Bipartite Multigraphs with Expander-Like Properties

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1 Introduction

A graph with vertex set $V$ and edge set $E$ is called a $(d, c)$-expander if the maximum degree of a vertex is $d$ and, for every set $W \subset V$ of cardinality at most $|V|/2$, the inequality $|\{w \in V \setminus W : \{v, w\} \in E\}| \geq c|V|$ holds. This note considers a related combinatorial question:

For which integers $d$ and functions $f_d$ does there exist, for every large enough $v$, a bipartite $d$-regular multigraph on $2v$ nodes with node sets $V$ and $W$ having the following property: For every $U \subseteq V$ and every $U \subseteq W$, the cardinality of the set of neighbours of $U$ is at least $f_d(|U|)$?

Graphs with the above property seem to behave well also with respect to other, more complicated, expansion-type properties. Indeed, the author was motivated to study this question by a paper communicated to him in May 2002 (the latest version of the paper is available from URL http://www-math.mit.edu/~vempala/papers/tspinapprox.ps). In this paper, Papadimitriou and Vempala established approximation hardness of TSP with triangle inequality using as a tool in their construction the fact that for $d = 6$ and

$$f_6(u) = \begin{cases} 2u & 0 \leq u \leq v/4, \\ u + v/4 & v/4 \leq u \leq v/2, \\ u/2 + v/2 & v/2 \leq u \leq v, \end{cases}$$

there exist bipartite multigraphs with the properties described in the above question. In this paper, we prove the following theorem:

**Theorem 1.** For $d \in \{5, 6, 7, 8\}$ and functions $f_d$ as described below, there exists, for every large enough $v$, a bipartite $d$-regular multigraph on $2v$ nodes
with node sets $V$ and $W$ having the property that for every $U \subseteq V$ and every $U \subseteq W$ the cardinality of the set of neighbours of $U$ is at least $f_d(|U|)$.

$$f_5(u) = \begin{cases} 
2u & 0 \leq u \leq 3v/20, \\
4u/3 + v/10 & 3v/20 \leq u \leq 3v/10, \\
u + v/5 & 3v/10 \leq u \leq v/2, \\
3u/4 + 13v/40 & v/2 \leq u \leq 7v/10, \\
u/2 + v/2 & 7v/10 \leq u \leq v,
\end{cases} \quad (1)$$

$$f_6(u) = \begin{cases} 
5u/2 & 0 \leq u \leq v/10, \\
5u/3 + v/12 & v/10 \leq u \leq v/4, \\
u + v/4 & v/4 \leq u \leq v/2, \\
3u/5 + 9v/20 & v/2 \leq u \leq 3v/4, \\
2u/5 + 3v/5 & 3v/4 \leq u \leq v,
\end{cases} \quad (2)$$

$$f_7(u) = \begin{cases} 
3u & 0 \leq u \leq v/10, \\
2u + v/10 & v/10 \leq u \leq 3v/20, \\
21u/15 + 19v/100 & 3v/20 \leq u \leq 3v/10, \\
u + 31v/100 & 30v/100 \leq u \leq 39v/100, \\
15u/21 + 59v/140 & 39v/100 \leq u \leq 3v/5, \\
u/2 + 11v/20 & 3v/5 \leq u \leq 7v/10, \\
u/3 + 2v/3 & 7v/10 \leq u \leq v.
\end{cases} \quad (3)$$

$$f_8(u) = \begin{cases} 
3u & 0 \leq u \leq v/10, \\
2u + v/10 & v/10 \leq u \leq v/5, \\
5u/4 + v/4 & v/5 \leq u \leq v/3, \\
4u/5 + 2v/5 & v/3 \leq u \leq v/2, \\
u/2 + 11v/20 & v/2 \leq u \leq 7v/10, \\
u/3 + 2v/3 & 7v/10 \leq u \leq v.
\end{cases} \quad (4)$$

The proof of the theorem relies partly on claims that have been verified by a computer. Hence, a second contribution in this paper is the description of a fairly general methodology for devising computer-assisted proofs for a wide class of mathematical claims.

