Existence results for cyclotomic orthomorphisms

David Fear and Ian M. Wanless
School of Mathematical Sciences
Monash University
VIC 3800 Australia
{david.fear, ian.wanless}@monash.edu

Abstract

An orthomorphism over a finite field $\mathbb{F}$ is a permutation $\theta : \mathbb{F} \mapsto \mathbb{F}$ such that the map $x \mapsto \theta(x) - x$ is also a permutation of $\mathbb{F}$. The orthomorphism $\theta$ is cyclotomic of index $k$ if $\theta(0) = 0$ and $\theta(x)/x$ is constant on the cosets of a subgroup of index $k$ in the multiplicative group $\mathbb{F}^*$. We say that $\theta$ has least index $k$ if it is cyclotomic of index $k$ and not of any smaller index. We answer an open problem due to Evans by establishing for which pairs $(q, k)$ there exists an orthomorphism over $\mathbb{F}_q$ that is cyclotomic of least index $k$.

Two orthomorphisms over $\mathbb{F}_q$ are orthogonal if their difference is a permutation of $\mathbb{F}_q$. For any list $[b_1, \ldots, b_n]$ of indices we show that if $q$ is large enough then $\mathbb{F}_q$ has pairwise orthogonal orthomorphisms of least indices $b_1, \ldots, b_n$. This provides a partial answer to another open problem due to Evans. For some pairs of small indices we establish exactly which fields have orthogonal orthomorphisms of those indices. We also find the number of linear orthomorphisms that are orthogonal to certain cyclotomic orthomorphisms of higher index.

Keywords: finite field, cyclotomic orthomorphism, orthogonal orthomorphisms, Weil’s Theorem.

1 Introduction

Suppose that $\mathbb{F}$ is a field of finite order $q \equiv 1 \mod k$. An orthomorphism over $\mathbb{F}$ is a permutation $\theta : \mathbb{F} \mapsto \mathbb{F}$ such that the map $x \mapsto \theta(x) - x$ is also a permutation of $\mathbb{F}$. Since the multiplicative group $\mathbb{F}^*$ is cyclic it has a unique subgroup $C_{k,0}$ of index $k$. The cyclotomy classes of index $k$ are the cosets of $C_{k,0}$ in $\mathbb{F}^*$. The orthomorphism $\theta : \mathbb{F} \mapsto \mathbb{F}$ is cyclotomic of index $k$ if $\theta(0) = 0$ and $\theta(x)/x$ is invariant on cyclotomy classes of index $k$. It is immediate from the definition that an orthomorphism that is cyclotomic of index $k$ is also cyclotomic of index $k'$ for any $k'$ such that $k | k'$ and $k' | (q - 1)$. Let $\mathcal{C}_k = \mathcal{C}_k(q)$ denote the set of all cyclotomic orthomorphisms of index $k$ over $\mathbb{F}$, and define

$$\mathcal{D}_k = \mathcal{C}_k \setminus \bigcup_{\ell < k} \mathcal{C}_\ell.$$  \hspace{1cm} (1)
Following Niederreiter and Winterhof [11], whose work we build on, we say that orthomorphisms in $\mathcal{O}_k$ have least index $k$. One aim of this paper is to establish exactly which fields possess an orthomorphism of least index $k$. In doing so we will answer this problem stated by Evans [6, Problem 28]:

**Problem 1.** If $a \mid b$ and $b \mid (q - 1)$ we know that $C_a(q) \subseteq C_b(q)$. When do we have equality? When do we have inequality?

Two orthomorphisms, $\theta$ and $\theta'$ over $\mathbb{F}$ are orthogonal if $\theta - \theta'$ is a permutation of $\mathbb{F}$. Orthogonal orthomorphisms are particularly useful for constructing orthogonal Latin squares [6]. Moreover, large sets of orthogonal Latin squares can be constructed from orthogonal cyclotomic orthomorphisms [7]. Our second main result shows that for any desired finite sequence of indices, if the field is large enough it will contain a set of pairwise orthogonal cyclotomic orthomorphisms of the desired indices. This gives a partial answer to another problem posed by Evans [6, Problem 27]:

**Problem 2.** For $p$ an odd prime, what types of orthomorphisms can be orthogonal to a non-linear cyclotomic orthomorphism of index $e$, where $1 < e < p - 1$?

The structure of the paper is as follows. In §2 we define the notation that we will use throughout, and show some basic results. In §3 we answer Problem 1 by constructing orthomorphisms with each plausible least index (except for some very small fields where some indices are not achievable). In §4 we give a partial solution to Problem 2 by showing that cyclotomic orthomorphisms of different indices can be orthogonal provided the field is large enough. For certain small indices of cyclotomic orthomorphisms it is possible to compute all cases not covered by the asymptotic results, and thereby state results that hold for all orders. Finally, in §5 we state some open problems and directions for future research.

## 2 The basics

We will be working in a finite field $\mathbb{F} = \mathbb{F}_q$ of order $q$. Suppose $k \mid (q - 1)$. We use $\omega_k$ to denote a complex primitive $k$-th root of unity, and $\eta_k$ to denote a multiplicative character of order $k$ in $\mathbb{F}$. For $0 \leq i < k$, we define $C_{k,i} = \eta_k^{-1}(\omega_k^i)$ to be the $i$-th cyclotomy class in $\mathbb{F}$. In particular, $C_{k,0}$ is the unique subgroup of index $k$ in $\mathbb{F}^*$, and $C_{k,i}$ is a coset of $C_{k,0}$ for each $i$.

