CONFORMAL INVARIANCE ON ORBIFOLDS
AND
EXCITATIONS OF SINGULARITY

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Abstract

We study conformal field theory on two-dimensional orbifolds and show this to be an effective way to analyze physical effects of geometric singularities with angular deficits. They are closely related to boundaries and cross caps. Representatives classes of singularities can be described exactly using generalizations of boundary states. From this we compute correlation functions and derive the spectra of excitations localized at the singularities.

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1 Introduction

Conformal invariance in two dimensions\cite{1} is a powerful tool for a wide range of physical problems where two-dimensional field theories are relevant\cite{2}, from condensed matter physics and statistical mechanical models to the perturbative formulation of string theory\cite{3}. Although field theories are most often formulated on a smooth manifold, in recent years it has become evident that geometric singularities play an essential role both in theory and application of conformal field theories. By far the most studied type of singularity is boundary. Boundary conformal field theory has been successfully applied to finite size effect\cite{4}, Kondo effect\cite{5}, and D-branes and orientfolds via open strings\cite{6}.

An important object in boundary conformal field theory is boundary state, which represents the geometric boundary algebraically as a generalization of quantum state\cite{7, 8}. It not only encodes physical information about a conformal boundary but also reveals deep structure of the bulk theory\cite{9, 10}. A similar notion is cross cap state, which is relevant for unoriented surfaces and open strings\cite{11, 12}.

In addition to boundary there are other types of singularities that are very common in the real world: conical defects and corners. They appear naturally in the geometry of physical models\cite{13}, open string field theory\cite{14}, and the possibility of rectangular open string\cite{15}. In this paper we present a general and systematic approach to study such singularities using orbifolds. Conical and corner defects can be as easily realized in orbifolds as boundaries and cross caps. In fact they are shown to be closely related and studied together in an unified manner. Generalizations of boundary states provide exact description of singularities representative of those from the mathematical classification of orbifolds\cite{16} and allows us to calculate correlation functions. We also analyze the degree of freedom localized at the singularity and show that their spectra correspond to those of the conformal fields on the bulk and boundary. We restrict the cases considered here to specific orbifolds whose covering space is a sphere. Technical detail and generality will appear separately.

2 Quasi-boundary

The simplest smooth two-dimensional surface of finite size is the sphere. Its isometry group is $O(3)$ and has three conjugacy classes of involutions. Each such involution generates one of the three simplest types of nontrivial orbifolds whose covering space is the sphere. Let
us parameterized the sphere with a complex coordinate $z$ using a stereographic projection. These orbifolds are shown in fig. 1 with their “primitive unit cell” (i.e. fundamental domain) shaded, borrowing a term from crystallography. We have chosen it to be the unit disc for all three cases.

Fig. 1a represents the disc and fig. 1b the cross cap. The corresponding orbifold actions are respectively $z \rightarrow 1/\bar{z}$ and $z \rightarrow -1/\bar{z}$. The former has a circle full of fixed points, $|z| = 1$, which is the boundary of the disc; the latter has no fixed point, but diametrically opposite points on the unit circle are identified.

Figure 1: $\mathbb{Z}_2$ orbifolds of sphere. (a): disc; (b): cross cap; (c): bicorne cap with singularity at $z = \pm 1$.

Fig. 1c represents the orbifold obtained with $z \rightarrow 1/z$. The points on the unit circle are identified by a reflection across its horizontal diameter, turning the circle into the line segment $S^1/\mathbb{Z}_2$. The two ends of the line segment, $z = \pm 1$, are the fixed points of the orbifold group. This orbifold has no boundary, but is singular at $z = \pm 1$ because they have angular deficits of $\pi$.

We shall collectively call the boundary of a chosen primitive cell of an orbifold in its covering space its quasi-boundary. In general, a quasi-boundary is not really a boundary of the orbifold. A physical theory on an orbifold is first formulated on its covering space and then the degree of freedom is restricted by choosing a primitive cell on the covering space, which amounts to choosing the quasi-boundary. Different choices of quasi-boundary in principle give equivalent results, but in practice random choices would lead to intractable calculation, while a judicious choice could allow efficient or exact analysis. This is why quasi-boundary is relevant and important. It plays a role similar to boundary, which is a special case.
Combining the above three involutions appropriately leads to quotients of the sphere with 4-element orbifold groups (fig. 2). We have again chosen to represent them all with the same geometric quasi-boundary, now made up of the upper unit semi-circle and the interval $[-1, 1]$. Their difference lies with the local geometries of the quasi boundaries. Fig. 2a shows a $Z_4$-orbifold generated by the map $z \rightarrow \bar{z}$ and $z \rightarrow 1/\bar{z}$. It has the topology of a disc so we call it an open disc. Here the quasi-boundary is a genuine boundary of the open disc here, but it has two singular points $z = \pm 1$ because the boundaries intersect there at right angle instead of $\pi$. It should be pointed out that the physical import of the open disc is not about an “open” CFT having boundary or an open string getting absorbed by D-brane. Those would be tautological and already accounted for by standard concepts in boundary CFT and open strings. Rather, the novelty here is the presence of singularities on the boundary of the worldsheet with angular defects.

