Quantumness protection for open systems in a double-layer environment

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We study the dynamics of two-level atomic systems (qubits) subject to a double-layer environment that consists of a network of single-mode cavities coupled to a common reservoir. A general exact master equation for the dynamics of a qubit system can be obtained by the quantum-state-diffusion (QSD) approach, which is extended to our spin-cavity-boson model. The quantumness of the atoms comprising coherence and entanglement is investigated for various configurations of the double-layer environment. The findings indicate that parametric control is available for the preservation and generation of system-quantumness by regulating the cavity network. Moreover, the underlying physics is profoundly revealed by an effective model obtained by a unitary transformation. Therefore, our work provides an interesting proposal to protect the quantumness of open systems in the framework of a double-layer environment containing bosonic modes.

quantum-state-diffusion equation, open quantum systems, decoherence, quantum noise

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1 Introduction

When modeling a realistic quantum system, it is necessary to take its coupling to the external environment [1] into consideration. Protecting the concerned system from decoherence along various channels that is caused by the environment is always a central issue or task for both quantum information processing and quantum computing [2]. Many approaches such as quantum error correction [3, 4], dynamical decoupling [5, 6], and the quantum Zeno effect [7, 8], have been applied to mitigate the leakage of quantum information from the system to the environment. These approaches primarily focus on operations directly performed on the concerned systems. Alternatively, improving both understanding and manipulation of the system-environment interaction in terms of coupling operators, coupling strength, and environmental configuration is also meaningful to protect the quantum property of the system.

With respect to the configurations of external environments, structured and hierarchical reservoirs have attracted extensive attention. For example, several independent reservoirs could simultaneously interact with a single central system [9]. In addition, the correlation between the sub-environments of the total system [10] might play an essential role in the decoherence dynamics of the system. In many realistic scenarios, quantum systems are surrounded by a hierarchical environment. In particular, the central system directly interacts with some components of the total environ-
ment that are further coupled to other components. In the quantum-dot systems, the electron spin is directly coupled to the positive-charge nuclei through the hyperfine interaction, while the phonon bath indirectly influences the momentum of the electron spin through spin-orbit interaction involving the electric field exerted by the nuclei [11]. The single-donor electron spin in silicon is also exposed to a similar configuration of the environment [12, 13]. In the NV center system [14], the central spin is strongly influenced by the surrounding electron spins of nitrogen impurities, which are simultaneously coupled to the nuclear spins of both nitrogen impurities and carbon\(^{13}\)s. In the spin-cavity-boson model [15], the spin-cavity interaction is used to establish the entanglement between two distant electron spins in diamond, while the photon loss due to the coupling between the cavity and the bosonic environment reduces the fidelity of the process.

Inspired by these facts, a system coupling to an environment with more than one layer has been theoretically proposed in recent work [16-18]. An interesting example [16] is that a two-level system interacts with a damping cavity mode. The dynamics of the concerned system is featured by the modified parameters of the cavity and the reservoir. Consequently, in refs. [17, 18], the authors suggested a more complicated hierarchical model in which one qubit is placed in a damping cavity that interacts with the other damping cavities. The contribution from the coupled cavities to the non-Markovian dynamics of the qubit has been discussed in detail. The results indicate that both structure and parameters of a multi-layer environment have a vital influence on the intrinsic behavior of the central system. In this study, we introduce an exact treatment to deal with a double-layer environment, and extend the model to a more realistic situation that is beyond the single-qubit case. We extend the quantum-state-diffusion (QSD) equation from application to specific multimode environments to solving a more general model with multi-layer environments.

Another motivation for this study is to find a way to protect the quantumness of the concerned system in a double-layer environment. We are interested in a model that is structured as follows. The concerned system consists of qubits that are individually placed at single-mode cavities. These cavities are mutually coupled to each other and embedded in a global multimode reservoir. Based on the exact dynamics of the system, the profound impact that the cavity-cavity coupling strength and the cavity-network size have on protecting the quantumness (including coherence and entanglement) of the concerned system in both Markovian and non-Markovian reservoirs has been demonstrated.

In our approach, although the qubit system and the single-mode cavity interact coherently, the cavity modes can be considered as the first layer of the total environment. The state of the concerned system at moment \(t\) is given by \(\rho_s(t) = \text{Tr}_E[U(t)\rho_s(0) \otimes \rho_E(0)U^\dagger(t)]\), where \(\rho_s\) and \(\rho_E\) represent the reduced density matrices of the open system and its environment, respectively. The direct product at \(t = 0\) means that the system and the environment are initially uncorrelated. It is necessary to emphasis here that the environment \(\rho_E\) contains both the cavity modes and the global reservoir. The latter is considered as the second layer of the total environment. We follow the exact method in ref. [19] and treat the states of all the environmental modes as complex Gaussian noises. Thus, the qubit system can be described by a special stochastic Schrödinger equation, i.e., the QSD equation.

The remainder of this paper is organized as follows. In sect. 2, we present the theoretical model in a general situation where \(M\) qubits are separately located at \(N \geq M\) damping cavities. In sect. 3, we present the QSD formation in our model, the ansatz as the \(O\)-operators in the equation of motion, and the consequent exact master equation. The detailed derivation can be found in A1. In sect. 4, we study the coherences of a single-qubit system and the entanglement of a double-qubit system. The effect of the coupled-cavity network is investigated in detail. Using an effective model described in A2, we explain the underlying physics in an alternative way in sect. 5. Finally we draw conclusions in sect. 6.

## 2 Theoretical model

In our model, the qubits intended to implement the tasks for quantum computation or information storage are individually located in certain nodes of a quantum network consisting of the mutually-coupled single-mode cavities. All the cavities serve as the nodes of the network that is embedded in a global reservoir. The configuration of the whole system is depicted in Figure 1. The number of cavities \(N\) is greater than or equal to the number of qubits \(M\). Figure 1 illustrates a configuration with 8 cavities, where the central qubit interacts with 4 of them. The other qubits are individually placed in the remaining cavities.

![Figure 1](Color online) Schematic diagram of our model. \(M\) qubits are individually placed at \(N\) cavities, \(N \geq M\). The connections among the cavities constitute a quantum network. All cavities are subject to a global reservoir.
to that of the qubits $M$, so that some of the cavities might be “empty” with respect to the qubits.

