Gravitational Waves from Relativistic Stars

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Stellar pulsations in rotating relativistic stars are reviewed. Slow rotation approximation is applied to solving the Einstein equations. The rotational effects on the non-axisymmetric oscillations are explicitly shown in the polar and axial modes.

§1. Introduction

The amplitude of gravitational waves is quite tiny, although they carry enormous amount of energy. They propagate without suffering any disturbance after leaving the sources. The observation gives the direct information of the emission regions. The gravitational wave astronomy in the next century will therefore be an invaluable tool to diagnose the interior structure of the relativistic stars. As one of the promising sources, stellar pulsations are important. The oscillations are excited in the newly born neutron stars just after supernova explosions, or in abrupt stellar quakes. The excitation mechanism is uncertain at present, much more the expected number of the sources. The direct observation only will give a clue to these questions. In addition to the dynamical oscillations, non-axisymmetric oscillations play an important role in rotating stars. They cause secular gravitational radiation-reaction instability, i.e., CFS-instability\(^1\)-\(^3\). The instability was so far studied in the polar f-modes. See e.g., Ref. \(^4\). Recently, the same mechanism was examined in the axial r-modes\(^5\),\(^6\). The estimates based on the Newtonian calculations suggest that the instability sets in even for smaller angular velocity. The further refinement of the axial modes is therefore crucial. The r-modes have been examined from various aspects\(^7\)-\(^12\). The whole of the works can not be reviewed here. See the some papers and references therein for the details.

In this paper, we consider the theoretical aspect of the stellar pulsations within general relativity. Astrophysical applications of the CFS instability may be given elsewhere\(^\ast\). In §2, the mode classification of the stellar pulsations is given from the viewpoint of the physical forces and mathematical displacements. This is useful for the following discussion. In §3, the rotational effects on the polar modes are reviewed. In §4, a method is given to solve the axial modes in slowly rotating stars. In §5, the method is applied to the axial oscillations without metric perturbations. The approximation is known as the Cowling approximation\(^13\), and gives good estimates for some pulsation modes. In §6, the method is also applied to axial oscillations with metric perturbations. Finally, §7 devotes to the summary. We use the geometrical units of \(c = G = 1\), in this paper.

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\(^\ast\ast\) For a recent overview see the paper by Friedman in this volume
§2. Mode classification of pulsations

2.1. Physical forces

Pulsations reveal various involved physics in different modes of oscillation patterns and the frequencies. In order to explain the some pulsation modes, we consider linearized equations of motion in the Newtonian gravity:

\[
\partial_t \delta v_i + v^k \nabla_k \delta v_i + \delta v^k \nabla_k v_i = -\nabla_i \left( \frac{\delta p}{\rho} \right) - \frac{\Gamma p}{p^2} \left( A_i \delta \rho + \xi^k A_k \nabla_i \rho \right) - \nabla_i \delta \phi, \quad (2.1)
\]

where \( \Gamma \) is adiabatic index, and \( A_i \) is the Schwarzchild discriminant defined by

\[
A_i = \frac{\nabla_i \rho}{\rho} - \frac{\nabla_i p}{\Gamma p}. \quad (2.2)
\]

The first term in the right hand side of eq.(2.1) represents the pressure force. This driving force causes the p-mode. The frequency \( \sigma_p \) of the p-mode is typically given by the inverse of the free fall time,

\[
\sigma_p \sim \left( \frac{M}{R^3} \right)^{1/2}, \quad (2.3)
\]

where \( M \) and \( R \) are the mass and the radius of the star. The frequency increases with the number of nodes. The f-mode can be regarded as the lowest node-less one. The second term in the right hand side of eq.(2.1) is related with the buoyant force. The mode associated with the buoyancy is called g-mode. The typical frequency \( \sigma_g \) of the g-mode is characterized by the Brunt-Väisälä frequency \( \omega_{BV} \),

