Analytic $\varepsilon-$ Expansion of the Scalar One-loop Bhabha Box Function *,**, J. Fleischer

Fakultät für Physik, Universität Bielefeld
Universitätsstr. 25, D-33615 Bielefeld, Germany

AND

T. Riemann, O.V. Tarasov

Deutsches Elektronen-Synchrotron DESY
Platanenallee 6, D–15738 Zeuthen, Germany

We derive the first three terms of the $\varepsilon-$ expansion of the scalar one-loop Bhabha box function from a representation in terms of three generalized hypergeometric functions, which is valid in arbitrary dimensions.

1. Introduction

One of the important problems in perturbative calculations is a precise determination of the cross section for Bhabha scattering. For this one has to determine the electroweak one-loop corrections in the Standard Model and some parametric enhanced contributions plus the complete photonic corrections to even higher orders. Here we are interested in a determination of photonic $\mathcal{O}(\alpha^2)$ corrections for this process in $d = 4 - 2\varepsilon, \varepsilon \to 0$, dimensions with account of the electron mass $m$ as a regulator of infrared singularities. These corrections naturally concern the virtual two-loop matrix element, which contributes to the cross section due to its interference with the Born matrix element. Of the same order is the absolute square of the one-loop amplitude $M_1$. The corresponding cross section contributions have been analytically determined recently [1].

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A peculiarity of the contribution from $M_1$ is the necessity to determine this amplitude as a function of the parameter $\varepsilon$ up to terms of order $\varepsilon$: $M_1(\varepsilon) = m_1/\varepsilon + m_0 + m_1\varepsilon$. The available numerical programs for the evaluation of one-loop functions are not sufficient for this task. In a series of papers, the possibility was studied to find some closed analytical expressions for one-loop 2-, 3- and 4-point functions for arbitrary dimension, external momenta and masses (in principle also complex ones) in terms of generalized hypergeometric functions with relatively simple integral representations \[2, 3, 4\]. The results immediately apply to a deduction of the coefficient $m_1$. Such formulae are also of great importance in the search of efficient algorithms for the calculation of 5-, 6- and higher point functions since these functions may be reduced to 4- and lower point functions with “unphysical” external kinematics.

In this contribution, we explicitly perform the $\varepsilon$-expansion of the most complicated part: The scalar one-loop box function $I_{1111}$ with two photons (taken here in the s-channel), as it is needed for the calculation of Bhabha scattering up to order $O(\varepsilon)$. Our starting point is the analytical expression as known from \[3, 4\]:

\[
\frac{(t - 4m^2)}{\Gamma\left(2 - \frac{d}{2}\right)} I_{1111} = \\
\frac{t - 4m^2}{in^{d/2}\Gamma\left(2 - \frac{d}{2}\right)} \int \frac{dk_1}{d^{d}k_1} \frac{d^dk_1}{(k_1^2 + 2q_4k_1)(k_1 + q_1 + q_4)^2(k_1^2 - 2q_3k_1)} = \\
-4m^{d-4}s \, F_2\left(\frac{d - 3}{2}, 1, 1, \frac{3}{2}, \frac{d - 2}{2}; \frac{t}{t - 4m^2}, -m^2Z\right) \\
+ 4m^{d-4}(d - 3)s \, F_{1; 2; 1}^{1; 1; 0}\left[\frac{d - 2}{2}, \frac{d - 1}{2}; \frac{1}{2}, \frac{1}{2}; -m^2Z, 1 - \frac{4m^2}{s}\right] \\
- \sqrt{\pi} (-s)^{\frac{d-4}{2}} \, \frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} F_1\left(\frac{d - 3}{2}, 1, \frac{1}{2}; \frac{d - 1}{2}; sZ, \frac{1}{4}, 1 - \frac{s}{4m^2}\right) \\
\tag{1.1}
\]

with $Z = \frac{4u}{s(4m^2 - t)}$, $q_i^2 = m^2$, $(q_1 + q_4)^2 = s$, $(q_1 + q_2)^2 = t$ and $s, t, u$ the usual Mandelstam variables. Here $F_1$, $F_2$ are Appell hypergeometric functions

