PLANAR OPEN BOOKS AND FLOER HOMOLOGY

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Abstract. Giroux has described a correspondence between open book decompositions on a 3–manifold and contact structures. In this paper we use Heegaard Floer homology to give restrictions on contact structures which correspond to open book decompositions with planar pages, generalizing a recent result of Etnyre.

1. Introduction

In [28], Thurston and Winkelnkemper showed that an open book decomposition of a 3–manifold $Y$ gives rise in a natural way to a contact structure over $Y$, and hence that every 3–manifold admits some contact structure. More recently, Giroux [10] obtained fundamental results stating a kind of converse to the Thurston–Winkelnkemper result, showing in effect that contact structures are in one-to-one correspondence with certain concretely describable equivalence classes of open book decompositions. This result brought about a revolution in contact geometry with repercussions throughout low-dimensional topology. More visibly, it is the inspiration for the recently proved embedding result of Eliashberg [3] and Etnyre [6] leading — among other results — to the proof that nontrivial knots have Property $P$ [13]. In another direction, it forms the foundations for the “contact invariant” in Heegaard Floer homology [20], a tool which has been helpful in the classification of contact structures over certain 3–manifolds [15, 16].

But Giroux’s construction also raises a number of questions: what contact geometric properties are reflected by topological properties of the open book decomposition? Or, more specifically, what types of contact structures correspond to open book decompositions whose pages are planar? For example, in [7], Etnyre shows that all overtwisted contact structures are compatible with planar open book decompositions, as are all tight structures on lens spaces [27]. Note also that the Weinstein conjecture has been verified for all contact structures which admit planar open books [1].

In [7], Etnyre gives the following constraints on contact structures compatible with planar open book decompositions. (Here, $b^+_2(X)$ resp. $b^0_2(X)$ denotes the maximal dimension of any subspace on which the intersection form is positive definite resp. identically zero.)

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Theorem 1.1 (Etnyre, [7]). If $X$ is a symplectic filling of a contact 3–manifold $(Y, \xi)$ which is compatible with a planar open book decomposition then $b_2^+(X) = b_2^-(X) = 0$, the boundary of $X$ is connected and if $Y$ is an integral homology sphere then the intersection form $Q_X$ is diagonalizable over the integers.

The above theorem can be used to show that many contact structures do not admit planar open book decompositions. For example, according to [4], a taut foliation on a 3–manifold $Y$ gives rise to a contact structure which admits a symplectic semi–filling by the 4–manifold $[0,1] \times Y$, and hence it admits no compatible planar open book decomposition.

The aim of the present article is to prove a result analogous to Theorem 1.1, using Heegaard Floer homology [25]. Recall that $HF^+(Y)$ is a 3–manifold invariant which is a module over the polynomial algebra $\mathbb{Z}[U]$. Moreover, the decreasing sequence of submodules $\{U^d \cdot HF^+(Y)\}_{d=0}^\infty$ stabilizes for sufficiently large $d$, c.f. Section 4 of [25].

Theorem 1.2. Suppose that the contact structure $\xi$ on $Y$ is compatible with a planar open book decomposition. Then its contact invariant $c^+(\xi) \in HF^+(\mathbb{R}Y)$ is contained in $U^d \cdot HF^+(\mathbb{R}Y)$ for all $d \in \mathbb{N}$.

Remark 1.3. Recall that $HF_{\text{red}}(-Y)$ is defined as $HF^+(-Y)/\text{Im}(U^d)$ for some sufficiently large $d$. Theorem 1.2 then translates to the statement that the contact invariant of a contact structure on a 3–manifold $Y$ compatible with a planar open book decomposition vanishes when regarded as an element of the quotient group $HF_{\text{red}}(-Y)$.

Etnyre’s theorem can be seen as a consequence of Theorem 1.2 (and its method of proof). There are, however, other contact structures which do not admit planar open books as a result of Theorem 1.2. We list here some consequences of Theorem 1.2.

Corollary 1.4. Suppose that $c^+(\xi) \neq 0$ and the associated spin$^c$ structure $s(\xi)$ is non–torsion (that is, $c_1(s(\xi))$ is not a torsion class). Then $\xi$ does not support a planar open book decomposition.

Corollary 1.5. Suppose that the contact 3–manifold $(Y, \xi)$ with $c_1(s(\xi)) = 0$ admits a Stein filling $(X, J)$ such that $c_1(X, J) \neq 0$. Then $\xi$ is not supported by a planar open book decomposition.

Corollary 1.6. Suppose that $L \subset (S^3, \xi_{st})$ is a Legendrian knot with zero Thurston–Bennequin invariant. Then the contact structure $\xi_L$ given by Legendrian surgery along $L$ is not supported by planar open book decomposition.

