WEIGHT STRUCTURES AND THE ALGEBRAIC $K$-THEORY OF
STABLE $\infty$-CATEGORIES

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Abstract. We introduce the notion of a bounded weight structure on a stable $\infty$-category
and use this to prove the natural generalization of Waldhausen’s sphere theorem: We show
that the algebraic $K$-theory of a stable $\infty$-category with a bounded non-degenerate weight
structure is equivalent to the algebraic $K$-theory of the heart of the weight structure.

1. Introduction

Algebraic $K$-theory is an invariant of rings that arose from considering Euler characteristics. The classical algebraic $K$-groups $K_0$, $K_1$, and $K_2$ were defined algebraically and were known to fit into exact sequences like those for cohomology theories. Quillen introduced a homotopical construction of an algebraic $K$-theory spectrum whose homotopy groups extended the classical definitions to define higher algebraic $K$-theory groups $K_i$ for $i \geq 0$ [Qui73].

The applications for algebraic $K$-theory span several fields. The algebraic $K$-theory of rings of integers in number fields contains arithmetic information: the class group, the Brauer group, and so on [Wei13]. On the other hand, the Whitehead torsion of a manifold $M$ is controlled by $K_1(\mathbb{Z}[\pi_1 M])$. In a vast generalization of this, Waldhausen showed that the algebraic $K$-theory of $\Sigma^\infty(\Omega M)_+$, the “spherical group ring of the loop space of $M$”, contains information about a stabilization of $B\text{Diff}(M)$ [Wal85].

$K$-theory is very difficult to compute. Trace methods compare it to topological cyclic homology and compute that in favorable conditions. Direct computations of $K$-theory involve reduction theorems, results proving that some $K$-theory is equivalent to the $K$-theory of something simpler. The chief examples of these reduction theorems are Quillen’s devissage theorem [Qui73, 4] and Waldhausen’s fibration [Wal85, 1.6.4], approximation [Wal85, 1.6.7], and sphere [Wal85, 1.7.1] theorems.

Following Quillen and Waldhausen, the modern view is that algebraic $K$-theory is a functor of modules categories (or their subcategories) to spectra. Recent work has extended the construction of algebraic $K$-theory to higher categories that behave like module categories and produced universal characterizations of the algebraic $K$-theory functor in this setting [BGT13, Bar16]. One consequence of this work is that most of Quillen and Waldhausen’s foundational theorems about the behavior of algebraic $K$-theory have been established in a very general context. In particular, this framework has permitted new localization [BM08, Bar16], devissage [Bar15], and approximation [Bar16, Fio13] results.

However, there has been no counterpart to the sphere theorem in the $K$-theory of higher categories. In Waldhausen’s setting, a category $\mathcal{C}$ equipped with cofibrations, weak equivalences, a cylinder functor, and a well-behaved homology theory is shown to have the same $K$-theory, after stabilizing, as the stable homology spheres. Waldhausen’s homology theories
are required to satisfy a condition which makes them display all objects of $\mathcal{C}$ as cell objects weakly built out of homology spheres. Essentially, the sphere theorem reduces the $K$-theory of a category to the $K$-theory of a subcategory when those objects can construct all the other objects in the category as finite cell complexes.

To provide such a theorem for higher categories, we need a suitable notion of cell object. We will use Bondarko’s theory of weight structures.

Weight structures were introduced by Bondarko in [Bon09] and studied extensively on triangulated categories [Bon10a, Bon10b, Bon13, Bon10b, Bon09]. A weight structure is a collection of data on a triangulated category which is designed to provide a weak notion of cellular filtrations. Advantageously, weight structures are specified entirely on the homotopy category of our stable $\infty$-category $\mathcal{C}$. The data of a weight structure consists of choices of subcategories of objects built with cells in degrees $\leq n$ (or $\geq n$) so that every object in $\mathcal{C}$ has at least one associated $n$-skeleton for all $n \in \mathbb{Z}$. These skeleta are not assumed to be functorial in any way and in fact rarely are. We emphasize that a weight structure should be thought of as providing weak $n$-skeleta for all objects in $\mathcal{C}$.

Familiar examples of weight structures include CW-structures on spectra and truncation on chain complexes of finitely-generated modules. The former is bounded on finite spectra and the latter is bounded when the chain complexes are bounded. Work of Bondarko has established a weight structure on Voevodsky’s category of effective motives whose heart consists of the Chow motives [Bon09]. On compact objects, this forms a bounded weight structure as well.

This paper essentially completes the program of lifting fundamental theorems of Quillen and Waldhausen to the algebraic $K$-theory of higher categories by providing an analog of Waldhausen’s sphere theorem [Wal85, 1.7.1]. Where Waldhausen approaches finite cell objects with homology functors, we use bounded weight structures.

**Theorem** (Theorem 5.1). If $\mathcal{C}$ is a stable $\infty$-category equipped with a bounded non-degenerate weight structure $w$, then the inclusion of the heart of the weight structure $\mathcal{C}_{\heartsuit} \hookrightarrow \mathcal{C} \rightarrow \mathcal{C}_{\heartsuit}$ induces an equivalence on algebraic $K$-theory $K(\mathcal{C}) \simeq K(\mathcal{C}_{\heartsuit})$.

The main theorem is an example of an equivalence between $K$-theory spectra that is not induced by an equivalence of derived categories. Quillen’s devissage theorem [Qui73, 4] provides such an equivalence. Blumberg and Mandell’s devissage theorem for ring spectra [BM08] and the closely-related theorem of the heart due to Barwick [Bar15] are the only other such non-trivial equivalences known to the author.

In addition to lifting Waldhausen’s sphere theorem to quasicategories, the main theorem generalizes several previous results in the literature. Bondarko proves a version of the theorem on $K_0$ for triangulated categories equipped with bounded weight structures [Bon10a, 5.3.1]. We reproduce his result as a corollary of our theorem.

**Corollary** (Corollary 5.2). If $\mathcal{T}$ is a triangulated category equipped with a non-degenerate bounded weight structure $w$, then $K_0(\mathcal{T}) \simeq K_0(\mathcal{T}_{\heartsuit})$.

The projective weight structure on bounded chain complexes gives an equivalence of $K(R)$ with the $K$-theory of the category of finitely-generate projective $R$-modules. Generalizing this result slightly, we produce a new proofs of the Gillet–Waldhausen Theorem and the Resolution Theorem for exact categories:
Corollary (Corollary 6.5, Gillet–Waldhausen Theorem, [TT90, 1.11.7]). For an exact category $\mathcal{E}$ which is idempotent-complete, Quillen’s algebraic $K$-theory $K(\mathcal{E})$ is homotopy equivalent to the Waldhausen algebraic $K$-theory of $\text{Ch}^{\text{bdd}}(\mathcal{E})$, the Waldhausen category of bounded chain complexes on $\mathcal{E}$ where cofibrations are taken to be admissible monomorphisms and weak equivalences are quasi-isomorphisms of chain complexes.

Corollary (Corollary 6.7, Resolution Theorem, [Wei13, Theorem V.3.1]). Let $\mathcal{P}$ be a full subcategory of an exact category $\mathcal{E}$ so that $\mathcal{P}$ is closed under extensions and under kernels of admissible surjections in $\mathcal{E}$. Suppose in addition that every object $M$ in $\mathcal{E}$ admits a finite $\mathcal{P}$-resolution:

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

then $K(\mathcal{P}) \simeq K(\mathcal{E})$.

The chief motivation for the main theorem is that $C_{\heartsuit}w$ is simpler than $C$ and so its $K$-theory can be described in an alternate manner. In particular, all cofiber sequences in the heart split in the homotopy category. Hence, $K(C_{\heartsuit}w)$ permits a description in the spirit of Quillen’s plus construction for $K$-theory:

Theorem (Corollary 5.16). If $C_{\heartsuit}w$ is the heart of a weight structure then all cofiber sequences split in the homotopy category and

$$K(C_{\heartsuit}w) \simeq K_0(C) \times (\text{hocolim}_{C_{\heartsuit}w, +} B\text{Aut}(X))^+$$

where $[X]$ is an equivalence class of objects in $C_{\heartsuit}w$ and $(-)^+$ denotes the group completion of the topological monoid.

1.1. Relation to other results. It would be appropriate at this point to issue a word of clarification about the various theorems relating to hearts. Neeman proved a theorem of the heart for the $K$-theory of triangulated categories [Nee98, Nee99, Nee01] which was later proven more generally for exact $\infty$-categories by Barwick [Bar15]. In both cases, the theorem said that a bounded $t$-structure on the homotopy category induced an equivalence $K(C) \simeq K(C_{\heartsuit}t)$ between the $K$-theory of the category and that of the heart of the $t$-structure. While there are superficial similarities between the theorems—and, as it turns out, the definitions of weight and $t$-structures—these theorems have little to do with one another. The theorems for $t$-structures generalize Quillen’s devissage theorem whereas the theorem for weight structures generalizes Waldhausen’s sphere theorem. A $t$-structure provides a Postnikov tower for every object of $C$ whereas a weight structure provides a cellular filtration. Furthermore, while $t$-structures (and Postnikov towers) are functorial, weight structures (and cellular filtrations) rarely are.

While this paper was transitioning from dissertation to journal form, several closely related results have appeared in the literature.

Sosnilo proves that the algebraic $K$-theory of $C_{\heartsuit}w$, or $K(Hw_{\infty})$ in his notation, is homotopy equivalent $K(C)$ [Sos17, 4.1]. Although this result appears identical to Theorem 5.1, his definition of $K(C_{\heartsuit}w)$ differs from ours. The $K$-theory functor in Sosnilo’s work is the nonconnective $K$-theory of stable $\infty$-categories of [BGT13]. Since $C_{\heartsuit}w$ is an additive $\infty$-category, Sosnilo’s $K(C_{\heartsuit}w)$ is defined to be the $K$-theory of the formal closure of $C_{\heartsuit}w$ under finite limits and colimits [Sos17, p. 15]. If $w$ is bounded, the closure of $C_{\heartsuit}w$ under finite limits and colimits is canonically equivalent to $C$. 

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In contrast, Theorem 5.1 uses Barwick’s $K$-theory functor for Waldhausen $\infty$-categories without passing through the $K$-theory of stable $\infty$-categories. A priori, Barwick’s construction for additive $\infty$-categories need not agree with the $K$-theory of their formal closure. Theorem 5.1 proves that it does in the presence of a bounded weight structure.

Heleodoro appears to have arrived independently at the proof of the sphere theorem, which appears as [Hel19, Theorem 5] and has a similar proof to the one presented in this paper. Mochizuki claims the sphere theorem and the theorem of the heart for $t$-structures as a consequence of a more general result [Moc19].

In private correspondence [Ant17], Ben Antieau describes an alternative approach to proving the sphere theorem with the technology of [EKMM97] which may provide an advantageous perspective for the reader. For any finite set of objects $X$ in $\mathcal{C}_{\mathcal{W}}$, let $E(X)$ be the endomorphism spectrum of the finite wedge of objects in $X$. Since the objects of $X$ lie in the heart of a weight structure, $E(X)$ is a connective ring spectrum. The additive closure of $X$ in $\mathcal{C}$ is a full subcategory of $\mathcal{C}_{\mathcal{W}}$ and can be identified with $\infty$-category of compact projective $E(X)$-modules, $\text{Proj}(E(X))$. The full idempotent-complete stable subcategory of $\mathcal{C}$ generated by $X$ can be identified with $\text{Perf}(E(X))$, the $\infty$-category of compact $E(X)$-modules. We can now utilize [EKMM97, IV.7.1] to conclude that $K(\text{Proj}(E(X))) \simeq K(\text{Perf}(E(X)))$.

Algebraic $K$-theory commutes with filtered colimits, so we take a colimit over all finite subsets $X$ of the heart. The colimit of the additive closures will be the heart $\mathcal{C}_{\mathcal{W}}$, and the colimit of the stable idempotent-complete closures will be $\mathcal{C}$ because the weight structure is bounded. We conclude $K(\mathcal{C}_{\mathcal{W}}) \simeq \text{colim}_X K(\text{Proj}(E(X))) \simeq \text{colim}_X K(\text{Perf}(E(X))) \simeq K(\mathcal{C})$. Instead of utilizing this approach, we present an internal proof to the result in Barwick’s algebraic $K$-theory machine that we hope will be of independent interest.

1.2. Outline of paper. In section 2 we provide a brief introduction Barwick’s Waldhausen $\infty$-categories and highlight the pair structures of interest for this paper.

In section 3 we define weight structures and recount some of their properties. We develop a yoga for manipulating weights and recount results of Bondarko on generating weight structures on categories. Finally, we compare the language of weight structures to Waldhausen’s formulation of the sphere theorem.

In section 4 we build the technical tools we require for the proof of the main theorem. We introduce cellular filtrations arising from weight structures and prove that they form Waldhausen $\infty$-categories. We show that localizing these categories at equivalences reflected from $\mathcal{C}$ induces a suitable model for the $K$-theory of $\mathcal{C}$.

Section 5 comprises the proof of the main theorem. We rely on tools developed in sections 3 and 4 as well as results of Barwick [Bar16]. Finally, we prove our version of the “$+ = \mathbb{Q}$” theorem: the $K$-theory of the heart of a weight structure admits a description analogous to Quillen’s plus construction.

Finally, in section 6 we enumerate examples of weight structures and explain applications of the main theorem in each case. The Gillet–Waldhausen Theorem and the Resolution Theorem appear as consequences of the main theorem. Several conjectural examples are mentioned that merit exploration in future work.

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2. Towards Waldhausen ∞-categories

Algebraic $K$-theory is a functor which takes homotopical data as input and produces a spectrum.

**Definition 2.1 (Wal85, 1.2).** A Waldhausen category is a pointed category $C$ equipped with subcategories, $wC$ (the weak equivalences, denoted $\sim$) and $cofC$ (the cofibrations, denoted $\hookrightarrow$), satisfying the following axioms

(C1) The isomorphisms of $C$ are contained in $cofC$.
(C2) For every object $A \in C$, the unique map $\ast \rightarrow A$ is a cofibration.
(C3) For every diagram

\[
\begin{array}{ccc}
A & \hookrightarrow & B \\
\downarrow\sim & & \downarrow\sim \\
A' & \hookrightarrow & B'
\end{array}
\]

the pushout $A \cup_B C$ exists in $C$ and $C \rightarrow A \cup_B C$ is a cofibration.

(W1) The isomorphisms of $C$ are contained in $wC$.
(W2) For each commutative diagram

\[
\begin{array}{ccc}
A & \hookrightarrow & B \\
\downarrow\sim & & \downarrow\sim \\
A' & \hookrightarrow & B'
\end{array}
\]

the induced map on pushouts $A \cup_B C \rightarrow A' \cup_{B'} C'$ is a weak equivalence.

Waldhausen categories encode homotopical information. Specifically, they encode the data of which quotients (or pushouts) in the category are homotopically meaningful. Axiom (C3) guarantees that pushouts of cofibrations exist while axiom (W2) implies that the weak equivalence type of those pushouts is not altered when the diagram is replaced by a weakly equivalent one.