\[
F_1\left(\frac{d - 3}{2}, 1, \frac{1}{2}; \frac{d - 1}{2}; x, y\right) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(d - 3)}{r + s} \frac{1}{2} \frac{x^r y^s}{(1)_s}, \quad \tag{1.2}
\]
The factor $\Gamma$ above is not quite straightforward. In particular since the re stands the steps we obtain a form in which one of the parameters of $F_2$
\[ F_2 \left( \frac{d-3}{2}, 1, \frac{1}{2}, \frac{d-2}{2}; x, y \right) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\left( \frac{d-3}{2} \right)_r \left( \frac{d-2}{2} \right)_s}{r! s!} x^r y^s \] (1.3)
and the Kampé de Féret function (KdF) \[ 5 \] is defined as
\[ F^{1,2;1}_{1,1;0} \left[ \frac{d-3}{2}, \frac{d-3}{2} - \frac{1}{2}; x, y \right] = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\left( \frac{d-3}{2} \right)_r \left( \frac{d-2}{2} \right)_s}{r! s!} x^r y^s. \] (1.4)

The $\varepsilon$- expansion of the generalized hypergeometric functions occurring above is not quite straightforward. In particular since there stands the factor $\Gamma \left( 2 - \frac{d}{2} \right) \sim \frac{1}{\varepsilon}$ in front of all of them, one needs their expansion up to order $\varepsilon^2$. We have to develop different techniques for each of them.

2. Expansion of $F_1$

We need to know the expansion
\[ F_1 \left( \frac{d-3}{2}, 1, \frac{1}{2}, \frac{d-1}{2}; x, y \right) = F_1^0 + \varepsilon F_1^1 + \varepsilon^2 F_1^2 + \cdots \] (2.1)

with the kinematics: $x = -\frac{y}{t-4m_\varepsilon^2} < 0, y = 1 - \frac{8}{4m_\varepsilon^2} < 0, |y| \gg 1$. In two steps we obtain a form in which one of the parameters of $F_1 \sim \varepsilon$. The transformations are the following:

\[ F_1 \left( \frac{d-3}{2}, 1, \frac{1}{2}, \frac{d-1}{2}; x, y \right) = \frac{\Gamma \left( \frac{d-1}{2} \right) \Gamma \left( \frac{8-d}{2} \right)}{\Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{d-2}{2} \right)} 2F_1 \left[ 1, \frac{d-3}{2}, \frac{3}{2}, 1 - z \right] \]
\[ + \frac{d-3}{(d-6)(-x)\sqrt{-y}} F_1 \left( \frac{6-d}{2}, 1, \frac{1}{2}, \frac{8-d}{2}, 1, 1, \frac{1}{y} \right) \] (2.2)

\( \left( \frac{\varepsilon}{y} \equiv z = 1 - \frac{m_\varepsilon}{(t-4m_\varepsilon^2)(t-4m_\varepsilon^2)} ; 0 < z, 1 - z < 1 \) \) and

\[ F_1 \left( \frac{6-d}{2}, 1, \frac{1}{2}, \frac{8-d}{2}, 1, \frac{1}{x}, \frac{1}{y} \right) = \frac{\Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{8-d}{2} \right)}{\Gamma \left( \frac{d-d}{2} \right)} y^{\frac{d-4}{2}} 2F_1 \left[ 1, \frac{6-d}{2}, \frac{7-d}{2}, 1, \frac{1}{z} \right] \]
\[ + \left( d-6)\sqrt{X-1} (Y-X) \right) \frac{X}{X} F_1 \left( 1, \frac{d-4}{2}, 1, \frac{3}{2}, X, Y \right) \] (2.3)

with $X = 1 - y = \frac{x}{4m_\varepsilon^2} \gg 1, Y = \frac{y-1}{x-1} = 1 - \frac{4}{4m_\varepsilon^2} \gg 1 \ (X > Y, 1 - \frac{1}{X} = \omega)$. Here we observe that the argument of the $2F_1$ as well as those of the $F_1$ function are larger than 1, i.e. both functions are complex and the imaginary
parts must cancel since the $F_1$ on the l.h.s. has arguments less than 0 and thus is real. For the imaginary part of the $F_1$ function we obtain

$$\text{Im } F_1 \left( 1, \frac{d-4}{2}, 1, \frac{3}{2}; X, Y \right) = \frac{\pi \sqrt{X}(X-1)^{\frac{d}{2}}}{2Y\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+1\right)} \left[ \frac{1}{2}, b \right].$$ (2.4)

