Financial Time Series and Statistical Mechanics

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Abstract. A few characteristic exponents describing power law behaviors of roughness, coherence and persistence in stochastic time series are compared to each other. Relevant techniques for analyzing such time series are recalled in order to distinguish how the various exponents are measured, and what basic differences exist between each one. Financial time series, like the JPY/DEM and USD/DEM exchange rates are used for illustration, but mathematical ones, like (fractional or not) Brownian walks can be used also as indicated.

1 Introduction

A great challenge in modern times is the construction of predictive theories for nonlinear dynamical systems for which the evolution equations are barely known, if known at all. General or so-called universal laws are aimed at from very noisy data. A universal law should hold for different systems characterized by different models, but leading to similar basic parameters, like the critical exponents, depending only on the dimensionality of the system and the number of components of the order parameter. This in fine leads to a predictive value or power of the universal laws.

In order to obtain universal laws in stochastic systems one has to distinguish true noise from chaotic behavior, and sort out coherent sequences from random ones in experimentally obtained signals $^\ddagger$. The stochastic aspects are not only found in the statistical distribution of underlying frequencies characterizing the Fourier transform of the signal, but also in the amplitude fluctuation distribution and high moments or correlation functions.

Following the scaling hypothesis idea, neither time nor length scales have to be considered $^\ddagger$.$^\ddagger$. Henceforth the fractal geometry is a perfect framework for studies of stochastic systems which do not appear at first to have underlying scales. A universal law can be a so-called scaling law if a $\log[y(x)]$ vs. $\log(x)$ plot gives a straight line (over several decades if possible) leading to a slope measurement and the exponent characterizing the power law.

When examining such phenomena, it is often recognized that some coherent factor is implied. Yet there are states which cannot be reached without going through intricate evolutions, implying concepts like transience and persistence, - well known if one recalls the turbulence phenomenon and its basic
theoretical understanding \[5\]. Finally, the apparent roughness of the signal can be put in mathematical terms. These concepts are briefly elaborated upon in Sect. 2.

In the \( y(x) \) function, \( y \) and \( x \) can be many "things". However the relevant outlined concepts can be well illustrated when \( x \) is the time \( t \) variable. In so doing time series serve as fundamental testing grounds. Several series can be found in the literature. Financial time series and mathematical ones based on the Weierstrass-Mandelbrot function \[6\] describing fractional (or not) Brownian walks can be used for illustration. The number of points should be large enough to obtain small error bars. A few useful references, among many others, discussing tests and other basic or technical considerations on non linear time series analysis are to be found in \[7,8,9,10,11\].

Mathematical series, like (fractional or not) Brownian motions (\( Bm \)) and practical ones, like financial time series, are thus of interest for discussion or illustrations as done in Sect.3. There are several papers and books for interest geared at financial time series analysis ... and forecasting. Again not all can be mentioned, though see \[13,14,15\]. On a more general basis, an introduction to financial market analysis \( \textit{per se} \), can be found in \[16\].

Nevertheless it should be considered that any "scaling exponent" should be robust in a statistical sense with respect to small changes in the data or in the data analysis technique. If this is so some physics considerations and modeling can be pursued. One question is often raised for statistical purposes whether the data is \( \textit{stationary} \) or not, i.e. whether the analyzed raw signal, or any of its combinations depend on the (time) origin of the series. This theoretical question seems somewhat practically irrelevant in financial, meteorological, ... sciences because the data is obviously \( \textit{never} \) stationary. In fact, in such new "exotic applications of physics" a restricted criterion for stationarity is thought to be sufficient: if the data statistical mean and the whatever-extracted-parameter do not change too much (up to some statistical significance \[7\]) the data is called \( \textit{quasi} \) − \( \textit{stationary} \). If so it can be next offered for fundamental investigations. Thereafter, the prefix "quasi" is immediately forgotten and not written anymore.

