HOMOLOGIES OF INVERSE LIMITS OF GROUPS

DANIL AKHTIAMOV

Abstract. Let $H_n$ be the $n$-th group homology functor (with integer coefficients) and let $\{G_i\}_{i \in \mathbb{N}}$ be any tower of groups such that all maps $G_{i+1} \to G_i$ are surjective. In this work we study kernel and cokernel of the following natural map:

$$H_n(\lim \leftarrow G_i) \to \lim \leftarrow H_n(G_i)$$

For $n = 1$ Barnea and Shelah [BS] proved that this map is surjective and its kernel is a cotorsion group for any such tower $\{G_i\}_{i \in \mathbb{N}}$. We show that for $n = 2$ the kernel can be non-cotorsion group even in the case when all $G_i$ are abelian and after it we study these kernels and cokernels for towers of abelian groups in more detail.

1. Introduction

It is well-known that $H_n$ commute with direct limits for any $n \geq 0$, where $H_n = H_n(-, \mathbb{Z})$ is a functor from topological space or from groups to abelian groups. But in general it is not true for projective limits. Moreover, it is quite difficult to understand whether they commute or not in some concrete cases. Some of these cases are connected with difficult problems. For example, it is known that $\lim \leftarrow H_3(F/\gamma_n(F)) = 0$ for finitely generated free group $F$. Although it turned out difficult to understand whether $H_3(\lim \leftarrow F/\gamma_n(F)) = 0$ or not and it is still an open problem. And $H_3(\lim \leftarrow F/\gamma_n(F)) \neq 0$ will imply that answer to the Strong Parafree Conjecture, which is also still open, is negative [Hill, p. 294]. Also in some cases it is quite easy to see that kernel of the natural map $H_n(\lim \leftarrow G_i) \to \lim \leftarrow H_n(G_i)$ is non-zero, but it is hard to say something about structure of this kernel. For example, let $n = 1$ and $G_i = F^{\times i}$, where, again, $F$ is a finitely generated free group. It is proven by Miasnikov and Kharlampovich [KM] that the kernel in this case contains $2$-torsion but their prove is quite difficult and uses very heavy machinery called "Non-commutative Implicit Function Theorem". And, additionally, question about $p$-torsions for prime $p > 2$ is still open.

This paper is an attempt to start systematic study of kernels and cokernels of maps $\mathfrak{g}(\lim \leftarrow G_i) \to \lim \mathfrak{g}(G_i)$, where $F$ is a functor, especially in the case $\mathfrak{g} = H_n$. It is well-known that for functor $\pi_n : Top_* \to Ab$ and for a tower of connected pointed spaces $X_i$, such that maps $X_{i+1} \to X_i$ are fibrations there is the following short exact sequence, called Milnor sequence (see [GJ, VI Proposition 2.15]):

$$0 \to \lim \leftarrow \pi_{n+1}(X_i) \to \pi_n(\lim \leftarrow X_i) \to \lim \leftarrow \pi_n(X_i) \to 0$$

The similar fact is true for the functor $H_n : Ch_{\mathbb{Z}} \to Ab$, where $Ch_{\mathbb{Z}}$ is the category of chain complexes of abelian groups and $H_n$ is $n$-th chain homology functor (see, for

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instance, [W, p. 94, prop. 3.5.8]): Let $C_i$ be a tower of chain complexes (of abelian groups) such that it satisfies degree-wise the Mittag-Leffler condition. Then there is the following short exact sequence:

$$0 \to \lim_{\leftarrow} H_{n+1}(C_i) \to H_n(\lim_{\leftarrow} C_i) \to \lim_{\leftarrow} H_n(C_i) \to 0$$

So, it is natural to ask if the same statement true for homologies of spaces with integer coefficients for a tower of connected pointed spaces $X_i$, such that maps $X_{i+1} \to X_i$ are fibrations. (Un)fortunately, the answer is no for two reasons. First, the kernel might be non-zero, while $\lim_{\leftarrow} H_{n+1}(X_i) = 0$. To see it let us consider $X_i = K(F_i, F)$, where $F$ is 2-generated free group. Using fibrant replacements, we can assume that all maps $X_{i+1} \to X_i$ are fibratons. Then, using Milnor sequences, we see that $\lim_{\leftarrow} X_i = K(\lim_{\leftarrow} F_i, 1)$. This case was studied by Bousfield and Kan deeply because of its connection with Bousfield’s completion of spaces. It is known that $\lim_{\leftarrow} H_{n+1}(X_i) = 0$ for every $n \geq 0$ [BK, p.123]. But it is proven by Bousfield in [Bous] that $H_2(F_i, F) \neq 0$ (and it implies that wedge of two circles is Z-bad space; actually wedge of two circles is also Q-bad and Z/p-bad space for $p > 2$, but it was proven much more later by Ivanov and Mikhailov in the papers [IM2], [IM3]). Second, the cokernel might be non-zero. Corresponding example actually was provided by Dwyer in [D] (Example 3.6). See Corollary 4 from my work for more details on this example.

**Definition.** We call an abelian group $A$ a cotorsion group if $A = \lim_{\leftarrow} B_i$ for some tower of abelian groups $B_i$.

**Remark.** Usually people define cotorsion groups in a different way but these definitions coincide because of [H] (Theorem 1). More detailed, there is the following equivalent definition of cotorsion groups:

**Definition.** We call an abelian group $A$ a cotorsion group if $\text{Ext}(C, A) = 0$ for any torsion-free abelian group $C$.

There is the following result which was proven by Shelah and Barnea ([BS], Corollary 0.0.9):

**Theorem.** Let $X_i$ be an inverse system of pointed connected spaces, such that all maps $X_{i+1} \to X_i$ are Serre fibrations, $\pi_1(X_i)$ and $\pi_2(X_i)$ satisfies the Mittag-Leffler condition. Then the natural map $H_1(\lim_{\leftarrow} X_i) \to \lim_{\leftarrow} H_1(X_i)$ is surjective and its kernel is a cotorsion group.