Transforming $2F_1\left[1, \frac{d-3}{2}, \frac{3}{2}; x, y\right]$ with argument $\frac{1}{2} > 1$ to a $2F_1$ function with argument $z < 1$, one shows that the imaginary parts cancel. Transforming (the real part) further to the argument $1 - z$ one finally obtains

$$F_1 \left( \frac{d-3}{2}, 1, \frac{1}{2} ; x, y \right) = -(d-3) \frac{Y}{\sqrt{X}} \text{ Re } F_1 \left( 1, \frac{d-4}{2}, 1, \frac{3}{2}; X, Y \right)$$

$$+ (d-3) \frac{\Gamma\left(\frac{d-3}{2}\right)\Gamma\left(\frac{d-3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \sin^2\left(\frac{\pi}{2} \frac{d}{2}\right) 2F_1 \left[ 1, \frac{d-3}{2}, \frac{3}{2}; 1 - z \right]$$

$$-(d-3) \frac{\pi}{2} \sin\left(\frac{\pi}{2} \frac{d}{2}\right)(-x) \frac{1}{\sqrt{-y(1-z)}}.$$ (2.5)

To expand up to the required order ($\sim \varepsilon^2$), it is sufficient to set $\sin\left(\frac{\pi}{2} \frac{d}{2}\right) = -\pi \varepsilon$ and

$$2F_1 \left[ 1, \frac{d-3}{2}, \frac{3}{2}; 1 - z \right] = \frac{1}{2 \sqrt{1 - \frac{1}{z}}} \ln\left( \frac{1 + \sqrt{1 - z}}{1 - \sqrt{1 - z}} \right) + O(\varepsilon).$$ (2.6)

This simplifies the expansion considerably and the $\text{Re } F_1 \left( 1, \frac{d-4}{2}, 1, \frac{3}{2}; X, Y \right)$ we take from [4]. Characteristic variables appearing in the result are

$$A = \sqrt{\frac{1 - \frac{1}{X} - 1}{1 - \frac{1}{X} + 1}} < 0, \quad B = \sqrt{\frac{1 - \frac{1}{Y} - 1}{1 - \frac{1}{Y} + 1}} < 0.$$ (2.7)

and introducing $a = \sqrt{1 - \frac{1}{X}}$, $b = \sqrt{1 - \frac{1}{Y}}$ we can write $(1 > a > b > 0)$

$$A = \frac{a-1}{a+1}, \quad B = \frac{b-1}{b+1}$$ and

$$F_1^0 = -\frac{m e}{\sqrt{s b}} \ln(-B),$$ (2.8)

yielding the correct $\frac{1}{2}$-term of $D_0$ [5]. Keeping only the leading terms, collecting the contributions, yields correspondingly

$$\frac{2}{s(t - 4m^2)} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} - \varepsilon\right)} \Gamma(1 - \varepsilon) \left( -\frac{s}{4} \right)^{-\varepsilon} \Gamma(\varepsilon) \frac{1}{b} \left[ \text{Re } \{ \ln(B) + \cdots \} \right.$$ 

$$- \pi^2 \varepsilon^2 \ln\left( \frac{1 - AB}{A - B} \right) - \pi^2 \varepsilon(1 + \varepsilon \ln\left( \frac{X}{Y} - 1 \right)) + 2\varepsilon^2 \pi^2 + O(\varepsilon^3) \right],$$ (2.9)
where the higher order terms in $\varepsilon$ of $\text{Re}\{\ln(B) + \cdots\}$ have to be taken from \[4\].

