Precise asymptotics on the Birkhoff sums for dynamical systems

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Received 10 December 2020, revised 30 May 2021
Accepted for publication 24 August 2021
Published 9 September 2021

Abstract

We establish two precise asymptotic results on the Birkhoff sums for dynamical systems. These results are parallel to that on the partial sums of independent and identically distributed random variables previously obtained by Hsu and Robbins, Erdős, Heyde. We apply our results to the Gauss map and obtain new precise asymptotics in the theorem of Lévy on the regular continued fraction expansion of irrational numbers in (0, 1).

Keywords: precise asymptotics, expanding Markov interval maps, continued fraction, Lévy’s theorem

Mathematics Subject Classification numbers: 37A44, 37A50, 60F15.

1. Introduction

It is of general interest to investigate various probabilistic limit laws as stochastic properties for deterministic dynamical systems. The current paper is a contribution to this topic. We establish two precise asymptotics on the Birkhoff sums for dynamical systems. Let us first introduce its background from probability theory.
1.1. Precise asymptotics for i.i.d. case

The concept of precise asymptotics was introduced by Hsu and Robbins [18] under the name of complete convergence. Since then, an abundance of papers have appeared in the setting of independent and identically distributed (i.i.d. for short) random variables, see a survey paper [16] for more details. Meanwhile, the researches on precise asymptotic topics also turn out to have important applications ranging from stochastic volatility models [12] to statistical analysis [30].

Let us briefly state the background on precise asymptotics for the i.i.d. case as follows. Denote by \(\{X_i\}_{i \in \mathbb{N}}\) a sequence of i.i.d. random variables with \(\mathbb{E}(X_i) = 0\), and write the partial sum \(S_n = \sum_{i=1}^{n} X_i\). The following theorem plays a fundamental role in precise asymptotics.

The first part was obtained by Hsu and Robbins [18], while the second part was obtained later by Erdős [11].

**Theorem 1.1** ([11, 18]). If \(\mathbb{E}(X_1^2) < \infty\), then for all \(\varepsilon > 0\),

\[
\sum_{n=1}^{\infty} \mathbb{P}(|S_n| \geq n \varepsilon) < \infty.
\]

Conversely, if the series is finite for some \(\varepsilon > 0\), then \(\mathbb{E}(X_1^2) < \infty\) and the series is finite for all \(\varepsilon > 0\).

Theorem 1.1 can be viewed as a refined result on the rate of convergence in the law of large numbers: not only the terms \(\mathbb{P}(|S_n| \geq n \varepsilon)\) have to tend to 0 as \(n\) tends to infinity, but the sum of them has to converge, which contains more information.

By using more general results linking the summability of the series to the rate of convergence in the law of large numbers, a series of paper (see for examples Spitzer [34], Katz [19]) pursued theorem 1.1 further. For \(0 < p < 2\) and \(r \geq p\), Baum and Katz [5] proved that \(\mathbb{E}(|X_1|^r) < \infty\) with \(r \geq 1\) if and only if

\[
\sum_{n=1}^{\infty} n^{1/p-2} \mathbb{P}(|S_n| \geq \varepsilon n^{1/p}) < \infty.
\]

In particular, for \(r = 2\) and \(p = 1\), the result reduces to theorem 1.1.

Another way to view these sums is to note that \(\sum_{n=1}^{\infty} \mathbb{P}(|S_n| \geq n \varepsilon)\) tends to infinity as \(\varepsilon\) goes to 0. It is therefore of interest to find the rate at which this occurs. This amounts to finding appropriate normalization of functions of \(\varepsilon\) that yield non-trivial limits. In this direction, Heyde [17] proved that

**Theorem 1.2** ([17]). If \(\mathbb{E}(X_1^2) < \infty\), then

\[
\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^{\infty} \mathbb{P}(|S_n| \geq \varepsilon n) = \mathbb{E}(X_1^2).
\]

Under appropriate moment conditions, extensions of theorem 1.2 for more general values of \(r\) and \(p\) have also been investigated in [9, 14, 35], which include:

(a) If \(\mathbb{E}(X_1^2) < \infty\), then

\[
\lim_{\varepsilon \to 0} \frac{1}{\log\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}(|S_n| \geq \varepsilon n) = 2.
\]
(b) Let $0 < p < 2$ and $r \geqslant 2$. If $\mathbb{E}(X_1^2) > 0$ and $\mathbb{E}(|X_1|') < \infty$, then
\[
\lim_{\varepsilon \to 0} \varepsilon^{2 - p} \sum_{n=1}^{\infty} n^{r - 2p - 2} \mathbb{P}(|S_n| \geqslant \varepsilon n^p) = \frac{p}{r - p} \mathbb{E}\left(\left|Y\right|^{2 - \frac{2}{r}}\right),
\]
where $Y$ denotes the normal distribution with mean 0 and variance $\mathbb{E}(X_1^2)$. We remark that there are no analogous results for $p = 2$. However, there are further results replacing $n^{1/p}$ by $\sqrt{n \log n}$ or $\sqrt{n \log \log n}$, see [15] for details.

