Research Article

The Properties of Maximal Filters in Multilattices

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Jacobson’s radical of a filter F is the intersection of all maximal filters containing F. We present several properties of maximal filters in multilattices. As a consequence of Zorn’s lemma, we prove that each proper filter is contained in a maximal filter. When the filter lattice is distributive, we prove that each maximal filter is prime. Finally, we determine Jacobson’s radical of filters in multilattices.

1. Introduction

We owe the multilattice theory to Benado who, in his work on multistructures, had the intuition to generalize lattices by replacing the existence of unique lower (resp. upper) bound by the existence of maximal lower (resp. minimal upper) bounds [1]. Several authors will contribute to the consolidation of this theory by enriching it with the study of mathematical concepts. It is in this perspective that Klaučová [2] and Hansen [3] propose many characterizations of multilattices. In the same vein, Johnston [4] identified three types of ideals and then showed that in multilattices, some concepts such as associativity and distributivity cannot be defined in a canonical way. Indeed, the notion associativity is replaced by a less natural notion of m-associativity and the notion of distributivity remains nontrivial.

Following the example of Medina et al. [5, 6], several authors will be interested in this theory because of its numerous applications in information systems with uncertainty. Cabrera et al. [7] proposed an algebraization of multilattices and advocated in [8], a new definition of ideal (resp. filter) suitable for congruences. Then, they proved that the set of all filters of a multilattice is a lattice with respect to inclusion. In [9], Awouafack et al. showed that for some multilattices, the filter lattice is distributive. This allowed them to define a notion of distributivity in multilattices which they used to establish the existence of prime filters.

This work is in the same register and aims at providing the main properties of maximal filters in multilattices. Some results are very close to the classical cases while others show that more caution is needed.

The paper is structured as follows: Section 2 recalls definitions and preliminary results necessary to understand the paper. Section 3 brings out the newly established results.

2. Preliminaries

Let \((M, \leq)\) be a poset and let \(X \subseteq M\) be a nonempty subset of \(M\). An element \(u \in M\) is said to be an upper bound of \(X\) if it satisfies \(x \leq u\) for all \(x \in X\). Dually, \(l \in M\) is a lower bound of \(X\) if it satisfies \(l \leq x\) for all \(x \in X\).

Definition 1. [1] A multisupremum (resp. multinfimum) of \(X\) is a minimal upper (resp. maximal lower) bound of \(X\).

The set of multisuprema (resp. multinfima) of \([x, y]\) will be denoted by \(x \cup Y\) (resp. \(x \cap Y\)). For all nonempty subsets \(X, Y\) of \(M\), \(X \cup Y\) and \(X \cap Y\) denote the following subsets:

\[
X \cup Y = \bigcup_{x \in X, y \in Y} x \cup y \quad \text{and} \quad X \cap Y = \bigcup_{x \in X, y \in Y} x \cap y.
\] (1)
Definition 2. [1] Let \((M; \leq)\) be a poset. \(M\) is said to be an ordered multilattice if the following conditions hold for all \(x, y, a \in M\):

\[ M_1: x \leq a \text{ and } y \leq a \Rightarrow \exists z \in x \cup y \text{ such that } z \leq a. \]

\[ M_2: a \leq x \text{ and } a \leq y \Rightarrow \exists z \in x \cap y \text{ such that } a \leq z. \]

The two hyperoperations \(\cup: M^2 \rightarrow \mathcal{P}(M)\) and \(\cap: M^2 \rightarrow \mathcal{P}(M)\) satisfy the following properties called axioms of multilattices [2, 3].

AM1: For all \(x \in M\), \(x \cup x = \{x\}\).

AM2: For all \(x, y \in M\), \(x \cup y = y \cup x\) and \(x \cap y = y \cap x\).

AM3: For all \(x, y, z \in M\), \(x \leq y \Rightarrow (x \cup y) \cup z \leq x \cup (y \cup z)\), and \((x \cap y) \cap z \leq x \cap (y \cap z)\).

AM4: For all \(x, y \in M\), \(x \cup (x \cap y) = x\) and \((x \cap y) \cup x = x\).

AM5: For all \(x, y \in M\), \(x \leq y \Leftrightarrow x \cap y = \{x\} \Leftrightarrow x \cup y = \{y\}\).

Definition 3. [8] An algebraic multilattice is any triple \((M; \cup, \cap)\) which satisfies the properties AM1–AM5.

