Invariance of the jacobian Newton diagram

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January 15, 2013

Abstract
We prove that the jacobian Newton diagram of the holomorphic mapping $(f, g) : (C^2, 0) \rightarrow (C^2, 0)$ depends only on the equisingularity class of the pair of curves $f = 0$ and $g = 0$.

1 Introduction
Write $R_+ = \{ x \in R : x \geq 0 \}$. The Newton diagram $\Delta_h$ of a power series $h(x, y) = \sum_{i,j} c_{ij} x^i y^j$ is by definition the convex hull of the union

$$\bigcup_{\{(i,j) : c_{ij} \neq 0\}} \{ (i, j) + R_+^2 \}.$$  

Example 1.1 The Newton diagram of $h(x, y) = y^5 + 2xy^3 - x^3y^2 + 3x^4y$ is drawn in the figure. Black dots are the points of the first quadrant $R_+^2$ corresponding to non-zero monomials of the series $h$.

Let $\phi : (C^2, 0) \rightarrow (C^2, 0)$, $\phi^{-1}(0, 0) = \{(0, 0)\}$ be a germ of a holomorphic mapping given by $(x, y) = (f(u, v), g(u, v))$. Let $\text{Jac} \phi = \frac{\partial f}{\partial u} \frac{\partial g}{\partial v} - \frac{\partial f}{\partial v} \frac{\partial g}{\partial u}$ be the usual jacobian determinant. The direct image of the curve germ Jac $\phi = 0$ by $\phi$ is called the discriminant curve of $\phi$ (see [Ca]). If $D(x, y) = 0$ is an analytic equation of the discriminant curve then the Newton diagram of $D$ is called the jacobian Newton diagram of $(f, g)$. We will write $N_J(f, g)$ for the jacobian Newton diagram.

Definition 1.2 Let $\xi, \xi', \nu, \nu'$ be germs of analytic curves in $(C^2, 0)$. We say that the pairs of curves $\xi, \nu$ and $\xi', \nu'$ are equisingular if there exists a homeomorphism $\Psi : (C^2, 0) \rightarrow (C^2, 0)$ preserving the multiplicity of each branch such that $\Psi(\xi) = \xi'$ and $\Psi(\nu) = \nu'$.
2 Main result

**Theorem 2.1** Let \((f, g) : (C^2, 0) \to (C^2, 0), (f, g)^{-1}(0, 0) = \{(0, 0)\}\) be a germ of a holomorphic mapping. Then the Jacobian Newton diagram \(N_J(f, g)\) depends only on the equisingularity class of the pair of curves \(f = 0\) and \(g = 0\).

The proof is in the last section.

We give a short survey of results related with Theorem 2.1. We need a few notions which will be used only in this section to explain connection between certain analytic factorizations of \(jac(f, g)\) and the Jacobian Newton diagram \(N_J(f, g)\).

The Minkowski sum of Newton diagrams \(\Delta_1\) and \(\Delta_2\) is by definition \(\Delta_1 + \Delta_2 = \{p + q : p \in \Delta_1, q \in \Delta_2\}\). The set of Newton diagrams is a semi-group with respect to Minkowski sum and the generators of this semi-group are elementary Newton diagrams illustrated in Figure 1.

![Figure 1: Elementary Newton diagrams](image)

The inclination of the elementary Newton diagram \(\{a\}\) is the quotient \(\frac{a}{b}\) with conventions \(\frac{a}{\infty} = \infty\) and \(\frac{\infty}{b} = 0\). For an arbitrary Newton diagram \(\Delta\) represented as a sum of elementary Newton diagrams let us denote \(I(\Delta)\) the set of inclinations of elementary Newton diagrams of the sum. It is easy to see that \(I(\Delta)\) does not depend on the choice of representation. Coming back to Example 1.1 the Newton diagram \(\Delta_h\) is the sum \(\{\frac{1}{2}\} + \{\frac{3}{2}\} + \{\frac{\infty}{1}\}\) and \(I(\Delta_h) = \{1/2, 3/2, \infty\}\).

For every irreducible factor \(h\) of \(jac(f, g)\) the Hironaka number \(q(h) = \frac{i_0(g, h)}{i_0(f, h)}\), where \(i_0(\cdot, \cdot)\) stands for the intersection multiplicity, is called the Jacobian quotient or the Jacobian invariant of \((f, g)\).

**Definition 2.2** Let \(jac(f, g) = J_1 \cdots J_n\) be an analytic factorization of the Jacobian.

We will call \(J_1 \cdots J_n\) a Hironaka factorization if for every \(J_i\) \((1 \leq i \leq n)\) the Hironaka number \(q(h)\) is constant for all irreducible factors \(h\) of \(J_i\).