### 1.1 The probabilistic method

The probabilistic method [1] is particularly well-suited for proving that there exists, in some large class of objects, an object with certain specified properties. Typically, the argument proceeds by first selecting a random object from the class and then estimating the probability that the selected object does not have the sought properties. If this probability can be shown to
be strictly less than one, the probability that the selected object has the property is strictly positive and hence there exists an object with the sought properties.

The method is highly non-constructive. For our case, we in fact have no idea whatsoever how to actually construct a graph with the sought properties in time polynomial in the number of vertices in the graph.

### 1.2 Computer-assisted proofs

As mentioned in the previous section, a critical component in the probabilistic method is to show that the probability of some event is strictly less than one. In our case, the function expressing this probability is fairly complicated, although continuously differentiable almost everywhere. In principle, straightforward but tedious analysis of first and second order derivatives of the function could be used to prove that it is strictly less than one. We feel, however, that the contribution to the community from such a proof is very minor. Instead we resort to a computer-assisted proof and argue that this method of proof should be accepted in cases like ours.

First, what is a computer assisted proof of some statement “\( f(x) < c \) for all \( x \in [a,b] \)”? Simply evaluating the function at some points is clearly not enough—the function may assume other, dangerous, values at the points where it was not evaluated. Given some bound, proven by a conventional mathematical proof, of the form “\( |f'(x)| < C \) for all \( x \in [a,b] \)”, we could argue that it is enough to evaluate the function at points that are sufficiently close since a Taylor expansion then bounds the value of the function at all points. However, floating point computations done by computers are not accurate, and the latter argument above fails to take into account possible influences of round-off errors.

A solution to the problem of round-off errors is to use interval arithmetic [2]. The main idea behind computations with interval arithmetic is to compute not with single numbers but rather with intervals. When some function \( f \) is applied to some interval \( I \), the result is an interval that contains \( f(x) \) for every \( x \in I \). Hence, interval arithmetic is particularly well suited for verifying claims that are of the form “\( f(x) < c \) for \( x \in I \)”. If the interval \( I \) is large, then the result of computing \( f(I) \) is usually also a large interval; in particular, \( I \) could also contain \( c \). The solution to this problem is to split \( I \) into sub-intervals that are sufficiently small and then compute \( f(I_k) \) for each sub-interval \( I_k \). The computer-assisted part of our proofs works precisely in this way and was inspired by Uri Zwick’s work on optimal approximation algorithms for certain constraint satisfaction problems [2].

We argue that computer-assisted proofs should be accepted in cases like ours. To examine and ascertain the correctness of the program presented in this paper is the same thing as examining a conventional mathematical proof. It is, of course, true that the correctness of the computer-assisted
verification of the claims that the program verifies, requires the assumption that the program is correctly compiled and that all used library routines are correctly written. It is in principle possible to construct a compiler that automatically includes logging facilities in the executable program; the output from such a system could then in principle be verified step by step to ascertain the correctness of the computation.

2 Proof of Theorem

We select a $d$-regular bipartite multigraph on $2v$ vertices by selecting one perfect matching in a bipartite graph on $dv + dv$ vertices uniformly at random. From this perfect matching, the $d$-regular bipartite graph is constructed by identifying groups of $d$ vertices in the “big” bipartite graph with single vertices in the sought $d$-regular bipartite graph.

The analysis of the construction proceeds by estimating the probability that such a randomly chosen graph does not have the desired properties, i.e., the probability that there is some $U \subseteq V$ or some $U \subseteq W$ such that the cardinality of the set of neighbours of $U$ is no more than $f_d(|U|)$.

