DOMINANCE OF A RATIONAL MAP TO THE COBLE QUARTIC

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Abstract. We show the dominance of the restriction map from a moduli space of stable sheaves on the projective plane to the Coble sixfold quartic. With the dominance and the interpretation of a stable sheaf on the plane in terms of hyperplane arrangements, we expect these tools to reveal the geometry of the Coble quartic.

1. Introduction

Let \( C \) be a smooth non-hyperelliptic curve of genus 3 over complex numbers. Then \( C \) is embedded into \( \mathbb{P}^2 \cong \mathbb{P} H^0(K_C)^* \) by canonical embedding as a plane quartic curve. The moduli space \( SU_C(2, K_C) \) of semistable vector bundles of rank 2 with canonical determinant over \( C \) is known to be a hypersurface in \( \mathbb{P}^7 \), called the ‘Coble quartic’, \([3]\), \([13]\). Let \( W_r \) be the closure of the following set
\[
\{ E \in SU_C(2, K_C) \mid h^0(C, E) \geq r + 1 \}.
\]
Then we have the following inclusions \([14]\) on the Brill-Noether loci,
\[
SU_C(2, K_C) \supseteq W \supseteq W_1 \supseteq W_2 \supseteq W_3 = \emptyset,
\]
where \( W = W^0 \). Many properties on the geometry of these Brill-Noether loci have been discovered in \([14]\).

Let \( \overline{M}(c_1, c_2) \) be the moduli space of stable sheaves of rank 2 with the Chern classes \( (c_1, c_2) \) on the projective plane. The dimension of this space is known to be \( 4c_2 - 3 \) if \( c_1 = 0 \) \([2]\) and \( 4c_2 - 4 \) if \( c_1 = -1 \) \([9]\). Then there exists a rational map \([8]\)
\[
\Phi_k : \overline{M}(1, k) \dashrightarrow SU_C(2, K_C) , \quad 1 \leq k \leq 4,
\]
defined by sending \( E \) to \( E|_C \). It is shown in \([8]\) that \( \Phi_k \) is a dominant map to \( W^2, W^1 \) and \( W \), for \( k = 1, 2, 3 \), respectively. In this article, we give a proof of the dominance of the rational map \( \Phi_4 \). This is equivalent to the dominance of the rational map from \( \overline{M}(3, 6) \) to \( SU_C(2, 3K_C) \) by twisting. For a general bundle \( E \in SU_C(2, 3K_C) \), we embed \( C \) with \( \mathbb{P}^2 \) into a Grassmannian \( Gr(5, 2) \) and take the pull-back of the universal quotient bundle of \( Gr(5, 2) \) to \( \mathbb{P}^2 \). This bundle is shown to be stable and have the Chern classes \((3, 6)\).
As a quick consequence, we can obtain the old result that $SU_C(2, K_C)$ is unirational since $M(1, 4)$ is rational. The unirationality implies the rationally connectedness. We see how we can obtain a rational curve through two general points of the Coble quartic in terms of hyperplane arrangements.

The restriction of vector bundles on $\mathbb{P}_2$ to plane curves was also studied in [7], where the author investigated the restriction of the tangent bundle of $\mathbb{P}_2$ to plane curves and gave the conditions for a vector bundle $E$ on a plane curve to be a pull-back of the tangent bundle of $\mathbb{P}_2$, twisted by $\mathcal{O}_{\mathbb{P}_1}(-1)$.

For the background on vector bundles, we suggest [12] as a good reference.

2. Embedding plane quartics in Grassmannians

Let $E$ be a semistable vector bundle of rank 2 with the determinant $3K_C$ over $C$, i.e. $E \in SU_C(2, 3K_C)$. By the following lemma, we can obtain a morphism

$$\varphi : C \to Gr(H^0(E), 2)$$

sending $p \in C$ to the 2-dimensional quotient space $E_p$ of $H^0(E)$.