The next theorem shows how to go back and forth from ordered multilattice to algebraic multilattice.

Theorem 1. [9] The following assertions are satisfied:

(i) If \((M; \leq)\) is an ordered multilattice, set

\[ x \cup y = \text{multisup}(x, y) \text{ and } x \cap y = \text{multinf}(x, y) \]

for all \(x, y \in M\).

\[ (2) \]

Then, \((M; \cup, \cap)\) is an algebraic multilattice.

(ii) Conversely if \((M; \cup, \cap)\) is a multilattice, set:

\[ x \leq y \Leftrightarrow x \cap y = \{x\} \Leftrightarrow x \cup y = \{y\}. \]

\[ (3) \]

Then, \((M; \leq)\) is an ordered multilattice.

We will simply write \(M\) instead of \((M; \leq)\), \((M; \cup, \cap)\) or \((M; \leq, \cup, \cap)\) assuming that the underlined order and the corresponding hyperoperations are understood.

Definition 4. [8]

(1) \(M\) is said to be a complete multilattice if every subset of \(M\) has at least one multisupremum and at least one multiminimum.

(2) \(A \cup\)-full multilattice (resp. \(\cap\)-full multilattice) is a multilattice in which \(x \cup y \neq \emptyset\) (resp. \(x \cap y \neq \emptyset\)) for all \(x, y \in M\). A multilattice is said to be full if it is both \(\cup\)-full and \(\cap\)-full.

(3) A coherent multilattice is a multilattice in which every chain is bounded.

In multilattices, there are several proposals for the definition of the ideal [4, 8]. In this paper, we use the one proposed in 2014 by Cabrera et al. [8] which is the only one adapted to the study of several mathematical concepts such as congruences and homomorphisms.

Definition 5. [8] A nonempty subset \(F\) of \(M\) is said to be a filter if it satisfies the following conditions:

\[ (F1): \text{ For all } x, y \in F, x \cap y \subseteq F; \]

\[ (F2): \text{ For all } a \in M \text{ and } x \in F, a \cup x \subseteq F; \]

\[ (F3): \text{ For all } x, y \in M \text{ such that } (x \cup y) \cap F \neq \emptyset, \]

\[ x \cup y \subseteq F. \]

The ideal is the dual concept of filter. So, a nonempty subset \(I\) of \(M\) is said to be an ideal if it satisfies the following conditions:

\[ (I1): \text{ For all } x, y \in I, x \cup y \subseteq I; \]

\[ (I2): \text{ For all } a \in M \text{ and } x \in I, a \cap x \subseteq I; \]

\[ (I3): \text{ For all } x, y \in M \text{ such that } (x \cap y) \cap I \neq \emptyset, \]

\[ x \cap y \subseteq I. \]

The set of all filters of \(M\) will be denoted by \(\mathcal{F}(M)\).

Proposition 1. [8] Let \(M\) be an \(\cup\)-full multilattice. Then, \(\mathcal{F}(M)\) is a lattice under the set inclusion.

The lattice \((\mathcal{F}(M), \vee, \wedge)\) is given by: \(F_1 \wedge F_2 = F_1 \cap F_2\) and \(F_1 \vee F_2\) is the smallest filter of \(M\) containing both \(F_1\) and \(F_2\). When \(M\) is not \(\cup\)-full, it is necessary to add the empty set to \(\mathcal{F}(M)\) (lifting) in order to obtain a lattice. That is \(\overline{\mathcal{F}}(M) = \mathcal{F}(M) \cup \{\emptyset\}\) is always a lattice ordered by set inclusion [9].

In [9], the authors described the filter generated by a nonempty subset by the substar operator as follows:

Theorem 2. [9] Let \(X\) a nonempty subset of \(M\). Set

\[ X_* = \bigcup \{a \cup b | (a \cup b) \cap (\{x\} \neq \emptyset, a, b \in M\}. \]

(4)

Also, define the sequence \((X_n)_{n \in \mathbb{N}}\) recursively as follows:

\[ X_0 = X, X_1 = X, \quad \forall n \geq 1, X_{n+1} = \{X_n \cap X_n\}^* \].

Then, the filter of \(M\) generated by \(X\), denoted by \(\langle X \rangle\) is given by: \(\langle X \rangle = \bigcup_{n \in \mathbb{N}} X_n\).