\[
\sigma_g \sim \omega_{BV} = (-A \cdot g)^{1/2}, \quad (2.4)
\]

where \( g \) is the local gravitational acceleration. The value \( A \) significantly depends on the position. The convectively unstable region corresponds to \( A \cdot g > 0 \) and stratified region corresponds to \( A \cdot g < 0 \). For isentropic stellar model, we have \( A \cdot g = 0 \) and hence \( \sigma_g = 0 \). Since the thermal force is much weaker than the dynamical pressure in general, the motion driven by the buoyancy is therefore slow, and the frequency of the g-mode is much smaller than that of the p-mode, \( \sigma_g \ll \sigma_p \).

These p-, g-modes exist in rather simple situation. If another physical ingredient is important, what happens? New effect changes the modes in general. For example, stellar rotation changes the frequency as

\[
\sigma = \sigma_0 + \sigma' \Omega + \cdots, \quad (2.5)
\]

where \( \Omega \) is angular velocity, and \( \sigma_0 \) is the frequency of the non-rotating star. In eq.(2.5), we assumed the slow rotation and considered the first-order effect only. In addition to the shift of the frequency, the rotation causes a new driving force, i.e., Coriolis force. The new mode associated with the force is called r-mode. The frequency is given by

\[
\sigma_r \sim \Omega. \quad (2.6)
\]
In the limit of no rotation, the frequency is zero. In this way, a new physical effect causes a new mode associated with it. Another good example of the existence of the new mode is w-mode\cite{14-17}, which is related with the gravitational wave in general relativity.

### 2.2. Oscillation pattern

From the mathematical point of view, the oscillation modes can be divided into two classes by parity. The displacements on a sphere can be expressed by the 'gradient' or 'rotation' of the spherical harmonics $Y_{lm}$. The polar (or poloidal) mode in the spherical star can be expressed by

$$
\xi_r = R(t, r)Y_{lm}(\theta, \phi), \quad \vec{\xi} = V(t, r)\vec{\nabla}Y_{lm}(\theta, \phi).
$$

(2.7)

The density and pressure perturbations are coupled in this mode. Another type of the displacements can be expressed by

$$
\xi_r = 0, \quad \vec{\xi} = U(t, r)(\hat{r} \times \vec{\nabla}Y_{lm}(\theta, \phi)).
$$

(2.8)

This class of displacements is called as the axial (or toroidal) mode. The density and pressure perturbations are zero on the spherical stars. Rotation however slightly induces them.

These types of the displacements cause different types of the gravitational radiation. The change of the mass moment plays an important role in the polar modes, whereas that of the mass current plays a role in the axial modes. Both modes are subject to the CFS instability due to the radiation reaction. The polar f-modes work only in the rotating stars with nearly break-up speed. On the other hand, recently discovered r-modes work even in slow rotation. The axial mode therefore deserves to be examined more extensively.

### §3. Rotational effects on the polar modes

In this section, the rotational effects on some p-modes are demonstrated. The formulation and the numerical results for normal frequencies of the slowly rotating stars are given in Refs.\cite{18-21}. The non-radial oscillations with index $l \geq 2$ are described by the coupled wave-equations inside the star. They are schematically written as

$$
(-\partial_t^2 + \partial_r^2)X + F(\partial_r X, \partial_r Y, X, Y) = 0,
$$

(3.1)

$$
(-c_s^2 \partial_t^2 + \partial_r^2)Y + G(\partial_r X, \partial_r Y, X, Y) = 0,
$$

(3.2)

where $c_s$ is the sound velocity. These two equations show that the perturbations propagate with the light velocity and sound velocity, respectively. The set of equations is solved as an eigen-value problem with $\exp\{-\imath(\sigma t - m\phi)\}$. We impose the out-going wave condition at infinity. The resultant eigen-value is a complex number. The imaginary part $\sigma_I$ represents the decay of the oscillations due to the gravitational radiation, if $\sigma_I > 0$. For the slowly rotating star, the frequency can be expanded with the rotational parameter $\varepsilon = \Omega \sqrt{R^3/M}$. As a result, the frequency is modified as

$$
\sigma = \sigma_R(1 + ma_R \varepsilon) - \imath \sigma_I(1 + ma_I \varepsilon).
$$

(3.3)
Note that the axisymmetric mode \((m = 0)\) is affected only from the second-order of the rotation. That is, the effect of the centrifugal force etc.

