RANK ONE SHEAVES OVER QUATERNION ALGEBRAS ON
ENRIQUES SURFACES

FABIAN REEDE

Abstract. Let \( X \) be an Enriques surface over the field of complex numbers. We prove that there exists a nontrivial quaternion algebra \( A \) on \( X \). Then we study the moduli scheme of torsion free \( A \)-modules of rank one. Finally we prove that this moduli scheme is an étale double cover of a Lagrangian subscheme in the corresponding moduli scheme on the associated covering K3 surface.

Introduction

A noncommutative variety is a pair \((X, A)\) consisting of a classical complex algebraic variety \( X \) and a sheaf of noncommutative \( \mathcal{O}_X \)-algebras \( A \) of finite rank as an \( \mathcal{O}_X \)-module.

The algebras of interest in this article are Azumaya algebras. These are algebras locally isomorphic to a matrix algebra \( M_r(\mathcal{O}_X) \) with respect to the étale topology. Especially interesting are the first nontrivial examples for \( r = 2 \), the so called quaternion algebras, Azumaya algebras of rank four. These are generalizations of the classical quaternions \( \mathbb{H} \).

Since the generic stalk of a quaternion algebra \( A \) is a central division ring over the function field of \( X \), locally projective left \( A \)-modules which are generically of rank one can be understood as line bundles on \((X, A)\). By \[9\] there is a quasi-projective moduli scheme for these line bundles, a noncommutative Picard scheme, which can be compactified to a projective moduli scheme \( \mathcal{M}_{A/X} \) by adding torsion free \( A \)-modules generically of rank one.

We study in detail the situation of Enriques surfaces. We prove that every Enriques surface \( X \) gives rise to a noncommutative Enriques surface \((X, A)\) with a quaternion algebra \( A \) on \( X \). The main results of this article can be summarized as follows

**Theorem.** Let \( X \) be an Enriques surfaces, then there is a quaternion algebra \( A \) on \( X \) representing the nontrivial class in \( Br(X) \). If \( X \) is very general then

i) The moduli scheme \( \mathcal{M}_{A/X} \) of torsion free \( A \)-modules of rank one is smooth.

ii) Every torsion free \( A \)-module of rank one can be deformed into a locally projective \( A \)-module, i.e. the locus \( \mathcal{M}_{A/X}^{lp} \) of locally projective \( A \)-modules is dense in \( \mathcal{M}_{A/X} \).

Let \( \overline{X} \) be the universal covering K3 surface of \( X \) and denote the pullback of the quaternion algebra to \( \overline{X} \) by \( \overline{A} \). For fixed Chern classes \( c_1 \) and \( c_2 \) we have

iii) \( \mathcal{M}_{A/X, c_1, c_2} \) is an étale double cover of a Lagrangian subscheme \( L \subset \mathcal{M}_{\overline{A}/\overline{X}, c_1, c_2} \).

The structure of this paper is as follows. We compare properties of modules over an Azumaya algebra on a smooth projective variety \( W \) to those of the pullbacks to an étale double cover \( \overline{W} \) in section 1. In section 2 we prove that a classical descent result for modules on the double cover is also true in the noncommutative setting. We look at the existence of Azumaya algebras on Enriques surfaces in section 3. In the final section 4 we study moduli schemes of sheaves generically of rank one on a noncommutative Enriques surface. Many of the results in the last section are noncommutative analogues of results found by Kim in [10]. We work over the field of complex numbers \( \mathbb{C} \).

1. Modules over an Azumaya algebra and double coverings

In this section \( W \) denotes a smooth projective complex variety of dimension \( d \) together with a nontrivial 2-torsion line bundle \( L \). By [3] I.17 there is an étale Galois double cover

\[ q : \overline{W} \to W \]
with covering involution $\iota : \overline{W} \to W$ such that

$$q_*\mathcal{O}_{\overline{W}} \cong \mathcal{O}_W \oplus L.$$

**Remark 1.1.** We make the following convention: for every coherent sheaf $E$ on $W$ we write $\overline{E}$ for the pullback to $\overline{W}$ along $q$, that is $\overline{E} := q^*E$.

**Definition 1.2.** A sheaf of $\mathcal{O}_W$-algebras $\mathcal{A}$ is called an Azumaya algebra if it is locally free of finite rank and for every point $w \in W$ the fiber $\mathcal{A}(w)$ is a central simple algebra over the residue field $\mathbb{C}(w)$. Such a sheaf is called a quaternion algebra if $rk(\mathcal{A}) = 4$. Furthermore a coherent $\mathcal{O}_W$-module $E$ is said to be an Azumaya module or an $\mathcal{A}$-module if $E$ is also a left $\mathcal{A}$-module.

Azumaya algebras on $W$ are classified up to similarity by the Brauer group $\text{Br}(W)$ of $W$. We say $\mathcal{A}$ is trivial if there is a locally free $\mathcal{O}_W$-module $P$ with $\mathcal{A} \cong \text{End}_W(P)$ or equivalently $[\mathcal{A}] = 0 \in \text{Br}(W)$. From now on, if not otherwise stated, by an Azumaya algebra $\mathcal{A}$ we mean a nontrivial Azumaya algebra. Furthermore we assume that there is a nontrivial Azumaya algebra $\mathcal{A}$ on $W$ such that $\mathcal{A}$ is nontrivial on $\overline{W}$.