Many of our proofs will be based on the following lemma.

Lemma 1. [9] Let \(x, y \in M\). Then, the following assertions are satisfied:

\[ (1) \ z \in x \cap y \Rightarrow \langle z \rangle = \langle x \rangle \cap \langle y \rangle; \]

\[ (2) \ z \in x \cup y \Rightarrow \langle z \rangle \subseteq \langle x \rangle \wedge \langle y \rangle; \]

\[ (3) \ z, z' \in x \cup y \Rightarrow \langle z \rangle = \langle z' \rangle. \]

Remark 1

(i) The inclusion of (2) of Lemma 1 is in general strict. For instance, in the multilattice of Example 1, we have \(\langle c_1 \cup c_2 \rangle = \{a_2\} \) but \(\langle c_1 \rangle \wedge \langle c_2 \rangle = M_1\).
(ii) The map defined on the power set of M by \( x \rightarrow \langle x \rangle \) is an algebraic closure operator. This gives us the means to understand some properties of the filter lattice.

(iii) However, the map \( \langle : M \rightarrow 2^M \rangle \) given by \( x \rightarrow \langle x \rangle \) is not an embedding, it will be an embedding if we are in front of a lattice. We can see this through Example 1. For instance, \( \langle c_1 \rangle = \langle c_2 \rangle = M \) but \( c_1 \neq c_2 \). So, according to Cabrera et al. [8], in multilattices there are more elements than filters.

Before introducing our results, let us recall some commonly used concepts. A filter (resp. an ideal) \( K \) is said to be prime if for two filters (resp. ideals) \( F, J \), if \( F \cap J \subseteq K \) then either \( F \subseteq K \) or \( J \subseteq K \).

A maximal filter is any maximal element of \( \mathcal{F}(M) \backslash \{M\} \).

3. Main Results

In a lattice, a filter is principal if and only if it is an upset whereas in multilattice the two notions appear to be distinct. The following result allows us to understand this difference. An illustration will be given in Example 1.

**Theorem 3.** Let \( M \) be a multilattice and let \( F \) be a filter of \( M \). Then, the following assertions are satisfied:

1. If \( F \) is finite, then \( F \) is an upset.
2. If \( F \) is finitely generated, then \( F \) is principal.
3. If \( F \) is generated by a subset with least element, then \( F \) principal.
4. If \( F \) is not finitely generated, then \( F \) contains an infinite chain without the least element.

**Proof.** (1) Let \( F = \{x_1, x_2, \ldots, x_n\} \). Since \( F \cap M \subseteq F \), then, for all \( x_i, x_j \in F \), there is \( x_k \in F \) such that \( x_k \leq x_i \) and \( x_k \leq x_j \). Thus, there exists \( i \in \mathbb{N} \) such that \( x_j \leq x_{i+1} \) and \( x_i \leq x_{i+1} \). Then, there exists \( i \in \mathbb{N} \) such that \( x_j \leq x_{i+1} \) and \( x_i \leq x_{i+1} \), by inference, there exists \( i \in \mathbb{N} \) such that \( x_j \leq x_{i+1} \) and \( x_i \leq x_{i+1} \). We claim that \( i \leq x_i \) for all \( i \in \mathbb{N} \). This implies \( F = \{x_i\} \).

For (2), let \( X = \{x_1, x_2, \ldots, x_n\} \), where \( x_1 < x_2 < \cdots < x_n \), such that \( \langle X \rangle = F \). We form a sequence \( \{x_i\} \) as follows: \( x_1 = x_0 \) and \( z_{i+1} = z_i \) for all \( i \geq 1 \). Then, \( X \leq \langle x \rangle \). We claim that \( \langle x \rangle = F \) and we obtain the desired conclusion.

For (3), let \( X \) be a subset of \( M \) with a least element \( a \). Then, \( X \leq \langle a \rangle \) and it follows that \( \langle x \rangle \leq \langle a \rangle \). Since \( a \in X \), we also have \( \langle a \rangle \leq \langle X \rangle \). This implies \( F = \langle a \rangle \).

For (4), let \( X = \{x_i\} \) be an infinite subset without least element such that \( \langle X \rangle \subseteq F \). We form a sequence, \( \{x_i\} \) as follows: \( z_0 = \varnothing \) and \( z_i = z_{i-1} \forall i \geq 1 \). Then, \( C = \{z_i\} \subseteq \langle X \rangle \) is an infinite chain of \( M \) contained in \( C \). However, if \( C \) is finite it will have least element. Thus, \( F \) contains an infinite chain without least element.