The total Hamiltonian in the rotating frame with respect to $H_0 = H_q + H_c$ where $H_c = \sum_{i=1}^N \omega_c a_i a_i^\dagger$ and $H_q = \sum_{k} \Omega b_k^\dagger b_k$ are the Hamiltonians of the cavities and the external global reservoir, respectively, can be written as:

$$H = H_q + H_{qc} + H_{cc} + H_{ce},$$

where

$$H_q = \sum_{m=1}^M \frac{\omega_{am}}{2} \sigma_z^{(m)},$$

$$H_{qc} = \sum_{m=1}^M \Omega_{am} \sigma_z^{(m)} a_m^\dagger e^{-i(\omega_{am} - \omega_c)n}\hbar + \text{h.c.},$$

$$H_{cc} = \sum_{p \neq q} \Omega_{pq} a_p^\dagger a_q e^{i(\omega_{pq} - \omega_c)n}\hbar,$$

$$H_{ce} = \sum_{n=1}^N \alpha_n e^{-i \omega_n t} \sum_k g_k b_k^\dagger e^{i \omega_n t} + \text{h.c.}.$$

It is composed of four parts. $H_q$ represents the concerned system of $M$ qubits. $\sigma_z^{(m)}$ and $\sigma_z^{(m)}$, $m \in [1, 2, \ldots, M]$, are the Pauli operators for the $m$th qubit with the transition frequency $\omega_{am}$. $H_{qc}$ describes the coupling between qubits and cavities with the coupling strength $\Omega_{am}$, whereas $a_m^\dagger (a_m)$ is the creation (annihilation) operator of the $m$th cavity mode with the eigen-frequency $\omega_{am}$. $H_{cc}$ involves the mutual connection among the cavities marked by $p \neq q \in [1, 2, \ldots, N]$ with different coupling strength $\Omega_{pq} (\Omega_{pq} = \Omega_{qp})$. $H_{ce}$ indicates the dissipation of cavities owing to their coupling to a global reservoir, where the coupling strength $g_k$ involves the $k$th environmental mode and $b_k^\dagger (b_k)$ represents its creation (annihilation) operator with the eigenfrequency $\omega_k$. Here the spontaneous emission of the qubit is omitted, which is physically reasonable in certain situations. For example, in a hybrid system where the artificial atoms or spins interact with the superconducting resonators, the coherence lifetime of atoms is much longer than that of resonators due to their comparatively weak interaction with the surrounding environment [20]. Moreover, the direct interactions between cavities are ignored since they are locally isolated by the cavities.

This structure of the total environment can be well-understood from a hierarchical environment. The correlated cavities are the first layer of environment, and the global reservoir serves as the second one. When the qubits contact with the leaking cavities, the quantum information of the qubit system will probabilistically travel across the first layer and the second one and then come back. Thus its dynamics is determined by the features (configuration and parameters) of the whole environment. After resolving the dynamical equation in a nonperturbative way, we would demonstrate that the coupling strength between the qubits and the cavities, the interaction between the qubits and the cavities, and the number of the cavities play inevitable roles in the dynamics of the qubit system.

3 Nonperturbative master equation of the qubit system

Recently an improved method based on the QSD approach was suggested to deal with the hybrid system consisting of atoms and cavities [19]. Its key idea is to treat the cavity as a constituent of the environment or noise, which follows a specific correlation function. Using this approach that is irrespective to the cavity number, one can deal with various configurations of cavity network or array. More importantly, exact solutions could be obtained for both Markovian and non-Markovian reservoirs. In the current section, we will derive an exact master equation for the qubit system in a general situation, where $M$ qubits are individually embedded in $N$ cavities ($M \leq N$).

Following the standard QSD approach [21, 22], the full wavefunction of the total system (including system and environment) $|\Psi(t)\rangle$ is projected to the Bargmann coherent states of the environment, yielding the following stochastic wavefunction of the system part:

$$|\psi_t(x^c, y^c)\rangle = \langle x^c | (|y^c\rangle - |\Psi(t)\rangle),$$

$$|\Psi(t)\rangle = \int \frac{d^2x}{\pi} e^{-|\langle x^c | (|y^c\rangle - |\Psi(t)\rangle)\rangle} \int \frac{d^2y}{\pi} e^{-|\langle y^c | (|y^c\rangle - |\Psi(t)\rangle)\rangle},$$

where $|x^c\rangle = \prod_{j=1}^N |x_j\rangle$ and $|y^c\rangle = \prod_{k=1}^N |y_k\rangle$ represent the random Bargmann coherent states for the modes of cavity $a_j$ and reservoir $b_k$, respectively.

The exact linear QSD equation can be formally written as (see A1):

$$\partial_t |\psi_t(x^c, y^c)\rangle = -i H_{\text{eff}} |\psi_t(x^c, y^c)\rangle,$$

where

$$H_{\text{eff}} = \sum_{m=1}^M \frac{\omega_{am}}{2} \sigma_z^{(m)} + \hat{H}_{qc} + \hat{H}_{cc} + \hat{H}_{ce},$$

$$\hat{H}_{qc} = \sum_{m=1}^M \Omega_{am} \sigma_z^{(m)} a_m^\dagger + \text{h.c.},$$

$$\hat{H}_{cc} = \Omega_{pq} \sum_{p \neq q=1}^N \sigma_x^{(p)} \sigma_x^{(q)} + \sigma_y^{(p)} \sigma_y^{(q)} + \text{h.c.},$$

$$\hat{H}_{ce} = \sum_{n=1}^N \left( x_n^c \hat{O}_x + \sigma_z^{(n)} y_n^c \right).$$

The terms $\hat{H}_{qc}$, $\hat{H}_{cc}$ and $\hat{H}_{ce}$ describe the interactions of qubit-cavity, cavity-cavity and cavity-reservoir, respectively. The
relevant $O$-operators are defined as:

$$O_{xp}(t,s)|\psi_{i}\rangle \equiv \frac{\delta}{\delta x_{p}^{*}}|\psi_{i}\rangle, \quad x_{p}^{*} \equiv -i x_{p} e^{i \omega_{p} t},$$

$$O_{y}(t,s)|\psi_{i}\rangle \equiv \frac{\delta}{\delta y_{s}^{*}}|\psi_{i}\rangle, \quad y_{s}^{*} \equiv -i \sum_{k} g_{k}^{*} y_{k} e^{i \omega_{k} t},$$

(4)

where $p$ runs from 1 to $N$ and the time-dependent complex Gaussian processes $x_{p}^{*}$ and $y_{s}^{*}$ respectively satisfy the following correlation functions:

$$M[x_{p}, x_{p}^{*}^{\dagger}]= \alpha_{p}(t, s) = e^{-i \omega_{p}(t-s)},$$

$$M[y_{s}, y_{s}^{*}^{\dagger}]= \beta(t, s) = \frac{\gamma}{2} e^{-\gamma|t-s|}$$

in statistics. Here $M[\cdot]$ stands for the ensemble average over the noise $x_{p}^{*}$ or $y_{s}^{*}$. In particular, the noise process $x_{p}^{*}$ arising from the single mode $a_{j}$ is crucial in applying the QSD approach to the spin-cavity-boson model. We have assumed that the spectrum density of reservoir follows a Lorenz form:

$$S(\omega) = \frac{1}{2 \pi} \frac{\Gamma \gamma^{2}}{\gamma^{2} + \omega^{2}},$$

(5)

where $\Gamma$ is the coupling strength between each cavity and the global reservoir, and $\gamma$ is related to the bandwidth of the spectrum measuring the reservoir memory capacity. The operators $\hat{O}_{xp}(t)$ and $\hat{O}_{y}(t)$ are respectively defined as:

$$\hat{O}_{xp}(t) = \int_{0}^{t} ds \alpha_{p}(t, s) O_{xp}(t, s),$$

$$\hat{O}_{y}(t) = \int_{0}^{t} ds \beta(t, s) O_{y}(t, s).$$

Furthermore, using the Novikov theorem, one can derive an exact master equation of the qubit system from the linear QSD equation under the one-exciton condition:

$$\dot{\rho}_{s} = M\left[|\psi_{i}\rangle \langle \psi_{i}| + |\psi_{j}\rangle \langle \psi_{j}|\right],$$

$$= -i \sum_{m=1}^{M} \frac{\omega_{am}}{2} \sigma_{+}^{(m)} \rho_{s} \sigma_{-}^{(m)} + \sum_{m=1}^{M} \left(\Omega_{am} [\hat{O}_{xp} \rho_{s}, \sigma_{+}^{(m)}] + \Omega_{am}^{*} [\rho_{s} \hat{O}_{xp}^{\dagger}]\right),$$

(6)

where $\rho_{s}$ is the density matrix of the concerned qubit system. The master equation involves explicitly with the $O$-operators arising from the cavities which are equipped with qubits, i.e., $O_{am}, m \in \{1, 2, \ldots, M\}$. Yet in fact all the $O$-operators including $O_{xp}, p \in \{1, 2, \ldots, N\}$ from the cavities and $O_{y}$ from the external reservoir (the second layer of the environment) are mutually correlated with each other, whose dynamical equations form a closed set. The details can be found in A1.

