Abstract—Convex relaxation methods have been studied and used extensively to obtain an optimal solution to the optimal power flow (OPF) problem. Meanwhile, convex relaxed power flow equations are also prerequisites for efficiently solving a wide range of problems in power systems including mixed-integer nonlinear programming (MINLP) and distributed optimization. When the exactness of convex relaxations is not guaranteed, it is important to recover a feasible solution for the convex relaxation methods. This paper presents an alternative convex optimization (ACP) approach that can efficiently recover a feasible solution from the result of second-order cone programming (SOCP) relaxed OPF in mesh networks. The OPF problem is first formulated as a difference-of-convex (DC) programming problem, then efficiently solved by a penalty convex concave procedure (CCP). CCP iteratively linearizes the concave parts of the power flow constraints and solves a convex approximation of the DCP problem. Numerical tests show that the proposed method can find a global or near-global optimal solution to the AC OPF problem, and outperforms those semidefinite programming (SDP) based algorithms.

Index Terms—Optimal power flow, mesh networks, convex optimization, mixed-integer nonlinear programming

NOMENCLATURE

A. Indices and Sets:

- \( \Phi_b \) Sets of all buses.
- \( \Phi_i \) Sets of all lines.
- \( K(i) \) Sets of buses connected to bus \( i \).

B. Parameters:

- \( G_{ij} \) Conductance of branch \( ij \).
- \( B_{ij} \) Susceptance of branch \( ij \).
- \( G_{m,i} \) Shunt conductance at bus \( i \).
- \( B_{s,i} \) Shunt susceptance at bus \( i \).
- \( p_i^a \) Active power demand at bus \( i \).
- \( q_i^a \) Reactive power demand at bus \( i \).
- \( p_i^q, p_i^c \) Active power capacity of generator at bus \( i \).
- \( q_i^q, q_i^c \) Reactive power capacity of generator at bus \( i \).
- \( \theta^a \) Maximum phase angle difference of each branch.

C. Variables:

- \( V_i \) Voltage magnitude of bus \( i \).
- \( \theta_i \) Phase angle of bus \( i \).
- \( \theta_{ij} \) Phase angle difference of branch \( ij \).
- \( p_{ij} \) Active power flow from bus \( i \) to bus \( j \).
- \( q_{ij} \) Reactive power flow from bus \( i \) to bus \( j \).
- \( p_i^g \) Active power provided by generator at bus \( i \).
- \( q_i^g \) Reactive power provided by generator at bus \( i \).
- \( U_i \) Square of \( V_i \).
- \( K_q \) Denotes \( V_i V_i \cos \theta_q \).
- \( L_q \) Denotes \( V_i V_i \cos \theta_q \).
- \( s_q \) Denotes \( \sin \theta_q \).
- \( c_q \) Denotes \( \cos \theta_q \).
- \( \varepsilon \) Slack variables in convex concave procedure.

I. INTRODUCTION

The AC optimal power flow (OPF) problem is essential for power systems to determine the operation point that best minimizes generation cost, power losses, voltage fluctuations, and other crucial outcomes. It is a typical nonconvex and NP-hard problem, for which the non-convexity mainly lies in the power flow equations. Traditional methods to solve OPF problems in transmission systems include linear approximations, the Newton-Raphson method and some heuristic algorithms, which either lack feasibility or cannot ensure optimality. With the increasing penetration of renewable generations, the OPF problem for power systems has drawn much attention in recent years. MINLP and decentralized optimization problems also require convex formulation of power flow equations so that the problem can be solved efficiently. The convex relaxation of OPF problems was first proposed in [1], [2], and has become an important research topic in the past five years.

Convex relaxation methods mainly include semidefinite
programming (SDP) relaxation [1] and second-order cone programming (SOCP) relaxation [2]. These methods can find a lower bound of the original minimization problem, and in certain circumstances, a feasible solution of the original problem can be recovered from the solution of convex relaxation methods. For SDP relaxation, if the rank one condition is satisfied, then the zero-duality gap can be guaranteed, hence a feasible solution is sure to be recovered [3]. For SOCP relaxation, a feasible solution can be recovered when the quadratic and arctangent equalities both hold [4], [5]. Under such circumstances, we say the convex relaxation is exact, and its solution is the global optimum of the original OPF problem. Many researchers have devoted efforts toward finding sufficient conditions for ensuring the exactness of, or strengthening, the convex relaxations.

In [6], a sufficient zero-duality-gap condition for SDP was found in resistive networks with active loads if over-satisfaction of the loads was permitted. In [7] and [8], sufficient conditions for SDP in radial and mesh networks were discussed. In [9], the exactness of SDP for mesh networks was related to the modelling of the capacity of a power line. In [10], the sufficient condition for SOCP in radial networks was proposed. If the objective function of the OPF problem is non-increasing in load, and there are no upper limits on load, then the solution of SOCP is exact for radial networks. In [11], three types of sufficient condition were discussed: power injections, voltage magnitudes, and voltage angles. A mild condition that only limits the power injections was proposed. In general, SOCP relaxation is excellent for solving OPF problems in radial networks, besides the sufficient conditions, its tightness can be checked a posteriori for many problems. In [12], the performance of SOCP in mesh networks was further studied. In [13], a cycle-based formulation of angle constraints was proposed to enhance SOCP relaxation. By exploring the fact that angle differences sum up to zero over each cycle, the angle constraints were transformed into bilinear constraints. However, there has not been a method that guarantees the exactness of SOCP relaxation in mesh networks, because the conic relaxation and the angle relaxation must both be exact to ensure the feasibility of the SOCP solution in mesh networks, but the angle constraints are difficult to deal with due to the trigonometric functions.

While the above literatures discussed the exactness of convex relaxation methods, it is still an important issue that how to recover a feasible solution of the original OPF problem when the exactness of convex relaxation is not guaranteed, especially for SOCP relaxation in mesh networks. The motivation for feasibility recovery is to make convex relaxation more practical in sophisticated problems based on OPF, such as MINLP or distributed optimization problems in power systems:

1) In MINLP problems such as transmission line switching [14] or voltage control considering the adjustment of transformer’s OLTC (On-Load Tap-Changer) [15], the power flow equations need to be convexified or linearized so that the problem can be to efficiently solved. To obtain a physically meaningful solution, feasibility recovery should be utilized.