**Lemma 1.3.** Assume $E$ and $F$ are $\mathcal{A}$-modules and $f : Z \to W$ is a flat morphism, then

$$\text{Hom}_{\mathcal{A}}(f^*E, f^*F) \cong f^*\text{Hom}_\mathcal{A}(E, F).$$

**Proof.** First we note that by [8, 0.4.4.6] there is a natural morphism

$$f^*\text{Hom}_\mathcal{A}(E, F) \to \text{Hom}_{f^*\mathcal{A}}(f^*E, f^*F).$$

So after a faithfully flat étale base change we may assume that $\mathcal{A}$ is trivial. Then Morita equivalence for $\mathcal{A} = \text{End}_W(P)$ reduces this problem to the case $\mathcal{A} = \mathcal{O}_W$. Now the lemma follows from [8, 0.6.7.6] since $f$ is flat by assumption.

**Lemma 1.4.** Assume $E$ and $F$ are $\mathcal{A}$-modules, then

$$\text{Hom}_{\mathcal{A}}(E, F) \cong \text{Hom}_\mathcal{A}(E, F) \oplus \text{Hom}_\mathcal{A}(E, F \otimes L).$$

**Proof.** By the previous Lemma 1.3 we have an isomorphism

$$\text{Hom}_{\mathcal{A}}(E, F) \cong \text{Hom}_\mathcal{A}(E, F).$$

This lemma is then a consequence of the following chain of isomorphisms, where the third line uses the projection formula for finite morphisms, [1, Lemma 5.7]:

$$q_*\text{Hom}_{\mathcal{A}}(E, F) \cong q_*q^*\text{Hom}_\mathcal{A}(E, F)$$

$$= q_*q^*\text{Hom}_\mathcal{A}(E, F)$$

$$\cong \text{Hom}_\mathcal{A}(E, F) \otimes q_*\mathcal{O}_{\overline{W}}$$

$$\cong \text{Hom}_\mathcal{A}(E, F) \oplus \text{Hom}_\mathcal{A}(E, F \otimes L).$$

**Corollary 1.5.** Assume $E$ is an $\mathcal{A}$-module. If $\overline{E}$ is a simple $\overline{\mathcal{A}}$-module, then $E$ is a simple $\mathcal{A}$-module and $\text{Hom}_\mathcal{A}(E, E \otimes L) = 0$.

**Proof.** As $\overline{E}$ is a simple $\overline{\mathcal{A}}$-module, we have $\text{End}_{\overline{\mathcal{A}}}(\overline{E}) \cong \mathbb{C}$. Lemma 1.4 gives

$$\text{End}_{\mathcal{A}}(\overline{E}) \cong \text{End}_\mathcal{A}(E) \oplus \text{Hom}_\mathcal{A}(E, E \otimes L)$$

and as $\text{id}_E \in \text{End}_\mathcal{A}(E)$ we find $\text{End}_\mathcal{A}(E) \cong \mathbb{C}$ and $\text{Hom}_\mathcal{A}(E, E \otimes L) = 0$.

**Proposition 1.6.** [9, Proposition 3.5.] Assume $E$ and $F$ are $\mathcal{A}$-modules, then there is the following variant of Serre duality:

$$\text{Ext}^d_{\mathcal{A}}(E, F) \cong \left( \text{Ext}^{d+1}_{\mathcal{A}}(F, E \otimes \omega_W) \right)^\vee.$$

We assume now furthermore that $\dim W = 2$. Denote the $\mathcal{O}_W$-double dual of $E$ by $E^{**}$.

**Lemma 1.7.** Assume $E$ is an $\mathcal{A}$-module which is torsion free as an $\mathcal{O}_W$-module. If $\overline{E}^{**}$ is a simple $\overline{\mathcal{A}}$-module, then

$$\text{Hom}_\mathcal{A}(E, E^{**} \otimes L) = 0.$$
Proof. We first observe that there is an isomorphism
\[ \text{End}_A(E^{**}) \cong \text{Hom}_A(E, E^{**}). \]

To see this, we note that there is an exact sequence of \( A \)-modules
\[ 0 \rightarrow E \rightarrow E^{**} \rightarrow T \rightarrow 0 \]
with \( \dim \text{supp}(T) = 0 \) as \( E \) is torsion free and \( \dim W = 2 \). It is known that \( E^{**} \) is a locally free \( \mathcal{O}_W \)-module, hence a locally projective \( A \)-module. This immediately implies \( \text{Hom}_A(T, E^{**}) = 0 \) since \( T \) is torsion. Furthermore this also shows the local-to-global spectral sequence and the fact that \( T \) is supported in dimension zero. Applying \( \text{Hom}_A(\cdot, E^{**}) \) to (1) and using the vanishing results gives the desired isomorphism.

Using the same argument for \( E \) shows that we also have an isomorphism
\[ \text{End}_A(E^{**}) \cong \text{Hom}_A(E, E^{**}) \]

since \( E^{**} \cong E^{**} \) by [8, 0.6.7.6.].

