ON THE SOLUTIONS OF SOME BOUNDARY VALUE PROBLEMS FOR THE GENERAL KDV EQUATION

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Abstract. This paper is concerned with nonlinear partial-differential equations from the Korteweg-de Vries hierarchy. A boundary value problem with inhomogeneous boundary conditions of a certain special form is studied. We construct some class of solutions of the problem using the inverse spectral method.

Key words: KdV hierarchy, boundary value problems, integrability, inverse spectral method
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1. Introduction

Boundary and initial-boundary value problems (BVPs and IBVPs) for integrable nonlinear partial-differential equations play a significant role in mathematical physics being a natural model for the wave processes in semi-bounded space developing under the influence of the boundary regime. The first studies in this area appeared as long ago as 1970-ies, just after the inverse scattering transform (IST) was developed for the Cauchy problems on the whole line (see, for instance, [1]). But BVPs were found to be much more complicated and classical IST failed being applied to them in a straightforward manner.

The further systematic studies on the adaption of the inverse spectral method to the BVPs and IBVPs yield some particular classes of boundary conditions, under which such problems demonstrate some features usually associated with ”integrability”. First, to the best of our knowledge, the special role of such particular boundary conditions were mentioned in [2], were they were treated from hamiltonian point of view as ”the boundary conditions, preserving integrability”. In subsequent papers (see, for instance, [3], [4], [5]) the BVPs with this special type of boundary conditions were shown to admit the wide classes of exact solutions that can be constructed using the appropriate version of the inverse spectral method.

In a framework of recently developed version of the IST for IBVPs (see [6], [7], [8] and references therein) the IBVPs with boundary conditions of the above mentioned particular type (now they are known as integrable or linearizable boundary conditions) also play a special role. Namely, for such problems it was found possible to reduce the method omitting the step dealing with some nontrivial nonlinear problem. For many classical integrable PDEs the corresponding IBVPs were solved completely [3], [5], [6], [7], also the results on the long–time asymptotics of solutions were obtained (see, for instance, [11], [10]).

Although the nature of ”integrability” of the boundary conditions is usually clear for each particular integrable PDE some technical aspects of the method require each particular integrable BVP (IBVP) to be considered separately. In this paper, we ask: Can the appropriate version of the inverse spectral method work with some class of integrable PDEs and some class of BVPs? In this study we start with revisiting the approach and the results of the above mentioned early works such as [3], [13] and examine the set of PDEs, associated in a framework of IST with a classical Sturm–Liouville spectral problem. We call each of these equations as general KdV equation (the explicit form of the general KdV equation is given by (2.1) below). The most known in this family is the classical KdV equation:

\[ q_t = 6qq_x - q_{xxx}. \tag{1.1} \]

In [12] the following boundary conditions:

\[ q(0, t) = a, \quad q_{xx}(0, t) = b \tag{1.2} \]
with real constants \(a, b\) were shown to be the only integrable boundary conditions for (1.1). The corresponding BVP was investigated in several works (see, for instance [13], [11], [14]). In particular, in [13] the following result was obtained. Given the function \(p(t)\) from a certain class and the number \(w_0\) from the certain interval, define \(M(T, \lambda)\) as a Weyl–Titchmarsh function for the Sturm–Liouville operator with the potential \(p(t + T)\) and calculate \(w(t)\) as a solution for the Cauchy problem \(\dot{w} + w^2 = p(t), \ w(0) = w_0\). Then the function \(m(t, \lambda)\) defined as:

\[
m(t, \lambda) = \frac{M(T, \lambda) - w(t)}{4 \lambda + 2a}, \quad f(\lambda) = 16 \lambda^3 - (12a^2 - 4b)\lambda - 2a(2a^2 - b)
\]

is a Weyl-Titchmarsh function for some Sturm-Liouville operator \(L(t) = -d^2/dx^2 + q(x, t)\) and \(q(x, t)\) is a solution of the BVP (1.1), (1.2). In this paper, we consider the general KdV equation together with boundary conditions of a certain special type (actually, a one-parametric set of nonhomogeneous BVPs for each fixed equation) and provide some extension of (1.3) that generates some class of solutions for this BVP. The exact formulation of our main result is contained in Theorem 4.1. Here we notice that the constructed class of solutions for the BVP contains, in particular, soliton and finite-gap solutions. Actually, we work with a wider class \(\tilde{B}\) introduced by V.A. Marchenko (see [15] for detailed discussion on the corresponding spectral theory and it's application to the KdV equation).

2. Formulation of the Problem. Notations and Assumptions

We consider the nonlinear partial-differential equation of the following form ("general KdV equation", see, for instance, [17]):

\[
\dot{q} = \sum_{\nu=0}^{s} C_\nu X_\nu(q)
\]

together with the boundary conditions:

\[
b_{2n}(q) = 0, \ n = 1, s - 1, \ b_{2n-1}(q) = a_n, \ n = 1, s + 1.
\]

Here:

- \(X_\nu(q)\) are polynomials of \(q\) and its derivatives defined as follows:
  \[
  X_\nu = -P'_{\nu+1}, \ P_1 = -\frac{1}{2} q, \ P'_{\nu+1} = H P_\nu,
  \]
  \[
  H = \frac{1}{2} \frac{d^3}{dx^3} + 2q \frac{d}{dx} + q'
  \]
  (here and below "prime" denotes the derivative with respect to \(x\) while "dot" denotes the derivative in \(t\)), \(C_\nu, \nu = 0, \ldots, s\) are real constants.
- \(b_n(q)\) are polynomials with respect to the values of \(q\) and it's derivatives at \(x = 0\) defined as \(b_n(q) := 2^{-n} \beta_n(0; q)\), where:
  \[
  \beta_1 = q, \ \ \beta_{n+1} = -\beta'_n - \sum_{\nu=1}^{n-1} \beta_{\nu} \beta_{n-\nu}.
  \]
- \(a_n, n = 1, s + 1\) are real constants, such that \(a = (a_1, \ldots, a_{s+1}) \in A\), where \(A\) is a certain one-parametric set in \(\mathbb{R}^{s+1}\) that we describe below in this section.

