Modified scattering for a wave equation with weak dissipation

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Abstract

We consider the Cauchy problem for the weakly dissipative wave equation

\[ \Box u + \frac{\mu}{1+t} u_t = 0 \]

with parameter \( \mu \geq 2 \).

Based on the explicit representations of solutions provided in [Math. Meth. Appl. Sci. 2004; 27:101-124] sharp decay estimates for data from a dense subspace of the energy space are derived. Furthermore, sharpness is discussed in terms of a modified scattering theory.

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1 Introduction

We are interested in a precise description of the behaviour of solutions to the Cauchy problem

\[ \Box u + \frac{\mu}{1+t} u_t = 0, \quad u(0, \cdot) = u_1, \quad u_t(0, \cdot) = u_2 \quad (1.1) \]

with \( \Box = \partial_t^2 - \Delta \) and for data \( u_1 \in H^1(\mathbb{R}^n) \), \( u_2 \in L^2(\mathbb{R}^n) \). It is well known that this problem is well-posed in the sense that it has a unique solution in the space

\[ C^1([0, \infty), L^2(\mathbb{R}^n)) \cap C([0, \infty), H^1(\mathbb{R}_+)). \]

It is known from the fundamental works of A. Matsumura, [1], and H. Uesaka, [2], that the energy of the solutions to this equation,

\[ E(u; t) = \frac{1}{2} \int (u_t^2 + |\nabla u|^2)dx, \quad (1.2) \]

satisfies an estimate of the form

\[ E(u; t) = O(t^{-\alpha}), \quad \alpha = \min\{2, \mu\}. \quad (1.3) \]

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Following F. Hirosawa and H. Nakazawa, it is possible to improve the last estimate in the case $\mu > 2$ to

$$E(u; t) = o(t^{-2}),$$

which means that the above given estimate is not sharp (with respect to the strong topology).

In [4] the author provided explicit representations of solutions to the Cauchy problem (1.1) in terms of special functions. Furthermore, from that treatment it follows that the estimate $E(u; t) = O(t^{-2})$ is sharp in the sense of a norm estimate for the energy operator

$$\mathbb{E}(t) : (\langle D \rangle u_1, u_2)^T \mapsto (\langle D \rangle u(t, \cdot), \partial_t u(t, \cdot))^T$$

associated to the solution representation. As usual we denote $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$.

In this paper we will use this explicit representation to describe a dense subspace of the energy space such that for all data from that space we obtain a sharp two-sided estimate of the energy in the form

$$E(u; t) \sim t^{-\mu}.$$  

(1.6)

The result can be formulated in terms of a modified scattering theory. Especially it describes the discrepancy between the estimates of Hirosawa/Nakazawa and the norm estimate from [4].

2 Preliminaries

Let us recall the representations of solutions to equation (1.1). With the notation

$$\Psi_{k,s,\rho,\delta}(t, \xi) = |\xi|^k \langle \xi \rangle^{s+1-k} \begin{bmatrix} \mathcal{H}_\rho^- (|\xi|) & \mathcal{H}_\rho^+ (|\xi|) \langle 1 + t |\xi| \rangle \\ \mathcal{H}_\rho^+ (|\xi|) & \mathcal{H}_\rho^- (1 + t |\xi|) \end{bmatrix},$$

(2.1a)

$$= 2i \csc(\rho \pi) |\xi|^k \langle \xi \rangle^{s+1-k} \begin{bmatrix} \mathcal{J}_{\rho}^- (|\xi|) & \mathcal{J}_{\rho}^- (1 + t |\xi|) \\ (-)^{s} \mathcal{J}_{\rho}^+ (|\xi|) & \mathcal{J}_{\rho}^+ (1 + t |\xi|) \end{bmatrix},$$

(2.1b)

the last line analytically continued to $\rho \in \mathbb{Z}$, the following theorem is valid, [4, Theorem 2.1].

