MÖBIUS INVARIANTS IN IMAGE RECOGNITION

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Abstract. In this paper rational differential invariants are used to classify various plane shapes as well as plane domains equipped with an additional geometrical object.

1. Introduction. In this paper we give a survey of applications of Möbius invariants of conformal and projective geometries to analysis of various shapes ([5], [6]).

Consider a connected and simply connected domain $D \subset \mathbb{CP}^1$ with smooth boundary $\gamma = \partial D$. Sharon E. and Mumford D. ([15]) suggested the following scheme to study shapes of the curve $\gamma$. By the Riemann theorem there are conformal mappings $\phi_i : D \to \mathbb{D}$ and $\phi_o : D^c \to \mathbb{D}^c$ of the inner and outer parts of $D$ on the inner and outer part of the unit disk $\mathbb{D}$. Restriction of $\phi_o \circ \phi_i^{-1}$ to the unit circle gives us a diffeomorphism $\psi : S^1 \to S^1$ which is defined up to $\mathbb{PSL}_2 (\mathbb{R}) \times \mathbb{PSL}_2 (\mathbb{R})$-action on the group of diffeomorphisms $\text{Diff} (S^1) : \psi \mapsto A \circ \psi \circ B^{-1}$.

In paper ([15]) some additional conditions on $\phi_o$ were imposed which lead to the right action of $\mathbb{PSL}_2 (\mathbb{R})$ and give the homogeneous space $\text{Diff} (S^1) / \mathbb{PSL}_2 (\mathbb{R})$. In paper ([6]) these conditions were revised in such a way that they gives another spaces $\text{SO} (2) \backslash \text{Diff} (S^1) / \mathbb{PSL}_2 (\mathbb{R})$ and $\mathbb{PSL}_2 (\mathbb{R}) \backslash \text{Diff} (S^1) / \mathbb{PSL}_2 (\mathbb{R})$.

The situation becomes to be much more rigid if we consider the domains $D \subset \mathbb{CP}^1$ equipped with some geometrical object, say set of points, curves, functions etc. We call such domains decorated. Applying once more the Riemann theorem we arrive at a classification problem of the same type objects on the Lobachevsky plane with respect to Lie group isometries $\mathbb{PSL}_2 (\mathbb{R})$.

In both these cases we use differential invariants to describe the corresponding orbit spaces. Remark that the actions of the above Lie groups on the corresponding jet bundles are algebraic and it allows us to use the Lie-Tresse theorem ([10]) to describe the field of rational differential invariants. These fields separate regular orbits in jet bundles and finally we use them to get the smooth classification of the corresponding geometrical objects.

In the first part of the paper we discuss shape spaces and related to them different types of fingerprints. In all cases we give complete description of fields of rational invariants and show their relations to the Hill equations and projective structures.

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on the projective line $\mathbb{P}^1$. For the cases of regular orbits these invariants are used to get smooth classification of fingerprints.

The second part of this paper is devoted to the case of decorated domains. We analyze three types of geometrical objects: functions, differential 1-forms and foliations. For all these cases the fields of Möbius invariants are found and used to get smooth classification of corresponding objects.

2. Shape spaces and fingerprints. Let $D \subset \mathbb{C}P^1$ be connected and simply connected domain with a smooth boundary $\gamma = \partial D$ (“shape”). It follows from the Riemann theorem that there is a conformal diffeomorphism $\phi_i : D \to \mathbb{D}$ of this domain on the unit disk. For the same reasons there is a conformal diffeomorphism $\phi_o : D^c \to D^c$ of the outer domain to the complement of the unit disk in $\mathbb{C}P^1$.

Moreover, two such diffeomorphisms differ on Möbius transformations, preserving of the unit disk, $A_i, A_o \in \mathbb{P}SL_2(\mathbb{R})$:

$$\phi_i \Rightarrow A_i \circ \phi_i \quad \text{and} \quad \phi_o \Rightarrow A_o \circ \phi_o.$$ 

Considering the restriction of $\phi_o \circ \phi_i^{-1}$ on the unit circle $S^1 = \partial D$ we get an orientation preserving diffeomorphism of the circle (a “fingerprint”):

$$\psi : S^1 \to S^1.$$ 

Different choices of the diffeomorphisms $\phi_i$ and $\phi_o$ lead us to the action

$$\psi \to A_o \circ \psi \circ A_i^{-1} \quad (1)$$

of the group $\mathbb{P}SL_2(\mathbb{R}) \times \mathbb{P}SL_2(\mathbb{R})$ on the group $\text{Diff} (S^1)$ of orientation preserving diffeomorphisms of the unit circle.

Recall that the Möbius transformations from the $\mathbb{P}SL_2(\mathbb{R})$ have the form

$$z \in S^1 \mapsto \frac{az + b}{bz + a} \in S^1,$$

where $a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1$.

The isomorphism of the group of matrices $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ with the standard group $SL_2(\mathbb{R})$ is given by the adjunction

$$X = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mapsto A^{-1} X A = \begin{pmatrix} Re(a + b) & Im(a - b) \\ -Im(a + b) & Re(a - b) \end{pmatrix} \in SL_2(\mathbb{R}),$$

where

$$A = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$ 

The corresponding to (2) Lie algebra $\mathfrak{sl}_2(\mathbb{R})$-action given by vector fields:

$$\mathfrak{sl}_2(\mathbb{R}) = \langle \cos^2(\phi) \partial_\phi, \sin(2\phi) \partial_\phi, \sin^2(\phi) \partial_\phi \rangle,$$

where $\phi$ is the angle.

It is worth to note that under the stereoprojection diffeomorphism $S^1 \to \mathbb{R}P^1$ the $\mathbb{P}SL_2(\mathbb{R})$-action (2) becomes to be the action by fractional transformation and the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$-action is given by the vector fields $\langle \partial_t, t \partial_t, t^2 \partial_t \rangle$, where $t$ is an affine coordinate.

We’ll consider the following specifications of action (1):

$$\langle \partial_1, t \partial_1, t^2 \partial_1 \rangle,$$
A-fingerprints: They were introduced by Mumford and Sharon (see, [15]) and they are defined by requirement $\phi_0(\infty) = \infty$ and $\phi_0'(\infty) \in \mathbb{R}$. In this case the space of a-fingerprints is the smooth Teichmüller space $\text{Diff}(S^1)/\text{PSL}_2(\mathbb{R})$ and the fingerprints define shapes up to translations and magnifications.