Fix a set $U \subseteq V$ of size $u$ and a set $N \subseteq W$ of size $n$. When $u > n$, the pigeon-hole principle implies that $U$ cannot have neighbours only in $N$. When $u \leq n$, the probability that $U$ has neighbours only in $N$ is

$$P(u, n) = \binom{v}{u} \binom{v}{n} \binom{dn}{du} / \binom{dv}{du}$$

where the factor $2$ above comes from the fact that we consider not only neighbour sets of $U \subseteq V$ but also neighbour sets of $U \subseteq W$. Let

$$P(u, n) = \binom{v}{u} \binom{v}{n} \binom{dn}{du} / \binom{dv}{du} = \binom{v}{u} \binom{v}{n} \frac{(dn)!(dv - du)!}{(dn - du)!(dv)!}.$$  (5)
Using $P$ and the $\Omega$’s, the probability that there is some $U \subseteq V$ or some $U \subseteq W$ such that the cardinality of the set of neighbours of $U$ is no more than $f_d(|U|)$ can be upper bounded by $2|\Omega\delta| \max_{(u,n)\in\Omega_d} P(u,n) + 2|\Omega\delta| \max_{(u,n)\in\Omega} P(u,n)$.

As can be seen from Figure 1, $2|\Omega\delta| \leq 2\delta f_d(\delta v)$ and $2|\Omega| \leq v^2$. The proof now proceeds by setting $\delta = 10^{-5}$ and then proving that

$$2\delta f_d(\delta v) \max_{(u,n)\in\Omega_d} P(u,n) < 1/2, \quad (6)$$

$$v^2 \max_{(u,n)\in\Omega} P(u,n) < 1/2. \quad (7)$$

This is enough to complete the proof, since in that case the probability that a randomly selected graph has the desired properties is non-zero and hence there exists at least one graph with the desired properties.

### 2.1 Analysis close to extreme points

Since both $P$ and our functions $f_d$ for $d \in \{5, 6, 7, 8\}$ are symmetric with respect to reflection around the line $u + n = v$, it is enough to consider pairs $(u,n) \in \Omega\delta$ such that $u + n \leq v$. For fixed $u \leq \delta v$,

$$\frac{P(u, n + 1)}{P(u, n)} = \frac{(v - n)}{(n + 1)} \prod_{i=1}^{d} \frac{dn + i}{dn - du + i} > 1,$$

therefore $P(u, n)$ is increasing in $n$. Hence, it suffices to bound $P(u, f_d(u))$.

The following lemma establishes a slightly more general result.

**Lemma 1.** For every integer $u$ such that $1 \leq u \leq 10^{-5} v$ and every $(k,d) \in \{(2,5), (3,6), (3,7), (3,8)\}$, $2k\delta v^2 P(u, ku) < 1/2$ where $P$ is defined by (5).

**Proof.** The proof is by induction on $u$. The base case is clear since

$$2k\delta v^2 P(1, k) = 2k(\delta v)^2 \binom{v}{k} \frac{(kd)!}{(dv)! (kd-d)!} < 2\delta^2 \frac{k^{d+1}}{k!} \cdot \frac{v^{3+k}}{(v-1)^d}.$$

For $d > 3 + k$ and $v$ large enough, the latter expression above is strictly less than $1/2$. For the cases when $d = 3 + k$, the latter expression is less than

$$10^{-10} \cdot 3^7 \cdot \frac{v^d}{(v-1)^d} < 1/2.$$

For the inductive step, we show that $P(u, ku)/P(u + 1, ku + k) > 1$. Since

$$\frac{P(u, ku)}{P(u + 1, ku + k)} = \frac{\binom{v}{u} \binom{v}{ku} \binom{kd}{du} / \binom{dv}{du}}{\binom{v}{u+1} \binom{v}{ku+k} \binom{kd+kd}{du+d} / \binom{dv}{du+d}}$$

