LEARNING FAMILIES OF ALGEBRAIC STRUCTURES
FROM INFORMANT

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Abstract. We combine computable structure theory and algorithmic learning theory to study learning of families of algebraic structures. Our main result is a model-theoretic characterization of the class $\text{InfEx}_\Sigma^2$, consisting of the structures whose isomorphism types can be learned in the limit. We show that a family of structures is $\text{InfEx}_\Sigma^2$-learnable if and only if the structures from $\mathfrak{A}$ can be distinguished in terms of their $\Sigma^2_{\inf}$-theories. We apply this characterization to familiar cases and we show the following: there is an infinite learnable family of distributive lattice; no pair of Boolean algebras is learnable; no infinite family of linear orders is learnable.

1. Introduction

In this paper we combine computable structure theory and algorithmic learning theory to study the question of extracting semantic knowledge from finite amount of structured data.

One of the equivalent definitions of computable structures states that those are the structures output by a Turing machine step by step, where the number of steps is potentially infinite (but at most countable). At each step we observe larger and larger finite pieces of the universe, and as soon as the algorithm outputs an element, it also reveals the relations between this element and all the elements appeared at previous stages. The algorithm can never change its mind whether a relation holds on particular elements or not. Formal definition follow in a further section.

This way of thinking about computable structures is in keeping with the ideas of the algorithmic learning theory (inductive inference) as introduced by Gold in [6]. Here a learner receives step by step more and more data (finite amount at each step) on an object to be learned, and outputs a sequence of hypotheses that converges to a description of the object. In general, learning can be viewed as a dialogue between a teacher and a learner, where the learner must succeed in learning, provided the teacher satisfies a certain protocol. The formalization of this idea has two aspects: convergence behavior and teacher constraints. Again, formal definitions follow below.

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Most work in inductive inference concerns either learning of formal languages or learning of total functions \([14, 8]\). The case of learning other structures has first been considered in \([5]\) and is surveyed in \([10]\). More recently, in \([7, 11, 16]\) Stephan and co-authors considered learnable ideals of rings, subgroups and submonoids of groups, subspaces of vector spaces and isolated branches on uniformly computable sequences of trees. They showed that different types of learnability of various classes of computable or computably enumerable structures have strong connections to their algebraic characterizations. The fact of such correspondence between learnability from different types of information and algebraic properties of structures is of big interest from mathematical point of view. In a sense, it is a way to study the interplay between algorithmic and algebraic properties of structures.

We employ a more general approach that can be applied to an arbitrary class of computable structures. The main idea is the following. Suppose we have a class of computable structures. And suppose we step by step get finite amounts of information about one of them. Then we learn the class, if after finitely many steps we correctly identify the structure we are observing. This is why, in this setting, we consider learning of a class of computable structures as a task of extracting semantic knowledge from finite amount of data.

In a recent paper \([4]\) Fokina, Kötzing and San Mauro considered learnable classes of equivalence structures. They reworked and extended the results appeared in \([5]\). In this paper we continue this line of investigation by applying the setup to other classes of structures. Our results are similar to Martin–Osherson approach \([10]\), but by using Turing computable embeddings, we can extract more information, in particular, we can control the complexity of a learner.

The paper is organized as follows. In Section 2 we give all the necessary definitions and useful facts from computable structure theory and learning theory. In Section 3 we prove our main result: a model-theoretic characterization of learnable families of structures. In Section 4 we apply the characterization from the previous section to get examples of learnable and non-learnable classes of natural structures.

2. Preliminaries

2.1. Computable structures. In the paper, we consider only finite signatures \(L\). When we talk about learnable families of \(L\)-structures, we assume that the domain of any countably infinite structure is equal to \(\omega\). This allows us to effectively identify sentences concerning such an \(L\)-structure with subsets of \(\omega\) through a fixed Gödel numbering. We can then define the (atomic) diagram of such an \(L\)-structure \(\mathcal{M}\) to be the set of \(n \in \omega\) such that \(n\) represents an atomic \(L_\mathcal{M}\)-sentence true in \(\mathcal{M}\) or the negation of an atomic \(L_\mathcal{M}\)-sentence that is false in \(\mathcal{M}\). We identify \(\mathcal{M}\) with its atomic diagram \(D(\mathcal{M})\) and measure the complexity of a structure via the complexity
of its diagram. Furthermore, we say that a structure $M$ is $d$-computable if $D(M)$ is a $d$-computable subset of $\omega$, where $d$ is a Turing degree. A presentation of a countable algebraic structure is an arbitrary isomorphic copy $M' \cong M$ with the universe a subset of $\omega$. We call a structure $M$ computably presentable if it has a presentation $M'$ which is computable. A structure is called $d$-computably presentable if for some $d_0 \leq d$ there exists a presentation $M' \cong M$ which is $d_0$-computable.

Any computable structure $A$ in a computable relational language can be presented as an increasing union of its finite substructures $A^0 \subseteq A^1 \subseteq \ldots \subseteq A^t \subseteq \ldots$ and $A = \bigcup_t A^t$.

By $\mathcal{K}_L$ we denote the class of all $L$-structures with domain $\omega$. We assume that every considered class of $L$-structures is closed under isomorphisms, modulo the restriction on domains. For a structure $S$, by $D(S)$ we denote the atomic diagram of $S$.

If $\mathcal{R}$ is a class of $L$-structures and $\sim$ is an equivalence relation on $\mathcal{R}$, then we always assume that for any $A$ and $B$ from $\mathcal{R}$,

$$A \cong B \Rightarrow A \sim B.$$ 

2.2. Informal discussion of our learning paradigm. Fokina, Kötzing, and San Mauro [4] introduced the paradigm of informant learning for families of computably presentable structures. Before delving into the formal details, we illustrate the paradigm by considering two simple examples.

Suppose that one wants to design an algorithm for learning the family $\mathcal{C}$, which consists of two countably infinite, undirected graphs:

1. $G_1$ which contains only cycles of size two, and
2. $G_2$ containing only 3-cycles.

Then the intuition behind a learning algorithm $A_\mathcal{C}$ is quite simple: Given a graph $H$ as input, we search for a cycle of size $n \in \{2, 3\}$ inside $H$. If $n = 2$, then $A_\mathcal{C}$ conjectures that $H$ is a copy of $G_1$. If $n = 3$, then $A_\mathcal{C}$ thinks that $H \cong G_2$.

More formally, the algorithm $A_\mathcal{C}$ is arranged as follows.

- Any possible input is an infinite binary string $I \in 2^\omega$. A string $I$ is treated as an object which encodes the atomic diagram of an undirected graph $G(I)$ on the domain $\omega$. The input $I$ can be incorrect: it is possible that some elements $x, y \in \omega$ satisfy one of the following conditions.
  - $I$ encodes neither $Edge(x, y)$, nor $\neg Edge(x, y)$. In other words, the information given by $I$ is incomplete.
  - $I$ says both $Edge(x, y)$ and $\neg Edge(x, y)$, i.e. the data provided by $I$ is inconsistent.

If $I$ is incorrect, then we do not really care about what $A_\mathcal{C}(I)$ will output.

- An input $I$ is processed by the algorithm $A_\mathcal{C}$ bit-by-bit: At a stage $s \in \omega$, we analyze the finite string $I[s]$, which contains the first
s + 1 bits of I, and we output our current conjecture $A_c(I[s])$. The possible conjectures are:

- “?” , which means that currently we have no clue about the isomorphism type of $G(I)$;
- “1”, i.e. now we think that the input I encodes (a presentation of) the graph $G_1$;
- “2”, i.e. we believe that I is a code for (a copy of) $G_2$.

We define $A_c(I[s]) = ?$. At a stage $s \geq 1$, proceed as follows: If $A_c(I[s]) \neq ?$, then just set $A_c(I[s + 1]) := A_c(I[s])$.

Otherwise, search for the least tuple $\bar{a}$ from $\omega$ such that the string $I[s] \omega$ contains the following data: the tuple $\bar{a}$ forms a cycle of size $n$, where $n \in \{2, 3\}$.

- If $n = 2$, then set $A_c(I[s + 1]) := 1$.
- If $n = 3$, then $A_c(I[s + 1]) := 2$.
- If there is no such $\bar{a}$, then define $A_c(I[s + 1]) := ?$.