The next corollary concerns the 2–plane fields underlying contact structures over rational homology 3–spheres $Y$. Recall that homotopy classes of 2–plane fields $\xi$ over $Y$ are classified by a pair of data, their induced Spin$^c$ structure $s(\xi)$, and a Hopf invariant $d_3(\xi)$, which takes values in $\mathbb{Q}$ when $b_1(Y) = 0$. 
According to a theorem of Kronheimer and Mrowka [12], there are only finitely many homotopy classes of 2–plane fields which represent symplectically fillable contact structures. We have the following refinement for contact structures which are both symplectically fillable and also compatible with planar open book decompositions.

**Corollary 1.7.** Suppose that $Y$ is a rational homology 3–sphere. The number of homotopy classes of 2–plane fields which admit contact structures which are both symplectically fillable and compatible with planar open book decompositions is bounded above by the number of elements in $H_1(Y;\mathbb{Z})$. More precisely, each Spin$^c$ structure $s$ is represented by at most one such 2–plane field, and moreover, the Hopf invariant of the corresponding 2–plane field must coincide with the “correction term” $d(-Y, s)$.

For the last part of the above statement, recall that on a rational homology 3–sphere $Y$ the Floer homology $HF^+(Y, s)$ admits a grading by $\mathbb{Q}$. The correction term referred to in the corollary is the function $d: \text{Spin}^c(Y) \to \mathbb{Q}$, which measures the minimal degree of any homogeneous element in $HF^+(Y, s)$ which lies in the image of $U^n$ for all non-negative integers $n$, cf. [21, Section 4]. This function is analogous to the gauge-theoretic invariant Frøyshov [9].

This paper is organized as follows. In Section 2, we review some of the background needed for the proof of Theorem 1.2: Giroux stabilizations and open book decompositions, Heegaard Floer homology, and the contact invariant. In Section 3 we turn to the proof of Theorem 1.2 and its corollaries. The proof of Theorem 1.2 falls naturally into two steps, the first of which is a result about monoid generators for the mapping class group of a planar surface, and the second of which is the calculation of the Heegaard Floer homology groups of a family of model 3–manifolds. With Theorem 1.2 in hand, we derive its corollaries stated above, and also give an alternate proof of Theorem 1.1.
The aim of this section is to review the relevant background needed for our present purposes. In Subsection 2.1 we describe the notion of “Giroux stabilization”, which gives the equivalence relation between open book decompositions inducing the same contact structure. For more on this, see [8, 10]. In Subsection 2.2 we review some basic facts regarding Heegaard Floer homologies and discuss a special class of 3–manifolds, called \(L\)-spaces, whose Heegaard Floer homology groups are as simple as possible. In Subsection 2.3, we describe the invariant \(c^+(ξ) ∈ HF^+(−Y)\) associated to a contact structure \(ξ\) over \(Y\), defined with the help of Giroux’s results.

2.1. Open books. Let \(φ\) be an automorphism of an oriented surface \(F\) with nonempty boundary, and suppose that \(φ\) fixes \(∂F\). We can form the mapping torus \(M_φ = F × [0, 1]/(φ(x), 0) ∼ (x, 1)\) to obtain a 3–manifold which fibers over the circle, and whose boundary is \(∂F × S^1\). There is a canonically associated closed 3–manifold \(Y\) obtained from \(M_φ\) by attaching solid tori \(∂F × D^2\) using the identifications suggested by the notation. The data \((F, φ)\) is called an open book decomposition of \(Y\), and \(φ\) is called the monodromy of the open book. Thus, an open book decomposition of \(Y\) gives rise to a link \(L ⊂ Y\) which is fibered, called the binding of the open book decomposition, while the fibers of \(M_φ\) are called its pages. An open book decomposition is said to be compatible with a contact structure \(ξ\) (or the given open book supports \(ξ\)) if there is a contact 1–form \(α ∈ Ω^1(Y)\) such that \(ker α\) is isotopic to \(ξ\), \(dα\) is a positive volume form on each page of the open book decomposition and \(α\) evaluates positively on a tangent vector of the binding \(L\) generated by the orientation of \(L\) compatible with the orientation of the pages.

The construction of Thurston and Winkelnkemper [28] associates a compatible contact structure to an open book decomposition of \(Y\). Indeed, according to recent work of Giroux [10], every contact structure is induced by an open book decomposition in this manner and, in fact, Giroux gives an explicit criterion for when two open books induce isotopic contact structures.

Let \(φ\) be an automorphism of \(F\) fixing its boundary. A Giroux stabilization \((F′, φ′)\) of \((F, φ)\) is a new surface-with-boundary \(F′\), equipped with an automorphism \(φ′\) obtained as follows. Let \(F′\) be obtained from \(F\) by attaching a 1–handle, and let \(γ\) be a curve which runs through the 1–handle geometrically once. The automorphism \(φ′\) is then obtained by extending \(φ\) over \(F′\) by the identity map over the 1–handle, and then composing by a right-handed Dehn twist \(t_γ\) along \(γ\), that is,

\[
φ′ = φ ∘ t_γ.
\]

It is not hard to see that Giroux stabilizations leave the 3–manifold and indeed the associated contact structure unchanged. Giroux’s theorem [10] states that two open
book decompositions of $Y$ are compatible with isotopic contact structures if and only if they can be connected by a sequence of Giroux stabilizations/destabilizations.

