NON-EMPTINESS OF BRILL-NOETHER LOCI IN $M(2, L)$

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Abstract. Let $C$ be a smooth projective complex curve of genus $g \geq 2$. We investigate the Brill-Noether locus consisting of stable bundles of rank 2 and determinant $L$ of odd degree $d$ having at least $k$ independent sections. This locus possesses a virtual fundamental class. We show that in many cases this class is non-zero, which implies that the Brill-Noether locus is non-empty. For many values of $d$ and $k$ the result is best possible.

1. Introduction

Let $C$ be a smooth projective complex curve of genus $g \geq 2$. Let $M(2, d)$ be the moduli space of stable bundles of rank 2 and degree $d$ and, for any line bundle $L$ of degree $d$, let $M(2, L)$ denote the moduli space of stable bundles of rank 2 and determinant $L$. The Brill-Noether locus $B(2, d, k) \subset M(2, d)$ is defined by

$$B(2, d, k) := \{ E \in M(2, d) \mid h^0(E) \geq k \}.$$  

Similarly

$$B(2, L, k) := B(2, d, k) \cap M(2, L).$$

If $d \leq k + 2g - 2$, then $B(2, d, k)$ is a degeneracy locus of expected dimension

$$\beta(2, d, k) := 4g - 3 - k(k - d + 2g - 2).$$

Similarly $B(2, L, k)$ is a degeneracy locus of expected dimension

$$\beta(2, d, k) - g = 3g - 3 - k(k - d + 2g - 2).$$

A great deal is known about $B(2, d, k)$ (see for example [18] and more recently [9] and [11]; also [17] and [4] for the case of general rank). Much less is known about $B(2, L, k)$, except when $L = K$, where $K$ is the canonical bundle on $C$ (see [12] for a recent result and further references). In [19] Teixidor obtained a sufficient condition for $B(2, L, k)$ to be non-empty and to have a component of dimension $\beta(2, d, k) - g$.

When $d = 2g - 1 - 2r$ with $r \geq 1$, this condition becomes

$$g \geq \begin{cases} \frac{k(k+2r-1)}{(k+1)(k+2r-1)} + 1 & \text{for } k \text{ even} \\ \frac{k(k+2r-1)}{2} + 1 & \text{for } k \text{ odd}. \end{cases}$$

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The proof uses degenerations of $C$ and assumes that $C$ and $L$ are both general; however, a semi-continuity argument then shows that the results for non-emptiness are valid for any $C$ and any $L$. Recent work of Osserman [15, 16] contains new information about the dimension of $B(2, L, k)$ and also a non-emptiness result for $k = 2$ [16, Theorem 1.3]. A complete solution is known for $k \leq 3$ (see [10] and Remark 5.4).

In this paper we use a different method to investigate the non-emptiness of $B(2, L, k)$ for $d$ odd. In this case, $M(2, d)$ and $M(2, L)$ are smooth projective varieties. Suppose $d = 2g - 1 - 2r$ with $r \geq 1$. Then

$$\beta(2, d, k) - g = 3g - 3 - k(k + 2r - 1)$$

and $B(2, L, k)$ possesses a virtual fundamental class $b(r, k)$ which is independent of the choice of $L$ with $L$ of degree $d$. If $b(r, k) \neq 0$, then certainly $B(2, L, k) \neq \emptyset$ for all $L$ of degree $d$. Equivalently, the projection $B(2, d, k) \to \text{Jac}^d(C)$ given by taking determinants is surjective. The converse is in general false, since it can (and very often does) happen that $B(2, L, k)$ has dimension $> \beta(2, d, k) - g$. The method is similar to that of [12].

Following some preliminaries in Section 2 concerning the cohomology of $M(2, L)$, we obtain a polynomial formula for the class $b(r, k)$ in Section 3. In Section 4, we compute certain values of this polynomial (Proposition 4.5; this depends on a combinatorial lemma (Lemma 4.4)).

As in [12], detailed calculations of $b(r, k)$ are easier if $g$ is a sufficiently large prime. In this way we prove in Section 5,

**Theorem 5.2.** Suppose $g$ is a prime with

$$g > \max \left\{ \frac{(k + 2r - 2)(k - 1)}{2}, \frac{k(k + 2r - 1)}{3} + 1, 2k + 2r - 1 \right\}.$$  

Then $b(r, k) \neq 0$.

**Theorem 5.3.** Let $g_{r,k}$ be the smallest prime such that

$$g_{r,k} > \max \left\{ \frac{(k + 2r - 2)(k - 1)}{2}, \frac{k(k + 2r - 1)}{3} + 1, 2k + 2r - 1 \right\}.$$  

Then, for $r \geq 1$ and $L$ a line bundle of degree $2g - 1 - 2r$,

$$B(2, L, k) \neq \emptyset \quad \text{and} \quad B(2, K \otimes L^*, k + 2r - 1) \neq \emptyset$$

for all $g \geq g_{r,k}$.

The condition on $g$ in the statements of the theorems is slightly less restrictive than that of [11]. Consequently in some cases we have improvements of the results of [19]. One can obtain much better results for small values of $r$ and $k$ using Maple. Let $g'_{r,k}$ be the smallest prime such that

$$g'_{r,k} \geq \frac{k(k + 2r - 1)}{3} + 1.$$
Note that this this inequality is equivalent to \( \beta(2, d, k) - g'_r, k \geq 0 \). Then we claim that \( b(r, k) \neq 0 \) for all \( g \geq g'_r, k \). The values of \( r \) and \( k \) for which we have verified this are listed in Remark 5.5.

In Section 6 we calculate \( b(1, k) \) exactly for \( k \leq 5 \) and consider the possible geometrical interpretation of these calculations. Finally, in the appendix, we give the proof of Lemma 4.4.

Throughout the paper \( C \) is a smooth projective complex curve of genus \( g \geq 2 \).

2. Preliminaries

Let \( M(2, L) \) and \( B(2, L, k) \) be as in the introduction, with \( L \) a line bundle of odd degree \( d < 2g - 2 \). Write also \( d = 2g - 1 - 2r \), where \( r \) is a positive integer. The moduli space \( M(2, L) \) supports a universal bundle \( E \) on \( C \times M(2, L) \) and \( B(2, L, k) \) can be viewed as a degeneracy locus in the following way. Choose an effective divisor \( D \) of degree \( g + r - 1 \) on \( C \). Denote also by \( D \) the pullback of \( D \) to \( C \times M(2, L) \) and consider the exact sequence

\[
0 \to E \to E(D) \to E|_D \to 0.
\]

Taking direct images via the projection \( p_2 : C \times M(2, L) \to M(2, L) \), we get the exact sequence

\[
0 \to p_2^*E(D) \to p_2^*E|_D \to R^1p_2^*E \to 0.
\]

The Brill-Noether locus \( B(2, L, k) \) is then the corank \( k \) degeneracy locus of the homomorphism \( p_2^*E(D) \to p_2^*E|_D \). Note that the vector bundle \( E := p_2^*E(D) \) is of rank \( 2 \deg D + 1 - 2r \) and \( F := p_2^*E|_D \) of rank \( 2 \deg D \), so the “expected dimension” of \( B(2, L, k) \) is

\[
\beta(2, d, k) - g = 3g - 3 - k(2r - 1).
\]

This means that every component of \( B(2, L, k) \) has dimension at least \( \beta(2, d, k) - g \), but it does not imply that all (or any) of its components are of this dimension or even that it is non-empty when \( \beta(2, d, k) - g \geq 0 \). However it does imply that \( B(2, L, k) \) possesses a virtual fundamental class

\[
b(r, k) \in H^{2k(k+2r-1)}(M(2, L), \mathbb{Z}).
\]

Moreover, if \( b(r, k) \neq 0 \), then \( B(2, L, k) \neq \emptyset \).

