Cosmological string backgrounds from super Poisson-Lie T-plurality

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We generalize the formulation of Poisson-Lie (PL) T-plurality proposed by R. von Unge [JHEP 07 (2002) 014] from Lie groups to Lie supergroups. By taking a convenient ansatz for metric of the $\sigma$-model in terms of the left-invariant one-forms of the isometry Lie supergroups $(C^3 + A)$ and $GL(1|1)$ we construct cosmological string backgrounds, including $(2 + 1|2)$-dimensional metric, time-dependent dilaton and vanishing torsion, in a way that they satisfy the one-loop beta-function equations. Starting from the decompositions of semi-Abelian Drinfeld superdoubles (DSDs) generated by the $(C^3 + \mathcal{A})$ and $gl(1|1)$ Lie super bi-algebras we find the conformal duality/plurality chains of $2 + 1$-dimensional cosmological string backgrounds coupling with two fermionic fields. In particular, the new backgrounds obtained by the super PL T-plurality remain conformally invariant at one-loop level.

Keywords: String duality, $\sigma$-model, String cosmology, Super Poisson-Lie T-plurality

1 Introduction

PL T-duality was proposed by Klimčík and Ševera [1–3] in 1995 as an extension of the Abelian T-duality [4,5] to non-Abelian groups of isometries. Already in their papers they considered the possibility of what is now called PL T-plurality but the first explicit formulas for the background and dilaton shift and a general discussion of all (possibly) conformal PL T-duality chains in three dimensions were given by R. von Unge [6]. He extended the path-integral formalism for PL T-duality to include the case of Drinfel’d doubles (DDs) which can be decomposed into Lie bi-algebras in more than one way. The possibility to decompose some DDs into more than two Manin triples enables us to construct more than two equivalent $\sigma$-models. This idea was explicitly realized in [6] and the classical equivalence of the $\sigma$-models was called the PL T-plurality. Various aspects of the PL T-plurality along with
some examples were discussed in [7–9] (see also [10, 11]). Furthermore, by using a suitable extension of DD it was obtained formulas for the PL T-plurality transformation in the presence of spectators for the σ-model background [12]. The PL T-plurality has also appeared as an $O(n, n)$ transformation in [13]. There, it has been shown that the double field theory equations of motion are indeed satisfied even in the presence of spectators; moreover, as a concrete example, the PL T-plurality transformation of $AdS_5 \times S^5$ solution has been studied. Lately, by using the PL T-plurals of cosmologies invariant with respect to non-simple Bianchi groups, it has been shown that [14] the resulting plural backgrounds together with dilaton field and a vector field $\mathcal{J}$ satisfy the generalized supergravity equations.

In this paper we firstly recall the definitions of Manin supertriple and DSD and briefly explain the construction of PL T-dual σ-models on Lie supergroups. We then generalize the formulation of PL T-plurality proposed by R. von Unge [6] from Lie groups (DDs) to Lie supergroups (DSDs). We introduce a σ-model constructing on $(2 + 1|2)$-dimensional supermanifold $\mathcal{M} \cong O \times G$ where $G$ is a four-dimensional Lie supergroup of the type $(2|2)$, and $O$ as the orbit of $G$ in $\mathcal{M}$ is a one-dimensional space with the time coordinate. Then, we take a convenient ansatz for metric of the model in terms of the left-invariant one-forms of Lie supergroup $G$. Notice that $G$ is considered to be as an isometry supergroup of the metric. Accordingly, with the help of the isometry Lie supergroups $(C^3 + A)$ and $GL(1|1)$ [15], we construct the cosmological string backgrounds including $(2 + 1|2)$-dimensional metric, time-dependent dilaton and vanishing torsion. These backgrounds are indeed conformally invariant up to the one-loop order. Our main results are presented in section 4. It is shown that the resulting backgrounds are equivalent to the ones of T-dual σ-models on $(C^3 + A)$ and $GL(1|1)$ [15], respectively, in terms of Manin supertriples. It turns out that for each of the Lie superalgebras $(C^3 + A)$ and $gl(1|1)$ there are several Manin supertriples and the possibility of embedding these Manin supertriples into the corresponding DSDs. The new backgrounds obtained by super PL T-plurality satisfy the one-loop beta-function equations which are the most important feature of obtained models.

2 The super PL T-duality/plurality

We shall consider σ-models without spectator fields, i.e. with isomorphic supermanifolds targeting a Lie supergroup. Let’s assume $G$ to be a Lie supergroup and $\mathcal{G}$ its Lie superalgebra. Suppose now that $G$ acts transitively and freely on $\mathcal{M}$, then σ-model having target space in
the Lie supergroup $G$, is given by the following action\(^1\)

\[ S = \frac{1}{2} \int_{\Sigma} d\tau^+ d\tau^- (-1)^{\lfloor a \rfloor} L_+^{(l)} a \ E_{ab}(g) \ L_-^{(l)b}, \quad (2.1) \]

where $\tau^\pm = \tau \pm \sigma$ are the standard light-cone variables on the worldsheet $\Sigma$, and $L_+^{(l)} a$ are components of the left-invariant Maurer-Cartan one-forms with left derivative which are defined by means of an element $g: \Sigma \to G$ in the following formula\(^2\)

\[ g^{-1} \partial \sigma^- g = (-1)^a L_+ a T_a, \quad (2.2) \]

in which $T_a, a = 1, ..., \dim G$ are the bases of Lie superalgebra $\mathcal{G}$, and $E_{ab}(g)$ is a certain bilinear form on the $\mathcal{G}$, to be specified below.

As noted explicitly in \([1,2]\), the algebraic structure underlying PL T-duality is the DD. In the super case \([20,21]\), the PL T-dual $\sigma$-models are also constructed by means of DSDs. A DSD is simply a Lie supergroup $D$ whose Lie superalgebra $D = (\mathcal{G}, \mathcal{G})$ admits a decomposition $D = \mathcal{G} \oplus \mathcal{G}$ into a pair of sub-superalgebras maximally isotropic with respect to a supersymmetric ad-invariant non-degenerate bilinear form $< . , . >$. Any such decomposition written as an ordered set $(D, \mathcal{G}, \mathcal{G})$ is called Manin supertriple.

The matrix $E(g)$ for $\sigma$-model (2.1) is of the form\(^3\)

\[ E(g) = (E_0^{-1} + \Pi(g))^{-1}, \quad \Pi(g) = b(g).a^{-1}(g), \quad (2.3) \]

where $E_0$ is a constant matrix, $\Pi(g)$ defines the super Poisson structure on the Lie supergroup $G$, and sub-matrices $a(g)$ and $b(g)$ are defined as\(^4\)

\[ g^{-1}T_a g = (-1)^c a^c_a(g) T_c, \quad g^{-1}\tilde{T}^a g = (-1)^c b'^c_a(g) T_c + (a^{-st})^c_a(g) \tilde{T}^c, \quad (2.4) \]

where $\tilde{T}^a$ are elements of dual bases in the dual Lie superalgebra $\tilde{\mathcal{G}}$. The dimension of sub-superalgebras have to be equal and one can choose a basis in each of the sub-superalgebras $T_a \in \mathcal{G}$ and $\tilde{T}^a \in \tilde{\mathcal{G}}$ such that

\[ < T_a, T_b > = 0, \quad < \tilde{T}^a, \tilde{T}^b > = 0, \quad < T_a, \tilde{T}^b > = \delta^a_b = (-1)^{ab} < T_b, \tilde{T}^a >. \quad (2.5) \]

The generators of the two sub-superalgebras satisfy the commutation relations

\[ [T_a, T_b] = f^{c}_{ab} T_c, \quad [\tilde{T}^a, \tilde{T}^b] = \tilde{f}^{ab}_{c} \tilde{T}^c. \quad (2.6) \]

\(^1\)Notice that the $|a|$ denotes the grading $a$ such that $|a| = 0$ for the bosonic coordinates and $|a| = 1$ for the fermionic ones. We identify the grading indices by the same indices in the power of $(-1)$, that is, we use $(-1)^a$ instead of $(-1)^{|a|}$; this notation has used by Dewitt in \([19]\). Throughout the paper we use this notation (see appendix A).

\(^2\)From now on we will omit the superscript $(l)$ on $L_+^{(l)} a$.

\(^3\)Here one must use the superinverse formula which has introduced in \([19]\).

