A novel nonequilibrium state of matter: a $d = 4 - \epsilon$ expansion study of Malthusian flocks

Leiming Chen (陈雷鸣)
School of Physical science and Technology, China University of Mining and Technology, Xuzhou Jiangsu, 221116, P. R. China

Chiu Fan Lee‡
Department of Bioengineering, Imperial College London, South Kensington Campus, London SW7 2AZ, U.K.

John Tone†
Department of Physics and Institute for Fundamental Science, University of Oregon, Eugene, OR

We show that “Malthusian flocks” – i.e., coherently moving collections of self-propelled entities (such as living creatures) which are being “born” and “dying” during their motion – belong to a new universality class in spatial dimensions $d > 2$. We calculate the universal exponents and scaling laws of this new universality class to $O(\epsilon)$ in a $d = 4 - \epsilon$ expansion, and find these are different from the “canonical” exponents previously conjectured to hold for “immortal” flocks (i.e., those without birth and death) and shown to hold for incompressible flocks with spatial dimensions in the range of $2 < d \leq 4$. We also obtain a universal amplitude ratio relating the damping of transverse and longitudinal velocity and density fluctuations in these systems. Furthermore, we find a universal separatrix in real ($r$) space between two regions in which the equal time density correlation $\langle \delta \rho(r,t) \delta \rho(0,t) \rangle$ has opposite signs. Our expansion should be quite accurate in $d = 3$, allowing precise quantitative comparisons between our theory, simulations, and experiments.

I. INTRODUCTION

“Active matter”, loosely defined as systems whose constituents have internal energy sources which drive motion, has been receiving intense attention in the physics community [1-4]. While one obvious motivation for this interest is its direct relevance to non-equilibrium physics and biophysics, active matter is also interesting because it exhibits a number of unusual phenomena. Among these is its ability to develop long-ranged orientational order in spatial dimension $d = 2$ [5-8], and the “anomalous hydrodynamics” exhibited by many of its ordered phases [6, 9, 10] even in spatial dimensions $d > 2$. By “anomalous hydrodynamics” we mean that the long-wavelength, long-time behavior of these systems can not be accurately described by a linear theory; instead, non-linear interactions between fluctuations must be taken into account, even to get the correct scaling laws. Indeed, it is the anomalous hydrodynamics in $d = 2$ that makes the existence of long-ranged order possible [6, 9, 10].

Of course, in addition to making active matter interesting, these intrinsically non-linear phenomena also makes it extremely difficult to treat analytically. How non-linear active matter is depends primarily on the symmetry of the state it is in, or, to borrow the language of equilibrium condensed matter physics, what “phase” it is in. The most non-linear phase found so far is what is known as the “polar ordered fluid” phase, which we will hereafter sometimes refer to as a “flock”. This is a phase of active (i.e., self-propelled) particles in which the only order is the alignment of the particles’ directions of motion, which breaks rotation invariance. Rotation invariance is “broken” because we consider systems whose underlying dynamics is rotation invariant. This aligning of the particles’ motion is the “polar order” of the phase’s name; the absence of other types of order (in particular, translational order) is the reason we describe this phase as “fluid”.

As always, the hydrodynamic (i.e., long length and time scale) behavior of polar ordered active fluids is determined by the symmetries and conservation laws of the system. Here, symmetries include not only the symmetries of the underlying microscopic dynamics, but the symmetries of the state as well. In particular, this means it depends on which symmetries are broken in the ordered state. Again, in polar ordered active fluids, the broken symmetry is rotational invariance.

Much of the past work [6, 9, 10] on polar ordered active fluids has focused on systems without momentum conservation, as is appropriate for active particles moving over a frictional substrate which can act as a momentum sink, but with number conservation. Hereafter we call these systems “immortal flocks”. For such systems, the density local number density $\rho$ of “flockers” (i.e., self-propelled particles) is a hydrodynamic variable. This considerably complicates the hydrodynamic theory; in particular, it gives rise to six additional relevant non-linearities [11], rendering the problem effectively intractable. All we know with any certainty about these systems is that they exhibit anomalous hydrodynamics in all spatial dimensions $d \leq 4$, and that this anomaly stabilizes long-ranged
orientational order (or, equivalently, makes it possible for an arbitrarily large flock to have a non-zero average velocity) in \( d = 2 \). A plausible but unproven conjecture \[11\] makes it possible to obtain exact scaling exponents characterizing the long-distance, long time scaling behavior for this system in \( d = 2 \). In other dimensions, in particular \( d = 3 \), little beyond the existence of anomalous hydrodynamics can be said.

Interestingly, one system about which more can be said is incompressible flocks \[12, 13\]: i.e., polar ordered active fluids in which the density is fixed, either by an infinitely stiff equation of state, or by long-ranged forces. For these systems, it is possible to obtain exact exponents for all spatial dimensions; as for number conserving systems with density fluctuations, these prove to be anomalous for spatial dimensions \( d \) in the range \( 2 \leq d \leq 4 \).

Specifically, there are three universal exponents characterizing the hydrodynamic behavior of these systems. One is the “dynamical exponent” \( z \), which gives the scaling of hydrodynamic time scales \( t(L_\perp) \) with length scale \( L_\perp \) perpendicular to the mean direction of flock motion (i.e., the direction of the average velocity \( \langle v \rangle \)); that is, \( t(L_\perp) \propto L_\perp^z \). Likewise, the scaling of characteristic hydrodynamic length scales \( L_\parallel \) along the direction of flock motion scale with those \( L_\perp \) perpendicular to that direction is given by an “anisotropy exponent” \( \zeta \) defined via \( L_\parallel(L_\perp) \propto L_\perp^\zeta \). Finally, fluctuations \( \chi_\perp \) of the local velocity perpendicular to its mean direction define a “roughness exponent” \( \chi \) via \( \chi_\perp \propto L_\perp^\chi/5 \). For incompressible flocks, these exponents are given by

\[
z = \frac{2(d+1)}{5}, \quad \zeta = \frac{d+1}{5}, \quad \chi = \frac{3-2d}{5}, \quad (I.1)
\]

for spatial dimensions satisfying \( 2 \leq d \leq 4 \), by \( z = 2, \zeta = 1, \text{ and } \chi = \frac{2-d}{5} \) for \( d > 4 \) (the latter range of \( d \) is obviously only of interest for simulations). For \( d = 2 \), the static properties of the ordered phase can be mapped onto the \((1+1)\)-dimensional Kardar-Parisi-Zhang model \[13\] and the exact scaling exponents are \( \zeta = 2/3 \) and \( \chi = -1/3 \), while the value of \( z \) remains unknown. We will hereafter refer to the exponents \( I.1 \) as the “canonical” exponents.

The exponents \( I.1 \) were originally asserted \[6, 9\] to hold for compressible, number conserving flocks, but this was later shown to be incorrect \[11\], due to the presence of the aforementioned extra non-linearities associated with the conserved density. If one conjectures that those extra non-linearities, which are relevant near the unstable linear fixed point near \( d = 4 \), are in fact irrelevant near the non-linear fixed point that controls the ordered phase in \( d = 2 \), then one obtains the “canonical” values

\[
z = \frac{6}{5}, \quad \zeta = \frac{3}{5}, \quad \chi = -\frac{1}{5}, \quad (I.2)
\]

In this paper, we will consider so-called “Malthusian flocks” \[14\]; that is, polar ordered active fluids with no conservation laws at all; in particular, particle number is not conserved. Such systems are readily experimentally realizable in experiments on a, e.g., growing bacteria colonies and cell tissues, and “treadmilling” molecular motor propelled biological macromolecules in a variety of intracellular structures, including the cytoskeleton, and mitotic spindles, in which molecules are being created and destroyed as they move.

In addition to describing biological and other active systems, our model for Malthusian flocks may also be viewed as a generic non-equilibrium \( d \)-dimensional \( d \)-component spin model in which the spin vector space \( s(r) \) and the coordinate space \( r \) are treated on an equal footing, and couplings between the two are allowed. In particular, terms like \( s \cdot \nabla s \) and \( (\nabla \cdot s)s \), will be present in the EOM that describes such a generic non-equilibrium system. As a result, the fluctuations in the system can propagate spatially in a spin-direction-dependent manner, but the spins themselves are not moving. Therefore, there are no density fluctuations and the only hydrodynamic variable is the spin field, the equation of motion (EOM) for which is exactly the same as the one we derive here for a Malthusian flock, with spin playing the role of the velocity field. (Of course, non-equilibrium spin systems in which the spins “live” in real space in the sense described here will not map onto Malthusian flocks if those spins live on a lattice, due to the breaking of rotation invariance by the lattice itself. There are, however, ways of eliminating these “crystal field” effects \[13\].

For Malthusian flocks, exact exponents can be obtained in \( d = 2 \) \[14\], and they again take on the “canonical” values \( I.2 \) implied by \( I.1 \) in \( d = 2 \).

Overall, the theoretical situation is therefore still quite unsatisfactory: we only have the scaling laws for flocks if they either are incompressible (which requires either infinitely strong, or infinitely ranged, interactions), or in \( d = 2 \). And in the cases in which we do know the exponents, their values are either the canonical ones \( I.1 \) \[12, 13\], or those from the \((1+1)\)-dimensional KPZ model \[13\].

It would clearly be desirable to find the scaling laws and exponents of some compressible three dimensional flocks, and to see if, as for incompressible flocks, they are also given by the canonical values \( I.1 \).

In this paper, we do so for Malthusian flocks in \( d > 2 \). Specifically, we study these systems in a \( d = 4 - \epsilon \) expansion. We find that they belong to a new universality class which does not have the canonical exponents \( I.1 \). Instead, we find, to leading order in \( \epsilon \),

\[
z = 2 - \frac{6\epsilon}{11} + \mathcal{O}(\epsilon^2), \quad (I.3)
\]
\[
\zeta = 1 - \frac{3\epsilon}{11} + \mathcal{O}(\epsilon^2), \quad (I.4)
\]
\[
\chi = -1 + \frac{6\epsilon}{11} + \mathcal{O}(\epsilon^2), \quad (I.5)
\]

which the interested reader can easily check do not agree with the “canonical” values \( I.1 \) near \( d = 4 \).
(i.e., for small \( \epsilon \)). However, it should also be noted that in \( d = 3 \) these values are not very different from the canonical values \([1.1]\); setting \( \epsilon = 1 \) in \([1.5]\) gives \( z = 16/11 = 1.45454 \), \( \zeta = 8/11 = 0.7272727 \), and \( \chi = -5/11 = -0.45454 \), which are fairly close to the “canonical” values \([1.1]\), which give, in \( d = 3 \), \( z = 8/5 = 1.6, \zeta = 4/5 = 0.8 \) and \( \chi = -3/5 = -0.6 \).

We have also estimated the exponents in \( d = 3 \) by applying the one-loop (i.e., lowest order in perturbation theory) perturbative renormalization group recursion relations in arbitrary spatial dimensions. This approach, although strictly speaking an uncontrolled approximation, can easily be shown to give exponents for the \( \mathcal{O}(n) \) model critical point in \( d = 3 \) that are at least as accurate as the first order in \( d = 4 - \epsilon \) expansion with \( \epsilon \) set to 1.

And there is reason to believe that this approach may be even more accurate for our problem: this “one-loop truncated” approach not only recovers the exact linear expansion results \([1.5]\), but it also recovers the exact results \([1.2]\) in \( d = 2 \). Thus, while uncontrolled, this approach should provide a very effective interpolation formula for \( d \) between 2 and 4, that should be quite accurate (indeed, probably more accurate than the \( \epsilon \) expansion) in \( d = 3 \).

Using this approach, we find

\[
\begin{align*}
  z &= 2 - \frac{2(4-d)(4d-7)}{14d-23}, \\
  \zeta &= 1 - \frac{(4-d)(4d-7)}{14d-23}, \\
  \chi &= -1 + \frac{2(4-d)(4d-7)}{14d-23},
\end{align*}
\]

which indeed recover our \( \epsilon \) expansion results near \( d = 4 \), and the exact results \([1.2]\) in \( d = 2 \), as the readers can verify for themselves.

In the physically interesting case \( d = 3 \), these give

\[
\begin{align*}
  z &= \frac{28}{19} \approx 1.47, \\
  \zeta &= \frac{14}{19} \approx 0.74, \\
  \chi &= -\frac{9}{19} \approx -0.47.
\end{align*}
\]

These are our best numerical estimates of the values of these exponents in \( d = 3 \). We suspect that they are accurate to \( \pm 1\% \), an error estimate which we will motivate in section [IV E] below. That is, the digits shown after the approximate equalities above are probably all correct.

These exponents govern the scaling behavior of the experimentally measurable velocity correlation function:

\[
C_u(r,t) \equiv \langle \mathbf{u}_\perp(r,t) \cdot \mathbf{u}_\perp(0,0) \rangle = r_\perp^{2\chi} F_u \left( \frac{ |x - \gamma t/\xi_x| }{r_\perp/\xi_\perp} \xi_\perp, \frac{ (t/\tau) }{r_\perp/\xi_\perp} \right)
\]

\[
\propto \begin{cases} 
  r_\perp^{2\chi}, & (r_\perp/\xi_\perp)^\zeta \gg |x - \gamma t/\xi_x|, (r_\perp/\xi_\perp)^\zeta \gg (|t|/\tau) \\
  |x - \gamma t|^{2\chi}, & |x - \gamma t|/\xi_x \gg (r_\perp/\xi_\perp)^\zeta, |x - \gamma t|/\xi_x \gg (|t|/\tau)^\zeta \\
  |t|^{2\chi}, & (|t|/\tau) \gg (r_\perp/\xi_\perp)^\zeta, (|t|/\tau) \gg (|x - \gamma t|/\xi_x)^\zeta
\end{cases}
\]

where \( F_u \) is a universal scaling function (i.e., the same for all Malthusian flocks), \( \gamma \) is a non-universal (i.e., system dependent) speed, \( \xi_{\perp,x} \) are non-universal lengths, and \( \tau \) is a non-universal time. We also note that fluctuations of the velocity field \( \mathbf{u}_\perp \) are always positively correlated, i.e., \( C_u \) is always positive.

Density correlations also obey a scaling law involving the same universal exponents \( z, \zeta, \) and \( \chi \), and non-universal lengths \( \xi_{\perp,x} \) and time \( \tau \):
\[ C_\rho(r, t) \equiv \langle \delta \rho(r, t) \delta \rho(0, 0) \rangle = r_\perp^{2(\chi-1)} F_\rho \left( \frac{(|x - \gamma t|/\xi_x)}{(r_\perp/\xi_\perp)^\chi}, \frac{(t/\tau)}{(r_\perp/\xi_\perp)^\gamma} \right) \]

\[
\propto \begin{cases} 
\frac{r_\perp^{2(\chi-1)}}{}, & (r_\perp/\xi_\perp)^\chi \gg \frac{|x - \gamma t|}{\xi_x}, \quad (r_\perp/\xi_\perp)^\gamma \gg \frac{(t/\tau)}{} \\
|t|^{-2(\chi-1)/\gamma} &, \quad (t/\tau) \gg (|x - \gamma t|/\xi_x)^{2/\gamma} \\
|t|^{-2(\chi-1)/\gamma} &, \quad (t/\tau) \gg (|x - \gamma t|/\xi_x)^{2/\gamma} 
\end{cases}
\]

(I.13)

also obtain a universal amplitude ratio. Section V summarizes our results and discusses their implications for experiments and simulations. In Appendix A, we present a simple DRG analysis that confirms the existence of the fixed point found in our more general treatment. Appendix B presents the lengthy and arduous details of the full DRG calculation, which shows that the fixed point found by the simplified analysis is the only stable fixed point for this problem, at least to one-loop order. We have also provided a list of useful formulae in Appendix C.

II. DERIVATION OF THE EQUATION OF MOTION

We begin by deriving the equation of motion. This derivation is virtually identical to that done in reference [14]; we review it here simply to make this paper self-contained. Our starting equation of motion for the velocity is identical to that of a flock with number conservation [6, 9, 11]:

\[ \partial_t v + \lambda_1 (v \cdot \nabla) v + \lambda_2 (\nabla \cdot v) v + \lambda_3 \nabla (|v|^2) = U(\rho, |v|) v - \nabla P_1 - v (v \cdot \nabla P_2(\rho, |v|)) + \mu_B \nabla (\nabla \cdot v) + \mu_T \nabla^2 v + \mu_A (v \cdot \nabla)^2 v + f. \]

(II.1)

In this equation, \( \lambda_i (i = 1 \rightarrow 3) \), \( U(\rho, |v|) \) and the “pressures” \( P_1, P_2 \) are, in general, functions of the flocker number density \( \rho \) and the magnitude \( |v| \) of the local velocity. We will expand all of them to the order necessary to include all terms that are “relevant” in the sense of changing the long-distance behavior of the flock.

This equation is derived purely from symmetry arguments [9, 11, 16]. However, each term in it has a simple physical interpretation, which we now give.

The \( U(\rho, |v|) \) term is responsible for spontaneous flock motion. Our analysis will apply to an extremely large class of \( U \)'s; specifically, to all of those that satisfy \( U(|v| < v_0) > 0 \), and \( U(|v| > v_0) < 0 \) in the ordered phase. This last condition insures that in the absence of fluctuations, the flock will move at a speed \( v_0 \).

The diffusion constants \( \mu_{B,T,A} \) reflect the tendency of flockers to follow their neighbors. The \( f \) term is a random Gaussian white noise, reflecting errors made by the flockers, with correlations:

\[ \langle f_i(r, t) f_j(r', t') \rangle = 2D \delta_{ij} \delta^d(r - r') \delta(t - t') \]

(II.2)
where we’ve defined \( \mu'_{B} \equiv \mu_{B} + \Delta \mu_{B} \).

In the ordered state (i.e., in which \( \langle \mathbf{v}(\mathbf{r}, t) \rangle = \mathbf{v}_{0}\hat{x} \), where we’ve chosen the spontaneously picked direction of mean flock motion as our \( x \)-axis), we can expand the \( \mathbf{v} \) EOM for small departures \( \mathbf{u}(\mathbf{r}, t) \equiv u_{x}\hat{x} + \mathbf{u}_{\perp}(\mathbf{r}, t) \) of \( \mathbf{v}(\mathbf{r}, t) \) from uniform motion with speed \( \mathbf{v}_{0} \):

\[
\mathbf{v}(\mathbf{r}, t) = (v_{0} + u_{x})\hat{x} + \mathbf{u}_{\perp}(\mathbf{r}, t) ,
\]

where, henceforth \( x \) and \( \perp \) denote components along and perpendicular to the mean velocity, respectively.

In this hydrodynamic approach, we’re interested only in fluctuations of \( \mathbf{u}(\mathbf{r}, t) \) that vary slowly in space and time. The component \( u_{x} \) of the fluctuation of the velocity \( \mathbf{v} \) along the direction of mean motion is \textit{not} such a fluctuation. Rather, like the density fluctuation \( \delta \rho \), it is a non-hydrodynamic or “fast” variable. It therefore can be eliminated from the equations of motion in much the same manner as we just eliminated the density fluctuations.

The details of this elimination are a bit tricky, and are discussed in detail in (II); here we will very briefly review the argument, as applied to our EOM (II.5).

To focus on fluctuations in the magnitude of the velocity (which are, strictly speaking, the fast variable here), we take the dot product of both sides of (II.5) with \( \mathbf{v} \) itself. This gives

\[
\partial_{t} \mathbf{v} + \lambda_{1}(\mathbf{v} \cdot \nabla)\mathbf{v} + \lambda_{2}(\nabla \cdot \mathbf{v})\mathbf{v} + \lambda_{3}\nabla(|\mathbf{v}|^{2}) = U(\rho, |\mathbf{v}|)\mathbf{v} - \mathbf{v}(\mathbf{v} \cdot \nabla P_{2}(\rho, |\mathbf{v}|)) + \mu'_{B}\nabla(\nabla \cdot \mathbf{v}) + \mu_{T}\nabla^{2}\mathbf{v} + \mu_{A}(\nabla \cdot \nabla)\mathbf{v} + \mathbf{f} ,
\]

where \( \mu'_{B} \equiv \mu_{B} + \Delta \mu_{B} \).

In the ordered state (i.e., in which \( \langle \mathbf{v}(\mathbf{r}, t) \rangle = \mathbf{v}_{0}\hat{x} \), we can expand the \( \mathbf{v} \) EOM for small departures \( \mathbf{u}(\mathbf{r}, t) \equiv u_{x}\hat{x} + \mathbf{u}_{\perp}(\mathbf{r}, t) \) of \( \mathbf{v}(\mathbf{r}, t) \) from uniform motion with speed \( \mathbf{v}_{0} \):

\[
\mathbf{v}(\mathbf{r}, t) = (v_{0} + u_{x})\hat{x} + \mathbf{u}_{\perp}(\mathbf{r}, t) ,
\]
\[
\frac{1}{2} \left( \partial_t |v|^2 + (\lambda_1 + 2\lambda_3)(v \cdot \nabla)|v|^2 + \lambda_2(\nabla \cdot v)|v|^2 \right) = U(|v|)|v|^2 - |v|^2 v \cdot \nabla P_2 + \mu_B v \cdot \nabla(\nabla \cdot v) + \mu_T v \cdot \nabla^2 v + \mu_A (v \cdot \nabla)^2 v + v \cdot f. \tag{II.7}
\]

where we have dropped the prime in \( \mathbf{r} \).

This equation will be the basis of our remaining theoretical analysis. Note that to obtain correlations in the original (unboosted) coordinate system, we need to take into account the boost \( \mathbf{R} \).

### III. LINEAR THEORY

#### A. Response functions

In this section we treat the linear approximation to the model \((\text{II.11})\). Keeping only the linear terms in \((\text{II.11})\), and writing the resultant EOM in Fourier space, we obtain

\[-i\omega \mathbf{u}_\perp(\mathbf{k}) = -\mu_1 k_z^2 \mathbf{u}_\perp(\mathbf{k}) - \mu_2 k_\perp \left( k_\perp \cdot \mathbf{u}_\perp(\mathbf{k}) \right) - \mu_3 k_\perp^2 \mathbf{u}_\perp(\mathbf{k}) + f_\perp(\mathbf{k}) \tag{III.1}\]

where \( \mathbf{k} \equiv (k, \omega) \), and

\[\mathbf{u}_\perp(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^d} \int dt d^d r \mathbf{u}_\perp(\mathbf{r}, t)e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})}. \tag{III.2}\]

The linear equation \((\text{III.1})\) can be easily solved by separating \( \mathbf{u}_\perp \) into its component along \( k_\perp \) (which we’ll hereafter call “longitudinal”) and its remaining \( d - 1 \) components perpendicular to \( k_\perp \) (which we’ll hereafter call “transverse”). (We remind the reader that \( \mathbf{u}_\perp \) has only \( d - 1 \) independent components, since it is by definition orthogonal to the mean direction of flock motion \( \mathbf{x} \).)

That is, we write:

\[\mathbf{u}_\perp(\mathbf{k}) = u_L(\mathbf{k}) \hat{k}_\perp + \mathbf{u}_T(\mathbf{k}), \tag{III.3}\]

with \( \mathbf{k}_\perp \cdot \mathbf{u}_T = 0 \) by definition. These components \( u_L \) and \( \mathbf{u}_T \) can be computed using

\[u_L(\mathbf{k}) = \mathbf{k}_\perp \cdot \mathbf{u}_\perp(\mathbf{k}) \tag{III.4}\]

and

\[u_T(\mathbf{k}) = P_{ij}^T(\mathbf{k}) u_j^L(\mathbf{k}), \tag{III.5}\]

where we’ve defined the “transverse projection operator”

\[P_{ij}^T(\mathbf{k}) \equiv \delta_{ij} - \frac{k_i^T k_j^L}{k_\perp^2}, \tag{III.6}\]

which projects any vector into the \((d - 2)\)-dimensional space orthogonal to both the direction of mean flock motion \( \mathbf{x} \) and \( \mathbf{k}_\perp \). We can decompose any vector in the space orthogonal to \( \mathbf{x} \), including, in particular, the random force \( f_\perp \), in exactly the same way.
We can now easily rewrite the EOM [III.1] for \( u_L \) as decoupled equations for \( u_L \) and \( u_T \). To obtain the former, we take the dot product of \( \mathbf{k}_\perp \) with (III.1); this gives a closed EOM for \( u_L \):

\[
-\omega u_L(\mathbf{k}) = -\mu_L k_\perp^2 u_L(\mathbf{k}) - \mu_x k_\perp^2 u_L(\mathbf{k}) + f_L(\mathbf{k}) , \quad (\text{III.7})
\]

where we have defined

\[
\mu_L \equiv \mu_1 + \mu_2 . \tag{III.8}
\]

Likewise, acting on both sides of (III.1) with the transverse projection operator (III.6) gives a closed EOM for \( u_T \):

\[
-\omega u_T(\mathbf{k}) = -\mu_T k_\perp^2 u_T(\mathbf{k}) - \mu_x k_\perp^2 u_T(\mathbf{k}) + f_T(\mathbf{k}) . \quad (\text{III.9})
\]

Before proceeding to solve these two simple linear equations for \( u_L \) and \( u_T \) in terms of the forces \( f_L \) and \( f_T \), it is informative to first determine the eigenfrequencies \( \omega(\mathbf{k}) \) of the normal modes of this system. These are clearly just

\[
\omega_L(\mathbf{k}) = -i \left( \mu_L k_\perp^2 + \mu_x k_\perp^2 \right) \quad (\text{II.10})
\]

for the longitudinal mode, and

\[
\omega_T(\mathbf{k}) = -i \left( \mu_T k_\perp^2 + \mu_x k_\perp^2 \right) \quad (\text{II.11})
\]

for the transverse mode. In order for the system to be stable, we must have the imaginary part \( I_{L,T}(\omega(\mathbf{k})) < 0 \) for both modes; this clearly requires that

\[
\mu_{L,1,2} > 0 . \quad (\text{III.12})
\]

Note that this condition (III.12) does not require \( \mu_2 > 0 \); using the definition (III.8) of \( \mu_L \) in (III.12) requires only that

\[
\mu_2 > -\mu_1 , \quad (\text{III.13})
\]

or, equivalently,

\[
\frac{\mu_2}{\mu_1} > -1 . \quad (\text{III.14})
\]

This last condition for stability was noted in the associated short paper [15].

Now we turn to the solutions of the EOMs (III.7) and (III.9). These can be immediately read off:

\[
u_L(\mathbf{k}) = G_L(\mathbf{k}) f_L(\mathbf{k}) , \quad (\text{III.15})
\]

\[
u_T(\mathbf{k}) = G_T(\mathbf{k}) f_T(\mathbf{k}) , \quad (\text{III.16})
\]

where we’ve defined the longitudinal and transverse “propagators”

\[
G_L(\mathbf{k}) = \frac{1}{-i\omega + \mu_L k_\perp^2 + \mu_x k_\perp^2} , \quad (\text{III.17})
\]

\[
G_T(\mathbf{k}) = \frac{1}{-i\omega + \mu_T k_\perp^2 + \mu_x k_\perp^2} . \quad (\text{III.18})
\]

These propagators will also have an important role to play in our DRG analysis later.

The solutions (III.15, III.16) for \( u_L \) and \( u_T \) can be summarized in a single equation using the relations (III.4) and (III.5) between \( u_L \) and its components \( u_L \) and \( f_T \), along with the analogous relations between \( f_L \) and \( f_T \); we obtain

\[
u_i^+(\mathbf{k}) = G_{ij}(\mathbf{k}) f_j^+(\mathbf{k}) , \quad (\text{III.19})
\]

where

\[
G_{ij}(\mathbf{k}) = L_{ij}^\perp(\mathbf{k}) G_L(\mathbf{k}) + P_{ij}^\perp(\mathbf{k}) G_T(\mathbf{k}) , \quad (\text{III.20})
\]

and we have defined the “longitudinal projection operator”

\[
L_{ij}^\perp(\mathbf{k}_\perp) \equiv \frac{k_i^+ k_j^+}{k_\perp^2} , \quad (\text{III.21})
\]

which projects any vector along \( \mathbf{k}_\perp \).

### B. Velocity correlation functions

Using (III.19), we obtain the autocorrelations:

\[
\left\langle u_i^+(\mathbf{k}) u_j^+(\mathbf{k}) \right\rangle = G_{im}(\mathbf{k}) G_{jn}(\mathbf{k}) \left\langle f_m^+(\mathbf{k}) f_n^+(\mathbf{k}) \right\rangle = 2D \delta_{mn} \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega') , \quad (\text{III.22})
\]

where in the second equality we have used the correlations of the noise in Fourier space:

\[
\left\langle f_m^+(\mathbf{k}) f_n^+(\mathbf{k}) \right\rangle = 2D \delta_{mn} \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega') , \quad (\text{III.23})
\]

and we’ve defined

\[
C_{ij}(\mathbf{k}) \equiv L_{ij}^\perp(\mathbf{k}) |G_L(\mathbf{k})|^2 + P_{ij}^\perp(\mathbf{k}) |G_T(\mathbf{k})|^2 . \quad (\text{III.24})
\]

In writing (III.22), we have also made liberal use of the property shared by both projection operators \( L_{ij}^\perp \) and \( P_{ij}^\perp \) that their squares are themselves.