B-fingerprints: They were introduced in ([5]) and they are defined by requirement $\phi_0(\infty) = \infty$. In this case the corresponding space of fingerprints is the double quotient $\text{SO}(2) \setminus \text{Diff}(S^1)/\text{PSL}_2(\mathbb{R})$ and the fingerprints define shapes up to similarity.

C-fingerprints: They were introduced in ([5]) and they do not impose any requirements on $\phi_0$. In this case the corresponding space of fingerprints is the double quotient $\text{PSL}_2(\mathbb{R}) \setminus \text{Diff}(S^1)/\text{PSL}_2(\mathbb{R})$ and the fingerprints define shapes up to Möbius transformations.

2.1. Differential invariants. Denote by $J^k$ the manifolds of $k$-jets of preserving orientation diffeomorphisms of the circle. Then action (1) of Lie group $\text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})$ induces the corresponding action on manifolds $J^k$. These manifolds are algebraic and the action is algebraic too.

A rational function on manifold $J^k$ we call differential invariant (of the corresponding action) of order $\leq k$ if it is an invariant of the corresponding action.

Due to Lie-Tresse theorem (see, [10]), the fields of invariants have a number of basic invariants and invariant derivations such that any invariant is a rational function of invariant derivatives of the basic invariants. Moreover, these fields separate regular orbits.

Below we use standard coordinates $(\phi, \psi_0, \ldots, \psi_k)$ in the $k$-jet manifolds. Thus, $k$-jet of diffeomorphism $f: \phi \mapsto \psi = f(\phi)$ has coordinates $(\phi, f(\phi), f'(\phi), \ldots, f^{(k)}(\phi))$.

2.1.1. Differential invariants for a-fingerprints. This case corresponds to the right $\text{PSL}_2(\mathbb{R})$-action

$$f \rightarrow f \circ A^{-1},$$

$A \in \text{PSL}_2(\mathbb{R})$.

It has ([5]) one basic (local) invariant of order 0 :

$$J_0 = \psi_0,$$

one basic invariant of order 3 (Schwartzian):

$$J_3 = \psi_1^{-4} \left( \psi_1 \psi_3 - \frac{3}{2} \psi_2^2 - 2 \psi_1^2 \right),$$

and invariant derivation

$$\nabla: J \rightarrow \psi_1^{-1} \frac{dJ}{d\phi},$$

where $\frac{d}{d\phi}$ is the total derivation.

Applying the Lie-Tresse theorem, we get the following result.

**Theorem 2.1.** ([5]) The field of rational differential invariants of $\text{PSL}_2(\mathbb{R})$-action ([6]) is generated over the field of rational functions in $(\cos \psi_0, \sin \psi_0)$ by the differential invariant $J_3$ of order 3 and by the invariant derivation $\nabla$. That is, any rational differential invariant of order $\leq k$ is a rational function in

$$\cos \psi_0, \sin \psi_0, J_3, \nabla(J_3), \nabla^2(J_3), \ldots, \nabla^{k-3}(J_3).$$
2.1.2. Differential invariants for b-fingerprints. This case corresponds to the $\text{SO}(2) \times \mathbb{PSL}_2(\mathbb{R})$-action:

$$f \to B \circ f \circ A^{-1},$$ (7)

where $A \in \mathbb{PSL}_2(\mathbb{R}), B \in \text{SO}(2)$.

It is also corresponds to the action of Lie algebra generated by

$$\mathfrak{sl}_2 \oplus \mathbb{R} = \langle \cos^2(\phi) \partial_\phi, \sin(2\phi) \partial_\phi, \sin^2(\phi) \partial_\phi, \partial_{\psi_0} \rangle.$$

Therefore, differential invariants of this action are invariants of action $I_3$, which are, in addition, invariant with respect to rotation $\partial_{\psi_0}$. Then the above theorem gives us the following result.

**Theorem 2.2.** (5) The field of rational differential invariants of $\text{SO}(2) \times \mathbb{PSL}_2(\mathbb{R})$-action (7) is generated by the differential invariant $J_3$ of order 3 and by the invariant derivation $\nabla$. That is, any rational differential invariant of order $\leq k$ is a rational function in $J_3$, $\nabla(J_3)$, $\nabla^2(J_3)$, ..., $\nabla^{k-3}(J_3)$.

All orbits in $\mathbf{J}^k$ are $\mathbb{PSL}_2(\mathbb{R})$-regular and the field of rational differential invariants separates them.

2.1.3. Differential invariants for c-fingerprints. This case corresponds to the $\mathbb{PSL}_2(\mathbb{R}) \times \mathbb{PSL}_2(\mathbb{R})$-action:

$$f \to B \circ f \circ A^{-1},$$

where $A, B \in \mathbb{PSL}_2(\mathbb{R})$.

The corresponding Lie algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$-action is given by the following vector fields:

$$\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 = \langle \cos^2(\phi) \partial_\phi, \sin(2\phi) \partial_\phi, \sin^2(\phi) \partial_\phi, \cos^2(\psi_0) \partial_{\psi_0}, \sin(2\psi_0) \partial_{\psi_0}, \sin^2(\psi_0) \partial_{\psi_0} \rangle.$$

The field of rational differential $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$-invariants is generated by differential invariants of action (6) which are invariants of $\mathbb{PSL}_2(\mathbb{R})$-action generated by vector fields

$$\mathfrak{sl}_2 = \langle \cos^2(\psi_0) \partial_{\psi_0}, \sin(2\psi_0) \partial_{\psi_0}, \sin^2(\psi_0) \partial_{\psi_0} \rangle.$$

**Theorem 2.3.** (5) The field of rational differential invariants of $\mathbb{PSL}_2(\mathbb{R}) \times \mathbb{PSL}_2(\mathbb{R})$-action (7) is generated by the differential invariant $I_5$ of order 5:

$$I_5 = \theta^{-2} \theta_2 - \frac{5}{4} \theta^{-3} \theta_1^2 - 4\theta^{-1},$$

where

$$\theta = J_3 + 2, \text{ and } \theta_1 = \nabla^i(\theta),$$

and invariant derivation $\delta$ of order 6:

$$\delta = \left( \theta^{-3} \theta_3 - \frac{9}{2} \theta^{-4} \theta_1 \theta_2 + \frac{15}{4} \theta^{-5} \theta_1^3 + 4\theta^{-3}\theta_1 \right) \nabla.$$

That is, any rational differential invariant is a rational function in $I_5, \delta(I_5), \delta^2(I_5), \ldots, \delta^k(I_5), \ldots$

This field separates regular orbits.