A *cyclotomic map* $\theta(x)$ of index $k$ is a map from $\mathbb{F}$ to $\mathbb{F}$ satisfying

$$
\theta(x) = \begin{cases} 
0 & \text{if } x = 0, \\
 a_i x & \text{if } x \in C_{k,i},
\end{cases}
$$

(2)

where $[a_0, a_1, \ldots, a_{k-1}]$ is a list of field elements, called *multipliers*, that defines $\theta$. We write $\theta = [a_0, \ldots, a_{k-1}]$. The map defined in (2) is a *cyclotomic orthomorphism of index $k$* if it is a bijection and

$$
x \mapsto \theta(x) - x = \begin{cases} 
0 & \text{if } x = 0, \\
 (a_i - 1)x & \text{if } x \in C_{k,i},
\end{cases}
$$

is also a bijection. From [6, p. 41] we have:
Lemma 1. A necessary and sufficient condition for \( \theta = [a_0, \ldots, a_{k-1}] \) to be an orthomorphism is that the maps \( C_{k,i} \mapsto a_iC_{k,i} \) and \( C_{k,i} \mapsto (a_i - 1)C_{k,i} \) both permute the cyclotomy classes. Moreover, orthomorphisms \( \theta = [a_0, \ldots, a_{k-1}] \) and \( \theta' = [a'_0, \ldots, a'_{k-1}] \) are orthogonal if and only if \( C_{k,i} \mapsto (a_i - a'_i)C_{k,i} \) permutes the cyclotomy classes.

One way to guarantee that a map \( C_{k,i} \mapsto \lambda_iC_{k,i} \) permutes the cyclotomy classes is to choose all the \( \lambda_i \) from the same cyclotomy class, say \( C_{k,\ell} \). In this case \( C_{k,i} \) is mapped to \( C_{k,(i+\ell) \mod k} \) for each \( i \), which necessarily produces a permutation. This observation will be particularly useful to us when applying Lemma 1 in §

All of the above notation depends on the field we are operating in, which will usually be implicitly understood. However, if the context is such that we are working with more than one field at a time, we adopt the notational convention of specifying the order of the relevant field in the superscript, for example, \( \eta^q \), \( C^q_{k,i} \) or \( \theta = [a_0, \ldots, a_{k-1}] \). Strictly speaking, the notation \( \theta = [a_0, \ldots, a_{k-1}] \) also conceals a dependence on the choice of the character \( \eta_k \), since choosing a different character might reorder the cyclotomy classes. However, this is not an important dependence for us, since it is just an issue of relabelling and we will never change our choice of \( \eta_k \). The sets we are most interested in, \( \mathcal{C}_k \) and \( \mathcal{D}_k \), do not depend on the choice of \( \eta_k \).

Next we look at the case when the list of multipliers for a cyclotomic orthomorphism is periodic.

Lemma 2. Let \( a, b \) be integers such that \( a \mid b \), and suppose \( \theta = [c_0, c_1, \ldots, c_{b-1}] \in \mathcal{C}_b \). Then \( \theta \in \mathcal{C}_a \) if and only if \( c_i = c_{i+a} \) for \( 0 \leq i < b - a \).

Proof. Suppose we have a character \( \eta_b \) of order \( b \). Then \( \eta_a = (\eta_b)^{b/a} \) is a character of order \( a \). Moreover, if \( x \in \eta_b^{-1}(\omega_b^{i+\lambda a}) \) for some integer \( \lambda \) then
\[
\eta_a(x) = (\eta_b(x))^{b/a} = (\omega_b^{i+\lambda a})^{b/a} = \omega_b^{ib/a+\lambda b} = \omega_b^{ib/a},
\]
which is independent of \( \lambda \). Hence \( C_{b,c} \cup C_{b,c+a} \cup C_{b,c+2a} \cup \cdots \cup C_{b,c+(b/a-1)a} \) is a cyclotomy class of index \( a \), for each \( 0 \leq c < a \). This result now follows from the definition of cyclotomic orthomorphisms.

Orthomorphisms in \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are called linear and quadratic, respectively [6]. We define a cyclotomic orthomorphism \( \theta = [a_0, \ldots, a_{k-1}] \) of index \( k \) to be near-linear if \( a_0 \neq a_1 = a_2 = \cdots = a_{k-1} \). For example, all non-linear quadratic orthomorphisms are near-linear. The name arises because, by Lemma 2 if the first multiplier of a near-linear orthomorphism was changed to agree with the later multipliers, the orthomorphism would become linear. Another immediate consequence of Lemma 2 is this:

Lemma 3. If \( \theta \) is a near-linear orthomorphism of index \( k \), then \( \theta \in \mathcal{D}_k \).

This last result, plus the fact that they are easy to construct, is the main reason for our interest in near-linear orthomorphisms. From Lemma 1 we have:

Lemma 4. The cyclotomic map \( \theta = [a_0, a_1, a_2, \ldots, a_1] \) of index \( k \) is a near-linear orthomorphism if and only if \( a_0 \neq a_1, \eta_k(a_0) = \eta_k(a_1) \) and \( \eta_k(a_0 - 1) = \eta_k(a_1 - 1) \).
In other words, to construct a near-linear orthomorphism we simply need to find two multipliers \( a_0 \) and \( a_1 \) from the same cyclotomy class such that \( a_0 - 1 \) and \( a_1 - 1 \) belong to the same cyclotomy class as each other. This is a simple task for a computer to do. There is one restriction, though, on when we might expect to find a near-linear orthomorphism.

There are no near-linear orthomorphisms of index \( q - 1 \) or \( (q - 1)/2 \) over \( \mathbb{F}_q \), for any \( q \). This is because a near-linear orthomorphism differs from a linear orthomorphism on a single cyclotomy class. Cyclotomy classes of index \( q \) or \( (q - 1)/2 \) have size 1 or 2 respectively.

