Research Article

The Numerical Class of a Surface on a Toric Manifold

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In this paper, we give a method to describe the numerical class of a torus invariant surface on a projective toric manifold. As applications, we can classify toric 2-Fano manifolds of the Picard number 2 or of dimension at most 4.

1. Introduction

The classification of smooth toric Fano $d$-folds is an important and interesting problem. They are classified for $d = 3$ by [1, 2], for $d = 4$ by [3, 4], and for $d = 5$ by [5]. In Øbro’s recent excellent paper [6], an algorithm which classifies all the smooth toric Fano $d$-folds for any given natural number $d$ was constructed. So, we can say that the classification of smooth toric Fano varieties is completed.

On the other hand, de Jong and Starr defined a special class of Fano manifolds called 2-Fano manifolds in [7] (see Definition 4.2). So, we consider the problem of the classification of toric 2-Fano manifolds as a next step. For this classification, we give a method to describe the numerical class of a 2-cycle on projective toric manifolds (see Section 3). This method makes calculations of intersection numbers much easier. As results, we obtain the classification of toric 2-Fano manifolds for the case of the Picard number $\rho(X) = 2$ and for the case of $\dim(X) \leq 4$. We remark that Nobili classified smooth toric 2-Fano 4-folds in [8] by using a Maple program.

The contents of this paper are as follows. In Section 2, we define the basic notation such as nef 2-cocycle and 2-Mori cone for our theory. In Section 3, we define a polynomial $I_{Y/X}$ for a torus invariant subvariety $Y \subset X$. This polynomial has all the information of intersection numbers of $Y$ on $X$. So, we can consider this polynomial as the numerical class
2. Preliminaries

In this section, we explain the notation and some basic facts of the toric geometry and the birational geometry used in this paper. See [9–11] for the details.

Let $X$ be a smooth projective toric $d$-fold. Put $Z^2(X)$ to be the free $\mathbb{Z}$-module of 2-cocycles on $X$ and $Z_2(X)$ the free $\mathbb{Z}$-module of 2-cycles on $X$. We define the numerical equivalence “$\equiv$” on $Z^2(X)$ and $Z_2(X)$. A 2-cocycle $E \in Z^2(X)$ is numerically equivalent to 0; that is, $E \equiv 0$ if the intersection number $(E \cdot S) = 0$ for any 2-cycle $S \in Z_2(X)$, while a 2-cycle $S \in Z_2(X)$ is numerically equivalent to 0; that is, $S \equiv 0$ if the intersection number $(E \cdot S) = 0$ for any 2-cocycle $E \in Z^2(X)$. We define $N^2(X) := (Z^2(X) / \equiv) \otimes \mathbb{R}$ and $N_2(X) := (Z_2(X) / \equiv) \otimes \mathbb{R}$.

The following definitions are similar to the case of divisors and curves.

**Definition 2.1.** A 2-cocycle $E \in Z^2(X)$ is a nef 2-cocycle if $(E \cdot S) \geq 0$ for any effective 2-cycle $S \in Z_2(X)$.

**Definition 2.2.** For a projective toric manifold $X$, let $\text{NE}_2(X) \subset N_2(X)$ be the cone of effective 2-cycles; namely,

$$\text{NE}_2(X) := \left\{ \left[ \sum a_i S_i \right] \in N_2(X) \mid a_i \geq 0 \right\}.$$ (2.1)

One calls $\text{NE}_2(X) \subset N_2(X)$ the 2-Mori cone of $X$.

We should remark that $N^l(X)$, $N_l(X)$, and $\text{NE}_l(X)$ can be defined for any $1 \leq l \leq d$ similarly.

The following is an immediate consequence of the projectivity of $X$.

**Proposition 2.3.** $\text{NE}_2(X)$ is a strongly convex cone.

**Proof.** Let $D$ be an ample divisor on $X$. Then, for any $S \in \text{NE}_2(X) \setminus \{0\}$, we have $(D^2 \cdot S) > 0$; namely, $\text{NE}_2(X)$ is strongly convex. \hfill \Box

On the other hand, for the toric case, the following is obvious.