The algebraic $K$-theory of a Waldhausen category $C$ is constructed out of a simplicial object encoding information about a sequence of cofibrations in $C$ and their successive quotients. This is Waldhausen’s $S_\bullet$-construction, and the algebraic $K$-theory of $C$ is defined to be the space

\[K(C) := \Omega|wS_\bullet C|\]

which deloops to form the connective $K$-theory spectrum which we will also denote $K(C)$ [Wal85, 1.3].

In essence, the algebraic $K$-theory functor takes as input some category with a notion of homotopy theory and information about the homotopically-meaningful quotients in that category. The modern perspective is that any model for a category equipped with a version of homotopy theory (such as a model category or a Waldhausen category) has an underlying $\infty$-category. In this paper, we make use of Barwick’s construction of algebraic $K$-theory [Bar16] for $\infty$-categories.

Specifically, Barwick defines $\infty$-categorical analogs for Waldhausen categories which are the input to his $K$-theory machine. These Waldhausen $\infty$-categories are pairs of $\infty$-categories (cf. Bar16 Definition 1.11) $(C, C_t)$ subject to several axioms. $C_t$ is a subcategory of $C$ that we will refer to as the ingressive morphisms of $C$. The ingressive morphisms are required to contain the maximal Kan subcomplex $\iota C$, and the Waldhausen structure on this pair requires a zero object to exist in $C$, all maps from zero objects to be ingressive, and
pushouts of ingressive morphisms to be ingressive. Functors which respect Waldhausen ∞-categories are called \textit{exact} and are required to take zero objects to zero objects, ingressives to ingressives, and pushouts squares of ingressives to pushout squares. (For a precise definition, cf. \cite[Definition 2.7]{Bar16}.)

An ∞-category with the minimal pair structure \( C^\dagger = \iota C \) will be Waldhausen when it is a contractible Kan complex. An ∞-category with the maximal pair structure \( C^\dagger = C \) will be Waldhausen when it has a zero object and admits finite colimits. A stable ∞-category is always a Waldhausen ∞-category via the maximal pair structure. The relative nerve can produce a Waldhausen ∞-category from an ordinary Waldhausen category by taking \( \operatorname{N}(C,wC)^\dagger \) to be the smallest subcategory of \( \operatorname{N}(C,wC) \) that contains the equivalences and the edges of \( NC \) corresponding to cofibrations in \( \operatorname{cof C} \). Barwick’s \( K \)-theory of this associated Waldhausen ∞-category agrees with Waldhausen’s \( K \)-theory for the original Waldhausen category.

In this paper, we are chiefly interested in the \( K \)-theory of a stable ∞-category \( C \) and a specific subcategory \( C^{\heartsuit} \), which is not closed under suspension. \( C \) will have the maximal pair structure where all maps are ingressives. \( C^{\heartsuit} \) will have the pair structure where maps are ingressives if they have cofiber in \( C^{\heartsuit} \).

3. Weight structures on stable ∞-categories

In this section, we define weight structures and provide an overview of their basic properties.

3.1. Definitions.

\textbf{Definition 3.1 (\cite{Bon10b}).} Let \( T \) be a triangulated category. A weight structure \( w \) on \( T \) is a pair of full subcategories \( T_{w \leq 0} \) and \( T_{w \geq 0} \) (closed under retract and finite coproducts) satisfying the following properties. We adopt the notation that \( T_{w \leq n} := \Sigma^n T_{w \leq 0} \) and \( T_{w \geq n} := \Sigma^n T_{w \geq 0} \).

1. We have the inclusions \( T_{w \geq 1} \subseteq T_{w \geq 0} \) and \( T_{w \leq -1} \subseteq T_{w \leq 0} \).
2. (Orthogonality) For \( X \in T_{w \leq 0} \) we have \( \operatorname{Hom}_T(X,Y) = 0 \) for any \( Y \in T_{w \geq 1} \).
3. (Weight decomposition) For any \( X \in T \) there exists a distinguished triangle

\[ X' \longrightarrow X \longrightarrow X'' \]

with \( X' \in T_{w \leq 0} \) and \( X'' \in T_{w \geq 1} \).

We will call \( T^{\heartsuit} := T_{w \leq 0} \cap T_{w \geq 0} \) the \textit{heart} of the weight structure.

We have chosen to use the homological sign convention for weight structures and all examples and statements will use that convention. Some papers in the literature (e.g., \cite{Bon10b}) opt to use cohomological signs, writing \( T^{w \geq 0} \) for what we denote by \( T_{w \leq 0} \). Due to a lack of consensus in the literature, we use the convention that appears more agreeable for homotopy theorists.

As defined, weight structures are overdetermined. Each subcategory of the weight structure \( T_{w \leq 0} \) or \( T_{w \geq 0} \) determines the other by orthogonality (cf. proposition 3.9). That is, \( T_{w \leq 0} \) is precisely the full subcategory on objects \( X \) with \( T(X,Y) = 0 \) for all \( Y \in T_{w \geq 1} \). By translating (through suspension), we can provide weight decompositions of an object \( X \) at any degree. That is, the degree-zero decomposition for \( \Sigma^{-n} X \), \( A \rightarrow \Sigma^{-n} X \rightarrow B \) provides a degree-\( n \)-decomposition \( \Sigma^n A \rightarrow X \rightarrow \Sigma^n B \) with \( \Sigma^n A \in T_{w \leq n} \) and \( \Sigma^n B \in T_{w \geq n+1} \).
Note that by suspending and desuspending we can provide weight decompositions of an object \( X \) at any degree. That is, the degree-zero decomposition for \( \Sigma^{-n}X \), \( A \to \Sigma^{-n}X \to B \) provides a degree-\( n \)-decomposition \( \Sigma^nA \to X \to \Sigma^nB \) where \( \Sigma^nA \in D_{w \leq n} \) and \( \Sigma^nB \in C_{w \geq n+1} \).

The homotopy category of a stable \( \infty \)-category is triangulated \cite[1.1.2.15]{Lurie2014}. Weight structures are defined on stable \( \infty \)-categories by way of the homotopy category.

**Definition 3.2.** Let \( \mathcal{C} \) be a stable \( \infty \)-category. A **weight structure** on \( \mathcal{C} \) will be a weight structure on the triangulated category \( h\mathcal{C} \).

Full subcategories of \( \infty \)-categories are specified by full subcategories of their homotopy categories. A weight structure on a stable \( \infty \)-category \( \mathcal{C} \) is equivalently defined by two full \( \infty \)-subcategories \( \mathcal{C}_{w \leq 0} \) and \( \mathcal{C}_{w \geq 0} \) (with \( \mathcal{C}_{w \geq n} \) and \( \mathcal{C}_{w \leq n} \) defined as above) and (co)fiber sequences for each object

\[
A \longrightarrow X \quad \longrightarrow \quad * \\
\downarrow \quad \quad \quad \quad \downarrow \\
* \longrightarrow B \longrightarrow \Sigma A
\]

where both squares are pushouts in \( \mathcal{C} \) and \( A \in \mathcal{C}_{w \leq n} \) and \( B \in \mathcal{C}_{w \geq n+1} \). Furthermore, the orthogonality condition requires that \( \pi_0\mathcal{C}(A, B) = 0 \) for all \( A \in \mathcal{C}_{w \leq n} \) and \( B \in \mathcal{C}_{w \geq n+1} \). We will further abuse notation by referring to \( \mathcal{C}_{\mathcal{C}} := \mathcal{C}_{w \leq 0} \cap \mathcal{C}_{w \geq 0} \) as the heart of the weight structure on \( \mathcal{C} \).

**Remark 3.3.** Throughout this paper, \( \mathcal{C}_{\mathcal{C}} \), the heart of a weight structure on a stable \( \infty \)-category, is defined to be a full \( \infty \)-subcategory of \( \mathcal{C} \). This conflicts with the notation used in triangulated categories, where the heart of the weight structure is the subcategory of the homotopy category of \( \mathcal{C} \). The latter we consider \( \pi_0\mathcal{C}_{\mathcal{C}} \).

**Definition 3.4.** We say that a weight structure \( \mathcal{w} \) on a triangulated category \( \mathcal{T} \) is non-degenerate if \( \bigcap_{n \to \infty} \mathcal{T}_{w \geq n} = 0 \) and \( \bigcap_{n \to -\infty} \mathcal{T}_{w \leq n} = 0 \).

A weight structure \( \mathcal{w} \) on a stable \( \infty \)-category \( \mathcal{C} \) is non-degenerate precisely when \( \bigcap_{n \to \infty} \mathcal{C}_{w \geq n} \) and \( \bigcap_{n \to -\infty} \mathcal{C}_{w \leq n} \) are equivalent to the subcategory of zero objects \( \mathcal{C}_0 \) in \( \mathcal{C} \).

Throughout the paper, we will view the stable \( \infty \)-category \( \mathcal{C} \) as a Waldhausen \( \infty \)-category equipped with the maximal pair structure in which all edges are ingressive. The heart \( \mathcal{C}_{\mathcal{C}} \) is given a sub-Waldhausen structure: an edge is ingressive only when its cofiber also lives in \( \mathcal{C}_{\mathcal{C}} \).

Several examples of weight structures are worked out in section 6. We mention the following example for the reader to keep in their mind.

**Example 3.5.** The Postnikov weight structure on finite spectra takes

\[
\text{Sp}_{w \geq n} = \{ E : \pi_\ast(E) = 0, \forall \ast < n \}
\]

that is, the \( n \)-connective spectra, and

\[
\text{Sp}_{w \leq n} = \{ E : H\mathbb{Z}_\ast(E) = 0, \forall \ast > n \text{ and } H\mathbb{Z}_n(E) \text{ free} \}.
\]

The heart \( \text{Sp}_{\mathcal{C}} \) consists spectra weakly equivalent to finite wedge sums of copies of \( S^0 \).
Weight structures generalize cellular structures. \( C_{w \leq n} \) is the subcategory of “cell-less-than-\( n \)” objects and \( C_{w \geq n} \) is the “cell-greater-than-\( n \)” subcategory. The weight decomposition is analogous to the inclusion of an \( n \)-skeleton into \( X \).

Weight structures are not formally dual to \( t \)-structures. The decompositions arising from \( t \)-structures are unique as can be summarized by the existence of a localizing “truncation” functor \( C \to C_{t \geq n} \) for all \( n \). In contrast, weight structure decompositions have no such unicity: there can be many choices of \( n \)-skeleta for each object. Instead for weight decompositions, we have the following result in the homotopy category \( hC \).

**Proposition 3.6.** Let \( A_{w \leq n} \to X \to B_{w \geq n+1} \) denote a weight decomposition of \( X \) at degree \( n \). If \( n \leq m \) then we get maps

\[
\begin{align*}
A_{w \leq n} & \longrightarrow X & \longrightarrow & B_{w \geq n+1} \\
\downarrow & & & \downarrow \\
A'_{w \leq m} & \longrightarrow X & \longrightarrow & B'_{w \leq m}
\end{align*}
\]

making the diagram commute. If \( n < m \) then the induced maps are unique.

As a consequence of this proposition, maps between skeleta are determined up to homotopy provided they map into a strictly “higher” skeleton.

**Remark 3.7.** As a subcategory of \( C \), \( C_{w \leq n} \) is closed under forming fibers. Likewise, \( C_{w \geq n} \) is closed under forming cofibers. If we consider \( C \) as a maximal exact \( \infty \)-category in the sense of [Bar15], \( C_{w \geq n} \) is a Waldhausen \( \infty \)-subcategory and \( C_{w \leq n} \) is a coWaldhausen \( \infty \)-subcategory.

**Definition 3.8.** The weight structure on \( C \) is **bounded** if

\[
\bigcup_{n \geq 0} (C_{w \geq -n} \cap C_{w \leq n}) = C.
\]

The weight structure defined above on the category of finite spectra is bounded. The same weight structure on the category of (not necessarily finite) spectra is not bounded but its “bounded closure” \( \bigcup_{n \geq 0} (C_{w \geq -n} \cap C_{w \leq n}) \) consists of the finite spectra.

### 3.2. Properties of weight structures

In this section we establish some basic properties of weight structures and how weights interact with forming fibers and cofibers. The punchline of the section is that a weight structure provides cellular decompositions of objects.

When convenient, we will use subscripts to denote the weights of given objects. That is, \( A_{w \leq n} \) will denote that the object \( A \) has weight \( w \leq n \) in \( C \).

**Proposition 3.9.** \( C_{w \leq n} \) determines \( C_{w \geq n+1} \) and vice-versa: \( C_{w \geq n+1} \) is precisely those \( X \in C \) with \( \text{Hom}(Y, X) = 0 \) for all \( Y \in C_{w \leq n} \).

**Proof.** If \( X \) is as given in the statement, then it’s weight decomposition at \( n \) is a fiber sequence

\[
X' \longrightarrow X \longrightarrow X''
\]

but \( X' \in C_{w \leq n} \) so \( X' \to X \) is the zero map in \( hC \). Thus \( X \simeq X'' \). The proof is identical to show that an object lives in \( C_{w \geq n+1} \).

**Proposition 3.10.** \( C_{w \geq n} \) is closed under retracts and cofibers.
Proof. Say $X$ is a retract of $Y$ in $\mathcal{C}$ and $Y$ lies in $\mathcal{C}_{w \leq n}$. Then it will be as well in $h\mathcal{C}$. In particular, fix $i : X \to Y$ and $r : Y \to X$ with $r \circ i = \text{id}_X$ in $h\mathcal{C}$. For any $Z \in \mathcal{C}_{w \geq n+1}$ we have induced maps
\[ h\mathcal{C}(X, Z) \to h\mathcal{C}(Y, Z) \to h\mathcal{C}(X, Z) \]
whose composite must be the identity. This demonstrates $h\mathcal{C}(X, Z)$ as a retract of $h\mathcal{C}(Y, Z)$. The latter is trivial so the former must be as well. The previous proposition concludes that $\mathcal{C}_{w \geq n}$ is closed under retracts.

Now suppose $X$ and $Y$ both live in $\mathcal{C}_{w \geq n}$ and
\[ X \xrightarrow{f} Y \xrightarrow{\text{cofiber}(f)} \]
is a cofiber sequence in $\mathcal{C}$. Rotating forward, we have a cofiber sequence $Y \to \text{cofiber}(f) \to \Sigma X$. Let $Z$ be any object of $\mathcal{C}_{w \geq n+1}$. Since $h\mathcal{C}(-, Z)$ carries cofiber sequences to fiber sequences, we have the following fiber sequence.
\[ h\mathcal{C}(\Sigma X, Z) \to h\mathcal{C}(\text{cofiber}(f), Z) \to h\mathcal{C}(Y, Z) \]
The axioms for a weight structure tell us that $\Sigma X \in \mathcal{C}_{w \geq n+1} \subseteq \mathcal{C}_{w \geq n}$ and thus all terms of this sequence must be trivial as the outer two are. The previous proposition concludes the proof. \qed

**Proposition 3.11.** $\mathcal{C}_{w \leq n}$ is closed under retracts and forming fibers.

**Proof.** Both proofs are nearly identical to those for the previous proposition when one replaces $h\mathcal{C}(-, Z)$ with $h\mathcal{C}(Z, -)$. $h\mathcal{C}(Z, -)$ carries fiber sequences to fiber sequences, and since $\mathcal{C}$ is stable, fiber and cofiber sequences coincide. Backing up a fiber sequence $\text{fiber}(f) \to X \to Y$ to produce $\Sigma^{-1}Y \to \text{fiber}(f) \to X$ and similar arguments conclude the proof. \qed

**Remark 3.12.** $\mathcal{C}_{w \geq n} \subset \mathcal{C}$ are Waldhausen subcategories of $\mathcal{C}$. If $\mathcal{C}$ is an exact stable $\infty$-category in the sense of [Bar15] with a weight structure then $\mathcal{C}_{w \leq n} \subset \mathcal{C}$ are coWaldhausen subcategories.