...
Figure 1. The level curve $Q(u, n) = 1$ and the function $f_d$ plotted for $d = 8$ with solid lines. The lower picture shows half of the set $\Omega_\delta$. 
we need estimates of the following form:

\[
\frac{v}{u} \left/ \frac{v}{u+1} \right. = \frac{u+1}{v-u} > \frac{u+1}{v},
\]

\[
\frac{v}{ku} \left/ \frac{v}{ku+k} \right. = \frac{\prod_{i=1}^{k} (ku+i)}{\prod_{i=0}^{k-1} (v-ku-i)} > \frac{(ku+1)^k}{(v-ku)^k} > \frac{(u+1)^k}{v^k},
\]

\[
\frac{dv}{du+d} \left/ \frac{dv}{du} \right. > \frac{(dv-du-d)^d}{(du+d)^d} > \frac{(v-2u)^d}{(u+1)^d};
\]

\[
\frac{kd\alpha}{d\alpha+kd} \left/ \frac{kd\alpha}{d\alpha+kd} \right. > \frac{(du)^d((k-1)du)^{(k-1)d}}{(kd\alpha+kd)^{kd}} = \frac{(k-1)^{(k-1)d}u^{kd}}{k^{kd}(u+1)^{kd}}.
\]

Put together, the above bounds imply that

\[
\frac{P(u, ku)}{P(u+1, ku+k)} > \frac{(v-2u)^d}{v^{k+1}(u+1)^{d-k-1}} \cdot \frac{(k-1)^{(k-1)d}}{kd^{kd}} \cdot \frac{u^{kd}}{(u+1)^{kd}}.
\]

For \( v > \delta^2 \) and \( u \) such that \( 1 \leq u \leq \delta v \), this is greater than

\[
\frac{(1-2\delta)^d}{(\delta+\delta^2)^{d-k-1}} \cdot \frac{(k-1)^{(k-1)d}}{kd^{kd}} \cdot \frac{\delta^{kd}}{(\delta+\delta^2)^{kd}} \quad \frac{\delta^{kd}}{(\delta+\delta^2)^{kd}} \quad \frac{(k-1)^{(k-1)d}}{kd^{kd}}.
\]

For \( (k, d) = (2, 5) \) the above ratio is at least \( 10^{10}2^{-10}(1-25\delta) > 1 \). For \( k = 3 \) the ratio is at least \( 10^{5d-20}2^{2d}3^{-3d}(1-6d\delta) \), which is strictly greater than one for the \( d \) considered.

2.2 The interior region

To bound \( v^2P(u, n) \) in \( \Omega' \), write \( u = \alpha v \) and \( n = \beta v \) and apply Stirling’s formula

\[
\left( \frac{v}{\alpha v} \right) = (\alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)})^v \text{poly}(v)
\]

to \( P(u, n) \) defined in (5):

\[
v^2P(\alpha v, \beta v) = \left( \frac{(1-\alpha)^{(d-1)(1-\alpha)}}{\alpha\alpha(1-\beta)(1-\beta)^{(d-1)^2}} \right)^v \text{poly}(v).
\]

Note that the above expression is valid also for \( u = n \), i.e., also for \( \alpha = \beta \), if we use the convention that \( 0^0 = 1 \). By the symmetry of the function \( P \) and the results obtained in §2.1 it is enough to consider pairs \( (\alpha, \beta) \) in the set

\[
A = \{(\alpha, \beta) : \beta - \alpha \geq 0 \land \beta v \leq f_d(\alpha v) \land \alpha \geq 10^{-5} \land \beta \leq 1 - 10^{-5}\}. \quad (8)
\]
Hence, it is sufficient to prove that there exists a universal constant $c < 1$, strictly bounded away from 1, such that

$$Q(\alpha, \beta) = \frac{(1 - \alpha)^{(d-1)(1-\alpha)} \beta^{(d-1)\beta}}{\alpha^\alpha (1 - \beta)^{(1-\beta)} (\beta - \alpha)^{(d-1)\beta}} < c \quad \text{for } (\alpha, \beta) \in A. \quad (9)$$

To this end, we first show that it is enough to consider the boundaries of $A$, i.e., the points where $\beta v = f_d(\alpha v)$, and then analyze the function $Q$ on those line segments.

