LATTICE POINTS IN STRETCHED FINITE TYPE DOMAINS

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Abstract. We study an optimal stretching problem, which is a variant lattice point problem, for convex domains in $\mathbb{R}^d$ ($d \geq 2$) with smooth boundary of finite type that are symmetric with respect to each coordinate hyperplane/axis. We prove that optimal domains which contain the most positive (or least nonnegative) lattice points are asymptotically balanced.

1. Introduction

The classical lattice point problem is about counting the number of lattice points $\mathbb{Z}^d$ in large domains in the Euclidean space $\mathbb{R}^d$. It has a long history which can be traced back to C.F. Gauss who studied the number of lattice points in large disks. In this paper we study the following variant lattice point problem, the so-called optimal stretching problem.

Let

$$A = \text{diag}(a_1, a_2, \ldots, a_d)$$

be a positive definite diagonal matrix with determinant 1. Let $\Omega \subset \mathbb{R}^d$ be a compact domain which contains the origin in its interior. A volume-preserving stretch of $\Omega$ by the stretching factor $A$ is a domain of the form

$$A\Omega = \{(a_1x_1, \ldots, a_dx_d) : (x_1, \ldots, x_d) \in \Omega\}.$$ 

One would like to know the limiting behaviour of $A$ (as $t$ goes to infinity) for those matrices $A$ such that the number of positive-integer lattice points in the enlarged stretch of $\Omega$, i.e. $\#(\mathbb{N}^d \cap tA\Omega)$, attains the largest value. A similar question can be asked for matrices $A$ such that $\#(\mathbb{Z}_+^d \cap tA\Omega)$ attains the smallest value where $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$.

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The optimal stretching problem was initiated by Antunes and Freitas, who considered in [1] the stretch of the unit disk in $\mathbb{R}^2$ and proved that among all ellipses of the same area, those that enclose the most lattice points in the first quadrant must be more and more “round”, as the area goes to infinity. In other words, the limit of the stretching factor is the identity matrix. Their motivation of such a study was a problem in spectral theory of minimizing Dirichlet eigenvalues of the Laplace operator among rectangles of equal area. In fact their result on asymptotically minimizing the $n$-th eigenvalue among rectangles of given area is equivalent to asymptotically maximizing the number of positive-integer lattice points in ellipses of given area.

We remark that the optimal stretching problem and closely related shape/eigenvalue optimization problems in spectral theory have been of large interest in recent years. For explanation on their connection and more results on the latter problems see for example van den Berg, Bucur and Gittins [3], van den Berg and Gittins [4], Gittins and Larson [6], Larson [10, 11] and references therein. In what follows we focus on the optimal stretching problem for domains more general than ellipses/ellipsoids.

In a pair of papers Laugesen and Liu [12] and Ariturk and Laugesen [2] extended the result of Antunes and Freitas by considering general planar domains (including $p$-ellipses $|sx|^p + |y/s|^p = t^p$ for $p \in (0, \infty) \setminus \{1\}$). They showed, among others, that under mild assumptions on the boundary curve optimal domains which contain the most positive (or least nonnegative) lattice points must be asymptotically balanced. (We recall that a domain in $\mathbb{R}^d$ ($d \geq 2$) is said to be balanced if the $(d - 1)$-dimensional measures of the intersections of the domain with each coordinate hyperplane are equal.) They also provided rates of convergence of optimal stretching factors. Notice that their results allow the curvature of the boundary curve to vanish or blow up at the intersection points with coordinate axes.

However, if the boundary is “too flat” the result could be very different—optimal domains needs not to be asymptotically balanced. For example, Marshall and Steinerberger [14] analyzed the case of triangles (namely the $p$-ellipses with $p = 1$). They showed that there are infinitely many optimal domains for arbitrarily large $t$.

The difference between these results is essentially a consequence of different curvature assumptions. This is not surprising since the lattice point counting is closely related to oscillatory integral estimates in which curvature plays a key role. The phenomenon of asymptotic balancing was further confirmed in Marshall [13] for convex domains in $\mathbb{R}^d$ with $C^{d+2}$ boundary and non-vanishing Gaussian curvature.
Naturally one may next ask if asymptotic balancing still occurs for the intermediate case between “non-flat” and “flat” cases, especially in high dimensions. A few attempts have been made. For example, in [8] the first author and Wang considered certain special convex domains of finite type $\mathbb{R}^d$ (including super spheres, i.e. high dimensional $p$-ellipsoids) and gave an affirmative answer. Later we slightly generalized this result in [7].

The goal of this paper is to prove the aforementioned asymptotic balancing phenomenon for arbitrary convex domains of finite type.

Let $\Omega$ be a convex domain with smooth boundary of finite type. Throughout this paper we set, for any $P \in \partial \Omega$, that

\begin{equation}
\nu_{\Omega}(P) = \sum_{i=1}^{d-1} a_i^{-1}
\end{equation}

and

\begin{equation}
\nu_{\Omega}^{(2)}(P) = \begin{cases} 
0 & \text{if } d = 2, \\
\sum_{i=2}^{d-1} a_i^{-1} & \text{if } d \geq 3,
\end{cases}
\end{equation}

where $a = (a_1, a_2, \ldots, a_{d-1})$ is the multitype (type if $d = 2$) of $\partial \Omega$ at the point $P$. See Iosevich, Sawyer and Seeger [9, P. 155–156] for the definition of multitype. We also set

\begin{equation}
\nu_{\Omega} = \min_{P \in \partial \Omega} \nu_{\Omega}(P)
\end{equation}

and

\begin{equation}
\mu_{\Omega} = \frac{1}{2} + \min_{P \in \partial \Omega} \nu_{\Omega}^{(2)}(P).
\end{equation}

For each $t \geq 1$ we define

\begin{equation}
\mathcal{A}_{\Omega}(t) = \arg\max_{A} \# \left( \mathbb{N}^d \cap tA\Omega \right)
\end{equation}

and

\begin{equation}
\widetilde{\mathcal{A}}_{\Omega}(t) = \arg\min_{A} \# \left( \mathbb{Z}_+^d \cap tA\Omega \right),
\end{equation}

where the argmax and argmin range over all positive definite diagonal matrices $A$ of determinant 1. The notation argmax$_x f(x)$ (resp. argmin$_x f(x)$) is the set of points $x$ for which $f(x)$ attains the function’s largest (resp. least) value. Note that optimal stretching factors in (1.5) and (1.6) are in general not unique. In what follows, when we write $A(t)$ in $\mathcal{A}_{\Omega}(t)$ (resp. $\widetilde{\mathcal{A}}_{\Omega}(t)$), we really mean that $A(t)$ is an arbitrary element in $\mathcal{A}_{\Omega}(t)$ (resp. $\widetilde{\mathcal{A}}_{\Omega}(t)$).

\footnote{That is, at each boundary point each tangent line has finite order of contact.}
For each $1 \leq j \leq d$, we let $\Omega_j$ be the intersection of $\Omega$ with the coordinate hyperplane $x_j = 0$.