The following is a lemma about triangulated categories that is surprisingly useful for manipulating weight structures.

**Lemma 3.13 ([Bon10a, 1.4.1]).** Let $X \to A \to B \to \Sigma X$ and $X' \to A' \to B' \to \Sigma X$ be two distinguished triangles in a triangulated category $h\mathcal{C}$.

1. If $h\mathcal{C}(B, \Sigma A') = 0$ then for any $g : X \to X'$ there exist $h : A \to A'$ and $i : B \to B'$ completing $g$ to a map of distinguished triangles.

2. If furthermore $h\mathcal{C}(B, A') = 0$ then $h$ and $i$ are unique.

**Proof.** By the axioms for a triangulated category it suffices to provide one of the two desired maps. Applying $h\mathcal{C}(B, -)$ to the second distinguished triangle yields the following exact sequence.
\[ h\mathcal{C}(B, A') \to h\mathcal{C}(B, B') \to h\mathcal{C}(B, \Sigma X') \to h\mathcal{C}(B, \Sigma A') \]
The assumption $h\mathcal{C}(B, \Sigma A') = 0$ lets us lift the composite $B \to \Sigma X \xrightarrow{\Sigma g} \Sigma X'$ to $i : B \to B'$. If the second assumption holds this map is determined uniquely by $g$. \qed
Now let $A_{w \leq n} \to X \to B_{w \geq n+1}$ and $A'_{w \leq m} \to X \to B_{w \geq m+1}$ denote two weight decomposition of $X$ at degrees $n$ and $m$, respectively.

**Corollary 3.14.** There are maps $a : A_{w \leq n} \to A'_{w \leq m}$ and $b : B_{w \geq n+1} \to B'_{w \geq m+1}$ that assemble into a map of distinguished triangles

$$
\begin{array}{ccc}
A_{w \leq n} & \longrightarrow & X \\
\downarrow^{a} & & \downarrow^{b} \\
A'_{w \leq m} & \longrightarrow & X
\end{array}
\quad
\begin{array}{ccc}
X & \longrightarrow & B_{w \geq m+1} \\
& & \downarrow^{b} \\
& & B'_{w \geq m+1}
\end{array}
$$

whenever $m \geq n$. If $m \geq n + 1$ then these maps are unique in $hC$.

**Proof.** Apply Lemma 3.13 to the sequences provided with the map id : $X \to X$. □

Note that the maps are unique up to choice of the two decompositions (which are not a priori unique).

**Corollary 3.15.** If $X$ has weight $w \geq n$ then for all $k \geq n - 1$ any weight decomposition

$$
A_{w \leq k} \longrightarrow X_{w \geq n} \longrightarrow B_{w \geq k+1}
$$

is equivalent to the trivial decomposition

$$
* \longrightarrow X_{w \geq n} \longrightarrow X_{w \geq n}
$$

**Proof.** Since $k + 1 \geq n$, $X$ lies in $C_{w \geq k+1}$. By the lemma we have maps between the two sequences. The unicity of maps into and out of $*$ makes these unique. Thus $A \simeq *$ and $B \simeq X$. □

**Corollary 3.16.** If $X$ has weight $w \leq n$ then for all $k \geq n$ any weight decomposition

$$
A_{w \leq k} \longrightarrow X_{w \leq n} \longrightarrow B_{w \geq k+1}
$$

is equivalent to the trivial decomposition

$$
X_{w \leq n} \longrightarrow X_{w \leq n} \longrightarrow *
$$

**Proof.** The proof is identical to the last corollary. □

**Proposition 3.17.** If $X$ has weight $w \geq n$ then for all $k \geq n$ any weight decomposition

$$
A_{w \leq k} \longrightarrow X_{w \geq n} \longrightarrow B_{w \geq k+1}
$$

has $A$ in $C_{w \geq n}$.

**Proof.** Note that $\Sigma^{-1}B$ has weight $w \geq k$ by the axioms. By assumption, this means that $\Sigma^{-1}B \in C_{w \geq n}$. Thus, rotating back the fiber sequence for the decomposition yields the fiber sequence

$$
(\Sigma^{-1}B)_{w \geq n} \longrightarrow A \longrightarrow X_{w \geq n}
$$

which demonstrates $A \in C_{w \geq n}$ by Proposition 3.9. □
Proposition 3.18. If $X$ has weight $w \leq n$ then for all $k \leq n - 1$ any weight decomposition
\[ A_{w \leq k} \longrightarrow X_{w \leq n} \longrightarrow B_{w \geq k + 1} \]
has $B$ in $\mathcal{C}_{w \leq n}$.

Proof. The proof is identical to that for the previous proposition. □

Proposition 3.19. Suppose $\mathcal{C}$ is a stable $\infty$-category with a non-degenerate weight structure. Maps are detected by maps into or out of the heart in the following sense.

For any $X \in \mathcal{C}$, $\pi_0h\mathcal{C}(X,Y) = 0$ for all $Y \in \mathcal{C}_{w \geq 0}$ if and only if $\pi_0h\mathcal{C}(X,Q) = 0$ for all $Q \in \mathcal{C}_{w = i}$ for $i \geq 0$.

Likewise, for any $Y \in \mathcal{C}$, $\pi_0h\mathcal{C}(X,Y) = 0$ for all $X \in \mathcal{C}_{w \leq 0}$ if and only if $\pi_0h\mathcal{C}(Q,Y) = 0$ for all $Q \in \mathcal{C}_{w = i}$ with $i \leq 0$.

Proof. Both proofs are identical so we check the first. The forward direction is trivial. For the reverse implication, fix a map $f : X \to Y$ in $h\mathcal{C}$. Pick a weight decomposition at degree 0 for $Y$. By proposition 3.17, this takes the form
\[ A_{w = 0} \longrightarrow Y_{w \geq 0} \longrightarrow B_{w \geq 1} \]
and applying $\pi_0h\mathcal{C}(X,-)$ produces a long exact sequence on mapping groups. By assumption, $\pi_0h\mathcal{C}(X,A) = \pi_0h\mathcal{C}(X,\Sigma A) = 0$, so we have that $\pi_0h\mathcal{C}(X,Y)$ is isomorphic to $\pi_0h\mathcal{C}(X,B)$ and $B$ has weight $w \geq 1$. We can iterate this argument: replace $Y$ with $B$ in this argument and take a weight decomposition at degree 1. Inductively, we can conclude that $\pi_0h\mathcal{C}(X,Y)$ is isomorphic to $\pi_0h\mathcal{C}(X,\tilde{B})$ where $\tilde{B}$ can be constructed in an arbitrarily high weight $w \geq n$. As $n \to \infty$, we conclude that $\pi_0h\mathcal{C}(X,Y) \cong 0$ as only zero objects have arbitrarily high weights due to the non-degeneracy of $w$. □

3.3. Generating weight structures. Suppose $\mathcal{C}$ is a stable $\infty$-category and $\mathcal{H}$ is a subcategory in $\mathcal{C}$. A natural question is whether there exists a weight structure on $\mathcal{C}$ with $\mathcal{H}$ as its heart. We will require that $\mathcal{H}$ is closed under retracts and finite coproducts.

Definition 3.20. We say that $\mathcal{H}$ weakly generates $\mathcal{C}$ if $X \in \mathcal{C}$ and
\[ \pi_0h\mathcal{C}(\Sigma^n S, X) = 0 \]
for all $n \in \mathbb{Z}$ and for all $S \in \mathcal{H}$, then $X$ is a zero object in $\mathcal{C}$.

Definition 3.21. We say that $\mathcal{H}$ is negative if for all $n > 0$ we have
\[ \pi_0h\mathcal{C}(S, \Sigma^n S') = 0 \]
for all $S, S' \in \mathcal{H}$.

For spectrally-enriched categories, Blumberg and Mandell introduce a very similar notion to a negative subcategory, namely a connective class. This definition is essential to their form of the sphere theorem which is discussed in [BM11 3.4].

Proposition 3.22 ([Bon10a 4.3.2.III(ii) and 4.5.2]). Suppose the objects of $\mathcal{H}$ are compact, $\mathcal{H}$ is negative, and $\mathcal{H}$ weakly generates $\mathcal{C}$. Suppose further that all finite cell complexes constructed from $\mathcal{H}$ exist in $\mathcal{C}$.

Let $\mathcal{C}^-$ be the full subcategory of $\mathcal{C}$ of objects $X$ so that $\forall S \in \mathcal{H}$ there exists a $N \in \mathbb{Z}$ so that $\pi_0\mathcal{C}(Y, \Sigma^n S) = 0$ for all $n > N$. Then $\mathcal{C}^-$ admits a weight structure with $\mathcal{H}$ contained in its heart.
We introduce two examples of such weight structures now and provide a more detailed discussion in section \ref{sec:weights}.

**Example 3.23.** Let \( R \) be a commutative ring. Let \( \mathcal{C} = \text{Ch}_R \) denote the stable \( \infty \)-category of bounded-above chain complexes of finitely-generated \( R \)-modules. There is a weight structure on \( \text{Ch}_R \) where \( \text{Ch}_{R,w \geq 0} \) contains complexes whose homology is concentrated in non-negative degrees, and \( \text{Ch}_{R,w \leq 0} \) contains complexes which are quasi-isomorphic to complexes of projectives whose homology is non-positive degrees (or, equivalently, complexes concentrated in non-positive degrees and projective in degree 0). The heart of this weight structure is the finitely-generated projective \( R \)-modules included as complexes concentrated in degree 0.

**Example 3.24.** Let \( \text{Sp}^{\text{fin}} \) denote the stable \( \infty \)-category of finite spectra. \( \text{Sp}^{\text{fin}} \) admits a weight structure generated by the sphere spectrum \( S^0 \). \( \text{Sp}^{\text{fin}}_{w \geq 0} \) consists of all connective spectra and \( \text{Sp}^{\text{fin}}_{w \leq 0} \) consists of spectra whose integral homology is concentrated in non-positive degrees. These are precisely spectra which occur as \( k \)-skeleta for other spectra for \( k \leq 0 \).

**Definition 3.25.** Let \( \mathcal{C} \) be a stable \( \infty \)-category equipped with a weight structure and a \( t \)-structure. We say that these structures are left adjacent (respectively, right adjacent) if \( C_{t \geq 0} = C_{w \geq 0} \) (respectively, \( C_{t \leq 0} = C_{w \leq 0} \)).

When a weight structure is left adjacent to a \( t \)-structure, the orthogonality relations of the two structures interact to permit more specific descriptions of the hearts \( C_{\leq w} \) and \( C_{\geq t} \).

If \( X \) is in the heart of the weight structure, then \( X \) is in \( C_{w \geq 0} = C_{t \geq 0} \) so for any \( Y \) in \( C_{t \leq -1} \) we have \( \pi_0 \mathcal{C}(X,Y) = 0 \). Likewise, \( X \) also lies in \( C_{w \leq 0} \) so the orthogonality relations for the weight structure make it admit no maps (on \( \pi_0 \)) to \( C_{w \geq 1} = C_{t \geq 1} \). As a consequence of the overdetermined nature of weight and \( t \)-structures (cf. Proposition \ref{prop:orthogonality}), the heart of the weight structure consists of precisely those objects \( X \) with \( \pi_0 \mathcal{C}(X,Y) = 0 \) for \( Y \in C_{t \geq 1} \cup C_{t \leq -1} \).

A similar analysis can be applied to \( C_{\leq t} \) to deduce that the heart can be detected by \( C_{w \leq 1} \) and \( C_{w \geq 1} \). Together with Proposition \ref{prop:orthogonality} we arrive at the following description.

**Proposition 3.26.** Suppose \( \mathcal{C} \) admits left adjacent weight- and \( t \)-structures. \( X \) is in \( C_{\leq w} \) if and only if \( \pi_0 \mathcal{C}(X,\Sigma^i B) = 0 \) for all \( B \in C_{\leq t} \) for \( i \neq 0 \). Likewise, \( Y \) is in \( C_{\geq t} \) if and only if \( \pi_0 \mathcal{C}(\Sigma^j A,Y) = 0 \) for all \( A \in C_{\leq w} \) for \( i \neq 0 \).

**Example 3.27.** In the category of finite spectra, the cellular weight structure is left adjacent to the Postnikov \( t \)-structure. The heart of the weight structure consists of wedge sums of the sphere spectrum and the heart of the \( t \)-structure is the Eilenberg–Mac Lane spectra. The proposition notes that the former are (equivalently) the spectra with whose cohomology is concentrated in degree 0 (for all \( HG \)), while the latter are precisely those spectra with homotopy groups concentrated in degree 0.

### 3.4. On weights and Waldhausen’s sphere theorem.

This section places Waldhausen’s original sphere theorem within our setting of weight structures on stable \( \infty \)-categories. Proposition \ref{prop:3.28} proves that our theorem from section \ref{sec:weights} generalizes Waldhausen’s. We take the rest of the section to explore the limits how analogous language can be lifted from Waldhausen’s setting to the world of weight structures.

As originally formulated in \cite[1.7]{Waldhausen}, Waldhausen’s sphere theorem applies to a Waldhausen category \( \mathcal{C} \) equipped with a cylinder functor that satisfies the cylinder axiom. The category must be further equipped with a \( \mathbb{Z} \)-graded homology functor \( H_* \) which carries
cofiber sequences to long exact sequences in some abelian target category. Furthermore, weak equivalences in $C$ are required to be precisely isomorphisms on homology. Finally, the hypothesis for the sphere theorem is that any $m$-connected map $X \to Y$ (with respect to $H_*$) can be factored as

$$X_m \to X_{m+1} \to \cdots \to X_n \to^\simeq Y$$

where the quotients $X_{k+1}/X_k$ are all homology spheres of dimension $k + 1$. In this case, the sphere theorem says that the $K$-theory of the stabilization of $C$ (under the suspension defined by the cylinder functor) is equivalent to the $K$-theory of the stabilization (under suspension again) of the homology spheres. In Waldhausen’s context, a homology $n$-sphere is an object $X$ whose homology $H_i(X)$ is 0 unless $i = n$ and then lies in some fixed full subcategory $E$ of the abelian target category of $H_*$ which is closed under extensions and retracts.

**Proposition 3.28.** If $C$ is a Waldhausen category satisfying the hypotheses of Waldhausen’s sphere theorem, then the stable $\infty$-category $\text{Stab}(C)$ admits a bounded and non-degenerate weight structure whose heart is equivalent to the stabilized homology spheres in $C$ if it has a set of compact generators which:

- generate the $\infty$-category under finite colimits,
- are homology 0-spheres, and
- form a negative class in $C$.

