Abstract

To pave the way for the journey from geometry to conformal field theory (CFT), these notes present the background for some basic CFT constructions from Calabi-Yau geometry. Topics include the complex and Kähler geometry of Calabi-Yau manifolds and their classification in low dimensions. I furthermore discuss CFT constructions for the simplest known examples that are based in Calabi-Yau geometry, namely for the toroidal superconformal field theories and their $\mathbb{Z}_2$-orbifolds. En route from geometry to CFT, I offer a discussion of K3 surfaces as the simplest class of Calabi-Yau manifolds where non-linear sigma model constructions bear mysteries to the very day. The elliptic genus in CFT and in geometry is recalled as an instructional piece of evidence in favor of a deep connection between geometry and conformal field theory.

Introduction

These lecture notes aim to make a contribution to paving the way from geometry to conformal field theory.

While two-dimensional conformal field theories can be defined abstractly in mathematics without reference to the algebraic geometry of complex manifolds of dimension greater than one, some of the most interesting examples are expected to arise geometrically from Calabi-Yau manifolds of complex dimension two or higher. Indeed, Calabi-Yau manifolds are the candidates for consistent geometric backgrounds for superstrings. Roughly, the string dynamics in such a geometric background are governed by so-called NON-LINEAR SIGMA MODELS, whose equations of motion imply conformal invariance. The
resulting quantum field theory on the world-sheet is thus a superconformal
field theory. From a mathematical point of view, this looks encouraging,
since conformal field theory allows an independent mathematical approach,
while string theory as a whole is not comprehensible to a mathematically
satisfactory degree, as yet. However, non-linear sigma model constructions
are far from well understood, mathematically. The only examples of (com-
 pact) Calabi-Yau manifolds where explicit constructions of a non-linear sigma
model are known are the complex tori and their orbifolds. It seems that we
are jumping out of the frying pan into the fire: By focusing on conformal
field theory instead of string theory, we trade one mathematically dissatisfac-
tory approach by a concept whose abstract definition prevents it from being
immediately applied. Though string theory in its full generality seems to
allow non-geometric phases, it is still crucially connected to geometry. To
advance the subject, I am convinced that it is a Sine qua non to get a better
understanding of the precise relation between geometry and conformal field
theory, beginning with those cases where constructions are explicitly known.

These lecture notes should be viewed as an invitation to this journey from
geometry to conformal field theory. They give a lightning introduction to the
subject, as do many other excellent sources, so I try to keep the exposition
somewhat complementary to existing works. As key examples, which are
both sufficiently simple and mysterious, K3 surfaces play a special role en
route from geometry to conformal field theory.

These notes are structured as follows:

Section 1 is devoted to some background in Calabi-Yau geometry. In Sec-

cion 1.1, I recall basic mathematical concepts leading to the definition
of Calabi-Yau manifolds, and some of their fundamental properties. Section
1.2 introduces a number of topological invariants of Calabi-Yau manifolds
and culminates in a summary of mathematical arguments that yield the
classification of Calabi-Yau manifolds of complex dimension \( D \leq 2 \). The
classification naturally leads to the definition of K3 surfaces. An important
topic whose discussion would naturally follow is the structure of the moduli
spaces of complex structures, Kähler structures and Hyperkähler structures
on K3 surfaces. However, since these topics have already been discussed
elsewhere, even by myself [NW01, Wen07], I omit them in these notes. In-
stead, Section 1.3 gives a detailed summary of the Kummer construction as
a classical example of a geometric orbifold procedure. In particular, I argue
that the Kummer surfaces constitute a class of K3 surfaces which are very well understood, because their geometric properties are entirely governed by much simpler manifolds, namely the underlying complex two-tori.

Section 2 discusses explicit examples of superconformal field theories that are obtained from non-linear sigma model constructions. Since I have done so elsewhere, in these notes I do not offer a proposal for the defining mathematical properties of conformal field theory. Instead, the recent review [Wen15] should be viewed as the conformal field theory companion to these lecture notes, while the present exposition, in turn, can be understood as the geometric companion of [Wen15]. Concretely, Section 2.1 gives a brief review of toroidal superconformal field theories. Lifting the Kummer construction of Section 1.3 to the level of conformal field theory, Section 2.2 addresses $\mathbb{Z}_2$-orbifold constructions of toroidal superconformal field theories. To substantiate the expectation that these orbifold conformal field theories are correctly interpreted as non-linear sigma models on Kummer surfaces, I include a brief discussion of elliptic genera: I introduce the conformal field theoretic elliptic genus as a counterpart of the geometric elliptic genus. I then show that the geometric elliptic genus of K3 surfaces agrees with the conformal field theoretic elliptic genus of the $\mathbb{Z}_2$-orbifold conformal field theory obtained from a toroidal superconformal field theory on a complex two-torus, recalling the known proof [EOTY89].

The final Section 3 places K3 en route from geometry to conformal field theory: I motivate and discuss the definition of K3 theories, which is formulated purely within representation theory. While this may be mathematically satisfying, it entails a consideration of K3 theories against non-linear sigma models on K3, for which explicit direct constructions on smooth K3 surfaces are lacking. I recall the role of the chiral de Rham complex in the context of elliptic genera of Calabi-Yau manifolds and conformal field theories, respectively. I conclude with a few speculations on the vertex algebra which can be obtained from the cohomology of the chiral de Rham complex, as candidate for a recovery of some of the conformal field theory structure from purely geometric ingredients.

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1 Calabi-Yau geometry

By definition, a **Calabi-Yau manifold** is a compact Kähler manifold with trivial canonical bundle. This section gives an introduction to the mathematical ingredients of this definition and illustrates its consequences.

Examples of Calabi-Yau manifolds are the complex tori, which also furnish the only examples where a non-linear sigma model construction immediately yields an associated conformal field theory (see Section 2.1). In Section 1.2 we will see that in complex dimension one and two, apart from tori, the only Calabi-Yau manifolds are the **K3 surfaces**. The final subsection is devoted to the **Kummer construction**, yielding a special class of K3 surfaces which are almost as well under control from a conformal field theoretic point of view as are the complex tori (see Section 2.2).

There are a number of excellent books on the topics presented in this section, see for example [BHPvdV04, GH78, Huy05, Joy00, Wel73]. In particular, if not stated otherwise, then proofs of the classical results below can be found in these references.

1.1 The Calabi-Yau condition

In the following, familiarity with the concepts of differential geometry and complex analysis is assumed. The definition of topological and Riemannian manifolds over $\mathbb{R}$ and of vector bundles is crucial and can be found in textbooks like [dC92, Lee09, Mor01, O'N83]. The concept of holomorphic

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1By **compact**, I mean bounded and closed; one also frequently finds definitions of non-compact Calabi-Yau manifolds, but those will not be of interest in these lectures.
functions and their special properties is the foundation of complex analysis, where good textbooks include [FB05, Lan99, Rem91].

Recall that a $D$-DIMENSIONAL COMPLEX MANIFOLD is a differentiable real $2D$-dimensional manifold $Y$ together with a holomorphic atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$, i.e. an open covering $\{U_\alpha \mid \alpha \in A\}$ of $Y$ with diffeomorphisms $\varphi_\alpha: U_\alpha \rightarrow V_\alpha$, $V_\alpha \subset \mathbb{C}^D$ open for all $\alpha \in A$, such that all coordinate changes $\varphi_\alpha \circ \varphi_\beta^{-1}: \varphi_\beta(U_\alpha \cap U_\beta) \rightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$ are holomorphic. Such a holomorphic atlas is said to define a COMPLEX STRUCTURE on the real manifold $Y$. There is an alternative description, which sometimes is more convenient, and which is motivated by the following observations:

Consider $y \in U_\alpha$ and note that due to $\varphi_\alpha(U_\alpha) = V_\alpha \subset \mathbb{C}^D$, for the tangent space $T_{\varphi_\alpha(y)}V_\alpha$ we have a natural identification $\mathbb{C}^D = T_{\varphi_\alpha(y)}V_\alpha$, such that multiplication by $i = \sqrt{-1}$ on $\mathbb{C}^D$ yields an endomorphism of $T_{\varphi_\alpha(y)}V_\alpha$. On the tangent space $T_yY$ this induces an endomorphism $I \in \operatorname{End}(T_yY)$ with $D\varphi_\alpha(Iv) = iD\varphi_\alpha(v) \in \mathbb{C}^D$ for all $v \in T_yY$. The endomorphism $I$ is checked to be independent of the choice of (holomorphic!) coordinates. Its $\mathbb{C}$-linear extension to the complexification $T^C Y := TY \otimes_\mathbb{R} \mathbb{C}$ of the tangent bundle $TY$ is also denoted by $I$. By construction, $I^2 = -I$, such that we have an eigenspace decomposition $T^C Y = T^{1,0}Y \oplus T^{0,1}Y$ for $I$, where $I$ acts on the fibers of $T^{1,0}Y$ by multiplication by $i$, and on those of $T^{0,1}Y$ by multiplication by $-i$. In fact, $T^{1,0}Y$ is a holomorphic vector bundle whose cocycles can be given by the Jacobians of the coordinate changes in any holomorphic atlas of $Y$, and $T^{0,1}Y = \overline{T^{1,0}Y}$ with the complex conjugation $v \otimes \lambda := v \otimes \overline{\lambda}$ for $y \in Y$, $v \in T_yY$, $\lambda \in \mathbb{C}$. The induced decomposition of the cotangent bundle yields a decomposition $d = \partial + \overline{\partial}$ of the exterior differential $d$, where on $\mathcal{A}^{p,q}(Y)$, the space of $(p,q)$-forms on $Y$, we have

$$\partial: \mathcal{A}^{p,q}(Y) \rightarrow \mathcal{A}^{p+1,q}(Y) \quad \text{and} \quad \overline{\partial}: \mathcal{A}^{p,q}(Y) \rightarrow \mathcal{A}^{p,q+1}(Y).$$

Since $T^{1,0}Y$ is a holomorphic vector bundle, it is closed under the Lie bracket (see e.g. [ON83, §1 and App. B] for a discussion of the Lie bracket).

Vice versa, on a differentiable real $2D$-dimensional manifold $Y$, a fiberwise endomorphism $I \in \operatorname{End}(TY)$ with $I^2 = -I$ is called an ALMOST COMPLEX STRUCTURE. Given an almost complex structure, the fiber-wise eigenspace decomposition $T^C Y = T^{1,0}Y \oplus T^{0,1}Y$ for $I$ holds as above, with complex vector bundles $T^C Y$, $T^{1,0}Y$ and $T^{0,1}Y$. The almost complex structure $I$ is called INTEGRABLE if the bundle $T^{1,0}Y$ closes under the Lie bracket. This condition is equivalent to $d = \partial + \overline{\partial}$ on each $\mathcal{A}^{p,q}(Y)$ as above, where
on $\mathcal{A}^{p,q}(Y)$, the operators $\partial, \bar{\partial}$ are obtained by composing the operator $d$ with the projector to $\mathcal{A}^{p+1,q}(Y)$ and $\mathcal{A}^{p,q+1}(Y)$, respectively. By the celebrated Newlander-Nirenberg theorem [NN75], every integrable almost complex structure on $Y$ is induced by a unique complex structure.

In summary, the choice of a complex structure on $Y$ is tantamount to the choice of an endomorphism $I$ which defines an integrable almost complex structure on $Y$.

Next, Riemannian geometry comes into play: We choose a Riemannian metric $g$ on our complex manifold $Y$, viewed as a real manifold, and discuss meaningful additional compatibility conditions. Let $g$ also denote the sesquilinear continuation of $g$ to $T^C Y$, where I use the convention

\[ \forall y \in Y, \ u, v \in T^C_y Y, \ \lambda, \mu \in \mathbb{C}: \ g(\lambda u, \mu v) = \lambda \mu g(u, v). \]

With respect to local complex coordinates $(z^1, \ldots, z^D)$ on $U \subset Y$, and restricting to real tangent vectors,

\[ g|_{TU \times TU} = \frac{1}{2} \sum_{j,k} \left( g_{j,k} dz^j \otimes d\bar{z}^k + g_{k,j} d\bar{z}^j \otimes dz^k + g_{j,k} dz^j \otimes d\bar{z}^k + g_{k,j} d\bar{z}^j \otimes dz^k \right), \]

where $g_{j,k} := 2g \left( \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k} \right)$, $g_{j,k} := 2g \left( \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial \bar{z}^k} \right)$, with a Hermitean matrix $(g_{j,k})_{j,k \in \{1, \ldots, D\}}$. The complex structure on $Y$ is said to be compatible with the metric $g$, if the corresponding almost complex structure $I \in \text{End}(TY)$ is orthogonal with respect to $g$, in other words if

\[ \forall y \in Y, \ u, v \in T^*_y Y: \ g(Iu, Iv) = g(u, v). \]

Then $g_{j,k} = 0$ for all $j, k \in \{1, \ldots, D\}$, so

\[ g|_{TU \times TU} = \frac{1}{2} \sum_{j,k} \left( g_{j,k} dz^j \otimes d\bar{z}^k + g_{k,j} d\bar{z}^j \otimes dz^k \right). \]

One checks directly that this compatibility condition implies that with respect to $g$, we have $T^{1,0}_y Y \perp T^{0,1}_y Y$ for all $y \in Y$.