Following [14] and noting that \( d = 2g - 1 - 2r \), we can write the Chern classes of \( E \) as

\[
c_1(E) = \alpha + (2g - 1 - 2r)\varphi,
\]

\[
c_2(E) = \frac{\alpha^2 - \beta}{4} + \psi + (g - r)\alpha \otimes \varphi.
\]

Here \( \alpha \) is the positive generator of \( H^2(M(2, L), \mathbb{Z}) \simeq \mathbb{Z} \),

\[
\beta \in H^1(M(2, L), \mathbb{Z}), \ \psi \in H^3(M(2, L), \mathbb{Z}) \otimes H^1(C, \mathbb{Z})
\]

and \( \varphi \) is the fundamental class of \( C \). We define \( \gamma \in H^6(M(2, L), \mathbb{Z}) \) by

\[
\psi^2 = \gamma \otimes \varphi.
\]

The subalgebra of \( H^*(M(2, L), \mathbb{Q}) \) generated by \( \alpha, \beta \) and \( \gamma \) can be written as \( \mathbb{Q}[\alpha, \beta, \gamma]/I_g \), and the ideal of relations \( I_g \) is explicitly described in [11]. This ideal
depends only on \( g \) provided \( \deg L \) is odd. For any polynomial \( f \in \mathbb{Q}[\alpha, \beta, \gamma] \), we denote by \((f)\) the corresponding cohomology class. It is proved in [11, Lemma 3.1] that, if \( g \geq g_0 \), then

\[
(2.1) \quad f \in I_g \implies f \in I_{g_0}.
\]

In general, it is quite complicated to determine whether a given polynomial \( f \) is in \( I_g \). However, Thaddeus [20] gave formulae for the intersection numbers \((\alpha^m \beta^n \gamma^p)\)

\[
(m + 2n + 3p = 3g - 3);
\]
we need only a particular deduction from these formulae, which was proved in [12].

**Lemma 2.1.** [12, Lemma 5.1] Suppose that \( g \) is an odd prime and \( m + 2n + 3p = 3g - 3 \). Then

\[
(\alpha^m \beta^n \gamma^p) \equiv \begin{cases} 
-1 \mod g & \text{if } p = 0 \text{ and } m = g - 1, 2g - 2 \text{ or } 3g - 3, \\
0 \mod g & \text{otherwise}.
\end{cases}
\]

Finally, recall that, if \( G \) is any vector bundle of rank 2 with Chern classes \( c_1, c_2 \), we can write formally

\[
1 + c_1 + c_2 = \left(1 + \frac{c_1 - \sqrt{c_1^2 - 4c_2}}{2}\right) \cdot \left(1 + \frac{c_1 + \sqrt{c_1^2 - 4c_2}}{2}\right)
\]

and then, for any \( n \geq 0 \), the Chern character of \( G \) is given by

\[
(2.2) \quad n! \, \text{ch}_n(G) = \left(\frac{c_1 - \sqrt{c_1^2 - 4c_2}}{2}\right)^n + \left(\frac{c_1 + \sqrt{c_1^2 - 4c_2}}{2}\right)^n.
\]

We shall write the right hand side of this formula for short in the form

\[
\left(\frac{c_1 - \sqrt{c_1^2 - 4c_2}}{2}\right)^n + (\).
\]

and do the same for other similar expressions.

### 3. The Fundamental Class

Recall the bundles \( E \) and \( F \) from Section [2] and write \( c_i := c_i(F - E) \). By the Porteous formula [2, II (4.2)], we have

\[
(3.1) \quad b(r, k) = \begin{vmatrix}
\begin{array}{cccc}
\alpha^r \beta^r & \alpha^r \beta^{r+1} & \ldots & \alpha^r \\
\alpha^r \beta^{r-1} & \alpha^r \beta^r & \ldots & \alpha^r \\
\ldots & \ldots & \ldots & \alpha^r \\
\alpha^r \beta^r & \alpha^r \beta^{r-1} & \ldots & \alpha^r \\
\end{array}
\end{vmatrix} \cdot \begin{vmatrix}
\begin{array}{cccc}
\beta + 2r & \beta + 2r & \ldots & \beta + 2r \\
\beta + 2r & \beta + 2r & \ldots & \beta + 2r \\
\ldots & \ldots & \ldots & \beta + 2r \\
\beta + 2r & \beta + 2r & \ldots & \beta + 2r \\
\end{array}
\end{vmatrix}.
\]

Our main object in this section is to compute the Chern classes \( c_i \). For this, note first that, if we choose \( D = q_1 + \ldots + q_{\deg D} \) with distinct points \( q_i \), then \( F \simeq \oplus_{i}^{\deg D} E_{\{q_i\} \times M(2, L)} \). Topologically the bundles \( E_{\{q_i\} \times M(2, L)} \) are all isomorphic and we denote any one of them by \( E_M \). We have then

\[
(3.2) \quad \text{ch}(F) = \deg(D) \text{ch}(E_M).
\]
**Lemma 3.1.** For $n \geq 1$,
\[
2^{n+1} \text{ch}_n(F - E) = \frac{1}{n!}(\alpha - \sqrt{\beta})^n(2r - 1) - \frac{1}{n!}(\alpha - \sqrt{\beta})^n \left(\frac{\alpha}{\sqrt{\beta}} + \frac{2\gamma}{\beta^2}\right) - \frac{1}{2(n - 1)!}(\alpha - \sqrt{\beta})^{n-1}\frac{4\gamma}{\beta} + (.).
\]

**Proof.** Using (3.2) and Grothendieck-Riemann-Roch, we obtain
\[
\text{ch}(F - E) = \deg(D) \text{ch}(E_M) - \deg(D) \text{ch}(E) + (g - 1) \text{ch}(E_M) - \text{ch}(E(C)),
\]
where
\[
\text{ch}(E) = \int_C \text{ch}(E).
\]

Now, by (2.2),
\[
n! \text{ch}_n E_M = \left(\frac{\alpha - \sqrt{\beta}}{2}\right)^n + \left(\frac{\alpha + \sqrt{\beta}}{2}\right)^n
\]
and
\[
(n + 1)! \text{ch}_{n+1} E = \left(\frac{\alpha + (2g - 1 - 2r)\varphi - \sqrt{\beta} - 4\psi - 2\alpha\varphi}{2}\right)^{n+1} + (.).
\]

Now expand
\[
\sqrt{\beta} - 4\psi - 2\alpha\varphi = \sqrt{\beta} \left(1 - \frac{4\psi}{\beta} - \frac{2\alpha\varphi}{\beta}\right)^{\frac{1}{2}}
\]
\[
= \sqrt{\beta} \left(1 - \frac{2\psi}{\beta} + \frac{\alpha\varphi}{\beta} - \frac{2\gamma\varphi}{\beta^2}\right).
\]

Hence
\[
(n + 1)! \text{ch}_n E(C) = \frac{1}{2^{n+1}} \int_C \left(\alpha - \sqrt{\beta} + \frac{2\psi}{\sqrt{\beta}} + \left(2g - 1 - 2r + \frac{\alpha}{\sqrt{\beta}} + \frac{2\gamma}{\beta^2}\right)\varphi\right)^{n+1} + (.)
\]
\[
= \frac{1}{2^{n+1}} \left[(n + 1)(\alpha - \sqrt{\beta})^n \left(2g - 1 - 2r + \frac{\alpha}{\sqrt{\beta}} + \frac{2\gamma}{\beta^2}\right) + \left(n + 1\right)\left(\alpha - \sqrt{\beta}\right)^{n-1}\frac{4\gamma}{\beta}\right] + (.).
\]
Since
\[
\text{ch}_n(F - E) = (g - 1) \text{ch}_n E_M - \text{ch}_n E(C),
\]
this implies the assertion. \qed

Turning now to Chern classes, we have of course $c_0 = 1$. We write also $c_n = 0$ for $n < 0$.

