Electromagnetic interaction in theory with Lorentz invariant CPT violation

Masud Chaichian*, Kazuo Fujikawa† and Anca Tureanu*

*Department of Physics, University of Helsinki, P.O.Box 64, FIN-00014
Helsinki, Finland
† Mathematical Physics Laboratory, RIKEN Nishina Center,
Wako 351-0198, Japan

Abstract

An attempt is made to incorporate the electromagnetic interaction in a Lorentz invariant but CPT violating non-local model with particle-antiparticle mass splitting, which is regarded as a modified QED. The gauge invariance is maintained by the Schwinger non-integrable phase factor but the electromagnetic interaction breaks C, CP and CPT symmetries. Implications of the present CPT breaking scheme on the electromagnetic transitions and particle-antiparticle pair creation are discussed. The CPT violation such as the one suggested here may open a new path to the analysis of baryon asymmetry since some of the Sakharov constraints are expected to be modified.

1 Introduction

The local field theory defined in Minkowski space-time is very successful, and CPT symmetry is a fundamental symmetry of any such theory [1]. Nevertheless, the possible breaking of CPT symmetry has also been discussed. One of the logical ways to break CPT symmetry is to make the theory non-local by preserving Lorentz symmetry, while the other is to break Lorentz symmetry itself. The Lorentz symmetry breaking scheme has been mainly studied in the past, including its physical implications [2, 3]. A proposal of Lorentz invariant CPT breaking scheme is relatively new [4] and its logical consistency has also been emphasized [5]. But its physical implications have not been analyzed except for the recent proposal of an explicit Lagrangian model of particle–antiparticle mass splitting [6] and its application to the neutrino–antineutrino...
mass splitting in the Standard Model [7]. It was emphasized there that only
the neutrino mass terms in the Standard Model can preserve the basic local
$SU(2)_L \times U(1)$ gauge symmetry in the Lorentz invariant non-local CPT
breaking scheme without introducing non-integrable phase factors. From the
point of view of particle phenomenology, this uniqueness of the neutrino mass
splitting in the Standard Model is quite interesting [8, 9, 10].

If one wants to accommodate the non-local Lorentz invariant CPT breaking
mechanism in the couplings of general elementary particles, one needs to go
beyond the conventional local gauge principle by incorporating the Schwinger
non-integrable phase factor. (This non-integrable phase factor is also known
as the Wilson-line integral in lattice gauge theory, and we use the terms
Schwinger’s factor, non-integrable phase factor and Wilson-line interchange-
ably in the present Letter.)

To be specific, we adopt the simplest Lorentz invariant and non-local CPT
breaking Hermitian Lagrangian [6]:

\[
S = \int d^4x \left\{ \bar{\psi}(x)i\gamma^\mu \partial_\mu \psi(x) - m\bar{\psi}(x)\psi(x) \right. \\
- i\mu \int d^4y \left[ \theta(x^0 - y^0) - \theta(y^0 - x^0) \right] \delta((x - y)^2 - l^2) \left[ \bar{\psi}(x)\psi(y) \right]
\]

as a model Lagrangian of a charged fermion ("electron") and study its electro-
magnetic interactions. For the real parameter $\mu$, the third term has $C = CP = CPT = -1$ and thus no symmetry to ensure the equality of particle and
antiparticle masses. The dimension of $\mu$ depends on the choice of the non-local
factor $\delta((x - y)^2 - l^2)$ and in the present case it is $[M]^3$, while $l$ has dimension
of length.