This result might give one a hope that, assuming towers $\pi_1(X_i), \pi_2(X_i), \ldots, \pi_k(X_i)$ for large enough (maybe infinite) $k$ satisfy Mittag-Leffler condition, we can ”fix” usual Milnor sequences and provide ”Milnor sequences for $H_n$”.

(UN)fortunately, the answer is no already for $n = 2$. Moreover, it is false even for homologies of abelian groups and it is proven in this work:

**Theorem 1.** There is an inverse system of abelian groups indexed by $\mathbb{N}$ such that all maps $A_{i+1} \to A_i$ are epimorphisms and kernel of the natural map $H_2(\lim_{\leftarrow} A_i) \to \lim_{\leftarrow} H_2(A_i)$ is not a cotorsion group.

This theorem shows that things are very difficult even for abelian groups, so the following result seems quite interesting:
Theorem 2. Let $A_i$ be an inverse system of torsion-free abelian groups indexed by $\mathbb{N}$ such that all maps $A_{i+1} \to A_i$ are epimorphisms. Then for any $n \in \mathbb{N}$ the natural map $H_n(\lim \leftarrow A_i) \to \lim \leftarrow H_n(A_i)$ is an embedding and its cokernel is a cotorsion group.

Theorem 3. Let $A_i$ be an inverse system of any abelian groups indexed by $\mathbb{N}$ such that all maps $A_{i+1} \to A_i$ are epimorphisms. Then:

1. Cokernel of the natural map $H_2(\lim \leftarrow A_i) \to \lim \leftarrow H_2(A_i)$ is a cotorsion group.

2. Suppose, additionally, that torsion subgroup of $A_i$ is a group of bounded exponent for any $i$. Then cokernel of the natural map $H_3(\lim \leftarrow A_i) \to \lim \leftarrow H_3(A_i)$ is a cotorsion group. In particular, it is true if all $A_i$ are finitely generated.

Also in this work there are two results not about homologies but about another functors and inverse limits of abelian groups:

Statement 2. Let $B$ be any abelian group and $B_i$ be an inverse system of abelian groups with surjective maps $f_i : B_{i+1} \to B_i$ between them. Then kernel of the map $\text{Tor}(B, \lim \leftarrow B_i) \to \lim \leftarrow \text{Tor}(B, B_i)$ is trivial.

Corollary 6. $\text{Tor}(\lim \leftarrow A_i, \lim \leftarrow A_i) \to \lim \leftarrow \text{Tor}(A_i, A_i)$ is embedding for any inverse system of abelian groups such that all maps $A_{i+1} \to A_i$ are surjective.

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2. Homologies of inverse limits of groups

A goal of this chapter is to prove the following results:

Theorem 1. There is an inverse system of abelian groups indexed by $\mathbb{N}$ such that all maps $A_{i+1} \to A_i$ are epimorphisms and kernel of the natural map $H_2(\lim \leftarrow A_i) \to \lim \leftarrow H_2(A_i)$ is not a cotorsion group.

Theorem 2. Let $A_i$ be an inverse system of torsion-free abelian groups indexed by $\mathbb{N}$ such that all maps $A_{i+1} \to A_i$ are epimorphisms. Then for any $n \in \mathbb{N}$ the natural map $H_n(\lim \leftarrow A_i) \to \lim \leftarrow H_n(A_i)$ is embedding and its cokernel is a cotorsion group.

Theorem 3. Let $A_i$ be an inverse system of any abelian groups indexed by $\mathbb{N}$ such that all maps $A_{i+1} \to A_i$ are epimorphisms. Then:

1. Cokernel of the natural map $H_2(\lim \leftarrow A_i) \to \lim \leftarrow H_2(A_i)$ is a cotorsion group.

2. Suppose, additionally, that torsion subgroup of $A_i$ is a group of bounded exponent for any $i$. Then cokernel of the natural map $H_3(\lim \leftarrow A_i) \to \lim \leftarrow H_3(A_i)$ is a cotorsion group. In particular, it is true if all $A_i$ are finitely generated.

Proof of Theorem 1. Let us consider

$$A_{i,p} = \mathbb{Z}^i \oplus \bigoplus_{i+1}^{\infty} \mathbb{Z}/p\mathbb{Z}, A_{i,p} := \bigoplus_{p \in \mathbb{P}} A_{i,p}, B := \bigoplus_{p \in \mathbb{P}} \mathbb{Z}/p\mathbb{Z}, A_i := A_i \times B$$

We are going to define maps $\psi_i : A_i \to A_{i-1}$ in the most natural way. Let us denote by $e^1_{i,p}, \ldots, e^l_{i,p}$ elements of basis of the free abelian summand of $A_{i,p}$ and let us denote by $e^{i+1}_{i,p}, e^{i+2}_{i,p}, \ldots$ elements of basis of the $\mathbb{Z}/p\mathbb{Z}$-vector space, which is the second direct summand of $A_{i,p}$. Now let us define $\psi_i(e^j_{i,p}) := e^j_{i-1,p}$. It is obvious that there is unique
ψ_i with such properties. And, finally, let φ_i : A_i → A_i−1 be the map defined by the matrix
\[
\begin{bmatrix}
\psi_i & 0 \\
0 & I_{dB}
\end{bmatrix}.
\]

Using Kunneth formula and using that all A'_i and B are abelian, we have:
\[
H_2(\varprojlim A_i) = H_2((\varprojlim A'_i) \times B) = H_2((\varprojlim A'_i) \times B) \oplus H_2(B) \oplus (\varprojlim A'_i \times B)
\]