If $I$ is compatible with $g$, then the $\mathbb{C}$-linear extension $\omega$ of the bilinear form

\[ \forall y \in Y, \ u, v \in T^*_y Y: \ \omega(u, v) := g(Iu, v) \]
to $T^C Y$ is called the Kähler form. By construction, $\omega$ is real, and by a direct calculation one checks that compatibility of $g$ with $I$ implies that $\omega$ is antisymmetric. Denoting by $A^k_\mathbb{R}(Y)$ the space of real $k$-forms on $Y$, and with respect to local complex coordinates $(z^1, \ldots, z^D)$ on $U \subset Y$ as above, we have

$$\omega \in A^{1,1}(Y) \cap A^2_\mathbb{R}(Y), \quad \omega|_U = \frac{i}{2} \sum_{j,k} g_{j,k} dz^j \wedge d\bar{z}^k.$$ 

It is important to keep in mind that $g$ is extended to $T^C Y$ as a Hermitean sesquilinear form, while $\omega$ is antisymmetric and $\mathbb{C}$-bilinear on $T^C Y$. Note that any two ingredients of the triple $(I, g, \omega)$ determine the third one.

The following additional condition on the metric turns out to have far-reaching consequences [Käh33]:

**Definition 1.1.1** Consider a Riemannian manifold $(Y, g)$, equipped with a compatible complex structure. Then the metric $g$ is called Kähler metric if and only if the Kähler form $\omega$ is a closed differential form: $d\omega = 0$.

A complex manifold $Y$ is called Kähler manifold if a Kähler metric exists on $Y$.

A harbinger of the fact that this condition is a very natural and interesting one is the following lemma, which states as many as six equivalent formulations of the Kähler condition, each emphasizing a different geometric aspect:

**Lemma 1.1.2** Consider a Riemannian manifold $(Y, g)$ with compatible complex structure $I$, such that $Y$ is a complex $D$-dimensional manifold and $\omega$ its Kähler form. Then the following are equivalent:

1. The metric $g$ is a Kähler metric.

2. For every $y \in Y$, there is an open neighborhood $U \subset Y$ of $y$ and a smooth function $f : U \to \mathbb{R}$, such that $\omega|_U = i\partial\bar{\partial}f$.

3. The Levi-Civita connection for $g$, that is, the unique torsion-free metric connection, agrees with the Chern (or holomorphic) connection for $I$, i.e., with the unique metric connection whose $(0,1)$ part is $\bar{\partial}$.

4. The almost complex structure $I$ of $Y$ is parallel with respect to the Levi-Civita connection for $g$. 
5. With respect to arbitrary holomorphic coordinates \((z^1, \ldots, z^D)\) on \(U \subset Y\), the coefficients of \(\omega | _U = \frac{i}{2} \sum_{j,k} g_{j,k} dz^j \wedge d\overline{z}^k\) obey \(\frac{\partial g_{k,l}}{\partial z^m} = \frac{\partial g_{m,l}}{\partial z^k}\) for all \(k, l, m \in \{1, \ldots, D\}\), or equivalently \(\frac{\partial g_{k,l}}{\partial z^m} = \frac{\partial g_{k,m}}{\partial z^l}\) for all \(k, l, m \in \{1, \ldots, D\}\).

6. The metric osculates to second order with the standard Euclidean metric, that is, for every \(y \in Y\), there are holomorphic coordinates \((z^1, \ldots, z^D)\) around \(y\) such that \(z(y) = (z^1, \ldots, z^D)(y) = 0\) and

\[
g \left( \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k} \right) = \delta_{j,k} + \mathcal{O}(z^2) \quad \forall j, k \in \{1, \ldots, D\}.
\]

7. The holonomy representation of \(Y\) is unitary on each tangent space.

In short, Kähler metrics are very similar to the Euclidean metric.

The Kähler condition has a number of important consequences. For example, the standard Laplace operators on a Kähler manifold obey a very simple relation: \(\Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}\), where \(\Delta_d = d^* + d\), \(\Delta_{\partial} = \partial^* + \partial\), and \(\Delta_{\bar{\partial}} = \overline{\partial^*} + \overline{\partial}\). Here, \(d^*, \partial^*\) and \(\overline{\partial^*}\) can be viewed as the \(L^2\)-duals of \(d, \partial\) and \(\overline{\partial}\) if \(Y\) is compact. One furthermore has

**Lemma 1.1.3** Consider a compact complex Kähler manifold \(Y\), and denote its de Rham and Dolbeault cohomology groups, respectively, by

\[
H^k(Y, \mathbb{C}) = \ker(d: \mathcal{A}^k(Y) \rightarrow \mathcal{A}^{k+1}(Y)) / \text{im}(d: \mathcal{A}^{k-1}(Y) \rightarrow \mathcal{A}^k(Y)),
\]

\[
H^{p,q}(Y, \mathbb{C}) = \ker(\overline{\partial}: \mathcal{A}^{p,q}(Y) \rightarrow \mathcal{A}^{p,q+1}(Y)) / \text{im}(\overline{\partial}: \mathcal{A}^{p,q-1}(Y) \rightarrow \mathcal{A}^{p,q}(Y))
\]

with \(\mathcal{A}^k(Y)\) the space of \(k\)-forms on \(Y\). Then for all \(k, p, q \in \mathbb{N}\),

\[
H^k(Y, \mathbb{C}) = \bigoplus_{r+s=k} H^{r,s}(Y, \mathbb{C}), \quad H^{p,q}(Y, \mathbb{C}) = \overline{H^{q,p}(Y, \mathbb{C})}.
\]

Apart from the Euclidean spaces and complex tori, important examples of Kähler manifolds are the complex projective spaces \(\mathbb{P}^D\). Indeed, the FUBINI-STUDY METRIC defines a Kähler metric on \(\mathbb{P}^D\); with respect to homogeneous coordinates \((z^0: \cdots: z^D)\), its Kähler form on the standard coordinate neighborhood \(U_j := \{(z^0: \cdots: z^D) \in \mathbb{P}^D \mid z^j \neq 0\}\) is

\[
\omega | _{U_j} = \frac{i}{2\pi} \partial \overline{\partial} \log \left( \sum_{k=0}^D \frac{|z_k|^2}{z_j} \right). \quad (1.1.1)
\]
Since the restriction of a Kähler metric on a complex manifold $Y$ to a complex submanifold yields a Kähler metric on the submanifold, this implies that all closed algebraic manifolds are compact Kähler manifolds.

Finally, we are ready to state

**Definition 1.1.4** A manifold $Y$ is called **Calabi-Yau manifold** if $Y$ is a compact Kähler manifold with trivial canonical bundle $K_Y$.

Recall that the canonical bundle $K_Y$ of a complex $D$-manifold $Y$ is the determinant line bundle of the holomorphic cotangent bundle $\Omega_Y := (T^{1,0}Y)^*$, i.e. $K_Y = \Lambda^D \Omega_Y$. Hence a Kähler manifold is Calabi-Yau if and only if $h^{D,0}(Y) := \dim_\mathbb{C} H^{D,0}(Y, \mathbb{C}) = 1$, or equivalently if the holonomy representation of $Y$ is special unitary on each tangent space.

Examples of Calabi-Yau manifolds include all complex tori $\mathbb{C}^D/L$ ($L \subset \mathbb{C}^D$ a lattice of rank $2D$) and the degree $D + 2$ hypersurfaces in $\mathbb{P}^{D+1}$.

It is natural to investigate whether a given Calabi-Yau manifold $Y$ possesses any Kähler metrics with particularly appealing properties. Indeed, the search within Kähler classes turns out to be fruitful. Here, I make use of the fact that the Kähler form $\omega$ of a Kähler metric on $Y$ is a real, closed $(1,1)$-form. Thus, $\omega$ represents a class $[\omega] \in H^{1,1}(Y, \mathbb{C}) \cap H^2(Y, \mathbb{R})$, which in fact is non-zero if $Y$ is compact. Another Kähler metric on $Y$ is said to belong to the same Kähler class as $g$, if its Kähler form represents the same class $[\omega] \in H^{1,1}(Y, \mathbb{C}) \cap H^2(Y, \mathbb{R})$. One now has the seminal

**Theorem 1.1.5 (Calabi-Yau Theorem [Cal54, Yau78])** Consider a Calabi-Yau manifold $Y$ with Kähler metric $g$. Then there exists a unique Ricci-flat Kähler metric in the Kähler class of $g$.

The proof of this theorem is non-constructive [Yau78]; with the exception of complex tori, explicit forms of Ricci-flat Kähler metrics on smooth Calabi-Yau manifolds are not known.

### 1.2 Classifying Calabi-Yau manifolds

A classification of Calabi-Yau $D$-manifolds turns out to be possible in dimension $D \leq 2$, as I shall explain in this section. In the following, if not stated otherwise, let $Y$ denote a connected Calabi-Yau $D$-manifold. I first recall some basic topological invariants of $Y$. 


The theory of elliptic differential operators ensures that the cohomology of \( Y \) is finite dimensional, \( h^{p,q}(Y) := \dim_{\mathbb{C}} H^{p,q}(Y, \mathbb{C}) < \infty \) for all \( p, q \in \mathbb{N} \). These so-called Hodge numbers are topological invariants which enjoy a number of constraints for our connected Calabi-Yau \( D \)-manifold \( Y \):

\[
\begin{align*}
h^{0,0}(Y) &= h^{D,D}(Y) = h^{0,D}(Y) = 1, \\
h^{p,q}(Y) &= h^{q,p}(Y) = h^{D-p,D-q}(Y) \quad \forall p, q \in \{0, \ldots, D\}. \tag{1.2.1}
\end{align*}
\]

Here, \( h^{p,q}(Y) = h^{q,p}(Y) \) is an immediate consequence of Lemma 1.1.3, while \( h^{0,q}(Y) = h^{D-p,D-q}(Y) \) is a consequence of the Serre duality [Ser55]. \( h^{0,0}(Y) = h^{D,D}(Y) = 1 \) is true by assumption, since \( Y \) is compact and connected, while \( h^{D,0}(Y) = h^{0,D}(Y) = 1 \) follows from the triviality of the canonical bundle.

Using the Hodge numbers, one obtains the following classical topological invariants:

**Definition 1.2.1** For a compact Kähler manifold \( Y \) of dimension \( D \), the Euler characteristic \( \chi(Y) \), holomorphic Euler characteristic \( \chi(O_Y) \) and signature \( \sigma(Y) \) are defined by

\[
\begin{align*}
\chi(Y) &:= \sum_{p,q=0}^{D} (-1)^{p+q} h^{p,q}(Y), \\
\chi(O_Y) &:= \sum_{q=0}^{D} (-1)^q h^{0,q}(Y), \\
\sigma(Y) &:= \sum_{p,q=0}^{D} (-1)^q h^{p,q}(Y).
\end{align*}
\]

For any holomorphic vector bundle \( E \to Y \),

\[
\chi(E) := \sum_{q=0}^{D} (-1)^q \dim H^q(Y, E)
\]

is the holomorphic Euler characteristic of \( E \).

By (1.2.1), the signature \( \sigma(Y) \) vanishes if the complex dimension \( D \) of \( Y \) is odd; I have thus trivially extended the traditional definition of the signature of oriented compact manifolds whose real dimension is divisible by 4 to all compact complex Kähler manifolds.

We are now in a position to state a first classification result:

**Theorem 1.2.2** Every connected Calabi-Yau one-manifold is biholomorphic to a torus \( \mathbb{C}/L \) with \( L \subset \mathbb{C} \) a lattice of rank 2.
Proof (sketch):

Since $D = 1$, the constraints (1.2.1) already fix all Hodge numbers. Hence by Definition (1.2.1) we have $\chi(Y) = 0$. Therefore, the claim follows from the classification of compact Riemann surfaces. A discussion of this deep and fundamental classification result can be found in textbooks on differential topology, for example [For81, Hir76]. In particular, one needs to use the fact that $\chi(Y)$ agrees with the Euler-Poincaré characteristic of $Y$ known from topology. This in turn is deeply linked to the relation between de Rham cohomology, Čech cohomology and singular homology, which is explained in depth in [Mor01].

□

Another type of topological invariants are the characteristic classes, all of which can be traced back to Chern classes for our Calabi-Yau manifold $Y$. Recall that by definition, a complex vector bundle $E \to Y$ of rank $r$ has Chern classes $c_k(E) \in H^{2k}(Y, \mathbb{Z})$, $k \in \{0, \ldots, D\}$, which are uniquely determined by the following four conditions (a)-(d):

(a). $c_0(E) = [1]$.

