Distributed Robust Output Regulation of Heterogeneous Uncertain Linear Agents by Adaptive Internal Model Principle

Satoshi Kawamura¹, Kai Cai¹, and Masako Kishida²

Abstract—We study a multi-agent output regulation problem, where not all agents have access to the exosystem’s dynamics. We propose a fully distributed controller that solves the problem for linear, heterogeneous, and uncertain agent dynamics as well as time-varying directed networks. The distributed controller consists of two parts: (1) an exosystem generator that locally estimates the exosystem dynamics, and (2) a dynamic compensator that, by locally approaching the internal model of the exosystem, achieves perfect output regulation. Moreover, we extend this distributed controller to solve an output synchronization problem where not all agents initially have the same internal model dynamics. Our approach leverages methods from internal model based controller synthesis and multi-agent consensus over time-varying directed networks; the derived result is a generalization of the (centralized) internal model principle to the distributed, networked setting.

I. INTRODUCTION

Over the past decade, many distributed control problems of networked multi-agent systems have been extensively studied; these include e.g. consensus, averaging, synchronization, coverage, and formation (e.g. [1]–[3]). Progressing beyond first/second-order and homogeneous agent dynamics, the distributed output regulation problem with general linear (time-invariant, finite-dimensional) and heterogeneous agent dynamics has received much recent attention (e.g. [4]–[9]). In this problem, a network of agents each tries to match its output with a reference signal, under the constraint that only a few agents can directly measure the reference. The reference signal itself is typically generated by an external dynamic system, called “exosystem”. The distributed output regulation problem not only subsumes some earlier problems such as (leader-following) consensus and synchronization, but also addresses issues of disturbance rejection and robustness to parameter uncertainty. Also see e.g. [10], [11] for further extensions of this problem that deal with nonlinear agent dynamics.

Output regulation has a well-studied centralized version: A single plant tries to match its output with a reference signal (while maintaining the plant’s internal stability) [12], [13]. In the absence of system parameter uncertainty, the solution of the “regulator equations”, embedding a copy of the exosystem dynamics, provides a solution to output regulation [13]. When system parameters are subject to uncertainty, however, a dynamic compensator/controller must be used embedding q-copy of the exosystem, where q is the number of (independent) output variables to be regulated. The latter is well-known as the internal model principle [12]. These methods for solving the centralized output regulation problem, however, cannot be applied directly to the distributed version, inasmuch as not all agents have direct access to the reference signal or the exosystem dynamics.

The distributed output regulation of networks of heterogeneous linear agents is studied in [5]. The proposed distributed controller consists of two parts: an exosystem generator and a controller based on regulator equation solutions. Specifically, the exosystem generator of each agent aims to (asymptotically) synchronize with the exosystem using consensus protocols, thereby creating a local estimate of the exosystem. Meanwhile each agent independently tracks the signal of its local generator, by applying standard centralized methods (in regulator equation solutions are applied). This approach effectively separates the controller synthesis into two parts – distributed exosystem generators by network consensus and local output regulation by regulator equation solution.

One important limitation, however, of the above approach is: in both the exosystem generator design and the regulator equation solution, it is assumed that each agent uses exactly the same dynamic model as that of the exosystem. This assumption may be unreasonable in the distributed network setting, because those agents that cannot directly measure the reference signal are unlikely to know the precise dynamic model of the exosystem. To deal with this challenge, [8] proposes (in the case of static networks) an “adaptive” exosystem generator and an adaptive solution to the regulator equations. In essence, each agent runs an additional consensus algorithm to update their “local estimates” of the exosystem dynamics.

All the regulator-equation based solutions above fall short in addressing the issue of system parameter uncertainty. In practice one may not have precise knowledge of some entries of the system matrices, or the values of some parameters may drift over time. The distributed output regulation problem considering parameter uncertainty is studied in [4], [6]. The proposed controller is based on the internal model principle, but does not employ the two-part structure mentioned above. It appears to be for this reason that restrictive conditions (acyclic graph or homogeneous nominal agent dynamics)
have to be imposed in order to ensure solving output regulation. Moreover, it is also assumed in [4], [6] that each agent knows the exact model of the exosystem dynamics.

In this paper, we provide a new solution to the distributed output regulation problem of heterogeneous linear agents, where the agents do not have an accurate dynamic model of the exosystem and the agent dynamics are subject to parameter uncertainty. In this setting, to our best knowledge, no solution exists in the literature. In particular, we propose to use the two-part structure of the distributed controller in the following manner: The first part is an exosystem generator that works over time-varying networks ( [14], [15]) and the second part is a dynamic compensator embedding an internal model of the exosystem that addresses parameter uncertainty. The challenge here is, in the design of the dynamic compensator, those agents that cannot directly measure the exosystem have no knowledge of the internal model of the exosystem; on the other hand, we know from [12] that a precise internal model is necessary to achieve perfect regulation with uncertain parameters. To deal with this challenge, we propose a novel consensus-based local internal model for each agent to estimate the internal model of the exosystem. For this time-varying local internal model, we moreover design novel strategies for its eigenvalues to avoid certain transmission zeros of the agents’ dynamics in order to guarantee the existence of a dynamic compensator for all time. In addition, we extend our new solution to solve a related problem of output synchronization [16], [17]. In this problem there is no exosystem; yet the outputs of all agents are required to converge to the same (dynamic) values.

The contributions of this paper are threefold. First, the proposed internal-model based distributed controller is the first solution to the multi-agent output regulation problem where the agents do not know a priori the internal model (q-copy) of the exosystem and the agent dynamics are uncertain. Concretely, the proposed distributed controller provably solves the multi-agent output regulation problem in which the following constraints/conditions simultaneously hold: (a) Unknown dynamic model of the exosystem. This is not considered in [4], [6]. (b) Parameter uncertainty of agent dynamics. This is not addressed in [5], [8]. (c) Non-minimum phase agent dynamics. This is not dealt with in [7]. (d) Time-varying directed networks. This is not addressed in [8]–[11]. (e) Heterogeneous agent dynamics. This is not dealt with in [9]. Second, our solution to the output synchronization problem improves the literature [16], [17] by providing capability of dealing with uncertain agent dynamics, and not requiring all agents initially to have the same internal model dynamics. These improvements allow easier implementation of the proposed controller in a distributed setting. As a third contribution, the core of our solution is the time-varying local internal model (q-copy), updated in the network setting, which is in itself new in the literature of the internal model principle (cf. [12], [13], [18], [19]) and generalizes the (static, centralized) internal model to the dynamic, distributed one.

In addition we note that [14] proposes a distributed controller to solve the consensus problem whose design idea is similar to ours. We point out, however, a few important differences. First, the consensus problem is different from the output regulation problem (the former is usually viewed as a special case of the latter with full-state observation). Second, while [14] deals with a class of nonlinear systems, the eigenvalues of the exosystem are required to be distinct. We do not make such an assumption; thus (i) the set of signals that can be generated by the exosystem is a strict superset of that in [14], and (ii) the minimal polynomial of the exosystem is generally different from the characteristic polynomial. Third, our designed distributed controller is based on the internal model principle, which is different from the controller designed in [14]. Finally, while the parameter uncertainty considered in [14] is represented by a vector, the uncertainty in this paper is represented by matrices.

The rest of the paper is organized as follows. Section II introduces the concept of communication graphs and formulates the robust output regulation problem. Section III presents the solution distributed controller, which consists of two parts – a distributed exosystem generator and a distributed dynamic compensator. Section IV states our main result and provides its proof. In Section V we design the more general distributed controller which addresses non-minimum phase agent dynamics with purely imaginary transmission zeros. Section VI extends our proposed controller to solve an output synchronization problem. Section VII illustrates our result by simulation examples. Finally, Section VIII states our conclusions.

II. PRELIMINARIES

In this paper, we will use the following notation. Let \( I_n := [1 \cdots 1]^\top \in \mathbb{R}^n \), and \( I_n \) be the \( n \times n \) identity matrix. For a complex number \( c \in \mathbb{C} \), denote its complex conjugate by \( c^* \). Write \( \mathbb{C}_+ \) for the closed right half (complex) plane; \( \sigma(A) \) for the set of all eigenvalues of \( A \). We say that a (square) matrix is stable if the real parts of all its eigenvalues are negative.

A. Agents and Exosystem

We consider a network of \( N \) agents that are linear, time-invariant, and finite-dimensional. The dynamics of each agent \( i (= 1, \ldots, N) \) is given by

\[
\dot{x}_i = A_i x_i + B_i u_i + P_i w_0 \quad (1)
\]

\[
z_i = C_i x_i + D_i u_i + Q_i w_0 \quad (2)
\]

\(^1\)This was apparently developed in [14] and in [15] independently. The first versions of [14] and [15] appeared on arXiv.org, with the former three months earlier than the latter. We thank Dr. Liu and Dr. Huang for in a correspondence bringing our attention to their work.

\(^2\)The conference version of this paper has been submitted to ACC’19. This paper improves the conference version in the following aspects. (i) A new problem of output synchronization is studied and solved by extending our controller design (Section IV). (ii) Elaborated simulations are provided including output regulation over a large-scale network and an output synchronization example. (iii) Detailed analyses and proofs are provided.
where \( x_i \in \mathbb{R}^{n_i} \) is the state vector, \( u_i \in \mathbb{R}^{m_i} \) the control input, \( z_i \in \mathbb{R}^{q_i} \) the output to be regulated, and \( w_0 \in \mathbb{R}^r \) the exogeneous signal generated by the exosystem

\[
\dot{w}_0 = S_0 w_0.
\]

(3)

Here \( A_i, B_i, C_i, D_i, P_i, Q_i \) and \( S_0 \) are real matrices of appropriate sizes. The signal \( w_0 \) represents reference to be tracked and/or disturbance to be rejected: \( P_i w_0 \) in (1) represents disturbance acting on the agent \( i \)'s dynamics and \( Q_i w_0 \) in (2) represents reference signals to be tracked by agent \( i \).

