A SURVEY ON SECOND-ORDER FREE BOUNDARY VALUE PROBLEMS MODELLING MEMS WITH GENERAL PERMITTIVITY PROFILE

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ABSTRACT. In this survey we review some recent results on microelectromechanical systems with general permittivity profile. Different systems of differential equations are derived by taking various physical modelling aspects into account, according to the particular application. In any case an either semi- or quasilinear hyperbolic or parabolic evolution problem for the displacement of an elastic membrane is coupled with an elliptic moving boundary problem that determines the electrostatic potential in the region occupied by the elastic membrane and a rigid ground plate. Of particular interest in all models is the influence of different classes of permittivity profiles.

The subsequent analytical investigations are restricted to a dissipation dominated regime for the membrane’s displacement. For the resulting parabolic evolution problems local well-posedness, global existence, the occurrence of finite-time singularities, and convergence of solutions to those of the so-called small-aspect ratio model, respectively, are investigated. Furthermore, a topic is addressed that is of note not till non-constant permittivity profiles are taken into account – the direction of the membrane’s deflection or, in mathematical parlance, the sign of the solution to the evolution problem. The survey is completed by a presentation of some numerical results that in particular justify the consideration of the coupled problem by revealing substantial qualitative differences of the solutions to the widely-used small-aspect ratio model and the coupled problem.

1. Introduction. Microelectromechanical systems (MEMS) is a generic term for miniaturised devices whose dimensions range between some micrometres and one millimetre. Being typically made up of a sensor, a transistor as well as a mechanical actuator, MEMS sense the environment and act on it by combining microelectronics with non-electronic activities from micromechanics, fluidics or optics. The range of applications for MEMS is enormous: As inertial sensors MEMS are used for the activation of airbags [3] and for the protection of hard disks or for mechanical image stabilisation in optic devices, to mention only a few examples. Furthermore, MEMS are applied as micro pumps [6] and micro valves [16] in micro fluidics.

In this survey we report on the analysis of idealised MEMS devices which perform mechanical motion by electrostatic actuation. More precisely, MEMS devices are considered which consist of a rigid ground plate and an elastic membrane that

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is suspended above the former and held fixed at its boundary. Moreover, the deformable elastic membrane is assumed to be of infinitely small thickness and features a certain dielectric permittivity profile. In order to cause a mechanical deflection of the latter, a voltage is applied across the device such that the ground plate and the membrane are at different electric potentials. This imbalance of potentials/charges acting on each other induces attractive or repulsive forces which are described by Coulomb's law and thus gives rise to a deformation of the membrane. A sketch of such a MEMS device is offered in Figure 1. A necessity in order to understand the mode of operation of the device is to gain knowledge about the membrane's deformation on the one hand and about the electrostatic potential in the region occupied by the ground plate and the membrane on the other hand.

![Figure 1. Sketch of the investigated idealised MEMS device.](image)

In fact, these two physical quantities obey a coupled system of nonlinear differential equations. More precisely, an elliptic problem is to be solved for the electrostatic potential in a domain whose boundary evolves with time as the membrane deflects with time. To describe the dynamics of the free boundary a further partial differential equation is to be specified.

In order to simplify the coupled system, heretofore the aspect ratio, i.e. the ratio of height and length of the device, being rather small has been exploited in a variety of applications. More precisely, the assumption of a negligibly small aspect ratio allows an explicit expression for the electrostatic potential, whereby the coupled problem is reduced to a single evolution equation whose right-hand side features a singularity in the moment the membrane touches down on the ground plate. However, it is worthwhile to note that the assumption of a vanishing aspect ratio is not reasonable in all applications [1]. As examples for MEMS devices with high aspect ratio turbines and micromotors may be mentioned. Moreover, qualitative differences between the coupled system for positive aspect ratios and the small-aspect ratio model have been revealed in [7].
Hitherto, various theoretical contributions have been dedicated to the analytical investigation of MEMS devices. Whereas a multitude of them treats the case of a vanishing aspect ratio (see for instance [13, 14, 16, 17, 20, 18, 19, 22, 25, 31, 36]), recent results on the coupled problem with constant permittivity profile \( p \equiv 1 \) go back to Escher, Laurençon and Walker. The reader shall be referred to the works [8, 9, 10, 24, 26, 27]. Further investigations of qualitative properties of MEMS systems may be found in [28, 29, 30]. In this survey we report on recent results obtained in [7, 11, 12, 32, 33, 34], treating non-constant permittivity profiles \( p = p(x, u(t, x)) \), which may depend on the spatial variable \( x \) and the membrane’s displacement \( u(t, x) \).

2. Modelling. It is the intention of this chapter to present a rather detailed derivation of the equations governing the dynamic behaviour of an idealised electrostatically actuated MEMS device with general permittivity profile. The type of MEMS devices under consideration consists of two quadrilateral components – a flat rigid ground plate and an elastic membrane that is suspended above the ground plate. Both components are assumed to be perfectly conducting and moreover the elastic membrane features in addition a certain dielectric permittivity profile.

The components of the investigated system are assumed to be spatially homogeneous in one lateral dimension so that we may in fact restrict the analysis to a cross section of the device. Denoting by \( \tilde{x} \) and \( \tilde{z} \) the horizontal and vertical direction, respectively, we consider the ground plate to be located at height \( \tilde{z} = -h \) and the undeflected membrane at \( \tilde{z} = 0 \), both having the length \( 2l \). Moreover, the length \( 2l \) of the device is assumed to be large compared to the gap size \( h \) of the undeformed configuration, which means that we are in the regime of a small aspect ratio \( \varepsilon = h/l \ll 1 \).

By applying a voltage \( V \) to the conducting film a deformation of the elastic membrane is induced, which is assumed to be only in \( \tilde{z} \)-direction. We denote the deformation at time \( \tilde{t} \geq 0 \) and position \( \tilde{x} \in L := (-l, l) \) by \( \tilde{u} = \tilde{u}(\tilde{t}, \tilde{x}) \). The second quantity of general interest, the electrostatic potential at a time \( \tilde{t} \geq 0 \) and a position \( (\tilde{x}, \tilde{z}) \) in the region between the ground plate and the elastic membrane is denoted by \( \tilde{\psi} = \tilde{\psi}(\tilde{t}, \tilde{x}, \tilde{z}) \). It is worthwhile to mention again that the shape of this region changes with time as the membrane deflects with time. Finally we denote the permittivity profile of the membrane by \( p = p(\tilde{x}, \tilde{u}(\tilde{t}, \tilde{x})) \).
2.1. A nonlinear elasticity model. A general model for the dynamic behaviour of an electrostatically actuated MEMS device is derived by means of nonlinear elasticity theory. Allowing also for large deflections of the membrane, it is the characteristic of the governing elasticity terms to be nonlinear.

For the nonce the time variable $t$ appears as a parameter, whence it is temporarily suppressed in the notation.

Governing equations for the electrostatic potential. Pursuant to Gauss’ law of electrodynamics the electrostatic potential is harmonic in the region $\hat{\Omega}(\hat{u}):=\{(\hat{x},\hat{z}); -l<\hat{x}<l, -h<\hat{z}<\hat{u}(\hat{x})\}$ between the rigid ground plate and the membrane, that is

$$\hat{\psi}_{\hat{x}\hat{x}} + \hat{\psi}_{\hat{z}\hat{z}} = 0, \quad (\hat{x},\hat{z}) \in \hat{\Omega}(\hat{u}).$$

Furthermore, the fixed plate at $\hat{z}=-h$ is grounded, i.e. at zero potential, whereas the membrane is at potential $V_p(\hat{x},\hat{u}(\hat{x}))$. These boundary conditions are expressed by the equations

$$\hat{\psi}(\hat{x}, -h) = 0, \quad \hat{\psi}(\hat{x}, \hat{u}(\hat{x})) = V_p(\hat{x}, \hat{u}(\hat{x})), \quad \hat{x} \in L,$$

c.f. [19, 20, 35, 37]...

Governing equations for the membrane’s deformation. By means of nonlinear elasticity theory we first derive the governing equations for the case of static plate deformations under the hypotheses of Love–Kirchhoff. In particular this includes the assumption that vectors normal to the middle surface remain normal to the middle surface after deformation (see i.e. [4]). We also assume the elastic plate to be infinitely thin. This reduces the general model for plate deformations to one describing deformations of elastic membranes. If no ambiguity is to be feared we use both expressions suitably.

The total potential energy $E_p$ of the configuration is constituted by the pointwise sum of stretching energy $E_s$, bending energy $E_b$ and electrostatic energy $E_e$, i.e. we have

$$E_p(\hat{u}) = E_s(\hat{u}) + E_b(\hat{u}) + E_e(\hat{u}).$$

Denoting by $\tau > 0$ the tension constant of the plate, the stretching energy is given by

$$E_s(\hat{u}) = \tau \int_{-l}^{l} \left( \sqrt{1 + (\hat{u}_{\hat{x}}(\hat{x}))^2} - 1 \right) d\hat{x}. \quad (1)$$

The integral describes the variation of the plate’s length from $2l$, i.e. from the situation in which deformation is absent.

The likewise involved bending energy is proportional to the $L_2$-norm of the plate’s curvature. More precisely, it is given by

$$E_b(\hat{u}) = \frac{b}{2} \int_{-l}^{l} \left( \partial_{\hat{x}} \left( \frac{\hat{u}_{\hat{x}}(\hat{x})}{\sqrt{1 + (\hat{u}_{\hat{x}}(\hat{x}))^2}} \right) \right)^2 \sqrt{1 + (\hat{u}_{\hat{x}}(\hat{x}))^2} d\hat{x},$$

where the coefficient $b$, describing the flexural rigidity of the plate, is defined as

$$b = \frac{2\alpha^3 Y}{3(1-\nu)^2}.$$

The parameters in this ratio denote the thickness $\alpha$ of the plate, the Young modulus $Y$ and the Poisson ratio $\nu$. 
Finally, the electrostatic energy is given by
\[ E_e(\tilde{u}) = -\frac{\varepsilon_0}{2} \int_{\Omega(\tilde{u})} \left( \nabla \tilde{\psi} (\tilde{x}, \tilde{z}) \right)^2 d(\tilde{x}, \tilde{z}), \]
with \( \varepsilon_0 \) being the permittivity of free space. The variation of \( E_e \) corresponds to the work of the force on the elastic plate that is induced by the electric field with potential \( \tilde{\psi}(\tilde{x}, \tilde{z}) \).