(b). For the dual $\mathcal{O}(1) \to \mathbb{P}^D$ of the tautological line bundle, $c_1(\mathcal{O}(1))$ is the Kähler class of the Fubini-Study metric on $\mathbb{P}^D$ with Kähler form (1.1.1).

(c). For smooth $f: X \to Y$ and $k \in \{0, \ldots, D\}$, one has $c_k(f^*E) = f^*c_k(E)$.

(d). The total Chern class $c(E) := \sum_{k=0}^{D} c_k(E) = \sum_{k=0}^{r} c_k(E)$ for line bundles $L_1, \ldots, L_r$ on $Y$ obeys

$$c(L_1 \oplus \cdots \oplus L_r) = c(L_1) \wedge \cdots \wedge c(L_r).$$

It follows that the total Chern class of every trivial bundle is $[1] \in H^0(Y, \mathbb{Z})$. By convention, the Chern classes $c_k(Y) \in H^{2k}(Y, \mathbb{Z})$ of a complex manifold $Y$ are the Chern classes of its holomorphic tangent bundle $T := T^{1,0}Y$. One checks that $c_1(Y) = -c_1(K_Y)$, where as above, $K_Y$ is the canonical bundle of $Y$. Hence every Calabi-Yau manifold $Y$ has vanishing first Chern class $c_1(Y) = 0$. Vice versa, using the exponential sheaf sequence one finds: If $Y$ is a compact Kähler manifold with $c_1(Y) = 0$ and vanishing first Betti
number, \( b_1(Y) = \dim_{\mathbb{C}} H^1(Y, \mathbb{C}) = 0 \), then \( Y \) is Calabi-Yau\(^2\).

If \( E \to Y \) is a complex vector bundle of rank \( r \), then by the so-called **splitting principle**, we may work with formal Chern roots \( e_1, \ldots, e_r \), such that \( c(E) = \prod_{j=1}^r (1 + e_j) \), where the \( e_j \) are elements of a ring extension of \( H^*(Y, \mathbb{R}) \). It then follows that \( c_k(Y) = \sigma_k(e_1, \ldots, e_r) \) with the elementary symmetric polynomials \( \sigma_k \). More generally, for an analytic function \( f \) on a neighborhood of 0 in \( \mathbb{C}^r \), by \( f(e_1, \ldots, e_r) \) one denotes the power series expansion of \( f \) about the origin with insertions \( (e_1, \ldots, e_r) \). This allows the definition of further topological invariants:

**Definition 1.2.3** Consider a compact complex \( D \)-manifold \( Y \) with holomorphic tangent bundle \( T := T^{1,0}Y \), and a complex vector bundle \( E \to Y \) of rank \( r \). Let \( y_1, \ldots, y_D \) and \( e_1, \ldots, e_r \) denote the formal Chern roots, such that \( c(Y) = \prod_{j=1}^D (1 + y_j) \) and \( c(E) = \prod_{j=1}^r (1 + e_j) \).

Then the Todd genus of \( Y \) is given by

\[
Td(Y) := \prod_{j=1}^D \frac{y_j}{1 - \exp(-y_j)} = c_0(Y) + \frac{1}{2}c_1(Y) + \frac{1}{12}(c_1(Y)^2 + c_2(Y)) + \cdots
\]

The Chern character of the bundle \( E \) is

\[
ch(E) := \sum_{j=1}^r \exp(e_j) = r c_0(Y) + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \cdots
\]

On first sight, the topological invariants that are obtained from the Hodge numbers in Definition 1.2.1 and those that are obtained from the Chern classes in Definition 1.2.3 have very different flavors: While the former are integers, the latter are cohomology classes. However, evaluation of such cohomology classes on the fundamental cycle of \( Y \), denoted by \( \int_Y \), yields integers from integral cohomology classes. Here, it is understood that \( \int_Y \alpha = 0 \) if \( \alpha \in H^k(Y, \mathbb{Z}) \) with \( k \neq 2D \). The **Atiyah-Singer Index Theorem**\(^{[AS63]}\) governs the deep relationship between the two types of invariants; in the context of complex manifolds and holomorphic bundles, which is relevant to our discussion, a precursor of this theorem is the following seminal

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\(^2\)Instead of Definition 1.1.4, some authors give a more restrictive definition of Calabi-Yau manifolds: A complex \( D \)-manifold \( Y \) is then called Calabi-Yau, if and only if \( Y \) is a compact Kähler manifold with \( h^{1,0}(Y) = 0 \) and \( c_1(Y) = 0 \) or, equivalently, such that the holonomy representation on every tangent space is an irreducible representation of \( SU(D) \). Hence complex tori are not Calabi-Yau according to this definition, whereas I view them as the simplest examples of Calabi-Yau manifolds.
Theorem 1.2.4 (Hirzebruch-Riemann-Roch Formula \cite{Hir54})  
Let \( E \rightarrow Y \) denote a holomorphic vector bundle on a compact complex \( D \)-manifold \( Y \). With notations as in Definitions 1.2.1 and 1.2.3,  
\[
\chi(E) = \int_Y Td(Y)ch(E).
\]

To appreciate this theorem, recall that representatives of the Chern classes \( c_k(Y), c_k(E) \) can be obtained in terms of the curvature forms of arbitrary Hermitean metrics on \( Y \) and \( E \), respectively (see e.g. \cite{LM89, BGV92} for a detailed discussion). The theorem thus yields the topological invariant \( \chi(E) \) in terms of the integral of a local curvature expression. The essence of this theorem is the fact that \( \mathcal{D}E = \partial_E + \overline{\partial}_E \) is a Dirac operator on \( E = E^+ \oplus E^- \), whose index \( \dim \ker(\mathcal{D}E|_{E^+}) - \dim \ker(\mathcal{D}E|_{E^-}) = \chi(E) \) can therefore alternatively be calculated by heat kernel methods in terms of an integral over local curvature data.

To recover the invariants of Definition 1.2.1 for the holomorphic Euler characteristic \( \chi(\mathcal{O}_Y) \) one uses the trivial bundle \( E \), so \( \mathcal{D}E = \partial + \overline{\partial} \) and \( Td(Y)ch(E) = Td(Y) \) with expansion into Chern classes as in Definition 1.2.3. The usual Euler characteristic \( \chi(Y) \) arises for the virtual bundle \( E = E^+ - E^- \) with \( E^+ = \oplus_{p=0}^\infty \Lambda^p T^*, E^- = \oplus_{p=1}^\infty \Lambda^p T^* \) and \( T = T^{1,0} Y \) as before. With \( \chi(E) := \chi(E^+) - \chi(E^-) = \chi(Y) \), and with notations as in Definition 1.2.3,  
\[
\begin{align*}
\text{ch}(E) &= \text{ch}(E^+) - \text{ch}(E^-) = \prod_{j=1}^D (1 - \exp(-y_j)), \\
Td(Y)\text{ch}(E) &= \prod_{j=1}^D y_j = c_D(Y).
\end{align*}
\]

Thus the Hirzebruch-Riemann-Roch Formula 1.2.4 yields  
\[
\begin{align*}
\chi(\mathcal{O}_Y) &= \int_Y \left( c_0(Y) + \frac{1}{2} c_1(Y) + \frac{1}{12} (c_1(Y)^2 + c_2(Y)) + \cdots \right), \\
\chi(Y) &= \int_Y c(Y).
\end{align*}
\]

To obtain the signature \( \sigma(Y) \) from the Hirzebruch-Riemann-Roch Formula, one uses the same total bundle \( E^+ \oplus E^- = \oplus_p \Lambda^p T^* \) with different \( \mathbb{Z}_2 \)-grading.
\[ \oplus_p \Lambda^p T^* = \tilde{E}^+ \oplus \tilde{E}^- \quad \text{and} \quad E = \tilde{E}^+ - \tilde{E}^- . \]

There is another relation between the Hodge numbers of \( Y \) and its top Chern class, due to the interpretation of \( \chi(Y) \) in terms of the Poincaré-Hopf Index Theorem [Poi85, Hop26]. One combines this classical result from differential topology, which is discussed in textbooks like [GP74, Mor01], with the celebrated Weitzenböck formula from differential geometry, see e.g. the textbook [Jos95]. Indeed, if \( \chi(Y) \neq 0 \), then the Poincaré-Hopf Index Theorem implies that every holomorphic one-form on \( Y \) has at least one zero. On the other hand, using a Ricci-flat metric on \( Y \), which exists by the Calabi-Yau Theorem 1.1.5, the Weitzenböck formula implies that every holomorphic one-form on \( Y \) has constant norm. In other words:

**Lemma 1.2.5** If a Calabi-Yau D-manifold \( Y \) has non-vanishing Euler characteristic, \( \chi(Y) \neq 0 \), then \( h^{1,0}(Y) = 0 \).

We are now ready to classify Calabi-Yau two-manifolds; the resulting Theorem 1.2.8 was first proved in [Kod64], but with more recent results the proof can be simplified. First note

**Lemma 1.2.6** Let \( Y \) denote a connected Calabi-Yau two-manifold. Then \( Y \) has Hodge numbers

\[ h^{1,0}(Y) = 2, \quad h^{1,1}(Y) = 4 \quad \text{or} \quad h^{1,0}(Y) = 0, \quad h^{1,1}(Y) = 20. \]

This fixes all Hodge numbers of \( Y \).

**Proof (sketch):**

Since \( D = 2 \), using (1.2.1) it is clear that the values of \( h^{1,0}(Y) \) and \( h^{1,1}(Y) \) determine all the Hodge numbers of \( Y \).

Generalizing Lemma 1.2.5, if a holomorphic \( k \)-form on \( Y \) has a zero, then it vanishes identically on \( Y \) [Bog74]. This implies \( h^{1,0}(Y) \leq 2 \). Since \( Y \) is Kähler, such that \( h^{1,1}(Y) \neq 0 \), one deduces that \( \chi(Y) = 0 \) can only hold if \( h^{1,0}(Y) = 2, \quad h^{1,1}(Y) = 4 \).

On the other hand, if \( \chi(Y) \neq 0 \), then Lemma 1.2.5 implies \( h^{1,0}(Y) = 0 \), so by (1.2.1) we have \( h^{1,1}(Y) = \chi(Y) - 4 \). Moreover, by (1.2.1) and Definition 1.2.1 \( Y \) has holomorphic Euler characteristic \( \chi(\mathcal{O}_Y) = 2 \). Hence

\[ \chi(Y) \overset{1.2.2}{=} \int_Y c_2(Y) \overset{1.2.2}{=} c_1(Y)=0, \quad 12\chi(\mathcal{O}_Y) = 24 \]

14
and therefore $h^{1,1}(Y) = \chi(Y) - 4 = 20$. □

If $\chi(Y) = 0$ for a Calabi-Yau two-fold $Y$, then one can use the Calabi-Yau Theorem 1.1.5 and a result by Bochner and Yano [YB53] to prove that $Y$ is a torus $\mathbb{C}^2/L$ with a lattice $L \subset \mathbb{C}^2$ of rank 4. If $\chi(Y) \neq 0$, then Lemma 1.2.6 implies that $Y$ is a K3 surface, according to

**Definition 1.2.7** A connected, compact complex surface $Y$ with trivial canonical bundle and $b_1(Y) = \dim_{\mathbb{C}} H^1(Y, \mathbb{C}) = 0$ is called a K3 surface.

It was conjectured independently by Andreotti and Weil [Wei58] and proved by Siu [Siu83] that all K3 surfaces are Kähler and thus Calabi-Yau. Using Lemma 1.2.6, I have thus summarized a derivation of the first claim of

**Theorem 1.2.8** ([Kod64]) If $Y$ is a connected Calabi-Yau two-manifold, then $Y$ is either a complex two-torus or a K3 surface. Viewed as real four-manifolds, all complex two-tori are diffeomorphic to one another, and all K3 surfaces are diffeomorphic to one another.

In summary, one main ingredient to the proof of the classification result Theorem 1.2.8 is the Atiyah-Singer Index Theorem in the form (1.2.2), which for Calabi-Yau two-manifolds $Y$ implies $\chi(Y) = 12\chi(O_Y)$. For Calabi-Yau three-manifolds $Y$, the corresponding formula does not suffice to fix the topological type of $Y$. Indeed, the problem of classifying all Calabi-Yau three-manifolds is wide open – the naive observation that there are more independent Chern classes to keep under control is precisely the source of the problem.