2) In distributed optimization problems, the convergence of distributed algorithms, such as ADMM [16], can only be guaranteed for convex problems [17]. In such problems, the convex relaxed power flow equations are employed. So, the solution of distributed OPF must be recovered to a feasible solution to make the strategies practical.

In [9], a penalized SDP method is proposed, the total amount of reactive power was added to the objective to force the rank to become one. In [20], the matrix rank is approximated by a continuous function and penalized in the objective function, then a majorization-minimization method is applied to solve the penalized SDP problem iteratively. In [21], moment relaxations were proposed for the OPF problem as a generalization of SDP relaxation, and had the potential to find a global optimal solution using polynomial optimization theory. Moment relaxations significantly increase the matrix size of semidefinite constraints, which is much more computational inefficient than SDP. In [22], a Laplacian-based approach was proposed to yield near-globally optimal solutions when SDP had a small optimality gap. In [23] and [24], instead of forcing matrix rank to be one, they employed the employed the quadratically constrained quadratic programming (QCQP) formulation of OPF problems, and applied convex concave procedure to deal with the indefinite coefficient matrix. In [25], when the SOCP relaxation is inexact, the OPF problem in radial networks was first formulated as a difference-of-convex programming problem (DCP), then solved as a sequence of convexified penalization problems.

However, there is no method yet available to recover a feasible and optimal or near-optimal solution for the SOCP relaxation in mesh networks. This paper applies the convex concave procedure (CCP) to the OPF problem in mesh networks and recovers a feasible and local optimal solution for SOCP relaxation. CCP is a powerful heuristic method for finding a local optimum of DCP problems [26], which was first introduced in [27] and [28]. It iteratively linearizes the concave parts of all constraints, thus solving a convex approximation of the DCP problem. In [29], penalty CCP was proposed to negate the need for an initial feasible point in the iteration. Penalty CCP usually benefits from a warm-start point, which makes good use of the solution solved by SOCP. The main contributions of this paper include:

1) An alternative convex optimization (ACP) algorithm is proposed that can efficiently recover a feasible solution from the result of SOCP relaxed OPF problem in mesh networks. The ACP algorithm first formulates OPF problem as a DCP problem, then solves the DCP problem by penalty CCP iteratively.

2) The convergence of ACP is proved. After ACP converges, if the slack variables all turn out to be zero, then the solution is guaranteed to be a KKT point of the original OPF problem. It is shown that ACP successfully converges to a KKT point of the original OPF problem in all the test cases.

3) Numerical tests are conducted on several benchmark systems using ACP and compared with other methods aimed to
recover feasible solutions for SDP relaxation. It is shown that the proposed algorithm can find a global or near-global optimal solution within a few iterations, but the other recovery methods may only find worse results. Its computation speed is comparable to SOCP, which is far beyond SDP-based recovery methods. Furthermore, an optimal control of OLTC test case is studied to show the availability of ACP in MINLP problems when other methods fail.

The remainder of this paper is organized as follows. Section II describes the original non-convex model of OPF in mesh networks and a tightened SOCP relaxation, and Section III details the DCP formulation of the OPF problem and the ACP algorithm for feasible solution recovery. Section IV outlines the test results of the algorithm using several IEEE test systems, and Section V concludes the paper.

II. OPF PROBLEM AND CONVEX RELAXATION

A. Original OPF problem

The OPF problem usually consists of convex functions of generator output, denoted by $C_i(p_i^e)$. This is described as:

\begin{align}
\text{min} & \quad \sum C_i(p_i^e) \\
\text{Subject to} & \quad \text{1) Branch power flow constraints} \\
& \quad p_{ij} = G_{ij}V_{i}^2 - G_{ij}V_{j}V_{i} \cos \theta_{ij} - B_{ij}V_{i}V_{j} \sin \theta_{ij}, \forall ij \in \Phi \\
& \quad q_{ij} = -B_{ij}V_{i}^2 + B_{ij}V_{i}V_{j} \cos \theta_{ij} - G_{ij}V_{i}V_{j} \sin \theta_{ij}, \forall ij \in \Phi \\
& \quad \theta_{ij} = \theta_{i} - \theta_{j}, \forall ij \in \Phi \\
\text{2) Active and reactive power balance constraints for buses} \\
& \quad p_i^e - p_i^d = G_{ai}V_i^2 + \sum_{j \in K(i)} p_{ij}, \forall i \in \Phi_b \\
& \quad q_i^e - q_i^d = -B_{ai}V_i^2 + \sum_{j \in K(i)} q_{ij}, \forall i \in \Phi_b \\
\text{3) Generator operation constraints} \\
& \quad p_i^l \leq p_i^e \leq p_i^u, \forall i \in \Phi_b \\
& \quad q_i^l \leq q_i^e \leq q_i^u, \forall i \in \Phi_b \\
\text{4) Phase angle difference limits} \\
& \quad -\theta^i \leq \theta_{ij} \leq \theta^i, \forall ij \in \Phi \\
\text{5) Branch thermal limits} \\
& \quad p_{ij}^2 + q_{ij}^2 \leq (S^t)^2, \forall ij \in \Phi \\
\text{6) Bus voltage limits} \\
& \quad V_i^l \leq V_i \leq V_i^u, \forall i \in \Phi_b
\end{align}

The original formulation of OPF problem is nonconvex and the non-convexity comes from branch power flow constraints (2) and (3). The challenge of non-convexity in realistic power systems OPF also comes from transformer taps, capacitor, etc., which has been discussed in [15] and [30]. So, in this paper, we mainly focus on the nonconvex power flow constraints.

By defining new variables $K_{ij} = V_iV_j \cos \theta_{ij}$, $L_{ij} = V_iV_j \sin \theta_{ij}$ and $U_i = V_i^2$, constraints (2), (3), (5), and (6) can be transformed into an alternative form:

\begin{align}
& \quad p_i^e - p_i^d = G_{ai}U_i + \sum_{j \in K(i)} p_{ij}, \forall i \in \Phi_b \\
& \quad p_{ij} = G_{ij}U_i - G_{ij}K_{ij} - B_{ij}L_{ij}, \forall ij \in \Phi \\
& \quad q_{ij} = -B_{ij}U_i + B_{ij}K_{ij} - G_{ij}L_{ij}, \forall ij \in \Phi \\
& \quad K_{ij}^2 + L_{ij}^2 = U_iU_j, \forall ij \in \Phi \\
& \quad \theta_{ij} = \arctan(L_{ij} / K_{ij}), \forall ij \in \Phi
\end{align}

For the OPF of a radial network, constraints (4) and (17) are not necessary because the optimal solution $K_{ij}$ and $L_{ij}$ will always recover a set of $\theta_{ij}$ and $\theta_{ij}$ that satisfy these two constraints. However, for the OPF of a meshed network, constraints (4) and (17) are necessary to ensure that $\theta_{ij}$ sums to zero over all cycles [13].

