First Steps in Algorithmic Real Fewnomial Theory

Frederic Bihan∗    J. Maurice Rojas†    Casey E. Stella‡

February 1, 2008

J. Maurice Rojas dedicates this paper to the memory of his dear friend, Richard Adolph Snavely, 1955–2005.

Abstract

Fewnomial theory began with explicit bounds — solely in terms of the number of variables and monomial terms — on the number of real roots of systems of polynomial equations. Here we take the next logical step of investigating the corresponding existence problem: Let FEASR denote the problem of deciding whether a given system of multivariate polynomial equations with integer coefficients has a real root or not. We describe a phase-transition for when m is large enough to make FEASR be NP-hard, when restricted to inputs consisting of a single n-variate polynomial with exactly m monomial terms: polynomial-time for m ≤ n + 2 (for any fixed n) and NP-hardness for m ≥ n + nε (for n varying and any fixed ε > 0). Because of important connections between FEASR and A-discriminants, we then study some new families of A-discriminants whose signs can be decided within polynomial-time. (A-discriminants contain all known resultants as special cases, and the latter objects are central in algorithmic algebraic geometry.) Baker’s Theorem from diophantine approximation arises as a key tool. Along the way, we also derive new quantitative bounds on the real zero sets of n-variate (n + 2)-nomials.

1 Introduction and Main Results

Let FEASR — a.k.a. the real feasibility problem — denote the problem of deciding whether a given system of polynomial equations with integer coefficients has a real root or not. While FEASR is arguably the most fundamental problem of real algebraic geometry, our current knowledge of its computational complexity is surprisingly coarse, especially for sparse polynomials. This is a pity, for in addition to numerous practical applications (see, e.g., [BGV03]), FEASR is also an important motivation behind effectivity estimates for the Real Nullstellensatz (e.g., [Ste74, Sch00]), the quantitative study of sums of squares [Ble04], and their connection to semi-definite programming [Par03]. Furthermore, efficient algorithms for FEASR are crucial for the tractability of harder problems such as quantifier elimination, and computing the closure and frontier of more general types of varieties such as sub-Pfaffian sets [GV04].
Before stating our main results, we will need to clarify some geometric notions concerning sparse polynomials.

**Definition 1** Let \( f(x) := \sum_{j=1}^{m} c_j x^{a_j} \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \), where \( x^{a_j} := x_1^{a_{1j}} \cdots x_n^{a_{nj}} \), \( c_j \neq 0 \) for all \( j \), and the \( a_j \) are distinct. We call such an \( f \) an \( n \)-variate \( m \)-nomial and we call \( \{a_1, \ldots, a_m\} \) the support of \( f \). Also, for any collection \( \mathcal{F} \) of polynomial systems with integer coefficients, let \( \text{FEAS}_{\mathbb{R}}(\mathcal{F}) \) denote the natural restriction of \( \text{FEAS}_{\mathbb{R}} \) to inputs in \( \mathcal{F} \). \( \diamond \)

Note that \( n \)-variate quadratic polynomials are a special case of \( n \)-variate \( m \)-nomials with \( m = \mathcal{O}(n^2) \).

**Definition 2** Let us use \( \#S \) for the cardinality of a set \( S \) and say that a subset \( A = \{a_1, \ldots, a_m\} \subset \mathbb{Z}^n \) with \( \#A = m \) is affinely independent iff the implication \( \{\sum_{j=1}^{m} \gamma_j a_j = 0 \text{ and } \sum_{j=1}^{m} \gamma_j = 0\} \Rightarrow \gamma_1 = \cdots = \gamma_m = 0 \) holds for all \( (\gamma_1, \ldots, \gamma_m) \in \mathbb{R}^m \). Also let \( \dim A \) denote the dimension of the subspace of \( \mathbb{R}^n \) generated by the set of all differences of vectors in \( A \). \( \diamond \)

Note in particular that \( \#A \geq 1 + \dim A \), with equality iff \( A \) is affinely independent. It is also easily checked that a random set of \( m \) points in \( \mathbb{R}^n \) (chosen, say, independently from any continuous probability distribution) will have dimension \( \min\{n, m-1\} \) with probability 1.

Clearly, any \( n \)-variate Laurent polynomial \( f \) always satisfies \( \dim \text{Supp}(f) \leq n \). Let \( \mathbb{R}^* := \mathbb{R} \setminus \{0\} \). It is not much harder to see that one can always find (even in an algorithmically efficient sense) a \( d \)-variate Laurent polynomial \( g \), with the same number of terms as \( f \) and \( d = \dim \text{Supp}(f) \), such that \( f \) vanishes in \( (\mathbb{R}^*)^n \) iff \( g \) vanishes in \( (\mathbb{R}^*)^d \) (see Corollary 1 of Section 2 below). In this sense, “almost all” \( n \)-variate \( m \)-nomials satisfy \( \dim \text{Supp}(f) = \min\{n,m-1\} \), and those that don’t are essentially just \( d \)-variate \( m \)-nomials (with \( d < n \)) in disguise.

Recall the containments of complexity classes \( \text{NC}_1 \subseteq \text{NC} \subseteq \text{P} \subseteq \text{RP} \subseteq \text{BPP} \cup \text{NP} \subseteq \text{PSPACE} \) (these complexity classes are reviewed briefly in Section 2 below). Roughly speaking, our first main result says that real feasibility for “honest” \( n \)-variate \( (n+k) \)-nomials is easy for \( k \leq 2 \), but \( \text{NP} \)-hardness kicks in quickly already for \( k \) a very slowly growing function of \( n \).

**Theorem 1** Let

\[
\begin{align*}
\mathcal{A} := & \{f \in \mathbb{Z}[x_1, \ldots, x_n] \mid \#\text{Supp}(f) = 1 + \dim \text{Supp}(f) \text{ and } n \in \mathbb{N}\}, \\
\mathcal{B}_n := & \{f \in \mathbb{Z}[x_1, \ldots, x_n] \mid \#\text{Supp}(f) = 2 + \dim \text{Supp}(f)\}, \\
\mathcal{C}_\varepsilon := & \left\{f \in \mathbb{Z}[x_1, \ldots, x_n] \mid \#\text{Supp}(f) \leq n + n^\varepsilon, \dim \text{Supp}(f) = n, n \in \mathbb{N}\right\}, \text{ and } f \text{ is a sum of squares of polynomials.} \\
\mathcal{S} := & \left\{(f_1, \ldots, f_k) \mid f_i \in \mathbb{Z}[x_1, \ldots, x_n] \text{ and } f_i \text{ is a linear trinomial or a } \text{binomial of degree } \leq 2 \text{ for all } i, \text{ and } k \geq n \geq 1.\right\}
\end{align*}
\]

Then, measuring the size of any polynomial \( f(x) = \sum_{j=1}^{m} c_j x^{a_j} \in \mathbb{Z}[x_1, \ldots, x_n] \) — denoted \( \text{size}(f) \) — as the total number of binary digits in the \( c_i \) and \( a_{i,j} \), we have:

1. \( \text{FEAS}_{\mathbb{R}}(\mathcal{A}) \in \text{NC}_1 \).
2. For any fixed \( n \in \mathbb{N} \), \( \text{FEAS}_{\mathbb{R}}(\mathcal{B}_n) \in \text{P} \).
3. \( \text{FEAS}_{\mathbb{R}}(\mathcal{S}) \) is \( \text{NP} \)-hard.
4. For any fixed \( \varepsilon > 0 \), \( \text{FEAS}_{\mathbb{R}}(\mathcal{C}_\varepsilon) \) is \( \text{NP} \)-hard.
Example 1 A very special case of Assertion (2) of Theorem 1 implies that one can decide — for any nonzero $c_1, \ldots, c_5 \in \mathbb{Z}$ and $D \in \mathbb{N}$ — whether
\[ c_1 + c_2 x^{999} + c_3 x^{73} z + c_4 y^D + c_5 z^D y^{3D} z^{9D} \]
has a root in $\mathbb{R}^3$, using a number of bit operations polynomial in $\log(D) + \log((|c_1|+1) \cdots (|c_5|+1))$.

The best previous results (e.g., via the critical points method, infinitesimals, and rational univariate reduction, as detailed in [BPR03]) would yield a bound polynomial in $D + \log((|c_1|+1) \cdots (|c_5|+1))$ instead. Assertion (2) also vastly generalizes an earlier analogous result for univariate trinomials [RY05].

The algorithm underlying Assertion (2) turns out to depend critically on the combinatorics of $\text{Supp}(f)$, particularly its triangulations. Furthermore, extending the polynomiality of $\text{FEAS}_\mathbb{R}(\mathcal{F})$ from $(n+1)$-nomials to $(n+2)$-nomials turns out to be surprisingly intricate, involving $A$-discriminants (cf. Section 1.1), Baker’s Theorem on Linear Forms in Logarithms (cf. Section 3), and Viro’s Theorem from toric geometry (see, e.g., [GKZ94, Thm. 5.6]). Theorem 1, along with some more technical strengthenings, is proved in Section 3.3 below.

Remark 1 There appears to have been no earlier explicit statement that $\text{FEAS}_\mathbb{R}(\mathcal{F}) \in \text{P}$ (or even $\text{FEAS}_\mathbb{R}(\mathcal{F}) \in \text{NP}$) for $\mathcal{F}$ some non-trivial family of $n$-variate $m$-nomials with $m=n+O(1)$. As for lower bounds, the best previous result for sparse polynomials appears to have been $\text{NP}$-hardness of $\text{FEAS}_\mathbb{R}(\mathcal{F})$ when $\mathcal{F}$ is the family of those $n$-variate $m$-nomials with $m=\Omega(n^3)$ (see, e.g., [RY03, discussion preceding Thm. 2]). Assertion (4) is therefore a considerable sharpening.\footnote{An earlier version of this paper proved $\text{NP}$-hardness of $\text{FEAS}_\mathbb{R}(\mathcal{F})$ when $\mathcal{F}$ is the family of $n$-variate $(6n+6)$-nomials, but an anonymous referee suggested an even easier proof for the stronger version of Assertion (4) we are now stating.}

Remark 2 Let $\mathcal{U}_m := \{ f \in \mathbb{Z}[x_1] | f \text{ has exactly } m \text{ monomial terms} \}$. While it has been known since the late 1980’s that $\text{FEAS}_\mathbb{R} \in \text{PSPACE}$ [Can88], it is already unknown whether $\text{FEAS}_\mathbb{R}(\mathcal{U}_1) \in \text{BPP} \cup \text{NP}$, or even whether $\text{FEAS}_\mathbb{R}(\mathbb{Z}[x_1, \ldots, x_n])$ is $\text{NP}$-hard for some particular value of $n$ [LM02, RY03]. (The latter reference nevertheless states certain analytic hypotheses under which it would follow that $\text{FEAS}_\mathbb{R}(\mathcal{U}_m) \in \text{P}$ for fixed $m$.) The role of sparsity in complexity bounds for univariate real feasibility is thus already far from trivial.\footnote{Quantitative results over $\mathbb{R}^n$ of course shape the kind of algorithms we can find over $\mathbb{R}^n$. In particular, Khovanski’s famous Theorem on Real Fewnomials implies an upper bound depending only on $n$ and $m$ — independent of the degree — for the number of connected components of the real zero set of any $n$-variate $m$-nomial [Kho91]. More recently, his bound has been improved from $2^{O(m^2)} n^{O(n)} m^{O(m)}$ in the smooth case [Kho91, Sec. 3.14, Cor. 5] to $2^{O(m^2)} 2^{O(n)} m^{O(m)}$ in complete generality [LRW03, Cor. 2] (see also [Per05] for further improvements).}

For $n$-variate $(n+2)$-nomials we can now make a dramatic improvement. Recall that a set $S \subseteq \mathbb{R}^n$ is convex iff for any $x, y \in S$, the line segment connecting $x$ and $y$ is also contained in $S$. Recall also that for any $A \subseteq \mathbb{R}^n$, the convex hull of $A$ — denoted $\text{Conv} A$ — is the smallest convex set containing $A$.\footnote{For $n$-variate $(n+2)$-nomials we can now make a dramatic improvement. Recall that a set $S \subseteq \mathbb{R}^n$ is convex iff for any $x, y \in S$, the line segment connecting $x$ and $y$ is also contained in $S$. Recall also that for any $A \subseteq \mathbb{R}^n$, the convex hull of $A$ — denoted $\text{Conv} A$ — is the smallest convex set containing $A$.}
Theorem 2 Let \( f \) be any \( n \)-variate \( m \)-nomial with \( m \leq 2 + \dim \text{Supp}(f) \), \( Z_+(f) \) its zero set in \( \mathbb{R}^n_+ \), and define \( N_{\text{comp}}(f) \) (resp. \( N_{\text{non}}(f) \)) to be the number of compact (resp. non-compact) connected components of \( Z_+(f) \). Then \( N_{\text{comp}}(f) \leq 1 \) (with examples attaining equality for each \( n \in \mathbb{N} \)), and \( N_{\text{comp}}(f) = 1 \implies \mathcal{Z}_+(f) \) is either a point, or isotopic to an \((n-1)\)-sphere. Also, \( N_{\text{non}}(f) \) is no more than 0, 2, 6, 9, or \( 2n+2 \), according as \( n \) is 1, 2, 3, 4, or \( \geq 5 \) (with examples attaining equality for each \( n \leq 2 \)). Finally, if \( \text{Supp}(f) \) is disjoint from the interior of its convex hull, then \( N_{\text{comp}}(f) = 0 \).