We obtain the following result as a consequence of (4) of Theorem 3.

**Corollary 1.** Any filter of a coherent multilattice is principal.

Let \( M \) be multilattice with bottom \( \bot \). Let \( \Delta \) denotes the set of minimal elements of \( M \).

**Proposition 2.** Let \( M \) be a maximal filter of \( M \). Then, \( M \) contains at most one element of \( \Delta \).

**Proof.** Let \( \alpha, \beta \in \Delta \) such that \( \alpha, \beta \in M \). Since \( M \) is a filter we have \( \alpha \cap \beta \subseteq M \). But necessarily, \( \alpha \cap \beta = \{\bot\} \) since \( \alpha \) and \( \beta \) are minimal in \( M \). This implies \( M = M \), a contradiction.

In the multilattice of Example 1, \( \cup_{i \in \mathbb{N}} a_i \) is a maximal filter which contains no element \( \Delta \).

**Corollary 2.** Let \( F \) be a filter of \( M \) such that \( F \cap \Delta = \{\alpha\} \). Then, \( F \) is a maximal filter of \( M \) verifying \( F = \{\alpha\} \).

**Corollary 3.** Let \( M \) be a coherent multilattice and let \( M \) be a maximal filter of \( M \). Then, there exists \( \alpha \in \Delta \) such that \( M \subseteq \{\alpha\} \).

**Example 1.** Let us consider the multilattice \( M_1 \) which is schematized in Figure 1:

\[
M_1 = \{a_i, i \in \mathbb{N}\} \cup \{b_j, j \in \mathbb{N}\} \cup \{c_k, k = 1, 2, 3, 4\} \cup \{\bot, \top\}.
\]

We can easily verify that

\( (i) \) \( F = \top \alpha, \alpha \in \{\top, a_i, b_j, \bot\} \) is a principal filter of \( M_1 \) generated by \{a\}. 

\( (ii) \) \( F = \cup_{i \in \mathbb{N}} \beta_k, \{\beta_k\} = \{a_i\} \) or \( \{\beta_k\} = \{b_j\} \) is a filter of \( M_1 \) which is not principal. It is generated by any unbounded subset of \( \{\beta_k\} \).

We notice that \( M_1 = \{c_i\} \neq \top c_i \), which means that in multilattices, we generally have \( \langle x \rangle \neq \top x \).

**Lemma 2.** Let \( M \) be a coherent multilattice and let \( M \) be a proper filter of \( M \). Then, \( M \) is a maximal filter if and only if for all \( x \in M \), there exists \( m \in M \) such that \( \langle x \cap m \rangle = M \).

**Proof.** \( (\Rightarrow) \) Suppose that \( M \) is maximal. Since \( M \) is coherent, there exists \( y \in M \) such that \( M = \langle y \rangle \). It follows that \( \langle x \rangle \cap \langle y \rangle = M \) for all \( x \in M \). From Lemma 1, we have \( \langle x \cap y \rangle = \langle x \rangle \cap \langle y \rangle = M \). \( (\Leftarrow) \) Suppose that for all \( x \in M \), \( \langle x \rangle \cap \langle y \rangle = M \). Then, \( m \in M \). Let \( F \) be a proper filter of \( M \) such that \( M \subseteq F \subseteq M \). Suppose that \( F \cap M \neq \emptyset \) and let \( x \in F \cap M \). Then, there exists \( m \in M \) such that \( \langle x \rangle \cap \langle m \rangle = M \). However, \( x, m \in F \) implies \( \langle x \rangle \cap \langle m \rangle \subseteq F \), so \( M = F \). This contradicts the fact that \( F \) is a proper filter of \( M \). Hence, \( F = M \) and it follows that \( F = M \).

**Proposition 3.** If \( \mathcal{F}(M) \) is distributive, then every maximal filter of \( M \) is prime.
In [10], Coquand et al. defined Jacobson’s radical of a proper ideal I of a distributive lattice L as the intersection of maximal ideals of L containing I. The duality filter/ideal allows us to use this definition for filters. After all, a filter of a multilattice M is nothing else than an ideal of the dual of M.

