Distributed Nash Equilibrium Seeking for Noncooperative Games of High-Order Nonlinear Multi-Agent Systems Over Weight-Unbalanced Digraphs

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Abstract

In this paper, we investigate the noncooperative games of multi-agent systems. Different from existing noncooperative games, our formulation involves the high-order nonlinear dynamics of players, and the communication topologies among players are weight-unbalanced digraphs. Due to the high-order nonlinear dynamics and the weight-unbalanced digraphs, existing Nash equilibrium seeking algorithms cannot solve our problem. In order to seek the Nash equilibrium of the noncooperative games, we propose two distributed algorithms based on state feedback and output feedback, respectively. Moreover, we analyze the convergence of the two algorithms with the help of variational analysis and Lyapunov stability theory. By the two algorithms, the high-order nonlinear players exponentially converge to the Nash equilibrium. Finally, two simulation examples illustrate the effectiveness of the algorithms.

Keywords: Noncooperative games, Nash equilibrium, multi-agent systems, high-order nonlinear systems, cyber-physical systems, weight-unbalanced digraphs.

1. Introduction

Noncooperative games arise widely in varieties of fields, such as economic markets, smart grids, communication networks, and social networks (see [1, 2, 3, 4, 5, 6]). In noncooperative games, each player aims to selfishly minimize its own cost function, depending on its own decisions and the decisions of other players (see [7, 8, 9, 10]). Consequently, Nash equilibrium seeking is a critical problem in noncooperative games, which has received more and more attention recently (see [11, 12, 13, 14]).

In the past few years, numbers of Nash equilibrium seeking algorithms have been proposed for noncooperative games. For example, [15] presented consensus-based seeking algorithms for unconstrained noncooperative games. [13] designed projection-based seeking algorithms for noncooperative games with convex set constraints. [12] proposed primal-dual based seeking algorithms for noncooperative games with inequality constraints. [14] developed gradient-based seeking algorithms for noncooperative games with differentiable cost functions. [16] exploited subgradient-based seeking algorithms for noncooperative games with nondifferentiable cost functions. [17] proposed internal model-based seeking algorithms for disturbed noncooperative games.

Communication topologies among players have a significant influence on the design and analysis of distributed algorithms. Most of existing distributed Nash equilibrium seeking algorithms for noncooperative games rely on undirected graphs or weight-balanced digraphs, such as [7, 9, 10, 11, 12, 13, 14, 15, 16, 17]. However, an algorithm may be ineffective over weight-unbalanced digraphs, even though it is convergent over undirected graphs or weight-balanced digraphs (see [18]). Moreover, it is well-known that weight-unbalanced digraphs are the extensions of undirected graphs and weight-balanced digraphs, which have wider applications and are easier to implement in engineering practices. Consequently, it is necessary to investigate the noncooperative games over weight-unbalanced digraphs.

As an integration of computation, communication, and physical processes, cyber-physical systems (CPSs) occur commonly in many engineering practices, such as transportation systems, power systems, and mobile sensor networks (see [19, 20, 21]). In the background of CPSs, physical systems are used to autonomously perform distributed tasks, and hence increasing attention has been paid to distributed algorithms combined with the dynamics of...
physical systems. For instance, [22, 23] developed distributed optimization algorithms for Euler-Lagrange systems. [24, 25] proposed distributed resource allocation algorithms for second-order and high-order linear systems, respectively. [26, 27] designed distributed Nash equilibrium seeking algorithms for second-order systems and high-order linear systems, respectively. On the other hand, high-order nonlinear systems can characterize many real physical systems, such as jerk systems [28], synchronous generators [29], vehicles [30], and turbine generators [31], which extend first-order and second-order nonlinear systems, respectively. However, to the best of our knowledge, there are no results about noncooperative games of high-order nonlinear multi-agent systems over weight-unbalanced digraphs. Furthermore, without furthering involving the control of high-order nonlinear dynamics and the weight-unbalanced digraphs, existing Nash equilibrium seeking algorithms for noncooperative games, including [3, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 26, 27], cannot solve the problem.

The objective of this paper is to investigate the noncooperative games of high-order nonlinear multi-agent systems over weight-unbalanced digraphs, and design distributed algorithms to seek the Nash equilibrium of the noncooperative games. The main contributions of this paper are summarized as follows.

(i) This paper studies a noncooperative game with high-order nonlinear players over weight-unbalanced digraphs. The formulation extends the well-known noncooperative games, such as [11, 17, 26, 27], by adding the high-order nonlinear dynamics of players. Moreover, in our problem, the communication networks among players are weight-unbalanced digraphs, which extend undirected graphs and weight-balanced digraphs required by many results, such as [7, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Owing to the high-order nonlinear dynamics and the weight-unbalanced digraphs, existing Nash equilibrium seeking algorithms, such as [3, 5, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 26, 27], cannot be applied to this problem.

(ii) The distributed algorithm design and analysis of our problem face the following challenges. (i) The dynamics of players are high-order and nonlinear, and the cost functions of players are also nonlinear. Hence, the whole closed-loop system is a complicated high-order nonlinear system; (ii) The convergence analysis of existing Nash equilibrium seeking algorithms over undirected graphs or weight-balanced digraphs depend on the conditions that $1^T L = 0_N$, $L^T L = L$, and the positive semi-definition of $L$ or $L + L^T$, while the Laplacian matrix $L$ of weight-unbalanced digraphs is nonsymmetric and nonpositive definite.

(iii) This paper proposes a distributed state-based algorithm and a distributed output-based algorithm for the high-order nonlinear players to seek the Nash equilibrium of noncooperative games. Moreover, this paper analyzes the convergence of the two algorithms. Under the two algorithms, the high-order nonlinear players exponentially converge to the Nash equilibrium, even the high-order nonlinear dynamics contain uncertain parameters. Furthermore, the distributed output-based algorithm only depends on the output variables of the high-order nonlinear players rather than all state information.

This paper is organized as follows. Section 2 introduces preliminaries, and formulates the problem. Section 3 presents two distributed Nash equilibrium seeking algorithms, and analyzes their convergence. Section 4 provides two examples to verify the algorithms. Finally, Section 5 summarizes the conclusion.

2. Preliminaries and Formulation

This section first presents some preliminaries, and then formulates the studied problem.

2.1. Preliminaries

Notations. $\mathbb{R}$ and $\mathbb{R}^n$ denote the set of real numbers and the $n$-dimensional Euclidean space, respectively. $\mathbb{Z}_+$ is the set of nonnegative integers. $\otimes$ represents the Kronecker product. $\| A \|$ and $\| x \|$ are the spectral norm of matrix $A$ and the standard Euclidean norm of vector $x$, respectively. $x^T$ is the transpose of $x$. $\text{col}(x_1, \ldots, x_n) = [x_1^T, \ldots, x_n^T]^T$ with $x_i \in \mathbb{R}^n$. $I_n$ and $0_{n \times n}$ denote the identity and zero matrices with $n \times n$ dimensions, respectively. $0_n$ and $1_n$ denote the column vectors of $n$ ones and zeros, respectively. $\text{diag}(v_1, v_2, \ldots, v_N)$ denotes a diagonal matrix with the elements $v_1, v_2, \ldots, v_N$ being all on its main diagonal.

