Star network synchronization led by strong coupling-induced frequency squeezing

Benedetto Militello$^{1,2,4}$, Dariusz Chruściński$^3$ and Anna Napoli$^{1,2}$

$^1$ Dipartimento di Fisica e Chimica, Università degli Studi di Palermo, Via Archirafi 36, I-90123 Palermo, Italy
$^2$ I.N.F.N. Sezione di Catania, Italy
$^3$ Institute of Physics, Faculty of Physics, Astronomy and Informatics Nicolaus Copernicus University, Grudziadzka 5/7, 87-100 Torun, Poland

E-mail: benedetto.militello@unipa.it

Received 6 July 2017, revised 31 October 2017
Accepted for publication 22 November 2017
Published 3 January 2018

Abstract

We consider a star network consisting of $N$ oscillators coupled to a central one which in turn is coupled to an infinite set of oscillators (reservoir), which makes it leaking. Two of the $N$ normal modes are dissipating, while the remaining $N-1$ lie in a frequency range which is more and more squeezed as the coupling strengths increase, which realizes synchronization of the single parts of the system.

Keywords: synchronization, star network, dissipation

(Some figures may appear in colour only in the online journal)

1. Introduction

The possibility of synchronizing the dynamics of two or more, different or not, physical systems, thus realizing coherent evolutions, has attracted a lot of attention in many different disciplines like physics, chemistry, biology but also social science [1]. Synchronization phenomenon is relevant also in view of applications in neurosciences and even in medicine [2, 3]. It is a rich and intriguing phenomenon that was originally studied in classical systems where it is effectively described by the Kuramoto model [4]. Detailed studies (based on the laws of classical mechanics) of occurrence of synchronization in classical systems, such as coupled pendula and metronomes, have been reported [5, 6].

During the few last decades, the search for synchronized behaviors has been extended to quantum platforms [7]. In this context, attempts to adapt the Kuramoto model to the quantum realm have been made [8]. Beyond the Kuramoto model, studies of synchronization in quantum systems have been developed [9–11], also with the aim of providing proper definition and measure tools of the degree of synchronization [12–14].

In the last few years, other than the natural archetypical quantum system, i.e. the harmonic oscillator, fundamental class of quantum systems have been considered in order to prove the possibility of realizing synchronization processes. A few uncoupled spins interacting with a common environment [15] as well as ensembles of dipoles [16] have been considered. More recently, collective behavior of many spins has been studied in order to establish a connection between synchronization processes and superradiance or subradiance [17]. A further extension present in the literature is the dynamical alignment of optomechanical systems [18–20], as well as hybrid systems like two-level atoms and oscillators [14, 21, 22]. Some works tracing back the origin of quantum synchronization to dissipation have appeared [9, 23, 24].

In this paper, we try to give a quantum counterpart of typical scenario of synchronization in classical mechanics. It is well known that two or more metronomes lying on a common platform which in turn can move (for example being place above two cans) and dissipate energy to the ground eventually synchronize. We then consider $N$ quantum harmonic oscillators (the metronomes counterparts) interacting with another oscillator (corresponding to the platform) which is coupled to a an environment consisting of an infinite set of harmonic oscillators (corresponding to the ground).
We will show that in the strong coupling limit (when the coupling of each oscillator with the leaking one is big) we obtain a twofold effect: on the one hand, two leaking normal modes appear, and on the other hand the remaining non-leaking modes occupy a frequency range whose amplitude becomes smaller and smaller when the strength of the coupling with the leaking mode increases. Our analysis thus allows us to bring into light, in a very clear way, the origin of the mechanism that, at least for the model envisaged in the paper, leads to synchronization phenomenon.

2. The model

Consider a system governed by the following Hamiltonian:

\[ H = H_S + H_B + H_I, \]  

with the system (a star network) Hamiltonian,

\[ H_S = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{p_i^2}{m} + k_i x_i^2 \right) + \frac{1}{2} \sum_{i=1}^{N} g_i (x_{N+1} - x_i)^2, \]

the Hamiltonian of the bath (reservoir),

\[ H_B = \sum_j \frac{p_j^2}{2 m_j} + \frac{1}{2} k_j y_j^2, \]

and the interaction Hamiltonian given by

\[ H_I = \gamma_j y_j, \]

where \( y_j \)'s are the coordinates referred to the bath oscillators.