**Assumption 1.** The polynomial

\[
\varphi(\rho) = -\frac{1}{2} \rho \sum_{\nu=0}^{s} C_\nu (2\rho^2)^\nu
\]

can be written in the following form:

\[
\varphi(\rho) = 4^s \rho \prod_{\nu=1}^{s} (\rho^2 - d_\nu)
\]

with \(d_n = -\delta_n^2, \ 0 < \delta_1 < \ldots < \delta_s\).
Now let us specify the set $A$. Define the polynomial:

$$f(\lambda) := 16^s \prod_{\nu=1}^{s} (\lambda - d_\nu)^2,$$

i.e., such that $(\varphi(\rho))^2 = f(\rho^2)$, and consider the equation:

$$f(\lambda) = \mu.$$  \hspace{1cm} (2.5)

Let $\mu^-$ be the greatest lower bound of the set of all real $\mu$ such that all the roots of (2.5) are real. Take an arbitrary $\mu \in (\mu^-, 0)$. Consider (2.5) and denote its roots as $0 > c_0 > c_1 > c'_1 > \ldots > c_s > c'_s$ (see figure 1). (In the sequel it will be convenient to assume that $c_\nu$ were defined as the roots of $f(\lambda)$ satisfying the condition: $f'(c_\nu) < 0$). Define the polynomial

$$g_\mu(\lambda) := 4^s \prod_{\nu=1}^{s} (\lambda - c_\nu)$$  \hspace{1cm} (2.6)

and the real numbers $a_n(\mu)$ as coefficients of the following Laurent series:

$$\frac{4^s \prod_{\nu=1}^{s} (\lambda - d_\nu)}{g_\mu(\lambda)} = \prod_{\nu=1}^{s} \frac{\lambda - d_\nu}{\lambda - c_\nu} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n a_n(\mu)}{\lambda^n}. \hspace{1cm} (2.7)$$

We set $a(\mu) = (a_1(\mu), \ldots, a_{s+1}(\mu))$ and $A = \{a(\mu), \mu \in (\mu^-, 0)\}$.

**Figure 1.**

Remark 2.1. In the framework of classical IST the equation (2.1) can be described as the equation under which evolution of the scattering data of the associated Sturm–Liouville operator $L(t) = -d^2/dx^2 + q(x, t)$ on the whole line has the form:

$$r(\rho, t) = r(\rho, 0)e^{2i\varphi(\rho)t}; \quad \alpha(\kappa, t) = \alpha(\kappa, 0)e^{2i\varphi(\kappa)t},$$

where $r(\rho, t)$ is a reflection coefficient and $\alpha(\kappa, t)$ is a normalizing constant corresponding to an eigenvalue $-\kappa^2$. This means, in particular, that specification of the particular equation of the form (2.1) is equivalent to the specification of the polynomial $\varphi(\cdot)$. The terms in the left-hand sides of the boundary conditions (2.2) can also be described in spectral terms connected with the corresponding Sturm–Liouville operator. Namely, let $m_+(\lambda)$ be a Weyl–Titchmarsh function for operator generated by the expression $\ell y = -y'' + q(x)y$ with real-valued $q \in C^\infty[0, \infty)$ and Dirichlet boundary condition $y(0) = 0$. Then the following asymptotic expansion holds:

$$m_+(\lambda) = i\rho + \sum_{n=1}^{\infty} \frac{b_n(q)}{(i\rho)^n}, \quad \lambda \to \infty, \arg \lambda \in [\epsilon, \pi - \epsilon], \lambda = \rho^2, \text{Im} \rho > 0. \hspace{1cm} (2.7)$$
3. Preliminary facts

First we recall some facts about the rapidly decreasing reflectionless potentials (see, for instance [18]). Consider on the real axis $-\infty < x < \infty$ the Sturm–Liouville operator of the form:

$$L = -\frac{d^2}{dx^2} + q(x)$$

with real–valued potential $q$ satisfying the condition:

$$\int_{-\infty}^{\infty} (1 + |x|) |q^{(k)}(x)| \, dx < \infty, \quad k = 0, 1, 2, \ldots$$

Denote by $B(\mu)$, $\mu > 0$ the set of all reflectionless potentials $q$ such that the spectrum of the operator $L$ is located on $[\mu, \infty)$ and define $B := \bigcup_{\mu<0} B(\mu)$.