**Lemma 2.1.** $H^1(C, E) = 0$ and $E$ is globally generated.

**Proof.** $H^1(E) \simeq H^0(E^* \otimes K_C) \neq 0$ implies the existence of a nonzero homomorphism $E \to \mathcal{O}_C(K_C)$ which contradicts the semistability of $E$. Now, by the same argument, we have $H^1(E(-p)) = 0$ for all $p \in C$. From the long exact sequence of the sequence

$$0 \to E(-p) \to E \to E_p \to 0,$$

we obtain the surjective evaluation map $H^0(E) \to E_p$, which implies the global generation of $E$. \qed

In fact, the morphism $\varphi$ fits in the diagram

$$\begin{array}{ccc}
C & \xrightarrow{\varphi} & Gr(H^0(E), 2) \\
|3K_C| & \downarrow & \downarrow \vartheta \\
\mathbb{P}H^0(3K_C)^* - \xrightarrow{\mathbb{P} \lambda^*} \mathbb{P}(\bigwedge^2 H^0(E)^*)
\end{array}$$

where $\vartheta$ is the Plucker embedding and $\mathbb{P} \lambda^*$ comes from the dual of the homomorphism

$$\lambda : \bigwedge^2 H^0(E) \to H^0(\bigwedge^2 E) \simeq H^0(3K_C).$$

By the following lemma, $\mathbb{P} \lambda^*$ is an embedding and so is $\varphi$ for general $E$.

**Lemma 2.2.** The homomorphism $\lambda$ is surjective for general $E \in SU_C(2, 3K_C)$.

**Proof.** If $E$ is stable, then by the Nagata-Severi theorem [13], we have the exact sequence for $E(-K_C)$,

$$0 \to \mathcal{O}(D) \to E(-K_C) \to \mathcal{O}(K_C - D) \to 0,$$

where $D$ is a divisor of degree 1. For general $E$, we have $H^0(E(-K_C)) = 0$, i.e. we can assume that $H^0(\mathcal{O}(D)) = 0$; i.e. $D$ is non-effective.

Let $L = \mathcal{O}(K_C + D)$ and $F = \mathcal{O}(2K_C - D)$. Then we have

$$0 \to L \to E \to F \to 0.$$
Note that \( h^0(L) = 3 \), \( h^0(F) = 5 \) and \( h^1(L) = h^1(F) = 0 \) and, from the long exact sequence of the above sequence, we have
\[
H^0(E) \simeq H^0(L) \oplus H^0(F),
\]
and hence it is enough to show the surjectivity of the map
\[
H^0(L) \otimes H^0(F) \to H^0(L \otimes F) \simeq H^0(3K_C).
\]
For every \( p \in C \), \( h^0(L(-p)) = 2 + h^1(L(-p)) = 2 + h^0(p - D) = 2 \) since \( D \) is not effective. Hence, we can have a map from \( C \) to \( Gr(2, H^0(L)) \) sending \( p \) to \( H^0(L(-p)) \). Since \( Gr(2, H^0(L)) \simeq \mathbb{P}_2 \), we can choose \( W \in Gr(2, H^0(L)) \) which is not the same as \( H^0(L(-p)) \) for any \( p \in C \). Then by the choice of \( W \), it does not have base locus on \( C \). Now consider the map
\[
W \otimes H^0(F) \to H^0(3K_C).
\]
By the Base-Point-Free Pencil Trick \([1]\), the kernel of this map is isomorphic to \( H^0(C, F \otimes L^{-1}) \), and this is isomorphic to \( H^0(K_C - 2D) \). Note that \( h^0(K_C - 2D) = h^0(2D) \) by the Riemann-Roch theorem. If \( h^0(2D) = 0 \), then \( W \otimes H^0(F) \) is isomorphic to \( H^0(3K_C) \) by the counting of the dimensions. Hence, it is enough to show that \( H^0(2D) = 0 \) for general \( E \).

Assume that \( h^0(2D) > 0 \), and then \( \mathcal{O}(2D) \) is an element of the theta divisor in \( \text{Pic}^1(C) \). The map
\[
\text{Pic}^1(C) \to \text{Pic}^2(C),
\]
derived by \( D \mapsto 2D \), is a finite surjective map of degree 64. Hence the subvariety of \( \text{Pic}^1(C) \) whose elements are \( D \) such that \( h^0(D) = 0 \) and \( h^0(2D) > 0 \) is of 2 dimensions. For these divisors \( D \), the extensions of \( \mathcal{O}(K_C - D) \) by \( \mathcal{O}(D) \) are parametrized by \( \mathbb{P}_3 \), which means that the vector bundles that do not satisfy \( h^0(2D) = 0 \) are of at most 5 dimensions. Hence \( h^0(2D) = 0 \) in general. \( \square \)