The $Z_2 \times Z_2$ orbifold group shown in fig. 2b is generated by $z \rightarrow \bar{z}$ and $z \rightarrow -1/z$. The line segment $[-1, 1]$ on the real axis remains a bona fide boundary of the orbifold, but the unit semi-circle on the upper complex plane does not. Similar to the bicorne cap, points on the semi-circle are subject to identification by reflection with respect to the imaginary axis. We call it the open bicorne cap. It also has the topology of a disc but with only one singular point at $z = i$.

Fig. 2c represents the $Z_4$-orbifold of the sphere generated by $z \rightarrow -z$ and $z \rightarrow 1/z$. It has no boundary at all and is topologically a sphere. Both the semi-circle and the interval are subject to a $Z_2$ identification as shown. There are three singular points, all with angular defect of $\pi$. It has the shape of a tortelli with three corners.

It is of course possible to consider more complicated orbifold groups as well as surface

![Figure 2: $Z_2 \times Z_2$ and $Z_4$ orbifolds of sphere. Singularities are marked with $\circ$. (a): open disc; (b): open bicorne cap; (c): an orbifold with 3 fixed points.](image)
of more complicated topology. One can already describe infinitely many other orbifolds by combining the varieties we have already considered. For example, any two quasi-boundaries from fig. 1 can be put together to describe a $\mathbb{Z}_2$ orbifold of the torus. On the other hand, a complete classification of 2d orbifold has been made by categorizing the singular locus \[16\]. The latter fall into three families: the mirror which is the fixed locus of reflection, elliptic points which are fixed points of $\mathbb{Z}_n$ rotation, and corner reflectors which are fixed point of dihedral groups $D_n$. Each family is already represented at least once in fig. 1 and fig. 2: the boundary is the mirror, present in fig. 1a, fig. 2a and b; the singular points in fig. 1c, fig. 2b and c are elliptic points, which are angular defects in the bulk; the singularities in fig. 2a are corner reflectors, which are angular defect on boundary. Hence the cases considered so far are representative.

3 Quasi-boundary state

In all the cases discussed so far we have chosen so that $|z| = 1$ overlaps with (part of) the quasi-boundary. In radial quantization, $\ln |z|$ is the “time.” Therefore (part of) the quasi-boundary sits at an instant in time. It therefore corresponds to a state-like object that we call quasi-boundary state. Note that there is no requirement for a quasi-boundary state to be normalizable because it does not in general represent a state of the physical system and has no intrinsic probabilistic interpretation; it is a state in the sense that the Hamiltonian can act on it. The description of such systems is therefore turned into a problem of finding the corresponding quasi-boundary state. The boundary and cross cap states are known to be solutions \[8\] to sets of equations involving the stress tensors and possibly other symmetry currents. Similarly, quasi-boundary state are solutions to a set of quasi-boundary conditions.

These conditions stipulate that the quasi-boundary state is preserved by the symmetries compatible with the orbifold condition. Suppose that the corresponding symmetry charges are $Q_m$. States have a well defined physical meaning only projectively, and the same should be true for quasi-boundary states. Therefore the general algebraic statement for a quasi-boundary condition is

$$ (Q_m - c_m \rangle \rangle) = 0. $$

where $c_m$ is a set of c-numbers.

Let us first consider the conformal symmetry. The general procedure to derive the
symmetry that survives the orbifolding is similar to that used in [17]. The charges are \( L_m - (\pm 1)^m \tilde{L}_m \), \( m \in \mathbb{Z} \) for the disc (+) and cross cap (-) respectively. For the bicorne cap the charges are \( l_m \equiv L_m - L_m, \) and \( \tilde{l}_m \equiv \tilde{L}_m - \tilde{L}_m, \) \( m > 0 \). Note that unlike the previous two cases, there is no mixing between the left and right chiral components, because its orbifold group consists of pure rotation. Although \( l_m \) had not been widely known, it originally appeared some time ago, probably in the context of open string field theory [18].