Several characteristics plots leading to \( \textit{universality considerations} \), thus fractal-like exponents are first to be recalled. They are obtained from different techniques which are briefly reviewed for completeness either in Sect. 3 or in Appendices. Some technical materials can be usefully found in \[17,18,19\]. This should serve to distinguish how the various exponents are measured, and what basic differences exist between each one. The numerical values pertaining to the words (i) persistence, (ii) coherence, and (iii) roughness will be given and related to each other. From a practical point of view, as illustrated in the exercise session which was taking place after the lecture, the cases of foreign exchange currency rates, i.e. DEM/USD and DEM/JPY are used. They are shown in Fig.1 and 2 for a time interval ranging from Jan. 01, 1993 till June 30, 2000.
For an adequate perspective, let it be recalled that such modern concepts of statistical physics have been recently applied in analyzing time series outside finance as well, like in particular those arising from biology [20,21], medicine [22], meteorology [23], electronics [24], image recognition [25], ... again without intending to list all references of interest as should be done in a (longer) review paper. Many examples can also be found in this book through contributions by world specialists of computer simulations.
2 Phase, amplitude and frequency revisited

2.1 Coherence

When mentioning the word coherence to any student in physics, he/she is immediately thinking about "lasers", that is where the word has been most striking in any scientific memory. It is recalled that the laser is a so interesting instrument because all photons are emitted in phase, more exactly the difference in phases between emitted waves is a constant in time. Thus there is a so-called coherence in the light beam. They are other cases in which phase coherence occurs, let it be recalled that light bugs are emitting coherently, young girls in dormitories have their period in a coherent way, driving conditions are best if some coherence is imposed, crystals have a better shape and properties if they are grown in a "coherent way"; sand piles and stock markets seem also to have coherent properties.

2.2 Roughness

No need to say that a wave is characterized by its amplitude which has also some known importance in measurements indeed, be it often the measure of an intensity (the square of the amplitude). More interestingly one can define the roughness of a profile by observing how the signal amplitude varies in time (and space if necessary), in particular the correlation between the various amplitude fluctuations.

2.3 Persistence

On the other hand, it can be easily shown that a periodic signal can be decomposed into a series of sin or cos for which the frequencies are in arithmetic order. Thus, the third "parameter" of the wave is its frequency. Some (regular) frequency effect is surely apparent in all cycling phenomena, starting from biology, climatology, meteorology, astronomy, but also stock markets, foreign currency exchange markets, tectonics events, traffic and turbulence, and politics. For non periodic signals, the Fourier transform has been introduced in order to sort out the distribution of frequencies of interest, i.e. the "density of modes". The distribution defines the sort of persistence of a phenomenon. It might be also examined whether the frequencies are distributed in a geometrical progression, rather than following an ordinary/usual arithmetic progression, i.e. whether the phenomena might be log-periodical, like in antennas, earthquakes and stock market crashes.

3 Power Law Exponents
3.1 Persistence and Spectral Density

Data from a (usually discrete) time series $y(t)$ are one dimensional sets and are more simple to analyze at first than spatial ones [22,33]. Below and for simplicity one considers that the measurements are taken at equal time intervals. Thus for financial time series, there is no holiday nor week-end. A more general situation is hardly necessary here. Two classic examples of a mathematical univariate stochastic time series are those resulting from Brownian motion and Levy walk cases [34]. In both cases, the power spectral density $S_1(f)$ of the (supposed to be self-affine) time series $y(t)$ has a single power-law dependence on the frequency $f$,

$$S_1(f) \sim f^{-\beta}, \quad (1)$$

following from the Fourier transform

$$S_1(f) = \int dt \, e^{if t} \, y(t). \quad (2)$$

For $y(t)$ one can use the Weierstrass-Mandelbrot (fractal) function \[1\]

$$W(t) = \sum_m \gamma^{(2-D)m} \left[ 1 - e^{i \gamma^n t} \right] e^{i \phi_n t}, \quad (3)$$

with $\gamma > 1$, and $1 < D < 2$. The phase $\phi_n$ can be stochastic or deterministic. For illustration, Berry and Lewis [3] took $\phi_n = n \mu$, with $\mu = 0$ or $\pi$. The function obeys

$$W(\gamma t) = \gamma^{(2-D)} e^{-i \gamma^n} \, W(t), \quad (4)$$

is stationary, and its trend, in the deterministic cases, is given by

$$W(t) \sim W(n) \sim t^{2-D}/\ln \gamma. \quad (5)$$

The power spectrum is easily calculated [3] to be

$$S_1(f) \sim \cdots (1/\ln \gamma)^{f^{2D-5}}. \quad (6)$$

One could search whether the function moments obey power laws with characterizing exponents. Equation(1) allows one to put the phenomena into the self-affine class of persistent phenomena characterized by the $\beta$ value. The range over which $\beta$ is well defined in Eq.(1) indicates the range of the persistence in the time series. A Brownian motion is characterized by $\beta = 2$, and a white noise by $\beta = 0$. See a very interesting set of such mathematical signals and the corresponding power spectrum in [12].