Consequently, the total potential energy of the system is given by
\[ E_p(\tilde{u}) = \tau \int_{-l}^{l} \left( \sqrt{1 + (\tilde{u}_x(\tilde{x}))^2} - 1 \right) d\tilde{x} + \frac{b}{2} \int_{-l}^{l} \left( \partial_x \left( \frac{\tilde{u}_x(\tilde{x})}{\sqrt{1 + (\tilde{u}_x(\tilde{x}))^2}} \right) \right)^2 \sqrt{1 + (\tilde{u}_x(\tilde{x}))^2} d\tilde{x} - \frac{\varepsilon_0}{2} \int_{\Omega(\tilde{u})} \left( \nabla \tilde{\psi} (\tilde{x}, \tilde{z}) \right)^2 d(\tilde{x}, \tilde{z}). \]

Derivation of the Euler–Lagrange equation. Due to Hamilton’s principle of least action the partial differential equation describing the dynamics of the plate’s deformation is the Euler–Lagrange equation which is obtained by minimising a suitable energy functional. In a first step we derive the Euler–Lagrange equation in terms of static deflections. To this end we define the Lagrangian \( \mathcal{L} : L \times W^1_2(L) \to \mathbb{R} \) by
\[ \mathcal{L}(\tilde{x}, \tilde{u}) = -\tau \left( \sqrt{1 + (\tilde{u}_x)^2} - 1 \right) - \frac{b}{2} \partial_x \left( \frac{\tilde{u}_x}{\sqrt{1 + (\tilde{u}_x)^2}} \right)^2 \sqrt{1 + (\tilde{u}_x)^2} + \frac{\varepsilon_0}{2} \int_{-h}^{h} \left( \tilde{\psi}_x(\tilde{x}, \tilde{z}) \right)^2 + \left( \tilde{\psi}_z(\tilde{x}, \tilde{z}) \right)^2 d\tilde{z} \]
and minimise the according energy functional
\[ \int_{-l}^{l} \mathcal{L}(\tilde{x}, \tilde{u}) d\tilde{x}. \]

To this end assume that \( \tilde{u} = \tilde{u}(\tilde{x}) \) is the current minimiser of (2), satisfying
\[ \tilde{u} \in W^1_2(L), \quad \tilde{u}(\pm l) = 0, \quad \tilde{u}(\tilde{x}) > -h, \quad \tilde{x} \in [-l, l]. \]

Then, given \( \sigma \in \mathbb{R} \) and a function \( v \in C_0^\infty(L) \), we introduce the notation
\[ w(\sigma)(\tilde{x}) := \tilde{u}(\tilde{x}) + \sigma v(\tilde{x}), \quad \tilde{x} \in [-l, l], \]
and derive the necessary condition for \( \tilde{u} \) being a minimiser of (2) by computing the first variation
\[ \delta E_p(\tilde{u}; v) = \frac{d}{d\sigma} E_p(\tilde{u} + \sigma v)|_{\sigma=0} = \delta \left( E_e(\tilde{u}; v) + E_b(\tilde{u}; v) + E_e(\tilde{u}; v) \right)|_{\sigma=0} \]
and finally checking the condition \( \delta E_p(\tilde{u}; v) = 0 \). For the stretching term one obtains

\[ 1 \text{In physics and engineering it is common to consider regularity assumptions as physically given and thus to presume the validity of the Euler–Lagrange equations. Mathematically speaking we therefore just verify the necessary condition for the existence of an extremum of the functional.} \]
\[
\frac{d}{d\sigma} E_s(\hat{u} + \sigma v) = \frac{d}{d\sigma} \left( \tau \int_{-l}^{l} \sqrt{1 + (w_\beta')^2} - 1 \, d\hat{x} \right) = \tau \int_{-l}^{l} \frac{\hat{u}_\beta v_\beta + \sigma (v_\beta)^2}{\sqrt{1 + (\hat{u}_\beta)^2}} \, d\hat{x},
\]
and therefore, using the fact that \( v \) is compactly supported in the interval \( L = (-l, l) \),
\[
\delta E_s(\hat{u}; v) = \tau \int_{-l}^{l} \frac{\hat{u}_\beta v_\beta}{\sqrt{1 + (\hat{u}_\beta)^2}} = -\tau \int_{-l}^{l} v \partial_\hat{x} \left( \frac{\hat{u}_\beta}{\sqrt{1 + (\hat{u}_\beta)^2}} \right) \, d\hat{x}. \tag{5}
\]
For the bending term, note that
\[
\partial_\hat{x} \left( \frac{w_\beta}{\sqrt{1 + (w_\beta)^2}} \right) = \frac{w_\beta}{(1 + (w_\beta)^2)^{3/2}},
\]
whence we may write
\[
\frac{d}{d\sigma} E_b(\hat{u} + \sigma v) = \frac{d}{d\sigma} \left( \frac{b}{2} \int_{-l}^{l} \left( \partial_\hat{x} \left( \frac{w_\beta}{\sqrt{1 + (w_\beta)^2}} \right) \right)^2 \sqrt{1 + (w_\beta)^2} \, d\hat{x} \right) = b \int_{-l}^{l} \frac{(\hat{u}_\beta + \sigma v_\beta) v_\beta}{(1 + (\hat{u}_\beta)^2 + 2\sigma \hat{u}_\beta v_\beta + \sigma^2 (v_\beta)^2)^{5/2}} \, d\hat{x} - \frac{5b}{2} \int_{-l}^{l} \frac{(\hat{u}_\beta v_\beta + \sigma (v_\beta)^2) (\hat{u}_{\beta \beta} + \sigma (v_{\beta \beta})^2)}{(1 + (\hat{u}_\beta)^2)^{7/2}} \, d\hat{x},
\]
and again using that \( v(\pm l) = 0 \) we obtain
\[
\delta E_b(\hat{u}; v) = b \int_{-l}^{l} \partial_\hat{x}^2 \left( \frac{\hat{u}_{\beta \beta}}{1 + (\hat{u}_\beta)^2)^{5/2}} \right) v \, d\hat{x} + \frac{5b}{2} \int_{-l}^{l} \partial_\hat{x} \left( \frac{\hat{u}_\beta v_\beta}{1 + (\hat{u}_\beta)^2)^{7/2}} \right) v \, d\hat{x}. \tag{6}
\]
It finally remains to take the electrostatic energy into account and to calculate \( \delta E_e(\hat{u} + \sigma v) \). In the sequel this is done by an application of the transport theorem, c.f. [2, XII, Theorem 2.11] or [21, Theorem 5.2.2] for instance. To this end, given \( \sigma \in \mathbb{R}, v \in C^\infty_c (L) \) and \( w(\sigma)(\hat{x}) = \hat{u}(\hat{x}) + \sigma v(\hat{x}) \) as above, we pick \( \sigma_0 > 0 \) such that the choice of \( \hat{u} \) as in (3) implies that \( w(\sigma)(\hat{x}) > -h \) for all \( \hat{x} \in [-l, l] \) and all \( \sigma \in [-\sigma_0, \sigma_0] \) and such that we may introduce the domain
\[
\tilde{\Omega}_\sigma := \{ (\hat{x}, \hat{z}) \in L \times (-h, \infty); -h < \hat{z} < w(\sigma)(\hat{x}) \}, \quad \sigma \in [-\sigma_0, \sigma_0].
\]
In addition, there exists a representation \( \tilde{\Omega}_\sigma = \phi(\sigma; \tilde{\Omega}(\hat{u})) \), of \( \tilde{\Omega}_\sigma \) via the (global) diffeomorphism
\[
\phi(\sigma; \hat{x}, \hat{z}) := \left( \hat{x}, \hat{z} + \sigma v(\hat{x}) \frac{h + \hat{z}}{h + \hat{u}(\hat{x})} \right), \quad (\hat{x}, \hat{z}) \in \tilde{\Omega}(\hat{u}) = \tilde{\Omega}_0.
\]
In order to be able to handle the electrostatic energy with a variational approach it is necessary to investigate the problem for \( \psi \) corresponding to the variation \( w \) of the minimiser \( \hat{u} \) in direction \( v \). For this purpose denote by \( \tilde{\psi}(\sigma; \hat{u}, v) \in W^2_2(\tilde{\Omega}_\sigma) \) the solution to
\[
\tilde{\psi}_{\hat{x} \hat{x}}(\sigma; \hat{u}, v) + \tilde{\psi}_{z z}(\sigma; \hat{u}, v) = 0, \quad (\hat{x}, \hat{z}) \in \tilde{\Omega}_\sigma, \tag{7}
\]
\[
\tilde{\psi}(\sigma; \hat{u}, v) = \frac{h + \hat{z}}{h + w(\sigma)(\hat{x})} Vp(\hat{x}, w(\sigma)(\hat{x})), \quad (\hat{x}, \hat{z}) \in \partial \tilde{\Omega}_\sigma. \tag{8}
\]
Moreover, we introduce the velocity \( V \) of the path \( \{ \tilde{\psi}(\sigma; \tilde{u}, v) : \sigma \in (-\sigma_0, \sigma_0) \} \), defined as

\[
V := \frac{d}{d\sigma} \psi(\sigma; \tilde{u}, v)|_{\sigma=0}, \quad (\tilde{x}, z) \in \tilde{\Omega}(\tilde{u}).
\]

(9)

and show that also \( V \) satisfies (7)–(8) in the limit \( \sigma = 0 \). To this end, observe that (7) is equivalent to

\[
\tilde{\psi}_{\tilde{x}\tilde{z}} \left( \sigma; \tilde{x}, \tilde{z} + \sigma v(\tilde{x}) \left( \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} \right) \right) + \tilde{\psi}_{\tilde{x}\tilde{z}} \left( \sigma; \tilde{x}, \tilde{z} + \sigma v(\tilde{x}) \left( \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} \right) \right) v(\tilde{x}) \left( \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} \right) = 0, \quad (\tilde{x}, \tilde{z}) \in \tilde{\Omega}(\tilde{u}),
\]

whence a differentiation of this equation with respect to \( \sigma \) yields

\[
\tilde{\psi}_{\tilde{x}\tilde{z}\sigma} \left( \sigma; \tilde{x}, \tilde{z} + \sigma v(\tilde{x}) \left( \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} \right) \right) + \tilde{\psi}_{\tilde{x}\tilde{z}\sigma} \left( \sigma; \tilde{x}, \tilde{z} + \sigma v(\tilde{x}) \left( \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} \right) \right) v(\tilde{x}) \left( \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} \right) + \tilde{\psi}_{\tilde{x}z\sigma} \left( \sigma; \tilde{x}, \tilde{z} + \sigma v(\tilde{x}) \left( \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} \right) \right) v(\tilde{x}) \left( \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} \right) = 0
\]

(10)

for all \((\tilde{x}, \tilde{z}) \in \tilde{\Omega}(\tilde{u})\). Then, letting \( \sigma \to 0 \) in (10), we first find that

\[
V_{\tilde{x}\tilde{z}}(\tilde{x}, \tilde{z}) + V_{\tilde{x}\tilde{z}}(\tilde{x}, \tilde{z}) + v(\tilde{x}) \left( \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} \right) \tilde{\psi}_{\tilde{x}\tilde{z}\tilde{z}}(\tilde{x}, \tilde{z}) + \tilde{\psi}_{\tilde{x}\tilde{z}\tilde{z}}(\tilde{x}, \tilde{z}) = 0, \quad (\tilde{x}, \tilde{z}) \in \tilde{\Omega}(\tilde{u}),
\]

whence by (7)

\[
V_{\tilde{x}\tilde{z}}(\tilde{x}, \tilde{z}) + V_{\tilde{x}\tilde{z}}(\tilde{x}, \tilde{z}) = 0, \quad (\tilde{x}, \tilde{z}) \in \tilde{\Omega}(\tilde{u}).
\]