**Assumption 1** The exosystem’s \( w_0 \) and \( S_0 \) are not (initially) known by the \( N \) agents.

Note that the agents are generally heterogeneous: Each of the matrices \( A_i, B_i, C_i, D_i, P_i, Q_i \) may have different dimensions and entries. Furthermore, we consider that the matrices may have uncertainty; namely

\[
A_i = A_{i0} + \Delta A_i, \quad B_i = B_{i0} + \Delta B_i, \quad C_i = C_{i0} + \Delta C_i,
\]

\[
D_i = D_{i0} + \Delta D_i, \quad P_i = P_{i0} + \Delta P_i, \quad Q_i = Q_{i0} + \Delta Q_i
\]

where \( A_{i0}, B_{i0}, C_{i0}, D_{i0}, P_{i0}, Q_{i0} \) are the nominal parts of agent \( i \) and \( \Delta A_i, \Delta B_i, \Delta C_i, \Delta D_i, \Delta P_i, \Delta Q_i \) are the uncertain parts. These uncertainty parts may represent measurement errors in the actual determination of the physical parameters, or the reality that these parameters may change with time due to wear and aging [18].

**B. Communication Digraphs**

Given a multi-agent system with \( N(\geq 1) \) agents and an exosystem, we represent the time-varying interconnection among the agents and the exosystem by a digraph \( \hat{G}(t) = (\hat{V}, \hat{E}(t)) \), where \( \hat{V} = V \cup \{0\} \), \( V = \{1, \ldots, N\} \), is the node set, and \( \hat{E}(t) \subseteq \hat{V} \times \hat{V} \) is the edge set. The node \( i, i = 1, \ldots, N \), represents the \( i \)th agent, and the node 0 the exosystem. Moreover, \( \hat{V} \) is the node set including the exosystem and \( \hat{V} \) is the node set except for the exosystem. The \( i \)th node receives information from the \( j \)th node at time \( t \) if and only if \( (j, i) \in \hat{E}(t) \). We consider the digraph \( \hat{G}(t) \) does not contain selfloop edges, i.e., \( (i, i) \notin \hat{E}(t) \) for all \( i \in \hat{V} \). Only those nodes \( i \in V \) such that \( (0, i) \in \hat{E}(t) \) can receive information from the exosystem 0 (i.e. \( w_0, S_0 \)) at time \( t \).

The union digraph for a time interval \( [t_1, t_2] \) is defined as

\[
\hat{G}([t_1, t_2]) := (\hat{V}, \bigcup_{t \in [t_1, t_2]} \hat{E}(t)).
\]

**Definition 1** The digraph \( \hat{G}(t) \) uniformly contains a spanning tree if there is \( T > 0 \) such that for every \( t \geq T \) the union digraph \( \hat{G}([t, t + T]) \) contains a spanning tree.

Further, we need the following notion. Consider a union digraph \( \hat{G}([t_1, t_2]) = (\hat{V}, \bigcup_{t \in [t_1, t_2]} \hat{E}(t)) \) (excluding the exosystem). Let \( \hat{V}_r \subseteq \hat{V} \) be a nonempty subset of \( \hat{V} \). Then the digraph \( \hat{G}_r([t_1, t_2]) = (\hat{V}_r, \hat{E}_r([t_1, t_2])) \), where \( \hat{E}_r([t_1, t_2]) := (\hat{V}_r \times \hat{V}_r) \cap (\bigcup_{t \in [t_1, t_2]} \hat{E}(t)) \), is said to be the induced subdigraph of \( \hat{G}([t_1, t_2]) \) by \( \hat{V}_r \).

**Definition 2** A strongly connected component \( \hat{G}_r([t_1, t_2]) = (\hat{V}_r, \hat{E}_r([t_1, t_2])) \) of a union digraph \( \hat{G}([t_1, t_2]) = (\hat{V}, \hat{E}(t)) \) is a maximal induced subdigraph of \( \hat{G}([t_1, t_2]) \) by \( \hat{V}_r \) which is strongly connected. Moreover, \( \hat{G}_r([t_1, t_2]) \) is a closed strongly connected component if for every \( i \in \hat{V}_r \) and every \( j \in \hat{V} \setminus \hat{V}_r, (j, i) \notin \hat{E}_r([t_1, t_2]) \).

**Definition 3** Consider the time-varying digraph \( \hat{G}(t) = (\hat{V}, \hat{E}(t)) \) and let \( \hat{V}_r \subseteq \hat{V} \) be nonempty. We say that \( \hat{G}(t) \) uniformly contains a spanning tree with respect to \( \hat{V}_r \) if there is \( T > 0 \) such that for every \( t \geq 0 \) the union digraph \( \hat{G}([t, t + T]) \) contains a spanning tree and there is a unique closed strongly connected component (subdigraph) induced by \( \hat{V}_r \).

We define the communication weight \( a_{ij}(t) \) by \( a_{ij}(t) \geq \epsilon \) (where \( \epsilon \) is a positive constant) if \( (j, i) \in \hat{E}(t) \), and \( a_{ij}(t) = 0 \) if \( (j, i) \notin \hat{E}(t) \). We assume that \( a_{ij}(t) \) is piecewise continuous and bounded for all \( t \geq 0 \) (a technical assumption to be used in Lemma 2 below). Note that the exosystem does not receive information from any agents, and thus \( a_{0j}(t) = 0 \) for all \( j \in \hat{V}, t \geq 0 \).

For time \( t \geq 0 \) and digraph \( \hat{G}(t) \), the graph Laplacian \( L(t) = [l_{ij}(t)] \in \mathbb{R}^{(N+1) \times (N+1)} \) is defined as

\[
l_{ij}(t) := \begin{cases} 
\sum_{j=0}^{N} a_{ij}(t), & i = j \\
-a_{ij}(t), & i \neq j 
\end{cases}
\]

where \( i, j \in \{0, \ldots, N\} \).

**C. Problem Statement**

We represent by \( \hat{G}(t) = (\hat{V}, \hat{E}(t)) \) the time-varying interconnection among the \( N \) agents and the exosystem as in the preceding subsection. In particular, at any time only a subset of agents (possibly different across time) can receive information from the exosystem. This differs the current problem from the traditional, centralized output regulation problem [12], [13], [18], [19]. Even if an agent receives information from the exosystem at some time, the agent does not know whether the information is from the exosystem or another agent. Namely we consider that the agents do not have the numbering information including the exosystem (numbered 0).

**Problem 1 (Distributed Output Regulation Problem)**

Given a network of agents (1), (2), (4) and an exosystem (3) with interconnection represented by \( \hat{G}(t) \) and with Assumption 1 design for each agent \( i \in \hat{V} \) a distributed controller such that

\[
\lim_{t \to \infty} z_i(t) = 0
\]

for all \( x_i(0), w_0(0) \).

In the next section we solve Problem 1 by designing an internal-model based distributed controller.

**D. Motivating Example**

Reference [8] considers Problem 1 but without the uncertainty part in (4), and proposes an effective solution based on regulator equations (for time-invariant digraphs). However, this solution cannot deal with uncertain agent dynamics, as we shall illustrate by an example.
The exosystem (node 0) is

\[ \dot{x}_0 = A_0 x_0 + B_0 u_0, \quad S_0 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}. \]

The agents \( i = 1, 2, 3, 4 \) are

\[ \dot{x}_i = A_i x_i + B_i u_i + P_i w_0, \quad z_i = C_i x_i + D_i u_i + Q_i w_0 \]

where

\[ A_i = A_{i0} + \Delta A_i, \quad B_i = B_{i0} + \Delta B_i, \]

\[ A_{i0} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ i & 0 & 0 \end{bmatrix}, \quad B_{i0} = \begin{bmatrix} 1 \\ 0 \\ (0.5 + 0.1i)^2 \end{bmatrix}, \]

\[ \Delta A_i = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0 \\ -0.5 & 0 & 0 \end{bmatrix}, \quad \Delta B_i = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \]

\[ C_i = [1 \ 0 \ 0], \quad D_i = 0, \quad P_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_i = [-1 \ 0]. \]

The initial states \( w_0(0), \ x_i(0) \) are selected uniformly at random from the interval \([-1, 1]\).

Fig. 2 shows the simulation result without uncertainty parts \( \Delta A_i \) and \( \Delta B_i \) using the solution in [8] and Fig. 3 shows that with uncertainty. Observe in Fig. 3 that \( z_i(i = i, \ldots, 4) \) do not converge to 0 due to the uncertainty parts \( \Delta A_i \) and \( \Delta B_i \). Thus Problem 1 cannot be solved by [8], and we are motivated to propose a new solution that is not based on regulator equations but based on the internal model (qi-copy).

III. STRUCTURE OF DISTRIBUTED CONTROLLER

At the outset we make the following (standard) assumptions.

**Assumption 2** The digraph \( \hat{G}(t) \) uniformly contains a spanning tree and its root is node 0 (the exosystem).

**Assumption 3** For each agent \( i \in \mathcal{V} \), \( (A_{i0}, B_{i0}) \) is stabilizable.

**Assumption 4** For each agent \( i \in \mathcal{V} \), \( (C_{i0}, A_{i0}) \) is detectable.

**Assumption 5** For each agent \( i \in \mathcal{V} \) and for every eigenvalues \( \lambda \) of \( S_0 \),

\[ \text{rank} \begin{bmatrix} A_{i0} - \lambda I_{n_i} & B_{i0} \\ C_{i0} & D_{i0} \end{bmatrix} = n_i + q_i. \]  \tag{5} \]

**Assumption 6** The real parts of all eigenvalues of \( S_0 \) are zeros.

**Remark 1** Assumptions 2 and Assumptions 3 are necessary conditions for consensus over time-varying networks [1] and for output regulation [12], respectively. Only Assumption 6 is a sufficient condition for (centralized) output regulation, but is commonly made for distributed output regulation (e.g. [7], [8]) such that the exogeneous signal does not diverge exponentially fast.