With the above notations, our main results can be stated as follows.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) be a convex compact domain, that is symmetric with respect to each coordinate hyperplane (axis if $d = 2$), with smooth boundary of finite type. If

$$A(t) = \text{diag}(a_1(t), \ldots, a_d(t)) \in \mathcal{A}_\Omega(t),$$

then

$$(1.7) \quad \left| a_j(t) - \frac{|\Omega_j|}{\sqrt{|\Omega_1||\Omega_2| \cdots |\Omega_d|}} \right| = O \left( t^{-\gamma} \right), \quad 1 \leq j \leq d,$$

where $|\Omega_j|$ is the $(d - 1)$-dimensional measure of $\Omega_j$ and

$$\gamma = \min \left\{ \frac{\nu_\Omega}{2}, \frac{\mu_\Omega}{2(d - \mu_\Omega)} \right\}.$$

Similarly, if

$$\widetilde{A}(t) = \text{diag}(\tilde{a}_1(t), \ldots, \tilde{a}_d(t)) \in \widetilde{\mathcal{A}}_\Omega(t),$$

then

$$(1.8) \quad \left| \tilde{a}_j(t) - \frac{|\Omega_j|}{\sqrt{|\Omega_1||\Omega_2| \cdots |\Omega_d|}} \right| = O \left( t^{-\gamma} \right), \quad 1 \leq j \leq d.$$

**Remark 1.2.** The key to prove this theorem is an application of a delicate estimate of the Fourier transform of surface carried measure obtained in Iosevich, Sawyer and Seeger [2].

Our main goal was to weaken curvature assumptions in high dimensions, namely to extend the results in [13, 8, 7] to arbitrary finite type domains. A further interesting question is whether the optimal balancing still occurs for infinite type domains. This may be a hard question noticing that there are very few results on counting lattice points in general convex domains of infinite type.

For convenience of stating our results we assume the domain’s boundary is smooth. However it suffices to assume sufficient smoothness.

Our results work for finite type domains in $\mathbb{R}^2$. We did not try to further weaken assumptions however. Comparing to the planar results in [12, 2], we allow the curvature to vanish at finitely many boundary points rather than just at points of intersection with coordinate axes. The results in [12, 2] have weaker regularity assumptions and are good for both convex and concave cases.
Notations: The Fourier transform of any function \( f \in L^1(\mathbb{R}^d) \) is \( \hat{f}(\xi) = \int f(x) \exp(-2\pi i x \cdot \xi) \, dx \). For functions \( f \) and \( g \) with \( g \) taking nonnegative real values, \( f \lesssim g \) means \( |f| \leq Cg \) for some constant \( C \). If \( f \) is nonnegative, \( f \gtrsim g \) means \( g \lesssim f \). The Landau notation \( f = O(g) \) is equivalent to \( f \lesssim g \). The notation \( f \eqsim g \) means that \( f \lesssim g \) and \( g \lesssim f \). We set \( \mathbb{R}^*_d = \mathbb{R}^d \setminus \{0\} \) and \( \mathbb{Z}^*_d = \mathbb{Z}^d \setminus \{0\} \).

2. Lattice point counting

Throughout this section, we denote by \( A = \text{diag}(a_1, \ldots, a_d) \) a positive definite diagonal matrix with determinant 1 and \( a_* = \|A^{-1}\|_\infty = \max\{a_1^{-1}, \ldots, a_d^{-1}\} \).

We first quote a result from [8] on two-term bounds for lattice point counting, which generalizes [12, Proposition 6 and 9] to the setting of strictly convex domains in \( \mathbb{R}^d \). We can indeed apply this result in this paper since convex domains of finite type are strictly convex.

**Lemma 2.1** ([8, Proposition 2.1]). Let \( \Omega \subset [-C, C]^d \subset \mathbb{R}^d \) be strictly convex, compact and symmetric with respect to each coordinate hyperplane (axis if \( d = 2 \)) with \( C^2 \) boundary. There is a positive constant \( c \) depending only on the domain \( \Omega \) such that if \( t/a_* \geq 1/C \) then
\[
\#(\mathbb{N}^d \cap tA\Omega) \leq 2^{-d} |\Omega| t^d - ca_* t^{d-1}
\]
and
\[
\#(\mathbb{Z}^d_+ \cap tA\Omega) \geq 2^{-d} |\Omega| t^d + ca_* t^{d-1}.
\]

We next quote some known results on the decay of the Fourier transform of surface carried measure. In order to state them we briefly recall some notations from [9, P. 155–156] that are related to the definition of multitype (see also [16, P. 1270]). For a convex compact domain \( \Omega \subset \mathbb{R}^d \) with smooth boundary of finite type and an arbitrarily fixed \( P \in \partial \Omega \), denote by \( T_P(\partial \Omega) \) the tangent plane (line if \( d = 2 \)) of \( \partial \Omega \) at \( P \). Let \( S_P^{m_j} \), \( 1 \leq j \leq k \), be the flag of subspaces of \( T_P(\partial \Omega) \), and \( W_j \) the orthogonal complement of \( S_P^{m_j} \) in \( S_P^{m_{j-1}} \), as defined in [9, P. 155–156]. One can choose an orthonormal basis \( \{V_1, \ldots, V_{d-1}\} \) of \( T_P(\partial \Omega) \) such that for any \( V = \sum_{i=1}^{d-1} x_i V_i \), the equality
\[
|\Pi_j^P V| = \sum_{i=d-\text{dim}\, S_P^{m_j}}^{d-1} x_i^2
\]

\(^2\)A domain \( \Omega \subset \mathbb{R}^d \) is said to be strictly convex if the line segment connecting any two points \( x \) and \( y \) in \( \Omega \) lies in the interior of \( \Omega \), except possibly for its endpoints.
holds, where $\Pi_P^j$ represents the orthogonal projection on $T_P(\partial \Omega)$ to $W_j$. Here $|\cdot|$ denotes the Euclidean distance in $W_j$. We notice that $|\Pi_P^j V|$ is independent of the choice of the orthonormal basis, hence we can apply [9, Proposition 1.2] with the above particularly chosen basis. Let $n(P)$ denote the unit exterior normal of $\partial \Omega$ at $P$. We may assume the basis \{ $V_1, \ldots, V_{d-1}, -n(P)$ \} has the same orientation as \{ $e_1, \ldots, e_d$ \}. There exists a rotation matrix $O = O(P)$ such that (2.1) $(e_1, \ldots, e_d) = (V_1, \ldots, V_{d-1}, -n(P))O$, namely

\[ e_j = \sum_{i=1}^{d-1} o_{ij} V_i - o_{dj} n(P), \quad \text{where } O = (o_{ij}). \]

Let $d\sigma$ be the surface measure carried on $\partial \Omega$. The following decay of its Fourier transform is known.