2.2. Equivalence. In this section we show how to use differential invariants of the above actions to classify circle diffeomorphisms.
2.2.1. A-fingerprints, Hill equations and projective structures.

Projective structures.

A projective structure on an interval $U \subset \mathbb{R}P^1$ consists of a covering $U$ by charts $(U_\alpha, t_\alpha)$, where $t_\alpha$ is a local coordinate on $U_\alpha$, such that on intersections $U_\alpha \cap U_\beta$ these coordinates are connected by fractional transformations:

$$t_\beta = \frac{a_\beta \alpha t_\alpha + b_\beta \alpha}{c_\beta \alpha t_\alpha + d_\beta \alpha},$$

with

$$\begin{bmatrix} a_\beta \alpha & b_\beta \alpha \\ c_\beta \alpha & d_\beta \alpha \end{bmatrix} \in \text{SL}_2(\mathbb{R}).$$

(8)

The standard projective structure on $\mathbb{R}P^1$ has the group $\text{SL}_2(\mathbb{R})$ as the group of projective transformations, with its Lie algebra $\text{sl}_2$ generated by vector fields:

$$\text{sl}_2 = \langle \partial_t, t\partial_t, t^2\partial_t \rangle,$$

where $t$ is an affine coordinate.

On the other hand, due to Sophus Lie theorem, any Lie algebra of vector fields on an interval $U$, which isomorphic to $\text{sl}_2$, is locally isomorphic to Lie algebra $\langle \partial_s, s\partial_s, s^2\partial_s \rangle$, for some local coordinate $s$. Moreover, any two such realizations are connected by fractional transformation of the corresponding local parameters.

Therefore, the above definition of the projective structure could be reformulated in the following way: for each $\alpha$ we define the Lie algebra $\mathfrak{g}_\alpha$ of vector fields on interval $U_\alpha$:

$$\mathfrak{g}_\alpha = \langle \partial_{t_\alpha}, t_{\alpha}\partial_{t_\alpha}, t_{\alpha}^2\partial_{t_\alpha} \rangle.$$

Then, the requirement that $t_\alpha$ and $t_\beta$ on the interval $U_\alpha \cap U_\beta$ are connected by a fractional transformation simply means that

$$\mathfrak{g}_\alpha = \mathfrak{g}_\beta \quad \text{on} \quad U_\alpha \cap U_\beta.$$

In other words, a projective structure could be defined by a covering $U_\alpha$ together with simple Lie algebras $\mathfrak{g}_\alpha$ of vector fields on $U_\alpha$ such that $\dim \mathfrak{g}_3 = 3$ and $\mathfrak{g}_\alpha = \mathfrak{g}_\beta$ on $U_\alpha \cap U_\beta$.

Hill equations.

Consider a Lie algebra $\mathfrak{g}$ of vector fields on an interval $U$, which is isomorphic to $\text{sl}_2$.

Let $A$, $B$, $H$ be the Chevalley basis in $\mathfrak{g}$:

$$[A, B] = H, \quad [H, A] = -2A, \quad [H, B] = 2B,$$

(9)

and let

$$A = a(t) \partial_t, \quad B = b(t) \partial_t, \quad H = h(t) \partial_t.$$

Then commutation relations (9) takes the form:

$$h = a'b - ab', \quad -2a = ha' - h'a, \quad 2b = hb' - h'b.$$

(10)

Multiplying the first equation by $h$, the second equation by $(-b)$, and the last one by $a$, we get $h^2 = 2ab$.

It could be shown (see, (7) for more details), that functions $a$ and $b$ can be chosen to be positive on the interval. Hence,

$$a = f^2, \quad b = g^2, \quad h = 2fg,$$

for some smooth functions $f$ and $g$. 

In this case system (10) of differential equations is equivalent to a single equation:
\[ f'g - fg' = 1. \]

Putting
\[ W = f'' \frac{f'}{f} - g'' \frac{g'}{g}, \]
we get the following result.

**Theorem 2.4.** (17) Any subalgebra \( g \cong \mathfrak{sl}_2 \) of vector fields on an interval \( U \) has the form
\[ g = \langle f^2 \partial_t, 2fg\partial_t, g^2 \partial_t \rangle, \]
where \( f \) and \( g \) is a fundamental system of solution of an equation
\[ y'' - Wy = 0, \quad (11) \]
with Wronskian \( f'g - fg' = 1. \)

**Remark 1.**
- Functions \( f^2, 2fg, g^2 \) also give a fundamental system of solutions for differential equation
\[ z''' - 4Wz' - 2W'z = 0. \]
(12)

Thus, elements of Lie algebra \( g \) are vector fields \( z(t) \partial_t \), where \( z(t) \) are solutions of (12).
- The standard projective structure on \( \mathbb{R}P^1 \) corresponds to the trivial potential \( W = 0 \).
- The diffeomorphism \( \eta : \mathbb{R}P^1 \rightarrow S^1 \), \( \eta : [x : y] \mapsto 2 \arctan \left( \frac{y}{x} \right) \) transforms the standard projective structure on \( \mathbb{R}P^1 \) to projective structure on the circle that corresponds to Lie algebra (15).

Consider an interval \( U \) equipped with the Lie algebra \( g = \langle f^2 \partial_t, 2fg\partial_t, g^2 \partial_t \rangle \) of vector fields.

Let’s choose the following covering of the interval by charts of two types: \((V, t_1 = \frac{2\pi}{\lambda})\), where \( f \neq 0 \), and \((V, t_2 = \frac{2\pi}{\lambda})\), where \( g \neq 0 \).

It is easy to check that \( g = \langle \partial_{t_1}, 2t_1 \partial_{t_1}, t_1^2 \partial_{t_1} \rangle \) in \( t_1 \)-charts and \( g = \langle -t_2^2 \partial_{t_2}, -2t_2 \partial_{t_2}, -\partial_{t_2} \rangle \) in \( t_2 \)-charts.

Therefore, the potential \( W \) defines a projective structure on the interval.

Let now \( f : S^1 \rightarrow S^1 \) be an orientation preserving diffeomorphism, and let \( \eta^{-1} \circ f : S^1 \rightarrow \mathbb{R}P^1 \), be a map of the form
\[ \eta^{-1} \circ f : \phi \mapsto [x(\phi) : y(\phi)] \]
for some periodic functions \( x \) and \( y \).

