Extremal graphs for edge blow-up of graphs

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Abstract

Given a graph $H$ and an integer $p$, the edge blow-up of $H$, denoted as $H^{p+1}$, is the graph obtained from replacing each edge in $H$ by a clique of size $p+1$ where the new vertices of the cliques are all different. The Turán numbers for edge blow-up of matchings were first studied by Erdős and Moon. In this paper, we determine the Turán numbers for edge blow-up of general graphs.

Key words: Turán number; Edge blow-up.

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1 Introduction

The Turán number of a graph $H$, $\text{ex}(n,H)$, is the maximum number of edges in a graph $G$ of order $n$ which does not contain $H$ as a subgraph. Denote by $\text{EX}(n,H)$ the set of graphs on $n$ vertices with $\text{ex}(n,H)$ edges containing no $H$ as a subgraph and call a graph in $\text{EX}(n,H)$ an extremal graph for $H$.

In 1941, Turán [18] proved that the extremal graph without containing $K_{p+1}$ as a subgraph is the complete $p$-partite graph on $n$ vertices which is balanced, in that the part sizes are as equal as possible (any two sizes differ by at most 1). This balanced complete $p$-partite graph on $n$ vertices is the Turán graph $T_p(n)$ and denote $t_p(n) = e(T_p(n))$.

In many ordinary extremal problems the minimum chromatic number plays a decisive role. Let $\mathcal{F}$ be a family of graphs, the subchromatic number $p(\mathcal{F})$ of $\mathcal{F}$ is defined by

$$p(\mathcal{F}) = \min \{\chi(F) : F \in \mathcal{F}\} - 1,$$

where $\chi(F)$ is the chromatic number of $F$. The classical Erdős-Stone-Simonovits theorem [5, 8] states that

$$\text{ex}(n, \mathcal{F}) = \left(1 - \frac{1}{p(\mathcal{F})}\right) \binom{n}{2} + o(n^2).$$

If $\mathcal{F}$ contains some bipartite graphs, then $p(\mathcal{F}) = 1$ and $\text{ex}(n, \mathcal{F}) = o(n^2)$. For this degenerate (bipartite) extremal graph problem, there is an excellent survey by Füredi and
Simonovits [10]. For non-bipartite graphs, let $G$ be a graph with $\chi(G) = p + 1$. If there is an edge $e$ such that $\chi(G - \{e\}) = p$, then we say that $G$ is edge-critical $(p + 1)$-chromatic and $e$ is a critical edge. The Turán number of those graphs are determined provided $n$ is sufficiently large. In 1968, Simonovits [15] proved the following theorems.

**Theorem 1.2** (Simonovits [15]) Let $F_1, \ldots, F_\ell$ be given graphs, such that $\chi(F_i) \geq p + 1$ $(i = 1, \ldots, \ell)$ but there are an $F_i$ and an edge $e$ in it such that $\chi(F_i - \{e\}) = p$. Then there exists an $n_0$ such that if $n > n_0$ then $T_p(n)$ is the only extremal graph for $F_1, \ldots, F_\ell$.

It is a challenge of determining the exact Turán function for more non-bipartite graphs, although the Turán function of non-bipartite graphs is asymptotically determined by Erdős-Stone-Simonovits theorem. There are only few graphs whose Turán number were determined exactly, including edge-critical graphs [15] and some special graphs [3, 17, 19].

**Definition 1.3** (Simonovits [16]) Given a family $\mathcal{F}$, let $M := \mathcal{M}(\mathcal{F})$ be the family of minimal graphs $M$ that satisfy the following: there exist an $F \in \mathcal{F}$ and a $t = t(F)$ such that $F \subset M + T_{p-1}(pt - t)$. We call $M$ the decomposition family of $\mathcal{F}$.

Thus, a graph $M$ is in $\mathcal{M}$ if the graph obtained from putting an $M$ (but not any of its proper subgraphs) into a class of a large $T_p(n)$ contains some $F \in \mathcal{F}$. If $F \in \mathcal{F}$ with minimum chromatic number $p + 1$, then $F \subset T_{p-1}(pt - t)$ for some $t \geq 1$, therefore the decomposition family $M$ always contains some bipartite graphs. A deep general theorem of Simonovits [16] shows that if the decomposition family $M(\mathcal{F})$ of $\mathcal{F}$ contains a graph $M$ which is a subgraph of a path, then the extremal graph for $\mathcal{F}$ have very simple and symmetric structure. Our theorems focus on the graphs whose decomposition family contains a matching. Hence it is a refinement of Simonovits’ theorem in a certain sense. The main purpose of this paper is to determine the Turán numbers of graphs whose decomposition family contains a matching and find new families of graphs whose extremal graphs are determined when the subchromatic number of the family of graphs is greater than one.

