Approximation and Localized Polynomial Frame on Double Hyperbolic and Conic Domains

Yuan Xu

Received: 23 May 2021 / Accepted: 13 January 2022 / Published online: 10 October 2022
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2022

Abstract

We study approximation and localized polynomial frames on a bounded double hyperbolic or conic surface and the domain bounded by such a surface and hyperplanes. The main work follows the framework developed recently in Xu (J Funct Anal 281(12):109257, 2021) for homogeneous spaces that are assumed to contain highly localized kernels constructed via a family of orthogonal polynomials. The existence of such kernels will be established with the help of closed-form formulas for the reproducing kernels. The main results provide construction of semi-discrete localized tight frame in weighted \( L^2 \) norm and characterization of best approximation by polynomials on our domains. Several intermediate results, including the Marcinkiewicz–Zygmund inequalities, positive cubature rules, Christoffel functions, and Bernstein inequalities, are shown to hold for doubling weights defined via the intrinsic distance on the domain.

Keywords Approximation · Conic domain · Fourier orthogonal series · Homogeneous spaces · Localized kernel · Orthogonal polynomials · Polynomial frame

Mathematics Subject Classification 41A10 · 41A63 · 42C10 · 42C40

1 Introduction

Recently a general framework based on highly localized polynomial kernels is developed in [29] for localizable homogeneous spaces and used for studying approximation and localized polynomial frames on a finite conic surface and the domain bounded by...
such a surface and a hyperplane. In the present work, we establish highly localized kernels and carry out analysis on bounded double hyperbolic or conic surface and the domain bounded by such a surface and hyperplanes.

1.1 Motivation

Let $\Omega$ be a set in $\mathbb{R}^d$, either an algebraic surface or a domain with a non-empty interior. A homogeneous space is a measure space $(\Omega, \varpi, d)$, where $\varpi$ is a nonnegative doubling weight function with respect to the metric $d(\cdot, \cdot)$ on $\Omega$. We call a homogeneous space $(\Omega, \varpi, d)$ localizable if it contains highly localized kernels.

The kernels are constructed using orthogonal polynomials. Let $dm$ be the Lebesgue measure on $\Omega$. Assume that the weight function $\varpi$ is regular so that

$$\langle f, g \rangle_{\varpi} := \int_{\Omega} f(x)g(x)\varpi(x)dm(x) \quad (1.1)$$

is a well-defined inner product on the space of polynomials restricted to $\Omega$. Let $V_n(\Omega, \varpi)$ be the space of orthogonal polynomials of degree $n$ with respect to this inner product. The projection operator $\text{proj}_n : L^2(\Omega, \varpi) \mapsto V_n(\Omega, \varpi)$ can be written as

$$\text{proj}_n(\varpi; f, x) = \int_{\Omega} f(y)P_n(\varpi; x, y)\varpi(y)dm(y), \quad f \in L^2(\Omega, \varpi). \quad (1.2)$$

where $P_n(\varpi; \cdot, \cdot)$ is the reproducing kernel of the space $V_n(\Omega, \varpi)$. Let $\hat{a}$ be a cut-off function, defined as a compactly supported function in $C^\infty(\mathbb{R}+)$. Then our highly localized kernels are of the form

$$L_n(\varpi; x, y) := \sum_{j=0}^{\infty} \hat{a}\left(\frac{j}{n}\right) P_j(\varpi; x, y).$$

The kernel is highly localized if it decays at rates faster than any inverse polynomial rate away from the main diagonal $y = x$ in $\Omega \times \Omega$ with respect to the distance $d$ on $\Omega$; see the definition in the next section. These kernels provide important tools for the analysis in regular domains, such as the unit sphere and the unit ball, and are essential ingredients in a recent study of approximation and localized polynomial frames; see, for example, [3, 8, 13, 14, 17–19] for some of the results on the spheres and balls and [1, 2, 7, 11, 12, 15, 24] for various applications.

The reason that highly localized kernels are known only on a few regular domains lies in the addition formula for orthogonal polynomials, which are closed-form formulas for the reproducing kernels of orthogonal polynomials. For spherical harmonics that are orthogonal on the unit sphere, which serves as a quintessential example for our study, the closed form formula is given by $Z_n(\langle x, y \rangle)$, where $Z_n$ is a Gegenbauer polynomial of one variable. The addition formulas are powerful tools when they exist.
Our recent work in [29] is prompted by two incentives. The first one is the realization that, if highly localized kernels are taken for granted, then much of the analysis can be developed within a general framework of homogeneous spaces. The second one is the possibility of establishing highly localized kernels and carrying out analysis thereafter on conic domains, which are domains largely untouched hitherto. The latter is made possible by recently discovered new addition formulas for orthogonal polynomials on conic domains [27, 28]. Altogether there are four types of conic domains and they are standardized as:

(1) conic surface \( V_{d+1} \) defined by
\[
V_{d+1} := \{ (x, t) \in \mathbb{R}^{d+1} : \|x\| = t, \ 0 \leq t \leq 1, \ x \in \mathbb{R}^d \};
\]

(2) solid cone \( V_{d+1} \) bounded by \( V_{d+1}^{0} \) and the hyperplane \( t = 1 \);

(3) two-sheets hyperbolic surface \( \varphi X_{d+1} \) defined by
\[
\varphi X_{d+1} := \{ (x, t) \in \mathbb{R}^{d+1} : \|x\|^2 = t^2 - \varphi^2, \ \varphi \leq |t| \leq 1 + \varphi, \ x \in \mathbb{R}^d \},
\]

where \( \varphi \geq 0 \), which becomes the double conic surface \( X_{d+1} \) when \( \varphi = 0 \);

(4) solid hyperboloid \( \varphi X_{d+1} \) bounded by \( \varphi X_{d+1}^{0} \) and the hyperplanes \( t = \pm 1 \), which becomes solid double cone \( X_{d+1} \) when \( \varphi = 0 \).

The first two types of conic domains are studied in [29]. The present paper deals with the other two cases: double hyperbolic surfaces and hyperboloid.

1.2 Main Results

We will follow the framework established in [29]. Assuming several assertions on highly localized kernels and fast-decaying polynomials, the framework provides a unified theory for two objectives. The first one leads to localized polynomial frames constructed via a semi-continuous Calderón type decomposition
\[
f = \sum_{j=0}^{\infty} L_{2j} \ast L_{2j} \ast f, \quad f \in L^2(\Omega, \omega),
\]
where \( L_n \ast f \) denotes the integral operator that has \( L_n(\omega; \cdot, \cdot) \) as its kernel,
\[
L_n \ast f(x) := \int_{\Omega} f(y) L_n(\omega; x, y) \omega(y) dy.
\]

Discretizing the integrals by appropriate positive cubature rules, we end up with a fame system \( \{ \psi_{\xi} \}_{\xi \in \Xi} \), indexed by a discrete set \( \Xi \) of well-separated points in \( \Omega \), where \( \psi_{\xi}(x) = \sqrt{\lambda_{\xi}} L_{2j}(\omega; x, \xi) \) with \( \lambda_{\xi} > 0 \) being the coefficients in the cubature,
and the frame is tight in the sense that, for all \( f \in L^2(\Omega, \omega) \),
\[
f = \sum_{\xi \in \Xi} \langle f, \psi_{\xi} \rangle \omega \psi_{\xi} \quad \text{and} \quad \int_{\Omega} |f(x)|^2 \omega(x) dx = \sum_{\xi \in \Xi} |\langle f, \psi_{\xi} \rangle|^2.
\]

This is an extension of the extensive work on the unit sphere and the unit ball in the literature described in the previous subsection. The second objective is to study the error of the best approximation by polynomials
\[
E_n(f)_{L^p(\Omega, \omega)} = \inf_{\deg g \leq n} \| f - g \|_{L^p(\Omega, \omega)}.
\]

The aim is to provide a characterization via a modulus of smoothness, defined as a multiplier operator, and an equivalent \( K \)-functional, defined by the spectral differential operator that has orthogonal polynomials as eigenfunctions. Such a characterization is in line with those on the unit sphere, see [8, 10, 20, 26] and the references therein. The characterization consists of a direct estimate, using the fast decaying of \( L_n(\omega; \cdot, \cdot) \), and an inverse estimate, using a Bernstein inequality established via highly localized kernels. It is worth mentioning that several intermediate results, such as the Marcinkiewicz–Zygmund inequalities, positive cubature rules, Christoffel functions, and Bernstein inequalities are of independent interest; furthermore, some of these intermediate results can be established for doubling weights, which extends the results on the interval and the unit sphere [4, 8, 16].

For the framework to work on a particular domain, substantial work is needed to fulfill the assertions and assumptions, which depend heavily on the geometry of the domain and the complexity of the orthogonal structure of the domain. For each conic domain, this starts with identifying an appropriate intrinsic distance on the domain. Establishing the highly localized kernels is delicate and fairly involved, relying on the structure of the addition formula and the distance function. The \( \varepsilon \)-separated set is assumed conceptually in the framework, an explicit construction is needed on each domain, which is also necessary for the computational purpose. Even though the roadmap is outlined by the framework, the actual work on each domain remains challenging and amounts to a thorough understanding of the intrinsic structure of the domain.

It should also be mentioned that the structures on double hyperbolic domains \( X_{d+1} \) and \( X_{d+1} \), which degenerate to double cones when \( \rho = 0 \), are distinctively different from those on the single conic domains \( Y_{d} \) and \( Y_{d+1} \). Indeed, the distance functions for these two types of domains are incomparable, around the conic apex, and the orthogonal structure on the double conic domains depends on the parity of the polynomials on the variable \( t \). For example, for the double conic surface \( X_{d+1} \), the closed form formula for the reproducing kernels is established for polynomials that are orthogonal with respect to \( w_{\beta, \gamma}(t) = |t|^{2\beta} (1 - t^2)^{\gamma-\frac{1}{2}} \), but only for the subspace of polynomials that are even in the \( t \) variable or odd in the \( t \) variable. This restriction requires us to restrict the class of functions that are even in the \( t \) variable, but the entire framework can still be carried out on the domain. Using the reproducing kernels in [28], we will show that \((X_{0}, w_{0,\gamma}, d_{X_{0}})\) is a localizable homogeneous...
space. Similar result holds for the solid double cone $\mathbb{R}^{d+1}$ with the weight function 
$W_{\beta,\gamma,\mu}(x, t) = |t|^\beta (1 - t)^\gamma (t^2 - \|x\|^2)^{\mu - \frac{1}{2}}$ and its own distance function. For these homogeneous spaces, we shall show that the framework in [29] on the localizable tight frame and the best approximation can be carried out completely.

1.3 Organization and Convention

The general framework on the localizable homogeneous space emerges from the study on the unit sphere and the unit ball, as explained in [29], and it consists of several topics. Because of its length, we shall refer to its discussion and formulation to [29]. The present work, however, is self-contained otherwise and can be read independently.

The paper is organized as follows. We will state the assertions, and just enough background for their statement, for the general framework in the next section. Whenever possible, we will not give definitions and properties for homogeneous spaces but only on conic domains and only when needed. For a more extensive discussion on the background materials, we also refer to [29]. The cases of hyperbolic surface and hyperboloid will be studied in Sects. 3 and 4, respectively, and two sections will have parallel structures, which is also the structure for the conic surface and the cone in [29], to emphasize what is required to carry out the program and for easier comparison between the cases. Each section will contain several sections and its organization will be described in the preamble of the section.

Throughout this paper, we will denote by $L^p(\Omega, w)$ the weighted $L^p$ space with respect to the weight function $w$ defined on the domain $\Omega$ for $1 \leq p \leq \infty$. Its norm will be denote by $\|\cdot\|_{p,w}$ for $1 \leq p \leq \infty$ with the understanding that the space is $C(\Omega)$ with the uniform norm when $p = \infty$.

Finally, we shall use $c, c', c_1, c_2$, etc. to denote positive constants that depend on fixed parameters and their values may change from line to line. Furthermore, we shall write $A \sim B$ if $c'A \leq B \leq cA$.

2 Preliminaries

In the first subsection, we collect all assertions needed for the general framework on the localized homogeneous space. Basics of classical orthogonal polynomials that will be used later are collected in the second subsection.

2.1 Assertions for the General Framework

Let $(\Omega, \sigma, d)$ be a homogeneous space. For a given set $E \subset \Omega$, we define $\sigma(E) = \int_E \sigma(x)dm(x)$. The weight function $\sigma$ is a doubling weight if there exists a constant $L > 0$ such that

$$\sigma(B(x, 2r)) \leq L \sigma(B(x, r)), \quad \forall x \in \Omega, \quad r \in (0, r_0),$$
where $r_0$ is the largest positive number such that $B(x, r) = \{y \in \Omega : d(x, t) < r\} \subset \Omega$. The smallest $L$ for doubling inequality to hold is called the doubling constant of $\varpi$.

Let $\varpi$ be a nonnegative weight function on $\Omega$ and let $\langle \cdot, \cdot \rangle_{\varpi}$ be the inner product defined by (1.1). For $n = 0, 1, 2, \ldots$, let $V_n(\varpi)$ be the space of orthogonal polynomials with respect to the inner product. If $\{P_{n, v} : 1 \leq v \leq \dim V_n(\varpi)\}$ is an orthogonal basis of $V_n(\varpi)$, then the reproducing kernel of $V_n(\varpi)$ is given by

$$P_n(\varpi; x, y) = \sum_{\nu=1}^{\dim V_n(\varpi)} \frac{P_{n, \nu}(x) P_{n, \nu}(y)}{\langle P_{n, \nu}, P_{n, \nu} \rangle_{\varpi}}.$$ (2.1)

It is the kernel of the projection operator $\text{proj}_n : L^2(\Omega, \varpi) \mapsto V_n(\varpi)$ stated in (1.2). The Fourier orthogonal series of $f \in L^2(\varpi)$ is given by

$$f = \sum_{n=0}^{\infty} \text{proj}_n(\varpi; x, y).$$ (2.2)

### 2.1.1 Highly Localized Kernels

A nonnegative function $\hat{a} \in C^\infty(\mathbb{R})$ is said to be admissible if it obeys either one of the conditions

(a) $\text{supp} \hat{a} \subset [0, 2]$ and $\hat{a}(t) = 1$, $t \in [0, 1]$; or

(b) $\text{supp} \hat{a} \subset [1/2, 2]$.

Given such a cut-off function, we define a kernel $L_n(\varpi; \cdot, \cdot)$ on $\Omega \times \Omega$ by

$$L_n(\varpi; x, y) := \sum_{j=0}^{\infty} \hat{a} \left( \frac{j}{n} \right) P_j(\varpi; x, y).$$

**Definition 2.1** The kernels $L_n(\varpi; \cdot, \cdot)$, $n = 1, 2, \ldots$, are called highly localized if they satisfy the following assertions:

**Assertion 1** For $\kappa > 0$ and $x, y \in \Omega$,

$$|L_n(\varpi; x, y)| \leq c_\kappa \frac{1}{\sqrt{\varpi(B(x, n^{-1}))} \sqrt{\varpi(B(x, n^{-1}))} (1 + nd(x, y))^\kappa}.$$ 

**Assertion 2** For $0 < \delta \leq \delta_0$ with some $\delta_0 < 1$ and $x_1 \in B(x_2, \frac{\delta}{n})$,

$$|L_n(\varpi; x_1, y) - L_n(\varpi; x_2, y)| \leq c_\kappa \frac{nd(x_1, x_2)}{\sqrt{\varpi(B(x_1, n^{-1}))} \sqrt{\varpi(B(x_2, n^{-1}))} (1 + nd(x_2, y))^\kappa}.$$
Assertion 3 For sufficient large $\kappa > 0$,

$$\int_{\Omega} \frac{\sigma(y)}{\sigma(B(y, n^{-1}))(1 + nd(x, y))^n} dm(y) \leq c.$$ 

The third assertion affirms the sharpness of the first two assertions, as can be seen from the following inequality [29, Lemma 2.1.4].

Lemma 2.2 Let $\sigma$ be a doubling weight that satisfies Assertion 3 with $\kappa > 0$. For $0 < p < \infty$, let $\tau = \kappa - \frac{p}{2} \alpha(\sigma)|1 - \frac{p}{2}| > 0$. Then, for $x \in \Omega$,

$$\int_{\Omega} \frac{\sigma(y)}{\sigma(B(y, n^{-1}))^\frac{p}{2}} (1 + nd(x, y))^\tau dm(y) \leq c \sigma(B(x, n^{-1}))^{1-\frac{p}{2}}. \quad (2.3)$$

In particular, the highly localized kernel satisfies

$$\int_{\Omega} |L_n(\sigma; x, y)|^p \sigma(s) dm(y) \leq c \left[ \sigma(B(x, n^{-1})) \right]^{1-p}. \quad (2.4)$$

The homogeneous space $(\Omega, \sigma, d)$ is called localizable if $\sigma$ is a doubling weight that admits highly localized kernels.

2.1.2 Well-Separated Set of Points

We will need well-distributed points for discretization, such as in the Marcinkiewicz–Zygmund inequality and the positive cubature rules, and our localized polynomial frame. The precise definition is given as follows.

Definition 2.3 Let $\Xi$ be a discrete set in $\Omega$.

(a) A finite collection of subsets $\{S_z : z \in \Xi\}$ is called a partition of $\Omega$ if $S_z \cap S_y = \emptyset$ when $z \neq y$ and $\Omega = \bigcup_{z \in \Xi} S_z$.

(b) Let $\varepsilon > 0$. A discrete subset $\Xi$ of $\Omega$ is called $\varepsilon$-separated if $d(x, y) \geq \varepsilon$ for every two distinct points $x, y \in \Xi$.

(c) $\Xi$ is called maximal if there is a constant $c_d > 1$ such that

$$1 \leq \sum_{z \in \Xi} \chi_{B(z, \varepsilon)}(x) \leq c_d, \quad \forall x \in \Omega, \quad (2.5)$$

where $\chi_E$ denotes the characteristic function of the set $E$.

The existence of $\varepsilon$-separated points is assumed in the general framework for homogeneous spaces in [29]. The construction of such points on a given domain depends on the geometry and the distance function, which is crucial for the explicit formulation and practical computation of localized frames. For conic domains $\mathbb{R}^{d+1}$, we will give a construction of such points in latter sections.
2.1.3 Fast Decaying Polynomials and Cubature Rules

Let \( w \) be a doubling weight on \( \Omega \). A cubature rule is a finite sum that discretizes a given integral,

\[
\int_\Omega f(z)w(x)\,dx = \sum_{k=1}^{N} \lambda_k f(x_k),
\]

where the equality holds for a given polynomial subspace. We will need positive cubature rules, for which all \( \lambda_k \) are positive, over \( \varepsilon \)-separated points. To quantify the coefficients \( \lambda_k \), which are used for our localized polynomial frame, we need to show that the Christoffel function \( \lambda_n(w; \cdot) \), defined by

\[
\lambda_n(w; x) := \inf_{g(x)=1} \int_\Omega |g(x)|^2w(x)\,dm(x),
\]

are bounded, where \( \Pi_n(\Omega) \) denotes the space of polynomials of degree \( n \) restricted on \( \Omega \). This is where our fourth assertion comes in, which ensures the existence of fast decaying polynomials on the domain \( \Omega \).

**Assertion 4.** Let \( \Omega \) be compact. For each \( x \in \Omega \), there is a nonnegative polynomial \( T_x \) of degree at most \( n \) that satisfies

1. \( T_x(x) = 1, T_x(y) \geq \delta > 0 \) for \( y \in B(x, \frac{1}{n}) \) for some \( \delta \) independent of \( n \), and, for each \( \gamma > 1 \),

\[
0 \leq T_x(y) \leq c_\gamma (1 + nd(x, y))^{-\gamma}, \quad y \in \Omega;
\]

2. there is a polynomial \( q_n \) such that \( q_n(x)T_x(y) \) is a polynomial of degree at most \( rn \), for some positive integer \( r \), in the \( x \) variable and \( c_1 \leq q_n(x) \leq c_2 \) for \( x \in \Omega \) for some positive numbers \( c_1 \) and \( c_2 \).

For conic domains, we will establish this assertion by an explicit construction.

2.1.4 Bernstein Inequality for the Spectral Operator

Our study of the best approximation by polynomials relies on the existence of a spectral operator for orthogonal polynomials.

**Definition 2.4** Let \( \varpi \) be a weight function on \( \Omega \). We denote by \( D_\varpi \) the second order derivation operator that has orthogonal polynomials with respect to \( \varpi \) as eigenfunctions; more precisely,

\[
D_\varpi Y = -\mu(n)Y, \quad \forall Y \in V_n(\Omega, \varpi),
\]

where \( \mu \) is a nonnegative quadratic polynomial.
For the unit sphere, this is the Laplace–Beltrami operator. Its analogs exist on the weighted sphere, ball, and simplexes (cf. [9]), as well as for conic domains [27, 28]. We will need the Bernstein inequality for the power of these operators. Since \( \mu(k) \geq 0 \), the operator \( -D^\varpi \) is a nonnegative operator. A function \( f \in L^p(\Omega; \varpi) \) belongs to the Sobolev space \( W^r_p(\Omega; \varpi) \) if there is a function \( g \in L^p(\Omega; \varpi) \), which we denote by \( (D^\varpi)^r f \), such that
\[
\text{proj}_n(\varpi; (D^\varpi)^r f) = \mu(n)^{r/2} \text{proj}_n(\varpi; f),
\]
where we assume that \( f, g \in C(\Omega) \) when \( p = \infty \). The fractional differential operator \( (D^\varpi)^r f \) is a linear operator on the space \( W^r_p(\Omega; \varpi) \) defined by (2.8).

For \( r > 0 \), we denote by \( L_n^{(r)}(\varpi; \cdot, \cdot) \) the kernel defined by
\[
L_n^{(r)}(\varpi; x, y) = \sum_{n=0}^{\infty} \hat{a}_{n}(k) \frac{\mu(k)}{\varpi} P_k(\varpi; x, y),
\]
which is the kernel \( D_x^{r/2} L_n(x, y) \) with \( D_x^{r/2} \) applying on the \( x \) variable. Our Bernstein inequality is proved under the following assumption on the decaying of this kernel.

**Assertion 5** For \( r > 0 \) and \( \kappa > 0 \), the kernel \( L_n^{(r)}(\varpi) \) satisfies, for \( x, y \in \Omega \),
\[
\left| L_n^{(r)}(\varpi; x, y) \right| \leq c_n \frac{n^r}{\sqrt{\varpi(B(x, n^{-1}))} \sqrt{\varpi(B(y, n^{-1}))}} (1 + nd(x, y))^{-\kappa}.
\]

For \( r = 0 \), this reduces to Assertion 1. We list it separately since Assertion 5 is not needed for the localized polynomial frames.

### 2.2 Classical Orthogonal Polynomials

In this subsection, we collect a few results on classical orthogonal polynomials that we will need. There are three families, Jacobi polynomials, spherical harmonics, and classical orthogonal polynomials on the ball. Each of the three families comes from an example of a localized homogeneous space, as explained in [29], and they have been extensively studied. We will recall only the basics.

