropy function.

**Example III.4.** For the 5-user system in [72] Example ??, we have:

$$Z_1 = (W_b, W_c, W_d, W_h, W_i),$$

$$Z_2 = (W_c, W_f, W_b, W_i),$$

$$Z_3 = (W_b, W_c, W_e, W_j),$$

$$Z_4 = (W_a, W_b, W_e, W_d, W_f, W_g, W_i, W_j),$$

$$Z_5 = (W_a, W_b, W_c, W_f, W_i, W_j),$$

where
with each $W_m$ being an independent and uniformly distributed random bit, it can be shown that

$$
\{X_s \subseteq V : |X_s| > 1, H(V) - H(X_s) + R_{ACO}(X_s) \leq R_{ACO}(V)\}
= \{X_s \subseteq V : |X_s| > 1, f_{Raco}(V)(X_s) = \hat{f}_{Raco}(V)(X_s)\}
= \{\{4, 5\}, \{1, 4\}, \{1, 4, 5\}, \{1, 2, 3, 4\}\},
$$
are all complimentary subsets for the asymptotic model and

$$
\{X_s \subseteq V : |X_s| > 1, H(V) - H(X_s) + R_{NCO}(X_s) \leq R_{NCO}(V)\}
= \{X_s \subseteq V : |X_s| > 1, f_{Raco}(V)(X_s) = \hat{f}_{Raco}(V)(X_s)\}
= \{\{1, 4\}, \{1, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{3, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}\},
$$
are all complimentary subsets for the non-asymptotic model.

Instead, a lower bound on $R_{ACO}(V)$ can be obtained as

$$\alpha = \sum_{i \in V} \frac{H(V) - H(i)}{|V| - 1} = 5.75 < R_{ACO}(V),$$

where we apply Lemma III.4 to find a complimentary subset

$$\{X_s \subseteq V : |X_s| > 1, f_{\alpha}(X_s) = \hat{f}_{\alpha}(X_s)\} = \{\{4, 5\}\}
$$
for the asymptotic model. Consider a tighter lower bound $\alpha' = \max_{P \in C^*} \sum_{C \in P} \frac{H(V) - H(C)}{|P| - 1} = 6$ on both $R_{ACO}(V)$ and $R_{NCO}(V)$. We have

$$\{X_s \subseteq V : |X_s| > 1, f_{\alpha'}(X_s) = \hat{f}_{\alpha'}(X_s)\}
= \{\{1, 4\}, \{4, 5\}, \{1, 4, 5\}\}
$$
being the complimentary subsets for both asymptotic and non-asymptotic models.

B. Existence of A Complimentary Subset

There are two observations in Example III.4 that need to be explored further. First, the number of complimentary subsets decreases with the value of lower bound $\alpha$. This is proved by the corollary below.

**Corollary III.5.** For any two lower bounds $\alpha$ and $\alpha'$ on the minimum sum-rate $R_{ACO}(V)$ for the asymptotic model, or on $R_{NCO}(V)$ for the non-asymptotic model, such that $\alpha < \alpha'$,

$$\{X_s \subseteq V : |X_s| > 1, f_{\alpha}(X_s) = \hat{f}_{\alpha}(X_s)\} \subseteq \{X_s \subseteq V : |X_s| > 1, f_{\alpha'}(X_s) = \hat{f}_{\alpha'}(X_s)\}.$$

**Proof:** As shown in the proof of Lemma III.3 for any $X_s \subseteq V$ such that $|X_s| > 1$, if $f_{\alpha}(X_s) = \hat{f}_{\alpha}(X_s)$, then $f_{\alpha}(P) - f_{\alpha}(X_s) \geq 0$, $f_{\alpha}(P) \in \Pi(X_s) : |P| > 1$ and inequality (2) holds. We necessarily have $f_{\alpha'}(X_s) = \hat{f}_{\alpha'}(X_s)$ so that $X_s \in \{X_s \subseteq V : |X_s| > 1, f_{\alpha'}(X_s) = \hat{f}_{\alpha'}(X_s)\}$. Corollary holds.

**Corollary III.5** means that all complimentary subsets for the asymptotic model are also complimentary in the non-asymptotic model (See Example III.4) since $R_{ACO}(V) \leq R_{NCO}(V)$. The second observation in Example III.4 is also based on Corollary III.5. Lemma III.3 just allows us to search a part of, rather than all, the complimentary subsets. As the number of complimentary subsets searched by Lemma III.3 shrinks to zero when $\alpha$ decreases, we should choose a lower bound $\alpha$ large enough to capture at least one of the complimentary subsets, if there exists one. For this purpose, the following lemma states that, for the lower bounds $\alpha = \sum_{i \in V} \frac{H(V) - H(i)}{|V| - 1}$ and $\alpha = \left[ \sum_{i \in V} \frac{H(V) - H(i)}{|V| - 1} \right]$ determined by the singleton partition $\{\{i\} : i \in V\}$ for the asymptotic and non-asymptotic models, respectively, the sufficient condition in Lemma III.3 is enough to prove the nonexistence of the complimentary subset.

**Lemma III.6.** There does not exist any complimentary subset in the asymptotic model if no $X_s \subseteq V$ such that $|X_s| > 1$ satisfies $f_{\alpha}(X_s) = \hat{f}_{\alpha}(X_s)$ for $\alpha = \sum_{i \in V} \frac{H(V) - H(i)}{|V| - 1}$; There does not exist any complimentary subset in the non-asymptotic model if no $X_s \subseteq V$ such that $|X_s| > 1$ satisfies $f_{\alpha}(X_s) = \hat{f}_{\alpha}(X_s)$ for $\alpha = \left[ \sum_{i \in V} \frac{H(V) - H(i)}{|V| - 1} \right]$.

The proof of Lemma III.6 is in Appendix III. An example to demonstrate Lemma III.6 for the asymptotic model is the independent source model with the terminals $Z_i$ being independent of each other, where we have only two trivial partitions $P^{(i)} = \{\{i\} : i \in V\}$ and $P^{(0)} = \{\{V\}\}$ in the PSP and $\alpha^{(i)} = R_{ACO}(V) = H(V)$. The lower bound in Lemma III.6 is $\alpha = \sum_{i \in V} \frac{H(V) - H(i)}{|V| - 1}$ and the partition $P^{(i)} = \{\{i\} : i \in V\}$ indicates that no $X_s \subseteq V$ such that $|X_s| > 1$ satisfies $f_{\alpha}(X_s) = \hat{f}_{\alpha}(X_s)$. Therefore, there does not exist any complimentary subset for the asymptotic model in the independent source. Lemmas III.6 and III.3 suggest an efficient method for searching the complimentary subset by the lower bound $\alpha$, which is applied to the two-stage SO in Section IV and the multi-stage SO in Section V.

