SL(2, R)-SYMMETRY AND NONCOMMUTATIVE PHASE SPACE IN (2+2) DIMENSIONS

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Abstract

We generalize the connection between 2t physics and noncommutative geometry. In particular, we apply our formalism to a target spacetime of signature (2+2). Specifically, we compute an algebra of a generalized SL(2, R)-Hamiltonian constraint, showing that it satisfies a kind of algebra associated with the noncommutative group $U_{*}(1, 1)$. We also comment about a possible connection between our formalism and nonsymmetric gravitational theory.

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It is known that noncommutative field theory of 2t physics [1-2] relies on a fundamental gauge symmetry principle based on the noncommutative group $U_*(1, 1)$ [3]. This approach originates from the observation that a worldline theory admits a Lie algebra $sl_*(2, R)$ gauge symmetry acting on phase space $(q, p)$ [4]. Local considerations of the general canonical transformations lead to the embedding of the corresponding noncommutative algebra $sl_*(2, R)$ into a bigger 4 parameter algebra $u_*(1, 1)$. However, it turns out that this noncommutative phase space symmetry is based on the usual noncommutative relation between $q$ and $p$ rather than on the full noncommutative phase space that includes the noncommutative configuration space of the $q$'s (and $p$'s) themselves. In this work we prove that it makes sense to consider this more general noncommutative phase space in 2t physics. We focus on the signature $2 + 2$ for at least two physical reasons; (1) it is the minimal possibility in 2t physics and (2) it is an exceptional signature [5]. In principle, however, our calculations also correspond to the more general case of $2 + d$ dimensions.

Let us start recalling the traditional transition from a classical to a quantum mechanical system. One may begin with the action

$$S[q] = \int_{t_i}^{t_f} dt L(q, \dot{q}),$$

(1)

where the Lagrangian $L$ is a function of the $q^i$-coordinates and the corresponding velocities $\dot{q}^i \equiv dq^i/dt$, with $i, j = 1, \ldots, n$. One then defines the canonical momentum $p_i$ conjugate to $q^i$ as follows;

$$p_i \equiv \frac{\partial L}{\partial \dot{q}^i},$$

(2)

and rewrites the action in the form

$$S[q, p] = \int_{t_i}^{t_f} dt (\dot{q}^i p_i - H(q, p)),$$

(3)

where $H = H(q, p)$ is the canonical Hamiltonian,

$$H(q, p) \equiv \dot{q}^i p_i - L.$$

(4)

The transition to quantum mechanics is made by promoting the Hamiltonian $H$ as an operator $\hat{H}$ via the nonvanishing commutator

$$[\hat{q}, \hat{p}] = -i,$$

(5)

with $\hbar = 1$, and by writing the quantum formula
\[ \hat{H} | \Psi > = 0, \quad (6) \]

which determines the physical states \(| \Psi >\) (see Refs. [6]-[8] for details). Here, 
\([\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}\) denotes a commutator for any arbitrary operators \(\hat{A}\) and \(\hat{B}\).

Recently, a new possibility to analyze the above program has emerged [9]. The key point for this new approach is the realization that since in the action (3) there is a hidden invariance \(SL(2, R) \approx Sp(2, R) \approx SU(1, 1)\) one may work in a unified canonical phase space of coordinates and momenta. Let us recall how this hidden invariance emerges [4]. Consider first the change of notation

\[ q^i_1 \equiv q^i, \tag{7} \]

and

\[ q^i_2 \equiv p^i. \tag{8} \]

Thus, by introducing the object \(q^i_a\) with \(a = 1, 2\) we see that these two formulae can be unified. The next step is to rewrite (3) in terms of \(q^i_a\) rather than in terms of \(q^i\) and \(p^i\). One find that up to a total derivative the action (3) becomes [4] (see also Ref. [9])

\[ S = \int_{t_i}^{t_f} dt \left( \frac{1}{2} \varepsilon^{ab}_i q^i_a q^i_b \delta_{ij} - H(q^i_a) \right). \tag{9} \]

Here, the symbol \(\delta_{ij}\) denotes a Kronecker delta and \(\varepsilon^{ab}_i = -\varepsilon^{ba}_i\), with \(\varepsilon^{12} = 1\), is the antisymmetric \(SL(2, R)\) invariant density. From this expression one observes that while the \(SL(2, R)\)-symmetry is hidden in (3) now it is manifest in the first term of (9). Thus, it is natural to require the same \(SL(2, R)\)-symmetry for the Hamiltonian \(H(q^i_a)\).