**Theorem 2.1.** The spatial Fourier transform of the solution to problem (1.1) can be represented in the form

$$\hat{u}(t, \xi) = \sum_{j=1,2} \Phi_j(t, \xi) \hat{u}_j(\xi),$$

(2.2)
where

\[ \Phi_1(t, \xi) = \frac{i\pi}{4} (1 + t)^\rho \Psi_{1,0,\rho-1,1}(t, \xi), \quad (2.3a) \]
\[ \partial_t \Phi_1(t, \xi) = \frac{i\pi}{4} \Psi_{2,1,\rho-1,0}(t, \xi), \quad (2.3b) \]
\[ \Phi_2(t, \xi) = -\frac{i\pi}{4} (1 + t)^\rho \Psi_{0,-1,\rho,0}(t, \xi), \quad (2.3c) \]
\[ \partial_t \Phi_2(t, \xi) = -\frac{i\pi}{4} (1 + t)^\rho \Psi_{1,0,\rho-1,0}(t, \xi) \quad (2.3d) \]

with \( \rho = (1 - \mu)/2 \).

We want to obtain estimates in \( L^2 \)-scale, which correspond by Plancherel’s theorem to \( L^\infty \) estimates for the Fourier multiplier. Therefore we recall the fundamental estimate from [4].

**Lemma 2.2.** It holds \( \Psi_{k,s,\rho,\delta}(t, \cdot) \in L^\infty(\mathbb{R}^n) \) for all \( t \) if and only if \( s \leq 0 \) and \( k \geq |\delta| \). Furthermore, the estimate

\[
|||\Psi_{k,s,\rho,\delta}(t, \cdot)|||_\infty \sim \begin{cases} (1 + t)^{-\frac{k}{2}}, & \rho \neq 0, |\rho| - k \leq -\frac{1}{2}, \\ (1 + t)^{|\delta| - k}, & \rho \neq 0, |\rho| - k \geq -\frac{1}{2}, \\ (1 + t)^{-k} \log(e + t), & \rho = 0, k \leq \frac{1}{2} \end{cases} \quad (2.4)
\]

is valid.

For later use we will introduce a notation. Let

\[
[\xi] = \frac{|\xi|}{\langle \xi \rangle} \quad (2.5)
\]

By the aid of this symbol we can control the vanishing order in the frequency \( \xi = 0 \). We denote

\[
[D]^\kappa L^2(\mathbb{R}^n) = \{ [D]^\kappa f \mid f \in L^2 \}, \quad ||g||_{[D]^\kappa L^2} = ||[D]^{-\kappa} g||_2. \quad (2.6)
\]

It turns out that the usage of date from these spaces allows an improvement of the decay order of the energy.

### 3 Improvements of the energy decay

**Theorem 3.1.** Let \( (D)u_1, u_2 \in [D]^\kappa L^2(\mathbb{R}^n) \). Then the solution \( u = u(t, x) \) to (1.1) with \( \mu > 2 \) satisfies

\[
||[D]u(t, \cdot)||_2 + ||u_1(t, \cdot)||_2 \lesssim ||(D)u_1, u_2||_{[D]^\kappa L^2} \begin{cases} (1 + t)^{-1}, & \kappa = 0, \\ (1 + t)^{-1-\kappa}, & 0 \leq \kappa \leq \frac{\mu-2}{2}, \\ (1 + t)^{-\frac{\mu-2}{2}}, & \kappa \geq \frac{\mu-2}{2} \end{cases} \quad (3.1)
\]

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Proof. The proof is a direct application of Lemma 2.2. We use the canonical isomorphism between $[D]^{\kappa}L^2$ and $L^2$ to simplify the estimates. The mapping $(\langle D \rangle u_1, u_2)^T \mapsto (\langle D \rangle u(t, \cdot), u(t, \cdot))^T$ can be represented as matrix Fourier multiplier in terms of $\Psi_{k,\rho,\delta}$, because it includes the identification of spaces we include the number $\kappa$ in the notation. It holds

$$E^{(\kappa)}(t, \xi) = \frac{i\pi}{4} (1 + t)^{\rho} \begin{pmatrix} \Psi_{2+\kappa,0,\rho-1,1}(t, \xi) & -\Psi_{1+\kappa,0,\rho,0}(t, \xi) \\ \Psi_{2+\kappa,0,\rho-1,0}(t, \xi) & -\Psi_{1+\kappa,0,\rho,-1}(t, \xi) \end{pmatrix}. $$

It remains to check the conditions on the indices. For $\mu > 2$ we have $\rho < -1/2$ and therefore we have to compare $1 + \kappa - \rho$ with $-1/2$.