While the bound \( N_{\text{non}}(f) \leq 2 \) (for \( n = 2 \)) was found earlier by Daniel Perrucci, all the other bounds of Theorem 2 are new, and the special case \( n = 3 \) improves Perrucci’s earlier bound for 3-variate 5-nomials by a factor of at least 2 (see [Per05, Thms. 4 & 5]). Except for an upper bound of \( 2^{O(n^2)} \) for the smooth case [Kho91, Sec. 3.14, Cor. 5], there appear to be no other earlier explicit bounds in the spirit of Theorem 2. Theorem 2 has recently been generalized to systems of \( k \) polynomials in \( \mathcal{B}_n \), with identical supports and \( k \in \{1, \ldots, n\} \) [BRS06]. (See also [BBS05, Bih05] for the opposite extreme to Theorem 2: bounding the number of isolated real roots of \( n \) polynomials in \( \mathcal{B}_n \) with the same support.) We prove Theorem 2 in Section 3.2.

It thus appears that, unlike algebraic geometry over \( \mathbb{C} \), large degree is potentially less of a complexity bottleneck over \( \mathbb{R} \). Considering the ubiquity of sparse real polynomial systems in engineering, algorithmic speed-ups in broader generality via sparsity are thus of the utmost interest. Furthermore, in view of the complexity threshold of Theorem 1, randomized, approximation, and/or average-case speed-ups appear to be the next key steps if we are to have a sufficiently general and useful algorithmic fewnomial theory over \( \mathbb{R} \). A promising step in this direction can be found in work of Barvinok [Bar02], but more work still needs to be done before we can assert significant new randomized algorithms — even for \( \text{FEAS}_R(\mathcal{U}_4) \).

Since the largest \( m \) for which \( \text{FEAS}_R \) is doable in polynomial-time — for input a single \( n \)-variate \( m \)-nomial — appears to be \( m = n + 2 \) (as of early 2006), we propose the following conjecture to address the cases \( m \geq n + 3 \).

Univariate Threshold Conjecture. For any \( m \in \mathbb{N} \), let

\[
\mathcal{U}_m := \{ f \in \mathbb{Z}[x_1] \mid f \text{ has exactly } m \text{ monomial terms} \}.
\]

Then \( \text{FEAS}_R(\mathbb{Z}[x_1]) \) is \( \text{NP} \)-hard but, for any fixed \( m \), there is a natural probability measure on \( \mathcal{U}_m \) so that \( \text{FEAS}_R(\mathcal{U}_m) \) has polynomial-time complexity on average.

Note that \( \text{FEAS}_R(\mathcal{U}_3) \in \text{P} \), thanks to Assertion (2) of Theorem 1 since \( \mathcal{U}_3 = \mathcal{B}_1 \). The latter “positive” part of the conjecture is meant to be reminiscent of Smale’s 17th Problem, which concerns the complexity of approximating complex roots of polynomial systems [Sma00, RY05].

That feasibility over \( \mathbb{R} \) may be \( \text{NP} \)-hard already for univariate sparse polynomials is suggested by a recent parallel over a different complete field: \( \mathbb{Q}_p \). In particular, (a) there is now a \textit{Theorem on \( p \)-adic Fewnomials} (due to the middle author [Roj04]), with significantly sharper bounds than Khovanski’s Theorem on Real Fewnomials, and (b) it is now known that \( \text{FEAS}_{\mathbb{Q}_p}(\mathbb{Z}[x_1]) \) — the natural \( p \)-adic analogue of \( \text{FEAS}_R(\mathbb{Z}[x_1]) \) — is doable in randomized polynomial-time only if an unlikely containment of complexity classes occurs: \( \text{NP} \subseteq \text{BPP} \) [PR05].
Possible alternative evidence for the Univariate Threshold Conjecture can be given via the $A$-discriminant, which is defined in Section 1.1 below. To set the stage, first recall the following elementary example.

**Example 2** Note that $f(x_1) := a + bx_1 + cx_1^2$ has either 0 or 2 real roots according as the discriminant $\Delta = b^2 - 4ac$ is negative or positive. Observe then that the real zero set, $\tilde{W} \subset \mathbb{R}^3$, of $\Delta$ can be identified with the collection of all quadratic polynomials possessing a degenerate root, and that $\Delta$ also defines a curve $W$ in the real projective plane $\mathbb{P}^2_{\mathbb{R}}$. Furthermore, $W$ is equivalent (under a linear change of variables over $\mathbb{Q}$) to a circle, and thus there are exactly 2 discriminant chambers. These 2 chambers correspond exactly to those quadratic polynomials possessing either 0 or 2 real roots. Finally, note that the support of $f$ is $\{0, 1, 2\}$ and this set admits exactly 2 triangulations with vertices in $\{0, 1, 2\}$. They are $\frac{\circ}{\circ}$ and $\frac{\circ}{\circ}$. ◦

### 1.1 Complexity and Topology of Certain $A$-Discriminants

The connection between computational complexity (e.g., of deciding membership in semi-algebraic sets [DL79] or approximating the roots of univariate polynomials [Sma87]) and the topology of discriminant complements dates back to the late 1970’s. Here, we point to the possibility of a more refined connection between $\text{FEAS}_\mathbb{R}$ and discriminant complements. (See also [DRRS05] for further results in this direction.) In particular, our last example was a special case of a much more general invariant attached to spaces of sparse multivariate polynomials.

**Definition 3** Given any $A = \{a_1, \ldots, a_m\} \subset \mathbb{Z}^m$ of cardinality $m$, define the set

$$\nabla_A^0 := \left\{ (c_1, \ldots, c_m) \in \mathbb{C}^m \mid f(x) := \sum_{j=1}^m c_j x^{a_j} \text{ has a degenerate root} \right\}.$$  

The $A$-discriminant is then the unique (up to sign) irreducible polynomial $\Delta_A \in \mathbb{Z}[c_1, \ldots, c_m] \setminus \{0\}$ whose complex zero set contains $\nabla_A^0$. (If $\text{codim} \nabla_A > 1$ then we set $\Delta_A := 1$.) For convenience, we will usually write $\Delta_A(f)$ in place of $\Delta_A(c_1, \ldots, c_m)$. Finally, we let $\nabla_A$ — the $A$-discriminant variety — denote the zero set of $\Delta_A$ in $\mathbb{C}^m$. ◦

**Example 3** If we take $A := \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ d \\ 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ e \\ 1 \end{bmatrix} \right\}$ then $\Delta_A(a_0, \ldots, a_d, b_0, \ldots, b_e)$ is exactly the classical Sylvester resultant of the univariate polynomials $a_0 + \cdots + a_dx^d$ and $b_0 + \cdots + b_ex^e$. This is a special case of a more general construction which shows that any multivariate toric resultant can be obtained as a suitable $A$-discriminant [GKZ94]. The Cayley Trick, Prop. 1.7, pp. 274]. ◦

**Definition 4** Let $\mathcal{F}_A := \{ f \in \mathbb{Z}[x_1, \ldots, x_n] \mid \text{Supp}(f) \subseteq A \}$, and let $\text{ADISCVAN}$ (resp. $\text{ADISCSIGN}$) denote the problem of deciding whether $\Delta_A(f)$ vanishes (resp. determining the sign of $\Delta_A(f)$) for an input $f \in \mathcal{F}_A$. Finally, let $\text{ADISCVAN}(\mathcal{F})$ (resp. $\text{ADISCSIGN}(\mathcal{F})$) be the natural restriction of $\text{ADISCVAN}$ (resp. $\text{ADISCSIGN}$) to input polynomials in some family $\mathcal{F}$. ◦

---

3That is, a root $\zeta$ of $f$ with $\frac{\partial f}{\partial x_1}|_{x=\zeta} = \cdots = \frac{\partial f}{\partial x_n}|_{x=\zeta} = 0$.
An intriguing link between \textbf{FEAS}_\mathbb{R} and discriminants is the fact that those \( A \) with \( \text{FEAS}_\mathbb{R}(\mathcal{F}_A) \in \mathbb{P} \) currently appear to coincide with those \( A \) with \( \text{ADISCVAN}(\mathcal{F}_A) \in \mathbb{P} \). This should not be too surprising in view of the following fact: If \( f \in \mathcal{F}_A \) has smooth complex zero set (and a similarly mild condition holds for its zero set at infinity), then \( f \) lies in some connected component \( C \) of \( (\mathbb{R}^*)^{\#A} \setminus \nabla_A \) (under a natural identification of coefficients of \( f \) and coordinates of \( (\mathbb{R}^*)^{\#A} \), and any other \( g \in C \) has real zero set isotopic to that of \( f \) (see, e.g., \cite{GKZ94} Ch. 11, Sec. 5A, Prop. 5.2, pg. 382). Let us call any such \( C \) a discriminant chamber (for \( A \)).

So deciding \( \text{FEAS}_\mathbb{R}(\mathcal{F}_A) \) for a given \( f \) is nearly the same as deciding whether \( f \) lies in a particular union of discriminant chambers, and it is thus natural to suspect that the following three situations may be equivalent in some rigorous and useful sense: (a) \((\mathbb{R}^*)^{\#A} \setminus \nabla_A \) has “few” connected components, (b) \( \text{FEAS}_\mathbb{R}(\mathcal{F}_A) \) is “easy”, (c) \( \text{ADISCVAN}(\mathcal{F}_A) \) is “easy”.

While connections between (a) and (c) are known (see, e.g., \cite{BCSS98} Ch. 16), the best current theorems appear to be too weak to yield any complexity lower bounds of use for the Univariate Threshold Conjecture. As for stronger connections between (b) and (c), concrete examples arise, for instance, from the following two facts: (1) \( \text{FEAS}_\mathbb{R}(\mathcal{Q}) \in \mathbb{P} \), where \( \mathcal{Q} := \{ f \in \mathbb{Z}[x_1, \ldots, x_n] \mid f \) is homogeneous and quadratic \} \par
(Bar93) and (2) for \( A \) the support of a quadratic polynomial \( f \), \( \Delta_A(f) \) is computable in polynomial-time. (The latter fact follows easily from an exercise in Cramer’s Rule for linear equations and the Newton identities \cite{BCSS98} Ch. 15, Pgs. 292–296.)

A new connection we can assert between the “easiness” of \( \text{FEAS}_\mathbb{R}(\mathcal{F}_A) \) and \( \text{ADISCVAN}(\mathcal{F}_A) \) is the case of \( n \)-variate \((n + 2)\)-nomials, thanks to Assertion (2) of Theorem \[1\] above, and the first part of our final main result below.

**Theorem 3** Following the notation of Theorem \[1\] and our last two definitions:

1. \( \text{ADISCVAN} \left( \bigcup_{n=1}^{\infty} \mathcal{B}_n \right) \in \mathbb{P} \).

2. For any fixed \( n \), \( \text{ADISCSIGN}(\mathcal{B}_n) \in \mathbb{P} \).

Note that Theorem \[3\] improves considerably on what can be done through quantifier elimination (e.g., \cite{Can88,BPR03}), because for fixed \( n \) these older methods already have complexity exponential in our notion of input size. Theorem \[3\] — proved in Section \[3.1\] — turns out to be a central tool in the algorithms behind the complexity upper bounds of Theorem \[1\] and is the main reason that diophantine approximation enters our scenery.

In light of the connections between the complexity of \( \text{FEAS}_\mathbb{R}(\mathcal{F}_A) \) and \( \text{ADISCVAN}(\mathcal{F}_A) \), we conclude our introduction with another possible piece of evidence in favor of the “negative” portion of the Univariate Threshold Conjecture:

**Karpinski-Shparlinski Theorem** \cite{KS99} \( \text{ADISCVAN}(\mathbb{Z}[x_1]) \) is computationally hard in the following sense: If \( \text{ADISCVAN}(\mathbb{Z}[x_1]) \subseteq \mathcal{C} \) for some complexity class \( \mathcal{C} \), then \( \mathbb{NP} \subseteq \mathcal{C} \cup \mathcal{RP} \). In particular, \( \text{ADISCVAN}(\mathbb{Z}[x_1]) \subseteq \mathbb{BPP} \implies \mathbb{NP} \subseteq \mathbb{BPP} \). \[4\]

\[4\]Barvinok actually proved the stronger fact that \( \text{FEAS}_\mathbb{R}(\mathcal{Q}_k) \in \mathbb{P} \) for any fixed \( k \), where \( \mathcal{Q}_k \) is the family of polynomial systems of the form \((f_1, \ldots, f_k) \) with \( f_i \in \mathcal{Q} \) for all \( i \).

\[5\]The paper \cite{KS99} actually asserts the stronger fact that \( \text{ADISCVAN}(\mathbb{Z}[x_1]) \) is \( \mathbb{NP} \)-hard, but without a proof. One of the authors of \cite{KS99} (Igor Shparlinski) has confirmed this oversight, along with the fact that it was also observed independently by Erich Kaltofen \cite{Shp06}. 

\[6\]
The containment $\text{NP} \subseteq \text{BPP}$ is widely disbelieved, so it would appear possible that $\text{ADISCVAN}(\mathbb{Z}[x_1])$ is not doable in randomized polynomial-time.