Some frequencies of \(l = 2\) mode are tabulated in Table I. The polytropic stellar model is adopted. The direct numerical results show that the corrections \(\sigma'_R\) and \(\sigma'_I\) are positive. This result means the counter-rotating mode \(m < 0\) beyond the critical velocity changes the sign in the pattern speed, i.e., the real part of the frequency and in the decay rate, i.e., the real part of it. This fact is the condition of the radiation reaction instability. In this way, the counter-rotating mode becomes unstable for large angular velocity \(\varepsilon\). Note that the f-mode is the most crucial among the p-modes, since the corrections are the largest and the f-mode becomes unstable first.

The frequencies and the rotational corrections are calculated for a wide range of stellar models. The numerical calculations\(^{[19]}\) show that these corrections increase with the relativistic factor \(M/R\). That is, fully relativistic calculation suggests that the critical angular velocity decreases. This means that the instability sets in even for smaller angular velocity, as the system becomes more relativistic.

| mode | \(\sigma_R\sqrt{R^2/M}\) | \(\sigma'_R\) | \(\sigma_I R^4/M^4\) | \(\sigma'_I\) |
|------|----------------|---------|----------------|---------|
| f    | 1.17           | 0.57    | 0.032          | 3.10    |
| p_1  | 2.70           | 0.32    | 0.006          | 1.78    |
| p_2  | 4.12           | 0.12    | 0.001          | 0.90    |

**§4. Perturbation scheme**

The axial oscillation is trivial in the non-rotating stars, since there is no restoring force in the fluid stars. The oscillation becomes possible in the rotating stars. We therefore have to consider the oscillation on the rotating stars, which will be a difficult task. We will consider the slow rotation approximation in the pulsation equations. That is, pulsation equations are expanded with respect to the rotation parameter. The method can be applied to the polar modes successfully. In the spherically symmetric case, the perturbations can be decoupled into the axial and polar perturbations with spherical harmonic index \((l, m)\). They are respectively described by the axial functions \(A_{lm} \equiv (U_{lm}, h_{0,lm}, h_{1,lm})\), and the polar functions \(P_{lm} \equiv (\delta p_{lm}, \delta \rho_{lm}, R_{lm}, V_{lm}, H_{0,lm}, H_{1,lm}, H_{2,lm}, K_{lm})\). In the presence of rotation, the perturbations are described by the mixed state of them. If the perturbation equations are expanded by the rotation parameter \(\varepsilon\), then the formal relation between the axial-led \(A_{lm}\) and the polar-led \(P_{lm}\) can be expressed as

\[
0 = [A_{lm}] + \mathcal{E} \times [P_{l \pm 1m}] + \mathcal{E}^2 \times [A_{lm}, A_{l \pm 2m}] + \cdots, \quad (4.1)
\]

\[
0 = [P_{lm}] + \mathcal{E} \times [A_{l \pm 1m}] + \mathcal{E}^2 \times [P_{lm}, P_{l \pm 2m}] + \cdots, \quad (4.2)
\]

where the symbol \(\mathcal{E}\) denotes some functions of order \(\varepsilon\), and the square bracket formally represents the relation among perturbation functions therein. We assume that the axial-led and polar-led functions are expanded as