IV. Two-Stage Successive Omniscience

The original SO problem in [8], [9] is outlined in two steps: select a complimentary subset $X_s$ and attain local omniscience in it; fuse all $i \in X_s$ into one super-user and consider the omniscience problem in the new system containing the fused super-user $X_s$ and the rest of the users in $i \in V \setminus X_s$. This two-stage SO approach can be applied recursively so that a sequence of local omniscience leads to the global one and hence the name ‘successive’. It is clear that the two-stage SO problem is solved if a complimentary $X_s$ is searched and the local omniscience problem in it is solved. While the sufficient condition in Lemma III.3 only determine a $X_s$, we show in this section that, when it is applied to the segmented $Q_{\alpha,V}$ and $r_{\alpha,V}$ in the PAR algorithm, not only a $X_s$, but also a local omniscience achievable rate vector $r_{X_s}$, can be determined at the same time for both asymptotic and non-asymptotic models.

The following lemma states that the existence of the complimentary subset and at least one of the complimentary subsets, if there exist one, can be determined by applying the lower bound $\alpha$ in Lemma III.6 to the PAR algorithm. A local omniscience achievable rate vector is also obtained at the same time. The proof is in Appendix C.

**Lemma IV.1.** Let $Q_{\alpha,V}$ and $r_{\alpha,V}$ be the segmented partition and rate vector, respectively, obtained at the end of any iteration $i \in \{2, \ldots, |V|\}$ of the PAR algorithm.

(a) For the asymptotic model, let $\alpha = \sum_{i \in V} \frac{H(V) - H(i)}{|V| - 1}$. Any nonsingleton $C \in Q_{\alpha,V}$ is a complimentary subset.
Algorithm 2: Two-stage Successive Omniscience (SO) by PAR Algorithm

input: \(H, V\) and (an arbitrarily chosen linear ordering) \(\Phi\).
output: a complimentary subset \(C\) and an optimal rate vector \(r_{\alpha,C}\) for attaining local omniscience in \(C\); Or, an empty set indicating no complimentary subset and \(Q_{\alpha,V}\) and \(r_{\alpha,V}\) that determines the optimal solution to the global omniscience.

1. \(\hat{\alpha} \leftarrow \sum_{i \in V} H(V) - H(i)\) for the asymptotic model or \(\tilde{\alpha} \leftarrow \left[ \sum_{i \in V} H(V) - H(i) \right]\) for the non-asymptotic model;
2. Call PAR\((H, V, \Phi)\);
3. for \(i = 2\) to \(|V|\) do
   4. if \(\exists C \in Q_{\alpha,V}: |C| > 1\) at some iteration \(i\) of PAR then
      break and return \(C\) and \(r_{\alpha,C}\) after the update in step 6 of the PAR algorithm, where
      \[\hat{\alpha} = \min \{\alpha \in \mathbb{R}: f_\alpha(C) = \hat{f}_\alpha(C)\},\]
      \[\alpha = \min \{\alpha \in \mathbb{Z}: f_\alpha(C) = \tilde{f}_\alpha(C)\},\]
      for the asymptotic and non-asymptotic models, respectively (see Remark IV.2 for how to obtain \(\hat{\alpha}\))
   endfor
6. return \(\emptyset, Q_{\alpha,V}\) and \(r_{\alpha,V}\) at the end of the PAR algorithm;

and \(r_{\alpha,C}\) for \(\hat{\alpha} = \min \{\alpha \in \mathbb{R}: f_\alpha(C) = \hat{f}_\alpha(C)\}\) is an optimal rate vector that attains local omniscience in \(C\) with the minimum sum-rate \(R_{ACO}(C)\).

(b) For the non-asymptotic model, let \(\tilde{\alpha} = \left[ \sum_{i \in V} H(V) - H(i) \right]\). Any non-singleton \(C \in Q_{\alpha,V}\) is a complimentary subset and \(r_{\alpha,C}\) for \(\alpha = \min \{\alpha \in \mathbb{Z}: f_\alpha(C) = \tilde{f}_\alpha(C)\}\) is an optimal rate vector that attains local omniscience in \(C\) with the minimum sum-rate \(R_{NCO}(C)\).

For both asymptotic and non-asymptotic models, if all subsets in \(Q_{\alpha,V}\) remain singleton \(Q_{\alpha,V} = \{\{m\}: m \in V\}\) until the \(|V|\)th iteration, there does not exist a complimentary subset.

Lemma IV.1 is implemented by Algorithm 2. When there does not exist a complimentary subset, Algorithm 2 outputs the results \(Q_{\alpha,V}\) and \(r_{\alpha,V}\) determining the optimal rate vector for the global omniscience problem in \(V\). That is, if there exists a complimentary subset, Algorithm 2 determines one such subset \(X\) and local omniscience rate vector \(r_{X}\) for the two-stage SO in \(O(|V| \cdot \text{SFM}(|V|))\) time; otherwise, the minimum sum-rate problem in \(V\) is solved in \(O(|V| \cdot \text{SFM}(|V|))\) time.

Remark IV.2 (Determining \(\hat{\alpha}\)). To obtain the value of \(\hat{\alpha}\) in Lemma IV.1, we just need to consider the range \([0, \alpha]\) for that \(\hat{\alpha} \leq \alpha\) must hold in both asymptotic and non-asymptotic models. For the asymptotic model, \(\hat{\alpha} = \min \{\alpha \in \mathbb{R}: f_\alpha(C) = \hat{f}_\alpha(C)\}\) is one of the critical points where the subsets in \(C\) merge to form \(C\), or the first time that \(C\) appears as an intact subset in \(Q_{\alpha,V}\). The reason is that: (i) for all \(\alpha < \hat{\alpha}\), \(f_\alpha(C) < \hat{f}_\alpha(C)\) and therefore \(C \notin \text{argmin}_{P \in H(V)} f_\alpha(P) = Q_{\alpha,V}\); (ii) for all \(\alpha \geq \hat{\alpha}\), \(f_\alpha(C) = \hat{f}_\alpha(C)\) so that \(C \subseteq C'\) for some \(C' \in Q_{\alpha,V}\). So, \(\hat{\alpha}\) must create a critical value in the segmented \(Q_{\alpha,V}\). Taking the least integer value that is no less than this critical value, we have \(\hat{\alpha}\) for the non-asymptotic model in Lemma IV.1(b).

Example IV.3 (two-stage SO by Algorithm 2). We run Algorithm 2 on the 5-user system in Example III.2 for linear ordering \(\Phi = (4, 5, 2, 3, 1)\). We set \(\alpha = \sum_{i \in V} H(V) - H(i) = 5.75\) for the asymptotic model. At the end of the 2nd iteration of the PAR algorithm, we have
\[
\hat{U}_{\alpha, V_2} = \begin{cases} 
\{\{5\}\} & \alpha \in [0, 4], \\
\{\{4, 5\}\} & \alpha \in (4, 10],
\end{cases}
\]
where \(\hat{U}_{\alpha, V_2} = \{\{4, 5\}\}\) is non-singleton so that \(C = \{4, 5\} \in Q_{\alpha,V}\) is a complimentary subset. To determine the optimal rate vector for the local omniscience in \(\{4, 5\}\), we use Remark IV.2 to search the value of \(\tilde{\alpha}\) in the segmented \(Q_{\alpha,V}\) and find that, at \(\alpha = 4\), the subsets \{4\} and \{5\} are merged to \{4, 5\}. So, \(\hat{\alpha} = 4\) is also an critical point in the segmented \(Q_{\alpha,V}\). See Fig 7. Considering the segmented rate vector
\[
r_{\alpha,V_2} = r_{\alpha,\{4,5\}} = \begin{cases} 
(\alpha - 2, \alpha - 4) & \alpha \in [0, 4], \\
(\alpha - 2, 0) & \alpha \in (4, 10],
\end{cases}
\]
determined by the PAR algorithm, we have \(r_{\alpha,\{4,5\}} = (2, 0)\) being the optimal rate vector that attains the local omniscience in \(\{4, 5\}\) with the minimum sum-rate \(R_{ACO}(\{4, 5\}) = 2\).