The free equation of motion for the fermion is

\[
i\gamma^\mu \partial_\mu \psi(x) = m\psi(x) \\
+i\mu \int d^4y \left[ \theta(x^0 - y^0) - \theta(y^0 - x^0) \right] \delta((x - y)^2 - l^2) \psi(y).
\]

By inserting an Ansatz for the possible solution, $\psi(x) = e^{-ipx}U(p)$, we obtain

\[
\not{p}U(p) = mU(p) + i\mu[f_+(p) - f_-(p)]U(p),
\]

where $f_\pm(p)$ is a Lorentz invariant "form factor" defined by

\[
f_\pm(p) = \int d^4z_1 e^{\pm ipz_1} \theta(z_1^0) \delta((z_1)^2 - l^2),
\]
which are inequivalent for time-like $p$ due to the factor $\theta(z_0)$. By assuming a time-like $p$, we go to the frame where $\vec{p} = 0$. Then the eigenvalue equation for the mass is given by

\[ p_0 = \gamma_0 \left[ m - 4\pi\mu \int_0^\infty dz z^2 \frac{\sin[p_0 \sqrt{z^2 + l^2}]}{\sqrt{z^2 + l^2}} \right], \tag{5} \]

where we used the explicit formula

\[ f_{\pm}(p^0) = 2\pi \int_0^\infty dz z^2 e^{\pm i p_0 \sqrt{z^2 + l^2}} \frac{\sqrt{z^2 + l^2}}{\sqrt{z^2 + l^2}}. \tag{6} \]

This eigenvalue equation under $p_0 \to -p_0$ becomes (after sandwiching by $\gamma_5$)

\[ p_0 = \gamma_0 \left[ m + 4\pi\mu \int_0^\infty dz z^2 \frac{\sin[p_0 \sqrt{z^2 + l^2}]}{\sqrt{z^2 + l^2}} \right], \tag{7} \]

which is not identical to the original equation in (5). This causes the mass splitting of particle and antiparticle in the sense of Dirac. One may solve the mass eigenvalue equations iteratively by assuming that the terms with the parameter $\mu$ are much smaller than $m$. One then obtains the mass eigenvalues at

\[ m_{\pm} \simeq m \pm 4\pi\mu \int_0^\infty dz z^2 \frac{\sin[m \sqrt{z^2 + l^2}]}{\sqrt{z^2 + l^2}}, \tag{8} \]

where the upper two (positive) components of the matrix $\gamma_0$ in (5) and (7) are used. See Ref. [6] for further details.

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\[ \int_0^\infty dz z^2 \frac{\sin[p_0 \sqrt{z^2 + l^2}]}{\sqrt{z^2 + l^2}} = -\frac{\partial^2}{\partial p_0^2} \int_0^\infty dz z^2 \frac{\sin[p_0 \sqrt{z^2 + l^2}]}{[z^2 + l^2]^{3/2}}. \]
2 Electromagnetic interaction – modified QED

To introduce the electromagnetic interaction in (1), we consider the simplest scheme (a modified QED):

\[ S = \int d^4x \left\{ \bar{\psi}(x)i\gamma^\mu D_\mu \psi(x) - m\bar{\psi}(x)\psi(x) \right\} \]

\[ -i\mu \int d^4y \left[ \theta(x^0 - y^0) - \theta(y^0 - x^0) \right] \delta((x - y)^2 - l^2) \]

\[ \times \bar{\psi}(x) \exp \left[ ie \int^x_y A_\mu(z)dz^\mu \right] \psi(y) \]

\[ -\frac{1}{4} \int d^4xF_{\mu\nu}(x)F^{\mu\nu}(x), \]

with

\[ D_\mu = \partial_\mu - ieA_\mu(x). \]

We added the Schwinger non-integrable phase factor

\[ \exp \left[ ie \int^x_y A_\mu(z)dz^\mu \right], \]

(11)

to make the non-local term gauge invariant. This action is invariant under the gauge transformation

\[ \psi(x) \rightarrow e^{i\alpha(x)}\psi(x), \]

\[ A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x), \]

(12)

and the C, CP and CPT transformation properties of each term in the action (8) are the same as in the theory without electromagnetic couplings.

It is natural to consider the non-integrable phase factor in (8) as an independent dynamical entity rather than a given external factor. In fact, Y. Nambu emphasized in many occasions the non-integrable phase factor as a manifestation of string-like objects appearing in the theory.