\[1\] Notice that the differences between adjacent values of a Brownian motion amplitude result in white noise.
Fig. 3. The power spectrum of the DEM/USD exchange rate for the time interval data in Fig.1

\[ S(f) = 10^\beta \pm 0.36 \]

\[ \beta = 1.85 \pm 0.36 \]

Fig. 4. The power spectrum of the DEM/JPY exchange rate for the time interval data in Fig.2

\[ S(f) = 10^\beta \pm 0.41 \]

\[ \beta = 1.82 \pm 0.41 \]
The Fourier transform, or power spectrum, of the financial signals used for illustration here are found in Fig. 3 and Fig. 4. Notice the large error bars, allowing to estimate that $\beta$ is about equal to 2, as for a trivial Brownian motion case. However the coherence and/or roughness aspect are masked in this one-shot analysis. Only the persistence behavior is touched upon.

If the distribution of fluctuations is not a power law, or if marked deviations exist, say the statistical correlation coefficient is less than 0.99, indicating that a mere power law for $S(f)$ is doubtful, a more thorough search of the basic frequencies is in order. A crucial step is to extract deterministic or stochastic components, e.g. the stochastic aspects found in the statistical distribution of values show its persistence to be either nonexistent (white noise case) or existent, i.e. $\beta \neq 2$. If so the persistence can be qualitatively thought to be strong or weak. The Fourier transform can sometimes indicate the presence of specific frequencies, much more abundant than others, in particular if cycles exist, as in meteorology and climatology.

The range of the persistence is obtained from the correlations between events. A "short" or "long range" is checked through the autocorrelation function, usually $c_1$. This function is a particular case of the so-called "$q$–th order structure function" \[\tau\] or "$q$–th order height-height correlation function" of the (normalized) time-dependent signal $y(t_i)$,

$$c_q(\tau) = \langle |y(t_{i+r}) - y(t_i)|^q \rangle / \langle |y(t_i)|^q \rangle_{\tau},$$

where only non-zero terms are considered in the average $\langle . \rangle_{\tau}$ taken over all couples $(t_{i+r}, t_i)$ such that $\tau = |t_{i+r} - t_i|$. In so doing one can obtain a set of exponents $\beta_q$.

If the autocorrelation is larger than unity for some long time $t$ one can talk about strong persistence, otherwise it is weak. This criterion defines the scale of time $t$ in $y(t)$ for which there is long or short (time) range persistence. Notice that the lower limit of the time scale is due to the discretization step, and this sets the highest frequency to be the inverse of twice the discretization interval. The upper limit is obvious.

3.2 Roughness, Fractal Dimension, Hurst exponent and Detrended Fluctuation Analysis

The fractal dimension is often used to characterize the roughness of profiles \[\text{cite refA1,addison,falconer,roughness} \] D is related to the exponent $\beta$ by

$$\beta = 5 - 2D.$$
A Brownian motion is characterized by \( D = 3/2 \), and a white noise by \( D = 2.0 \) \[12\].

Another "measure" of a signal roughness is sometimes given by the Hurst \( H_u \) exponent, first defined in the "rescale range theory" (of Hurst \[37,38\] ) who measured the Nile flooding and drought amplitudes. The Hurst method consists in listing the differences between the observed value at a discrete time \( t \) over an interval with size \( N \) on which the mean has been taken. The upper (\( y_M \)) and lower (\( y_m \)) values in that interval define the range \( R_N = y_M - y_m \). The root mean square deviation \( S_N \) being also calculated, the "rescaled range" is \( R_N/S_N \) is expected to behave like \( N^{H_u} \). This means that for a (discrete) self-affine signal \( y(t) \), the neighborhood of a particular point on the signal can be rescaled by a factor \( b \) using the roughness (or Hurst \[3,4\]) exponent \( H_u \) and defining the new signal \( b^{-H_u}y(bt) \). For the exponent value \( H_u \), the frequency dependence of the signal so obtained should be undistinguishable from the original one, i.e. \( y(t) \).

