On the generating function of weight multiplicities for the representations of the Lie algebra $C_2$

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Abstract
We use the generating function of the characters of $C_2$ to obtain a generating function for the multiplicities of the weights entering in the irreducible representations of that simple Lie algebra. From this generating function we derive some recurrence relations among the multiplicities and a simple graphical recipe to compute them.

PACS: 02.20.Qs, 02.30.Ik, 03.65.Fd.

Key words: Lie algebras, representation theory, weight-multiplicities
1 Introduction

Each irreducible representation of a simple Lie algebra is defined by a set of weights which, for rank two algebras, can be conveniently arranged in a two-dimensional weight diagram. These weights result from successive applications of the lowering operators $E_{-\alpha}$ corresponding to the positive roots of the algebra to the highest weight of the representation. As there are, in general, several ways by which a particular weight can be obtained in this form, the weights forming the representation enter in it with some multiplicity. The computation and understanding of weight multiplicities has been a subject of much research along the years [1]–[5] and, as it is a rule when dealing with Lie algebra representations, one of the most efficient tools available to address the question is the theory of characters. In a recent paper [6], we have presented a general method for computing the generating function of the characters of simple Lie algebras which is based on the theory of the quantum trigonometric Calogero-Sutherland system [7]–[10] (see also [11, 12] for other approaches to that problem). In particular, we have applied the method to the cases of the Lie algebras $A_2$ and $C_2$. The aim of this note is to supplement the results of [6] by showing how they can be used to obtain some useful generating functions for weight multiplicities. In doing so, we will specialize to the case of the algebra $C_2$, given that the case of the generating function for multiplicies of $A_2$ has been soundly treated in reference [13].

Let us recall, to begin with, the way in which characters and weight multiplicities are related. Let $A$ be a simple Lie algebra of rank $r$ with fundamental weights $\lambda_1, \lambda_2, \ldots, \lambda_r$ and let us denote $R_\lambda$ the irreducible representation of $A$ with highest weight $\lambda = p_1 \lambda_1 + p_2 \lambda_2 + \cdots + p_r \lambda_r$. The character of this representation is defined as

$$\chi_{p_1, p_2, \ldots, p_r} = \sum_w \mu_w e(w)$$

where the sum extends to all weights $w$ entering in the representation, $\mu_w$ is the multiplicity of the weight $w$ and, if $w = m_1 \lambda_1 + m_2 \lambda_2 + \cdots + m_r \lambda_r$, then $e(w)$ is

$$e(w) = \exp \left( i \sum_{l=1}^{r} m_l \varphi_l \right) = x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r},$$

where $\varphi_1, \varphi_2, \ldots, \varphi_r$ are angular coordinates on the maximal torus and $x_l = e^{i \varphi_l}$. The multiplicity $\mu_{p_1, p_2, \ldots, p_r}(m_1, m_2, \ldots, m_r)$ of the weight $w \equiv (m_1, m_2, \ldots, m_r)$ in the representation $R_\lambda$, $l \equiv (p_1, p_2, \ldots, p_r)$, can be computed as

$$\mu_{p_1, p_2, \ldots, p_r}(m_1, m_2, \ldots, m_r) = \frac{1}{(2\pi)^r} \int_0^{2\pi} d\varphi_1 e^{-im_1 \varphi_1} \int_0^{2\pi} d\varphi_2 e^{-im_2 \varphi_2} \cdots \int_0^{2\pi} d\varphi_r e^{-im_r \varphi_r} \chi_{p_1, \ldots, p_r}$$

$$= \frac{1}{(2\pi)^r} \oint dx_1 \oint dx_2 \cdots \oint dx_r \frac{\chi_{p_1, \ldots, p_r}}{x_1^{1+m_1} x_2^{1+m_2} \cdots x_r^{1+m_r}}; \quad (1)$$

where the integrals in the second line are along the unit circles on the $r$ complex planes parametrized by the complex coordinates $x_1, x_2, \ldots, x_r$. 

