APPLICATIONS OF THE DUALITY BETWEEN THE COMPLEX MONGE-AMPÈRE EQUATION AND THE HELE-SHAW FLOW

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ABSTRACT. We give two applications of the duality between the complex Homogeneous Monge-Ampère Equation (HMAE) and the Hele-Shaw flow. First, we prove existence of smooth boundary data for which the weak solution to the Dirichlet problem for the HMAE over \( \mathbb{P}^1 \times \partial \mathbb{D} \) is not twice differentiable at a given collection of points, and also examples that are not twice differentiable along a set of codimension one in \( \mathbb{P}^1 \times \partial \mathbb{D} \). Second we discuss how to obtain explicit families of smooth geodesic rays in the space of Kähler metrics on on \( \mathbb{P}^1 \) and on the unit disc \( \mathbb{D} \) that are constructed from an exhausting family of increasing smoothly varying simply connected domains.

1. INTRODUCTION

The purpose of this paper is to give two applications of previous work of the authors that describes a duality between a certain Dirichlet problem for the complex Homogeneous Monge-Ampère Equation (HMAE) and a free boundary problem in the plane called the Hele-Shaw flow [19]. First, for any finite set of points in \( \mathbb{P}^1 \times \partial \mathbb{D} \), where \( \mathbb{D} \subset \mathbb{C} \) denotes the unit disc, we give examples of smooth boundary data for which the weak solution to this Dirichlet problem over \( \mathbb{P}^1 \times \partial \mathbb{D} \) is not twice differentiable at these points. We also produce such examples that are not twice differentiable along a set of codimension one in \( \mathbb{P}^1 \times \partial \mathbb{D} \). Second, we use this duality to produce families of regular solutions to this Dirichlet problem over the punctured disc \( \mathbb{D}^\times \), giving explicit families of smooth geodesic rays in the space of Kähler metrics on \( \mathbb{P}^1 \) and on \( \mathbb{D} \).

1.1. Regularity of the Dirichlet problem for the HMAE over the disc. The setup for the first application is as follows. Fix a chart \( 0 \in \mathbb{C} \subset \mathbb{P}^1 \) with coordinate \( z \) and let \( \omega \) denote the Fubini-Study form. Choose a Kähler potential \( \phi \in C^\infty(\mathbb{P}^1) \), by which we mean \( \omega + dd^c \phi \) is a Kähler form, and let \( \pi_{\mathbb{P}^1} : \mathbb{P}^1 \times \partial \mathbb{D} \to \mathbb{P}^1 \) be the projection. Consider the envelope

\[
\Phi := \sup \left\{ \psi : \mathbb{P}^1 \times \partial \mathbb{D} \to \mathbb{R} \cup \{-\infty\} : \psi \text{ is usc, } \pi_{\mathbb{P}^1}^* \omega + dd^c \psi \geq 0 \right\},
\]

which is the weak solution to the Dirichlet problem

\[
\Phi(z, \tau) = \phi(\tau z) \text{ for } (z, \tau) \in \mathbb{P}^1 \times \partial \mathbb{D},
\]

\[
\pi_{\mathbb{P}^1}^* \omega + dd^c \Phi \geq 0,
\]

\[
(\pi_{\mathbb{P}^1}^* \omega + dd^c \Phi)^2 = 0.
\]

The following is a preliminary version of what we shall prove:

**Theorem A.** Let \( S \) be a union of finitely many points and non-intersecting smooth curve segments in \( \mathbb{P}^1 \setminus \{0\} \). Then there exist Kähler potentials such that the above weak solution \( \Phi \) to the HMAE is not twice differentiable at any \( (\tau^{-1} z, \tau), z \in S, |\tau| = 1 \).
The question of regularity of solutions to the HMAE has a long history, and has proved to be a difficult problem that depends subtly on the boundary data (see, for example, Lempert [12], Bedford-Demailly [2] or Błocki [3]). As is well known, if \( D \) is replaced by a closed annulus in \( \mathbb{C} \), and \( P_1 \) is replaced by any Kähler manifold \( X \), the above Dirichlet problem with \( S^1 \)-invariant boundary data corresponds to finding a geodesic segment in the space of Kähler potentials on \( X \) (and similarly if \( D \) is replaced by the punctured disc \( D^\times \) it corresponds to finding a geodesic ray). The regularity of these geodesics has been of intense interest ever since this space was considered by Mabuchi [15] Semmes [22] and Donaldson [7]. However it is only since the relatively recent work of Lempert-Vivas [13] and Lempert-Darvas [14] that we have known that it is not always possible to join two potentials by a geodesic segment that lies in the class \( C^2 \).

What we have here is similar in spirit, but is in a sense stronger that the result of Lempert-Vivas in that we are able to prescribe the location of the singular locus (which need not consist of isolated points), as well as see exactly how the regularity fails; we are not aware of any similar result in the theory of the HMAE in which this precise information about the weak solution is available, other than the toric case [20, 21].

What permits us to have such a good understanding of the singularities of \( \Phi \) is the connection with the Hele-Shaw flow. To define this, suppose \((X, \omega)\) is a one-dimensional Kähler manifold, which we will usually take to be either \( \mathbb{P}^1 \) with its Fubini-Study form \( \omega_{FS} \), or the open unit disc \( D \subset \mathbb{C} \) with the Poincaré form \( \omega_P \). In the first case we use the convention that \( \mathbb{P}^1 \) has area one (of course the Poincaré metric on the disc has infinite area), so

\[
V := \int_X \omega \in \{1, \infty\}.
\]

The unit disc has the origin as a distinguished point, and when \( X = \mathbb{P}^1 \) we fix a point that we denote by \( 0 \in \mathbb{P}^1 \).

Given any \( \phi \in C^\infty(X) \) (if \( X \) is noncompact we also assume \( \phi \) to be bounded) such that \( \omega + dd^c \phi \) is a Kähler form, the Hele-Shaw flow consists of an increasing collection of sets

\[
\Omega_t \subset X \text{ for } t \in (0, V)
\]

such that \( \Omega_t \) has area \( t \) with respect to \( \omega + dd^c \phi \). It is defined by setting

\[
\Omega_t := \{ z \in X : \psi_t(z) < \phi(z) \}
\]

where

\[
\psi_t := \sup \{ \psi : \psi \text{ is usc and } \psi \leq \phi \text{ and } \omega + dd^c \psi \geq 0 \text{ and } \nu_0(\psi) \geq t \}.
\]

By this we mean the supremum is over all upper semicontinuous (usc) functions \( \psi : X \to \mathbb{R} \cup \{-\infty\} \) with these properties, and \( \nu_0(\psi) \) denotes the order of the logarithmic singularity (Lelong number) of \( \psi \) at \( 0 \in X \).

What is proved\(^1\) in [19] is that this flow is intimately connected to the weak solution \( \tilde{\Phi} \) to the Dirichlet problem for the complex HMAE on \( X \times \mathbb{D}^\times \) with boundary data the pullback of \( \phi \) to \( X \times \partial \mathbb{D} \) and a certain prescribed singularity at \((0, 0)\); in fact \( \psi_t \) is the Legendre transform of \( \tilde{\Phi} \). Moreover there is a simple way to transform between \( \Phi \) and \( \tilde{\Phi} \), and thus each contain the same information as the Hele-Shaw flow (we shall recall this in more detail in Section 4).

\(^1\)Strictly speaking only the case that \( X = \mathbb{P}^1 \) is considered in [19], but the proof works for \( X = \mathbb{D} \).
To state our first result more precisely we need to consider flows of sets that develop singularities in a particularly simple way. Let $S$ be the union of finitely many points and non-intersecting smooth curve segments in $\mathbb{P}^1 \setminus \{0\}$.

**Definition 1.1.** We say that the Hele-Shaw flow develops tangency along $S$ if there exists a $T \in (0,1)$ such that (1) $\Omega_t$ is smoothly bounded, simply connected and varies smoothly for $t < T$ and (2) $\partial \Omega_T$ is the image of a smooth locally embedded curve intersecting itself tangentially precisely along $S$ (see Figure 1).

![Figure 1. Developing tangency along $S$](image)

**Theorem B.** Let $\phi \in C^\infty(\mathbb{P}^1)$ be a Kähler potential and suppose the Hele-Shaw flow develops tangency along $S$. Then the weak solution $\Phi$ from (1) to the Dirichlet problem for the HMAE on $\mathbb{P}^1 \times \mathbb{D}$ with boundary data $(z, \tau) \mapsto \phi(\tau z)$ is not twice differentiable at the points $(\tau^{-1}z, \tau)$, $z \in S$, $|\tau| = 1$.

We note that actually we know more, and from the discussion below it will be apparent that there is an explicit open set in $\mathbb{P}^1 \times \mathbb{D}$ on which $\Phi$ smooth. With more work it may be possible to describe precisely where (and how) $\Phi$ fails to be twice differentiable, but we shall not consider that further in this paper.