2.2. Heegaard Floer homologies. Let $W$ be a connected, oriented four-manifold with two boundary components, $\partial W = -Y_1 \cup Y_2$. Then, we denote this cobordism by $W : Y_1 \rightarrow Y_2$. Recall [18] that there is an induced $\mathbb{Z}[U]$–equivariant map

$$F^+_W : HF^+(Y_1) \rightarrow HF^+(Y_2)$$

on Heegaard Floer homology. Recall also [25] that the Heegaard Floer homology $HF^+(Y)$ of a 3–manifold $Y$ naturally splits into summands indexed by Spin$^c$ structures over $Y$.

Heegaard Floer homology groups are hard to determine in general. One very useful calculational tool is the surgery exact triangle which relates the Heegaard Floer homology groups of three suitably related three-manifolds. More specifically, the three 3–manifolds $Y_1, Y_2, Y_3$ form a triad if there is a knot $K \subset Y_1$ such that $Y_2, Y_3$ can be given by integer surgeries along $K$, and the framing for producing $Y_3$ is one higher than the one giving rise to $Y_2$. Let $W_1 : Y_1 \rightarrow Y_2$ denote the cobordism specified by the original framing of $K$ in $Y_1$, and $W_2 : Y_2 \rightarrow Y_3$ specified by $(-1)$–framed surgery along a normal circle $C$ to $K$. Finally, $W_3 : Y_3 \rightarrow Y_1$ denotes the cobordism induced by $(-1)$–surgery along a normal circle $D$ to $C$. Then the surgery exact triangle (Theorem 9.12 of [24]) takes the form

![Surgery Exact Triangle Diagram]

An $L$–space is a rational homology 3–sphere with the property that the map $U : HF^+(Y) \rightarrow HF^+(Y)$ (and hence $U^d$ for all $d \in \mathbb{N}$) is surjective. The lens space $L(p, q)$ is an $L$–space, for example, and the connected sum of two $L$–spaces is also an $L$–space. For a more thorough discussion on $L$–spaces see [19]. In particular, in [19, Proposition 2.1], the following result is proved using the surgery triangle:

**Lemma 2.1.** If a triad $(Y_1, Y_2, Y_3)$ of rational homology spheres satisfy that $Y_1, Y_3$ are $L$–spaces and the cobordism $W_3 : Y_3 \rightarrow Y_1$ has $b_2^+(W_3) = 1$ then $Y_2$ is an $L$–space. $\blacksquare$

We will use this principle in proving the following:

**Theorem 2.2.** The 3–manifold $Z$ given by the Kirby diagram of Figure 1 with $p_i, q_i \geq 1$ is an $L$–space.

**Proof.** We will verify the statement by induction first on the number $n$ of 0–framed unknots and then on $q_n$. Notice first that in the case of a single 0–framed knot (by the
assumption $q_1 \geq 1$) the 3–manifold $Z$ is a lens space, hence the statement easily follows. Suppose now that the theorem holds for all 3–manifolds of the type given by Figure 1 involving at most $(n - 1)$ 0–framed unknots, and consider $Z$ built with $n$ of those. Suppose by induction that for $q_n - 1$ the statement is true, and consider the triad given by the last $(-1)$–framed unknot $K$ meridional to the $n^{th}$ 0–framed unknot. If $q_n - 1$ is still at least 1, then the first element of the triad is an $L$–space by induction on $q_n$, while the third manifold in the triad (when doing 0–surgery on the knot $K$) can be given by a surgery diagram of the type given in Figure 1, now with $(n - 1)$ unknots with 0–framing. Hence our inductive hypothesis shows that it is an $L$–space. For $q_n = 1$ we can easily observe that the first manifold in the triad (i.e, when we delete the single meridional curve linking the last 0–framed unknot) admits a presentation of the type of Figure 1, since the last 0–framed unknot can be canceled against one of the $(-1)$–framed circles. Therefore, in the light of Lemma 2.1 the proof of the theorem follows once we check the condition on the cobordism $W$. In this case it is given by the 2–handle attachment along the dashed curve $D$ of Figure 2. (The solid curves represent a Kirby diagram for $Y_3$ in the triad.) Blow down $C$ and slide $D$ over the $n^{th}$ 0–framed unknot, and finally cancel $K$ against the $n^{th}$ 0–framed unknot. In the resulting diagram (shown by Figure 3) the dashed curve represents the cobordism $W$. Blow down all $(-1)$–circles from the diagram; simple linear algebra shows that the resulting 4–manifold $X$ will be positive definite. Since $W$ was shown to admit an embedding into $X$, it follows that $b_2^+(W) = 1$, hence the proof of the theorem is complete.

