1. Introduction

The scissors congruence group, or pre-Bloch group, $\mathcal{P}(F)$ of a field $F$ with at least 4 elements (see [2]) is an abelian group defined by an explicit presentation, whose subgroup

\begin{abstract}
We give a definition of (refined) Bloch groups of general commutative rings which agrees with the standard definition in the case of local rings whose residue field has at least 4 elements. Under appropriate conditions on a ring $A$, satisfied by any field or local ring, these groups are closely related to third homology of $\text{SL}_2(A)$ and to indecomposable $K_3$ of $A$. We analyze these conditions. We calculate the Bloch groups of $F_2$, $F_3$, $Z$ and $Z[\frac{1}{2}]$.
\end{abstract}
the Bloch group, $\mathcal{B}(F)$, describes the indecomposable $K_3$ of the field, $K_3^{ind}(F)$, modulo some known torsion coming from the group $\mu_F$ of roots of unity ([18]). (See section 7 below for a more precise statement.) The refined scissors congruence group $\mathcal{RP}(F)$ and the refined Bloch group $\mathcal{RBP}(F)$ were introduced in [7] to help to understand the kernel of the natural surjective homomorphism $H_3(\text{SL}_2(F), \mathbb{Z}) \to K_3^{ind}(F)$. These groups and their properties were then used to calculate $H_3(\text{SL}_2(F), \mathbb{Z}[\frac{1}{2}])$ for local fields $F$ as well as as for $F = \mathbb{Q}$ ([8],[10]). The definitions and techniques in these results extend readily from fields to local rings with sufficiently large residue fields and thus allow us to describe $K_3^{ind}(A)$ and $H_3(\text{SL}_2(A), \mathbb{Z}[\frac{1}{2}])$ for many local rings $A$ ([13],[9])

However, the definitions of these (refined) pre-Bloch groups do not give anything useful in the case of the fields $\mathbb{F}_2$ and $\mathbb{F}_3$ or in the case of local rings with small residue field. In [8] an ad-hoc definition of Bloch groups of the fields $\mathbb{F}_2$ and $\mathbb{F}_3$ was given so that the main results could also be stated and proved in these cases. The ad hoc nature of these definitions is unsatisfactory, however. Furthermore, it was not clear how to define Bloch groups for local rings with small residue fields. In [9] the main results are stated and proved only for local rings whose residue field is sufficiently large.

The purpose of the present article is to extend the definition and fundamental properties of (pre-)Bloch groups to all local rings. In fact we define functorially the pre-Bloch group and refined pre-Bloch group of an arbitrary commutative ring and develop the main properties to be used in applications in this generality (section 3 below). It turns out that some known properties admit simpler proofs in this more general setting. In particular, for any commutative ring $A$ we can define, in the refined pre-Bloch group $\mathcal{RP}(A)$, the constant $C_\mathcal{A}$ and the elements $\psi_1(u)$ and $\psi_2(u)$, $u \in A^\times$ and show that these satisfy certain fundamental algebraic identities, including the key identity $2 \langle u \rangle C_\mathcal{A} = \psi_1(u) - \psi_2(u)$. This identity is the fundamental starting point in using the refined pre-Bloch group to calculate the third homology of $\text{SL}_2(A)$ for local fields and local rings $A$, as well as in calculating $H_3(\text{SL}_2(\mathbb{Q}), \mathbb{Z}[\frac{1}{2}])$. The original proof of this identity for fields with at least 4 elements, in [8], is long, complicated and obscure. Its extension to local rings in [9] was similarly complicated and again required the residue field to have at least 5 elements. By contrast, the proof of this identity given below is short, direct and valid for all commutative rings.

Although we define the pre-Bloch group and refined pre-Bloch group for arbitrary commutative rings $A$, the definition is homological in nature and, in this generality, is not given by an explicit presentation. Furthermore, for a general commutative ring $A$ the Bloch group and refined Bloch group may not be a homomorphic image of $K_3^{ind}(A)$ or of $H_3(\text{SL}_2(A), \mathbb{Z})$ respectively. In sections 5 and 6 below we clarify the conditions on a ring $A$ under which (i) the refined pre-Bloch group is naturally a quotient of $H_3(\text{SL}_2(A), \mathbb{Z})$ and (ii) the pre-Bloch group admits its classical presentation. These conditions are expressed in terms of exactness properties of a certain augmented complex of abelian groups $\cdots \to L_1(A) \to L_0(A) \to \mathbb{Z} \to 0$ that is functorially associated to $A$ (see section 2). If this complex is exact in dimensions 0 and 1 (a condition valid for all fields and local rings), then the refined pre-Bloch group of $A$ is a quotient of $H_3(\text{SL}_2(A), \mathbb{Z})$, at least after tensoring with $\mathbb{Z}[\frac{1}{2}]$. If the complex is exact in dimensions 0, 1 and 2, then the refined pre-Bloch group $\mathcal{RP}(A)$ and the pre-Bloch group $\mathcal{P}(A)$ are generated by the elements $[u]$ where $u(1-u) \in A^\times$. If the complex is exact in dimensions 0, 1, 2 and 3, then $\mathcal{RP}(A)$ and $\mathcal{P}(A)$ admit the standard presentation known from earlier works. It has been shown in [9] that the complex is exact in dimensions $0, \ldots, n$ if $A$ is a local ring whose residue field has at least $n+1$ elements. The question of the exactness in dimensions 0 and 1
for general rings $A$ has been addressed by the second author in [11]. In particular, the complex is exact in dimensions 0 and 1 if $A$ is a $K_2(F)$-ring such that $K_2(Z, A)$ is generated by symbols. This applies to the rings $Z$ and $Z[1/\ell]$, for example, but not to the Euclidean domain $Z[1/\ell]$ (see section 3).

In the final section of the article, we verify that the Bloch groups and refined Bloch groups of $F_2$ and $F_3$ have the expected relation to $K_2^\text{ind}(F_2)$ and $K_3^\text{ind}(F_3)$ and to $H_3(SL_2(F_2), Z)$ and $H_3(SL_2(F_3), Z)$ (and, furthermore, that they agree with the ad hoc definitions mentioned above). We show that the pre-Bloch group of $Z$ is cyclic of order 6, generated by $C_Z$ while the Bloch group of $Z$, $B(Z)$, is the cyclic subgroup of order 3 in $P(Z)$. We show that $B(Z[1/\ell])$ is cyclic of order 6 with generator $C_{Z[1/\ell]}$ (and is isomorphic to $B(Q)$).

There are many remaining questions about these general Bloch groups which we hope to address in future. For example:

1. In each of the examples of section 3 it turns out that $RP(A)$ and $P(A)$ are generated by the known (classes of) elements: $C_A$, $\psi(u)$, $u \in A^\times$ and $[u, u(1 - u)] \in A^\times$ and these are subject only to the universal relations itemized and proved in Section 3. What is the most general class of rings for which such a statement is true?

2. If $A$ is a local ring with residue field $F_3$ then $RP(A)$ and $P(A)$ are generated by the elements $[u, u(1 - u)] \in A^\times$. Are there any relations other than those of Section 3? What is a minimal set of relations?

3. If $A$ is a local ring with residue field $F_2$ is it true that $RP(A)$ and $P(A)$ are generated by the elements $C_A$ and $\psi(u)$, $u \in A^\times$?

4. Is it true that for any local ring $A$ one has exact sequences

$$0 \longrightarrow \text{tor}(\mu_2, \mu_2) \longrightarrow H_3(SL_2(A), Z) \longrightarrow RB(A) \longrightarrow 0$$

and

$$0 \longrightarrow \text{tor}(\mu_2, \mu_2) \longrightarrow K_3^\text{ind}(A) \longrightarrow B(A) \longrightarrow 0$$

(at least over $Z[1/\ell]$)?

1.1. Notation and conventions. In this article all rings are commutative and have a unit. For a ring $A$, $A^\times$ denotes the group of units. $R_A$ denotes the group ring $Z[A^\times/(A^\times)^2]$ of the group of square classes of units. For a unit $u \in A^\times$, the image in $R_A$ will be denoted $\langle u \rangle$. The augmentation ideal $I_A$ of the ring $R_A$ is the additive subgroup generated by the elements $\langle u \rangle := \langle u \rangle - 1$, $u \in A^\times$.

2. The complexes $L_\bullet$ and $L^\bullet$

Let $A$ be a ring and let $\Gamma(A)$ be the following associated graph. The vertices of $\Gamma(A)$ are equivalence classes, $[u]$, of unimodular rows $u = (u_1, u_2) \in A^2$ under scalar multiplication by units in $A$. The pair $\{[u], [v]\}$ is an edge in $\Gamma(A)$ if the matrix

$$M = \begin{bmatrix} u \\ v \end{bmatrix}$$

lies in $GL_2(A)$; i.e., if $\det(M) \in A^\times$.

Recall that a subset $S$ of the vertices of a graph $\Gamma$ is a clique if every pair of elements of $S$ is an edge of the graph. We let $X_1(A)$ denote the set of vertices of $\Gamma(A)$ and for $n \geq 1$, we let

$$X_n = X_n(A) := \{ (x_0, \ldots, x_n) \in X_1(A)^{n+1} | \{x_0, \ldots, x_n\} \text{ is a clique in } \Gamma(A) \}.$$
For $n \geq 0$, let $L_n = L_n(A) := \mathbb{Z}[X_{n+1}(A)]$. These groups form a complex $L_\bullet$ by equipping them with the standard simplicial boundary:

$$d_n : L_n \to L_{n-1}, \quad (x_0, \ldots, x_n) \mapsto \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_n).$$

This is the (oriented) clique complex of the graph $\Gamma(A)$.

The sets $X_n(A)$ are naturally right $\text{PGL}_2(A)$-sets, and thus $L_\bullet$ is a complex of right $\text{PGL}_2(A)$-modules (and hence also a complex of right modules over each of $\text{GL}_2(A)$, $\text{PSL}_2(A)$ and $\text{SL}_2(A)$).

Let $L^r_\bullet := L^r_\bullet(A)$ be the following truncation of the complex $L_\bullet$:

$$L^r_n(A) = \begin{cases} L_n(A), & n \leq 2 \\ \ker(L_2(A) \to L_1(A)), & n = 3 \\ 0, & n > 3 \end{cases}$$

(where the map $L^r_2(A) \to L^r_3(A) = L_2(A)$ is the inclusion homomorphism).

Let $\epsilon : K_0(A) = L_0(A) = \mathbb{Z}[X_1] \to \mathbb{Z}$ be the homomorphism sending each $x \in X_1$ to 1. If we regard $\mathbb{Z}$ as a complex of (trivial) $\text{PGL}_2(A)$-modules concentrated in dimension 0, then $\epsilon$ induces maps of complexes $\epsilon : L^r_\bullet \to \mathbb{Z}$ and $\epsilon : L_\bullet \to \mathbb{Z}$.

Observe that $L_\bullet$ and $L^r_\bullet$ define functors from commutative rings to complexes and that there is a natural map of complexes $\epsilon : L^r_\bullet \to L_\bullet$.

Observe that if $M$ is any right $\text{PGL}_2(A)$-module then the module of coinvariants $M_{\text{SL}_2(A)} = M_{\text{PSL}_2(A)}$ is a module over the group $\text{PGL}_2(A)/\text{PSL}_2(A) \cong A^\times/(A^\times)^2$ and hence over the group ring $R_A := \mathbb{Z}[A^\times/(A^\times)^2]$. Here, the square class $\langle a \rangle$ acts via right-multiplication by $X$ where $X \in \text{GL}_2(A)$ is any matrix with determinant $a$.

We will use the following notations: For any $a \in A$, $a_+$ denotes (the class of) $(a, 1)$ in $X_1(A)$ and $a_-$ denotes the class of $(1, a)$. Thus if $a$ is a unit then $a_- = (a^{-1})_+$ in $X_1(A)$. Furthermore, we set

$$0 := 0_+ = (0, 1), \quad \infty := 0_- = (1, 0), \quad 1 := 1_+ = 1_-, \quad -1 := (-1)_+ = (-1)_- \in X_1(A).$$

Note that it is true for any commutative ring $A$ that the three vertices $0, 1, \infty \in X_1(A)$ are connected to each other by edges; i.e., $\{0, 1, \infty\}$ is a 3-clique. Thus we always have at least 6 elements in $X_2(A)$; namely, the orbit of $(0, \infty, 1)$ under $S_3$.

Note that for $x \in A$, $\{0, x_+\}$ is edge of $\Gamma(A)$ if and only if $x \in A^\times$ and $\{1, x_+\}$ is an edge if and only if $1-x \in A^\times$. For any commutative ring $A$, we let $\mathcal{W}_A := \{x \in A \mid x(1-x) \in A^\times\}$. Observe that if $x \in \mathcal{W}_A$ then $x^{-1}, 1-x \in \mathcal{W}_A$ and hence $\mathcal{W}_A$ is naturally acted on by the nonabelian group of order 6 generated by these two involutions. Thus if $x \in \mathcal{W}_A$, then the following 6 elements also lie in $\mathcal{W}_A$:

$$x, \quad \frac{1}{x}, \quad 1-x, \quad \frac{x-1}{x}, \quad \frac{1}{1-x}, \quad \frac{x}{x-1}.$$

We summarize the basic facts about the $\text{PSL}_2(A)$-sets $X_n(A)$ (details can be found in [9, Section 3]):

For a commutative ring $A$ let

$$B = B_A := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \text{GL}_2(A) \right\}, \quad T = T_A := \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \text{GL}_2(A) \right\}.$$

For any subgroup $G$ of $\text{GL}_2(A)$, let $B(G) = G \cap B$ and $T(G) = G \cap T$. We let $Z = Z(\text{GL}_2(A)) = A^\times \cdot I$. 


Proposition 2.1. Let \( A \) be a ring.

1. \( \text{SL}_2(A) \) acts transitively on \( X_1(A) \). The stabilizer of \((0)\) is the subgroup \( B(\text{SL}_2(A)) \).
2. \( \text{SL}_2(A) \) acts transitively on \( X_2(A) \). The stabilizer of \((0, \infty)\) is \( T(\text{SL}_2(A)) \).
3. For all \( n \geq 3 \), \( \text{PSL}_2(A) \) acts freely on \( X_n(A) \) and there is a bijection of \( A^\times/(A^\times)^2 \)-sets

\[
X_n(A) \leftrightarrow (A^\times/(A^\times)^2) \times Z_{n-3}(A)
\]

\[
([u_0], \ldots, [u_n]) \mapsto \left(\frac{d(u_1, u_2)d(u_2, u_3)}{d(u_1, u_3)}\right) \left(\frac{d(u_2, u_3)d(u_1, u_4)}{d(u_1, u_3)d(u_2, u_4)}\right) \cdots \left(\frac{d(u_2, u_3)d(u_1, u_n)}{d(u_1, u_3)d(u_2, u_n)}\right)
\]

\((0, \infty, a_+, (az_1)_+, \ldots, (az_{n-3})_+) \leftrightarrow (a, (z_1, \ldots, z_{n-3}))\)

Corollary 2.2. Let \( A \) be a ring.

1. The modules \((L_0(A))_{\text{SL}_2(A)}\) and \((L_1(A))_{\text{SL}_2(A)}\) are isomorphic to the trivial \( R_A \)-module \( \mathbb{Z} \).
2. For all \( n \geq 2 \), there are isomorphisms of \( R_A \)-modules \((L_n(A))_{\text{SL}_2(A)} \cong R_A[Z_{n-2}] \).

The corresponding facts about the action of \( \text{GL}_2(A) \) on \( X_n(A) \) easily follow:

Proposition 2.3. Let \( A \) be a ring.

1. \( \text{GL}_2(A) \) acts transitively on \( X_1(A) \). The stabilizer of \((0)\) is the subgroup \( B \).
2. \( \text{GL}_2(A) \) acts transitively on \( X_2(A) \). The stabilizer of \((0, \infty)\) is \( T \).
3. For all \( n \geq 3 \), \( \text{PGL}_2(A) \) acts freely on \( X_n(A) \) and there is a bijection of sets

\[
X_n(A) \leftrightarrow Z_{n-3}(A)
\]

\[
([u_0], \ldots, [u_n]) \mapsto \left(\frac{d(u_1, u_2)d(u_1, u_4)}{d(u_1, u_3)d(u_2, u_4)}\right) \cdots \left(\frac{d(u_1, u_3)d(u_1, u_n)}{d(u_1, u_3)d(u_2, u_n)}\right)
\]

\((0, \infty, 1, (z_1)_+, \ldots, (z_{n-3})_+) \leftrightarrow (z_1, \ldots, z_{n-3})\)

Corollary 2.4. Let \( A \) be a ring.

1. The abelian groups \((L_0(A))_{\text{GL}_2(A)}\) and \((L_1(A))_{\text{GL}_2(A)}\) are isomorphic to \( \mathbb{Z} \).
2. For all \( n \geq 2 \), there are isomorphisms of abelian groups \((L_n(A))_{\text{GL}_2(A)} \cong \mathbb{Z}[Z_{n-2}] \).

Corollary 2.5. If \( A \) is a ring satisfying \( \mathcal{W}_A = \emptyset \) then \( L_n(A) = 0 \) for all \( n \geq 3 \).

For example, \( L_n(\mathbb{Z}) = 0 \) for all \( n \geq 3 \).

3. Pre-Bloch groups of rings

For any ring \( A \), we define

\[
\mathcal{R} \mathcal{P}(A) := L^\times_3(A)_{\text{PSL}_2(A)} \quad \text{and} \quad \mathcal{P}(A) := L^\times_3(A)_{\text{PGL}_2(A)}.
\]
Given $x \in W_A$, we have $(0, \infty, 1, x_+) \in X_4(A)$. We let $[x]'$ denote the corresponding class in each of $L_3(A)_{\text{SL}_2(A)}$ and $L_3(A)_{\text{GL}_2(A)}$.

Thus $d_3((0, \infty, 1, x_+)) = (\infty, 1, x_+) - (0, 1, x_+) + (0, \infty, x_+) - (0, \infty, 1) \in L_2^3(A)$. We denote by $[x]$ the corresponding class in each of $\mathcal{RP}(A)$ and $\mathcal{P}(A)$. So the natural maps induced by $d_3$

$$L_3(A)_{\text{SL}_2(A)} \to \mathcal{RP}(A) \text{ and } L_3(A)_{\text{GL}_2(A)} \to \mathcal{P}(A)$$

send $[x]'$ to $[x]$.

3.1. **The element $D_A$.** Let $\{x, y, z\}$ be a 3-clique in $\Gamma(A)$. Then

$$d_2((x, y, z) + (x, z, y)) = (y, z) + (z, y) = d_2((y, z, x) + (z, y, x)) \text{ in } L_1(A).$$

It follows that

$$W(x, y, z) := (y, z, x) + (z, y, x) - (x, y, z) - (x, z, y) \in \text{Ker}(d_2) = K_3(A).$$

Let $w(x, y, z)$ denote the resulting class in $\mathcal{RP}(A)$ (and also in $\mathcal{P}(A)$).

**Lemma 3.1.** For any 3-clique $\{x, y, z\}$ in $\Gamma(A)$, we have $3 \cdot w(x, y, z) = 0$ in $\mathcal{RP}(A)$ (and in $\mathcal{P}(A)$).

**Proof.** By Proposition 2.1 (3), $(x, y, z), (x', y', z') \in X_3(A)$ lie in the same orbit of the (right) action of $\text{SL}_2(A)$ if and only if

$$d(x, z)d(x, y)d(y, z) \equiv d(x', z')d(x', y')d(y', z') \pmod{(A^\times)^2}.$$

It follows that $(x, y, z)$ and $(y, z, x)$ lie in the same $\text{SL}_2(A)$-orbit (using $d(v, u) = -d(u, v)$).