\(^4\)“st” denotes supertransposition.
One can use the super ad-invariance of $<\ ,\ >$ to show the remaining commutation relations must be [22]

$$[T_a, \bar{T}^b] = (-1)^b \bar{f}^{bc}{}_a T_c + (-1)^a f^{bc}{}_a \bar{T}^c. \quad (2.7)$$

It should be noted that the Lie superalgebra structure defined by relations (2.6) and (2.7) is called the DSD $\mathcal{D}$. We have used the left-invariant one-forms to write the model in coordinates $\mathcal{E}_{\mu\nu}(x) = (-1)^{\mu} L_\mu^a E_{ab}(g) (L^b)^{\nu}_\nu$. Thus, we have

$$S = \frac{1}{2} \int d\sigma^+ d\sigma^- (-1)^\mu \partial_+ x^\mu \mathcal{E}_{\mu\nu}(x) \partial_- x^\nu, \quad (2.8)$$

where the functions $x^\mu: \Sigma \to \mathbb{R}$, $\mu = 1,..., dimG$ are the coordinates of the target supermanifold which is here isomorphic to the $G$. We say that the background $\mathcal{E}_{\mu\nu}(x)$ has super PL symmetry if [20]

$$\mathcal{L}_{V_a} \mathcal{E}_{\mu\nu} = (-1)^{a+\lambda} c_0 a_\mu + c_\rho \bar{T}^{bc} \mathcal{E}_{\mu\rho} V_c^b \mathcal{E}_{\lambda\nu}, \quad (2.9)$$

for some left-invariant supervector fields $V_a$ satisfying $[V_a, V_b] = f^{c}{}_{ab} V_c$. The integrability condition on the Lie derivative, $[\mathcal{L}_{V_a}, \mathcal{L}_{V_b}] = \mathcal{L}_{[V_a, V_b]}$, then implies the mixed super Jacobi identities [20] showing that this construction leads naturally to the DSD. It is possible to define an equivalent but dual $\sigma$-model by the exchange of $G \leftrightarrow \tilde{G}$, $\mathcal{D} \leftrightarrow \mathcal{D}$, $E_a \leftrightarrow E_a^{-1}$ and $\Pi(g) \leftrightarrow \tilde{\Pi}(\tilde{g})$.

As noted explicitly in [6], the possibility to decompose some DDs into more than two Manin triples leads to the notion of PL T-plurality. Below, we shall generalize the formulation of PL T-plurality to the DSDs case and then call it *super PL T-plurality*. We use the fact that there are, in general, several decompositions (Manin supertriples) of a DSD. Let $X_A = \{T_a, \bar{T}^b\}$, $a, b = 1,..., dimG$ be generators of Lie sub-superalgebras $\mathcal{D}$ and $\tilde{\mathcal{D}}$ of a DSD $\mathcal{D}$ associated with the $\sigma$-model (2.1), and $X'_A = \{U_a, \bar{U}^b\}$ are generators of some other Manin supertriple $(\mathcal{D}'_U, \tilde{\mathcal{D}}')$ in the same DSD $(\mathcal{D} \cong \mathcal{D}'_U)$ so that they also satisfy equations (2.6) and (2.7), as well as the bilinear form (2.5). We say that $\mathcal{D}$ and $\mathcal{D}'_U$ are isomorphic to each other iff there is an invertible supermatrix $C_A{}^B$ such that the linear map given by $X_A = (-1)^B C_A{}^B X'_B$ transforms the Lie multiplication of $\mathcal{D}$ into that of $\mathcal{D}'_U$ and preserves the canonical form of the bilinear form $<\ ,\ >$. Notice that the generators $\{T_a, \bar{T}^b\}$ define a canonical decomposition of the double. We define the isomorphism transformation between the $\mathcal{D}$ and $\mathcal{D}'_U$ in the matrix form as [6]

$$\begin{pmatrix} T \\ \bar{T} \end{pmatrix} = \begin{pmatrix} F & G \\ H & K \end{pmatrix} \begin{pmatrix} U \\ \bar{U} \end{pmatrix}. \quad (2.10)$$

One finds the transformation between the bases of $\mathcal{D}$ and $\mathcal{D}'_U$ in explicit components form as follows:

$$T_a = (-1)^{c} F^{c}{}_{a} U_{c} + G_{ac} \bar{U}_{c}, \quad (2.11)$$

$$\bar{T}^b = (-1)^{c} H^{b}{}_{c} U_{c} + K_{b}{}^{c} \bar{U}_{c}. \quad (2.12)$$
The supergroup associated with the generators $U_\sigma$ will be the supergroup over which the $\sigma$-model is defined. In fact, the transformed model is then given by the same form as (2.1) but with $E(g)$ replaced by [6]

$$E(g_U) = (NM^{-1} + \Pi(g_U))^{-1} = ((E_{0g})^{-1} + \Pi(g_U))^{-1}. \quad (2.13)$$

One may express $M, N$ and $\Pi(g_U)$ in the form of their components as

$$M_{ab} = (-1)^c (K^{st})_a^c E_{0cb} - (G^{st})_{ab}, \quad (2.14)$$

$$N^a_b = (F^{st})^a_b - (-1)^c (H^{st})^{ac} E_{0cb}, \quad (2.15)$$

$$\Pi^{ab}(g_U) = (-1)^c b^{ac}(g_U) (a^{-1})_c^b(g_U), \quad (2.16)$$

where $a(g_U)$ and $b(g_U)$ is defined as in (2.4) by replacing $g$ with $g_U$ and $T_a(\tilde{T}^a)$ with $U_a(\tilde{U}^a)$. As shown in [6], the plurality transformation must be supplemented by a correction that comes from integrating out the fields on the dual group in path-integral formulation in such a way that it can be absorbed at the one-loop level into the transformation of the dilaton field. Following [6] the formula for the transformation of the dilaton on Lie supergroups is given by

$$\Phi_U = \phi^{(0)} + \frac{1}{2} \log |\det(aE_b(g_U))| - \frac{1}{2} \log |\det(a M_b)| + \frac{1}{2} \log |\det(a^{-1}b(g_U))|, \quad (2.17)$$

where $\phi^{(0)}$ is the dilaton that makes the original $\sigma$-model conformal and may depend on the coordinates of $G_U$. The PL T-duality transformation is a special case of the plurality transformation, because one may obtain the canonical decomposition by choosing the decomposition matrix (2.10) as block diagonal, i.e., $F = K = 1, G = H = 0$. With this choice one obtains $N^a_b = \delta^a_b, M_{ab} = E_{0ab}$, then we arrive at (2.3), as well as,

$$\Phi = \phi^{(0)} + \frac{1}{2} \log |\det(aE_b(g))| - \frac{1}{2} \log |\det(a E_{0b})| + \frac{1}{2} \log |\det(a^{-1}b(g))|, \quad (2.18)$$

From the combination of two formulas (2.17) and (2.18) one gets the dilaton transformation between the doubles $\mathcal{D}$ and $\mathcal{D}_U$, giving

$$\Phi_U = \Phi + \frac{1}{2} \log |\det(\delta^a_b + \Pi^{ac}(g), E_{0b})| - \frac{1}{2} \log |\det(N^a_b + \Pi^{ac}(g_U), M_b)|$$

$$- \frac{1}{2} \log |\det(a^{-1}b(g))| + \frac{1}{2} \log |\det(a^{-1}b(g_U))|. \quad (2.19)$$

The super plurality transformation (2.11)-(2.16) also includes the canonical dual by choosing the decomposition matrix (2.10) as block antidiagonal, that is, for $F = K = 0, G = H = 1$ we get the dual model with $E_{0U} = E_0 E_0^{-1}$, corresponding to the interchange $\mathcal{D} \leftrightarrow \mathcal{D}$; moreover, one gets the background $\tilde{E}(g) = (E_0 + \tilde{\Pi}(g))^{-1}$.

In the following, we will apply the above formulas in order to construct super PL T-plurals of cosmologies invariant with respect to the $(C^3 + A)$ and $GL(1|1)$ Lie supergroups.
3 Cosmological string backgrounds on supermanifolds

As in conventional general relativity, homogeneous backgrounds in string cosmology may be defined as those $3 + 1$-dimensional spacetime manifolds which admit a 3-parameter group of isometries. It has been shown that [23–26] Bianchi-type string cosmology involves $3 + 1$-dimensional spatially homogeneous spacetimes which satisfy at least the lowest-order string beta-function equations. These string solutions offer prototypes for studying spatial anisotropy and understanding the impacts of anisotropy on the dynamics of early universe. Bianchi-type cosmologies can be generally defined in terms of a three-dimensional real Lie group of isometries that act simply-transitively on three-dimensional, space-like orbits [23,26] (see also [27,28] and references therein). These models were then generalized to $4 + 1$-dimensional cosmological models with four-dimensional real Lie groups whose spatial hypersurfaces are (simply) connected homogeneous Riemannian manifolds [29] (see also [30]).

Here we present some new solutions of string cosmological models on supermanifolds characterized by $(2 + 1|2)$-dimensional metric (with four-dimensional Lie supergroups of the type $(2|2)$ as isometry supergroups), a dilaton field at most a function of $t$ only and vanishing torsion.