The roughness (Hurst) exponent \( H_u \) can be calculated from the height-height correlation function \( c_1(\tau) \) supposed to behave like

\[
c_1(\tau) = \langle |y(n_{r+\tau}) - y(n_i)| \rangle_\tau \sim \tau^{H_1}
\]

whereas

\[
H_u = 1 + H_1,
\]

rather than from the box counting method. For a persistent signal, \( H_1 > 1/2 \); for an anti-persistent signal, \( H_1 < 1/2 \). Flandrin has theoretically proved \[39\] that

\[
\beta = 2H_u - 1,
\]

thus \( \beta = 1 + 2H_1 \). This implies that the classical random walk (Brownian motion) is such that \( H_u = 3/2 \). It is clear that

\[
D = 3 - H_u.
\]

Fractional Brownian motion values are practically found to lie between 1 and 2 \[17,18,28\]. Since a white noise is a truly random process, it can be concluded that \( H_u = 1.5 \) implies an uncorrelated time series \[34\].

Thus \( D > 1.5 \), or \( H_u < 1.5 \) implies antipersistence and \( D < 1.5 \), or \( H_u > 1.5 \) implies persistence. From preimposed \( H_u \) values of a fractional Brownian motion series, it is found that the equality here usually holds true in a very limited range and \( \beta \) only slowly converges toward the value \( H_u \) \[12\].

The inertia axes of the 2-variability diagram \[13,14,15\] seem to be related to these values and could be used for fast measurements as well.

The above results can be compared to those obtained from the Detrended Fluctuation Analysis \[23,13\] (DFA) method. DFA \[20\] consists in dividing a random variable sequence \( y(n) \) over \( N \) points into \( N/\tau \) boxes, each containing
τ points. The best linear trend \( z(n) = an + b \) in each box is defined. The fluctuation function \( F(\tau) \) is then calculated following

\[
F^2(\tau) = \frac{1}{\tau} \sum_{n=(k-1)\tau+1}^{k\tau} |y(n) - z(n)|^2, \quad k = 1, 2, \cdots, N/\tau. \tag{13}
\]

Averaging \( F(\tau)^2 \) over the \( N/\tau \) intervals gives the fluctuations \( \langle F(\tau)^2 \rangle \) as a function of \( \tau \). If the \( y(n) \) data are random uncorrelated variables or short range correlated variables, the behavior is expected to be a power law

\[
\langle F^2 \rangle^{1/2} \sim \tau^{H_a} \tag{14}
\]

with \( H_a \) different from 0.5.

The exponent \( H_a \) is so-labelled for Hausdorff \([3,34]\). It is expected, not always proved as emphasized by \([6,44]\) that

\[
H_a = 2 - D, \tag{15}
\]

where \( D \) is the self-affine fractal dimension \([3,34]\). It is immediately seen that

\[
\beta = 1 + 2H_a. \tag{16}
\]

For Brownian motion, \( H_a = 0.5 \), while for white noise \( H_a = 0 \) and \( D = 2 \).

![Fig. 5. The DFA result for the DEM/USD exchange rate for the time interval data in Fig.1](image)

The \( DFA \) log-log plots of the DEM/JPY and DEM/USD exchange rates are given in Fig. 5 and Fig. 6. It is seen that the value of \( H_a \) fulfills the above
Fig. 6. The DFA result for the DEM/JPY exchange rate for the time interval data in Fig.2

relations for the DEM/JPY and DEM/USD data, since $H_a = 0.55$. Both $H_a$ and $D$ readily measure the roughness and persistence strength. Fractional Brownian motion $H_a$ values are found to lie between 0 and 1. The effect of a trend is supposedly eliminated here. However only the linear or cubic detrending have been studied to my knowledge [43]. Other trends, emphasizing some characteristic frequency, like seasonal cycles, could be further studied.

A generalized Hurst exponent $H(q)$ is defined through the relation

$$c_q(\tau) \propto \tau^{qH(q)}, \quad q \geq 0$$

where $c_q(\tau)$ has been defined here above

The intermittency of the signal can be studied through the so-called singular measure analysis of the small-scale gradient field obtained from the data through

$$\varepsilon(r;l) = \frac{r^{-1} \sum_{i=r}^{l+r-1} |y(t_{i+r}) - y(t_i)|}{<|y(t_{i+r}) - y(t_i)|>}$$

with

$$i = 0, \ldots, A - r$$

and

$$r = 1, 2, \ldots, A = 2^m ,$$

where $m$ is an integer. The scaling properties of the generating function are then searched for through the equation

$$\chi_q(\tau) = <\varepsilon(r;l)^q > \sim \tau^{-K(q)}, \quad q \geq 0,$$
with $\tau$ as defined above.