Consider the usual Hamiltonian for a free nonrelativistic point particle

\[ H = \frac{p^i p^j \delta_{ij}}{2m}, \tag{10} \]

with \(i = \{1, 2, 3\}\). According to the notation (7)-(8) this becomes

\[ H = \frac{q^i_a p^i q^j_a \delta_{ij}}{2m}. \tag{11} \]

This Hamiltonian is not, of course, \(SL(2, R)\)-invariant. Adding a potential \(V(q)\) to \(H\) does not modify this conclusion. Thus, a Hamiltonian of the form \(H = \frac{p^i p^j \delta_{ij}}{2m} + V(q)\) does not admit a \(SL(2, R)\)-invariant formulation. It turns
out that the same conclusion can be obtained when one considers the relativistic Hamiltonian constraint \( H = p^i p_i + m^2 = 0 \), where in this case \( i \) runs from 0 to 3.

The simplest example of \( SL(2,R) \)-invariant Hamiltonian seems to be

\[
H = \frac{1}{2} \lambda^{ab} q_a^i q_b^j \eta_{ij},
\]

which can be understood as the Hamiltonian associated with a relativistic harmonic oscillator in phase space. Here, we assume that \( \lambda^{ab} = \lambda^{ba} \) is a Lagrange multiplier and \( \eta_{ij} = \text{diag}(-1, -1, 1, 1) \). (Notice that we are considering the special case of \( 2 + 2 \) signature. The reason for this is that the symmetry of \( SL(2,R) \) requires necessarily two times and two times with two space coordinates provide an exceptional signature \([5]\). However, most our calculations below can be easily generalized to \( 2 + d \) dimensions.) Indeed, the Hamiltonian (12) is a total Hamiltonian according to the terminology of the Dirac’s constraint Hamiltonian systems formalism \([10]\) (see also Refs. \([6]-[8]\)). Let us write (12) in the form

\[
H = \frac{1}{2} \lambda^{ab} Q_{ab},
\]

where

\[
Q_{ab} = q_a^i q_b^j \eta_{ij}
\]

can be identified as the constraint of the physical system. Observe that the constraint \( Q_{ab} \approx 0 \) is symmetric in the indices \( a \) and \( b \), that is, \( Q_{ab} = Q_{ba} \). (Here the symbol ”\( \approx \)” means weakly equal to zero \([6]-[8]\).) Moreover, \( Q_{ab} \) is a first class constraint and this can be verified as follows. First note that using the definitions (7) and (8) we can write the usual Poisson bracket, for arbitrary functions \( f(q,p) \) and \( g(q,p) \) of the canonical variables \( q \) and \( p \),

\[
\{ f, g \} = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i},
\]

in the form

\[
\{ f, g \} = \varepsilon_{ab} \eta^{ij} \frac{\partial f}{\partial q_a^i} \frac{\partial g}{\partial q_b^j}.
\]

Thus, from (16) one discovers that

\[
\{ q_a^i, q_b^j \} = \varepsilon_{ab} \eta^{ij}.
\]
Now, using the formula (17) it is straightforward to compute \( \{ Q_{ab}, Q_{cd} \} \). Explicitly, we find

\[
\{ Q_{ab}, Q_{cd} \} = \{ q_a^i q_b^j \eta_{ij}, q_c^k q_d^l \eta_{kl} \}
\]

\[
= (\{ q_a^i, q_c^k \} q_b^j q_d^l + \{ q_a^i, q_d^l \} q_b^j q_c^k + \{ q_b^j, q_c^k \} q_a^i q_d^l + \{ q_b^j, q_d^l \} q_a^i q_c^k) \eta_{ij} \eta_{kl}
\]

\[
= (\varepsilon_{ac} \eta_{ik} q_b^j q_d^l + \varepsilon_{ad} \eta_{il} q_b^j q_c^k + \varepsilon_{bc} \eta_{jk} q_a^i q_d^l + \varepsilon_{bd} \eta_{jl} q_a^i q_c^k) \eta_{ij} \eta_{kl}.
\]

This implies the algebra,

\[
\{ Q_{ab}, Q_{cd} \} = \varepsilon_{ac} Q_{bd} + \varepsilon_{ad} Q_{bc} + \varepsilon_{bc} Q_{ad} + \varepsilon_{bd} Q_{ac}.
\]  

(19)

Thus, since we are assuming \( Q_{ab} \approx 0 \), one sees that \( \{ Q_{ab}, Q_{cd} \} \approx 0 \) which means that \( Q_{ab} \) is a first class constraint. It turns out that \( Q_{ab} \) can be also understood as the gauge generator of the \( SL(2, R) \)-symmetry which is in fact determined by the algebra (19) (see Ref. [11])

