On Lenglart’s Theory of Meyer-σ-fields and El Karoui’s Theory of Optimal Stopping

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Abstract

We summarize the general results of El Karoui [1981] on optimal stopping problems for processes which are measurable with respect to Meyer-σ-fields. Meyer-σ-fields are due to Lenglart [1980] and include the optional and predictable σ-field as special cases. Novel contributions of our work are path regularity results for Meyer measurable processes and limit results for Meyer-projections. We also clarify a minor issue in the proof of El Karoui’s optimality result. These extensions were inspired and needed for the proof of a stochastic representation theorem in Bank and Besslich [2018a]. As an application of this theorem, we provide an alternative approach to optimal stopping in the spirit of Bank and Föllmer [2003].

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1 Introduction

In recent work on stochastic optimal control problems ([Bank and Besslich 2018b]), we found it useful to model information flows that allow one to interpolate between predictable and optional controls. In fact, such a situation is perfectly natural when exogenous shocks are pre-announced by possibly noisy signals. For continuous-time models, Meyer-σ-fields turn out to be the tool of choice to capture such phenomena in a most flexible and rigorous manner.

Meyer σ-fields where introduced by [Lenglart 1980] and used in [El Karoui 1981] for a most general theory of optimal stopping. This paper gives a survey of both of these papers, giving a condensed account of the main concepts and results we find most important for our optimal control purposes. Moreover, it provides supplementary results to both Lenglart’s and El Karoui’s theory which shed extra light on the regularity properties of Meyer-measurable processes and allow us to clarify minor issues in the proof of El Karoui’s optimal stopping result.

From Lenglart’s theory, we recall the definition of Meyer-σ-fields, discuss their completion and how they are embedded between predictable and optional fields. Also, accessible and totally inaccessible Meyer-stopping times are discussed as are the fundamental tools provided by Meyer’s section and projection theorems.

Our survey of El Karoui’s theory of optimal stopping starts with a discussion of Meyer-supermartingales and the structure of Snell envelopes. We explain how she relaxes the optimal stopping problem by introducing divided stopping times and show how this relaxation ensures existence of optimizers. To round off this survey, we recall how the results simplify in the classically considered optional case.

Apart from this survey section, we will clarify a minor issue arising in the analysis in [El Karoui 1981]. Moreover, we give extensions to existing results concerning the testing of path properties for Meyer-measurable processes and a result concerning the right- and left-upper-semicontinuous envelopes of Meyer-projections. Those extensions are not just for the sake of mathematical generality, but were inspired and needed in our paper [Bank and Besslich 2018a], where we prove some extension of a representation result for stochastic processes from [Bank and El Karoui 2004].
This result in turn is used in Bank and Besslich [2018b] to construct solutions to irreversible investment problems for which Meyer $\sigma$-fields offer a novel way to model information dynamics.

For another application, we show how the solution to the representation theorem of Bank and Besslich [2018a] can be used as a universal stopping signal which allows one to characterize optimal divided stopping times for a suitable parametric family of optimal stopping problems.

The rest of the paper is organized as follows. In Section 2.1 we follow El Karoui [1981] to give a summary of the key results on Meyer-$\sigma$-fields from Lenglart [1980]. In Section 2.2 we state the results of El Karoui [1981] concerning optimal stopping problems and in Section 3 we state our extensions and clarifications on El Karoui [1981]. Section 4 studies the universal signals for optimal stopping as specified by a solution to a representation problem.

2 Survey of results on Meyer-$\sigma$-fields and general optimal stopping

This section gives a condensed account of the results from Lenglart’s general theory of Meyer-$\sigma$-fields and El Karoui’s general theory of optimal stopping that we found most useful for our own work in the companion papers Bank and Besslich [2018a,b].

2.1 Lenglart’s theory of Meyer-$\sigma$-fields

Lenglart [1980] gives a very thorough and comprehensive account of the theory of Meyer-$\sigma$-fields. Let us recall some of his results by following the outline given in the introduction of El Karoui [1981], p.118-121.

2.1.1 Basic definition, examples and characterization result

We will start with the definition of Meyer-$\sigma$-fields and some fundamental examples.

Definition 2.1 (Lenglart [1980], Definition 2, p.502). A $\sigma$-field $\Lambda$ on $\Omega \times [0, \infty)$ is called a Meyer-$\sigma$-field, if the following conditions hold:

(i) It is generated by some right-continuous, left-limited (rcll or càdlàg in short) processes.

(ii) It contains $\{\emptyset, \Omega\} \times \mathcal{B}([0, \infty))$, where $\mathcal{B}([0, \infty))$ denotes the Borel-$\sigma$-field on $[0, \infty)$.

(iii) It is stable with respect to stopping at deterministic time points, i.e. for a $\Lambda$-measurable process $Z$, $s \in [0, \infty)$, also the stopped process $(\omega, t) \mapsto Z_{t \wedge s}(\omega)$ is $\Lambda$-measurable.

Example 2.2 (El Karoui [1981], Remark, p.p.118). Assume we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Then the optional $\sigma$-field with respect to the
filtration \((\mathcal{F}_t)_{t \geq 0}\), i.e. the \(\sigma\)-field generated by all càdlàg, \((\mathcal{F}_t)_{t \geq 0}\)-adapted processes and the predictable \(\sigma\)-field with respect to the filtration \((\mathcal{F}_t)_{t \geq 0}\), i.e. the \(\sigma\)-field generated by all continuous, \((\mathcal{F}_t)_{t \geq 0}\)-adapted processes are both Meyer-\(\sigma\)-fields.

So Meyer-\(\sigma\)-fields can emerge from suitable processes adapted to a given filtration. Conversely, Meyer-\(\sigma\)-fields also induce a filtration:

**Definition 2.3** (Compare El Karoui [1981], p.119). For a Meyer-\(\sigma\)-field \(\Lambda\) we define its associated filtration \(\mathcal{F}^\Lambda := (\mathcal{F}^\Lambda_t)_{t \geq 0}\) by setting

\[
\mathcal{F}^\Lambda_t := \sigma(Z_t \mid Z \text{ \(\Lambda\)-measurable}), \quad t \in [0, \infty).
\]

In addition, one can choose a \(\sigma\)-field \(\mathcal{F}^\Lambda_0^- \subset \mathcal{F}^\Lambda_0\) and we put \(\mathcal{F}^\Lambda_\infty := \bigvee_{t=1}^{\infty} \mathcal{F}^\Lambda_t\) as well as

\[
\mathcal{F}^\Lambda_{t+} := \bigcap_{s > t} \mathcal{F}^\Lambda_s, \quad t \geq 0, \quad (\mathcal{F}^\Lambda_0^+) := \mathcal{F}^\Lambda_0^-.
\]

A process \(Z : \Omega \times [0, \infty] \to \mathbb{R}\) is called \(\Lambda\)-measurable, if \(Z_\infty\) is \(\mathcal{F}^\Lambda_\infty\)-measurable and the restriction \(Z|_{\Omega \times [0, \infty)}\) is \(\Lambda\)-measurable.

The next theorem gives us some idea what Meyer-\(\sigma\)-fields look like.

**Theorem 2.4** (Compare El Karoui [1981], Definition 2.22.2, p.119). A Meyer-\(\sigma\)-field contains the predictable \(\sigma\)-field \(\mathcal{P}(\mathcal{F}^\Lambda)\) relative to the filtration \((\mathcal{F}^\Lambda_t)_{t \geq 0}\) and it is contained in the optional \(\sigma\)-field \(\mathcal{O}(\mathcal{F}^\Lambda)\) relative to \((\mathcal{F}^\Lambda_t)_{t \geq 0}\).

Conversely, a \(\sigma\)-field on \(\Omega \times [0, \infty)\) generated by càdlàg processes is a Meyer-\(\sigma\)-field, if it lies between the predictable and the optional \(\sigma\)-field of some filtration.

### 2.1.2 Stopping times corresponding to a Meyer-\(\sigma\)-field \(\Lambda\)

Next we will give a definition for the concept of stopping times when using general Meyer-\(\sigma\)-fields.

**Definition 2.5** (El Karoui [1981], Definition 2.22.2, p.119). A random variable \(S\) with values in \([0, \infty]\) is a \(\Lambda\)-stopping time, if

\[
[[S, \infty]] := \{(\omega, t) \in \Omega \times [0, \infty) \mid S(\omega) \leq t\} \in \Lambda.
\]

The set of all \(\Lambda\)-stopping times is denoted by \(\mathcal{S}_\Lambda\). Additionally we define for each mapping \(S : \Omega \to [0, \infty]\) a \(\sigma\)-field

\[
\mathcal{F}^\Lambda_S := \sigma(Z_S \mid Z \text{ \(\Lambda\)-measurable process})
\]

This concept of \(\Lambda\)-stopping times extends classical notions of stopping times in a natural way:

**Example 2.6** (Compare El Karoui [1981], p.119). For a filtration \((\mathcal{F}_t)_{t \geq 0}\) and \(\Lambda = \mathcal{O}(\mathcal{F})\), a \(\Lambda\)-stopping time \(S\) is a classical stopping time associated to the filtration \((\mathcal{F}_t)_{t \geq 0}\) and \(\mathcal{F}^\Lambda_S = \mathcal{F}_S\). For \(\Lambda = \mathcal{P}(\mathcal{F})\) by contrast a \(\Lambda\)-stopping time \(S\) is a predictable stopping time associated to the filtration \((\mathcal{F}_t)_{t \geq 0}\).
Remark 2.7 (Compare El Karoui [1981], p.119). The $\sigma$-fields from Definition 2.5 satisfy
\[ \mathcal{F}^\Lambda_S^{-} \subset \mathcal{F}^\Lambda_S \subset \mathcal{F}^\Lambda_S^{+}, \]
where \( \mathcal{F}^\Lambda_S^{+} := (\mathcal{F}^\Lambda)_{S^+} \) with \( \mathcal{F}^\Lambda_{S^+} \) the right-continuous filtration defined in Definition 2.3 and \( \mathcal{F}^\Lambda_S^{-} := (\mathcal{F}^\Lambda)_{S}^\pi(\mathcal{F}^\Lambda) \), which is in the case of an \( \mathcal{F}^\Lambda_{S^+} \)-stopping time \( S \) equal to the $\sigma$-field generated by \( \mathcal{F}^\Lambda_0^{-} \) and the sets \( A \cap \{ t < S \} \), \( t \geq 0 \), \( A \in \mathcal{F}^\Lambda_t \) (see Lenglart [1980], Remark, p.505). By Lenglart [1980], Theorem 4.1, p.505, we have for a $\Lambda$-stopping time \( S \)
\[ \mathcal{F}^\Lambda_S = \{ H \in \mathcal{F}^\Lambda_\infty \mid S_H \in \mathcal{F}^\Lambda \}, \]
where \( S_H \) is defined to be
\[ S_H := \begin{cases} S & \text{on } H, \\ \infty & \text{on } H^c. \end{cases} \]

2.1.3 Meyer’s section and projection theorems

Next we state the key section and projection theorems, which are well known for the optional and the predictable $\sigma$-field, but actually still hold for Meyer-$\sigma$-fields. We will fix for the rest of this section a complete probability space \((\Omega, \mathcal{F}, P)\) and a Meyer-$\sigma$-field \( \Lambda \) (not necessarily complete; see Definition 2.15 below), which is contained in \( \mathcal{F} \otimes \mathcal{B}([0, \infty)) \).

Theorem 2.8 (Meyer Section Theorem, El Karoui [1981], Theorem 2.23.1, p.120). Let \( B \) be an element of \( \Lambda \). For every \( \varepsilon > 0 \), there exists \( S \in \mathcal{F}^\Lambda \) such that \( B \) contains the graph of \( S \), i.e.
\[ B \supset \text{graph}(S) := \{ (\omega, S(\omega)) \in \Omega \times [0, \infty) \mid S(\omega) < \infty \} \]
and
\[ P(S < \infty) > P(\pi(B)) - \varepsilon, \]
where \( \pi(B) := \{ \omega \in \Omega \mid (\omega, t) \in B \text{ for some } t \in [0, \infty) \} \) denotes the projection of \( B \) onto \( \Omega \).

Remark 2.9. The projection \( \pi(B) \) of a set \( B \in \Lambda \) is an element of \( \mathcal{F} \) as the probability space is assumed to be complete. In general we would have to replace \( P(\pi(B)) \) by \( P^*(\pi(B)) \), where \( P^* \) denotes the outer measure of \( P \) (see Dellacherie and Meyer [1978], Footnote (1), p.137).

An important consequence of Theorem 2.8 is the following corollary:

Corollary 2.10 (Lenglart [1980], p. 507). If \( Z \) and \( Z' \) are two \( \Lambda \)-measurable processes, such that for each bounded \( T \in \mathcal{F}^\Lambda \) we have \( Z_T \leq Z'_T \) a.s. (resp. \( Z_T = Z'_T \) a.s.), then the set \( \{ Z > Z' \} \) is evanescent (resp. \( Z \) and \( Z' \) are indistinguishable).

Next it is also possible to project a suitable process into the space of \( \Lambda \)-measurable processes:
Theorem 2.11 (Projection Theorem, compare Lenglart [1980], Theorem 8, p.511). For any bounded or positive $\mathbb{F} \otimes \mathcal{B}([0,\infty))$-measurable process $Z$, there exists a $\Lambda$-measurable process $\Lambda Z$, unique up to indistinguishability, such that

$$\Lambda Z_S = \mathbb{E} [Z_S \mid \mathcal{F}_S^\Lambda] \text{ \ P-a.s. for any finite } S \in \mathcal{F}^\Lambda.$$ 

This process is called the $\Lambda$-projection of $Z$.

Example 2.12. If $\Lambda$ is the optional or predictable $\sigma$-field with respect to a filtration $(\mathcal{F}_t)_{t \geq 0}$ then $\Lambda Z$ coincides with the well known optional and predictable projection, respectively.

Remark 2.13. By Lenglart [1980], Theorem 11, p.513, we can use Theorem 2.11 also for processes $Z$ of class(D$_\Lambda$), i.e. when $\{Z_T \mid T \in \mathcal{F}^\Lambda\}$ is uniformly integrable.

The Meyer Section Theorem and the definition of $\Lambda$-stopping times yield the following equivalent characterization:

Theorem 2.14. For any $\mathbb{F} \otimes \mathcal{B}([0,\infty))$-measurable process $Z \geq 0$, the $\Lambda$-projection $\Lambda Z$ is the, unique up to indistinguishability, process satisfying

$$\mathbb{E} \left[ \int_{[0,\infty)} Z_s dA_s \right] = \mathbb{E} \left[ \int_{[0,\infty)} \Lambda Z_s dA_s \right]$$

for any càdlàg, $\Lambda$-measurable, increasing process $A$.