**Lemma 2.2** ([5, 15, 9]). Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) be a convex compact domain with smooth boundary of finite type and $P \in \partial \Omega$. Then there is a neighborhood $U_P \subset \partial \Omega$ of $P$ and a conic neighborhood $V_P \subset \mathbb{R}^d$ of \{ $\pm n(P)$ \} such that for all $\chi \in C_0^\infty (U_P)$ and all $\xi \in V_P$, we have

\[ |\hat{\chi}d\sigma(\xi)| \lesssim \min \left\{ |\xi| - \nu_\Omega(P), |\xi|^{-\frac{1}{2}} \nu_\Omega^{(2)}(P) \left( \sum_{i=1}^{d-1} \left( \frac{|O_i \xi|}{|\xi|} \right)^{a_{d-1}} \right)^{\frac{1}{a_{d-1}} - \frac{1}{2}} \right\}, \]

where $\nu_\Omega(P)$ and $\nu_\Omega^{(2)}(P)$ are defined by (1.1) and (1.2) respectively, $O_i$ is the $i$-th row vector of the matrix $O = O(P)$ defined by (2.1), $a = (a_1, \ldots, a_{d-1})$ is the multitype (type if $d = 2$) of $\partial \Omega$ at $P$, and the implicit constant may depend on the domain $\Omega$ and upper bounds of $\chi$ and finitely many derivatives of $\chi$.

The first bound on the right hand side is standard, which follows easily from [5, P. 335–336, Theorem B]. The second one follows from [15, Lemma 1] in dimension two and [9, Proposition 1.2] in higher dimensions.

In the rest of this section we establish results on lattice point counting in stretched finite type domains. Recall that $\nu_\Omega$ and $\mu_\Omega$ are defined by (1.3) and (1.4) respectively.

**Proposition 2.3.** Let $\Omega \subset [-C, C]^d \subset \mathbb{R}^d$ ($d \geq 2$, $C > 0$) be a convex compact domain, which contains the origin as an inner point, with smooth boundary of finite type. If $t/\alpha_* \geq 1/C$, then

\[ \# (\mathbb{Z}^d \cap tA\Omega) = |\Omega|^d + O \left( \alpha_*^{d^2-d+1} \left( t^{d-1-\nu_\Omega} + t^{d-1-\frac{\mu_\Omega}{d-1}} \right) \right), \]

where the implicit constant depends only on the domain $\Omega$.\]
Remark 2.4. If $A$ is a fixed matrix, the above result is given directly by [9, Theorem 1.3]. For our need, $A$ is allowed to change. Hence we have to track the impact of $A$ and modify the proof of [9, Theorem 1.3] accordingly.

We did not try to find the smallest exponent of the $a^*$ term since it does not matter in the study of the optimal stretching problem. Indeed, we will manage to show that $a^*$ is uniformly bounded in Section 3 hence the $a^*$ term is bounded by a constant after all.

Proof of Proposition 2.3. Let $0 \leq \rho \in C_0^\infty(\mathbb{R}^d)$ be a cut-off function with supp $\rho \subset B(0, 1)$ and $\int_{\mathbb{R}^d} \rho(x) \, dx = 1$. Set $\rho_\varepsilon(x) = \varepsilon^{-d} \rho(\varepsilon^{-1} x)$, $0 < \varepsilon < 1$, and

$$N_{A, \varepsilon}(t) = \sum_{k \in \mathbb{Z}^d} \chi_{tA\Omega} * \rho_\varepsilon(k),$$

where $\chi_{tA\Omega}$ denotes the characteristic function of $tA\Omega$. It is a standard result that there exists a constant $c > 0$ depending only on the domain $\Omega$ such that

$$N_{A, \varepsilon}(t - ca^* \varepsilon) \leq \#(\mathbb{Z}^d \cap tA\Omega) \leq N_{A, \varepsilon}(t + ca^* \varepsilon).$$

By using the Poisson summation formula we have

$$N_{A, \varepsilon}(t) = t^d \sum_{k \in \mathbb{Z}^d} \hat{\chi}_\Omega(tAk) \hat{\rho}(\varepsilon k) = |\Omega| t^d + R_{A, \varepsilon}(t)$$

with

$$R_{A, \varepsilon}(t) = t^d \sum_{k \in \mathbb{Z}^d} \hat{\chi}_\Omega(tAk) \hat{\rho}(\varepsilon k).$$

Let $\Gamma$ denote the set of points $P \in \partial \Omega$ at which all principal curvatures vanish. It is known that $\Gamma$ is a finite set (see [10, P. 164]). For each $P \in \Gamma$, choose an open conic symmetric neighborhood $V_P$ of the normals $\{\pm n(P)\}$. If two points in $\Gamma$ have parallel normals we choose the same conic neighborhood for both of them. We may shrink these neighborhoods so that they are disjoint pairwise.

Let $\text{dist}_{\infty}$ denote the distance taken with respect to the $\ell^\infty$ metric in $\mathbb{R}^d$. For $P \in \Gamma$ let

$$\mathcal{N}_P^1 = \{x \in V_P : \text{dist}_{\infty}(x, \mathbb{R} n(P)) \leq 3/4\},$$

$$\mathcal{N}_P^2 = \{x \in V_P : \text{dist}_{\infty}(x, \mathbb{R} n(P)) > 3/4\}$$

and

$$\mathcal{M} = \{x \in \mathbb{R}^d : x \notin \bigcup_{P \in \Gamma} V_P\}.$$

To estimate $R_{A, \varepsilon}(t)$ we just need to estimate

$$S_{A, \varepsilon}(t) = t^d \sum_{k \in A^{-1} \mathcal{M}} \hat{\chi}_\Omega(tAk) \hat{\rho}(\varepsilon k).$$
\[ S_P^2 = t^d \sum_{k \in A^{-1}N_p^1} \hat{\chi}_\Omega(tAk) \hat{\rho}(\varepsilon k) \]

and

\[ S_0 = t^d \sum_{k \in A^{-1}N} \hat{\chi}_\Omega(tAk) \hat{\rho}(\varepsilon k). \]

To the sum \( S_P^1 \) we apply the bound

\[ \hat{\chi}_\Omega(\xi) \lesssim |\xi|^{-1-\nu_\Omega} \quad \text{if} \quad \xi \in N_p^1. \]

To verify this bound, by the divergence theorem, we have

\[ \hat{\chi}_\Omega(\xi) = \frac{i}{|\xi|} \int_{\partial \Omega} \frac{\xi}{|\xi|} \cdot n(x) e^{-2\pi i \langle x, \xi \rangle} \, d\sigma(x). \]

Let \( P' \in \partial \Omega \) be the boundary point whose outward normal is along \(-n(P)\). Following a standard argument from the oscillatory integral theory, we split the above integral over \( \partial \Omega \) into three parts over a neighborhood about \( P \), a neighborhood about \( P' \) and the rest respectively. The former two parts are of size \( O(|\xi|^{-\nu_\Omega}) \), both yielded by the first bound of Lemma 2.2 (we may shrink the conic neighborhood \( V_p \) if necessary). The third part is of size \( O(|\xi|^{-N}) \), given by a simple integration by parts. The bound (2.5) then follows easily.