Let $e^{\pm}(x, \rho)$, $\pm \text{Im} \rho \geq 0$ be the Jost solutions for $L$ normalized with the asymptotics:

$$e^{\pm}(x, \rho) = e^{i\rho x}(1 + o(1)), \quad x \to \pm \infty$$

and $\psi(x, \rho)$ be the Weyl–Marchenko function defined as follows:

$$\psi(x, \rho) = e^{\pm}(x, \rho), \quad \pm \text{Im} \rho > 0.$$

The function $m(\rho) := \psi'(0, \rho)$ is called the Weyl–Marchenko function. If $q \in B(-a^2)$, $a > 0$ then the Weyl–Marchenko function admits the representation:

$$m(\rho) = i\rho + i \int_{-a}^{a} \frac{d\sigma(\xi)}{\rho - i\xi}$$

(3.2)

where $d\sigma$ is a discrete measure concentrated at the finite set of points $\Lambda_0(q) = \{\xi_k\}_{k=1}^{n_0}$. Inversely, any function $m(\rho)$ of the form (3.2) is a Weyl–Marchenko function for some $q \in B$; moreover $q \in B(-\mu^2), \mu^2 = a^2 + \int_{-\infty}^{\infty} d\sigma(\xi)$.

Let $\{-\kappa_k^2\}_{k=1}^{n}$, $\kappa_k > 0$ be the set of eigenvalues of $L$. Define $\Lambda(q) = \{\kappa_k\}_{k=1}^{n}$. It is known that $n_0 = n$ and the ordering can be chosen such that

$$0 \leq |\xi_1| \leq \kappa_1 \leq |\xi_2| \leq \ldots \leq |\xi_n| \leq \kappa_n.$$

Moreover, if we split $\Lambda(q) = \Lambda_1(q) \cup \Lambda_2(q)$, $\Lambda_2(q) = \Lambda(q) \setminus \Lambda_0(q)$, $\Lambda_1(q) = \Lambda(q) \setminus \Lambda_2(q)$ then for any $\xi \in \Lambda_2(q)$ we necessarily have $-\xi \in \Lambda_0$ and inversely: if $\xi \neq 0$ and both $\pm \xi \in \Lambda_0$ then $|\xi| \in \Lambda_2$. Also one can notice that the set $\{\rho = \pm \rho : \kappa \in \Lambda_1(q)\}$ coincides with the set of nonzero roots of the equation:

$$m(\rho) = m(-\rho).$$

(3.3)

For $\kappa \in \Lambda(q)$ we denote as $\alpha(\kappa)$ the normalizing constant:

$$e^{-}(x, -i\kappa) = \alpha(\kappa)e^{+}(x, i\kappa).$$

(3.4)

Normalizing constants can be represented in terms of $\Lambda(q)$, $\Lambda_0(q)$ and $d\sigma$ as follows:

$$\alpha(\kappa) = \prod_{\xi \in \Lambda_1(q)} \frac{\kappa + \xi}{\kappa - \xi}, \quad \kappa \in \Lambda_1(q),$$

(3.5)

and

$$\alpha(\kappa) = -\frac{d\sigma(-\kappa)}{d\sigma(\kappa)} \prod_{\xi \in \Lambda_0(q)} \frac{\kappa + \xi}{\kappa - \xi}, \quad \kappa \in \Lambda_2(q),$$

(3.6)

where $\Lambda_0(q) := \{\xi \in \Lambda_0(q) : |\xi| \notin \Lambda_2(q)\}$.

Now we consider the set $B = \bigcup_{\mu<0} B(\mu)$, where the closure is considered in the topology of uniform convergence of functions on each compact set of the real axis. For $q \in \overline{B}$ we define the Weyl–Marchenko solution $\psi(x, \rho)$ as

$$\psi(x, \rho) = \psi^{\pm}(x, \rho^2), \quad \pm \text{Im} \rho > 0,$$

where $\psi^{\pm}(x, \lambda)$ are the Weyl solutions on the semi-axes $\pm x \in (0, \infty)$ normalized as $\psi^{\pm}(0, \lambda) = 1$, and the Weyl–Marchenko function $m(\rho)$ as $m(\rho) := \psi'(0, \rho)$. If $q \in \overline{B(-a^2)}, a > 0$ then the Weyl–Marchenko function admits the representation (3.2) with some measure $d\sigma$ concentrated on
\(-a, a\) and satisfying the estimate \(\int_{-a}^{a} d\sigma(\xi) < a^2\). Inversely, any function of the form (3.2) with an arbitrary measure \(d\sigma\) is a Weyl–Marchenko function for some \(q \in \hat{B}\); moreover \(q \in B(\mu^2)\), \(\mu^2 = a^2 + \int_{-a}^{a} d\sigma(\xi)\). It is clear from the representation (3.2) that for \(q \in \hat{B}\) the corresponding Weyl–Marchenko function \(m(\rho)\) is holomorphic outside some finite segment of the imaginary axis, the corresponding Laurent series has the same form as an asymptotic expansion (2.7), namely:

\[
m(\rho) = i\rho + \sum_{n=1}^{\infty} \frac{b_n(q)}{(i\rho)^n},
\]

(3.7)

4. Main result

Let us choose an arbitrary \(\mu^* \in (\mu^-, 0)\) and consider the problem (2.1), (2.2) with \((a_1, \ldots, a_{s+1}) = a(\mu^*) \in \mathcal{A}^s\).