There is a single mode of the main system—the one corresponding to \( x_{N+1} \)—which is a leaking one. We assume that all the masses of the \( N + 1 \) oscillators are equal, a condition that we can always realize through a suitable canonical transformation. In general, the coupling constants \( g_i \)'s can be positive or negative, in the latter case describing repulsive interactions. In our specific case, since to obtain synchronization we will consider the large \( g_i \) limit, in order to prevent instability of the system we will assume \( g_i \geq 0 \) \( \forall i \).

3. Normal modes analysis

After ordering the \( N + 1 \) coordinates as \( (x_1, \ldots, x_N, x_{N+1}) \), the whole potential in (2) can be considered as a quadratic form

\[ \frac{1}{2} \sum_{i=1}^N \frac{p_i^2}{m_i} + \sum_{i=1}^N k_0 x_i^2, \]

associated to the following matrix:

\[ V = \begin{pmatrix} k_1 + g_1 & 0 & \cdots & -g_1 \\ 0 & k_2 + g_2 & \cdots & -g_2 \\ 0 & 0 & \ddots & \vdots \\ -g_1 & -g_2 & \cdots & k_N + g_N -g_N & k_{N+1} + Ng_{N+1} \end{pmatrix}, \]

with the averaged coupling constant

\[ g_{av} = \frac{1}{N} \sum_{j=1}^{N} g_j. \]  

This matrix can be reorganized as:

\[ V = (k_{av} + g_{av})I + G + D, \]

with the diagonal matrix

\[ D = \begin{pmatrix} \delta k_1 + \delta g_1 & 0 & \cdots & 0 \\ 0 & \delta k_2 + \delta g_2 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \delta k_N + \delta g_N \end{pmatrix}, \]

\[ G = \begin{pmatrix} 0 & 0 & \cdots & -g_1 \\ 0 & 0 & \cdots & -g_2 \\ 0 & 0 & \ddots & \vdots \\ -g_1 & -g_2 & \cdots & -g_N \end{pmatrix}, \]

where we have introduced the averaged Hook’s constant,

\[ k_{av} = \frac{1}{N + 1} \sum_{j=1}^{N+1} k_j, \]

and

\[ \delta k_j \equiv k_j - k_{av}, \quad \delta g_j = g_j - g_{av}, \]

together with

\[ \Delta = 2k_{av} + (N - 1)g_{av}. \]

In what follows we consider the following regime:

\[ |\delta g_j|, |\delta k_j| \ll g_{min}, \]

with \( g_{min} = \min g_j \), one may treat \( D \) matrix as a perturbation to \( (k_{av} + g_{av})I + G \), that is, to diagonalize \( V \) one diagonalizes \( G \) and then looks for perturbative corrections induced by \( D \).

3.1. Diagonalization of \( G \)

The eigenvalues of \( G \) are: \( G_{\pm} = 0 \) (\( N - 1 \) eigenstates), and two non degenerate eigenvalues

\[ G_{\pm} = \frac{1}{2} (\Delta \pm \sqrt{\Delta^2 + 4N^2}) \]

with \( N^2 = \sum_j g_j^2 \). The corresponding ‘eigenvectors’ are:

\[ \bar{x}_{\pm} = \sum_{j=1}^{N} \alpha_{j,N+1}^+ x_j + \alpha_{N+1}^+ x_{N+1}, \]

with

\[ \alpha_j^+ / \alpha_{N+1}^+ = -g_j / G_{\pm}, \quad j = 1, \ldots, N \]
and for the degenerate \((N - 1)\)-dimensional subspace we can take the following orthonormal set:

\[
\begin{align*}
\bar{x}_{0,1} &= \cos\theta_1 x_1 + \sin\theta_1 x_2, \\
\bar{x}_{0,2} &= \cos\theta_2 (-\sin\theta_1 x_1 + \cos\theta_1 x_2) + \sin\theta_2 x_3, \\
\bar{x}_{0,3} &= \cos\theta_3 (-\sin\theta_2 (-\sin\theta_1 x_1 + \cos\theta_1 x_2) + \cos\theta_2 x_3) + \sin\theta_3 x_4, \\
&\vdots
\end{align*}
\]

