SUMMING THE DERIVATIVE EXPANSION OF THE EFFECTIVE ACTION

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Abstract

The derivative expansion of the effective action is a perturbative development in derivatives of the fields. The expansion breaks down when some of the derivatives are too large. We show how to sum exactly the first and second derivatives and treat perturbatively derivatives higher than second.
I. INTRODUCTION

The effective action incorporates the effects of closed loops in a quantum field theory. This object contains all the information relevant for the low-energy effective limit. However, in general it cannot be evaluated exactly, and one has to rely on some sort of approximation.

Since the effective action is the generating functional of the one-particle-irreducible diagrams, we can develop it quite naturally in powers of the momenta flowing into the diagram vertices. This is equivalent to expand the effective Lagrangian in powers of the derivatives of the background field, an approximation that is known as the derivative expansion.

The leading term of this derivative expansion is the effective potential, i.e., the term with no derivatives that corresponds to a constant background field case. The next orders in the expansion are obtained by developing the effective Lagrangian about this constant field case. The development is perturbative and, as such, is valid only when the field is slowly varying.

There might be instances where all the field derivatives are small except for some of them. Then, the derivative expansion breaks down and cannot be used. We have studied how to rescue such a case. To be precise, we have found a method to sum exactly the first and the second derivatives in the derivative expansion, with the higher derivatives (third, etc.) treated perturbatively. We have used a simple scalar theory to explain the details of our work, although it can be extended to more realistic theories, like QED.

Our method is developed in momentum-space. We concentrate our efforts to calculate the momentum-space Green’s function $G(p)$ since, as we will see, the knowledge of $G(p)$ allows to calculate the effective Lagrangian $L_{eff}$ quite trivially. We reobtain the derivative expansion in a very simple way. We then show how to determine an expression for $G(p)$ and $L_{eff}$ that is exact in first and second derivatives and perturbative in higher derivatives.

II. THE EFFECTIVE ACTION

We start with the action for a scalar field

$$S[\phi] = \int d^4x \left\{ \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 - V(\phi) \right\}$$

It can be expanded around a classical background field $\phi_c$,

$$\phi(x) = \phi_c(x) + \omega(x)$$
where \( \omega(x) \) is a quantum field, and we obtain

\[
S[\phi] = S[\phi_c] + \Delta S[\phi_c, \omega] + \ldots
\]  

(3)

where \( \Delta S \) is defined as the piece that contains the terms bilinear in \( \omega \)

\[
\Delta S[\phi_c, \omega] = \int d^4x \left( \frac{1}{2} \omega(x) [-\partial^2 - m^2 - V''(\phi_c)] \omega(x) \right)
\]  

(4)

and the dots in (3) correspond to higher powers in \( \omega \).

At the one-loop level, the effective action \( S_{\text{eff}} \) is defined after integrating out the \( \omega \)-field

\[
e^{\frac{i}{\hbar} S_{\text{eff}}[\phi_c]} = N \int D[\omega] e^{\frac{i}{\hbar} \Delta S[\phi_c, \omega]}
\]  

(5)

\( (N \) is a normalization constant).

As we said, we will apply momentum-space methods to calculate the effective action. We follow \cite{1} and first differentiate (5) by \( m^2 \)

\[
\frac{i}{\hbar} \frac{\partial}{\partial m^2} S_{\text{eff}} = -\frac{i}{2\hbar} \frac{\int D[\omega] \omega^2 e^{\frac{i}{\hbar} \Delta S}}{\int D[\omega] e^{\frac{i}{\hbar} \Delta S}}
\]

\[
= -\frac{1}{2} \int d^4x \ G(x, x)
\]

\[
= -\frac{1}{2} \int d^4x \int \frac{d^4p}{(2\pi)^4} G(p)
\]  

(6)

Here \( G(x, x') \) is the Green’s function satisfying

\[
[\partial_x^2 + m^2 + V''(\phi_c(x))] G(x, x') = \delta^4(x, x')
\]  

(7)

and \( G(p) \) is its Fourier-transformed

\[
G(p) = \int d^4x \ e^{ip(x-x')} G(x, x')
\]  

(8)

The effective action and Lagrangian are formally obtained integrating the relation (6)

\[
S_{\text{eff}} = \int d^4x \ L_{\text{eff}}
\]

\[
L_{\text{eff}} = \frac{i\hbar}{2} \int dm^2 \int \frac{d^4p}{(2\pi)^4} G(p)
\]  

(9)

In the following section we will work out the equation for \( G(p) \) and its perturbative solution. From (5), it is clear that the knowledge of \( G(p) \) leads easily to the effective action \( S_{\text{eff}} \). In general, one has constants of integration in (5) which are determined by demanding \( S_{\text{eff}} \rightarrow 0 \) when \( \phi_c \rightarrow 0 \).
III. THE DERIVATIVE EXPANSION

We will expand $V''(\phi_c)$ in (7) around a reference point $x_o$

$$V''(\phi_c(x)) = V''(\phi_c(x_o)) + (\partial_\mu V'')_o (x - x_o)^\mu + \frac{1}{2} (\partial_\mu \partial_\nu V'')_o (x - x_o)^\mu (x - x_o)^\nu$$

$$+ \frac{1}{3!} (\partial_\mu \partial_\nu \partial_\sigma V'')_o (x - x_o)^\mu (x - x_o)^\nu (x - x_o)^\sigma + \ldots $$  \hspace{1cm} (10)

where

$$(\partial_\mu V'')_o = \partial_\mu V''(\phi_c(x)) \big|_{x=x_o}$$  \hspace{1cm} (11)

etc.