\[
A_{lm} = A_{lm}^{(1)} + \varepsilon^2 A_{lm}^{(2)} + \cdots, \quad P_{lm} = \varepsilon(P_{lm}^{(1)} + \varepsilon^2 P_{lm}^{(2)} + \cdots). \quad (4.3)
\]
Substituting these functions into eqs. (4.1)-(4.2), and comparing each order of $\varepsilon$, we have the following equations of $\varepsilon^n (n = 0, 1, 2)$,

\begin{align*}
0 &= [A_{lm}^{(1)}], \\
0 &= [\varepsilon \mathcal{P}_{l \pm 1m}^{(1)} + \mathcal{E} \times A_{lm}^{(1)}], \\
0 &= [\varepsilon^2 A_{lm}^{(2)}] + \mathcal{E} \times [\varepsilon \mathcal{P}_{l \pm 1m}^{(1)}] + \mathcal{E}^2 \times [A_{lm}^{(1)} A_{l \pm 2m}^{(1)}] \\
&= [\varepsilon^2 A_{lm}^{(2)} + \mathcal{E}^2 \times A_{lm}^{(1)}].
\end{align*}

We have here assumed that the perturbation is described by a single spherical harmonic in the lowest order, that is, $A_{l'm}^{(1)} = 0$, for $l' \neq l$, and used eq. (4.5) in eq. (4.6). Equation (4.4) represents the axial oscillation at the lowest order. Equation (4.7) is the second-order form of it, and the term $\mathcal{E}^2 \times A_{lm}^{(1)}$ can be regarded as the rotational corrections. The method to solve the equations is straightforward. The first-order equations are solved by the axial-led functions. The polar-led functions are expressed using them. We have the second-order equations with the corrections expressed by the axial-led functions at the lowest-order. These equations are successively solved in the following sections. In the actual calculations, we also assume that the time variation of the oscillation is slow and proportional to $\Omega$, i.e., $\partial_t \sim \Omega \sim O(\varepsilon)$. This is true in the r-mode oscillation, as will be confirmed soon.

§5. Axial oscillations in the Cowling approximation

5.1. First-order solution

We will apply the perturbation scheme described in the previous section to the problem of axial oscillations. In this section, we neglect the gravitational perturbation, that is, we consider the Cowling approximation. The details of the calculations are explained in Ref. [1]. The leading order equation of $\delta T^\mu_{\nu; \mu} = 0$ is reduced to

\begin{equation}
(\partial_T - im\chi)U_{lm}^{(1)} = 0,
\end{equation}

where

\begin{equation}
\chi = \frac{2}{l(l+1)}\omega = \frac{2}{l(l+1)}(\Omega - \omega),
\end{equation}

and $\partial_T$ denotes time derivative in the co-rotating frame, i.e., $\partial_T U_{lm} = (\partial_t + im\Omega)U_{lm}$. It is easy to observe that the motion is trivial for the non-rotating case, i.e., $\partial_t U_{lm} = 0$.

If one solves eq. (5.1) by the eigen-value problem, then the solution is expressed by a delta function. Instead, we will examine the evolution of the perturbation by the initial value problem. The solution of eq. (5.1) is

\begin{equation}
U_{lm}^{(1)}(t, r) = \int f_{lm}^{(1)}(r) \frac{e^{st}}{s + im(\Omega - \chi)} ds = f_{lm}^{(1)}(r)e^{-im(\Omega - \chi)t}H(t),
\end{equation}

where $H(t)$ is the Heaviside step function. We will consider $t > 0$ region only, so that the function $H(t)$ may well be omitted from now on. The function $f_{lm}^{(1)}$ describes the initial disturbance at $t = 0$. 