### 1.3 The Kummer construction

In the previous section, I stated that all K3 surfaces are diffeomorphic to one and the same real four-manifold $X$ [Kod64]. The choice of a complex

---

3It is not clear to me who introduced the name “K3 surface”, and when. The standard explanation and first mention in writing that I am aware of is due to Weil, who in [Wei58], declares “Dans la seconde partie de mon rapport, il s'agit des variétés Kähleriennes dites K3, ainsi nommées en l'honneur de Kummer, Kodaira, Kähler et de la belle montagne K2 au Cachemire.” The explanation of course relies on the Kähler property for all K3 surfaces, which at the time was only conjectural. It probably also relies on the fact that among mountaineers, K2 is often understood as the most challenging summit to the very day, which however had been vanquished only a few years before Weil completed his report [Wei58], namely in 1954.
structure and Kähler class on $X$, of course, greatly influences its geometric properties, for example the symmetries of the K3 surface. The Kummer construction, which shall be discussed in the present section, amounts to a special choice of complex structure and (degenerate) Kähler class, governed by the geometry of an underlying complex two-torus $T_L$:

**Definition 1.3.1** Let $T_L = \mathbb{C}^2/L$ denote a complex two-torus, where $L \subset \mathbb{C}^2$ is a lattice of rank 4, and $T_L$ carries the complex structure and Kähler metric induced from the standard complex structure and Euclidean metric on $\mathbb{C}^2$. This Calabi-Yau two-manifold enjoys a biholomorphic isometry $\kappa \in \text{Aut}(T_L)$ of order 2 which is induced by $z \mapsto -z$ on $\mathbb{C}^2$.

The quotient $T/L_2$ of $T_L$ by the group $\{I, \kappa\} \cong \mathbb{Z}_2$ is called the singular Kummer surface with underlying torus $T_L$.

The singular Kummer surface $T_L/Z_2$ is indeed singular: We choose $\epsilon > 0$ and denote by $B_\epsilon(0) \subset \mathbb{C}^3$ the open ball of radius $\epsilon$ with respect to the standard Euclidean metric on $\mathbb{C}^3$. Then the map $u : \mathbb{C}^2 \longrightarrow \mathbb{C}^3$, $u(z^1, z^2) := ((z^1)^2, (z^2)^2, z^1 z^2)$
descends to an open neighborhood $U_\epsilon \subset T_L/Z_2$ of 0 in the singular Kummer surface, where it is also denoted $u$, such that $u$ bijectively maps $U_\epsilon$ to $u(U_\epsilon) = \{u = (u^1, u^2, u^3) \in B_\epsilon(0) \mid u^1 u^2 = (u^3)^2\}$.

The map $u$ is biholomorphic upon restriction to $U_\epsilon \setminus \{0\} \xrightarrow{u} u(U_\epsilon) \setminus \{0\}$.

The equation $u^1 u^2 = (u^3)^2$ of $u(U_\epsilon)$ in $B_\epsilon(0)$ immediately shows that $u(U_\epsilon)$ is a double cone with an isolated singularity at $u = 0$. Let us define the minimal resolution of this singularity:

**Definition 1.3.2** With notations as above, the point $0 \in U_\epsilon$, and equivalently its image $0 \in u(U_\epsilon)$, is called a singularity of type $A_1$. Let $K_\epsilon := u(U_\epsilon) \setminus \{0\}$, and with homogeneous coordinates $v = (v^1 : v^2 : v^3) \in \mathbb{P}^2$,

$$W_\epsilon := \{(u, v) \in K_\epsilon \times \mathbb{P}^2 \mid u^j v^k = u^k v^j \quad \forall j, k \in \{1, 2, 3\}\}.$$ 

Moreover, let $V_\epsilon$ denote the interior of the closure $\overline{W}_\epsilon$ of $W_\epsilon \subset \mathbb{C}^3 \times \mathbb{P}^2$. Then $\sigma : V_\epsilon \longrightarrow u(U_\epsilon)$, $\sigma(u, v) := u$
is called the blow-up of the singularity $0 \in u(U_\epsilon)$ of type $A_1$, and $E := \sigma^{-1}(0)$ is its exceptional divisor.
If \((u, v) \in W_\varepsilon\), then \(u \neq 0\), and the defining equations of \(W_\varepsilon\) imply \(u^1 u^2 = (u^3)^2\) and \((v^1: v^2: v^3) = (u^1: u^2: u^3)\). Denote this by \(v = [u] \in \mathbb{P}^2\) and observe that \(v^1 v^2 = (v^3)^2\) follows. Then for \(t \in \mathbb{R}\) close to \(t = 0\), say \(t \in (0, \delta)\) for an appropriate \(\delta > 0\), the map \(t \mapsto \gamma_\varepsilon(t) := (tu, [u]) \in W_\varepsilon\) yields a smooth curve with \(\lim_{t \to 0} \gamma_\varepsilon(t) = (0, [u]) \in E\). In fact,

\[
E = \{(u, v) \in \mathbb{C}^3 \times \mathbb{P}^2 \mid u = 0, v^1 v^2 = (v^3)^2\},
\]

so the exceptional divisor \(E\) is biholomorphic to \(\mathbb{P}^1\),

\[
\mathbb{P}^1 \cong E \quad \text{under} \quad (t^1: t^2) \mapsto (0, ((t^1)^2: (t^2)^2: t^1 t^2)).
\]

The most important properties of the blow-up of a singularity of type \(A_1\) are summarized in the following

**Proposition 1.3.3** The resolution \(\sigma: V_\varepsilon \to u(U_\varepsilon)\) of the singularity \(0 \in U_\varepsilon\) of type \(A_1\) yields a smooth complex two-manifold \(V_\varepsilon\). The restriction \(\sigma|_{V_\varepsilon \setminus E}: V_\varepsilon \setminus E \to u(U_\varepsilon) \setminus \{0\}\) is biholomorphic, and the exceptional divisor \(E \subset V_\varepsilon\) is biholomorphic to \(\mathbb{P}^1\). Moreover, \(V_\varepsilon\) has trivial canonical bundle.

**Proof (sketch):**

The proof of Proposition 1.3.3 can be performed by a direct calculation.

For example, smoothness of \(V_\varepsilon\) close to the point \(p_1 := (0, (1: 0: 0)) \in E\) can be checked in the chart \(U_1 := \{(u, v) \in V_\varepsilon \mid v^1 \neq 0\}\). Indeed, \((u^1, v^3) \mapsto ((u^1, u^1(v^3)^2, u^1 v^3), (1: (v^3)^2: v^3))\) yields a smooth parametrization of \(U_1\) near \(p_1\). Analogously, one obtains smoothness everywhere, and holomorphism of changes of coordinates is immediate, as are the claims about biholomorphism of \(\sigma|_{V_\varepsilon \setminus E}\) and \(E \cong \mathbb{P}^1\) by what was said above.

The triviality of the canonical bundle of \(V_\varepsilon\) follows since \(\eta(z_1, z_2) = dz_1 \wedge dz_2\) on \(\mathbb{C}^2\) descends to a section of the canonical bundle of \(V_\varepsilon\) over \(V_\varepsilon \setminus E\). On \(U_1\) and with respect to coordinates \((u^1, v^3)\) as above, we have

\[
\eta(u^1, v^3) = \frac{du^1 \wedge dv^3}{2u^1} = \frac{1}{2} du^1 \wedge dv^3
\]
as long as \(u^1 \neq 0\). However, this implies that \(\eta\) has a holomorphic continuation to all of \(U_1\), which never vanishes. By proceeding similarly for other coordinate neighborhoods, \(\eta\) can be continued to a nowhere vanishing global section of the canonical bundle of \(V_\varepsilon\), which thus is trivial.

The Kummer construction is now summarized in
Theorem 1.3.4 (Kummer construction) The singular Kummer surface $T_L/\mathbb{Z}_2$ of Definition 1.3.1 has 16 singularities of type $A_1$, situated at $L/2L \subset T_L/\mathbb{Z}_2$. The complex surface $X$ obtained by blowing up each of these singularities is a K3 surface.

Proof (sketch):
One immediately checks that the singular points of $T_L/\mathbb{Z}_2$ are precisely the images of the fixed points of $\mathbb{Z}_2$ in $T_L$ under the quotient $T_L \rightarrow T_L/\mathbb{Z}_2$. If $[z]$ denotes the image of $z \in \mathbb{C}^2$ under the natural projection $\mathbb{C}^2 \rightarrow T_L$, then $[z]$ is fixed under $\mathbb{Z}_2$ if and only if $[z] = [-z]$, that is, if and only if $2z \in L$. Since the lattice $L$ has rank 4, we find that $L/2L \cong \mathbb{F}_4^2$ contains 16 points. That each of these singularities is of type $A_1$ is also immediate — one translates the coordinates induced from $\mathbb{C}^2$ on an open neighborhood of $y \in L/2L \subset T_L$ by $-y$ and finds that a punctured neighborhood of the origin is then biholomorphically mapped to $U_\epsilon \setminus \{0\}$ as in Definition 1.3.2.

Now $X$ is obtained from the singular Kummer surface by blowing up all the singularities, that is, by replacing a neighborhood $U_\epsilon$ of each of the singular points by a copy of the blow-up $V_\epsilon$. Since $U_\epsilon \setminus \{0\}$ is biholomorphic to $V_\epsilon \setminus \{0\}$ and according to Proposition 1.3.3 $X$ is a smooth complex surface with trivial canonical bundle. Moreover, $X$ is compact and connected by construction. I claim that $b_1(X) = 0$. Indeed, $\kappa \in \mathbb{Z}_2$ acts by multiplication by $-1$ on $H^1(T_L, \mathbb{C})$, such that none of the classes in $H^1(T_L, \mathbb{C})$ can descend to $H^1(X, \mathbb{C})$. Furthermore, according to Proposition 1.3.3 the exceptional divisor of each blow-up is biholomorphic to $\mathbb{P}^1$ with $h^{p,q}(\mathbb{P}^D) = \delta_{p,q}$, and thus the blow-ups only contribute to the cohomology of $X$ in even degree.

In summary, $X$ is a connected, compact complex surface with trivial canonical bundle and $b_1(X) = 0$. By Definition 1.2.7 I have shown that $X$ is a K3 surface. 

Note that the standard Kähler metric on $\mathbb{C}^2$, which is the Euclidean one, induces a degenerate Kähler metric on $X$. Indeed, the induced metric assigns vanishing volume to each irreducible component $E \cong \mathbb{P}^1$ of the exceptional divisor and thus does not correspond to a smooth, Riemannian metric on $X$.

Definition 1.3.5 The K3 surface $X$ obtained from a complex torus $T_L = \mathbb{C}^2/L$ by the Kummer construction of Theorem 1.3.4 is called a Kummer surface. In other words, a Kummer surface carries the complex structure
and (degenerate) Kähler structure induced from the respective structures inherited by $T_L$ from the standard ones on $\mathbb{C}^2$.

By [Nik75], every K3 surface $X$ which is obtained from some singular surface by the minimal resolution of 16 distinct singularities of type $A_1$ is biholomorphic to a Kummer surface. In other words, there exists a rank 4 lattice $L \subset \mathbb{C}^2$ such that $X$ is biholomorphic to the minimal resolution of the singular Kummer surface $T_L/\mathbb{Z}_2$. There are many examples of singular quartic hypersurfaces in $\mathbb{P}^3$ with 16 singularities of type $A_1$, e.g. the one defined by

$$\left\{ z = (z^0: \cdots : z^3) \in \mathbb{P}^3 \left| \sum_{j=0}^3 (z^j)^4 - 4 \prod_{j=0}^3 z^j \right. \right\}.$$ 

Such singular Kummer surfaces were first studied by Kummer [Kum64]. The construction has become a classical one by now and is described in detail, for example, in [BHPvdV04].

## 2 Conformal field theory

As argued in the introduction, the Calabi-Yau geometry which the previous section was devoted to plays a crucial role in string theory. Indeed, a so-called NON-LINEAR SIGMA MODEL CONSTRUCTION is predicted to yield a SUPERCONFORMAL FIELD THEORY as the world-sheet theory for superstrings in a Calabi-Yau target geometry.

Unfortunately, for a generic Calabi-Yau manifold it is still hopeless to attempt a non-linear sigma model construction explicitly. The only exceptions, in general, are the complex tori, which carry flat metrics, such that non-linear sigma models are only little more complicated than free field theories. Furthermore, orbifold constructions, which can be viewed as generalizations of the Kummer construction of Section 1.3, are reasonably well understood.

Non-linear sigma models on K3 surfaces, which will be discussed in Section 3, furnish an intermediate case in that many K3 THEORIES are accessible through orbifold procedures, and in that the moduli space of these theories has been determined globally under a few additional assumptions [Sei88, Cec90, AM94, NW01]. That superconformal field theories allow an independent, mathematical approach is a crucial ingredient to that result. The current section therefore gives an overview on aspects of conformal field theory related to the particular Calabi-Yau geometries that the previous section was focused on, namely the non-linear sigma models on complex tori.
(Section 2.1) and their \(\mathbb{Z}_2\)-orbifolds (Section 2.2). Due to restrictions of space and time, I content myself with giving an overview and pointing to some relevant literature.