At the quantum level we promote \( Q_{ab} \) as an operator \( \hat{Q}_{ab} \) and write

\[
[\hat{Q}_{ab}, \hat{Q}_{cd}] = -i(\varepsilon_{ac} \hat{Q}_{bd} + \varepsilon_{ad} \hat{Q}_{bc} + \varepsilon_{bc} \hat{Q}_{ad} + \varepsilon_{bd} \hat{Q}_{ac})
\]  

(20)

and

\[ \hat{Q}_{ab} | \Psi > = 0. \]  

(21)

Explicitly, the nonvanishing brackets of the algebra (20) can decomposed in the form

\[
[\hat{Q}_{11}, \hat{Q}_{22}] = -4i \hat{Q}_{12},
\]

(22)

\[
[\hat{Q}_{11}, \hat{Q}_{12}] = -2i \hat{Q}_{11},
\]

(23)

and

\[
[\hat{Q}_{22}, \hat{Q}_{12}] = +2i \hat{Q}_{22}.
\]

(24)

By writing \( \hat{J}_3 = -\frac{1}{2} \hat{Q}_{12}, \hat{J}_1 = \frac{1}{4}(\hat{Q}_{11} + \hat{Q}_{22}) \) and \( \hat{J}_2 = \frac{1}{4}(\hat{Q}_{11} - \hat{Q}_{22}) \) one finds the result

\[
[\hat{J}_1, \hat{J}_2] = -i \hat{J}_3
\]

(25)

\[
[\hat{J}_1, \hat{J}_3] = i \hat{J}_2,
\]

(26)

and

\[
[\hat{J}_2, \hat{J}_3] = i \hat{J}_1,
\]

(27)
which can be obtained from the algebra

\[ [\hat{J}_{AB}, \hat{J}_{CD}] = i(\eta_{AC}\hat{J}_{BD} - \eta_{AD}\hat{J}_{BC} + \eta_{BD}\hat{J}_{AC} - \eta_{BC}\hat{J}_{AD}), \]  

(28)

where \( \eta_{AB} = (-1, 1, 1), \hat{J}_{AB} = -\hat{J}_{BA} \) and \( \hat{J}^A = \frac{1}{2} \epsilon^{ABC} \hat{J}_{BC} \), with \( \epsilon^{123} = 1 \) and \( \epsilon_{123} = -1 \). This is one way to see that \( SL(2, R) \) is in fact the covering group of \( SO(1, 2) \).

We would like now to generalize (17) in the form

\[ \{q^i_a, q^j_b\} = \varepsilon_{ab} \eta^{ij} + g_{ab} \Omega^{ij}. \]  

(29)

Here, \( \Omega^{ij} \) is skew-simplectic form which can be chosen as

\[
\Omega_{ij} = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]  

(30)

By convenience in (29) we choose \( \eta_{ij} = \text{diag}(-1, 1, -1, 1) \) rather than \( \eta_{ij} = \text{diag}(-1, -1, 1, 1) \) and \( g_{ab} = \text{diag}(\theta, \phi) \). Here \( \theta \) and \( \phi \) are two constant parameters. Note that \( \eta_{ij} \) corresponds to a flat signature. The reason for this choice for \( \eta_{ij} \), among other things, is because using the signature \((1 + 1) + (1 + 1)\) instead of \((2+2)\) some calculations are simplified. It is important to mention that the expression (30) differs from the usual simplectic structure \( \begin{pmatrix} 0 & \delta_{ij} \\ -\delta_{ij} & 0 \end{pmatrix} \) by a change of bases. So, one can also think on (30) as a consequence of the Darboux theorem.

One can prove that

\[
\Omega^{ij} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}.
\]  

(31)

In particular, using (30) and (31) we find the result \( \Omega^{ij} \Omega^{kl} \eta_{jl} = -\eta^{ik} \).

The generalization (29) motivates us to modify also the Hamiltonian constraint (13) as follows

\[ \mathcal{H} = \frac{1}{2} \Lambda^{ab} \Sigma_{ab}, \]

(32)

where \( \Lambda^{ab} \) are new Lagrange multipliers no necessary symmetric in the indices \( a \) and \( b \) and
Here, $\xi$ is another constant parameter. Of course, the constraint $\Sigma_{ab}$ reduces to $Q_{ab}$ when $\xi \to 0$. We have

$$\{\Sigma_{ab}, \Sigma_{cd}\} = \{q^i_a q^j_b \gamma_{ij}, q^k_a q^l_b \gamma_{kl}\}$$

$$= ((q^i_a, q^k_c)q^j_d + (q^i_a, q^l_d)q^j_c + (q^j_b, q^k_c)q^i_d + (q^j_b, q^l_d)q^i_c)\gamma_{ij}\gamma_{kl}$$