2.1.4 Completion of Meyer-$\sigma$-fields

Let us next analyze the influence of $\mathbb{P}$-nullsets on $\Lambda$-stopping times and $\Lambda$-measurable process. Recall that we have fixed a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and a Meyer-$\sigma$-field $\Lambda$, which is contained in $\mathbb{F} \otimes \mathcal{B}([0,\infty))$.

Definition 2.15 (Compare Lenglart [1980], Definition and Theorem 2, p.507). A Meyer-$\sigma$-field $\Lambda$ is called $\mathbb{P}$-complete if and only if one of the following equivalent conditions is fulfilled:

1. Every mapping $T : \Omega \rightarrow [0,\infty]$, which is almost surely equal to a $\Lambda$-stopping time is a $\Lambda$-stopping time.
2. Every $\mathbb{F} \otimes \mathcal{B}([0,\infty))$-measurable process, which is indistinguishable from a $\Lambda$-measurable process is itself $\Lambda$-measurable.

The next statement shows how to obtain a completion of a Meyer-$\sigma$-field: Meyer-$\sigma$-field $\Lambda$:

Definition and Theorem 2.16 (Compare Lenglart [1980], p.507-509). Define $\Lambda^\mathbb{P}$ as the $\sigma$-field generated by the stochastic intervals $[T, \infty]$ for random variables $T : \Omega \rightarrow [0,\infty]$ which almost surely coincide with $\Lambda$-stopping time. Then the following results hold true:

1. $\Lambda^\mathbb{P}$ is the smallest $\mathbb{P}$-complete Meyer-$\sigma$-field containing $\Lambda$. 
(ii) The mapping \( T : \Omega \to [0, \infty] \) is a \( \Lambda^P \)-stopping time if and only if the graph of \( T \) is contained in \( \Lambda^P \).

(iii) A random variable \( T : \Omega \to [0, \infty] \) is a \( \Lambda^P \)-stopping time if and only if it is a.s. equal to a \( \Lambda \)-stopping time.

(iv) Fix a \( \Lambda^P \)-stopping time \( T \) and take any corresponding \( \Lambda \)-stopping time \( \tilde{T} \) with \( T = \tilde{T} \) almost surely. Then \( \mathcal{F}_T^{\Lambda^P} = \mathcal{F}_T^{\Lambda} \), where \( \mathcal{F}_T^{\Lambda} \) denotes the \( \sigma \)-field generated by \( \mathcal{F}_T^{\Lambda} \) and all \( P \)-nullsets.

We call \( \Lambda^P \) the \( P \)-completion of \( \Lambda \).

Analogously to Theorem 2.4, the following theorem characterizes the \( P \)-complete Meyer-\( \sigma \)-fields:

**Theorem 2.17** (Lenglart [1980], Theorem 5, p.509). A \( \sigma \)-field \( \Lambda \) generated by c\'{a}dl\'{a}g processes is a \( P \)-complete Meyer-\( \sigma \)-field if and only if \( \Lambda \) is in between the predictable and optional \( \sigma \)-field of a filtration, which is right-continuous and \( P \)-complete, i.e. which fulfills the usual conditions.

### 2.1.5 Inaccessible stopping times and connection to Meyer-measurable processes

This paragraph follows Lenglart [1980], Section 3, p.510, and Lenglart [1980], Section 4, p.513. We give a small extension to those results in Proposition 2.21.

**Definition 2.18** (Lenglart [1980], Definition, p.510). For a Meyer-\( \sigma \)-field \( \Lambda \) a random variable \( T : \Omega \to [0, \infty] \) is called \( \Lambda \)-accessible if there exists a sequence of \( \Lambda \)-stopping times \((T_n)_{n \in \mathbb{N}}\) such that

\[
\mathbb{P} \left( \bigcup_{n \in \mathbb{N}} \{ T_n = T < \infty \} \right) = \mathbb{P}(T < \infty).
\]

The random variable \( T \) is called totally \( \Lambda \)-inaccessible if \( \mathbb{P}(S = T < \infty) = 0 \) for all \( \Lambda \)-stopping times \( S \).

Now we have the following decomposition result:

**Theorem 2.19** (Lenglart [1980], Theorem 6, p.510). Let \( \Lambda \) be a Meyer-\( \sigma \)-field contained in another Meyer-\( \sigma \)-field \( \tilde{\Lambda} \). If \( T \) is a \( \tilde{\Lambda} \)-stopping time there exists a partition of \( \{ T < \infty \} \) into \( A, I \in \mathcal{F}_T^{\tilde{\Lambda}} \) such that \( T_A \) is a \( \tilde{\Lambda} \)-stopping time \( \Lambda \)-accessible and \( T_I \) is a \( \tilde{\Lambda} \)-stopping time totally \( \Lambda \)-inaccessible. The sets \( A, I \) are unique up to \( P \)-nullsets.

The next result characterizes \( \Lambda \)-measurability of a process via its sections at certain stopping times.

**Theorem 2.20** (Compare Lenglart [1980], Theorem 13, p.513). Let \( \Lambda \) be \( P \)-complete and denote by \( \mathcal{F} := (\mathcal{F}_t)_{t \geq 0} \) a filtration satisfying the usual conditions such that \( \mathcal{P}(\mathcal{F}) \subset \Lambda \subset \mathcal{O}(\mathcal{F}) \) (see Theorem 2.17).

A c\'{a}dl\'{a}g process \( X \) is \( \Lambda \)-measurable if and only if it satisfies the following two conditions:
(i) For any finite \( \Lambda \)-stopping time \( T \), \( X_T \) is \( \mathcal{F}_T^\Lambda \)-measurable.

(ii) For any totally \( \Lambda \)-inaccessible \( \mathcal{F} \)-stopping time \( T \), \( \Delta X_T = 0 \) a.s..

As a generalization of (ii) in the above result we note the following observation:

**Proposition 2.21.** In the setting of Theorem 2.20, we have \( \mathbb{P} \)-almost surely for any \( \Lambda \)-measurable bounded process \( C \)

\[
C_T = (\mathcal{P}C)_T
\]

at any totally \( \Lambda \)-inaccessible \( \mathcal{F} \)-stopping time \( T \).

**Proof.** The result follows by a monotone class argument, if we can show the result for \( C := \alpha \mathbb{1}_{[S,\infty]} \) for \( S \in \mathcal{F}^{\Lambda} \) and positive, bounded \( \alpha \in \mathcal{F}_S^{\Lambda} \). In this situation, we can find by Theorem 2.19 two disjoint sets \( A, I \in \mathcal{F}_S^{\Lambda} \) such that \( \{ S < \infty \} = A \cup I \), \( I \) is totally \( \mathcal{P} \)-inaccessible and \( S_A \) is \( \mathcal{P} \)-accessible. Hence, there exists a sequence of \( \mathcal{F} \)-predictable stopping times \((S_n)_{n \in \mathbb{N}}\) such that

\[
\mathbb{P} (\bigcup_{n \in \mathbb{N}} \{ S_n = S_A < \infty \}) = \mathbb{P} (S_A < \infty).
\]

We assume without loss of generality that the graphs of the \((S_n)_{n \in \mathbb{N}}\) are disjoint, because otherwise we could consider the sequence of \( \mathcal{F} \)-predictable stopping times \( \tilde{S}_1 := S_1, \tilde{S}_n := (S_n)_{\bigcup_{k=1}^{n-1} \{ S_n \neq S_k \}} \), \( n = 2, 3, \ldots \).

Now we have

\[
\mathcal{P} C = \mathcal{P} (\alpha \mathbb{1}_{[S_A]} + \mathbb{1}_{[S_1]} + \alpha \mathbb{1}_{[S,\infty]}).
\]

As \( S_I \) is totally \( \mathcal{P} \)-inaccessible we get at any predictable stopping time \( T \)

\[
(\alpha \mathbb{1}_{[S_1]})_T = \mathbb{1}_{\{ S_I = T \}} = 0 \quad \text{a.s.,}
\]

and so, by Corollary 2.10 to Meyer’s Section Theorem:

\[
\mathcal{P} (\alpha \mathbb{1}_{[S_1]}) \equiv 0.
\]

Moreover, by monotonicity and additivity of the \( \mathcal{P} \)-projection and because the graphs of the \((S_n)_{n \in \mathbb{N}}\) are disjoint we obtain

\[
0 \leq \mathcal{P} (\alpha \mathbb{1}_{[S_A]}) = \sum_{n=1}^{\infty} \mathcal{P} (\alpha \mathbb{1}_{[S_n]}),
\]

where we have used in the last step that \( \mathbb{1}_{[S_n]} \) is predictable. Hence, we obtain at the totally \( \mathcal{P} \)-inaccessible \( \mathcal{F} \)-stopping time \( T \) that a.s.

\[
0 \leq \mathcal{P} (\alpha \mathbb{1}_{[S_A]})_T = \sum_{n=1}^{\infty} \mathcal{P} (\alpha \mathbb{1}_{[S_n]})(T) = 0.
\]

Combining (1) with (2) and (3) gives us

\[
(\mathcal{P} C)_T = \alpha \mathbb{1}_{[S,\infty]}(T) = C_{T^-} = C_T,
\]

where the final identity follows from Theorem 2.20 as \( T \) is totally \( \Lambda \)-inaccessible. \( \square \)
2.1.6 A Riesz representation

Let us conclude this section with a Riesz representation stated in Lenglart [1980], p.516, which was originally proven in the optional case in Dellacherie and Meyer [1982], Theorem 2, p.184.

We denote by \( \mathcal{G} \) a \( \Lambda \)-stable vector space of processes, which satisfies the following:

(i) \( \mathcal{G} \) contains the almost constant processes, i.e. processes of the form \( a1_{(0,\infty]} \), \( a \in \mathbb{R} \).

(ii) All \( Z \in \mathcal{G} \) are càglàd with a limit at infinity such that \( Z_+ \) is \( \Lambda \)-measurable.

(iii) For any \( \Lambda \)-stopping time \( T \) the process \( 1_{[T,\infty]} \) is contained in \( \mathcal{G} \).

Then we have the following result:

**Theorem 2.22** (Lenglart [1980], Theorem 18, p.516). Let \( J \) be a positive linear form on \( \mathcal{G} \) with the following property: For any non-increasing sequence \( (Z^n)_{n \in \mathbb{N}} \) of positive elements of \( \mathcal{G} \), such that

\[
\lim_{n \to \infty} \sup_{t \in [0,\infty]} Z^n_t = 0
\]

we have

\[
\lim_{n \to \infty} J(Z^n) = 0.
\]

Then there exists two increasing, right-continuous processes \( A, B \) with \( \mathbb{E}[A_\infty] < \infty \), and \( \mathbb{E}[B_\infty] < \infty \), where \( A \) is \( \mathcal{F}^\Lambda \)-predictable, \( A_0 = 0 \) and \( B \) is \( \Lambda \)-measurable, purely discontinuous with \( \lim_{t \to \infty} B_t = B_\infty \) such that for any \( Z \in \mathcal{G} \) we have

\[
J(Z) = \mathbb{E} \left[ \int_{[0,\infty]} Z_s dA_s + \int_{[0,\infty]} Z_s+ dB_s \right].
\]

The representing processes \( A, B \) are unique up to indistinguishability.

2.2 El Karoui’s general theory of optimal stopping

Let us henceforth consider a fixed \( \mathbb{P} \)-complete Meyer-\( \sigma \)-field \( \Lambda \subset \mathcal{F} \otimes \mathcal{B}([0,\infty)) \) with a given complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). The upcoming part is based on El Karoui [1981], p.117-143, and we will just give a summary of the results stated in there for the special case where the considered “weak chronology” is given by \( \mathcal{F}^\Lambda \).

2.2.1 \( \Lambda \)-supermartingales

In this section we develop the notion of supermartingales for Meyer-\( \sigma \)-fields. As we will only state a small part of the results on such \( \Lambda \)-supermartingales we refer the interested reader to Lenglart [1980], Chapter III, for a full account.

**Definition 2.23** (El Karoui [1981], Definition 2.25.2, p.121). A family of random variables \( (Z(S))_{S \in \mathcal{F}^\Lambda} \) is a \( \mathcal{F}^\Lambda \)-system, if
(i) For all $S$ and $T$ in $\mathcal{F}^{\Lambda}$, we have $Z(S) = Z(T)$ a.s. on $\{S = T\}$.

(ii) $Z(S)$ is $\mathcal{F}_S^{\Lambda}$-measurable for all $S \in \mathcal{F}^{\Lambda}$.

A process $Z : \Omega \times [0, \infty] \to \mathbb{R}$ aggregates a given $\mathcal{F}^{\Lambda}$-system $(Z(S))_{S \in \mathcal{F}^{\Lambda}}$, if it is $\Lambda$-measurable and $Z_S = Z(S)$ almost surely for all $S \in \mathcal{F}^{\Lambda}$.

Remark 2.24 (El Karoui [1981], p.121). The process which aggregates $(Z(S))_{S \in \mathcal{F}^{\Lambda}}$ is unique up to indistinguishability by Corollary 2.10.

Next we introduce the notion of super- and submartingales for the previous sets of random variables.

Definition 2.25 (El Karoui [1981]Definition 2.25.32). a) We call an $\mathcal{F}^{\Lambda}$-system $(Z(S))_{S \in \mathcal{F}^{\Lambda}}$ an $\mathcal{F}^{\Lambda}$-supermartingale system, if

(i) $Z(S)$ is integrable for all $S \in \mathcal{F}^{\Lambda}$.

(ii) $Z(S) \geq \mathbb{E}[Z(T) | \mathcal{F}_S^{\Lambda}]$ a.s. for all $S, T \in \mathcal{F}^{\Lambda}$ with $S \leq T$.

Analogously we define $\mathcal{F}^{\Lambda}$-martingale systems, if we have equality in (ii).

b) A $\Lambda$-measurable process $Z : \Omega \times [0, \infty] \to \mathbb{R}$ is called a $\Lambda$-supermartingale if the $\mathcal{F}^{\Lambda}$-system $(Z_S)_{S \in \mathcal{F}^{\Lambda}}$ is an $\mathcal{F}^{\Lambda}$-supermartingale system. Analogously, we call $Z$ a $\Lambda$-martingale if $(Z_S)_{S \in \mathcal{F}^{\Lambda}}$ is an $\mathcal{F}^{\Lambda}$-martingale system.