Transforming the above correlation function back to spatio-temporal domain, we obtain the velocity correlation in real space and time. First let’s calculate the equal-time correlation function:

\[
\left\langle u_\perp(0, \mathbf{r}) \cdot u_\perp(0, 0) \right\rangle = \frac{1}{(2\pi)^{d+1}} \int \frac{d\omega d\omega'}{d^d k d^d k'} \left\langle u_\perp(\mathbf{k}) \cdot u_\perp(\mathbf{k}') \right\rangle e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{2D}{(2\pi)^{d+1}} \int \frac{d\omega d\omega'}{d^d k d^d k'} \left[ \frac{1}{\omega^2 + (\mu_L k_\perp^2 + \mu_x k_\perp^2)^2} + \frac{d - 2}{\omega^2 + (\mu_T k_\perp^2 + \mu_x k_\perp^2)^2} \right] = D[U_L(\mathbf{r}) + (d - 2)U_T(\mathbf{r})] . \quad (\text{III.25})
\]
where
\[ U_L(r) = \frac{1}{(2\pi)^d} \int \frac{d^d k}{\mu_L k_\perp^2 + \mu_x k_\parallel^2} e^{i k r} \quad \text{(III.26)} \]
\[ U_T(r) = \frac{1}{(2\pi)^d} \int \frac{d^d k}{\mu_1 k_\perp^2 + \mu_x k_\parallel^2} e^{i k r} \quad \text{(III.27)} \]
Clearly, \( U_{L,T}(r) \) satisfy the anisotropic Poisson equations:
\[ (\mu_L \nabla_\perp^2 + \mu_x \nabla_\parallel^2) U_L(r) = -\delta^d(r), \quad \text{(III.28)} \]
\[ (\mu_1 \nabla_\perp^2 + \mu_x \nabla_\parallel^2) U_T(r) = -\delta^d(r). \quad \text{(III.29)} \]
The solutions to the above equations are, for \( d > 2 \),
\[ U_L(r) = \left( \frac{\mu_L x^2 + r^2_\perp}{S_d(d-2)\sqrt{\mu_x}\mu_L} \right)^{(2-d)/2}, \quad \text{(III.30)} \]
\[ U_T(r) = \left( \frac{\mu_1 x^2 + r^2_\perp}{S_d(d-2)\sqrt{\mu_x}\mu_1} \right)^{(2-d)/2}, \quad \text{(III.31)} \]
where \( S_d \) is the surface area of a \( d \)-dimensional unit sphere. Inserting the above results into Eq. (III.25) we get
\[ \langle u_\perp(t, r) \cdot u_\perp(0, 0) \rangle \propto r^{-(d-2)}. \quad \text{(III.32)} \]
Now we calculate the temporal correlation. Setting the spatial distance to zero in the correlation function to get
\[ \langle u_\perp(t, 0) \cdot u_\perp(0, 0) \rangle = \frac{1}{(2\pi)^{d+1}} \int d\omega d\omega' d^d k d^d k' \langle u_\perp(k) \cdot u_\perp(k') \rangle e^{-i\omega t} \]
\[ = \frac{2D}{(2\pi)^{d+1}} \int d\omega d\omega' d^d k e^{-i\omega t} \left[ \frac{1}{\omega^2 + (\mu_L k_\perp^2 + \mu_x k_\parallel^2)^2} + \frac{d-2}{\omega^2 + (\mu_1 k_\perp^2 + \mu_x k_\parallel^2)^2} \right] \]
\[ = \frac{D}{(2\pi)^d} \int d^d k \left[ \frac{e^{-(\mu_L k_\perp^2 + \mu_x k_\parallel^2)|t|}}{\mu_L k_\perp^2 + \mu_x k_\parallel^2} + \frac{(d-2)e^{-(\mu_1 k_\perp^2 + \mu_x k_\parallel^2)|t|}}{\mu_1 k_\perp^2 + \mu_x k_\parallel^2} \right] \]
\[ \propto |t|^{-\frac{d-2}{2}}, \quad \text{(III.33)} \]
where in the penultimate equality we have made the change of vectorial variable, \( \mathbf{q} = |t|^{\frac{1}{2}} \mathbf{k} \), while in the ultimate proportionality we have used the fact that the integral over \( \mathbf{q} \) is a finite constant (i.e., independent of time \( t \)).

We can easily generalize these results to arbitrary spatio-temporal separations. We start with
\[ \langle u_\perp(t, r) \cdot u_\perp(0, 0) \rangle = \frac{1}{(2\pi)^{d+1}} \int d\omega d\omega' d^d k d^d k' \langle u_\perp(k) \cdot u_\perp(k') \rangle e^{i(kr_\perp - r_\parallel)} \]
\[ = \frac{2D}{(2\pi)^{d+1}} \int d\omega d\omega' d^d k e^{i(kr_\perp - r_\parallel)} \left[ \frac{1}{\omega^2 + (\mu_L k_\perp^2 + \mu_x k_\parallel^2)^2} + \frac{d-2}{\omega^2 + (\mu_1 k_\perp^2 + \mu_x k_\parallel^2)^2} \right]. \quad \text{(III.34)} \]
Changing the variables of integration from \( \mathbf{k} \) and \( \omega \) to \( \mathbf{Q} \) and \( \Upsilon \):
\[ k_\perp = Q_\perp / r_\perp, \quad k_\parallel = Q_\parallel / r_\perp, \quad \omega = \Upsilon / r_\perp^2, \quad \text{(III.35)} \]
we obtain
\[ \langle u_\perp(t, r) \cdot u_\perp(0, 0) \rangle = r_\perp^{-(d-2)} H_u \left( \frac{x}{r_\perp}, \frac{t}{r_\perp^2} \right) \]
\[ \propto \begin{cases} r^{-(d-2)} & \text{if } |t| \ll |r|^{\frac{1}{2}} \\ |t|^{-(d-2)} & \text{if } |t| \gg |r|^{\frac{1}{2}} \end{cases}, \quad \text{(III.36)} \]
where we’ve defined the scaling function
\[ H_u(a, b) = \frac{2D}{(2\pi)^{d+1}} \int d\Upsilon d^d Q e^{iQ_\perp \cdot \hat{x} + Q_\parallel \cdot \hat{r}} \times \]
\[ \left[ \frac{1}{\Upsilon^2 + (\mu_L Q_\perp^2 + \mu_x Q_\parallel^2)^2} + \frac{d-2}{\Upsilon^2 + (\mu_1 Q_\perp^2 + \mu_x Q_\parallel^2)^2} \right]. \quad \text{(III.37)} \]

C. Density correlations

Although it is not a “soft mode” of Malthusian flocks, since it is not conserved in these systems, the density \( \rho \) nonetheless exhibits long-ranged spatio-temporal correlations by virtue of being enslaved to the slow \( u_x \) field via (II.4). Using this relation in Fourier space, we obtain
\[ \langle \delta\rho(k) \delta\rho(k') \rangle = \frac{2D' k^2_\perp \delta(k + k') \delta(\omega + \omega')}{\omega^2 + (\mu_L k^2_\perp + \mu_x k^2_\parallel)^2}, \quad \text{(III.38)} \]
where we’ve defined
\[ D' \equiv D \left( \frac{\rho_0}{c'(\rho_0)} \right)^2. \quad \text{(III.39)} \]

The spatio-temporal correlations can be calculated by Fourier transforming (III.38) back to real space and time. In particular, the equal time correlation function is
\[ \langle \delta\rho(0, r) \delta\rho(0, 0) \rangle = \frac{1}{(2\pi)^{d+1}} \int d\omega d\omega' d^d k d^d k' \langle \delta\rho(k) \delta\rho(k') \rangle e^{i k \cdot r} \]
\[ = \frac{1}{(2\pi)^d} \int d^d k \frac{D' k^2_\perp}{\mu_L k^2_\perp + \mu_x k^2_\parallel} e^{i k \cdot r}, \quad \text{(III.40)} \]
where in the last equality we have used (III.38). To calculate this correlation function we write
\[
\langle \delta \rho(t, r) \delta \rho(t, 0) \rangle = -D' \nabla_x^2 U_L(r), \tag{III.41}
\]
where \(U_L(r)\) is given in (III.30). Inserting (III.30) into the above expression gives
\[
\langle \delta \rho(t, r) \delta \rho(t, 0) \rangle = \left( \frac{\rho_0}{d-k_0^2/\rho_0} \right)^{2/d} \frac{D}{S_d} \frac{\mu_{d-1}}{\mu_L} \left[ \frac{\mu_L (d-1) x^2 - \mu_x r_+^2}{(\mu_L x^2 + \mu_x r_+^2)^{2+2d/2}} \right] \propto r^{-d}.
\tag{III.42}
\]

In particular, for \(d = 3\), we have
\[
\langle \delta \rho(t, r) \delta \rho(t, 0) \rangle \sim r^{-3}. \tag{III.43}
\]

It is clear from (III.42) that the equal time correlation function of the density fluctuation \(\delta \rho\) vanishes on the surface
\[
x = \pm \left( \frac{\mu_x}{\sqrt{\mu_L (d-1)}} \right) r_+ \tag{III.44}
\]
which, in \(d = 3\), is a cone. For \(|x| > \sqrt{\mu_x/\mu_L (d-1)} r_+\), \(\langle \delta \rho(t, r) \delta \rho(t, 0) \rangle\) is positive; otherwise, the correlation is negative.

The qualitative shape of the regions of positive and negative density correlations can be understood heuristically as follows. We first recall that in the hydrodynamic limit, we can ignore velocity fluctuations in the \(x\) direction. Hence, equation (III.4) implies that \(\delta \rho \propto \nabla_x \cdot \mathbf{u}_x\). That is, a positive \(\delta \rho(r_0)\), results from a positive divergence of \(\mathbf{u}_x\) at \(r_0\). Therefore, a positive \(\delta \rho(r_0)\) will occur if, e.g., \(u_x(r_0 - \epsilon \mathbf{y}) > u_x(r_0 + \epsilon \mathbf{y})\), where \(\epsilon\) is a small distance. Since we know that the equal-time correlation of \(\mathbf{u}_x\) is always positive, we expect in this situation that \(u_y(r_0 - \epsilon \mathbf{y}) > u_y(r_0 + \epsilon \mathbf{y})\) will remain positive even if \(r_0\) is shifted along the \(x\) direction. Therefore, we expect that, more often than not, \(\delta \rho(t, \mathbf{A} x) > 0\) if \(\delta \rho(t, \mathbf{0}) > 0\). Thus, this case will make a positive contribution to \(\langle \delta \rho(t, \mathbf{0}) \delta \rho(t, \mathbf{A} \mathbf{x}) \rangle\) where \(A\) is any positive or negative number.

One can make a similar argument for the case in which \(\delta \rho(r_0) < 0\), and conclude that usually \(\delta \rho(t, \mathbf{A} x) < 0\) if \(\delta \rho(t, \mathbf{0}) < 0\). Thus, this case will also make a positive contribution to \(\langle \delta \rho(t, \mathbf{0}) \delta \rho(t, \mathbf{A} \mathbf{x}) \rangle\).

This explains the positive region of the density correlation. Now, as the equal-time density correlation function is in the form of the Laplacian of a function (III.41), the overall spatial integral of the correlation function must be zero. Therefore, there must be a separatrix that separates the positive region and the negative region, which is the region roughly perpendicular to the \(x\) direction.

In section (III.35) we will show that the shape of this separatrix will be modified if we go beyond the linear theory.

Now we turn to the temporal correlations:
\[
\langle \delta \rho(t, \mathbf{0}) \delta \rho(t, 0) \rangle = \frac{1}{(2\pi)^{d+1}} \int d\omega d\omega' d^d k d^d k' \left\langle \delta \rho(\mathbf{k}) \delta \rho(\mathbf{k}') \right\rangle e^{-i \omega t}
\]
\[
= \frac{1}{(2\pi)^{d+1}} \int d\omega d^d k \frac{2D' k_2^2 e^{-i \omega t}}{\omega^2 + (\mu_L k_2^2 + \mu_x k_2^2)^2}
\]
\[
= \int d^d k \ D' k_2^2 e^{-\frac{\mu_L (k_2^2 + \mu_x k_2^2) t}{2}} \propto |t|^{-d/2} \int d^d q \ D' q_2^2 e^{-\frac{\mu_L q_2^2 + \mu_x q_2^2}{2}} \mu_L q_2^2 + \mu_x q_2^2 \]  
\tag{III.45}
\]
where, again, in the penultimate equality we have made the change of variable, \(q = |t|^{1/2} k\), and in the ultimate proportionality we have used the fact that the integral over \(q\) is a finite constant (i.e., independent of time \(t\)).

For arbitrary spatio-temporal separations, the correlation function is given by
\[
\langle \delta \rho(t, r) \delta \rho(0, 0) \rangle = \int \frac{1}{(2\pi)^{d+1}} d\omega d\omega' d^d k d^d k' \left\langle \delta \rho(\mathbf{k}) \delta \rho(\mathbf{k}') \right\rangle e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}
\]
\[
= \frac{1}{(2\pi)^{d+1}} \int d\omega d\omega' d^d k d^d k' \frac{2D' k_2^2 e^{-i (\mathbf{k} \cdot \mathbf{r} - \omega t)}}{\omega^2 + (\mu_L k_2^2 + \mu_x k_2^2)^2}, \tag{III.46}
\]
Making the changes of variables of integration prescribed by (III.35) we obtain
\[
\langle \delta \rho(t, r) \delta \rho(0, 0) \rangle = r_{-d} H_{\rho} \left( \frac{x}{r_+}, \frac{t}{r_+} \right) \propto \begin{cases} r_{-d}, & r \gg |t|^{1/2} \medskip \medskip |t| \gg r^2. \end{cases}, \tag{III.47}
\]
where we’ve defined the scaling function
\[
H_{\rho}(u, v) \equiv \frac{2D'}{(2\pi)^{d+1}} \int dY dQ Q_2^2 \frac{e^{i[(Q_2 + Q_2 + Q_x + Q_y) t - Y^2 + (\mu_L Q_2 + \mu_x Q_2^2) t]}}{Y^2 + (\mu_L Q_2^2 + \mu_x Q_2^2)^2}. \tag{III.48}
\]

In any spatial dimension \(d\), these correlations decay too rapidly to give rise to giant number fluctuations (GNF) \cite{5, 19}; that is, they are not sufficiently long-ranged to make the rms number fluctuations \(\delta N \equiv \sqrt{(\langle N - \langle N \rangle \rangle^2)}\) in a large region grow more rapidly than the square root of the mean number \(\sqrt{\langle N \rangle}\). However, they are sufficiently long-ranged to make \(\delta N\) depend on the shape of the region in which the particle number \(N\) is being counted \cite{20}.

Unfortunately, as we will see in the next section, these scaling laws, in particular the power law with which correlations decay with distance \(r\), are changed by non-linear effects, leading to a more rapid decay which eliminates this shape dependence. Nonetheless, the strange power law dependence of density correlations persists (albeit with different exponents than found here in the linear theory), and still displays universal exponents which can be readily measured in experiments and simulations.
IV. NONLINEAR EFFECTS AND DYNAMIC RG ANALYSIS

A. Full nonlinear equation of motion in Fourier space

We write the full model \( \Pi_{11} \) in Fourier space in tensor form:

\[
-\omega u^+ \equiv \left( \mu_1 k^2 + \mu_2 k^4 \right) u^+ + f (\tilde{k}) - \mu k^2 u^+ \equiv \frac{\lambda \sqrt{2}}{\pi^{d+1}} \times
\int \left[ u^+ (\tilde{k} - \tilde{q}) \cdot q \right] u^+ (\tilde{q}) , \quad (IV.1)
\]

where \( \tilde{q} \equiv (q, \Omega) \) and \( \int_\Omega \equiv \int d\Omega \int dq \). Going through essentially the same calculation as the one which leads to \( \Pi_{19} \) we get

\[
u_1 (\tilde{k}) = G_{ij} (\tilde{k}) \left\{ f_j (\tilde{k}) - \frac{\lambda \sqrt{2}}{\pi^{d+1}} \times
\int \left[ u^+ (\tilde{k} - \tilde{q}) \cdot q \right] u^+ (\tilde{q}) \right\} . \quad (IV.2)
\]

B. Dynamical Renormalization Group I: recursion relations

To probe what happens for \( d > 2 \), we use a DRG analysis together with the \( \epsilon \)-expansion method to one-loop level \( 21 \). Readers interested in a more complete and pedagogical discussion of the DRG are referred to \( 21 \) for the details of this general approach, including the use of Feynmann graphs in it.

First we decompose the Fourier modes \( u_1 (\tilde{k}) \) into a rapidly varying part \( u_1 (\tilde{k}) \) and a slowly varying part \( u_1 (\tilde{k}) \) in \( (IV.1) \). The rapidly varying part is supported in the momentum shell \( -\infty < k_x < \infty, \Delta e^{-\delta t} < k_x < \Lambda \), where \( \delta t \) is an infinitesimal and \( \Lambda \) is the ultraviolet cutoff. The slowly varying part is supported in \( -\infty < k_x < \infty, 0 < k_x < \Lambda e^{-\delta t} \).

The DRG procedure then consists of two steps. In step 1, we eliminate \( u_1 (\tilde{k}) \) from \( (IV.1) \). We do this by solving \( (IV.2) \) iteratively for \( u_1 (\tilde{k}) \). This solution is a series of \( \lambda \) which depends on \( u_1 (\tilde{k}) \). We substitute this solution into \( (IV.1) \) and average over the short wavelength components \( f^\gamma (\tilde{k}) \) of the noise \( f \), which gives a closed EOM for \( u_1 (\tilde{k}) \).

Step 2, rescale the length and time as the following

\[
r \rightarrow e^{\delta t} r , \quad x \rightarrow e^{\delta t} x , \quad t \rightarrow e^{\delta t} t \ , \quad u \rightarrow e^{\delta t} u , \quad (IV.3)
\]

which restores the ultraviolet cutoff back to \( \Lambda \). We reorganize the resultant EOM so that it has the same form as \( (IV.1) \) but with various coefficients renormalized.

The calculation of the renormalization of the coefficients arising from the process of eliminating \( u_1 (\tilde{k}) \) can be represented by graphs. The basic rules for the graphical representation are illustrated in Fig. 2.

Following these rules and the prescription of \( 21 \), the renormalization of the linear terms and the noise to one-loop order are represented by the graphs in Fig. 2 and Fig. 3 respectively. For example, Fig. 3a represents a linear term in the EOM for \( u_1 (\tilde{k}) \) given by

\[
-2 D \lambda^2 k^4 u_1 (\tilde{k}) \left( 2 \pi \right)^{d+1} \int_\Omega (k^4 - q^4) C_{ij}(\tilde{q}) G_{\delta t}(\tilde{k} - \tilde{q}) , \quad (IV.4)
\]

where

\[
\int_\Omega \equiv \int_{-\infty}^{+\infty} d\Omega \int_{-\infty}^{+\infty} dq_x \int_{\left| \epsilon^{-\delta t} q_x < \Delta \right.}^{q_x < \Delta} dq_x \int_{\left| \epsilon^{-\delta t} q_x < \Delta \right.}^{q_x < \Delta} d^{d-1} q \ . \quad (IV.5)
\]

By expanding the integrand to \( O(\epsilon) \) we show in appendix B.1 that \( (IV.3) \) gives contributions to the two linear terms \( k^2 u_1 (\tilde{k}) \) and \( k^2 k^4 u_1 (\tilde{k}) \), which lead respectively to the renormalization of \( \mu_1 \) and \( \mu_2 \).

We iterate the DRG procedure repeatedly, which leads to the following flow equations of the coefficients to one-
FIG. 4. Graphical representation of the correction to the noise correlator $\langle f_\ell(\mathbf{k}) f_u(-\mathbf{k}) \rangle$.

loop order:

$$\frac{1}{D} \frac{d D}{d \ell} = z - 2\chi - d + 1 - \zeta + g_1 G_D(g_2), \quad (\text{IV.6})$$

$$\lambda \frac{d \lambda}{d \ell} = z + \chi - 1, \quad (\text{IV.7})$$

$$\mu_x \frac{d \mu_x}{d \ell} = z - 2\zeta, \quad (\text{IV.8})$$

$$\mu_1 \frac{d \mu_1}{d \ell} = z - 2 + g_1 G_{\mu_1}(g_2), \quad (\text{IV.9})$$

$$\mu_2 \frac{d \mu_2}{d \ell} = z - 2 + g_1 G_{\mu_2}(g_2), \quad (\text{IV.10})$$

where we’ve defined

$$g_1 \equiv \frac{D \chi^2}{\sqrt{\mu_x \mu_1^2}} \frac{S_{d-1}}{(2\pi)^{d-1}} \Lambda^{d-4}, \quad g_2 \equiv \frac{\mu_2}{\mu_1}, \quad (\text{IV.11})$$

where $S_{d-1}$ is the surface area of a $d-1$-dimensional unit sphere, and $G_D, G_{\mu_1, \mu_2}$ are all functions of $g_2$. They are

$$G_D(g_2) \equiv \frac{(d-2)}{2(d-1) g_2^2} \left[ 1 + \frac{1}{\sqrt{g_2+1}} - \frac{2\sqrt{2}}{\sqrt{g_2+2}} \right] \quad (\text{IV.12})$$

$$= \frac{1}{g_2^2} \left[ \frac{1}{3} + \frac{1}{3\sqrt{g_2+1}} - \frac{2\sqrt{2}}{3\sqrt{g_2+2}} \right], \quad (d = 4) \quad (\text{IV.13})$$

$$G_{\mu_1}(g_2) \equiv \frac{2}{d^2-1} \left( \frac{2d^2 - 6d + 3}{32} + \frac{(d+3)\sqrt{2}}{g_2^2(g_2+2)^{3/2}} - \frac{1}{g_2^2} + \frac{d+1}{2g_2\sqrt{g_2+1}} + \frac{d-3}{2g_2} + \frac{d+15}{2\sqrt{2}g_2(g_2+2)^{3/2}} \right)$$

$$+ \frac{3}{\sqrt{2}(g_2+2)^{3/2}} - \frac{d+1}{4g_2\sqrt{g_2+1}} + \frac{3-d}{2\sqrt{2}g_2\sqrt{g_2+2}} \right) \quad (\text{IV.14})$$

$$= \frac{2}{15} \left( \frac{11}{32} + \frac{7\sqrt{2}}{g_2^2(g_2+2)^{3/2}} - \frac{1}{g_2^2} - \frac{5}{2g_2\sqrt{g_2+1}} + \frac{1}{2g_2} + \frac{19}{2\sqrt{2}g_2(g_2+2)^{3/2}} + \frac{3}{\sqrt{2}(g_2+2)^{3/2}} \right)$$

$$- \frac{5}{4g_2\sqrt{g_2+1}} - \frac{1}{2\sqrt{2}g_2\sqrt{g_2+2}} \right), \quad (d = 4) \quad (\text{IV.15})$$
The fact that there are no graphical corrections to $\lambda$ is not an accident, nor an artifact of our one loop approximation. Rather, it is a consequence of the fact that $\lambda$ is “protected” by a pseudo-Galilean symmetry. That is, the EOM is invariant under the substitutions: $x_\perp \mapsto x_\perp + t\lambda w$ and $u_\perp \mapsto u_\perp + w$ for some arbitrary constant vector $w$ perpendicular to the mean velocity $\langle u \rangle$. Since this exact symmetry involves $\lambda$, and the renormalization group preserves the underlying symmetries of the problem, it follows that $\lambda$ can not be renormalized (except trivially by rescaling): its graphical corrections must vanish in any dimension $d$.

The absence of graphical corrections to $\mu_x$, on the other hand, is likely an artifact of the one-loop approximation, which we will discuss in later sections.

Note that the appearance of negative powers of $g_2$ in the expressions \( \text{IV.12} - \text{IV.17} \) is somewhat misleading: despite those negative powers, none of these functions diverges at $g_2 = 0$; in fact, these singularities all cancel, and $G_D$, $G_{\mu_1}$, and $G_{\mu_2}$ are all smooth, analytic, and finite for all finite $g_2$ \textit{(including} $g_2 = 0$, \textit{that satisfy the stability constraint} $g_2 \geq -1$.

From Eqs \( \text{IV.6 - IV.10} \), we obtain the closed flow equations for $g$'s:

\[
\frac{1}{g_1} \frac{dg_1}{d\ell} = \epsilon + g_1 (G_D - \frac{5}{2} G_{\mu_1}) \equiv \epsilon + g_1 G_{g_1}(g_2)
\]

\[
\frac{1}{g_2} \frac{dg_2}{d\ell} = g_1 (G_{\mu_2} - G_{\mu_1}) \equiv g_1 G_{g_2}(g_2),
\]

where

\[
G_{g_1}(g_2) = \frac{(-10d^2 + 30d - 15)}{32(d^2 - 1)} + \frac{(d^2 - d + 8)}{2(d^2 - 1) g_2^2} - \frac{(2d^2 + 3d + 11)\sqrt{2}}{2(d^2 - 1) g_2^2 (g_2 + 2)^{3/2}} + \frac{(d + 3)}{2(d - 1) g_2^2 \sqrt{g_2 + 1}} + \frac{(15 - 5d)}{2(d^2 - 1) g_2^2}
\]

\[
- \frac{\sqrt{2}(4d^2 - 9d + 97)}{4(d^2 - 1) g_2^2 (g_2 + 2)^{3/2}} + \frac{(5d - 45)}{2\sqrt{2}(d^2 - 1) (g_2 + 2)^{3/2}} + \frac{5}{4(d - 1) g_2 \sqrt{g_2 + 1}}
\]

\[
= \frac{-11}{96} + \frac{2}{3 g_2^2} - \frac{11\sqrt{2}}{3 g_2^2 (g_2 + 2)^{3/2}} + \frac{7}{6 g_2 \sqrt{g_2 + 1}} - \frac{1}{6 g_2} - \frac{25}{6 \sqrt{2} g_2 (g_2 + 2)^{3/2}} - \frac{5}{6 \sqrt{2} (g_2 + 2)^{3/2}}
\]

\[
+ \frac{5}{12 g_2 \sqrt{g_2 + 1}}, \quad (d = 4)
\]

\[
G_{g_2}(g_2) = \frac{2}{(d^2 - 1) g_2^2} - \frac{(d - 1)\sqrt{2}}{g_2^2 (g_2 + 2)^{3/2}} + \frac{(d + 1)}{2 g_2^2 \sqrt{g_2 + 1}} + \frac{(d^2 - 4d + 3)\sqrt{2}}{4 g_2^2 \sqrt{g_2 + 2}} - \frac{(d^2 - 7d + 8)}{4 g_2^2}
\]

\[
+ \frac{d + 1}{64(g_2 + 1)^{3/2}} + \frac{13 - 15d}{4\sqrt{2} g_2 (g_2 + 2)^{3/2}} - \frac{3(d - 1)}{\sqrt{2}(g_2 + 2)^{3/2}} + \frac{(d + 1)}{4 g_2^2 \sqrt{g_2 + 2}} + \frac{2d^2 - 9d + 11}{32}
\]

\[
= \frac{2}{15 g_2} \left( -\frac{11\sqrt{2}}{g_2^2 (g_2 + 2)^{3/2}} + \frac{3}{g_2^2} + \frac{5}{2 g_2 \sqrt{g_2 + 1}} + \frac{3\sqrt{2}}{4 g_2 \sqrt{g_2 + 2}} + \frac{1}{g_2} + \frac{5}{64(g_2 + 1)^{3/2}} \right)
\]

\[
- \frac{47}{2 \sqrt{2} g_2 (g_2 + 2)^{3/2}} - \frac{9}{\sqrt{2}(g_2 + 2)^{3/2}} + \frac{5}{4 g_2 \sqrt{g_2 + 1}} + \frac{7}{32} \right), \quad (d = 4)
\]
Finding the fixed points of these two flow equations is equivalent to finding the fixed points of the original flow equations of the coefficients. We turn to this calculation in the next subsection.

C. Renormalization Group fixed points in the $\epsilon$-expansion

We now seek fixed points of these recursion relations to linear order in $\epsilon$. To this order, it is sufficient to evaluate the graphical corrections $G_{g_1}$ in precisely $d = 4$. We start with the recursion relations (IV.19) for $g_2$. In (Fig. 5) we plot $\frac{dg_2}{d\ell}$ versus $g_2$ for fixed $g_1$ and spatial dimension $d = 4$. As shown in the figure, the only point at which $\frac{dg_2}{d\ell}$ vanishes is $g_2 = 0$. Hence, the fixed points in the $(g_1, g_2)$ plane must lie at $g_2 = 0$. Furthermore, since $\frac{dg_2}{d\ell} > 0$ for $g_2 < 0$, and $\frac{dg_2}{d\ell} < 0$ for $g_2 > 0$, these fixed points at $g_2 = 0$ are stable with respect to $g_2$. (We will later demonstrate more thoroughly the stability of these fixed points.)