Our main results are proved in Section 3, after the development of some necessary theory in Section 2 below. A useful elementary result on the real zero sets of $n$-variate $(n+1)$-nomials is then proved in Section 4.

2 Background and Ancillary Results

Let us first informally review some well-known complexity classes (see, e.g., [Pap95] for a complete and rigorous description).

**Remark 3** Throughout this paper, our algorithms will always have a notion of input size that is clear from the context, and our underlying computational model will always be the classical Turing model [Pap95]. Thus, appellations such as “polynomial-time” are to be understood as “having bit-complexity polynomial in the underlying input size”, and the underlying polynomial and/or $O$-constants depend only on the algorithm, not on the specific instance being solved. The same of course applies to “linear-time”, “exponential-time”, etc.

- **$\text{NC}_1$** The family of decision problems which can be done within time $O(\log \text{InputSize})$, using a number of processors linear in the input size.
- **$\text{NC}$** The family of decision problems which can be done within time poly-logarithmic in the input size, using a number of processors polynomial in the input size.
- **$\text{P}$** The family of decision problems which can be done within polynomial-time.
- **$\text{RP}$** The family of decision problems admitting randomized polynomial-time algorithms for which a ‘‘Yes’’ answer is always correct but a ‘‘No’’ answer is wrong with probability $\frac{1}{2}$.
- **$\text{BPP}$** The family of decision problems admitting randomized polynomial-time algorithms that terminate with an answer that is correct with probability at least $\frac{5}{6}$.
- **$\text{NP}$** The family of decision problems where a ‘‘Yes’’ answer can be certified within polynomial-time.
- **$\text{PSPACE}$** The family of decision problems solvable within polynomial-time, provided a number of processors exponential in the input size is allowed.

Recall also that even the containment $\text{P} \subseteq \text{PSPACE}$ is still an open problem (as of early 2006).

A very useful and simple change of variables is to replace variables by monomials in new variables. Please note that in what follows, we will sometimes use real exponents.

\[6\text{It is easily shown that we can replace } \frac{2}{3} \text{ by any constant strictly greater than } \frac{1}{2} \text{ and still obtain the same family of problems [Pap95].}\]
Definition 5 For any ring $R$, let $R^{m \times n}$ denote the set of $m \times n$ matrices with entries in $R$. For any $M = [m_{ij}] \in R^{n \times n}$ and $y = (y_1, \ldots, y_n)$, we define the formal expression $y^M := (y_1^m_{11}, \ldots, y_1^{m_{1n}}, \ldots, y_n^m_{11}, \ldots, y_n^{m_{nn}})$. We call the substitution $x := y^M$ a monomial change of variables. Also, for any $z := (z_1, \ldots, z_n)$, we let $xz := (x_1z_1, \ldots, x_nz_n)$. Finally, let $\mathrm{GL}_n(Z)$ denote the set of all matrices in $Z^{n \times n}$ with determinant $\pm 1$ (the set of unimodular matrices).

Proposition 1 (See, e.g., [LRW08, Prop. 2].) For any $U, V \in R^{n \times n}$, we have the formal identity $(xy)^UV = (x^U)V(y^V)$. Also, if $\det U \neq 0$, then the function $e_U(x) := x^U$ is an analytic automorphism of $R^{\mathbb{N}_+}$, and preserves smooth points and singular points of zero sets of analytic functions. Finally, $U \in \mathrm{GL}_n(Z)$ implies that $e_U^{-1}(R^{\mathbb{N}_+}) = R^{\mathbb{N}_+}$ and that $e_U$ maps distinct open orthants of $R^n$ to distinct open orthants of $R^n$.

Via a simple application of Hermite factorization (see Definition 5 and Lemma 1 below), we can derive the following corollary which reveals why the dimension related hypotheses of Theorem 1 are mild and necessary.

Corollary 1 Given any $n$-variate $m$-nomial $f$ with $d = \dim \text{Supp}(f) < n$, we can find (within $P$) a $U \in \mathrm{GL}_n(Z)$ such that $g(y) := f(y^U)$ is a $d$-variate $m$-nomial with $\dim \text{Supp}(f) = d$, and $g$ vanishes in $(\mathbb{R}^*)^d$ iff $f$ vanishes in $(\mathbb{R}^*)^n$. Moreover, there is an absolute constant $c$ such that $\text{size}(g) = O(\text{size}(f)^c)$.

Definition 6 [Ilis89, Sto98] Given any $M \in Z^{m \times n}$, the Hermite factorization of $M$ is an identity of the form $UM = H$ where $U \in \mathrm{GL}_m(Z)$ and $H = [h_{ij}] \in Z^{n \times n}$ is nonnegative and upper triangular, with all off-diagonal entries smaller than the positive diagonal entry in the same column. Finally, the Smith factorization of $M$ is an identity of the form $UMV = S$ where $U \in \mathrm{GL}_m(Z)$, $V \in \mathrm{GL}_n(Z)$, and $S = [s_{ij}] \in Z^{m \times n}$ is diagonal, with $|s_{ij}| |s_{i+1,j+1}|$ for all $i$.

Lemma 1 [Ilis89, Sto98] For any $M = [m_{ij}] \in Z^{n \times n}$, the Hermite and Smith factorizations of $M$ exist uniquely, and can be computed within $O(n^4 \log^3(n \max_{i,j} |m_{ij}|))$ bit operations. Furthermore, in the notation of Definition 6, the entries of $U$, $V$, $S$, and $H$ all have bit size $O(n^3 \log^2(2n + \max_{i,j} |m_{ij}|))$.

To prove Theorems 1 and 2, we will first need some tricks for efficiently deciding when polynomials in $A$ have roots in $(\mathbb{R}^*)^n$ or $\mathbb{R}^n$.

Lemma 2 Suppose $f$ is an $n$-variate $(n + 1)$-nomial with affinely independent support $A = \{a_1, \ldots, a_{n+1}\} \subset \mathbb{R}^n$, with $a_j = (a_{1,j}, \ldots, a_{n,j})$ for all $j$. Then

1. $f$ has a root in $\mathbb{R}^n$ iff not all the coefficients of $f$ have the same sign.

2. If $A \subset \mathbb{Z}^n$ and all the coefficients of $f$ have the same sign, then $f$ has a root in $(\mathbb{R}^*)^n$ iff there are indices $i, j, j'$ with $a_{i,j} - a_{i,j'}$ odd iff there are indices $i, j$ with $a_{i,j} - a_{i,1}$ odd.

While this last lemma is ultimately elementary, we were unable to find any similar explicit statement in the literature. So we supply a proof in Section 1. Another tool we will need is a description of certain real zero sets “at infinity”.

8
Definition 7 Given any compact $S \subset \mathbb{R}^n$ and $w \in \mathbb{R}^n$, the face of $S$ with inner normal $w$—denoted $S^w$—is the set of all $x \in S$ that minimize the inner product $x \cdot w$. In particular, a facet of $S$ is a face $S^w$ for some $w$ with $\dim S^w = \dim S - 1$. ◻

Definition 8 The Newton polytope of $f$, $\text{Newt}(f)$, is the convex hull of the support of $f$. Also, for any $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$, the initial term function of $f(x) = \sum_{a \in A} c_a x^a$ with respect to the weight $w$ is $\text{In}_w(f) := \sum_{a \in A} c_a x^a$. ◻

Theorem 4 [LRW03, Theorem 3 and Lemmata 14 and 15] Let $f$ be any $n$-variate $m$-nomial $f$ with $n$-dimensional Newton polytope and define $\mathcal{N}(f)$ to be the number of connected components of $f$. Then

1. If $Z_+(\text{In}_w(f))$ is smooth for all $w \in \mathbb{R}^n \setminus \{0\}$, then $\mathcal{N}_{\text{non}}(f) \leq \sum_{w \text{ a unit inner facet of } \text{Newt}(f)} \mathcal{N}(\text{In}_w(f))$.

2. $[\mathcal{N}(\text{In}_w(f)) > 0 \text{ and } Z_+(\text{In}_w(f)) \text{ smooth}]$ for some $w \in \mathbb{R}^n \setminus \{0\} \implies Z_+(f)$ has a non-compact connected component.

3. $[f \in \mathbb{Z}[x_1, \ldots, x_n] \text{ and } w = (0, \ldots, 0, 1)] \implies$ for $\varepsilon > 0$ sufficiently small, $Z_+(f)$ has no more than $\mathcal{N}(\text{In}_w(f) - \varepsilon)$ non-compact connected components with limit points in $\mathbb{R}^{n-1}_+ \times \{0\}$.

4. $\mathcal{N}_{\text{non}}(f) \geq 1 \implies \mathcal{N}(\text{In}_w(f)) \geq 1$ for some $w \in \mathbb{R}^n \setminus \{0\}$. ◻

The initial term functions above are also known as initial term polynomials or face polynomials when the exponents are integral.

3 The Proofs of Our Main Results: Theorems 3, 2, and 1

We will use $[k]$ in place of $\{1, \ldots, k\}$ throughout. Let us start by highlighting two of the most important theoretical tools behind our proofs: the binomial formula for circuit discriminants and Baker’s famous result on approximating linear forms in logarithms.

First recall that $A \subset \mathbb{R}^n$ is a circuit iff $A$ is affinely dependent, but every proper subset of $A$ is affinely independent. Also, we say that $A$ is a degenerate circuit iff $A$ contains a point $a$ and a proper subset $B$ such that $a \in B$, $A \setminus a$ is affinely independent, and $B$ is a circuit. For instance, $\circlearrowleft$ and $\circlearrowright$ are respectively a circuit and a degenerate circuit.

We can now summarize what we need to know about $A$-discriminants when $A$ is a circuit:

Lemma 3 Suppose $A = \{a_1, \ldots, a_{n+2}\} \subset \mathbb{Z}^n$ is a circuit of cardinality $n + 2$. Then, defining $\text{size}(A)$ to be the sum of the sizes of the coordinates of $A$, and letting $\text{Z}_\mathbb{R}(f)$ (resp. $\text{Z}_\mathbb{C}(f)$) denote the zero set of $f$ in $(\mathbb{R}^*)^n$ (resp. $(\mathbb{C}^*)^n$), we have:

\[\text{size}(A) = \sum_{i=1}^{n+2} a_i \quad \text{and} \quad \text{Z}_\mathbb{R}(f) = \text{Z}_\mathbb{C}(f) = \{a_1, \ldots, a_{n+2}\} .\]

\[\text{This terminology comes from matroid theory and has nothing to do with circuits from complexity theory.}\]
1. There is a unique (up to sign) vector \( m = (m_1, \ldots, m_{n+2}) \in \mathbb{Z}^{n+2} \) such that (a) the coordinates of \( m \) are all nonzero and have gcd 1, (b) \( \sum_{j=1}^{n+2} m_j a_j = 0 \), (c) \( \sum_{j=1}^{n+2} m_j = 0 \), and (d) \( |m_j| = \frac{\mathrm{Vol} \left( \text{Conv}(A) \setminus \{a_j\} \right)}{\gcd \left( \frac{\mathrm{Vol} \left( \text{Conv}(A) \setminus \{a_j\} \right)}{i \in [n+2]} \right)} \) for all \( j \).

2. The vector \( m \) from Assertion (1) can be found in time polynomial in size(\( A \)).

3. Permuting (if necessary) the \( a_j \) so that \( m_j > 0 \) for all \( j \in [k] \) and \( m_j < 0 \) for all \( j \in \{k + 1, \ldots, n + 2\} \), we have
   \[
   \Delta_A(c_1, \ldots, c_{n+2}) = \left( \prod_{i=1}^{k} m_i^{m_i} \right) \left( \prod_{i=k+1}^{n+2} c_i^{-m_i} \right) - \left( \prod_{i=k+1}^{n+2} m_i^{-m_i} \right) \left( \prod_{i=1}^{k} c_i^{m_i} \right).
   \]

4. \( \Delta_A(c_1, \ldots, c_{n+2}) = 0 \) for some \( c' = (c_1', \ldots, c_{n+2}') \in (\mathbb{C}^*)^{n+2} \iff Z_A^c(f) \) contains a degenerate point \( \zeta \) (and any such \( \zeta \) satisfies \( \zeta_a^i = \frac{\partial \Delta_A}{\partial c_i} \bigg|_{c=c'} \) for all \( i \)). In particular, \( Z_A^c(f) \) has at most one degenerate point per open orthant of \( (\mathbb{R}^c)^n \).