In momentum space

$$(x - x_o)^\mu \rightarrow -i \frac{\partial}{\partial p_\mu}$$  \hspace{1cm} (12)

So that the equation for $G(p)$ is

$$(-p^2 + \alpha + K) \ G(p) = 1$$  \hspace{1cm} (13)

where

$$\alpha = m^2 + V''(\phi_c(x_o))$$  \hspace{1cm} (14)

and the field derivatives are in the operator $K$:

$$K = -i(\partial_\mu V'')_o \frac{\partial}{\partial p_\mu}$$

$$- \frac{1}{2} (\partial_\mu \partial_\nu V'')_o \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial p_\nu}$$

$$+ i \frac{1}{3!} (\partial_\mu \partial_\nu \partial_\sigma V'')_o \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial p_\nu} \frac{\partial}{\partial p_\sigma}$$

$$+ \ldots $$  \hspace{1cm} (15)

When $K = 0$, one can solve trivially for $G_o = G$

$$(-p^2 + \alpha) \ G_o(p) = 1$$  \hspace{1cm} (16)

$$G_o(p) = \frac{1}{-p^2 + \alpha}$$  \hspace{1cm} (17)

At this point we could find the effective action in the case that the scalar field is constant, i.e., when all the derivatives are zero. Thus we would find $\mathcal{L}_{eff}$, and therefore the effective potential, introducing the Green’s function (17) in (9). We shall not recalculate the effective potential since it is a very well-known object; instead, we will next work out the case $K \neq 0$. 

In this general case, $K \neq 0$, having a solution for $G(p)$ is equivalent to determine the inverse of the operator acting on $G(p)$ in (13), namely,

$$
\frac{1}{-p^2 + \alpha + K}
$$

We now use the following expansion

$$
\frac{1}{-p^2 + \alpha + K} = \frac{1}{-p^2 + \alpha} - \frac{1}{-p^2 + \alpha} \frac{K}{-p^2 + \alpha} + \frac{1}{-p^2 + \alpha} \frac{K}{-p^2 + \alpha} \frac{1}{-p^2 + \alpha} - \ldots
$$

so that

$$
G(p) = G_o(p) - G_o(p)KG_o(p) + G_o(p)KG_o(p)KG_o(p) + \ldots
$$

$$
= G_o(p) \sum_{m=0}^{\infty} (-1)^m (KG_o(p))^m
$$

This is an infinite expansion of $G(p)$ in powers of derivatives. With (20) we can calculate $G(p)$ up to any desired order and then integrate it as we indicate in (9) and obtain the effective Lagrangian $L_{eff}$.

This development of the effective Lagrangian is called the derivative expansion and was first obtained by [2] and by [3] (see also [4], [5]). We think our method to derive this expansion is quite simple.

IV. SUMMING THE DERIVATIVE EXPANSION

The derivative expansion is a perturbative expansion in the field derivatives. As such, it may be useful when all the derivatives are small, i.e.,

$$
\frac{1}{\alpha^{1+n/2}} \frac{\partial^n V''}{\partial x^{\mu_1} \partial x^{\mu_2} \ldots \partial x^{\mu_n}} \ll 1
$$

$$
n = 1, 2, \ldots
$$

Actually, there has been much discussion in the literature about the convergence of the derivative expansions [3, 4, 8, 9, 10]. Of course it would be interesting to sum exactly some of the derivative terms. We know that one cannot sum all the derivative expansion for an arbitrary theory, so we have studied which part of the expansion we can sum. The conclusion of our study is that we are able to sum exactly the terms in $\alpha$, eq.(14), and in the first and second derivatives

$$
\beta_\mu = (\partial_\mu V'')_o
$$

$$
\gamma^2_{\mu\nu} = 2(\partial_\mu \partial_\nu V'')_o
$$

(22)
and leave the other terms, i.e., the higher-than-second derivatives, as a perturbation expansion. In the rest of the paper we will show how to do it.