The fact that the fixed points are at $g_2 = 0$ may seem like rather miraculous result, given the complexity of the recursion relations (IV.18) for $g_1$ and (IV.19) for $g_2$. We note here that we do not expect this result to persist to higher loop orders. We will discuss the implications of

$$G_{g_2}(g_2) = \left( \frac{2}{d^2 - 1} \right) \left( \frac{(1 - 3d)\sqrt{2}}{g_2^2(g_2 + 2)^{3/2}} + \frac{(d - 1)}{g_2}\sqrt{g_2 + 1} \right) + \frac{(d + 1)}{2g_2\sqrt{g_2 + 1}} \frac{1}{2\sqrt{2g_2(g_2 + 2)^{3/2}}} + \frac{(d - 19d)}{2\sqrt{2g_2^2(g_2 + 2)^{3/2}}} + \frac{(d^2 - 4d + 3)}{2\sqrt{2g_2^2(g_2 + 2)^{3/2}}} + \frac{(d^2 - 7d + 4)}{4g_2^2}$$

$$+ \frac{3(d + 1)}{4g_2^2\sqrt{g_2 + 1}} - \frac{3\sqrt{2}}{2(g_2 + 2)^{3/2}} + \frac{(9 + 7d)}{2\sqrt{2g_2(g_2 + 2)^{3/2}}} + \frac{(2d^2 - 25d + 59)}{32g_2} + \frac{d + 1}{64g_2(g_2 + 1)^{3/2}} + \frac{(d + 1)}{4g_2\sqrt{g_2 + 1}}$$

$$+ \frac{(d - 3)}{2\sqrt{2g_2\sqrt{g_2 + 2}}} - \frac{(2d^2 - 6d + 3)}{32}$$

(IV.22)

$$= -\frac{22\sqrt{2}}{15g_2^2(g_2 + 2)^{3/2}} + \frac{2}{5g_2^2} + \frac{1}{3g_2^3\sqrt{g_2 + 1}} - \frac{\sqrt{2g_2^2(g_2 + 2)^{3/2}}}{3} + \frac{5}{5\sqrt{2g_2^2\sqrt{g_2 + 2}}} + \frac{1}{15g_2^2}$$

$$+ \frac{1}{2g_2\sqrt{g_2 + 1}} - \frac{1}{5(g_2 + 2)^{3/2}} - \frac{\sqrt{2}}{15\sqrt{2g_2^2(g_2 + 2)^{3/2}}} - \frac{3}{80g_2} + \frac{1}{96g_2(g_2 + 1)^{3/2}} + \frac{1}{6g_2\sqrt{g_2 + 1}}$$

(IV.23)
this in section [IV.1].

To find the value of \( g_1 \) at these fixed points we take the slightly tricky limit \( g_2 \to 0 \) in our expression \( (IV.21) \) for \( G_{g_1} \) in \( d = 4 \). This gives

\[
G_{g_1}(g_2 = 0) = -\frac{11}{64}. \tag{IV.24}
\]

Inserting this value of \( G_{g_1} \) into the recursion relation \( (IV.18) \) for \( g_1 \) and finding the values of \( g_1 \) at which \( \frac{dg_1}{d\ell} = 0 \) (a value that we’ll refer to as \( g_1^* \)) gives two solutions: \( g_1^* = 0, g_2^* = 0 \), which is just the Gaussian fixed point, and obviously unstable, and a stable non-Gaussian fixed point at:

\[
g_1^* = \frac{64}{11} \epsilon + \mathcal{O}(\epsilon^2), \quad g_2^* = 0. \tag{IV.25}
\]

Note that the value of \( g_1 \) at this non-Gaussian fixed point is \( \mathcal{O}(\epsilon) \), so our perturbation theory, which is valid for small \( g_1 \), should be accurate for small \( \epsilon \); i.e., for spatial dimensions near the upper critical dimension \( d = 4 \). This validity for small \( \epsilon \) is, of course, a standard feature of all \( \epsilon \) expansions.

To demonstrate the stability of this fixed point, we show that small departures from it decay to zero upon renormalization. Specifically, we linearize the recursion relations \( (IV.18) \) and \( (IV.19) \) around the fixed point, writing

\[
g_1(\ell) = g_1^* + \delta g_1(\ell) \tag{IV.26}
\]

and expanding the recursion relations \( (IV.18) \) and \( (IV.19) \) to linear order in \( \delta g_1(\ell) \) and \( g_2(\ell) \). This leads to the recursion relations:

\[
\frac{d\delta g_1}{d\ell} = -\epsilon \delta g_1(\ell) - \frac{160}{121} \epsilon^2 g_2 \tag{IV.27}
\]

\[
\frac{dg_2}{d\ell} = -\frac{16}{33} \epsilon g_2(\ell), \tag{IV.28}
\]

from which it is obvious that the fixed point \( (IV.25) \) is stable.

The full renormalization group flows in the \( g_1 \)-\( g_2 \) plane for small \( \epsilon \) are illustrated in (Fig. 6).

D. Scaling exponents

With the location of the fixed point \( (IV.25) \) in hand, we can now easily find the universal scaling exponents governing the behavior of all properties (in particular, correlation functions) of Malthusian flocks.

The most direct way to do this is to choose the heretofore arbitrary RG rescaling exponents — that is, the dynamical exponent \( z \), the anisotropy exponent \( \zeta \), and the roughness exponent \( \chi \) — to keep all of the other important parameters (i.e., the noise strength \( D \), the diffusion constants \( \mu_{z,1} \) \cite{22}, and the convective nonlinearity \( \lambda \)) fixed.

Keeping the noise strength \( D \) fixed, leads, via \( (IV.6) \), to the condition

\[
z - 2\chi - d + 1 - \zeta + g_1^* G_{g_1}^* = 0, \tag{IV.29}
\]

where we’ve defined \( G_{g_1}^* \equiv G_{\hat{g}}(g_2 = 0) \); i.e., the value of \( G_{\hat{g}}(g_2) \) at the fixed point \( g_2 = 0 \). From our expression \( (IV.13) \) for \( G_{\hat{g}}(g_2) \), it is relatively simple to take the limit \( g_2 \to 0 \) and obtain, in \( d = 4 \),

\[
G_{\hat{g}}^* = G_{\hat{g}}(g_2 = 0) = \frac{1}{16}. \tag{IV.30}
\]

Inserting this, the fixed point value \( (IV.25) \) of \( g_1^* \), and \( d = 4 - \epsilon \) into \( (IV.29) \) gives

\[
z - 2\chi - \zeta = 3 - \frac{15}{11} \epsilon. \tag{IV.31}
\]

From the above, we can obtain two more linear conditions on our three exponents \( z, \zeta \) and \( \chi \), by requiring that \( \mu_x \) and \( \lambda \) remain fixed. The former condition leads to

\[
z = 2\zeta, \tag{IV.32}
\]

while the latter implies

\[
z + \chi = 1. \tag{IV.33}
\]

The three linear equations \( (IV.31), (IV.32), \) and \( (IV.33) \) are easily solved to give:

\[
z = 2 - \frac{6\epsilon}{11} + \mathcal{O}(\epsilon^2) \tag{IV.34}
\]

\[
\chi = -1 + \frac{6\epsilon}{11} + \mathcal{O}(\epsilon^2) \tag{IV.35}
\]

\[
\zeta = 1 - \frac{3\epsilon}{11} + \mathcal{O}(\epsilon^2). \tag{IV.36}
\]

To the best of our knowledge, the above fixed point and the associated scaling exponents characterize a previously undiscovered universality class.

E. Beyond linear order in \( \epsilon \)

Our results so far are based on a one-loop calculation, which is exact to linear order in \( \epsilon \). However, since all of our expressions for \( G_{\hat{g}_{1,2}} \) are evaluated for general \( d \), one can potentially extrapolate our results to arbitrary \( d \) based on our one-loop calculation, ignoring higher loop graphs. We must emphasize that this is a uncontrolled approximation, since the higher loop graphs are of higher order in \( g_1 \), but \( g_1 \) is not small at the fixed point once \( d \) is far from 4. Nonetheless, there are two limits in which this approach will recover exact results: 1) near \( d = 4 \), where it will automatically recover the exact \( 4 - \epsilon \) expansion results we’ve just obtained, and 2) in precisely \( d = 2 \), where, as we’ll show below, this approach reproduces the known exact “canonical” exponents \( [11] \) \cite{14}. 


Given these constraints, it’s quite likely that the exponents obtained by this uncontrolled approximation are extremely close to the actual values.

This truncated one-loop calculation for general $d$ now proceeds in much the same way as our small $\epsilon$ approach. We start by noting that once again, as for $d$ near 4, $\frac{dg}{dl}$ looks like figure 5 in particular, $\frac{dg}{dl} > 0$ for $g_2 < 0$, and $\frac{dg}{dl} < 0$ for $g_2 > 0$. Hence, as for small $\epsilon$, in our current uncontrolled one-loop approximation, we again have two fixed points, which are both at $g_2 = 0$, and of which again only the non-Gaussian one is stable. (We will do a more thorough analysis of the stability of this fixed point for general $d$ later.)

Since the fixed point value $g_2^*$ of $g_2$ is zero, we again only need the values of $G_{g_1,z}$ at $g_2 = 0$, but now for general $d$. With a bit more assistance from le Marquis de l’Hôpital, we find, for the non-Gaussian fixed point,

$$G'_{g_1} = G_{g_1}(g_2 = 0) = \frac{23 - 14d}{64(d - 1)}, \quad (IV.37)$$

$$G'_{g_2} = G_{g_2}(g_2 = 0) = \frac{5(4 - d - d^2)}{64(d^2 - 1)}. \quad (IV.38)$$

Using the first of these in the recursion relations $[IV.18]$ for $g_1$, and expressing the fixed point values of $G'_{g_1}$ and $G'_{g_1}$ in terms of $d$ (instead of $\epsilon$), we have

$$g_1^* = \frac{64(4 - d)(d - 1)}{14d - 23}. \quad (IV.39)$$

To demonstrate the stability of this fixed point, we show that small departures from it decay to zero upon renormalization. Specifically, we linearize the recursion relations $[IV.18]$ and $[IV.19]$ around the fixed point, writing

$$g_1(\ell) = g_1^* + \delta g_1(\ell) \quad (IV.40)$$

and expanding the recursion relations $[IV.18]$ and $[IV.19]$ to linear order in $\delta g_1(\ell)$ and $g_2(\ell)$. This leads to the recursion relations:

$$\frac{d\delta g_1}{d\ell} = (d - 4)\delta g_1(\ell) + (g_1^*)^2 G'_{g_1}(g_2 = 0) g_2$$

$$= (d - 4)\delta g_1(\ell) - \frac{160(d - 1)(3d^2 - 8d - 1)}{(14d - 23)^2(d + 1)}(4 - d)^2 g_2$$

$$\frac{dg_2}{dl} = G_{g_2}(g_2 = 0) g_2^* g_2$$

$$= \left(\frac{5(4 - d - d^2)}{(d + 1)(14d - 23)}\right) \epsilon g_2. \quad (IV.42)$$

Because $d^2 + d - 4 > 0$ for all spatial dimensions $d$ in the range of interest $2 \leq d \leq 4$, it is obvious from $[IV.41]$ and $[IV.42]$ that the fixed point $[IV.39]$ is stable.

For this uncontrolled one-loop approximation the full renormalization group flows in the $g_1$-$g_2$ plane still looks qualitatively like Fig. 6.

We can now determine the scaling exponents $z$, $\zeta$, and $\chi$, as we did in the $\epsilon$ expansion, by choosing them to keep $D$, $\mu_x$, and $\lambda$ fixed. This leads to the same conditions $[IV.29]$, $[IV.32]$ and $[IV.33]$ as in the $\epsilon$ expansion, but now in $[IV.29]$ we use the value

$$G_{g_2}^* = G_{g_2}(g_2 = 0) = \frac{3(d - 2)}{32(d - 1)} \quad (IV.43)$$

which arises from our one-loop truncation in arbitrary dimension $d$. Solving these three linear equations $[IV.29]$, $[IV.32]$ and $[IV.33]$ for the three exponents now gives

$$z = 2 - \frac{2(4 - d)(4d - 7)}{14d - 23}, \quad (IV.44)$$

$$\zeta = 1 - \frac{(4 - d)(4d - 7)}{14d - 23}, \quad (IV.45)$$

$$\chi = -1 + \frac{2(4 - d)(4d - 7)}{14d - 23}, \quad (IV.46)$$

which are the results in general dimension $d$ quoted in the introduction.

As noted earlier, these exponents $[IV.44]$, $[IV.45]$, and $[IV.46]$, in addition to automatically recovering the exact linear order in $\epsilon = 4 - d$ behavior that we found earlier, also become exact in $d = 2$. The reason for this is simple: as can be seen by inspecting our one-loop recursion relations, they correctly recover the exact fact that, in $d = 2$, $\lambda$, $\mu_x$, and $D$ are unrenormalized graphically. Keeping them fixed therefore leads to three very simple linear equations for $z$, $\zeta$, and $\chi$, whose solutions are the “canonical” exponents $[12]$. 

---

**FIG. 7.** Graphical summary of our results for the dynamic exponent $z$ as a function of the spatial dimension $d$. The result $[IV.33]$ based on the $\epsilon$-expansion method to $O(\epsilon)$ is shown by the broken black line, while the extrapolation to arbitrary $d$ based on our one-loop result is shown in the blue line $[IV.44]$, which converges to the known exact value (red square) in 2D. The dashed red line is the “canonical” value $z = \frac{2(d + 1)}{5}$. 

---
Why are these parameters exactly unrenormalized in $d = 2$? For the non-linearity $\lambda$, this is because it is unrenormalized in any dimension due to the pseudo-Galilean invariance of (II.11), for which we have given a detailed argument in section IV.B.

The absence of graphical corrections to $\mu_x$ is, as we'll discuss below, highly likely an artifact of the one-loop approximation, except in $d = 2$, where it becomes exact because the sole non-linearity in the problem is $\lambda(u_\perp \cdot \nabla_\perp)u_\perp$ term in (II.11) — becomes a total $y$-derivative: $\lambda(u_\perp \cdot \nabla_\perp)u_\perp = \lambda u_{\perp z} \partial_{y_\perp} y = \lambda (\partial_{y_\perp} u_{\perp z}^2/2)\hat{y}$. This implies that this non-linearity can only generate terms in the EOM that involve $y$ derivatives. Since the $\mu_x$ term only involves $x$ derivatives, it cannot be renormalized in $d = 2$.

In our one-loop calculation, the graphical correction $G_D(g_2)$ to $D$ vanishes due to the explicit factor of $d - 2$ in our expression (IV.12) for $G_D$. The presence of this factor is not an accident; rather, it reflects the same fact that implied $D$ is unrenormalized: in $d = 2$, the non-linearity can only generate terms involving $y$ derivatives. Since the noise correlation $D$ has weight at $q = 0$, it cannot be renormalized, to all orders in a loop expansion. The factor of $d - 2$ in equation (IV.12) for $G_D$ is simply explicit confirmation of this fact at one-loop order.

To summarize, all of the properties required to obtain the canonical exponents (I.2) in $d = 2$ are correctly reproduced by the uncontrolled, truncated one-loop approach. This is why it reproduces the exact exponents in $d = 2$.

Note that the predicted values of the scaling exponents in 3D obtained from these two approaches ($\epsilon$ expansion and one-loop in arbitrary $d$) are in fact very quantitatively similar (Fig. 7). For example, the value of $z$ obtained from the $\epsilon$ expansion in $d = 3$, obtained from equation (IV.34) by setting $\epsilon = 1$, is $z_\epsilon = 16/17$, while that obtained from our uncontrolled one-loop approximation is $z_\mu = 28/19$. The difference between these is $z_\mu - z_\epsilon = 4/399$, which is only $\frac{1}{4}$ of $z_\mu$. The other exponents are comparably close. Furthermore, since we know that the uncontrolled exponents approach the exact answer in $d = 2$, they are probably closer to the exact answer in $d = 3$ than the difference between themselves and the $\epsilon$ expansion result. We thereby conclude that the values given by the uncontrolled approximation in $d = 3$, namely

$$z = \frac{28}{19} \approx 1.47, \quad (\text{IV.47})$$
$$\zeta = \frac{14}{19} \approx 0.74, \quad (\text{IV.48})$$
$$\chi = -\frac{9}{19} \approx -0.47. \quad (\text{IV.49})$$

are likely accurate to $\pm 1\%$. As noted in the introduction, this implies that the digits shown after the approximate equalities above are probably all correct.

F. Beyond one-loop order

In this section, we discuss what features of the above results are artifacts of the one-loop truncation. Aside from small quantitative corrections to the precise values of the exponents, which we have just argued are small, there are two more significant changes that we expect will occur in a higher order calculation (which, we should emphasize, we have not done!).

The first of these is that the diffusion constant $\mu_x$ will no longer be unrenormalized at higher order. We expect this to be the case because there is no symmetry that “protects” $\mu_x$ from renormalizing. Its failure to renormalize at one-loop order is therefore to some extent simply a coincidence, and almost certainly an artifact of the one-loop approximation.

The second change that will occur at higher order is that the fixed point will no longer be at $\mu_x \neq 0$. This is because, as for the renormalization of $\mu_x$, there is no symmetry that prevents a non-zero $\mu_x$ from being generated, even when the initial (bare) $\mu_x = 0$.

As a result, the recursion relation for $g_2$ near $g_1 = 0$ will, at two-loop order, become

$$\frac{dg_2}{d\ell} = g_1 G_{g_2}(g_2)g_2 + g_1^2 H(g_2), \quad (\text{IV.51})$$

where $H(g_2)$ is a function of $g_2$ that will presumably be even more formidable than $G_{g_2}(g_2)$. More importantly, it will be non-zero at $g_2 = 0$. Expanding the right hand side of (IV.51) for small $g_2$ and $\epsilon$ gives

$$\frac{dg_2}{d\ell} = g_1 g_2 G_{g_2}(g_2 = 0) + g_1^2 H(g_2 = 0), \quad (\text{IV.52})$$

where the alert reader will recognize the first term on the right hand side from our linearized recursion relation
to one-loop order. Solving for the fixed point value \( g_2^* \) of \( g_2 \) by setting \( \frac{dg}{dt} = 0 \) and \( g_1 = g_1^* \) gives

\[
g_2^* = - \frac{H(g_2 = 0)}{G_{g_2}(g_2 = 0)} g_2^* = \mathcal{O}(\epsilon), \tag{IV.53}
\]

where in the last equality we have used the fact that \( g_1^* = \mathcal{O}(\epsilon) \). Thus \( g_2^* \) is non-zero, and \( \mathcal{O}(\epsilon) \), once higher loop corrections are taken into account. Unfortunately, it is impossible to say much more about the value of \( g_2^* \), other than that it is non-zero, without actually doing the two-loop calculation necessary to determine the function \( H(g_2) \) in equation (IV.51). We have not attempted this formidable calculation, and so can say no more except note that \( g_2^* \) will be non-zero. (Frustratingly, we can not even determine its sign!)

This has experimental consequences, because, as we'll show in section V below, the value of \( g_2^* \) determines a universal amplitude ratio that appears in the velocity correlation function.

\[
C_u \left( r_\perp, x - \gamma t, t; \{ D_0, \mu_{x0}, \mu_{10}, \mu_{20}, \lambda_0 \} \right) = \langle \mathbf{u}_\perp (\mathbf{r} \cdot t) \cdot \mathbf{u}_\perp (0, 0) \rangle \tag{V.1}
\]

of the original system (whose parameters – the “bare” parameters – are denoted by the subscript 0) can be related to that of the system after a renormalization group time \( \ell \) has elapsed via

\[
C_u \left( r_\perp, x - \gamma t, t; \{ D_0, \mu_{x0}, \mu_{10}, \mu_{20}, \lambda_0 \} \right) = \exp \left[ 2 \int_0^\ell \chi (\ell') d\ell' \right] 
\times C_u \left( r_\perp e^{-\chi (\ell)} \mathbf{r} - \gamma t \exp \left( - \int_0^\ell \zeta (\ell') d\ell' \right), t \exp \left( - \int_0^\ell z (\ell') d\ell' \right); \{ D(\ell), \mu_x(\ell), \mu_1(\ell), \mu_2(\ell), \lambda(\ell) \} \right). \tag{V.2}
\]

In this relation the combination \( x - \gamma t \) appears rather than \( x \) due to the boost (II.10) we performed to obtain the model equation (II.11) which we actually used for the renormalization group.

The relation (V.2) holds for an arbitrary choice of the rescaling exponents \( \chi(\ell), \zeta(\ell), \) and \( z(\ell) \); they need not be the special choice (IV.34), (IV.35) and (IV.36) that we made earlier to produce fixed points. Indeed, as our notation suggests, we can even choose different values for these exponents at different renormalization group times \( \ell \). We will take advantage of this freedom to use (V.2) to derive the scaling relation (I.12). We will do so by choosing the rescaling exponents \( \chi(\ell), \zeta(\ell), \) and \( z(\ell) \) according to the following scheme: for \( \ell < \ell^* \), where \( \ell^* \) is the renormalization group time at which \( g_1, 2 \) get close to their fixed point values, we will choose these exponents so that at \( \ell = \ell^* \), the parameters \( D(\ell^*), \mu_x(\ell^*), \) and \( \mu_1(\ell^*) \) take the values \( D(\ell^*) = D_{\text{ref}}, \mu_x(\ell^*) = \mu_1(\ell^*) = \mu_{\text{ref}}, \)

V. EXPERIMENTAL CONSEQUENCES

A. Scaling laws for velocity and density correlations

The scaling exponents \( z, \zeta, \) and \( \chi \) just determined control the scaling properties of velocity and density correlations, as embodied in equations (I.12) and (I.13) for the velocity and density autocorrelations, respectively. This can be seen by using the “trajectory integral matching formalism” [24], which is simply a fancy way of describing the process of undoing all of the variable and coordinate rescaling done in the renormalization group process. This implies, for example, that the velocity autocorrelation function
rameters $\mu_2(\ell^*)$ and $\lambda(\ell^*)$ are also determined, the former by the relation $g_2 = \frac{\mu_2}{\mu_1}$, the latter by the definition (IV.11) of $g_1$. Combining this fact with $g_1(\ell^*) = g_1^*$ and $g_2(\ell^*) = g_2^*$, (which follows our definition of $\ell^*$ as the renormalization group time at which we get close to the fixed point), we have that

$$\mu_2(\ell^*) = g_2^* \mu_{\text{ref}} \quad , \quad \lambda(\ell^*) = \sqrt{\frac{g_1^* \mu_{\text{ref}}^2 (2\pi)^{d-1} \Lambda^{4-d}}{D_{\text{ref}} S_{d-1}}} ,$$

(V.3)

Hereafter we also refer to $\lambda(\ell^*)$ as $\lambda_{\text{ref}}$.

Note finally that the value of $\ell^*$ at which we get close to the fixed point is unaffected by our arbitrary choice of the rescaling exponents $\chi(\ell)$, $\zeta(\ell)$, and $z(\ell)$, since these do not enter the recursion relations for $g_{1,2}$.

For $\ell > \ell^*$, we will choose the rescaling exponents $\chi(\ell)$, $\zeta(\ell)$, and $z(\ell)$ to take on the values (V.34), (V.35) and (IV.36) that we showed earlier keep all of the parameters fixed once $g_{1,2}$ have flowed to their fixed point values. For the remainder of this subsection, we will refer to these values of $\chi$, $\zeta$, $z$ as the “fixed point” values $\chi_{\text{FP}}$, $\zeta_{\text{FP}}$, and $z_{\text{FP}}$.

This choice will, for all $\ell > \ell^*$, keep all of the parameters fixed at the “reference” values we have just described.

With this choice of $\chi(\ell)$, $\zeta(\ell)$, and $z(\ell)$, we can rewrite equation (V.2) as

$$C_u \left( r_\perp, x - \gamma t, t; \{ D_0, \mu_{x0}, \mu_{10}, \mu_{20}, \lambda_0 \} \right) = \exp \left[ 2 \int_0^{\ell^*} \chi(\ell')d\ell' + 2\chi_{\text{FP}}(\ell - \ell^*) \right]$$

$$\times C_u \left( r_\perp e^{-\ell}, (x - \gamma t)e^{-\ell}, \left[ -\int_0^{\ell^*} \zeta(\ell')d\ell' - \zeta_{\text{FP}}(\ell - \ell^*) \right], t \exp \left[ -\int_0^{\ell^*} z(\ell')d\ell' - z_{\text{FP}}(\ell - \ell^*) \right]; \{ D_{\text{ref}}, \mu_{\text{ref}}, \mu_{\text{ref}}, g_2^* \mu_{\text{ref}}, \lambda_{\text{ref}} \} \right).$$

(V.4)

To derive our scaling law (I.12) for $C_u$, we simply apply this relation (V.4) at particular value of $\ell$, which we’ll call $\ell(r_\perp)$, determined by

$$e^{-\ell(r_\perp)}r_\perp = a \equiv \frac{1}{\Lambda}.$$  

(V.5)

$$C_u \left( r_\perp, x - \gamma t, t; \{ D_0, \mu_{x0}, \mu_{10}, \lambda_0 \} \right) = Ar^{2\chi_{\text{FP}}} C_u \left( a, a, \frac{|x - \gamma t|/\xi_x}{(r_\perp/\xi_\perp)^{\xi_{\text{FP}}}}, \frac{(t/\tau)\tau_0}{(r_\perp/\xi_\perp)^{\xi_{\text{FP}}}}; \{ D_{\text{ref}}, \mu_{\text{ref}}, \mu_{\text{ref}}, g_2^* \mu_{\text{ref}}, \lambda_{\text{ref}} \} \right)$$

$$\equiv r^{2\chi_{\text{FP}}} F_u \left( \frac{|x - \gamma t|/\xi_x}{(r_\perp/\xi_\perp)^{\xi_{\text{FP}}}}, \frac{(t/\tau)}{(r_\perp/\xi_\perp)^{\xi_{\text{FP}}}}; \{ D_{\text{ref}}, \mu_{\text{ref}}, \mu_{\text{ref}}, g_2^* \mu_{\text{ref}}, \lambda_{\text{ref}} \} \right) ,$$

(V.6)

where we’ve defined scaling function

$$F_u \equiv AC_u \left( a, a, \frac{|x - \gamma t|/\xi_x}{(r_\perp/\xi_\perp)^{\xi_{\text{FP}}}}, \frac{(t/\tau)\tau_0}{(r_\perp/\xi_\perp)^{\xi_{\text{FP}}}}; \{ D_{\text{ref}}, \mu_{\text{ref}}, \mu_{\text{ref}}, g_2^* \mu_{\text{ref}}, \lambda_{\text{ref}} \} \right) ,$$

(V.7)

the constant

$$A \equiv a^{-2\chi_{\text{FP}}} \exp \left[ 2 \int_0^{\ell^*} (\chi(\ell') - \chi_{\text{FP}})d\ell' \right],$$

(V.8)

and the non-universal “non-linear lengths” $\xi_{\perp,x}$ to satisfy

$$\xi_{\perp} = e^{\ell^*} a ,$$

(V.9)
and
\[ \frac{\xi_{FP}}{\xi_x} = \exp \left[ \int_0^\ell (\zeta_{FP} - \zeta(\ell)) d\ell \right] a^{\xi_{FP}^{-1}} , \] (V.10)
and the non-universal “non-linear time” \( \tau \) to satisfy
\[ \frac{\xi_{FP}^2}{\tau} \tau_0 = \exp \left[ \int_0^\ell (z_{FP} - z(\ell)) d\ell \right] a^{z_{FP}} . \] (V.11)

Here the value of the characteristic time \( \tau_0 \) is not arbitrary, but set by the cutoff length \( a \) and the \( \mu_1 \) of the rescaled system, namely \( \mu_{ref} \). Specifically it is given by
\[ \tau_0 = \frac{a^2}{\mu_{ref}} . \] (V.12)

Note that, like the reference values of other parameters, this characteristic time is the same for all systems, regardless of the bare values of the parameters.

Because the parameters appearing in \( C_\rho \) on the right hand side of (V.7), namely, \( a, \tau_0, D_{ref}, \mu_{ref}, \eta_1^2 \mu_{ref}, \) and \( \lambda_{ref} \) are all independent of the initial system under consideration, the scaling function \( F_\xi \) is, as claimed in the introduction, a universal function of its arguments \((a, \xi, /\xi_{FP}) \) and \((\ell/\tau)\), up to the non-universal multiplicative factor \( A \), which is given by (V.8).