1.2. Statements of main results
As a comparison, we will adapt some of the precise asymptotic results mentioned above to dynamical systems setting. To be more precise, we will develop an ‘axiomatic approach’ by considering a measure-preserving dynamical system $(X, T, \mu)$ and an observable $f : X \to \mathbb{R}$. They yield a stationary stochastic process $X_n := f \circ T^{n-1}$, $n \geqslant 1$, whose finite-dimensional marginal distributions are $\mathbb{P}(x_0)^{\delta_{T_1(\cdots T_{n-1}(\cdots T_{n})(\cdots)})}$ with $\delta$: the Dirac measure at $z$. In general $\{X_n\}_{n \in \mathbb{N}}$ is not independent, as $X_n$ and $X_{n+1}$ are correlated by the deterministic action of $T$. Assume that $(X, T, \mu)$ is ergodic and $f \in L^1(\mu)$ with $\int f \, d\mu = 0$. Write $S_n f$ for the Birkhoff sum $\sum_{j=0}^{n-1} f \circ T^j$. By Birkhoff’s ergodic theorem,
\[
\lim_{n \to \infty} \frac{1}{n} S_n f(x) = 0 \quad \mu \text{-a.e. } x \in X.
\]
For any $\varepsilon > 0$ and $n \in \mathbb{N}$, let $\Lambda_n(\varepsilon) = \Lambda_n^+(\varepsilon) + \Lambda_n^-(\varepsilon)$, where
\[
\Lambda_n^+(\varepsilon) = \mu \left\{ x \in X : \frac{1}{n} S_n f(x) \geqslant \varepsilon \right\} \quad \text{and} \quad \Lambda_n^-(\varepsilon) = \mu \left\{ x \in X : \frac{1}{n} S_n f(x) \leqslant -\varepsilon \right\}.
\]
With these notations, the main theorems are as follows.

**Theorem 1.3 (Main theorem).** Suppose the following assumptions hold:

- **(CLT)** $\frac{S_n f}{\sqrt{n \log n}}$ converges in law to the normal distribution with mean 0 and variance 1 as $n \to \infty$;
- **(LD)** There exist constants $\delta > 0$, $M > 0$, $C > 0$ and a $C^2$ function $I : (-\delta, \delta) \to [0, \infty)$ such that:
  1. $I(0) = 0$, $I'(0) = 0$, $I''(0) > 0$;
  2. For any $\varepsilon \in (0, \delta)$ and any integer $n > M/\varepsilon$,
     \[
     \Lambda_n^+(\varepsilon) \leqslant C e^{-\varepsilon^m} \quad \text{and} \quad \Lambda_n^-(\varepsilon) \leqslant C e^{-\varepsilon^{-m}}.
     \]

Then the following hold:

- **(a)**
  \[
  \lim_{\varepsilon \to 0} \frac{1}{\sqrt{\log n}} \sum_{n=1}^{\infty} \Lambda_n(\varepsilon) = \sigma^2;
  \]
- **(b)**
  \[
  \lim_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \sum_{n=1}^{\infty} \frac{\Lambda_n(\varepsilon)}{n} = 2.
  \]
Let us comment on the two key assumptions in theorem 1.3. First, (CLT) requires that the central limit theorem holds with the positive limiting variance \( \sigma \), which in almost all known cases is of the form

\[
\sigma^2 = \int f^2 \, d\mu + 2 \sum_{j=1}^{\infty} \int (f \circ T^j) f \, d\mu,
\]

where \( f \) needs to belong to a suitable Banach space. (CLT) is a natural assumption in that it is verified for various dynamical systems and sufficiently regular observables. See the survey paper of Gouëzel [13] for more details. One way to verify the positivity of the limiting variance is to use Livšić theorems on measurable rigidity (see e.g. [23, 24, 29]). For example, for expanding Markov interval maps with infinitely many branches (see section 3.1 for the definition) this can be shown by the Livšić theorem of Aaronson and Denker [1]. For these maps, Morita [27] verified the positivity of the limiting variance and (CLT) for a large class of observables.

Next, we comment on (LD) from two aspects. On the one hand, (LD) is stronger than the usual large deviation bounds which are currently available for various dynamical systems with some amount of hyperbolicity. For a uniformly hyperbolic dynamical system \( T \) of class \( C^2 \) of a compact Riemannian manifold \( X \) and the equilibrium state \( \mu \) for a Hölder continuous potential [7], it is a consequence of the large deviation principle [21, 28, 36, 38] that for any continuous observable \( f : X \to \mathbb{R} \) with mean 0, there exists a rate function \( I_f : \mathbb{R} \to [0, \infty] \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \log \Lambda_n^+(\epsilon) = I_f(\epsilon) \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \log \Lambda_n^-(\epsilon) = I_f(-\epsilon), \tag{1.1}
\]