**Remark 2.3.** The surgery diagram of $Z$ determines a planar graph as follows: Substitute each 0–framed circle by a vertex $v_i$ ($i = 1, \ldots, n$), and connect $v_i$ and $v_{i+1}$ with $p_i$ edges if there are $p_i$ $(-1)$–circles linking both the $i^{th}$ and the $(i + 1)^{st}$ unknot. Finally take an extra vertex $v_{n+1}$ and connect it with $q_i$ edges to $v_i$ where $q_i$ is the number of $(-1)$–framed normals to the $i^{th}$ 0–framed surgery curve. This planar graph gives rise to a connected projection of an alternating link $L$ through the black graph of the

![Kirby diagram for the 3–manifold $Z$](image)
projection. It can be shown that $Z$ is diffeomorphic to the double branched cover $\Sigma(L)$ of $S^3$ branched along the alternating link $L$. From this observation the proof of Theorem 2.2 is a simple application of [22, Proposition 3.3], where it is proved that the double branched cover of $S^3$ along a link admitting a connected alternating projection is an $L$–space.

2.3. The contact invariant. In order to define the contact invariant $c^+(\xi)$, we need one more observation from Heegaard Floer theory. Let $Y_0$ be a 3–manifold which fibers over the circle, with fiber $F$. Given $i \in \mathbb{Z}$, we can consider

$$HF^+(Y_0, i) = \sum_{\{t \in \text{Spin}^c(Y_0) \mid \langle c_1(t), [F] \rangle = 2i\}} HF^+(Y_0, t).$$

When the genus $g(F) > 1$, then it follows from [20] that $HF^+(Y_0, g - 1) \cong \mathbb{Z}$, endowed with the trivial action by $\mathbb{Z}[U]$ (i.e., $HF^+(Y_0, g - 1) \cong \mathbb{Z}[U]/U \cdot \mathbb{Z}[U]$). Thus, there is a
canonical (up to sign) Heegaard Floer homology class in $HF^+(Y_0)$, which corresponds to a generator of the summand $HF^+(Y_0, g - 1) \subset HF^+(Y_0)$.

Suppose now that $Y$ is a 3–manifold equipped with a contact structure $\xi$. We can consider a compatible open book decomposition. After taking repeated Giroux stabilizations if necessary, we obtain a new open book decomposition $(F, \phi)$ whose binding is connected, and whose genus $g(F)$ is greater than one. By performing a canonical zero–framed surgery along this connected binding, we obtain a 3–manifold $Y_0$ which fibers over the circle, and also a cobordism (obtained by a single 2–handle addition) $Y \to Y_0$. By turning this cobordism upside down, we can view it as $W: -Y_0 \to -Y$.

The image of a generator of $HF^+(-Y_0, g - 1) \subset HF^+(-Y_0)$ under $F^+_W$ in $HF^+(-Y)$ is the element denoted $c^+(F, \phi)$. It is shown in [20] that this element is invariant under Giroux stabilizations and hence, according to Giroux’s theorem, it depends only on the isotopy class of the underlying contact structure $\xi$ on $Y$; correspondingly, we denote this element by $c^+(\xi)$. (Note that in [20] the primary object of study is a lift of $c^+(\xi)$ from $HF^+(-Y)$ to $\hat{HF}(-Y)$; we do not need this refinement for our present applications, however.)

The importance of $c^+(\xi)$ stems from the fact that it seems to capture interesting contact geometric properties of $\xi$; for example, $c^+(\xi) = 0$ for an overtwisted contact structure, while $c^+(\xi) \neq 0$ once $\xi$ is Stein fillable.

The computation of $c^+(\xi)$ can be a very delicate problem. A successful scheme of computation rests on the following result. To set the stage, let $(F, \phi)$ be a given open book decomposition for $Y$ with binding $L$, and fix a curve $\gamma \subset Y - L$ supported in a page of the open book decomposition, which is not homotopic (in the page) to the boundary. Let $Y_{+1}$ denote the 3–manifold obtained by doing $+1$–surgery along $\gamma$ (with respect to the framing $\gamma$ inherits from the page), and denote the cobordism defined by this surgery by $X$. Notice that $Y_{+1}$ carries a natural open book decomposition $(F, \phi \circ t_{\gamma}^{-1})$ and that $-X$ (the 4–manifold $X$ with its reversed orientation) provides a cobordism $-X: -Y \to -Y_{+1}(\gamma)$. The following is proved in [20, Theorem 4.2]:

