AN ANALOGUE OF KUMMER’S RELATION BETWEEN THE IDEAL CLASS NUMBER AND THE UNIT INDEX OF CYCLOTOMIC FIELDS

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Abstract. In this paper, we obtain a new formula for the special value of Dirichlet $L$-function $L(s, \chi)$ at $s = 1$. This leads to another class number formula of $\mathbb{Q}(\mu_m)^+$, the maximal real subfield of $m$th cyclotomic field. From this formula, we construct a new type of cyclotomic units in $\mathbb{Q}(\mu_p^n)$, which implies a similar Kummer’s relation between the ideal class number of $\mathbb{Q}(\mu_p^n)^+$ and the unit index.

1. Introduction

In this paper, we obtain a new formula for the special value of Dirichlet $L$-function $L(s, \chi)$ at $s = 1$ (see Theorem 1.5). This leads to another class number formula of $\mathbb{Q}(\mu_m)^+$, the maximal real subfield of $m$th cyclotomic field (see Theorem 1.6). From this formula, we construct a new type of cyclotomic units in $\mathbb{Q}(\mu_p^n)$ (see Eq. (1.12)), which implies a similar Kummer’s relation between the class number of $\mathbb{Q}(\mu_p^n)^+$ and the unit index (see Theorem 1.7).

1.1. Background. Let $\chi$ be a Dirichlet character of conductor $f$ and $L(s, \chi)$ be the Dirichlet $L$-function ($L$-series) defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where the series on the right is absolutely convergent for $\text{Re}(s) > 1$ and is conditionally convergent for $\text{Re}(s) > 0$ for non-principal $\chi$ (see [3]). Let

$$\zeta_f = e^{\frac{2\pi i}{f}} = \cos \left(\frac{2\pi}{f}\right) + i \sin \left(\frac{2\pi}{f}\right)$$

be a primitive $f$th root of unity and

$$\tau(\chi) = \sum_{r \,(\text{mod} \, f)} \chi(r) \zeta_f^r,$$

be the Gauss sum corresponding to the character $\chi$, where $r$ runs through a full (or a reduced) system of residues modulo $f$. If $\chi$ is non-principal, the following result on the special value of $L(s, \chi)$ at $s = 1$ is well-known.

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Theorem 1.1 (See Washington [7, p.38, Theorem 4.9]).

\[ L(1, \chi) = -\frac{2\pi(\chi)}{f} \sum_{1 \leq k < f/2} \overline{\chi}(k) \log \sin \left( \frac{k\pi}{f} \right) \]
\[ = -\frac{\tau(\chi)}{f} \sum_{k=1}^{f} \overline{\chi}(k) \log |1 - \zeta_{f}^k|, \quad \text{if } \chi \text{ is even, i.e. } \chi(-1) = 1. \]

\[ L(1, \chi) = \frac{\pi i \tau(\chi)}{f^2} \sum_{k=1}^{f} \overline{\chi}(k)k, \quad \text{if } \chi \text{ is odd, i.e. } \chi(-1) = -1. \]

This formula has an important application in algebraic number theory by connecting \( L(1, \chi) \) with the class number formula of abelian fields \( K/\mathbb{Q} \), that is:

Theorem 1.2 (See Lang [5, p.77]).

\[ 2^{r_1}(2\pi)^{r_2}hR \overline{w}d^{1/2} = \prod_{\chi \neq 1} L(1, \chi), \]

where the product is taken over all the primitive characters induced by the characters of \( \text{Gal}(K/\mathbb{Q}) \) and

- \( w = w_k \) is the number of roots of unity in \( K \).
- \( h = h_k \) is the class number of \( K \).
- \( R = R_K \) is the regulator of \( K \).
- \( d = d_K \) is the absolute value of the discriminant.

If \( K \) is real, then \( r_1 = [K : \mathbb{Q}] \); if \( K \) is not real, then \( r_1 = 0 \) and \( r_2 = \frac{1}{2}[K : \mathbb{Q}] \).