Now, for the 5-dimensional subspace \( V \subset H^0(E) \), we have the following diagram:

\[
\begin{array}{ccc}
\mathbb{P}^0 & \xrightarrow{\varphi} & \text{Gr}(V, 2) \\
|3K_C| \downarrow & & \downarrow \theta \\
\mathbb{P}H^0(3K_C)^* & \xrightarrow{P\lambda^*} & \mathbb{P}(\wedge^2 V^*)
\end{array}
\]

Consider a natural map
\[
\begin{array}{ccc}
\mathbb{P}(\wedge^2 \mathcal{E}) & \xrightarrow{f} & \mathbb{P}(\wedge^2 V_7) \\
\downarrow & & \downarrow \\
\text{Gr}(5, V_7), & &
\end{array}
\]
where \( \mathcal{E} \) is the universal subbundle, \( V_7 \) is a 7-dimensional vector space and \( \text{Gr}(5, V_7) \) is the Grassmannian of 5-dimensional subspaces of \( V_7 \). Over \( [V_7] \in \text{Gr}(5, V_7) \), the fibre \( \wedge^2 V_7 \) is linearly embedded into \( \wedge^2 V_7 \).

**Lemma 2.3.** The image of \( f \) is the secant variety of \( \text{Gr}(2, V_7) \subset \mathbb{P}(\wedge^2 V_7) \), and its dimension is equal to 17.
Proof. Let \([x] \in \text{Im}(f)\); i.e. there exists a \(V_5\) such that \(x \in \wedge^2 V_5\). Consider \(G = \text{Gr}(2, V_5) \subseteq \mathbb{P}(\wedge^2 V_5)\). Since the secant variety of \(G\) is \(\mathbb{P}(\wedge^2 V_5)\), we can express \(x\) by
\[(v \wedge w)\text{ or } (v_1 \wedge v_2 + v_3 \wedge v_4),\]
which proves that \(\text{Im}(f)\) is contained in the secant variety of \(\text{Gr}(2, V_7)\).

Now we show the inclusion \(\text{Sec}(\text{Gr}(2, V_7)) \hookrightarrow \text{Im}(f)\). Assume that \(x\) is a general point in the secant variety. This means that
\[x = v_1 \wedge v_2 + v_3 \wedge v_4,\]
where \(U = \langle v_1, v_2, v_3, v_4 \rangle\) is a 4-dimensional space. For any \(V_5 \supset U\), we have \(x \in \wedge^2 V_5\). This shows that
\[\text{Sec}(\text{Gr}(2, V_7)) = \text{Im}(f),\]
since both sides are closed subvarieties of \(\mathbb{P}(\wedge^2 V_7)\). Also the set of such \(V_5\) is 2-dimensional and \(\dim f^{-1}([x]) = 2\). Hence the dimension of \(\text{Im}(f)\) is 17, since \(\dim(\mathbb{P}(\wedge^2 E)) = 19\). □

Remark 2.4. \(\text{Gr}(2, V_7)\) is a Scorza variety of defect \(\delta = 4\) [16]. So, it is known that \(\dim \text{Sec}(\text{Gr}(2, V_7)) = 17\).

Lemma 2.5. For general \(E \in \text{SU}_C(2, 3K_C)\) and general 5-dimensional vector subspace \(V \subset H^0(E)\), the restriction of \(\lambda\) to \(\wedge^2 V\),
\[
\lambda: \wedge^2 V \to H^0(3K_C),
\]
is an isomorphism.

Proof. In the proof of (2.2), let \(V_7 := W \oplus V_5\), where \(V_5 \cong H^0(F)\). In fact, we can take any \(V_5 \subset V_7\) with \(V_5 \cap H^0(L) = 0\). Then, the restriction of \(\lambda\) to \(\wedge^2 V_7\) is also surjective. Let \(K = \ker(\lambda)\) be the 11-dimensional subspace of \(\wedge^2 V_7\). Consider an incidence variety \(\mathcal{R} \subset \text{Gr}(5, V_7) \times \mathbb{P}(K)\),
\[
\mathcal{R} = \{(V_5, [x]) \mid x \in \wedge^2 V_5 \cap K\}.
\]
We have the following diagram:
\[
\begin{array}{ccc}
\text{Gr}(5, V_7) & \xrightarrow{pr_1} & \mathcal{R} \\
& \xrightarrow{pr_2} & \mathbb{P}(K).
\end{array}
\]
It is enough to show that the map \(pr_1\) is not dominant, which means that for the general \(V_5 \subset V_7\) not in the image of \(pr_1\), we have the surjection in the assertion. Assume that \(pr_1\) is dominant, then
\[
\dim(\mathcal{R}) \geq 10.
\]
If we consider again the map
\[
\begin{array}{ccc}
\mathbb{P}(\wedge^2 E) & \xrightarrow{f} & \mathbb{P}(\wedge^2 V_7) \\
\downarrow & & \downarrow \\
\text{Gr}(5, V_7), & &
\end{array}
\]
then the image of $pr_2$ in $\mathbb{P}(K)$ is the intersection of $\text{Im}(f) = \text{Sec}(Gr(2, V_7))$ with $\mathbb{P}(K)$ in $\mathbb{P}(\wedge^2 V_7) \simeq \mathbb{P}_{20}$. Since $\text{Im}(f)$ is 17-dimensional, we have