Thus our map \( H_2(\varprojlim A_i) \to \varprojlim H_2(A_i) \) is a map from \( H_2((\varprojlim A'_i) \times B) \oplus (\varprojlim A'_i \times B) \) to \( \varprojlim H_2(A'_i) \). Analyzing maps from the Kunneth formula, we see that this map is given by a diagonal matrix, the corresponding maps \( H_2((\varprojlim A'_i) \times B) \to \varprojlim H_2(A'_i) \) and \( (\varprojlim A'_i) \times B \to (\varprojlim A'_i \times B) \) coincide with obvious maps which come from definition of inverse limit and the corresponding map \( H_2(B) \to H_2(B) \) is isomorphism. Thus kernel of the natural map \( (\varprojlim A'_i) \times B \to (\varprojlim A'_i \times B) \) is direct summand of kernel of the natural map \( H_2((\varprojlim A'_i) \to \varprojlim H_2(A'_i) \). So it is enough to prove that kernel of the natural map \( (\varprojlim A'_i) \times B \to (\varprojlim A'_i \times B) \) is not a cotorsion group and now we will do it. Let us note that the map \( (\varprojlim A'_i) \times B \to (\varprojlim A'_i \times B) \) can be decomposed in the following way:

\[
(\varprojlim A'_i) \times B = \bigoplus_{p \in P} \varprojlim A'_i \otimes \mathbb{Z}/p\mathbb{Z} \to \bigoplus_{p \in P} \varprojlim (A'_i \otimes \mathbb{Z}/p\mathbb{Z}) \to \varprojlim \left( \bigoplus_{p \in P} A'_i \otimes \mathbb{Z}/p\mathbb{Z} \right) = \varprojlim (A'_i \times B)
\]

It is easy to see that the map \( \bigoplus_{p \in P} \varprojlim (A'_i \otimes \mathbb{Z}/p\mathbb{Z}) \to \varprojlim \left( \bigoplus_{p \in P} A'_i \otimes \mathbb{Z}/p\mathbb{Z} \right) \) is injective.

Then kernel of the map \( (\varprojlim A'_i) \times B \to (\varprojlim A'_i \times B) \) equals \( \bigoplus_{p \in P} \text{Ker}[(\varprojlim A'_i) \times \mathbb{Z}/p\mathbb{Z} \to \varprojlim (A'_i \otimes \mathbb{Z}/p\mathbb{Z})] \). It is proven at [IM] (Corollary 2.5) that \( \text{Ker}[(\varprojlim A'_i) \otimes \mathbb{Z}/p\mathbb{Z} \to \varprojlim (A'_i \otimes \mathbb{Z}/p\mathbb{Z})] \cong \lim_{i \to \infty} \text{Tor}(A'_i, \mathbb{Z}/p\mathbb{Z})) \). Let us note that \( \text{Tor}(A'_i, \mathbb{Z}/p\mathbb{Z}) = \bigoplus_{i \geq 1} \mathbb{Z}/p\mathbb{Z} \).

So \( \lim_{i \to \infty} \text{Tor}(A'_i, \mathbb{Z}/p\mathbb{Z}) \cong \lim_{i \to \infty} \bigoplus_{i \geq 1} \mathbb{Z}/p\mathbb{Z} \), which is, obviously, p-torsion abelian group and which is nonzero because of [MP] (p. 330, Proposition A.20). Then, finally, we have that kernel of the map \( (\varprojlim A'_i) \otimes B \to (\varprojlim A'_i \times B) \) is not a cotorsion group because of [B] (Theorem 8.5). Q.E.D.

We will need the following lemmas and statements in order to prove Theorem 2 and Theorem 3. All inverse systems supposed to be indexed by \( \mathbb{N} \).

**Statement 1.** Let \( C \) be any category with projective limits and let \( B_i \) be a tower of objects from \( C \). Let \( F : C \to \text{Ab} \) be a functor such that \( \varprojlim Coker[F(\varprojlim B_i) \to F(B_n)] = 0 \). Then cokernel of the natural map \( F(\varprojlim B_i) \to \varprojlim F(B_i) \) is a cotorsion group.

This statement was proved in the third version of Barnea’s and Shelah’s preprint [BS] in the case when functor \( F \) preserves surjection. However, their proof was quite complicated. Our proof of the statement is straightforward.
Proof. Let denote by \( \Phi_i := \text{Ker}(F(\lim_i B_i) \to F(B_i)) \) and by \( \Psi_i := \text{Im}(F(\lim_i B_i) \to F(B_i)) \). Then we have the following exact sequences:

\[
0 \to \Phi_i \to F(\lim_i B_i) \to \Psi_i \to 0
\]

and

\[
0 \to \Psi_i \to F(B_i) \to \text{Coker}[F(\lim_i B_i) \to F(B_i)] \to 0.
\]

From the second sequence we get by the assumption \( \lim_n \text{Coker}[F(\lim_i B_i) \to F(B_n)] = 0 \) \( \lim \Psi_i = \lim F(B_i) \). From the first we deduce a new exact sequence:

\[
0 \to \lim \Phi_i \to F(\lim_i B_i) \to \lim \Psi_i \to \lim \Phi_i \to 0
\]

Thus finally we have the following exact sequence

\[
0 \to \lim \Phi_i \to F(\lim_i B_i) \to \lim F(B_i) \to \lim \Phi_i \to 0
\]

Now it is enough to note that \( \lim \Psi_i \) of any inverse system of abelian groups is a cotorsion group by \([H]\) (Theorem 1). Q.E.D.

**Corollary 1.** Let \( B_i \) be an inverse system of groups (respectively abelian groups) and any maps between them. Let \( F : \text{Grp} \to \text{Ab} \) (respectively \( F : \text{Ab} \to \text{Ab} \)) be a functor such that \( \lim_n \text{Coker}[F(\lim_i B_i) \to F(B_n)] = 0 \). Then kernel of the map \( F(\lim_i B_i) \to \lim F(B_i) \) equals \( \lim \Phi_i \), where \( \Phi_n = \text{Ker}[F(\lim_i B_i) \to F(B_n)] \).

