EXPLICIT FORMULA FOR THE SOLUTION OF THE SZEGŐ EQUATION ON THE REAL LINE AND APPLICATIONS

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Abstract. We consider the cubic Szegő equation

\[ i\partial_t u = \Pi(|u|^2 u) \]

in the Hardy space \( L^2_+(\mathbb{R}) \) on the upper half-plane, where \( \Pi \) is the Szegő projector. It is a model for totally non-dispersive evolution equations and is completely integrable in the sense that it admits a Lax pair. We find an explicit formula for solutions of the Szegő equation. As an application, we prove soliton resolution in \( H^s \) for all \( s \geq 0 \), for generic data. As for non-generic data, we construct an example for which soliton resolution holds only in \( H^s \), \( 0 \leq s < 1/2 \), while the high Sobolev norms grow to infinity over time, i.e. \( \lim_{t \to \pm \infty} \| u(t) \|_{H^s} = \infty \), \( s > 1/2 \). As a second application, we construct explicit generalized action-angle coordinates by solving the inverse problem for the Hankel operator \( H_u \) appearing in the Lax pair. In particular, we show that the trajectories of the Szegő equation with generic data are spirals around Lagrangian toroidal cylinders \( \mathbb{T}^N \times \mathbb{R}^N \).

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1. Introduction

1.1. Cubic Szegő equation. One of the most important properties in the study of the nonlinear Schrödinger equations (NLS) is dispersion. It is often exhibited in the form of the Strichartz estimates of the corresponding linear flow. In case of the cubic NLS:

\[ i\partial_t u + \Delta u = |u|^2 u, \quad (t, x) \in \mathbb{R} \times M, \]

Gérard and Grellier [17] remarked that there is a lack of dispersion when \( M \) is a sub-Riemannian manifold (for example, the Heisenberg group). In this situation, many of the classical arguments used in the study of NLS no longer hold. As a consequence, even the problem of global well-posedness of (1.1) on a sub-Riemannian manifold still remains open. In [16, 17], Gérard and Grellier introduced a model of a non-dispersive Hamiltonian equation called the cubic Szegő equation. (See (1.2) below.) The study of this equation is expected to give new tools to be used in understanding existence and other properties of smooth solutions of NLS in the absence of dispersion.

In this paper we will consider the Szegő equation on the real line. The space of solutions in this case is the Hardy space \( L^2_+ (\mathbb{R}) \) on the upper half-plane \( \mathbb{C}_+ = \{ z; \text{Im} z > 0 \} \), defined by

\[ L^2_+ (\mathbb{R}) = \left\{ f \text{ holomorphic on } \mathbb{C}_+; \| g \|_{L^2_+(\mathbb{R})} := \sup_{y>0} \left( \int_{\mathbb{R}} |g(x+iy)|^2 dx \right)^{1/2} < \infty \right\}. \]

In view of the Paley-Wiener theorem, we identify this space of holomorphic functions on \( \mathbb{C}_+ \) with the space of their boundary values:

\[ L^2_+ (\mathbb{R}) = \{ f \in L^2(\mathbb{R}); \text{supp } \hat{f} \subset [0, \infty) \}. \]

The corresponding Sobolev spaces \( H^s_+(\mathbb{R}) \), \( s \geq 0 \) are defined by:

\[ H^s_+(\mathbb{R}) = \{ h \in L^2_+(\mathbb{R}); \| h \|_{H^s_+} := \left( \frac{1}{2\pi} \int_0^\infty (1 + |\xi|^2)^s |\hat{h}(\xi)|^2 d\xi \right)^{1/2} < \infty \}. \]

Similarly, we define the homogeneous Sobolev norm for \( h \in \dot{H}^s_+ \) by

\[ \| h \|_{\dot{H}^s_+} := \left( \frac{1}{2\pi} \int_0^\infty |\xi|^{2s} |\hat{h}(\xi)|^2 \right)^{1/2} < \infty. \]

Endowing \( L^2(\mathbb{R}) \) with the usual scalar product \( (u, v) = \int_{\mathbb{R}} uv \), we define the Szegő projector \( \Pi : L^2(\mathbb{R}) \to L^2_+(\mathbb{R}) \) to be the projector onto the non-negative frequencies,

\[ \Pi(f)(x) = \frac{1}{2\pi} \int_0^\infty e^{ix\xi} \hat{f}(\xi) d\xi. \]

