Separation of Quantization and Control in Optimal Analog to Digital Converters

Mitra Osqui†  Alexandre Megretski‡

Abstract

In this paper we prove optimality of a certain class of Analog to Digital Converters (ADCs), which can be viewed as generalized Delta-Sigma Modulators (DSMs), with respect to a performance measure that can be characterized as the worst-case average intensity of the signal representation error. An analytic expression for the ADC performance is given. Furthermore, our result proves separation of quantization and control for this class of ADCs subject to some technical conditions.

I. INTRODUCTION AND MOTIVATION

Analog to Digital Converters (ADCs) act as the interface between the analog world and digital processors. They are present in almost all digital control and communication systems and modern high-speed data conversion and storage systems. Naturally, the design and analysis of ADCs have, for many years, attracted the attention and interest of researchers from various disciplines across academia and industry. Despite the progress that has been made in this field, the design of optimal ADCs remains an open challenging problem, and the fundamental limitations of their performance are not well understood. This paper is concerned with the latter problem.

A particular class of ADCs primarily used in high resolution applications is the Delta-Sigma Modulator (DSM). Fig. 1 illustrates the classical first-order DSM [1], where $Q$ is a quantizer with uniform step size.

An extensive body of research on DSMs has appeared in the signal processing literature. One well known approach is based on linearized additive noise models and filter design for noise shaping [1]-[6]. The underlying assumption for validity of the linearized additive noise model is availability of a

†Mitra Osqui is currently a Ph.D. candidate at the department of EECS, Laboratory for Information and Decision Systems (LIDS) at the Massachusetts Institute of Technology, Cambridge, MA. E-mail: mitra@mit.edu
‡ Alexandre Megretski is currently a professor of EECS at LIDS at MIT, Cambridge, MA. E-mail: ameg@mit.edu.
Fig. 1. Classical First-Order Sigma-Delta Modulator

relatively high number of bits. Alternative approaches based on a formalism of the signal transformation performed by the quantizer have been exploited for deterministic analysis in [7]-[9]. Some other works that do not use linearized additive noise models are reported in [10]-[12].

In control literature, [13]-[15] find performance bounds and suboptimal policies for linear stochastic control problems using Bellman inequalities with quadratic value functions. The problem is relaxed and solved using linear matrix inequalities and semidefinite programming. For references on quantized control, please see [16]-[18].

In [19] and [20] we provided a characterization of the solution to the optimal ADC design problem and presented a generic methodology for numerical computation of sub-optimal solutions along with computation of a certified upper bound and lower bound on the performance, respectively.

Fig. 2 illustrates the setup we use for measuring the performance of the ADC. The performance of an ADC is evaluated with respect to a cost function which is a measure of the intensity of the error signal $e$ (the difference between the input signal $r$ and its quantized version $u$) for the worst case input sequence. The error signal is passed through a shaping filter which dictates the frequency region in which the error is to be minimized. Furthermore, we show that the dynamical system within the optimal ADC is a copy of the shaping filter used to define the performance criteria.

Fig. 2. Setup Used for Measuring the Performance of the ADC

In [19] we also presented an exact analytical solution to the optimal ADC for first-order shaping filters, and showed that the classical first-order DSM (Figure 1) is identical to our optimal ADC. This result proved the optimality of the classical first-order DSM with respect to the adopted performance measure, and was a step towards understanding the limitations of performance. In this paper we provide
the optimal solution for higher order shaping filters subject to certain technical conditions and prove optimality of some higher order DSMs.

Notation and Terminology:
• Given a set $P$, $\ell_+(P)$ is the set of all one-sided sequences $x$ with values in $P$, i.e. functions $x : \mathbb{Z}_+ \rightarrow P$.

II. Problem Formulation

The problem setup in this section is taken from [19].

A. Analog to Digital Converters

In this paper, a general ADC is viewed as a causal, discrete-time, non-linear system $\Psi$, accepting arbitrary inputs in the $[-1, 1]$ range and producing outputs in a fixed finite subset $U \subset \mathbb{R}$, as shown in Fig. 3. We assume $\max U > 1$ and $\min U < -1$.

Equivalently, an ADC is defined by a sequence of functions $\Upsilon_n : [-1, 1]^{n+1} \mapsto U$ according to

$$\Psi : u[n] = \Upsilon_n (r[n], r[n-1], \cdots, r[0]) , \quad n \in \mathbb{Z}_+. \quad (1)$$

The class of ADCs defined above is denoted by $\mathcal{Y}_U$.