In addition, one can infer from the boundary condition (8) that

\[
\tilde{\psi}(\sigma; \tilde{x}, \tilde{z} + \sigma v(\tilde{x}) \left( \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} \right)) = \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} V p(\tilde{x}, \tilde{u}(\tilde{x}) + \sigma v(\tilde{x})), \quad (\tilde{x}, \tilde{z}) \in \partial\tilde{\Omega}(\tilde{u}),
\]

and differentiating this identity with respect to \( \sigma \) yields

\[
\tilde{\psi}_{\sigma} \left( \sigma; \tilde{x}, \tilde{z} + \sigma v(\tilde{x}) \left( \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} \right) \right) + \tilde{\psi}_{\sigma} \left( \sigma; \tilde{x}, \tilde{z} + \sigma v(\tilde{x}) \left( \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} \right) \right) v(\tilde{x}) \left( \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} \right) = \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} V p_0(\tilde{x}, \tilde{u}(\tilde{x}) + \sigma v(\tilde{x}))(\tilde{x}), \quad (\tilde{x}, \tilde{z}) \in \tilde{\Omega}(\tilde{u}).
\]

for \((\tilde{x}, \tilde{z}) \in \partial\tilde{\Omega}(\tilde{u})\). Finally, as \( \sigma \to 0 \), we find that

\[
V(\tilde{x}, \tilde{z}) = v(\tilde{x}) \left( \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} \right) \left( V p_0(\tilde{x}, \tilde{u}(\tilde{x})) - \tilde{\psi}_{\tilde{z}}(\tilde{x}, \tilde{z}) \right), \quad (\tilde{x}, \tilde{z}) \in \partial\tilde{\Omega}(\tilde{u}).
\]

(11)

Since \( v \in C_\infty(L) \) and \( h + \tilde{z} = 0 \) for \( \tilde{z} = -h \), one may in particular extract from equation (11) the identities

\[
V(\pm l, \tilde{z}) = 0, \quad \tilde{z} \in (-h, 0),
\]

\[
V(\tilde{x}, -h) = 0, \quad \tilde{x} \in L,
\]

\[
V(\tilde{x}, \tilde{u}(\tilde{x})) = v(\tilde{x}) \left( V p_0(\tilde{x}, \tilde{u}(\tilde{x})) - \tilde{\psi}_{\tilde{z}}(\tilde{x}, \tilde{u}(\tilde{x})) \right), \quad \tilde{x} \in L.
\]

\[\text{Note that in fact } V \text{ is a function of the variables } \tilde{x} \text{ and } \tilde{z} \text{ in the sense that } V(\tilde{x}, \tilde{z}) = \frac{d}{d\sigma} \psi(\sigma; \tilde{u}, v)|_{\sigma=0}(\tilde{x}, \tilde{z}).
\]

\[\text{In fact } \psi(\sigma; \tilde{u}, v) \text{ is a function of the variables } \tilde{z} \text{ and } \tilde{z}. \text{ For the sake of simplicity we suppress the dependence of } \psi \text{ on } \tilde{u} \text{ and } v \text{ and use the notation } \tilde{\psi}(\sigma; \tilde{x}, \tilde{z}).\]
We are finally prepared to consider the energy
\[ E_e(\tilde{u} + \sigma v) = -\varepsilon_0 \int_{\tilde{\Omega}_2} (\tilde{\psi}_2(\sigma; \tilde{x}, \tilde{z}))^2 + (\tilde{\psi}_2(\sigma; \tilde{x}, \tilde{z}))^2 \, d(\tilde{x}, \tilde{z}). \]

Firstly, invoking [21, Thm. 5.2.2] yields the identity
\[
\delta E_e(\tilde{u}; v) = -\varepsilon_0 \int_{\tilde{\Omega}(\tilde{u})} \tilde{\psi}_2(\tilde{x}, \tilde{z}) \varphi(\tilde{x}, \tilde{z}) \, d(\tilde{x}, \tilde{z})
\]
\[
-\varepsilon_0 \int_{\tilde{\Omega}(\tilde{u})} \text{div} \left( \left( \tilde{\psi}_2(\tilde{x}, \tilde{z}) \right)^2 + \left( \tilde{\psi}_2(\tilde{x}, \tilde{z}) \right)^2 \right) \phi(0, \tilde{x}, \tilde{z}) \, d(\tilde{x}, \tilde{z}) \tag{13}
\]
and using (7) one can readily see that
\[
\text{div} \left( \nabla \tilde{\psi}_2 \right) = \nabla \tilde{\psi}_2 + \nabla \tilde{\psi}_2 \tag{14}
\]
holds true for all \((\tilde{x}, \tilde{z}) \in \tilde{\Omega}(\tilde{u})\). Then, fusing the findings (13) and (14) leads to the equation
\[
\delta E_e(\tilde{u}; v) = -\varepsilon_0 \int_{\tilde{\Omega}(\tilde{u})} \text{div} \left( \nabla(\tilde{x}, \tilde{z}) \tilde{\psi}_2(\tilde{x}, \tilde{z}) \right) \, d(\tilde{x}, \tilde{z})
\]
\[
-\varepsilon_0 \int_{\tilde{\Omega}(\tilde{u})} \text{div} \left( \left( \tilde{\psi}_2(\tilde{x}, \tilde{z}) \right)^2 + \left( \tilde{\psi}_2(\tilde{x}, \tilde{z}) \right)^2 \right) \phi(0, \tilde{x}, \tilde{z}) \, d(\tilde{x}, \tilde{z}). \]

Allowing for the identity
\[
\phi(0; \tilde{x}, \tilde{z}) = \left( 0, v(\tilde{x}) \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} \right), \quad (\tilde{x}, \tilde{z}) \in \tilde{\Omega}(\tilde{u}),
\]
an application of the Green–Riemann integration theorem reveals
\[
\delta E_e(\tilde{u}; v) = -\varepsilon_0 \int_{\partial \tilde{\Omega}(\tilde{u})} \nabla(\tilde{x}, \tilde{z}) \tilde{\psi}_2(\tilde{x}, \tilde{z}) \, d\tilde{z} + \varepsilon_0 \int_{\partial \tilde{\Omega}(\tilde{u})} \nabla(\tilde{x}, \tilde{z}) \tilde{\psi}_2(\tilde{x}, \tilde{z}) \, d\tilde{z}
\]
\[
+\frac{\varepsilon_0}{2} \int_{\partial \tilde{\Omega}(\tilde{u})} v(\tilde{x}) \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{x})} \left( \left( \tilde{\psi}_2(\tilde{x}, \tilde{z}) \right)^2 + \left( \tilde{\psi}_2(\tilde{x}, \tilde{z}) \right)^2 \right) \, d\tilde{x}.
\]

Then, exploiting the relations \(v(\pm L) = 0, h + \tilde{z} = 0\) for \(\tilde{z} = -h\), and the boundary conditions (12) for \(\nabla\), the above integrals vanish at the lateral boundaries and on the ground plate at \(\tilde{z} = -h\), whereby we obtain
\[
\delta E_e(\tilde{u}; v) = \varepsilon_0 V \int_{-L}^{L} v(\tilde{x}) p_x(\tilde{x}, \tilde{u}(\tilde{x})) \left( \tilde{\psi}_2(\tilde{x}, \tilde{u}(\tilde{x})) \tilde{\psi}_2(\tilde{x}, \tilde{u}(\tilde{x})) - \tilde{\psi}_2(\tilde{x}, \tilde{u}(\tilde{x})) \right) \, d\tilde{x}
\]
\[
-\varepsilon_0 \int_{-L}^{L} v(\tilde{x}) \left( \tilde{\psi}_2(\tilde{x}, \tilde{u}(\tilde{x})) \tilde{\psi}_2(\tilde{x}, \tilde{u}(\tilde{x})) \tilde{\psi}_2(\tilde{x}, \tilde{u}(\tilde{x})) - \tilde{\psi}_2(\tilde{x}, \tilde{u}(\tilde{x})) \right) \, d\tilde{x}
\]
\[
-\frac{\varepsilon_0}{2} \int_{-L}^{L} v(\tilde{x}) \left( (\tilde{\psi}_2(\tilde{x}, \tilde{u}(\tilde{x})))^2 + (\tilde{\psi}_2(\tilde{x}, \tilde{u}(\tilde{x})))^2 \right) \, d\tilde{x}.
\]

From the boundary condition \(\tilde{\psi}(\tilde{x}, \tilde{u}(\tilde{x})) = Vp(\tilde{x}, \tilde{u}(\tilde{x}))\), \(\tilde{x} \in L\), we may deduce the relation
\[
\tilde{\psi}_2(\tilde{x}, \tilde{u}(\tilde{x})) \tilde{\psi}_2(\tilde{x}, \tilde{u}(\tilde{x})) = V \left( p_x(\tilde{x}, \tilde{u}(\tilde{x})) + p_u(\tilde{x}, \tilde{u}(\tilde{x})) \right) \tilde{\psi}_2(\tilde{x}, \tilde{u}(\tilde{x})) - \tilde{\psi}_2(\tilde{x}, \tilde{u}(\tilde{x})},
\]
and it follows that
\[
\delta E_v(\ddot{u}; v) = \frac{\varepsilon_0}{2} \int_{-l}^l v(\dddot{x}) \left( (\dddot{\psi}_x(x, \dddot{u}(x)))^2 + (\dddot{\psi}_z(x, \dddot{u}(x)))^2 \right) d\dddot{x} \\
- \varepsilon_0 V \int_{-l}^l v(\dddot{x}) \left( \dddot{\psi}_x(\dddot{x}, \dddot{u}(\dddot{x})))p_x(\dddot{x}, \dddot{u}(\dddot{x})) + \dddot{\psi}_z(\dddot{x}, \dddot{u}(\dddot{x})p_z(\dddot{x}, \dddot{u}(\dddot{x})) \right) d\dddot{x}.
\]

Recalling (4) as well as the equality
\[
\delta E_{\rho}(\dddot{u}; v) = \delta \left( E_{\rho}(\dddot{u}; v) + E_{\varepsilon}(\dddot{u}; v) + E_v(\dddot{u}; v) \right) = 0
\]
as a necessary condition for \( \dddot{u} \) being a minimiser of the energy functional (2), we may see by (5), (6) and (15) that this is satisfied for all suitable functions \( v \), if and only if \( \dddot{u} \) complies with the Euler–Lagrange equation
\[
0 = \tau \partial \left( \frac{\dddot{u}_x}{\sqrt{1 + (\dddot{u}_x)^2}} \right) - b \partial \left( \frac{\dddot{u}_x}{(1 + (\dddot{u}_x)^2)^{3/2}} \right) - \frac{5b}{2} \partial \left( \frac{\dddot{u}_x(\dddot{u}_x)^2}{(1 + (\dddot{u}_x)^2)^{3/2}} \right) \\
\quad - \frac{\varepsilon_0}{2} \left( (\dddot{\psi}_x(\dddot{x}, \dddot{u}(\dddot{x})))^2 + (\dddot{\psi}_z(\dddot{x}, \dddot{u}(\dddot{x}))^2 \right) \\
\quad + \varepsilon_0 V \left( \dddot{\psi}_x(\dddot{x}, \dddot{u}(\dddot{x}))p_x(\dddot{x}, \dddot{u}(\dddot{x})) + \dddot{\psi}_z(\dddot{x}, \dddot{u}(\dddot{x}))p_z(\dddot{x}, \dddot{u}(\dddot{x})) \right).
\]