**Proof:** Assertions (1) and (3) of Lemma 3 follow immediately from [GKZ94, Prop. 1.8, Pg. 274].

Assertion (2) follows easily from Lemma 3 upon observing that \( m \) is merely the generator of the integral kernel of a suitable integral matrix.

To prove Assertion (4), observe that by Assertion (3) and Proposition 4 \( \Delta_A(c_1, \ldots, c_{n+2}) = 0 \) for some \( c := (c_1, \ldots, c_{n+2}) \in (\mathbb{C}^*)^{n+2} \Rightarrow c \) is a smooth point on a hypersurface defined by a real binomial equation. Via [GKZ94, Thm. 1.5, Ch. 1, pp. 16], we then obtain that \( Z_A^c \left( \sum_{j=1}^{n+2} c_j x^{a_j} \right) \) has a degeneracy \( \zeta \), and any such \( \zeta \) must satisfy the binomial system stated above. (In particular, the coordinates are rational in \( c_1, \ldots, c_{n+2} \) if \( A \) generates \( \mathbb{Z}^n \) as a lattice, thanks to Lemma 1.) Moreover, by Proposition 1 and Lemma 3 we easily derive that a real binomial system can have no more than 1 solution per open orthant of \( (\mathbb{R}^c)^n \).

Conversely, \( Z_A^c \left( \sum_{j=1}^{n+2} c_j x^{a_j} \right) \) has a degenerate point \( \Rightarrow \Delta_A(c_1, \ldots, c_{n+2}) = 0 \), by the definition of \( \Delta_A \).

**Baker’s Theorem (Special Case) [Bak77]** Suppose \( c_0, \ldots, c_N \in \mathbb{Z} \), \( \alpha_1, \ldots, \alpha_N \in \mathbb{N} \), \( C := \max\{4, c_0, \ldots, c_N\} \), and \( A := \log^N \max\{4, \alpha_1, \ldots, \alpha_N\} \). Also let \( \Lambda := c_0 + c_1 \log(\alpha_1) + \cdots + c_N \log(\alpha_N) \). Then \( \Lambda \neq 0 \iff |\Lambda| > (CA)^{-(16N)^{200N} A \log A} \).

### 3.1 The Proof of Theorem 3

Theorem 3 will follow easily from the two algorithms we state immediately below, once we prove their correctness and verify their polynomial-time complexity. However, we will first need to recall the concept of a gcd-free basis. In essence, a gcd-free basis is nearly as powerful as factorization into primes, but is far easier to compute.

**Definition 9** [BS92, Sec. 8.4] For any subset \( \{\alpha_1, \ldots, \alpha_N\} \subset \mathbb{N} \), a **gcd-free basis** is a pair of sets \( \{\gamma_i\}_{i=1}^n \), \( \{e_{ij}\}_{(i,j) \in [N] \times [N]} \) such that (1) \( \gcd(\gamma_i, \gamma_j) = 1 \) for all \( i \neq j \), and (2) \( \alpha_i = \prod_{j=1}^n \gamma_i^{e_{ij}} \) for all \( i \).
Theorem 5  \cite{BS96}  Cor. 4.8.2 and Thm. 4.8.7 of Sec. 4.8| Following the notation of Definition \ref{def:pair} there is a gcd-free basis for \{\(\alpha_1, \ldots, \alpha_N\}\}, with \(\eta\), size(\(\gamma_i\)), and size(\(e_{ij}\)) each polynomial in \(\sum_{\ell=1}^N\) size(\(\alpha_\ell\)), for all \(i\) and \(j\). Moreover, one can always find such a gcd-free basis using just \(O\left( \left( \sum_{\ell=1}^N\text{size}(\alpha_\ell) \right)^2 \right)\) bit operations. \hfill \blacksquare

Algorithm BinomialVanish
Input:  Positive integers \(\alpha_1, \beta_1, u_1, v_1, \ldots, \alpha_N, \beta_N, u_N, v_N\).
Output:  A true declaration as to whether \(\alpha_1^{u_1} \cdots \alpha_N^{u_N} = \beta_1^{v_1} \cdots \beta_N^{v_N}\).

Description:
1. Let \(C := \log^2 N \max\{4, \alpha_1, \beta_1, \ldots, \alpha_N, \beta_N\}\), \(M := \max\{4, u_1, v_1, \ldots, u_N, v_N\}\), and \(E := \frac{1}{3}(CM)^{-32N}\)\(900N^2 C \log C\).
2. For all \(i \in [N]\), let \(A_i\) (resp. \(B_i\)) be an approximation of \(\log \alpha_i\) (resp. \(\log \beta_i\)) within \(\frac{E}{2NM}\) (using, say, Arithmetic-Geometric Mean Iteration \cite{Ber03}).
3. Output the sign of \(\left( \sum_{i=1}^N u_i A_i \right) - \left( \sum_{i=1}^N v_i B_i \right)\) and stop.

Note that while we can certainly compute \(A := \alpha_1^{u_1} \cdots \alpha_N^{u_N}\) using a number of arithmetic operations polynomial in \(s := \text{size}(u_1) + \cdots + \text{size}(u_N)\), the bit size of \(A\) is already exponential in \(s\); hence the need for our last two algorithms.

Lemma 4  Algorithms BinomialVanish and BinomialSign are both correct. Moreover, Algorithm BinomialVanish runs in time polynomial in
\[I := \sum_{i=1}^N (\text{size}(\alpha_i) + \text{size}(\beta_i) + \text{size}(u_i) + \text{size}(v_i))\]
and, if \(N\) is fixed, Algorithm BinomialSign runs in time polynomial in \(I\) as well.

Proof of Lemma 4  That Algorithm BinomialVanish is correct and runs in time polynomial in \(I\) follows directly from Theorem 5. That Algorithm BinomialSign is correct and runs in time polynomial in \(I\) (for fixed \(N\)) follows easily from Baker’s Theorem: First, taking logarithms, observe that the sign of

\(^8\)Other approximation techniques can be used as well: It is sufficient to use any algorithm that can find the \(b\) leading bits of \(\log N\) within a number of bit operations polynomial in \(b + \log N\).
\[ \alpha_1^{u_1} \cdots \alpha_N^{u_N} - \beta_1^{m_1} \cdots \beta_N^{m_N} \] is the same as the sign of \[ S := \left( \sum_{i=1}^{N} u_i \log \alpha_i \right) - \left( \sum_{i=1}^{N} v_i \log \beta_i \right). \]

Clearly then, \[ |S - \left( \left( \sum_{i=1}^{N} u_i A_i \right) - \left( \sum_{i=1}^{N} v_i B_i \right) \right)| < E, \] so Baker’s Theorem tells us that Step (3) of Algorithm BinomialSign indeed computes the sign of \( S \). So we have correctness.

To see that Algorithm BinomialSign runs in time polynomial in \( I \) for fixed \( N \), first note that the \( A_i \) and \( B_i \) each require \( O(\log(M) + \log(N) - \log E) = O\left((32N)^{400N}C \log(C) \log(CM)\right) \) bits of accuracy.\(^9\) So, via our chosen method for approximating logarithms \cite{Ber03}, we see that the complexity of our algorithm is polynomial in \( O\left((N(32N)^{400N}C \log(C) \log(CM) + \sum_{i=1}^{N} \log(a_i \beta_i)\right) \]

\[ = 2^{O(N \log N)} \left( \sum_{i=1}^{N} \text{size}\left(\alpha_i\right) + \text{size}\left(\beta_i\right) \right)^{O(N)} \sum_{i=1}^{N} \left( \text{size}\left(\alpha_i\right) + \text{size}\left(\beta_i\right) \right), \]

and we are done. \(\blacksquare\)

**The Proof of Theorem 3** First note that if our input \( A \) is not a circuit,\(^10\) then \( A \) is a degenerate circuit, and \( \Delta_A(f) \) is then identically 1. This is because \( Z_{\mathbb{R}}(f) \) is smooth when the support of \( f \) is a degenerate circuit (see, e.g., the proof of Case 2 of Theorem 2 in Section 3.2). So we can assume that \( A = \{a_1, \ldots, a_{n+2}\} \) is a circuit, \( f \) has support \( A \), and that \( c_j \) is the coefficient of \( x^{a_j} \) in all \( j \).

Thanks to Lemma 3 Assertion (1) (resp. Assertion (2)) follows straightforwardly from the complexity bound for Algorithm BinomialVanish (resp. BinomialSign) we just proved in Lemma 4. In particular, the latter lemma tells us that the complexity of ADISCVAN, for an input \( (A, c_1, \ldots, c_{n+2}) \), is polynomial in \( \sum_{i=1}^{n+2} \log(c_i m_i) \) (following the notation of Lemma 3); and the same is true for ADISCSIGN provided \( n \) is fixed. The classical Hadamard matrix inequality \cite{Mig92} tells us that \( \text{size}(m_i) = O(n \log(n \max_{j,k} \{a_{jk}\})) \), so the complexity of ADISCVAN is indeed polynomial in \( \text{size}(f) \); and the same holds for ADISCSIGN when \( n \) is fixed. \(\blacksquare\)

**3.2 Deforming to Polyhedra: The Proof of Theorem 2**

Let \( A = \{a_1, \ldots, a_m\} \) be the support of \( f \) and and write \( f(x) = \sum_{j=1}^{m} c_j x^{a_j} \). Since \( Z_{\mathbb{R}}(f) \) is unaffected if \( f \) is replaced by a monomial multiple of \( f \), we can clearly assume without loss of generality that \( a_1 = 0 \).

If \( n = 1 \) then \( f \) must be a univariate trinomial and Theorem 2 follows immediately from Descartes’ Rule. So let us assume henceforth that \( n \geq 2 \).

**Case 1: A affinely independent:** Letting \( A’ \) denote the matrix whose columns are \( a_2, \ldots, a_m \), observe that \( f(y) := f(y^{A’}) = c_1 + c_2 y_1 + \cdots + c_m y_m, \) with \( m \leq n \). Moreover, thanks to Proposition 4, \( Z_{\mathbb{R}}(f) \) (the intersection of a hyperplane with the positive orthant of \( \mathbb{R}^n \)) and \( Z_{\mathbb{R}}(f) \) are diffeomorphic, so we are done. In particular, we see that \( Z_{\mathbb{R}}(f) \) is either empty or a connected, open, \( C^\infty \), real \( (n - 1) \)-manifold. \(\blacksquare\)

**Remark 4** Note that we can in fact allow arbitrary real exponents for \( f \) in our proof above. \(\diamond\)

\(^{9}\)Note also that the true number of bits of accuracy we would use in practice is \( 2^\mu \), where \( \mu \) is the smallest integer with, say, \( 2^\mu > (32N)^{400N}C \log(C) \log(CM) \). This is because while it may be non-trivial to compute \( E \) exactly, finding \( 2^\mu \) is easy via the old trick of recursive squaring.

\(^{10}\)One can in fact check in polynomial-time whether \( A \) is a circuit. See, e.g., Proposition 2 of Section 3.2.
**Case 2: A is a degenerate circuit:** Suppose, without loss of generality, that \(B = \{0, a_2, \ldots, a_{\ell}\}\) (with \(\ell < m \leq n + 2\)) is a circuit, and \(a_\ell\) in the relative interior of \(\text{Conv}B\) if \(B\) intersects the relative interior of \(\text{Conv}B\).

Let \(A''\) be the \(n \times (m - 2)\) matrix whose columns are \(a_{m-1}, \ldots, a_2\) and (via Lemma 11) define \(U\) to be any \(n \times n\) unimodular matrix \(U\) such that \(UA''\) is lower triangular, with nonnegative diagonal. Defining \(\bar{f}(y) := f(y^U)\), it is then easily checked that \(\bar{f}(y)\) is of the form \(c_1 + c_2y_2^2 + \cdots + cm y_m^m\) with \(a_j \in \{0\}^n \times \mathbb{Z}^m\) for all \(j \in \{2, \ldots, m - 1\}\), and \(a''_m \in \{0\}^{n+2-m} \times (\mathbb{Z} \setminus \{0\}) \times \mathbb{Z}^{m-3}\). Moreover, Proposition 11 tells us that \(Z_+(f)\) and \(Z_+(\bar{f}) \times \mathbb{R}^{n-m}\) (when \(Z_+(\bar{f})\) is considered as a subset of \(\mathbb{R}^m_+\)) are diffeomorphic.