We can now conclude as follows: by Lemma 1.4 we have
\[ \text{Hom}_A(E, E^{**}) \cong \text{Hom}_A(E, E^{**}) \oplus \text{Hom}_A(E, E^{**} \otimes L). \]

As \( E^{**} \) is simple then by the previous observation and Corollary 1.5 we get
\[ \text{Hom}_A(E, E^{**}) \cong \mathbb{C} \text{ and } \text{Hom}_A(E, E^{**}) \cong \mathbb{C}. \]

\[ \Box \]

2. Noncommutative descent

We use the same notation as in the previous section. We have the étale Galois double cover \( q : \overline{W} \rightarrow W \) with \( \text{Aut}(\overline{W}/W) \) generated by the covering involution \( \iota \):

\[ \begin{array}{ccc}
\overline{W} & \xrightarrow{\iota} & \overline{W} \\
\downarrow q & & \downarrow q \\
W & & W
\end{array} \]

Definition 2.1. We say a coherent sheaf \( F \) of \( \mathcal{O}_{\overline{W}} \)-modules on \( \overline{W} \) descends to \( W \), if there is a coherent sheaf \( E \) of \( \mathcal{O}_W \)-modules on \( W \) together with an isomorphism \( F \cong E \).

Since \( q : \overline{W} \rightarrow W \) is an étale Galois double cover with \( \text{Aut}(\overline{W}/W) = \langle \iota \rangle \cong \mathbb{Z}/2\mathbb{Z} \), the descent condition for a coherent sheaf \( F \) on \( \overline{W} \), see [16, Lemma 0D1V], reduces to the existence of an isomorphism \( \varphi_\iota : F \rightarrow \iota^*F \) such that (using \( \varphi_\iota^2 = \text{id} \)):
\[ \iota^*\varphi_\iota \circ \varphi_\iota = \text{id}. \]

But we have \( \iota^*\varphi_\iota \circ \varphi_\iota : F \rightarrow \iota^*\iota^*F \cong F \). So, for example, if \( F \) is simple, then any isomorphism \( \varphi_\iota \) satisfies \( \iota^*\varphi_\iota \circ \varphi_\iota \in \text{End}_{\overline{W}}(F) = \mathbb{C} \cdot \text{id}_F \). Hence after multiplication with an appropriate scalar, \( \varphi_\iota \) satisfies (2) and \( F \) descends. Summing up:

Proposition 2.2. Assume \( F \) is a simple coherent \( \mathcal{O}_{\overline{W}} \)-module on \( \overline{W} \) together with an isomorphism \( F \cong \iota^*F \), then \( F \) descends to \( W \).

In the rest of this section we want to prove a similar results for \( \overline{A} \)-modules on \( \overline{W} \). For this we need some notation: let \( p : Y \rightarrow W \) be the Brauer-Severi variety of \( A \), see [12] for more information. By functoriality the Brauer-Severi variety \( p : Y \rightarrow W \) of \( \overline{A} \) is given by \( Y = Y \times_W \overline{W} \) and thus \( q : Y \rightarrow Y \) is also an étale Galois double cover with covering involution \( \tau \). All this fits in to the following diagram with both squares cartesian:

\[ \begin{array}{ccc}
\overline{Y} & \xrightarrow{\tau} & \overline{Y} \\
\downarrow \pi & & \downarrow \pi \\
\overline{W} & \xrightarrow{q} & W
\end{array} \]
The Brauer-Severi variety of \(A\) has the property that \(A_Y := p^* A\) is split, more exactly we have

\[
A_Y^{\op} \cong \text{End}_Y(G)
\]

for a locally free sheaf \(G\) on \(Y\), which can be described explicitly, see \cite{12} Remark 1.8.

In the following we will frequently use, without further mention, the fact that a coherent left \(\mathcal{A}\)-module is the same as a coherent right \(\mathcal{A}^{\op}\)-module. Denote these isomorphic categories by \(\text{Coh}_r(W,A)\) and \(\text{Coh}_r(W,A^{\op})\) respectively.

We also define

\[
\text{Coh}(Y,W) = \left\{ E \in \text{Coh}(Y) \mid p^* p_*(E \otimes G^*) \cong E \otimes G^* \right\}.
\]

Then by \cite{12} Lemma 1.10] we have the following equivalences

\[
\phi : \text{Coh}_r(W,A^{\op}) \to \text{Coh}(Y,W), \quad E \mapsto p^* E \otimes \mathcal{A}^{\op}_r G
\]
\[
\psi : \text{Coh}(Y,W) \to \text{Coh}_r(W,\mathcal{A}^{\op}), \quad E \mapsto p_*(E \otimes G^*)
\]

We have similar equivalences \(\overline{\phi}\) and \(\overline{\psi}\) involving \(\overline{\mathcal{A}}^{\op}_Y \cong \text{End}_{\overline{\mathcal{Y}}} (\overline{q}^* G), \overline{Y}\) and \(\overline{W}\).

**Lemma 2.3.** Assume \(F\) is an \(\overline{\mathcal{A}}\)-module, then

\[
\text{End}_{\overline{\mathcal{A}}}(F) \cong \text{End}_{\overline{\mathcal{Y}}} (\overline{\phi}(F)).
\]

**Proof.** Using \(\text{End}_{\overline{\mathcal{A}}}(F) = \text{End}_{\overline{\mathcal{A}}^{\op}}(F)\), the following chain of isomorphisms gives the result:

\[
\text{End}_{\overline{\mathcal{A}}^{\op}}(F) \cong \overline{p}_* \text{End}_{\overline{\mathcal{Y}}} (\overline{p}^* F) \quad \text{by \cite{12} Lemma 1.6]}
\]
\[
\cong \overline{p}_* \text{End}_{\overline{\mathcal{Y}}} (\overline{p}^* \phi(F)) \quad \text{by Lemma 1.3]}
\]
\[
\cong \overline{p}_* \text{End}_{\overline{\mathcal{Y}}} (\overline{\phi}(F)) \quad \text{by Morita equivalence}
\]
\[
\cong \text{End}_{\overline{\mathcal{Y}}} (\overline{\phi}(F)).
\]

\(\square\)

**Lemma 2.4.** Assume \(F\) is a \(\overline{\mathcal{A}}\)-module such that there is an isomorphism \(F \cong \iota^* F\) of \(\overline{\mathcal{A}}\)-modules, then \(\overline{\phi}(F) \cong \overline{\iota^*}(\overline{\phi}(F))\) as \(\overline{\mathcal{Y}}\)-modules.

