Understanding Several Power Spectrum Density Estimation Algorithms Based on the Problems Solved

Qun WAN*, Jin NIEa, Ji-Hao YINb, Lin ZOUC and Xian-Sheng GUOd

School of Information and Communication Engineering, University of Electronic Science and Technology of China, Chengdu 611731, China

*wanqun@uestc.edu.cn, b)jhyin@uestc.edu.cn, c)weight_0@163.com, d)xsguo@uestc.edu.cn

Keywords: Spectral estimation; Estimation criterion; Solving problem.

Abstract. Starting from the estimation criterion and the essential problem of algorithmic computing, this paper introduces four kinds of spectral estimation methods commonly used in teaching: Periodogram to fit frequency vector under least square criterion, AR model estimation converts spectrum estimation into parameter estimation, MVDR converts spectrum estimation into filter weight coefficient estimation, and MUSIC converts spectral estimation into subspace estimation. Then we analyze and solve the equation, derive the assumption of signal and the defects caused by this assumption.

Introduction

In the field of signal and information processing, spectral estimation is a very important technology [1, 2].

The power spectral density $S(\omega)$ of signal $x(n)$ characterizes the frequency content of the signal. In theory, the power spectral density function is defined as

$$S(\omega) = \lim_{N \to \infty} \left\{ \frac{1}{N} \sum_{n=1}^{N} |x(n)e^{-jn\omega}|^2 \right\}.$$  \hspace{1cm} (1)

where $S(\omega)$ can be understood as the expectation of the mean value of the power spectrum of $x(n)$ [3]. But in actual signal processing, we only have $N$ observation data of the $x(n)$. Next, we will introduce four methods to process these data. Unlike traditional teaching, we will not deduce the algorithm in detail, but focus on the key expressions of the algorithm to start the analysis. It is hoped that learners will have a deeper understanding of the algorithm and understand under what conditions it is optimal to use these algorithms. The specific principle of the algorithm can be referred to the relevant literature: periodogram [3, 4], AR model estimation [5], MVDR [6, 7], MUSIC [8, 10].

Periodogram—The Solution to Signal Fitting

The principle of periodogram estimation is based on the traditional Fourier transform, which is also the deduction criterion of power spectrum definition formula (1). Here we interpret the power spectrum as the solution of fitting the signal with the signal frequency vector under the least square criterion, i.e., solving the formula:

$$\min_{s(\omega)} \|x(n) - a(\omega)s(\omega)\|.$$ \hspace{1cm} (2)

where $x(n)$ is the signal to be estimated, $a(\omega)$ is the signal frequency vector

The solution can be described as
\[
\hat{S}_{\text{PER}}(\omega) = \frac{1}{N} \left| \sum_{n=1}^{N} x(n) e^{-j\omega n} \right|^2.
\]  

(3)

We could get the hypothesis condition of periodogram by comparing equation (1) and (3). The equation (3) drop the \(\lim_{N \to \infty}\) and \(E\{\star\}\), which means the periodogram method is a generalization of the definition of power spectrum with finite data. The signal should satisfy assumption:

\[x(n) = 0, \quad n > N.\]  

(4)

In this case, the estimation is optimal. Otherwise, periodogram estimation is equivalent to truncation of infinite time series, which will lead to widening of main lobe and spectrum leakage.

In addition, we find that the spectral estimation of the signal \(x(n)\) is equal to that of the time transposed conjugation \(x^*(N-n)\) substitution, which indicates that the estimation can utilize less data.

**AR Model Estimation—The Solution to Parametric Estimation**

The basic criterion of spectral estimation of parametric model is that the signal \(x(n)\) is the response of white noise through LTI discrete-time system. The power spectrum can be obtained by estimating the model parameters using the observed samples. We only take AR model as an example to introduce parameter estimation. The P-order AR model can be described as

\[x(n) = -\sum_{k=1}^{p} a_k x(n-k) + v(n).\]  

(5)

where \(a_k\) is the parameter to be estimated and \(v(n)\) is white noise. The method of parameter estimation is to solve Yule-Walker equation:

\[
\min_{\theta_p} \left\| \hat{R}_{p+1} \begin{bmatrix} 1 \\ \theta_p \end{bmatrix} - \begin{bmatrix} \sigma_v^2 \\ \theta_p \end{bmatrix} \right\|.
\]  

(6)

where

\[\theta_p = [a_1 \ a_2 \ \cdots \ a_p]^T.\]  

(7)

Matrix \(\hat{R}_{p+1}\) is defined as

\[
\hat{R}_{p+1} = \begin{bmatrix}
r_x(0) & r_x(-1) & \cdots & r_x(-p) \\
r_x(1) & r_x(0) & \cdots & r_x(-p+1) \\
\vdots & \vdots & \ddots & \vdots \\
r_x(p) & r_x(p-1) & \cdots & r_x(0)
\end{bmatrix}
\]

(8)

\(r_x(m)\) is a signal autocorrelation function, which can be estimated by the observed samples \(x(n)\).