for all \( \epsilon > 0 \) for which \( I_f \) is bounded on \([-\epsilon, \epsilon]\]. The rate function is unique and given by the Fenchel–Legendre transform of the cumulant generating function

\[
L(t) = \lim_{n \to \infty} \frac{1}{n} \log \int e^{tf} \, d\mu,
\]

see e.g. [10, section 4]. If \( f \) is sufficiently regular, then the regularity of the rate function around 0 can be analyzed with a spectral theory of Perron–Frobenius type operators, yielding \( I'_f(0) = 0 \) and \( I''_f(0) = 1 / \sigma^2 \). For a class of non-uniformly hyperbolic systems admitting Markov tower extensions, a spectral analysis can also be used to show the existence of rate functions as in (1.1), see e.g. [25, theorem 2.1 and corollary 2.6], [33, theorems A and B]. It is clear that (1.1) does not imply the exponential decay condition in (LD) with \( I = I_f \).

On the other hand, (LD) is known to hold for a large class of uniformly hyperbolic systems. For an expanding Markov interval map \( T \) with finitely many branches and a Hölder continuous observable \( f \) with mean 0, Chazottes and Collet [8, lemma A.1] obtained stronger bounds than (LD) under the assumption (CLT): for all sufficiently small \( \epsilon > 0 \), \( \Lambda_n(\epsilon) \) is bounded from both sides by constant multiples of \( e^{-I_f(\epsilon)/\sqrt{n}} \). These bounds are in agreement with the i.i.d. case in [4, theorem 1]. In [37], (LD) was verified for the Gauss map, that is an expanding Markov interval map with infinitely many branches. For a class of non-uniformly hyperbolic systems including well-known intermittent interval maps, Melbourne and Nicol [25, theorems 3.5 and 4.3] showed that the limits in (1.1) are identically zero for all sufficiently small \( \epsilon \), which therefore implies the breakdown of (LD).

This paper is organized as follows. Section 2 provides the proof of theorem 1.3. Our strategy is to modify the proofs by Heyde [17] and Spătaru [35] in the i.i.d. case, by using (LD) and (CLT) to compensate the lack of independence. One key step in their proofs is to deduce...
accurate upper bounds of $P(|S_n| \geq n\varepsilon)$, see [17, p 175], [35, lemma 2]. These bounds heavily rely on the independence, and it is difficult to check their validity in the dynamical systems setting. Therefore, we put (LD) and (CLT) as assumptions, and deduce a new upper bound of $P(|S_n| \geq n\varepsilon)$.

Section 3 provides applications of our results to expanding Markov interval maps including the Gauss map. We will apply theorem 1.3 to this setting and obtain a new precise asymptotics in Lévy’s theorem [22] for the regular continued fraction expansion of irrational numbers in $(0,1)$, see theorem 3.2.

2. Proof of main results

This section is devoted to the proof of theorem 1.3. Let us begin with some useful lemmas. The first one is the classical Euler–Maclaurin formula, see [2, theorem 7.13].

**Lemma 2.1 (The Euler–Maclaurin formula).** Let $a, b \in \mathbb{Z}$ with $a < b$. Assume that $f$ has a continuous derivative $f'$ on $[a, b]$. Then we have

$$\sum_{n=a}^{b} f(n) = \int_{a}^{b} f(x)dx + \int_{a}^{b} f'(x)\psi(x)dx + \frac{1}{2}(f(a) + f(b)),$$

where $\psi(x) = x - \lfloor x \rfloor - 1/2$. Furthermore, if the improper integrals $\int_{a}^{\infty} f(x)dx$ and $\int_{a}^{\infty} f'(x)\psi(x)dx$ are convergent and $f(x) \to 0$ as $x \to \infty$, then

$$\sum_{n=a}^{\infty} f(n) = \int_{a}^{\infty} f(x)dx + \int_{a}^{\infty} f'(x)\psi(x)dx + \frac{f(a)}{2}.$$

The second lemma is Pólya’s theorem, see [3, theorem 9.1.4].

**Lemma 2.2 (Pólya’s theorem).** Let $Y$ be a random variable and $\{Y_n\}_{n \in \mathbb{N}}$ be a sequence of random variables. Assume that for any $x \in \mathbb{R}$, $F_n(x) \to F(x)$ as $n \to \infty$, where $F_n$ and $F$ are distribution functions of $Y_n$ and $Y$ respectively. If $F$ is a continuous function, then

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| = 0.$$

We denote by $\Phi(\cdot)$ the distribution function of the standard normal random variable, namely

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt = \frac{1}{\sqrt{2\pi}} \int_{-x}^{\infty} e^{-t^2/2} dt.$$

The next lemma gives lower and upper bounds for $\Phi(-x)$. See [6, lemma 6.1.6].

**Lemma 2.3.** For all $x > 0$,

$$\frac{x}{x^2 + 1} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \leq \Phi(-x) \leq \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

2.1. Proof of theorem 1.3(a)

We will first prove the following lemmas.

