NONPARAMETRIC STUDY OF THE EVOLUTION OF THE COSMOLOGICAL EQUATION OF STATE WITH SNeIa, BAO, AND HIGH-REDSHIFT GRBs

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ABSTRACT
We study the dark energy equation of state as a function of redshift in a nonparametric way, without imposing any a priori \( w(z) \) (ratio of pressure over energy density) functional form. As a check of the method, we test our scheme through the use of synthetic data sets produced from different input cosmological models that have the same relative errors and redshift distribution as the real data. Using the luminosity–time \( L_X-T_{\text{obs}} \) correlation for gamma-ray burst (GRB) X-ray afterglows (the Dainotti et al. correlation), we are able to utilize GRB samples from the Swift satellite as probes of the expansion history of the universe out to \( z \approx 10 \). Within the assumption of a flat Friedmann–Lemaître–Robertson–Walker universe and combining supernovae type Ia (SNeIa) data with baryonic acoustic oscillation constraints, the resulting maximum likelihood solutions are close to a constant \( w = -1 \). If one imposes the restriction of a constant \( w \), we obtain \( w = -0.99 \pm 0.06 \) (consistent with a cosmological constant) with the present-day Hubble constant as \( H_0 = 70.0 \pm 0.6 \text{ km s}^{-1} \text{ Mpc}^{-1} \) and density parameter as \( \Omega_{\Lambda 0} = 0.723 \pm 0.025 \), while nonparametric \( w(z) \) solutions give us a probability map that is centered at \( H_0 = 70.04 \pm 1 \text{ km s}^{-1} \text{ Mpc}^{-1} \) and \( \Omega_{\Lambda 0} = 0.724 \pm 0.03 \). Our chosen GRB data sample with a full correlation matrix allows us to estimate the amount, as well as quality (errors), of data needed to constrain \( w(z) \) in the redshift range extending an order of magnitude beyond the farthest SNeIa measured.

Key words: cosmological parameters – cosmology: miscellaneous – cosmology: observations – cosmology: theory – dark energy – dark matter

1. INTRODUCTION

Recent advances in precision cosmology and astronomical observations have yielded large amounts of data against which cosmological models can be calibrated and tested. Among them are supernovae type Ia (SNeIa) catalogs, galaxy surveys, baryonic acoustic oscillation (BAO) measurements (e.g., Tegmark et al. 2006), temperature fluctuations of the cosmic microwave background (CMB), observational Hubble data from differential ages of galaxies (e.g., Moresco et al. 2012; Stern et al. 2010a, 2010b), and ages of globular clusters (GC; e.g., Mackey & Gilmore 2003a, 2003b, 2003c).

In principle, taking the above as statistically independent singular event measurements, the expansion history of the universe and its temporal evolution can be inferred. Most inferences are observationally limited to the region \( z < 2 \), at best, with reliable low-error observations gradually disappearing after \( z = 1.5 \).

Given the much larger redshift range over which gamma-ray bursts (GRBs) can be observed extending out to \( z \approx 10 \) (e.g., Izzo et al. 2009) it has long been tempting to include GRBs as cosmological probes, extending the redshift range by almost an order of magnitude farther than available SNeIa. To this end it is necessary to find scaling relations among GRB observables.

A crucial breakthrough in this field has been the observation of GRBs by the Swift satellite, which provides a rapid follow-up of the afterglows in several wavelengths with better coverage than previous missions. Swift revealed a more complex behavior of the light curves, rather different from the broken power laws assumed in the past. The light curves observed by Swift identify two, three, and sometimes even more regions in the afterglows. The second segment, a temporally extended phase of close to constant luminosity, is called the plateau region. Here we make use of a proposal by some of us, who in Dainotti et al. (2008, 2010, 2011a, 2011b) discovered that a tight correlation exists in GRB light curves between the X-ray luminosity at the end of the plateau phase, \( L_X \), and the rest frame temporal duration of this phase, \( T_\text{obs}^* = T_\text{obs} / (1 + z) \).

In this paper we develop a new fully Bayesian (reduced to maximum likelihood) methodology that we apply to the problem of inferring the evolution of the dark energy (DE) equation of state (EOS), \( w(z) \), firmly based upon the best existing \( z < 2 \) observations and traced for the first time out to \( z \approx 9 \) through the use of a carefully selected subsample of 68 long GRBs. An advantage of our statistical approach is that no a priori functional form for \( w(z) \) is imposed; the method proceeds from a large number of randomly chosen \( w(z) \) functions, which are refined to yield an optimum maximum likelihood answer, together with corresponding confidence interval bands.

The nonparametric reconstruction of the DE EOS has been recently addressed by several authors (e.g., Su et al. 2012; Barboza et al. 2009; Holslaw et al. 2010, 2011; Adak et al. 2011; Barboza & Alcaniz 2012). Sophisticated Bayesian analysis was done by Vazquez et al. (2012) to reconstruct the DE EOS, and they found a mild indication of time dependence within the low-redshift range available to non-GRB observations. Our sequential Bayesian analysis method arises from similar work being done in reconstructing the equation of state of dense nuclear matter from observational
properties of neutron stars. The method needs no binning or predetermined \( w(z) \) proposals and so uses fully all the information available in the data, which are then the only factor driving the solution. The use of the recently established correlation for a subsample of GRBs then in principle allows for a reconstruction of the evolution history of the DE EOS out to an unprecedented redshift of 8.2. Unfortunately, the current number of high-redshift events and their large error bars permit only very crude \( w(z) \) inferences at high redshift.

Still, we obtain a maximum likelihood solution for the optimum \( w(z) \) model, remaining within the assumptions of spatial flatness, standard general relativity (GR), and Friedmann–Lemaitre–Robertson–Walker (FLRW) models yielding highly accurate determinations for \( H_0 \) and \( \Omega_{m0} \). Although the resulting confidence interval bands allow for some minor variations for the low redshift \( w(z) \) of \( \pm 0.3 \), the result we obtain is consistent with a classical cosmological constant \( w(z) = -1 \). We note that our results, as is always the case, do not represent an absolute confirmation of the assumptions made, in this case a classical GR universe with spatial flatness, which would have to be validated or excluded using a more extended approach that considers also other cosmological models such as those related to extended theories of gravity (Capozziello & Francaviglia 2008; Capozziello & de Lauretis 2011).