(17)

with

\[
\begin{align*}
\tan\theta_1 &= -\frac{g_1}{g_2} \\
\tan\theta_2 &= -\sqrt{\frac{g_1^2 + g_2^2}{g_3}} \\
\tan\theta_3 &= -\sqrt{\frac{g_1^2 + g_2^2 + g_3^2}{g_4}} \\
&\vdots
\end{align*}
\]

(18)

The 0-modes do not involve the coordinate \(x_{N-1}\) and then are not subjected to decay processes, therefore we will call them ‘protected’ modes.

### 3.2. Perturbation treatment of \(D\)

In order to complete our analysis we need to treat \(D\) perturbatively.

First of all, we evaluate the eigenvalue zeroth-order correction related to the normal modes \(\bar{x}_\pm\), which is easily done by evaluating the relevant diagonal matrix elements:

\[
\Delta k_\pm \equiv (\bar{x}_\pm, D \bar{x}_\pm)
\]

\[
= (\alpha^\pm_N)^2 \Delta + \sum_{j=1}^{N} (\alpha^+_j)^2 (\delta k_j + \delta g_j).
\]

(19)

Then we should proceed by evaluating the first order correction to the ‘eigenvectors’. Such corrections are of the order \((\delta k_j + \delta g_j)/g_{av}\), so that we can write:

\[
\bar{x}_\pm = x_\pm + o(\xi),
\]

(20)

where

\[
\xi = \max_j |\delta k_j| + |\delta g_j|/g_{av}.
\]

(21)

For the protected modes we need to diagonalize the restriction of \(D\) in the relevant eigenspace. Following this procedure, we will obtain a correction to the ‘eigenvectors’:

\[
\bar{x}_{0,j} \rightarrow \left\{ \bar{x}_{0,j} = \sum_k O_k \bar{x}_{0,k} \right\}
\]

(22)

and a correction to the eigenvalues, \(\Delta k_{0,j}\). Therefore the eigenvectors must be corrected to the first order, then obtaining:

\[
\bar{x}_{0,j} = x_{0,j} + o(\xi).
\]

(23)

The corrected Hook’s constants are then given by:

\[
k_\pm = g_{av} + k_{av} + G_\pm + \Delta k_\pm,
\]

(24)

and, for the protected modes:

\[
k_{0,j} = g_{av} + k_{av} + \Delta k_{0,j}.
\]

(25)

A very special case is that of two oscillators. Indeed, if \(N = 2\) then the 0-mode subspace is a singlet (then \(\bar{x}_{0,j} = \bar{x}_{0,j}\)) and the correction of the eigenvalues requires only the evaluation of the relevant diagonal matrix element of \(D\):

\[
\Delta k_{0,1} \equiv (\bar{x}_{0,1}, D \bar{x}_{0,1}) = \cos^2 \theta_1 (\delta k_1 + \delta g_1) + \sin^2 \theta_1 (\delta k_2 + \delta g_2)
\]

\[
= \frac{g_1^2 (\delta k_1 + \delta g_1)}{g_1^2 + g_2^2} + \frac{g_2^2 (\delta k_2 + \delta g_2)}{g_1^2 + g_2^2},
\]

(26)

\[
\bar{x}_{0,1} = x_{0,1} + o(\xi).
\]

(27)

### 4. Uniform versus almost uniform model

Let us consider the case of \(N\) different oscillators (that is different \(k_i’s\)) but with the same coupling strengths to the leaking mode \((g_i = g, \forall i)\), which means \(\delta g = 0, \forall i\).

Now, whatever it is the specific value of \(g\), the structure of the ‘eigenvectors’ in (15) and (17) does not change, as well as the matrix \(D\) is not modified.

