ABSOlutely Continuous Invariant MeasUres for PIECEwise $C^2$ AND EXPANDING MAPPINGS IN HIGHER DIMENSIONS

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Abstract. In this paper, by using a trace theorem in the theory of functions of bounded variation, we prove the existence of absolutely continuous invariant measures for a class of piecewise expanding mappings of general bounded domains in any dimension.

1. Introduction. Since the pioneering work of Lasota and Yorke [7] in 1973 on the existence of absolutely continuous invariant measures for a class of piecewise $C^2$ and stretching mappings of an interval, there have been several approaches in generalizing their work to multi-dimensional mappings. A major contribution to the existence result on multi-dimensional absolutely continuous invariant measures was the paper [5] by Góra and Boyarsky in 1989, following the earlier approach of using the modern notion of variation in [3] on rectangular partitions of the domain. In their work the theory of distributions for multi-variable functions was employed to prove the existence of fixed densities for Frobenius-Perron operators associated with piecewise $C^2$ and expanding mappings in $\mathbb{R}^N$ with the assumption that where the $C^2$ pieces of the boundaries meet, the angle subtended by the tangents to these pieces at the point of contact is bounded away from zero. Recently Adl-Zarabi [1] has extended the result of [5] to more general domains with cusps on the boundaries, based on the idea of Keller for two dimensional mappings. To our knowledge, all the previous approaches were by means of investigating the behavior of the function value on the boundary of a domain under various conditions through complicated analysis.

In this paper we want to clarify some points not very clearly stated previously by providing a general variation approach to proving the existence result via directly employing the trace theorem in the theory of functions of bounded variation that was proved in such books as [4] and [9]. While we do not explore function values on the boundary of the partition of the domain for different cases, our purpose here is to give a general condition for the existence of absolutely continuous invariant measures for a domain in any dimension so that the previous work reduces to

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checking the condition in different cases. However we point out that estimating the constant in the trace theorem for more general cases is a difficult problem that requires delicate analysis from geometric measure theory.

In the next section we briefly review the concept of variation and its basic properties, and in Section 3 we give the unified approach to establishing the existence theorem.

2. Preliminaries. The concept of variation plays an important role in the compactness argument in $L^1$ spaces (see [4] and [9] for more details). In this section we first introduce the modern notion of variation for functions of several variables and present some results for later use.

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. As standard notation in functions theory, $C^m(\Omega)$ is the space of functions which have continuous derivatives up to order $m$ in $\Omega$, and $C^m_0(\Omega)$ consists of those in $C^m(\Omega)$ with compact support. $C^m(\Omega; \mathbb{R}^N)$ and $C^m_0(\Omega; \mathbb{R}^N)$ denote the corresponding vector functions spaces, respectively.

Let $m$ denote the Lebesgue measure on $\mathbb{R}^N$, and let $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the usual Euclidean inner product and the Euclidean 2-norm for $\mathbb{R}^N$, respectively. The space and norm notations for different functions spaces, such as the Sobolev spaces $W^{1,1}(\Omega)$ and $W^{0,1}(\Omega) \equiv L^1(\Omega)$, are used in a standard way as in [4] or [9].

Definition 2.1. ([4]) Let $f \in L^1(\Omega)$. The number (may be $\infty$)

$$V(f; \Omega) = \sup \{ \int_{\Omega} f \, \text{div} \, g \, dm : g \in C^1_0(\Omega; \mathbb{R}^N), \ |g(x)| \leq 1, \ x \in \Omega \}$$

is called the variation of $f$. Here $\text{div} = \sum_{i=1}^N \frac{\partial g_i}{\partial x_i}$ is the divergence of $g$.

Remark 2.1. The gradient of $f \in L^1(\Omega)$ in the sense of distribution will be denoted by $Df$ [4], and so we can write $V(f; \Omega)$ as $\int_{\Omega} |Df|$. The latter notation for the variation of $f$ comes from the fact that if $f \in C^1(\Omega)$, then

$$\int_{\Omega} |Df| = \int_{\Omega} |\text{grad} f| \, dm$$

where $\text{grad} f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_N)$ is the gradient of $f$ in the classic sense, which is deduced from integration by parts,

$$\int_{\Omega} f \, \text{div} \, g \, dm = -\int_{\Omega} \sum_{i=1}^N \frac{\partial f}{\partial x_i} g_i \, dm$$

for $g \in C^1_0(\Omega; \mathbb{R}^N)$. More generally, if $f \in W^{1,1}(\Omega)$, then

$$\int_{\Omega} |Df| = \int_{\Omega} |\text{grad} f| \, dm$$

where $\text{grad} f = (f_1, \ldots, f_N)^T$ and $f_i$ are the weak derivatives of $f$.