The described algorithm $A_c$ learns the class $C$ in the following sense. Suppose that an input I encodes a structure $M$, which is isomorphic to either $G_1$ or $G_2$. Then there is a stage $s_0$ such that for any $s \geq s_0$, we have $A_c(I[s]) = A_c(I[s_0])$. Moreover, the conjecture $A_c(I[s_0])$ correctly identifies the isomorphism type of the graph $M$.

Note the following features of the algorithm $A_c$:

(a) An arbitrary copy $H$ of, say, $G_1$ with $\text{dom}(H) = \omega$ can be encoded via an appropriate correct input $I_H$. Hence, roughly speaking, the correct inputs are countably infinite graphs of arbitrary Turing complexity.

(b) If $H$ is an undirected graph which contains a 2-cycle or a 3-cycle, then the conjectures $A_c(I_H[s])$ eventually stabilize. Thus, all graphs which do not belong to $C$ can be roughly divided into two classes:

(i) “False-positive” graphs $F$: We have $F \notin C$, but nevertheless, the learning algorithm $A_c$ identifies $F$ as a member of $C$.

(ii) All other graphs $M$: In our setting, this case means that

$$\lim_{s \to \infty} A_c(I_M[s]) = ?.$$ 

We note that in more intricate learning algorithms, the sequence $A_c(I_M[s])$ can diverge.

Our second example is a generalization of the first one. Consider an infinite family $\mathcal{D}$, which consists of the following undirected graphs: for each $i \geq 1$, the graph $G_i$ contains infinitely many $(i + 1)$-cycles. As in the previous example, the intuition behind the desired learning algorithm $A_{\mathcal{D}}$ is pretty straightforward: Given a graph $H$, search for a cycle of some size $l + 1$ inside it. When the first such cycle is found, start outputting the conjecture “$H$ is a copy of $G_i$.”

The main technical problem of the algorithm $A_{\mathcal{D}}$ can be formulated as follows (note that this problem is already implicit even in our first example):
How does one formally define the set of possible conjectures?

We discuss two possible solutions of the problem, and both of them seem to be pretty natural.

As before, we assume that every conjecture is an element of the set $\omega \cup \{?\}$.

**First Solution.** One can assume that for any $m \in \omega$, the conjecture “$m$” means that “$H \cong G_{m+1}$.”

We call this solution an *honest learning*: by default, here we assume that every input $I$ is honest, i.e. $I$ describes a graph from the class $\mathcal{D}$.

The honest learning algorithm $A^h_\mathcal{D}$ is a straightforward modification of the algorithm $A_C$. At a stage $s+1$, $A^h_\mathcal{D}$ searches for the least tuple $\bar{a}$ such that the string $I[r_s]s$ encodes the following data: the tuple $\bar{a}$ forms a cycle of some size $n \geq 2$. When such $\bar{a}$ is found, the algorithm starts outputting the conjecture “$n-1$.”

Notice that every infinite graph $F$, which contains a cycle, is false-positive for the algorithm $A^h_\mathcal{D}$.

**Second Solution.** Fix an effective list $(\mathcal{M}_e)_{e \in \omega}$ of all computable undirected graphs. W.l.o.g., one may assume that $\mathcal{M}_0 \notin \mathcal{D}$ and $\mathcal{M}_{i(0)} \cong G_i$ for all $i \geq 1$. We assume that the conjecture “$m$” means that “$H \cong \mathcal{M}_m$.”

This solution can be called an *index learning*: We assume that an input $I$ can describe an arbitrary infinite graph $H$, and we try to guess an index $e$ such that $\mathcal{M}_e \cong H$. We require that our guesses must be eventually correct only for graphs $H \in \mathcal{D}$, and we do not care much about other isomorphism types.

The index learning algorithm $A^i_\mathcal{D}$ works on an input $I$ as follows:

(a) First, as in the honest $A^h_\mathcal{D}$, we search for a cycle of some size $n \geq 2$.

When the cycle is found, start outputting the conjecture “$\langle n-1, 0 \rangle$.”

(b) After that stage, assume that we find a finite piece of evidence (provided by $I$) showing that $G(I) \notin G_{n-1}$: e.g., we see that

- $G(I)$ contains a component of size at least $n + 1$, or
- $G(I)$ contains a vertex of degree at least 3, or
- $G(I)$ contains a cycle of size at most $n - 1$.

Then we start outputting the conjecture “0.”

It is clear that some of the false-positive graphs are successfully eliminated by the algorithm $A^i_\mathcal{D}$. Nevertheless, note that $A^i_\mathcal{D}$ still admits some false-positive graphs: e.g., the graph $H$ containing infinitely many singletons and one 3-cycle.

The learning algorithms $A^h_\mathcal{D}$ and $A^i_\mathcal{D}$ can be unified in a general framework as follows. One can consider an arbitrary superclass $\mathcal{R} \supseteq \mathcal{D}$. We assume that the class $\mathcal{R}$ is *uniformly enumerable*, i.e. there is a uniformly computable sequence of structures $(\mathcal{N}_e)_{e \in \mathbb{N}}$ such that:

1. Any structure from $\mathcal{R}$ is isomorphic to some $\mathcal{N}_e$.

2. For every $e$, $\mathcal{N}_e$ belongs to $\mathcal{R}$.
Then for a number \( e \in \omega \), the conjecture “\( e \)” is interpreted as “the input structure is isomorphic to \( N_e \).”

The formal description of this framework is given in Section 2.4 (see the notion of InfEx\( \nu \)–r\( \nu \)s-learning in Definition 2.4).

2.3. Infinitary formulas. Suppose that \( X \subseteq \omega \) is an oracle, and \( \alpha \) is an \( X \)-computable non-zero ordinal. Following Chapter 7 of [2], we describe the class of \( X \)-computable infinitary \( \Sigma^c_\alpha \) formulas (or \( \Sigma^c_{\alpha} \) formulas, for short) in a signature \( L \).

(a) \( \Sigma^c_0(X) \) and \( \Pi^c_0(X) \) formulas are quantifier-free first-order \( L \)-formulas.
(b) A \( \Sigma^c_\alpha(X) \) formula \( \psi(x_0, \ldots, x_m) \) is an \( X \)-computably enumerable (\( X \)-c.e.) disjunction

\[
\bigwedge_{i \in I} \exists \bar{y}_i \xi_i(\bar{x}, \bar{y}_i),
\]

where each \( \xi_i \) is a \( \Pi^c_\beta(X) \) formula, for some \( \beta_i < \alpha \).
(c) A \( \Pi^c_\alpha(X) \) formula \( \psi(\bar{x}) \) is an \( X \)-c.e. conjunction

\[
\bigwedge_{i \in I} \forall \bar{y}_i \xi_i(\bar{x}, \bar{y}_i),
\]

where each \( \xi_i \) is a \( \Sigma^c_\beta(X) \) formula, for some \( \beta_i < \alpha \).

In the paper, we mainly work with \( \Sigma^c_\alpha(X) \) formulas for finite ordinals \( \alpha \) (even more, for \( \alpha \leq 2 \)). Henceforth, in this section we assume that \( \alpha = n \) is a natural number.

Infinitary \( \Sigma_n \) formulas (or \( \Sigma^\text{inf}_n \) formulas, for short) are defined in the same way as above, modulo the following modification: infinite disjunctions and conjunctions are not required to be \( X \)-c.e. It is clear that a formula \( \psi \) is logically equivalent to a \( \Sigma^\text{inf}_n \) formula if and only if \( \psi \) is equivalent to a \( \Sigma^c_n(X) \) formula for some oracle \( X \). A similar fact holds for \( \Pi^\text{inf}_n \) formulas. For more details on infinitary formulas, we refer the reader to [2].

As per usual, the \( \Sigma^\text{inf}_n \)-theory of an \( L \)-structure \( S \) is the set

\[
\Sigma^\text{inf}_n \text{-} \text{Th}(S) = \{ \psi : \psi \text{ is a } \Sigma^\text{inf}_n \text{ sentence true in } S \}.
\]

2.4. Learning families of structures: Formal details. Here we give the necessary formal preliminaries on our learning paradigm.

Let \( L = \{ P^{n_0}_0, P^{m_1}_1, \ldots, P^{n_k}_k \} \) be a relational signature. An \( L \)-informant is a function

\[
I : \omega \to (\omega^{n_0} \times \{0, 1\}) \times (\omega^{m_1} \times \{0, 1\}) \times \cdots \times (\omega^{n_k} \times \{0, 1\}).
\]

For a number \( m \), the value \( I(m) \) is treated as a \((k + 1)\)-tuple

\[
I(m) = (I_0(m), I_1(m), \ldots, I_k(m)),
\]

where \( I_j(m) \in \omega^{n_j} \times \{0, 1\} \). Let \( \text{content}^+_j(I) := \{ \bar{a} \in \omega^{n_j} : (\bar{a}, 1) \in \text{range}(I_j) \} \).