For the non-asymptotic model, we have the lower bound \(\tilde{\alpha} = \left[ \sum_{i \in V} H(V) - H(i) \right] = 6\). Again, \(\{4, 5\} \in Q_{\alpha,V}\) is complimentary and \(\tilde{\alpha} = \min \{\alpha \in \mathbb{Z}: f_\alpha(\{4, 5\}) = \tilde{f}_\alpha(\{4, 5\})\}\) for the local omniscience in \(\{4, 5\}\). Consider the optimal rate vector \((1, 0.5, 0.5, 4.5, 0.0)\) for the global omniscience obtained in [22] Example 2, we have \(1, 0.5, 0.5, 4.5, 0.0 = (0, 0, 0, 2, 0) + (1, 0.5, 0.5, 2.5, 0)\), which means that, by letting users transmit at rates \((1, 0.5, 0.5, 2.5, 0)\) after reaching the local omniscience in \(\{4, 5\}\), the global omniscience is still attained with the minimum sum-rate \(R_{ACO}(V) = 6.5\). Alternatively, it also means
that the optimal rate vector \((1, 0.5, 0.5, 4.5, 0)\) can be implemented in a successive manner so that the local omniscience in \(\{4, 5\}\) can be attained first. In fact, the necessary and sufficient condition for a nonsingleton \(X_1 \subseteq V\) to be complimentary in [2] Theorems 4.2 and 5.2] are derived in terms of the existence of the successive optimal rates: \(X_1\) is complimentary if and only if there exists an optimal rate vector \(r_V \in \mathcal{R}_{\text{ACO}}(V)\) such that \(r_V^{(1)} = r_V^{(2)} = r_V\) with \(r_X^{(1)} \in \mathcal{R}_{\text{ACO}}(X_1)\) and \(r_i^{(1)} = 0\) for all \(i \in V \setminus X_1\). Likewise, the optimal rate vector \((0, 1, 1, 5, 0)\) for the non-asymptotic model can be implemented in a successive manner as \((0, 1, 1, 5, 0) = (0, 0, 0, 2, 0) + (0, 1, 1, 3, 0)\), where the local omniscience in \(\{4, 5\}\) can be attained first. It should be noted that Lemma \(6.7\) (a) and (b) holds for any lower bound \(\alpha\). The following example shows that a larger complimentary subset can be found with a tighter lower bound. With \(\alpha = 6.25\), we apply Algorithm 2 to the asymptotic model for another linear ordering \(\Phi = (5, 1, 4, 2, 3)\). We have \(U_{6.25, V}\) remains singleton until the end of the \(2\)-stage algorithm, where \(U_{6.25, V} = \{1, 4, 5\}\) and the segmented rate vector \(r_{\alpha, V}\) and partition \(Q_{\alpha, V}\) are respectively

\[
\begin{align*}
\mathbf{r}_{\alpha, (1, 4, 5)} &= \left\{ \begin{array}{ll}
(\alpha - 5, \alpha - 2, \alpha - 4) & \alpha \in [0, 4], \\
(\alpha - 5, 2, \alpha - 4) & \alpha \in [4, 6], \\
(\alpha - 5, 3, \alpha - 4) & \alpha \in [6, 7], \\
(2, 1, \alpha - 4) & \alpha \in (7, 10],
\end{array} \right. \\
Q_{\alpha, (1, 4, 5)} &= \{\{1\}, \{4\}, \{5\}\} \quad \alpha \in [0, 4], \\
\{\{4, 5\}, \{1\}\} \quad \alpha \in [4, 6], \\
\{\{1, 4, 5\}\} \quad \alpha \in (6, 10].
\end{align*}
\]

In this case, we have the nonsingleton \(U_{6.25, (1, 4, 5)} = \{1, 4, 5\} \in Q_{6.25, (1, 4, 5)}\), \(\hat{\alpha} = \min\{\alpha \in \mathbb{R}: f_\alpha(1, 4, 5) = 6\}\) and \(r_{\alpha, (1, 4, 5)} = (1, 2, 2) \in \mathcal{R}_{\text{ACO}}^*(1, 4, 5)\) being the optimal rate vector that attains local omniscience in \(\{1, 4, 5\}\) with the minimum sum rate \(R_{\text{ACO}}(1, 4, 5) = 5\).

A. Achievability

For \(K \leq |V| - 1\), let \(\{X_i^{(k)}, r_i^{(k)}: k \in \{1, \ldots, K\}\}\) denote the \(K\)-stage SO, where all \(X_i^{(k)}\) are nonsingleton subsets of \(V\). The following proposition states the achievability of the \(K\)-stage SO.

**Proposition V.1.** In the asymptotic model, a \(K\)-stage SO \(\{X_i^{(k)}, r_i^{(k)}: k \in \{1, \ldots, K\}\}\) is achievable if \(X_i^{(k)}\) for all \(k \in \{1, \ldots, K\}\) is complimentary and forms a set sequence chain

\[
\emptyset \subseteq X_i^{(1)} \subseteq X_i^{(2)} \subseteq \ldots \subseteq X_i^{(K)} = V.
\]

For all \(k \in \{1, \ldots, K - 1\}\),

(a) \(r_i^{(k)} \in \mathcal{R}_{\text{ACO}}(X_i^{(k)})\) and \(r_i^{(k)} = 0\) for all \(i \in V \setminus X_i^{(k)}\);

(b) \(r_i(1)X_i^{(k)} - r_i(k)(X_i^{(k)}) \geq 0\),

and \(r_V^{(K)} \in \mathcal{R}_{\text{ACO}}^*(V)\). In the non-asymptotic model, a \(K\)-stage SO is achievable if the above conditions hold for \(r_V^{(k)} \in \mathcal{R}_{\text{ACO}}^*(V)\) and \(r_V^{(k)} \in \mathcal{R}_{\text{ACO}}^*(X_i^{(k)} \cap \mathbb{Z}^{|V|})\) for all \(k \in \{1, \ldots, K - 1\}\).

**Proof:** The nesting subset chain \(5\) and the nondecreasing sum-rate \(r_i(k)(X_i^{(k)})\) in \(k\) in (b) are the necessary conditions for local omniscience in \(X_i^{(k)}\) by the rest of users overhearing. The optimality of the final rate vector is ensured by \(r_V^{(K)} \in \mathcal{R}_{\text{ACO}}^*(V)\). By restricting these conditions hold for the integer-valued rate vectors \(r_i^{(k)}\) for all \(k \in \{1, \ldots, K\}\), the proposition applies to the non-asymptotic model.