This concludes our derivation of the scaling law for velocity correlations. The derivation of the density correlations is almost identical. The only difference lies in the field rescaling. Since \( \delta \rho \) is enslaved to \( u_z \) by the relation (II.4), the rescaling exponent for \( \delta \rho \) is \( \chi - 1 \) instead of \( \chi \), the rescaling exponent of \( u_z \). Therefore, in analogy to (V.2), we get the following relation between density correlations in the original system and the rescaled system:

\[ C_\rho \left( r_{FP} = r_{FP} \right) = \left( \frac{\xi_{FP}^2}{\tau} \right) \exp \left[ 2 \int_0^\ell \gamma(\ell') - 1 d\ell' \right] \]
\[ \times C_\rho \left( r_{FP} \left( x - \gamma t, t \right) \right) \exp \left( - \int_0^\ell \gamma(\ell') d\ell' \right) t \exp \left( - \int_0^\ell z(\ell') d\ell' \right) \]. (V.13)

From here on the derivation is virtually identical to that of the velocity correlations, which we will not repeat. The final result is given by (I.13) in the introduction.

### B. Calculation of the non-linear lengths and times

There are two independent ways of calculating the non-linear lengths and times appearing in the scaling functions (V.6) and (I.13) just derived. One way is to continue with the RG approach just presented. We take this approach in the next subsection. An alternative approach, which we present as a check on the RG approach, is to calculate perturbative corrections to the linear theory and calculate the length and time scales on which they become appreciable. These length and time scales prove to be precisely the lengths \( \xi_{FP} \) and \( \xi_x \), and the time \( \tau \).

We’ll begin here with the RG calculation; then, in the next subsection, we’ll present the perturbation theory approach.

1. **RG calculation**

The conditions (V.10) and (V.11) can be solved for the non-linear length \( \xi_{FP} \) and non-linear time \( \tau \), giving

\[ \xi_x = a \left( \frac{\xi_{FP}}{a} \right)^{\xi_{FP}} \exp \left[ - \xi_{FP} \ell^* + \int_0^\ell \zeta(\ell') d\ell' \right] , \] (V.14)

and

\[ \tau = \tau_0 \left( \frac{\xi_{FP}}{a} \right)^{\xi_{FP}} \exp \left[ - z_{FP} \ell^* + \int_0^\ell z(\ell') d\ell' \right] . \] (V.15)

Using our expression (V.9) for \( \xi_{FP} \) in these expressions simplifies them to

\[ \xi_x = a \exp \left[ \int_0^\ell \zeta(\ell') d\ell' \right] , \] (V.16)

and

\[ \tau = \tau_0 \exp \left[ \int_0^\ell z(\ell') d\ell' \right] . \] (V.17)

The alert reader may be alarmed by the apparent dependence of \( \xi_{FP} \) and \( \tau \) in the arbitrary choices of \( \zeta(\ell) \) and \( z(\ell) \). But those choices are not completely arbitrary, since they must lead to the parameters \( D \) and \( \mu_{1,FP} \) flowing to their reference values. This requirement proves to
constrain the very integrals that appear in (V.16) and (V.17) to (non-universal) values that are determined entirely by the bare parameters of the model. Likewise, the non-universal overall scale factor $A$ in the correlation function (V.7), while apparently dependent on our arbitrary choice of the velocity rescaling exponent $\chi(\ell)$, in fact does not, and is, instead, also determined solely by the non-universal values of the bare parameters of the model, as we’ll show now.

The requirement that $\mu_x(\ell)$ and $\mu_1(\ell)$ reach equality at $\ell = \ell^*$ constrains the integral of $\zeta(\ell)$ in (V.16). To see this, consider the recursion relations for $\mu_x$ and $\mu_1$. In complete generality, to arbitrary order in perturbation theory, these can be written:

\begin{align*}
\frac{1}{\mu_x} \frac{d\mu_x}{d\ell} &= z - 2\zeta(\ell) + Y_2(g_1(\ell), g_2(\ell)), \quad (V.18) \\
\frac{1}{\mu_1} \frac{d\mu_1}{d\ell} &= z - 2 + Y_1(g_1(\ell), g_2(\ell)). \quad (V.19)
\end{align*}

To one-loop order, $Y_2 = 0$ and $Y_1 = g_1 G_{\mu_1}(g_2)$; here we’ll use this more general form to demonstrate that our conclusion is not an artifact of the one-loop approximation, or, indeed, any approximation at all.

The recursion relations (V.18) and (V.19) taken together imply that the logarithm of the ratio $\frac{\mu_x}{\mu_1}$ obeys the recursion relation

\begin{equation}
\ln \left( \frac{\mu_x(\ell)}{\mu_1(\ell)} \right) = \frac{1}{\mu_x} \frac{d\mu_x}{d\ell} - \frac{1}{\mu_1} \frac{d\mu_1}{d\ell} = 2(1 - \zeta(\ell) + Y_2(g_1(\ell), g_2(\ell)) - Y_1(g_1(\ell), g_2(\ell))). \quad (V.20)
\end{equation}

The solution of this is

\begin{equation}
\ln \left( \frac{\mu_x(\ell)}{\mu_1(\ell)} \right) = 2\ell^* - \frac{2}{\ell^*} \int_0^{\ell^*} d\ell \zeta(\ell) + 2\Phi(g_{10}, g_{20}), \quad (V.22)
\end{equation}

where we’ve defined

\begin{equation}
\Phi(g_{10}, g_{20}) \equiv \frac{1}{2} \int_0^{\ell^*} \left[ Y_2(g_1(\ell'), g_2(\ell')) - Y_1(g_1(\ell'), g_2(\ell')) \right] d\ell'. \quad (V.23)
\end{equation}

Note that, as our notation suggests, $\Phi$ is completely determined by the bare values $g_{10,20}$ of $g_{1,2}$; in particular, it is independent of the arbitrary choice of the rescaling exponents $\chi(\ell)$, $\zeta(\ell)$, and $z(\ell)$. This is because the recursion relations for $g_{1,2}$ are independent of those exponents; so their solutions $g_{1,2}(\ell)$ are determined entirely by the initial conditions $g_{1,2}(\ell = 0) = g_{10,20}$. Once those solutions are determined, the integrand in (V.23) is also fully determined (since it depends only on $g_{1,2}(\ell)$). Furthermore, the limits on the integral are completely determined by $g_{10,20}$ as well, since $\ell^*$ is. Hence, $\Phi$ is completely determined by $g_{10,20}$, as claimed.

The condition (V.22) can be rewritten as

\begin{equation}
\int_0^{\ell^*} \zeta(\ell) d\ell = \frac{1}{2} \ln \left( \frac{\mu_0}{\mu_{10}} \right) + \ell^* + \Phi(g_{10}, g_{20}). \quad (V.24)
\end{equation}

Using this in (V.16) gives

\begin{equation}
\xi_x = ae^{\ell^*} e^{\Phi} \sqrt{\frac{\mu_0}{\mu_{10}}} = \xi_\perp e^{\Phi} \sqrt{\frac{\mu_0}{\mu_{10}}}, \quad (V.25)
\end{equation}

where in the last equality we have used our expression (V.9) for $\xi_\perp$.

Note that the ratio of $\xi_x$ to $\xi_\perp$ implied by (V.25) depends only on $g_{10,20}$ and $\mu_{x0,10}$, and not at all on the exact choice of the functional dependence rescaling exponent $\zeta(\ell)$ on $\ell$: any choice that leads to $\mu_x(\ell^*) = \mu_1(\ell^*)$ gives the same answer.

A similar argument can be applied to the time scale $\tau$. We start by solving the recursion relation (V.19) for $\mu_1(\ell^*)$:

\begin{equation}
\ln \left( \frac{\mu_{10}}{\mu_{10}} \right) = \int_0^{\ell^*} z(\ell) d\ell + 2\ell^* + \Phi_{\mu_1}(g_{10}, g_{20}) \quad (V.26)
\end{equation}

where we’ve defined

\begin{equation}
\Phi_{\mu_1}(g_{10}, g_{20}) \equiv \int_0^{\ell^*} Y_1(g_1(\ell), g_2(\ell)) d\ell'. \quad (V.27)
\end{equation}

Note that, like $\Phi$, $\Phi_{\mu_1}$ is completely determined by the bare values $g_{10,20}$, and is independent of the arbitrary choice of the rescaling exponents $\chi(\ell)$, $\zeta(\ell)$, and $z(\ell)$ for the same reasons as before: both integrand and the limits
of integration in (V.27) depend only on $g_{10,20}$. Solving (V.26) for $\int_0^{\ell_*} z(\ell) d\ell$ gives

$$\int_0^{\ell_*} z(\ell) d\ell = \ln \left( \frac{\mu_{\text{ref}}}{\mu_{10}} \right) + 2\ell^* - \Phi_{\mu_1}(g_{10}, g_{20}).$$  (V.28)

Inserting this result into (V.17), and using (V.9) and (V.12) gives

$$\tau = \frac{\xi^2}{\mu_{10}} e^{-\Phi_{\mu_1}}.$$  (V.29)

If the bare parameter $g_{10}$ is small, then, up to factors of $O(1)$, we can take $\Phi$ and $\Phi_{\mu_1}$ to be zero, which reduces (V.25) to

$$\xi_x = \xi_\perp \sqrt{\frac{\mu_{x0}}{\mu_{10}}},$$  (V.30)

and (V.29) to

$$\tau = \frac{\xi^2_x}{\mu_{10}}.$$  (V.31)

We can also determine $\xi_\perp$ in this limit by noting that, for small $g_1$, the recursion relation (IV.18) for $g_1$ becomes simply

$$\frac{dg_1}{d\ell} = \epsilon g_1,$$  (V.32)

which is easily solved to give

$$g_1(\ell) = g_{10} e^{\epsilon \ell}.$$  (V.33)

Setting $g_1(\ell^*) = 1$ and solving for $\ell^*$ gives

$$\ell^* = (g_{10})^{-\frac{1}{\epsilon}}.$$  (V.34)

Using our expression (IV.11) for $g_1$, evaluated with the bare parameters, this gives

$$e^{\ell^*} = \Lambda \left( \frac{\mu_{x0} \mu_{10}^5}{D_0^2 \lambda_0^4} \right)^{\frac{1}{\epsilon}}.$$  (V.35)

Using this in turn in (V.9) gives

$$\xi_\perp = \left( \frac{\mu_{x0} \mu_{10}^5}{D_0^2 \lambda_0^4} \right)^{\frac{1}{\epsilon}}.$$  (V.36)

Note that $\xi_\perp$ is independent of the ultraviolet cutoff $\Lambda$ in this case, which is to be expected, since the divergent renormalization of the parameters is an infrared phenomenon.

In $d = 3$, where $\epsilon = 1$, this becomes

$$\xi_\perp = \sqrt{\frac{\mu_{x0} \mu_{10}^5}{D_0 \lambda_0^2}}, \quad d = 3.$$  (V.37)

Using this in (V.30) and (V.31) gives respectively

$$\xi_x = \frac{\mu_{x0} \mu_{10}^2}{D_0 \lambda_0^2}, \quad \tau = \frac{\mu_{x0} \mu_{10}^4}{D_0 \lambda_0^4}, \quad d = 3.$$  (V.38)

In $d = 2 (\epsilon = 2)$, we obtain

$$\xi_\perp = \left( \frac{\mu_{x0} \mu_{10}^5}{D_0 \lambda_0^2} \right)^{\frac{1}{2}}, \quad d = 2,$$  (V.39)

and

$$\xi_x = \frac{\mu_{x0} \mu_{10}^2}{\sqrt{2} D_0 \lambda_0}, \quad \tau = \frac{\mu_{x0} \mu_{10}^4}{\sqrt{2} D_0 \lambda_0^2}, \quad d = 2.$$  (V.40)

We now turn to the last remaining concern about the scaling form of the correlation function $C_u$. We will now show that this is also independent of the arbitrary rescaling choices, and we’ll also calculate it for small bare coupling $g_{10}$.

To do this, we see from (V.6) that we need to calculate the value of the integral $\int_0^\ell 2\chi(\ell) d\ell$. We can obtain this by integrating the recursion relation (IV.6) for the noise strength $D$ from $\ell = 0$ to $\ell^*$, and using the fact that $D(\ell^*) = D_{\text{ref}}$. This gives

$$\ln \left( \frac{D_{\text{ref}}}{D_0} \right) = \int_0^{\ell^*} \left[ z(\ell) - 2\chi(\ell) - \zeta(\ell) \right] d\ell + (1 - d)\ell^* + \Phi_D(g_{10}, g_{20}),$$  (V.41)

where we’ve defined

$$\Phi_D(g_{10}, g_{20}) \equiv \int_0^{\ell^*} g_1(\ell) G_D(g_2(\ell)) d\ell,$$  (V.42)

which again, like all of our $\Phi$‘s, depends only on the bare couplings $g_{10,20}$.

Solving (V.41) for the integral $\int_0^{\ell^*} 2\chi(\ell) d\ell$ gives
\[
\int_0^{\ell^*} 2\chi(\ell) d\ell = \int_0^{\ell^*} [z(\ell) - \zeta(\ell)] d\ell + \ln \left( \frac{D_0}{D_{\text{ref}}} \right) + (1 - d)\ell^* + \Phi_D(g_{10}, g_{20}).
\]

Using our results (V.28) and (V.24) for \( \int_0^{\ell^*} z(\ell) d\ell \) and

\[
\int_0^{\ell^*} 2\chi(\ell) d\ell = \ln \left( \frac{D_0}{\sqrt{\mu_{10}\mu_{x0}}D_{\text{ref}}} \right) + (2 - d)\ell^* + \Phi_A(g_{10}, g_{20}),
\]

where we've defined
\[
\Phi_A \equiv \Phi_D - (\Phi_{\mu_1} + \Phi).
\]

Using this and our expression (V.9) for \( \xi_\perp \) in equation (V.8) for \( A \) gives

\[
A = a^{-2\xi_{FP}} \left( \frac{\xi_\perp}{a} \right)^{(2 - d - 2\chi_{FP})} \frac{\mu_{\text{ref}}D_0}{\sqrt{\mu_{10}\mu_{x0}}D_{\text{ref}}} e^{\Phi_A}.
\]

It is clear from this expression that, as claimed earlier, the value of \( A \) depends only on the parameters of the bare model, not on the arbitrary choice of rescaling exponents.

For a model with the bare coupling \( g_{10} \ll 1 \), we can set \( \Phi_A = 0 \) in (V.40), and obtain an explicit expression for \( A \) in terms of the bare parameters:

\[
A = a^{-2\xi_{FP}} \left( \frac{\xi_\perp}{a} \right)^{(2 - d - 2\chi_{FP})} \frac{\mu_{\text{ref}}D_0}{\sqrt{\mu_{10}\mu_{x0}}D_{\text{ref}}}. \tag{V.47}
\]

Arguments virtually identical to those just presented can be used to show the scaling form of the density correlation function given by (I.13).

The lengths \( \xi_\perp \), \( \xi_\parallel \), and the time \( \tau \), have significance beyond their appearance in the scaling laws (I.12) and (I.13); they are also the 'non-linear lengths and time'. By this, we mean that, if all of the distances \( r_\perp, |x - \gamma t| \), and the time \( t \), are much smaller than the corresponding length or time - that is, if \( r_\perp \ll \xi_\perp, |x - \gamma t| \ll \xi_\parallel \), and \( t \ll \tau \), the linear theory results of section III will apply. This can be seen either by a renormalization group argument, or perturbation theory.

The renormalization group argument starts with the general trajectory integral matching expression (V.2). We then note that, if all the conditions \( r_\perp \ll \xi_\perp, |x - \gamma t| \ll \xi_\parallel \), and \( t \ll \tau \) are satisfied, we can always choose to evaluate the right hand side at a value of \( \ell < \ell^* \) at which all the arguments \( r_\perp e^{-\ell}, (x - \gamma t) \exp \left( - \int_0^{\ell} \zeta(\ell') d\ell' \right), \)

and \( t \exp \left( - \int_0^{\ell} z(\ell') \right) \) are microscopic. The \( C_u \) on the right hand side of (V.2) can then simply be treated as a finite constant since it is evaluated at short distances and times, and so will be unaffected by any infrared divergences.

We'll now illustrate this for the special case \( r_\perp \ll \xi_\perp, x = 0, t = 0 \). Again we choose \( \ell = \ln (r_\perp/a) \). We will also choose \( \chi, z, \) and \( \zeta \) to keep \( \mu_{1,2,3} \) and \( D \) fixed at their initial values. Since \( \ell \ll \ell^* \) and \( r_\perp \ll \xi_\perp, g_1(\ell) \) is small if \( g_{10} \) is small. Therefore, the graphical corrections in (IV.6, IV.9, IV.10) are negligible. Then our special choices of the scaling exponents become

\[
\chi = \frac{2 - d}{2} \quad z = 2, \quad \zeta = 1. \tag{V.48}
\]

Inserting these exponents and \( \ell = \ln (r_\perp/a) \) into (V.2) we obtain

\[
C_u \left( r_\perp, 0, 0; \left\{ D_0, \mu_{x0}, \mu_{10}, \mu_{20} \right\} \right)
= \left( \frac{r_\perp}{a} \right)^{2 - d} C_u \left( a, 0, 0; \left\{ D_0, \mu_{x0}, \mu_{10}, \mu_{20} \right\} \right)
\propto r_\perp^{2 - d}, \tag{V.49}
\]

which agrees with the linear theory (III.36). This argument can be easily extended to correlation functions with more general spatial and temporal separations, and to the density-density correlation function. Therefore, the conclusion that the linear theory results of section (III), in particular equations (III.36) and (III.47) for the velocity-density and density-density correlation functions, hold if all distances and times are short compared to the corresponding non-linear lengths or times; that is, if the conditions \( r_\perp \ll \xi_\perp, |x - \gamma t| \ll \xi_\parallel \), and \( t \ll \tau \) are satisfied.

2. Perturbation theory approach

In this subsection, we present the perturbation theory approach to the calculation of the non-linear lengths and time. We remind the reader that we do this by calculating perturbative corrections to the linear theory and finding the length and time scales on which they become appreciable. These prove to be precisely the lengths \( \xi_\perp \), and \( \xi_\parallel \), and the time \( \tau \) that we have just derived from the RG approach, thereby confirming the validity of that approach.

The perturbation theory calculation is very similar to the step 1 of the DRG procedure which we described in section (V.B) and can also be represented by graphs.
Here we focus on the correction to $\mu_1$ obtained from one particular graph, Fig. 3a, which we have evaluated in detail in appendices B1a and A1a. Using different one-loop graphs, or considering renormalization of different parameters, will lead to the same estimates of the non-linear lengths and times, up to factors of $O(1)$. We also simplify our calculation by considering the case $\mu_2 = 0$: taking a non-zero $\mu_2$ only modifies the lengths $\xi_\perp$, and $\xi_\|$, and the time $\tau$ by an $O(1)$ multiplicative factor.

Our strategy is to crudely estimate the non-linear lengths $\xi_\perp$, $\xi_\|$ and the non-linear time $\tau$ by using their inverses as infra-red cutoffs of the infra-red divergent integrals that appear in a perturbation theory calculation of the renormalized $\mu_1$. We’ll then determine the values of $\xi_\perp$ and $\xi_\|$, and $\tau$, as the values of these cutoffs for which the correction to $\mu_1$ becomes comparable to its bare value $\mu_{10}$. As mentioned earlier, we would get the same values for $\xi_\perp$, $\xi_\|$, and $\tau$ had we chosen to apply this logic to one of the parameters (i.e., $D$ or $\mu_\|$) instead.

The graph Fig. 3a represents a correction to $\partial_t u_\perp$ of the form

$$ \Delta (\partial_t u_\perp)_{\mu,a} = -2D_0\lambda^2 k_\perp u_\perp (\tilde{k}) \int_q (k_\perp - q_\perp^R) C_{iu}(\tilde{q}) G_{jc}(\tilde{k} - \tilde{q}) $$

$$ \equiv -2D_0\lambda^2 k_\perp u_\perp (\tilde{k}) \left[ (I_1^{\mu,a})_{cju}(\tilde{k}) - (I_2^{\mu,a})_{cju}(\tilde{k}) \right], \quad \text{(V.50)} $$

where $\tilde{k} = (\omega, \tilde{k})$, $\tilde{q} = (\Omega, q)$,

$$(I_1^{\mu,a})_{cju}(\tilde{k}) \equiv \frac{k_\perp}{(2\pi)^{d+1}} \int_q C_{iu}(\tilde{q}) G_{jc}(\tilde{k} - \tilde{q}), \quad \text{(V.51)} $$

$$(I_2^{\mu,a})_{cju}(\tilde{k}) \equiv \frac{1}{(2\pi)^{d+1}} \int_q q_\perp C_{iu}(\tilde{q}) G_{jc}(\tilde{k} - \tilde{q}), \quad \text{(V.52)} $$

$$ G_{jc}(\tilde{q}) \equiv G_T(\tilde{q}) \delta_{jc} $$

$$ \equiv \frac{\delta_{jc}}{-i\Omega + \mu_{10} q_\perp^2 + \mu_{x0} q_\parallel^2}, \quad \text{(V.53)} $$

$$ C_{iu}(\tilde{q}) \equiv |G_T(\tilde{q})|^2 \delta_{iu} $$

$$ = \frac{\delta_{iu}}{\Omega^2 + (\mu_{10} q_\perp^2 + \mu_{x0} q_\parallel^2)}, \quad \text{(V.54)} $$

and the superscripts “$\mu,a$” indicate that this correction comes from the renormalization of the $\mu$ terms due to Fig. 3a. Note that we have replaced all of the parameters $\lambda$, $D$, $\mu_1$, and $\mu_\|$ by their bare values $\lambda_0$, $D_0$, $\mu_{10}$, and $\mu_{x0}$, since we are now doing perturbation theory, rather than the renormalization group.

Inserting (V.53, V.54) into (V.51, V.52) we get

$$(I_1^{\mu,a})_{cju}(\tilde{k}) = \frac{k_\perp}{(2\pi)^{d+1}} \int_q |G_T(\tilde{q})|^2 G_T(\tilde{k} - \tilde{q}) $$

$$ (I_2^{\mu,a})_{cju}(\tilde{k}) = \frac{\delta_{jc}}{(2\pi)^{d+1}} \int_q q_\perp |G_T(\tilde{q})|^2 G_T(\tilde{k} - \tilde{q}) $$

Since (V.50) has already a factor $k_\perp^2$ in front of it, and we are only interested in terms of $O(k^2)$ (since only these will be relevant at small $k$, that being the order of the $\mu_1$ terms in the EOM), to get relevant contributions to the linear terms of the EOM we can set the external frequency $\omega = 0$ in $(I_1^{\mu,a},2)_{cju}(\tilde{k})$, and expand both of them to $O(k)$. This gives

$$ (I_1^{\mu,a})_{cju}(\tilde{k}) = \frac{k_\perp}{(2\pi)^{d+1}} \int_q |G_T(\tilde{q})|^2 G_T(-\tilde{q}) = \frac{k_\perp}{(2\pi)^{d+1}} \int_q \left( \frac{1}{\Omega^2 + (\mu_{10} q_\perp^2 + \mu_{x0} q_\parallel^2)^2} \right) |G_T(\tilde{q})|^2 G_T(-\tilde{q}), \quad \text{(V.55)} $$

$$ (I_2^{\mu,a})_{cju}(\tilde{k}) = \frac{2\mu_{10} k_\perp}{(2\pi)^{d+1}} \int_q q_\perp k_\perp^2 |G_T(\tilde{q})|^2 |G_T(\tilde{q})|^2 = \frac{2\mu_{10} k_\perp}{(2\pi)^{d+1}} \int_q \left( \frac{q_\perp q_\parallel^2}{\Omega^2 + (\mu_{10} q_\perp^2 + \mu_{x0} q_\parallel^2)^2} \right)^2. \quad \text{(V.57)} $$

To calculate the non-linear length along $\perp$ directions, $\xi_\perp$, we impose an infra-red cutoff $|q|_{\text{min}} = \xi_\perp^{-1}$ on the $q_\perp$
The integrals in this expression. That is, we define

$$\int_{\Omega} = \int_{-\infty}^{\infty} d\Omega \int_{-\infty}^{\infty} dq_x \int_{\Lambda > |q_x| > \xi^{-1}} d^{d-1}q \quad \text{(V.59)}$$

The integrals over $\Omega$ and $q_x$ in (V.57) and (V.58) are straightforward, particularly if done in that order (i.e., integrating first over $\Omega$, then over $q_x$). The results are:

$$(I_1^{\mu,a})_{cju}(k) = \frac{k_u \delta^{\perp}_{\perp}}{16\sqrt{\mu_0 \mu_0^+} (2\pi)^{d-1}} \int_{\Lambda > |q_x| > \xi^{-1}} \frac{d^{d-1}q \perp}{q \perp} \quad \text{(V.60)}$$

and

$$(I_2^{\mu,a})_{cju}(k) = \frac{3k_u \delta^{\perp}_{\perp}}{32\sqrt{\mu_0 \mu_0^+} (2\pi)^{d-1}} \int_{\Lambda > |q_x| > \xi^{-1}} \frac{d^{d-1}q \perp}{q \perp} \quad \text{(V.61)}$$

where, in the penultimate equality, we have used

$$\int d\zeta_{\perp} q_x^{\perp} q_x^{\perp} = S_{d-1} \frac{q_x^{\perp}}{d-1} \quad \text{(V.62)}$$

where $d\zeta_{\perp}$ denotes an integral over the $(d-1)$-dimensional solid angle associated with $q_x$. In the ultimate equality, we have used $\xi^{-1} \ll \Lambda$.

Inserting (V.60) (V.61) into (V.50) we obtain a correction to $\partial_t u_j$ given by

$$\Delta(\partial_t u_j)_{\mu,a} = -\frac{D_0 \lambda_0^2}{\sqrt{\mu_0 \mu_0^+}} \frac{S_{d-1}}{(2\pi)^{d-1}} \left[ \frac{1}{8} - \frac{3}{16(d-1)} \right] k_u^2 u_j \quad \text{(V.63)}$$

where the factor of 2 takes into account the fact that the Brillouin zone in $q_x$, with this infrared cutoff, consists of two disjoint sections, one running from $\xi^{-1}$ to $\infty$, the other running from $-\infty$ to $-\xi^{-1}$. These two regions make exactly equal contributions; hence the factor of 2 above.

Then we obtain for the integrals in (V.57) (V.58):

$$(I_1^{\mu,a})_{cju}(k) = \frac{k_u \delta^{\perp}_{\perp}}{2(2\pi)^{d-1}} \int_{\xi^{-1}}^{\infty} dq_x \int \frac{d^{d-1}q \perp}{(\mu_0 q_x^2 + \mu_0^+ q_x^2)^2} \quad \text{(V.64)}$$

and

$$\int_{\Omega} = 2 \int_{-\infty}^{\infty} d\Omega \int_{|q_x| < \infty} d^{d-1}q \perp \int_{\xi^{-1}}^{\infty} dq_x \quad \text{(V.66)}$$

From the form of this correction (i.e., the fact that it is proportional to $k_u^2 u_j$), we recognize this as a perturbative correction to $\mu_1$:

$$(\Delta \mu_1)_{\mu,a} = \frac{D_0 \lambda_0^2}{\sqrt{\mu_0 \mu_0^+}} \frac{S_{d-1}}{(2\pi)^{d-1}} \left[ \frac{1}{8} - \frac{3}{16(d-1)} \right] k_u^2 u_j \quad \text{(V.65)}$$

This agrees with our earlier RG result (V.36).

To calculate the non-linear length $\xi_x$ along $x$, we now introduce $\xi_x^{-1}$ as an infrared cutoff on the integrals over $q_x$, and allow $q_x$ and $\Omega$ to run free. That is, we set the limits on our integrals as follows:

$$(H_1^{\mu,a})_{cju}(k) = \frac{k_u \delta^{\perp}_{\perp}}{2(2\pi)^d} \int_{\xi^{-1}}^{\infty} dq_x \int \frac{d^{d-1}q \perp}{(\mu_0 q_x^2 + \mu_0^+ q_x^2)^2} \quad \text{(V.64)}$$

where

$$\int d\zeta_{\perp} q_x^{\perp} q_x^{\perp} = S_{d-1} \frac{q_x^{\perp}}{d-1} \quad \text{(V.62)}$$

where $d\zeta_{\perp}$ denotes an integral over the $(d-1)$-dimensional solid angle associated with $q_x$. In the ultimate equality, we have used $\xi^{-1} \ll \Lambda$.

Inserting (V.60) (V.61) into (V.50) we obtain a correction to $\partial_t u_j$ given by

$$\Delta(\partial_t u_j)_{\mu,a} = -\frac{D_0 \lambda_0^2}{\sqrt{\mu_0 \mu_0^+}} \frac{S_{d-1}}{(2\pi)^{d-1}} \left[ \frac{1}{8} - \frac{3}{16(d-1)} \right] k_u^2 u_j \quad \text{(V.63)}$$

where the factor of 2 takes into account the fact that the Brillouin zone in $q_x$, with this infrared cutoff, consists of two disjoint sections, one running from $\xi^{-1}$ to $\infty$, the other running from $-\infty$ to $-\xi^{-1}$. These two regions make exactly equal contributions; hence the factor of 2 above.