The shortcoming of AR models is the denominator of spectrum can’t be zero. The power spectrum estimated by AR model can be described as

\[
\hat{S}_{\text{AR}}(\omega) = \frac{\sigma_v^2}{1 + \sum_{k=1}^{p} a_k e^{-j\omega k}}.
\]  

(9)

Formula (9) shows that AR model can double computable data by the time transposed conjugation of the signal.
The AR model of the signal is
\[
x^*(n-p) = a_1 x^*(n-p+1) + \ldots + a_{n-2} x^*(n-2) + a_{n-1} x^*(n-1) + a_n x^*(n) - v^*(n).
\]

In this way, we can get more prior information and get better estimation results.

Observation of \( \hat{S}_{AR}(\omega) \) shows that due to the existence of noise, its molecule can’t be zero, its denominator is less than the square of the parameter modes, so there is a lower bound:
\[
\hat{S}_{AR}(\omega) \geq \frac{\sigma_v^2}{1 + \sum_{k=1}^{\infty} |a_k|^2}.
\]

which shows that the existence of noise will make the power spectrum relatively flat and reduce the resolution.

**MVDR—The Solution to Filter Weight Estimation**

We first introduce the signal model used in the algorithm.
\[
x(n) = \sum_{k=1}^{K} a_k e^{j\omega_k n} + v(n).
\]

Define the signal vector as
\[
x(n) = As(n) + v(n) \in \mathbb{C}^{M+1}.
\]

where
\[
A = \begin{bmatrix} a(\omega_1) & a(\omega_2) & \cdots & a(\omega_K) \end{bmatrix}.
\]

\[
a(\omega) = \begin{bmatrix} 1 \\
e^{-j\omega} \\
\vdots \\
 e^{-(M-1)j\omega} \end{bmatrix}, s(n) = \begin{bmatrix} s_1(n) \\
s_2(n) \\
\vdots \\
s_K(n) \end{bmatrix}, v(n) = \begin{bmatrix} v(n) \\
v(n-1) \\
\vdots \\
v(n-M+1) \end{bmatrix}.
\]

MVDR estimation criterion converts spectral estimation into filter weight estimation. Its constraints include two parts: minimum variance constraints and distortionless response constraints, which can be described as
\[
\min_w w^H R w, \quad \text{st. } w^H a(\omega) = 1.
\]

where \( w \) is the weight coefficient vector of the filter and \( R \) is the autocorrelation matrix of the input signal. \( a(\omega) \) is the frequency vector of the desired signal. Constrained \( w^H a(\omega) \) is designed to ensure that the desired signal can pass through the filter without distortion, and that the minimum output power of constrained \( w^H R w \) can achieve the purpose of suppressing other frequency signals and noise. By solving (17), the optimal weight vector estimation can be obtained.

\[
\hat{w}_{MVDR} = \frac{\hat{R}'a(\omega)}{a^H(\omega)\hat{R}'a(\omega)}.
\]

Next, we analyze the assumptions that the signal should satisfy.
\( \mathbf{R} \) can be described as

\[
\mathbf{R} = E\left\{ \mathbf{x}(n)\mathbf{x}^H(n) \right\} = \mathbf{A}\mathbf{P}\mathbf{A}^H + \sigma^2 \mathbf{I} \tag{19}
\]

where

\[
\mathbf{P} = E\left\{ \mathbf{s}(n)\mathbf{s}^H(n) \right\} \tag{20}
\]

We find that the input signal must be incoherent, otherwise the minimum variance constraint of the algorithm will be invalid. Suppose that the signal frequency vectors \( \mathbf{a}(\omega_1) \) and \( \mathbf{a}(\omega_2) \) correspond to the signal coherence, such as

\[
s_2(n) = \beta s_1(n) \tag{21}
\]

The signal frequency matrix becomes

\[
\mathbf{A}' = \begin{bmatrix} \mathbf{a}(\omega_1) + \beta \mathbf{a}(\omega_2) & \mathbf{a}(\omega_3) & \cdots & \mathbf{a}(\omega_K) \end{bmatrix} \tag{22}
\]

We find that the signal frequency matrix for minimum variance constraints may not contain frequency vectors \( \mathbf{a}(\omega_1) \) and \( \mathbf{a}(\omega_2) \) and can’t achieve the purpose of constraints. Moreover, \( \mathbf{A}' \) becomes K-1 column, which will lead \( \mathbf{R} \) to ill-conditioned matrix, unstable inversion and large estimation error.