Defining \((a''_{m,j}, \ldots, a''_{n,j}) := a''_j\) for all \(j\) and \(\bar{f}_j(y) = \frac{- (c_1 + c_2 y_2^2 + \cdots + c_{m-1} y_{m-1}^{m-1})}{y_1 a''_{m,1} \cdots y_{m-1} a''_{m,m-1}}\), observe then that \(\bar{f}_j \in \mathbb{Z}[y_1, \ldots, y_{m-1}]\) and, for all \(y = (y_1, \ldots, y_m) \in \mathbb{R}_+^m\), we have \(\bar{f}(y) = 0 \iff \bar{f}_1 y_{m} = y_m\). So we see that \(Z_+(\bar{f})\) is exactly the positive part of the graph of the \(C^\infty\) function \(\bar{f}_1^{1/a_{m,m}} : \mathbb{R}^{n-m} \rightarrow \mathbb{R}\). Clearly then, \(Z_+(\bar{f})\) consists of a union of connected, open, \(C^\infty\), real \((m - 1)\)-manifolds, and \(Z_+(f) \approx Z_+(\bar{f}) \times \mathbb{R}^{n-m}\) is thus smooth and has no compact connected components. So let us now bound \(\mathcal{N}_{\text{non}}(\bar{f})\).

If \(\bar{f}_1\) is always positive on \(\mathbb{R}^{n-1}_+\) then \(Z_+(\bar{f})\) clearly consists of a single non-compact connected component. So we can henceforth assume that \(Z_+(\bar{f})\) is non-empty, which in turn implies that every connected component \(C\) of \(Z_+(\bar{f})\) has a limit point in \(\mathbb{R}^{n-1}_+ \times \{0\}\). Assertion (3) of Theorem 11 then tells us that \(\mathcal{N}_{\text{non}}(\bar{f}) \leq \mathcal{N}(\bar{f}_1 - \varepsilon)\) for some \(\varepsilon > 0\). So, by induction (reducing to lower-dimensional instances of Cases 2 or 3), we are done.

At this point, we must recall a result of Viro on the classification of certain real algebraic hypersurfaces. In what follows, we liberally paraphrase from [GKZ94] Thm. 5.6].

**Definition 10** Given any finite point set \(A \subset \mathbb{R}^n\), let us call any function \(\omega : A \rightarrow \mathbb{R}\) a lifting, denote by \(\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n\) the natural projection which forgets the last coordinate, and let \(\hat{\mathcal{A}} := \{(a, \omega(a)) | a \in A\}\). We then say that the polyhedral subdivision \(\Sigma_\omega\) of \(A\) defined by \(\{Q | Q \text{ a lower }^{10} \text{ face of Conv} \hat{\mathcal{A}} \text{ of dimension } \dim A\}\) is induced by the lifting \(\omega\), and we call \(\Sigma_\omega\) a triangulation induced by a lifting iff every cell of \(\Sigma_\omega\) is a simplex. Finally, given any \(f(x) = \sum_{a \in A} c_a x^a \in \mathbb{Z}[x_1, \ldots, x_n]\), we define \(f_{\omega,\varepsilon}(x) := \sum_{a \in A} c_a \varepsilon^{\omega(a)} x^a\) to be the toric perturbation of \(f\) (corresponding to the lifting \(\omega\)).

**Definition 11** Following the notation above, suppose \(\dim A = n\) and \(A\) is equipped with a triangulation \(\Sigma\) induced by a lifting and a function \(s : A \rightarrow \{\pm\}\) which we will call a distribution of signs for \(A\). We then define a locally piece-wise linear manifold — the Viro diagram \(\mathcal{V}(\Sigma, s)\) — in the following local manner: For any \(n\)-cell \(C \in \Sigma\), let \(L_C\) be the convex hull of the set of midpoints of edges of \(C\) with vertices of opposite sign, and then define \(\mathcal{V}(\Sigma, s) := \bigcup_{C \text{ an } n\text{-cell}} L_C\).

**Example 4** The following figure illustrates 6 circuits of cardinality 4, each equipped with a triangulation induced by a lifting, and a distribution of signs. The corresponding (possibly empty) Viro diagrams are drawn in the lightest color visible (yellow on the color version of

---

10 A lower face is simply a face which has an inner normal with positive last coordinate.
Theorem 6  Suppose $f(x) = \sum_{a \in A} c_a x^a \in \mathbb{Z}[x_1, \ldots, x_n]$ with $\text{Supp}(f) = A$ and $\dim A = n$, $\omega$ is any lifting of $A$, and define $s_f(a) = \text{sign}(c_a)$ for all $a \in A$. Then for any sufficiently small $\varepsilon > 0$, $Z_+(f_{\omega,\varepsilon})$ is isotopic to $\mathcal{V}(\Sigma_\omega, s_f) \setminus \partial \text{Conv } A$. In particular, $\mathcal{V}(\Sigma_\omega, s_f)$ is a disjoint finite union of piece-wise linear manifolds, each possibly having a non-empty boundary.

Lemma 5  Suppose $A$ is a circuit, $\Sigma$ is a triangulation of $A$, $n = \dim A$, and $s$ is any distribution of signs on $A$. Call any point $a$ in the relative interior of $A$ with $s(a)$ opposite $s(a')$ for all $a' \in A \setminus \{a\}$ a caged alternation of $(A, s)$. Then
1. $Z_+(f)$ smooth $\implies Z_+(f)$ is isotopic to $\mathcal{V}(\Sigma, s_f) \setminus \text{Conv } A$ for some $\Sigma$.
2. $\mathcal{V}(\Sigma, s)$ has no boundary iff $\mathcal{V}(\Sigma, s)$ is the boundary of an $n$-simplex iff $A$ has a caged alternation $a$ and $\Sigma$ is the triangulation obtained by the lifting that sends $a \mapsto 0$ and $a' \mapsto 1$ for all $a' \in A \setminus \{a\}$.

Proof of Lemma $\|$  By Lemma $\|$ and Lemma $\|$ it easily follows that $A$ has at most 2 discriminant chambers in $\mathbb{R}^{n+2}_+$, and each such chamber contains a unique toric perturbation. Since the topology of $Z_+(f)$ is constant on any discriminant chamber containing $f$ (e.g., [GKZ94, Ch. 11, Sec. 5A, Prop. 5.2, pg. 382]), we obtain Assertion (1).

Now note that by definition, any triangulation of a circuit $A$ obtained by lifting has at most $n+1$ top-dimensional cells (since it is the projected lower hull of a $(n+1)$-simplex). So then, if $\mathcal{V}(\Sigma, s)$ has no boundary, its convex hull must clearly have dimension $n$, in which case we see that $\mathcal{V}(\Sigma, s)$ is a union of at least $n+1$ simplices of dimension $n-1$. So $\mathcal{V}(\Sigma, s)$ is the union of exactly $n+1$ simplices of dimension $n-1$, equal to the boundary of its convex hull, i.e., $\mathcal{V}(\Sigma, s)$ is the boundary of a $n$-simplex. This proves the first rightward implication of Assertion (2), and the converse is obvious.

Now if $\mathcal{V}(\Sigma, s)$ is the boundary of a $n$-simplex, then $A$ clearly intersects its relative interior, which in turn implies that $\Sigma$ must be the specified triangulation. (Indeed, it is a standard fact that any circuit has exactly 2 triangulations. So the only other possible $\Sigma$ has exactly 1 $n$-cell and could not possibly give the $\mathcal{V}(\Sigma, s)$ we desire.) Furthermore, $A$ must then clearly contain a caged alternation, for otherwise $\mathcal{V}(\Sigma, s)$ would no longer be the boundary of an $n$-simplex. This proves the second rightward implication of Assertion (2), and the converse is obvious. ■

We are now ready to return to our proof of Theorem $\|$ and finish the remaining special case.
Case 3: A is a circuit: First note that by Corollary 1 we can apply a monomial change of variables and assume \( m = n + 2 \) without loss of generality. (Moreover, if \( m < n + 2 \) initially, then every connected component of \( Z_+(f) \) must be non-compact, thanks to Proposition 1.)

Let us first bound \( \mathcal{N}_{\text{non}}(f) \): First note that since \( A \) is a circuit, every facet of \( \text{Conv}(A) \) has affinely independent vertex set. So by Case 1, the facet functions of \( f \) each automatically have smooth zero sets which are either empty or consist of a single non-compact connected component. The Upper Bound Theorem of polyhedral combinatorics \([\text{Ede87}]\) then implies that \( \text{Conv}(A) \) has no more facets than a moment \( n \)-polytope with \( n + 2 \) vertices. The latter polytope has exactly 4, 6, or 9 facets, according as \( n = 2, 3, \) or 4 \([\text{Ede87}]\). So Assertion (1) of Theorem 4 then directly implies our stated bound on \( \mathcal{N}_{\text{non}}(f) \), for \( n \in \{3, 4\} \). That \( \mathcal{N}_{\text{non}}(f) \leq 2 \) for \( n = 2 \) follows from earlier work of Daniel Perrucci using a different argument, involving a detailed analysis of the central special case \( 1 + x + y + Ax^ay^b \) \([\text{Per05}, \text{Thm. 4, Assertion (4)]}\).

Note in particular that when \( n = 1 \), \( Z_+(f) \) has no non-compact components unless \( f \) is identically zero. Note also that \( f(x, y) := (x - 1)(y - 1) = xy - x - y + 1 \) has support a circuit and exactly two connected components (each non-compact) for its zero set in \( \mathbb{R}^2_+ \). So our stated bounds for \( \mathcal{N}_{\text{non}}(f) \) are indeed tight for \( n \in \{1, 2\} \).

Assume now that \( n \geq 5 \). If \( Z_+(f) \) is smooth, then Assertion (1) of Lemma 5 tells us that \( Z_+(f) \) is isotopic to some Viro diagram. As observed within the proof of Lemma 5, a triangulation \( \Sigma \) of \( A \) contains no more than \( d + 1 \) \( d \)-simplicies, and thus a Viro diagram of the form \( \mathcal{V}(\Sigma, s) \) consists of no more than \( d + 1 (d - 1) \)-simplicies. Since any such \( (d - 1) \)-simplex can belong to at most one connected component of \( \mathcal{V}(\Sigma, s) \), and since \( \mathcal{V}(\Sigma, s) \) is a union of piece-wise linear manifolds with boundary, we see that \( \mathcal{V}(\Sigma, s) \setminus \text{Conv}A \) has at most \( d + 1 \) non-compact connected components. So by Theorem 6, \( \mathcal{N}_{\text{non}}(f) \leq n + 1 \) in the smooth case.

Now recall the following two standard inequalities:

\[
\begin{align*}
(N_c) & \quad \mathcal{N}_{\text{comp}}(f) \leq \mathcal{N}_{\text{comp}}(f - \varepsilon) + \mathcal{N}_{\text{comp}}(f + \varepsilon) \\
(N_n) & \quad \mathcal{N}_{\text{non}}(f) \leq \mathcal{N}_{\text{non}}(f - \varepsilon) + \mathcal{N}_{\text{non}}(f + \varepsilon)
\end{align*}
\]

for \( \varepsilon > 0 \) sufficiently small. Moreover, \( Z_+(f \pm \varepsilon) \) is smooth for all \( \varepsilon > 0 \) sufficiently small. (See, e.g., [\text{Bas99}, Lemma 2].)

Having proved \( \mathcal{N}_{\text{non}}(f) \leq n + 1 \) in the smooth case, Inequality (\( N_n \)) then immediately implies that \( \mathcal{N}_{\text{non}}(f) \leq 2n + 2 \), even in the presence of singularities for \( Z_+(f) \).

To bound \( \mathcal{N}_{\text{comp}}(f) \), let \( n \) be arbitrary once again. Observe then that Lemma 5 implies that \( \mathcal{N}_{\text{comp}}(f) \leq 1 \), with equality only if \( Z_+(f) \) is isotopic to an \((n - 1)\)-sphere, as long as \( Z_+(f) \) is smooth. So our bound for \( \mathcal{N}_{\text{comp}}(f) \) holds if \( Z_+(f) \) is smooth.

On the other hand, if \( Z_+(f) \) has a singularity, consider first the special case where \((A, s_f)\) does not have a caged alternation. Lemma 5 and (\( N_c \)) then imply that \( \mathcal{N}_{\text{comp}}(f) = 0 \). So we have in fact proved a strengthening of the final assertion of Theorem 2.

As for the case where \((A, s_f)\) has a caged alternation, assume without loss of generality that it is \( a_{n+2} \). Lemma 5 then tells us that \( \mathcal{N}_{\text{comp}}(f) \leq 1 \) (with \( Z_+(f) \) isotopic to an \((n - 1)\)-sphere if \( \mathcal{N}_{\text{comp}}(f) = 1 \)), provided \( Z_+(f) \) is smooth. So we have our assertion for compact components of \( Z_+(f) \) in the smooth case.