3.1 The model setup

Let us suppose that $\mathcal{M}$ be a (pseudo-)Riemannian target supermanifold of superdimension $(d_B | d_F)$ with the coordinates $x_M$, where $d_B$ is the dimension of the bosonic directions, while $d_F$ denotes the dimension of the fermionic ones. Notice that because of invertibility of the metric of supermanifold $\mathcal{M}$, $d_F$ must be even [19]. Consider the propagation of a bosonic string in the presence of arbitrary backgrounds of three fields: the supersymmetric metric $G_{MN}$, the super antisymmetric tensor field $B_{MN}$ and dilaton $\Phi$ where the labels $M$ and $N$ run from 1 to $d_B + d_F$. The string tree level effective action on supermanifold $\mathcal{M}$ for these background fields has the form [21]

$$S_{\text{eff}} = \int d^M x \sqrt{G} e^{-2\Phi} \left[ R + 4 \nabla^M \Phi \nabla_M \Phi + \frac{1}{12} H_{MNP} H^{PNM} - 4 \Lambda \right], \quad (3.1)$$

where $G$ stands for the superdeterminant of $G_{MN}$. The covariant derivative $\nabla_M$ and scalar curvature $R$ are calculated from the metric $G_{MN}$ which is also used for lowering and raising indices and $H_{MNP}$ defined by

$$H_{MNP} = B_{MN} \frac{\partial}{\partial x^P} + (-1)^{M(N+P)} B_{NP} \frac{\partial}{\partial \Phi^M} + (-1)^{P(M+N)} B_{PM} \frac{\partial}{\partial \Phi^N}$$

$$= (-1)^M \frac{\partial}{\partial x^M} B_{NP} + (-1)^{N+M(N+P)} \frac{\partial}{\partial x^P} B_{PM} + (-1)^{P(1+M+N)} \frac{\partial}{\partial x^M} B_{MN}, \quad (3.2)$$

$^5$The relation between the left partial differentiation and right one has been discussed in appendix A.
is the torsion of the field $B_{MN}$, and $\Lambda$ is a cosmological constant. The effective action (3.1) leads to the following equations of motion \[ R_{MN} + \frac{1}{4} H_{MPQ} H^{PQ}_{\quad N} + 2 \nabla_M \nabla_N \Phi = 0, \] \[ (-1)^F \nabla^F (e^{-2\Phi} H_{PMN}) = 0, \] \[ 4\Lambda - R - \frac{1}{12} H^N_{\quad MNP} H^{PQM} + 4 \nabla_M \Phi \nabla_N \Phi - 4 \nabla_M \nabla_N \Phi = 0, \] where $R_{MN}$ is the Ricci tensor of the metric (see appendix A). These equations can be also obtained as the conditions of vanishing one-loop beta-functions in the corresponding two-dimensional $\sigma$-model \[ S = \frac{1}{2} \int_{\Sigma} \left[ (1)^M \partial_+ x^M \left[ G_{MN}(x) + B_{MN}(x) \right] \partial_- x^N \right] \] \[ - \frac{1}{4\pi} \int_{\Sigma} d\sigma^+ d\sigma^- R^{(2)} \Phi(x), \] where $R^{(2)}$ is the worldsheet curvature.

To continue, we consider the action (3.6) on $(2 + 1|2)$-dimensional supermanifold $\mathcal{M} \simeq O \times G$ where $G$ is a four-dimensional Lie supergroup of the type $(2|2)$ (here we shall use the $(C^3 + A)$ and $GL(1|1)$ Lie supergroups [15]), while $O$ as the orbit of $G$ in $\mathcal{M}$ is a one-dimensional space with time coordinate $y^i = \{t\}$. We note that the time coordinate does not participate in the PL T-duality transformations and is therefore called spectator [31]. As an ansatz for the metric of the model we consider \[ ds^2 = (1)^M dx^M G_{MN} dx^N = -dt^2 + (1)^{\mu} dx^\mu g_{\mu\nu}(t, x) dx^\nu = -dt^2 + (1)^{a} dx^a \mu L^a_{\quad \mu} E_{oab}(t) (L^t)^b_{\quad \nu} dx^\nu, \] where $x^{\mu}$'s are the coordinates of $(2|2)$-dimensional isometry Lie supergroups with corresponding left-invariant one-forms $\mu L^a$. Moreover, we want the tensor field $B_{MN}$ to be absent. Then, the action (3.6) turns into \[ S = \frac{1}{2} \int_{\Sigma} d\sigma^+ d\sigma^- \left[ - \partial_+ t \partial_- t + (1)^{a} \partial_+ x^a \mu L^a_{\quad \mu} E_{oab}(t) (L^t)^b_{\quad \nu} \partial_- x^\nu \right] \] \[ - \frac{1}{4\pi} \int_{\Sigma} d\sigma^+ d\sigma^- R^{(2)} \Phi(t). \] Notice that $E_{oab}(t)$ is, in the presence of the tensor field $B_{MN}$, replaced by $E(g, t)$. For the dilaton field we have considered a cosmological, i.e. time-dependent dilaton. By a suitable choice of the matrix $E_{oab}(t)$ we can construct the metric (3.7) over a particular Lie supergroup. We then find a $(2 + 1|2)$-dimensional metric on the supermanifold $\mathcal{M}$ so that $E_{oab}(t)$ has explicitly not determined yet. To determine $E_{oab}(t)$ and also time-dependent dilaton $\Phi(t)$ we solve the field equations (3.3)-(3.5). In this way, one gets $(2 + 1|2)$-dimensional cosmological string backgrounds which are conformally invariant up to the one-loop order.
3.2 Cosmological string backgrounds from the \((C^3 + A)\) Lie supergroup

In this subsection we firstly construct the metric (3.7) using the left-invariant one-forms of the \((C^3 + A)\) Lie supergroup. Then, we solve the field equations (3.3)-(3.5) with the resulting metric so that the torsion is absent here (i.e., \(H_{MNP} = 0\)). Before proceeding to construct the metric, let us introduce the \((C^3 + A)\) Lie superalgebra \([15]\). The \((C^3 + A)\) is a four-dimensional Lie superalgebra of the type \((2|2)\) which has supersymmetric, ad-invariant and non-degenerate metric on its structure \([32]\). It is spanned by the set of generators \(\{T_1, T_2, T_3, T_4\}\) with gradings; \(\text{grade}(T_1) = \text{grade}(T_2) = 0\) and \(\text{grade}(T_3) = \text{grade}(T_4) = 1\), which fulfill the following non-zero \((anti)\)commutation rules \([15]\):

\[
[T_1, T_4] = T_3, \quad \{T_4, T_4\} = T_2.
\] (3.9)

We parametrize an element of the \((C^3 + A)\) as

\[
g = e^{xT_4} e^{yT_1} e^{xT_2} e^{yT_3},
\] (3.10)

where \((y, x)\) are bosonic fields, while \((\psi, \chi)\) are fermionic ones. Then, using equation (2.2) we obtain

\[
g^{-1} \partial_\pm g = \partial_\pm y T_1 + (\partial_\pm x - \partial_\pm \chi \frac{\chi}{2}) T_2 + (\partial_\pm \psi - \partial_\pm \chi y) T_3 + \partial_\pm \chi T_4,
\] (3.11)

such that the corresponding left-invariant one-forms components take the following matrix form

\[
\mu L^a = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & -\frac{1}{2} & y & -1
\end{pmatrix}.
\] (3.12)

In order to make the metric (3.7) on the \((C^3 + A)\) Lie supergroup one may use the following ansatz

\[
E_{oab}(t) = \begin{pmatrix}
a_1(t) & 0 & 0 & 0 \\
0 & a_2(t) & 0 & 0 \\
0 & 0 & 0 & a_3(t) \\
0 & 0 & -a_3(t) & 0
\end{pmatrix},
\] (3.13)

then,

\[
ds^2 = -dt^2 + a_1(t) dy^2 + a_2(t) dx^2 + a_2(t) \chi dx d\chi - 2a_3(t) d\psi d\chi.
\] (3.14)

As mentioned above, we shall solve the field equations (3.3)-(3.5) for the background including the metric (3.14) and a time-dependent dilaton field. Since the background is torsionless, the
equation (3.4) is satisfied. Finally, from equations (3.3) and (3.5) the cosmological string background coupled with two fermionic fields is read

\[ ds^2 = -dt^2 + e^t \left[ dy^2 + dx^2 + \chi \, dx d\chi - 2e^{-2t}d\psi d\chi \right], \]
\[ B = 0, \]
\[ \Phi(t) = t + \varphi_0, \]

where \( \varphi_0 \) is an arbitrary constant. For this solution one finds that the cosmological constant is zero. The metric is flat in the sense that its scalar curvature is \( R = 4 \). In addition, one may obtain another solution with a constant dilaton field and vanishing cosmological constant in the form

\[ ds^2 = -dt^2 + a_0 dy^2 + (t + \frac{b_0}{2})^2 \left( dx^2 + \chi \, dx d\chi \right) - 2c_0 d\psi d\chi, \]
\[ B = 0, \]
\[ \Phi(t) = \varphi_0. \]

for some constants \( a_0, b_0 \) and \( c_0 \). The metric of this solution is also flat in the sense that its Ricci tensor and scalar curvature vanish.