Thus there exists $\beta \in \text{SL}_2(A)$ such that $x\beta = y, y\beta = z$ and $z\beta = x$ in $X_1(A)$. In $\mathcal{RP}(A)$, we thus have

$$3w(x, y, z) = w(x, y, z) + w(x, y, z)\beta + w(x, y, z)\beta^2$$

$$= (y, z, x) + (z, y, x) - (x, y, z) - (x, z, y)$$

$$+ (z, x, y) + (x, z, y) - (y, z, x) - (y, x, z)$$

$$+ (x, y, z) + (y, x, z) - (z, x, y) - (z, y, x)$$

$$= 0.$$

In particular, we define $D_A := w(1, 0, \infty) \in \mathcal{RP}(A)$ for any ring $A$. So

$$D_A = (0, \infty, 1) + (\infty, 0, 1) - (1, 0, \infty) - (1, \infty, 0) \text{ in } \mathcal{RP}(A)$$

and we always have $3D_A = 0$.

We also observe:

**Lemma 3.2.** Let $A$ be a ring. Then $\langle -1 \rangle D_A = D_A$ in $\mathcal{RP}(A)$.

**Proof.** In $L_2^3(A)$, we have $W(1, 0, \infty) \text{diag}(-1, 1) = W(-1, 0, \infty) = W(1, 0, \infty) \cdot \omega$, which implies that $\langle -1 \rangle w(1, 0, \infty) = w(-1, 0, \infty) = w(1, 0, \infty)$ in $\mathcal{RP}(A)$.

**Lemma 3.3.** For any $(x, y, z) \in X_3(A)$, there exist $a \in A^\times$ such that

$$w(x, y, z) = \langle a \rangle D_A \text{ in } \mathcal{RP}(A).$$
Corollary 3.4. For any \((x, y, z) \in X_3(A)\) we have \(w(x, y, z) \cdot \alpha = w(x\alpha, y\alpha, z\alpha) = (\det(\alpha)) w(x, y, z)\) in \(\mathcal{RP}(A)\).

Since \(GL_2(A)\) acts transitively on \(X_3(A)\) ([9], Corollary 3.7] again), for any \((x, y, z) \in X_3(A)\) there exists \(\alpha \in GL_2(A)\) with \((x, y, z) = (1, 0, \infty)\alpha\). Hence \(w(x, y, z) = (a) D_A\) in \(\mathcal{RP}(A)\), where \(a = \det(\alpha)\).

**Corollary 3.4.** For any \((x, y, z) \in X_3(A)\) we have \(w(x, y, z) = D_A\) in \(\mathcal{P}(A)\).

### 3.2. The elements \(\psi_1(x)\). If \(x \in \mathcal{W}_A\), then there is a corresponding element \([x]' + \langle -1 \rangle [x^{-1}]'\) in \(L_3(A)_{SL_2(A)}\). Now \(\langle -1 \rangle [x^{-1}]'\), in \(L_3(A)_{SL_2(A)}\), is represented by

\[
(0, \infty, 1, x_+^{-1}) \text{diag}(-1, 1) = (0, \infty, -1, -x_+^{-1}).
\]

This latter element in turn is equal to the class of

\[
(0, \infty, -1, -x_+^{-1}, \omega) = (\infty, 0, 1, x_+)
\]

in \(L_3(A)_{SL_2(A)}\). Thus the image of \([x]' + \langle -1 \rangle [x^{-1}]'\) in \(\mathcal{RP}(A)\) is represented by

\[
d_3((0, \infty, 1, x_+) + (\infty, 0, 1, x_+)) = (\infty, 1, x_+ - (0, 1, x_+) + (0, \infty, x_+) - (0, \infty, 1)
\]

\[
+ (0, 1, x_+) - (\infty, 1, x_+) + (\infty, 0, x_+) + (\infty, 0, 1)
\]

\[
= (0, \infty, x_+) + (\infty, 0, x_+) - (0, \infty, 1) - (\infty, 0, 1).
\]

Note, however, that this last expression is well-defined and lies in \(L_3^+(A)\) for any \(x \in A^\times\) (i.e., we don’t need \(1 - x \in A^\times\) also).

Thus, for \(x \in A^\times\), we will define the element

\[
\psi_1(x) := (0, \infty, x_+) + (\infty, 0, x_+) - (0, \infty, 1) - (\infty, 0, 1)\]

in \(\mathcal{RP}(A)\).

**Lemma 3.5.** Let \(A\) be a ring.

1. For all \(x \in A^\times\), \(\langle -1 \rangle \psi_1(x) = \psi_1(x^{-1})\).
2. For all \(x, y \in A^\times\) we have

\[
\psi_1(xy) = \langle x \rangle \psi_1(y) + \psi_1(x).
\]

In other words, the map \(\psi_1 : A^\times \to \mathcal{RP}(A)\) is a 1-cocycle for the natural \(A^\times\)-action on \(\mathcal{RP}(A)\).

**Proof.**

1. A representative of \(\langle -1 \rangle \psi_1(x)\) in \(\mathcal{RP}(A)\) is obtained by multiplying the representative of \(\psi_1(x)\) by \(\text{diag}(-1, 1)\). Thus \(\langle -1 \rangle \psi_1(x)\) is represented by

\[
(0, \infty, -x_+) + (\infty, 0, -x_+) - (0, \infty, 1) - (\infty, 0, -1).\]

Multiplying this in turn by \(\omega\) gives the representative

\[
(\infty, 0, x_+^{-1}) + (0, \infty, x_+^{-1}) - (\infty, 0, 1) - (0, \infty, 1).
\]

2. \(\langle x \rangle \psi_1(y)\) is represented by

\[
\psi_1(y) \text{diag}(x, 1) = ((0, \infty, y_+) + (\infty, 0, y_+) - (0, \infty, 1) - (\infty, 0, 1)) \text{diag}(x, 1)
\]

\[
= (0, \infty, xy_+) + (\infty, 0, xy_+) - (0, \infty, x_+ - (\infty, 0, x_+).
\]

It follows that \(\langle x \rangle \psi_1(y) + \psi_1(x)\) is represented by

\[
(0, \infty, xy_+) + (\infty, 0, xy_+) - (0, \infty, x_+ - (\infty, 0, x_+ + (\infty, 0, x_+)) - (0, \infty, 1) - (\infty, 0, 1)
\]

\[
= (0, \infty, xy_+) + (\infty, 0, xy_+) - (0, \infty, 1) - (\infty, 0, 1),
\]

and hence is equal to \(\psi_1(xy)\), as claimed. \(\square\)
**Corollary 3.6.** Let $A$ be a ring.

1. For all $x \in A^\times$, $\psi_1(x) + \psi_1(-1) = \psi_1(-x^{-1})$.
2. $2\psi_1(-1) = 0$.

**Proof.**

1. Let $x \in A^\times$. Then $\psi_1(x) + \psi_1(-1) = (-1)\psi_1(x^{-1}) + \psi_1(-1) = \psi_1(-1 \cdot x^{-1})$ by Lemma 3.5.
2. Take $x = -1$ in (1), and note that $\psi_1(1) = 0$.

We denote the image of $\psi_1(x)$ in $\mathcal{P}(A)$ by $\psi(x)$.

Thus for any $x \in \mathcal{W}_A$, we have $\psi(x) = [x] + [x^{-1}]$ in $\mathcal{P}(A)$. In particular, $\psi(-1) = 2[-1]$ in $\mathcal{P}(A)$ whenever $-1 \in \mathcal{W}_A$; i.e., whenever $2 \in A^\times$.

**Corollary 3.7.** Let $A$ be a ring. We have the following identities in $\mathcal{P}(A)$.

1. For all $x, y \in A^\times$, $\psi(xy) = \psi(x) + \psi(y)$.
2. For all $x \in A^\times$, $2\psi(x) = \psi(x^2) = 0$.
3. $4[-1] = 0$ if $2 \in A^\times$.

**Proof.**

1. This follows immediately from the cocycle property of $\psi_1(x)$.
2. For all $x \in A^\times$, $\psi(x^{-1}) = \psi(x)$ by Lemma 3.6 (1), and hence $0 = \psi(x \cdot x^{-1}) = \psi(x) + \psi(x^{-1}) = 2\psi(x)$.
3. Take $x = -1$ in (2).

**Remark 3.8.** On the other hand, for any ring $A$ the elements $\psi_1(x) \in \mathcal{RP}(A)$ are usually non-torsion: see Proposition 3.24 below.

**Proposition 3.9.** Let $A$ be a ring and let $x \in \mathcal{W}_A$. Then

$$D_A = [1 - x^{-1}] + (-1)[1 - x] - \psi_1(x) \text{ in } \mathcal{RP}(A).$$

**Proof.** Since $x \in \mathcal{W}_A$, $\{0, \infty, 1, x_+\}$ is a 4-clique in $\Gamma(A)$ and we have

$$D_A = (0, \infty, 1) + (\infty, 0, 1) - (1, 0, \infty) - (1, \infty, 0)$$

$$= d_3(-(0, \infty, 1, x_+) - (\infty, 0, 1, x_+) + (1, 0, \infty, x_+) + (1, \infty, 0, x_+)).$$

If $w = \{a_1, a_2, a_3, a_4\}$ is any 4-clique in $\Gamma(A)$, then, by Proposition 2.1 (3), the image of $(a_1, a_2, a_3, a_4)$ in $L_3(A)_{SL_2(A)}$ is

$$\text{cr}(w) := \langle d(a_1, a_2)d(a_1, a_2)d(a_2, a_3) \rangle \left[\frac{d(a_1, a_4)d(a_2, a_3)}{d(a_1, a_3)d(a_2, a_4)}\right].$$

(cr stands for ‘cross ratio’.)

Note that for $x, y \in A$ we have

$$d(\infty, x_+) = 1, \ d(x_+, \infty) = -1, \ \text{and} \ d(x_+, y_+) = x - y.$$

Thus

$$\text{cr}(1, 0, \infty, x_+) = [1 - x^{-1}] \quad \text{cr}(1, \infty, 0, x_+) = (-1)[1 - x]$$

$$\text{cr}(0, \infty, 1, x) = [x] \quad \text{cr}(0, \infty, 1, x) = (-1)[x^{-1}].$$

So the image of $-(0, \infty, 1, x_+) - (\infty, 0, 1, x_+) + (1, 0, \infty, x_+) + (1, \infty, 0, x_+)$ in $L_3(A)_{SL_2(A)}$ is $[1 - x^{-1}] + (-1)[1 - x] - [x] - (-1)[x^{-1}]$. Applying $d_3$ give the result.
Taking the image in $\mathcal{P}(A)$, we deduce:

**Corollary 3.10.** Let $A$ be a commutative ring and let $x \in \mathcal{W}_A$. Then

$$D_A = \left[1 - x^{-1}\right] + [1 - x] + \psi(x) \text{ in } \mathcal{P}(A).$$

3.3. **The element $C_A$.** For a commutative ring $A$ we define

$$C_A := (1,0,\infty) + (1,\infty,0) - (0,\infty,-1) - (\infty,0,-1) \in RP(A).$$

From the expressions for $D_A$ and $\psi_1(x)$ above, we immediately deduce

$$C_A = -D_A - \psi_1(-1) \in RP(A)$$

for any commutative ring $A$.

From the definitions and results above, we have the identities

$$6C_A = 0, \quad 2C_A = D_A, \quad 3C_A = \psi_1(-1), \quad C_A + D_A + \psi_1(-1) = 0$$

in $RP(A)$.

**Proposition 3.11.** Let $A$ be a commutative ring and let $x \in \mathcal{W}_A$. Then

$$C_A = [x] + (-1)[1 - x] + \langle[1 - x]\rangle \psi_1(x) \text{ in } RP(A).$$

**Proof.** By Proposition 3.9 $D_A = \left[1 - x^{-1}\right] + (-1)[1 - x] - \psi_1(x)$ for any $x \in \mathcal{W}_A$. Since $1 - x^{-1} \in \mathcal{W}_A$ also, we can replace $x$ by $1 - x^{-1}$ in this expression:

$$D_A = \left[\frac{1}{1 - x}\right] + (-1)\left[\frac{1}{x}\right] - \psi_1 \left(1 - x^{-1}\right).$$

Thus

$$C_A = \quad -D_A - \psi_1(-1)$$

$$= \quad - \left[\frac{1}{1 - x}\right] - (-1)\left[\frac{1}{x}\right] + \psi_1 \left(1 - x^{-1}\right) - \psi_1(-1)$$

$$= \quad (-1)[1 - x] + [x] + \psi_1 \left(1 - x^{-1}\right) - \psi_1(-1) - \psi_1 \left(\frac{1}{1 - x}\right) - \psi_1(x).$$

(Here, the last equality follows from the identities

$$- [a] = (-1) [a^{-1}] - \psi_1(a) \text{ and } - (-1) [a] = [a^{-1}] - \psi_1(a^{-1})$$

for $a \in \mathcal{W}_A$.)

This last expression in turn is equal to

$$[x] + (-1)[1 - x] + \psi_1 \left(1 - x^{-1}\right) - \psi_1(x - 1) - \psi_1(x)$$

using $\psi_1(-1) + \psi_1(a) = \psi_1(-a^{-1})$ (by Corollary 3.6 (1)).

Finally, the cocycle property of $\psi_1$ gives

$$\psi_1 \left(1 - x^{-1}\right) = \psi_1 \left(\frac{x - 1}{x}\right) = \langle x - 1 \rangle \psi_1(x^{-1}) + \psi_1(x - 1)$$

$$= \langle 1 - x \rangle \psi_1(x) + \psi_1(x - 1)$$

and hence

$$\psi_1 \left(1 - x^{-1}\right) - \psi_1(x - 1) - \psi_1(x) = \langle 1 - x \rangle \psi_1(x) - \psi_1(x) = \langle[1 - x]\rangle \psi_1(x).$$

\[\square\]
Lemma 3.15. Let $A$ be a commutative ring and let $x \in \mathcal{W}_A$. Then
\[ C_A = [x] + [1 - x] \text{ in } \mathcal{P}(A). \]

Corollary 3.12. The element $C_Z \in \mathcal{RP}(\mathbb{Z})$ has order 6.

Proof. Certainly $6C_Z = 0$. On the other hand, under the functorial maps $\mathcal{RP}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{R})$, $C_Z$ maps to $C_R$. By Corollary 3.12, $C_R = [x] + [1 - x]$ for any $x \in \mathbb{R} \setminus \{0, 1\} = \mathcal{W}_R$. By [18, Section 1], $C_R$ has exact order 6 and hence so does $C_Z$. \qed

3.4. The elements $\psi_2(x)$ and the key identity. For any ring $A$, and for any unit $x \in A^\times$ we define
\[ \psi_2(x) := (x_+, 0, \infty) + (x_+, \infty, 0) - (1, 0, \infty) - (1, \infty, 0) \in \mathcal{RP}(A). \]

Lemma 3.14. Let $A$ be a ring. For all $x \in \mathcal{W}_A$,
\[ \psi_2(x) = \langle x^{-1} - 1 \rangle [x] + \langle 1 - x \rangle [x^{-1}] \in \mathcal{RP}(A). \]

Proof. The term $\langle x^{-1} - 1 \rangle [x]$ is represented by
\[ d_3((0, \infty, 1, x_+) \diag(1 - x, x)) = d_3((0, \infty, a(x), b(x))) \]
\[ = (\infty, a(x), b(x)) - (0, a(x), b(x)) + (0, \infty, b(x)) - (0, \infty, a(x)). \]

where $a(x) := (x^{-1} - 1)_+$, $b(x) := (1 - x)_+$ in $X_1(A)$. The term $\langle 1 - x \rangle [x^{-1}]$ is represented by
\[ d_3((0, \infty, 1, x^{-1}_+) \diag(1 - x, 1)) = d_3((0, \infty, b(x), a(x))) \]
\[ = (\infty, b(x), a(x)) - (0, b(x), a(x)) + (0, \infty, a(x)) - (0, \infty, b(x)). \]

Adding these two representatives gives the expression
\[ Z := (\infty, a(x), b(x)) + (\infty, b(x), a(x)) - (0, a(x), b(x)) - (0, b(x), a(x)). \]

Now consider
\[ \tau(x) := \begin{bmatrix} x & 1 \\ x^{-1} & 1 \end{bmatrix} \in \text{SL}_2(A). \]

Then $\psi_2(x)$ is represented by $Z \cdot \tau(x)$. Since
\[ \infty \cdot \tau(x) = x_+, \ 0 \cdot \tau(x) = 1, \ a(x) \cdot \tau(x) = 0, \ b(x) \cdot \tau(x) = \infty, \]
we see that $\langle x^{-1} - 1 \rangle [x] + \langle 1 - x \rangle [x^{-1}]$ is represented by
\[ (x_+, 0, \infty) + (x_+, \infty, 0) - (1, 0, \infty) - (1, \infty, 0) \]
as required. \qed

Analogously to the elements $\psi_1(x)$, we have:

Lemma 3.15. Let $A$ be a ring.

1. For all $x \in A^\times$, $\langle -1 \rangle \psi_2(x) = \psi_2(x^{-1})$.
2. For all $x, y \in A^\times$ we have
\[ \psi_2(xy) = \langle x \rangle \psi_2(y) + \psi_2(x). \]

In other words, the map $\psi_2 : A^\times \to \mathcal{RP}(A)$ is a 1-cocycle for the natural $A^\times$-action on $\mathcal{RP}(A)$.

Proof. The proof is almost identical to that for $\psi_1(x)$. \qed
Thus $\psi_1$ and $\psi_2$ both define 1-cocycles in $\mathcal{RP}(A)$. Both are lifts of the homomorphism $\psi : A^\times \to \mathcal{P}(A)$. In general, however, they do not coincide. In fact they differ by a 1-coboundary, as the following key identity shows:

**Theorem 3.16.** Let $A$ be a ring. For all $x \in A^\times$ we have
\[
\langle x \rangle D_A = \psi_1(x) - \psi_2(x).
\]

**Proof.** For $x \in A^\times$, $\langle x \rangle D_A$ is represented by
\[
W(1, 0, \infty) \text{diag}(x, 1) = (0, \infty, x_+) + (\infty, 0, x_+) - (x_+, 0, \infty) - (x_+, \infty, 0).
\]
Therefore $D_A + \psi_1(x) - \langle x \rangle D_A$ is represented by
\[
(0, \infty, 1) + (\infty, 0, 1) - (1, 0, \infty) - (1, 0, 0)
\]
\[
\quad + (0, \infty, x_+) + (\infty, 0, x_+) - (0, \infty, 1) - (\infty, 0, 1)
\]
\[
\quad - (0, \infty, x_+) - (\infty, 0, x_+) + (x_+, 0, \infty) + (x_+, \infty, 0)
\]
\[
\quad = (x_+, 0, \infty) + (x_+, \infty, 0) - (1, 0, \infty) - (1, 0, 0),
\]
which is $\psi_2(x)$. \hfill \Box

**Remark 3.17.** This identity was previously known for fields with at least 4 elements and for local rings whose residue field has at least 5 elements. It is the starting point for the calculation of the third homology of $\text{SL}_2$ of local fields and local rings as well of $\mathbb{Q}$ in the papers [8], [9] and [10].

**Corollary 3.18.** Let $A$ be a ring. Then $\psi_1(-1) = \psi_2(-1)$ and $(-1) D_A = D_A$ in $\mathcal{RP}(A)$.

**Proof.** In the identity $\langle -1 \rangle D_A = \psi_1(-1) - \psi_2(-1)$ the left side is annihilated by 3 and the right side by 2. So both sides are zero. \hfill \Box

3.5. The map $\lambda_1$. By Corollary 2.2, $L_2^x(A)_{\text{SL}_2(A)} \cong R_A$, via a map, $D$ say, sending $([u], [v], [w])$ to the square class $\langle d(u, v)d(u, w)d(v, w) \rangle$. We let $\lambda_1$ denote the $R_A$-module map
\[
\mathcal{RP}(A) = L_3^x(A)_{\text{SL}_2(A)} \to L_2^x(A)_{\text{SL}_2(A)} \cong R_A
\]
induced by the inclusion $L_3^x(A) \to L_2^x(A)$.