Next we get a statement concerning aggregation and decomposition of $\Lambda$-supermartingales.

Proposition 2.26 (Compare El Karoui [1981], Proposition 2.26, p.123). Let $(Z(S))_{S \in \mathcal{F}^{\Lambda}}$ be an $\mathcal{F}^{\Lambda}$-supermartingale system.

(i) There exists a $\Lambda$-supermartingale $Z$ unique up to indistinguishability, which aggregates $(Z(S))_{S \in \mathcal{F}^{\Lambda}}$, i.e. for all $S \in \mathcal{F}^{\Lambda}$ we have $Z_S = Z(S)$ almost surely.

(ii) Assume that the $\mathcal{F}^{\Lambda}$-system $(Z(S))_{S \in \mathcal{F}^{\Lambda}}$ is of class($D^{\Lambda}$), i.e. $\{Z(S) | S \in \mathcal{F}^{\Lambda}\}$ is uniformly integrable. Then the $\Lambda$-supermartingale $Z$ from (i) is of class($D^{\Lambda}$), i.e. $(Z_S)_{S \in \mathcal{F}^{\Lambda}}$ is of class($D^{\Lambda}$), and it has the following unique decomposition

\[ Z = M - A - B, \]

where $M : \Omega \times [0, \infty] \to \mathbb{R}$ is a $\Lambda$-martingale of class($D^{\Lambda}$), $A$ is a non-decreasing, right-continuous process which is $\mathcal{F}^{\Lambda}$-predictable with $A_0 = 0$, $\mathbb{E}[A_\infty] < \infty$ and $B$ is a non-decreasing, right-continuous, $\Lambda$-measurable process which is purely discontinuous, $B_{0-} = 0$, $B_\infty = B_{\infty-}$, and $\mathbb{E}[B_\infty] < \infty$.

Notation: If a process $Z$ has a left- or a right limit, then we define by $Z_+$ the right-limit process and by $Z_-$ the left-limit process $Z_{\infty+} := Z_\infty$. More generally, if a process $Z$ is just $\Lambda$-measurable, we will often use for $t \in [0, \infty)$ the notation

\[ Z_t^*(\omega) := \limsup_{s \downarrow t} Z_s(\omega) := \lim_{n \to \infty} \sup_{s \in (t, t+\frac{1}{n})} Z_s(\omega), \quad Z_{\infty}^*(\omega) := Z_\infty(\omega), \]

\[ Z_{t-}(\omega) := \liminf_{s \downarrow t} Z_s(\omega) := \lim_{n \to \infty} \inf_{s \in (t, t+\frac{1}{n})} Z_s(\omega), \quad Z_{\infty-}(\omega) := Z_\infty(\omega), \quad (4) \]
and, for $t \in (0, \infty)$,

\[
\begin{align*}
^*Z_t(\omega) & := \limsup_{s \uparrow t} Z_s(\omega) := \lim_{n \to \infty} \sup_{s \in (t-\frac{1}{n}, t) \cap (0, \infty)} Z_s(\omega), \\
^*Z_0(\omega) & := Z_0(\omega), \quad ^*Z_{\infty}(\omega) := \limsup_{t \uparrow \infty} Z_t(\omega) := \lim_{n \to \infty} \sup_{s \in [n, \infty)} Z_s(\omega), \\
^*_0(\omega) & := Z_0(\omega), \quad ^*_\infty(\omega) := \liminf_{t \uparrow \infty} Z_t(\omega) := \lim_{n \to \infty} \inf_{s \in [n, \infty)} Z_s(\omega), \\
\end{align*}
\]

Furthermore we denote by $^\Lambda Z$ the $^\Lambda$-projection and by $^\Theta Z$ the $^\Theta$-predictable projection of $Z$ if they exist; we furthermore follow the convention $^\Lambda Z_\infty := Z_\infty$.

Remark 2.27. By Dellacherie and Meyer [1978], Theorem 90, p.143, the process $^*Z$ is $^\Lambda$-progressively measurable and $^*Z$ is an $^\Lambda$-predictable process. In general, $^*Z$ is not $^\Lambda$-measurable, not even $^\Lambda$-optional, anymore, which can be seen by Dellacherie and Meyer [1978], Remark 91 (b), p.144.

In the last proposition of this section we state results on the path of a $^\Lambda$-supermartingale and detailed results on its points of discontinuity.

**Proposition 2.28** (Compare El Karoui [1981], Proposition 2.27, p.125). (i) We have that every $^\Lambda$-martingale $M : [0, \infty] \to \mathbb{R}$ is $^\Lambda$-adl`ag. The process $M_+$ is an $^\Lambda$-martingale whose $^\Lambda$-projection is equal to $M$, i.e.

\[ M = ^\Lambda (M_+). \]

(ii) Every $^\Lambda$-supermartingale $Z$ of class($D^\Lambda$) is $^\Lambda$-adl`ag. Furthermore such a $Z$ is the $^\Lambda$-projection of an $^\Theta(O^\Lambda)$-supermartingale $\hat{Z}$ and

\[ \hat{Z}_+ = Z_+. \]

For the discontinuities $\Delta A := A_- - A_-$ and $\Delta B := B_- - B_-$ of the non-decreasing, right-continuous processes $A$ and $B$ of the decomposition $Z = M - A - B_-$ from Proposition 2.26, we have

\[ \Delta A = Z_- - ^\Theta Z, \quad \Delta B = Z - ^\Lambda(Z_+). \]

In particular, $Z_- \geq ^\Theta Z$ and $Z \geq ^\Lambda(Z_+)$. 

2.2.2 Snell envelope

Next we introduce a classical process in the context of optimal stopping:

**Theorem 2.29** (El Karoui [1981], Theorem 2.28, p.126). Let $(Z(T))_{T \in \mathcal{F}^\Lambda}$ be a positive $\mathcal{F}^\Lambda$-system. The maximal conditional gain

\[ \bar{Z}(S) := \esssup_{T \geq S, T \in \mathcal{F}^\Lambda} \mathbb{E}\left[ Z(T) \mid \mathcal{F}_S^\Lambda \right], \quad S \in \mathcal{F}^\Lambda, \]

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is an $\mathcal{F}^\Lambda$-supermartingale system, which is aggregated by a $\Lambda$-supermartingale $\tilde{Z}$. This $\tilde{Z}$ is the smallest among all positive $\Lambda$-supermartingales $\tilde{Z}$ dominating $Z$ in the sense that $\tilde{Z}_T \geq Z(T)$ for all $T \in \mathcal{F}^\Lambda$.

**Definition 2.30** (El Karoui [1981], Remark, p.127). The process $\tilde{Z}$ is called the $\Lambda$-Snell envelope or just Snell envelope of the $\mathcal{F}^\Lambda$-system $(Z(T))_{T \in \mathcal{F}^\Lambda}$.

As it is important to know in which situations $\tilde{Z}$ is of class$(D^\Lambda)$ we need the following result.

**Proposition 2.31** (El Karoui [1981], Proposition 2.29, p.127). If the given $\mathcal{F}^\Lambda$-system $(Z(T))_{T \in \mathcal{F}^\Lambda}$ of Theorem 2.29 is of class$(D^\Lambda)$, then also its Snell envelope $\tilde{Z}$ is of class$(D^\Lambda)$. In that case, $\tilde{Z}$ has the decomposition $\tilde{Z} = \tilde{M} - \tilde{A} - \tilde{B}$, with processes $\tilde{M}, \tilde{A}, \tilde{B}$ defined as in Proposition 2.26 (ii).

### 2.2.3 Optimality criterion and an approximation of the Snell envelope

For an arbitrary positive $\mathcal{F}^\Lambda$-system $(Z(S))_{S \in \mathcal{F}^\Lambda}$ one can formulate the following optimal stopping problem

$$\text{Maximize } \mathbb{E}[Z(S)] \text{ over all } S \in \mathcal{F}^\Lambda. \quad (6)$$

The following theorem uses the Snell envelope to give necessary and sufficient conditions for a stopping time to be optimal, i.e. to attain the maximal value in (6).

**Theorem 2.32** (El Karoui [1981], Theorem 2.31, p.129). Let $(Z(S))_{S \in \mathcal{F}^\Lambda}$ be a positive $\mathcal{F}^\Lambda$-system of class$(D^\Lambda)$ and let $\bar{Z}$ denote its $\Lambda$-Snell envelope. Then $\bar{U} \in \mathcal{F}^\Lambda$ is optimal for (6) if and only if

(i) $Z(\bar{U}) = \bar{Z}\bar{U}$ $\mathbb{P}$-a.s.,

(ii) $(\bar{Z}_{t\wedge \bar{U}})_{t \in [0, \infty]}$ is a $\Lambda$-martingale.

The next proposition introduces the entry time of the event that the Snell envelope is close to the $\mathcal{F}^\Lambda$-system and gives more precise information about the processes $\tilde{A}$ and $\tilde{B}$ of the decomposition of the Snell envelope of a $\Lambda$-measurable process introduced in Proposition 2.31.

**Proposition 2.33** (Compare El Karoui [1981], Proposition 2.32, p.130 and El Karoui [1981], Proposition 2.34, p.131). Let $(Z(T))_{T \in \mathcal{F}^\Lambda}$ be a positive $\mathcal{F}^\Lambda$-system of class$(D^\Lambda)$, aggregated by a $\Lambda$-measurable process $Z$ and denote by $\tilde{Z}$ its $\Lambda$-Snell envelope with $\tilde{Z} = \tilde{M} - \tilde{A} - \tilde{B}$ the decomposition of $\tilde{Z}$ from Proposition 2.31. Furthermore consider for $\lambda \in (0, 1)$ the $\Lambda$-measurable set

$$E^\Lambda := \{(\omega, t) \in \Omega \times [0, \infty) \mid \lambda\tilde{Z}_t(\omega) \leq Z_t(\omega)\}.$$ 

and let

$$T^\Lambda_S(\omega) := \inf \{t \geq S(\omega) \mid (\omega, t) \in E^\Lambda\}$$

denote the entry time of $E^\Lambda$ after some given $S \in \mathcal{F}^\Lambda$. Then we have

$$\mathbb{E}[\tilde{Z}_S] = \mathbb{E}\left[\tilde{Z}_{T^\Lambda_S}1_{\{T^\Lambda_S \in E^\Lambda\}} + \tilde{Z}_{T^\Lambda_S + 1}1_{\{T^\Lambda_S \notin E^\Lambda\}}\right]$$

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and

\[ \tilde{A}_S = \tilde{A}_{T_S}, \quad \tilde{B}_{S-} = \tilde{B}_{T_S-} \mathbb{1}_{\{T_S \in E\}} + \tilde{B}_{T_S} \mathbb{1}_{\{T_S \notin E\}} \quad \text{a.s.} \]

In particular, we have up to evanescent sets that \( \{ \tilde{A} > \tilde{A}_- \} \subset \{ \tilde{Z}_- = *Z \} \), \( \{ \tilde{B} > \tilde{B}_- \} \subset \{ \tilde{Z} = Z \} \) and

\[ \tilde{Z} = \Lambda(\tilde{Z}_+) \lor Z \quad \text{and} \quad \tilde{Z}_- = (\mathcal{P} \tilde{Z}) \lor *Z. \]

**Remark 2.34.** The stopping time \( T^\Lambda_S \) from the previous proposition may not be a \( \Lambda \)-stopping time, very much like a level passage time for a predictable process may not itself be predictable.

### 2.2.4 Relaxed optimal stopping

In this subsection we state results on some stopping time which “nearly” solves the optimal stopping problem introduced in (6). This result will give rise to the notion of divided stopping times to be discussed thereafter.

For now we assume that the \( \mathcal{S}^\Lambda \)-system \( (Z(T))_{T \in \mathcal{S}^\Lambda} \) can be aggregated by some \( \Lambda \)-measurable process \( Z \).

**Proposition 2.35** (El Karoui [1981], Proposition 2.35 and 2.36, p.133 and 135 and El Karoui [1981], Remark, p.136). We use the notations and hypotheses from Proposition 2.31 and 2.33. For any \( S \in \mathcal{S}^\Lambda \), the family of \( \mathcal{F}^+_{\Lambda} \)-stopping times \( (T^\Lambda_S)_{\lambda \in (0,1)} \) is non-decreasing in \( \lambda \in [0,1) \) and we denote its limit by \( T_S := \lim_{\lambda \uparrow 1} T^\Lambda_S \). We have

\[ H^-_S := \{ T^\Lambda_S < T_S \text{ for every } \lambda \in [0,1) \} \subset \{ \tilde{Z}_{T_S-} = *Z_{T_S} \}, \]

\[ H^+_S := (H^-_S)^c \cap \{ \tilde{Z}_{T_S} \leq Z_{T_S} \} \subset \{ \tilde{Z}_{T_S} = Z_{T_S} \}, \]

\[ H^+_S := (H^-_S)^c \cap \{ \tilde{Z}_{T_S} > Z_{T_S} \} \subset \{ \tilde{Z}_{T_S+} = Z_{T_S} \} \]

and

\[ T_S = \inf \{ t \geq S \mid Z_t = \tilde{Z}_t \lor *Z_t = \tilde{Z}_t+ \text{ or } Z_t^* = \tilde{Z}_t^- \}. \quad (7) \]

The sets \( H^-_S \) and \( H^+_S \) are contained in \( \mathcal{F}^+_{T_S} \) and \( H^-_S \in \mathcal{F}^+_{T_S} \). Moreover \( (T_S)_{T_H^\Lambda} \) is a predictable \( \mathcal{F}^+_{T_S} \)-stopping time, \( (T_S)_{H^\Lambda_S} \) is a \( \Lambda \)-stopping time, \( (T_S)_{H^\Lambda_S} \) is an \( \mathcal{F}^+_{T_S} \)-stopping time and for each \( S \in \mathcal{S}^\Lambda \) we get that

\[ \tilde{Z}_S = \mathbb{E} \left[ *Z_{T_S} \mathbb{1}_{H^-_S} + Z_{T_S} \mathbb{1}_{H^+_S} + Z^*_{T_S} \mathbb{1}_{H^+_S} \mid \mathcal{F}^+_{S} \right] \quad (8) \]

and, in particular,

\[ \mathbb{E}[\tilde{Z}_S] = \mathbb{E} \left[ *Z_{T_S} \mathbb{1}_{H^-_S} + Z_{T_S} \mathbb{1}_{H^+_S} + Z^*_{T_S} \mathbb{1}_{H^+_S} \right]. \]
2.2.5 General Divided Stopping Times

Even in deterministic examples it is easy to see that it is not always possible to solve the optimal stopping problem (6). Proposition 2.35 though gives a good idea how to relax this problem suitably.