**Remark 2** By [20], Assumption 5 means that the transmission zeros of agent \( i \) are disjoint from all eigenvalues of \( S_0 \), and also implies that the number of outputs is no more than that of inputs, i.e. \( m_i \geq q_i \). A transmission zero \( \zeta \in \mathbb{C} \) of agent \( i \) is such that

\[ \text{rank} \begin{bmatrix} A_{i0} - \zeta I_{n_i} & B_{i0} \\ C_{i0} & D_{i0} \end{bmatrix} < n_i + q_i. \]

Because not all agents can access the exosystem (i.e. \( w_0 \) cannot be measured by all agents), we cannot use (2) directly. Instead we consider the following (estimated) error vector

\[ e_i = C_i x_i + D_i u_i + Q_i w_1. \]

In order to solve Problem 1 we present a controller that consists of two parts: (1) distributed exosystem generator and (2) distributed dynamic compensator.

A. Distributed exosystem generator

It is reasonable for each agent \( i \in \mathcal{V} \) to have a local estimate of the exosystem’s dynamics since not all agents can access the exosystem. Let \( S_i(t) \in \mathbb{R}^{r \times r} \) be the estimate of \( S_0 \) and consider

\[ \dot{S}_i(t) = \sum_{j=0}^{N} a_{ij}(t) (S_j(t) - S_i(t)), \]  \tag{7} \]

\[ \dot{w}_i(t) = S_i(t) w_i(t) + \sum_{j=0}^{N} a_{ij}(t) (w_j(t) - w_i(t)). \]  \tag{8} \]

By using (7) and (8), it is guaranteed under Assumption 2 that

\[ \lim_{t \to \infty} (S_i(t) - S_0) = 0, \quad \lim_{t \to \infty} (w_i(t) - w_0(t)) = 0 \]

for all \( S_i(0), w_i(0) \) and all \( i \in \mathcal{V} \). We show this statement in detail in Section IV below.
This protocol is used to approximate the exosystem for each agent $i \in \mathcal{V}$. Thus we call \[7\] and \[8\] the exosystem generator.

Equations \[7\] and \[8\] have also been used in \[14\] for the adaptive distributed observer (see Footnote 1 above), and first proposed in \[8\] but for time-invariant networks.

B. Distributed dynamic compensator

We consider the following dynamic compensator
\[
\begin{align*}
\dot{\xi}_i &= E_i(t)\xi_i + F_i(t)e_i \\
u_i &= K_i(t)\xi_i
\end{align*}
\]
where $\xi_i$ is the state of the dynamic compensator and $e_i$ is defined in \[6\].

In order to specify the matrices $E_i(t), F_i(t), K_i(t)$ in \[9\], we extend the internal model control design in \[19, Section 1.3\] to the multi-agent system setting. Let $\lambda_{0,1}, \ldots, \lambda_{0,k}, k \leq r$ be the roots of the minimal polynomial of $S_0$. Note that $\{\lambda_{0,1}, \ldots, \lambda_{0,k}\} \subseteq \sigma(S_0)$. Then we define $\lambda_0 := [\lambda_{0,1} \cdots \lambda_{0,k}]^T$. Let $c_{0,d}(\lambda_0), d = 1, \ldots, k$ be the estimated coefficients of the polynomial satisfying
\[
s^k + c_{0,1}(\lambda_0)s^{k-1} + \cdots + c_{0,k-1}(\lambda_0)s + c_{0,k}(\lambda_0) = \prod_{d=1}^{k} (s - \lambda_{0,d}(t)).
\]
For each agent $i \in \mathcal{V}$, let $\lambda_i(t) := [\lambda_{i,1}(t) \cdots \lambda_{i,k}(t)]^T$ be a local estimate of $\lambda_0$, and $c_{i,d}(\lambda_i), d = 1, \ldots, k$, the estimated coefficients generated by $\lambda_i(t)$ that satisfy
\[
s^k + c_{i,1}(\lambda_i)s^{k-1} + \cdots + c_{i,k-1}(\lambda_i)s + c_{i,k}(\lambda_i) = \prod_{d=1}^{k} (s - \lambda_{i,d}(t)).
\]
Consider the following consensus algorithm:
\[
\dot{\lambda}_i(t) = \sum_{j=0}^{N} a_{ij}(t) (\lambda_j(t) - \lambda_i(t)), \quad \lambda_i(0) \in \mathbb{R}^k.
\]
It follows from Assumption \[2\] that $\lambda_i(t) \to \lambda_0$ as $t \to \infty$. As a result, the coefficient $c_{i,d}(\lambda_i) \to c_{0,d}(\lambda_0)$ as $t \to \infty$ for each $d = 1, \ldots, k$. Note that by Assumption \[4\] the entries of $\lambda_0$ are purely imaginary, and hence we only need to consider the initial condition $\lambda_i(0) \in \mathbb{R}^k$ (thus $\lambda_i(t) \in \mathbb{R}^k$ for all $t \geq 0$).

Since we consider that the agents’ dynamics have uncertainty, the regulator equation approach (e.g. \[8\]) does not work. Thus for the robust output regulation problem, we consider the $q_i$-copy internal model as \[19, Section 1.3\]. In the case where an agent has multiple outputs, we need to assign the internal model to each output. Let $G_i(\lambda_i) := I_{q_i} \otimes G_i(\lambda_i)$, $H_i := I_{q_i} \otimes H_i^r$ be the $q_i$-copy internal model (\otimes denotes Kronecker product), where
\[
G_i(\lambda_i) := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -c_{i,k}(\lambda_i) & -c_{i,k-1}(\lambda_i) & \cdots & -c_{i,1}(\lambda_i) \end{bmatrix}
\]
\[
H_i := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.
\]

We state the following lemma using the above matrices.

**Lemma 1** Assume that Assumptions \[3\] and \[5\] hold. Then the following pair of matrices
\[
\begin{bmatrix} A_{i0} & 0 \\ H_iC_{i0} & G_i(\lambda_i) \end{bmatrix}, \begin{bmatrix} B_{i0} \\ H_iD_{i0} \end{bmatrix}
\]
is stabilizable if
\[
\text{rank} \begin{bmatrix} A_{i0} - \lambda_{i,d}I_{q_i} & B_{i0} \\ C_{i0} & D_{i0} \end{bmatrix} = n_i + q_i \quad (14)
\]
for each $\lambda_{i,d} \in \mathbb{C}_+, d = 1, \ldots, k$.

**Proof:** Refer to the proof of \[19, Lemma 1.26\] with the condition \[14\] and the fact that in \[12\] $\lambda_{i,d}(t) \in \mathbb{R}$ and thus $\lambda_{i,d}(t) \in \mathbb{C}_+$ for all $t$.

In Lemma \[1\] the sufficient condition \[14\] means that every $\lambda_{i,d}$ does not correspond to transmission zeros of agent $i$. In \[14\], $\lambda_{i,d}$ is time-varying because it is updated according to \[12\]. Since $\lambda_{i,d}(t) \in \mathbb{R}$ for all $t$, if agent $i$’s dynamics has purely imaginary transmission zeros, it is possible that \[14\] is violated. In order to satisfy \[14\] for all $t \geq 0$, we make the following (simplifying) assumption.

**Assumption 7** For every agent $i \in \mathcal{V}$, there are no transmission zeros on the imaginary axis, i.e.
\[
\text{rank} \begin{bmatrix} A_{i0} - \lambda I_{n_i} & B_{i0} \\ C_{i0} & D_{i0} \end{bmatrix} = n_i + q_i \quad (14)
\]
for all $\lambda \in \mathbb{R}$.

If every agent $i \in \mathcal{V}$ is minimum-phase, then Assumption \[7\] is satisfied. In addition, this assumption allows transmission zeros to be on the open right (complex) plane, thus admitting non-minimum-phase system. In the case where Assumption \[7\] does not hold, it is a challenge to ensure that \[14\] holds for all $t \geq 0$. Nevertheless, in Section \[V\] below we shall present a novel strategy to guarantee \[14\] even in the presence of purely imaginary transmission zeros.

From Lemma \[1\] and Assumption \[7\] we may synthesize $[K_{i1}(\lambda_i) \ K_{i2}(\lambda_i)]$ such that the matrix
\[
\begin{bmatrix} A_{i0} & 0 \\ H_iC_{i0} & G_i(\lambda_i) \end{bmatrix} + \begin{bmatrix} B_{i0} \\ H_iD_{i0} \end{bmatrix} [K_{i1}(\lambda_i) \ K_{i2}(\lambda_i)]
\]
is stable for all $t \geq 0$. In addition, we choose $L_i$ such that the matrix $A_{i0} - L_iC_{i0}$ is stable under Assumption \[4\].
Now we are ready to present the matrices $E_i(t)$, $F_i(t)$ and $K_i(t)$ in the dynamic compensator (9):

$$E_i(\lambda_i) := \begin{bmatrix} A_{i0} - L_i C_{i0} & 0 \\ 0 & G_i(\lambda_i) \end{bmatrix} + \begin{bmatrix} B_{i0} - L_i D_{i0} \\ 0 \end{bmatrix} K_i(\lambda_i),$$

$$F_i := \begin{bmatrix} L_i \\ H_i \end{bmatrix},$$

$$K_i(\lambda_i) := [K_{i1}(\lambda_i) \ K_{i2}(\lambda_i)].$$

(16)

Note that in (16), $E_i$ and $K_i$ are time-varying as $\lambda_i$ is time-varying, while $F_i$ is time-invariant; and by (12) there hold

$$G_i(\lambda_i) \to G_i(\lambda_0)$$

$$K_i(\lambda_i) \to K_i(\lambda_0)$$

$$E_i(\lambda_i) \to E_i(\lambda_0).$$

Using the distributed dynamic compesator (9), we will show in the next section that the estimated error $e_i$ and the output $z_i$ to be regulated converge to 0.