Then we obtain for the integrals in (V.57) (V.58):

$$(I_1^{\mu,a})_{cju}(k) = \frac{k_u \delta^{\perp}_{\perp}}{2(2\pi)^d} \int_{\xi^{-1}}^{\infty} dq_x \int \frac{d^{d-1}q \perp}{(\mu_0 q_x^2 + \mu_0^+ q_x^2)^2} \quad \text{(V.64)}$$

and

$$\int_{\Omega} = 2 \int_{-\infty}^{\infty} d\Omega \int_{|q_x| < \infty} d^{d-1}q \perp \int_{\xi^{-1}}^{\infty} dq_x \quad \text{(V.66)}$$

From the form of this correction (i.e., the fact that it is proportional to $k_u^2 u_j$), we recognize this as a perturbative correction to $\mu_1$:

$$(\Delta \mu_1)_{\mu,a} = \frac{D_0 \lambda_0^2}{\sqrt{\mu_0 \mu_0^+}} \frac{S_{d-1}}{(2\pi)^{d-1}} \left[ \frac{1}{8} - \frac{3}{16(d-1)} \right] k_u^2 u_j \quad \text{(V.65)}$$

This agrees with our earlier RG result (V.36).

To calculate the non-linear length $\xi_x$ along $x$, we now introduce $\xi_x^{-1}$ as an infrared cutoff on the integrals over $q_x$, and allow $q_x$ and $\Omega$ to run free. That is, we set the limits on our integrals as follows:

$$(H_1^{\mu,a})_{cju}(k) = \frac{k_u \delta^{\perp}_{\perp}}{2(2\pi)^d} \int_{\xi^{-1}}^{\infty} dq_x \int \frac{d^{d-1}q \perp}{(\mu_0 q_x^2 + \mu_0^+ q_x^2)^2} \quad \text{(V.64)}$$

where $\int d\zeta_{\perp} q_x^{\perp} q_x^{\perp} = S_{d-1} \frac{q_x^{\perp}}{d-1} \quad \text{(V.62)}$
and
\[(P_d,a)_{\gamma} \tilde{k} (\tilde{k}) = \frac{\mu_0}{16} \frac{e^{-\frac{1}{4}}}{(2\pi)^{d+1}} \int_{\xi^{-1}}^{\infty} dq_x \left| q_{+} Q_{+} \right|^3 \left( \frac{q_{+} Q_{+}}{\mu_{10} Q_{+}^2 + \mu_{10} Q_{+}^2} \right)^{\frac{1}{2}} dq_x \]  

where \( H_{1,2}(d) \) are finite, \( O(1) \) constants given by
\[H_{1}(d) = \int_{0}^{\infty} \frac{y^{d-2} dy}{(1 + y^2)^3} = \frac{\pi}{4} (d - 3) \sec \left( \frac{\pi d}{2} \right), \]  

and
\[H_{2}(d) = \int_{0}^{\infty} \frac{y^{d} dy}{(1 + y^2)^3} = \frac{\pi}{16} (d - 3) (1 - d) \sec \left( \frac{\pi d}{2} \right). \]  

Note that, appearances to the contrary, neither \( H_{1}(d = 3) \) nor \( H_{2}(d = 3) \) vanishes; instead \( H_{1}(3) = 1/2 \) and \( H_{2}(3) = 1/4 \), as can be verified either by taking the singular limit \( d \rightarrow 3 \) in \((V.69)\) and \((V.70)\), or by evaluating the corresponding integrals in exactly \( d = 3 \).

Inserting \((V.67)-(V.68)\) into \((V.50)\) we obtain the perturbative correction to \( \mu_{1} \):
\[(\Delta_{\mu_{1}})_{\mu_{a}} = 8 D_{0} \frac{\lambda_0}{\mu_{10}^2} \frac{\xi_{x}}{\tau} \frac{1}{(2\pi)^{d+1}} \frac{\tau}{4 - d} \frac{d - 2}{d} H_{3}(d). \]  

Equating this to \( \mu_{10} \) gives
\[(\xi_{x} = \left( \frac{\mu_{0} \mu_{10}}{D_{0} \lambda_0^2} \right)^{\frac{1}{d-1}} \frac{\mu_{x}}{\mu_{10}} \frac{\tau}{4 - d} \frac{1}{O(1)} = \xi_{x} \frac{\mu_{0}}{\mu_{10}} \times O(1), \]

which agrees with our earlier RG result \((V.30)\).

Now we turn to the non-linear time scale \( \tau \). We impose a lower limit \( 1/\tau \) on the frequency integral in \((V.50)\) and let the wave vectors be completely free:
\[\int_{\tilde{q}_{x}} = 2 \int_{T^{-1}}^{\infty} d\Omega \int_{q_{\perp} < \infty} d^{d-1} q_{\perp} \int_{-\infty}^{\infty} dq_x, \]  

where, much as in our treatment of the integral over \( q_{x} \) earlier, the factor of 2 takes into account the fact that the region of integration over \( \Omega \), with this infrared cutoff, consists of two disjoint sections, one running from \( -\tau^{-1} \) to \( \infty \), the other running from \( -\infty \) to \( -\tau^{-1} \). These two regions also make exactly equal contributions; hence the factor of 2 above.

In this case we do the integral over wave vectors first. We obtain
\[(P_{d,a})_{\gamma} \tilde{k} (\tilde{k}) = \frac{2H_{3}(d)}{(2\pi)^{d+1}} \frac{\tau}{4 - d} \frac{d - 2}{d} \int_{\frac{1}{4}}^{\infty} \omega^{2/3 - d} d\omega \]
and
\[\left( P_{d,a} \right)_{\gamma} \tilde{k} (\tilde{k}) = \frac{4H_{3}(d)}{(2\pi)^{d+1}} \frac{\tau}{4 - d} \frac{d - 2}{d} \int_{\frac{1}{4}}^{\infty} \omega^{2/3 - d} d\omega \]
where \( H_{3}(d) \) is a finite, \( O(1) \) constant given by
\[H_{3}(d) = \frac{Q^{2} d^{d} Q}{16 (1 + Q^{4})^{2}} = \frac{2 - d}{16} \frac{\pi d}{4} S_{d} \pi \sec \left( \frac{\pi d}{4} \right). \]

Inserting \((V.74)-(V.75)\) into \((V.50)\) we obtain the perturbative correction to \( \mu_{1} \):
\[\left( \Delta_{\mu_{1}} \right)_{\mu_{a}} = 8 D_{0} \frac{\lambda_0}{\mu_{10}^2} \frac{\xi_{x}}{\tau} \frac{1}{(2\pi)^{d+1}} \frac{\tau}{4 - d} \frac{d - 2}{d} H_{3}(d). \]  

Equating this to \( \mu_{10} \) gives
\[\xi_{x} = \left( \frac{\mu_{0} \mu_{10}}{D_{0} \lambda_0^2} \right)^{\frac{1}{d-1}} \frac{\mu_{x}}{\mu_{10}} \times O(1) = \xi_{x} \frac{\mu_{0}}{\mu_{10}} \times O(1), \]

which agrees with \((V.31)\). This completes our calculation of the crossover length and time scales between linear and non-linear theories using perturbation theory.

C. Universal amplitude ratio

The fact that there is a fixed point value of \( g_{2} \), even if it’s not zero at higher loop orders, implies an experimentally observable universal amplitude ratio. Specifically, it is the ratio of the damping of the transverse and the longitudinal modes, obtained as follows. For the longitudinal mode, we look at the equal-time correlation:
\[C_{L}(t = 0, k) = \frac{D(k)}{\mu_{L}(k)k_{x}^2 + \mu_{L}(k)k_{x}^2}, \]  

where we have explicitly shown the dependencies of the coefficients on the wavevector \( k \). A similar expression can be obtained for \( C_{T} \).

If one considers a generic direction of \( k \) (i.e., any direction for which \( k_{x}^2 \geq k_{y} \), we can ignore the second term in the above denominator and the ratio of \( C_{T}(t = 0, k)/C_{L}(t = 0, k) \) is thus
\[\lim_{k \rightarrow 0} \frac{\mu_{L}(k)}{\mu_{L}(k)} = 1 + g_{2}^2 = 1 + O(e^2), \]

which is a universal number.
D. Separatrix between positive and negative density correlations

In section III C we have shown using linear theory that the sign of the equal time density correlation function depends on the spatial difference between the two correlating points \( \mathbf{r} \). Specifically, in \( \mathbf{r} \)-space the positive and the negative regions of the density correlations are separated by a cone-shaped locus given by (III.44), which, up to a \( O(1) \) factor, can be rewritten in term of the non-linear propagators as

\[
|x| \approx \frac{r_\perp}{\xi_x} \quad (\text{V.81})
\]

For \( x/\xi_x \gg r_\perp/\xi_\perp \) the density correlations are positive, while for \( x/\xi_x \ll r_\perp/\xi_\perp \) they become negative. This result only holds for small distances (i.e., \( x \ll \xi_x \) and \( r_\perp \ll \xi_\perp \)) since linear theory is only valid at short length scales.

At large distances we expect a similar separatrix in \( \mathbf{r} \)-space which separates the regions with different signs of the density correlations. The scaling form of these correlations (I.13) shows that sign of the equal time correlations is determined by the ratio \( |x|/\xi_x \) instead of \( (r_\perp/\xi_\perp)^2 \). This implies at large distances (i.e., \( x \gg \xi_x \) or \( r_\perp \gg \xi_\perp \)) the positive and the negative density correlations are separated by a locus given by

\[
|x| \approx \left( \frac{r_\perp}{\xi_\perp} \right)^\zeta \quad (\text{V.82})
\]

Note that (V.81) and (V.82) connect right at \( |x| = \xi_x \).

The regions in \( \mathbf{r} \)-space with different signs of the density correlations are illustrated in Fig. 1.

VI. SUMMARY & OUTLOOK.

Focusing on the ordered phase of a generic Malthusian flock in dimensions \( d > 2 \), we have used dynamic renormalization group analysis to reveal a novel universality class that describes the system’s hydrodynamic properties. In particular, the predicted scaling exponents were shown to converge to the known exact result in 2D. Our work highlights another instance in which an active system can be fundamentally different from known equilibrium and non-equilibrium systems. Looking ahead, we believe that much analytical work is needed to verify whether novel universality class indeed underlie some of the phenomenology reported from simulation work \[26\]–[29].

Appendix A: One-loop RG calculation with \( \mu_2 \) set to zero

In this appendix, we derive the dynamical renormalization group recursion relations for the special case of \( \mu_2 = 0 \). This restriction immensely simplifies the calculation, by making the propagators \( G_{ij} \) and correlation functions \( C_{ij} \) diagonal. It also proves to be sufficient to explore this region, since it contains the only stable fixed point in the problem, which we can find, and the exponents of which we can calculate, using this restricted approach. However, to demonstrate the stability of this fixed point against non-zero \( \mu_2 \), and to show that it is the only stable fixed point (and, indeed, the only fixed point other than the unstable Gaussian fixed point), it is necessary to extend these calculations to non-zero \( \mu_2 \), which we do in the next Appendix.

For \( \mu_2 = 0 \), the EOM simplifies to:

\[
-i\omega \mathbf{u}_i^\perp = - \left( \mu_1 k_\perp^2 + \mu_x k_x^2 \right) \mathbf{u}_i^\perp - \frac{i\lambda}{\sqrt{2\pi}} \int_{\mathbf{q}} \left[ \mathbf{u}_\perp (\mathbf{q}) \cdot (\mathbf{k}_\perp - \mathbf{q}_\perp) \right] \mathbf{u}_i^\perp (\mathbf{k} - \mathbf{q}) + f_i^\perp \quad (A.1)
\]

We can formally solve this equation for \( \mathbf{u} \), which gives

\[
\mathbf{u}_i^\perp (\mathbf{k}) = G_{ij}(\mathbf{k}) \left\{ f_j^\perp (\mathbf{k}) - \frac{i\lambda}{\sqrt{2\pi}} \int_{\mathbf{q}} \left[ \mathbf{u}_\perp (\mathbf{k}_\perp - \mathbf{q}_\perp) \cdot \mathbf{q}_\perp \right] \mathbf{u}_j^\perp (\mathbf{q}) \right\} , \quad (A.2)
\]

where

\[
G_{ij}(\mathbf{k}) = G_T(\mathbf{k}) \delta_{ij} = \frac{\delta_{ij}^\perp}{-i\omega + \mu_1 k_\perp^2 + \mu_x k_x^2} \quad (A.3)
\]

is diagonal, as noted above.

We now apply the dynamical renormalization group procedure of [21] to this equation. As described in section IV, the first step of this procedure consists of averaging the above solution over the short wavelength components \( f^\perp (\mathbf{k}) \) of the noise \( f \), which gives a closed EOM for \( \mathbf{u}_\perp (\mathbf{k}) \). This step can be represented by graphs. The basic rules for the graphical representation are illustrated in Fig. 2. We will now evaluate these graphs, each of which can be interpreted as adding a term to the equation of motion for \( \mathbf{u}_\perp (\mathbf{k}) \). We begin with the graph in Fig. 3(a).
1. Renormalizations of the $\mu$'s

a. Graph in Fig. 3(a)

The graph in Fig. 3(a) gives a contribution $\Delta(\partial_t u_j^<)_{\mu,a}$ to the EOM for $u_j^<$ ($k$):

$$\Delta(\partial_t u_j^<)_{\mu,a} = -\frac{2D\lambda^2 k_u^+ u_j^<(k)}{(2\pi)^{d+1}} \int_{\tilde{q}}^> (k_i^+ - q_i^+) C_{iu}(\tilde{q}) G_{jc}(\tilde{k} - \tilde{q}) \equiv -2D\lambda^2 k_u^+ u_j^<(k) \left( (I_1^{\mu,a})_{cu}^{\tilde{q}}(\tilde{k}) - (I_2^{\mu,a})_{cu}^{\tilde{q}}(\tilde{k}) \right),$$

(A.4)

where

$$\int_{\tilde{q}}^> \equiv \int_{\Lambda > |q_\perp| > \Lambda e^{-\epsilon t}} d^{d-1}q_\perp \int_{-\infty}^\infty d\Omega \int_{-\infty}^\infty dq_x,$$

(A.5)

$$C_{iu}(\tilde{q}) \equiv | G_T(\tilde{q}) |^2 \delta_{iu} = \frac{\delta_{iu}}{\omega^2 + (\mu_1 q_\perp^2 + \mu_x q_z^2)^2},$$

(A.6)

$$(I_1^{\mu,a})_{cu}^{\tilde{q}}(\tilde{k}) \equiv -\frac{k_i^+}{(2\pi)^{d+1}} \int_{\tilde{q}}^> C_{iu}(\tilde{q}) G_{jc}(\tilde{k} - \tilde{q}),$$

(A.7)

$$(I_2^{\mu,a})_{cu}^{\tilde{q}}(\tilde{k}) \equiv \frac{1}{(2\pi)^{d+1}} \int_{\tilde{q}}^> q_i^+ C_{iu}(\tilde{q}) G_{jc}(\tilde{k} - \tilde{q}) \cdot \ldots,$$

(A.8)

Since we are interested in terms only up to $O(k^2)$, since that is the order of the $\mu$ terms in the EOM, and we already have an explicit factor $k_u^+$, we need only expand these integrals $(I_1^{\mu,a})_{cu}^{\tilde{q}}$ up to linear order in $k$. Doing so, and inserting (A.3), (A.5), and (A.6) into (A.7) and (A.8), we obtain to this order

$$(I_1^{\mu,a})_{cu}^{\tilde{q}}(\tilde{k}) = \frac{k_i^+ \delta_{jc}}{(2\pi)^{d+1}} \int_{\tilde{q}}^> | G_T(\tilde{q}) |^2 G_T(\tilde{k} - \tilde{q}) = \frac{k_i^+}{16} \frac{D\lambda^2}{\sqrt{\mu_x \mu_1}} \frac{S_{d-1}}{(2\pi)^{d-1}} \Lambda^{d-4} d\ell,$$

(A.9)

$$(I_2^{\mu,a})_{cu}^{\tilde{q}}(\tilde{k}) = \frac{\delta_{jc}}{(2\pi)^{d+1}} \int_{\tilde{q}}^> q_i^+ | G_T(\tilde{q}) |^2 G_T(\tilde{k} - \tilde{q})$$

$$= \frac{\delta_{jc}}{(2\pi)^{d+1}} \int_{\tilde{q}}^> q_i^+ | G_T(\tilde{q}) |^2 [G_T(\tilde{q})]^2 (2\mu_1 q_\perp \cdot k_\perp + 2\mu_x q_x k_x)$$

$$= \frac{3k_i^+ \delta_{jc}}{64(d-1)} \frac{D\lambda^2}{\sqrt{\mu_x \mu_1}} \frac{S_{d-1}}{(2\pi)^{d-1}} \Lambda^{d-4} d\ell,$$

(A.10)

where we have only kept terms up to $O(k)$. In the last equality in (A.10), we have used the fact that the second ($2\mu_x$) term in the parenthesis is odd in $q_x$, and so integrates to zero. We have also evaluated the first term by using the fact that

$$\int d\Omega_{\perp} q_\perp^+ k_\perp^+ = \int d\Omega_{\perp} q_\perp^+ q_\perp^+ \cdot k_\perp^+ = S_{d-1} \delta_{\perp} k_\perp^+ \frac{q_\perp^2}{d-1} = S_{d-1} k_u^+ \frac{q_\perp^2}{d-1},$$

(A.11)

where $\int d\Omega_{\perp} q_\perp$ denotes an integral over the $d-1$-dimensional solid angle associated with $q_\perp$.

Inserting (A.9) and (A.10) into (A.4), we find:

$$\Delta(\partial_t u_j^<)_{\mu,a} = -\left( \frac{1}{32} \frac{D\lambda^2}{\sqrt{\mu_x \mu_1}} \frac{S_{d-1}}{(2\pi)^{d-1}} \Lambda^{d-4} d\ell \left[ 4 - \frac{3}{(d-1)} \right] \right) = -\left[ \frac{1}{8} - \frac{3}{32(d-1)} \right] (g_1 \mu_1 d\ell) k^2 u_j^<$$

(A.12)

where in the second equality we have used our earlier definition (IV.11) of $g_1$. Since this contribution to $\partial_t u_j^<$ has the same form as the $\mu_1$ term already present, we can absorb it into a renormalization of $\mu_1$:

$$(\delta \mu_1)_{\mu,a} = \frac{1}{32} \frac{D\lambda^2}{\sqrt{\mu_x \mu_1}} \frac{S_{d-1}}{(2\pi)^{d-1}} \Lambda^{d-4} d\ell \left[ 4 - \frac{3}{(d-1)} \right] = \left[ \frac{1}{8} - \frac{3}{32(d-1)} \right] g_1 \mu_1 d\ell.$$

(A.13)
b. Graph in Fig. 3(b)

The graph in Fig. 3(b) gives a contribution $\Delta(\partial_t u_j^\varphi)_{\mu,b}$ to the EOM for $u_j^\varphi(\tilde{k})$:

$$
\Delta(\partial_t u_j^\varphi)_{\mu,b} = -\frac{2\lambda^2 Dk_\mu^+ u_j^\varphi(\tilde{k})}{(2\pi)^{d+1}} \int_\mathbb{R}^d q_\mu^+ C_{ju}(\tilde{q})G_{ic}(\tilde{k} - \tilde{q}) = -2\lambda^2 Dk_\mu^+ u_j^\varphi(\tilde{k}) (I_{\mu,b}^{\varphi})_{cju}(\tilde{k}) ,
$$

(A.14)

where

$$(I_{\mu,b}^{\varphi})_{cju}(\tilde{k}) = \frac{1}{(2\pi)^{d+1}} \int_\mathbb{R}^d q_\mu^+ C_{ju}(\tilde{q})G_{ic}(\tilde{k} - \tilde{q}) .
$$

(A.15)

Inserting (A.3, A.6) into (A.15) we get

$$(I_{\mu,b}^{\varphi})_{cju}(\tilde{k}) = \frac{\delta_{\mu,j}}{(2\pi)^{d+1}} \int_\mathbb{R}^d q_\mu^+ | G_T(\tilde{q}) | 2 G_T(\tilde{k} - \tilde{q})$$

$$= \frac{\delta_{\mu,j}}{(2\pi)^{d+1}} \int_\mathbb{R}^d q_\mu^+ | G_T(\tilde{q}) | 2 | G(-\tilde{q}) |^2 (2\mu_1 q_\perp \cdot k_\perp + 2\mu_2 q_\perp k_x)
$$

$$= \frac{3\delta_{\mu,j} k_\perp^2}{64(d-1)} \frac{D\lambda^2}{\mu_2} S_{d-1} \Lambda^{d-4} d\ell .
$$

(A.16)

where we have only kept terms up to $O(k)$, and we have again used (A.11) to evaluate the angular integrals, and thrown out odd terms that integrate to zero.

Inserting (A.17) into (A.14) we obtain a modification to the equation of motion for $u_j^\varphi$:

$$\Delta(\partial_t u_j^\varphi)_{\mu,b} = -\left(\frac{3}{32(d-1)} \frac{D\lambda^2}{\mu_2} \frac{S_{d-1}}{(2\pi)^{d-1}} \Lambda^{d-4} d\ell \right) k_\perp^2 k_\perp u_j^\varphi .
$$

(A.18)

From the form of this correction (namely, the fact that it is proportional to $k_\perp^2 k_\perp u_j^\varphi$), we can identify this as a correction to $\mu_2$ (which we remind the reader is the parameter whose bare value we took to be zero):

$$(\delta\mu_2)_{\mu,b} = \frac{3}{32(d-1)} \frac{D\lambda^2}{\mu_2} \frac{S_{d-1}}{(2\pi)^{d-1}} \Lambda^{d-4} d\ell .
$$

(A.19)

Thus, it would appear at this point that, even starting as we have with a model in which $\mu_2 = 0$, we generate a non-zero $\mu_2$ upon renormalization. This proves to not be the case, at least to one loop order. Instead, to this order, (A.19) is exactly cancelled by other graphs, as we shall now see.

c. Graph in Fig. 3(c)

The graph in Fig. 3(c) gives a contribution $\Delta(\partial_t u_j^\varphi)_{\mu,c}$ to the EOM for $u_j^\varphi(\tilde{k})$:

$$
\Delta(\partial_t u_j^\varphi)_{\mu,c} = \frac{2\lambda^2 Du_j^\varphi(\tilde{k})}{(2\pi)^{d+1}} \int_\mathbb{R}^d (k_\mu^+ - q_\mu^+) q_\mu^+ C_{i\ell}(\tilde{q})G_{j\ell}(\tilde{k} - \tilde{q}) \equiv 2\lambda^2 Du_j^\varphi(\tilde{k}) \left[ (I_{1}^{\mu,c})_{j\ell}(\tilde{k}) + (I_{2}^{\mu,c})_{j\ell}(\tilde{k}) \right] ,
$$

(A.20)

where

$$(I_{1}^{\mu,c})_{j\ell}(\tilde{k}) = \frac{k_\mu^+}{(2\pi)^{d+1}} \int_\mathbb{R}^d q_\mu^+ C_{i\ell}(\tilde{q})G_{j\ell}(\tilde{k} - \tilde{q})
$$

(A.21)

$$(I_{2}^{\mu,c})_{j\ell}(\tilde{k}) = -\frac{1}{(2\pi)^{d+1}} \int_\mathbb{R}^d q_\mu^+ q_\mu^+ C_{i\ell}(\tilde{q})G_{j\ell}(\tilde{k} - \tilde{q}) .
$$

(A.22)

Inserting (A.6, A.3) into (A.21) leads to

$$(I_{1}^{\mu,c})_{j\ell}(\tilde{k}) = \frac{k_\mu^+}{(2\pi)^{d+1}} \int_\mathbb{R}^d q_\mu^+ | G_T(\tilde{q}) | 2 G_T(\tilde{k} - \tilde{q})$$

$$= \frac{k_\mu^+}{(2\pi)^{d+1}} \int_\mathbb{R}^d q_\mu^+ | G_T(\tilde{q}) | 2 | G_T(-\tilde{q}) |^2 (2\mu_1 q_\perp \cdot k_\perp + 2\mu_2 q_\perp k_x)
$$

$$= \frac{3k_\mu^+ k_\perp^2}{64(d-1)} \frac{D\lambda^2}{\mu_2} S_{d-1} \Lambda^{d-4} d\ell .
$$

(A.23)
We deliberately leave \((I_{j}^{\mu,d})_{\mu,u}(\tilde{k})\) untouched since will show later in next section that this piece is canceled out by that from \(\text{Fig. 3(d).}\)

Inserting only this \((I_{j}^{\mu,d})_{\mu,u}\) contribution into \((A.20)\) leads to a term in the equation of motion for \(u_{j}^{\perp}\):\[
\Delta(\partial_{t}u_{j}^{\perp})_{\mu,c} = \left(\frac{3}{32(d-1)} \frac{D\lambda^{2}}{\sqrt{\mu_{a}h_{1}}}(2\pi)^{d-1} \Lambda d^{-4} d\ell\right) k_{j}^{\perp} k_{u}^{\perp} u_{u} \tag{A.24}.\]

which, as before, can be interpreted as a correction to \(\mu_{2}\):
\[
\delta\mu_{2} = -\left(\frac{3}{32(d-1)} \frac{D\lambda^{2}}{\sqrt{\mu_{a}h_{1}}}(2\pi)^{d-1} \Lambda d^{-4} d\ell\right). \tag{A.25}\]

Note that this exactly cancels the contribution to \(\mu_{2}\) from \(\text{Fig. 3(d)}\) that we just calculated.

d. Graph in \(\text{Fig. 3(d)}\)

The graph in \(\text{Fig. 3(d)}\) gives a contribution \(\Delta(\partial_{t}u_{j}^{\perp})_{\mu,d}\) to the EOM for \(u_{j}^{\perp}(\tilde{k})\):
\[
\Delta(\partial_{t}u_{j}^{\perp})_{\mu,d} = \frac{2\lambda^{2} D u_{\perp}^{+}(\bar{k})}{(2\pi)^{d+1}} \int_{\mathcal{q}\bar{q}} q_{\perp}^{+} q_{\perp}^{+} C_{j\ell}(\bar{q}) G_{\ell\ell}(\bar{k} - \bar{q}) \equiv 2\lambda^{2} D u_{\perp}^{+}(\bar{k})(I^{\mu,d})_{uj}(\bar{k}), \tag{A.26}\]

where \((I^{\mu,d})_{uj}(\bar{k}) = \frac{1}{(2\pi)^{d+1}} \int_{\mathcal{q}\bar{q}} q_{\perp}^{+} q_{\perp}^{+} C_{j\ell}(\bar{q}) G_{\ell\ell}(\bar{k} - \bar{q}).\)

This contribution cancels out the \((I_{j}^{\mu,c})_{\mu,u}\) contribution from \(\text{Fig. 3(b)}\) above exactly, leaving no correction to \(\mu_{2}\) at all, to one loop order. Thus, to this order, \(\mu_{2} = 0\) is a fixed point.

2. Noise renormalization

The graphs in \(\text{Fig. 4(a)}\) and \(\text{Fig. 4(b)}\) represent the following two corrections to the noise correlator \(\langle f_{\ell}(\bar{k}) f_{u}(-\tilde{k}) \rangle\):
\[
\Delta \langle f_{\ell}(\bar{k}) f_{u}(-\tilde{k}) \rangle_{D,a} = \frac{2\lambda^{2} D^{2}}{(2\pi)^{d+1}} \int_{\mathcal{q}\bar{q}} q_{\perp}^{+} q_{m}^{+} C_{\ell m}(\bar{k} - \bar{q}) C_{\ell u}(\bar{q}), \tag{A.28}\]
\[
\Delta \langle f_{\ell}(\bar{k}) f_{u}(-\tilde{k}) \rangle_{D,b} = \frac{2\lambda^{2} D^{2}}{(2\pi)^{d+1}} \int_{\mathcal{q}\bar{q}} q_{\perp}^{+} (k_{\perp} - q_{\perp}^{+}) C_{\ell u}(\bar{k} - \bar{q}) C_{\ell m}(\bar{q}). \tag{A.29}\]

Since the noise strength \(D\) is the value of this correlation at \(k = 0\), we set \(\bar{k} = 0\) in \((A.28, A.29)\) to get
\[
\Delta \langle f_{\ell}(\bar{k}) f_{u}(-\tilde{k}) \rangle_{D,a} = \frac{2\lambda^{2} D^{2}}{(2\pi)^{d+1}} \int_{\mathcal{q}\bar{q}} q_{\perp}^{+} q_{m}^{+} C_{\ell m}(\bar{q}) C_{\ell u}(\bar{q}), \tag{A.30}\]
\[
\Delta \langle f_{\ell}(\bar{k}) f_{u}(-\tilde{k}) \rangle_{D,b} = \frac{2\lambda^{2} D^{2}}{(2\pi)^{d+1}} \int_{\mathcal{q}\bar{q}} q_{\perp}^{+} (-q_{m}^{+}) C_{\ell u}(\bar{k} - \bar{q}) C_{\ell m}(\bar{q}). \tag{A.31}\]

Inserting our expression \((A.6)\) for the correlation function into the above two formulae we get
\[
\Delta \langle f_{\ell}(\bar{k}) f_{u}(-\tilde{k}) \rangle_{D,a} = \frac{2\lambda^{2} D^{2}}{(2\pi)^{d+1}} \delta_{\ell u} \int_{\mathcal{q}\bar{q}} q_{\perp}^{+} | G_{T}(\bar{q}) |^{4} = \frac{3}{32} \frac{D^{2} \lambda^{2}}{\sqrt{\mu_{a}h_{1}}}(2\pi)^{d-1} \Lambda d^{-4} d\ell \delta_{\ell u}, \tag{A.32}\]
\[
\Delta \langle f_{\ell}(\bar{k}) f_{u}(-\tilde{k}) \rangle_{D,b} = -\frac{2\lambda^{2} D^{2}}{(2\pi)^{d+1}} \delta_{\ell u} \int_{\mathcal{q}\bar{q}} q_{\perp}^{+} | G_{T}(\bar{q}) |^{4} = -\frac{3}{32} \frac{D^{2} \lambda^{2}}{\sqrt{\mu_{a}h_{1}}}(2\pi)^{d-1} \Lambda d^{-4} d\ell \delta_{\ell u}, \tag{A.33}\]

where in the first equality of \((A.33)\) we have again used the angular average \((C.24)\) derived in appendix \((C)\).