**Lemma 3.19.** Let $A$ be a ring.

1. The image of $\lambda_1$ is contained in $\mathcal{I}_A$.
2. For all $a \in A^\times$,
\[
\lambda_1(\psi_1(a)) = \lambda_1(\psi_2(a)) = -\langle -a \rangle \langle a \rangle = (1 + \langle -1 \rangle) \langle a \rangle.
\]
3. For all $a \in \mathcal{W}_A$, $\lambda_1([a]) = -\langle a \rangle \langle 1 - a \rangle$.

**Proof.**

1. The map $R_A \cong L_2^x(A)_{\text{SL}_2(A)} \to L_1^x(A)_{\text{SL}_2(A)} \cong \mathbb{Z}$ induced by the map $d_2 : L_2^x(A) \to L_1^x(A)$ is easily seen to be the augmentation homomorphism $\epsilon : R_A \to \mathbb{Z}$. Thus the image of $\lambda_1$ is contained in $\text{Ker}(\epsilon) = \mathcal{I}_A$.
2. Let $a \in A^\times$.
   \[
   D(0, \infty, a_+) = \langle d(0, \infty)d(0, a_+)d(\infty, a_+) \rangle = \langle -1 \cdot -a \cdot 1 \rangle = \langle a \rangle
   \]
   \[
   D(\infty, 0, a_+) = \langle d(\infty, 0)d(\infty, a_+)d(0, a_+) \rangle = \langle 1 \cdot 1 \cdot -a \rangle = \langle -a \rangle.
   \]
Taking \( a = 1 \), \( D(0, \infty, 1) = \langle 1 \rangle = 1 \) and \( D(\infty, 0, 1) = \langle -1 \rangle \). So
\[
\lambda_1(\psi_1(a)) = \langle a \rangle + \langle -a \rangle - 1 - \langle -1 \rangle = (1 + \langle -1 \rangle) \langle a \rangle = -\langle a \rangle \langle -a \rangle.
\]

Similarly, \( \psi_2(a) = (a_+, 0, \infty) + (a_+, \infty, 0) - (1, 0, \infty) - (1, \infty, 0) \). Now \( D(a_+, 0, \infty) = \langle a \rangle \) while \( D(a_+, \infty, 0) = \langle -a \rangle \), so that \( \lambda_1(\psi_2(a)) = \langle a \rangle + \langle -a \rangle - 1 - \langle -1 \rangle \) as before.

(3) For \( a \in W_A \), \([a] \) is represented by \((\infty, 1, a_+) - (0, 1, a_+) + (0, \infty, a_+) - (0, \infty, 1)\). Now
\[
D(\infty, 1, a_+) = \langle d(\infty, 1)d(\infty, a_+)d(1, a_+) \rangle = \langle 1 \cdot 1 \cdot (1 - a) \rangle = \langle 1 - a \rangle
\]
\[
D(0, 1, a_+) = \langle d(0, 1)d(0, a_+)d(1, a_+) \rangle = \langle -1 \cdot -a \cdot (1 - a) \rangle = \langle a(1 - a) \rangle.
\]

Furthermore, \( D(0, \infty, a_+) = \langle a \rangle \) as above. So
\[
\lambda_1([a]) = \langle 1 - a \rangle - \langle a(1 - a) \rangle + \langle a \rangle - 1 = -\langle a \rangle \langle 1 - a \rangle
\]
as required.

\[
\square
\]

For any ring \( A \), we set
\[
\mathcal{RP}_1(A) := \text{Ker}(\lambda_1) = \text{Ker}(L^*_3(A)_{\text{SL}_2(A)} \to L^*_2(A)_{\text{SL}_2(A)}) = H_3 \left( L^*_\bullet(A)_{\text{SL}_2(A)} \right).
\]

**Remark 3.20.** If we consider \( L^*_\bullet(A)_{\text{GL}_2(A)} \) instead, note that the map \( L^*_3(A)_{\text{GL}_2(A)} \to L^*_3(A)_{\text{GL}_2(A)} \) (induced by inclusion \( L^*_3(A) \to L^*_2(A) \)) is the zero map, since the map \( \mathbb{Z} \cong L^*_2(A)_{\text{GL}_2(A)} \to L^*_1(A)_{\text{GL}_2(A)} \cong \mathbb{Z} \) is the identity map. Thus we have
\[
\mathcal{P}(A) = H_3 \left( L^*_\bullet(A)_{\text{GL}_2(A)} \right)
\]
for any ring \( A \).

### 3.6. The modules \( \mathcal{K}^{(i)}_A \)

Let \( A \) be any ring. Recall that the 1-cocycles \( \psi_i : A^\times \to \mathcal{RP}(A), i = 1, 2 \) satisfy
\[
\psi_i(1) = 0 \quad \text{and} \quad \langle -1 \rangle \psi_i(a) = \psi_i(a^{-1}) \quad \text{for all} \quad a \in A^\times.
\]

**Lemma 3.21.** Let \( M \) be an \( R_A \)-module and let \( \phi : A^\times \to M \) be a 1-cocycle satisfying \( \phi(1) = 0 \) and \( \langle -1 \rangle \phi(a) = \phi(a^{-1}) \) for all \( a \in A^\times \). Then

1. \( \langle a \rangle \phi(b) = \langle b \rangle \phi(a) \) for all \( a, b \in A^\times \).
2. \( \phi(ab^2) = \phi(a) + \phi(b^2) \) for all \( a, b \in A^\times \).
3. \( 2\phi(-1) = 0 \).
4. \( \phi(a^2) = \langle a \rangle \phi(-1) = \langle -1 \rangle \phi(a) \) for all \( a \in A^\times \).
5. \( 2\phi(a^2) = 0 \) for all \( a \in A^\times \). If \(-1 \in (A^\times)^2\), then \( \phi(a^2) = 0 \) for all \( a \in A^\times \).

**Proof.**

1. This follows from the cocycle condition since
\[
\phi(ab) = \langle a \rangle \phi(b) + \phi(a) = \langle b \rangle \phi(a) + \phi(b).
\]
2. \( \phi(b^2a) = \langle b^2 \rangle \phi(a) + \phi(b^2) = \phi(a) + \phi(b^2) \) since \( \langle b^2 \rangle \) acts trivially on \( M \) by assumption.
3. \( 0 = \phi(1) = \phi(-1) \cdot -1 = \langle -1 \rangle \phi(-1) + \phi(-1) = 2\phi(-1) \).
4. For any \( a \in A^\times \) we have
\[
\phi(a) = \phi(a^2 \cdot a^{-1}) = \langle a^2 \rangle \phi(a^{-1}) + \phi(a^2) = \langle -1 \rangle \phi(a) + \phi(a^2).
\]

Thus \( \phi(a^2) = -\langle -1 \rangle \phi(a) = -\langle a \rangle \phi(-1) = \langle a \rangle \phi(-1) \).
Let Example 3.23. Since \( \langle \psi^K \rangle \text{Lemma 3.21 (2),(4)}. \) Furthermore, for any \( a \), \( \phi(a^2) = \langle -1 \rangle \phi(a) \) and this vanishes if \(-1\) is a square in \( A \).

\[ \square \]

**Corollary 3.22.** Let \( A \) be a ring. Suppose that \( u \in A \) satisfies \(-u^2 \in \mathcal{W}_A \). Then \( \langle 1 + u^2 \rangle \psi_i (-1) = \psi_i (-1) \) in \( \mathcal{R}\mathcal{P}(A) \).

**Proof.** For \( i = 1, 2 \), we have \( \psi_i (-u^2) = \psi_i (-1) + \psi_i (a^2) = \psi_i (-1) + \langle u \rangle \psi_i (-1) = \langle u \rangle \psi_i (-1). \) Thus \( \psi_i (-u^2) = \psi_2 (-u^2) = \langle u \rangle \psi_1 (-1) \) by Corollary 3.18. Now

\[
\psi_2 (-u^2) = \langle -u^2 - 1 \rangle [ -u^2 ] + \langle 1 + u^2 \rangle [ -u^2 ] = \langle -(1 + u^2) \rangle [ -u^2 ] + \langle (1 + u^2) \rangle [ -u^2 ] = \langle -(1 + u^2) \rangle [ [ -u^2 ] + \langle -1 \rangle [ -u^2 ] ] = \langle -(1 + u^2) \rangle \psi_1 (-u^2) = \langle -(1 + u^2) \rangle \psi_2 (-u^2).
\]

Thus \( \langle -(1 + u^2) \rangle = \langle -1 \rangle \langle 1 + u^2 \rangle \) acts trivially on \( \psi_2 (-u^2) \), and hence also \( \psi_2 (-1). \) Since \( \langle -1 \rangle \) also acts trivially on \( \psi_2 (-1) \), the result follows. \[ \square \]

**Example 3.23.** Let \( p \) be a prime number satisfying \( p \equiv 1 \mod 4 \). Then \( \langle p \rangle \psi_1 (-1) = \psi_1 (-1) \) in \( \mathcal{R}\mathcal{P}(\mathbb{Q}) \): There exist \( x, y \in \mathbb{Z} \) with \( p = x^2 + y^2 \) and hence \( py^2 = (x/y)^2 + 1 \) in \( \mathbb{Q}^\times \). Thus \( \langle p \rangle = \langle py^2 \rangle = \langle (x/y)^2 + 1 \rangle. \)

For \( i = 1, 2 \) we let \( \mathcal{K}_A^{(i)} \) denote the \( \mathcal{R}_A \)-submodule of \( \mathcal{R}\mathcal{P}(A) \) generated by the set \( \{ \psi_i (a) \mid a \in A^\times \} \).

**Proposition 3.24.** Let \( A \) be a ring. For \( i = 1, 2 \) there is a short exact sequence of \( \mathcal{R}_A \)-modules

\[
0 \longrightarrow (\mathcal{K}_A^{(i)})_{\text{tors}} \longrightarrow \mathcal{K}_A^{(i)} \longrightarrow (1 + \langle -1 \rangle)\mathcal{I}_A \longrightarrow 0
\]

and \( (\mathcal{K}_A^{(1)})_{\text{tors}} = (\mathcal{K}_A^{(2)})_{\text{tors}} = \mathcal{R}_A \psi_1 (-1) \) (which is annihilated by \( 2 \)).

**Proof.** The image of \( \lambda_1 : \mathcal{K}_A^{(i)} \to \mathcal{R}_A \) is \( (1 + \langle -1 \rangle)\mathcal{I}_A \) by Lemma 3.19 (2).

Recall that \( \psi_1 (-1) = \psi_2 (-1) \) by Corollary 3.18. Now \( \lambda_1 (\psi_1 (-1)) = (1 + \langle -1 \rangle) \langle -1 \rangle = \langle \langle -1 \rangle \rangle + \langle \langle 1 \rangle \rangle - \langle \langle -1 \rangle \rangle = 0 \). So \( \mathcal{R}_A \psi_1 (-1) \subset \text{Ker}(\lambda_1 : \mathcal{K}_A^{(i)} \to (1 + \langle -1 \rangle)\mathcal{I}_A) \).

For any \( a \in A \), we have \( (1 + \langle -1 \rangle) \langle a \rangle \rangle = \langle \langle a \rangle \rangle + \langle \langle -a \rangle \rangle - \langle \langle -1 \rangle \rangle \). It follows at once that the abelian group \( (1 + \langle -1 \rangle)\mathcal{I}_A \) is free with basis \( \{(1 + \langle -1 \rangle) \langle a \rangle \} \mid a \in S \} \) where \( S := A^\times / \pm (A^\times)^2 \setminus \{1\} \).

Now, for any \( a, b \in A^\times \), we have \( \psi_i (ab^2) = \psi_i (a) + \psi_i (b^2) = \psi_i (a) + \langle b \rangle \psi_1 (-1) \) by Lemma 3.21 (2),(4). Furthermore, for any \( a \in A^\times \), \( \psi_i (-a) = \psi_i (a \cdot -1) = \langle a \rangle \psi_1 (-1) + \psi_i (a) \). Recall also that \( \langle a \rangle \psi_i (b) = \psi_i (ab) - \psi_i (a) \) for any \( a, b \in A^\times \). It follows that \( \mathcal{K}_A^{(i)} / (\mathcal{R}_A \psi_1 (-1)) \) is generated, as an abelian group, by the elements \( \{ \psi_i (a) \mid a \in S \} \). Since these elements map under \( \lambda_1 \) to a basis of \( (1 + \langle -1 \rangle)\mathcal{I}_A \), the result follows. \[ \square \]

**Remark 3.25.** Since \( (1 + \langle -1 \rangle)\mathcal{I}_A \) is a free abelian group, the short exact sequence of Proposition 3.24 is split as a sequence of abelian groups.
Corollary 3.26. Let $A$ be a ring. There is an isomorphism of $R_A$-modules $\mathcal{K}_A^{(1)} \to \mathcal{K}_A^{(2)}$ sending $\psi_1(a)$ to $\psi_2(a)$ for $a \in A^\times$.

Proof. Since $\mathcal{K}_A^{(2)}$ contains no non-trivial 3-torsion, $\mathcal{K}_A^{(2)} \cap I_A D_A = 0$. By Theorem 3.16, the inclusion map $\mathcal{K}_A^{(1)} \to \mathcal{K}_A^{(2)} + I_A D_A$ induces an isomorphism $\mathcal{K}_A^{(1)} \to (\mathcal{K}_A^{(2)} + I_A D_A)/(I_A D_A) \cong \mathcal{K}_A^{(2)}$ sending $\psi_1(a)$ to $\psi_2(a)$. □

Remark 3.27. However, in general $\psi_1(a) \neq \psi_2(a)$ and $\mathcal{K}_A^{(1)} \neq \mathcal{K}_A^{(2)}$.

By Theorem 3.16 and Proposition 3.24, the image of $\mathcal{K}_A^{(2)}$ in $\mathcal{R} \mathcal{P}(A)/\mathcal{K}_A^{(1)}$ is isomorphic to the module $I_A D_A$. Consider now the case $A = \mathbb{Q}$. Then $I_\mathbb{Q} D_\mathbb{Q}$ is an infinite-dimensional $\mathbb{F}_3$-vector space with basis $\{ \langle p \rangle : p \equiv -1 \pmod{3} \}$ (see [10, Section 6.2]).

Remark 3.28. In general, the module $\left( \mathcal{K}_A^{(i)} \right)_{\text{tors}}$ is nonzero. It always contains the element $\psi_1(-1)$. Since $C_Z$ had order 6, it follows that $\psi_1(-1) \neq 0$ in $\mathcal{R} \mathcal{P}(\mathbb{Z})$.

Now let $\widetilde{\mathcal{R}} \mathcal{P}(A)$ be the $R_A$-module $\mathcal{R} \mathcal{P}(A)/\mathcal{K}_A^{(1)}$. Let $\tilde{\lambda}_1$ be the induced homomorphism $\widetilde{\mathcal{R}} \mathcal{P}(A) \to I_A/(1 + \langle -1 \rangle) I_A$ and define $\mathcal{R} \mathcal{P}_1(A) := \text{Ker}(\tilde{\lambda}_1)$.

Proposition 3.29. There is a natural short exact sequence of $R_A$-modules

$$0 \to R_A \psi_1(-1) \to \mathcal{R} \mathcal{P}_1(A) \to \widetilde{\mathcal{R}} \mathcal{P}_1(A) \to 0.$$ 

Proof. This follows from the following commutative diagram in which the lower two rows and all columns are exact:

$$
\begin{array}{cccccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & R_A \psi_1(-1) & \mathcal{R} \mathcal{P}_1(A) & \widetilde{\mathcal{R}} \mathcal{P}_1(A) & 0 \\
\downarrow \downarrow \downarrow & \downarrow \downarrow \downarrow & \downarrow \downarrow \downarrow & \downarrow \downarrow \downarrow \\
0 & \mathcal{K}_A^{(1)} & \mathcal{R} \mathcal{P}(A) & \widetilde{\mathcal{R}} \mathcal{P}(A) & 0 \\
\downarrow \downarrow \downarrow \downarrow \downarrow & \downarrow \downarrow \downarrow \downarrow \downarrow & \downarrow \downarrow \downarrow \downarrow \downarrow & \downarrow \downarrow \downarrow \downarrow \downarrow \\
0 & (1 + \langle -1 \rangle) I_A & I_A & I_A/(1 + (-1)) I_A & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

□

4. Bloch groups of rings and the hyperhomology spectral sequence $E(G, L^r)$

Let $A$ be a ring. For any subgroup $G$ of $\text{GL}_2(A)$, we let $T(G, L^r)$ denote the total complex of the double complex

$$D_{**} = D_{*,*}(G, L^r) := L^r \otimes_{\mathbb{Z}[G]} F_*$$

where $F_*$ is a fixed projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$. The homology of this total complex is, by definition, the hyperhomology $H_{**}(G, L^r)$. 

More precisely, we set $D_{p,q} = L^r_q \otimes F_p$. Filtering the double complex vertically then gives a spectral sequence $E^r(G, L^r)$ satisfying

$$E^1_{p,q}(G, L^r) = H_p \left( G, L^r_q \right) \implies H_{p+q} (G, L^r).$$

We thus have:

**Lemma 4.1.** Let $A$ be a ring. $E^2_{0,3}( \text{SL}_2(A), L^r) = \mathcal{R} \mathcal{P}_1(A)$ and $E^2_{0,3}( \text{GL}_2(A), L^r) = \mathcal{P}(A)$.

Furthermore, there are natural edge homomorphisms

$$H_3( \text{SL}_2(A), L^r) \to \mathcal{R} \mathcal{P}_1(A), \quad H_3( \text{GL}_2(A), L^r) \to \mathcal{P}(A).$$

For $G = \text{GL}_2(A)$, let $\mathcal{R} = \mathcal{R}_G := \mathbb{Z}_\epsilon$, equipped with augmentation $\epsilon = \text{id} : \mathbb{Z} \to \mathbb{Z}$.

For $G = \text{SL}_2(A)$, let $\mathcal{R} = \mathcal{R}_G := R_A$, the group ring $\mathbb{Z}[A^\times/(A^\times)^2]$ with its natural augmentation.

**Lemma 4.2.** For $G = \text{SL}_2(A)$ or $\text{GL}_2(A)$, the $E^1$-page of the spectral sequence $E(G, L^r)$ has the form

\[
\begin{array}{cccccc}
 & & & & & \\
 & & & & & \\
 \downarrow & \downarrow & \downarrow & \cdots & & \\
 & & & & & \\
 0 & 0 & 0 & \cdots & & \\
 & & & & & \\
 (L^r_3)_G & H_1(G, L^r_3) & H_2(G, L^r_3) & \cdots & & \\
 & & & & & \\
 \downarrow d^1 & \downarrow d^1 & \downarrow d^1 & & & \\
 & & & & & \\
 \mathcal{R} & \mathcal{R} \otimes \mathbb{Z} H_1(Z(G), \mathbb{Z}) & \mathcal{R} \otimes \mathbb{Z} H_2(Z(G), \mathbb{Z}) & \cdots & & \\
 & & & & & \\
 \downarrow \epsilon & \downarrow \epsilon \otimes H_1(\text{inc}) & \downarrow \epsilon \otimes H_2(\text{inc}) & & & \\
 & & & & & \\
 \mathbb{Z} & H_1(T(G), \mathbb{Z}) & H_2(T(G), \mathbb{Z}) & H_3(T(G), \mathbb{Z}) & & \\
 & & & & & \\
 0 & H_1(\text{inc}) \circ (C_\omega - \text{id}) & H_2(\text{inc}) \circ (C_\omega - \text{id}) & H_3(\text{inc}) \circ (C_\omega - \text{id}) & & \\
 & & & & & \\
 \mathbb{Z} & H_1(B(G), \mathbb{Z}) & H_2(B(G), \mathbb{Z}) & H_3(B(G), \mathbb{Z}) & & \\
 & & & & & \\
 \end{array}
\]

where $C_\omega$ denotes the map induced on homology by conjugation by $\omega := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

**Proof.** By Propositions 2.1 and 2.3 for $G = \text{GL}_2(A)$ or $\text{SL}_2(A)$ we have $L^r_0 \cong \mathbb{Z}[B(G)\backslash G]$ and hence $E^1_{p,0}(G, L^r) = H_p(\mathbb{Z}[B(G)\backslash G])$ by Shapiro’s Lemma.

