A model theoretic Rieffel’s theorem of quantum 2-torus

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Abstract

We defined a notion of quantum 2-torus $T_\theta$ in [1] and studied its model theoretic property. In this note we associate quantum 2-tori $T_\theta$ with the structure over $C_{\theta} = (\mathbb{C}, +, \cdot, y = x^\theta)$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and introduce the notion of geometric isomorphisms between such quantum 2-tori.

We show that this notion is closely connected with the fundamental notion of Morita equivalence of non-commutative geometry. Namely, we prove that the quantum 2-tori $T_{\theta_1}$ and $T_{\theta_2}$ are Morita equivalent if and only if $\theta_2 = \frac{a\theta_1 + b}{c\theta_1 + d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z})$ with $|ad - bc| = 1$. This is our version of Rieffel’s Theorem [3] which characterises Morita equivalence of quantum tori in the same terms.

The result in essence confirms that the representation $T_\theta$ in terms of model-theoretic geometry [1] is adequate to its original definition in terms of non-commutative geometry.

1 Introduction

We introduce the notion of geometric transformation from a quantum 2-torus into another which fixes the underlying field structure and gives a one-to-one correspondence between the canonical bases of the
modules constituting the quantum 2-tori. When there is a geometric transformation, say \( L \), from \( T_1 \) to \( T_2 \). In this case we say that the quantum 2-tori \( T_1 \) and \( T_2 \) are \textit{geometrically isomorphic}.

Our main result establishes a direct correspondence between the notion of geometric isomorphism of tori and the well-known notion of \textit{Morita equivalence} of quantum 2-tori given in terms of their “coordinate” algebras.

Recall that two algebras \( A \) and \( B \) are said to be Morita equivalent if the categories \( A\text{-mod} \) and \( B\text{-mod} \) of modules are equivalent.

For quantum tori this notion was studied by M.Rieffel and in the particular case of 2-tori we have the following

\textbf{Theorem 1 (Rieffel [3])} Let \( A_{\theta_1} \) and \( A_{\theta_2} \) be (the coordinate algebras of) quantum 2-tori Tori. Then \( A_{\theta_1} \) and \( A_{\theta_2} \) are Morita equivalent if and only if there exist integers \( a, b, c, d \) such that \( ad - bc = \pm 1 \) and

\[ \theta_2 = \frac{a \theta_1 + b}{c \theta_1 + d}. \]

We also say in this case that the quantum 2-tori \( T_{\theta_1} \) and \( T_{\theta_2} \) are Morita equivalent.

In section 4 we prove Theorem 12 stating that: \( T_{\theta_1} \) and \( T_{\theta_2} \) are Morita equivalent if and only if \( T_{\theta_1} \) and \( T_{\theta_2} \) are geometrically isomorphic.

Of course, in light of Rieffel’s theorem it is enough to prove that the geometric isomorphism of \( T_{\theta_1} \) and \( T_{\theta_2} \) amounts to the condition

\[ \theta_2 = \frac{a \theta_1 + b}{c \theta_1 + d} \]

for some \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}_2(\mathbb{Z}) \) with \(|ad - bc| = 1\).

In section 2 we review quickly the construction of quantum 2-tori defined in [1]. In section 3, we introduce the notion of \textit{Morita transformation} and \textit{Morita equivalence} and prove basic properties. In section 4, we characterise the property of functions giving rise to Morita transformations and prove Theorem 12.

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2 Quick review of the construction of a quantum 2-torus

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and put $q = \exp(2\pi i \theta)$. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Consider a $\mathbb{C}^*$-algebra $\mathcal{A}_q$ generated by operators $U, U^{-1}, V, V^{-1}$ satisfying

$$VU = q_1 UV, \quad UU^{-1} = U^{-1}U = VV^{-1} = V^{-1}V = I.$$ 

Let $\Gamma_\theta = q^\mathbb{Z} = \{q^n : n \in \mathbb{Z}\}$ be a cyclic multiplicative subgroup of $\mathbb{C}^*$. From now on in this note we work in an uncountable $\mathbb{C}$-module $\mathcal{M}$ such that $\dim \mathcal{M} \geq |\mathbb{C}|$.

