Advances in development of the analytical dynamics of the hereditary discrete systems

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Abstract. Research results in area of dynamics of hereditary discrete systems, obtained by authors of this paper, are presented and generalized in the paper, which contains first completed presentation of the analytical dynamics of hereditary discrete systems. Two classes of dynamically defined and undefined hereditary systems are defined and considered by introducing corresponding restrictions. Main results of dynamics of hereditary discrete systems are presented with new applications important to engineering and sciences. Analogy between hereditary interactions and reactive forces in systems of automatic control is identified and a possibility to extend theory of analytical dynamics of hereditary systems to mechanical systems with automatic control is pointed out.

1. Introduction

Methods of Analytical Dynamics nowadays obtain largest use in many of scientific research area and applications to the different models of real world, from engineering system dynamics along natural sciences from dynamics of mechanical systems to the dynamics of microworld. In Reference [7] by He was written: “On 3 October 2006, the Royal Swedish Academy of Sciences decided to award the Nobel Prize in Physics for 2006 jointly to John C. Mather and George F. Smoot "for their discovery of the blackbody form and anisotropy of the cosmic microwave background radiation”. Cosmic microwave background radiation is probably more related to Nonlinear Dynamics than the Big Bang Theory. However the vital question is: when will nonlinear dynamics claim the same prize?”

1.1. Development of Analytical Dynamics.

Analytical dynamics as general science of mechanical system motions was founded by Lagrange (Joseph Luis Lagrange (1735-1813)) in the period of his work at Berlin Academy. The Lagrange’s book “Mécanique Analytique” [18] contains basic analytical methods of mechanics and was published in France in 1788. Introduced analytical methods in Mechanics by Lagrange are main and first base of analytical mechanics in general. Lagrange’s equations of second kind and Lagrange’s equations of
first kind with unwoven Lagrange’s multipliers of constraints are main fundament of Analytical Dynamics.

After first period of Analytical Mechanics foundation, canonical equations obtained by Hamilton (Wiliam Rowan Hamilton (1805 – 1865)) were main advanced results of fundamental progress of analytical methods in mechanics.

In Lagrange’s opinion his equations first and second kind are universal and applicable to all mechanical systems. In 1985 С.А.Чаплыгин analysis of a paper written by Э.Линделеф follows to the conclusions that Lagrange’s equations are not applicable to nonholonomic mechanical systems [19]. Than, С.А.Чаплыгин proposed beginning of research in a new area of Analytical mechanics under the name Nonholonomic Mechanics – Mechanics of nonholonomic mechanical systems.

Lagrange’s equations of first kind with unknown Lagrange’s multipliers of constraints are really general and universal. At the beginning of the XX century nonholonomic mechanics was founded as a separate science discipline. Equations of С.А.Чаплыгин, В.Вольтерра, П.В.Воронца, Г.Маджи, П.Аппел and other are considered to be great contributions in the area of Analytical dynamics of nonholonomic systems. Separate parts of mechanics of nonholonomic systems are applicable to the control of systems and to systems containing deformable bodies.

In current literature term “hereditary” and “rheological” systems are equivalent. In opinion of Работнов Ю.Н. [1], the name “hereditary” system or continuum proposed by В.Вольтерра, is more precise as well as suitable. By using this name, the property of rheological systems “to remember” history of loading is fully described. By series of fundamental papers and monographs, Mechanics of hereditary continuum is presented. Also, in numerous references, many examples with applications in engineering, biological and other areas [2] are published. Pioneer research results in area of mechanics of discrete hereditary systems are presented in a publication written by a talented scientist А.Р.Ржаницин [3]. Also, numerous applications in this area grow.

Dynamics of discrete hereditary systems up to a few years before was presented only by separate single papers [3] and containing only solutions of partial problems.

Research results in area of dynamics of hereditary discrete systems, obtained by authors of this paper, are generalized and presented in the monograph [4], published in 2001, which contains first presentation of analytical dynamics of hereditary discrete systems. We can conclude that this monograph contains complete foundation of analytical dynamics theory of discrete hereditary systems and by using these results, numerous examples are obtained and solved (see Refs. [5-17]). In this analytical mechanics of hereditary discrete systems, modified Lagrange’s differential equations second kind in the form of differential and integro-differential forms with kernels of relaxation or rheology are derived.

This paper based on research results from monograph [4] and new authors’ research and advanced published [5-7] and unpublished results.

1.2. Terminology of Analytical Dynamics.

As we say, in the name “hereditary” properties of rheological body “to remember” history of loading is fully described for both cases: for first case of the short time loading period when rheological body possesses property to obtain quickly previous unloaded body form after unloading and second case of the long time loading when rheological body possesses property to obtain previous unloaded body form after unloading for long period when the material property “to remember” is present as history of loading, and name “hereditary elasticity” is the corresponding name.

2. Models of hereditary elements in analytical dynamics of hereditary discrete systems

Hereditary system is every system which contains mutual hereditary interaction between material particles in the form of one or more constraints with hereditary properties.