Similarly, $L^r_1 \cong \mathbb{Z}[T(G)\backslash G]$ and hence $E^1_{p,1}(G, L^r) \cong H_p(T(G), \mathbb{Z})$.

By Proposition 2.3 again, $L^r_2 \cong \mathbb{Z}[Z\backslash \text{GL}_2(A)]$ and hence $E^1_{p,2}(\text{GL}_2(A), L^r) \cong H_p(Z, \mathbb{Z}) = \mathcal{R} \otimes \mathbb{Z} H_p(Z(G), \mathbb{Z})$.

By Proposition 2.1 for $G = \text{SL}_2(A)$, there is a $Z[G]$-decomposition

$$L^r_2 \cong \bigoplus_{(a) \in A^\times/(A^\times)^2} \mathbb{Z}[Z(G)\backslash G] \cdot (0, \infty, 1, a)$$

and hence

$$E^1_{p,2}(G, L^r) = H_p(\mathbb{Z}[B(G)\backslash G]) \cong \bigoplus_{(a) \in A^\times/(A^\times)^2} H_p(Z(G), \mathbb{Z}) \cong \mathcal{R} \otimes \mathbb{Z} H_p(Z(G), \mathbb{Z}).$$

This accounts for the $E^1$-terms in the spectral sequences. The calculation of the differentials is straightforward.

**Remark 4.3.** When $G = \text{SL}_2(A)$ all terms and differentials in the spectral sequence $E^r(G, L^r)$ are naturally $R_A$-modules and homomorphisms respectively.
Corollary 4.4. Let $A$ be a ring. For $G = \text{GL}_2(A)$ or $\text{SL}_2(A)$, $E_{1,1}^2(G, L^\tau) = 0$.

Proof. $E_{1,1}^2(G, L^\tau)$ is the homology of

$$E_{1,2}^1(G, K) \xrightarrow{d^1} E_{1,1}^1(G, K) \xrightarrow{d^1} E_{1,0}^1(G, K).$$

This is thus the homology of

$$\mathcal{R} \otimes Z(G) = \mathcal{R} \otimes H_1(Z(G), \mathbb{Z}) \xrightarrow{\epsilon \otimes \text{inc}} T(G) = H_1(T(G), \mathbb{Z}) \xrightarrow{\text{inc} \circ (c_\omega - \text{id})} H_1(B(G), \mathbb{Z})$$

where $c_\omega$ denotes conjugation by $\omega$.

Since $\text{inc} : T(G) = H_1(T(G), \mathbb{Z}) \to H_1(B(G), \mathbb{Z})$ is a split injection, this is equal to the homology of

$$\mathcal{R} \otimes Z(G) \xrightarrow{\epsilon \otimes \text{inc}} T(G) \xrightarrow{c_\omega - \text{id}} T(G),$$

which is easily verified to be 0 for $G = \text{GL}_2(A)$ or $G = \text{SL}_2(A)$. \qed

It follows that for $G = \text{GL}_2(A)$ or $\text{SL}_2(A)$ the differential $d^2 : E_{0,3}^2 \to E_{1,1}^3$ is the zero map.

Corollary 4.5. Let $A$ be a ring. Then $E_{0,3}^3(\text{SL}_2(A), L^\tau) = \mathcal{RP}_1(A)$ and $E_{0,3}^3(\text{GL}_2(A), L^\tau) = \mathcal{P}(A)$.

The Bloch groups of the ring are defined to be the $E_{0,3}^\infty$-terms:

Let $A$ be a ring. We define $\mathcal{RB}(A)$ to be the $R_A$-module $E_{0,3}^\infty(\text{SL}_2(A), L^\tau)$. Thus

$$\mathcal{RB}(A) := \text{Ker}(d^3 : \mathcal{RP}_1(A) \to E_{2,0}^3(\text{SL}_2(A), L^\tau)).$$

We define $\mathcal{B}(A)$ to be the abelian group $E_{0,3}^\infty(\text{GL}_2(A), L^\tau)$. Thus

$$\mathcal{B}(A) := \text{Ker}(d^3 : \mathcal{P}(A) \to E_{2,0}^3(\text{GL}_2(A), L^\tau)).$$

From the definitions, we have

**Lemma 4.6.** For any ring $A$ there are commutative diagrams of $R_A$-modules

$$\begin{array}{ccc}
H_3(\text{SL}_2(A), L^\tau) & \longrightarrow & \mathcal{RB}(A) \\
\downarrow & & \downarrow \\
H_3(\text{GL}_2(A), L^\tau) & \longrightarrow & \mathcal{B}(A)
\end{array}$$

where the horizontal arrows are surjections.

For use in calculations below, we record the following descriptions of $d^2$ differentials in the spectral sequence $E(\text{SL}_2(A), L^\tau)$:

**Lemma 4.7.** Let $A$ be a ring.

1. We have $E_{1,0}^2 = B^{ab}/T^2$ and $E_{0,2}^2$ is naturally a quotient of $\mathcal{I}_A \subset R_A = E_{1,2}^1$. The composite homomorphism

$$\mathcal{I}_A \longrightarrow E_{0,2}^2 \xrightarrow{d^2} E_{1,0}^2 = B^{ab}/T^2$$

sends $\langle \langle u \rangle \rangle$ to the class of the matrix $\begin{bmatrix} u & 0 \\ 3(1 - u) & u^{-1} \end{bmatrix}$.
(2) The map $H_2(B, \mathbb{Z}) \to H_2(T, \mathbb{Z})$ induces a well-defined homomorphism $E^2_{2,0} \to H_2(T, \mathbb{Z}) = T \wedge T$. $E^2_{1,2}$ is naturally a quotient of $I_A \otimes \mu_2 \subset R_A \otimes \mu_2 = E^1_{1,2}$. The composite homomorphism

$$I_A \otimes \mu_2 \xrightarrow{d^2} E^2_{1,2} \xrightarrow{d^2} E^2_{2,0} \to T \wedge T$$

sends $\langle \langle u \rangle \rangle \otimes -1$ to $u \wedge -1$.

5. Connection with $H_3(\text{SL}_2(A), \mathbb{Z})$

For the purposes of these notes, we introduce the following terminology:

Let us say that a commutative ring $A$ is $L_\bullet$-acyclic in dimensions $\leq r$ (resp. $L^\bullet_\bullet$-acyclic in dimensions $\leq r$) if the map $\varepsilon : L_\bullet(A) \to \mathbb{Z}$ (resp. $\varepsilon : L^\bullet_\bullet(A) \to \mathbb{Z}$) induces an isomorphism on homology in dimensions $\leq r$. Equivalently, $A$ is $L_\bullet$-acyclic in dimensions $\leq r$ if the sequence

$$L_{r+1}(A) \to L_r(A) \to \cdots \to L_0(A) \to \mathbb{Z} \to 0$$

is exact.

We will say that $A$ is $L_\bullet$-acyclic if it is $L_\bullet$-acyclic in dimensions $\leq r$ for all $r \geq 0$. Note that $A$ is $L^\bullet_\bullet$-acyclic if and only if it is $L_\bullet$-acyclic in dimensions $\leq 1$ (since $L^1_2(A) := \text{Ker}(d_2)$).

From the definition, we immediately have

**Corollary 5.1.** Let $A$ be a $L^\bullet_\bullet$-acyclic ring and let $G$ be a subgroup of $\text{GL}_2(A)$. There are isomorphisms

$$H_n(G, L^\bullet_\bullet) \cong H_n(G, \mathbb{Z})$$

for $n \geq 0$.

Thus, the spectral sequence $E^r(G, L^\bullet_\bullet)$ converges to $H_\bullet(G, \mathbb{Z})$.

**Proof.** The equivalence $L^\bullet_\bullet \to \mathbb{Z}$ induces an isomorphism $H_\bullet(G, L^\bullet_\bullet) \cong H_\bullet(G, \mathbb{Z})$. □

**Theorem 5.2.** A local ring $[k]$ with residue field $k$ is $L_\bullet$-acyclic in dimensions $\leq |k| - 1$. In particular, every local ring is $L^\bullet_\bullet$-acyclic.

**Proof.** We have $H_r(L_\bullet(A)) = 0$ for $r < |k|$ by [9, Lemma 3.21].

Furthermore

$$L_1(A) \xrightarrow{\varepsilon} L_0(A) \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{0}$$

is exact: $\ker(\varepsilon)$ is generated by the elements $\bar{u} - \bar{v}$, $\bar{u} \neq \bar{v} \in X_1$. Here $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are elements of $U_2(A)$ representing $\bar{u}$ and $\bar{v}$ in $X_1$. Let $k$ be the residue field of $A$ and let $x, y$ be the images respectively of $\bar{u}, \bar{v}$ in $X_1(k) = \mathbb{P}^1(k)$. Choose $z \in \mathbb{P}^1(k) \setminus \{x, y\}$ and let $\bar{w}$ be an inverse image of $z$ in $X_1(A)$. Then $(\bar{u}, \bar{w}), (\bar{v}, \bar{w}) \in X_2(A)$ and $d_1((\bar{u}, \bar{w}) - (\bar{v}, \bar{w})) = \bar{u} - \bar{v} \in L_0^\bullet$. □

The conditions under which a ring $A$ is $L^\bullet_\bullet$-acyclic have been determined in [11]:

Recall that a ring $A$ is a $\text{GE}_2$-ring (P. M. Cohn,[4]) if $\text{SL}_2(A) = E_2(A)$ where $E_2(A)$ is the subgroup generated by elementary matrices.

For a ring $A$, $K_2(2, A)$ denotes the rank one $K_2$ of $A$ and $C(2, A)$ denotes the subgroup of $K_2(2, A)$ generated by symbols (see, for example, [11, Appendix]).

---

1For convenience, the term “local ring $A$ with residue field $k$” will include the case when $A = k$ is a field.
Corollary 5.4. The ring $\mathbb{Z}$ is a GE-domain. Thus $\mathbb{Z}$ is isomorphic to $\mathbb{Z}[E_2(A) \setminus \text{SL}_2(A)]$.

Theorem 5.3 ([11, Theorem 3.3, Theorem 7.2]). Let $A$ be a ring. Then $H_0(L_\bullet(A)) \cong \mathbb{Z}[E_2(A) \setminus \text{SL}_2(A)]$ and

$$H_1(L_\bullet(A)) \cong \left( \frac{K_2(2, A)}{C(2, A)} \right)^{ab}.$$  

Corollary 5.4. The ring $A$ is $L_\bullet^*$-acyclic if and only if $A$ is a GE$_2$-ring and the group $K_2(2, A)/C(2, A)$ is perfect.

Example 5.5. Note that any Euclidean domain is a GE$_2$-ring, but there are many examples of PIDs which are not GE$_2$-rings (see, for example, [3]).

Example 5.6. L. Vaserstein ([19]) has shown that a ring of $S$-integers in a number field with infinitely many units is a GE$_2$-ring.

Example 5.7. $\mathbb{Z}$ is a Euclidean domain and $K_2(2, \mathbb{Z})$ is generated by the symbol $c(-1, -1)$. Thus $\mathbb{Z}$ is $L_\bullet^*$-acyclic.

Example 5.8. Morita ([14, Proposition 2:13]) has shown that if $D$ is Dedekind domain and if $\pi \in D$ is prime element for which $D^{\times}$ surjects onto $(D/\langle \pi \rangle)^{\times}$ and if $K_2(2, D)$ is generated by symbols then the same holds for the ring $D[\frac{1}{\pi}]$.

This fact, and an argument by induction, shows that the Euclidean domains $\mathbb{Z}[\frac{1}{m}]$ are $L_\bullet^*$-acyclic whenever there exist primes $p_1, \ldots, p_t$ such that $m = p_1^{a_1} \cdots p_t^{a_t}$ and $(\mathbb{Z}/p_i)^{\times}$ is generated by $\{-1, p_1, \ldots, p_i-1\}$ for all $i \leq t$. In particular, $\mathbb{Z}[\frac{1}{2}]$ and $\mathbb{Z}[\frac{1}{3}]$ are $L_\bullet^*$-acyclic.

Example 5.9. On the other hand, consider the ring $\mathbb{Z}[\frac{1}{p}]$, $p \geq 5$ is a prime. The calculations of Morita ([15]) imply that the groups $\left( K_2(2, \mathbb{Z}[\frac{1}{p}])/C(2, \mathbb{Z}[\frac{1}{p}]) \right)^{ab}$ are non zero (see [11, Lemma 6.15]). Thus $\mathbb{Z}[\frac{1}{p}]$ is not $L_\bullet^*$-acyclic for any $p \geq 5$.

In fact, Morita writes down explicit elements of infinite order in $\left( K_2(2, \mathbb{Z}[\frac{1}{p}])/C(2, \mathbb{Z}[\frac{1}{p}]) \right)^{ab}$. This allows us to write down explicit cycles in $L_1(\mathbb{Z}[\frac{1}{p}])$ which represent homology classes of infinite order ([11, Example 7.4]). For example the cycle

$$(\infty, 0) + \left( 0, -\frac{1}{10} \right) + \left( -\frac{1}{10}, -\frac{3}{5} \right) + \left( -\frac{3}{5}, \infty \right) \in L_1 \left( \mathbb{Z} \left[ \frac{1}{5} \right] \right)$$

represents a homology class of infinite order.

Example 5.10. For any field $k$, the ring of Laurent polynomials $k[t, t^{-1}]$ is a Euclidean domain. Furthermore, $K_2(2, k[t, t^{-1}])$ is generated by symbols (see [16]). Thus the ring $k[t, t^{-1}]$ is $L_\bullet^*$-acyclic for any field $k$.

6. Classical Bloch Group and Classical Refined Bloch Group

Recall that, for a commutative ring $A$, $\mathcal{W}_A := \{a \in A^\times \mid 1 - a \in A^\times\}$. Let $\mathcal{P}(A)$ denote the group with generators $[a], a \in \mathcal{W}_A$ and relations

$$0 = [x] - [y] + \left[ \frac{y}{x} \right] - \left[ \frac{1 - x^{-1}}{1 - y^{-1}} \right] + \left[ \frac{1 - x}{1 - y} \right] \quad \text{for all } x, y, x/y \in \mathcal{W}_A.$$

Similarly, let $\mathcal{RP}(A)$ be the $\mathbb{R}_A$-module generated by $[a], a \in \mathcal{W}_A$ subject to the relations

$$0 = [x] - [y] + \langle x \rangle \left[ \frac{y}{x} \right] - \langle x^{-1} - 1 \rangle \left[ \frac{1 - x^{-1}}{1 - y^{-1}} \right] + \langle 1 - x \rangle \left[ \frac{1 - x}{1 - y} \right] \quad \text{for all } x, y, x/y \in \mathcal{W}_A.$$
Theorem 6.1. Suppose that the ring $A$ is $L^\bullet$-acyclic in dimensions $\leq 2$. (Recall that this implies that $A$ is $L^\bullet$-acyclic. This condition is satisfied by any local ring whose residue field has order at least 3.)

For $a \in W_A$, let $[a] := d_3(0_+, 0_-, 1, a) \in L^+_3(A) \subset L^+_5(A)$.

1. The $R_A$-homomorphism $\alpha : \mathcal{RP}(A) \to \mathcal{RP}(A), [a] \mapsto [a]$ and the $\mathbb{Z}$-module homomorphism $\beta : \mathcal{P}(A) \to \mathcal{P}(A), [a] \mapsto [a]$ are both surjective.

2. If furthermore $A$ is $L^\bullet$-acyclic in dimension $\leq 3$ (e.g., if $A$ is a local ring with residue field of size $\geq 4$) then the homomorphisms $\alpha$ and $\beta$ induce isomorphisms

$$\mathcal{RP}(A) \cong \mathcal{RP}(A) \text{ and } \mathcal{P}(A) \cong \mathcal{P}(A).$$

Proof. By the results of [9, Section 3], we have

$$\mathcal{RP}(A) \cong \frac{L_3(A)_{SL_2(A)}}{d_4(L_4(A)_{SL_2(A)})} \text{ and } \mathcal{P}(A) \cong \frac{L_3(A)_{GL_2(A)}}{d_4(L_4(A)_{GL_2(A)})}$$

via $[a] \leftrightarrow$ the class of $(0_+, 0_-, 1, a), a \in W_A$ in both cases.

We have a complex of $GL_2(A)$-modules

$$L_4(A) \xrightarrow{d_4} L_3(A) \xrightarrow{d_3} L^+_5(A) \to 0.$$

For any subgroup $G$ of $GL_2(A)$ this induces a complex

$$L_4(A)_G \xrightarrow{d_4} L_3(A)_G \xrightarrow{d_3} L^+_5(A)_G \to 0$$

which induces a homomorphism

$$\alpha_G : \frac{L_3(A)_G}{d_4(L_4(A)_G)} \to L^+_5(A)_G.$$

When $G = SL_2(A)$, we have $\alpha_G = \alpha$. When $G = GL_2(A), \alpha_G = \beta$.

1. If $A$ is $L^\bullet$-acyclic in dimensions $\leq 2$ the map $L_3(A) \to L^+_5(A)$ is surjective. It follows that the map $\alpha_G$ is surjective for any $G$.

2. If $A$ is $L^\bullet$-acyclic in dimensions $\leq 3$ then the sequence $L_4(A) \to L_3(A) \to L^+_5(A) \to 0$ is exact. Taking $G$-coinvariants, the sequence remains exact. Hence the map $\alpha_G$ is an isomorphism in this case.

Example 6.2. $W_{\mathbb{F}_3} = \{1\}$. Thus $R\mathcal{P}(\mathbb{F}_3)$ is a cyclic $R_{\mathbb{F}_3}$-module generated by $[-1]$ and $\mathcal{P}(\mathbb{F}_3)$ is a cyclic $\mathbb{Z}$-module generated by $[-1]$.

(Note, more generally, that if $A$ is a local ring with residue field $\mathbb{F}_3$, then $W_A = -1 + M_A = -1 \cdot U_1$.)

Let $A$ be a ring. Let $\lambda_1 : R\mathcal{P}(A) \to R_A$ be the $R_A$-module homomorphism $[a] \mapsto \langle a \rangle \langle 1 - a \rangle$ for $a \in W_A$. Let $\lambda : \mathcal{P}(A) \to S^2_2(A^\times)$ be the $\mathbb{Z}$-module homomorphism $[a] \mapsto a \circ (1 - a)$ for $a \in W_A$. Let $\lambda_2 : R\mathcal{P}(A) \to S^2_2(A^\times)$ denote the composite homomorphism

$$R\mathcal{P}(A) \longrightarrow \mathcal{P}(A) \longrightarrow S^2_2(A^\times).$$

We define the $R_A$-modules

$$R\mathcal{P}_1(A) := \text{Ker}(\lambda_1), \quad R\mathcal{B}(A) := \text{Ker}(\lambda_2 : R\mathcal{P}_1(A) \to S^2_2(A^\times)).$$
We define the $\mathbb{Z}$-module
\[ \mathcal{B}(A) := \text{Ker}(\lambda : \mathcal{P}(A) \to S^2_\mathbb{Z}(A^x)) \].