2.1 $\Gamma$-sets, $\Gamma$-bundles, line-bundles

For each pair $(u, v) \in \mathbb{C}^* \times \mathbb{C}^*$, we will construct two $\mathcal{A}_q$-modules $M_{(u,v)}$ and $M_{(v,u)}$ so that both $M_{(u,v)}$ and $M_{(v,u)}$ are sub-modules of $\mathcal{M}$.

The module $M_{(u,v)}$ is generated by linearly independent elements labelled $\{u(\gamma u, v) : \gamma \in \Gamma_\theta\}$ satisfying

$$U : u(\gamma u, v) \mapsto \gamma u(\gamma u, v),$$
$$V : u(\gamma u, v) \mapsto vu(q^{-1}\gamma u, v).$$ \hfill (1)

Next let $\phi : \mathbb{C}^*/\Gamma_\theta \to \mathbb{C}^*$ such that $\phi(x\Gamma_\theta) \in x\Gamma_\theta$ for each $x\Gamma_\theta \in \mathbb{C}^*/\Gamma_\theta$. Put $\Phi = \text{ran}(\phi)$. We call $\phi$ a choice function and $\Phi$ the system of representatives.

Set for $\langle u, v \rangle \in \Phi^2$

$$\Gamma \cdot u(u,v) := \{\gamma u(u,v) : \gamma \in \Gamma_\theta\},$$
$$U_{(u,v)} := \bigcup_{\gamma \in \Gamma_\theta} \Gamma \cdot u(u,v) = \{\gamma_1 \cdot u(\gamma_2 u, v) : \gamma_1, \gamma_2 \in \Gamma_\theta\}. \hfill (2)$$

And set

$$U_\phi := \bigcup_{(u,v) \in \Phi^2} U_{(u,v)} = \{\gamma_1 \cdot u(\gamma_2 u, v) : \langle u, v \rangle \in \Phi^2, \gamma_1, \gamma_2 \in \Gamma_\theta\},$$
$$F^*U_{\phi_1} := \{x \cdot u(\gamma u, v) : \langle u, v \rangle \in \Phi^2, x \in F^*, \gamma \in \Gamma_\theta\}. \hfill (3)$$

We call $\Gamma \cdot u(u,v)$ a $\Gamma$-set over the pair $(u,v)$, $U_\phi$ a $\Gamma$-bundle over $\mathbb{C}^* \times \mathbb{C}^*/\Gamma_\theta$, and $\mathbb{C}^*U_\phi$ a line-bundle over $\mathbb{C}^*$. Notice that $U_\phi$ can also be seen as a bundle inside $\bigcup_{(u,v)} M_{(u,v)}$. Notice also that the line bundle $\mathbb{C}^*U_\phi$ is closed under the action of the operators $U$ and $V$ satisfying the relations (1).
We define the module $M_{[v,u]}$ generated by linearly independent elements labelled \( \{ v(\gamma v, u) \in M : \gamma \in \Gamma \} \) satisfying
\[
U : v(\gamma v, u) \mapsto uv(q\gamma v, u), \\
V : v(\gamma v, u) \mapsto \gamma v(v(\gamma v, u)),
\]
and also
\[
U^{-1} : u(\gamma u, v) \mapsto \gamma^{-1}u^{-1}u(\gamma u, v), \\
V^{-1} : u(\gamma u, v) \mapsto v^{-1}u(q\gamma u, v).
\]

Similarly a $\Gamma$-set $\Gamma \cdot v(u, v)$ over the pair $(v, u)$, a $\Gamma$-bundle $V_\phi$ over $\mathbb{C}^*/\Gamma \times \mathbb{C}^*$, and $\mathbb{C}^*V_\phi$ a line-bundle over $\mathbb{C}^*$ are defined.