Constraint (17) is equivalent to:

\begin{align}
& \quad \sin \theta_{ij}K_{ij} = \cos \theta_{ij}L_{ij}, \forall ij \in \Phi \\
\end{align}

By introducing new variables $s_{ij}$, $c_{ij}$, (18) is equivalent to:

\begin{align}
& \quad s_{ij} = \sin \theta_{ij}, \forall ij \in \Phi \\
& \quad c_{ij} = \cos \theta_{ij}, \forall ij \in \Phi \\
& \quad s_{ij}^2 + c_{ij}^2 = 1, \forall ij \in \Phi \\
& \quad s_{ij}K_{ij} = c_{ij}L_{ij}, \forall ij \in \Phi
\end{align}

With the above transformation, the OPF problem (Model 1) is equivalent to:

\begin{align}
\text{min} & \quad \sum C_i(p_i^e) \\
\text{Subject to} & \quad (4), (7)–(16), \text{and (19)–(22)}
\end{align}

B. Tightened SOCP relaxation

The OPF problem (Model 2) is nonconvex due to constraints (16) and (19)–(22). Constraint (16) can be relaxed to a second-order cone constraint [10]:

\begin{align}
& \quad \|2K_{ij}\| \leq U_i + U_j, \forall ij \in \Phi \\
& \quad \|U_i - U_j\| \leq 2L_{ij}
\end{align}

Constraints (19) and (20) can be relaxed by convex envelopes for sine and cosine functions [31]:

\begin{align}
& \quad s_{ij} \leq \cos(\theta_{ij})/2(\theta_{ij} - \sqrt{\theta_{ij}^2 - 4}), \forall ij \in \Phi \\
& \quad s_{ij} \geq \cos(\theta_{ij})/2(\theta_{ij} + \sqrt{\theta_{ij}^2 - 4}), \forall ij \in \Phi \\
& \quad c_{ij} \leq 1 - (1 - \cos(\theta_{ij}))\theta_{ij}^2 / (\theta_{ij}^2)^2, \forall ij \in \Phi \\
& \quad c_{ij} \geq \cos(\theta_{ij}), \forall ij \in \Phi
\end{align}

Constraint (21) can be relaxed to:

\begin{align}
& \quad s_{ij}^2 + c_{ij}^2 \leq 1, \forall ij \in \Phi \\
\end{align}

For constraint (22), by introducing new variables $m_{ij}$ and $n_{ij}$, it is equivalent to:

\begin{align}
& \quad m_{ij} = s_{ij}K_{ij}, \forall ij \in \Phi \\
& \quad n_{ij} = c_{ij}L_{ij}, \forall ij \in \Phi
\end{align}
\( m_y = n_y, \forall ij \in \Phi_i \) \hspace{1cm} (32)

Constraints (30) and (31) can be relaxed by McCormick envelopes for bilinear terms \([32]\):

\[
m_y \geq s^t K_y + s_y K^t - s^t K^t, \forall ij \in \Phi_i
\]

\[
m_y \geq s^t K_y + s_y K^t - s^t K^t, \forall ij \in \Phi_i
\]

\[
m_y \leq s^t K_y + s_y K^t - s^t K^t, \forall ij \in \Phi_i
\]

\[
m_y \leq s^t K_y + s_y K^t - s^t K^t, \forall ij \in \Phi_i
\]

\[
 n_y \geq c^t L_y + c_y L_c^t - c^t L_c^t, \forall ij \in \Phi_i
\]

\[
 n_y \geq c^t L_y + c_y L_c^t - c^t L_c^t, \forall ij \in \Phi_i
\]

\[
 n_y \leq c^t L_y + c_y L_c^t - c^t L_c^t, \forall ij \in \Phi_i
\]

\[
 n_y \leq c^t L_y + c_y L_c^t - c^t L_c^t, \forall ij \in \Phi_i
\]

Thus, the tightened SOCP relaxed OPF problem (SOCPT) is expressed as follows:

(Model 3) \( \min \sum C_i(p_i) \) \hspace{1cm} (41)

subject to (4), (7)–(15), (24)–(29), and (32)–(40)

### III. Feasible Solution Recovery Algorithm

#### A. Difference-of-convex formulation

The relaxation exactness is barely guaranteed by convex relaxed Model 3, because equality (19), (20) and (22) are hard to be satisfied by convex envelopes (25)-(28) and McCormick relaxation (33)–(40), so that a feasible solution cannot be recovered from the solution of Model 3 directly. On the other hand, if the bilinear constraints (16), (21) and (22) are satisfied and the trigonometric functions (19) and (20) are well approximated, then the solution will be feasible to the original OPF problem.

In order to satisfy the bilinear constraints (16), (21) and (22), we formulate them as difference-of-convex constraints, which can be solved by DCP algorithms effectively. In such formulation, the equalities are not easy to loosen as in convex relaxation. Take constraint (22) as an example, it can be written in an alternative form:

\[
(s_y + K_y)^2 - (s_y - K_y)^2 = (c_y + L_y)^2 - (c_y - L_y)^2
\]

which is equivalent to two difference-of-convex constraints:

\[
(s_y + K_y)^2 + (c_y - L_y)^2 - (s_y - K_y)^2 - (c_y + L_y)^2 \leq 0
\]

\[
(s_y - K_y)^2 + (c_y + L_y)^2 - (s_y + K_y)^2 - (c_y - L_y)^2 \leq 0
\]