Let us start discussing first the case that $K$ in (13) has only first and second derivatives,

$$K = -i\beta_\mu \frac{\partial}{\partial p_\mu} - \frac{1}{4} \gamma^{\mu\nu}_{\mu} \frac{\partial^2}{\partial p_\mu \partial p_\nu} \equiv K_2$$

(23)

so that our eq.(13) reads

$$(-p^2 + \alpha + K_2) G_2(p) = 1$$

(24)

The finding that (24) has an exact solution goes back to the work of Schwinger [11], but we will follow the momentum-space methods of [1]. We will seek solutions of the type

$$G(p) = \int_0^\infty ds \ e^{F(s)} e^{-\alpha s}$$

(25)

($s$ is called proper time). When this ansatz is introduced in eq.(13), we end up with

$$\int_0^\infty ds \ H(s) e^{F(s)} e^{-\alpha s} = 1$$

(26)

where

$$H(s) e^{F(s)} = (-p^2 + \alpha + K)e^{F(s)}$$

(27)

If a solution to (26) exists, it fulfills

$$H(s) = -\frac{\partial}{\partial s} (F - \alpha s)$$

(28)

with boundary conditions

$$F - s\alpha \rightarrow 0 \quad \text{when} \quad s \rightarrow 0$$

$$F - s\alpha \rightarrow -\infty \quad \text{when} \quad s \rightarrow \infty$$

(29)

When $K = 0$, the ansatz (25) for $G_o$ corresponds to

$$F = s \ p^2$$

(30)

clearly, then we have

$$G_o = \int_0^\infty ds \ e^{(p^2 - \alpha)s} = \frac{1}{-p^2 + \alpha}$$

(31)

in agreement with (17).

When $K = K_2$, we follow reference [1] and generalize (30)

$$F = A_{2\mu\nu} \ p^\mu p^\nu + B_{2\mu} \ p^\mu + C_2 \equiv F_2$$

(32)
where $A_2$, $B_2$, $C_2$ are functions of $s$ to be determined. The function $H(s)$ in (26) is easily calculated

$$H = -p^2 + \alpha - i\beta \mu B_2^\mu - \frac{1}{2} \gamma_{\mu\nu} A_2^{\mu\nu} - \frac{1}{4} \gamma_{\mu\nu} B_2^\mu B_2^\nu - \left[2i\beta \nu A_2^{\nu\mu} + \gamma_{\nu\rho} B_2^{\nu} A_2^{\rho\mu} \right] p_\mu - \gamma_{\rho\sigma} A_2^{\rho\mu} A_2^{\sigma\nu} p_\mu p_\nu$$

$$\equiv H_2$$

(33)

Both $F_2$ and $H_2$ are polynomials of second order in $p$ and thus we can use (28) and equate equal powers of $p$ on both sides. We obtain the following differential equations

$$\frac{\partial}{\partial s} A_{2\mu\nu} = g_{\mu\nu} + \gamma_{\rho\sigma} A_{2\rho} A_{2\sigma}^{\mu\nu}$$

$$\frac{\partial}{\partial s} B_{2\mu} = 2i\beta \nu A_{2\nu}^{\mu} + \gamma_{\rho\mu} B_{2}^{\nu} A_{2\nu}^{\rho\mu}$$

$$\frac{\partial}{\partial s} C_2 = i\beta \mu B_{2}^{\mu} + \frac{1}{2} \gamma_{\mu\nu} A_{2}^{\mu\nu} + \frac{1}{4} \gamma_{\mu\nu} A_{2}^{\rho} A_{2}^{\nu} B_{2}^{\rho}$$

(34)

Taking into account the boundary conditions (29), the solutions are

$$A_{2\mu\nu} = \gamma^{-1}_{\mu\rho}(\tan \gamma s)^\rho$$

$$B_{2\mu} = -2i\gamma^{-2}_{\mu\rho} [g_{\rho} - (\sec \gamma s)^\rho \beta_{\rho}]$$

$$C_2 = -\frac{1}{2} tr \ln(\cos \gamma s) - \beta_{\mu} \gamma_{\mu\rho} A_{2}^{\rho\nu}(\tan \gamma s - \gamma s)^\nu$$

(35)

Putting these solutions in (32), this determines

$$G_2(p) = \int_0^\infty ds e^{F_2} e^{-\alpha s}$$

(36)

This expression for $G_2$ is valid for arbitrary values of the first and second derivatives $\beta$ and $\gamma$. From $G_2$ one can get, using (31), the effective Lagrangian that we call $L_2$ (see below eq. (48)).