There are three mathematical forms for description of the constitutive relations of hereditary properties of the hereditary interaction [4-5], in the building of the hereditary system’s mechanics. These forms are:
1* Differential equation, expressed in the form of dependence reaction force \( P \) of the rheological coordinate \( x \), usually presented as deformation or relative displacement of the hereditary constraint:

\[
\sum_{k=1}^{n} a_k \frac{d^k P}{dt^k} + P(t) = b_0 x + \sum_{k=1}^{n} b_k \frac{d^k x}{dt^k}
\]  \( \text{(1)} \)

2* Integral equation, expressed in the form of dependence reaction force \( P \) of the rheological coordinate \( x \), usually presented as deformation or relative displacement of the hereditary constraint.

\[
P(t) = c \left( x(t) - \int_{0}^{t} R(t-\tau)x(\tau)d\tau \right)
\]  \( \text{(2)} \)

By this integral equation, the relaxation of the reaction force \( P \) depending on the rheological coordinate \( x \), is presented and expressed.

Integral equation, expressed in the form of dependence rheological coordinate \( x \), usually presented deformation or relative displacement of the hereditary constraint and reaction force \( P \):

\[
x(t) = \frac{1}{c} \left[ P(t) + \int_{0}^{t} K(t-\tau)P(\tau)d\tau \right].
\]  \( \text{(3)} \)

By this integral equation, the retardation of the rheological coordinate \( x \) of the reaction force \( P \) is presented and expressed. In the previous integral equations \( R(t-\tau) \) and \( K(t-\tau) \) - are relaxational and rheological kernel

\[
R(t-\tau) = \frac{c - \tilde{c}}{nc} e^{-\frac{1}{n}(t-\tau)}
\]  \( \text{(4)} \)

is relaxation kernel, and \( \beta = \frac{1}{n} \) is coefficient of the element relaxation

\[
K(t-\tau) = \frac{c - \tilde{c}}{nc} e^{-\frac{2}{n}(t-\tau)}
\]  \( \text{(5)} \)

is kernel of rheology and \( \beta_i = \frac{\tilde{c}}{nc} \) is the coefficient of the creep or retardation or rheology.

Previous integral equations (2) and (3), can be expressed in following form:

\[
P(t) = c \left( x(t) - \int_{0}^{t} R(t-\tau)x(\tau)d\tau \right)
\]  \( \text{and} \)

\[
x(t) = \frac{1}{c} \left[ P(t) + \int_{0}^{t} K(t-\tau)P(\tau)d\tau \right]
\]  \( \text{(6)} \)

where by integral operators, the histories of the previously interactions of the hereditary constraints are expressed.

3* Fractional -differential equation, expressed by fractional order derivative and in the form of dependence reaction force \( P \) of the rheological coordinate \( x \), usually presented as deformation or relative displacement of the hereditary creep constraint:

\[
P(t) = -\left[ c_0 x(t) + c_\alpha D_\alpha^\alpha [x(t)] \right]
\]  \( \text{(7)} \)

where \( D_\alpha^\alpha [\bullet] \) fractional order \( 0 \leq \alpha \leq 1 \) differential operator in the form:
\[ D_\alpha^\Gamma [x(t)] = \frac{d^\alpha x(t)}{dt^\alpha} = x^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{\alpha-1} d\tau \]

(8)

and \( \alpha \) is a rational constant depending on material properties of material creep prosperous; \( c \) and \( \varepsilon \) rigidity of creep material, momentary and prophetic one; \( \Gamma(1-\alpha) \) Gamma Euler function.

3. Universal form of Lagrange’s equations for hereditary discrete systems

In the basis of the construction of the Lagrange’s mechanics of the hereditary discrete systems, the classical mechanics principles are used. These principles are: Principle of work of forces along corresponding possible system displacements, as well as Principle of dynamical equilibrium.

Using the Principle of work of system forces along corresponding possible system displacements we can write the following equation:

\[ \sum_{i=1}^{3N} (m_i \ddot{x}_i - X_i + \sum_{k=1}^{K} P_k \dot{e}_{ik}) \dot{x}_i = 0, \]

(9)

where force of rheological reaction \( P_k \) of the rheological interactions between material particles are present because constraint is no ideal.

Lagrange’s equations second kind for hereditary systems are in the form:

\[ \frac{d}{dt} \frac{\partial T}{\partial q_j} - \frac{\partial T}{\partial q_j} = \frac{\partial H}{\partial q_j} + \sum_{k=1}^{K} b_{kj} P_k, \quad j = 1, 2, \ldots, n \]

(10)

From previous equations, it is necessary to eliminate reaction forces \( P_k \) by use corresponding numbers of the integral equations expressed in the form of dependence reaction force \( P_k \) of the corresponding rheological coordinates in the form (2) or (6) or (7) and with corresponding change of the coordinate from Descartes to generalized. Equations (9) are universal because their applications are possible in the case of the arbitrary number of the hereditary interactions between material particles in the system, and also in the sense when number of the hereditary interactions is larger than number of the system degree of freedom, \( K > n \).