Section 2 discusses the use of the \( L_X-T_\alpha \) correlation for a subset of GRBs, and Section 3 gives an introduction to the basic sequential Bayesian approach and describes the cosmological model under which we will be working, a FLRW model. The maximum likelihood \( w(z) \) reconstruction methodology applied to the available data sets is developed in Section 4, where it is also tested through the use of synthetic data samples constructed for a series of test input cosmological models. In this same section we detail the construction of a distance ladder in order to use high-redshift GRBs and give also the treatment of correlations among GRB data points. Section 5 gives our results using the real data samples, and Section 6 presents our conclusions.

2. GRBs AS DISTANCE ESTIMATORS

Despite the extraordinary redshift range over which GRBs are observed, their use as cosmological references has been hindered by the fact that they are not inherently standard candles of fixed intrinsic luminosity. Numerous efforts have been made in recent years to look for correlations between the parameters of GRB light curves that might be used to turn GRBs into cosmological measurement tools. A number of published correlations (e.g., \( E_{iso}-E_{peak} \), Lloyd & Petrosian 2000; Amati et al. 2009; \( E_{\gamma}-E_{peak} \), Ghirlanda et al. 2004, 2006; \( L-E_{peak} \), Schaefer 2003; Yonetoku et al. 2004; and \( L-V \), Ramirez-Ruiz & Fenimore 2000; Reichart et al. 2001) and other proposed luminosity indicators (Norris et al. 2000; Liang & Zhang 2005, 2006), however, suffer from the problem of large data scatter (e.g., Butler et al. 2009; Yu et al. 2009). Also, the probable impact of detector thresholds on cosmological standard candle calibrations (Shahmoradi 2013; Shahmoradi & Nemiroff 2009, 2011) is an issue that has been debated (e.g., Cabrera et al. 2007) and should be taken into account.

The underlying problem of the scatter in the above correlations is that it is larger than the spread expected from the redshift dependence alone. GRBs can be seen over a large fraction of the visible universe, up to \( z = 9.4 \) (Cucchiara et al. 2011). The luminosity spread due exclusively to the luminosity distance squared dependence gives, for the limiting redshifts, a factor of \( D_L^2(9.4)/D_L^2(0.085) = 6.4 \times 10^4 \), while the actual spread in luminosity is on the order of eight orders of magnitude, from \( 10^{46} \text{ erg s}^{-1} \) to \( 10^{54} \text{ erg s}^{-1} \). It is not clear what is responsible for such a large dynamical range. Moreover, the correlations listed above are affected by selection biases, especially at high redshifts where the most luminous events are detected (Malmquist effect). Last, in the application of GRBs to cosmology, possible logical circularity problems should be carefully taken into account and explicitly excluded (e.g., Petrosian et al. 2009).

To overcome this problem, Dainotti et al. (2013b) demonstrated through the Efron and Petrosonian method (Efron & Petrosian 1992) that the Dainotti correlation is not an apparent correlation but is due to the intrinsic properties of GRBs, at a 12σ level, with an intrinsic slope of \( b_{\text{intrinsic}} = -1.07^{+0.09}_{-0.14} \) (see Equation (1)). This is a very relevant finding since the Dainotti et al. correlation in Equation (1) not only is a useful tool for the testing of theoretical GRB physical models (e.g., Cannizzo & Gehrels 2009; Cannizzo et al. 2011; Dall’Oso et al. 2011; Bernardini et al. 2012; Rowlinson et al. 2010; Yamazaki 2009; Ghisellini et al. 2008) but also has been applied to cosmology in, e.g., Cardone et al. (2009, 2010). Moreover, Dainotti et al. (2013b) identified a caveat of the use of nonintrinsic correlations to constraining cosmological parameters by showing how systematics can lead to errors in the evaluation of the cosmological parameters (Dainotti et al. 2013a).

This correlation can be stated as

\[
\lg L_X = \lg a + b \lg T_\alpha^*,
\]

The normalization \( a \) and the slope parameter \( b \) are constants obtained through the D’Agostini fitting method (D’Agostini 2005). We determine the power-law slope \( b \) in Equation (1) after properly correcting for possible data selection due to instrumental threshold effects. This step is necessary to make the Dainotti correlation a useful distance estimator and an effective cosmological tool. Notice crucially that the correlations present in the literature listed above have \( E_{iso} \) as one variable, and because of that they suffer from a double truncation due to detection selection thresholds (Lloyd & Petrosian 1999). Thus, these correlations bring with them a dual-selection bias problem at both low energy and high energy. In the Dainotti et al. correlation, by involving time, which does not depend on the detector threshold, and depending on the luminosity of the plateau \( L_X \), the only problem is the detection threshold for faint plateau cases. Thus, selection biases are significantly reduced compared to other correlations.

Regarding the sample analyzed, it is constituted by all GRB X-ray afterglows with known redshifts detected by \textit{Swift} from 2005 January up to 2011 May, for which the light curves include early X-ray Telescope data and therefore can be fitted using Willingale’s phenomenological model (Willingale et al. 2007). We used the redshifts available in the literature (Xiao & Schaefer 2009) and at http://www.mpe.mpg.de/~jcq/grbgen.html, after excluding all GRBs with nonspectroscopic redshifts.