**Corollary 2.** Cokernel of the map \( \Lambda^n(\lim_i B_i) \to \lim \Lambda^n(B_i) \) is a cotorsion group for any inverse system \( B_i \), such that \( B_{i+1} \to B_i \) are epimorphisms.

Proof. Since \( B_{i+1} \to B_i \) are epimorphisms, the maps \( \Lambda^n(\lim_i B_i) \to \Lambda^n(B_i) = 0 \) are also epimorphisms (it easily follows from constructive description of projective limits in the category of groups). Then \( \Lambda^n(\lim_i B_i) \to \Lambda^n(B_i) \) is epimorphism, because \( \Lambda^n \) is right-exact. Then \( \text{Coker}[\Lambda^n(\lim_i B_i) \to \Lambda^n(B_i)] = 0 \) and we are done because of Statement 1.

**Corollary 3.** Cokernel of the map \( H_2(\lim_i B_i) \to \lim H_2(B_i) \) is a cotorsion group for any inverse system of abelian groups \( B_i \), such that \( B_{i+1} \to B_i \) are epimorphisms.

Proof. It is well-known \([Breen, \text{section} \ 6]\) that \( H_2 \) is naturally isomorphic to \( \Lambda^2 \) in the category of abelian groups, so the corollary follows from Corollary 2.

Definition. Let \( G \) be any group. Let \( \gamma_1(G) := G \) and \( \gamma_{i+1}(G) := [G, \gamma_i(G)] \). Series \( \gamma_i(G) := G \) are called lower central series of a group \( G \). Let denote \( \hat{G} := \lim G/\gamma_i(G) \).

**Corollary 4.** Cokernel of the map \( H_2(\hat{G}) \to \lim H_2(G/\gamma_i(G)) \) is a cotorsion group for any group \( G \).

Proof. It is clear that the map \( H_2(G) \to H_2(G/\gamma_i(G)) \) factors through the map \( H_2(\hat{G}) \to H_2(G/\gamma_i(G)) \). Then it is enough to prove that \( \lim \text{Coker}[H_2(G) \to H_2(G/\gamma_i(G))] = 0 \). Let note that maps between groups \( \text{Coker}[H_2(G) \to H_2(G/\gamma_i(G))] \) are zero. Really, using that \( H_1(G) = H_1(G/\gamma_i(G)) \), we see from the 5-term exact sequence \([Brown, \text{p.47, exercise} \ 6a]\) that these cokernels are equal to \( \frac{\gamma_i(G)[G,G]}{[\gamma_i(G),G]} = \frac{\gamma_i(G)}{\gamma_{i+1}(G)} \). Q.E.D.

Following statement is not really necessary for a proof of the Theorems, but it is interesting by itself and makes clearer what is happening.
Statement 2. Let $B$ be any abelian group and $B_i$ be an inverse system of abelian groups with surjective maps $f_i : B_{i+1} \to B_i$ between them. Then kernel of the map $\text{Tor}(B, \varprojlim B_i) \to \varprojlim \text{Tor}(B, B_i)$ is trivial.

Proof. It is proven in [IM] (Proposition 2.6) that for any free resolution $P_\bullet$ of $B$ there are the following exact sequences (take $\Lambda = \mathbb{Z}$ and use a fact which states that ring $\mathbb{Z}$ has a global dimension one):

$$0 \to H_1(\lim \leftarrow P_\bullet \otimes B_i) \to \lim \leftarrow \text{Tor}(B, B_i) \to 0$$

So, we understood that $H_1(\lim \leftarrow P_\bullet \otimes B_i)$ is isomorphic to $\lim \leftarrow \text{Tor}(B, B_i)$.

Let take $P_\bullet$ be a minimal resolution, i.e. such that $P_s = 0$ when $s > 1$. It is possible because global dimension of $\mathbb{Z}$ equals 1.

Let us note that the maps $P_s \otimes \lim \leftarrow B_i \to \lim \leftarrow (P_s \otimes B_i)$ are embeddings because of Lemma 1 ($s = 0, 1$).

Let consider a short exact sequence of complexes ($C_\bullet$ is defined from this sequence):

$$0 \to P_\bullet \otimes (\lim \leftarrow B_i) \to \lim \leftarrow (P_\bullet \otimes B_i) \to C_\bullet \to 0$$

Since $P_s=0$ for $s > 1$, note that $C_s = 0$ for $s > 1$. Thus we have the following exact sequence:

$$0 \to H_1(P_\bullet \otimes (\lim \leftarrow B_i)) \to H_1(\lim \leftarrow (P_\bullet \otimes B_i))$$

But $H_1(P_\bullet \otimes (\lim \leftarrow B_i)) = \text{Tor}(B, \lim \leftarrow B_i)$ by definition and we already have got that $H_1(\lim \leftarrow P_\bullet \otimes B_i) = \lim \leftarrow \text{Tor}(B, B_i)$ so we are done. Q.E.D.

Corollary 5. $\text{Tor}(\lim \leftarrow A_i, \lim \leftarrow A_i) \to \lim \leftarrow \text{Tor}(A_i, A_i)$ is an embedding for any inverse system of abelian groups such that all maps $A_{i+1} \to A_i$ are epimorphic.