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In view of (1), the generating function for the multiplicities of the weight \( w \) in all the representations of \( \mathfrak{A} \)

\[
A_{m_1,m_2,\ldots,m_r}(t_1,t_2,\ldots,t_r) = \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_r=0}^{\infty} t_1^{p_1} t_2^{p_2} \cdots t_r^{p_r} \mu_{p_1,\ldots,p_r}(m_1,\ldots,m_r)
\]

(2)

comes from the formula

\[
A_{m_1,\ldots,m_r}(t_1,t_2,\ldots,t_r) = \frac{1}{(2\pi i)^r} \int dx_1 \int dx_2 \cdots \int dx_r \frac{G(t_1,t_2,\ldots,t_r;z_1,z_2,\ldots,z_r)}{x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}},
\]

(3)

where \( G(t_1,t_2,\ldots,t_r;z_1,z_2,\ldots,z_r) \) is the generating function of the characters

\[
G(t_1,t_2,\ldots,t_r;z_1,z_2,\ldots,z_r) = \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_r=0}^{\infty} t_1^{p_1} t_2^{p_2} \cdots t_r^{p_r} \chi_{p_1,\ldots,p_r}(z_1,z_2,\ldots,z_r)
\]

and we have chosen to express the latter by means of a set of variables \( z_1, z_2, \ldots, z_r \) which coincide with the characters of the representations corresponding to the fundamental weights. Even for low-rank algebras and small values of the indices \( m_j \) the integrand in (3) is a quite complicated rational function but, nevertheless, the integral can be evaluated by iterated application of the Cauchy’s residue theorem in each complex plane.

Let us consider, for instance, the case of the generating function of zero weight multiplicities for the Lie algebra \( \mathfrak{A}_2 \). According to [14], see also [6], the fundamental characters are

\[
z_1 = x_1 + \frac{x_2}{x_1}, \quad z_2 = x_2 + \frac{x_1}{x_2},
\]

whereas the generating function \( G \) is [6]

\[
G(t_1,t_2;z_1,z_2) = \frac{1-t_1t_2}{(1-t_1z_1+t_1^2z_2-t_1^3)(1-t_2z_2+t_2^2z_1-t_2^3)}
\]

\[
= \frac{1-t_1t_2}{(1-t_1t_2)x_1^2 x_2^2} \frac{(t_2-x_1)(t_1 x_1-1)(t_1-x_2)(t_2 x_2-1)(t_1 x_2-x_1)(t_2 x_2-1)}{(t_2-x_1)(t_1 x_1-1)(t_1-x_2)(t_2 x_2-1)(t_1 x_2-x_1)(t_2 x_2-1)}.
\]

Then we have to compute

\[
A_{0,0}(t_1,t_2) = \frac{1}{(2\pi i)^2} \int dx_1 \int dx_2 \frac{(t_2-x_1)(t_1 x_1-1)(t_1-x_2)(t_2 x_2-1)(t_1 x_2-x_1)(t_2 x_2-1)}{(1-t_1t_2)x_1 x_2}
\]

and we choose to perform the \( x_1 \) integral first. As \( |x_1| = |x_2| = 1 \) and \( t_1, t_2 < 1 \), there are poles inside the unit circle for \( x_1 = t_2 \) and \( x_1 = t_1 x_2 \). Thus, by computing the residues, we find

\[
J_1(t_1,t_2;x_2) = \frac{1}{2\pi i} \int \frac{G(t_1,t_2;z_1,z_2)}{x_1 x_2} dx_2 \frac{(t_2-x_1)(t_1 x_1-1)(t_1-x_2)(t_2 x_2-1)(t_1 x_2-x_1)(t_2 x_2-1)}{(1-t_1t_2)x_2^2} = \frac{(1+t_1t_2)x_2}{(t_1-x_2)(t_2 x_2-1)(t_2 x_2-1)}.
\]

Now, integrating \( J_1(t_1,t_2;x_2) \), which has poles inside the \( x_2 \) unit circle at \( x_2 = t_1 \) and \( x_2 = t_2 \), we finally obtain the generating function for zero weight multiplicities as

\[
A_{0,0}(t_1,t_2) = \frac{1}{2\pi i} \int dx_2 J_1(t_1,t_2,x_2) = \frac{1-t_1^2 t_2^2}{(1-t_1^3)(1-t_1 t_2)^2(1-t_2^3)}.
\]
2 The generating function $A_{m,n}(t_1, t_2)$ for $C_2$

