No-Interaction Theorem
without Hamiltonian and Lagrangian Formalism

--- Invariant Momentum on Null Cones ---

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In a previous paper [G. Yoneda, Proc. R. Soc. London A445 (1994), 221], we proved the no-interaction theorem for four particles with the assumption that the (linear and angular) momentum on space-like planes is invariant. In this paper, we assume that the momentum on null cones is invariant and prove that there is no interaction for four particles.

§ 1. Introduction

The no-interaction theorem states that there is no interaction in a relativistically invariant system of particles in the framework of Hamiltonian formalism. Proofs of the no-interaction theorem have been given by many authors in the frameworks of Hamiltonian and Lagrangian formalism. These proofs are based on the assumption that there is a Hamiltonian or Lagrangian describing the interaction between particles.

In a previous paper, we proved this theorem without the use of Hamiltonian or Lagrangian formalism. The assumption in the theorem, relativistic invariance, is that the sums of the linear and angular momenta of individual particles on a space-like plane are independent of the choice of the plane. Also, we proved that the worldlines of the individual particles are straight.

In this paper, we use a null cone instead of a space-like plane. Namely, we prove the no-interaction theorem with the assumption of an invariant momentum on null cones. This form of the no-interaction theorem is simple, like the previous form.

We consider the meaning of the no-interaction theorem to be summarized by the following.

(a) Newtonian momentum invariance regarding only particles’ momenta is meaningless in special relativity.
(b) In general relativity, the flatness of Minkowski space is usually accounted for by the observation that zero curvature makes the worldlines of particles straight. However, with the no-interaction theorem, we can account for the flatness of Minkowski space by noting that the invariance of the total momentum makes the worldlines of particles straight. In other words, the no-interaction theorem implies the flatness of Minkowski space kinematically.

Thus we believe that obtaining no-interaction theorem in simple form is meaningful.

Next we comment on the difference between the assumptions of this and previous

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papers. Two assumptions are different with regard to geometrical meaning. A spacelike plane can be interpreted as representing simultaneity, while a null cone does not have such a Newtonian interpretation. A null cone can be interpreted as the set of all events that can be observed at some time and place. However, in the above statement (b) concerning the meaning of the no-interaction theorem, the invariance of null cones is a more elegant assumption in view of relativistic invariance.

The plan of this paper is as follows. In §2, we introduce the notation and present the main theorem. In §3, we give a proof of this theorem. Section 4 is devoted to discussion. In the Appendix, we prove lemmas that are used in §3.

§2. Theorem

The stage of the theory is four-dimensional Minkowski affine space \((M, V)\), where the base set \(M\) is the set of all events, and the four-dimensional vector space \(V\) is the set of all parallel transformations of \(M\). The metric of \(V\) is of type \((+, −, −, −)\) and denoted by \(\langle | \rangle\). The ordered pair \((m, X)\) of a positive number \(m\) and a smooth timelike worldline\(^7\) \(X\) is called “the particle”. In this paper, the particle has positive constant rest mass \(m\) and is structureless. The worldline \(X\) is parametrized by its proper time \(\tau\). Thus \(\dot{X}(\tau)\) is the proper velocity, which is a future time-like unit, \(\ddot{X}\) is the proper acceleration, \(m\dot{X}(\tau)\) is the linear momentum, and \(m\overrightarrow{OX}(\tau) \wedge \dot{X}(\tau)\) is the angular momentum with respect to the origin \(O \in M\), where \(\wedge\) is the ordinary wedge product. (The arrow denotes a vector.) The set of particles \((m_i, X_i) (i = 1, 2, \cdots, N)\) is called the system of \(N\) particles if they do not collide with each other, i.e., \(X_i \cap X_j = \emptyset\) for \(i \neq j\).

The past null cone with apex \(x \in M\) is defined as

\[ C(x) = \{ y \in M : \overrightarrow{yx} \text{ is future null} \}. \]

For a system of \(N\) particles, we define the linear and angular momenta on the null cones \(C(x)\) as the instantaneous sums over the respective quantities of individual particles:

\[
\vec{P}(x) = \sum_{i=1,\cdots,N} m_i \dot{X}_i(\tau_i), \\
\vec{A}(x) = \sum_{i=1,\cdots,N} m_i \overrightarrow{OX}_i(\tau_i) \wedge \dot{X}_i(\tau_i).
\]

Here \(X_i(\tau_i) \in C(x)\). (The boldface denotes a rank-two tensor.) Now we assume that the worldlines intersect \(C(x)\). (There is the possibility that worldlines do not intersect \(C(x)\), but such a case is not realistic.)\(^8\) The no-interaction theorem in this paper is the following.