B. Asymptotic Weighted Average Intensity (AWAI) of a Signal

Let $\phi : \mathbb{R} \mapsto \mathbb{R}_+$ be an even, non-negative, and monotonically nondecreasing function on the positive reals; and $G(z)$ be the transfer function of a strictly causal LTI dynamical system $L_G$ with input $w$ and output $q$:

$$L_G : \begin{cases} x[n+1] = Ax[n] + Bw[n], & x[0] = 0, \\ q[n] = Cx[n] \end{cases} \quad (2)$$
where $A$, $B$, $C$ are given matrices of appropriate dimensions. The Asymptotic Weighted Average Intensity $\eta_{G,\phi}(w)$ of signal $w$ with respect to $G(z)$ and $\phi$ is given by:

$$
\eta_{G,\phi}(w) = \lim_{N \to \infty} \sup \frac{1}{N} \sum_{n=0}^{N-1} \phi(q[n]).
$$

(3)

Examples of functions $\phi$ to consider are: $\phi(q) = |q|$ and $\phi(q) = |q|^2$. We assume without loss of generality that $CB \neq 0$. Indeed, since $\eta_{G,\phi}$ does not change if $G(z)$ is replaced by $zG(z)$, i.e. if $q[n]$ is replaced with $q[n+1]$ in (2), the case when $CB = 0$ can be reduced to the case $CB \neq 0$ by extracting a delay from $L_G$.

C. ADC Performance Measure

The setup that we use to measure the performance of an ADC is illustrated in Fig. 4. The performance measure of $\Psi \in \mathcal{Y}_U$, denoted by $J_{G,\phi}(\Psi)$, is the worst-case AWAI of the error signal for all input sequences $r \in \ell_+([-1, 1])$, that is:

$$
J_{G,\phi}(\Psi) = \sup_{r \in \ell_+([-1, 1])} \eta_{G,\phi}(r - \Psi(r)).
$$

(4)

Fig. 4. Setup Used for Measuring the Performance of the ADC

D. ADC Optimization

Given $L_G$ and $\phi$, we consider $\Psi_o \in \mathcal{Y}_U$ an optimal ADC if $J_{G,\phi}(\Psi_o) \leq J_{G,\phi}(\Psi)$ for all $\Psi \in \mathcal{Y}_U$. The corresponding optimal performance measure $\gamma_{G,\phi}(U)$ is defined as

$$
\gamma_{G,\phi}(U) = \inf_{\Psi \in \mathcal{Y}_U} J_{G,\phi}(\Psi).
$$

(5)

III. OUR APPROACH

We search for the optimal ADC within the class of time invariant state-space models and associate the optimal ADC design problem with a full-information feedback control problem. We show for a certain class of ADCs that the setup depicted in Figure 5 is an optimal ADC architecture. The function
$K : \mathbb{R}^m \times [-1, 1] \mapsto U$ is said to be an admissible controller if there exists $\gamma \in [0, \infty)$ such that every triplet of sequences $(x_\Psi, u, r)$ satisfying

\begin{equation}
    x_\Psi [n + 1] = Ax_\Psi [n] + Br [n] - Bu [n], \quad x_\Psi [0] = 0,
\end{equation}

\begin{equation}
    u [n] = K (x_\Psi [n], r [n]),
\end{equation}

\begin{equation}
    q_\Psi [n] = C x_\Psi [n],
\end{equation}

also satisfies the dissipation inequality

\begin{equation}
    \sup_{N, r \in \ell_+([-1, 1])} \sum_{n=0}^{N-1} (\phi (q_\Psi [n]) - \gamma) < \infty
\end{equation}

Note that if (9) holds subject to (6)-(8), then $J_{G, \phi} (\Psi) \leq \gamma$. Let $\gamma_o$ be the maximal lower bound of $\gamma$, for which an admissible controller exists. Then $K$ is said to be an optimal controller if (9) is satisfied with $\gamma = \gamma_o$.

![Fig. 5. Full State-Feedback Control Setup](image)