$$= ((\varepsilon_{ac} \eta^{jk} + g_{ac} \Omega^{jk})q^j_d q^k_d + (\varepsilon_{ad} \eta^{jl} + g_{ad} \Omega^{jl})q^l_d q^k_d)\eta_{ij} + i\xi \Omega_{ij})(\eta_{kl} + i\xi \Omega_{kl}),$$

where we used the definition

$$\gamma_{ij} = \eta_{ij} + i\xi \Omega_{ij},$$

which can be understood as an Hermitian metric since $\gamma_{ij}^\dagger = \gamma_{ij}$. After some straightforward algebra we get

$$\{\Sigma_{ab}, \Sigma_{cd}\} = (1 + \xi^2)\varepsilon_{ac} M_{bd} + (1 - \xi^2)\varepsilon_{ad} M_{bc} + (1 - \xi^2)\varepsilon_{bc} M_{ad}$$

$$+ (1 + \xi^2)\varepsilon_{bd} M_{ac} + 2i\xi \varepsilon_{ad} S_{cb} + 2i\xi \varepsilon_{bc} S_{ad} + (1 + \xi^2)g_{ac} S_{bd} + (1 - \xi^2)g_{ad} S_{bc}$$

$$+ (1 + \xi^2)g_{bd} S_{ac} + (1 - \xi^2)g_{bc} S_{ad} - 2i\xi g_{ad} M_{bc} + 2i\xi g_{bc} M_{ad}.$$  

Here, we define $M_{ab} = q^i_a q^j_b \eta_{ij} = M_{ba}$ and $S_{ab} = q^i_a q^j_b \Omega_{ij} = -S_{ba}$. In other words we have

$$\Sigma_{ab} = M_{ab} + i\xi S_{ab}.$$  

Since in two dimensions we can always write

$$S_{ab} = \kappa \varepsilon_{ab},$$

with $\kappa = \frac{1}{2} \varepsilon^{ab} q^i_a q^j_b \Omega_{ij}$, we find that (36) is simplified in the form

$$\{\Sigma_{ab}, \Sigma_{cd}\} = (1 + \xi^2)\varepsilon_{ac} M_{bd} + (1 - \xi^2)\varepsilon_{ad} M_{bc} + (1 - \xi^2)\varepsilon_{bc} M_{ad}$$

$$+ (1 + \xi^2)\varepsilon_{bd} M_{ac} + (1 + \xi^2)g_{ac} S_{bd} + (1 - \xi^2)g_{ad} S_{bc}$$

$$+ (1 - \xi^2)g_{bd} S_{ac} + (1 + \xi^2)g_{bc} S_{ad} - 2i\xi g_{ad} M_{bc} + 2i\xi g_{bc} M_{ad}.$$  

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It is not difficult to see that if the parameter \( \xi \) and the quantity \( \kappa \) vanish then (39) is reduced to (19). Thus, the expression (39) provides a generalization of the algebra \( sl(2, \mathbb{R}) \). In fact, (39) seems to correspond to the algebra associated with the noncommutative group \( U(1, 1) \). One way to understand this conclusion it is by observing that using (38) the expression (37) can be written as

\[
\Sigma_{ab} = M_{ab} + i\omega_0 \varepsilon_{ab},
\]

with \( \omega_0 = \xi \kappa \). It turns out that according to reference [1] these are precisely the algebra generators associated with the noncommutative group \( U(1, 1) \) (see Ref. [3] for details). However, in the way the expression (39) it is written, it is not clear whether it is a closed algebra. In order to clarify this point we shall use the property that in two dimensions it is always possible to choose a basis such that

\[
M_{ab} = \rho g_{ab},
\]

where \( \rho \neq 0 \) is an arbitrary function of the coordinates \( q^i \). Using this choice for \( M_{ab} \) one finds that the last two terms of (39) vanish. Thus, one discovers that (39) can be reduced in the form

\[
\{\Sigma_{ab}, \Sigma_{cd}\} = (\rho + \kappa)[(1 + \xi^2)(\varepsilon_{ac} g_{bd} + \varepsilon_{bd} g_{ac}) + (1 - \xi^2)(\varepsilon_{ad} g_{bc} + \varepsilon_{bc} g_{ad})].
\]