IV. MAIN RESULT

Our main result is the following.

**Theorem 1** Consider the multi-agent system (1), (2), (4) and the exosystem (3), and suppose that Assumptions 1-7 hold. Then for each agent $i \in V$, the distributed exosystem generator (7) and (8), and the distributed dynamic compensator (9) with (12), (16) solve Problem 1.

Several remarks on Theorem 1 are in order.

**Remark 3** Theorem 1 asserts that the proposed two-part distributed controller – the distributed exosystem generator (7), (8) and the distributed dynamic compensator (9) – provides the first solution to the multi-agent output regulation problem where the agents have no initial knowledge of the exosystem’s internal model and the agent dynamics are uncertain. The key of our solution is the time-varying $q_i$-copy internal model, updated locally based on only information received from neighbors, which eventually converges to the exact internal model of the exosystem.

** Remark 4** When there is only one agent (i.e. $N = 1$), the problem is specialized to the centralized output regulation, and Theorem 1 is thus an extension of the conventional results in [12], [13], [18], [19]. Even if the (single) agent does not know the exosystem dynamics initially, the output regulation problem is solvable by the exosystem generator (7), (8) and dynamic compensator (9).

**Remark 5** If the exosystem is a leader agent that possesses computation and communication abilities, then the leader can compute the roots of its own minimal polynomial and send the information to other connected agents. If the exosystem is some entity that cannot compute or communicate, then those agents that can measure the exosystem (in particular know $S_0$) compute the corresponding minimal polynomial and the roots, and send the information to the rest of the network.

**Remark 6** For each agent to ‘learn’ the internal model of the exosystem, our strategy is to make the agents reach consensus by (12) for the roots of the exosystem’s minimal polynomial (i.e. eigenvalues of $S_0$). It might appear more straightforward to reach consensus for the coefficients of the exosystem’s minimal polynomial; the advantage of updating $\lambda_i$ with (12), nevertheless, is that we may directly guarantee the equality in (5) in Assumption 5.

**Remark 7** In (7), there are $r \times r$ entries to update and communicate. If the minimal polynomial of $S_0$ equals its characteristic polynomial ($k = r$) and $S_0$ is in the companion form

$$S_0 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

where $c_{0,d}(\lambda_0), d = 1, \ldots, r$, and $\lambda_0$ are as defined in (10), then each agent does not need to exchange and update the whole $S_i$. Each agent only needs to exchange and update $\lambda_i = [\lambda_{i,1}, \ldots, \lambda_{i,k}]^\top$ by (12) and make $S_i$ also in the companion form.

**Remark 8** In the equation (12), we do not need to use all entries of $\lambda_i = [\lambda_{i,1}, \ldots, \lambda_{i,k}]^\top$, because the eigenvalues of the real matrices $S_i$ must be in conjugate pairs. Indeed, for all $i \in \hat{V}$ we may write $\lambda_i$ in the following form

$$\lambda_i(t) = \begin{cases} \hat{\lambda}_i(t)^\top \hat{\lambda}_i(t)^\top^\top & \text{if } k \text{ is an even number} \\ \hat{\lambda}_i(t)^\top \hat{\lambda}_i(t)^\top 0^\top & \text{if } k \text{ is an odd number} \end{cases}$$

where $\hat{\lambda}_i \in \mathbb{R}^{k/2}$. From this form, each agent can make their entire $\lambda_i$ after exchanging and updating only $\hat{\lambda}_i$.

To prove Theorem 1 we need the following two lemmas. Their proofs are presented in Appendix (alternative proofs can also be found in [14]).

The first lemma states a stability result for a particular type of time-varying systems.

**Lemma 2** Consider

$$\dot{x}(t) = A_1(t)x(t) + A_2(t)x(t) + A_3(t)$$

where $A_1(t), A_2(t), A_3(t)$ are piecewise continuous and bounded on $[0, \infty)$. Suppose that the origin is a uniformly exponentially stable equilibrium of $\dot{x} = A_1(t)x$, and $A_2(t) \to 0$, $A_3(t) \to 0$ as $t \to \infty$. Then $x(t) \to 0$ as $t \to \infty$.

The second lemma asserts that the distributed exosystem generators proposed in Section 11A synchronize with the exosystem.
Lemma 3 Consider the distributed exosystem generator (7) and (8). If Assumption 2 holds, then
\[
\lim_{t \to \infty} (S_i(t) - S_0) = 0, \quad \lim_{t \to \infty} (w_i(t) - w_0) = 0
\]
for all \( S_i(0), w_i(0) \).

Now we are ready to prove Theorem 1.

**Proof of Theorem 1** Let \( \vec{\eta}_i := [x_i^T \xi_i^T]^T \) be the combined state. From (11), (12), (15) and (16), we derive
\[
\dot{\vec{\eta}}_i = M_i(\lambda_i)\vec{\eta}_i + \begin{bmatrix} 0 \\ F_i Q_i \end{bmatrix} w_i + \begin{bmatrix} P_i \\ 0 \end{bmatrix} w_0 - X_i(\lambda_0) S_i w_0 \quad (18)
\]
\[
z_i = [C_i D_i K_i(\lambda_i)]\vec{\eta}_i + Q_i w_0. \quad (19)
\]
where
\[
M_i(\lambda_i) := \begin{bmatrix} A_i & B_i K_i(\lambda_i) \\ F_i C_i & E_i(\lambda_i) + F_i D_i K_i(\lambda_i) \end{bmatrix}
\]
\[
M_{i0}(\lambda_i) := \begin{bmatrix} A_{i0} & B_{i0} K_{i0}(\lambda_i) \\ L_i C_{i0} & A_{i0} - L_i C_{i0} B_{i0} K_{i0}(\lambda_i) & B_{i0} K_{i0}(\lambda_i) \\ H_i D_{i0} K_i(\lambda_i) & G_i(\lambda_i) & H_i D_{i0} K_{i0}(\lambda_i) \end{bmatrix}
\]
\[
\Delta M_i(\lambda_i) = \begin{bmatrix} \Delta A_i & \Delta B_i K_i(\lambda_i) & \Delta B_i K_{i0}(\lambda_i) \\ L_i \Delta C_i & L_i \Delta D_i K_i(\lambda_i) & L_i \Delta D_i K_{i0}(\lambda_i) \\ -\Delta A_i + L_i \Delta C_i & -L_i \Delta D_i K_i(\lambda_i) & -L_i \Delta D_i K_{i0}(\lambda_i) \end{bmatrix}
\]
\[
\Delta M_{i0}(\lambda_i) = \begin{bmatrix} \Delta A_{i0} & \Delta B_{i0} K_{i0}(\lambda_i) \\ L_i C_{i0} & A_{i0} - L_i C_{i0} B_{i0} K_{i0}(\lambda_i) & B_{i0} K_{i0}(\lambda_i) \\ H_i D_{i0} K_{i0}(\lambda_i) & G_{i0}(\lambda_i) & H_i D_{i0} K_{i0}(\lambda_i) \end{bmatrix}
\]

First, we define
\[
T := \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ -I & I & 0 \end{bmatrix}
\]
and obtain
\[
T M_{i0}(\lambda_i) T^{-1} = \begin{bmatrix} A_{i0} + B_{i0} K_{i0}(\lambda_i) & B_{i0} K_{i0}(\lambda_i) & B_{i0} K_{i0}(\lambda_i) \\ L_i C_{i0} + H_i D_{i0} K_{i0}(\lambda_i) & G_{i0}(\lambda_i) & H_i D_{i0} K_{i0}(\lambda_i) \\ 0 & 0 & 0 \end{bmatrix}
\]
Its upper-left submatrix equals (15) and thus the submatrix is stable. Also \( A_i - L_i C_i \) is stable. Moreover \( M_{i0}(\lambda_i) T^{-1} \) are similar. Therefore \( M_{i0}(\lambda_i) \) is stable for all \( t \geq 0 \). By the continuity of eigenvalues, if the term \( \Delta M_i(\lambda_i) \) is sufficiently small, then \( M_i(\lambda_i) \) remains stable for all \( t \geq 0 \). Indeed, as long as the uncertainty parts \( \Delta A_i, \Delta B_i, \Delta C_i, \Delta D_i, \Delta P_i, \Delta Q_i \) in (4) are such that the term \( \Delta M_i(\lambda_i) \) does not perturb the stable eigenvalues of \( M_{i0}(\lambda_i) \) to the closed right-hand-side of the complex plane, \( M_i(\lambda_i) \) remains stable for all \( t \geq 0 \).

Next, from Assumption 3 and the above statement \( \sigma(S_0) \cap \sigma(M_i(\lambda_0)) = \emptyset \), and thus the following equations
\[
\begin{align*}
X_i(\lambda_0) S_0 = & M_i(\lambda_0) X_i(\lambda_0) + \begin{bmatrix} P_i \\ F_i Q_i \end{bmatrix} w_i \\
0 = & [C_i D_i K_i(\lambda_0)] X_i(\lambda_0) + Q_i 
\end{align*}
\]
have a unique solution \( X_i(\lambda_0) \) from (18, Appendix A). Let \( \vec{\eta}_i := \dot{\vec{\eta}}_i - X_i(\lambda_0) w_0 \), \( \vec{M}_i := M_i(\lambda_i) - M_i(\lambda_0) \). Then from (18) and (21), we obtain
\[
\vec{\eta}_i = \dot{\vec{\eta}}_i - X_i(\lambda_0) w_0 \\
= M_i(\lambda_i) \vec{\eta}_i + \begin{bmatrix} 0 \\ F_i Q_i \end{bmatrix} w_i + \begin{bmatrix} P_i \\ 0 \end{bmatrix} w_0 - X_i(\lambda_0) S_i w_0 \\
= (\vec{M}_i + M_i(\lambda_0)) (\vec{\eta}_i + X_i(\lambda_0) w_0) + \begin{bmatrix} 0 \\ F_i Q_i \end{bmatrix} w_i \\
+ \begin{bmatrix} P_i \\ 0 \end{bmatrix} w_0 - X_i(\lambda_0) S_0 w_0 \\
= M_i(\lambda_0) \vec{\eta}_i + \vec{M}_i \vec{\eta}_i + M_i X_i(\lambda_0) w_0 + \begin{bmatrix} 0 \\ F_i Q_i \end{bmatrix} (w_i - w_0) \\
+ \left[M_i(\lambda_0) X_i(\lambda_0) + \begin{bmatrix} P_i \\ F_i Q_i \end{bmatrix} - X_i(\lambda_0) S_0 \right] w_0 \\
= (M_i(\lambda_0) \vec{\eta}_i + \vec{M}_i \vec{\eta}_i) + M_i X_i(\lambda_0) w_0 + \begin{bmatrix} 0 \\ F_i Q_i \end{bmatrix} (w_i - w_0).
\]
From Lemma 3 \( w_i - w_0 \to 0 \). Moreover, \( M_i(\lambda_0) \) is stable and \( \vec{M}_i \to 0 \). Therefore, \( \vec{\eta}_i \to 0 \) from Lemma 2.