$$7 \leq \dim \text{Im}(pr_2) \leq 10.$$ 

It is clear that $\mathbb{P}(K)$ contains a point in $\text{Sec}(Gr(2, V_7))$, but not in $Gr(2, V_7)$. The fibre over this point in $\mathcal{R}$ is isomorphic to $Gr(1, 3) \simeq \mathbb{P}_2$. Thus the dimension of $\text{Im}(pr_2)$ is greater than 7.

Now assume that $\dim \mathbb{P}(K) \cap \text{Sec}(Gr(2, 7)) \geq 8$. In the proof of (2.2), we have

$$K \cap (W \wedge V_5) = (0),$$

if $V_5 \cap W = (0)$. If $V_5 \cap W \neq (0)$, the intersection is always $[\wedge^2 W]$. Let us consider the canonical map

$$s : W \otimes V_7/W \rightarrow \wedge^2 V_7/\wedge^2 W.$$ 

For all $V_5$ with $V_5 \cap W = (0)$, the images in $\wedge^2 V_7/\wedge^2 W$ are the same as a 10-dimensional vector space. If we take the preimage of this space in $\wedge^2 V_7$, then it is the union of $W \wedge V_5$ for all $V_5$, which is now an 11-dimensional space. Note that $K \cap (W \wedge V_5) = [\wedge^2 W]$ if $W \cap V_5 \neq (0)$. Let us denote by $D$ the projectivization of the preimage of $s(W \otimes V_7/W)$ in $\wedge^2 V_7$. Then $D$ is a 10-dimensional subvariety of $\mathbb{P}(\wedge^2 V_7)$ and it intersects with $\mathbb{P}(K)$ at the unique point $[\wedge^2 W]$. In fact, $D$ is the projective tangent space $\mathbb{P}T_{[W]}Gr(2, V_7)$ of $Gr(2, V_7)$ at $[W]$ in $\mathbb{P}(\wedge^2 V_7)$. Recall that

$$\begin{align*}
T_{[W]}Gr(2, V_7) &= \text{Hom}(W, V_7/W) \simeq W^* \otimes V_7/W \\
T_{[\wedge^2 W]}\mathbb{P}(\wedge^2 V_7) &= \text{Hom}(\wedge^2 W, \wedge^2 V_7/\wedge^2 W).
\end{align*}$$

The differential map of the Plücker embedding at $[W]$ is defined as follows: $x = w^* \otimes e \in T_{[W]}Gr(2, V_7)$ is sent to the map

$$w_1 \wedge w_2 \mapsto s((w^*(w_1)w_2 - w_1w^*(w_2)) \otimes e),$$

where $W = \langle w_1, w_2 \rangle$. This explains the assertion.

Now since the union of the secant lines of $Gr(2, V_7)$ passing through $[\wedge^2 W]$ is 11-dimensional and $\mathbb{P}(K) \cap \text{Sec}(Gr(2, V_7))$ is of dimension $\geq 8$, we can pick an element $[U] \in \mathbb{P}(K) \cap Gr(2, V_7)$, and then the secant line $[U][W]$ lies in $\mathbb{P}(K)$. From the condition on $W$, $U$ and $W$ span a 4-dimensional subspace of $V_7$. In particular, general points on the secant line $[U][W]$ are indecomposable. Let $p$ be such a point. Since $\text{Sing}(\text{Sec}(Gr(2, V_7))) = Gr(2, V_7)$ [133], the dimension of $T_p(\text{Sec}(Gr(2, V_7)))$ is 17. Note that

$$T_p(\text{Sec}(Gr(2, V_7))) = \langle T_{[W]}G, T_{[U]}G \rangle.$$ 

Since

$$T_p(\mathbb{P}(K) \cap \text{Sec}(Gr(2, V_7))) = \mathbb{P}(K) \cap T_p(\text{Sec}(Gr(2, V_7)))$$