### 3.3 Cosmological string backgrounds from the \( GL(1|1) \) Lie supergroup

Analogously, the \( gl(1|1) \) Lie superalgebra has also supersymmetric, ad-invariant and non-degenerate metric [33], and is defined by the following non-zero (anti)commutations relations

\[ [T_1, T_3] = T_3, \quad [T_1, T_4] = -T_4, \quad \{T_3, T_4\} = T_2, \]

where \( (T_1, T_2) \) and \( (T_3, T_4) \) are bosonic and fermionic bases, respectively. In Backhouse’s classification [15], the \( gl(1|1) \) has been labeled by \( \left( \mathcal{C}_2^1 + \mathcal{A} \right) \). The parametrization of a general element of \( GL(1|1) \) we choose as in (3.10), giving

\[ g^{-1}\partial_+ g = \partial_+ y \, T_1 + (\partial_+ x - \partial_+ \chi \, \psi e^y) \, T_2 + (\partial_+ y \, \psi + \partial_+ \psi) \, T_3 + \partial_+ \chi \, e^y \, T_4. \]

In this case one may choose the following ansatz

\[ E_{\alpha \beta}(t) = \begin{pmatrix} a_1(t) & b(t) & 0 & 0 \\ b(t) & a_2(t) & 0 & 0 \\ 0 & 0 & a_3(t) & 0 \\ 0 & 0 & -a_3(t) & 0 \end{pmatrix}. \]

Using (3.18) and (3.19) together with (3.7) we obtain

\[ ds^2 = -dt^2 + a_1(t) dy^2 + a_2(t) dx^2 + 2b(t) dy dx + 2\psi e^y (b(t) - a_3(t)) dy d\chi + 2\psi e^y a_4(t) dx d\chi - 2e^y a_4(t) d\psi d\chi. \]
The resulting solution to one-loop beta-function equation is

\[ ds^2 = -dt^2 - \frac{1}{2}t^2 dy^2 + 2b_0 dydx - 2b_0 e^\psi d\psi d\chi, \]

\[ B = 0, \]

\[ \Phi(t) = \varphi_0, \]  

(3.21)

for some constants \( b_0 \) and \( \varphi_0 \). The metric is flat in the sense that its Ricci tensor and scalar curvature vanish. Moreover, for this solution we get \( \Lambda = 0 \). Another solution for which the dilaton field is time-dependent exists and it is given by

\[ ds^2 = -dt^2 + (t + e^t) dy^2 + 2b_0 dydx - 2b_0 e^\psi d\psi d\chi, \]

\[ B = 0, \]

\[ \Phi(t) = \frac{1}{2} t + \varphi_0. \]  

(3.22)

For the metric of this solution one finds that the only non-zero component of the Ricci tensor is \( R_{yy} = (1 + e^t)/2 \), then, \( R = 0 \). Moreover, the corresponding cosmological constant to this solution is \( \Lambda = 1/4 \). The resulting conformal backgrounds will be useful in the next section. We will show that these backgrounds are equivalent to the ones of T-dual \( \sigma \)-models constructing on semi-Abelian DSDs \((\mathfrak{c}^3 + \mathfrak{a}), \mathcal{I}_{(2|2)}\) and \((\mathfrak{g}l(1|1), \mathcal{I}_{(2|2)})\).

4 Super PL T-plurality of cosmologies invariant with respect to the \((\mathfrak{c}^3 + A)\) and \( GL(1|1) \) Lie supergroups

In this section, we present forms of the conformally invariant backgrounds with vanishing torsion and a dilaton field at most a function of \( t \) only on semi-Abelian DSDs \((\mathfrak{c}^3 + \mathfrak{a}), \mathcal{I}_{(2|2)}\) and \((\mathfrak{g}l(1|1), \mathcal{I}_{(2|2)})\) in a way that the resulting backgrounds are in agreement with those of section 3. Then, using the formulation of super PL T-plurality we obtain the conformal duality chains of cosmological string backgrounds with non-vanishing torsion on the other Manin supertriples in the corresponding DSDs so that they satisfy one-loop beta-function equations.

One can find the classification of real \((4|4)\)-dimensional DSDs generated by the \( \mathfrak{g}l(1|1) \) Lie super bi-algebras and their decompositions into six classes of non-isomorphic Manin supertriples in Ref. [18]. Here we shall present only those occurring in this paper, i.e., only first class including the following two isomorphic Manin supertriples:

\[ (\mathfrak{g}l(1|1), \mathcal{I}_{(2|2)}) \cong (\mathfrak{g}l(1|1), \mathfrak{c}^2_{p=-1} \oplus \mathfrak{a}_{1,1,ii}). \]  

(4.1)

In addition, in Ref. [16] we have performed complete classification of Lie super bi-algebra structures on the \((\mathfrak{c}^3 + \mathfrak{a})\) Lie superalgebra. Lately, we have listed the corresponding \((4|4)\)-dimensional DSDs including all possible decompositions into the Manin supertriples and have
obtained 24 non-isomorphic classes [17]. Here is one of those classes that we need in studying the conformal duality chain

\[
((C^3 + A), \mathcal{I}_{(2|2)}) \cong ((C^3 + A), C^3 \oplus A_{1,1} i) \cong ((C^3 + A), (C^3 + A)^{k=4-1}).
\] (4.2)

It should be noted that the label of each Manin supertriple, e.g. \((gl(1|1), C^3_{p=-1} \oplus A_{1,1} ii)\), indicates the structure of the first sub-superalgebra \(A\), e.g. \(gl(1|1)\), the structure of the second sub-superalgebra \(\mathcal{G}\), e.g. \(C^3_{p=-1} \oplus A_{1,1} ii\); roman numbers \(i, ii\) etc. (if present) distinguish between several possible pairings \(<.,.>\) of the sub-superalgebras \(A\) and \(\mathcal{G}\), and the parameter \(k\) (e.g. in (4.2)) indicates the Manin supertriples differing in the rescaling of \(<.,.>\).

### 4.1 Conformal duality chain starting from \(((C^3 + A), \mathcal{I}_{(2|2)})\)

- \(((C^3 + A), \mathcal{I}_{(2|2)})\):

  In order to obtain a possible conformal duality chain on the non-isomorphic DSDs (4.2) we take \(((C^3 + A), \mathcal{I}_{(2|2)})\) as the canonical decomposition whose Lie superalgebra is given by [17]

\[
\begin{align*}
[T_1, T_4] &= T_3, \quad \{T_2, T_4\} = T_2, \quad [T_1, \tilde{T}^3] = -\tilde{T}^4, \\
[T_4, \tilde{T}^4] &= -\tilde{T}^4, \quad \{T_4, \tilde{T}^3\} = -\tilde{T}^1.
\end{align*}
\] (4.3)

Here and henceforth \((T_1, T_2, \tilde{T}^1, \tilde{T}^2)\) and \((T_3, T_4, \tilde{T}^3, \tilde{T}^4)\) are bosonic and fermionic bases, respectively. As mentioned in section 2, to obtain the canonical decomposition we choose the decomposition matrix (2.10) as block diagonal \((F = K = 1, G = H = 0)\), then, we arrive at \(N^a_b = \delta^a_b\) and \(M_{ab} = E_{a \bar{b}}\). We use the parametrization of a general group element of \((C^3 + A)\) as in (3.10), then, one utilizes (2.4) in order to calculate the matrices \(a(g)\) and \(b(g)\) for this decomposition, giving

\[
a^b_a (g) = \begin{pmatrix}
1 & 0 & -\chi & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & -\chi & y & -1
\end{pmatrix}, \quad b^{ab} (g) = 0.
\] (4.4)

Since \(b(g) = 0\) there is no super antisymmetric tensor field \(B_{MN}\). This enables us to obtain the background (3.15) from a \(\sigma\)-model constructing on the double \(((C^3 + A), \mathcal{I}_{(2|2)})\). To this end, we have to choose the matrix \(E_0\) in the following form

\[
E_{0 \bar{a}b}(t) = \begin{pmatrix}
e^t & 0 & 0 & 0 \\
0 & e^t & 0 & 0 \\
0 & 0 & 0 & e^{-t} \\
0 & 0 & -e^{-t} & 0
\end{pmatrix}.
\] (4.5)

By making use of (4.4) it then follows that \(\Pi(g) = 0\), which gives us \(E = E_0\) as expected for the canonical decomposition. The dilaton field that makes the original \(\sigma\)-model conformal is
found by using equation (2.18) to be $\Phi = \phi^{(0)}$. However, according to (3.15) since we want the total dilaton to be $\Phi(t) = t + \varphi_0$ we need to choose $\phi^{(0)} = t + \varphi_0$ in which $\varphi_0$ is an arbitrary constant. On the other hand, the corresponding left-invariant one-forms have been given in equation (3.12). Finally, using (3.8) one can construct the original $\sigma$-model on the double $((\mathcal{E}^3 + \mathcal{A}), T_{(2/2)})$ whose background is

$$
\begin{align*}
 ds^2 &= -dt^2 + e^t [dy^2 + dx^2 + \chi \, dx \, d\chi - 2e^{-2t} \, d\psi \, d\chi], \\
 B &= 0, \\
 \Phi(t) &= t + \varphi_0.
\end{align*}
$$

Indeed, this background is nothing but (3.15).