The $K(q)$-exponent is closely related to the generalized dimensions $D_q = 1 - K(q)/(q - 1)$ [45]. The nonlinearity of both characteristics exponents, $qH(q)$ and $K(q)$, describes the multifractality of the signal. If a linear dependence is obtained, then the signal is monofractal or in other words, the data follows a simple scaling law for these values of $q$. Thus the exponent [35,46]

$$C_1 = \frac{dK_q}{dq} \bigg|_{q=1}$$

(22)

is a measure of the intermittency lying in the signal $y(n)$ and can be numerically estimated by measuring $K_q$ around $q = 1$. Some conjecture on the role/meaning of $H_1$ is found in [28]. From some financial and political data analysis it seems that $H_1$ is a measure of the information entropy of the system.

4 Conclusion

It has been emphasized that to analyze stochastic time series, like those describing fractional Brownian motion and foreign currency exchange rates reduces to examining the distribution of and correlations between amplitudes, frequencies an phases of harmonic-like components of the signal. Due to some scaling hypothesis, characteristic exponents can be obtained to describe power laws. The usefulness of such exponents serves in determining universality classes, and in fine building physical or algorithmic models. In the case of financial times series, the exponents can even serve into imagining some investment strategy [28]. Notice that due to the non stationarity of the data, such exponents vary with time, and multifractal concepts must be brought in at a refining stage, - including in an investment strategy.

In that spirit, let it be emphasized here the analogy between a $H_1, C_1$ diagram and the $\omega, k$ diagram of dynamical second order phase transitions [47]. In the latter the frequency and the phase of a time signal are considered on the same footing, and encompass the critical and hydrodynamical regions. In the present cases an analogous diagram relates the roughness and intermittency. This has been already examined in [48,49].

Among other various physical data analysis techniques which have been recently presented in order to obtain some information on the deterministic and/or chaotic content of univariate data, let us point out the wavelet technique which has been considerably used (several references exist, see below) including for DNA and meteorology studies [46,50,51,52]. The $H_1, C_1$ technique used in turbulence and meteorology [53] is also somewhat appropriate.

There are many other techniques which are not mentioned here, like the weighted fixed point [55], and the time-delay embedding [56]. Surely several others have been used, but only those relevant for the present purpose have
Table 1. Values of the most relevant exponents in various regimes (i.e., stationary, persistent, antipersistent) of univariate stochastic series: $D$: fractal dimension; $H_a$: Hausdorff measure; $H_u$: Hurst exponent; $\alpha$: from DFA technique; $\beta$: power spectrum exponent; $WN$: white noise, $(f)Bm$: (fractional) Brownian motion; $flat$: flat spectrum.

| Signal name | $D$ | $H_a$ | $H_u$ | $\alpha$ | $\beta$ |
|-------------|-----|-------|-------|---------|--------|
|             |     |       |       |         |        |
| $ WN $      | 2   | 0     | 1     |         |        |
| $ fBm $     | 3/2 | 0.5   | 3/2   |         |        |
| $ Bm $      |     | 0.5   | 2     |         |        |
| $ flat $    | 1   | 1     | 2     | $superpersistence$ | 3 | - |

been fully mentioned hereabove. As a summary of the above, Table 1 indicates the range of values found for different signals and their relationship to stationarity, persistence and coherence. In the many years to come, it seems relevant to ask for more data on multivariate functions, thus extending the above considerations to other real mathematical and physical cases in higher dimensions.

Finally, two short Appendices should follow in order to remain consistent with the oral lectures. It was shown that another time series “analysis” technique is often used by experts for some predictability purpose, i.e. the moving average technique. It is briefly discussed in Appendix A. It has served in the Übungen. Also, these lecture notes would be incomplete without mentioning the intrinsic discrete scale invariance implication in time series. Such a substructure leads to log-periodic oscillations in the time series, whence to fascinating effects and surprises in predicting crash-like events. This is mentioned in Appendix B.

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5 Appendix A : moving averages

To take into account the trend can be shown to be irrelevant for such short range correlation events. The trend is anyway quite ill defined since it is a statistical mean, and thus depends on the size of the interval, i.e. the number of data points which is taken into account. Nevertheless many technical analyses rely on signal averages over various time intervals, like the moving average method \[57\]. These methods should be examined from a physical point of view. An interesting observation has resulted from checking the density of intersections of such mean values over different time interval windows, which are continuously shifted. This corresponds to obtaining a spectrum of the so-called moving averages \[57\], used by analysts in order to point to "gold" or "death" crosses in a market. The density $\rho$ of crossing points between any two moving averages is obviously a measure of long-range power-law correlations in the signal. It has been found that $\rho$ is a symmetric function of $\Delta T$, i.e. the difference between the interval sizes on which the averages are taken, and it has a simple power law form \[58,59\]. This leads to a very fast and rather reliable measure of the fractal dimension of the signal. The method can be easily implemented for obtaining the time evolution of $D$, thus for elementary investment strategies.