When higher derivatives are considered there is no longer an exact solution for $G(p)$ and $L_{eff}$. However, we will show that we can get a solution valid for arbitrary $\alpha$, $\beta$, and $\gamma$ and perturbative in higher derivatives. For the sake of clarity, let us explicitly demonstrate this in the case of having a third derivative:

$$K = -i\beta \mu \frac{\partial}{\partial p_\mu} - \frac{1}{4} \gamma_{\mu\nu} \frac{\partial^2}{\partial p_\mu \partial p_\nu} + i \sum g_{\mu\rho} \delta_{\mu\rho} \frac{\partial^3}{\partial p_\mu \partial p_\nu \partial p_\rho}$$

with $g_{\mu\rho} = \frac{1}{3!} \partial_{\mu} \partial_{\nu} \partial_{\rho} V''\big|_{\nu,\rho}$. We adopt the ansatz (23) with

$$F = A_{\mu\nu} p^\mu p^\nu + B_{\mu} p^\mu + C + g_{\mu\rho} D_{\mu\rho} p^\mu p^\nu p^\rho$$

(38)
where $A$, $B$, $C$ and $D$ are $s$-functions to be determined. When the ansatz is introduced in (13) we get equation (26) with

$$H = -p^2 + \alpha - i \beta_\alpha B^\alpha - \frac{1}{4} \gamma_{\alpha\beta}(B^\alpha B^\beta + 2A^{\alpha\beta}) - (\gamma_{\alpha\beta} B^\beta + 2i \beta_\alpha)A^{\alpha\mu}p_\mu - \gamma_{\alpha\beta} A^{\alpha\mu} A^{\beta\nu} p_\mu p_\nu + \left\{ i \delta_{\alpha\beta\gamma} B^\alpha (B^\beta B^\gamma + 6 A^{\alpha\gamma}) - \left[ \frac{3}{2} \gamma_{\alpha\beta} D^{\alpha\beta} - 6i \delta_{\alpha\beta\gamma} A^{\gamma\mu} (B^\alpha B^\beta + 2A^{\alpha\beta}) \right] p_\mu - \left[ \frac{3}{2} (\gamma_{\alpha\beta} B^\beta + 2i \beta_\alpha) D^{\alpha\mu} - 12i \delta_{\alpha\beta\gamma} B^\alpha A^{\beta\nu} A^{\gamma\mu} \right] p_\mu p_\nu - \left[ 3 \gamma_{\alpha\beta} A^{\alpha\mu} D^{\beta\nu\rho} - 8i \delta_{\alpha\beta\gamma} A^{\alpha\mu} A^{\beta\nu} A^{\gamma\rho} \right] p_\mu p_\nu p_\rho \right\} + O(g^2) \quad (39)$$

Here we have used the fact that $A_{\mu\nu}$ and $D_{\mu\nu\rho}$ are symmetric tensors.

In order that (28) has a solution, it is a necessary condition that the two polynomials in $p$, $F$ and $H$, have the same degree. $F$ is of order $p^3$. However, the $O(g^2)$ term in $H$ in (39) is of order $p^6$, and this is why we are not able to obtain an exact solution for all $g$. At first order in $g$, $H$ is of order $p^2$ and we can equate equal powers of $p$ in eq.(28). In this way, we are able to find equations for $A$, $B$, $C$, and $D$. It is convenient to write

$$A = A_2 + g \Delta A + O(g^2)$$
$$B = B_2 + g \Delta B + O(g^2)$$
$$C = C_2 + g \Delta C + O(g^2) \quad (40)$$

Then, using (28) and (34), we find that the unknown $s$-functions $\Delta A$, $\Delta B$, $\Delta C$, and $D$ satisfy the differential equations

$$\frac{\partial}{\partial s} \Delta A_{\mu\nu} = 2 \gamma_{\alpha\beta} (A^\alpha_{\mu\nu} \Delta A^\beta_{\mu\nu} + A^\beta_{\mu\nu} \Delta A^\alpha_{\mu\nu}) + \frac{3}{2} (\gamma_{\alpha\beta} B^\beta_{\mu\nu} + 2i \beta_\alpha) D_{\mu\nu}^\alpha - 12i \delta_{\alpha\beta\gamma} B^\beta_{\mu\nu} A^\gamma_{\alpha\beta\gamma} \quad (41)$$
$$\frac{\partial}{\partial s} \Delta B_{\mu} = \gamma_{\alpha\beta} A^\beta_{\mu\nu} \Delta B^\alpha + (\gamma_{\alpha\beta} B^\beta_{\mu\nu} + 2i \beta_\alpha) \Delta A^\alpha_{\mu} + \frac{3}{2} \gamma_{\alpha\beta} D_{\mu\nu}^\alpha - 6i \delta_{\alpha\beta\gamma} B^\gamma_{\mu\nu} (B^\beta_{\mu\nu} + 2A^\beta_{\mu\nu}) \quad (42)$$
$$\frac{\partial}{\partial s} \Delta C = \frac{1}{2} \gamma_{\alpha\beta} \Delta A^{\alpha\beta} + \frac{1}{2} (\gamma_{\alpha\beta} B^\beta_{\mu\nu} + 2i \beta_\alpha) \Delta B^\alpha - i \delta_{\alpha\beta\gamma} B^\gamma_{\mu\nu} (B^\beta_{\mu\nu} + 6 A^\beta_{\mu\nu}) \quad (43)$$
$$\frac{\partial}{\partial s} D_{\mu\nu\rho} = 3 \gamma_{\alpha\beta} (A^\alpha_{\mu\nu} D^\beta_{\mu\nu} + A^\beta_{\mu\nu} D^\alpha_{\mu\nu} + A^\gamma_{\mu\nu} D^\beta_{\mu\nu}) - 8i \delta_{\alpha\beta\gamma} A^{\alpha\mu} A^{\beta\nu} A^{\gamma\rho} \quad (44)$$