Furthermore from (19) and (21), we obtain
\[
z_i = [C_i D_i K_i(\lambda_i)] \vec{\eta}_i + ([C_i D_i K_i(\lambda_i)] X_i(\lambda_0) + Q_i) w_0.
\]
Since \( \vec{\eta}_i \to 0 \) and
\[
[C_i D_i K_i(\lambda_i)] X_i(\lambda_0) + Q_i = [C_i D_i K_i(\lambda_i)] X_i(\lambda_0) + Q_i = 0,
\]
we conclude that \( z_i(t) \to 0 \) as \( t \to \infty \).

**Remark 9** Note from the proof above that the uncertainty parts \( \Delta A_i, \Delta B_i, \Delta C_i, \Delta D_i, \Delta P_i, \Delta Q_i \) need not be small. In particular, the matrices \( \Delta P_i, \Delta Q_i \) can be arbitrary, and the matrices \( \Delta A_i, \Delta B_i, \Delta C_i, \Delta D_i \) have only to satisfy that the matrix \( M_i(\lambda_i) \) in (20) is stable for all \( t \geq 0 \).

This requirement on the uncertainty parts is standard for centralized output regulation as well [20], and ideally should have been stated as a sufficient condition in Theorem 1. Inasmuch as this condition is rather clumsy (involving the matrix (20)) and for clarity of presentation, we choose to state it here as a remark.

V. PURELY IMAGINARY TRANSMISSION ZEROS

In this section, we generalize Theorem 1 by designing a distributed controller for the case where Assumption 7 does not hold, i.e., there exist transmission zeros of the agents on the imaginary axis. In this case, because each (vector) \( \lambda_i = [\lambda_{i,1}, \ldots, \lambda_{i,k}]^T \) is updated continuously, entries of \( \lambda_i \) may coincide with the transmission zeros of agent \( i \), which would violate (14). Consequently we cannot design \( K_i(\lambda_i) \) with Lemma 1.

In order to choose the local estimate \( \lambda_i \) satisfying the condition (14) and design \( K_i(\lambda_i) \), \( \lambda_i \) must converge to \( \lambda_0 \) and at the same time avoid the transmission zeros of the agent \( i \). Fig. 4 shows examples of the trajectory of \( \lambda_{i,d}(d \in \{1, \ldots, k\}) \). The circles and the crosses represent respectively the transmission zeros of the agent \( i \) and the
eigenvalue $\lambda_{0,d}$ of $S_0$. The initial value $\lambda_{i,d}(0)$ is in $\mathbb{R}$ and $\lambda_{i,d}(t)$ moves toward $\lambda_{0,d}$. We di vide the arrangement of transmission zeros into three cases:

(i) If there is no purely imaginary transmission zero of agent $i$ (see Fig. 4(i)), then Assumption 7 holds and we need no further control design.

(ii) If there is a purely imaginary transmission zero of agent $i$, and $\lambda_{i,d}(t)$ moves close to it (see Fig. 4(ii)), then $\lambda_{i,d}(t)$ should move in a semicircle dented to the right around the transmission zero. By moving to the right, $\lambda_{i,d}(t)$ is always in $\mathbb{C}_+$ and thus Lemma 1 is guaranteed for all $t \geq 0$.

(iii) If there is a purely imaginary transmission zero of agent $i$, and there are also other transmission zeros on the open right-half-plane (see Fig. 4(iii)), the radius of semicircle should be smaller than (e.g. half of) the distance between these transmission zeros.

To formalize the above idea, we define several quantities. Let

$$\Pi_i := \left\{ s \in \mathbb{C}_+ \mid \text{rank} \begin{bmatrix} A_{i0} - sI_{n_i} & B_{i0} \\ C_{i0} & D_{i0} \end{bmatrix} \leq n_i + q_i \right\}$$

be the set of closed right-half-plane transmission zeros of agent $i \in \mathcal{V}$

and

$$\tilde{\Pi}_i := \left\{ s \in \Pi_i \mid \text{Re}(s) = 0 \right\}$$

the subset of purely imaginary transmission zeros. We do not need to avoid open left-half-plane transmission zeros because $\lambda_{i,d}(t) \in \mathbb{C}_+$. Note that Assumption 5 and $\Pi_i \cap \sigma(S_0) = \emptyset$ are equivalent, and Assumption 7 holds if and only if $\tilde{\Pi}_i = \emptyset$.

We define a new function. For two sets $C_1, C_2 \subseteq \mathbb{C}$ of finite number of complex numbers, define the distance between $C_1$ and $C_2$ by

$$\text{dist}(C_1, C_2) := \min \left\{ |c_1 - c_2| \mid c_1 \in C_1, c_2 \in C_2 \right\}$$

Then define the radius $\rho_i \geq 0$ of the semicircle shown in Fig. 4 as

$$\rho_i := \left\{ \begin{array}{ll} 0, & \tilde{\Pi}_i = \emptyset \\ \frac{1}{2} \min \{\text{dist}(\sigma(S_0), \tilde{\Pi}_i), \text{dist}(\Pi_i \setminus \tilde{\Pi}_i), \text{dist}(\Pi_i \setminus \tilde{\Pi}_i)\}, & \text{otherwise} \end{array} \right.$$
Remark 10 As in Remark 9 we do not need to use all entries of $\beta_i$ in the equation (26). For all $i \in \mathcal{V}$ write $\beta_i$ in the following form

$$
\beta_i(t) = \begin{cases} 
[\hat{\beta}_i(t)^T - \hat{\beta}_i(t)]^T, & k \text{ is an even number} \\
[\hat{\beta}_i(t)^T - \hat{\beta}_i(t)^0]^T, & k \text{ is an odd number} 
\end{cases}
$$

where $\hat{\beta}_i \in j\mathbb{R}^{[k/2]}$. From this form, each agent can make their entire $\beta_i$ after exchanging and updating only $\hat{\beta}_i$.

Remark 11 As in Remark 9 the uncertainty parts $\Delta A_i, \Delta B_i, \Delta C_i, \Delta D_i, \Delta P_i, \Delta Q_i$ need not be small.

VI. OUTPUT SYNCHRONIZATION

In this section, we extend our approach to solve an output synchronization problem. Notable results of this problem are reported in [16], [17]. In [16], the synchronization problem is solved for homogeneous agents. The result of [16] is extended by [17] to deal with heterogeneous agents. However, the approaches in [16], [17] cannot deal with uncertain agent dynamics and moreover assume certain common information available to all agents. We address these issues by extending the approach developed in the previous sections.

The output synchronization problem differs from the output regulation problem studied previously in that, there is no exosystem (node 0). Consider $N$ heterogeneous agents whose dynamics are given by

$$
\dot{x}_i = A_ix_i + B_iu_i \\
y_i = C_ix_i + D_iu_i
$$

where $x_i \in \mathbb{R}^{m_i}$ is the state vector, $u_i \in \mathbb{R}^{m_i}$ the control input, $y_i \in \mathbb{R}^q$ the output for $i = 1, \ldots, N$. Matrices $A_i, B_i, C_i, D_i$ may have different dimensions and entries and also have uncertainty as in the distributed output regulation problem, namely

$$
A_i = A_{i0} + \Delta A_i, \quad B_i = B_{i0} + \Delta B_i, \\
C_i = C_{i0} + \Delta C_i, \quad D_i = D_{i0} + \Delta D_i
$$

where $A_{i0}, B_{i0}, C_{i0}, D_{i0}$ and $\Delta A_i, \Delta B_i, \Delta C_i, \Delta D_i$ are given in (4).

Since there does not exist the exosystem, we represent the time-varying interconnection among the $N$ agents by $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$. We make the following assumption.

Assumption 8 There is a fixed subset of nodes $\mathcal{V}_r \subseteq \mathcal{V}$ such that the digraph $\mathcal{G}(t)$ uniformly contains a spanning tree with respect to $\mathcal{V}_r$.

Remark 12 In the distributed output regulation problem, it is necessary that the digraph $\mathcal{G}(t)$ uniformly contains a single spanning tree whose root is the exosystem. By contrast, in the output synchronization problem, the digraph $\mathcal{G}(t)$ may uniformly contain multiple spanning trees with multiple roots. This is more general, although we require by Assumption 8 that these roots be time-invariant.

The output synchronization problem is the following:

Problem 2 (Output Synchronization Problem) Given a network of agents (29), (30) and (31) with interconnection represented by $\mathcal{G}(t)$, design for each agent $i \in \mathcal{V}$ a distributed controller such that

$$
\lim_{t \to \infty} (y_i(t) - y_j(t)) = 0 
$$

for all $i \neq j \in \mathcal{V}$ and all $x_i(0), x_j(0)$.