The positive content of the informant \( I \) is the tuple

\[
\text{content}^+(I) = (\text{content}^+_0(I), \text{content}^+_1(I), \ldots, \text{content}^+_k(I)).
\]
For an $L$-informant $I$ and an $L$-structure $S = (\omega; P_0, P_1, \ldots, P_k)$, we say that $I$ is an informant for $S$ if for every $i \leq k$, content$^+(I) = P_i$. By $\text{Inf}(S)$ we denote the set of all informants for the structure $S$.

If a signature $L$ contains functional symbols and/or constants, then one can use a standard convention from computable structure theory: by replacing functions with their graphs, we can treat any $L$-structure as a relational one. If a signature $L$ is clear from the context, then we will talk about informants without specifying their prefix $L$.

For a number $n$ and a function $f$ with $\text{dom}(f) = \omega$, by $f[n]$ we denote the finite sequence $f(0), f(1), \ldots, f(n-1)$.

A learner is a function $M$ mapping initial segments of informants to conjectures (elements of $\omega \cup \{?\}$). The learning sequence of a learner $M$ on an informant $f$ is the function $p: \omega \to \omega \cup \{?\}$ such that $p(n) = M(f[n])$ for every $n$.

Let $\sigma$ be an initial part of an $L$-informant. By $A_\sigma$ we denote the finite structure which is defined as follows: The domain of $A_\sigma$ is the greatest (under set-theoretic inclusion) set $D \subseteq \omega$ with the following properties:

(a) Every $x \in D$ is mentioned in $\sigma$, i.e. there are numbers $m < |\sigma|$, $j \leq k$, and a tuple $\bar{a}$ such that $x$ occurs in $\bar{a}$ and $\sigma_j(m) = \bar{a}$ and $\sigma_j(m)$ is a tuple from $\sigma_j$ that for all $s \geq s_0$, we have $A_I[s] \neq \emptyset$. Furthermore, it is clear that

$$A_I[s] \subseteq A_I[s+1] \text{ and } B = \bigcup_{s \in \omega} A_I[s].$$

Definition 2.1. Let $\mathcal{K}$ be a class of $L$-structures. An effective enumeration of the class $\mathcal{K}$ is a function $\nu: \omega \to \mathcal{K}$ with the following properties:

1. The sequence of $L$-structures $(\nu(e))_{e \in \omega}$ is uniformly computable.
2. For any $A \in \mathcal{K}$, there is an index $e$ such that the structures $A$ and $\nu(e)$ are isomorphic.

In other words, the map $\nu$ effectively lists all isomorphism types from the class $\mathcal{K}$.

Sometimes we abuse our notations: we assume that the notions “enumeration” and “effective enumeration” are synonymous. If $\nu$ and $\mu$ are two
enumerations, then a new enumeration $\nu \oplus \mu$ is defined as follows.\\

$$(\nu \oplus \mu)(2n) := \nu(n), \quad (\nu \oplus \mu)(2n + 1) := \mu(n).$$

**Definition 2.2.** Let $\nu$ be an effective enumeration of a class $\mathfrak{K}$, and let $A$ be a structure from $\mathfrak{K}$. The index set of the structure $A$ w.r.t. $\nu$ is defined as follows:

$$\text{Ind}(A; \nu) = \{ e \in \omega : \nu(e) \simeq A \}.$$ 

We say that an effective enumeration $\nu$ is **decidable** if the set 

$$\{(i, j) : \nu(i) \simeq \nu(j)\}$$

is computable. An effective enumeration $\nu$ is **Friedberg** if $\nu(i) \neq \nu(j)$ for all $i \neq j$.

**Remark 2.3.** Note that any Friedberg enumeration is decidable. Moreover, if $\nu$ is a decidable enumeration of a class $\mathfrak{K}$, then for any $A \in \mathfrak{K}$, its index set $\text{Ind}(A; \nu)$ is computable.

Now we are ready to give the notion of informant learning:

**Definition 2.4.** Let $\mathfrak{K}$ be a class of $L$-structures, and let $\nu$ be an effective enumeration of $\mathfrak{K}$. Suppose that $C$ is a subclass of $\mathfrak{K}$. We say that $C$ is **InfEx**-learnable if there is a learner $M$ with the following property: If $I$ is an informant for a structure $A \in C$, then there are $e$ and $s_0$ such that $\nu(e) \simeq A$ and $M(I[s]) = e$ for all $s \geq s_0$. In other words, in the limit, the learner $M$ learns all isomorphism types from $C$.

2.5. **Locking sequences.** The paper [4] is focused on different versions of learning for various classes of equivalence structures. Here we briefly recap the results of [4] on locking sequences, but now we formulate them for arbitrary classes of structures.

We say that a finite sequence $\sigma$ describes a finite part of an $L$-structure $A$ if $\sigma$ is an initial segment of some $L$-informant for the structure $A$. Note that since we are working with informant learning, $\sigma$ contains both positive and negative data about the structure $A$.

**Definition 2.5** ([4, Definition 17]). Suppose that $M$ is a learner and $A$ is an $L$-structure. A sequence $\sigma$ describing a finite part of $A$ is a **weak informant locking sequence of $M$ on $A$** if for every $\tau \supseteq \sigma$ describing a finite part of $A$, we have $M(\tau) = M(\sigma)$. 
Theorem 2.6 ([4 Theorem 18]). Let \( \nu \) be an effective enumeration of a class \( \mathfrak{K} \), and let \( A \) be a structure from \( \mathfrak{K} \). Suppose that a learner \( M \) \( \text{InfEx}_{\geq}[\nu] \)-learns the structure \( A \). Let \( \sigma_0 \) be a sequence which describes a finite part of \( A \). Then there is a finite sequence \( \sigma \supseteq \sigma_0 \) such that \( \sigma \) is a weak informant locking sequence of \( M \) on \( A \). Furthermore, \( \nu(M(\sigma)) \approx A \).

Proof Sketch. Towards contradiction, assume that there is \( \sigma_0 \) with no weak locking sequence \( \sigma \supseteq \sigma_0 \). Then for any \( \sigma \supseteq \sigma_0 \) describing a finite part of \( A \), there is a string \( \text{ext}(\sigma) \supseteq \sigma \) such that \( \text{ext}(\sigma) \) also describes a finite part of \( A \), and \( M(\text{ext}(\sigma)) \neq M(\sigma) \).

Fix an informant \( I \) for \( A \). Then one can produce a new informant \( I' \) for \( A \) such that the learner \( M \) does not correctly converge on \( I' \): Just “alternate” between the data given by \( I \) and “bad” extensions \( \text{ext}(\sigma) \), in an appropriate way. \( \square \)

Definition 2.7 ([4 Definition 19]). Let \( M \) be a learner and \( A \) be an \( L \)-structure. We say that \( M \) is informant locking on \( A \) if for every informant \( I \) for \( A \), there is \( n \) such that \( I[n] \) is a weak informant locking sequence for \( M \) on \( A \).

Suppose that a class \( \mathfrak{A} \) is \( \text{InfEx}_{\geq}[\nu] \)-learnable. A learner \( M \) which \( \text{InfEx}_{\geq}[\nu] \)-learns \( \mathfrak{A} \) is informant locking if it is informant locking for every \( A \in \mathfrak{A} \).

Theorem 2.8 (see Theorem 20 in [4]). If a class \( \mathfrak{A} \) is \( \text{InfEx}_{\geq}[\nu] \)-learnable, then there is an informant locking learner \( M \) which \( \text{InfEx}_{\geq}[\nu] \)-learns \( \mathfrak{A} \).

3. Learning from informant, and infinitary \( \Sigma_2 \)-theories

In this section, we offer a model-theoretic characterization of what families of structures are \( \text{InfEx}_{\geq}[\nu] \)-learnable: Informally speaking, we show that a family of structures \( \mathfrak{K} \) is \( \text{InfEx}_{\geq}[\nu] \)-learnable if and only if the (isomorphism types of) structures from \( \mathfrak{K} \) can be distinguished in terms of their \( \Sigma^\inf_2 \)-theories.