Assume that \( m \) is odd or \( m \equiv 0 \pmod{4} \), \( K = \mathbb{Q}(\mu_m) \) and \( K^+ = \mathbb{Q}(\mu_m)^+ \) be the \( m \)-th cyclotomic field and its maximal real subfield, respectively; \( h \) and \( h^+ \) be the class number of \( K \) and \( K^+ \), respectively.

For convenience of the notations, we also denote \( \mathbb{Z}/m\mathbb{Z} \) by \( \mathbb{Z}(m) \). Let \( f_{\chi} \) be the conductor of \( \chi \). Introduce the group

\[ G = \mathbb{Z}(m)^*/\pm 1 \] and \( G_{\chi} = \mathbb{Z}(f_{\chi})^*/\pm 1 \]

Combining (1.4) and (1.5), we have other class number formula of \( K^+ \).

Theorem 1.3 (See Lang [5, p.81]).

\[ h^+ = \frac{1}{R^+} \prod_{\chi \neq 1} \sum_{k \in G_{\chi}} -\overline{\chi}(k) \log |1 - \zeta_{f_{\chi}}^k|, \]

where the product over \( \chi \neq 1 \) is taken over the non-trivial characters of \( G \), or equivalently, the non-trivial even characters \( \mathbb{Z}(m)^* \).

\( \zeta_m \) denotes a primitive \( m \)-th roots of unity. For \( k \) prime to \( m \), we let

\[ g_k = \frac{\zeta_m^k - 1}{\zeta_m - 1}. \]
Then $g_k$ is called a cyclotomic unit. It is easy to see that $g_k$ is equal to a real unit times a root of unity. Since $\zeta_m^k$ only depends on the residue class of $k \mod m$, without loss of generality, we may assume that $k$ is odd. Then

$$\zeta_m^{-v} g_k \text{ for } v = \frac{k - 1}{2}$$

is real (i.e. fixed under $\sigma_{-1}$), and call it the real cyclotomic unit.

Let $E$ be the group of unit in $K$ and $\mathcal{E}$ be the subgroup of $E$ generated by the roots of unity and the cyclotomic units. Let $E^+$ be the group of unit in $K^+$ and $\mathcal{E}^+$ be the subgroup of $E^+$ generated by $\pm 1$ and real cyclotomic units. Assume $m = p^n$ is a prime power, we have

$$E/\mathcal{E} \cong E^+/\mathcal{E}^+.$$  

(see [7, p. 40, Corollary 4.13]). From (1.7) and the Dedekind determinant formula (see Theorem 4.1), we have the following Kummer’s result related to the class number $h^+$ and the unit indexes $(E : \mathcal{E}) = (E^+ : \mathcal{E}^+)$. 

**Theorem 1.4** (See Lang [5, p.85, Theorem 5.1]). Let $K = \mathbb{Q}(\mu_m), K^+ = \mathbb{Q}(\mu_m)^+$ and $h^+$ be the class number of $K^+$. Assume $m = p^n$ is a prime power. Then

$$h^+ = (E : \mathcal{E}) = (E^+ : \mathcal{E}^+)$$

1.2. **Our results.** Assume $\chi$ is a non-principal Dirichlet character. By using an alternating form of Dirichlet $L$-function:

$$L_E(s, \chi) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \chi(n)}{n^s}, \Re(s) > 0,$$

we give another formula of $L(1, \chi)$ (comparing with Theorem 1.1 above).

**Theorem 1.5.**

$$L_E(s, \chi) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \chi(n)}{n^s}, \Re(s) > 0,$$

$$L_E(s, \chi) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \chi(n)}{n^s}, \Re(s) > 0.$$
The above $\eta(s)$ is a particular case of Witten’s zeta functions in mathematical physics (see [6, p. 248, (3.14)]) and it has been used by Euler to obtain a functional equation of Riemann zeta function $\zeta(s)$ (see [8, p.273–276]).

