We study the volume averaging of inhomogeneous metrics within GR and discuss its shortcomings such as gauge dependence, singular behavior as a result of caustics, and causality violations. To remedy these shortcomings, we suggest some modifications to this method. As a case study we focus on the inhomogeneous structured FRW model based on a flat LTB metric. The effect of averaging is then studied in terms of an effective backreaction fluid. It is shown that, contrary to the claims in the literature, the backreaction fluid behaves like a dark matter component, instead of dark energy, having a density of the order of $10^{-5}$ times the matter density, and most importantly, it is gauge dependent.

**Keywords**: Inhomogeneous Models; Averaging in Cosmology, LTB Metric.

1. **Introduction**

The amazing success of the FRW model of the universe has for years overshadowed the fact that we have devised it for a smoothed out geometry of the actual universe which shows inhomogeneity at different scales. The dark energy concept cannot be properly understood until the effect of inhomogeneities in the observational parameters and the role of geometrical averaging is understood. While studies of inhomogeneous models are progressing (see Refs. [1] [2] for extensive literature review, and also Refs. [3] [4]), the question of how to average the geometry is still an open question. Is it possible to write the Einstein equations for an inhomogeneous
universe, make then an averaging of the geometry, and obtain an effective FRW model of certain type? What would then be the difference between the result of averaging and a $\Lambda CDM$ model of the universe\(^5\,6\)?

There have been different attempts to answer these questions. The inhomogeneities maybe considered as a perturbation to the FRW models; an averaging process then leads to backreaction and a modified FRW universe\(^2\,7\,8\,9\,10\,11\,12\). The perturbative approach is likely to diverge due to the growth of perturbation at the epoch of structure formation, i.e. exactly the epoch of interest related to dark energy, as has been shown in \(^12\) \(\text{(for a review of different approaches see Ref. 2)}\). In the non-perturbative approach\(^14\,15\,16\,17\), a spatial volume averaging is devised to smooth out the inhomogeneities of the geometry as well as the fluid content of the universe leading to a non-standard homogeneous FRW model. Although the methods are non-perturbative, the difference between the real universe and the averaged one is usually called backreaction too.

In this paper we are dealing with the Buchert’s non-perturbative approach to the averaging problem in general relativity, which is based on the averaging formalism within the Newtonian gravity\(^15\). While the Newtonian case, being non-relativistic, is well defined, the general relativistic case is to be applied with care\(^19\). The main critique to the non-perturbative procedures is the fact that any inhomogeneous cosmological solution leads quickly to singularities making the formalism invalid\(^19\).

Using flat LTB cosmological models, we will see in detail how the singularities affect the volume averaging procedure. Based on resulting insights, we use the structured FRW model of the universe (SFRW)\(^20\) to propose and apply two different modifications of the volume averaging methods along the lightcone. It is then shown, that the averaging and the resulting backreaction is coordinate dependent, corresponds to dark matter instead of dark energy, and its corresponding density is 4 to 5 order of magnitudes less than the mean density of the universe.

Section 2 is devoted to flat LTB metrics, a short introduction to the structured FRW model of the universe, and a discussion on the place of the past light cone and the SFRW singularities within it. After introducing the volume averaging method within GR in section 3, we continue with a critique of the Buchert’s averaging method elaborating its shortcomings, suggest some modifications to it, and calculate the backreaction in SFRW model for different methods. We conclude then in section 4.

2. The structured FRW model of the universe

2.1. Overview

The structured FRW (SFRW) model proposed recently\(^20\) is a suitable case to study different averaging methods in general relativistic cosmological models. The basic idea in developing the SFRW model is the mere fact that the events outside the past lightcone of the observer have had no influence on the past events observable to us. One may then ignore the possible inhomogeneities outside the lightcone and model
the universe outside the lightcone as a FRW one. It then has to be matched to an
inhomogeneous universe inside the past lightcone. As the structures are effectively
influencing the universe much later than the last scattering surface, SFRW model
is applied to the matter dominated and pressure-less universe, and up to those $z$-
values corresponding to the length scales of at least 1000 Mpc in which the universe
is almost homogeneous. In contrast to the familiar usage of LTB metric to represent
over- or under-density bubbles in the constant time slices of the universe, in SFRW
the LTB junction to FRW is adapted along the past lightcone.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{$t_n$ vs $r$ for $t_n = \alpha/(r^4 + r^2 + 1)$.}
\end{figure}