It remains to comment that to get Theorem A from Theorem B we have to show that given such a set $S$ it is possible to find a Kähler potential whose Hele-Shaw flow develops tangency along $S$. To do this we first choose $\Omega_T$ as in Definition 1.1 and we aim to find a Kähler potential for which we can understand the Hele-Shaw flow backwards for a small time, say for $t \in [T - \epsilon, T]$. As is well known in the Hele-Shaw literature it is not normally the case that the strong Hele-Shaw flow exists backwards in time starting at some $\Omega_T$ (for instance if $\omega$ is analytic then a necessary condition is that $\Omega_T$ has analytic boundary). However, using a previous result of the authors [18], we shall see that this assumption is not necessary as long as one allows a (smooth) modification of the area form near $\Omega_T$ (said another way, we make a smooth modification of the permeability that governs the flow). We may then shrink $\Omega_{T-\epsilon}$ down to 0, and expand $\Omega_T$ out to $\infty$, so as to obtain a flow of sets $\{\Omega_t\}_{t \in (0,1)}$ with properties that ensure that it is the Hele-Shaw flow for some Kähler potential that can be constructed from the flow. Details can be found in Section 3.

**Families of Geodesic Rays.** As already mentioned, the weak solution $\tilde{\Phi}$ to the Dirichlet problem for the HMAE over the punctured disc $\mathbb{D}^\times$ with $S^1$-invariant boundary data is by definition a weak geodesic ray in the space of positive potentials on $X$. If this solution is
regular (by which we mean it is smooth and strictly $\omega$-plurisubharmonic along the fibres over $\mathbb{C}^N$) then it gives a genuine geodesic in this space, i.e. a smooth geodesic in the space of Kähler metrics. For this reason regularity of the weak geodesic ray is of interest, and following [19] we know that on $\mathbb{P}^1$ and $\mathbb{D}$, this regularity is intimately related to the topology of the Hele-Shaw flow. To state our theorems in the simplest way, let $B(t)$ denote the geodesic ball in $X$ centred at 0 with area $t$ taken with respect to the metric $\omega$.

**Definition.** Let $a \in [0, V]$. We say that a collection of subsets $\{\Omega_t\}_{t \in (0, V)}$ of $X$ is standard as $t$ tends to $a$ if there exist $\epsilon > 0$ such that $\Omega_t = B(t)$ for $|t - a| < \epsilon$.

**Theorem C.** Let $X = \mathbb{P}^1$ or $X = \mathbb{D}$ and suppose the flow Hele-Shaw $\{\Omega_t\}_{t \in (0, V)}$ for a Kähler form $\omega + dd^c \phi$ satisfies

1. $\{\Omega_t\}_{t \in (0, V)}$ is smoothly bounded and varies smoothly with non-vanishing normal velocity,
2. $\Omega_t$ is simply connected for all $t \in (0, V)$,
3. if $X = \mathbb{P}^1$ then $\{\Omega_t\}_{t \in (0, V)}$ is standard as $t$ tends to 1.

Then the weak geodesic ray obtained as the Legendre transform of the Hele-Shaw envelopes $\{\psi_t\}$ is regular, and so defines a smooth geodesic ray in the space of Kähler metrics on $X$.

Of course, for this theorem to have any content we must be able to provide examples of potentials $\phi$ for which the Hele-Shaw has these properties. An interesting case of this is given by a result of Hedenmalm-Shimorin:

**Theorem HS.** (Hedenmalm-Shimorin [10]) Let $(X, \omega) = (\mathbb{D}, \omega_{\mathbb{D}})$ and suppose that $\phi$ is taken so that the Kähler form $\omega_{\mathbb{D}} + dd^c \phi$ is analytic and hyperbolic. Then the Hele-Shaw flow $\{\Omega_t\}$ for $\omega + dd^c \phi$ is smoothly bounded, smoothly varying, and simply connected for all $t \in (0, \infty)$.

Another class of examples can be constructed from an observation due to Berndtsson (following a question of Zelditch) which says that any reasonable smooth increasing family of simply connected domains is the Hele-Shaw flow for some smooth Kähler potential, see Theorem 3.1 (we observe that what we then get for $\mathbb{P}^1$ turns out to be essentially the same geodesic ray as that described by Donaldson [6, p24]).

Of course it is trivial to construct families of domains $\{\Omega_t\}$ that satisfy the required hypotheses, and thus combining with Theorem C gives an easy way to construct explicit families of smooth geodesic rays in the space of Kähler metrics on $\mathbb{P}^1$ (resp. on $\mathbb{D}$). In particular we have that any hyperbolic analytic Kähler metric on $\mathbb{D}$ is the starting point for some canonical smooth geodesic ray.

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2. The Hele Shaw Flow

2.1. Definition and Preliminaries. Suppose \((X, \omega)\) is a one-dimensional simply-connected Kähler manifold and \(\phi \in C^\infty(X) \cap L^\infty(X)\) is such that \(\omega_\phi := \omega + dd^c \phi\) is Kähler. Let \(V := \int_X \omega \in (0, \infty)\) and fix an origin \(0 \in X\). We use the convention \(dd^c = \frac{1}{2\pi} (\overline{\partial} - \partial)\) so \(dd^c \log |z|^2 = \delta_0\). On \(\mathbb{P}^1\) we always having in mind a chart \(0 \in \mathbb{C} \subset \mathbb{P}^1\) with coordinate \(z\) so the Fubini-Study metric \(\omega_{FS}\) has local potential \(\log(1 + |z|^2)\) on \(\mathbb{C}\) giving \(\mathbb{P}^1\) area 1. We let \(dA\) denote the Lebesgue measure on \(\mathbb{C}\).

Definition 2.1. For \(t \in (0, V)\) set
\[
\psi_t := \sup \{ \psi : X \to \mathbb{R} \cup \{-\infty\} : \psi \text{ is usc and } \psi \leq \phi \text{ and } \omega + dd^c \psi \geq 0 \text{ and } \nu_0(\psi) \geq t \}.
\]

Here \(\nu_0\) denotes the Lelong number at 0, so \(\nu_0(\psi) \geq t\) means that \(\psi(z) \leq t \ln |z|^2 + O(1)\) near 0. As the upper semi-continuous regularisation of \(\psi_t\) is itself a candidate for the envelope defining \(\psi_t\), we see that \(\psi_t\) is usc.

Definition 2.2. For \(t \in (0, V)\) set
\[
\Omega_t := \{ z \in X : \psi_t(z) < \phi(z) \}. \tag{2}
\]

It is easy to see that if \(\phi\) is replaced by \(\phi + h\) for some harmonic function \(h\) then \(\psi_t\) is replaced by \(\psi_t + h\). Thus \(\Omega_t\) depends only on \(\omega_\phi\).

Definition 2.3. (Hele-Shaw flow) We refer to collection of sets \(\{\Omega_t\}_{t \in (0, V)}\) as the Hele-Shaw flow associated to \((X, \omega_\phi)\) and the collection \(\{\psi_t\}_{t \in (0, V)}\) as the Hele-Shaw envelopes associated to \((X, \omega, \phi)\).

Remark 2.4. What we have called the Hele-Shaw flow is often called the “weak Hele-Shaw flow”. If \((a, b) \subset (0, V)\) we will also refer to the subcollection \(\{\Omega_t\}_{t \in (a, b)}\) as the Hele-Shaw flow and similarly for the envelopes.

We first record some basic properties of this flow:

Proposition 2.5.

1. \(\Omega_t\) is an open connected set containing the origin for all \(t \in (0, V)\).
2. \(\partial \Omega_t\) has measure zero.
3. \(\psi_t\) is \(C^{1,\omega}\) on \(X \setminus \{0\}\).
4. \(\omega_{\psi_t} = (1 - \chi_{\Omega_t})\omega_\phi + t\delta_0\)
in the sense of currents. Here \(\chi_A\) denotes the characteristic function of a set \(A\), and \(\delta_0\) the Dirac delta.
5. For \(t \in (0, V)\) we have
\[
\int_{\Omega_t} \omega_\phi = t.
\]

Proof. This is standard material for the Hele-Shaw flow, and the details are given in [19, Proposition 1.1] (the cited reference is for \(X = \mathbb{P}^1\), but the same proof applies for \(\mathbb{D}\) or \(\mathbb{C}\)). By uniformization, \(X\) is conformally equivalent to one of these three, which is enough to prove the statement in general. \(\square\)

Our next Lemma says that the Hele-Shaw flow is local, by which we mean \(\Omega_t\) depends only one the restriction of \(\omega_\phi\) to a neighbourhood of \(\Omega_t\).
Lemma 2.6 (Locality of the Hele-Shaw Flow). Let \( X' \subset X \) be an open simply-connected set containing 0. Suppose \( \omega_\phi \) and \( \omega_\tilde{\phi} \) are Kähler forms on \( X \) such that \( \phi|_{X'} = \tilde{\phi}|_{X'} \) and denote the Hele-Shaw flow for \((X, \omega_\phi)\) and \((X, \omega_\tilde{\phi})\) by \( \Omega_t \) and \( \tilde{\Omega}_t \) respectively (and similarly for \( \psi_t \) and \( \tilde{\psi}_t \)).