We relate these modules to the hyperhomology groups $H_n(G, L_\bullet)$ for certain subgroups $G$ of $\text{GL}_2(A)$:

**Proposition 6.3.** Let $A$ be a ring. Let $T := T(\text{SL}_2(A))$ and $B := B(\text{SL}_2(A))$. For any subgroup $G$ of $\text{GL}_2(A)$ there is a spectral sequence $E_{p,q}^r(G, L) \Longrightarrow H_{p+q}(G, L_\bullet)$.

1. $\mathcal{RP}_1(A) \cong E^3_{0,3}(\text{SL}_2(A), L)$, and $\mathcal{P}(A) \cong E^3_{0,3}(\text{GL}_2(A), L)$.
2. If the inclusion $T \to B$ induces an isomorphism $H_2(T, \mathbb{Z}) \to H_2(B, \mathbb{Z})$ then $\mathcal{RB}(A) = E^\infty_{0,3}(\text{SL}_2(A), L)$.
3. If the inclusion $T_A \to B_A$ induces an isomorphism $H_2(T_A, \mathbb{Z}) \cong H_2(B_A, \mathbb{Z})$ then $\mathcal{B}(A) = E^\infty_{0,3}(\text{GL}_2(A), L)$.

**Proof.** (1) Note that $E^2_{0,3}(G, L) = (L^3_\mathbb{Z}(A))_G/(d_4(L^4_\mathbb{Z}(A)))_G$ for any subgroup $G$ of $\text{GL}_2(A)$ and hence $E^2_{0,3}(\text{SL}_2(A), L) = \mathcal{RP}_1(A)$ and $E^2_{0,3}(\text{GL}_2(A), L) = \mathcal{P}(A)$.

However, $E^2_{0,1}(G, L) = 0$ when $G = \text{SL}_2(A)$ or $\text{GL}_2(A)$ (as in the proof of Corollary 4.4) and hence $\mathcal{RP}_1(A) \cong E^3_{0,3}(\text{SL}_2(A), L)$ and $\mathcal{P}(A) \cong E^3_{0,3}(\text{GL}_2(A), L)$ as claimed.

(2) We consider the relevant terms and differentials in the spectral sequence $E(\text{SL}_2(A), L)$.

Recall that
\[ E^\infty_{0,3} = E^4_{0,3} = \text{Ker}(d^4 : E^3_{0,3} = \mathcal{RP}_1(A) \to E^3_{2,0}). \]

We first calculate $E^3_{2,0}$: The term $E^3_{1,2}$ is the homology of the sequence
\[ R_A[Z_1] \otimes H_1(\mu_2(A), \mathbb{Z}) \overset{\partial \otimes \text{id}}{\longrightarrow} R_A \otimes H_1(\mu_2(A), \mathbb{Z}) \overset{\epsilon \otimes \text{id}}{\longrightarrow} H_1(T, \mathbb{Z}) \]
where $\partial : R_A[Z_1] \to R_A$ is the $R_A$-module map $[u] \mapsto \langle u \rangle \langle 1 - u \rangle$ and $\epsilon : R_A \to \mathbb{Z}$ is the augmentation homomorphism. It follows that $E^3_{1,2} = I(A) \otimes \mu_2(A)$ where $I(A) := \text{Ker}(\bar{\epsilon}) : R_A/J_A \to \mathbb{Z}$ with $J_A$ equal to the ideal of $R_A$ generated by the elements $\langle u \rangle \langle 1 - u \rangle, u \in Z_1 = \mathcal{W}_A$.

Now $E^3_{2,0} = E^3_{2,0} = H_2(B, \mathbb{Z}) = H_2(T, \mathbb{Z}) \cong A^x \wedge A^x$ and the differential
\[ d^2 : E^2_{1,2} = I(A) \otimes \mu_2(A) \to A^x \wedge A^x \cong E^2_{2,0} \]
sends $\langle u \rangle \otimes \eta$ to $u \wedge \eta$. Thus $E^3_{2,0} \cong (A^x \wedge A^x)/(A^x \wedge \mu_2(A))$.

The inclusion map $\text{SL}_2(A) \to \text{GL}_2(A)$ induces a map of spectral sequences and a commutative diagram
\[ \begin{array}{ccc}
E^3_{0,3} = \mathcal{RP}_1(A) & \longrightarrow & \mathcal{P}(A) = E^3_{0,3}(\text{GL}_2(A), L) \\
\downarrow d^3 & & \downarrow d^3 \\
\mathcal{A}^x \wedge \mathcal{A}^x & \longrightarrow & E^3_{2,0}(\text{GL}_2(A), L)
\end{array} \]

Now $E^3_{2,0}(\text{GL}_2(A), L)$ is a quotient of $E^3_{1,0}(\text{GL}_2(A), L) = H_2(B_A, \mathbb{Z})$. Let $\pi_A : H_2(T_A, \mathbb{Z}) = T_A \cap T_A \to S^2_\mathbb{Z}(A^x)$ be the (surjective) homomorphism sending $\text{diag}(a, b) \wedge \text{diag}(c, d)$ to $a \circ d + b \circ c$. Let $p_A : B_A \to T_A$ be the homomorphism
\[ \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \mapsto \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}. \]
Then the map $\pi_A \circ (p_A)_* : H_2(B_A, \mathbb{Z}) \to S_2^2(A^\times)$ induces a map $\bar{\pi} : E_{2,0}^3(\text{GL}_2(A), L) \to S_2^2(A^\times)$ such that the diagram

$$
\begin{array}{ccc}
P(A) & \xrightarrow{d^3} & E_{2,0}^3(\text{GL}_2(A), L) \\
\downarrow & & \downarrow \\
S_2^2(A^\times)
\end{array}
$$

commutes (see, for example, [5, p188]). Now the inclusion $T \to T_A$ induces the homomorphism

$$j : A^\times \wedge A^\times = H_2(T, \mathbb{Z}) \to H_2(T_A, \mathbb{Z}) \to S_2^2(A^\times), u \wedge v \mapsto -2(u \circ v)$$

which in turn induces the injective map $\bar{j} : E_{3,0}^3 = (A^\times \wedge A^\times)/(A^\times \wedge \mu_2(A)) \to S_2^2(A^\times)$. Since $\bar{j}$ is injective, the commutativity of the diagram

$$
\begin{array}{ccc}
\mathcal{RP}_1(A) & \xrightarrow{d^3} & \mathcal{P}(A) \\
\downarrow & & \downarrow \\
E_{2,0}^3 & \xrightarrow{\bar{j}} & S_2^2(A^\times)
\end{array}
$$

establishes the result, since $\text{Ker}(d^3) = \text{Ker}(\lambda_1)$.

(3) Under the hypothesis $H_2(T, \mathbb{Z}) \cong H_2(B, \mathbb{Z})$, the map $\bar{\pi} : E_{2,0}^3(\text{GL}_2(A), L) \to S_2^2(A^\times)$ is an isomorphism (see [5, p188] again) and so the differential $d^3 : E_{3,0}^3(\text{GL}_2(A), L) \to E_{2,0}^3(\text{GL}_2(A), L)$ is equal to $\lambda$ (up to sign).

\[\square\]

**Corollary 6.4.** If $A$ is $L_\bullet$-acyclic in dimensions $\leq 3$ and if $H_2(T, \mathbb{Z}) \cong H_2(B, \mathbb{Z})$ for the ring $A$, then there are natural isomorphisms $\mathcal{RB}(A) \cong \mathcal{RB}(A)$ and $\mathcal{B}(A) \cong \mathcal{B}(A)$.

If the ring $A$ is not $L_\bullet$-acyclic in dimensions $\leq 3$ then there is not necessarily a map $H_3(\text{SL}_2(A), \mathbb{Z}) \to \mathcal{RB}(A)$. However, the following lifting result is useful for establishing the existence of certain homology classes:

**Corollary 6.5.** Let $A$ be a ring and let $A \to B$ be a ring homomorphism where $B$ is $L_\bullet$-acyclic in dimensions $\leq 3$. Suppose that $H_2(T, \mathbb{Z}) \cong H_2(B, \mathbb{Z})$ for both $A$ and $B$. Let $\alpha \in \text{Im}(\mathcal{RB}(A) \to \mathcal{RB}(B))$.

Then $\alpha \in \text{Im}(H_3(\text{SL}_2(A), \mathbb{Z}) \to H_3(\text{SL}_2(B), \mathbb{Z}) \to \mathcal{RB}(B))$.

**Proof.** This follows from the commutative diagram

$$
\begin{array}{ccc}
H_3(\text{SL}_2(A), \mathbb{Z}) & \xleftarrow{\cong} & H_3(\text{SL}_2(A), L_\bullet) \xrightarrow{\alpha} \mathcal{RB}(A) \\
\downarrow & & \downarrow \\
H_3(\text{SL}_2(B), \mathbb{Z}) & \xrightarrow{\cong} & H_3(\text{SL}_2(B), L_\bullet) \xrightarrow{\mathcal{RB}(B)} \mathcal{RB}(B)
\end{array}
$$

(where the two rightmost horizontal arrows are surjective and the bottom left horizontal arrow is an isomorphism).

\[\square\]

The hypothesis of Proposition 6.3 (2) applies, for example, to many subrings of $\mathbb{Q}$:

**Lemma 6.6.** Let $A$ be a subring of $\mathbb{Q}$ satisfying $6 \in A^\times$. Then the natural map

$$H_2(T, \mathbb{Z}) \to H_2(B, \mathbb{Z})$$

is an isomorphism.
Proof. By considering the Hochschild-Serre spectral sequence associated to the group extension

\[ 1 \to U \cong A \to B \to T \to 1, \]

it is enough to establish the vanishing of \( H_0(T, H_2(U, \mathbb{Z})) \) and \( H_1(T, H_1(U, \mathbb{Z})) \). Furthermore, since \( A \) is a colimit of infinite cyclic groups, \( H_2(A, \mathbb{Z}) = 0 \). We need only prove the vanishing of \( H_1(T, H_1(U, \mathbb{Z})) = H_1(A^\times, A) \) where \( u \in A^\times \) acts on \( A \) as multiplication by \( u^2 \). However, the pair of maps \( (\iota_u, u \cdot) : (A^\times, A) \to (A^\times, A) \) induces the identity on the groups \( H_i(A^\times, A) \), where \( \iota_u \) is conjugation by \( u \). Taking \( u = 2 \in A^\times \), we deduce that multiplication by \( 4 = 2^2 \) is the identity map on \( H_1(A^\times, A) \) and hence that this group is annihilated by \( 3 \). But \( 3 \in A^\times \), by hypothesis, and so acts invertibly on \( H_1(A^\times, A) \). Thus \( H_1(A^\times, A) = 0 \) as required. \( \square \)

7. Review: The Bloch group and indecomposable \( K_3 \) of fields

For any field \( F \) there is a natural surjective homomorphism \( H_3(SL_2(F), \mathbb{Z}) \to K_3^{\text{ind}}(F) \). (For infinite fields, see [8, Lemma 5.1]. For the case of finite fields, see [7, Corollary 3.9].) Furthermore, there is a natural map \( K_3^{\text{ind}}(F) \to \mathcal{B}(F) \) fitting into a commutative diagram

\[
\begin{array}{ccc}
H_3(SL_2(F), \mathbb{Z}) & \longrightarrow & \mathcal{RB}(F) \\
\downarrow & & \downarrow \\
K_3^{\text{ind}}(F) & \longrightarrow & \mathcal{B}(F).
\end{array}
\]

We note that the inclusion map \( \mu_F \to SL_2(F), \zeta \mapsto \text{diag}(\zeta, \zeta^{-1}) \) induces a homomorphism

\[ \text{tor}(\mu_F, \mu_F) = H_3(\mu_F, \mathbb{Z}) \to H_3(SL_2(F), \mathbb{Z}) \to K_3^{\text{ind}}(F). \]

For a finite cyclic group \( C \) of even order, there is a unique nontrivial extension \( \bar{C} \) of \( C \) by \( \mathbb{Z}/2 \). If \( C \) is cyclic of odd order, let \( \bar{C} = C \). Since \( \mu_F \), and hence \( \text{tor}(\mu_F, \mu_F) \), is a union of finite cyclic groups we can define \( \text{tor}(\mu_F, \mu_F) \) to be the union of \( \text{tor}(\mu_n(F), \mu_n(F)) \).

Theorem 7.1. For any field \( F \), there is a natural short exact sequence

\[ 0 \to \text{tor}(\mu_F, \mu_F) \to K_3^{\text{ind}}(F) \to \mathcal{B}(F) \to 0. \]

Proof. For infinite fields, this is [18, Theorem 5.2]. For case of finite fields with at least 4 elements, see [7, Corollary 7.5]. The result is extended to the fields \( \mathbb{F}_2 \) and \( \mathbb{F}_3 \) in [8]. (The ad hoc definitions of \( \mathcal{B}(\mathbb{F}_2) \) and \( \mathcal{B}(\mathbb{F}_3) \) which are given in that paper are justified below.) \( \square \)

We will use the following observation below:

Lemma 7.2. Let \( F \) be a field with \( \text{char}(F) \neq 2 \) and \( \mu_F = \mu_2 \). Let \( \omega \) be the element

\[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\]

of order 4 in \( SL_2(F) \).

Then the map \( H_3(\langle \omega \rangle, \mathbb{Z}) \to H_3(SL_2(F), \mathbb{Z}) \to K_3^{\text{ind}}(F) \) induces an isomorphism

\[ H_3(\langle \omega \rangle, \mathbb{Z}) \cong \text{tor}(\mu_F, \mu_F) \subset K_3^{\text{ind}}(F). \]

Proof. Let \( E = F(\sqrt{-1}) \). \( \omega \) has eigenvalues \( \pm \sqrt{-1} \) in \( E \). It is conjugate in \( GL_2(E) \) to \( J := \text{diag}(\sqrt{-1}, -\sqrt{-1}) \). We consider \( SL_n(F) \) as a subgroup of \( SL_{n+1}(F) \) via the
homomorphism which sends the matrix \( M \in \text{SL}_n(E) \) to \( \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix} \in \text{SL}_{n+1}(E) \). Then \( \omega \) is conjugate to \( J \) in \( \text{SL}_3(E) \). (If \( C \in \text{GL}_2(E) \) conjugates \( \omega \) to \( J \) in \( \text{GL}_2(E) \), then \( \begin{bmatrix} C & 0 \\ 0 & \det(C)^{-1} \end{bmatrix} \) conjugates \( \omega \) to \( J \) in \( \text{SL}_3(E) \).) Thus the image of \( H_3(\langle \omega \rangle, \mathbb{Z}) \) is equal to the image of \( H_3(\langle J \rangle, \mathbb{Z}) \) in \( H_3(\text{SL}_2(E), \mathbb{Z}) \).

For any field \( L \), the map \( H_3(\text{SL}_2(L), \mathbb{Z}) \to K_3^{\text{ind}}(L) \) factors through a map \( H_3(\text{SL}_n(L), \mathbb{Z}) \to K_3^{\text{ind}}(L) \) for all \( n \geq 2 \). It thus follows that the image of \( H_3(\langle \omega \rangle, \mathbb{Z}) \) is equal to the image of \( H_3(\langle J \rangle, \mathbb{Z}) \) in \( K_3^{\text{ind}}(E) \). This image is \( \text{tor}(\mu_4, \mu_4) \), which is the unique subgroup of order 4 in \( \text{tor}(\mu_E, \mu_E) \).

On the other hand, the commutativity of the diagram

\[
\begin{array}{ccc}
\text{tor}(\mu_E, \mu_E) & \longrightarrow & K_3^{\text{ind}}(E) \\
\downarrow & & \downarrow \\
\text{tor}(\mu_F, \mu_F) & \longrightarrow & K_3^{\text{ind}}(F)
\end{array}
\]

shows that the image of \( \text{tor}(\mu_F, \mu_F) \) in \( K_3^{\text{ind}}(E) \) is also the unique subgroup of \( \text{tor}(\mu_E, \mu_E) \) order 4. Since the map \( K_3^{\text{ind}}(F) \to K_3^{\text{ind}}(E) \) is injective (see [12, Corollary 4.6]), the result follows.

We will also need:

**Lemma 7.3.** There does not exist a homology class in \( H_3(\text{SL}_2(\mathbb{Z}), \mathbb{Z}) \) which maps to \( \psi(-1) \in \mathcal{B}(\mathbb{Q}) \) under the composite map
\[
H_3(\text{SL}_2(\mathbb{Z}), \mathbb{Z}) \to H_3(\text{SL}_2(\mathbb{Q}), \mathbb{Z}) \to \mathcal{R}\mathcal{B}(\mathbb{Q}) \to \mathcal{B}(\mathbb{Q}).
\]

**Proof.** There is an exact sequence
\[
0 \to \mathbb{Z}/4 \cong \text{tor}(\mu_2, \mu_0) \to K_3^{\text{ind}}(\mathbb{Q}) \to \mathcal{B}(\mathbb{Q}) \to 0.
\]
Here \( K_3^{\text{ind}}(\mathbb{Q}) \) is cyclic of order 24 and hence \( \mathcal{B}(\mathbb{Q}) \) is cyclic of order 6. Since \( C_{\mathbb{Q}} \in \mathcal{B}(\mathbb{Q}) \) has order 6, it generates \( \mathcal{B}(\mathbb{Q}) \). Thus \( 3C_{\mathbb{Q}} = \psi(-1) = 2[-1] \) is the unique element of order 2 in \( \mathcal{B}(\mathbb{Q}) \). It follows that any element of \( K_3^{\text{ind}}(\mathbb{Q}) \) mapping to \( \psi(-1) \) has order 8. It follows, in turn, that any element of \( H_3(\text{SL}_2(\mathbb{Q}), \mathbb{Z}) \) mapping to \( \psi(-1) \) will have order divisible by 8. Since \( H_3(\text{SL}_2(\mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}/12 \) has no elements of order 8, the lemma is proved.

\[\square\]

## 8. Examples and applications

### 8.1. The Bloch group of \( \mathbb{F}_2 \)
Since \( G := \text{PSL}_2(\mathbb{F}_2) = \text{SL}_2(\mathbb{F}_2) = \text{GL}_2(\mathbb{F}_2) \) it follows that \( \mathcal{R}\mathcal{P}(\mathbb{F}_2) = \mathcal{R}\mathcal{P}_1(\mathbb{F}_2) = \mathcal{P}(\mathbb{F}_2) \) and \( \mathcal{R}\mathcal{B}(\mathbb{F}_2) = \mathcal{B}(\mathbb{F}_2) \). \( G \) is non-abelian of order 6. More precisely, it is generated by \( \omega = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \), of order 2, and \( \gamma = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \), of order 3, satisfying \( \gamma \omega = \beta = \omega \gamma^2 \) (of order 2). The group \( B = B_{\mathbb{F}_2} \) is cyclic of order 2 generated by \( \beta \) and \( T = Z = \{1\} \). By Proposition 2.11 the exact sequence \( 0 \to L_3^1(\mathbb{F}_2) \to \cdots \to L_0^6(\mathbb{F}_2) \to \mathbb{Z} \to 0 \) has the form
\[
0 \to L_3^1(\mathbb{F}_2) \to \mathbb{Z}[G] \to \mathbb{Z}[G] \to \mathbb{Z}[B\setminus G] \to \mathbb{Z} \to 0.
\]
Counting ranks, it follows that \( L_3^1(\mathbb{F}_2) \) is a free abelian group of rank 2.
Now $E := W(1, 0, \infty), F := W(1, 0, \infty) \gamma \in L_3^-(F_2)$ and it is easily verified that these elements form a basis. Note that the right $\mathbb{Z}[G]$-module structure is determined by the identities:

$$E\gamma = F, \quad E\omega = E, \quad F\gamma = -(E + F) = F\omega.$$ 

If we let $\bar{E}, \bar{F}$ denote the images of these elements in $(L_3^-(F_2))G = \mathcal{P}(F_2) = \mathcal{RP}(F_2) = \mathcal{RP}_1(F_2)$ we deduce that

$$\bar{E} = \bar{F} = -\bar{E} - \bar{F}$$

and hence $\mathcal{RP}(F_2)$ is cyclic of order 3 with generator $\bar{E} = \bar{F} = w(1, 0, \infty) = C_{F_2}$.