Given a graph $H$ and an integer $p$, the edge blow-up of $H$, denoted by $H^{p+1}$, is the graph obtained from replacing each edge in $H$ by a clique of order $p + 1$ where the new vertices of the cliques are all different. The subscript in the case of graphs indicates the number of vertices, e.g., denote by $K_n$ a complete graph on $n$ vertices, $K_i$ a path on $i$ vertices, $K_{i_1, \ldots, i_p}$ the complete $p$-partite graph with part sizes $n_1, \ldots, n_p$. A matching in $G$ is a set of edges from $E(G)$, no two of which share a common vertex, and the matching number of $G$, denoted by $\nu(G)$, is the number of edges in a maximum matching. Denote by $M_{2k}$ the disjoint union of $k$ disjoint copies of edges.

In 1959, Erdős and Gallai [6] determined the extremal graphs for $M_{2k}$. Later, Erdős [7] determined the extremal graphs for $M_{2k}^p$ and Moon [14] determined the extremal graphs for $M_{2k}^{p+1}$ for infinite value of $n$ when $p \geq 3$. Simonovits [15] determined the extremal graphs for $M_{2k}^{p+1}$ when $p \geq 3$ and $n$ is sufficiently large. Erdős, Füredi, Gould and Gunderson [9] determined the Turán number of $S^{p+1}$ when $p \geq 3$. Glebov [12] determined the extremal graphs for edge blow-up of paths. Later, Liu [13] generalized Glebov’s result to edge blow-up of paths, cycles and a class of trees. We will generalize their results.
To describe our main theorems and related results, we first introduce more results about the degrees and the matchings of graphs.

Define \( f(\nu, \Delta) = \max\{|E(G)| : \nu(G) \leq \nu, \Delta(G) \leq \Delta\} \). In 1972, Abbott, Hanson and Sauer [1] determined \( f(k - 1, k - 1) \). Later Chvátal and Hanson [4] proved the following theorem.

**Theorem 1.4** [4] For every \( \nu \geq 1 \) and \( \Delta \geq 1 \),

\[
 f(\nu, \Delta) = \nu \Delta + \left\lfloor \frac{\nu}{\Delta/2} \right\rfloor \left\lfloor \frac{\nu}{\Delta/2} \right\rfloor \leq \nu \Delta + \nu.
\]

In 2009, based on Gallai’s Lemma [11], Balachandran and Khar [2] gave a more ‘structural’ proof of this result. Hence they gave a simple characterization of all the cases where the extremal graph is unique. Denote by \( E_{\nu,\Delta} \) the set of the extremals graph in Theorem 1.4.

Denote \( H(n, p, s, \nu, \Delta, B) \) the set of graphs which are obtained by taking an \( H'(n, p, s) \), putting an \( E_{\nu,\Delta} \in E_{\nu,\Delta} \) in one class of \( T_p(n - s + 1) \) and putting a \( Q_{s-1} \in EX(s-1, B) \) in \( K_{s-1} \).

In the rest of this paper, for any connected bipartite graph \( G \), denote by \( A \) and \( B \) its two color class with \(|A| \leq |B|\). Moreover, if \( G \) is disconnected, we always chose \( A \) such that \(|A|\) is as small as possible. We will establish the following theorems.

**Theorem 1.5** Let \( G \) be a bipartite graph, \( M = M(G^{p+1}) \) and \( q(M) = q \). Let \( k = \min\{d_H(x) : x \in S, S \in S(M)\} \). Let \( n \) be sufficiently large.

(i). If \( q = |A| \), then

\[
 h'(n, p, |A|) \leq ex(n, G^{p+1}) \leq h(n, p, |A|) + f(k - 1, k - 1).
\]
Furthermore, both bounds are best possible.

(ii). Let $B = B(M)$. If $q < |A|$, then

$$\text{ex}(n, G^{p+1}) = h'(n, p, q) + \text{ex}(q - 1, B)$$

**Theorem 1.6** Let $G$ be a non-bipartite graph with $3 \leq \chi(G) \leq p - 1$ and $M = M(G^{p+1})$. Let $q = q(M)$ and $B = B(M)$. If $n$ is sufficiently large, then

$$\text{ex}(n, G^{p+1}) = h'(n, p, q) + \text{ex}(q - 1, B),$$

Moreover, the graphs in $H(n, p, q, 0, 0, B)$ are the only extremal graphs for $G^{p+1}$.