**Theorem 4.1.** Let \(Q\) be an arbitrary function from \(\overline{B(\mu^*)}\), \(\mu^* \in (\mu^+, 0)\). Denote by \(M(T, \cdot)\), \(T\) the Weyl–Marchenko function for \(Q_T(t) := Q(t + T)\). Let \(w\) be a solution of the Cauchy problem:

\[
\dot{w} + w^2 = Q(t) - \mu^*, \quad w(0) = w_0
\]

(4.1)

with an arbitrary \(w_0 \in (M(0, i\kappa^*), M(0, -i\kappa^*))\), \(\kappa^* := \sqrt{-\mu^*}\). Denote \(g(\lambda) = g_{\mu^*}(\lambda)\).

Then the function \(m(t, \cdot)\) defined as:

\[
m(t, \rho) := \frac{M(t, \varphi(\rho)) - w(t)}{g(\rho^2)},
\]

(4.2)

is a Weyl–Marchenko function for some function \(q(\cdot, t) \in \hat{B}\) and the function \(q(x, t)\) is a solution of the boundary value problem (2.1), (2.2) on each of the two semi-axes \((-\infty, 0), (0, \infty)\).

**Proof.** We start with the following remark. One can easily show that \(M(0, i\kappa^*) < M(0, -i\kappa^*)\) (it is clear for \(q \in B(\mu^*)\), for \(q \in \overline{B(\mu^*)}\) it can be proved via the limiting procedure). This means that the interval \((M(0, i\kappa^*), M(0, -i\kappa^*))\) specified in Theorem is not empty. Moreover, the comparison theorem for the Riccati equation yields the estimate \(M(T, i\kappa^*) < w(t) < M(T, -i\kappa^*)\), and since \(M(T, \pm i\kappa^*)\) are finite for all \(T\) we can conclude the \(w(T)\) is also finite for all \(T\). This means, in particular, that the function \(m(t, \rho)\) is correctly defined via (4.2).

The further proof will be divided into several steps.

1) Let us show that under the conditions of Theorem for any fixed \(T m(t, \cdot)\) is a Weyl–Marchenko function for some \(q(\cdot, t) \in \overline{B(\lambda^*)}\), where \(\lambda^*\) depends only upon \(\mu^*\). Moreover, \(q(\cdot, t)\) satisfies the boundary conditions (2.2).

First we use the relation

\[
\frac{1}{g(\rho^2)} = \sum_{\nu=1}^{s} \frac{1}{g'(c_{\nu})(\rho^2 - c_{\nu})} = \sum_{\nu=1}^{s} \frac{i}{2\gamma_{\nu}g'(c_{\nu})} \left[ -\frac{1}{\rho - i\gamma_{\nu}} + \frac{1}{\rho + i\gamma_{\nu}} \right],
\]

where \(\gamma_{\nu} = \sqrt{-c_{\nu}}\) and rewrite (4.2) into the following form:

\[
m(\rho) = \sum_{\nu=1}^{s} \frac{i}{2\gamma_{\nu}g'(c_{\nu})} \left[ \frac{w - M(\varphi)(\rho) - w}{\rho - i\gamma_{\nu}} + \frac{M(\varphi(\rho)) - w}{\rho + i\gamma_{\nu}} \right],
\]

(here and everywhere throughout this consideration for the sake of brevity we omit the parameter \(t\) in all the arguments). Then we note that \(\varphi(\pm i\gamma_{\nu}) = \pm (-1)^{\nu - 1}i\kappa^*\) and rewrite this representation as follows:

\[
m(\rho) = m_1(\rho) + m_2(\rho),
\]

(4.3)

where

\[
m_1(\rho) = \sum_{\nu=1}^{s} \frac{i}{2\gamma_{\nu}g'(c_{\nu})} \left[ \frac{M(\varphi(\gamma_{\nu})) - M(\varphi)(\rho) + M(\varphi(\rho)) - M(\varphi(-i\gamma_{\nu}))}{\rho - i\gamma_{\nu}} \right],
\]

(4.4)

\[
m_2(\rho) = \sum_{\nu=1}^{s} \frac{i}{2\gamma_{\nu}g'(c_{\nu})} \left[ \frac{w - M((-1)^{\nu - 1}i\kappa^*) + M((-1)^{\nu}i\kappa^*) - w}{\rho + i\gamma_{\nu}} \right].
\]

(4.5)
Now we use the representation (3.2) for $M(T, \cdot)$:

$$M(\mu) = i\mu + i \int_{-\kappa}^{\kappa} \frac{d\theta(\eta)}{\mu - i\eta},$$

to rewrite (4.4) into the following form:

$$m_1(\rho) = m_{10}(\rho) + \sum_{\nu=1}^{s} \frac{i}{2\gamma_\nu g'(c_\nu)} \left\{ \frac{i}{\rho - i\gamma_\nu} \int_{-\kappa}^{\kappa} \left[ \frac{1}{\varphi(i\gamma_\nu) - i\eta} - \frac{1}{\varphi(\rho - i\eta)} \right] d\theta(\eta) + \frac{i}{\rho + i\gamma_\nu} \int_{-\kappa}^{\kappa} \left[ \frac{1}{\varphi(\rho) - i\eta} - \frac{1}{\varphi(-i\gamma_\nu) - i\eta} \right] d\theta(\eta) \right\},$$

(4.6)

where

$$m_{10}(\rho) = \sum_{\nu=1}^{s} \frac{i}{2\gamma_\nu g'(c_\nu)} \left\{ \frac{i\varphi(\rho) - \varphi(i\gamma_\nu)}{\rho - i\gamma_\nu} + \frac{i\varphi(i\gamma_\nu) - \varphi(\rho)}{\rho + i\gamma_\nu} \right\}.$$