#### 2.3 Jacobi Polynomials

For \( \alpha, \beta > -1 \), the Jacobi weight function is defined by
\[
w_{\alpha,\beta}(t) := (1 - t)^{\alpha}(1 + t)^{\beta}, \quad -1 < t < 1.
\]

Its normalization constant \( c'_{\alpha,\beta} \), defined by \( c'_{\alpha,\beta} \int_{-1}^{1} w_{\alpha,\beta}(x)dx = 1 \), is given by
\[
c'_{\alpha,\beta} = \frac{1}{2\alpha+\beta+1} c_{\alpha,\beta} \quad \text{with} \quad c_{\alpha,\beta} := \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)}.
\]

\( \odot \) Springer
The Jacobi polynomials $P_n^{(\alpha,\beta)}$ satisfy the orthogonal relations \[ (4.3.3) \]

\[
\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) w_{\alpha,\beta}(x) P_m^{(\alpha,\beta)}(x) \, dx = h_n^{(\alpha,\beta)} \delta_{m,n},
\]

where $h_n^{(\alpha,\beta)}$ is the square of the $L^2$ norm that satisfies

\[
h_n^{(\alpha,\beta)} = \frac{(\alpha + 1)_n(\beta + 1)_n(\alpha + \beta + n + 1)}{n!(\alpha + \beta + 2)_n(\alpha + \beta + 2n + 1)}.
\]

Let $\hat{a}$ be an admissible cut-off function. We defined a polynomial $L_n^{(\alpha,\beta)}$ by

\[
L_n^{(\alpha,\beta)}(t) = \sum_{j=0}^{\infty} \hat{a}\left(\frac{j}{n}\right) \frac{P_j^{(\alpha,\beta)}(t) P_j^{(\alpha,\beta)}(1)}{h_j^{(\alpha,\beta)}}.
\]

The estimate stated below ([3] and [8, Theorem 2.6.7]) will be used several times.

**Theorem 2.5** Let $\ell$ be a positive integer and let $\eta$ be a function that satisfy, $\eta \in C^{3\ell-1}(\mathbb{R})$, $\text{supp} \eta \subset [0,2]$ and $\eta^{(j)}(0) = 0$ for $j = 0, 1, 2, \ldots, 3\ell - 2$. Then, for $\alpha \geq \beta \geq -\frac{1}{2}$, $t \in [-1,1]$ and $n \in \mathbb{N}$,

\[
\left| \frac{d^m}{dt^m} L_n^{(\alpha,\beta)}(t) \right| \leq c_{\ell,m,\alpha} \left\| \eta^{(3\ell-1)} \right\|_{\infty} \left( \frac{n^{2\alpha+2m+2}}{(1 + n\sqrt{1-t})^{\ell}} \right), \quad m = 0, 1, 2, \ldots (2.12)
\]

The Jacobi polynomials with equal parameters are the Gegenbauer polynomials $C_n^{\lambda}$, which are orthogonal with respect to $w_{\lambda}(x) = (1-x^2)^{\lambda - \frac{1}{2}}$,

\[
c_\lambda \int_{-1}^{1} C_n^{\lambda}(x) C_m^{\lambda}(x) w_{\lambda}(x) \, dx = h_n^{\lambda} \delta_{n,m},
\]

where the normalization constant $c_\lambda$ of $w_{\lambda}$ and the norm square $h_n^{\lambda}$ are given by

\[
c_\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\frac{1}{2}) \Gamma(\lambda + \frac{1}{2})} \quad \text{and} \quad h_n^{\lambda} = \frac{\lambda}{n + \lambda} C_n^{\lambda}(1) = \frac{\lambda}{n + \lambda} \frac{(2\lambda)_n}{n!}.
\]

**2.3.1 Spherical Harmonics**

Spherical harmonics are homogeneous polynomials that are orthogonal on the unit sphere. Let $\mathcal{H}_n(\mathbb{S}^{d-1})$ be the space of spherical harmonics of degree $n$ in $d$ variables. Then $\dim \mathcal{H}_n(\mathbb{S}^{d-1}) = \binom{n+d-2}{n} + \binom{n+d-3}{n-1}$. For $n \in \mathbb{N}_0$ let \( Y_\ell^n : 1 \leq \ell \leq \binom{n+d-2}{n} + \binom{n+d-3}{n-1} \).
\[ \dim \mathcal{H}_n(S^{d-1}) \] be an orthonormal basis of \( \mathcal{H}_n(S^{d-1}) \); then

\[ \frac{1}{\omega_d} \int_{S^{d-1}} Y^n_\ell(\xi) Y^m_\ell(\xi) d\sigma(\xi) = \delta_{n,m}, \]

where \( d\sigma \) is the surface measure of \( S^{d-1} \) and \( \omega_d \) denotes the surface area \( \omega_d = 2\pi^{\frac{d}{2}}/\Gamma\left(\frac{d}{2}\right) \) of \( S^{d-1} \). The spherical harmonics satisfy two characteristic properties. In terms of the orthonormal basis \( \{Y^n_\ell\} \), the reproducing kernel \( P_n(\xi, \eta) \) satisfies the addition formula

\[ P_n(\xi, \eta) = \sum_{\ell=1}^{\dim \mathcal{H}_n(S^{d-1})} Y^n_\ell(\xi) Y^n_\ell(\eta) = Z_n^{d-2}(\langle \xi, \eta \rangle), \quad \xi, \eta \in S^{d-1}, \tag{2.15} \]

where \( Z_n^\lambda \) is defined in terms of the Gegenbauer polynomial by

\[ Z_n^\lambda(t) = \frac{n + \lambda}{\lambda} C_n^\lambda(t), \quad \lambda = \frac{d - 2}{2}. \tag{2.16} \]

Furthermore, spherical harmonics are the eigenfunctions of the Laplace–Beltrami operator \( \Delta_0 \), which is the restriction of the Laplace operator \( \Delta \) on the unit sphere. More precisely (cf. [8, (1.4.9)])

\[ \Delta_0 Y = -n(n + d - 2) Y, \quad Y \in \mathcal{H}^d_n. \tag{2.17} \]

### 2.3.2 Orthogonal Polynomials on the Unit Ball

The classical weight function on the unit ball \( \mathbb{B}^d \) of \( \mathbb{R}^d \) is defined by

\[ W_\mu(x) = (1 - \|x\|)^{\mu - \frac{1}{2}}, \quad x \in \mathbb{B}^d, \quad \mu > -\frac{1}{2}. \tag{2.18} \]

Its normalization constant is \( b^d_\mu = \Gamma(\mu + \frac{d+1}{2})/\Gamma\left(\frac{d}{2}\right) \Gamma(\mu + \frac{1}{2}) \). Let \( V^n_d(\mathbb{B}^d, W_\mu) \) be the space of orthogonal polynomials of degree \( n \) with respect to \( W_\mu \). An orthogonal basis of \( V^n_d(\mathbb{B}^d, W_\mu) \) can be given explicitly in terms of the Jacobi polynomials and spherical harmonics. For \( 0 \leq m \leq n/2 \), let \( \{Y^n_{n-2m} : 1 \leq \ell \leq \dim \mathcal{H}_{n-2m}(S^{d-1})\} \) be an orthonormal basis of \( \mathcal{H}^d_{n-2m} \). Define [9, (5.2.4)]

\[ P^n_{\ell,m}(x) = P_m^{\left(\mu - \frac{1}{2}, n-2m+\frac{d-2}{2}\right)} \left(2\|x\|^2 - 1\right) Y^n_{n-2m}(x). \tag{2.19} \]

Then \( \{P^n_{\ell,m} : 0 \leq m \leq n/2, 1 \leq \ell \leq \dim \mathcal{H}_{n-2m}(S^{d-1})\} \) is an orthogonal basis of \( V^n_d(W_\mu) \). Let \( P_n(W_\mu, \cdot, \cdot) \) denote the reproducing kernel of the space \( V^n_d(\mathbb{B}^d, W_\mu) \).
This kernel satisfies an analog of the addition formula [25]: for \( x, y \in \mathbb{B}^d \),

\[
P_n(W_\mu; x, y) = c_{\mu-\frac{1}{2}} \int_{-1}^{1} Z_n^{d-1/2} \left( \langle x, y \rangle + u \sqrt{1 - \|x\|^2} \sqrt{1 - \|y\|^2} \right) \times (1 - u^2)^{\mu-1} du,
\]

where \( \mu > 0 \) and it holds for \( \mu = 0 \) under the limit

\[
\lim_{\mu \to 0^+} c_{\mu-\frac{1}{2}} \int_{-1}^{1} f(t)(1 - t^2)^{\mu-1} du = \frac{f(1) + f(-1)}{2}.
\]

The orthogonal polynomials with respect to \( W_\mu \) on the unit ball are eigenfunctions of a second order differential operator [9, (5.2.3)]: for all \( u \in V_n(\mathbb{B}^d, W_\mu) \),

\[
\left( \Delta - \langle x, \nabla \rangle^2 - (2\mu + d - 1)\langle x, \nabla \rangle \right) u = -n(n + 2\mu + d - 1)u
\]

### 3 Homogeneous Space on Double Conic and Hyperbolic Surfaces

In this chapter, we work in the setting of homogeneous space on the surface

\[
\mathbb{X}_{d+1}^0 = \left\{ (x, t) : \|x\|^2 = t^2 - \varrho^2, \ x \in \mathbb{R}^d, \ \varrho \leq |t| \leq \sqrt{\varrho^2 + 1} \right\},
\]

which is a hyperbolic surface of two sheets when \( \varrho > 0 \) and a double conic surface when \( \varrho = 0 \). We shall verify that this domain admits a localized homogeneous space for a family of weight functions \( W_{\rho, \gamma}^0 \), which are related to the Gegenbauer weight functions and are even in the \( t \) variable.

For \( \varrho = 0 \), the upper part of \( \mathbb{X}_{d+1}^0 \) with \( t \geq 0 \) is exactly the conic surface \( \mathbb{V}_{d+1}^0 \).

The analysis on \( \mathbb{X}_{d+1}^0 \), however, is of a different character. As a starter, the distance on \( \mathbb{X}_{d+1}^0 \) is comparable to the Euclidean distance, in contrast to the distance on \( \mathbb{V}_{d+1}^0 \). Moreover, the space of orthogonal polynomials is divided naturally into two orthogonal subspaces, consisting of polynomials even in the \( t \) variable or odd in the \( t \) variable. We consider only orthogonal polynomials even in the \( t \) variable since they alone satisfy the two characteristic properties: the addition formula and the differential operator having orthogonal polynomials as eigenfunctions. As a consequence, we consider approximation and localized frames for functions that are even in the \( t \) variable. Many estimates and computations are easier to handle when compared with the conic surface because of the simpler distance function and a less cumbersome addition formula. The structure of the chapter follows that of the previous two chapters, with contents arranged in the same order under similar section names.
### 3.1 Distance on Double Conic and Hyperbolic Surfaces

Whenever it is necessary to emphasize the dependence on \( \varrho \), we write \( \varrho^\mathbb{R}^{d+1} \) instead of \( \mathbb{R}^{d+1} \). In most places, however, we use \( \mathbb{R}^{d+1}_0 \) to avoid excessive notations.

The surface \( \mathbb{R}^{d+1}_0 \) can be decomposed as an upper part and a lower part,

\[
\mathbb{R}^{d+1}_0 = \mathbb{R}^{d+1}_{0,+} \cup \mathbb{R}^{d+1}_{0,-} = \left\{ (x, t) \in \mathbb{R}^{d+1}_0 : t \geq 0 \right\} \cup \left\{ (x, t) \in \mathbb{R}^{d+1}_0 : t \leq 0 \right\}.
\]

For \( \varrho = 0 \), the upper part \( \mathbb{R}^{d+1}_{0,+} \) is exactly the conic surface \( \mathbb{R}^{d+1}_0 \).

As \( \mathbb{R}^{d+1}_0 \) is a bounded domain, its distance function should take into account the boundary at the two ends, but it does not regard the apex as a boundary point because of symmetry. We first define a distance on the double conic surface; that is when \( \varrho = 0 \).

**Definition 3.1** Let \( \varrho = 0 \). For \( (x, t), (y, s) \in \mathbb{R}^{d+1}_0 \), define

\[
d_{\mathbb{R}^{d+1}_0}((x, t), (y, s)) = \arccos \left( \langle x, y \rangle + \sqrt{1 - t^2} \sqrt{1 - s^2} \right).
\]

To see that \( d_{\mathbb{R}^{d+1}_0} (\cdot, \cdot) \) is indeed a distance on the double conic surface \( \mathbb{R}^{d+1}_0 \), we let \( X = (x, \sqrt{1 - t^2}) \) and \( Y = (y, \sqrt{1 - s^2}) \), so that \( X, Y \in \mathbb{S}^d \) if \( x, y \in \mathbb{R}^{d+1}_0 \). Then

\[
d_{\mathbb{R}^{d+1}_0}((x, t), (y, s)) = d_{\mathbb{S}^d}(X, Y), \quad (x, t), (y, s) \in \mathbb{R}^{d+1}_0,
\]

from which it follows readily that \( d_{\mathbb{R}^{d+1}_0} \) is indeed a distance on \( \mathbb{R}^{d+1}_0 \).

This distance is quite different from the distance \( d_{\mathbb{V}^{d+1}_0} (\cdot, \cdot) \) of \( \mathbb{V}^{d+1}_0 \) defined by

\[
d_{\mathbb{V}^{d+1}_0}((x, t), (y, s)) = \arccos \left( \sqrt{\frac{t s + \langle x, y \rangle}{2} + \sqrt{1 - t^2} \sqrt{1 - s^2}} \right), \quad (x, t), (y, s) \in \mathbb{V}^{d+1}_0
\]

in [29] and it resembles the distance \( d_{\mathbb{S}} (\cdot, \cdot) \) of the unit ball which is of the same form but with \( t \) and \( s \) replaced by \( \|x\| \) and \( \|y\| \). The distance \( d_{\mathbb{R}^{d+1}_0} \) is related to the distance on the sphere, defined by

\[
d_{\mathbb{S}}(\xi, \eta) = \arccos \langle \xi, \eta \rangle, \quad \xi, \eta \in \mathbb{S}^{d-1}, \quad (3.1)
\]

and the distance on the interval \([-1, 1]\), defined by

\[
d_{[-1,1]}(t, s) = \arccos \left( t s + \sqrt{1 - t^2} \sqrt{1 - s^2} \right), \quad t, s \in [-1, 1]. \quad (3.2)
\]

**Proposition 3.2** Let \( \varrho = 0 \) and \( d \geq 2 \). For \( (x, t), (y, s) \in \mathbb{R}^{d+1}_0 \), write \( x = t \xi \) and \( y = s \eta \) with \( \xi, \eta \in \mathbb{S}^{d-1} \), it holds

\[
1 - \cos d_{\mathbb{R}^{d+1}_0}((x, t), (y, s)) = 1 - \cos d_{[-1,1]}(t, s) + ts \left( 1 - \cos d_{\mathbb{S}}(\xi, \eta) \right). \quad (3.3)
\]
In particular,

\[ c_1 d_{X_0}((x, t), (y, s)) \leq d_{[-1,1]}(t, s) + \sqrt{ts} d_{\mathbb{S}}(\xi, \eta) \leq c_2 d_{X_0}((x, t), (y, s)). \quad (3.4) \]

**Proof** Using \((x, y) \text{sign}(ts) = ts(\xi, \eta)\), we obtain

\[ 1 - \cos d_{X_0}((x, t), (y, s)) = 1 - ts(\xi, \eta) - \sqrt{1 - t^2} \sqrt{1 - s^2}. \]

Hence, \((3.3)\) follows immediately from \((3.2)\) and \((3.1)\). Moreover, using \((3.3)\), estimate \((3.4)\) follows from \(1 - \cos \theta = 2 \sin^2 \theta/2, \quad 1/\pi \theta \leq \sin \theta/2 \leq \theta/2\) for \(0 \leq \theta \leq \pi\), and \((a + b)^2/2 \leq a^2 + b^2 \leq (a + b)^2\) for \(a, b \geq 0\).

Remark 3.3 In \([29, \text{Remark } 4.1]\), it is pointed out that the distance to the lower conic surface; in fact, the line passes through \((\xi, 1)\) on the rim of the upper conic surface to \((-\xi, -1)\) on the opposite rim of the lower conic surface.

In particular, the distance on the linear segment \(l_\xi = \{(t\xi, t) : -1 \leq t \leq 1\}\) for any fixed \(\xi \in \mathbb{S}^{d-1}\) on the double conic surface becomes

\[ d_{X_0}((t\xi, t), (s\xi, s)) = d_{[-1,1]}(t, s). \]

We note that the line \(l_\xi\) passes through the origin when it passes from the upper conic surface to the lower conic surface; in fact, the line passes through \((\xi, 1)\) on the rim of the upper conic surface to \((-\xi, -1)\) on the opposite rim of the lower conic surface.

Lemma 3.4 Let \(\varphi = 0\) and \(d \geq 2\). If \((x, t), (y, s)\) both in \(X_0^{d+1}\) or both in \(X_0^{d+1}\), then

\[ |t - s| \leq d_{X_0}((x, t), (y, s)) \quad \text{and} \quad \sqrt{1 - t^2} - \sqrt{1 - s^2} \leq d_{X_0}((x, t), (y, s)). \]

**Proof** It suffices to consider the case \(t, s \geq 0\). Let \(t = \cos \theta\) and \(s = \cos \phi\) with \(0 \leq \theta, \phi \leq \pi/2\). It follows readily that \(\cos d_{X_0}((x, t), (y, s)) \leq ts + \sqrt{1 - t^2} \sqrt{1 - s^2} = \cos(\theta - \phi)\), which is equivalently to \(|\theta - \phi| \leq d_{X_0}((x, t), (y, s))\). Hence, the stated inequalities follow from

\[ |t - s| = |\cos \theta - \cos \phi| = 2 \sin \frac{\theta + \phi}{2} \sin \frac{|\theta - \phi|}{2} \leq |\theta - \phi|, \]

and, similarly,

\[ \sqrt{1 - t^2} - \sqrt{1 - s^2} = |\sin \theta - \sin \phi| = 2 \cos \frac{\theta + \phi}{2} \sin \frac{|\theta - \phi|}{2} \leq |\theta - \phi|. \]
for all $t, s \in [-1, 1]$. This completes the proof. □

For $\varrho > 0$, the hyperbolic surface $X^{d+1}_0 = X^{d+1}_{0,+} \cup X^{d+1}_{0,-}$ consists of two disjoint parts. For all practical purposes, it is sufficient to consider the distance between points that lie in the same part. For $(x, t)$ and $(y, t)$ both in $X^{d+1}_{0,+}$ or both in $X^{d+1}_{0,-}$, we define

$$d_{X_0}^\varrho ((x, t), (y, s)) = \arccos \left( \langle x, y \rangle + \sqrt{1 + \varrho^2 - t^2} \sqrt{1 + \varrho^2 - s^2} \right)$$

$$= d_{X_0} \left( (x, \sqrt{t^2 - \varrho^2}), (y, \sqrt{s^2 - \varrho^2}) \right). \quad (3.5)$$

It is easy to see that this is a distance function and we can derive its properties as we did for the distance on the double conic surface.

### 3.2 A Family of Doubling Weights

For $d \geq 2$, $\beta > -\frac{1}{2}$ and $\gamma > -\frac{1}{2}$, let $w_{\beta, \gamma}^\varrho$ be the weight function defined on the hyperbolic surface $X^{d+1}_0$ by

$$w_{\beta, \gamma}^\varrho (t) = |t|(t^2 - \varrho^2)^{-\frac{1}{2}}(1 - t^2)^{-\frac{1}{2}}, \quad \varrho \geq 0, \quad (3.6)$$

for $\varrho \leq |t| \leq \sqrt{\varrho^2 + 1}$. When $\varrho = 0$, or on the double conic surface, it becomes

$$w_{\beta, \gamma}^0 (t) = |t|2^\beta (1 - t^2)^{-\frac{1}{2}},$$

which is integrable if $\beta > -\frac{d+1}{2}$ on $X^{d+1}_0$. When $\beta = 0$, $w_{0, \gamma}^0$ is the classical weight function for the Gegenbauer polynomial $C_n^\gamma$. The two cases are related,

$$w_{\beta, \gamma}^\varrho (t) = |t|w_{\beta - \frac{1}{2}, \gamma}^0 \left( \sqrt{t^2 - \varrho^2} \right).$$

Let $d\sigma_\varrho$ denote the surface measure on $eX^{d+1}_0$. Then $d\sigma_\varrho (x, t) = d\omega \sqrt{t^2 - \varrho^2}(x)dt$, where $d\omega_r$ denotes the surface measure on the sphere $S^{d-1}_r$ of radius $r$. It follows then

$$\int_{eX^{d+1}_0} f(x, t)|t|d\sigma_\varrho (x, t) = \int_{|t| \leq \varrho + 1} |t| \int_{\|x\|=\sqrt{t^2 - \varrho^2}} f(x, t)d\omega \sqrt{t^2 - \varrho^2}(x)dt$$

$$= \int_{|s| \leq 1} \int_{\|x\|=|s|} f(x, \sqrt{s^2 + \varrho^2})|s|d\omega|_s (x)ds$$

$$= \int_{eX^{d+1}_0} f(x, \sqrt{s^2 + \varrho^2})|s|d\sigma (x, s). \quad (3.7)$$
Using this relation, it is easy to verify that the normalization constant $b^\rho_{\beta,\gamma}$ of the weight function $w^\rho_{\beta,\gamma}$ is given by

$$b^\rho_{\beta,\gamma} = b_{\beta,\gamma} = \frac{\Gamma(\beta + \gamma + \frac{d+1}{2})}{\sigma_d \Gamma(\beta + \frac{d}{2}) \Gamma(\gamma + \frac{1}{2})}.$$

For $r > 0$, $(x, t)$ on the hyperbolic surface $\mathbb{H}_0^{d+1}$ and $(x, t) \neq (0, 0)$, we denote the ball centered at $(x, t)$ with radius $r$ by

$$c_\rho((x, t), r) := \{(y,s) \in \mathbb{H}_0^{d+1} : d_{\mathbb{H}_0}((x, t), (y, s)) \leq r \}.$$

**Lemma 3.5** For $\rho \geq 0$ and $(x, t) \in \mathbb{H}_0^{d+1}$,

$$w^\rho_{\beta,\gamma}(c_\rho((x, t), r)) = b_{\beta,\gamma} \int_{c_\rho((x, t), r)} w^\rho_{\beta,\gamma}(s) d\sigma_\rho(y, s).$$

**Proof** By the definition of $w(E)$ and (3.5),

$$w^\rho_{\beta,\gamma}(c_\rho((x, t), r)) = b_{\beta,\gamma} \int_{c_\rho((x, t), r)} w^\rho_{\beta,\gamma}(s) d\sigma_\rho(y, s) = b_{\beta,\gamma} \int_{d_{\mathbb{H}_0}((x, \sqrt{t^2 - \rho^2}), (y, s)) \leq r} w^\rho_{\beta,\gamma}(s) d\sigma_\rho(y, s) = b_{\beta,\gamma} \int_{d_{\mathbb{H}_0}((x, \sqrt{t^2 - \rho^2}), (y, s)) \leq r} w^\rho_{\beta,\gamma}(s) d\sigma_\rho(y, s) = w^\rho_{\beta,\gamma}(c_0 \left((x, \sqrt{t^2 - \rho^2}), r \right)).$$

where we have used (3.7) in the second to last step.