IV. MAIN RESULT

Consider the ADC optimization problem presented in Section II with $L_G$ defined by (2) with $CB \neq 0$. For $\delta \in (0, 2]$ and $M \in \mathbb{N} \cup \{\infty\}$, define the set $U_M$ and function $K_M : \mathbb{R} \mapsto U_M$ as

\begin{equation}
    U_M = \{m \delta \mid m \in \mathbb{Z}, \quad |m| \leq M\}
\end{equation}

\begin{equation}
    K_M (\theta) = \min \left\{ \arg \min_{u \in U_M} |\theta - u| \right\}.
\end{equation}
Consider the ADC $\hat{\Psi} \in \mathcal{Y}_{U_M}$ defined by

$$L_{\hat{\Psi}} : \begin{cases} x_{\hat{\Psi}}[n + 1] = Ax_{\hat{\Psi}}[n] + Br[n] - Bu[n], \\ q_{\hat{\Psi}}[n] = Cx_{\hat{\Psi}}[n] \\ x_{\hat{\Psi}}[0] = 0 \end{cases} \quad (12)$$

with the control law

$$u[n] = K_M\left((CB)^{-1}CAx_{\hat{\Psi}}[n] + r[n]\right). \quad (13)$$

We show in Theorem 1 below that if $M$ is large enough and $\delta$ is small enough, then the ADC defined above is optimal. The control decision $u[n]$ in (13) minimizes $|q_{\hat{\Psi}}[n + 1]|$. An interpretation of Theorem 1 is that a greedy algorithm is optimal subject to certain conditions. Let $q_{\hat{\Psi}}[n + 1] = \sum_{i=0}^{k} a_i q_{\hat{\Psi}}[n - i] + \sum_{j=0}^{k} b_j (r[n - j] - u[n - j])$. \quad (14)

be the difference equation which is equivalent to (12). Let $F$ be the causal LTI system with transfer function

$$F(z) = \frac{1}{k \sum_{j=0}^{k} b_j z^{-j}}. \quad (15)$$

Let $\{c_l\}_{l=0}^{\infty}$ be the unit sample response of system (14), i.e.

$$F(z) = \sum_{l=0}^{\infty} c_l z^{-l}, \quad \text{for } |z| > R_0 \quad (16)$$

where $R_0 \in \mathbb{R}$ is the maximal absolute value of the largest pole of $F(z)$ in (15).

Theorem 1: Let $\hat{\Psi} \in \mathcal{Y}_{U_M}$ be the ADC defined by (12)–(13) with $CB \neq 0$ and $K_M$ defined by (10)–(11). Let

$$\beta = \left[|CB| \frac{\delta}{2} \left(\sum_{i=0}^{k} |a_i| + 1 \right) + \sum_{j=0}^{k} |b_j| \right] \sum_{l=0}^{\infty} |c_l|,$$

where $\{a_i\}_{i=0}^{k}$ and $\{b_j\}_{j=0}^{k}$ are defined by (14) and $\{c_l\}_{l=0}^{\infty}$ is defined by (15)–(16). Let $M\delta$ be such that $M\delta > 1$ and

$$M\delta > \beta - \delta. \quad (17)$$

Let $f : [0, \infty) \to [0, \infty)$ be a monotonically nondecreasing function and $\phi(q) = f(|q|)$. Then $\hat{\Psi}$ is an optimal ADC in the sense that

$$J_{G,\phi}(\Psi) \geq J_{G,\phi}(\hat{\Psi}) = \phi(|CB|\delta/2) \quad \forall \Psi \in \mathcal{Y}_{U_M}. \quad (18)$$
Proof: Please see the Appendix.

Remark 1: We showed in [19] that the first-order DSM in Figure 1 is optimal with respect to the shaping filter \( L_G = 1/(z - 1) \) with any uniform quantizer \( Q \) with \( M\delta > 1 \).

Remark 2: For \( L_G = z/(z-1)^2 \) with any uniform quantizer \( Q \) with step size \( \delta \leq 2 \) and the magnitude of the largest value of the quantizer being larger than \( 1 + \delta \), the second-order DSM is optimal.

The optimal ADC architecture presented in Figure 5 along with the optimal control law given in (13) can be equivalently represented by Figure 6 and equation (19), where \( Q \) is a uniform quantizer with step size \( \delta \) and saturation level \( M\delta \) satisfying (17) and \( G(z) \) is the transfer function of the shaping filter \( L_G \). Furthermore, Figure 6 has a DSM architecture, thus with a proper selection of \( L_G \) as the shaping filter, many standard DSMs that satisfy the conditions in Theorem 1 are proven optimal.

![Diagram of optimal ADC Architecture](image)

Fig. 6. Optimal ADC Architecture, where \( G(z) = C(zI - A)^{-1}B \) is the transfer function of \( L_G \).

\[
H(z) = (CB)^{-1}zG(z) - 1 = (CB)^{-1}C(zI - A)^{-1}AB
\]  

(19)

That is, if the magnitude of the largest value of the quantizer output is large enough and quantization step size is small enough, then the greedy algorithm is the optimal output for the ADC. This shows separation of quantization and control for this problem, subject to inequality (17).