**Lemma 2.4.** We have

$$\lim_{\rho \to 0^+} \rho^2 \sum_{n=0}^{\infty} \Phi(-\rho \sqrt{n}) = \frac{1}{2}.$$
Proof. Let $\rho > 0$ be small. It follows from lemma 2.3 that the improper integrals
\[
\int_{0}^{\infty} \Phi(-\rho \sqrt{x}) dx \quad \text{and} \quad \int_{0}^{\infty} x^{-1/2} \Phi'(-\rho \sqrt{x}) \psi(x) dx
\]
are convergent as $|\psi(x)| \leq 1/2$. Since $\Phi(\cdot)$ has continuous derivative on $\mathbb{R}$ and $\Phi(-y) \to 0$ as $y \to \infty$, applying $f(x) = \Phi'(-\rho \sqrt{x})$ to lemma 2.1, we have
\[
\sum_{n=0}^{\infty} \Phi \left( -\rho \sqrt{n} \right) = \int_{0}^{\infty} \Phi \left( -\rho \sqrt{x} \right) dx - \frac{\rho}{2} \int_{0}^{\infty} x^{-1/2} \Phi' \left( -\rho \sqrt{x} \right) \psi(x) dx + \frac{1}{2}.
\]
(2.1)

From lemma 2.3, we see that $x \cdot \Phi(-\rho \sqrt{x}) \to 0$ as $x \to \infty$. Using the integration by parts to the first term in the right-hand side of (2.1), we obtain
\[
\int_{0}^{\infty} \Phi \left( -\rho \sqrt{x} \right) dx = 0 + \int_{0}^{\infty} \Phi' \left( -\rho \sqrt{x} \right) \cdot \frac{\rho \sqrt{x}}{2} dx
\]
\[
= \frac{1}{2 \sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{t^2}{2}} \rho \sqrt{x} dx = \frac{1}{\rho^2} \cdot \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t^2/2} dt = \frac{1}{2\rho^2}.
\]
For the second term in the right-hand side of (2.1), note that $|\psi(x)| \leq 1/2$, we have
\[
\left| \frac{\rho}{2} \int_{0}^{\infty} x^{-1/2} \Phi' \left( -\rho \sqrt{x} \right) \psi(x) dx \right| \leq \int_{0}^{\infty} \Phi' \left( -\rho \sqrt{x} \right) dx \cdot (\rho \sqrt{x}) = \frac{1}{2}.
\]
Therefore,
\[
\rho^{-2}/2 \leq \sum_{n=0}^{\infty} \Phi \left( -\rho \sqrt{n} \right) \leq \frac{1}{2\rho^2} + 1.
\]
Multiplying $\rho^2$ and letting $\rho \to 0$ yields the desired equation. \qed

Lemma 2.5. Let $K > 0$ be a constant. Then
\[
\lim_{\rho \to 0^+} \rho^2 \cdot \sum_{n \geq K / \rho^2} \Phi \left( -\rho \sqrt{n} \right) = 0.
\]
Proof. We derive from lemma 2.3 that
\[
\Phi \left( -\rho \sqrt{n} \right) \leq \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\rho \sqrt{n}} \cdot e^{-\rho^2 n/2}
\]
and then
\[
\rho^2 \cdot \sum_{n \geq K / \rho^2} \Phi \left( -\rho \sqrt{n} \right) \leq \frac{1}{\sqrt{2\pi}} \sum_{n \geq K / \rho^2} \frac{\rho}{\sqrt{n}} \cdot e^{-\rho^2 n/2}.
\]
Note that
\[
\int_{K}^{\infty} \frac{1}{\sqrt{y}} e^{-y^2/2} dy < \infty
\]
and
\[
\sum_{n \geq K / \rho^2} \rho \cdot \frac{1}{\sqrt{n}} \cdot e^{-\rho^2 n/2} \leq C \int_{K / \rho^2}^{\infty} \frac{\rho}{\sqrt{x}} \cdot e^{-\rho^2 x/2} dx = \frac{C}{2} \cdot \rho \cdot \int_{K}^{\infty} \frac{1}{\sqrt{y}} e^{-y^2/2} dy,
\]
where $C$ is a constant.
where $C_K > 0$ is a constant depending on $K$. Hence we obtain

$$\rho^2 \cdot \sum_{n \geq K/\rho^2} \Phi \left( -\rho \sqrt{n} \right) \leq \rho \cdot \frac{C_K}{2 \sqrt{2 \pi}} \int_{K}^{\infty} \frac{1}{\sqrt{y}} e^{-y^2/2} dy.$$  

Taking the limit $\rho \to 0$, the desired result follows. \hfill \Box

To complete the proof of theorem 1.3(a), it suffices to show that

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^{\infty} \Lambda^+_{n}(\varepsilon) = \frac{\sigma^2}{2} \quad \text{and} \quad \lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^{\infty} \Lambda^-_{n}(\varepsilon) = \frac{\sigma^2}{2}.$$  