### 2.2.1 It is enough to consider the boundary

To achieve the first goal, we prove that $\ln Q$ is convex along lines of the form $\beta - \alpha = y$ for non-negative $y$. This amounts to substituting, e.g., $\alpha = (1 + x - y)/2$ and $\beta = (1 + x + y)/2$ in the expression for $Q$ and then consider the resulting expression as a function of $x$ for arbitrary fixed $y$.

**Lemma 2.** For every fixed $y \in [0, 1 - 2/d]$, the function

$$q(x; y) : x \mapsto \ln Q((1 + x - y)/2, (1 + x + y)/2)$$

is convex in the interval

$$|x| \leq \frac{d(1-y) - 2}{d - 2}.$$

**Proof.** Straightforward substitution shows that $q(x; y) = -dy(\ln 2 + \ln y) - (d - 2)\ln 2 + \frac{1}{2}g(x; y)$ where

$$g(x; y) = (d-1)(1+x+y)\ln(1+x+y) + (d-1)(1-x+y)\ln(1-x+y) - (1+x-y)\ln(1+x-y) - (1-x-y)\ln(1-x-y).$$

Hence the derivative of $q$ with respect to $x$ is

$$g'(x; y) = (d-1)\ln\frac{1+x+y}{1-x+y} - \ln\frac{1+x-y}{1-x-y}$$

and the second derivative is, consequently,

$$g''(x; y) = \frac{(d-1)(2+2y)}{(1+x+y)(1-x+y)} - \frac{2-2y}{(1-x-y)(1+x-y)}.$$

We now rewrite the second derivative as

$$g''(x; y) = 2\frac{(d-2)(1-x^2-y^2) - dy(1+x^2-y^2)}{(1+x+y)(1-x+y)(1-x-y)(1+x-y)}$$

and obtain that $g''(x; y)$ is non-negative when

$$(d-2)(1-x^2-y^2) \geq dy(1+x^2-y^2),$$

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or, equivalently,

\[ x^2(d(1 + y) - 2) \leq (d(1 - y) - 2)(1 - y^2). \]

It is now straightforward to see by substitution that the above inequality is satisfied as soon as \( |x| \leq (d(1 - y) - 2)/(d - 2) \) and \( y \in [0, 1 - 2/d] \). Finally, we remark that the same result is valid also for the case when \( y = 0 \): \( q(x; 0) \) is convex for \( |x| \leq 1 \).

To summarize, the function \( q \) considered in the above lemma is convex along lines parallel to the \( x \)-axis inside a triangle with corners \( (x, y) \in \{(1, 0), (-1, 0), (0, 1 - 2/d)\} \). Translated to \((\alpha, \beta)\)-coordinates, the lemma therefore implies that it is enough to bound the function \( Q(\alpha, \beta) \) on the boundaries of \( A \) as soon as \( \Omega' \) is contained inside the triangle with corners \( (\alpha, \beta) \in \{(1, 1), (0, 0), (1/d, 1 - 1/d)\} \). For our case, it can be seen without much ado that \( \Omega' \) is indeed contained in this triangle: the functions \( f_d \) are piecewise linear and the slopes of the line segments are all strictly less than the slope of the line from \((0, 0)\) to \((1/d, 1 - 1/d)\).

### 2.2.2 Bounding the function on the boundary

Since the function \( Q(\alpha, \beta) \) is symmetric with respect to reflection around the line \( \alpha + \beta = 1 \), it is sufficient to prove that \( Q(\alpha, \beta) \) is strictly less than one on the first three “legs” of \( f_5 \), the first three “legs” of \( f_6 \), the first four “legs” of \( f_7 \), and the first three “legs” of \( f_8 \). In principle, this can be done by substituting \( \beta = f_d(\alpha) \) and then analyzing the resulting function using calculus. Since this is extraordinarily tedious—both for the author and for the reader—we instead choose a computer-assisted method of proof using interval arithmetic \[2\]. Specifically, we present a computer program that verifies the following claims:

**Claim 1.** For \( d = 5 \) and \( \alpha \in [10^{-5}, 0.5] \), \( Q(\alpha, \frac{1}{v}f_5(\alpha v)) \leq 0.9999 \) where \( Q \) is defined by (9) and \( f_5 \) by (1).