Although Waldhausen does not require these additional assumptions, they are true in the cases he studies.

We prove this proposition by defining a weight structure on $\text{Stab}(C)$ where objects are in weight $w \geq n$ if their homology is concentrated in degrees $* \geq n$ and in weight $w \leq n$ if their homology is concentrated in degrees $* \leq n$ and $H_n(X) \in E$. Under the hypotheses listed, we can generate a weight structure on $\text{Stab}(C)$ using proposition 3.22. The heart is precisely the homology $n$-spheres as claimed.

We can transplant Waldhausen’s language to the setting of weight structures on stable $\infty$-categories. Specifically, we can view weight structures as providing a language for discussing connectivity of maps without specifying compact generators whose (co)homology theories measure connectivity.

**Definition 3.29.** A map $f : X \to Y$ in $C$ with cofiber $Cf$ will be called $n$-connected if $Cf$ lives in $\mathcal{C}_{w \geq n+1}$.

**Proposition 3.30.** The composite of two $n$-connected maps is $n$-connected.

**Proof.** Say $f : X \to Y$ and $g : Y \to Z$ are $n$-connected. Write $D$ for the given pushout in the following diagram.
Cf and Cg are the respective cofibers and D is evidently the cofiber of the composite \( g \circ f \). Since the top and outer squares on the right of the diagram are pushouts, so is the lower square. The lower square induces a distinguished triangle \( Cf_{w \geq n+1} \to D \to Cg_{w \geq n+1} \) in \( hC \) and thus \( D \) lies in \( C_{w \geq n+1} \). □

This proposition implies that a weight decomposition at degree \( k \leq n \) is guaranteed to yield a degree-\( k \) decomposition for \( Y \) as well after composing with an \( n \)-connected map \( f : X \to Y \).

**Proposition 3.31.** If the weight structure on \( C \) is bounded, any \( n \)-connected map \( f : X \to Y \) factors

\[
X = X_n \longrightarrow X_{n+1} \longrightarrow \cdots X_m \xrightarrow{\sim} Y
\]

with \( X_k/X_{k-1} \) in \( C_{w=k} \) for \( n+1 \leq k \leq m \).

**Proof.** We will induct up until the cofiber must be concentrated in weight \( w = m \) due to boundedness of the weight structure. The induction essentially proceeds by providing a cellular filtration for \( Cf \) (see section [4]). Fix a diagram in \( C \) for the cofiber sequence for \( Cf \).

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow g & & \downarrow g \\
0 & \xrightarrow{h} & Cf \\
\end{array}
\]

We will write \( X_n = X \) to start the induction as \( Cf \) has weight \( w \geq n+1 \) by assumption.

Fix \( A_k \to Cf \to B_{k+1} \) weight decompositions at \( k \) for all \( k \). Proposition 3.17 tells us that \( A_k \) lives in weight \( n+1 \leq w \leq k \). In particular, \( A_{n+1} \) lives in \( C_{w=n+1} \). Fix a lift of the map \( a : A_{n+1} \to Cf \) to \( C \). Form \( X_{n+1} \) as the cofiber of the composite

\[
\Sigma^{-1}A_{n+1} \xleftarrow{\Sigma^{-1}a} \Sigma^{-1}Cf \xrightarrow{\Sigma^{-1}h} X_n
\]

By construction, the cofiber of the map \( X_n \to X_{n+1} \) will be equivalent to \( A_{n+1} \) which is in \( C_{w=n+1} \) as desired. It remains to show that there is an \((n+1)\)-connected map \( X_{n+1} \to Y \) to complete the induction.

The composite \( \Sigma^{-1}A_{n+1} \to Y \) is homotopic to the zero map because it factors through two consecutive maps in a cofiber sequence.

\[
\Sigma^{-1}A_{n+1} \longrightarrow \Sigma^{-1}Cf \longrightarrow X_n \longrightarrow Y
\]

Thus \( Y \) admits a map from \( X_n \). This leads us to consider the following diagram.

\[
\begin{array}{ccc}
X_n & \longrightarrow & X_{n+1} \longrightarrow Y \\
\downarrow & & \downarrow \\
0 & \longrightarrow & A_{n+1} \longrightarrow Cf \\
\downarrow & & \downarrow \\
0 & \longrightarrow & Cf_{n+1}
\end{array}
\]
We know that the outer upper square is a pushout along with the upper left. This implies that the upper right one is as well. We form the lower square as the cofiber of the map \( A_{n+1} \rightarrow Cf \). The outer right square is thus also a pushout and identifies the lower right square as the cofiber of \( f_{n+1} : X_{n+1} \rightarrow Y \). The lower right square now tells us that \( Cf_{n+1} \) lives in \( C_{w \geq n+2} \) as desired. The relevant cofiber sequence with weights marked is indicated below.

\[
(A_{n+1})_{w=n+1} \longrightarrow (Cf)_{w \geq n+1} \longrightarrow Cf_{n+1} \longrightarrow (\Sigma A_{n+1})_{w=n+2}
\]

\[
\square
\]

4. Bounded cell complexes

In this section, we define cellular weight filtrations and develop some of their properties. The proof of our main theorem relies on careful manipulation of these cellular filtrations. Throughout, we will assume \( C \) is a stable \( \infty \)-category, viewed as a Waldhausen \( \infty \)-category equipped with the maximal pair structure, and \( w \) is a bounded weight structure on \( C \).

4.1. Definitions. In preparation for the proof of the main theorem, we study an ancillary object: the \( \infty \)-category of bounded cell complexes in \( C \).

**Definition 4.1.** Suppose \( C \) is a stable Waldhausen \( \infty \)-category. A relative cell complex in \( C \) is a functor \( A : (\mathbb{N})^2 \rightarrow C \) of Waldhausen \( \infty \)-categories so that any quotient \( A_{i}/A_{i-1} \) is in \( C_{w=i} \). \( \lim_{Z} \) and \( \text{colim}_{Z} \) define functors from the category of relative cell complexes to \( C \). A cell complex will be a relative cell complex which \( \lim_{Z} \) takes to a zero object of \( C \). Write \( \text{Cell} C \subset \text{Fun}_{\text{Wald} \infty}((\mathbb{N})^2,C) \) for the full \( \infty \)-subcategory of cell complexes in \( C \). We will write \( A_{\infty} \) for \( \text{colim}_{Z} A \) and \( A_{-\infty} \) for \( \lim_{Z} A \) and will say that \( A \) is a filtration for \( A_{\infty} \).

By definition, all the morphisms \( A_{n} \rightarrow A_{m} \) in the diagram for a cell complex \( A \) are ingressions in \( C \). Furthermore, two cell complexes \( A_{\bullet} \) and \( B_{\bullet} \) in \( \text{Cell} C \) are equivalent if there is a map between them that restricts levelwise to equivalences in \( C \), i.e., levelwise these edges must lie in \( iC \).

Let \( i_{\leq n} : \mathbb{Z}_{\leq n} \rightarrow \mathbb{Z} \) be the inclusion of the poset of integers \( \leq n \). \( i_{\leq n} \) induces a functor \( i_{\leq n}^{\ast} : \text{Fun}_{\text{Wald} \infty}((\mathbb{N})^2,C) \rightarrow \text{Fun}_{\text{Wald} \infty}((\mathbb{N}Z_{\leq n})^2,C) \) which admits a left adjoint. The adjoint is induced by the map \( p_{\leq n} : \mathbb{Z} \rightarrow \mathbb{Z}_{\leq n} \) which is the identity on \( \mathbb{Z}_{\leq n} \) and collapses all larger integers to \( n \). Write \( \text{tr}_{n} \) for the composite \( i_{\leq n}^{\ast} \circ p_{\leq n}^{\ast} \), the degree-\( n \) truncation of a (relative) cell complex. Likewise \( i_{\geq n} : \mathbb{Z}_{\geq n} \rightarrow \mathbb{Z} \) induces a functor on relative cell complexes which admits a right adjoint induced by the map \( p_{\geq n} : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq n} \) which is the identity \( \geq n \) and collapses all integers below \( n \) to \( n \). Write \( \text{cotr}_{n} \) for the composite \( p_{\geq n}^{\ast} \circ i_{\geq n}^{\ast} \), the degree-\( n \) cotruncation.

**Definition 4.2.** We call a (relative) cell complex \( A \) bounded if \( A \simeq \text{tr}_{n} A \simeq \text{cotr}_{m} A \) for some finite \( n \) and \( m \). Write \( \text{Cell}^{\text{bdd}} C \) for the full \( \infty \)-subcategory of \( \text{Cell} C \) on bounded cell complexes. If a bounded complex \( A \) is equivalent to its \( n \)-truncation \( \text{tr}_{n} A \simeq A \), then we say that \( A \) has degree \( \leq n \). Write \( \text{Cell}_{n}^{\text{bdd}} C \) for the full \( \infty \)-subcategory of \( \text{Cell}^{\text{bdd}} C \) on degree \( \leq n \) cell complexes.

A cell complex \( A \) with \( A \simeq \text{cotr}_{n} A \) must have

\[
\lim_{Z} A \simeq \lim_{Z} \text{cotr}_{n} A \simeq A_{n}
\]
as a zero object. Thus, bounded cell complexes are finite-stage cellular constructions in
the weight structure on $C$ that begin with a zero object. The subcategories $Cell_{n}^{bdd} C$ filter
$Cell^{bdd} C$. That is, under the inclusion maps $\text{colim}_{n} Cell_{n}^{bdd} C \simeq Cell^{bdd} C$.

**Proposition 4.3.** If $A$ is a bounded cell complex in $C$, then $A_i$ is in $C_{w \leq i}$ for all $i$. If $A_n$ is
in $C_{w \geq n}$ then $A_{n-1}$ is in $C_{w \geq n-1}$ as well. In particular, if $A \in Cell_{n}^{bdd} C$ and $A_{\infty} \in C_{w=n}$ then
$A_i \in C_{w=i}$ for all $i$.

**Proof.** Induct up the filtration starting with a zero object $A_{\leq -N}$ in $C_{\leq -N}$ as in §3. For the second part of the proposition, use proposition 3.10 for the cofiber sequence $A_n \to A_n/A_{n-1} \to \Sigma A_{n-1}$. The final statement follows by induction down from $n$. □

**Proposition 4.4.** If $C$ is a stable $\infty$-category equipped with a bounded and non-degenerate
weight structure, then every object in $C$ admits a bounded cellular filtration.

**Proof.** Suppose $X$ is an object of $C$. Then even if the weight structure on $C$ is not bounded, we can fix weight decompositions $A_{w \leq n} \to X$ for all $n \in \mathbb{Z}$. Corollary 3.14 implies the existence of maps $A_n \to A_m$ in the homotopy category for $n \leq m$. These can be lifted to a coherent diagram $NZ \to C$ but for a bounded weight structure this is even simpler. In this case, $X$ has weight $-N \leq w \leq N$ for some $N \geq 0$. Set $A_i = X$ for $i \geq N$ and use the weight decomposition starting with $A_N = X$ to inductively find weight decompositions for $A_n$ at weight $n-1$ to get $A_{n-1} \to A_n$. Proposition 3.18 implies that the fiber $A_n/A_{n-1}$ is in weight $w = n$ as desired. Each of these maps can be lifted from the homotopy category to $C$. For $i \leq -N$, $A_i$ is a zero object of $C$ by the non-degeneracy of the weight structure. The compositions of these maps and the retraction of $Z$ onto $\Delta^N$ as the interval $[-N,N]$ induce the desired functor $NZ \to C$. □

**4.2. Waldhausen structure on cell complexes.** We want to give $Cell^{bdd} C$ and $Cell_{n}^{bdd} C$
compatible Waldhausen $\infty$-category structures. This amounts to selecting ingressive edges. If we think of cell complexes as diagrams in $C$, an edge in $Cell^{bdd} C$ is a diagram

$$
\cdots \Rightarrow A_{i-1} \Rightarrow A_{i} \Rightarrow A_{i+1} \Rightarrow \cdots \\
\cdots \Rightarrow B_{i-1} \Rightarrow B_{i} \Rightarrow B_{i+1} \Rightarrow \cdots
$$

and we have a choice of which edges to make ingressive. Just requiring that all vertical maps
are ingressions in $C$ does not imply that the induced maps $A_{j}/A_{i} \to B_{j}/B_{i}$ are ingressions.
As noted in [Bar16, 5.6] and [Wal85, 1.1.2], we need a latching condition on the diagrams:
that for any $i < j$, the map from $A_{j} \cup A_{i}$ to $B_{j}$ is an ingressive in $C$.

![Diagram](image-url)
This result follows by considering the following commuting cube.

If \( A_j \cup A_i \to B_j \) is ingressive then so is the dotted edge.

We require further that the cofiber of an ingressive map \( A \to B \) in \( \text{Cell}_{n}^{\text{bdd}} \mathcal{C} \) is also a degree-\( n \) bounded cell complex in \( \mathcal{C} \). The cofiber is computed levelwise and we want, in particular, for the cofiber of \( B_{i-1}/A_{i-1} \to B_i/A_i \) to have weight \( w = i \). This cofiber is identified with the cofiber of the map \( A_i \cup A_{i-1} B_{i-1} \to B_i \) which we will require to have weight \( w = i \). More generally, an ingression \( A \to B \) will be levelwise ingressions \( A_i \to B_i \) with the map \( A_j \cup A_i \to B_j \) an ingestion in \( \mathcal{C} \) with cofiber of weight \( j + 1 \leq w \leq i \).

Following Barwick (see [Bar16, 5.6] and the following discussion), it is easier to define the Waldhausen structure on \( \text{Cell}_{n}^{\text{bdd}} \mathcal{C} \) as follows.

**Definition 4.5.** For \( i \leq j \), write \( e_{i,j} : \Delta^1 \to N \mathbb{Z} \) for the map hitting \( i \) and \( j \). Let \( (\text{Cell}_{n}^{\text{bdd}} \mathcal{C})_\dagger \) be the smallest subcategory spanned by the edges \( f : \Delta^1 \to \text{Cell}_{n}^{\text{bdd}} \mathcal{C} \), which we will write \( A \to B \), for which the square \( e_{i,j}^* f : \Delta^1 \times \Delta^1 \to \mathcal{C} \) is either of the form

\[
\begin{array}{ccc}
A_i & \xrightarrow{g} & A_j \\
\downarrow & & \downarrow \\
B_i & \xrightarrow{h} & B_j
\end{array}
\]

where all the edges are ingressive and the square is a pushout square in \( \mathcal{C} \), or of the form

\[
\begin{array}{ccc}
A_i & \xrightarrow{g} & A_j \\
\downarrow & & \downarrow \\
B_i & \xrightarrow{h} & B_j
\end{array}
\]

with the cofiber of the map \( A_j \to B_j \) having weight \( i + 1 \leq w \leq j \), where here the left arrow is an equivalence in \( \mathcal{C} \) and the right is an ingestion.