To conclude, assume that \( Z_+(f) \) has a singularity. Lemma 5 then tells us that this singularity is unique and that \( \left( \prod_{i=1}^{n+1} m_i \right)^{n-m_{n+2}} = m^{n+1} \prod_{i=1}^{n+1} c_i^{m_i} \). So \( f - \varepsilon \) and \( f + \varepsilon \) thus lie in \textit{opposite} discriminant chambers for any \( \varepsilon > 0 \). (Note that we are still assuming that \((A, s_f)\) has a caged alternation.) So one of \( Z_+(f - \varepsilon) \) or \( Z_+(f + \varepsilon) \) is empty. By Lemma
and inequalities \((N_c)\) and \((N_n)\), we thus obtain that \(N_{\text{comp}}(f) \leq 1\) and \(N_{\text{non}}(f) = 0\). In particular, \(Z_+(f)\) must have consisted of a single point, for otherwise, both \(Z_+(f - \varepsilon)\) and \(Z(f + \varepsilon)\) would have been non-empty (by the Implicit Function Theorem).

Having proved our upper bound for \(N_{\text{comp}}(f)\), we need only exhibit examples proving tightness for each \(n \geq 1\). For \(n = 1\) there is the obvious example of \((x - 1)^2 = x^2 - 2x + 1\). As for \(n \geq 2\), Theorem \(\text{[4]}\) tells us that it suffices to use \(\varepsilon(1 + x_1^{2n} + \cdots + x_n^{2n}) - x_1 \cdots x_n\), for any \(\varepsilon > 0\) sufficiently small. So we are done. ■

### 3.3 Phase Transitions: The Proof of Theorem \(\text{[1]}\)

The complexity lower bounds of Theorem \(\text{[1]}\) — Assertions (3) and (4) — are the easiest to prove, so we start there:

**The Proof of Assertion (3):** Recall that \(3\text{CNFSAT}\) is the problem of deciding whether a Boolean formula of the form \(B(X) = C_1(X) \land \cdots \land C_k(X)\) has a satisfying assignment, where \(C_i\) is of one of the following forms:

\[
X_i \lor X_j \lor X_k, \quad \neg X_i \lor X_j \lor X_k, \quad \neg X_i \lor \neg X_j \lor X_k, \quad \neg X_i \lor \neg X_j \lor \neg X_k,
\]

\(i, j, k \in \{3, N\}\), and a satisfying assignment consists of an assignment of values from \(\{0, 1\}\) to the variables \(X_1, \ldots, X_{3N}\) which makes the equality \(B(X) = 1\) true. \(3\text{CNFSAT}\) is one of the most basic \(\text{NP}\)-complete problems \(\text{[GJ79]}\). In particular, for our purposes, let us measure the size of a \(3\text{CNFSAT}\) instance such as the one above as \(N\).

Let us now observe that \(\text{FEAS}_R(M)\) is \(\text{NP}\)-hard, where \(M\) is the family of all polynomial systems of the form \((f_1, \ldots, f_k)\) where, for all \(i, f_i \in \mathbb{Z}[x_1, \ldots, x_{3N}]\) involves no more than 3 variables and has degree \(\leq 1\) with respect to each of them.\(^{11}\) To see why, first observe that any Boolean formula \(B(X)\) with no occurrence of “\(\land\)” can be converted into a polynomial \(f_B(x) \in \mathbb{Z}[x_1, \ldots, x_{3N}]\) via the following table of substitutions:

\[
\begin{align*}
X_i & \mapsto x_i \\
\neg X_i & \mapsto 1 - x_i \\
C_i(X) \lor C_j(X) & \mapsto f_{C_i}(x) + f_{C_j}(x) - f_{C_i}(x)f_{C_j}(x)
\end{align*}
\]

For instance, the formula \(X_3 \land \neg X_3 \land X_8\) becomes

\[(x_1 + (1 - x_3) - x_1(1 - x_3)) + x_8 - (x_1 + (1 - x_3) - x_1(1 - x_3))x_8 = 1 - x_3 + x_1x_3 + x_3x_8 - x_1x_3x_8.
\]

To any \(3\text{CNFSAT}\) instance as above, we can then associate the polynomial system

\[
(f_{C_1} - 1, \ldots, f_{C_k} - 1, x_1(1 - x_1), \ldots, x_{3N}(1 - x_{3N})),
\]

and it is easily checked that \(B\) has a satisfying assignment iff \(F_B\) has a real root. (Moreover, any root of \(F_B\) clearly lies in \(\{0, 1\}^{3N}\).) Note also that the size of \(F_B\) is clearly \(O(N)\), and that \(F_B\) has at least as many equations as variables. Clearly then, \(\text{FEAS}_R(M) \in \text{P} \implies 3\text{CNFSAT} \in \text{P}\), and thus \(\text{FEAS}_R(M)\) is \(\text{NP}\)-hard.

To conclude, we need only recall that any system of polynomials \(F\) chosen from \(\mathbb{Z}[x_1, \ldots, x_n]\) can always be converted to its **Shor Normal Form** \(S_F\) \(\text{[Sho91]}\). In particular, \(F\) has a real root iff \(S_F\) has a real root, the size of \(S_F\) is linear in the size of \(F\), \(S_F\) consists of linear trinomials and/or binomials of degree \(\leq 2\), and the number of new equations introduced is the same as the number of new variables introduced. (More concretely, substitutions like \(x_2 - x_1^2 \mapsto (y_1 - x_1^2, y_2 - y_1^2, y_3 - y_2y_1, x_2 - y_3x_1)\) can be used to reduce all powers to 2 or less,

\hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm}

\(^{11}\) While this construction is well-known in some circles, we have included a detailed explanation for the convenience of the reader.
and substitutions like \( y_k = c_i x^{a_i} + c_j x^{a_j} \) can be used to reduce any polynomial to a collection of polynomials, each with 3 or fewer monomial terms.) So the number of variables of \( S_F \) is bounded above by the number of equations of \( S_F \), and we are done. ■

**The Proof of Assertion (4):** By our proof of Assertion (3), and replacing any system \((f_1, \ldots, f_k) \in S\) with the polynomial \( f_1^2 + \cdots + f_k^2 \), it is easy to see that \( \text{FEAS}_R(\mathcal{E}) \) is NP-hard, where \( \mathcal{E} \) is the family of \( n \)-variate 11\( n \)-nomials that are sums of squares. Indeed, by our earlier reduction to 3CNFSAT, we can additionally assume that any polynomial in \( \mathcal{E} \cap \mathbb{Z}[x_1, \ldots, x_n] \) contains \( x_1^2(1-x_1)^2, \ldots, x_n^2(1-x_n)^2 \) as summands. The latter assumption easily implies that \( \dim \text{Supp}(f) = n \) for any \( n \)-variate polynomial \( f \in \mathcal{E} \), for then \( \text{Newt}(f) \) must contain a line segment parallel to each and every coordinate axis of \( \mathbb{R}^n \).

To reduce to the family \( \mathcal{C} \), simply note that from any \( f \in \mathcal{E} \) and \( \varepsilon > 0 \), we can form the new polynomial \( g_\varepsilon(x, y) := f(x) + y_1^2 + \cdots + y_N^2 \), where \( N := \lceil 10n^{1/\varepsilon} \rceil \). Observe then that \( g \) involves exactly \( n + N \) variables, \( g \) has exactly \( 11n + N \) monomial terms, and \( \dim \text{Supp}(g) = n + N \). In particular,

\[
11n + N - (n + N) = 10n \leq 10n^{1/\varepsilon} \leq (n + N)^\varepsilon,
\]

for all \( n \geq 1 \). Moreover, for any fixed \( \varepsilon > 0 \), the size of \( g \) is clearly polynomial in the size of \( f \). So we then clearly obtain that \( \text{FEAS}_R(\mathcal{C}_\varepsilon) \in \mathbb{P} \implies \text{FEAS}_R(\mathcal{E}) \in \mathbb{P} \), and we are done. ■

The complexity upper bounds of Theorem — Assertions (1) and (2) — then follow easily from the two final algorithms we state below. In what follows, we let \( Z_R(f) \) denote the real zero set of \( f \) and \( A := \text{Supp}(f) \).

**Algorithm DimPlus1Case**

**Input:** A polynomial \( f \in \mathbb{Z}[x_1, \ldots, x_n] \) with affinely independent support \( A = \{a_1, \ldots, a_m\} \) of cardinality \( m \), and \( c_j \) the coefficient of \( x^{a_j} \) in \( f \) for all \( j \).

**Output:** True declarations as to whether \( Z_+(f) \), \( Z_R^+(f) \), and \( Z_R(f) \) are, respectively, empty or not.

**Description:**

1. If the \( c_i \) are **not** all of the same sign, then output
   
   ‘‘\( Z_+(f) \), \( Z_R^+(f) \), and \( Z_R(f) \) are all non-empty.’’
   
   and stop.

2. If any one of \( a_2 - a_1, \ldots, a_m - a_1 \) has an odd coordinate, then output
   
   ‘‘\( Z_R^+(f) \) and \( Z_R(f) \) are non-empty, but \( Z_+(f) \) is empty.’’
   
   and stop.

3. Output ‘‘\( Z_+(f) \) and \( Z_R^+(f) \) are empty.’’.

4. If \( O \notin A \) then output ‘‘\( Z_R(f) \) is non-empty.’’ and stop.

5. Output ‘‘\( Z_R(f) \) is empty.’’ and stop. ■

**Remark 5** Note that \( \dim \text{Supp}(f) \) can be computed in \( \mathbb{NC}_1 \) as follows: compute the rank of the matrix whose columns are \( a_2 - a_1, \ldots, a_m - a_1 \) via the parallel algorithm of Csanky [Cs76]. So checking whether a given \( f \) is a valid input to Algorithm DimPlus1Case can be done within \( \mathbb{NC}_1 \). ◊
The Proof of Assertion (1): Here we simply apply Algorithm DimPlus1Case. Assuming Algorithm DimPlus1Case is correct, the complexity upper bound then follows trivially, since one can check all the necessary signs and/or parities in logarithmic time, using a number of processors linear in size($f$). (Note also that we can check within NC$_1$ whether an input $f$ has Supp($f$) affinely independent, thanks to Remark 5.) So we need only show that Algorithm DimPlus1Case is correct.

To prove the latter, note that Lemma 2 implies that the existence of a pair of coefficients of $f$ with opposite sign is the same as $Z_+(f)$ being non-empty. So Step (1) is correct, and we can assume henceforth that all the coefficients of $f$ have the same sign. By Lemma 2 again, the existence of indices $i, j$ with $a_{i,j} - a_{i,1}$ odd implies that $Z^*_R(f) \setminus Z_+(f)$ is non-empty. So Step (2) is correct, assuming its hypothesis is true.

So let us now assume that $a_{i,j}$ has the same parity as $a_{i,1}$ for all $i, j$. Note that if $f$ has a root in $\mathbb{R}^n$ then this root must lie in some coordinate subspace $L$ of minimal positive dimension. So, by our initial hypotheses, on $L$, the polynomial $f$ will restrict to an $n'$-variate $m'$-nomial with $m' \leq n' + 1$ and affinely independent support a subset of $A$. In particular, since we now have that all the coefficients of $f$ have the same sign, and since $a_{i,j}$ has the same parity as $a_{i,1}$ for all $i, j$, one final application of Lemma 2 implies that $Z_R(f)$ is empty. So Step (2) is also correct when its hypothesis is false.

Steps (3)–(4) are clearly correct since $f$ vanishes at $O$ iff $f$ does not have a nonzero constant term.

Remark 6 Let FEAS$_R^+$ and FEAS$_R^*$ denote the obvious analogues of FEAS$_R$ where we respectively restrict to roots in $\mathbb{R}_+^n$ or $(\mathbb{R}^*)^n$. Also, paralleling our earlier notation, let $\text{FEAS}_R^+(\mathcal{F})$ and $\text{FEAS}_R^*(\mathcal{F})$ be the corresponding natural restrictions of FEAS$_R^+$ and FEAS$_R^*$ to inputs in some family $\mathcal{F}$. Our preceding proof then clearly implies that $\text{FEAS}_R^+(A) \in \text{NC}_1$ (even if real exponents are allowed) and $\text{FEAS}_R^*(A) \in \text{NC}_1$.

We can now describe our final algorithm:

Algorithm DimPlus2Case

Input: A polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$ with support $A = \{a_1, \ldots, a_{n+2}\}$ of cardinality $n + 2$, dim $A = n$, $a_1 = O$, and an $\ell \in \{3, \ldots, n + 2\}$ such that $B := \{a_1, \ldots, a_\ell\}$ is a circuit and $a_\ell$ lies in the relative interior of Conv$B$ if $B$ intersects the relative interior of Conv$B$.

Output: True declarations as to whether $Z_+(f)$ has compact and/or non-compact connected components, along with notification as to whether $Z_+(f)$ is a point.

Description:

0. Let $W$ denote the set of all inner facet normals, with integer coordinates having no common factor, of $A$. Also let $c_j$ be the coefficient of $x^{a_j}$ in $f$ for all $j$, and let $A''$ be the $n \times n$ matrix whose columns are $a_{n+1}, \ldots, a_2$.