**Proposition 3.2.** Let $c_i = c_i(F - E)$. Then, for every integer $n$,
\[
(n + 4)c_{n+4} + (2n + 6 - r)\alpha c_{n+3} + \left[(n + 2 - r)\alpha^2 + (2n + 5 - 2r)\frac{\alpha^2 - \beta}{4}\right]c_{n+2}
\]
\[
+ \left[2n + 3 - r\alpha\frac{\alpha^2 - \beta}{4} + \frac{\gamma}{2}\right]c_{n+1} + \frac{1}{16}(\alpha^2 - \beta)^2(n + 1 - 2r)c_n = 0.
\]
Proof. Let \( c(t) = \sum_0^\infty c_n t^n \). Consider
\[
c(t) = \exp(\log(c(t)) = \exp \left( c_1(F - E)t + c_2(F - E)t^2 + \cdots + (-1)^{n-1}(n-1)! c_n(F - E)t^n + \cdots \right)
\]
\[
= \exp \left[ (2r - 1) \sum_{n=1}^\infty \frac{(-1)^{n-1}}{2^{n+1}} (\alpha - \sqrt{\beta})^n t^n \right]
\]
\[
- \sum_{n=1}^\infty \frac{(-1)^{n-1}}{2^n} (\alpha - \sqrt{\beta})^n \left( \frac{\alpha}{\sqrt{\beta}} + \frac{2\gamma}{\beta^2} \right) t^n - \sum_{n=1}^\infty \frac{(-1)^{n-1}}{2^n} (\alpha - \sqrt{\beta})^{n-1} \gamma t^n + (.) \right].
\]
So
\[
\frac{d}{dt}(c(t)) = c(t) \left[ (2r - 1) \sum_{n=1}^\infty \frac{(-1)^{n-1}}{2^{n+1}} (\alpha - \sqrt{\beta})^n t^{n-1}
\right.
\]
\[
- \sum_{n=1}^\infty \frac{(-1)^{n-1}}{2^n} (\alpha - \sqrt{\beta})^n \left( \frac{\alpha}{\sqrt{\beta}} + \frac{2\gamma}{\beta^2} \right) t^{n-1}
\]
\[
- \sum_{n=1}^\infty \frac{(-1)^{n-1}}{2^n} (\alpha - \sqrt{\beta})^{n-1} \gamma t^{n-1} + (.) \right]
\]
\[
= c(t) \left[ \frac{2r - 1}{4} (\alpha - \sqrt{\beta}) \frac{1}{1 + \alpha - \sqrt{\beta} t} - \frac{1}{4} (\alpha - \sqrt{\beta}) \frac{\alpha}{\sqrt{\beta}} \frac{1}{1 + \alpha - \sqrt{\beta} t}
\right.
\]
\[
- \frac{\gamma}{2\beta} \frac{1}{(1 + \alpha - \sqrt{\beta} t)^2} + (.) \right].
\]
Substituting \( c(t) = \sum_0^\infty c_n t^n \), multiplying by
\[
\left( 1 + \frac{\alpha - \sqrt{\beta}}{2} t \right)^2 \cdot \left( 1 + \frac{\alpha + \sqrt{\beta}}{2} t \right)^2
\]
and comparing the coefficients of \( t^{n+3} \) gives the result (after some algebraic manipulation). \( \square \)

Proposition 3.2 allows us to consider \( c_i(F - E) \) as a polynomial \( c_i(\alpha, \beta, \gamma) \). We can therefore define a polynomial
\[
P_k(\alpha, \beta, \gamma) :=
\]
\[
\begin{vmatrix}
c_k(2r-1) & c_k(2r) & \cdots & c_k(2r-2) \\
c_{k+2r-2} & c_{k+2r-1} & \cdots & c_{k+2r-3} \\
\cdots & \cdots & \cdots & \cdots \\
c_2 & c_{2r+1} & \cdots & c_{k+2r-1}
\end{vmatrix}
\]
such that
\[
(P_k(\alpha, \beta, \gamma)) = b(r, k).
\]
In the next proposition, we obtain a simpler recurrence relation for \( c_i(\alpha, \beta, 0) \).
Proposition 3.3. Let $c_i = c_i(\alpha, \beta, 0)$. Then, for every integer $n$,

$$ (n+2)c_{n+2} + (n+1-r)\alpha c_{n+1} + (n+1-2r)\frac{\alpha^2 - \beta}{4} c_n = 0. $$

Proof. By definition, $c_0 = 1$. From Lemma 3.1 we get $c_1 = r\alpha$. If $\gamma = 0$, we have the equation

$$ (1 + \frac{\alpha t}{2})^2 - \frac{\beta}{4} t^2 \sum_{n=1}^{\infty} nc_n t^{n-1} $$

$$ = \left[ \frac{1}{4} \left( 1 + \frac{\alpha + \sqrt{\beta}}{2} t \right) (\alpha - \sqrt{\beta}) \left( 2r - 1 - \frac{\alpha}{\sqrt{\beta}} \right) + (.) \right] \sum_{n=0}^{\infty} c_n t^n. $$

Comparing the coefficients of $t^{n+1}$ gives (3.3). □

Corollary 3.4. If $\beta = \alpha^2$, then $c_n = 0$ for $n \geq r + 1$.

Proof. This follows immediately from (3.3). □

4. Computation of $P_k$

We would like to prove that $P_k(\alpha, \beta, 0)$ (or equivalently $P_k(1, \beta, 0)$) is not identically zero. We prove in fact the stronger statement that $P_k(0, \beta, 0) \neq 0$.

For this, consider $\tilde{c}_i := c_i(0, \beta, 0)$. The recurrence relation for the $\tilde{c}_i$ is

$$ \tilde{c}_0 = 1, \quad \tilde{c}_1 = 0 \quad \text{and} \quad (n+2)\tilde{c}_{n+2} = \frac{\beta}{4}(n+1-2r)\tilde{c}_n $$

for $n \geq 0$.

Lemma 4.1. For all $n$ we have $\tilde{c}_{2n+1} = 0$ and, for $n \geq r$,

$$ \tilde{c}_{2n} = (-1)^r \frac{(2r-1)(2r-3) \cdots 1}{2^{2n-r} n(n-1) \cdots (n-r+1)} \cdot \frac{(2n-2r)!}{((n-r)!)^2} \left( \frac{\beta}{4} \right)^n. $$

Furthermore, if $\beta = 4$, then for any odd prime $p > \max\{2r-1, n\}$,

$$ \tilde{c}_{2n} \equiv (-1)^n e_n \mod p $$

where $e_n$ is defined by $(1+t)^{\frac{2n+2r-1}{2}} = \sum_{i=0}^{\frac{2n+2r-1}{2}} e_i t^i$.

Proof. The fact that $\tilde{c}_{2n+1} = 0$ follows directly from the recurrence relation. For $n \geq r$ we can solve the recurrence relation for $\tilde{c}_{2n}$ giving

$$ \tilde{c}_{2n} = \frac{2n-1-2r}{2n} \cdot \frac{2n-3-2r}{2n-2} \cdots \frac{1-2r}{2} \left( \frac{\beta}{4} \right)^n $$

$$ = (-1)^r (2r-1)(2r-3) \cdots 1 \cdot \frac{(2n-2r)!}{2^{2n-r} n(n-1) \cdots (n-r+1)}. $$

This gives the second assertion.

If $\beta = 4$, as in the proof of [12, Lemma 4.4] we see that

$$ \left( \frac{2^{n}}{n} \right) \equiv (-1)^n \frac{(2n)!}{2^{2n(n-1)!}^2} \mod p $$
for an odd prime, \( p > n \). So, for \( p > \max\{2r - 1, n\} \),

\[
\tilde{c}_{2n} \equiv (-1)^n \frac{(2r - 1)(2r - 3) \cdots 1}{2^r n(n - 1) \cdots (n - r + 1)} \binom{p - 1}{n - r} \pmod{p},
\]

\[
\equiv (-1)^n \frac{(2r - 1)(2r - 3) \cdots 1}{(p - 1 + 2r)(p - 3 + 2r) \cdots (p + 1)} \binom{p + 2r - 1}{n} \mod{p},
\]

giving the last assertion.