To define the line bundles $\mathbb{C}^*U_\phi$ and $\mathbb{C}^*V_\phi$, we do not need any particular properties of the element $q = \exp(2\pi i\theta)$ or the choice function $\phi$. Therefore we have:

**Proposition 2 (Proposition 2 [1])** Let $F$, $F'$ be fields and $q \in F$, $q' \in F'$ such that there is an field isomorphism $i$ from $F$ to $F'$ sending $q$ to $q'$. Then $i$ can be extended to an isomorphism from the $\Gamma$-bundle $U_\phi$ to the $\Gamma'$-bundle $U_{\phi'}$ and also from the line-bundle $F^*U_\phi$ to the line-bundle $(F'^*)'U_{\phi'}$. The same is true for the line-bundles $F^*V_\phi$ and $(F'^*)'V_{\phi'}$.

In particular the isomorphism type of $\Gamma$-bundles and line-bundles does not depend on the choice function.

**Proof:** Let $i$ be an isomorphism from $F$ to $F'$ sending $q$ to $q'$. Set $i(x \cdot u(\gamma u, v)) = i(x) \cdot u(i(\gamma u), i(v))$. Then this defines an isomorphism from $F^*U_\phi$ to $(F'^*)'U_{\phi'}$. $lacksquare$

### 2.2 Pairing function

Recall next the notion of pairing function $\langle \cdot | \cdot \rangle$ which plays the rôle of an inner product of two $\Gamma$-bundles $U_\phi$ and $V_\phi$:
\[
\langle \cdot | \cdot \rangle : \left( V_\phi \times U_\phi \right) \cup \left( U_\phi \times V_\phi \right) \to \Gamma.
\]

having the following properties:

1. $\langle u(v, u)\rangle = 1$,
2. for each $r, s \in \mathbb{Z}$, $\langle U^r V^s u(v, u) | U^r V^s v(v, u) \rangle = 1$, 


3. for $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma$:
\[\langle \gamma_1 u(\gamma_2 u, v) \mid \gamma_3 v(\gamma_4 v, u) \rangle = \langle \gamma_3 v(\gamma_4 v, u) \mid \gamma_1 u(\gamma_2 u, v) \rangle,\]

4. $\langle \gamma_1 u(\gamma_2 u, v) \mid \gamma_3 v(\gamma_4 v, u) \rangle = \gamma_1^{-1}\gamma_3(u(\gamma_2 u, v) \mid v(\gamma_4 v, u))$, and

5. for $v' \notin \Gamma \cdot v$ or $u' \notin \Gamma \cdot u$, $\langle q^s v(v', u) \mid q^r u(u', v) \rangle$ is not defined.

**Proposition 3 (Proposition 3 [1])** The pairing function (6) defined above satisfies the following: for any $m, k, r, s \in \mathbb{N}$ we have
\[\langle q^s v(q^m v, u) \mid q^r u(q^k u, v) \rangle = q^{r-s-km},\] (7)

and
\[\langle q^r u(q^k u, v) \mid q^s v(q^m v, u) \rangle = q^{k+m-s-r} = \langle q^s v(q^m v, u) \mid q^r u(q^k u, v) \rangle^{-1}.\] (8)

We call the three sorted structure $\langle U_\phi, V_\phi, \langle \cdot \mid \cdot \rangle \rangle$ a quantum 2-torus and denoted by $T_\theta$.

From Proposition 2 we know that the structure of the line-bundles does not depend on the choice function. The next proposition tells us that the structure of the quantum 2-torus $T_\theta^2(\mathbb{C})$ depends only on $\mathbb{C}$, $q$ and not on the choice function.

**Proposition 4 (cf. Proposition 4.4, [5])** Given $q \in \mathbb{F}^*$ not a root of unity, any two structures of the form $T_\theta^2(\mathbb{F})$ are isomorphic over $\mathbb{F}$. In other words, the isomorphism type of $T_\theta^2(\mathbb{F})$ does not depend on the system of representatives $\Phi$.

3 Geometrically isomorphic quantum 2-tori

From now on we work in the structure $\mathbb{C}^\theta = (\mathbb{C}, +, \cdot, 1, x^\theta)$ (raising to real power $\theta$ in the complex numbers).