**Definition 2.36** (El Karoui [1981], Definition 2.37, p.136-137). A quadruple $\sigma := (T, W^-, W, W^+)$ is called a divided stopping time, if $T$ is an $\mathcal{F}_+^{\Lambda}$-stopping time and $W^-, W, W^+$ form a partition of $\Omega$ such that

(i) $W^- \in (\mathcal{F}_+)^T$ and $W^- \cap \{T = 0\} = \emptyset$,

(ii) $W \in \mathcal{F}_+^{\Lambda T}$,

(iii) $W^+ \in \mathcal{F}_+^{\Lambda T}$ and $W^+ \cap \{T = \infty\} = \emptyset$,

(iv) $T_{W^-}$ is an $\mathcal{F}_+^{\Lambda}$-predictable stopping time,

(v) $T_{W^+}$ is a $\Lambda$-stopping time.

The set of all divided stopping times will be denoted as $\mathcal{S}_\Lambda^{\text{div}}$. For a $\Lambda$-measurable positive process $Z$, we define the value attained at a divided stopping time $\sigma = (T, W^-, W, W^+)$ as

$Z_\sigma := Z_T 1_{W^-} + Z_T 1_{W} + Z_T^* 1_{W^+}$.

**Remark 2.37.** Proposition 2.35 shows that $\delta_S := (T_S, H_S^-, H_S, H_S^+)$ is a divided stopping time.

The $\Lambda$-(super)martingale property can be extended to accommodate divided stopping times:

**Lemma 2.38** (El Karoui [1981], Lemma 2.38, p.137). Let $\sigma = (T, W^-, W, W^+)$ be a divided stopping time and $S$ a $\Lambda$-stopping time such that $\sigma \geq S$, i.e. $T \geq S$ and $T > S$ on $W^-$. Then we have for every positive $\Lambda$-martingale $M : \Omega \times [0, \infty] \to \mathbb{R}$

$M_S = \mathbb{E}[M_\sigma \mid \mathcal{F}^\Lambda_S]$

and for every positive $\Lambda$-supermartingale $Z : \Omega \times [0, \infty] \to \mathbb{R}$ that

$Z_S \geq \mathbb{E}[Z_\sigma \mid \mathcal{F}^\Lambda_S]$.

By contrast to problem (6) its relaxation for divided stopping times

Maximize $\mathbb{E}[Z_\sigma]$ over $\sigma \in \mathcal{S}_\Lambda^{\text{div}}$, (9)

always have a solution:

**Theorem 2.39** (El Karoui [1981], Theorem 2.39, p.138). Let $Z$ be a positive $\Lambda$-measurable process of class $(D^\Lambda)$ and $\bar{Z}$ its Snell envelope. Then for every $S \in \mathcal{S}^\Lambda$

$\mathbb{E}[\bar{Z}_S] = \mathbb{E}[\bar{Z}_{\delta_S}] = \mathbb{E}[\bar{Z}_{\delta_S}] = \sup_{S \geq S, \sigma \in \mathcal{S}_\Lambda^{\text{div}}} \mathbb{E}[Z_\sigma]$,
where \( \delta_S := (T_S, H^{-}_S, H^{+}_S) \) is the divided stopping time given by Proposition 2.35 and where the supremum is taken over all divided stopping times \( \sigma = (T, W^{-}, W, W^{+}) \) such that \( T \geq S \) and \( T > S \) on \( W^{-} \). In particular, the divided stopping time \( \delta_0 \) is optimal for (9).

The optimal divided stopping time \( \delta_0 \) is constructed with the help of \( T_0 \), which is constructed by using part (i) of Theorem 2.32. It is also possible to construct a second optimal divided stopping time with the help of the second condition as the next theorem shows.

**Theorem 2.40** (El Karoui [1981], Theorem 2.40, p.138). Let \( Z \) be a positive \( \Lambda \)-measurable process of class \((D^{\Lambda})\) and denote by \( \bar{Z} = M - \bar{A} - \bar{B} \) its Snell envelope with decomposition from Proposition 2.31. Define

\[
T^S := \inf \{ u \geq S \mid \bar{A}_u + \bar{B}_u > \bar{A}_S + \bar{B}_{S^{-}} \}
\]

and the \( \mathcal{F}^\Lambda_{T^S+} \)-measurable sets

\[
K^{-}_S := \{ \bar{A}_{T^S} > \bar{A}_S \},
\]

\[
K_S := \{ \bar{A}_{T^S} = \bar{A}_S, \quad \bar{B}_{T^S} > \bar{B}_{S^{-}} \},
\]

\[
K^{+}_S := \{ \bar{A}_{T^S} + \bar{B}_{T^S} = \bar{A}_S + \bar{B}_{S^{-}} \}.
\]

The quadruple \( \sigma_S := (T^S, K^{-}_S, K_S, K^{+}_S) \) is a divided stopping time satisfying

(i) \( \bar{Z}_{\sigma_S} = Z_{\sigma_S} \),

(ii) \( \mathbb{E}[\bar{Z}_S] = \mathbb{E}[Z_{\sigma_S}] \) for all \( S \in \mathcal{S}^\Lambda \).

In particular, the divided stopping time \( \sigma_0 \) is optimal for (9).

**Remark 2.41** (El Karoui [1981], Remark, p.140). The optimality conditions given in Theorem 2.32 show that no optimal stopping time for (6) can be smaller than \( \delta_0 \) and none can be larger than \( \sigma_0 \). Indeed, \( \bar{Z} \) looses the martingale property after \( T_0 \) and therefore \( T_0 \) is by Theorem 2.32 dominating all possible optimal stopping times. Taking a closer look at the set \( K^{-}_0 \), we see that actually on this set \( \bar{Z} \) has already lost the martingale property at \( T_0 \) and therefore \( T_0 \) is strictly larger than all optimal stopping times for (6) on this set. On the other hand, we see that \( T_0 \) is smaller than the entry time of the set \( \{ \bar{Z} = Z \} \), which we denote by \( \bar{T}_0 \), and \( T_0 \) is strictly smaller than \( \bar{T}_0 \) on \( H^{-}_0 \cap \{ T_0 < \bar{T}_0 \} \). Hence, the divided stopping time \( (T_0, H^{-}_0 \cap \{ T_0 < \bar{T}_0 \}, (H^{-}_0 \cap \{ T_0 = \bar{T}_0 \}) \cup H_0, H^{+}_0) \) is smaller than all optimal stopping times for (6).

### 2.2.6 Conditions for optimality in the optional case

In this section, let us consider the classical case where \( \Lambda = \mathcal{O}(\mathcal{F}) \) is the optional \( \sigma \)-field of a right-continuous filtration \((\mathcal{F}_t)_{t \geq 0}\) with \( \mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t \) and \( \mathcal{F}_{\infty}^{-} \subset \mathcal{F}_0 \) a \( \mathbb{P} \)-complete \( \sigma \)-field. Then \( \mathcal{S}^\Lambda \) coincides with the set of “classical” stopping times
with respect to \((\mathcal{F}_t)_{t \geq 0}\), which we denote by \(\mathcal{I}^\theta\). Furthermore we assume, that \(Z\) is an optional, positive process of class(D). The optimal stopping problem is then to

\[
\text{Maximize } \mathbb{E}[Z_T] \text{ over } T \in \mathcal{I}^\theta
\]

and we get the following first optimality result:

**Theorem 2.42** (El Karoui [1981], Theorem 2.41, p.140). Let \(\Lambda = \mathcal{O}(\mathfrak{F})\). Assume \(Z\) is an optional, positive process of class(D) and denote by \(\bar{Z}\) its Snell envelope.

(i) If the process \(Z\) is upper-semicontinuous from the right and from the left, i.e. \(\ast Z \leq Z\) and \(Z \leq \ast Z\), then the entry time \(\bar{T}_0\) of the set \(\{Z = \bar{Z}\}\) is optimal for (11) and it is equal to \(T_0\) from (7). In particular, it is the smallest optimal stopping time.

(ii) If \(\bar{Z}\) satisfies \(\bar{Z} = \mathcal{P}\bar{Z}\) and \(Z\) is upper-semicontinuous from the right, then the entry time \(\bar{T}^0\) of the set \(\{\bar{M} \neq \bar{Z}\}\) is optimal for (11) with \(\bar{M}\) from Proposition 2.31 and \(\bar{T}^0 = T^0\) with \(T^0\) from (10). In particular, \(T^0\) is the largest optimal stopping time.

**Remark 2.43.** (i) The proof of the previous result is mainly based on the results on divided stopping times, which can also be used for general Meyer-\(\sigma\)-fields. But the reason why we cannot get the same general results for those more general \(\sigma\)-fields is that \(T_S\) and \(T^S\) are not necessarily Meyer-stopping times and, therefore we cannot use the \(\Lambda\)-supermartingale property of \(\bar{Z}\) in the proof.

(ii) The condition \(\bar{Z} = \mathcal{P}\bar{Z}\), in part (ii) of the previous theorem, means that the Snell envelope \(\bar{Z}\) of \(Z\) has to be left-upper-semicontinuous in expectation (see Lemma 3.4 (ii) below). It will hold when \(Z\) is upper-semicontinuous in expectation. Note that \(\bar{Z} = \mathcal{P}\bar{Z}\) is always fulfilled since \(\bar{Z}\) is a supermartingale.

The following result gives a nice equivalence for the previous pathwise properties of \(Z\):

**Proposition 2.44** (Compare Bismut and Skalli [1977], Theorem II.1, p.305). An optional positive process \(Z : \Omega \times [0, \infty) \to \mathbb{R}\) of class(D) satisfies

\[
\mathbb{E}[Z_T] \geq \limsup_{n \to \infty} \mathbb{E}[Z_{T_n}]
\]

for every monotone sequences \(T_n\) of stopping times converging to \(T\) if and only if \(Z\) has upper-semicontinuous paths from the right on \([0, \infty)\) and

\[
\mathcal{P}Z_t \ast Z_t \text{ for } t \in (0, \infty) \text{ almost surely.}
\]

**Remark 2.45.** Bismut and Skalli [1977] state in the introduction to chapter I, p.301, that we can use processes \(Z\) of class(D) in their Theorem II.1, p.305, instead of merely bounded processes.

The previous proposition gives us the second optimality result:
Theorem 2.46 (El Karoui [1981], Theorem 2.43, p.142). Let $Z$ be an optional, positive process of class(D).

(i) Assume $Z$ is upper-semicontinuous in expectation, i.e. for every monotone (not necessarily strict) sequence $(T_n)_{n\in\mathbb{N}}$ with limit $T$, we have
\[ \mathbb{E}[Z_T] \geq \limsup_{n\to\infty} \mathbb{E}[Z_{T_n}] . \]
Then the entry time $\bar{T}_0$ of the set $\{Z = \bar{Z}\}$ is optimal as is the entry time $\bar{T}^0$ of the set $\{\bar{Z} \neq \bar{Z}\}$ with $\bar{M}$ from the decomposition of $\bar{Z}$ given by Proposition 2.31.

(ii) If for every non-increasing sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ with limit $T$ we have
\[ \mathbb{E}[Z_T] \geq \limsup_{n\to\infty} \mathbb{E}[Z_{T_n}] \]
and if for every non-decreasing sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ with limit $T$ we have
\[ \lim_{n\to\infty} \sup_{T_n \leq R \in \mathcal{F}_\sigma} \mathbb{E}[Z_R] = \sup_{T \leq R \in \mathcal{F}_\sigma} \mathbb{E}[Z_R], \]
then the entry time $\bar{T}^0$ of the set $\{\bar{Z} \neq \bar{Z}\}$ with $\bar{M}$ from the decomposition of $\bar{Z}$ given by Proposition 2.31 is optimal.

Remark 2.47. In the original article of El Karoui [1981] it is claimed that part (i) of the previous theorem follows directly by the equivalence in Proposition 2.44 and the statement in Theorem 2.42. But as one can see we do not get upper-semi-continuity from the left of the process $Z$, but just of the process $Z$. These two processes are not the same in general and hence the proof is incomplete. This gap will be closed by Proposition 3.6 in the next section.

3 Supplementary results to Lenglart’s and to El Karoui’s theory

In this section we will prove some additional results which are not given in El Karoui [1981], but which we find useful. Throughout this section we will fix a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and a Meyer-$\sigma$-field $\Lambda \subset \mathcal{F} \otimes \mathcal{B}([0, \infty))$ with $\mathcal{F}_\infty := \bigvee_t \mathcal{F}_t \subset \mathcal{F}$, $\mathcal{F}_0$- some $\mathbb{P}$-complete $\sigma$-field contained in $\mathcal{F}_0$ and $\mathcal{F}$ is satisfying the usual conditions of completeness and right-continuity.

3.1 Special case of an embedded Meyer-$\sigma$-field

Lemma 3.1. Assume that the given Meyer-$\sigma$-field $\Lambda$ is embedded in the sense that:
\[ \mathcal{P}(\mathcal{F}) \subset \Lambda \subset \mathcal{O}(\mathcal{F}), \quad (12) \]
where $\mathcal{P}(\mathcal{F})$ and $\mathcal{O}(\mathcal{F})$ denote, respectively, the predictable and the optional $\sigma$-field associated with $\mathcal{F}$.

Then we have $\mathcal{F}_t^- = \mathcal{F}_t^\Lambda$ for $t > 0$, $\mathcal{F}_t^\Lambda = \mathcal{F}_t$ for $t \geq 0$, and $\Lambda$ is a $\mathbb{P}$-complete Meyer-$\sigma$-field. In particular, this gives us $\mathcal{F}_t^- \subseteq \mathcal{F}_t^\Lambda \subseteq \mathcal{F}_t$, $t > 0$, and $\mathcal{P}(\mathcal{F}) = \mathcal{P}(\mathcal{F}_t^\Lambda)$, $\mathcal{O}(\mathcal{F}) = \mathcal{O}(\mathcal{F}_t^\Lambda)$.