We write \( (\text{Cell}_{n}^{\text{bdd}} \mathcal{C})_{\dagger} \) for the subcategory of ingressions in \( \text{Cell}_{n}^{\text{bdd}} \mathcal{C} \) and \( (\text{Cell}^{\text{bdd}} \mathcal{C})_{\dagger} \) for the ingressions in \( \text{Cell}^{\text{bdd}} \mathcal{C} \).
Lemma 4.6. An edge \( f \) of \( \text{Cell}_{n}^{\text{bdd}} \mathcal{C} \) is ingressive if and only if for any \( e_{i,j} : \Delta^{1} \to N \mathbb{Z} \) and any diagram \( X \) from the pair \( \infty \)-category

\[
\begin{array}{ccc}
  0 & \hookrightarrow & 1 \\
  \downarrow & \downarrow & \downarrow \\
  2 & \hookrightarrow & \infty'
\end{array}
\]

where \( X|_{0,1,\infty} \) is a pushout square, the marked edges are ingressions, and \( X|_{0,1,2,\infty} = e_{i,j}^{*} f : (\Delta^{1})^{2} \times (\Delta^{1})^{2} \to \mathcal{C} \), then \( X(\infty) \to X(\infty') \) is an ingress in \( \mathcal{C} \) with cofiber in \( \mathcal{C}_{i+1 \leq w \leq j} \).

**Proof.** Ingressions on \( \text{Cell}_{n}^{\text{bdd}} \mathcal{C} \) are defined by their restrictions along the \( e_{i,j} \). The resulting types of squares in the definition all admit the desired property: in the first case we merely note that zero objects are in every weight and the second case the requirement on the vertical map in the square is precisely what is required for the map from the pushout. Hence all ingressive maps satisfy the lemma.

For the converse, we can factor any map satisfying the condition into a composite of maps satisfying the definition. Say \( f : A \to B \) satisfies the lemma. There is some \( k \) so that \( \text{cotr}_{k} A \simeq A \) and \( \text{cotr}_{k} B \simeq B \). Then \( A_{k} \) and \( B_{k} \) are both zero objects in \( \mathcal{C} \), so the map \( \text{tr}_{k} f : \text{tr}_{k} A \to \text{tr}_{k} B \) is an equivalence and hence is ingressive in \( \text{Cell}_{n}^{\text{bdd}} \mathcal{C} \). Form the pushout \( \text{tr}_{k} B \cup \text{tr}_{k} A \) \( A \) levelwise. The map from \( A \) to this pushout is directly an ingression.

\[
\begin{array}{c}
  A_{k} \twoheadrightarrow A_{k+1} \twoheadrightarrow A_{k+2} \twoheadrightarrow \cdots \\
  \downarrow \simeq \downarrow \simeq \downarrow \\
  B_{k} \twoheadrightarrow B_{k} \cup A_{k} \twoheadrightarrow B_{k} \cup A_{k} \twoheadrightarrow \cdots
\end{array}
\]

All the squares in this diagram are pushouts by [Lur09 4.4.2.1], and hence the map directly satisfies the definition of ingressive. \( f \) induces a map \( \text{tr}_{k} B \cup_{\text{tr}_{k} A} A \to \text{tr}_{k+1} B \cup_{\text{tr}_{k+1} A} A \) which we write below as the second row of maps.

\[
\begin{array}{c}
  A_{k} \twoheadrightarrow A_{k+1} \twoheadrightarrow A_{k+2} \twoheadrightarrow \cdots \\
  \downarrow \simeq \downarrow \simeq \downarrow \\
  B_{k} \twoheadrightarrow B_{k} \cup A_{k} \twoheadrightarrow B_{k} \cup A_{k} \twoheadrightarrow \cdots \\
  \downarrow \simeq \downarrow \simeq \downarrow \\
  B_{k} \twoheadrightarrow B_{k+1} \twoheadrightarrow B_{k+1} \cup A_{k+1} \twoheadrightarrow \cdots
\end{array}
\]

Repeated application of [Lur09 4.4.2.1] demonstrates that the marked squares are pushouts, and application of the hypothesis shows that the dotted arrow induced by \( f \) is ingressive in \( \mathcal{C} \) and has a cofiber of the appropriate weight. Induction now factors \( f \) as a composite of ingressions \( A \to \text{tr}_{k} B \cup_{\text{tr}_{k} A} A \to \cdots \to \text{tr}_{n} B \cup_{\text{tr}_{n} A} A \simeq B \) as \( \text{tr}_{n} B \simeq B \) and \( \text{tr}_{n} A \simeq A \). \( \square \)

**Proposition 4.7.** The pair \( \infty \)-category \( (\text{Cell}_{n}^{\text{bdd}} \mathcal{C}, (\text{Cell}_{n}^{\text{bdd}} \mathcal{C})_{1}) \) of bounded cell complexes and the pair \( \infty \)-category \( (\text{Cell}_{n}^{\text{bdd}} \mathcal{C}, (\text{Cell}_{n}^{\text{bdd}} \mathcal{C})_{1}) \) of bounded and \( n \)-truncated cell complexes each form a Waldhausen \( \infty \)-category.
Proof. As $\text{Cell}_n^{\text{bdd}} \mathcal{C}$ is the colimit of the subcategories $\text{Cell}_n^{\text{bdd}} \mathcal{C}$ and the same is true for the ingessions, it suffices to check that the pair $(\text{Cell}_n^{\text{bdd}} \mathcal{C}, (\text{Cell}_n^{\text{bdd}} \mathcal{C})_i)$ form a Waldhausen $\infty$-category.

As zero objects in $\mathcal{C}$ are in all weights, the constant diagram at a zero object in $\mathcal{C}$ forms a zero object in $\text{Cell}_n^{\text{bdd}} \mathcal{C}$. For any $A$ in $\text{Cell}_n^{\text{bdd}} \mathcal{C}$, the cofiber of the map $A_i \rightarrow A_j$ has weight $i + 1 \leq w \leq j$, so the map $0 \rightarrow A$ satisfies the lemma above and hence is ingressive.

Now suppose $A \hookrightarrow B$ is an ingression in $\text{Cell}_n^{\text{bdd}} \mathcal{C}$ and $A \rightarrow C$ is an arbitrary map. The pushout in diagrams $B \cup_A C$ is formed levelwise with $(B \cup_A C)_i = B_i \cup_{A_i} C_i$. The maps $B_i \cup_{A_i} C_i \rightarrow B_j \cup_{A_j} C_j$ are ingressions in $\mathcal{C}$. As equivalences are checked levelwise, $\text{tr}_n(B \cup_A C) \simeq \text{tr}_n B \cup_{\text{tr}_n A} \text{tr}_n C$ is equivalent to $B \cup_A C$. Since the same holds for cotruncation, if $B \cup_A C$ is a cellular complex it will lie in $\text{Cell}_n^{\text{bdd}} \mathcal{C}$. It remains to show that $B \cup_A C$ is a cellular complex and the map $C \rightarrow B \cup_A C$ is an ingression in $\text{Cell}_n^{\text{bdd}} \mathcal{C}$. Both amount to checking that certain

Write $D$ for the pushout $B \cup_A C$. As pushouts commute, $D_j/D_i \simeq (B_j/B_i) \cup_{A_j/A_i} (C_j/C_i)$.

\[
\begin{array}{ccc}
A_j/A_i & \hookrightarrow & B_j/B_i \\
\downarrow & & \downarrow \\
C_j/C_i & \hookrightarrow & D_j/D_i
\end{array}
\]

Hence, the cofiber of the top and bottom map are equivalent in $h\mathcal{C}$, so

\[
(D_j/D_i)/(C_j/C_i) \simeq (B_j/B_i)/(A_j/A_i)
\]

which is equivalent to $B_j/(A_j \cup_A B_i)$ by commuting pushouts again. The latter is in weight $i + 1 \leq w \leq j$ by assumption on the map $A \rightarrow B$. We note that since weights (both $\mathcal{C}_{w \geq i+1}$ and $\mathcal{C}_{w \leq j}$) are closed under extension by definition, the cofiber sequence $C_j/C_i \hookrightarrow D_j/D_i \rightarrow (D_j/D_i)/(C_j/C_i)$ now shows that $D_j/D_i$ also has weight $i + 1 \leq w \leq j$ as desired. Hence, $D$ lies in $\text{Cell}_n^{\text{bdd}} \mathcal{C}$ and the map $C \rightarrow D$ is an ingression. We note that this analysis did not require the particular model of $D$ as the levelwise pushout, so we also conclude that any pushout of an ingression in $\text{Cell}_n^{\text{bdd}} \mathcal{C}$ is also an ingression. \qed

4.3. Localizing cell complexes. By construction, the equivalences in $\text{Cell} \mathcal{C}$ are those maps which induce equivalences in $\mathcal{C}$ degewise. This is too rigid: two cell complexes are only equivalent if all the $n$-skeleta are equivalent. We would like to make all cell complexes for a single object in $\mathcal{C}$ equivalent to each other.

We regard the functor $\text{colim}_Z$ as taking a cell complex to the object in $\mathcal{C}$ it models. We are primarily interested in bounded complexes, which are filtered by the subcategories $\text{Cell}_n^{\text{bdd}} \mathcal{C}$ of complexes $A$ that precisely carry the data of a cellular filtration for $A_\infty \simeq A_n$.

Denote by $v \text{Cell} \mathcal{C}$ (or $\text{vCell}_n^{\text{bdd}} \mathcal{C}$ or $v \text{Cell}^{\text{bdd}} \mathcal{C}$) the subcategory of $\text{Cell} \mathcal{C}$ which $\text{colim}_Z$ takes to equivalences in $\mathcal{C}$. We’d like to localize the bounded cell complexes at $v \text{Cell}^{\text{bdd}} \mathcal{C}$ which will require Barwick’s labeled Waldhausen $\infty$-categories [Bar16, 2.9]. The virtual Waldhausen $\infty$-category given by $\text{Cell}^{\text{bdd}} \mathcal{C}$ labeled by $v \text{Cell}^{\text{bdd}} \mathcal{C}$ will be our surrogate for $\mathcal{C}$.

In the proof of the main theorem, we compare the $K$-theory of the localization

\[
(v \text{Cell}^{\text{bdd}} \mathcal{C})^{-1} \text{Cell}^{\text{bdd}} \mathcal{C}
\]

to that of the unlocalized cell complexes $\text{Cell}^{\text{bdd}} \mathcal{C}$. The following result compares this directly to $K(\mathcal{C})$.
Proposition 4.8. The $K$-theory of the localization is equivalent to that of $\mathcal{C}$.

$$K((v \text{Cell}^{\text{bdd}} \mathcal{C})^{-1} \text{Cell}^{\text{bdd}} \mathcal{C}) \simeq K(\mathcal{C})$$

Proof. Not every map in $(v \text{Cell}^{\text{bdd}} \mathcal{C})^{-1} \text{Cell}^{\text{bdd}} \mathcal{C}$ is necessarily ingressive. However, we can apply Fiore’s approximation theorem [Fio13] in this situation. Write $F$ for the functor induced by colim$_\mathbb{Z}$ from the localization $(v \text{Cell}^{\text{bdd}} \mathcal{C})^{-1} \text{Cell}^{\text{bdd}} \mathcal{C}$ to $\mathcal{C}$. By proposition 4.4, $F$ is essentially surjective. By construction of $v \text{Cell}^{\text{bdd}} \mathcal{C}$, $F$ reflects equivalences in $\mathcal{C}$. Finally, any diagram indexed by a finite poset in the maximal Kan subcategory of $(v \text{Cell}^{\text{bdd}} \mathcal{C})^{-1} \text{Cell}^{\text{bdd}} \mathcal{C}$ admits a colimit in $(v \text{Cell}^{\text{bdd}} \mathcal{C})^{-1} \text{Cell}^{\text{bdd}} \mathcal{C}$ which can be constructed as the colimit of a diagram indexed on a finite poset in $\text{Cell}^{\text{bdd}} \mathcal{C}$ where all maps induce equivalences on colim$_\mathbb{Z}$. These colimits are constructed levelwise and since the poset is finite there is some $N$ where all terms achieve colim$_\mathbb{Z}$, so we directly see that colim$_\mathbb{Z}$ preserves those colimits. Hence $F$ satisfies the hypotheses of [Fio13, 4.5] to show that it induces an equivalence of homotopy categories on the subcategories of ingressions. The approximation theorem [Fio13, 4.10] now implies that $F$ induces an equivalence on $K$-theory.

□

5. The main theorem

This section provides the proof of theorem 5.1 and a description of $K(\mathcal{C}_{\mathbb{Q}_w})$ analogous to the “plus-equals-$Q$” theorem. Throughout, we let $\mathcal{C}$ denote a fixed stable $\infty$-category equipped with a bounded weight structure $w$.

Theorem 5.1. If $\mathcal{C}$ is a stable $\infty$-category equipped with a bounded non-degenerate weight structure $w$, then the inclusion of the heart of the weight structure $\mathcal{C}_{\mathbb{Q}_w} \hookrightarrow \mathcal{C}$ induces an equivalence on algebraic $K$-theory

$$K(\mathcal{C}) \simeq K(\mathcal{C}_{\mathbb{Q}_w}).$$

Recall that $\mathcal{C}_{\mathbb{Q}_w}$ is given a Waldhausen $\infty$-category structure where ingressions are precisely those maps admitting cofibers in $\mathcal{C}_{\mathbb{Q}_w}$.

The proof for the theorem studies bounded cellular filtrations of objects in $\mathcal{C}$ with respect to the weight structure $w$. Call this category $\text{Cell}^{\text{bdd}}(\mathcal{C})$ and let $\text{Cell}^{\text{bdd}}_{\text{triv}}(\mathcal{C})$ denote the subcategory of cellular filtrations of zero objects in $\mathcal{C}$. The localization theorem gives a fiber sequence

$$K(\text{Cell}^{\text{bdd}}_{\text{triv}}(\mathcal{C})) \longrightarrow K(\text{Cell}^{\text{bdd}}(\mathcal{C})) \longrightarrow K(\mathcal{C})$$

and additivity identifies the left and middle terms of this sequence as infinite products of $K(\mathcal{C}_{\mathbb{Q}_w})$. By filtering the category of cellular filtrations, we show that the left product has one fewer copy of $K(\mathcal{C}_{\mathbb{Q}_w})$

$$\prod_{\text{one fewer}} K(\mathcal{C}_{\mathbb{Q}_w}) \longrightarrow \prod K(\mathcal{C}_{\mathbb{Q}_w}) \longrightarrow K(\mathcal{C})$$

which implies $K(\mathcal{C}_{\mathbb{Q}_w}) \simeq K(\mathcal{C})$ as desired.

Bondarko proves this theorem on $K_0$-groups by considering bounded weight structures on triangulated categories [Bon10a, 5.3.1]. We independently reproduce his result by passing to the underlying stable $\infty$-category, applying our theorem, and taking $\pi_0$. 

20
Corollary 5.2 (cf. [Bon10a, 5.3.1]). If \( \mathcal{T} \) is a triangulated category equipped with a non-degenerate bounded weight structure \( w \), then the inclusion of the heart \( \mathcal{T}_w \) into \( \mathcal{T} \) induces an equivalence on K-theory, \( K_0(\mathcal{T}) \simeq K_0(\mathcal{T}_w) \).

At the end of the section, we study the K-theory of the heart of a weight structure. All ingressions in \( \mathcal{C}_w \) split in the homotopy category, so the K-theory admits a description in the style of Quillen’s plus construction for K-theory.