**Theorem 2.4.** Under the above circumstance, for the map

$$F^+_X: HF^+(-Y) \to HF^+(-Y_{+1}(\gamma))$$

we have that

$$F^+_X(c^+(F, \phi)) = \pm c^+(F, \phi \circ t_{\gamma}^{-1}).$$

□

A contact structure $\xi$ over a 3–manifold $Y$ induces a Spin$^c$ structure $\mathfrak{s}(\xi)$, whose first Chern class $c_1(\mathfrak{s}(\xi))$ is the first Chern class of the oriented 2–plane field underlying the contact structure $\xi$. It is shown in [20] that

$$c^+(\xi) \in HF^+(-Y, \mathfrak{s}(\xi)) \subset HF^+(-Y).$$
Moreover, since the maps induced by cobordisms are $\mathbb{Z}[U]$-equivariant, it also follows that $U \cdot c^+(\xi) = 0$. In the case where $c_1(\xi)$ is torsion, hence $HF^+(−Y, s(\xi))$ admits a $\mathbb{Q}$-grading (defined in [18, Section 7], see also [21]), the element $c^+(\xi)$ is a homogeneous element of degree $−d_3(\xi)$ in $HF^+(−Y, s(\xi))$, where $d_3(\xi)$ is the Hopf invariant of the oriented 2-plane field underlying $\xi$. If $(X, \omega)$ is a symplectic filling of $(Y, \xi)$ and $c_1(s(\xi))$ is torsion then $d_3(\xi)$ can be captured as

$$\frac{1}{4}(c_1^2(X, \omega) − 3\sigma(X_0) − 2\chi(X_0)),$$

where $X_0$ is gotten from $X$ by deleting an open 4-ball from its interior.
3. Proof of Theorem 1.2.

In view of the naturality of the contact invariant under left–handed Dehn twists, the proof of Theorem 1.2 breaks into two basic steps: first, we give a simple set of monoid generators of the mapping class group of a genus zero surface, so that the contact invariant of any planar open book decomposition is the image of a Floer homology class of certain model 3–manifolds, and second, we verify that those model 3–manifolds are \( L \)-spaces. These two steps are the subjects of the next two subsections; in Subsection 3.3, we turn to the proof of Theorem 1.2 and its corollaries.

3.1. Monoid generators for the mapping class group of a planar surface. Let \( S \) be a compact, planar surface with \( n + 1 \) boundary components \( B_0, \ldots, B_n \). Let \( \Gamma_S \) denote the mapping class group of \( S \), consisting of diffeomorphisms of \( S \) which pointwise fix \( \partial S \), modulo isotopies, which pointwise fix \( \partial S \). Let \( \delta_i \) denote the right–handed Dehn twist along a circle parallel to \( B_i \) and let \( \gamma_i \) denote the right–handed Dehn twist along a circle encircling the \( i \) boundary components \( B_1, \ldots, B_i \), cf. Figure 4. (Notice that \( \delta_1 = \gamma_1 \) and \( \delta_0 = \gamma_n \).)

![Figure 4](image)

Figure 4. The planar surface \( S \), in the case where \( n = 3 \). Here, the automorphisms \( \delta_i \) and \( \gamma_j \) are right–handed Dehn twists along the indicated curves.

**Theorem 3.1.** For an element \( x \in \Gamma_S \) there is a decomposition

\[
x = \Pi_{i=1}^{n} \delta_i^{n_i} \cdot \Pi_{j=2}^{n} \gamma_j^{m_j} \cdot y
\]

where \( n_i, m_j \) are positive integers and \( y \) can be written as a product of left–handed Dehn twists.

**Proof.** It is known that \( \Gamma_S \) is generated by Dehn twists. Let \( x \in \Gamma_S \) and write it as a product of Dehn twists

\[
x = t_1 \cdots t_m \in \Gamma_S,
\]
where \( t_i \) denotes a right– or left–handed Dehn twist along a simple closed curve in \( S \). Suppose that \( t_{i_1}, \ldots, t_{i_k} \) are right–handed Dehn twists. The idea of the proof is that by using the “lantern relation” [11] in some related surfaces \( S' \) we replace \( t_{i_j} \) with products of left–handed Dehn twists, some other right–handed ones which have smaller “complexity” and with powers of \( \delta_i, \gamma_j \). An inductive argument then provides an expression for \( x \) involving left–handed Dehn twists and powers of \( \delta_i \)’s and \( \gamma_j \)’s only.

By complexity we mean the following: suppose that \( t_\alpha \) is a Dehn twist along \( \alpha \subset S \) encircling the boundary components \( B_{a_1}, \ldots, B_{a_h} \) (\( a_1 < \ldots < a_h \)). Let \( c(t_\alpha) \) be the maximal element in the difference \( \{1, 2, \ldots, a_h\} - \{a_1, \ldots, a_h\} \). In particular, if \( c(t_\alpha) = -\infty \) then \( t_\alpha = \gamma_{a_h} \). Among Dehn twists with the same complexity we say that one is simpler than the other if the maximal index appearing among the boundary components encircled by it is smaller than the similar index for the other one. Now using the lantern relation we can replace any right–handed Dehn twist with left–handed ones, and with right–handed ones either with smaller complexity or which are simpler:

**Lemma 3.2.** Let \( t_\alpha \) be a given right–handed Dehn twist in \( \Gamma_S \). Then

\[
t_\alpha = t_1 \cdot t_2 \cdot \Pi_{i=1}^d d_i
\]

where \( d_i \) are either left–handed or of the form \( \delta_j \), \( t_1 \) has smaller complexity than \( t_\alpha \) and either \( c(t_2) < c(t_\alpha) \) or \( c(t_2) = c(t_\alpha) \) and \( t_2 \) is simpler than \( t_\alpha \).