We define
\[x^\theta = \exp(\theta \cdot (\ln x + 2\pi i \mathbb{Z})) = \{\exp(\theta \cdot (\ln x + 2\pi i k)) : k \in \mathbb{Z}\},\]
as a multi-valued function and by $y = x^\theta$ we mean the relation $\exists z (x = \exp(z) \land y = \exp(z\theta))$. 


Notation 5 $C_\theta(x, y)$ denotes the binary relation $y = x^\theta$ as defined above.

Let $\theta_1, \theta_2 \in \mathbb{R} \setminus \mathbb{Q}$. Set $q_1 = \exp(2\pi i \theta_1)$ and $q_2 = \exp(2\pi i \theta_2)$. Put $\Gamma_{q_1} = \langle q_1 \rangle$ and $\Gamma_{q_2} = \langle q_2 \rangle$.

Let $\Phi_1$ be the system of representatives for a choice function $\phi_1 : \mathbb{C}^*/\Gamma_{q_1} \to \mathbb{C}^*$. Let $T_{q_2}$ be quantum 2-tori constructed as explained in the previous section.

Suppose $(u, v) \in (\Phi_1)^2$. We identify the modules $M_{|u,v\rangle}$ constitutes the quantum 2-torus $T_{q_1}$ with its canonical basis denoted by $E_{|u,v\rangle}$.

We see the $\Gamma_{q_1}$-bundle $U_{\phi_1}$ as a bundle inside $\bigcup_{(u,v) \in (\Phi_1)^2} M_{|u,v\rangle}$. Thus knowing the set of bases of $U_{\phi_1}$ that is the set $\bigcup_{(u,v) \in (\Phi_1)^2} E_{|u,v\rangle}$, we can determine the quantum 2-torus $T_{q_1}$ which we denote $T_{\theta_1}$.

Let $\Phi_2$ be the system of representatives for a choice function $\phi_2 : \mathbb{C}^*/\Gamma_{q_2} \to \mathbb{C}^*$. Let $T_{q_2}$ be quantum 2-tori constructed as explained in the previous section.

We define a similar set $E_{|u',v'\rangle}$ which is a canonical basis for $M_{|u',v'\rangle}$ where $(u', v') \in (\Phi_2)^2$ and the set $\bigcup_{(u',v') \in (\Phi_2)^2} E_{|u',v'\rangle}$ determines the quantum 2-torus $T_{q_2}$ which we denote $T_{\theta_2}$.

We now introduce the notion called Morita equivalence between quantum 2-tori.

Definition 6 Let $a, b \in \mathbb{C}^*$.

(1) We say that $C_\theta$ sends the coset $a \cdot \Gamma_{q_1}$ of $\Gamma_{q_1}$ to the coset $b \cdot \Gamma_{q_2}$ of $\Gamma_{q_2}$ if

$$\forall x' \in a \cdot \Gamma_{q_1} \forall y' \in \mathbb{C}^* \left( y' \in b \cdot \Gamma_{q_2} \iff C_\theta(x', y') \right).$$

(2) We say that $C_\theta$ gives rise to a one-to-one correspondence from the cosets of $\Gamma_{q_1}$ to the cosets of $\Gamma_{q_2}$.

Definition 7 (Geometric isomorphism) We say that the quantum 2-torus $T_{\theta_1}$ is geometrically isomorphic to $T_{\theta_2}$, written $T_{\theta_1} \simeq_{\theta} T_{\theta_2}$, if

(1) $C_\theta$ sends the cosets of $\Gamma_{q_1}$ to the cosets of $\Gamma_{q_2}$, and
(2) there is a one-to-one correspondence $L_\theta$ from $\bigcup_{(u,v)} E_{[u,v]}$ to $\bigcup_{(u',v')}$ such that for each $(u,v) \in (\Phi_1)^2$ and $(u',v') \in (\Phi_2)^2$ satisfying $C_\theta(u,u')$ and $C_\theta(v,v')$ we have

$$L_\theta(q_1^n u(q_1^n u, v)) = q_2^n u(q_2^n u', v').$$

We call $L_\theta$ a geometric transformation from $\bigcup_{(u,v)} E_{[u,v]}$ to $\bigcup_{(u',v')}$ and we simply write as