2.1.1. Graph Theory (see [34])

A directed graph (or simple a digraph) of $N$ nodes is denoted by $G := \{V, E, A\}$, where $V = \{1, \ldots, N\}$, $E \subseteq V \times V$ are the node and edge sets, respectively, and $A$ is the adjacency matrix. $(i, j) \in E$ is an edge of $G$ if node $i$ can receive information from node $j$. Denote adjacency matrix $A := [a_{ij}]_{N \times N}$, where $a_{ij} > 0$ if $(i, j) \in E$, and $a_{ij} = 0$, otherwise. Here $a_{ii} = 0$ for any $i \in V$, which indicates no self-connection in the graph. Besides,
for an edge \((i, j) \in E\), \(i\) is called the out-neighbor of \(j\), and \(j\) is called the in-neighbor of \(i\). The weighted in-degree and weighted out-degree of node \(i\) are \(d^i_{\text{in}} = \sum_{j=1}^{N} a_{ij}\) and \(d^i_{\text{out}} = \sum_{j=1}^{N} a_{ji}\), respectively. Note that \(d^i_{\text{in}}\) may not equal to \(d^i_{\text{out}}\) for weight-unbalanced digraphs. A directed path from \(v_0\) and \(v_l\) is defined as a sequence of nodes \(v_0, v_1, \ldots, v_{l-1}, v_l\), such that \((v_k, v_{k+1}) \in E, \forall k \in \{0, 1, \ldots, l-1\}\).

Particularly, a digraph \(G\) is said to be strongly connected if there exists a directed path between any two distinct nodes. The Laplacian matrix of \(G\) is \(L := D_{\text{in}} - A\), where \(D_{\text{in}} := \text{diag}\{d^1_{\text{in}}, \ldots, d^n_{\text{in}}\} \in \mathbb{R}^{N \times N}\). Obviously, \(L1_N = 0_N\).

For a weighted-unbalanced digraph \(G\), we have the following results.

**Lemma 1.** (see [35]) Suppose the weight-unbalanced digraph \(G\) is strongly connected with the Laplacian matrix \(L\), and \(M\) is a diagonal matrix with nonnegative diagonal elements and at least one diagonal element being positive. Then, the following conditions hold.

(i) \(L + M\) is positive definite;

(ii) There exists a positive definite matrix \(Q \in \mathbb{R}^{N \times N}\) such that \(Q(L + M) + (L + M)^T Q = I_N\).

### 2.1.2. Variational Analysis (see [36])

A function \(f: \mathbb{R}^m \rightarrow \mathbb{R}\) is convex if

\[
 f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) \quad \forall x, y \in \mathbb{R}^m, \forall \alpha \in [0, 1].
\]

A map \(F: \mathbb{R}^m \rightarrow \mathbb{R}^m\) is \(\omega\)-strongly monotone (\(\omega > 0\)) if

\[
 (x - y)^T (\nabla F(x) - \nabla F(y)) \geq \omega \|x - y\|^2, \quad \forall x, y \in \mathbb{R}^m.
\]

A map \(F: \mathbb{R}^m \rightarrow \mathbb{R}^m\) is \(\theta\)-Lipschitz (\(\theta > 0\)) if

\[
 \|F(x) - F(y)\| \leq \theta \|x - y\|, \quad \forall x, y \in \mathbb{R}^m.
\]

For a continuously differentiable map \(F: \mathbb{R}^m \rightarrow \mathbb{R}^m\), it is (strongly) monotone if and only if its Jacobian matrix \(JF\) is (uniformly) positive definite (see [36]).

The solution \(x \in \mathbb{R}^m\) of the variational inequality \(VI(\mathbb{R}^m, F), F: \mathbb{R}^m \rightarrow \mathbb{R}^m\), satisfies the following condition (see [36]).

\[
 (y - x)^T F(x) \geq 0, \quad \forall y \in \mathbb{R}^m. \tag{1}
\]

In addition, \(x\) satisfies the condition (1) if and only if \(x\) is the solution of \(F(x) = 0_m\).

The following lemma is about the uniqueness of the solution of \(VI(\mathbb{R}^m, F)\).

**Lemma 2.** (see [37]) If a continuous map \(F: \mathbb{R}^m \rightarrow \mathbb{R}^m\) is \(\omega\)-strongly monotone, then \(VI(\mathbb{R}^m, F)\) has a unique solution.

### 2.2. Problem Formulation

Consider a noncooperative game of \(N\) players over a weight-unbalanced digraph \(G\). Player \(i \in V\) has a differentiable cost function \(J_i(x_i, x_{-i}) : \mathbb{R}^{Nm} \rightarrow \mathbb{R}\), where \(x_i \in \mathbb{R}^m\) is the decision of player \(i\), \(x_{-i} = \text{col}(x_{1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)\). The purpose of player \(i\) is to minimize its own cost function \(J_i(x_i, x_{-i})\) by changing its own decision \(x_i\). Specifically, player \(i\) faces the following optimization problem.

\[
 \min_{x_i \in \mathbb{R}^m} J_i(x_i, x_{-i}). \tag{2}
\]

The Nash equilibrium of the noncooperative game (2) is defined as follows (referring to [8, 11]).

**Definition 1.** A decision profile \(x^* := (x_i^*, x_{-i}^*)\) is a Nash equilibrium of the noncooperative game (2) if

\[
 J_i(x_i^*, x_{-i}) \leq J_i(x_i, x_{-i}), \forall x_i \in \mathbb{R}^m, i \in V.
\]

The above definition indicates that, at a Nash equilibrium, no player can decrease its cost function by changing its own decision unilaterally.

Player \(i \in V\) has the following uncertain \(n\)-th-order nonlinear dynamics.

\[
 \dot{x}_i = x_i^{(1)}
\]

\[
 : \quad x_i^{(n-1)} = x_i^{(n)}
\]

\[
 x_i^{(n)} = f_i(x_i, x_i^{(1)}, \ldots, x_i^{(n-1)}, w_l) + u_i
\]

where \(x_i^{(l)}, l \in \{1, \ldots, n\}\), is the \(l\)-th order derivative of \(x_i\); \(f_i(\cdot, \ldots, \cdot, w_l) : \mathbb{R}^m \times \cdots \times \mathbb{R}^m \rightarrow \mathbb{R}^m\) is the nonlinear function; \(w_l \in \mathbb{R}^{m_w}\); is the unknown constant parameter with \(n_w \in \mathbb{Z}^+_+\); and \(u_i \in \mathbb{R}^m\) is the control input of player \(i\).

**Remark 1.** In engineering, the dynamics of many physical systems, such as jerk systems [28], synchronous generators [29], vehicles [30], and turbine generators [31], can be described by (3). Besides, the first-order and second-order nonlinear systems, and high-order linear systems investigated in [17, 26, 27] are the special cases of the high-order nonlinear system (3). Therefore, our results can be straightforwardly applied to these systems.

The aim of this paper is to design distributed algorithms such that the output of high-order nonlinear player (3) converges to the Nash equilibrium of the noncooperative game (2).
Remark 2. In contrast to well-studied noncooperative games, such as [5, 7, 13, 15], this formulation involves the high-order nonlinear dynamics of players, which is in keeping with the development of CPSs. Moreover, the communication topologies among players in our formulation are weight-unbalanced digraphs, which are weaker than undirected graphs and weight-balanced digraphs required by [7, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Without considering the control of high-order dynamics and the weight-unbalanced digraphs, existing Nash equilibrium seeking algorithms are ineffective for our problem, such as [3, 5, 7, 10, 11, 12, 13, 14, 15, 16, 17, 26, 27]. Because of the high-order nonlinear dynamics, the nonlinear cost functions, and the weighted-unbalance digraphs, it is not easy to design distributed Nash equilibrium seeking algorithms for our problem and analyze their convergence.

Some assumptions about the communication topologies, the high-order nonlinear dynamics, and the cost functions are presented as follows.

Assumption 1. The weight-unbalanced digraph $G$ is strongly connected.

Assumption 2. $f_i(\cdot)$ satisfies the following Lipschitz-like condition.
\[
\|f_i(x_i, x_i^{(1)}, \ldots, x_i^{(n-1)}, w_i) - f_i(\tilde{x}_i, \tilde{x}_i^{(1)}, \ldots, \tilde{x}_i^{(n-1)}, w_i)\| \\
\leq l_x \|x_i - \tilde{x}_i\| + \sum_{i=1}^{n-1} l_{x(i)} \|x_i^{(i)} - \tilde{x}_i^{(i)}\|
\]
where $l_x, l_{x(i)} \geq 0$.