## 2 Several technical lemmas.

Given a graph $H$, a vertex split on some vertex $v \in V(H)$ is defined as follows: replace $v$ by an independent set of size $d(v)$, say $v_1, v_2, \ldots, v_{d(v)}$, in which each vertex is adjacent to exactly one distinct vertex in $N_H(v)$. Denote by $H(H)$ the family of graphs that can be obtained from $H$ by applying vertex split on some $U \subseteq V(H)$. Obviously each graph in $H(H)$ has $e(H)$ number of edges. Note that $U$ could be empty, therefore $H \in H(H)$. For example, $H(P_{k+1})$ is the family of all linear forests with $k$ edges and $H(C_k)$ is consist of $C_k$ and all linear forests with $k$ edges (a linear forest is a forest whose connected components are paths).

The following lemma is proved in [13].

**Lemma 2.1** (Liu [13]) Given $p \geq 3$ and any graph $H$ with $\chi(H) \leq p - 1$, we have $\mathcal{M}(H^{p+1}) = H(H)$, in particular, a matching of size $e(H)$ is in $\mathcal{M}(H^{p+1})$.

**Proposition 2.2** Let $G$ be a bipartite graph. Then $q(G) = |A|$.

**Proof.** Since $A$ is an independent vertex set which covers $G$, we have $q(G) \leq |A|$. Suppose $G$ is connected, the independent vertex covering set must contain all the vertices of $A$ or all the vertices of $B$. In fact, let $A_1 \subseteq A$, $B_1 \subseteq B$ be two non-empty vertex sets and $A_1 \cup B_1$ be an independent vertex covering set of $G$. Let $A_2 = A - A_1$ and $B_2 = B - B_1$. Since $G$ is connected, there is some edge between $A_2$ and $B_2$, contradicting that $A_1 \cup B_1$ is a vertex covering set of $G$. Hence we have $q(G) = |A|$. If $G$ is disconnected, the result follows easily by studying each component of $G$ (recall that we always partition $G$ with $|A|$ as small as possible). The proof is completed. □

**Lemma 2.3** Let $G$ be a bipartite graph, $\mathcal{M} = \mathcal{M}(G^{p+1})$ and $q(\mathcal{M}) = q$. Let $k = \min\{d_H(x) : x \in S, S \in S(\mathcal{M})\}$ and $n$ be sufficiently large. If $q = |A|$, then

$$h'(n, 1, q) \leq \text{ex}(n, \mathcal{M}) \leq h(n, 1, q) + f(k - 1, k - 1).$$

Furthermore, both bounds are best possible. If $q < |A|$, then

$$\text{ex}(n, \mathcal{M}) = h'(n, 1, q) + \text{ex}(q - 1, B).$$

**Proof.** Let $G'$ be a graph on $n$ vertices which does not contain any graph in $\mathcal{M}$ as a subgraph. For the upper bound of (1), suppose that

$$e(G') \geq \left(\frac{q - 1}{2}\right) + (q - 1)(n - q + 1) + f(k - 1, k - 1).$$

(3)
First there are at most $|A| - 1$ vertices of $G'$ with degree more than $e(G)$, otherwise $G'$ contains an $H \in \mathcal{M}$ as a subgraph which is obtained by splitting all vertices in $H \setminus S$, where $S$ is an independent covering vertex set of $H \in \mathcal{M}$ with order $q$ with a vertex $x \in S$ such that $d_H(x) = k$, a contradiction. Suppose that the number of vertices of $G'$ with degree more than $e(G)$ is less than $|A| - 1$. By Lemma 2.1, $\mathcal{M}$ contains a matching with size $e(G)$. Since $n$ is sufficiently large,

$$e(G') \leq (|A| - 2)(n - 1) + f(e(G), e(G)) < \binom{|A| - 1}{2} + (|A| - 1)(n - |A| + 1) + f(k - 1, k - 1),$$

contradicting (3). Let $X = \{x_1, x_2, \ldots, x_{|A| - 1}\}$ be the vertices with degree more than $e(G)$ and $\tilde{G} = G' - X$. Then $\tilde{G}$ cannot contain $S_{k+1}$ nor $M_{2k}$ as a subgraph, otherwise $G'$ contains a copy of $H_1$ or $H_2$ in $\mathcal{M}$, where $H_1$ is obtained by splitting the neighbours of $x$ and $H_2$ is obtained by splitting $x$ and its neighbours. Hence by Theorem 1.4, we have $e(\tilde{G}) \leq f(k - 1, k - 1)$. Then by (3), we have that $e(\tilde{G}) = f(k - 1, k - 1)$ and each vertex in $X$ has degree $n - 1$. Thus we have $G' \in \mathcal{H}(n, 1, q, k - 1 - K_q)$. The lower bound of (1) follows form that $H'(n, 1, q)$ does not contain any graph in $\mathcal{M}$. Note that if $q < |A|$, then, by Proposition 2.2, $H$ is obtained by splitting some vertices of $G$. Thus we have $k = 1$. Hence we have $e(\tilde{G}) = 0$. So we finish the proof of (2) by the definition of $B$.