**Theorem** For a system of four particles, if \(\vec{P}(x)\) and \(\vec{A}(x)\) are independent of \(x\), the worldline of every particle is straight.

In this paper, we assume an invariant momentum on null cones instead of on space-like planes, which we assumed in the previous paper.
§3. Proof of the theorem

We can regard $\tau_i$, defined by $X_i(\tau_i) \in C(x)$, as a scalar field $M \to \mathbb{R}$. From the null condition $0 = \langle X_i(\tau_i) x | X_i(\tau_i) x \rangle$, we obtain the gradient of $\tau_i$:

$$\text{grad } \tau_i = \frac{X_i(\tau_i) x}{\langle X_i(\tau_i) x | X_i(\tau_i) x \rangle}.$$  \hspace{1cm} (1)

We denote the invariance of the linear momentum in tensor form as follows. We will prove $\ddot{\tau}$ of the theorem.

We denote the invariance of the angular momentum in tensor form as well. We denote the invariance of the linear momentum in tensor form as follows. We choose $x$ so that $\ddot{\tau}$ (4). Thus $\ddot{\tau} = 0$ is sufficient to obtain the conclusion of the theorem.

We may choose $x$ so that $\ddot{\tau} = 0$. The number $\tau_i = 1$ and proper time $\tau_i$ do not have special meaning. Thus the proof of $\ddot{\tau} = 0$ is sufficient to obtain the conclusion of the theorem.

We may choose $x$ so that $\ddot{\tau}$ is parallel with neither $X_2 x$, $X_3 x$ nor $X_4 x$. (See Lemma (4).) It is useful to consider the independence of the null vectors $\ddot{X}_1 x$, $\ddot{X}_2 x$, $\ddot{X}_3 x$ and $\ddot{X}_4 x$ by considering four cases: (a) dim$[\ddot{X}_2 x, \ddot{X}_3 x, \ddot{X}_4 x] = 1$; (b) dim$[\ddot{X}_2 x, \ddot{X}_3 x, \ddot{X}_4 x] = 2$; (c) dim$[\ddot{X}_1 x, \ddot{X}_2 x, \ddot{X}_3 x, \ddot{X}_4 x] = 4$ (so dim$[\ddot{X}_2 x, \ddot{X}_3 x, \ddot{X}_4 x] = 3$); (d) dim$[\ddot{X}_2 x, \ddot{X}_3 x, \ddot{X}_4 x] = 3$ and dim$[\ddot{X}_1 x, \ddot{X}_2 x, \ddot{X}_3 x, \ddot{X}_4 x] = 3$.

For the cases (a), (b) and (c), we see that there exists $\ddot{r} \in V$ such that

$$\langle \ddot{X}_1 x | \ddot{r} \rangle \neq 0 \quad \text{and} \quad \langle \ddot{X}_2 x | \ddot{r} \rangle = \langle \ddot{X}_3 x | \ddot{r} \rangle = \langle \ddot{X}_4 x | \ddot{r} \rangle = 0.$$  \hspace{1cm} (4)

Thus in these cases, using $Q(\ddot{r}, \cdot) = 0$, we have $\ddot{w}_i = 0$, and thus $\ddot{X}_1(\tau_i) = 0$.

For the case (a), the three vectors $\ddot{X}_2 x$, $\ddot{X}_3 x$ and $\ddot{X}_4 x$ are parallel to each other, and not parallel to $\ddot{X}_1 x$. Thus $\ddot{r} = \ddot{X}_2 x$ satisfies (4).
For the case (b), some pair of \( \overrightarrow{X_2x}, \overrightarrow{X_3x} \) and \( \overrightarrow{X_4x} \) are parallel to each other, while the remaining one is not parallel to the others, because three non-parallel null vectors are linearly independent (Lemma 3). We assume, for example, \( \overrightarrow{X_2x} \parallel \overrightarrow{X_3x} \parallel \overrightarrow{X_4x} \).

Then let us set \( \vec{r} := -\langle \overrightarrow{X_2x} \mid \overrightarrow{X_3x} \rangle \overrightarrow{X_1x} + \langle \overrightarrow{X_1x} \mid \overrightarrow{X_3x} \rangle \overrightarrow{X_2x} + \langle \overrightarrow{X_1x} \mid \overrightarrow{X_2x} \rangle \overrightarrow{X_3x} \).

We then easily see this \( \vec{r} \) satisfies (4).

For the case (c), let us consider a nonzero vector \( \vec{r} \in [\overrightarrow{X_2x}, \overrightarrow{X_3x}, \overrightarrow{X_4x}]^\perp \) (orthogonal space). We remark that \( V = [\overrightarrow{X_1x}, \overrightarrow{X_2x}, \overrightarrow{X_3x}, \overrightarrow{X_4x}] \). If we assume, on the contrary, that \( \langle \vec{r} \mid \overrightarrow{X_1x} \rangle = 0 \), the non-zero vector \( \vec{r} \) is perpendicular to all vectors of \( V \), which contradicts the non-degeneracy of the metric. Therefore we see that \( \vec{r} \) satisfies (4).