Some remarks for the interpretation of the result are necessary. The decay rate $t^{-\mu/2}$ corresponds to the rate for high frequencies. The cut-off by $|\xi|^\kappa$ damps out the small frequencies around the exceptional frequency $\xi = 0$.

For $\mu > 2$ the decay rate obtained in [4] was determined by the small frequencies around $\xi = 0$. This is the reason for the improvement of this theorem. An improvement over the decay rate for high frequencies is not possible.

4 Sharpness

The aim of this section is to discuss the sharpness of the estimate in the limit case $\kappa = (\mu - 2)/2$. For this we compare the operator family $E^{(\kappa)}(t)$ with the unitary evolution $E_0(t)$ of free waves in energy space,

$$E_0(t, \xi) = \begin{pmatrix} \cos t|\xi| & \sin t|\xi| \\ -\sin t|\xi| & \cos t|\xi| \end{pmatrix}, \quad (4.1)$$

modified by the decay rate.

**Theorem 4.1.** Let $\kappa = \frac{\mu - 2}{2}$ and $\mu \geq 2$. Then the strong limit

$$Z_+ = s-lim_{t \to \infty} (1 + t)^{\mu/2} E_0(-t)E^{(\kappa)}(t) \quad (4.2)$$

exists in the operator space $[D]^{\kappa}L^2 \to L^2$.

Let us first explain the main strategy of the proof. Theorem 3.1 yields a uniform bound for the operator family. In order to obtain the strong convergence we employ Banach-Steinhaus theorem for the dense subspace

$$M = \{ f \in L^2 \mid 0 \notin \text{supp } \hat{f} \}.$$ 

For this it is sufficient to prove (uniform) convergence of the Fourier multiplier for $|\xi| \geq c$ with $c > 0$. This can be done by the given explicit representations using known asymptotic expansions for Bessel functions, [5]. Because of its simplicity we will use the representation by real valued functions valid for $\rho \notin \mathbb{Z}$. The multiplier $Z_+$ will be analytic in $\rho$ and therefore the statement will follow in all cases.

The result is based on the following asymptotic formula, [3].
Proposition 4.2. For Bessel functions of first kind it holds

\[ \mathcal{J}_\rho(z) = \sqrt{\frac{2}{\pi z}} \cos \left( z - \frac{\rho \pi}{2} - \frac{\pi}{4} \right) + \mathcal{O}(z^{-3/2}), \quad z \to \infty. \]  

(4.3)

Proof of Theorem 4.1. Let \(|\xi| \geq c > 0\). Then it holds

\[
(1 + t)^{\nu/2} \mathcal{E}(t, \xi) = \frac{i \pi}{4} (1 + t)^{1/2} \begin{pmatrix}
\Psi_{2+,0,0+1,0}(t, \xi) & -\Psi_{1+,0,0+0,0}(t, \xi) \\
\Psi_{2+,0,0+0,0}(t, \xi) & -\Psi_{1+,0,0+1,0}(t, \xi)
\end{pmatrix}
\]

such that

\[
(1 + t)^{\nu/2} \mathcal{E}_0(-t, \xi) \mathcal{E}(t, \xi) = \frac{i \pi}{4} (1 + t)^{1/2} \begin{pmatrix}
\cos t|\xi| - \sin t|\xi| \\
\sin t|\xi| \cos t|\xi|
\end{pmatrix}
\]

\[
\begin{pmatrix}
\Psi_{2+,0,0+1,0}(t, \xi) & -\Psi_{1+,0,0+0,0}(t, \xi) \\
\Psi_{2+,0,0+0,0}(t, \xi) & -\Psi_{1+,0,0+1,0}(t, \xi)
\end{pmatrix}
\]

\[
= \left( m_{1,1}(t, \xi) \ m_{1,2}(t, \xi) \right) \left( m_{2,1}(t, \xi) \ m_{2,2}(t, \xi) \right).
\]