Two different orthomorphisms must differ on at least 3 elements of their domain (c.f. [3, Thm 5]). Another way to see that near-linear orthomorphisms cannot have index \( q - 1 \) or \( (q - 1)/2 \) is to apply the following result by Niederreiter and Winterhof [11].

**Theorem 1.** The number of near-linear orthomorphisms over \( \mathbb{F}_q \) of a given index \( k \) is \( (q - 1 - k)(q - 1 - 2k)/k^2 \).

We will not explicitly need the following two results, but they seem worth recording.

**Lemma 5.** Let \( \theta = [a_1, \ldots, a_n]^F \) be a cyclotomic orthomorphism over a field \( F \). Let \( E \) be an extension field of \( F \). Then \( \phi = [a_1, \ldots, a_n]^E \) is an orthomorphism over \( E \).

*Proof.* Immediate from Lemma [1].

**Lemma 6.** Let \( E \) be an extension field of a field \( F \), where \( |F| \geq 4 \). Let \( a_1, a_2 \) be distinct elements of \( F \setminus \{0, 1\} \). Define \( k = (|E| - 1)/(|F| - 1) \). Then \( \theta = [a_1, a_2, \ldots, a_2]^E \) (where \( a_2 \) occurs \( k - 1 \) times) is an orthomorphism over \( E \) of least index \( k \).

*Proof.* The cyclotomy class \( C_{k,0} \) in \( E \) is the non-zero elements of the subfield \( F \). Since \( a_1, a_2, a_1 - 1 \) and \( a_2 - 1 \) are all in \( C_{k,0} \), Lemma [1] tells us that \( \theta \) is an orthomorphism. Also, \( \theta \) has least index \( k \), by Lemma [6].

## 3 Existence of cyclotomic orthomorphisms

The aim of this section is to establish exactly which fields have cyclotomic orthomorphisms of a given least index. In doing so we will completely answer Problem [1]. It is easily checked that \( C_1(3) = C_2(3) \), \( C_1(4) = C_3(4) \), \( C_1(5) = C_2(5) = C_4(5) \) and \( C_1(7) = C_3(7) \).

We will show:

**Theorem 2.** Let \( q \) be any prime power. Except for the cases just listed, \( C_a(q) \neq C_b(q) \) whenever \( a, b \) are distinct integers satisfying \( a \mid b \) and \( b \mid (q - 1) \).

Note that to prove \( C_a \neq C_b \) it suffices to show that \( \mathcal{D}_b \) is non-empty. Theorem [1] together with Lemma [8] does this for all \( b < (q - 1)/2 \). This fact was noted in [11], leaving open the cases \( b = (q - 1)/2 \) and \( b = q - 1 \). These two cases will be handled respectively by Theorems [9] and [1] below. Theorem [2] will then follow immediately.

If \( q \) is an odd prime power then the cyclotomy classes \( C_{(q-1)/2,i} \) each have the form \( \{\pm x\} \) for some \( x \in \mathbb{F}_q \). We next give a method for building cyclotomic orthomorphisms of index \( (q - 1)/2 \).
Lemma 7. Suppose there exists an integer \( h \), permutations \( \rho, \tau \in S_h \), a vector \( \sigma \in \{ \pm 1 \}^h \) and nonzero field elements \( r, m_i, v_i \in \mathbb{F}_q^* \) such that

\[
\sigma(i)m_iv_i = rv_{\rho(i)} \tag{3}
\]

\[
(m_i - 1)v_i = (r - 1)v_{\tau(i)} \tag{4}
\]

for \( 1 \leq i \leq h \). Suppose also that

\[
0 \neq (m_1 - r) \prod_{i=2}^h (m_1 - m_i) \prod_{1 \leq i < j \leq h} (v_i^2 - v_j^2). \tag{5}
\]

Then there exists an orthomorphism \( \phi \in \mathcal{D}_{(q-1)/2} \).

Proof. We first note that \( m_1 \neq r \) by (5), which means that \( r \neq 1 \) and \( m_i \neq 1 \) for \( 1 \leq i \leq h \), by (4). Define a map \( \phi : \mathbb{F}_q \to \mathbb{F}_q \) by

\[
\phi(x) = \begin{cases} 
  m_ix & \text{if } x = \pm v_i \text{ for some } 1 \leq i \leq h, \\
  rx & \text{otherwise.}
\end{cases}
\]

Condition (5) ensures that \( \phi \) is a well-defined cyclotomic map of index \((q-1)/2\). Condition (4) and (3) ensure respectively that the maps \( x \mapsto \phi(x) \) and \( x \mapsto \phi(x) - x \) are permutations of \( \mathbb{F}_q \). Hence \( \phi \in \mathcal{C}_{(q-1)/2} \). Finally, (5) ensures that \( m_1 \) is distinct from \( m_2, \ldots, m_h \) and \( r \), which implies that \( \phi \in \mathcal{D}_{(q-1)/2} \) by Theorem 2. \( \square \)

It is routine to check that each of the following examples satisfy the hypotheses of Lemma 7.