is at least 8-dimensional, $\mathbb{P}(K)$ intersects $T_{[W]}G$ along at least 1-dimensional subspace, which is a contradiction because $\mathbb{P}(K) \cap D$ is a single point. \[\square\]
From the previous lemma, we have the commutative diagram

\[
\begin{array}{ccc}
P^2 & \xrightarrow{v_3} & \mathbb{P}H^0(3K_C)^* \\
\\
\cap & \downarrow & \downarrow \\
C & \rightarrow & \text{Gr}(H^0(E), 2) \cap \mathbb{P}(\wedge^2 H^0(E)^*) \\
\\
& & \downarrow \\
& & \text{Gr}(V, 2) \cap \mathbb{P}(\wedge^2 V^*)
\end{array}
\]

where the composite of the two vertical maps on the right,

\[
\mathbb{P}H^0(3K_C)^* \hookrightarrow \mathbb{P}(\wedge^2 H^0(E)^*) \rightarrow \mathbb{P}(\wedge^2 V^*),
\]

is an isomorphism and \(v_3\) is the 3-tuple Veronese embedding; i.e. \(v_3\) is given by the complete linear system \(|O_{P^2}(3)|\). In particular, \(C\) is embedded into \(\text{Gr}(V, 2)\). Note that \(C\) is non-degenerate in \(P^9 \simeq \mathbb{P}(\wedge^2 V^*)\) due to the Riemann-Roch theorem and the Noether theorem.

**Corollary 2.6.** General element \(E\) in \(SU_C(2, 3K_C)\) is generated by a 5-dimensional subspace of \(H^0(E)\).

### 3. Embedding the Projective Plane into Grassmannian

In the diagram (10), the projective plane \(\mathbb{P}H^0(K_C)^* \simeq P^2\) is embedded into the projective space \(\mathbb{P}(\wedge^2 V^*) \simeq P^9\) by the Plücker embedding.

**Lemma 3.1.** For general \(E \in SU_C(2, 3K_C)\), there exists a 5-dimensional vector subspace \(V \subset H^0(E)\) such that \(\mathbb{P}H^0(K_C)^*\) is embedded into \(\text{Gr}(V, 2)\) in the diagram (10).

**Proof.** Let \(V \subset H^0(E)\) be a 5-dimensional subspace selected in \([15]\) and assume that \(\mathbb{P}H^0(K_C)^*\) is not embedded into \(\text{Gr}(V, 2)\). Recall that \(\text{Gr}(V, 2)\) is cut out by the 4-dimensional projectively linear family of quadrics of rank 6 in \(P^9\) whose singular locus is \(P_3\) contained in \(\text{Gr}(V, 2)\) as the Schubert variety of lines through a point corresponding to the quadric in \(P_4\) \([15]\). Let \(Q(p)\) be one of the quadrics of rank 6 containing \(\text{Gr}(V, 2)\) which does not contain \(S\), where \(p\) is a point in \(P_4\) and \(S\) is the image of \(P_2\) by \(v_3\). Since \(v_3^{-1}(Q(p))\) is a plane sextic curve, we have

\[v_3^{-1}(Q(p)) = C + C',\]

where \(C'\) is a conic. First, assume that \(\text{Gr}(V, 2) \cap S = C + C'\). If we consider the incidence variety \(Z_C = \{(l, x)| x \in l\} \subset C \times P_4\), we have a diagram

\[
\begin{array}{c}
\mathbb{P}_4 \\
\downarrow \\
C \\
\downarrow \\
Z_C \\
\downarrow \\
p \\
\downarrow \\
q \\
\end{array}
\]