Considering constraints (16), (21), and (22), we can define the following convex functions:

\[
f_{y,1}(x) = (U_i + U_j)^2
\]

\[
f_{y,2}(x) = 1
\]

\[
g_{y,1}(x) = (s_y + K_y)^2 + (c_y - L_y)^2
\]

\[
g_{y,2}(x) = (s_y + K_y)^2 + (c_y - L_y)^2
\]

\[
g_{y,3}(x) = (s_y - K_y)^2 + (c_y + L_y)^2
\]

\[
g_{y,4}(x) = (s_y - K_y)^2 + (c_y + L_y)^2
\]

\[
g_{y,5}(x) = (s_y - K_y)^2 + (c_y + L_y)^2
\]

\[
g_{y,6}(x) = (s_y - K_y)^2 + (c_y + L_y)^2
\]

Thus, constraints (16), (21), and (22) can be expressed as difference-of-convex constraints:

\[
g_{y,m}(x) - f_{y,m}(x) \leq 0, \forall ij \in \Phi_i, m = 1, 2, 3
\]

\[
f_{y,m}(x) - g_{y,m}(x) \leq 0, \forall ij \in \Phi_i, m = 1, 2, 3
\]

For the precise approximation of trigonometric functions (19) and (20), sixth-order Taylor expansion of the cosine function is utilized as follows:

\[
c_y = 1 - \theta_y^2 / 2 + \theta_y^4 / 24 - \theta_y^6 / 720
\]

By introducing \( \alpha_y = \theta_y^2, \beta_y = \theta_y^4, \gamma_y = \theta_y^6 \), and define the following convex functions:

\[
f_{y,4}(x) = \alpha_y, g_{y,4}(x) = \theta_y^2
\]

\[
f_{y,5}(x) = \beta_y, g_{y,5}(x) = \alpha_y^2
\]

\[
f_{y,6}(x) = (\alpha_y + \gamma_y)^2, g_{y,6}(x) = (\alpha_y - \gamma_y)^2 + (2\beta_y)^2
\]

(53) can be written in a difference-of-convex form:

\[
c_y = 1 - \alpha_y / 2 + \beta_y / 24 - \gamma_y / 720
\]

\[
g_{y,m}(x) - f_{y,m}(x) \leq 0, \forall ij \in \Phi_i, m = 4, 5, 6
\]

\[
f_{y,m}(x) - g_{y,m}(x) \leq 0, \forall ij \in \Phi_i, m = 4, 5, 6
\]

Here, only the cosine function (20) needs to be approximated because equation (21) is satisfied by difference-of-convex constraints (51) and (52).

It should be noted that in realistic power systems operation, \( \theta^o \) is usually very small, i.e., less than \( S^t \). In this situation, directly using \( \sin(\theta_y) = \theta_y \) will also be a good approximation.

With difference-of-convex formulation (51), (52) for bilinear terms and accurate approximations (57)-(59) for trigonometric functions, OPF problem (Model 2) can be formulated as a DCP problem:

(Model 4) \( \min \sum C_i(p_i) \) \hspace{1cm} (60)

subject to (4), (7)–(15), (51), (52), (57)–(59)

Here, the constraints in (51) corresponding to \( m = 1, 2 \) and the constraints in (58) are convex. However, the constraints in (52) and (59) are nonconvex.

Comparing Model 2 and Model 4, the only different constraints are (19), (20) and (57)-(59). Whether the solution of Model 4 is feasible to Model 2 is depended on the quality of approximations (57)-(59). The approximation error between (57) and (19), denoted by \( |\cos \theta - c_y| \), is less than \( 10^{-10} \) when \( \theta_y < 10^{-6} \) and less than \( 10^{-3} \) when \( \theta_y < 90^{-6} \), which applies to power systems in most cases.

#### B. Penalty convex-concave procedure

The OPF problem is formed as a DCP problem (Model 4) in part A; thus, penalty CCP can be applied to find a local optimum of Model 4. The procedure for penalty CCP is in two parts:

1) Tighten the difference-of-convex constraints via partial linearization. For example, \( g_{y,m}(x) \) can be linearized around point \( x^{(0)} \) as

\[
\hat{g}_{y,m}(x, x^{(0)}) = g_{y,m}(x^{(0)}) - \nabla g_{y,m}(x^{(0)})^T(x - x^{(0)})
\]
Since \( g_{ij,m}(x) \) is convex, we have \( g_{ij,m}(x) \geq \hat{g}_{ij,m}(x,x^{(0)}) \), and (52) can be tightened into a convex constraint
\[
f_{ij,m}(x) - \hat{g}_{ij,m}(x,x^{(k)}) \leq \varepsilon \tag{62}
\]
Constraint (62) reduces the feasible region of the original problem, which may lead to infeasibility, so part 2) is needed.

2) Relax constraint (62) by adding slack variables
\[
f_{ij,m}(x) - \hat{g}_{ij,m}(x,x^{(k)}) \leq \varepsilon \tag{63}
\]
and penalize the sum of constraint violations in the objective function. By doing so, the problem is always feasible.

The steps for ACP are described in Algorithm 1.

**Algorithm 1: ACP**

**Initialization:**
1. Set the value of \( x^{(0)} \) to the solution of Model 3.
2. Set \( \tau^{(0)} > 0 \), \( \tau_{\text{max}} \), \( \mu > 1 \) and \( k = 0 \).

**Repeat**
1. Convexify
\[
\hat{g}_{ij,m}(x(x^{(k)})) - \nabla g_{ij,m}(x(x^{(k)}))^T(x-x^{(k)}), m = 1, \ldots, 6
\]
\[
f_{ij,m}(x(x^{(k)})) - \hat{f}_{ij,m}(x^{(k)})^T(x-x^{(k)})
\]
2. Set the value of \( x^{(k+1)} \) to the solution of
\[
(\text{Model 5}) \quad \min \left\{ \sum_{\ell=0} \mathcal{C}(p_{\ell}^{(k)}) + \tau^{(k)} \sum_{m=1} \sum_{i=0} \varepsilon_{ij,m}^{(k)} \right\}
\]
subject to
\[
(4), (7) -(15), (51), (57)
\]
\[
f_{ij,m}(x) - \hat{g}_{ij,m}(x,x^{(k)}) \leq \varepsilon_{ij,m}^{(k)}, \forall ij \in \Phi_i, m = 1, \ldots, 6 \tag{65}
\]
\[
g_{ij,3}(x) - \hat{f}_{ij,3}(x^{(k)}) \leq \varepsilon_{ij,7}, \forall ij \in \Phi_i \tag{66}
\]
\[
g_{ij,m}(x) - \hat{f}_{ij,m}(x^{(k)}) \leq 0, \forall ij \in \Phi_i, m = 1, 2, 4, 5, 6 \tag{67}
\]
\[
\varepsilon_{ij,m}^{(k)} \geq 0, \forall ij \in \Phi_i, m = 1, \ldots, 7 \tag{68}
\]
3. Update \( \tau^{(k+1)} = \min(\mu \tau^{(k)}, \tau_{\text{max}}) \)
4. Update iteration \( k = k + 1 \).