3. Expansion of $F_2$

We need to know the expansion

$$F_2\left(\frac{d-3}{2}, 1, 1, \frac{3}{2}, \frac{d-2}{2}; \omega, z\right) = F_2^0 + \varepsilon F_2^1 + \varepsilon^2 F_2^2 + \cdots \quad (3.1)$$

with the kinematics: $\omega = \frac{t}{t-4m_e^2}, z = -4m_e^2 \left(\frac{1}{s} + \frac{1}{t-4m_e^2}\right)$. At first we perform the following Euler transformation:

$$F_2\left(\frac{d-3}{2}, 1, 1, \frac{3}{2}, \frac{d-2}{2}; \omega, z\right) = (1 - z)^{\frac{3-d}{2}} F_2\left(\frac{d-3}{2}, 1, \frac{d-4}{2}, \frac{3}{2}, \frac{d-2}{2}; \omega, z - \frac{z}{1-z}\right). \quad (3.2)$$

The factor $(1 - z)^{\frac{3-d}{2}}$ will be dropped in what follows and is taken into account again when collecting the results. Introducing $\frac{\omega}{1-z} = \omega'$ and $-\frac{z}{1-z} = z'$, we have

$$0 \leq \omega' + z' < 1. \quad (3.3)$$

With $\alpha = \frac{1}{2} - \varepsilon, \beta = 1, \beta' = -\varepsilon, \gamma = \frac{3}{2}$ and $\gamma' = 1 - \varepsilon$

$$F_2(\alpha, \beta, \beta', \gamma, \omega', z') = 2 F_1(\alpha, \beta, \gamma; \omega') + \beta' S(\alpha, \beta, \beta', \gamma, \omega', z'), \quad (3.4)$$

where $\gamma' = 1 + \beta'$ has been used and

$$S(\alpha, \beta, \beta', \gamma, \omega', y) = \sum_{n=1}^{\infty} \frac{(\alpha)_n}{\beta + n \cdot n!} y^n 2 F_1(\alpha + n, \beta, \gamma; \omega') \quad (3.5)$$

In order to get rid of the denominator $\beta' + n$ we differentiate $S$ w.r.t. $y$ and use \[7\]

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n(\delta)_n}{n!(\delta)_n} y^n 2 F_1(\alpha + n, \beta, \gamma; \omega') = F_2(\alpha, \beta, \delta, \gamma, \delta'; \omega', y) \quad (3.6)$$

with $\delta = \delta'$. Applying again the Euler relation,

$$F_2(\alpha, \beta, \delta, \gamma, \omega', y) = (1 - y)^{-\alpha} F_2(\alpha, \beta, 0, \gamma, \delta; \frac{\omega'}{1-y}, -\frac{y}{1-y}) = (1 - y)^{-\alpha} 2 F_1(\alpha, \beta; \frac{\omega'}{1-y}), \quad (3.7)$$
we finally have \((\beta = 1, \gamma = \frac{3}{2} \) inserted)

\[
S(\alpha, \beta', \omega', z') = \int_{y=0}^{z'} \frac{\partial S(\alpha, \beta', \omega', y)}{\partial y} dy = S_0(\frac{1}{2}, 0, \omega', z') + \varepsilon S_1(\frac{1}{2}, 0, \omega', z') + O(\varepsilon^2), \quad (3.8)
\]

with

\[
\frac{\partial S}{\partial y} = \frac{1}{y} \left[ \frac{1}{(1-y)^\alpha} \begin{array}{c}
2F_1(\alpha, 1, \frac{3}{2}; \frac{\omega'}{1-y}) - 2F_1(\alpha, 1, \frac{3}{2}; \omega')
\end{array}
\right]
+ \frac{\varepsilon}{y} S(\frac{1}{2}, 0, \omega', y) + O(\varepsilon^2). \quad (3.9)
\]

To complete the \(\varepsilon\)-expansion, we need (with \(z = \frac{\omega'}{1-y}\))

\[
2F_1(\alpha, 1, \frac{3}{2}; z) = 2F_1(\frac{1}{2}, 1, \frac{3}{2}; z) + \varepsilon \delta^{(1)}(z, u) + \varepsilon^2 \delta^{(2)}(z, u) + \cdots \quad (3.10)
\]

with the abbreviation \(u = \frac{1+\sqrt{z}}{1-\sqrt{z}}\). Explicitely \(u = \frac{\sqrt{z}+w}{\sqrt{z}-w}\) with \(w = \sqrt{\omega'}\),

\[
\delta^{(1)}(z, u) = \frac{1}{2\sqrt{z}} \left[ 2Li_2(-\frac{1}{u}) - 2\ln(u)\ln(1+u) + \frac{3}{2}\ln^2(u) + \zeta(2) \right] \quad (3.11)
\]