Suppose that \( \mathfrak{K}_0 \) is a class of \( L \)-structures, and \( \nu \) is an effective enumeration of the class \( \mathfrak{K}_0 \).

Theorem 3.1. Let \( \mathfrak{K} = \{ B_i : i \in \omega \} \) be a family of structures such that \( \mathfrak{K} \subseteq \mathfrak{K}_0 \), and the structures \( B_i \) are infinite and pairwise non-isomorphic. Then the following conditions are equivalent:

1. The class \( \mathfrak{K} \) is \( \text{InfEx}_{\geq}[\nu] \)-learnable;
2. There is a sequence of \( \Sigma^\inf_2 \) sentences \( \{ \psi_i : i \in \omega \} \) such that for all \( i \) and \( j \), we have \( B_j \models \psi_i \) if and only if \( i = j \).

Remark 3.2. Note that Theorem 3.7 talks about classes \( \mathfrak{K} \) which contain infinitely many isomorphism types. Nevertheless, one can easily formulate (and prove) an analogous result for classes with only finitely many isomorphism types: Just work with a family \( \mathfrak{K} = \{ B_0, B_1, \ldots, B_n \} \) and the corresponding finite sequence of \( \Sigma^\inf_2 \) sentences \( \{ \psi_0, \psi_1, \ldots, \psi_n \} \).
The proof of Theorem 3.1 is organized as follows. Section 3.1 discusses the necessary preliminaries on Turing computable embeddings, which constitute one of the main ingredients of the proof. In Section 3.2, we give a result (Proposition 3.6) which provides a connection between InfEx\_\text{c}\_\text{e}-learnability and Turing computable embeddings. Section 3.3 finishes the proof.

3.1. Turing computable embeddings. When we are working with Turing computable embeddings, we consider structures $S$ such that the domain of $S$ is an arbitrary subset of $\omega$. In contrast, recall that our learning paradigm applies only to structures with domain equal to $\omega$. As before, any considered class of structures is closed under isomorphisms, modulo the domain restrictions.

Let $K_0$ be a class of $L_0$-structures, and $K_1$ be a class of $L_1$-structures.

**Definition 3.3** ([3, 9]). A Turing operator $\Phi = \varphi_e$ is a Turing computable embedding of $K_0$ into $K_1$, denoted by $K_0 \leq_{tc} K_1$, if $\Phi$ satisfies the following:

1. For any $A \in K_0$, the function $\varphi_{\Phi(A)}$ is the characteristic function of the atomic diagram of a structure from $K_1$. This structure is denoted by $\Phi(A)$.
2. For any $A, B \in K_0$, we have $A \equiv B$ if and only if $\Phi(A) \equiv \Phi(B)$.

The term “Turing computable embedding” is often abbreviated as $tc$-embedding. One of the important results in the theory of $tc$-embeddings is the following:

**Theorem 3.4** (Pullback Theorem; Knight, Miller, and Vanden Boom [9]). Suppose that $K_0 \leq_{tc} K_1$ via a Turing operator $\Phi$. Then for any computable infinitary sentence $\psi$ in the signature of $K_1$, one can effectively find a computable infinitary sentence $\psi^*$ in the signature of $K_0$ such that for all $A \in K_0$, we have $A \models \psi^*$ if and only if $\Phi(A) \models \psi$. Moreover, for a non-zero $\alpha < \omega_1^{CK}$, if $\psi$ is a $\Sigma_{\alpha}$ formula ($\Pi_{\alpha}$ formula), then so is $\psi^*$.

An analysis of the proof of Theorem 3.4 shows that this result admits a full relativization as follows.

Fix an oracle $X \subseteq \omega$. In a natural way, a Turing $X$-relativized operator $\varphi_{e,X}$ can be defined as follows: for a set $Z \subseteq \omega$ and a natural number $k$, let $
abla_{\varphi_{e,X}}(k) := \varphi_{e}^{Z\oplus X}(k)$.

Informally speaking, one can identify a Turing $X$-relativized operator with a Turing machine which has three tapes: the input tape, the output tape, and the oracle tape, where the oracle tape always contains the characteristic function of $X$.

In a straightforward way, one can use the notion of a Turing $X$-relativized operator to introduce Turing $X$-computable embeddings. If there is a Turing $X$-computable embedding from $K_0$ into $K_1$, then we write $K_0 \leq_{tc}^X K_1$.

One can obtain the following consequence of Theorem 3.4.
Corollary 3.5 (Relativized Pullback Theorem). Suppose that $X \subseteq \omega$, and $\mathfrak{R}_0 \leq^X \mathfrak{R}_1$ via an operator $\Phi_{[X]}$. Then for any $X$-computable infinitary sentence $\psi$ in the signature of $\mathfrak{R}_1$, one can find, effectively with respect to $X$, an $X$-computable infinitary sentence $\psi^*$ in the signature of $\mathfrak{R}_0$ such that for all $A \in \mathfrak{R}_0$, we have $A \models \psi^*$ if and only if $\Phi_{[X]}(A) \models \psi$. Furthermore, for a non-zero $\alpha < \omega^X_1$, if $\psi$ is a $\Sigma^c_\alpha(X)$ formula (or $\Pi^c_\alpha(X)$ formula), then so is $\psi^*$.

3.2. Connecting $\text{InfEx}_\omega$-learnability and $tc$-embeddings. Let $L$ be a finite signature, and $\mathfrak{R}_0$ be a class of $L$-structures. Let $\nu$ be an effective enumeration of the class $\mathfrak{R}_0$.

Suppose that $\mathfrak{R} = \{B_i : i \in \omega\}$ is a family of $L$-structures with the following properties:

(a) $\mathfrak{R}$ is a subclass of $\mathfrak{R}_0$. All $B_i$ are infinite and pairwise non-isomorphic.

(b) There is a learner $M$ which $\text{InfEx}_\omega[\nu]$-learns the class $\mathfrak{R}$.

We choose the oracle $X$ as follows:

(1) $X := M \oplus \{\langle i, k \rangle : i \in \omega, k \in \text{Ind}(B_i; \nu)\} \oplus \{j : \exists i (j \in \text{Ind}(B_i; \nu))\}$.

Consider a signature

$L_{st} := \{\leq\} \cup \{P_i : i \in \omega\},$

where every $P_i$ is a unary relation. For $i \in \omega$, we define an $L_{st}$-structure $S_i$ as follows: All $P_j$ are disjoint. For $j \neq k$, if $x \in P_j$ and $y \in P_k$, then $x$ and $y$ are incomparable under $\leq$. Every $P_j$, $j \neq i$, contains a $\leq$-structure isomorphic to the ordering of rationals $\eta$. The relation $P_i$ contains a copy of $1 + \eta$.

Let $\mathfrak{R}_{st}$ denote the class $\{S_i : i \in \omega\}$.

Proposition 3.6. There is a Turing $X$-computable embedding $\Phi_{[X]}$ from $\mathfrak{R}$ into $\mathfrak{R}_{st}$ such that for any $i \in \omega$, we have $\Phi_{[X]}(B_i) \cong S_i$.

Proof. Let $C$ be a structure such that $C$ is isomorphic to some $B_i$, and $\text{dom}(C) \subseteq \omega$.

It is not hard to show that there is a Turing operator $\Psi$ with the following property: If $\mathcal{E}$ is a countably infinite $L$-structure with $\text{dom}(\mathcal{E}) \subseteq \omega$, then $\Psi^{D(\mathcal{E})}$ is the atomic diagram of a structure $\mathcal{E}_1$ such that $\text{dom}(\mathcal{E}_1) = \omega$ and $\mathcal{E}_1$ is $D(\mathcal{E})$-computably isomorphic to $\mathcal{E}$.

The existence of the operator $\Psi$ implies that w.l.o.g., we may assume that the domain of our $\mathcal{C}$ is equal to $\omega$. For simplicity, we assume that $L = \{Q_0, Q_1, \ldots, Q_l\}$, where each $Q_i$ has arity $i + 1$. For $i \leq l$, fix a computable bijection $\gamma_i : \omega \to \omega^{i+1}$.