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The implications of the result (5.3) are as follows. In the Newtonian stars, the sinusoidal time-dependence can be described by a single frequency,

\[ m(\Omega - \chi) \rightarrow \sigma_N = \left( 1 - \frac{2}{l(l+1)} \right) m\Omega. \] (5.4)

This is the r-mode frequency measured in the non-rotating frame. In the relativistic stars, \( \varpi \) is monotonically increasing function of \( r \), \( \varpi_0 \leq \varpi \leq \varpi_R \). The possible frequency range is then spread out,

\[ \left( 1 - \frac{2}{l(l+1)} \frac{\varpi_R}{\Omega} \right) m\Omega \leq \sigma \leq \left( 1 - \frac{2}{l(l+1)} \frac{\varpi_0}{\Omega} \right) m\Omega. \] (5.5)

In both cases, Newtonian and relativistic stars, the radial dependence \( f_{lm}^{(1)} \) is arbitrary at this order. The function \( f_{lm}^{(1)} \) in eq.(5.3) is constrained by the equation of motions for the polar part, as will be shown in the subsequent subsections. In this meaning, the scheme (4.4)-(4.7) is degenerate perturbation scheme.

5.2. Second-order equations

According to (4.5), the polar-led functions \( (\delta \rho_{l\pm 1m}, \delta \rho_{l\pm 1m}, R_{l\pm 1m}, V_{l\pm 1m}) \) are expressed by \( U_{lm}(1) \). These corrections affect the axial parts with indices \((l \pm 2,m)\) and \((l,m)\). We consider the equation with \((l,m)\) only. Corresponding to eq.(4.7), the axial equation with the corrections up to \( O(\varepsilon^3) \) is

\[ 0 = (\partial_T - im\chi)U_{lm}^{(2)} + \mathcal{L}[\partial_T U_{lm}^{(1)}], \] (5.6)

where \( \mathcal{L} \) is the Sturm-Liouville differential operator defined by

\[ \mathcal{L}[\partial_T U_{lm}^{(1)}] = 8c_3 \varpi e^{-\lambda/2-\nu}(\rho_0 + p_0) \left[ \frac{r^2 e^{\lambda-\nu}/2}{A\nu'(\rho_0 + p_0)(e^{\nu/2}\varpi\partial_T U_{lm}^{(1)})'} \right] - (F + G)\partial_T U_{lm}^{(1)}, \] (5.7)

\[ F = -4c_3 \varpi^2 e^{-\lambda/2-\nu} \left( \frac{r^2 e^{\lambda/2} \rho_0'}{A\nu'} \right)' \]

\[-4c_2 \varpi^2 e^{-\lambda/2-\nu} \left( \frac{e^{(\lambda+3\nu)/2}}{A\nu\varpi} (e^{\nu/2}\varpi)'(\rho_0 + p_0) \right) \]

\[ + \left( \frac{2c_1 e^\nu}{A\nu r^2} \right) [r^2 e^{-\nu}]^2, \] (5.8)

\[ G = -8c_3 \varpi^2 e^{-\lambda/2-\nu} \left( \frac{r^2 e^{\lambda/2}}{\nu'} \right)' + \frac{4c_2}{\nu r^2} (e^{\nu/2}r^4 e^{-\nu})' \]

\[-3c_1 \left[ r e^{-\lambda/2} \left( \frac{e^{\nu/2} \xi_2}{r} \right)' - \frac{3 \varpi'}{2 \varpi} \xi_2 - k_2 + \frac{e^\lambda}{r} m_2 + \frac{5W_3}{\varpi} \right] \]

\[-3m^2 \left[ \frac{\xi_2}{r} + \frac{1}{2 \varpi} \frac{\xi_2'}{\varpi} + k_2 - \frac{5W_3}{\varpi} \right] - \left( \frac{W_1}{\varpi} + \frac{6W_3}{\varpi} \right). \] (5.9)
5.3. **Solution of the radial function - mode specification -**

In order to solve eq. (5.6), we introduce a complete set of functions $y_\kappa(r)$ with eigenvalue $-\kappa$ with respect to the Sturm-Liouville operator $\mathcal{L}$,

$$\mathcal{L}[y_\kappa] + \kappa y_\kappa = 0. \quad (5.10)$$

The second-order equation (5.6) can be integrated with $y_\kappa = im\chi f_{lm;\kappa}^{(1)}(r)$ and unknown function $f_{lm}^{(2)}(r)$ as

$$U_{lm}^{(2)} = (im\kappa \chi t f_{lm;\kappa}^{(1)} + f_{lm}^{(2)}) e^{-im(\Omega - \chi)t}. \quad (5.11)$$