**Proof.** Suppose that the circle \( \alpha \) encircles the boundary components \( B_{a_1}, \ldots, B_{a_h} \), and its complexity \( c(\alpha) \neq -\infty \). Denote the circle encircling the same boundary components as \( \alpha \) except \( B_{a_h} \) by \( \beta \). Define \( S' \) as the planar surface we get by substituting the side of \( \beta \) not containing \( B_0 \) with an annulus. In \( S' \) we can write down the lantern relation

\[
t_\alpha \cdot t_{\lambda_1} \cdot t_{\lambda_2} = t_\beta \cdot t_{B_{a_h}} \cdot t_{B_{c(\alpha)}} \cdot t_{\lambda_0}
\]

where we use right–handed Dehn twists everywhere. (Here \( \lambda_1, \lambda_2 \) are the circles encircling \( \{\beta, B_{a_h}\} \) and \( \{B_{c(\alpha)}, B_{a_h}\} \) respectively, while \( \lambda_0 \) encircles \( \{\beta, B_{c(\alpha)}, B_{a_h}\} \).) Notice that since this identity holds in \( S' \), it will hold in \( S \) which can be given by gluing in an appropriately punctured disk along the boundary component of \( S' \) corresponding to \( \beta \).

After ordering the identity of (1) we get an expression for \( t_\alpha \) as a product of two left–handed Dehn twists \( (t_{\lambda_1}^{-1} \text{ and } t_{\lambda_2}^{-1}) \), two others of the form \( \delta_j \), and \( t_\beta, t_{\lambda_0} \). For these latter two, the complexity of \( \lambda_0 \) is smaller than that of \( \alpha \), while for \( \beta \) we dropped \( B_{a_h} \) from the encircled boundary components, hence either \( c(\beta) < c(\alpha) \) or in case \( c(\beta) = c(\alpha) \) then \( \beta \) is simpler. \( \square \)

Now induction shows that \( x \) can be written as a product of Dehn twists which are all left–handed except possibly powers of \( \delta_i \) and \( \gamma_j \). Since we can conjugate powers of \( \delta_i \) and \( \gamma_j \) to the front, and \( f^{-1}D_\alpha f = D_{f(\alpha)} \) for any mapping class \( f \in \Gamma_S \) and simple closed curve \( \alpha \), we get the desired expression. In addition, by inserting \( \delta_i \delta_i^{-1} \) or \( \gamma_j \gamma_j^{-1} \) if
necessary, we can assume that \( n_i > 0 \) and \( m_j > 0 \). (Recall that \( \delta_i, \gamma_j \) are right-handed Dehn twists, hence \( \delta_i^{-1}, \gamma_j^{-1} \) are left-handed.)

3.2. Model 3–manifolds.

**Theorem 3.3.** Consider a 3–manifold \( Z \) which admits a planar open book decomposition whose page \( S \) has \((n + 1)\) boundary components and the monodromy map \( \phi \in \Gamma_S \) is of the form

\[
\Pi_{i=1}^{n} \delta_i^{n_i} \cdot \Pi_{j=2}^{m_j} \gamma_j^{m_j}
\]

with \( n_i \) and \( m_j \) positive. This 3–manifold is an L-space.

**Proof.** We give a Kirby calculus description of \( Z \). To this end, consider first the case when \( \phi \) is the identity map. Then the Kirby diagram consists of \( n \) 0–framed unknots, which can be seen as follows. Assume first that \( S = D^2 \) (i.e., \( n = 0 \)), in which case the open book decomposition is simply the standard genus–1 Heegaard decomposition of \( S^3 \). 0–surgery on \( n \) parallel copies of the core circle of the mapping torus \( M_1 = S^1 \times D^2 \subset S^3 \) then provides the 3–manifold corresponding to general \( S \) and \( \phi = 1 \). Since the page \( S \) can be constructed by puncturing a disk orthogonal to the surgery curves at the intersections, the curves giving rise to the Dehn twists \( \delta_i, \gamma_j \) are explicitly visible in the picture, cf. Figure 5. It is known that multiplication of the monodromy by the right–handed Dehn twists \( \delta_i \) or \( \gamma_j \) changes the 3–manifold by a \((-1)\)–surgery along the corresponding curve. This observation provides a surgery presentation of \( Z \). Recall that the boundary components of \( S \), and so the 0–framed unknots were indexed by \( \{1, \ldots, n\} \). To get a more convenient presentation of \( Z \), slide the \( i^{th} \) 0–framed unknot over the \((i + 1)^{st} \) (where \( i \) runs from 1 to \( n - 1 \)) in the way that with the natural compatible orientations we subtract the corresponding homology classes. The resulting Kirby diagram is now of the type considered in Theorem 2.2, where it is shown to give rise to an L–space, hence the proof is complete.
Remark 3.4. Notice that the mapping class \( \phi \) also provides a contact structure \( \xi_\phi \) on \( Z \). Since \( \phi \) is given as a product of right–handed Dehn twists, it follows [10] that \( \xi_\phi \) is Stein fillable, in particular its contact invariant \( c^+(\xi_\phi) \) is nontrivial in \( HF^+(\neg Z) \). This observation accords with the fact we will prove in the next Subsection, stating that contact invariants of contact structures compatible with planar open book decompositions are images of \( c^+(\xi_\phi) \)'s.