5.1. Proving the main theorem. Using the technology of cellular filtrations constructed in section 4, we study the K-theory of \( \mathcal{C} \) through the labeled pair Waldhausen \( \infty \)-category \( \text{Cell}^{\text{bdd}} \mathcal{C}, v \text{Cell}^{\text{bdd}} \mathcal{C} \). We use the localization theorem to relate the algebraic K-theory of this pair to the algebraic K-theory of \( \text{Cell}^{\text{bdd}} \mathcal{C} \).

Theorem 5.3 ([Bar16, 9.24]). Suppose \((A, wA)\) is a labeled Waldhausen \( \infty \)-category that has enough cofibrations. Suppose \( \phi : \text{Wald}_\infty \rightarrow E \) is an additive theory with left derived functor \( \Phi \). Then the inclusion \( i : A^w \rightarrow A \) and the morphism of virtual Waldhausen \( \infty \)-categories \( e : A \rightarrow B(A, wA) \) give rise to a fiber sequence

\[
\begin{align*}
\phi(A^w) & \longrightarrow \phi(A) \\
\downarrow & \downarrow \\
* & \Phi(B(A, wA)).
\end{align*}
\]

Here, \( B(A, wA) \) is the virtual Waldhausen \( \infty \)-category corresponding to the pair \((A, wA)\). For our result \( \phi \) will be K-theory, and the derived K-theory on the virtual Waldhausen \( \infty \)-category of the pair \( K(B(A, wA)) \) will be written simply as the K-theory of the pair \( K(A, wA) \). In this theorem, \( A^w \) denotes the full subcategory of \( w \)-acyclic objects in \( A \). In our setting, the \( v \)-acyclic objects of \( \text{Cell}^{\text{bdd}} \mathcal{C} \) are classified by the following computation.

Proposition 5.4. If \( A \) is a \( v \)-acyclic object of \( \text{Cell}^{\text{bdd}} \mathcal{C} \) then \( A_n \) has weight \( w = n \) in \( \mathcal{C} \) for all \( n \).

Lemma 5.5. If

\[
A \longrightarrow B \longrightarrow C
\]

is a cofiber sequence in \( \mathcal{C} \) with \( B \in \mathcal{C}_{w \geq m} \) and \( C \in \mathcal{C}_{w \geq m+1} \) and \( m \geq n + 1 \) then \( A \in \mathcal{C}_{w \geq n} \).

Proof. For any \( X \in \mathcal{C}_{w \leq n-1} \) we obtain the following exact sequence of mapping spaces in \( h\mathcal{C} \).

\[
hC(X, \Sigma^{-1}C) \longrightarrow hC(X, A) \longrightarrow hC(X, B)
\]

Now since \( C \in \mathcal{C}_{w \geq n+1} \), \( \Sigma^{-1}C \) lives in \( \mathcal{C}_{w \geq n} \) so the left mapping space is trivial. The same is true of the right mapping space since \( B \in \mathcal{C}_{w \geq m} \) and \( m \geq n + 1 \geq n \). We conclude that \( A \) is in \( \mathcal{C}_{w \geq n} \).

Proof of proposition 5.4. The proof follows from induction down from the finite stage where \( A \) achieves its colimit, a zero object. Say that \( A \in \text{Cell}_{n}^{\text{bdd}} \mathcal{C} \) so that we have an equivalence \( * \simeq A_n \), letting us conclude that \( A_n \) has weight \( w = n \). Induction using the lemma above when \( m = k+1 \) lets us conclude that \( A_k \) has weight \( w \geq k \) for all \( k \). A similar induction from below demonstrates that \( A_n \in \mathcal{C}_{w \leq n} \) for any sequence in \( A \). This completes the proof. \( \square \)
Remark 5.6. As \( C_{w=n} \) consists of the “pure” objects in the weight structure, \((\text{Cell}^{\text{bdd}} \mathcal{C})^v\) can be thought of as “finite Koszul resolutions” between zero objects in \( \mathcal{C} \). We note that \( \mathcal{C}_{w=0} \) is equivalent (via suspension) to \( \mathcal{C}_{w=n} \) for any \( n \).

To use the localization theorem, we must check that the marked cell filtrations satisfy the technical hypothesis of having enough cofibrations.

**Proposition 5.7.** When \( \mathcal{C} \) is a stable \( \infty \)-category equipped with the maximal Waldhausen \( \infty \)-category structure, the labeled Waldhausen \( \infty \)-category \((\text{Cell}^{\text{bdd}} \mathcal{C}, v \text{Cell}^{\text{bdd}} \mathcal{C})\) has enough cofibrations.

**Proof.** By [Bar16, 9.22], it is sufficient to construct a functorial mapping cylinder \( M \) on arrows

\[
M : \text{Fun}(\Delta^1, \text{Cell}^{\text{bdd}} \mathcal{C}) \to \text{Fun}(\Delta^1, \text{Cell}^{\text{bdd}} \mathcal{C})
\]

that produces ingressive arrows, preserves \( v \)-equivalences, and comes with a natural transformation \( \eta : \text{id} \to M \) which is an objectwise labeled by \( v \)-equivalences.

We construct \( M \) for arrows in \( \text{Fun}(\mathbb{N} \mathbb{Z}, \mathcal{C}) \) and show that it produces arrows in \( \text{Cell}^{\text{bdd}} \mathcal{C} \). Let \( \text{sh}_{-1} \) denote the functor induced on \( \text{Fun}(\mathbb{N} \mathbb{Z}, \mathcal{C}) \) by the map \( z \mapsto z - 1 \) on \( \mathbb{Z} \). Note that \( \text{sh}_{-1}(A)_i = A_{i-1} \). Also note that there is a natural transformation \( \text{sh}_{-1} \to \text{id} \) whose levelwise maps are just the structure maps, i.e., \( A_{i-1} \to A_i \).

For \( f : A \to B \) an arrow, \( Mf \) will be defined to be the pushout

\[
\begin{array}{ccc}
\text{sh}_{-1}(A) & \xrightarrow{\text{sh}_{-1}(f)} & \text{sh}_{-1}(B) \\
\downarrow & & \downarrow \\
A & \xrightarrow{\gamma} & Mf \\
\end{array}
\]

and since colimits are computed levelwise on diagram categories, we observe that

\[
(Mf)_i \simeq B_i \cup_{A_{i-1}} A_i.
\]

We regard \( M(f) \) as the arrow \( A \to Mf \) and will write \( Mf \) only for the target object. We note that the construction is functorial on the arrow category for \( \text{Fun}(\mathbb{N} \mathbb{Z}, \mathcal{C}) \).

First we check that \( Mf \) is an endofunctor for arrows in \( \text{Cell}^{\text{bdd}}(\mathcal{C}) \). By commuting pushouts, we find that \( (Mf)_{i+1}/(Mf)_i \) is the (homotopy) pushout

\[
\begin{array}{ccc}
A_i/A_{i-1} & \xrightarrow{\gamma} & B_{i+1}/B_i \\
\downarrow & & \downarrow \\
A_{i+1}/A_i & \xrightarrow{\gamma} & (Mf)_{i+1}/(Mf)_i \\
\end{array}
\]

but due to the weights of the objects, the top and left map are both 0, so \( (Mf)_{i+1}/(Mf)_i \) splits up as a wedge sum

\[
(Mf)_{i+1}/(Mf)_i \simeq A_{i+1}/A_i \lor B_{i+1}/B_i \lor \Sigma(A_i/A_{i-1})
\]

which has weight \( w = i + 1 \) as desired. Since \( A \) and \( B \) are bounded, so will \( Mf \). Hence \( M(f) \) is an arrow in \( \text{Cell}^{\text{bdd}}(\mathcal{C}) \) if \( f \) is as well.
Next we check that $M(f)$ is a cofibration in $\text{Cell}^{\text{bdd}}(C)$. It suffices to check the latching condition for the map $g$.

From the construction of $Mf$, we have the following pushout squares

$$
\begin{array}{ccc}
A_{i-1} & \longrightarrow & A_i \\
\downarrow & & \downarrow \\
B_i & \longrightarrow & (Mf)_i \\
\end{array}
\quad
\begin{array}{ccc}
A_i & \longrightarrow & A_{i+1} \\
\downarrow & & \downarrow \\
B_{i+1} & \longrightarrow & M_{i+1} \\
\end{array}
$$

and the outer square is also a pushout square by [Lur09, 4.4.2.1]. We map the outer pushout square to

$$
\begin{array}{ccc}
A_i/A_{i-1} & \longrightarrow & * \\
\downarrow & & \downarrow \\
B_{i+1}/B_i & \longrightarrow & M_{i+1}/P \\
\end{array}
$$

The weights of $A_i/A_{i-1}$ and $B_{i+1}/B_i$ imply that $M_{i+1} \simeq B_{i+1}/B_i \vee \Sigma(A_i/A_{i-1})$ which lives in weight $w = i + 1$ as desired.

Now we claim that $M$ is a mapping cylinder for $f$ with respect to the $v$-equivalences. That is, we show that the natural map $B \to Mf$ is a $v$-equivalence. Denote this map $\phi : B \to Mf$. The maps $f : A \to B$ and $\text{id} : B \to B$ also induce a map $\psi : Mf \to B$. Levelwise, these maps appear as

$$
\begin{array}{ccc}
A_{i-1} & \longrightarrow & B_i \\
\downarrow & & \downarrow f_i \\
A_i & \longrightarrow & (Mf)_i \\
\end{array}
\quad
\begin{array}{ccc}
& & \phi \\
\downarrow & & \downarrow \text{id} \\
& & \psi \\
f_i & \longrightarrow & B_i \\
\end{array}
$$

and $\psi \circ \phi \simeq \text{id}$. $\psi$ is a right inverse for $\phi$ as well if $f_i$ factors the map from $A_i \to (Mf)_i$ through $B_i$ via $\phi$. This will be satisfied once $A$ and $B$ both achieve their limits. Equivalently,
the vertical cofibers in the square

\[ \begin{array}{ccc} A_{i-1} & \longrightarrow & B_i \\
\downarrow & \searrow & \downarrow \\
A_i & \longrightarrow & (Mf)_i \end{array}\]

must be equivalent. The left is \(A_i/A_{i-1}\) and the right is \((Mf)_i/B_i\). Once \(A\) no longer has any cells, we conclude that \(\phi_i : B_i \to (Mf)_i\) is an equivalence. Hence, \(\phi\) is a \(v\)-equivalence. Since \(\phi\) induces the desired natural transformation \(id \to M\), we conclude that the pair \((\text{Cell}^\text{bdd} C, v \text{Cell}^\text{bdd} C)\) has enough cofibrations. \(\square\)

We apply Barwick's localization theorem to the algebraic \(K\)-theory of the labeled Waldhausen \(\infty\)-category \((\text{Cell}^\text{bdd} C, v \text{Cell}^\text{bdd} C)\) to produce the following pushout diagram.

\[ \begin{array}{ccc} K((\text{Cell}^\text{bdd} C)^v) & \longrightarrow & K(\text{Cell}^\text{bdd} C) \\
\downarrow \quad \quad & \searrow \quad \downarrow \\
* & \longrightarrow & K(\text{Cell}^\text{bdd} C, v \text{Cell}^\text{bdd} C) \end{array}\]

The top map is induced by the inclusion of the \(v\)-acyclics into \(\text{Cell}^\text{bdd} C\). The proof will proceed by factoring this map on \(K\)-theory and analyzing the resulting diagram.

Integral to our argument are two functorial ways of embedding \(C_{\text{cell}}\) into cell complexes. On one hand, we can include an \(n\)-spherical object \(a_n\) as the cell filtration concentrated in degree \(n\), where we attach \(a_n\) to a zero object and then keep the filtration constant. This functor essentially includes \(a_n\) as the filtration \(0 \to a_n\) living in \(\text{Cell}^\text{bdd} C\). We will abuse notation and write the resulting complex simply as \(a_n\) when its weight is clear. We can also include \(a_n\) into the \(v\)-acyclic cell filtrations by including it as the filtration \(0 \to a_n \to 0'\) where \(a_n\) is attached to 0 and then immediately killed at the next level. This will be referred to as the \(\text{cone} \) on \(a_n\), written \(\text{cone}(a_n)\), and includes \(a_n\) into \((\text{Cell}_{n+1} C)^v\). Coherent functoriality of these maps is ensured by the following construction.

By [Lur09, 3.2.2], the source map \(s : \text{Fun}(\Delta^1, C_{w=i}) \to C_{w=i}\) is a cartesian fibration and if we denote the full subcategory of zero objects in \(C\) by \(C_0\), pulling back \(s\) over the inclusion of \(C_0 \to C_{w=i}\) yields a cartesian fibration \(C_0 \times_s \text{Fun}(\Delta^1, C_{w=i})\) which at a zero object \(*\) classifies the \(\infty\)-category of arrows \(* \to X\) in \(C\) where \(X\) has weight \(w = i\). Let \(p_i : \mathbb{N}Z \to \Delta^1\) be defined by \(p_i(j) = 0\) for \(j < i\) and \(p_i(j) = 1\) for \(j \geq i\). Pullback along \(p_i\) and inclusion into \(C\) induces a map \(C_0 \times_s \text{Fun}(\Delta^1, C_{w=i}) \to \text{Fun}(\mathbb{N}Z, C)\). This carries an arrow \(* \to X\) with target in weight \(w = i\) to a filtration that is evidently cellular and bounded as the level quotients are all zero objects except for at degree \(i\) where it is equivalent to \(X\). We will denote the resulting functor from \(C_{w=i} \to \text{Cell}^\text{bdd} C\) by \(c_i\), or, when clear, with no decoration, as this is the natural way to include \(C_{w=i}\) into \(\text{Cell}^\text{bdd} C\).

To produce the cone functor, we use that the maps \(ev_0\) and \(ev_1 : \text{Fun}(\Delta^2, C_{w=i}) \to C_{w=i}\) are also cartesian fibrations. We pull back the map \((ev_0, ev_2)\) from \(\text{Fun}(\Delta^2, C_{w=i}) \to C_{w=i} \times C_{w=i}\) along the inclusion of \(C_0 \times C_0\). Above a pair of zero objects this classifies composable pairs of arrows \(0 \to X \to 0'\) with \(X\) in \(C_{w=i}\). We pull back along the map \(\mathbb{N}Z \to \Delta^2\) which collapses \(Z\) onto the interval \([i-1, i+1]\) as above. Observe that the resulting cell filtration
lives in \( \text{Cell}_{i+1}^\text{bdd} \) and \( \lim_{\mathbb{Z}} \) takes it to a zero object. Denote the corresponding functor \( C_{w=i} \rightarrow (\text{Cell}_{i+1}^\text{bdd})^v \) by cone.