The Hironaka factorization \(J_1 \cdots J_n\) will be called minimal if Hironaka numbers of irreducible factors of \(J_1\) and \(J_k\) are different for \(1 \leq l < k \leq n\).

Let \(jac(f, g) = h_1 \cdots h_n\) be the factorization of the Jacobian into irreducible factors. It is easy to check (cf. [10]) that

\[
N_J(f, g) = \sum_{i=1}^{n} \left\{ \frac{i_0(g, h_i)}{i_0(f, h_i)} \right\}.
\]
It follows directly from the above formula that

- the set of jacobian quotients of \((f, g)\) is the set of inclinations of \(\mathcal{N}_J(f, g)\),
- if \(J_1 \cdots J_r\) is a Hironaka factorization of \(\text{jac}(f, g)\) then
\[
\mathcal{N}_J(f, g) = \sum_{i=1}^{r} \left\{ \frac{i_0(g, J_i)}{i_0(f, J_i)} \right\},
\]
- if \(\mathcal{N}_J(f, g) = \sum_{i=1}^{s} \left\{ \frac{a_i}{b_i} \right\}\) with inclinations \(\frac{a_i}{b_i}\) pairwise different then \(\text{jac}(f, g)\) has the minimal Hironaka factorization \(J_1 \cdots J_s\) such that \(i_0(g, J_i) = a_i\) and \(i_0(f, J_i) = b_i\) for \(i = 1, \ldots, s\).

Take a germ of a holomorphic mapping \((l, f) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)\) such that \(f = 0\) is a curve germ without multiple components and \(l = 0\) is a smooth curve. Under these assumptions \(\text{jac}(l, f) = 0\) is called the polar curve of \(f\) with respect to \(l\) and jacobian quotients of \((l, f)\) are called polar quotients. A survey of recent results concerning polar curves is in \([GLP]\).

In \([KL]\) the authors described the contact orders of Newton-Puiseux roots of \(f'(x, y) = 0\) with the Newton-Puiseux roots of \(f(x, y) = 0\). They constructed the tree model \(T(f)\) which encodes these contact orders. Using Kuo-Lu tree \(T(f)\) one can compute the set of polar quotients of \((y, f)\). One can also give a formula for the jacobian Newton diagram of \((y, f)\) in terms of \(T(f)\) (see the last line before Example 5.2 in \([GG]\)).

Merle in \([Me]\) obtained the minimal Hironaka decomposition of the polar curve of the irreducible curve germ \(f = 0\) with respect to a smooth curve \(l = 0\) transverse to \(f = 0\). Merle’s results are rewritten in \([Ma]\) as a formula for the jacobian Newton diagram of \((l, f)\) (see also \([GLP]\), Theorem 4.1).

In \([Eg]\) the author found a Hironaka factorization of the polar curve of a many-branched curve \(f = 0\). He associated the factors with vertexes of a new type of tree \(E(f)\) called now Eggers tree. Eggers found also the intersection multiplicities of every factor with \(l\) and \(f\). Since the Eggers tree \(E(f)\) depends only on the equisingularity class of \((l, f)\), Theorem 2.1 in this particular case follows from \([Eg]\).

Consider now a general case of a holomorphic mapping germ \((f, g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)\), where \((f, g)^{-1}(0, 0) = \{(0, 0)\}\).

In \([KP]\) the authors additionally assumed that the curve \(fg = 0\) has no multiple components. They introduced the Eggers tree model \(E(f, g)\) of a pair \((f, g)\). They obtained the Hironaka factorization of the jacobian associated with vertexes of \(E(f, g)\). However they did not compute the intersection multiplicities of some factors (factors associated with collinear bars in terminology of \([KP]\)) with \(f\) and \(g\). Hence Theorem 4.1 does not follow from \([KP]\).

In \([Ma]\) and \([Mi]\) the authors resolved singularities of the curve \(fg = 0\). Then they distinguished some subsets of the exceptional divisor called rupture zones and associated with every rupture zone a factor of the jacobian \(\text{jac}(f, g)\). Maugendre found in \([Ma]\) using topological methods the set of jacobian quotients.
(see also [Ca1] for an algebraic proof) and Michel completed the work computing intersection multiplicity of every factor with \(f\) and \(g\). Since the decomposition of the jacobian obtained by Michel is a Hironaka factorization, Theorem 2.1 follows from [Mi]. However my proof is much simpler as it uses only Theorem 3.1 and Theorem 2.1 of [Ca].