**Lemma 4.2.** Suppose \( \beta = 4 \). For integers \( u \geq v \geq r \) let

\[
A_{u,v} = \begin{pmatrix}
\tilde{c}_{2u} & \tilde{c}_{2u+2} & \cdots & \tilde{c}_{4u-2v} \\
\tilde{c}_{2u-2} & \tilde{c}_{2u} & \cdots & \tilde{c}_{4u-2v-2} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{c}_{2v} & \tilde{c}_{2v+2} & \cdots & \tilde{c}_{2u}
\end{pmatrix}.
\]

Then for any odd prime \( p > \max\{2u - v, 2r + 2u - 2v - 1\} \),

\[
\det A_{u,v} \not\equiv 0 \pmod{p}.
\]

**Proof.** Lemma 4.1 gives

\[
\det A_{u,v} \equiv (-1)^\delta \begin{vmatrix}
e_u & e_{u+1} & \cdots & e_{2u-v} \\
e_{u-1} & e_u & \cdots & e_{2u-v-1} \\
\vdots & \vdots & \ddots & \vdots \\
e_v & e_{v+1} & \cdots & e_u
\end{vmatrix} \mod{p},
\]

where \( u \) is repeated \( u - v + 1 \) times and

\[
\delta = \begin{cases} 
-1 & \text{if } u \text{ and } v \text{ are both odd,} \\
+1 & \text{otherwise.}
\end{cases}
\]

Now

\[
\Delta_{u,v}(e_i) \equiv S_{u-v+1,\ldots,u-v+1,0,\ldots,0}(1,\ldots,1) \mod{p}
\]

where \( u - v + 1 \) is repeated \( u \) times, \( 0 \) is repeated \( \frac{p+2r-1}{2} - u \) times and \( S \) is the Schur polynomial (see [8, equation (A.6)]). Using [8, Exercise A.30] we see that for \( p > 2r + 2u - 2v - 1 \),

\[
S_{u-v+1,\ldots,u-v+1,0,\ldots,0}(1,\ldots,1) \not\equiv 0 \pmod{p}.
\]

This implies the assertion.

**Proposition 4.3.** \( P_k(0,4,0) \not\equiv 0 \pmod{p} \) for any odd prime \( p > k + 2r - 2 \).
Proposition 4.5. For some non-zero constant \( c \),

\[
P_k(1, \beta, 0) = c \prod_{i=1}^{r} \left( \beta - \frac{1}{(2i - 1)^2} \right)^{k-1} \prod_{i=1}^{k-1} \left( \beta - \frac{1}{(2r + 2i - 1)^2} \right)^{k-i}
\]

Proof. Note first that if \( c_{2r}(1, \beta, 0) = 0 \), then by Lemma 4.3, \( c_n(1, \beta, 0) = 0 \) for all \( n \geq 2r \). So by Lemma 4.4 the matrix defining \( P_k(1, \beta, 0) \) is the zero matrix for \( \beta = \frac{1}{(2i-1)^2} \) for \( 1 \leq i \leq r \). This gives the first product in formula (4.1).

Let \( \ell \) be an integer \( \geq r + 1 \). We use the recurrence relation

\[
(n + 2)d_{n+2} + (n + 1 - r)d_{n+1} + (n + 1 - 2r)\frac{\ell(\ell - 1)}{(2\ell - 1)^2}d_n = 0
\]

for \( n \geq 2r - 1 \). Note that, for any value of \( d_{2r} \), there is a unique solution for \( d_n \) for \( n \geq 2r \). We claim that for \( n \geq 2r \),

\[
d_n = s \left( -\frac{\ell - 1}{2\ell - 1} \right)^n (a_0 + a_1n + \cdots + a_{\ell-1}n^{\ell-1})
\]

for some constants \( s, a_0, \ldots, a_{\ell-1} \) with \( a_0, \ldots, a_{\ell-1} \) not all zero.

We need to show that there exist constants \( a_0, \ldots, a_{\ell-1} \), not all zero, such that

\[
\left( -\frac{\ell - 1}{2\ell - 1} \right)^{n+2} (n + 2)(a_0 + a_1(n + 2) + \cdots + a_{\ell-1}(n + 2)^{\ell-r-1})
\]

\[
+ \left( -\frac{\ell - 1}{2\ell - 1} \right)^{n+1} (n + 1 - r)(a_0 + a_1(n + 1) + \cdots + a_{\ell-1}(n + 1)^{\ell-r-1})
\]
for all $n$, i.e.

$$(n + 2)(a_0 + a_1(n + 2) + \cdots + a_{\ell - 1}(n + 2)^{\ell - 1})$$

$\frac{-2\ell - 1}{\ell - 1}(n + 1 - r)(a_0 + a_1(n + 1) + \cdots + a_{\ell - 1}(n + 1)^{\ell - 1})$

$+ \frac{\ell}{\ell - 1}(n + 1 - 2r)(a_0 + a_1n + \cdots + a_{\ell - 1}n^{\ell - 1}) = 0.$

One checks that the coefficients of $n^{\ell - r}$ and $n^{\ell - r - 1}$ are both zero. This leaves us with $\ell - r - 1$ homogeneous linear equations in $a_0, \ldots, a_{\ell - 1}$ which have a non-trivial solution. The claim follows. Note that, if $d_{2r} = 0$, then $d_n = 0$ for all $n \geq 2r$, which is impossible unless $s = 0$.

According to (3.3), $c_n = c_n \left(1, \frac{1}{(2r - 1)^2}, 0\right)$ satisfies the recurrence relation

$$(n + 2)c_{n+2} + (n + 1 - r)c_{n+1} + (n + 1 - 2r)\frac{\ell(\ell - 1)}{(2\ell - 1)^2}c_n = 0.$$ 

Now choose $s$ such that $c_{2r} = d_{2r}$. Then $c_n = d_n$ for all $n \geq 2r$. It follows that the rows of the matrix defining $P_k(1, \frac{1}{(2r - 1)^2}, 0)$ lie in a $\mathbb{Q}$-vector space of dimension $\leq \ell - r$. So $\frac{1}{(2r - 1)^2}$ is a zero of multiplicity at least $k - \ell + r$ of the polynomial $P_k(1, \beta, 0)$. This gives the second product in formula (4.1). Since the degree of $P_k(1, \beta, 0)$ is $\frac{k(k + 2r - 1)}{2}$ by Proposition 4.3, this completes the proof of the proposition.

5. Main Theorem

For $g \geq 2k + 2r - 1$, define

$$w := ((g - 1)!2^{g-1})^k P_k(\alpha, \beta, \gamma) \in \mathbb{Z}[\alpha, \beta, \gamma]$$

and write

$$w = \sum_{j \geq 0} M_j \beta^j \alpha^{k + 2r - 1 - 2j} + \gamma R(\alpha, \beta, \gamma)$$

with $M_j \in \mathbb{Z}$. Then, writing

$$e := 3g - 3 - k(2r - 1),$$

we define

$$w_0 := \alpha^e w = \sum_{j \geq 0} M_j \beta^j \alpha^{3g - 3 - 2j} + \gamma \tilde{R}(\alpha, \beta, \gamma),$$

and

$$w_{\ell} := \alpha^{e - 2\ell} \beta^\ell w,$$

for $1 \leq \ell \leq \frac{e}{2}$. If $g$ is a prime, then, according to Lemma 2.1

$$(w_0) \equiv -M_0 - M_{2r - 1} - M_{g - 1} \mod g$$

and

$$(w_\ell) \equiv -M_{\frac{e}{2} - \ell} - M_{g - 1 - \ell} \mod g.$$ 

Note that, if $(w_0) \not\equiv 0 \mod g$ or $(w_\ell) \not\equiv 0 \mod g$, then $b(r, k) \neq 0$. 


Define as in [12, Section 5], for $0 \leq i < \frac{g-1}{2}$,
\[ M_i' := M_i + M_{i+\frac{g-1}{2}} + M_{i+g-1} \mod g \]
with $0 \leq M_i' \leq g - 1$ and consider
\[ q(\beta) := M_0' + M_1' \beta + \cdots + M_{\frac{g-1}{2}}' \beta^{\frac{g-3}{2}} \in \mathbb{F}_g[\beta]. \]

**Lemma 5.1.** Suppose $g$ is a prime. If
\[ g > \max \left\{ \frac{k(k + 2r - 1)}{3} + 1, 2k + 2r - 1 \right\}, \]
then $q(\beta)$ is not identically zero. Moreover, $q$ has $k + r - 1$ distinct zeros different from 0.