In the case of $C_2$, the fundamental characters are [13]

$$z_1 = x_1 + \frac{1}{x_1} + \frac{x_1}{x_2} + \frac{x_2}{x_1}, \quad z_2 = 1 + x_2 + \frac{1}{x_2} + \frac{x_2}{x_1},$$

and the generating function of the characters is [6]

$$G(t_1, t_2; z_1, z_2) = \frac{1 + t_2 - z_1 t_1 t_2 + t_1^2 t_2 + t_1^2 t_2^2}{(1 - (t_1 + t_1^3) z_1 + t_1^2 (z_2 + 1) + t_1^4)((1 - (t_2 + t_2^3) (z_2 - 1) + t_2^2 (z_2^2 - 2 z_2) + t_2^4))}.$$}

Thus, for $m_1 = m_2 = 0$, the poles of integrand $G/x_1 x_2$ in [3] are easy to identify, and going through the steps seen in the previous example, we eventually find that the generating function for zero-weight multiplicities for $C_2$ is

$$A_{0,0}(t_1, t_2) = \frac{1 + t_1^2 t_2}{(1 - t_1^2)(1 - t_2)(1 - t_2^2)}.$$}

The form of $A_{0,0}(t_1, t_2)$ is simple enough to allow us to go one step further. We can expand $A_{0,0}(t_1, t_2)$ as a sum of partial fractions

$$A_{0,0}(t_1, t_2) = \frac{1}{2} \left[ \frac{1}{(1 - t_1^2)(1 - t_2)} + \frac{1 + t_1^2}{(1 - t_1^2)(1 - t_2^2)} \right]$$

whose Taylor series are quite simple. Matching coefficients yields the general formula for the multiplicities $\mu_{p,q}(0,0)$ of the zero weight as

$$\mu_{p,q}(0,0) = \frac{1}{2} \varepsilon_p \varepsilon_q + (p + 1)(q + 1)$$

where $\varepsilon_p = 1$ for $p$ even, or $\varepsilon = 0$ for $p$ odd.

The calculations needed to obtain the generating functions for the multiplicities of other low-lying weights go along the same lines and we list some results in the Appendix. However, using directly formula [3] to find the generating function of the multiplicities of a general weight $(m, n)$ seems to be quite involved. In order to make progress, it is more convenient to introduce a new generating function $H(t_1, t_2; y_1, y_2)$ defined as

$$H(t_1, t_2; y_1, y_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \mu_{p,q}(m, n) y_1^m y_2^n t_1^p t_2^q.$$
which collects the multiplicities \( \mu_{p,q}(m,n) \) of all weights \( m\lambda_1 + n\lambda_2 \) in all the representations \( R_{p\lambda_1+q\lambda_2} \) of \( C_2 \). Expressing \( \mu_{p,q}(m,n) \) as was done in ④, the sums in the indices \( m \) and \( n \) yield geometric series, leading to the formula

\[
H(t_1, t_2; y_1, y_2) = \frac{1}{(2\pi i)^2} \oint dx_1 \oint dx_2 \frac{G(t_1, t_2; z_1, z_2)}{(x_1 - y_1)(x_2 - y_2)},
\]

which, after substitution of ④ and ⑤, takes the form of a rational integral to be evaluated by means of Cauchy’s theorem as in the previous examples. The result is

\[
H(t_1, t_2; y_1, y_2) = \frac{a + b_1 y_1 + b_2 y_2 + c_{1,2} y_1 y_2 + dy_1^2 + ey_2^2}{(1 - t_1^2)(1 - t_2^2)(1 - t_2)(1 - t_1 y_1)(1 - t_2^2 y_1^2)(1 - t_2 y_2)(1 - t_2 y_2)}
\]

(6)

where

\[
\begin{align*}
a & = 1 + t_1^2 t_2, \\
b_1 & = t_1 t_2(1 - t_1^2), \\
b_2 & = -t_1 t_2(t_1^3 + t_1 t_2), \\
c_{1,2} & = t_1 t_2(t_1^2 - t_2^2), \\
d & = -t_1^2 t_2(1 + t_2), \\
e & = t_1^2 t_2(t_1^2 + t_1 t_2 + t_2^2 - 1).
\end{align*}
\]