Now we will show that both bounds are possible. For the upper bound, we present the graph $q \cdot S_{k+1}$. The graphs in $\mathcal{H}(n, 1, q, k - 1 - K_q)$ do not contain any graph in $\mathcal{M}(q \cdot S_{k+1})$. For the lower bound of (1), we present the graph $S_{q, q}$ obtained by taking two copies of $S_q$ and joining the center of them with a new edge. It is not hard to show that $\text{ex}(n, \mathcal{M}(S_{q, q})) = h'(n, 1, q)$.

Let $\mathcal{H}_n$ be a set of graphs on $n$ vertices with same number of edges. Let $n_1 \geq \ldots \geq n_p$. Denote by $K_{n_1, \ldots, n_p}(n, \mathcal{H}_n)$ the set of graphs which are obtained by embedding an $H_n \in \mathcal{H}_n$ in the largest partite set of the completed $p$-partite graph $K_{n_1, \ldots, n_p}$. If $\mathcal{H}_n = \{H_n\}$, we use $K_{n_1, \ldots, n_p}(n, H_n)$ instead of $K_{n_1, \ldots, n_p}(n, \{H_n\})$.

**Proposition 2.4** Let $\mathcal{F}$ be a family of graphs with $p(\mathcal{F}) = p$ and $G'$ be an extremal graph for $\mathcal{M}(\mathcal{F})$ on $n_1$ vertices. Then $K_{n_1, \ldots, n_p}(n, G')$ does not contain any $F \in \mathcal{F}$ as a subgraph.

**Lemma 2.5** Let $G$ be a bipartite graph, $\mathcal{M} = \mathcal{M}(G^{p+1})$ and $q(\mathcal{M}) = q$. Let $k = \min\{d_H(x) : x \in S, S \in \mathcal{S}(\mathcal{M})\}$. Let $H$ be a graph with a partition of vertices into $p + 1$ parts

$$V(H) = V_0 \cup V_1 \cup V_2 \cup \ldots \cup V_p.$$

Let $|V_0| = q - 1$, $V'_i \subseteq V_i$, $V''_i = V_i \setminus V'_i$, $|V''_i| = a \geq e(G)$ and $\mathcal{H}[V'_1 \cup \ldots \cup V''_p] = T_p(ap)$. Each vertex of $V''_i$ is joint to each vertex of $V'_i \setminus V''_i$ and each vertex of $V_0$ is joint to each vertex of $V'_i$ for $i = 1, 2, \ldots, p$. If there exists an $x \in V'_i$, such that the following hold:

$$\sum_{j \neq i} \nu(H[V''_j]) \geq k \text{ or } \Delta(H[V''_i]) \geq k \text{ or } d_H[V''_i](x) + \sum_{j \neq i} \nu(H[N(x) \cap V''_j]) \geq k,$$

then $H$ contains a copy of $G^{p+1}$ for $p \geq 3$.

**Proof.** Let $V'_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,a}\}$ for $i = 1, 2, \ldots, p$ and $H' = H - V_0$. Since each vertex of $V_0$ is joint to each vertex of $V'_i$, $|V_0| = |A| - 1$ and $a \geq |B| + 1$, it is enough to show that $H'$ contains a copy of $S^{p+1}_{k+1}$ with the following property: each clique of $S^{p+1}_{k+1}$ with order $p$ without containing the center of $S^{p+1}_{k+1}$ contains one vertex in $\bigcup_{i=1}^{p} V'_i$ (the center of $S^{p+1}_{k+1}$ is the vertex in $S^{p+1}_{k+1}$ with degree $pk$). We will prove the lemma in the following three cases.
Case 1. $\sum_{j \neq i} \nu(H[V_j]) \geq k$. Without loss of generality, let $\sum_{j \neq i} \nu(H[V_j]) \geq k$. Let $y_1z_1, y_2z_2, \ldots, y_kz_k$ be a matching in $\bigcup_{j \neq i} H[V_j]$ and

$$H_s = H[x_{1,1}, y_s, z_s, x_{2,s}, x_{3,s}, \ldots, x_{p,s}]$$

for $s = 1, 2, \ldots, k$. Clearly each $H_s$ contains a copy of $K_{p+1}$ which contains the vertex $x_{1,1}$, and $H_1, \ldots, H_k$ intersect the unique vertex $x_{1,1}$, the result follows.