**Proposition 3.6** Let $r > 0$ and $(x, t) \in \mathbb{H}_0^{d+1}$. For $\beta > -\frac{d+1}{2}$ and $\gamma > -\frac{1}{2}$,

$$w^0_{\beta,\gamma}(c_0((x, t), r)) = b^0_{\beta,\gamma} \int_{c_0((x, t), r)} w^0_{\beta,\gamma}(s) d\sigma(y, s) \sim r^d \left(1 + r^2\right)^{\frac{d}{2}} \left(1 - r^2 + r^2\right)^{\frac{d}{2}}. \quad (3.8)$$

In particular, $w^0_{\beta,\gamma}$ is a doubling weight on the double conic surface and the doubling index is given by $\alpha(w^0_{\beta,\gamma}) = d + 2 \max\{0, \beta\} + 2 \max\{0, \gamma\}$.
Proof Without loss of generality, we assume $r \leq \frac{\pi}{12}$ and $0 \leq t \leq 1$. Since $w_{0,\gamma}^0$ is even in $t$, we only need to work on $\mathbb{R}^{d+1}_0 = \mathbb{R}^{d+1}_0$. Let $x = t\xi$ and $y = s\eta$ for $\xi, \eta \in \mathbb{S}^{d-1}$. By inequality (3.4), from $d_{X_0}(x, t, (y, s)) \leq r$ we obtain $d_{[-1,1]}(t, s) \leq r$; moreover, denote $	au_r(t, s) = (\cos r - \sqrt{1 - t^2} \sqrt{1 - s^2})/(ts)$ and $\theta_r(t, s) = \arccos \tau_r(t, s)$, we also have $d_{\mathbb{S}}(\xi, \eta) \leq \frac{1}{2} \arccos \tau_r(t, s)$. Then it is easy to see, since $d\sigma(y, s) = s^{d-1}d\sigma_{\mathbb{S}}(\eta)ds$, that

$$w_{0,\gamma}^0(c((x, t), r)) = \int_{d_{[-1,1]}(t, s) \leq r} s^{d-1} \int_{d_{\mathbb{S}}(\eta) \leq \frac{1}{2} \theta_r(t, s)} w_{0,\gamma}^0(s\eta, s)d\sigma_{\mathbb{S}}(\eta)ds.$$ 

By symmetry, we can choose $\xi = (1, 0, \ldots, 0)$ and use the identity (cf. [8, (A.5.1)])

$$\int_{\mathbb{S}^{d-1}} g((\xi, \eta))d\sigma(\eta) = \omega_{d-1} \int_{0}^{\pi} g(\cos \theta)(\sin \theta)^{d-2}d\theta$$

(3.9)

with $\omega_{d-1}$ being the surface area of $\mathbb{S}^{d-2}$, to obtain that

$$w_{0,\gamma}^0(c((x, t), r)) = \omega_{d-1} \int_{d_{(0,1)}(t, s) \leq r} s^{d-1} w_{0,\gamma}^0(s) \int_{0}^{\frac{1}{2} \theta_r(t, s)} (\sin \theta)^{d-2}d\theta ds.$$ 

Since $\theta \sim \sin \theta \sim \sqrt{1 - \cos \theta}$, it follows then

$$w_{0,\gamma}^0(c_{0}((x, t), r)) \sim \int_{d_{[-1,1]}(t, s) \leq r} s^{d-1} w_{0,\gamma}^0(s)(1 - \tau_r(t, s))^{\frac{d-1}{2}} ds. \quad (3.10)$$

We now need to consider three cases. If $3r < t < 1 - 3r$, then we can use Lemma 3.4 to conclude that $s^2 \sim t^2 + r^2$ and $1 - s^2 \sim 1 - t^2 + r^2$, so that

$$w_{0,\gamma}^0(c_{0}((x, t), r)) \sim (t^2 + r^2)^{\beta}(1 - t^2 + r^2)^{\gamma} \times \int_{d_{[-1,1]}(t, s) \leq r} (\cos(d_{[-1,1]}(t, s)) - \cos r)^{\frac{d-1}{2}} \frac{ds}{\sqrt{1 - s^2}}.$$ 

Setting $t = \cos \theta$ and $s = \cos \phi$ so that $d_{[-1,1]}(t, s) = |\theta - \phi|$, then the last integral is easily seen to be

$$\int_{|\theta - \phi| \leq r} (\cos(\theta - \phi) - \cos r)^{\frac{d-1}{2}} d\phi = c \int_{|\xi| \leq r} (\sin \frac{\xi - r}{2} \sin \frac{\xi + r}{2})^{\frac{d-1}{2}} d\xi \sim r^{d}.$$
This completes the proof of the first case. If \( t \leq 3r \), then \(|t - s| \leq d[[-1, 1]](t, s) \leq r\) so that \( s \leq 4r \). Evidently \( 1 - s^2 \sim 1 - t^2 \sim 1 \) in this case. Furthermore, let \( t = \sin \theta \) and \( s = \sin \phi \); then \(|\theta - \phi| = d[[-1, 1]](t, s) \leq r\), which is easily seen to be equivalent to \(|\tau_r(t, s)| \leq 1\). Hence, using \( 1 - \tau_r(t, s) \leq 2 \) and \( s \leq 4r \), we obtain by (3.10) and \( \sin \phi \sim \phi \),

\[
W_{\beta, \gamma}(\mathcal{C}(x, t)) \leq c \int_{d[[-1, 1]](t, s) \leq r} s^{d+2\beta-1} \sim r^{2\beta+d},
\]

which proves the upper bound in (3.8). For the lower bound, we consider a subset of \( \mathcal{C}(x, t) \) with \( d[[-1, 1]](t, s) \leq r/2 \). Using the upper bound of \( s \) and \( t \), we then deduce

\[
1 - \tau_r(t, s) = \frac{\cos d[[-1, 1]](t, s) - \cos r}{ts} \geq \frac{\cos \frac{s}{2} - \cos r}{12r^2} \geq \frac{1}{8\pi^2},
\]

where in the last step we have used \( \sin \theta \geq \frac{2}{\pi} \theta \), which shows that

\[
W_{\beta, \gamma}(\mathcal{C}(x, t)) \geq c \int_{d[[-1, 1]](t, s) \leq r} s^{d+2\beta-1} \sim r^{2\beta+d}.
\]

Finally, if \( t \geq 1 - 3r \), then we have \( s \sim t \sim 1 \) for \((y, s) \in \mathcal{C}(x, t))\). Since \( d[[-1, 1]](t, s) = d[[-1, 1]](\sqrt{1 - t^2}, \sqrt{1 - s^2}) \), changing variable \( s \mapsto \sqrt{1 - s^2} \) in (3.10), we see that this case can be reduced to that of the second case. This completes the proof of (3.8). \( \square \)

For \( \sigma > 0 \) and \( \beta, \gamma > -\frac{1}{2} \), the weight function \( W_{\beta, \gamma}^{\sigma} \) is a doubling weight on the hyperbolic surface \( H^d_{\beta, \gamma} \) by Lemma 3.5.

**Corollary 3.7** For \( \sigma > 0 \), \( d \geq 2 \), and \( \beta, \gamma > -\frac{1}{2} \), the space \((H^{d+1}_{\sigma 0}, W_{\beta, \gamma}^{\sigma}, d_{\sigma 0}^{\sigma})\) is a homogeneous space. If \( \sigma = 0 \), the restriction on \( \beta \) can be relaxed to \( \beta > -\frac{d+1}{2} \).

When \( \sigma = 0 \), \( \beta = 0 \) and \( \gamma = \frac{1}{2} \), relation (3.8) is for the Lebesgue measure \( \sigma_0 \) on the double conic surface; in particular, if \( \beta = 0 \) and \( \gamma = 0 \), then \( W_{0,0}^{\sigma}(t) = (1-t^2)^{-\frac{1}{2}} \) is the Chebyshev weight and \( W_{0,0}^{0}(\mathcal{C}_0(x, t)) \sim r^d \). The apex point \( t = 0 \) does not appear as a boundary point of \( H^{d+1}_{\sigma 0} \).

For convenience, we will introduce the function \( W_{\beta, \gamma}(n; t) \) defined by

\[
W_{\beta, \gamma}(n; t) = n^{-d}W_{\beta, \gamma}(\mathcal{C}_0((x, t), n^{-1})) = (t^2 - \sigma^2 + n^{-2})^{\beta} (1 - t^2 + \sigma^2 + n^{-2})^{\gamma}
\] (3.11)
on the hyperbolic surface \( H^{d+1}_{\sigma 0} \) and use it in the latter sections.
3.3 Orthogonal Polynomials on Hyperbolic Surfaces

For $\gamma > -\frac{1}{2}$, we define the inner product on the hyperbolic surface by

$$\langle f, g \rangle_{W_{\beta, \gamma}} = b_{\beta, \gamma} \int_{\mathbb{H}^{d+1}} f(x, t) g(x, t) w_{\beta, \gamma}(t) d\sigma_e(x, t),$$

where $\beta > -\frac{1}{2}$ if $\varrho > 0$ and $\beta > -\frac{d+1}{2}$ if $\varrho = 0$. The orthogonal polynomials with respect to this inner product are studied in [28]. Let $V_n(\mathbb{H}^{d+1}_0, w_{\beta, \gamma})$ be the space of these polynomials with respect to this inner product, which has the same dimension as the space of spherical harmonics $\mathcal{H}^{d+1}_0$. Because the weight function $w_{\beta, \gamma}$ is even in $t$, this space can be factored as

$$V_n(\mathbb{H}^{d+1}_0, w_{\beta, \gamma}) = V_n^E(\mathbb{H}^{d+1}_0, w_{\beta, \gamma}) \bigoplus V_n^O(\mathbb{H}^{d+1}_0, w_{\beta, \gamma}),$$

where the subspace $V_n^E(\mathbb{H}^{d+1}_0, w_{\beta, \gamma})$ consists of orthogonal polynomials that are even in the $t$ variable, and the subspace $V_n^O(\mathbb{H}^{d+1}_0, w_{\beta, \gamma})$ consists of orthogonal polynomials that are odd in the $t$ variable. Moreover,

$$\dim V_n^E(\mathbb{H}^{d+1}_0, w_{\beta, \gamma}) = \binom{n + d - 1}{n}, \quad \dim V_n^O(\mathbb{H}^{d+1}_0, w_{\beta, \gamma}) = \binom{n + d - 2}{n - 1}.$$

It turns out that an orthogonal basis can be given in terms of the spherical harmonics and the Jacobi polynomials for the subspace $V_n^E(\mathbb{H}^{d+1}_0, w_{\beta, \gamma})$ for all $\varrho \geq 0$, but for the subspace $V_n^O(\mathbb{H}^{d+1}_0, w_{\beta, \gamma})$ only when $\varrho = 0$. For example, let $\{Y_{\ell}^{m}: 1 \leq \ell \leq \dim \mathcal{H}_m^d\}$ denote an orthonormal basis of $\mathcal{H}_d^m$. Then the polynomials

$$C_{m, \ell}(x, t) = P_k^\gamma(1, n-2k+\beta+\frac{d-2}{2})(2\ell^2 - 2\varrho^2 - 1)Y_{\ell}^{n-2k}(x), \quad (3.12)$$

where $1 \leq \ell \leq \dim \mathcal{H}_m^{d-2k}$ and $0 \leq k \leq n/2$, form an orthogonal basis of $V_n^E(\mathbb{H}^{d+1}_0, w_{\beta, \gamma})$. Since we will not work directly with explicit bases, we refer to [28] for further information, where these polynomials are called, when $\beta = 0$, the Gegenbauer polynomials on the hyperbolic surface or the double conic surface when $\varrho = 0$.

Let $P_n^E(w_{\beta, \gamma}; \cdot, \cdot)$ be the reproducing kernel of $V_n^E(\mathbb{H}^{d+1}_0, w_{\beta, \gamma})$, which can be written in terms of basis (3.12) as

$$P_n^E(w_{\beta, \gamma}; (x, t), (y, s)) = \sum_{m=0}^{n} \sum_{\ell=1}^{\dim \mathcal{H}_m^{d-2m}} C_{m, \ell}(x, t) C_{m, \ell}(y, s) \langle C_{m, \ell}, C_{m, \ell} \rangle_{w_{\beta, \gamma}}.$$
Let \( \text{proj}_n^E (w_{\beta,\gamma}^E) : L^2(\mathbb{X}_0^{d+1}, w_{\beta,\gamma}^E) \mapsto \gamma_n^E(\mathbb{X}_0^{d+1}, w_{\beta,\gamma}^E) \) be the orthogonal projection operator. Then it can be written as

\[
\text{proj}_n^E (w_{\beta,\gamma}^E ; f) = b_{\beta,\gamma} \int_{\mathbb{X}_0^{d+1}} f(y) P_n^E (w_{\beta,\gamma}^E ; \cdot, (y, s)) w_{\beta,\gamma}(s) dy ds.
\]

If \( f \) is a function that is even in the variable \( t \) on \( \mathbb{X}_0^{d+1} \), then its orthogonal projection on \( V_{\mathbb{O}}^n(\mathbb{X}_0^{d+1}, w_{\beta,\gamma}^E) \) becomes zero, so that its Fourier orthogonal expansion is given by

\[
f = \sum_{n=0}^{\infty} \text{proj}_n^E (w_{\beta,\gamma}^E ; f).
\] (3.13)

Hence, the kernel \( P_n^E (w_{\beta,\gamma}^E ; \cdot, \cdot) \) is meaningful for studying the Fourier orthogonal expansions on the hyperboloid.

If \( \varrho = 0 \), then the upper part \( \mathbb{X}_0^{d+1} = \mathbb{Y}_0^{d+1} \) is the upper conic surface. The function \( f(x, t) \) that is even in the \( t \) variable can be regarded as defined on \( \mathbb{Y}_0^{d+1} \) or as the even extension in the \( t \) variable of a function defined on the upper conic surface. Consequently, the Fourier expansion (3.13) works for the function \( f \) defined on the upper conic surface \( \mathbb{X}_0^{d+1} \) when \( \varrho = 0 \). The latter, however, is different from the Fourier expansions in the Jacobi polynomials on the conic surface \( \mathbb{Y}_0^{d+1} \) discussed in [29].

The case \( \beta = 0 \) is the most interesting one since the orthogonal polynomials for \( w_{0,\gamma} \) enjoy two characteristic properties. The first one is the spectral operator that has orthogonal polynomials as eigenfunctions.

**Theorem 3.8** Let \( \varrho > 0 \) and \( \gamma > -\frac{1}{2} \). Then for \( x = t \xi, \xi \in \mathbb{S}^{d-1} \), define the differential operator

\[
\Delta_{0,\gamma}^E = (1 + \varrho^2 - t^2) \left( 1 - \frac{\varrho^2}{t^2} \right) \partial_t^2 + \left( (1 + \varrho^2 - t^2) \frac{\varrho^2}{t^2} - (2\gamma + d)(t^2 - \varrho^2) \right) \frac{1}{t} \partial_t + \frac{d - 1}{t} \partial_t + \frac{1}{t^2 - \varrho^2} \Delta_0^{(\xi)}.
\]

Then the polynomials in \( V_n^E(\mathbb{Y}_0^{d+1}, w_{0,\gamma}^E) \) are eigenfunctions of \( \Delta_{0,\gamma}^E \),

\[
\Delta_{0,\gamma}^E u = -n(n + 2\gamma + d - 1)u, \quad \forall u \in V_n^E(\mathbb{Y}_0^{d+1}, w_{0,\gamma}^E).
\] (3.14)

The second one is the addition formula for the reproducing kernel \( P_n^E (w_{\beta,\gamma}^E ; \cdot, \cdot) \), which is of the simplest form when \( \beta = 0 \).

**Theorem 3.9** Let \( d \geq 2 \) and \( \varrho \geq 0 \). Then
(a) For $\beta, \gamma > -\frac{1}{2}$,

$$p_n^E(w_{\beta,\gamma}^0; (x,t), (y,s)) = p_n^E(w_{\beta,\gamma}^0; (x,\sqrt{t^2 - \rho^2}), (y,\sqrt{s^2 - \rho^2})).$$

(3.15)

(b) For $\rho = 0$, $\beta = 0$ and $\gamma \geq 0$,

$$
p_n^E(w_{0,\gamma}^0; (x,t), (y,s)) = c_{\gamma} \int_{-1}^{1} Z_n^{\gamma + \frac{d-1}{2}}(\xi(x, t, y, s; v))(1 - v^2)^{\gamma - 1} dv,
$$

(3.16)

where $Z_n^\lambda$ is defined in (2.16), $c_{\gamma} = c_{\gamma-1,\gamma-1}$ and

$$\xi(x, t, y, s; v) = \langle x, y \rangle \text{sgn}(ts) + v\sqrt{1 - t^2}\sqrt{1 - s^2},$$

and the case $\gamma = 0$ holds under limit (2.21).

The closed form formula (3.16) is essential for studying highly localized kernels.

### 3.4 Highly Localized Kernels

Let $\hat{a}$ be a cut-off function. For $(x, t), (y, s) \in \mathbb{R}^{d+1}_0$, define the kernel $L_n^E(w_{0,\gamma}^0)$ by

$$L_n^E(w_{0,\gamma}^0; (x,t), (y,s)) = \sum_{j=0}^{\infty} \hat{a} \left( \frac{j}{n} \right) p_j^E(w_{0,\gamma}^0; (x,t), (y,s)).$$

The kernel uses only orthogonal polynomials that are even in the $t$ and the $s$ variables so that it is even in both the $t$ and $s$ variables. We show that this kernel is highly localized when $(x, t)$ and $(y, s)$ are either both in $\mathbb{R}^{d+1}_0$ or both in $\mathbb{R}^{d+1}_0$. For $\gamma \geq 0$, recall by (3.11),

$$w_{0,\gamma}^0(n; t) := (1 + q^2 - t^2 + n^{-2})^\gamma.$$

**Theorem 3.10** Let $d \geq 2$ and $\gamma \geq 0$. Let $\hat{a}$ be an admissible cutoff function. Then, for any $\kappa > 0$, either $(x, t), (y, s)$ both in $\mathbb{R}^{d+1}_0$ or both in $\mathbb{R}^{d+1}_0$,

$$\left| L_n^E(w_{0,\gamma}^0; (x,t), (y,s)) \right| \leq \frac{c_{\kappa} n^d}{\sqrt{w_{0,\gamma}^0(n; t)} \sqrt{w_{0,\gamma}^0(n; s)}} \left( 1 + n d \tilde{c}_{\kappa} ((x, t), (y, s)) \right)^{-\kappa},$$

where we assume $t$ and $s$ have the same sign when $\rho > 0$. 

\[\square\] Springer
Proof By (3.15), it is sufficient to consider the case $\varrho = 0$. The proof follows the similar procedure as in the case of the conic surface, so we shall be brief. By (3.16) we can write $L^E_n (\mathcal{W}_{0, y})$ in terms of the kernel for the Jacobi polynomials. Let $\lambda = \gamma + \frac{d-1}{2}$. Then

$$L^E_n (\mathcal{W}_{0, y}; (x, t), (y, s)) = c_y \int_{-1}^{1} L_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})} (\zeta(x, t, y, s; v))(1 - v^2)^{\gamma - 1} dv.$$  

Applying (2.12) with $m = 0$ and $\alpha = \beta = \lambda - 1/2$, we then obtain

$$\left| L^E_n (\mathcal{W}_{0, y}; (x, t), (y, s)) \right| \leq c n^{2\lambda + 1} \int_{-1}^{1} \frac{1}{(1 + n \sqrt{1 - \zeta(x, t, y, s; v)})^{\lambda + 1}} \sqrt{1 - v^2}^{\gamma - 1} dv.$$  

By the definition of $d_{\mathcal{X}_0} (\cdot, \cdot)$, we have

$$1 - \zeta(x, t, y, s; t) = 1 - \cos d_{\mathcal{X}_0} ((x, t), (y, s)) + (1 - v) \sqrt{1 - t^2} \sqrt{1 - s^2}. \quad (3.17)$$

In particular, $1 - \zeta(x, t, y, s; t)$ is bounded below by either the first term or the second term in the right-hand side of (3.17), which leads to, in particular, the estimate

$$\left| L^E_n (\mathcal{W}_{0, y}; (x, t), (y, s)) \right| \leq c n^{2\lambda + 1} \int_{0}^{1} \frac{1}{(1 + n d_{\mathcal{X}_0} ((x, t), (y, s))))^{\lambda + \gamma}} \times c_y \int_{0}^{1} \frac{(1 - v^2)^{\gamma - 1}}{\sqrt{1 - v} \sqrt{1 - t^2} \sqrt{1 - s^2}}^{2\gamma + 1} dv,$$

where we have used $1 - \cos \theta \sim \theta^2$ and the symmetry of the integral. The last integral is bounded by 1 and it can be estimated by using the inequality [8, (13.5.8)]

$$\int_{0}^{1} \frac{(1 - t)^{a - 1} dt}{(1 + n \sqrt{B + A(1 - t)})^b} \leq c \frac{n^{-2a}}{A^a (1 + n \sqrt{B})^{b - 2a - 1}}, \quad (3.18)$$

which holds for $A > 0$, $B \geq 0$, $a > 0$ and $b \geq 2a + 1$, which leads to the estimate

$$c_y \int_{0}^{1} \frac{(1 - v^2)^{\gamma - 1}}{(1 + n \sqrt{1 - t^2} \sqrt{1 - s^2})^{2\gamma + 1}} dv \leq c \frac{n^{-2\gamma}}{(\sqrt{1 - t^2} \sqrt{1 - s^2} + n^{-1})^{\gamma}} \leq c \frac{n^{-2\gamma}}{\sqrt{W^0_{0, y}(n; t)} \sqrt{W^0_{0, y}(n; t)}} \left(1 + n d_{\mathcal{X}_0} ((x, t), (y, s)))^{\gamma},
$$
where the second inequality follows from the elementary identity \[8, (11.5.13)]
\[(a + n^{-1})(b + n^{-1}) \leq 3(ab + n^{-2})(1 + n|b - a|)\] (3.19)
with \(a = \sqrt{1 - t^2}\) and \(b = \sqrt{1 - s^2}\) and Lemma 3.4. Putting the last two displayed inequalities together complete the proof. \(\square\)

This theorem establishes the Assertion 1 on \(X_0^{d+1}\). We now turn to Assertion 2.

**Theorem 3.11** Let \(d \geq 2\) and \(\gamma \geq 0\). For either \((x_i, t_i), (y, s)\) all in \(X_0^{d+1}\) or all in \(X_0^{d-1}\), and \((x_1, t_1) \in c_\rho((x_2, t_2), c^*n^{-1})\) with \(c^*\) sufficiently small and for any \(\kappa > 0\),
\[
\left| L_n^E (w_{0, \gamma}; (x_1, t_1), (y, s)) - L_n^E (w_{0, \gamma}; (x_2, t_2), (y, s)) \right| \\
\leq \frac{c_\kappa n^{d+1} d_{\kappa_0}^p ((x_1, t_1), (x_2, t_2))}{\sqrt{w_{0, \gamma}^p (n; t_2)} \sqrt{w_{0, \gamma}^p (n; s)} (1 + n d_{\kappa_0}^p ((x_2, t_2), (y, s)))^\kappa}. \tag{3.20}
\]

**Proof** Again, it is sufficient to consider \(\varrho = 0\). Denote the left-hand side of (3.20) by \(K\). Let \(\partial L(u) = L'(u)\). Using the integral expression of \(L_n^E (w_{-1, \gamma})\), we obtain
\[
K \leq 2 \int_{-1}^{1} \left\| \partial L_n^{\lambda - \frac{1}{2}, \lambda - \frac{1}{2}} \right\|_{L^\infty(I_v)} |\xi_1(v) - \xi_2(v)|(1 - v^2)^{\gamma - 1} dv,
\]
where \(|\xi_1(v) - \xi_2(v)| = \xi(x_i, t_i, y, s; v), \) and \(I_v\) is the interval with endpoints \(\xi_1(v)\) and \(\xi_2(v)\). We claim that
\[
|\xi_1(v) - \xi_2(v)| \leq d_{\kappa_0}((x_1, t_1), (x_2, t_2))[\Sigma_1 + \Sigma_2(v)], \tag{3.21}
\]
where
\[
\Sigma_1 = d_{\kappa_0}((x_2, t_2), (y, s)) + d_{\kappa_0}((x_1, t_1), (x_2, t_2)), \\
\Sigma_2(v) = (1 - v)\sqrt{1 - s^2}.
\]
To see this, we first use (3.17) to write that
\[
\xi_1(v) - \xi_2(v) = \cos d_{\kappa_0}((x_1, t_1), (y, s)) - \cos d_{\kappa_0}((x_2, t_2), (y, s)) + (1 - v) \left(\sqrt{1 - t_1^2} - \sqrt{1 - t_2^2}\right) \sqrt{1 - s^2}.
\]
Denote temporarily \(\alpha_i = d_{\kappa_0}((x_1, t_1), (y, s))\) for \(i = 1, 2\). The identity
\[
\cos \alpha_1 - \cos \alpha_2 = 2 \sin \frac{\alpha_1 - \alpha_2}{2} \left(2 \sin \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} - \sin \frac{\alpha_2 - \alpha_1}{2}\right),
\]
implies that \(| \cos \alpha_1 - \cos \alpha_2 | \leq | \alpha_1 - \alpha_2 | \left( | \alpha_1 | + \frac{1}{2} | \alpha_1 - \alpha_2 | \right) \), which leads to the estimate for the \( \Sigma_1 \) term by the triangle inequality of \( d_{\Sigma_0} \) and Lemma 3.4. The estimate for the \( \Sigma_2 \) terms is trivial. Hence, (3.21) holds as claimed.