Adding these two pieces together, and identifying the coefficient of \(\delta_{\ell u}\) as a correction to \(D\) gives the total correction \(\delta D\) to \(D\) to one loop order:
\[
\delta D = \frac{3}{32} \left(1 - \frac{1}{d-1}\right) \frac{D^{2} \lambda^{2}}{\sqrt{\mu_{a}h_{1}}}(2\pi)^{d-1} \Lambda d^{-4} d\ell = \frac{3}{32} \left(1 - \frac{1}{d-1}\right) q_{\ell} D d\ell. \tag{A.34}\]
3. Summary of all corrections to one loop order in the $\mu_2 = 0$ limit

Adding up the results obtained in previous sections gives the total one loop graphical corrections to the various parameters when $\mu_2 = 0$:

\[
\delta \mu_1 = \left[ \frac{1}{8} - \frac{3}{32(d-1)} \right] g_1 \mu_1 d\ell, \tag{A.35}
\]
\[
\delta D = \frac{3}{32} \left( 1 - \frac{1}{(d-1)} \right) g_1 D d\ell, \tag{A.36}
\]
\[
\delta \mu_2 = 0, \tag{A.37}
\]
\[
\delta \mu_x = 0. \tag{A.38}
\]

Dividing both sides of each of these equations by $d\ell$, we obtain the graphical contributions to the recursion relations for the parameters of our model, in the special case $\mu_2 = 0$:

\[
\left( \frac{d\mu_1}{d\ell} \right)_{\text{graph}} = \left[ \frac{1}{8} - \frac{3}{32(d-1)} \right] g_1 \mu_1, \tag{A.39}
\]
\[
\left( \frac{dD}{d\ell} \right)_{\text{graph}} = \frac{3}{32} \left( \frac{d-2}{d-1} \right) g_1 D, \tag{A.40}
\]
\[
\left( \frac{d\mu_2}{d\ell} \right)_{\text{graph}} = 0, \tag{A.41}
\]
\[
\left( \frac{d\mu_x}{d\ell} \right)_{\text{graph}} = 0. \tag{A.42}
\]

The vanishing of the correction to $\mu_2$ at this one loop order, in the restricted model in which the bare $\mu_2 = 0$, tells us that at one loop order $\mu_2 = 0$ is a fixed point. It requires analysis at non-zero $\mu_2$, which we perform in the next appendix, to show that this $\mu_2 = 0$ fixed point is actually stable, and furthermore, is the only fixed point in the problem.

Since $g_2 \equiv \frac{\mu_2}{\mu_1}$ vanishes when $\mu_2 = 0$, our results (B.47-B.50) constrain the values of $G_{\mu_1,2,D}(g_2 = 0)$ in equations (IV.9), (IV.10), and (IV.6) as

\[
G_{\mu_1}(g_2 = 0) = \frac{1}{8} - \frac{3}{32(d-1)}, \tag{A.43}
\]
\[
\left( \mu_2 G_{\mu_2}(g_2 = 0) \right)_{\mu_2=0} = 0, \tag{A.44}
\]
\[
G_D(g_2 = 0) = \frac{3}{32} \left( \frac{d-2}{d-1} \right). \tag{A.45}
\]

These in turn fix the value of $G_{g_1}$ as

\[
G_{g_1}(\mu_2 = 0) = G_D(\mu_2 = 0) - \frac{5}{2} G_{\mu_1}(\mu_2 = 0) = \frac{23 - 14d}{64(d-1)}, \tag{A.46}
\]
\[
(\text{A.47})
\]

which is the value that we used in our analysis of the fixed points and exponents in section [IV]. Note that we can not, by this $\mu_2 = 0$ analysis, which only gives us equation (A.44), say anything about the behavior of $G_{g_2}(g_2) = G_{\mu_2}(g_2) - G_{\mu_1}(g_2)$ in the limit $g_2 \to 0$, other than that $G_{g_2}(g_2)g_2 \to 0$ as $g_2 \to 0$, which can be satisfied if $G_{g_2}(g_2)$ approached any finite limit as $g_2 \to 0$.

We obtain the same values for $G_{\mu_1,D}(g_2 = 0)$ and $G_{g_1}(\mu_2 = 0)$, albeit with much greater effort, by taking the tricky limit $\mu_2 \to 0$ of the recursion relations for the full problem with $\mu_2 \neq 0$, which we’ll derive in the next appendix. This provides a reassuring check on the accuracy of those far more difficult calculations, to which we now, with some trepidation, turn.

Appendix B: One-loop graphical corrections with non-zero $\mu_2$

We now turn to the calculation of the graphical corrections to the parameters for the full model with $\mu_2 \neq 0$. The reasoning of this section is exactly the same as that of the previous section; the only difference is that the algebra is
complicated by the non-zero value of $\mu_2$.

The origin of this complication lies in the more complicated form of the propagators and correlation functions. Instead of the simple, diagonal expressions (A.3) and (A.6) that we have when $\mu_2 = 0$, we now, as shown in section III, have, for the propagators:

$$G_{ij}(\hat{k}) \equiv L_{ij}(k_\perp)G_L(\hat{k}) + P_{ij}(k_\perp)G_T(\hat{k}),$$

(B.1)

with

$$G_L(\hat{k}) = \frac{1}{-i\omega + \mu_1 k_\perp^2 + \mu_x k_x^2},$$

(B.2)

$$G_T(\hat{k}) = \frac{1}{-i\omega + \mu_1 k_\perp^2 + \mu_x k_x^2},$$

(B.3)

and the longitudinal and transverse projection operators $L_{ij}^\perp(k_\perp)$ and $P_{ij}^\perp(k_\perp)$, respectively, defined as and we have defined the “longitudinal projection operator”

$$L_{ij}^\perp(k_\perp) \equiv k_i^\perp k_j^\perp / k_\perp^2,$$

(B.4)

which projects any vector along $k_\perp$, and

$$P_{ij}^\perp(k_\perp) \equiv \delta_{ij} k_i^\perp k_j^\perp / k_\perp^2,$$

(B.5)

which projects any vector onto the space orthogonal to both the mean direction of flock motion $\hat{x}$ and $k_\perp$.

We now also have similar decompositions for the correlation functions:

$$C_{ij}(\hat{k}) \equiv L_{ij}(k_\perp)|G_L(\hat{k})|^2 + P_{ij}(k_\perp)|G_T(\hat{k})|^2.$$

(B.6)

With these in hand, we’ll now calculate the graphical corrections to the various parameters in the full model with $\mu_2 \neq 0$. Note that all of the graphs are exactly the same as those we evaluated in the previous section; all that will change is their values, because we are now taking $\mu_2 \neq 0$.

1. Renormalizations of $\mu_{1,2,x}$

   a. Graph in Fig. 3(a)

The graph in Fig. 3(a) gives a contribution $\Delta(\partial_t u_j^\perp)_{\mu,a}$ to the EOM for $u_j^\perp(\hat{k})$:

$$\Delta(\partial_t u_j^\perp)_{\mu,a} = -\frac{2D\lambda^2 k_\perp^2 u_j^\perp(\hat{k})}{(2\pi)^{d+1}} \int Q^\perp(k_\perp^2 - q_i^2) C_{iu}(\hat{q}) G_{jc}(\hat{k} - \hat{q}) \equiv -2D\lambda^2 k_\perp^2 u_c^\perp(\hat{k}) \left[ (I_1^{\mu,a})_{cju}(\hat{k}) - (I_2^{\mu,a})_{cju}(\hat{k}) \right],$$

(B.7)

where

$$(I_1^{\mu,a})_{cju}(\hat{k}) \equiv \frac{k_i^\perp}{(2\pi)^{d+1}} \int Q^\perp C_{iu}(\hat{q}) G_{jc}(\hat{k} - \hat{q}),$$

(B.8)

$$(I_2^{\mu,a})_{cju}(\hat{k}) \equiv \frac{1}{(2\pi)^{d+1}} \int Q^\perp q_i^\perp C_{iu}(\hat{q}) G_{jc}(\hat{k} - \hat{q}).$$

(B.9)

Note that the overall correction already has a common factor $k_\perp$ outside the loop integral. That means for both $(I_1^{\mu,a})_{cju}$ and $(I_2^{\mu,a})_{cju}$ we can set $\omega = 0$ inside the loop integral since expanding the loop integral to $O(\omega)$ or higher orders in $\omega$ only gives terms irrelevant compared to $-i\omega u_\perp$, which is already present in the EOM (IV.1). This also means that we can obtain the renormalization of the $\mu$’s, which enter the EOM at $O(k^2)$, by setting $k = 0$ inside the integral for $(I_1^{\mu,a})_{cju}$, and expanding the integral $(I_2^{\mu,a})_{cju}$ to $O(k)$. 
Let’s calculate \((P_{1u}^{\mu})_{cju}(k)\) first. Using the integrals and angular averages given in appendix [C] we obtain

\[
(P_{1u}^{\mu})_{cju}(k) = \frac{k_1^+ K_d d \Omega}{(2\pi)^{d+1}} \int_{\mathbf{q}} \left[ G_L(\mathbf{q})^2 L_{\mu c}^{\perp}(\mathbf{q}) + |G_T(\mathbf{q})|^2 P_{\mu c}^{\perp}(\mathbf{q}) \right] \left[ G_L(-\mathbf{q})L_{jc}^{\perp}(-\mathbf{q}) + G_T(-\mathbf{q})P_{jc}^{\perp}(-\mathbf{q}) \right]
\]

\[
= \frac{k_1^+}{(2\pi)^{d-1}} \int_{\mathbf{q}} \left[ \frac{1}{16\sqrt{\mu_z \mu_L}} L_{\mu c}^{\perp}(\mathbf{q}) L_{jc}^{\perp}(\mathbf{q}) + \frac{1}{16\sqrt{\mu_z \mu_L}} \right] \frac{P_{\mu c}^{\perp}(\mathbf{q}) P_{jc}^{\perp}(\mathbf{q})}{q_\perp^4} + A(\mu_L, \mu_1) \frac{P_{\mu c}^{\perp}(\mathbf{q}) L_{jc}^{\perp}(\mathbf{q})}{q_\perp^4} \right] \Bigg]
\]

\[
= k_1^+ K_d A^{d-4} d \Omega \left[ \frac{1}{16\sqrt{\mu_z \mu_L}} \frac{\Pi_{\mu j c}^{\perp}}{(d-1)(d+1)} + \frac{1}{16\sqrt{\mu_z \mu_L}} \frac{(d-3)\delta_{\mu j c}^\perp \delta_{i u c}^\perp}{d-1} + \frac{\Pi_{\mu j c}^{\perp}}{(d-1)(d+1)} \right]
\]

\[
+ A(\mu_L, \mu_1) \left[ \frac{\delta_{\mu j c}^\perp}{d-1} - \frac{\Pi_{\mu j c}^{\perp}}{(d-1)(d+1)} \right] + A(\mu_1, \mu_L) \left[ \frac{\delta_{\mu j c}^\perp}{d-1} - \frac{\Pi_{\mu j c}^{\perp}}{(d-1)(d+1)} \right]
\]

\[
= K_d A^{d-4} d \Omega \left[ k_1^+ \delta_{\mu c}^\perp \left( \frac{1}{16\sqrt{\mu_z \mu_L}} + \frac{1}{16\sqrt{\mu_z \mu_L}} \right) - 2A(\mu_L, \mu_1) \right]
\]

\[(B.10)\]

where the function \(A(x, y)\) is defined in appendix [C] and \(\Pi_{\mu c d e} \equiv \delta_{\mu \nu} \delta_{c d} + \delta_{\mu \nu} \delta_{a d} + \delta_{\mu \nu} \delta_{a c} \). (Note that the two “\(dA\)”s in the penultimate line above represent spatial dimension \(d\) times \(A\), not the differential of \(A\).) We have also equated \(k_1^+ \delta_{\mu c}^\perp \) with \(k_c^+ \delta_{\mu j u}^\perp \) in the last step as they lead to the same correction when contracted with the prefactor \(k_1^+ \) outside the integral.

Now we turn to \((P_{2}^{\mu})_{cju}(k)\). We need to expand the integrand up to \(O(k)\). Note that the integration of the zeroth-order part of the integrand gives 0 since it is odd in \(q_\perp \) while the integration region is isotropic in \(q_\perp \). Therefore we only keep the \(O(k)\) part of

\[
(P_{2}^{\mu})_{cju}(k) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbf{q}} \frac{q_\perp^\perp}{|G_L(\mathbf{q})|^2} \left[ G_L(\mathbf{k} - \mathbf{q})L_{jc}^{\perp}(\mathbf{k} - \mathbf{q}) + G_T(\mathbf{k} - \mathbf{q})P_{jc}^{\perp}(\mathbf{k} - \mathbf{q}) \right]
\]

\[
= \frac{1}{(2\pi)^{d+1}} \int_{\mathbf{q}} \left[ (2\mu_L q_{\perp} \cdot k_{\perp} + 2\mu_q q_{\perp} k_{\perp}) G_L(-\mathbf{q})^2 L_{jc}^{\perp}(\mathbf{q}) + (2\mu_1 q_{\perp} \cdot k_{\perp} + 2\mu_q q_{\perp} k_{\perp}) \times \right]
\]

\[
G_T(-\mathbf{q})^2 P_{jc}^{\perp}(\mathbf{q}) + \left( \frac{2L_{jc}^{\perp}(\mathbf{q}) q_{\perp} \cdot k_{\perp} - k_1^+ q_\perp^\perp - k_c^+ q_\perp^\perp}{q_\perp^4} \right) \left[ G_L(-\mathbf{q}) - G_T(-\mathbf{q}) \right]
\]

\[
= K_d A^{d-4} d \Omega \left[ 3L_{jc}^{\perp}(\mathbf{q}) q_{\perp} \cdot k_{\perp} - k_1^+ q_\perp^\perp - k_c^+ q_\perp^\perp \right] \left[ \frac{1}{16\sqrt{\mu_z \mu_L}} - A(\mu_L, \mu_1) \right]
\]

\[(B.11)\]

where the function \(B(x, y)\) is defined in appendix [C]
where

\[ \Delta \left( \partial_t u_j^\perp \right)_{\mu,a} = -2 \lambda^2 D k_u^+ u_c^+ (\tilde{k}) \left[ (I_1^{\mu,a})_{cju}(\tilde{k}) - (I_2^{\mu,a})_{cju}(\tilde{k}) \right] \]

\[ = - \frac{2 \lambda^2 D K_d \Lambda d^{-4} d \ell k_u^+ u_c^+ (\tilde{k})}{(d-1)(d+1)} \times \left[ k_u^+ \delta_{j\ell} \left( - \frac{7}{64 \sqrt{\mu_x \mu_L^3}} + \frac{d^2 - 2d - 2}{16 \sqrt{\mu_x \mu_L^3}} + (d + 2) A(\mu_L, \mu_1) + dA(\mu_1, \mu_L) - 2d \mu_1 B(\mu_L, \mu_1) + k_j^+ \delta_{\mu,b} \left( - \frac{32}{32 \sqrt{\mu_x \mu_L^3}} + \frac{1}{8 \sqrt{\mu_x \mu_L^3}} \right) - 2dA(\mu_L, \mu_1) - 2A(\mu_1, \mu_L) + 4 \mu_1 B(\mu_L, \mu_1) \right] \]

\[ = - \frac{2 \mu_1 \gamma_d d \ell}{(d-1)(d+1)} \left[ k_u^+ \delta_{j\ell} (\tilde{k}) \left( - \frac{7}{64} (1 + g_2)^{-3/2} + \frac{d^2 - 2d - 2}{16} + (d + 2) \sqrt{\mu_x \mu_1^2} A(\mu_L, \mu_1) \right) \right. \]

\[ + d \sqrt{\mu_x \mu_1^2} A(\mu_1, \mu_L) - 2d \sqrt{\mu_x \mu_1^2} B(\mu_L, \mu_1) \left. + k_j^+ k_u^+ (\tilde{k}) \left( \frac{4d - 3}{32} (1 + g_2)^{-3/2} + \frac{1}{8} \right) \right] \]

\[ \Delta \left( \partial_t u_j^\perp \right)_{\mu,b} \text{ to the EOM for } u_j^\perp (\tilde{k}) : \]

\[ \Delta \left( \partial_t u_j^\perp \right)_{\mu,b} = - \frac{2 \lambda^2 D k_u^+ u_c^+ (\tilde{k})}{(2\pi)^{d+1}} \int_\mathcal{Q} q_i^+ C_{j\mu}(\tilde{q}) G_{ic}(\tilde{k} - \tilde{q}) = - \frac{2 \lambda^2 D k_u^+ u_c^+ (\tilde{k}) (I_{\mu,b}^{\perp})_{cju}(\tilde{k})}{(2\pi)^{d+1}} \int_\mathcal{Q} q_i^+ C_{j\mu}(\tilde{q}) G_{ic}(\tilde{k} - \tilde{q}) , \]

where

\[ (I_{\mu,b}^{\perp})_{cju}(\tilde{k}) \equiv \frac{1}{(2\pi)^{d+1}} \int_\mathcal{Q} q_i^+ C_{j\mu}(\tilde{q}) G_{ic}(\tilde{k} - \tilde{q}) . \]

Again, since there’s already a factor \( k_u^+ \) outside the loop integral, we can set \( \omega = 0 \) in the integrand. Also we only need to expand the integrand up to \( O(k) \) and keep the \( O(k) \) part, since the integration of the zeroth-order part is odd.
in \( q_i^+ \), and hence vanishes. Therefore, we have

\[
(I_{\mu}^b)_{qju}(k) = \frac{1}{(2\pi)^{d+1}} \int_\mathbf{q}^+ \left\{ G_L(\mathbf{q}) \left[ \frac{1}{2} L_{ju}(\mathbf{q}) + \frac{1}{2} P_{ju}(\mathbf{q}) \right] \times \right.
\]
\[
\left[ (2\mu_L \mathbf{q}_\perp \cdot \mathbf{k}_\perp + 2\mu_x q_x k_x) G_L(-\mathbf{q})^2 \frac{2L_{ij}^+(\mathbf{q})}{q_i^+} \cdot \mathbf{k}_\perp - k_i^+ q_i^+ q_c^+ - k_c^+ q_i^2}{q_i^+} [G_L(-\mathbf{q}) - G_T(-\mathbf{q})] \right]
\]
\[
= \frac{1}{(2\pi)^{d+1}} \int_\mathbf{q}^+ \left\{ G_L(\mathbf{q}) G_L(-\mathbf{q}) L_{ju}(\mathbf{q}) \times \right.
\]
\[
\left[ (2\mu_L \mathbf{q}_\perp \cdot \mathbf{k}_\perp + 2\mu_x q_x k_x) G_L(-\mathbf{q})^2 q_c^+ + \frac{2q_c^+ \mathbf{q}_\perp \cdot \mathbf{k}_\perp - k_i^+ q_i^+ q_c^+ - k_c^+ q_i^2}{q_i^+} [G_L(-\mathbf{q}) - G_T(-\mathbf{q})] \right]
\]
\[
+ G_T(\mathbf{q}) G_L(-\mathbf{q}) P_{ju}(\mathbf{q}) \times \left[ (2\mu_L \mathbf{q}_\perp \cdot \mathbf{k}_\perp + 2\mu_x q_x k_x) G_L(-\mathbf{q})^2 q_c^+ + \frac{2q_c^+ \mathbf{q}_\perp \cdot \mathbf{k}_\perp - k_i^+ q_i^+ q_c^+ - k_c^+ q_i^2}{q_i^+} [G_L(-\mathbf{q}) - G_T(-\mathbf{q})] \right]\}
\]
\[
= \frac{1}{(2\pi)^{d+1}} \int_\mathbf{q}^+ \frac{1}{q_i^+} \left\{ (2\mu_L \mathbf{q}_\perp \cdot \mathbf{k}_\perp + 2\mu_x q_x k_x) \frac{3q_c^+ L_{ju}^+(\mathbf{q})}{128 \sqrt{\mu_x \mu_L}} + (2q_c^+ \mathbf{q}_\perp \cdot \mathbf{k}_\perp - k_i^+ q_i^+ q_c^+ - k_c^+ q_i^2)}{q_i^+} \times \right.
\]
\[
\left( \frac{L_{ju}^+(\mathbf{q})}{16 \sqrt{\mu_x \mu_L}} - L_{ju}^+(\mathbf{q}) A(\mu_L, \mu_1) \right) \left( 2\mu_L \mathbf{q}_\perp \cdot \mathbf{k}_\perp + 2\mu_x q_x k_x \right) q_c^+ P_{ju}^+(\mathbf{q}) B(\mu_1, \mu_L) + \right.
\]
\[
(2q_c^+ \mathbf{q}_\perp \cdot \mathbf{k}_\perp - k_i^+ q_i^+ q_c^+ - k_c^+ q_i^2) \left( P_{ju}^+(\mathbf{q}) A(\mu_L, \mu_1) - \frac{P_{ju}^+(\mathbf{q})}{16 \sqrt{\mu_x \mu_1}} \right) \left\} \right.
\]
\[
= K_d A^{d-4} d \ell \frac{1}{(d+1)} \left\{ \frac{3 \Pi_{\mu_{cj}}^+ k_c^+}{64 \sqrt{\mu_x \mu_L}} \right. + \left( \frac{\Pi_{\mu_{cj}}^+ k_c^+}{d+1} - \delta_{j}^+ k_c^+ \right) \left( \frac{1}{16 \sqrt{\mu_x \mu_1}} - A(\mu_L, \mu_1) \right)
\]
\[
+ \frac{2\mu_L [d+1] \delta_{j}^+ k_c^+ - \Pi_{\mu_{cj}}^+ k_c^+]}{(d+1)} B(\mu_1, \mu_L) + \left( 2 - d \right) \delta_{j}^+ k_c^+ - \left( 2 - d \right) + \delta_{j}^+ k_c^+ \right) \times \right.
\]
\[
\left( A(\mu_L, \mu_1) - \frac{1}{16 \sqrt{\mu_x \mu_1}} \right) \left\} \right.
\]
\[
= K_d A^{d-4} d \ell \frac{1}{(d+1)} \left\{ k_c^+ \delta_{j}^+ \left[ \frac{7}{64 \sqrt{\mu_x \mu_L}} + \frac{1}{16 \sqrt{\mu_x \mu_1}} - A(\mu_L, \mu_1) - A(\mu_1, \mu_L) - 2 \mu_L B(\mu_1, \mu_L) \right] \right.
\]
\[
+ \frac{5 - 2d}{32 \sqrt{\mu_x \mu_L}} + \frac{d^2 - 2d - 1}{16 \sqrt{\mu_x \mu_1}} - (d^2 - 2d - 1) A(\mu_1, \mu_L) - (1 - d) A(\mu_1, \mu_1) + \right.
\]
\[
2(d-1) \mu_L B(\mu_1, \mu_L) \right\} .
\]

(B.15)
The overall correction $\Delta(\partial_t u_j^c)_{\mu,b}$ to the EOM for $u_j^c(\tilde{k})$ is therefore:

$$
\Delta(\partial_t u_j^c)_{\mu,b} = -2\lambda^2 Dk^\mu_u u^c_{\nu} (\tilde{k})(I^\mu_{\nu,b})_{\epsilon j u}(\tilde{k})
$$

$$
= -2\lambda^2 Dk\Delta t\Delta^d(\tilde{k}) \left\{ k^\mu_u \delta_{j c} \left[ \frac{7}{64\sqrt{\mu_c^2}} + \frac{1}{16\sqrt{\mu_c^2}} - A(\mu_L, \mu_1) - A(\mu_1, \mu_L) - 2\mu_L B(\mu_1, \mu_L) \right] \right\}
$$

$$
\kappa^\mu_c \delta_{j u} \left[ \frac{5 - 2d}{32\sqrt{\mu_c^2}} + \frac{d^2 - 2d - 1}{16\sqrt{\mu_c^2}} - (d^2 - 2d - 1)A(\mu_1, \mu_L) - (1 - d)A(\mu_L, \mu_1) + 2(d - 1)\mu_L B(\mu_1, \mu_L) \right] \right\}
$$

$$
- \frac{2\mu_1 g_1 d\ell}{(d - 1)(d + 1)} \left\{ k^\mu_j u^j_{\nu} (\tilde{k}) \left[ \frac{7}{64} (1 + g_2)^{-3/2} + \frac{1}{16} \sqrt{\mu_c^2} A(\mu_1, \mu_L) - \sqrt{\mu_c^2} A(\mu_L, \mu_1) - 2\sum_{\mu_c}^3 \B_1^c \mu_1 \mu_L B(\mu_1, \mu_L) \right] \right\}
$$

$$
+ \kappa^\mu_j \kappa^\mu_c u^j_{\nu} (\tilde{k}) \left[ \frac{5 - 2d}{32} (1 + g_2)^{-3/2} + \frac{d^2 - 2d - 1}{16} - (d^2 - 2d - 1)\sqrt{\mu_c^2} A(\mu_1, \mu_L) - (1 - d)\sqrt{\mu_c^2} A(\mu_L, \mu_1) + 2(d - 1)\sqrt{\mu_c^2} \mu_L B(\mu_1, \mu_L) \right] \right\}.
$$

\hspace{1cm} (B.16)

\hspace{1cm} c. Graph in Fig. 3(c)

The graph in Fig. 3(c) gives a contribution $\Delta(\partial_t u_j^c)_{\mu,c}$ to the EOM for $u_j^c(\tilde{k})$:

$$
\Delta(\partial_t u_j^c)_{\mu,c} = \frac{2\lambda^2 Dq_{\nu} u^\nu_{\nu} (\tilde{k})}{(2\pi)^d+1} \int_\mathbb{R} (k^\mu_j - q^\mu_j) q_{\nu} C_{\ell\delta}(\tilde{\mathbf{q}}) G_{j\ell}(\tilde{\mathbf{k}} - \tilde{\mathbf{q}}) \equiv 2\lambda^2 Dq_{\nu} u^\nu_{\nu} (\tilde{k}) \left[ (I^\mu_{\nu,c})_{j u}(\tilde{k}) + (I_2^\mu_{\nu,c})_{j u}(\tilde{k}) \right],
$$

\hspace{1cm} (B.17)

where

$$
(I^\mu_{\nu,c})_{j u}(\tilde{k}) \equiv \frac{1}{(2\pi)^d+1} \int_\mathbb{R} q_{\nu} C_{\ell\delta}(\tilde{\mathbf{q}}) G_{j\ell}(\tilde{\mathbf{k}} - \tilde{\mathbf{q}})
$$

\hspace{1cm} (B.18)

$$
(I_2^\mu_{\nu,c})_{j u}(\tilde{k}) \equiv -\frac{1}{(2\pi)^d+1} \int_\mathbb{R} q_{\nu} q_{\nu} C_{\ell\delta}(\tilde{\mathbf{q}}) G_{j\ell}(\tilde{\mathbf{k}} - \tilde{\mathbf{q}})
$$

\hspace{1cm} (B.19)