**Proof.** There are the following isomorphisms:

\[
\overline{\iota^*}(\overline{\phi}(F)) = \overline{\iota^* (\overline{p}^* F \otimes_{\overline{\mathcal{A}^{\op}}} \overline{q}^* G)}
\]
\[
\cong \overline{\iota^* \overline{p}^* F \otimes_{\overline{\mathcal{A}^{\op}}} \overline{q}^* G} \quad \text{by \cite{8} 4.3.3]}
\]
\[
\cong \overline{\iota^* \overline{p}^* F \otimes_{\overline{\mathcal{A}^{\op}}} \overline{q}^* G} \quad \text{by \cite{3]}
\]
\[
\cong \overline{\iota^* \overline{p}^* F \otimes_{\overline{\mathcal{A}^{\op}}} \overline{q}^* G}
\]
\[
= \overline{\phi}(F).
\]

\(\square\)

**Lemma 2.5.** Assume \(F\) is a \(\overline{\mathcal{A}}\)-module such that there is \(M \in \text{Coh}(Y)\) with \(\overline{\phi}(F) \cong \overline{\iota^*} M\), then \(M \in \text{Coh}(Y,W)\).

**Proof.** We have to prove that the canonical morphism

\[
p^* p_*(M \otimes G^*) \to M \otimes G^*
\]

is an isomorphism. It is enough to prove this after the faithfully flat base change \(\overline{\iota} : \overline{Y} \to Y:\)

\[
\overline{\iota^*} (p^* p_*(M \otimes G^*)) \to \overline{\iota^*} (M \otimes G^*)
\]
\[
\cong \overline{\iota^*} q^* p_*(M \otimes G^*) \to \overline{\iota^*} M \otimes (\overline{q}^* G)^* \quad \text{by \cite{3] and \cite{8} 0.6.7.6]}
\]
\[
\cong \overline{\iota^* p}_* (\overline{\iota^*} M \otimes (\overline{q}^* G)^*) \to \overline{\iota^*} M \otimes (\overline{q}^* G)^* \quad \text{by \cite{3] and \cite{16} Lemma 02KH]}
\]
\[
\cong \overline{\iota^* p}_* (\overline{\phi}(F) \otimes (\overline{q}^* G)^*) \to \overline{\phi}(F) \otimes (\overline{q}^* G)^*
\]

But \(\overline{\phi}(F) \in \text{Coh}(\overline{Y}, \overline{W})\), so the last morphism is an isomorphism, hence so is (4).

\(\square\)

We can now prove the main result of this section:
Theorem 2.6. Assume \( F \) is a simple \( \mathcal{A} \)-module with an isomorphism \( F \cong \iota^* F \) of \( \mathcal{A} \)-modules, then there is an \( \mathcal{A} \)-module \( E \) and an isomorphism of \( \mathcal{A} \)-modules \( F \cong \overline{\mathcal{E}} \).

Proof. Since \( F \) satisfies \( F \cong \iota^* F \), by Lemma 2.4 we get an isomorphism \( \overline{\phi}(F) \cong \overline{\tau}(\overline{\phi}(F)) \). Since furthermore the \( \mathcal{O}_Y \)-module \( \overline{\phi}(F) \) is simple using Lemma 2.3, it descends to \( Y \), so \( \overline{\phi}(F) \cong \overline{\tau} \mathcal{M} \) for some coherent \( \mathcal{O}_Y \)-module \( M \). But then \( M \in \text{Coh}(Y, \mathcal{W}) \) due to Lemma 2.5. Define \( E := \psi(M) \) then \( E \in \text{Coh}(Y, \mathcal{A}) \) and \( \overline{\mathcal{E}} \cong F \) since:

\[
\overline{\mathcal{E}} = \overline{\psi}(M) = \overline{\psi}(\mathcal{M} \otimes \overline{\mathcal{T}^* \mathcal{G}}) \cong \overline{\mathcal{M}}(\overline{\tau} \mathcal{M} \otimes (\overline{\mathcal{T}^* \mathcal{G}})^*) \cong \overline{\mathcal{M}}(\overline{\phi}(F) \otimes (\overline{\mathcal{T}^* \mathcal{G}})^*) \cong F.
\]

\( \square \)

3. Quaternion algebras on Enriques surfaces

Definition 3.1. A smooth projective surface \( X \) is called an Enriques surface if it satisfies

- \( H^1(X, \mathcal{O}_X) = 0 \)
- \( \omega_X \) is 2-torsion, i.e. \( \omega_X / \mathcal{O}_X \) but \( \omega_X \otimes \omega_X \cong \mathcal{O}_X \).