Now, from (40) and (41) one sees that \( \Sigma_{ab} = \rho g_{ab} + i\omega_0 \varepsilon_{ab} \). This implies that (42) can be written as

\[
\{\Sigma_{ab}, \Sigma_{cd}\} = \frac{\rho + \kappa}{\rho} [(1 + \xi^2)(\varepsilon_{ac} \Sigma_{bd} + \varepsilon_{bd} \Sigma_{ca}) + (1 - \xi^2)(\varepsilon_{ad} \Sigma_{bc} + \varepsilon_{bc} \Sigma_{da})]
\]

or

\[
\{\Sigma_{ab}, \Sigma_{cd}\} = C^{ef}_{abcd} \Sigma_{ef},
\]

where

\[
C^{ef}_{abcd} = \frac{(\rho + \kappa)}{\rho} [(1 + \xi^2)(\varepsilon_{ac} \delta^e_a \delta^f_d + \varepsilon_{bd} \delta^e_c \delta^f_a) + (1 - \xi^2)(\varepsilon_{ad} \delta^e_b \delta^f_c + \varepsilon_{bc} \delta^e_d \delta^f_a)].
\]

Thus, we have shown that the algebra (39) can be written in the closed form (44), with \( C^{ef}_{abcd} \) playing the role of the structure "constants". Of course, since
we are assuming \( \Sigma_{ab} \approx 0 \), from (40) one sees that \( \{ \Sigma_{ab}, \Sigma_{cd} \} \approx 0 \) which means that \( \Sigma_{ab} \) is a first class constraint.

Let us make some final analysis. First, consider the most general prescription of the canonical variables \( q^i_a \),

\[
\{ q^i_a, q^j_b \} = \theta^{ij}_{ab},
\]  

(46)

which can be obtained from the generalized bracket

\[
\{ f, g \} = \theta^{ij}_{ab} \frac{\partial f}{\partial q^i_a} \frac{\partial g}{\partial q^j_b}.
\]  

(47)

Since we can always decompose any matrix \( B^{ij} = B^{(ij)} + B^{[ij]} \) in its symmetric \( B^{(ij)} \) and antisymmetric \( B^{[ij]} \) parts, one finds that (47) can also be written as

\[
\{ f, g \} = (\theta^{(ij)}_{(ab)} + \theta^{[ij]}_{(ab)} + \theta^{[ij]}_{[ab]}) \frac{\partial f}{\partial q^i_a} \frac{\partial g}{\partial q^j_b}.
\]  

(48)

It is not difficult to realize that if we want that the bracket \( \{ f, g \} \) determines a simplectic structure we must set \( \theta^{(ij)}_{(ab)} = 0 \) and \( \theta^{[ij]}_{[ab]} = 0 \). Therefore, (48) can be reduced to

\[
\{ f, g \} = (\theta^{(ij)}_{(ab)} + \theta^{[ij]}_{(ab)}) \frac{\partial f}{\partial q^i_a} \frac{\partial g}{\partial q^j_b}.
\]  

(49)

This can be simplified further by considering that in two dimensions we can always write \( \theta^{(ij)}_{[ab]} = \epsilon_{ab} \eta^{ij} \) where we assumed a flat "spacetime" \( \eta^{ij} \) (Of course, in a more general case one can assume a curved metric \( g^{ij} \).) Similarly, in two dimensions we can write \( \theta^{[ij]}_{(ab)} = g_{ab} \Omega^{ij} \), where \( g_{ab} = g_{ba} \) is a two dimensional metric and \( \Omega^{ij} = -\Omega^{ji} \). In order to distinguish between \( q \)'s and \( p \)'s we choose a basis such that \( g_{ab} = \text{diag}(\theta, \phi) \), but in principle in two dimensions one can always find a basis such that \( g_{ab} \to \sigma \delta_{ab} \), where \( \sigma \) is a constant conformal factor. Thus, we have proved that the most general meaningful simplectic structure is provided by the bracket

\[
\{ f, g \} = (\epsilon_{ab} \eta^{ij} + g_{ab} \Omega^{ij}) \frac{\partial f}{\partial q^i_a} \frac{\partial g}{\partial q^j_b}.
\]  

(50)

It turns out that this expression leads precisely to our generalized bracket (29). Notice that the above calculation is true for any even spacetime dimension. In terms of \( q^i \) and \( p^j \) one finds that the algebra (29) becomes

\[
\{ q^i, p^j \} = \eta^{ij},
\]  

(51)
\[\{q^i, q^j\} = \theta \Omega^{ij},\]  
(52)

and

\[\{p^i, p^j\} = \phi \Omega^{ij}.\]  
(53)

(See Refs. [12] and [13] for an alternative construction.) We still need to justify that the quantities \(\theta\) and \(\phi\) can be chosen as a constant parameters. Let us first introduce new canonical variables \(\tilde{q}^i_a\) such that

\[q_{ai} = a_{aij} \tilde{q}_j^a,\]  
(54)

where \(q_{ai} = \eta_{ij} q^j_a\) and \(a_{aij} = g_{ac} a_{cj}^b\). Writing \(a_{ij}^{ab} = g_{ij}^{ab} + A_{ij}^{ab}\) with \(g_{ij}^{ab} = \frac{1}{2}(a_{ij}^{ab} + a_{ij}^{ba})\) and \(A_{ij}^{ab} = \frac{1}{2}(a_{ij}^{ab} - a_{ij}^{ba})\) we see that one can always write (54) as

\[q_{ai} = a_{aij} \tilde{q}_j^a + A_{aij} \tilde{q}_j^a.\]  
(55)