1. If $n \geq 2$, decide (using Algorithm DimPlus1Case) whether there is a $w \in W$ affinely independent and $Z_+(\ln_w(f))$ non-empty. If so, then output “$Z_+(f)$ is non-empty and all its connected components are non-compact.”, and stop.

2. If $A$ is a degenerate circuit then do the following:
(a) If $B$ intersects the relative interior of $\text{Conv}B$ then do the following:

i. Find (via Lemma 1 of Section 2) a unimodular matrix $U$ such that $UA^\prime$ is lower triangular and has a nonnegative diagonal with exactly one zero entry. Then, replacing $f$ by $-f$ if necessary, assume that all the $c_j$ (except possibly $c_\ell$) are positive. Finally, define $h(x) := f(x) - c_{\ell+1}x^{\ell+1} - \cdots - c_{n+2}x^{n+2}$ and $g(y) := h(y^\prime)$.

ii. Decide, via a lower-dimensional instance of Algorithm DimPlus2Case, whether $Z_+(g)$ contains at least 2 points. If so, then output 

\begin{quote}
\text{``$Z_+(f)$ is non-empty, smooth, and all its connected components are non-compact.''}
\end{quote}
and stop.

3. If $A$ intersects the interior of $\text{Conv}A$ then let $s_f(a) := \text{sign}(c_a)$ for all $a \in A$, order the sequence of signs $s_f(a)$ in increasing order of $a$ (if $n=1$), and do the following:

(a) If $n=1$ and $s_f$ has exactly one sign alternation then output 

\begin{quote}
\text{``$Z_+(f)$ is a point of multiplicity 1.''}
\end{quote}
and stop.

(b) If $(A, s_f)$ has a caged alternation then do the following:

i. Decide, via Algorithm BinomialVanish, whether $\Delta_A(c_1, \ldots, c_{n+2}) = 0$. If so, then output 

\begin{quote}
\text{``$Z_+(f)$ has exactly one connected component, and it is a singular point.''}
\end{quote}
and stop.

ii. Decide, via Algorithm BinomialSign, whether $\text{sign}(\Delta(g)) = (-1)^{m_{n+2}}$. If so, then output 

\begin{quote}
\text{``$Z_+(f)$ has exactly one connected component, and it is smooth and isotopic to an $(n-1)$-sphere.''}
\end{quote}
and stop.

4. Output \text{``$Z_+(f)$ is empty.''} and stop.

It will be useful to observe that the input hypotheses to the preceding algorithm can be checked within $\mathbf{P}$:

\begin{proposition} \label{prop:check}
Given any finite set $A \subset \mathbb{Z}^n$ with $\#A \leq \dim A + 2$, we can decide if $A$ contains a circuit $B$, and find the unique such $B$ should it exist, in time polynomial in \text{size}(A). Moreover, if $B$ contains a circuit and $B$ intersects the relative interior of $\text{Conv}B$, then we can find the unique point of this intersection also in time polynomial in \text{size}(A).
\end{proposition}

\begin{proof}[Proof of Proposition \ref{prop:check}]
First, one simply checks via the method of Remark 5 if $\#A = \dim A + 2$ (for if not, $A$ cannot contain a circuit). Then, via Cramer’s Rule and the Newton identities [BCSS98, Ch. 15, Pgs. 292–296], one simply checks which subsets of $A$ of cardinality $\dim A$ form facets of $A$. (Overlaps can be distinguished via a computation of the underlying inner facet normals.) If all the facets of $A$ have cardinality $\dim A$, then $A$ itself is a circuit and we are done. Otherwise, some facet $S$ of $A$ has cardinality $> \dim A$, and we use the same method recursively to find the unique circuit of $S$. Via [BCSS98, Prop. 21, Ch. 15, pp. 295], it is then easily checked that the bit complexity of this method is no worse than $O(\text{size}(A)^8)$, where \text{size}(A) is the sum of the sizes of the coordinates of the points of $A$.
\end{proof}
To efficiently check whether $B$ intersects the relative interior of Conv$B$, we can employ any linear programming algorithm with polynomial-time bit complexity as follows: express — whenever possible — each $b_i \in B$ as a convex linear combination of points in $B \setminus \{b_i\}$. If some point $b_i$ can be expressed in this way, then this point is unique, and we can permute the entries of $A$ so that $a_1 = 0$, $a_\ell = b_i$, and $a_j = b_j$ for all $j \in [\ell] \setminus \{i\}$. Otherwise, $B$ does not intersect its relative interior.

**The Proof of Assertion (2):** First note that $f$ has a nonzero constant term iff $f$ does not have $0$ as a root, and this can be checked with just 1 bit operation. So we can assume henceforth $f$ has a nonzero constant term. Also note that the polynomial obtained from $f$ by setting any subset of its variables to 0 lies in $B_{n'} \cup A$ for some $n' < n$. Moreover, since we can apply changes of variables like $x_i \mapsto -x_i$ in $P$, and since there are exactly $3^n$ sequences of the form $(\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_i \in \{0, \pm 1\}$ for all $i$, it suffices at this point to show that FEAS$^+_P(B_n) \in P$ for fixed $n$.

This will be accomplished by Algorithm DimPlus2Case, and via Proposition and a monomial change of variables (employing Lemma), we can indeed prepare $f$ to be a suitable input to this algorithm. So we can now assume that $f$ satisfies these input hypotheses. Assertion (2) then follows immediately — assuming that Algorithm DimPlus2Case is correct and runs in $P$ for fixed $n$. So let us now prove correctness and analyze the complexity along the way.

**Correctness and Complexity of Steps (0)–(1):** The correctness of Steps (0)–(1) follows immediately from Theorem. Note also that $W$ consists of no more than $\binom{n+2}{n} = (n+2)(n+1)/2$ normals, and each such normal can be constructed (employing Cramer’s Rule and the Newton identities [BCSS98, Ch. 15, Pgs. 292–296]) via $n-1$ determinants of $(n-1) \times (n-1)$ matrices, followed by a gcd computation (see, e.g., [BS96] for a detailed exposition on near optimal gcd algorithms). Since we’ve already proved that Algorithm DimPlus1Case runs in $P$, Steps (0)–(1) clearly run in $P$, and no further complexity analysis is needed if the hypothesis of Step (1) is satisfied.

If the hypothesis of Step (1) is false, then (applying Lemma again) this means that the $c_j$ with $a_j$ a vertex of Conv$A$ have the same sign. So we can assume without loss of generality that the $c_j$ are all positive and continue to Step (2).

**Correctness and Complexity of Step (2):** First let us define $h_j(x) := f(x) - \sum_{i=n+3-j}^{n+2} c_i x^a_i$ and $g_j(y) := h_j(y^\ell)$ for all $j \in [n+2-\ell]$. Note that $g_j \in \mathbb{R}[x_1, \ldots, x_{n-j}]$ for all $j \in [n+2-\ell]$.

By the Case 2 portion of the Proof of Theorem it immediately follows that $Z_+(f)$ is smooth and diffeomorphic to the positive part of the graph of $-g_1$ (as a function on $\mathbb{R}_{n-1}^+$. Clearly then, every connected component of $Z_+(f)$ is non-compact. Furthermore, since $c_1, \ldots, c_{\ell-1} > 0$ (and $c_\ell > 0$ as well, if $a_\ell$ is a vertex of Conv$A$), Lemma and Assertion (2) of Theorem imply that every connected component must have a limit point on $\mathbb{R}_{n-1}^+ \times \{0\}$. Moreover, it follows easily from the Implicit Function Theorem and Assertion (4) of Lemma that $Z_+(g_1)$ has at least 2 points $\implies Z_+(f)$ is non-empty. So by induction, $Z_+(f)$ is non-empty iff $Z_+(g_{n+1-\ell})$ is non-empty (provided $\ell < n+1$).

\footnote{See, e.g., [BCSS98, Ch. 15] for a nice description of a barrier method employing Newton’s method.}

\footnote{We should note that if dim Supp($f$) < $n$ initially then Corollary implies that every connected component of $Z_+(f)$ will be non-compact.}
Note also that by construction, \( \text{Supp}(g_j) \) is a degenerate circuit for all \( j \in [n+1-\ell] \) (provided \( \ell < n+1 \)). So, to simplify notation, we can clearly assume without loss of generality that \( \ell = n+1 \) and that \( \text{Supp}(g) \) is a circuit. Just as in the last paragraph, we can still assert the two implications (a) \( Z_+(g) \) has at least 2 points \( \implies Z_+(f) \) is non-empty, and (b) \( Z_+(g) \) empty \( \implies Z_+(f) \) is empty. So we are left with the special case where \( Z_+(g) \) is a point. Moreover, unless \( (B, s_g) \) has a caged alternation, the coefficients of \( g \) will all be positive, thus making \( Z_+(f) \) empty (and this will be correctly declared later in Step (4)). So we can assume that \( Z_+(g) \) is a point and \( (B, s_g) \) has a caged alternation.

Define \( g_\varepsilon(y) := \varepsilon \left( \prod_{i=1}^{\ell} c_i y^{a_i} \right) + c_\ell y^{a_\ell} \). Assertion (3) of Lemma 3 then immediately implies that for \( \delta, \varepsilon > 0 \) sufficiently small, \( \text{sign}(\Delta_B(g - \delta)) = \text{sign}(\Delta_B(g_\varepsilon)) \). In particular, \( g - \delta \) must then lie in the same \( B \)-discriminant chamber as \( g_\varepsilon \), and Theorem 3 then implies that the graph of \(-g\) attains a maximum value of 0 within \( \mathbb{R}^{n-1}_+ \). In other words, \( Z_+(f) \) is empty, and this would be correctly declared later in Step (4). So Step (2) is correct, and its complexity is dominated by a single instance of Algorithm \text{DimPlus2Case}, for some \( n' < n \).

We can thus assume now that \( A \) is a circuit and continue to Step (3). ■

**Steps (3) and (4):** The correctness of Step (3-a) follows immediately from Descartes’ Rule of Signs, combined with the observation that the derivative of \( f(x) \) (or \( x^{a_3} f(1/x) \)) is nonzero on \( \mathbb{R}_+ \).

If the hypothesis of Step (3-b) is violated, then all the coefficients of \( f \) must have the same sign and \( Z_+(f) \) must then be empty (and this will be correctly declared later in Step (4)). So we may assume that \( (A, s_f) \) has a caged alternation. Moreover, we may also assume without loss of generality that \( c_1, \ldots, c_{n+1} > 0 \) and \( c_{n+2} < 0 \).

If \( Z_+(f) \) is smooth then Assertion (1) of Lemma 3 implies that \( Z_+(f) \) is isotopic to one of two possible Viro diagrams, easily seen to be either empty or the boundary of an \( n \)-simplex. Moreover, by Theorem 3 \( Z_+(f) \) is isotopic to the latter diagram iff \( \text{sign}(\Delta_A(f_\varepsilon)) = \text{sign}(\Delta_A(f_\varepsilon)) \) for all \( \varepsilon > 0 \) sufficiently small, where \( f_\varepsilon(x) := \varepsilon \left( \prod_{i=1}^{n+1} c_i x^{a_i} \right) + c_{n+2} x^{a_{n+2}} \). A simple calculation from Assertion (3) of Lemma 3 then tells us that \( \text{sign}(\Delta_A(f_\varepsilon)) = (-1)^{m_{n+2}} \) for all \( \varepsilon > 0 \). In other words, Step (3-b-ii) is correct.

So now assume \( Z_+(f) \) has a singularity \( \zeta \). Then, Assertion (4) of Lemma 3 implies that \( \zeta \) is the only singularity of \( Z_+(f) \). Moreover, since \( (A, s_f) \) has a caged alternation, \( Z_+(f) \) must then be exactly \( \{ \zeta \} \), as already proved toward the end of the Case 3 portion of the proof of Theorem 2. So Step (3-b-i) is correct.

To conclude, observe that any case not satisfying the hypotheses of any of our steps results in \( Z_+(f) \) being empty, and this is correctly declared by Step (4). Furthermore, we see that the complexity of Steps (3)–(4) is dominated by a single instance of Algorithm Binomial\text{Vanish} and a single instance of Binomial\text{Sign}, for input \( f \). ■

## 4 The Proof of Lemma 2

As before, we can assume without loss of generality that \( a_1 = 0 \) and \( c_1 = 1 \) by dividing by a suitable monomial. Furthermore, we can clearly permute the \( a_i \) so that \( c_i > 0 \) iff \( i \leq k \), for some \( k \leq n + 1 \). Let \( A' \) be the matrix whose columns are \( a_2, \ldots, a_{n+1} \). Then, via the change of variables \( x = z/(|c_2|, \ldots, |c_{n+1}|)^{A-1} \) (which clearly preserves the existence of roots of \( f \) in
any open orthant of \((\mathbb{R}^*)^n\), we can then clearly assume that \(f(x) = 1 + x^{a_2} + \cdots + x^{a_k} - x^{a_{k+1}} - \cdots - x^{a_{n+1}}\).