**Proposition 2.4.** Let $X$ be a smooth projective toric $d$-fold. Then, $\text{NE}_2(X)$ is a polyhedral cone.

Thus, $\text{NE}_2(X)$ is a strongly convex polyhedral rational cone similarly as $\text{NE}(X)$. 


We end this section by giving the following simple examples.

**Example 2.5.** (1) If $X = \mathbb{P}^d$, then

$$\text{NE}_2(X) = \mathbb{R}_{\geq 0}[S],$$

where $S$ is a plane in $X$.

(2) If $X = \mathbb{P}^1 \times \mathbb{P}^3$, then

$$\text{NE}_2(X) = \mathbb{R}_{\geq 0}[(\text{a point}) \times \mathbb{P}^2] + \mathbb{R}_{\geq 0}[\mathbb{P}^1 \times \mathbb{P}^1].$$

(3) If $X = \mathbb{P}^2 \times \mathbb{P}^2$, then

$$\text{NE}_2(X) = \mathbb{R}_{\geq 0}[(\text{a point}) \times \mathbb{P}^2] + \mathbb{R}_{\geq 0}[(\mathbb{P}^1 \times \mathbb{P}^1)] + \mathbb{R}_{\geq 0}[(\mathbb{P}^2 \times \text{a point})].$$

### 3. Combinatorial Descriptions

In this section, we establish a method to describe the numerical class of a torus invariant subvariety. We assume that $X = X_\Sigma$ is a smooth projective toric variety.

Let $Y = Y_\sigma \subset X$ be a torus invariant subvariety of dim $Y = l$ associated to a cone $\sigma \in \Sigma$ and $G(\Sigma) = \{x_1, \ldots, x_m\}$. Put

$$I_{Y/X} = I_{Y/X}(X_1, \ldots, X_m) := \sum_{1 \leq i_1, \ldots, i_l \leq m} (D_{x_{i_1}} \cdots D_{x_{i_l}} \cdot Y) X_{i_1} \cdots X_{i_l} \in \mathbb{Z}[X_1, \ldots, X_m],$$

where $D_{x_i}$ is the torus invariant prime divisor corresponding to $x_i$, while $X_i$ is defined to be the independent variable corresponding to $x_i$. We will use this notation throughout this paper.

**Remark 3.1.** $I_{Y/X}$ has all the informations of intersection numbers of $Y$ on $X$. So, we can consider $I_{Y/X}$ as the numerical class of $Y \in N_l(X)$.

**Example 3.2.** Let $C = C_\tau \subset X$ be a torus invariant curve, where $\tau$ is a $(d - 1)$-dimensional cone, that is, a wall in $\Sigma$. In this case,

$$I_{C/X} = \sum_i (D_i \cdot C) X_i$$

is a polynomial of degree 1. On the other hand,

$$\sum_i (D_i \cdot C) x_i = 0$$

is the so-called *Reid’s wall relation* associated to the wall $\tau$ (see [12]); namely, $I_{C/X}$ is calculated from the wall relation immediately.
Example 3.3. When \( Y = X \), \( I_{X/X} \) sometimes becomes a simple shape as follows.

1. **Projective spaces.** Let \( X \) be the \( d \)-dimensional projective space \( \mathbb{P}^d \) and \( G(\Sigma) = \{ x_1 := e_1, \ldots, x_d := e_d, x_{d+1} := -(e_1 + \cdots + e_d) \} \). Then,
   \[
   I_{X/X} = (X_1 + \cdots + X_{d+1})^d.
   \] (3.4)

2. **Hirzebruch surfaces.** Let \( X \) be the Hirzebruch surface \( F_\alpha \) of degree \( \alpha \) and \( G(\Sigma) = \{ x_1 := e_1, x_2 := e_2, x_3 := -e_1 + a e_2, x_4 = -e_2 \} \). Then,
   \[
   I_{X/X} = \alpha(X_2 + X_4)^2 + 2(X_2 + X_4)(X_1 + X_3 - \alpha X_2).
   \] (3.5)

Let \( X \) be a smooth projective toric variety and \( S \subset X \) a torus invariant surface. For some special cases, \( I_{S/X} \) is simply calculated as follows. These are the main theorems of this paper.

**Theorem 3.4.** Suppose \( S \equiv \mathbb{P}^2 \). Let \( C \subset S \) be a torus invariant curve. Then, \( I_{S/X} = (I_{C/X})^2 \).