Remark 3. Assumption 2 generalizes the Lipschitz condition satisfied by numerous well-known systems, such as Chen systems, Chua’s circuits, Lorenz systems, pendulum systems and car-like robots (see [38] and references therein). What is more, if the partial derivatives of $f_i(\cdot)$ are uniformly bounded, such as (piecewise-) linear continuous functions, then Assumption 2 holds.

Assumption 3. The cost function $J_i(x_i, x_{-i})$ is continuously differentiable in $x$ and convex in $x_i$ for every fixed $x_{-i}$.

Assumption 4. The pseudo-gradient $F(x)$ is $\omega$-strongly monotone and $\theta$-Lipschitz continuous, where $F(x) = \text{col}(\nabla_{x_1} J_1(x_1, x_{-1}), \ldots, \nabla_{x_N} J_N(x_N, x_{-N}))$. (4)

Remark 4. Assumptions 3 and 4 indicate the existence and uniqueness of the Nash equilibrium of the noncooperative game (2), which were widely used in noncooperative games (see [11, 15, 17, 26]). Also, Assumptions 3 and 4 hold in a variety of practical engineering problems, such as Nash-Cournot games of generation systems (see [11]), energy consumption games in smart grids (see [3]), and rate allocation games in communications (see [39]).

With Assumptions 3 and 4, we have the following results about the Nash equilibrium of the noncooperative game (2).

Lemma 3. (see [40]) Under Assumption 3, the solution of $VI(\mathbb{R}^{Nm}, F)$ coincides with the Nash equilibrium of the noncooperative game (2).

Lemma 4. Under Assumption 3, $x^* = \text{col}(x_1^*, \ldots, x_N^*)$ is the Nash equilibrium of the noncooperative game (2) if and only if
\[
\nabla_{x_i} J_i(x_i^*, x_{-i}^*) = 0, \forall i \in \mathcal{V} \text{ or } F(x^*) = 0_{Nm}.
\]

Proof. Based on Lemma 3, the solution of $VI(\mathbb{R}^{Nm}, F)$ is equivalent to the Nash equilibrium of the noncooperative game (2), and the converse is true. Furthermore, the solution of $VI(\mathbb{R}^{Nm}, F)$ satisfies $F(x) = 0_{Nm}$. Consequently, we obtain the conclusion. $\square$

3. Main Results

This section proposes two distributed Nash equilibrium seeking algorithms for the noncooperative game (2) with the high-order nonlinear player (3), and then analyzes their convergence.

3.1. Distributed State-Based Algorithm

Before giving our algorithms, the following characteristic polynomial associated with real coefficients $(k_1, \ldots, k_{n-1})$ is defined such that its roots are in the open left half plane (LHP).
\[
p(s) := s^{n-1} + k_{n-1}s^{n-2} + \ldots + k_2s + k_1.
\]

The above definition indicates that the following companion matrix $A$ is Hurwitz.
\[
A = \begin{bmatrix}
0_{n-2} & I_{n-2} \\
-k_1 & -k_2 & \ldots & -k_{n-1}
\end{bmatrix}.
\]

The following well-known lemma about the companion matrix $A$ is used later.

Lemma 5. (see [41]) There exists a positive definite symmetric matrix $P := [p_{ij}]_{(n-1)\times(n-1)}$ such that $PA + A^TP = -I_{n-1}$ is satisfied.

When the players’ state variables are obtainable, the distributed Nash equilibrium seeking algorithm for player $i \in \mathcal{V}$ is designed as follows.
\[
u_i = -\sum_{l=1}^{n-1} \varepsilon^{n-1-l} k_l x_i^{(l)} - \alpha_1 \nabla_{x_i} J_i(x_i, \tilde{x}_{-i}) - \alpha_2 y_i
\]
\[
\dot{y}_i = \sum_{l=1}^{n-1} \varepsilon^{l-1} k_l x_i^{(l)} + \frac{\alpha_1}{\varepsilon^{n-1}} \nabla_x J_i(x_i, \hat{x}_{-i}) \quad (8b)
\]
\[
\dot{x}_j = -\alpha_3 \left( \sum_{k=1}^{N} a_{jk} (\hat{x}_j - \hat{x}_k) + a_{ij} (\hat{x}_i - x_j) \right) \quad (8c)
\]
where \( \hat{x}_{-i} = \text{col}(\hat{x}_1, \ldots, \hat{x}_{i-1}, \hat{x}_{i+1}, \ldots, \hat{x}_N) \) with \( \hat{x}_j \) being the estimate of player \( i \) on \( x_j \) of player \( j \); \( \alpha_1, \alpha_2, \alpha_3 \) and \( \varepsilon \) are parameters (to be determined later); \( k_1, \ldots, k_{n-1} \) are the coefficients of the characteristic polynomial \( p(s) \) defined in (6) with roots in the open LHP.

Substituting (8) into (3), we have the following closed-loop system.
\[
\dot{x} = x^{(1)}
\]
\[
x^{(n)} = f(x, \ldots, x^{(n-1)}, w) - \sum_{l=1}^{n-1} \varepsilon^{n-l} k_l x^{(l)}
\]
\[
- \alpha_1 F(x, \hat{x}) - \alpha_2 y
\]
\[
\dot{y} = -\alpha_3 \left( \sum_{l=1}^{n-1} \varepsilon^{l-1} k_l x^{(l)} \right) + \frac{\alpha_1}{\varepsilon^{n-1}} F(x, \hat{x})
\]
\[
\dot{x} = -\alpha_3 \left( (L \otimes I_N) \otimes I_m \right) \hat{x} + (M \otimes I_m) (\hat{x} - 1_N \otimes x) \quad (9c)
\]
where \( x = \text{col}(x_1, \ldots, x_N) \); \( x^{(l)} = \text{col}(x_1^{(l)}, \ldots, x_N^{(l)}) \) with \( l \in \{1, \ldots, n\} \); \( y = \text{col}(y_1, \ldots, y_N) \); \( \hat{x} = \text{col}(\hat{x}_1, \ldots, \hat{x}_N) \);
\[
f(x, \ldots, x^{(n-1)}, w) = \text{col}(f_1(x_1, \ldots, x_{i-1}, w_1), \ldots, f_N(x_N, \ldots, x_{N-(n-1)}, w_N));
\]
\[
F(x, \hat{x}) = \text{col}(\nabla_x J_1(x_1, \hat{x}_{-1}), \ldots, \nabla_x J_N(x_N, \hat{x}_{-N}));
\]
and \( M = \text{diag}(M_1, \ldots, M_N) \) with \( M_i = \text{diag}(a_{1i}, a_{2i}, \ldots, a_{Ni}) \). In what follows, we use \( f \) to denote \( f(x, \ldots, x^{(n-1)}, w) \), if there is no ambiguity.

For the system (9), we have the following theorem.

**Theorem 1.** Under Assumptions 1 and 3, if \((x^*, x^{*(1)}, \ldots, x^{*(n-1)}, y^*, \hat{x}^*)\) is an equilibrium point of (9), then \( x^* \) is a Nash equilibrium of the noncooperative game (2). Conversely, if \( x^* \) is a Nash equilibrium of the noncooperative game (2), there exists \((x^{*(1)}, \ldots, x^{*(n-1)}, y^*, \hat{x}^*) \in \mathbb{R}^{Nm} \times \cdots \times \mathbb{R}^{Nm} \times \mathbb{R}^{Nm} \times \mathbb{R}^{N^2m} \) such that \((x^*, x^{*(1)}, \ldots, x^{*(n-1)}, y^*, \hat{x}^*) \) is an equilibrium point of (9).