In previous papers (Dainotti et al. 2008, 2010; Willingale et al. 2007), the \textit{Swift} Burst Alert Telescope (BAT) + X-Ray Telescope (XRT) light curves of GRBs were fitted with a two-component model assuming that the rise time of the afterglow, \( t_\alpha \), started at the time of the beginning of the decay phase of the prompt emission, \( T_p \), namely, \( t_\alpha = T_p \). Here we search for an independent measure of the above parameters of the afterglow, thus leaving \( t_\alpha \) as a free parameter. In the majority of cases we have \( t_\alpha \geq 0 \). We have created a semiautomatic analyzer with...
the Mathemtica 9 software, which allows a computation of the best-fit parameters.\footnote{ASCII tables with all the quantities needed for the analysis and the Mathematica codes used are available on request.}

The source rest frame luminosity in the \textit{Swift} XRT bandpass, $(E_{\text{min}}, E_{\text{max}}) = (0.3, 10)$ keV, is

\begin{equation}
L_X(E_{\text{min}}, E_{\text{max}}, t) = 4 \pi d(T_a)^2 F_X(E_{\text{min}}, E_{\text{max}}, t) \cdot K,
\end{equation}

where $d(T_a)(z)$ is the GRB luminosity distance for the redshift $z$ within a flat cosmological model with any chosen $\Omega_M$, $w(z)$, and $h$ parameters. $F_X$ is the measured X-ray energy flux in erg cm$^{-2}$ s$^{-1}$ and $K$ is the $K$ correction for cosmic expansion. Details on the fitting procedure and the computation of the power-law spectrum can be found in Evans et al. (2009), Dainotti et al. (2010).

The complete sample of GRBs analyzed contains 101 events, covering the redshift range $0.033 \leq z \leq 9.4$. In our analysis, we take a subsample of GRBs within the SNeIa overlap redshift range of $z < 1.4$, and we use them to derive the best-fit parameters $a$ and $b$ through the D'Agostini method for the $L_X - T_a$ (LT) correlation. We found for this subsample $b_{\text{obs}} = -1.51_{-0.27}^{+0.26}$, however, we are aware from Dainotti et al. (2013a) that the steepening of the slope is an effect of time evolution; therefore, we use in the analysis the intrinsic slope $b$. This intrinsic slope has been evaluated assuming an underlying flat cosmology for the universe. The small number of GRBs in the overlap region prevents us from repeating the Efron & Petrosian (1992, EP) test. However, we also point out that in Dainotti et al. (2013b) we have demonstrated that the calibration of the correlation with the present data set is independent of the underlying cosmology within the error ranges of the parameters themselves. Further, within the overlap region, SNeIa and a host of other independent cosmological tracers robustly fix the low-$z$ cosmological model. This make our assumption of using $b_{\text{int}}$ robust for our purposes. One can then use the same distance luminosity we have obtained from the SNeIa sample with these parameters calculated for $z < 1.4$. Thus, we assume the cosmology inferred from the SNeIa is valid in the low-redshift range, and we then use the GRB correlation to calculate distance luminosities for the high-redshift sample. In this way, the circularity problem is eliminated because no assumptions have been made for the high-redshift universe. As will be shown explicitly in Section 5, we can confirm that the sample selection is not biasing the results obtained, while it still covers an ample redshift range of $0.49 \leq z \leq 8.2$.

3. SEQUENTIAL BAYESIAN ANALYSIS AND COSMOLOGICAL ASSUMPTIONS

In this section we outline the basic probabilistic and cosmological assumptions of our work. Bayes’ theorem (Bayes 1763), an application of the formula of conditional probability to data sets as arising from a given underling model, can be formally written in the following way:

\begin{equation}
P(H_j|D) = P(D|H_j)P(H_j)/P(D),
\end{equation}

where $H_j$ is the hypothesis (model) to be tested against a given data set $D$ and $P(H_j)$ is the probability of the hypothesis being valid, in the absence of data, any prior one might want to introduce. $P(D|H_j)$ is the probability of obtaining the data set $D$ assuming $H_j$ to be valid; the likelihood function, $P(D|H_j)$, is the probability of the validity of $H_j$ given the occurrence of the data $D$ (the posterior); and $P(D) = \sum H_j P(D|H_j)P(H_j)$ is the relevant normalization. The key in Bayesian analysis is the introduction of an objective and insightful quantitative assignment of probability in order to take maximum advantage of all the information available in observations, mainly central values and confidence intervals of measurements. Our likelihood assignment will be explained in the following section.

If more statistically independent data become available, the theorem can be applied sequentially to adjust the probability of the hypothesis. A sequential formulation has the advantage of being an adaptive approach in analyzing data, consequentially optimizing computational time. The approach is highly flexible with respect to handling new data, which enter as refinements on the previous solution, with no need to redo the full analysis.

The sequential form of Bayes’ theorem becomes

\begin{equation}
P(H_j[D_1, D_{-1}, ...]) \propto P([D_1, D_{-1}, ...]|H_j)P(H_j|D_{-1}, D_{-2}, ...)),
\end{equation}

all new data correct and evolve the probability of the hypothesis being valid, with previous results effectively becoming a prior on the likelihood assignment associated with the new extended data sample. The theorem can thus be thought of as a quantitative formulation of Occam’s razor.

In this work, the hypothesis $H_j$ refers to a particular $w(z)$ function proposed, together with the choice of two cosmological parameters: the present Hubble constant $H_0$ and the present DE density parameter $\Omega_{DE}$. We retain the assumption of isotropy for the cosmological model, impose reasonable bounds on the DE EOS (see Figure 1), and assume also a constant value for $w(z)$ for the very low $z < 0.01$ redshift range. We shall be considering a very large number of very general randomly chosen $w(z)$ models restricted only to be continuous in value, and in their first derivative, no violent or rapid transitions are considered.

