REGULARITY OF PRIMES ASSOCIATED WITH POLYNOMIAL PARAMETRISATIONS

FRANCESCA CIOFFI AND ALDO CONCA

Abstract. We prove a doubly exponential bound for the Castelnuovo-Mumford regularity of prime ideals defining varieties with polynomial parametrisation.

1. Introduction

Let $I$ be a homogeneous ideal in the polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$. The Castelnuovo-Mumford regularity $\text{reg}(I)$ of $I$ has been introduced in [9] and [17] as an algebraic counterpart of the corresponding notion introduced by Mumford [15] for coherent sheaves over projective spaces. It quickly became one of the most important homological and cohomological invariants of $I$. It is defined in terms of the graded Betti numbers $\beta_{i,j}(I)$ as

$$\text{reg}(I) = \max\{j - i : \beta_{i,j}(I) \neq 0\}$$

as well as in terms of the graded local cohomology modules $H^i_m(I)$ as

$$\text{reg}(I) = \max\{j + i : H^i_m(I)_j \neq 0\},$$

where $m = (x_1, \ldots, x_n)$. It is known [1, 4] that $\text{reg}(I) \leq (2u)^{2n^2}$, where $u$ is the largest degree of a generator of $I$. When $P$ is a prime homogeneous ideal one expects better bounds for $\text{reg}(P)$, see [1] and the recent [14] for an overview. On the other hand, McCullough and Peeva proved in [13] that $\text{reg}(P)$ cannot be bounded above by any polynomial in the degree (or multiplicity) $\deg(R/P)$ hence disproving the long-standing Eisenbud-Goto conjecture [9]. Later on Caviglia, Chardin, McCullough, Peeva and Vbaro [5] proved that $\text{reg}(P)$ can be actually bounded by (a highly exponential) function in $\deg(R/P)$.

In this short note we prove a doubly exponential bound for $\text{reg}(P)$ when $P$ defines a variety with a polynomial parametrisation that does not involve the degree of the generators of $P$ but only the numerical data of the parametrisation.

Theorem 1.1. Let $P$ be the kernel of a $K$-algebra map $\phi : K[x_1, \ldots, x_n] \to K[y_1, \ldots, y_m]$ with $\phi(x_i) = f_i$ homogeneous polynomials of degree $d > 0$. Then one has:

$$\text{reg}(P) \leq d^{n^2m^2 - 1}.$$ 

We remark that a “combinatorial” bound for $\text{reg}(P)$ in the case of curves (i.e. $m = 2$) with a monomial parametrisation has been obtained in [12, Prop. 5.5] and that it would be very interesting to obtain similar bounds for higher dimensional toric ideals. We thank two anonymous referees for helpful comments and for suggesting that a bound comparable with that of Theorem [14] might be obtained using the techniques and the ideas of [5] combined with other estimates on the regularity and on the degree.

2. Flat extensions, regularity and elimination

For the proof of Theorem [14] we need to collect some ingredients.

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2.1. Regularity and flat extensions. Let \( R = K[x_1, \ldots, x_n] \) with its standard graded structure. Let \( d \) be a positive integer and let \( \alpha : R \to R \) be the \( K \)-algebra map defined by \( \alpha(x_i) = x_i^d \) for \( i = 1, \ldots, n \). For a homogeneous ideal \( I \) of \( S \) we set \( I' = \alpha(I)R \). By construction \( I' \) is homogeneous and we have:

**Lemma 2.1.** \( \text{reg}(I) \leq \text{reg}(I')/d. \)

**Proof.** By \([10]\) the map \( \alpha \) is flat. Hence if \( F \) is a minimal graded free resolution of \( I \) then \( \alpha(F)R \) is a minimal graded free resolution of \( I' \). Therefore the graded Betti numbers of \( I \) and \( I' \) are related as follows: \( \beta_{i,j}(I') = \beta_{i,j}(I) \) for all \( i,j \) and \( \beta_{i,j}(I') = 0 \) if \( d \) does not divide \( j \). For \( i = 0, \ldots, \text{projdim}(I) \) set \( t_i(I) = \max\{j : \beta_{i,j}(I) \neq 0\} \). Then we have \( t_i(I') = dt_i(I) \). By definition \( \text{reg}(I) = \max\{t_i(I) - i : i = 0, \ldots, \text{projdim}(I)\} \). Let \( p \) be the largest integer \( i \) such that \( \text{reg}(I) = t_i(I) - i \). Then

\[
\text{reg}(I') \geq t_p(I') - p = dt_p(I) - p = d(\text{reg}(I) + p) - p = d \text{reg}(I) + p(d-1).
\]

It follows that

\[
\text{reg}(I')/d \geq \text{reg}(I) + p(d-1)/d
\]

and, in particular,

\[
\text{reg}(I')/d \geq \text{reg}(I).
\]