Heretofore, static deflections of the elastic plate are discussed and it remains to take the dynamics into account. This means that from now on the time variable \( \dddot{t} \) explicitly returns to the notation. More precisely, denoting by \( \rho \) the mass density per unit volume of the plate and recalling that \( \alpha \) denotes its thickness, due to Newton’s Second Law the sum of all forces is equal to \( \rho \alpha \dddot{u}(\dddot{t}, \dddot{x}) \) which is linearly proportional to the velocity \( \dddot{u}(\dddot{t}) \) with a positive damping constant \( a \). That is, we obtain
\[
\rho \alpha \dddot{u}(\dddot{t}, \dddot{x}) + a \dddot{u}(\dddot{t}) - \tau \partial \left( \frac{\dddot{u}_x}{\sqrt{1 + (\dddot{u}_x)^2}} \right) + b \partial \left( \frac{\dddot{u}_x}{(1 + (\dddot{u}_x)^2)^{3/2}} \right) \\
\quad + \frac{5b}{2} \partial \left( \frac{\dddot{u}_x(\dddot{u}_x)^2}{(1 + (\dddot{u}_x)^2)^{3/2}} \right) \\
\quad = - \frac{\varepsilon_0}{2} \left( (\dddot{\psi}_x(\dddot{x}, \dddot{u}(\dddot{x})))^2 + (\dddot{\psi}_z(\dddot{x}, \dddot{u}(\dddot{x}))^2 \right) \\
\quad + \varepsilon_0 V \left( \dddot{\psi}_x(\dddot{x}, \dddot{u}(\dddot{x}))p_x(\dddot{x}, \dddot{u}(\dddot{x})) + \dddot{\psi}_z(\dddot{x}, \dddot{u}(\dddot{x}))p_z(\dddot{x}, \dddot{u}(\dddot{x})) \right).
\]

Fusing the above considerations we end up with the following coupled system of partial differential equations. The elliptic free boundary value problem for the electrostatic potential in the region determined by the grounded plate at \( \dddot{z} = -h \) and the membrane at \( \dddot{z} = \bar{u}(\dddot{t}, \dddot{x}) \), both of length \( 2l \), reads
\[
\dddot{\psi}_{xx} + \dddot{\psi}_{zz} = 0, \quad \dddot{t} > 0, (\dddot{x}, \dddot{z}) \in \dddot{\Omega}(\dddot{u}),
\]
\[
\dddot{\psi}(\dddot{t}, \dddot{x}, \dddot{z}) = \frac{h + \dddot{z}}{h + \bar{u}(\dddot{t}, \dddot{x})} p(\dddot{x}, \dddot{u}(\dddot{t}, \dddot{x})), \quad \dddot{t} > 0, (\dddot{x}, \dddot{z}) \in \partial \dddot{\Omega}(\dddot{u}),
\]

where the conditions \( \dddot{\psi} = 0 \) and \( \dddot{\psi} = V p(\dddot{x}, \dddot{u}(\dddot{t}, \dddot{x})) \) on the ground plate and the membrane, respectively, are continuously extended to the lateral boundaries.
(±l, ̂z), ̂z ∈ (−h, 0). The dynamics of the deflection ̂u is thus described by the fourth-order equation

\[ \rho \alpha \ddot{u}_x + a \dddot{u}_x + A_1(\ddot{u}) = -\varepsilon_0 \left( \left( \ddot{\psi}_z (x, \ddot{u}_x) \right)^2 + \left( \dot{\psi}_z (x, \dot{u}_x) \right)^2 \right) \]

\[ + \varepsilon_0 V \left( \ddot{\psi}_z (x, \ddot{u}_x) p_x (x, \ddot{u}_x) + \dot{\psi}_z (x, \dot{u}_x) p_u (x, \dot{u}_x) \right), \]  

where ̂A_1(\ddot{u}) is the quasilinear fourth-order differential operator defined by

\[ ̂A_1(\ddot{u}) := -\tau \partial_x \left( \frac{\ddot{u}_x}{\sqrt{1 + (\ddot{u}_x)^2}} \right) + b \partial_x^2 \left( \frac{\ddot{u}_x}{(1 + (\ddot{u}_x)^2)^{5/2}} \right) + \frac{5b}{2} \partial_x^4 \left( \frac{\ddot{u}_x (\dddot{u}_x)^2}{(1 + (\ddot{u}_x)^2)^{7/2}} \right). \]

Furthermore, we assume the membrane to be clamped at its boundary (±l, 0) and to have a certain initial deflection ̂u_*(x̂) at time ̂t = 0. This is expressed by the clamped boundary conditions

\[ ̂u(̂t, ±l) = ̂u_x(̂t, ±l) = 0, \quad ̂t > 0, \]

and the initial conditions

\[ ̂u(0, x̂) = ̂u_*(x̂), \quad ̂u_x(0, x̂) = ̂u_*(x̂), \quad x̂ \in L, \]

respectively.

**Remark 1.** We briefly discuss two variants of the above modelling by distinguishing energy conserving and energy dissipating systems. Whereas the first occurs when damping effects are neglected, the latter corresponds to the case of no inertial effects.

1. We assume to be in an energy conserving Hamiltonian regime in which damping is not taken into account. The total energy of the system is defined as the pointwise difference of kinetic energy ̂E_k and potential energy ̂E_p. The kinetic energy at any instant in time is described by the functional

\[ ̂E_k(̂u) = \frac{\rho \alpha}{2} \int_{-l}^{l} (\ddot{u}_x)^2 d̂x. \]

Immediately taking dynamics into account, given 0 < t_1 < t_2 < ∞, Hamilton’s principle means to minimise the action of the system, i.e. the double integral

\[ \int_{t_1}^{t_2} \int_{-l}^{l} L(̂t, ̂x, ̂u) d̂x d̂u, \]

where the Lagrangian ̂L is now given by 4 ̂L : (t_1, t_2) × L × W_2^{2,4}((t_1, t_2) × L) → \mathbb{R},

\[ ̂L(̂t, ̂x, ̂u) = \frac{\rho \alpha}{2} \ddot{u}_x^2 - \tau \left( \sqrt{1 + (\ddot{u}_x)^2} - 1 \right) - \frac{b}{2} \frac{\ddot{u}_x^2}{(1 + (\ddot{u}_x)^2)^{5/2}} + \frac{\varepsilon_0}{2} \int_{-h}^{h} \left( \ddot{\psi}_z (x, ̂z) \right)^2 + \left( \dot{\psi}_z (x, ̂z) \right)^2 d̂z, \]

and the corresponding Euler–Lagrange equation, obtained by a straightforward adaption of the above calculations, reads

\[ \rho \alpha \ddot{u}_x + ̂A_1(̂u) = -\varepsilon_0 \left( \left( \ddot{\psi}_z (x, \ddot{u}_x) \right)^2 + \left( \dot{\psi}_z (x, \dot{u}_x) \right)^2 \right) \]

\[ + \varepsilon_0 V \left( \ddot{\psi}_z (x, \ddot{u}_x) p_x (x, \ddot{u}_x) + \dot{\psi}_z (x, \dot{u}_x) p_u (x, \dot{u}_x) \right), \]

4Note that W_2^{2,4}((t_1, t_2) × L) denotes the usual anisotropic Sobolev space with respect to ̂t and ̂x.
(2) Being in the energy dissipating regime where inertial effects are neglected, we shall see in the following that the corresponding evolution equation may formally be perceived as a gradient flow system.

(i) Let $H$ be a Hilbert space over $\mathbb{R}$ with inner product $(\cdot, \cdot)_H$ and let $E \in C(H, \mathbb{R})$ denote a continuous functional on $H$. Given $v \in H$, assume that

$$\delta E(v; w) := \frac{d}{d\sigma}E(v + \sigma w)|_{\sigma=0}$$

exists in $H$ for all $w \in H$. Under this hypothesis assume in addition that there is a $z(v) \in H$ such that

$$(z(v), w)_H = \delta E(v; w), \quad w \in H.$$ 

Note that $z(v)$ is uniquely determined if it exists. We call $z(v)$ the generalised gradient of $E$ at $v$ and use the notation $\nabla E(v) := z(v)$. If $E \in C^1(H, \mathbb{R})$ then $\nabla E(v)$ exists for all $v \in H$ with

$$DE(v)w = (\nabla E(v), w)_H, \quad w \in H.$$ 

(ii) Given $T > 0$, consider $v \in C^1((0, T), H)$ and assume that $\nabla E(v(t))$ exists in $H$ for all $t \in (0, T)$. If $v$ complies with the equation

$$v'(t) = -\nabla E(v(t)), \quad t \in (0, T), \quad (19)$$

then we say that $v$ is a solution to the gradient flow system associated to $E$ on $(0, T)$.

(iii) Suppose that $E$ belongs to $C^1(H, \mathbb{R})$ and $v \in C^1((0, T), H)$ is a solution to $(19)$ on $(0, T)$. Then $E(v(t))$ is decreasing on $(0, T)$. Indeed $E(v(\cdot))$ is differentiable on $(0, T)$ and the chain rule yields

$$\frac{d}{dt}E(v(t)) = (\nabla E(v(t)), v'(t))_H = -\|\nabla E(v(t))\|_H^2, \quad t \in (0, T). \quad (20)$$

Interpreting $E$ as an energy the last equation reveals the energy dissipation of the system. Moreover, if the path $v(t)$ avoids any critical point of $E$ the dissipation is strict.

(iv) Taking $H = L_2(L)$ and $E(\tilde{u}) = E_p(\tilde{u})$ with $\tilde{u} \in W_2^2(L)$ and $\tilde{u}(\pm l) = 0$ we deduce from $(5)$, $(6)$ and $(15)$ that formally

$$\nabla E_p(\tilde{u}) = -A_1(\tilde{u}) - \frac{e_n}{2} \left( (\tilde{\psi}_{\tilde{x}}(\tilde{x}, \tilde{u}(\tilde{x})))^2 + (\tilde{\psi}_{\tilde{x}}(\tilde{x}, \tilde{u}(\tilde{x})))^2 \right)$$

$$+ e_0 V \left( \tilde{\psi}_{\tilde{x}}(\tilde{x}, \tilde{u}(\tilde{x})) p_{\tilde{x}}(\tilde{x}, \tilde{u}(\tilde{x})) + \tilde{\psi}_{\tilde{x}}(\tilde{x}, \tilde{u}(\tilde{x})) p_{\tilde{x}}(\tilde{x}, \tilde{u}(\tilde{x})) \right).$$

This means that if $\rho = 0$ and $a = 1$ equation $(18)$ may be perceived as the gradient flow system associated to $E_p$ in $L_2(L)$.

**Remark 2.** Observe that the above reasoning is formal in the sense that several regularity properties are used which are not verified rigorously, e.g.

- the Gâteaux-differentiability of $E_p$ at $\tilde{u}$ in $(4)$;
- the differentiability of the path $\psi(\sigma; u); \sigma \in (-\sigma_0, \sigma_0)$ with respect to $\sigma$, c.f. $(9)$;
- the additional spatial regularity of $\psi$ used to derive $(10)$.