To begin, we take no initial priors, i.e., each random EOS is assumed to be equally probable with respect to the others:

\begin{equation}
P(H_j) = 1/N_{Hj},
\end{equation}

where $N_{Hj}$ is the total number of generated curves. Each curve is labeled by an index $n$ and receives initial likelihood $P(D_1|H_{ym})$ calculated from the analysis of the first set of data with

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Examples of randomly generated equations of state inside the chosen boundaries (dashed lines). Dotted lines represent curves generated by choosing a random initial point, slope, and line segment length. The solid line is the result of applying the Gaussian smoothing filter from Equation (15) with $\ell = 4$.}
\end{figure}
its errors included (represented by $D_i$). As more statistically independent data sets are added to the method (represented by $D_i$), the analysis is repeated for new data by the sequential multiplication prescribed in Equation (4). As the number of data sets included increases, the recovered $w(z)$ curves with higher relative probabilities will begin to narrow into a band about the optimal $w(z)$. This approach allows us to consider independent data sets and to check the different restrictions each imposes onto the best recovered $w(z)$ model and its confidence bands, as they are sequentially added to the full data considered: SNeIa samples having a low redshift coverage, BAO restrictions with no redshift resolution but high local constraining power, and the high-redshift GRB sample.

We introduce the standard assumptions of spatial flatness, a negligible relativistic component, and pressureless matter until the end in order to have a framework of a very general applicability ready. In the case of a redshift-independent effective EOS, the dimensionless luminosity distance $D_L(z)$ gives the usual luminosity distance $d_L(z)$ and magnitude $m$, as inferred from an observed photon flux,

$$d_L(z) = (c/H_0)(1 + z)D_L(z),$$

$$\mu(z) = 5 \log(d_L(z)) + 25,$$

where $d_L$ has units of Mpc, $H_0$ is the present Hubble constant, and function $D_L(z)$ is the dimensionless luminosity distance.

At this point it is necessary to note that in addition to specifying the relevant equations of state, the full cosmological model also depends on four parameters. In the general case of three components (matter, radiation, and DE), these parameters are the present Hubble constant $H_0$ and the present-day total, radiation, and DE density parameters, $\Omega_{\text{tot}}$, $\Omega_r$, and $\Omega_\Lambda$, respectively. These are, in principle, independent of any given DE EOS. $D_L$ is affected by the $\Omega$’s, while $H_0$ enters $\mu$ to give the dimensions of distance in Equation (7). Therefore, for every set of $\Omega$’s one requires a new solution to the differential equation, while different values of $H_0$ will just rescale the distance modulus.

We introduce the DE EOS as

$$w_\Lambda \equiv \frac{p_\Lambda}{\epsilon_\Lambda},$$

where $\epsilon_\Lambda = \Lambda/(8\pi G N)$. Considering matter to be pressureless ($p_m = 0$) allows to express the effective $\tilde{w}$ in terms of $w_\Lambda$ as

$$\tilde{w}(x) = \frac{p_m + p_\Lambda + p_r}{\epsilon} = w_\Lambda(x) + \frac{1}{3} \frac{\Omega_r(x)}{\Omega_\Lambda(x)} + w_\Lambda(x)$$

$$+ \left(1 - 3 w_\Lambda \Omega_r(x) - w_\Lambda (\Omega_{\text{tot}} - \Omega_\Lambda - \Omega_r) x^3\right)$$

$$\frac{f(x) + (\Omega_{\text{tot}} - 1) x^2}{f(x) + (\Omega_{\text{tot}} - 1) x^2},$$

where $x = 1 + z$ and $f(x) = (H(z)/H_0)^2$. For the case of a constant EOS $w_\Lambda(x) = w_\Lambda$, and $\Omega_\text{tot} = \Omega_\Lambda - \Omega_\Lambda - \Omega_r$.

We shall now neglect the energy contribution from radiation ($\Omega_r = 0$), valid for the present expansion epoch, and split the cosmological fluid into two components: pressureless matter ($w_m = 0$) and DE ($\Lambda$). Thus,

$$\Omega_{\text{tot}} = \Omega_\Lambda + \Omega_m$$

and therefore, Equation (9) becomes

$$\tilde{w}(x) = \frac{0 + p_m}{\epsilon} = w_m(x) \frac{\Omega_\Lambda(x)}{\Omega_{\text{tot}}(x)}$$

$$= w_m(x) \left(1 - \frac{(\Omega_{\text{tot}} - \Omega_\Lambda x^3)}{f(x) + (\Omega_{\text{tot}} - 1) x^2}\right).$$

We now introduce the assumption of a flat cosmology, namely justified by observations of the CMB anisotropy and power spectrum inferences from luminous red galaxies in the Sloan Digital Sky Survey (e.g., Tegmark et al. 2006), giving $\Omega_\text{tot} = 1.0$. With the above, Equation (11) simplifies to

$$\tilde{w}(x) = w_m(x) \left(1 - \frac{(\Omega_\Lambda x^3)}{f_{\text{flat}}(x)}\right),$$

where $f_{\text{flat}}(x)$ is the solution to the evolution equation $f_{\text{flat}}(x) x = 3 (1 + \tilde{w}(x)) f_{\text{flat}}(x)$.

Therefore, in the particular case of a flat universe we have only two scalar parameters to vary, $\Omega_\Lambda$ and $H_0$, in addition to the DE EOS, the function $w(z)$.

In general, the EOS $p_\epsilon$ adiabatic parameter $w$ is defined as the ratio of the pressure $p$ to the energy density $\epsilon$ of any given component,

$$w \equiv \frac{p_\epsilon}{\epsilon}.$$ 

Given that the only unknown EOS remaining in the model is that of the dark matter component, from this point onward we sometimes drop the subscript from the DE EOS.

To generate a set of random $w(z)$ functions, we begin with randomly choosing an initial point $w_0 = w(z = 0)$. As already mentioned, any EOS will be assumed to be constant at $w(z) = w_0$ for $z < 0.01$. Next, we discretize the EOS space in the range of interest, $-2 \leq \lg(z) \leq 0.2$, working in a $w$, $\lg(z)$ plane. Note the use of $\lg$ for base 10 logarithms and of $\ln$ for natural logarithms. In this plane we generate curves through linear segments with random slope $a_s$, and random length at every step $s$,

$$w_s(z) = w_{s-1}(z) + a_s (\lg(z) - \lg(z_s)),$$

where $z_s \leq z < z_{s+1}$. A class EOSs constant throughout the full redshift range of data is also trivially generated and added to the full set of random walk curves treated.