Since \( \max_{r \in I_0}(1 + n \sqrt{1 - r^2})^{-\kappa} \) is attained at one of the endpoints of the interval, it follows from (2.12) with \( m = 1 \) and \( \lambda = \gamma + \frac{d-1}{2} \) that

\[
\left| \mathcal{L}_n^E(w_{0, \gamma}^0(x_1, t_1), (y, s)) - \mathcal{L}_n^E(w_{0, \gamma}^0(x_2, t_2), (y, s)) \right|
\leq c \, d_{\Sigma_0}(x_2, t_2) \int_{-1}^{1} \left[ \frac{n^{2\lambda+3}}{(1 + n \sqrt{1 - \zeta_i(v)^2})^{\kappa(\gamma)}} + \frac{n^{2\lambda+3}}{(1 + n \sqrt{1 - \zeta_2(v)^2})^{\kappa(\gamma)}} \right]
\times (\Sigma_1 + \Sigma_2(v))(1 - v^2)^{\gamma-1} \, dv,
\]

where we choose \( \kappa(\gamma) = \kappa + 3\gamma + 2 \). Since \( (x_1, t_1) \in C((x_2, t_2), c^n n^{-1}) \), \( \Sigma_1 \) is bounded by \( \Sigma_1 \leq c n^{-1}(1 + nd_{\Sigma_0}(x_2, t_2), (y, s)) \). Hence, using the two lower bounds of \( 1 - \zeta_i(v) \) given by the right-hand side of (3.17), we obtain

\[
\int_{-1}^{1} \frac{n^{2\lambda+3}}{(1 + n \sqrt{1 - \zeta_i(v)^2})^{\kappa(\gamma)}} \Sigma_1(1 - v^2)^{\gamma-1} \, dv
\leq \frac{c n^{2\lambda+2}}{(1 + nd_{\Sigma_0}(x_1, t_1), (y, s))^{\kappa+\gamma}} \int_{0}^{1} \frac{(1 - v^2)^{\gamma-1}}{(1 + n \sqrt{1 - v^2})^{1 - \frac{1}{2}}} \, dv
\leq \frac{c \kappa n^{d+1}}{\sqrt{w_{0, \gamma}^0(n; t_1) \sqrt{w_{0, \gamma}^0(n; s)(1 + nd_{\Sigma_0}(x_1, t_1), (y, s))^{\kappa}}}},
\]

where the second step follows from the estimate of the last integral in the proof of Theorem 3.10. Since \( w_{0, \gamma}^0(n, t_1) \sim w_{0, \gamma}^0(n, t_2) \) and \( d_{\Sigma_0}(x_1, t_1), (y, s) + n^{-1} \sim d_{\Sigma_0}(x_2, t_2), (y, s) + n^{-1} \) by Lemma 3.4, we can replace \( (x_1, t_1) \) in the right-hand side by \( (x_2, t_2) \). This shows that the integral containing \( \Sigma_1 \) has the desired estimate.

For the integral that contains \( \Sigma_2(v) = (1 - v) \sqrt{1 - s^2} \), the factor \( 1 - v \) increases the power of the weight to \( (1 - v_1)^{\gamma} \), so that we can follow the estimate for the integral with \( \Sigma_1 \) but using \( (1 - v)^{\gamma} \), which leads to

\[
\int_{-1}^{1} \frac{n^{2\lambda+3}}{(1 + n \sqrt{1 - \zeta_i(v)^2})^{\kappa(\gamma+\frac{1}{2})}} \Sigma_2(v_1)(1 - v^2)^{\gamma-1} \, dv
\leq \frac{c n^{d+1} n^{-1} \sqrt{1 - s^2}}{\sqrt{w_{0, \gamma+\frac{1}{2}}^0(n; s) \sqrt{w_{0, \gamma+\frac{1}{2}}^0(n; t_2)(1 + nd_{\Sigma_0}(y, s, x_i, t_i))^{\kappa}}}}
\leq \frac{c n^{d+1}}{\sqrt{w_{\gamma, d}^0(n; s) \sqrt{w_{\gamma, d}^0(n; t_2)(1 + nd_{\Sigma_0}(y, s, x_i, t_i))^{\kappa}}}},
\]

\( \copyright \) Springer
where the last step uses the inequality \( n^{-1} \sqrt{1 - s^2} \leq (\sqrt{1 - t_i^2} + n^{-1})(\sqrt{1 - s^2} + n^{-1}) \). This takes care of the integral with \( \Sigma_2(v) \) and completes the proof. \( \Box \)

The case of \( p = 1 \) of the following lemma establishes Assertion 3 for \( w^0_{\beta,\gamma} \).

Lemma 3.12 Let \( d \geq 2 \), \( \beta > \frac{1}{2} \) and \( \gamma > \frac{1}{2} \). For \( 0 < p < \infty \), assume \( \kappa > \frac{2d + 2}{p} + 2(\beta + \gamma)|\frac{1}{p} - \frac{1}{2}| \). Then for \( (x, t) \in \mathcal{X}^{d+1}_0 \),

\[
\int_{\mathcal{X}^{d+1}_0} \frac{w^0_{\beta,\gamma}(s)}{w^0_{\beta,\gamma}(n; s)(1 + n\sigma^2_{\mathcal{X}^d_0}((x, t), (y, s)))^{\kappa p}} d\sigma_0(y, s) \leq c n^{-d} w^0_{\beta,\gamma}(n; t)^{1 - \frac{p}{2}}.
\]

(3.22)

Proof We again only need to consider \( q = 0 \). Let \( J_p \) denote the left-hand side of (3.22). As shown in the proof of [29, Lemma 2.4], it is sufficient to prove the case \( p = 2 \). Furthermore, the integral over \( \mathcal{X}^{d+1}_0 \) is a sum of two integrals over \( \mathcal{X}^{d+1}_0 \) and \( \mathcal{X}^{d+1}_{0,-} \), respectively. We only need to estimate one of them. Denote the integral over \( \mathcal{X}^{d+1}_0 \) by \( J_{2,+} \). Then

\[
J_{2,+} = \int_{\mathcal{X}^{d+1}_0} \frac{w^0_{\beta,\gamma}(s)}{w^0_{\beta,\gamma}(n; s)(1 + n\sigma^2_{\mathcal{X}^d_0}((x, t), (y, s)))^{2\kappa}} d\sigma_0(y, s).
\]

Let \( x = t\xi \) and \( y = s\eta \). Using (3.9), we obtain

\[
J_{2,+} \leq c \int_0^1 \int_{-1}^1 \frac{s^{d-1}w^0_{\beta,\gamma}(s)(1 - u^2)^{\frac{d-3}{2}}}{w^0_{\beta,\gamma}(n; s)(1 + n\sqrt{1 - tsu - \sqrt{1 - t^2\sqrt{1 - s^2}}})^{2\kappa}} du ds.
\]

Hence, using \( w^0_{\beta,\gamma}(s) \leq c w^0_{\beta,\gamma}(n; s)(1 - s^2 + n^{-2})^{-\frac{1}{2}} \), making another change of variable \( u \mapsto v/s \) and simplifying, it follows that

\[
J_{2,+} \leq c \int_0^1 \int_{-s}^s \frac{s (s^2 - v^2)^{\frac{d-3}{2}}}{(1 - s^2 + n^{-2})^{\frac{1}{2}}(1 + n\sqrt{1 - tv - \sqrt{1 - t^2\sqrt{1 - s^2}}})^{2\kappa}} dv ds
\]

\[
\leq cn d \int_{-1}^1 \int_{|v|}^1 \frac{(s^2 - v^2)^{\frac{d-3}{2}}}{(1 + n\sqrt{1 - tv - \sqrt{1 - t^2\sqrt{1 - s^2}}})^{2\kappa}} ds dv.
\]
where we changed the order of integration in the second step. A further change of variable $s \mapsto \sqrt{1 - \|u\|^2}$ shows then

$$J_{2,+} \leq c \int_{-1}^{1} \int_{0}^{\sqrt{1 - v^2}} \frac{(1 - u^2 - v^2)^{d-3}}{(1 + n\sqrt{1 - tv - \sqrt{1 - t^2u}})^2} du dv,$$

which is an integral over the right half $\{(u, v) \in B^2 : v \geq 0\}$ of the unit disk $B^2$. Setting $z = tv + \sqrt{1 - t^2}u$ and $w = -\sqrt{1 - t^2}v + tu$ in the integral, which is an orthogonal transformation, and enlarging the integral domain while taking into account that $z \geq 0$, it follows that

$$J_{2,+} \leq c \int_{0}^{1} \frac{1}{1 + n\sqrt{1 - z}}^2 dz \int_{\sqrt{1 - z^2}}^{\frac{1}{\sqrt{1 - z^2}}} (1 - z^2 - w^2)^{d-3} dw,$$

by setting $r = n\sqrt{1 - z}$ and recalling that $\kappa > d$. This completes the proof. $\square$

**Proposition 3.13** For $\gamma \geq 0$ and $(x, t) \in \mathcal{X}^{d+1}_0$,

$$\int_{\mathcal{X}^{d+1}_0} |L^E_n (w^\gamma_{0,y}; (x, t), (y, s))|^p w^\gamma_{0,y} (s) d\sigma_{\rho} (y, s) \leq \left( \frac{n^d}{w^\gamma_{0,y} (n; t)} \right)^{p-1}.$$

This follows by applying Lemma 3.12 on the estimate of Theorem 3.10.

We have established Assertions 1 and 3 for $L^E_n (w^\gamma_{\rho,y}; \cdot, \cdot)$ and also Assertion 2 for $L^E_n (w^\gamma_{0,y}; \cdot, \cdot)$. The kernel uses, however, only polynomials that are even in the $t$ and in the $s$ variables. Consequently, we have proved the following:

**Corollary 3.14** For $d \geq 2$, $\rho \geq 0$ and $\gamma \geq 0$, the space $(\mathcal{X}^{d+1}_0, w^\gamma_{0,y}, c^E_{\rho,\gamma})$ is a localizable homogeneous space, where its localized kernels are defined for polynomials even in the $t$ and in the $s$ variables.

### 3.5 Maximal $\varepsilon$-Separated Sets and MZ Inequality

We provide a construction of maximal $\varepsilon$-separated set, as defined by Definition 2.3, on the double conic and hyperbolic surfaces.

We first consider the double conic surface; that is, $\rho = 0$. We shall need maximal $\varepsilon$-separated sets on the unit sphere. We adopt the following notation. For $\varepsilon > 0$, we denote by $\Xi_B (\varepsilon)$ a maximal $\varepsilon$-separated set on the unit sphere $S^{d-1}$ and we let $S^\varepsilon_\rho$.
be the subsets in $\mathbb{S}^{d-1}$ so that the collection $\{S_\xi(\epsilon) : \xi \in \Xi(\epsilon)\}$ is a partition of $\mathbb{S}^{d-1}$, and we assume
\[
c_\mathcal{S}(\xi, \epsilon_1) \subset S_\xi(\epsilon) \subset c_\mathcal{S}(\xi, \epsilon_2), \quad \xi \in \Xi(\epsilon),
\] (3.23)
where $c_\mathcal{S}(\xi, \epsilon)$ denotes the spherical cap centered at $\xi$ with radius $\epsilon$, $c_1$ and $c_2$ depending only on $d$. Such a $\Xi(\epsilon)$ exists for all $\epsilon > 0$, see for example [8, Section 6.4], and its cardinality satisfies
\[
c_d \epsilon^{-d+1} \leq \#\Xi(\epsilon) \leq c_d \epsilon^{-d+1}.
\] (3.24)

Let $\epsilon > 0$. We let $N = 2 \lceil \frac{\pi}{2} \epsilon^{-1} \rceil$, so that $N$ is an even integer. For $1 \leq j \leq N$ we define
\[
\theta_j := \frac{(2j-1)\pi}{2N}, \quad \theta_j^+ := \theta_j - \frac{\pi}{2N} \quad \text{and} \quad \theta_j^- := \theta_j + \frac{\pi}{2N}.
\]
Let $t_j = \cos \theta_j$, $t_j^- = \cos \theta_j^-$ and $t_j^+ = \cos \theta_j^+$. Thus, $t_1^+ = 1$ and $t_N^- = -1$ and
\[
1 > t_1 > t_2 > \cdots > t_N > 0 = t_N^+ > t_{N+1}^+ > t_{N+2}^+ > \cdots > t_N > -1.
\]
In particular, $t_{j-1}^- = t_j^-$ and we can partition $\mathcal{X}_0^{d+1}$ as the disjoint union of
\[
\mathcal{X}_0^{(j)} := \{(x, t) \in \mathcal{X}_0^{d+1} : t_j^- < t \leq t_j^+\}, \quad 1 \leq j \leq N.
\]
Furthermore, the upper and lower surfaces $\mathcal{X}^{d+1}_{0,+}$ and $\mathcal{X}^{d+1}_{0,-}$ can be partitioned by
\[
\mathcal{X}^{d+1}_{0,+} = \bigcup_{j=1}^{N/2} \mathcal{X}_0^{(j)} \quad \text{and} \quad \mathcal{X}^{d+1}_{0,-} = \bigcup_{j=N/2+1}^N \mathcal{X}_0^{(j)}.
\]
Let $\epsilon_j := \pi \epsilon/(2t_j)$. Then $\Xi(\epsilon_j)$ is the maximal $\epsilon_j$-separated set of $\mathbb{S}^{d-1}$ such that $\{S_\xi(\epsilon_j) : \xi \in \Xi(\epsilon_j)\}$ is a partition $\mathbb{S}^{d-1} = \bigcup_{\eta \in \Xi(\epsilon_j)} S_\eta(\epsilon_j)$, and $\#\Xi(\epsilon_j) \sim \epsilon_j^{-d+1}$. For each $j = 1, \ldots, N$, we decompose $\mathcal{X}_0^{(j)}$ by
\[
\mathcal{X}_0^{(j)} = \bigcup_{\xi \in \Xi(\epsilon_j)} X_0(\xi, t_j), \quad \text{where} \quad X_0(\xi, t_j) := \{(t, t) : t_j^- < t \leq t_j^+, \eta \in S_\xi(\epsilon_j)\}.
\]
Finally, we define the subset $\Xi_0$ of $\mathcal{X}_0^{d+1}$ by
\[
\Xi_0 = \{(t_j \xi, t_j) : \xi \in \Xi(\epsilon_j), 1 \leq j \leq N\}.
\]
Proposition 3.15 Let $\varepsilon > 0$ and $N = 2\lceil \frac{\pi}{d} \varepsilon^{-1} \rceil$. Then $\Xi \chi_0$ is a maximal $\varepsilon$-separated set of $\mathbb{X}_d+1$ and $\{\mathbb{X}_0(\xi, t) : (t_j \xi, t_j) \in \Xi \chi_0\}$ is a partition

$$
\mathbb{X}_d+1 = \bigcup_{(\xi, t) \in \Xi \chi_0} \mathbb{X}_0(\xi, t) = \bigcup_{j=1}^{N} \bigcup_{\xi \in \chi_0(\varepsilon_j)} \mathbb{X}_0(\xi, t_j).
$$

Moreover, there are positive constants $c_1$ and $c_2$ depending only on $d$ such that

$$
c_0((t_j \xi, t_j), c_1\varepsilon) \subset \mathbb{X}_0(\xi, t_j) \subset c_0((t_j \xi, t_j), c_2\varepsilon), \quad (3.25)
$$

and $c'_d$ and $c_d$ such that

$$
c'_d \varepsilon^{-d} \leq \# \Xi \chi_0 \leq c_d \varepsilon^{-d}. \quad (3.26)
$$

**Proof** Let $(t_j \xi, t_j)$ and $(t_k \eta, t_k)$ be two distinct points in $\Xi \chi_0$. By its definition, $d_{[-1,1]}(t_j, t_k) = |t_j - t_k| \geq \frac{\pi}{N} \geq \varepsilon$ if $j \neq k$. Hence,

$$
d_{\mathbb{X}_0}(t_j \xi, t_j, t_k) \geq d_{[0,1]}(t_j, t_k) \geq \varepsilon, \quad j \neq k.
$$

If $j = k$, then $\xi$ and $\eta$ are both elements of $\mathbb{S}(\varepsilon_j)$, so that $d_{\mathbb{S}}(\xi, \eta) \geq \varepsilon_j$. Hence, using $\frac{2}{\pi} \phi \leq \sin \phi \leq \phi$, we deduce from (3.3) that

$$
d_{\mathbb{X}_0}((t_j \xi, t_j), (t_j \eta, t_j)) \geq \frac{2}{\pi} t_j d_{\mathbb{S}}(\xi, \eta) \geq \frac{2}{\pi} t_j \varepsilon_j = \varepsilon.
$$

Hence, $\Xi \chi_0$ is $\varepsilon$-separated. Moreover, since $\# \chi_0(\varepsilon_j) \sim \varepsilon_j^{-d+1}$,

$$
\# \Xi \chi_0 = \sum_{j=1}^{N} \# \chi_0(\varepsilon_j) \sim \sum_{j=1}^{N} \varepsilon_j^{-d+1} \sim \varepsilon^{-d+1} \sum_{j=1}^{N} \varepsilon_j^{-d} \sim \varepsilon^{-d}.
$$

For the proof of (3.25), we only need to consider $\mathbb{X}_0(\xi, t_j)$ in the upper part of the double cone, which means $1 \leq j \leq N/2$. If $d_{[-1,1]}(s, t_j) \leq \delta/N$ with $\delta \leq 1/2$, then

$$
|t_j - s| \leq d_{[-1,1]}(t_j, s) \leq \frac{\delta}{N} \leq \delta \sin \frac{\pi}{2N} = \delta t_j \frac{\pi}{2} \leq \delta t_j.
$$

Hence, by $\delta \leq \frac{1}{2}$, we obtain $s \geq t_j/2$. Similarly, we see that if $s \in c_{[-1,1]}(t_j, \pi/N)$, then $s \leq c_{+}t_j$. By (3.23), there are constants $b_1 > 0$ and $b_2 > 0$ such that $c_{\mathbb{S}}(\xi, b_1 \varepsilon_j) \subset \mathbb{S}(\varepsilon_j) \subset c_{\mathbb{S}}(\xi, b_2 \varepsilon_j)$. We claim that (3.25) holds for some $c_1 < \delta$ and some $c_2 > b_2$. Indeed, if $(y, \eta) \subset c_0((t_j \xi, t_j), c_1 \varepsilon)$, then $d_{[-1,1]}(s, t_j) \leq c_1 \varepsilon \leq \delta/N$; moreover, by $s \geq t_j/2$ and $(st)^{1/2} d_{\mathbb{S}}(\xi, \eta) \leq c c_1 \varepsilon$ by (3.4), we see that $d_{\mathbb{S}}(\xi, \eta) \leq \frac{1}{2} c c_1 \varepsilon / \sqrt{t_j} \leq b_1 \varepsilon_j$ by choosing $c_1$ small. This establishes the left-hand side inclusion of (3.25). The right-hand side inclusion can be similarly established. The proof is completed. \(\square\)
For \( \varrho > 0 \), the point set on the hyperbolic surface \( e^{x_d+1}_0 \) can be deduced easily from that on the double conic surface.

**Proposition 3.16** For \( \varepsilon > 0 \), let \( \Xi_{X_0^\varepsilon} \) be a maximal \( \varepsilon \)-separated set in \( X_0^{d+1} \). Define

\[
\Xi_{X_0^\varepsilon}^e := \left\{ (x, \sqrt{t^2 - \varrho^2}) : (x, t) \in \Xi_{X_0^\varepsilon} \right\}.
\]

Then \( \Xi_{X_0^\varepsilon}^e \) is a maximal \( \varepsilon \)-separated set in \( X_0^{d+1} \).

This is an immediate consequence of (3.5). In particular, if \( \Xi_{X_0^0} \) is the set given in Proposition 3.15, then we can define \( \Xi_{X_0^\varepsilon}^e(\xi, t_j) \) accordingly so that both (3.25) and (3.26) hold.

**Definition 3.17** Let \( \Xi_{X_0^\varepsilon}^e \) be a set on \( X_0^{d+1} \) that does not contain \((0, 0)\). Define

\[
\Xi_{X_0^\varepsilon}^e_{X_0^0, +} = \{(x, t) \in \Xi_{X_0^\varepsilon}^e : t > 0\} \quad \text{and} \quad \Xi_{X_0^\varepsilon}^e_{X_0^0, -} = \{(x, t) \in \Xi_{X_0^\varepsilon}^e : t < 0\}.
\]

We call the set \( \Xi_{X_0^\varepsilon}^e \) evenly symmetric on \( X_0^{d+1} \) if

\[
\Xi_{X_0^\varepsilon}^e_{X_0^0, -} = \left\{ (-x, -t) : (x, t) \in \Xi_{X_0^\varepsilon}^e_{X_0^0, +} \right\}.
\]

By definition, \( \Xi_{X_0^0} = \Xi_{X_0^0, +} \cup \Xi_{X_0^0, -} \) and the two subsets are disjoint. For the set \( \Xi_{X_0^\varepsilon} \) in Proposition 3.15, the points \( t_j, 1 \leq j \leq N \), are zeros of the Chebyshev polynomial of the first kind defined by \( T_N(t) = \cos N \arccos(x) \) and we have

\[
\Xi_{X_0^\varepsilon, +} = \{(t_j, t_j) \in \Xi_{X_0^\varepsilon} : 1 \leq j \leq N \}, \quad \Xi_{X_0^\varepsilon, -} = \{(t_j, t_j) \in \Xi_{X_0^\varepsilon} : \frac{N}{2} + 1 \leq j \leq N \}.
\]

Since \( t_j = -t_{N-j} \) for \( 0 \leq j \leq \frac{N}{2} \), it follows that the set in Proposition 3.15 is evenly symmetric on \( X_0^{d+1} \), so is its analogue on \( \rho X_0^{d+1} \).

We further notice that, for the set \( \Xi_{X_0^\varepsilon} \) in Proposition 3.15, with either + or −, \( \Xi_{X_0^\varepsilon, \pm} \) is a maximal \( \varepsilon \)-separated sets of \( X_0^{d+1} \) and the set \( \{\Xi_{X_0^\varepsilon}(\xi, t_j) : (t_j \xi, t_j) \in \Xi_{X_0^\varepsilon, \pm} \} \) is a partition of \( X_0^{d+1} \). Comparing with the maximal \( \varepsilon \)-separated set \( \Xi_{V_0} \) constructed on \( \chi^{d+1} \) in [29], we see that the points in \( \Xi_{V_0} \) congest towards the apex, with a rate \( t_j \sim N^{-2} \), whereas the points in \( \Xi_{X_0^\varepsilon} \) do not.