6 Appendix B : discrete scale invariance

It has been proposed that an economic index $y(t)$ follows a complex power law \[60,61\], i.e.

$$y(t) = A + B (t_c - t)^{-m} \left[ 1 + C \cos(\omega \ln((t_c - t)/t_c) + \phi) \right]$$  \hspace{1cm} \hspace{1cm} (23)

for $t < t_c$, where $t_c$ is the crash-time or rupture point, $A$, $B$, $m$, $C$, $\omega$, $\phi$ are parameters. This index evolution is a power law ($m$) divergence (for $m > 0$) on which log-periodic ($\omega$) oscillations are taking place. The law for $y(t)$ diverges at $t = t_c$ with an exponent $m$ (for $m > 0$) while the period of the oscillations converges to the rupture point at $t = t_c$. This law is similar to that of critical points at so-called second order phase transitions \[17\], but generalizes the scaleless situation for cases in which discrete scale invariance is presupposed \[62\]. This relationship was already proposed in order to fit experimental measurements of sound wave rate emissions prior to the rupture of heterogeneous composite stressed up to failure \[63\]. The same type of complex power law behavior has been observed as a precursor of the Kobe earthquake in Japan \[54\].

Fits using Eq.(23) were performed on the S&P500 data \[60,61\] for the period preceding the 1987 October crash. The parameter values have not been found to be robust against small perturbations, like a change in the phase of the signal. It is known in fact that a nonlinear seven parameter fit
is highly unstable from a numerical point of view. Indeed, eliminating the contribution of the oscillations in Eq.(1), i.e. setting $C=0$, implies that the best fit leads to an exponent $m = 0.7$ quite larger than $m = 0.33$ for $C \neq 0$ \cite{10}. Feigenbaum and Freund \cite{11} also reported various values of $m$ ranging from 0.53 to 0.06 for various indexes and events (upsurges and crashes).

Universality in this case means that the value of $m$ should be the same for any crash and for any index. In so doing a single model should describe the phase transition, and the exponent would define the model and be the only parameter. A limiting case of a power law behavior is the logarithmic behavior, corresponding to $m = 0$, i.e. the divergence of the index $y$ for $t$ close to $t_c$ should be

$$y(t) = A + B \ln((t_c - t)/t_c) \left[1 + C \cos(\omega \ln((t_c - t)/t_c) + \phi)\right]$$  \hspace{1cm} (24)

This logarithmic behavior is known in physics as characterizing the specific heat ("four point correlation function") of the Ising model, and the Kosterlitz-Thouless phase transition \cite{15} in spatial dimensions equal to two. They are thus specific to systems with a low order dimension of the order parameter. It is nevertheless a smooth transition. The mean value of the order parameter \cite{17} is not defined over long range scales, but a phase transition nevertheless exists because there is some ordered state on small scales. In addition to the physical interpretation of the latter relationship, the advantages are that (i) the number of parameters is reduced by one, and (ii) the log-divergence seems to be close to reality.

In order to test the validity of Eq.(24) in the vicinity of crashes, we have separated the problems of the divergence itself and the oscillation convergences on the other hand, in order to extract two values for the rupture point $t_c$: (i) $t_{c, div}$ for the power (or logarithmic) divergence and (ii) $t_{c, osc}$ for the oscillation convergence. In so doing the long range and short range fluctuation scales are examined on an equal footing. The final $t_c$ is obtained at the intersection of two straight lines, by successive iteration fits. The results of the fit as well as the correlation fitting factor $R$ have been given in \cite{66,67}.

A technical point is in order: The rupture point $t_{c, osc}$ is estimated by selecting the maxima and the minima of the oscillations through a double envelope technique \cite{67}. Finally notice that the oscillation basic frequency depends on the connectivity of the underlying space \cite{18,19}. The log-periodic behavior also corresponds to a complex fractal dimension \cite{62,68}.