### 4 Quantumness of the qubit system

In this section, we are working in the conditions in which the qubits are resonant with the cavities, i.e., $\omega_{am} = \omega_{cn} = \omega$, and the coupling strengths between qubit and cavity as well as those among cavities are isotropic, i.e., $\Omega_{a1} = \Omega_{a2} = \cdots = \Omega_{aM} = \Omega$ and $\Omega_{pq} = \Omega$. Note our approach is insensitive to the choice of the parameters. For simplicity, throughout this paper we suppose the initial state of the total system is given by $\rho = \rho_{s} \otimes |0_{1}0_{2} \ldots 0_{N}\rangle \langle 0_{1}0_{2} \ldots 0_{N}| \otimes |00 \cdots 0\rangle \langle 00 \cdots 0|$, where $|0_{1}0_{2} \ldots 0_{N}\rangle$ and $|00 \cdots 0\rangle$ indicate that there is no photon in both the cavity network and reservoir when $t = 0$.

In the remainder part of this section, we will investigate the quantumness of one-qubit and two-qubit systems, focusing on the effect from the double-layer environment.

#### 4.1 Coherence of one-qubit system

As one of the most important quantum resources, quantum coherence has been investigated intensively [23-25]. Here we employ the relative entropy as the measure of coherence, whose definition is [23]

$$\text{Coh}(\rho_{s}) = S(\rho_{s}^{\rho}) - S(\rho_{s}),$$

(7)

where $S(\rho) \equiv -\text{Tr}(\rho \ln \rho)$ is the von Neumann entropy, and $\rho_{s}^{\rho}$ is obtained by cancelling all the off-diagonal elements in the density matrix of $\rho_{s}$. We present the coherence dynamics of the one-qubit system based on the results of exact master eq. (6) with $M = 1$.

Consider firstly a situation in which the global reservoir is of the Markovian type, i.e., in the correlation function eq. (5) $\gamma/T \rightarrow \infty$, and the cavity number is chosen as $N = 3$. The initial state of qubit is prepared as $|\psi_{s}\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$ and then its coherence is $\sqrt{2}/2$ at $t = 0$. In Figure 2(a), it is shown that the dynamics of the qubit coherence can be modified by regulating the coupling strengths $\Omega$ and $\Omega$. The solid black line shows that if the coupling between the qubit and cavity $\Omega$ is strong while the cavity-cavity coupling strength $\Omega$ is comparatively weak, the coherence will quickly decay, and then rapidly revive. Afterwards such an intensive fluctuation repeats itself with an asymptotically decreasing magnitude. With the increasing cavity-cavity coupling strength $\Omega$, the dashed blue line indicates that the fluctuation magnitude is reduced and the coherence asymptotically follows almost the same decay pattern. In the dotted red line with a smaller $\Omega$, the coherence fluctuation is further suppressed and the coherence magnitude asymptotically decays at a much slower rate with time.

The average coherence $M[\text{Coh}] \equiv \frac{1}{T} \int_{0}^{T} ds \text{Coh}(s)$ reveals more about the coherence in the parameter space of $\Omega$ and $\Omega$. 

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In Figure 2(b), it is shown that during a desired evolution period $\Gamma t = [0, 200]$, the average coherence can approach a high value by enhancing $\Omega$ while reducing $\Omega_a$. When the qubit-cavity coupling strength $\Omega_a$ is comparatively weak, the average coherence can be sensitively manipulated by the cavity-cavity coupling strength $\Omega$. It is found that the effects from both $\Omega$ and $\Omega_a$ on coherence are irrelevant to the choice of the evolution period in quality, although the average coherence will be changed in quantity.

Consider the strong system-environment coupling in some realistic systems, in which the effect of the non-Markovian environment due to its memory capacity can lead to more protection of the central system. In the case of a strictly Markovian reservoir, the correlation function corresponding to the Lorentz spectrum eq. (5) will become proportional to the delta function when $\gamma/\Gamma \to \infty$; while in contrast it will manifest a strong non-Markovian effect when $\gamma/\Gamma \to 0$. Thus through manipulating $\gamma$, one can realize the transition of reservoir from the Markovian to the non-Markovian regimes. In Figure 3, we fix the coupling strengths and the number of cavities while changing $\gamma$. It is shown that the coherence revival can be promoted by reducing $\gamma$ yet the coherence fluctuation is insensitive to the value of $\gamma$, which means the non-Markovian reservoir has a positive effect to protect the coherence of the qubit.

In a larger size of quantum network with more cavities, the qubit-system dynamics becomes more interesting. In Figure 4, we fix the coupling parameters as $\Omega_a/\Gamma = 0.1$ and $\Omega/\Gamma = 0.2$ and present the coherence dynamics with $N = 2, 4, 8$ under the Markovian reservoir. It is obvious that increasing size of the cavity network in the first layer would facilitate to suppress the decoherence rate. The magnitude of coherence fluctuation is also enhanced by the network size. This result could be understood that a larger first-layer environment provides more chances for the photon or exciton reabsorbed by the central qubit system in the presence of the irreversible loss due to the second-layer environment.

4.2 Entanglement of a two-qubit system

As an important physical model for the superposed state transfer [26] and preparation, the dynamics of two-qubit

![Figure 2](image)

Figure 2 (Color online) (a) The coherence dynamics of the qubit under various pairs of the qubit-cavity coupling strength $\Omega_a$ and the cavity-cavity coupling strength $\Omega$. (b) The average coherence during the desired evolution period $\Gamma t = [0, 200]$ as the function of $\Omega_a$ and $\Omega$. Here we have a Markovian reservoir and $N = 3$ cavities in the cavity-network. The initial state of the qubit is prepared in $\frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$.

![Figure 3](image)

Figure 3 (Color online) The coherence dynamics of the qubit under different $\gamma$. The other parameters are fixed as $\Omega_a/\Gamma = 0.1$, $\Omega/\Gamma = 0.2$, and $\omega_a/\Gamma = \omega/\Gamma = 5$. The initial state of the qubit is prepared in $\frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$.