\((n-1)\)-truncation \( \tau_{n-1} \) is an exact functor \( \text{Cell}_{n-1}^\text{bdd} \) to \( \text{Cell}_{n-1}^\text{bdd} \) by lemma 4.6. Suppressing the inclusion \( \text{Cell}_{n-1}^\text{bdd} \) to \( \text{Cell}_{n-1}^\text{bdd} \) as well as the restriction to the subcategory, we consider \( \tau_{n-1} \) as an endofunctor of \( \text{Cell}_{n-1}^\text{bdd} \). The truncation \( \tau_{n-1} \) comes with a natural transformation to \( \text{id} \) by construction. We take the (homotopy) cofiber of this map of endofunctors \( \tau_{n-1} \rightarrow \text{id} \) in the arrow \( \infty \)-category of functors. Denote the resulting endofunctor of \( \text{Cell}_{n-1}^\text{bdd} \) by \( q_n \).

The functor \( q_n \) is constructed to be a homotopy-coherent model for the top level quotient \( A \mapsto A_n/A_{n-1} \). In particular, since \( \lim_{\mathbb{Z}} \tau_{n-1} A \) is equivalent to \( A_{n-1} \) and \( \lim_{\mathbb{Z}} A \) is equivalent to \( A \), we observe that \( \lim_{\mathbb{Z}} q_{n-1} A \simeq A_n/A_{n-1} \) in the homotopy category of \( C \). Alternatively, the cofiber is computed pointwise, and we observe that the image of \( q_n \) is equivalent to \( C_{w=n} \) included as constant cell complexes concentrated in degree \( n \).

**Proposition 5.8.** The map

\[
K(\text{Cell}_{n-1}^\text{bdd} C) \times K(C_{w=n}) \xrightarrow{\vee} K(\text{Cell}_{n}^\text{bdd} C)
\]

is an equivalence, where we include \( C_{w=n} \) as constant cell complexes concentrated in degree \( n \).

**Proof.** The pushout square of endofunctors

\[
\begin{array}{ccc}
\tau_{n-1} & \longrightarrow & \text{id} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & q_n
\end{array}
\]

implies by [Bar16 7.4.(5)] that \( K(\tau_{n-1} \oplus q_n) \) is equivalent to \( K(\text{id}) \) on \( K(\text{Cell}_{n}^\text{bdd} C) \). Hence the inclusion \( (\text{Cell}_{n}^\text{bdd} C)^v \rightarrow \text{Cell}_{n}^\text{bdd} C \) factors on \( K \)-theory as

\[
K((\text{Cell}_{n-1}^\text{bdd} C) \times \text{im} q_n) \simeq K((\text{Cell}_{n-1}^\text{bdd} C) \times C_{w=n})
\]

where the middle term is determined by the images of \( \tau_{n-1} \) and \( q_n \), which on \( (\text{Cell}_{n}^\text{bdd} C)^v \) are an \((n-1)\)-truncated cell complex and a cell complex concentrated in degree \( n \), which is the essential image of the weight-\( n \)-spheres under the constant-cell-complex functor \( C_{w=n} \rightarrow \text{Cell}_{n}^\text{bdd} C \). The vertical map \( \vee \) is the wedge of the inclusion and the constant-cell-filtration functor from \( C_{w=n} \).

Furthermore, the wedge product map is a right inverse to \( \tau_{n-1} \oplus q_n \) before taking \( K \)-theory, so we conclude that all the maps are equivalences on \( K \)-theory and hence the wedging map is an equivalence on \( K \)-theory. \( \square \)

We can induct down on \( n \) to arrive at the following result.

**Corollary 5.9.** \( K(\text{Cell}_{n}^\text{bdd} C) \) is equivalent to \( \prod_{i \leq n} K(C_{w=i}) \) under the map induced by \( q = q_n \oplus q_{n-1} \oplus q_{n-2} \oplus \cdots \).
Now we can factor the pushout square from the fibration theorem into two pushout squares

$$K((\text{Cell}^\text{bdd}_{n} C)^{v}) \longrightarrow K(\text{Cell}^\text{bdd}_{n-1} C) \times K(C_{w=n})$$

$$\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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and we note that the vertical maps are all $v$-equivalences, hence equivalences in the localization. The lower zig-zags are hit by the localization functor. Hence, any sequence of maps in $S_{\bullet}((v \text{ Cell}^\bdd_n \mathcal{C})^{-1} \text{Cell}^\bdd_n \mathcal{C})$ receives a map from a sequence of maps in the image of $L$. We conclude that $L$ induces a weak equivalence of the nerves as desired. \hfill \square

Claim 5.11 will follow from proving that the truncation functor induces an equivalence on $K$-theory from $(\text{Cell}^\bdd_n \mathcal{C})^v$ to $\text{Cell}^\bdd_{n-1} \mathcal{C}$. We will prove this by identifying $K((\text{Cell}^\bdd_n \mathcal{C})^v)$ with $\prod_{i \leq n-1} K(C_{w=i})$ in $K(\text{Cell}^\bdd_n \mathcal{C})$ which is identified with $\prod_{i \leq n} K(C_{w=i})$ using corollary 5.9.

**Lemma 5.12.** The truncation functor $\text{tr}_{n-1} : (\text{Cell}^\bdd_n \mathcal{C})^v \to \text{Cell}^\bdd_{n-1} \mathcal{C}$ induces an equivalence on $K$-theory.

**Proof.** The truncation functor maps the acyclic $n$-cell complexes fully and faithfully into the $(n-1)$-cell complexes by forgetting the zero object at degree $n$. By corollary 5.9 $q : \text{Cell}^\bdd_{n-1} \mathcal{C} \to \prod_{i \leq n-1} C_{w=i}$ induces an equivalence on $K$-theory whose inverse is induced by the map $W$ which forms the wedge sum of constant cell filtrations. Hence it suffices to check that $\text{tr}_{n-1}$ is essentially surjective onto the image of $W$ to induce an equivalence of $K$-theory. This will follow by inducting down on cells. In particular, we have the following diagram of functors

\[
\begin{array}{ccc}
(\text{Cell}^\bdd_n \mathcal{C})^v & \xrightarrow{\text{tr}_{n-2}} & \text{Cell}^\bdd_{n-2} \times C_{w=n-1} \mathcal{C} \\
\downarrow^{\text{incl}} & & \downarrow^{q_{\text{id} \times W}} \\
(\text{Cell}^\bdd_n \mathcal{C})^v & \xrightarrow{\text{tr}_{n-1}} & \text{Cell}^\bdd_{n-1} \\
\end{array}
\]

where all functors in the right square induce equivalences on $K$-theory. A bounded cell filtration in $\text{Cell}^\bdd_{n-1} \mathcal{C}$ corresponds to a sequence of level quotients $(a_i) \in \prod_{i \leq n-1} C_{w=i}$ where all but finitely many are zero objects. $W$ sends this to the wedge $\bigvee_{i \leq n-2} a_i$ where the weight of each $a_i$ indicates in which degree its filtration is concentrated in $\text{Cell}^\bdd_{n-1} \mathcal{C}$.

If we inductively assume that $\bigvee_{i \leq n-2} a_i$ is in the essential image of $\text{tr}_{n-2}$ in $\text{Cell}^\bdd_{n-2} \mathcal{C}$, then observe that cone$(a_{n-1})$ is sent to $a_{n-1}$ under $\text{tr}_{n-1}$, so $a_{n-1} \bigvee \bigwedge_{i \leq n-2} a_i$ will also be in the essential image. Boundedness implies that this induction terminates in finite steps once all cells are coned off. \hfill \square

The homotopy fibers of the vertical maps in the pushout square

\[
K((\text{Cell}^\bdd_n \mathcal{C})^v) \xrightarrow{\gamma} K(\text{Cell}^\bdd_{n-1} \mathcal{C}) \\
\downarrow & & \downarrow \\
* & \xrightarrow{\sim} & P
\]

must agree, so in light of the previous lemma, we conclude that $P \simeq K(C_{w=n})$. This completes the proof of claim 5.11 as well as the proof of the main theorem.

**5.2. On the $K$-theory of the heart of a weight structure.** Let $\mathcal{C}_{w}$ denote the heart of a weight structure on a stable $\infty$-category $\mathcal{C}$. In this section, we describe the $K$-theory of $\mathcal{C}_{w}$ in terms of equivalence classes of objects and their automorphisms.
One key observation about $C_{\mathcal{W}}$ allows its $K$-theory to be simply described: cofiber sequences in $C_{\mathcal{W}}$ all split.

**Proposition 5.13.** If $f : A \to B$ is an ingression in $C_{\mathcal{W}}$, then $f$ splits in the homotopy category $hC$.

**Proof.** Write $A \to B \to C$ for the cofiber sequence associated to $f$, considered as an exact triangle in $hC$. By assumption, $C$ lies in $C_{\mathcal{W}}$ as well. Lemma 3.13 implies that there are maps $g$ and $h$ extending $id : A \to A$ to a map of exact triangles.

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{id} & & \downarrow{g} \\
0 & \xrightarrow{id} & \Sigma A
\end{array} \\
\begin{array}{ccc}
A & \xrightarrow{id} & 0 \\
\downarrow{h} & & \downarrow{id} \\
\Sigma A & \xrightarrow{f} & C
\end{array}
$$

Thus $g \circ f \simeq id_A$ in $hC$. □

Recall here that ingressions in $C_{\mathcal{W}}$ are those maps which are ingressions in $C$ with cofiber in $C_{\mathcal{W}}$.

This case was studied by Waldhausen in [Wal85, §1.8] and his approach applied directly. Write $\mathcal{D}$ for any pointed Waldhausen $\infty$-category with finite coproducts where all ingressions split up to homotopy, even though the only case of interest to us at this point is $\mathcal{D} = C_{\mathcal{W}}$.

Define $N_* \mathcal{D}$ to be the nerve of $\mathcal{D}$ with respect to the coproduct, so $N_n \mathcal{D}$ is $\mathcal{D}^n$ and the face map is

$$d_i(X_1, \ldots, X_n) = \begin{cases} (X_2, \ldots, X_n) & i = 0 \\ (X_1, \ldots, X_i \vee X_{i+1}, \ldots, X_n) & 0 < i < n \\ (X_1, \ldots, X_{n-1}) & i = n \end{cases}$$

and the degeneracies wedge with a zero object. There is an inclusion $N_* \mathcal{D} \to S_* \mathcal{D}$ that takes $(X_1, \ldots, X_n)$ to

$$
\begin{array}{c}
\ast \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_1 \vee \cdots \vee X_n \\
\downarrow & & \downarrow \\
\ast \rightarrow X_2 \rightarrow \cdots \rightarrow X_2 \vee \cdots \vee X_n \\
& \vdots & \\
\ast \rightarrow \cdots \rightarrow X_n \\
\downarrow & \\
\ast
\end{array}
$$

But the assumption that sequences of ingressions split in $\mathcal{D}$ implies that this inclusion induces an equivalence $wN_* \mathcal{D} \to wS_* \mathcal{D}$. Waldhausen’s own argument for this is [Wal85, Proposition 1.8.7] and reduces to inductively studying the map.

$$wN_* S_n \mathcal{D} \to wN_* S_{n-1} \mathcal{D} \times wN_* \mathcal{D}$$

$$(A_1 \rightarrow \cdots \rightarrow A_n) \leftrightarrow ((A_1 \rightarrow \cdots \rightarrow A_{n-1}), A_n/A_{n-1})$$

This is an equivalence if for every object $X$ in $\mathcal{D}$, the map $j : \mathcal{D} \to \mathcal{D}_X$, the category of ingressions under $X$, that sends $A$ to $X \mapsto X \vee A$ induces an equivalence $wN_* \mathcal{D} \to wN_* \mathcal{D}_X$. 

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This is ensured by the splitting hypothesis on \( \mathcal{D} \), since any \( X \rightarrow A \) is equivalent via a zig-zag to \( X \rightarrow X \vee A/X \) and the nerve \( N_* \) identifies \( X \rightarrow X \rightarrow X \rightarrow X \vee A/X \) by the face corresponding to \((X, X \vee A/X)\). We summarize Waldhausen’s equivalence.

**Proposition 5.14.** If every ingress splits in \( \mathcal{D} \), a Waldhausen \( \infty \)-category, then \( wN_*\mathcal{D} \simeq wS_*\mathcal{D} \).

Let \([X]\) denote an equivalence class of objects in \( \mathcal{D} \). \( \text{Aut}(X) \) is a topological monoid with classifying space \( B\text{Aut}(X) \). Since ingressions split in \( \mathcal{D} \), we have a map \( \text{Aut}(X) \rightarrow \text{Aut}(X') \) for any ingress \( X \rightarrow X' \), taking \( f : X \rightarrow X \) to \( f \vee \text{id} \) from \( X \vee X'/X \rightarrow X \vee X'/X \). We consider the homotopy colimit \( \text{hocolim}_{[X]\in\mathcal{D}} B\text{Aut}(X) \), which we will denote \( \text{hocolim}_\mathcal{D} B\text{Aut}(X) \). For example, when \( \mathcal{D} = \text{Proj}(R) \) is the category of finitely-generated projective \( R \)-modules, one has the cofinal collection of objects \( \{R^n\} \) with block-sum maps \( \text{GL}_n(R) = \text{Aut}(R^n) \rightarrow \text{GL}_{n+1}(R) = \text{Aut}(R^{n+1}) \) and \( \text{hocolim}_{\text{Proj}(R)} B\text{Aut}(X) \simeq B\text{GL}(R) \).

We form the group completion \( (\text{hocolim}_\mathcal{D} B\text{Aut}(X))^+ \) and observe that

\[
K_0(\mathcal{D}) \times (\text{hocolim}_\mathcal{D} B\text{Aut}(X))^+
\]

is equivalent to \( \Omega|wN_*\mathcal{D}| \) following [Seg74]. We now have the desired description.

**Theorem 5.15.** If \( \mathcal{D} \) is a Waldhausen \( \infty \)-category where all ingressions split up to homotopy, then

\[
K(\mathcal{C}) \simeq K_0(\mathcal{C}) \times (\text{hocolim}_\mathcal{D} B\text{Aut}(X))^+
\]

**Corollary 5.16** (“plus equals \( Q \)” for weight structures). If \( \mathcal{C}_{\wedge w} \) is the heart of a weight structure on a stable \( \infty \)-category,

\[
K(\mathcal{C}_{\wedge w}) \simeq K_0(\mathcal{C}) \times (\text{hocolim}_{\mathcal{C}_{\wedge w}} B\text{Aut}(X))^+
\]

### 6. Examples of Weight Structures and Applications

In this section, we provide an overview of several examples of weight structures. We produce applications of our main theorem and conjecture about some future directions for research.

#### 6.1. The stable category and cellular truncation

Let \( \text{Sp}^\text{fin} \) denote the category of finite spectra. There is a standard Postnikov \( t \)-structure on \( h\text{Sp}^\text{fin} \) where \( \text{Sp}^\text{fin} \) contains all connective ((\(-1\))-connected) spectra and \( \text{Sp}^\text{fin}_{\leq 0} \) consists of all 1-coconnective spectra (i.e., those with homotopy groups concentrated in degrees \( \leq 0 \)). The \( t \)-structure decompositions are provided by taking \( n \)-connected covers and truncating homotopy groups at degree \( n-1 \). The heart of this \( t \)-structure is the abelian category of finitely-generated groups included as the Eilenberg–Maclane spectra.