To solve Problem 2 we employ the same controller structure: (i) the distributed exosystem generator (8) and (ii) the distributed dynamic compensator.

A. Distributed exosystem generator

First, we consider the distributed exosystem generator. We define $r \geq 1$ as the dimension of the distributed exosystem generator as in Section III-A. A natural choice for $r$ is $r = q$, the dimension of the output of each agent. The more general case where $r$ be different from $q$ can generate more diverse (interesting) synchronized patterns. For example, when $q = 1$ and $r = 2$, we can make the final synchronized patterns to be constant, ramp, or sinusoidal by choosing suitable second-order exosystem generators. An illustrating example is provided below in Section VII-B.

To solve Problem 2 we consider again the distributed exosystem generator:

$$
\dot{S}_i(t) = \sum_{j=1}^{N} a_{ij}(t) (S_j(t) - S_i(t)), \\
\dot{w}_i(t) = S_i(t)w_i(t) + \sum_{j=1}^{N} a_{ij}(t) (w_j(t) - w_i(t))
$$

where for each $i \in \mathcal{V}$, $S_i(0) = S_i^* \in \mathbb{R}^{r \times r}$ is a fixed matrix; for each $i \in \mathcal{V} \setminus \mathcal{V}_r$, $S_i(0) \in \mathbb{R}^{r \times r}$ is arbitrary; and for each $i \in \mathcal{V}$, $w_i(0) \in \mathbb{R}^r$ is arbitrary. For the fixed $S^*$ we need the following assumption similar to Assumption 8 (for $S_0$ in the distributed output regulation problem).

Assumption 9 The real parts of all eigenvalues of $S^*$ are zeros.

We require that the agents in $\mathcal{V}_r$ have the same initial dynamics $S^*$. This is to derive the following convergence result, as for the distributed exosystem generator (7) and (8). This requirement on the initial condition might be stringent, but it already relaxes the requirement in the literature [16], [17] where all agents must have the same dynamics for synchronization.

Lemma 5 Consider the distributed exosystem generator (32). If Assumption 8 holds, then

$$
\lim_{t \to \infty} (S_i(t) - S^*) = 0, \quad \lim_{t \to \infty} (w_i(t) - w_j(t)) = 0
$$

for $(\forall i \in \mathcal{V}_r) S_i(0) = S^*, (\forall i \in \mathcal{V} \setminus \mathcal{V}_r) S_i(0)$, and $(\forall i, j \in \mathcal{V}) w_i(0), w_j(0)$.

It is more appropriate to call this generator a “distributed synchronizer generator”, as there is no exosystem in the current problem. However, since we use basically the same design as before, we simply inherit the same name.
The proof is in Appendix.

B. Distributed dynamic compensator

As in the output regulation problem, we solve the output synchronization problem by reducing the error between the output \( y_i \in \mathbb{R}^q \) of each agent and the exogeneous signal \( w_i \in \mathbb{R}^r \).

When \( q = r \), the error is simply \( e_i = y_i - w_i \). Since we consider the more general case where \( q \) need not be equal to \( r \), we define the error to be

\[
e_i = y_i + Q_i w_i.
\]

Here \( Q_i \in \mathbb{R}^{q \times r} \) and may be different for different agents. Thus for output synchronization, it is important that \( Q_i, w_i \) converge to the same vector for all agents. Since \( w_i \) do so by Lemma 5, we propose the following consensus update for \( Q_i \):

\[
\dot{Q}_i = \sum_{j=1}^{N} a_{ij}(t) (Q_j(t) - Q_i(t)), \quad Q_i(0) \in \mathbb{R}^{q \times r}
\]

for all \( i, j \in \mathcal{V} \).

**Lemma 6** Consider the distributed exosystem generator (32) and (34). If Assumption 8 holds, then

\[
\lim_{t \to \infty} (Q_i(t)w_i(t) - Q_j(t)w_j(t)) = 0
\]

for all \( i, j \in \mathcal{V} \) and all \( Q_i(0), Q_j(0), w_i(0), w_j(0) \).

**Proof:** From [21, Theorem 1] and (34), \( Q_i(i \in \mathcal{V}) \) reach consensus for all \( Q_i(0) \). Let \( Q^* \) be the consensus value of \( Q_i(t) \). Then

\[
Q_iw_i - Q_jw_j = (Q_i - Q^*)w_i + Q^*w_i - (Q_j - Q^*)w_j - Q^*w_j
\]

Since \( (w_i - w_j) \to 0 \) for all \( w_i(0), w_j(0) \) from Lemma 5 and \( (Q_i - Q^*) \to 0 \), \( (Q_j - Q^*) \to 0 \), we ensure \( Q_i(t)w_i(t) - Q_j(t)w_j(t) \to 0 \) as \( t \to \infty \).

We consider again the dynamic compensator

\[
\dot{\xi}_i = E_i(t)\xi_i + F_i e_i(t)\]
\[
\epsilon_i = K_i(t)\xi_i
\]

where \( \xi_i \) is the state of the dynamic compensator and \( e_i \) is defined in (33). The matrices \( E_i(t), F_i, K_i(t) \) are as specified in (16), (25), (26) and (27). As in Section VII, such a dynamic compensator can deal with purely imaginary transmission zeros. Note that in (26), the initial values of \( \beta_i \) are

\[
(\forall i \in \mathcal{V}_r) \beta_i(0) = [\text{Im}\{\lambda_1(S^*)\}] \cdots [\text{Im}\{\lambda_r(S^*)\}]^T
\]

\[
(\forall i \in \mathcal{V} \setminus \mathcal{V}_r) \beta_i(0) \in \mathbb{R}^r.
\]

In the next subsection, we present the result of the output synchronization problem.

C. Result

Our result for the output synchronization problem is the following.

**Theorem 3** Consider the multi-agent system (29), (30), (31), and suppose that Assumptions 3, 4, 5, 8, 9 hold. Then for each agent \( i \in \mathcal{V} \), the distributed exosystem generator (32), and the distributed dynamic compensator (35) with (16), (25), (26) and (34) solve Problem 2.

Theorem 3 improves the results of [17] in the following aspects. (i) While [17] requires \( S^* \) and \( Q^* \) to be known as common information by all agents, we allow \( Q_i(i \in \mathcal{V}) \) and \( S_i(i \in \mathcal{V} \setminus \mathcal{V}_r) \) to be different. This makes our solution more suitable for distributed implementation. Only in the special case when \( \mathcal{V}_r = \mathcal{V} \) does our requirement on \( S_i \) become the same as [17]. (ii) While [17] cannot deal with uncertain agent dynamics, we address uncertainty by the (q-copy) internal model principle.

**Proof of Theorem 3** From Lemma 4 with (25), (26), (27), for all \( i \in \mathcal{V} \), \( \lambda_i(t) \) achieve consensus while avoiding transmission zeros of agent \( i \) and the consensus value is the vector of eigenvalues of \( S^* \). From Lemma 6, we have \( Q_iw_i \to w_{\text{ref}}^i \in \mathbb{R}^r, i \in \mathcal{V} \) where \( w_{\text{ref}}^i \) is some constant vector. In the same way as in the proof of Theorem 1 with \( P_i = 0 \), we obtain \( e_i \to 0 \) and

\[
y_i = e_i - Q_iw_i \to -w_{\text{ref}}^i, i \in \mathcal{V}.
\]

Therefore \( (y_i(t) - y_j(t)) \to 0 \).

Note that as in Remark 2, the uncertainty parts \( \Delta A_i, \Delta B_i, \Delta C_i, \Delta D_i \) need not be small, and only need to satisfy that the matrix \( M_i(\lambda_i) \) in (20) is stable for all \( t \geq 0 \).

VII. SIMULATION EXAMPLES

In this section, we illustrate the designed distributed controller by applying it to solve distributed output regulation problems, as well as an output synchronization problem.

A. Distributed Output Regulation Problem

1) Example in Section VII-D continued: Consider the time-varying network as displayed in Fig. 5. The network periodically switches between \( \hat{G}_1 \) and \( \hat{G}_2 \) every 10 seconds. Thus this network uniformly contains a spanning tree and the root is node 0 (indeed the network in Fig. 1). Therefore Assumption 2 holds.

Also consider the exosystem and 4 agents with exactly the same parameters as Section VII-D. Then \( \beta_0 \) in (26) is \( \beta_0 = [2 - 2]^T \). It is checked that \( (A_{00}, B_{00}) \) and \( (C_i, A_{in}) \) are
We also apply the distributed dynamic compensator (9), \(z\) (in this simulation, \(\beta\) set signal the dotted curve represents the first element of exosystem’s \(w\) imprecise internal model of the exosystem.

Therefore, all the transmission zeros are on the imaginary axis and Assumption 7 does not hold.

We choose \(w_0(0)\) uniformly at random from the interval \([-1, 1]\). We apply the distributed exosystem generator (7) and (8) with the initial conditions \(w_i(0)\) selected uniformly at random from the interval \([-1, 1]\), and set \(S_i(0) = \begin{bmatrix} 0 & 0.5i \\ -0.5i & 0 \end{bmatrix}\).

The simulation result is displayed in Fig. 6. In Fig. 6(a), the dotted curve represents the first element of exosystem’s signal \(w_{0,1}\) and others represent the estimated exogeneous signals \(w_{i,1}, i = 1, 2, 3, 4\). Observe that all \(w_{i,1}\) synchronize with \(w_{0,1}\). Thus the distributed exosystem generators effectively create a local copy of the exosystem, despite that not all agents have access to the exosystem and the network is time-varying.

Fig. 6(b) shows the regulated output \(z_i\) of each agent (in this simulation, \(z_i = x_{i,1} - w_{0,1}\)). Observe that all \(z_i\) converge to 0. This demonstrates the effectiveness of the distributed dynamic compensators for achieving perfect regulation, despite of the parameter perturbation and initially imprecise internal model of the exosystem.