Therefore, by making a first order Taylor expansion in the expression of the frequencies of the protected modes, we get:

\[
\omega_{0,j} = \sqrt{\frac{k_{0,j}}{m}} \approx \sqrt{g + k_{av}} \left[ 1 + \frac{\Delta k_{0,j}}{2(g + k_{av})} \right]
\]

(28)

Let us define: \(\Delta k = \max|\Delta k_{0,j}|, \delta k = \max_j|\delta k_j|\), which are of the same order (see for example the special case in (26) and put \(\delta g = 0\)). We can then say that \(\Delta \omega\) (the amplitude of the frequency range where all the corrected frequencies of the 0-mode multiplet lie) is of the order of \(\Delta k/(\sqrt{m(g + k_{av})})\). This quantity is smaller than the original frequency range, which is approximately given by \(\delta k/\sqrt{mk_{av}}\). The higher \(g\), the tighter is the frequency distribution of the ‘preserved’ modes.

This squeezing of the frequency range is probably the main result of this paper. Indeed, the decay of the leaking normal modes is not enough to justify the occurrence of synchronization. It is also necessary that the surviving modes are characterized by the same frequency. This essentially happens when \(g\) is assumed to be much larger than all \(\delta k_j\), for the uniform model.

The prediction of a frequency squeezing, though obtained for the uniform model, can be easily obtained for a non-uniform model provided it satisfies the condition that all \(\delta g\) are kept smaller than a certain quantity, say \(\delta g\), in spite of the fact that each \(g_i\) can increase (this can be easily obtained for example when the \(g_i’s\) are equally lifted: \(g_i \rightarrow \xi + g_i\), so that in the limit \(g_i \rightarrow \infty\) one has \(\delta g / g_i \rightarrow 0\). Under such hypothesis, it is evident that the frequency squeezing in (28) still holds.

### 5. Evolution of quantum states

To complete our analysis we want to apply the previous theory to the study of an evolution.
5.1. Canonical transformation

Consider the following canonical transformations of the \( N + 1 \) coordinates and momenta:

\[
\bar{x}_j = \sum_l b^{(j)}_l x_l \Leftrightarrow x_l = \sum_j e^{(j)}_l \bar{x}_j, \tag{29}
\]

\[
\bar{p}_j = \sum_l c^{(j)}_l p_l \Leftrightarrow p_l = \sum_j \eta^{(j)}_l \bar{p}_j, \tag{30}
\]

with \( j = 1, \ldots N + 1 \) and \( l = \pm, \, 0, \, n' \), where the coefficients should satisfy the following conditions to preserve the commutation relations:

\[
\sum_l b^{(j)}_l c^{(j')}_l = \delta_{jj'}, \quad \forall j, j', \tag{31}
\]

\[
\sum_l \eta^{(j)}_l \epsilon^{(j')}_l = \delta_{jj'}, \quad \forall j, j'. \tag{32}
\]

Consequently, the annihilation operators are transformed as follows:

\[
a_j = \sum_k \left[ \frac{c^{(j)}_k}{\omega_k} \bar{a}_k + \frac{\omega_k}{\omega_k} \bar{a}_k \right] a_k + \sum_k \left[ \frac{\epsilon^{(j)}_k}{\omega_k} - \frac{\omega_k}{\omega_k} \right] \bar{a}_k \right], \tag{33}
\]

and

\[
\bar{a}_j = \sum_k \left[ \frac{b^{(j)}_k}{\omega_k} \bar{a}_k + \frac{\omega_k}{\omega_k} \bar{a}_k \right] a_k + \sum_k \left[ \frac{b^{(j)}_k}{\omega_k} - \frac{\omega_k}{\omega_k} \right] \bar{a}_k \right], \tag{34}
\]

where \( \tilde{\omega}_j \) denotes the corresponding frequencies of the eigenmodes.

5.2. Markovian evolution of the star network

To derive the Markovian master equation for the evolution of the system one may follow the standard approach assuming for example weak coupling between system and the bath and assume that initially the bath was in the thermal state at the temperature \( T \). Since the 0-modes (i.e., the protected modes) involve the modes \( \hat{x}_\pm \) to the order \( \xi \ll 1 \), deriving the master equation in the normal mode representation one finds that the 0-modes are characterized by decay rates which are of the order \( \xi \) with respect to the decay rates related to the \( \pm \) modes. Therefore, with a good degree of approximation, the complete evolution of the system may be evaluated through the following Markovian master equation:

\[
\dot{\rho} = \mathcal{L}_0 \rho + \mathcal{L}_1 \rho + \xi \mathcal{D}_0 \rho \approx \mathcal{L}_0 \rho + \mathcal{L}_1 \rho, \tag{35}
\]