We describe the construction of the $L_{st}$-structure $\Phi_{[X]}(C)$. First, define an $L$-informant $I^C$ as follows. For $i \leq l$ and $m \in \omega$, set:

$I^C_i(m) = \begin{cases} (\gamma_i(m), 1), & \text{if } C \models Q_i(\gamma_i(m)), \\ (\gamma_i(m), 0), & \text{if } C \models \neg Q_i(\gamma_i(m)). \end{cases}$
Fix a computable copy $\mathcal{M}$ of the ordering $\eta$, and choose a computable descending sequence $q_0 >_\mathcal{M} q_1 >_\mathcal{M} q_2 >_\mathcal{M} \ldots$.

The construction of the structure $\mathcal{E} = \Phi_{[X]}(\mathcal{C})$ proceeds in stages.

**Stage 0.** Put inside every $P^\mathcal{C}_j$, $j \in \omega$, a computable copy of the interval $(q_0; \infty)_\mathcal{M}$.

**Stage $s + 1$.** Recall that the learner $M$ InfEx$_s$[$\nu$]-learn the class $\mathfrak{R}$. Compute the value $t := M(\mathcal{I}^\mathcal{C}_{s + 1})$. Using the oracle $X$, one can find whether the number $t$ is a $\nu$-index for some $\mathcal{B}_j$, $j \in \omega$.

If $t$ is not a $\nu$-index for any $\mathcal{B}_j$, then extend every $P^\mathcal{C}_k$, $k \in \omega$, to a copy of $(q_{s + 1}; \infty)_\mathcal{M}$.

Otherwise, assume that $t$ is an index for $\mathcal{B}_j$. If $P^\mathcal{C}_j[s]$ has the least element, then do not change $P^\mathcal{C}_j[s]$. If $P^\mathcal{C}_j[s]$ has no least element, then define $P^\mathcal{C}_j[s + 1]$ as a copy of the interval $[q_{s + 1}; \infty)_\mathcal{M}$. Note that this interval is isomorphic to $1 + \eta$. In any case, extend every $P^\mathcal{C}_k[s]$, $k \neq j$, to a copy of the open interval $(q_{s + 1}; \infty)_\mathcal{M}$.

This concludes the description of the construction. It is not hard to show that the construction gives a Turing $X$-computable operator $\Phi_{[X]}$. Moreover, if the input structure $\mathcal{C}$ is isomorphic to $\mathcal{B}_i$, then there is a stage $s_0$ such that for any $s \succeq s_0$, we have $M(\mathcal{I}^\mathcal{C}_s) = M(\mathcal{I}^\mathcal{C}_{s_0})$ is a $\nu$-index of the structure $\mathcal{B}_i$. Hence, $P^\mathcal{C}_i \Phi_{[X]}(\mathcal{C})$ contains a copy of $1 + \eta$, and for every $j \neq i$, $P^\mathcal{C}_j \Phi_{[X]}(\mathcal{C})$ copies $\eta$. Thus, $\Phi_{[X]}(\mathcal{C})$ is isomorphic to $\mathcal{S}_i$.

Proposition 3.6 is proved. \qed

### 3.3. Proof of Theorem 3.1

**Proof.** (1) $\Rightarrow$ (2): Choose an oracle $X$ according to equation (1). By Proposition 3.6, there is a Turing $X$-computable embedding $\Phi_{[X]} : \mathfrak{R} \leq^X \mathfrak{R}_{st}$ such that $\Phi_{[X]}(\mathcal{B}_i)$ is a copy of $\mathcal{S}_j$.

Consider an $\exists \forall$-sentence in the signature $L_{st}$

$$\xi_i := \exists x \forall y [P_i(y) \rightarrow (x \leq y)].$$

Note that $\mathcal{S}_j \models \xi_i$ if and only if $i = j$. By Corollary 3.5, we obtain a sequence of $X$-computable infinitary $\Sigma_2$ sentences $(\xi^*_i)_{i \in \omega}$. Clearly, this sequence has the desired properties.

(2) $\Rightarrow$ (1): W.l.o.g., for all $i$, assume that $\psi_i := \exists x_1, \ldots, x_{n_i} \bigwedge_{j \in J_i} \forall y_1, \ldots, y_{m_{i,j}} \varphi_{i,j}(x_1, \ldots, x_{n_i}, y_1, \ldots, y_{m_{i,j}})$, where every $\varphi_{i,j}$ is a quantifier-free formula.

Let $\mathcal{C}$ be a finite structure, and $i \in \omega$. We say that the formula $\psi_i$ is $\mathcal{C}$-compatible via a tuple $\bar{a} \in \omega^{n_i}$ if there is no pair $(j, \bar{b}) \leq \text{dom}(\mathcal{C})$, with $j \in J_i$ and $\bar{b} \in \omega^{m_{i,j}}$, such that $\mathcal{C} \models \neg \varphi_{i,j}(\bar{a}, \bar{b})$. 

We fix a sequence $(e_i)_{i \in \omega}$ such that for every $i$, the structure $\nu(e_i)$ is a copy of $B_i$.

A learner $M$ for the class $\mathcal{K}$ can be arranged as follows: Suppose that $M$ reads a string $\sigma$, which is an initial part of some $L$-informant. Then we search for the least pair $\langle i, \bar{a} \rangle$ such that the formula $\psi_i$ is $A_\sigma$-compatible via the tuple $\bar{a}$. If the pair $\langle i, \bar{a} \rangle$ is found, then set $M(\sigma) := e_i$. Otherwise, define $M(\sigma) := 0$.

**Verification.** Fix $j \in \omega$. Let $I$ be an informant for the structure $B_j$. Recall that $B_j = \bigcup_{s \in \omega} A_{I[s]}$ and $A_{I[s]} \subseteq A_{I[s+1]}$.

We note the following simple fact: Suppose that a formula $\psi_i$ is not $A_{I[t]}$-compatible via a tuple $\bar{d}$. Then for any $t \geq t_0$, $\psi_i$ also cannot be $A_{I[t]}$-compatible via $\bar{d}$.

Recall that $B_j \models \psi_i$ if and only if $i = j$. Hence, there exists the least tuple $\bar{a} \in \omega^{\nu_j}$ with the following property: there is a stage $s_0$ such that for every $s \geq s_0$, the formula $\psi_j$ is $A_{I[s]}$-compatible via the tuple $\bar{a}$. Furthermore, it is not difficult to see that

$$B_j \models \neg\psi_i \iff (\forall \bar{c} \in \omega^{\nu_j})(\exists s_1)(\psi_i \text{ is not } A_{I[s_1]} \text{-compatible via } \bar{c}).$$

Hence, for every number $\langle k, \bar{e} \rangle < \langle j, \bar{a} \rangle$, there is a stage $t_1$ such that for any $t \geq t_1$, the formula $\psi_k$ is not $A_{I[t]}$-compatible via $\bar{c}$. This means that there is $t^*$, such that the current conjecture $M(I[t^*])$ is correct (i.e. $\nu(M(I[t^*]))$ is a copy of $B_j$), and our learner $M$ does not change its mind after the stage $t^*$.

Therefore, the class $\mathcal{K}$ is $\text{InfEx}_{\geq}[\nu]$-learnable by the learner $M$. This concludes the proof of Theorem 3.1.

4. Applications of the main result

The first application gives an upper bound for the Turing complexity of learners. A straightforward analysis of the proof of Theorem 3.1 provides us with the following:

**Corollary 4.1.** Let $X \subseteq \omega$ be an oracle. Let $\mathcal{K}_0$ be a class of countably infinite $L$-structures, and $\nu$ be an effective enumeration of $\mathcal{K}_0$. Assume that either $I = \omega$, or $I$ is a finite initial segment of $\omega$. Consider a subclass $\mathcal{K} = \{B_i : i \in I\}$ inside $\mathcal{K}_0$. Suppose that:

(i) There is uniformly $X$-computable sequence of $\Sigma^0_2(X)$-sentences $(\psi_i)_{i \in I}$ such that:

$$B_j \models \psi_i \iff i = j.$$

(ii) There is an $X$-computable sequence $(e_i)_{i \in I}$ such that $\nu(e_i) \equiv B_i$ for all $i$. Note that if the set $I$ is finite, then one can always choose this sequence in a computable way.

Then the class $\mathcal{K}$ is $\text{InfEx}_{\geq}[\nu]$-learnable via an $X$-computable learner.

The rest of the section discusses applications of Theorem 3.1 and Corollary 4.1 to some familiar classes of algebraic structures.
4.1. Simple examples of learnable classes. Here we give two examples of learnable infinite families.

The first one deals with distributive lattices. We treat lattices as structures in the signature $\mathcal{L}_{\text{lat}} := \{ \lor, \land \}$.