**Proof.** (i) The equilibrium point of (9) satisfies the following conditions.
\[
0_{Nm} = x^{*(l)}, l \in \{1, \ldots, n-1\} \quad (10a)
\]
\[
0_{Nm} = f^* - \sum_{l=1}^{n-1} \varepsilon^{n-l} k_l x^{*(l)} - \alpha_1 F(x^*, \hat{x}^*) - \alpha_2 y^*
\]
\[
= \sum_{i=1}^{N} \| \nabla_x J_i(x_i, \hat{x}_{-i}) - \nabla_x J_i(x_i, \tilde{y}_{-i}) \|^2 \leq \theta^2 \| \hat{x}_i - \tilde{y}_i \|^2 \leq \theta^2 \| x_i - y_i - \tilde{y}_i \|^2 \leq \theta^2 \| \hat{x}_i - \tilde{y}_i \|^2 \quad \forall i \in \{1, \ldots, N\},
\]
where \( \hat{x}_i = \text{col}(\hat{x}_1, \ldots, x_i, \ldots, \hat{x}_N) \) and \( \tilde{y}_i = \text{col}(\tilde{y}_1, \ldots, x_i, \ldots, \tilde{y}_N) \). Therefore, it is not hard to obtain \( \| F(x, \tilde{x}) - F(x, \hat{y}) \|_N \leq \theta \| x_i - y_i \|^2 \)
\[ \sum_{i=1}^{N} \theta^2 \| \dot{x}_i - \dot{y}_i \|^2 = \theta^2 \| \dot{x} - \dot{y} \|^2, \] which implies that 
\[ F(x, \dot{x}) \] is \( \theta \)-Lipschitz in \( \dot{x} \).

By virtue of Lemma 7, the following result is obtained.

**Theorem 2.** Under Assumptions 1, 2, 3 and 4, the high-order nonlinear player \((3)\) with the algorithm \((8)\) exponentially converges to the Nash equilibrium of the noncooperative game \((2)\).

**Proof.** Without loss of generality, let \( m = 1 \) for simplicity. Let 
\[ \dot{x} = x - x^*, \quad \ddot{y} = y - y^*, \quad \dddot{x} = x^{(l)} - x^{(r)}, \quad \dddot{x} = \dddot{x} - I_N \otimes x, \quad f(\dddot{x}, \dddot{x}(n-1), w) = f(x, \ldots, x^{(n-1)}, w) - f(x^*, \ldots, x^{(n-1)}, w), \]
where \( l \in \{1, \ldots, n-1\} \). For simplicity, in what follows, we use \( f \) to represent \( f(\dddot{x}, \dddot{x}(n-1), w) \).

Combining (9) with (10), we have
\[
\dddot{x} = \dddot{x}(1) \\
\dddot{x}(n) = \dddot{x} - \sum_{i=1}^{n-1} \varepsilon^{n-1} k_i \dddot{x}(i) - \alpha_1 h - \alpha_2 \dddot{y} \\
\dddot{y}(1) = \dddot{y} + \sum_{i=1}^{n-1} \varepsilon^{n-1} k_i \dddot{x}(i) + \frac{\alpha_1}{\varepsilon^{n-1}} \dddot{y} \\
\dddot{y}(n) = -\alpha_3 ((L \otimes I_N) + M) \dddot{x} - 1_N \otimes \dddot{x}
\]
where \( h = F(x, \dddot{x}) - F(x^*, \dddot{x}^*) \).

After the above transformation, the equilibrium point of (11) is the origin. Let 
\[ \dddot{x} = \text{col}(\dddot{x}(1), \ldots, \dddot{x}(n-1)), \]
\[ \dddot{x}(l) = \text{col}(\dddot{x}(l), \ldots, \dddot{x}(n-1)), \]
where \( \dddot{x}(l) \) with \( l \in \{1, \ldots, n-1\} \).

Thus (11) can be rewritten as
\[
\dddot{x} = \varepsilon \dddot{x}(1) \\
\dddot{x} = \varepsilon (A \otimes I_N) \dddot{x} + \frac{1}{\varepsilon^{n-1}} (b \otimes I_N)(\dddot{f} - \alpha_1 \dddot{h} - \alpha_2 \dddot{y}) \\
\dddot{y} = \sum_{i=1}^{n-1} \varepsilon^{n-1} k_i \dddot{x}(i) + \frac{\alpha_1}{\varepsilon^{n-1}} \dddot{y} \\
\dddot{y} = -\alpha_3 ((L \otimes I_N) + M) \dddot{x} - 1_N \otimes \dddot{x}
\]
where \( b = [0_{n-2}^T 1]^T \) and \( A \) is defined in (7). Clearly, it is equivalent between (11) and (12).

Take the following Lyapunov function for (12).
\[ V_1 = \dddot{x}^T (P \otimes I_N) \dddot{x} + \dddot{x}^T Q \dddot{x} + \frac{1}{2} \| \dddot{x}(n-1) \|^2 + \frac{1}{2} \| \dddot{x}(n-1) + \dddot{y} \|^2 \]
where \( P \) and \( Q \) are positive definite matrices such that 
\[ PA + A^T P = -I_{n-1} \] (see Lemmas 5) and 
\[ Q(L \otimes I_N + M) + (L \otimes I_N + M)^T Q = I_{n^2} \] (see Lemma 1), respectively.

The derivative of \( V_1 \) along (12) is
\[ \dot{V}_1 = -\alpha_3 \| \dddot{x} \|^2 - 2 \dddot{x}^T (1_N \otimes \dddot{x}) - \| \dddot{y} \|^2 - \frac{\alpha_2^2}{\varepsilon^{n-1}} \| \dddot{y} \|^2 \\
+ \frac{1}{\varepsilon^{n-1}} \dddot{y}^T \dddot{f} + \frac{k_1}{\varepsilon^{n-1}} \dddot{x}^T (f - \alpha_1 \dddot{h} - \alpha_2 \dddot{y}) \\
+ \frac{1}{\varepsilon^{n-1}} \left( \sum_{l=1}^{n-2} (2p_{l, n-1} + k_{l+1}) \dddot{x}(l) \right) (\dddot{f} - \alpha_2 \dddot{y}) \\
- \frac{\alpha_1}{\varepsilon^{n-1}} \left( \sum_{l=1}^{n-2} (2p_{l, n-1} + k_{l+1}) \dddot{x}(l) \right) (\dddot{f} - \alpha_2 \dddot{y}) \\
+ (2p_{n-1, n-1} + 1) \dddot{x}(n-1) \right) \right) \right)^T h. \]

Owing to \( F(x) = F(x, 1_N \otimes x) \) and \( F(x^*) = F(x^*, \dddot{x}^*) \), based on the \( \omega \)-strongly monotonicity of \( F(x) \) (see Assumption 4), and the \( \theta \)-Lipschitz continuity of \( F(x, \dddot{x}) \) (see Lemma 7), we obtain
\[
\frac{1}{\varepsilon^{n-1}} \dddot{y}^T \dddot{f} \leq \frac{1}{\varepsilon^{n-1}} \| \dddot{y} \| (l_x \| \dddot{x} \| + \sum_{i=1}^{n-1} l_{x(i)} \| \dddot{x}(i) \|) \\
\leq \frac{1}{2 \varepsilon^{n-1}} \left( l_x + \sum_{i=1}^{n-1} l_{x(i)} \varepsilon^{2l + 1 - n} \| \dddot{y} \|^2 \right) \\
+ l_x \| \dddot{x} \|^2 + \sum_{i=1}^{n-1} l_{x(i)} \varepsilon^{n-1} \| \dddot{x}(i) \|^2 \right) \\
\leq \frac{k_1}{\varepsilon^{n-1}} \dddot{x}^T \dddot{f} \leq \frac{k_1}{\varepsilon^{n-1}} \| \dddot{x} \| (l_x \| \dddot{x} \| + \sum_{i=1}^{n-1} l_{x(i)} \| \dddot{x}(i) \|) \\
\leq \frac{1}{2 \varepsilon^{n-1}} \left( 2l_x + \sum_{i=1}^{n-1} l_{x(i)} \varepsilon^{2l + 1 - n} \| \dddot{x} \|^2 \right) \\
+ \sum_{i=1}^{n-1} l_{x(i)} \varepsilon^{n-1} \| \dddot{x}(i) \|^2 \right) \right) \right)^2. \] (13)