There is also a connection between $L_E(s, \chi)$ and the ideal class group of the $p^{n+1}$-th cyclotomic field where $p$ is a prime number. For details, we refer to a recent paper [4], especially [4, Propositions 3.2 and 3.4].

Assume that $m$ is odd or $m \equiv 0 \pmod{4}$, $K = \mathbb{Q}(\mu_m)$ and $K^+ = \mathbb{Q}(\mu_m)^+$ be the $m$-th cyclotomic field and its maximal real subfield, respectively; $h$ and $h^+$ be the class number of $K$ and $K^+$, respectively. Denote by

$$\eta = \prod_{\chi \neq 1} (1 - \chi(2)),$$

where the product over $\chi \neq 1$ is taken over the non-trivial characters of $G$, or equivalently, the non-trivial even characters $\mathbb{Z}(m)^*$. Combing (1.9) and the class number formula of abelian fields (see (1.5)), we also have another class number formula of $K^+$ (comparing with Theorem 1.3 above).

**Theorem 1.6.**

(1.11) 
$$\eta h^+ = \frac{1}{R^+} \prod_{\chi \neq 1} \sum_{k \in G_\chi} \chi(k) \log |1 + \zeta_k|^2,$$

where the product over $\chi \neq 1$ is taken over the non-trivial characters of $G$, or equivalently, the non-trivial even characters $\mathbb{Z}(m)^*$.

For $k$ prime to $m$, we define a new type of cyclotomic units to be

(1.12) 
$$\tilde{g}_k = \frac{\zeta_k}{\zeta_m} + 1$$

(comparing with the definition of $g_k$ above). Then $\tilde{g}_k$ is equal to a real unit times a root of unity. Since $\zeta_k$ only depends on the residue class of $k$ mod $m$, without loss of generality, we may assume that $k$ is odd. Then

$$\zeta_m^{-v} \tilde{g}_k$$

for $v = \frac{k - 1}{2}$

is real (i.e. fixed under $\sigma_{-1}$), and we call it a new type of cyclotomic units.

Let $E$ be the group of unit in $K$ and $\tilde{E}$ be the the subgroup of $E$ generated by the roots of unity and the new type cyclotomic units defined above. Let $E^+$ be the group of unit in $K^+$ and $\tilde{E}^+$ be the the subgroup of $E^+$ generated by $\pm 1$ and the new type real cyclotomic units introduced above.

Assume $m = p^n$ is a prime power, we have

$$E/\tilde{E} \cong E^+/\tilde{E}^+.$$

From (1.11) and the Dedekind determinant formula (see Theorem 1.1), we also have a new formula related to $h^+$ and the unit indexes $(E : \tilde{E})$ and $(E^+ : \tilde{E}^+)$ (comparing with Theorem 1.4 above).

**Theorem 1.7.** Let $K = \mathbb{Q}(\mu_m)$, $K^+ = \mathbb{Q}(\mu_m)^+$ and $h^+$ be the class number of $K^+$. Assume $m = p^n$ is an odd prime power and -1, 2 generate the group $\mathbb{Z}(m)^*$. Then

$$|\eta|h^+ = (E : \tilde{E}) = (E^+ : \tilde{E}^+).$$
2. Proof of Theorem 1.5

Introduce

\[ L_E(s, \chi) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\chi(n)}{n^s}, \quad \text{Re}(s) > 0. \]

It is easy to see that

\[
(1 - \chi(2)2^{1-s})L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} - 2 \sum_{n=1}^{\infty} \frac{\chi(2n)}{(2n)^s}
\]

\[ = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\chi(n)}{n^s} = L_E(s, \chi) \]

for \( \text{Re}(s) > 1 \). Then by analytic continuation, the equality

\[ (1 - \chi(2)2^{1-s})L(s, \chi) = L_E(s, \chi) \]

is established in the whole complex plane (see e.g. [2, Proof of Lemma 2.1]). Thus

\[ (1 - \chi(2))L(1, \chi) = L_E(1, \chi). \]