We have shown that \( (\chi^{d+1}_0, w_{\varepsilon, y}, d_{\varepsilon}^{\varrho}) \) is a localizable homogeneous space with the highly localized kernels \( L_n^E(w_{\varepsilon, y}, \cdot, \cdot) \). As part of the framework in [29], we can then state the Marcinkiewicz–Zygmund inequality for a doubling weight on \( X_0^{d+1} \), which holds under the following constraints: the weight function \( w \) needs to be even in the \( t \) variable so that the integral of polynomials in \( \Pi_n^E(X_0^{d+1}) \) can be written as over \( X_0^{d+1} \), the maximal \( \frac{\delta}{n} \)-separated set \( \Xi_{X_0^\varepsilon} \) need to be symmetric, and it works for polynomials even in the \( t \) variable. Let us define, for \( n = 0, 1, 2, \ldots \),

\[
\Pi_n^E(X_0^{d+1}) = \left\{ p \in \Pi_n(X_0^{d+1}) : p(x, t) = p(x, -t), \forall (x, t) \in X_0^{d+1} \right\}.
\]
Theorem 3.18 Let \( w \) be an doubling weight on \( \mathcal{X}^{d+1}_0 \) such that \( w(x, t) = w(x, -t) \). Let \( \Sigma_\mathcal{X}^E \) be a symmetric maximal \( \frac{d}{n} \)-separated subset of \( \mathcal{X}^{d+1}_0 \) and \( 0 < \delta \leq 1 \).

(i) For all \( 0 < p < \infty \) and \( f \in \Pi_m^E(\mathcal{X}^{d+1}_0) \) with \( n \leq m \leq cn \),

\[
\sum_{z \in \Sigma} \left( \max_{(x, t) \in c_\mathcal{X}((z, r), \frac{\delta}{n})} |f(x, t)| \right)^p w(c_\mathcal{X}((z, r), \frac{\delta}{n})) \leq c_{w,p} \|f\|_{p,w}^p
\]

where \( c_{w,p} \) depends on \( p \) when \( p \) is close to 0 and on the doubling constant of \( w \).

(ii) For \( 0 < r < 1 \), there is a \( \delta_r > 0 \) such that for \( \delta \leq \delta_r \), \( r \leq p < \infty \) and \( f \in \Pi_n^E(\mathcal{X}^{d+1}_0) \),

\[
\|f\|_{p,w}^p \leq c_{w,r} \sum_{z \in \Sigma} \left( \min_{(x, t) \in c_\mathcal{X}((z, r), \frac{\delta}{n})} |f(x, t)| \right) w(c_\mathcal{X}((z, r), \frac{\delta}{n}))
\]

where \( c_{w,r} \) depends only on the doubling constant of \( w \) and on \( r \) when \( r \) is close to 0.

This is a consequence of [29, Theorem 2.15]; its proof remains valid for polynomials in \( \Pi_m^E(\mathcal{X}^{d+1}_0) \) under the assumptions on symmetry.

### 3.6 Positive Cubature Rules

The Marcinkiewicz–Zygmund inequality is used to establish the positive cubature rule in the general framework. To quantify the coefficients of the cubature rule, we will need Assertion 4. This is given by the fast decaying polynomials on the hyperbolic surface given below.

Lemma 3.19 Let \( d \geq 2 \) and \( c \geq 0 \). For each \( (x, t) \in \mathcal{X}^{d+1}_0 \) and for every \( \kappa > 0 \), there is a polynomial \( T^E_{(x,t)} \) in \( \Pi_n^E(\mathcal{X}^{d+1}_0) \) that satisfies

(1) \( T^E_{(x,t)}(x, t) = 1 \), \( T^E_{x,t}(y, s) \geq c > 0 \) if \( (y, s) \in c_\mathcal{X}((x, t), \frac{\delta}{n}) \),

\[
0 \leq T^E_{(x,t)}(y, s) \leq c_\kappa \left( 1 + d^E_{\mathcal{X}_0}((x, t), (y, s)) \right)^{-\kappa}, \quad (y, s) \in \mathcal{X}_0^{d+1}.
\]

(2) There is a polynomial \( q(t) \) of degree \( 4n \) such that \( q(t)T^E_{(x,t)} \) is a polynomial of degree \( 5n \) in the \( (x, t) \) variables and \( 1 \leq q_n(t) \leq c \).

**Proof** For positive integer \( n \), let \( m = \lfloor \frac{d}{r} \rfloor + 1 \) and define

\[
S_n(\cos \theta) = \left( \frac{\sin(m + \frac{1}{2}) \theta}{(m + \frac{1}{2}) \sin \frac{\theta}{2}} \right)^2, \quad 0 \leq \theta \leq \pi.
\]

Then \( S_n(z) \) is an even algebraic polynomial of degree at most \( 2n \) and it satisfies

\[
S_n(1) = 1, \quad 0 \leq S_n(\cos \theta) \leq c(1 + n\theta)^{-2r}, \quad 0 \leq \theta \leq \pi. \quad (3.27)
\]

\( \Box \) Springer
To construct polynomials on hyperbolic surface, we define, for \((x, t), (y, s) \in \mathbb{X}_{0}^{d+1}\),

\[
S((x, t), (y, s)) = S_n \left( \langle x, y \rangle + \sqrt{1 + q^2 - t^2} \sqrt{1 + q^2 - s^2} \right)
\]

\[+ S_n \left( \langle x, y \rangle - \sqrt{1 + q^2 - t^2} \sqrt{1 + q^2 - s^2} \right).
\]

Since \(S_n\) is an even polynomial, it follows that \(S((x, t), (y, s))\) is a polynomial of degree \(n\) in either the \((x, t)\) or in the \((y, s)\) variables. Moreover, since \(\|y\|^2 = s^2 - q^2\), it also follows that \(S((x, t), \cdot)\) is even in the \(s\) variable. Define

\[
T^Q_{(x, t)}(y, s) = \frac{S((x, t), (y, s))}{1 + S_n(2t^2 - 2q^2 - 1)}, \quad (y, s) \in \mathbb{X}_{0}^{d+1}.
\]

Then \(T^Q_{(x, t)} \in \Pi_n^E(\mathbb{X}_{0}^{d+1})\) and \(T^Q_{(x, t)}(x, t) = 1\). If \(0 \leq \theta \leq \frac{2\pi}{2m+1}\), then it follows from \(\sin \theta \geq \frac{2}{\pi} \theta\) for \(0 \leq \theta < \pi/2\) that \(S_n(\cos \theta) \geq \left(\frac{2}{\pi}\right)^2\). Hence, since \(0 \leq S_n(2t-1) \leq 1\), it follows that

\[
T^Q_{(x, t)}(y, s) \geq \frac{S_n \left( \langle x, y \rangle + \sqrt{1 + q^2 - t^2} \sqrt{1 + q^2 - s^2} \right)}{1 + S_n(2t^2 - 2q^2 - 1)} \geq \frac{1}{2} \left(\frac{2}{\pi}\right)^2 r
\]

for \((y, s) \in \mathbb{C}((x, t), \frac{2\pi}{2m+1})\). Next, we use the estimate \(S_n(z) \leq c(1 + n\sqrt{1 - t^2})^{-2r}\) for \(|t| \leq 1\) to obtain an upper bound for \(T_{(x, t)}\). Since \(1 - \langle x, y \rangle + \sqrt{1 - t^2} \sqrt{1 - s^2} \geq 1 - \langle x, y \rangle - \sqrt{1 - t^2} \sqrt{1 - s^2} \geq 0\),

\[
0 \leq T^Q_{(x, t)}(y, s) \leq c \left(1 + n\sqrt{1 - \langle x, y \rangle - \sqrt{1 - t^2} \sqrt{1 - s^2}}\right)^{-2r}
\]

\[
= c \left(1 + n\sqrt{1 - \cos d_{\mathbb{X}_0}((x, t), (y, s))}\right)^{-2r} \sim (1 + nd_{\mathbb{X}_0}((x, t), (y, s)))^{-2r}
\]

using \(1 - \cos \theta \sim \theta^2\). This completes the proof of item (1). The item (2) follows from setting \(q(t) = 1 + S_n(2t^2 - 2q^2 - 1)\), which is a polynomial of degree \(4n\) and \(1 \leq q(t) \leq c\). This completes the proof. \(\square\)

The lemma establishes Assertion 4 with a polynomial in \(\Pi_n^E(\mathbb{X}_{0}^{d+1})\). It allows us to follow the general framework to bound the Christoffel function in \(\Pi_n^E(\mathbb{X}_{0}^{d+1})\). Let \(w\) be a doubling weight function, even in the \(t\) variable, on \(\mathbb{X}_{0}^{d+1}\). Let

\[
\lambda_n^E(w; x, t) := \inf_{g \in \Pi_n^E(\mathbb{X}_{0}^{d+1})} \int_{\mathbb{X}_{0}^{d+1}} |g(x, t)|^2 w(x, t) d\sigma(x, t),
\]

(3.28)
which is the Christoffel function for the space $\Pi_n^{E}(\mathcal{X}_0^{d+1})$. Let $K_n^E(w; \cdot, \cdot)$ denote the kernel of the $n$-th partial sum operator of the series (3.13). Then

$$K_n^E(w; (x, t), (y, s)) = \sum_{k=0}^{n} P_k^E(w; (x, t), (y, s)).$$

By the proof of [9, Theorem 4.6.6], it is related to the Christoffel function by

$$\lambda_n^E(w; x, t) = \frac{1}{K_n^E(w; (x, t), (x, t))}, \quad (x, t) \in \mathcal{X}_0^{d+1}.$$

Using Lemma 3.19, we can adopt [29, Propositions 2.17 and 2.18] to bound $\lambda_n^E(w)$.

**Corollary 3.20** Let $w$ be a doubling weight function on $\mathcal{X}_0^{d+1}$ such that $w(x, t) = w(x, -t)$ for all $(x, t) \in \mathcal{X}_0^{d+1}$. Then

$$\lambda_n^E(w; (x, t)) \leq c w(c_\theta ((x, t), \frac{1}{n})).$$

Moreover, for $\gamma \geq 0$,

$$\lambda_n^E(w_0^\rho; (x, t)) \geq c' w_0^\rho(c_\theta ((x, t), \frac{1}{n})) = c' n^{-d} w_0^\rho(n; t).$$

We can now state the positive cubature rule for the hyperbolic surface, which holds for polynomials in $\Pi_n^{E}(\mathcal{X}_0^{d+1})$ under the assumption of symmetry for both the weight $w$ and the set $\Xi_{\mathcal{X}_0}$.

**Theorem 3.21** Let $d \geq 2$ and $\varrho \geq 0$. Let $w$ be a doubling weight on $\mathcal{X}_0^{d+1}$ such that $w(x, t) = w(x, -t)$ for all $(x, t) \in \mathcal{X}_0^{d+1}$. Let $\Xi_{\mathcal{X}_0}$ be a symmetric maximum $\delta n$-separated subset of $\mathcal{X}_0^{d+1}$. There is a $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ there exist positive numbers $\lambda_{z,r}$, $(z, r) \in \Xi_{\mathcal{X}_0}$, so that

$$\int_{\mathcal{X}_0^{d+1}} f(x, t)w(x, t)d\sigma(x, t) = \sum_{(z, r) \in \Xi_{\mathcal{X}_0}} \lambda_{z,r} f(z, r), \quad \forall f \in \Pi_n^{E}(\mathcal{X}_0^{d+1}). \quad (3.29)$$

Moreover, $\lambda_{z,r} \sim w(c_\theta ((z, r), \frac{\delta}{n}))$ for all $(z, r) \in \Xi_{\mathcal{X}_0}$.

This is [29, Theorem 2.20] when the domain becomes $\mathcal{X}_0^{d+1}$ and it remains valid under the assumptions on symmetry.
### 3.7 Localized Polynomial Frame

The localized polynomials are constructed using the highly localized kernel defined with a cut-off function \( \hat{a} \) of type (b) that satisfies

\[
\begin{align*}
\hat{a}(t) \geq \rho > 0, & \quad \text{if } t \in [3/5, 5/3], \\
[\hat{a}(t)]^2 + [\hat{a}(2t)]^2 = 1, & \quad \text{if } t \in [1/2, 1].
\end{align*}
\]

For the hyperbolic or double conic surface, we will also require symmetry for the weight and the \( \varepsilon \)-separated subset. Let \( \varrho \geq 0 \) and let \( w \) be a doubling weight on \( X_{d+1} \) and assume that it is even in the \( t \) variable. Let \( \mathbb{L}_E^n(w) \ast f \) denote the near best approximation operator from \( \Pi^E_{2n}(X_{d+1}) \) defined by

\[
\mathbb{L}_E^n(w) \ast f(x) := \int_{X_{d+1}} f(y, s) \mathbb{L}_E^n(w; (x, t), (y, s)) w(y, s) d\sigma_{\varrho}(y, s). \tag{3.31}
\]

For \( j = 0, 1, \ldots \), let \( \Xi^\varrho_j \) be a symmetric maximal \( \frac{\delta}{2^j} \)-separated subset in \( X_{d+1} \), so that

\[
\int_{X_{d+1}} f(x, t) w(x, t) d\sigma_{\varrho}(x, t) = \sum_{(z, r) \in \Xi^\varrho_j} \lambda_{(x, r), j} f(z, r), \quad f \in \Pi^E_{2j}(X_{d+1}).
\]

For \( j = 1, 2, \ldots \), define the operator \( F^\varrho_j(w) \) by

\[
F^\varrho_j(w) \ast f = \mathbb{L}_{2j-1}^E(w) \ast f
\]

and define the frame elements \( \psi_{(z, r), j} \) for \( (z, r) \in \Xi^\varrho_j \) by

\[
\psi_{(z, r), j}(x, t) := \sqrt{\lambda_{(z, r), j}} F^\varrho_j((x, t), (z, r)), \quad (x, t) \in X_{d+1}.
\]

Then \( \Phi = \{ \psi_{(z, r), j} : (z, r) \in \Xi^\varrho_j, \ j = 1, 2, 3, \ldots \} \) is a tight frame. Following [29, Theorem 2.21] of the general framework, we have the following:

**Theorem 3.22** Let \( w \) be a doubling weight on \( X_{d+1} \) even in its \( t \) variable. If \( f \in L^2(X_{d+1}, w) \) and \( f \) is even in the \( t \) variable, then

\[
f = \sum_{j=0}^{\infty} \sum_{(z, r) \in \Xi^\varrho_j} \langle f, \psi_{(z, r), j} \rangle w \psi_{(z, r), j} \quad \text{in } L^2(X_{d+1}, w)
\]
and
\[ \| f \|_{2,w} = \left( \sum_{j=0}^{\infty} \sum_{(z,r) \in \Xi_j} |\langle f, \psi(z,r), j \rangle_w|^2 \right)^{1/2}. \]

Furthermore, for \( \gamma \geq -\frac{1}{2} \), the frame for \( w_0^\rho,0,\gamma \) is highly localized in the sense that, for every \( \sigma > 0 \), there exists a constant \( c_\sigma > 0 \) such that
\[ |\psi(z,r), j(x,t)| \leq c_\sigma \frac{2^{j/2}}{\sqrt{w_0^\rho,0,\gamma(2j; t)(1 + 2^jd_\sigma^\rho,0,\gamma((x,t),(z,r)))}} \]  
(3.32)

The frame elements involve only orthogonal polynomials even in the \( t \) variable and they are well defined for all doubling weight that is even in its \( t \) variable. Localization (3.32) follows from Theorem 3.10 and \( \lambda(z,r), j \sim 2^{-jd_\rho,0,\gamma(2j; t)} \), which holds for \( w_0^\rho,0,\gamma \) by Corollary 3.20 and (3.8). It is worthwhile to point out that the localized frame is established for the Lebesgue measure on \( \mathbb{X}_0^{d+1} \), which is the case \( w_0, \frac{1}{2} \), and in particular for the Lebesgue measure on the upper conic surface \( \mathbb{W}_0^{d+1} \), in contrast to the localized frame established in [29, Section 2.7].

### 3.8 Characterization of Best Approximation

For \( f \in L^p(\mathbb{X}_0^{d+1}, w_0^\rho,0,\gamma) \), we denote by \( E_n(f)_{p, w_0^\rho,0,\gamma} \) the best approximation to \( f \) from \( \Pi_n(\mathbb{X}_0^{d+1}) \), the space of polynomials of degree at most \( n \) restricted on the surface \( \mathbb{X}_0^{d+1} \), in the norm \( \| \cdot \|_{p, w_0^\rho,0,\gamma} \); that is,
\[ E_n(f)_{p, w_0^\rho,0,\gamma} := \inf_{g \in \Pi_n(\mathbb{X}_0^{d+1})} \| f - g \|_{p, w_0^\rho,0,\gamma}, \quad 1 \leq p \leq \infty. \]

If \( f \) is even in the \( t \) variable, then the triangle inequality and changing variable \( t \mapsto -t \) show, by the symmetry of the integrals,
\[ \| f(x,t) - \frac{1}{2} (g(x,t) + g(x,-t)) \|_{p, w_0^\rho,0,\gamma} \leq \| f - g \|_{p, w_0^\rho,0,\gamma}. \]

Hence, we can choose the polynomial of best approximation from \( \Pi_n(\mathbb{X}_0^{d+1}) \) when \( f \) is symmetric in the \( t \) variable. Following the general framework in [29], we can give a characterization of the best approximation by polynomials for functions even in the \( t \) variable.

We define a \( K \)-functional and a modulus of smoothness. In terms of the fractional differential operator \( (-\Delta_0^\rho,0,\gamma)^z \) and a doubling weight \( w \), even in the \( t \) variable on
\( K_r(f; \rho; \omega_{0,0}^\rho) := \inf_{g \in W^r_{p}(\mathbb{R}^{d+1}, w)} \left\{ \|f - g\|_{p,w} + r^{1/p} \left\| (-\Delta_{0,y}^{\rho})^{\frac{\rho}{2}} f \right\|_{p,w} \right\}, \)

where the Sobolev space \( W^r_{p}(\mathbb{R}^{d+1}, w) \) is the space that consists of \( g \in L^p(\mathbb{R}^{d+1}, w) \), even in the \( t \) variable, so that \( \left\| (-\Delta_{0,y}^{\rho})^{\frac{\rho}{2}} g \right\|_{p,w} \) is finite. The \( K \)-functional is well defined, as shown in [29, Section 3.3], where we need to require the functions to be even in the \( t \) variable in the proof. Moreover, the modulus of smoothness is defined by

\[
\omega_r(f; \rho; \omega_{0,0}^\rho) = \sup_{0 \leq \theta \leq \rho} \left\| (I - S_{\theta,\omega_{0,0}^\rho})^{r/2} f \right\|_{p,\omega_{0,0}^\rho}, \quad 1 \leq p \leq \infty,
\]

where the operator \( S_{\theta,\omega_{0,0}^\rho} \) is defined by, for \( n = 0, 1, 2, \ldots \) and \( \lambda = \gamma + \frac{d-1}{2} \),

\[
\text{proj}_n^E(\omega_{0,0}^\rho; S_{\theta,\omega_{0,0}^\rho} f) = \frac{C_n^{\lambda - \frac{1}{2}}(\cos \theta)}{C_n^{\lambda - \frac{1}{2}}(1)} \text{proj}_n^E(\omega_{0,0}^\rho; f).
\]

The operator is well defined since the above relations determine it uniquely among functions even in the \( t \) variable and in \( L^p(\mathbb{R}^{d+1}, \omega_{0,0}^\rho) \) and the modulus of smoothness satisfies the usual properties under its name. Such a modulus of smoothness is in line with those defined on the unit sphere and the unit ball [8, 20, 26] but its structure is more complicated. There are recent results for modulus of smoothness on fairly general domains in \( \mathbb{R}^d \) [5, 6, 22, 23], defined via simple difference operators of functions, but conic domains are not included.

By [29, Theorem 3.1.2], the characterization of the best approximation holds under Assertions 1, 3, and 5. For \( \omega_{0,0}^\rho \) on \( \mathbb{R}^{d+1} \), we have already established Assertions 1 and 3. We now establish Assertion 5, again for kernels even in the \( t \) and \( s \) variables.

By (3.14), the kernel \( L_n^{(r)}(\sigma) \) in Assertion 5 becomes

\[
L_n^{(r)}(\omega_{0,0}^\rho; (x, t), (y, s)) = \sum_{k=0}^{\infty} \tilde{a} \left( \frac{k}{n} \right) (k(k + 2\gamma + d - 1))^\frac{\rho}{2} \text{proj}_k^E(\omega_{0,0}^\rho; (x, t), (y, s)).
\]

**Lemma 3.23** Let \( \gamma \geq -\frac{1}{2} \) and \( \kappa > 0 \). Then, for \( r > 0 \) and \( (x, t), (y, s) \in \mathbb{R}^{d+1} \),

\[
\bigg| L_n^{(r)}(\omega_{0,0}^\rho; (x, t), (y, s)) \bigg| \leq c_\kappa \frac{n^{d+r}}{\sqrt{\omega_{0,0}^\rho(n; t) \sqrt{\omega_{0,0}^\rho(n; s)}}} \left( 1 + n\omega_{0,0}^\rho((x, t), (y, s)) \right)^{-\kappa}.
\]
Proof By (3.15) it suffices to consider the case $\rho = 0$. By (3.16), the kernel can be written as

$$L^{(r)}(\mathcal{W}^0_{0,\gamma}; (x, t), (y, s)) = c_\gamma \int_{-1}^1 L_{n,r} (\zeta (x, t, y, s; v) (1 - v^2)^{\gamma - 1} dv,$$

where $L_{n,r}$ is defined by, with $\lambda = \gamma + d - 1/2$,

$$L_{n,r}(t) = \sum_{k=0}^{\infty} \tilde{a} \left( \frac{k}{n} \right) (k(k + 2\gamma + d - 1))^{r/2} \frac{P_n(\lambda - \frac{1}{2}, \frac{\sqrt{\lambda - 1} - \frac{1}{2}}{\lambda - \frac{1}{2}})}{h(\lambda - \frac{1}{2}, \frac{\sqrt{\lambda - 1} - \frac{1}{2}}{\lambda - \frac{1}{2}})}.$$ 

Applying (2.12) with $\eta(t) = \tilde{a}(t) (t(t + n^{-1}(2\gamma + d - 1)))^{r/2}$ and $m = 0$, we obtain

$$|L_{n,r}(t)| \leq cn^{r} \frac{n^{2\lambda + 1}}{(1 + n\sqrt{1 - t})^\ell}.$$ 

Using this estimate, it is easy to see that the proof follows from the estimate already established in the proof of Theorem 3.10. \hfill \Box

We are now in a position to state the characterization of the best approximation by polynomials for the hyperbolic surface, following [29, Theorem 3.12].

**Theorem 3.24** Let $\rho \geq 0$. For $\gamma \geq 0$, let $f \in L^p(\mathbb{H}^{d+1}_0, \mathcal{W}^0_{0,\gamma})$ if $1 \leq p < \infty$ and $f \in C(\mathbb{H}^{d+1}_0)$ if $p = \infty$. Assume that $f$ satisfies $f(x, t) = f(x, -t)$. Let $r > 0$ and $n = 1, 2, \ldots$. Then

(i) **Direct estimate**

$$E_n(f)_{p, \mathcal{W}^0_{0,\gamma}} \leq c K_r(f; n^{-1})_{p, \mathcal{W}^0_{0,\gamma}}.$$ 

(ii) **Inverse estimate**

$$K_r(f; n^{-1})_{p, \mathcal{W}^0_{0,\gamma}} \leq cn^{-r} \sum_{k=0}^{n} (k + 1)^{-1} E_k(f)_{p, \mathcal{W}^0_{0,\gamma}}.$$ 

For $w = \mathcal{W}_{0,\gamma}$, the $K$-functional is equivalent to the modulus of smoothness.