The remaining case is (d). The proof for this case is more complicated than the others. Since \( \dim[\overrightarrow{X_2x}, \overrightarrow{X_3x}, \overrightarrow{X_4x}] = 3 \), we see that none of \( \overrightarrow{X_2x}, \overrightarrow{X_3x} \) and \( \overrightarrow{X_4x} \) are parallel to each other. Thus none of \( \overrightarrow{X_1x}, \overrightarrow{X_2x}, \overrightarrow{X_3x} \) and \( \overrightarrow{X_4x} \) are parallel to each other. Since \( \dim[\overrightarrow{X_1x}, \overrightarrow{X_2x}, \overrightarrow{X_3x}, \overrightarrow{X_4x}] = 3 \), there exist nonzero real \( a_i \) (\( i = 1, 2, 3, 4 \)) such that \( \sum a_i \overrightarrow{X_i} = 0 \). Note that if we have \( \sum b_i \overrightarrow{X_i} = 0 \), then there exists \( b \in \mathbb{R} \) such that \( b_i = a_i b \), because of the linear independence of \( \overrightarrow{X_2x}, \overrightarrow{X_3x} \) and \( \overrightarrow{X_4x} \). (Lemma 3)

From \( Q = 0 \), there exists \( \vec{w} \in V \) such that \( \vec{w}_i = a_i \vec{w} \). (Use Lemma 3 for every component of \( \vec{w}_i \).) By substitution of \( \vec{w}_i = a_i \vec{w} \) into \( B = 0 \), we have

\[
0 = B = \sum \overrightarrow{X_i} \otimes O \overrightarrow{X_i} \wedge a_i \vec{w}.
\]

Thus we see that there exists \( C \in V \wedge V \) such that \( O \overrightarrow{X_i} \wedge a_i \vec{w} = a_i C \) (\( i = 1, 2, 3, 4 \)). (Use Lemma 3 for every component of \( O \overrightarrow{X_i} \wedge a_i \vec{w} \)). Therefore we see that \( O \overrightarrow{X_i} \wedge \vec{w} \) is independent of \( i \). Thus we have \( \overrightarrow{X_i} \overrightarrow{X_j} \wedge \vec{w} = 0 \) (\( i, j = 1, 2, 3, 4 \)). This implies that \( \overrightarrow{X_i} \parallel \vec{w} \) (\( i, j = 1, 2, 3, 4 \)). Noting that \( \overrightarrow{X_1x} \parallel \overrightarrow{X_2x} \) is not parallel to \( \overrightarrow{X_1x} \) (If they are parallel, \( \overrightarrow{X_1x}, \overrightarrow{X_2x} \) and \( \overrightarrow{X_3x} \) are dependent, which contradicts Lemma 3), we get \( \vec{w} = 0 \). Thus we have \( \overrightarrow{X_1} = 0 \), so that \( \overrightarrow{X_1} = 0 \). This proves the theorem.

§4. Discussion

We succeeded in proving the no-interaction theorem with the assumption of the invariance of the momentum on null cones. Because of differences in the assumptions, we do not give another proof of the no-interaction theorems of the papers cited. However, we believe that our theorems (in this and previous papers) follow the spirit of the Currie no-interaction theorem. The reason that the proof can be carried out in the same way as that for the previous paper’s theorem is that there are four particles. The following two questions are still not answered and are being studied: (1) Is the theorem satisfied in the case of more than five particles? (2) Is the assumption for the angular momentum necessary?
Appendix

Lemma 1 There exists $x_0 \in M$ such that $\overrightarrow{X_1(\tau_1)x_0}$ is not parallel with $\overrightarrow{X_2(\tau_2)x_0}$, $
\overrightarrow{X_3(\tau_3)x_0}$ nor $\overrightarrow{X_4(\tau_4)x_0}$, where $X_1(\tau_1), X_2(\tau_2), X_3(\tau_3), X_4(\tau_4) \in C(x_0)$.