**Proof.** First of all, we have by Theorem 2.17 that $\Lambda$ is $\mathcal{P}$-complete by the properties of $\mathcal{F}$. Next we get by Example 2.6 and Remark 2.7 that

\[ \mathcal{F}_t^- = \mathcal{P}(\mathcal{F}_t) \subset \mathcal{F}_t^\Lambda \subset \mathcal{O}(\mathcal{F}_t) = \mathcal{F}_t \]

for $t > 0$. Furthermore, we get for $t > 0$ by (13) that

\[ \mathcal{F}_t^- = \sigma\left( \bigcup_{s \in [0,\infty), s < t} \mathcal{F}_s \right) = \sigma\left( \bigcup_{s \in [0,\infty), s < t} \mathcal{F}_{(s+t^{-})} \right) \]

and on the other hand we have

\[ \mathcal{F}_t^\Lambda = \sigma\left( \bigcup_{s \in [0,\infty), s < t} \mathcal{F}_s^\Lambda \right) \subset \sigma\left( \bigcup_{s \in [0,\infty), s < t} \mathcal{F}_s \right) = \mathcal{F}_t^- . \]

Combining the previous two results proves $\mathcal{F}_t^\Lambda = \mathcal{F}_t^\Lambda$ for $t > 0$. Additionally we see that for $t \geq 0$

\[ \mathcal{F}_t^- = \bigcap_{r \in [0,\infty), t < r} \mathcal{F}_r^- \subset \bigcap_{r \in [0,\infty), t < r} \mathcal{F}_r = \mathcal{F}_t = \mathcal{F}_t^\Lambda \]

and

\[ \mathcal{F}_t \subset \bigcap_{r \in [0,\infty), t < r} \sigma\left( \bigcup_{s \in [0,\infty), s < r} \mathcal{F}_s \right) = \bigcap_{r \in [0,\infty), t < r} \mathcal{F}_r^- \subset \bigcap_{r \in [0,\infty), t < r} \mathcal{F}_r^\Lambda = \mathcal{F}_t^\Lambda , \]

which implies $\mathcal{F}_t^\Lambda = \mathcal{F}_t$. As we have proven that the filtration $(\mathcal{F}_t)_{t \geq 0}$ and $(\mathcal{F}_t^\Lambda)_{t \geq 0}$ are the same, we see that $\mathcal{O}(\mathcal{F})$ and $\mathcal{O}(\mathcal{F}_t^\Lambda)$ (resp. $\mathcal{P}(\mathcal{F})$ and $\mathcal{P}(\mathcal{F}_t^\Lambda)$) are generated by the same processes, which shows in particular that they are the same. \hfill $\Box$

### 3.2 Approximating the lim sup

It is sometimes convenient to know that the lim sup from the right or from the left of a stochastic process at a stopping time can be realized by a suitable sequence of larger or smaller stopping times converging to it. Such a result was proven by Dellacherie and Lenglart [1982], Lemma 1, p.300, for the case of optional processes. We extend their argument to cover $\Lambda$-measurable processes:
Proposition 3.2. Assume that $\Lambda$ is embedded in the sense of (12). Let $Z$ be a $\Lambda$-measurable process with $Z_\infty = 0$ and denote by $Z^*$ and, respectively, $^*Z$ the right- and the left-upper-semicontinuous envelope of $Z$, which are defined in (4) and (5). Now we have the following two results:

(i) For every given $\mathcal{F}$-stopping time $T$, there exists a non-increasing sequence $(T_n)_{n \in \mathbb{N}} \subset \mathcal{F}_\Lambda$ with $T_n \geq T$, $\infty > T_n > T$ on $\{T < \infty\}$ and $\lim_{n \to \infty} T_n = T$ such that $Z^*_T = \lim_{n \to \infty} Z_{T_n}$ almost surely.

(ii) For any predictable $\mathcal{F}$-stopping time $T > 0$, there exists a sequence of $\Lambda$-stopping times $(T_n)_{n \in \mathbb{N}}$ with $T_n < T$ and $\lim_{n \to \infty} T_n = T$ such that $^*Z_T = \lim_{n \to \infty} Z_{T_n}$ almost surely.

Remark 3.3. (i) Result (ii) of Proposition 3.2 cannot be generalized to an $\mathcal{F}$-stopping time $T$ in the way it is possible for result (i), because the sequence $(T_n)_{n \in \mathbb{N}}$ is an advertising sequence, which would directly imply that $T$ is predictable.

(ii) Obviously one can get analogously to Proposition 3.2 the same result for the right- and left-lower-semicontinuous envelope of a given $\Lambda$-measurable process $Z$ by using Proposition 3.2 for $-Z$.

Proof. Proof of (i): We will prove this result by adapting the proof of Dellacherie and Lenglart [1982], Lemma 1, p.300. Assume without loss of generality that $\mathbb{P}(T < \infty) > 0$, because for $T = \infty$ a.s. we could set $T_n = \infty$ as $Z_\infty = 0 = Z^*_\infty$. Next we see that $Z^*$ is $\mathcal{F}$-progressively measurable by Dellacherie and Meyer [1982], Theorem 90, p.143, and, therefore, $Z^*_T \in \mathcal{F}_T$ by Dellacherie and Meyer [1982], Theorem 64 (b), p.122. Furthermore, we can assume that $Z$ and $Z^*_T$ are bounded by replacing $Z$ with $\frac{Z}{1+|Z|}$. Now we set $T := \infty$ and we define inductively $S_n$, $n = 2, 3, \ldots$, as a $\Lambda$-stopping time from the Meyer-Section Theorem 2.8, which we apply for $\varepsilon_n := 2^{-n}$ and

$$B_n := \{T, \infty \} \cap \left[ 0, \min \left( T + \frac{1}{n}, S_{n-1} \right) \right] \cap \left\{ |Z - Z^*_T| < \frac{1}{n} \right\}. \quad (14)$$

We just have to prove $B_n \in \Lambda$ which we will do below. Granted $B_n \in \Lambda$, we can define $T_1 := T + 1$ and $T_n := \min(S_n, T_{n-1}, T + 1)$, $n = 2, 3, \ldots$, and the sequence $(T_n)_{n \in \mathbb{N}} \subset \mathcal{F}_\Lambda$ will satisfy the desired properties as $\pi(B_n) = \{T < \infty\}$. Indeed, a Borel-Cantelli argument applied to $\{S_n = \infty\} \cap \{T < \infty\}$, $n \in \mathbb{N}$, shows that for $\mathbb{P}$-almost every $\omega \in \{T < \infty\}$ there exists $N_\omega$ such that for $n \geq N_\omega$ we have $S_n(\omega) < \infty$ for $n \geq N_\omega$. In particular, $T(\omega) < S_n(\omega) < T(\omega) + \frac{1}{n}$, $|Z_{S_n}(\omega) - Z^*_T(\omega)| < \frac{1}{n}$ eventually. For $\omega \in \{T = \infty\}$, we get that $\omega \notin \pi(B_n)$ for $n \in \mathbb{N}$ and hence $S_n(\omega) = \infty$ for all $n \in \mathbb{N}$.

Proof of $B_n \in \Lambda$: We will argue that each of the three sets in the specification (14) of $B_n$ is contained in $\Lambda$. First, we have $\{T, \infty \} \in \Lambda$ by Lenglart [1980], Theorem 2.1 and Corollary 1.1), p.503-504, as $T$ is an $\mathcal{F}_\Lambda$-stopping time by (12). Next, we see that for any $n \in \mathbb{N}$ the stopping time $T + \frac{1}{n}$ is a predictable $\mathcal{F}$-stopping time, which implies by $\mathcal{P}(\mathcal{F}) \subset \Lambda$ that it is a $\Lambda$-stopping time. Analogously, $S_1$ is a
\[0, \min \left( T + \frac{1}{n}, S_{n-1} \right) \in \Lambda. \]

Finally, we see that \( Z^*_T 1_{[T, \infty]} \) is \( \mathcal{F} \)-predictable, because it is left-continuous and \( \mathcal{F} \)-adapted. Hence, again by (12) it is a \( \Lambda \)-measurable process. As \( Z \) is \( \Lambda \)-measurable also \( Z 1_{[T, \infty]} \) is a \( \Lambda \)-measurable process. Therefore, \( (Z - Z^*_T) 1_{[T, \infty]} \) is \( \Lambda \)-measurable and, as a consequence,

\[\| T, \infty \cap \left\{ |Z - Z^*_T| < \frac{1}{n} \right\} \in \Lambda \]

for every \( n \in \mathbb{N} \).

Proof of (ii): We assume w.l.o.g. that \( T(\omega) > 0 \) for all \( \omega \in \Omega \) as we could replace \( T \) by \( T_{\{T > 0\}} \) with \( T = T_{\{T > 0\}} \) almost surely. For \( T \) there exists an announcing sequence \((\hat{T}_n)_{n \in \mathbb{N}}\) by Dellacherie and Meyer [1982], Theorem 77, p.132, i.e a sequence of predictable \( \mathcal{F} \)-stopping times \((\hat{T}_n)_{n \in \mathbb{N}}\) with \( \hat{T}_n(\omega) < T(\omega) \) and \( \lim_{n \to \infty} \hat{T}_n(\omega) = T(\omega) \) for all \( \omega \in \Omega \). Now we adapt the proof of (i) with

\[\mathcal{B}_n := \mathbb{E}[\hat{T}_n, T \cap \left\{ |Z - \Lambda^{*}Z_T| < \frac{1}{n} \right\}],\]

which satisfies \( \mathbb{P}(\pi(\mathcal{B}_n)) = 1 \), to obtain again a sequence of \( \Lambda \)-stopping times \( S_n \) by the Meyer Section Theorem and the desired sequence is given by \( T_n := \inf_{k \geq n} S_k \).

Indeed, again a Borel-Cantelli argument applied to \( \{S_n = \infty\}, n \in \mathbb{N} \), shows that for almost every \( \omega \in \Omega \) there exists \( N_\omega \) such that for \( n \geq N_\omega \) we have \( S_n(\omega) < \infty \). In particular \( \hat{T}_n(\omega) < S_n(\omega) < T(\omega) \) and \( |Z_{S_n}(\omega) - \Lambda^{*}Z_T(\omega)| < \frac{1}{n} \) for \( n \geq N_\omega \). As \((T_n)_{n \in \mathbb{N}}\) is non-decreasing to \( T \), we see that \( T_n(\omega) = \inf_{k \geq n} S_k(\omega) \) is actually a minimum over finitely many elements. Hence, \( T_n \) inherits the desired properties from \( S_k, k \geq n \).

\[ \square \]

3.3 Path-properties of a process under the assumption of upper-semicontinuity in expectation

In this subsection we will extend Theorem II.1 of Bismut and Skalli [1977] to the case of \( \Lambda \)-measurable processes.

Lemma 3.4. Assume that \( \Lambda \) is embedded in the sense of (12). Let \( Z \) be a \( \Lambda \)-measurable process of class(\( D^\Lambda \)) with \( Z_\infty = 0 \). Then the following holds:

(i) For any \( S \in \mathcal{F}^\Lambda \), the following assertions are equivalent:

(a) \( Z \) is right-upper-semicontinuous in expectation in \( S \) in the sense that for any non-increasing sequence \((S_n)_{n \in \mathbb{N}} \subset \mathcal{F}^\Lambda \) with \( S_n \geq S, \infty > S_n > S \) on \( \{S < \infty\}, \lim_{n \to \infty} S_n = S, \lim_{n \to \infty} Z_{S_n} = Z^*_S \) and for any \( A \in \mathcal{F}^\Lambda \) we have

\[\mathbb{E}[Z_{S_n}] \geq \lim_{n \to \infty} \mathbb{E}[Z_{(S_n)_A}].\]
(b) \( Z_S \geq ^\Lambda (Z^*)_S \) almost surely.

In particular, \( Z \) is right-upper-semicontinuous in expectation in all \( S \in \mathcal{F}^\Lambda \) if and only if the set \( \{ Z < ^\Lambda (Z^*) \} \) is evanescent.

(ii) For \( S \in \mathcal{F}^\rho \), the following assertions are equivalent:

(a) For any non-decreasing sequence \( (S_n)_{n\in\mathbb{N}} \subset \mathcal{F}^\Lambda \) with \( S_n < S \) on \( \{ S > 0 \} \), \( \lim_{n\to\infty} S_n = S \), we have

\[
\mathbb{E}[Z_S] \geq \limsup_{n\to\infty} \mathbb{E}[Z_{S_n}].
\]

(b) For any non-decreasing sequence \( (S_n)_{n\in\mathbb{N}} \subset \mathcal{F}^\Lambda \) with \( S_n < S \) on \( \{ S > 0 \} \), \( \lim_{n\to\infty} S_n = S \), \( \lim_{n\to\infty} Z_{S_n} = Z_S^* \) and \( A \in \bigcup_{m\in\mathbb{N}} \mathcal{F}^\Lambda_{S_m} \) we have

\[
\mathbb{E}[Z_{S^*_A}] \geq \lim_{n\to\infty} \mathbb{E}[Z_{S_n^*A}].
\]

(c) \( ^\rho Z_S \geq Z_S^* \) almost surely.

If the process \( Z \) satisfies one of the three equivalent conditions then we call \( Z \) left-upper-semicontinuous in expectation at \( S \).

In particular, the process \( Z \) is left-upper-semicontinuous in expectation at every \( S \in \mathcal{F}^\rho \) if and only if the set \( \{ ^\rho Z < Z^* \} \) is evanescent and we have

\[
Z^*_\infty \leq ^\rho Z^*_\infty = 0.
\]

Remark 3.5 (Right-upper-semicontinuity in expectation). (i) For the class of optional processes, Bismut and Skalli [1977], p.306, have shown that even \( ^\rho (Z^*)_S \geq Z^*_S \) (see also Proposition 3.7). The analogous statement does not hold true for general \( \Lambda \)-measurable process. Pathwise upper-semicontinuity of \( \Lambda \)-measurable processes thus has to be ensured by further assumptions (see Besslich [2019], Example 2.11, p.47).

(ii) If \( Z \) is right-upper-semicontinuous in expectation in all \( S \in \mathcal{F}^\Lambda \), the initial definition of upper-semicontinuity in expectation becomes easier to state. In fact, it amounts to the requirement that for all \( S \in \mathcal{F}^\Lambda \) we want that for any non-increasing sequence \( S_n \in \mathcal{F}^\Lambda \) with \( S_n \geq S \), \( S_n > S \) on \( \{ S < \infty \} \), \( \lim_{n\to\infty} S_n = S \), \( \lim_{n\to\infty} Z_{S_n} = Z^*_S \) we have

\[
\mathbb{E}[Z_S] \geq \lim_{n\to\infty} \mathbb{E}[Z_{S_n}].
\]

This follows by the fact that if \( A \in \mathcal{F}^\Lambda_S \subset \mathcal{F}^\Lambda_{S_n} \) also \( S_A, (S_n)_A \in \mathcal{F}^\Lambda \) and \( \lim_{n\to\infty} (S_n)_A = S_A \).