*Proof.* Let \( \tau = \mathbb{R}_{\geq 0} x_1 + \cdots + \mathbb{R}_{\geq 0} x_{d-2} \in \Sigma \) be the \((d - 2)\)-dimensional cone associated to \( S = S_\tau \), where \( \tau \cap G(\Sigma) = \{ x_1, \ldots, x_{d-2} \} \). Then, there exist exactly three maximal cones \( \tau + \mathbb{R}_{\geq 0} y_1, \tau + \mathbb{R}_{\geq 0} y_2, \) and \( \tau + \mathbb{R}_{\geq 0} y_3 \in \Sigma \) which contain \( \tau \). Put
   \[
   y_1 + y_2 + y_3 + a_1 x_1 + \cdots + a_{d-2} x_{d-2} = 0
   \] (3.6)

to be the wall relation corresponding to \( C \). For the proof, it is sufficient to show that
   \[
   D_x D_w S = a_x a_w,
   \] (3.7)
for any \( z, w \in G(\Sigma) \), where \( D_x \) is the prime torus invariant divisor corresponding to \( z \), while \( a_x \) is the coefficient of \( z \) in the above wall relation.

Suppose that \( z \) or \( w \notin \{ x_1, \ldots, x_{d-2}, y_1, y_2, y_3 \} \); namely, \( a_x = 0 \) or \( a_w = 0 \). In this case, trivially, \( D_x S = 0 \) or \( D_w S = 0 \). So, \( D_x D_w S = a_x a_w = 0 \).

For any \( 1 \leq i, j \leq 3 \),
   \[
   D_{y_i} D_{y_j} S = (D_{y_i} |_S) \left( D_{y_j} |_S \right) = C^2 = 1.
   \] (3.8)

So, the remaining case is \( z \) or \( w \in \{ x_1, \ldots, x_{d-2}\} \). By calculating the rational functions associated to a \( \mathbb{Z} \)-basis \( \{ x_1, \ldots, x_{d-2}, y_1, y_2 \} \) for \( N \), we have the relations
   \[
   D_{x_1} - a_1 D_{y_1} + E_d = 0, \ldots, D_{x_{d-2}} - a_{d-2} D_{y_1} + E_{d-2} = 0,
   \]
   \[
   D_{y_1} - D_{y_3} + E_{d-1} = 0, \quad D_{y_1} - D_{y_2} + E_d = 0
   \] (3.9)
in $\text{Pic} \, X$, where $E_1, \ldots, E_d$ are torus invariant divisors such that $\text{Supp} \, E_i \cap S = \emptyset$ for any $1 \leq i \leq d$. Therefore, we have

$$D_{x_1}S = a_1D_{y_1}S, \ldots, D_{x_{d-2}}S = a_{d-2}D_{y_3}S.$$  \hspace{1cm} (3.10)

By these relations, the equality $D_zD_wS = azaw$ is obvious.

**Theorem 3.5.** Suppose $S \cong F_\alpha$, that is, a Hirzebruch surface of degree $\alpha$. Let $C_{\text{fib}} \subset S$ be a fiber of the projection $S = F_\alpha \rightarrow \mathbb{P}^1$, while let $C_{\text{neg}}$ be the negative section of $S$. Then, $I_S/X = \alpha(I_{C_{\text{fib}}/X})^2 + 2I_{C_{\text{fib}}/X}I_{C_{\text{neg}}/X}$.

**Proof.** Let $\tau = \mathbb{R}_{\geq 0}x_1 + \cdots + \mathbb{R}_{\geq 0}x_{d-2} \in \Sigma$ be the $(d - 2)$-dimensional cone associated to $S = S_\tau$, where $\tau \cap G(\Sigma) = \{x_1, \ldots, x_{d-2}\}$. Then, there exist exactly four maximal cones $\tau + \mathbb{R}_{\geq 0}y_1$, $\tau + \mathbb{R}_{\geq 0}y_2$, $\tau + \mathbb{R}_{\geq 0}y_3$, and $\tau + \mathbb{R}_{\geq 0}y_4 \in \Sigma$ which contain $\tau$. Put