We require that the new variables \(\tilde{q}^i_a\) satisfy the usual canonical algebra

\[\{\tilde{q}^i_a, \tilde{q}^j_b\} = \varepsilon_{ab} \eta^{ij};\]  
(56)

or

\[\{\tilde{q}^i, \tilde{p}^j\} = \eta^{ij},\]  
(57)

and

\[\{\tilde{q}^i, \tilde{q}^j\} = 0,\]  
(58)

and

\[\{\tilde{p}^i, \tilde{p}^j\} = 0,\]  
(59)

where we used the corresponding definitions (7) and (8) for \(\tilde{q}^i_a\). The authors of Ref. [14] have shown that a meaningful result can be obtained if \(g_{ij}^{ab} = g_{ij}^{ba}\) and \(A_{ij}^{ab} = -A_{ij}^{ba}\), that is, if \(g_{ij}^{ab}\) is symmetric in both kind of indices \(a, b\) and \(i, j\), and \(A_{ij}^{ab}\) is antisymmetric in both kind of indices \(a, b\) and \(i, j\). Moreover, these authors show that one can assume \(g_{1ij}^1 = \alpha \eta_{ij}, g_{2ij}^2 = \beta \eta_{ij}\), while \(g_{1ij}^2 = g_{2ij}^1 = 0\).

Here, \(\alpha\) and \(\beta\) are, in principle, two different constant parameters. In addition, one can take \(A_{ij}^{ab} = \varsigma \varepsilon^{ab} \Omega_{ij}\), with \(\varsigma = (\alpha \beta - 1)^{1/2}\). Thus, these results can be summarized by writing (55) in the form

\[q^i = \alpha \tilde{q}^i - \varsigma \Omega^{ij} \tilde{p}_j\]  
(60)

and
\[ p^i = \beta \tilde{p}^i + \zeta \Omega^{ij} \tilde{q}_j. \] (61)

Solving \( \tilde{q}_i \) and \( \tilde{p}_i \) in terms of \( q^i \) and \( p^j \) one finds \([15]\)

\[ \tilde{q}^i = \frac{1}{\rho}(\beta q^i + \zeta \Omega^{ij} p_j) \] (62)

and

\[ \tilde{p}^i = \frac{1}{\rho}(\alpha p^i - \zeta \Omega^{ij} q_j). \] (63)

Here, \( \rho = 2\alpha\beta - 1 \).

Using (57)-(61) we obtain

\[ \{q^i, p^j\} = \eta^{ij}, \] (64)

\[ \{q^i, q^j\} = 2\alpha\zeta \Omega^{ij} \] (65)

and

\[ \{p^i, p^j\} = 2\beta\zeta \Omega^{ij}. \] (66)

By comparing (64)-(66) with (51)-(53) we see that one must set \( \theta = 2\alpha\zeta = 2\alpha(\alpha\beta - 1)^{1/2} \) and \( \phi = 2\beta\zeta = 2\beta(\alpha\beta - 1)^{1/2} \). Thus, one discovers that these results not only prove that it makes sense to choose \( \theta \) and \( \phi \) as a constant parameters but also assure that due to (57)-(59) any three arbitrary functions \( f(\tilde{q}_a), g(\tilde{q}_a) \) and \( h(\tilde{q}_a) \) satisfy automatically the Jacobi identity. In turn, this implies that the variables \( \Sigma_{ab} \) must also satisfy the Jacobi identity.

Another possible consequence of our formalism is that we can develop noncommutative field theory by defining the noncommutative Moyal star product as follows

\[ (F \star G)(q^i_a) = \exp(\theta^{ij}_{ab} \frac{\partial}{\partial q^i_a} \frac{\partial}{\partial \tilde{q}^j_b})F(q^i_a)G(\tilde{q}^i_a) \Big|_{\tilde{q}^i_a = \tilde{q}^i_a}, \] (67)

with \( \theta^{ij}_{ab} = \varepsilon_{ab}\eta^{ij} + g_{ab}\Omega^{ij} \). Consequently, one can define the star commutator between any two field \( F(q^i_a) \) and \( G(q^i_a) \) as

\[ [F, G]_\star = F \star G - G \star F. \] (68)