Example 1: Let \( q > 10 \) be an odd prime power. Suppose \( \mathbb{F}_q \) is a field containing an element \( \xi \) such that \( \xi^2 = -3 \). Let \( h = 3, \rho = [2, 3, 1], \tau = [3, 1, 2], \sigma = [1, 1, 1], r = (1 + \xi)/2 \) and

\[
v_1 = (v_3 - v_3\xi + 1 + \xi)/2, \quad m_1 = \frac{1 + \xi}{v_3 - v_3\xi + 1 + \xi}, \quad m_2 = \frac{(1 + \xi)v_3/2}{2v_3}, \quad m_3 = \frac{2v_3 - 1 + \xi}{2v_3},
\]

where \( v_3 \) is any field element that is not a root of the polynomial

\[
x(x^2 - 1)(2x + \xi + 1)(2x + \xi - 1)(\xi x + x - 2)(\xi x - x - 2) \times (\xi x - x - 1 - \xi)(\xi x - x - 3 - \xi)(\xi x - 3x - 1 - \xi).
\]

There is a choice for \( v_3 \) available, given that \( q > 10 \).

Example 2: Let \( q > 12 \) be an odd prime power. Suppose \( \mathbb{F}_q \) is a field containing an element \( \xi \) such that \( \xi^2 = -1 \). Let \( h = 4, \rho = [2, 3, 4, 1], \tau = [4, 1, 2, 3], \sigma = [1, 1, 1, 1], \)

\[
5
\[ r = (\xi + 1)/2, \]
\[ v_1 = (v_3 - 1 + \xi)/\xi, \]
\[ v_2 = 1, \]
\[ v_4 = (v_3 + v_3\xi - 1)/\xi, \]
\[ m_1 = \frac{\xi - 1}{2(v_3 - 1 + \xi)}, \]
\[ m_2 = \frac{(1 + \xi)v_3/2}{2v_3 - 1 + \xi}, \]
\[ m_3 = \frac{v_3 - 1 + \xi}{2v_3}, \]
\[ m_4 = \frac{v_3 + v_3\xi - 2}{2(v_3 + v_3\xi - 1)}, \]

where \( v_3 \) is any field element that is not a root of the polynomial
\[
x(x^2 - 1)(x + \xi)(x + \xi - 1)(2x + \xi - 1)(x + 2\xi - 1)
\times (x + \xi x - 2)(x + \xi x - 1)(x + \xi x - 1 + \xi)(2x + \xi + \xi x - 2)(2\xi x + x - 1).\]

There is a choice for \( v_3 \) available, given that \( q > 12 \).

**Example 3:** Let \( q \) be a power of a prime \( p > 3 \). Suppose \( \mathbb{F}_q \) is a field containing an element \( \xi \) such that \( \xi^2 = -2 \) (in characteristic 11, we must be careful to choose \( \xi = 3 \) since \( \xi = -3 \) results in \( v_1 = -v_2 \)). Let \( h = 4, \rho = [4, 3, 2, 1], \tau = [3, 1, 4, 2], \sigma = [-1, -1, -1, 1], r = 1 - \xi, \) and

\[ v_1 = -2, \]
\[ v_2 = -\xi - 1, \]
\[ v_3 = 1, \]
\[ v_4 = \xi, \]
\[ m_1 = \xi/2 + 1, \]
\[ m_2 = \frac{\xi + 2}{\xi - 2}, \]
\[ m_3 = 3, \]
\[ m_4 = \xi + 2. \]

**Example 4:** Suppose \( \mathbb{F}_q \) is a field of odd characteristic other than 3, 7 or 17. Further suppose that \( \mathbb{F}_q \) contains an element \( \xi \) such that \( \xi^2 = 2 \) (if \( \mathbb{F}_q \) has characteristic 23, we take \( \xi = -5 \), since \( \xi = 5 \) results in \( v_3 = v_6 \)). Let \( h = 6, \rho = [5, 4, 2, 1, 6, 3], \tau = [2, 3, 1, 6, 4, 5], \sigma = [1, -1, -1, 1, 1, -1], r = \xi, \) and

\[ v_1 = -1 + \xi, \]
\[ v_2 = 1, \]
\[ v_3 = -3 + \xi, \]
\[ v_4 = 4 - 3\xi, \]
\[ v_5 = 2 - 2/\xi, \]
\[ v_6 = 2, \]
\[ m_1 = 2, \]
\[ m_2 = 6 - 4\xi, \]
\[ m_3 = \frac{\xi}{3 - \xi}, \]
\[ m_4 = \frac{\xi - 2}{3\xi - 4}, \]
\[ m_5 = 2 + 2\xi, \]
\[ m_6 = -1 + 3\xi/2. \]

**Theorem 3.** There exists an orthomorphism \( \phi \in \mathcal{D}_{(q-1)/2} \) for all odd prime powers \( q \notin \{5, 7\} \).
Proof. It is easy to check by exhaustion that \( \mathcal{D}_{(q-1)/2} = \emptyset \) when \( q \in \{5, 7\} \). For \( q = 3 \) the map \( x \mapsto 2x \) is in \( \mathcal{D}_{(q-1)/2} \). For \( q = 9 \) we use Example 1 with \( \xi = 0 \) and \( v_3 = 1 + \sqrt{-1} \) in \( \mathbb{F}_9 = \mathbb{Z}_3[\sqrt{-1}] \). Henceforth we assume that \( q > 11 \). For \( \mathbb{F}_q \) of characteristic 3 or 7 we can use Example 1, and for \( q \equiv 1 \mod 4 \) we can use Example 2. So suppose that \( q \equiv 3 \mod 4 \) where \( \mathbb{F}_q \) has characteristic at least 11. Since \(-1\) is not a quadratic residue in \( \mathbb{F}_q \), we know that either \(-2\) is a quadratic residue or \( 2 \) is a quadratic residue. In the former case we use Example 3 and in the latter case we use Example 4.