**Claim 2.** For \( d = 6 \) and \( \alpha \in [10^{-5}, 0.5] \), \( Q(\alpha, \frac{1}{v}f_6(\alpha v)) \leq 0.9999 \) where \( Q \) is defined by (9) and \( f_6 \) by (2).

**Claim 3.** For \( d = 7 \) and \( \alpha \in [10^{-5}, 0.39] \), \( Q(\alpha, \frac{1}{v}f_7(\alpha v)) \leq 0.9999 \) where \( Q \) is defined by (9) and \( f_7 \) by (3).

**Claim 4.** For \( d = 8 \) and \( \alpha \in [10^{-5}, 0.34] \), \( Q(\alpha, \frac{1}{v}f_8(\alpha v)) \leq 0.9999 \) where \( Q \) is defined by (9) and \( f_8 \) by (4).

There is built-in support for interval arithmetic in some compilers for some programming languages. The author has constructed a C++ program that uses the interval arithmetic routines built into Sun One Studio 7 \[3\]. Complete source code of the program is given in the appendix. Most of it
is self-explanatory, the exception being, maybe, the recursive function that verifies that some function \( f(x, y) \) is strictly less than some given upper bound \( b \) along the line \( y = y(x) \):

```cpp
bool check(const AffineFunction& y, const dim& i) {
    if(clt(f(i,y(i)), b)) {
        log << i << ": " << sup(f(i,y(i))) << std::endl;
        return true;
    } else {
        return check(y, interval_hull(di(inf(i)), di(mid(i)))) &&
          check(y, interval_hull(di(mid(i)), di(sup(i))));
    }
}
```

This function computes an interval \( I \) containing \{ \( f(x, y(x)) : x \in i \) \}. If that interval is certainly less than an interval \( b \) that contains the desired upper bound, i.e., if \( \forall z \in I \forall w \in b \ [z < w] \), the claim has been verified on the interval \( i \). Otherwise, the verification proceeds recursively: Two intervals that together cover \( i \) are constructed, and the verification continues on those intervals. A transcript of the verified subintervals is written to a log file.

When compiled with Sun One Studio 7 and executed on a Sun workstation, the program verified Claims 1–4 in a couple of seconds. The number of subintervals used were 332, 391, 857, and 261, respectively. The only option given to the compiler was “-xia”, which enables support for interval arithmetic.

3 Conclusions

The methods described in this paper are fairly general and can be applied to larger values of \( d \) without any complications. For smaller \( d \), the analysis of the function \( P \) close to the extreme points needs to be adapted.

Our functions \( f_d \) are all symmetric along the line \( u + n = v \). This property follows from our use of the probabilistic method; the involved probabilities have the same symmetry. One possible direction for future work could be to improve the behaviour of the functions \( f_d \) in the region where \( u \) is close to \( v \). Indeed, for large \( d \), one would expect that the neighbour set of any 0.99\( v \) vertices is the entire other side of the bipartite graph.

4 Acknowledgments

Per-Olof Persson suggested, with great insight, to the author that he should write the program verifying Claims 1–4 in C++ instead of Fortran. In addition, Staffan Gustafsson gave many helpful comments that made the C++ program more readable.
References

