An odd \([1, b]\)-factor in regular graphs from eigenvalues

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Abstract

An odd \([1, b]\)-factor of a graph \(G\) is a spanning subgraph \(H\) such that for each vertex \(v \in V(G)\), \(d_H(v)\) is odd and \(1 \leq d_H(v) \leq b\). Let \(\lambda_3(G)\) be the third largest eigenvalue of the adjacency matrix of \(G\). For positive integers \(r \geq 3\) and even \(n\), Lu, Wu, and Yang [10] proved a lower bound for \(\lambda_3(G)\) in an \(n\)-vertex \(r\)-regular graph \(G\) to guarantee the existence of an odd \([1, b]\)-factor in \(G\). In this paper, we improve the bound; it is sharp for every \(r\).

Keywords: Odd \([1, b]\)-factor, eigenvalues

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1 Introduction

In this paper we deal only with finite and undirected graphs without loops or multiple edges. The adjacency matrix \(A(G)\) of \(G\) is the \(n\)-by-\(n\) matrix in which entry \(a_{i,j}\) is 1 or 0 according to whether \(v_i\) and \(v_j\) are adjacent or not, where \(V(G) = \{v_1, \ldots, v_n\}\). The eigenvalues of \(G\) are the eigenvalues of its adjacency matrix \(A(G)\). Let \(\lambda_1(G), \ldots, \lambda_n(G)\) be its eigenvalues in nonincreasing order. Note that the spectral radius of \(G\), written \(\rho(G)\) equals \(\lambda_1(G)\).

The degree of a vertex \(v\) in \(V(G)\), written \(d_G(v)\), is the number of vertices adjacent to \(v\). An odd (or even) \([a, b]\)-factor of a graph \(G\) is a spanning subgraph \(H\) of \(G\) such that for each vertex \(v \in V(G)\), \(d_H(v)\) is odd (or even) and \(a \leq d_H(v) \leq b\); an \([a, a]\)-factor is called the \(a\)-factor. For a positive integer \(r\), a graph is \(r\)-regular if every vertex has the same degree \(r\). Note that \(\lambda_1(G) = r\) if \(G\) is \(r\)-regular. Many researchers proved the conditions for a graph

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to have an $a$-factor, or (even or odd) $[a, b]$-factor. (See [2, 9, 11, 12]) Brouwer and Haemers started to investiage the relations between eigenvalues and the existence of 1-factor.

In fact, they [5] proved that if $G$ is an $r$-regular graph without an 1-factor, then

$$\lambda_3(G) > \begin{cases} 
  r - 1 + \frac{3}{r+1} & \text{if } r \text{ is even}, \\
  r - 1 + \frac{3}{r+2} & \text{if } r \text{ is odd}
\end{cases}$$

by using Tuttes 1-Factor Theorem [13], which is a special case of Berge-Tutte Formula [3]. Cioabă, Gregory, and Haemers [6] improved their bound and in fac t proved that if $G$ is an $r$-regular graph without an 1-factor, then

$$\lambda_3(G) \geq \begin{cases} 
  \theta = 2.85577... & \text{if } r = 3, \\
  \frac{1}{2}(r - 2 + \sqrt{r^2 + 12}) & \text{if } r \geq 4 \text{ is even}, \\
  \frac{1}{2}(r - 3 + \sqrt{(r + 1)^2 + 16}) & \text{if } r \geq 5 \text{ is odd},
\end{cases}$$

where $\theta$ is the largest root of $x^3 - x^2 - 6x + 2 = 0$. More generally, O and Cioabă [7] determined connections between the eigenvalues of a $t$-edge connected $r$-regular graph and its matching number when $1 \leq t \leq r - 2$. In 2010, Lu, Wu, and Yang [10] proved that if an $r$-regular graph $G$ with even number of vertices has no odd $[1, b]$-factor, then