3 Invariance of a generic curve of the pencil

**Theorem 3.1** Let \((f, g) : (C^2, 0) \to (C^2, 0), (f, g)^{-1}(0, 0) = \{(0, 0)\}\) be a germ of a holomorphic mapping. Then for all \(t \in C\) but a finite number the equisingularity class of the curve \(f(x, y) - tg(x, y) = 0\) depends only on the equisingularity class of the pair of curves \(f = 0\) and \(g = 0\).

I guess that the above theorem is a known result. However I did not find any reference and I decided to prove it.

**Proof.** Our main reference is Chapter III of [Oka]. Let \(R : M \to (C^2, 0)\) be the minimal good resolution of singularities of the curve \(fg = 0\). The set \(R^{-1}\{(fg = 0)\}\) can be written as the union of irreducible components \(E_1 \cup \ldots \cup E_n \cup E_{n+1} \cup \ldots \cup E_m\), where \(E = E_1 \cup \ldots \cup E_n\) is the exceptional divisor \(R^{-1}(0)\) and \(E_{n+1}, \ldots, E_m\) are non-compact curves corresponding with branches of the curve \(fg = 0\). Put \(\tilde f = f \circ R\) and let \(a_i = \text{order of } \tilde f\) along \(E_i\), \(b_i = \text{order of } \tilde g\) along \(E_i\) for \(i = 1, \ldots, m\). Then, after renumbering \(E_1, \ldots, E_m\) if necessary, the total dual resolution graph as well as the numbers \(a_i\) and \(b_i\) for \(i = 1, \ldots, m\) depend only on the equisingularity class of the pair of curves \(f = 0\) and \(g = 0\).

Consider the meromorphic function \(\tilde g/\tilde f : M \setminus E \to C \cup \{\infty\}\). We will check that this function extends analytically to the whole \(M\) but a finite number of points. Let \(\partial\) denotes any germ of a holomorphic function \(u(x, y)\) such that \(u(0, 0) \neq 0\).

First take \(P \in E_i\) \((1 \leq i \leq n)\) which is not an intersection point with another component \(E_j\) for \(1 \leq j \leq m\). Then there exists a local analytical coordinate system \((x, y)\) centered at \(P\) such that \(E_i\) has an equation \(x = 0\). In these coordinates \(\tilde f = \partial x^{a_i}\) and \(\tilde g = \partial x^{b_i}\). We get \(\tilde g/\tilde f = \partial x^{b_i-a_i}\).

Now take the intersection point \(P\) of \(E_i\) with another component \(E_j\). Choose a local analytical coordinate system \((x, y)\) centered at \(P\) such that \(E_i\) has an equation \(x = 0\) and \(E_j\) has an equation \(y = 0\). In these coordinates \(\tilde f = \partial x^{a_j}y^{b_j}\) and \(\tilde g = \partial x^{a_i}y^{b_i}\). We get \(\tilde g/\tilde f = \partial x^{b_i-a_i}y^{b_j-a_j}\).

Let \(H\) be an analytic extension of \(\tilde g/\tilde f\). Divide the set \(\{E_1, \ldots, E_m\}\) into three subsets \(A_+ = \{E_i : b_i - a_i > 0\}\), \(A_0 = \{E_i : b_i - a_i = 0\}\) and \(A_- = \{E_i : b_i - a_i < 0\}\). It follows from above description of \(\tilde g/\tilde f\) near \(E\) that \(H\) is not defined only at intersection points of components from \(A_+\) with components from \(A_-\).

Let \(E_i \in A_0\). Consider the restriction \(H|_{E_i}\) of the meromorphic function \(H\) to \(E_i\). Then \(P \in E_i\) is a zero of \(H|_{E_i}\) if and only if \(\{P\} = E_i \cap E_j\) for some \(E_j \in A_+\). Moreover \(\text{ord}_PH|_{E_i} = b_j - a_j\). Hence the topological degree of \(H|_{E_i}\)
is the number \( d_i = \sum (b_j - a_j) \) where the sum runs over all \( j \) such that \( E_j \in A_+ \) and the intersection \( E_i \cap E_j \) is nonempty.

Choose any complex number \( t \) which is different from

- any critical value of meromorphic functions \( H|_{E_i} \) where \( E_i \in A_0 \),
- any value \( H(P) \) where \( P \) is the intersection point of some \( E_i \in A_0 \) with some \( E_j, j \neq i \).

Let \( \Gamma \) be the proper preimage of the curve \( f - tg = 0 \). The curve \( \Gamma \) has an equation \( H = t \) at every point where \( H \) is defined. Hence \( \Gamma \) intersects transversally every \( E_i \in A_0 \) at \( d_i \) points and none of these points belongs to \( \bigcup_{j \neq i} E_j \).