Proof. Let consider following commutative diagramm with $\psi$ being isomorphism:

$$\begin{array}{ccc}
\text{Tor}(\lim \leftarrow A_i, \lim \leftarrow A_j) & \xrightarrow{\varphi} & \lim \leftarrow \text{Tor}(A_i, A_i) \\
\downarrow f & & \downarrow \psi \\
\lim \leftarrow \text{Tor}(A_i, \lim \leftarrow A_j) & \xrightarrow{g} & \lim \leftarrow \text{Tor}(A_i, A_j)
\end{array}$$

But $f$ is monomorphism because of Statement 3 and $g$ is monomorphism because of exactness of $\lim \leftarrow$. Q.E.D.

Statement 3. Let $B$ be a cotorsion group and $A$ be another abelian group. Let suppose that there are $i : A \to B$ and $\pi : B \to A$, such that $\pi i = n \text{Id}_A$ for $n \geq 1$. Then $A$ is a cotorsion group.

Proof. Since $\text{Ext}(\mathbb{Q}, B) = 0$, a map $\pi i = n \text{Id}_A$ induces zero endomorphism of $\text{Ext}(\mathbb{Q}, A)$. Let consider the following exact sequences (we denote $n$-torsion subgroup of $A$ by $A_n$):

$$0 \to A_n \to A \to nA \to 0$$

$$0 \to nA \to A \to A/nA \to 0$$
It is known that any $n$-torsion group is a cotorsion group (see [B], Theorem 8.5). Thus the maps $A \to nA$ and $nA \to A$ induce isomorphisms on $Ext(Q, -)$, and then so do $nId_A$. But it induces zero endomorphism of $Ext(Q, A)$, so we are done. Q.E.D.

**Proof of Theorem 3.** We already proved the first point of this theorem (Corollary 2), so let us prove second point of the theorem. It is known [Breen, section 6] that for any abelian group $A$ there is the following short exact sequence (here $L_1\Lambda^2$ is first derived functor of functor $\Lambda^2$):

$$0 \to \Lambda^3(A) \to H_3(A) \to L_1\Lambda^2(A) \to 0$$

Then for $A = \lim A_i$ we have the following sequence:

$$0 \to \Lambda^3(\lim A_i) \to H_3(\lim A_i) \to L_1\Lambda^2(\lim A_i) \to 0$$

Since $\Lambda^3(\lim A_{i+1}) \to \Lambda^3(\lim A_i)$ is an epimorphism for any $i$, $\lim^1\Lambda^3(\lim A_i) = 0$ and we have the following sequence:

$$0 \to \lim \Lambda^3(A_i) \to \lim H_3(A_i) \to \lim L_1\Lambda^2(A_i) \to 0$$

Then $\ker(L_1\Lambda^2(\lim A_i) \to \lim L_1\Lambda^2(A_i)) \subseteq \ker(Tor(\lim A_i, \lim A_i) \to \lim Tor(A_i, A_i)) = 0$ (here we used Corollary 5). Then, using Snake Lemma, we have the following short exact sequence:

$$0 \to \text{Coker}[\Lambda^3(\lim A_i \to \lim \Lambda^3(A_i)) \to \text{Coker}[H_3(\lim A_i \to \lim H_3(A_i)) \to \text{Coker}[L_1\Lambda^2(\lim A_i) \to \lim L_1\Lambda^2(A_i)]$$

It is obvious that any extension of a cotorsion group by a cotorsion group is a cotorsion group. And we already know that $\text{Coker}[\Lambda^3(\lim A_i \to \lim \Lambda^3(A_i))$ is a cotorsion group. Then it is enough to prove that $\text{Coker}[L_1\Lambda^2(\lim A_i) \to \lim L_1\Lambda^2(A_i)]$ is a cotorsion group.

It is known [Breen, sections 4 and 5] that for any abelian group $A$ there are maps $L_1\Lambda^2(A) \to Tor(A, A)$ and $Tor(A, A) \to L_1\Lambda^2(A)$, such that their composition equals to $2Id_{L_1\Lambda^2(A)}$. Then they induce natural maps $\text{Coker}[L_1\Lambda^2(\lim A_i) \to \lim L_1\Lambda^2(A_i)] \to \text{Coker}[Tor(\lim A_i, \lim A_i) \to \lim Tor(A_i, A_i)]$ and $\text{Coker}[L_1\Lambda^2(\lim A_i) \to \lim L_1\Lambda^2(A_i)] \to \text{Coker}[Tor(\lim A_i, \lim A_i) \to \lim Tor(A_i, A_i)]$, such that their composition is multiplication by two. Then, using Statement 3, we see that it is enough to prove that $\text{Coker}[Tor(\lim A_i, \lim A_i) \to \lim Tor(A_i, A_i)]$ is a cotorsion group. But if torsion subgroups of $A_i$ are groups of bounded exponent, then $Tor(A_i, A_i)$ are torsion groups of bounded exponent. Hence they are cotorsion groups [B, Theorem 8.5], and $\lim Tor(A_i, A_i)$ is a cotorsion group, because it is an inverse limit of cotorsion groups (it is obvious from our second definition of cotorsion groups that an inverse limit of cotorsion groups is itself a cotorsion group). Thus $\text{Coker}[Tor(\lim A_i, \lim A_i) \to \lim Tor(A_i, A_i)]$ is a cotorsion group as an image of a cotorsion group and we are done. Q.E.D.

We need the following statement in order to prove Theorem 2:
Statement 4. Let $A_i$ be an inverse system of torsion-free abelian groups indexed by $\mathbb{N}$ such that all maps $A_{i+1} \to A_i$ are epimorphisms and let $B$ be any torsion-free abelian group. Then the natural map $B \otimes \varprojlim A_i \to \varprojlim (B \otimes A_i)$ is an embedding.