(iii) The classical definition of right-upper-semicontinuity given by Dellacherie and Lenglart [1982], p.303, implies our definition and in the case of optional processes they are equivalent. Indeed, for optional processes Dellacherie and Lenglart [1982],
Lemma 3.4, p.303, implies that their definition of right-upper-semicontinuity implies \( Z_S \geq Z^*_S \). Hence

\[
Z_S = Z_S \mathbb{1}_{\{S < \infty\}} = \mathbb{E}[Z_S \mathbb{1}_{\{S < \infty\}} | \mathcal{F}_S] \geq \mathbb{E}[Z^*_S \mathbb{1}_{\{S < \infty\}} | \mathcal{F}_S] = \lambda(Z^*)_S \mathbb{1}_{\{S < \infty\}} = \lambda(Z^*)_S,
\]

and this is equivalent to our definition of right-upper-semicontinuity.

(iv) Analogously to Remark 3.3 (ii), Lemma 3.4 also characterizes right- and left-lower-semicontinuity.

**Proof of Lemma 3.4.** The proof of (i) is mainly the proof in Dellacherie and Lenglart [1982], Theorem 6, p.303, and the proof of (ii) will be accomplished by adapting the same argument to the setting of predictable stopping times.

**Part (i):** 
- **a) “⇒”** b): Assume \( Z \) is right-upper-semicontinuous in expectation in \( S \). By Proposition 3.2 (i), there exists a non-increasing sequence \((S_n)_{n \in \mathbb{N}} \subset \mathcal{P}^S\) such that \( S_n \geq S, \infty > S_n > S \) on \( \{S < \infty\} \), \( \lim_{n \to \infty} S_n = S \) and \( \lim_{n \to \infty} Z_{S_n} = Z^*_S \) almost surely. Let \( A \in \mathcal{F}_S^A \). Then we get by \( Z_\infty = Z = 0 \) and because \( Z \) is of class \((D^A)\) that

\[
\mathbb{E}[Z_S \mathbb{1}_{\{S < \infty\}} \mathbb{1}_A] = \mathbb{E}[Z_{S_n} \mathbb{1}_A] \geq \lim_{n \to \infty} \mathbb{E}[Z_{S_n} \mathbb{1}_A] = \mathbb{E}[\lim_{n \to \infty} Z_{S_n} \mathbb{1}_{\{S < \infty\}} \mathbb{1}_A] = \mathbb{E}[Z^*_S \mathbb{1}_{\{S < \infty\}} \mathbb{1}_A] = \lambda(Z^*)_S \mathbb{1}_{\{S < \infty\}} \mathbb{1}_A.
\]

As \( A \in \mathcal{F}_S^A \) is arbitrary this shows that \( Z_S \mathbb{1}_{\{S < \infty\}} \geq \mathbb{E}[Z^*_S \mathbb{1}_{\{S < \infty\}} | \mathcal{F}_S^A] \). Hence

\[
Z_S \mathbb{1}_{\{S < \infty\}} \geq \mathbb{E}[Z^*_S \mathbb{1}_{\{S < \infty\}} | \mathcal{F}_S^A] = \lambda(Z^*)_S \mathbb{1}_{\{S < \infty\}} \mathbb{1}_A
\]

and the rest follows by \( Z_\infty = \lambda(Z^*)_\infty = 0 \).

- **b) “⇒” a):** Assume now that we have \( Z_S \geq \lambda(Z^*)_S \) and let \((S_n)_{n \in \mathbb{N}} \subset \mathcal{P}^A\) be a sequence such that \( S_n \geq S, \infty > S_n > S \) on \( \{S < \infty\} \), \( \lim_{n \to \infty} S_n = S \) and \( \lim_{n \to \infty} Z_{S_n} = Z^*_S \) almost surely. Then we obtain for \( A \in \mathcal{F}_A^S \) that

\[
\mathbb{E}[Z_{S_n} A] = \mathbb{E}[Z_S \mathbb{1}_{\{S < \infty\}} \mathbb{1}_A] \geq \mathbb{E}[\lambda(Z^*_S) \mathbb{1}_{\{S < \infty\}} \mathbb{1}_A] = \mathbb{E}[Z^*_S \mathbb{1}_A] = \mathbb{E}[\lim_{n \to \infty} Z_{S_n} \mathbb{1}_A] = \lim_{n \to \infty} \mathbb{E}[Z_{S_n} \mathbb{1}_A],
\]

which finishes the proof of (i).

**Part (ii):** We can assume \( S > 0 \) as we could replace \( S \) by \( S_{\{S > 0\}} \) because \( Z_0 = Z_0 = *Z_0 \).

- **a) ⇒ b):** Assume we have a non-decreasing sequence \((S_n)_{n \in \mathbb{N}} \subset \mathcal{P}^A\) with \( S_n < S \), \( \lim_{n \to \infty} S_n = S \), \( \lim_{n \to \infty} Z_{S_n} = *Z_S \) and \( A \in \mathcal{F}_S^A \) for some \( m \in \mathbb{N} \). Define \( \tilde{S}_n := S_{n+m} \) and observe that because \( A \in \mathcal{F}_S^A \) also \((\tilde{S}_n)_n \in \mathbb{N} \subset \mathcal{P}^A\) and this sequence satisfies the conditions of (a). Hence

\[
\mathbb{E}[Z_S \mathbb{1}_A] \geq \lim_{n \to \infty} \mathbb{E}[Z_{\tilde{S}_n} \mathbb{1}_A] = \lim_{n \to \infty} \mathbb{E}[Z_{\tilde{S}_n} \mathbb{1}_A],
\]

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which we wanted to show.

$b) \Rightarrow c):$ By Proposition 3.2 (ii), there exists a sequence \((S_n)_{n \in \mathbb{N}} \subset \mathcal{F}_\Lambda, S_n < S, \lim_{n \to \infty} S_n = S\) and \(*Z_S = \lim_{n \to \infty} Z_{S_n}\) almost surely. Hence we get for \(A \in \mathcal{F}_{S_m}, m \in \mathbb{N},\) that

\[
\mathbb{E} \left[ Z_S 1_{\{S<\infty\}} 1_A \right] = \mathbb{E} [Z_{S_A}] \geq \lim_{n \to \infty} \mathbb{E} [Z_{(S_n)_A}]
\]

\[
= \mathbb{E} \left[ \lim_{n \to \infty} Z_{(S_n)_A} \right] = \mathbb{E} [^*Z_{S_A}] = \mathbb{E} [^*Z_S 1_A].
\]

Since \(A \in \mathcal{F}_{S_m}\) is arbitrary, this implies that

\[
\mathbb{E} [Z_S 1_{\{S<\infty\}} | \mathcal{F}_{S_m}] \geq \mathbb{E} [^*Z_S | \mathcal{F}_{S_m}]
\]

for all \(m \in \mathbb{N}.\) Thus, Dellacherie and Meyer [1982], Theorem 31, p.26, allows us to conclude for \(m \to \infty\) that

\[
\mathbb{E} [Z_S 1_{\{S<\infty\}} | \mathcal{F}_{S_m}] \geq \lim_{m \to \infty} \mathbb{E} [^*Z_S | \mathcal{F}_{S_m}] = \mathbb{E} [^*Z_S | \mathcal{F}_{S_m}] = ^*Z_S
\]

and, due to \(^*Z_\infty = 0,\) we get

\[
^pZ_S = ^pZ_S 1_{\{S<\infty\}} = \mathbb{E} [Z_S 1_{\{S<\infty\}} | \mathcal{F}_{S_m}] \geq ^*Z_S,
\]

which we wanted to show.

c) \Rightarrow a): Assume now that \(^pZ_S \geq ^*Z_S\) holds and let \((S_n)_{n \in \mathbb{N}} \subset \mathcal{F}_\Lambda\) be a non-decreasing sequence such that \(S_n < S, \lim_{n \to \infty} S_n = S.\) Then we have

\[
\mathbb{E} [Z_S] = \mathbb{E} [Z_S 1_{\{S<\infty\}}] = \mathbb{E} [^pZ_S 1_{\{S<\infty\}}] = \mathbb{E} [^pZ_S]
\]

\[
\geq \mathbb{E} [^*Z_S] \geq \mathbb{E} \left[ \limsup_{n \to \infty} Z_{S_n} \right] \geq \limsup_{n \to \infty} \mathbb{E} [Z_{S_n}],
\]

which finishes our proof. \(\square\)

In the next proposition we see that under some regularity conditions we can get more information about the sets \(H^-_S, H_S\) and \(H^+_S\) of Proposition 2.35 and we can indeed close the small gap in El Karoui’s proof of Theorem 2.46 (see Remark 2.47).

**Proposition 3.6.** We use the notation from Proposition 2.33 and 2.35. Let \(Z\) be a positive \(\Lambda\)-measurable process of class \((D^\Lambda),\) which is left-upper-semicontinuous in expectation at every \(S \in \mathcal{F}_\Lambda\) (see Lemma 3.4 (ii)). Then we have for any fixed \(S \in \mathcal{F}_\Lambda,\) that \(H^-_S \in \mathcal{F}_{T_{S^-}}\) and

\[
H^-_S \subset \{ \bar{Z}_{T_S} = Z_{T_S} \}
\]

up to a \(\mathbb{P}\)-nullset. In particular, we get, up to \(\mathbb{P}\)-nullsets, that

\[
H^-_S \cup H_S = \{ \bar{Z}_{T_S} = Z_{T_S} \}, \quad H^+_S = \{ \bar{Z}_{T_S} < Z_{T_S} \}.
\]
In the optional case $\Lambda = \mathcal{O}$ and if in addition $Z$ is right-upper-semicontinuous in expectation at all stopping times (see Lemma 3.4 (i)), then the stopping time

$$\bar{T}_S := \inf \{ t \geq S \mid \bar{Z}_t = Z_t \}.$$  

satisfies

$$Z_{\bar{T}_S} = \bar{Z}_{\bar{T}_S} \quad \text{and} \quad \mathbb{E}[Z_{\bar{T}_S} \mid \mathcal{F}_S^\Lambda] = \bar{Z}_S.$$  

In particular, $\bar{T}_0$ is optimal for (11).

Proof. First we get by Proposition 2.35 that $R := (T_S)_{H_S^-}$ is a predictable $\mathcal{F}_+^\Lambda$-stopping time and, hence, by Lemma 3.1, it is a predictable $\mathcal{F}$-stopping time and a $\Lambda$-stopping time. Furthermore, Dellacherie and Meyer [1978], Theorem 56, p.118, yields again with Lemma 3.1

$$H_S^- = \bigcap_{n=1}^{\infty} \left\{ T_{\bar{T}_S, \frac{n}{\Gamma} < T_S, \in \mathcal{F}_T = \mathcal{F}_S^\Lambda. \right\}$$

Now we want to show that $H_S^- \subset \{ \bar{Z}_{T_S} = Z_{T_S} \}$ for $S \in \mathcal{F}_S^\Lambda$. By Lemma 3.4 (ii), we know $\bar{Z} \geq *Z$ and, by Dellacherie and Meyer [1982], Remark (a), p.104, we have $\bar{Z} \leq \bar{Z}$ since $Z \leq \bar{Z}$. Hence, $*Z \leq \bar{Z}$ and, by Proposition 2.33, this yields $\bar{Z} = (\bar{Z}) \lor *Z = \bar{Z}$. Combining this with the predictability of $R$ and $\bar{Z}_R^- = *Z_R$ (Proposition 2.35) gives us

$$\mathbb{E} \left[ Z_{\bar{R} \uparrow_{R<\infty}} \right] = \mathbb{E} \left[ \bar{Z}_{\bar{R} \uparrow_{R<\infty}} \mid \mathcal{F}_R^- \right] = \mathbb{E} \left[ \bar{Z}_{\bar{R} \uparrow_{R<\infty}} \right] = \mathbb{E} \left[ \bar{Z}_{\bar{R} \uparrow_{R<\infty}} \mid \mathcal{F}_R^- \right]$$

By $\bar{Z}_\infty = Z_\infty = 0$ we can conclude

$$\mathbb{E} \left[ Z_{\bar{T}_S \uparrow_{H_S^-}} \right] = \mathbb{E} \left[ Z_{\bar{R} \uparrow_{R<\infty}} \right] + \mathbb{E} \left[ \bar{Z}_{\bar{R} \uparrow_{R<\infty}} \right] \leq \mathbb{E} \left[ Z_{\bar{T}_S \uparrow_{H_S^-}} \right]$$

and, because $Z \leq \bar{Z}$, we get

$$\bar{Z}_{\bar{T}_S \uparrow_{H_S^-}} = Z_{\bar{T}_S \uparrow_{H_S^-}}.$$  

The “in particular part” follows because $H_S \subset \{ \bar{Z}_{T_S} = Z_{T_S} \}$, $H_S^+ \subset \{ \bar{Z}_{T_S} < Z_{T_S} \}$ and $H_S^- \cup H_S^+ \cup H_S^+ = \Omega$ (see Proposition 2.35).