![Figure 4](image)

Figure 4 (Color online) The coherence dynamics of the qubit with different cavity numbers $N$. The other parameters are fixed at $\Omega_a/\Gamma = 0.1$ and $\Omega/\Gamma = 0.2$. The reservoir is supposed to be of Markovian type. The initial state of the qubit is prepared in $\frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$.
system has been widely studied. In this section, we focus on the entanglement dynamics of the two-qubit system attained by the master eq. (6) with $M = 2$. We adopt the concurrence as a measure of entanglement. It is defined as [27]:

$$C(\rho) = \max \left\{ 0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \right\},$$

(8)

where $\lambda_i$s are the eigenvalues of the decreasing order of the matrix $\rho_s(\sigma_x^{(1)} \otimes \sigma_y^{(2)}) \rho_s(\sigma_y^{(1)} \otimes \sigma_x^{(2)})$ and $\rho_s$ is the density operator of the two-qubit system. Inspired by the results in the one-qubit case, here we still focus on the effects from the coupling strengths $\Omega_s$ and $\Omega_s$ of the two-qubit system attained by the master eq. (6) with the entanglement dynamics of the two-qubit system. Thus, we define the average concurrence by

$$M[C] \equiv \frac{1}{\Gamma} \int_0^\Gamma ds C(s)$$

during a desired evolution period $\Gamma t = [0, 200]$. In Figure 5(b), by continuously regulating the values of $\Omega_s$ and $\Omega_s$, it is shown that fixing $\Omega_s$, a larger $\Omega_s$ can maintain the average concurrence at a higher level. However, with an increasing $\Omega_s$, it becomes more difficult to protect the qubits entanglement by increasing $\Omega_s$. As a whole, it is inferred that the entanglement lifetime can be extended by strengthening the couplings among the cavities $\Omega_s$ while maintaining the coupling between qubit and cavity $\Omega_s$. Strong couplings among the cavities can suppress the magnitude of the concurrence fluctuation, which is consistent with one of the conclusions in a previous study about the spin environment [28]. Both the pair of the solid black line and the dotted blue line and the pair of the dashed black line and the dot-dashed blue line with circles suggest that a weaker qubit-cavity coupling $\Omega_s$ yields a slower decay rate.

To see more general situations in the parameter space of $\Omega_s$ and $\Omega_s$, we define the average concurrence by $M[C] = \frac{1}{\Gamma t} \int_0^{\Gamma t} ds C(s)$ as the function of parameters $\Omega_s$ and $\Omega_s$ in Figure 5(b). In Figure 5(b), by continuously regulating the values of $\Omega_s$ and $\Omega_s$, it is shown that fixing $\Omega_s$, a larger $\Omega_s$ can maintain the average concurrence at a higher level. However, with an increasing $\Omega_s$, it becomes more difficult to protect the qubits entanglement by increasing $\Omega_s$. As a whole, it is inferred that the entanglement lifetime can be extended by strengthening the couplings among the cavities $\Omega_s$ while maintaining the coupling between qubit and cavity $\Omega_s$. Strong couplings among the cavities can suppress the magnitude of the concurrence fluctuation, which is consistent with one of the conclusions in a previous study about the spin environment [28]. Both the pair of the solid black line and the dotted blue line and the pair of the dashed black line and the dot-dashed blue line with circles suggest that a weaker qubit-cavity coupling $\Omega_s$ yields a slower decay rate.

We start again from the situation that the global reservoir is of Markovian type and the number of cavities is $N = 3$. The system state is initially prepared in a symmetric Bell state $\frac{1}{\sqrt{2}}(|eg\rangle + |ge\rangle)$. In Figure 5(a), four combinations of coupling strengths $\Omega_s$ and $\Omega_s$ are selected to demonstrate their respective effects on the entanglement dynamics. On comparing the two black lines or the two blue lines, it is shown that the entanglement lifetime can be extended by strengthening the couplings among the cavities $\Omega_s$ and cavity $\Omega_s$. Strong couplings among the cavities can suppress the magnitude of the concurrence fluctuation, which is consistent with one of the conclusions in a previous study about the spin environment [28]. Both the pair of the solid black line and the dotted blue line and the pair of the dashed black line and the dot-dashed blue line with circles suggest that a weaker qubit-cavity coupling $\Omega_s$ yields a slower decay rate.

Under the non-Markovian reservoir, the combined effect of the parameters $\Omega_s$ and $\gamma$ on the qubit entanglement is demonstrated in Figure 6. The solid black line and the dashed blue line, where the cavity-cavity coupling strength is fixed at $\Omega_s/\Gamma = 0.2$, indicate that a small $\gamma/\Gamma$ or a long reservoir memory time is effective to protect the qubit entanglement. Furthermore, in the dotted red line and the dash-dotted orange line with a larger $\Omega_s/\Gamma$, the concurrence value is promoted to a higher level. Therefore, we can infer that the memory effect of the reservoir facilitates the quantumness feedback from the cavity network to the qubit system. From an alternative perspective based on the pseudo-mode method [29, 30], a reservoir with a spectral density function in the Lorentz form could be considered as a single mode coupled to a Markovian reservoir. Consequently, in our model, employment of the non-Markovian reservoir transforms the double-layer environment to a triple-layer environment. In this scenario, the qubit system is effectively coupled to a cavity network, and the network is simultaneously coupled to another single mode with strength proportional to $\sqrt{\Gamma \gamma}$, which is further coupled to a Markovian reservoir with decay rate $\gamma$. Compared with the Markovian scenario, the coherent flow of quantum information between the qubit and cavities has to travel a longer path before it irreversibly leaks to the external reservoir. Thus, the leaking rate can be effectively suppressed by reducing $\gamma$.

In Figure 7, we investigate the size effect of the cavity network on the concurrence under the Markovian reservoir. The parameters are chosen as $\Omega_s/\Gamma = 0.1$ and $\Omega_s/\Gamma = 0.2$. In the case of $N = 2$, which means there are only two nodes in the cavity-network, the concurrence decreases exponentially with time exhibiting no fluctuations. However, the dynamics...
becomes oscillatory when \( N > 2 \), and the results resemble those for the one-qubit situation, where a larger \( N \) is more beneficial to maintain the level of quantumness. This phenomenon indicates that the presence of the “empty” cavities (i.e., the cavities that do not contain a qubit) causes the quantumness of a qubit system to spend more time leaking to the second layer. In addition, the fluctuation magnitude of the system entanglement is enhanced as the number of the “empty” cavities increases, which actually displays the feedback of quantumness from the first-layer environment.

5 Discussion

To dig more about the underlying physics, we can construct an effective model through a set of unitary transformations as illustrated in A2. Using this method, the structure of the full Hamiltonian (1) can be remarkably simplified. Particularly in the two-qubit case with \( N \geq 3 \), the effective Hamiltonian under the isotropic parametric condition can be given by combining eq. (a31) with \( M = 2 \) and eq. (a38):

\[
H' = H_q' + H_{qc}' + H_{cc}' + H_{ee}',
\]

\[
H_q' = \frac{\omega_q}{2} (\hat{\sigma}_z^A + \hat{\sigma}_z^B),
\]

\[
H_{qc}' = \left[ \Omega_q^c \hat{\sigma}_z^A \hat{a}_3^\dagger + \Omega_q^c \hat{\sigma}_y^B \left( \sqrt{2N} \hat{a}_1^\dagger + \frac{\sqrt{N^2 - 2N}}{N} \hat{a}_2^\dagger \right) \right] + \text{h.c.,}
\]

\[
H_{cc}' = [\omega + (N - 1)\Omega] \hat{a}_1^\dagger \hat{a}_1 + \sum_{n=2}^{N} (\omega - \Omega) \hat{a}_n^\dagger \hat{a}_n,
\]

\[
H_{ee}' = \sqrt{N} \hat{a}_1 \sum_k g_k |b_k^\dagger e^{-i\omega t} + \text{h.c.,}
\]

where the atoms and cavity modes have been respectively renormalized to a set of effective qubits \( \hat{\sigma} \) and modes \( \hat{a} \). The details can be found in A2. Now we have a model in which one qubit (qubit-A) is coupled to a single mode (\( \hat{a}_1 \)) and another qubit (qubit-B) is coupled to the other two modes (\( \hat{a}_1 \) and \( \hat{a}_2 \)) with renormalized coupling strengths. The mode \( \hat{a}_1 \) is the unique leaking mode that is coupled to the second-layer environment (reservoir), and other cavity-modes are decoupled from it. Thus, in this transformed picture, the qubit system is at most coupled to 3 cavities regardless of the physical size of the cavity-network.