Applying (2.5) yields

\[ S_P^1 \lesssim t^{d-1-\nu_\Omega} \sum_{k \in A^{-1}N_p^1} |Ak|^{-1-\nu_\Omega}. \]

We split the above sum on the right into two sums depending on whether \( |\langle n(P), Ak \rangle| \) is \( \geq C \) or \( \leq C \) for an absolute constant \( C \). If \( C \) is large, a comparison with an integral yields that the sum with \( |\langle n(P), Ak \rangle| \geq C \) is

\[ \lesssim a^d \int_{T} |x|^{-1-\nu_\Omega} \, dx \lesssim a^d, \]

where \( T \) represents a tubular neighborhood of a line away from the origin. Trivial estimate gives that the sum with \( |\langle n(P), Ak \rangle| \leq C \) is

\[ \lesssim a^d \int T |x|^{-1+\nu_\Omega} \, dx = a^d. \]

Therefore

\[ S_P^1 \lesssim a^{d+\nu_\Omega} t^{d-1-\nu_\Omega}. \]

To the sum \( S_0 \) we apply the bound

\[ \hat{\chi}_\Omega(\xi) \lesssim |\xi|^{-1-\mu_\Omega} \quad \text{if} \quad \xi \in \mathcal{M}, \]
which follows from the divergence theorem, the Bruna-Nagel-Wainger estimate (in [5]) and an integration by parts argument. Hence

\[
S_0 \lesssim t^d \sum_{k \in A^{-1}\mathfrak{M}} |tAk|^{-1-\mu} |\hat{\rho}(\varepsilon k)|
\lesssim A^1 \mu t^{-1-\mu} \varepsilon^{1+\mu}^{-d}.
\]

(2.7)

For the sum \(S_2^P\) we handle \(\hat{\chi}_\Omega(\xi), \xi \in N^2_P\), similarly as in the proof of (2.5), except that we now use the second bound of Lemma 2.2. As before, the estimate of \(\hat{\chi}_\Omega\) is reduced to the Fourier transform of the surface carried measure \(d\sigma\), which is then split into three parts. We apply the second bound of Lemma 2.2 to the first part (over a neighborhood about \(P\)). We may assume \(P' \in \Gamma\) without loss of generality, thus the second part (over a neighborhood about \(P'\)) is of size \(O(|\xi|^{-N})\) by the Bruna-Nagel-Wainger estimate. The third part is of size \(O(|\xi|^{-N})\) by integration by parts. To conclude we obtain the bound

\[
\hat{\chi}_\Omega(\xi) \lesssim |\xi|^{-1-\mu} \left( \sum_{l=1}^{d-1} \left( \frac{|O_l|}{|\xi|} \right)^{a'_l} \right)^{\frac{1}{a_1} - \frac{1}{2}},
\]

if \(\xi \in N^2_P\),

where \(a'_l = a_l/(a_l - 1)\) with \((a_1, \ldots, a_{d-1})\) the multitype (type if \(d = 2\)) of \(\partial \Omega\) at \(P\), and \(O_l\) is the \(l\)-th row vector of the matrix \(O = O(P)\) defined by (2.1).

Applying the above bound yields

\[
S_2^P \lesssim t^{d-1-\mu} \sum_{k \in A^{-1}\mathfrak{M}} |Ak|^{-1-\mu} \left( \sum_{l=1}^{d-1} \left( \frac{|O_l|}{|Ak|} \right)^{a'_l} \right)^{\frac{1}{a_1} - \frac{1}{2}} (1 + |\varepsilon k|)^{-N}.
\]

Notice that

\[
4 \leq a_1 \leq a_2 \leq \cdots \leq a_{d-1},
\]

\[
|k| = |A^{-1}Ak| \geq a_1^{-d} |Ak|
\]

and if \(\xi \in \mathfrak{N}^2_P\) (i.e. \(\xi \in V_P\) and \(\text{dist}_{\infty}(\xi, \mathbb{R}n(P)) > 3/4\)) and \(V_P\) is sufficiently narrow then \(|\xi| \geq 1\). A dyadic decomposition on the size of \(|Ak|\) then yields

\[
S_2^P \lesssim \sum_{s=0}^{\infty} \frac{t^{d-1-\mu} \sum_{k \in A^{-1}\mathfrak{M}} |Ak|^{-1-\mu} \left( \sum_{l=1}^{d-1} \left( \frac{|O_l|}{|Ak|} \right)^{a'_l} \right)^{\frac{1}{a_1} - \frac{1}{2}}}{(1 + a_1^{-d} \varepsilon^{2s})^N}. \]
We claim that

\[
\sum_{k \in A^{-1} \mathcal{N}_P^2 \setminus |Ak| \geq \lambda} |Ak|^{-1-\mu} \left( \sum_{i=1}^{d-1} \left( \frac{|O_i Ak|}{|Ak|} \right)^{a'_i} \right)^{\frac{1}{a'_1}} \lesssim a_*^d \lambda^{d-1-\mu}. \tag{2.8}
\]

We will prove (2.8) later. Using (2.8) we get

\[
S_P^2 \lesssim a_*^d t^{d-1-\mu} \sum_{s=0}^{\infty} \frac{(2s)^{d-1-\mu}}{(1 + a_*^{-d} 2^s)^N} \lesssim a_*^{(d-1)(d-1-\mu_1)} + d t^{d-1-\mu_1} \varepsilon^{1+\mu_0-d}. \tag{2.9}
\]

Using bounds (2.6), (2.7) and (2.9), we obtain that

\[
R_{A,\varepsilon}(t) \lesssim a_*^{(d-1)(d-1-\mu_1)} + t^{d-1-\mu_1} + t t^{d-1-\mu_1} \varepsilon^{1+\mu_0-d}. \tag{2.10}
\]

If \( t/a_* \) is sufficiently large, we take

\[
\varepsilon = (t/a_*)^{-\frac{\mu_0}{d-\mu_1}}.
\]

Then \( t/2 \leq t \pm c a_* \varepsilon \leq 3t/2 \). Combining (2.3), (2.4) and (2.10) yields

\[
\left| \# \left( \mathbb{Z}^d \cap tA \Omega \right) - |\Omega| t^d \right| \lesssim a_* t^{d-1} \varepsilon + |R_{A,\varepsilon}(t \pm c a_* \varepsilon)| \lesssim a_*^{d^2-d+1} \left( t^{d-1-\nu_1} + t t^{d-1-\mu_1} \varepsilon^{1+\mu_0-d} \right),
\]

which is (2.2).