$$y_1 + y_3 - \alpha y_2 + a_1x_1 + \cdots + a_{d-2}x_{d-2} = 0$$ \hspace{1cm} (3.11)

to be the wall relation corresponding to $C_{\text{neg}}$, while

$$y_2 + y_4 + b_1x_1 + \cdots + b_{d-2}x_{d-2} = 0$$ \hspace{1cm} (3.12)

to be the wall relation corresponding to $C_{\text{fib}}$. As in the proof of Theorem 3.4, by calculating the rational functions associated to a $\mathbb{Z}$-basis $\{x_1, \ldots, x_{d-2}, y_1, y_2\}$ for $N$, we have the relations

$$D_{x_1}S = a_1D_{y_1}S + b_1D_{y_3}S, \ldots, D_{x_{d-2}}S = a_{d-2}D_{y_3}S + b_{d-2}D_{y_3}S,$$

$$D_{y_1}S = D_{y_2}S, \quad D_{y_2} = -\alpha D_{y_3}S + D_{y_4}S.$$ \hspace{1cm} (3.13)

First, we remark that, for any $1 \leq i, \ j \leq d$, $j \leq 4$,

$$D_{y_i}D_{y_j}S = (D_{y_i}|_S)(D_{y_j}|_S)$$ \hspace{1cm} (3.14)

on $S$. So, these intersection numbers can be recovered from $I_S/S$ (see Example 3.3).

The above relations say that, for any $1 \leq i, \ j \leq d - 2$,

$$D_{x_i}D_{x_j}S = ab\delta_{ij} + a_i b_j + a_j b_i,$$ \hspace{1cm} (3.15)

while for any $1 \leq i \leq d - 2$,

$$D_{y_i}D_{x_i} = b_i, \quad D_{y_2}D_{x_i} = a_i, \quad D_{y_3}D_{x_i} = b_i, \quad D_{y_4}D_{x_i} = a_i + ab_i.$$ \hspace{1cm} (3.16)
On the other hand, put \( f_1 = f_1(X_1, \ldots, X_{d-2}) := a_1X_1 + \cdots + a_{d-2}X_{d-2} \) and \( f_2 = f_2(X_1, \ldots, X_{d-2}) := b_1X_1 + \cdots + b_{d-2}X_{d-2} \). Then,

\[
\alpha(I_{\text{Coh}/X})^2 + 2I_{\text{Coh}/X}I_{\text{Cone}/X} = \alpha(Y_2 + Y_4 + f_1)^2 + 2(Y_2 + Y_4 + f_1)(Y_1 + Y_3 - aY_2 + f_2) \\
= I_{S/X}(Y_1, Y_2, Y_3, Y_4) + \alpha f_2^2 + 2f_1 f_2 \\
+ 2Y_1 f_2 + 2Y_2 f_1 + 2Y_3 f_2 + Y_4(2f_1 + 2\alpha f_2).
\]

This coincides with \( I_{S/X} \) by the above calculations. \( \square \)

4. 2-Fano Manifolds

As an application of Section 3, we study on toric 2-Fano manifolds in this section. The notion of 2-Fano manifolds was introduced in [7].

Definition 4.1. A smooth projective algebraic variety \( X \) is a Fano manifold if its first Chern class \( c_1(X) = -K_X \) is an ample divisor.

Definition 4.2 (see [7]). A Fano manifold \( X \) is a 2-Fano manifold if its second Chern character \( \text{ch}_2(X) = (1/2)(c_1(X)^2 - 2c_2(X)) \) is a nef 2-cocycle.

Remark 4.3. Since a 2-Fano manifold is a Fano manifold by the definition, for the classification of toric 2-Fano manifolds, all we have to do is to check the list of toric Fano manifolds. The classification of toric Fano manifolds can be done by the algorithm of Øbro [6] for any dimension.

For a projective toric manifold \( X \), one can easily see that \( \text{ch}_2(X) = (1/2) \sum_{i=1}^m D_i^2 \) where \( D_1, \ldots, D_m \) are the torus invariant prime divisors. So, the following is immediate.

Proposition 4.4. For a torus invariant surface \( S \subset X \), put \( I_{S/X} := \sum_{i,j} a_{ij} X_i X_j \). Then, \( (\text{ch}_2(X) \cdot S) = (1/2) \sum_{i=1}^m a_{ii} \).