Although, FRW and LTB could in principle be any of the three cases of open,
flat, or close, it has been shown in Ref. [21] that the only meaningful matching of
these two spaces along a null boundary is the flat-flat case. Given the cosmological
preferences, we are therefore assuming both metrics to be flat. As a consequence of
this matching along the past lightcone, the observed density of the universe along
the past lightcone has to be set equal to the density of FRW metric everywhere out-
side the lightcone. [21] The inhomogeneities are therefore just in our neighbourhood
within the lightcone up to distances of the order of 1000 Mpc.

We have to stress however that SFRW is just a toy model to understand the
impact of nearby inhomogeneities on our observations inside our past lightcone; one
is free to use it as a base for a Swiss cheese model, or add extra perturbations to it.
It should not be considered as a model for structure formation similar to the onion
model [13]
2.2. The Flat LTB solution

According to the SFR W model, a flat FRW universe outside the past lightcone is matched to a marginally bound or flat LTB metric inside the lightcone. Therefore, we restrict ourselves in this paper to the marginally bound LTB case. These are solutions of Einstein equations described by the metric

$$ds^2 = -c^2 dt^2 + R'(r,t)^2 dr^2 + R^2(r,t)(d\theta^2 + \sin^2 \theta d\phi^2),$$

in which overdot and prime (will thereafter) denote partial differentiation with respect to $t$ and $r$, respectively. The corresponding Einstein equations turn out to be

$$\dot{R}'(r,t) = \frac{2GM(r)}{R},$$

$$4\pi \rho(r,t) = \frac{M'(r)}{R^2 R'}.$$

The density $\rho(r,t)$ is in general an arbitrary function of $r$ and $t$, and the integration time-independent function $M(r)$ is defined as

$$M(r) \equiv 4\pi \int_0^{R(r,t)} \rho(r,t) R^2 dR = \frac{4\pi}{3} \bar{\rho}(r,t) R^3,$$

where $\bar{\rho}$, as a function of $r$ and $t$, is the average density up to the radius $R(r,t)$. The metric (1) can also be written in a form similar to the Roberts on-Walker metric. The following definition

$$a(t,r) \equiv \frac{R(t,r)}{r},$$

brings the metric into the form

$$ds^2 = -c^2 dt^2 + a^2 \left[ \left( 1 + \frac{a'}{a} \right)^2 dr^2 + r^2 d\Omega^2 \right].$$

For a homogeneous universe, $a$ doesn’t depend on $r$ and we get the familiar Robertson-Walker metric. The corresponding field equations can be written in the following familiar form

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3} \frac{\rho_c(r)}{\dot{a}^3},$$

where we have introduced $\rho_c(r) \equiv 6M(r)/r^3$ indicating a quasi comoving $r$-dependent density. These are very similar to the familiar Friedman equations, except for the $r$-dependence of different quantities. The solutions to the field equations can be written in the form

$$R(r,t) = \left[ \frac{9GM(r)}{2} \right]^{\frac{1}{3}} |t - t_n(r)|^{\frac{2}{3}},$$

$$a(r,t) = \left[ \frac{3}{4} G\rho_c(r) \right]^{\frac{1}{3}} (t - t_n(r))^{\frac{4}{3}}.$$
These are by now standard definitions of the flat LTB metric as a solution of Einstein equations \textsuperscript{20}.

