Two types irregular labelling on dodecahedral modified generalization graph

Nuradin Hinding a,*, Kiki A. Sugeng b, Nurlindah c, Timothy J. Wahyudi d, Rinovia Simanjuntak c

a Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Hasanuddin, Indonesia
b Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Indonesia, Indonesia
Undergraduate’s Program in Mathematics, Faculty of Mathematics and Natural Sciences, Hasanuddin University, Indonesia
Undergraduate’s Program in Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Indonesia, Indonesia
Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Indonesia

Abstract

Irregular labelling on graph is a function from component of graph to non-negative natural number such that the weight of all vertices, or edges are distinct. The component of graph is a set of vertices, a set of edges, or a set of both. In this paper we study two types of irregular labelling on dodecahedral modified generalization graph. We determined the total vertex irregularity strength and the modular irregularity strength of dodecahedral modified generalization graph. These results are important because there many classes of graph have the same structure with modified dodecahedral graphs. These results can be used to determine the total vertex irregularity strength and the modular irregularity strength of other graphs that have the similar structure with modified dodecahedral graph.

1. Introduction

Irregular labelling was introduced by Chartrand, et al. in 1988 [5]. Let $G(V, E)$ be a graph where $V$ and $E$ are vertex set and edge set, respectively. In this paper we investigate a connected graph no multiple edges, no loops, and no component graph of order 2.

A function is a mapping that maps elements of graph (vertex or edge) to a natural number is called graph labelling. There are many kinds of graph labelling. There are two types of them will be discussed, namely vertex irregular total $k$-labelling and modular irregular $k$-labelling, where $k$ is a natural number. Bača et al. introduced the vertex irregular total $k$-labelling in 2007 [2] and the modular irregular $k$-labelling in 2020 [4]. The results of the study are written in Theorem 1 and Theorem 2.

2. Dodecahedral modified generalization graph

The dodecahedral graph is the platonic graph corresponding to the connectivity of the vertices of dodecahedron, where the platonic graph is a polyhedral graph corresponding to the skeleton of a platonic solid. A dodecahedral graph is a 3-connected planar graph on 20 vertices and 30 edges as in Fig. 1(a) [8].

In this paper, we modified the dodecahedral graph by adding 10 edges such that the number of vertices of degree 3 is equal to the number of vertices of degree 5 which are 10 as in Fig. 1(b) and then we generalize it into a new by add some vertices as in Fig. 1(c).

The dodecahedral modified generalization graph denoted by $GD_n$ with the number of vertices is $2n$. We have the vertices $u_i$ for $1 \leq i \leq n$ forming a cycle called the outer cycle. We also have an inner cycle using the vertices $v_i$ for $1 \leq i \leq n$. Additionally we connect vertices with the same second index from both cycles. Furthermore, we add edges connecting vertices from the inner cycle if they are at distance two away from each other. We have the edge in the following notation $E = \{u_iu_{i+1}, v_iu_j, v_iy_{i+2}, u_iu_j | 1 \leq i \leq n\}$, with the index of all vertices be in modulo $n$.

3. Total vertex irregular labelling on dodecahedral modified generalization graph

Total vertex irregular labelling on any graph developed by Bača et al. in 2007. This labelling is a development of irregular labelling. Formally, total vertex irregular labelling on any graph defined as follows.
Definition 1. [2] Suppose $G = (V, E)$ is a graph. A total labelling $f : V \cup E \rightarrow \{1, 2, 3, \ldots, k\}$ named total vertex irregular $k$-labelling of $G$ if $\Delta f(x) \neq \Delta f(y)$ for every two vertices $x \neq y$, where $\Delta f(x) = f(x) + \sum_{y \in V} f(xy)$. The minimum positive integer number $k$ such that $G$ have a total vertex irregular $k$-labelling is named total vertex irregularity strength of $G$ and denoted by $\text{tvsl}(G)$.

Some results about this parameter. Nurdin, et al. give lower bound of the total vertex irregularity strength of any graph as in Theorem A.

Theorem A. [10] Let $G$ be a connected graph having $n$ vertices of degree $i$ $(i = \delta, \delta + 1, \delta + 2, \ldots, \Delta)$ where $\delta$ and $\Delta$ are the minimum and the maximum degree of $G$, respectively. Then

$$\text{tvsl}(G) \geq \max \left\{ \left\lfloor \frac{\delta + n_1}{\delta + 1} \right\rfloor, \left\lfloor \frac{\delta + n_2 + n_{k+1}}{\delta + 2} \right\rfloor, \ldots, \left\lfloor \frac{\delta + \sum_{i=1}^{\Delta} n_i}{\Delta + 1} \right\rfloor \right\}.$$  

All graphs whose the total vertex irregularity strength are known support this theorem, such as caterpillar graph [11], diamond graph [6], and generalized prism graph [9]. Recently, we have found the total vertex irregularity strength of hexagonal cluster graphs [7].

Theorem 1. The dodecahedral modified generalization graph $GD_n$ has total vertex irregularity strength $\text{tvsl}(GD_n) = \left\lceil \frac{2n+3}{6} \right\rceil$ for even $n \geq 6$.

