Automatic Feynman diagram calculation with \textit{xloops}
\hspace{1cm}–a short overview

L. Brücher*
27.3.1998

Abstract

\textit{xloops} is a program package that calculates Feynman diagrams by using computer algebra systems. In this paper it is shown which problems to be solved by computer algebra arise during such calculations, and how this problems are handled in the framework of \textit{xloops}.

1 Introduction

The increasing precision of experimental data in High energy Physics forces theorists to give very exact predictions for cross section of reactions between elementary particles. Calculating such cross sections efforts taking hundreds or thousands of Feynman diagrams into account. Every such diagram represents mathematically a 4-fold integral over internal momenta. Of course this

*email:bruecher@thep.physik.uni-mainz.de
calculations cannot be done by hand any more. So the usage of a computer algebra system is vital.

2 Structure of xloops

Building a computer program doing automatic calculations of Feynman diagrams efforts knowledge in different sections of science. Beside physical understanding knowledge in mathematics and computer science is necessary to solve the arising problems. This together with just the number of problem to be solved makes it necessary to give xloops a structure with well defined interfaces between different programs. How this is realized in xloops can be seen in fig. 2. There can also be seen that xloops uses different programming languages for different tasks. First the graphical user interface (GUI) allows the user to input the process he is interested in, as easy and comfortable as possible. The main evaluation of the Feynman diagram is done by the MAPLE part of the program. Its major task is to produce either the full result for a Feynman diagram or a result that can be integrated numerically. Therefore xloops reduces the whole integral to sets of standard integrals, which can be solved in an algorithmic manner. To do so, xloops has to handle non-commutative objects like elements of the Dirac algebra and be able to evaluate traces of such objects. Finally these objects have to be exported to C++ code, so that they are ready for numerical analysis.

3 Physical and mathematical background

To clarify the physical background of Feynman diagram evaluation the following section will give a brief overview.

The quantized structure of micro cosmos is visualized by Feynman diagram containing closed loops of internal particles. But such a Feynman diagram is also a prescription for evaluating its contribution to the cross section or decay width of a physical process. Every line (propagator)
and every point (vertex) corresponds to a mathematical expression as described by the so called Feynman rules. Multiplying these terms gives the whole contribution of the diagram. Closed loops of particles require integration over the momenta of the particles forming the loop. So every Feynman diagram corresponds to a $n$-fold integral, with $n$ depending on dimension in space time and number of closed loops. The principal analytical structure of these diagrams is:

$$
\int d^D l \frac{l_\mu l_\nu + /l + m_1 + \cdots}{[(l + q_1)^2 - m_1^2] \cdots [l^2 - m_n^2]},
$$

where $l$ represents a Lorenz vector and $l_\mu$ represents its components. Expanding the numerator and writing all loop-momentum independent variables in front of each integral lead to an integral containing just products of components of loop momentum in the numerator and the so called Propagators $P_i = [(l + q_i)^2 + m_i^2]$ in the denominator. These integrals are called tensor integrals in contradiction to scalar integrals, which have no loop momentum in the numerator. They can now be further reduced to a number of standard integrals with different techniques.

The standard technique, called Passarino Veltman method, tries to reduce these integrals to scalar integrals. In an intermediate step a linear system of equations is built and has to be solved. This linear system of equations can be rather big and so the most time consuming part of tensor integral calculation.

An alternative technique splits the Minkowski space time into the space spanned by the external momenta (parallel space) and its orthogonal complement (orthogonal space). The integration over the orthogonal space can now be transformed to an rather simple integration over the surface of a hyper-sphere and a remaining radius integration. Beside integrations done with residue theorem the integral formula

$$
\int_0^\infty x^{a-1} \prod_{i=1}^k (z_i + x)^{-b_i} \, dx = B(\beta - \alpha, \alpha) R_{\alpha - \beta}(b, z)
$$

expresses the result in $R$-functions. Between these functions exist a lot of relations, which allow to reduce all integrals to principally one basic $R$-function.

A third technique uses the fact that in parallel and orthogonal space every scalar product between an outer momentum $q_i$ and a loop momentum $l$ just projects out one component of the loop momentum. Every such product occurs in one of the propagators. So every component of $l_\mu$ from parallel space can be expressed in terms of propagators and masses $m_i$. The remaining components from orthogonal space can be expressed in terms of the last propagator from (1), which has no external momentum. So a recursive definition, which reduces every tensor integral to its corresponding scalar integral and number of integral with less propagators is gained.

4 Exemplary CA problem: reduction of tensor integrals

In this section some techniques used by xloops should be shown in more detail. As example serve to different methods for evaluating tensor integrals with xloops. The standard technique (Passarino Veltman method) is not used by xloops, as solving linear systems of equations with computer algebra methods is much slower than the algorithms discussed below.