$$\lambda_3(G) > \begin{cases} 
  r - \left\lfloor \frac{b}{r+1} \right\rfloor - 1 + \frac{1}{(r+1)(r+2)} & \text{if } r \text{ is even and } \left\lfloor \frac{b}{r} \right\rfloor \text{ is even}, \\
  r - \left\lfloor \frac{b}{r+1} \right\rfloor + 1 & \text{if } r \text{ is even and } \left\lfloor \frac{b}{r} \right\rfloor \text{ is odd}, \\
  r - \left\lfloor \frac{b}{r+1} \right\rfloor - 1 & \text{if } r \text{ is odd and } \left\lfloor \frac{b}{r} \right\rfloor \text{ is even}, \\
  r - \left\lfloor \frac{b}{r+1} \right\rfloor - 1 & \text{if } r \text{ is odd and } \left\lfloor \frac{b}{r} \right\rfloor \text{ is odd}.
\end{cases}$$

To prove the above bounds in the paper [10], they used Amahashi’s result.

**Theorem 1.1.** [1] Let $G$ be a graph and let $b$ be a positive odd integer. Then $G$ contains an odd $[1, b]$-factor if and only if for every subset $S \subseteq V(G)$, $o(G - S) \leq b|S|$, where $o(H)$ is the number of odd components in a graph $H$.

Thoerem 1.1 guarantees that if there is no odd $[1, b]$-factor in an $r$-regular graph, then there exists a subset $S \subseteq V(G)$ such that $o(G - S) > b|S|$. By counting the number of edges between $S$ and $G - S$, we can show that $G - S$ has at least three odd components $Q_1, Q_2, Q_3$ such that $|[V(Q_i), S]| \leq r - 1$ (see the proof of Theorem [10] or Theorem 3.2). Then they found lower bounds for the largest eigenvalue in a graph in the family $F_{r,b}$, where $F_{r,b}$ is a family of such a possible component depending on $r$ and $b$, and those bounds are appeared above.

In this paper, we improve their bound and in fact prove that if $G$ is an $n$-vertex $r$-regular graph without an odd $[1, b]$-factor, then

$$\lambda_3(G) \geq \rho(r, b),$$
where
\[
\rho(r, b) = \begin{cases} 
- \frac{r-2+\sqrt{(r+2)^2-4\left(\left\lceil \frac{r}{b} \right\rceil -2\right)}}{2} & \text{if both } r \text{ and } \left\lceil \frac{r}{b} \right\rceil \text{ are even,} \\
- \frac{r-2+\sqrt{(r+2)^2-4\left(\left\lceil \frac{r}{b} \right\rceil -1\right)}}{2} & \text{if } r \text{ is even and } \left\lceil \frac{r}{b} \right\rceil \text{ is odd,} \\
- \frac{r-3+\sqrt{(r+3)^2-4\left(\left\lceil \frac{r}{b} \right\rceil -2\right)}}{2} & \text{if both } r \text{ and } \left\lceil \frac{r}{b} \right\rceil \text{ are odd,} \\
- \frac{r-3+\sqrt{(r+3)^2-4\left(\left\lceil \frac{r}{b} \right\rceil -1\right)}}{2} & \text{if } r \text{ is odd and } \left\lceil \frac{r}{b} \right\rceil \text{ is even.}
\end{cases}
\]

The bounds that we found are sharp in a sense that there exists a graph \( H \) in \( F_{r,b} \) such that \( \lambda_1(H) = \rho(r, b) \).

For undefined terms, see West [14] or Godsil and Royle [8].

2 Construction

Suppose that \( \varepsilon = \begin{cases} 
2 & \text{if } r \text{ and } \left\lceil \frac{r}{b} \right\rceil \text{ has same parity} \\
1 & \text{otherwise}
\end{cases} \) and \( \eta = \left\lceil \frac{r}{b} \right\rceil - \varepsilon \). In this section, we provide graphs \( H_{r,\eta} \) such that \( \lambda_1(H_{r,\eta}) = \rho(r, b) \). These graphs show that the bounds in Theorem 3.2 are sharp.