$$L_\theta : E_{[u,v]} \mapsto E_{[u',v']}.$$

For a geometric transformation $L_\theta$, we have the following diagrams, for each $(u,v) \in (\Phi_1)^2$ and $(u',v') \in (\Phi_2)^2$:

$$u((q_1^n u, v) \xrightarrow{L_\theta} u((q_2^n u', v'))$$

$$\downarrow U \quad \bigcirc \quad \downarrow U$$

$$(q_1^n u u((q_1^n u, v) \xrightarrow{L_\theta} (q_2^n u u((q_2^n u', v'))$$

and

$$u((q_1^n u, v) \xrightarrow{L_\theta} u((q_2^n u', v'))$$

$$\downarrow V \quad \bigcirc \quad \downarrow V$$

$$v u((q_1^n u, v) \xrightarrow{L_\theta} v' u((q_2^n u', v'))$$

Conversely, the existence of such diagrams is sufficient for $L_\theta$ to be a geometric transformation.

Remark. Note that for corresponding $(u,v) \in (\Phi_1)^2$ and $(u',v') \in (\Phi_2)^2$ such diagram to exist it is enough to have isomorphism between the groups $\Gamma_{q_1}$ and $\Gamma_{q_2}$. This is clearly the case when $q_1$ and $q_2$ are of infinite order.

In order to show that a geometric transformation gives rise to a geometric isomorphism between quantum 2-tori, we need to show that it preserves the values of pairing functions $\langle \cdot \mid \cdot \rangle_{\theta_1}$ in $T_{\theta_1}$ and and the pairing function $\langle \cdot \mid \cdot \rangle_{\theta_2}$ in $T_{\theta_2}$.

Lemma 8 A geometric transformation preserves the values of pairing functions $\langle \cdot \mid \cdot \rangle_{\theta_1}$ and $\langle \cdot \mid \cdot \rangle_{\theta_2}$. More precisely we have:

$$L_\theta (\langle \cdot \mid \cdot \rangle_{\theta_1}) = \langle L_\theta(\cdot) \mid L_\theta(\cdot) \rangle_{\theta_2}.$$
Proof: We show that the five properties of pairing function are preserved by geometric transformation.

1. 
\[ L_\theta (\langle u(u, v) | v(u, v) \rangle_{\theta_1}) = \langle L_\theta(u(u, v)) | L_\theta(v(v, u)) \rangle_{\theta_2} \]
\[ \parallel L_\theta(1) \parallel \parallel \langle u(u', v') | v(v', u') \rangle_{\theta_2} \parallel \]
\[ L_\theta(1) \]

2. It suffices to note that we have for each \( r, s \in \mathbb{Z} \),
\[ L_\theta(U^r V^s u(u, v)) = U^r V^s (L_\theta(u(u, v))) = U^r V^s (u(u', v')) \]
and the same equation for \( v(v, u) \).

3., 4., 5., are proved by similar computations.

Knowing the modules \( M_{u,v} \) for each \( (u, v) \in (\Phi_1)^2 \) and the modules \( M_{u',v'} \) for each \( (u', v') \in (\Phi_2)^2 \) we can determine the structure of quantum 2-tori \( T_{\theta_1} \) and \( T_{\theta_2} \). Thus we have

Lemma 9 A geometric transformation from \( \bigcup_{(u,v) \in (\Phi_1)^2} E_{[u,v]} \) to \( \bigcup_{(u',v') \in (\Phi_2)^2} E_{[u',v']} \) induces a geometric isomorphism between \( T_{\theta_1} \) and \( T_{\theta_2} \).

4 Relations giving rise to geometric transformations

Proposition 10 For each \( \left( \begin{array}{cc} m_{11} & m_{12} \\ m_{21} & m_{22} \end{array} \right) \in \text{GL}_2(\mathbb{Z}) \), the binary relation
\[ C_\Theta(x, y), \quad \Theta = \frac{m_{11} \theta + m_{12}}{m_{21} \theta + m_{22}} \]
corresponding to
\[ y = x^{m_{11} \theta + m_{12}} \]
is positive quantifier-free definable in the structure \( C_\theta \).