**Until the stopping criterion is satisfied.**

Model 5 is convex, and the number of second-order cone constraints, as well as quadratic constraints, grows linearly with the number of branches in the system. So, Model 5 can be solved easily and quickly using software packages such as Gourbi, CPLEX, or MOSEK. As for the convergence of ACP, it can be proved that the objective value will converge.

**Proposition I:** The objective value of Model 5 will converge.

**Proof:** Suppose \((x^{(k)}, \varepsilon^{(k)})\) is the optimal solution to Model 5 in iteration \(k\).

We will first prove that \((x^{(k)}, \varepsilon^{(k)})\) is a feasible solution to Model 5 in iteration \(k + 1\). Since the different constraints in iteration \(k\) and \(k + 1\) are (65) and (66), it suffices to show that \((x^{(k)}, \varepsilon^{(k)})\) satisfies (65) and (66) in iteration \(k + 1\). That is to prove:
\[
f_{ij,m}(x^{(k)}) - \hat{g}_{ij,m}(x^{(k)}, x^{(k)}) \leq \varepsilon_{ij,m}^{(k)}, m = 1, 2, 3 \tag{69}
\]
\[
g_{ij,3}(x^{(k)}) - \hat{f}_{ij,3}(x^{(k)}, x^{(k)}) \leq \varepsilon_{ij,7}^{(k)} \tag{70}
\]
As \((x^{(k)}, \varepsilon^{(k)})\) is the optimal solution to iteration \(k\), we have
\[
f_{ij,m}(x^{(k)}) - \hat{g}_{ij,m}(x^{(k)}, x^{(k)}) \leq \varepsilon_{ij,m}^{(k)} \tag{71}
\]
The convexity of \(g_{ij,m}(x)\) gives
\[
f_{ij,m}(x^{(k)}) - \hat{g}_{ij,m}(x^{(k)}, x^{(k)}) \leq \varepsilon_{ij,m}^{(k)} \tag{72}
\]
Substituting \(g_{ij,m}(x^{(k)}) = \hat{g}_{ij,m}(x^{(k)}, x^{(k)})\) into (72), together with (71), we have
\[
f_{ij,m}(x^{(k)}) - \hat{g}_{ij,m}(x^{(k)}, x^{(k)}) \leq \varepsilon_{ij,m}^{(k)} \tag{73}
\]
Thus, (69) holds, and (70) can be proved in a similar way. So \((x^{(k)}, \varepsilon^{(k)})\) is a feasible solution to Model 5 in iteration \(k + 1\).

We will now show that the objective value is non-increasing. Let \(v^{(k)}(x, \varepsilon)\) denote the objective function of Model 5 in iteration \(k\). When \(k > \log_\mu (\tau_{\text{max}} / \tau^{(0)})\), \(\tau^{(k)} = \tau_{\text{max}}\), the objective function (64) will not change, which means
\[
v^{(k+1)}(x^{(k+1)}, \varepsilon^{(k+1)}) = v^{(k)}(x^{(k)}, \varepsilon^{(k)}) \tag{74}
\]
Since \((x^{(k)}, \varepsilon^{(k)})\) is a feasible solution to Model 5 in iteration \(k + 1\) and \((x^{(k+1)}, \varepsilon^{(k+1)})\) is the optimal solution, it follows that
\[
v^{(k+1)}(x^{(k+1)}, \varepsilon^{(k+1)}) \leq v^{(k)}(x^{(k)}, \varepsilon^{(k)}) = v^{(k)}(x^{(k)}, \varepsilon^{(k)}) \tag{75}
\]
This shows that the objective value is non-increasing. Since both \(\sum \mathcal{C}(p_{\ell}^{(k)})\) and \(\varepsilon\) have lower bounds, the objective value will converge, which completes the proof.

According to Proposition 1, the stopping criterion of ACP can be chosen as:
\[
v^{(k+1)}(x^{(k+1)}, \varepsilon^{(k+1)}) - v^{(k)}(x^{(k)}, \varepsilon^{(k)}) \leq \delta_l \tag{76}
\]
when \(k > \log_\mu (\tau_{\text{max}} / \tau^{(0)})\), which indicates that the objective value converges. Or \(\sum \sum \varepsilon_{ij,m}^{(k)} \leq \delta_2 \approx 0\), which means \(x^{(k)}\) is already feasible for Model 4.

When ACP converges, if the slack variables all turn out to be zero, then the solution of Model 5 is a feasible solution to Model 4. In this situation, the feasible set of Model 5 is a subset of Model 4, and the solution will be a local optimum to Model 4, which also means it is a KKT point to Model 4. As long as the approximations (57)-(59) are accurate enough, the solution will be a KKT point to the original OPF problem.

Although the objective value of Model 5 will converge, it may converge to an infeasible point of the original OPF problem if the slack variables are not equal to zero. The convergence behavior of ACP depends mainly on two points:

1) The penalty parameter \(\tau_{\text{max}}\).
\(\tau_{\text{max}}\) should not be too small, because this leads easily to nonzero slack variables, nor too large, which may cause numerical problems.

2) The initial point.
A good starting point helps ACP finding a solution that all slack variables equal to zero. Since ACP aims to recover a feasible solution for SOCP relaxation, the initial point is chosen to be the result of convex relaxed OPF Model 3, which is actually a good choice considering both optimality and computation speed. It should be clarified that the initial point for ACP means \(x^{(k)}\) in (65) and (66) used for linearization, while
initial values for the whole problem is not needed because SOCP and ACP are both convex optimization problems. It is shown in the test results that, by choosing $r_{\text{max}}$ and the initial point appropriately, ACP always converges to a feasible point where all the slack variables are equal to zero within a few iterations.

IV. NUMERICAL RESULTS

In this section, IEEE benchmark test systems were used to demonstrate the effectiveness of the proposed algorithm. First, the nonlinear solver IPOPT was applied to find a local optimum of the original OPF problem. Then, Three SDP-based heuristic models aiming to recover feasible solutions for SDP relaxation were created to show whether they can achieve a feasible solution to the original OPF problem. Finally, the proposed ACP algorithm was tested to show its ability to recover a feasible solution for SOCP relaxation of the original OPF problem and compared with the other heuristic methods.