Controllable and observable, respectively, and therefore stabilizable and detectable, respectively; thus Assumptions 8 hold. Then we choose \(L_i\) such that the eigenvalues of \(A_{0i} - L_i C_i\) are \([-1, -2, -3]\).

The transmission zeros of the agents are

\[
P_i = \Pi_i = \{ \pm (0.5 + 0.1i) \} \tag{36}\]

We also apply the distributed dynamic compensator (9), (16), (23), (26), (27) with initial conditions \(x_i(0)\) and \(\xi_i(0)\) selected uniformly at random from the interval \([-1, 1]\), and set \(\beta_i(0) = 0\) for all \(i \in \mathcal{V}\). From (36), \(\rho_i\) and \(\gamma_{i,d}(t)\) in (27) are

\[
\rho_i = \frac{(2 - (0.5 + 0.1i))/2}{\gamma_{i,1}(t) = |0.5 + 0.1i - \beta_{i,1}(t)|} \quad \gamma_{i,2}(t) = |-(0.5 + 0.1i) - \beta_{i,2}(t)|.
\]

The simulation result is displayed in Fig. 6. In Fig. 6(a), the dotted curve represents the first element of exosystem’s signal \(w_{0,1}\) and others represent the estimated exogeneous signals \(w_{i,1}, i = 1, 2, 3, 4\). Observe that all \(w_{i,1}\) synchronize with \(w_{0,1}\). Thus the distributed exosystem generators effectively create a local copy of the exosystem, despite that not all agents have access to the exosystem and the network is time-varying.

Fig. 6(b) shows the regulated output \(z_i\) of each agent (in this simulation, \(z_i = x_{i,1} - w_{0,1}\)). Observe that all \(z_i\) converge to 0. This demonstrates the effectiveness of the distributed dynamic compensators for achieving perfect regulation, despite of the parameter perturbation and initially imprecise internal model of the exosystem.

We next examine the parameters in the distributed dynamic compensator. Define \(\hat{\alpha}_i\) as the real part of local estimate \(\hat{\lambda}_i\) made in the same as (27) with \(\hat{\beta}_i\) in (28). In this example, \(\hat{\alpha}_i, \hat{\beta}_i \in \mathbb{R}, \alpha_i = [\hat{\alpha}_i, \hat{\alpha}_i]^{T} \in \mathbb{R}^{2}\) and \(\beta_i = [\hat{\beta}_i, -\hat{\beta}_i]^{T} \in \mathbb{R}^{2}\). Fig. 7(a) and (b) show \(\hat{\beta}_i\) and \(\hat{\alpha}_i\), respectively. Each \(\hat{\beta}_i\) converges to \(\beta_0\), and each \(\hat{\alpha}_i\) becomes positive exactly when the distance between \(\hat{\beta}_i\) and the closest transmission zeros to \(\hat{\beta}_i\), namely \(\gamma_{i,d}(t)\), is less than \(\rho_i\).

The internal model \(G_i(t)\)’s entries contain \(\alpha_i(t)\) and \(\beta_i(t)\). In this example, \(G_i(t)\) is in the form

\[
G_i(t) = \begin{bmatrix} 0 & 1 \\ -\left(\hat{\alpha}_i(t)^{2} + \hat{\beta}_i(t)^{2}\right) & 2\hat{\alpha}_i(t) \end{bmatrix},
\]

and we choose the matrix \(K_i(t)\) such that the eigenvalues of (15) are \([-0.4, -0.8, -1.2, -1.6, -2.0]\).

Figs. 8 - 11 show the trajectories of all elements of \(K_i(t) = [k_{i,1}(t) \cdots k_{i,5}(t)], i = 1, \ldots, 4\) in this example, respectively. Observe that each entry of \(K\) changes
significantly exactly when $\alpha_i$ becomes positive (i.e. avoiding transmission zeros).

2) Output regulation of two-dimensional outputs over a large scale network: Consider the large scale time-varying network as displayed in Fig. 12. The network periodically switches among $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ every 2, 3 and 5 seconds, respectively. Thus this network uniformly contains a spanning tree and the root is node 0. Therefore Assumption 2 holds.

The exosystem (node 0) is

$$
\dot{w}_0(t) = S_0 w_0, \quad S_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
$$

Excluding the exosystem the network contains 155 agents and they are classified into five types:

$$
\begin{align*}
\dot{x}_i &= A_i x_i + B_i u_i + P_i w_0 \\
z_i &= C_i x_i + D_i u_i + Q_i w_0
\end{align*}
$$

where

$$
A_i = A_{i0} + \Delta A, \quad B_i = B_{i0} + \Delta B,
$$

$$
A_{i0} = \begin{bmatrix} 0 & 1 \\ m_i & 2 \end{bmatrix}, \quad \Delta A = \begin{bmatrix} 0 & 0 \\ 0 & -0.1 \end{bmatrix},
$$

$$
B_{i0} = \begin{bmatrix} 1 \\ (0.1 m_i + 0.2)^2 + m_i + 1 \\ 1 \end{bmatrix}, \quad \Delta B = \begin{bmatrix} 0 & 0 \\ -0.1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
$$

$$
D_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_i = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},
$$

and $m_i = 1$ for $i = 1, 6, 11, \ldots, m_i = 2$ for $i = 2, 7, 12, \ldots, m_i = 3$ for $i = 3, 8, 13, \ldots, m_i = 4$ for $i = 4, 9, 14, \ldots$ and $m_i = 5$ for $i = 5, 10, 15, \ldots$. It is checked that $(A_{i0}, B_{i0})$ and $(C_i, A_{i0})$ are controllable and observable, respectively, and therefore stabilizable and detectable, respectively; thus Assumptions 3, 4 hold. Then we choose $L_i$ such that the eigenvalues of $A_{i0} - L_i C_i$ are $\{-1.20, -1.21\}$.

The transmission zeros of the agents are

$$
\Pi_i = \bar{\Pi}_i = \{\pm (0.1 m_i + 0.2) j\}
$$

Therefore, all the transmission zeros are on the imaginary axis and Assumption 7 does not hold.

We choose $w_0(0)$ uniformly at random from the interval $[-1, 1]$. We apply the distributed exosystem generator (7) and (8) with the initial conditions $w_i(0)$ selected uniformly at random from the interval $[-1, 1]$, and set

$$
S_i(0) = 0_{2 \times 2}.
$$

We also apply the distributed dynamic compensator (9), (10), (25), (26), (27) with initial conditions $x_i(0), \xi_i(0)$ selected uniformly at random from the interval $[-1, 1]$ and $\beta_i(0) = 0$ for all $i \in \mathcal{V}$. We choose the matrix $K_i(t)$ such that the eigenvalues of (15) are $\{-0.70, -0.71, -0.72, -0.73, -0.74, -0.75\}$.

The simulation result is displayed in Fig. 13. This figure shows the regulated output $z_{i1}, z_{i2}$ of each agent (in this simulation, $z_{i1} = x_{i1} - w_{01}$ and $z_{i2} = x_{i2} - w_{02}$). Observe that all $z_{i1}$ and $z_{i2}$ converge to 0 for all $i \in \mathcal{V}$. This demonstrates the effectiveness of $q_i$-copy internal model for robust regulation of higher dimensional outputs over large scale networks.

B. Distributed Output Synchronization Problem

Consider the time-varying network in Fig. 14. The network periodically switches between $\mathcal{G}_1$ and $\mathcal{G}_2$ every 3 seconds. Thus this network uniformly contains a spanning tree with respect to $\mathcal{V}_r = \{1, 2, 3\}$. Therefore Assumption 8 holds.
The agents \((i = 1, \ldots, 5)\) are
\[
\dot{x}_i = A_i x_i + B_i u_i \\
y_i = C_i x_i + D_i u_i
\]
where
\[
A_i = A_i0 + \Delta A, \quad B_i = B_i0 + \Delta B,
\]
\[
A_{i0} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad B_{i0} = [0 \ -0.99 i]^T,
\]
\[
\Delta A = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Delta B = \begin{bmatrix} 0.1 & 0 \end{bmatrix}, \quad C_i = [1 \ 0], \quad D_i = 1.
\]
It is checked that \((A_{i0}, B_{i0})\) and \((C_i, A_{i0})\) are controllable and observable, respectively, and therefore stabilizable and detectable, respectively; thus Assumptions 3.4 hold. Then we choose \(L_i\) such that the eigenvalues of \(A_{i0} - L_i C_i\) are \([-0.7, -0.8]\).

The transmission zeros of the agents are
\[
\Pi_i = \hat{\Pi}_i = \{ \pm 0.1 i \}
\]
Therefore all the transmission zeros are on the imaginary axis and, set \(\rho_i = (1 - 0.1 i)/2\). Although the output of each agent is one dimensional, i.e. \(q = 1\), we define the dimension of the distributed exosystem generator as \(r = 2\) and for the agent \(i \in \mathcal{V}_r\), set
\[
S_i(0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\
\hat{\beta}_i(0) = 1.
\]
Then we set \(S_i(0) = 0_{2 \times 2} \) \((i \in \mathcal{V} \setminus \mathcal{V}_r)\) and choose \(\hat{\beta}_i(0) \) \((i \in \mathcal{V} \setminus \mathcal{V}_r)\) uniformly at random from the interval \([-1, 1]\) and \(w_i(0), x_i(0), \xi_i(0) \) \((i \in \mathcal{V})\) uniformly at random from the interval \([-1, 1]\), and set \(Q_i(0) = [-i 0]\) for all \(i \in \mathcal{V}\). Then we apply the distributed exosystem generator 3.2 and the distributed dynamic compensator 3.4. We choose the matrix \(K_i(t)\) such that the eigenvalues of \((15)\) are \([-0.7, -0.8, -0.9, -1.0]\).

The simulation result is displayed in Fig. 15. This figure shows the outputs \(y_i\) of all agents. Observe that all outputs synchronize. This example illustrates the effectiveness of our proposed controller for achieving robust output synchronization.