The sum of the first and second order forms is approximated as

$$U_{lm}^{(1)} + U_{lm}^{(2)} = \left[ (1 + im\kappa \chi t) f_{lm;\kappa}^{(1)} + f_{lm}^{(2)} \right] e^{-im(\Omega - \chi)t} \quad (5.12)$$

$$= \left[ f_{lm;\kappa}^{(1)} + f_{lm}^{(2)} \right] e^{-im(\Omega - (1+\kappa)\chi)t}, \quad (5.13)$$

where we have exploited the freedom of $f_{lm}^{(2)}$ to eliminate the unphysical growing term in eq.(5.12). The value $\kappa$ originated from fixing of the initial data becomes evident for large $t$, since the accumulation of small effects from the higher order terms is no longer neglected. As a result, the frequency should be adjusted with the second-order correction to be a good approximation even for slightly large $t$, as eq.(5.13). This remedy is known as the renormalization of the frequency, or strained coordinate for $t$ in the perturbation method.

We here summarize the calculations up to the second-order by the perturbation scheme (4.4)-(4.7). In the lowest-order calculation, the first-order spatial function is not determined, whereas the time-dependence is fixed. By considering the next order, the first-order spatial function $f_{lm;\kappa}^{(1)}$ is specified with the second-order correction of the frequency $\kappa$. The range and the nature of the correction $\kappa$ can not be explored without explicitly solving the eigen-value problem (5.10), which significantly depends on the equilibrium state, in particular, $A = 0$ or $A \neq 0$. For the barotropic case ($A = 0$), eq.(5.6) is modified.

§6. **Axial oscillations including gravitational perturbations**

6.1. **Lowest-order calculation**

In this section, we incorporate the metric perturbations. The method to solve the axial oscillations is the same as in the Cowling approximation. We have additionally six metric functions $(h_{0 \, lm}, h_{1 \, lm}, H_{0 \, lm}, H_{1 \, lm}, H_{2 \, lm}, K_{lm})$. In the lowest-order, the calculation is rather simple, since only two components, $h_{0 \, lm}$ and $h_{1 \, lm}$ are relevant. We define a function $\Phi_{lm}$ as

$$\Phi_{lm} = \frac{h_{0 \, lm}}{r^2}. \quad (6.1)$$
The relation between the metric functions is given by

\[ h_{1 \text{lm}} = \frac{r^4 e^{-\nu}}{(l-1)(l+2)} \left[ (\partial_T - im \varpi) \Phi'_{\text{lm}} - \frac{2im\omega'}{l(l+1)} \Phi_{\text{lm}} \right]. \quad (6.2) \]

The axial velocity function is expressed by two ways:

\[
(\partial_T - im\chi) U_{\text{lm}} = -4\pi (\rho + p) r^2 e^{-\nu} \partial_T \Phi_{\text{lm}}, \quad (6.3)
\]

\[
U_{\text{lm}} = \frac{r^2 j^2}{4} \left[ \frac{1}{jr^4} (j r^4 \Phi'_{\text{lm}})' - (v + 16\pi (\rho + p) e^\lambda) \Phi_{\text{lm}} \right], \quad (6.4)
\]

where

\[
v = \frac{e^\lambda}{r^2} \left[ l(l+1) - 2 \right], \quad (6.5)
\]

\[
j = e^{-(\lambda + \nu)/2}. \quad (6.6)
\]