The SDP-based heuristic methods used in this paper are described as follows:

1) Penalized SDP Relaxation in [9] (PSDP1). In this method, the total amount of reactive power is added to the objective function to force the matrix rank to become one.

2) Penalized SDP Relaxation in [20] (PSDP2). In this method, the matrix rank is approximated by a continuous function and penalized in the objective function. The penalized SDP problem is solved by majorization-minimization method iteratively.

3) Difference-of-convex Programming in [25] (DSDP). In this method, the matrix rank one constraints are formulated as difference-of-convex inequalities, thus solved by convex-concave procedure iteratively. When the rank one equality is satisfied, SDP is identical to QCQP, so that DSDP can be regarded as a specific formulation of the method in [23].

The ACP algorithm, along with PSDP1, PSDP2, and DSDP, was implemented using YALMIP and MATLAB R2016a software. The SDP relaxation was implemented using sparse technique [34]. All the models were solved by MOSEK. Numerical tests were performed on a computer with an Intel® Core™ i5 (2.30 GHz) processor and 8 GB RAM. The original OPF problem was solved by MATPOWER using an IPOPT solver.

A. 9-bus test system

The IEEE 9-bus system consists of three generators and nine branches. The branch, bus, generator and generator cost data of the system were taken from MATPOWER. There were three generators connected to buses 1, 2, and 3, and the total real and reactive power capacity were 0 to 820 MW and -900 to 900 MVar, respectively. The voltage of bus 1 was set to $1.0 \times 0^\circ$. The lower and upper bounds of system bus voltages were 0.9 p.u. and 1.1 p.u., and the maximum phase angle difference was $10^\circ$.

The solution of MATPOWER is assumed to be a benchmark solution to the original OPF problem, and the sub-optimality gap of the heuristic methods are defined as:

$$\text{Gap} = \frac{v_{\text{other}} - v_{\text{MP}}}{|v_{\text{MP}}|} \times 100\%$$

where $v_{\text{MP}}$ is the objective value of the MATPOWER solution and $v_{\text{other}}$ is the objective value of ACP, PSDP1, PSDP2, and DSDP.

To demonstrate the effectiveness of the proposed algorithm, different scenarios were tested:

1) Generation cost minimization (congested operation)

In this test case, we considered the cost of generators under congested operating conditions. The maximum apparent power for each branch was set to 120MVA, and MATPOWER indicated that three branches were reaching its limit. Table I shows that ACP could recover a feasible solution to the original OPF problem from the result of tightened SOCP relaxation, as well as PSDP1, PSDP2, and DSDP, which were able to recover feasible solutions from SDP relaxations. While ACP and PSDP1 reached zero sub-optimality gap in this case, PSDP2 only recovered a near-global solution and DSDP recovered solutions far from the global optimum.

The computation time for each method is listed in Table I. For the iterative methods, the first iteration was SOCPT or SDP relaxation that aimed to obtain an initial point for the heuristic methods. It can be observed that among these methods, ACP consumed much less computation time than the other three methods, because in each iteration, ACP solved a SOCP optimization problem, while PSDP2 and DSDP solved a SDP optimization problem. Although PSDP1 only needed to solve SDP once, the computation burden of SDP was much larger than SOCPT even in a single iteration.

Fig. 1 shows the objective values and sums of slack variables generated by ACP in each iteration: the parameters were set to $r_{(0)} = r_{\text{max}} = 10^5$ and $\delta_{ij} = 10^{-6}$. It can also be seen that ACP converged in three iterations with the sum of slack variables converged to zero, and the objective value generated by the ACP is nonincreasing. Iteration 0 performed a SOCP relaxation to obtain an initial point for ACP, which was not part of the ACP nonincreasing sequence.

### Table I

**Numerical Results of 9-Bus System**

| Method  | Obj. Value | Gap (%) | Rank | Iteration | Solver time (s) |
|---------|------------|---------|------|-----------|-----------------|
| MP      | 5412.98    | -       | -    | -         | 0.75            |
| ACP     | 5412.98    | 0.00    | 4    | 1         | 0.18            |
| PSDP1   | 5413.38    | 0.01    | 1    | 2         | 1.09            |
| PSDP2   | 5412.98    | 0.00    | 1    | 2         | 3.25            |
| DSDP    | 5430.72    | 0.33    | 5    | 1         | 5.38            |

Fig. 1. Convergence behavior of ACP for generation cost minimization

2) Generation cost minimization (large phase angle difference)

The feasibility of ACP is dependent on the accuracy of approximations (57)-(59), to test the behavior of ACP when
there exists large phase angle differences in power systems, the reactance of branch 1-4 is changed from 0.0576p.u. to 0.576p.u., so that the phase angle difference across this branch can be as large as 33.25°. In this test case, ACP converged in three iterations with slack variables all converged to zero and obtained the same objective as MATPOWER. The voltage magnitude and phase angle data of each bus are shown in Table II. Comparing the results of MATPOWER and ACP, the mismatches in voltage magnitudes and phase angles were both very small, which proved that the solution recovered by ACP was a feasible solution.

3) Loss minimization (with transformer)

In this test case, the OLTC of transformer was considered to demonstrate the effectiveness of ACP in MINLP. A transformer was added to line 1-4, the turns ratio was between 0.9 to 1.1 with 0.2 per tap. Exact linearization of transformer [16] introduced binary variables to all the OPF models. The power loss and transformer turns ratio obtained by ACP and SOCPT is shown in Table III. The actual power loss is obtained by running power flow with generator output and turns ratio derived from these two methods. Since SOCPT is not exact, its control effect differs from the optimization results. To verify the results of ACP, the turns ratio of line 1-4 is set to 1.1 manually, and MATPOWER obtained the same result as ACP. The convergence behavior of ACP is shown in Fig. 2, with \( r^{(0)} = r^{\text{max}} = 10 \) and \( \delta_i = 10^{-5} \), ACP converged in three iterations and the slack variables all converged to zero.

In the MINLP test case, only ACP could recover a feasible solution, whereas the original mixed-integer nonconvex OPF model could not be solved by IPOPT, so well as mixed-integer SDP problems could not be solved by Mosek.