and

\[
\delta^{(2)}(z, u) = \frac{1}{2\sqrt{z}} \left[ -4S_{1,2}(-\frac{1}{u}) - (\ln(u) + 2\ln(1 + \frac{1}{u}))\zeta(2) - 4\ln(1 + \frac{1}{u})Li_2(-\frac{1}{u}) + 2\ln(u)\ln^2(1 + \frac{1}{u}) + \ln^2(u)\ln(1 + \frac{1}{u}) + \frac{1}{6}\ln^3(u) + 2\zeta(3) + 2Li_3(-\frac{1}{u}) \right]. \quad (3.12)
\]

3.1. Order \(\varepsilon\) of \(F_2\)

In this order we have

\[
F_2(\alpha, \beta, \beta', \gamma, \gamma', \omega', z') = 2F_1(\frac{1}{2}, 1, \frac{3}{2}; \omega')
+ \varepsilon \delta^{(1)}(\omega', u_0) - \varepsilon S(\frac{1}{2}, 0, \omega', z') + O(\varepsilon^2),
\equiv F_2^0 + \varepsilon F_2^1 + O(\varepsilon^2), \quad (3.13)
\]
where the ‘scale’ \( u_0 = u(y = 0) = \frac{1+w}{1-w} \sim \frac{s}{m^2} \frac{1}{1-4m^2} \). \( u_0 \) is large for \( s \gg m^2 \) and \(-t \gg 4m^2\) and sets the scale for the variable \( u \) in general. Further we introduce \( u_1 = u(y = z') < u_0 \). Thus we can write

\[
F^0_2 = \frac{1}{2w} \ln(u_0),
\]

\[
S(\frac{1}{2}, 0, \omega', y) = \frac{1}{2w} \int_{y=0}^{y} \frac{\ln(u)}{u_0} dy
\]

\[
= \frac{1}{2w} \int_{u}^{u_0} \left[ \frac{2}{u-1} - \frac{1}{u-u_0} - \frac{1}{u-u_0} \right] \ln\left(\frac{u}{u_0}\right) du
\]

\[
\equiv \frac{1}{2w} S_0(u_0, u). \quad (3.14)
\]

and \( S(\frac{1}{2}, 0, \omega', z') = \frac{1}{2w} S_0(u_0, u_1) \). The integration yields:

\[
S_0(u_0, u) = 2 \text{Li}_2\left(\frac{1}{u}\right) + \text{Li}_2\left(\frac{u}{u_0}\right) - 2 \text{Li}_2\left(-\frac{1}{u_0}\right) - \zeta(2) - \ln\left(\frac{u}{u_0}\right) \left[ 2 \ln(u - 1) - \ln(u_0u - 1) - \ln(1 - \frac{u}{u_0}) - \frac{1}{2} \ln\left(\frac{u}{u_0}\right) \right]. \quad (3.15)
\]

3.2. Order \( \varepsilon^2 \) of \( F_2 \)

The next order can be written in the form

\[
F_2^2 = \delta^{(2)}(\omega', u_0) - S_1(\frac{1}{2}, 0, \omega', z') \quad (3.16)
\]

with

\[
S_1(\frac{1}{2}, 0, \omega', z') \equiv S_1(u_0, u_1) = \int_{y=0}^{y'} dy \left\{ \frac{1}{2w} \ln(1 - y) \ln(u) + \delta^{(1)}(u_0, \omega', y) \right\} \quad (3.17)
\]

The curly bracket in the above integrand finally reads

\[
\{ \cdots \} = \text{Li}_2\left(\frac{u}{u_0}\right) - \text{Li}_2(1) + \text{Li}_2\left(\frac{1}{u^2}\right) - \text{Li}_2\left(\frac{1}{u_0u}\right) + 2 \ln(u) \ln\left(\frac{u_0 - 1}{u - 1}\right) - 2 \ln(u_0 + 1) \ln\left(\frac{u}{u_0}\right) + \frac{3}{2} \ln(u_0u) \ln\left(\frac{u}{u_0}\right) - \ln\left(\frac{u}{u_0}\right)
\]