Eliminating \( U_{\text{lm}} \) in eqs.(6.3)-(6.4), we have the master equation as

\[
(\partial_T - im\chi) \left[ \frac{1}{jr^4} (j r^4 \Phi'_{\text{lm}})' - v \Phi_{\text{lm}} \right] = -16\pi im\chi (\rho + p) e^\lambda \Phi_{\text{lm}}. \quad (6.7)
\]

This equation is reduced to an eigen-value equation\(^{\text{b)}}, by assuming \( \exp(-i\sigma t) :\)

\[
(\varpi - \mu) \left[ \frac{1}{jr^4} (j r^4 \Phi'_{\text{lm}})' - v \Phi_{\text{lm}} \right] = q \Phi_{\text{lm}}, \quad (6.8)
\]

where

\[
\mu = -\frac{l(l+1)}{2m} (\sigma - m\Omega), \quad (6.9)
\]

\[
q = \frac{1}{jr^4} \left( j r^4 \varpi' \right)' = 16\pi (\rho + p) e^\lambda \varpi \geq 0. \quad (6.10)
\]

Equation (6.8) is called singular eigen-value equation, since it has a singular point \( r_0 \) unless \( q(r_0) = 0 \), corresponding to the real value of \( \mu = \varpi(r_0) \). This kind of singular eigen-value equation is studied for the incompressible shear flow. See the Appendix for the Rayleigh equation in two dimensional parallel flows. Some important conclusion can be derived from the behavior of the background flow, i.e., the function \( \varpi \) in this problem. For example, the necessary condition of the instability is that the function \( \varpi \) has inflection point, i.e., \( q = 0 \). This condition is never satisfied inside the star. As a result, the frequency is real and the flow is stable in this order.

6.2. From the Rayleigh to Orr-Sommerfeld equations

In this section, the second-order corrections will be included. The calculation is straightforward, but very complicated in actual. We only consider the term with the highest rank of derivative with respect to \( r \). The rank of the derivative is important

\(^{\text{b)}\) There is a misprint in the previous paper\(^{\text{b)}}.\)
factor to determine the type of equations. The term is origin
ated from the second-order derivative of $U_{lm}$ as eq.(5.6). Since $U_{lm}$ is expressed by the second-order derivative of $\Phi_{lm}$, we have the fourth-order derivative of $\Phi_{lm}$. The term with the highest rank of derivative is explicitly given by

$$D_0[\Phi_{lm}^{(1)}] = 8c_3 \frac{\omega e^{-\nu}}{jr^2} \left\{ \frac{(\rho_0 + p_0)r^2}{jAv'} \left[ \frac{jr^4}{(\rho_0 + p_0)r^2} \left( jr^4 \partial_T \Phi_{lm}^{(1)} \right) \right]' \right\}'. \quad (6.11)$$

The second-order equation with this correction can be written as

$$(\partial_T - im\chi) \left[ \frac{1}{jr^2} \left( jr^4 \Phi_{lm}^{(2)} \right) ' - v\Phi_{lm}^{(2)} \right] = -16\pi im\chi(\rho + p)e^{\lambda} \Phi_{lm}^{(2)} + D_0[\Phi_{lm}^{(1)}]. \quad (6.12)$$

The term (6.11) effectively gives the ‘viscosity’ like the Orr-Sommerfeld equation in the incompressible shear flow. (See Appendix.) The viscosity is important for the stability of the flows. For the small Reynolds number, the laminar flow is realized, whereas the flow becomes turbulence above a critical Reynolds number. The effective Reynolds number $R_e$ in eq.(6.12) is estimated from dimensional argument as

$$R_e \sim \frac{Av'}{\omega^2} \sim \frac{\omega^2 V^2}{\omega^2}. \quad (6.13)$$

The viscosity term will play a key role on the singular point of the first-order equation, but the consequence is not clear at moment. It is necessary to explore further how the effective Reynolds number should operate in the stability and so on.