Finally we prove the assertion in the optional case. By Proposition 2.44 we get that $Z$ is pathwise upper-semicontinuous from the right and by repeating the arguments in (15) we have with $Z_\infty = 0$ that

$$\mathbb{E} \left[ *Z_{\bar{T}_S \uparrow_{H_S^-} \uparrow_{T_S<\infty}} \mid \mathcal{F}_S^\Lambda \right] = \mathbb{E} \left[ Z_{\bar{T}_S \uparrow_{H_S^-} \uparrow_{T_S<\infty}} \mid \mathcal{F}_S^\Lambda \right]$$

(16)

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Moreover, we get by Lemma 3.4 (ii) that $^*Z_\infty \leq \mathcal{P}Z_\infty = 0$, which implies by (16)

$$\mathbb{E}\left[^*Z_{T_S}^{\mathbb{1}_{H_S^+}} \mid \mathcal{F}_S^\Lambda\right] \leq \mathbb{E}\left[Z_{T_S}^{\mathbb{1}_{H_S^+}} \mid \mathcal{F}_S^\Lambda\right].$$

(17)

Combining this result with (8), the pathwise upper-semicontinuity from the right of $Z$ and $Z^*_\infty = 0$, we obtain

$$\bar{Z}_S^{(8)} \leq \mathbb{E}\left[^*Z_{T_S}^{\mathbb{1}_{H_S^+}} + Z_{T_S}^{\mathbb{1}_{H_S^+}} + Z^*_{T_S}^{\mathbb{1}_{H_S^+}} \mid \mathcal{F}_S^\Lambda\right] \leq \mathbb{E}\left[Z_{T_S} \mid \mathcal{F}_S\right] \leq \mathbb{E}\left[\bar{Z}_{T_S} \mid \mathcal{F}_S\right] \leq \bar{Z}_S.$$

Here, we have used in the last step that $\bar{Z}$ is a supermartingale and $T_S$ is a $\Lambda$-stopping time as $\Lambda = \emptyset$. This shows $\mathbb{E}[Z_{T_S} \mid \mathcal{F}_S^\Lambda] = \mathbb{E}[\bar{Z}_{T_S} \mid \mathcal{F}_S^\Lambda] = \bar{Z}_S$, which in particular implies, by $Z \leq \bar{Z}$, that $Z_{T_S} = \bar{Z}_{T_S}$ almost surely and hence $T_S = T_S$. □

### 3.4 Limit results for Meyer projections

Assume that $\Lambda$ is embedded in the sense of (12). The next result can be viewed as a version of Fatou’s lemma for Meyer projections.

**Proposition 3.7.** Assume that $\Lambda$ is embedded in the sense of (12). Let $Z$ be an $(\mathcal{F} \otimes \mathcal{B}([0, \infty)))$-measurable process of class $(D^\Lambda)$ with $Z_\infty = 0$. Then the following two assertions hold true:

(i) For an arbitrary $\mathcal{F}$-stopping time $T$, we have

$$^\emptyset(Z_*)_T \leq (\Lambda Z)_T^* \leq (\Lambda Z)_T \leq ^\emptyset(Z_*)_T,$$

where $^\emptyset$ is the optional-$\sigma$-field with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$. In particular, if $Z$ has right-limits then we get

$$(\Lambda Z)_T^* = ^\emptyset(Z_*)_T.$$

(ii) For an arbitrary predictable $\mathcal{F}$-stopping time $T$, we have

$$\mathcal{P}(Z)_T \leq ^* (\Lambda Z)_T \leq (\Lambda Z)_T \leq ^\mathcal{P}(Z)_T,$$

where $\mathcal{P}$ denotes the predictable-$\sigma$-field with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$. In particular, if $Z$ has left-limits then we get

$$(\Lambda Z)_T^- = \mathcal{P}(Z)_T^-.$$

**Remark 3.8.** One should remark that the above results for example do not imply that $\{ (\Lambda Z)^* > ^\emptyset(Z^*)\}$ is an evanescent set as one can not apply in general the Meyer section theorem to $(\Lambda Z)^*$. 

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Proof. Part (i): Let $T$ be an arbitrary $\mathcal{F}$-stopping time. Then there exists by Proposition 3.2 (i) a sequence $(T_n)_{n \in \mathbb{N}} \subset \mathcal{F}_T^A$ such that $T_n \geq T$, $T < T_n < \infty$ on \{\(T < \infty\}\}, \(\lim_{n \to \infty} T_n = T\) and \((^\Lambda Z)^*_T = \lim_{n \to \infty} (^\Lambda Z)_{T_n}\). Hence, we get on \(\{T < \infty\}\)
\[
(^\Lambda Z)^*_T = \lim_{n \to \infty} (^\Lambda Z)_{T_n} = \lim_{n \to \infty} \mathbb{E} \left[ Z_{T_n} \mid \mathcal{F}_{T_n}^A \right],
\]
where \((^\Lambda Z)^*_T\) is $\mathcal{F}_T$-measurable. Indeed, by Dellacherie and Meyer [1978], Theorem 90, p.143 the process \((^\Lambda Z)^*_T\) is $\mathcal{F}$-progressively measurable and therefore by Dellacherie and Meyer [1978], Theorem 64, p.122 we have \((^\Lambda Z)^*_T\) is $\mathcal{F}_T$-measurable. On the other hand we have on \(\{T < \infty\}\) that
\[
\theta(Z^*_T) = \mathbb{E} \left[ Z^*_T \mid \mathcal{F}_T \right] = \mathbb{E} \left[ Z^*_T \mid \mathcal{F}_{T^*_T}^A \right],
\]
where we have used that $\mathcal{F}_T = \mathcal{F}_{T^*_T}^A$ by Lemma 3.1. For $A \in \mathcal{F}_T = \mathcal{F}_{T^*_T}^A \subset \mathcal{F}_{T^*_T}^A$, we can now use Fatou’s Lemma and that $Z$ is of class($D^A$) to conclude
\[
\mathbb{E} \left[ \lim_{n \to \infty} \mathbb{E} \left[ Z_{T_n} \mid \mathcal{F}_{T_n}^A \right] 1_A \right] = \lim_{n \to \infty} \mathbb{E} \left[ \mathbb{E} \left[ Z_{T_n} \mid \mathcal{F}_{T_n}^A \right] 1_A \right] = \lim_{n \to \infty} \mathbb{E} \left[ Z_{T_n} 1_A \right] \leq \mathbb{E} \left[ Z_T 1_A \right].
\]
As $A \in \mathcal{F}_T = \mathcal{F}_{T^*_T}^A$ was arbitrary, equation (20) leads to
\[
\lim_{n \to \infty} \mathbb{E} \left[ Z_{T_n} \mid \mathcal{F}_{T_n}^A \right] \leq \mathbb{E} \left[ Z_T \mid \mathcal{F}_{T^*_T}^A \right],
\]
which proves \(\theta(Z^*_T) \leq \theta(Z^*_T)\) by (18) and (19). Analogously we get that \(\theta(Z_n) \leq \theta(Z_{T_n})\) by using Fatou’s Lemma into the other direction and Remark 3.3 (ii).

Part (ii): Let $T$ be an arbitrary predictable $\mathcal{F}$-stopping time, where we assume $T > 0$ as we could replace $T$ by $T_{(T > 0)}$ as we defined \(Z_0 = \ast Z_0 = 0\). Then there exists by Proposition 3.2 a sequence $(T_n)_{n \in \mathbb{N}} \subset \mathcal{F}_T^A$ such that $T_n < T$, $\lim_{n \to \infty} T_n = T$ and \(\ast(^\Lambda Z)_T = \lim_{n \to \infty} (^\Lambda Z)_{T_n}\). Hence, we get
\[
\ast(^\Lambda Z)_T = \lim_{n \to \infty} (^\Lambda Z)_{T_n} = \lim_{n \to \infty} \mathbb{E} \left[ Z_{T_n} \mid \mathcal{F}_{T_n}^A \right].
\]
For $A \in \mathcal{F}_{T_m}^A$, with $m \in \mathbb{N}$ fixed, we can use Fatou’s Lemma and that $Z$ is of class($D^A$) to obtain
\[
\mathbb{E} \left[ \ast(^\Lambda Z)_T 1_A \right] = \mathbb{E} \left[ \lim_{n \to \infty} \mathbb{E} \left[ Z_{T_n} \mid \mathcal{F}_{T_n}^A \right] 1_A \right] = \lim_{n \to \infty} \mathbb{E} \left[ \mathbb{E} \left[ Z_{T_n} \mid \mathcal{F}_{T_n}^A \right] 1_A \right] = \lim_{n \to \infty} \mathbb{E} \left[ Z_{T_n} 1_A \right] \leq \mathbb{E} \left[ \ast Z_T 1_A \right].
\]
As $A \in \mathcal{F}_{T_m}$ was arbitrary, equation (21) leads to
\[
\mathbb{E} \left[ \ast(^\Lambda Z)_T \mid \mathcal{F}_{T_m}^A \right] \leq \mathbb{E} \left[ \ast Z_T \mid \mathcal{F}_{T_m}^A \right].
\]
Now we get by Dellacherie and Meyer [1982], Theorem 31, p.26, for \( m \to \infty \) and \( \mathcal{F}^\Lambda_T = \mathcal{F}_T^- \) (see Lemma 3.1) that
\[
*(\Lambda Z)_T = \lim_{m \to \infty} \mathbb{E} \left[ *(\Lambda Z)_T | \mathcal{F}^\Lambda_{Tm} \right] \leq \lim_{m \to \infty} \mathbb{E} \left[ *Z_T | \mathcal{F}^\Lambda_{Tm} \right] = \mathbb{P}(Z)_T,
\]
which proves the first part of our claim. Here we have used for the identity that by Dellacherie and Meyer [1978], Theorem 89, p.143, \( *(\Lambda Z) \) is predictable and hence \( *(\Lambda Z)_T \) is \( \mathcal{F}_T^- \)-measurable. Analogously, we get the other direction by using Fatou’s Lemma for the limes inferior and by recalling Remark 3.3 (ii).

Let us give another limit theorem for Meyer projections and show that for a continuously parametrized family of stochastic processes, one can choose the corresponding Meyer projections so that they inherit this continuity.

**Lemma 3.9.** Let \((Z^\ell_t)_{t \geq 0}, \ell \in \mathbb{R}, \) be a family of processes such that for \( \mathbb{P}\)-almost every \( \omega \in \Omega \) we have locally uniform continuity with respect to \( \ell \) in the sense that
\[
\lim_{\delta \downarrow 0} \sup_{\ell, \ell' \in C} \sup_{t \in [0, \infty]} |Z^\ell_t(\omega) - Z^{\ell'}_t(\omega)| = 0 \quad \text{for any compact } C \subset \mathbb{R}. \tag{22}
\]
Assume in addition that, for any such \( C, \)
\[
\mathbb{E} \left[ \sup_{\ell, \ell' \in C} \sup_{t \in [0, \infty]} |Z^\ell_t(\omega) - Z^{\ell'}_t(\omega)| \right] < \infty. \tag{23}
\]
Then the family of \( \Lambda \)-projections \( \Lambda(Z^\ell_t)_{t \geq 0}, \ell \in \mathbb{R}, \) can be chosen such that for any \( \omega \in \Omega \) we also have local uniform continuity with respect to \( \ell \in \mathbb{R} \) for also for the family of \( \Lambda \)-projections, i.e.
\[
\lim_{\delta \downarrow 0} \sup_{\ell, \ell' \in C} \sup_{t \in [0, \infty]} \left| \Lambda(Z^\ell_t)(\omega) - \Lambda(Z^{\ell'}_t)(\omega) \right| = 0 \quad \text{for any compact } C \subset \mathbb{R}. \tag{24}
\]

**Proof.** Noting the arguments of Kiiski and Perkkiö [2017] only require Meyer’s section and projection theorem, which also hold for Meyer-\( \sigma \)-fields, we first obtain that the \( \Lambda \)-projections of \( Z^\ell, \ell \in \mathbb{R}, \) can be chosen such that
\[
\lim_{\delta \to 0} \sup_{\ell, \ell' \in C} \sup_{t \in [0, \infty]} \left| \Lambda(Z^\ell_t)(\omega) - \Lambda(Z^{\ell'}_t)(\omega) \right| = 0 \quad \text{for all } \ell \in \mathbb{R}, (\omega, t) \in \Omega \times [0, \infty]. \tag{24}
\]
To deduce uniform convergence as claimed, fix in the following a compact interval \( I \subset \mathbb{R} \) and observe that
\[
Z(\delta) := \sup_{\ell, \ell' \in I} \sup_{t \in [0, \infty]} |Z^\ell_t - Z^\ell'_t|, \quad \delta > 0,
\]
and observe that
converges to zero almost surely and in $L^1(\mathbb{P})$ as $\delta \downarrow 0$ because of (22) and (23). Furthermore for fixed $\ell, \ell' \in I$ with $|\ell - \ell'| \leq \delta$ and any $T \in \mathcal{F}^\Lambda$ we have
\[
\left| \Lambda(Z^{\ell'})_T - \Lambda(Z^\ell)_T \right| \leq \mathbb{E} \left[ \left| Z^{\ell'}_T - Z^\ell_T \right| \middle| \mathcal{F}^\Lambda_T \right] \\
\leq \mathbb{E} \left[ Z(\delta) \left| \mathcal{F}^\Lambda_T \right] = \Lambda Z(\delta)_T \text{ a.s. on } \{T < \infty\}.
\]
Hence, by the Meyer Section Theorem, $|\Lambda(Z^{\ell'})_t - \Lambda(Z^\ell)_t| \leq \Lambda Z(\delta)_t$ up to an evanescent set for any fixed $\ell, \ell' \in I$ with $|\ell - \ell'| \leq \delta$. Therefore, almost surely,
\[
\sup_{t \in [0, \infty]} \sup_{|\ell' - \ell| \leq \delta} \left| \Lambda(Z^{\ell'})_t - \Lambda(Z^\ell)_t \right| \leq \sup_{t \in [0, \infty]} \Lambda Z(\delta)_t.
\]
Interchanging its two suprema, the left-hand side can be rewritten as
\[
\sup_{t \in [0, \infty]} \sup_{|\ell' - \ell| \leq \delta} \left| \Lambda(Z^{\ell'})_t - \Lambda(Z^\ell)_t \right| = \sup_{t \in [0, \infty]} \sup_{|\ell' - \ell| \leq \delta} \left| \Lambda(Z^{\ell'})_t - \Lambda(Z^\ell)_t \right| = \sup_{t \in [0, \infty]} \sup_{|\ell' - \ell| \leq \delta} \left| \Lambda(Z^{\ell'})_t - \Lambda(Z^\ell)_t \right|
\]
where we used the pointwise continuity (24) in the last equality. Therefore we have almost surely
\[
\sup_{t \in [0, \infty]} \sup_{|\ell' - \ell| \leq \delta} \left| \Lambda(Z^{\ell'})_t - \Lambda(Z^\ell)_t \right| \leq \sup_{t \in [0, \infty]} \Lambda Z(\delta)_t,
\]
and now it suffices to argue that $\Lambda Z(\delta)_t \to 0$ almost surely uniformly in $t \in [0, \infty]$. For this, we use Doob's maximal martingale inequality, suitably generalized for $\Lambda$-martingales (Dellacherie and Meyer [1982], Appendix 1, (3.1), p.394). More precisely, for any $\lambda \in [0, \infty)$ we have by dominated convergence that
\[
\lambda \mathbb{P} \left( \sup_{t \in [0, \infty]} \Lambda Z(\delta)_t > \lambda \right) \leq \mathbb{E} \left[ \Lambda Z(\delta)_\infty \right] = \mathbb{E} \left[ Z(\delta) \right] \delta \lambda \to 0,
\]
which finishes our proof.