Selivanov [15] constructed a strongly computable family $\{ \mathcal{D}_i : i \in \omega \}$ of finite distributive lattices with the following property: If $i \neq j$, then there is no isomorphic embedding from $\mathcal{D}_i$ into $\mathcal{D}_j$ (see Figure 4.1).

For $i \in \omega$, we define a countably infinite poset $\mathcal{B}_i$. Informally speaking, $\mathcal{B}_i$ is a direct sum of the lattice $\mathcal{D}_i$ and the linear order $\omega$. More formally, we set:

- $\text{dom}(\mathcal{B}_i) = \{ \langle x, 0 \rangle : x \in \mathcal{D}_i \} \cup \{ \langle y, 1 \rangle : y \in \omega \}$.
- We always assume that $\langle x, 0 \rangle \leq \langle y, 1 \rangle$. The ordering of the elements $\langle x, 0 \rangle$ is induced by $\mathcal{D}_i$. We have $\langle y, 1 \rangle \leq \langle z, 1 \rangle$ if and only if $y \leq \omega z$.

It is not hard to show that $\mathcal{B}_i$ is a distributive lattice, thus, we will treat $\mathcal{B}_i$ as an $L_{\text{lat}}$-structure.

Let $\mathcal{K}_{\text{lat}}$ denote the class $\{ \mathcal{B}_i : i \in \omega \}$. It is clear that one can build a Friedberg effective enumeration $\nu_{\text{lat}}$ as follows: just define $\nu_{\text{lat}}(i)$ as a natural computable copy of $\mathcal{B}_i$.

**Proposition 4.2.** The class $\mathcal{K}_{\text{lat}}$ is $\text{InfEx}_{\geq}[\nu_{\text{lat}}]$-learnable via a computable learner.

**Proof.** For $i \in \omega$, one can easily define a first-order $\exists$-sentence $\psi_i$ in the signature $L_{\text{lat}}$, which means the following: for a structure $\mathcal{S}$, $\mathcal{S} \models \psi_i$ iff the finite lattice $\mathcal{D}_i$ can be isomorphically embedded into $\mathcal{S}$.
Note the following properties of the considered objects:

- $D_i$ embeds into $B_j$ if and only if $i = j$.
- The sequence $\{\psi_i\}_{i \in \omega}$ is uniformly computable.
- For every $i$, $\nu_{\text{lat}}(i) \cong B_i$.

Therefore, one can apply Corollary 4.1 with a computable oracle $X$. Proposition 4.2 is proved.

Recall that $\mathbb{K}_{\text{lat}}$ is the class of all countably infinite $\text{Lat}$-structures.

**Corollary 4.3.** Suppose that $\nu$ is an arbitrary effective enumeration of the class $\mathbb{K}_{\text{lat}}$. Then the following holds:

- (a) The class $\mathfrak{K}_{\text{lat}}$ is $\text{InfEx} \equiv[\nu]$-learnable. Note that here the complexity of the learner depends only on the complexity of the sequence $(e_i)_{i \in \omega}$ from Corollary 4.1.
- (b) $\mathfrak{K}_{\text{lat}}$ is $\text{InfEx} \equiv[\nu \oplus \nu_{\text{lat}}]$-learnable by a computable learner.

Our second example deals with abelian $p$-groups. We treat abelian groups as structures in the signature $L_{\text{ag}}$:

''$t`, $0$u

For a number $i \in \omega$, define the group $A_i := \bigoplus_{j \in \omega} \mathbb{Z}(p^{i+1})$.

We set $\mathfrak{K}_{\text{ag}} := \{A_i : i \in \omega\}$, and we construct a Friedberg effective enumeration $\nu_{\text{ag}}$ of $\mathfrak{K}_{\text{ag}}$ in a straightforward way.

**Proposition 4.4.** The class $\mathfrak{K}_{\text{ag}}$ is $\text{InfEx} \equiv[\nu_{\text{ag}}]$-learnable by a computable learner.

**Proof Sketch.** For $i \in \omega$, one can define a first-order sentence $\psi_i$ which means the following: $\mathcal{S} \models \psi_i$ if and only if $\mathbb{Z}(p^{i+1})$ is a subgroup of $\mathcal{S}$, but $\mathbb{Z}(p^{i+2})$ is not a subgroup of $\mathcal{S}$. Clearly, $\psi_i$ is logically equivalent to a conjunction of an $\exists$-formula and an $\forall$-formula. After that, just follow the lines of the proof of Proposition 4.2.

**Corollary 4.5.** Suppose that $\nu$ is an arbitrary effective enumeration of the class $\mathbb{K}_{\text{ag}}$. Then the following holds:

- (a) The class $\mathfrak{K}_{\text{ag}}$ is $\text{InfEx} \equiv[\nu]$-learnable.
- (b) $\mathfrak{K}_{\text{ag}}$ is $\text{InfEx} \equiv[\nu \oplus \nu_{\text{ag}}]$-learnable by a computable learner.

**4.2. Boolean algebras.** Proposition 4.2 provides us with an example of an infinite learnable family of distributive lattices. Here we show that in the realm of Boolean algebras, the situation is dramatically different: informally speaking, one cannot learn even two different isomorphism types of infinite Boolean algebras.

Let $A$ and $B$ be structures in the same signature, and $n$ be a non-zero natural number. We write $A \leq_n B$ is every infinitary $\Pi_n$ sentence true in $A$ is also true in $B$. The relation $\leq_n$ is usually called the $n$-th back-and-forth relation.
For a Boolean algebra $C$, let $\#_{\text{atom}}(C)$ denote the cardinality of the set of atoms of $C$.

**Proposition 4.6.** Let $\mathcal{K}$ be some class of infinite Boolean algebras, and let $\nu$ be an effective enumeration of $\mathcal{K}$. Suppose that $\mathcal{C}$ is a subclass of $\mathcal{K}$ such that $\mathcal{C}$ contains at least two non-isomorphic members. Then the class $\mathcal{C}$ is not $\text{InfEx}_{\geq}[\nu]$-learnable.

**Proof.** Suppose that $A$ and $B$ are structures from the class $\mathcal{C}$ such that $A \equiv_{2} B$. Using the description of the back-and-forth relations on Boolean algebras [2, §15.3.4], one can prove the following fact: The condition $A \leq_{2} B$ holds if and only if $\#_{\text{atom}}(A) \geq \#_{\text{atom}}(B)$ (see, e.g., Lemma 11 in [1] for more details).

This fact implies that at least one of the following two conditions must be true:

\[
\Sigma_{2}^{\text{inf}} \cdot \text{Th}(A) \subseteq \Sigma_{2}^{\text{inf}} \cdot \text{Th}(B) \quad \text{or} \quad \Sigma_{2}^{\text{inf}} \cdot \text{Th}(B) \subseteq \Sigma_{2}^{\text{inf}} \cdot \text{Th}(A).
\]

Therefore, by Theorem 3.1, we deduce that the class $\mathcal{C}$ is not $\text{InfEx}_{\geq}[\nu]$-learnable. $\square$

**4.3. Linear orders.** First, we show that linear orders exhibit learning properties, which cannot be witnessed by Boolean algebras.

**Proposition 4.7.** Let $n \geq 2$ be a natural number. Then there is a class of computable infinite linear orders $\mathcal{C}$ with the following properties:

(a) $\mathcal{C}$ contains precisely $n$ isomorphism types.
(b) Suppose that $\mathcal{K}$ is a superclass of $\mathcal{C}$, and $\nu$ is an effective enumeration of $\mathcal{K}$. Then the class $\mathcal{C}$ is $\text{InfEx}_{\geq}[\nu]$-learnable by a computable learner.

**Proof Sketch.** We show how to build a family $\mathcal{C}$ containing precisely four non-isomorphic structures. We set

\[
\mathcal{C} = \{4 + \eta + 1; 3 + \eta + 2; 2 + \eta + 3; 1 + \eta + 4\}.
\]

We also define first-order $\exists \forall$-sentences $\psi_i$ as follows: for a linear order $\mathcal{L}$,

1. The sentence $\psi_1$ says that $\mathcal{L}$ has four consecutive elements in the beginning, i.e. there are elements $a_0 < a_1 < a_2 < a_3$ such that $a_0$ is the least element and $a_{i+1}$ is the immediate successor of $a_i$, for every $i \leq 3$.
2. $\psi_2$ says that $\mathcal{L}$ has three consecutive elements in the beginning and two consecutive elements in the end (i.e. there are $b_1 < b_0$ such that $b_0$ is the greatest and $b_1$ is the immediate predecessor of $b_0$).
3. $\psi_3$ says that $\mathcal{L}$ has two consecutive elements in the beginning and three consecutive elements in the end.
4. $\psi_4$ says that $\mathcal{L}$ has four consecutive elements in the end.
We apply Corollary 4.1 to the class $C$ and the sequence $\{\psi_i\}_{1 \leq i \leq 4}$. Thus, we obtain the desired learnability via a computable learner. Proposition 4.7 is proved.