Remark 2.2. $Df$ is a vector valued Radon measure $\omega = (\omega_1, \ldots, \omega_N)$ for $f \in BV(\Omega)$ and $\int_{\Omega} |Df|$ is actually the total variation of $\omega$ on $\Omega$. More generally $\int_A |Df|$ is the total variation of $\omega$ on $A \subset \Omega$ (see Remark 1.5 of [4]; also [8]). Also the
notation $\int_A <Df, g> \mathds{1}$ means the integral of the vector function $g = (g_1, \ldots, g_N)$ such that $g_i \in L^1(\omega_i)$ over $A \subset \Omega$ with respect to $\omega$, i.e.,

$$\int_A <Df, g> \mathds{1} = \int_A g \cdot d\omega = \sum_{i=1}^N \int_A g_i d\omega_i.$$ 

Finally $V(f; \Omega) = \sup \{ \int_\Omega <Df, g> : g \in C^1(\Omega; R^N), |g(x)| \leq 1, x \in \Omega \} = \sup \{ \int_\Omega <Df, g> : g \in C^1(\Omega; R^N), |g(x)| \leq 1, x \in \Omega \}.$

Let $BV(\Omega) = \{ f \in L^1(\Omega) : V(f; \Omega) < \infty \}$. Each $f \in BV(\Omega)$ is said to have bounded variation in $\Omega$. $BV(\Omega)$ is a Banach space with norm $\|f\|_{BV} = \|f\|_{0,1} + V(f; \Omega)$.

Now we introduce the definition of the trace of a function in $BV(\Omega)$.

**Definition 2.2.** The trace of $f \in BV(\Omega)$ is defined as

$$tr_\Omega f(x) = \lim_{r \to 0} \frac{1}{m(\overline{B(x,r) \cap \Omega})} \int_{\overline{B(x,r) \cap \Omega}} f dm$$

for $x \in \partial \Omega$ a.e. with respect to the $(N-1)$-dimensional Hausdorff measure $H$, where $\overline{B(x,r)}$ is the open ball centered at $x$ with radius $r$.

In this paper we assume that $\Omega$ and any $\Omega_0 \subset \Omega$ involved are open, bounded, and admissible (see [9]), say a domain with piecewise Lipschitz continuous boundary, so that for any $f \in BV(\Omega_0)$ the trace $tr_{\Omega_0} f$ is well defined and the following three results are valid (see [4] or [9]).

**Proposition 2.3.** Let $\Omega_0 \subset \Omega$. If $f \in BV(\Omega)$, then for $g \in C^1_0(\Omega; R^N)$,

$$\int_{\Omega_0} \text{div} g dm = -\int_{\Omega_0} <Df, g> + \int_{\partial \Omega_0} tr_{\Omega_0} f <g, n> dH,$$

where $n$ is the outward unit normal vector to $\partial \Omega_0$.

**Proposition 2.4.** There exists a constant $\rho(\Omega)$ such that

$$\int_{\partial \Omega} |tr_\Omega f| dH \leq \rho(\Omega) \|f\|_{BV}, \forall f \in BV(\Omega).$$

**Proposition 2.5.** The closed unit ball of $(BV(\Omega), \|\cdot\|_{BV})$ is compact in $L^1(\Omega)$.

We need the following result which was proved in [1] since it is needed to investigate the Frobenius-Perron operator expression for the class of mappings studied in this paper (see (3.7) in the next section). This result is basically a generalization of the familiar change of variable formula for smooth functions to weakly differentiable functions. For a mapping $T$, we denote by $J_T$ the Jacobian matrix of $T$ and $|J_T|$ the absolute value of the determinant of $J_T$. If $A$ is a matrix, then $\|A\|$ is defined to be the Euclidean 2-norm of $A$. Note that $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$.

**Lemma 2.6.** Let $\Omega_1, \Omega_2, \tilde{\Omega}_1, \tilde{\Omega}_2$ be bounded open sets in $R^N$, $\Omega_1 \subset \tilde{\Omega}_1, \Omega_2 \subset \tilde{\Omega}_2$, let $S = (S_1, \cdots, S_N)^T : \Omega_1 \to \Omega_2$ be a diffeomorphism of class $C^2$ with $S(\Omega_1) = \Omega_2$, and let $f \in BV(\tilde{\Omega}_1)$. Then there hold
In this section, we prove the existence result for a class of piecewise $C^2$ mappings in any dimension. First we prove two useful results.