Our proposal here is to replace the non-integrable phase factor in (8) by a first quantized very massive particle propagation defined by the covariant path integral

\[ \exp \left[ ie \int^x_y A_\mu(z)dz^\mu \right] \delta_{\alpha,\beta} \Rightarrow \]

\[ \int Dz^\mu \exp \left\{ i \int^x_y \frac{1}{2} \left[ (\dot{z}^\mu)^2 + M^2 \right] d\tau + ie \int^x_y A_\mu(z)\frac{dz^\mu}{d\tau} d\tau \right\} \delta_{\alpha,\beta}, \]
where the factor $\delta_{\alpha, \beta}$ contracts the spinor indices, an analogue of the Chan-Paton factor in string theory. In this way, the non-integrable phase factor becomes more dynamical and the flow of the charge is visualized in Feynman diagrams, although the second quantized particle and the first quantized particle appear in a mixed manner in Feynman diagrams. This use of a semi-static massive particle for the non-integrable phase factor is common in lattice gauge theory.

As for the quantization of the theory non-local in time, we adopt the path integral on the basis of Schwinger’s action principle, which is based on the equations of motion $\Box$.

3 Current conservation and Ward–Takahashi identity

One may examine the fermion pair creation through the lowest order electromagnetic interaction, for example, to study the implications of the fermion and antifermion mass splitting on the pair production. To the lowest order in $O(\epsilon)$, one may expand the non-integrable phase factor as

$$\exp \left[ ie \int_y^x A_\mu(z) dz^\mu \right] = 1 + ie \int_y^x A_\mu(z) dz^\mu.$$  

The interaction part for the lowest order pair creation is given by

$$S_I = e \int d^4 x \bar{\psi}(x) \gamma^\mu A_\mu(x) \psi(x)$$

$$+ e \mu \int d^4 x d^4 y [\theta(x^0 - y^0) - \theta(y^0 - x^0)] \delta((x - y)^2 - l^2)$$

$$\times \bar{\psi}(x) \left[ \int_y^x A_\mu(z) dz^\mu \right] \psi(y)$$

$$\times \bar{\psi}(x) \int_{\tau_y}^{\tau_x} \left[ \delta^4(z(\tau) - w) \frac{dz^\mu}{d\tau} \right] d\tau \psi(y),$$  

and the electromagnetic current is

$$J^\mu(w) = \frac{\delta}{\delta A_\mu(w)} S_I$$

$$= e \bar{\psi}(w) \gamma^\mu \psi(w)$$

$$+ e \mu \int d^4 x d^4 y [\theta(x^0 - y^0) - \theta(y^0 - x^0)] \delta((x - y)^2 - l^2)$$

$$\times \bar{\psi}(x) \int_{\tau_y}^{\tau_x} \left[ \delta^4(z(\tau) - w) \frac{dz^\mu}{d\tau} \right] d\tau \psi(y),$$  

$$\Box$$.
where \( z^\mu(\tau) \) stands for the coordinate of the massive particle. The current conservation condition becomes

\[
\partial_\mu J^\mu(w) = e \partial_\mu [ \bar{\psi}(w) \gamma^\mu \psi(w) ]
\]

\[
+ \frac{e}{\mu} \int d^4x d^4y [ \theta(x^0 - y^0) - \theta(y^0 - x^0) ] \delta((x - y)^2 - l^2) \times \bar{\psi}(x) \int_\tau \left[ \frac{\partial}{\partial \mu} \delta^4(z(\tau) - w) \frac{dz^\mu}{d\tau} \right] d\tau \psi(y)
\]

\[
= e[ \bar{\psi}(w) \partial \psi(w) ] - e(-\partial_\mu \bar{\psi}(w) \gamma^\mu \psi(w))
\]