Proof. First let prove the statement for $B = \mathbb{Q}$. Let us consider the following short exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

After applying the functor $- \otimes A_i$ for this sequence we get:

$$0 \to Tor(\mathbb{Q}/\mathbb{Z}, A_i) \to A_i \to \mathbb{Q} \otimes A_i \to \mathbb{Q}/\mathbb{Z} \otimes A_i \to 0$$

Since $Tor(\mathbb{Q}/\mathbb{Z}, A) = t(A)$ for any abelian group $A$, we have:

$$0 \to t(A_i) \to A_i \to \mathbb{Q} \otimes A_i \to \mathbb{Q}/\mathbb{Z} \otimes A_i \to 0$$

Then, using left-exactness of $\varprojlim$, we have:

$$0 \to \varprojlim t(A_i) \to \varprojlim A_i \to \varprojlim (\mathbb{Q} \otimes A_i)$$

Let us apply the exact functor $\mathbb{Q} \otimes -$ for this sequence:

$$0 \to \mathbb{Q} \otimes \varprojlim t(A_i) \to \varprojlim \mathbb{Q} \otimes A_i \to \mathbb{Q} \otimes \varprojlim (\mathbb{Q} \otimes A_i)$$

Let us note that $\mathbb{Q} \otimes \varprojlim (\mathbb{Q} \otimes A_i) \cong \varprojlim (\mathbb{Q} \otimes A_i)$, because $\varprojlim (\mathbb{Q} \otimes A_i)$ is a $\mathbb{Q}$-vector space. It means that we got the following:

$$0 \to \mathbb{Q} \otimes \varprojlim t(A_i) \to \varprojlim \mathbb{Q} \otimes A_i \to \varprojlim (\mathbb{Q} \otimes A_i)$$

But in the our case $t(A_i) = 0$ for any $i$. So we have:

$$0 \to \mathbb{Q} \otimes \varprojlim A_i \to \varprojlim (\mathbb{Q} \otimes A_i)$$

Now let us prove the statement in the case $B = \mathbb{Q}^{\oplus I}$, where $I$ is any cardinal. Using the sequence for $B = \mathbb{Q}$ we get:

$$0 \to (\mathbb{Q} \otimes \varprojlim t(A_i))^{\oplus I} \to (\mathbb{Q} \otimes \varprojlim A_i)^{\oplus I} \to (\varprojlim (\mathbb{Q} \otimes A_i))^{\oplus I}$$

Since it is obvious that the map $(\varprojlim (\mathbb{Q} \otimes A_i))^{\oplus I} \to \varprojlim ((\mathbb{Q} \otimes A_i)^{\oplus I})$ is injective, we have:

$$0 \to (\mathbb{Q} \otimes \varprojlim t(A_i))^{\oplus I} \to (\mathbb{Q} \otimes \varprojlim A_i)^{\oplus I} \to \varprojlim ((\mathbb{Q} \otimes A_i)^{\oplus I})$$

But in the our case $t(A_i) = 0$ for any $i$. So we have the exactness of the following sequence:

$$0 \to (\mathbb{Q} \otimes \varprojlim A_i)^{\oplus I} \to \varprojlim ((\mathbb{Q} \otimes A_i)^{\oplus I})$$

Since the functor $- \otimes A$ commutes with direct sums for any $A$, we proved the statement in the case when $B$ is any $\mathbb{Q}$-vector space.

Now let us prove the statement for any torsion-free $B$. Let $Q_B$ be the injective hull of $B$. Since $B$ is torsion-free, $Q_B = \mathbb{Q}^{\oplus I}$ for some cardinal $I$. Let us consider the following short exact sequence ($B'$ is defined from this sequence):

$$0 \to B \to Q_B \to B' \to 0$$

Since $\varprojlim A_i$ is torsion-free and $\varprojlim A_i = 0$ because the maps $B \otimes A_{i+1} \to B \otimes A_i$ are epimorphisms, it gives us two sequences:

$$0 \to B \otimes \varprojlim A_i \to Q_B \otimes \varprojlim A_i \to B' \otimes \varprojlim A_i \to 0$$
0 \to \varprojlim (B \otimes A_i) \to \varprojlim (Q_B \otimes A_i) \to \varprojlim (B' \otimes A_i) \to 0

Then, using Snake Lemma for this two sequences and natural maps between their elements, we have:

0 \to \text{Ker}[B \otimes \varprojlim A_i \to \varprojlim (B \otimes A_i)] \to \text{Ker}[Q_B \otimes \varprojlim A_i \to \varprojlim (Q_B \otimes A_i)]

Since \(Q_B\) is a \(\mathbb{Q}\)-vector space, we already proved that \(\text{Ker}[Q_B \otimes \varprojlim A_i \to \varprojlim (Q_B \otimes A_i)] = 0\). Then \(\text{Ker}[B \otimes \varprojlim A_i \to \varprojlim (B \otimes A_i)] = 0\) and we are done. Q.E.D.

**Proof of Theorem 2.** Let consider following commutative diagram with \(\psi\) being isomorphism.

\[
\begin{array}{ccc}
\varprojlim A_i \otimes \varprojlim A_j & \xrightarrow{\varphi} & \varprojlim (A_i \otimes A_i) \\
\downarrow f & & \downarrow \psi \\
\varprojlim (A_i \otimes \varprojlim A_j) & \xrightarrow{g} & \varprojlim \varprojlim (A_i \otimes A_j)
\end{array}
\]

Then we see that \(f\) is embedding because of Statement 4 and \(g\) is embedding because of Statement 4 and left-exactness of \(\varprojlim\). This implies that \(\varphi\) is also embedding. Then, since there is a natural embedding \(\Lambda^2(A) \to A \otimes A\) and \(H_2\) is naturally isomorphic to \(\Lambda^2\) [Breen, section 6] for abelian groups, we proved the theorem for \(n = 2\).

Now let us note that \(H_n\) is naturally isomorphic to \(\Lambda^n\) on the category of torsion-free abelian groups also for any \(n > 2\) [Breen, p. 214, (1.12)].

Let us prove the theorem by induction on \(n\).