**Proof.** Let $x \in \mathbb{Z}$, $1 \leq x \leq g - 1$. Using the fact that $x^{g-1} \equiv 1 \mod g$ we see that
\[ P_k(1, x^2, 0) \equiv q(x^2) \mod g, \]
since $M_i = 0$ for $i > \frac{3g-3}{2}$. This is true, since $\frac{3g-3}{2} > \frac{k(k+2r-1)}{2}$ by hypothesis.

By Proposition 4.5, $P_k(1, x^2, 0)$ has precisely $k + r - 1$ distinct zeros. The field $\mathbb{F}_g$ contains $\frac{g-1}{2}$ non-zero squares. Since $k + r - 1 < \frac{g-1}{2}$ by hypothesis, there exists an integer $x$, $0 < x < g - 1$, such that
\[ P_k(1, x^2, 0) \not\equiv 0 \mod g. \]
Both assertions now follow from (5.3). \hfill \Box

**Theorem 5.2.** Suppose $g$ is a prime with
\[ g > \max \left\{ \frac{(k + 2r - 2)(k - 1)}{2}, \frac{k(k + 2r - 1)}{3} + 1, 2k + 2r - 1 \right\}. \]
Then $b(r, k) \neq 0$.

**Proof.** If $M_0' \neq 0$, then $b(r, k) \neq 0$ by (5.1). If $M_0' = 0$, then $M_{k_0}' \neq 0$ for some $k_0 \geq k + r$ by Lemma 5.1. We have $k_0 < \frac{g-1}{2}$ and we claim that
\[ g - 1 - k_0 \leq \frac{e}{2}. \]
In fact, this is equivalent to $g - 1 - 2k_0 \leq 3g - 3 - k(k + 2r - 1)$ which is true if $g \geq \frac{(k+2r-2)(k-1)}{2}$. The last inequality is true by hypothesis.

So consider $w_\ell$ with $\ell = \frac{g-1}{2} - k_0$. Note that
\[ M_{k_0} + M_{\frac{g-1}{2}+k_0} \equiv M_{k_0}' \mod g, \]
provided that $M_{g-1+k_0} \equiv 0 \mod g$. This is true if $g - 1 + k_0 > \frac{k(k+2r-1)}{2}$ which holds by hypothesis, since $k_0 \geq k + r$. So $b(r, k) \neq 0$ by (5.2). \hfill \Box

**Theorem 5.3.** Let $g_{r,k}$ be the smallest prime such that
\[ g_{r,k} > \max \left\{ \frac{(k + 2r - 2)(k - 1)}{2}, \frac{k(k + 2r - 1)}{3} + 1, 2k + 2r - 1 \right\}. \]
Then, for \( r \geq 1 \) and \( L \) a line bundle of degree \( 2g - 1 - 2r \),

\[
B(2, L, k) \neq \emptyset \quad \text{and} \quad B(2, K \otimes L^*, k + 2r - 1) \neq \emptyset
\]

for all \( g \geq g_{r,k} \).

**Proof.** For \( B(2, L, k) \) this follows from Theorem 5.2 and (2.1). The last part of the assertion follows from Serre duality. \( \square \)

**Remark 5.4.** For \( k = 1, 2, 3 \) precise conditions for the non-emptiness of \( B(2, L, k) \) are known.

- For \( k = 1 \): \( B(2, L, k) \neq \emptyset \) if and only if \( d \geq 1 \) or equivalently \( g \geq r + 1 \) (see [8]).
- For \( k = 2 \): \( B(2, L, k) \neq \emptyset \) for all \( L \) of degree \( d \) if and only if \( d \geq g \) or equivalently \( g \geq 2r+1 \). This follows from [10] Corollary 3.8 or [16] Theorem 1.3.
- For \( k = 3 \): \( B(2, L, k) \neq \emptyset \) for all \( L \) of degree \( d \) if and only if \( d \geq g + 2 \) or equivalently \( \beta(2, d, k) - g \geq 0 \). This follows from [10] Corollary 3.11.

The results for \( k = 1 \) and \( k = 2 \) imply that \( b(r, 1) = 0 \) if \( g < r + 1 \) and \( b(r, 2) = 0 \) if \( g < 2r + 1 \). These facts must correspond to relations in the cohomology rings.

For \( k \geq 4 \) the last two terms in the maximum of Theorems 5.2 and 5.3 are needed only for \( k = 4 \) and \( 1 \leq r \leq 5 \) and for \( k = 5 \) and \( r = 1 \) or 2.

**Remark 5.5.** For \( k \geq 4 \) we can improve the above results using Maple. Note that the definitions of \( w_0 \) and \( w_1 \) require only that \( g \) be a prime number with \( g > 2k+2r-2 \) and \( \ell \geq 1 \). Let \( g'_{r,k} \) be the smallest prime such that

\[
g'_{r,k} \geq \frac{k(k + 2r - 1)}{3} + 1.
\]

Note that for \( k \geq 4 \) we have \( \frac{k(k + 2r - 1)}{3} + 1 \geq 2k + 2r - 1 \) except when \( k = 4 \) and \( r = 1 \) or 2. In these cases we find that (5.4) implies that \( g'_{r,k} \geq 2k + 2r - 1 \). So this holds always.

Suppose we can prove directly that [5.1] or [5.2] gives an integer which is not congruent to 0 modulo \( g'_{r,k} \). Then it follows by (2.1) that \( b(r, k) \neq 0 \) and \( B(2, L, k) \neq \emptyset \) for all \( g \geq g'_{r,k} \) and for every line bundle \( L \) on \( C \) of degree \( 2g - 2r - 1 \). We carried this out for

\[
r = 1, \quad 4 \leq k \leq 17 \quad \text{and} \quad 2 \leq r \leq 5, \quad 4 \leq k \leq 10.
\]

For \( (r, k) = (1, 5), (1, 9), (1, 12), (1, 13), (1, 14), (1, 17), (2, 4), (2, 6), (2, 8), (2, 9), (3, 4), (3, 6), (3, 7), (3, 9), (4, 8), (5, 4), (5, 6) \) and \( (5, 8) \), this gives the best possible result for \( b(r, k) \neq 0 \), namely that \( b(r, k) \neq 0 \) whenever \( \beta(2, d, k) - g \geq 0 \).

**6. Calculations for \( r = 1 \)**

Ideally, we would like to prove that \( b(r, k) \neq 0 \) whenever

\[
g \geq g^0_{r,k} := \left\lceil \frac{k(k + 2r - 1)}{3} \right\rceil + 1,
\]

for all \( g \geq g_{r,k} \).
since this is equivalent to $\beta(2, d, k) - g \geq 0$. If $g_{r,k}^0$ is prime, we have $g_{r,k}^0 = g_{r,k}^0$ and the methods of Section 5 apply. Otherwise, the calculations become much more complicated. However, a complete calculation of the cohomology class $b(r, k)$ would be of interest not only for proving that $b(r, k) \neq 0$ but for investigating the geometry of the Brill-Noether locus. With the help of Maple, using (3.1), Proposition 3.2 and Thaddeus' formulae for the intersection numbers \[20\], we have carried out the computation for $r = 1$ (that is, $d = 2g - 3$) and $k \leq 5$. For $k \leq 3$ (and partially for $k = 4$), we can interpret these results geometrically.

For the remainder of this section, we suppose that $r = 1$. For $k = 1$, we have $g_{1,1}^0 = 2$ and it is easy to see by hand that $P_1(\alpha, \beta, \gamma) = \frac{1}{2}(\alpha^2 - \beta)$ and that the intersection number $(\alpha \cdot P_1(\alpha, \beta, \gamma))$ is 1. Geometrically, it is well known that $M(2, L)$ is a smooth intersection of quadrics in $\mathbb{P}^5$ and that $B(2, L, 1)$ is a line contained in this intersection \[13\] Theorem 2]. The elements of $B(2, L, 1)$ can be written as non-trivial extensions

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow L \longrightarrow 0.$$  

This works for all $L$.