**Lemma 3.1.** Let $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ with $\Gamma = \partial \Omega_1 \cap \partial \Omega_2$. Then for any $f \in BV(\Omega)$, there hold

\[
\int_{\Gamma} |Df| = \int_{\Gamma_1} |\text{tr}_{\Omega_1} f - \text{tr}_{\Omega_2} f| dH, \tag{3.3}
\]

and

\[
V(f; \Omega) = V(f; \Omega_1) + V(f; \Omega_2) + \int_{\Gamma} |\text{tr}_{\Omega_1} f - \text{tr}_{\Omega_2} f| dH. \tag{3.4}
\]

**Proof:** Let $g \in C^1_0(\Omega; R^N)$. Then from Proposition 2.3,

\[
\int_{\Omega_1} f \ div g dm = - \int_{\Omega_1} <Df, g> + \int_{\partial \Omega_1} \text{tr}_{\Omega_1} f < g, n_1 > dH,
\]

and

\[
\int_{\Omega_2} f \ div g dm = - \int_{\Omega_2} <Df, g> + \int_{\partial \Omega_2} \text{tr}_{\Omega_2} f < g, n_2 > dH,
\]

where $n_i$ is the outward unit normal vector to $\partial \Omega_i$, $i = 1, 2$. From the above and using the fact that $n_1 = -n_2$ along $\Gamma$, we have

\[
\int_{\Omega} f \ div g dm = - \int_{\Omega_1} <Df, g> - \int_{\Omega_2} <Df, g> + \int_{\Gamma} (\text{tr}_{\Omega_1} f - \text{tr}_{\Omega_2} f) < g, n_1 > dH.
\]

Hence,

\[
- \int_{\Gamma} <Df, g> = \int_{\Gamma} (\text{tr}_{\Omega_1} f - \text{tr}_{\Omega_2} f) < g, n_1 > dH.
\]

Taking supremum over all $g \in C^1_0(\Omega; R^N)$ such that $|g(x)| \leq 1$, $x \in \Omega$, we obtain (3.3).

Now, from Remark 2.2, we see that

\[
\int_{\Omega} |Df| = \int_{\Omega_1} |Df| + \int_{\Omega_2} |Df| + \int_{\Gamma} |Df|
\]

\[
= \int_{\Omega_1} |Df| + \int_{\Omega_2} |Df| + \int_{\Gamma} |\text{tr}_{\Omega_1} f - \text{tr}_{\Omega_2} f| dH.
\]

**Corollary 3.2.** Let $\Omega_0 \subset \Omega$. If $f \in BV(\Omega)$, then

\[
V(f; \Omega_0; \Omega) = V(f; \Omega_0) + \int_{\partial \Omega_0 \setminus \partial \Omega} |\text{tr}_{\Omega_0} f| dH. \tag{3.5}
\]
It is time to study the existence problem of absolutely continuous invariant measures for a class of piecewise $C^2$ and expanding mappings in $\mathbb{R}^N$. Specifically this class of mappings is given in the next definition [1].

**Definition 3.3.** Let $S : \Omega \to \Omega$ and let $\{\Omega_1, \cdots, \Omega_r\}$ be a partition of $\Omega$. Denote $S_i = S|_{\Omega_i}$. We say that $S$ is piecewise $C^2$ and $\alpha$-expanding if each $S_i$ is $C^2$ and one-to-one on $\bar{\Omega}_i$, can be extended as a $C^2$ mapping on $\bar{\Omega}_i$ (i.e., $C^2$ on an open neighborhood of $\bar{\Omega}_i$), and satisfies
\[
\sup_{x \in S(\Omega_i)} \|J_{S_i^{-1}}(x)\| \leq \alpha^{-1}, \text{ for } i = 1, 2, \cdots, r.
\]

For the mapping $S : \Omega \to \Omega$, the corresponding Frobenius-Perron operator $P : L^1(\Omega) \to L^1(\Omega)$ defined by
\[
\int_A Pf dm = \int_{S^{-1}(A)} f dm, \forall \text{ } m \text{- measurable } A \subset \Omega \tag{3.6}
\]
has the expression
\[
Pf = \sum_{i=1}^{r} (f \circ S_i^{-1})|J_{S_i^{-1}}|\chi_{S(\Omega_i)} \tag{3.7}
\]
From (3.6) it is clear that any fixed density $f^*$ of $P$ defines an absolutely continuous invariant measure
\[
\mu_{f^*}(A) = \int_A f^* dm, \forall \text{ } m \text{- measurable } A \subset \Omega.
\]

See [2] and [6] for more details about properties and applications of Frobenius-Perron operators.