Then \( \Omega_t = \tilde{\Omega}_t \) and \( \psi_t = \tilde{\psi}_t \) as long as \( \Omega_t \in X' \).

Proof. Define \( \gamma \) to be equal to \( \psi_t \) on \( X' \) and equal to \( \tilde{\phi} \) on \( X \setminus X' \). Since \( \Omega_t \) is relatively compact in \( X' \) we see that \( \gamma = \phi = \tilde{\phi} \) on \( X \setminus \Omega_t \). Clearly \( \gamma \leq \phi \) on \( X' \), so as \( \phi|_{X'} = \tilde{\phi}|_{X'} \) we have \( \gamma \) is bounded above by \( \tilde{\phi} \) and clearly \( \omega + dd^c \gamma \geq 0 \).

Hence \( \gamma \) is a candidate for the envelope defining \( \tilde{\psi}_t \) giving \( \gamma \leq \tilde{\psi}_t \). Thus

\[
\psi_t|_{X'} \leq \tilde{\psi}_t|_{X'}.
\]

This implies \( X' \setminus \Omega_t \subset X' \setminus \tilde{\Omega}_t \). So as \( \tilde{\Omega}_t \) is connected by Proposition 2.3 and intersects \( \Omega_t \) (they both contain the origin) we must actually have \( \Omega_t \subset \tilde{\Omega}_t \). In particular, \( \tilde{\Omega}_t \) is relatively compact in \( X' \), so we may run the argument with \( \Omega_t \) and \( \tilde{\Omega}_t \) swapped to conclude \( \Omega_t \subset \tilde{\Omega}_t \). \( \square \)

2.2. The Strong Hele-Shaw Flow. We shall also need the notion of a strong solution to the Hele-Shaw flow. We shall only consider this in the plane, so suppose \( \{ \Omega_t \}_{t \in (a, b)} \) is a smooth increasing family of domains of \( \mathbb{C} \). By this we mean each \( \Omega_t \) is smoothly bounded and varies smoothly, so locally \( \partial \Omega_t \) is the graph of a smooth function that varies smoothly with \( t \). So if \( n \) denotes the outward unit normal vector field \( n \) on \( \partial \Omega_{t_0} \) for some \( t_0 \), then for \( t \) close to \( t_0 \) we can write \( \partial \Omega_t = \{ x + f(x, t) n_x : x \in \partial \Omega_{t_0} \} \) for some smooth function \( f_t(x) = f(x, t) \) on \( \partial \Omega_{t_0} \) that is positive for \( t > t_0 \) and negative for \( t < t_0 \). Then the normal velocity of \( \partial \Omega_{t_0} \) is defined to be

\[
V_{t_0} := \frac{df}{dt} \bigg|_{t=0} n.
\]

Now assume also each \( \Omega_t \) contains the origin. For each \( t \) let \( p_t(z) := -G_{\Omega_t}(z, 0) \) where \( G_{\Omega_t} \) denotes the Green’s function for \( \Omega_t \) with logarithmic singularity at the origin. Thus \( p_t = 0 \) on \( \partial \Omega_t \) and \( \Delta p_t = -\delta_0 \).

The statement that \( p_t \) exists and is smooth on \( \overline{\Omega}_t \setminus \{0\} \) is classical (this follows immediately from regularity of the Dirichlet problem for the Laplacian (e.g. [11] Proposition 1.3.11) which can be found, for instance, in [9] Chapter 6)). We also fix a smooth area form \( \eta \) on \( \mathbb{C} \) which we write as

\[
\eta = \frac{1}{\kappa} dA
\]

where \( dA \) is the Lebesgue measure and \( \kappa \) is a strictly positive real-valued smooth function on \( \mathbb{C} \).

Definition 2.7. (Strong Hele-Shaw flow) We say that \( \{ \Omega_t \}_{t \in (a, b)} \) is the strong Hele-Shaw flow if

\[
V_t = -\kappa \nabla p_t \text{ on } \partial \Omega_t \text{ for } t \in (a, b)
\]

where \( V_t \) is the normal velocity of \( \partial \Omega_t \).

Remark 2.8. The strong Hele-Shaw flow has an interpretation as the flow of a fluid moving between two plates in a medium which has a permeability encoded by the function \( \kappa \), under injection of fluid at the origin (see [13] for a discussion, and also [8] for a comprehensive account of the subject, which for the most part considers the case where \( \kappa \equiv 1 \)). When
Then by construction $\omega \{ \}$ and assume increasing simply connected domains that is the strong Hele-Shaw flow with respect to $\kappa$ (Gustafsson) Suppose that Proposition 2.11.

Taking the limit as $t \to 0$ we have

Proof. We compute

$$\frac{d}{dt} \int_{\Omega_t} h^\frac{1}{\kappa} dA = \int_{\partial \Omega_t} h V_t \kappa ds = -\int_{\partial \Omega_t} h \frac{\partial p_t}{\partial n} ds$$

$$= \int_{\Omega_t} (p_t \Delta(h) - h(\Delta p_t)) dA - \int_{\partial \Omega_t} p_t \frac{\partial h}{\partial n} ds \geq h(0)$$

since $\Delta h \geq 0$ and $p_t = 0$ on $\partial \Omega_t$ and $\Delta p_t = -\delta_0$. □

Corollary 2.10. With the assumption of the above lemma, suppose that $a = 0$ and $\Omega_t$ tends to $\{0\}$ as $t \to 0$, that is given any neighbourhood $U$ of the origin $\Omega_t \subset U$ for $t$ sufficiently small. Then for any integrable subharmonic function $h$ on $\Omega_t$, we have

$$\int_{\Omega_t} h^\frac{1}{\kappa} dA \geq th(0).$$

and moreover equality holds if $h$ is holomorphic.

Proof. Taking the limit as $t_0 \to 0$ in the above Lemma gives the first statement. The second follows as if $h$ is holomorphic then $h$ and $-h$ are subharmonic. □

Proposition 2.11. (Gustafsson) Suppose that $\{\Omega_t\}_{t \in (0, b)}$ is a smooth family of strictly increasing simply connected domains that is the strong Hele-Shaw flow with respect to $\kappa$, and assume $\{\Omega_t\}_{t \in (0, b)}$ tends to $\{0\}$ as $t \to 0$. Set

$$\phi(z) = \int_{\Omega} \log |z - \zeta|^2 \frac{dA_\kappa}{\kappa(\zeta)} - \log(1 + |z|^2) \text{ for } z \in \mathbb{C}.$$ 

Then $\{\Omega_t\}_{t \in (0, b)}$ is the Hele-Shaw flow with respect to $\omega_\phi := d\phi'(\log(1 + |z|^2) + \phi)$.

Proof. For the proof we shall write $\Omega^\omega_t := \{z \in X : \psi_t(z) < \phi(z)\}$ for the Hele-Shaw flow with respect to $\omega_\phi$, so the goal is to prove $\Omega^\omega_t = \Omega_t$. Define

$$\tilde{\psi}_t(z) := \int_{\Omega_t} \log |z - \zeta|^2 \frac{dA_\kappa}{\kappa(\zeta)} - \log(1 + |z|^2) + t \ln |z|^2.$$ 

Then by construction $\omega_{\tilde{\psi}_t} \geq 0$ and $\nu_0(\tilde{\psi}_t) = t$. As $h(\zeta) := \log |z - \zeta|^2$ is subharmonic and integrable, we get from the previous Corollary that for all $z \in \mathbb{C}$

$$\phi(z) - \tilde{\psi}_t(z) = \int_{\Omega_t} \log |z - \zeta|^2 \frac{dA_\kappa}{\kappa(\zeta)} - t \ln |z|^2 \geq 0. \quad (5)$$
Remark 2.12. Although we will shall not really need it, we remark that there is a \(2.5(4)\) with Lelong number one at \(\Omega\) as well. Thus we conclude
\[
\psi_t = \psi_t = \phi \text{ on } \Omega_t^c.
\]
Now \(\psi_t + \log(1 + |z|^2)\) and \(\psi_t + \log(1 + |z|^2)\) are both harmonic on \(\Omega_t \setminus \{0\}\) (by Proposition 2.5(4)) with Lelong number one at 0. Hence by the maximum principle \(\psi_t = \psi_t \text{ on } \Omega_t \setminus \{0\}\) as well. Thus we conclude \(\Omega_t^w = \Omega_t\) as desired. \(\square\)

Finally, we state two previous results of the authors that give existence results for the strong Hele-Shaw flow. The first says this flow always exists for small time.