V. CONCLUSION

In this paper, we showed optimality of a certain class of ADCs (which were shown to have DSM like architecture) subject to some conditions and provided an analytic expression for the performance. We showed that there is separation of quantization and control, i.e. in the absence of quantization, the obvious choice for the optimal control law is proven to be the optimal control law given quantization, when certain technical conditions are met.
VI. APPENDIX

Proof of Theorem 1: Let us begin by showing that with the control law given in (13) with $M = \infty$ we have:

$$|q_\tilde{\Psi}[n]| \leq |CB|\delta/2, \quad \forall n \in \mathbb{Z}_+,$$  \hspace{1cm} (20)

Indeed, for $n = 0$, inequality (20) follows from the initial condition in (12). For $n > 0$,

$$q_\tilde{\Psi}[n + 1] = CB(w[n] - K(w[n])),$$

where $w[n] = (CB)^{-1}CAx_\tilde{\Psi}[n] + r[n]$. Since $|\theta - K(\theta)| \leq \delta/2$ for all $\theta \in \mathbb{R}$, we have (20) for all $n \geq 0$.

The next step is to use the bound $|q_\tilde{\Psi}[n]| \leq |CB|\delta/2$ to show that $|u[n]| \leq \beta$. Rearranging (14), taking absolute value from both sides, and using the triangle inequality yields:

$$\left| \sum_{j=0}^{k} b_j u[n - j] \right| \leq |CB|\delta \left( \sum_{i=0}^{k} |a_i| + 1 \right) + \sum_{j=0}^{k} |b_j|$$

If $\sum_{j=0}^{k} b_j u[n - j]$ is the input signal to the system $F$ with transfer function $F(z)$ defined in (15), then the output $u[n]$ is bounded in magnitude by

$$|u[n]| \leq \beta$$  \hspace{1cm} (21)

A sufficient condition for $|u[n]| \leq M\delta$, is given by (17), (21), and $u \in U_\infty$. Therefore (17) implies (20).

Since both systems $L_G$ and $L_\tilde{\Psi}$ have the same input and $x_\tilde{\Psi}[0] = x[0] = 0$, condition (20) implies that

$$|q[n]| \leq |CB|\delta/2, \quad \forall n \in \mathbb{Z}_+.$$

Therefore,

$$\sup_{N,r \in [-1,1]} \sum_{n=0}^{N} (\phi(q[n]) - \phi(|CB|\delta/2)) \leq 0 < \infty,$$

which implies that

$$\mathcal{J}_{G,\phi}(\tilde{\Psi}) \leq \phi(|CB|\delta/2).$$  \hspace{1cm} (22)

In order to complete the proof, we need to show that no ADC can achieve a better performance than $\phi(|CB|\delta/2)$. It is sufficient to show that for all $\Psi \in \mathcal{Y}_U$, there exists an input sequence $r$ such that

$$|q_\Psi[n]| \geq |CB|\delta/2, \quad \forall n \in \mathbb{Z}_+ \setminus \{0\}.$$  \hspace{1cm} (23)
Define function \( \rho : \mathbb{R}^m \rightarrow \mathbb{Z} \) by

\[
\rho(x) = \min \left\{ \arg \min_{k \in \mathbb{Z}} \left[ \frac{2k + 1}{2} \delta - (CB)^{-1}CAx \right] \right\}.
\] (24)

When \( r[n] \) is given by

\[
r[n] = \frac{2\rho(x[n]) + 1}{2} \delta - (CB)^{-1}CAx[n],
\] (25)

we have \( r[n] \in [-1, 1] \) (since \( \delta \in (0, 2) \)) and

\[
|q_{\Psi}[n + 1]| = \left| CB \left( \frac{2\rho(x[n]) + 1}{2} \delta - u[n] \right) \right| \geq |CB|\delta/2
\] (26)

for all \( n \in \mathbb{Z}_+ \), because \( u[n] \in k\delta \). Hence

\[
\mathcal{J}_{G,\phi}(\hat{\Psi}) \geq \phi \left(|CB|\delta/2\right).
\] (27)

Inequalities (22) and (27) complete the proof.

REFERENCES

[1] A. V. Oppenheim, R. W. Schafer, and J. R. Buck, *Discrete-Time Signal Processing*. Prentice-Hall, 1999.

[2] M. Derpich, E. Silva, D. Quevedo, and G. Goodwin, “On optimal perfect reconstruction feedback quantizers,” *IEEE Transactions on Signal Processing*, vol. 56, no. 8, pp. 3871–3890, Aug 2008.