In what follows, we only prove the first equation since the second one can be obtained by means of similar arguments. To this end, we write

$$\varepsilon^2 \sum_{n=1}^{\infty} \Lambda^+_{n}(\varepsilon) = \varepsilon^2 \sum_{n=1}^{\infty} \left( \Lambda^+_{n}(\varepsilon) - \Phi \left( -\varepsilon \sqrt{n}/\sigma \right) \right) + \varepsilon^2 \sum_{n=1}^{\infty} \Phi \left( -\varepsilon \sqrt{n}/\sigma \right). \quad (2.2)$$  

From lemma 2.4, it then follows that the second term on the right-hand side of (2.2) converges to $\sigma^2/2$ as $\varepsilon \to 0$. So we only need to prove that the first term on the right-hand side of (2.2) converges to 0 as $\varepsilon \to 0$. To this end, let $K > 0$ be a large constant, whose precise value will be exhibited later, and let

$$K(\varepsilon) := \left\lfloor \frac{K}{\varepsilon^2} \right\rfloor. \quad (2.3)$$  

We first treat the partial sum

$$\varepsilon^2 \sum_{n=1}^{K(\varepsilon)} \left( \Lambda^+_{n}(\varepsilon) - \Phi \left( -\varepsilon \sqrt{n}/\sigma \right) \right).$$  

Define

$$\Delta_n = \sup_{y \in \mathbb{R}} \left| \mu \left\{ x \in X : \frac{(S_n f)(x)}{\sigma \sqrt{n}} \leq y \right\} - \Phi(y) \right|. \quad (2.4)$$  

By lemma 2.2 it follows that $\Delta_n \to 0$ as $n \to \infty$. Combining this with the definition of $K(\varepsilon)$, we see that

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^{K(\varepsilon)} \left| \Lambda^+_{n}(\varepsilon) - \Phi \left( -\varepsilon \sqrt{n}/\sigma \right) \right| \leq K \cdot \limsup_{\varepsilon \to 0} \frac{1}{K(\varepsilon)} \sum_{n=1}^{K(\varepsilon)} \Delta_n = 0.$$  

We are now in a position to show that

$$\lim_{K \to \infty} \limsup_{\varepsilon \to 0} \varepsilon^2 \sum_{n > K(\varepsilon)} \left| \Lambda^+_{n}(\varepsilon) - \Phi \left( -\varepsilon \sqrt{n}/\sigma \right) \right| = 0.$$  

From lemma 2.5, we have

$$\lim_{K \to \infty} \limsup_{\varepsilon \to 0} \varepsilon^2 \sum_{n > K(\varepsilon)} \Phi \left( -\varepsilon \sqrt{n}/\sigma \right) = 0.$$  

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It remains to prove
\[
\limsup_{K \to \infty} \limsup_{\epsilon \to 0} \epsilon^2 \sum_{n > K(\epsilon)} \Lambda_n^+(\epsilon) = 0.
\]
Recall that \(\delta\) and \(M\) are constants given in (LD). Let \(0 < \epsilon < \min\{1, \delta\}\) and \(K > M + 1\). Then \(K(\epsilon)\) in (2.3) satisfies \(K(\epsilon) \geq M/\epsilon\). For any \(n > K(\epsilon)\), \(\Lambda_n^+(\epsilon) \leq C e^{-K(\epsilon)/\epsilon}\) holds, which implies that
\[
\epsilon^2 \sum_{n > K(\epsilon)} \Lambda_n^+(\epsilon) \leq C \cdot \frac{\epsilon^2}{1 - e^{-K(\epsilon)/\epsilon}} \cdot e^{-K(\epsilon)/\epsilon}. \quad (2.5)
\]
Meanwhile, (LD) gives \(I(\epsilon) \to 0\), \(I'(\epsilon) \to 0\), \(I''(\epsilon) \to I''(0) > 0\) as \(\epsilon \to 0\). Hence
\[
\lim_{\epsilon \to 0} \frac{1 - e^{-L(\epsilon)}}{\epsilon^2} = \lim_{\epsilon \to 0} \frac{I(\epsilon)}{\epsilon^2} = \frac{I''(0)}{2} > 0. \quad (2.6)
\]
Combining (2.5) and (2.6), we deduce that
\[
\limsup_{K \to \infty} \limsup_{\epsilon \to 0} \epsilon^2 \sum_{n > K(\epsilon)} \Lambda_n^+(\epsilon) \leq C \limsup_{K \to \infty} e^{-K(\epsilon)/\epsilon} = 0.
\]
This completes the proof of theorem 1.3(a). \(\square\)