Theorem 2.13. \([18, \text{Theorem } 2.1]\) The Hele-Shaw flow for any Kähler form \(\omega + dd^c \phi\) is a smoothly bounded Jordan domain for some \(T \in (0, V)\) then there is an \(\epsilon > 0\) such that \(\{\Omega_t\}_{t \in (T-\epsilon, T+\epsilon)}\) is actually the strong Hele-Shaw flow. Thus the hypothesis that \(\{\Omega_t\}\) varies smoothly in Theorem [C] (as well in Definition [1.1] is redundant. The proof of this statement follows easily from the work in [18]; specifically from [18, Remark 3.12] the Hele-Shaw domains \(\Omega_t\) all lift to holomorphic curves \(\Sigma_T\) in \(\mathbb{C} \times \mathbb{P}^1\) with boundary contained in the submanifold given as the graph of \(\frac{\partial \phi}{\partial z}\). The hypothesis on \(\Omega_T\) imply that \(\Sigma_T\) is a holomorphic disc, at which point we can run the proof of [18, Theorem 2.2].

The second says that any simply connected bounded Jordan domain \(\Omega\) is part of a strong Hele-Shaw flow, both backwards and forwards in time, as long as one allows a modification of the area form inside \(\Omega\).

Theorem 2.14. \([18, \text{Theorem } 2.2, \text{Remark } 7.1]\) Let \(\eta\) be a smooth area form on \(\mathbb{C}\) and \(\Omega\) a smoothly bounded Jordan domain containing the origin. Then there exists a smooth area form \(\eta'\) on a neighbourhood of \(\Omega\) such that \(\eta = \eta'\) on \(\Omega\) and so that \(\Omega = \Omega_T\) is part of a strong Hele-Shaw flow \(\{\Omega_t\}_{t \in (T-\epsilon, T+\epsilon)}\) with respect to \(\eta'\).

3. Designer Potentials

In this section we show how to produce potentials with particular prescribed properties (we do this only on \(\mathbb{P}^1\) but a similar story holds for \(\mathbb{D}\)). We first show that any (reasonable) strictly increasing family of smooth domains in \(\mathbb{P}^1\) is the Hele-Shaw flow for some smooth Kähler potential. Recall \(B(t)\) denotes the geodesic ball centred at the origin of area \(t\) taken with respect to \(\omega_{FS}\).

Theorem 3.1. Suppose \(\{\Omega_t\}_{t \in (0,1)}\) is a family of subsets of \(\mathbb{P}^1\) that is

1. smoothly bounded, varies smoothly, and is simply connected for all \(t\),
2. increasing, i.e. \(\Omega_t \subsetneq \Omega_{t'}\) for \(t < t'\), with non-vanishing normal velocity of the boundary \(\partial \Omega_t\), and
3. standard as \(t\) tends to 0 and as \(t\) tends to 1.

Then there exists a smooth \(\phi \in C^\infty(\mathbb{P}^1)\) such that \(\{\Omega_t\}_{t \in (0,1)}\) is the Hele-Shaw flow with respect to the Kähler form \(\omega_{FS} + dd^c \phi\).
\textbf{Proof.} The idea of the proof is to construct a smooth function $\kappa$ on $\mathbb{C}$ such that $\{\Omega_t\}$ is the strong Hele-Shaw flow with respect to the permeability $\kappa$. Since $\{\Omega_t\}_{t \in (0,1)}$ is assumed to be standard as $t$ tends to 1 we have $\Omega_t \subset \mathbb{C}$ for all $t \in (0,1)$ and so by Lemma 2.6 we may as well consider the Hele-Shaw flow as taking place in $\mathbb{C}$. Let $p_t$ satisfy $$ p_t = 0 \text{ on } \partial \Omega_t \text{ and } \Delta p_t = -\delta_0. $$ As already mentioned, the fact that $p_t$ exists and is smooth on $\overline{\Omega_t} \setminus \{0\}$ is classical. What is also true is that $p_t$ varies smoothly with $t$; this is presumably also well-known in some circles, but since we were not able to find a convenient reference we give a proof in Appendix A.

Assuming this smoothness for now, we use $p_t$ to define a function $\kappa$ by requiring that $$ V_t = -\kappa \nabla p_t \text{ on } \partial \Omega_t \text{ for } t \in (0,1) \tag{6} $$ where $V_t$ is the normal velocity of $\partial \Omega_t$. Since $\{\Omega_t\}_{t \in (0,1)}$ is increasing smoothly and $V_t$ was assumed to be non-vanishing we see that $\kappa$ is a well-defined strictly positive smooth function on $\mathbb{C} \setminus \{0\}$.

Now we use the assumption that $\{\Omega_t\}_{t \in (0,1)}$ is standard as $t$ tends to zero to deduce that $\kappa$ extends to a smooth function over 0. Assume $t \ll 1$. By explicit calculation with the Fubini-Study metric one computes that $\Omega_t = \{z \in \mathbb{C} : |z| < R_t\}$ where $$ R_t := \left(\frac{t}{1-t}\right)^{1/2}. $$ Thus $p_t(z) = -\frac{1}{2\pi}(\log |z|^2 - \log(R_t^2))$. Clearly $\kappa$ is radially symmetric near 0, so it is sufficient to compute it at a point $z_t := (R_t,0)$ for small $t$. To do so observe that at $z_t$ we have $$ \nabla p_t = -\frac{1}{2\pi R_t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. $$ On the other hand the normal velocity of $\partial \Omega_t$ at the point $z_t$ is $\frac{dR_t}{dt}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix})$ and so the defining equation (6) for $\kappa$ becomes $$ \frac{1}{2R_t} \frac{1}{(1-t)^2} = \frac{\kappa(z_t)}{2\pi R_t}. $$ After some calculation this yields $$ \kappa(z) = \pi(1+|z|^2)^2 \text{ near } z = 0 \tag{7} $$ which clearly extends smoothly over $z = 0$.

Now define $$ \phi(z) = \int_{\mathbb{C}} \log |z - \zeta|^2 \frac{dA_\zeta}{\kappa(\zeta)} - \log(1+|z|^2) \text{ for } z \in \mathbb{C} \tag{8} $$ which is a smooth function on $\mathbb{C}$ chosen so that $$ dd^c \log(1+|z|^2) + \phi = \frac{dA}{\kappa} \text{ on } \mathbb{C}. \tag{9} $$

Using that $\{\Omega_t\}_{t \in (0,1)}$ is standard as $t$ tends to infinity we have that (7) also holds for $|z|$ sufficiently large. We claim this implies $\phi$ extends to a smooth function on $\mathbb{P}^1$ and $\omega_\phi$ is Kähler on $\mathbb{P}^1$. To see this, start with the identity $$ \frac{1}{\pi} \int_{\mathbb{C}} \frac{\log |z - \zeta|^2}{(1+|\zeta|^2)^2} dA_\zeta = \log(1+|z|^2) $$
(this can be seen by noting that the difference is harmonic on \( \mathbb{C} \) bounded and equal to zero at \( z = 0 \).) Now the same calculation as above means the assumption that \( \{ \Omega_t \} \) is standard as \( t \) tends to 1 implies \( C > 0 \) such that \( 7 \) holds on \( \{ |z| > C \} \). Therefore

\[
\phi(z) = \frac{1}{\pi} \int_{|z|<C} \log |z - \zeta|^2 \left( \frac{\pi}{\kappa(\zeta)} - \frac{1}{1 + |\zeta|^2} \right) d\Lambda \zeta
\]

which one sees extends smoothly over \( z = \infty \) in such a way that makes \( \omega_\phi \) Kähler as claimed. (We remark that this can also be seen abstractly, since the flow being standard near 0 and \( \infty \) means that \( \kappa \) has to agree with the permeability for the standard flow for \( (\mathbb{P}^1, \omega_{FS}) \).

Now by construction \( \{ \Omega_t \} \) is the strong Hele-Shaw flow with respect to \( \kappa \), and hence by Proposition \([2.11]\) is the strong Hele-Shaw flow for \( \omega_{FS} + dd^c \phi \) as desired. \( \square \)

**Remark 3.2.** We observe that the above proof actually shows slightly more, namely that if \( \{ \Omega_t \}_{t \in (0,T)} \) is a smooth family of strictly increasing domains that is standard as \( t \to 0 \) then setting \( X' := \Omega_T \) there exists a \( \phi \in C^\infty(X') \) such that \( \{ \Omega_t \}_{t \in (0,T)} \) is the Hele-Shaw flow for \( (X', \omega_\phi) \).

Now let \( S \) be a finite union of points and non-intersecting smooth embedded curve segments in \( \mathbb{P}^1 \setminus \{ 0 \} \). Using similar ideas to above we now show that there are Kähler potentials whose Hele-Shaw flow is smoothly bounded and simply connected until it develops a tangency along \( S \).

**Proposition 3.3.** There exists a \( \phi \in C^\infty(\mathbb{P}^1) \) such that \( \omega_\phi \) is Kähler and whose associated Hele-Shaw flow develops tangency along \( S \).