For $k = 2$, we have $g_{1,2}^0 = 3$ and $d = 3$. This time $b(r, k)$ is itself a top dimensional class and is numerically equal to 1, so in particular there exists $E \in B(2, L, 2)$. Certainly $E$ has no line subbundle of degree $\geq 2$ and hence no line subbundle with $h^0 = 2$. Hence $E$ is generically generated and there is an exact sequence

$$0 \longrightarrow \mathcal{O}^2 \longrightarrow E \longrightarrow T \longrightarrow 0,$$

where $T$ is a torsion sheaf with associated line bundle $L$. Suppose that $h^0(L) = 1$ and that $L \simeq \mathcal{O}(p + q + r)$ with $p, q, r$ all different. The extensions of $T$ by $\mathcal{O}^2$ are classified (up to automorphisms of $T$) by points $(x, y, z) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Stability of $E$ implies that $x, y, z$ are all distinct (for example, if $x = y$, then $\mathcal{O}(p + q)$ is a subbundle of $E$). There is just one orbit of points of this type for the action of $\text{Aut}(\mathcal{O}^2) = \text{GL}(2, \mathbb{C})$, so $E$ is uniquely determined.

Another way of proving that $B(2, L, 2)$ consists of just one point is to look at extensions

$$0 \longrightarrow \mathcal{O}(p) \longrightarrow E \longrightarrow \mathcal{O}(q + r) \longrightarrow 0.$$  

Any non-trivial extension defines a stable bundle $E$. Moreover $h^0(E) = 2$ if and only if the element classifying the extension belongs to

$$\text{Ker} : H^1(\mathcal{O}(p - q - r)) \longrightarrow \text{Hom}(H^0(\mathcal{O}(q + r)), H^1(\mathcal{O}(p))).$$

It is easy to show that this kernel has dimension 1. Replacing $p$ by $q$ or $r$ could conceivably give up to 3 points in $B(2, L, 2)$. Since $b(r, k) = 1$, the 3 points must coincide. It is also easy to see directly that the 3 bundles are the same.

For $k = 3$, we have $g_{1,3}^0 = 5$ and $d = 7$. Again $b(r, k)$ is top dimensional and equal to 1. Let $L$ be a generated line bundle of degree 7 with $h^0(L) = 3$ (this is true generically) and consider the bundle $E$ defined by the evaluation sequence

$$0 \longrightarrow E_1^* \longrightarrow H^0(L) \otimes \mathcal{O} \longrightarrow L \longrightarrow 0.$$  

If $C$ has Clifford index 2, then $E_L$ is stable. In fact, if $M$ is any quotient line bundle of $E_L$, then $M$ is generated and $h^0(M^*) = 0$. So $h^0(M) \geq 2$ and hence $\text{deg} M \geq 4$. 

In order to show that $B(2, L, 3)$ consists of one point, it remains to show that there are no bundles $E \in B(2, L, 3)$ which are not generated. Certainly $E$ is generically generated since it cannot have a line subbundle of degree $\geq 4$ and hence no subbundle with $h^0 \geq 2$. Let $E'$ be the subsheaf of $E$ generated by its sections and suppose that $\deg E' \leq 6$. Since $h^0(E') \geq 3$, we can choose a 3-dimensional subspace of $H^0(E')$ which generates $E'$. Dualising the evaluation sequence

$$0 \longrightarrow \det E'' \longrightarrow V \otimes \mathcal{O} \longrightarrow E' \longrightarrow 0,$$

we see that $h^0(\det E') = 3$ and hence $\deg E' \geq 6$. So $\deg E' = 6$ and $E' = L(-p)$ for some $p$. Since $\dim B(1, 6, 3) = 2$, this is impossible for general $L$.

The case $k = 4$ is particularly interesting. Here $g^0_{1,k} = 8$, $d = 13$ and the expected dimension of the Brill-Noether locus is 1. It turns out that $(\alpha \cdot P_1(\alpha, \beta, \gamma))$ is equal to 13. This proves firstly that $B(2, L, 4)$ is non-empty, which was not previously known (neither Section 5 nor [19] applies). Secondly, recall that the unique line bundle on $M(2, L)$ with $c_1 = \alpha$ is very ample [3]. One might therefore expect that, for general $L$, $B(2, L, 4)$ is a curve whose degree with respect to this line bundle is 13. The construction of bundles $E \in B(2, L, 4)$ is much harder than for the cases considered above ($k \leq 3$). There is, however, one method that should give a 1-parameter family of such bundles. Let $C$ be a general curve of genus 8 and $L$ a general line bundle of degree 13 on $C$; in particular, $L$ is generated with $h^0(L) = 6$. Consider the canonical map

$$\psi : S^2 H^0(L) \longrightarrow H^0(L^2),$$

whose kernel is the Koszul cohomology group $K_{1,1}(C, L)$. We have $h^0(L^2) = 19$ by Riemann-Roch and $\dim \mathcal{K}^2 H^0(L) = 21$. For any non-zero element of $K_{1,1}(C, L)$, one can construct a rank 2 bundle $E$ with determinant $L$ and $h^0(E) \geq 4$ using [1] Theorem 3.4] and it can be shown that in general $E$ is generated and stable.

This construction can be carried out in a more geometrical fashion by the method used in the proof of [9] Theorem 3.2(ii)]. Let $\psi_L : C \rightarrow \mathbb{P}^5 = \mathbb{P}(h^0(L)^*)$ be the morphism defined by evaluation of sections of $L$. The fact that $\dim \ker \psi \geq 2$ means that $\psi_L(C)$ is contained in a pencil of quadrics. If we choose a 2-dimensional subspace $W$ of $H^0(L)$ such that the plane in $\mathbb{P}^5$ orthogonal to $W$ lies on one of the quadrics and does not meet $\psi_L(C)$, then $W$ generates $L$ and we can define $E$ by the evaluation sequence

$$0 \longrightarrow E^* \longrightarrow W \otimes \mathcal{O} \longrightarrow L \longrightarrow 0.$$

Clearly $E$ is generated. One can check firstly that $h^0(E) \geq 4$ and then that $E$ is stable and $h^0(E) = 4$. Dimensional calculations suggest that this should give a 1-parameter family of bundles $E \in B(2, L, 4)$. Whether this is the whole of $B(2, L, 4)$ requires further investigation.

For $k = 5$, we have $g^0_{1,5} = 11$ and $b(r, k)$ is a top dimensional class numerically equal to 23. In this case, we already know that $b(r, k) \neq 0$. A full investigation of the geometry is likely to be complicated.
Appendix. Proof of Lemma 4.4

For any integer \( r \geq 1 \), consider the sequence of polynomials with integer coefficients \( \tilde{c}(n, r, b) \) defined recursively by

\[
\tilde{c}(0, r, b) = 1, \quad \tilde{c}(1, r, b) = r
\]

and for \( n \geq 2 \),

\[
\tilde{c}(n, r, b) = (r + 1 - n)\tilde{c}(n - 1, r, b) + b(2r + 1 - n)(n - 1)\tilde{c}(n - 2, r, b).
\]

Lemma A.1. Lemma 4.4 is a consequence of the following equation,

\[
(A.1) \quad \tilde{c}(2r, r, b) = \prod_{j=1}^{r} [(2j - 1)^2 \cdot b - j(j - 1)].
\]

Proof. Inserting \( b = \frac{1-\beta}{4} \) we have

\[
\tilde{c}(2r, r, b) = \frac{1}{2^r} \prod_{j=1}^{r} (1 - (2j - 1)^2 \beta).
\]

On the other hand, \( \tilde{c}(n, r, b) = n!c_n(1, \beta, 0) \), since both sides satisfy the same recurrence relation and have the same initial values. \( \square \)

For the proof of (A.1) we need some preliminaries. Consider the following matrix.