We skip these technicalities here but refer the reader to [26, Proposition 2.2] for a rigorous analysis of the regularity properties of the energy functional in the case $p \equiv 1$. 
Scaling – introduction of dimensionless variables. Now dimensionless variables are introduced and the above terms and equations are rewritten in dimensionless form. To that effect, the electrostatic potential is scaled with the applied voltage,

$$\psi = \frac{\psi}{V},$$

the time is scaled with a damping timescale of the system,

$$t = \frac{\tau}{a \ell^2 \tilde{t}},$$

and the variables $\tilde{x}$ and $\tilde{z}$ as well as $\tilde{u}$ are scaled with the length $l$ and the gap size $h$ of the undeflected configuration, respectively,

$$x = \frac{\tilde{x}}{l}, \quad z = \frac{\tilde{z}}{h}, \quad u = \frac{\tilde{u}}{h}. \tag{21}$$

Furthermore, the aspect ratio of the device is denoted by $\varepsilon = h/l$. The rescaled dimensionless problem for the electrostatic potential thus reads

$$\varepsilon^2 \psi_{xx} + \psi_{zz} = 0, \quad t > 0, \ (x, z) \in \Omega(u(t)),
\psi(t, x, z) = \frac{1 + z}{1 + u(t, x)} p(x, u(t, x)), \quad t > 0, \ (x, z) \in \partial \Omega(u(t)),$n

where the region $\Omega(u(t))$ is now given by

$$\Omega(u(t)) = \{(x, z) \in (-1, 1) \times (-1, \infty); -1 < z < u(t, x)\}.$$

In dimensionless form the evolution of the membrane’s deflection is specified by the equation

$$\rho \frac{\tau^2}{a^2 \ell^2} u_{tt} + \frac{h \tau}{l^2} u_t + A_1(u) = -\frac{\varepsilon_0 V^2}{2} \left( \frac{1}{l^2} (\psi_x(x, u(x)))^2 + \frac{1}{h^2} (\psi_z(x, u(x)))^2 \right)
+ \varepsilon_0 V^2 \left( \frac{1}{l^2} \psi_x(x, u(x)) p_x(x, u(x)) + \frac{1}{h^2} \psi_z(x, u(x)) p_u(x, u(x)) \right), \tag{22}$$

with $A_1(u)$ given by

$$A_1(u) = -\frac{\tau \varepsilon}{l} \partial_x \left( \frac{u_x}{\sqrt{1 + \varepsilon^2 (u_x)^2}} \right) + \frac{b \varepsilon}{l^2} \partial_x \left( \frac{u_{xx}}{(1 + \varepsilon^2 (u_x)^2)^{5/2}} \right)
+ \frac{5b \varepsilon^3}{2l^3} \partial_x \left( \frac{u_x (u_{xx})^2}{(1 + \varepsilon^2 (u_x)^2)^{7/2}} \right).$$

Multiplying (22) by $l^2/h \tau$ and using the definition of $\varepsilon$ then leads to the equation

$$\rho \frac{\tau^2}{a^2 \ell^2} u_{tt} + u_t + A(u) = -\frac{\varepsilon_0 V^2}{2} \varepsilon^2 \frac{1}{h^2 \tau} \left( \varepsilon^2 (\psi_x(x, u(x)))^2 + (\psi_z(x, u(x)))^2 \right)
+ \frac{\varepsilon_0 V^2}{\varepsilon^2 h \tau} \left( \varepsilon^2 \psi_x(x, u(x)) p_x(x, u(x)) + \psi_z(x, u(x)) p_u(x, u(x)) \right),$$

with the rescaled quasilinear fourth-order differential operator

$$A(u) := \frac{l^2}{h \tau} A_1(u) = -\partial_x \left( \frac{u_x}{\sqrt{1 + \varepsilon^2 (u_x)^2}} \right) + \frac{b}{l^2 \tau} \partial_x \left( \frac{u_{xx}}{(1 + \varepsilon^2 (u_x)^2)^{5/2}} \right)
+ \frac{5b \varepsilon^2}{2l^3 \tau} \partial_x \left( \frac{u_x (u_{xx})^2}{(1 + \varepsilon^2 (u_x)^2)^{7/2}} \right).$$
Lastly, by introduction of the parameters
\[ \gamma := \frac{\sqrt{\rho\alpha\tau}}{al}, \quad \beta := \frac{b}{\tau^2}, \quad \lambda = \lambda(\varepsilon) := \frac{\varepsilon_0 V^2}{2\varepsilon^2 b\tau}, \]
the deflection of the thin elastic plate in terms of nonlinear elasticity may be determined by the evolution equation
\[ \gamma^2 u_{tt} + u_t + A(u) = f_{\varepsilon, \lambda}(u) \]
with
\[ A(u) = -\partial_x \left( \frac{u_x}{\sqrt{1 + \varepsilon^2 (u_x)^2}} \right) + \beta \partial_x^2 \left( \frac{u_{xx}}{(1 + \varepsilon^2 (u_x)^2)^{3/2}} \right) + \frac{5}{2} \beta \varepsilon^2 \partial_x \left( \frac{u_x(u_{xx})^2}{(1 + \varepsilon^2 (u_x)^2)^{7/2}} \right), \]
the right-hand side
\[ f_{\varepsilon, \lambda}(u) := -\lambda \left( \varepsilon^2 (\psi_x(x, u(x))^2 + (\psi_z(x, u(x))^2) + 2\lambda \varepsilon^2 \psi_x(x, u(x))p_x(x, u(x)) + \psi_z(x, u(x))p_u(x, u(x)) \right) \]
and the according boundary and initial conditions
\[ u(t, \pm 1) = u_x(t, \pm 1) = 0, \quad t > 0, \]
\[ u(0, x) = u_*(x), \quad u_t(0, x) = u_{**}(x), \quad x \in (-1, 1). \]

Here, \( \gamma \) is the system’s quality factor\(^5\), \( \beta \) measures the relative importance of tension and rigidity and \( \lambda \) is a ratio of a reference electrostatic force to a reference elastic force. It is proportional to the square of the applied voltage and serves as a tuning parameter for the system.

2.2. A simplified linear elasticity model. In many engineering applications it is reasonable to only require the device to feature small membrane deflections and thus to restrict the mathematical investigations to a linear elasticity model. It is the purpose of this section to derive the analogon of the above model by means of linear elasticity theory.

Starting from the unscaled regime, in a first step, we assume \((\tilde{u}_x)^2\) to be small, i.e. \((\tilde{u}_x)^2 \ll 1\), and consider the first-order Taylor approximation \(\sqrt{1 + (\tilde{u}_x)^2} \approx 1 + (\tilde{u}_x)^2/2\) for small values of \(|u_x|\). The linearised stretching energy may then be written as
\[ E_s(\tilde{u}) = \frac{\tau}{2} \int_{-l}^l (\tilde{u}_x(\tilde{x}))^2 d\tilde{x}. \]
As before, given \(\sigma \in \mathbb{R}\) and a function \(v \in C^\infty_c(L)\), we introduce for \(\tilde{x} \in [-l, l]\) the variation \(w(\sigma)(\tilde{x}) = \tilde{u}(\tilde{x}) + \sigma v(\tilde{x})\) of \(\tilde{u}(\tilde{x})\) in the direction of \(v\). We then find that
\[ \frac{d}{d\sigma} E_s(\tilde{u} + \sigma v) = \frac{d}{d\sigma} \left( \frac{\tau}{2} \int_{-l}^l (w_x)^2 d\tilde{x} \right) = \tau \int_{-l}^l \tilde{u}_x v_x + \sigma (v_x)^2 d\tilde{x} \]
whence, using that \(v\) is compactly supported in \(L\),
\[ \delta E_s(\tilde{u}; v) = \tau \int_{-l}^l \tilde{u}_x v_x d\tilde{x} = -\tau \int_{-l}^l \tilde{u}_{xx} v d\tilde{x}. \]

\(^5\)Recall that \(\gamma = \sqrt{\rho\alpha\tau}/al\) is a measure for the damping of an oscillating system. Small values \(\gamma\) refer to strongly damped systems and thus indicate a large rate of decay of oscillations.
We proceed similarly in order to obtain the linearised version of the bending term. Again requiring \((\tilde{u}_z)^2 \ll 1\) to be small, we consider the Taylor series expansion
\[
\left( \partial_{\tilde{z}} \left( \frac{\tilde{u}_z}{\sqrt{1 + (\tilde{u}_z)^2}} \right) \right)^2 \sqrt{1 + (\tilde{u}_z)^2} = \frac{(\tilde{u}_{zz})^2}{(1 + (\tilde{u}_z)^2)^{3/2}} \approx (\tilde{u}_{zz})^2 + \ldots
\]
around zero, whence the linearised bending energy reads
\[
E_b(\tilde{u}) = \frac{b}{2} \int_{-l}^{l} (\tilde{u}_{zz}(\tilde{x}))^2 d\tilde{x}.
\]
Therefore, we find that
\[
\frac{d}{d\sigma} E_b(\tilde{u} + \sigma v) = \frac{d}{d\sigma} \left( \frac{b}{2} \int_{-l}^{l} (w_{zz})^2 d\tilde{x} \right) = b \int_{-l}^{l} \tilde{u}_{zz} v_{zz} + \sigma (v_{zz})^2 d\tilde{x}
\]
and thus finally
\[
\delta E_b(\tilde{u}; v) = b \int_{-l}^{l} \tilde{u}_{zz} v_{zz} d\tilde{x} = b \int_{-l}^{l} \tilde{u}_{zzzz} v d\tilde{x}.
\]

With the same scaling as above, the Euler–Lagrange equation in the regime of linear elasticity reads
\[
\gamma^2 u_{tt} + u_t - u_{xx} + \beta u_{xxxx} = -\lambda \left( c^2 (\psi_x(x,u(x)))^2 + (\psi_z(x,u(x)))^2 \right) + 2\lambda \left( \varepsilon^2 \psi_x(x,u(x)) p_x(x,u(x)) + \psi_z(x,u(x)) p_u(x,u(x)) \right)
\]
for \(t > 0\) and \(x \in (-1,1)\).

### 2.3. Simplified models

It is the purpose of the present section to give a brief overview of the different variants of the general nonlinear and linear elasticity models, respectively, which reflect different physical assumptions, as they are adequate for different applications. Even if there are more variants conceivable, the presented elaboration is restricted to those models which are considered more detailed subsequently.

To this end, denoting by \(u = u(t,x), \ t > 0, \ x \in I := (-1,1)\), the membrane’s deformation, the elliptic problem governing the electrostatic potential of the system at any instant \(t \geq 0\) of time always reads
\[
\psi_{xx} + \psi_{zz} = 0, \quad t > 0, \ (x,z) \in \Omega(u), \quad (24)
\]
\[
\psi(t,x,z) = \frac{1 + z}{1 + u(t,x)} p(x,u(t,x)), \quad t > 0, \ (x,z) \in \partial \Omega(u), \quad (25)
\]
where the region \(\Omega(u(t))\) between the rigid ground plate at \(z = -1\) and the elastic membrane at \(z = u(t,x)\) at any instant of time is given by
\[
\Omega(u) = \{(x,z) \in (-1,1) \times (-1,\infty); \ -1 < z < u(t,x)\}.
\]

Independent of the choice of the evolution equation for the membrane’s displacement, only the permittivity profile might vary, being either a function \(p = p(x), \ p = p(u(t,x))\) or \(p = p(x,u(t,x))\). In the above elliptic moving boundary value problem this does only influence the boundary condition accordingly.

The situation is more involved for the choice of an appropriate model describing the dynamics of the thin elastic plate’s displacement. Within the scope of the linear as well as the nonlinear elasticity approach we now make two physical assumptions...
which have significant effects on the mathematical classification of the resulting equations.