Then, because \( f' > 0 \), one could choose a function \( \lambda(\phi) > 0 \), such that functions
\[ X = \lambda x, \quad Y = \lambda y \]
satisfy relation
\[ X'Y - XY' = 1. \]

Then, \( X'' = Y'' \) and we get the following result.

**Lemma 2.5.** Any orientation preserving diffeomorphism \( f : S^1 \rightarrow S^1 \) can be presented in the form
\[ f : \phi \mapsto 2 \arctan \left( \frac{u(\phi)}{v(\phi)} \right), \]
where \( u(\phi) \) and \( v(\phi) \) is a fundamental solution of a Hill equation
\[ y'' - Wy = 0 \]
on the circle, with Wronskian $u'v - uv' = 1$.

Change of the fundamental system corresponds to $\mathbb{PSL}_2(\mathbb{R})$-action

$$f \rightarrow A \circ f,$$

where $A \in \mathbb{PSL}_2(\mathbb{R})$.

**Equivalence of A-fingerprints.**

In this section we use the representation of the circle diffeomorphisms, given in the above lemma, to get a classification of a-fingerprints.

Let's consider an orientation preserving diffeomorphism $f : S^1 \rightarrow S^1$, $f : \phi \rightarrow \psi = f(\phi)$ and let the inverse diffeomorphism be of the form:

$$\psi \mapsto \phi = 2 \arctan \left( \frac{u(\psi)}{v(\psi)} \right),$$

where functions $u$ and $v$ is a fundamental solution of some Hill equation

$$y'' - W(\psi) y = 0,$$

with Wronskian equals 1.

Then the straightforward computations show that the value of differential invariant $J_3$ on $f$ is proportional to the potential:

$$J_3(f) = 2W(f).$$

Therefore, given diffeomorphism $f$, we’ll find the potential $W(\psi)$, using formula (16).

This potential defines the inverse $f^{-1}$ up to $\mathbb{PSL}_2(\mathbb{R})$-action (14):

$$f^{-1} \rightarrow A \circ f^{-1},$$

or defines the diffeomorphism $f$ itself up to $\mathbb{PSL}_2(\mathbb{R})$-action (6).

Summarizing, we get the following result.

**Theorem 2.6.** The potential $W$, given by formula (16), defines a-fingerprints up to $\mathbb{PSL}_2(\mathbb{R})$-action (6). That is, two $\mathbb{PSL}_2(\mathbb{R})$-equivalent diffeomorphisms have the same potentials and if two diffeomorphisms have the same potential then they are $\mathbb{PSL}_2(\mathbb{R})$-equivalent.

Remark also that the functions $W$, which we call potentials of the corresponding diffeomorphisms, are not arbitrary. First of all, the corresponding Hill equations should have the trivial monodromy. Secondly, mappings (15) are local diffeomorphisms $S^1 \rightarrow S^1$, for monodromy free potentials, and to have diffeomorphism, one should require that their degree equals one.

**Remark 2.** Comparing solutions of the Hill equation with solutions of the base equation $y'' + y = 0$ and using the Sturm comparison theorem ([3]) we see that non trivial solutions of the Hill equation have at least two zeroes if $W < -1$ or $J_3(f) + 2 < 0$.

**Equivalence of B-fingerprints.**

Any diffeomorphism $f : S^1 \rightarrow S^1$ defines a map

$$\sigma_f : S^1 \rightarrow \mathbb{R}^2,$$

$$\sigma_f = (J_3(f), J_4(f)),$$

where $J_4(f) = \nabla(J_3(f))$, $J_5(f) = \nabla(J_4(f))$, etc.
The differential of this map is non trivial if
\[ J^2_\sigma(f) + J^2_{\bar{\sigma}}(f) \neq 0, \]
and mapping \( \sigma_f \) is embedding in this case.

We'll assume in addition that \( \sigma_f \) is a diffeomorphism of \( S^1 \) on \( \Sigma_f = \text{Im} \sigma_f \).

Then the following result holds (5).

**Theorem 2.7.**
1. If diffeomorphisms \( f : S^1 \to S^1 \) and \( g : S^1 \to S^1 \) are \( \text{SO}(2) \times \text{PSL}_2(\mathbb{R}) \)-equivalent, then \( \Sigma_f = \Sigma_g \).
2. If \( \Sigma_f = \Sigma_g = \Sigma \) and mappings \( \sigma_f \) and \( \sigma_g \) are diffeomorphisms between \( S_1 \) and \( \Sigma \), then diffeomorphisms \( f \) and \( g \) are \( \text{SO}(2) \times \text{PSL}_2(\mathbb{R}) \)-equivalent.

**Equivalence of C-fingerprints.**

In this case we associate mapping \( \bar{\sigma}_f : S^1 \to \mathbb{R}^2 \) with a diffeomorphism \( \sigma : S^1 \to S^1 \) as follows:
\[
\bar{\sigma}_f = (I_6(f), I_7(f)),
\]
where \( I_6(f) = \delta(I_5(f)), I_7(f) = \delta(I_6(f)), \ldots, \) etc.

For this case the differential of this map is non trivial if
\[ I^2_6(f) + I^2_7(f) \neq 0, \]
and we assume in addition that \( \bar{\sigma}_f \) is a diffeomorphism of \( S^1 \) on \( \bar{\Sigma}_f = \text{Im} \bar{\sigma}_f \).

**Theorem 2.8.**
1. If diffeomorphisms \( f : S^1 \to S^1 \) and \( g : S^1 \to S^1 \) are \( \text{PSL}_2(\mathbb{R}) \)-equivalent, then \( \bar{\Sigma}_f = \bar{\Sigma}_g \).
2. If \( \bar{\Sigma}_f = \bar{\Sigma}_g = \bar{\Sigma} \) and mappings \( \bar{\sigma}_f \) and \( \bar{\sigma}_g \) are diffeomorphisms between \( S_1 \) and \( \bar{\Sigma} \), then diffeomorphisms \( f \) and \( g \) are \( \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R}) \)-equivalent.

3. **Decorated shapes.** In the previous section we've studied plane simply connected domains with smooth boundaries up to conformal diffeomorphisms. In this section we study decorated domains, i.e. connected and simply connected domains with smooth boundaries equipped with some geometrical objects such as set of points, function, differential form, foliation, etc. We'll consider such domains up to conformal equivalence too.