First of all, we classify toric 2-Fano manifolds of Picard number 2. So, let \( X \) be a complete toric manifold of \( \rho(X) = 2 \). In this case, the structure of \( X \) is very simple as follows.

Theorem 4.5 (see [13]). Every complete toric manifold of the Picard number 2 is a projective space bundle over a projective space.

By Theorem 4.5, we can put

\[
X = X_\Sigma = \mathbb{P}_{\mathbb{P}^{d-1}}(\mathcal{O} \oplus \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{m-1})),
\]

where \( a_1 \geq \cdots \geq a_{m-1} \geq 0 \), \( m + n - 2 = d := \dim X \). Let

\[
x_1 + \cdots + x_m = 0, \quad y_1 + \cdots + y_n = a_1 x_1 + \cdots + a_{m-1} x_{m-1}
\]
be the wall relations of $\Sigma$ which correspond to the extremal rays of $\text{NE}(X)$, where
\[ G(\Sigma) = \{ x_1, \ldots, x_m, y_1, \ldots, y_n \}. \]  
(4.4)

Let $C_1$ and $C_2$ be the extremal torus invariant curves corresponding to the wall relations (4.2) and (4.3), respectively.

First, we determine the extremal rays of $\text{NE}_2(X)$. By calculating the rational functions for a $\mathbb{Z}$-basis $\{ x_1, \ldots, x_{m-1}, y_1, \ldots, y_{n-1} \}$, we have the relations
\begin{align*}
D_1 - D_m + a_1 E_n = 0, & \quad \ldots, D_{m-1} - D_m + a_{m-1} E_n = 0, \\
E_1 - E_n = 0, & \quad \ldots, E_{n-1} - E_n = 0
\end{align*}
(4.5)
in $\mathbb{N}^1(X)$, where $D_1, \ldots, D_m, E_1, \ldots, E_n$ are torus invariant prime divisors corresponding to $x_1, \ldots, x_m, y_1, \ldots, y_n$. Therefore, for $1 \leq i, j \leq m-1$,
\begin{align*}
D_j = D_i + (a_i - a_j) E_n, \\
E_1 = E_2 = \cdots = E_n.
\end{align*}
(4.6)

On the other hand, every $(d-2)$-dimensional cone $\tau \in \Sigma$ is expressed as
\[ \tau = \mathbb{R}_{\geq 0} x_{i_1} + \cdots + \mathbb{R}_{\geq 0} x_{i_k} + \mathbb{R}_{\geq 0} y_{j_1} + \cdots + \mathbb{R}_{\geq 0} y_{j_l}, \]  
(4.7)
for some $1 \leq i_1 < \cdots < i_k \leq m$, $1 \leq j_1 < \cdots < j_l \leq n$ such that $k < m$, $l < n$, and $k + l = d - 2$. So, the corresponding torus invariant surface $S_\tau$ is expressed as
\[ S_\tau = D_{i_1} \cdots D_{i_k} E_{j_1} \cdots E_{j_l} \in \text{N}_2(X). \]  
(4.8)

By using (4.6), any $S_\tau$ is expressed as a linear combination of 2-cycles:
\[ D_1 \cdots D_p E^q \quad (p \leq m - 1, \ q \leq n - 1, \ p + q = d - 2), \]  
(4.9)
whose coefficients are nonnegative, because $i < j$ implies $a_i - a_j \geq 0$. Moreover, since $D_1 \cdots D_m = E_1 \cdots E_n = 0$ by wall relations (4.2) and (4.3), the possibilities for the generators of $\text{NE}_2(X)$ are
\begin{align*}
S_1 := D_1 \cdots D_{m-3} E^{n-1}, & \quad S_2 := D_1 \cdots D_{m-2} E^{n-2}, \quad \text{or} \\
S_3 := D_1 \cdots D_{m-1} E^{n-3}.
\end{align*}
(4.10)