The 2-torsion element \( \omega_X \in \text{Pic}(X) \) induces an étale Galois double cover

\[
\pi : \overline{X} \rightarrow X.
\]

It is well known that \( \overline{X} \) is a K3 surface hence \( \pi \) is a universal cover of \( X \). Denote the associated involution by \( \iota : \overline{X} \rightarrow \overline{X} \).

By results of Cossec and Dolgachev, see [6, Theorem 1.1.3., Corollary 5.7.1.] we have:

Theorem 3.2. Assume \( X \) is an Enriques surface over \( \mathbb{C} \), then

\[
\text{Br}(X) \cong \mathbb{Z}/2\mathbb{Z}.
\]

This result shows that there is one nontrivial element \( b_X \) in the Brauer group of an Enriques surface. The first question is if we can find a representative of this class in terms of Azumaya algebras.

Proposition 3.3. The nontrivial element in the Brauer group of \( X \) can be represented by a quaternion algebra \( \mathcal{A} \) on \( X \).

Proof. The result of Cossec and Dolgachev shows that the nontrivial element \( b_X \in \text{Br}(X) \) has order two. As \( X \) is smooth by [5] Théorème 2.4. the restriction to the generic point \( \eta \) gives an injection

\[
r_\eta : \text{Br}(X) \hookrightarrow \text{Br}(\mathbb{C}(X)).
\]

So the image \( r_\eta(b_X) \) has order two in \( \text{Br}(\mathbb{C}(X)) \).

The field \( \mathbb{C}(X) \) has property \( C_2 \), see [13] II.4.5.(b)]. By a result of Platonov (simultaneously found by Artin and Harris) the Brauer class \( r_\eta(b_X) \) can be represented by a quaternion algebra \( \mathcal{A} \) over \( \mathbb{C}(X) \), see [13] Theorem 5.7. ([2] Theorem 6.2.).

Since the class \([\mathcal{A}]\) comes from \( \text{Br}(X) \) it is unramified at every point of codimension one in \( X \), and thus by [5, Théorème 2.5.] there is a quaternion algebra \( \mathcal{A} \) on \( X \) with \( \mathcal{A} \otimes \mathbb{C}(X) = \mathcal{A} \) such that \([\mathcal{A}] = b_X \).

One natural question to ask then: Is the pullback of the nontrivial class still nontrivial in \( \text{Br}(\overline{X}) \), i.e. is \( \pi^* : \text{Br}(X) \rightarrow \text{Br}(\overline{X}) \) injective? Beauville gives a complete answer to this question, see [4] Corollary 4.3., Corollary 5.7., Corollary 6.5.:

Theorem 3.4. The morphism \( \pi^* : \text{Br}(X) \rightarrow \text{Br}(\overline{X}) \) is trivial if and only if there is \( L \in \text{Pic}(\overline{X}) \) with \( \iota^* L = L^{-1} \) and \( c_1(L)^2 \equiv 2 \pmod{4} \). The surfaces \( X \) with \( \pi^* b_X = 0 \) form an infinite, countable union of (non-empty) hypersurfaces in the moduli space \( \mathcal{M} \) of Enriques surfaces.

Thus if \( X \) is a very general Enriques surface (in the sense of the previous theorem) then the pullback of the quaternion algebra \( \mathcal{A} \) constructed in Proposition 3.3 represents the nontrivial class \( \pi^* b_X \in \text{Br}(\overline{X}) \).

Remark 3.5. For a description of the (non)triviality of \( \pi^* : \text{Br}(X) \rightarrow \text{Br}(\overline{X}) \) using lattice theory, group cohomology and the Hochschild-Serre spectral sequence, see [11].
4. Moduli schemes of sheaves over quaternion algebras

Assume $W$ is a smooth projective $d$-dimensional variety and $\mathcal{A}$ is an Azumaya algebra on $W$, then we can think of the pair $(W, \mathcal{A})$ as a noncommutative version of $W$. In this section, we want to study moduli schemes of sheaves on such noncommutative pairs.

**Definition 4.1.** A sheaf $E$ on $W$ is called a generically simple torsion free $\mathcal{A}$-module, if $E$ is a left $\mathcal{A}$-module such that $E$ is coherent and torsion free as an $\mathcal{O}_W$-module and the stalk $E_\eta$ over the generic point $\eta \in W$ is a simple module over $\mathcal{A}_\eta$. If furthermore $\mathcal{A}_\eta$ is a division ring over $\mathbb{C}(W)$ then such a module is also called a torsion free $\mathcal{A}$-module of rank one.

**Remark 4.2.** A generically simple torsion free $\mathcal{A}$-module $E$ is simple, see [9].

Apart from being simple, these modules share many properties with classical stable sheaves, for example we have

**Lemma 4.3.** Assume $E$ and $F$ are generically simple torsion free $\mathcal{A}$-modules with the same Chern classes, then $\text{Hom}_{\mathcal{A}}(E, F) \neq 0$ implies $E \cong F$.

**Proof.** A nontrivial $\mathcal{A}$-morphism $\phi$ must be generically bijective as $E$ and $F$ are generically simple. As $E$ and $F$ are torsion free this implies that $\phi$ is injective, so we get an exact sequence with $Q = \text{Coker}(\phi)$:

$$0 \longrightarrow E \xrightarrow{\phi} F \longrightarrow Q \longrightarrow 0$$

But $E$ and $F$ have the same Chern classes, so $Q = 0$ and hence $E \cong F$. □

By fixing the Hilbert polynomial $P$ of such sheaves, Hoffmann and Stuhler showed that these modules are classified by a moduli scheme, see [9, Theorem 2.4. iii), iv]):

**Theorem 4.4.** There is a projective moduli scheme $M_{\mathcal{A}/W, P}$ classifying generically simple torsion free $\mathcal{A}$-modules with Hilbert polynomial $P$ on $W$.