Let \(S_C\) be the image of \(q\) in \(P_4\). If \(S_C\) is degenerate, i.e. there exists a hyperplane \(P_3 \subset P_4\) containing \(S_C\), then \(C\) is contained in some Grassmannian \(\text{Gr}(4, 2) \subset \text{Gr}(V, 2)\) and, in particular, \(C\) is contained in \(P_3\), the Plücker space of \(\text{Gr}(4, 2)\),
which is a contradiction to the non-degeneracy of $C$ in $\mathbb{P}_9$. Similarly we can define $Z_{C'}$ and $S_{C'}$. Recall the well known fact that
\[ \deg(C) = \deg(S_C) \cdot \deg(q). \]
If $\deg(S_C) = 1$, i.e. $S_C$ is a plane in $\mathbb{P}_4$, then $C$ must be contained in $\mathbb{P}_3(p)$, the singular locus of a quadric $Q(p)$ for $p \in S_C$, which is a contradiction to the fact that $C \subset \mathbb{P}_9$ is nondegenerate. Hence $\deg(S_C) \geq 2$ and so $\deg(q) \leq 6$. This implies that the number of points in $\mathbb{P}_3(p) \cap C$ is less than 7 for $p \in S_C$. Since the intersection of $S_C$ and $S_{C'}$ is at most 1-dimensional in $S_C$, we still have 2-dimensional choices for $p$ for which $\mathbb{P}_3(p) \cap (C + C') = \mathbb{P}_3(p) \cap C$ is less than 7 points. We can also have the same conclusion on the intersection number of $\mathbb{P}_3(p) \cap (C + C')$ in the case when $Gr(V, 2) \cap S$ is the proper subset of $C + C'$ since it still contains $C$. Now choose $p \in \mathbb{P}_4$ such that the singular locus $\mathbb{P}_3(p)$ of $Q(p)$ meets $C + C'$ with $k$ points where $0 < k < 7$. We have the commutative diagram
\[
\begin{array}{c}
\mathbb{P}_3(p) \\
\downarrow \\
Gr(V, 2) \\
\downarrow \\
Gr(4, 2) \\
\downarrow \\
\mathbb{P}_3 \\
\downarrow \\
S \\
\downarrow \\
C + C',
\end{array}
\]
where $\overline{S}$, $C + C'$ are the images of $S$, $C + C'$, respectively, via the projection, and the image of $Gr(V, 2)$ lies in the image of the quadric $Q$, i.e. the Grassmannian $Gr(4, 2) \subset \mathbb{P}_5$. Let $Q'$ be another quadric cutting $Gr(V, 2)$ with singular locus $\mathbb{P}_3'$. Since $\mathbb{P}_3 \cap \mathbb{P}_3'$ is a single point, the image of $Q'$ by the projection is $\mathbb{P}_5$. Thus the image of $Gr(V, 2)$ is $Gr(4, 2)$. Note that the degree of $C + C'$ is $18 - k$ and the degree of $\overline{S}$ is $9 - k$ since $\mathbb{P}_3(p) \cap S = \mathbb{P}_3(p) \cap (C + C')$. If $Q(p)$ contains $S$ for all such $p \in S_C$, then all quadrics containing $Gr(V, 2)$ of rank 6 should contain $S$ since $S_C$ is nondegenerate in $\mathbb{P}_4$. In particular, $Gr(V, 2)$ should contain $S$, which is against the assumption. So there exists a $p \in S_C$ for which $S$ is not contained in $Q(p)$. Thus the image of $S$ by the projection is also not contained in the image of $Q(p)$, i.e. $Gr(4, 2)$. But the degree of intersection $Gr(4, 2) \cap \overline{S}$ is $2 \times (9 - k) < 18 - k$, which is a contradiction to the fact that this intersection contains $C + C'$. \hfill \Box

Let $U_V$ and $\overline{U_V}$ be the universal subbundle and quotient bundle of $Gr(V, 2)$, respectively. With the condition on $V$ in the previous lemma, let
\[ E_V := v_3^* U_V, \]
which implies that the restriction of $E_V$ to $C$ is $E$, i.e. $E_V|_C = E$.

**Lemma 3.2.** $E_V$ is stable with the Chern classes $(3, 6)$, i.e. $E_V \in \overline{M}(3, 6)$.

**Proof.** Since the first Chern class of $\overline{U_V}$ is the hyperplane section of $Gr(V, 2)$ in $\mathbb{P}(\wedge^2 V^*)$ and $v_3$ is the 3-tuple Veronese embedding, we get $c_1(E_V) = 3$.