In fact, the following hold:

\begin{align*}
\text{NE}_2(X) &= \mathbb{R}_{\geq 0} S_1 + \mathbb{R}_{\geq 0} S_2 + \mathbb{R}_{\geq 0} S_3 \quad \text{if } m \geq 3, \ n \geq 3. \\
\text{NE}_2(X) &= \mathbb{R}_{\geq 0} S_2 + \mathbb{R}_{\geq 0} S_3 \quad \text{if } m = 2, \ n \geq 3. \\
\text{NE}_2(X) &= \mathbb{R}_{\geq 0} S_1 + \mathbb{R}_{\geq 0} S_2 \quad \text{if } m \geq 3, \ n = 2.
\end{align*}
(4.11)
For each case, \( \dim N_2(X) = 3 \), \( \dim N_2(X) = 2 \), and \( \dim N_2(X) = 2 \), respectively. So, \( \text{NE} X \) is a \textit{simplicial} cone for each case, and \( S_1 \), \( S_2 \), and \( S_3 \) are extremal surfaces.

Next, we will check when \( X \) becomes a 2-Fano manifold.

So, let \( C_2 \) be the torus invariant curve which generates the extremal ray corresponding to the wall relation (4.3). Then,

\[
(-K_X \cdot C_2) = n - (a_1 + \cdots + a_{m-1}).
\]  

Therefore, \( X \) is a Fano manifold if and only if

\[
n - (a_1 + \cdots + a_{m-1}) > 0.
\]

Since \( S_1 \equiv S_2 \equiv \mathbb{P}^2 \), \( (\text{ch}_2(X) \cdot S_1) \geq 0 \) and \( (\text{ch}_2(X) \cdot S_3) \geq 0 \) are trivial by Theorem 3.4. On the other hand, we can easily check that \( S_2 \equiv F_{a_{m-1}} \). By Theorem 3.5, we have

\[
I_{S_2} = a_{m-1}(I_{C_2})^2 + 2I_{C_1}I_{C_2} = a_{m-1}(X_1 + \cdots + X_m)^2 + 2(X_1 + \cdots + X_m)(Y_1 + \cdots + Y_n - (a_1X_1 + \cdots + a_{m-1}X_{m-1})).
\]

So, we obtain

\[
(\text{ch}_2(X) \cdot S_2) = ma_{m-1} - 2(a_1 + \cdots + a_{m-1}).
\]  

(4.15)

In (4.15), suppose that \( m \geq 3 \) and \( (\text{ch}_2(X) \cdot S_2) \geq 0 \). Then,

\[
(\text{ch}_2(X) \cdot S_2) = (m - 2)a_{m-1} - 2(a_1 + \cdots + a_{m-2}).
\]  

(4.16)

The assumption \( a_1 \geq \cdots \geq a_{m-1} \geq 0 \) says that \( a_1 = \cdots = a_{m-1} = 0 \); that is, \( X \equiv \mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \). On the other hand, suppose that \( m = 2 \) in (4.15). Then, \( (\text{ch}_2(X) \cdot S_2) = 0 \); that is, \( \text{ch}_2(X) \) is nef.

By (4.13), we can summarize as follows.

**Theorem 4.6.** If \( X \) is a toric 2-Fano manifold of the Picard number 2, then \( X \) is one of the following:

1. A direct product of projective spaces,
2. \( \mathbb{P}_{r-1}(O \oplus O(a)) \) (1 \( \leq a \leq d - 1 \)).

**Remark 4.7.** This calculation shows that there exist infinitely many projective toric manifolds of fixed dimension \( d \) whose second Chern character is nef.

Next, we consider the classification of toric 2-Fano manifolds of a fixed dimension \( d \). For \( d \leq 4 \), fortunately, these classifications can be done by only Theorems 3.4 and 3.5. Table 1 is the classification list (see [8] for the detail).

Since there exist 124 smooth toric Fano 4-folds, it is hard to check all the smooth toric Fano 4-folds. However, by using the following trivial Lemma 4.8, we can omit a large part of the calculations.
Lemma 4.8. Let $X$ be a 4-dimensional toric 2-Fano manifold. Then,

$$c_1^4(X) - 2c_2^2(X)c_2(X) \geq 0. \quad (4.17)$$

For any smooth toric Fano 4-fold $X$, $c_1^4(X)$ and $c_2^2(X)c_2(X)$ are calculated in [3]. One can see that for 52 smooth toric Fano 4-folds, they are not 2-Fano manifolds by Lemma 4.8.

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