**Theorem 3.25** Let $\gamma \geq 0$ and $f \in L^p(\mathbb{H}^{d+1}_0, \mathcal{W}^0_{0,\gamma})$, $1 \leq p \leq \infty$. Then for $0 < \theta \leq \pi/2$ and $r > 0$

$$c_1 K_r(f; \theta)_{p, \mathcal{W}^0_{0,\gamma}} \leq \omega_r(f; \theta)_{p, \mathcal{W}^0_{0,\gamma}} \leq c_2 K_r(f; \theta)_{p, \mathcal{W}^0_{0,\gamma}}.$$ 

\copyright Springer
Consequently, the characterization in Theorem 3.24 can be stated in terms of the modulus of smoothness in place of $K$-functional.

Finally, we mention that, if $\tilde{a}$ is an admissible cut-off function of type (a), then the operator $L^E_n (\varpi) * f$ defined in (3.31) is the near best approximation in the sense that

$$\|L^E_n (w^{\varpi}_{0,\gamma}) - f\|_{p,w} \leq c E_n(f)_{p, w^0_{\varpi,\gamma}}, \quad 1 \leq p \leq \infty$$

for all $f \in L^r_p (X^{d+1}_0, w)$, where $w$ is a doubling weight by [29, Theorem 3.15].

4 Homogeneous Space on Double Cone and Hyperboloid

We work in the setting of homogeneous space on the solid domain defined by

$$X^{d+1} = \left\{ (x, t) : \|x\|^2 \leq t^2 - \varrho^2, \ x \in \mathbb{R}^d, \ \varrho \leq |t| \leq \sqrt{\varrho^2 + 1} \right\},$$

which is a double hyperboloid when $\varrho > 0$ and a double cone when $\varrho = 0$, and it is bounded by $X^{d+1}_0$ and the hyperplanes $t = \pm \sqrt{\varrho^2 + 1}$ of $\mathbb{R}^{d+1}$. The analysis on $X^{d+1}$ differs substantially from that on the cone $V^{d+1}$ because the distance function is defined differently. We shall verify that the framework for homogeneous space is applicable in this domain for a family of weight functions related to the Gegenbauer weight and the classical weight on the unit ball, following the study on the hyperbolic surface closely. The structure of this section is parallel to that of the previous one, with contents arranged in the same order and under similar section names. Furthermore, part of the proof and development follows from the counterpart on $X^{d+1}_0$; hence, the proof is often brief or omitted.

4.1 Distance on the Solid Double Cone and Hyperboloid

We write $\varrho X^{d+1}$ whenever it is necessary to emphasize the dependence on $\varrho$, but will use $X^{d+1}$ most of the time. The domain $X^{d+1}_0$ can be decomposed as an upper part and a lower part,

$$X^{d+1} = X^{d+1}_+ \cup X^{d+1}_- = \{(x, t) \in X^{d+1} : t \geq 0\} \cup \{(x, t) \in X^{d+1} : t \leq 0\}.$$ 

For $\varrho = 0$, the upper part is the solid cone $X^{d+1}_+ = V^{d+1}$. The distance function on the hyperboloid needs to take into account the boundary behavior of the domain. Like the case of the hyperbolic surface, the boundary, in this case, is the intersection of the surface with the hyperplanes $t = 1$ and $t = -1$. In particular, the apex point is not considered a boundary point. We first define the distance function on the double cone, that is, when $\varrho = 0$. 

\textcopyright Springer
Definition 4.1 Let $\varrho = 0$. For $(x, t), (y, s) \in \mathbb{X}^{d+1}$, define
\[
d_{\mathbb{X}}((x, t), (y, s)) = \arccos \left( \langle x, y \rangle + \sqrt{t^2 - \|x\|^2} \sqrt{s^2 - \|y\|^2} + \sqrt{1 - t^2} \sqrt{1 - s^2} \right).
\]
Then $d_{\mathbb{X}}(\cdot, \cdot)$ is a distance function on the double cone $\mathbb{X}^{d+1}$.

Let $X = (x, \sqrt{t^2 - \|x\|^2})$ and $Y = (y, \sqrt{s^2 - \|y\|^2})$, then $|t|^{-1}X$ and $|s|^{-1}Y$ belong to $\mathbb{B}^{d+1}$, so that $(X, t), (Y, s) \in \mathbb{X}_0^{d+2}$ and
\[
d_{\mathbb{X}^{d+1}}((x, t), (y, s)) = d_{\mathbb{X}_0^{d+2}}((X, t), (Y, s)). \tag{4.1}
\]

In particular, it follows that $d_{\mathbb{X}}(\cdot, \cdot)$ defines a distance on the solid double cone $\mathbb{X}^{d+1}$.

This distance function, however, is different from the distance $d_{\mathbb{Y}}(\cdot, \cdot)$ defined in for $\mathbb{Y}^{d+1}$ in [29]. It is closely related to the distance functions $d_{[-1, 1]}(\cdot, \cdot)$ of $[-1, 1]$ and the distance function on $\mathbb{B}^d$ defined by
\[
d_{\mathbb{B}}(x', y') := \arccos \left( \langle x', y' \rangle + \sqrt{1 - \|x'\|^2} \sqrt{1 - \|y'\|^2} \right), \quad x', y' \in \mathbb{B}^d. \tag{4.2}
\]

Proposition 4.2 Let $\varrho = 0$ and $d \geq 2$. For $(x, t), (y, s) \in \mathbb{X}^{d+1}$, write $x = tx'$ and $y = sy'$ with $x', y' \in \mathbb{B}^d$. Then
\[
1 - \cos d_{\mathbb{X}}((x, t), (y, s)) = 1 - \cos d_{[-1, 1]}(t, s) + ts \left( 1 - \cos d_{\mathbb{B}}(x', y') \right). \tag{4.3}
\]

In particular, if $t$ and $s$ have the same sign, then
\[
c_1 d_{\mathbb{X}}((x, t), (y, s)) \leq d_{[-1, 1]}(t, s) + \sqrt{ts} d_{\mathbb{B}}(x', t') \leq c_2 d_{\mathbb{X}}((x, t), (y, s)). \tag{4.4}
\]

The proof is similar to that of Proposition 3.2, using (3.2) and (4.2). In particular, it follows that the distance on the line segment $l_x = \{(tx', t) : -1 \leq t \leq 1\}$, where $x' \in \mathbb{B}^d$, of the double cone becomes $d_{[-1, 1]}(t, s)$ as expected. We will also need the following lemma

Lemma 4.3 Let $\varrho = 0$ and $d \geq 2$. For $(x, t), (y, s)$ either both in $\mathbb{X}^{d+1}_+$ or both in $\mathbb{X}^{d+1}_-$,
\[
|t - s| \leq d_{\mathbb{X}}((x, t), (y, s)) \quad \text{and} \quad \sqrt{1 - t^2} - \sqrt{1 - s^2} \leq d_{\mathbb{X}}((x, t), (y, s)).
\]

and
\[
\sqrt{t^2 - \|x\|^2} - \sqrt{s^2 - \|y\|^2} \leq (\sqrt{2} + \pi) d_{\mathbb{X}}((x, t), (y, s)).
\]

Proof We consider only $t, s \geq 0$. Let $x = tx'$ and $y = sy'$ with $x', y' \in \mathbb{B}^d$. Setting $t = \cos \theta$ and $s = \cos \phi$. Using the inequality $|\langle x', y' \rangle + \sqrt{1 - \|x'\|^2} \sqrt{1 - \|y'\|^2}| \leq 1,$
it follows readily that \( \cos d_{X_0}(x, t, y, s) \leq ts + \sqrt{1 - t^2}\sqrt{1 - s^2} = \cos(\theta - \phi) \), which allows us to follow the proof of Lemma 3.4 to establish the first two inequalities. For the third inequality, we assume without loss of generality that \(|t| \geq |s|\). Then,

\[
|\sqrt{t^2 - \|x\|^2} - \sqrt{s^2 - \|y\|^2}| = |t|\sqrt{1 - \|x'\|^2} - |s|\sqrt{1 - \|y'\|^2} \\
\leq |t - s|\sqrt{1 - \|x'\|^2} + |s|\left|\sqrt{1 - \|x'\|^2} - \sqrt{1 - \|y'\|^2}\right| \\
\leq \sqrt{2}d_X((x, t), (y, s)) + \sqrt{ts}d_B(x', y').
\]

where the last step uses the inequality \([8, (A.1.4)]\)

\[
|\sqrt{1 - \|x'\|^2} - \sqrt{1 - \|y'\|^2}| \leq \sqrt{2}d_B(x', y').
\]

Hence, the third inequality follows from (4.4). This completes the proof. \(\Box\)

For \(\varrho > 0\), the two parts of the solid hyperboloid, \(X^{d+1}_+\) and \(X^{d+1}_-\), are disjoint. It is sufficient to consider the distance between points that lie in the same part. We define

\[
d_{\varrho X}(x, t, y, s) = \arccos \left( \langle x, y \rangle + \sqrt{1 - \|x\|^2} \sqrt{1 - \|y\|^2} + \sqrt{1 + \varrho^2 - t^2} \sqrt{1 + \varrho^2 - s^2} \right). \tag{4.5}
\]

If \((x, t)\) and \((y, t)\) are both in \(X^{d+1}_+\) or both in \(X^{d+1}_-\), then

\[
d_{\varrho X}(x, t, y, s) = d_X \left( (x, \sqrt{t^2 - \varrho^2}), (y, \sqrt{s^2 - \varrho^2}) \right). \tag{4.6}
\]

It is easy to see that this is a distance function and, evidently, its properties follow from those of the distance on the double cone.

### 4.2 A Family of Doubling Weights

For \(d \geq 2, \beta > -\frac{1}{2}, \gamma > -\frac{1}{2}\) and \(\mu > -\frac{1}{2}\), let \(W_{\varrho, \beta, \gamma, \mu}^d\) be the weight function defined on the hyperboloid \(X^{d+1}\) by

\[
W_{\varrho, \beta, \gamma, \mu}^d(x, t) = b_{\varrho, \beta, \gamma, \mu}^d |t|^{\beta} (1 + \varrho^2 - t^2)^{\gamma - \frac{1}{2}} (t^2 - \|x\|^2)^{\mu - \frac{1}{2}}. \tag{4.7}
\]

When \(\varrho = 0\) or the double cone, the weight function becomes

\[
W_{\beta, \gamma, \mu}^0(x, t) = b_{\beta, \gamma, \mu}^0 |t|^{2\beta} (1 - t^2)^{\gamma - \frac{1}{2}} (t^2 - \|x\|^2)^{\mu - \frac{1}{2}},
\]

\(\copyright\) Springer
which remains integrable over $\mathbb{X}^{d+1}$ if $\beta > -\frac{d+1}{2}$. Using the identity

$$
\int_{\mathbb{X}^{d+1}} f(x, t)|t| \, dx \, dt = \int_{\mathbb{X}^{d+1}} |t| \, \int_{\|x\| \leq \sqrt{t^2 - \varrho^2}} f(x, t) \, dx \, dt
$$

$$
= \int_{|s| \leq 1} \int_{\|x\| \leq |s|} f(x, \sqrt{s^2 + \varrho^2}) \, dy \, ds
$$

$$
= \int_{0}^{|s| \leq 1} f(x, \sqrt{s^2 + \varrho^2}) |s| \, dy \, ds, \quad (4.8)
$$

it is easy to see that the normalization constant $b_{\varrho, \beta, \gamma, \mu}^0$ of $W_{\varrho, \beta, \gamma, \mu}(x, t)$ satisfies

$$
b_{\varrho, \beta, \gamma, \mu}^0 = b_{\varrho, \beta, \gamma, \mu}^0 = \frac{b_{\varrho} B_{\mu}}{\Gamma(\frac{\beta + \mu + \gamma + d + 1}{2})},
$$

where $b_{\varrho} B_{\mu}$ is the normalization constant for the weight function $\varrho_{\mu}$ in (2.18) on $\mathbb{B}^d$.

For $r > 0$ and $(x, t)$ on the solid hyperboloid $\varrho_{\mathbb{X}^{d+1}}$, we denote the ball centered at $(x, t)$ with radius $r$ by

$$
c_{\varrho}((x, t), r) := \left\{ (y, s) \in \mathbb{X}^{d+1} : d_{\varrho}((x, t), (y, s)) \leq r \right\}.
$$

The following lemma is an analog of Lemma 3.5 and can be proved similarly.

**Lemma 4.4** For $\varrho > 0$ and $(x, t) \in \mathbb{X}^{d+1}$,

$$
W_{\varrho, \beta, \gamma, \mu}(c_{\varrho}((x, t), r)) = W_{\varrho, \beta, \gamma, \mu}^0(c_0((x, \sqrt{t^2 - \varrho^2}), r)).
$$

**Proposition 4.5** Let $r > 0$ and $(x, t) \in \mathbb{X}^{d+1}$. Then for $\beta > -\frac{d+1}{2}$ and $\gamma > -1$ and $\mu \geq 0$,

$$
W_{\varrho, \beta, \gamma, \mu}^0(c_0((x, t), r)) := b_{\varrho, \beta, \gamma, \mu}^0 \int_{c_0((x, t), r)} W_{\varrho, \beta, \gamma, \mu}^0(y, s) \, dy \, ds
$$

$$
\sim r^{d+1} \left(t^2 + r^2\right)^\beta \left(1 - t^2 + r^2\right)^\gamma \left(t^2 - \|x\|^2 + r^2\right)^\mu.
$$

In particular, $W_{\varrho, \beta, \gamma, \mu}^0$ is a doubling weight on the double cone and the doubling index is given by $\alpha(W_{\varrho, \beta, \gamma, \mu}^0) = d + 1 + 2\mu + 2 \max\{0, \beta\} + 2 \max\{0, \gamma\}$. 

\(\square\) Springer
Proof Let $\tau_r(t, s)$ and $\theta_r(t, s) = \arccos \tau_r(t, s)$ be as in the proof of Proposition 3.6. From $d_X((x, t), (y, s)) \leq r$, we obtain $d_{[-1,1]}(t, s) \leq r$ and, by (4.1)

$$d_B(x', y') \leq \arccos \left(2[\tau_r(t, s)]^2 - 1\right) = \frac{1}{2} \arccos \tau_r(t, s),$$

where $d_B(\cdot, \cdot)$ is the distance on the unit ball $B^d$. Hence, it follows that

$$W^0_{\beta, \gamma, \mu}(c_0((x, t), r)) = \int_{d_{[-1,1]}(t, s) \leq r} s^d \int_{d_B(x', y') \leq \frac{1}{2} \theta_r(t, s)} W^0_{\beta, \gamma, \mu}(y, s) dy ds \sim \int_{d_{[-1,1]}(t, s) \leq r} s^{2\beta + 2\mu + d - 4} (1 - s^2)^{\gamma - \frac{1}{2}} ds \int_{d_B(x', y') \leq \frac{1}{2} \theta_r(t, s)} (1 - ||y'||^2)^{\frac{1}{2}} dy.$$

For $\mu \geq 0$ and $0 < \rho < 1$, it is known [19, Lemma 5.3] or [8, p. 107] that

$$\int_{d_B(x', y') \leq \rho} (1 - ||y'||^2)^{\mu - \frac{1}{2}} dy' \sim (1 - ||x'||^2 + \rho^2)^{\mu} \rho^d,$$

which implies, together with $\theta_r(t, s) \sim \sqrt{1 - \tau_r(t, s)}$, that

$$W^0_{\beta, \gamma, \mu}(c_0((x, t), r)) \sim \int_{d_{[-1,1]}(t, s) \leq r} s^{2\beta + 2\mu + d - 1} (1 - s^2)^{\gamma - \frac{1}{2}} \times (1 - ||x'||^2 + 1 - \tau_r(t, s))^{\mu} (1 - \tau_r(t, s))^d ds.$$

If $t \geq 3r$, then $s \sim t + r$ and, by $1 - \tau_r(t, s) = (\cos d_{[-1,1]}(t, s) - \cos r)/(ts)$, it follows that

$$s^{2\mu} (1 - ||x'||^2 + 1 - \tau_r(t, s))^\mu \sim (r^2 - ||x||^2 + r^2)^\mu.$$

With this term removed, the integral of the remaining integrand can be estimated by following the estimates of Case 1 and Case 3 of the proof of Proposition 3.6. If $t \leq 3r$, then $t^2 - ||x||^2 + r^2 \sim r^2$. We use $1 - ||x'||^2 + 1 - \tau_r(t, s) \leq 2$ for the upper bound and $1 - \tau_r(t, s) \geq 1/(8\pi^2)$ on the subset $d_{[-1,1]}(t, s) \leq r/2$, proved in the Case 2 of the proof of Proposition 3.6, to remove the term $(1 - ||x'||^2 + 1 - \tau_r(t, s))^\mu$ from the integral. The rest of the proof then follows from that of Case 2 of the proof of Proposition 3.6. This completes the proof. 

Corollary 4.6 For $d \geq 2$, $\beta > -\frac{d+1}{2}$ and $\gamma > -\frac{1}{2}$, the space $(\mathcal{W}^{d+1}, W^0_{\beta, \gamma, \mu}, d_X)$ is a homogeneous space.
When $\varrho = 0$, $\beta = 0$, $\mu = 0$ and $\gamma = \frac{1}{2}$, relation (4.9) is for the Lebesgue measure $dm$ on the double cone,

$$m(c_0((x, t), r)) \sim r^{d+1}(1 - t^2 + r^2)^{\frac{1}{2}}(t^2 + \|x\|^2 + n^{-2})^{\frac{1}{2}}.$$ 

Furthermore, $W_{0,0,0}^0(c_0((x, t), r)) \sim r^{d+1}$ and $W_{0,0,0}^0(x, y) = (1 - t^2)^{-\frac{1}{2}}(t^2 + \|x\|^2)^{-\frac{1}{2}}$.

### 4.3 Orthogonal Polynomials on the Hyperboloid

With respect to the weight function $W^\varrho_{\beta, \gamma, \mu}$, we defined the inner product

$$\langle f, g \rangle_W = \int_{\mathbb{H}^{d+1}} f(x, t)g(x, t)W^\varrho_{\beta, \gamma, \mu}(x, t)dxdt.$$ 

Let $V_n(\mathbb{H}^{d+1}, W^\varrho_{\beta, \gamma, \mu})$ be the space of these orthogonal polynomials of degree $n$, which has the dimension $\dim V_n(\mathbb{H}^{d+1}, W) = \binom{n+d}{n}$. Like the decomposition on the surface $\mathbb{H}^d_0$, this space satisfies

$$V_n(\mathbb{H}^{d+1}, W^\varrho_{\beta, \gamma, \mu}) = V_n^E(\mathbb{H}^{d+1}, W^\varrho_{\beta, \gamma, \mu}) \bigoplus V_n^O(\mathbb{H}^{d+1}, W^\varrho_{\beta, \gamma, \mu}),$$ 

where the subspace $V_n^E(\mathbb{H}^{d+1}, W^\varrho_{\beta, \gamma, \mu})$ consists of orthogonal polynomials that are even in the $t$ variable, whereas the subspace $V_n^O(\mathbb{H}^{d+1}, W^\varrho_{\beta, \gamma, \mu})$ consists of orthogonal polynomials that are odd in the $t$ variable.

An orthogonal basis can be given explicitly in terms of the Jacobi polynomials and classical orthogonal polynomials on the unit ball for the subspace $V_n^E(\mathbb{H}^{d+1}, W^\varrho_{\beta, \gamma, \mu})$ for all $\varrho \geq 0$, but for the subspace $V_n^O(\mathbb{H}^{d+1}, W^\varrho_{\beta, \gamma, \mu})$ only when $\varrho = 0$. For example, let $\{P_{n-2k}^k : |k| = n - 2k, k \in \mathbb{N}^d\}$ denote an orthonormal basis of $V_{n-2k}(\mathbb{B}^d, W_\mu)$. Then the polynomials

$$C_{n-2k, k}^n(x, t) = P_k^{(\frac{1}{2}, n-2k+\beta+\mu+d-2)}(2t^2 - 2q^2 - 1)$$

$$\times (t^2 - q^2)^{\frac{n-2k}{2}}P_k^{n-2k}\left(\frac{x}{\sqrt{t^2 - q^2}}\right)$$

(4.10)

with $|k| = n - 2k$ and $0 \leq k \leq n/2$ form an orthogonal basis of $V_n^E(\mathbb{H}^{d+1}, W^\varrho_{\beta, \gamma, \mu})$. We call these polynomials generalized Gegenbauer polynomials on the solid hyperboloid.

We will not work with the basis directly; see [28] for the basis in other cases.
Let $P_n^E(W^\varrho_\beta,\gamma,\mu;\cdot,\cdot)$ denote the reproducing kernel of $V_n^E(\mathbb{K}^{d+1}, W^\varrho_\beta,\gamma,\mu)$. In terms of the basis of (4.10), we can write

$$
P_n^E(W^\varrho_\beta,\gamma,\mu; (x, t), (y, s)) = \sum_{m=0}^{n} \sum_{|k|=m} \mathcal{C}_{n-2k,k}^n(x, t) \mathcal{C}_{n-2k,k}^n(y, s) \mathcal{C}_{n-2k,k}^n W^\varrho_\beta,\gamma,\mu.
$$

Just like on the surface $\mathbb{K}^{d+1}_0$, the kernel $P_n^E(W^\varrho_\beta,\gamma,\mu; \cdot, \cdot)$ can be used to study the Fourier orthogonal series of any function $f$ that is even in the variable $t$ on $\mathbb{K}^{d+1}$. For such a function, $f(x, t) = f(x, -t)$, its projection on $V_n^E(\mathbb{K}^{d+1}, W^\varrho_\beta,\gamma,\mu)$ becomes zero, so that its Fourier orthogonal expansion is given by

$$f = \sum_{n=0}^{\infty} \text{proj}_n^E(W^\varrho_\beta,\gamma,\mu; f),$$

where the projection $\text{proj}_n^E(W^\varrho_\beta,\gamma,\mu; \cdot, \cdot) : L^2(\mathbb{K}^{d+1}, W^\varrho_\beta,\gamma,\mu) \rightarrow V_n^E(\mathbb{K}^{d+1}, W^\varrho_\beta,\gamma,\mu)$ can be written in terms of the kernel $P_n^E(W^\varrho_\beta,\gamma,\mu; \cdot, \cdot)$ as

$$\text{proj}_n(W^\varrho_\beta,\gamma,\mu; f) = c_w \int_{\mathbb{K}^{d+1}} f(y) P_n^E(W^\varrho_\beta,\gamma,\mu; \cdot, (y, s)) W^\varrho_\beta,\gamma,\mu(s) dyds.$$

Moreover, since $f$ is even in the $t$ variable, we can regard it as the even extension of a function $f$ defined on the upper hyperboloid $\mathbb{K}^{d+1}_+, which is the cone $\mathbb{K}^{d+1}$ when $\varrho = 0$. In particular, this provides a Fourier orthogonal series for functions on the cone $\mathbb{K}^{d+1}_+$, which is, however, different from the Fourier orthogonal series in the Jacobi polynomials on the cone discussed in [29].

The most interesting case on $\mathbb{K}^{d+1}_+$ is $\beta = \frac{1}{2}$. To simplify the notation, we shall denote $W^\varrho_{\frac{1}{2},\gamma,\mu}$ by $W^\varrho_{\gamma,\mu}$ throughout the rest of the section; that is,

$$W^\varrho_{\gamma,\mu}(x, t) := |t|(1 + \varrho^2 - t^2)^{\gamma - \frac{1}{2}} (t^2 - \varrho^2 - ||x||^2)^{\mu - \frac{1}{2}}.$$

The orthogonal polynomials with respect to $W_{\gamma,\mu}$ also possess two characteristic properties: the first one is the spectral operator that has orthogonal polynomials as eigenfunctions [28, Theorem 4.8].