First of all, by using the Riemann conformal mapping theorem, we establish conformal isomorphism this domain with the unit disk. Secondly, two such conformal diffeomorphisms differ on conformal transformation that preserve the unit disk, i.e. on transformations from the Möbius group \( \text{PSL}_2(\mathbb{R}) \), and, therefore, conformal classification of decorated domains transforms to the classification problem of the corresponding geometrical objects on the Lobachevsky plane with respect to Möbius group \( \text{PSL}_2(\mathbb{R}) \), or the group of isometries of the Lobachevsky plane.

For example, if we consider finite sets of points, then the classification problem becomes to be a part of the classical problem on classification of binary forms (11).

After such a reformulation we use the Lie-Tresse theorem to find (we call them Möbius) rational differential invariants of the \( \text{PSL}_2(\mathbb{R}) \)-actions. These fields of Möbius differential invariants separates regular \( \text{PSL}_2(\mathbb{R}) \)-orbits in the corresponding jet spaces and give us complete solutions of the of the equivalence problems in regular cases.

**Lobachevskian geometry.**

Let \( D \subset \mathbb{C}, D = \{x^2 + y^2 < 1\} \), be the unit disk. The Möbius group \( \text{PSL}_2(\mathbb{R}) \) acts on \( D \) in the following way:
\[ z \in \mathbb{D} \mapsto \frac{az + b}{\overline{bz} + \overline{a}} \in \mathbb{D}, \]

where \( a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1. \)

The corresponding representation of Lie algebra \( \mathfrak{sl}_2 \) is given by the following vector fields:

\[ X = y \partial_x - x \partial_y, \quad Y = t \partial_x + 2xy \partial_y, \quad Z = -2xy \partial_x + t \partial_y, \quad (17) \]

where \( t = 1 - x^2 - y^2. \)

It is well known that the Möbius transformations of \( \mathbb{PSL}_2(\mathbb{R}) \) are isometries of the metric

\[ g = \frac{dx^2 + dy^2}{t^2}, \]

and preserve, in addition, the symplectic form

\[ \Omega = \frac{dx \wedge dy}{t^2}, \]

and complex structure

\[ I = \partial_y \otimes dx - \partial_x \otimes dy. \quad (18) \]

3.1. Functions. Denote by \( J^k \) the manifolds of \( k \)-jets of smooth functions, defined on the unit disk \( \mathbb{D} \), and by \((x, y, u, u_{10}, u_{01}, u_{11}, u_{12}, u_{22}, \ldots)\) we denote the standard canonical coordinates in \( J^k \). The value of \( u_{ij} \) at the \( k \)-jet \([f]_a^k\) of function \( f \) at point \( a \in \mathbb{D} \) is equal to \( \frac{\partial^{i+j} f}{\partial x^i \partial y^j} (a) \).

Remark that the Möbius group \( \mathbb{PSL}_2(\mathbb{R}) \) is algebraic, manifolds of \( k \)-jets \( J^k \) are algebraic and irreducible, and \( \mathbb{PSL}_2(\mathbb{R}) \)-action in \( J^k \) is algebraic too.

By Möbius differential invariant of order \( \leq k \) we mean rational function defined on \( J^k \) and invariant under induced \( \mathbb{PSL}_2(\mathbb{R}) \)-action. Due to Lie-Tresse theorem the fields of Möbius differential invariants separate regular orbits in the jet spaces, and the field of all Möbius differential invariants is generated by some number basic invariants and invariant derivations.

By invariant derivation here we mean a total derivation of the form

\[ A \frac{d}{dx} + B \frac{d}{dy}, \]

which commutes with \( \mathbb{PSL}_2(\mathbb{R}) \)-action.

Here \( A, B \) are rational functions on some \( k \)-jet space \( J^k \) and \( \frac{d}{dx}, \frac{d}{dy} \) are the total derivations in \( x \) and \( y \) respectively.

3.1.1. Invariant frames. First of all, let’s remark that the Liouville differential 1-form on the manifold 1-jets:

\[ \rho = u_{10} dx + u_{01} dy \]

is invariant under prolongations of all diffeomorphisms of the unit disk.

Secondly, using invariant complex structure \( I \), (18), we get another differential 1-form on the 1-jet space:

\[ I\rho = -u_{01} dx + u_{10} dy, \]

which is a conformal invariant.

Therefore, horizontal differential 1-forms

\[ \omega_1 = \rho, \quad \omega_2 = I\rho \]
define a $\mathbb{PSL}_2(\mathbb{R})$-invariant coframe in the domain, where
\[ T = u_{10}^2 + u_{01}^2 \neq 0. \]

The structure equations for this coframe has the form
\[ \hat{d}\omega_1 = 0, \]
\[ \hat{d}\omega_1 = J_2\omega_1 \wedge \omega_2, \]
where $\hat{d}$ is the total differential and
\[ J_2 = \frac{u_{20} + u_{02}}{u_{10}^2 + u_{01}^2} \]
is a second order M"obius invariant.

The frame, dual to coframe $(\omega_1, \omega_2)$, is formed by the following $\mathbb{PSL}_2(\mathbb{R})$-invariant derivations:
\[ \nabla_1 = \frac{u_{10}}{u_{10}^2 + u_{01}^2} \frac{d}{dx} + \frac{u_{01}}{u_{10}^2 + u_{01}^2} \frac{d}{dy}, \]
\[ \nabla_2 = \frac{-u_{01}}{u_{10}^2 + u_{01}^2} \frac{d}{dx} + \frac{u_{10}}{u_{10}^2 + u_{01}^2} \frac{d}{dy}. \]
These derivations are satisfied the following commutation relation:
\[ [\nabla_1, \nabla_2] + J_2\nabla_2 = 0. \]

### 3.1.2. M"obius invariants for functions.

Two invariants of order 0 and 1 are obvious. They are
\[ J_0 = u, \]
\[ J_1 = g(\omega_1, \omega_1) = \left(1 - x^2 - y^2\right)^2 \left(u_{10}^2 + u_{01}^2\right). \]

Therefore, dimensions of regular orbits in $J^k$, when $k \geq 1$, equal 3 and we expect $(k + 1)$ independent invariants of pure order $k$, when $k \geq 2$.