First of all, we restrict the further investigations to \textit{viscosity-dominated systems}, i.e. to a setting in which damping effects dominate over inertial effects. More precisely, this means that the parameter \( \gamma = \sqrt{\rho\alpha T/\omega l} \) appearing in front of the inertial term is assumed to be very small, whence we ignore the inertial term \( \gamma^2 u_{tt} \) in the equations. Note that the highest-order time derivative thus appears as the damping term \( u_t \) which is of first order. This restriction is of course not relevant for all possible MEMS models but for instance for those describing the dynamic behaviour of micro pumps [6] or micro grippers [39].

One may furthermore act on the assumption that membranes or infinitely thin plates do not resist bending, i.e. they have no flexural rigidity. This is the case if

\[
b = \frac{2\alpha^3 Y}{3(1 - \nu)^2} = 0 \quad \Rightarrow \quad \beta = \frac{b}{\tau l} = 0,
\]

and thus the spatial higher-order terms

\[
\beta \partial_x^2 \left( \frac{u_{xx}}{(1 + \varepsilon^2 (u_x)^2)^{5/2}} \right) + \frac{5}{2} \beta \varepsilon^2 \partial_x \left( \frac{u_x (u_{xx})^2}{(1 + \varepsilon^2 (u_x)^2)^{7/2}} \right) \quad \text{or} \quad \beta u_{xxxx},
\]

in the nonlinear or linear elasticity regime, respectively, are eliminated. Deformations due to bending are thus neglected, which means that the governing equations are reduced from spatially fourth-order to spatially second-order equations. Combining the above two physical assumptions we end up with a model influenced by stretching, damping and electrostatic forces.

It finally remains to take different varying dielectric permittivity profiles into account. This profile is modeled as a function depending either on the spatial variable \( x \in I \), the membrane’s displacement \( u = u(t, x) \), or even both.

For the most general of these cases, i.e. when \( p = p(x, u) \), the right-hand side of the evolution equation is given by

\[
f_{\varepsilon, \lambda}(u) := - \lambda \left( \varepsilon^2 (\psi_x(x, u))^2 + (\psi_z(x, u))^2 \right) \\
+ 2\lambda (\varepsilon^2 \psi_x(x, u)p_x(x, u) + \psi_z(x, u)p_u(x, u)),
\]

(26)

\( p_x \) and \( p_u \) denoting the partial derivatives of \( p \) with respect to its first and second variable, respectively.

Reviewing the above considerations as a whole, we end up with the quasilinear parabolic initial-boundary value problem

\[
u_t - \partial_x \left( \frac{u_x}{\sqrt{1 + \varepsilon^2 (u_x)^2}} \right) = f_{\varepsilon, \lambda}(u) \quad t > 0, \ x \in I,
\]

(27)

\[
u(t, \pm 1) = 0, \quad t > 0,
\]

(28)

\[
u(0, x) = u_*(x), \quad x \in I,
\]

(29)

in the regime of nonlinear elasticity, whereby in the linear elasticity setting the analogue problem is a semilinear parabolic initial-boundary value problem which reads

\[
u_t - u_{xx} = f_{\varepsilon, \lambda}(u) \quad t > 0, \ x \in I,
\]

(30)

\[
u(t, \pm 1) = 0, \quad t > 0,
\]

(31)

\[
u(0, x) = u_*(x), \quad x \in I,
\]

(32)
with a right-hand side (26) according to the choice of the permittivity profile \( p \).

3. **Existence and uniqueness results.** As a first aspect in the mathematical analysis of the coupled systems derived in Section 2 we address the questions of existence and uniqueness of solutions. Both when the membrane’s displacement is determined in the semilinear regime (30)–(32) as well as when it is described by the quasilinear problem (27)–(29), it turns out that the answers to those questions strongly depend on the applied voltage. More precisely, we show that the systems possess locally in time existing unique solutions for all arbitrarily large values \( \lambda \) of the applied voltage, and that solutions exist even globally in time, provided that the applied voltage does not exceed a certain critical value \( \lambda_* \).

We consider the elliptic moving boundary value problem
\[
\varepsilon^2 \psi_{xx} + \psi_{zz} = 0, \quad t > 0, \quad (x, z) \in \Omega(u(t)),
\]
\[
\psi(t, x, z) = \frac{1 + z}{1 + u(t, x)} p, \quad t > 0, \quad (x, z) \in \partial \Omega(u(t)),
\]
coupled with an either semi- or quasilinear initial boundary value problem for the displacement \( u \) of the membrane. In the sequel this evolution problem for \( u \) is rewritten as the general abstract parameter-dependent Cauchy problem
\[
u_t + A(u)u = f_{\varepsilon, \lambda}(u), \quad t > 0,
\]
\[
u(0) = u_*,
\]
where for a given \( v \in W^{2}_q(I) \) the differential operator \( A(v) \in \mathcal{L}(W^{2}_q(I), L_q(I)) \), \( q > 2 \), is defined as
\[
A(v)u := -\frac{u_{xx}}{1 + \varepsilon^2 v_x^2}^{3/2}, \quad u \in W^{2}_q(I),
\]
in the quasilinear case arising from the nonlinear elasticity theory, whereas for the semilinear case, arising from a linear elasticity approach, we set \( A(v) \equiv A(0) \) for all \( v \in S_q(\kappa) \) and obtain
\[
A(0)u := -u_{xx}, \quad u \in W^{2}_q(I).
\]
The exact structure of the right-hand side \( f_{\varepsilon, \lambda}(u) \) is then determined by the choice of the permittivity profile \( p \).

**Theorem 3.1** (Local Well-Posedness, [32, 34]). Let \( q \in (2, \infty) \), \( \varepsilon > 0 \), \( \lambda > 0 \), \( p \in C^3([-1, 1] \times \mathbb{R}, \mathbb{R}) \), and an initial value \( u_* \in W^{2}_q(I) \) be given such that \( u_*(\pm 1) = 0 \) and \(-1 < u_*(x) \) for \( x \in I \). Then there is a unique \( T > 0 \) and a unique non-extendable solution \( (u, \psi) \) to (33)–(36). This means that \( u \) is unique in the class
\[
C^1([0, T), L_q(I)) \cap C([0, T), W^{2}_q(I)).
\]
satisfying (35)–(36) together with
\[
u(t, x) > -1, \quad (t, x) \in [0, T) \times I,
\]
and \( \psi(t) \in W^{2}_q(\Omega(u(t))) \) uniquely solves (33)–(34) for each \( t \in [0, T) \).

Results on local and global well-posedness of (33)–(36) with a constant permittivity profile \( p \equiv 1 \) have recently been established in [9, 10].

**Theorem 3.2** (Global Existence, [32, 34]). Let \( q \in (2, \infty), \varepsilon > 0 \) and \( \lambda > 0 \). Furthermore, given \( p \in C^3([-1, 1] \times \mathbb{R}, \mathbb{R}) \) and \( u_* \in W^{2}_q(I) \) satisfying \( u_*(\pm 1) = 0 \), \( u_*(x) \geq -1 + \kappa \) for some \( \kappa \in (0, 1) \) and \( x \in I \), let \( (u, \psi) \) denote the corresponding solution to (33)–(36) on the maximal interval \([0, T)\) of existence. Then there exist
\[ \lambda_* = \lambda_*(\kappa) > 0 \text{ and } c = c(\kappa) > 0 \text{ such that } T = \infty \text{ with } u(t, x) \geq -1 + \kappa \text{ for all } t \in [0, \infty) \text{ and } x \in I, \text{ provided that } \lambda \in (0, \lambda_*) \text{ and } \|u_*\|_{W^2_0(I)} \leq c(\kappa). \text{ In that case } u \text{ is bounded in the sense that } u \in BC([0, \infty), W^2_q(I)). \]

It is worthwhile to mention that temporally global solutions \( u \) do never touch down on the ground plate, not even in infinite time.

4. The small-aspect ratio limit. As mentioned in the introduction the analysis of coupled systems of partial differential equations has only recently become part of the mathematical investigation of microelectromechanical systems. Irrespective of the precise physical regime for an adequate choice of the governing equations for \( u \) the displacement of the elastic membrane causes a change of the shape of the domain \( \Omega(u(t)) \) occupied by the ground plate and the overlying membrane. This gives rise to a coupling between the problem for the electrostatic potential \( \psi \) and the membrane’s displacement \( u \). Roughly speaking, this coupling between the two problems makes their mathematical analysis rather complex, whereby it has heretofore been and it still is a quite common approach in MEMS research to make an assumption which reduces the initial nonlocal coupled problem to an uncoupled semilinear evolution equation for \( u \).

For pioneering contributions to the understanding of the full coupled problem the reader is again referred to the works [24] and [9]. It is the intention of this chapter to generalise a convergence result on the small-aspect ratio limit obtained in [9] for \( p \equiv 1 \) to the case of a general permittivity profile \( p = p(x, u(t, x)) \). To this end, consider the system

\[ \begin{align*}
  u_t - u_{xx} &= -\lambda \left( \epsilon^2 (\psi_x(x, u))^2 + (\psi_z(x, u))^2 \right) \\
  &\quad + 2\lambda \left( \epsilon^2 \psi_x(x, u)p_x(x, u) + \psi_z(x, u)p_u(x, u) \right), \quad t > 0, \quad x \in I, \quad (37) \\
  u(t, \pm 1) &= 0, \quad t > 0, \\
  u(0, x) &= u_*(x), \quad x \in I, \quad (38)
\end{align*} \]

arising from the linear elasticity approach for the membrane’s displacement, together with the elliptic moving boundary problem

\[ \begin{align*}
  \epsilon^2 \psi_{xx} + \psi_{zz} &= 0, \quad (x, z) \in \Omega(u(t)), \\
  \psi(t, x, z) &= \frac{1 + z}{1 + u(t, x)} p(x, u(t, x)), \quad (x, z) \in \partial\Omega(u(t)), \quad (40)
\end{align*} \]

for the electrostatic potential in the region

\[ \Omega(u(t)) := \{(x, z) \in I \times (-1, \infty); -1 < z < u(t, x)\}. \]

A common approach in order to decouple the problems (37)–(39) and (40)–(41) is to consider the aspect ratio \( \epsilon \) of the respective MEMS device to be fairly small, i.e. \( \epsilon \ll 1 \), or in fact even \( \epsilon = 0 \). In this case the potential is computed as if the two plates were locally parallel and the resulting explicit expression for \( \psi \) avoids the coupling via the right-hand side of (37). The full coupled problem is then reduced to a semilinear parabolic initial boundary value problem possessing a singularity in the instant the elastic membrane touches down on the ground plate. In detail, setting \( \epsilon = 0 \) in (40) yields the reduced problem

\[ \psi_{zz}(t, x, z) = 0, \quad t > 0, \quad (x, z) \in \Omega(u(t)), \quad (42) \]
\[ \psi(t, x, z) = \frac{1 + z}{1 + u(t, x)} p(x, u(t, x)), \quad t > 0, (x, z) \in \partial \Omega(u(t)), \quad (43) \]

for the electrostatic potential whose solution \( \psi := \psi_0 \) may be explicitly stated as

\[ \psi(t, x, z) = \frac{1 + z}{1 + u(t, x)} p(x, u(t, x)), \quad t > 0, (x, z) \in I \times (-1, 0). \quad (44) \]

Inserting the likewise computable partial derivative

\[ \psi_z(t, x, z) = \frac{p(x, u(t, x))}{1 + u(t, x)} \]

into the evolution equation for the membrane’s displacement in the case \( \varepsilon = 0 \), i.e. into the equation

\[ u_t - u_{xx} = -\lambda (\psi_z(t, x, u))^2 + 2\lambda \psi_z(t, x, u) p_u(x, u), \quad t > 0, x \in I, \quad (45) \]

implies that the displacement \( u := u_0 \) finally satisfies the so-called small-aspect ratio model

\[ u_t - u_{xx} = -\lambda \left( \frac{p(x, u(t, x))}{1 + u(t, x)} \right)^2 + 2\lambda \frac{p(x, u(t, x))}{1 + u(t, x)} p_u(x, u(t, x)), \quad t > 0, x \in I, \quad (46) \]

\[ u(t, \pm 1) = 0, \quad t > 0, \quad (47) \]

\[ u(0, x) = u_*(x), \quad x \in I. \quad (48) \]

Problem (46)–(48) is a reduced model for the elastic behaviour of the system. It is uncoupled from the potential equation and may be solved independently. However, note that the evolution equation (46) is still nonlinear.