The only remaining case of Theorem 2 is when \( b = q - 1 \). We say that an orthomorphism over \( \mathbb{F}_q \) that lies in \( \mathcal{D}_{q-1} \) is non-cyclotomic, since it makes no use of the cyclotomic structure of \( \mathbb{F}_q \). Also, following the terminology and notation of Evans [6], we define a translation \( T_g \) of an orthomorphism \( \theta \) to be the orthomorphism \( T_g[\theta](x) = \theta(x+g) - \theta(g) \).

In any field of prime order, translating an orthomorphism in \( \mathcal{D}_c \) where \( 1 < c < q - 1 \) by a nonzero \( g \) will produce a non-cyclotomic orthomorphism. However, the same is not guaranteed in fields of composite order \([6, \text{p. 48}]\). Nevertheless, translation will still be a useful tool in our next proof.

**Theorem 4.** There are non-cyclotomic orthomorphisms over every field \( \mathbb{F}_q \) of order \( q > 5 \).

Proof. First, we consider the case where \( \mathbb{F}_q \) is a field of characteristic 2, with \( q \geq 8 \). The construction for this case was shown to us by A.B. Evans. Pick \( a \in \mathbb{F}_q \setminus \{0,1\} \). Set \( H = \{0, 1, a, a+1\} \subseteq \mathbb{F}_q \) and choose \( c \not\in H \). Let

\[
\theta(x) = \begin{cases} 
ax + a(a+1) & \text{if } x \in H + c, \\
ax & \text{else.}
\end{cases}
\]  

(6)

Now \( ax \in a(H + c) + a(a+1) \) if and only if \( x \in H + c \), which implies that \( \theta \) is a permutation. Also, \( (a-1)x \in (a-1)(H + c) + a(a+1) \) if and only if \( x \in H + c \), and hence \( \theta \) is an orthomorphism. Note that \( \theta(0) = 0 \) and there are exactly four nonzero elements \( e \in \mathbb{F}_q \) such that \( \theta(e)/e \neq a \). Since 4 is coprime to \( q - 1 \), it follows that \( \theta \) is non-cyclotomic.

Next, we consider the case where \( \mathbb{F}_q \) is a field of characteristic \( c > 2 \). We note that every orthomorphism is defined by a permutation polynomial. By [6, p. 47] or [11, Thm 1], an orthomorphism \( \theta \in \mathcal{D}_k \) corresponds to a polynomial of the form

\[
\theta(x) = \sum_{i=0}^{k-1} a_i x^{i(q-1)/k+1},
\]  

(7)

where \( a_i \in \mathbb{F}_q \) for \( 0 \leq i < k \). Let \( \theta \) be an orthomorphism such that \( \theta \in \mathcal{D}_2 \) (we know such \( \theta \) exists, for example, by Theorem 1). Hence, \( \theta \) is of the form \( \theta(x) = a_1 x^{(q+1)/2} + a_0 x \) where \( a_1 \neq 0 \). In particular, \( T_1[\theta] \) is defined as:

\[
T_1[\theta](x) = a_1 x^{(q+1)/2} + \frac{a_1(a+1)}{2} x^{(q-3)/2} + O(x^{(q-3)/2}),
\]

and since this is not of the form in equation (7) for any \( k < q - 1 \), we must have \( T_1[\theta] \in \mathcal{D}_{q-1} \). 

This completes the proof.
In our proof of Theorem 4, we made use of translation to construct non-cyclotomic orthomorphisms. However, these still have the property that they are easily transformed into a cyclotomic orthomorphism, and as such still possess an underlying cyclotomic structure. We say that an orthomorphism $\theta$ is \textit{irregular} if $T_g[\theta]$ is non-cyclotomic for all $g \in \mathbb{F}_q$. We conjecture that irregular orthomorphisms exist over all sufficiently large fields. However, here we only prove the much weaker statement that there are arbitrarily large fields that have irregular orthomorphisms.

**Theorem 5.** Let $\mathbb{F}_q$ be a field of order $q = 2^{2k+1}$ for some positive integer $k$. Then there exists an irregular orthomorphism $\theta$ over $\mathbb{F}_q$.

**Proof.** Consider the translates $T_g[\theta]$ of the orthomorphism $\theta$ constructed in (6). We will consider two cases.

Case 1: $g \in H + c$.
If $x + g \in H + c$ then

$$T_g[\theta](x) = \theta(x + g) - \theta(g) = a(x + g) + a(a + 1) + ag + a(a + 1) = ax.$$ 

If $x + g \not\in H + c$ then

$$T_g[\theta](x) = \theta(x + g) - \theta(g) = ax + ag - ag - a(a + 1) = ax + a(a + 1).$$

In this case we have exactly 3 non-zero elements satisfying $T_g[\theta](x) = ax$, and since 3 is relatively prime to $q - 1$ (given that $q = 2^{2k+1} \equiv 2 \mod 3$), $T_g[\theta]$ is non-cyclotomic.

Case 2: $g \not\in H + c$.
If $x + g \in H + c$ then

$$T_g[\theta](x) = \theta(x + g) - \theta(g) = a(x + g) + a(a + 1) - ag = ax + a(a + 1).$$

If $x + g \not\in H + c$ then

$$T_g[\theta](x) = \theta(x + g) - \theta(g) = ax + ag - ag = ax.$$ 

In this case, there are exactly $q - 5$ non-zero elements $x$ satisfying $T_g[\theta](x) = ax$. Since $q - 5$ is coprime to $q - 1$ when $q$ is a power of 2, the translate $T_g[\theta]$ is non-cyclotomic.