where

\[
\mathcal{L}_0 \rho = -i[H_S, \rho], \tag{36}
\]

\[
\mathcal{L}_1 \rho = -i[H_S, \rho] + \mathcal{D}_0 \rho + \mathcal{D}_1 \rho, \tag{37}
\]

with the corresponding Hamiltonians:

\[
H_S^0 = \sum_j \tilde{\omega}_j \bar{a}^\dagger_j \bar{a}_j, \tag{38}
\]

\[
H_S^1 = \tilde{\omega}_+ \bar{a}_+^\dagger \bar{a}_+ + \tilde{\omega}_- \bar{a}_-^\dagger \bar{a}_-, \tag{39}
\]

and dissipators in the dissipative sectors:

\[
\mathcal{D}_0 \rho = \gamma_+ (\tilde{\omega}_+) \left[ a_+^\dagger a_- + \frac{1}{2} (a_-^\dagger a_-, \rho) \right] + \gamma_- (\tilde{\omega}_-) \left[ a_-^\dagger a_+ + \frac{1}{2} (a_+^\dagger a_-, \rho) \right]. \tag{40}
\]

Finally, \( \mathcal{D}_0 \) is a dissipator describing the decay of the 0-modes, which we neglect being of \( \xi \) order.

To better understand the origin of this microscopic master equation, consider the Hamiltonian in equation (4) which is responsible for the decay of the \( x_N \) mode. Once the variable \( x_N \) is expressed as a linear combination of the normal mode coordinates—\( x_N = \cos \theta \hat{x}_+ + \sin \theta \hat{x}_- + o(\xi) \) with \( \tan \theta = -G/\Gamma_s \), coming from equations (15) and (20)—we can see that the dissipators associated to the modes \( \hat{x}_+ \) and \( \hat{x}_- \) naturally emerge from the standard derivation of the microscopic Markovian master equation of a damped harmonic oscillator [25, 26]. At the same time, the negligibility of the dissipators related to the 0-modes is well visible. In fact, we have \( N + 1 \) uncoupled oscillators, each one interacting with the environment, two of them with significant strengths, \( N - 1 \) with negligible strengths. It is worth noting that, generally speaking, since the modes \( \hat{x}_+ \) and \( \hat{x}_- \) interact with the same bath, cross terms could appear in the dissipator. However, since we are deriving the master equation in the standard Born–Markov approximation, which requires also the secular approximation, and since the two oscillators have quite different frequencies (see equation (24)), such cross terms disappear. Finally, we again underline that our master equation is correct up to terms of the order \( \xi \), because this is the precision of our derivation of the normal modes.

The evolution evaluated on the basis of the previous master equation can be factorized as follows:

\[
\rho(0) \rightarrow \rho(t) = \mathcal{S}(t) \rho(0) = \mathcal{S}_0(t) \mathcal{S}_1(t) \rho(0), \tag{41}
\]

with \( \mathcal{S}_0(t) \) and \( \mathcal{S}_1(t) \) generated by \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \), respectively: \( \mathcal{S}_j = \exp(\mathcal{L}_j t) \), with \( j = 0, 1 \).

Suppose now that the system is prepared in a pure quantum state

\[
|\psi(0)\rangle = \sum c_{|n_0|} |n_0\rangle |\{n_\pm\}\rangle. \tag{42}
\]

Here \( |\{n_0\}\rangle |\{n_\pm\}\rangle \) denotes a Fock state of the \( N + 1 \) normal modes, and in particular \( |\{n_0\}\rangle \) is a Fock state of the protected modes, while \( |\{n_\pm\}\rangle \) is a Fock state of the leaking modes. The relevant density operator reads

\[
\rho(0) = |\psi(0)\rangle \langle \psi(0)| = \sum c_{|n_0|} |n_0\rangle \langle n_0| c_{|n_\pm|}^* |n_\pm\rangle \langle n_\pm| + \sum c_{|n_\pm|} |n_\pm\rangle \langle n_\pm| c_{|n_0|}^* |n_0\rangle \langle n_0|. \tag{43}
\]