On the other hand, the next result shows that one still cannot learn infinite families of linear orders.

**Theorem 4.8.** Let $K$ be some class of infinite linear orders, and let $\nu$ be an effective enumeration of $K$. Suppose that $C$ is a subclass of $K$ such that $C$ contains infinitely many pairwise non-isomorphic members. Then the class $C$ is not InfEx$[\nu]$-learnable.

**Proof.** The key ingredient of the proof is an analysis of $\Sigma^\inf_2$ formulas for linear orders $L$. First, we define the following auxiliary relations on $L$:

- A first-order $\forall$-formula $\text{First}(x)$ says that $x$ is the least element of $L$.
- An $\forall$-formula $\text{Last}(x)$ says that $x$ is the greatest element of $L$.
- An $\forall$-formula $\text{Succ}(x, y)$ says that $x$ and $y$ are consecutive elements, i.e. $(x < y) \land \neg \exists z(x < z < y)$.
- A $\Sigma^\inf_2$ formula $\text{Block}(x, y)$ says the following: either $x = y$, or there are only finitely many elements $z$ between $x$ and $y$ in $L$. The block of an element $x \in L$ is the set $\text{Block}_L[x] := \{y : L \models \text{Block}(x, y)\}$.

**Lemma 4.9 ([13]).**

1. In the class of countably infinite linear orders, every $\Pi^\inf_1$ formula in the signature $\{\leq\}$ is logically equivalent to a $\Sigma^\inf_1$ formula in the signature $\{\leq, \text{First}, \text{Last}, \text{Succ}\}$.

(2) Let $A$ and $B$ be countably infinite linear orders. Then we have:

$$A \leq_2 B \iff (A, \text{First}, \text{Last}, \text{Succ}) \leq_1 (B, \text{First}, \text{Last}, \text{Succ}).$$

**Proof Sketch.** The proof of (1) can be recovered from [13, p. 871], see also Lemma II.43 in [12]. Item (2) easily follows from (1) and the following fact: Every first-order $\exists$-formula in the signature $\{\leq, \text{First}, \text{Last}, \text{Succ}\}$ is logically equivalent to a first-order $\exists\forall$-formula in the signature $\{\leq\}$. □

Towards contradiction, we assume that there is a family of infinite linear orders $C = \{C_i : i \in \omega\}$ such that $C$ is InfEx$[\nu]$-learnable and the structures $C_i$ are pairwise non-isomorphic. Then by Theorem 3.1, there is a sequence of $\Sigma^\inf_2$ sentences $\{\psi_i\}_{i \in \omega}$ such that $C_i \models \psi_j \iff i = j$.

We apply Lemma 4.9(1), and for every $i$, we obtain a $\Sigma^\inf_1$ sentence $\xi_i$ in the signature $\{\leq, \text{First}, \text{Last}, \text{Succ}\}$, which is equivalent to $\psi_i$. W.l.o.g., one can choose $\xi_i$ as a finitary $\exists$-sentence. Thus, the intuition behind $\xi_i$ can be explained as follows. The sentence $\xi_i$ describes a finite substructure $F_i \subseteq (C_i, \text{First}, \text{Last}, \text{Succ})$ such that $F_i$ cannot be isomorphically embedded into $C_j$, for $j \neq i$. 
Clearly, at least one of the following four cases is satisfied by infinitely many $C_i$:

1. $C_i$ has neither least nor greatest elements;
2. $C_i$ has the least element, but there is no greatest one;
3. $C_i$ has the greatest element, but there is no least;
4. $C_i$ has both.

Thus, w.l.o.g., one may assume that every $C_i$ has both least and greatest elements. All other cases can be treated in a way similar to the exposition below.

We give an excerpt from the description \[13, p. 872\] of the relation $\leq_2$ for linear orders.

Let $A$ be a countably infinite linear order. We define:

- Let $t_0(A) = n$ if $A = n + A_1$, where $n \in \omega$ and the order $A_1$ has no least element. Set $t_0(A) = \infty$ if $A = \omega + A_1$, where $A_1$ has no least element.
- Define $t_2(A) = m$ if $A = A_2 + m$, where $m \in \omega$ and $A_2$ has no greatest element. Let $t_2(A) = \infty$ if $A = A_2 + \omega^*$, where $A_2$ has no greatest element.

As per usual, we assume that $\infty$ is greater than every natural number.

We write $A \equiv_2 B$ if $A \leq_2 B$ and $B \leq_2 A$.

**Lemma 4.10 (13).** Let $A$ and $B$ be countably infinite linear orders.

1. Suppose that $\max(t_0(A), t_2(A)) = \infty$. Then, independently of $B$, we have

   $$A \leq_2 B \iff t_0(A) \geq t_0(B) \text{ and } t_2(A) \geq t_2(B).$$

2. Suppose that $A = n_0 + A_1 + n_2$ and $B = m_0 + B_1 + m_2$, where $n_0, n_2, m_0, m_2 \in \omega$, and both $A_1$ and $B_1$ have no endpoints. Then

   $$A \leq_2 B \iff (n_0 \geq m_0) \text{ and } (A_1 \leq_2 B_1) \text{ and } (n_2 \geq m_2).$$

3. Suppose that both $A$ and $B$ have no endpoints. Then:

   3.1 If for every non-zero $n \in \omega$, $A$ has a tuple of $n$ consecutive elements, then $A \leq_2 B$.

   3.2 Suppose that $m$ is a non-zero natural number, and both $A$ and $B$ do not have tuples of $m + 1$ consecutive elements. If $A$ has infinitely many tuples of $m$ consecutive elements, then $A \leq_2 B$.

Lemma 4.10(1) implies the following: if $t_0(A) = t_0(B) = \infty$, then we always have either $A \leq_2 B$ or $B \leq_2 A$. Hence, we deduce that there is at most one structure $C_i$ with $t_0(C_i) = \infty$.

A similar argument shows that there is at most one $C_i$ with $t_2(C_i) = \infty$. Therefore, w.l.o.g., one can assume that for every $i \in \omega$, both values $t_0(C_i)$ and $t_2(C_i)$ are finite. Let

$$C_i = m_i + D_i + n_i,$$
where \( m_i, n_i \in \omega \), and the order \( D_i \) has no endpoints. For \( i \in \omega \), we define
\[
g_i := \sup\{\text{card}(\text{Block}_{D_i}[x]) : x \in D_i\}.
\]

**Claim 4.11.** There are only finitely many \( i \) with \( q_i = \infty \).

**Proof.** For simplicity of exposition, towards contradiction, assume that every \( q_i \) is infinite. Note that Lemma 4.10(3.1) shows that \( D_i \equiv D_j \) for all \( i \) and \( j \).

Since for every \( j \neq 0 \), we have \( C_j \nless C_0 \), by Lemma 4.10(2), we obtain that \( C_j \) satisfies at least one of the following two conditions: \( m_j < m_0 \) or \( n_j < n_0 \). W.l.o.g., we assume that there are infinitely many \( j \) with \( m_j < m_0 \). Then there is a number \( m^* < m_0 \) and an infinite sequence \( j[0] < j[1] < j[2] < \ldots \) such that \( m_j[k] = m^* \) for all \( k \).

Recall that \( C_j[k] \nless C_j[0] \) for all \( k \neq 0 \). By Lemma 4.10(2), we have \( n_j[k] < n_j[0] \) for every non-zero \( k \). Hence, there is a number \( n^* < n_j[0] \) such that \( n_j[k] = n^* \) for infinitely many \( k \). Clearly, if \( k \neq k' \) are such numbers, then \( C_j[k] \equiv C_j[k'] \), which gives a contradiction. \( \square \)

By Claim 4.11 one can assume that \( q_i < \infty \) for every \( i \).