By Assumption 2, we have
\[

16

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By Young’s inequality, we have

\[ -2\tilde{x}^T(1_N \otimes \hat{\theta}) \leq \varepsilon^2 \| \tilde{x} \|^2 + N \| \tilde{x}^{(1)} \|^2 \]  
\[ -k_1 \alpha_2 \varepsilon \| \tilde{x} \|^2 \leq \frac{k_1 \alpha_2}{2\varepsilon n-1} \| \tilde{x} \|^2 + \alpha_2 \frac{2n}{2\varepsilon n-1} \| \bar{y} \|^2 \]  

\[ \left( \sum_{i=1}^{n-2}(2p_{i(n-1)} + k_{i+1})\tilde{x}^{(i)} \right)^T \tilde{f} \]
\[ + (2p_{i(n-1)(n-1)} + 2)\tilde{x}^{(n-1)} \]  
\[ \leq \sum_{i=1}^{n-2}(2p_{i(n-1)} + k_{i+1})\tilde{x}^{(i)} \left( I_x \| \bar{x} \| + \sum_{i=1}^{n-1} I_{x(i)} \| \bar{x}^{(i)} \| \right) \]
\[ + |2p_{i(n-1)(n-1)} + 2| \| \tilde{x}^{(n-1)} \| \left( I_x \| \bar{x} \| + \sum_{i=1}^{n-1} I_{x(i)} \| \bar{x}^{(i)} \| \right) \]  
\[ \leq \frac{1}{2} \left( I_x + \sum_{i=1}^{n-1} I_{x(i)} \varepsilon^{2l+1-n} \right) \left( \sum_{i=1}^{n-2}(2p_{i(n-1)} + k_{i+1})^2 \| \tilde{x}^{(i)} \|^2 \right) + (2p_{i(n-1)(n-1)} + 2)^2 \| \tilde{x}^{(n-1)} \|^2 \]
\[ + \frac{n-1}{2} \left( I_x \| \bar{x} \|^2 + \sum_{i=1}^{n-1} I_{x(i)} \varepsilon^{n-1} \| \tilde{x}^{(i)} \|^2 \right) \]  

(15a)

(15b)

(15c)

\[ (\sum_{i=1}^{n-2}(2p_{i(n-1)} + k_{i+1})\tilde{x}^{(i)} \right)^T h \]
\[ \leq \left( \sum_{i=1}^{n-2}(2p_{i(n-1)} + k_{i+1})\tilde{x}^{(i)} \right)^T \| \tilde{x} \| \]
\[ + \left(2p_{i(n-1)(n-1)} + 1\right)\tilde{x}^{(n-1)} \]  
\[ \leq \alpha_2 \theta \left( \sum_{i=1}^{n-2}(2p_{i(n-1)} + k_{i+1})^2 \| \tilde{x}^{(i)} \|^2 \right) + (2p_{i(n-1)(n-1)} + 1)^2 \| \tilde{x}^{(n-1)} \|^2 + (n-1)\| \tilde{x} \|^2 \]  

(15d)

\[ \leq \frac{n\alpha_2}{2} \left( \sum_{i=1}^{n-2}(2p_{i(n-1)} + k_{i+1})^2 \| \tilde{x}^{(i)} \|^2 \right) + (2p_{i(n-1)(n-1)} + 2)^2 \| \tilde{x}^{(n-1)} \|^2 \]
\[ + \frac{(n-1)\alpha_2}{2n} \| \bar{y} \|^2. \]

Combining (13)-(15), we have

\[ \bar{V}_i \leq -\rho_1 \| \bar{x} \|^2 - \rho_2 \| \bar{x} \|^2 - \rho_3 \| \bar{y} \|^2 - \rho_4 \| \tilde{x} \|^2 \]

where

\[ \rho_1 = \frac{1}{2\varepsilon n-1}(k_1 \omega_1 \alpha_1 - k_1^2 \alpha_2 - k_1^1 \bar{l}_x(2k_1 + n)) \]
\[ - \sum_{i=1}^{n-1} k_{i} I_{x(i)} \varepsilon^{2l+1-n} \]
\[ \rho_2 = \frac{1}{2\varepsilon n-1}(2\varepsilon n - 2N \varepsilon^{n-1} - \bar{r}_x \varepsilon^{n-1}(n + k_1) \]
\[ - \bar{p}(I_x + \sum_{i=1}^{n-1} I_{x(i)} \varepsilon^{2l+1-n} + \theta \alpha_1 + n\alpha_2) \]
\[ \rho_3 = \frac{1}{2\varepsilon n-1} \alpha_2 - \frac{n}{\rho} \bar{l}_x + \sum_{i=1}^{n-1} I_{x(i)} \varepsilon^{2l+1-n} \]
\[ \rho_4 = \alpha_3 - \frac{\theta}{\rho} \alpha_1 + \frac{n}{\varepsilon n-1} (n - 1 + \frac{k_1 \theta}{\omega}) \]

\[ p_{i(n-1)} \] is the last element of the ith row vector of matrix P; \( \bar{p} = \max\{ (2p_{i(n-1)} + k_2)^2, \ldots, (2p_{i(n-2)(n-1)} + k_{n-1})^2, (2p_{i(n-1)(n-1)} + 2)^2 \} \); and \( \bar{r}_x = \max\{ l_x, I_{x(i)}, \ldots, I_{x(n-1)} \} \).

Obviously, by choosing suitable sufficiently large positive constants \( \varepsilon, \alpha_1, \alpha_2 \) such that \( \varepsilon^{n-1} < \alpha_2 < \alpha_1 < \varepsilon^n \), we have \( \rho_1, \rho_2, \rho_3, \rho_4 > 0 \). Further, \( \rho_4 > 0 \) holds by taking an appropriate \( \alpha_3 \). Hence, there exist positive constants \( \varepsilon, \alpha_1, \alpha_2 \) and \( \alpha_3 \) such that \( \bar{V}_i \leq 0 \). Moreover, because \( V_1 \) and \( \bar{V}_1 \) are quadratic form, the high-order nonlinear player (3) with the algorithm (8) globally exponentially converges to the Nash equilibrium of the noncooperative game (2). \( \square \)

3.2. Distributed Output-Based Algorithm

When only the players’ output variables are obtainable, the distributed Nash equilibrium seeking algorithm for player \( i \in V \) is designed as follows.

\[ u_i = -\sum_{i=1}^{n-1} \varepsilon^{n-i} k_1 z_i^{(i)} - \alpha_1 \nabla x_i J_i(x_i, \bar{x}_i) - \alpha_2 y_i \]  

(16a)

\[ \dot{y}_i = \sum_{i=1}^{n-1} \varepsilon^{n-i} k_1 z_i^{(i)} + \frac{\alpha_1}{\varepsilon^{n-1}} \nabla x_i J_i(x_i, \bar{x}_i) \]  

(16b)

\[ \dot{z}_i = \frac{\alpha_1}{\mu} (x_i - z_i) \]  

(16c)

\[ \dot{z}_i^{(1)} = \frac{\alpha_2}{\mu^2} (x_i - z_i) \]  

(16d)
\[
\frac{\dot{z}_i^{(n)}}{\mu^n} = \varepsilon^n \beta_n \left( x_i - z_i \right)
\]
(16e)

\[
\dot{x}_j = -\alpha_3 \sum_{k=1}^{N} a_{jk} (\hat{x}_j - \tilde{x}_j^2) + a_{ij} (\hat{x}_j - x_j)
\]
(16f)

where \(k_1, \ldots, k_{n-1}\) are the coefficients of the characteristic polynomial \(p(s)\) defined in (6) with roots in the open LHP; \(\beta_1, \ldots, \beta_n\) are the coefficients of \(s^n + \beta_1 s^{n-1} + \ldots + \beta_{n-1} s + \beta_n = 0\) with roots in the open LHP; \(\alpha_1, \alpha_2, \alpha_3, \varepsilon\) and \(\mu\) are parameters (to be determined later).