4.1 A fast technique: recurrence relations with $R$-functions

In the case of the one-loop two-point function the resulting $R$-functions gained by integration fulfill the following recurrence relation:

$$
(b_1 + b_2) R_t(b_1, b_2; z_1, z_2) = (b_1 + b_2 + t) R_t(b_1 + 1, b_2; z_1, z_2) - t z_1 R_{t-1}(b_1, b_2; z_1, z_2)
$$

(3)
resulting in the basic $\mathcal{R}$-function:

$$\mathcal{R}_\epsilon(-\frac{1}{2} + \epsilon, 1; z_1, z_2) = \frac{1}{2} z_1^{-\epsilon} \left( \frac{z_1}{z_2} \right)^{1-\epsilon} \left[ \left( 1 + \sqrt{1 - \frac{z_1}{z_2}} \right)^{-1+2\epsilon} + \left( 1 - \sqrt{1 - \frac{z_1}{z_2}} \right)^{-1+2\epsilon} \right] + O(\epsilon^2)$$

This can of course be directly translated into the following MAPLE-code:

R2 := proc(ind,b1,b2,z1,z2) local erg;

# *** Parameter increasing ***
else b1<0 then
  erg := ( (2*b1+2*b2+2*ind) * R2(ind,b1+1,b2,z1,z2)
- 2*ind*z1* R2(ind-1,b1+1,b2,z1,z2) )/(2*b1+2*b2);

elif ind=0 then
  erg := z2^eps/(2*z1^(2*eps)) * 
( (1+Sqrt(1-z1/z2))*(1-Sqrt(1-z1/z2))^(2*eps) 
+ (1-Sqrt(1-z1/z2))*(1+Sqrt(1-z1/z2))^(2*eps) )
RETURN( erg );
fi;
end:

The recurrence relations are for most cases twice as fast as solving the linear system of equations, where as intermediate step matrices have to inverted.

### 4.2 Even faster technique: cancellation of Propagators

Parallel and orthogonal space splitting lead for the one-loop two-point function to the following relation (here just for parallel space component of loop momentum):

$$l_\parallel = \frac{1}{2 q_\parallel} [P_1 - P_2 + C]$$

in terms of integrals this means:

$$\int \frac{l_\parallel^n}{P_1 P_2} = \frac{1}{2 q_\parallel} \left[ \int \frac{l_\parallel^{n-1}}{P_2} - \int \frac{l_\parallel^{n-1}}{P_1} + C \int \frac{l_\parallel^{n-1}}{P_1 P_2} \right]$$

with a integration independent constant $C$. So this equation reduces the exponent of loop momentum of a tensor integral by one and produces additional, simpler integrals. Applying this formula $n$ times ends with the scalar two-point function. This can also be written in a iterative algorithm:

FastTensor2Pt := proc(p0,p1,q0,m_1,m_2,eps)
local C1,C2,DM,i0,i1,i0s,cf1,cf2,cf3,cf4,cf5,cf6,m1,m2,temp0;
global rho;
if not type(p1,even) then RETURN( 0 ); fi;
cf4 := 0;
for i0 from 1 to p0 do
  i0s := trunc((i0-1)/2);
cf3 := 0;
  for i1 from 0 to i0s do
    cf1 := binomial(i0-1,2*i1)*(2*q0)^(2*i1-p0)*Two2OnePtFactor(2*i1,p1,eps);
    cf2 := (C1)^(i0-1-2*i1)*Tadpole(m2,1+i1+p1/2-eps)
    +(-1)^i0*C2^(i0-1-2*i1)*Tadpole(m1,1+i1+p1/2-eps)
    RETURN( erg );
  fi;
end:

The recurrence relations are for most cases twice as fast as solving the linear system of equations, where as intermediate step matrices have to inverted.
5 Conclusion and outlook

In high energy physics different techniques of symbolic calculation are used. Some of them used by xloops have been pointed out here in detail. Moreover xloops uses the following elements and algorithm of computer algebra systems:

- user defined recursive functions
- power series expansion
- fast list evaluation
- algebraic term collection

The hole package is still a developing project. It can be obtained from its homepage on the WWW:

http://wwwthep.physik.uni-mainz.de/~xloops

References

[1] J. C. Collins. Renormalization. Cambridge University Press (1984)

[2] B. C. Carlson. Special Functions of Applied Mathematics. Academic Press (1977)

[3] L. Brücher, J. Franzkowski, D. Kreimer. Introduction to xloops. Nucl.Instrum.Meth. A 389 (1997) 323–342

[4] L. Brücher, J. Franzkowski, D. Kreimer. The xloops Manual. Eprint hep-ph/9710484

[5] L. Brücher, J. Franzkowski, D. Kreimer. xloops: Automated Feynman diagram calculation. Comput. Phys. Commun. 115 (1998) 140

[6] L. Brücher, J. Franzkowski, D. Kreimer. Loop Integrals, R Functions and their Analytic Continuation. Mod. Phys. Lett. A9 (1994) 2335–2346

[7] L. Brücher and J. Franzkowski. Mod. Phys. Lett. A14 (1999) 881;

[8] L. Brücher, J. Franzkowski and D. Kreimer. Comp. Phys. Comm. 85 (1995) 153–165;

[9] L. Brücher, J. Franzkowski, D. Kreimer. Comp. Phys. Comm. 107 (1997) 281–291.