B. Asymptotic Model

We propose Algorithm 5 that uses \(P_j\)'s in the PSP and the corresponding rate vector \(r_{\alpha,j,V}\) to build a \(p\)-stage SO that iteratively attains the local omniscience in all nonsingleton subsets in each partition \(P_j\). The achievability of this \(p\)-stage SO is stated below. The proof in Appendix 1 is based on the monotonic sum-rate for all \(X \in P_j\) in [2] Lemma ??(c).

**Corollary V.2.** For the asymptotic model, all \(X_i^{(k)}\) and \(r_i^{(k)}\) at the end of Algorithm 3 constitute an achievable p-stage SO \(\{X_i^{(k)}, r_i^{(k)}: k \in \{1, \ldots, p\}\}\), where \(r_i^{(p)} \in \mathcal{R}_{\text{ACO}}(V)\),

\[
\emptyset \subseteq X_1^{(1)} \subseteq \ldots \subseteq X_1^{(p)} = V
\]

and, for all \(k \in \{1, \ldots, p\}\), all \(C \in X_i^{(k)}\) are complimentary. For each \(C \in X_i^{(k)}\), the local omniscience in each \(C\) is attained by an optimal rate vector \(r_C^{(k)} \in \mathcal{R}_{\text{ACO}}(C)\).

**Remark V.3.** We remark the followings about Algorithm 3

(a) The output \(r_V^{(k)}\) is nondecreasing in \(k\), i.e., \(r_V^{(k-1)} \geq r_V^{(k)}\) for all \(k \in \{2, \ldots, p\}\). Therefore, \(r_V^{(k)}\) is not necessarily the same as \(r_{\alpha,p-k+1,V}\) in that \(r_{\alpha,V}\) returned by the PAR algorithm is not monotonic in general, e.g., \(r_{\alpha,3}\) in (6) is not nondecreasing in \(\alpha\).

5We say \(r_i \geq r_i^*\) if \(r_i \geq r_i^*\) for all \(i \in V\) with at least one of these inequalities holding strictly.
Algorithm 3: Multi-stage Successive Omniscience (SO) by the PAR Algorithm for the Asymptotic Model

input : H, V and Φ.
output: an achievable p-stage SO
\{(X^{(k)}_i, r^{(k)}_i): k \in \{1, \ldots, p\}\}.
1 Call PAR(H, V, Φ) to obtain the segmented Q_{α,V} and r_{α,V} for all α;
2 Initialize r^{(0)}_i \leftarrow (0, \ldots, 0);
3 for k = 1 to p do
4 \quad r^{(k)}_i \leftarrow r^{(k-1)}_i;
5 \quad X^{(k)}_i \leftarrow \{C \in P^{(p-k)}: |C| > 1\};
6 \quad \textbf{for each } C \in X^{(k)}_i \textbf{ do}
7 \quad \quad \textbf{if } C \neq \langle C \rangle_{P(p-k+1)} \textbf{ then for each } C' \in \langle C \rangle_{P(p-k+1)} \textbf{ randomly select user } i \in C' \textbf{ and let }
8 \quad \quad \quad \Delta r \leftarrow r^{(k)}_{r_i(p-k+1)}(C') - r^{(k)}_{i}(C') \textbf{ and }
9 \quad \quad \quad r_i \leftarrow r_i + \Delta r;
10 \quad \textbf{end}
11 \textbf{end}
12 \textbf{return } X^{(k)}_{i} \textbf{ and } r^{(k)}_i \textbf{ for all } k \in \{1, \ldots, p\};

(b) Unlike Proposition [71] where only one complimentary subset is chosen each time, Algorithm 3 allows more than one complimentary subset to attain local omniscience at each stage. Since all \( C \) in \( X^{(k)}_i \) are disjoint, the local omniscience in step 2 can be attained simultaneously if the broadcasts between subsets do not cause interference, e.g., via orthogonal wireless channels in CCDE.

(c) \( \Delta r \) in (5) is interpreted as, in addition to the rates for attaining the local omniscience in \( C' \), how many transmissions is required from the super-user \( C' \) for attaining the local omniscience in \( C \). Since all users in \( C' \) have recovered \( Z_{C'} \) in previous stages, \( \Delta r \) can be assigned to any one of them. Apart from the random selection in step 2 we can moderate \( \Delta r \) to the users i with the lowest \( r_i^{(k)} \) for improving fairness. See Example V.4.

Intuitively, the \( p \)-stage SO \( \{(X^{(k)}_i, r^{(k)}_i): k \in \{1, \ldots, p\}\} \) results in an agglomerative SO tree that converges to the global omniscience. See Fig. 2. This SO tree is exactly the hierarchical clustering result determined by the PSP of \( V \) as described in [12] Section 7?]. This bottom-up approach can also be considered as an opposite process of the divide-and-conquer algorithm in [24], where the ground set \( V \) is recursively split into subsets until the optimal rates \( r_{Recov(V,i)} \) are determined for all users \( i \in V \). However, the complexity is much reduced: while the complexity of divide-and-conquer algorithm [24] is \( O(|V|^3 \cdot \text{SFM}(|V|)) \), Algorithm 3 completes in \( O(|V| \cdot \text{SFM}(|V|)) \) time.6

Example V.4. We apply Algorithm 3 to the 5-user system in

Example III.4. The call \( \text{PAR}(H, V, \{4, 5, 2, 3, 1\}) \) returns

\[
\begin{align*}
\alpha \in [0, 4], & \quad r_{\alpha,V} = \begin{cases}
(\alpha - 5, \alpha - 6, \alpha - 6, \alpha - 2, \alpha - 4) & \alpha \in [0, 4], \\
(\alpha - 5, \alpha - 6, \alpha - 6, \alpha - 2, 0) & \alpha \in [4, 6], \\
(1 - \alpha - 6, \alpha - 6, \alpha - 2, 0) & \alpha \in [6, 6.5], \\
(1 - 2\alpha - 6, \alpha - 6, \alpha - 2, 0) & \alpha \in [6.5, 7], \\
(0, \alpha - 6, \alpha - 2, 0) & \alpha \in [7, 8], \\
(0, 2, 0, \alpha - 2, 0) & \alpha \in [8, 10],
\end{cases}
\end{align*}
\]

and \( Q_{α,V} \) characterized by the PSP: \( P^{(3)} = \{\{1\}, \ldots, \{5\}\} \) \( P^{(2)} = \{\{4, 5\}, \{1\}, \{2\}, \{3\}\} \) and \( P^{(1)} = \{\{1, 4, 5\}, \{2\}, \{3\}\} \) with \( p = 3 \) critical values \( α^{(3)} = 4, α^{(2)} = 6 \) and \( α^{(1)} = 6.5 \). Let \( r_{0,0}^{(0)} = (0, \ldots, 0) \).