- $((\mathcal{E}^3 + \mathcal{A}), \mathcal{E}^3 \oplus \mathcal{A}_{1,1,1})$:

Considering the linear map $X_A = (-1)^B \, C_A \, X'_B$ transforming the Lie multiplication of $\mathcal{D}$ into that of $\mathcal{D}_U$ and preserving the canonical form of the bilinear form $\langle \cdot \rangle$, we find the following non-trivial decomposition matrix between the doubles $((\mathcal{E}^3 + \mathcal{A}), T_{(2/2)})$ and $((\mathcal{E}^3 + \mathcal{A}), \mathcal{E}^3 \oplus \mathcal{A}_{1,1,1})$ [17]

$$
\begin{pmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
\tilde{T}_1 \\
\tilde{T}_2 \\
\tilde{T}_3 \\
\tilde{T}_4
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
U_1 \\
U_2 \\
U_3 \\
U_4 \end{pmatrix},
$$

where $\{U_a, \tilde{U}^b\}, a, b = 1, ..., 4$ are elements of bases of the double $((\mathcal{E}^3 + \mathcal{A}), \mathcal{E}^3 \oplus \mathcal{A}_{1,1,1})$ whose Lie superalgebra is defined by the following non-zero (anti)commutation relations [17]:

$$
\begin{align*}
[U_1, U_4] &= U_3, \quad \{U_4, U_4\} = U_2, \quad [\tilde{U}^2, \tilde{U}^3] = \tilde{U}^4, \\
[U_1, \tilde{U}^3] &= -\tilde{U}^4, \quad [U_4, \tilde{U}^2] = U_3 - \tilde{U}^4, \quad \{U_4, \tilde{U}^3\} = U_2 - \tilde{U}^1.
\end{align*}
$$

Here and henceforth $(U_1, U_2, \tilde{U}^1, \tilde{U}^2)$ are bosonic bases, while $(U_3, U_4, \tilde{U}^3, \tilde{U}^4)$ are fermionic ones. One utilizes equations (2.14) and (2.15) together with (4.5) to find the matrices $M_{ab}$ and $N^a_{\ b}$, obtaining

$$
M_{ab} =
\begin{pmatrix}
e^t & 0 & 0 & 0 \\
0 & e^t & 0 & 0 \\
0 & 0 & e^{-t} & 0 \\
0 & 0 & -e^{-t} & 0
\end{pmatrix},
N^a_{\ b} =
\begin{pmatrix}
1 & -e^t & 0 & 0 \\
e^t & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

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Since the first sub-superalgebra is \((\mathcal{C}^3 + \mathcal{A})\), the left-invariant one-forms and the matrix 
\[ a^b_a (g_U) \] are the same forms as in (3.12) and (4.4), respectively. Using (2.4), (2.16), (3.10) and 
(4.8) we then find that

\[
\Pi^{ab} (g_U) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & x & 0 \\
0 & -x & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

These results give us the background

\[
E_{ab} (g_U) = \begin{pmatrix}
\frac{e^{-t}}{\Delta(t)} & \frac{1}{\Delta(t)} & 0 & \frac{x e^{-t}}{\Delta(t)} \\
-\frac{1}{\Delta(t)} & \frac{x e^{-t}}{\Delta(t)} & 0 & \frac{x e^{-2t}}{\Delta(t)} \\
0 & 0 & 0 & e^{-t} \\
x e^{-t} \Delta(t) & -x e^{-2t} \Delta(t) & -e^{-t} & 0
\end{pmatrix},
\]

\[
\Phi_U = \phi^{(o)} - t - \frac{1}{2} \log (\Delta(t)),
\]

where \(\Delta(t) = e^{-2t} + 1\). Finally we get the dilaton by remembering that \(\phi^{(o)} = t + \varphi_0\) which gives the final result

\[
\Phi_U = \varphi_0 - \frac{1}{2} \log (\Delta(t)).
\]

The supersymmetric part of the matrix \(E_{ab} (g_U)\) gives the metric in the coordinate basis, whereas the super antisymmetric part of \(E_{ab} (g_U)\) gives the super antisymmetric tensor. Thus, the background in the coordinate basis is read off

\[
ds^2 = -dt^2 + \frac{e^{-t}}{\Delta(t)} \left[ dy^2 + dx^2 - 2x dy d\chi + x dx d\chi \right] - 2e^{-t} d\psi d\chi,
\]

\[
B = \frac{1}{\Delta(t)} \left[ dy \wedge dx + \frac{1}{2} \chi dy \wedge d\chi - \chi e^{-2t} dx \wedge d\chi \right],
\]

\[
\Phi(t) = \varphi_0 - \frac{1}{2} \log (\Delta(t)).
\]

One immediately finds that the scalar curvature of the metric is \(R = 6e^{-2t}/\Delta^2(t)\). Looking at

the one-loop beta-function equations one verifies the conformal invariance conditions of the

background (4.13) with vanishing cosmological constant.

\[
\bullet \ (\mathcal{C}^3 + \mathcal{A}, \mathcal{C}^3 + \mathcal{A})_{k=4}^{x-1}:
\]
The Lie superalgebra of the double \( ((\mathcal{E}^3 + \mathcal{A}), (\mathcal{E}^3 + \mathcal{A})_{\text{a} = 4}) \) obeys the following set of non-trivial (anti)commutation relations [17]:

\[
[U_1, U_4] = U_3, \quad \{U_4, U_4\} = U_2, \quad [\tilde{U}^2, \tilde{U}^4] = -\tilde{U}^4, \\
\{\tilde{U}^3, \tilde{U}^3\} = 4\tilde{U}^4, \quad [U_4, \tilde{U}^3] = -U_3 - \tilde{U}^4, \quad \{U_3, \tilde{U}^3\} = -U_2 - \tilde{U}^4, \\
[U_1, \tilde{U}^3] = -4U_3 - \tilde{U}^4.
\tag{4.14}
\]

The isomorphism transformation between the doubles \( ((\mathcal{E}^3 + \mathcal{A}), \mathcal{I}_{(a=2)}) \) and \( ((\mathcal{E}^3 + \mathcal{A}), (\mathcal{E}^3 + \mathcal{A})_{\text{a} = 4}) \) is given by the following decomposition matrix [17]:

\[
\begin{pmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
\tilde{T}_1 \\
\tilde{T}_2 \\
\tilde{T}_3 \\
\tilde{T}_4
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
U_1 \\
U_2 \\
U_3 \\
U_4 \\
\tilde{U}_1 \\
\tilde{U}_2 \\
\tilde{U}_3 \\
\tilde{U}_4
\end{pmatrix}.
\tag{4.15}
\]

In this case, the matrix \( a_a^b(g_U) \) is the same form as in (4.4). Calculating the matrices \( M_{ab}^1, N^a_b \) and \( \Pi_{ab}^b(g_U) \) for this decomposition we then get

\[
NM^{-1} = \begin{pmatrix}
e^{-t} & -1 & 0 & 0 \\
1 & e^{-t} & 0 & 0 \\
0 & 0 & e^t - 2 & 0 \\
0 & 0 & -(e^t + 2) & 0
\end{pmatrix}, \quad \Pi(g_U) = \begin{pmatrix}0 & 0 & 0 & 0 \\
0 & 0 & -\chi & 0 \\
0 & \chi & -4y & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\tag{4.16}
\]

leading to a background

\[
E_{ab}(g_U) = \begin{pmatrix}
\frac{e^{-t}}{\Delta(t)} & \frac{1}{\Delta(t)} & 0 & -\frac{\chi}{(e^t + 2)\Delta(t)} \\
\frac{1}{\Delta(t)} & \frac{e^{-t}}{\Delta(t)} & 0 & -\frac{\chi}{(e^t + 2)\Delta(t)} \\
0 & 0 & 0 & \frac{1}{(e^t + 2)} \\
-\frac{\chi}{(e^t - 2)\Delta(t)} & -\frac{\chi e^{-t}}{(e^t - 2)\Delta(t)} & -\frac{1}{(e^t - 2)} & \frac{4y}{(e^t - 4)}
\end{pmatrix},
\]

\[
\Phi_U = \phi^{(0)} - 2t + \frac{1}{2} \log \left( \frac{e^{2t} - 4}{\Delta(t)} \right).
\tag{4.17}
\]