If \( t/a_* \approx 1 \) then \( ta_i \geq t/a_* \geq 1 \). Note that \( tA \Omega \) is contained in an enlarged rectangular box with side lengths \( O(ta_1), \ldots, O(ta_d) \). Since such a box contains at most \( O(t^d) \) lattice points by trivial estimate, we get

\[
\# \left( \mathbb{Z}^d \cap tA \Omega \right) \lesssim t^d,
\]

which leads to (2.2) trivially. This finishes the proof. \( \square \)

**Proof of (2.8).** If \( \xi \in \mathcal{N}_P^2 \) and \( \text{dist}_\infty(\xi, y) \leq 1/(2a_*) \) then \( |\xi| \simeq |y| \) and

\[
\| (O_1 \xi, \ldots, O_{d-1} \xi) \|_\infty = \text{dist}_\infty(\xi, \mathbb{R}^n(P)) > 3/4,
\]

where we have used the definition (2.1) of the matrix \( O \) to obtain the above equality. As a consequence we have

\[
\sum_{i=1}^{d-1} \left( \frac{|O_i \xi|}{|\xi|} \right)^{a'_i} \lesssim \sum_{i=1}^{d-1} \left( \frac{|O_i y|}{|y|} \right)^{a'_i}.
\]

Denote by \( C_\xi \) the open cube in \( \mathbb{R}^d \) with center \( \xi \), side length \( 1/a_* \) and all sides parallel to coordinate axes. It is clear that \( \{ C_{Ak} : k \in \mathbb{Z}^d \} \)
are disjoint cubes. Comparing the sum on the left side of (2.8) with an integral, followed by a proper rotation, yields that

$$\sum_{k \in A - 1} |Ak|^{-\mu} \left( \sum_{l=1}^{d-1} \left( \frac{|O_l Ak|}{|Ak|} \right) \right)^{1/4} \lesssim \lambda |Ak|^{-1} - \mu \Omega \left( \frac{d}{2} \sum_{l=1}^{d-1} \left( \frac{|O_l|}{|y|} \right) \right)^{1/4} a_1^{1/2} \lesssim a^d_\star \int_{|y| > \lambda} \int_{|y'| \leq \lambda} |y|^{-\mu} \left( \sum_{l=1}^{d-1} \left( \frac{|y_l|}{|y|} \right) \right)^{1/4} \lesssim a^d_\star \lambda^{d-1-\mu} \Omega,$$

as desired. \(\square\)

Recall that in Section 1 we denote by \(\Omega_j \subset \mathbb{R}^d\) the intersection of \(\Omega\) with the coordinate hyperplane \(x_j = 0\) and by \(|\Omega_j|\) the \((d-1)\)-dimensional measure of \(\Omega_j\). We sometimes naturally treat \(\Omega_j\) as a subset of \(\mathbb{R}^{d-1}\). The following result on the number of lattice points in \(tA \cup \bigcup_{j=1}^d \Omega_j\) is a consequence of the previous proposition.

**Proposition 2.5.** Let \(\Omega \subset [-C, C]^d \subset \mathbb{R}^d\) \((d \geq 2, C > 0)\) be a convex compact domain, which contains the origin as an inner point, with smooth boundary of finite type. If \(t/a_\star \geq 1/C\), then

$$\#(\mathbb{Z}^d \cap tA \bigcup_{j=1}^d \Omega_j) \leq d \sum_{j=1}^d \#(\mathbb{Z}^{d-1} \cap tA_j \Omega_j) = \sum_{1 \leq j < l \leq d} \#(\mathbb{Z}^{d-2} \cap tA_{j,l} \Omega_{j,l}) + O \left( a^d_\star \lambda^{d-1-\mu} \left( \frac{t^{d-1-\mu} + t^{d-1-\mu} \Omega}{d-\mu} \right) \right),$$

where the implicit constant depends only on the domain \(\Omega\).

**Proof.** For \(1 \leq j \neq l \leq d\), let \(A_j\) be the \((d-1) \times (d-1)\) matrix obtained from \(A\) by deleting the \(j\)-th row and column, and \(A_{j,l}\) be the \((d-2) \times (d-2)\) matrix by deleting the \(j\)-th and \(l\)-th rows and columns. Denote by \(\Omega_{j,l}\) the intersection of \(\Omega\) with the hyperplane \(x_j = 0\) and \(x_l = 0\). We sometimes treat \(\Omega_{j,l}\) as a subset of \(\mathbb{R}^{d-2}\). Hence \(A_j \Omega_j\) and \(A_{j,l} \Omega_{j,l}\) make sense.

It is geometrically evident that

$$\sum_{j=1}^d \#(\mathbb{Z}^{d-1} \cap tA_j \Omega_j) - \sum_{1 \leq j < l \leq d} \#(\mathbb{Z}^{d-2} \cap tA_{j,l} \Omega_{j,l}) \leq \#(\mathbb{Z}^d \cap tA \bigcup_{j=1}^d \Omega_j) \leq \sum_{j=1}^d \#(\mathbb{Z}^{d-1} \cap tA_j \Omega_j).$$
Hence it suffices to find the asymptotics of $\#(\mathbb{Z}^{d-1} \cap tA_j \Omega_j)$ (by Proposition 2.3) and estimate the size of $\#(\mathbb{Z}^{d-2} \cap tA_j,t \Omega_{j,l})$. Combining with the above inequality, we will then get the desired asymptotics (2.11).

If $d \geq 3$, applying Proposition 2.3 to the domain $\Omega_j \subset \mathbb{R}^{d-1}$ yields

\[
\left| \# \left( \mathbb{Z}^{d-1} \cap tA_j \Omega_j \right) - a_j^{-1} |\Omega_j| t^{d-1} \right| \\
= \left| \# \left( \mathbb{Z}^{d-1} \cap \left( t a_j^{-\frac{1}{\alpha_j}} \right) \left( a_j^{-\frac{1}{\alpha_j}} A_j \right) \Omega_j \right) - a_j^{-1} |\Omega_j| t^{d-1} \right| \\
\lesssim a_*^{d-d+1} \left( t^{d-2-\nu_{\Omega_j}} + t^{d-2-\frac{\mu_{\Omega_j}}{d-1-\mu_{\Omega}}} \right),
\]

where

\[
\nu_{\Omega_j} = \min_{P \in \partial \Omega_j} \nu_{\Omega_j}(P) \quad \text{and} \quad \mu_{\Omega_j} = 1/2 + \min_{P \in \partial \Omega_j} \nu_{\Omega_j}^{(2)}(P).
\]

By Lemma A.1 we have

\[
1 + \nu_{\Omega_j}(P) > \nu_\Omega(P) \quad \text{for any } P \in \partial \Omega_j,
\]

which gives

\[
d - 2 - \nu_{\Omega_j} < d - 1 - \nu_\Omega.
\]

We also have

\[
d - 2 - \frac{\mu_{\Omega_j}}{d - 1 - \mu_{\Omega_j}} < d - 2 < d - 1 - \frac{\mu_\Omega}{d - \mu_\Omega}
\]

since $\mu_\Omega \leq (d - 1)/2$. We thus readily get

\[
\left| \# \left( \mathbb{Z}^{d-1} \cap tA_j \Omega_j \right) - a_j^{-1} |\Omega_j| t^{d-1} \right| \\
\lesssim a_*^{d-d+1} \left( t^{d-1-\nu_\Omega} + t^{d-1-\frac{\mu_\Omega}{d-1-\mu_\Omega}} \right).
\]

Note that (2.12) holds trivially if $d = 2$. This provides the asymptotics of $\#(\mathbb{Z}^{d-1} \cap tA_j \Omega_j)$ we need.