Now we calculate \( L_E(1, \chi) \). Recall \( \chi \) is a Dirichlet character of conductor \( f \), we rearrange the terms in the series for \( L_E(s, \chi) \) according to the residue classes mod \( f \). That is, we write

\[ n = qf + r, \quad \text{where } 1 \leq r \leq f \text{ and } q = 0, 1, 2, \ldots, \]

and obtain

\[ -L_E(s, \chi) = \sum_{r=1}^{f} \chi(r) \sum_{q=0}^{\infty} \frac{(-1)^{qf+r}}{(qf+r)^s} \]

\[ = \sum_{(r,f)=1} \chi(r) \sum_{n \equiv r \pmod{f}} \frac{(-1)^n}{n^s}, \]

since \( \chi(r) = 0 \) if \( (r, f) > 1 \). The inner series can be written in the form

\[ \sum_{n=1}^{\infty} \frac{(-1)^n c_n}{n^s}, \]

where

\[ c_n = \begin{cases} 1 & \text{for } n \equiv r \pmod{f} \\ 0 & \text{for } n \not\equiv r \pmod{f}. \end{cases} \]

To find a convenient way of writing the coefficients \( c_n \), we consider the following formula:

\[ \sum_{k=0}^{f-1} \zeta_f^k = \begin{cases} f & \text{for } \ell \equiv 0 \pmod{f} \\ 0 & \text{for } \ell \not\equiv 0 \pmod{f}, \end{cases} \]
where
\[(2.7)\quad \zeta_f = \cos \left( \frac{2\pi}{f} \right) + i \sin \left( \frac{2\pi}{f} \right) = e^{\frac{2\pi i}{f}} \]
is a primitive \(f\)th root of unity. We remark
\[(2.8)\quad c_n = \frac{1}{f} \sum_{k=0}^{f-1} \zeta_f^{(r-n)k} \]
(see [1, p. 332]). Therefore, combining with (2.5), (2.6) and (2.8) we have the identity
\[(2.9)\quad -L_E(s, \chi) = \sum_{(r,f) = 1} \chi(r) \sum_{n=1}^{\infty} \frac{1}{f} \sum_{k=0}^{f-1} \zeta_f^{(r-n)k} \frac{(-1)^n}{n^s} \]
\[(2.10)\quad \sum_{r \ (\text{mod} \ f)} \chi(r) \zeta_f^k = \overline{\chi}(k) \tau(\chi). \]
Using (2.10) we can write (2.9) in the form
\[(2.11)\quad -L_E(s, \chi) = \frac{\tau(\chi)}{f} \sum_{k=1}^{f-1} \overline{\chi}(k) \sum_{n=1}^{\infty} \frac{(-1)^n \zeta_f^{-nk}}{n^s}. \]
The above series (2.11) converges for \(0 < s < \infty\) and represents a continuous function of \(s\). Hence we may set \(s = 1\) in this last equation and obtain
\[(2.12)\quad -L_E(1, \chi) = \frac{\tau(\chi)}{f} \sum_{k=1}^{f-1} \overline{\chi}(k) \sum_{n=1}^{\infty} \frac{(-1)^n \zeta_f^{-nk}}{n}. \]
To find the sum of the inner series on the right in (2.12), we consider
\[(2.13)\quad \log(1 + z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n, \quad \text{convergent for } |z| < 1. \]
It is well known that the analytic function \(\log(1 + z)\) defined by (2.13) has \(z = -1\) as its only singular point at finite distance. Since this series also converges at the point \(z = \zeta_f^{-k}\) (on the unit circle), then by Abel’s theorem, we have
\[(2.14)\quad \sum_{n=1}^{\infty} \frac{(-1)^n \zeta_f^{-nk}}{n} = -\log(1 + \zeta_f^{-k}) \]
and hence

\[ L_E(1, \chi) = \frac{\tau(\chi)}{f} \sum_{k=1}^{f} \chi(k) \log(1 + \zeta_f^{-k}), \]

thus we have obtained a finite expression for the series \( L_E(1, \chi) \).