We want to study these moduli schemes for a noncommutative Enriques surfaces $(X, \mathcal{A})$, where $X$ is a very general Enriques surface and $\mathcal{A}$ is a quaternion algebra representing the nontrivial class in $\text{Br}(X)$. Note that the $\mathcal{O}_X$-rank of a torsion free $\mathcal{A}$-module of rank one is four in this case.

We also have an associated noncommutative K3 surface $(\overline{X}, \overline{\mathcal{A}})$. Now we first recall some facts about the moduli schemes for such pairs, see [9, Theorem 3.6.]:

**Theorem 4.5.** Let $\overline{X}$ be a K3 surface which is a double cover of a very general Enriques surface $X$ and let $\overline{\mathcal{A}}$ be the quaternion algebra coming from the quaternion algebra on $X$ which represents the nontrivial class in $\text{Br}(X)$.

i) The moduli scheme $M_{\overline{\mathcal{A}}/\overline{X}}$ of torsion free $\overline{\mathcal{A}}$-modules of rank one is smooth.

ii) There is a nowhere degenerate alternating 2-form on the tangent bundle of $M_{\overline{\mathcal{A}}/\overline{X}}$.

iii) Every torsion free $\overline{\mathcal{A}}$-module of rank one can be deformed into a locally projective $\overline{\mathcal{A}}$-module, i.e. the locus $M_{\overline{\mathcal{A}}/\overline{X}}^{lp}$ of locally projective $\overline{\mathcal{A}}$-modules is dense in $M_{\overline{\mathcal{A}}/\overline{X}}$.

iv) For fixed Chern classes $\overline{c}_1$ and $\overline{c}_2$ we have

$$\dim M_{\overline{\mathcal{A}}/\overline{X}, \overline{c}_1, \overline{c}_2} = \overline{\Delta} - c_2(\overline{\mathcal{A}}) - 6$$

where $\overline{\Delta} = 8\overline{c}_2 - 3\overline{c}_1^2$ is the discriminant and $\overline{c}_i = \pi^* c_i$.

By using the $\overline{\mathcal{A}}$-Mukai vector we even get by [12, Theorem 2.11]:

**Theorem 4.6.** Let the pair $(\overline{X}, \overline{\mathcal{A}})$ be as in Theorem 4.5. Assume $\pi$ is a fixed primitive $\overline{\mathcal{A}}$-Mukai vector, then $M_{\overline{\mathcal{A}}/\overline{X}, \pi}$ is an irreducible holomorphic symplectic manifold deformation equivalent to $\text{Hilb}^{\overline{\mathcal{A}}} (\overline{X})$.

The covering involution $\iota : \overline{X} \to \overline{X}$ induces an involution $\iota^* : M_{\overline{\mathcal{A}}/\overline{X}, \overline{c}_1, \overline{c}_2} \to M_{\overline{\mathcal{A}}/\overline{X}, \overline{c}_1, \overline{c}_2}$, $[F] \mapsto [\iota^* F]$. 
Lemma 4.7. The involution $\iota^*$ is antisymplectic, that is if we denote the symplectic form on the tangent bundle of $M_{\mathcal{A}/X}$ by $\omega$, then $\omega(\iota^* f_1, \iota^* f_2) = -\omega(f_1, f_2)$.

Proof. By [9] Theorem 3.6. ii), and similar to Mukai’s construction, after the identification $T[F]M_{\mathcal{A}/X} \cong \text{Ext}^1_{\mathcal{A}}(F, F)$ the symplectic form is defined by the Yoneda product

$$\text{Ext}^1_{\mathcal{A}}(F, F) \times \text{Ext}^1_{\mathcal{A}}(F, F) \to \text{Ext}^2_{\mathcal{A}}(F, F).$$

composed with the trace map $tr_{\mathcal{A}}: \text{Ext}^2_{\mathcal{A}}(F, F) \to H^2(X, \mathcal{O}_X)$.

Using the functoriality of the Yoneda pairing (the cup product) we get the following commutative diagram

$$\begin{array}{ccc}
\text{Ext}^1_{\mathcal{A}}(F, F) \times \text{Ext}^1_{\mathcal{A}}(F, F) & \longrightarrow & \text{Ext}^2_{\mathcal{A}}(F, F) \\
\iota^* \downarrow & & \downarrow \iota^* \\
\text{Ext}^1_{\mathcal{A}}(\iota^* F, \iota^* F) \times \text{Ext}^1_{\mathcal{A}}(\iota^* F, \iota^* F) & \longrightarrow & \text{Ext}^2_{\mathcal{A}}(\iota^* F, \iota^* F)
\end{array}$$

According to the definition in [9] the trace map $tr_{\mathcal{A}}$ is the composition of the forgetful functor from $\mathcal{A}$-modules to $\mathcal{O}_X$-modules and the usual trace map $tr_{\mathcal{O}_X}$, so $tr_{\mathcal{A}}$ is also functorial and we get the following commutative diagram

$$\begin{array}{ccc}
\text{Ext}^2_{\mathcal{A}}(F, F) & \longrightarrow & H^2(X, \mathcal{O}_X) \\
\iota^* \downarrow & & \downarrow \iota^* \\
\text{Ext}^2_{\mathcal{A}}(\iota^* F, \iota^* F) & \longrightarrow & H^2(X, \mathcal{O}_X)
\end{array}$$

But $\iota^*: H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X)$ is multiplication by $-1$. This follows from the identification $H^2(X, \mathcal{O}_X) \cong \mathbb{C}$ by using $H^0(X, \omega_X) = \mathbb{C}$ and the fact that $\iota^*$ is antisymplectic with respect to $\sigma$ as $H^0(X, \omega_X) = 0$.