2.2. Proof of theorem 1.3(b)

It suffices to show that
\[
\lim_{\epsilon \to 0} \frac{1}{- \log \epsilon} \sum_{n=1}^{\infty} \frac{\Lambda_n^+(\epsilon)}{n} = 1. \quad (2.7)
\]
Split
\[
\sum_{n=1}^{\infty} \frac{\Lambda_n^+(\epsilon)}{n} = I(\epsilon) + II(\epsilon) - III(\epsilon) + IV(\epsilon), \quad (2.8)
\]
where
\[
I(\epsilon) := \sum_{n=L(\epsilon)}^{L(\epsilon)} \frac{1}{n} \left( \Lambda_n^+(\epsilon) - \Phi \left( -\epsilon \sqrt{n}/\sigma \right) \right);
\]
\[
II(\epsilon) := \sum_{n > L(\epsilon)} \frac{\Lambda_n^+(\epsilon)}{n};
\]
\[
III(\epsilon) := \sum_{n > L(\epsilon)} \frac{1}{n} \Phi \left( -\epsilon \sqrt{n}/\sigma \right);
\]
with \(L(\epsilon) = \left\lfloor \epsilon^{-2} \right\rfloor\), and
\[
IV(\epsilon) := \sum_{n=1}^{\infty} \frac{1}{n} \Phi \left( -\epsilon \sqrt{n}/\sigma \right).
\]
We will deal with these four terms one by one. The condition (CLT) will only be used for an estimation of $I(\varepsilon)$ and (LD) will only be used for an estimate of $\Pi(\varepsilon)$.

By lemma 2.2, for $\Delta_n$ in (2.4) we have $\Delta_n \to 0$ as $n \to \infty$. Therefore
\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\Delta_k}{k} = 0,
\]
and so we have
\[
\limsup_{\varepsilon \to 0} \frac{|I(\varepsilon)|}{-\log \varepsilon} \leq \limsup_{\varepsilon \to 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{L(\varepsilon)} \frac{\Delta_n}{n} = \limsup_{\varepsilon \to 0} \frac{\log L(\varepsilon)}{-\log \varepsilon} \limsup_{\varepsilon \to 0} \frac{1}{\log L(\varepsilon)} \sum_{n=1}^{L(\varepsilon)} \frac{\Delta_n}{n} = 0.
\]

Let $0 < \varepsilon < \min\{\delta, 1, M^{-1}\}$. For any $n > L(\varepsilon)$, it follows from (LD) that
\[
\Lambda_n^+(\varepsilon) \leq C e^{-L(\varepsilon)n} < \frac{C}{L(\varepsilon)n},
\]
which implies that
\[
\limsup_{\varepsilon \to 0} \Pi(\varepsilon) = \limsup_{\varepsilon \to 0} \sum_{n > L(\varepsilon)} \frac{\Lambda_n^+(\varepsilon)}{n} \leq \limsup_{\varepsilon \to 0} \frac{C}{L(\varepsilon)} \sum_{n > L(\varepsilon)} \frac{1}{n^2} \leq \limsup_{\varepsilon \to 0} \frac{C}{L(\varepsilon)L(\varepsilon)} \leq \frac{2C}{L^2(0)}.
\]
The last inequality is deduced from (2.6). Hence $\lim_{\varepsilon \to 0} \Pi(\varepsilon)/(-\log \varepsilon) = 0$.

Let $0 < \varepsilon < 1/2$ and put $\rho = \varepsilon/\sigma$. The upper bound in lemma 2.3 gives
\[
\Phi\left(-\rho \sqrt{n}\right) \leq \frac{1}{\rho \sqrt{n}} e^{-\frac{\rho^2}{2}} \leq \frac{1}{\rho^3 \sqrt{2\pi}} \cdot \frac{2}{\rho n \sqrt{n}},
\]
and hence
\[
\Pi(\varepsilon) = \sum_{n > L(\varepsilon)} \frac{\Phi\left(-\rho \sqrt{n}\right)}{n} \leq \frac{2}{\rho^3 \sqrt{2\pi}} \cdot \frac{1}{L(\varepsilon) \sqrt{L(\varepsilon)}} + \frac{1}{L(\varepsilon) \sqrt{L(\varepsilon)}} \leq \frac{2}{\rho^3 \sqrt{2\pi}} \cdot 2\varepsilon^3 = \frac{4\sigma^3}{\sqrt{2\pi}}.
\]
Then $\lim_{\varepsilon \to 0} \Pi(\varepsilon)/(-\log \varepsilon) = 0$. Finally, from [35, proposition 1] it follows that $\lim_{\varepsilon \to 0} IV(\varepsilon)/(-\log \varepsilon) = 1$. Combining these estimates on the four terms in (2.8) we obtain (2.7). This completes the proof of theorem 1.3(b). \hfill \square

3. Expanding Markov interval maps and continued fractions

In this section we provide applications of theorem 1.3 to expanding Markov interval maps including the Gauss map. As a consequence we obtain a new precise asymptotics in Lévy’s theorem [22] for the regular continued fraction expansion of irrational numbers in $(0, 1)$. 