Applications of the Lagrange’s equations second kind for hereditary systems in the form (10) with use of the rheological constitutive equations of the rheological interactions in the form (1) or (2) is possible for the case “dynamical defined systems” for which determinant satisfy the following condition \( \det \left| b_{kj} \right| \neq 0, \quad K \leq n \). For these cases Lagrange’s equations second kind is easy to solve with respect to the reactions \( P_k \) by use equations (1) or (2) or (7).

For description of properties of the dynamics of the hereditary system by use relaxational or rheological kernel (resolvent), these kernels are expressed by exponential or fractional-exponential forms [4]. Description of the hereditary properties of the system by use differential forms (1) and integral form (2) and (3) with exponential kernels is equivalent. For the case of the fractional-exponential forms of the kernel in the integral form corresponding equivalent differential forms not exist.

4. Forms of initial conditions for solving equations of the hereditary system dynamics

Initial condition for solving integro-differential equations (16) or (17) are in the classical form

\[ y_i(0) = y_{0i}, \quad \dot{y}_i(0) = \dot{y}_{0i} \]

(11)
In these cases initial conditions are defined in classical way by initial position $y_i(0)$ and initial velocity $\dot{y}_i(0)$ of every from the material particle system.

History of rheological standard element loadings in these integro-differential equations is taken into account by integral members in the period of integration $(-\infty, 0)$. For solving differential equation (15) in every case, initial conditions are defined by three initial conditions $y_i(0), \dot{y}_i(0)$, and $\ddot{y}_i(0)$ for every generalized coordinate of the system dynamics. In these cases initial conditions are defined by initial position $y_i(0)$, initial velocity $\dot{y}_i(0)$, and initial acceleration $\ddot{y}_i(0)$ of material particles. Last initial conditions - initial accelerations $\ddot{y}_i(0)$ of material particles are directly defined from stress-strain states of the corresponding standard light hereditary (rheological) elements on the basis of element loading histories. Particular examples to obtain or to define the third type initial conditions in accordance with the different loaded elements histories are presented in the Refs. [4]. In Ref. [4] a detailed schema for to obtain initial conditions of the hereditary oscillator in the case of the impulse external excitations is presented.

5. Dynamically defined hereditary discrete systems
From all previously considered theories [4] and [5] of hereditary discrete systems and numerous examples as well as presented in the main key points hereditary discrete systems we must separate two groups of hereditary discrete systems on the basis of possibilities to solve governing equations (9) or (10) with respect to rheological reactions $P_i$. Solving of governing equations (9) with respect to rheological reactions $P_i$ is possible under the following two conditions:

1* Number $K$ of the rheological elements must be less or equal to the number $n$ of the degrees of the hereditary discrete system freedom, $K \leq n$.

2* Structure of the mechanical hereditary discrete system must be like that, that there are possibility of the choices of the generalized coordinate that is possible to obtain inequities with zero defined by

$$\left| p_i(q) \right| = \left| \sum_{J=1}^{3N} e_{ij}(q) \frac{\partial \phi_j}{\partial q_i} \right| \neq 0 \quad \text{for } K \leq n \quad (12)$$

These conditions are generalization of the known conditions of the static defined mechanical system, applied widely for solving problems in the strength of materials.

6. Analogy between the hereditary interactions and reactive force and the systems of the automatic control
Equivalency of the hereditary interactions and reactive forces in the systems of the automatic control gives possibility to extend theory of the analytical dynamics of the hereditary systems to the mechanical systems with automatic control. For example, automat with transfer function presented in the following form

$$\frac{W(p)}{1} = \frac{b_0 + b_1 p + \ldots + b_n p^n}{1 + a_1 p + \ldots + a_n p^n}$$

present a hereditary interaction (1) between material particles of the mechanical system.

7. Concluding Remarks
In the basis of the construction of analytical dynamics of hereditary discrete systems, the classical mechanics principles are used. These principles are: Principle of the work of forces along corresponding possible system displacements, as well as Principle of dynamical equilibrium. Using the
Principle of the work of system forces along corresponding possible system displacements we obtain governing system equations of the hereditary discrete system dynamics.

For the class of dynamically defined hereditary systems, it is possible to eliminate reactions of rheological elements and to obtain modified Lagrange’s equations in differential and integro-differential forms. In the case that stress-strain relations of hereditary elements can be expressed in all three forms, as also only in the forms by used relaxational kernels it is possible to construct Lagrange’s equations for every arbitrary type of hereditary systems.

Also, a class of dynamically undefined hereditary systems is defined and considered in the basic.

Initial conditions of hereditary system dynamics are very important, containing the history of rheological interactions of the system. It is also important to take into account stress-strain history of viscoelastic elements – interactions between hereditary system material particles.

A hereditary discrete system with $n$ degrees of freedom which contains rheological elements of Maxwell’s type, with stress-strain states described by differential equations is considered.

The Lagrange’s mechanics of the hereditary systems is extended and generalized to the thermo-rheological and piezo-rheological [14-18] mechanical systems. Methods for solving problems of dynamics of the hereditary systems are considered with special Euler’s Gama functions [4-5]. Approximation of the expressions for the coefficients of the damping and corresponding decrements as well as for frequency of the oscillations hereditary oscillatory systems are obtained with high accuracy.

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