**MUSIC—The Solution to Subspace Estimation**

MUSIC uses the same signal model as MVDR. The estimation criterion of MUSIC is to convert power spectrum estimation into subspace estimation. Its estimation principle can be described as

\[
\min_{\mathbf{G}} \| \mathbf{G}^H\mathbf{R} \| \tag{23}
\]

where \( \mathbf{R} \) is the autocorrelation matrix of signal \( \mathbf{x}(n) \) and \( \mathbf{G} \) is the noise subspace matrix. The power spectrum estimation function can be expressed as

\[
\hat{\mathbf{P}}_{\text{MUSIC}}(\omega) = \frac{1}{\mathbf{a}^H(\omega)\mathbf{G}\mathbf{G}^H\mathbf{a}(\omega)}, \omega \in [-\pi, \pi] \tag{24}
\]

The analysis of \( \hat{\mathbf{P}}_{\text{MUSIC}}(\omega) \) shows that MUSIC assumes the same signal condition as MVDR, requiring that the signal is irrelevant. The analysis method is as same as the previous introduction. Suppose that the signal frequency vectors \( \mathbf{a}(\omega_1) \) and \( \mathbf{a}(\omega_2) \) correspond to the signal coherence, such as

\[
s_2(n) = \beta s_1(n) \tag{25}
\]

The signal frequency matrix becomes

\[
\mathbf{A}' = \begin{bmatrix} \mathbf{a}(\omega_1) + \beta \mathbf{a}(\omega_2) & \mathbf{a}(\omega_3) & \cdots & \mathbf{a}(\omega_K) \end{bmatrix} \tag{26}
\]

Then the corresponding noise subspace \( \mathbf{G}' \) should satisfy

\[
\mathbf{G}'^H\mathbf{A}' = 0 \tag{27}
\]

We find that because of signal coherence, the noise subspace may no longer be orthogonal to the frequency vector \( \mathbf{a}(\omega_1), \mathbf{a}(\omega_2) \) and can’t distinguish the frequency of the corresponding signal. The
results of other signal coherence cases are similar. Signal coherence results in that the noise subspace may no longer be orthogonal to the corresponding signal frequency vector and the signal frequency can’t be estimated.

Summary
In learning and analyzing different spectral estimation algorithms, we should grasp the essence of their solution. When power spectrum estimation is used in practical problems, we can choose the appropriate algorithm through the assumptions of different algorithms. In learning or teaching power spectrum estimation, we pay attention to the overall idea of analyzing the spectrum estimation problem. As introduced in this paper, starting from the estimation criteria, we find the solution of the essence of the algorithm, analyze the signal form theoretically, and find the algorithm defects. In this way, learners will have a clearer idea when comparing different methods or practical applications.

References
[1] S.L. Marple, Jr., Digital Spectral Analysis with Applications, Englewood Cliffs, NJ: Prentice-Hall, 1987.
[2] S.M. Kay, Modern Spectral Estimation: Theory and Application. Englewood Cliffs, NJ: Prentice-Hall, 1988.
[3] Steven M. Kay, Stanley Lawrence Marple, Spectrum Analysis—A Modern Perspective, Proceedings of the IEEE, Vol. 69, No. 11, November 1981.
[4] Welch P D, The use of fast Fourier transform for the estimation of power spectra: a method based on time averaging over short modified periodograms, IEEE Trans. Audio Electroacoustics, 1967, 15(2):70-73.
[5] James A. CADzoW, Spectral Estimation: An Overdetermined Rational Model Equation Approach, Proceedings of the IEEE, Vol. 70, No. 9, September 1982.
[6] Jacob Benesty, Jingdong Chen, Yiteng (Arden) Huang, A Generalized MVDR Spectrum, in: B.S. Jones, R.Z. Smith (Eds.), Introduction to the Electronic Age, E-Publishing Inc., New York, 1999, pp. 281-304.
[7] Capon J, High-resolution frequency-wavenumber spectrum analysis, Proc. IEEE, 1969, 57(8):1408-1418
[8] R.O. Schmidt, Multiple emitter location and signal parameter estimation, in Proc. RADC Spectral Estimation Workshop, Rome, NY, 1979, pp. 243-258.
[9] P.G. Clem, M. Rodriguez, Music, Maximum Likelihood and Cramer-Rao Bound, IEEE Transactions on Acoustics. Speech. and Signal Processing. Vol 17. No 5.
[10] Thomson-CSF ASM Division, Cagnes-Sur-Mer, Optimality of high resolution array processing using the eigen system approach, IEEE Transactions on Acoustics, Speech, and Signal Processing Volume: 31, Issue: 5, Oct 1983