We proceed with calculating \( m_{1,1}(t, \xi) \). It holds

\[
m_{1,1}(t, \xi) = \frac{\pi}{2} \csc(\rho \pi) |\xi|^2 \mathcal{E}(\xi)^{-\kappa}(1 + t)^{1/2}
\]

\[
\left( \cos t|\xi| (\mathcal{J}_{1-\rho}(|\xi|) \mathcal{J}_\rho((1 + t)|\xi|) + \mathcal{J}_{\rho-1}(|\xi|) \mathcal{J}_{1-\rho}((1 + t)|\xi|))
\]

\[
- \sin t|\xi| (\mathcal{J}_{1-\rho}(|\xi|) \mathcal{J}_{\rho-1}((1 + t)|\xi|) - \mathcal{J}_{\rho-1}(|\xi|) \mathcal{J}_{1-\rho}((1 + t)|\xi|)) \right)
\]

\[
= \sqrt{\frac{\pi}{2}} \csc(\rho \pi) |\xi|^2 \mathcal{E}(\xi)^{-\kappa}(1 + t)^{1/2}
\]

\[
\left( \cos t|\xi| (\mathcal{J}_{1-\rho}(|\xi|) \cos((1 + t)|\xi| - \frac{\rho \pi}{2} - \frac{\pi}{4})
\]

\[
+ \mathcal{J}_{\rho-1}(|\xi|) \cos((1 + t)|\xi| + \frac{\rho \pi}{2} - \frac{\pi}{4}))
\]

\[
- \sin t|\xi| (\mathcal{J}_{1-\rho}(|\xi|) \cos((1 + t)|\xi| - \frac{\rho - 1}{2} \pi - \frac{\pi}{4})
\]

\[
- \mathcal{J}_{\rho-1}(|\xi|) \cos((1 + t)|\xi| - \frac{1 - \rho}{2} \pi - \frac{\pi}{4})) \right)
\]

\[
+ \mathcal{O}((1 + t)^{-1}|\xi|^{-1})
\]

\[
= \sqrt{\frac{\pi}{2}} \csc(\rho \pi) |\xi|^{\kappa+1} |\xi|^{1/2}
\]

\[
\left( \cos(|\xi| - \frac{\rho \pi}{2} - \frac{\pi}{4}) \mathcal{J}_{1-\rho}(|\xi|) - \cos(|\xi| + \frac{\rho \pi}{2} - \frac{\pi}{4}) \mathcal{J}_{\rho-1}(|\xi|) \right)
\]

\[
+ \mathcal{O}((1 + t)^{-1}|\xi|^{-1})
\]
as \( t \to \infty \), \(|\xi| \geq c\). Hence, as \( t \to \infty \) the function \( m_{1,1}(t, \xi) \) tends uniformly (on \(|\xi| \geq c\)) to the limit \( m_{1,1}(\infty, \xi) \) which extends continuously (and analytically) up to \( \xi = 0 \). The last statement follows from the generalized power series expansion of \( J_\rho(z) = z^\rho \Delta_\rho(z), \Delta_\rho(0) \neq 0 \) together with \( \kappa + 3/2 = (\mu + 1)/2 = 1 - \rho \).

A similar calculation yields for the other entries of the matrix limit expressions of the form

\[
m_{1,2}(\infty, \xi) = \sqrt{\frac{\pi}{2}} \csc(\rho \pi)|\xi|^{\kappa}|\xi|^{1/2} \left( \cos(|\xi| - \frac{\rho \pi}{2} - \frac{\pi}{4})J_\rho(|\xi|) + \cos(|\xi| + \frac{\rho \pi}{2} - \frac{\pi}{4})J_{-\rho}(|\xi|) \right)
\]

\[
m_{2,1}(\infty, \xi) = \sqrt{\frac{\pi}{2}} \csc(\rho \pi)|\xi|^{\kappa+1}|\xi|^{1/2} \left( \sin(|\xi| - \frac{\rho \pi}{2} - \frac{\pi}{4})J_\rho(|\xi|) + \sin(|\xi| + \frac{\rho \pi}{2} - \frac{\pi}{4})J_{-\rho}(|\xi|) \right)
\]

\[
m_{2,2}(\infty, \xi) = \sqrt{\frac{\pi}{2}} \csc(\rho \pi)|\xi|^{\kappa}|\xi|^{1/2} \left( \sin(|\xi| - \frac{\rho \pi}{2} - \frac{\pi}{4})J_{-\rho}(|\xi|) - \sin(|\xi| + \frac{\rho \pi}{2} - \frac{\pi}{4})J_\rho(|\xi|) \right).
\]