In order 2 we have 3 independent invariants:
\[ J_2, \quad J_{11} = \nabla_1(J_1), \quad J_{12} = \nabla_2(J_1), \]
where
\[ J_{11} = \frac{2u_{10}u_{20} + 2u_{10}u_{01}u_{11} + u_{01}^3u_{02} + 2(1 - x^2 - y^2) (xu_{10} + yu_{01})}{u_{10}^2 + u_{01}^2}, \]
\[ J_{21} = \frac{2u_{10}u_{01} (u_{02} - u_{20}) + (u_{10}^2 - u_{01}^2) u_{11} + 2(1 - x^2 - y^2) (yu_{10} - xu_{01})}{u_{10}^2 + u_{01}^2}. \]

All other invariants we can get by taking invariant derivatives of invariants $J_1$ and $J_2$. Thus in order $k$, we get $k$ invariants by taking invariant derivatives of $J_1$ of order $(k - 1)$ and $(k - 1)$ invariants by taking invariant derivatives of $J_2$ of order $(k - 2)$. Therefore, we get $(2k - 1) - (k + 1) = k - 2$ syzygies in pure order $k$. In particular, we have only one syzygy in order 3.

Hence, all syzygy could be obtained by taking invariant derivatives of the syzygy of order 3.

**Theorem 3.1.** The field of M"obius differential invariants for the $\mathbb{PSL}_2(\mathbb{R})$-action on the jet spaces of functions on the unit disk is generated by invariants $J_0, J_1, J_2$ and their invariant derivatives. This field separates orbits, where $u_{10}^2 + u_{01}^2 \neq 0$. 
3.1.3. Möbius equivalence of functions.

Tresse derivatives.

Assume that given two functions $F_1$ and $F_2$ on the $k$-jet space, such that their total differentials

$$\hat{d}F_i = \frac{dF_i}{dx} dx + \frac{dF_i}{dy} dy$$

are linear independent in a domain $O$ in $J^{k+1}$:

$$\hat{d}F_1 \wedge \hat{d}F_2 \neq 0.$$ 

Then, for the total differential $\hat{d}G$ of any function on jet space $J^l$, one has decomposition,

$$\hat{d}G = \lambda_1 \hat{d}F_1 + \lambda_2 \hat{d}F_2,$$

over domain $O$.

These functions $\lambda_i$, which are functions on $J^m$, where $m = \max(k+1,l+1)$, are called Tresse derivatives and will be denoted as $DG/DF_i$.

Remark that under condition that functions $F_1$, $F_2$ and $G$ are invariants their Tresse derivatives are invariants also.

In other words, derivations

$$\frac{D}{DF_i} : G \mapsto \frac{DG}{DF_i}$$

are invariant.

In our case we have also two invariant derivations $\nabla_1$ and $\nabla_2$.

Therefore Tresse derivatives $\frac{D}{DJ_0}, \frac{D}{DJ_1}$ are linear combinations of $\nabla_1, \nabla_2$, and vice versa.

One has

$$\frac{D}{DJ_0} = \nabla_1 - \frac{J_{11}}{J_{12}} \nabla_2, \quad \frac{D}{DJ_1} = \frac{1}{J_{12}} \nabla_2,$$

and

$$\nabla_1 = \frac{D}{DJ_0} + J_{11} \frac{D}{DJ_1}, \quad \nabla_2 = J_{12} \frac{D}{DJ_1},$$

in domain where $J_1 \neq 0$ and $J_{12} \neq 0$.

The above theorem, therefore, could be reformulated in terms of Tresse derivatives in the following way.

**Theorem 3.2.** The field of Möbius invariants of functions, defined on the unit disk, is generated by invariants $(J_0, J_1, J_2, J_{11}, J_{12})$ and their Tresse derivatives [19].

We say that a function $f$ on the unit disk is regular if values of differential invariants $J_1(f)$ and $J_{12}(f)$ do not take zero values.

It is easy to check that for regular functions the mapping

$$\sigma_f : \mathbb{D} \to \mathbb{R}^2,$$

$$\sigma_f = (J_0(f), J_1(f))$$

is a local diffeomorphism.

Let $\Sigma_f = \text{Im} \sigma_f$. 
Therefore, for regular functions, value of invariants of the second order, \(J_2, J_{11}, J_{12}\), are functions of \((J_0 (f), J_1 (f)):\)

\[
J_2 = A (J_0 (f), J_1 (f)), \quad J_{11} = B (J_0 (f), J_1 (f)), \quad J_{12} = C (J_0 (f), J_1 (f)).
\]

Let’s consider system (20) as a system of differential equations of the second order:

\[
J_2 = A (J_0, J_1), \quad J_{11} = B (J_0, J_1), \quad J_{12} = C (J_0, J_1).
\]

This is overdetermined system of the Frobenius type, having symmetry group \(\mathbb{PSL}_2 (\mathbb{R})\) and the function \(f\) is one of its solutions.

We show that (20) is an automorphic system in the sense that the symmetry group acts in a transitive way on the solution space.

First of all remark, that system (20) is completely integrable, and the integrability conditions are exactly the syzygy for differential invariants, which are satisfied by the construction of functions \(A, B, C\).

The solution space of this system could be identified with space \(J^1 a\) of 1-jets of functions at a point \(a \in \mathbb{D}\).

Assume that \(g\) is another regular function and solution of (21) and let \(\Sigma_f = \Sigma_g\). Then, up to transformation from \(\mathbb{PSL}_2 (\mathbb{R})\), we can assume that \(f (a) = g (a)\) and \(J_1 (f) (a) = J_1 (g) (a)\).

Stationary subgroup of the point \(a\) acts in a transitive way on \(T^*_a \mathbb{D} \setminus \{0\}\) and therefore there exist an element of group \(\mathbb{PSL}_2 (\mathbb{R})\) that transforms \(f\) to \(g\).

**Theorem 3.3.** Two regular functions \(f\) and \(g\) are \(\mathbb{PSL}_2 (\mathbb{R})\)-equivalent if and only if they have the same representative functions \(A, B, C\) and \(\Sigma_f = \Sigma_g\).

**3.1.4. Möbius invariants for differential 1-forms.** Here we discuss the conformal equivalence of simply connected domains \(D \subset \mathbb{C}\) equipped with differential 1-forms \(\theta \in \Omega^1 (D)\).

As above this problem is equivalent to the problem of classification of differential 1-forms defined on the unit disk \(D\) with respect to the Möbius group \(\mathbb{PSL}_2 (\mathbb{R})\).