[3] S. Ardalan and J. Paulos, “An analysis of nonlinear behavior in delta-sigma modulators,” *IEEE Transactions on Circuits and Systems*, vol. 34, no. 6, pp. 593–603, jun 1987.

[4] A. Marques, V. Peluso, M. S. Steyaert, and W. M. Sansen, “Optimal parameters for \( \Delta \Sigma \) modulator topologies,” *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 45, no. 9, pp. 1232–1241, Sep. 1998.

[5] R. Schreier and G. Temes, *Understanding Delta-Sigma Data Converters*. IEEE Press, John Wiley and Sons, Inc, 2005.

[6] S. Norsworthy, R. Schreier, and G. C. Temes, *Delta-Sigma Data Converters: Theory, Design, and Simulation*. IEEE Press, John Wiley and Sons, Inc, 1997.

[7] N. T. Thao and M. Vetterli, “A deterministic analysis of oversampled A/D conversion and \( \Sigma \Delta \) modulation,” *IEEE International Conference on Acoustics, Speech, and Signal Processing*, vol. 3, pp. 468–471, Apr. 1993.

[8] ——, “Deterministic Analysis of Oversampled A/D Conversion and Decoding Improvement Based on Consistent Estimates,” *IEEE Transactions on Signal Processing*, vol. 42, no. 3, pp. 519–531, Mar. 1994.

[9] N. T. Thao, “The Tiling Phenomenon in \( \Sigma \Delta \) Modulation,” *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 51, no. 7, pp. 1365 – 1378, Jul. 2004.

[10] D. Quevedo and G. Goodwin, “Multistep optimal analog-to-digital conversion,” *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 52, no. 3, pp. 503 – 515, march 2005.

[11] P. Steiner and W. Yang, “A framework for analysis of high-order sigma-delta modulators,” *Circuits and Systems II: Analog and Digital Signal Processing, IEEE Transactions on*, vol. 44, no. 1, pp. 1 –10, jan 1997.

[12] H. Wang, “A geometric view of sigma; delta; modulations,” *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 39, no. 6, pp. 402 –405, jun 1992.
[13] Y. Wang and S. Boyd, “Performance bounds and suboptimal policies for linear stochastic control via LMIs,” International Journal of Robust and Nonlinear Control, vol. 21, no. 14, pp. 1710–1728, 2011, available: http://dx.doi.org/10.1002/rnc.1665 [Online]. Available: http://www.stanford.edu/~boyd/papers/gen_ctrl_bnds.html

[14] ——, “Performance bounds for linear stochastic control,” Systems and Control Letters, vol. 58, no. 3, pp. 178 – 182, 2009.

[15] ——, “Approximate dynamic programming via iterated bellman inequalities,” April 2010. [Online]. Available: http://www.stanford.edu/~boyd/papers/adp_iter_bellman.html

[16] F. Bullo and D. Liberzon, “Quantized control via locational optimization,” IEEE Transactions on Automatic Control, vol. 51, no. 1, pp. 2 – 13, jan. 2006.

[17] R. Brockett and D. Liberzon, “Quantized feedback stabilization of linear systems,” IEEE Transactions on Automatic Control, vol. 45, no. 7, pp. 1279 –1289, Jul. 2000.

[18] N. Elia and S. Mitter, “Stabilization of linear systems with limited information,” IEEE Transactions on Automatic Control, vol. 46, no. 9, pp. 1384 –1400, sep 2001.

[19] M. Osqui, A. Megretski, and M. Roozbehani, “Optimality and Performance Limitations of Analog to Digital Converters,” Conference on Decision and Control, pp. 7527–7532, Dec. 2010.

[20] ——, “Lower bounds on the performance of analog to digital converters,” pp. 1036 –1041, dec. 2011.

[21] A. Megretski, “Robustness of finite state automata,” in Multidisciplinary Research in Control: The Mohammed Dahleh Symposium 2002, ser. Lecture Notes in Control and Information Sciences, L. Giarre and B. Bamieh, Eds. Springer, 2003, vol. 289, pp. 147–160.

[22] M. Osqui, M. Roozbehani, and A. Megretski, “Semidefinite Programming in Analysis and Optimization of Performance of Sigma-Delta Modulators for Low Frequencies,” American Control Conference, pp. 3582–3587, Jul. 2007.

[23] K. Zhou, J. C. Doyle, and K. Glover, Robust and Optimal Control. Prentice Hall, 1996.

[24] T. Basar and P. Bernhard, $H^\infty$ - Optimal Control and related Minimax Design Problems: A Dynamic Game Approach. Birkhauser, 1995.