For \( k = 1 \), first assign \( r_{1}^{(1)} = (0, \ldots, 0) \). We get \( X^{(1)}_i = \{C \in P^{(2)}: |C| > 1\} = \{\{4, 5\}\} \) such that \( \{\{4, 5\}\}_{P^{(3)}} \neq \{\{4, 5\}\} \neq \{\{4, 5\}\} \). This means local omniscience has not attained in \{4, 5\} before. We then assign rates in step 2 as

\[
\begin{align*}
r_{1}^{(1)} = r_{1}^{(3),4} & = 2 \quad \text{and} \quad r_{5}^{(1)} = r_{5}^{(3),5} = 0 \quad \text{so that } r_{1}^{(1)} = (0, 0, 0, 2, 0).
\end{align*}
\]

Finally, \( \Delta r = (0, 0, 0, 2, 4) \) that attains the local omniscience in \{4, 5\}, users 4 and 5 need to transmit 2 more times for attaining the local omniscience in \{1, 4, 5\}. In this case, we choose user 5 to transmit \( \Delta r \) so that \( r_{5}^{(2)} = 2 + 2 = 4 \); For \( C' = \{1\} \) being singleton, we haven’t assigned any rates to user 1 before and therefore \( r_{1}^{(2)} = r_{a(2),1} = 1. \) So, \( r_{1}^{(2)} \) is updated to \( (1, 0, 0, 4, 0) \).

For \( k = 3 \), assign \( r_{3}^{(3)} = r_{2}^{(2)} = (1, 0, 0, 4, 0) \) and get \( X^{(1)}_i = \{C \in P^{(0)}: |C| > 1\} = \{\{1, 5\}\} \), where \( \{\{1, 5\}\}_{P^{(3)}} \neq \{\{4, 5\}\} \neq \{\{4, 5\}\} \). For the super-user formed by \( C' = \{1, 4, 5\} \), we have \( r_{a(3)}^{(1)} = 5.5 \) and \( \Delta r = r_{a(3)}^{(1)}(\{1, 4, 5\}) - r_{3}^{(3)}(\{1, 4, 5\}) = 0.5 \). Still choose user 4 so that \( r_{4}^{(3)} = 4 + 0.5 = 4.5 \); For singletons \( \{2\} \) and \( \{3\} \), assign \( r_{3}^{(3)} = r_{5}^{(3),3} = 0.5 \). So, \( r_{3}^{(3)} \) is updated to \( (1, 0, 0, 4, 5, 0) \), which is an optimal rate vector in \( S^{\text{ACO}}(V) \) for attaining the global omniscience. Finally, we have a 3-stage SO \( \{(X^{(k)}_i, r^{(k)}_i): k \in \{1, \ldots, 3\}\} \) such that \( X^{(1)} \subseteq X^{(2)} \subseteq X^{(3)} = V \) and \( r_{1}^{(3)} \leq r_{2}^{(2)} \leq r_{3}^{(3)} \).

While the above procedure outputs \( r^{(k)}_i = r_{a(p-k+1),V} \) for all \( k \), we show that a fair allocation of \( \Delta r \) in Remark [V.3] results in a different \( r^{(k)}_i \). If we assign \( \Delta r = r_{a(1)}^{(1)}(\{4, 5\}) - r_{2}^{(2)}(\{4, 5\}) = 2 \) to user 5 in stage \( k = 2 \) and \( \Delta r = r_{a(1)}^{(1)}(\{1, 4, 5\}) - r_{3}^{(3)}(\{1, 4, 5\}) = 0.5 \) to user 1, we have a fairer rate vector sequence \( r_{1}^{(3)} = (0, 0, 0, 2, 0), r_{2}^{(2)} = (1, 0, 0, 2, 2) \) and \( r_{3}^{(3)} = (1, 5, 0, 0, 2, 2) \in S^{\text{ACO}}(V) \). However, in general, this approach does not necessarily result in the fairest optimal rate vector \( S^{\text{ACO}}(V) \) at the end of final
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Fig. 2. The 3-stage agglomerative SO tree outlined in Example V.4 by applying Algorithm III to the 5-user system for the asymptotic model in Example III.3. Here, the users/super-users at each stage k correspond to a \(\mathcal{P}(k)\) in the PSP of \(V\), which characterizes the segmented partition \(Q_{\alpha,V}\). The rates \(r^{(k)}\) are determined by the segmented rate vector \(r_{\alpha,V}\) in (6). The 3-stage SO for the non-asymptotic model determined in Example V.6 by Algorithm III is also shown, which only differs from the one for the asymptotic model at the last stage \(k = 3\), where the super-user 145 and users 2 and 3 merges at \(\alpha = 7\) instead of \(\alpha = 6.5\).

stage \(k = p\)

The 3-stage SO above can be presented as the agglomerative tree diagram in Fig. 2 which provides a more intuitive interpretation of what Algorithm III does: attain local omniscience in all subsets merged from \(P^{(k)}\) to \(P^{(k+1)}\) at each \(\alpha^{(k)}\). In Fig. 2 increasing \(\alpha\) from 0 at the bottom, at \(\alpha = 3\) = 4, the value of \(Q_{\alpha,V}\) changes from \(P^{(3)} = \{\{1\}, \{5\}\} \) to \(P^{(2)} = \{\{4,5\}, \{1\}, \{2\}, \{3\}\}\) where users 4 and 5 attain omniscience and merge to the super-user 45; at \(\alpha^{(2)} = 6\), \(Q_{\alpha,V}\) changes from \(P^{(2)} = \{\{4,5\}, \{1\}, \{2\}, \{3\}\}\) where super-users 45 and user 1 attain omniscience and merge to the super-user 145; at \(\alpha^{(1)} = 6.5\), \(Q_{\alpha,V}\) changes from \(P^{(1)} = \{\{4,5\}, \{2\}, \{3\}\}\) to \(P^{(0)} = \{\{1, \ldots , 5\}\}\) where super-users 145, user 2 and user 3 attain omniscience and merge to the super-user 12345 so that global omniscience is attained.

C. Non-asymptotic Model

Although all nonsingleton \(C \in Q_{\alpha,V}\) for each integer-valued \(\alpha \in \{0, \ldots , R_{\text{NCO}}(V)\}\) are complimentary based on Lemma III.3 the rate vector \(r_{\alpha,V}\) is not necessarily nondecreasing in integer-valued \(\alpha\). This means that Algorithm III cannot be applied to the non-asymptotic model by simply running each stage \(k\) of Algorithm III at the integer-valued critical points \([\alpha^{(p-k+1)}]\). In this section, we show that an achievable \(K\)-stage SO \((X^{(k)}_\alpha, r^{(k)}_V)\) with the integer-valued \(r^{(k)}_V\) attaining local omniscience in each complimentary subset \(X^{(k)}_\alpha\) can be searched by no more than two calls of the PAR algorithm.