Furthermore, the expansion and shear are defined in the following way:

\[ \Theta_{ij} = -K_{ij} = h^i_i h^j_j u_{\mu;\nu}, \]
\[ h_{\alpha\beta} = g_{\alpha\beta} + u_{\alpha} u_{\beta}; \quad h_{ij} = g_{ij}, \]

For the LTB metric (1) we obtain

\[ \Theta_j^i = \left[ \frac{\dot{R}}{R}, \frac{\dot{R}}{R}, \frac{\ddot{R}}{R} \right], \]
\[ \theta = tr\Theta_j^i = \frac{\dot{R}}{R} + 2 \frac{\ddot{R}}{R}, \]
\[ \sigma_j^i = \Theta_j^i - \frac{1}{3} \theta \delta_j^i, \]
\[ \sigma_1^1 = -2\sigma_2^2, \]
\[ \sigma_2^2 = \sigma_3^3 = \frac{1}{3} \left( \frac{\dot{R}}{R} - \frac{\ddot{R}}{R} \right), \]
\[ \sigma^2 = \frac{1}{3} \left( \frac{\dot{R}}{R} - \frac{\ddot{R}}{R} \right)^2. \]

Now, the flat LTB part of the SFRW model is defined by its bang time which is assumed to be

\[ t_n = \frac{\alpha}{r^4 + r^2 + 1}, \]

where \( r \) is scaled to 100 Mpc, as a typical inhomogeneity scale. For \( r \)-values much larger than 100 Mpc the bang function \( t_n \) tends to zero and, therefore, the LTB tends to a FRW metric. The special form of the bang function has been chosen such that there should be no singularity on the past lightcone. The constant \( \alpha = 10^{17} \text{ sec} \) is chosen such that the age of the universe at the last scattering surface corresponds to that of the standard cosmological model \textsuperscript{20}. The comoving coordinate \( r \) is used alternatively in the scaled form or not, and the reader may simply see from the context which one is meant. Figure 1 shows \( t_n \) as a function of \( r \). The bang time is almost everywhere zero except in our vicinity, reflecting the desired feature of SFRW. It can also be seen that for large \( r \) corresponding to the redshifts \( z > 1 \), we have almost FRW. Now, for this bang time we have

\[ t_n^\prime|_{r=0} = 0. \]

Therefore, there is no weak singularity at the origin \textsuperscript{22}. In fact, for the LTB domain with this bang time, there is no singularity in the vicinity of the lightcone, as it is shown in figure 2. No invariant of the metric has a singularity within the domain of our interest. All of the quantities appearing in the averaging process...
behave regularly at $t = 3t_n$ where the Kretschmann invariant is singular. The singularities of LTB metric, which are more sophisticated than in the case of the Robertson-Walker metric, have been discussed extensively in the literature (see for example Ref. [22– 23]). Vanishing of each of the metric functions and its derivatives $R, R', \ddot{R}, \dot{R}'$ may lead to different singularities. In a general LTB metric there is another singularity, the event horizon, related to zero of $1 + E$, where $E(r)$ is the energy function of the LTB metric absent in our flat LTB case. The place of different singularities are summarized as follows:

$$
R', R'' = \infty \quad \rightarrow \quad t = t_n,
$$

$$
R = 0 \quad \rightarrow \quad t = t_n,
$$

$$
R' = 0 \quad \rightarrow \quad t = t_n + \frac{2}{3}rt_n',
$$

$$
\dot{R}' = 0 \quad \rightarrow \quad t = t_n - \frac{1}{3}rt_n',
$$

$$
R'' = 0 \quad \rightarrow \quad t = \frac{3rt_n t_n'' + 6t_n t_n' - rt_n'^2}{6t_n' + 3rt_n''}.
$$

It is obvious that for $r \ll 1$, the bang time approaches a constant, and for $r \gg 1$
it approaches a constant, in fact zero, meaning that for large $r$ we have effectively FRW metric again.

\[ R' \] is plotted versus $r$ for different time values. Looking at $R'$ as an effective scale factor, it shows that the scale of the universe increases with time, although the rate of cosmic expansion is different at different places.

The zero’s of $R''$ are also sketched in the mentioned figures. The corresponding curve intersects the past lightcone on a point which corresponds to a local maximum of $R'$ as can be seen from figure 3 and none of the metric invariants, including that of Kretschman, have a singularity at this event. Therefore, $R'' = 0$ does not correspond to a metric singularity.

Having established the fact that singularities disturbing the averaging process happen around $t = t_n$, we note that the time constant slices with no singularities corresponds to redshift values less than $z = 1.45$. Therefore, any averaging over a fixed domain includes singularities except for $z$-values less than 1.45. Note that even in domains of no singularity, where the averaged values may be defined, the domain gets extended beyond the lightcone and we have to face superluminal effects. Finally, we stress again that for this special choice of the bang time the condition (18) is valid and therefore, there is no weak singularity at the origin 22.