**Proof.** It is clear that one can find a simply connected domain \( \Omega \) containing \( 0 \) such that \( \partial \Omega \) is the image of a smooth locally embedded curve \( \gamma \) intersecting itself tangentially precisely along \( S \) and so \( \Omega_t \setminus S \) is connected as in Figure 1 (use induction on the number of components of \( S \)). Let

\[
T := \int_\Omega \omega_{FS}.
\]

We construct the Hele-Shaw flow backwards starting at \( \Omega_T := \Omega \).

Pick a point \( z_i \) in each connected component of \( \mathbb{P}^1 \setminus \Omega_T \), and let \( \pi \) be the projection from the universal cover \( \Sigma \) of \( \mathbb{P}^1 \) with the points \( z_i \) removed. Then \( \gamma \) lifts to a smooth embedded curve in \( \Sigma \) and so \( \pi^{-1}(\Omega_T) \) is a disjoint union of copies of \( \Omega_T \). We pick one of them and call it \( \Omega' \) which is smoothly bounded and simply connected. Then Theorem \([2.14]\) implies that there exists a smooth area form \( \eta' \) on a neighbourhood of \( \Sigma \setminus \Omega' \), equal to \( \eta := \pi^* \omega_{FS} \) on \( \Sigma \setminus \Omega' \), such that the strong Hele-Shaw flow exists starting from \( \Omega' \) with respect to \( \eta' \) for a short while backwards in time. We denote the projection of this Hele-Shaw flow to \( \mathbb{P}^1 \) by \( \{ \Omega_t \}_{t \in (T-\epsilon, T)} \).

We then extend this to a family of domains \( \Omega_t, t \in (0,T) \), in \( \mathbb{P}^1 \), with the properties as in Theorem \([3.1]\) so by this Theorem and Remark \([3.2]\) we have an area form \( \omega' \) on \( \Omega_T \) such that \( \{ \Omega_t \}_{t \in (0,T)} \) is a strong Hele-Shaw flow with respect to \( \omega' \). We also have that \( \omega' = \omega_{FS} \) on \( \Omega_T \setminus \Omega_{T-\epsilon} \). We can thus extend \( \omega' \) to a smooth Kähler form on \( \mathbb{P}^1 \) by letting it be equal to \( \omega_{FS} \) on \( \mathbb{P}^1 \setminus \Omega_T \). Thus \( \{ \Omega_t \}_{t \in (0,T)} \) is the strong Hele-Shaw flow with respect to the area form \( \omega' \) on \( \mathbb{P}^1 \), and thus also the Hele-Shaw flow by Proposition \([2.11]\).

On the other hand, by the continuity of the Hele-Shaw flow (applied on \( \Sigma \)) it follows that \( \Omega_T \) is the Hele-Shaw domain of \( \omega' \) at time \( T \). Thus if \( \phi \) is a smooth function so that \( \omega' = \omega_{FS} + dd^c \phi \) we get that the Hele-Shaw flow with respect to \( \phi \) develops a tangency along \( S \) at time \( T \). \( \square \)
Remark 3.4. If we assume in addition that $S$ is such that one can find such an $\Omega_T$ with real-analytic boundary, then instead of the previous result of the authors (Theorem 2.14) one can use the classical short-time existence result of the Hele-Shaw backwards starting with simply connected domain with real analytic boundary.

4. DIRICHLET PROBLEM FOR THE HOMOGENEOUS MONGE-AMPERE EQUATION

4.1. Preliminary definitions. We first recall two versions of the Dirichlet Problem for the complex homogeneous Monge-Ampère Equation, first over the disc and second over the punctured disc. In the following $(X,\omega)$ will be either $\mathbb{P}^1$ with the Fubini-Study metric $\omega_{FS}$ with area 1, or $X$ is the unit disc $\mathbb{D}$ with the Poincaré metric $\omega_P$. Again we let $\phi \in C^\infty(X)$ be such that $\omega + dd^c\phi$ is Kähler and $\pi_X : X \times \mathbb{M} \to X$ and $\pi_D : X \times \mathbb{D} \to \mathbb{D}$ be the projections.

Definition 4.1. (Weak Solution) Let

\[ \Phi := \sup \left\{ \psi : X \times \mathbb{D} \to \mathbb{R} \cup \{-\infty\} : \psi \text{ is usc, } \pi_X^*\omega + dd^c\psi \geq 0 \text{ and } \psi(z,\tau) \leq \phi(\tau z) \text{ for } (z,\tau) \in X \times \partial \mathbb{D}. \right\}. \]  

(10)

(2) Let

\[ \tilde{\Phi} := \sup \left\{ \psi : X \times \mathbb{D} \to \mathbb{R} \cup \{-\infty\} : \psi \text{ is usc, } \pi_X^*\omega + dd^c\psi \geq 0 \text{ and } \psi(z,\tau) \leq \phi(\tau z) \text{ for } (z,\tau) \in X \times \partial \mathbb{D} \text{ and } \nu_{(0,0)}(\psi) \geq 1 \right\}. \]  

(11)

So the difference between these two definitions is that in the second the boundary data is $S^1$-invariant but has an additional requirement of giving a prescribed singularity at the point $(0,0)$. However these two quantities carry the same information as given by:

Proposition 4.2. We have that

\[ \Phi(z,\tau) + \ln |\tau|^2 + \ln(1 + |z|^2) = \tilde{\Phi}(\tau z,\tau) + \ln(1 + |\tau z|^2) \text{ for } (z,\tau) \in \mathbb{P}^1 \times \mathbb{D}^\times. \]

Proof. This is proved in [19] Proposition 2.3 using a blowup, but it can also be seen directly from the definition that $\Phi(z,\tau) + \ln |\tau|^2 + \ln(1 + |z|^2) - \ln(1 + |\tau z|^2)$ is a candidate for the envelope defining $\Phi(\tau z,\tau)$, giving one inequality and the other inequality is proved similarly.

Definition 4.3. (Regular solution) We say that $\Phi$ is regular on an open subset $S \subset X \times \mathbb{D}$ if it is smooth on $S$ and the restriction of $\pi_X^*\omega + dd^c\Phi$ to $S_\tau := \pi_{\mathbb{P}^1}^{-1}(\tau) \cap S$ is strictly positive for all $\tau \in \mathbb{D}$. Similarly we say $\Phi$ is regular on $S$ if it is smooth on $S \setminus \{0\}$ and the restriction of $\pi_X^*\omega + dd^c\Phi$ to $S_\tau$ is strictly positive for all $\tau \in \mathbb{D}^\times$.

Finally we say that $\Phi$ (resp. $\tilde{\Phi}$) is regular if it is regular on all of $X \times \mathbb{D}$ (resp. $X \times \mathbb{D}^\times$).

By well-known arguments, $\Phi$ is usc, $\pi_X^*\omega + dd^c\Phi \geq 0$ and $(\pi_X^*\omega + dd^c\Phi)^2 = 0$ away from $(0,0)$ and $\tilde{\Phi}(z,\tau) = \phi(z)$ for $\tau \in \partial \mathbb{D}$. Moreover it is not hard to show that $\tilde{\Phi}$ is locally bounded away from $(0,0)$ and $\lim_{\nu_{(0,0)}(\tilde{\Phi}) = 1}$. Thus $\tilde{\Phi}$ is the weak solution to Dirichlet problem to the Homogeneous Monge-Ampère Equation with boundary data consisting of $\phi(z)$ on $X \times \partial \mathbb{D}$, and this prescribed singularity at $(0,0)$. Thinking of $s := -\ln |\tau|^2$ for $\tau \in \mathbb{D}^\times$ as a time variable let $\Phi_s(\cdot) = \Phi(\cdot, s)$. Then the map

\[ s \mapsto \omega + dd^c\Phi_s \]

is a weak geodesic ray in the space of weak Kähler metrics that starts with $\omega + dd^c\phi$ and has limit the singular potential $\omega + dd^c \ln |z|^2$ as $s$ tends to infinity. Moreover if $\tilde{\Phi}$ is regular this is a smooth geodesic in the space of Kähler metrics.
Similarly \( \Phi \) is the weak solution to the same Dirichlet problem over \( X \times \mathbb{D} \) with prescribed boundary \( \phi(\tau z) \) over \( X \times \partial \mathbb{D} \).

4.2. The Duality Theorem. The duality between \( \tilde{\Phi} \) and the Hele-Shaw envelopes \( \psi_t \) is provided by the following:

**Theorem 4.4.** (Ross-Witt-Nyström [19, Theorem 2.7]) Let \( \psi_t \) be the Hele-Shaw envelopes associated to \((X, \omega, \phi)\) and \( \tilde{\Phi} \) be the weak solution to the Homogeneous Monge-Ampère Equation as defined in (11). Then

\[
\psi_t(z) = \inf_{|\tau| > 0} \{ \tilde{\Phi}(z, \tau) - (1 - t) \ln |\tau|^2 \}
\]

and

\[
\tilde{\Phi}(z, \tau) = \sup_t \{ \psi_t(z) + (1 - t) \ln |\tau|^2 \}.
\]

**Remark 4.5.** In [19] this theorem is proved when \( X = \mathbb{P}^1 \), but precisely the same case works when \( X \) is the disc.