For \( u \in L^2_+(\mathbb{R}) \), we consider the Szegő equation on the real line:

\[ i\partial_t u = \Pi(|u|^2 u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \]

Endowing \( L^2_+ \) with the symplectic structure \( \omega(u, v) = 4\text{Im} \int_{\mathbb{R}} uv \), we have that the Szegő equation is a Hamiltonian evolution associated to the Hamiltonian

\[ E(u) = \int_{\mathbb{R}} |u|^4 dx \]

defined on \( L^2_+(\mathbb{R}) \). From this structure, we obtain the formal conservation law of the energy \( E(u(t)) = E(u(0)) \). The invariance under translations and under modulations provides
two more conservation laws, the mass \( Q(u(t)) = Q(u(0)) \) and the momentum \( M(u(t)) = M(u(0)) \), where

\[
Q(u) = \int_{\mathbb{R}} |u|^2 \, dx \quad \text{and} \quad M(u) = \int_{\mathbb{R}} \bar{u} D u \, dx, \quad \text{with} \quad D = -i \partial_x.
\]

Noting that \( Q(u) + M(u) = \|u\|^2_{H^1_+} \), we have \( \|u(t)\|_{H^1_+}^2 = \|u(0)\|_{H^1_+}^2 \). Hence, \( H^1_+ \) is the natural space for studying the well-posedness of the equation. In [29 Theorem 1.1], it was shown that the Szegö equation on the real line is globally well-posed in \( H^1_+ (\mathbb{R}) \) and satisfies the persistence of regularity, i.e. if \( u_0 \in H^s_+ (\mathbb{R}) \) for some \( s > \frac{1}{2} \), then \( u \in C(\mathbb{R}, H^s_+ (\mathbb{R})) \).

First of all we recall some notions and properties concerning the Szegö equation. We refer the readers to [29] for more details. The main property of the Szegö equation is that it is completely integrable in the sense that it possesses a Lax pair structure [29, Proposition 1.4]. We first define two important classes of operators on \( L^2_+ \), the Hankel and Toeplitz operators. The Lax pair is given in terms of these operators in Proposition 1.4.

A Hankel operator \( H_u : L^2_+ \to L^2_+ \) of symbol \( u \in H^1_+ \) is defined by

\[
H_u(h) = \Pi(uh).
\]

Then, as it was shown in Lemma 3.5 in [29], \( H_u \) is Hilbert-Schmidt and \( \mathbb{C} \)-antilinear. Moreover, it satisfies the following identity:

\[
(H_u(h_1), h_2) = (H_u(h_2), h_1).
\]

As a consequence, \( H^2_u \) is a self-adjoint linear operator. A Toeplitz operator \( T_b : L^2_+ \to L^2_+ \) of symbol \( b \in L^\infty (\mathbb{R}) \) is defined by

\[
T_b(h) = \Pi(bh).
\]

Then, \( T_b \) is \( \mathbb{C} \)-linear and bounded. Moreover, \( T_b \) is self-adjoint if and only if \( b \) is real-valued.

**Proposition 1.1** (Proposition 1.5 in [29]). Let \( u \in C(\mathbb{R}; H^s_+) \) for some \( s > \frac{1}{2} \). The cubic Szegö equation (1.2) is equivalent to the following evolution equation:

\[
\frac{d}{dt} H_u = [B_u, H_u],
\]

where

\[
B_u = \frac{i}{2} H^2_u - iT_u^2.
\]

In other words, the pair \((H_u, B_u)\) is a Lax pair for the cubic Szegö equation on the real line.

According to the classical theory developed by Lax [21], a direct consequence of the above proposition is the following corollary:

**Corollary 1.2.** Let \( U(t) \) be an operator on \( H^1_+ \) defined by:

\[
\frac{d}{dt} U(t) = B_u(t) U(t), \quad U(0) = I.
\]

Then, \( U(t) \) is a unitary operator and if \( u \) is a solution of the Szegö equation (1.2) with initial condition \( u_0 \), we have:

\[
H_u(t) = U(t) H_{u_0} U(t)^*.
\]

This yields

\[
U(t) (\ker(H_{u_0})) \subset \ker(H_u(t)), \quad U(t) (\operatorname{ran}(H_{u_0})) \subset \operatorname{ran}(H_u(t)).
\]
Another consequence of the Lax pair structure is the existence of an infinite sequence of conservation laws. More precisely, the following corollary holds.