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3.1. Expanding Markov interval maps

Let $S$ be a countable set and let $m$ be the Lebesgue measure on $[0,1]$. An expanding Markov interval map is a map $T : \bigcup_{a \in S} J_a \to [0,1]$ such that the following hold:

(a) $\{J_a\}_{a \in S}$ is a family of subintervals of $[0,1]$ with pairwise disjoint interiors such that $m(\bigcup_{a \in S} J_a) = 0$;
(b) For each $a \in S$, $T|_{J_a}$ is a $C^2$ diffeomorphism onto its image with bounded derivatives;
(c) There exist an integer $p \geq 1$ and a constant $\lambda > 1$ such that
$$\inf_{a \in S} \inf_{x \in J_a} \left| (T^p)'x \right| > \lambda;$$
(d) (Rényi’s condition)
$$\sup_{a \in S} \sup_{x \in J_a} \left| T''x \right| < \infty.$$

An expanding Markov map $T$ is said to be with finitely many branches if $S$ is a finite set. Otherwise it is said to be with infinitely many branches.

It is known as a folklore theorem originating in the 1950s that expanding Markov interval maps admit a unique invariant probability measure $\nu$ that is absolutely continuous with respect to $m$, see for example [32]. Moreover, $\nu$ is ergodic.

Let $T$ be an expanding Markov interval map with finitely many branches. From [8, lemma A.1], (LD) holds for a Hölder continuous observable $f$ with $\int f \, d\nu = 0$ under the assumption of (CLT). It is well-known that $\sigma > 0$ holds if and only if the cohomological equation $f = \psi \circ T - \psi + \int f \, d\nu$ has no solution in $L^2(\nu)$. Since $f$ is Hölder continuous, by the Livšic theorem [23, 24], any solution of the cohomological equation in $L^2(\nu)$ has a version which is Hölder continuous. It follows that $\sigma = 0$ holds if and only if $f$ is cohomologous to a constant. In the case $f = \log |T'|$, $\sigma = 0$ holds if and only if $\nu$ is the measure of maximal entropy [7].

For maps with infinitely many branches we have the following result.

**Theorem 3.1.** Let $T$ be an expanding Markov interval map with infinitely many branches, and let $\nu$ be the $T$-invariant probability measure that is ergodic and absolutely continuous with respect to $m$. Assume $\int \log |T'| \, d\nu < \infty$. Then
$$\lim_{\varepsilon \to 0} \varepsilon \sum_{n=1}^{\infty} \Lambda_n(\varepsilon) = \sigma^2 \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{\infty} \frac{\Lambda_n(\varepsilon)}{n} = 2,$$
where
$$\Lambda_n(\varepsilon) = m\left\{ x \in (0,1) : \frac{1}{n} \log \left| (T^n)'(x) \right| - \int \log |T'| \, d\nu \geq \varepsilon \right\}.$$

**Proof.** The (CLT) for $f = \log |T'| - \int \log |T'| \, d\nu$ holds as a consequence of the result of Morita [27, theorem 4.1], or Aaronson and Denker [1, corollary 2.3]. The argument in the proof of theorem 3.2 below to show the (LD) for the Gauss map works verbatim to show (LD) in this general setting. Hence, theorem 1.3 yields the desired equalities. □
3.2. The Gauss map and continued fractions

An interesting example of an expanding Markov interval map with infinitely many branches is the Gauss map

$$G : x \in (0, 1] \mapsto \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \in [0, 1).$$

Each $x \in (0, 1)$ admits a continued fraction expansion of the form

$$x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \ldots}}} := [a_1(x), a_2(x), \ldots],$$

where $a_n(x)$ are positive integers. This representation of $x$ can be generated by the Gauss map $G$, in the sense that

$$a_1(x) = \lfloor 1/x \rfloor$$

and

$$a_{n+1}(x) = a_1(G^n(x))$$

for all $n \geq 1$. For any $x \in (0, 1)$, its continued fraction expansion is finite (i.e., there exists $k \geq 1$ such that $G^k(x) = 0$) if and only if $x$ is rational. For any irrational number $x \in (0, 1)$, we denote by

$$p_n(x) = [a_1(x), a_2(x), \ldots, a_n(x)]$$

the $n$th convergent of $x$, with $n \geq 1$ and $p_n(x)$ and $q_n(x)$ are relatively prime positive integers. Moreover, it is well known that

$$\frac{1}{2q_n^2(x)} \leq \left| x - \frac{p_n(x)}{q_n(x)} \right| \leq \frac{1}{q_n^2(x)}.$$

In other words, the order of $q_n^{-2}(x)$ dominates the speed of $p_n(x)/q_n(x)$ approximation. The result of Lévy [22] states that

$$\lim_{n \to \infty} \frac{1}{n} \log q_n(x) = \frac{\pi^2}{12 \log 2} =: \gamma \quad m = \text{a.e. } x \in (0, 1). \quad (3.1)$$

We obtain precise asymptotics on $q_n$ beyond (3.1). For $\varepsilon > 0$ and $n \geq 1$, put

$$\Gamma_n(\varepsilon) = m \left\{ x \in (0, 1) : \left| \frac{1}{n} \log q_n(x) - \gamma \right| \geq \varepsilon \right\}.$$