Proof. To find the lower bound we use Theorem A. Since the number vertices of degree 3 and of degree 5 is $n$, then

$$\text{tvsl}(GD_n) \geq \left\lceil \frac{2n+3}{6} \right\rceil.$$  

(1)

To prove that $\text{tvsl}(GD_n) \leq \left\lceil \frac{2n+3}{6} \right\rceil$, we construct a labelling function on $GD_n$. To constructs a labelling function on $GD_n$ we need to rename the vertex to make the formula simpler. Define

$$x_i = \begin{cases} \frac{u+1}{2} & \text{for } i \text{ is odd and } 1 \leq i \leq \frac{n}{2}, \\ \frac{u+1}{2} + 1 & \text{for } i \text{ is even and } \frac{n}{2} + 1 \leq i \leq n, \end{cases}$$

and

$$y_j = \begin{cases} \frac{v+1}{2} & \text{for } j \text{ is odd and } 1 \leq j \leq \frac{n}{2}, \\ \frac{v+1}{2} + 1 & \text{for } j \text{ is even and } \frac{n}{2} + 1 \leq j \leq n. \end{cases}$$

Thus, we have the edge set of $GD_n$ as follows:

$$E = \{x_i x_{i+1}, x_i y_i, x_i y_{i+1}, y_i y_{i+1}, y_i y_{i+2}, y_{i+1} y_{i+2} | 1 \leq i \leq n-2\}$$

$$\cup \{x_i y_{i+1} | 1 \leq i \leq n\} \cup \{y_i y_{i+1} | i = 2, n-2\} \cup \{y_{i+1} y_{i+2} | i = 1, n-3\}$$

$$\cup \{y_{i+1} y_{i+2} | i = 1, 2, \ldots, n-4\}.$$  

For $n = 6$ given a labelling $GD_n$ as shown in Fig. 2.

The labelling function on $GD_n$ has been constructed for $n \geq 8$ as follows:

$$f(x_i) = \begin{cases} 1 & \text{for } i = 1, 2, 3, \\ 2 & \text{for } i = 4, 5. \end{cases}$$

For $n = 6k + 2$ for some $k$ positive integer,

$$f(x_i) = \begin{cases} 6k + 2 & \text{for } i = 6t \text{ where } t = 1, 2, \ldots, k, \\ 6k + 4 & \text{for } i = 6t + 1 \text{ where } t = 1, 2, \ldots, k, \\ 6k + 6 & \text{for } i = 6t + 2 \text{ where } t = 1, 2, \ldots, k. \end{cases}$$

For $n = 6k + 4$ for some $k$ positive integer,

$$f(x_i) = \begin{cases} \frac{4}{3} + 1 & \text{for } i = 6t \text{ where } t = 1, 2, \ldots, k, \\ \frac{4}{3} + 2 & \text{for } i = 6t + 1 \text{ where } t = 1, 2, \ldots, k, \\ \frac{4}{3} + 3 & \text{for } i = 6t + 2 \text{ where } t = 1, 2, \ldots, k. \end{cases}$$

For $n = 6k + 6$ for some $k$ positive integer,

$$f(x_i) = \begin{cases} \frac{6}{3} + 1 & \text{for } n = 6k + 2, \\ \frac{6}{3} + 2 & \text{for } n = 6k + 4, \\ \frac{6}{3} + 3 & \text{for } n = 6k + 6. \end{cases}$$

Fig. 1. Dodecahedral graph and its modifications.

Fig. 2. The total vertex irregular 3-labelling of $GD_n$. 
\[ f(y) = \begin{cases} 1 & \text{for } i = 1, 2, \\ 2 & \text{for } i = 3, 4, \\ 3 & \text{for } i = 5. \\ \end{cases} \]

For \( n = \begin{cases} 6k + 2 \\ 6k + 4 & \text{for some } k \text{ positive integer}, \\ 6k + 6 & \end{cases} \)

\[ f(y) = \begin{cases} \frac{n}{6} + 1 & \text{for } n = 6k + 2, \\ \frac{n}{6} + 2 & \text{for } n = 6k + 4, \\ \frac{n}{6} + 3 & \text{for } n = 6k + 6. \\ \end{cases} \]

\[ f(y_n) = \begin{cases} \frac{n}{6} + 2 & \text{for } n = 6k + 2, \\ \frac{n}{6} + 2 & \text{for } n = 6k + 4, \\ \frac{n}{6} + 3 & \text{for } n = 6k + 6. \\ \end{cases} \]

\[ f(x_i) = \begin{cases} 1 & \text{for } i = 1, 2, \\ 2 & \text{for } i = 3. \\ \end{cases} \]