![Fig. 2. Convergence behavior of ACP for loss minimization with OLTC](image)

### Table II

| Bus | MATPOWER | ACP |
|-----|----------|-----|
|     | V (p.u.) | θ (degree) | V (p.u.) | θ (degree) |
| 1   | 1.000    | 0       | 1.000    | 0       |
| 2   | 1.100    | -26.310 | 1.100    | -26.305 |
| 3   | 1.100    | -28.267 | 1.100    | -28.262 |
| 4   | 0.931    | -33.255 | 0.931    | -33.251 |
| 5   | 0.942    | -35.485 | 0.942    | -35.482 |
| 6   | 1.054    | -31.047 | 1.054    | -31.043 |
| 7   | 1.030    | -32.903 | 1.030    | -32.899 |
| 8   | 1.047    | -30.568 | 1.047    | -30.564 |
| 9   | 0.926    | -36.330 | 0.926    | -36.327 |

Maximum relative error (%) 0.00 0.02

### Table III

| Test case | Obj. Value (MATPOWER) | Sub-optimality gap (%) | Opt. Gap (SOCPT) | Solver Time (s) |
|-----------|-----------------------|------------------------|------------------|-----------------|
| ACP       | PSDP1                 | PSDP2                  | PSDP1            | PSDP2           | PSDP3           | PSDP4           |
| Loss minimization problem (without transformer) (MW) | | | | |
| 9         | 3.546                 | 0.00                   | 0.00             | 0.11            | 7.87            | 0.69            | 0.11            | 0.80            | 5.61            | 4.65 |
| 14        | 0.635                 | 0.00                   | 0.00             | 0.00            | 0.00            | 0.70            | 0.15            | 0.83            | 13.00           | 4.08 |
| 30        | 1.777                 | 0.00                   | 0.00             | 4.46            | 0.00            | 0.93            | 0.18            | 1.35            | 30.42           | 6.79 |
| 57        | 12.148                | 0.00                   | 0.00             | 1.20            | 0.12            | 0.80            | 0.41            | 2.52            | 1883.82         | 26.05 |
| 118       | 10.667                | 0.00                   | 0.00             | *               | 0.83            | 1.09            | 0.92            | 10.96           | >4×10^4 [20]   | 55.13 |

Generation cost minimization problem (congested operation) ($/h)

| Test case | Obj. Value (MATPOWER) | Sub-optimality gap (%) | Opt. Gap (SOCPT) | Solver Time (s) |
|-----------|-----------------------|------------------------|------------------|-----------------|
| ACP       | PSDP1                 | PSDP2                  | PSDP1            | PSDP2           |
| Loss minimization problem (without transformer) (MW) | | | | |
| 9         | 5329.53               | 0.00                   | 0.01             | 0.00            | 0.33            | 0.75            | 0.18            | 1.09            | 3.05            | 5.38 |
| 14        | 9252.28               | 0.00                   | 0.00             | ×               | 0.00            | 0.77            | 0.32            | 0.94            | 5.48            | 4.28 |
| 30        | 582.79                | 0.00                   | 0.07             | 0.02            | 0.14            | 1.13            | 5.31            | 2.23            | 58.05           | 38.46 |
| 57        | 43697.64              | 0.00                   | 0.00             | ×               | 0.00            | 0.76            | 1.46            | 4.08            | 1945.05         | 14.5 |
| 118       | 134007.40             | 0.01                   | ×                | *               | 0.12            | 1.06            | 2.73            | 10.26           | >4×10^5 [20]   | 62.99 |

*– Infeasible solution, *– numerical problems.
V. CONCLUSION

In this paper, an ACP algorithm was proposed to recover a global or near-global optimal solution for SOCP relaxed OPF problem in mesh networks when the convex relaxation method is not exact. The OPF problem was first formulated as a DCP problem to maintain equality in the nonconvex power flow equations, then solved efficiently by penalty CCP. A tightened SOCP relaxation of the OPF problem in mesh networks was also proposed to provide a good initial point for the ACP algorithm. Numerical results showed that the proposed algorithm could recover global or near-global optimal solutions for SOCP relaxation with various objective functions and generally performed better than the SDP-based recovery methods in solution quality. The computational efficiency of the proposed algorithm was comparable to SOCP, which was far beyond the SDP-based methods.

Since every iteration of ACP is a convex optimization problem, the proposed method is suitable for more complicated optimization problems in power systems such as MINLP or distributed control optimization which require a convex formulation of power flow equations. We have demonstrated the capability of ACP in MINLP with a test case considering transformer turns ratio in this paper. The application of ACP in other problems deserve further investigation in our future research.