**Corollary 1.3.** Define \( J_n(u) := (u, H_n^{−2}u) \) for all \( n \geq 2 \). Then \( J_{2k}(u) \), \( k \in \mathbb{N}^* \), are conserved quantities for the Szegő equation. In particular, \( J_2(u) = Q(u), \ J_4(u) = \frac{E(u)}{2} \), and we recover the conservation laws of the mass and energy.

**Remark 1.4.** Using the Mikhlin multiplier theorem, we can prove that \( J_{2k}(u) \leq \|u\|_{L^2}^{2k} \). Then, by the Sobolev embedding we have that \( J_{2k}(u) \leq \|u\|_{H^{k/2}}^{2k} \). This shows that the strongest conservation law for the Szegő equation is the \( H^{1/2}_{\pm} \)-norm.

### 1.2. Main results.

It turns out that rational functions play an important role in studying the Hankel operators, and thus the Szegő equation. In the following, we first consider solutions for the Szegő equation with rational function initial data \( u_0 \in \mathcal{M}(N) \), where \( \mathcal{M}(N) \) is defined below.

**Definition 1.1.** Let \( N \in \mathbb{N}^* \). We denote by \( \mathcal{M}(N) \) the set of rational functions of the form

\[
\frac{A(z)}{B(z)},
\]

where \( A \in \mathbb{C}_{N−1}[z], B \in \mathbb{C}_{N}[z], 0 \leq \deg(A) \leq N − 1, \deg(B) = N, B(0) = 1, B(z) \neq 0, \) for all \( z \in \mathbb{C} \cup \mathbb{R} \), and \( A \) and \( B \) have no common factors.

Note that \( \mathcal{M}(N) \) is a \( 4N \)-dimensional real manifold, \( \mathcal{M}(N) \subset H^s(\mathbb{R}) \) for all \( s \geq 0 \), and that \( \bigcup_{N=1}^{\infty} \mathcal{M}(N) \) is dense in \( L^2_{\pm} \) [26 Lemma 6.2.1]. Moreover, they remain invariant under the flow.

**Proposition 1.5.** The manifolds \( \mathcal{M}(N) \) are invariant under the flow of the Szegő equation.

In order to prove this statement, we recall a Kronecker-type theorem.

**Proposition 1.6** (Theorem 2.1 in [29]). Let \( u \in H^1_{\pm} \). Then \( u \in \mathcal{M}(N) \) if and only if \( \text{rk}(H_u) = N \). Moreover, if \( u = \frac{A}{B} \in \mathcal{M}(N) \), where \( A \) and \( B \) are relatively prime, \( B(0) = 1, B(x) = (x−p_1)^{m_1} \cdots (x−p_k)^{m_k}, m_1 + \cdots + m_k = N, \) and \( \text{Im}(p_j) < 0 \) for all \( j = 1, 2, \ldots, k \), then we have that

\[
\text{Ran}(H_u) = \text{span}_\mathbb{C} \left\{ \frac{1}{(x−p_j)^{l_j}}, j = 1, 2, \ldots, k \text{ and } l_j = 1, 2, \ldots, m_j \right\}.
\]

**Proof of Proposition 1.6** By equation (1.6) and Proposition 1.6, we have that if \( u_0 \in \mathcal{M}(N) \), then \( \text{rk}(H_{u(t)}) = \text{rk}(H_{u_0}) = N \). Thus the corresponding solution \( u(t) \in \mathcal{M}(N) \) for all \( t \in \mathbb{R} \). \( \square \)

As a corollary of the Kronecker-type theorem [29 Remark 2.2], we also have that if \( u \in \mathcal{M}(N) \) then \( u \in \text{Ran}(H_u) \), i.e. there exists a unique element \( g \in \text{Ran}(H_u) \) such that

\[
u = H_u(g).
\]