As all its translates are non-cyclotomic, $\theta$ is irregular. \qed

4 Orthogonal cyclotomic orthomorphisms

This section is devoted to the study of orthogonality between cyclotomic orthomorphisms with given least indices. Our main result (Theorem 6) will give a partial answer to Problem 2. The proof will need the following Lemma from [1]:

**Lemma 8.** Suppose $k, t \geq 2$ and $k \mid (q-1)$. Let $a_1, \ldots, a_t$ be distinct elements of the finite field $\mathbb{F}_q$. Then the number of solutions $x \in \mathbb{F}_q$ to the system of equations $\eta_k(a_i + x) = 1$ where $i = 1, \ldots, t$ is between $qk^{-t} - t\sqrt{q}$ and $qk^{-t} + t\sqrt{q}$. 

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Actually, closer inspection of the proof in [1] shows that the lower bound can be improved somewhat, which will leave us with fewer cases to check in some of our computations.

**Lemma 9.** Suppose $k, t \geq 2$ and $k \mid (q-1)$. Let $a_1, \ldots, a_t$ be distinct elements of the finite field $\mathbb{F}_q$. Then the number of solutions $x \in \mathbb{F}_q$ to the system of equations $\eta_k(a_i + x) = 1$ where $i = 1, \ldots, t$ is at least $qk^{-t} - (t - 1 - t/k + k^{-t})\sqrt{q} - t/k$.

**Proof.** We use similar notation and logic to [1]. Let $N$ denote the number of solutions to the given system of equations. Let $\eta_k$ be a multiplicative character of order $k$. Let $\Psi^*$ denote the set of functions $\psi : \{1, \ldots, t\} \to \{0, \ldots, k - 1\}$ that are not everywhere zero. Let

$$S = q + \sum_{\psi \in \Psi^*} \sum_{x \in \mathbb{F}_q} \eta_k\left(\prod_{i=1}^t (a_i + x)^{\psi(i)}\right).$$

Using Weil’s theorem [1] Thm 3.7 to bound the terms in $S$, but bounding each term according to its number $d$ of distinct roots, we have

$$|S| \geq q - \sum_{d=1}^t \left(\frac{t}{d}\right)(k - 1)^d(d - 1)\sqrt{q} = q - (t - 1)k^t - tk^{t-1} + 1)\sqrt{q}.$$

Hence, using the relationship between $N$ and $S$ from [1],

$$N \geq \frac{|S| - tk^{t-1}}{k^t} = qk^{-t} - (t - 1 - t/k + k^{-t})\sqrt{q} - t/k,$$

as claimed. \hfill \square

**Theorem 6.** Let $B = b_1, \ldots, b_n$ be a finite sequence of positive integers, and let $c$ be a positive integer. Define $t = 2 + \sum_{i=1}^n b_i$ and $k = \text{lcm}(b_1, \ldots, b_n, c)$ and let $q_0$ be the smallest positive integer satisfying $q_0k^{-t} - (t - 1 - t/k + k^{-t})\sqrt{q_0} - t/k > 1$. Let $\mathbb{F}_q$ be a field such that $k \mid (q-1)$ and $q \geq q_0$. Given any set of orthomorphisms $T = \{\theta_1, \ldots, \theta_n\}$ over $\mathbb{F}_q$, where $\theta_i \in \mathcal{C}_{b_i}$ for $1 \leq i \leq n$, there exists a near-linear orthomorphism, $\phi \in \mathcal{D}_c$, which is orthogonal to each of the orthomorphisms in $T$.

**Proof.** Since $q \geq q_0$ and $t - 1 - t/k + k^{-t} \geq 0$, we have

$$qk^{-t} - (t - 1 - \frac{t}{k} + k^{-t})\sqrt{q} - \frac{t}{k} \geq \left(q_0k^{-t} - (t - 1 - \frac{t}{k} + k^{-t})\sqrt{q_0} - \frac{t}{k}\right)\frac{q}{q_0} > 1. \tag{8}$$

Suppose that $\theta_i = [a_{i,0}, a_{i,1}, \ldots, a_{i,b_i-1}]$. Define $A_i = \{-a_{i,0}, -a_{i,1}, \ldots, -a_{i,b_i-1}\}$ and $A = \bigcup_{i=1}^n A_i \cup \{0, -1\}$. Consider the system of equations $\eta_k(x + a) = 1$ for each $a \in A$. By (3) and Lemma 3 we can find two distinct solutions $x = x_1$ and $x = x_2$. Define $\phi = [x_1, x_2, \ldots, x_2]$. By choice, $\eta_k(x_1) = \eta_k(x_2) = 1 = \eta_k(x_1 - 1) = \eta_k(x_2 - 1)$, so $\phi \in \mathcal{D}_c$, by Lemma 3 and Lemma 4. Similarly, since $\eta_k(x_i - a_{i,j}) = \eta_k(x_2 - a_{i,j}) = 1$ for $1 \leq i \leq n$ and $0 \leq j < b_i$, we see by Lemma 4 that $\phi$ is orthogonal to each $\theta_i \in T$. \hfill \square

Repeatedly applying the method in the proof of Theorem 6 and noting that the set $A$ grows by only 2 elements at each iteration, we get:
Corollary 1. Let \( B = b_1, b_2, \ldots, b_n \) be a finite sequence of positive integers. Define \( t = 2n \) and \( k = \text{lcm}(b_1, \ldots, b_n) \). Let \( q_0 \) be the smallest positive integer satisfying the inequality \( q_0 k^{-t} - (t - 1 - t/k + k^{-t}) \sqrt{q_0} - t/k > 1 \). Let \( F_q \) be a field such that \( k \mid (q - 1) \) and \( q \geq q_0 \). Then there exists a set of mutually orthogonal strong orthomorphisms \( \{\theta_1, \ldots, \theta_n\} \) over \( F_q \), where \( \theta_i \in \mathcal{D}_b \) for \( 1 \leq i \leq n \).