Let’s denote by \(\tau^* : T^* \mathbb{D} \rightarrow \mathbb{D}\) the cotangent bundle over the unit disk with canonical coordinates \((x, y, u, v)\) and denote by \(J^k (\tau^*)\) the bundle of \(k\)-jets of sections of \(\tau^*\) (differential 1-forms) with standard canonical coordinates \((x, y, u, v, u_{10}, u_{01}, v_{10}, u_{011}, ... )\).

Thus differential 1-form \(\theta = a (x, y) dx + b (x, y) dy\) defines a section \(S_\theta : (x, y) \mapsto (u = a (x, y), v = b (x, y))\) of the bundle \(\tau^*\) and its \(k\)-jet prolongation \(u_{ij} = \frac{\partial^{i+j} \theta}{\partial x^i \partial y^j} (x, y), \quad \nu_{ij} = \frac{\partial^{i+j} \mu}{\partial x^i \partial y^j} (x, y)\).

The universal Liouville form \(\rho = u dx + v dy \in \Omega^1 (T^* \mathbb{D})\) gives the isomorphism between sections of the cotangent bundle \(\tau^*\) and differential forms on the unit disk \(D\): \(\theta = S^*_\rho (\rho)\).

The \(\mathbb{PSL}_2 (\mathbb{R})\)-action on the unit disk has the natural prolongation to \(\mathbb{PSL}_2 (\mathbb{R})\)-action on the cotangent bundle \(T^* \mathbb{D}\).

The correspondent realization of this action by vector fields is given by the following Hamiltonian vector fields:

\[
\begin{align*}
\mathcal{X} &= X - v \partial_u + u \partial_v, \\
\mathcal{Y} &= Y - 2 (xu + yv) \partial_u - 2 (xv - yu) \partial_u, \\
\mathcal{Z} &= Z + 2 (xv - yu) \partial_u - 2 (xu + yv) \partial_v,
\end{align*}
\]

where vectors fields \((X, Y, Z)\) correspond to \(\mathbb{PSL}_2 (\mathbb{R})\)-action (17) on \(\mathbb{D}\).
As above, using the Liouville form \( \rho \) and its image \( I \rho \) we get \( \text{PSL}_2(\mathbb{R}) \)-invariant coframe on \( T^* \mathbb{D} \):

\[
\begin{align*}
\omega_1 &= \rho = u dx + vdy, \\
\omega_2 &= I \rho = -v dx +udy.
\end{align*}
\]

(22)

(23)

The structure equations for this coframe are

\[
\hat{d} \omega_1 = \frac{v_{10} - u_{01}}{u^2 + v^2} \omega_1 \wedge \omega_2,
\]

\[
\hat{d} \omega_2 = \frac{u_{10} + v_{01}}{u^2 + v^2} \omega_1 \wedge \omega_2.
\]

They give us two Möbius invariants of order one:

\[
\begin{align*}
J_{11} &= \frac{v_{10} - u_{01}}{u^2 + v^2}, \\
J_{12} &= \frac{u_{10} + v_{01}}{u^2 + v^2}.
\end{align*}
\]

In addition to them we have also obvious invariant of order zero:

\[
J_0 = g(\rho, \rho) = (1 - x^2 - y^2)^2 \left( u^2 + v^2 \right).
\]

This invariant separates regular orbits in \( J^0 \tau^* \).

The frame, dual to coframe (22), has the form

\[
\begin{align*}
\delta_1 &= \frac{u}{u^2 + v^2} \frac{d}{dx} + \frac{v}{u^2 + v^2} \frac{d}{dy}, \\
\delta_2 &= -\frac{v}{u^2 + v^2} \frac{d}{dx} + \frac{u}{u^2 + v^2} \frac{d}{dy}.
\end{align*}
\]

Dimensions of regular orbits in \( J^0 \tau^* \) equal \( \text{dim} \ \text{PSL}_2(\mathbb{R}) = 3 \). Therefore, in pure order 1 we expect 4 independent invariants and they are

\[
\begin{align*}
J_{11}, \quad J_{12}, \quad \delta_1 (J_0), \quad \delta_2 (J_0).
\end{align*}
\]

Applying the invariant derivations to them we get 7 invariants of pure order 2 and therefore one syzygy in order 2.

Straightforward computations show that this syzygy is

\[
2J_0^2 \left( \delta_2 \left( J_{11}^2 \right) + \delta_1 \left( J_{12}^2 \right) + J_{11}^2 + J_{12}^2 \right) +
+ J_0 \left( \delta_1^2 + \delta_2^2 + J_{11} \delta_2 - J_{12} \delta_1 \right) (J_0) + \delta_1 (J_0)^2 + \delta_2 (J_0)^2 - 8J_0 = 0.
\]

(24)

In order \( k \geq 3 \), we have \( 2(k+1) \) invariants of pure order \( k \).

Differentiations of invariant \( J_0 \) of \( k \) times give us \( (k+1) \) invariants and differentiations of invariants \( J_{11}, J_{12} \) of \( (k-1) \) times give us \( 2k \) invariants of pure order \( k \).

Therefore, we have \( (k-1) \) syzygies, which could be obtained from (24) by \( (k-2) \) differentiations.

Finally, we get the following result.

**Theorem 3.4.** *The field of Möbius invariants for differential 1-forms on the unit disk is generated by invariants \( J_0, J_{11}, J_{12} \) and derivations \( \delta_1, \delta_2 \). All syzygies are generated by (24). The field separates orbits where \( J_0 \neq 0 \).*
3.1.5. Möbius equivalence of differential forms. Let \( \theta \in \Omega^1 (\mathbb{D}) \) be a differential 1-form on the unit disk and let \( A, B \) be two invariants of order \( \leq 1 \). We say that 1-form \( \theta \) is regular if

\[ J_0 (\theta) \neq 0 \quad \text{and} \quad dA (\theta) \wedge dB (\theta) \neq 0 \]
on the unit disk.

In this case, to know restriction of the Möbius invariant field on \( \theta \), we have to know values of the basic invariants: \( J_0 (\theta), J_{11} (\theta), J_{12} (\theta) \) as well as action of invariant derivations \( \delta_1 \) and \( \delta_2 \) on invariants \( A, B: \delta_1 (A) (\theta), \delta_2 (A) (\theta), \delta_1 (B) (\theta), \delta_2 (B) (\theta) \).