Now we compute the equation of \( \Gamma \) at points where \( H \) is not defined. Take \( E_i \in A_+, E_j \in A_- \) with nonempty intersection and denote \( P_{i,j} \) their intersection point. There exists a local analytical coordinate system \( (x,y) \) centered at \( P_{i,j} \) such that \( E_i \) has an equation \( x = 0 \) and \( E_j \) has an equation \( y = 0 \). In these coordinates \( \tilde{f} - \tilde{t}g = o \cdot x^{a_j}y^{a_j} - t \cdot o \cdot x^{b_j}y^{b_j} = x^{a_j}y^{b_j} (o \cdot y^{a_j} - b_j - t \cdot o \cdot x^{b_j} - a_i) \) hence \( \Gamma \) has the equation \( o \cdot y^{a_j} - b_j - t \cdot x^{b_j} - a_i = 0 \).

We want to resolve singularities of the curve \( \Gamma \) to obtain a good (not necessarily minimal) resolution of singularities of \( f - tg = 0 \). The functions \( h_{i,j}(x,y) = o \cdot y^{a_j} - b_j - t \cdot x^{b_j} - a_i \) are nondegenerate with Newton diagrams \( \left\{ \frac{b_j - a_i}{a_j - b_j} \right\} \). Hence by Theorem 4.3 of [Oka] there exists a canonical toric resolution of \( h_{i,j}(x,y) = 0 \) at the origin, that is the resolution of \( \Gamma \) at \( P_{i,j} \), which depends only on the Newton diagram of \( h_{i,j} \). Applying such a toric resolution at every point \( P_{i,j} \) described above we obtain a good resolution of \( f - tg = 0 \). Moreover the total dual resolution graph of this resolution depends only on the total dual resolution graph of \( R \) and on the numbers \( a_i \) and \( b_i \) for \( i = 1, \ldots, m \). Since the total dual resolution graph of the plane curve singularity determines its equisingularity class, the proof is finished. ■

4 Proof of the main result

For all holomorphic functions \( h_i \) \( (i = 1, 2) \) defined in the neighborhood of the origin of \( \mathbb{C}^2 \) we will denote \( \mu_0(h_1, h_2) \) the intersection multiplicity of curves \( h_1 = 0 \) and \( h_2 = 0 \) at zero and \( \mu_0(h_1) \) the Milnor number of the curve \( h_1 = 0 \) at zero. For every Newton diagram \( \Delta \) and for every \( \vec{v} = (v_1, v_2) \), where \( v_1 > 0, v_2 > 0 \) we define

\[ l(\vec{v}, \Delta) = \min\{ v_1 i + v_2 j : (i, j) \in \Delta \} \]

Lemma 4.1 Let \( \vec{v} = (m, n) \), where \( n, m \) are co-prime positive integers. Then for generic \( t \in \mathbb{C} \)

\[ l(\vec{v}, \mathcal{N}_J(f, g)) = \mu_0(f^n - tg^m) - \mu_0(f, g)(m - 1)(n - 1) - 1 \]
Proof. Let $D = 0$, where $D(x,y) = \sum c_{ij}x^iy^j$ be the equation of the discriminant curve of $\phi = (f,g) : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$. Take a curve $x^n - ty^m = 0$.

Claim. For generic $t \in \mathbb{C}$ there is $i_0(x^n - ty^m, D) = l(\vec{v}, N_J(f,g))$.

Let $\tau = \sqrt[\bar{\bar{y}}]{t}$. Then $x = \tau s^m$, $y = s^n$ is a parametrization of the branch $x^n - ty^m = 0$. By the classical formula for the intersection multiplicity

$$i_0(x^n - ty^m, D) = \text{ord}_s D(\tau s^m, s^n) = \text{ord}_s \sum c_{ij} \tau^i s^{mi+nj} = l(\vec{v}, N_J(f,g))$$

provided $\tau$ is sufficiently general so that the sum $\sum_{mi+nj=l(\vec{v},N_J(f,g))} c_{ij} \tau^i$ is nonzero. The Claim is proved.

The pull-back of a curve $x^n - ty^m = 0$ by $\phi$ has an equation $f^n - tg^m = 0$. Thus by Theorem 3.2 of [Ca] there is

$$\mu_0(f^n - tg^m) - 1 = i_0(f,g)[\mu_0(x^n - ty^m) - 1] + i_0(x^n - ty^m, D)$$

which gives the Lemma because $\mu_0(x^n - ty^m) = (m-1)(n-1)$. ■

Proof of Theorem 2.1. It follows from Lemma 4.1 and Theorem 3.1 that for every vector $\vec{v} = (m,n)$, where $m$, $n$ are co-prime positive integers, the number $l(\vec{v}, N_J(f,g))$ depends only on the equisingularity class of the pair $f = 0$ and $g = 0$. This proves the Theorem. ■

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