References

* GRASP = Group for Research in Applied Statistical Physics;
SUPRAS = Services Universitaires Pour la Recherche et les Applications en Supraconductivité
1. C.J. Cellucci, A.M. Albano, P.E. Rapp, R.A. Pittenger, and R.C. Josiassen, \textit{Chaos} \textbf{7}, 414 (1997)
2. M. Schroeder, *Fractals, Chaos and Power Laws*, (W.H. Freeman and Co., New York, 1991)
3. P. S. Addison, *Fractals and Chaos*, (Inst. of Phys., Bristol, 1997)
4. K. J. Falconer, *The Geometry of Fractal Sets*, (Cambridge Univ. Press, Cambridge, 1985)
5. P. Bergé, Y. Pomeau, and Ch. Vidal, *L'ordre dans le chaos*, (Hermann, Paris, 1984).
6. M.V. Berry and Z.V. Lewis, *Proc. R. Soc. Lond. A* 370, 459 (1980)
7. S. L. Meyer, *Data Analysis for Scientists and Engineers*, (Wiley, New York, 1975).
8. P.E. Rapp, *Integrat. Physiol. Behav. Sci*. 29, 311 (1994)
9. Th. Schreiber, *Phys. Rep*. 308, 1 (1999)
10. H. Kantz and Th. Schreiber, *Nonlinear Time Series Analysis*, (Cambridge Univ. Press, Cambridge, 1997).
11. C. Diks, *Nonlinear Time Series Analysis*, (World Scient., Singapore, 1999).
12. B.D. Malamud and D.L. Turcotte, *J. Stat. Plann. Infer.* 80, 173 (1999)
13. P.J. Brockwell and R.A. Davis, *Introduction to time series and forecasting*, Springer Text in Statistics, (Springer, Berlin, 1998).
14. Ph. Franses, *Time Series Models for Business and Economic Forecasting*, (Cambridge U. Press, Cambridge, 1998)
15. Ch. Gourieroux and A. Monfort, *Time Series and Dynamic Models*, (Cambridge U. Press, Cambridge, 1997)
16. D. Blake, *Financial Market Analysis*, (Wiley, New York, 2000).
17. M. Ausloos, N. Vandewalle and K. Ivanova, in *Noise of frequencies in oscillators and dynamics of algebraic numbers*, M. Planat, Ed. (Springer, Berlin, 2000) pp. 156-171.
18. M. Ausloos, N. Vandewalle, Ph. Boveroux, A. Minguet, and K. Ivanova, in *Applications of Statistical Physics*, A. Gadomski, J. Kertesz, H. E. Stanley and N. Vandewalle, *Physica A* 274 229 (1999)
19. M. Ausloos, in A. Pekalski Ed., Proc. Ladek Zdroj Conference on *Exotic Statistical Physics, Physica A*, 285 48 (2000)
20. R.N.Mantegna, S.V.Buldyrev, A.L.Goldberger, S.Havlin, C.-K.Peng, M.Simmons and H.E.Stanley, *Phys. Rev. Lett.* 73, 3169 (1994)
21. S. Mercik, K. Weron and Z. Siwy, *Phys. Rev. E* 60, 743 (1999)
22. B.J. West, R. Zhang, A.W. Sanders, S. Miniyar, J. H. Zuckerman, B.D. Levine, *Physica A* 270 552 (1999)
23. K. Ivanova and M. Ausloos *Physica A* 274 349 (1999)
24. N. Vandewalle, M. Ausloos, M. Houssa, P.W. Mertens and M.M. Heyns, *Appl. Phys. Lett*. 74 1579 (1999)
25. T. Lundahl, W.J. Ohley, S.M. Kay, and R. Siffert *IEEE Trans. Med. Imag. MI-5*, 152 (1986)
26. J. F. Lennon, in *Pour la Science*, (special issue, Jan. 1995 ) p. 111
27. as on belgian highways to and from the seashore during summer time at peak hours
28. N. Vandewalle and M. Ausloos, *Physica A* 246, 454 (1997)
29. R. Mantegna and N. Vandewalle, *Physica A* xxx, in press (2000)
30. A. Johansen, D. Sornette, H. Wakita, U. Tsunogai, W.I. Newman, and H. Saleur, *J. Phys. I France* 6, 1391 (1996)
31. N. Vandewalle, M. Ausloos, Ph. Boveroux and A. Minguet, *Eur. J. Phys. B* 9, 355 (1999)
32. S. Prakash and G. Nicolis, *J. Stat. Phys.* **82**, 297 (1996)