5. Regularity of Geodesic Rays

Let \((X, \omega)\) be either \((\mathbb{P}^1, \omega_{FS})\) or \((\mathbb{D}, \omega_P)\) with \( V := \int_X \omega \in \{ 1, \infty \} \), and let \( \phi \in C^\infty(X) \) be such that \( \omega + dd^c \phi \) is Kähler (when \( X = \mathbb{D} \) we also assume \( \phi \) to be bounded).

We continue with the notation from the previous section, so \( \hat{\Phi} \) is as defined in (11).

**Definition.** Let \( f : \mathbb{D} \to X \) be holomorphic. We say that the graph of \( f \) is a harmonic disc for \( \tilde{\Phi} \) if \( \tilde{\Phi} \) is \( \pi_X^\ast \omega \)-harmonic along the graph of \( f|_D \). That is, the restriction of \( \pi_X^\ast \omega + dd^c \tilde{\Phi} \) to \((f(\tau), \tau) : \tau \in D, \tau \neq 0\) vanishes.

**Definition 5.1.** We define

\[
H(z, \tau) := \frac{\partial}{\partial s} \tilde{\Phi}(z, e^{-s/2}) \quad (z, \tau) \in X \times \mathbb{D}^\times
\]

where \( s := -\ln |\tau|^2 \) (so when \( |\tau| = 1 \) and thus \( s = 0 \) we take the right derivative).

We recall that by a result of Chen [5], with complements by Błocki [4], the function \( \tilde{\Phi} \) is \( C^{1,1} \) and thus this definition makes sense, and \( H \) is continuous (even Lipschitz but we will not use this).

**Lemma 5.2.** The function \( H \) is constant along any harmonic disc of \( \tilde{\Phi} \).

**Proof.** Let \( \Omega = \pi_X^\ast \omega_{FS} + dd^c \tilde{\Phi} \). Then one calculates that \( dH = \iota_\zeta \Omega \) where \( \zeta \) is the infinitesimal action of the natural \( S^1 \)-action given by \( e^{i\theta} \cdot (z, \tau) = (e^{i\theta} z, e^{-i\theta} \tau) \) for \((z, \tau) \in X \times \mathbb{D}^\times \) (this is essentially as in [17, Theorem 3.14]). This proves that \( H \) is constant along any harmonic leaf as required.

The connection with the Hele-Shaw flow is given by:

**Proposition 5.3.**

\[
H(z, 1) + 1 = \sup \{ t : \psi_t(z) = \phi(z) \} = \sup \{ t : z \notin \Omega_t \}.
\]

**Proof.** This is [19, Proposition 2.8] and for convenience we repeat the proof here. From (13) if \( \psi_t(z) = \phi(z) \) then

\[
\tilde{\Phi}(z, e^{-s/2}) \geq (t - 1)s + \phi(z)
\]

and thus

\[
H(z, 1) \geq \sup \{ t : \psi_t(z) = \phi(z) \} = 1.
\]
Suppose \( \psi_t(z) \leq \phi(z) + a \) for some \( a < 0 \). One can easily check that for a fixed \( z \) the function \( t' \mapsto \psi_{t'}(z) \) is concave and decreasing in \( t' \), so for \( t \leq t' < V \) and \( s \geq 0 \) we have
\[
\psi_{t'}(z) + (t' - 1)s \leq \phi(z) + a.
\]
On the other hand we always have \( \psi_{t'} \leq \phi \) so if \( 0 \leq t' \leq t \) then
\[
\psi_{t'}(z) + (t' - 1)s \leq \phi(z) + (t - 1)s.
\]
Putting this together with (13) gives
\[
\hat{\Phi}(z, e^{-s/2}) \leq \phi(z) + \max((t - 1)s, a)
\]
and so \( H(z, 1) \leq t - 1 \), which proves the proposition.

**Theorem C.** Suppose the flow Hele-Shaw \( \{\Omega_t\}_{t \in (0, V)} \) for a Kähler form \( \omega + dd^c \phi \) satisfies
\begin{enumerate}
\item \( \{\Omega_t\}_{t \in (0, V)} \) is smoothly bounded and varies smoothly with non-vanishing normal velocity,
\item \( \Omega_t \) is simply connected for all \( t \in (0, V) \),
\item if \( X = \mathbb{P}^1 \) then \( \{\Omega_t\}_{t \in (0, V)} \) is standard as \( t \) tends to 1.
\end{enumerate}
Then the weak geodesic ray (13) obtained as the Legendre transform of the Hele-Shaw envelopes \( \{\psi_t\} \) is regular, and so defines a smooth geodesic ray in the space of Kähler metrics on \( X \).

**Proposition 5.4.** With the assumptions as in Theorem C let \( f_t : \mathbb{D} \to \Omega_t \) be a Riemann-mapping for \( \Omega_t \) that maps 0 to 0. The the graph of \( f_t \) is a harmonic disc for \( \hat{\Phi} \). Another harmonic disc is given by the graph of the constant function \( f_0(\tau) \equiv 0 \) and when \( X = \mathbb{P}^1 \) yet another is given by the graph of \( f_\infty(\tau) \equiv \infty \). Finally the union of all these harmonic discs form a smooth foliation of \( X \times \mathbb{D}^\times \).

**Proof.** From (13) we have that
\[
\hat{\Phi}(f(\tau), \tau) \geq \psi_t(f(\tau)) + (1 - t) \ln |\tau|^2.
\]
Now the left hand side is \( \pi_X^* \omega \)-subharmonic whilst the right hand side is \( \pi_X^* \omega \)-harmonic for \( \tau \neq 0 \). Since \( f(\tau) \) escapes to infinity in \( \Omega_t \) as \( |\tau| \to 1 \) the left hand side approaches the right hand side as \( |\tau| \to 1 \). Hence by the maximum principle we deduce that in fact
\[
\hat{\Phi}(f(\tau), \tau) = \psi_t(f(\tau)) + (1 - t) \ln |\tau|^2.
\]
which is \( \pi_X^* \omega \)-harmonic for \( \tau \neq 0 \). The fact that the graph of \( f_0 \) is a harmonic disc is clear as the boundary data is invariant for the \( S^1 \)-action \( e^{i\theta} \cdot (z, \tau) = (e^{i\theta} z, e^{-i\theta} \tau) \), and hence so is \( \hat{\Phi} \), meaning that \( \hat{\Phi} \) is actually constant along the graph of \( f_0 \) (and similarly for \( f_\infty \) when \( X = \mathbb{P}^1 \)).

That these harmonic discs \( X \times \mathbb{D} \) do not intersect for different values of \( t \) is clear from Lemma 5.2 and Proposition 5.3 (which together say that \( H \equiv t - 1 \) on \( f_t \) for \( t \in (0, V) \)) and the union of all these discs cover \( X \times \mathbb{D} \) since the union of the \( \Omega_t \) cover \( X \).

Now by Theorem 2.13 the foliation is diffeomorphic to the standard one \( B(t) \) for \( t \in (0, \epsilon) \) for some \( \epsilon > 0 \). This proves the foliation by harmonic discs is smooth in a neighbourhood of \( \{0\} \times \mathbb{D} \). When \( X = \mathbb{P}^1 \), the assumption that the Hele-Shaw flow is standard as \( t \) tends to 1 ensures the foliation is also smooth near \( \{\infty\} \times \mathbb{D} \). That these harmonic discs give a smooth foliation in the remaining part of \( X \times \mathbb{D} \) is immediate from the hypothesis that the normal velocity of \( \{\Omega_t\} \) is non-vanishing, as we can arrange so that the Riemann mapping \( f_t : \mathbb{D} \to \Omega_t \) that takes 0 to 0 varies smoothly with \( t \) since \( \Omega_t \) varies smoothly in \( t \).

**Remark 5.5.** We remark that the first part of the above is the easy direction of Theorem 3.1 which actually gives a complete characterisation of harmonic discs in terms of the simply connectedness of the Hele-Shaw flow domains.
Remark 5.6. One can avoid using the previous work of the authors (Theorem 2.13) if one assumes in addition that \( \{\Omega_t\} \) is standard as \( t \) tends to 0.