Now, trading the factors \( (1 - t_1 y_1)(1 - t_2^2 y_1^2)(1 - t_2 y_2)(1 - t_2 y_2) \) in the denominator by geometric series in \( y_1 \) and \( y_2 \), we can rewrite \( H(t_1, t_2; y_1, y_2) \) as a series

\[
H(t_1, t_2; y_1, y_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n}(t_1, t_2) y_1^m y_2^n
\]

such that the coefficients are precisely the generating functions for weight multiplicities which we are seeking for. From ⑥, and after some tedious algebra, one can obtain the explicit form of these generating functions as

\[
A_{m,n}(t_1, t_2)
\]

\[
= \frac{t_1^{m+2n+2} - t_1^{m+2n+1}(1 - t_1^2) - t_2^{m+2n+1}(1 - t_2^2) - t_1^{m+n+1}(t_1^2 - t_2)(1 - t_1^2) - t_2^{m+n+1}(t_1^2 - t_2)(1 - t_2^2) f(t_1, t_2)}{(1 - t_1^2)(1 - t_2^2)(1 - t_2)(1 - t_1 y_1)(1 - t_2^2 y_1^2)(1 - t_2 y_2)(1 - t_2 y_2)}
\]

(7)

with

\[
f(t_1, t_2) = \begin{cases} t_1^2 + t_2, & \text{for } m \text{ even} \\ t_1(1 + t_2), & \text{for } m \text{ odd} \end{cases}
\]

thus generalizing the results for low-lying multiplicities explained before ⑧.

This expression of the generating function differs from the examples of the Appendix by the factors \((t_1^2 - t_2^2)(t_1^2 - t_2)\) in the denominator. In fact, some further simplification work shows that

\[
1 \text{After this work was completed we learned about the very interesting paper by Doković [15] in which he obtains the generating functions for weight multiplicities for the simple Lie algebras of rank 2. Unfortunately, the result quoted in that paper for } B_2 \equiv C_2 \text{ and } m \text{ even is not correct.}
\]
these factors cancel out, giving

\[
A_{m,n}(t_1, t_2) = \frac{1}{D} \left[ (1 - t_2^2) \sum_{j=0}^{n-1} t_1^{m+2n-2j} t_2^j + (1 - t_1^2 + t_2 - t_1^3 t_2^2) \sum_{j=0}^{m-1} t_1^{2j} t_2^{m+n-2j} \right. \\
- \left. (1 - t_1^2 - t_2^2) t_1^{m+n+1} + t_1^{m+2} t_2^n \right]
\]

for even \( m \) and

\[
A_{m,n}(t_1, t_2) = \frac{1}{D} \left[ (1 - t_2^2) \sum_{j=0}^{n-1} t_1^{m+2n-2j} t_2^j + (1 + t_2 - t_2^2 - t_1^2 t_2) \sum_{j=0}^{m-1} t_1^{2j+1} t_2^{m+n-2j-1} \\
+ (1 + t_2 - t_2^2) t_1^{m+n} + t_1 t_2^{m+n+1} \right]
\]

for odd \( m \), with \( D = (1 - t_1^2)^2(1 - t_2^2)(1 - t_2) \).

After the generating functions are known some other interesting results come from them. In particular, looking at their form for low \( m \) and \( n \), one can identify two different recurrence relations among the multiplicities \( \mu_{p,q}(m,n) \), and with some additional labour, it is possible to show that these recurrence relations are valid in general. This is described in the next two sections.

### 3 The first recurrence relation

The first recurrence relation is among the multiplicities of a fixed weight \( ml_1 + nl_2 \) in different representations. Let us call

\[
X_{m,n}(t_1, t_2) = (1 - t_1^2)(1 - t_2)A_{m,n}(t_1, t_2).
\]  

(8)