This gives a partial answer to Problem 2 (in fact, we do not use the assumption in Problem 2 that we are working in a field of odd prime order). The answer is that any conceivable set of orthogonals is achievable, provided the field is large enough. The measure of “large enough” we have given is likely to be quite conservative in the sense that orthogonal orthomorphisms will often exist for much smaller orders. For example, [8, 31], [14, 44, 44], [47, 11, 11, 11, 11] are orthogonal near-linear orthomorphisms of indices 2, 3, 5 over \( F_{61} \). Also [165, 121], [111, 326, 326], [90, 132, 132, 132, 132], [47, 175, 175, 175, 175, 175], [57, 21, 21, 21, 21, 21], are orthogonal near-linear orthomorphisms of indices 2, 3, 5, 7 over \( F_{421} \). Nonetheless, some small fields do not possess orthogonal orthomorphisms of all the indices they plausibly might, so asymptotic results are probably the best we can hope for in general. The main opportunity for improving the scope of our results might be to prove existence of sets of orthogonal cyclotomic orthomorphisms, at least one of which has an index comparable to \( \sqrt{n} \).

A similar result to Theorem 6 holds if we impose more structure on our orthomorphisms. An orthomorphism \( \theta \) is called a strong orthomorphism (also called a complete mapping) if it is an orthomorphism over a field \( F \) and if \( x \mapsto \theta(x) + x \) is also a permutation of \( F \). We say that a strong orthomorphism is a cyclotomic strong orthomorphism of index \( k \) if it is both a strong orthomorphism, and a cyclotomic orthomorphism of index \( k \). For a survey of results on strong orthomorphisms see [8] and for a proof of the existence of cyclotomic strong orthomorphisms of index 2, see [2]. We can easily adapt Theorem 6 to prove the asymptotic existence of cyclotomic strong orthomorphisms of any given least index.

Theorem 7. Let \( B = b_1, b_2, \ldots, b_n \) be a finite sequence of positive integers, and let \( c \) be a positive integer. Define \( t = 3 + \sum b_i \) and \( k = \text{lcm}(b_1, \ldots, b_n, c) \) and let \( q_0 \) be the smallest positive integer such that \( q_0 k^{-t} - (t - 1 - t/k + k^{-t}) \sqrt{q_0} - t/k > 1 \). Let \( F_q \) be a field such that \( k \mid (q - 1) \) and \( q \geq q_0 \). Given any set of orthomorphisms \( T = \{\theta_1, \ldots, \theta_n\} \) over \( F_q \), where \( \theta_i \in \mathcal{D}_b \) for \( 1 \leq i \leq n \), there exists a near-linear strong orthomorphism, \( \phi \in \mathcal{D}_c \), which is orthogonal to each of the orthomorphisms in \( T \).

Proof. The proof is the same as for Theorem 6 except we put \( A = \cup_{i=1}^{n} A_i \cup \{0, -1, 1\} \). \( \square \)

Corollary 2. Let \( B = b_1, b_2, \ldots, b_n \) be a finite sequence of positive integers. Define \( t = 2n + 1 \) and \( k = \text{lcm}(b_1, \ldots, b_n) \). Let \( q_0 \) be the smallest positive integer satisfying the inequality \( q_0 k^{-t} - (t - 1 - t/k + k^{-t}) \sqrt{q_0} - t/k > 1 \). Let \( F_q \) be a field such that \( k \mid (q - 1) \) and \( q \geq q_0 \). Then there exists a set of mutually orthogonal strong orthomorphisms \( \{\theta_1, \ldots, \theta_n\} \) over \( F_q \), where \( \theta_i \in \mathcal{D}_b \) for \( 1 \leq i \leq n \).

The previous section was devoted to working out when \( \mathcal{D}_b \) is nonempty. As a side remark, we note two alternate proofs that \( \mathcal{D}_b \) is non-empty for \( q \) large relative to \( b \). The first is by applying Theorem 6 in the case where \( B \) is the empty sequence. The second
is to use recent work by Bell [2]. He showed that \(|c_k| = k!^2 k^{-2} q^k (1 + O(q^{-1/2}))\) for \(k\) fixed, as \(q \to \infty\) with \(q \equiv 1 \mod k\). It follows that

\[ |D_k| = |c_k| - \sum_{c \neq k} |c_{c|k}| = k!^2 k^{-2} q^k (1 + O(q^{-1/2})). \quad (9) \]

See [1, 10, 12] for some congruences satisfied by the number of orthomorphisms of fields of prime order. For non-trivial bounds on the number of all orthomorphisms see [3, 9, 10, 13], although the known lower bounds only apply to fields of prime order. It would be of interest to find analogous lower bounds for fields of composite order. In that direction, we note the following direct corollary of [11, Thm 3].

**Theorem 8.** Let \(q\) be a prime power such that \(q \equiv 1 \mod k\) for some \(k > 2\). Then \(\mathbb{F}_q\) has at least \(2^{(q-1)/k}\) orthomorphisms.