This yields \( \Pi(u(1−g)) = 0 \), which gives:

\[
\bar{u}(1−g) \in L^2_+.
\]

An important property of Hankel operators, that will be a key point in this paper, is their characterization using the shift operators \( \tilde{T}_\lambda : L^2_+ \to L^2_+, \lambda > 0, \)

\[
\tilde{T}_\lambda f(x) = e^{i\lambda x} f(x).
\]
More precisely, the bounded operator $H : L^2_+ \to L^2_+$ is a Hankel operator if and only if
\begin{equation}
\tilde{T}^*_\lambda H = HT^*_\lambda
\end{equation}
for all $\lambda > 0$, [26, p. 273]. The adjoint $\tilde{T}^*_\lambda : L^2_+ \to L^2_+$, defined by
\begin{equation}
\tilde{T}^*_\lambda f(x) = e^{-i\lambda x} (f \ast F^{-1}(\chi_{[\lambda, \infty)}))(x),
\end{equation}
is very inconvenient to use. Then, for rational functions $u$, we define the infinitesimal shift operator $T : \text{Ran}(H_u) \to \text{Ran}(H_u)$,
\begin{equation}
T f(x) = x f(x) - \left( \lim_{x \to \infty} x f(x) \right) (1 - g(x))
\end{equation}
and prove that
\begin{equation}
T^* H_u = H_u T.
\end{equation}

In the general case, when $u$ is not a rational function, $u$ does not always belong to $\text{Ran}(H_u)$. Thus, $g$ satisfying (1.13) does not always exist. If such $g$ does not exist, the above definition (1.12) of $T$ does not make sense. We then propose, in Section 3, to extend a definition for $T^*$ (see 3.3 below) and pursue our work using $T^*$ rather than $T$.

Next, we recall the definition and the characterization of soliton solutions for the Szegö equation. See [29] for details.

**Definition 1.2.** A soliton for the Szegö equation on the real line is a solution $u$ with the property that there exist $c, \omega \in \mathbb{R}$, $c \neq 0$ such that
\begin{equation}
 u(t, x) = e^{-it\omega} u_0(x - ct).
\end{equation}

In [29, Theorem 2] it was proved that all the solitons for the Szegö equation on $\mathbb{R}$ are of the form
\begin{equation}
 u(t, x) = e^{-i\omega t} \phi_{C, p}(x - ct),
\end{equation}
where $\phi_{C, p} = \frac{c}{x - p}$, $\omega = \frac{|C|^2}{\pi \text{ Imp}^2}$, $c = \frac{|C|^2}{2 \text{ Imp}}$, $C, p \in \mathbb{C}$, and $\text{Imp} < 0$. Hence, a soliton of the Szegö equation on $\mathbb{R}$ is a simple fraction $u(t, x) = \frac{C e^{-i\omega t}}{x - ct - p} \in \mathcal{M}(1)$, where $\text{Imp}(p) < 0$.

We are now ready to state the main results of this paper. In the first place we find an explicit formula for the solutions of the Szegö equation with rational function data.

**Theorem 1.7** (Explicit formula in the case of rational function data). Suppose that $u_0 \in \mathcal{M}(N)$ and $H^2_{u_0}$ has positive eigenvalues $\lambda_1^2 \leq \lambda_2^2 \leq \cdots \leq \lambda_N^2$. We will assume that $\lambda_j > 0$ for all $j = 1, 2, \ldots, N$. Choose a complex orthonormal basis $\{e_j\}_{j=1}^N$ of $\text{Ran}(H_{u_0})$, consisting of eigenvectors of $H^2_{u_0}$ such that $H_{u_0} e_j = \lambda_j e_j$ for all $j = 1, 2, \ldots, N$. Let $W(t) = e^{i\frac{t}{2}H^2_{u_0}}$ and $\beta_j = (g_0, e_j)$.