A similar result to the following one was presented in [1] for some domain. But we intend to give a complete proof for a very general domain and then combine it with the trace theorem to get Lemma 3.5.

**Lemma 3.4.** If $S : \Omega \to \Omega$ is piecewise $C^2$ and $\alpha$-expanding, then for any $f \in BV(\Omega)$,
\[
V(Pf; \Omega) \leq \alpha^{-1}V(f; \Omega) + C''\|f\|_{0,1} + \alpha^{-1} \sum_{i=1}^{r} \int_{\partial \Omega \setminus S_i^{-1}(\partial \Omega)} |\text{tr}_{\Omega_i}f| \, dH,
\]
where $C'' > 0$ is a constant independent of $f$.

**Proof:** By (3.7) and Corollary 3.2,
\[
V(Pf; \Omega) \leq \sum_{i=1}^{r} V[(f \circ S_i^{-1})|J_{S_i^{-1}}|\chi_{S(\Omega_i)}; \Omega] \\
\leq \sum_{i=1}^{r} \{V[(f \circ S_i^{-1})|J_{S_i^{-1}}|; S(\Omega_i)] \\
+ \int_{\partial S(\Omega_i) \setminus \partial \Omega} |\text{tr}_{\Omega_i}[(f \circ S_i^{-1})|J_{S_i^{-1}}]| \, dH \}. \tag{3.8}
\]

By Definition 2.1 and Remark 2.2,
\[
V[(f \circ S_i^{-1})|J_{S_i^{-1}}|; S(\Omega_i)] \\
= \sup \{- \int_{S(\Omega_i)} \chi_{S(\Omega_i)} < D[(f \circ S_i^{-1})|J_{S_i^{-1}}|, g > : \\
g \in C_0^1(S(\Omega_i); R^N), |g(x)| \leq 1, \ x \in S(\Omega_i)\}.
\]
By (i) of Lemma 2.6, for \( g \in C^1_0(S(\Omega_\alpha), R^N) \) such that \( |g(x)| \leq 1, \forall x \in S(\Omega_\alpha) \),

\[
\int_{S(\Omega_\alpha)} < D[(f \circ S^{-1}_i)|J_{S^{-1}_i}^{-1}], g > = \int_{\Omega_\alpha} < Df, (J_{S^{-1}_i}g) \circ S_i > + \int_{\Omega_\alpha} f \sum_{j,k=1}^N (g_j \circ S_i) \frac{\partial (S^{-1}_i)^k_j}{\partial y_j} \circ S_i \, dm.
\]

Let \( \phi \equiv \alpha g \). Then the above two equalities give

\[
V[(f \circ S^{-1}_i)|J_{S^{-1}_i}^{-1}; S(\Omega_\alpha)] = \sup \{-\alpha^{-1} \int_{\Omega_\alpha} < Df, (J_{S^{-1}_i} \phi) \circ S_i > - \int_{\Omega_\alpha} f \sum_{j,k=1}^N (g_j \circ S_i) \frac{\partial (S^{-1}_i)^k_j}{\partial y_j} \circ S_i \, dm : g \in C^1_0(S(\Omega_\alpha), R^N), |g(x)| \leq 1, x \in S(\Omega_\alpha) \}.
\]

If we let \( C' \) be the \( L^\infty \)-norm of

\[
\sum_{j,k=1}^N (g_j \circ S_i) \frac{\partial (S^{-1}_i)^k_j}{\partial y_j} \circ S_i,
\]

then

\[
- \int_{\Omega_\alpha} f \sum_{j,k=1}^N (g_j \circ S_i) \frac{\partial (S^{-1}_i)^k_j}{\partial y_j} \circ S_i \, dm \leq C' \int_{\Omega_\alpha} |f| \, dm.
\]

On the other hand, since \( |(J_{S^{-1}_i} \phi)(x)| \leq ||J_{S^{-1}_i}|||\phi(x)| \leq 1, \)

\[
\int_{\Omega_\alpha} < Df, (J_{S^{-1}_i} \phi) \circ S_i > \leq V(f; \Omega_\alpha).
\]

Therefore, combining (3.9) through (3.11), we obtain

\[
V[(f \circ S^{-1}_i)|J_{S^{-1}_i}^{-1}; S(\Omega_\alpha)] \leq \alpha^{-1} V(f; \Omega_\alpha) + C' \int_{\Omega_\alpha} |f| \, dm.
\]