As to the size of $\#(\mathbb{Z}^{d-2} \cap tA_j,t \Omega_{j,l})$, we observe that $tA_j,t \Omega_{j,l}$ (as a subset of $\mathbb{R}^{d-2}$) is contained in a rectangular box with side lengths $O(t\alpha_1), \ldots, O(t\alpha_{j-1}), O(t\alpha_{j+1}), \ldots, O(t\alpha_{l-1}), O(t\alpha_{l+1}), \ldots, O(t\alpha_l)$. We also know that $t\alpha_i \geq t/a_* \geq 1/C$ and $\det A = 1$. By trivial estimate we have

\[
\#(\mathbb{Z}^{d-2} \cap tA_j,t \Omega_{j,l}) \lesssim (a_j t\alpha_l)^{-1} t^{d-2} \\
\lesssim a_*^{d-d+1} \left( t^{d-1-\nu_\Omega} + t^{d-1-\frac{\mu_\Omega}{d-1-\mu_\Omega}} \right).
\]

This provides the size of $\#(\mathbb{Z}^{d-2} \cap tA_j,t \Omega_{j,l})$, thus finishes the proof. \(\square\)
Theorem 2.6. Let $\Omega \subset [-C, C]^d \subset \mathbb{R}^d$ ($d \geq 2$, $C > 0$) be a convex compact domain with smooth boundary of finite type that is symmetric with respect to each coordinate hyperplane (axis if $d = 2$). If $t/a_* \geq 1/C$, then

\[
\# (\mathbb{N}^d \cap t A \Omega) = 2^{-d} |\Omega|^d - 2^{-d} \sum_{j=1}^{d} a_j^{-1} |\Omega_j|^d t^{d-1} + O \left( a_*^{d^2-d+1} \left( t^{d-1-n_\Omega} + t^{d-1-\frac{\mu_\Omega}{d-1}} \right) \right)
\]  

(2.14)

and

\[
\# (\mathbb{Z}_+^d \cap t A \Omega) = 2^{-d} |\Omega|^d + 2^{-d} \sum_{j=1}^{d} a_j^{-1} |\Omega_j|^d t^{d-1} + O \left( a_*^{d^2-d+1} \left( t^{d-1-n_\Omega} + t^{d-1-\frac{\mu_\Omega}{d-1}} \right) \right),
\]

(2.15)

where implicit constants depend only on the domain $\Omega$.

Proof. Since the domain $\Omega$ is symmetric we have

\[
\# (\mathbb{N}^d \cap t A \Omega) = 2^{-d} \left( \# (\mathbb{Z}^d \cap t A \Omega) - \# \left( \mathbb{Z}^d \cap t A \bigcup_{j=1}^{d} \Omega_j \right) \right).
\]

Then (2.14) can be obtained from (2.2) and (2.11).

It remains to prove (2.15). For $1 \leq j \leq d$ let $P_j(1, 2, \ldots, d)$ be the collection of all subsets of $\{1, 2, \ldots, d\}$ having exactly $j$ elements. For any $S \in P_j(1, 2, \ldots, d)$ let

\[
k(S) = \{ (k_1, \ldots, k_d) \in \mathbb{Z}_+^d : k_i = 0 \text{ if } i \in S; k_i \in \mathbb{N} \text{ otherwise} \}.
\]

Then

\[
\# (\mathbb{Z}_+^d \cap t A \Omega) = \# (\mathbb{N}^d \cap t A \Omega) + \sum_{j=1}^{d} \sum_{S \in P_j(1, \ldots, d)} \# (k(S) \cap t A \Omega).
\]

Notice that

\[
\sum_{S \in P_1(1, \ldots, d)} \# (k(S) \cap t A \Omega) = \sum_{j=1}^{d} \# (\mathbb{N}^{d-1} \cap t A_j \Omega_j).
\]

By the symmetry of $\Omega_j$, (2.12) and (2.13), we have

\[
\sum_{j=1}^{d} \# (\mathbb{N}^{d-1} \cap t A_j \Omega_j)
\]
\[
= \sum_{j=1}^{d} 2^{-(d-1)} a_j^{-1} |\Omega_j| t^{d-1} + O \left( a^*_{d-2} t^{-1} + t^{d-1} \frac{\mu_\Omega}{d-\nu_\Omega} \right).
\]

By (2.13) we also have
\[
\sum_{j=2}^{d} \sum_{S \in P_j(1, \ldots, d)} \# (k(S) \cap tA\Omega) \lesssim \sum_{1 \leq l < m \leq d} \# \left( \mathbb{Z}^{d-2} \cap tA_{l,m}\Omega_{l,m} \right) \\
\lesssim a^*_{d-2} t^{-1} + t^{d-1} \frac{\mu_\Omega}{d-\nu_\Omega}.
\]

Then (2.15) follows from the above four equalities and (2.14). \(\square\)

3. Proof of Theorem 1.1

With results of lattice point counting established, we follow a standard procedure to prove Theorem 1.1. We refer readers to [12] and also [8, Section 4] for this procedure.

We first consider the case \(A(t) \in \mathfrak{A}_\Omega(t)\). We set a diagonal matrix
\[
(3.1) \quad B = \text{diag} \left( \frac{|\Omega_1|}{\sqrt{|\Omega_1| \cdots |\Omega_d|}}, \ldots, \frac{|\Omega_d|}{\sqrt{|\Omega_1| \cdots |\Omega_d|}} \right).
\]

Applying (2.14) with this stretching factor yields
\[
\# \left( \mathbb{N}^d \cap tB\Omega \right) = 2^{-d} |\Omega| t^d - 2^{-d} d \sqrt{|\Omega_1| \cdots |\Omega_d|} t^{d-1} \\
+ O \left( t^{d-1} - \nu_\Omega + t^{d-1} \frac{\mu_\Omega}{d-\nu_\Omega} \right),
\]
which leads to
\[
(3.2) \quad \# \left( \mathbb{N}^d \cap tB\Omega \right) \geq 2^{-d} |\Omega| t^d - 2^{-d} d \sqrt{|\Omega_1| \cdots |\Omega_d|} t^{d-1}
\]
for sufficiently large \(t\).

Since \(A(t) \in \mathfrak{A}_\Omega(t)\), we have
\[
t/a_*(t) \geq 1/C,
\]
where \(a_*(t) = \|A(t)^{-1}\|_\infty\) and \(C > 0\) is a constant satisfying \(\Omega \subset [-C, C]^d\), otherwise \(tA(t)\Omega\) does not contain any positive lattice point. Then Lemma 2.1 gives
\[
(3.4) \quad \# \left( \mathbb{N}^d \cap tA(t)\Omega \right) \leq 2^{-d} |\Omega| t^d - c a_*(t) t^{d-1},
\]
where \(c\) is a positive constant depending only on the domain \(\Omega\).