\[
- e\mu \int d^4x d^4y [ \theta(x^0 - y^0) - \theta(y^0 - x^0) ] \delta((x - y)^2 - l^2) \times \bar{\psi}(x) \delta^4(x - w) - \delta^4(y - w) \psi(y)
\]

\[
= e[ \bar{\psi}(w) \partial \psi(w) ] - e(-\partial_\mu \bar{\psi}(w) \gamma^\mu \psi(w))
\]

\[
- e\mu \int d^4y [ \theta(w^0 - y^0) - \theta(y^0 - w^0) ] \delta((w - y)^2 - l^2) \bar{\psi}(w) \psi(y)
\]

\[
+ e\mu \int d^4x [ \theta(x^0 - w^0) - \theta(w^0 - x^0) ] \delta((x - w)^2 - l^2) \bar{\psi}(x) \psi(w),
\]

which in fact vanishes if one uses the free equation of motion for the fermion in (2) and its conjugate. Here we used the relation

\[
\frac{\partial}{\partial \mu} \delta^4(z(\tau) - w) \frac{dz^\mu}{d\tau} = -\frac{d}{d\tau} \delta^4(z(\tau) - w).
\]

If one remembers the inclusion of the path integral for the (free) massive particle, one should actually write in the current (15)

\[
\delta^4(z(\tau) - w) \frac{dz^\mu}{d\tau} \Rightarrow \frac{1}{Z} \int d^4z' \delta^4(z'(\tau) - w) \langle x, \tau_x | z', \tau' \frac{dz'^\mu}{d\tau} \rangle \langle z', \tau | y, \tau_y \rangle
\]

\[
= \frac{1}{Z} \langle x, \tau_x | \delta^4(\hat{z}(\tau) - w) \frac{d}{d\tau} \hat{z}^\mu(\tau) | y, \tau_y \rangle,
\]

where \( Z = \langle x, \tau_x | y, \tau_y \rangle \) is the normalization factor of the path integral partition function, and the last expression is given in the operator notation in the interaction picture, since we are working in the lowest order of the electromagnetic coupling in (14) in the first quantized path integral. Note that

\[
\hat{z}^\mu(\tau) = e^{i\hat{H}_0 \tau} \hat{z}^\mu(0) e^{-i\hat{H}_0 \tau},
\]

\[
\hat{H}_0 = \frac{1}{2} \left[ \hat{P}_\mu(0) - M^2 \right],
\]

(19)
with $[\hat{P}_\mu(0), \hat{z}'(0)] = i\hbar g_{\mu\nu}$. With the replacement in (18), one still obtains the same conservation relation as in (16) if one notes the relation

$$\delta^4(\hat{z}(\tau_y) - w)|y, \tau_y\rangle = \delta^4(y - w)|y, \tau_y\rangle,$$

for example.

The first term in (14) is invariant under C (and in fact under CP and CPT), while the second term is confirmed to be odd under the conventional charge conjugation symmetry C (and in fact odd under CP and CPT also). Thus the electromagnetic interaction breaks those basic symmetries slightly. Nevertheless, this interaction is invariant under the gauge transformation $A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x)$ if one uses the equation of motion for the free fermion field in (2) and its conjugate, as was already explained. This gauge invariance ensures the Ward–Takahashi identity for the three-point vertex function in the form

$$\langle T^* \psi(u) S_I(A_\mu = \frac{1}{e} \partial_\mu \alpha(x) \psi(w) \rangle = \int d^4x \langle T^* \psi(u) \bar{\psi}(x) \rangle \frac{1}{e} \alpha(x) \delta(x - w)$$

$$- \int d^4x \delta(u - x) \frac{1}{e} \alpha(x) \langle T^* \psi(x) \bar{\psi}(w) \rangle$$

in the interaction picture, since the interaction part $S_I$ in (14) is defined in the lowest order in the electromagnetic coupling and the current conservation condition (16) is satisfied by using the free equations of motion of the fermion field. The free propagator is defined by the inverse of the free equation of motion in (2), namely,

$$\langle T^* \psi(x) \bar{\psi}(y) \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{\not{p} - m + i\epsilon - i\mu [f_+(p) - f_-(p)]}.$$