In particular, it may be interesting for further research to apply this construction to the cases of gravity \([16]-[18]\), self-dual gravity \([19]\) and area-preserving diffeomorphisms in gauge theory on a non-commutative plane \([20]\).
It seems also interesting to generalize the constraint Hamiltonian (32) to a curved spacetime as follows

\[ \mathcal{H} = \frac{1}{2} \Lambda^a_{\phantom{a}b} q^i_b q^j_c (g_{ij}(q^k_c) + iA_{ij}(q^k_c)). \]  

(69)

Here \( g_{ij}(q^k_c) = g_{ji}(q^k_c) \) is a curved spacetime metric and \( A_{ij} = -A_{ji} \) is antisymmetric gauge field. This generalizes the metric (35) in the form

\[ \varphi_{ij}(q^k_c) = g_{ij}(q^k_c) + iA_{ij}(q^k_c), \]  

(70)

which is also a Hermitian metric. It turns out that this kind of metric is the main mathematical object in nonsymmetric gravitational theories (see Refs. [21]-[23] and references therein). But of course our metric refers to the phase space rather to the configuration space itself. At this respect it is worth mentioning that in Ref. [24] it is provided evidence for a position and momentum dependent metric in 2t physics.

Moreover, it turns out interesting to write (70) in the alternative vielbeins form

\[ \varphi_{ij}(q^k_c) = e^{(m)}_i(q^k_c)e^{(n)}_j(q^k_c)\eta_{(mn)} + if^{(m)}_i(q^k_c)f^{(n)}_j(q^k_c)\Omega_{(mn)}, \]  

(71)

with \( g_{ij} = e^{(m)}_i(q^k_c)e^{(n)}_j(q^k_c)\eta_{(mn)} \) and \( A_{ij} = f^{(m)}_i(q^k_c)f^{(n)}_j(q^k_c)\Omega_{(mn)} \). This way to write (70) it suggests to consider a star product deformation \( E_i^{(m)} \star E_i^{(n)} \), where we have introduced the complex vielbein field \( E_i^{(m)} = e^{(m)}_i + if^{(m)}_i \). This should lead of course to an infinite number of corrections to (70). Moreover, we should mention that the Moyal product in curved phase space [25] has already been studied by a number of authors, including Fedosov [26] and Kontsevich [27] (see also Ref. [28]). However, it seems that the particular case of 2+2 dimensions has not been considered. In any case the metric (70) seems to determine a bridge between our formalism and nonsymmetric gravitational theory, which we expect to explore in more detail in the coming future.

The present work it might be also relevant in connection with the Ref. [29] where there is a region with two-times in \( U_s(1,1) \times U_s(1,1) \) noncommutative gauge theory formulation of 3D gravity.

It has been established [30] a connection between 2t physics and oriented matroid theory [31] (for a connection between oriented matroid theory and other scenarios in high energy physics see Refs. [32]-[33] and references therein). Since the 2 + 2 signature is linked to brane physics [34] which in turn it is connected to oriented matroids [35-36] it may also be physically interesting for further directions to investigate the relation between the present formalism and all these scenarios in the context of oriented matroid theory.
We should mention a number of interesting topics that may be related to
the present formalism. It is known that in the Yang’s algebra the coordinates
and the momenta are also not commuting [37]-[39]. The relevant group in
this case is the conformal group $SO(2,4)$ which has also an important status
in 2t physics (see Refs. [1], [2] and [4]). Another direction for extensions of
our calculations is the possibility to include in the discussion the quantum
group concept. At this respect the work by Majid [40] may be of special
help since as this author emphasize "Lie groups are the simplest Riemann
manifolds and quantum groups are the simplest noncommutative spaces". So,
quantum groups are deeply connected with noncommutative geometry and in
this direction the Refs. [41] and [42] may be specially useful. Finally, bi-
Hamiltonian structure for integrable models (see Refs [43]-[45] and references
there in) may require also two times and one wonders whether our formalism
may also find an important application in such a subject.

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References

[1] I. Bars, Phys. Rev. D 64, 126001 (2001); hep-th/0106013.
[2] I. Bars, and S. J. Rey, Phys. Rev. D 64, 046005 (2001); hep-th/0104135.
[3] I. Bars, Sheikh-Jabbari and M. A. Vasiliev Phys. Rev. D 64, 086004
(2001); hep-th/0103209.
[4] I. Bars, Class. Quant. Grav. 18 (2001) 3113; hep-th/0008164.
[5] J. A. Nieto, Nuovo Cim. B 120, 135 (2005); hep-th/0410003.
[6] M. Henneaux and C. Teitelboim, Quantization of Gauge Systems (Princeto-
on University Press, Princeton, New Jersey, 1992).
[7] J. Govaerts, *Hamiltonian Quantisation and Constrained Dynamics* (Leuven University Press, Leuven, 1991).