\[
\left[ 2 \ln(u - 1) - \ln(u_0u - 1) - \ln(1 - \frac{u}{u_0}) - \frac{1}{2} \ln\left(\frac{u}{u_0}\right) \right]. \quad (3.18)
\]
There is no problem to perform the final integration, but the expressions blow up considerably. Therefore we confine ourselves here to the leading terms only by considering \( u(u_0) \) as large and drop the small quantities. Then the integral can be written in the simplified form (\( \frac{1}{u_0} = v \) the new integration variable)

\[
S_1(u_0, u_1) = \frac{1}{2w} \int_{v=r}^{1} \left[ \frac{1}{v} + \frac{1}{1 - v} \right] \left\{ \text{Li}_2(v) - \text{Li}_2(1) - \ln^2(v) - \ln(u_0)\ln(v) + \ln(v)\ln(1 - v) \right\}
+ \ln(v)\ln(1 - v) \right\} = \frac{1}{2w} \left[ -\text{Li}_3(1 - r) + \ln(1 - r)\text{Li}_2(r) + \frac{1}{3}\ln^3(r) \right.
+ \frac{1}{2}(\ln(u_0) - \ln(1 - r))\ln^2(r) + \ln^2(1 - r)\ln(r) - \zeta(2)\ln(\frac{1}{r} - 1) \right]
\]

(3.19)

with \( 0 < r = \frac{u_1}{u_0} < 1 \). If one wants higher precision, it is easier to expand in \( \frac{1}{u} (\frac{1}{u_0}) \) instead of performing all integrals analytically, which is possible nevertheless. Collecting the results, we have

\[
- \frac{2(m_e^2)^{-\varepsilon}}{s(t - 4m_e^2)} (1 - z)\varepsilon \Gamma(\varepsilon) \frac{1}{\sqrt{\omega}} (\ln(u_0) + \cdots) \right)
\]

(3.20)

4. Expansion of the Kampé de Fériet function

We need to know the expansion

\[
\left[ \begin{array}{c}
\frac{d_1 - 3}{d_1}; \frac{d_1 - 3}{d_1}; 1; \frac{1}{1 - \varepsilon} \gamma(x, y) \\
\frac{d_2 - 2}{d_2}; \frac{d_2 - 2}{d_2}; -; \varepsilon; y
\end{array} \right] = K^0 + \varepsilon K^1 + \varepsilon^2 K^2 + \cdots.
\]

(4.1)

with the kinematics: \( x = -4m_e^2 \left( \frac{1}{s} + \frac{1}{t - 4m_e^2} \right); y = 1 - \frac{4m_e^2}{s} \). In this case we begin with the integral representation of the KdF function:

\[
\left[ \begin{array}{c}
\frac{d_1 - 3}{d_1}; \frac{d_1 - 3}{d_1}; 1; \frac{1}{1 - \varepsilon} \gamma(x, y) \\
\frac{d_2 - 2}{d_2}; \frac{d_2 - 2}{d_2}; -; \varepsilon; y
\end{array} \right]
= \frac{d_2 - 3}{2} \int_0^1 dt \frac{t^{d_2 - 3}}{1 - t y} \text{ } 2F\left[ 1, \frac{d_1 - 3}{d_1}, \frac{d_2 - 2}{d_2}, x t \right]
\]

(4.2)

Again we perform a shift such that one of the parameters of the \( 2F_1 \sim \varepsilon \):

\[
2F\left[ 1, \frac{d_1 - 3}{d_1}, \frac{d_2 - 2}{d_2}, x t \right] \right) = (1 - x t)^{-\frac{d_1 - 3}{d_1}} 2F\left[ -\varepsilon, \frac{1}{2}, 1 - \varepsilon, x t \right]
\]

\[
= \left( 1 - x t \right)^{-\frac{d_1 - 3}{d_1}} \left[ 1 - \varepsilon S(x t) \right].
\]

(4.3)
with
\[ S(x t) = \sum_{n=1}^{\infty} \frac{1}{n - \varepsilon} \frac{(\frac{1}{x})^n}{n!} (x t)^n. \] (4.4)