Since $E_{\mathbb{Z}}^2$ is a quotient of $H_2(B, \mathbb{Z}) = 0$, it follows that $\mathcal{RP}(F_2) = \mathcal{RP}_1(F_2) = \mathcal{RB}(F_2) = (\mathbb{Z}/3) \cdot C_{F_2}$.

From the spectral sequence it now follows that we have a short exact sequence

$$0 \to H_3(B, \mathbb{Z}) \to H_3(SL_2(F_2), \mathbb{Z}) \to \mathcal{RB}(F_2) \to 0$$

(and hence $H_3(SL_2(F_2), \mathbb{Z})$ is cyclic of order 6 as expected).

**Corollary 8.1.** Let $A$ be an $F_2$-algebra. Then the induced homomorphism $\mathcal{RB}(F_2) \to \mathcal{RB}(A)$ sends the generator $C_{F_2}$ of $\mathcal{RB}(F_2)$ to $C_A$.

**Corollary 8.2.** The natural map $K_3^\text{ind}(F_2) \to B(F_2)$ is an isomorphism.

**Proof.** In [7, Corollary 3.9] it is shown that for a finite field $F$ of characteristic $p$, the natural homomorphism $H_3(SL_2(F), \mathbb{Z}) \to K_3^\text{ind}(F)$ induces an isomorphism (on tensoring by $\mathbb{Z}[1/p]$) $H_3(SL_2(F), \mathbb{Z}[1/p]) \cong K_3^\text{ind}(F)$. By the calculations above, however,

$$H_3(SL_2(F_2), \mathbb{Z}[1/2]) \cong B(F_2).$$

\[\square\]

### 8.2. The Bloch group of $F_2$.

$F_2$ is $L_\bullet$ acyclic in dimensions $\leq 2$. So $\mathcal{RP}(F_3)$ is generated as an $R_{F_2}$-module by $[-1]$ (since $W_{F_2} = \{-1\}$). By Corollary 3.6 (2) we have $0 = 2\psi_1(-1) = 2([-1] + [-1] [-1])$ in $\mathcal{RP}(F_3)$.

**Proposition 8.3.** $\mathcal{RP}(F_3)$ is the cyclic $R_{F_2}$-module with generator $[-1]$ subject to the relation $2([-1] + [-1] [-1]) = 0$.

**Proof.** Let $A$ denote the cyclic $R_{F_2}$-module generated by the symbol $[-1]$, subject to the relation $2\psi_1(-1) = 0$. Thus we have a surjective homomorphism of $R_{F_2}$-modules $A \to \mathcal{RP}(F_3)$. However the natural map $F_3 \to F_{27}$ induces an homomorphism of $R_{F_2}$-modules $\mathcal{RP}(F_3) \to \mathcal{RP}(F_{27})$ which sends $[-1]$ to $[-1]$. By the results of [7, Section 7], $[-1]$ has infinite order in $\mathcal{RP}(F_{27})$ and $\psi_1(-1)$ has order 2. It follows that the composite homomorphism $A \to \mathcal{RP}(F_{27})$ is injective and hence $A \cong \mathcal{RP}(F_3)$.

Taking $F_3^\times$-coinvariants, we deduce:

**Corollary 8.4.** $\mathcal{P}(F_3)$ is a cyclic abelian group of order 4 with generator $[-1]$.

**Corollary 8.5.** $\mathcal{RP}_1(F_3)$ is a trivial $R_{F_2}$-module of order 2 generated by $\psi_1(-1)$.

**Proof.** By Lemma 3.19 (3), the map $\lambda_1 : \mathcal{RP}(F_3) \to R_{F_2}$ sends $[-1]$ to $-\langle(-1)\rangle^2 = 2 \langle(-1)\rangle$, which has infinite order. It sends $\psi_1(-1)$ to 0. It follows that $\mathcal{RP}_1(F_3) = \text{Ker}(\lambda_1)$ is generated by $\psi_1(-1)$.

\[\square\]
The group $B = B(\text{SL}_2(\mathbb{F}_3))$ is nonabelian of order 6. Thus we have $E_{2,0}^1(\text{SL}_2(\mathbb{F}_3), L^\tau) = H_2(B, \mathbb{Z}) = 0$ and $E_{1,0}(\text{SL}_2(\mathbb{F}_3), L^\tau) = H_3(B, \mathbb{Z}) = \mathbb{Z}/6$. In particular, the map $d^3 : E_{3,0}^3 = R\mathcal{P}_1(\mathbb{F}_3) \to E_{2,0}^3$ is the zero homomorphism. We immediately conclude:

**Corollary 8.9.** $R\mathcal{B}(\mathbb{F}_3) = R\mathcal{P}_1(\mathbb{F}_3)$ is cyclic of order 2 generated by $\psi_1(−1)$.

**Corollary 8.7.** $B(\mathbb{F}_3)$ is cyclic of order 2 with generator $\psi(−1) = 2 [−1]$.

**Proof.** The map $R\mathcal{P}(\mathbb{F}_3) \to \mathcal{P}(\mathbb{F}_3)$ induces a map $R\mathcal{B}(\mathbb{F}_3) \to B(\mathbb{F}_3)$ sending the generator $\psi_1(−1)$ to $2 [−1]$. Thus $B(\mathbb{F}_3)$ contains the element $2 [−1]$ of order 2.

Now $B(\mathbb{F}_3) = \text{Ker}(d^3 : \mathcal{P}(\mathbb{F}_3) \to E_{2,0}^3(\text{GL}_2(\mathbb{F}_3), L^\tau))$. The map $\mathbb{F}_3 \to \mathbb{F}_{27}$ induces a commutative diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathbb{F}_3) & \xrightarrow{d^3} & E_{2,0}^3 \\
\downarrow & & \downarrow \\
\mathcal{P}(\mathbb{F}_{27}) & \xrightarrow{\lambda_2} & S_2^2(\mathbb{F}_{27}^\times) = E_{2,0}^3(\text{GL}_2(\mathbb{F}_{27}), L^\tau)
\end{array}
$$

where $\lambda_2([-1]) = -1 \circ -1 \in S_2^2(\mathbb{F}_{27}^\times)$ which has order 2. It follows that $d^3([-1]) \neq 0$ and the result follows.

**Proposition 8.8.** There is a natural commutative diagram with exact rows

$$
\begin{array}{cccc}
0 & \xrightarrow{\mathcal{P}(\mathbb{F}_3)} & H_3(\text{SL}_2(\mathbb{F}_3), \mathbb{Z}) & \xrightarrow{R\mathcal{B}(\mathbb{F}_3)} & 0 \\
\downarrow & & \downarrow & & \downarrow_{\cong} \\
0 & \xrightarrow{\mathcal{P}(\mathbb{F}_{27})} & K_3^{\text{ind}}(\mathbb{F}_3) & \xrightarrow{B(\mathbb{F}_3)} & 0.
\end{array}
$$

**Proof.** A direct calculation, shows that $H_3(\text{SL}_2(\mathbb{F}_3), \mathbb{Z}) \cong \mathbb{Z}/24$ (see, for example, [7, Section 3]). Furthermore, the natural map $H_3(\text{SL}_2(\mathbb{F}_3), \mathbb{Z}) \to K_3^{\text{ind}}(\mathbb{F}_3)$ induces an isomorphism $\mathbb{Z}/8 \cong H_3(\text{SL}_2(\mathbb{F}_3), \mathbb{Z}[1/3]) \cong K_3^{\text{ind}}(\mathbb{F}_3)$ (again, see [7, Section 3]).

We will use the following fact in our calculation of $R\mathcal{B}(\mathbb{Z})$ below:

**Corollary 8.9.** In the spectral sequence $E(\text{SL}_2(\mathbb{F}_3), L^\tau)$ we have $\mathbb{Z}/2 \cong E_{1,2}^2 = E_{1,2}^\infty$.

**Proof.** $E_{1,2}^2$ is the homology of

$$
H_1(\text{SL}_2(\mathbb{F}_3), L_1^\tau) \xrightarrow{d^1} R\mathbb{F}_3 \otimes \mu_2 \xrightarrow{d^2} \mu_2
$$

where the second differential is surjective. Thus $E_{1,2}^2$ has order 2 or 0.

Now the terms $E_{0,3}^\infty, E_{1,2}^\infty, E_{2,1}^\infty, E_{3,0}^\infty$ are the terms of a graded abelian group associated to a filtration on $H_3(\text{SL}_2(\mathbb{F}_3), \mathbb{Z}) \cong \mathbb{Z}/24$. Since $E_{2,1}^1 = H_2(\mu_2, \mathbb{Z}) = 0$, we have $E_{2,1}^\infty = 0$. Furthermore, $E_{0,3}^\infty = R\mathcal{B}(\mathbb{F}_3) \cong \mathbb{Z}/2$ and $E_{3,0}^\infty = H_3(B, \mathbb{Z}) \cong \mathbb{Z}/6$. It follows that $E_{1,2}^\infty = \mathbb{Z}/2$ as required.

**8.3. The Bloch group of $\mathbb{Z}$.** We begin by recalling the relevant classical facts about the structure and homology of the group $G = \text{SL}_2(\mathbb{Z})$ (see, for example, [17]):
Theorem 8.10. \( \text{SL}_2(\mathbb{Z}) \cong \mathbb{Z}/4 \times \mathbb{Z}/6 \) where the first factor is generated by 
\( \omega := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) and the second by \( \tau := \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \). The homomorphism \( \text{SL}_2(\mathbb{Z}) \to \mathbb{Z}/12 \) 
sending \( \omega \) to 3 and \( \tau \) to 2 induces an isomorphism on homology 
\[
H_n(\text{SL}_2(\mathbb{Z}), \mathbb{Z}) \cong H_n(\mathbb{Z}/12, \mathbb{Z}) = \begin{cases} 
\mathbb{Z}, & n = 0 \\
\mathbb{Z}/12, & n \text{ odd} \\
0, & n > 0 \text{ even} 
\end{cases}
\]

Now we have the spectral sequence \( E_1^{p,q} (G, L^\tau) \Rightarrow H_{p+q}(G, L^\tau) \cong H_{p+q}(G, \mathbb{Z}) \) for any subgroup \( G \subset \text{GL}_2(\mathbb{Z}) \). We consider the case \( G = \text{SL}_2(\mathbb{Z}) \).

Note that \( T(\text{SL}_2(\mathbb{Z})) = \mu_2 := \{ \pm I \} = Z(\text{SL}_2(\mathbb{Z})) \). Furthermore, \( B := B(\text{SL}_2(\mathbb{Z})) = \mu_2 \times \rho \) where
\[
\rho = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.
\]

Since \( B \) is abelian, \( H_1(B, \mathbb{Z}) = B \). By the Künneth formula, \( H_2(B, \mathbb{Z}) \cong \mu_2 \otimes \mathbb{Z} = \mathbb{Z}/2 \) and the inclusion \( \mu_2 \to B \) induces an isomorphism \( \mathbb{Z}/2 = H_3(\mu_2, \mathbb{Z}) \cong H_3(B, \mathbb{Z}) \).

Thus the \( E_1 \)-page of the spectral sequence has the form

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & \cdots \\
\mathcal{R}P(\mathbb{Z}) & H_1(G, L_3^\tau) & H_2(G, L_3^\tau) & \cdots \\
\downarrow d^1 & \downarrow d^1 & \downarrow d^1 & \\
\mathbb{Z} & \mathcal{R} \otimes \mathbb{Z} \mu_2 & 0 & \cdots \\
\downarrow \iota & \downarrow \iota \otimes \text{id} & & \downarrow \iota \\
\mathbb{Z} & \mu_2 & 0 & \mu_2 \\
\downarrow & & & \\
\mathbb{Z} & B & \mathbb{Z}/2 & \mathbb{Z}/2 \\
\end{array}
\]

and hence the \( E_2 \)-page looks (in part) like:

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & \cdots \\
\mathcal{R}P_1(\mathbb{Z}) & E_1^{2,0} & E_2^{2,0} & \cdots \\
\downarrow \mathcal{I}_Z/\text{Im}(d^1) & \downarrow d^2 & \downarrow 0 & \cdots \\
\downarrow 0 & \downarrow d^2 & \downarrow 0 & \cdots \\
\downarrow \mathbb{Z} & B & \mathbb{Z}/2 & \mathbb{Z}/2 \\
\end{array}
\]

Lemma 8.11. In this spectral sequence, \( E_3^{2,0} = H_3(\mu_2, \mathbb{Z}) = H_3(B, \mathbb{Z}) = E_{3,0}^\infty \).
Proof. Since the inclusion map \( \mathbb{Z}/2 \to \mathbb{Z}/12 \) induces an injective map \( H_3(\mathbb{Z}/2, \mathbb{Z}) \to H_3(\mathbb{Z}/12, \mathbb{Z}) \), it follows from Theorem 8.10 that the inclusion map \( \mu_2 : \text{SL}_2(\mathbb{Z}) \to H_3 (\mathbb{Z} \to \text{SL}_2(\mathbb{Z}), \mathbb{Z}) \).

Lemma 8.12. The map \( d^1 = \lambda_1 : \mathcal{RP}(\mathbb{Z}) \to \mathbb{R} \) is the zero map, and hence \( \mathcal{RP}_1(\mathbb{Z}) = \mathcal{RP}(\mathbb{Z}) \).

Proof. By Lemma 4.7 (1), the composite map \( \mathcal{I}_3 \to \mathcal{I}_2/\text{Im}(d^1) = E^2_{3,2} = E^2_{1,0} = B \) sends \( \langle -1 \rangle \) to the matrix \[
\begin{bmatrix}
-1 & 0 \\
6 & -1
\end{bmatrix}.
\]

Since \( \mathcal{I}_2 = \mathbb{Z} \cdot \langle -1 \rangle \cong \mathbb{Z} \), this map is injective. It follows that \( \text{Im}(d^1) = 0 \).

Lemma 8.13. \( \psi_1 (-1) \in \mathcal{RP}_1(\mathbb{Z}) \setminus \mathcal{RB}(\mathbb{Z}) \) while \( D_2 \in \mathcal{RB}(\mathbb{Z}) \).

Proof. Certainly \( \psi_1 (-1) \in \mathcal{RP}_1(A) \) for any ring \( A \). Suppose, for the sake of contradiction, that \( \psi_1 (-1) \in \mathcal{RB}(\mathbb{Z}) \). Since \( H_3(\text{SL}_2(\mathbb{Z}), \mathbb{Z}) \to \mathcal{RB}(\mathbb{Z}) \) is surjective, there is then a homology class mapping to \( \psi_1 (-1) \). But then the image of this homology class in \( B(Q) \) would be \( \psi(-1) \), contradicting Lemma 7.3.

\( \mathcal{RB}(\mathbb{Z}) \) is the kernel of \( d^3 : \mathcal{RP}_1(\mathbb{Z}) \to E^3_{2,1} \) and this latter term is a quotient of \( \mathbb{Z}/2 \).

Since \( D_2 \) has order 3, it follows that \( D_2 \in \mathcal{RB}(\mathbb{Z}) \).

Corollary 8.14. Let \( A \) be a ring. Then \( D_A \in \mathcal{RB}(A) \). More generally, \( \langle u \rangle D_A \in \mathcal{RB}(A) \) for all \( u \in A^\times \).

Proof. \( D_A \) is the image of \( D_2 \) under the homomorphism \( \mathcal{RB}(\mathbb{Z}) \to \mathcal{RB}(A) \). The second statement holds because \( \mathcal{RB}(A) \) is a \( R_A \)-module.

Lemma 8.15. In the spectral sequence \( E(\text{SL}_2(\mathbb{Z}), L^r) \), we have \( \mathbb{Z}/2 \cong E^2_{1,2} = E^\infty_{1,2} \).

Proof. The map \( \mathbb{Z} \to \mathbb{F}_3 \) induces a map of spectral sequences \( E(\text{SL}_2(\mathbb{Z}), L^r) \to E(\text{SL}_2(\mathbb{F}_3), L^r) \).

Thus we have a commutative diagram

\[
\begin{array}{ccc}
E^1_{1,3} = H_1(\text{SL}_2(\mathbb{Z}), L_3) & \longrightarrow & H_1(\text{SL}_2(\mathbb{F}_3), L_3) \\
\downarrow d^1 & & \downarrow d^1 = 0 \\
E^1_{1,2} = \mathbb{R}_2 \otimes \mu_2 & \cong & \mathbb{R}_{\mathbb{F}_3} \otimes \mu_2 = E^2_{1,2} \\
\epsilon \otimes \text{id} & & \epsilon \otimes \text{id} \\
E^1_{1,1} = \mu_2 & \cong & \mu_2 = E^1_{1,1}
\end{array}
\]

which induces an isomorphism \( E^2_{1,2}(\text{SL}_2(\mathbb{Z}), L^r) \cong E^2_{1,2}(\text{SL}_2(\mathbb{F}_3), L^r) = \mathbb{Z}/2 \) (using Corollary 8.9).

Now in \( E(\text{SL}_2(\mathbb{Z}), L^r) \), we have \( E^\infty_{1,2} = \text{Ker}(d^2 : E^2_{1,2} \to E^2_{2,0}) \). Since \( E^2_{2,0} = H_2(B, \mathbb{Z}) \) has order 2 and since Lemma 8.13 shows that \( d^3 : \mathcal{RP}_1(\mathbb{Z}) \to E^3_{2,1} \) is not the zero map, it follows that \( E^3_{2,0} = E^2_{2,0} = \mathbb{Z}/2 \) and hence the map \( d^2 : E^2_{1,2} \to E^2_{2,0} \) is trivial.

\( \square \)

Theorem 8.16. There is a short exact sequence

\[
0 \to \mathbb{Z}/4 \cong H_3(\langle \omega \rangle, \mathbb{Z}) \to H_3(\text{SL}_2(\mathbb{Z}), \mathbb{Z}) \to \mathcal{RB}(\mathbb{Z}) \to 0
\]

and \( \mathcal{RB}(\mathbb{Z}) \) is cyclic of order 3, generated by the element \( D_2 = 2C_2 \).
Remark 8.19. The corresponding statement for $RB$ the generator $ightarrow RB$ sequence $0$

Proof. Recall that $B$. Thus $C$. Where $Z$

It follows that $H$ is cyclic in dimension $4$ and that $RB(Z) \cong Z/3$ as required. 

Remarked 8.17. Note that it follows that $Z$ is not $L_\ast$-acyclic in dimension $2$. For $W_Z = \emptyset$ and hence $X_3(Z) = \emptyset$ (i.e., there are no 4-cliques in the graph $\Gamma(Z)$), but $RP(Z) \neq 0$ and hence $L_3^\ast(Z) \neq 0$.

Corollary 8.18. Let $A$ be a ring and let $A \rightarrow F_2$ be a ring homomorphism. Then the induced (functorial) homomorphism $RB(A) \rightarrow RB(F_2)$ is surjective.

Proof. The constant $D_A$, which is the image of $D_Z$, maps to the generator $D_{F_2} = 2C_{F_2}$ of $RB(F_2)$.

Remark 8.19. The corresponding statement for $F_3$ is false, as the example $A = Z$ shows: the generator $D_Z$ of $RB(Z)$ maps to $D_{F_3} = 0$ in $RB(F_3)$.

Corollary 8.20. $RP(Z) = RP_1(Z) = P(Z)$ is cyclic of order $6$ generated by the element $C_Z$.