[8] A. Hanson, T. Regge and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Roma, 1976).

[9] V. M. Villanueva, J. A. Nieto, L. Ruiz and J. Silvas, J. Phys. A 38, 7183 (2005); [hep-th/0503093].

[10] P. A. M. Dirac, *Lectures on Quantum Mechanics* (New York: Yeshiva UP, 1964).

[11] J. M. Romero and A. Zamora, Phys. Rev. D 70, 105006 (2004); [hep-th/0408193].

[12] R. Banerjee, Mod. Phys. Lett. A 17, 631 (2002); [hep-th/0106280].

[13] V. Cuesta, M. Montesinos and J. D. Vergara, Phys. Rev. D 76, 025025 (2007).

[14] K. Li, J. Wang and Ch. Chen, Mod. Phys. Lett. A 20, 2165 (2005); [hep-th/0409234].

[15] V. Cuesta, M. Montesinos and J. D. Vergara, ”Gauge systems with noncommutative phase space”, [hep-th/0611333].

[16] A. H. Chamseddine, G. Felder and J. Frohlich, Comm. Math. Phys. 155, 205 (1993), [hep-th/9209044]; Comm. Math. Phys. 218, 283 (2001); [hep-th/0005222].

[17] P. Aschieri, M. Dimitrijevic, F. Meyer and J. Wess, Class. Quant. Grav. 23, 1883 (2006); [hep-th/0510059].

[18] C. Castro, Adv. Studies in Theor. Phys. 2, no. 7, 309 (2008).

[19] S. Estrada-Jimenez, H. García-Compean, O. Obregón and C. Ramirez, Phys. Rev. D 78, 124008 (2008); [arXiv:0808.0211].

[20] W. Bietenholz, A. Bigarini and A. Torrielli, JHEP 0708, 041 (2007); [arXiv:0705.3536].

[21] J. Moffat, J. Math. Phys. 36, 5897 (1995); Phys.Lett. B 491, 345 (2000); [hep-th/0007181].

[22] C. Castro, Phys. Lett. B 668, 442 (2008).
[23] J.A. Nieto, ”Is nonsymmetric gravity related to string theory?”,
hep-th/9610160.

[24] W. Chagas-Filho, JHEP 0810, 060 (2008); arXiv:0802.2840.

[25] R. Coquereaux and A. Jadczyk Rev. Math. Phys. 2, 1 (1990).

[26] B. Fedosov, J. Diff. Geometry, 40, 213 (1994).

[27] M. Kontsevich, Lett. Math. Phys. 66, 157 (2003); q-alg/9709040

[28] C. Castro, J. Geom. and Phys, 33 173 (2000); hep-th/9802023.

[29] H. C. Kim, M. I. Park, C. Rim and J. H. Yee, JHEP 0810, 060 (2008);
arXiv:0710.1362.

[30] J. A. Nieto, Rev. Mex. Fis. E 51, 5 (2005); hep-th/0407093

[31] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. M. Ziegler,
Oriented Martroids, (Cambridge University Press, Cambridge, 1993).

[32] J. A. Nieto and M.C. Marín, J. Math. Phys. 41, 7997 (2000);
hep-th/0005117

[33] J. A. Nieto, J. Math. Phys. 45, 285 (2004); hep-th/0212100.

[34] J. A. Nieto, Mod. Phys. Lett. A 22, 2453 (2007); hep-th/0606219.

[35] J. A. Nieto, Adv. Theor. Math. Phys. 10, 747 (2006); hep-th/0506106.

[36] J. A. Nieto, Adv. Theor. Math. Phys. 8, 177 (2004); hep-th/0310071

[37] C. N. Yang, Phys. Rev. 72, 847 (1947).

[38] S. Tanaka, Nuovo Cim. B 114, 49 (1999), hep-th/9808064 ”Yang’s quantized
space-time algebra and holographic hypothesis” hep-th/0303105

[39] C. Castro, Phys. Letts B 626, 209 (2005).

[40] S. Majid, Lect. Notes Phys. 541, 227 (2000); hep-th/0006166

[41] M. S. Engeli, ”Quantization of SL(2,R)* as Bialgebra”; math/0211448

[42] L. Castellani, Class. Quant. Grav. 17, 3377 (2000); hep-th/0005210

[43] W. Ma and B Fuchssteiner, Physics Letters A 213, 49 (1996)
[44] W Oevel and W Strampp, Comm.Math. Phys. 157, 51 (1993).
[45] W Oevel and Z Popowicz, Comm. Math. Phys 39 441 (1991).