The configuration of the effective model is depicted in Figure 8. In the renormalized representation, qubit-A and qubit-B share the same ground state \( |eg\rangle_{12} \). And their excited states are described by the dressed states \( |e_A\rangle = \frac{1}{\sqrt{2}} (|eg\rangle_{12} - |ge\rangle_{12}) \) and \( |e_B\rangle = \frac{1}{\sqrt{2}} (|eg\rangle_{12} + |ge\rangle_{12}) \), respectively. Therefore, the dynamics of the interested initial state \( \frac{1}{\sqrt{2}} (|eg\rangle_{12} + |ge\rangle_{12}) \) is fully determined by the evolution of qubit-B. The Hamiltonian \( H_{cc}' \) in eq. (12) determines that a large cavity-cavity coupling strength \( \Omega \) can enhance the detuning between the effective qubits and cavities. According to the Fermi’s Golden rule, it will considerably reduce the transition rate between the qubit system and the cavity network as the first-layer environment. It is shown by the Hamiltonian \( H_{qc}' \) in eq. (11) that the qubit-cavity coupling strength \( \Omega_q \) plays the role of the coupling strength between qubit-B and the leaky cavity mode \( \hat{a}_1 \). It causes the decay of the qubit entanglement. With fixed parameters \( \Omega_q \) and \( \Omega \), one can find from eq. (11) that the effective coupling strength between qubit-B and the leaky cavity mode \( \hat{a}_2 \) will be enhanced with the increasing size of the cavity-network \( N \). This interaction term results in the significant fluctuating behavior as shown by the two lines with \( N = 4 \) and \( N = 8 \) in Figure 7, even when the external reservoir is of Markovian type. The cavity-network size remarkably affects the evolution of the qubit system in terms of period and fluctuation magnitude. In particular, the effective coupling strength between qubit-B and the leaky cavity mode...
$\bar{a}_1$ decreases in the order $\frac{1}{\sqrt{N}}$, which justifies that a larger cavity network facilitates to protect the quantumness. In the limit of $N \to \infty$, the coupling approaches zero indicating a perfect protection. In contrast, with $N = 2$, the Hamiltonian $H'_c$ in eq. (11) becomes

$$H'_c = (\Omega^a_2\sigma^+_B\bar{a}^*_1 + \Omega^B_2\sigma^-_B\bar{a}^*_1) + \text{h.c.},$$

(14)

in which qubit-B interacts only with the lossy cavity $\bar{a}_1$. Then it is easy to understand that the evolution of qubit-system entanglement inevitably follows an exponential decay evolution (see the solid black line in Figure 7) under a Markovian reservoir.

The effective model described by the Hamiltonian (9) can also explain the quantumness generation of the qubit system starting from separable states. We consider the product state $|eg\rangle_{12}$ as the initial state, whose degree of entanglement is zero at $t = 0$. One can observe the entanglement generation upon the indirect coupling between the two qubits, which is induced by their direct coupling to the cavity network. In Figure 9, we show the entanglement generation process under the non-Markovian reservoirs with various memory parameter $\gamma$. Under a strong non-Markovian environment with $\gamma/\Gamma = 0.1$, the qubit entanglement can achieve almost the maximal value and quasi-periodically evolve with time. With the increasing $\gamma$, the quasi-period is enlarged and the peak value of the entanglement revival gradually decays with time. In the long time limit, the concurrence approaches $1/2$.

Now we turn to the transformed picture, where the state $|eg\rangle_{12}$ can be expressed by $|eg\rangle_{12} = \frac{1}{\sqrt{2}}(|e_A\rangle + |e_B\rangle)$. Based on eq. (11), qubit-A is decoupled from the reservoir, so the state $|e_A\rangle$ is free of dissipation. However, due to the indirect coupling with reservoir, the state $|e_B\rangle$ is expected to eventually decay to the ground state. Then after a long time, the final state should collapse into the mixed state of the dressed state $|e_A\rangle = \frac{1}{\sqrt{2}}(|eg\rangle_{12} - |ge\rangle_{12})$ and the ground state $|gg\rangle_{12}$ of equal probability. This result is independent on the values of $\Omega$, $\Omega_a$, $N$ and $\gamma$. The same steady-state has been reported in ref. [31].

**6 Conclusion**

In summary, we have investigated an open-quantum-system model with a double-layer environment consisting of a cavity network and a global multi-mode reservoir. Based on the exact master equation obtained by the quantum-state-diffusion approach, which is extended to the spin-cavity-bosons model, we discuss the protection of the quantumness (coherence and entanglement) of the qubit system through manipulating the parameters of the double-layer environment, including the qubit-cavity and cavity-cavity coupling strengths $\Omega_a$ and $\Omega$, the cavity network size $N$ and the reservoir memory parameter $\gamma$. It is found that the quantumness of an open system could have a long lifetime under any of the following three conditions: (1) the system-environment interaction strength is reduced (by a weak qubit-cavity coupling); (2) the inner-coupling among environmental modes is enhanced (by a strong cavity-cavity coupling and/or a sizable cavity-network); (3) the external reservoir as the second-layer environment is structured (by a non-Markovian reservoir with a long memory time).

Besides the numerical evaluation, we established an effective model in a transformed picture to uncover the underlying physics of our open-quantum-system model with double-layer quantum noises. The qubit system is found to be effectively coupled to a very limited number of tilde cavity-modes, despite in the original picture they are embedded in a leaky network with $N$ nodes. Only one tilde cavity-mode is subject to the second-layer environment in the transformed picture. Our study therefore paves an extendable way to understand and control the dynamics of the central qubit system surrounded by a multilayer environment. It should be emphasized that the results in this paper are not limited to the spin-boson-environment models, but also serve as a clue for other models including the spin-spin-environment model that is popular in the solid-state quantum devices.

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![Figure 8](image-url) (Color online) Schematic diagram of the effective model for $N \geq 3$. Here $\omega_1 = \omega + (N - 1)\Omega$ and $\omega_2 = \omega_1 + \omega - \Omega$.