\[
D(2n; z, a) :=
\begin{pmatrix}
z(a + n) & 2n - 1 & 0 & 0 & \ldots & 0 & 0 \\
1 & z(a + n - 1) & 2n - 2 & 0 & \ldots & 0 & 0 \\
0 & 2 & z(a + n - 2) & 2n - 3 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 2n - 2 & z(a - n + 2) & 1 \\
0 & 0 & 0 & 0 & 0 & 2n - 1 & z(a - n + 1)
\end{pmatrix}.
\]

We claim that for the proof of (A.1) it suffices to show that

\[
(A.2) \quad \det D(2n; z, a) = \prod_{j=1}^{n} [z^2(a + j)(a - j + 1) - (2j - 1)^2].
\]

Proof of the claim. For this consider 3-band-matrices

\[
\tilde{C}(n, r, b) = \left( \tilde{C}(n, r, b)_{i,j} \right)_{1 \leq i, j \leq n},
\]

where

\[
\tilde{C}(n, r, b)_{i,j} = \begin{cases} 
  r + 1 - i & \text{if } j = i \ (1 \leq i \leq n) \\
  \sqrt{b} \cdot (i - 1) & \text{if } j = i - 1 \ (1 < i \leq n) \\
  \sqrt{b} \cdot (2r - i) & \text{if } j = i + 1 \ (1 \leq i < n) \\
  0 & \text{otherwise}
\end{cases}
\]
For $1 \leq k < n$, the matrix $\tilde{C}(k, r, b)$ is the principal sub-minor of size $k \times k$ of $\tilde{C}(n, r, b)$ (taking elements in the first $k$ rows and columns). Hence, for $n \geq 4$,

$$\tilde{C}(n, r, b) = \begin{pmatrix} \tilde{C}(n-1, r, b) & 0 \\ 0 & \sqrt{b(n-1)} & r+1-n \\ 0 & 0 & \sqrt{b(2r+1-n)} & r+1-n \\ 0 & 0 & 0 & \sqrt{b(2r+1-n)} & r+1-n \end{pmatrix},$$

and expanding $\det \tilde{C}(n, r, b)$ starting from the lower right corner of the matrix gives

$$\det \tilde{C}(n, r, b) = (r+1-n) \det \tilde{C}(n-1, r, b) - b(n-1)(2r+1-n) \det \tilde{C}(n-2, r, b),$$

which is the recurrence relation defining $\tilde{c}(n, r, -b)$. Checking initial values then gives

$$\tilde{c}(n, r, b) = \det \tilde{C}(n, r, -b)$$

by induction.

Comparing the matrices $\tilde{C}(2n, n, b)$ and $D(2n, b^{1/2}, 0)$ one sees that

$$\tilde{C}(2n, n, b) = b^{1/2} \cdot D(2n, b^{1/2}, 0).$$

and hence using (A.2),

$$\det \tilde{C}(2n, n, b) = b^n \cdot \det D(2n, b^{1/2}, 0)$$

$$= b^n \cdot \prod_{1 \leq i \leq n} \left[ \frac{1}{b} \cdot j \cdot (-j+1) - (2j-1)^2 \right]$$

$$= \prod_{1 \leq i \leq n} \left[ j \cdot (-j+1) - b \cdot (2j-1)^2 \right]$$

and finally

$$\tilde{c}(2n, n, b) = \det \tilde{C}(2n, n, -b) = \prod_{1 \leq i \leq n} \left[ b \cdot (2j-1)^2 - j \cdot (j-1) \right].$$

It remains to prove (A.2). For this consider the following matrices:

(1)

$$A_n = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n-1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & n-1 & n \end{pmatrix}.$$
$A_n$ has simple eigenvalues 1, 2, \ldots, $n$ and the matrix of (non-orthogonal) left eigenvectors (i.e., $A_n$ is multiplied from the right) is the binomial matrix

\[
\begin{pmatrix}
(i - 1) \\
(j - 1)
\end{pmatrix}_{i,j=1,\ldots,n}.
\]

(2)

$$B_n = \begin{pmatrix}
2 & -(n - 1) & 0 & 0 & \cdots & 0 & 0 \\
1 & 4 & -(n - 2) & 0 & \cdots & 0 & 0 \\
0 & 2 & 6 & -(n - 3) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 2(n - 1) & -1 \\
0 & 0 & 0 & 0 & \cdots & n - 1 & 2n
\end{pmatrix}.$$  

$B_n$ has $n+1$ as its only eigenvalue, which is $n$-fold and maximally degenerate (i.e., the eigenspace of $n+1$ is one-dimensional). The eigenspace is given by the vector

\[
\begin{pmatrix}
(n - 1) \\
0 \\
1 \\
2 \\
\vdots \\
\vdots \\
\vdots \\
(n - 1)
\end{pmatrix}
\]

which is indeed the same as the last eigenvector of $A_n$.

There is an intimate relation between $A_n$ and $B_n$: the matrix $\tilde{B}_n = (n+1)I_n - B_n$ maps the eigenvectors of $A_n$ as follows. For $1 \leq k \leq n-1$, let

$$\alpha_k := \begin{pmatrix}
(k - 1) \\
0 \\
1 \\
2 \\
\vdots \\
\vdots \\
\vdots \\
(n - 1)
\end{pmatrix}$$

denote the left eigenvectors of $A_n$. Then, for $1 \leq k \leq n-1$,

$$\alpha_k \mapsto \alpha_k \tilde{B}_n = (n - k) \alpha_{k+1}$$

and (obviously)

$$\alpha_n \mapsto \alpha_n \tilde{B}_n = 0.$$  

**Lemma A.2.** Let $s$ be a real parameter. The characteristic polynomial of the matrix $C_n(s) := (n+1) \cdot A_n + s \cdot B_n$ is

$$\chi(C_n(s); z) = \prod_{1 \leq k \leq n} (z - (s + k)(n + 1))$$

**Proof.** For the proof we consider the matrix of the transformation given by $C_n(s)$ in the eigenbasis $\alpha_1, \ldots, \alpha_n$ of $A_n$. We have

$$C_n(s) = (n + 1)(A_n + s I_n) - s\tilde{B}_n$$

and hence, for $1 \leq k \leq n$,

$$C_n(s) : \alpha_k \mapsto (n + 1)(k + s) \alpha_k - s(n - k) \alpha_{k+1}.$$
This shows that the transformation $C_n(s)$ in the basis $\alpha_1, \ldots, \alpha_n$ is
\[
\begin{pmatrix}
(n+1)(1+s) & -s(n-1) & 0 & 0 & \cdots & 0 \\
0 & (n+1)(2+s) & -s(n-2) & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -s \\
0 & 0 & 0 & 0 & \cdots & (n+1)(n+s)
\end{pmatrix}
\]
and the eigenvalues are simply the diagonal elements $(n+1)(s+k)$ $(1 \leq k \leq n)$. This implies the assertion.

From now on we assume that $n$ is even.

**Lemma A.3.** *The characteristic polynomial of the matrix* $L_n(a)$ *is* 
\[
L_n(a) = \begin{pmatrix}
a & n-1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 2a & n-2 & 0 & 0 & \cdots & 0 \\
0 & 2 & 3a & n-3 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & n-1 & an
\end{pmatrix}
\]

($a$ is a parameter) is
\[
\Lambda_n(a; z) := \prod_{1 \leq i < j \leq n} \left[ (z - i \cdot a)(z - j \cdot a) - (i - j)^2 \right].
\]

**Proof.** Let $a_\ell = \frac{k - \ell}{\sqrt{k \ell}}$ where $k, \ell$ are real parameters. Multiplying by $\sqrt{k \ell}$ gives
\[
\sqrt{k \ell} \cdot L_n(a_\ell) = \begin{pmatrix}
k - \ell & (n-1)\sqrt{k \ell} & 0 & 0 & 0 & \cdots & 0 \\
k \sqrt{k \ell} & 2(k - \ell) & (n-2)\sqrt{k \ell} & 0 & 0 & \cdots & 0 \\
0 & 2\sqrt{k \ell} & 3(k - \ell) & (n-3)\sqrt{k \ell} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & (n-2)\sqrt{k \ell} & (n-1)(k - \ell) & \sqrt{k \ell} \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & (n-1)\sqrt{k \ell} & n(k - \ell)
\end{pmatrix}.
\]

This matrix is similar to
\[
\begin{pmatrix}
k - \ell & (n-1)\ell & 0 & 0 & 0 & \cdots & 0 \\
k & 2(k - \ell) & (n-2)\ell & 0 & 0 & \cdots & 0 \\
0 & 2k & 3(k - \ell) & (n-3)\ell & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & (n-2)k & (n-1)(k - \ell) & \ell \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & (n-1)k & n(k - \ell)
\end{pmatrix}
\]

If $k$ and $\ell$ are related by $k + \ell = n + 1$, then this is just the matrix $C_n(-\ell)$ we considered above.