REFERENCES

[1] X. Bai, H. Wei, K. Fujisawa, and Y. Wang, “Semidefinite programming for optimal power flow problems,” Int. J. Electr. Power Energy Syst., vol. 30, no. 6-7, pp. 383-392, March 2008.
[2] R. A. Jabr, ”Radial Distribution Load Flow Using Conic Programming,” IEEE Trans. Power Syst., vol. 21, no. 3, pp. 1458-1459, Aug. 2006.
[3] B. C. Lesieutre, D. K. Moltzahn, A. R. Borden and C. L. DeMarco, ”Examining the limits of the application of semidefinite programming to power flow problems,” 2011 49th Annual Allerton Conference on Communication, Control, and Computing (Allerton), Monticello, IL, 2011, pp. 1492-1499.
[4] R. A. Jabr, ”A Conic Quadratic Format for the Load Flow Equations of Meshed Networks,” IEEE Trans. Power Syst., vol. 22, no. 4, pp. 2285-2286, Nov. 2007.
[5] R.A. Jabr, ”Optimal Power Flow Using an Extended Conic Quadratic Formulation,” IEEE Trans. Power Syst., vol. 23, no. 3, pp. 1000-1008, 2008.
[6] J. Lavaei and S. H. Low, ”Zero Duality Gap in Optimal Power Flow Problem,” IEEE Trans. Power Syst., vol. 27, no. 1, pp. 92-107, Feb. 2012.
[7] S. H. Low, ”Convex Relaxation of Optimal Power Flow—Part I: Formulations and Equivalence,” IEEE Trans. Control Netw. Syst., vol. 1, no. 1, pp. 15-27, March 2014.
[8] S. H. Low, ”Convex Relaxation of Optimal Power Flow—Part II: Exactness,” IEEE Trans. Control Netw. Syst., vol. 1, no. 2, pp. 177-189, June 2014.
[9] R. Madani, S. Sojoudi and J. Lavaei, ”Convex Relaxation for Optimal Power Flow Problem: Mesh Networks,” IEEE Trans. Power Syst., vol. 30, no. 1, pp. 199-211, Jan. 2015.
[10] M. Farivar and S. H. Low, ”Branch Flow Model: Relaxations and Convexification—Part I,” IEEE Trans. Power Syst., vol. 28, no. 3, pp. 2554-2564, Aug. 2013.
[11] L. Gan, N. Li, U. Topcu, and S. H. Low, ”Exact Convex Relaxation of Optimal Power Flow in Radial Networks,” IEEE Trans. Autom. Control, vol. 60, no. 1, pp. 72-87, 2015.
[12] M. Farivar and S. H. Low, ”Branch Flow Model: Relaxations and Convexification—Part II,” IEEE Trans. Power Syst., vol. 28, no. 3, pp. 1655-1672, 2013.
[13] B. Kocuk, S. S. Dey, and X. A. Sun, ”Strong SOCP relaxations for optimal power flow,” Oper. Res., vol. 64, no. 6, pp. 1177-1196, May 2016.
[14] B. Kocuk, S. S. Dey and X. A. Sun, ”New Formulation and Strong MISOCP Relaxations for AC Optimal Transmission Switching Problem,” IEEE Trans. Power Syst., vol. 32, no. 6, pp. 4161-4170, Nov. 2017.
[15] W. Wu, Z. Tian and B. Zhang, ”An Exact Linearization Method for OLTc of Transformer in Branch Flow Model,” IEEE Trans. Power Syst., vol. 32, no. 3, pp. 2475-2476, May 2017.
[16] E. Dall’Anese, H. Zhu and G. B. Giannakis, ”Distributed Optimal Power Flow for Smart Microgrids,” IEEE Trans. Smart Grid, vol. 4, no. 3, pp. 1464-1475, Sept. 2013.
[17] W. Zheng, W. Wu, B. Zhang, H. Sun and Y. Liu, ”A Fully Distributed Reactive Power Optimization and Control Method for Active Distribution Networks,” IEEE Trans. Smart Grid, vol. 7, no. 2, pp. 1021-1033, March 2016.
[18] C. Zhao, J. Wang, J. P. Watson and Y. Guan, ”Multi-Stage Robust Unit Commitment Considering Wind and Demand Response Uncertainties,” IEEE Trans. Power Syst., vol. 28, no. 3, pp. 2708-2717, Aug. 2013.
[19] B. Zeng and L. Zhao, ”Solving two-stage robust optimization problems using a column-and-constraint generation method,” Oper. Res. Lett., vol. 41, no. 5, pp. 457–461, Jun. 2013.
[20] T. Liu; B. Sun; D. H. K. Tsang, ”Rank-one Solutions for SDP Relaxation of QCQPs in Power Systems,” IEEE Trans. Smart Grid, to be published, doi: 10.1109/TSG.2017.2729082.
[21] D. K. Moltzahn and I. A. Hiskens, ”Moment-based relaxation of the optimal power flow problem,” 2014 Power Systems Computation Conference, Wroclaw, 2014, pp. 1-7.
[22] D. K. Moltzahn, C. Josz, I. A. Hiskens and P. Panciatici, ”A Laplacian-Based Approach for Finding Near Globally Optimal Solutions to OPF Problems,” IEEE Trans. Power Syst., vol. 32, no. 1, pp. 305-315, Jan. 2017.
[23] A. S. Zamzam; N. D. Sidirooulos; E. Dall’Anese,”Beyond Relaxation and Newton-Raphson: Solving AC OPF for Multi-phase Systems with Renewables,” IEEE Trans. Smart Grid, to be published, doi: 10.1109/TSG.2016.2645220.
[24] S. Merkli; A. Domahidi; J. Jerez; M. Morari; R. S. Smith,”Fast AC Power Flow Optimization Using Difference of Convex Functions Programming,” IEEE Trans. Power Syst., to be published, doi: 10.1109/TPWRS.2017.2688329.
[25] W. Wei, J. Wang, N. Li and S. Mei,”Optimal Power Flow of Radial Networks and its Variations: A Sequential Convex Optimization Approach,” IEEE Trans. Smart Grid, to be published, doi: 10.1109/TSG.2017.2684183.
[26] J. Park and S. Boyd,”General Heuristics for Nonconvex Quadratically Constrained Quadratic Programming,” arXiv preprint arXiv:1703.07870, 2017.
[27] L. Yuille and A. Rangarajan,”The concave-convex procedure,” Neural Comput., vol. 15, no. 4, pp. 915–936, Apr. 2003.
[28] J. Smola, S. V. N. Vishwanathan, and T. Hofmann,”Kernel methods for missing variables,” Proc. 10th Int. Workshop Artif. Intell. Stat., Mar. 2005, pp. 325–332.
[29] T. Lipp and S. Boyd,”Variations and extension of the convex-concave procedure,” Optim. Eng., vol. 17, no. 2, pp. 263–287, 2016.
[30] W. Wu, Z. Tian and B. Zhang,”An Exact Linearization Method for OLTc of Transformer in Branch Flow Model,” IEEE Trans. Power Syst., vol. 32, no. 3, pp. 2475-2476, May 2017.
[31] H. Hijazi, C. Coffrin, and P. Van Hentenryck,”Convex quadratic relaxations of mixed-integer nonlinear programs in power systems,” NICTA, Canberra, ACT Australia, Tech. Rep., 2013.
[32] G. McCormick,”Computability of global solutions to factorable nonconvex programs: Part I convex underestimating problems,” Math. Program., vol. 10, no. 1, pp. 147-175, 1976.
[33] S. Kim and M. Kojima,”Exact solutions of some nonconvex quadratic optimization problems via SDP and SOCP relaxations,” Computational Optimization and Applications, vol. 26, no. 2, pp. 143–154, Nov. 2003.
[34] R. Madani, M. Ashraphijuo and J. Lavaei,”Promises of Conic Relaxation for Contingency-Constrained Optimal Power Flow Problem,” IEEE Trans. Power Syst., vol. 31, no. 2, pp. 1297-1307, March 2016.