By the choice of $V$, we have an exact sequence
\[
0 \to G \to V \otimes \mathcal{O}_{\mathbb{P}_4} \to E_V \to 0,
\]
where $G$ is the kernel of the surjection $V \otimes \mathcal{O}_{\mathbb{P}_4} \to E_V$ and $V$ is a 5-dimensional vector subspace of $H^0(E_V)$. In particular, $h^0(E_V) \geq 5$. By the choice of $E$, we
have \( h^0(E_V(-1)|_C) = 0 \). From the long exact sequence of cohomology of the exact sequence
\[
0 \to E_V(-5) \to E_V(-1) \to E_V(-1)|_C \to 0,
\]
we have
\[
H^0(E_V(-5)) \simeq H^0(E_V(-1)).
\]

For a line \( H \subset \mathbb{P}_2 \), \( E_V|_H \simeq \mathcal{O}_H(a) \oplus \mathcal{O}_H(3-a) \) for \( a = 2 \) or \( 3 \) since \( E_V \) is globally generated. In particular, \( h^0(E_V(-k)|_H) = 0 \) for \( k \geq 4 \). From the long exact sequence of cohomology of the exact sequence
\[
0 \to E_V(-k-1) \to E_V(-k) \to E_V(-k)|_H \to 0,
\]
we have \( h^0(E_V(-k-1)) = h^0(E_V(-k)) \) for all \( k \geq 4 \). Since \( h^0(E_V(-k)) = 0 \) for sufficiently large \( k \), we have \( h^0(E_V(-k)) = 0 \) for \( k \geq 4 \) and in particular, \( h^0(E_V(-1)) = h^0(E_V(-5)) = 0 \); i.e. \( h^0(E_V(-k)) = 0 \) for all \( k \geq 1 \). Hence the vector bundle \( E_V \) is stable.

Again, let \( H \) be a line in \( \mathbb{P}_2 \). From the exact sequence
\[
0 \to E_V(-1) \to E_V \to E_V|_H \to 0,
\]
we get \( h^0(E_V) \leq h^0(E_V|_H) \). Since \( E_V|_H \simeq \mathcal{O}_H(a) \oplus \mathcal{O}_H(3-a) \) for \( a = 2 \) or \( 3 \), \( h^0(E_V|_H) = 5 \) and so \( h^0(E_V) \leq 5 \). Thus we obtain \( h^0(E_V) = \dim V = 5 \).

From the long exact sequence of cohomology of (13), we have \( h^0(\mathbb{P}_2, G) = 0 \). If we twist (13) by \(-1\), we have \( h^1(\mathbb{P}_2, G(-1)) = 0 \). For any line \( l \subset \mathbb{P}_2 \), consider the exact sequence
\[
0 \to G(-1) \to G \to G|_l \to 0.
\]
From the above statement, we get \( H^0(G|_l) = 0 \). Since \( c_1(G) = -c_1(E_V) = -3 \), we have \( G|_l \simeq \mathcal{O}_l(a) \oplus \mathcal{O}_l(b) \oplus \mathcal{O}_l(c) \) with \( a + b + c = -3 \). The only choice from the vanishing of \( H^0(G|_l) \) is \( (a, b, c) = (-1, -1, -1) \). Hence \( G \) is a uniform vector bundle of rank 3 on \( \mathbb{P}_2 \) with the splitting type \((-1, -1, -1)\). From the classification of such bundles [3], we have
\[
G \simeq \mathcal{O}_{\mathbb{P}_2}(-1)^{\oplus 3}.
\]
In particular, \( c_2(G) = 3 \) and so \( c_2(E_V) = 6 \).

Since we can pick an element \( E_V \in \overline{M}(3, 6) \) mapping to a general element \( E \in SU_C(2, 3K_C) \), the rational map
\[
(14) \quad \overline{M}(3, 6) \dashrightarrow SU_C(2, 3K_C)
\]
is dominant. By twisting the map (14) with \( \mathcal{O}_{\mathbb{P}_2}(-1) \) and \( \mathcal{O}_C(-K_C) \), we have the following main theorem.

**Theorem 3.3.** The restriction map
\[
\Phi_4 : \overline{M}(1, 4) \dashrightarrow SU_C(2, K_C)
\]
is dominant.

**Remark 3.4.** Dolgachev and Kapranov [4] showed that the logarithmic bundles \( E(\mathcal{H}) \) attached to the general hyperplane arrangement \( \mathcal{H} = (H_1, \cdots, H_6) \) in \( \mathbb{P}_2 \) form an open Zariski subset \( U \subset \overline{M}(3, 6) \). For these bundles \( E(\mathcal{H}) \), we have a Steiner resolution
\[
0 \to \mathcal{O}_{\mathbb{P}_2}(-1)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}_2}^{\oplus 5} \to E(\mathcal{H}) \to 0.
\]
From this, we have a 5-dimensional space \( V = H^0(\mathbb{P}_2, E(\mathcal{H})) \). Tensoring the exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}_2}(-4) \to \mathcal{O}_{\mathbb{P}_2} \to \mathcal{O} \to 0
\]

by \( E(\mathcal{H}) \), we can consider \( V \) as a subspace of \( H^0(C, E(\mathcal{H})|_C) \), which is 8-dimensional. As we have seen already in the proof of \( \text{[23]} \), the bundle \( E_V \) has a Steiner resolution, pulled back from the universal exact sequence on the Grassmannian \( Gr(V, 2) \). This motivates the whole argument in this paper.