These are linear differential equations and can be solved, although the general solution is extremely complicated. We will discuss this issue in the next section. Here we will assume we got the solution of (11)-(14) and we will work out the form of the effective Lagrangian.
The effective Lagrangian is obtained from \( G(p) \) by means of eq.(9). We see we need to evaluate
\[
\mathcal{L}_{\text{eff}} = \frac{i\hbar}{2} \int_{0}^{\infty} ds \int dm^2 \, e^{-\alpha s} \int \frac{d^4p}{(2\pi)^4} \, e^{F(s)}
\]  
(45)

Since we work in perturbation theory at first order in the coupling \( g \), the expression of \( \mathcal{L}_{\text{eff}} \) has to be of the form
\[
\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + g \Delta \mathcal{L}_3
\]  
(46)

That this is indeed the case can be seen by expanding the integrand
\[
e^{F} = e^{F_2} \left[ 1 + g \left( \Delta A^\mu p_\mu p_\nu + \Delta B^\mu p_\mu + \Delta C + D^{\mu\nu\rho} p_\mu p_\nu p_\rho \right) + O(g^2) \right]
\]  
(47)

and inserting the expansion in (45). To integrate in momentum, we shift \( p \) in such a way that \( F_2 \), eq.(32), is quadratic in the shifted momentum and the integral becomes a simple Gaussian. We also integrate in \( m^2 \). In this integration we obtain an integration constant that will be fixed by demanding \( \mathcal{L}_{\text{eff}} \to 0 \) when \( \phi_c \to 0 \). After the integration we obtain
\[
\mathcal{L}_2 = \frac{\hbar}{2(4\pi)^2} \int_{0}^{\infty} ds \, \frac{1}{s^3} \left[ \frac{e^{-\alpha s} s^2}{\sqrt{\det A_2}} e^{-\frac{1}{4}B_2 A_2^{-1} B_2 + C_2} - e^{-m^2 s} \right]
\]  
(48)

and
\[
\Delta \mathcal{L}_3 = \frac{\hbar}{2(4\pi)^2} \int_{0}^{\infty} ds \, \frac{e^{-\alpha s}}{s} \frac{1}{\sqrt{\det A_2}} e^{-\frac{1}{4}B_2 A_2^{-1} B_2 + C_2} \Delta I
\]

\[
\Delta I = \frac{1}{2} \Delta A^\mu \left[ \frac{1}{2} \left( A_2^{-1} B_2 \right)_\mu \left( A_2^{-1} B_2 \right)_\nu - A_2^{-1} \right] - \frac{1}{2} \Delta B^\mu \left( A_2^{-1} B_2 \right)_\mu + \Delta C
\]
\[
+ \frac{1}{4} D^{\mu\nu\rho} \left[ 3 A_2^{-1} \left( A_2^{-1} B_2 \right)_\rho - \frac{1}{2} \left( A_2^{-1} B_2 \right)_\mu \left( A_2^{-1} B_2 \right)_\nu \left( A_2^{-1} B_2 \right)_\rho \right]
\]  
(49)

We have determined the expression for the effective Lagrangian (48), that is valid for arbitrary \( \alpha, \beta, \gamma \) and first-order in the third derivative \( g \delta \). To \( \mathcal{L}_2 \) one should add counter terms coming from the renormalization of the mass and coupling constants in \( V(\phi_c) \). These make \( \mathcal{L}_2 \) finite.

We could obtain similar expressions when higher derivatives are considered. To show this, first consider the case that we have a \( n \)-th order derivatives instead of third order
\[
K = i\beta \frac{\partial}{\partial p} - \frac{1}{4} \gamma^2 \frac{\partial^2}{\partial p^2} + g \rho \frac{\partial^n}{\partial p^n}
\]  
(50)

(in order to simplify notation we write without indices; \( \rho \) would be a tensor with \( n \) indices, etc.).
The ansatz that we have to consider still is eq. (25) with
\[ F = A p^2 + B p + C + g D_3 p^3 + \ldots + g D_n p^n \] (again no indices appear; \( D_i \) is a tensor with \( i \)-indices, etc).