**Definition 6.** For every proper filter F of M, let \( \mathcal{F}(F) = \{ \mathcal{M} | \mathcal{M} \text{ is a maximal filter of } M \text{ containing } F \} \). Jacobson’s radical of F is the filter of M denoted \( J_M(F) \) and defined as follows: \( J_M(F) = \bigcap_{\mathcal{M} \in \mathcal{F}(F)} \mathcal{M} \). It is the intersection of all maximal filters of M containing F.

Notice that \( \mathcal{F}(F) \) is not empty according to Theorem 4. It is therefore a filter since any intersection of filters is a filter.

**Theorem 5.** Let F be a proper filter of M. Then,

\[
J_M(F) = \{ x \in M | \forall y \in M, (\langle x \rangle \lor \langle y \rangle) = M \Rightarrow \exists z \in F, (\langle z \rangle \lor \langle y \rangle) = M \}.
\]

**Proof.** For simplicity, let us note \( \mathcal{M} \) be a maximal filter of M containing F and let \( F, F' \) be two filters of M such that \( F \land F' \subseteq \mathcal{M} \) and neither \( F \cap \mathcal{M} \) nor \( F' \cup \mathcal{M} \). Then, \( F \lor \mathcal{M} = F' \lor \mathcal{M} = M \) due to the maximality of \( \mathcal{M} \). It follows that \((F \lor \mathcal{M}) \lor (F' \lor \mathcal{M}) = M\). That is \((F \land F') \lor \mathcal{M} = M\), which implies \( \mathcal{M} = M \), a contradiction. Therefore, \( \mathcal{M} \) is a prime filter of M. \( \square \)

The previous result is no longer true if \( \mathcal{F}(M) \) is not distributive. For instance, for the multilattice of Figure 2, \( \mathcal{F}(M_2) \) is not distributive since it contains a copy of the pentagon. We have \((\{a_4\} \land \{b_2\}) \subseteq \{c_4\}\) but neither \( \{a_4\} \subseteq \{c_4\} \) nor \( \{b_2\} \subseteq \{c_4\} \).

**Theorem 4.** Every proper filter of M is contained in a maximal filter.

**Proof.** Let \( \mathcal{H} = \{ F \subseteq J \subseteq M | J \text{ is a proper filter of } M \text{ containing } F \} \). Clearly, \( F \in \mathcal{H} \) so \( \mathcal{H} \) is not empty. Let \( \mathcal{C} \) be a chain in \( \mathcal{H} \) and let \( K = \bigcup_{J \in \mathcal{C}} J \). It is obvious that \( F \subseteq K \). We claim that K is a proper filter of M. Let \( a, x, y \in K \), if \( x, y \in K \), then \( x \in J_1 \) and \( y \in J_2 \) for some \( J_1, J_2 \in \mathcal{C} \), since \( \mathcal{C} \) is a chain, either \( J_1 \subseteq J_2 \) or \( J_2 \subseteq J_1 \), if, say \( J_1 \subseteq J_2 \), then \( x, y \in J_2 \) and since \( J_2 \) is a filter, \( x \lor y \subseteq J_2 \subseteq K \); if \( x \in K \), then \( x \in J \) for some \( J \in \mathcal{C} \), thus \( a \lor x \subseteq J \subseteq K \); if \( \forall j \in J \), then \( (a \lor x) \land K \neq \emptyset \), then \( x \lor y \subseteq J \) for some \( J \in C \) and then \( x \lor y \subseteq K \). Therefore, K is a proper filter of M containing F. By Zorn’s Lemma, \( \mathcal{H} \) has a maximal element, which is clearly a maximal filter of M. \( \square \)

**Corollary 4.** Let F be a proper filter of M. If \( y \in M \) such that \( \langle x \rangle \lor \langle y \rangle \neq M \) for all \( x \in F \). Then, there exists a maximal filter of M containing F and y.

The intersection of all maximal filters of M containing a.

In particular, if \( F = \{ \top \} \), then,

\[
J_M(\top) = \{ x \in M | \forall y \in M, (\langle x \rangle \lor \langle y \rangle) = M \Rightarrow \langle y \rangle = M \}.
\]

is the intersection of all maximal filters of M.