The full set of Ward-Takahashi identities, i.e. the relations among different Green’s functions for the presented modified QED, can be derived formally in the present path integral quantization, but the exact current becomes more involved than (15) due to the presence of the non-integrable phase factor in the full action (9). An analysis of higher order effects in the electromagnetic coupling in the presence of the non-integrable phase factor even in the lowest order of the CPT-violation parameter $\mu$ is an interesting subject of future study.

## 4 Fermion pair creation

We now consider the charged particle pair creation from a virtual photon

$$\gamma(k) \rightarrow e(p) + \bar{e}(\bar{p}).$$

(23)
The current matrix element in the momentum space is given by using the current in (15)

\[ J^\mu(k) = \int d^4w e^{-ikw} \langle p, \bar{p} | J^\mu(w) | 0 \rangle \]

\[ = (2\pi)^4 \delta(k - p - \bar{p}) e\bar{u}(p) \gamma^\mu v(\bar{p}) \]

\[ + e\mu \int d^4x d^4y e^{i\bar{p}y + ipx} \theta(x^0 - y^0) - \theta(y^0 - x^0)) \delta((x - y)^2 - l^2) \]

\[ \times \bar{u}(p) \int_{\tau_y}^{\tau_x} e^{-ikz(\tau)} \frac{dz^\mu}{d\tau} d\tau v(\bar{p}), \] (24)

where we used the solutions \( u(p) \) and \( v(\bar{p}) \) of the modified Dirac equation (3) with masses in (8), and the representation

\[ \delta^4(z(\tau) - w) = \int \frac{d^4q}{(2\pi)^4} e^{iq[z(\tau) - w]}, \] (25)

Here we use the original expression of the current in (15) without the quantum fluctuation of the non-integrable phase factor; this is because the dependence of the path integral normalization factor \( Z = \langle x, \tau_x | y, \tau_y \rangle \) on the coordinates of the end-points complicates the evaluation, although it does not make it impossible. We thus employ the straight-line path between the two end-points:

\[ z^\mu(\tau) = (x^\mu - y^\mu) \frac{\tau - \tau_y}{\tau_x - \tau_y} + y^\mu \] (26)
and we evaluate
\[
\int d^4x d^4y e^{ipx} [\theta(x^0 - y^0) - \theta(y^0 - x^0)] \delta((x - y)^2 - l^2) \tag{27}
\]
\[
\times \int\limits_{\tau_y}^{\tau_x} d\tau \exp\{-i[(kx - ky) \frac{\tau - \tau_y}{\tau_x - \tau_y} + ky]\} \left\{\frac{x^\mu - y^\mu}{\tau_x - \tau_y}\right\}
\]
\[
= \int_0^1 d\eta \int d^4x d^4y e^{ipx} [\theta(x^0 - y^0) - \theta(y^0 - x^0)] \delta((x - y)^2 - l^2)
\times \exp\{-i[k(x - y)\eta + ky]\}(x^\mu - y^\mu)
\]
\[
= -i \left(\frac{\partial}{\partial p_\mu} - \frac{\partial}{\partial \bar{p}_\mu}\right) \int_0^1 d\eta \int d^4x d^4y e^{ipx} [\theta(x^0 - y^0) - \theta(y^0 - x^0)]
\times \delta((x - y)^2 - l^2) \exp\{-i[k(x - y)\eta + ky]\}
\]
\[
= -i(2\pi)^4 \delta(k - p - \bar{p}) \left(\frac{\partial}{\partial p_\mu} - \frac{\partial}{\partial \bar{p}_\mu}\right) \int_0^1 d\eta \int d^4u [\theta(u^0) - \theta(-u^0)] \delta(u^2 - l^2) \exp\{-iku(\eta - 1) - i\bar{p}u\}
\]
\[
= (2\pi)^4 \delta(k - p - \bar{p}) \left(-i \frac{\partial}{\partial \bar{p}_\mu}\right) \int_0^1 d\eta [f_+(k(\eta - 1) + \bar{p}) - f_-(k(\eta - 1) + \bar{p})],
\]
where we defined \(\eta = \frac{\tau - \tau_y}{\tau_x - \tau_y}\) and \(u = x - y\), and used the form factor defined in (4). We thus have the current (by suppressing the factor \((2\pi)^4 \delta(k - p - \bar{p}))
\[
J^\mu(k) = e\bar{u}(p)\gamma^\mu v(\bar{p}) + e\bar{\mu}\bar{u}(p)F^\mu(p, \bar{p})v(\bar{p})
\tag{28}
\]
with
\[
F^\mu(p, \bar{p}) \equiv \left\{-i \frac{\partial}{\partial \bar{p}_\mu}\right\} \int_0^1 d\eta [f_+(k(\eta - 1) + \bar{p}) - f_-(k(\eta - 1) + \bar{p})] \bigg|_{k=p+\bar{p}}. \tag{29}
\]
We have a small correction \(F^\mu(p, \bar{p})\) to the electromagnetic current, which flips chirality (and thus it is similar to the Pauli term) and violates C, CP and CPT. Note that the first term in (28) alone is not conserved due to the mass splitting, but the first and second terms in (28) put together are conserved, as eq. (16) indicates.