In order to evaluate the total dilatonic contribution we have to use \( \phi^{(0)} = t + \varphi_0 \) which gives the final result

\[
\Phi_U = \varphi_0 - t + \frac{1}{2} \log \left( \frac{e^{2t} - 4}{\Delta(t)} \right).
\tag{4.18}
\]
To go to the coordinate basis we have to use the left-invariant one-forms (3.12) to write the background as
\[
\begin{align*}
    ds^2 &= -dt^2 + \frac{e^{-t}}{\Delta(t)} \left[ dy^2 + dx^2 + \frac{2\chi e^{2t}}{e^{2t} - 4} \, dy \, d\chi + \frac{\chi (e^{2t} - 8)}{e^{2t} - 4} \, dx \, d\chi \right] - \frac{2e^t}{e^{2t} - 4} \, d\psi \, d\chi, \\
    B &= \frac{1}{\Delta(t)} \left[ dy \wedge dx + \frac{\chi (e^{2t} - 8)}{2(e^{2t} - 4)} dy \wedge d\chi + \frac{\chi}{e^{2t} - 4} dx \wedge d\chi \right] \\
    &\quad + \frac{2}{e^{2t} - 4} \left( d\psi \wedge d\chi - 2y \, d\chi \wedge d\chi \right), \\
    \Phi(t) &= \varphi_0 - t + \frac{1}{2} \log \left( \frac{e^{2t} - 4}{\Delta(t)} \right). \tag{4.19}
\end{align*}
\]

By taking into consideration the scalar curvature of the metric which is
\[
R = -50 \frac{(e^{2t} + 4e^{-2t} - 2)}{(e^{2t} - 4)^2 \Delta^2(t)}, \tag{4.20}
\]
one verifies the one-loop conformal invariance conditions of the background (4.19) with vanishing cosmological constant. We thus studied a concrete example of the super PL T-plurality.

Starting with the double \((g(1|1), \mathcal{I}_{(2|2)})\) we found the conformal duality chain of \(2 + 1\)-dimensional cosmological string backgrounds coupled with two fermionic fields in the form of equations (4.6), (4.13) and (4.19).

### 4.2 Conformal duality chain starting from \((g(1|1), \mathcal{I}_{(2|2)})\)

- \((g(1|1), \mathcal{I}_{(2|2)})\):

We take \((g(1|1), \mathcal{I}_{(2|2)})\) as the canonical decomposition. The corresponding Lie superalgebra is given by [18]
\[
\begin{align*}
    [T_1, T_3] &= T_3, & [T_1, T_4] &= -T_4, & \{T_3, T_4\} &= T_2, \\
    [T_1, \tilde{T}_3] &= -\tilde{T}_3, & [T_1, \tilde{T}_4] &= \tilde{T}_4, & [\tilde{T}_3, T_3] &= \tilde{T}_4, \\
    [\tilde{T}_2, T_4] &= \tilde{T}_3, & \{T_3, \tilde{T}_3\} &= -\tilde{T}_1, & \{T_4, \tilde{T}_4\} &= \tilde{T}_3. \tag{4.21}
\end{align*}
\]

Choosing the parametrization of a general group element of \(GL(1|1)\) as in (3.10) and employing (2.4) we obtain
\[
\begin{align*}
    a^b_a (g) &= \begin{pmatrix}
        1 & -\psi \chi e^y & -\psi & \chi e^y \\
        0 & 1 & 0 & 0 \\
        0 & -\chi & -e^{-y} & 0 \\
        0 & -\psi e^y & 0 & -e^y
    \end{pmatrix}, &
    b^{ab} (g) &= 0. \tag{4.22}
\end{align*}
\]

then, it is simply followed that \(\Pi(g) = 0\). Also, in the canonical decomposition we have \(N^a_b = \delta^a_b\) and \(M_{ab} = E_{0ab}\) and thus \(E = E_0\) as this was expected. The dilaton field is found
by the use of equation (2.18) to be \( \Phi = \phi^{(0)} \). Here we want the total dilaton to be constant, so we choose \( \phi^{(0)} = \varphi_0 \). Our goal is now to obtain the background (3.21) from a \( \sigma \)-model constructing on the double \( (\mathfrak{gl}(1|1), \mathcal{I}_{(2|2)}) \). To do so, we have to choose the \( E_o \) as follows:

\[
E_{oab}(t) = \begin{pmatrix}
-\frac{1}{2}t^2 & b_0 & 0 & 0 \\
b_0 & 0 & 0 & 0 \\
0 & 0 & 0 & b_0 \\
0 & 0 & -b_0 & 0
\end{pmatrix},
\] 

(4.23)

where \( b_0 \) is a constant that differs from \{0, 1, -1\}. Inserting (4.23) and the corresponding left-invariant one-forms given by equation (3.18) into action (3.8) one gets

\[
ds^2 = -dt^2 - \frac{1}{2}t^2 dy^2 + 2b_0 dy dx - 2b_0 e^\psi dy^2 d\chi,
B = 0,
\Phi(t) = \varphi_0,
\] 

(4.24)

which is nothing but the background (3.21).

- \( (\mathfrak{gl}(1|1), \mathcal{C}_{p=-1}^2 \oplus \mathcal{A}_{1,1}^{ii}) \):

The Lie superalgebra of the double \( (\mathfrak{gl}(1|1), \mathcal{C}_{p=-1}^2 \oplus \mathcal{A}_{1,1}^{ii}) \) obeys the following set of non-trivial (anti)commutation relations [18]:

\[
\begin{align*}
[U_1, U_3] &= U_3, & [U_1, U_4] &= -U_3, & \{U_3, U_4\} &= U_2, \\
[\tilde{U}^2, \tilde{U}^3] &= \tilde{U}^3, & [\tilde{U}^2, \tilde{U}^4] &= -\tilde{U}^4, & [U_1, \tilde{U}^3] &= -\tilde{U}^3, \\
[U_1, \tilde{U}^4] &= \tilde{U}^4, & [\tilde{U}^2, U_4] &= U_4 + \tilde{U}^3, & [U_3, \tilde{U}^3] &= U_3 - U_4, \\
\{U_3, \tilde{U}^3\} &= U_2 - \tilde{U}^1, & \{U_4, \tilde{U}^3\} &= -(U_2 - \tilde{U}^1).
\end{align*}
\] 

(4.25)

The decomposition matrix between the doubles \( (\mathfrak{gl}(1|1), \mathcal{I}_{(2|2)}) \) and \( (\mathfrak{gl}(1|1), \mathcal{C}_{p=-1}^2 \oplus \mathcal{A}_{1,1}^{ii}) \) is given by [18]:

\[
\begin{pmatrix}
T_1 \\
T_2 \\
T_3 \\
\hat{T}_1 \\
\hat{T}_2 \\
\hat{T}_3 \\
\hat{T}_4
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
U_1 \\
U_2 \\
U_3 \\
\hat{U}_1 \\
\hat{U}_2 \\
\hat{U}_3 \\
\hat{U}_4
\end{pmatrix}.
\] 

(4.26)
Employing (2.4), (4.23) and (4.25) the matrices (2.14), (2.15) and (2.16) are obtained for this decomposition, giving

\[ M_{ab} = \begin{pmatrix} -\frac{t^2}{2} & b_0 & 0 & 0 \\ b_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_0 \\ 0 & 0 & b_0 & 0 \end{pmatrix}, \quad N^a_{\ b} = \begin{pmatrix} 1 - b_0 & 0 & 0 & 0 \\ -\frac{t^2}{2} & 1 + b_0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \]

\[ \Pi^{ab}(g_U) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \psi e^y & -\chi \\ 0 & -\psi e^y & 0 & 0 \\ 0 & \chi & 0 & 0 \end{pmatrix}. \] (4.27)

Using equations (2.13) and (2.17), it is then followed that

\[ E_{ab}(g_U) = \begin{pmatrix} \frac{1}{b_0 - 1} \left( \frac{t^2}{2} + 2b_0^3 \psi \chi e^y \right) & b_0 & b_0^2 \chi & b_0^2 \psi e^y \\ -b_0 & 0 & 0 & 0 \\ b_0^2 \chi & 0 & b_0 & 0 \\ b_0^2 \psi e^y & 0 & -b_0 & 0 \end{pmatrix}, \]

\[ \Phi_U = \phi^{(0)} - \frac{1}{2} \log |b_0^2 - 1|. \] (4.28)

Finally, background including the supersymmetric metric, the super antisymmetric tensor field and dilaton field is, in the coordinate basis, read off

\[ ds^2 = -dt^2 + \frac{1}{b_0^2 - 1} \left[ \left( \frac{t^2}{2} + 2b_0^3 \psi \chi (e^y + 1) \right) dy^2 - 2b_0 \ dy dx - 2b_0^3 \chi \ dy \ dy \psi - 2b_0^3 \psi e^y (e^y + 1) \ dy \ dy \chi \right] - 2b_0 e^y \ dy \ dy \chi, \]

\[ B = \frac{b_0^2}{(b_0^2 - 1)} \left[ dy \ \wedge \ dx + \chi \ dy \ \wedge \ dy \psi + \psi e^y (e^y + 1) \ dy \ \wedge \ dy \chi \right], \]

\[ \Phi(t) = \varphi_0 - \frac{1}{2} \log |b_0^2 - 1|. \] (4.29)