**Theorem 4.7** Let $\varrho \geq 0$, $\gamma$, $\mu > -\frac{1}{2}$. Define the differential operator

$$\mathfrak{D}^\varrho_{\gamma,\mu} := (1 + \varrho^2 - t^2) \left( 1 - \frac{\varrho^2}{t^2} \right) \partial_t^2 + \Delta_x - \langle x, \nabla_x \rangle^2 + \langle x, \nabla_x \rangle$$

$$+ \frac{2}{t} (1 + \varrho^2 - t^2) \langle x, \nabla_x \rangle \partial_t + \left( (1 + \varrho^2 - t^2) \frac{\varrho^2}{t^2} + 2\mu + d \right) \frac{1}{t} \partial_t$$

$$- (2\gamma + 2\mu + d + 1) \left( \left( 1 - \frac{\varrho^2}{t^2} \right) t \partial_t + \langle x, \nabla_x \rangle \right)$$

where $\Delta_x$ denotes the Laplacian on the upper hyperboloid $\mathbb{K}^{d+1}_+$.
Then the polynomials in $\mathcal{V}^E_n(\mathbb{X}^{d+1}, W^e_{\gamma,\mu})$ are eigenfunctions of $\mathcal{D}^E_{\gamma,\mu}$,

$$
\mathcal{D}^E_{\gamma,\mu}u = -n(n + 2\gamma + 2\mu + d)u, \quad \forall u \in \mathcal{V}^E_n(\mathbb{X}^{d+1}, W^e_{\gamma,\mu})
$$

(4.12)

The second one is the addition formula for the reproducing kernel $\mathbf{P}^E_n(W^0_{\beta,\gamma,\mu}; \cdot, \cdot)$, which is of the simplest form when $\beta = \frac{1}{2}$ [28, Prop. 5.7 and Cor. 5.6].

**Theorem 4.8** Let $d \geq 2$ and $\varrho \geq 0$. Then

(a) For $\beta, \gamma, \mu > -\frac{1}{2}$,

$$
\mathbf{P}^E_n(W^0_{\beta,\gamma,\mu}; (x, t), (y, s)) = \mathbf{P}^E_n(W^0_{\beta,\gamma,\mu}; (x, \sqrt{t^2 - \varrho^2}), (y, \sqrt{s^2 - \varrho^2})).
$$

(4.13)

(b) For $\varrho = 0$, and $\gamma, \mu \geq 0$,

$$
\mathbf{P}^E_n(W^0_{\gamma,\mu}; (x, t), (y, s)) = b_{\gamma,\mu} \int_{-1}^{1} \int_{-1}^{1} Z^\gamma_{n+\mu+\frac{d}{2}}(\xi(x, t, y, s; u, v))
\times (1 - v^2)^{\gamma-1}(1 - u^2)^{\mu-1} du dv,
$$

(4.14)

where $Z^\gamma_n$ is defined in (2.16), $b_{\gamma,\mu} = c_{\gamma-1,\gamma-1}c_{\mu-1,\mu-1}$ with $c_{a,b}$ defined as in (2.10) and

$$
\xi(x, t, y, s; u, v) = \left(\langle x, y \rangle + u\sqrt{t^2 - \|x\|^2}\sqrt{s^2 - \|y\|^2}\right)\text{sign}(st) + v\sqrt{1 - s^2}\sqrt{1 - t^2},
$$

and identity (4.14) holds under the limit when $\mu = 0$ or $\gamma = 0$.

### 4.4 Highly Localized Kernels

Let $\hat{a}$ be a cut-off function. For $(x, t), (y, s) \in \mathbb{X}^{d+1}$, the localized kernel $\mathbf{L}^E_n(W^e_{\gamma,\mu}; \cdot, \cdot)$ is defined by

$$
\mathbf{L}^E_n(W^e_{\gamma,\mu}; (x, t), (y, s)) = \sum_{j=0}^{\infty} \hat{a}\left(\frac{j}{n}\right) \mathbf{P}^E_j(W^e_{\gamma,\mu}; (x, t), (y, s)).
$$

We show that this kernel is highly localized when $(x, t)$ and $(y, s)$ are either both in $\mathbb{X}_+^{d+1}$ or both in $\mathbb{X}_-^{d+1}$. For $\mu, \gamma \geq 0$, define

$$
W^e_{\gamma,\mu}(n; x, t) := (1 + \varrho^2 - t^2 + n^{-2})^\gamma (t^2 - \varrho^2 - \|x\|^2 + n^{-2})^\mu.
$$
Theorem 4.9 Let $d \geq 2$, $\mu, \gamma \geq 0$. Let $\widehat{a}$ be an admissible cutoff function. Then for any $\kappa > 0$, and $(x, t), (y, s)$ either both in $\mathbb{R}^d_+$ or both in $\mathbb{R}^d_-$,

$$\left| L_n^E (W^0_{\gamma, \mu}; (x, t), (y, s)) \right| \leq \frac{c_n d^{d+1}}{\sqrt{W^0_{\gamma, \mu}(n; x, t)} \sqrt{W^0_{\gamma, \mu}(n; y, s)}} (1 + nd_{\mathbb{R}^d}((x, t), (y, s)))^{-\kappa}.$$ 

Proof Again, it is sufficient to consider the case $\varrho = 0$ and we shall be brief. By (4.14) we can write $L_n^E (W^0_{\gamma, \mu})$ in terms of the kernel for the Jacobi polynomials. Let $\lambda = \gamma + \mu + \frac{d}{2}$. Then

$$L_n^E (W^0_{\gamma, \mu}; (x, t), (y, s)) = b_{\gamma, \mu} \int_{-1}^{1} \int_{-1}^{1} L_n^{(\lambda-\frac{1}{2}, \lambda-\frac{1}{2})} (\xi(x, t, y, s; u, v)) \times (1 - v^2)^{\gamma-1}(1 - u^2)^{\mu-1} du dv.$$ 

Hence, applying (2.12) with $m = 1$ and $\alpha = \beta = \lambda - 1/2$, we obtain

$$\left| L_n^E (W^0_{\gamma, \mu}; (x, t), (y, s)) \right| \leq c n^{2d+1} \int_{-1}^{1} \int_{-1}^{1} \frac{1}{(1 + n \sqrt{1 - \xi(x, t, y, s; u, v)})^{\kappa + 3\gamma + 3\mu + 2}} \times (1 - v^2)^{\gamma-1}(1 - u^2)^{\mu-1} du dv.$$ 

Since $t$ and $s$ have the same sign, it is easy to verify that

$$1 - \xi(x, t, y, s; u, v) = 1 - \cos d_{\mathbb{R}^d}((x, t), (y, s)) + (1 - u)\sqrt{t^2 - \|x\|^2} \sqrt{s^2 - \|y\|^2} + (1 - v)\sqrt{1 - t^2} \sqrt{1 - s^2}.$$ 

The entries in both two lines on the right-hand side of the above identity are lower bounds of $1 - \xi(x, t, u, s; u, v)$. Using the first one, we obtain the estimate

$$\left| L_n^E (W^0_{\gamma, \mu}; (x, t), (y, s)) \right| \leq c n^{2d+1} \frac{1}{(1 + n d_{\mathbb{R}^d}((x, t), (y, s)))^{\kappa + \gamma + \mu}} I(x, t, y, s),$$ 

where, using the second one and the symmetry of the integral, the integral $I(x, t, y, s)$ is given by

$$I(x, t, y, s) = c_{\gamma-\frac{1}{2}, \mu-\frac{1}{2}} \int_{0}^{1} \int_{0}^{1} \frac{(1 - u^2)^{\mu-1}(1 - v^2)^{\gamma-1}}{(1 + n \sqrt{A(1 - u) + B(1 - v)})^{2\gamma + 2\mu + 2}} du dv$$ 

with $A = \sqrt{t^2 - \|x\|^2} \sqrt{s^2 - \|y\|^2}$ and $B = \sqrt{1 - t^2} \sqrt{1 - s^2}$. The integral $I(x, t, y, s)$ is bounded by 1 and it can also be bounded by applying (3.18) twice.
Carrying out the estimates, we conclude that

\[
I(x, t, y, s) \leq c \frac{n^{-2\gamma-2\mu}}{(A + n^{-1})^\mu (B + n^{-1})^\gamma} \leq c \frac{n^{-2\gamma-2\mu}}{\sqrt{W_\gamma^0(n; t)} \sqrt{W_\gamma^0(n; t)}} (1 + n\mathbf{d}_\mathcal{X}((x, t), (y, s)))^{\gamma+\mu},
\]

where the second inequality follows from (3.19) and Lemma 4.3. Putting the last two displayed inequalities together, we have established (ii). \(\square\)

This establishes Assertion 1. The next theorem establishes Assertion 2.

**Theorem 4.10** Let \(d \geq 2\), \(\mu, \gamma \geq -\frac{1}{2}\). Then for \((x_1, t_1)\) and \((y, s)\) that are either all in \(X_{\gamma}^{d+1}\) or all in \(X_{-\gamma}^{d+1}\), \((x_1, t_1) \in \mathcal{C}_d(x_2, t_2), c_\gamma n^{-1}\) with \(c_\gamma\) small and any \(\kappa > 0\),

\[
\left| \mathbf{L}_n^E (W_\gamma^0; (x_1, t_1), (y, s)) - \mathbf{L}_n^E (W_\gamma^0; (x_2, t_2), (y, s)) \right| \leq \frac{c_\gamma n^{d+1} \mathbf{d}_\mathcal{X}^\gamma((x_1, t_1), (x_2, t_2))}{\sqrt{W_\gamma^0(n, t_2)} \sqrt{W_\gamma^0(n; s)} (1 + n\mathbf{d}_\mathcal{X}^\gamma((x_2, t_2), (y, s)))^\kappa}.
\]

**Proof** Again it suffices to prove the case \(\gamma = 0\). Let \(\xi_i(u, v) = \xi(x_i, t_i, y, s; u, v)\). From (4.3), it is easy to see that

\[
\xi_1(u, v) - \xi_2(u, v) = \cos \mathbf{d}_\mathcal{X}((x_1, t_1), (y, s)) - \cos \mathbf{d}_\mathcal{X}((x_2, t_2), (y, s)) + (1 - u) \left( \sqrt{t_2^2 - \|x_2\|^2} - \sqrt{t_1^2 - \|x_1\|^2} \right) \sqrt{s^2 - \|y\|^2} + (1 - v) \left( \sqrt{1 - t_2^2} - \sqrt{1 - t_1^2} \right) \sqrt{1 - s^2}.
\]

By Lemma 4.3 and the proof of Theorem 3.11, this leads to

\[
|\xi_1(u, v) - \xi_2(u, v)| \leq \mathbf{d}_\mathcal{X}((x_1, t_1), (x_2, t_2)) \left[ \Sigma_1 + \Sigma_2(u) + \Sigma_3(v) \right],
\]

where

\[
\Sigma_1 = \mathbf{d}_\mathcal{X}((x_2, t_2), (y, s)) + \mathbf{d}_\mathcal{X}((x_1, t_1), (x_2, t_2)), \\
\Sigma_2(u) = (1 - u) \sqrt{s^2 - \|y\|^2}, \\
\Sigma_3(v) = (1 - v) \sqrt{1 - s^2}.
\]

Hence, following the proof of Theorem 3.11, we see that, with \(\lambda = \gamma + \mu + \frac{d}{2}\),

\[
\left| \mathbf{L}_n (W_\gamma^0; (x_1, t_1), (y, s)) - \mathbf{L}_n (W_\gamma^0; (x_2, t_2), (y, s)) \right|
\]

Springer
The integrals that contain $\Sigma_1$ and $\Sigma_3(v)$ can be estimated exactly as in the proof of Theorem 3.11. The integral that contains $\Sigma_2(u)$ does not cause an additional problem and can be handled just as the integral containing $\Sigma_3(v)$. We omit the details.  

The case $p = 1$ of the following lemma establishes Assertion 3 for $W_{\gamma, \mu}^0$.

**Lemma 4.11** Let $d \geq 2$, $\gamma > -\frac{1}{2}$ and $\mu > -\frac{1}{2}$. For $0 < p < \infty$, assume $\kappa > \frac{2d+2}{p} + 2(\beta + \gamma)\frac{1}{p} - \frac{1}{2}$. Then for $(x, t) \in \mathbb{R}^{d+1}$,

$$\int_{\mathbb{R}^{d+1}} \frac{W_{\gamma, \mu}^0(y, s)}{W_{\gamma, \mu}^0(n; y, s)} \left(1 + nd_X((x, t), (y, s))\right)^{2k} \frac{dy ds}{\kappa^p} \leq c n^{d-1} W_{\gamma, \mu}^0(n; x, t)^{1 - \frac{2}{p}}.$$

**Proof** Again, it suffices to consider $\varrho = 0$. Let $J_p$ denote the left-hand side of the inequality to be proved. As in the proof of Lemma 3.12, we only need to estimate $J_2$. Furthermore, following the proof of Lemma 3.12, we only need to estimate the integral in $J_2$ over either $\mathbb{R}_+^{d+1}$ or $\mathbb{R}_-^{d+1}$, which we choose as $\mathbb{R}_+^{d+1} = \mathbb{V}^{d+1}$ and denote it by $J_{2,+}$. Then

$$J_{2,+} = \int_0^1 s^d \int_{\mathbb{R}^d} \frac{W_{\gamma, \mu}^0(sy', s)}{W_{\gamma, \mu}^0(n; sy', s)(1 + nd_X((x, t), (sy', s)))^{2k}} dy' ds.$$

Let $x = tx'$ and $y = sy'$ with $x', y' \in \mathbb{R}^d$. Using $w_{0,\gamma}^0(t) = (1 - t^2)^{\gamma - \frac{1}{2}}$ and $w_{\beta,\gamma}^0(n; s) = (1 - t^2 + n^{-2})^{\gamma'}$, we can easily verify that

$$\frac{W_{\gamma, \mu}^0(y, s)}{W_{\gamma, \mu}^0(n; y, s)} \leq c \frac{w_{0,\gamma}^0(s)}{\sqrt{1 - ||y'||^2} w_{0,\gamma}^0(n; s)^2},$$

which leads to

$$J_{2,+} \leq \int_0^1 s^d \int_{\mathbb{R}^d} \frac{w_{0,\gamma}^0(s)}{w_{0,\gamma}^0(n; s)(1 + nd_X((x, t), (sy', s)))^{2k}} \frac{dy'}{\sqrt{1 - ||y'||^2}} ds.$$

Setting $x = tx'$, $X = (x', \sqrt{1 - ||x'||^2})$ and $Y = (y', \sqrt{1 - ||y'||^2})$, so that $d_{\mathbb{R}^d}(x', y') = d_{\mathbb{R}_+^d}(X, Y)$, we use the identity

$$\int_{\mathbb{R}^d} g \left(y', \sqrt{1 - ||y'||^2}\right) \frac{dy'}{\sqrt{1 - ||y'||^2}} = \int_{\mathbb{R}_+^d} g(y) d\sigma(y), \quad (4.15)$$

\phantomsection
where $S^d_+$ denotes the upper hemisphere of $S^d$, which allows us to follow the proof of Lemma 3.12 to obtain

$$J_{2,+} \leq c \int_0^1 \int_{-1}^1 \frac{s^d W_0^0(s)(1-u^2)^{d-2}}{W_0^0(n; s) \left( 1 + n \sqrt{1 - t s u} - \sqrt{1 - t^2} \sqrt{1 - s^2} \right)} \, dw \, ds.$$

The integral in the right-hand side with $d$ replaced by $d + 1$ appeared in the proof of Lemma 3.12, and it is bounded by $c n^{-d-1}$ accordingly.

**Proposition 4.12** For $\gamma \geq 0$, $\mu \geq 0$ and $(x, t) \in X_d^{+1}$,

$$\int_{X_d^{+1}} \left| L_E^n (W_{\gamma, \mu} (x, t), (y, s)) \right|^p W_{\gamma, \mu}^0 (y, s) \, dy \, ds \leq \left( \frac{n^d}{W_{\gamma, \mu}^0 (n; t)} \right)^{p-1}.$$

This follows by applying Lemma 4.11 on the estimate in Theorem 4.9.

We have established Assertions 1 – 3 for $L_E^n (W_{\gamma, \mu} ; \cdot, \cdot)$. The kernel uses, however, only polynomials that are even in the $t$ and in the $s$ variable.

**Corollary 4.13** For $d \geq 2$, $\varrho \geq 0$, $\gamma \geq 0$ and $\mu \geq 0$, the space $(X_d^{+1}, W_{\gamma, \mu}^0, d^\varrho)$ is a localizable homogeneous space, where its localized kernels are defined for polynomials even in the $t$ and in the $s$ variables.

### 4.5 Maximal $\varepsilon$-Separated Sets on the Hyperbolic Surface

We give a construction of maximal $\varepsilon$-separated on the hyperboloid and the double cone, following the definition of Definition 2.3. Our construction follows the one on $X_0^{+1}$ in Sect. 3.5. We first need $\varepsilon$-separated set on the unit ball $B^d$. We adopt the following notation. For $\varepsilon > 0$, we denote by $B_B (\varepsilon)$ a maximal $\varepsilon$-separated set on the unit ball $B^d$ and we let $B_B (u, \varepsilon)$ be the subsets in $B^d$ so that the collection $\{ B_B (u, \varepsilon) : u \in B_B (\varepsilon) \}$ is a partition of $B^d$, and we assume

$$c_B (u, c_1 \varepsilon) \subset B_B (\varepsilon) \subset c_B (u, c_2 \varepsilon), \quad u \in B_B (\varepsilon), \quad (4.16)$$

where $c_B (u, \varepsilon)$ denotes the ball centered at $u$ with radius $\varepsilon$ in $B^d$, $c_1$ and $c_2$ depend only on $d$. It is known (see, for example, [19]) that such a $B_B (\varepsilon)$ exists for all $\varepsilon > 0$ and its cardinality satisfies

$$c'_d \varepsilon^{-d} \leq \# B_B (\varepsilon) \leq c_d \varepsilon^{-d} \quad (4.17)$$

For the hyperboloid $X^{+1}$, we denote by $B_X = \Xi_X (\varepsilon)$ a maximum $\varepsilon$-separated set and, furthermore, denote by $\{ X(u, t) : (tu, t) \in B_X \}$ a partition of $X^{+1}$. We give one construction of such sets below.
Let \( \varepsilon > 0 \) and let \( N = 2 \lceil \pi \varepsilon^{-1} \rceil \). We define \( t_j = \cos \theta_j \) and \( t_j^+ \) and \( t_j^- \), \( 1 \leq j \leq N \), as in Sect. 3.5. Then \( \mathcal{X}^{d+1} \) can be partitioned by

\[
\mathcal{X}^{(j)} := \left\{ (x, t) \in \mathcal{X}^{d+1} : t_j^- < t \leq t_j^+ \right\}, \quad 1 \leq j \leq N.
\]

Furthermore, the upper and lower hyperboloid \( \mathcal{X}^{d+1}_+ \) and \( \mathcal{X}^{d+1}_- \) can be partitioned by

\[
\mathcal{X}^{d+1}_+ = \bigcup_{j=1}^{N/2} \mathcal{X}^{(j)} \quad \text{and} \quad \mathcal{X}^{d+1}_- = \bigcup_{j=N/2+1}^{N} \mathcal{X}^{(j)}.
\]

Let \( \varepsilon_j := \pi \varepsilon/(2t_j) \). Then \( \mathcal{X}_n(\varepsilon_j) \) is the maximal \( \varepsilon_j \)-separated set of \( \mathbb{B}^d \) such that, for each \( j \geq 1 \), \( \{ \mathbb{B}_u(\varepsilon_j) : u \in \mathcal{X}_n(\varepsilon_j) \} \) is a partition of \( \mathbb{B}^d \) and \( \# \mathcal{X}_n(\varepsilon_j) \sim \varepsilon_j^{-d} \). For each \( j = 1, \ldots, N \), we decompose \( \mathcal{X}^{(j)} \) by

\[
\mathcal{X}^{(j)} = \bigcup_{u \in \mathcal{X}_n(\varepsilon_j)} \mathcal{X}(u, t_j), \quad \text{where} \quad \mathcal{X}(u, t_j) := \left\{ (tv, t) : t_j^- < t \leq t_j^+, \ v \in \mathbb{B}_u(\varepsilon_j) \right\}.
\]

Finally, we define the set \( \mathcal{X}_n \) of \( \mathcal{X}^{d+1} \) by

\[
\mathcal{X}_n = \left\{ (t_j u, t_j) : u \in \mathcal{X}_n(\varepsilon_j), \ 1 \leq j \leq N \right\}.
\]

**Proposition 4.14** Let \( \varepsilon > 0 \) and \( N = 2 \lceil \pi \varepsilon^{-1} \rceil \). Then \( \mathcal{X}_n \) is a maximal \( \varepsilon \)-separated set of \( \mathcal{X}^{d+1} \) and \( \{ \mathcal{X}(t_j u, t_j) : u \in \mathcal{X}_n(\varepsilon_j), \ 1 \leq j \leq N \} \) is a partition

\[
\mathcal{X}^{d+1} = \bigcup_{(u,t) \in \mathcal{X}_n} \mathcal{X}(u, t) = \bigcup_{j=1}^{N} \bigcup_{u \in \mathcal{X}_n(\varepsilon_j)} \mathcal{X}(u, t_j).
\]

Moreover, there are positive constants \( c_1 \) and \( c_2 \) depending only on \( d \) such that

\[ c_0 ((t_j u, t_j), c_1 \varepsilon) \subset \mathcal{X}(u, t_j) \subset c_0 ((t_j u, t_j), c_2 \varepsilon), \quad (t_j u, t_j) \in \mathcal{X}_n, \quad (4.18) \]

and \( c'_d \) and \( c_d \) depending only on \( d \) such that

\[ c'_d \varepsilon^{-d-1} \leq \# \mathcal{X}_n \leq c_d \varepsilon^{-d-1}. \]

**Proof** The proof is parallel to that of Proposition 3.15 and follows almost verbatim. We omit the details. \( \square \)

**Proposition 4.15** For \( \varepsilon > 0 \), let \( \mathcal{X}_n \) be a maximal \( \varepsilon \)-separated set in \( \mathcal{X}^{d+1} \). Define

\[
\mathcal{X}_n^0 := \left\{ \left( x, \sqrt{t^2 - \rho^2} \right) : (x, t) \in \mathcal{X}_n \right\}.
\]
Then $\Xi^0_\mathbb{X}$ is a maximal $\epsilon$-separated set in the solid hyperboloid $\mathbb{X}^{d+1}$.

This is an immediate consequence of (3.5). In particular, for $\Xi_\mathbb{X}$ defined in Proposition 4.14, both (3.25) and (3.26) extend to $\mathbb{X}^{d+1}$ as well. Moreover, this set is also evenly symmetric, where the notion of evenly symmetric set on $\mathbb{X}^{d+1}$ is defined analogously as in Definition 3.17 with $\Xi_0$ replaced by $\Xi$.

With Assertions 1–3 established for $W_{\gamma,\mu}$ for $\gamma \geq 0$ and $\mu \geq 0$, we can now apply [29, Theorem 2.15] to state the Marcinkiewicz–Zygmund inequality. However, following the consideration in the case of $\mathbb{X}^{d+1}$, our result holds under symmetry assumption on the weight $W$ and the set $\Xi_\mathbb{X}$ and is restricted to the subspace of polynomials

$$
\Pi^E_n(\mathbb{X}^{d+1}) = \left\{ p \in \Pi_n^{d+1} : p(x, t) = p(x, -t), \forall (x, t) \in \mathbb{X}^{d+1} \right\}.
$$

**Theorem 4.16** Let $W$ be a doubling weight on $\mathbb{X}^{d+1}$ such that $W(x, t) = W(x, -t)$ for all $(x, t) \in \mathbb{R}^{d+1}$. Let $\Xi^0_\mathbb{X}$ be a symmetric maximal $\frac{\delta}{n}$-separated subset of $\mathbb{X}^{d+1}$ and $0 < \delta \leq 1$.