![Figure 9](image-url) (Color online) The entanglement generation for the initial state $|eg\rangle_{12}$ under the non-Markovian reservoirs with different memory parameter $\gamma$. The rest parameters are fixed as $\Omega_a/\Gamma = 0.1$, $\Omega/\Gamma = 0.5$, $\omega_a/\Gamma = \omega/\Gamma = 5$ and $N = 3$. 
Appendix A1

A1 Exact master equation for a double-layer environment

In this appendix, we will first derive eq. (3) based on the standard QSD method, and then obtain the exact master eq. (6) from the linear QSD equation. The solutions are explicitly presented for the single- and double-qubit cases. Using the definition of the Bargmann coherent state,\n\[ |z\rangle \equiv \frac{1}{\sqrt{n!}} \sum_{n=0}^{\infty} \frac{d^n}{d z^n} |n\rangle = e^{\frac{1}{2} z^2} |z\rangle, \]
where |z\rangle, z = x_j, y_j is the normal coherent state for the relevant bosonic mode (see eq. (2) and the explanations in the main text), it is straightforward to find that\n\[ \langle x_j | a_j^\dagger = \frac{\partial}{\partial x_j^*} \langle x_j |, \quad \langle x_j | a_j = x_j^* \langle x_j |, \]
\[ \langle y | b_k^\dagger = \frac{\partial}{\partial y_k^*} \langle y |, \quad \langle y | b_k = y_k^* \langle y |. \]

From the Schrödinger equation \[ \partial_t |\psi(t)\rangle = -i[H|\psi(t)\rangle] \] with the total Hamiltonian (1) and the projection by the bra state of the Bargmann coherent state for the modes of the two-layer environment, one can directly get a stochastic Schrödinger equation for the central system:
\[ \partial_t |\psi^{(s)}(x^*, y^*)\rangle = -i[H_q + \sum_{m=1}^{M} (\Omega_{am} L_m x^*_m e^{i\omega_m t}) + \Omega_{am} L_m x^*_m \frac{\partial}{\partial x_m^*} + \Omega_{pq} \sum_{p \neq q} e^{-i(\omega_p - \omega_q) t} \frac{\partial}{\partial x_p^*} + \sum_{n,k} \left( g_k^* e^{-i(\omega_n - \omega_k) t} y_n \frac{\partial}{\partial y_k^*} + g_k e^{i(\omega_n - \omega_k) t} y_n \frac{\partial}{\partial y_k} \right) |\psi^{(s)}(x^*, y^*)\rangle, \]
where \( L_m \equiv \sigma_m^{(n)} \). Regarding \( \exp{\hat{a}}_n = -i \chi_n e^{i\omega_n t} \), \( n = 1, 2, \ldots, N \), as a special “noise process”, its correlation function is \( \alpha_n(t, s) = M[x_n, \chi_n^*] = e^{-i\omega_n(t-s)} \), \( y_n \equiv -i \sum_k g_k^* e^{i\omega_n t} \), and its correlation function is assumed to be \( \beta(t, s) = M[y_n y_n^*] = \sum_k |g_k|^2 e^{-i(\omega_n - \omega_k) t} = \frac{\Gamma}{2} e^{-|\Gamma| t} \).

With the chain rule, one can define the following functional derivatives:
\[ \frac{\partial}{\partial x^*_j} = \int_0^t ds \frac{\partial}{\partial x^*_j} e^{i\omega_n t} = \int_0^t ds \frac{\partial}{\partial x^*_j} e^{i\omega_n t}, \]
\[ \frac{\partial}{\partial y_k} = \int_0^t ds \frac{\partial}{\partial y_k} e^{i\omega_n t} = \int_0^t ds \frac{\partial}{\partial y_k} e^{i\omega_n t}. \]

Thus eq. (a3) can be rewritten as:
\[ \partial_t |\psi^{(s)}(x^*, y^*)\rangle = -i[H_q + \sum_{m=1}^{M} (\Omega_{am} L_m x^*_m e^{i\omega_m t}) \frac{\partial}{\partial x^*_j} + \Omega_{am} L_m x^*_m \frac{\partial}{\partial x^*_j} + \Omega_{pq} \sum_{p \neq q} e^{-i(\omega_p - \omega_q) t} \frac{\partial}{\partial x_p^*} + \sum_{n,k} \left( g_k^* e^{-i(\omega_n - \omega_k) t} y_n \frac{\partial}{\partial y_k^*} + g_k e^{i(\omega_n - \omega_k) t} y_n \frac{\partial}{\partial y_k} \right) |\psi^{(s)}(x^*, y^*)\rangle, \]
\[\begin{align*}
&\lim_{\Delta t \to 0} \left( \frac{\partial}{\partial \Delta t} \right) \left[ \int_0^t ds \alpha(t, s) \frac{\delta}{\delta x(t, s)} \right] \psi(t), \quad (a5) \\
&\text{The ansatz of } O\text{-operators are then introduced by} \\
&O_{\text{mn}}(t, s) |\psi(t), y\rangle \equiv \frac{\delta}{\delta x_{ns}^m} |\psi(t), y\rangle, \\
&O_{\text{ns}}(t, s) |\psi(t), y\rangle \equiv \frac{\delta}{\delta y_{ns}^m} |\psi(t), y\rangle. \\
&\text{Afterwards we formally obtain} \\
&\partial_t |\psi(t), y\rangle = -i H_{\text{eff}} |\psi(t), y\rangle \\
&= \left[ -i H_q + \sum_{m=1}^{M} \Omega_{\text{pot}} L_m L_m^\dagger \Omega_{\text{pot}} \right] t \\
&- i \Omega_{pq} \sum_{p \neq q}^{N} x_p y_q \tilde{O}_{xp} - i \sum_{m=1}^{N} \left( y^\dagger_{ns} \tilde{O}_{sn} + x^\dagger_{ns} \tilde{O}_{n} \right) |\psi(t), y\rangle, \\
&\text{where} \\
&\tilde{O}_{xp} \equiv \tilde{O}_{xp}(t) = \int_0^t ds \alpha_p(t, s) \Omega_{xp}(t, s), \\
&\tilde{O}_{n} \equiv \tilde{O}_{n}(t) = \int_0^t ds \beta(t, s) \Omega_{n}(t, s). \\
&\text{Eq. (a8) is just eq. (3) in the main text. According to the consistency condition } \partial_t \delta_{\alpha} = \delta_{\beta} \partial_t \text{ and } \partial_t \delta_{\gamma} = \delta_{\gamma} \partial_t, \text{ one can obtain the equations of motion for the } O\text{-operators,} \\
&\partial_t O_{\text{mn}}(t, s) = -i [H_{\text{eff}}, O_{\text{mn}}] - i \delta_{\alpha} H_{\text{eff}} \\
&= -i [H_{\text{eff}}, O_{\text{mn}}] - \sum_{m=1}^{M} \Omega_{\text{pot}} L_m L_m^\dagger \Omega_{\text{pot}} \\
&- i \Omega_{pq} \sum_{p \neq q}^{N} x_p \delta_{\alpha} y_q \delta_{\beta} \\
&- i \sum_{m=1}^{N} \left( y^\dagger_{ns} \delta_{\alpha} x_{ns} + x^\dagger_{ns} \delta_{\alpha} x_{ns} \right), \\
&\text{where} \\
&\Omega_{\text{pot}} L_m L_m^\dagger \Omega_{\text{pot}}. \\
&\text{One can consequently obtain a general exact master eq. (6) from eq. (a8)} \\
&\tilde{\rho}_s = M[|\psi(t), y\rangle |\psi(t), y\rangle] \\
&= -i H_q \tilde{\rho}_s + \sum_{m=1}^{M} \left( \Omega_{\text{pot}} [O_{\text{pot}}, \sigma_{\gamma}^m] \right) \hbar. \\
&\text{The example considered in sect. 4.1 is a system of } M = 1 \text{ qubit embedded in a cavity network consisting of } N \text{ leaky cavities. The } O\text{-operators can then be given by} \\
&O_{\text{mn}}(t, s) = f_{\text{mn}}(t, s) \sigma_{\gamma} - O_{\text{ns}}(t, s) = f_{\text{ns}}(t, s) \sigma_{\gamma}. \\
&\text{Inserting eq. (a16) into eqs. (a10) and (a11) and using the definitions in eq. (a14) and the spectral function (5), one can obtain the differential equations for the coefficients in the } O\text{-operators:} \\
&\frac{dF_{\text{mn}}}{dt} = \Omega^*_{\text{mn}} \sigma_{\gamma} + i \Omega_{\text{pot}} \sum_{m=1}^{M} F_{\text{nm}} + i (\omega_{\text{mn}} - \omega_{\text{nm}}) F_{\text{mn}}, \\
&\text{and} \\
&\frac{dF_{\text{ns}}}{dt} = -i (\omega_{\text{ns}} - \gamma) F_{\text{ns}} + i \Omega^*_{\text{mn}} F_{\text{ns}} - i \gamma, \\
&\text{The exact master equation for the one-qubit system can be obtained from eq. (a15) for } M = 1 \text{ and eq. (a16)} \\
&\tilde{\rho}_s = -i \left( \frac{\omega_{\gamma}}{2} \sigma_{\gamma} - \tilde{\rho}_s \right) + \Omega_{\text{pot}} F_{\text{ns}}^* \sigma_{\gamma} + i \Omega_{\text{pot}} F_{\text{ns}}^* \sigma_{\gamma} + i \Omega_{\text{pot}} F_{\text{ns}}^* \sigma_{\gamma}. \\
&\text{In the two-qubit case (} M = 2 \text{), the } O\text{-operators have a more complex formation. Based on eq. (a13), we have} \\
&O_{\text{ns}}(t, s) = f_{\text{ns}}^{11}(t, s) \sigma_{\gamma}^{(1)} + f_{\text{ns}}^{12}(t, s) \sigma_{\gamma}^{(2)} + f_{\text{ns}}^{21} \sigma_{\gamma}^{(2)} \sigma_{\gamma}^{(1)} + f_{\text{ns}}^{22} \sigma_{\gamma}^{(2)} \sigma_{\gamma}^{(1)} - \end{align*}\]
\[ O_j(t,s) = \int f_{j1}^{(1)}(t,s) \sigma_\uparrow^{(1)} + f_{j2}^{(1)}(t,s) \sigma_\downarrow^{(1)} + f_{j1}^{(2)}(t,s) \sigma_\uparrow^{(2)} + f_{j2}^{(2)}(t,s) \sigma_\downarrow^{(2)}. \]  
\[ (a20) \]