Starting from the decomposition of semi-Abelian DSD \((gl(1|1), \mathcal{I}_{(2|2)})\) we could calculate the super PL T-plural of cosmology invariant with respect to the \(GL(1|1)\) Lie supergroup. In fact, we obtained another conformally invariant duality chain of \(2 + 1\)-dimensional cosmological string backgrounds coupled with two fermionic fields in the form of equations (4.24) and (4.29).
5 Conclusion

We have generalized the formulation of PL T-plurality from Lie groups to Lie supergroups, more strictly speaking, from DDs to DSDs. Using the left-invariant one-forms of the \((C^3 + A)\) and \(GL(1|1)\) Lie supergroups we have obtained some new cosmological string backgrounds including \((2 + 1|2)\)-dimensional metric, a dilaton field at most a function of \(t\) only and vanishing torsion which are conformally invariant up to the one-loop order. The metrics of these backgrounds are flat in the sense that their scalar curvature is zero (more precisely, constant). We have then shown that the resulting backgrounds are equivalent to the ones of T-dual \(\sigma\)-models constructing on semi-Abelian DSDs \((\mathcal{C}^3 + \mathcal{A}, \mathcal{I}_{(2|2)})\) and \((\mathcal{g}l(1|1), \mathcal{I}_{(2|2)})\). Most importantly, starting from the above-mentioned decompositions of semi-Abelian DSDs we have found the conformal duality/plurality chains of \(2 + 1\)-dimensional cosmological string backgrounds coupled with two fermionic fields. We have furthermore checked that the backgrounds obtained by super PL T-plurality remain conformally invariant at one-loop level. Our current goal was to get better understanding of super PL T-plurality through investigation of examples presented in section 4 of the results.

As mentioned at the beginning of section 3, the Bianchi-type string cosmologies could be generalized to \(4 + 1\)-dimensional cosmological models whose spatial hypersurfaces are (simply) connected homogeneous Riemannian manifolds \([29]\). Our results can be also generalized to higher dimensions. One may employ the formulation of super PL T-plurality in order to obtain the conformal duality chains of cosmological string backgrounds in higher dimensions of the type \((m + 1|2n)\). For instance, in dimension five one must use the decompositions of DSDs generated by Lie super bi-algebras of the type \((3|2)\) \([34]\). We intend to address this problem in the future.

We don’t know at the moment whether the resulting backgrounds have meaningful physical interpretation. But, we hope that in future it will be possible to find super PL T-plural models even for physically interesting metrics.

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A Some properties of supermatrices and tensors on supervector space

In this appendix we give a few relevant details concerning properties of matrices and tensors on supervector space which feature in the main text, appear as supertranspose, superdeterminant, supertrace, etc \([19]\).

Let \(\mathcal{G}\) be a supervector space with the bases \(e_M\), and \(e'_N\) be its dual bases. The transformation between the bases can be written as follows:

\[
e_M = (-1)^N K_M^N e'_N. \tag{A.1}
\]
One may consider the standard bases for the supervector space $\mathcal{G}$ such that in writing the bases as a column matrix, one first presents the bosonic bases, then the fermionic ones. In this way, the transformation matrix $K$ has the following block diagonal representation [19]

$$
K = \begin{pmatrix}
A & C \\
D & B
\end{pmatrix},
$$

where $A, B$ and $C$ are real sub-matrices while $D$ is pure imaginary sub-matrix. Here we consider the matrices and tensors having a form with all upper and lower indices written in the right hand side. The transformation properties of upper and lower right indices to the left one are, for general tensors, given by

$$
^pT_{mn...}^q = T_{mn...}^{pq}, \quad n^T_{l...}^m = (-1)^n T_{ml...}^{pq}.
$$

Let $K, L, P$ and $Q$ be the matrices whose elements indices have different positions. Then, we define the supertranspose for these matrices in the following forms

$$
(K^{st})_{MN}^M = (-1)^{MN} K_N^M, \quad (L^{st})_M^N = (-1)^{MN} L_N^M,
$$

$$
(P^{st})_{MN}^M = (-1)^{MN} P_{NM}, \quad (Q^{st})_{MN} = (-1)^{MN} Q_{NM}^{MN}.
$$

For the matrix $K$ whose elements, $_{M}K_{N}^{M}$, have the left index in the lower position and the right index in the upper position, one defines the supertrace as

$$
strK = (-1)^{M} K_{M}^{M}.
$$

When the matrix $K$ is expressed in the block form (A.2) the supertrace becomes

$$
strK = trA - trB,
$$

where “tr” denotes the ordinary trace.

If the sub-matrix $B$ in the block form (A.2) be a non-singular one, then the superdeterminant for the matrix $K$ is defined by

$$
sdet \begin{pmatrix} A & C \\
D & B
\end{pmatrix} = det(A - CB^{-1}D)(detB)^{-1},
$$

and if the sub-matrix $A$ be non-singular, then

$$
sdet \begin{pmatrix} A & C \\
D & B
\end{pmatrix} = (det(B - DA^{-1}C))^{-1} (detA).
$$

If both $A$ and $B$ are non-singular, then the inverse matrix for (A.2) has the following form:

$$
\left( \begin{array}{c|c}
A & C \\
\hline
D & B
\end{array} \right)^{-1} = \left( \begin{pmatrix}
(1_m - A^{-1}CB^{-1}D)^{-1}A^{-1} \\
-(1_m - B^{-1}DA^{-1}C)^{-1}B^{-1}DA^{-1}
\end{pmatrix} \begin{pmatrix}
-(1_m - A^{-1}CB^{-1}D)^{-1}A^{-1}CB^{-1} \\
(1_m - B^{-1}DA^{-1}C)^{-1}B^{-1}
\end{pmatrix} \right), \quad (A.9)
$$
where \( m \) and \( n \) are dimensions of sub-matrices \( A \) and \( B \), respectively.

If \( F(x) \) be a differentiable function on \( \mathbb{R}^m \times \mathbb{R}^n \) where \( \mathbb{R}^m \) are a subset of all real numbers (c-numbers) with dimension \( m \) and \( \mathbb{R}^n \) are a subset of all odd Grassmann variables (a-numbers) with dimension \( n \), then relation between the left and right partial differentiations is given by

\[
\frac{\partial}{\partial x^M} F = (-1)^{|F|+1} \frac{\partial}{\partial x^M},
\]

(A.10)

where \( |F| \) indicates the grading \( F \).

If \( f \) be a scalar field, \( V^M = \frac{\partial}{\partial x^M} \) a contravariant vector field and \( \omega = \omega_M dx^M \) a covariant vector field, then one finds covariant derivative in explicit components form as follows:

\[
f \nabla_M = (-1)^{|f|} \nabla_M f = f \frac{\partial}{\partial x^M},
\]

(A.11)

\[
V^M \nabla_N = (-1)^{|V|+|M|} \nabla_N V^M = V^M \frac{\partial}{\partial x^N} + (-1)^{(P+1)} V^P \nabla_P M_N,
\]

(A.12)

\[
\omega_M \nabla_N = (-1)^{|\omega|+|M|} \nabla_N \omega_M = \omega_M \frac{\partial}{\partial x^N} - \omega_P \nabla_P M_N,
\]

(A.13)

where \( \Gamma_{MN}^P \) are called the components of the connection \( \nabla \).