Corollary V.5. In the non-asymptotic model, for any nonsingleton subset sequence \(X^{(1)}_\alpha \subseteq \ldots \subseteq X^{(K)}_\alpha = V\) and the integer-valued sequence \(\alpha^{(1)} < \ldots < \alpha^{(K)} = R_{\text{NCO}}(V)\) such that \(\alpha^{(k)} \in [\alpha^{(j)}, \alpha^{(j-1)}]\) for some \(j \in \{1, \ldots , p\}\) and \(X^{(k)}_\alpha \in \mathcal{P}(j-1)\), let \(\Phi = (\phi_1, \ldots , \phi_V)\) be a linear ordering such that

\[ \phi_k < \phi_{k'}, \forall \phi_k \in X^{(k)}_\alpha, \phi_{k'} \in X^{(k')}_\alpha, k < k'. \] (7)

and \(r_{\alpha,V}\) be the rate vector returned by the call \(\text{PAR}(V, H, \Phi)\). For all \(k \in \{1, \ldots , K\}\), the integer-valued \(r^{(k)}_{\alpha^{(k)},X^{(k)}_\alpha}\) attains local omniscience in \(X^{(k)}_\alpha\); for all \(k \in \{1, \ldots , K\}\), \(r^{(k+1)}_{\alpha^{(k+1)},X^{(k+1)}_\alpha} \geq r^{(k)}_{\alpha^{(k)},X^{(k)}_\alpha}\).

Corollary V.5 holds the achievability of the multi-stage SO in Proposition V.1 for the non-asymptotic model. The proof is in Appendix E. We propose Algorithm III for determining this multi-stage SO, where the purpose of steps 8 to 10 is to obtain the values of \(X^{(k)}_\alpha, \alpha^{(k)}\) and \(\Phi\) in Corollary V.5. We explain these steps as follows. While each \(\alpha^{(k)}\) can be any integer-valued \(\alpha \in [\alpha^{(j)}, \alpha^{(j-1)}]\), we choose \(\alpha^{(k)}\) to be the minimum integer in the range \([\alpha^{(j)}, \alpha^{(j-1)}]\) so that the last value is \(\alpha^{(K)} = \min\{\alpha : \alpha \in [\alpha^{(1)}, \alpha^{(0)}] \cap \mathbb{Z} = R_{\text{NCO}}(V)\}\). Note, we could have the number of stages \(K < p\) if there is no integer in some range \([\alpha^{(j)}, \alpha^{(j-1)}]\). Once \(\alpha^{(1)}\) is determined, randomly select any nonsingleton \(C \in \mathcal{P}(p-1)\) and assign to \(X^{(1)}_\alpha\) and, for each \(k > 1\), choose \(X^{(k)}_\alpha\) as any subset \(C \in \mathcal{P}(j-1)\) such that \(X^{(k-1)}_\alpha \subseteq C\) so that \(X^{(1)}_\alpha \subseteq \ldots \subseteq X^{(k-1)}_\alpha \subseteq C = X^{(K)}_\alpha\) maintains for all \(k\) with the last subset being \(X^{(K)}_\alpha = V\) necessarily.

A linear ordering \(\Phi = (\phi_1, \ldots , \phi_V)\) satisfying (7) can be constructed by keeping each \(X^{(k)}_\alpha\) the first |\(X^{(k)}_\alpha|\) elements \(\phi_1, \ldots , \phi_{|X^{(k)}_\alpha|}\). To construct such a linear ordering \(\Phi\), let \(\{\phi_1, \ldots , \phi_{|X^{(k)}_\alpha|}\} = X^{(1)}_\alpha\) and \(\{\phi_{|X^{(k)}_\alpha|+1}, \ldots , \phi_{|X^{(k+1)}_\alpha|}\} = X^{(k+1)}_\alpha \setminus X^{(k)}_\alpha\) for all \(k \in \{1, \ldots , K\}\). For example, if \(X^{(1)}_\alpha = \{3, 4\}\) and \(X^{(2)}_\alpha = \{1, \ldots , 4\}\), we could have \(\Phi\) being \((3, 4, 1, 2), (3, 4, 2, 1), (4, 3, 1, 2)\) or \((4, 3, 2, 1)\), since all of them hold \(V_2 = X^{(1)}_\alpha = \{3, 4\}\) and \(V_4 = X^{(2)}_\alpha = \{1, \ldots , 4\}\).

Example V.6. We apply Algorithm III to the 5-user system in Example III.3. The call \(\text{PAR}(V, H, (4, 5, 2, 3, 1))\) returns the same \(Q_{\alpha,V}\) and \(r_{\alpha,V}\) as in Example V.4. For the three critical points \(\alpha^{(3)} = 4, \alpha^{(2)} = 6\) and \(\alpha^{(1)} = 6.5\), we need to search the integer valued \(\alpha\) for three regions \([4, 6]\), \([6, 6.5]\) and \([6.5, 10]\).

For region \([\alpha^{(3)}, \alpha^{(2)}] = [4, 6]\), we have \(\alpha^{(1)} = \min\{\alpha : \alpha \in [\alpha^{(3)}, \alpha^{(2)}] \cap \mathbb{Z}\} = 4\), where we assign \(X^{(1)}_\alpha = \{4, 5\}\) since it is the only nonsingleton subset in

10The linear ordering \(\Phi\) satisfying (7) ensures \(V_{|X^{(k)}_\alpha|} = X^{(k)}_\alpha\) for all \(k \in \{1, \ldots , K\}\).
Algorithm 4: Multi-stage Successive Omnicience (SO) by the PAR Algorithm for the Non-Asymptotic Model

\begin{algorithm}
\KwIn{$H$, $V$, and $\Phi$}
\KwOut{an achievable $K$-stage SO $(\{X_{(k)}^i, r_{(k)}^i\} : k \in \{1, \ldots, K\})$ in the non-asymptotic model.}

1 Arbitrarily choose a linear ordering $\Phi$ and call PAR($H, V, \Phi$) to obtain the segmented $Q_{\alpha,V}$ and $r_{\alpha,V}$ for all $\alpha$; $2 \quad k \leftarrow 1$ and $X^{(0)} \leftarrow \emptyset$;

3 for $j = p$ decreasing from $p$ to $1$ do

4 if $[\alpha^{(j)}, \alpha^{(j-1)}] \cap \mathbb{Z} \neq \emptyset$ then

5 $\bar{\alpha}^{(k)} \leftarrow \min \{\alpha : \alpha \in [\alpha^{(j)}, \alpha^{(j-1)}] \cap \mathbb{Z}\}$;

6 $X^{(k)} \leftarrow C$, where $C \in \mathcal{P}^{(j-1)}$ such that $X^{(k-1)} \subseteq C$;

7 $k \leftarrow k + 1$

8 $(\bar{\phi}_{X^{(k-1)}}, \ldots, \bar{\phi}_{X^{(1)}}) \leftarrow X^*_k \setminus X^{(k-1)}$;

9 endfor

10 $(Q_{\alpha,V}, \bar{r}_{\alpha,V}) \leftarrow \text{PAR}(H, V, \Phi)$;

11 Initiate $r^{(0)}_V \leftarrow (0, \ldots, 0)$;

12 for $k = 1$ to $K$ do

13 $r^{(k)}_V \leftarrow r^{(k-1)}_V$;

14 Randomly select user $i \in X^{(k-1)}$ and let $\Delta r \leftarrow r_{\bar{\alpha}^{(k)}}(X^{(k-1)}) - r_{\bar{\alpha}^{(k)}}(X^{(k-1)})$ and $r^{(k)}_i \leftarrow r^{(k)}_i + \Delta r$;

15 foreach $i \in X^{(k)} \setminus X^{(k-1)}$ do let user $i$ transmit at rate $r_{\bar{\phi}^{(k)}_i}$ and $r^{(k)}_i \leftarrow r^{(k)}_i$;

16 return $X^{(k)}_V$ and $r^{(k)}_V$ for all $k \in \{1, \ldots, K\}$;

17

endfor

end algorithm

\end{algorithm}
It is also of interest to see how the results on the non-asymptotic model can be applied to practical CCDE systems, e.g., apart from random linear network coding (RLNC) [27] in the recursive two-stage SO in Section IV-A.