**Claim 4.12.** There is a number \( r \in \omega \) such that \( q_i \leq r \) for every \( i \).

**Proof.** Again, for simplicity of exposition, assume that \( q_0 < q_1 < q_2 < \ldots \). Recall that \( C_j \models \xi_0 \) for all \( j \neq 0 \). Suppose that the finite structure \( F_0 \) associated with the \( \exists \)-sentence \( \xi_0 \) contains precisely \( t_0 \) elements.

Choose \( j^* \) such that \( q_{j^*} > 2t_0 \). Clearly, for every \( j \geq j^* \), the order \( D_j \) contains at least one block of size at least \( 2t_0 \). Thus, \( F_0 \) cannot be embedded into \( C_j \) only because of one of the following two obstacles:

- \( m_j < m_0 \), i.e. the size of the first (under \( \leq_{C_j} \)) block in \( C_j \) is too small for an appropriate embedding; or
- \( n_j < n_0 \), i.e. the size of the last block in \( C_j \) is too small.

The relation \( \text{Succ}^{C_j} \) won’t give us any problems, since one can embed all the \( F_0 \)-blocks (except the first one and the last one) inside a \( D_j \)-block of size \( \geq 2t_0 \).

As in Claim 4.11 we can assume that there is a number \( m^* < m_0 \) such that \( m_j = m^* \) for infinitely many \( j \geq j^* \). Form an increasing sequence \( j[0] < j[1] < j[2] < \ldots \) of these \( j \). Recall that \( q_{j[l]} < q_{j[l+1]} \) for all \( l \in \omega \). Re-iterating the argument above, we obtain that there is a number \( n^* < n_j[0] \) such that there are infinitely many \( l \) with \( n_{j[l]} = n^* \). Choose a sequence \( l[0] < l[1] < l[2] < \ldots \) of these \( l \). Suppose that the structure \( F_{j[l][0]} \) contains precisely \( t_1 \) elements.

Find the least \( l^* = l[s^*] \) with \( q_{j[l^*]} \geq 2t_1 \). Recall that we have \( m_{j[l^*]} = m_{j[l][0]} = m^* \) and \( n_{j[l^*]} = n_{j[l][0]} = n^* \). Thus, as before, it is not hard to show that the structure \( F_{j[l][0]} \) can be embedded into \( C_{j[l^*]} \). This shows that \( C_{j[l^*]} \models \xi_{j[l][0]} \), which gives a contradiction. \( \square \)
Moreover, we will assume that $q_i = r$ for all $i \in \omega$: indeed,

- If there are only finitely many $i$ with $q_i = r$, then we just delete the corresponding structures $D_i$. After that the value $r$ goes down.
- If there are already infinitely many $i$ with $q_i = r$, then we delete all $D_j$ with $q_j < r$.

**Claim 4.13.** There are only finitely many $i$ such that the order $D_i$ has infinitely many blocks of size $r$.

**Proof.** Again, for simplicity, assume that every $D_i$ has infinitely many blocks of size $r$. Since $q_i = r$ for all $i$, Lemma 4.10(3.2) implies that $D_i \equiv_2 D_j$ for all $i$ and $j$.

As in Claim 4.12, $F_0$ is not embeddable into $C_j$, $j \neq 0$, and this is witnessed by one of the following: either $m_j < m_0$ or $n_j < n_0$. We recover a number $m^* < m_0$ and a sequence $j[0] < j[1] < j[2] < \ldots$ such that $m_{j[l]} = m^*$ for all $l$.

The finite structure $F_{j[0]}$ is not embeddable into $C_{j[l]}$, $l \neq 0$. This implies that $n_{j[l]} < n_{j[0]}$ for non-zero $l$. Again, there is a number $n^* < n_{j[0]}$ and a sequence $l[0] < l[1] < l[2] < \ldots$ such that $n_{j[l[s]]} = n^*$ for all $s$. This shows that $C_{j[1]} \models \xi_{j[0]}$, and this yields a contradiction. \qed

Claim 4.13 implies that one may assume the following: each $D_i$ has only finitely many blocks of size $r = q_i$.

The rest of the proof is only sketched, since all the key ideas are already present. Let $\#(r; i)$ denote the number of blocks of size $r$ inside $D_i$.

**Claim 4.14.** There is a number $N$ such that $\#(r; i) \leq N$ for all $i$.

**Proof.** Assume that $\#(r; i) < \#(r; i + 1)$ for all $i$. As before, the finite structure $F_0$ cannot be embedded into $C_j$, where $j$ is large enough, and this can be witnessed only by one of the following conditions: $m_j < m_0$ or $n_j < n_0$ for such $j$. Hence, we assume that there is a sequence $j[0] < j[1] < j[2] < \ldots$ with $m_{j[l]} = m^* < m_0$ for all $l$. By considering possible embeddings of the finite structure $F_{j[0]}$, we recover a sequence $l[0] < l[1] < l[2] < \ldots$ with $n_{j[l[s]]} = n^* < n_{j[0]}$ for all $l$. Clearly, $F_{j[l[0]]}$ can be embedded into any $C_{j[l[s]]}$, where $s$ is large enough, and this produces a contradiction. \qed

By Claim 4.14, one can assume that $\#(r; i) = N < \infty$ for all $i$. For simplicity, consider $N = 2$. Then every $D_i$ can be presented in the following form:

$$D_i = D_{i,0} + r + D_{i,1} + r + D_{i,2},$$

- every $D_{i,0}$ does not have endpoints, and
- every block inside $D_{i,j}$ has size at most $r - 1$. 
After that, one needs to write a cumbersome proof by recursion in \( r \). The arrangement of this recursion can be recovered from the ideas from [13, p. 872].

In our case, the first stage of recursion will roughly consist of the following claims:

(a) We say that a block of size \((r - 1)\) is \textit{large}. Then one can prove that there are only finitely many \( i \) such that every \( D_{i,j} \) contains infinitely many large blocks.

(b) If there are infinitely many \( i \) such that, say, both \( D_{i,0} \) and \( D_{i,1} \) contain infinitely many large blocks, then one can assume that there is a number \( N_1 \) such that every \( D_{i,2} \) has at most \( N_1 \) large blocks. In this case, the next stage of recursion will play essentially only with \( D_{i,2} \).

(c) Assume that there are infinitely many \( i \) such that \( D_{i,0} \) has infinitely many large blocks, but every \( D_{i,1} \) and \( D_{i,2} \) has only finitely many large blocks. Then there are three main variants:

(c.1) There is a number \( N_2 \) and a sequence \( i_0 < i_1 < i_2 < \ldots \) such that for every \( k \), \( D_{i_k,1} \) has precisely \( N_2 \) large blocks and \( D_{i_k,2} \) contains, say, at least \( k \) large blocks. Then one needs to invoke recursion for \( D_{i,1} \).

(c.2) A case similar to the previous one, but here we require that every \( D_{i_k,1} \) has at least \( k \) large blocks. Then one can obtain a contradiction.

(c.3) There is a number \( N_3 \) such that every \( D_{i,1} \) or \( D_{i,2} \) has at most \( N_3 \) large blocks. Then proceed to the next recursion stage by considering both \( D_{i,1} \) and \( D_{i,2} \) simultaneously.

(d) Assume that each \( D_{i,j} \) has only finitely many large blocks. The main cases are as follows:

(d.1) There is a number \( N_4 \) and a sequence \( i_0 < i_1 < i_2 < \ldots \) such that for every \( k \), \( D_{i_k,0} \) contains precisely \( N_4 \) large blocks and each of \( D_{i_k,1} \) and \( D_{i_k,2} \) has at least \( k \) large blocks. Then one calls recursion for \( D_{i,0} \).

(d.2) A case similar to the previous one, but now we require that \( D_{i_k,1} \) always keeps precisely \( N_5 \) large blocks. Then the next recursion stage will work with \( D_{i,0} \) and \( D_{i,1} \) simultaneously.

(d.3) Every block \( D_{i_k,j} \) contains at least \( k \) large blocks. This leads to a contradiction.

(d.4) There is a number \( N_6 \) such that each \( D_{i,j} \) contains at most \( N_6 \) large blocks. Then we go to the next stage of recursion, and we have to consider all \( D_{i,j} \) simultaneously.

When the outlined recursion procedure finishes, we will get a contradiction in all considered cases. This implies that the class \( \mathcal{C} \) cannot be \( \text{InfEx}_{\geq [\nu]} \)-learnable. Theorem 4.8 is proved. \( \square \)
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