The curves are then smoothed through the use of a Gaussian filter. The coefficients of the filter are chosen to be

$$c_j = e^{-\frac{(j - \frac{l}{2})^2}{\sigma^2}} \sum_{j=-l}^{l} c_j,$$

where $i = -l, -l + 1, \ldots, l$, with the sum a normalization. Hence, smoothing by the filter function $f(x)$ is done using $2l + 1$ neighboring points $x_j$ at every point $x_j$.

$$f_s(x_j) = \sum_{i=-l}^{l} c_i f(x_{j+i}).$$

We choose 9 neighbor points out of 10 per segment, $l = 4$ for our purpose. Examples of a few randomly generated EOS are shown in Figure 1.
4. BAYESIAN $w(z)$ INFERENCEs FROM SNeIa, BAO, AND GRB CONSTRAINTS AND TESTS OF THE METHOD

To begin, we consider data from a recent sample of SNeIa "standard candles." We take $N_{\text{SN}} = 580$ SN events from the Union 2.1 compendium (Suzuki et al. 2012). As a first test, we treat our $N_{\text{GRB}} = 54$ GRBs assuming the proposed correlation and, only here, also a standard flat $w = -1$ cosmology (e.g., Capozziello et al. 2012a, 2012b; Izzo et al. 2009) to obtain a first GRB $\mu, z$ catalog. Luminosity distance moduli $\mu$ are defined in Equation (7). Redshifts and $\mu$ values for both samples are shown in Figure 2, where we see that despite the larger error bars associated with the GRB sample, no discontinuity is evident between the two data sets. Using the correlation coefficients inferred for the SNeIa overlap region, where both data samples are clearly compatible, has yielded a consistent Hubble plot extending out to $z = 8.2$. Notice the close to one order of magnitude extension in redshift range provided by the GRB sample, which in turn permits the possibility of tracing the cosmological model out to the extremely large redshift of 8.2. Another observational constraint comes from BAO measurements, essentially a "standard ruler" at redshift $z_{\text{BAO}} = 0.35$, characterized by the dimensionless parameter $A$ (Tegmark et al. 2006; Adak et al. 2011):

$$A = \sqrt{\Omega_{m0}} \left( \frac{D_L(z_{\text{BAO}})}{z_{\text{BAO}}} \right)^{1/3}, \quad (17)$$

where $\Omega_{m0}$ is value of the present-day matter density parameter and

$$A = 0.469 \pm 0.017. \quad (18)$$

Thus, $A_{\text{BAO}} = 0.469$ and $\Delta A_{\text{BAO}} = 0.017$ represent the central value and a 1σ confidence interval on the BAO observations. Data points for the SNeIa sample are magnitudes $\mu_j$ with error bars $\Delta \mu_j$ at redshift $z_j$ (indexed by $j = 1, \ldots, N_D$), with $N_D = N_{\text{SN}}$ being the total number of available events.

To every proposed model (identified by the index $n$), i.e., a set of $(H_0, \Omega_{\Lambda 0})$ values and a corresponding generated $w(z)$ curve within the redshift interval chosen, a probability related to $\chi^2$ is assigned as a measure of how well each model represents the various data sets, the SNeIa, and the BAO constraint. Assuming each data set to be statistically independent, this $\chi^2$ takes the form

$$\chi^2_n = \chi^2_{n, \text{SN}} + \chi^2_{n, \text{BAO}}, \quad (19)$$

where we define

$$\chi^2_{n, \text{SN}} = \frac{1}{N_D} \sum_{j=1}^{N_D} (\mu_{j,n} - \mu_{j}(z_j))^2/\Delta \mu_j^2, \quad (20)$$

and

$$\chi^2_{n, \text{BAO}} = (A_{\text{BAO}} - A_{n,0})^2/\Delta A_{\text{BAO}}^2. \quad (21)$$

Note that although the BAO observation has the same statistical weight as one SN, the fact that this is very localized in redshift space and the tight confidence interval for $\Delta A_{\text{BAO}} = 0.017$ make it a valuable constraint, effectively narrowing substantially the range of $\Omega_{\Lambda 0}$ and allowed $w(z)$ solutions at the BAO redshift.

Since $\mu$ scales with $H_0$ in the logarithm, a further simplification is possible. Let us define

$$\mu_H = 5 \log(c/H_0), \quad (22)$$

and introduce the following parameters:

$$\alpha = \sum_{j} \frac{\mu_{j}}{\Delta \mu_{j}}, \quad (24)$$

$$\beta_n = \sum_{j} \frac{\tilde{\mu}_{n,j}(z_j)}{\Delta \mu_{j}}, \quad (25)$$

$$\gamma = \sum_{j} \frac{1}{\Delta \mu_{j}}. \quad (26)$$

We use them to define

$$\mu_{D,n} = \frac{\alpha - \beta_n}{\gamma}, \quad (27)$$

$$\Delta \mu = 1/\sqrt{\gamma}; \quad (28)$$

therefore, we can split as

$$\chi^2_{n, \text{SN}} = \chi^2_{n, \Omega} + \chi^2_{n, H}/N_D, \quad (29)$$

where

$$\chi^2_{n, H} = \frac{(\mu_H - \mu_{D,n})^2}{\Delta \mu^2}, \quad (30)$$

and

$$\chi^2_{n, \Omega} = \frac{1}{N_D} \sum_{j} \left( \frac{\tilde{\mu}_{n,j}(z_j) - \mu_{j}(z_j)}{\Delta \mu_j^2} - \mu_{D,n}^2/N_D\Delta \mu^2 \right)^2. \quad (31)$$

Now

$$H_0 = c 10^{-\mu_{D,0}/5}, \quad (32)$$

gives the value of $H_0$, which minimizes $\chi^2$ for a given $w(z)$ and $\Omega_{\Lambda 0}$.