(i) For all $0 < p < \infty$ and $f \in \Pi^E_n(\mathbb{X}^{d+1})$ with $n \leq m \leq n$,

$$
\sum_{z \in \Xi} \left( \max_{(x, t) \in \mathcal{E}_d((z, \frac{\delta}{n}))} |f(x, t)|^p \right) W(c_d((z, \frac{\delta}{n}))) \leq c_{W, p} \|f\|_p^p,
$$

where $c_{W, p}$ depends on $p$ when $p$ is close to 0 and on the doubling constant of $W$.

(ii) For $0 < r < 1$, there is a $\delta_r > 0$ such that for $\delta \leq \delta_r$, $r \leq p < \infty$ and $f \in \Pi^E_n(\mathbb{X}^{d+1})$,

$$
\|f\|_p^p \leq c_{W, r} \sum_{z \in \Xi} \left( \min_{(x, t) \in \mathcal{E}_d((z, \frac{\delta}{n}))} |f(x, t)|^p \right) W(c_d((z, \frac{\delta}{n})))
$$

where $c_{W, r}$ depends only on the doubling constant of $W$ and on $r$ when $r$ is close to 0.

### 4.6 Positive Cubature Rules

We need fast decaying polynomials on the solid hyperboloid, which will verify Assertion 4.

**Lemma 4.17** Let $d \geq 2$ and $q \geq 0$. For each $(x, t) \in \mathbb{X}^{d+1}$, there is a polynomial $T^E_{x, t}$ in $\Pi^E_n$ that satisfies

(I) $T^E_{x, t}(x, t) = 1$, $T^E_{x, t}(y, s) \geq c > 0$ if $(y, s) \in \mathcal{C}((x, t), \frac{\delta}{n})$, and for every $\kappa > 0$,

$$
0 \leq T^E_{x, t}(y, s) \leq c_{\kappa} \left( 1 + d^E_{\mathbb{X}}((x, t), (y, s)) \right)^{-\kappa}, \quad (y, s) \in \mathbb{X}^{d+1}.
$$
there is a polynomial $q$ of degree $4n$ such that $q(x, t)T_{x, t}^E$ is a polynomial of degree $5n$ in $(x, t)$ and $1 \leq q(x, t) \leq c$.

**Proof** We construct our polynomial based on the one given in Lemma 3.19 by following the approach in Lemma 3.19. For $(x, t), (y, s) \in \mathbb{X}^{d+1}$, we introduce the notation $X = (x, \sqrt{t^2 - q^2 - \|x\|^2})$ and $Y = (y, \sqrt{s^2 - q^2 - \|y\|^2})$. Moreover, denote $X_\ast = (x, -\sqrt{t^2 - q^2 - \|x\|^2})$ and $Y_\ast = (y, -\sqrt{s^2 - q^2 - \|y\|^2})$. Then $(X, t), (Y, s)$ and $(X_\ast, t)$ and $(Y_\ast, s)$ are all elements of $\mathbb{X}_0^{d+2}$. Let $T_{(X,t)}^E$ denote the polynomial of degree $n$ on $\mathbb{X}_0^{d+2}$ defined in Lemma 3.19. We now define

$$T_{x, t}^E(y, s) := \frac{T_{(X,t)}^E(Y, s) + T_{(X,t)}^E(Y_\ast, s)}{1 + T_{(X,t)}^E(X_\ast, t)}.$$ 

Since $T_{(X,t)}^E(x, t) = 1$, it follows that $T_{(X,t)}^E(x, t) = 1$. Moreover, since

$$T_{(X,t)}^E(X_\ast, t) = \frac{S_n(2\|x\|^2 + 1 + 2q^2 - 2t^2) + S_n(2\|x\|^2 - 1)}{1 + S_n(2t^2 - 2q^2 - 1)}$$

and $0 \leq S_n(t) \leq c$, we see that $0 \leq T_{(X,t)}(X_\ast, s) \leq c$. In particular, it follows that

$$T_{x, t}^E(y, s) \geq c T_{(X,t)}^E(Y, s) \geq c > 0, \quad (y, s) \in c((x, t), \frac{\delta}{n}),$$

since $d_{\mathbb{X}}((x, t), (y, s)) = d_{\mathbb{X}_0^{d+2}}((X, t), (Y, s))$. Furthermore, since $\cos d_{\mathbb{X}_0}((X, t), (Y, s) \geq \cos d_{\mathbb{X}_0}((X, t), (Y_\ast, s))$, we obtain

$$d_{\mathbb{X}_0}((X, t), (Y_\ast, s)) \geq d_{\mathbb{X}_0}((X, t), (Y, s)) = d_{\mathbb{X}}((x, t), (y, s)).$$

Hence, using the estimate of $T_{(X,t)}^E$ in Lemma 3.19, we conclude that

$$0 \leq T_{x, t}^E(y, s) \leq c \left[ (1 + n d_{\mathbb{V}_0}((X, t), (Y, s)))^{-K} + (1 + n d_{\mathbb{V}_0}((X, t), (Y_\ast, s)))^{-K} \right]^{-K} \leq c(1 + n d_{\mathbb{V}}((x, t), (y, s)))^{-K}.$$

Finally, let $q(x, t) = (1 + S_n(2t^2 - 2q^2 - 1))T_{(X, t)}^E(X_\ast, t)$. Then $q(x, t)$ is a polynomial of degree at most $4n$, so that $q(x, t)T_{x, t}^E$ is a polynomial of degree at most $5n$ and $1 \leq q(x, t) \leq c$. This completes the proof. □

The lemma establishes Assertion 4 with a polynomial in $\Pi_q^E(\mathbb{X}^{d+1})$. Let $W$ be a doubling weight function that is even in the $t$ variable on $\mathbb{X}^{d+1}$. We define

$$\lambda_n^E(W; x, t) := \inf_{g(x, t)=1} \int_{g \in \Pi_q^E(\mathbb{X}^{d+1}) \mathbb{X}_{d+1}} |g(x, t)|^2 W(x, t) dx dt.$$ 

(4.19)
which is the Christoffel function for the space $\Pi^E_n(\mathbb{X}^{d+1})$. Just like the case of $\mathbb{X}^d_0$, this Christoffel function is related to the kernel $K_n^E(W) = \sum_{k=0}^n P_k^E(W)$ by

$$\lambda_n^E(W; x, t) = \frac{1}{K_n^E(W; (x, t), (x, t))}, \quad (x, t) \in \mathbb{X}^{d+1}. $$

Moreover, [29, Propositions 2.17 and 2.18] remain valid for $\lambda_n^E(W)$ on $\mathbb{X}^{d+1}$ by using Lemma 4.17. Hence, we obtain the following corollary.

**Corollary 4.18** Let $W$ be a doubling weight function on $\mathbb{X}^{d+1}$ such that $W(x, t) = W(x, -t)$ for all $(x, t) \in \mathbb{X}^{d+1}$. Then

$$\lambda_n^E(W; (x, t)) \leq c W(\mathbf{c}_\epsilon ((x, t), \frac{1}{n})). $$

Moreover, for $\gamma \geq 0$ and $\mu \geq 0$,

$$\lambda_n^E(W_{\gamma, \mu}; (x, t)) \geq c W_{\gamma, \mu}(\mathbf{c}_\epsilon ((x, t), \frac{1}{n})) = cn^{-d} W_{\gamma, \mu}(n; x, t).$$

We are now in a position to state the positive cubature rule for the solid hyperboloid, following which holds for polynomials in $\Pi^E_n(\mathbb{X}^{d+1})$ and under the symmetry assumptions.

**Theorem 4.19** Let $d \geq 2$ and $\varrho \geq 0$. Let $W$ be a doubling weight on $\mathbb{X}^{d+1}$ such that $W(x, t) = W(x, -t)$ for all $(x, t) \in \mathbb{X}^{d+1}$. Let $\mathcal{E}$ be a symmetric maximum $\frac{s}{n}$-separated subset of $\mathbb{X}^{d+1}$. There is a $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ there exist positive numbers $\lambda_{z,r}$, $(z, r) \in \mathcal{E}$, so that

$$\int_{\mathbb{X}^{d+1}} f(x, t)W(x, t)dxdt = \sum_{(z, r) \in \mathcal{E}} \lambda_{z,r} f(z, r), \quad \forall f \in \Pi^E_n(\mathbb{X}^{d+1}). \quad (4.20)$$

Moreover, $\lambda_{z,r} \sim W(\mathbf{c}_\epsilon((z, r), \frac{\delta}{n}))$ for all $(z, r) \in \mathcal{E}$.

Again, this follows from [29, Theorem 2.20] when the domain becomes $\mathbb{X}^{d+1}_0$ and it remains valid under symmetry assumptions on the weight and the polynomials.

### 4.7 Localized Tight Frames

The symmetry assumption for the weight and the $\varepsilon$-separated subset also carries over to the local frame on the solid hyperboloid. Let $\varrho \geq 0$ and let $W$ be a doubling weight on $\mathbb{X}^{d+1}$ that is even in the $t$ variable. Let $L_n^E(W) * f$ denote the operator in $\Pi^E_{2n}(\mathbb{X}^{d+1})$ defined by

$$L_n^E(W) * f(x) := \int_{\mathbb{X}^{d+1}} f(y, s)L_n^E(W; (x, t), (y, s))W(y, s)dyds,$$
where $L_{E}^{j}(W; \cdot, \cdot)$ is the highly localized kernel defined via a cut-off function $\hat{a}$ that satisfies (3.30). For $j = 0, 1, \ldots$, let $\Xi_{j}^{0}$ be a symmetric maximal $\frac{\delta}{2j}$-separated subset in $\mathbb{X}^{d+1}$, so that

$$\int_{\mathbb{X}^{d+1}} f(x,t)W(x,t)dxdt = \sum_{(z,r)\in \Xi_{j}^{0}} \lambda_{(x,r),j} f(z,r), \quad f \in \Pi_{2j-1}^{E}(\mathbb{X}^{d+1}).$$

For $j = 1, 2, \ldots$, define the operator $F_{j}^{0}(W)$ by

$$F_{j}^{0}(W) * f = L_{2j-1}^{E}(W) * f$$

and define the frame elements $\psi_{(z,r),j}$ for $(z, r) \in \Xi_{j}^{0}$ by

$$\psi_{(z,r),j}(x,t) := \sqrt{\lambda_{(z,r),j} F_{j}^{0}((x,t),(z,r))}, \quad (x,t) \in \mathbb{X}^{d+1}.$$ 

Then $\Phi = \{\psi_{(z,r),j} : (z, r) \in \Xi_{j}^{0}, \ j = 1, 2, 3, \ldots\}$ is a tight frame by [29, Theorem 2.21].

**Theorem 4.20** Let $W$ be a doubling weight on $\mathbb{X}^{d+1}$ even in its $t$ variable. If $f \in L^{2}(\mathbb{X}^{d+1}, W)$ and $f$ is even in the $t$ variable, then

$$f = \sum_{j=0}^{\infty} \sum_{(z,r)\in \Xi_{j}^{0}} (f, \psi_{(z,r),j})_{W} \psi_{(z,r),j} \quad \text{in } L^{2}(\mathbb{X}^{d+1}, W)$$

and

$$\|f\|_{2,W} = \left(\sum_{j=0}^{\infty} \sum_{(z,r)\in \Xi_{j}^{0}} |(f, \psi_{(z,r),j})_{W}|^{2}\right)^{1/2}.$$ 

Furthermore, for $\gamma \geq 0$ and $\mu \geq 0$, the frame for $W_{\gamma,\mu}^{0}$ is highly localized in the sense that, for every $\kappa > 0$, there exists a constant $c_{\kappa} > 0$ such that

$$|\psi_{(z,r),j}(x,t)| \leq c_{\kappa} \frac{2^{j(d+1)/2}}{W_{\gamma,\mu}^{0}(2^{j};x,t)(1 + 2^{j}d_{X}^{0}((x,t),(z,r)))^{\kappa}}, \quad (x,t) \in \mathbb{X}^{d+1}. \quad (4.21)$$

Like the case on the hyperbolic surface, the frame elements involve only orthogonal polynomials even in the $t$ variable. Localization (4.21) follows from Theorem 4.9 and $\lambda_{(z,r),j} \sim 2^{-j(d+1)} W_{\gamma,\mu}^{0}(2^{j};x,t)$ which holds for $W_{\gamma,\mu}^{0}$ by Corollary 4.18 and (4.9). We note, however, that the localized tight frame holds for the weight function $W_{\gamma,\mu}^{0}$, which does not include the Lebesgue measure on $\mathbb{X}^{d+1}$. 

\(\square\) Springer
4.8 Characterization of Best Approximation

For \( f \in L^p(\mathbb{X}^{d+1}, W^p_{\gamma, \mu}) \), we denote by \( E_n(f)_{p, W^p_{\gamma, \mu}} \) the best approximation to \( f \) from \( \Pi^d_{n+1}(\mathbb{X}^{d+1}) \) in the norm \( \| \cdot \|_{p, W^p_{\gamma, \mu}} \); that is,

\[
E_n(f)_{p, W^p_{\gamma, \mu}} := \inf_{g \in \Pi^d_{n}(\mathbb{X}^{d+1})} \| f - g \|_{p, W^p_{\gamma, \mu}}, \quad 1 \leq p \leq \infty.
\]

As in the case of hyperbolic surface, if \( f \) is even in the \( t \) variable, then we can choose the polynomial of best approximation from \( \Pi^d_{n}(\mathbb{X}^{d+1}) \). We can give a characterization of this quantity for functions that are even in the \( t \) variable.

For \( f \in L^p(\mathbb{X}^{d+1}, W^p_{\gamma, \mu}) \) and \( r > 0 \), the modulus of smoothness is defined by

\[
\omega_r(f; \rho)_{p, W^p_{\gamma, \mu}} = \sup_{0 \leq \rho \leq \rho} \left\| \left( I - S_{\theta, W^p_{\gamma, \mu}} \right)^{r/2} f \right\|_{p, W^p_{\gamma, \mu}}, \quad 1 \leq p \leq \infty,
\]

where the operator \( S_{\theta, W^p_{\gamma, \mu}} \) is defined by, for \( n = 0, 1, 2, \ldots \) and \( \lambda = \gamma + \mu + \frac{d}{2} \),

\[
\text{proj}_{n}^{E} \left( W^p_{\gamma, \mu}; S_{\theta, W^p_{\gamma, \mu}}, f \right) = K_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(\cos \theta) \text{proj}_{n}^{E} \left( W^p_{\gamma, \mu}, f \right).
\]

In terms of the fractional differential operator \((-\mathcal{D}_{\gamma, \mu}^\varrho)^\varrho \), the \( K \)-functional for a doubling weight \( W \), even in the \( t \) variable on \( \mathbb{X}^{d+1} \), is defined by

\[
K_r(f, t)_{p, W} := \inf_{g \in W^r_p(\mathbb{X}^{d+1}, W)} \left\{ \| f - g \|_{p, W} + t^r \left\| (-\mathcal{D}_{\gamma, \mu}^\varrho)^\varrho f \right\|_{p, W} \right\},
\]

where the Sobolev space \( W^r_p(\mathbb{X}^{d+1}, W) \) is the space that consists of \( f \in L^p(\mathbb{X}^{d+1}, W) \), even in the \( t \) variable, so that \( \left\| (-\mathcal{D}_{\gamma, \mu}^\varrho)^\varrho f \right\|_{p, W} \) is finite.

For \( W^p_{\gamma, \mu} \), Assertions 1 and 3 hold and we now verify that Assertion 5 holds as well. By (4.12), the kernel \( L_n^{(r)}(\varrho) \) in Assertion 5 becomes

\[
L_n^{(r)}(W^p_{\gamma, \mu}; (x, t), (y, s)) = \sum_{k=0}^{\infty} \hat{a} \left( \frac{k}{n} \right) (k(k + 2\gamma + 2\mu + d))^{\varrho} \mathcal{P}_k^E(W^p_{\gamma, \mu}; (x, t), (y, s)).
\]

Lemma 4.21 Let \( \gamma, \mu \geq -\frac{1}{2} \) and \( \kappa > 0 \). Then, for \( r > 0 \) and \( (x, t), (y, s) \in \mathbb{X}^{d+1} \),

\[
\left| L_n^{(r)}(W^p_{\gamma, \mu}; (x, t), (y, s)) \right| \leq c_{\kappa} \frac{n^{d+r+1}}{\sqrt{W_{\gamma, \mu}^p(n; t) W_{\gamma, \mu}^p(n; s)}} \left( 1 + nd^{2}_{\gamma}(x, t), (y, s)) \right)^{-\kappa}.
\]

Proof This can be proved similarly as in the proof of Lemma 3.23 and using the estimate in Theorem 4.9. We omit the details.
We can now state the characterization of the best approximation by polynomials on the hyperboloid, following [29, Theorem 3.12].

**Theorem 4.22** Let $\rho \geq 0$. For $\gamma, \mu \geq 0$, let $f \in L^p(\mathbb{H}^{d+1}, W^\rho_{\gamma,\mu})$ if $1 \leq p < \infty$ and $f \in C(\mathbb{H}^{d+1})$ if $p = \infty$. Assume that $f$ satisfies $f(x,t) = f(x,-t)$. Let $r > 0$ and $n = 1, 2, \ldots$. Then

(i) **Direct estimate**

$$E_n(f)_{p,W^\rho_{\gamma,\mu}} \leq c K_r(f; n^{-1})_{p,W^\rho_{\gamma,\mu}};$$

(ii) **Inverse estimate**

$$K_r(f; n^{-1})_{p,W^\rho_{\gamma,\mu}} \leq cn^{-r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_{p,W^\rho_{\gamma,\mu}}.$$

For $W = W^\rho_{\gamma,\mu}$, the $K$-functional is equivalent to the modulus of smoothness.

**Theorem 4.23** Let $\rho \geq 0$, $\gamma, \mu \geq 0$ and $f \in L^p(\mathbb{H}^{d+1}, W^\rho_{\gamma,\mu}), 1 \leq p \leq \infty$. Then for $0 < \theta \leq \pi/2$ and $r > 0$

$$c_1 K_r(f; \theta)_{p,W^\rho_{\gamma,\mu}} \leq \omega_r(f; \theta)_{p,W^\rho_{\gamma,\mu}} \leq c_2 K_r(f; \theta)_{p,W^\rho_{\gamma,\mu}}.$$

In particular, the characterization in Theorem 4.22 can be stated in terms of the modulus of smoothness instead.

**References**

1. Baldi, P., Kerkyacharian, G., Marinucci, D., Picard, D.: Asymptotics for spherical needlets. Ann. Stat. 37, 1150–1171 (2009)
2. Baldi, P., Kerkyacharian, G., Marinucci, D., Picard, D.: Adaptive density estimation for directional data using needlets. Ann. Stat. 37, 3362–3395 (2009)
3. Brown, G., Dai, F.: Approximation of smooth functions on compact two-point homogeneous spaces. J. Funct. Anal. 220, 401–423 (2005)
4. Dai, F.: Multivariate polynomial inequalities with respect to doubling weights and $A_\infty$ weights. J. Funct. Anal. 235, 137–170 (2006)
5. Dai, F., Prymak, A.: $L^p$-Bernstein inequalities on $C^2$ domains applications to discretization. Trans. Am. Math. Soc. 375, 1933–1976 (2022)
6. Dai, F., Prymak, A.: On directional Whitney inequality. Can. J. Math. 74(2022), 833–857 (2020)
7. Dai, F., Wang, H.: Optimal cubature formulas in weighted Besov spaces with $A_\infty$ weights on multi-variate domains. Constr. Approx. 37, 167–194 (2013)
8. Dai, F., Xu, Y.: Approximation Theory and Harmonic Analysis on Spheres and Balls. Springer Monographs in Mathematics. Springer, Berlin (2013)
9. Dunkl, C.F., Xu, Y.: Orthogonal Polynomials of Several Variables. Encyclopedia of Mathematics and its Applications, vol. 155. Cambridge University Press, Cambridge (2014)
10. DeVore, R.A., Lorentz, G.G.: Constructive approximation. Grundlehren der Mathematischen Wissenschaften, vol. 303. Springer, Berlin (1993)
11. Ivanov, K., Petrushev, P.: Fast memory efficient evaluation of spherical polynomials at scattered points. Adv. Comput. Math. 41, 191–230 (2015)
12. Ivanov, K., Petrushev, P.: Highly effective stable evaluation of bandlimited functions on the sphere. Numer. Algorithms 71, 585–611 (2016)
13. Ivanov, K., Petrushev, P., Xu, Y.: Decomposition of spaces of distributions induced by tensor product bases. J. Funct. Anal. 263, 1147–1197 (2012)
14. Kyriazis, G., Petrushev, P., Xu, Y.: Decomposition of weighted Triebel–Lizorkin and Besov spaces on the ball. Proc. Lond. Math. Soc. 97, 477–513 (2008)
15. Le Gia, Q.T., Sloan, I.H., Wang, Y.G., Womersley, R.S.: Needlet approximation for isotropic random fields on the sphere. J. Approx. Theory 216, 86–116 (2017)
16. Mastroianni, G., Totik, V.: Weighted polynomial inequalities with doubling and $A_\infty$ weights. Constr. Approx. 16, 37–71 (2000)
17. Narcowich, F.J., Petrushev, P., Ward, J.D.: Localized tight frames on spheres. SIAM J. Math. Anal. 38, 574–594 (2006)
18. Narcowich, F.J., Petrushev, P., Ward, J.D.: Decomposition of Besov and Triebel–Lizorkin spaces on the sphere. J. Funct. Anal. 238, 530–564 (2006)
19. Petrushev, P., Xu, Y.: Localized polynomial frames on the ball. Constr. Approx. 27, 121–148 (2008)
20. Rustamov, K.P.: On the approximation of functions on a sphere (Russian). Izv. Ross. Akad. Nauk Ser. Mat. 57, 127–148 (1993). Translation in Russian Acad. Sci. Izv. Math. 43(2), 311–329 (1994)
21. Szegő, G.: Orthogonal Polynomials, 4th edn. American Mathematical Society, Providence (1975)
22. Totik, V.: Polynomial approximation on polytopes. Mem. Am. Math. Soc. 232(1091), vi+112 (2014)
23. Totik, V.: Polynomial approximation in several variables. J. Approx. Theory 252, 105364 (2020)
24. Wang, Y.G., Le Gia, Q.T., Sloan, I.H., Womersley, R.S.: Fully discrete needlet approximation on the sphere. Appl. Comput. Harmon. Anal. 43, 292–316 (2017)
25. Xu, Y.: Summability of Fourier orthogonal series for Jacobi weight on a ball in $\mathbb{R}^d$. Trans. Am. Math. Soc. 351, 2439–2458 (1999)
26. Xu, Y.: Weighted approximation of functions on the unit sphere. Constr. Approx. 21, 1–28 (2005)
27. Xu, Y.: Orthogonal polynomials and Fourier orthogonal series on a cone. J. Fourier Anal. Appl. 26(6), 42 (2020)
28. Xu, Y.: Orthogonal structure and orthogonal series in and on a double cone or a hyperboloid. Trans. Am. Math. Soc. 374, 3603–3657 (2021)
29. Xu, Y.: Approximation and localized polynomial frame on conic domains. J. Funct. Anal. 281 (12), Paper No. 109257, 94 pp. (2021)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.