From the definition of \( A_{m,n}(t_1, t_2) \), one has

\[
X_{m,n}(t_1, t_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} [\mu_{p,q}(m,n) - \mu_{p-2,q}(m,n) - \mu_{p,q-1}(m,n) + \mu_{p-2,q-1}(m,n)] t_1^p t_2^q
\]

while the explicit expressions given above yield

\[
X_{m,n}(t_1, t_2) = \frac{(1 - t_2^2) \sum_{j=0}^{n-1} t_1^{m+2n-2j} t_2^j + (1 - t_2^2)(1 + t_2) \sum_{j=0}^{m-1} t_1^{2j} t_2^{m+n-2j} - t_2^{m+n+1}}{(1 - t_1^2)(1 - t_2^2)}
\]

\[
+ \frac{t_1^{m+n+1} + t_1^{m+n+2}}{(1 - t_1^2)(1 - t_2^2)}
\]
for $m$ even and

$$X_{m,n}(t_1, t_2) = \frac{(1 - t_2^2)^{n-1} \sum_{j=0}^{n-1} t_1^{m+2n-2j} t_2^j + (1 - t_2^2)(1 + t_2)^2 \sum_{j=0}^{m-1} t_1^{2j+1} t_2^{m+n-1-2j} + t_1^m t_2^n}{(1 - t_1^2)(1 - t_2^2)}$$

for $m$ odd.

Let us consider the formula for $m$ even and compare it with the diagram of Figure 1, which shows all representations $R_{pl_1+ql_2}$ with nonzero multiplicity for a weight $ml_1 + nl_2$. The labels $(m, n), (p, q)$ etc., represent coordinates in the non-Euclidean $(l_1, l_2)$-plane. The first term in $X_{m,n}(t_1, t_2)$ can be expanded as a geometric series which contains, always with coefficient equal to one, all the products $t_1^p t_2^q$ for $p$ and $q$ corresponding to points in the diagonals beginning in the segment $AB$. In a similar way, the second term in $X_{m,n}(t_1, t_2)$ gives all such products for the points in the diagonals normal to the line from $A$ to $C$, the third corresponds to the diagonals beginning in $D', E', F', \ldots$, etc, and the fourth to the diagonals from $D, E, F, \ldots$ etc. From this and an analogous analysis for $m$ odd, we can finally conclude that

$$\mu_{p,q}(m, n) - \mu_{p-2,q}(m, n) - \mu_{p,q-1}(m, n) + \mu_{p-2,q-1}(m, n) = y_{p,q}(m, n)$$

(9)

where $y_{p,q}(m, n) = 1$ if $(p, q)$ labels a irreducible representation of $C_2$ containing the weight $ml_1 + nl_2$ (except for $m$ even, $p = 0$ and $q$ of opposite parity to $n$, which gives $y_{p,q}(m, n) = 0$) and $y_{p,q}(m, n) = 0$ if the weight $ml_1 + nl_2$ is not in $R_{pl_1+ql_2}$.

**Remark.** A geometric interpretation can be given to the function $X_{m,n}(t_1, t_2)$ taking into account that, after direct substitution of the expression (2) of the generating function $A_{m,n}$, we can write it as a combination of sums of infinite geometric series, namely

$$X_{m,n}(t_1, t_2) = \sum_{j,k=0}^{\infty} t_1^{(m+2n)+2j-2k} t_2^{j+k} - \sum_{j,k=0}^{\infty} t_1^{(m-2)-2j-2k} t_2^{(n+1)+2j+k}$$

$$- \sum_{j,k=0}^{\infty} t_1^{-2j} t_2^{(m+n+1)+2j+2k} - \sum_{j,k=0}^{\infty} t_1^{-2-2j} t_2^{(m+n+2)+2j+2k}.$$

Each term represents the contribution of the points, with coefficient +1 or −1, in a bidimensional lattice obtained after translating a point $(p_0, q_0)$ along some independent directions in the $(l_1, l_2)$-plane. For instance, the first sum represents the total contribution to $X_{m,n}$ of the lattice generated from the basis $\{2l_1, -2l_1 + l_2\}$, with origin the point $(m+2n, 0)$ and nonnegative integers $j, k$. All the points thus generated have coefficient +1. With reference to the example in Figure 1, this is the lattice obtained by translating first the point $B(m+2n, 0)$ along the $l_1$-axis with step $2l_1$; the linear lattice is then translated along $BA$ with step $-2l_1 + l_2$. 