Now, we define the graph \( H_{r,\eta} \) as follows:
\[
H_{r,\eta} = \begin{cases} 
K_{r+1-\eta} \lor \frac{2K_r}{2} & \text{if } r \text{ is even,} \\
C_{\eta} \lor \frac{r+2-\eta}{2}K_2 & \text{if } r \text{ is odd.}
\end{cases}
\]

To compute the spectral radius of \( H_{r,\eta} \), the notion of equitable partition of a vertex set in a graph is used. Consider a partition \( V(G) = V_1 \cup \cdots \cup V_s \) of the vertex set of a graph \( G \) into \( s \) non-empty subsets. For \( 1 \leq i, j \leq s \), let \( q_{i,j} \) denote the average number of neighbours in \( V_j \) of the vertices in \( V_i \). The quotient matrix of this partition is the \( s \times s \) matrix whose \((i,j)\)-th entry equals \( q_{i,j} \). The eigenvalues of the quotient matrix interlace the eigenvalues of \( G \). This partition is equitable if for each \( 1 \leq i, j \leq s \), any vertex \( v \in V_i \) has exactly \( q_{i,j} \) neighbours in \( V_j \). In this case, the eigenvalues of the quotient matrix are eigenvalues of \( G \) and the spectral radius of the quotient matrix equals the spectral radius of \( G \) (see [4], [8] for more details).

**Theorem 2.1.** For \( r \geq 3 \) and \( b \geq 1 \), we have \( \lambda_1(H_{r,\eta}) = \rho(r, b) \).

**Proof.** We prove this theorem only in the case when \( r \) is odd because the proof of the other case is similar.

Consider the vertex partition \( \{V(C_\eta), V(\frac{r+2-\eta}{2}K_2)\} \) of \( H_{r,\eta} \). The quotient matrix of the vertex partitions equals
\[
Q = \begin{pmatrix} 
\eta - 3 & r + 2 - \eta \\
\eta & r - \eta
\end{pmatrix}
\]

The characteristic polynomial of \( Q \) is
\[
p(x) = (x - \eta + 3)(x - r + \eta) - (r + 2 - \eta)\eta.
\]
Since the vertex partition is equitable, the largest root of the graph $H_{r, \eta}$ equals the largest root of the polynomial, which is $\lambda_1(Q) = \frac{r-3+\sqrt{(r+3)^2-4\eta}}{2}$. \hfill \square

3 Main results

In this section, we prove an upper bound for $\lambda_3(G)$ in an $r$-regular graph $G$ with even number of vertices to guarantee the existence of an odd $[1, b]$-factor by using Theorem 1.1 and Theorem 3.1.

**Theorem 3.1.** [4, 8] If $H$ is an induced subgraph of a graph $G$, then $\lambda_i(H) \leq \lambda_i(G)$ for all $i \in \{1, \ldots, |V(H)|\}$.

**Theorem 3.2.** Let $r \geq 3$, and $b$ be a positive odd integer less than $r$. If $\lambda_3(G)$ of an $r$-regular graph $G$ with even number of vertices is smaller than $\rho(r, b)$, then $G$ has an odd $[1, b]$-factor.

**Proof.** We prove the contrapositive. Assume that an $r$-regular graph $G$ with even number of vertices has no odd $[1, b]$-factor. By Theorem 1.1, there exists a vertex subset $S \subseteq V(G)$ such that $o(G - S) > b|S|$. Note that since $|V(G)|$ is even, $b$ is odd, and $o(G - S) \equiv |S| \pmod{2}$, we have $o(G - S) \geq b|S| + 2$. Let $G_1, \ldots, G_q$ be the odd components of $G - S$, where $q = o(G - S)$.

**Claim 1.** There are at least three odd components, say $G_1, G_2, G_3$, such that $|[V(G_i), S]| < \left\lceil \frac{r}{b} \right\rceil$ for all $i \in \{1, 2, 3\}$.

Assume to the contrary that there are at most two such odd components in $G - S$. Since $G$ is $r$-regular, we have

$$r|S| \geq \sum_{i=1}^{q} |[V(G_i), S]| \geq \left\lceil \frac{r}{b} \right\rceil (q-2) + 2 \geq \left\lceil \frac{r}{b} \right\rceil b|S| + 2 \geq r|S| + 2,$$

which is a contradiction.