Under the action of \( \mathcal{S}_1(t) \) for a sufficient large time all the terms with \( |n_\pm| \approx |n_0| \) go to zero (all coherences are destroyed by the dissipative evolution) while the whole set of...
the terms with \(\{n_a\} = \{n_\perp\} \) is eventually mapped to the thermal state of the two leaking modes \(\rho^{(T)}_{\perp}\). Therefore, after a sufficient long time one has:

\[
S(t)\rho(0) \approx \left(\sum_{\{n_a\}} c_{\{n_a\}} c^{\dagger}_{\{n_a\}}|n_a\rangle \langle n_a|\right) S_0(t) \\
\times |\{n_o\}\rangle \langle \{n_o\}| \otimes \rho^{(T)}_{\perp} \\
+ \left(\sum_{\{n_a\}} c_{\{n_a\}} c^{\dagger}_{\{n_a\}}|n_a\rangle \langle n_a|\right) U_0(t) \\
\times |\{n_o\}\rangle \langle \{n_o\}| U^\dagger(t) \otimes \rho^{(T)}_{\perp}
\]

(43)

with \(U_0(t) = \exp(-iH_0^0 t)\).

In particular for \(N = 2\) one has only a single protected mode \(|n_0\rangle\) indicates its generic Fock state and hence:

\[
S(t)\rho(0) \approx U_0(t) \left(\sum_{n_a, n_\perp} \alpha_{n_a n_\perp} |n_0\rangle \langle n_0| U_0^\dagger(t) \otimes \rho^{(T)}_{\perp}\right),
\]

where

\[
\alpha_{n_a n_\perp} = c_{n_a n_\perp} c^{\dagger}_{n_a n_\perp} + c_{n_a n_\perp} c^{\dagger}_{n_a n_\perp}.
\]

5.3. Expectation values

After a long time, the expectation value of the position operator of any of the protected modes is \(\langle \hat{x} (t) \rangle = \langle \hat{x}_j (0) \rangle \cos(\hat{\omega} t)\), where \(\langle \hat{x}_j (0) \rangle = \langle n_0\rangle \langle n_0| \langle \hat{a}_j + \hat{a}_j^\dagger \rangle \psi(0)\rangle\).

Taking into account of the transformation laws in (29) one easily finds:

\[
\langle x_j (t) \rangle = \sum_j e^{i\hat{\omega}_j t} \langle \hat{x}_j (0) \rangle \cos(\hat{\omega} t).
\]

A similar result holds for \(\langle p_j (t) \rangle\).

It is worth recalling that, at the order we have developed our analysis, the protected modes do not involve the two leaking modes in their transformation. Moreover, the frequencies of the protected modes are very close, in the strong coupling limit \(g_{av} \gg g_{21} \gg g_{12}\). The difference between any couple of such frequencies is of the order of \(\max(g_{21}, g_{12}) \sqrt{\hat{k}_{av} + \hat{g}_{av}}\) and then becomes smaller and smaller as \(g_{av}\) increases. This makes both \(\langle x_j (t) \rangle\) and \(\langle p_j (t) \rangle\) essentially oscillate at a given frequency which is \(\hat{\omega} = \sqrt{\hat{k}_{av} + \hat{g}_{av}} \frac{f}{m}\).

Again, after a long time, the expectation values of the two leaking normal modes are zero \(\langle \hat{x}_+ \rangle = \langle \hat{x}_- \rangle = 0\), then we can get information about the mean values of \(\chi_{N+1}\) and \(\sum_{j=1}^N g_j x_j\).

In order to demonstrate the synchronization and the frequency squeezing in a very simple case, let us consider for example the simple case of three oscillators all coupled to a fourth one. According to the previous analysis, we obtain, as shown in figure 1 that the frequencies of the preserved modes get closer and closer as the coupling constants with the fourth oscillator increase.