\( \square \)

**Remark 2.2.** The proof of Lemma 2.1 shows that the inequality in Lemma 2.1 is strict unless \( d = 1 \) (which is obvious) or the index \( p \) defined in the proof is \( 0 \) and this happens only if \( I \) is principal. If \( R/I \) is Cohen-Macaulay then \( p = \text{height}(I) - 1 \) and in Eq. (1) one has equality. For example if \( I \) is a complete intersection of \( c \) forms of degree \( s \) then \( \text{reg}(I) = sc - (c - 1) \) and \( \text{reg}(I') = dsc - (c - 1) \) so that

\[
\text{reg}(I')/d - \text{reg}(I) = (c - 1)(d - 1)/d.
\]

In general however in Eq. (1) one does not have equality. For example for an ideal \( I \) with \( \text{projdim}(I) = 4 \) and

\[
(t_0(I), t_1(I), t_2(I), t_3(I), t_4(I)) = (2, 4, 5, 5, 6)
\]

one has \( \text{reg}(I) = 3 \), \( p = 2 \) and \( \text{reg}(I') = 6d - 4 \) for \( d > 1 \). Hence the inequality in Eq. (1) is strict for \( d > 2 \). An ideal with invariants as in (2) is, for example, \( I = (x_1)x_1, x_2, \ldots, x_5) + (x_2^2, x_3^2) \).

2.2. Flat extensions and elimination. Now let \( \ell \geq n \) and \( S = K[x_1, \ldots, x_\ell] \). Let \( J \) be an ideal of \( S \) and \( I = J \cap R \). Let \( d_1, \ldots, d_\ell \in \mathbb{N}_{>0} \) and \( \varphi : S \to S \) be the \( K \)-algebra map defined by \( \varphi(x_i) = x_i^{d_i} \) and let \( \alpha \) be the restriction of \( \varphi \) to \( R \). Let \( J \) be an ideal of \( S \) and \( I = J \cap R \). Set

\[
I' = \alpha(I)R \text{ and } J' = \varphi(J)S.
\]

Let \( < \) be the lexicographic order associated with \( x_\ell > x_{\ell-1} > \cdots > x_1 \). Recall that \( < \) is an elimination term order for the variables \( x_{n+1}, \ldots, x_\ell \). In particular, if \( G \) is a Gröbner basis of \( J \) with respect to \( < \), then \( G \cap R \) is a Gröbner basis of \( I \) with respect to \( < \) restricted to \( R \), see for example [7], Thm. 2, Sect. 3.1.

**Lemma 2.3.** With the notation above we have:

(i) If \( G \) is a Gröbner basis of \( J \) with respect to \( < \), then \( \varphi(G) = \{ \varphi(g) : g \in G \} \) is a Gröbner basis of \( J' \) with respect to \( < \).

(ii) \( I' = J' \cap R \).
Proof. Firstly we observe that, since we deal with the lex order, for every pair of monomials \( \tau \) and \( \sigma \) of \( S \), we have \( \tau < \sigma \) if and only if \( \varphi(\tau) < \varphi(\sigma) \). In particular, \( \text{in}(\varphi(f)) = \varphi(\text{in}(f)) \) for any non-zero \( f \in S \). Secondly, we observe that \( \varphi \) is compatible with the Buchberger criterion. Indeed denoting by \( S(f, h) \) the \( S \)-polynomial of two polynomials \( f \) and \( h \) of \( S \), one has that \( \varphi(S(f, h)) = S(\varphi(f), \varphi(h)) \). Furthermore if \( h = \sum_{g \in G} p_g g \) is a division with remainder 0 of \( h \) with respect to \( G \), then \( \varphi(h) = \sum_{g \in G} \varphi(p_g) \varphi(g) \) is a division with remainder 0 of \( \varphi(h) \) with respect to \( \varphi(G) \). By the Buchberger criterion [8], this is enough to conclude that (i) holds.

To prove (ii) we observe that \( I \) is generated by \( G \cap R \) so that \( I' \) is generated by the set \( \alpha(G \cap R) \). On the other hand, by (i), \( J' \cap R \) is generated by the set \( \alpha(G) \cap R \). Clearly \( \alpha(G \cap R) = \varphi(G) \cap R \) and hence it follows that \( I' = J' \cap R \).

2.3. Regularity and elimination. We keep the notation above, i.e. \( \ell \geq n \) and \( S = K[x_1, \ldots, x_n] \supseteq R = K[x_1, \ldots, x_n] \). Let \( J \) be a homogeneous ideal of \( S \) and \( I = J \cap R \). In general it can happen that \( \text{reg}(I) > \text{reg}(J) \) or \( \text{reg}(I) < \text{reg}(J) \).