If the supersymmetric matrix \( G^P \) (its inverse denotes to \( A^B \), and \( G^{AB} = (-1)^{AB} G^{BA} \)) be the components of metric tensor field on a Reimannian supermanifold, then, in a coordinate basis, the components of the connection and Reimann tensor field are given by

\[
\Gamma_{NP}^M = (-1)^Q G^{MQ} \Gamma_{QNP} = (-1)^Q G^{MQ} \Gamma_{QNP} \frac{\partial}{\partial x^M} \frac{\partial}{\partial x^N} - (-1)^Q G^{NP} \frac{\partial}{\partial x^M} \frac{\partial}{\partial x^Q},
\]

(A.14)

\[
R^I_{JKL} = -\Gamma^I_{JK} + (-1)^{KL} \Gamma^I_{JL} \frac{\partial}{\partial x^K} + (-1)^{K(I+1)} \Gamma^I_{MK} \Gamma^M_{JL} - (-1)^{(J+K+M)} \Gamma^I_{ML} \Gamma^M_{JK}.
\]

(A.15)

In addition, for the curvature tensor field, the Ricci tensor and the curvature scalar field we have

\[
R_{IJKL} = G_{IM} R^M_{JKL},
\]

(A.16)

\[
R_{IJ} = (-1)^{K(I+1)} R^K_{IKJ},
\]

(A.17)

\[
R = R^M_M = str(R^N_{MN} G^{NM}).
\]

(A.18)

Accordingly, one may use the above definitions and formulas to rewrite the one-loop beta-
function equations (3.3)-(3.5) in the following form

\[
R_{MN} - \frac{1}{4}(-1)^{D+BN} H_{MBA} G^{AD} H_{DNS} G^{SB} \\
+ 2 \left[ (-1)^{M+N} \frac{\partial}{\partial x^M} \left( \frac{\partial \Phi}{\partial x^N} \right) - (-1)^K \frac{\partial}{\partial x^K} \Gamma^K_{MN} \right] = 0, \tag{A.19}
\]

\[
(-1)^{1+M+Q} \frac{\partial}{\partial x^Q} G^{QM} H_{MNP} + (-1)^{M+L+LN+LP} G^{LM} \left[ (-1)^{L(1+M+N+P)} \frac{\partial}{\partial x^P} H_{MNP} \right] \\
- (-1)^{(P+N)(M+Q)} H_{QNP} \Gamma^Q_{ML} - (-1)^{P(N+Q)} H_{MQP} \Gamma^{Q}_{NL} - H_{MNQ} \Gamma^{Q}_{PL} = 0, \tag{A.20}
\]

\[
4\Lambda - R + \frac{1}{12}(-1)^{M+BN} H_{GBA} G^{AM} H_{MNP} G^{PB} G^{NC} + 4(-1)^M \left( \frac{\partial}{\partial x^M} \right) G^{MN} \left( \frac{\partial}{\partial x^N} \right) \\
- 4G^{NM} \frac{\partial}{\partial x^M} \left( \frac{\partial \Phi}{\partial x^N} \right) + 4(-1)^{K+M+N} \Gamma^K_{MN} = 0, \tag{A.21}
\]

where \(R_{MN}\) and \(R\) are defined according to equations (A.17) and (A.18), respectively. It’s worth mentioning that all the lowering and raising of the indices will be done with respect to the tensor fields \(G_{MN}\) and \(G^{MN}\). As an example, for the components of tensor \(T\) we have

\[
T_{A_1, \ldots, A_r}^N B_1, \ldots, B_S = (-1)^{(M+N)(B_1, \ldots, B_S)} T_{A_1, \ldots, A_r, M, B_1, \ldots, B_S} G^{MN}. \tag{A.22}
\]

References

[1] C. Klimčík and P. Ševera, Dual non-Abelian duality and the Drinfeld double, Phys. Lett. B 351 (1995) 455, arXiv:hep-th/9502122.

[2] C. Klimčík, Poisson-Lie T-duality, Nucl. Phys. (Proc. Suppl.) B 46 (1996) 116, arXiv:hep-th/9509095.

[3] C. Klimčík and P. Ševera, Poisson-Lie T-duality and loop groups of Drinfeld doubles, Phys. Lett. B 372 (1996) 65, arXiv:hep-th/9512040.

[4] T. Buscher, A symmetry of the string background field equations, Phys. Lett. B 194 (1987) 59.

[5] T. Buscher, Path-integral derivation of quantum duality in non-linear sigma-models, Phys. Lett. B 201 (1988) 466.

[6] R. von Unge, Poisson-Lie T-plurality, J. High Energy Phys. 07 (2002) 014, arXiv:hep-th/0205245.

[7] L. Hlavatý and L. Šnobl, Poisson-Lie T-plurality of three-dimensional conformally invariant sigma models, J. High Energy Phys. 05 (2004) 010, arXiv:hep-th/0403164.
[8] L. Hlavatý and L. Šnobl, *Poisson-Lie T-plurality of three-dimensional conformally invariant sigma models II: non-diagonal metrics and dilaton puzzle*, J. High Energy Phys. 10 (2004) 045, arXiv:hep-th/0408126.

[9] L. Hlavatý, J. Hybl and M. Turek, *Classical solutions of sigma models in curved backgrounds by the Poisson-Lie T-plurality*, Int. J. Mod. Phys. A 22 (2007) 1039, arXiv:hep-th/0608069.

[10] L. Hlavatý and L. Šnobl, *Poisson-Lie T-plurality as canonical transformation*, Nucl. Phys. B 768 (2007) 209, arXiv:hep-th/0608133.

[11] L. Hlavatý and I. Petr, *Poisson-Lie T-plurality revisited. Is T-duality unique?*, J. High Energy Phys. 04 (2019) 157, arXiv:1811.12235 [hep-th].

[12] L. Hlavatý, I. Petr and V. Štěpán, *Poisson-Lie T-plurality with spectators*, J. Math. Phys. 50 (2009) 043504.

[13] Y. Sakatani, *Type II DFT solutions from Poisson-Lie T-duality/plurality*, Prog. Theor. Exp. Phys. 073B04 (2019), arXiv:1903.12175 [hep-th].

[14] L. Hlavatý and I. Petr, *Poisson-Lie plurals of Bianchi cosmologies and Generalized Supergravity Equations*, arXiv:1910.08436 [hep-th].

[15] N. Backhouse, *A classification of four-dimensional Lie superalgebras*, J. Math. Phys. 19 (1978) 2400.

[16] A. Eghbali and A. Rezaei-Aghdam, *Lie superbialgebra structures on the Lie superalgebra \((C^3 + A)\) and deformation of related integrable Hamiltonian systems*, J. Math. Phys. 58 (2017) 063514, arXiv:1606.04332 [math-ph].

[17] Ali Eghbali and Adel Rezaei-Aghdam, *A hierarchy of WZW models related to super Poisson-Lie T-duality*, Work in progress.

[18] A. Eghbali and A. Rezaei-Aghdam, *The gl(1|1) Lie superbialgebras*, J. Geom. Phys. 65 (2013) 7, arXiv:1112.0652 [math-ph].

[19] B. DeWitt, *Supermanifolds*, Cambridge University Press (1992).

[20] A. Eghbali and A. Rezaei-Aghdam, *Poisson-Lie T-dual sigma models on supermanifolds*, J. High Energy Phys. 09 (2009) 094, arXiv:0901.1592 [hep-th].

[21] A. Eghbali and A. Rezaei-Aghdam, *String cosmology from Poisson-Lie T-dual sigma models on supermanifolds*, J. High Energy Phys. 01 (2012) 151, arXiv:1107.2041 [hep-th].

[22] A. Eghbali, A. Rezaei-Aghdam and F. Heidarpour, *Classification of two and three dimensional Lie superbialgebras*, J. Math. Phys. 51 (2010) 073503, arXiv:0901.4471 [math-ph].
[23] G. F. R. Ellis and M. A. H. MacCallum, *A class of homogeneous cosmological models*, Commun. Math. Phys. 12 (1969) 108.

[24] C. G. Callan, D. Friedan, E. Martinec and M. J. Perry, *String in background fields*, Nucl. Phys. B 262 (1985) 593.

[25] E. J. Copeland, Amitabha Lahiri and David Wands, *String cosmology with a time-dependent antisymmetric tensor potential*, Phys. Rev. D 51 (1995) 1569, arXiv:hep-th/9410136.

[26] M. Gasperini and R. Ricci, *Homogeneous conformal string backgrounds*, Class. Quantum Grav. 12 (1995) 677, arXiv:hep-th/9501055.

[27] N. A. Batakis and A. A. Kehagias, *Anisotropic space-times in homogeneous string cosmology*, Nucl. Phys. B 449 (1995) 248, arXiv:hep-th/9502007.

[28] N. A. Batakis, *A new class of homogeneous string backgrounds*, Phys. Lett. B 353 (1995) 450, arXiv:hep-th/9503142.

[29] S. Hervik, *Multidimensional cosmology: spatially homogeneous models of dimension 4+1*, Class. Quantum Grav. 19 (2002) 5409.

[30] B. Mojaveri and A. Rezaei-Aghdam, *4 + 1 dimensional homogeneous anisotropic string cosmological models*, Int. J. Mod. Phys. A 27 (2012) 1250032, arXiv:1106.1795 [hep-th].

[31] K. Sfetsos, *Poisson-Lie T-duality and supersymmetry*, Nucl. Phys. (Proc. Suppl.) B 56 (1997) 302, arXiv:hep-th/9611199.

[32] A. Eghbali and A. Rezaei-Aghdam, *WZW models as mutual super Poisson-Lie T-dual sigma models*, J. High Energy Phys. 07 (2013) 134, arXiv:1303.4069 [hep-th].

[33] A. Eghbali and A. Rezaei-Aghdam, *Super Poisson-Lie symmetry of the GL(1|1) WZNW model and worldsheet boundary conditions*, Nucl. Phys. B 866 (2013) 26, arXiv:1207.2304 [hep-th].

[34] C. Juszczak and J. T. Sobczyk, *Classification of low-dimensional Lie super-bialgebras*, J. Math. Phys. 39 (1998) 4982, arXiv:q-alg/9712015.