3. Averaging in cosmology

3.1. Volume averaging in GR

The cosmic fluid is assumed to be perfect and irrotational. A flow-orthogonal foliation of spacetime, i.e. synchronous slicing, with the metric $ds^2 = -dt^2 + g_{ij}dx^i dx^j$ is then assumed. We should note that the synchronous coordinates, being not necessary for the following definitions, are however suitable for our problem. One could equally choose Newtonian gauge which is more suitable to study backreaction of
inhomogeneities considered as perturbation to FRW models.

The volume-averaging is based on the following definition. Let \( f(\vec{r}, t) \) be an arbitrary function of coordinates. Its average is defined by

\[
\langle f \rangle \equiv \frac{1}{V_D} \int_D dV f,
\]

(20)

where \( dV \) is the proper volume element of the 3-dimensional inhomogeneous domain \( D \) we are considering and \( V_D \) is its volume. The space-volume average of the function \( f(\vec{r}, t) \) does not commute with its time derivative:

\[
\langle f \rangle \cdot \langle \dot{f} \rangle = \langle f \theta \rangle - \langle f \rangle \langle \theta \rangle,
\]

(21)

where the expansion scalar \( \theta \), being equal to the minus of the trace of the second fundamental form of the hypersurface \( t = \text{const.} \), is now a function of \( r \) and \( t \). The right hand side trivially vanishes for a FRW universe because of the homogeneity. The averaged scale factor is then defined using the volume of our domain \( D \) by

\[
a_D \equiv V_D(t)^{\frac{1}{3}}.
\]

Now it can be shown that

\[
\theta_D \equiv \langle \theta \rangle \equiv \frac{\dot{V}}{V} = 3 \frac{\dot{a}_D}{a_D} = 3H_D,
\]

(22)

where we have used the notation \( \dot{a}_D \equiv da_D/dt \), and denoted the average Hubble function as \( H_D \). Therefore, all the derived quantities should be based on the average value \( a_D \). This is why one usually takes the above definition for the mean Hubble parameter and not \( \langle \dot{a}/a \rangle \), which is different from \( \dot{a}_D/a_D \). A similar difference holds for the second derivative of \( a \):

\[
\langle \ddot{a} \rangle / \langle a \rangle \neq \left( \frac{\ddot{a}_D}{a_D} \right).
\]

(23)

Therefore, the definition of the averaged deceleration parameter is not without ambiguity, specially because there is no firm relation like (22) for the deceleration parameter. This has motivated many authors so far to make the following definition for the deceleration parameter:

\[
q_D = \frac{\ddot{a}_D a_D}{\dot{a}_D^2} = -\frac{\dot{a}_D}{a_D} \frac{1}{H_D^2}.
\]

(24)

Now, the averages of the Einstein equations, using the Hamiltonian constraint and the Raychaudhuri equation, is written in the following form

\[
\left( \frac{\ddot{a}_D}{a_D} \right)^2 = \frac{8\pi G}{3} (\rho_D + \rho_\Sigma + \rho_R),
\]

(25)

\[
\frac{\ddot{a}_D}{a_D} = -\frac{4\pi G}{3} (\rho_D + 4\rho_\Sigma),
\]

(26)

where we have set \( \langle \rho \rangle = \rho_D \), \( \rho_\Sigma = \Sigma/8\pi G \), \( \rho_R = -\mathcal{R}_D/16\pi G \), \( \mathcal{R}_D \) is the backreaction term due to the average of three dimensional Ricci scalar, and \( \Sigma \) is the backreaction term corresponding to the non-vanishing shear defined as follows:

\[
\Sigma \equiv \langle \sigma^2 \rangle - \frac{1}{3} \langle (\theta - \langle \theta \rangle)^2 \rangle = \langle \sigma^2 \rangle - \frac{1}{3} \left[ \langle \theta^2 \rangle - \langle \theta \rangle^2 \right],
\]