## 5 Discussion

It is interesting that the gauge invariance is maintained in Lorentz invariant non-local theory (9) by a scheme apparently different from that in local theory.
The crucial point is that the equality of masses does not play any essential role in this analysis of gauge invariance. This gauge invariance is somewhat analogous to the gauge invariance of the Pauli term in the ordinary coupling of the photon to charged fermions. Another interesting aspect is that the fermion line is not continuous in space-time in the non-local term, as is seen from the expression of \( S_I \) in (14), which is something new in view of the Feynman’s picture of charged particle propagation in space-time. To reconcile this discontinuous world-line in space-time with Feynman’s picture, we have suggested that the non-integrable phase factor represents a very massive semi-static fermion propagation in the first quantization picture, which is common in lattice gauge theory. In this way, one can maintain the manifestly continuous flow of the charged particle and thus the continuous flow of the electric current.

Physically one may argue that the presence of this (indefinite) very massive particle in the intermediate stage of the flow of the charge allows for the possible appearance of the mass difference between the particle and antiparticle, which is consistent with gauge invariance, in our Lorentz invariant CPT breaking model.

As for practical implications of CPT breaking in the present modified QED, the search for the mass splitting of particle and antiparticle, just as the search for the neutrino antineutrino mass splitting in oscillation experiments [10], is interesting [12, 13]. In the atomic transitions of the matter or antimatter systems, the frequency differences caused by the small mass difference between the “electron” and “positron” such as in (8) will be important. Other possibilities are to look for the possible small C and CP breaking in electromagnetic interactions other than those caused by weak interactions. The effects of unitarity breaking are expected to be minimal if one considers the processes lowest order in the small CPT breaking non-local term.

The analysis in this Letter suggests the following questions:

i) The parameter \( \mu \) in eqs. (1) and (9) controls C, CP and CPT breaking,
while Lorentz and gauge invariance are maintained. However, C violation has not been observed in electromagnetic interactions. What bound does this impose on the mass splitting?

\textit{ii)} Is this bound also reflected as a lower mass for the static heavy fermion?

\textit{iii)} Can this mechanism be used to generate the baryon asymmetry of the Universe in equilibrium (with CPT violation, the three Sakharov conditions can be ignored) and what does such a case imply for the parameters of the model?

We thank the anonymous Referee for raising the above basic questions, and we plan to address those issues in a future communication.

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