For the rest of the proof we assume now $k + \ell = n + 1$. Then Lemma A.2 implies that the characteristic polynomial of $\sqrt{k \ell} \cdot L_n(a_\ell)$ is $\prod_{1 \leq m \leq n} [z - (n+1)(m-\ell)]$. 
which immediately gives: the matrix $L_n(a_\ell)$ has characteristic polynomial

$$\prod_{1 \leq m \leq n} \left[ z - (n + 1) \frac{m - \ell}{\sqrt{k\ell}} \right].$$

Now $\Lambda_n(a); z)$ is a monic polynomial of degree $n$ in the parameter $a$. For showing that $\Lambda_n(a; z)$ is the characteristic polynomial of $L_n(a)$, it suffices to show that for the $n$ interpolation points $a_\ell$ ($1 \leq \ell \leq n$) the polynomial $\Lambda_n(a_\ell; z)$ is indeed the characteristic polynomial of $L_n(a_\ell)$, i.e. that, for $1 \leq \ell \leq n$,

$$\Lambda_n(a_\ell; z) = \prod_{1 \leq m \leq n} \left[ z - (n + 1) \frac{m - \ell}{\sqrt{k\ell}} \right].$$

Now both sides are monic polynomials of degree $n$ in $z$. Since the expression on the right hand side vanishes at the $n$ interpolation points

$$\xi_{\ell,m} := (n + 1) \frac{m - \ell}{\sqrt{k\ell}} (1 \leq m \leq n),$$

it suffices to show that for $1 \leq m \leq n$,

$$\Lambda_n(a_\ell; \xi_{\ell,m}) = 0.$$

We write explicitly

$$\Lambda_n(a_\ell; \xi_{\ell,m}) = \prod_{1 \leq i < j \leq n} \left[ (n + 1) \frac{(m - \ell)^2}{k\ell} - (n + 1) \frac{(m - \ell)(k - \ell)}{k\ell} + ij \frac{(k - \ell)^2}{k\ell} - (i - j)^2 \right].$$

We have to show that (at least) one of the bracketed terms under the product vanishes. Now we use both conditions $k + \ell = n + 1$ and $i + j = n + 1$ crucially!

From

$$(k - \ell)^2 = (n + 1)^2 - 4k\ell$$

we have

$$ij \frac{(k - \ell)^2}{k\ell} - (i - j)^2 = ij \frac{(n + 1)^2}{k\ell} - 4ij - (i^2 - 2ij + j^2)$$

$$= ij \frac{(n + 1)^2}{k\ell} - (n + 1)^2$$

$$= (n + 1)^2 \left( \frac{ij}{k\ell} - 1 \right)$$
and (since $n$ is even)

$$
\Lambda_n(a, \xi_{\ell, m}) = \left( \frac{n+1}{\sqrt{k\ell}} \right)^n \prod_{i+j=n+1, 1 \leq i < j \leq n} \left[ (m-\ell)^2 - (m-\ell)(k-\ell) + ij - k\ell \right]
$$

$$
= \left( \frac{n+1}{\sqrt{k\ell}} \right)^n \prod_{i+j=n+1, 1 \leq i < j \leq n} \left[ m^2 - (k+\ell)m + ij \right]
$$

$$
= \left( \frac{n+1}{\sqrt{k\ell}} \right)^n \prod_{i+j=n+1, 1 \leq i < j \leq n} \left[ m^2 - (i+j)m + ij \right]
$$

$$
= \left( \frac{n+1}{\sqrt{k\ell}} \right)^n \prod_{i+j=n+1, 1 \leq i < j \leq n} \left[ (m-i)(m-j) \right].
$$

This shows that for all $1 \leq m \leq n$ and $z = \xi_{\ell, m}$ the quadratic factor belonging to $m = i$ and $m = j$ vanishes. \hfill \Box

Lemma A.3 can be stated in the following (elegant) way (again, for even $n$). For parameters $a, b$, let $\varepsilon_{a,b}$ be the affine function $\varepsilon_{a,b}(x) := a \cdot x + b$. Then

$$
\det \begin{pmatrix}
\varepsilon_{a,b}(1) & n-1 & 0 & 0 & \ldots & 0 & 0 \\
1 & \varepsilon_{a,b}(2) & n-2 & 0 & \ldots & 0 & 0 \\
0 & 2 & \varepsilon_{a,b}(3) & n-3 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \ldots & n-2 & \varepsilon_{a,b}(n-1) & 1 & 0 & 0 \\
0 & \ldots & 0 & n-1 & \varepsilon_{a,b}(n) & 0 & 0
\end{pmatrix} = \prod_{i+j=n+1, 1 \leq i < j \leq n} \left[ \varepsilon_{a,b}(i) \cdot \varepsilon_{a,b}(j) - (i-j)^2 \right].
$$

Now we are in a position to complete the proof of (A.2).

Proof of equation (A.2). For $n$ even we have to show

$$
\det \begin{pmatrix}
z(a+n/2) & n-1 & 0 & \ldots & 0 & 0 \\
1 & z(a+n/2-1) & n-2 & \ldots & 0 & 0 \\
0 & 2 & z(a+n/2-2) & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 & z(a-n/2+2) \\
0 & 0 & 0 & \ldots & 0 & n-1
\end{pmatrix}
$$

$$
= \prod_{j=1}^{n/2} \left[ z^2(a+j)(a-j+1) - (2j-1)^2 \right]
$$

Replacing $a$ by $a + n/2$, the matrix becomes

$$
\begin{pmatrix}
z(a+n) & n-1 & 0 & \ldots & 0 & 0 \\
1 & z(a+n-1) & n-2 & \ldots & 0 & 0 \\
0 & 2 & z(a+n-2) & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 & z(a+2) \\
0 & 0 & 0 & \ldots & 0 & n-1
\end{pmatrix}
$$
and this is (up to reordering rows and columns)

\[
\begin{pmatrix}
\varepsilon_{z,az}(1) & n-1 & 0 & 0 & \ldots \\
1 & \varepsilon_{z,az}(2) & n-2 & 0 & \ldots \\
0 & 2 & \varepsilon_{z,az}(3) & n-3 & \ldots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & n-2 & \varepsilon_{a,az}(n-1) & 1 \\
0 & \ldots & 0 & n-1 & \varepsilon_{z,az}(n)
\end{pmatrix}
\]

According to Lemma A.3 the determinant of the last matrix is

\[
\prod_{1 \leq i < j \leq n \atop i+j=n+1} \left[ \varepsilon_{z,az}(i) \cdot \varepsilon_{z,az}(j) - (i-j)^2 \right]
\]

\[
= \prod_{1 \leq i < j \leq n \atop i+j=n+1} \left[ (z \cdot i + a \cdot z)(z \cdot j + a \cdot z) - (i-j)^2 \right]
\]

\[
= \prod_{1 \leq i < j \leq n \atop i+j=n+1} \left[ z^2(a^2 + (i+j)a + ij) - (i-j)^2 \right]
\]

\[
= \prod_{1 \leq i \leq n/2 \atop 1 \leq i \leq n/2} \left[ z^2(a+i)(a+n-i+1) - (n+1-2i)^2 \right]
\]

\[
= \prod_{1 \leq i \leq n/2} \left[ z^2(a+n/2 - i + 1)(a+n/2 + i) - (2i-1)^2 \right]
\]

Replacing back \( a + \frac{n}{2} \) by \( a \), this gives the assertion. □

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