The formula \( (2.15) \) can be further investigated and considerably simplified as follows. Let

\[ S_\chi = \sum_{k=1}^{f} \chi(k) \log(1 + \zeta_f^{-k}), \]

where \( k \) running through a reduced system of residues modulo \( f \). From \( (2.7) \), the number \( 1 + \zeta_f^{-k} \) (for \( 0 < k < f \)) can be represented as

\[ 1 + \zeta_f^{-k} = 1 + e^{-\frac{2\pi ki}{f}} = 2 \cos \left( \frac{\pi k}{f} \right) \left( \cos \left( \frac{-\pi k}{f} \right) + i \sin \left( \frac{-\pi k}{f} \right) \right), \]

which is equivalent to the relation

\[ 1 + \zeta_f^{-k} = 2 \cos \left( \frac{\pi k}{f} \right) \left( \cos \left( \frac{-\pi k}{f} \right) + i \sin \left( \frac{-\pi k}{f} \right) \right). \]

Therefore

\[ \log(1 + \zeta_f^{-k}) = \begin{cases} 
\log |1 + \zeta_f^{-k}| - i \left( \frac{\pi k}{f} \right), & 0 < k < \frac{f}{2}, \\
\log |1 + \zeta_f^{-k}| + i \left( \frac{\pi k}{f} \right), & \frac{f}{2} < k < f.
\end{cases} \]

Further, since \( 1 + \zeta_f^{-k} \) and \( 1 + \zeta_f^{k} \) are conjugate, we have

\[ \log(1 + \zeta_f^{k}) = \begin{cases} 
\log |1 + \zeta_f^{k}| + i \left( \frac{\pi k}{f} \right), & 0 < k < \frac{f}{2}, \\
\log |1 + \zeta_f^{k}| + i \left( \frac{\pi k}{f} - \pi \right), & \frac{f}{2} < k < f.
\end{cases} \]

Now assume that the character \( \chi \) (and hence also \( \overline{\chi} \)) is even. Interchanging \( k \) and \( -k \) in \( (2.16) \), we have

\[ S_\chi = \sum_{k=1}^{f} \overline{\chi}(-k) \log(1 + \zeta_f^{-k}) = \sum_{k=1}^{f} \overline{\chi}(k) \log(1 + \zeta_f^{k}), \]

and \( (2.16), (2.18), (2.19) \) and \( (2.20) \) this yields

\[ 2S_\chi = \sum_{k=1}^{f} \overline{\chi}(k) \left[ \log(1 + \zeta_f^{-k}) + \log(1 + \zeta_f^{k}) \right] \]

\[ = 2 \sum_{k=1}^{f} \overline{\chi}(k) \log |1 + \zeta_f^{k}| \]

\[ = 2 \sum_{k=1}^{f} \overline{\chi}(k) \log \left| 2 \cos \left( \frac{\pi k}{f} \right) \right|. \]
Thus

\[ S_\chi = \sum_{k=1}^{f} \bar{\chi}(k) \log |1 + \zeta_f^k| \]

(2.23)

\[ = \sum_{k=1}^{f} \bar{\chi}(k) \log \left| \cos \left( \frac{\pi k}{f} \right) \right|. \]

since \( \sum_{k=1}^{f} \bar{\chi}(k) = 0 \). Then combing with (2.4), (2.15) and (2.16), we have

\[ (1 - \chi(2))L(1, \chi) = \frac{\tau(\chi)}{f} \sum_{k=1}^{f} \bar{\chi}(k) \log \left| \cos \left( \frac{\pi k}{f} \right) \right| \]

(2.24)

\[ = \frac{\tau(\chi)}{f} \sum_{k=1}^{f} \bar{\chi}(k) \log |1 + \zeta_f^k|, \]

which is the desired result if \( \chi \) is a even character.