APPENDIX A

PROOF OF COROLLARY II.2

Consider the asymptotic model first. Based on Theorem II.1 $R_{ACO}(X_\alpha) \leq R_{ACO}(V) - H(V) + H(X_\alpha)$ is the necessary and sufficient condition for $X_\alpha$ to be complimentary. We also have $R_{ACO}(X_\alpha) \geq \sum_{C \in P} H(X_\alpha | H(C))$, $\forall P \in \Pi(X_\alpha) : |P| > 1$. So, $\sum_{C \in P} H(X_\alpha | H(C)) \leq R_{ACO}(V) - H(V) + H(X_\alpha), \forall P \in \Pi(X_\alpha) : |P| > 1$, which is equivalent to $f_{R_{ACO}(V)}(X_\alpha) \leq \sum_{C \in P} f_{R_{ACO}(V)}(C), \forall P \in \Pi(X_\alpha)$, i.e., $f_{R_{ACO}(V)}(X_\alpha) = \hat{f}_{R_{ACO}(V)}(X_\alpha)$. In the same way, one can prove that $X_\alpha$ is complimentary in the non-asymptotic model if and only if $f_{R_{ACO}(V)}(X_\alpha) = \hat{f}_{R_{ACO}(V)}(X_\alpha)$.

APPENDIX B

PROOF OF LEMMA III.6

The proof is based on [12 Lemma 2]. If $\hat{X}_\alpha \subseteq V$ such that $|\hat{X}_\alpha| > 1$, we must have $f_\alpha(\hat{X}_\alpha) > f_{\alpha}(\hat{X}_\alpha)$ for some $\alpha \in \Pi(\hat{X}_\alpha) : |\Pi(\hat{X}_\alpha)| > 1$ such that $|\hat{X}_\alpha| > 1$. This necessarily means that $Q_\alpha(V) = \{ i : i \in \hat{X}_\alpha \}$. In this case, the value of $\alpha$ in the lemma is in fact the minimum sum-rate for the asymptotic model, i.e., $\alpha(\hat{X}_\alpha) = R_{ACO}(V) = \alpha(\hat{X}_\alpha)$ where the necessary and sufficient condition in Corollary II.2 does not hold, i.e., $\hat{X}_\alpha \subseteq V$ such that $|\hat{X}_\alpha| > 1$, therefore, there is no complimentary subset for $\hat{X}_\alpha$. Similarly, for $\hat{\alpha}_\alpha = \left[ \sum_{i \in V} H(X_\alpha | H(i)) \right], \forall \alpha \in \Pi(V)$ for the non-asymptotic model, we have $Q_{\hat{\alpha}_\alpha}(V) = \{ i : i \in V \}$. Due to the property of the PŚ $Q_\alpha(V) \subseteq Q_{\hat{\alpha}_\alpha}(V)$ for $\alpha \leq \hat{\alpha}_\alpha$ (see Section II-B). This necessarily means $\alpha \leq \hat{\alpha}_\alpha$, and there is no complimentary subset in the non-asymptotic model.

APPENDIX C

PROOF OF LEMMA IV.2

Based on [12 Lemma 2], for $\alpha = \sum_{i \in \hat{V}} \frac{H(V) - H(i)}{|\hat{V}|-1}$, $f_\alpha(C) = f_{\hat{\alpha}_\alpha}(C)$ for all $C \subseteq Q_\alpha(V)$. Also note that we must have $C \subseteq \hat{V}$ for all $C \in Q_\alpha(V)$. Then, Lemma II.3 holds for all $C \in Q_\alpha(V)$ such that $|C| > 1$ in the asymptotic model and therefore there is at least one complimentary subset. On the contrary, if the partition $Q_\alpha(V) = \{m : m \in V \}$ only contains singleton subsets at the end of last iteration, it means no subset $X_\alpha \subseteq V$ such that $|X_\alpha| > 1$ holds $f_\alpha(X_\alpha) = f_{\hat{\alpha}_\alpha}(X_\alpha)$. Based on Lemma III.6 there is no complimentary subset in the asymptotic model. The same statement holds for $\alpha = \left[ \sum_{i \in V} \frac{H(V) - H(i)}{|\hat{V}|-1} \right]$ for the non-asymptotic model can be proved in the same way. We prove the optimality of $r_{\alpha,C}$ as follows.

In the asymptotic model, consider the value of $\alpha$ satisfying $f_{\alpha}(C) = f_{\hat{\alpha}_\alpha}(C)$. We have $H(C) + \alpha - H(V) \leq H[P | (\alpha - H(V)), \forall P \in \Pi(C)$, which can be rewritten as $H(C) - H[P] \leq (|P| - 1)(\alpha - H(V))$ and converted to $\alpha \geq H(V) - H(C) + \sum_{P \in \Pi(C)} \frac{H(P)}{|P| - 1}$. Therefore, $\hat{\alpha} = \min\{ \alpha : f_{\alpha}(C) = f_{\hat{\alpha}_\alpha}(C) \}$

$= H(V) - H(C) + \max_{P \in \Pi(C) : |P| > 1} \sum_{X_\alpha \in \hat{V}} \frac{H(C) - H(X)}{|P| - 1}$

$= H(V) - H(C) + R_{ACO}(C)$.

Also, we have $r_{\alpha,C} \in B(f_{\alpha,C}^{(\alpha)} \subseteq C)$ because $r_{\alpha,C} \in P(f_{\alpha,C}^{(\alpha)} \subseteq C)$ based on Lemma II.2 and so, $R_{ACO}(C) = f_{\alpha,C}^{(\alpha)}(C)$.

So, the inequality $r_{\alpha,C}(X) \leq f_{\alpha}(C) = H(V) + \hat{\alpha} - H(V)$ holds for all $X \subseteq C$ and the equality $r_{\alpha,C}(X) = f_{\alpha,C}^{(\alpha)}(C) = H(V) - H(C) + R_{ACO}(C)$ holds for the sum-rate in $C$, i.e., the rate vector $r_{\alpha,C} \in R_{ACO}(C)$ is an optimal rate vector that attains the omniscience in $C$ with the minimum sum-rate $R_{ACO}(C)$. In the same way, we can prove that $\hat{\alpha} = \min\{ \alpha : f_{\alpha}(C) = f_{\hat{\alpha}_\alpha}(C) \}$ is an optimal rate vector for the non-asymptotic model.