If preferred, the simpler bound from Lemma 8 can be used instead of the bound from Lemma 9 in Theorem 6, Theorem 7 and their corollaries. However, if \(k, t\) are given even then in the more complicated bound the value of \(q_0\) is easily found by solving the quadratic equation in \(\sqrt{q_0}\). By explicit computations in fields of orders less than \(q_0\) we were able to establish:

**Theorem 9.** Let \(q\) be a prime power such that \(q \equiv 1 \mod c\) where \(1 < a < b\) and \(c = \text{lcm}(a, b) \leq 6\). Then there exists a pair of orthogonal orthomorphisms \(\theta \in D_a^q\) and \(\theta' \in D_b^q\) if and only if \(q > 7\) and

\[(q, a, b) \notin \{(9, 2, 4), (13, 2, 3), (13, 2, 6)\}.\]

Note that Corollary 1 proves Theorem 9 for all \(q \geq 9154945\) (whereas using the bound from Lemma 8 would require us to consider \(q\) up to 26876448). For \(q < 9154945\) we first searched for near-linear \(\theta, \theta'\). In the few cases where near-linear examples did not exist we did an exhaustive search for \(\theta, \theta'\). The exhaustive search was quick, since it was only ever required for \(q \leq 19\).

We omitted the case \(a = 1\) from Theorem 9 since in that case we can obtain a stronger result. It is known (see e.g. [6, p. 39]) that every orthomorphism over \(\mathbb{F}_q\) of least index 2 is orthogonal to exactly \((q - 7)/2\) linear orthomorphisms. More generally, we have:

**Theorem 10.** Let \(q \equiv 1 \mod d\) where \(d \geq 2\). Any near-linear \(\theta \in D_d^q\) is orthogonal to precisely \((q - 3d - 1)/d\) linear orthomorphisms.

**Proof.** Suppose \(\theta = [a, b, \ldots, b]\), where \(a \neq b\), \(\eta_d(a) = \eta_d(b)\) and \(\eta_d(a - 1) = \eta_d(b - 1)\). We wish to find all \(c \notin \{0, 1\}\) for which \([c]\) is orthogonal to \(\theta\). The number of such \(c\) is

\[ |\{c \in \mathbb{F}_q : \eta_d(c - a) = \eta_d(c - b)\}| - 2 = \left| \{c \in \mathbb{F}_q \setminus \{a\} : \eta_d\left(\frac{c - b}{c - a}\right) = 1\} \right| - 2 = (q - 1)/d - 3, \]

since \((c - b)/(c - a)\) may take any value in \(C_{d,0} \setminus \{1\}\), and for each such value there is exactly one \(c\) that achieves it. The result follows. \(\square\)
Theorem 10 does not generalise beyond the near-linear case, at least not in an obvious fashion. For example, consider $\mathbb{F}_{31}$ and choose $\eta_3$ such that $3 \in C_{31}$. Then $[3, 9, 2] \in D_3$ is orthogonal to only one linear orthomorphism, namely $[8]$, whereas $[3, 9, 16] \in D_3$ is orthogonal to five different linear orthomorphisms. By comparison, all near-linear orthomorphisms in $D_3$ are orthogonal to seven different linear orthomorphisms, by Theorem 10.

We remark that Theorem 10 is vacuous if $q < 3d + 1$, by Theorem 1. However, for larger $q$ it helps us to show:

**Theorem 11.** Let $q$ be a prime power and let $d$ be a positive integer dividing $q - 1$. There exists an orthomorphism $\theta \in D_q^d$ which is orthogonal to some linear orthomorphism, except when

$$(q, d) \in \{(2, 1), (3, 1), (3, 2), (4, 3), (5, 2), (5, 4), (7, 2), (7, 3), (7, 6), (13, 4)\},$$

and possibly when $d > 50$ and either $q = 2d + 1 \equiv 3 \mod 4$ or $q = 3d + 1$.

**Proof.** Suppose $q = ed + 1$. If $e \geq 4$ then by Theorem 1 there is a near-linear orthomorphism in $D_q^d$, and it is orthogonal to a positive number of linear orthomorphisms by Theorem 10. Hence we may assume that $e \leq 3$.

Next we discuss the case $q = d + 1$. Assuming $q \geq 11$, then by Theorem 1 and Theorem 10 we can find an orthogonal pair of orthomorphisms $\theta_1 \in D_1$ and $\theta_2 \in D_2$. Let $g \in \mathbb{F}_q \setminus \{0\}$. As per the proof of Theorem 4, we know that $T_g[\theta_2] \in D_{q-1}$, whereas $T_g[\theta_1] = \theta_1$. Since translation preserves orthogonality, these two orthomorphisms provide the example we need.

Now suppose $13 \leq q = 2d + 1 \equiv 1 \mod 4$. In this case, we use the orthomorphism $\phi$ from Lemma 7 constructed using Example 2. It is not hard to check that $\phi$ is orthogonal to the linear orthomorphism $[(q + 1)/2]$, given that $(m_i - 1/2)v_i = (r - 1/2)v_{\pi(i)}$ where $\pi = [3, 4, 1, 2]$.

For small orders not covered by the above results, exhaustive computations reveal the exceptions quoted in the theorem, with no other exceptions for $d \leq 50$.

We conjecture that (10) is the full list of exceptions.

## 5 Concluding remarks

The success of Evans [7] in constructing large maximal sets of mutually orthogonal Latin squares using orthogonal linear and quadratic orthomorphisms motivates further study of cyclotomic orthomorphisms and their orthogonality properties.

We have answered Problem 1 completely and Problem 2 partially. However, several open questions were raised. In §3 we formulated the conjecture that irregular orthomorphisms exist over all large enough fields. We also noted in passing that the exponential lower bound on the number of orthomorphisms given in [3] applies only to fields of prime order. In Theorem 8 we gave a similar exponential lower bound for some fields of composite order. However, these results could surely be improved.

Our results in §4 showed existence of orthomorphisms of fixed least indices when the field is large. Future work could pursue similar results for orthogonality between orthomorphisms with relatively large least index. It would also be of interest to resolve the cases $q = 2d + 1 \equiv 3 \mod 4$, and $q = 3d + 1$ which were left unsolved in Theorem 11.
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