If the character \( \chi \) is odd, then interchanging \( k \) and \( -k \) in (2.16), we have

\[ S_\chi = -\sum_{k=1}^{f} \bar{\chi}(k) \log(1 + \zeta_f^k), \]

and by (2.19) and (2.20), we have

\[ 2S_\chi = \sum_{k=1}^{f} \bar{\chi}(k) \left[ \log(1 + \zeta_f^{-k}) - \log(1 + \zeta_f^k) \right] \]

(2.25)

\[ = -2 \left[ \sum_{1 \leq k < f/2} \bar{\chi}(k) \frac{\pi k}{f} + \sum_{f/2 < k < f} \bar{\chi}(k) i \left( \frac{\pi k}{f} - \pi \right) \right] \]

and

\[ S_\chi = -\frac{\pi i}{f} \left[ \sum_{1 \leq k < f/2} \bar{\chi}(k) k + \sum_{f/2 < k < f} \bar{\chi}(k) (k - f) \right]. \]

Then combing with (2.4), (2.15) and (2.16), we have

\[ (1 - \chi(2))L(1, \chi) = -\frac{\pi i \tau(\chi)}{f^2} \left[ \sum_{1 \leq k < f/2} \bar{\chi}(k) k + \sum_{f/2 < k < f} \bar{\chi}(k) (k - f) \right], \]

which is the desired result if \( \chi \) is an odd character.

3. Proof of Theorem 1.6

For \( K^+ = \mathbb{Q}(\mu_m)^+ \) and \( m = p^n \), we have \([K^+ : \mathbb{Q}] = \varphi(p^n)/2\), where \( \varphi \) is Euler-phi function. Let \( d \) be the absolute value of the discriminant of
$K^+$. Since $\prod \tau_\chi = d^{1/2}$ ([4] p.36, Corollary 4.6]), \(d = \prod \chi f_\chi ([4] p.36]) and $f_\chi = f_\chi$, by the class number formula ([4]) and Theorem 1.5 we have

\[
(3.1) \quad \eta 2^{\varphi(p^n)/2} h^+ R^+ = \prod_{\chi \neq 1} \sum_{k=1}^f \chi(k) \log |1 + \zeta_k^k|,
\]

where the product over $\chi \neq 1$ is taken over the non-trivial characters of $G$, or equivalently, the non-trivial even characters $\mathbb{Z}(m)^*$.

Since

\[
\sum_{k=1}^f \chi(k) \log |1 + \zeta_k^k| = 2 \sum_{k \in G} \chi(k) \log |1 + \zeta_k^k|
\]

and there are exactly $\varphi(p^n)/2$ even characters and $\varphi(p^n)/2 - 1$ non-trivial even characters, we have

\[
(3.2) \quad \eta h^+ R^+ = \prod_{\chi \neq 1} \sum_{k \in G} \chi(k) \log |1 + \zeta_k^k|,
\]

which is the desired result.

4. Proof of Theorem 1.7

The Galois group of $\mathbb{Q}(\mu_m)$ over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}(m)^*$ under the map:

\[
a \mapsto \sigma_a
\]

where $\sigma_a : \zeta_m \mapsto \zeta_m^a$. If $\epsilon_1, \ldots, \epsilon_r$ is a basis for $E^+$ (mod roots of unity), then the regulator $R^+$ is the absolute value of the determinant

\[
R(E) = R^+ = \pm \det_{a,j} \log |\sigma_a \epsilon_j|,
\]

where $j = 1, \ldots, r$ and $a \in \mathbb{Z}(m)^*/\pm 1$ and $a \neq \pm 1$ (mod $m$) (see [5] p. 85]). Let $G = \mathbb{Z}(m)^*/\pm 1$ and view $a \in G$ and $a \neq 1$ in $G$. Using the cyclotomic units introduced in (1.12), we also form a new type of cyclotomic regulator as follows

\[
(4.1) \quad R(\widetilde{\mathcal{E}}) = \widetilde{R}_{cyc} = \pm \det_{a,k} \log |\sigma_a \tilde{y}_k|
\]

with $a, k \in G$.