Proof of Theorem 4 This is an almost immediate consequence of the existence of the smoothly varying foliation by harmonic discs provided by Proposition 4.4 (this is essentially contained in [7]). Let \( D = \{(f(\tau), \tau)\} \) be such a harmonic disc away from \( \{0\} \times \mathbb{D} \) (and away from \( \{\infty\} \times \mathbb{D} \) when \( X = \mathbb{P}^1 \)). Then \( \bar{\Phi}(f(\tau), \tau) \) is harmonic along \( D \) with Lelong number one. Thus by the mean-value property of harmonic functions, \( \bar{\Phi} |_D \) can be expressed as an integral of \( \bar{\Phi} \) over \( \partial D \). But \( \bar{\Phi} = \phi \) over \( \partial D \) (which is smooth) and the foliation varies smoothly, from which we conclude that \( \bar{\Phi} \) must in fact be smooth. Since the foliation is diffeomorphic to the trivial foliation near \( \{0\} \times \mathbb{D} \) one sees that in fact \( \bar{\Phi} \) is also smooth near \( \{0\} \times \mathbb{D} \) (and similarly for \( \{\infty\} \times \mathbb{D} \) when \( X = \mathbb{P}^1 \)). This proves smoothness of \( \bar{\Phi} \) over \( X \times \mathbb{D}^\times \).

For the regularity we argue as follows. For \( \tau \neq 0 \) let \( T_\tau : \pi^{-1}_D(1) \to \pi^{-1}_D(\tau) \) be the flow along the leaves of the above foliation and set \( \Omega_\tau := \pi_\Omega^* \omega_F S + dd^c \bar{\Phi}|_{\pi^{-1}_D(\tau)} \). Then by what is now considered a classical calculation (originally due to Semmes [22] and Donaldson [7], see also [17] Proposition 3.4) we know \( T_\tau^* \Omega_\tau = \Omega_1 \). But \( \Omega_1 = \omega_\phi \) is certainly strictly positive, and hence \( \Omega_\tau \) is strictly positive as well.

6. Explicit Singularities

We now give a proof of Theorem 4 and show that a potential whose Hele-Shaw flow that develops a tangency along a set \( \Sigma \) gives a singularity of the associated weak solution.

Example 6.1. The reader may find the following simple example instructive. Suppose \( \phi \) develops tangency at a single point \( S = \{z_0\} \). Then we may find smooth coordinates \( (x, y) \) centered at \( z_0 \) such that

\[
\partial \Omega_t = \{y = x^2 + (t_0 - t)\} \cup \{y = -x^2 - (t_0 - t)\}
\]

near \( z_0 \).

Let \( h := H(\cdot, 1) \) where \( H \) is as defined in (5.1). Notice that if \( |y| \) is sufficiently small then \( (0, \pm y) \) lies in \( \Omega_t \) for \( t \) sufficiently close to \( t_0 \). Thus Proposition 5.3 says that for \( |y| \) sufficiently small

\[
h(0, y) = \begin{cases} 
  t_0 - y - 1 & y > 0 \\
  t_0 + y - 1 & y < 0 
\end{cases}
\]

and so \( \frac{\partial h}{\partial y} \) does not exist at the origin. Hence \( \bar{\Phi} \) is not \( C^2 \) at the point \( (z_0, 1) \), and by Proposition 4.4 the same must be true for \( \Phi \).

Theorem B. Let \( S \) be a finite union of points and curve segments in \( \mathbb{P}^1 \setminus \{0\} \). Let \( \phi \in C^\infty(\mathbb{P}^1) \) be a Kähler potential and suppose the Hele-Shaw for \( \omega + dd^c \phi \) develops tangency along \( S \). Then the weak solution \( \Phi \) from (1) to the Dirichlet problem for the HMAE on \( \mathbb{P}^1 \times \mathbb{D} \) with boundary data \( (z, \tau) \mapsto \phi(\tau z) \) is not twice differentiable at the points \( (\tau^{-1} z, \tau) \), \( z \in S, |\tau| = 1 \).

Proof of Theorem 4 and Theorem B Suppose first that \( \phi \) is as produced by Proposition 3.3. That is, we have picked points \( z_i \) in each component of \( \mathbb{P}^1 \setminus \Omega_0 \) and \( \pi : \Sigma \to \mathbb{P}^1 \setminus \{z_i\} \) is the universal cover, and \( \Omega_{t \in (T - \varepsilon, T]} \) is the pushforward of a strong Hele-Shaw flow on \( \Sigma \). This implies that the normal velocity of the boundary of \( \Omega_t \) as \( t \) tends to \( T \) from below is nowhere vanishing.

Now let \( z \in S \), so \( \Omega_T \) has boundary tangent to itself at \( z \), and so \( \Omega_T \) splits locally into two pieces, call them \( P_1 \) and \( P_2 \). Working on \( P_1 \), the combination of Proposition 5.3 (which says that \( \partial \Omega_t \) are the level sets of \( H(\cdot, 1) - 1 \)) and the fact that \( \Omega_t \) varies smoothly
imply the partial derivative of $H(\cdot, 1)$ in the normal direction to $\Omega_t$ is strictly negative at $z$ (see Example 5.1). Since the analogous statement is true for $P_2$, this proves that $H$ is not differentiable at $(z, 1)$. Thus $\Phi'$ is not twice differentiable at the point $(z, 1)$, and by Proposition 4.2 the same is true for $\Phi$. Then by $S^1$-invariance we see that $\Phi$ cannot be twice differentiable at any point of the form $(\tau^{-1}z, \tau)$ for $z \in S, |\tau| = 1$.

Now if $\phi$ is any Kähler potential whose Hele-Shaw $\{\Omega_t\}$ develops tangency along $S$ then it is not hard to see from the proof of Proposition 3.3 that $\{\Omega_t\}$ is the pushforward of some Hele-Shaw flow on $\Sigma$ call it $\{\Omega'_t\}$. The hypothesis on $\Omega_T$ ensures that $\Omega'_T$ is smoothly bounded, and hence by the argument in Remark 2.12 we conclude that the normal velocity is non-vanishing as $t$ tends to $T$ from below (the reader who prefers not to invoke this argument may prefer to make this non-vanishing as part of the hypothesis of what it means to develop a tangency along $S$). The proof of the Theorem then follows as before.

Finally Theorem A follows from Theorem B and Proposition 3.3.

7. AN EXTENSION

So far we have been working under the hypothesis that our Hele-Shaw flow $\{\Omega_t\}_{t \in (0, V)}$ is standard as $t$ tends to $0$ (and also as $t$ tends to $1$ when $X = \mathbb{P}^1$). We did this to ensure regularity of the associated potential near the point $0$ (resp. $\infty$) which we achieved by direct computation. In this section we explain how this hypothesis can be relaxed. For simplicity we work only with $(X, \omega) = (\mathbb{P}^1, \omega_{FS})$ but a similar story holds for the disc.

**Definition 7.1.** Let $\text{Diff}_0(\mathbb{P}^1)$ be the group of diffeomorphisms of $\mathbb{P}^1$ such that $\alpha(0) = 0$.

Given $\alpha \in \text{Diff}_0(\mathbb{P}^1)$ we define

$$\Omega_t = \alpha(B(t)) \quad \text{for } t \in (0, 1)$$

where, we recall, $B(t)$ denotes the geodesic ball centred at $0$ with area $t$ with respect to $\omega_{FS}$. Clearly $\{\Omega_t\}_{t \in (0, 1)}$ is a strictly increasing, smoothly varying family of smoothly bounded simply connected domains in $\mathbb{P}^1$ that tends to zero as $t$ tends to $0$. We claim that for $\{\Omega_t\}_{t \in (0, 1)}$ constructed in this way the conclusion of Theorem C and Theorem 3.1 still hold; that is, there exists a Kähler potential $\phi$ such that $\{\Omega_t\}_{t \in (0, 1)}$ is the Hele-Shaw flow for $\omega_\phi$, and that weak geodesic obtained as the Legendre transform of the Hele-Shaw envelopes $\{\psi_t\}$ is regular.

We sketch why this is the case. Observe that the only place in which we used that $\{\Omega_t\}_{t \in (0, 1)}$ is standard as $t$ tends to $0$ and $1$ in the proof of Theorem C was to ensure that $\omega_\phi$ was a smooth Kähler form at $0$ and at $\infty$. So assume instead that (14) holds. Let $\alpha_0 = \text{id}_{\mathbb{P}^1} \in \text{Diff}_0(\mathbb{P}^1)$, whose associated flow is $\{B(t)\}_{t \in (0, 1)}$ which is the Hele-Shaw flow associated to $\omega_{FS}$ and, as we saw in (7), is the classical Hele-Shaw flow on $\mathbb{C}$ with permeability $\kappa_0(z) := \pi(1 + |z|^2)^2$. Without loss of generality say $\alpha(z) = z + O(|z|^2)$ near $z = 0$. Then $\alpha$ is $C^\infty$ close to $\alpha_0$ in a neighbourhood of $z = 0$ which implies that $\Omega_t$ is $C^\infty$-close to $B(t)$ for $t$ sufficiently small. In turn this implies the permeability $\kappa$ as defined in (6) is $C^\infty$ close to $\kappa_0$ in a punctured neighbourhood of $0$, which is enough to imply it extends across $0$ to a smooth strictly positive function. The argument near $\infty$ is similar: again without loss of generality say $\alpha(\infty) = \infty$ locally given by $\alpha(1/z) = 1/z + O(1/|z|^2)$ near $z = \infty$. Given a small neighbourhood $U$ of $\infty$ we can construct an $\alpha_1$ that is equal to $\alpha$ on $\mathbb{P}^1 \setminus U$ and is equal to $\alpha_0$ near $\infty$. Thus $\alpha$ is $C^\infty$ close to $\alpha_1$ and the same argument then applies to deduce that $\phi$ extends smoothly across $\infty$ and $\omega_\phi$ is Kähler. Hence Theorem C still holds.