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Figure 1. The highest weights (modulo Weyl reflections) corresponding to representations of $C_2$ containing the weight $ml_1 + nl_2$ ($m = 6, n = 4$ in the example). Here, $\alpha_1, \ldots, \alpha_4$ are the positive roots and $l_1, l_2$ are the fundamental weights, of magnitude $|l_1| = 1$ and $|l_2| = \sqrt{2}$).

The remaining contributions have the same interpretation, differing only in the fact that the coefficient is now $-1$; this means that points obtained from two contributions of different sign are superposed to give a null contribution. This way of counting gives the same lattice (Figure 1) as before, for both $m$ even or odd.

4 The second recurrence relation

There is a second recurrence relation, this time among the multiplicities of weights of the same representation. Let $P_{m,n}(t_1, t_2)$ be the function defined as

$$P_{m,n}(t_1, t_2) = A_{m,n}(t_1, t_2) - A_{m+2,n}(t_1, t_2) - A_{m,n+1}(t_1, t_2) + A_{m+2,n+1}(t_1, t_2).$$ (10)
Then, from the previous expressions of $A_{m,n}(t_1, t_2)$, one can obtain
\[
P_{m,n}(t_1, t_2) = \frac{1}{1 - t_2} \sum_{j=0}^{n} t_1^{m+2n-2j} t_2^j + \frac{1}{1 - t_2} \sum_{j=0}^{m-1} t_1^{2j} t_2^{m+n-2j} - \frac{t_1^{m+n+1}}{(1 - t_1^2)(1 - t_2)}
\]
for $m$ even and
\[
P_{m,n}(t_1, t_2) = \frac{1}{1 - t_2} \sum_{j=0}^{n} t_1^{m+2n-2j} t_2^j + \frac{1}{1 - t_2} \sum_{j=0}^{m-1} t_1^{2j+1} t_2^{m+n-1-2j} - \frac{t_1^{m+n+1}}{(1 - t_1^2)(1 - t_2)}
\]
for $m$ odd. Expanding the denominators as geometric series and using the definition of $A_{m,n}(t_1, t_2)$, one finds that for both parities
\[
\mu_{p,q}(m, n) - \mu_{p,q}(m+2, n) - \mu_{p,q}(m, n+1) + \mu_{p,q}(m+2, n+1) = \varepsilon_{p,q}(m, n)
\]
where the right-hand member is a sum of three terms, $\varepsilon_{p,q}(m, n) = X + Y - Z$, which are zero except for the cases
- $X = 1$ when $m \leq p$ and $p \leq m + 2n \leq p + 2q$
- $Y = 1$ when $m \geq p + 2$ and $m + n \leq p + q$
- $Z = 1$ when $m + n \leq q - 1$.

As $X = 1$ and $Y = 1$ do not occur at the same time, $\varepsilon_{p,q}(m, n)$ is always 1, 0 or $-1$. To describe the domains $D_1$, $D_0$ and $D(-1)$ in which $\varepsilon_{p,q}(m, n)$ takes these values, let us consider the weight diagram of the representation $\rho l_1 + ql_2$ using now Cartesian coordinates, denoted as $[x, y]$ with $x = m$, $y = m + 2n$, instead of the $m$ and $n$ labels of the weights. The weights entering in the diagram form a square lattice with a spacing equal to $|l_2|$ which includes the highest weight of the representation and is contained in the polygon of vertices $[0, p+2q]$, $[p, p+2q]$, $[p+q, p+q]$ and $[0, 0]$. If we call $P_X$, $P_Y$ and $P_Z$ the regions of the diagram in which, respectively, $X = 1$, $Y = 1$ and $Z = 1$, it follows that:

- If $2q - 2 < p$, $P_X \cup P_Y$ does not intersect $P_Z$. Thus, $D_1 = P_X \cup P_Y$, $D(-1) = P_Z$, and $D0$ the remaining weights. So, in this case $D1$ is the upper region of the diagram, starting from the horizontal line $y = p$, $D0$ is the area below that line and above the diagonal $y = 2q - 2 - x$, and $D(-1)$ includes that diagonal and the weights below it, see Figure 2.