Although complex probability density functions could be introduced at this point to take advantage of the Bayesian approach, for this introductory work (and lacking any evidence to the contrary), every data point $j$ (with measured redshift $z_j$)
is treated as a normal probability density function \( f_j \) with mean value \( y_j \) and standard deviation \( \Delta y_j \), taken from observations,

\[
f_j(y) = \frac{e^{-(y-y_j)^2/(2\Delta y_j)^2}}{\sqrt{2\pi\Delta y_j}}, \tag{33}
\]

therefore, Bayesian analysis turns into maximum likelihood with minimization of \( \chi^2 \). A particular full cosmological model will now yield a fixed curve \( \mu_n(z; H_0, \Omega_{\Lambda0}, w(z)) \), with a likelihood assignment at redshift \( z_j \) given by

\[
p(D_j|H_{yn}) = f_j(\mu_n(z_j; H_0, \Omega_{\Lambda0}, w(z))), \tag{34}
\]

where \( p \) is thus the relative probability of this model with respect to another one. Sequential application on all data points of Equation (4) results in the total likelihood given by the product

\[
p(SN|H_{yn}) = \prod_j f_j(\mu_n(z_j; H_0, \Omega_{\Lambda0})), \tag{35}
\]

and for BAO

\[
p(\text{BAO}|H_{yn}) = \frac{e^{-(A_{\text{BAO}}-A_0)^2/2(\Delta A_{\text{BAO}})^2}}{\sqrt{2\pi}\Delta A_{\text{BAO}}}. \tag{36}
\]

We next apply Equations (35) and (36) to all generated curves. The result is the hypothesis \( H_y \) (a \( w(z) \) curve with particular \( H_0 \) and \( \Omega_{\Lambda0} \) parameters values) acquires the probability

\[
p(H_{yn}|D) \propto p(\text{BAO}|H_{yn})p(SN|H_{yn}) \propto e^{-\chi^2_{\text{BAO}} - \chi^2_{\text{SN}}} e^{-N_0 \chi^2_{\text{SN}}} \tag{37}
\]

Notice that for the particular case of a normal error distribution function, the dependence of the probability on the Hubble constant can be calculated analytically, saving computation time.

Next, we choose the curve with maximum probability, \( p_{\text{max}} \), and order the rest in descending order by value of relative probability \( p \to p/p_{\text{max}} \), which is also used for scaling in gray (black corresponds to the maximum). This helps visualize the regions through which the curves with the highest probability go and therefore where the real DE EOS is most likely located.

The size and shape of the region depends on the number of data points, their quality (errors), and their distribution (in redshift bins).

Finally, we must derive confidence intervals around the optimal solution; for this we shall construct a band of EOSs encompassing the most likely curves and bounding the variations allowed by the data. We begin by taking the maximum probability EOS \( w_{\text{max}}(z) \) and constructing boundary curves that accumulate 68.2% of the total probability \( P_t = \sum_n p_n \); following in the sum a descending order in \( p_n \), we have constructed

\[
w_+(z) = \max(w_n(z)) \sum_{\text{ordered}} p_n / P_t \leq 0.682, \tag{38}
\]

\[
w_-(z) = \min(w_n(z)) \sum_{\text{ordered}} p_n / P_t \leq 0.682.
\]

To construct a relative probability map for parameters \( H_0 \) and \( \Omega_{\Lambda0} \), we marginalize as

\[
p(H_0, \Omega_{\Lambda0}) = \sum_n p(H_0, \Omega_{\Lambda0}, EOS_n). \tag{39}
\]

We locate the maximum probability model as well as the maximum point \( (H_0, \Omega_{\Lambda0})_{\text{max}} \) after marginalization. In order to estimate the \( 1\sigma \) and \( 2\sigma \) contours we locate the contours where probability accumulates 68% and 95% of the total, respectively.

Since every model of a particular \( w(z) \) includes a set of initial cosmological parameters, we marginalize over them to get the probability band for EOS only:

\[
p(EOS_n) = \sum_{\Omega_{\Lambda0}} \int_{H_0} p(H_0, \Omega_{\Lambda0}, EOS_n). \tag{40}
\]

The code begins by generating 5000 random \( w(z) \) curves as described in Section 3, together with 2000 constant \( w \) curves with values constrained to the range \( -4 \leq w \leq 2 \), including one curve for a classical cosmological constant EOS, \( w(z) = -1 \), resulting in 7000 curves in total. The length of every line segment is randomly distributed between 0.1 and 0.5, and the angle of every segment is randomly chosen between \(-89^\circ\) and \(+89^\circ\), in the \( \ln(g(z)) \) versus \( w \) plane.

The two scalar parameters of the flat universe run through 50 discrete values in the range \( \Omega_{\Lambda0} = 0, \ldots, 1 \) and 50 in \( \Omega_{\Lambda0} = 0.5, \ldots, 0.9 \). To avoid large quantities of low-probability cases, we prune the initial random EOSs to those within \( N_\sigma = 3 \), estimating \( n\sigma \) cuts through

\[
\ln(p) > \ln(p_{\text{max}}) - N_\sigma^2/2, \tag{41}
\]

on the basis of a correspondence with the maximum likelihood method.

In this section we shall perform a number of tests designed to assess the robustness of the methodology presented, first, in a case where the answer is actually known, i.e., through the use of synthetic data samples constructed using a particular cosmological model, an input \( w(z) \) assumption. In this way, the results and their corresponding confidence intervals can be compared to the input model for a variety of controlled possibilities.

Redshifts and normal error amplitudes are taken from the original SNeIa data set. The results of the test are shown in Figure 3. We see the recovered EOS, given by the white curves, is always very close to the input value, shown by the thin white lines. The region between the dashed curves gives the estimated \( 1\sigma \) (cumulative 68%) confidence band. We see that even though the method intrinsically samples nonlinear \( w(z) \), the data drive the solution to a linear value, coinciding with the input one. Notice also that the final inferred \( H_0 \) and \( \Omega_{\Lambda0} \) values appear very close to the input ones (see bottom row in Figure 3).