Our criteria for checking the existence of roots of \(f\) in \(\mathbb{R}_+^n\) or \((\mathbb{R}^*)^n\) then clearly reduce to checking whether \(k < n\) or whether \(A'\) has an odd entry. So let us prove that the latter conditions correctly characterize the existence of roots of \(f\) in \(\mathbb{R}_+^n\) and \((\mathbb{R}^*)^n\).

First note that \(f(x) = 0\) for some \(x \in (\mathbb{R}^*)^n\) iff

\[(*) \quad x^{A'} = \alpha \quad \text{and} \quad 1 + \alpha_1 + \cdots + \alpha_k - \alpha_{k+1} - \cdots - \alpha_n = 0,
\]

for some \(\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{R}^*)^n\). Assertion (1) then follows almost trivially: Assuming \(x \in \mathbb{R}_+^n\), the equality \(k = n\) and Proposition 1 imply that \(f(x) = 1 + \alpha_1 + \cdots + \alpha_n > 0\), so there can be no roots for \(f\) in \(\mathbb{R}_+^n\). Taking the inverse implication, suppose \(k < n\). Then we can set

\[
\alpha := \left(1, \ldots, 1, \frac{k+1}{n-k}, \ldots, \frac{k+1}{n-k} \right)
\]

to obtain \(1 + \alpha_1 + \cdots + \alpha_k - \alpha_{k+1} - \cdots - \alpha_n = 0\). So if we can solve \(x^{A'} = \alpha\) over \(\mathbb{R}_+^n\), we will have found a root in \(\mathbb{R}_+^n\) for \(f\). Proposition 1 tells us that we can indeed (since \(\det A' \neq 0\)), so we are done.

We now focus on Assertion (2). Letting \(y := x^V\), note that

\[
x^{A'} = \alpha \iff y^S = y^{UA'V} = (x^{A'})^V = \alpha^V,
\]

thanks to Proposition 1 where \(S = [s_{ij}]\) is an \(n \times n\) diagonal matrix with \(s_{1,1}|s_{2,2}| \cdots |s_{n,n}\).

So we’ll be able to find a root in \((\mathbb{R}^*)^n\) for \(f\) iff the hypothesis is invariant under a common translation of \(a_1, \ldots, a_{n+1}\). So we can assume \(a_1 = 0\) (and \(k = n\) as given), and our hypothesis then translates into \(A' \in \mathbb{Z}^n\) having at least one odd entry. \(A'\) having at least one odd entry then implies that the mod 2 reduction of \(A'\) has positive \((\mathbb{Z}/2\mathbb{Z})\)-rank, and this in turn implies that \(s_{1,1}\) is odd. (Since left and right multiplication by matrices in \(\mathbb{GL}_n(\mathbb{Z})\) preserves \((\mathbb{Z}/2\mathbb{Z})\)-rank, and \(s_{1,1}|s_{2,2}| \cdots |s_{n,n}\).) Since the map \(e_V(x) := x^V\) is clearly an automorphism of the open orthants of \((\mathbb{R}^*)^n\) (provided \(V \in \mathbb{GL}_n(\mathbb{Z})\)), there must then clearly be some open orthant (having exactly \(j\) positive coordinates) which is mapped bijectively onto \(\mathbb{R}_- \times \mathbb{R}_+^{n-1}\) under \(e_V\). So then define \(\alpha\) to be any permutation of the vector

\[
\left(1, \ldots, 1, \frac{-j+1}{n-j}, \ldots, \frac{-j+1}{n-j} \right)
\]

such that \(\text{sign}(\alpha^V) = (-1, 1, \ldots, 1)\). Clearly then, \(y^S = \alpha^V\) has a solution in \((\mathbb{R}^*)^n\) and thus, by (\(\bigtriangledown\)) and our choice of \(\alpha\), \(f\) indeed has a root in \((\mathbb{R}^*)^n\).

(\(\iff\)): We will prove the contrapositive. Via translation invariance again, in the notation above, we see that our hypothesis is equivalent to all the entries of \(A'\) being even, and thus all the \(s_{i,j}\) must be even. We then obtain, via Proposition 1 that \(y^S = \alpha^V\) has no roots in \((\mathbb{R}^*)^n\) unless \(\alpha \in \mathbb{R}_+^n\). But then \(\alpha \in \mathbb{R}_+^n\) implies that \(1 + \alpha_1 + \cdots + \alpha_n > 0\) (since \(k = n\) by assumption), so there can be no roots for \(f\) in \((\mathbb{R}^*)^n\).
Acknowledgements

The authors thank Francisco Santos for discussions on counting regular triangulations, and Frank Sottile for pointing out reference [BBS05]. Thanks also to Dima Pasechnik for discussions, and Sue Geller and Bruce Reznick for detailed commentary, on earlier versions of this work. The authors also thank the anonymous referees for their suggestions, especially the considerable improvement of Assertion (4) of Theorem 1.

References

[BS96] Bach, Eric and Shallit, Jeff, Algorithmic Number Theory, Vol. I: Efficient Algorithms, MIT Press, Cambridge, MA, 1996.

[Bak77] Baker, Alan, “The Theory of Linear Forms in Logarithms,” in Transcendence Theory: Advances and Applications: proceedings of a conference held at the University of Cambridge, Cambridge, January – February, 1976, Academic Press, London, 1977.

[Bar93] Barvinok, Alexander I., “Feasibility testing for systems of real quadratic equations,” Discrete Comput. Geom. 10 (1993), no. 1, pp. 1–13.

[Bar02] Barvinok, Alexander I., “Estimating $L^\infty$ norms by $L^{2k}$ norms for functions on orbits,” Foundations of Computational Mathematics, 2 (2002), pp. 393–412.

[Bas99] Basu, Saugata, “On Bounding the Betti Numbers and Computing the Euler Characteristic of Semi-Algebraic Sets,” Journal of Discrete and Computational Geometry, 22:1-18, (1999).

[BGV03] Basu, Saugata and Gonzalez-Vega, Laureano, Algorithmic and Quantitative Real Algebraic Geometry, Papers from the DIMACS Workshop on Algorithmic and Quantitative Aspects of Real Algebraic Geometry in Mathematics and Computer Science held at Rutgers University, Piscataway, NJ (March 12–16, 2001), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, 60.

[BPR03] Basu, Saugata; Pollack, Ricky; and Roy, Marie-Francoise, Algorithms in Real Algebraic Geometry, Algorithms and Computation in Mathematics, vol. 10, Springer-Verlag, 2003.

[Ber03] Bernstein, Daniel J., “Computing Logarithm Intervals with the Arithmetic-Geometric Mean Iterations,” available from http://cr.yp.to/papers.html.

[BBS05] Bertrand, Benoit; Bihan, Frederic; and Sottile, Frank, “Polynomial Systems with Few Real Zeroes,” Math Z., to appear.

[Bih05] Bihan, Frederic, “Polynomial Systems Supported on Circuits and Dessins D’Enfants,”, Math ArXiV paper math.AG/0509219.

[BRS06] Bihan, Frederic; Rojas, J. Maurice; Sottile, Frank, “Gale Duality and Fewnomial Systems,” preprint.
[Ble04] Blekherman, Grigoriy, “Convexity properties of the cone of nonnegative polynomials,” Discrete Comput. Geom. 32 (2004), no. 3, pp. 345–371.

[BCSS98] Blum, Lenore; Cucker, Felipe; Shub, Mike; and Smale, Steve, Complexity and Real Computation, Springer-Verlag, 1998.

[Can88] Canny, John F., “Some Algebraic and Geometric Computations in PSPACE,” Proc. 20th ACM Symp. Theory of Computing, Chicago (1988), ACM Press.

[Csak76] Csanky, L., “Fast Parallel Matrix Inversion Algorithms,” SIAM J. Comput. 5 (1976), no. 4, pp. 618–623.

[DRRRS05] Dickenstein, Alicia; Rojas, J. Maurice; Rusek, Korben; and Shih, Justin, “A-Discriminants and Extremal Real Algebraic Geometry,” preprint.

[DL79] Dobkin, David and Lipton, Richard, “On the Complexity of Computations Under Varying Sets of Primitives,” J. of Computer and System Sciences 18, pp. 86–91, 1979.

[Ede87] Edelsbrunner, Herbert, Algorithms in combinatorial geometry, EATCS Monographs on Theoretical Computer Science, 10, Springer-Verlag, Berlin, 1987.

[GV04] Gabrielov, Andrei and Vorobjov, Nicolai, “Complexity of computations with Pfaffian and Noetherian functions,” Normal Forms, Bifurcations and Finiteness Problems in Differential Equations, pp. 211–250, Kluwer, 2004.

[GJ79] Garey, Michael R. and Johnson, David S. Computers and Intractability: A Guide to the Theory of NP-Completeness, A Series of Books in the Mathematical Sciences, W. H. Freeman and Co., San Francisco, Calif., 1979, x+338 pp.

[GKZ94] Gel’fand, Israel Moseyevitch; Kapranov, Misha M.; and Zelevinsky, Andrei V.; Discriminants, Resultants and Multidimensional Determinants, Birkhäuser, Boston, 1994.

[Ili89] Iliopoulos, Costas S., “Worst Case Complexity Bounds on Algorithms for Computing the Canonical Structure of Finite Abelian Groups and the Hermite and Smith Normal Forms of an Integer Matrix,” SIAM Journal on Computing, 18 (1989), no. 4, pp. 658–669.

[KS99] Karpinski, Marek and Shparlinski, Igor, “On the computational hardness of testing square-freeness of sparse polynomials,” Applied algebra, algebraic algorithms and error-correcting codes (Honolulu, HI, 1999), pp. 492–497, Lecture Notes in Comput. Sci., 1719, Springer, Berlin, 1999.

[Kho91] Khovanski, Askold, Fewnomials, AMS Press, Providence, Rhode Island, 1991.

[LM01] Lickteig, Thomas and Roy, Marie-Francoise, “Sylvester-Habicht Sequences and Fast Cauchy Index Computation,” J. Symbolic Computation (2001) 31, pp. 315–341.

[LRW03] Li, Tien-Yien; Rojas, J. Maurice; and Wang, Xiaoshen, “Counting Real Connected Components of Trinomial Curve Intersections and m-nomial Hypersurfaces,” Discrete and Computational Geometry, 30 (2003), no. 3, pp. 379–414.
[Mig92] Mignotte, Maurice, *Mathematics for Computer Algebra*, translated from the French by Catherine Mignotte, Springer-Verlag, New York, 1992.

[Pap95] Papadimitriou, Christos H., *Computational Complexity*, Addison-Wesley, 1995.

[Par03] Parrilo, Pablo A., “Semidefinite programming relaxations for semialgebraic problems,” Algebraic and geometric methods in discrete optimization, Math. Program. 96 (2003), no. 2, Ser. B, pp. 293–320.

[Per05] Perrucci, Daniel, “Some Bounds for the Number of Connected Components of Real Zero Sets of Sparse Polynomials,” Discrete and Computational Geometry, vol. 34, no. 3 (sept. 2005), pp. 475–495.

[PR05] Poonen, Bjorn and Rojas, J. Maurice, “From Algebraic to Quantum Complexity via Sparse Polynomials,” preprint.

[Roj04] Rojas, J. Maurice, “Arithmetic Multivariate Descartes’ Rule,” American Journal of Mathematics, vol. 126, no. 1, February 2004, pp. 1–30.

[RY05] Rojas, J. Maurice and Ye, Yinyu, “On Solving Sparse Polynomials in Logarithmic Time,” Journal of Complexity, special issue for the 2002 Foundations of Computation Mathematics (FOCM) meeting, February 2005, pp. 87-110.

[Sch00] Schmid, Joachim, “On the Complexity of the Real Nullstellensatz in the 0-Dimensional Case,” J. Pure Appl. Algebra 151 (2000), no. 3, pp. 301–308.

[Sho91] Shor, Peter, “Stretchability of Pseudolines is NP-hard,” Applied Geometry and Discrete Mathematics – The Victor Klee Festschrift (P. Gritzmann, B. Sturmfels, eds.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Amer. Math. Soc., Providence, RI, 4 (1991), pp. 531–554.

[Shp06] Shparlinski, Igor, e-mail communication, received January 6, 2006.

[Sma87] Smale, Steve, “On the Topology of Algorithms I,” Journal of Complexity 3 (1987), no. 2, pp. 81–89.

[Sma00] Smale, Steve, “Mathematical Problems for the Next Century,” Mathematics: Frontiers and Perspectives, pp. 271–294, Amer. Math. Soc., Providence, RI, 2000.

[Sto98] Storjohann, Arne, “Computing Hermite and Smith normal forms of triangular integer matrices,” Linear Algebra Appl. 282 (1998), no. 1–3, pp. 25–45.

[Ste74] Stengle, G., “A nullstellensatz and a positivstellensatz in semialgebraic geometry,” Math. Ann. 207 (1974) pp. 87–97.