3.3. Proof of Theorem 1.2 and its corollaries. With all the pieces in place, we now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. Let \( \xi \) be a contact structure on \( Y \) specified by a planar open book decomposition. Combining Theorem 3.1 with Theorem 2.4, it follows that there is a 3–manifold \( Z \) which admits a planar open book decomposition whose monodromy \( \phi \) has the form

\[
\Pi_{i=1}^{n_i} \delta_{n_i} \cdot \Pi_{j=2}^{m_j} \gamma_{m_j} \cdot y
\]

with \( n_i, m_j > 0 \) and which has the property that \( c^+(\xi) \) is the image of the contact invariant \( c^+(\xi_\phi) \in HF^+(\neg Z) \) under the map induced by some cobordism \( W: \neg Z \longrightarrow \neg Y \). By Theorem 3.3, \( Z \) is an L–space, i.e.

\[
U^d: HF^+(\neg Z) \longrightarrow HF^+(\neg Z)
\]

is surjective for all \( d \in \mathbb{N} \). Since maps induced by cobordisms are \( \mathbb{Z}[U] \)–module homomorphisms, it follows at once that \( c^+(\xi) \in HF^+(\neg Y) \) is in the image of

\[
U^d: HF^+(\neg Y) \longrightarrow HF^+(\neg Y),
\]

as stated. \( \square \)

Although Theorem 1.1 is not a formal consequence of Theorem 1.2, it does follow from the method of proof, as we illustrate here:

Proof of Theorem 1.1. Suppose that \( \xi \) is compatible with the planar open book decomposition \( (S, \phi) \), where \( \phi = \Pi_{i=1}^{n_i} \delta_{n_i} \cdot \Pi_{j=2}^{m_j} \gamma_{m_j} \cdot y \) with \( y \) being the product of left–handed Dehn twists in \( \Gamma_S \). Therefore, the multiplication of \( \phi \) by \( y^{-1} \) can be achieved by a sequence of Legendrian surgeries along \( (Y, \xi) \), providing a Stein cobordism \( W \) from \( (Y, \xi) \) to some \( (Z, \xi') \), where an open book decomposition compatible with \( \xi' \) is given by \( (S, \phi \cdot y^{-1}) \). This observation provides an embedding of the filling \( (X, \omega) \) into a filling of \( (Z, \xi') \). On the other hand, the manifold \( Z \) was proved to be an L–space in Theorem 2.2, hence by [23, Theorem 1.4] it can be filled only with symplectic 4–manifolds admitting connected boundary and vanishing \( b_2^+ \), implying \( b_2^+(X) = b_2^0(X) = 0 \) and the connectedness of \( \partial X \).

Suppose now that \( Y \) is an integral homology sphere and consider the cobordism \( W: Y \rightarrow Z \) found above. Notice that by blowing down the \( (-1) \)–curves in the diagram of Figure 1 we see that \( Z \) can be given as the boundary of a positive definite 4–manifold, therefore \( \neg Z \) can be considered as the boundary of a 4–manifold \( U \) with
negative-definite intersection form. Then the closed 4–manifold $X \cup Y \cup W \cup Z \cup U$ has negative-definite intersection form $Q_{X \cup Y \cup W \cup Z \cup U}$, and hence by Donaldson’s theorem [2], the form is diagonalizable. Since $Y$ is an integral homology sphere, we can split the intersection form $Q_{X \cup Y \cup W \cup Z} = Q_X \oplus Q_{W \cup Z}$. It is an easy consequence of a theorem of Elkies [5] that $Q_X$ has diagonalizable intersection form, as well. This final observation concludes the proof of the theorem.

Lemma 3.5. Suppose that the contact structure $\xi$ on the rational homology 3–sphere $Y$ is compatible with a planar open book decomposition. If $X$ is a symplectic filling of $(Y, \xi)$ then $b_1(X) = 0$.

Proof. Suppose that $b_1(X) > 0$ for a symplectic filling $X$. If $|H_1(Y; \mathbb{Z})| = n$ then $X$ admits a connected $(n+1)$–fold unramified cover $\tilde{X}$ which is the trivial $(n+1)$–fold cover when restricted to $\partial X$. By capping off $n$ of the components of $\partial \tilde{X}$ with concave fillings of positive $b_2^+$–invariants, we get a symplectic filling $\check{X}$ of $(Y, \xi)$ with $b_2^+(\check{X}) > 0$, contradicting our previous result.