As will be seen from Figure 4, once correlation coefficients are calculated from the overlap region (33 GRBs) with SNeIa and then used to get the luminosity distance for higher \( z \) (68 GRBs), they are quite scattered and hard to fit well (\( \chi^2 \) is large) with a randomly generated smooth
Figure 3. Reconstruction of the test EOS using simulated SNeIa and BAO data. Three cases are shown by the straight white lines: rising, falling, and mixed (from left to right). Corresponding parameter maps are shown below. Crossing lines are input values, squares give the maximum probability cases, and solid dots show the maximum after marginalization.

Figure 4. Distance ladder. GRBs in the SNeIa overlap redshift range, where cosmology is well constrained, are used to calculate the GRB intrinsic correlation coefficients. This correlation is then used to calculate the luminosity distance for high-redshift GRBs from their X-ray afterglow luminosity curves. Standard constant $w$ solutions are shown for reference. A vertical dashed line marks the farthest SNIa event. The inset to the right shows a histogram of our GRB sample distribution in redshift. The inset to the left shows the resulting most probable EOS, together with a small sample of models probed; confidence intervals are so large that only extreme variations with respect to $w = -1$ can be excluded.

$w(z)$ curve (see the inset). Also, a large number of GRBs are detected in $z < 4$, as the other inset with the histogram shows. Using the correlation to calculate the luminosity distance of GRBs beyond SNeIa introduces a correlation between the data points of the different GRB events. Once a correlation matrix $C_{ij}$ is known, Equation (20) has to be modified into

$$\chi^2_{\nu,GRB} = \frac{1}{N_D} \sum_{i,j} N_D \left( \mu_i - \mu_n(z_i) \right) C_{ij} \left( \mu_j - \mu_n(z_j) \right),$$

(42)

where now $N_D = N_{GRB} = 68$. What one can do is impose $\mu$ to lie in the standard case of $w = -1$ and study the constraining power of the correlated errors as well as the effect of data distribution in the redshift bins.

5. RESULTS

To begin, we analyze only 2000 constant EOSs evenly spaced in the range $-4 \leq w_\Lambda \leq 2$. Starting with only the SNeIa data sample, we will study the effect of adding the BAO constraint later. Results are shown in the left column of Figure 5. We see a very well defined solution consistent with a cosmological constant and a narrow confidence interval, $w = -0.99 \pm 0.2$. Adding the BAO constraint (right panel) does not change the central result but considerably reduces the confidence interval, resulting now in $w = -0.99 \pm 0.06$. The inferred scalar parameters appear in the bottom row, where, again, we see the inclusion of the BAO constraint significantly tightening the confidence region, particularly in the case of the inferred DE density parameter, yielding the very precise estimate of $\Omega_{\Lambda 0} = 0.723 \pm 0.025$. In the bottom row we see, in general, the very high resolution with which the method infers the present-day cosmological parameters $H_0$ and $\Omega_{\Lambda 0}$ once all the data are taken into account.

If we now allow the data to select the $w(z)$ model that best reproduces the luminosity distance and redshifts of the SNeIa sample and the observed BAO constraint, we obtain results shown in Figures 6 and 7 for the inferred $w(z)$ DE EOS. Every EOS is shaded in gray according to its relative probability $p/p_{\text{max}}$, where 1 is black and 0 is white. It is seen that using only SNeIa data, at higher redshift ($\lg(z) > 0$), curves explore a wide range of possibilities, not restricted by any data. The further addition of the BAO constraint significantly tightens the confidence band of the allowed $w(z)$ curves. Our principal result is now given in terms of the maximum probability DE $w(z)$ and the corresponding confidence interval bands covered by the distribution of lower-probability equations of state. We see our inference lying extremely close to the classical cosmological constant of $w(z) = -1$ once the BAO constraint is included. In Figures 6 and 7, the maximum probability curves are given by the white dashed line, while a representative sample of the other $w(z)$ curves explored is given in shades of gray.
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Figure 5. EOSs shaded according to relative probability after applying our Bayesian analysis to the class of constant curves with only SNeIa data (left column) and with the addition of the BAO constraint (right column). The white central lines are the maximum probability \( w_{\text{max}}(z) \), whereas the dashed lines give the confidence intervals (cumulative 68%). Vertical dashed lines mark the data range. The bottom row gives corresponding probability maps for the scalar cosmological parameters of the fit. White dots mark the marginalized maxima, and crossed lines show the maximum probability cases.

Figure 6. Resulting relative probability (shaded in gray) for \( w(z) \) when only SNeIa data are analyzed. Marginalization of Equation (40) is applied. Maximum probability (white) and cumulative 68% boundaries (striped) are shown. Vertical dashed lines show the SNeIa data range.

Figure 7. Same as Figure 6, but after the BAO constraint is included in the analysis.

The corresponding scalar parameter \( H_0 \) and \( \Omega_{\Lambda 0} \) inferences for the method applied to the SNeIa data sample and the BAO constraint, allowing for free \( w(z) \) variations, are shown in Figures 8 and 9.

The BAO constraint was sequentially added to the data set considered. In all cases above, the two contours give 1\( \sigma \) and 2\( \sigma \) confidence regions. Notice that in all our results, once the data samples are considered, in terms of our inferred \( w(z) \) and \( H_0 \) and \( \Omega_{\Lambda 0} \) values, probability distributions peak far from the borders of the parameter region sampled; in all cases our confidence intervals are closed, showing that the inferred model is not only the best fit to the data among the many models sampled but also
Having said that, there is much expectation for the 12 curves always yield ages above 13 Gyr and hence are fully consistent with this extra condition. Figure 10, left panel) distribution for the different EOSs tested. A very flat probability (seen as the uniform black shading in Figure 9, right panel) distribution and error bars as the real data but sampled directly from a $w = -1$ universe (left panel). We see only very extreme $w(z)$ variations are excluded, and within the confidence region, there is an almost flat (constant color) likelihood surface. We can explore what additional constraining power a better future sample might yield by repeating the experiment and reducing the errors by a factor of four (right panel). Even without considering any expected increase in the number of GRB events, this would permit much more interesting high-redshift DE constraints. We note Dainotti et al. (2010) showed that such a decrease in the errors of GRB data is certainly feasible.