APPENDIX D

PROOF OF COROLLARY V.2

Based on Lemma IV.1(a), $\chi_{\alpha}^{(k)}$ obtained in step 5 contains all nonsingleton subsets of $P(p-k)$ that are complimentary. For all $k \in \{1, \ldots, p\}$ and each $C \subseteq \chi_{\alpha}^{(k)}$, $\hat{r}(C) = \min\{ \alpha : f_{\alpha}(C) = f_{\hat{\alpha}_\alpha}(C) \}$ so that $r_{\hat{\alpha}_\alpha}(C) \in \beta_{\alpha}(C)$ with sum-rate $r_{\hat{\alpha}_\alpha}(p-k+1)(C) = R_{ACO}(C)$ according to Lemma IV.1(a) and Remark IV.2.

For $k = 1$, since $P(p) = \{ i : i \in V \}$, we have $C \subseteq \chi_{\alpha}^{(1)} = C \subseteq \hat{V}$. In this case, $r_{\hat{\alpha}_\alpha}(C) \in \beta_{\alpha}(C)$ with the dimensions $r_{\hat{\alpha}_\alpha}(i) \geq 0$ for all $i \in C$ [28 Lemma 2.3] and the monotonicity in Proposition V.2(b) holds. After step 7 the local omniscience is attained in each $C \subseteq \chi_{\alpha}^{(1)}$ with $r_{\hat{\alpha}_\alpha}(i) = r_{\hat{\alpha}_\alpha}(i)$ for all $i \in C$, i.e., the sum-rate $r_{\hat{\alpha}_\alpha}(1)(C)$ is assigned to the users in $C$.

By recursion, before step 7 of Algorithm 3 in iteration $k$, we have all nonsingleton $C' \subseteq \chi_{\alpha}^{(k)}$ attain local omniscience by an optimal rate vector $C'' = \hat{r}_{\hat{\alpha}_\alpha}(p-k+1)(C'' \subseteq \beta_{\alpha}(C'') \subseteq \beta_{\alpha}(C'' | (p-k+1))$ at some previous stage $k' < k$. So, all $C'$ can be treated as superusers with the index $C''$ and, for the problem of attaining the local omniscience in $C$, it suffices to consider the super-user system $C' = C'' \subseteq \beta_{\alpha}(C'')$. For $r_{\hat{\alpha}_\alpha}(p-k+1)(C') \subseteq \beta_{\alpha}(C') \subseteq \beta_{\alpha}(C')$ we have $r_{\hat{\alpha}_\alpha}(p-k+1)(C') \subseteq \beta_{\alpha}(C')$ reduce to $r_{\hat{\alpha}_\alpha}(p-k+1)(C') \subseteq \beta_{\alpha}(C')$ with $r_{\hat{\alpha}_\alpha}(p-k+1)(C') = R_{ACO}(C') = r_{\hat{\alpha}_\alpha}(p-k+1)(C')$.

12For an $\alpha, f_{\hat{\alpha}_\alpha}^{(\alpha)} : 2^C \to R$ such that $f_{\hat{\alpha}_\alpha}^{(\alpha)}(X) = f_{\alpha}(X)$ for all $X \subseteq C$ is the reduction of $f_{\alpha}$ on $C$ [28 Section 3.1(a)].
Therefore, we just need to assign the rates $r_{\alpha(p-k+1)}(C')$ to the users in $C'$, where the monotonicity in Proposition [11]b also holds for all $C'$: based on [12] Lemma ??(c),
\[
\Delta r = r_{\alpha(p-k+1)}(C') - r^{(k)}(C') = r_{\alpha(p-k+1)}(C') - r_{\alpha(p-k+1)}(C') > 0
\]
(8)
since $\alpha(p-k+1) > \alpha(p-k+1)$ and $C' \in \mathcal{P}(p-k+1)$ so that $(C')^{\mathcal{P}(p-k+1)} \subseteq \mathcal{P}(p-k+1)$ for all $k' < k$; For single-ton $C' \in \mathcal{C}(\mathcal{P}(p-k+1))$, we have $r^{(k)}(C') = 0$ so that \(\Delta r = r_{\alpha(p-k+1)}(C') - r^{(k)}(C') > 0\) [28] Theorem 9], i.e., Proposition [11]b holds. Note, the above rate updates only need to be considered for all $C \in \mathcal{A}(k)$ such that $C \neq \mathcal{C}(\mathcal{P}(p-k+1))$. This is because, if $C = \mathcal{C}(\mathcal{P}(p-k+1))$, the local omniscience in $C$ has already been attained in the previous stages.

At the end of the last stage $k = p$, $r_{\alpha(p-k+1)}(C) \in \mathcal{A}(p)$ and the rate vector $r_{\alpha,V}$ returned by the call $PAR(H,V,\Phi)$. Consider the rate vector $\mathcal{R}_{\alpha,V}$ returned by the call PAR($H,V,\Phi$), taking integer values for integer-valued entropy function $H$ and $v$ in the non-asymptotic model, we have $r_{\alpha,V} \in \mathbb{Z}^{[p]}$ for all integer-valued $\alpha$, i.e., $r_{\alpha(V)} \in \mathbb{Z}^{[p]}$ for all $k$. If $\alpha(k) = \alpha(k)^*$, we have shown in the proof of Corollary [7] that $\mathcal{R}_{\alpha(k),X^*}$ is an optimal rate vector that attains local omniscience in $X^*$ with the minimum sum-rate $R_{\alpha,k}(X^*) = R_{\alpha,k}(X^*)$; When $\alpha(k) \in \{\alpha(j), \alpha(j-1)\} \cap \mathbb{Z}$, while Lemma [7]b states that $\mathcal{R}_{\alpha(k),X^*} \in \mathcal{A}(\mathcal{A}(k))$ if $\alpha(k) = \min(\alpha: \alpha \in [\alpha(j), \alpha(j-1)] \cap \mathbb{Z})$, it can be proven in the same way that $\mathcal{R}_{\alpha(k),X^*} \in \mathcal{A}(\mathcal{A}(k)) \cap \mathbb{Z}^{[p]}(X^*)$ for any $\alpha \in [\alpha(j), \alpha(j-1)]$, i.e., $\mathcal{R}_{\alpha(k),X^*}$ is achievable, but may not be optimal. Thus, $\mathcal{R}_{\alpha(k),X^*}$ attains the local omniscience in $X^*$.

For the linear ordering $\Phi$ satisfying [7], we have $V[X^*] = X^*$ for all $k \in \{1, \ldots, K\}$. Based on [12] Lemma ??(a),
\[
\text{the call PAR($V,H,\Phi$) outputs a rate vector $r_{\alpha,V}$ such that $r_{\alpha}(X^*) = f_{\alpha}(V[X^*]) = f_{\alpha}(V[X^*]) = f_{\alpha}(X^*)$ for all $\alpha$. Then, according to [12] Lemma ??(c), for all $k \in \{1, \ldots, K-1\}$, since $\alpha(k) < \alpha(k)^*$, $\mathcal{R}_{\alpha(k),X^*} - \mathcal{R}_{\alpha(k)^*,X^*} > 0$.}

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