Since \( \overline{M}(1, 4) \) is rational and the map \( \Phi_4 \) is dominant, \( SU_C(2, K_C) \) is unirational. It implies that \( SU_C(2, K_C) \) is rationally connected and so rationally chain-connected. Let \( \mathcal{H} = (H_0, \cdots, H_6) \) be a general arrangement of 6 lines on \( \mathbb{P}_2 \) and then we can associate a logarithmic bundle \( E(\mathcal{H}) \in \overline{M}(3, 6) \) to \( \mathcal{H} \). It is known \( \text{[24]} \) that the logarithmic bundles \( E(\mathcal{H}) \) form an open Zariski subset of \( \overline{M}(3, 6) \) and, after twisting by \( \mathcal{O}_{\mathbb{P}_2}(-1) \), \( \overline{M}(1, 4) \). Let \( \mathcal{F} \) be a family of arrangements of 6 lines on \( \mathbb{P}_2 \) and let \( E(\mathcal{F}) \) be the closure of the subvariety of \( \overline{M}(1, 4) \) whose closed points correspond to \( E(\mathcal{H}) \otimes \mathcal{O}_{\mathbb{P}_2}(-1) \) with \( \mathcal{H} \in \mathcal{F} \).

**Proposition 3.5.** \( SU_C(2, K_C) \) is rationally chain-connected. In fact, any two general points in \( SU_C(2, K_C) \) can be connected by at most 6 rational curves which can be described explicitly.

**Proof.** Let us consider a special type of arrangement of 6 lines. Let \( H_0, H_1, \cdots, H_5 \) be 6 lines in general position on \( \mathbb{P}_2 \) and let \( p \) be a fixed point on \( H_0 \) in general position. If we fix \( H_1, \cdots, H_5 \), then we have a 1-dimensional family \( \mathcal{F} \) of 6 lines with \( H_0 \) moving. Consider a map

\[
\Psi : \mathbb{P}_1(\mathcal{F}) \to SU_C(2, K_C),
\]

sending \( \mathcal{H} \) to \( E(\mathcal{H})(-1)|_C \). Since \( SU_C(2, K_C) \) is projective, this map is a morphism \( \text{[6]} \). Clearly \( \Psi \) is not a constant map; otherwise \( \Phi_4 \) is also a constant map, which is not true. From the fact that logarithmic bundles associated to 6 lines in general position form an open Zariski subset of \( \overline{M}(3, 6) \) and \( \Phi_4 \) is dominant, we can find a 1-dimensional family of 6 lines \( \mathcal{F} \) which maps to a rational curve on \( SU_C(2, K_C) \) via \( \Psi \) for a general element of \( SU_C(2, K_C) \). Furthermore, for two general elements \( E_1, E_2 \in \overline{M}(3, 6) \), we can find 6 families of 6 lines \( \mathcal{F}_i, 1 \leq i \leq 6 \), as above such that the arrangements corresponding to \( E_1, E_2 \) lie in \( \mathcal{F}_1, \mathcal{F}_6 \) respectively and \( \mathcal{F}_i \cap \mathcal{F}_{i+1} \neq \emptyset \). From this fact with the dominance of \( \Phi_4 \), we can find 6 rational curves passing through two general points on \( SU_C(2, K_C) \).

**Remark 3.6.** Note that we can choose these rational curves not contained in the singular locus of \( SU_C(2, K_C) \) which is the Kummer variety of \( \text{Pic}^2(C) \). Let \( \widetilde{S} \) be a desingularization by the blow-up \( \text{[10]} \) and consider the proper transform of the previous 6 rational curves on \( SU_C(2, K_C) \). It shows the rationally chain-connectedness of \( \widetilde{S} \), and since \( \widetilde{S} \) is smooth, it implies the rational connectedness; i.e. the chain of these 6 curves can be deformed to a rational curve and its image on \( SU_C(2, K_C) \) will give us a rational curve through two general points.
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