\section{A stochastic representation theorem and universal stopping signals}

In this section we want to study an optimal stopping problem over divided stopping times, where we try to find the optimal time to collect a terminal reward given by a $\Lambda$-measurable $(X_t)_{t \in [0, \infty)}$ when before one receives “running rewards” represented by some function $g(\ell)$. Specifically, we want to solve for any $\ell \in \mathbb{R}$ the optimal stopping problem with value
\[
\sup_{\tau \in \mathcal{\mathcal{F}}^\Lambda, \text{div}} \mathbb{E} \left[ X_\tau + \int_{(0, \tau)} g(\ell) \mu(dt) \right]. \tag{25}
\]
To make this precise, let us next make the following assumptions. As in the previous chapters, \( \Lambda \) is a \( \mathcal{F} \)-complete Meyer-\( \sigma \)-field and we denote by \( \mathcal{F} := (\mathcal{F}_t)_{t \geq 0} \) a filtration satisfying the usual conditions such that \( \mathcal{P}(\mathcal{F}) \subset \Lambda \subset \mathcal{O}(\mathcal{F}) \) (see Theorem 2.17). The \( \mu \) on \([0, \infty)\) and the random field \( g : \Omega \times [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} \) satisfy the following assumption:

**Assumption 4.1.**

(i) \( \mu \) is a random Borel-measure on \([0, \infty)\) with \( \mu(\{\infty\}) := 0 \).

(ii) The random field \( g : \Omega \times [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} \) satisfies:

(a) For each \( \omega \in \Omega, t \in [0, \infty) \), the function \( g_t(\omega, \cdot) : \mathbb{R} \rightarrow \mathbb{R} \) is continuous and strictly increasing from \(-\infty\) to \(\infty\).

(b) For each \( \ell \in \mathbb{R} \), the process \( g(\cdot, \ell) : \Omega \times [0, \infty) \rightarrow \mathbb{R} \) is \( \mathcal{F} \otimes \mathcal{B}([0, \infty)) \)-measurable with

\[
\mathbb{E}\left[\int_{[0, \infty)} |g_t(\ell)| \mu(dt)\right] < \infty.
\]

Moreover, the \( \Lambda \)-measurable process \( X \) is chosen in such a way that there exists a \( \Lambda \)-measurable process \( L \) satisfying

\[
X_S = \mathbb{E}\left[\int_{[S, \infty)} g_t \left( \sup_{v \in [S, t]} L_v \right) \mu(dt) \right| \mathcal{F}_S^\Lambda], \quad S \in \mathcal{S}^\Lambda,
\]

(26)

\[
\mathbb{E}\left[\int_{[S, \infty)} g_t \left( \sup_{v \in [S, t]} L_v \right) \mu(dt) \right| \mathcal{F}_S^\Lambda] < \infty \quad \text{for any } S \in \mathcal{S}^\Lambda.
\]

(27)

In the representation theorem of Bank and Besslich [2018a] one can find (rather mild) sufficient conditions on \( X \) such that the previous representation is possible. The optimal stopping problem (25) in the optional case, is discussed in Bank and Föllmer [2003], Theorem 2, p.6, albeit only for atomless \( \mu \) with full support. In this case a sufficient condition for a representation as in (26) was proven in Bank and El Karoui [2004]. Without these regularity properties, optimal stopping times can no longer be expected from a representation as in (26). But we still can describe optimal divided stopping times in terms of the representing process \( L \) and thus provide an optimal stopping characterization alternative to the Snell envelope approach of Theorem 2.39:

**Theorem 4.2.** Let \( \Lambda \) be a \( \mathcal{F} \)-complete Meyer-\( \sigma \)-field, \( g \) and \( \mu \) satisfy Assumption 4.1. Suppose furthermore that \( X \) is a \( \Lambda \)-measurable process of class(\( D^\Lambda \)), which is left-upper-semicontinuous in expectation at every \( S \in \mathcal{S}^\mathcal{P} \) (see Lemma 3.4 (ii)), and that \( X \) allows a representation by a \( \Lambda \)-measurable process \( L \) with (26) and (27).

Then \( X \) is right-upper-semicontinuous in expectation at every \( S \in \mathcal{S}^\Lambda \) (see Lemma 3.4 (i)) and \( L \) is a universal stopping signal for (25) in the sense that, for any \( \ell \in \mathbb{R} \), the divided stopping times

\[
\tau_{\ell}^{(i)} := (T_{\ell}^{(i)}, \emptyset, H_{\ell}^{(i)}, (H_{\ell}^{(i)})^c), \quad i = 1, 2,
\]

(28)
with
\[
H^{(1)}_\ell := \{ L_{T^{(1)}_\ell} \geq \ell \} \supset \{ L_{T^{(2)}_\ell} > \ell \} =: H^{(2)}_\ell
\]
\[
T^{(1)}_\ell := \inf \left\{ t \geq 0 \mid \sup_{v \in [0,t]} L_v \geq \ell \right\} \leq \inf \left\{ t \geq 0 \mid \sup_{v \in [0,t]} L_v > \ell \right\} =: T^{(2)}_\ell
\]
attain the supremum in (25).

Proof. We first show that \( X \) is right-upper-semicontinuous in expectation. For that we will adapt the proof of Bank and El Karoui [2004], Theorem 2, p.1048, to our stochastic setting. Define for arbitrary \( S \in \mathcal{S} \Lambda \) the process \( i^S \): \( \Omega \times [0, \infty) \to [0, \infty] \) by
\[
i^S_t(\omega) := \mathbf{1}_{[S, \infty]}(\omega, t) g_t \left( \omega, \sup_{v \in [S, \omega]} L_v(\omega) \right) \leq 0 \lor g_t \left( \omega, \sup_{v \in [S, t]} L_v(\omega) \right).
\] (29)

Consider now a sequence \( (S_n)_{n\in\mathbb{N}} \subset \mathcal{S} \Lambda ([S, \infty]) \) with \( \lim_{n \to \infty} \mu([S, S_n]) = 0 \) almost surely. Combining the estimate (29) with (27), we may use Fatou’s lemma to obtain from the representation property of \( L \) that
\[
\limsup_{n \to \infty} \mathbb{E}[X_{S_n}] = \limsup_{n \to \infty} \mathbb{E} \left[ \int_{[0, \infty)} i^S_n \mu(dt) \right]
\]
\[
\leq \mathbb{E} \left[ \limsup_{n \to \infty} \int_{[0, \infty)} g_t \left( \sup_{v \in [S, t]} L_v \right) \mathbf{1}_{[S_n, \infty)}(t) \mu(dt) \right]
\]
\[
= \mathbb{E} \left[ \int_{[S, \infty)} g_t \left( \sup_{v \in [S, t]} L_v \right) \mu(dt) \right] = \mathbb{E}[X_S].
\]
which we wanted to show.

The next arguments are inspired by Bank and Föllmer [2003], Theorem 2, p.6. As a first step we get by Lemma 3.4 (ii), that left-upper-semi-continuity in expectation is equivalent to \(^*X \leq \mathcal{P}X\) up to an evanescent set. Therefore, for any \( \tau = (T, H^-, H, H^+) \in \mathcal{F}^{\Lambda, \text{div}} \) the alternative divided stopping time \( \tilde{\tau} := (T, \emptyset, H^- \cup H, H^+) \) yields at least as high a value in (25) as \( \tau \) does. Here, \( \tilde{\tau} \) is indeed a divided stopping time as by \( \mathcal{P} \)-completeness of \( \Lambda \) and Theorem 2.17 there exists a filtration \( \mathcal{F} := (\mathcal{F}_t)_{t \geq 0} \) satisfying the usual conditions such that
\[
\mathcal{P}(\mathcal{F}) \subset \Lambda \subset \mathcal{O}(\mathcal{F}).
\]

By Lemma 3.1 we then have \( \mathcal{P}(\mathcal{F}^{\Lambda}_+) = \mathcal{P}(\mathcal{F}) \subset \Lambda \), which implies that the predictable \( \mathcal{F}^{\Lambda}_+\)-stopping time \( T_{H^-} \) is also a \( \Lambda \)-stopping time. This shows that \( \tilde{\tau} \in \mathcal{F}^{\Lambda, \text{div}} \) as the other parts of the definition of a divided stopping times are satisfied by having \( \tau \in \mathcal{F}^{\Lambda, \text{div}} \). Hence, we can confine ourselves to considering divided stopping times with \( H^- = \emptyset \).
Next, we show that $\tau^{(1)}_\ell$ is a divided stopping time. To this end note that by \{sup$_{v \in [0,\cdot)} L_v < \ell$\} $\in \mathcal{F}_+$, the set $A_\ell := \{L \geq \ell\} \cap \{\text{sup}_{v \in [0,\cdot)} L_v < \ell\}$ is $\Lambda$-measurable and satisfies for $\omega \in \Omega$ that

$$\langle T_\ell \rangle_{\{L^{(1)}_{\tau^{(1)}_\ell} \geq \ell\}}(\omega) = \inf\{t \geq 0 | (\omega, t) \in A_\ell\}.$$ 

From Dellacherie and Meyer [1978], Theorem 50, p.116, we obtain that $T^{(1)}_\ell$ is a $\mathcal{F}$-stopping time and by Lenglart [1980], Theorem 2, p.503, it is a stability time. As $A_\ell$ contains the graph of $(T^{(1)}_\ell)_{\{L^{(1)}_{\tau^{(1)}_\ell} \geq \ell\}}$ we get by an application of Lenglart [1980], Corollary 2, p.504, that $(T^{(1)}_\ell)_{\{L^{(1)}_{\tau^{(1)}_\ell} \geq \ell\}}$ is a $\Lambda$-stopping time. It follows that $\tau^{(1)}_\ell$ is indeed a divided stopping time. One proves analogously that $\tau^{(2)}_\ell \in \mathcal{F}_{\Lambda, \text{div}}$.

Now it remains to prove that $\tau^{(i)}_\ell, i = 1, 2$, attains the supremum in (25). Let us first argue why the representation property (26) allows us to conclude that for any $\tau = (T, \emptyset, H, H^c) \in \mathcal{F}_{\Lambda, \text{div}}$ we have

$$\mathbb{E} \left[ X_\tau + \int_{(0,\tau]} g_\ell(t) \mu(dt) \right] \leq \mathbb{E} \left[ 1_{H} \int_{(T,\infty)} g_\ell \left( \sup_{v \in [T,t]} L_v \right) - g_\ell(\ell) \right] \mu(dt) + \mathbb{E} \left[ 1_{H^c} \int_{(T,\infty)} g_\ell \left( \sup_{v \in [T,t]} L_v \right) - g_\ell(\ell) \right] \mu(dt) + C \quad (30)$$

with $C := \mathbb{E} \left[ \int_{(0,\infty)} g_\ell(t) \mu(dt) \right]$. Indeed, first we obtain by (26) and because $H \in \mathcal{F}_T$ with $T_H \in \mathcal{F}_{\Lambda}$ that

$$\mathbb{E} \left[ 1_{H} X_T \right] = \mathbb{E} \left[ 1_{H} \int_{(T,\infty)} g_\ell(t) \left( \sup_{v \in [T,t]} L_v \right) \mu(dt) \right].$$

Second, we obtain by Proposition 3.2 (i), that there exists a sequence of $\Lambda$-stopping times $(T_n)_{n \in \mathbb{N}}$ such that $T_n \geq T_{H^c}, \infty > T_n > T_{H^c} \in \{T_{H^c} < \infty\}, \lim_{n \to \infty} T_n = T_{H^c}$ and $X^{+}_{T_{H^c}} = \lim_{n \to \infty} X_{T_n}$. Hence, again by (26) and using that $X$ is of class$(D^\Lambda)$ with $X_\infty = X^{+} = 0$, we get by Fatou’s lemma that

$$\mathbb{E} \left[ 1_{H} X_T^{+} \right] = \lim_{n \to \infty} \mathbb{E} \left[ 1_{H} X_{T_n} \right] \leq \mathbb{E} \left[ 1_{H} \int_{(T,\infty)} g_\ell \left( \sup_{v \in [T,t]} L_v \right) \mu(dt) \right], \quad (31)$$

which finishes the proof of (30).

Choose $i \in \{1, 2\}$ arbitrary. Now, one can use the monotonicity of $g$ to check
that our divided level passage time \( \tau^{(i)}_\ell \) of (28) satisfies

\[
\mathbb{1}_H \int_{[T,\infty)} \left( g_t \left( \sup_{v \in [T,t]} L_v \right) - g_t(\ell) \right) \mu(dt)
+ \mathbb{1}_{H_c} \int_{[T,\infty)} \left( g_t \left( \sup_{v \in [T,t]} L_v \right) - g_t(\ell) \right) \mu(dt)
\]

\[
\leq \mathbb{1}_H \int_{[T,\infty)} \left( g_t \left( \sup_{v \in [T,t]} L_v \right) - g_t(\ell) \right) \lor 0 \mu(dt)
+ \mathbb{1}_{H_c} \int_{[T,\infty)} \left( g_t \left( \sup_{v \in [T,t]} L_v \right) - g_t(\ell) \right) \lor 0 \mu(dt)
\]

\[
\leq \mathbb{1}_{H^{(i)}} \int_{[T^{(i)},\infty)} \left( g_t \left( \sup_{v \in [T^{(i)},t]} L_v \right) - g_t(\ell) \right) \lor 0 \mu(dt)
+ \mathbb{1}_{(H^{(i)})^c} \int_{[T^{(i)},\infty)} \left( g_t \left( \sup_{v \in [T^{(i)},t]} L_v \right) - g_t(\ell) \right) \lor 0 \mu(dt)
\]

with equality in both steps for \( \tau \) replaced by \( \tau^{(i)}_\ell \). For optimality of \( \tau^{(i)}_\ell \), it thus suffices to show that we also have equality in (30) for \( \tau = \tau^{(i)}_\ell \). Recall that the inequality there is due to the application of Fatou’s Lemma in (31). For \( \tau = \tau^{(i)}_\ell \), the integrand is bounded from below by \( g(\ell) \) and from above by \( g(\sup_{v \in [0,\cdot]} L_v) \), which justifies the application of dominated convergence by assumptions on \( g \) and (27). This finishes our proof.

\[\square\]

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