Combining (3.3), (3.4) and
\[
(3.5) \quad \# \left( \mathbb{N}^d \cap tB\Omega \right) \leq \# \left( \mathbb{N}^d \cap tA(t)\Omega \right)
\]
yields that
\[
a_*(t) \leq 2^{1-d} d \sqrt{|\Omega_1| \cdots |\Omega_d|}/c,
\]
namely, $a_*(t)$ is uniformly bounded from above for sufficiently large $t$.

Applying (2.14) with the stretching factor $A(t)$ gives

\[
\# \left( \mathbb{N}^d \cap tA(t)\Omega \right) = 2^{-d}|\Omega|t^d - 2^{-d} \sum_{j=1}^{d} a_j(t)^{-1}|\Omega_j|t^{d-1} + O \left( t^{d-1-\nu_1} + t^{d-1-\frac{\nu_1}{d}} \right).
\]

(3.6)

Combining (3.2), (3.6) and (3.5) yields that

\[
\sum_{j=1}^{d} a_j(t)^{-1} \frac{|\Omega_j|}{\sqrt{|\Omega_1| \cdots |\Omega_d|}} \leq d + O \left( t^{-\nu_1} + t^{-\frac{\nu_1}{d}} \right).
\]

Then the desired convergence (1.7) follows easily from an elementary result in [8, Lemma B.1]. This completes the proof of the first case.

The second case $\tilde{A}(t) \in \tilde{\Omega}(t)$ can be proved similarly. We sketch its proof. Applying (2.15) with the matrix $B$ (defined by (3.1)) yields

\[
\# \left( \mathbb{Z}_+^d \cap tB\Omega \right) \leq 2^{-d}|\Omega|t^d + 2^{1-d}d^d \sqrt{|\Omega_1| \cdots |\Omega_d|}t^{d-1}
\]

for sufficiently large $t$. We also have

\[
\# \left( \mathbb{Z}_+^d \cap t\tilde{A}(t)\Omega \right) \geq \# \left( \mathbb{Z}_+^d \cap t\tilde{A}(t)\Omega \right).
\]

(3.8)

Let $\tilde{a}_*(t) = ||\tilde{A}(t)^{-1}||_\infty$. We claim that if $t$ is sufficiently large, then

\[t/\tilde{a}_*(t) \geq 1/C\]

with the same constant $C$ aforementioned. Indeed, if $t/\tilde{a}_*(t) < 1/C$ then $\mathbb{Z}_+^d \cap t\tilde{A}(t)\Omega$ is contained in $t\tilde{A}_j(t)\Omega_j$ for some $1 \leq j \leq d$, where $\tilde{A}_j(t)$ is the $(d-1) \times (d-1)$ matrix obtained from $\tilde{A}(t)$ by removing its $j$-th row and column. Hence

\[
\# \left( \mathbb{Z}_+^d \cap t\tilde{A}(t)\Omega_j \right) \geq 2^{-(d-1)}|t\tilde{A}_j(t)\Omega_j|
\]

\[> 2^{1-d}C|\Omega_j|t^d,
\]

where in the second inequality we have used $t/\tilde{a}_*(t) < 1/C$. Since $\Omega \subseteq \Omega_j \times [-C,C]$, we have $2C|\Omega_j| > |\Omega|$. If $t$ is sufficiently large we then have

\[
\# \left( \mathbb{Z}_+^d \cap t\tilde{A}(t)\Omega \right) \geq 2^{-d}|\Omega|t^d + 2^{1-d}d^d \sqrt{|\Omega_1| \cdots |\Omega_d|}t^{d-1} \geq \# \left( \mathbb{Z}_+^d \cap tB\Omega \right)
\]

by (3.7). This contradicts with (3.8).

It is then easy to show that $\tilde{a}_*(t)$ is uniformly bounded, as a consequence of (3.7), (3.8) and Lemma 2.1. We apply (2.15) to $\#(\mathbb{Z}_+^d \cap t\tilde{A}(t)\Omega)$ and use (3.8) and (3.9) to finish the proof. \square
Appendix A. Multitype

In the appendix we compare the multitypes of $\partial \Omega_j$ and $\partial \Omega$ at a common point $P \in \partial \Omega_j \subset \partial \Omega$. The result is a direct consequence of the definition of multitype (see for example [9, P. 155–156]), which says that the $i$-th component of the multitype of $\partial \Omega_j$ at $P$ is not greater than the $(i+1)$-th component of the multitype of $\partial \Omega$ at $P$.

Lemma A.1. Let $\Omega \subset \mathbb{R}^d$ ($d \geq 3$) be a convex compact domain with smooth boundary of finite type and $\Omega_j \subset \mathbb{R}^{d-1}$, $1 \leq j \leq d$, the intersection of $\Omega$ with the coordinate hyperplane $x_j = 0$. If $(\tilde{a}_1, \ldots, \tilde{a}_{d-2})$ and $(a_1, \ldots, a_{d-1})$ are multitypes of $\partial \Omega_j$ and $\partial \Omega$ at $P \in \partial \Omega_j$ respectively, then for any $1 \leq i \leq d-2$ we have

$$\tilde{a}_i \leq a_{i+1}.$$ 

Proof. We first briefly recall the definition of multitype. Let

$$\{ u_1, \ldots, u_{d-1}, -n(P) \}$$

be an orthonormal basis with the same orientation as $\{ e_1, \ldots, e_d \}$ and $u_1, \ldots, u_{d-2} \in T_P(\partial \Omega_j)$ and $u_{d-1} \in T_P(\partial \Omega)$. Then the boundary $\partial \Omega$ in a small neighborhood of $P$ can be parameterized by

\[
\Gamma(V) = P + V - \Phi(V)n(P),
\]

where $V = \sum_{i=1}^{d-1} x_i u_i \in T_P(\partial \Omega)$ and

$$\Phi(V) = \Phi(x_1, \ldots, x_{d-1}).$$

It is obviously that $\Phi(0) = \nabla \Phi(0) = 0$. For any $m \geq 2$, define

$$S^m_P = \left\{ x \in \mathbb{R}^{d-1} : \sum_{j=2}^{m} \left| D^j_x \Phi(0) \right| = 0 \right\},$$

where

$$D^j_x \Phi(0) = \left. \left( \frac{\partial}{\partial t} \right)^j \Phi(tx) \right|_{t=0}$$

is the $j$-th derivative of $\Phi$ at the origin in the direction $V = \sum_{i=1}^{d-1} x_i u_i$. Then there are at most $d - 1$ even numbers

$$2 \leq m_1 < m_2 < \cdots < m_k, \quad 1 \leq k \leq d - 1$$

such that the sequence

\[
\{0\} = S^m_P \subsetneq \cdots \subsetneq S^{m_1}_P \subsetneq S^{m_0}_P = \mathbb{R}^{d-1}
\]

is maximal in the sense that $S^m_P = S^{m_j-1}_P$ if $m_{j-1} \leq n < m_j$ (see [16, P. 1270]). Here $m_0 = m_1 - 1$. For $1 \leq j \leq k$ we define

$$a_i = m_j \quad \text{if} \quad d - 1 - \dim S^{m_{j-1}}_P < i \leq d - 1 - \dim S^{m_j}_P.$$
Then \( \mathbf{a} = (a_1, \ldots, a_{d-1}) \) is the multitype of \( \partial \Omega \) at \( P \). Notice that the multitype \( \mathbf{a} \) is independent of the choice of the orthonormal basis \( \{u_1, \ldots, u_{d-1}\} \). Furthermore, the convexity of \( \Phi \) makes \( S^m_{P_j} \)'s the linear subspaces of \( \mathbb{R}^{d-1} \). Let \( W_j \) be the orthogonal complement of \( S^m_{P_j} \) in \( S^m_{P_{j-1}} \). Then

(A.3) \[
S^m_P = S^m_{P_j} \oplus W_j \oplus W_{j-1} \oplus \cdots \oplus W_1.
\]

We observe that the dimension of \( W_j \) is the number of \( m_j \) appearing in the multitype \( \mathbf{a} \).