Case 2. $\Delta(H[V_i]) \geq k$. Without loss of generality, let $\Delta(H[V_i]) \geq k$, $x$ be a vertex in $V_i'$ with $d_H[V_i](x) \geq k$ and $x_1, x_2, \ldots, x_k$ be the neighbours of $x$ in $V_i'$. Let

$$H_s = H[x, x_s, x_{2,s}, x_{3,s}, \ldots, x_{p,s}]$$

for $s = 1, 2, \ldots, k$. Clearly each $H_s = K_{p+1}$ and $H_1, \ldots, H_k$ intersect the unique vertex $x$, the result follows.

Case 3. $d_H[V_i](x) + \sum_{j \neq i} \nu(H[N(x) \cap V_j]) \geq k$. Without loss of generality, let $d_H[V_i](x) + \sum_{j \neq i} \nu(H[N(x) \cap V_j]) \geq k$. Let $d_H[V_i](x) = t < k$, $x_1, x_2, \ldots, x_t$ be the neighbours of $x$ in $H[V_i']$ and $y_1z_1, y_2z_2, \ldots, y_kz_k$ be a matching in $\bigcup_{j \neq i} H[N(x) \cap V_j]$. Let

$$H_s = \begin{cases} H[x, x_s, x_{2,s}, x_{3,s}, \ldots, x_{p,s}] & \text{for } s = 1, 2, \ldots, t, \\ H[x, y_s, z_s, x_{2,s}, x_{3,s}, \ldots, x_{p,s}] & \text{for } s = t+1, t+2, \ldots, k. \end{cases}$$

Clearly, each $H_s$ contains a copy of $K_{p+1}$ which contains the vertex $x$ and $H_1, \ldots, H_k$ intersect the unique vertex $x$, the result follows. □

Let $G$ be a graph with a partition of the vertices into $p \geq 3$ non-empty parts

$$V(G) = V_1 \cup V_2 \cup \ldots \cup V_p.$$ 

Let $G_i = G[V_i]$ for $i = 1, 2, \ldots, p$ and define

$$G_{cr} = (V(G), \{v_i v_j : v_i \in V_i, v_j \in V_j, i \neq j\}),$$

where “cr” denotes “crossing”. The following lemma is proved in [3].

**Lemma 2.6** (Chen, et al. [3]) Let $G$ be a graph on $n$ vertices. Suppose $G$ is partitioned as above so that

\begin{align*}
\sum_{j \neq i} \nu(G[V_j]) \leq k - 1 \text{ and } \Delta(G[V_i]) \leq k - 1; \\
d_{G[V_i]}(x) + \sum_{j \neq i} \nu(G[N(x) \cap V_j]) \leq k - 1.
\end{align*}

are satisfied. If $G$ does not contain a copy of $S_{k+1}^{p+1}$, then

$$\sum_{i=1}^{p} |E(G_i)| - \left(\sum_{1 \leq i < j \leq p} |V_i||V_j| - |E(G_{cr})|\right) \leq f(k - 1, k - 1).$$

Moreover, if the equality holds, then

$$\sum_{1 \leq i < j \leq p} |V_i||V_j| = |E(G_{cr})|, \quad e(G[V_i]) = f(k - 1, k - 1), \quad e(G[V_i \cup V_{\xi_i}]) = 0,$$

and $G[V_i] \in E_{k-1,k-1}$ for some $i \in \{1, \ldots, p\}$.

**Remark.** Thought the proof of Lemma 2.6 in [3], it is not difficult to see that if the equality holds in (6), then (7) is satisfied and $G[V_i] \in E_{k-1,k-1}$ which is not appeared in the original description of Lemma 2.6 in [3]. See Lemma 2.7 in [19].
3 Proof of the main theorems

In 1968, Simonovits \[15\] introduced the so-called progressive induction which is similar to the mathematical induction and Euclidean algorithm and combined from them in a certain sense.

Lemma 3.1 (Simonovits \[12\]) Let \( \mathcal{U} = \bigcup_1^\infty \mathcal{U}_n \) be a set of given elements, such that \( \mathcal{U}_n \) are disjoint subsets of \( \mathcal{U} \). Let \( B \) be a condition or property defined on \( \mathcal{U} \) (i.e. the elements of \( \mathcal{U}_n \) may satisfy or not satisfy \( B \)). Let \( \Delta(n) \) be a function defined also on \( \mathcal{U} \) such that \( \Delta(n) \) is a non-negative integer and \( (a) \) if \( a \) satisfies \( B \), then \( \Delta(a) \) vanishes. \( (b) \) there is an \( M_0 \) such that if \( n > M_0 \) and \( a \in \mathcal{U}_n \) then either \( a \) satisfies \( B \) or there exist an \( n' \) and an \( a' \) such that

\[
\frac{n}{2} < n' < n, a' \in \mathcal{U}_{n'} \text{ and } \Delta(a) < \Delta(a').
\]

Then there exists an \( n_0 \) such that if \( n > n_0 \), from \( a \in \mathcal{U}_n \) follows that \( a \) satisfies \( B \).