By Theorem 3.1, we have

$$\lambda_3(G) \geq \lambda_3(G_1 \cup G_2 \cup G_3) \geq \min_{i \in \{1, 2, 3\}} \lambda_1(G_i). \quad (1)$$

Now, we prove that if $H$ is an odd component of $G - S$ such that $|[V(H), S]| < \left\lceil \frac{r}{b} \right\rceil$, then $\lambda_1(H) \geq \rho(r, b)$.

**Claim 2.** If $H$ is an odd components of $G - S$ such that $|[V(H), S]| < \left\lceil \frac{r}{b} \right\rceil$ and if $\lambda_1(H) \leq \lambda_1(H')$ for all odd components $H'$ in $G - S$ such that $|[V(H'), S]| < \left\lceil \frac{r}{b} \right\rceil$, then we have

$$|V(H)| = \begin{cases} r + 2 & \text{if } r \text{ is odd}, \\ r + 1 & \text{if } r \text{ is even} \end{cases}, \quad \text{and} \quad 2|E(H)| = \begin{cases} r(r+2) - \eta & \text{if } r \text{ is odd}, \\ r(r+1) - \eta & \text{if } r \text{ is even}. \end{cases}$$
Let \( x = \begin{cases} 1 \text{ if } r \text{ is odd}, \\ 0 \text{ if } r \text{ is even}. \end{cases} \) Since \(|V(H), S| < \lceil \frac{x}{2} \rceil < r\) and \(G\) is \(r\)-regular, we have \(|V(H)| \geq r + 1 + x\) since \(H\) has an odd number of vertices. If \(|V(H)| > r + 1 + x\), then we have \(|V(H)| \geq r + 3 + x\) since \(H\) has an odd number of vertices. Thus it suffices to show \(\rho(r, b) < \lambda_1(H)\) if \(|V(H)| \geq r + 3 + x\). By using the fact that \(\lambda_1(G) \geq \frac{2|E(G)|}{|V(G)|}\) for any graph \(G\), we have

\[
\lambda_1(H) > \frac{r|V(H)| - \eta}{|V(H)|} \geq \frac{r(r + 3 + x) - \eta}{r + 3 + x} > \frac{r - 2 - x + \sqrt{(r + 2 + x)^2 - 4\eta}}{2}.
\]

Now, we prove this theorem by considering two cases depending on the parity of \(r\).

**Case 1.** \(r\) is even. By Claim 2, assume that \(H\) is an odd component of \(G - S\) such that \(|V(H), S| < \lceil \frac{x}{2} \rceil\), \(|V(H)| = r + 1\), and \(2|E(H)| = r(r + 1) - \eta\). Then there are at least \(r + 1 - \eta\) vertices of degree \(r\). Let \(V_1\) be a set of vertices with degree \(r\) such that \(|V_1| = r + 1 - \eta\), and let \(V_2\) be the remaining vertices in \(V(H)\). Then the quotient matrix of the vertex partition \(\{V_1, V_2\}\) of \(H\) equals

\[
\begin{pmatrix}
   r - \eta & \eta \\
   r + 1 - \eta & \eta - 2
\end{pmatrix}
\]

whose characteristic polynomial is \(p(x) = (x - r + \eta)(x - \eta + 2) - \eta(r + 1 - \eta)\). Since the largest root of \(p(x)\) equals \(\rho(r, b)\), we have \(\lambda_1(H) \geq \rho(r, b)\).