Let's calculate $(I^\mu_{\nu,c})_{j u}$ first. Again, since there's already a factor $k^\mu_j$ outside the integral, we can set $\omega = 0$ in the integrand. Also we only need to expand the integrand up to $O(k)$ and keep the $O(k)$ part, since the integration of
the zeroth-order part gives 0. Therefore, we have

\[(I_{\mu,c}^{\mu,c})_{ju}(\mathbf{k}) = \frac{k_+^4}{(2\pi)^{d+1}} \int_{\mathbf{q}}^{>} q_u^+ \left[ |G_L(\mathbf{q})|^2 L_{\mu,c}^+(\mathbf{q}) + |G_T(\mathbf{q})|^2 P_{\mu,c}^+(\mathbf{q}) \right] \left\{ G_L(-\mathbf{q}) \frac{2L_{\mu,c}^+(\mathbf{q}) q_m^+ k_m^+ - k_+^2 q_+^+ - k_+^2 q_+^+}{q_+^2} 
+ G_L(-\mathbf{q})^2 L_{\mu,c}^+(\mathbf{q}) (2\mu_L q_m^+ k_m^+ + 2\mu_q q_x k_x) 
+ G_T(-\mathbf{q})^2 P_{\mu,c}^+(\mathbf{q}) (2\mu_1 q_m^+ k_m^+ + 2\mu_q q_x k_x) \right\}
= \frac{k_+^4}{(2\pi)^{d+1}} \int_{\mathbf{q}}^{>} q_u^+ \left\{ |G_L(\mathbf{q})|^2 G_L(-\mathbf{q}) \frac{2L_{\mu,c}^+(\mathbf{q}) q_m^+ k_m^+ - k_+^2 q_+^+ - L_{\mu,c}^+(\mathbf{q}) k_+^2 q_+^+}{q_+^2} 
+ G_L(-\mathbf{q})^2 L_{\mu,c}^+(\mathbf{q}) (2\mu_L q_m^+ k_m^+ - k_+^2 q_+^+ - k_+^2 q_+^+ L_{\mu,c}^+(\mathbf{q})) 
+ |G_T(\mathbf{q})|^2 \left[ -G_L(-\mathbf{q}) k_+^2 q_+^+ P_{\mu,c}^+(\mathbf{q}) + G_T(-\mathbf{q}) k_+^2 q_+^+ P_{\mu,c}^+(\mathbf{q}) \frac{q_+^2}{q_+^2} + G_T(-\mathbf{q})^2 P_{\mu,c}^+(\mathbf{q}) (2\mu_1 q_m^+ k_m^+) \right] \right\}
= \frac{k_+^4}{(2\pi)^{d+1}} \int_{\mathbf{q}}^{>} q_u^+ \left\{ \frac{2L_{\mu,c}^+(\mathbf{q}) q_m^+ k_m^+ - k_+^2 q_+^+ - L_{\mu,c}^+(\mathbf{q}) k_+^2 q_+^+}{16\sqrt{\mu_x \mu_L^2}} + \frac{3 (\mu_L q_m^+ k_m^+) L_{\mu,c}^+(\mathbf{q})}{64\sqrt{\mu_x \mu_L^2}} 
- [2L_{\mu,c}^+(\mathbf{q}) q_m^+ k_m^+ - k_+^2 q_+^+ - k_+^2 q_+^+ L_{\mu,c}^+(\mathbf{q})] A(\mu_L, \mu_1) 
- k_+^2 q_+^+ P_{\mu,c}^+(\mathbf{q}) A(\mu_1, \mu_L) + k_+^2 q_+^+ P_{\mu,c}^+(\mathbf{q}) + \frac{3 P_{\mu,c}^+(\mathbf{q}) (\mu_1 q_m^+ k_m^+)}{64\sqrt{\mu_x \mu_L^2}} \right\}
= \frac{k_d A^{d-4} d\ell k_+^4}{(d-1)(d+1)} \left\{ \frac{2\Pi_{\mu,i,j} k_m^+ - (d+1) \delta_{i,j} k_+^4}{16\sqrt{\mu_x \mu_L^2}} + \frac{3 \mu_L \Pi_{\mu,i,j} k_m^+}{64\sqrt{\mu_x \mu_L^2}} - [\Pi_{\mu,i,j} k_m^+ - (d+1) \delta_{i,j} k_+^4] A(\mu_L, \mu_1) 
+ (-\Pi_{\mu,i,j} k_m^+ - (d+1) \delta_{i,j} k_+^4) \left( -A(\mu_1, \mu_L) + \frac{1}{16\sqrt{\mu_x \mu_L^2}} \right) + (-\Pi_{\mu,i,j} k_m^+ - (d+1) \delta_{i,j} k_+^4) \frac{3}{64\sqrt{\mu_x \mu_L^2}} \right\}
= \frac{k_d A^{d-4} d\ell}{(d-1)(d+1)} \left\{ \frac{7 \delta_{i,j}^4}{64\sqrt{\mu_x \mu_L^2}} - A(\mu_1, \mu_L) - dA(\mu_1, \mu_L) + \frac{4d-3}{64\sqrt{\mu_x \mu_L^2}} \right\}
+ k_+^4 k_+ \left[ \frac{5 - 2d}{32\sqrt{\mu_x \mu_L^2}} + (d-1) A(\mu_1, \mu_L) + 2 A(\mu_1, \mu_L) + \frac{3d-11}{64\sqrt{\mu_x \mu_L^2}} \right] \right\}. \quad (B.20)\]

Now we turn to \((I_{\mu,c}^{\mu,c})_{ju}(\mathbf{k})\). In appendix \ref{appB1} we show that the sum of \((I_{2}^{\mu,c})_{ju}(\mathbf{k})\) and \((I_{\mu,c}^{\mu,c})_{ju}(\mathbf{k})\), which is introduced in evaluating Fig. \ref{fig3}, is at most \(O(k^2)\). This means we can set \(\omega = 0\) and focus on the \(O(k^2)\) part when evaluating \((I_{\mu,c}^{\mu,c})_{ju}(\mathbf{k})\) and \((I_{\mu,c}^{\mu,c})_{ju}(\mathbf{k})\), since their lower order parts (i.e., \(O(1)\) and \(O(k)\)) all cancel out. We will use this knowledge in the following calculations.

\[(I_{\mu,c}^{\mu,c})_{ju}(\mathbf{k}) = - \frac{1}{(2\pi)^{d+1}} \int_{\mathbf{q}}^{>} q_u^+ q_+^+ G_L(\mathbf{q}) G_L(-\mathbf{q}) \left( - (\mu_L k_+^2 + \mu_q k^2) G_L(-\mathbf{q})^2 L_{\mu,c}^+(\mathbf{q}) + \left( 4 \mu_1^2 (\mathbf{q} \cdot \mathbf{k})^2 + 4 \mu_2^2 q_+^2 k_+^2 \right) - L_{\mu,c}^+(\mathbf{q}) \left( L_{\mu,c}^+(\mathbf{q}) - k_+^2 q_+^+ + 4 (\mathbf{q} \cdot \mathbf{k})^2 \right) \right)
+ G_T(-\mathbf{q})^2 L_{\mu,c}^+(\mathbf{q}) \left[ L_{\mu,c}^+(\mathbf{q}) - \frac{k_+^2 q_+^+}{q_+^2} + \frac{4 (\mathbf{q} \cdot \mathbf{k})^2}{q_+^2} \right] - \left[ L_{\mu,c}^+(\mathbf{q}) - \frac{k_+^2 q_+^+}{q_+^2} + \frac{4 (\mathbf{q} \cdot \mathbf{k})^2}{q_+^2} \right] \right]\]
\[\times G_L(-\mathbf{q}) \left( 2 \mu_L q_m^+ k_m^+ - k_+^2 q_+^+ - k_+^2 q_+^+ \right) q_+^2 \]
\[- G_T(-\mathbf{q}) \left[ L_{\mu,c}^+(\mathbf{q}) - \frac{k_+^2 q_+^+}{q_+^2} + \frac{4 (\mathbf{q} \cdot \mathbf{k})^2}{q_+^2} \right] \]
\[- \left[ L_{\mu,c}^+(\mathbf{q}) - \frac{k_+^2 q_+^+}{q_+^2} + \frac{4 (\mathbf{q} \cdot \mathbf{k})^2}{q_+^2} \right] \right]\]
-\(2\mu_1 q_m^+ k_m^+ G_T(-\mathbf{q})^2 \left( 2 \mu_L q_m^+ k_m^+ - k_+^2 q_+^+ - k_+^2 q_+^+ \right) q_+^2 \right\}. \quad (B.21)\]
where we have discarded terms linear in \( q_x \) since the integration of these terms gives 0. Performing the \( \Omega \) and \( q_x \) integrals in (B.21), as described in detail in appendix (C), we have

\[
(I_{2,c}^{\mu,\nu})_{j,u}(\mathbf{k}) = -\frac{1}{(2\pi)^{d-1}} \int_{q_\perp}^{q_\perp} q_{\perp u}^{1/2} \left\{ \frac{3\delta\ell}{128 \sqrt{\mu_L \mu_1^L}} + \frac{(5\mu_L^2 q_m^{\perp} + \mu_L k_\perp^2) \delta_{\perp}}{128 \sqrt{\mu_L \mu_1^L} q_{\perp}^{2}} \right\}
\]

\[
+ \left[ \delta\ell \left( -k_\perp^2 + 4\left( \frac{q_{\perp} \cdot \mathbf{k}_{\perp}}{q_{\perp}^2} \right)^2 \right) \right] + \frac{q_{\perp}^2}{q_{\perp}^2} \left( \frac{2\delta\ell q_{\perp} \cdot \mathbf{k}_{\perp} - k_\perp^2 q_{\perp} - 2q_{\perp} \cdot \mathbf{k}_{\perp}(k_\perp^2 q_{\perp} + k_\perp^2 q_{\perp}^2) }{q_{\perp}^2} \right)
\]

\[
- \frac{3\mu_L q_{\perp} \cdot \mathbf{k}_{\perp}}{64 \sqrt{\mu_L \mu_1^L}} \left( 2\delta\ell q_{\perp} \cdot \mathbf{k}_{\perp} - k_\perp^2 q_{\perp} - 2q_{\perp} \cdot \mathbf{k}_{\perp}(k_\perp^2 q_{\perp} + k_\perp^2 q_{\perp}^2) \right) A(\mu_L, \mu_1)
\]

\[
- 2\mu_L q_{\perp} \cdot \mathbf{k}_{\perp} B(\mu_L, \mu_1) \left( 2\delta\ell q_{\perp} \cdot \mathbf{k}_{\perp} - k_\perp^2 q_{\perp} - 2q_{\perp} \cdot \mathbf{k}_{\perp}(k_\perp^2 q_{\perp} + k_\perp^2 q_{\perp}^2) \right)
\]

\[
= -\frac{K_d \Delta^{4-d} \ell}{(d-1)(d+1)} \left\{ \frac{3(d+1) \delta_{\perp}^j_{\perp}(\mu_L k_\perp^2 + \mu_\perp k_\perp^2)}{128 \sqrt{\mu_L \mu_1^L}} + \frac{5\mu_L^2 \Pi_{mnj_u} k_{m,n}^+ k_{m,n}^+}{128 \sqrt{\mu_L \mu_1^L}} + (d+1) \delta_{\perp}^j_{\perp} \mu_L \mu_1 k_\perp^2 \right\}
\]

\[
+ (-d+1) \delta_{\perp}^j_{\perp} k_\perp^2 + \Pi_{mnj_u} k_{m,n}^+ k_{m,n}^+ - (d+1) \delta_{\perp}^j_{\perp} \mu_L \mu_1 \left[ \frac{1}{16 \sqrt{\mu_L \mu_1^L}} - A(\mu_L, \mu_1) \right] \right\}
\]

\[
+ \left( \Pi_{mnj_u} k_{m,n}^+ k_{m,n}^+ - (d+1) \delta_{\perp}^j_{\perp} \mu_L \mu_1 \right) \left( \frac{3\mu_L}{64 \sqrt{\mu_L \mu_1^L}} - 2\mu_1 B(\mu_L, \mu_1) \right) \right\}
\]

\[
= -\frac{K_d \Delta^{4-d} \ell}{(d-1)(d+1)} \left\{ -\frac{k_\perp^2 \delta^j_{\perp} (d+1) \sqrt{\mu_L}}{64 \mu_1^L} - \frac{k_\perp^2 \delta_{\perp}^j_{\perp}}{128 \sqrt{\mu_L \mu_1^L}} - dA(\mu_L, \mu_1) + 2\mu_1 B(\mu_L, \mu_1) \right\}
\]

\[
+ \frac{2\delta_{\perp}^j_{\perp} u_{\perp}^j(\mathbf{k})}{64 \sqrt{\mu_L \mu_1^L}} + \frac{3d-10}{64 \mu_1^L} \right\}
\]

\[
\Delta(\delta t u_{\perp}^j)_{\mu,c} = 2\lambda^2 D u_{\perp}^j(\mathbf{k}) \left[ (I_{1,c}^{\mu,\nu})_{j,u}(\mathbf{k}) + (I_{2,c}^{\mu,\nu})_{j,u}(\mathbf{k}) \right]
\]

\[
= \frac{2\lambda^2 D K_d \Delta^{4-d} u_{\perp}^j(\mathbf{k}) \ell}{(d-1)(d+1)} \left\{ \frac{k_\perp^2 \delta_{\perp}^j_{\perp} (d+1) \sqrt{\mu_L}}{64 \mu_1^L} + \frac{k_\perp^2 \delta_{\perp}^j_{\perp}}{128 \sqrt{\mu_L \mu_1^L}} - dA(\mu_L, \mu_1) + 2\mu_1 B(\mu_L, \mu_1) \right\}
\]

\[
+ \frac{4d-3}{64 \sqrt{\mu_L \mu_1^L}} + \frac{3d-10}{64 \sqrt{\mu_L \mu_1^L}} \right\}
\]

\[
= \frac{2\mu_1 g \delta_{\perp}^j_{\perp} \ell}{(d-1)(d+1)} \left\{ \frac{k_\perp^2 u_{\perp}^j(\mathbf{k})}{64 \sqrt{\mu_L \mu_1^L}} + \frac{11d-2}{128} (1+g_2)^{-3/2} - d \sqrt{\mu_L \mu_1^3} A(\mu_L, \mu_1) \right\}
\]

\[
+ 2\sqrt{\mu_L \mu_1^3} B(\mu_L, \mu_1) - d \sqrt{\mu_L \mu_1^3} A(\mu_L, \mu_1) + \frac{4d-3}{64} + k_\perp^2 u_{\perp}^j(\mathbf{k}) \left[ \frac{3d-10}{64} - (1+g_2)^{-3/2} \right]
\]

\[
+ 2\sqrt{\mu_L \mu_1^3} A(\mu_L, \mu_1) + 2(1-d) \sqrt{\mu_L \mu_1^3} B(\mu_L, \mu_1) + 2 \sqrt{\mu_L \mu_1^3} A(\mu_L, \mu_1) + \frac{3d-11}{64} \right\}.
\]
Clearly the zeroth-order parts of the two integrands cancel out. Thus the sum of the two integrands is of $O(k^2)$. Furthermore, it is easy to see that the integrand of $(I_{\mu,d})_{j_u}(\tilde{k})$ can be rewritten as

$$\Delta(\partial_t u_j^-)_{\mu,d} = \frac{2\lambda^2 D_{u \nu}(\tilde{k})}{(2\pi)^{d+1}} \int_{\mathbf{q}} q_i^+ q_u^+ C_{ij\ell}(\tilde{q}) G_{i\ell}(\tilde{k} - \tilde{q}) \equiv 2\lambda^2 D_{u \nu}(\tilde{k})(I_{\mu,d})_{j_u}(\tilde{k}), \quad \text{(B.24)}$$

where

$$(I_{\mu,d})_{j_u}(\tilde{k}) \equiv \frac{1}{(2\pi)^{d+1}} \int_{\mathbf{q}} q_i^+ q_u^+ C_{ij\ell}(\tilde{q}) G_{i\ell}(\tilde{k} - \tilde{q}). \quad \text{(B.25)}$$

First let’s show that the sum of $(I_{\mu,c})_{j_u}(\tilde{k})$ and $(I_{\mu,d})_{j_u}(\tilde{k})$ is at most of $O(k^2)$. The integrand of $(I_{\mu,c})_{2j_u}(\tilde{k})$ can be rewritten as

$$-q_i^+ q_u^+ C_{ij\ell}(\tilde{q}) G_{i\ell}(\tilde{k} - \tilde{q}) = -q_u^+ |G_L(\tilde{q})|^2 q_i^+ G_{i\ell}(\tilde{k} - \tilde{q})$$
$$= -q_u^+ |G_L(\tilde{q})|^2 (q_i^+ - k_i^+) G_{i\ell}(\tilde{k} - \tilde{q}) - q_u^+ k_i^+ |G_L(\tilde{q})|^2 G_{i\ell}(\tilde{k} - \tilde{q})$$
$$= -q_u^+ |G_L(\tilde{q})|^2 (q_i^+ - k_i^+) G_{i\ell}(\tilde{k} - \tilde{q}) - q_u^+ k_i^+ |G_L(\tilde{q})|^2 G_{i\ell}(\tilde{k} - \tilde{q})$$
$$= -q_u^+ q_i^+ |G_L(\tilde{q})|^2 G_{i\ell}(\tilde{k} - \tilde{q}) + q_i^+ q_u^+ |G_L(\tilde{q})|^2 G_{i\ell}(\tilde{k} - \tilde{q}) - k_i^+ q_u^+ |G_L(\tilde{q})|^2 G_{i\ell}(\tilde{k} - \tilde{q}); \quad \text{(B.26)}$$

The integrand of $(I_{\mu,d})_{j_u}(\tilde{k})$ can be rewritten as

$$q_i^+ q_u^+ C_{ij\ell}(\tilde{q}) G_{i\ell}(\tilde{k} - \tilde{q})$$
$$= q_u^+ C_{ij\ell}(\tilde{q})(q_i^+ - k_i^+) G_{i\ell}(\tilde{k} - \tilde{q}) + k_i^+ q_u^+ C_{ij\ell}(\tilde{q}) G_{i\ell}(\tilde{k} - \tilde{q})$$
$$= q_u^+ C_{ij\ell}(\tilde{q})(q_i^+ - k_i^+) G_{i\ell}(\tilde{k} - \tilde{q}) + k_i^+ q_u^+ C_{ij\ell}(\tilde{q}) G_{i\ell}(\tilde{k} - \tilde{q})$$
$$= q_u^+ q_i^+ |G_L(\tilde{q})|^2 G_{i\ell}(\tilde{k} - \tilde{q}) - k_i^+ q_u^+ C_{ij\ell}(\tilde{q}) G_{i\ell}(\tilde{k} - \tilde{q}) + k_i^+ q_u^+ C_{ij\ell}(\tilde{q}) G_{i\ell}(\tilde{k} - \tilde{q}). \quad \text{(B.27)}$$

Clearly the zeroth-order parts of the two integrands cancel out. Thus the sum of the two integrands is of $O(k)$. Furthermore, it is easy to see that the $O(k)$ parts vanish after the integration. This proves our earlier claim.

In the following calculations we will focus on the $O(k^2)$ parts. Again we will omit terms linear in $q_x$ since the
\[ (I^{\mu, d})_{J\mu}(q) = \frac{1}{(2\pi)^{d+1}} \int_{q}^L q_+^d q_+^d \left\{ |G_L(q)|^2 L_{ij}(q) \right. \]

\[ \left. \times G_L(-q)^3 L_{ij}(q) + \left[ -4q_+^d q_+^d - (\mu_L k_L^2 + \mu_x k_L^2) G_L(-q)^3 L_{ij}(q) + (4\mu_L^2 (q_\perp \cdot k_\perp)^2 + 4\mu_x^2 q_\perp k_L^2) \right] \right. \]

\[ \left. \times G_L(-q)^3 L_{ij}(q) + \left[ -2q_+^d \cdot k_\perp G_L(-q)^2 \left( \frac{2L_{ij}(q)q_\perp \cdot k_\perp - k_L^2 q_\perp^2}{q_\perp^2} \right) \right] \right. \]

\[ \left. \times G_T(q)^2 P_{ij}(q) \times \left( -2\mu_L q_+^d \cdot k_\perp G_L(-q)^2 \left( \frac{k_\perp^2 q_\perp^2}{q_\perp^2} \right) + 2\mu_L q_+^d \cdot k_\perp G_T(q)^2 \left( \frac{k_\perp^2 q_\perp^2}{q_\perp^2} \right) \right) \right\} \]

\[ = \frac{1}{(2\pi)^{d-1}} \int_{q}^L q_+^d q_+^d \left\{ \left( -\frac{3L_{ij}(q) \left( \mu_L k_L^2 + \mu_x k_L^2 \right)}{128 \sqrt{\mu_x \mu_L}} + \frac{5\mu_L^2 q_\perp^2 k_m^2 k_n^2 + \mu_x \mu_L k_L^2 q_\perp^2}{128 \sqrt{\mu_x \mu_L} q_\perp^2} \right) L_{ij}(q) \right. \]

\[ + \left. \left[ L_{ij}(q) \left( -k_L^2 + \frac{4(q_\perp \cdot k^2)}{q_\perp^2} \right) + k_L^2 k_L L_{ij}(q) - \frac{2q_+^d \cdot k_\perp (k_\perp^2 q_\perp^2 + k_L^2 q_\perp^2)}{q_\perp^2} \right] \right. \]

\[ \left. \left( \frac{1}{16 \sqrt{\mu_x \mu_L} A(\mu_L, \mu_1) - 2q_\perp \cdot k_\perp \left( \frac{3\mu_L}{128 \sqrt{\mu_x \mu_L}} - A(\mu_L, \mu_1) \right) \right) \right. \]

\[ \left. \left( 2L_{ij}(q) q_\perp \cdot k_\perp - k_L^2 q_\perp^2 \right) \right. \]

\[ \left. \left( A(\mu_L, \mu_1) \right) - \frac{3\mu_1}{128 \sqrt{\mu_x \mu_L}} \right) \left( \frac{k_\perp^2 q_\perp^2}{q_\perp^2} \right) \]

\[ P_{ij}(q) \right\} \]

\[ = K_d \frac{\Delta_{d-1} dL}{(d-1)(d+1)} \left\{ -\frac{3\delta_{ju}(d+1)(mu_L k_L^2 + \mu_x k_L^2)}{128 \sqrt{\mu_x \mu_L}} + \frac{5\mu_L^2 \Pi_{mnju}^l k_m^2 k_n^2 + \mu_x \delta_{ju}(d+1) \mu_L k_x^2}{128 \sqrt{\mu_x \mu_L}} \right. \]

\[ + \left. (\Delta_{ju}(d+1) k_L^2 + \Pi_{mnju}^l k_m^2 k_n^2) \left( \frac{1}{16 \sqrt{\mu_x \mu_L}} - A(\mu_L, \mu_1) \right) + (-(d+1)k_u^2 k_L^2 + \Pi_{mnju}^l k_m^2 k_n^2) \right. \]

\[ \left. + \left( A(\mu_L, \mu_1) - \frac{3\mu_1}{128 \sqrt{\mu_x \mu_L}} \right) - 2 \right\} \left( (d+1)k_u^2 k_L^2 - \Pi_{mnju}^l k_m^2 k_n^2 \right) \left( \mu_L B(\mu_1, \mu_L) - \frac{3\mu_1}{128 \sqrt{\mu_x \mu_L}} \right) \right\}. \]
The total contribution $\Delta(\partial_t u_j^\mp)_{\mu,d}$ of the graph Fig. 3(d) to the equation of motion is therefore:

$$
\Delta(\partial_t u_j^\mp)_{\mu,d} = 2\lambda^2 D_{\mu} u_j^\mp \right( \partial_{\mu} \partial_{\nu} \left( J_{\mu}^{\parallel d} \right) \bigg|_{\mu d} \right) \bigg( \phi \bigg)
$$

$$
\frac{2\lambda^2 D_{\mu} u_j^\mp (\hat{k}) K_d A^{d-4} d\ell}{(d-1)(d+1)} \left\{ -\delta^d_{\mu} k_\perp^d \frac{(d+1) \sqrt{\mu_\perp}}{64 \sqrt{\mu_L^3}} + \delta^d_{\mu} k_\perp^d \left[ \frac{2 - 11d}{128 \sqrt{\mu_L^3}} - 2A(\mu_L, \mu_1) + (1 - d)A(\mu_1, \mu_L) \right] 
\right.
$$

$$
- \left. \frac{7}{64 \sqrt{\mu_L^3}} + 2\mu_L B(\mu_1, \mu_L) \right) + k_\perp u_j^\mp (\hat{k}) \left[ \frac{13}{64 \sqrt{\mu_L^3}} - 2\mu_L B(\mu_1, \mu_L) \right] + (1 - d) \sqrt{\mu_\perp^3 A(\mu_L, \mu_1)} + \frac{7(d-1)}{64} + 2(1 - d) \sqrt{\mu_\perp^3 B(\mu_1, \mu_L)} \bigg\} .
$$

2. Overall propagator renormalization

Summing over all the contributions from the one-loop diagrams in Fig. 3 we find two types of terms: $k_\perp^2 u_j^\mp (\hat{k})$ and $k_\perp^d k_\perp u_j^\mp (\hat{k})$. The sum of the coefficients of the former gives the correction to $-\mu_1$; that of the latter gives the correction to $-\mu_2$. There is no correction to $\mu_x$ to one-loop order, since no terms proportional to $k_\perp^2 u_j$ survive to this order.

Thus the graphical correction $\delta \mu_1$ to $\mu_1$ is

$$
\delta \mu_1 = \frac{2\mu_1 g_1 d\ell}{(d-1)(d+1)} \left\{ -\frac{7}{64} (1 + g_2)^{-3/2} + \frac{d^2 - 2d - 2}{16} + (d + 2) \sqrt{\mu_\perp^3 A(\mu_L, \mu_1)} + d \sqrt{\mu_\perp^3 A(\mu_1, \mu_L)} - 2d \sqrt{\mu_\perp^3 B(\mu_L, \mu_1)} \right\}
$$

$$
\left[ \frac{11d - 2}{128} + \frac{7(d-1)}{64} \right] + \left[ \frac{2 - 11d}{128} + \frac{7(d-1)}{64} \right]
$$

$$
\left\{ -\frac{2d^2 - 6d + 3}{32} + (1 + d) \sqrt{\mu_\perp^3 A(\mu_\perp, \mu_1)} + 2(d-1) \sqrt{\mu_\perp^3 A(\mu_1, \mu_L)} - \frac{7}{64} + 2 \sqrt{\mu_\perp^3 B(\mu_1, \mu_L)} \right\}. \quad (B.31)
$$
where

\begin{align}
\sqrt{\mu_x \mu_1^b} A(\mu_L, \mu_1) &= \frac{1}{4g_2} \left( \frac{\sqrt{2}}{\sqrt{1 + g_2}} - \frac{1}{\sqrt{2 + g_2}} \right) \quad \text{(B.32)}
\end{align}

\begin{align}
\sqrt{\mu_x \mu_1^b} A(\mu_1, \mu_L) &= \frac{1}{4g_2} \left( 1 - \frac{\sqrt{2}}{\sqrt{2 + g_2}} \right) \quad \text{(B.33)}
\end{align}

\begin{align}
\sqrt{\mu_x \mu_1^b} B(\mu_L, \mu_1) &= \frac{2(2 + g_2)^{3/2} - \sqrt{2(1 + g_2)} (4 + g_2)}{8(1 + g_2)(2 + g_2)^{3/2} g_2^2} \quad \text{(B.34)}
\end{align}

\begin{align}
\sqrt{\mu_x \mu_1^b} B(\mu_1, \mu_L) &= (1 + g_2) \frac{2(2 + g_2)^{3/2} - \sqrt{2(4 + g_2)}}{8(2 + g_2)^{3/2} g_2^2} \quad \text{(B.35)}
\end{align}

Using these, we can, after considerable algebra, rewrite (B.31) as

\begin{align}
\delta \mu_1 &= g_1 \mu_1 G_{\mu_1}(g_2) d\ell, \quad \text{(B.36)}
\end{align}

with \( G_{\mu_1}(g_2) \) given by \[\text{(IV.14)}\]. The graphical correction \( \delta \mu_2 \) to \( \mu_2 \) is

\begin{align}
\delta \mu_2 &= \frac{2 \mu_2 g_1 d\ell}{(d - 1)(d + 1) g_2} \left\{ \frac{4d - 3}{32} (1 + g_2)^{-3/2} + \frac{1}{8} - 2d \sqrt{\mu_x \mu_1^b} A(\mu_L, \mu_1) - 2 \sqrt{\mu_x \mu_1^b} A(\mu_1, \mu_L) + 4 \sqrt{\mu_x \mu_1^b} B(\mu_L, \mu_1) \right\} \\
&\quad + \left[ 5 - \frac{2d - 2d - 1}{16} - (d - 2d - 1) \sqrt{\mu_x \mu_1^b} A(\mu_1, \mu_L) - (1 - d) \sqrt{\mu_x \mu_1^b} A(\mu_L, \mu_1) \right] \\
&\quad + 2(d - 1) \sqrt{\mu_x \mu_1^b} B(\mu_L, \mu_1) - \left[ \frac{3d - 10}{64} (1 + g_2)^{-3/2} + 2 \sqrt{\mu_x \mu_1^b} A(\mu_1, \mu_1) + 2(1 - d) \sqrt{\mu_x \mu_1^b} B(\mu_L, \mu_1) + 2 \sqrt{\mu_x \mu_1^b} A(\mu_1, \mu_L) + 2 \sqrt{\mu_x \mu_1^b} A(\mu_L, \mu_1) + 2 \sqrt{\mu_x \mu_1^b} B(\mu_1, \mu_L) \right] \\
&\quad + \frac{7(d - 1)}{64} + 2(1 - d) \sqrt{\mu_x \mu_1^b} B(\mu_1, \mu_L) \} \\
&= \frac{2 \mu_2 g_1 d\ell}{(d - 1)(d + 1) g_2} \left[ \frac{1 + d}{64} (1 + g_2)^{-3/2} + \frac{2d - 9d + 11}{32} - (d + 1) \sqrt{\mu_x \mu_1^b} A(\mu_1, \mu_1) - \\ \left( d^2 - 3d + 4 \right) \sqrt{\mu_x \mu_1^b} A(\mu_1, \mu_1) + 2(d + 1) \sqrt{\mu_x \mu_1^b} B(\mu_1, \mu_1) + 4(d - 1) \sqrt{\mu_x \mu_1^b} B(\mu_1, \mu_1) \right] \\
&= g_1 \mu_2 G_{\mu_2}(g_2), \quad \text{(B.37)}
\end{align}

where \( G_{\mu_2}(g_2) \) is given in \[\text{(IV.16)}\].