Proof of Corollary 1.4. According to [17, Proposition 2.3], if $s$ is a Spin$^c$ structure whose first Chern class is non–torsion, then there is some integer $m$ with the property that $U^m \cdot HF^+(Y, s) = 0$. On the other hand, according to Theorem 1.2, $c^+(\xi)$ must lie in this group if $\xi$ supports a planar open book decomposition. For $c^+(\xi) \neq 0$ this provides a contradiction, verifying the corollary.

Corollary 1.5 follows from Theorem 1.2, together with known properties of the contact invariants of Stein manifolds (cf. [14] and [26]):

Proof of Corollary 1.5. Suppose that $(X, J)$ is a Stein manifold with contact boundary $(Y, \xi)$, and assume that $c_1(X, J)$ is nonzero in $H^2(X; \mathbb{Z})$ while $c_1(\xi) = 0$. Let $\overline{J}$ denote the conjugate complex structure, inducing $\overline{\xi}$ on $Y$. For the induced Spin$^c$ structure $s(J)$ on $\check{X}$ the condition $c_1(X, J) \neq 0$ readily implies $s(J) \neq s(\overline{J})$, hence by [14] we get that $\xi$ and $\overline{\xi}$ are not isotopic, in fact, according to [26] we also know that $c^+(\xi) \neq c^+(\overline{\xi})$. On the other hand, the assumption $c_1(\xi) = 0$ implies that $\xi$ and $\overline{\xi}$ induce the same Spin$^c$ structure $s(\xi) \in$ Spin$^c(Y)$. Since the quotient map

$$R: HF^+(-Y, s(\xi)) \to HF_{\text{red}}(-Y, s(\xi))$$

has 1–dimensional kernel when restricted to ker $U$, we get that at most one of $c^+(\xi)$ and $c^+(\overline{\xi})$ can map to zero under $R$. Therefore one of $\xi$ and $\overline{\xi}$ is not compatible with planar open book decomposition. Since an open book decomposition of $\overline{\xi}$ can be given by taking an open book decomposition $(F, \phi)$ of $\xi$, reversing the orientation of the page $F$ and inverting $\phi$, we see that $\xi$ and $\overline{\xi}$ admit planar open book decompositions at the same time. Since one of them is not compatible with a planar open book decomposition, the proof is complete.
Proof of Corollary 1.6. Since \( \text{tb}(L) = 0 \), we have that \( \text{rot}(L) \) is odd, in particular it is not zero. Notice that contact \((-1)\)-surgery on \( L \) also provides a Stein filling \((X, J)\) for \((S^3_1(L), \xi_L)\) with \( c_1(X, J) \) evaluating on the generator of \( H_2(X; \mathbb{Z}) \) as \( \text{rot}(L) \neq 0 \). Since \( H_1(S^3_1(L); \mathbb{Z}) = 0 \), it follows that \( c_1(\xi_L) = 0 \), hence the application of Corollary 1.5 implies the result.

Proof of Corollary 1.7. The contact invariant \( c^+(\xi) \) is a homogeneous element of degree given by \(-d_3(\xi)\), with \( U \cdot c^+(\xi) = 0 \). According to Theorem 1.2, the hypothesis that \( \xi \) is compatible with a genus zero open book ensures that \( c^+(\xi) \) lies in \( U^m HF^+(-Y, s(\xi)) \) for all \( m \), while the hypothesis that \( \xi \) is fillable ensures that \( c^+(\xi) \) is non-trivial homology class, according to [23, Theorem 4.2]. On the other hand, for a rational homology sphere, a non-trivial homogeneous homology class in \( HF^+(-Y, s) \) which lies simultaneously in the kernel of \( U \) and the image of \( U^m \) is supported in degree \( d(-Y, s) \), [21, Section 4].

As an illustration of the above results, let \( L \) be a \((pq - p - q)\)-fold stabilization of the Legendrian positive torus knot \( T_{p,q} \) with (maximal) Thurston–Bennequin number \( pq - p - q \). According to Corollary 1.6, the contact structure \( \xi_L \) on \( S^3_1(T_{p,q}) \) obtained by Legendrian surgery on \( L \) is not compatible with a genus zero open book decomposition. (In fact, in the cases where \( \text{rot}(L) \neq \pm 1 \), it is easy to see that the degree of the contact invariant \(-d(\xi_L)\) is different from zero, which is \(-d(S^3_1(T_{p,q}))\); and hence for those Legendrianizations, one can appeal to the slightly more elementary Corollary 1.7.)

Note that Theorem 1.1 does not apply to these contact structures directly, unless someone finds a symplectic filling of \((S^3_1(L), \xi_L)\) with positive \( b^+_2 \). We could neither find nor rule out the existence of such a filling.
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