Remark. In our problems, \( \mathcal{U}_n \) is the set of graphs with \( n \) vertices such that the graph in \( \mathcal{U}_n \) already satisfies some properties (e.g., if \( u_n \in \mathcal{U}_n \), then \( u_n \) does not contain a \( K_p,a \) as a subgraph), \( B \) is some property defined on the graphs, such as the number of edges, the chromatic number or some special structure of graphs (e.g., the graph is a complete \( p \)-partite graph).

Now, we are able to prove the main theorems.

Proof of Theorem 1.5 (i):

Proof. Lemma 2.4 together with Proposition 2.1 implies the low bound, and that both bounds are best possible when we determine the upper bound. We will prove this theorem by progressive induction. Suppose \( S_n \) is an extremal graph for \( G^{p+1} \) on \( n \) vertices. It will be shown that, if \( n \) is sufficiently large, then \( e(S_n) \leq h(n, p, q) + f(k - 1, k - 1) \). Let \( H_n \in H(n, p, q, k - 1, k - 1, K_p) \). If \( e(S_n) < e(H_n) = h(n, p, q) + f(k - 1, k - 1) \), we are done. Let

\[
e(S_n) \geq e(H_n) \geq t_p(n).
\]  

(8)

Hence \( \Delta(n) = e(S_n) - e(H_n) \) is a non-negative integer. The theorem will be proved by progressive induction, where \( \mathcal{U}_n \) is the set of extremal graphs for \( G^{p+1} \) on \( n \) vertices. \( B \) states that \( e(S_n) \leq e(H_n) \), and \( \Delta(n) \) is a non-negative integer. According to the lemma of progressive induction, it is enough to show that if \( e(S_n) > e(H_n) \), then there exists an \( n' < n \) such that \( \Delta(n') > \Delta(n) \) provided \( n \) is sufficiently large. By Theorem 1.1 and 8, there is an \( n_1 \), if \( n > n_1 \), then \( S_n \) contains \( T_p(n_2) \) (\( n_2 \) is sufficiently large) as a subgraph.

By Lemma 2.1 \( \mathcal{M} \) contains a matching \( M_{2k_1} \), where \( k_1 = e(G) \). Each partite class of \( T_p(n_3) \) can not contain \( M_{2k_1} \), otherwise \( S_n \) contains a copy of \( G^{p+1} \), a contradiction. Hence there is an induced subgraph \( T_p(n_3) \) (\( n_3 \) is sufficiently large) of \( S_n \) with partite set \( B_1, B_2, \ldots, B_p \) (\( n_3 \geq n_2 - 2k_1 \)). In fact, let \( x_1, x_2, x_3, \ldots, x_{k_1+y_1} \) be a maximal matching in one class, say \( B'_1 \), of \( T_p(n_3) \) and \( \tilde{B}_1 = B'_1 - \{x_1, y_1, \ldots, x_{k_1}, y_{k_1}\} \). There is no edge in \( S_n[\tilde{B}_1] \). Hence there is an induced subgraph \( T_p(n_3) \) of \( S_n \).

Let \( c \) be sufficiently small and \( \tilde{S} = S_n - T_p(n_3) \). We partition \( \tilde{S} \) by the following produce. If there is an \( x_1 \in \tilde{S} \) joining to all the classes of \( T_p(n_3) = T_0 \) by more than \( c^2 n_3 \) vertices, then \( T_0 \) contains a \( T_1 = T_p(c^2 n_3) \) each vertex of which is joint to \( x_1 \); \ldots If there is an \( x_i \) joint to at least \( c^2 n_3 \) vertices of each class of \( T_{i-1} \), then there is a \( T_i = T_p(c^2 n_3) \) each vertices of which is joint to all the vertices \( x_1, x_2, \ldots, x_i \). Thus we may define recursively a sequence of graphs. However, this process stops at last after the
construction of $T_{q-1}$. Since if we could find a $T_q \subseteq S_n$, then the induced subgraph of $S_n$ on $B_1 \cup \{x_1, x_2, \ldots, x_q\}$, where $B_1$ is a partite set of $T_q$, is a graph with $q + c^{2q}n_3 > n_0$ vertices and at least $c^{2q}n_3g$ edges. Since $c^{2q}n_3g > \left(9 - \frac{1}{2}\right) + (q - 1)(c^{2q}n_3 + 1) + f(k - 1, k - 1)$, provided $n_3$ is sufficiently large, by Lemma 2.3 this induced subgraph contains a copy of $H \in \mathcal{M}$. Note that each vertex of this induced subgraph is joint to each vertex of $B_2 \cup \ldots B_p$. $S_n$ contains a copy of $G^{p+1}$, a contradiction.