proof Let us choose $\tau'_1, \tau'_3, \tau'_4$ so that $X_2(\tau'_1), X_3(\tau'_3), X_4(\tau'_4) \in C(X_1(\tau_1))$. We can choose the future null vector $\vec{p}$ except for the future null directions $\overrightarrow{X_2(\tau'_1)X_1(\tau_1)}$, $\overrightarrow{X_3(\tau'_3)X_1(\tau_1)}$ and $\overrightarrow{X_4(\tau'_4)X_1(\tau_1)}$. Also, we set $x_0 = X_1(\tau_1) + \vec{p}$. Then $X_1(\tau_1) \in C(x_0)$. Next, choose $\tau_2, \tau_3, \tau_4$ such that $X_2(\tau_2), X_3(\tau_3), X_4(\tau_4) \in C(x_0)$. If we assumed $\overrightarrow{X_1(\tau_1)x_0} \parallel \overrightarrow{X_2(\tau_2)x_0}$, by contrast, we see $\overrightarrow{X_2(\tau_2)X_1(\tau_1)} \parallel \overrightarrow{X_1(\tau_1)x_0} = \vec{p}$, which contradicts the way in which $\vec{p}$ was selected. Hence we conclude $\overrightarrow{X_1(\tau_1)x_0} \not\parallel \overrightarrow{X_2(\tau_2)x_0}$.

Similarly we see $\overrightarrow{X_1(\tau_1)x_0} \not\parallel \overrightarrow{X_3(\tau_3)x_0}$, $\overrightarrow{X_1(\tau_1)x_0} \not\parallel \overrightarrow{X_4(\tau_4)x_0}$. □

Lemma 2 Three null vectors that are not parallel to each other are linearly independent.

proof For three non-parallel null vectors $\vec{p}_1$, $\vec{p}_2$ and $\vec{p}_3$, we assume on the contrary that $\vec{p}_1 = a\vec{p}_2 + b\vec{p}_3$ for some nonzero $a, b \in \mathbb{R}$. Then we would see $\langle \vec{p}_1 | \vec{p}_1 \rangle = 2ab\langle \vec{p}_2 | \vec{p}_3 \rangle \neq 0$, which is a contradiction. Thus we see that the vectors $\vec{p}_1$, $\vec{p}_2$ and $\vec{p}_3$ are linearly independent. □

Lemma 3 For a vector $\vec{p}_1$ and three independent vectors $\vec{p}_2$, $\vec{p}_3$ and $\vec{p}_4$, we assume that there exists a nonzero real $a_i$ ($i = 1, 2, 3, 4$) such that $\Sigma a_i \vec{p}_i = 0$. If we have $\Sigma b_i \vec{p}_i = 0$, then there exists $b \in \mathbb{R}$ such that $b_i = a_ib$ ($i = 1, 2, 3, 4$).

proof We may assume $b_1 \neq 0$, because if $b_1 = 0$, we see that $b_2 = b_3 = b_4 = 0$ by the independence of $\vec{p}_2$, $\vec{p}_3$ and $\vec{p}_4$. Then we can set $b = 0$. From $\Sigma a_i \vec{p}_i = 0$ and $\Sigma b_i \vec{p}_i = 0$, we have

\[ \vec{p}_1 = \frac{a_2}{a_1} \vec{p}_2 - \frac{a_3}{a_1} \vec{p}_3 - \frac{a_4}{a_1} \vec{p}_4 = -\frac{b_2}{b_1} \vec{p}_2 - \frac{b_3}{b_1} \vec{p}_3 - \frac{b_4}{b_1} \vec{p}_4, \]

\[ 0 = \left( \frac{a_2}{a_1} \frac{b_2}{b_1} \right) \vec{p}_2 + \left( \frac{a_3}{a_1} \frac{b_3}{b_1} \right) \vec{p}_3 + \left( \frac{a_4}{a_1} \frac{b_4}{b_1} \right) \vec{p}_4. \]

Since the vectors $\vec{p}_2$, $\vec{p}_3$ and $\vec{p}_4$ are linearly independent, we get $a_2/a_1 = b_2/b_1$, $a_3/a_1 = b_3/b_1$ and $a_4/a_1 = b_4/b_1$. Hence we obtain $b_1/a_1 = b_2/a_2 = b_3/a_3 = b_4/a_4$. Therefore we can set $b = b_1/a_1$. □

References

[1] D. G. Currie, T. F. Jordan and E. C. G. Sudarshan, Rev. Mod. Phys. 35 (1963), 350.
[2] J. T. Cannon and T. F. Jordan, J. Math. Phys. 5 (1964), 299.
[3] H. Leutwyler, Nouvo Cim. 37 (1965), 556.
[4] A. P. Balachandran, G. Marmo and A. Stern, Nouvo Cim. 69 (1982), 175.
[5] G. Marmo, N. Mukunda and E. C. G. Sudarshan, Phys. Rev. D30 (1984), 2110.
[6] G. Yoneda, Proc. R. Soc. London A445 (1994), 221
[7] G. Yoneda, Y. Ishigami and S. Arima, J. Phys. Soc. Jpn. 62 (1993), 1495.