For \( n = \begin{cases} 6k + 2 \\ 6k + 4 & \text{for some } k \text{ positive integer}, \\ 6k + 6 & \end{cases} \)

\[ f(x_iy_i) = \begin{cases} \frac{n}{6} + 1 & \text{for } n = 6k + 2, \\ \frac{n}{6} + 1 & \text{for } n = 6k + 4, \\ \frac{n}{6} + 1 & \text{for } n = 6k + 6, \\ \end{cases} \]
For $n = \begin{cases} 6k + 2 \\ 6k + 4 \text{ where } k = 1,2,\ldots, \\ 6k + 6 \end{cases}$

$$f(y_{n-1}y_n) = f(y_{n-4}y_n) = \begin{cases} 2n + 2 & \text{for } n = 6k + 1, \\ 2n + 4 & \text{for } n = 6k + 4, \\ 2n + 6 & \text{for } n = 6k + 6. \end{cases}$$

Hence, we have weights of size at least $\left\lceil \frac{2n+1}{2} \right\rceil$ to get one vertex to have weight 2 (mod 2n).

Now we provide a construction for a modular irregular $k$-labelling with $k = \left\lceil \frac{2n+1}{2} \right\rceil$. First, we describe the graph $GD_n$ in the following way.

Now, for the collection of the $|u_{i+1}|$ edges, we label the edges from 1 to a maximum of $x - y$. Mathematically we write it as $f(u_{i+1}) = \begin{cases} i & \text{for } 0 \leq i \leq x - y - 1, \\ n - i + 1 & \text{for } n - x + y + 2 \leq j \leq n, \\ x - y & \text{for others.} \end{cases}$

Next, we label the edges of the form $e_{i+3}$ as follows,

$$f(e_{i+3}) = f(u_{i+1}) + y \text{ for all } 1 \leq j \leq n.$$
\[ n - 2x + 2y + 2 = 4y + 2 = 2n - 4x + 2 = 5 \left( \frac{2n + 2}{5} \right) - 5x + x \\
\]

For the second case, we have a minimum at \( i = n - x + y + 1 \) with \( f(v_{i}v_{i+2}) = 3 \). We also have a maximum at \( i = \frac{3}{2} + 1 \) with \( f(v_{i}v_{i+2}) = n - 2x + 2y + 1 \). Note that this is smaller than the maximum in the previous case.

Hence this is a modular irregular labelling for \( GD_n \) with maximum label \( x = \left\lceil \frac{2n+2}{5} \right\rceil \) as required. □

**Declarations**

**Author contribution statement**

Nurdin Hinding: Conceived and designed the experiments; Analyzed and interpreted the data; Wrote the paper. Kiki A. Sugeng, R novia Simanjuntak: Analyzed and interpreted the data. Nurlindah, Timothy J. Wahyudi: Performed the experiments.

**Funding statement**

Dr. Nurdin Hinding was supported by Universitas Hasanuddin [16/EI/KPT/2020].

**Data availability statement**

Data included in article/supp. material/referenced in article.

**Declaration of interests statement**

The authors declare no conflict of interest.

**Additional information**

No additional information is available for this paper.

**References**

[1] M. Bača, M. Imran, A. Semaničová-Fečcová, Irregularity and modular irregularity strength of wheels, Mathematics 9 (2021) 2710.
[2] M. Bača, S. Jendrow, M. Miller, J. Ryan, On irregular total labellings, Discrete Math. 307 (2007) 1378–1388.
[3] M. Bača, Z. Kimáková, M. Lascáková, A. Semaničová-Fečcová, The irregularity and modular irregularity strength of fan graphs, Symmetry 13 (2021) 605.
[4] M. Bača, R. Mathugumupackiam, K.M. Kathiresan, S. Ramya, Modular irregularity strength of graphs, Electron. J. Graph Theory Appl. 8 (2) (2020) 435–443.
[5] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz, F. Saba, Irregular networks, Congr. Numer. 81 (1986) 113–119.
[6] N. Hinding, D. Firmayasari, H. Banir, M. Bača, A. Semaničová-Fečcová, On irregularity strength of diamond network, AKCE Int. J. Graphs Comb. 15 (2018) 291–297.
[7] N. Hinding, H.K. Kim, N. Sunusi, R. Mise, On total vertex irregularity strength of hexagonal cluster graphs, Int. J. Math. Math. Sci. 2021 (2021) 1–9.
[8] https://mathworld.wolfram.com/PlatonicGraph.html.
[9] Muhammad Imran, Ali Ahmad, Muhammad Kamran Siddiqui, Tariq Mehmood, Total vertex irregularity strength of generalized prism graphs, J. Discrete Math. Sci. Cryptogr. (2021).
[10] Nurdin, E.T. Bankoro, A.N.M. Salman, N.N. Gaon, On the total vertex irregularity strength of trees, Discrete Math. 310 (2010) 3043–3048.
[11] Nurdin, M. Zakir, Firman, Vertex-irregular labeling and vertex-irregular total labeling on caterpillar graph, Int. J. Appl. Math. Stat. 40 (10) (2013) 99–105.
[12] K.A. Sugeng, Z.Z. Barack, N. Hinding, R. Simanjuntak, Modular irregular labeling on double-star and friendship graphs, J. Math. 2021 (4746609) (2021) 1–6.