From the consistency conditions (eqs. (a10) and (a11)), these coefficients follow a close group of differential equations ($\kappa = x_n, y$):

\[ \partial_t f_{jn}^{(1)} = \iota \omega_{jn} f_{jn}^{(1)} + \Omega_{1n} f_{jn}^{(1)} F_{x1}^{(1)} + \Omega_{2n} f_{jn}^{(2)} F_{x2}^{(2)} - \Omega_{2n} f_{jn}^{(2)} F_{x1}^{(2)} + \Omega_{2n} f_{jn}^{(1)} F_{x2}^{(1)} + \Omega_{2n} f_{jn}^{(2)} F_{x1}^{(2)} + \Omega_{2n} f_{jn}^{(1)} F_{x2}^{(2)} + \Omega_{2n} f_{jn}^{(2)} F_{x1}^{(2)} \]
\[ (a21) \]

\[ \partial_x f_{jn}^{(1)} = \iota \omega_{jn} f_{jn}^{(1)} - \Omega_{1n} f_{jn}^{(1)} F_{x1}^{(1)} + \Omega_{2n} f_{jn}^{(2)} F_{x2}^{(2)} - \Omega_{2n} f_{jn}^{(2)} F_{x1}^{(2)} + \Omega_{2n} f_{jn}^{(1)} F_{x2}^{(2)} + \Omega_{2n} f_{jn}^{(2)} F_{x1}^{(2)} + \Omega_{2n} f_{jn}^{(1)} F_{x2}^{(2)} + \Omega_{2n} f_{jn}^{(2)} F_{x1}^{(2)} \]
\[ (a22) \]

\[ \partial_x f_{jn}^{(2)} = \iota \omega_{jn} f_{jn}^{(2)} - \Omega_{1n} f_{jn}^{(1)} F_{x1}^{(1)} + \Omega_{2n} f_{jn}^{(2)} F_{x2}^{(2)} - \Omega_{2n} f_{jn}^{(2)} F_{x1}^{(2)} + \Omega_{2n} f_{jn}^{(1)} F_{x2}^{(2)} + \Omega_{2n} f_{jn}^{(2)} F_{x1}^{(2)} + \Omega_{2n} f_{jn}^{(1)} F_{x2}^{(2)} + \Omega_{2n} f_{jn}^{(2)} F_{x1}^{(2)} \]
\[ (a23) \]

\[ \partial_x f_{jn}^{(2)} = \iota \omega_{jn} f_{jn}^{(2)} - \Omega_{1n} f_{jn}^{(1)} F_{x1}^{(1)} + \Omega_{2n} f_{jn}^{(2)} F_{x2}^{(2)} - \Omega_{2n} f_{jn}^{(2)} F_{x1}^{(2)} + \Omega_{2n} f_{jn}^{(1)} F_{x2}^{(2)} + \Omega_{2n} f_{jn}^{(2)} F_{x1}^{(2)} + \Omega_{2n} f_{jn}^{(1)} F_{x2}^{(2)} + \Omega_{2n} f_{jn}^{(2)} F_{x1}^{(2)} \]
\[ (a24) \]

And the boundary condition can be obtained from eq. (a12).

\[ f_{jn}^{(1)}(t) = \Omega_{yn} \delta_{j,n} - i \Omega_{mn} \sum_{m=1}^{N} F_{mn}^{(1)}(t) - i F_{jn}^{(1)}(t), \]
\[ (a25) \]

\[ f_{jn}^{(2)}(t) = -i \Omega_{mn} \sum_{m=1}^{N} F_{mn}^{(2)}(t) - i F_{jn}^{(2)}(t), \]

\[ f_{jn}^{(2)}(t) = -i \Omega_{mn} \sum_{m=1}^{N} F_{mn}^{(2)}(t) - i F_{jn}^{(2)}(t), \]

\[ f_{jn}^{(2)}(t) = \Omega_{2n} \delta_{j,n} - i \Omega_{mn} \sum_{m=1}^{N} F_{mn}^{(2)}(t) - i F_{jn}^{(2)}(t), \]

\[ f_{jn}^{(p,q)}(t) = -i \sum_{n=1}^{N} F_{jn}^{(p,q)}(t), \]

\[ p, q = 1, 2. \]

The exact master equation of the two-qubit system (coupled to N leaking cavities) is given by

\[ \dot{\rho}_s = -i \omega_{a1} \sigma_\uparrow^{(1)} + i \omega_{a2} \sigma_\downarrow^{(2)} + \rho_s \]
\[ + \sum_{j=1}^{2} \left( \Omega_{1j}^{(1)} \rho_s \sigma_\uparrow^{(1)} + \Omega_{2j}^{(2)} \rho_s \sigma_\downarrow^{(2)} \right). \]  
\[ (a26) \]