It follows from (3.8), (3.12), Lemma 3.1, and (ii) of Lemma 2.6 that

\[
V(Pf; \Omega) \leq \sum_{i=1}^r (\alpha^{-1} V(f; \Omega_i) + C' \int_{\Omega_i} |f| \, dm) + \sum_{i=1}^r \alpha^{-1} \int_{\partial \Omega_i \setminus S^{-1}_i(\partial \Omega)} |\text{tr}_\Omega f| \, dH
\]

\[
\leq \alpha^{-1} V(f; \Omega) + C \|f\|_{0,1} + \alpha^{-1} \sum_{i=1}^r \int_{\partial \Omega_i \setminus S^{-1}_i(\partial \Omega)} |\text{tr}_\Omega f| \, dH.
\]

This completes the proof.

**Lemma 3.5.** If \( S : \Omega \to \Omega \) is piecewise \( C^2 \) and \( \alpha \)-expanding, then

\[
V(Pf; \Omega) \leq \alpha^{-1} (1 + \rho_{\alpha}(S)) V(f; \Omega) + C \|f\|_{0,1}, \forall f \in BV(\Omega),
\]

where \( \rho_{\alpha}(S) = \max_{1 \leq i \leq m} \rho(\Omega_i) \) and \( C > 0 \) is a constant independent of \( f \).
**Proof:** From Lemma 3.4, we have
\[
V(Pf; \Omega) \leq \alpha^{-1}V(f; \Omega) + C'\|f\|_{0,1} + \alpha^{-1} \sum_{i=1}^{r} \int_{\partial \Omega_i \setminus S_i^{-1}(\partial \Omega)} |\tr_{\Omega_i} f| \, dH
\]
\[
\leq \alpha^{-1}V(f; \Omega) + C'\|f\|_{0,1} + \alpha^{-1} \sum_{i=1}^{r} |\tr_{\Omega_i} f| \, dH.
\]
Note that Proposition 2.4 and Lemma 3.1 imply
\[
\sum_{i=1}^{r} \int_{\partial \Omega_i} |\tr_{\Omega_i} f| \, dH \leq \sum_{i=1}^{r} \rho(\Omega_i) \left( V(f; \Omega_i) + \int_{\Omega_i} |f| \, dm \right) \leq \rho(\Omega)(V(f; \Omega) + \|f\|_{0,1}).
\]
Thus we get
\[
V(Pf; \Omega) \leq \alpha^{-1}V(f; \Omega) + C'\|f\|_{0,1} + \alpha^{-1} \rho(\Omega)(V(f; \Omega) + \|f\|_{0,1})
\]
\[
\leq \alpha^{-1}(1 + \rho(\Omega)) V(f; \Omega) + C\|f\|_{0,1},
\]
where \(C = C' + \alpha^{-1} \rho(\Omega)\).

**Theorem 3.6.** Let \(S : \Omega \to \Omega \subset \mathbb{R}^N\) be piecewise \(C^2\) and \(\alpha\)-expanding. If \(\alpha^{-1}(1 + \rho(\Omega)) < 1\), then \(S\) admits an absolutely continuous invariant measure.

**Proof:** From Lemma 3.5, the assumption implies that the sequence
\[
\{\|P^n 1\|_{BV}\}_{n \geq 1}
\]
is uniformly bounded, where 1 is the constant function 1. Hence the sequence \(\{P^n 1\}_{n \geq 1}\) is precompact in \(L^1(\Omega)\) by Proposition 2.5, and it follows from the Kakutani-Yosida theorem [6] that \(P\) has a nontrivial fixed point which is the density of an absolutely continuous invariant measure.

**Remark 3.1.** Under the condition of Theorem 3.6, it can be shown with a standard technique that \(P : BV(\Omega) \to BV(\Omega)\) is quasi-compact [2]. Hence many statistical properties of the dynamical system, including the exponential decay of correlation, can be obtained.

**Corollary 3.7.** Let \(S : \Omega \to \Omega \subset \mathbb{R}^N\) be piecewise \(C^2\). If some iterate \(S^k\) is piecewise \(\alpha\)-expanding and satisfies \(\alpha^{-1}(1 + \rho(\Omega(S^k))) < 1\), then \(S\) admits an absolutely continuous invariant measure.

Finally, it should be mentioned that since no additional conditions are assumed for \(\Omega\), the existence results under different assumptions on \(\Omega\) obtained before in, e.g., [1, 5] can be viewed as consequences of our general framework after giving an upper bound for \(\rho(\Omega)\) in different cases.

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