(4.7)

Now consider (4.7) in details. One can easily notice that its right-hand side is a polynomial while taking the limit as $\rho \to \infty$ we obtain:

$$m_{10}(\rho) = \sum_{\nu=1}^{s} \frac{i}{2\gamma_\nu g'(c_\nu)} \left[ \frac{i\varphi(\rho) - \varphi(i\gamma_\nu)}{\rho - i\gamma_\nu} + O \left( \frac{1}{\rho} \right) \right] + O \left( \frac{1}{\rho} \right) = i\rho + O \left( \frac{1}{\rho} \right).$$

Thus we have $m_{10}(\rho) = i\rho$ and we can return to (4.6) and write it in the following way:

$$m_1(\rho) = i\rho + \int_{-\kappa}^{\kappa} \sum_{\nu=1}^{s} \frac{i}{2\gamma_\nu g'(c_\nu)} \left\{ \frac{i}{\rho - i\gamma_\nu} \varphi(i\gamma_\nu) - \frac{i\varphi(\rho) - \varphi(i\gamma_\nu)}{(\varphi(\rho) - i\eta)(\varphi(i\gamma_\nu) - i\eta)} \right\} \left[ \frac{1}{\varphi(\rho) - i\eta} - \frac{1}{\varphi(-i\gamma_\nu) - i\eta} \right] d\theta(\eta) + \frac{i}{\rho + i\gamma_\nu} \left[ \frac{\varphi(-i\gamma_\nu) - \varphi(\rho)}{(\varphi(-i\gamma_\nu) - i\eta)(\varphi(\rho) - i\eta)} \right] d\theta(\eta).$$

Note that the integrand is a sum of some meromorphic functions vanishing at infinity with poles that are the roots of the equation $\varphi(\rho) = i\eta$. Clear that for $\eta \in [-\kappa, \kappa]$ all these roots are pure imaginary and can be written in the form $i\xi_j = i\xi_j(\eta)$, $j = -s, s$, where $\xi_0 \in (\gamma_0, \gamma_0)$, $\xi_j \in (\gamma_j, \gamma_j')$, $j = \overline{1, s}$, $\xi_j \in \left(\gamma_j, -\gamma_j\right)$, $j = \overline{1, s}$ (see figure 2).

Using the representations:

$$\frac{i}{\rho - i\gamma_\nu} \cdot \frac{\varphi(\rho) - \varphi(i\gamma_\nu)}{(\varphi(i\gamma_\nu) - i\eta)(\varphi(\rho) - i\eta)} = \sum_{j=-s}^{s} \frac{1}{\varphi'(i\xi_j)(\rho - i\xi_j)} \cdot \frac{1}{\gamma_\nu - \xi_j},$$

$$\frac{i}{\rho + i\gamma_\nu} \cdot \frac{\varphi(-i\gamma_\nu) - \varphi(\rho)}{(\varphi(-i\gamma_\nu) - i\eta)(\varphi(\rho) - i\eta)} = \sum_{j=-s}^{s} \frac{1}{\varphi'(i\xi_j)(\rho - i\xi_j)} \cdot \frac{1}{\gamma_\nu + \xi_j},$$

we obtain:

$$m_1(\rho) = i\rho + \int_{-\kappa}^{\kappa} \sum_{j=-s}^{s} \frac{1}{\varphi'(i\xi_j)(\rho - i\xi_j)} \frac{1}{\gamma_\nu - \xi_j(\eta)} + \frac{1}{\gamma_\nu + \xi_j(\eta)} \right\} d\theta(\eta)$$

that yields after some algebra:

$$m_1(\rho) = i\rho + i \sum_{j=-s}^{s} \int_{-\kappa}^{\kappa} \frac{d\theta(\eta)}{(\rho - i\xi_j(\eta))} \varphi'(i\xi_j(\eta)) g(-\xi_j^2(\eta)).$$

(4.8)

In each integral we make a change of variable $\xi_j(\eta) = \xi$ and arrive at:

$$m_1(\rho) = i\rho + i \sum_{j=-s}^{s} \int_{\xi_j}^\kappa \frac{d\sigma_j(\xi)}{\rho - i\xi_j},$$

(4.9)
where $I_j$, $j = -s, s$ are some segments such that $I_j \subset (\gamma_j, \gamma'_j)$, $I_{-j} \subset (-\gamma'_j, -\gamma_j)$, $j = 1, s$, $I_0 \subset (-\gamma_0, \gamma_0)$ and $\sigma_j(\xi)$ are nondecreasing functions on $I_j$ defined as:

$$
d\sigma_j(\xi) = \frac{(-1)^j d(\theta(i\varphi(\xi)))}{\varphi'(i\xi)g(-\xi^2)}.
$$

Now let us return to representation (4.5). From the estimate $M(i\kappa^*) < w < M(-i\kappa^*)$ mentioned above we get

$$
\text{sgn}(w - M((-1)^{\nu-1}i\kappa^*)) = \text{sgn}(M((-1)^{\nu}i\kappa^*) - w) = \text{sgn}g'(c_\nu) = (-1)^{\nu-1}.
$$

This means that in the representation:

$$
m_2(\rho) = i \int_{-\gamma_2}^{\gamma_2} \frac{d\sigma^0(\xi)}{\rho - i\xi},
$$