Denoting for \( \varepsilon > 0 \) the solution to (37)–(41) by \((u_\varepsilon, \psi_\varepsilon)\), one may prove that as \( \varepsilon \) tends to zero, the corresponding sequence \((u_\varepsilon, \psi_\varepsilon)\) converges in a certain sense to the solution \((u_0, \psi_0)\) of the small-aspect ratio model (46)–(48) with \( \psi_0 \) given in (44).

**Theorem 4.1** (Small- Aspect Ratio Limit, [32, Theorem 4.1]). Let \( \lambda > 0, q \in (2, \infty), p \in C^3([-1, 1] \times \mathbb{R}, \mathbb{R}) \), and let \( u_* \in W^{2, q}_q(I) \) with \( u_*(\pm 1) = 0 \) and \( u_*(x) > -1 \). For \( \varepsilon > 0 \), let \((u_\varepsilon, \psi_\varepsilon)\) be the unique solution to (37)–(41) on the maximal interval \([0,T)\) of existence. Then there are \( \tau > 0 \) and \( \varepsilon_* \in (0, 1) \) such that \( T \geq \tau \) for all \( \varepsilon \in (0, \varepsilon_*) \). Moreover, the small-aspect ratio problem (46)–(48) has a unique solution

\[ u_0 \in C^4([0, \tau], L^q_q(I)) \cap C([0, \tau], W^{2,q}_q(I)), \quad (49) \]

satisfying \( u_0(t, x) > -1 \) for all \( t \in [0, \tau] \), \( x \in [-1, 1] \), and such that the convergences

\[ u_\varepsilon \longrightarrow u_0 \quad \text{in} \quad C^{1-\theta}([0, \tau], W^{2,q}_q(I)), \quad \theta \in (0,1), \quad (50) \]

and

\[ \psi_\varepsilon(t) \chi_{\Omega(u_\varepsilon(t))} \longrightarrow \psi_0(t) \chi_{\Omega(u_0(t))} \quad \text{in} \quad L^2(I \times (-1, 0), \mathbb{R}), \quad t \in [0, \tau], \quad (51) \]

hold true as \( \varepsilon \to 0 \). Here, \( \psi_0 \) is the potential given in (44). Furthermore, there exists a \( \Lambda > 0 \) such that the above results hold true for each \( \tau > 0 \) provided that \( \lambda \in (0, \Lambda) \).

The result for \( p \equiv 1 \) may be found in [9].
5. Qualitative properties of solutions. In the previous parts of this survey we have seen different mathematical models for the characterisation of the dynamic behaviour of MEMS devices. In addition to choosing an either linear or nonlinear elasticity approach, we have in particular distinguished between the small-aspect ratio model and the full problem, coupling the moving boundary problem for the potential $\psi$ with an either semi- or quasilinear evolution problem for the membrane’s displacement $u$. Moreover, different permittivity profiles $p$ give rise to different equations and might thus have a certain influence on the qualitative behaviour of solutions. It turns out, that there exist indeed qualitative differences of the solutions to the different systems and that these differences become apparent not till non-constant permittivity profiles are taken into account.

This section is divided into two parts. The first one is devoted to sign-properties of the solution $u$ to the evolution problem for the displacement of the elastic membrane. It deals with the question, if the membrane always deflects towards the ground plate or if other scenarios, such as a sign-changing or a positive deflection, are possible. The second part is concerned with the phenomenon of the so-called pull-in instability, i.e. with the situation in which the pull-in voltage exceeds a certain critical value and thus causes a singularity of the solution after finite time.

5.1. Non-positivity of the membrane’s displacement. Since parabolic comparison principles are available for both settings the semilinear as well as the quasilinear evolution problem (see i.e. [15, 38, 23]) we do not explicitly distinguish between these two cases. In the regime of a positive aspect ratio $\varepsilon > 0$ we thus consider the moving boundary problem

$$\varepsilon^2 \psi_{xx} + \psi_{zz} = 0, \quad t > 0, \quad (x, z) \in \Omega(u(t)),$$

$$\psi(t, x, z) = \frac{1 + z}{1 + u(t, x)} p, \quad t > 0, \quad (x, z) \in \partial \Omega(u(t)),$$

coupled with the either semi- or quasilinear parameter-dependent Cauchy problem

$$u_t + A(u)u = f_{\varepsilon, \lambda}(u), \quad t > 0,$$

$$u(0) = u_*,$$

where for a given $v \in W^q_0(I)$ the differential operator $A(v) \in \mathcal{L}(W^2_q(I), L^q(I))$, $q > 2$, is introduced in Section 3. The exact structure of the right-hand side $f_{\varepsilon, \lambda}(u)$ is determined by the choice of the permittivity profile $p$. Almost the same notation is used for the small-aspect ratio model, i.e. in the situation of a formally vanishing aspect ratio $\varepsilon = 0$. As we have seen in the previous section, given an explicit expression for the potential $\psi$, the small-aspect ratio model may be rewritten as

$$u_t + A(u)u = f_{0, \lambda}(u), \quad t > 0,$$

$$u(0) = u_*.$$

In the same way as for $\varepsilon > 0$ the structure of the right-hand side $f_{0, \lambda}$ might vary, depending on the choice of the function $p$. In order to be able to reveal sign-properties of the according solutions by means of the parabolic comparison principle, the challenge is thus to investigate the respective right-hand side $f_{\varepsilon, \lambda}(u)$ or $f_{0, \lambda}(u)$ of the evolution equation regarding its sign. It is worthwhile to explicitly mention again, that this challenge strongly depends on the choice of the permittivity profile $p$. More precisely, in the case of a constant permittivity profile $p \equiv 1$ the evolution
equation (52) reads
\[ u_t + A(u)u = -\lambda \left( \varepsilon^2 (\psi_x(x,u))^2 + (\psi_z(x,u))^2 \right), \quad t > 0. \] (56)

With \( \psi(t, x, z) = (1+z)/(1+u(t, x)) \) for \( t > 0, \ (x, z) \in I \times (-1, 0) \), the corresponding small-aspect ratio equation (54) is given by
\[ u_t + A(u)u = -\frac{\lambda}{(1+u)^2}, \quad t > 0. \] (57)

It may be readily deduced from the parabolic comparison principle that, given a non-positive initial value \( u_* \leq 0 \), both equations (56) and (57) always provide non-positive solutions \( u \). In other words, a constant permittivity profile \( p \equiv 1 \) immediately implies that the membrane always deflects towards the ground plate.

The situation is rather different in the case of a spatially varying permittivity profile \( p = p(x) \). Denoting by \( p'(x) \) the derivative of \( p \) with respect to \( x \), the evolution equation (52) is given by
\[ u_t + A(u)u = -\lambda \left( \varepsilon^2 (\psi_x(x,u))^2 + (\psi_z(x,u))^2 - 2\varepsilon^2 \psi_x(x,u)p'(x) \right), \quad t > 0, \] (58)

whereas the according small-aspect ratio equation (54) for a computed \( \psi(t, x, z) = p(x)(1+z)/(1+u) \), \( t > 0, \ (x, z) \in I \times (-1, 0) \), reads
\[ u_t + A(u)u = -\lambda \left( \frac{p(x)}{1+u} \right)^2, \quad t > 0. \] (59)

Invoking again the parabolic maximum principle, one may observe that for \( p = p(x) \) the small-aspect ratio model always possesses non-positive solutions, provided that the initial value \( u_* \) is non-positive. On the other hand, due to the additional term \( 2\varepsilon^2 \psi_x(x,u)p'(x) \) in (58), this is not at all clear for the coupled problem. Although the initial deflection \( u_* \) is non-positive, after a certain time the deflection might become positive or change its sign.

In the setting where \( p \) depends only on the deformation \( u \) of the membrane, i.e. when \( p = p(u) \) the evolution equations (52) reads
\[ u_t + A(u)u = -\lambda \left( \varepsilon^2 (\psi_x(x,u))^2 + (\psi_z(x,u))^2 - 2\varepsilon^2 \psi_x(x,u)p'(u) \right), \quad t > 0, \] (60)

with \( p'(u) \) denoting the derivative of \( p \) with respect to \( u \). With \( \psi(t, x, z) = p(u)(1+z)/(1+u) \), \( t > 0, \ (x, z) \in I \times (-1, 0) \), the associated small-aspect ratio equation is given by
\[ u_t + A(u)u = -\lambda \left( \left( \frac{p(u)}{1+u} \right)^2 - 2\frac{p(u)}{1+u} p'(u) \right), \quad t > 0. \] (61)

One may thus observe that in the case \( p = p(u) \) neither in the coupled setting nor in the small-aspect ratio regime an immediate statement about the sign of the solution \( u \) is possible. Additional information on the potential \( \psi \) and on the permittivity profile \( p \) is necessary in order to deduce a statement from the comparison principle.

The situation is similar when the permittivity profile \( p \) depends on both \( x \) and \( u \). The equation (52) is then given by
\[ u_t + A(u)u = -\lambda \left( \varepsilon^2 (\psi_x(x,u))^2 + (\psi_z(x,u))^2 ight. \\
\left. + 2\varepsilon^2 \psi_x(x,u)p_x(x,u) + \psi_z(x,u)p_u(x,u) \right), \quad t > 0, \] (62)
whereas the small-aspect ratio equation reads
\[ u_t + A(u)u = -\lambda \left( \left( \frac{p(x, u)}{1 + u} \right)^2 - 2 \frac{p(x, u)}{1 + u} p_u(x, u) \right), \quad t > 0. \] (63)

The corresponding results are published in [33] for the case of a spatially varying permittivity \( p = p(x) \), in [11] for the case in which \( p \) depends on the membrane’s displacement, and in [34] for \( p = p(x, u) \). In this survey we only state the result on the most general setting \( p = p(x, u) \) and discuss the other cases briefly in the subsequent remark.