Now suppose the above progress ends at $T_{\ell}$ with $0 \leq \ell \leq q - 1$. Denote by $x_1, x_2, \ldots, x_\ell$ the vertices joining to all the vertices of $T_{\ell}$ and $B_{\ell 1}, B_{\ell 2}, \ldots, B_{\ell p}$ the classes of $T_{\ell}$. Partition the remaining vertices into the following vertex sets: If $x$ is joint to less than $c^{2\ell+2}n_3$ vertices of $B_{\ell i}$ and is joint to more than $(1 - c)c^{2\ell}n_3$ vertices of $B_{i \neq \ell i}$, then $x \in C_{\ell i}$. If $x$ is joint to less than $c^{2\ell+2}n_3$ vertices of $B_{\ell i}$ and is joint to less than $(1 - c)c^{2\ell}n_3$ vertices of some of $B_{i \neq \ell i}$, then $x \in D$. Obviously, this is a partition of $S_n - T_{\ell} - \{x_1, x_2, \ldots, x_\ell\}$. Since $\mathcal{M}$ contains a matching with size $k_1$ and each vertex of $C_{\ell i}$ is joint to less than $c^{2\ell+2}n_3$ vertices of $B_{\ell i}$, there are $c^{2\ell}n_3(1 - c^2k_1)$ vertices of $B_{\ell i}$ which is not joint to any vertices of $C_{\ell i}$.

In fact, there are at most $k_1$ independent edges in $B_{\ell j} \cup C_{\ell i}$, otherwise, $S_n$ contains a copy of $G^{p+1}$. Consider the edges joining $B_{\ell i}$ and $C_{\ell i}$ and select a maximal set of independent edges, say $x_1y_1, \ldots, x_qy_q$, $x_i \neq B_{\ell i}$, $y_i \in C_{\ell i}$, $1 \leq i' \leq q \leq k_1$, among them, then the number of vertices of $B_{\ell i}$ joining to at least one of $y_1, y_2, \ldots, y_q$ is less than $c^{2\ell+2}n_3$, and the remaining vertices of $B_{\ell i}$ is not joint to any vertex of $C_i$ by the maximal of $x_1y_1, \ldots, x_qy_q$. Hence we can move $c^{2\ell+2}n_3k_1$ vertices of $B_{\ell i}$ to and $C_{\ell i}$, obtain $B_{\ell i}$ and $C_{\ell i}$ such that $B_{\ell i} \subseteq B_{\ell i}$, $C_{\ell i} \subseteq C_i$ and there is no edge between $B_{\ell i}$ and $C_{\ell i}$. Let $\ell' = (1 - c^2k_1)c^{2\ell}n_3$. We conclude that $T_\ell' = T_p(\ell' p)$ with classes $B_1, \ldots, B_p$ is an induced subgraph of $S_n$ satisfying the following conditions:

Let $\tilde{S} = S_n - T_\ell'$, the vertices of $\tilde{S}$ can be partitioned into $p + 2$ classes $C_1, \ldots, C_p$, $D$ and $E$ such that

- Each $x \in E$ is joint each vertex of $T_\ell'$ and $|E| = \ell$.
- If $x \in C_i$ then $x$ is joint to at least $(1 - c - c^2k_1)c^{2\ell}n_3$ vertices of $B_{i \neq \ell i}$ and is joint to no vertex of $B_i$.
- If $x \in D$ then there are two different classes of $T_\ell'$: $B_{i(x)}$ and $B_{j(x)}$ such that $x$ is joint to less than $(1 - c)c^{2\ell}n_3$ vertices of $B_{i(x)}$ and less than $c^{2\ell+2}n_3$ vertices of $B_{j(x)}$.