33. J. E. Wesfreid and S. Zaleski *Cellular Structures in Instabilities*, Lect. Notes Phys. bf 210 (Springer, Berlin, 1984)
34. B. J. West and B. Deering, *The Lure of Modern Science: Fractal Thinking*, (World Scient., Singapore, 1995)
35. A. L. Barabási and T. Vicsek, *Phys. Rev. A* **44**, 2730 (1991)
36. B.B. Mandelbrot, D.E. Passoja, and A.J. Paulay, *Nature* **308**, 721 (1984)
37. H. E. Hurst, *Trans. Amer. Soc. Civ. Engin.* **116**, 770 (1951)
38. H. E. Hurst, R.P. Black, and Y.M. Simaika, *Long Term Storage*, (Constable, London, 1965)
39. P. Flandrin, *IEEE Trans. Inform. Theory*, **35** (1989).
40. K. Ivanova and M. Ausloos, *Physica A* **265**, 279 (1999)
41. A. Babloyantz and P. Maurer, *Phys. Lett. A* **221**, 43 (1996)
42. K. Ivanova, M. Ausloos, A.B. Davis, and T.P. Ackerman, *Physica A* **272**, 269 (1999)
43. N. Vandewalle and M. Ausloos, *Int. J. Comput. Anticipat. Syst.* **1**, 342 (1998)
44. B.B. Mandelbrot, *Proc. Natn. Acad. Sci. USA* **72**, 3825 (1975)
45. H. P. G. E. Hentschel and I. Procaccia, *Physica D* **8**, 435 (1983)
46. A. Davis, A. Marshak and W. Wiscombe, in *Wavelets in Geophysics*, E. Foufoula-Georgiou and P. Kumar, Eds. (Academic Press, New York, 1994) p. 249; see also A. Marshak, A. Davis, R. Cahalan, and W. Wiscombe *Phys. Rev. E* **49**, 55 (1994)
47. H. E. Stanley, *Phase transitions and critical phenomena*, (Oxford Univ. Press, Oxford, 1971)
48. B. Mandelbrot and J. R. Wallis, *Water Resour. Res.* **5**, 967 (1969)
49. J. B. Bassingthwaigte and G. M. Raymond, *Ann. Biomed. Engin* **22**, 432 (1994)
50. E. Bacry, J.F. Muzy and A. Arneodo, *J. Stat. Phys.* **70**, 635 (1993)
51. A. Arneodo, E. Bacry and J. F. Muzy, *Physica A* **213**, 232 (1995)
52. Z. R. Struzik, in *Fractals: Theory and Applications in Engineering*, M. Dekking, J. Levy-Vehel, E. Lutton, and C. Tricot, Eds (Springer, Berlin, 1999)
53. E. Koscielny-Bunde, A. Bunde, S. Havlin, H. E. Roman, Y. Goldreich, and H.-J. Schellnhuber, *Phys. Rev. Lett.* **81**, 729 (1998)
54. K. Ivanova and T. Ackerman, *Phys. Rev. E*, **59**, 2778 (1999).
55. V.I. Yukalov and S. Gluzman, *Int. J. Mod. Phys. B* **13**, 463 (1999)
56. L. Cao, *Physica A* **247**, 473 (1997)
57. A.G. Ellinger, *The Art of Investment*, (Bowers & Bowers, London, 1971)
58. N. Vandewalle and M. Ausloos, *Phys. Rev. E* **58**, 6832 (1998)
59. N. Vandewalle, M. Ausloos, and Ph. Boveroux, *Physica A* **269**, 170 (1999)
60. D. Sornette, A. Johansen and J.-P. Bouchaud, *J. Phys. I France* **6**, 167 (1996)
61. J.A. Feigenbaum and P.G.O. Freund, *Int. J. Mod. Phys. B* **10**, 3737 (1996)
62. D. Sornette, *Phys. Rep.* **297**, 239 (1998)
63. J.C. Anifrani, C. Le Floc’h, D. Sornette and B. Souillard, *J. Phys. I France* **5**, 631 (1995)
64. A. Johansen, D. Sornette, H. Wakita, U. Tsunogai, W.I. Newman, and H. Saleur, *J. Phys. I France* **6**, 1391 (1996)
65. J.M. Kosterlitz and D.J. Thouless, *J. Phys. C* **6**, 1181 (1973)
66. N. Vandewalle and M. Ausloos, *Eur. J. Phys. B* **4**, 139 (1998)
67. N. Vandewalle, M. Ausloos, Ph. Boveroux and A. Minguet, *Eur. J. Phys. B* **9**, 355 (1999)
68. D. Bessis, J.S. Geronimo and P. Moussa, *J. Physique-LETTERS* **44**, L-977 (1983)