**Theorem 3.2.** We have

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^{\infty} \Gamma_n(\varepsilon) = \sigma^2 \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \log \varepsilon \sum_{n=1}^{\infty} \frac{\Gamma_n(\varepsilon)}{n} = 2.$$

**Proof.** We view $G$ as a dynamical system acting on the set of irrational numbers in $(0, 1)$. Then $G$ leaves invariant the Gauss measure

$$d\mu_G = \frac{1}{\log 2} \cdot \frac{dx}{1 + x}.$$

By Lévy’s theorem and Birkhoff’s ergodic theorem,

$$\int \log |G'| d\mu_G = 2\gamma.$$
We now apply theorem 1.3 to the dynamical system \(((0, 1) \setminus \mathbb{Q}, G, \mu_G)\) and the observable \(\log|G^n|\). (CLT) was established by Misevičius [26]. To verify (LD), we introduce the Lyapunov spectrum \(\alpha \in [2 \log((\sqrt{5} + 1)/2), \infty) \mapsto b(\alpha) \in [0, \infty)\) by

\[
b(\alpha) = \dim_H \left\{ x \in (0, 1) \setminus \mathbb{Q} : \lim_{n \to \infty} \frac{1}{n} \log |G^n|(x) = \alpha \right\},
\]

where \(\dim_H\) denotes the Hausdorff dimension on \([0, 1]\). The Lyapunov spectrum for the Gauss map was analyzed by Kesseböhmer and Stratmann [20], Pollicott and Weiss [31]. It was shown to be analytic, \(b(\alpha) = 0\) if and only if \(\alpha = 2\gamma\). Using the Lyapunov spectrum, we define \(I : [2 \log((\sqrt{5} + 1)/2) - 2\gamma, \infty) \mapsto [0, \infty)\) by

\[
I(\varepsilon) = (\varepsilon + 2\gamma)(1 - b(\varepsilon + 2\gamma)).
\]

Then \(I\) is analytic and \(I(0) = 0, I'(0) = 0\). By these and [37, main theorem], the function \(I\) in (3.2) satisfies all the conditions in (LD) but \(I'(0) > 0\), which we verify below.

**Lemma 3.3.** \(I'(0) > 0\).

**Proof.** A direct calculation gives \(I''(\varepsilon) = -2b'(\varepsilon + 2\gamma) - b''(\varepsilon + 2\gamma)(\varepsilon + 2\gamma)\). Substituting \(\varepsilon = 0\) gives

\[
I''(0) = -2b''(2\gamma)\gamma.
\]

To evaluate \(b''(2\gamma)\), we introduce a pressure function \(\beta \in (1/2, \infty) \mapsto P(\beta)\) by

\[
P(\beta) = \sup \left\{ h(\nu) - \beta \int \log |G'|d\nu : \nu \in \mathcal{M}(G), \int \log |G'|d\nu < \infty \right\},
\]

where \(\mathcal{M}(G)\) denotes the set of \(G\)-invariant Borel probability measures. The pressure function is convex and analytic [20, 31]. For each \(\alpha > 2 \log((\sqrt{5} + 1)/2)\), let \(\beta(\alpha)\) denote the solution of the equation \(P'(\beta(\alpha)) + \alpha = 0\). We have

\[
b(\alpha) = \frac{1}{\alpha}(P(\beta(\alpha)) + \alpha \beta(\alpha)).
\]

Differentiating (3.4) twice gives

\[
b''(\alpha) = -\frac{1}{\alpha^2}(-\beta'(\alpha)\alpha^3 - 2P'(\beta(\alpha))\alpha).
\]

By the implicit function theorem applied to the function \(P'(\beta) + \alpha, \alpha \mapsto \beta(\alpha)\) is differentiable and \(\beta'(\alpha) = -1/P''(\beta(\alpha)) < 0\). Since \(P(\beta(2\gamma)) = 0\), substituting \(\alpha = 2\gamma\) into (3.5) we obtain

\[
b''(2\gamma) = -\frac{\beta''(2\gamma)}{2\gamma} < 0,
\]

and therefore \(I''(0) > 0\).

Since the Radon–Nikodym derivative \(\frac{d\mu_G}{d\mu}\) is bounded from above and zero, and \(\log q_n(x)/\log(|G'|x)\) is uniformly bounded from above and zero over all \(n\) and \(x\), theorem 3.2 follows from theorem 3.1.
Acknowledgments

We would like to thank two anonymous referees for their invaluable comments and suggestions that have helped to improve the original manuscript. We would also like to thank Shanghai Center for Mathematics Science, and the 2019 Fall Program of Low Dimensional Dynamics, where part of this work was written. L Fang is supported by NSFC No. 11801591. H Takahasi is supported by the JSPS KAKENHI 19K21835 and 20H01811. Y Zhang is supported by NSFC Nos. 11701200, 11871262, and Hubei Key Laboratory of Engineering Modeling and Scientific Computing in HUST.

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