**Corollary 6.** Let \( F_1, F_2 \) be two filters of M. Then, the following condition holds:
Jacobson’s radicals of these filters are listed below:

\[ \text{Figure 2: The multilattice } M_2 \text{ and its filter lattice } \mathcal{F}(M_2). \]

\[ F_1 \subseteq F_2 \Rightarrow F_1 \subseteq J_M(F_1) \subseteq J_M(F_2). \quad (10) \]

**Remark 2.** Keeping the same notations as in Proposition 2, we can easily verify that \( J_M(\top) \subseteq \cap_{a \in \Lambda} \top \alpha \) whenever \( M \) is a coherent multilattice.

**Remark 3.** If \( M \) has a bottom \( \bot \), then from Lemma 1, the set \( L \) we use in Theorem 5 can alternatively be defined as follows:

\[ L = \{ x \in M | \forall y \in M, (\bot \in \langle x \sqcap y \rangle \Rightarrow \exists z \in F, \bot \in \langle z \sqcap y \rangle) \}. \quad (11) \]

**Example 2.** The multilattice \( M_2 \) whose Hasse diagram is given by Figure 2 has 13 filters of which 3 are maximal. Jacobson’s radicals of these filters are listed below:

(i) For \( F \in \{ \top, \uparrow b_3, \uparrow b_4 \} \), \( J_M(F) = \uparrow a_4 \sqcap \uparrow b_2 \sqcap c_4 = \uparrow b_4 \);

(ii) For \( F \in \{ \uparrow a_{13}, \uparrow a_{11}, \uparrow a_7, \uparrow a_4 \} \), \( J_M(F) = \uparrow a_4 \);

(iii) For \( F \in \{ \uparrow b_1, \uparrow b_2 \} \), \( J_M(F) = \uparrow b_2 \);

(iv) For \( F \in \{ \uparrow c_{12}, \uparrow c_7, \uparrow c_4 \} \), \( J_M(F) = \uparrow c_4 \).

One of the fundamental properties of the Jacobson’s radical is the following:

**Proposition 4.** If \( M \) is coherent and \( \mathcal{F}(M) \) is distributive, then,

\[ J_M(F_1 \cap F_2) = J_M(F_1) \cap J_M(F_2). \quad (12) \]

**Proof.** Let \( x \in J_M(F_1) \cap J_M(F_2) \), that is \( x \in J_M(F_1) \) and \( x \in J_M(F_2) \) and let \( y \in M \) such that \( \langle x \rangle \vee \langle y \rangle = M \). Then, there exists \( z_1 \in F_1 \) and \( z_2 \in F_2 \) such that \( \langle z_1 \rangle \vee \langle y \rangle = M = \langle z_2 \rangle \vee \langle y \rangle \). Since \( M \) is coherent, there exists \( z \in M \) such that \( \langle z \rangle = \langle z_1 \rangle \land \langle z_2 \rangle \subseteq F_1 \cap F_2 \). Therefore, \( \langle z \rangle \lor \langle y \rangle = (\langle z_1 \rangle \land \langle z_2 \rangle) \lor \langle y \rangle = (\langle z_1 \rangle \lor \langle y \rangle) \land (\langle z_2 \rangle \lor \langle y \rangle) = M \) since \( \mathcal{F}(M) \) is distributive. This implies \( x \in J_M(F_1) \cap J_M(F_2) \), and we obtain the inclusion \( J_M(F_1) \cap J_M(F_2) \subseteq J_M(F_1 \cap F_2) \). The reverse inclusion holds from Corollary 6 since \( F_1 \cap F_2 \subseteq F_1 \) and \( F_1 \cap F_2 \subseteq F_2 \).

The previous results show that when \( \mathcal{F}(M) \) is a distributive lattice, the binary relation \( \triangleleft \) defined on \( M \) by \( a \triangleleft b \) if and only if \( J_M(b) \subseteq J_M(a) \) is a preorder. We can look for the properties of this preorder in order to define on the multilattice a quotient which is neither a quotient by a filter nor a quotient by an ideal. Such an investigation could lead to the Heitmann dimension of multilattice.

**4. Conclusion and Perspectives**

Throughout this paper, we have studied the properties of maximal filters in multilattices. In addition to the classical properties related among others to the distributivity, we have introduced Jacobson’s radical of a filter. This gives tracks to define a particular quotient of multilattice and opens to the study of the Heitmann dimension of multilattice which we leave in perspective. With the ambition to study the representation of multilattices, especially Priestley’s representation, the study of the properties of maximal filters has been an important part of our project.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.
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