- If $p \leq 2q - 2 \leq 2p$ the intersection $P_X \cap P_Z$ is the (possibly degenerate) triangle $T$ of vertices $[0, 2q - 2]$, $[0, p]$ and $[2q - 2 - p, p]$. In this case $D1 = P_X \cup P_Y - T$, $D(-1) = P_Z - T$, and $D0$ the remaining weights. Now, as one can see in Figure 3, some weights above $y = p$ located near the $y$ axis are in $D0$ instead of in $D1$.

- If $2q - 2 > 2p$ the intersection $(P_X \cap P_Y) \cap P_Z$ is the quadrilateral $K$ of vertices $[0, p]$, $[0, 2q - 2]$, $[q - 1, q - 1]$ and $[p, p]$. Therefore $D1 = P_X \cup P_Y - C$, $D(-1) = P_Z - C$, and $D0$ the remaining weights, see Figure 4.
Figure 2. The weights of the representation $R_{10\lambda_1+5\lambda_2}$ in the domains $D1$, $D0$ and $D(-1)$ are marked, respectively, with black dots, circles and encircled black dots. The region $P_X \cup P_Y$ has perimeter $ABCDEA$, while $P_Z$ is contained in $FGOF$. The number over a weight means its multiplicity.

Figure 3. Weights of the representation $R_{10\lambda_1+9\lambda_2}$. The region $P_X \cup P_Y$ is bounded by $AFBCDEGA$, $P_Z$ is into $FGHOAF$ and the triangle $T$ is $FGAF$. 
Figure 4. Weights of the representation $P_{5l_1 + 10l_2}$. The region $P_X \cup P_Y$ is bounded by $AFBCDGEAF$, $P_Z$ is into $FGEOAF$ and the quadrilateral $K$ is $FGEAF$.

Figure 5. The multiplicity of $ml_1 + nl_2$ in the representation $R_{pl_1 + ql_2}$ is given by the number of weights marked with a square. By parity, the weight signaled with $X$ is excluded.
5 An application

The recurrence relations of the previous sections provide useful information on the multiplicities and, in fact, can be used to devise some simple rules to compute the multiplicity of any desired weight on a given representation. As an example, let us take the case of \( m \) even and consider a situation with \((m, n)\) and \((p, q)\) as given in the figure 5. We can write the first recurrence relation \((9)\) as

\[
\mu_{p, q}(m, n) - \mu_{p, q-1}(m, n) = y_{p, q}(m, n) + \mu_{p-2, q}(m, n) - \mu_{p-2, q-1}(m, n)
\]

and iterating we find

\[
\mu_{p, q}(m, n) = y_{p, q}(m, n) + y_{p-2, q}(m, n) + \cdots + y_{2, q}(m, n) + y_{0, q}(m, n)
\]

so that finally

\[
\mu_{p, q}(m, n) = \sum_{\beta \in R} y_{\beta}(m, n)
\]

where the sum is over to the set \( R \) of weights marked in the figure. Thus, \( \mu_{p, q}(m, n) \) can be obtained by simply counting the number of points in \( R \) except those in the vertical axis with opposite parity of \( q \) and \( n \), which have \( y_{\beta}(m, n) = 0 \). This rule can be used to obtain some explicit formulae. For instance, for the case \( p \) and \( q \) even and \( q \leq p/2 \), one finds that the multiplicities of the weights on borders of the diagram are given by

\[
\mu_{p, q}(p + q - 2s, 0) = (s + 1)^2 \quad \text{for} \quad 0 \leq s \leq \frac{q}{2}
\]

\[
\mu_{p, q}(p - 2s, 0) = \left(\frac{q}{2} + 1\right)^2 + s(q + 1) \quad \text{for} \quad 0 \leq s \leq \frac{p - q}{2}
\]

\[
\mu_{p, q}(2s, 0) = \mu_{p, q}(0, 0) - s^2 \quad \text{for} \quad 0 \leq s \leq \frac{q}{2}
\]

and

\[
\mu_{p, q}\left(\frac{p}{2} + q - s\right) = \frac{(s + 1)(s + 2)}{2} \quad \text{for} \quad 0 \leq s \leq q
\]

\[
\mu_{p, q}\left(\frac{p}{2} - s\right) = \frac{(s + 1)(s + 2)}{2} + s(q + 1) \quad \text{for} \quad 0 \leq s \leq \frac{p}{2} - q
\]

\[
\mu_{p, q}(0, s) - \mu_{p, q}(0, s + 1) = s + 1 - \theta(s + 1) \quad \text{for} \quad 0 \leq s \leq q - 1
\]

where \( \theta(r) \) is one (zero) for \( r \) even (odd). The combination of these formulas with the recurrence relation \((11)\) can be used as another alternative to compute the multiplicities of the inner weights.