We can assume that \(n \geq 3\). Let us note that the map \((\varprojlim A_i)^{\otimes n} \to \varprojlim (A_i)^{\otimes n}\) may be decomposed up to isomorphism in the following way:

\((\varprojlim A_i)^{\otimes n} \cong \varprojlim A_i \otimes (\varprojlim A_j)^{\otimes (n-1)} \to \varprojlim A_i \otimes \varprojlim (A_j)^{\otimes (n-1)} \to \varprojlim \varprojlim A_i \otimes (A_j)^{\otimes (n-1)} \cong \varprojlim (A_i)^{\otimes n}\). But all maps in the decomposition are monic because of inductive assumption, left-exactness of \(\varprojlim\) and exactness of \(- \otimes \mathbb{B}\) for torsion-free \(\mathbb{B}\). Then the map \((\varprojlim A_i)^{\otimes n} \to \varprojlim (A_i)^{\otimes n}\) is also monic, and then so is \(H_n(\varprojlim A_i) \to \varprojlim H_n(A_i)\). Q.E.D.

3. Applications to topology

**Theorem 4.** Let \(X_i\) be an inverse system of pointed connected spaces, such that all maps \(X_{i+1} \to X_i\) are Serre fibrations, all \(\pi_1(X_i)\) are abelian, all maps \(\pi_1(X_{i+1}) \to \pi_1(X_i)\) are epimorphisms and \(\pi_2(X_i)\) satisfy the Mittag-Leffler condition. Then:

1. Cokernel of the natural map \(H_2(\varprojlim X_i) \to \varprojlim H_2(X_i)\) is a cotorsion group.

2. Suppose, additionally, that all \(\pi_1(X_i)\) are torsion-free. Then kernel of the natural map \(H_2(\varprojlim X_i) \to \varprojlim H_2(X_i)\) is a cotorsion group.

3. Suppose that condition (2) is satisfied and, additionally, \(\pi_3(X_i)\) satisfy the Mittag-Leffler condition. Then the natural map \(H_2(\varprojlim X_i) \to \varprojlim H_2(X_i)\) is embedding.

Proof. Let us denote \(\Pi_2(X) := \text{Im}[\pi_2(X) \to H_2(X)]\). It is obvious that \(\Pi_2\) is a functor. Then we have the following sequence:

\[0 \to \Pi_2(X) \to H_2(X) \to H_2(\pi_1(X)) \to 0\]
Then we get the following sequences, feeding \( X = \varprojlim X_i \) and \( X = X_i \):

\[
0 \to \Pi_2(X_i) \to H_2(X_i) \to H_2(\pi_1(X_i)) \to 0
\]

It gives us the following sequence:

\[
0 \to \varprojlim \Pi_2(X_i) \to \varprojlim H_2(X_i) \to \varprojlim H_2(\pi_1(X_i)) \to \varprojlim^1 \Pi_2(X_i) \to \varprojlim^1 H_2(X_i) \to \varprojlim^1 H_2(\pi_1(X_i)) \to 0
\]

Since \( \pi_2(X_i) \to \Pi_2(X_i) \) is epimorphism, \( \varprojlim^1 \pi_2(X_i) \to \varprojlim^1 \Pi_2(X_i) \) it is well-known (e.g. see [Brown, p. 42, Theorem 5.2]) that for any space \( X \) there is the following exact sequence:

\[
\pi_2(X) \to H_2(X) \to H_2(\pi_1(X)) \to 0
\]

Also we have the following sequence:

\[
0 \to \varprojlim \Pi_2(X_i) \to \varprojlim H_2(X_i) \to \varprojlim H_2(\pi_1(X_i)) \to 0
\]

We have the following sequences, because \( X_{i\in\mathbb{N}} \) is a tower of pointed fibrations [GJ, VI Proposition 2.15]:

\[
0 \to \varprojlim^1 \pi_2(X_i) \to \pi_1(\varprojlim X_i) \to \varprojlim \pi_1(X_i) \to 0
\]

Since \( \pi_2(X_i) \) satisfy the Mittag-Leffler condition, we have:

\[
\pi_1(\varprojlim X_i) \cong \varprojlim \pi_1(X_i)
\]

Finally, we have two sequences:

\[
0 \to \varprojlim \Pi_2(X_i) \to \varprojlim H_2(X_i) \to \varprojlim H_2(\pi_1(X_i)) \to 0
\]

\[
0 \to \Pi_2(\varprojlim X_i) \to H_2(\varprojlim X_i) \to H_2(\varprojlim \pi_1(X_i)) \to 0
\]

Let us note that \( \text{Coker}[\varprojlim \Pi_2(X_i) \to \Pi_2(\varprojlim X_i)] = 0 \) and \( \text{Ker}[\varprojlim \Pi_2(X_i) \to \Pi_2(\varprojlim X_i)] \cong \varprojlim^1 \pi_3(X_i) \). It follows from the following sequence:

\[
0 \to \varprojlim^1 \pi_3(X_i) \to \pi_2(\varprojlim X_i) \to \varprojlim \pi_2(X_i) \to 0
\]

Then, using Snake Lemma for this two sequences and the natural maps between them, we have:

\[
0 \to \varprojlim^1 \pi_3(X_i) \to \text{Ker}[\varprojlim H_2(X_i) \to H_2(\varprojlim X_i)] \to \text{Ker}[\varprojlim H_2(\pi_1(X_i)) \to H_2(\varprojlim \pi_1(X_i))] \to 0
\]

and

\[
\text{Coker}[H_2(\varprojlim X_i) \to \varprojlim H_2(X_i)] \cong \text{Coker}[H_2(\varprojlim \pi_1(X_i)) \to \varprojlim H_2(\varprojlim X_i)]
\]

Then point (1) of the Theorem follows from point (1) of Theorem 1. Let us prove points (2) and (3) of the Theorem. It follows from Theorem 2 for \( n = 2 \) that \( \text{Ker}[\varprojlim H_2(\pi_1(X_i)) \to H_2(\varprojlim \pi_1(X_i))] = 0 \). Then we have:

\[
\text{Ker}[\varprojlim H_2(X_i) \to H_2(\varprojlim X_i)] \cong \varprojlim^1 \pi_3(X_i)
\]

So we proved point (3) and point (2). Q.E.D.
Theorem 5. Let \( Y_i \) be a sequence of spaces. Then cokernel of the map \( \lim_{n} H_k(\prod_{i=1}^{\infty} Y_i) \rightarrow \lim_{n} H_k(\prod_{i=1}^{\infty} Y_i) \) is a cotorsion group for every \( k \).