Correspondingly, by our choice of \( \{u_1, \ldots, u_{d-2}\} \), the boundary \( \partial \Omega_j \) in a small neighborhood of \( P \) can be parameterized by

\[
\Gamma(V) = P + V - \tilde{\Phi}(V)n(P),
\]

where \( V = \sum_{i=1}^{d-2} x_i u_i \in T_P(\partial \Omega_j) \) and

\[
\tilde{\Phi}(V) = \tilde{\Phi}(x_1, \ldots, x_{d-2}) = \Phi(x_1, \ldots, x_{d-2}, 0).
\]

Notice that for any \( \tilde{x} = (x_1, \ldots, x_{d-2}) \in \mathbb{R}^{d-2} \) and \( j \geq 2 \),

\[
D^j_x \tilde{\Phi}(0) = \left( \frac{\partial}{\partial t} \right)^j \Phi(tx_1, \ldots, tx_{d-2}, 0) \bigg|_{t=0} = D^j_x \Phi(0),
\]

where \( x = (x_1, \ldots, x_{d-2}, 0) \), namely the \( j \)-th derivative of \( \tilde{\Phi} \) at the origin in the direction \( V = \sum_{i=1}^{d-2} x_i u_i \) equals the \( j \)-th derivative of \( \Phi \) at the origin in the direction \( V = \sum_{i=1}^{d-2} x_i u_i + 0 u_{d-1} \). Hence all \( \tilde{a}_j \)'s are chosen from \( \{m_1, \ldots, m_k\} \) by the maximization of the space sequence \( (A.2) \) and the definition of multitype. For every \( m_j \), let

\[
\tilde{S}^m_P = \left\{ \tilde{x} \in \mathbb{R}^{d-2} : \sum_{j=2}^{m_j} |D^j_x \tilde{\Phi}(0)| = 0 \right\}
\]

and \( \tilde{W}_j \) be the orthogonal complement of \( \tilde{S}^m_P \) in \( \tilde{S}^m_{P_{j-1}} \). Then we have

(A.4) \[
\tilde{S}^m_P = \tilde{S}^m_{P_j} \oplus \tilde{W}_j \oplus \tilde{W}_{j-1} \oplus \cdots \oplus \tilde{W}_1.
\]

Notice that

(A.5) \[
\dim S^m_P = d - 1, \quad \dim \tilde{S}^m_P = d - 2
\]

and for any \( 1 \leq j \leq k \),

(A.6) \[
\dim S^m_{P_j} \geq \dim \tilde{S}^m_{P_j}.
\]

Then combining \( (A.3)-(A.6) \) yields that for any \( 1 \leq j \leq k \),

\[
\dim(W_1 \oplus \cdots \oplus W_j) - 1 \leq \dim(\tilde{W}_1 \oplus \cdots \oplus \tilde{W}_j),
\]

namely the number of \( m_1 \) appearing in \( \tilde{\mathbf{a}} \) is no less than the number of \( m_1 \) appearing in \( \mathbf{a} \) minus 1, and the same is true for the number
of $m_1, m_2$ and more generally for the number of $m_1, m_2, \ldots, m_j$ with $1 \leq j \leq k$. Hence we obtain the desired result. Notice that in some special cases we may have $\dim \tilde{W}_j = 0$. Then $m_j$ will not appear in $\tilde{a}$. But this does not affect our conclusion. □

References

[1] P. R. S. Antunes and P. Freitas, Optimal spectral rectangles and lattice ellipses, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 469 (2013), no. 2150, 20120492, 15 pp.
[2] S. Ariturk and R. S. Laugesen, Optimal stretching for lattice points under convex curves, *Port. Math.*, 74 (2017), no. 2, 91–114.
[3] M. van den Berg, D. Bucur and K. Gittins, Maximising Neumann eigenvalues on rectangles, *Bull. Lond. Math. Soc.*, 48 (2016), no. 5, 877–894.
[4] M. van den Berg and K. Gittins, Minimizing Dirichlet eigenvalues on cuboids of unit measure, *Mathematika*, 63 (2017), no. 2, 469–482.
[5] J. Bruna, A. Nagel and S. Wainger, Convex hypersurfaces and Fourier transforms, *Ann. of Math. (2)*, 127 (1988), no. 2, 333–365.
[6] K. Gittins and S. Larson, Asymptotic behaviour of cuboids optimising Laplacian eigenvalues, *Integral Equations Operator Theory*, 89 (2017), no. 4, 607–629.
[7] J. Guo and T. Jiang, A note on lattice points and optimal stretching, *Colloq. Math.*, 157 (2019), no. 1, 65–82.
[8] J. Guo and W. Wang, Lattice points in stretched model domains of finite type in $\mathbb{R}^d$, *J. Number Theory*, 191 (2018), 273–288.
[9] A. Iosevich, E. Sawyer and A. Seeger, Two problems associated with convex finite type domains, *Publ. Mat.*, 46 (2002), no. 1, 153–177.
[10] S. Larson, Maximizing Riesz means of anisotropic harmonic oscillators, *Ark. Mat.*, 57 (2019), no. 1, 129–155.
[11] S. Larson, Asymptotic shape optimization for Riesz means of the Dirichlet Laplacian over convex domains, *J. Spectr. Theory*, 9 (2019), no. 3, 857–895.
[12] R. S. Laugesen and S. Liu, Optimal stretching for lattice points and eigenvalues, *Ark. Mat.*, 56 (2018), no. 1, 111–145.
[13] N. F. Marshall, Stretching convex domains to capture many lattice points, *Int. Math. Res. Not. IMRN*, 2020, no. 10, 2918–2951.
[14] N. F. Marshall and S. Steinerberger, Triangles capturing many lattice points, *Mathematika*, 64 (2018), no. 2, 551–582.
[15] B. Randol, On the Fourier transform of the indicator function of a planar set, *Trans. Amer. Math. Soc.*, 139 (1969), 271–278.
[16] H. Schulz, Convex hypersurfaces of finite type and the asymptotics of their Fourier transforms, *Indiana Univ. Math. J.*, 40 (1991), no. 4, 1267–1275.