2.1 Toroidal superconformal field theories

In this treatise, I do not attempt to give a definition of conformal field theory (CFT), though hopefully these notes are useful also for the CFT-novice, in that they discuss some of the basic ingredients to CFT. I have summarized my own view on a definition of CFT elsewhere, see e.g. [Wen10]. To make best use of the restricted amount of space, I refer the reader to the recent review [Wen15] as a companion paper to the present lecture notes. Indeed, while in [Wen15] the main emphasis lies on CFT aspects, here I focus on the geometric point of view. In particular, for the notions of holomorphic fields, operator product expansions (OPEs) and normal ordered products, following [LW78, FK81, Bor86, Kac98, FBZ04], see [Wen15, Sect. 2.1], and for a summary of a definition of conformal and superconformal field theories, see [Wen15, Sect. 2.2]. As in this reference, in what follows, I solely discuss two-dimensional Euclidean unitary conformal field theories. All superconformal field theories (SCFTs) are assumed to be non-chiral and to enjoy space-time supersymmetry as well as \(N = (2,2)\) world-sheet supersymmetry.

An important ingredient to SCFT is the representation theory of the (super-) Virasoro algebra. Indeed, the space of states of any of our SCFTs is a \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-graded complex vector space

\[
\mathcal{H} = \mathcal{H}^{NS} \oplus \mathcal{H}^R, \quad \text{where} \quad \mathcal{H}^{NS} = \mathcal{H}_b^{NS} \oplus \mathcal{H}_f^{NS}, \quad \mathcal{H}^R = \mathcal{H}_b^R \oplus \mathcal{H}_f^R,
\]

\[
\mathcal{H}_b := \mathcal{H}_b^{NS} \oplus \mathcal{H}_b^R, \quad \mathcal{H}_f := \mathcal{H}_f^{NS} \oplus \mathcal{H}_f^R.
\]

The subspaces \(\mathcal{H}^{NS}\) and \(\mathcal{H}^R\) are called Neveu-Schwarz (NS) and Ramond (R) sectors, while \(\mathcal{H}_b\) and \(\mathcal{H}_f\) are the subspaces of bosonic and fermionic states, respectively. The space \(\mathcal{H}\) carries a representation of two copies of a super-Virasoro algebra at central charges \((c,\bar{c})\), where for all examples discussed here we have \(c = \bar{c}\). As usual, the zero-modes of left- and right-moving Virasoro fields and \(U(1)\)-currents are denoted \(L_0, \bar{L}_0\) and \(J_0, \bar{J}_0\), respectively. Then space-time supersymmetry implies that \((-1)^F := e^{\pi i (J_0 - \bar{J}_0)}\)

\[4\]

As in [Wen15, Sect. 2.2], I assume a compactness condition for our SCFTs, namely that the operators \(L_0, \bar{L}_0, J_0, \bar{J}_0\) are simultaneously diagonalizable, and that simultaneous
acts as identity operator $\mathbb{1}$ on $\mathbb{H}_b$ and as $-\mathbb{1}$ on $\mathbb{H}_f$.

Every SCFT possesses a modular invariant partition function $Z(\tau, z)$ in two complex variables $\tau, z \in \mathbb{C}$ with $\Im(\tau) > 0$,

$$ Z(\tau, z) = \text{tr}_{\mathbb{H}_b} \left( q^{L_0-c/24} y^{L_0-2\tau/24} \bar{y}^{L_0-\tau/24} \right). $$

Space-time supersymmetry is tantamount to a decomposition of the partition function into four sectors,

$$ Z(\tau, z) = \frac{1}{2} \left( Z_{NS}(\tau, z) + Z_{\overline{NS}}(\tau, z) + Z_R(\tau, z) + Z_{\overline{R}}(\tau, z) \right), $$

for $S \in \{NS, R\}$:

$$ Z_S(\tau, z) := \text{tr}_{\mathbb{H}_S} \left( q^{L_0-c/24} y^{L_0-2\tau/24} \bar{y}^{L_0-\tau/24} \right), $$

with $Z_R(\tau, z) = (q\bar{y})^{c/24}(y\bar{y})^{c/6} Z_{NS}(\tau, z + \frac{1}{2})$, (2.1.1)

$$ Z_{\overline{S}}(\tau, z) := \text{tr}_{\mathbb{H}_S} \left( (-1)^{J_0-J_\overline{0}} q^{L_0-c/24} y^{L_0-2\tau/24} \bar{y}^{L_0-\tau/24} \right) = Z_S(\tau, z + \frac{1}{2}). $$

Equation (2.1.1) reflects an isomorphism $\mathbb{H}^{NS} \cong \mathbb{H}^R$ of representations of the $N = (2, 2)$ superconformal algebra, known as SPECTRAL FLOW, which holds due to our assumption of space-time supersymmetry.

The simplest example of such an SCFT is a TOROIDAL $N = (2, 2)$ SUPERCONFORMAL FIELD THEORY, see [Wen15, Def. 5] for a more detailed account. At central charges $(c, \overline{c}) = (3D, 3D)$, $D \in \mathbb{N}$, such a theory is in particular characterized by the fact that the chiral algebras on the left and on the right contain a $\mathfrak{u}(1)^{2D}$-current algebra each, along with $D$ left-moving and $D$ right-moving Dirac fermions, the superpartners of the $\mathfrak{u}(1)$-currents. The corresponding fields $j^k(z), k \in \{1, \ldots, 2D\}$ and $\psi^\pm_l(z), l \in \{1, \ldots, D\}$, whose only non-vanishing OPEs are

$$ \forall k, l \in \{1, \ldots, 2D\}: \quad j^k(z)j^l(w) \sim \frac{\delta^{k,l}}{(z-w)^2}, $$

$$ \forall k, l \in \{1, \ldots, D\}: \quad \psi^+_k(z)\psi^-_l(w) \sim \frac{\delta^{k,l}}{(z-w)}, $$

(2.1.2)

along with their right-moving analogues are called the BASIC FIELDS. They are examples of so-called FREE FIELDS (see e.g. [Kac98]).
The space of states $H$ of a toroidal SCFT arises as Fock space representation of the modes of these basic fields, that is, of the super-Lie algebra generated by $I$ and $a_n^k$, \((\psi^\pm)_m \in \text{End}(H^S), k \in \{1, \ldots, 2D\}, l \in \{1, \ldots, D\}, S \in \{NS, R\}\), where $n \in \mathbb{Z}$, and on the NS-sector $H^{NS}$, $m \in \mathbb{Z} + \frac{1}{2}$, while on the R-sector $H^R$, $m \in \mathbb{Z}$,

\[
[a_m^j, a_n^k] := a_m^j a_n^k - a_n^k a_m^j = m\delta^{jk}\delta_{m+n,0}I,
\]

\[
\{\psi_m^j, \psi_n^k\} := \psi_n^k \psi_m^j + \psi_m^j \psi_n^k = \delta^{jk}\delta_{m+n,0}I \quad \forall j, k, m, n,
\]

and analogously for the right-movers. All other (super-)commutators between the $a_n^k$, \((\psi^\pm)_m\)$ and their right-moving analogues vanish. One then has $H^{NS} = \oplus_{\gamma \in \Gamma} H_\gamma$ with $\Gamma$ the so-called charge lattice, and $H_\gamma$ a lowest weight representation of the above super-Lie algebra of modes, with lowest weight vector $|\gamma\rangle \in H_\gamma$ such that

\[
\forall j \in \{1, \ldots, 2D\}: \quad a_0^j |\gamma\rangle = \gamma^j |\gamma\rangle,
\]

\[
\forall k \in \{1, \ldots, D\}: \quad a_m^k |\gamma\rangle = 0 \quad \forall m \in \mathbb{Z} \text{ with } m < 0,
\]

and analogously for the right-moving modes. Here, $\Gamma \subset \mathbb{R}^{2D,2D}$, by which I mean $\mathbb{R}^{2D,2D} = \mathbb{R}^{2D} \oplus \mathbb{R}^{2D}$ and

\[
\forall \gamma \in \Gamma: \quad \gamma = (\gamma_L, \gamma_R) \text{ with } \gamma_L, \gamma_R \in \mathbb{R}^{2D}, \quad \langle \gamma, \gamma \rangle = \gamma_L \cdot \gamma_L - \gamma_R \cdot \gamma_R
\]

with the standard Euclidean scalar product on $\mathbb{R}^{2D}$ and $\gamma_L = (\gamma^1, \ldots, \gamma^{2D})^T, \gamma_R = (\gamma^{2D+1}, \ldots, \gamma^{4D})^T$. The representation $H^R$ is obtained from $H^{NS}$ by spectral flow, as was mentioned in the discussion of equation (2.1.1). A counting argument then shows

\[
\text{tr}_{H_\gamma} \left( q^{L_0-c/24} j_0 \bar{q}^{L_0-\tau/24\bar{y}_0} \right) = q^{\frac{1}{2} \gamma_L \cdot \gamma_L} \eta^2(\tau) \frac{\vartheta_3(\tau, z)^{2D}}{\eta(\tau)^{|\gamma_R|}}
\]

with the Dedekind eta function $\eta(\tau)$ and the standard Jacobi theta functions $\vartheta_k(\tau, z)$, $k \in \{1, \ldots, 4\}$. Here, the charge lattice $\Gamma$ is an even selfdual lattice of signature $(2D, 2D)$, given in terms of an embedding into $\mathbb{R}^{2D,2D}$, which is specified by the decomposition $\gamma = (\gamma_L, \gamma_R)$ for every $\gamma \in \Gamma$ as above. Generalizing Kac’s holomorphic lattice algebras [Kac98] to the non-holomorphic case, in [KO03] the lattice vertex operator algebra corresponding to such an indefinite lattice is described. One has:
Proposition 2.1.1 ([CENT85, Nar86]) A toroidal superconformal field theory is uniquely characterized by its charge lattice \( \Gamma \subset \mathbb{R}^{2D,2D} \). For a theory with charge lattice \( \Gamma \), setting
\[
Z_{\Gamma}(\tau) := \sum_{\gamma=(\gamma_L,\gamma_R) \in \Gamma} q^{\frac{1}{2} \gamma_L \cdot \gamma_L} q^{\frac{1}{2} \gamma_R \cdot \gamma_R} |\eta(\tau)|^{4D},
\]
the four sectors of the partition function \( Z(\tau, z) \) are
\[
Z_{NS}(\tau, z) = Z_{\Gamma}(\tau) \cdot \left| \frac{\vartheta_3(\tau, z)}{\eta(\tau)} \right|^{2D}, \quad Z_{\tilde{NS}}(\tau, z) = Z_{\Gamma}(\tau) \cdot \left| \frac{\vartheta_4(\tau, z)}{\eta(\tau)} \right|^{2D},
\]
\[
Z_R(\tau, z) = Z_{\Gamma}(\tau) \cdot \left| \frac{\vartheta_2(\tau, z)}{\eta(\tau)} \right|^{2D}, \quad Z_{\tilde{R}}(\tau, z) = Z_{\Gamma}(\tau) \cdot \left| \frac{\vartheta_1(\tau, z)}{\eta(\tau)} \right|^{2D},
\]
hence
\[
Z(\tau, z) = Z_{\Gamma}(\tau) \cdot Z_f(\tau, z) \quad \text{with} \quad Z_f(\tau, z) := \frac{1}{2} \sum_{k=0}^{2D} \left| \frac{\vartheta_k(\tau, z)}{\eta(\tau)} \right|^{2D}.
\]

From Proposition 2.1.1 one also reads the moduli space of such theories [CENT85, Nar86], see [Wen15, Thm. 1].

As mentioned above, the complex tori are the unique Calabi-Yau manifolds for which a non-linear sigma model construction can be performed directly and explicitly; in fact, for a torus \( T_L = \mathbb{C}^D/L \) with \( L \subset \mathbb{C}^D \) a lattice of rank \( 2D \), the non-linear sigma model construction yields a toroidal SCFT as discussed above. Geometrically, each \( u(1) \)-current \( j^k(z), k \in \{1, \ldots, 2D\} \), as in (2.1.2), arises as the field corresponding to a parallel tangent vector field, given by a standard Euclidean coordinate vector field in \( \mathbb{R}^{2D} \approx \mathbb{C}^D \). The real and imaginary parts of its fermionic superpartners are the fields corresponding to the dual cotangent vector fields. The resulting non-linear sigma model also depends on the choice of a B-field, which in this setting can be described by a constant, skew-symmetric endomorphism of \( \mathbb{R}^{2D} \). Identifying \( \mathbb{R}^{2D} \approx \mathbb{C}^D \) with its dual by means of the standard Euclidean scalar product, the dual lattice of \( L \) is given by
\[
L^* := \left\{ \alpha \in \mathbb{R}^{2D} \mid \alpha \cdot \lambda \in \mathbb{Z} \ \forall \lambda \in L \right\},
\]
and the charge lattice of the resulting toroidal SCFT is
\[
\Gamma = \left\{ \frac{1}{\sqrt{2}}(\mu - B\lambda + \lambda, \mu - B\lambda - \lambda) \mid \lambda \in L, \ \mu \in L^* \right\}.
\]
Vice versa, every toroidal SCFT allows a geometric interpretation in terms of a non-linear sigma model, see e.g. [NW01].
Definition 2.1.2 Consider a toroidal $N = (2,2)$ SCFT at central charges $(c,\overline{c}) = (3D,3D)$ with charge lattice $\Gamma \subset \mathbb{R}^{2D,2D}$. A geometric interpretation of the theory is any choice of complementary 2D-dimensional subspaces $Y, Y^0 \subset \mathbb{R}^{2D,2D}$, such that $\mathbb{R}^{2D,2D} = Y \oplus Y^0$, both $Y$ and $Y^0$ are null, and both $\Gamma \cap Y$ and $\Gamma \cap Y^0$ are rank 2D lattices.