The weight structure is generated by the sphere spectrum. Let \( B \) denote the collection of finite wedges of the sphere spectrum \( S^0 \). \( \text{Sp}_{w \geq 0} \) is defined to be those spectra \( E \) with \( h\text{Sp}(X, E) = 0 \) for any \( X \in \Sigma^k B \) for \( k < 0 \). In other words, \( \text{Sp}_{w \geq 0} \) is the subcategory of connective spectra. \( \text{Sp}_{w \leq 0} \) is the defined via the orthogonality condition: \( E \in \text{Sp}_{w \leq 0} \) if \( h\text{Sp}(E, Y) = 0 \) for all \( Y \in \text{Sp}_{w \geq 1} \). \( \text{Sp}_{w \leq 0} \) turns out to contain those spectra which are 0-skeleta—which can be distinguished by their homology: \( E \in \text{Sp}_{w \leq 0} \) if and only if \( HZ_\ast(E) = 0 \) for \( \ast > 0 \) and \( HZ_0(E) \) is a free abelian group. The heart \( \text{Sp}_{w \wedge} \) of this weight structure consists of all spectra weakly equivalent to finite wedges of \( S^0 \).
Corollary 6.1. The $K$-theory of the sphere spectrum, $K(S^0) := K(\text{Sp}_{\text{fin}})$, is equivalent to the $K$-theory of the full additive $\infty$-subcategory on finite wedge sums of the sphere spectrum $S^0$. The latter has a plus-construction description $K(S^0) \simeq K(\mathbb{Z}) \times B\text{Aut}(S^0)^+$, where $\text{Aut}(S^0)$ is the homotopy colimit $\text{hocolim}_{n \in (\text{FinSet}, \text{Inj})} B\text{Aut}(\bigvee_n S^0, \bigvee_n S^0)$ over the skeleton of the category of finite sets and injections.

The cellular truncation applies to many other cases where there is already a robust notion of cellularity. For example, when $R$ is a connective ring spectrum, every finite cell spectrum over $R$ admits a CW-cell structure. Hence, there is a weight structure on $f\text{C}_R$, the $\infty$-category of finite cell $R$-modules (in the sense of [EKMM97]), where weight decompositions are given by truncating with a CW-skeleton. Likewise, the $\infty$-category $f\text{CW}_R$ of finite CW-spectra over $R$ admits a cellular truncation weight structure as well. The heart of both weight structures will consist of retracts of finite wedge sums of copies of $R$.

Corollary 6.2 (see also [EKMM97, IV.3.1]). If $R$ is a connective ring spectrum, then the $K$-theory of finite CW-spectra over $R$ and finite cell spectra over $R$ are equivalent $K(f\text{CW}_R) \simeq K(f\text{C}_R)$ and are both equivalent to the $K$-theory of the full additive $\infty$-subcategory of retracts of finite wedge sums of $R$.

6.2. Chain complexes, the brutal truncation, and a theorem of Gillet and Waldhausen. Let $\mathcal{E}$ be an exact category (in the sense of Quillen). We consider the Waldhausen category $\text{Ch}^{\text{bdd}}(\mathcal{E})$ of bounded chain complexes in $\mathcal{E}$. Cofibrations in this category are admissible monomorphisms and weak equivalences are chain homotopy equivalences. The relative nerve produces a corresponding stable Waldhausen $\infty$-category which also denote $\text{Ch}^{\text{bdd}}(\mathcal{A})$ by abuse of notation. We defend this notation with the fact that the algebraic $K$-theory spectra of these objects coincide [Bar16, 10.16].

$\text{Ch}^{\text{bdd}}(\mathcal{E})$ admits a bounded weight structure where $\text{Ch}^{\text{bdd}}(\mathcal{E})_{w \geq 0}$ consists of chain complexes (chain) homotopy equivalent to ones concentrated in nonnegative degrees. Similarly, $\text{Ch}^{\text{bdd}}(\mathcal{E})_{w \leq 0}$ consists of complexes homotopy equivalent to complexes concentrated in nonpositive degrees. The heart of this weight structure is chain complexes homotopy equivalent to complexes concentrated in degree 0, hence it is equivalent to $\mathcal{E}$ where morphisms between objects are replaced by morphisms in $\text{Ch}^{\text{bdd}}(\mathcal{E})$. Weight decompositions are provided by the brutal truncation:

$$
\begin{align*}
M_{w \leq n} & \quad (\cdots \leftarrow M_{n-1} \leftarrow M_n \leftarrow 0 \leftarrow 0 \leftarrow \cdots) \\
M & \quad (\cdots \leftarrow M_{n-1} \leftarrow M_n \leftarrow M_{n+1} \leftarrow M_{n+2} \leftarrow \cdots) \\
M_{w \geq n+1} & \quad (\cdots \leftarrow 0 \leftarrow 0 \leftarrow M_{n+1} \leftarrow M_{n+2} \leftarrow \cdots)
\end{align*}
$$

If we write $\text{Ch}^{\text{bdd}}(\mathcal{E})_{\triangleleft w}$ for the heart of this weight structure, our main theorem implies the following.
Corollary 6.3. The algebraic $K$-theory of $\text{Ch}^{\text{bdd}}(\mathcal{E})$ is equivalent to Quillen's algebraic $K$-theory of $\mathcal{E}$.

Proof. The main theorem implies that $K(\text{Ch}^{\text{bdd}}(\mathcal{E})\upharpoonright_{\text{cu}}) \simeq K(\text{Ch}^{\text{bdd}}(\mathcal{E}))$ since the brutal truncation provides a bounded weight structure on bounded complexes. The heart $\text{Ch}^{\text{bdd}}(\mathcal{E})\upharpoonright_{\text{cu}}$ is equivalent to the simplicial nerve of $\mathcal{E}$ (morphisms between objects in $\mathcal{E}$ are given by simplicial mapping spaces of morphisms between their chain complexes). By [Bar13, 3.11], the Barwick $K$-theory of this (exact) $\infty$-category coincides with that from Quillen’s $Q$-construction. □

Lemma 6.4. Let $\mathcal{E}$ be an exact category which is idempotent-complete and let $\text{Ch}^{\text{bdd}}(\mathcal{E})$ denote the Waldhausen category of bounded chain complexes on $\mathcal{E}$ with cofibrations level-wise admissible monomorphisms. The $K$-theory spectrum $K(\text{Ch}^{\text{bdd}}(\mathcal{E}))$ is equivalent whether chain homotopy equivalences of quasi-isomorphisms are taken as weak equivalences.

Proof. Let $v_{\text{ch}}$ and $v_{\text{qi}}$ denote the class of chain homotopy equivalences and quasi-isomorphisms, respectively. Consider the localization sequence

$$K(\text{Ch}_{\text{ac}}^{\text{bdd}}(\mathcal{E}), v_{\text{ch}}) \to K(\text{Ch}^{\text{bdd}}(\mathcal{E}), v_{\text{ch}}) \to K(\text{Ch}^{\text{bdd}}(\mathcal{E}), v_{\text{qi}})$$

where $\text{Ch}_{\text{ac}}^{\text{bdd}}(\mathcal{E})$ denotes the full Waldhausen subcategory on acyclic bounded complexes. The lemma follows from proving the $K$-theory of the bounded acyclics is trivial.

It suffices to check for non-negative chain complexes. Filter the acyclics by subcategories $\text{Ch}_{\text{ac}}^{\text{bdd}, \leq n}(\mathcal{E})$ of acyclic complexes concentrated in degrees $0 \leq * \leq n$. The short exact sequence of complexes

$$
\begin{array}{cccccc}
0 & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \\
M_n & \simeq & M_n & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\text{im}(d_n) & \hookrightarrow & M_{n-1} & \to & \text{coker}(d_n) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \to & M_{n-2} & \to & M_{n-2} \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\end{array}
$$

gives a short exact sequence of endofunctors on the acyclics $\text{Ch}_{\text{ac}}^{\text{bdd}, \leq n}(\mathcal{E})$. We note that $\mathcal{E}$ is required to be closed under idempotents for these to take values in chain complexes on $\mathcal{E}$. If $\mathcal{E}$ is not idempotent-complete, one can pass to it’s idempotent closure or Karoubi envelope to apply this result.

The first column is an elementary acyclic complex composed of single isomorphism. Elementary acyclic complexes are all chain homotopy equivalent to the zero complex. Hence, the algebraic $K$-theory of the elementary acyclics is trivial. Since the right column takes values in $\text{Ch}_{\text{ac}}^{\text{bdd}, \leq n-1}(\mathcal{E})$, we can apply the additivity theorem to conclude $K(\text{Ch}_{\text{ac}}^{\text{bdd}, \leq n}(\mathcal{E})) \simeq K(\text{Ch}_{\text{ac}}^{\text{bdd}, \leq n-1}(\mathcal{E}))$. Since $\text{Ch}_{\text{ac}}^{\text{bdd}, \leq 1}(\mathcal{E})$ is equivalent to the category of elementary acyclics
concentrated in degrees 0 and 1, we conclude \( K(\text{Ch}_{\text{ac}}^{\text{bdd}, \leq n}(E)) \simeq * \) for all \( n \) and conclude that \( K(\text{Ch}_{\text{ac}}^{\text{bdd}}(E)) \simeq * \) since \( K \)-theory commutes with directed colimits. □

We now arrive at a new proof of the Gillet–Waldhausen Theorem.

**Corollary 6.5** (Gillet–Waldhausen Theorem, [TT90, 1.11.7]). For an exact category \( \mathcal{E} \) which is idempotent-complete, Quillen’s algebraic \( K \)-theory \( K(\mathcal{E}) \) is homotopy equivalent to the Waldhausen algebraic \( K \)-theory of \( \text{Ch}_{\text{bdd}}^{\text{bdd}}(\mathcal{E}) \), the Waldhausen category of bounded chain complexes on \( \mathcal{E} \) where cofibrations are taken to be admissible monomorphisms and weak equivalences are quasi-isomorphisms of chain complexes.

**Proof.** Corollary 6.3 implies that Quillen’s \( K(\mathcal{E}) \) is equivalent to \( K(\text{Ch}_{\text{bdd}}^{\text{bdd}}(\mathcal{E}), v_{\text{ch}}) \) and Lemma 6.4 implies that \( K(\text{Ch}_{\text{bdd}}^{\text{bdd}}(\mathcal{E}), v_{\text{qi}}) \simeq K(\text{Ch}_{\text{bdd}}^{\text{bdd}}(\mathcal{E}), v_{\text{ch}}) \) as desired. □

**Corollary 6.6.** If \( \text{Proj}^{fg}(R) \) denotes the exact category of finitely-generated projective \( R \)-modules, \( K(\text{Proj}^{fg}(R)) \simeq K(\text{Ch}_{\text{bdd}}^{\text{bdd}}(\text{Proj}^{fg}(R))) \).

6.3. **The Resolution Theorem for exact categories and \( G = K \) by way of the brutal truncation.** For a ring \( R \), the \( G \)-theory of \( R \) is defined to be algebraic \( K \)-theory of the exact category of finitely-generated modules over \( R \).

**Corollary 6.7** (Resolution Theorem, [Wei13, Theorem V.3.1]). Let \( \mathcal{P} \) be a full subcategory of an exact category \( \mathcal{E} \) so that \( \mathcal{P} \) is closed under extensions and under kernels of admissible surjections in \( \mathcal{E} \). Suppose in addition that every object \( M \) in \( \mathcal{E} \) admits a finite \( \mathcal{P} \)-resolution:

\[
0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0
\]

then \( K(\mathcal{P}) \simeq K(\mathcal{E}) \).

**Proof.** The hypotheses imply that every bounded chain complex in \( \mathcal{E} \) admits a bounded quasi-isomorphic \( \mathcal{P} \)-replacement. Put a weight structure on \( \text{Ch}_{\text{bdd}}^{\text{bdd}}(\mathcal{E}) \) by fixing \( \mathcal{P} \)-replacements and applying brutal truncation. The heart of this weight structure is equivalent to \( \mathcal{P} \), included as complexes concentrated in degree 0. Applying the sphere theorem, we get that \( K(\mathcal{P}) \simeq K(\text{Ch}_{\text{bdd}}^{\text{bdd}}(\mathcal{E})) \). The standard brutal truncation on \( \text{Ch}_{\text{bdd}}^{\text{bdd}}(\mathcal{E}) \) produces an equivalence \( K(\text{Ch}_{\text{bdd}}^{\text{bdd}}(\mathcal{E})) \simeq K(\mathcal{E}) \) by Corollary 6.3. □

For Noetherian regular rings, all modules admit finite projective resolutions. The standard corollary of the resolution theorem follows.

**Corollary 6.8** (“\( G = K \)”). When \( R \) is a Noetherian regular ring, \( K(R) \simeq G(R) \).

**Proof.** The resolution theorem implies \( K(\text{Proj}^{fg}(R)) \simeq K(\text{Mod}^{fg}(R)) \). □

6.4. **Categories of motives.** In [Bon09], Bondarko establishes a weight structure on Voevodsky’s triangulated category of motives. These are constructed from the bounded chain complexes of smooth varieties (with smooth correspondences as morphisms) by localizing and forming the idempotent completion. Within this category, the Chow motives are cut out by smooth projective varieties (with morphisms smooth correspondences modulo rational equivalences). Bondarko builds a “Chow” weight structure on \( \text{DM}^{\text{eff}}_{\text{gm}} \) the category of effective geometric motives whose heart is the effective Chow motives [Bon10a]. His \( K_0 \) version of the sphere theorem computes \( K_0(\text{DM}^{\text{eff}}_{\text{gm}}) \simeq K_0(\text{Chow}^{\text{eff}}) \).
Effective geometric motives are constructed as a localization of presheaves of abelian groups on smooth schemes. Let $\text{DM}_{gm,\infty}^{\text{eff}}$ denote the stable $\infty$-category produced by this construction, so that the triangulated category of effective geometric motives is its homotopy category. Bondarko’s weight structure produces a weight structure on $\text{DM}_{gm,\infty}^{\text{eff}}$ whose heart is the full additive $\infty$-subcategory on the effective Chow motives, which we denote $\text{Chow}_{\infty}^{\text{eff}}$. Applying the sphere theorem gives the following new result.

Corollary 6.9. There is an equivalence of connective $K$-theory spectra

$$K(\text{DM}_{gm,\infty}^{\text{eff}}) \simeq K(\text{Chow}_{\infty}^{\text{eff}}).$$

On $\pi_0$, this result reproduces that of Bondarko.

6.5. Conjectural weight structures. Blumberg, Gepner, and Tabuada introduce a category $\mathcal{M}_{\text{loc}}$ of “localizing noncommutative (spectral) motives” in [BGT13]. This category is constructed from the category of spectrum-valued presheaves on the $\infty$-category of small stable $\infty$-categories. This is the category where non-connective $K$-theory is co-represented. Blumberg has conjectured that $\mathcal{M}_{\text{loc}}$ admits a weight structure whose heart is the dualizable objects—the smooth and proper dg-categories.

Following Hill–Hopkins–Ravenel, the category of genuine $G$-spectra admits interesting “slice filtrations”. These are equivariant analogues for Postnikov towers. The author conjectures that there are adjacent slice weight structures generated by wedge sums of the regular representation spheres. The heart of this weight structure would contain finite wedge sums of all finite-dimensional representation spheres concentrated in virtual degree 0.

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