By analytic continuation these formulas are valid for all \( \xi \in \mathbb{R}^n \) and \( m_{i,j}(\infty, \cdot) \in L^\infty(\mathbb{R}^n) \). Furthermore, all entries are non-zero for \( \xi = 0 \) (due to the exact cancellation).

Furthermore the representations are analytic in \( \rho \) using the definition of the Weber functions \( \psi_\rho(z) = \csc(\rho \pi)(J_\rho(z) \cos(\rho \pi) - J_{-\rho}(z)) \).

An application of Liouville theorem to the differential equation for \((|D|u, u_1)^T\) yields

\[
\begin{vmatrix}
\Psi_{1,0,\rho,-1} & -\Psi_{1,0,\rho,0} \\
\Psi_{1,0,\rho,-1,0} & -\Psi_{1,0,\rho,-1}
\end{vmatrix} = (1 + t)^{-\mu} \quad (4.4)
\]

and therefore \( \det(1 + t)^{\mu/2}\Gamma(\kappa)(t, \xi) = |\xi|^{1+2\kappa} \).

**Corollary 4.3.** The operator \( Z_+ : [|D|]^{\kappa}L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) is injective and its symbol satisfies \( \det Z_+(\xi) = |\xi|^{1+2\kappa} \).

Remark. What have we obtained so far? The existence of nontrivial Cauchy data \((|D|u_1, u_2)\) in the null space of \( Z_+ \) is equivalent to the fact that the energy of the corresponding solution \( \|E(t)(|D|u_1, u_2)^T\|_2^2 \) decays faster than \( t^{-\mu} \). But we have shown, that such Cauchy data do not exist. This means, we have proven a two-sided energy estimate.

**Corollary 4.4.** Let \((|D|u_1, u_2) \in [|D|]^{\kappa}L^2(\mathbb{R}^n) \). Then

\[ E(u; t) \sim (1 + t)^{-\mu}. \quad (4.5) \]
Furthermore, the solutions behave in energy space as solutions to the free wave equation multiplied by the decay rate \((1 + t)^{-\mu/2}\). In the case of high frequencies we can even say more. The operator \(Z_+\) almost preserves high frequencies. Using again the asymptotic representation of Bessel functions for large arguments, Proposition \(4.2\) we conclude

**Corollary 4.5.** It holds \(\lim_{|\xi|\to\infty} Z_+(\xi) = I\).

For \(\mu = 2\) we have even more. In this case the Bessel functions are trigonometric ones and \(Z_+(\xi) = I\).

In order to conclude this article we will give one further interpretation to the assumptions on the Cauchy data we made. In case \(0 \leq \kappa < \frac{n}{2}\) the Sobolev-Hardy inequality can be used to obtain an embedding of the space of weighted \(L^2\)-functions into \([D]^{\kappa}L^2\).

**Lemma 4.6.** Let \(0 \leq \kappa < \frac{n}{2}\). Then \((x)^{-\kappa}L^2(\mathbb{R}^n) \subseteq [D]^{\kappa}L^2(\mathbb{R}^n)\).

**Proof.** Let \(\chi \in C_0\infty(\mathbb{R}^n)\) satisfy \(\chi(\xi) = 1\) near \(\xi = 0\). Then

\[
||f||_{[D]^{\kappa}L^2} = |||\xi|^{-\kappa} \hat{f}||_2 \sim |||\xi|^{-\kappa} \chi(\xi) \hat{f}||_2 + ||(1 - \chi(\xi))f||_2 \lesssim ||\hat{f}||_{H^\kappa} + ||\hat{f}||_2 \sim ||x|^{\kappa}f||_2
\]

by Sobolev-Hardy inequality and Plancherel’s theorem.

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