The terminology deserves explanation: Given a geometric interpretation $\mathbb{R}^{2D,2D} = Y \oplus Y^0$ of a theory with charge lattice $\Gamma \subset \mathbb{R}^{2D,2D}$, without loss of generality we can set

$$\Gamma \cap Y = \left\{ \frac{1}{\sqrt{2}}(\mu,\mu) \mid \mu \in L^* \right\}, \quad \Gamma \cap Y^0 = \left\{ \frac{1}{\sqrt{2}}(\lambda - B\lambda, -\lambda - B\lambda) \mid \lambda \in L \right\}$$

for some rank 2D lattice $L \subset \mathbb{R}^{2D} \cong \mathbb{C}^D$, $L^* \subset \mathbb{R}^{2D}$ its dual, and $B$ some skew-symmetric linear endomorphism of $\mathbb{R}^{2D}$. By Proposition 2.1.1 and the explanations preceding Definition 2.1.2, our theory then agrees with a non-linear sigma model on $T_L = \mathbb{C}^D/L$ with B-field $B$.

Given such a geometric interpretation of a toroidal SCFT, it is now also clear how certain geometric symmetries of the torus $T_L$ may induce symmetries of a toroidal SCFT on $T_L$: If a symmetry of $T_L$ is given in terms of $A \in \text{End}(\mathbb{C}^D)$, then $A$ has to act as lattice automorphism of $L$. Hence $A \in O(2D)$ and $A$ also acts as lattice automorphism of $L^*$. If in addition, $AB = BA$, then $A$ acts as lattice automorphism on $\Gamma$ which respects the embedding $\Gamma \subset \mathbb{R}^{2D,2D}$. One checks that the induced action on the space of states $H$ yields a symmetry of the toroidal SCFT.

2.2 $\mathbb{Z}_2$-orbifold conformal field theories

In Section 1.3, I presented the classical Kummer construction, which yields a K3 surface from the much simpler complex two-torus $T_L = \mathbb{C}^2/L$ by $\mathbb{Z}_2$-ORBIFOLDING. The construction begins with the projection to $T_L/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by the symmetry $\kappa$ of $T_L$ which is induced from $-I$ on $\mathbb{C}^2$.

By the discussion at the end of the previous section, $\kappa$ also acts as symmetry on the non-linear sigma model constructed on $T_L$ with an arbitrary B-field $B$. The present section is devoted to the lift of this $\mathbb{Z}_2$-orbifold procedure to the level of CFT, to construct a new superconformal field theory from the much simpler toroidal one. This string theory procedure was inspired by the techniques that were originally developed in the context of MONSTROUS MOONSHINE [FLM84] for holomorphic vertex algebras, and it can be carried
out in much greater generality [DHVW85, DHVW86], that is, in arbitrary dimensions and with more general orbifolding-groups. Indeed, already the Kummer construction can be generalized to obtain K3 surfaces by orbifolding an underlying complex two-torus \( T_L = \mathbb{C}^2 / L \) with the appropriate symmetry by groups \( \mathbb{Z}_3, \mathbb{Z}_4 \) or \( \mathbb{Z}_6 \), or even by certain examples of non-Abelian groups, see [Wen01] and references therein. In the following I provide an overview of the basic ideas behind the orbifold procedure in CFT. For the sake of brevity, I content myself with the discussion of \( \mathbb{Z}_2 \)-orbifolds as an instructive example.

If not stated otherwise, in the following I consider a toroidal superconformal field theory at central charges \((c, \bar{c}) = (3D, 3D)\) with geometric interpretation on a complex torus \( T_L = \mathbb{C}^D / L \) and with some B-field \( B \) as discussed in Section 2.1. As before, the space of states of the SCFT is denoted by \( \mathbb{H} = \mathbb{H}_b \oplus \mathbb{H}_f \), and \( \kappa \) denotes the symmetry of order 2 of this theory which is induced by \( \mathbb{C}^D \to \mathbb{C}^D, z \mapsto -z \). It acts by multiplication by \(-1\) on the basic fields (2.1.2) and on the charge lattice \( \Gamma \), and therefore as linear involution on the bosonic and fermionic spaces of states \( \mathbb{H}_b \) and \( \mathbb{H}_f \), respectively.

Let us investigate the \( \kappa \)-invariant subsector \( \mathbb{H}_b^{Z_2} \) of the bosonic space of states. With notations as in Proposition 2.1.1 a counting argument shows

\[
\text{tr}_{\mathbb{H}_b^{Z_2}} \left( q^{L_0-c/24} y^{J_0} q^{-\tau/24} y^{\tau} \right) = \frac{1}{2} \left( Z_f(\tau, z) + Z_{-1}(\tau, z) \right)
\]

with \( Z_{-1}(\tau, z) = \left| \frac{2\eta(\tau)}{\vartheta_2(\tau, 0)} \right|^{2D} \cdot Z_f(\tau, z) \).

This expression is not modular invariant, since \( Z_{-1}(\tau, z) \) isn’t, while \( Z_f(\tau) \) and \( Z_f(\tau, z) \) are. Hence \( \mathbb{H}_b^{Z_2} \) is not the space of bosonic states of a full-fledged superconformal field theory. However, one can construct a twisted sector \( \mathbb{H}_b^{Z_2} \) such that \( \mathbb{H}_b^{Z_2} \oplus \mathbb{H}_b^{Z_2} \) has this property. To obtain information about such a would-be twisted sector, we observe that \( Z_{-1}(\tau, z) \) has a natural modular invariant completion

\[
Z_{-1}(\tau, z) + Z_{-1} \left( -\frac{1}{\bar{\tau}}, \frac{z}{\bar{\tau}} \right) + Z_{-1} \left( -\frac{1}{\tau+1}, \frac{z}{\tau+1} \right).
\]

On the basis of a path integral interpretation of each summand, see e.g. [Gin88, §8.3], modular invariance of this expression is expected; it is shown to hold by a direct calculation. The key idea behind the orbifolding procedure
is to interpret the three summands of the above expression as follows:

\[
Z_{-1}(\tau, z) = \text{tr}_{\mathbb{H}_b} \left( \kappa q^{L_0-c/24} y^J \tau^0 - \tau^{24} \right),
\]

\[
Z_{-1} \left(-\frac{1}{\tau}, \frac{z}{\tau} \right) = \text{tr}_{\mathbb{H}_b} \left( q^{L_0-c/24} y^J \tau^0 - \tau^{24} \right),
\]

\[
Z_{-1} \left(-\frac{1}{\tau+1}, \frac{z}{\tau+1} \right) = \text{tr}_{\mathbb{H}_b} \left( \kappa q^{L_0-c/24} y^J \tau^0 - \tau^{24} \right).
\]

Here, \(\mathbb{H}_b^{tw}\) needs to be constructed as some representation of the super-Virasoro algebras that enjoys a symmetry \(\kappa\) of order 2 such that \(\mathbb{H}_b^{tw}\) is the \(\kappa\)-invariant subspace of \(\mathbb{H}_b^{tw}\). Indeed, in the present example an appropriate space \(\mathbb{H}_b^{tw}\) can be found, that is, such that \(\mathbb{H}_b^{Z2} \oplus \mathbb{H}_b^{tw}\) can be interpreted as bosonic space of states of a full-fledged superconformal field theory. The details of the construction are quite technical but have been worked out, see [FFRS10] and references therein.

Proposition 2.2.1Consider a toroidal \(N=(2,2)\) superconformal field theory at central charges \((c, \tilde{c}) = (3D, 3D)\) with charge lattice \(\Gamma \subset \mathbb{R}^{2D,2D}\) and space of states \(\mathbb{H}\). Let \(\kappa\) denote the symmetry of order 2 which acts by multiplication by \(-1\) on \(\Gamma\) and on the basic fields \(2.1.2\).

Then there exists a \(Z_2\)-orbifold conformal field theory of this toroidal theory, whose space of states is \(\mathbb{H}^{Z2} \oplus \mathbb{H}^{tw}\), where \(\mathbb{H}^{Z2} \subset \mathbb{H}\) denotes the \(\kappa\)-invariant subspace. The four sectors of the partition function \(Z^{orb}(\tau, z)\) of this theory are obtained from its \(\tilde{R}\)-sector by application of the spectral flow formulas \(2.1.1\). With notations as in Proposition 2.1.1 we have

\[
Z^{orb}_{\tilde{R}}(\tau, z) = \frac{1}{2} \cdot \left( Z_{\Gamma}(\tau) \cdot \left| \frac{\vartheta_1(\tau, z)}{\eta(\tau)} \right|^{2D} + \frac{2\vartheta_2(\tau, z)}{\vartheta_2(\tau, 0)}^{2D} + \frac{2\vartheta_4(\tau, z)}{\vartheta_4(\tau, 0)}^{2D} + \frac{2\vartheta_3(\tau, z)}{\vartheta_3(\tau, 0)}^{2D} \right).
\]

If the \(Z_2\)-orbifold CFT of Proposition 2.2.1 is obtained from a toroidal SCFT with geometric interpretation on \(T_L = \mathbb{C}^D/L\) with some B-field \(B\), then it is believed that the orbifold theory can be obtained by a non-linear sigma model construction from the orbifold limit of the Calabi-Yau \(D\)-manifold obtained.

\[\text{Also see Yi-Zhi Huang’s blog \[Hua14\] and the references therein to appreciate the mathematical problems that the introduction of a SCFT on \(\mathbb{H}_b^{Z2} \oplus \mathbb{H}_b^{tw}\) poses.}\]
from blowing up all singularities in $T_L/\mathbb{Z}_2$. To support this belief, one checks from the partition function $Z_{\text{orb}}(\tau,z)$ given in Proposition 2.2.1 that the dimension of the space of twisted ground states in $\mathbb{H}^{tw}$ is $2^{2D}$. This is precisely the number of singular points in $T_L/\mathbb{Z}_2$. For more complicated orbifolding-groups, each singular point corresponds to a higher dimensional subspace of twisted ground states, depending on the order of the stabilizer group. This supports the idea that, at least naively, the introduction of twisted sectors corresponds to the resolution of the singular points in the non-linear sigma model on $T_L/\mathbb{Z}_2$.

If $D=2$, then by Theorem 1.3.4 the resolution of all singularities in $T_L/\mathbb{Z}_2$ yields a K3 surface. We should hence expect the $\mathbb{Z}_2$-orbifold conformal field theory of Proposition 2.2.1 to allow a non-linear sigma model interpretation on a Kummer surface in this case. In [NW01], a map between the respective moduli spaces of conformal field theories is constructed which is compatible with this expectation. Earlier evidence in favor of the prediction arises from a calculation of the so-called CONFORMAL FIELD THEORETIC ELLIPTIC GENUS [EOTY89], according to

**Definition 2.2.2** Consider the $\tilde{R}$-sector $Z_{\tilde{R}}$ of the partition function of an $N=(2,2)$-superconformal field theory, viewed as a function of four complex variables $Z_{\tilde{R}} = Z_{\tilde{R}}(\tau,z;\overline{\tau},\overline{z})$. The CONFORMAL FIELD THEORETIC ELLIPTIC GENUS of the theory is

$$E(\tau, z) := Z_{\tilde{R}}(\tau, z; \overline{\tau}, \overline{z} = 0).$$

For a CFT that arises as non-linear sigma model on some Calabi-Yau $D$-manifold $Y$, it is expected that the conformal field theoretic elliptic genus agrees with the GEOMETRIC ELLIPTIC GENUS of $Y$. The latter can be defined as the holomorphic Euler characteristic of a formal vector bundle $\mathbb{E}_{y-y}$ on $Y$, more precisely a formal power series in $q = e^{2\pi i \tau}$ and $y = e^{2\pi iz}$ whose coefficients are holomorphic vector bundles on $Y$. Using the notations $\Lambda_x E$, $S_x E$ for any vector bundle $E \to Y$ and a formal variable $x$ with

$$\Lambda_x E := \bigoplus_{p=0}^{\infty} x^p \Lambda^p E, \quad S_x E := \bigoplus_{p=0}^{\infty} x^p S^p E,$$