$\sigma^0$ corresponds to a discrete measure concentrated at the points $\pm\gamma_\nu$, $\nu = 1, s$ and

$$
d\sigma^0(\pm\gamma_\nu) = \pm \frac{(w - M(\pm(-1)^{\nu-1}i\kappa^*))}{2\gamma_\nu g'(c_\nu)} > 0.
$$

Finally, gathering together (4.3), (4.9)-(4.12) we can conclude that $m(\rho)$ can be represented in the form:

$$
m(\rho) = i\rho + i \int_{-\gamma_2}^{\gamma_2} \frac{d\sigma(\xi)}{\rho - i\xi},
$$

where

$$
\sigma(\xi) = \sigma^0(\xi) + \sum_{j=-s}^{s} \chi_j(\xi)\sigma_j(\xi),
$$

and $\chi_j$ are the characteristic functions of the segments $I_j$. Since $\mu_* < \mu^*$ the endpoints of the segments $I_j$ are on some positive distance (depending only upon $\mu^*$ and $\mu_*$) from $\gamma_j$, $\gamma'_j$. This means that all the denominators in (4.10) can be estimated from below by some positive constant that depends on $\{d_\nu\}$ and $\{c_\nu\}$ but not on the function $Q$. Thus we can estimate

$$
\int_{I_j} d\sigma_j(\xi) < C \int_{-\kappa_*}^{\kappa_*} d\theta(\eta).
$$
Further, using, for instance, [13], Lemma 2.4 one can show that for any $Q \in \overline{B(\mu_*)}$ the Weyl–Marchenko function $M(t, \mu)$ is bounded for $t \in [0, \infty)$ and any fixed $|\mu| > \sqrt{\|\mu\|}$ with some constant that depends only upon $\mu, \mu_*$ and not on $Q \in \overline{B(\mu_*)}$. This means that $|\nu(t) - M(t, \pm i\kappa)| < c_0$, i.e.

$$\gamma' \int d\sigma^0(\xi) < C_0.$$  \hfill (4.16)

The required assertion follows now from (4.13)–(4.16), relation (4.2) and the Laurent expansion (3.7).

2) Now we show that under the conditions of Theorem if $Q(\cdot) \in B(\mu_*)$ then corresponding $q(\cdot, \cdot)$ satisfies the equation (2.1). Consider (for any fixed $T$) the function $m(T, \cdot)$. It is already shown to be the Weyl–Marchenko function for some $q(\cdot, T)$. From (4.10), (4.12) shows that all $\gamma \in \Lambda(\cdot)$ and consequently the set $\Lambda(\cdot)$ and we have finally:

$$\Lambda_0(q(\cdot, T)) = \{\xi_j(\eta), j = -s, s, \eta \in \Lambda_0(Q_T)\} \cup \{\pm \gamma_{\nu}\}_{\nu = 1}^s.$$  \hfill (4.17)

The total number of the jump points of the function $\sigma(\cdot)$ is $N = (2s + 1)n + 2s$, where $n = \text{card}(\Lambda(Q))$.

Further, since $\varphi(\cdot)$ is odd $M(T, -i\eta) = M(T, i\eta)$ implies $m(t, -i\xi_j(\pm\eta)) = M(t, i\xi_j(\pm\eta))$, $j = -s, s$ and thus for any $\eta \in \Lambda_1(\cdot)$ all corresponding $|\xi_j(\eta)|$, $j = -s, s$ belong to $\Lambda_1(q(\cdot, T))$. The same arguments show that for any $\eta \in \Lambda_2(\cdot)$ all the $|\xi_j(\eta)|$, $j = -s, s$ belong to $\Lambda_2(q(\cdot, T))$. Furthermore, (4.17) shows that all $\gamma_{\nu}$, $\nu = 1, s$ belong to $\Lambda_2(q(\cdot, T))$ and (4.2) shows that all $\delta_{\nu}$, $\nu = 0, s$ belong to $\Lambda_1(q(\cdot, T))$.

Now if we count all the points that are already shown to belong to $\Lambda(q(\cdot, T))$ we obtain $(2s + 1)n + s + s = N$. This means that we have found all the elements of this set and we have finally:

$$\Lambda_1(q(\cdot, T)) = \{|\xi_j(\eta)|, j = -s, s, \eta \in \Lambda_1(Q_T)\} \cup \{\delta_{\nu}\}_{\nu = 1}^s,$$
$$\Lambda_2(q(\cdot, T)) = \{|\xi_j(\eta)|, j = -s, s, \eta \in \Lambda_2(Q_T)\} \cup \{\gamma_{\nu}\}_{\nu = 1}^s.$$  \hfill (4.18), (4.19)

One important observation can be already made: it follows from (4.18), (4.19) that:

$$\Lambda(q(\cdot, T)) = \{|\xi_j(\eta)|, j = -s, s, \eta \in \Lambda(Q_T)\} \cup \{\delta_{\nu}\}_{\nu = 1}^s \cup \{\gamma_{\nu}\}_{\nu = 1}^s$$

and consequently the set $\Lambda(q(\cdot, T))$ does not depend upon $T$. Our next goal is to observe the evolution in $T$ of the normalizing constants $\alpha(\xi, T), \xi \in \Lambda(q(\cdot, T))$. There are four different possibilities that require to be considered separately.