6. Conclusions

In this paper, we have considered a system composed by \(N\) oscillators coupled to a central one, which in turn interacts with an infinite set of oscillators (reservoir). This system can be seen as a quantum counterpart of the classical system consisting of two or more metronomes lying in the same platform. We have shown that, under appropriate conditions, it is possible to foresee synchronization phenomena in the dynamics of the system. In particular, the analysis developed clearly puts into light how the original intrinsic frequencies of the oscillators are modified by the interaction with the central one leading to a common effective frequency. In more detail we have demonstrated that two of the \(N + 1\) normal modes of the system are dissipating modes, while the remaining \(N - 1\) lie in a frequency range, which is more and more squeezed as the coupling strength increases. It is just the frequency squeezing phenomenon that allows to the \(N\) oscillators to evolve by swinging in unison. It is worth mentioning that, though also in the case of two oscillators coupled to a third one the system reaches a synchronized regime because of the presence of a single stable mode, the phenomenon of frequency squeezing is visible only with more than two oscillators coupled to the leaking one.

References

[1] Pikovsky A, Rosenblum M and Kurths J 2003

Synchronization: A Universal Concept in Nonlinear Sciences (Cambridge Nonlinear Science Series) (Cambridge: Cambridge University Press)

[2] Strogatz S H and Stewart I 1993 Sci. Am. 269 102

[3] Angelini L, Lattanzi G, Maestri R, Mannozzo D, Nardulli G, Nitti L, Pellicoro M, Pinna G D and Stramaglia S 2004 Phys. Rev. E 69 061923

[4] Acebrón J A, Bonilla L L, Vicente C J P, Ritort F and Spigler R 2005 Rev. Mod. Phys. 77 137
[5] Maianti M, Pagliara S, Galimberti G and Parmigiani F 2009 *Am. J. Phys.* 77 834
[6] Pantaleone J 2002 *Am. J. Phys.* 70 992
[7] Galve F, Giorgi G L and Zambrini R 2017 *QuantumCorrelations and Synchronization Measures (Lectures on General Quantum Correlations and their Applications)* ed F Fanchini, D Soares-Pinto and G Adesso (Berlin: Springer)
[8] Hermoso de Mendoza I, Pachon L A, Gomez-Gardenes J and Zueco D 2014 *Phys. Rev.* E 90 052904
[9] Manzano G, Galve F, Giorgi G L, Harnández-Garcia E and Zambrini R 2013 *Sci. Rep.* 3 1439
[10] Giorgi G L, Galve F, Manzano G, Colet P and Zambrini R 2012 *Phys. Rev.* A 85 052101
[11] Galve F, Giorgi G L and Zambrini R 2010 *Phys. Rev.* A 81 062117
[12] Mari A, Farace A, Didier N, Giovannetti V and Fazio R 2013 *Phys. Rev. Lett.* 111 103605
[13] Ameri V, Eghbali-Arani M, Mari A, Farace A, Kheirandish F, Giovannetti V and Fazio R 2015 *Phys. Rev.* A 91 012301
[14] Hush M R, Li W, Genway S, Lesanovsky I and Armour A D 2015 *Phys. Rev.* A 91 061401
[15] Giorgi G L, Plastina F, Francica G and Zambrini R 2013 *Phys. Rev.* A 88 042115
[16] Zhu B, Schachenmayer J, Xu M, Herrera F, Restrepo J G, Holland M J and Rey A M 2015 *New J. Phys.* 17 083063
[17] Bellomo B, Giorgi G L, Palma G M and Zambrini R 2017 *Phys. Rev.* A 95 043807
[18] Li W, Li C and Song H 2016 *Phys. Rev.* E 93 062221
[19] Li W, Li C and Song H 2017 *Phys. Rev.* E 95 022204
[20] Bemani F, Motazedifard A, Roknizadeh R, Naderi M H and Vitali D 2017 *Phys. Rev.* A 96 023805
[21] Qiu H, Zambrini R, Polls A, Martorell J, Juliá-Díaz B and Song H 2015 *Phys. Rev.* A 92 043619
[22] Padmanaban E, Boccaletti S and Dana S K 2015 *Phys. Rev.* E 91 022920
[23] Manzano G, Galve F and Zambrini R 2013 *Phys. Rev.* A 87 032114
[24] Militello B, Nakazato H and Napoli A 2017 *Phys. Rev.* A 96 023862
[25] Breuer H-P and Petruccione F 2002 *The Theory of Open Quantum Systems* (Oxford: Oxford University Press)
[26] Gardiner C W and Zoller P 2000 *Quantum Noise* (Berlin: Springer)