Proof. Since composition of the natural maps \( H_k(\prod_{i=1}^{\infty} Y_i) \rightarrow H_k(\prod_{i=1}^{\infty} Y_i) \) and \( H_k(\prod_{i=1}^{\infty} Y_i) \rightarrow H_k(\prod_{i=1}^{\infty} Y_i) \) is the identity map, the map \( H_k(\prod_{i=1}^{\infty} Y_i) \rightarrow H_k(\prod_{i=1}^{\infty} Y_i) \) is surjective. So we are done because of Statement 1. Q.E.D.

Remark. It is easy to see from this from this proof that the same fact holds for any category which has infinite products instead of category of spaces and for and functor \( F \) from this category to \( Ab \) instead of \( H_k \). But I formulated it in this way because it seems more natural in this section.

We will also prove the following Theorem, which does not follow from our previous results but follows from Shelah’s and Barnea’s. It is connected with Theorem 3, so I found quite natural to formulate and prove it here. This Theorem is a generalization of Shelah’s and Barnea’s [Corollary 0.0.9. BS]. Actually we show that it is not necessary to assume that \( \pi_2(X_i) \) satisfies the Mittag-Leffler condition.

Theorem 6. Let \( X_i \) be an inverse system of pointed connected spaces, such that all maps \( X_{i+1} \rightarrow X_i \) are Serre fibrations and \( \pi_1(X_i) \) satisfies the Mittag-Leffler condition. Then the natural map \( H_1(\lim_{\leftarrow} X_i) \rightarrow \lim_{\leftarrow} H_1(X_i) \) is surjective and its kernel is a cotorsion group.

Proof. Let us consider the following exact sequence:

\[
0 \rightarrow \lim_{\leftarrow} \pi_1(X_i) \rightarrow \pi_0(\lim_{\leftarrow} X_i) \rightarrow \lim_{\leftarrow} \pi_0(X_i) \rightarrow 0
\]

Then \( \lim_{\leftarrow} X_i \) is also connected and we have:

\[
H_1(\lim_{\leftarrow} X_i) \cong \pi_1(\lim_{\leftarrow} X_i)_{ab}
\]

It follows that:

\[
Ker[H_1(\lim_{\leftarrow} X_i) \rightarrow \lim_{\leftarrow} H_1(X_i)] = Ker[(\pi_1(\lim_{\leftarrow} X_i))_{ab} \rightarrow \lim_{\leftarrow} (\pi_1(X_i))_{ab}]
\]

and

\[
Coker[H_1(\lim_{\leftarrow} X_i) \rightarrow \lim_{\leftarrow} H_1(X_i)] = Coker[(\pi_1(\lim_{\leftarrow} X_i))_{ab} \rightarrow \lim_{\leftarrow} (\pi_1(X_i))_{ab}]
\]

Also we have the following sequence:

\[
0 \rightarrow \lim_{\leftarrow} \pi_2(X_i) \rightarrow \pi_1(\lim_{\leftarrow} X_i) \rightarrow \lim_{\leftarrow} \pi_1(X_i) \rightarrow 0
\]

Thus we have the following exact sequence from 5-term exact sequence:

\[
\lim_{\leftarrow} \pi_2(X_i) \rightarrow (\pi_1(\lim_{\leftarrow} X_i))_{ab} \rightarrow (\lim_{\leftarrow} \pi_1(X_i))_{ab} \rightarrow 0
\]

Then, since \( \lim_{\leftarrow} \pi_2(X_i) \) is a cotorsion group by the Theorem of Huber(Theorem 1, [H]) and any quotient group of a cotorsion group is itself a cotorsion group, \( Ker[(\pi_1(\lim_{\leftarrow} X_i))_{ab} \rightarrow (\lim_{\leftarrow} \pi_1(X_i))_{ab}] \) is a cotorsion group. Now consider the map \( (\pi_1(\lim_{\leftarrow} X_i))_{ab} \rightarrow (\lim_{\leftarrow} \pi_1(X_i))_{ab} \).

It is surjective and its kernel is a cotorsion group by Theorem 0.0.1 of Shelah and Barnea from [BS]. Now note that we can decompose the map \( (\pi_1(\lim_{\leftarrow} X_i))_{ab} \rightarrow (\lim_{\leftarrow} \pi_1(X_i))_{ab} \) in the following way:

\[
(\pi_1(\lim_{\leftarrow} X_i))_{ab} \rightarrow (\lim_{\leftarrow} \pi_1(X_i))_{ab} \rightarrow (\lim_{\leftarrow} \pi_1(X_i))_{ab}
\]
Then the map is surjective as a composition of two surjective maps and its kernel is an extension of a cotorsion group by a cotorsion group. Thus it is obvious from our second definition of cotorsion groups that the kernel itself is a cotorsion group. Q.E.D.

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Laboratory of Modern Algebra and Applications, St. Petersburg State University, 14th Line, 29b, Saint Petersburg, 199178, Russia

E-mail address: akhtyamoff1997@gmail.com