As pointed out by Serre to Lang (see [5] p. 90]), the following determinant relation is due to Dedekind, February 1896, who communicated it to Frobenius in March.

**Theorem 4.1** (Dedekind determinant formula, [5] p.90, Theorem 6.2]). Let $G$ be a finite group and $f$ be any (complex valued) function on $G$. Then

\[
\det_{a,b} f(ab^{-1}) = \left[ \sum_{a \in G} f(a) \right] \det_{a,b \neq 1} [f(ab^{-1}) - f(a)].
\]

Therefore, for a finite abelian group $G$,

\[
\prod_{\chi \neq 1} \sum_{a \in G} \chi(a) f(a^{-1}) = \det_{a,b \neq 1} [f(ab^{-1}) - f(a)].
\]
The above Dedekind determinant relation implies the following lemma.

**Lemma 4.2.** We have for $G = \mathbb{Z}(m)^*/\pm 1$,

$$\pm \det a_{a,k} \log |\sigma_a g_k| = \prod_{\chi \neq 1} \sum_{k \in G} \chi(k) \log |1 + \zeta_m^k|.$$  

**Lemma 4.3.** Let $G_\chi = \mathbb{Z}(f_\chi)^*/\pm 1$. For prime power $m = p^n$, we have

$$\sum_{k \in G_\chi} \chi(k) \log |1 + \zeta_f^k| = \sum_{k \in G} \chi(k) \log |1 + \zeta_m^k|.$$  

**Proof.** Let $f_\chi = p^s$. We write the residue classes in $\mathbb{Z}(p^n)$ in the form $y = k + cp^s$, with $0 \leq c < p^{n-s}$, and $k$ ranges over a fixed set of representatives for residue classes of $\mathbb{Z}(p^s)^*$, then we have

$$\sum_{y \mod p^n} \chi(y) \log |1 + \zeta_m^p| = \sum_{k \mod p^s} \sum_{c=0}^{p^{n-s}-1} \chi(k) \log |1 + \zeta_{p^n}^k (\zeta_{p^n}^c)|$$  

$$= \sum_{k \mod p^s} \chi(k) \sum_{c=0}^{p^{n-s}-1} \log |1 + \zeta_{p^n}^k (\zeta_{p^n}^c)|$$

Since

$$\prod_{\chi^{p^{n-s}} = 1} (X - \lambda Y) = X^{p^{n-s}} - Y^{p^{n-s}},$$

we have

$$\prod_{c=0}^{p^{n-s}-1} (1 + \zeta_{p^n}^k (\zeta_{p^n}^c)^c) = \prod_{c=0}^{p^{n-s}-1} (1 - (-\zeta_{p^n}^k)(\zeta_{p^n}^c)^c)$$  

$$= 1 - (-\zeta_{p^n}^k)^{p^{n-s}}$$  

$$= 1 + \zeta_{p^n}^k$$

since $p$ is odd. By (4.4) and (4.3), we obtain

$$\sum_{y \mod p^n} \chi(y) \log |1 + \zeta_m^p| = \sum_{k \mod p^s} \chi(k) \sum_{c=0}^{p^{n-s}-1} \log |1 + \zeta_{p^n}^k (\zeta_{p^n}^c)|$$  

$$= \sum_{k \mod p^s} \chi(k) \log |1 + \zeta_{p^n}^k|.$$  

Recall

$$G = \mathbb{Z}(p^n)^*/\pm 1$$

and $G_\chi = \mathbb{Z}(f_\chi)^*/\pm 1 = \mathbb{Z}(p^s)^*/\pm 1$.