We define an operator $S(t)$ on $\text{Ran}(H_{u_0})$ in the following way. Fix $j \in \{1, \ldots, N\}$, and let $\lambda_j^2$ be an eigenvalue of multiplicity $m_j$. Moreover, let $M_j \subset \mathbb{N}$ be the set of all indices $k$ such that $H_{u_0} e_k = \lambda_j e_k$. Then, $S(t)$ in the basis $\{e_j\}_{j=1}^N$ is defined by the matrix
\begin{equation}
S(t)_{k,j} = \begin{cases}
\frac{\lambda_j}{2\pi i(\lambda_k^2 - \lambda_j^2)} \left( \lambda_j e^{i\frac{t}{2}(\lambda_k^2 - \lambda_j^2)} \beta_k - \lambda_k e^{i\frac{t}{2}(\lambda_j^2 - \lambda_k^2)} \beta_j \right), & \text{if } k \in \{1, \ldots, N\} \setminus M_j, \\
\frac{\lambda_j^2}{2\pi} \beta_j \beta_k t + (T e_j, e_k), & \text{if } k \in M_j.
\end{cases}
\end{equation}
Then, we have the following explicit formula for the solution of the Szegö equation:

$$u(t, x) = \frac{i}{2\pi} \left( u_0, W(t)(S - xI)^{-1}W(t)g_0 \right), \text{ for all } (t, x) \in \mathbb{R} \times \mathbb{R}.$$ 

We extend the explicit formula to more general initial data, that are not necessarily rational functions, in the following corollary.

**Corollary 1.8** (A first generalization of the explicit formula). Let \( u_0 \in H^{1/2}_+ \) be a general initial condition. Denote by \( \{\lambda_j^2\}_{j=1}^\infty \) the positive eigenvalues of the operator \( H^{2}_{u_0} \). We assume that \( \lambda_j > 0 \) for all \( j \in \mathbb{N} \). Choose a complex orthonormal basis \( \{e_j\}_{j=1}^\infty \) of \( \text{Ran}(H_{u_0}) \) consisting of eigenvectors of \( H^{2}_{u_0} \) such that \( H_{u_0}e_j = \lambda_j e_j \) for all \( j \in \mathbb{N}^* \). Denote \( W(t) = e^{i\frac{t}{2}H^{2}_{u_0}} \) and \( \beta_j = \frac{1}{\lambda_j}(e_j, u_0) \).

We define an operator \( S(t) \) on \( \text{Ran}(H_{u_0}) \) in the following way. Fix \( j \in \mathbb{N}^* \), and let \( \lambda_j^2 \) be an eigenvalue of multiplicity \( m_j \). Moreover, let \( M_j \subset \mathbb{N}^* \) be the set of all indices \( k \) such that \( H_{u_0}e_k = \lambda_j e_k \). Then, \( S(t) \) is defined by

\[
(S(t)e_j, e_k) = \begin{cases} 
\lambda_j \frac{\lambda_k}{2\pi i(\lambda_k^2 - \lambda_j^2)}(\lambda_j e^{i\frac{t}{2}(\lambda_j^2 - \lambda_k^2)}\beta_j \beta_k - \lambda_k e^{i\frac{t}{2}(\lambda_k^2 - \lambda_j^2)}\beta_j \beta_k), & \text{if } k \in \mathbb{N} \setminus M_j, \\
\lambda_j^2 \beta_k t + T(e_j, e_k), & \text{if } k \in M_j.
\end{cases}
\]

Denote by \( \hat{S} \) the closure of the operator \( S \).

If the sequence \( \{\beta_j\}_{j=1}^N \in l^2 \), then there exists \( g_0 \in \text{Ran}(H_{u_0}) \) such that \( u_0 = H_{u_0}(g_0) \). Moreover, for \( \text{Im} \zeta > 0 \), the following formula for the solution of the Szegö equation with initial condition \( u_0 \) holds:

$$u(t, z) = \frac{i}{2\pi} \left( u_0, W(t)(S - zI)^{-1}W(t)g_0 \right).$$

The condition \( \{\beta_j\}_{j=1}^N \in l^2 \) characterizes all initial data satisfying \( u_0 \in \text{Ran}H_{u_0} \). In particular, by \([1.9]\), it is satisfied by all rational functions. However, simple non-rational functions, like \( \frac{x^n}{1 + x^n} \) with \( \alpha > 0 \), do not satisfy it, and hence Corollary 1.8 is not applicable. In the following theorem, we extend the explicit formula to even more general initial data.