Acknowledgement

J.F.N. acknowledges financial support from MTM2012-33575 project, SGPI-DGICT(MEC), Spain.
Appendix

We list here some generating functions of multiplicities of low-lying weights of $C_2$ up to $m+n = 4$ as obtained by computing the corresponding integrals in formula (3).

\[
A_{1,0}(t_1, t_2) = \frac{t_1}{(1 - t_1^2)^2(1 - t_2)^2}
\]
\[
A_{0,1}(t_1, t_2) = \frac{t_1^2 + t_2}{(1 - t_1^2)^2(1 - t_2)(1 - t_2^2)}
\]
\[
A_{2,0}(t_1, t_2) = \frac{t_2^2 + t_1^2(1 + t_2 - t_2^2)}{(1 - t_1^2)^2(1 - t_2)^2(1 + t_2)}
\]
\[
A_{1,1}(t_1, t_2) = \frac{t_1^3(1 - t_2) + t_1 t_2}{(1 - t_1^2)^2(1 - t_2)^2}
\]
\[
A_{0,2}(t_1, t_2) = \frac{t_1^2 t_2 + t_2^3 + t_1^4(1 - t_2^2)}{(1 - t_1^2)^2(1 - t_2)^2(1 + t_2)}
\]
\[
A_{3,0}(t_1, t_2) = \frac{t_1 t_2^2 + t_1^3(1 - t_2^2)}{(1 - t_1^2)^2(1 - t_2)^2}
\]
\[
A_{2,1}(t_1, t_2) = \frac{t_2^3 + t_1^4(1 - t_2^2) + t_1^3 t_2(1 + t_2 - t_2^2)}{(1 - t_1^2)^2(1 - t_2)^2(1 + t_2)}
\]
\[
A_{1,2}(t_1, t_2) = \frac{t_1^5(1 - t_2) + t_1^4 t_2(1 - t_2) + t_1 t_2^3}{(1 - t_1^2)^2(1 - t_2)^2}
\]
\[
A_{0,3}(t_1, t_2) = \frac{t_1^2 t_2^2 + t_2^3 + t_1^5(1 - t_2^2) + t_1^4 t_2(1 - t_2^2)}{(1 - t_1^2)^2(1 - t_2)^2(1 + t_2)}
\]
\[
A_{4,0}(t_1, t_2) = \frac{t_1^4 + t_1^4(1 - t_2) + t_1^3 t_2(1 + t_2)^2 + t_1^2 t_2^2(1 + t_2 - t_2^2)}{(1 - t_1^2)^2(1 - t_2)^2(1 + t_2)}
\]
\[
A_{3,1}(t_1, t_2) = \frac{t_1^5(1 - t_2) + t_1^4 t_2^2 + t_1^3 t_2(1 - t_2^2)}{(1 - t_1^2)^2(1 - t_2)^2}
\]
\[
A_{2,2}(t_1, t_2) = \frac{t_1^4 + t_1^4(1 - t_2^2) + t_1^3 t_2(1 - t_2^2) + t_1^2 t_2^2(1 + t_2 - t_2^2)}{(1 - t_1^2)^2(1 - t_2)^2(1 + t_2)}
\]
\[
A_{1,3}(t_1, t_2) = \frac{t_1^5(1 - t_2) + t_1^4 t_2(1 - t_2) + t_1^3 t_2^2(1 - t_2) + t_1 t_2^3}{(1 - t_1^2)^2(1 - t_2)^2}
\]
\[
A_{0,4}(t_1, t_2) = \frac{t_1^2 t_2^3 + t_2^4 + t_1^5(1 - t_2^2) + t_1^4 t_2(1 - t_2^2) + t_1^3 t_2^2(1 - t_2)}{(1 - t_1^2)^2(1 - t_2)^2(1 + t_2)}
\]
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