Case 1. $\xi^0 = |\xi_0(\eta^0)|$, $\eta^0 \in \Lambda_1(Q_T)$. In this case we should use the relation (3.5) that yields:

$$\alpha(\xi^0, T) = \prod_{\xi \in \Lambda_1(Q_T)} \frac{\xi^0 + \xi}{\xi^0 - \xi} = \prod_{\eta \in \Lambda_1(Q_T)} \frac{\xi^0 + \xi_0(\eta)}{\xi^0 - \xi_0(\eta)} = \prod_{\eta \in \Lambda_1(Q_T)} \frac{\varphi(i\xi^0) + i\eta}{\varphi(i\xi^0) - i\eta}.$$  \hfill (4.17)

Further, we obtain

$$\alpha(\xi^0, T) = \prod_{\eta \in \Lambda_1(Q_T)} \frac{\eta^0 + \eta}{\eta^0 - \eta} = A(\eta^0, T),$$

if $i\eta^0 = \varphi(\xi^0)$,

$$\alpha(\xi^0, T) = \prod_{\eta \in \Lambda_1(Q_T)} \frac{-\eta^0 + \eta}{-\eta^0 - \eta} = A^{-1}(\eta^0, T),$$

if $i\eta^0 = -\varphi(\xi^0)$ and in both cases $A(\eta^0, T)$ denotes the normalizing constant in (3.4) for the potential $Q_T$. Since $A(\eta^0, T) = A(\eta^0, 0)e^{-2\eta^0 T}$ we obtain $\alpha(\xi^0, T) = \alpha(\xi^0, 0)e^{2\varphi(i\xi^0) T}$.

Case 2. $\xi^0 = \delta_k$. Proceeding as above we obtain

$$\alpha(\delta_k, T) = \prod_{\eta \in \Lambda_1(Q_T)} \frac{\varphi(\delta_k) + i\eta}{\varphi(\delta_k) - i\eta}.$$  \hfill (4.18)

Since $\varphi(\delta_k) = 0$ this yields: $\alpha(\delta_k, T) = \alpha(\delta_k, 0)$ that can be written in the same form as in case 1: $\alpha(\delta_k, T) = \alpha(\delta_k, 0)e^{2\varphi(i\delta_k) T}$. 

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Case 3. $\xi^0 = |\xi_k(\eta^0)|$, $\eta^0 \in \Lambda_2(Q_T)$. In this case we use the relation (3.6) that yields:

$$\alpha(\xi^0, T) = -\frac{d\sigma(-\xi^0, T)}{d\sigma(\xi^0, T)} \prod_{\xi \in \Lambda_2(Q_T)} \frac{\xi^0 + \xi}{\xi^0 - \xi} = -\frac{d\sigma(-\xi^0, T)}{d\sigma(\xi^0, T)} \prod_{\eta \in \Lambda_2(Q_T)} \prod_{\xi_0 + \xi_j(\eta)} =$$

$$\frac{d\sigma(-\xi^0, T)}{d\sigma(\xi^0, T)} \prod_{\eta \in \Lambda_2(Q_T)} \frac{\varphi(i\xi^0) + i\eta}{\varphi(i\xi^0) - i\eta}$$

Using (4.10) we obtain:

$$\alpha(\xi^0, T) = -\frac{d\theta(-\eta^0)}{d\theta(\eta^0)} \prod_{\eta \in \Lambda_2(Q_T)} \frac{\eta^0 + \eta}{\eta^0 - \eta} = A(\eta^0, T),$$

if $i\eta^0 = \varphi(i\xi^0)$,

$$\alpha(\xi^0, T) = -\frac{d\theta(\eta^0)}{d\theta(-\eta^0)} \prod_{\eta \in \Lambda_2(Q_T)} \frac{-\eta^0 + \eta}{-\eta^0 - \eta} = A^{-1}(\eta^0, T),$$

if $i\eta^0 = -\varphi(i\xi^0)$ and in both cases we obtain $\alpha(\xi^0, T) = \alpha(\xi^0, 0)e^{2i\varphi(i\xi^0)T}$.

Case 4. Consider the points $\gamma_k, k = 1, s$. Here (3.6) yields:

$$\alpha(\gamma_\nu, T) = -\frac{d\sigma(-\gamma_\nu, T)}{d\sigma(\gamma_\nu, T)} \prod_{\xi \in \Lambda_2(Q_T)} \frac{\gamma_\nu + \xi}{\gamma_\nu - \xi} =$$

$$\prod_{\eta \in \Lambda_2(Q_T)} \frac{\varphi(T, -1)^{-1}iK^*}{\psi(T, -1)^{-1}iK^*)} \prod_{\eta \in \Lambda_2(Q_T)} \frac{\varphi(T, -1)^{-1}iK^*)}{\psi(T, -1)^{-1}iK^*)},$$

$$\frac{d\sigma(-\gamma_\nu, T)}{d\sigma(\gamma_\nu, T)} = \frac{w(T) - M(T, (-1)^{-1}iK^*)}{w(T) - M(T, (-1)^{-1}iK^*)} =$$

$$\frac{w(T) - M(T, (-1)^{-1}iK^*)}{w(T) - M(T, (-1)^{-1}iK^*)} =$$

$$\frac{\psi(T, -1)^{-1}iK^*)}{\psi(T, -1)^{-1}iK^*)},$$

where $\psi(t, \rho)$ is the Weyl–Marchenko solution for $Q$.