Denote by $e_s$ the number of the edges joining $\tilde{S}$ and $T_\ell'$. Clearly

$$e(S_n) = e(T_\ell') + e_S + e(\tilde{S}).$$  \hspace{1cm} (9)

Select an induced $T_\ell'$ in $H_n$, let $H_n - e_p = H_n - T_\ell'$ and $e_T$ be the number of edges of $H_n$ joining $T_\ell'$ and $H_n - e_p$, then we have

$$e(H_n) = e(T_\ell') + e_T + e(H_n - e_p).$$  \hspace{1cm} (10)

Since $\tilde{S}$ does not contain a copy of $G^{p+1}$, we have $e(\tilde{S}) \leq e(S_n - e_p)$, where $S_n - e_p$ is an extremal graph for $G^{p+1}$ on $n - \ell' p$ vertices. By (9), (10), we have

$$\Delta(n) = e(S_n) - e(H_n) = e(T_\ell') - e(T_\ell') + (e_S - e_T) + e(\tilde{S}) - e(H_n - e_p) \leq \Delta(n) \leq e_S - e_T + e(S_n - e_p) - e(H_n - e_p) = (e_S - e_T) + \Delta(n - \ell' p).$$

If $e_S - e_T < 0$, then $\Delta(n) < \Delta(n - \ell' p)$, we are done. Hence we may assume $e_S - e_T \geq 0$. Since $c$ is sufficiently small, we have

$$e_S - e_T \leq \ell' \cdot \ell' p + (n - \ell - \ell' p - |D|) \cdot \ell' p - (q - 1) \cdot \ell' p \leq 0,$$

since $c$ is sufficiently small, we have

$$e_S - e_T \leq \ell' \cdot \ell' p + (n - \ell - \ell' p - |D|) \cdot \ell' p - (q - 1) \cdot \ell' p \cdot \ell' p \leq 0,$$

\begin{align*}
\end{align*}
with the equality holds if and only if \(|D| = 0\), \(\ell = q - 1\), and every \(x \in C_i\) is joint to all the vertices of \(B_{j \neq i}\) and is joint to no vertex of \(B_i\) for \(i = 1, 2, \ldots, p\). Since \(e(S_n) \geq h(n, p, q) + f(k-1, k-1)\) and \(T_p(n-q+1-\ell'p)\) has more edges than any other \(p\)-chromatic graph on \(n-q+1-\ell'p\) vertices, by Lemmas 2.5 and 2.6 we have \(e(S_n) = e(H_n)\). The proof is completed. \(\square\)

Proof of Theorem 1.5 (ii):
Proof. The proof is essentially the same as the proof of Theorem 1.5 (i) and omitted. \(\square\)

Proof of Theorem 1.6:
Proof. Let \(H \in \mathcal{M}\) with \(q(H) = q(\mathcal{M})\) and \(S\) be an independent covering vertex set of \(H\) of order \(q\). Since \(G\) is non-bipartite, there is an \(x \in S\) with \(d_H(x) = 1\) which is obtained by splitting a vertex in \(G\). By Lemma 2.1, \(\mathcal{M}\) contains a matching with size \(e(G)\). Hence one can get the result by similar method in Theorem 1.5 (note that \(d_H(x) = 1\)). \(\square\)

4 Corollaries
By similar method in the proof of Theorem 1.5, one can get the following corollaries.

**Corollary 4.1** Let \(F\) be a family of graphs with \(p(F) = p\). If \(\mathcal{M}(F) = \{M_{2s}\}\), then
\[
ex(n, F) = h(n, p, s),
\]
provided \(n\) is sufficiently large. Moreover, the unique graph in \(\mathcal{H}(n, p, s, 0, 0, K_s)\) is the unique extremal graph for \(F\).

**Remark.** There are many interesting graphs belong to the graphs set of Corollary 4.1 including the Petersen graph, vertex-disjoint union of cliques, dodecahedron \(D_{20}\) and so on.

**Corollary 4.2** Let \(F\) be a family of graphs with \(p(F) = p\). If \(\text{EX}(n, \mathcal{M}(F)) = \{\overline{K}_{s-1} + \overline{K}_{n-s+1}\}\), provided \(n\) is sufficiently large, and \(\mathcal{M}(F)\) contains a matching, then
\[
ex(n, F) = h'(n, p, s),
\]
provided \(n\) is sufficiently large. Moreover, the unique graph in \(\mathcal{H}(n, p, s, 0, 0, K_2)\) is the unique extremal graph for \(F\).

**Corollary 4.3** Let \(F = \{s \cdot K_{p+1}, K_{p+2}\}\). If \(n\) is sufficiently large, then
\[
ex(n, F) = h'(n, p, s),
\]
Moreover, the unique graph in \(\mathcal{H}(n, p, s, 0, 0, K_2)\) is the unique extremal graph for \(F\).

**Proof.** Obviously, \(\mathcal{M}(F) = \{K_3, M_{2s}\}\). It is easy to prove that \(\text{EX}(n, \{K_3, M_{2s}\}) = \{\overline{K}_{s-1} + \overline{K}_{n-s+1}\}\) provided \(n\) is sufficiently large. Hence, the corollary follows from Corollary 4.2. \(\square\)
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