(27)
where $\sigma$ is the shear scalar. It may be written more suitably in the form
\[
\Sigma = \langle \sigma^2 \rangle - \frac{1}{3} \langle (\theta^2) \rangle - \theta_D^2 = \left[ \langle \sigma^2 \rangle - \frac{1}{3} \langle (\theta^2) \rangle \right] + \frac{1}{3} \theta_D^2 = \left[ \langle \sigma^2 \rangle - \frac{1}{3} \langle (\theta^2) \rangle \right] + 3H_D^2 = A + 3H_D^2, \tag{28}
\]
where
\[
A \equiv \left[ \langle \sigma^2 \rangle - \frac{1}{3} \langle (\theta^2) \rangle \right]. \tag{29}
\]
Substituting from (12)-(16) we obtain
\[
A = \left\langle \left[ -2\dot{R}\dot{R} - \frac{\dot{R}^2}{R^2} \right] \right\rangle. \tag{30}
\]
Note that in the synchronous gauge we have chosen, the so-called dynamical backreaction term vanishes and $\Sigma$, the kinematical backreaction, is equal to the total backreaction \cite{17, 21}. We use the backreaction here for the geometric term $\langle G_{\mu\nu} \rangle - G_{\mu\nu}$, where $G_{\mu\nu}$ is the Einstein tensor and the averaging is understood to smooth out the inhomogeneities. Therefore, the Einstein tensor $G_{\mu\nu}$ is divided into a “homogeneous” part corresponding to the FRW universe, and a remaining part which could be brought to the rhs of the Einstein equations as backreaction of the inhomogeneities and interpreted as the energy-momentum tensor of a “geometric fluid”. Furthermore, the total density $\rho$, and the backreaction density $\rho_b$, may be defined in the following way:
\[
\rho = \rho_D + \rho_b, \tag{31}
\]
\[
\rho_b = \rho_\Sigma + \rho_\mathcal{R} = \frac{-1}{8\pi G} \left( \Sigma + \frac{1}{2} \mathcal{R}_D \right). \tag{32}
\]
According to this definition, the field equations lead to the following effective backreaction pressure:
\[
p_b = \rho_\Sigma - \frac{1}{3} \rho_\mathcal{R}. \tag{33}
\]
Although $\theta_D$ and $H_D$ are proportional, $\langle \theta^2 \rangle$ and $\langle H^2 \rangle$ are not. Hence, \cite{28} and \cite{29} cannot be written in terms of $H$. In the case of vanishing the three-dimensional Ricci scalar, as it is the case in the flat LTB, the equation of state reduces to
\[
\rho_b = \rho_\Sigma; \quad p_b = \rho_\Sigma. \tag{34}
\]
Therefore, in the flat LTB case the backreaction is defined effectively by an ideal fluid having the equation of state
\[
\rho_b = p_b = \rho_\Sigma; \quad w = 1. \tag{35}
\]
It is clear that it can not be interpreted as dark energy. We have not verified the gauge dependence of the above equation of state, as it is beyond the scope
of this paper. It may be different in Newtonian gauge, which would be another sign of coordinate dependence of the averaging method and its consequence for the backreaction.

3.2. Critique of the averaging method and modifying suggestions

The averaging process so far described and used in literature has already been criticized in Ref. [19]. Here we review different shortcomings of the method:

i) It is generally gauge dependent. One then has to check each time how far the result in a specific gauge is viable. Note that the Eqs. (26-27) are the result of a non-covariant integration over the Raychaudhuri equation. This has not to be confused with the assumption of homogeneity of the universe and the resulting FRW universe: the FRW universe is a solution of the Einstein equations based on the homogeneity assumption.

ii) Even if there is some natural choice of coordinates, like the comoving synchronous gauge, i.e. slices orthogonal to the world lines of the dust, it usually breaks down due to the formation of caustics [19].

iii) The domain of integration over which the average values are defined is fixed. In cosmological models of interest, even in regions without any caustics, the domain $D$ extends definitely outside the lightcone for the most interesting range of time or redshift values, independent of how small the chosen range of $r$ is. Integrating over distances outside the lightcone is, however, equivalent to taking into account superluminal effects. This is clearly a noncausal procedure which should not be implemented in the relativistic equations. It is equivalent to assigning a value to the density at a point on the light cone, or within it, depending on events causally disconnected to it, similar to the horizon problem in standard cosmology.