Let us discuss the order in \( p \) of \( H \) in (26). With the action of the first and second derivatives in (27) we obtain contributions to \( H \) that have maximum degree
\[ g p^n \] at first order in \( g \). There are terms of higher degree in \( p \), but are \( O(g^2) \) and we neglect them. When the \( n \)-th derivative in \( K \) acts on \( G \) we obtain again (52) as the maximum degree. There are also higher degrees in \( p \), but \( O(g^2) \). Since \( F \) in (51) is order \( p^n \), eq. (28) is consistent. Equating equal powers of \( p \) we obtain \( n + 1 \) linear differential equations for the functions \( A, B, C, D_3, \ldots, D_n \). For a general tensor \( \rho \) in (50), to find a solution for these functions can be a painful task. Our point is that the solutions for \( A, B, C, D_3, \ldots, D_n \) could be found and they determine \( G(p) \) that leads to
\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_2 + g \Delta \mathcal{L}_n \] (53)

Now that we have discussed what happens when we have the \( n \)-th derivative, eq. (50), it is easy to see what happens in the general case of a finite number of derivatives: first, second and then from third derivative until \( n \)-th derivative. Let each of these higher than second derivatives be proportional to a coupling constant: from \( g_3 \) until \( g_n \). Since we work at first-order in the coupling constants \( g_i \), the overall modification to \( \mathcal{L}_2 \) is simply the sum of the individual modifications coming from each \( g_i \). Finally we would obtain
\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \sum_{i=3}^{n} g_i \Delta \mathcal{L}_i \] (54)
where each \( \Delta \mathcal{L}_i \) is obtained like in (53). The effective Lagrangian (54) is exact in \( \alpha, \beta, \gamma \) and first-order in \( g_3, \ldots, g_n \).

Until now we have limited ourselves to first-order perturbation theory. Our method is really perturbative and thus we can go to higher orders in the coupling constants. Let us show this by going back to \( K \) with a \( n \)-th derivative, eq. (50), and work at second order in \( g \).

We use the ansatz (23) with
\[ F = A p^2 + B p + C + (g D_3 + g^2 E_3) p^3 + \ldots + (g D_n + g^2 E_n) p^n + g^2 D_{n+1} p^{n+1} + \ldots + g^2 D_{2(n-1)} p^{2(n-1)} \] (55)
and

\[ A = A_2 + g \Delta A + g^2 \Delta \tilde{A} + O(g^3) \]
\[ B = B_2 + g \Delta B + g^2 \Delta \tilde{B} + O(g^3) \]
\[ C = C_2 + g \Delta C + g^2 \Delta \tilde{C} + O(g^3) \]  \hspace{1cm} (56)

The functions \( D_3, \ldots, D_n, \Delta A, \Delta B, \Delta C \) contain the modifications arising at first-order in \( g \). Let us concentrate in the second-order corrections. It is not difficult to see that, at order \( g^2 \), the action of \( K \) leads to terms in \( H \) that have a maximum degree of

\[ g^2 p^{2(n-1)} \]  \hspace{1cm} (57)

and thus we have from eq.(28) consistent equations for \( D_{n+1}, \ldots, D_{2(n-1)}, E_3, \ldots, E_n, \Delta \tilde{A}, \Delta \tilde{B}, \Delta \tilde{C} \). The final effective Lagrangian will have the form

\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_2 + g \Delta \mathcal{L}_n + g^2 \Delta \tilde{\mathcal{L}}_n \]  \hspace{1cm} (58)

The perturbative expansion could be calculated up to any order in the coupling constants corresponding to a finite number of derivatives.

V. EXPLICIT SOLUTIONS

The linear differential equations (41-44) can be solved in the standard way. It is easy to see that we have to solve them in the following order. First, eq.(44) has the solutions

\[ D_{\mu\nu\rho} = -2i \left( \frac{1}{\gamma \cos \gamma s} \right)_\mu \left( \frac{1}{\gamma \cos \gamma s} \right)_\nu \left( \frac{1}{\gamma \cos \gamma s} \right)_\rho \delta_{\sigma\pi\tau} \sum_{i=0}^{3} \theta_i \left[ \frac{\cos(T_i s) - 1}{T_i} \right]_{\lambda\theta\omega} \]  \hspace{1cm} (59)

where

\[ T_{i, \lambda\sigma\theta\pi\omega\tau} = a^i \gamma_{\lambda\sigma} \otimes g_{\theta\pi} \otimes g_{\omega\tau} + b^i g_{\lambda\sigma} \otimes \gamma_{\theta\pi} \otimes g_{\omega\tau} + c^i g_{\lambda\sigma} \otimes g_{\theta\pi} \otimes \gamma_{\omega\tau} \]  \hspace{1cm} (60)

with

\[ a^1 = 1 \quad i \neq 1, \quad a^1 = -1 \]
\[ b^1 = 1 \quad i \neq 2, \quad b^2 = -1 \]
\[ c^1 = 1 \quad i \neq 3, \quad c^3 = -1 \]
\[ \theta_i = -1 \quad i \neq 0, \quad \theta_0 = 1 \]  \hspace{1cm} (61)
Having $D$ one can solve (11)

$$\Delta A_{\mu\nu} = \left( \frac{1}{\gamma \cos s} \right)_\mu^\alpha \left( \frac{1}{\gamma \cos s} \right}_\nu^\beta \left\{ \int ds \left( \cos \gamma s \right)_\alpha^\alpha \left( \cos \gamma s \right)_\beta^\beta M_{\sigma\gamma} + K_{\alpha\beta} \right\}$$  (62)

$$M_{\mu\nu} = 3i \left[ \beta^\alpha \left( \sec s \right)_\alpha^\alpha D_\beta_{\mu\nu} - 4 \delta_{\alpha\beta} B_2^\alpha A_2^\alpha A_2^\gamma \right]$$

The constant of integration $K_{\alpha\beta}$ has to be chosen in such a way that the boundary conditions (29) are satisfied. $A_2$ and $B_2$ are given by (33).