### 3. Noise renormalization

**a. Graph in Fig. 4(a)**

The graph in Fig. 4(a) represents the following correction to the noise correlator \( \langle f_\ell(\vec{k}) f_u(-\vec{k}) \rangle \):

\begin{align}
\Delta \langle f_\ell(\vec{k}) f_u(-\vec{k}) \rangle_{D,\alpha} &= \frac{2 \lambda^2 D^2}{(2\pi)^{d+1}} \int_{\vec{q}} q^+_i q^+_m C_{im}(\vec{k} - \vec{q}) C_{\alpha u}(\vec{q}) \equiv 2 \lambda^2 D^2 (I^{D,\alpha})_{\ell u}(\vec{k}), \quad \text{(B.38)}
\end{align}

where

\begin{align}
(I^{D,\alpha})_{\ell u}(\vec{k}) &\equiv \frac{1}{(2\pi)^{d+1}} \int_{\vec{q}} q^+_i q^+_m C_{im}(\vec{k} - \vec{q}) C_{\alpha u}(\vec{q}). \quad \text{(B.39)}
\end{align}
Since the noise strength $D$ is the value of this correlation at $\mathbf{k} = \mathbf{0}$, we can evaluate $(I^{D,a})_{\ell u}(\mathbf{k})$ at $\mathbf{k} = \mathbf{0}$. This gives

$$(I^{D,a})_{\ell u}(\mathbf{0}) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbf{q}} q_i^+ q_m^+ C_{i m}(-\mathbf{q}) C_{\ell u}(\mathbf{q})$$

$$= \frac{1}{(2\pi)^{d+1}} \int_{\mathbf{q}} q_i^+ q_m^+ \left[ |G_L(\mathbf{q})|^2 L_{i m}^+(\mathbf{q}) + |G_T(\mathbf{q})|^2 P_{i m}^+(\mathbf{q}) \right] \left[ |G_L(\mathbf{q})|^2 L_{\ell u}^+(\mathbf{q}) + |G_T(\mathbf{q})|^2 P_{\ell u}^+(\mathbf{q}) \right]$$

$$= \frac{1}{(2\pi)^{d+1}} \int_{\mathbf{q}} q_i^+ q_m^+ \left[ |G_L(\mathbf{q})|^2 L_{i m}^+(\mathbf{q}) + |G_T(\mathbf{q})|^2 L_{\ell u}^+(\mathbf{q}) + |G_T(\mathbf{q})|^2 P_{\ell u}^+(\mathbf{q}) \right]$$

$$= \frac{1}{(2\pi)^{d+1}} \int_{\mathbf{q}} q_i^+ q_m^+ \left[ \frac{3}{64 \sqrt{\mu_x \mu_L}} L_{\ell u}^+(\mathbf{q}) + \frac{1}{4 \sqrt{\mu_x} (\mu_L - \mu_1)^2} \left( \frac{1}{\sqrt{\mu_L}} + \frac{1}{\sqrt{\mu_1}} - \frac{2 \sqrt{2}}{\sqrt{\mu_L} + \mu_1} \right) P_{\ell u}^+(\mathbf{q}) \right]$$

$$= \left[ \frac{3}{64 \sqrt{\mu_x \mu_L} (d-1)} + \frac{1}{4 \sqrt{\mu_x} (\mu_L - \mu_1)^2} \left( \frac{1}{\sqrt{\mu_L}} + \frac{1}{\sqrt{\mu_1}} - \frac{2 \sqrt{2}}{\sqrt{\mu_L} + \mu_1} \right) \right] \frac{1}{d-1} \frac{D^2 \lambda^2 S_{d-1}}{(2\pi)^{d-1}} \Lambda^{d-4} d\ell .$$

(B.40)

Identifying the coefficient of $\delta_{\ell u}^+$ as a correction (δD)_{D,a} to D gives:

$$(\delta D)_{D,a} = 2 \left[ \frac{3}{64 \sqrt{\mu_x \mu_L} (d-1)} + \frac{1}{4 \sqrt{\mu_x} (\mu_L - \mu_1)^2} \left( \frac{1}{\sqrt{\mu_L}} + \frac{1}{\sqrt{\mu_1}} - \frac{2 \sqrt{2}}{\sqrt{\mu_L} + \mu_1} \right) \right] \frac{1}{d-1} \frac{D^2 \lambda^2 S_{d-1}}{(2\pi)^{d-1}} \Lambda^{d-4} d\ell$$

$$= 2 g_1 D d\ell \left[ \frac{3}{64 (1 + g_2)^{5/2}} \left( \frac{1}{d-1} + \frac{1}{4 g_2} \right) + \frac{1}{d-1} \frac{2}{d-1} \right] .$$

(B.41)

b. Graph in Fig. 7(b)

The second correction to D comes from the diagram in Fig. 4(b), which represents the following correction to the noise correlator $\langle f_i(\mathbf{k}) f_u(-\mathbf{k}) \rangle$:

$$\Delta \langle f_i(\mathbf{k}) f_u(-\mathbf{k}) \rangle_{D,b} = \frac{2 \lambda^2 D^2}{(2\pi)^{d+1}} \int_{\mathbf{q}} q_i^+ (k_m^+ - q_m^+) C_{i u}(\mathbf{k} - \mathbf{q}) C_{\ell m}(\mathbf{q}) \equiv 2 \lambda^2 D^2 (I^{D,b})_{\ell u}(\mathbf{k}) ,$$

where

$$(I^{D,b})_{\ell u}(\mathbf{k}) \equiv (2\pi)^{d+1} \int_{\mathbf{q}} q_i^+ (k_m^+ - q_m^+) C_{i u}(\mathbf{k} - \mathbf{q}) C_{\ell m}(\mathbf{q}) .$$

(B.43)

Again we only need to calculate $(I^{D,b})_{\ell u}(\mathbf{0})$ to get the relevant correction to the noise strength D.

$$(I^{D,b})_{\ell u}(\mathbf{0}) = - \frac{1}{(2\pi)^{d+1}} \int_{\mathbf{q}} q_i^+ q_m^+ C_{i u}(\mathbf{q}) C_{\ell m}(\mathbf{q})$$

$$= - \frac{1}{(2\pi)^{d+1}} \int_{\mathbf{q}} q_i^+ q_m^+ \left[ |G_L(\mathbf{q})|^2 L_{i u}^+(\mathbf{q}) + |G_T(\mathbf{q})|^2 P_{i u}^+(\mathbf{q}) \right] \left[ |G_L(\mathbf{q})|^2 L_{\ell u}^+(\mathbf{q}) + |G_T(\mathbf{q})|^2 P_{\ell u}^+(\mathbf{q}) \right]$$

$$= - \frac{1}{(2\pi)^{d+1}} \int_{\mathbf{q}} q_i^+ q_m^+ \left[ |G_L(\mathbf{q})|^4 \right]$$

$$= - \frac{3}{64 \sqrt{\mu_x \mu_L}} \frac{1}{(2\pi)^{d-1}} \int_{\mathbf{q}} q_i^+ q_m^+$$

$$= - \frac{3 \delta_{\ell u}^+}{64 (d-1) \sqrt{\mu_x \mu_L}} \frac{S_{d-1}}{(2\pi)^{d-1}} \Lambda^{d-4} d\ell .$$

(B.44)

Identifying the coefficient of $\delta_{\ell u}^+$ as a correction (δD)_{D,b} to D gives:

$$(\delta D)_{D,b} = - \frac{3 D^2 \lambda^2 S_{d-1}}{32 (d-1) \sqrt{\mu_x \mu_L}} \Lambda^{d-4} d\ell = \frac{3 g_1 D d\ell}{32 (d-1) (1 + g_2)^{5/2}} .$$

(B.45)
Combining the two corrections to $D$ from the two diagrams in Fig. 4, we obtain the total one loop correction $\delta D$ to $D$:

$$\delta D = (\delta D)_{6a} + (\delta D)_{6b} = \frac{g_1 D d \ell}{2g_2^2} \left( \frac{d-2}{d-1} \left[ 1 + \frac{1}{\sqrt{1+g_2}} - \frac{2\sqrt{2}}{\sqrt{2+g_2}} \right] = g_1 D d \ell G_D(g_2), \right.$$

$$(B.46)$$

where $G_D(g_2)$ is given by [IV.12].

4. Summary of all corrections to one loop order

Adding up the results obtained in previous sections gives the total one loop graphical corrections to the various parameters:

$$\delta \mu_1 = g_1 \mu_1 G_{\mu_1}(g_2) d \ell,$$

$$\delta \mu_2 = g_1 \mu_2 G_{\mu_2}(g_2) d \ell,$$

$$\delta D = g_1 D G_D(g_2) d \ell,$$

$$\delta \mu_x = 0.$$

(B.47) \hspace{1cm} (B.48) \hspace{1cm} (B.49) \hspace{1cm} (B.50)

Dividing both sides of each of these equations by $d \ell$, we obtain the graphical contributions to the recursion relations for the parameters of our model:

$$\left( \frac{d\mu_1}{d\ell} \right)_{\text{graph}} = g_1 \mu_1 G_{\mu_1}(g_2),$$

$$\left( \frac{d\mu_2}{d\ell} \right)_{\text{graph}} = g_1 \mu_2 G_{\mu_2}(g_2),$$

$$\left( \frac{dD}{d\ell} \right)_{\text{graph}} = g_1 D G_D(g_2),$$

$$\left( \frac{d\mu_x}{d\ell} \right)_{\text{graph}} = 0.$$

(B.51) \hspace{1cm} (B.52) \hspace{1cm} (B.53) \hspace{1cm} (B.54)

where the functions $G_{\mu_1}(g_2)$, $G_{\mu_2}(g_2)$, and $G_D(g_2)$ are given respectively by equations [IV.14], [IV.16], and [IV.12] of section [IV]. Combining these with the RG rescalings discussed in that section lead to the full recursion relations [IV.6]-[IV.10] of that section.

It is also a straightforward, but tedious, exercise in the application of l’Hopital’s rule to show that, in the limit $g_2 \to 0$, all of the apparent singularities at small $g_2$ in $G_{\mu_1}(g_2)$, $G_{\mu_2}(g_2)$, and $G_D(g_2)$ exactly cancel, leaving precisely the finite results [A.43]-[A.45] obtained in the previous appendix for the $\mu_2 = 0$ case.

Appendix C: Useful Formulae

In this appendix, we summarize the integrals needed for the graphical calculations done in the preceding appendices. Throughout this section, we will for convenience define the wavevector dependent dampings:

$$\Gamma_L \equiv \frac{1}{\mu_L q_\perp^2 + \mu_x q_x^2}, \quad \Gamma_T \equiv \frac{1}{\mu_T q_\perp^2 + \mu_x q_x^2}.$$

(C.1)

where we remind the reader of our definition $\mu_L \equiv \mu_1 + \mu_2$.

We begin with:
1. Integrations over $\Omega$ and $q_x$

\[
\frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} d\Omega \ G_L(\hat{q})G_L(-\hat{q}) = \frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} d\Omega \ \frac{1}{\Omega^2 + \Gamma_L(q)} = \frac{1}{(2\pi)^{d}} \int_{-\infty}^{\infty} dq_x \ \frac{1}{2\Gamma_L(q)} = \frac{1}{(2\pi)^{d-1}} \frac{1}{4\sqrt{\mu_x \mu_L}} \frac{1}{q_\perp^4},
\]

where both the integrals over $\Omega$ and $q_x$ can be done either by simple complex contour techniques, or by even simpler trigonometric substitutions. The same statement applies to all of the integrals that follow here; we will therefore simply quote the results for the remainder:

\[
\frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} d\Omega \ G_L(\hat{q})G_L(-\hat{q})^2 = \frac{1}{(2\pi)^{d}} \int_{-\infty}^{\infty} dq_x \ \frac{1}{4\Gamma_L(q)^2} = \frac{1}{16\sqrt{\mu_x \mu_L}} \frac{1}{(2\pi)^{d-1}} \frac{1}{q_\perp^3}. \tag{C.3}
\]

\[
\frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} d\Omega \ G_L(\hat{q})G_L(-\hat{q})^3 = \frac{1}{(2\pi)^{d}} \int_{-\infty}^{\infty} dq_x \ \frac{1}{8\Gamma_L(q)^3} = \frac{3}{128\sqrt{\mu_x \mu_L}} \frac{1}{(2\pi)^{d-1}} \frac{1}{q_\perp^2}. \tag{C.4}
\]

\[
\frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} d\Omega \ G_L(\hat{q})G_L(-\hat{q})^4 = \frac{1}{(2\pi)^{d}} \int_{-\infty}^{\infty} dq_x \ \frac{1}{16\Gamma_L(q)^4} = \frac{5}{512\sqrt{\mu_x \mu_L}} \frac{1}{(2\pi)^{d-1}} \int_{q_\perp} \frac{1}{q_\perp^5}. \tag{C.5}
\]

\[
\frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} d\Omega \ G_L(\hat{q})G_L(-\hat{q})^5 = \frac{1}{(2\pi)^{d}} \int_{-\infty}^{\infty} dq_x \ \frac{1}{64\Gamma_L(q)^5} = \frac{1}{512\sqrt{\mu_x \mu_L}} \frac{1}{(2\pi)^{d-1}} \int_{q_\perp} \frac{1}{q_\perp^6}. \tag{C.6}
\]

\[
\frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} d\Omega \ G_L(\hat{q})G_L(-\hat{q})^2 = \frac{1}{(2\pi)^{d}} \int_{-\infty}^{\infty} dq_x \ \frac{1}{4\Gamma_T(q)^2} = \frac{3}{64\sqrt{\mu_x \mu_L}} \frac{1}{(2\pi)^{d-1}} \int_{q_\perp} \frac{1}{q_\perp^3}. \tag{C.7}
\]

\[
\frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} d\Omega \ G_T(\hat{q})G_T(-\hat{q})^2 = \frac{1}{(2\pi)^{d}} \int_{-\infty}^{\infty} dq_x \ \frac{1}{16\Gamma_T(q)^2} = \frac{1}{512\sqrt{\mu_x \mu_L}} \frac{1}{(2\pi)^{d-1}} \frac{1}{q_\perp^3}. \tag{C.8}
\]

\[
\frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} d\Omega \ G_T(\hat{q})G_T(-\hat{q})^3 = \frac{1}{(2\pi)^{d}} \int_{-\infty}^{\infty} dq_x \ \frac{1}{8\Gamma_T(q)^3} = \frac{3}{128\sqrt{\mu_x \mu_L}} \frac{1}{(2\pi)^{d-1}} \frac{1}{q_\perp^2}. \tag{C.9}
\]

\[
\frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} d\Omega \ G_T(\hat{q})G_T(-\hat{q})^4 = \frac{1}{(2\pi)^{d}} \int_{-\infty}^{\infty} dq_x \ \frac{1}{16\Gamma_T(q)^4} = \frac{5}{512\sqrt{\mu_x \mu_L}} \frac{1}{(2\pi)^{d-1}} \int_{q_\perp} \frac{1}{q_\perp^5}. \tag{C.10}
\]

\[
\frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} d\Omega \ G_T(\hat{q})G_T(-\hat{q})^5 = \frac{1}{(2\pi)^{d}} \int_{-\infty}^{\infty} dq_x \ \frac{1}{64\Gamma_T(q)^5} = \frac{1}{512\sqrt{\mu_x \mu_L}} \frac{1}{(2\pi)^{d-1}} \frac{1}{q_\perp^6}. \tag{C.11}
\]

\[
\frac{1}{(2\pi)^{d+1}} \int_{q_\perp} G_T(\hat{q})G_T(-\hat{q})^2 = \frac{1}{(2\pi)^{d}} \int_{-\infty}^{\infty} dq_x \ \frac{1}{4\Gamma_T(q)^2} = \frac{3}{64\sqrt{\mu_x \mu_L}} \frac{1}{(2\pi)^{d-1}} \int_{q_\perp} \frac{1}{q_\perp^3}. \tag{C.12}
\]
\[
\frac{1}{(2\pi)^{d+1}} \int_{\mathbf{q}} G_L(\mathbf{q}) G_L(-\mathbf{q}) G_T(\mathbf{q}) G_T(-\mathbf{q}) = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} dq_x \frac{1}{2\Gamma_L(\mathbf{q}) \Gamma_T(\mathbf{q}) (\Gamma_L(\mathbf{q}) + \Gamma_T(\mathbf{q}))} = \frac{1}{2\sqrt{\mu_L(\mu_L - \mu_1)^2} + \frac{2\sqrt{2}}{\sqrt{\mu_L} - \sqrt{\mu_L + \mu_1}} - \frac{1}{(2\pi)^{d-1}} \int q_{\perp} \frac{1}{q_{\perp}^5}.
\]

(C.13)

\[
\frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} d\Omega \ G_L(\mathbf{q}) G_L(-\mathbf{q}) G_T(-\mathbf{q}) = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} dq_x \frac{1}{4\Gamma_L(\mathbf{q}) (\Gamma_L(\mathbf{q}) + \Gamma_T(\mathbf{q}))} = \frac{1}{4\sqrt{\mu_L(\mu_L - \mu_1)^2} + \frac{2\sqrt{2}}{\sqrt{\mu_L} - \sqrt{\mu_L + \mu_1}} + \frac{1}{(2\pi)^{d-1}} \int q_{\perp} \frac{1}{q_{\perp}^5}.
\]

(C.14)

\[
\frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} d\Omega \ G_L(\mathbf{q}) G_L(-\mathbf{q}) G_T(-\mathbf{q})^2 = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} dq_x \frac{1}{2\Gamma_L(\mathbf{q}) (\Gamma_L(\mathbf{q}) + \Gamma_T(\mathbf{q}))^2} = \frac{1}{2\sqrt{\mu_L(\mu_L +\mu_1)^2} - \frac{2\mu_L}{\mu_L + 3\mu_1} + \frac{1}{8\mu_L(\mu_L +\mu_1)^2} (\mu_L - \mu_1)^2} (2\pi)^{d-1} \int q_{\perp} \frac{1}{q_{\perp}^5}.
\]

(C.15)

2. Integrals over \(q_{\perp}\)

After performing the integrals over \(\Omega\) and \(q_x\), our last step is performing the remaining integral over \(q_{\perp}\) in

\[
\int_{\mathbf{q}} \overrightarrow{\mathbf{q}} \equiv \int_{\Lambda_{\mathbf{q}_{\perp} > \Lambda e^{-d\ell}}} d^{d-1} q_{\perp} \int_{-\infty}^{\infty} d\Omega \int_{-\infty}^{\infty} dq_x.
\]

(C.16)

The simplest such integral that arises is:

\[
\frac{1}{(2\pi)^{d-1}} \int_{\Lambda_{\mathbf{q}_{\perp} > \Lambda e^{-d\ell}}} d^{d-1} q_{\perp} q_{\perp}^5 = \frac{1}{(2\pi)^{d-1}} \int_{\Lambda e^{-d\ell}} d\xi_{\perp} q_{\perp}^{d-5},
\]

(C.17)

where \(d\xi_{\perp}\) denotes an integral over the \(d-1\)-dimensional solid angle associated with \(q_{\perp}\). Since the integrand is independent of the direction of \(q_{\perp}\), we can do this integral trivially; it simply gives a multiplicative factor of \(S_{d-1}\), the surface area of a unit \(d-1\)-dimensional sphere. Thus we have

\[
\frac{1}{(2\pi)^{d-1}} \int_{\Lambda_{\mathbf{q}_{\perp} > \Lambda e^{-d\ell}}} d^{d-1} q_{\perp} q_{\perp}^5 = \frac{S_{d-1}}{(2\pi)^{d-1}} \int_{\Lambda e^{-d\ell}} d\xi_{\perp} q_{\perp}^{d-5} = \frac{S_{d-1}}{(2\pi)^{d-1}} \Lambda^{d-1} d\ell,
\]

(C.18)

where in the last step we have used the fact that \(d\ell\) is infinitesimal to write \(1 - e^{-d\ell} = d\ell\).

A slightly harder integral that arises in our calculations is

\[
I_{iu} = \frac{1}{(2\pi)^{d-1}} \int_{q_{\perp}} q_{\perp}^{d-1} q_{\perp}^{d-1}.
\]

(C.19)

This can, however, be done using symmetry arguments. Since the integrand is odd if \(i \neq u\), this integral can only be non-zero if \(i = u\). Furthermore, by spherical symmetry, if \(i = u\), we should get the same value for this integral regardless of the value of \(i\). Hence, this integral must be proportional to \(d_{iu}^2\):

\[
I_{iu} = A d_{iu}^2,
\]

(C.20)

where the constant \(A\) remains to be determined. This can be done by taking the trace of (C.20) over \(iu\), which gives

\[
I_{ii} = A(d - 1).
\]

(C.21)
On the other hand, taking the trace over $iu$ of (C.19) gives

$$I_{ii} = \frac{1}{(2\pi)^{d-1}} \int_{q^+} \frac{1}{q^+} = \frac{S_{d-1}}{(2\pi)^{d-1}} \Lambda^{d-4} d\ell ,$$

(C.22)

where in the second equality we have used (C.18).

Equating (C.22) and (C.21) and solving for $A$ gives

$$A = \frac{S_{d-1}}{(d-1)(2\pi)^{d-1}} \Lambda^{d-4} d\ell ,$$

(C.23)

whence it follows from (C.20) that

$$\frac{1}{(2\pi)^{d-1}} \int_{q^+} \frac{q^+_{ij} q^+_{kl}}{q^+_{ij} q^+_{kl}} = \frac{\delta_{ij} \delta_{kl} - \delta_{ij} \delta_{jl} - \delta_{ik} \delta_{j+} - \delta_{il} \delta_{j+} - \delta_{ik} \delta_{jl}}{\Lambda^{d-4} d\ell} .$$

(C.24)

Very similar reasoning can be used to show that

$$\frac{1}{(2\pi)^{d-1}} \int_{q^+} \frac{q^+_{ij} q^+_{kl}}{q^+_{ij} q^+_{kl}} = \frac{\delta_{ij} \delta_{kl} + \delta_{ij} \delta_{jl} + \delta_{ik} \delta_{j+} + \delta_{il} \delta_{j+} + \delta_{ik} \delta_{jl}}{\Lambda^{d-4} d\ell} .$$

(C.25)

3. Expansions with respect to $k$

At many points in our calculations, we have to expand the propagators and projection operators in powers of the external wavevector. These expansions are:

$$L_{ij}(k - q) = \frac{(k_i - q_i)(k_j - q_j)}{|k - q|^2} = \frac{(k_i - q_i)(k_j - q_j)}{q^2} \left( 1 + \frac{2q \cdot k - k^2}{q^2} \right) + O(k^3)$$

$$= L_{ij}(q) + \frac{2L_{ij}(q)q \cdot k - k^2 q_i - k^2 q_j}{q^2} + \frac{L_{ij}(q)q_i k_j + L_{ij}(q)q_j k_i - 2q \cdot k(k_i q_j + k_j q_i)}{q^2} + O(k^3) .$$

(C.26)

$$P_{ij}(k - q) = \frac{1}{i\omega + \mu L(k - q)} = \frac{1}{i\omega + \mu L(k - q)} \left( 1 + \frac{2\mu \cdot q - k^2}{q^2} \right) + \frac{4\mu^2 q^2 k^2}{q^2} + O(k^3)$$

$$= G_L(-q) + \left( 2\mu \cdot q \cdot k + 2\mu q \cdot k + \mu k^2 \right) G_L(-q) - \frac{1}{i\omega - \Omega + \Gamma(q)} \left( 1 + \frac{2\mu \cdot q \cdot k + 2\mu q \cdot k^2}{i\omega - \Omega + \Gamma(q)} \right) + \frac{4\mu^2 q^2 k^2}{q^2} + O(k^3) .$$

(C.27)

$$G_L(-\tilde{q}) = \frac{1}{i\omega + \mu L(-\tilde{q})} = \frac{1}{i\omega - \Omega + \Gamma(q)} \left( 1 + \frac{2\mu \cdot q \cdot k + 2\mu q \cdot k + \mu k^2}{i\omega - \Omega + \Gamma(q)} \right) + \frac{4\mu^2 q^2 k^2}{q^2} + O(k^3) .$$

(C.28)
\[ G_L(\tilde{k} - \tilde{q})L_{\tilde{q}}(\mathbf{k} - \mathbf{q}) = G_L(-\tilde{q})L_{\tilde{q}}(\mathbf{q}) + (2\mu L\mathbf{q} \cdot \mathbf{k} + 2\mu_{\perp}q_xk_x)G_L(-\tilde{q})^2L_{\tilde{q}}(\mathbf{q}) + \frac{2L_{\tilde{q}}(\mathbf{q})\mathbf{q} \cdot \mathbf{k} - k^+_Lq^+_L - k^+_Lq^+_L}{q^+_L}G_L(-\tilde{q}) - (\mu Lk^2 + \mu_{\perp}k^2)L_{\tilde{q}}(\mathbf{q}) + \left[ \left(4\mu_L^2(q_{\perp} \cdot k_{\perp})^2 + 4\mu_{\perp}^2q_{\perp}^2k_x^2 + 4\mu_{\perp}\mu_L(q_{\perp} \cdot k_{\perp})q_xk_x \right) G_L(-\tilde{q})^3L_{\tilde{q}}(\mathbf{q}) + \frac{L_{\tilde{q}}(\mathbf{q}) \left( \frac{k^2_{\perp}}{q^2_{\perp}} + \frac{4(q_{\perp} \cdot k_{\perp})^2}{q^2_{\perp}} \right) + \frac{k^+_Lk^+_L}{q^+_L} - \frac{2q_{\perp} \cdot k_{\perp}}{q^+_L} \left( k^+_Lq^+_L + k^+_Lq^+_L \right) }{q^+_L} G_L(-\tilde{q}) + (2\mu_Lq_{\perp} \cdot k_{\perp} + 2\mu_{\perp}q_xk_x) G_L(-\tilde{q})^2 \left( \frac{2L_{\tilde{q}}(\mathbf{q})q_{\perp} \cdot k_{\perp} - k^+_Lq^+_L - k^+_Lq^+_L}{q^+_L} \right) \right]. \] (C.29)

\[ G_T(\tilde{k} - \tilde{q})P_{\tilde{q}}(\mathbf{k} - \mathbf{q}) = G_T(-\tilde{q})P_{\tilde{q}}(\mathbf{q}) + (2\mu_{\perp}q_{\perp} \cdot k_{\perp} + 2\mu_{\perp}q_xk_x) G_T(-\tilde{q})^2 P_{\tilde{q}}(\mathbf{q}) - \frac{2L_{\tilde{q}}(\mathbf{q})q_{\perp} \cdot k_{\perp} - k^+_Lq^+_L - k^+_Lq^+_L}{q^+_L} G_T(-\tilde{q}) - (\mu_{\perp}k^2 + \mu_{\perp}k^2) G_T(-\tilde{q})^2 P_{\tilde{q}}(\mathbf{q}) - \left[ \left(4\mu_L^2(q_{\perp} \cdot k_{\perp})^2 + 4\mu_{\perp}^2q_{\perp}^2k_x^2 + 4\mu_{\perp}\mu_L(q_{\perp} \cdot k_{\perp})q_xk_x \right) G_T(-\tilde{q})^3 P_{\tilde{q}}(\mathbf{q}) - \frac{L_{\tilde{q}}(\mathbf{q}) \left( \frac{k^2_{\perp}}{q^2_{\perp}} + \frac{4(q_{\perp} \cdot k_{\perp})^2}{q^2_{\perp}} \right) + \frac{k^+_Lk^+_L}{q^+_L} - \frac{2q_{\perp} \cdot k_{\perp}}{q^+_L} \left( k^+_Lq^+_L + k^+_Lq^+_L \right) }{q^+_L} G_T(-\tilde{q}) - (2\mu_{\perp}q_{\perp} \cdot k_{\perp} + 2\mu_{\perp}q_xk_x) G_T(-\tilde{q})^2 \left( \frac{2L_{\tilde{q}}(\mathbf{q})q_{\perp} \cdot k_{\perp} - k^+_Lq^+_L - k^+_Lq^+_L}{q^+_L} \right) \right]. \] (C.30)

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