**Note however, at least in the $D=2$-dimensional case, a subtlety concerning the B-field on the orbifold [NW01], which turns out to be non-zero on every irreducible component of the exceptional divisor in the blow-up.**
where $\Lambda^p E, S^p E$ denote the $p$th exterior and symmetric powers of $E$, and with $T := T^{1,0} Y$ the holomorphic tangent bundle of $Y$,

$$
\mathbb{E}_{q,-y} = y^{-D/2} \bigotimes_{n=1}^{\infty} (\Lambda_{-y} q^{n-1} T^* \otimes \Lambda_{-y}^{-1} q^n T \otimes S_{q^n} T^* \otimes S_{q^n} T^*). \tag{2.2.1}
$$

The holomorphic Euler characteristic of Definition 1.2.1 is naturally extended to formal power series with coefficients in holomorphic vector bundles on $Y$,

$$
\chi \left( \sum_{n=0}^{\infty} x^n F_n \right) := \sum_{n=0}^{\infty} x^n \chi(F_n).
$$

We then have:

**Definition 2.2.3** For a Calabi-Yau $D$-manifold $Y$ with holomorphic tangent bundle $T := T^{1,0} Y$, the geometric elliptic genus $\mathcal{E}_Y(\tau, z)$ is the holomorphic Euler characteristic of the bundle $\mathbb{E}_{q,-y}$ introduced in (2.2.1),

$$
\mathcal{E}_Y(\tau, z) := \chi(\mathbb{E}_{q,-y}).
$$

Following [Wit87], the geometric elliptic genus can be interpreted as a regularized version of a $U(1)$-equivariant index of a Dirac operator on the loop space of $Y$. It is a topological invariant of $Y$ with many beautiful mathematical properties. In particular, its modular transformation properties agree with those of the conformal field theoretic elliptic genus of Definition 2.2.2 for a theory at central charges $c = \frac{\tau}{2}$ (with a character, if $D$ is odd) $\text{[Hir88, Wit88, EOTY89, Kri90, DY93, Wit94, BL00]}$. See Section 8 and [Wen15, Sect. 2.4] for a more detailed discussion of the two versions (geometric vs. conformal field theoretic) of the elliptic genus and their expected relationship, and for further references.

From Propositions 2.1.1 and 2.2.1 one immediately finds the conformal field theoretic elliptic genera of the non-linear sigma model on a complex two-torus $T_L = \mathbb{C}^2 / L$ and of its $\mathbb{Z}_2$-orbifold CFT:

$$
\mathcal{E}_{T_L}(\tau, z) = 0, \\
\mathcal{E}_{K3}(\tau, z) = 8 \left( \frac{\vartheta_2(\tau, z)}{\vartheta_2(\tau, 0)} \right)^2 + 8 \left( \frac{\vartheta_3(\tau, z)}{\vartheta_3(\tau, 0)} \right)^2 + 8 \left( \frac{\vartheta_4(\tau, z)}{\vartheta_4(\tau, 0)} \right)^2. \tag{2.2.2}
$$

These functions agree with the known geometric elliptic genera of $T_L$ and a Kummer surface, respectively, and thereby of all complex two-tori resp. K3 surfaces [EOTY89]:

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Proposition 2.2.4 Consider a toroidal $N = (2, 2)$ superconformal field theory at central charges $c = \overline{c} = 6$ with geometric interpretation on a complex two-torus $T_L = \mathbb{C}^2/L$ with some $B$-field $B$. Then its conformal field theoretic elliptic genus agrees with the geometric elliptic genus of $T_L$, and thereby of every complex two-torus. The conformal field theoretic elliptic genus of the $\mathbb{Z}_2$-orbifold CFT of this toroidal theory agrees with the geometric elliptic genus of any (and thereby every) K3 surface.

3 Outlook: Towards superconformal field theory on K3 and beyond

The discussion in the previous section foreshadows a rather subtle relation between geometry and conformal field theory. This final section of these lecture notes gives a rough overview and outlook on attempts to get a better understanding of this relation. Special attention is paid to K3 surfaces and conformal field theories associated to them, because as we have seen, the K3 surfaces furnish the simplest examples where non-linear sigma model constructions are not fully understood.

From a mathematical point of view, an abstract approach to CFT, based in representation theory, is desirable. However, then the route back to geometry is not immediate. As mentioned before, direct constructions of superconformal field theories from Calabi-Yau data are sparse – they are essentially restricted to toroidal SCFTs and their orbifolds.

However, if the existence of a non-linear sigma model on a Calabi-Yau $D$-manifold $Y$ is assumed, then a number of additional properties are known for the resulting SCFT. First, it enjoys $N = (2, 2)$ (world-sheet) supersymmetry at central charges $c = \overline{c} = 3D$ as well as space-time supersymmetry. Second, all eigenvalues of the linear operators $J_0$ and $\overline{J}_0$ on the space of states $H = H^{NS} \oplus H^R$ are integral in the Neveu-Schwarz sector $H^{NS}$.

At small central charges, more precisely at $c = \overline{c} = 3$ and $c = \overline{c} = 6$, these

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7Specifically concerning the discussion of the “geometry of CFTs”, see also Yi-Zhi Huang’s blog [Hua14] and the references therein.

8Since space-time supersymmetry implies an equivalence of representations of the $N = (2, 2)$ superconformal algebra $H^{NS} \cong H^R$ under spectral flow, this integrality condition implies that $J_0 - \overline{J}_0$ has only integral eigenvalues on all of $H$, and that the eigenvalues of $J_0$ and $\overline{J}_0$ in the Ramond sector $H^R$ lie in $\frac{1}{2} + \mathbb{Z}$.
conditions severely restrict the types of theories that can occur. One first observes that the conformal field theoretic elliptic genus of Definition 2.2.2 either vanishes, or it agrees with the geometric elliptic genus $E_{K3}(\tau, z)$ of K3 surfaces as in (2.2.2): For $c = \overline{c} = 3$ this follows immediately, since the space of weak Jacobi forms of weight 0 and index $\frac{1}{2}$ is trivial, for $c = \overline{c} = 6$ see [Wen15, Prop. 2(1)]. This reproduces the classification of Calabi-Yau $D$-manifolds with $D \leq 2$ according to Theorems 1.2.2 and 1.2.8 on the level of the elliptic genus. If the conformal field theoretic elliptic genus vanishes for a theory at $c = \overline{c} = 3$ or $c = \overline{c} = 6$ which obeys all the above assumptions on supersymmetry and on the $J_0$, $\overline{J}_0$-eigenvalues, then one actually finds that the underlying theory is toroidal (see [Wen15, Prop. 2(2)] for the case $c = \overline{c} = 6$ – the case $c = \overline{c} = 3$ is in fact simpler and can be treated analogously). This motivates the following definition, see also [Wen15, Def. 8]:

**Definition 3.1** A superconformal field theory is called a K3 theory, if the following conditions hold: The CFT is an $N = (2, 2)$ superconformal field theory at central charges $c = \overline{c} = 6$ with space-time supersymmetry, all the eigenvalues of $J_0$ and of $\overline{J}_0$ are integral, and the conformal field theoretic elliptic genus of the theory agrees with the geometric elliptic genus of K3 surfaces $E_{K3}(\tau, z)$ as in (2.2.2).

By Proposition 2.2.4, all $\mathbb{Z}_2$-orbifold conformal field theories obtained from toroidal superconformal ones at $c = \overline{c} = 6$ are examples of K3 theories.

One needs to appreciate that Definition 3.1 does not make use of non-linear sigma model assumptions. This also means that one needs to carefully distinguish between K3 theories and conformal field theories on K3. It is wide open, yet interesting and important, whether all K3 theories are theories on K3 in the sense that they can be constructed as non-linear sigma models. At least to our knowledge, no counter example is known.

There are various geometric properties of K3 surfaces that can be recovered from abstractly defined K3 theories. Let us only mention that under few additional assumptions, listed in [Sei88, Cee09, AM94, NW01], each connected component of the moduli space of K3 theories agrees with the moduli space expected for non-linear sigma models on K3. From this identification one obtains the notion of geometric interpretation for K3 theories in the spirit of Definition 2.1.2 see [AM94]. For the so-called

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9This can only happen if $c = \overline{c} = 6$. 
Gepner models, this allows to interpret certain symmetries of these theories in terms of geometric symmetries of an underlying K3 surface, see e.g. \[EOTY89, NW01, Wen02, Wen07, Wen06\]. The precise role of geometric symmetries for the properties of the elliptic genus is central to more recent discussions \[TW13a, TW13c, TW13b\] of the Mathieu Moonshine phenomena, in search of a geometric explanation for the seminal observations \[EOT11, Gan12\], see \[Wen15, Sect. 4\] for a summary. Using mirror symmetry, one can also explicitly construct a family of K3 theories with a geometric interpretation on a family of smooth K3 surfaces, which are expected to provide examples of non-linear sigma models on such smooth manifolds \[Wen06\].

It is natural, but much more subtle, to reconstruct aspects of the vertex algebra which underlies a superconformal field theory from geometric data, if the theory is expected to have a non-linear sigma model interpretation. From a string theory perspective, one should introduce free fields on local coordinate patches of \(Y\), analogously to the fields \(j^k(z), \psi_{i}^{\pm}(z)\) of (2.1.2), however it is unclear how to relate these “locally defined” fields to the ones that are used in abstract conformal field theories.

A more “rigid” approach, inspired by and closely related to TOPOLOGICAL FIELD THEORY, uses \(bc - \beta\gamma\) systems on holomorphic coordinate patches of a compact, connected complex manifold \(Y\). Roughly, one replaces the local holomorphic coordinate functions and holomorphic tangent vector fields by free bosonic fields of scaling dimensions zero and one, respectively, (the \(bc\)-fields), while cotangent vector fields correspond to fermionic fields (the \(\beta\gamma\)-fields). For these fields, transition functions between holomorphic coordinate patches can be defined according to the known geometric transformation rules. By this construction, one arrives at a sheaf of conformal vertex algebras on \(Y\), which allows the definition of a natural \(\mathbb{Z}\)-grading and a differential of degree one. The resulting complex is known as the CHIRAL DE RHAM COMPLEX \[MSV99\] because it is quasi-isomorphic to the classical (Dolbeault-) de Rham complex on \(Y\).

The construction of the chiral de Rham complex is “rigid” or topological in that an underlying conformal covariance manifests itself in a global section of the chiral de Rham complex, which yields a Virasoro field at central charge zero. One can perform a topological twist to a traditional \(N = 2\) superconformal structure at central charge \(c = 3D\) if \(Y\) is a Calabi-Yau \(D\)-manifold. This structure is found to descend to the sheaf cohomology of the chiral de Rham complex, which carries the structure of a superconformal ver-
tex operator algebra \[\text{[Bor01, BL00]}\]. Thereby, the above-mentioned problem is solved, that the geometrically motivated fields are only defined locally on coordinate patches, while abstractly defined conformal field theories do not exhibit such “locally defined” fields.

Armed with this success, one would naturally expect the vertex algebra that is obtained from the chiral de Rham complex to be closely related to the one which underlies a non-linear sigma model on \(Y\). This expectation is reinforced by the fact that almost by construction, the graded Euler characteristic of the chiral de Rham complex on a Calabi-Yau \(D\)-manifold \(Y\) yields the geometric elliptic genus \(\mathcal{E}_Y(\tau, z)\) of Definition 2.2.3 \[\text{[Bor01, BL00]}\]. However, the precise relation between the relevant vertex algebras turns out to be more subtle. In particular, the construction of the chiral de Rham complex does not depend on the choice of the Kähler class on \(Y\), in contrast to what one expects for the vertex algebras obtained in non-linear sigma models. According to \[\text{[Kap05]}\], this problem can be solved by performing a large volume limit to identify the BRST-cohomology of a topologically half-twisted non-linear sigma model on \(Y\) with the sheaf cohomology of the chiral de Rham complex.

Returning to K3 theories, where the notion of geometric interpretations is well understood, one may hope for more concrete results. Indeed, for generic Calabi-Yau \(D\)-manifolds it is notoriously hard to calculate the sheaf cohomology of the chiral de Rham complex and its superconformal vertex algebra structure. On the other hand, for toroidal SCFTs and their \(\mathbb{Z}_2\)-orbifolds at \(c = \overline{c} = 6\), I expect that a direct calculation should show that the vertex algebra obtained from the sheaf cohomology of the chiral de Rham complex in fact agrees with the one obtained by a topological half-twist from the respective superconformal field theories \[\text{[CGW15]}\]. Intriguingly, our calculations seem to indicate that although the construction of the chiral de Rham complex crucially depends on the choice of complex structure on \(Y\), the resulting vertex algebras show no dependence on the choice of complex structure (nor Kähler structure) within these two classes of examples.