Proof: Observe the following immediate equivalences:
- \( y = x^{m\theta} \equiv C_\theta(x^m, y) \)
- \( y = x^{m\theta+n} \equiv C_\theta(x^m, yx^{-n}) \)
\[ y = x^1 \equiv C_\theta(y, x) \]
\[ y = x^{m_\theta + n} \equiv x = y^{m_\theta + n} \equiv C_\theta(y^m, xy^{-n}) \]

It follows
\[ y = x^{m_{11}\theta + m_{12}} = y^{m_{21}\theta + m_{22}} = x^{m_{11}\theta + m_{12}} \]
\[ \equiv (y^{m_{21}x^{-m_{11}}})^\theta = x^{m_{12}y^{-m_{22}}} \]
\[ \equiv C_\theta(y^{m_{21}x^{-m_{11}}}, x^{m_{12}y^{-m_{22}}}) \]

\[ \square \]

**Lemma 11** Suppose that \( C_\theta \) sends the cosets of \( \Gamma_{q_1} \) to the cosets of \( \Gamma_{q_2} \). Then there is a geometric transformation from \( T_{\theta_1} \) to \( T_{\theta_2} \), hence we have \( T_{\theta_1} \simeq \theta T_{\theta_2} \).

**Proof:** Once we know the correspondence between the cosets of \( \Gamma_{q_1} \) and the cosets of \( \Gamma_{q_2} \), by the remark above we can define a geometric transformation \( L_\theta \) from \( T_{\theta_1} \) to \( T_{\theta_2} \), and we have \( T_{\theta_1} \simeq \theta T_{\theta_2} \) \( \square \)

### 4.1 Main theorem

We now show the main theorem.

**Theorem 12** Let \( \theta_1, \theta_2 \in \mathbb{R} \setminus \mathbb{Q} \). Then \( T_{\theta_1} \simeq \theta T_{\theta_2} \) if and only if \( \theta_2 = \frac{a\theta_1 + b}{c\theta_1 + d} \) for some \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}_2(\mathbb{Z}) \) with \( |ad - bc| = 1 \).

**Proof:** By Lemma 11 \( T_{\theta_1} \simeq \theta T_{\theta_2} \) if and only if \( C_\theta \) sends cosets of \( \Gamma_{q_1} \) to \( \Gamma_{q_2} \). In particular, \( C_\theta \) induces a group isomorphism \( \Gamma_{q_1} = \langle q_1 \rangle \) to \( \Gamma_{q_2} = \langle q_2 \rangle \):

\[ \exp(2\pi i(\mathbb{Z}\theta_1 + \mathbb{Z})) \xrightarrow{\theta} \exp(2\pi i((\mathbb{Z}\theta_1 + \mathbb{Z})\theta)) = \exp(2\pi i(\mathbb{Z}\theta_2 + \mathbb{Z})). \]

The isomorphism is completely determined by the images of \( q_1 = \exp(2\pi i\theta_1) \) and \( 1 \) both in \( \Gamma_{q_1} \). Thus it suffices to know the images of \( \theta_1 \) and \( 1 \) by this isomorphism i.e., multiplication by \( \theta \). Hence we have

\[
\begin{aligned}
\theta_1 & \xrightarrow{\theta} \theta_1\theta = a\theta_2 + b, \\
1 & \xrightarrow{\theta} \theta = c\theta_2 + d
\end{aligned}
\]

where \( a, b, c, d \in \mathbb{Z} \) and \( |ad - bc| = 1 \).
It follows that
\[ \theta = \frac{a\theta_2 + b}{\theta_1} = c\theta_2 + d. \] (9)

Solving for \(\theta_2\) we get
\[ \theta_2 = \frac{d\theta_1 - b}{-c\theta_1 + a}. \] (10)

Since \(|ad - bc| = 1\) we have
\[ \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in \text{GL}_2(\mathbb{Z}). \]

And this completes the proof.

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