There are several reasons for the low $w(z)$ constraining power of high-redshift events: the expansion of the universe becomes dominated by matter, and hence, the sensitivity of distance luminosity to the dark component is quite small. Also, the low number of events in the SNeIa overlap region implies large error bars on the inferred GRB correlation coefficients, while the large error bars, despite their correlations, for high-$z$ GRBs produce a very flat probability (seen as the uniform black shading in Figure 10, left panel) distribution for the different EOSs tested. Having said that, there is much expectation for the $1 < z < 4$ region once the GRB sample is increased and its quality is improved.

We next check the look-back time for the different cosmological models tested, which is consistent with a comparison to a lower age limit coming from GC ages of 13 Gyr (Mackey & Gilmore 2003a, 2003b, 2003c). The maximum probability curves always yield ages above 13 Gyr and hence are fully consistent with this extra condition.

Finally, we test the robustness of our results to the details of the $w(z)$ sampling method used. We now implement an independent curve-generating algorithm based on a Markov chain formalism. Similar to what was done previously, a chain DE EOS is formed from piecewise curves with end points chosen at redshifts $\lg (z_0) = -2$ and $\lg (z_{\text{nodes}+1}) = 1$, with $N_{\text{nodes}}$ in between.

At each step we slightly modify the curve by randomly choosing a node (including the end points) and randomly moving it in the EOS plane, where the end points are only allowed to move vertically, stepping in $w_\Lambda$. The resulting curve is then smoothed using the same Gaussian filter used previously in both the $\lg (z)$ and $w_\Lambda$ directions; a sample of such curves is shown in Figure 11. These final curves are then passed to the same likelihood-calculating procedure described previously, using both the SNeIa data and the BAO constraint. To these we add a synthetic sample of high-redshift GRBs that have the same redshift distribution and errors as the real data and are generated for a $w(z) = -1$ universe in order to assess the constraining power of our current GRB sample.

If the Bayesian probability of the modified curve is increased relative to the previous one, then this new curve is picked for new adjustments at the next step; otherwise, a random number between 0 and 1 is selected, with the new EOS rejected if the following condition holds:

$$r_{\text{dec}[0...1]} > p_{\text{modified}} / p_{\text{initial}}.$$  \hspace{1cm} (43)

where $p$ is the Bayesian probability of the corresponding curve.

In this way we explore the EOS space close to the most probable cases and spend less time on less probable ones. After a significant number of steps, the result is independent of the particular EOS curve chosen to begin with.

We have run the alternative Markov chain method with over 10,000 steps for $N_{\text{nodes}} = 1, 3,$ and 10, and results are shown in Figures 12 and 13. For one node we also tested for convergence using different initial conditions, which indeed were seen to quickly converge to consistent solutions. Increasing the number of nodes naturally increases the sensitivity of the method to redshift variations in $w(z)$ but also results in broader confidence intervals. Within the low redshift range, the final maximum probability EOSs and their confidence bands are very similar to what was obtained previously for each of the three cases considered in terms of the sequential data sets introduced and remain consistent with the cosmological constant case of $w_\Lambda = -1$. The same applies to the probability contours in the $H_0, \Omega_M$ plane, hence proving the method to be fully robust with respect to the random $w(z)$ generation procedure and the
subsequent exploration of the EOS plane; it is the data through the full Bayesian probability assignment that drive the inference. Regarding the high-redshift $w(z)$ constraints, we see the results as being consistent with the input $w = -1$ model but having associated confidence intervals that flare beyond the $z = 1.4$ region.

6. CONCLUSIONS

We have developed and tested a sequential Bayesian analysis method tuned to the nonparametric inference of the evolution of the DE EOS.

Complementary SNeIa samples and BAO constraints with new GRB cosmological distance estimators will allow an order of magnitude extension in the redshift range over which cosmological and DE physics can be traced once error bars can be somewhat reduced, and the sample extended. We obtain results consistent with a cosmological constant $\Lambda$CDM model, i.e., $w(z) = -1$ not requiring any redshift evolution out to $z = 1.4$, although the confidence intervals obtained do allow for small variations of ±0.3 at most throughout the redshift range sampled.

Construction of a distance ladder and use of the full correlation matrix has been performed on a subclass of GRBs. Analysis of our GRB sample and extrapolation of the local $w(z)$ EOS to high redshifts emphasizes the need to increase the number of data points (at least for $z < 4$ and in the overlap region with SNeIa data), and to improve their quality, it is sufficient to reduce the error bars by a factor of four. Let us note here that such a subsample with low error bars is indeed realistic. In fact, in Dainotti et al. (2010), we demonstrated that the error bars of the GRB observables can be reduced by factors of 10 and more.

The method simultaneously yields optimal maximum likelihood inferences for the scalar parameters of the cosmological model, considering the most general case of possible $w(z)$: $H_0 = 70.04 \pm 1 \text{ km s}^{-1} \text{ Mpc}^{-1}$ and $\Omega_\Lambda = 0.724 \pm 0.03$.

This result has a twofold interpretation: on the one hand, our approach cannot select among similar concurrent DE models at very high redshift. This could mean that we need further indicators and a wider redshift sample. On the other hand, this result could be read as the first indication of the fact that
Figure 13. Recovered parameters using the Markov chain method with cumulative 68.2% (solid line) and 95.4% (dashed line) contours. Three cases are shown: $N_{node} = 1, 3,$ and 10. Circles mark the locations of the maximum probability.

The cosmological constant problem persists also at very high redshift. At a fundamental level, this feature would have severe consequences because of the difficulty in connecting the very tiny value of the observed cosmological constant to the vacuum state of the gravitational field at cosmological scales.

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