Putting both diagrams together, we see that $\iota^*$ is in fact antisymplectic.

□

Corollary 4.8. The locus of fixed points of the involution

$$\text{Fix}(\iota^*) \subset M_{\mathcal{A}/X, \mathcal{O}_X}$$

is a smooth isotropic projective subscheme.

Proof. Fix($\iota^*$) is smooth and projective by [11] 3.1., 3.4. The previous Lemma 4.7 shows that is also isotropic.

□

For the rest of this section we need the following

Remark 4.9. For a torsion free $\mathcal{A}$-module $E$ of rank one on $X$, the $\mathcal{A}$-modules $E^{**}$ and $E \otimes L$ for $L \in \text{Pic}(X)$ are also torsion free of rank one. In addition $\overline{E}$ is a torsion free $\mathcal{A}$-module of rank one on $X$ since $\pi$ is flat.

Theorem 4.10. Let $X$ be a very general Enriques surfaces and let $\mathcal{A}$ be a quaternion algebra on $X$ representing the nontrivial class in $\text{Br}(X)$.

i) The moduli scheme $M_{\mathcal{A}/X}$ of torsion free $\mathcal{A}$-modules of rank one is smooth.

ii) Every torsion free $\mathcal{A}$-module of rank one can be deformed into a locally projective $\mathcal{A}$-module, i.e. the locus $M_{\mathcal{A}/X}^{lp}$ of locally projective $\mathcal{A}$-modules is dense in $M_{\mathcal{A}/X}$.

iii) For fixed Chern classes $c_1$ and $c_2$ we have

$$\dim M_{\mathcal{A}/X, c_1, c_2} = \frac{\Delta}{4} - c_2(\mathcal{A}) - 3$$

where $\Delta = 8c_2 - 3c_1^2$ is the discriminant.

Proof. i) For a given point $[E] \in M_{\mathcal{A}/X}$ we have to show that all obstruction classes in $\text{Ext}^2_{\mathcal{A}}(E, E)$ vanish. But by Proposition 1.6 we have:

$$\text{Ext}^2_{\mathcal{A}}(E, E) \cong (\text{Hom}_{\mathcal{A}}(E, E \otimes \omega_X))^\vee.$$

As $\overline{E}$ is a simple $\mathcal{A}$-module, we get $\text{Hom}_{\mathcal{A}}(E, E \otimes \omega_X) = 0$ by Corollary 1.5. Thus all obstructions vanish and $M_{\mathcal{A}/X}$ is smooth at $[E]$. 

ii) The proof of [9, Theorem 3.6.iii)] carries over to our situation with one small change: the surjectivity of the connecting homomorphisms $\delta$ in the diagram:

$$
\begin{array}{ccc}
\text{Ext}^1_A(E, E) & \xrightarrow{\delta} & \text{Ext}^2_A(T, E) \\
\downarrow{\pi^*} & & \downarrow{\text{c}} \\
\text{Ext}^2_A(T, E^**) & \bigoplus_{i=1}^l & \text{Ext}^2_A(T_{x_i}, E^**) \\
\end{array}
$$

follows from the fact that $\text{Ext}^2_A(E^**, E) = 0$.

This vanishing can be seen as follows: using Proposition 1.6 we have

$$
\text{Ext}^2_A(E^**, E) \cong (\text{Hom}_A(E, E^{**} \otimes \omega_X))^\vee.
$$

But the last space is zero by Lemma 1.7. The rest of the proof works unaltered.

iii) Using ii) is suffices to compute the dimension of $T[E]M_{A/X} \cong \text{Ext}^1_A(E, E) \cong H^1(X, \text{End}_A(E))$ for a locally projective $A$-module $E$ of rank one.

Again as in [9, Theorem 3.6.iv)] we have:

$$
c_1(\text{End}_A(E)) = 0 \quad \text{and} \quad c_2(\text{End}_A(E))) = \frac{\Delta}{4} - c_2(A)
$$

where $\Delta$ is the discriminant of $E$. So by Hirzebruch-Riemann-Roch:

$$
\chi(X, \text{End}_A(E)) = \frac{-\Delta}{4} + c_2(A) + 4\chi(X, O_X)
$$

Using $\text{End}_A(E) \cong \mathbb{C}$, $\text{Ext}^2_A(E, E) = 0$ and $\chi(X, O_X) = 1$ we get our result.

\begin{remark}
4.11. The proof of i) also implies $E \not\cong E \otimes \omega_X$ for all torsion free $A$-modules of rank one.