A2 Derivation of the effective Hamiltonian

In this appendix, we use the unitary transformation to simplify the total system indicated by eq. (1). To be simple but with no loss of generality, we assume an isotropic condition: the qubits have the same transition frequency $\omega_{a1} = \omega_{a2} = \cdots = \omega_a$, the frequencies of cavities are also identical $\omega_{c1} = \omega_{c2} = \cdots = \omega$, and the coupling strengths of atom-cavity and cavity-cavity satisfy $\Omega_{c1} = \Omega_{c2} = \Omega_c$ and $\Omega_{pq} = \Omega_c$, respectively. Under this condition (note it is more general than the resonant condition $\omega_a = \omega$ in the main text) and the single-exciton assumption, we can set up an effective model in which the qubit system is merely coupled to a few tilde modes. The transformations read

\[ \begin{bmatrix} a_1, a_2, \cdots, a_N \end{bmatrix}^T = U^{(N)} [\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_N]^T, \]
\[ [\sigma_1, \sigma_2, \cdots, \sigma_M]^T = U^{(M)} [\tilde{\sigma}_1, \tilde{\sigma}_2, \cdots, \tilde{\sigma}_M]^T, \]  
\[ (a27) \]

where $U^{(N)}$ is a real unitary matrix used to correlate the original operators $a_i (\sigma_{ij}^{(2)})$ and the tilde operators $\tilde{a}_i (\tilde{\sigma}_{ij}^{(2)})$. To explicitly obtain $U^{(N)}$, one can regard the following procedure or restrictions: (1) The matrix is unitary so that every row or column is normalized. (2) To make sure only one effective cavity mode (here it is marked as $\tilde{a}_i$) is directly coupled to the reservoir and to decouple all the tilde modes, the first column of $U^{(N)}$ is set to be $1/\sqrt{N} \begin{bmatrix} 1, 1, \cdots, 1 \end{bmatrix}^T$ and all the summations of the elements along the rest columns are required to be zero. (3) When $N \geq 3$, to simplify the interaction formation between the tilde qubit and the tilde cavity mode, one can fix that $U^{(N)}_{23} = -U^{(N)}_{13} = \frac{1}{\sqrt{2}}$ and $U^{(N)}_{1m} = U^{(N)}_{2m} = 0$ with $m > 3$. All elements of the matrix $U^{(N)}$ can then be straightforwardly yet maybe not uniquely obtained. Particularly, for $N = 2, 3, 4$, the matrices read:

\[ U^{(2)} = \begin{pmatrix} 1 & -1 \\ \sqrt{2} & \sqrt{2} \end{pmatrix}, \]  
\[ (a28) \]

\[ U^{(3)} = \begin{pmatrix} \sqrt{3} & \sqrt{6} & -1 \\ 3 & 6 & \sqrt{2} \end{pmatrix}, \]  
\[ (a29) \]

\[ U^{(4)} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & \sqrt{2} \\ 2 & 2 & \sqrt{2} \end{pmatrix}, \]  
\[ (a30) \]

It can be checked that these new operators satisfy all the required commutation relations for Pauli matrices: $[\tilde{a}_i^{(1)}, \tilde{\sigma}_i^{(1)}] = 0$ and $[\tilde{a}_i^{(1)}, \tilde{\sigma}_i^{(2)}] = \tilde{\sigma}_i^{(2)} \delta_{ij}$. The commutation relations for bosonic modes are also conserved: $[\tilde{a}_i, \tilde{a}_j] = 0$ and $[\tilde{a}_i, \tilde{a}_j^+] = \delta_{ij}$. 
Based on eq. (a27), the original Hamiltonian (1) is converted to

\[ H' = H'_q + H'_{qc} + H'_{cc} + H'_{ce}, \]  

(a31)

\[ H'_q = \sum_{i=1}^{M} \frac{\omega_i}{2} a_i^\dagger a_i, \]  

(a32)

\[ H'_{qc} = \Omega^* a \sum_{m=1}^{M} \sum_{i=1}^{N} U_{mi} U_{mj}^* \tilde{a}_i^\dagger a_j + \text{h.c.}, \]  

(a33)

\[ H'_{cc} = [\omega + (N-1)\Omega] \tilde{a}_1^\dagger \tilde{a}_1 + \sum_{n=2}^{N} (\omega - \Omega) \tilde{a}_n^\dagger \tilde{a}_n, \]  

(a34)

\[ H'_{ce} = \sqrt{N} \tilde{a}_1 \sum_{k} g_k b_k e^{-i\delta_k} + \text{h.c.}. \]  

(a35)

In the transformed picture, it is clear that the mutual interaction among the physical cavities vanishes, and the renormalized frequency of the artificial cavities is determined by the cavity-cavity coupling strength \( \Omega \). In addition, through the transformation, only one artificial cavity-mode represented by \( \tilde{a}_1 \) is left to be subject to the global reservoir. Thus an artificial qubit can be isolated from the reservoir if it has no direct coupling with the artificial mode \( \tilde{a}_1 \). Then the excited-state population of the decoupled qubits as well as the relevant quantumness therein is free of the effective interaction Hamiltonian \( H'_{qc} \) in eq. (a33).

In the single-qubit case, i.e., \( M = 1 \), it is not necessary to transform the qubit operator. The qubit-cavity interaction Hamiltonian (a33) for \( N \geq 3 \) can then be written as:

\[ H'_{qc} = \left( \frac{1}{\sqrt{N}} \Omega^* \tilde{a}_1^\dagger + \frac{\sqrt{2N^2 - 4N}}{2N} \Omega^* \tilde{a}_2^\dagger - \frac{1}{\sqrt{2}} \Omega^* \tilde{a}_3^\dagger \right) \sigma_- + \text{h.c.}. \]  

(a36)

If there are only two cavities, then eq. (a33) becomes

\[ H'_{qc} = \left( \frac{1}{\sqrt{2}} \Omega^* \tilde{a}_1^\dagger - \frac{1}{\sqrt{2}} \Omega^* \tilde{a}_2^\dagger \right) \sigma_- + \text{h.c.}. \]  

(a37)

In the double-qubit case, i.e., \( M = 2 \), we can get a compact form of eq. (a33) for \( N \geq 3 \):

\[ H'_{qc} = \left[ \Omega^* \tilde{\sigma}_A \tilde{a}_1^\dagger + \Omega^* \tilde{\sigma}_B \left( \frac{\sqrt{2N}}{N} \tilde{a}_1^\dagger + \frac{\sqrt{N^2 - 2N}}{N} \tilde{a}_2^\dagger \right) \right] + \text{h.c.}, \]  

(a38)

where the two artificial qubits are labeled by A and B. One can see that merely three mutually-decoupled cavities take part into the dynamics of qubit system. In particular, the qubit described by \( \tilde{\sigma}_B \) interacts with two cavities, one of which is leaky and the other is lossless. In the limit of a large \( N \), qubit-B interacts only with the lossless cavity-mode \( \tilde{a}_2 \), and another qubit \( \tilde{\sigma}_A \) interacts with a lossless cavity by a strength \( \Omega_a \). Therefore, the quantumness contained in qubit-A is immune to the decoherence process caused by the external environment.