In short, there are many intriguing mathematical and physical mysteries left when it comes to the journey from geometry to conformal field theory, even en route at the potentially simpler K3 geometry and K3 theories.
References

[AM94] P.S. Aspinwall and D.R. Morrison, *String theory on K3 surfaces*, in: Mirror symmetry II, B. Greene and S.T. Yau, eds., AMS, 1994, pp. 703–716; hep-th/9404151.

[AS63] M.F. Atiyah and I.M. Singer, *The index of elliptic operators on compact manifolds*, Bulletin American Mathematical Society 69 (1963), 322–433.

[BGV92] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Springer-Verlag, Berlin Heidelberg New York, 1992.

[BHPvdV04] W. Barth, K. Hulek, C. Peters, and A. van de Ven, *Compact Complex Surfaces*, Second enlarged ed., Springer-Verlag, Berlin Heidelberg, 2004.

[BL00] L.A. Borisov and A. Libgober, *Elliptic genera of toric varieties and applications to mirror symmetry*, Invent. Math. 140 (2000), no. 2, 453–485.

[Bog74] F.A. Bogomolov, *Kähler manifolds with trivial canonical class*, Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 11–21.

[Bor86] R.E. Borcherds, *Vertex algebras, Kac–Moody algebras and the monster*, Proc. Nat. Acad. Sci. U.S.A. 83 (1986), 3068–3071.

[Bor01] L.A. Borisov, *Vertex algebras and mirror symmetry*, Commun. Math. Phys. 215 (2001), no. 2, 517–557; math.AG/9809094.

[Cal54] E. Calabi, *The space of Kähler metrics*, Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, vol. II, Groningen-Amsterdam, pp. 206–207.

[Cec90] S. Cecotti, *N = 2 supergravity, type IIB superstrings and algebraic geometry*, Commun. Math. Phys. 131 (1990), 517–536.
A. Casher, F. Englert, H. Nicolai, and A. Taormina, Consistent superstrings as solutions of the $D = 26$ bosonic string theory, Phys. Lett. B162 (1985), 121–126.

S. Carnahan, F. Grimm, and K. Wendland, work in progress.

M.P. do Carmo, Riemannian geometry, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992, Translated from the second Portuguese edition by Francis Flaherty.

L.J. Dixon, J. Harvey, C. Vafa, and E. Witten, Strings on orbifolds, Nucl. Phys. B261 (1985), 678–686.

L.J. Dixon, J. Harvey, C. Vafa, and E. Witten, Strings on orbifolds II, Nucl. Phys. B274 (1986), 285–314.

P. Di Francesco and S. Yankielowicz, Ramond sector characters and $N = 2$ Landau-Ginzburg models, Nucl. Phys. B409 (1993), 186–210; hep-th/9305037.

T. Eguchi, H. Ooguri, and Y. Tachikawa, Notes on the $K3$ surface and the Mathieu group $M_{24}$, Exp. Math. 20 (2011), no. 1, 91–96; arXiv:1004.0956 [hep-th].

T. Eguchi, H. Ooguri, A. Taormina, and S.-K. Yang, Superconformal algebras and string compactification on manifolds with $SU(n)$ holonomy, Nucl. Phys. B315 (1989), 193–221.

E. Freitag and R. Busam, Complex analysis, Universitext, Springer-Verlag, Berlin, 2005, Translated from the 2005 German edition by Dan Fulea.

E. Frenkel and D. Ben-Zvi, Vertex algebras and algebraic curves, second ed., Mathematical Surveys and Monographs, vol. 88, American Mathematical Society, Providence, RI, 2004.

J. Fröhlich, J. Fuchs, I. Runkel, and Ch. Schweigert, Defect lines, dualities and generalised orbifolds, in: XVIth International Congress on Mathematical Physics, World Sci.
Publ., Hackensack, NJ, 2010, pp. 608–613; arXiv:0909.5013 [math-ph].

[FK81] I.B. Frenkel and V.G. Kac, Basic representations of affine Lie algebras and dual resonance models, Invent. Math. 62 (1980/81), no. 1, 23–66.

[FLM84] I. Frenkel, J. Lepowsky, and A. Meurman, A natural representation of the Fischer-Griess Monster with the modular function $J$ as character, Proc. Nat. Acad. Sci. U.S.A. 81 (1984), no. 10, Phys. Sci., 3256–3260.

[For81] O. Forster, Lectures on Riemann surfaces, Graduate Texts in Mathematics, vol. 81, Springer-Verlag, New York-Berlin, 1981, Translated from the German by Bruce Gilligan.

[Gan12] T. Gannon, Much ado about Mathieu; arXiv:1211.5531 [math.RT].

[GH78] P. Griffiths and J. Harris, Principles of algebraic geometry, John Wiley & Sons, New York, 1978.

[Gin88] P. Ginsparg, Applied conformal field theory, Lectures given at the Les Houches Summer School in Theoretical Physics 1988 (Les Houches, France), pp. 1–168.

[GP74] V. Guillemin and A. Pollack, Differential topology, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974.

[Hir54] F. Hirzebruch, Arithmetic genera and the theorem of Riemann-Roch for algebraic varieties., Proc. Nat. Acad. Sci. U.S.A. 40 (1954), 110–114.

[Hir76] M.W. Hirsch, Differential topology, Springer-Verlag, New York-Heidelberg, 1976, Graduate Texts in Mathematics, No. 33.

[Hir88] F. Hirzebruch, Elliptic genera of level $N$ for complex manifolds, in: Differential geometric methods in theoretical physics, K. Bleuler and M. Werner, eds., Kluwer Acad. Publ., 1988, pp. 37–63.
[Hop26] H. Hopf, *Vektorfelder in n-dimensionalen Mannigfaltigkeiten*, Math. Ann. 96 (1926), 225–250.

[Hua14] Y.-Z. Huang, *Conformal Field Theory*; https://qcft.wordpress.com/2014/09/16/a-program-to-construct-and-study-conformal-field-theories/.

[Huy05] D. Huybrechts, *Complex geometry*, Universitext, Springer-Verlag, Berlin, 2005.

[Jos95] J. Jost, *Riemannian geometry and geometric analysis*, Universitext, Springer-Verlag, Berlin, 1995.

[Joy00] D.D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[Kac98] V.G. Kac, *Vertex algebras for beginners*, second ed., University Lecture Series, vol. 10, American Mathematical Society, Providence, RI, 1998.

[Käh33] E. Kähler, *Über eine bemerkenswerte Hermitesche Metrik*, Abh. math. Sem. Univ. Hamburg 9 (1933), 173–186.

[Kap05] A. Kapustin, *Chiral de Rham complex and the half-twisted sigma-model*; hep-th/0504074.

[KO03] A. Kapustin and D. Orlov, *Vertex algebras, mirror symmetry, and D-branes: the case of complex tori*, Commun. Math. Phys. 233 (2003), no. 1, 79–136; hep-th/0010293.

[Kod64] K. Kodaira, *On the structure of compact complex analytic surfaces*, I. Am. J. Math. 86 (1964), 751–798.

[Kri90] I. Krichever, *Generalized elliptic genera and Baker-Akhiezer functions*, Math. Notes 47 (1990), 132–142.

[Kum64] E.E. Kummer, *Über die Flächen vierten Grades mit sechzehn singulären Punkten*, in: Gesamtwerk, 1864, pp. 246–260, Monatsbericht der Königlich Preußischen Akademie der Wissenschaften zu Berlin aus dem Jahre 1864; 18. April. Sitzung der physikalisch-mathematischen Klasse.
[Lan99] S. Lang, Complex analysis, Graduate Texts in Mathematics, vol. 103, Springer-Verlag, New York, 1999.

[Lee09] J.M. Lee, Manifolds and differential geometry, Graduate Studies in Mathematics, vol. 107, American Mathematical Society, Providence, RI, 2009.

[LM89] H.B. Lawson and M.-L. Michelsohn, Spin Geometry, Princeton University Press, Princeton, New Jersey, 1989.

[LW78] J. Lepowsky and R.L. Wilson, Construction of the affine Lie algebra \( A_1(1) \), Commun. Math. Phys. 62 (1978), no. 1, 43–53.

[Mor01] S. Morita, Geometry of differential forms, Translations of Mathematical Monographs, vol. 201, American Mathematical Society, Providence, RI, 2001, Translated from the two-volume Japanese original (1997, 1998) by Teruko Nagase and Katsumi Nomizu, Iwanami Series in Modern Mathematics.

[MSV99] F. Malikov, V. Schechtman, and A. Vaintrob, Chiral de Rham complex, Commun. Math. Phys. 204 (1999), no. 2, 439–473; math.AG/9803041.

[Nar86] K.S. Narain, New heterotic string theories in uncompactified dimensions < 10, Phys. Lett. 169B (1986), 41–46.

[Nik75] V.V. Nikulin, On Kummer surfaces, Math. USSR Isv. 9 (1975), 261–275.

[NN75] A. Newlander and C. Nirenberg, Complex analytic coordinates in almost complex manifolds, Ann. Math. 65 (1975), 391–404.

[NW01] W. Nahm and K. Wendland, A hiker’s guide to K3 – Aspects of \( N = (4,4) \) superconformal field theory with central charge \( c = 6 \), Commun. Math. Phys. 216 (2001), 85–138; hep-th/9912067.

[O’N83] B. O’Neill, Semi-Riemannian geometry with applications to relativity, Pure and Applied Mathematics, vol. 103, Academic
Press, Inc. (Harcourt Brace Jovanovich, Publishers), New York, 1983.

[Poi85] H. Poincaré, *Sur les courbes définies par les équations différentielles (3ème partie)*, Journal de mathématiques pures et appliquées 4 (1885), 167–244.

[Rem91] R. Remmert, *Theory of complex functions*, Graduate texts in mathematics, vol. 122, Springer-Verlag, 1991.

[Sei88] N. Seiberg, *Observations on the moduli space of superconformal field theories*, Nucl. Phys. B303 (1988), 286–304.

[Ser55] J.-P. Serre, *Un théorème de dualité*, Comment. Math. Helv. 29 (1955), 9–26.

[Siu83] Y.T. Siu, *Every K3 surface is Kähler*, Invent. Math. 73 (1983), no. 1, 139–150.

[TW13a] A. Taormina and K. Wendland, *The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}*, JHEP 08 (2013), 125; arXiv:1107.3834 [hep-th].

[TW13b] ——, *Symmetry-surfing the moduli space of Kummer K3s*, accepted for publication in Proc. of the Conference String-Math 2012, PSPUM; arXiv:1303.2931 [hep-th].

[TW13c] ——, *A twist in the M_{24} moonshine story*, accepted for publication in Confluentes Mathematici; arXiv:1303.3221 [hep-th].

[Wei58] A. Weil, *Final report on contract AF 18(603)-57*, in: Oeuvres scientifiques - Collected Works, vol. II (1979), Springer Verlag, 1958, pp. 390–395.

[Wei73] R.O. Wells Jr., *Differential analysis on complex manifolds*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1973, Series in Modern Analysis.

[Wen01] K. Wendland, *Consistency of orbifold conformal field theories on K3*, Adv. Theor. Math. Phys. 5 (2001), no. 3, 429–456; hep-th/0010281.
[Wen02] —, Orbifold constructions of $K3$: A link between conformal field theory and geometry, in: Orbifolds in Mathematics and Physics, vol. 310 of Contemporary Mathematics, AMS, Providence R.I., 2002, pp. 333–358; hep-th/0112006.

[Wen06] —, A family of SCFTs hosting all ‘very attractive’ relatives of the $(2)^4$ Gepner model, JHEP 03 (2006), 102; hep-th/0512223.

[Wen07] —, On superconformal field theories associated to very attractive quartics, Frontiers in Number Theory, Physics and Geometry II (P. Cartier, B. Julia, P. Moussa, and P. Vanhove, eds.), Springer 2007, pp. 223–244; hep-th/0307066.

[Wen10] —, On the geometry of singularities in quantum field theory, Proceedings of the International Congress of Mathematicians, Hyderabad, August 19-27, 2010, Hindustan Book Agency, pp. 2144–2170.

[Wen15] —, Snapshots of conformal field theory, in: Mathematical Aspects of Quantum Field Theories, D. Calaque and Th. Strobl, eds., Springer-Verlag, 2015, pp. 89–129; arXiv:1404.3108 [hep-th].

[Wit87] E. Witten, Elliptic genera and quantum field theory, Commun. Math. Phys. 109 (1987), 525–536.

[Wit88] —, The index of the Dirac operator in loop space, in: Elliptic curves and modular forms in algebraic geometry, P. Landweber, ed., Springer-Verlag, 1988, pp. 161–181.

[Wit94] —, On the Landau-Ginzburg description of $N = 2$ minimal models, Int. J. Mod. Phys. A9 (1994), 4783–4800; hep-th/9304026.

[Yau78] S.T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I, Comm. Appl. Math. 31 (1978), 339–411.

[YB53] K. Yano and S. Bochner, Curvature and Betti numbers, Annals of Mathematics Studies, No. 32, Princeton University Press, Princeton, N. J., 1953.