For both the proper boundary state and the cross cap state, consistency requires that \( c_m = 0 \). However, for the bicorne cap, the most general solutions are

\[
l_m - 2m(h_+ + (-1)^m h_-), \quad \tilde{l}_m - 2m(\tilde{h}_+ + (-1)^m \tilde{h}_-).
\]

Therefore the bicorne cap state is the solution of

\[
0 = (L_m - L_m - 2m(h_+ + (-1)^m h_-)) \]

\[
= (\tilde{L}_m - \tilde{L}_m - 2m(\tilde{h}_+ + (-1)^m \tilde{h}_-)) \] (2)

for \( m > 0 \). The parameters \( h_\pm \) and \( \tilde{h}_\pm \) have physical meaning that will be discussed below along with excitations.

The above procedure for finding the symmetry charges can be applied to all types of orbifold. Here we briefly describe a couple more examples. For the case of open disc (fig. 2a), the \( z \to \bar{z} \) action reduces the conformal algebra to a single copy of Virasoro algebra \( L_m \). Then \( z \to 1/\bar{z} \) further reduces it to \( l_m \). The boundary condition is then just the first line of (eq. 2). In a formal sense, this quasi-boundary state is the tensor square-root of the bicorne cap state. However, unlike the bicorne cap, in this case it is in fact a bona fide boundary state, that is, of the open string [14, 15], and two specific values of \( h_\pm \) can be related to Dirichlet and Neumann boundary conditions respectively.

For the case of open bicorne cap (fig. 2c), the first step is the same but then the single copy of Virasoro algebra is subject to \( z \to -1/z \) instead. This leads to the boundary condition

\[
(L_m - L_m - 2m(i^m h_+ + (-i)^m h_-)) \]

(3)

One can also consider theories with bigger chiral algebra. For such case, analogous to boundary and cross cap states, one has the choice of requiring the quasi-boundary (state) to preserve either just the relevant conformal symmetry or also (part of) the other chiral
symmetry. In the latter case, the conformal boundary conditions such as (eq. 2) is supplemented by additional conditions from the generators of chiral symmetry. Here we gave the result for the case of affine Lie symmetries on the bicorne cap. For detailed analysis and other cases such as superconformal symmetry see [19]. Let \( J^a_m \) denote the affine current mode associated with some simple Lie algebra \( \mathfrak{g} \). The relevant boundary conditions are

\[
(J^i_m + U^i_j J^j_{-m}) |\rangle = 0
\]

(4)

where \( U \) is an involutary automorphism of \( \mathfrak{g} \), i.e. it satisfies \( U^2 = 1 \) as well as \( U^i_a U^j_b f^{ab}_k = f^{ij}_c U^c_k \) for \( \mathfrak{g} \)'s structure constant \( f^{ij}_k \). For bosonic string or affine-U(1) currents \( \alpha^\mu_m \), the conditions is

\[
(\alpha^\mu_m - R^\mu_\nu \alpha^\nu_{-m}) |\rangle = 0
\]

(5)

where \( R^\mu_\nu \) is an involution orthogonal with respect to the target space metric.

4 Excitation at the singularity

A physical system defined on a space with singularity introduces the possibility of localized degree of freedom, which are excitations at the singularity. A well known example is boundary, which allows boundary fields describing changes of boundary types. Boundary fields must be distinguished from the regular local scaling fields in the bulk, because the former are only defined on a boundary component. As a result they only transform under the subalgebra of the conformal/chiral algebra preserved by the boundary conditions.

The bicorne cap state and the likes provide an exact description of singularities. Therefore one expects to find a relation between these quasi boundary states and excitations at the singularities. Here we derive the correct relation directly using (eq. 2). Excitation at a singularity can be induced by a bulk field approaching it. Let \( \phi \) be a bulk primary field of conformal bi-weights \((\Delta, \bar{\Delta})\) such that

\[
\psi_{\pm} \equiv \lim_{z\to \pm 1} \lim_{\bar{z} \to \pm 1} (z \mp 1)^{\Delta \pm} (\bar{z} \mp 1)^{\bar{\Delta} \pm} \phi(z, \bar{z})
\]

is finite. Then

\[
[l_m, \psi_{\pm}] = 2m(\pm 1)^m d_{\pm} \psi_{\pm},
\]

\[
[l_m, \psi_{\pm}] = 2m(\pm 1)^m \bar{d}_{\pm} \psi_{\pm}.
\]
Here \( d_{\pm} = \Delta - \lambda_{\pm} \) and \( \tilde{d}_{\pm} = \tilde{\Delta} - \tilde{\lambda}_{\pm} \) characterize the effect of \( \psi_{\pm} \) on the singularity.

Therefore \( \psi_{\pm} \) act as shift operators for the parameters \( h_{\pm} \) in (eq. \[2\]). Since \( \psi_{\pm} \) is localized at \( z = \pm 1 \), it follows that \( h_{\pm} \) characterize the singularity \( z = \pm 1 \) respectively. Each such point is therefore characterized by a pair of parameters \((h, \tilde{h})\), and every excitation there is characterized by a pair \((d, \tilde{d})\). The excitation \( \psi_{\pm} \) changes the singularity by a shift \( h \rightarrow h + d, \tilde{h} \rightarrow \tilde{h} + \tilde{d} \). This is reminiscent of the relation between boundary and boundary operator [9].