The algorithm (16) is similar to the algorithm (8), except from \(z_i\) and \(z_i^{(l)}\). In the algorithm (16), \(z_i\) and \(z_i^{(l)}\) are used to estimate the state variables.

Combining (3) with (16), we obtain the following closed-loop system.

\[
\dot{x} = x^{(1)}
\]
(17a)

\[
x^{(n)} = f - \sum_{l=1}^{n-1} \varepsilon^{n-l} k_l z^{(l)} - \alpha_1 F(x, \hat{x}) - \alpha_2 y
\]
(17b)

\[
\dot{y} = \sum_{l=1}^{n-1} \varepsilon^{n-l} k_l z^{(l)} + \frac{\alpha_1}{\varepsilon^{n-1}} F(x, \hat{x})
\]
(17c)

\[
\dot{z} = \varepsilon^{n} \beta_n (x - z)
\]
(17d)

\[
\dot{z}^{(2)} = \frac{\varepsilon^{n-1} \beta_1}{\mu} (x - z)
\]
(17e)

\[
z^{(n)} = \frac{\varepsilon^n \beta_n}{\mu^n} (x - z)
\]
(17f)

\[
\dot{x} = -\alpha_3 ((L \otimes I_N) \otimes I_m) \hat{x} + (M \otimes I_m) (\hat{x} - N \otimes x)
\]
(17g)

where \(z = \text{col}(z_1, \ldots, z_N)\) and \(z^{(l)} = \text{col}(z_1^{(l)}, \ldots, z_N^{(l)})\) with \(l \in \{1, \ldots, n\}\).

There are following results about the system (17).

**Theorem 3.** Under Assumptions 1 and 3, if \((x^*, x^{*(1)}, \ldots, x^{*(n-1)}, z^*, z^{*(1)}, \ldots, z^{*(n-1)}, y^*, \hat{x}^*)\) is an equilibrium point of (17), \(x^*\) is a Nash equilibrium of the noncooperative game (2). Conversely, if \(x^*\) is a Nash equilibrium of the noncooperative game (2), there exists \((x^{*(1)}, \ldots, x^{*(n-1)}, z^*, z^{*(1)}, \ldots, z^{*(n-1)}, y^*, \hat{x}^*)\) in \(\mathbb{R}^{N-m} \times \cdots \times \mathbb{R}^{N-m} \times \mathbb{R}^{N-m} \times \cdots \times \mathbb{R}^{N-m} \times \mathbb{R}^{N-m} \times \mathbb{R}^{N-m} \times \mathbb{R}^{N-m} \times \mathbb{R}^{N-m}\) such that \((x^*, x^{*(1)}, \ldots, x^{*(n-1)}, z^*, z^{*(1)}, \ldots, z^{*(n-1)}, y^*, \hat{x}^*)\) is an equilibrium point of (17).

**Proof.** At the equilibrium point of (17), we have \(x^* = z^*\) and \(x^{*(l)} = z^{*(l)} = 0_{N-m}, l \in \{1, \ldots, n-1\}\). Therefore, similarly to the proof of Theorem 1, the conclusion is yielded.

**Lemma 8.** The high-order nonlinear player (3) converges to the Nash equilibrium of the noncooperative game (2) under the algorithm (16), if the system (17) is stabilized to its equilibrium point.

**Proof.** Similarly to the proof of Lemma 6, the conclusion can be directly deduced.

With Lemma 8, we can obtain the following result.

**Theorem 4.** Under Assumptions 1, 2, 3 and 4, the high-order nonlinear player (3) with the algorithm (16) exponentially converges to the Nash equilibrium of the noncooperative game (2).

**Proof.** Without loss of generality, for simplicity, let \(m = 1\). Transforming the equilibrium point of (17) into the origin, we have

\[
\dot{\hat{x}} = \hat{x}^{(1)}
\]
(18a)

\[
\dot{x}^{(n)} = \hat{f} - \sum_{l=1}^{n-1} \varepsilon^{n-l} k_l z^{(l)} - \sum_{l=1}^{n-1} \varepsilon^{n-l} k_l \hat{x}^{(l)} - \alpha_1 h - \alpha_2 \hat{y}
\]
(18b)

\[
\dot{y} = \sum_{l=1}^{n-1} \varepsilon^{n-l} k_l z^{(l)} + \sum_{l=1}^{n-1} \varepsilon^{n-l} k_l \hat{x}^{(l)} + \frac{\alpha_1}{\varepsilon^{n-1}} h
\]
(18c)

\[
\dot{z} = -\frac{\varepsilon^n \beta_1}{\mu} (z - z^{(1)}) + \frac{1}{\mu} z^{(2)}
\]
(18d)

\[
\dot{z}^{(2)} = -\frac{\varepsilon^n \beta_2}{\mu^2}z^{(2)} + \frac{1}{\mu} z^{(2)}
\]
(18e)

\[
z^{(n)} = \frac{\varepsilon^n \beta_n}{\mu^n} (z - z^{(1)})
\]
(18f)

\[
\dot{\hat{x}} = -\alpha_3 ((L \otimes I_N) + M) \hat{x} - N \otimes \hat{x}
\]
(18g)

where \(\hat{z} = \frac{1}{\mu^n}(z - x)\) and \(\hat{z}^{(l)} = \frac{1}{\mu^{l+1}}(z^{(l)} - x^{(l)})\) with \(l \in \{1, \ldots, n-1\}\). Let

\[
\Delta \hat{x} = \text{col}(\hat{z}, \Delta \hat{z}^{(1)}, \ldots, \Delta \hat{z}^{(n-1)})
\]
\[
\Delta \hat{x}^{(l)} = \text{col}(\Delta \hat{z}^{(l)}, \ldots, \Delta \hat{z}^{(n)})
\]

where \(\Delta \hat{z}^{(l)} = \frac{1}{\mu} \hat{z}^{(l)}\) with \(l \in \{1, \ldots, n-1\}\). Then, (18) can be rewritten as

\[
\dot{\hat{x}} = \varepsilon \hat{z}^{(1)}
\]
(19a)
\[
\dot{x} = \varepsilon (A \otimes I_N) \dot{x} + \frac{1}{\varepsilon n-1} (b \otimes I_N)(\tilde{f} - \alpha_1 h - \alpha_2 \tilde{y}) - \sum_{l=1}^{n-1} \frac{\varepsilon^n k_l}{\mu^{l+1-n}} \Delta \tilde{x}^l
\]

\[
\dot{y} = \sum_{l=1}^{n-1} \varepsilon k_l \dot{x}^l + \sum_{l=1}^{n-1} \frac{\varepsilon^n k_l}{\mu^{l+1-n}} \Delta \tilde{x}^l + \frac{\alpha_1}{\varepsilon n-1} h
\]

\[
\Delta \dot{x} = \frac{\varepsilon}{\mu} (B \otimes I_N) \Delta \dot{x} - \frac{1}{\varepsilon n-1} (b_n \otimes I_N)(\tilde{f} - \alpha_1 h - \alpha_2 \tilde{y}) - \sum_{l=1}^{n-1} \frac{\varepsilon^n k_l}{\mu^{l+1-n}} \Delta \tilde{x}^l - \sum_{l=1}^{n-1} \varepsilon^n k_l \tilde{x}^{(l)}
\]

\[
\dot{\tilde{x}} = -\alpha_3 ((L \otimes I_N) + M) \tilde{x} - 1_N \otimes \dot{x}
\]

where \(\mu = \max \{0, k_1 n^{n-2}, \ldots, k_{n-2} n^{2}, k_{n-1} n\}\), \(b_n = [\bar{0}^T_{n-1}, 1]^T\) and

\[
B = \begin{bmatrix}
-\beta_1 & \cdots & -\beta_{n-1} \\
-\beta_n & 0_{n-1}
\end{bmatrix}
\]

It is obviously that \(B\) is Hurwitz, and hence there exists a positive definite matrix \(R\) such that \(RB + B^TR = -I_n\). Take the following Lyapunov function for (19).