Similar to the involution $\iota$, using Remark 4.9 the projection $\pi : X \rightarrow X$ induces a morphism

$$
\pi^* : M_{A/X,c_1,c_2} \rightarrow M_{A/X,\pi_1 c_2}, \quad [E] \mapsto [E].
$$

Our goal is to understand this morphism:

\begin{theorem}
4.12. Let the pair $(X, A)$ be as in Theorem 4.10. The pullback map

$$
\pi^* : M_{A/X,c_1,c_2} \rightarrow M_{A/X,\pi_1 c_2}
$$

factors through $\text{Fix}(\iota^*)$ restricting to an étale double cover

$$
\varphi : M_{A/X,c_1,c_2} \rightarrow \text{Fix}(\iota^*).
$$

\end{theorem}

\begin{proof}
We have

$$
\iota^* \overline{E} = \iota^* \pi^* E \cong (\pi \circ \iota)^* E = \pi^* E = \overline{E}.
$$

So $\text{Im}(\pi^*) \subset \text{Fix}(\iota^*)$ and hence $\pi^*$ factors through $\text{Fix}(\iota^*)$ giving rise to

$$
\varphi : M_{A/X,c_1,c_2} \rightarrow \text{Fix}(\iota^*).
$$

By Theorem 2.6 we also have $\text{Fix}(\iota^*) \subset \text{Im}(\pi^*)$. So $\text{Im}(\pi^*) = \text{Fix}(\iota^*)$ and the morphism $\varphi$ is surjective.

Assume $\varphi([E]) = \varphi([F])$ that is $\overline{E} \cong \overline{F}$ and $\text{Hom}_A(E, \overline{F}) \neq 0$. Then Lemma 1.4 says

$$
\text{Hom}_A(E, \overline{F}) \cong \text{Hom}_A(E, F) \oplus \text{Hom}_A(E, F \otimes \omega_X)
$$

and so by Lemma 4.3 and Remark 4.9 we have

$$
E \cong F \text{ or } E \cong F \otimes \omega_X
$$

but not both by Remark 4.11. So $\varphi$ is an unramified 2 : 1-morphism. Moreover the computations also shows that $\varphi$ is a flat morphism by [14, Lemma, p.675], hence $\varphi$ is étale.

\end{proof}
Corollary 4.13. The locus of fixed points of the involution
\[ \text{Fix}(\iota^*) \subset M_{\mathcal{X}/X, c_1, c_2} \]
is a Lagrangian subscheme.

Proof. The previous theorem 4.12 shows
\[ \dim \text{Fix}(\iota^*) = \dim M_{\mathcal{A}/X, c_1, c_2}. \]
On the other hand by Theorem 4.5 and the fact that \( \pi \) is of degree 2 we have
\[ \dim M_{\mathcal{A}/X, c_1, c_2} = \frac{\Delta}{4} - c_2(\mathcal{A}) - 6 = 2(\frac{\Delta}{4} - c_2(\mathcal{A}) - 3) = 2 \dim M_{\mathcal{A}/X, c_1, c_2} \]
Both results together give
\[ \dim \text{Fix}(\iota^*) = \frac{1}{2} \dim M_{\mathcal{X}/X, c_1, c_2} \]
By Corollary 4.8 we already know that \( \text{Fix}(\iota^*) \) is an isotropic subscheme, so it is in fact a Lagrangian subscheme of \( M_{\mathcal{X}/X, c_1, c_2} \). \qed

References

[1] Donu Arapura. Frobenius amplitude and strong vanishing theorems for vector bundles. Duke Math. J., 121(2):231–267, 2004.
[2] M. Artin. Brauer-Severi varieties. In Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981), volume 917 of Lecture Notes in Math., pages 194–210. Springer, Berlin-New York, 1982.
[3] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. Compact complex surfaces. Springer-Verlag, Berlin, second edition, 2004.
[4] Arnaud Beauville. On the Brauer group of Enriques surfaces. Math. Res. Lett., 16(6):927–934, 2009.
[5] Jean-Louis Colliot-Thélène. Algèbres simples centrales sur les corps de fonctions de deux variables (d’après A. J. de Jong). Number 307, pages Exp. No. 949, ix, 379–413. 2006. Séminaire Bourbaki. Vol. 2004/2005.
[6] François R. Cossec and Igor V. Dolgachev. Enriques surfaces. I, volume 76 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1989.
[7] Bas Edixhoven. Néron models and tame ramification. Compositio Math., 81(3):291–306, 1992.
[8] A. Grothendieck. Éléments de géométrie algébrique. I. Le langage des schémas. Inst. Hautes Études Sci. Publ. Math., (4):228, 1960.
[9] Norbert Hoffmann and Ulrich Stuhler. Moduli schemes of generically simple Azumaya modules. Doc. Math., 10:369–389, 2005.
[10] Hoil Kim. Moduli spaces of stable vector bundles on Enriques surfaces. Nagoya Math. J., 150:85–94, 1998.
[11] Hermes Martínez. The Brauer group of K3 covers. Rev. Colombiana Mat., 46(2):185–204, 2012.
[12] Fabian Reede. The symplectic structure on the moduli space of line bundles on a noncommutative Azumaya surface. Beitr. Algebra Geom., 60(1):67–76, 2019.
[13] V. G. Sarkisov. On conic bundle structures. Izv. Akad. Nauk SSSR Ser. Mat., 46(2):371–408, 432, 1982.
[14] Mary Schaps. Deformations of Cohen-Macaulay schemes of codimension 2 and non-singular deformations of space curves. Amer. J. Math., 99(4):669–685, 1977.
[15] Jean-Pierre Serre. Galois cohomology. Springer-Verlag, Berlin, 1997.
[16] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2019.

Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany
E-mail address: reede@math.uni-hannover.de