The shift \( d \) and \( \tilde{d} \) depend not only on the conform weights of \( \phi \) but also on the order of the zero/pole needed to extract \( \psi \). It should be emphasized that \( \lambda \) and \( \tilde{\lambda} \) depend on the specific theory and details of the singularity, so they are not determined solely by \( \Delta \) and \( \tilde{\Delta} \).

In fact the complete information is encoded in a bulk-singularity expansion that generalizes the bulk-bulk and bulk-boundary operator product expansions. Analogous result holds for the open bicorne. For open boundary, the situation is similar except that instead of a chiral/anti-chiral pair, there is just a single parameter associated with each corner reflector and its excitation.

In boundary conform field theory, not only are the boundary operators labeled by the conformal weights, but also the boundaries themselves. This is because there is a type of boundary corresponding to the identity field. The same holds here. Even though it might seem that \( h \) has no preferred value because it can be shifted by excitation, we have found on general and reliable ground that it has a special value at \(-c/8\) that correspond to “vacuum,” \( c \) being the central charge. For unitary theories, this is a lower bound. Therefore the spectra of excitations at elliptic and corner singularities respectively correspond to those of the bulk and boundary fields and have the meaning of scaling dimension.

5 Explicit solutions

Quasi-boundary states not only provide an exact description of orbifold singularity, but also allows explicit calculations in the operator formalism. Here we gave several examples of correlation functions of vertex operators in the theory of \( D \) free bosons (the Gaussian model):

\[
\langle V_{k_1}(z_1, \tilde{z}_1) \ldots V_{k_n}(z_n, \tilde{z}_n) \rangle
\]
Detailed derivation will be given in [19]. For this theory solutions of quasi-boundary states can be explicitly constructed. For example,

\[
\left( \exp\left( \sum_m G_{\mu\nu}\alpha^{-m}_\mu\alpha^{\nu}_m \right) |0\rangle \right) \otimes |\chi\rangle
\]

is a bicorne cap state. Here \( G_{\mu\nu} \) is the metric for the free boson, \(|0\rangle\) is the oscillator vacuum, and \(|\chi\rangle\) is the quantum state for the string center of mass. Using this state we find the vertex operator correlation function on the bicorne cap to be

\[
\prod_{1 \leq a < b \leq n} \left| \frac{z_a - z_b}{1 - z_a z_b} \right|^{2k_a k_b} \prod_{a=1}^{n} \left| 1 - z_a^2 \right|^{-k_a^2} 2\pi \chi_k^*,
\]

where \( k = \sum_{a=1}^{n} k_a \) and \( \chi_k \) is the Fourier component of the wave function \( \chi(x) \) for momentum \( k \).

For the open disc, the correlation function is

\[
\prod_{1 \leq a < b \leq n} \left| \frac{(z_a - z_b)(z_a - \bar{z}_b)}{(1 - z_a z_b)(1 - z_a \bar{z}_b)} \right|^{2k_a k_b} \prod_{a=1}^{n} \left| 1 - |z_a|^2 \right| -k_a^2 2\pi \chi_k^*.
\]

And for the open bicorne cap, the correlation function is

\[
\prod_{1 \leq a < b \leq n} \left| \frac{(z_a - z_b)(z_a - \bar{z}_b)}{(1 + z_a z_b)(1 + z_a \bar{z}_b)} \right|^{2k_a k_b} \prod_{a=1}^{n} \left| 1 + |z_a|^2 \right| -k_a^2 2\pi \chi_k^*.
\]

6 Conclusion

We conclude this paper by remarking on the difference between the spacetime orbifolds that appear in string theory, and general conformal field theory on two-dimensional orbifolds considered here. At the formal level, while here we simply formulate the usual conformal field theory on a given orbifold, spacetime orbifold comes out of the procedure of “orbifolding,” where a discrete symmetry is gauged and at the same time the so-called twisted sectors are added to obtain a different theory [20].
More concretely, they differ in the way in which orbifold geometry, especially its singularities, is represented in physical quantities. For string theory, the worldsheet is smooth while the spacetime is classically a singular orbifold. However, the stringy effect of the twisted sectors masks the singularity of the orbifold and yield nonsingular amplitudes. This is considered to be a positive attribute of string theory because it modifies the small scale geometry of spacetime. By contrast, the zeros and poles in the correlation functions (eq. 789) clearly indicates the presence and location of singularity. This does not mean the theory is sick, but instead shows that singularities of the underlying geometry are fully revealed by the field theory.

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