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# A Source code

```cpp
#include <fstream>
#include <suninterval.h>
typedef SUNW_interval::interval<double> di;

class Q
{
    const di d;
    const di one;
public:
    Q(const int degree) : d(degree), one("[1]") {} 
    di operator()(const di& alpha, const di& beta) const {
        return (pow(one-alpha, (d-one)*(one-alpha)) * pow(beta,(d-one)*beta)) 
            / (pow(alpha,alpha) * pow(one-beta,one-beta) 
                * pow(beta-alpha, d*(beta-alpha)));
    }
};

class AffineFunction
{
    const di k;
    const di m;
public:
   AffineFunction(const di& slope, const di& offset)
        : k(slope), m(offset) {} 
    AffineFunction(const di& x0, const di& y0, const di& x1, const di& y1) 
        : k((y1-y0)/(x1-x0)), m(y0-k*x0) {}
    di operator()(const di& x) const { return k*x+m; }
};

class Segment
{
    const di i;
    const AffineFunction f;
public:
   Segment(const di& preimage, const di& slope, const di& offset) 
            : i(preimage), f(slope, offset) {} 
   Segment(const di& x0, const di& y0, const di& x1, const di& y1) 
            : i(interval_hull(x0,x1)), f(x0, y0, x1, y1) {}
    const di& preimage() const { return i; }
    const AffineFunction& function() const { return f; }
};

class Checker
{
    const di b;
    const Q& f;
    std::ofstream log;
    bool check(const AffineFunction& y, const di& i) {
        if(clt(f(i,y(i)), b)) {
            log << i << " : " << sup(f(i,y(i))) << std::endl;
            return true;
        }
    }
};
```

else {
    return check(y, interval_hull(di(inf(i)), di(mid(i)))) &&
      check(y, interval_hull(di(mid(i)), di(sup(i))));
}
}

public:
  Checker(const di& bound, const Q& fun, const char *file)
    : b(bound), f(fun), log(file) {}
  bool operator()(const Segment& s) {
      return check(s.function(), s.preimage());
  }
};

int main()
{
    const di bound("[0.9999]");
    // Verify claim for d=5
    Checker C5(bound, Q(5), "Q5.txt");
    Segment L51(di("[1e-5,0.15]"), di("[2]"), di("[0]"));
    Segment L52(di("[0.15]"), di("[0.30]"), di("[0.30]"), di("[0.50]")));
    Segment L53(di("[0.30]"), di("[0.50]"), di("[0.50]"), di("[0.70]")));
    if(C5(L51) && C5(L52) && C5(L53)) {
      std::cout << "Claim is true for d==5." << std::endl;
    }
    // Verify claim for d=6
    Checker C6(bound, Q(6), "Q6.txt");
    Segment L61(di("[1e-5,0.10]"), di("[2.5]"), di("[0]"));
    Segment L62(di("[0.10]"), di("[0.25]"), di("[0.25]"), di("[0.50]")));
    Segment L63(di("[0.25]"), di("[0.50]"), di("[0.50]"), di("[0.75]")));
    if(C6(L61) && C6(L62) && C6(L63)) {
      std::cout << "Claim is true for d==6." << std::endl;
    }
    // Verify claim for d=7
    Checker C7(bound, Q(7), "Q7.txt");
    Segment L71(di("[1e-5,0.10]"), di("[3]"), di("[0]")));  
    Segment L72(di("[0.10]"), di("[0.15]"), di("[0.40]")));  
    Segment L73(di("[0.15]"), di("[0.40]"), di("[0.61]")));  
    Segment L74(di("[0.30]"), di("[0.61]"), di("[0.39]"), di("[0.70]")));  
    if(C7(L71) && C7(L72) && C7(L73)) {
      std::cout << "Claim is true for d==7." << std::endl;
    }
    // Verify claim for d=8
    Checker C8(bound, Q(8), "Q8.txt");
    Segment L81(di("[1e-5,0.10]"), di("[3]"), di("[0]")));  
    Segment L82(di("[0.10]"), di("[0.10]"), di("[0.20]"), di("[0.50]")));  
    Segment L83(di("[0.20,0.34]"), di("[1.25]"), di("[0.25]")));  
    if(C8(L81) && C8(L82) && C8(L83)) {
      std::cout << "Claim is true for d==8." << std::endl;
    }
}