Proof. Recall that $C_Z \in RP_1(Z) = RP(Z)$ has order $6$. Since there is a short exact sequence $0 \rightarrow RB(Z) \rightarrow RP_1(Z) \rightarrow E_3^{3,0} = Z/2 \rightarrow 0$, where $RB(Z)$ is cyclic of order $3$, it follows that $RP_1(Z)$ has order $6$ and is generated by $C_Z$. Thus $RP(Z)$ is a trivial $R_{Z}$-module and hence $RP(Z) = P(Z)$ also.

Lemma 8.21. Let $B = B(SL_2(Z))$ as above and let $\tilde{B} := B(GL_2(Z))$. Then the inclusion homomorphism $B \rightarrow \tilde{B}$ induces an injective map $H_2(B, Z) \rightarrow H_2(\tilde{B}, Z)$.

Proof. Consider the map of (split) group extensions

\[
\begin{array}{c}
1 \rightarrow U \rightarrow B \rightarrow Z \rightarrow 1 \\
1 \rightarrow U \rightarrow \tilde{B} \rightarrow C_1 \times C_2 \rightarrow 1
\end{array}
\]

where $Z \cong U := \langle \rho \rangle$, $\mu_2 \cong Z$ is generated by $\text{diag}(-1, -1)$, $C_1$ is generated by $\sigma_1 := \text{diag}(-1, 1)$ and $C_2$ is generated by $\text{is generated by} \sigma_2 := \text{diag}(1, -1)$. This induces a
map of the associated Hochschild-Serre spectral sequences. Considering the terms \( E_{i,j}^2 = H_i(Z, H_j(U, Z)) \) of the spectral sequence of the top extension, we deduce that there is an isomorphism \( E_{i,1}^2 = H_1(Z, U) \cong H_2(B, Z) \). Likewise, the spectral sequence for the lower extension yields a (split) short exact sequence

\[
0 \rightarrow E_{1,1}^2 = H_1(C_1 \times C_2, U) \rightarrow H_2(B, Z) \rightarrow H_2(C_1 \times C_2, Z) \rightarrow 0.
\]

Thus the Lemma is equivalent to the statement that the map \( H_1(Z, U) \rightarrow H_1(C_1 \times C_2, U) \) is injective. To see this, consider the (again split) extension

\[
1 \rightarrow Z \rightarrow C_1 \times C_2 \xrightarrow{p} C \rightarrow 1
\]

where \( C \) is cyclic of order 2 generated by \( \sigma \) and \( p(\sigma_1, \sigma_2^j) := \sigma^{i+j} \). The exact sequence of terms of low degree for the associated spectral sequence (for the module \( U \)) has the form

\[
H_2(C_1 \times C_2, U) \rightarrow H_2(C, U) \rightarrow H_1(Z, U) \rightarrow H_1(C_1 \times C_2, U) \rightarrow \cdots
\]

But the leftmost arrow is a surjection since the morphism of pairs \( (C_1 \times C_2, U) \rightarrow (C, U) \) has a right-inverse. Thus the third arrow is an injection as claimed. \( \square \)

**Corollary 8.22.** The natural map \( RB(Z) \rightarrow B(Z) \) is an isomorphism.

**Proof.** The functorial maps \( RB(Z) \rightarrow B(Z) \rightarrow B(Q) \) sends \( D_Z \) to \( D_Q \neq 0 \). So the map \( RB(Z) \rightarrow B(Z) \) is injective.

Next we show that \( \psi(-1) \in P(Z) \setminus B(Z) \).

Recall that \( B(Z) := E_{0,3}^3(GL_2(Z), L^\tau) = \text{Ker}(d^3 : P(Z) \rightarrow E_{3,0}^3(GL_2(Z), L^\tau)) \). We first note that in the spectral sequence \( E(GL_2(Z), L^\tau) \), we have

\[
H_2(B, Z) = E_{2,0}^1 = E_{2,0}^2 = E_{2,0}^3:
\]

(i) The map \( d^1 : E_{1,2}^1 = H_2(T, Z) \rightarrow H_2(B, Z) \) is the map \( c_\omega - \text{id} \) followed by corestriction. Here \( c_\omega \) is the map induced by conjugation by \( \omega \). This map is inversion on \( T \) and hence induces the identity on \( H_2(T, Z) \). So the \( d^1 \) map is trivial and \( E_{2,0}^1 = E_{2,0}^2 \).

(ii) The map \( d^1 : E_{1,2}^1 = H_1(\mu_2, Z) \rightarrow H_1(T, Z) = E_{1,1}^1 \) is just the inclusion map and hence \( E_{1,2}^1 = 0 \). It follows of course that \( d^2 : E_{1,2}^2 \rightarrow E_{2,0}^2 \) is the zero map and hence \( E_{2,0}^2 = E_{2,0}^3 \).

Thus we have a commutative diagram

\[
\begin{array}{ccc}
E_{0,3}^3(SL_2(Z), L^\tau) & \xrightarrow{\cong} & \mathcal{P}(Z) = E_{0,3}^3(GL_2(Z), L^\tau) \\
\downarrow d^3 & & \downarrow d^3 \\
E_{2,0}^3(SL_2(Z), L^\tau) = H_2(B, Z) & \rightarrow & H_2(B, Z) = E_{2,0}^3(GL_2(Z), L^\tau).
\end{array}
\]

where the lower horizontal arrow is an injection by Lemma 8.21. It follows that \( \psi(-1) \) is not in the kernel of the right-hand vertical map, since \( \psi_1(-1) \) is not in the kernel of the left-hand vertical map; i.e., \( \psi_1(-1) \not\in B(Z) \). Since \( P(Z) \) is cyclic of order 6 generated by \( C_Z \), it follows that \( B(Z) \) is cyclic of order 3 generated by \( 2C_Z = D_Z \). \( \square \)
Corollary 8.23. There is a commutative diagram with exact rows and injective vertical arrows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H_3(\langle \omega \rangle, \mathbb{Z}) & \longrightarrow & H_3(\text{SL}_2(\mathbb{Z}), \mathbb{Z}) & \longrightarrow & B(\mathbb{Z}) & \longrightarrow & 0 \\
\downarrow & & \cong & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{tor}(\mu_2, \mu_2) & \longrightarrow & K_{3}^{\text{ind}}(\mathbb{Q}) & \longrightarrow & B(\mathbb{Q}) & \longrightarrow & 0.
\end{array}
\]

Proof. The diagram commutes, the bottom row is exact and the left hand vertical arrow is an isomorphism by Theorem [7.1] and Lemma [7.2]. The top row is exact and the map \( \mathcal{R}B(\mathbb{Z}) = B(\mathbb{Z}) \rightarrow B(\mathbb{Q}) \) is injective by Theorem 8.10. \( \square \)

For future reference, it is useful to note that there is an explicit homology class in \( H_3(\text{SL}_2(\mathbb{Z}), \mathbb{Z}) \) which maps to the generator \( D_3 \in \mathcal{R}B(\mathbb{Z}) = B(\mathbb{Z}) \):

**Proposition 8.24.** Let \( t := \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) \). Let \( G \) be the cyclic group (of order 3) generated by \( t \).

The inclusion map \( G \rightarrow \text{SL}_2(\mathbb{Z}) \) induces a composite homomorphism, \( \phi \) say, \( H_3(G, \mathbb{Z}) \rightarrow H_3(\text{SL}_2(\mathbb{Z}), \mathbb{Z}) \rightarrow \mathcal{R}P(\mathbb{Z}) \).

Consider \( 1 \in \mathbb{Z}/3 \cong H_3(G, \mathbb{Z}) \). Then \( \phi(1) = D_3 \).

Proof. We can calculate \( \phi \) as follows (see, for example, [7, p 56]): Let \( C_3 \) be the standard periodic resolution of \( \mathbb{Z} \) as a \( \mathbb{Z}[G] \)-module. Let \( \beta : C_3 \rightarrow L^\bullet_3 \) be an augmentation-preserving map of \( \mathbb{Z}[G] \)-complexes. Then \( \beta \) induces a map

\[
H_3(G, \mathbb{Z}) = H_3(\langle C_3 \rangle, \mathbb{Z}) \rightarrow H_3(\langle L^\bullet_3 \rangle, \mathbb{Z}) \rightarrow H_3(\langle L^\bullet_3 \rangle, \text{SL}_2(\mathbb{Z})) = \mathcal{R}P_1(\mathbb{Z}) \subseteq \mathcal{R}P(\mathbb{Z})
\]

which coincides with the map \( \phi \).

Now it is straightforward to verify that \( \beta \) can be given as follows:

\[
\begin{align*}
\beta_0 : C_0 &= \mathbb{Z}[G] \rightarrow L^0_3(\mathbb{Z}), 1 \mapsto (1), \\
\beta_1 : C_1 &= \mathbb{Z}[G] \rightarrow L^1_3(\mathbb{Z}), 1 \mapsto (1, \infty), \\
\beta_2 : C_2 &= \mathbb{Z}[G] \rightarrow L^2_3(\mathbb{Z}), 1 \mapsto (0, 1, \infty) + (1, \infty, 0) + (1, 0, \infty), \\
\beta_3 : C_3 &= \mathbb{Z}[G] \rightarrow L^3_3(\mathbb{Z}), 1 \mapsto (\infty, 0, 1) + (\infty, 1, 0) - (0, 1, \infty) - (1, 0, \infty).
\end{align*}
\]

Since this last term represents \( D_3 \) and since \( 1 \in H_3(G, \mathbb{Z}) \) is represented by \( 1 \in \mathbb{Z}[G] = C_3 \), the result follows. \( \square \)

**Corollary 8.25.** Let \( A \) be a commutative ring. Let \( G, t \) be as in Proposition 8.24. Let \( \phi_A \) be the composite homomorphism \( H_3(G, \mathbb{Z}) \rightarrow \mathcal{R}P(\mathbb{Z}) \rightarrow \mathcal{R}P(A) \). Then \( \phi_A(1) = D_A \).

**Remark 8.26.** For any \( L^\bullet_3 \)-acyclic ring \( A \) it is natural to also let \( D_A \in H_3(\text{SL}_2(\mathbb{A}), \mathbb{Z}) \) denote the image of \( 1 \in H_3(G, \mathbb{Z}) \) in \( H_3(\text{SL}_2(\mathbb{A}), \mathbb{Z}) \).

8.4. The Bloch group of \( \mathbb{Z}[\frac{1}{2}] \). By Example 5.8 the ring \( \mathbb{Z}[\frac{1}{2}] \) is \( L^\bullet_3 \)-acyclic. So there is a natural surjective map \( H_3(\text{SL}_2(\mathbb{Z}[\frac{1}{2}]), \mathbb{Z}) \rightarrow \mathcal{R}B(\mathbb{Z}[\frac{1}{2}]) \).

We will rely on the following description of the structure of \( H_3(\text{SL}_2(\mathbb{Z}[\frac{1}{2}]), \mathbb{Z}) \) as a \( \mathbb{Z} \)-module, due to Adem and Naffah [11]:

**Proposition 8.27.** \( H_3(\text{SL}_2(\mathbb{Z}[\frac{1}{2}]), \mathbb{Z}) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3. \)

We also note:

**Lemma 8.28.** If \( A = \mathbb{Z}[\frac{1}{2}] \) then \( H_2(T, \mathbb{Z}) \cong H_2(B, \mathbb{Z}) \).
Proof. As in the proof of Lemma 6.6, it is enough to show that $H := H_1(A^\times, A) = 0$. Since $2 \in A^\times$, we again deduce that $H$ is annihilated by 3. We have a (split) short exact sequence of groups $1 \to \mu_2 \to A^\times \to 2Z \to 1$, where here $2Z := \{2^i : i \in \mathbb{Z}\} \cong \mathbb{Z}$. Since 2 acts invertibly on $A$, the groups $H_i(\mu_2, A)$ vanish for $i \geq 1$ and the Hochschild-Serre spectral sequence induces and isomorphism $H_i(A^\times, A) \cong H_i(2Z, A)$ for all $i$. In particular, $H_1(A^\times, A) = H_1(2Z, A) = \{a \in A \mid 3a = 0\} = 0$. □

Lemma 8.29. The element $\psi_1(-1)$ lies in $\mathcal{RB}(\mathbb{Z}[\frac{1}{2}])$ and has order 2. There exists a class $X$ of order 8 in $H_3(\text{SL}_2(\mathbb{Z}[\frac{1}{2}]), \mathbb{Z})$ mapping to $\psi_1(-1)$.

Proof. By direct calculation, we have $[-1] + (-1) [-1] \in \mathcal{RB}(\mathbb{Z}[\frac{1}{2}]) = E_{0,3}^\infty(\text{SL}_2(\mathbb{Z}[\frac{1}{2}]), L)$ (using Proposition 6.3 (2) for this last equality). The image of this element under the map $E_{0,3}^\infty(\text{SL}_2(\mathbb{Z}[\frac{1}{2}]), L) \to E_{0,3}^\infty(\text{SL}_2(\mathbb{Z}[\frac{1}{2}]), L^*) = \mathcal{RB}(\mathbb{Z}[\frac{1}{2}])$ is $\psi_1(-1)$. Furthermore, $\psi_1(-1)$ has order 2 since its image under the map $\mathcal{RB}(\mathbb{Z}[\frac{1}{2}]) \to \mathcal{RB}(\mathbb{Q}) \to \mathcal{B}(\mathbb{Q})$ is $\psi(-1) = 2 [-1]$ which has order 2.

Thus, there exists $X$ in $H_3(\text{SL}_2(\mathbb{Z}[\frac{1}{2}]), \mathbb{Z})$ mapping to $\psi_1(-1)$ in $\mathcal{RB}(\mathbb{Z}[\frac{1}{2}])$. Replacing $X$ by $3X$, if needed, we may assume $X$ has order dividing 8. On the other hand, the image of $X$ under the map $H_3(\text{SL}_2(\mathbb{Z}[\frac{1}{2}]), \mathbb{Z}) \to H_3(\text{SL}_2(\mathbb{Q}), \mathbb{Z}) \to K_3^{\text{ind}}(\mathbb{Q})$ maps to $\psi(-1) \in \mathcal{B}(\mathbb{Q})$ and hence this image has order 8 by Theorem 7.1. Thus $X$ has order 8. □

Lemma 8.30. The elements $D_{\mathbb{Z}[\frac{1}{2}]}$, $\langle 2 \rangle D_{\mathbb{Z}[\frac{1}{2}]} \in \mathcal{RB}(\mathbb{Z}[\frac{1}{2}])$ generate a subgroup of type $\mathbb{Z}/3 \oplus \mathbb{Z}/3$.

Proof. Let $T$ denote the composite homomorphism $\mathcal{RB}(\mathbb{Z}[\frac{1}{2}]) \to \mathcal{RB}(\mathbb{Q}) \to \mathcal{B}(\mathbb{Q})$. Note that $T$ is a $\mathbb{R}_{\mathbb{Z}[\frac{1}{2}]}$-homomorphism where $\mathcal{B}(\mathbb{Q})$ has the trivial module structure. Thus $T(D_{\mathbb{Z}[\frac{1}{2}]}) = T(\langle 2 \rangle D_{\mathbb{Z}[\frac{1}{2}]}) = D_Q$, which has order 3. Thus $D_{\mathbb{Z}[\frac{1}{2}]}$ and $\langle 2 \rangle D_{\mathbb{Z}[\frac{1}{2}]}$ each have order 3 in $\mathcal{RB}(\mathbb{Z}[\frac{1}{2}])$.

Now give $\mathcal{P}(\mathbb{F}_2)$ the $\mathbb{R}_{\mathbb{Q}}$-module structure: $\langle q \rangle x := (-1)^{v_2(q)} x$ for $q \in \mathbb{Q}^\times$, $x \in \mathcal{P}(\mathbb{F}_2)$. Recall that $\mathcal{P}(\mathbb{F}_2)$ is cyclic of order 3 with generator $C_{\mathbb{F}_2} = -D_{\mathbb{F}_2}$. There is a well-defined $\mathbb{R}_{\mathbb{Q}}$-homomorphism

$$S_2 : \mathcal{RB}(\mathbb{Q}) \to \mathcal{P}(\mathbb{F}_2), [q] \mapsto \begin{cases} C_{\mathbb{F}_2}, & v_2(q) > 0 \\ 0, & v_2(q) = 0 \\ -C_{\mathbb{F}_2}, & v_2(q) < 0. \end{cases}$$

Composing this with the map $\mathcal{RB}(\mathbb{Z}[\frac{1}{2}]) \to \mathcal{RB}(\mathbb{Q})$ gives a $\mathbb{R}_{\mathbb{Z}[\frac{1}{2}]}$-homomorphism $S' : \mathcal{RB}(\mathbb{Z}[\frac{1}{2}]) \to \mathcal{P}(\mathbb{F}_2)$. We have $S'(D_{\mathbb{Z}[\frac{1}{2}]}) = S_2(D_Q) = D_{\mathbb{F}_2}$. By the module structure on $\mathcal{P}(\mathbb{F}_2)$ it follows that

$$S'(\langle 2 \rangle D_{\mathbb{Z}[\frac{1}{2}]}) = S'(\langle 2 \rangle D_{\mathbb{Z}[\frac{1}{2}]}) = S'(\langle 2 \rangle D_{\mathbb{Z}[\frac{1}{2}]}) = \langle 2 \rangle D_{\mathbb{F}_2} = -2D_{\mathbb{F}_2} = D_{\mathbb{F}_2}$$

has order 3. □

Proposition 8.31. We have $\mathcal{RB}(\mathbb{Z}[\frac{1}{2}]) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$ with generators $\psi_1(-1)$, $D_{\mathbb{Z}[\frac{1}{2}]}$ and $\langle 2 \rangle D_{\mathbb{Z}[\frac{1}{2}]}$. The square class $\langle -1 \rangle$ acts trivially on $\mathcal{RB}(\mathbb{Z}[\frac{1}{2}])$. The square class $\langle 2 \rangle$ is trivial on the first factor and interchanges the last two factors.
There is a commutative diagram of $R_{Z[\frac{1}{2}]}$-modules with exact rows (where $R_{Z[\frac{1}{2}]}$ acts trivially on the terms of the bottom row) and surjective vertical arrows:

$$
\begin{array}{cccc}
0 & \longrightarrow & H_3((\omega), Z) & \longrightarrow H_3(SL_2(\mathbb{Z}[\frac{1}{2}]), Z) & \longrightarrow R\mathcal{B}(Z[\frac{1}{2}]) & \longrightarrow 0 \\
\hspace{1cm} & \searrow & \downarrow \cong & \downarrow & \downarrow & \\
0 & \longrightarrow & \text{tor}(\mu_2, \mu_2) & \longrightarrow & K_{3}^{\text{ind}}(Q) & \longrightarrow & B(Q) & \longrightarrow 0.
\end{array}
$$

Proof. By Lemmas 8.29 and 8.30 $R\mathcal{B}(Z[\frac{1}{2}])$ contains a subgroup generated by $\psi_1(-1)$, $D_{Z[\frac{1}{2}]}$ of type $\mathbb{Z}/2 \oplus /\mathbb{Z}/3 \oplus \mathbb{Z}/3$. The map $\mathbb{Z}/4 \cong H_3((\omega), Z) \rightarrow H_3(SL_2(\mathbb{Z}[\frac{1}{2}]), Z)$ is injective since the injective map $H_3((\omega), Z) \rightarrow K_{3}^{\text{ind}}(Q)$ factors through this map. Furthermore, the image of this map is in the kernel of the surjective map $H_3(SL_2(\mathbb{Z}[\frac{1}{2}]), Z) \rightarrow R\mathcal{B}(Z[\frac{1}{2}])$. By Proposition 8.27 it follows that the subgroup in question is all of $R\mathcal{B}(Z[\frac{1}{2}])$, and that the top row is a short exact sequence.