3.2.1. Gauge dependence

According to the Stewart-Walker lemma [25], any average quantity is gauge invariant if the zeroth order part vanishes on the background, or is a constant scalar field there, or is a linear combination of products of Kronecker deltas with constant coefficients. The theorem has been used in Ref. [26] to show the gauge invariance of the quantities they are calculating. One should however be aware of the fact that the theorem is valid only for a perturbed metric with respect to a background. Therefore, this theorem can not be applied to the general case of an exact solution of Einstein equations such as the LTB metric we are using here. In fact, it is not clear at all if and when the LTB metric can be written as a perturbation to FRW metric for cases where backreaction maybe significant [27]. We will show in the section 3.2.2 that the backreaction is non-zero for the structured FRW model in the gauge chosen, although it is exactly zero in another gauge [11], indicating the gauge dependence of the averaging method.
3.2.2. Modified methods: Lightcone averaging formalism

We suggest two different alternatives to the volume averaging method in general relativity [10] to the aim of circumventing the shortcomings of the method elaborated in the section (3.2). The general motivation is to remedy the non-causal input to the averaging integrals through the fixed domain and also to avoid singularities. The first is accomplished by a time constant integral up to the lightcone. To avoid the singularities, we consider a region bounded by the lightcone and a time-like hypersurface within it at a distance of the order of 1000 Mpc to be sure that outside this inhomogeneous domain we may consider the universe as homogeneous again. Note that the distance from any point on the curve corresponding to \( t = 3t_n \), i.e. the Kretschmann singular curve, is at least of the inhomogeneity order of 1000 Mpc (see fig. 2). We may therefore assume that the domain within the lightcone but outside the Kretschmann curve is effectively homogeneous. We have already mentioned that the density for the events on the Kretschmann curve are regular. Therefore, these events are no obstacle to define the averaging if necessary.

Both modified domains are bounded by the lightcone. To distinguish them we call the averaging by using the first domain the in-lightcone and the second one the on-lightcone averaging method. Let us summarize the three different averaging procedures:

1) **Volume averaging using fixed domain**

This is the standard averaging formalism used in literature [10].

2) **In-lightcone averaging formalism**

The domain of integration is extended from a fixed \( r \)-value, say \( r = 0 \), to the corresponding \( r \) at the lightcone for a fixed time. Clearly the non-causal character of the standard volume averaging is absent here, but the domain may still include singular points. From what has been outlined in sections 2.1 and 2.2, both the volume averaging with fixed domain and the in-lightcone averaging method may include caustics. Hence, these methods are only applicable to small \( z \) values where no singularity is seen within the domain of integration. In our model we are bounded to the redshifts \( z < 1.45 \).

3) **On-lightcone averaging formalism**

A time-like hypersurface within the lightcone, far from the singularities but at a distance from the lightcone of the order of the inhomogeneity scale such as the Kretschmann curve is chosen. The domain of integration is then defined for a constant time from a point on this hypersurface up to the lightcone. By this on-lightcone formalism we have secured the causal implications of the averaging integrals and also a singularity-free domain of integration.

The first two methods are applicable along the past lightcone just up to \( z \)-values where the density is regular for any \( r \) value corresponding to \( z \approx 1.45 \). As it is seen from the figure 2, all the metric singularities lie under the curve corresponding to \( t = t_n(r) \). Therefore, the hypersurface necessary for the on-lightcone method may be any time-like one within the lightcone bounded by the curve \( t = t_n \) such as the
Fig. 4. Comparison of the backreaction density as a function of redshift by the three averaging methods for low redshifts. The value of $\rho_\Sigma$ is not defined for redshifts higher than $\sim 0.8$ due to the singularities in standard and In-lightcone methods.

Kretschmann curve. The averaging integrals in this case are well-defined and maybe handled numerically.

The on-lightcone averaging method is, therefore, free from both shortcomings and we may define all along the lightcone without incorporating any caustics or non-causal effects.