As $\chi$ is even, dividing both sides of (4.4) by 2, we get the desired formula. □

Combining Theorem 1.6, (4.1), Lemma 4.2 and Lemma 4.3, we have the following result related to $h^+$ and $\hat{R}_{cyc}/R^+$.  

Proposition 4.4. Let $K = \mathbb{Q}(\mu_m)$, $K^+ = \mathbb{Q}(\mu_m)^+$ and $h^+$ be the class number of $K^+$. Assume $m = p^n$ is an odd prime power. Then
\[ |\eta|h^+ = \overline{R_{cyc}}/R^+. \]

Lemma 4.5. For $m = p^n$ being an odd prime power, $\chi(2) \neq 1$ for any non-trivial character $\chi$ of $G = \mathbb{Z}(m)^*/\pm 1$, or equivalently, for any non-trivial even characters $\mathbb{Z}(m)^*$ if and only if $\overline{R_{cyc}} \neq 0$.

Proof. By (4.1), Lemmas 4.2 and 4.3 and Theorem 1.6 we have
\[ \overline{R_{cyc}} = \pm \det_{a,k} \log|\sigma_a g_k| = \prod_{\chi \neq 1 \in G_X} \chi(k) \log|1 + \zeta_k^k| = \eta R^+ h^+. \]

Since $\eta = \prod_{\chi \neq 1} (1 - \chi(2))$, the assertion follows from the above equality. □

If $m = p^n$ and $\chi(2) \neq 1$ for any non-trivial character $\chi$ of $G = \mathbb{Z}(m)^*/\pm 1$, then from Lemma 4.5 $\overline{R_{cyc}} \neq 0$, which shows that the subgroup $\hat{E}^+$ has finite index in the group $E^+$, and by [7, p.41, Lemma 4.15] we have
\[ (E : \hat{E}) = (E^+ : \hat{E}^+) = \overline{R_{cyc}}/R^+. \]

Thus from Proposition 4.4 we also have a new formula related to $h^+$ and the unit indexes $(E : \hat{E})$ and $(E^+ : \hat{E}^+)$. 

Corollary 4.6. Let $K = \mathbb{Q}(\mu_m)$, $K^+ = \mathbb{Q}(\mu_m)^+$ and $h^+$ be the class number of $K^+$. Assume $m = p^n$ is an odd prime power and $\chi(2) \neq 1$ for any non-trivial character $\chi$ of $G = \mathbb{Z}(m)^*/\pm 1$. Then
\[ |\eta|h^+ = (E : \hat{E}) = (E^+ : \hat{E}^+). \]

The following lemma is concerning the vanish of $1 - \chi(2)$.

Lemma 4.7. For $m = p^n$ being an odd prime power, $\chi(2) \neq 1$ for any non-trivial character $\chi$ of $G = \mathbb{Z}(m)^*/\pm 1$ if and only if $-1, 2$ generate the group $\mathbb{Z}(m)^*$.

Proof. $\chi(2) \neq 1$ for any non-trivial character $\chi$ of $G = \mathbb{Z}(m)^*/\pm 1$ \iff $\chi(2) \neq 1$ for any non-trivial even character $\chi$ of $\mathbb{Z}(m)^*$ \iff for any non-trivial character $\chi$ of $\mathbb{Z}(m)^*$, $\chi(-1) = 1$ and $\chi(2) = 1$ can not hold simultaneously \iff the character group of $\mathbb{Z}(m)^*/\langle -1, 2 \rangle$ is trivial \iff $-1, 2$ generate the group $\mathbb{Z}(m)^*$. □

Finally by Corollary 4.6 and Lemma 4.7 we get Theorem 1.7.

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