**Theorem 1.9** (Explicit formula for general data). Let \( u_0 \in H^s_+ \), \( s > \frac{1}{2} \), \( xu_0 \in L^\infty(\mathbb{R}) \). With the notations in Corollary 1.8, we define an operator \( S^*(t) \) on \( \text{Ran}(H_{u_0}) \) in the following way. Fix \( j \in \mathbb{N}^* \). If \( \lambda_j^2 \) is an eigenvalue of multiplicity \( m_j \) and \( M_j \subset \mathbb{N} \) is the set of all indices \( k \) such that \( H_{u_0}e_k = \lambda_j e_k \), then

\[
(S^*(t)e_j, e_k) = \begin{cases} 
\lambda_k \frac{\lambda_j}{2\pi i(\lambda_k^2 - \lambda_j^2)}(\lambda_j e^{i\frac{t}{2}(\lambda_j^2 - \lambda_k^2)}\beta_j \beta_k - \lambda_k e^{i\frac{t}{2}(\lambda_k^2 - \lambda_j^2)}\beta_j \beta_k), & \text{if } k \in \mathbb{N} \setminus M_j, \\
\lambda_j^2 \beta_k t + (T^*e_j, e_k), & \text{if } k \in M_j.
\end{cases}
\]

Let \( A \) be the closure of \( S^* \). Then, for \( \text{Im} \zeta > 0 \), the solution of the Szegö equation writes

$$u(t, z) = \lim_{\varepsilon \to 0} \frac{i}{2\pi} \left( W^*(t)(A - zI)^{-1}W^*(t)u_0, \frac{1}{1 - i\varepsilon z} \right).$$

Let \( S^*_\alpha \) be the semi-group of contractions whose infinitesimal generator is \(-iA\). Then, the above formula is equivalent to

$$\hat{u}(t, \lambda) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \left( W^*(t)S^*_\alpha(t)W^*(t)u_0, \frac{1}{1 - i\varepsilon x} \right), \text{ a.e. } \lambda \in \mathbb{R}. $$
**Definition 1.3.** A function \( u_0 \in \mathcal{M}(N) \) is called *generic* if the operator \( H^2_{u_0} \) has simple eigenvalues \( 0 < \lambda_1^2 < \lambda_2^2 < \cdots < \lambda_N^2 \) and \( |(u_0,e_j)| \neq 0 \), for all \( j = 1,2,\ldots,N \). We denote by \( \mathcal{M}(N)_{\text{gen}} \) the set of generic rational functions in \( \mathcal{M}(N) \).

A function \( u_0 \) is called *strongly generic* if it is generic and, in addition, \( |(u_0,e_j)| \neq |(u_0,e_k)| \) for all \( k \neq j \). We denote by \( \mathcal{M}(N)_{\text{sgen}} \) the set of strongly generic rational functions in \( \mathcal{M}(N) \).

The sets \( \mathcal{M}(N)_{\text{gen}} \) and \( \mathcal{M}(N)_{\text{sgen}} \) are indeed generic, in the sense that they are open, dense subsets of \( \mathcal{M}(N) \). As in [16] Theorem 7.1, we have that \( \det(J_{2(m+n)})_{1 \leq m,n \leq N} \neq 0 \) if and only if \( H^2_{u_k}(g), k = 1,2,\ldots,n \), are linearly independent. Decomposing \( g, H^2g, \ldots, H^2(N-1)g \) in the basis \( \{e_j\}_{j=1}^N \), we obtain that the determinant of the matrix which contains these vectors as columns is:

\[
\begin{vmatrix}
\nu_1 & \lambda_1^2 \nu_1 & \cdots & \lambda_1^{2(N-1)} \nu_1 \\
\nu_2 & \lambda_2^2 \nu_2 & \cdots & \lambda_2^{2(N-1)} \nu_2 \\
\vdots & \vdots & \ddots & \vdots \\
\nu_N & \lambda_N^2 \nu_N & \cdots & \lambda_N^{2(N-1)} \nu_N
\end{vmatrix}
= \nu_1 \cdots \nu_N
\begin{vmatrix}
1 & \lambda_1^2 & \cdots & \lambda_1^{2(N-1)} \\
1 & \lambda_2^2 & \cdots & \lambda_2^{2(N-1)} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \lambda_N^2 & \cdots & \lambda_N^{2(N-1)}
\end{vmatrix},
\]