**Theorem 5.1** (Non-Positivity of \( u \), [34]). Let \( p \in C^1([-1,1] \times \mathbb{R}, \mathbb{R}) \) be positive and assume that the boundary conditions
\[ \psi_{zz}(x, -1) \geq 0 \quad \text{and} \quad \psi_{zz}(x, v(x)) \geq 0, \quad x \in I, \] (64)
hold true for the solution \( \psi \) to (50)–(51). Then, if
\[ 0 < \varepsilon^2 \leq \min_{x \in [-1,1]} \frac{(p(x, r))^2 - 4(p_x(x, r))^2}{2(p_x(x, r))^2}, \] (65)
and \( u_*(x) \leq 0, \ x \in I \), the unique solution \( u \) to (52)–(53) satisfies
\[ u(t, x) \leq 0, \quad (t, x) \in [0, T) \times I. \]

**Remark 3.** It is worthwhile to briefly discuss the main assumptions of Theorem 5.1.

1. Clearly (64) is a strong condition as it is to be satisfied a priori by the electrostatic potential \( \psi \) as a part of the solution. On the other hand this condition holds true in the reduced small-aspect ratio model as \( \psi \) is there affine in the \( z \)-variable. In contrast to the small-aspect ratio model and the case \( p \equiv 1 \) in the coupled system, both providing inherently non-positive deflections, (64) is hitherto the only existing attempt to recover the fact that in general not all solutions deflect towards the ground plate.

2. We also discuss the condition (65) on the aspect ratio \( \varepsilon \) by considering different permittivity profiles.
   (i) If \( p = p(x) \), then (65) reduces to
   \[ 0 < \varepsilon^2 \leq \min_{x \in [-1,1]} \frac{p(x)}{\sqrt{2|p'(x)|}}. \]
   Thus, if \( p \) satisfies (65), then the very same is true for \( p(x) := p(x) + c \) with any positive constant \( c \).
   (ii) If \( p \) depends only on the membrane’s displacement, then (65) formally implies that the numerator of the right-hand side has to be positive and hence (65) modifies to
   \[ \min_{r \in [-1,0]} p(r) > \min_{r \in [-1,0]} 2|p'(r)|. \]
   Apparently this condition does not depend on \( \varepsilon > 0 \). Examples for permittivity profiles complying with (65) are then (c.f. [12]) \( p(r) := 3 - r \) and \( p(r) := 1 + r^2/4 \).
   (iii) If \( p = p(x, u) \) depends on both the spatial variable \( x \) and the membrane’s displacement, then the fact that \( p = p(x, r) \) complies with (65) implies that also \( p(x, r) := p(x, r) + c \) complies with (65).
5.2. Finite-time singularities. Depending on the individual application of the MEMS-based device it might be either an explicitly desired effect to apply a voltage value that leads to a touchdown of the membrane on the ground plate, or, in contrast, the contact of the two plates could damage the device. The understanding of this touchdown behaviour is one of the major objectives in the mathematical investigation of MEMS-based devices. We know from Section 3 that in the semilinear as well as in the quasilinear setting there exists a critical value $\lambda^* > 0$ such that the unique solution $(u, \psi)$ to the coupled problem exists forever, provided that the applied voltage $\lambda > 0$ is smaller than $\lambda^*$. In this case we have uniform bounds on $u$ in the $W^2_q(I)$-norm and the membrane does never touch down on the ground plate, not even in infinite time. Contrariwise, there is another critical value $\lambda^* \geq \lambda^*$ such that the solution $u$ ceases to exists after a finite time $T$ of existence, provided that $\lambda > \lambda^*$, $\varepsilon$ is small enough and $u$ is non-positive. In this case the membrane’s displacement develops a singularity in the sense that one of the following two phenomena may be observed. Either the membrane touches down on the ground plate, i.e.

$$\lim \inf_{t \to T} \min_{x \in [-1,1]} u(t, x) = -1,$$

or it becomes unbounded in the $W^2_q(I)$-norm, i.e.

$$\lim \sup_{t \to T} \|u(t)\|_{W^2_q(I)} = \infty.$$

For a constant permittivity profile $p \equiv 1$ this is shown in [9]. For non-constant permittivity profiles the current literature covers the following settings. Spatially varying permittivity profiles $p = p(x)$, $x \in I$, for both, the semilinear and the quasilinear case are treated in [33] and [12], respectively.

**Theorem 5.2** (Finite-Time Singularities; $p = p(x)$; semilinear case; [33]). Let $p \in C^2([-1,1])$ be positive with $p(-1) = p(1)$ and denote by $(u, \psi)$ the unique solution to (50)-(53) on the maximal interval $[0, T)$ of existence. Assume in addition that

$$u(t, x) \leq 0, \quad (t, x) \in [0, T) \times I.$$

Then there is $\lambda^* > 0$ such that $T < \infty$, provided that $\lambda > \lambda^*$ and $\varepsilon \in (0, 1/\sqrt{\lambda}]$. That is, either (66) or (67) takes place.

**Theorem 5.3** (Finite-Time Singularity; $p = p(x)$; quasilinear case; [12]). Let $\varepsilon > 0$ and $\lambda > 0$. Moreover, given a positive $p \in C^1([-1,1])$, denote by $(u, \psi)$ the unique solution to (50)-(53) on the maximal interval $[0, T)$ of existence and assume that the following conditions hold true:

$$(A_1) \quad \max_{x \in [-1,1]} p(x) < \sqrt{2}\min_{x \in [-1,1]} p(x);$$

$$(A_2) \quad \min_{x \in [-1,1]} p(x) = p(-1) = p(1);$$

$$(A_3) \quad u(t, x) \leq 0, \quad (t, x) \in [0, T) \times I.$$

Then there exists $\varepsilon^* > 0$ and $\lambda^* = \lambda^*(\varepsilon^*) > 0$ such $T < \infty$, provided that $\varepsilon \in (0, \varepsilon^*)$ and $\lambda > \lambda^*$. In this case either (66) or (67) takes place.

Permittivity profiles $p = p(u)$ are treated for the quasilinear case in [11].

**Theorem 5.4** (Finite-Time Singularity; $p = p(u(t, x))$; quasilinear case; [11]). Let $\varepsilon > 0$ and $\lambda > 0$. Moreover, given a positive $p \in C^2([-1,0])$, denote by $(u, \psi)$ the unique solution to (50)-(53) on the maximal interval $[0, T)$ of existence and assume that the following conditions hold true:

$$(A_1) \quad \max_{x \in [-1,0]} p(r) < \sqrt{2}\min_{r \in [-1,0]} p(r);$$
\( (A_2) \) \( p'(r) \leq 0, \quad r \in [-1, 0], \)
\( (A_3) \) \( u(t, x) \leq 0, \quad (t, x) \in [0, T] \times I; \)
\( (A_4) \) \( \psi_{zz}(x, -1) \geq 0 \quad \text{and} \quad \psi_{zz}(x, v(x)) \geq 0, \quad x \in I. \)

Then we have \( T < \infty. \) More precisely, there exists \( \lambda^* > 0 \) such that either (66) or (67) for all \( \lambda > \lambda^* \).

**Remark 4.** It is worthwhile to compare the assumptions of Theorem 5.4 with those of Theorem 5.3, where the case \( p = p(x) \) is treated.

1. **Theorem 5.4** holds true for any \( \varepsilon > 0 \) (provided that \( \lambda \) is accordingly large enough). This is in contrast to Theorem 5.3, where we have to assume that \( \varepsilon > 0 \) is small (and \( \lambda \) accordingly large enough).
2. Note that the condition \( (A_4) \) on \( \psi_{zz} \) to be non-negative on the membrane and on the ground plate is crucial in order to prove non-positivity of the membrane’s displacement \( u(t, x) \). But, moreover, in the case where the permittivity profile \( p \) depends on \( u \), this assumption on \( \psi_{zz} \) is also necessary in order to verify the occurrence of finite-time singularities, even if we already know that \( u(t, x) \leq 0 \), c.f. the proof of [11, Theorem 3.4].

6. **Numerical results.** In this final section we report on some numerical results obtained in [7] for the semilinear models with a spatially varying profile \( p = p(x) \). These results indicate a significant difference in the qualitative behaviour of solutions to the coupled model and the small-aspect ratio model, respectively. In fact, for fairly small \( \varepsilon > 0 \) positive deformations \( u \) of the membrane can be observed for solutions emerging from the initial condition \( u_* \equiv 0 \) of no deflection. In contrast to that any solution \( u_0 \) to the small-aspect ratio model is non-positive as a consequence of the comparison principle, c.f. (58) and (59). Note that this phenomenon may not be observed when assuming a constant permittivity profile.

In the following the results of [7] for the data
\[
p(x) := x^8 + 0.1 \quad \text{and} \quad u_*(x) = 0, \quad x \in [-1, 1],
\]
are illustrated.

Figures 3–5 show the approximate solution to (52)–(53) at different levels of time for a decreasing aspect ratio \( \varepsilon \) and fixed voltage \( \lambda = 1 \). The curves represent the approximate membrane’s displacement at every tenth time step, to be read from bottom up.

For \( \varepsilon \in \{0.4, 0.6\} \) the membrane’s deflection instantaneously becomes positive at all interior points \( x \in I \) and it increases in time as to be observed in Figure 3.

Figures 4 and 5, considering \( \varepsilon \in \{0.1, 0.15, 0.2\} \), reveal a different behaviour. While for \( \varepsilon = 0.2 \) the membrane’s deflection is still increasing in time, although developing initially also negative values, it becomes strictly positive at all interior points after a certain time, c.f. Figure 4(a). Contrariwise, the aspect ratio \( \varepsilon = 0.1 \) leads to a monotonically decreasing displacement of the membrane. It becomes instantaneously negative at all interior points \( x \in I \), see Figure 4(b).

The intermediate value \( \varepsilon = 0.15 \) also seems to be remarkable.

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6The small-aspect ratio model is able to capture various qualitative properties of the coupled system, such as spatial symmetry, the existence of a critical pull-in voltage, as well as global existence for small values of the applied voltage.

7For the numerical computations in [7] the final time is \( T = 1 \) and the time increment is \( 1/100 \).
Figure 3. Membrane’s deflection $u$ of the coupled system for $p(x) = x^8 + 0.1$ with $u_* \equiv 0$, $\lambda = 1$, and $\epsilon \in \{0.4, 0.6\}$.

Figure 4. Membrane’s deflection $u$ of the coupled system for $p(x) = x^8 + 0.1$ with $u_* \equiv 0$, $\lambda = 1$, and $\epsilon \in \{0.1, 0.2\}$.

The approximate solution at time $t = 1/100$ is represented by the magenta curve in Figure 5, while all the other curves at all further time levels $t = 10/100, 20/100, 30/100, \ldots$ seem to coincide in the single blue curve.

The picture for the coupled system with different positive aspect ratios $\epsilon > 0$, see Figure 5, is finally compared with the dynamics of the solution to the small-aspect ratio model.

Reading Figure 6 top down, the membrane’s deflection (also emerging from the initial configuration $u_* \equiv 0$) becomes instantaneously negative at all interior points $x \in I$ and decreases monotonically in time. Note that Figures 4(b) and 6 are in compliance with Theorem 4.1.

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Figure 5. Membrane’s deflection $u$ of the coupled system for $p(x) = x^8 + 0.1$ with $u_* \equiv 0$, $\lambda = 1$, and $\varepsilon = 0.15$.

Figure 6. Approximate solution $u$ to the small-aspect ratio model with $u_* \equiv 0$ for $p(x) = x^8 + 0.1$ and $\lambda = 1$.

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