The \(\varepsilon\)-expansion of \(S(x t)\) can again be obtained by first differentiating \(S\):
\[
\frac{\partial S}{\partial x} = \frac{1}{x} \left( \frac{1}{\sqrt{1 - x t}} - 1 \right) + \frac{\varepsilon}{x} \sum_{n=1}^{\infty} \frac{1}{n} \frac{(\frac{1}{x})^n}{n!} (x t)^n + O(\varepsilon^2)
\]
\[
= \frac{1}{x} \left( \frac{1}{\sqrt{1 - x t}} - 1 \right) + \frac{\varepsilon}{x} S(x t)|_{\varepsilon=0} + O(\varepsilon^2).
\] (4.5)

In order to obtain \(S\) to \(O(\varepsilon)\), we need the following integrals:

\[ S(x t)|_{\varepsilon=0} = \int_0^x \frac{dx}{x} \left( \frac{1}{\sqrt{1 - x t}} - 1 \right) = 2\ln(1 + \frac{1}{v}) \] (4.6)

and

\[ \int_0^x \frac{dx}{x} S(x t)|_{\varepsilon=0} = -2\text{Li}_2(-\frac{1}{v}) - 2\ln^2(1 + \frac{1}{v}), \] (4.7)

where we introduced \(v = \frac{1 + \sqrt{1 - x t}}{1 - \sqrt{1 - x t}}\). Thus the above integral reads
\[
\int_0^1 \frac{dt}{1 - t} \frac{1}{\sqrt{1 - x t}} \left\{ 1 - \varepsilon \ln\left( \frac{4}{v x} \right) - \varepsilon^2 \left[ -2\text{Li}_2(-\frac{1}{v}) - \frac{1}{2} \ln^2\left( \frac{4}{v x} \right) \right] \right\} \] (4.8)

After a variable transformation we can write

\[ \int_0^1 \frac{dt}{1 - t} \frac{1}{\sqrt{1 - x t}} f(v) = -\frac{1}{\sqrt{y - x}} \int_0^1 dt \left\{ b_1 \left[ \frac{1}{1 + b_2 t} + \frac{1}{1 - b_2 t} \right] \right\} f\left( \frac{v_1}{t^2} \right) \] (4.9)

with \(v_1 = v(t = 1) = \frac{1 + \sqrt{1 - x t}}{1 - \sqrt{1 - x t}}\), \(b_1 = \frac{1}{\sqrt{v_0}}\), \(v_1\) and \(v_0\) both being large, results in \(b_1 \ll 1\) and \(b_2 < 1\) but very close to 1. Taking again the attitude to keep only leading contributions, the \(b_1\)-contribution can be dropped. The \(\text{Li}_2\)-function in the second order term can be written as
\[
\text{Li}_2\left( \frac{t^2}{v_1} \right) = 2 \left[ \text{Li}_2\left( i \frac{t}{\sqrt{v_1}} \right) + \text{Li}_2\left( -i \frac{t}{\sqrt{v_1}} \right) \right] \] (4.10)
so that integration is possible. We do get, however, relatively complicated complex conjugate contributions. On the other hand since \( v_1 \gg 1 \) this contribution is small from the very beginning and can be well approximated by expanding the \( \text{Li}_2 \)-function. Here it is dropped altogether. Thus we are left with the following contributions:

\[
K^0 = \frac{d - 3}{2\sqrt{\omega}} \ln(u_0),
\]

\[
K^1 = \frac{d - 3}{2\sqrt{\omega}} \left[ \ln\left(\frac{1 + b_2}{1 - b_2}\right) \ln\left(\frac{v_1 x}{4}\right) + 2(\text{Li}_2(b_2) - \text{Li}_2(-b_2)) \right]
\]

\[
K^2 = \frac{d - 3}{2\sqrt{\omega}} \left[ \frac{1}{2} \ln\left(\frac{1 + b_2}{1 - b_2}\right) \ln^2\left(\frac{v_1 x}{4}\right) + 2\ln\left(\frac{v_1 x}{4}\right) (\text{Li}_2(b_2) - \text{Li}_2(-b_2)) \\
+ 4(\text{Li}_3(b_2) - \text{Li}_3(-b_2)) \right]
\]

Collecting the results we obtain

\[
\frac{2(m_e^2)^{-\varepsilon}}{s(t - 4m_e^2)} \Gamma(\varepsilon) \frac{1}{\sqrt{\omega}} \ln(u_0) + \cdots
\]

We see that the \( \frac{1}{\varepsilon} \)-term cancels against the one from the \( F_2 \) contribution.