On the other hand the Jost solution $e^+(t, \rho)$ for $Q$ admits the representation (18):

$$e^+(t, \rho) = e^{i\rho t} \prod_{\eta \in \Lambda(Q)} \frac{\rho - i\eta}{\rho + i\eta},$$

that yields, in particular, for any $\tau : \pm \tau \notin \Lambda(Q) \cup \Lambda_0(Q_T)$:

$$\prod_{\eta \in \Lambda(Q_T)} \frac{\tau + \eta}{\tau - \eta} = C(\tau) \cdot e^{2i(\tau t)} \frac{e^+(T, -i\tau)}{e^+(T, i\tau)}.$$

Gathering together the relations (4.20)-(4.24) we obtain:

$$\alpha(\gamma_\nu, T) = C_\nu \cdot \frac{\psi(T, (-1)^{-1}iK^*)}{\psi(T, (-1)^{-1}iK^*)} \cdot e^{2i(-1)^{-1}iK^*)T} \cdot e^+(T, (-1)^{-1}iK^*) \cdot e^+(T, (-1)^{-1}iK^*) =$$

Since Jost $e^+(t, \rho)$ and Weyl–Marchenko $\psi(t, \rho)$ solutions are proportional finally this yields $\alpha(\gamma_\nu, T) = \alpha(\gamma_\nu, 0)e^{2i\varphi(i\xi^0)T}$.

Thus we have $\alpha(\xi, T) = \alpha(\xi, 0)e^{2i\varphi(i\xi^0)T}$ for all $\xi \in \Lambda(g(\cdot, T))$ and this means that $q$ satisfies equation (2.1).

3) Now we take an arbitrary $Q \in \overline{B(\mu_.)}$ and define $m(t, \cdot)$ via (4.2). As it was already shown in the first part of this proof $m(t, \cdot)$ is a Weyl–Marchenko function for some $g(\cdot, T) \in \overline{B(\lambda_*)}$.
On the other hand $Q$ can be obtained as the limit of the sequence $Q_N \in B(\mu_\ast)$ in the topology of $\mathcal{B}(\mu_\ast)$ (i.e., in the topology of uniform convergence of the functions and all their derivatives on any compact set) and the corresponding Weyl–Marchenko functions $M_N(t, \rho)$ converge to the Weyl–Marchenko function $M(t, \rho)$ (at least in a pointwise sense). Let us define

$$m_N(t, \rho) := \frac{M_N(t, \varphi(\rho)) - w_N(t)}{g(\rho^2)},$$

where $w_N$ is a solution of the Cauchy problem:

$$w_N + w_N^2 = Q_N(t) - \mu^* \quad \text{and} \quad w_N(0) = w_0.$$

Then $m_N(t, \cdot)$ are the Weyl–Marchenko functions for $q_N(\cdot, t) \in B(\lambda_\ast)$, $q_N$ satisfies general KdV equation (2.1) and

$$m(t, \rho) = \lim_{N \to \infty} m_N(t, \rho), \quad \rho \in \mathbb{C} \setminus [-i\tau_\ast, i\tau_\ast]$$

with some fixed $\tau_\ast$.

Let $B(\lambda_\ast)$ be the set of all the solutions of (2.1) that belong to $B(\lambda_\ast)$ with the topology of the convergence of functions with all their derivatives uniform on any compact set of the $(x, t)$ -plane. As in proof of [18], Theorem 2.3 $B(\lambda_\ast)$ can be shown to be a precompact set. So there exists $q^\ast$ such that some subsequence $q_{N_n}(x, t)$ converges to $q^\ast(x, t)$ as $n \to \infty$ together with all the derivatives uniformly on any compact set of the $(x, t)$ - plane. Clear that $q^\ast$ satisfies the equation (2.1). At the same time it is clear that for any fixed $t$ $q_{N_n}(\cdot, t) \to q^\ast(\cdot, t)$ in the topology of $B(\lambda_\ast)$. This means that $q^\ast(\cdot, t) \in B(\lambda_\ast)$ and without loss of generality we can assume that the corresponding Weyl–Marchenko function $m^\ast(t, \rho) = \lim_{n \to \infty} m_{N_n}(t, \rho)$. Together with (4.25) this yields $m^\ast(t, \rho) \equiv m(t, \rho)$, consequently, $q^\ast = q$ and the Theorem is proved. \hfill \Box

Remark 4.1 One can easily notice from the proof of Theorem 4.1 that the procedure presented in theorem allows to construct, in particular, the soliton solutions for the problem (2.1), (2.2): it is sufficient to choose $Q$ from the proper class of reflectionless potentials. In an analogous way the finite-gap solutions for the problem can be constructed via the same procedure. For this purpose one should choose $Q$ as a finite-gap potential with all the gaps lying on the interval $(\mu^*, 0)$ and set $w_0 = M(0, -i\kappa^*)$ or $w_0 = M(0, i\kappa^*)$ (one can easily show that the assertion of the Theorem remains true in this case).

Remark 4.2 In our considerations we treated the function $Q$ as a free parameter but it can also be described in terms of boundary values of the solution $q(x, t)$. Namely, as it follows from (4.2) $w(t)$ (which is a solution of the Cauchy problem (4.1)) can be written as $w(t) = (-1)^\ast \cdot 14^\ast b_{2s}(q^\ast(\cdot, t))$.

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