The argument for Theorem 3.1 is similar, as near $\{0\} \times \mathbb{D}$ the foliation by harmonic discs provided by Proposition 5.4 for $\alpha$ is (in the obvious sense) $C^\infty$-close to that provided by
\[ \alpha_0, \text{ and this is enough to prove that } \tilde{\Phi} \text{ is smooth over } \{0\} \times \overline{D}. \text{ Arguing similarly with } \alpha_1 \text{ near } \{\infty\} \times \overline{D} \text{ we conclude that Theorem 3.1 still holds as well.} \]

Accepting this argument, we see that to any \( \alpha \in \text{Diff}_0(\mathbb{P}^1) \) we have an associated smooth geodesic ray in the space of Kähler metrics on \( \mathbb{P}^1 \) that starts at \( \omega_\phi \) and has limit \( \omega + dd^c \ln |z|^2 \) at infinity (i.e. as \( \tau \) tends to zero). Of course different \( \alpha \) can give rise to the same flow, but the ambiguity is precisely coming from the subgroup of “angular diffeomorphisms” given by

\[ \Gamma := \{ \alpha \in \text{Diff}_0(\mathbb{P}^1) : \alpha(B(t)) = B(t) \text{ for all } t \}. \]

Moreover this process can be reversed, since any smooth geodesic joining \( \omega_\phi \) to \( \omega + dd^c \ln |z|^2 \) comes from a regular solution to the complex Monge-Ampère Equation, and thus gives rise to a foliation by harmonic discs. By the harder direction of [19, Theorem 3.1] we know that such discs can only be those described in Proposition 5.4. Finally it is clear from the proof of Theorem 3.1 that different Hele-Shaw flows give rise to different \( \omega_\phi \) and vice versa. Thus in all we have the following explicit description of all smooth geodesics rays in the space of Kähler metrics on \( \mathbb{P}^1 \) that have limit \( \omega + dd^c \log |z|^2 \) as follows:

**Theorem 7.2.** The duality that associates a weak geodesic ray to the Hele-Shaw flow gives a bijection between \( \text{Diff}_0(\mathbb{P}^1)/\Gamma \) and

\[ \{ \phi \in C^\infty(X) : \exists \text{ a smooth geodesic ray starting } \omega_\phi \text{ with limit } \omega_{FS} + dd^c \ln |z|^2 \}. \]

**Remark 7.3.** Equivalently one can encode such a flow \( \{\Omega_t\} \) by the smooth function on \( \mathbb{P}^1 \) whose level sets are \( \partial \Omega_t \) (this is the approach taken in [6]).

**Appendix A. Smoothness of Green’s Functions**

We collect some regularity results for elliptic operators, all of which is essentially standard. Suppose \( I \subset \mathbb{R} \) is an open interval and \( \{L_t\}_{t \in I} \) is a smoothly varying family of strictly elliptic operators on the unit disc \( D \) with uniform ellipticity constant. That is, we suppose

\[ L_t u = a^{ij}(x,t) D_{ij} u + b^i(x,t) D_i u + c(x,t) u \text{ for } t \in I \quad (15) \]

where \( a^{ij}, b^i, c \in C^\infty(\overline{D} \times I) \) and \( u \) is a function defined on \( D \), such that there is a \( \lambda > 0 \) such that \( a^{ij}(x,t) \xi_i \xi_j \geq \lambda |\xi|^2 \) for all \( (x,t) \in \overline{D} \times I \) and \( \xi \in \mathbb{R}^N \). We assume also \( c(x,t) \leq 0 \) for \( (x,t) \in \overline{D} \times I \).

Suppose now \( \varphi \in C^\infty(\partial D \times I) \), and we write \( \varphi_t(\cdot) = \varphi(\cdot, t) \). Then for each \( t \in I \) standard elliptic theory says [9, Corollary 6.9, Theorem 6.19] there exists a unique \( u_t \in C^\infty(\overline{D}) \) that solves

\[ L_t u_t = 0 \text{ and } u_t|_{\partial D} = \varphi_t. \]

We claim that \( u_t \) is also smooth in the \( t \)-variable. To prove this it is sufficient to show it at \( t = 0 \) assuming \( 0 \in I \). Then expanding \( a^{ij}, b^i, c \) in \( t \) we can write

\[ L_t u = L_0 u + t M_1 u + \cdots + t^N M_N u + O(t^{N+1}) \]

for some operators \( M_i \) that are independent of \( t \). Here and henceforth we work in the \( C^\infty \)-topology so the \( O(t^{N+1}) \) error terms means that for all \( k \in \mathbb{N} \) there exists a \( C_k \) such that this term is bounded by \( C_k |t|^{N+1} \) in the \( C^k(\overline{D}) \)-norm. We wish to find an expansion for \( u_t \) in \( t \), say

\[ u_t = u_0 + tv_1 + \cdots + t^N v_N + O(t^{N+1}) \quad (16) \]
where \( v_i \in C^\infty(\mathbb{D}) \). To do so expand \( \varphi = \varphi_0 + t\sigma_1 + \cdots + t^N\sigma_N + O(t^{N+1}) \) where \( \sigma_i \in C^\infty(\partial\mathbb{D}) \). Then comparing coefficients of \( t \) forces the \( v_i \) to satisfy
\[
L_0v_1 + M_1u_0 = 0 \quad \text{and} \quad v_1|_{\partial\mathbb{D}} = \sigma_1
\]
\[
L_0v_2 + M_1v_1 + M_2u_0 = 0 \quad \text{and} \quad v_2|_{\partial\mathbb{D}} = \sigma_2
\]
and so forth. So starting with \( u_0 \) we may inductively define \( v_i \), and as \( L_0 \) is elliptic, the same elliptic regularity guarantees \( v_i \in C^\infty(\bar{\mathbb{D}}) \). To see that \([16]\) does actually hold, observe that by construction the difference \( w_1 := u_t - u_0 - tv_1 - \cdots - v_Nt^N \) satisfies
\[
L_tw_1 = O(t^{N+1}) \quad \text{and} \quad w_1|_{\partial\mathbb{D}} = O(t^{N+1}).
\]
Then, by elliptic theory again \([9, \text{Corollary 8.7, Theorem 8.13}]\) this implies \( w_t = O(t^{N+1}) \) in the \( C^\infty(\bar{\mathbb{D}}) \) topology (here we are using that the elliptic constant for \( L_t \) is uniform over \( t \in I \) to apply \([9, \text{Corollary 8.7}]\) uniformly over \( I \)), which gives \([16]\). As this holds for all \( N \), the map \( t \mapsto u_t \) is smooth in \( t \), which implies \( u \in C^\infty(\mathbb{D} \times I) \) as claimed.

For the application we have in mind, suppose that \( \{\Omega_t\}_{t \in I} \) is a smoothly varying family of smoothly bounded simply connected domains in \( \mathbb{D} \) containing \( 0 \). We may assume \( \Omega_t = \alpha_t(\mathbb{D}) \) where \( \alpha : \mathbb{D} \times I \to \mathbb{C} \) is smooth. Then set
\[
L_t(u) := (\Delta(u \circ \alpha_t^{-1})) \circ \alpha_t
\]
where \( u \) is a function on \( \mathbb{D} \) and \( \Delta \) is the standard Laplacian on \( \mathbb{C} \). Then \( \{L_t\}_{t \in I} \) is a smoothly varying family of elliptic operators with uniform ellipticity constant, as in \([15]\).

Set \( \Gamma(z) := \log |z|^2 \) and \( \varphi_t = \Gamma \circ \alpha_t \) so \( \varphi \in C^\infty(\bar{\mathbb{D}} \times I) \). Then by the above discussion we know there exists a \( u \in C^\infty(\bar{\mathbb{D}} \times I) \) such that
\[
L_tu_t = 0 \quad \text{and} \quad u_t|_{\partial\mathbb{D}} = \Gamma \circ \alpha_t.
\]
Finally set
\[
p_t := u_t \circ \alpha_t^{-1} - \Gamma
\]
so by construction \( \Delta p_t = 0 \) on \( \Omega_t \) and \( p_t|_{\partial\Omega_t} = 0 \), that is \( p_t \) is minus the Greens function for \( \Omega_t \) with logarithmic pole at 0. From this we see that \( p_t \) varies smoothly in \( t \), in particular the quantity \( \nabla p_t \) on \( \partial\Omega_t \) is a smooth vector field on \( \bigcup_{t \in I} \Omega_t \) which is precisely what we used in the proof of Theorem \([3,1]\).

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