Example 2.4. Let \( J = (x_1^2, x_2^2) \) and \( I = J \cap K[x_1] = (x_1^2) \). Then \( \text{reg}(J) = 3 \) and \( \text{reg}(I) = 2 \). Let \( J = (x_1x_2 + x_2x_3, x_1x_3, x_3^2) \) and \( I = J \cap K[x_1, x_2] = (x_1^2x_2) \). Then \( \text{reg}(J) = 2 \) and \( \text{reg}(I) = 3 \).

However we can get a bound for \( \text{reg}(I) \) in terms of the regularity of the lexicographic ideal \( \text{Lex}(J) \) of the ideal \( J \). We refer the reader to [3] for generalities on the lexicographic ideal \( \text{Lex}(J) \). Here we just recall that \( \text{Lex}(J) \) is defined as \( \oplus_{i \in \mathbb{N}} \text{Lex}(J_i) \) where \( \text{Lex}(J_i) \) is the \( K \)-vector space generated by the largest \( \ell_i \) monomials of degree \( i \) with respect to the lexicographic order. Macaulay proved that \( \text{Lex}(J) \) is actually an ideal of \( S \) and Bigatti [2], Hulett [11] and Pardue [18] proved that \( \beta_{ij}(J) \leq \beta_{ij}(\text{Lex}(J)) \) for all \( i, j \). In particular one has \( \text{reg}(J) \leq \text{reg}(\text{Lex}(J)) \).

Proposition 2.5. With the notations above we have \( \text{reg}(I) \leq \text{reg}(\text{Lex}(J)) \).

Proof. Let \( G \) be a Gröbner basis of \( J \) with respect to the lex order with \( x_\ell > x_{\ell-1} > \cdots > x_1 \). The elements of \( G \) that belong to \( R \) form a Gröbner basis of \( I \). In particular \( \text{in}(I) = \text{in}(J) \cap R \). It is known that \( \text{reg}(I) \leq \text{reg}(\text{in}(I)) \) [8]. Furthermore by [16] Cor. 2.5 we have \( \text{reg}(\text{in}(I)) \leq \text{reg}(\text{in}(J)) \) since \( R/\text{in}(I) \) is an algebra retract of \( S/\text{in}(J) \). Finally \( \text{Lex}(J) = \text{Lex}(\text{in}(J)) \) because \( J \) and \( \text{in}(J) \) have the same Hilbert function and \( \text{reg}(\text{in}(J)) \leq \text{reg}(\text{Lex}(J)) \) by the Bigatti-Hulett-Pardue theorem mentioned above. Summing up,

\[
\text{reg}(I) \leq \text{reg}(\text{in}(I)) \leq \text{reg}(\text{in}(J)) \leq \text{reg}(\text{Lex}(J))
\]

and we are done.

3. Proof of the main theorem

We have collected all the ingredients for the proof of Theorem 1.1.

Proof of Theorem 1.1. One has \( P = J \cap K[x_1, \ldots, x_n] \) with \( J = (x_i - f_i : i = 1, \ldots, n) \subset S = K[x_1, \ldots, x_n, y_1, \ldots, y_m] \). Consider \( \varphi : S \rightarrow S \) defined by \( \varphi(x_i) = x_i^d \) and \( \varphi(y_i) = y_i \) and let \( \alpha : R \rightarrow R \) be the restriction of \( \varphi \) to \( R \), so that \( \alpha(x_i) = x_i^d \).

In this setting, we have \( J' = \varphi(J)S = (x_i^d - f_i : i = 1, \ldots, n) \) and \( P' = \alpha(P)R \). The ideal \( J' \) is a complete intersection of \( n \) forms of degree \( d \) and dimension \( m \) so its Hilbert function and hence \( \text{Lex}(J') \) just depend on \( n, d, m \). Set \( G_{n,d,m} = \text{reg}(\text{Lex}(J')) \).

By Lemma 2.3(ii) we have \( P' = J' \cap R \) and thanks to Proposition 2.5,

\[
\text{reg}(P') \leq G_{n,d,m}.
\]
Since by Lemma 2.1 \( \text{reg}(P) \leq \text{reg}(P')/d \) we conclude
\[
\text{reg}(P) \leq G_{n,d,m}/d.
\]
Taking into account that by [6, Corollary 3.4] we have \( G_{n,d,m} \leq d^{n^2-1} \), we obtain the desired inequality. \( \square \)

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DIPARTIMENTO DI MATEMATICA E APPLICAZIONI DELL’UNIVERSITÀ DI NAPOLI FEDERICO II, VIA CINTIA, 80126 NAPOLI, ITALY

Email address: cioffifr@unina.it

DIPARTIMENTO DI MATEMATICA DELL’UNIVERSITÀ DI GENOVA, VIA DODECANESO 35, 16146 GENOVA, ITALY

Email address: conca@dima.unige.it