Afterwards, we can solve (12)

$$\Delta B_\mu = \left( \frac{1}{\gamma \cos s} \right)_\mu^\alpha \left\{ \int ds \left( \cos \gamma s \right)_\alpha^\beta V_\beta + K_\alpha \right\}$$  (63)

$$V_\mu = 2i \beta_\alpha \left( \sec s \right)_\alpha^\beta \Delta A_{\beta\mu} + \frac{3}{2} \gamma_{\alpha\beta}^2 D_\mu^\beta - 6i \delta_{\alpha\beta} A_2^\gamma (2A_2^\alpha + B_2^\alpha B_2^\gamma)$$

and finally (13)

$$\Delta C = \int ds \left\{ \frac{1}{2} \gamma_{\alpha\beta}^2 \Delta A_{\alpha\beta} + i \beta^\alpha \left( \sec s \right)_\alpha^\beta \Delta B_\beta - i \delta_{\alpha\beta} B_2^\alpha (B_2^\beta B_2^\gamma + 6A_2^\alpha) \right\} + K$$  (64)

As before, $K_\alpha$ in (63) and $K$ in (64) are fixed by demanding (29).

We have evaluated the expressions for $D$, $\Delta A$, $\Delta B$ and $\Delta C$ for general $\delta_{\alpha\beta\gamma}$, but they are rather long and not particularly illuminating. We do not display them here. In practice, we may be interested in cases where some of the $\delta_{\alpha\beta\gamma}$ elements are zero and then the expressions for $D$, $\Delta A$, $\Delta B$ and $\Delta C$ are much easier to handle. For example, let us work out the case that $\delta_{\alpha\beta\gamma}$ is diagonal. First, we notice that we can always work in a basis where $\gamma_{\mu\nu}^2$ in (22) is diagonal,

$$\gamma_{\mu\nu} = \begin{pmatrix} \gamma_o & 0 & 0 \\ 0 & \gamma_1 & 0 \\ 0 & 0 & \gamma_2 \end{pmatrix}$$

Then, since $\delta_{\alpha\beta\gamma}$ is diagonal, it follows that $D_{\mu\nu}$, $\Delta A_{\mu\nu}$ are diagonal and given by

$$D_{\mu\nu} = \frac{8i \delta_{\mu\nu}}{3 \gamma_o^2} \left( \frac{3}{\cos^2(s \gamma_o)} - \frac{2}{\cos^3(s \gamma_o)} - 1 \right)$$  (66)

$$\Delta A_{\mu\nu} = \frac{4 \beta_\mu \gamma_{\mu\nu}}{\gamma_o^2} \left( \frac{3 \tan(s \gamma_o)}{\gamma_o} - \frac{3 s}{\cos^2(s \gamma_o)} + \frac{4 \tan(s \gamma_o)}{\gamma_\alpha \cos^2(s \gamma_o)} - \frac{4 \tan(s \gamma_o)}{\gamma_o \cos(s \gamma_o)} \right)$$  (67)

We have only displayed the temporal diagonal terms. To get the other elements, we have to change all the _o subindices by the desired one. Although we are using the Minkowskian metric (+, −, −, −) there is no change of sign.
We also have

\[
\Delta B_o = \frac{8 i \beta_o^2 \delta_{oo}}{\gamma_o^6} \left( \frac{2}{\cos^3(s \gamma_o)} - 3 - \frac{1}{\cos^2(s \gamma_o)} + \frac{2}{\cos(s \gamma_o)} - \frac{3 s \gamma_o \tan(s \gamma_o)}{\cos(s \gamma_o)} \right) \\
+ \frac{8 i \beta_o^2 \delta_{oo}}{\gamma_o^6} \left( \frac{\gamma_o^2}{\beta_o^2} \tan(s \gamma_o) - \frac{\gamma_o^2}{\beta_o^2} \cos(s \gamma_o) \right) \\
\Delta C = \frac{4 \beta_o \delta_{oo}}{\gamma_o^4} \left( \frac{2}{\cos^3(s \gamma_o)} - \frac{1}{\cos(s \gamma_o)} + \frac{2 s \beta_o^2}{\gamma_o^2} + \frac{3 s \beta_o^4}{\gamma_o^2 \cos^2(s \gamma_o)} - \frac{11 \beta_o^2 \tan(s \gamma_o)}{3 \gamma_o^3} \right) \\
- \frac{4 \beta_o^2 \tan(s \gamma_o)}{\gamma_o^3 \cos^2(s \gamma_o)} - \frac{3 s \gamma_o \tan(s \gamma_o)}{2} - 1 \right) + \\
\sum_{i=1}^{3} \frac{4 \beta_i \delta_{iii}}{\gamma_i^4} \left( \frac{1}{\cos(s \gamma_i)} - \frac{2}{\cos^2(s \gamma_i)} + \frac{2 s \beta_i^2}{\gamma_i^2} + \frac{3 s \beta_i^4}{\gamma_i^2 \cos^2(s \gamma_i)} - \frac{11 \beta_i^2 \tan(s \gamma_i)}{3 \gamma_i^3} \right) \\
- \frac{4 \beta_i^2 \tan(s \gamma_i)}{3 \gamma_i^3 \cos^2(s \gamma_i)} + \frac{3 s \gamma_i \tan(s \gamma_i)}{2} + 1 \right)
\]