Using functions \((A (\theta), B (\theta))\) as coordinates on the disk, we get relations:

\[
\begin{align*}
J_0 &= F_0 (A, B), \\
J_{11} &= F_{11} (A, B), \\
J_{12} &= F_{12} (A, B), \\
\delta_i (A) &= G_i (A, B), \\
\delta_i (B) &= H_i (A, B), \quad i = 1, 2,
\end{align*}
\]
on the form \( \theta \).

Let \( \sigma_\theta : \mathbb{D} \to \mathbb{R}^2 \) be the mapping,

\[ \sigma_\theta = (A (\theta), B (\theta)), \]
and

\[ \Sigma_\theta = \text{Im} \sigma_\theta. \]

Then, as above, consider \( \Sigma_\theta \) as a system of differential equations for differential 1-forms on the disk.

By the construction this is completely integrable \( \mathbb{PSL}_2 (\mathbb{R}) \)-invariant system with 3 dimensional space (each solution is determined by its 1-jet at a point) of solutions that are images of \( \theta \) under the \( \mathbb{PSL}_2 (\mathbb{R}) \)-action.

Summarising, we arrive at the following result.

**Theorem 3.5.** Two regular differential 1-forms \( \theta_1, \theta_2 \) are \( \mathbb{PSL}_2 (\mathbb{R}) \)-equivalent if \( \Sigma_{\theta_1} = \Sigma_{\theta_2} \) and they have the same determinative functions \( \Sigma_\theta \).

3.1.6. Möbius invariants for foliations. In the last section we consider conformal classification of oriented foliations defined in simply connected domains, or equally \( \mathbb{PSL}_2 (\mathbb{R}) \)-classification of oriented foliations on the unit disk. Remark, that any such a foliation is defined by some differential 1-form \( \theta \) of the length 1 : \( g (\theta, \theta) = 1 \).

Denote by \( T^*_\pi (\mathbb{D}) \subset T^* (\mathbb{D}) \) the subbundle of covectors, having length 1, and let \( \pi : T^*_\pi (\mathbb{D}) \to \mathbb{D} \) be the corresponding bundle. Sections of this bundle could be identified with oriented foliations.

Denote also by \( \mathcal{J}^k (\pi) \) the bundles of \( k \)-jets of sections of \( \pi \). Thus, they are jets of oriented foliations.

The \( \mathbb{PSL}_2 (\mathbb{R}) \)-action on \( \mathbb{D} \) in a natural way could be lifted on the bundle \( \pi \), and prolong to actions on the bundles of \( k \)-jets. Rational invariants of these actions we call Möbius invariants of foliations.

Remark that \( \mathcal{J}^k (\pi) \subset \mathcal{J}^k (\pi^*) \) are subbundles with \( \mathbb{PSL}_2 (\mathbb{R}) \)-action induced by the \( \mathbb{PSL}_2 (\mathbb{R}) \)-action on the orbit \( J_0 = 1 \). This subbundle contains regular orbits and therefore Möbius invariants of foliations could be found as the restrictions of invariants differential 1-forms.

Denote by \((x, y, w, w_{10}, w_{01}, ...)\) the standard coordinates in the jet spaces \( \mathcal{J}^k (\pi) \), where \( w \) is the angle.

Then, the restriction on the regular orbit \( J_0 = 1 \), gives us the following:
• Invariant coframe:
  \[
  \omega_1 = t^{-1} \left( \sin w \, dx + \cos w \, dy, \right), \\
  \omega_2 = t^{-1} \left( -\cos w \, dx + \sin w \, dy \right).
  \]

• Invariant derivations:
  \[
  \delta_1 = t \left( \sin w \, \frac{d}{dx} + \cos w \, \frac{d}{dy} \right), \\
  \delta_2 = t \left( -\cos w \, \frac{d}{dx} + \sin w \, \frac{d}{dy} \right).
  \]

• Structure equations:
  \[
  \widehat{d}\omega_1 + J_1 \omega_1 \wedge \omega_2 = 0, \\
  \widehat{d}\omega_2 + J_2 \omega_1 \wedge \omega_2 = 0.
  \]

• Möbius invariants of order 1:
  \[
  J_1 = t \left( \sin w \, w_{10} + \cos w \, w_{01} \right) - 2 \left( x \cos w - y \sin w \right), \\
  J_2 = t \left( \cos w \, w_{10} - \sin w \, w_{01} \right) + 2 \left( x \sin w + y \cos w \right).
  \]

In these formulae
  \[
  t = 1 - x^2 - y^2.
  \]

**Theorem 3.6.** 1. The field of Möbius invariants of oriented foliations on the unit disk is generated by basic invariants \( J_1, J_2 \) of order 1 and invariant derivations \( \delta_1, \delta_2 \).

2. This field separates \( \mathbb{PSL}_2(\mathbb{R}) \)-orbits of foliations.

3. Differential syzygies are generated by the following syzygy in order 2:
   \[
   \delta_1 (J_2) + \delta_2 (J_1) + J_1^2 + J_2^2 - 4 = 0.
   \]

**Equivalence.**

We say that a foliation \( \theta \) is regular if
\[
  dJ_1 (\theta) \wedge dJ_2 (\theta) \neq 0
\]
on the unit disk.

It follows from the above theorem that the restriction of the filed of Möbius invariants on a regular foliation is completely defined by invariants \( \delta_i (J_j) \), considering as functions of \( J_1, J_2 \):
\[
  \delta_i (J_j) (\theta) = F_{ij} (J_1 (\theta), J_2 (\theta)).
\]
Let
\[
  \sigma_\theta : \mathbb{D} \to \mathbb{R}^2_{(\alpha, \beta)},
\]
be the mapping
\[
  \alpha = J_1 (\theta), \beta = J_2 (\theta),
\]
and let
\[
  \Sigma_\theta = \text{Im} \sigma_\theta.
\]
Define in the domain \( \Sigma_\theta \) two vector fields
\[
  V_1 (\theta) = F_{11} (\alpha, \beta) \, \partial_\alpha + F_{12} (\alpha, \beta) \, \partial_\beta, \\
  V_2 (\theta) = F_{21} (\alpha, \beta) \, \partial_\alpha + F_{22} (\alpha, \beta) \, \partial_\beta.
\]
Then the above theorem shows that foliations $\theta_1$ and $\theta_2$ are $\text{PSL}_2(\mathbb{R})$-equivalent if and only if the geometrical images coincide:

$$(\Sigma_{\theta_1}, V_1(\theta_1), V_2(\theta_1)) = (\Sigma_{\theta_2}, V_1(\theta_2), V_2(\theta_2)).$$

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