3.3. Results

Now, we have all the prerequisites to do the averaging. The results are plotted in the form of backreaction density $\rho_\Sigma$ and the ratio of backreaction to the matter density $\rho_\Sigma/\rho_z$ versus redshift for the bang time $t_n=17$ (figures 4, 6). Figure 4 shows the backreactions up to $z \approx 0.8$ for the three methods. Larger $z$-values leads to singularities and, therefore, the averaging cannot be executed in the volume averaging and in-lightcone methods. This shows again the restricted applicability of the volume averaging method. In the case of on-light-cone method we have been able to integrate up to the surface of the last scattering corresponding to $z \approx 1100$. Therefore, any meaningful statement about the effect of backreaction can only be made for this method. For the sake of completeness, and as a reference to our numerical calculation, we have also calculated numerically the case of bang time $t_n=0$ equivalent to FRW with vanishing backreaction plotted in (figures 7, 8). The vanishing of backreaction in this case is easily seen up to the numerical errors. In all the cases under consideration, the backreaction density and pressure are roughly 5 order of magnitudes smaller than the dark energy needed to explain the acceleration of the universe. Hence, it cannot be concluded that the backreaction has any effect towards explaining the mysterious dark energy. Note that the backreaction density for a LTB universe using the volume averaging for a fixed domain (figure 1), when
defined, is of the same order of magnitude as the other cases and negative.

A noticeable effect is the increase of the backreaction with redshift as shown in figure 5. A more meaningful parameter is the relative backreaction density $\rho / \rho$ shown in figure 6. There we see that the backreaction has a maximum at about $z = 0.4$ and then decrease to a minimum by $z \approx 4$, before start increasing. This is in contrast to the motivation of SFRW in choosing the bang time $t_0$. One would expect that the backreaction would decrease all along the past lightcone. To understand this
behavior, we have plotted the bang time (17) versus the redshift in figure 9. It shows that the bang time has a minimum at roughly the same redshift value of the minimum of the backreaction. This redshift value corresponds to \( r \approx 5000 \) Mpc which is of the order of magnitude of the inhomogeneity of the universe. It is, therefore, clear why the backreaction increases with increasing redshift after that minimum: the bang time, and as a consequence the deviation of LTB from FRW, increases. The selected bang time (17) seems to be well-behaved up to this inhomogeneity scale. We know already that the universe is homogeneous at that scale to a very good approximation, and is modeled by FRW standard cosmology. Hence, we may assume that within the SFRW philosophy the metric after \( z \approx 4 \) is given by FRW. For \( z \)-values smaller than the maximum in figure 6, i.e. \( z \approx 0.4 \) we are faced with compact structures and therefore the model is not applicable.

After publishing the first version of this work, a study 4 appeared, discussing an averaging approach along the past lightcone, in contrast to time constant domains. The authors define a new procedure for averaging in cosmology not related to the volume averaging and its modifications we have proposed in this paper. A separate study on the backreaction in a flat LTB model universe 28 was published, in which the authors suggest a so-called 'running averaging scale' to modify the volume averaging method, using a gauge similar to that of Ref. 11 and, as in Ref. 11, show the vanishing of the backreaction in it. They are mainly interested in the modification of the luminosity distance. A recent paper on the cosmological backreaction from perturbations 24 also appeared. Working in the Newtonian gauge, the authors announce a backreaction of the order of \( 10^{-5} \) times the matter density, similar to our result, but with an effective equation of state \( w \approx -1/19 \).
4. Conclusion

We have explored the possibility of explaining, at least partially, the dark energy using the volume averaging of Einstein equations for a specific model based on the flat LTB inhomogeneous solutions. We were able to define two modifications to the familiar method with a fixed domain of integration without the usual shortcomings such as the caustics and non-causal implications. It turned out that the backreaction in the gauge chosen mimics a normal matter with positive pressure, and a density roughly 4 to 5 order of magnitudes less than the matter density. This is in contrast to vanishing of backreaction in other gauges, and confirm the claims that
backreaction is gauge dependent. We therefore conclude that, although the method can be made free of singularities and superluminal effects, the volume averaging in the SFRW toy model leads to a non-vanishing addition to the normal matter five order of magnitudes less than the matter density, indicating a kind of dark matter component, but no sign of any dark energy component.

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