**Case 2.** \(r\) is odd. By Claim 2, assume that \(H\) is an odd component of \(G - S\) such that \(|V(H), S| < \lceil \frac{x}{2} \rceil\), \(|V(H)| = r + 2\), and \(2|E(H)| = r(r + 2) - \eta\). Then there are at least \(r + 2 - \eta\) vertices of degree \(r\). Let \(V_1\) be a set of vertices with degree \(r\) such that \(|V_1| = r + 2 - \eta\), and let \(V_2\) be the remaining vertices in \(V(H)\). Suppose that there are \(m_{12}\) edges between \(V_1\) and \(V_2\). Note that \((r + 2 - \eta)(\eta - 1) \leq m_{12} \leq (r + 2 - \eta)\eta\). Then the quotient matrix of the vertex partition \(\{V_1, V_2\}\) of \(H\) equals

\[
\begin{pmatrix}
   r - \frac{m_{12}}{r + 2 - \eta} & \frac{m_{12}}{r + 2 - \eta} \\
   \frac{m_{12}}{\eta} & r - 1 - \frac{m_{12}}{r + 2 - \eta}
\end{pmatrix}
\]

whose characteristic polynomial is \(q(x) = (x - r + \frac{m_{12}}{r + 2 - \eta})(x - r + 1 + \frac{m_{12}}{\eta}) - \frac{m_{12}^2}{(r + 2 - \eta)\eta}\).

Note that since \((r + 2 - \eta)(\eta - 1) \leq m_{12} \leq (r + 2 - \eta)\eta\), \(m_{12}\) can be expressed \(m_{12} = (r + 2 - \eta)\eta - t\), where \(0 \leq t \leq r + 2 - \eta\). Thus we have

\[
q(x) = x^2 - (r - 3 + \frac{t(r + 2)}{(r + 2 - \eta)\eta})x - 3r + \eta - \frac{t}{r + 2 - \eta} + \frac{tr(r + 2)}{(r + 2 - \eta)\eta}.
\]

\[
= x^2 - (r - 3)x - 3r + \eta - \frac{t(r + 2)}{(r + 2 - \eta)\eta}x - \frac{t}{r + 2 - \eta} + \frac{tr(r + 2)}{(r + 2 - \eta)\eta}.
\]
Note that $q(\rho(r, b)) = -\frac{t(r+2)}{(r+2-\eta)\eta} (\rho(r, b) + \frac{\eta}{r+2} - r) \leq 0$, since $\eta \geq 1$ and $0 \leq t \leq r + 2 - \eta$.

\[ \square \]

References

[1] A. Amahashi, On factors with all degrees odd, *Graphs Combin.*, 1 (1985), 111–114.

[2] K. Ando, A. Kaneko, T. Nishimura, A degree condition for the existence of 1-factors in graphs or their complements. *Discrete Math.* 203 (1999), no. 1–3, 1–8.

[3] C. Berge, Sur le couplage maximum dun graphe, *C. R. Acad. Sci. Paris*, 247 (1958), pp. 258–259

[4] A.E. Brouwer and W.H. Haemers, *Spectra of Graphs*, Springer, New York, (2011).

[5] A.E. Brouwer and W.H. Haemers, Eigenvalues and perfect matchings. *Linear Algebra Appl.*, 395 (2005), 155–162.

[6] S.M. Cioabă, D.A. Gregory, W.H. Haemers, Matchings in Regular Graphs from Eigenvalue. *J. Combin. Theory Ser. B* 99 (2009), 287–297.

[7] S.M. Cioabă, S. O, Edge-connectivity, Matchings, and Eigenvalues in Regular Graphs. *SIAM J. Discrete Math.* 22 (2010), 1470–1481.

[8] C. Godsil and G. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics, 207. Springer-Verlag, New York, 2001.

[9] M. Kouider, Sufficient condition for the existence of an even $[a, b]$-factor in graph. *Graphs Combin.* 29 (2013), no. 4, 1051–1057.

[10] H. Lu, Z. Wu, and X. Yang, Eigenvalues and $[1, n]$-odd factors. *Linear Algebra Appl.*, 433 (2010), 750–757.

[11] H. Matsuda, Ore-type conditions for the existence of even $[2, b]$-factors in graphs. *Discrete Math.* 304 (2005), no. 1–3, 51–61.

[12] L. Nebesky, Some sufficient conditions for the existence of a 1-factor. *J. Graph Theory* 2 (1978), no. 3, 251–255.

[13] W.T. Tutte, The factorization of linear graphs, *J. Lond. Math. Soc.* 22 (1947) 107–111.

[14] D.B. West, *Introduction to Graph Theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 2001.