§7. Summary - pulsations and gravitational waves -

The polar modes, which exist in the spherical non-rotating stars, are extensively studied so far. The angular dependence of the modes is specified by a single spherical harmonic index. The decoupled radial equation can be calculated as an eigen-value problem. The rotational effects are also examined within the first-order, and the corrections are calculated as $\sigma = \sigma_0 + m\varepsilon_1 \sigma_1 + \cdots$. The relation between the pulsation and the gravitational radiation is evident in the modes, that is, the gravitational emission gives the imaginary part of the frequency. The gravitational waves would therefore give a good insight into the stellar interiors, when observed.

Axial modes never exist in the non-rotating fluid stars, but exist in rotating stars. The oscillations are calculated for the slow-rotation approximation. When the mode is calculated by the eigen-value problem, large number of spherical harmonics are required in general[12]. In this paper, we considered the mode whose angular dependence is dominated for a single spherical harmonic. As a result, the solution is constructed by an initial-value problem. The temporal dependence can be written as an infinite sum of the Fourier mode $\exp(-i\sigma t)$. The rotational corrections are calculated up to the third-order as $\sigma = \varepsilon_1 \sigma_1(r) + \varepsilon_2 \sigma_3(r) + \cdots$. At present, the relation to the gravitational radiation is not clear. From the Newtonian estimates[6], the radiation reaction affects the oscillations in order $\varepsilon^{2m+2}$. The reaction term should be even power of $\varepsilon$, which is time-asymmetric and originated from the radiative boundary condition at infinity. Rather, the equations describing the axial oscillations are clearly related with the vortex as the Rayleigh or Orr-Sommerfeld equations.
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Appendix A

Incompressible shear flow

In this appendix, linear perturbation equations of the two-dimensional flows are summarized for the comparison. The detailed arguments can be written in some books of hydrodynamics. The basic parallel flow of incompressible matter is assumed to \((0, u(x))\). We consider the perturbation within the inviscid theory, and assume the stream function in the form as \(\Phi(x) \exp\{-ik(y - ct)\}\), then we have

\[
(u - c) \left[ \Phi'' - k^2 \Phi \right] = u'' \Phi. \tag{A.1}
\]

This is called Rayleigh’s stability equation. Some theorems are proved for the equation. For example, Rayleigh’s inflexion-point theorem is stated as follows. A necessary condition for instability is that the basic velocity profile should have an inflexion point, i.e., \(u'' = 0\). However this condition is not sufficient. The sufficient condition is not yet found.

Viscosity cannot be neglected in some flows. If we consider the problem within the viscous theory, the perturbation equation of the two-dimensional flows is reduced to the Orr-Sommerfeld equation. Using appropriate normalization, we can express it as

\[
(\partial_t + u \partial_y) \left( \partial_x^2 + \partial_y^2 \right) \Phi = u'' \partial_y \Phi + R_e^{-1} (\partial_x^2 + \partial_y^2) \partial_y \Phi. \tag{A.2}
\]

In eq.(A.2), \(R_e\) is the Reynolds number defined by \(R_e = u_0 \ell / \nu\), where \(\nu\) is kinematical viscosity, and \(u_0\) and \(\ell\) are typical scales of the velocity and length. Using the Fourier mode, we have

\[
(u - c) \left[ \Phi'' - k^2 \Phi \right] = u'' \Phi - \frac{1}{ikR_e} \left[ \frac{d^2}{dx^2} - k^2 \right]^2 \Phi. \tag{A.3}
\]

Formally, the Rayleigh equation (A.1) can be regarded as the leading term in the expansion of \((kR_e)^{-1}\). The Rayleigh equation is a good approximation, if the second and fourth order derivatives can be neglected. When the condition is no longer valid in some regions, we have to take account of the viscosity term there. The stability of the viscous flow significantly depends on the Reynolds number. For the flow with low Reynolds number, the flow is laminar, whereas it becomes turbulent for the high Reynolds number. More detailed arguments with boundary conditions are indispensable to find out the critical number, flow pattern and so on.

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