5. Expansion of the scalar box function with Feynman parameters

In order to have a completely independent numerical check of the above results, we derived a Feynman parameter integral representation for the \( \varepsilon \)-expansion. We follow closely [8], where the scalar four-point integral was treated with a finite photon mass in \( d = 4 \) dimensions.

The function to be calculated is, in LoopTools notations [9]:

\[
J = D_0(m^2, m^2, m^2, m^2 \mid t, s \mid m^2, 0, m^2, 0)
\]

\[
= \frac{(2\pi \mu)^{2\varepsilon}}{i\pi^2} \int \frac{d^dk}{k^2(k^2 + 2kq_4)(k^2 - 2kq_3)(k + q_1 + q_4)^2}.
\]

A constant transforms the normalization of \( D_0 \) to that of \( I^{(d)}_{1111} \):

\[
D_0 = (4\pi \mu^2)^\varepsilon I^{(d)}_{1111}.
\]
The infrared singularity may be isolated in a 3-point function $C_0$ by redefining

$$J = \frac{2}{s} (F + C_0), \quad (5.3)$$

with

$$C_0 = C_0(t, \mu^2, m^2 | m^2, \mu^2, 0) = \frac{(2\pi\mu)^{2\varepsilon}}{i\pi^2} \int \frac{d^dk}{k^2(k^2 + 2kq_4)(k^2 - 2kq_3)}, \quad (5.4)$$

and with

$$F = \frac{(2\pi\mu)^{2\varepsilon}}{i\pi^2} \int \frac{d^dk}{k^2(k^2 + 2kq_4)(k^2 - 2kq_3)(k + q_1 + q_4)^2} \quad (5.5)$$

being a finite scalar four point function.

The $\varepsilon$-expansions may be easily derived now starting from

$$C_0 = \Gamma(\varepsilon) \int_0^1 dx \frac{2p^2}{2p^2_x} \left[ \frac{4\pi\mu^2}{p^2_x} \right]^{\varepsilon}, \quad (5.6)$$

$$F = \Gamma(2 + \varepsilon) \int_0^1 dx dy dz \frac{y^2z}{(M^2)^2} \left( \frac{1}{2} y z s + \frac{1 - 2\varepsilon}{1 + \varepsilon} M^2 \right) \left[ \frac{4\pi\mu^2}{p^2_x} \right]^{\varepsilon}$$

$$= \Gamma(2 + \varepsilon) \left[ I_0 + \varepsilon I_1 + \varepsilon I_L \right] + \ldots, \quad (5.7)$$

with

$$p^2_x = -x(1-x)t + m^2 - i\varepsilon, \quad (5.8)$$

$$M^2 = y[yz^2(p^2_x - (1-y)(1-z)s]. \quad (5.9)$$

Thus, the four-point function may be represented as follows:

$$I_0 = -\int_0^1 \frac{dx}{2p^2_x} \ln(-A), \quad (5.10)$$

$$I_1 = -3 \int_0^1 \frac{dx}{p^2_x} \frac{dz}{N(z)} \left( \frac{z}{N(z)} + \frac{Az(1-z)}{(1-z)(-A)} \right) \ln \frac{z^2}{(1-z)(-A)}, \quad (5.11)$$

with

$$A = \frac{s}{p^2_x}, \quad (5.12)$$

$$N(z) = z^2 + (1-z)A. \quad (5.13)$$
The last function will be given here in short as a three-fold integral. But it is evident that the $y$-integration leads to simple integrals in terms of dilogarithms or simpler functions:

$$I_L = \int_{0}^{1} \frac{dx}{p_z^2} dz dy \left[ \frac{Ay}{2z^2 K(y)^2} + \frac{y}{zK(y)} \right] \ln \frac{4\pi\mu^2 A}{z^2 y K(y) s}, \quad (5.14)$$

with

$$K(y) = y - (1 - y) \frac{1 - z}{z^2} A. \quad (5.15)$$

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