\[
V_2 = V_1 + \Delta \tilde{x}^T (R \otimes I_N) \Delta \tilde{x}
\]

The derivative of \(V_2\) along (19) is

\[
\dot{V}_2 = -\alpha_3 \|\tilde{x}\|^2 - 2\tilde{x}^T (I_N \otimes \tilde{x}) - \varepsilon \|\tilde{x}\|^2
\]

\[
- \frac{\alpha_2}{\varepsilon n-1} \|\tilde{y}\|^2 - \frac{\varepsilon}{\mu} \|\Delta \tilde{x}\|^2 + \frac{1}{\varepsilon n-1} \tilde{y}^T \tilde{f} + \frac{k_1}{\varepsilon n-1} \tilde{x}^T (\tilde{f} - \alpha_1 h - \alpha_2 \tilde{y}) - \sum_{l=1}^{n-1} \frac{\varepsilon^n k_l}{\mu^{l+1-n}} \Delta \tilde{x}^l
\]

\[
+ \frac{1}{\varepsilon n-1} \left( \sum_{l=1}^{n-2} (2p_{(n-1)} + k_{l+1}) \tilde{x}^{(l)} + (2p_{(n-1)(n-1)} + 2\tilde{x}^{(n-1)}) \right)^T (\tilde{f} - \alpha_2 \tilde{y})
\]

\[
- \frac{1}{\varepsilon n-1} \left( \sum_{l=1}^{n-2} (2p_{(n-1)} + k_{l+1}) \tilde{x}^{(l)} + (2p_{(n-1)(n-1)} + 1) \tilde{x}^{(n-1)} \right)^T (\alpha_1 h + \sum_{l=1}^{n-1} \frac{\varepsilon^n k_l}{\mu^{l+1-n}} \Delta \tilde{x}^l)
\]

\[
- \frac{1}{\varepsilon n-1} \left( \sum_{l=1}^{n-2} (2p_{(n-1)} + k_{l+1}) \tilde{x}^{(l)} + (2p_{(n-1)(n-1)} + \tilde{x}^{(n-1)}) \right)^T (\tilde{f} - \alpha_1 h - \alpha_2 \tilde{y}) - \sum_{l=1}^{n-1} \frac{\varepsilon^n k_l}{\mu^{l+1-n}} \Delta \tilde{x}^l - \sum_{l=1}^{n-1} \varepsilon^n k_l \tilde{x}^{(l)}
\]

By Assumption 2, we have

\[
\begin{align*}
- \left( 2 \bar{r}_{1n} \tilde{z} + \sum_{l=1}^{n-1} 2 \bar{r}_{(l+1)n} \Delta \tilde{x}^{(l)} \right)^T \tilde{f} \\
\leq \frac{1}{2} \left( l_x + \sum_{l=1}^{n-1} l_{x^{(l)}} \varepsilon^{2l+1-n} \left( 4 \bar{r}_{1n}^2 \|\tilde{z}\|^2 + \sum_{l=1}^{n-1} 4 \bar{r}_{(l+1)n}^2 \|\Delta \tilde{x}^{(l)}\|^2 \right) \right)
\end{align*}
\]

\[
+ \frac{n}{2} \left( l_x \|\tilde{x}\|^2 + \sum_{l=1}^{n-1} l_{x^{(l)}} \varepsilon^{n-1} \|\tilde{x}^{(l)}\|^2 \right).
\]

(20)

Under Assumption 4, we obtain

\[
\alpha_1 \left( 2 \bar{r}_{1n} \tilde{z} + \sum_{l=1}^{n-1} 2 \bar{r}_{(l+1)n} \Delta \tilde{x}^{(l)} \right)^T h
\]

\[
\leq \frac{\alpha_2}{2} \left( 4 \bar{r}_{1n}^2 \|\tilde{z}\|^2 + \sum_{l=1}^{n-1} 4 \bar{r}_{(l+1)n}^2 \|\Delta \tilde{x}^{(l)}\|^2 + n \|\tilde{x}\|^2 \right).
\]

(21)

Besides, we have

\[
\begin{align*}
- \frac{k_1 \alpha_2}{\varepsilon n-1} \tilde{x}^T \tilde{y} & \leq \frac{k_1^2 \alpha_2}{\varepsilon n-1} \|\tilde{x}\|^2 + \frac{\alpha_2}{4 \varepsilon n-1} \|\tilde{y}\|^2 \\
- \sum_{l=1}^{n-1} \frac{\varepsilon k_l}{\mu^{l+1-n}} \tilde{x}^T \Delta \tilde{x}^l & \leq \frac{n k_1}{2} \|\tilde{x}\|^2 + \frac{k_1^2 \varepsilon \mu}{2} \|\Delta \tilde{x}\|^2
\end{align*}
\]

(22a)

\[
- \alpha_2 \left( \sum_{l=1}^{n-2} (2p_{(n-1)} + k_{l+1}) \tilde{x}^{(l)} + (2p_{(n-1)(n-1)} + 2) \tilde{x}^{(n-1)} \right)^T (2p_{(n-1)} + k_{l+1}) \tilde{x}^{(l)}
\]

\[
- \frac{\alpha_2}{2} \left( n \left( \sum_{l=1}^{n-2} (2p_{(n-1)} + k_{l+1}) \|\tilde{x}^{(l)}\|^2 + (2p_{(n-1)(n-1)} + 2) \|\tilde{x}^{(n-1)}\|^2 \right) + \frac{n-1}{4n} \|\tilde{y}\|^2 \right)
\]

(22b)

\[
\begin{align*}
- \left( \sum_{l=1}^{n-2} (2p_{(n-1)} + k_{l+1}) \tilde{x}^{(l)} + (2p_{(n-1)(n-1)} + 2) \tilde{x}^{(n-1)} \right)^T & \sum_{l=1}^{n-1} \frac{\varepsilon k_l}{\mu^{l+1-n}} \Delta \tilde{x}^{(l)}
\end{align*}
\]

(22c)
\[
V_2 \leq -\hat{\rho}_1 \|ar{x}\|^2 - \hat{\rho}_2 \|ar{x}\|^2 - \hat{\rho}_3 \|ar{y}\|^2 - \hat{\rho}_4 \|ar{\rho}_5 \|ar{x}\|^2
\]

where
\[
\hat{\rho}_1 = \frac{1}{2e^{-n-1}} (k_1 \omega \alpha_1 - 2k_2^2 \alpha_2 - 2l_x (n + k_1)) - \sum_{i=1}^{n-1} k_1 l_{x(i)} e^{2l+1-n} - n k_1 \mu e^{n-1},
\]
\[
\hat{\rho}_2 = \frac{1}{2e^{-n-1}} (e^n - e^{-n-1} \left( \frac{\bar{p}}{16} + 2N (k_1 + 2n))l_x d_{\alpha_{\infty}} \right) - \bar{p} \theta_\alpha_1 + 2n \alpha_2 + l_x + \sum_{i=1}^{n-1} l_{x(i)} e^{2l+1-n}),
\]
\[
\hat{\rho}_3 = \frac{1}{2e^{-n-1}} \left( n + \frac{1}{2n} \alpha_3 - l_x - \sum_{i=1}^{n-1} l_{x(i)} e^{2l+1-n} \right),
\]
\[
\hat{\rho}_4 = \alpha_3 - \frac{\bar{p}}{\mu} (2n - 1 + k_1 \theta) + \sum_{i=1}^{n-1} l_{x(i)} e^{2l+1-n},
\]
\[
\hat{\rho}_5 = \frac{e}{\mu} \left( k_1 + \frac{n \bar{p}}{\varepsilon} + 8n(n-1) \mu \right) - \bar{p} \theta_\alpha_1 + 2n \alpha_2 + \sum_{i=1}^{n-1} 2n e^n k_i^2 + l_x + \sum_{i=1}^{n-1} l_{x(i)} e^{2l+1-n}.
\]

Similarly to the proof of Theorem 2, we can take suitable \(\alpha_1, \alpha_2, \alpha_3\) and \(\varepsilon\) such that \(\hat{\rho}_2, \hat{\rho}_3, \hat{\rho}_4 > 0\). Then, by selecting a small enough \(\mu\), we have \(\rho_1, \rho_5 > 0\). Therefore, there exist positive constants \(\alpha_1, \alpha_2, \alpha_3, \varepsilon\) and \(\mu\) such that \(V_2 \leq 0\), which indicates that the high-order nonlinear player (3) with the algorithm (16) exponentially converges to the Nash equilibrium of the noncooperative game (2).