(68)

(69)

As before, if we want the other elements of \( \Delta B \) we have to change all the \( _o \) subindices, but in this case we also have to change the sign of the second parenthesis in (68).

With these solutions we easily get the corresponding effective Lagrangian, using eqs. (16), (18), and (19). We find the following result

\[
\Delta I = \frac{2 \beta_o \delta_{oo}}{\gamma_o^6} \left( 4 s \beta_o^2 + 6 s \beta_o^2 \sec^2\left(\frac{s \gamma_o}{2}\right) - 7 \gamma_o^2 + 4 \gamma_o^2 \sec^2\left(\frac{s \gamma_o}{2}\right) + \frac{3}{2} s \gamma_o \cot\left(\frac{s \gamma_o}{2}\right) \right) \\
- \frac{44 \beta_o^2 \tan\left(\frac{s \gamma_o}{2}\right)}{3 \gamma_o} - \frac{16 \beta_o^2 \sec^2\left(\frac{s \gamma_o}{2}\right) \tan\left(\frac{s \gamma_o}{2}\right)}{3 \gamma_o} - \frac{3}{2} s \gamma_o \tan\left(\frac{s \gamma_o}{2}\right) \right) + \\
\sum_{i=1}^{3} \frac{2 \beta_i \delta_{iii}}{\gamma_i^6} \left( 4 s \beta_i^2 + 6 s \beta_i^2 \sec^2\left(\frac{s \gamma_i}{2}\right) + 7 \gamma_i^2 - 4 \gamma_i^2 \sec^2\left(\frac{s \gamma_i}{2}\right) - \frac{3}{2} s \gamma_i \cot\left(\frac{s \gamma_i}{2}\right) \right) \\
- \frac{44 \beta_i^2 \tan\left(\frac{s \gamma_i}{2}\right)}{3 \gamma_i} - \frac{16 \beta_i^2 \sec^2\left(\frac{s \gamma_i}{2}\right) \tan\left(\frac{s \gamma_i}{2}\right)}{3 \gamma_i} + \frac{3}{2} s \gamma_i \tan\left(\frac{s \gamma_i}{2}\right) \right)
\]

(70)

Another particular case that leads to a simple solution is when \( \gamma^2 = 0 \), i.e., the second derivative vanishes. In that case, from (33) we have

\[
A_{2\mu} = s g_{\mu\nu} \\
B_{2\mu} = i s \beta_{\mu} \\
C_2 = -\frac{s^3}{3} \beta_{\mu} \beta^\mu
\]

and then we get

\[
D_{\mu\nu} = -2i s^4 \delta_{\mu\nu} \\
\Delta A_{\mu\nu} = \frac{18}{5} s \beta^\alpha \delta_{\alpha\mu

\]

13
\[ \Delta B_\mu = \mu \delta_{\mu \nu} \left( \frac{11}{5} s^3 \beta^\nu \beta^\rho - 4 g^\nu\rho \right) \]

\[ \Delta C = \frac{5}{2} s^4 \delta^\alpha_{\alpha \mu} \beta^\mu - \frac{16}{35} s^7 \delta_{\mu \nu \rho \beta} \beta^\mu \beta^\nu \beta^\rho \]  

(72)

When we put these expressions in (49) and then in (46), we get the effective Lagrangian for this particular case

\[ \mathcal{L}_{\text{eff}} = \frac{\hbar}{32\pi^2} \int_0^\infty \frac{ds}{s^3} \left\{ e^{-\alpha s} \frac{1}{12} s^3 \left[ 1 + g \left( g^\mu\nu - \frac{s^3}{28} \beta^\mu \beta^\nu \right) \delta_{\mu \nu \rho} \right] - e^{-m^2 s} \right\} \]  

(73)

Note added

We would like to comment that it is possible to sum the derivative expansion in QED using other methods [12, 13]. Also, we would like to thank A. Zelnikov for pointing out to us references that also deal with the problem of derivative summation [14, 15].

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