**REFERENCES**

[1] W. Ren and R. Beard, *Distributed Consensus in Multi-Vehicle Cooperative Control*. Communications and Control Engineering Series, Springer-Verlag, 2008.

[2] F. Bullo, J. Cortés, and S. Martínez, *Distributed Control of Robotic Networks*. Princeton University Press, 2009.

[3] Z. Lin, L. Wang, Z. Han, and M. Fu, “Distributed formation control of multi-agent systems using complex laplacian,” *IEEE Transactions on Automatic Control*, vol. 59, no. 7, pp. 1765–1777, 2014.

[4] X. Wang, Y. Hong, J. Huang, and Z. P. Jiang, “A distributed control approach to a robust output regulation problem for multi-agent linear systems;,” *IEEE Transactions on Automatic Control*, vol. 55, no. 12, pp. 2891–2895, 2010.

[5] Y. Su and J. Huang, “Cooperative output regulation of linear multi-agent systems,” *IEEE Transactions on Automatic Control*, vol. 57, no. 4, pp. 1062–1066, 2012.

[6] Y. Su, Y. Hong, and J. Huang, “A general result on the robust cooperative output regulation for linear uncertain multi-agent systems,” *IEEE Transactions on Automatic Control*, vol. 58, no. 5, pp. 1275–1279, 2013.

[7] W. Liu and J. Huang, “Cooperative robust output regulation of linear minimum-phase multi-agent systems under switching network,” in *Proc. 10th Asian Control Conference (ASCC)*, 2015, pp. 1–5.

[8] H. Cai, F. L. Lewis, G. Hu, and J. Huang, “Cooperative output regulation of linear multi-agent systems by the adaptive distributed observer,” in *Proc. 54th IEEE Conference on Decision and Control (CDC)*, 2015, pp. 5432–5437.

[9] Y. Yan and J. Huang, “Cooperative robust output regulation problem for discrete-time linear time-delay multi-agent systems via the distributed internal model,” in *Proc. 56th IEEE Conference on Decision and Control (CDC)*, 2017, pp. 4680–4685.

[10] Y. Su and J. Huang, “Cooperative global output regulation of heterogeneous second-order nonlinear uncertain multi-agent systems,” *Automatica*, vol. 49, p. 33453350, 2013.

[11] Y. Dong, J. Chen, and J. Huang, “Cooperative robust output regulation for second-order nonlinear multi-agent systems with an unknown exosystem,” in *Proc. 56th IEEE Conference on Decision and Control (CDC)*, 2017, pp. 3431–3436.

[12] B. A. Francis and W. M. Wonham, “The internal model principle of control theory,” *Automatica*, vol. 12, no. 5, pp. 457–465, 1976.

[13] B. A. Francis, “The linear multivariable regulator problem,” *SIAM Journal on Control and Optimization*, vol. 15, no. 3, pp. 486–505, 1977.
Appendix

Proof of Lemma 2. Since the origin is a uniformly exponentially stable equilibrium of $\dot{x} = A_1(t)x$, there exist bounded and positive definite matrices $P_1(t), Q_1(t)$ (for all $t \geq 0$) such that

$$ P_1(t) + P_1(t)A_1(t) + A_1(t)^{T}P_1(t) = -Q_1(t). $$

Then $V_1(x, t) := x^{T}P_1(t)x$ is a quadratic Lyapunov function for $\dot{x} = A_1(t)x$, and there exist constants $c_1, c_2, c_3, c_4$ such that the following are satisfied (globally):

$$
\begin{align*}
&c_1\|x\|^2 \leq V_1(x, t) \leq c_2\|x\|^2 \\
&\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x} A_1(t)x \leq -c_3\|x\|^2 \\
&\|\frac{\partial V_1}{\partial x}\| \leq c_4\|x\|.
\end{align*}
$$

Now consider $\dot{x} = A_1(t)x + A_2(t)x$. The term $A_2(t)x$ satisfies the inequality

$$ ||A_2(t)x|| \leq ||A_2(t)|| \cdot ||x||. $$

Since $A_2(t) \to 0$, we have $||A_2(t)|| \to 0$. Hence viewing $A_2(t)x$ as a vanishing perturbation to $\dot{x} = A_1(t)x$, it follows from [22, Corollary 9.1 and Lemma 9.5] that the origin is also a uniformly exponentially stable equilibrium of $\dot{x} = A_1(t)x + A_2(t)x$. In turn, there exist bounded and positive definite matrices $P_2(t), Q_2(t)$ (for all $t \geq 0$) such that

$$ P_2(t) + P_2(t)(A_1(t) + A_2(t)) + (A_1(t) + A_2(t))^T P_2(t) = -Q_2(t). $$

Let $V_2(x, t) := x^T P_2(t)x$ be a candidate Lyapunov function for (17). Then

$$
\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x} (A_1(t)x + A_2(t)x + A_3(t)) = -x^T Q_2(t)x + 2x^T P_2(t)A_3(t)x \\
\leq -\left(||Q_2(t)|| - \frac{1}{\epsilon}||x||^2 + \epsilon||P_2(t)||^2||A_3(t)||^2\right) \\
\leq -\left(||Q_2(t)|| - \frac{1}{\epsilon}||x||^2 + \epsilon||P_2(t)||^2||A_3(t)||^2\right).
$$

Let $\epsilon$ be such that $\epsilon > 0$ and $||Q_2(t)|| = \frac{1}{\epsilon} > 0$. Then it follows from [23, Theorem 5] that (17) is input-to-state stable, with $A_3(t)$ the input. Since $A_3(t) \to 0$ (uniformly exponentially), as a consequence of input-to-state stability ([23, Section 3.1], [22, Section 4.9]) we conclude that $x(t) \to 0$ (uniformly exponentially) as $t \to \infty$. □

Proof of Lemma 3. From [21, Theorem 1] and (7), $S_i(t \in \bar{V})$ reach consensus for all $S_i(0)$. Since Assumption 2 holds, the consensus value is $S_0$, i.e., $S_i(t) \to S_0$.

To show $w_i \to 0$, we consider

$$
\dot{w}_i = S_0w_i + \sum_{j=0}^{N} a_{ij}(t)(w_j - w_i) - S_0w_0,
$$

where $w_i := w_i - w_0$. Derivate

$$
\dot{w}_i = S_0w_i + \sum_{j=0}^{N} a_{ij}(t)(w_j - w_i) - S_0w_0,
$$

and in a compact form,

$$
\dot{\tilde{w}} = (I_N \otimes S_0 - L^{-}(t) \otimes I_r)\tilde{w}
$$

where $\tilde{w} := [\tilde{w}_1^T \cdots \tilde{w}_N^T]^T$. Since $\dot{\tilde{w}} \to 0$, the origin is a uniformly exponentially stable equilibrium of (40).

Now consider (8). Using $\tilde{S}_i := S_i - S_0$,

$$
\dot{\tilde{w}}_i = S_i(t)w_i + \sum_{j=0}^{N} a_{ij}(t)(w_j - w_i) - S_0w_0,
$$

and in a compact form,

$$
\dot{\tilde{w}} = (I_N \otimes S_0 - L^{-}(t) \otimes I_r)\tilde{w} + \text{diag}(\tilde{S}_1, \ldots, \tilde{S}_N)\tilde{w} + \text{diag}(\tilde{S}_1, \ldots, \tilde{S}_N)(I_N \otimes w_0)
$$

where $\tilde{w} := [\tilde{w}_1^T \cdots \tilde{w}_N^T]^T$. Since $\tilde{S}_i \to 0$, $\tilde{w} \to 0$ from Lemma 2. Therefore $w_i \to w_0$ as $t \to \infty$ for all $i \in V$. □

Proof of Lemma 5. Without loss of generality, we reorder the index of agents as $V_r = \{1, \ldots, k\}$ and $V \setminus V_r = \{k+1, \ldots, N\}$. Let $\tilde{S} = [S_1(t)^T \cdots S_N(t)^T]^T \in \mathbb{R}^{N \times r}$ be a bundled variable. In a compact form with respect to $S_i$, (32) can be written as

$$
\dot{\tilde{S}} = -\text{diag}(L_r(t) \otimes I_r)\tilde{S},
$$

$$
\tilde{S}(0) = [S_1^{(0)}^T \cdots S_N^{(0)}^T \cdots S_{k+1}^{(0)}^T \cdots S_N^{(0)}^T]^T
$$
where

\[
L_r(t) = \begin{bmatrix}
L_1(t) & 0 \\
L_2(t) & L_3(t)
\end{bmatrix}.
\]

From [21, Theorem 1] and Assumption 8, every \( S_i(i \in \mathcal{V}_r) \) is such that \( S_i(t) = S^* \) for all \( t \geq 0 \), and every \( S_i(i \in \mathcal{V} \setminus \mathcal{V}_r) \) is such that \( S_i(t) \to S^* \) as \( t \to \infty \). Therefore every \( S_i(i \in \mathcal{V}) \) reaches consensus for \( S_i(0) = S^*(i \in \mathcal{V}_r) \) and arbitrary \( S_i(0)(i \in \mathcal{V} \setminus \mathcal{V}_r) \), and the consensus value is \( S^* \).

To show \((w_i - w_j) \to 0\), we again consider (38) in the proof of Lemma 3 above. From the proof of Lemma 1 of [16] and by Assumption 8 \( w_i \to w^* \) as \( t \to \infty \) for all \( i \in \mathcal{V} \). Here \( w^* \) is a (virtual) signal generated by \( \dot{w}^* = S^*w^* \) and \( w^*(0) \) is related only to \( w_i(0), i \in \mathcal{V}_r \).

As with the proof of Lemma 3 using \( w^* \) instead of \( w_0 \), it is again derived that \( w_i \to w^* \) i.e. \((w_i - w_j) \to 0\) for all \( i, j \in \mathcal{V} \).