With regard to the $R_{Z[\frac{1}{2}]}$-module structure on $R\mathcal{B}(Z[\frac{1}{2}])$, it is known (see section 3) that $\langle -1 \rangle$ acts trivially on $\psi_1(-1)$ and $D_{Z[\frac{1}{2}]}$. Clearly $\langle 2 \rangle$ must fix the unique element $\psi_1(-1)$ of order 2. (Alternatively, deduce this from Corollary 3.22) Finally, $\langle 2 \rangle$ interchanges the last two factors by Lemma 8.30.

Finally, the bottom row is known to be exact and the diagram clearly commutes. The left-hand vertical arrow is an isomorphism by Lemma 7.2. The right-hand vertical arrow is surjective since $\psi_1(-1) - D_{Z[\frac{1}{2}]} = C_{Z[\frac{1}{2}]} \in R\mathcal{B}(Z[\frac{1}{2}])$ maps to the generator $C_{Q}$ of $B(Q)$. □

**Theorem 8.32.** The $R_{Z[\frac{1}{2}]}$-module $R\mathcal{P}(Z[\frac{1}{2}])$ is generated by $[-1]$, $[2]$ and $[\frac{1}{2}]$; i.e., it is generated by $\{[u] \mid u \in \mathcal{W}_{Z[\frac{1}{2}]}\}$. As an abelian group it has the structure

$$
R\mathcal{P}(Z[\frac{1}{2}]) = R\mathcal{B}(Z[\frac{1}{2}]) \oplus \mathbb{Z} \cdot [-1] \oplus \mathbb{Z} \cdot [\frac{1}{2}] \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z} \oplus \mathbb{Z}
$$

where the generators of the factors are $\psi_1(-1)$, $D_{Z[\frac{1}{2}]}$, $\langle 2 \rangle D_{Z[\frac{1}{2}]}$, $[-1]$ and $[\frac{1}{2}]$ respectively.

The theorem is proven in Lemmas 8.33 to 8.40.

In Lemmas 8.33 to 8.40 $R$ will denote $R_{Z[\frac{1}{2}]}$, $D$ will denote $D_{Z[\frac{1}{2}]}$, and $C$ will denote $C_{Z[\frac{1}{2}]}$.

**Lemma 8.33.** Let $\epsilon := \langle -1 \rangle [-1] - [-1] \in R \cdot [-1] \subset R\mathcal{P}(Z[\frac{1}{2}])$. Then $\langle -1 \rangle [u] = \epsilon$ in $R\mathcal{P}(Z[\frac{1}{2}])$ for $u = -1, 2$ and 1/2.

Proof. Certainly $\langle -1 \rangle [-1] = \epsilon$ by definition. By Corollary 3.22 we have $\langle 2 \rangle \psi_1(-1) = 0 = \langle -1 \rangle \psi_1(2)$. By Proposition 3.11 we have

$$
C = [2] + \langle -1 \rangle [-1] + \langle -1 \rangle \psi_1(2) = [-1] + \langle -1 \rangle [2] + \langle 2 \rangle \psi_1(-1)
$$

which thus implies $\langle -1 \rangle [2] = \langle -1 \rangle [-1] = \epsilon$.

Again $\langle -1 \rangle \psi_1(2)$ gives $\langle -1 \rangle ([2] \langle -1 \rangle [\frac{1}{2}]) = [2] \langle -1 \rangle [\frac{1}{2}]$ which implies $\langle -1 \rangle [\frac{1}{2}] = \langle -1 \rangle [2] = \epsilon$. □

**Lemma 8.34.** $D = [-1] - [2]$ in $R\mathcal{P}(Z[\frac{1}{2}])$.

Proof. We have $-D + \psi_1(-1) = C = [2] + \langle -1 \rangle [-1] = ([2] - [-1]) + \psi_1(-1)$. □

**Lemma 8.35.** The ring homomorphism $Z[\frac{1}{2}] \rightarrow \mathbb{F}_3$ induces an isomorphism of $R$-modules $R \cdot [-1] \cong R\mathcal{P}(\mathbb{F}_3)$.
Lemma 8.37. \[
\langle 2 \rangle D = \psi_1(2) - \psi_2(2) = ([2] + \langle -1 \rangle \left[ \frac{1}{2} \right] - (\langle -2 \rangle [2] + \langle -1 \rangle \left[ \frac{1}{2} \right] ) = [2] - \langle -2 \rangle [2].
\]

Thus \( \langle 2 \rangle ([-1] - [2]) = [2] - \langle -2 \rangle [2] \) by Lemma 8.34. This gives
\[
\langle 2 \rangle [-1] - [-1] = \langle 2 \rangle [2] - \langle -2 \rangle [2] = \langle -2 \rangle \langle -1 \rangle [2] = \langle -2 \rangle \langle -1 \rangle [-1] \text{ by Lemma } 8.33
\]
and thus \( \langle -2 \rangle [-1] = [-1] \) in \( \mathcal{R} \mathcal{P}(\mathbb{Z}[\frac{1}{2}]) \). It follows that the \( \mathcal{R} \)-module structure of \( \mathcal{R} \cdot [-1] \) factors through \( \mathcal{R}_3 \).

The result now follows since \( \mathcal{R} \mathcal{P}(\mathbb{F}_3) \) is generated by \( [-1] \) as an \( \mathcal{R}_3 \)-module subject only to the relation \( 2\psi_1(-1) = 0 \).

Now let \( \mathcal{M} \) denote the \( \mathcal{R} \)-submodule \( \mathcal{R} \mathcal{B}(\mathbb{Z}[\frac{1}{2}]) + \mathcal{R} \cdot [-1] = \mathcal{R} \mathcal{B}(\mathbb{Z}[\frac{1}{2}]) \oplus \mathbb{Z} \cdot [-1] \) of \( \mathcal{R} \mathcal{P}(\mathbb{Z}[\frac{1}{2}]) \).

Lemma 8.36. \( \mathcal{M} \) is the submodule generated by \( [2] \).

Proof. From the description of \( \mathcal{R} \mathcal{B}(\mathbb{Z}[\frac{1}{2}]) \), \( \mathcal{M} \) is generated by \([-1]\) and \( D \). Since \([-1] = [2] + D \), it is enough to show that \( D \in \mathcal{R} \cdot [2] \). Since \([2] = [-1] - D \) and since \( \langle -1 \rangle D = D \), we have \( [2] + \langle -1 \rangle [2] = \psi_1(-1) - 2D = \psi_1(-1) + D \). Multiplying both sides by 2 shows that \( 2D = -D \in \mathcal{R} \cdot [2] \).

Lemma 8.37. The element \( \left[ \frac{1}{2} \right] \) has infinite order in \( \mathcal{R} \mathcal{P}(\mathbb{Z}[\frac{1}{2}]) \) and \( \mathbb{Z} \cdot \left[ \frac{1}{2} \right] \cap \mathcal{M} = 0 \). Furthermore, \( \mathcal{N} := \mathcal{M} + \mathbb{Z} \cdot \left[ \frac{1}{2} \right] = \mathcal{M} \oplus \mathbb{Z} \cdot \left[ \frac{1}{2} \right] \) is an \( \mathcal{R} \)-submodule.

Proof. We use the \( \mathcal{R} \)-homomorphism \( \lambda_1 : \mathcal{R} \mathcal{P}(\mathbb{Z}[\frac{1}{2}]) \to \mathcal{I}_{\mathbb{Z}[\frac{1}{2}]} \). Recall that \( \mathcal{I}_{\mathbb{Z}[\frac{1}{2}]} \) is a free \( \mathbb{Z} \)-module with basis \( \{ \langle -1 \rangle, \langle 2 \rangle, \langle -2 \rangle \} \). Since \( \lambda_1 \left( \left[ \frac{1}{2} \right] \right) = \langle \left[ \frac{1}{2} \right] \rangle^2 = \langle \langle 2 \rangle \rangle^2 = -2 \langle \langle 2 \rangle \rangle \) it follows that \( \left[ \frac{1}{2} \right] \) has infinite order in \( \mathcal{R} \mathcal{P}(\mathbb{Z}[\frac{1}{2}]) \). Since \( \lambda_1(x) = 0 \) for \( x \in \mathcal{R} \mathcal{B}(\mathbb{Z}[\frac{1}{2}]) \) and since \( \lambda_1([-1]) = \langle -1 \rangle \langle 2 \rangle = \langle -1 \rangle - \langle -1 \rangle \langle 2 \rangle \), the second statement follows.

Finally, we must prove that \( \mathcal{R} \cdot \left[ \frac{1}{2} \right] \subset \mathcal{N} \). Certainly \( \langle -1 \rangle \left[ \frac{1}{2} \right] = \left[ \frac{1}{2} \right] + \epsilon \in \mathcal{N} \). The cocycle relation gives the identity \( \langle 2 \rangle \psi_1 \left( \frac{1}{2} \right) = \psi_1(1) - \psi_1(2) = -\psi_1(2) \). Thus, by definition, \( \langle 2 \rangle \left[ \frac{1}{2} \right] + \langle -2 \rangle [2] = -[2] - \langle -1 \rangle [2] \) and hence \( \langle 2 \rangle \left[ \frac{1}{2} \right] \in \mathcal{N} \) since the other three terms lie in \( \mathcal{N} \).

Lemma 8.38. In the spectral sequence \( E(\text{SL}_2(\mathbb{Z}[\frac{1}{2}]), L^\gamma) \) we have \( E^0_{0,2} = 0 \). Hence \( \mathcal{R} \mathcal{P}_1(\mathbb{Z}[\frac{1}{2}]) = \mathcal{R} \mathcal{B}(\mathbb{Z}[\frac{1}{2}]) \).

Proof. \( E^0_{3,0} = H_2(B, \mathbb{Z}) \cong H_2(T, \mathbb{Z}) \) (by Lemma 8.28) \( \cong H_2(\mathbb{Z}[\frac{1}{2}])^\times, \mathbb{Z}) \cong \Lambda^2(\mathbb{Z}[\frac{1}{2}])^\times = -1 \wedge \mathbb{Z}[\frac{1}{2}]^\times \). Since the differential \( d^2 : E^2_{1,2} = \mathcal{I}_{\mathbb{Z}[\frac{1}{2}]} \otimes \mu_2 \to E^2_{2,0} \) sends \( \langle u \rangle \otimes -1 \) to \( -1 \wedge u \), the first statement follows. The second statement now follows since \( \mathcal{R} \mathcal{B}(A) = \ker(d^3 : \mathcal{R} \mathcal{P}_1(A) \to E^3_{2,0}) \).
Corollary 8.39. \( \mathcal{I}^2_{Z[\frac{1}{2}]} / \text{Im}(\lambda_1) \cong \mathbb{Z} \).

Proof. Since, in the spectral sequence \( E^0_{1,1} = E^\infty_{2,0} = 0 \) it follows that \( H_2(\mathbb{SL}_2(Z[\frac{1}{2}]), \mathbb{Z}) \cong E^\infty_{0,2} \). Now \( H_2(\mathbb{SL}_2(Z[\frac{1}{2}]), \mathbb{Z}) \cong \mathbb{Z} \) (see, for example, [1]) and \( E^\infty_{0,2} = \text{Ker}(d^2) / \text{Im}(\lambda_1) \) where \( d^2 : E^2_{0,2} = \mathbb{I}_{Z[\frac{1}{2}]} \rightarrow E^1_{1,0} = T/2T \cong Z[\frac{1}{2}]^x / (Z[\frac{1}{2}]^x)^2 \) sends \( \langle \langle u \rangle \rangle \rightarrow \tilde{u} \). Since the kernel of the homomorphism \( d^2 \) is \( \mathcal{I}^2_{Z[\frac{1}{2}]} \), the result follows. \( \square \)

We complete the proof of Theorem 8.32.

Lemma 8.40. \( \mathcal{N} = \mathcal{R}\mathcal{P}(Z[\frac{1}{2}]) \).

Proof. \( \mathcal{I}^1_{Z[\frac{1}{2}]} \) is a free \( \mathbb{Z} \)-module with basis \( \langle \langle -1 \rangle \rangle \langle \langle 2 \rangle \rangle, 2 \langle \langle -1 \rangle \rangle, \langle \langle 2 \rangle \rangle \). We have \( \lambda_1([-1]) = \langle \langle -1 \rangle \rangle \langle \langle 2 \rangle \rangle \) and \( \lambda_1([\frac{1}{2}]) = \langle \langle 2 \rangle \rangle^2 = -2 \langle \langle 2 \rangle \rangle \). Since these two elements already generate a free summand with quotient \( \mathbb{Z} \), it follows from Corollary 8.39 that they generate the whole image of \( \lambda_1 \). Since \( [-1], [\frac{1}{2}] \in \mathcal{N} \), it follows that \( \lambda_1(\mathcal{N}) = \lambda_1(\mathcal{R}\mathcal{P}(Z[\frac{1}{2}])) \).

But the kernel of \( \lambda_1 : \mathcal{R}\mathcal{P}(Z[\frac{1}{2}]) \rightarrow \mathcal{I}_{Z[\frac{1}{2}]} \) is \( \mathcal{R}\mathcal{P}_1(Z[\frac{1}{2}]) = \mathcal{R}\mathcal{B}(Z[\frac{1}{2}]) \) by Lemma 8.38 while the kernel of \( \lambda_1 : \mathcal{N} \rightarrow \mathcal{I}_{Z[\frac{1}{2}]} \) is \( \mathcal{R}\mathcal{B}(Z[\frac{1}{2}]) \) also. The result follows. \( \square \)

Remark 8.41. In all of these calculations, it turns out that \( \mathcal{R}\mathcal{P}(A) \) is generated by known generic elements \( C_A, \psi_1(u), u \in A^*, [u], u \in \mathcal{W}_A \) and subject only to the generic classes of relations enumerated in section 3.

Corollary 8.42. The abelian group \( \mathcal{P}(Z[\frac{1}{2}]) \cong \mathbb{Z}/12 \oplus \mathbb{Z}/2 \) has generators \( [2], [\frac{1}{2}] \), both of order 12 and satisfying \( 2([2] + [\frac{1}{2}]) = 0 \). We have \( C_{Z[\frac{1}{2}]} = \langle [2] \rangle \) of order 6, \( D_{Z[\frac{1}{2}]} = 4 \langle [2] \rangle \) of order 3 and \( [-1] = 9 \langle [2] \rangle \) of order 4.

Proof. By Theorem 8.32, \( \mathcal{R}\mathcal{P}(Z[\frac{1}{2}]) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z} \oplus \mathbb{Z} \) where the generators of the factors are \( \psi_1(-1), D_{Z[\frac{1}{2}]}, \langle [2] \rangle D_{Z[\frac{1}{2}]}, [1] \) and \( [\frac{1}{2}] \) respectively. Let \( P \) be the abelian group \( \mathbb{Z}/12 \oplus \mathbb{Z}/2 \) and let \( f : \mathcal{R}\mathcal{P}(Z[\frac{1}{2}]) \rightarrow P \) be the group homomorphism \( f(\psi_1(-1)) = (6, 0), f(D_{Z[\frac{1}{2}]}) = f([2] D_{Z[\frac{1}{2}]}) = (4, 0), f([-1]) = (9, 0) \) and \( f([\frac{1}{2}]) = (-1, 1) \). So \( f([2]) = f([-1] + D_{Z[\frac{1}{2}]}) = (9, 0) + (4, 0) = (1, 0) \) and \( f([2] + [\frac{1}{2}]) = (0, 1) \).

Now \( \mathcal{I}_{Z[\frac{1}{2}]} \mathcal{R}\mathcal{P}(Z[\frac{1}{2}]) \) is generated, as a \( \mathbb{Z} \)-module, by the following elements:

\[ \langle \langle -1 \rangle \rangle \langle -1 \rangle = \langle \langle 2 \rangle \rangle \langle -1 \rangle = \langle \langle -1 \rangle \rangle \langle \frac{1}{2} \rangle = \epsilon = \psi_1(-1) - 2 \langle -1 \rangle, \]
\[ \langle \langle 2 \rangle \rangle D_{Z[\frac{1}{2}]} = - \langle \langle 2 \rangle \rangle \langle [2] \rangle D_{Z[\frac{1}{2}]} = \langle [2] \rangle D_{Z[\frac{1}{2}]} - D_{Z[\frac{1}{2}]} \]
\[ \langle \langle 2 \rangle \rangle \langle \frac{1}{2} \rangle = 2 \langle -1 \rangle - 2 \langle \frac{1}{2} \rangle - D_{Z[\frac{1}{2}]} - \langle [2] \rangle D_{Z[\frac{1}{2}]}, \] (Note that \( \langle \langle -1 \rangle \rangle \psi_1(-1) = 0 = \langle [2] \rangle \psi_1(-1) \) and \( \langle \langle -1 \rangle \rangle D_{Z[\frac{1}{2}]} = 0 \).)

Since these elements generate \( \text{Ker}(f) \), it follows that \( \text{Ker}(f) = \mathcal{I}_{Z[\frac{1}{2}]} \mathcal{R}\mathcal{P}(Z[\frac{1}{2}]) \) and hence \( f \) induces an isomorphism

\[ \mathcal{P}(Z[\frac{1}{2}]) = \frac{\mathcal{R}\mathcal{P}(Z[\frac{1}{2}])}{\mathcal{I}_{Z[\frac{1}{2}]} \mathcal{R}\mathcal{P}(Z[\frac{1}{2}])} \cong P. \]

\( \square \)

Corollary 8.43. The ring homomorphism \( Z[\frac{1}{2}] \rightarrow \mathbb{Q} \) induces an injective map
\( \mathcal{P}(Z[\frac{1}{2}]) \rightarrow \mathcal{P}(\mathbb{Q}) \) and an isomorphism \( \mathcal{B}(Z[\frac{1}{2}]) \cong \mathcal{B}(\mathbb{Q}) \). In particular, \( \mathcal{B}(Z[\frac{1}{2}]) \) is cyclic of order 6 with generator \( C_{Z[\frac{1}{2}]} \).
Proof. We note first that \( [2] \in \mathcal{P}(\mathbb{Q}) \) has order 12 since \( C_{\mathbb{Q}} = [2] + [-1] \) and hence \( [2] = C_{\mathbb{Q}} - [-1] \) where \( 3C_{\mathbb{Q}} = \psi(-1) = 2[-1] \) and \( 2\psi(-1) = 0 \). Furthermore, in \( \mathcal{P}(\mathbb{Q}) \) we have \( 0 = 2\psi(2) = 2([2] + \left[ \frac{1}{2} \right]) \).

Consider now the homomorphism \( \lambda : \mathcal{P}(\mathbb{Q}) \to E_{2,0}^3(\text{GL}_2(\mathbb{Q}), L) = S_2^2(\mathbb{Q})^\times, [a] \mapsto a \circ (1 - a) \).

Since \( \lambda([2]) = 2 \circ -1 \) and \( \lambda([\frac{1}{2}]) = \frac{1}{2} \circ \frac{1}{2} = 2 \circ 2 \neq 2 \circ -1 \), it follows that \( [2] \neq [\frac{1}{2}] \) in \( \mathcal{P}(\mathbb{Q}) \). Thus the map \( \mathcal{P}(\mathbb{Z}[\frac{1}{2}]) \to \mathcal{P}(\mathbb{Q}) \) is injective. It follows that the induced map \( \mathcal{B}(\mathbb{Z}[\frac{1}{2}]) \to \mathcal{B}(\mathbb{Q}) \) is injective. But \( C_{\mathbb{Z}[\frac{1}{2}]} \in \mathcal{RB}(\mathbb{Z}[\frac{1}{2}]) \) and hence \( C_{\mathbb{Z}[\frac{1}{2}]} \in \mathcal{B}(\mathbb{Z}[\frac{1}{2}]) \). Since this maps to the generator \( C_{\mathbb{Q}} \) of \( \mathcal{B}(\mathbb{Q}) \), the result follows. \( \square \)

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