Remark 5. Compared with the results in [7, 11, 13, 17], the algorithms (8) and (16) are exponentially convergent, even in the presence of uncertain parameters. Furthermore, the output-based algorithm (16) only relies on the output variables of high-order nonlinear players instead of all state variables.

\begin{figure}[h]
\centering
\includegraphics[width=0.5	extwidth]{network}
\caption{The communication network of ten vehicles.}
\end{figure}

4. Simulations

In this section, simulation examples are given to illustrate the algorithms (8) and (16).

Example 1: Formation control of vehicles.

Consider the formation problem of ten vehicles over a weight-unbalanced digraph described as Fig. 1. The desired formation is constituted by the vehicles, if the relative positions between vehicles are the desired values. That is, \(lim_{t \to \infty} (p_i(t) - p_j(t)) = d_{ij}, \forall i, j \in V\), where \(p_i, p_j\) are the position of vehicles \(i\) and \(j\), respectively; \(d_{ij} \in \mathbb{R}^n\) is the desired relative position between vehicles \(i\) and \(j\). According to [42], the vehicles can form the desired formation by seeking the Nash equilibrium of the following game.

\[
\min_{\rho_i, \in \mathbb{R}^2} J_i(p_i, p_{-i})
\]

where \(J_i(p_i, p_{-i}) := \frac{1}{2N} ||p_i - 2d_i||^2 + \frac{1}{2} p_i^T \sum_{j=1}^{N} p_j\) are the cost function and the position of vehicle \(i\), respectively; \(d_i := [d_{xi} d_{yi}]^T \in \mathbb{R}^2\) and \(d_j := [d_{xj} d_{yj}]^T \in \mathbb{R}^2\) are constant vectors such that \(d_i - d_j = d_{ij}\) holds.

The vehicle \(i\) has following dynamics (see [30]).

\[
\dot{p}_i = v_i
\]
\[
\dot{v}_i = -\frac{\rho A_i C d_i}{2m_i} v_i^2 - \frac{d_{mi}}{m_i} + u_i
\]
where $v_i$ is the speed of vehicle $i$, $u_i$ is the force produced by the engine; $m_i$, $A_i$ and $C_{di}$ are the mass, the cross-sectional area, and the drag coefficient of vehicle $i$, respectively; $\rho$ is air density; and $d_{mi}$ is a constant representing the amplitude of the mechanical drag force.

The parameters of ten vehicles are given in Table 1, and $\rho = 1.225 m/s^3$. The desired formation is a five-pointed star.

Fig. 2 shows the simulation results of algorithm (8) with parameters $\alpha_1 = 3$, $\alpha_2 = 2.2$, $\alpha_3 = 18$ and $\varepsilon = 2$. Fig. 3 displays the simulation results of algorithm (16) with parameters $\alpha_1 = 3$, $\alpha_2 = 2.2$, $\alpha_3 = 18$, $\varepsilon = 2$ and $\mu = 0.02$. In Figs. 2 and 3, the triangles and the dots are the initial and final positions of the vehicles, respectively, and the solid lines are the position trajectories of vehicles. As shown in Figs. 2 and 3, the ten vehicles converge to a five-pointed star under the algorithms (8) and (16), respectively. By comparing Fig. 2 with Fig. 3, the ten vehicles under the algorithm (8) form the desired formation more directly than that under the algorithm (16).

Table 1: The parameters of ten vehicles

| Vehicle $i$ | $m_i (kg)$ | $A_i (m^2)$ | $C_{di}$ | $d_{mi} (N)$ |
|-------------|------------|-------------|----------|--------------|
| 1           | 1800       | 2.180       | 1.526    | 6.412        |
| 2           | 1775       | 2.165       | 1.649    | 5.241        |
| 3           | 825        | 1.634       | 1.052    | 2.466        |
| 4           | 1025       | 1.746       | 1.281    | 3.969        |
| 5           | 1200       | 1.844       | 1.359    | 4.113        |
| 6           | 1450       | 1.983       | 1.420    | 4.755        |
| 7           | 970        | 1.715       | 1.138    | 2.842        |
| 8           | 1500       | 2.011       | 1.409    | 4.672        |
| 9           | 1320       | 1.911       | 1.389    | 4.263        |
| 10          | 1670       | 2.107       | 1.514    | 5.038        |

Example 2: Electricity market games of turbine-generator systems.

In electricity markets, the competition among distributed energy resources can be described by noncooperative games (see [43]). Consider a noncooperative game of six turbine-generator systems over a weight-unbalanced digraph depicted as Fig. 4. The turbine-generator system $i \in V$ faces the following game (see [11]).

$$\min_{P_i \in \mathbb{R}} J_i(P_i, P_{-i})$$

where $J_i : \mathbb{R} \rightarrow \mathbb{R}$ and $P_i$ are the cost function and the output power of the turbine-generator system $i$, respectively; $P_{-i} = \text{col}(P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_N)$. Specifically,
the cost function of turbine-generator system $i$ is

$$J_i(P_i, P_{-i}) = c_i(P_i) - p(\sigma)P_i$$

where $c_i(P_i) := \gamma_1 + \gamma_2 P_i + \gamma_3 P_i^2$ is the generation cost with $\gamma_1$, $\gamma_2$, $\gamma_3$ being the characteristics of the generation system $i$ presented in Table 2 (see [44]); $p(\sigma) := 200 - 0.1N\sigma$ is the electricity price with $\sigma(P) := \frac{1}{N} \sum_{i=1}^{N} P_i$.

When the valve positions of governors are fixed, the turbine-generator system $i$ have fourth-order dynamics $P_i^{(4)} = u_i$ (see [25, 31]).

![Figure 5: The evolutions of output powers under algorithm (8).](image)

Figure 5: The evolutions of output powers under algorithm (8).

Fig. 5 displays the simulation results of algorithm (8) with parameters $\alpha_1 = 500$, $\alpha_2 = 400$, $\alpha_3 = 400$ and $\varepsilon = 20$. Fig. 6 shows the simulation results of algorithm (16) with parameters $\alpha_1 = 500$, $\alpha_2 = 400$, $\alpha_3 = 400$, $\varepsilon = 20$ and $\mu = 0.01$. In Figs. 5 and 6, the solid lines and the dotted lines are the evolutions of output powers and the Nash equilibrium, respectively. It is obvious from Figs. 5 and 6 that the output powers of six turbine-generator systems converge to the Nash equilibrium under the algorithms (8) and (16). By comparing Fig. 5 with Fig. 6, the algorithm (8) has better performance than the algorithm (16), since the fluctuation of simulation results under algorithm (16) is more drastic than that under algorithm (8), especially in initial stage. These simulation results verify the effectiveness of the algorithms (8) and (16).

5. Conclusions

This paper has studied the noncooperative games of multi-agent systems over weight-unbalanced digraphs, where the players have high-order uncertain nonlinear dynamics. In order to guarantee that the high-order nonlinear players can autonomously seek the Nash equilibrium of the game, this paper has developed a distributed state-based algorithm and a distributed output-based algorithm. The state-based algorithm uses all state variables of high-order nonlinear players, and the output-based algorithm only needs the output variables. Moreover, the paper has analyzed the globally exponentially convergence of the two algorithms. Finally, the two algorithms have been verified by two examples about the formation problem of vehicles and the electricity market games of smart grids.

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