where \( \nu_j := \frac{1}{\epsilon_j}(u,e_j) \). Thus, the fact that \( g, H^2g, \ldots, H^2(N-1)g \) are linearly independent is equivalent to \( (u,e_j) \neq 0, j = 1,2,\ldots,N \) and \( \lambda_j \) are all distinct. Therefore,

\[\mathcal{M}(N)_{\text{gen}} = \{ u_0 \in \mathcal{M}(N) \mid \det(J_{2(m+n)})_{1 \leq m,n \leq N} \neq 0 \}\]

is an open, dense subset of \( \mathcal{M}(N) \). By Theorem 1.14 below, we obtain that \( \chi : \mathcal{M}(N)_{\text{gen}} \to \Omega \) (see [16] below) is a diffeomorphism. Since \( \mathcal{M}(N)_{\text{sgen}} \) corresponds, through \( \chi \), to an open dense subset of \( \Omega \), it results that \( \mathcal{M}(N)_{\text{sgen}} \) is also generic.

**Definition 1.4.** We say that *soliton resolution* holds in \( H^s \) for a solution \( u(t) \) of the Szegö equation, if \( u(t) \) can be written as the sum of a finite number of solitons and a remainder \( \varepsilon(t,x) \) with the property that \( \lim_{t \to \pm \infty} \|\varepsilon(t,x)\|_{H^s} = 0 \).

Using the above explicit formula for the solution, we prove the following result:

**Theorem 1.10 (Soliton resolution for strongly generic data).** Let \( u_0 \in \mathcal{M}(N)_{\text{sgen}} \) be a strongly generic initial data for the Szegö equation. Then, the corresponding solution satisfies the property of soliton resolution in \( H^s \) for all \( s \geq 0 \). More precisely, with the notations in Theorem 1.7, we have

\[u(t,x) = \sum_{j=1}^N e^{-i\lambda_j^2 t} \phi_{C_j,p_j}(x - \frac{\lambda_j^2}{2\pi} t) + \varepsilon(t,x),\]

where \( C_j = \frac{i\lambda_j \bar{\gamma}^2}{2\pi}, \ p_j = \text{Re}(c_j(0)) - i\frac{\bar{\gamma}^2}{4\pi}, \) and \( \lim_{t \to \pm \infty} \|\varepsilon(t,x)\|_{H^s} = 0 \) for all \( s \geq 0 \).

Studying the case of non-generic initial data \( u_0 \in \mathcal{M}(2) \), such that \( H^2_{u_0} \) has a double eigenvalue \( \lambda_1^2 = \lambda_2^2 \), we can prove that the soliton resolution holds in \( H^s \) only for \( 0 \leq s < 1/2 \). It turns out that the \( H^s \)-norms with \( s > 1/2 \) of such non-generic solutions grow to \( \infty \) as \( t \to \pm \infty \).

**Theorem 1.11 (Partial soliton resolution for non-generic data).** Let \( u_0 \in \mathcal{M}(2) \) be such that \( H^2_{u_0} \) has a double eigenvalue \( \lambda^2 > 0 \). Then the corresponding solution satisfies the
property of soliton resolution in $H^s$ for $0 \leq s < 1/2$. More precisely,
\[
u(t, x) = e^{-\imath \lambda x^3} \phi(t, x) + \varepsilon(t, x),
\]
where the first term is a soliton with $|C| = \frac{\|u_0\|_2^2}{\sqrt{\pi} \|u_0\|_H^{1/2}}$, $\text{Im}(p) = -\left(\frac{\|u_0\|_2^2}{\|u_0\|_H^1}\right)^2$, and $\varepsilon(t, x) \to 0$ in all the $H^s$-norms with $0 \leq s < 1/2$.

However, $\varepsilon(t, x)$ stays away from zero and is bounded in the $L^\infty$-norm and $H^{1/2}$-norm. Moreover, $\lim_{t \to \pm \infty} \|\varepsilon(t, x)\|_{H^s} = \infty$ if $s > 1/2$.

As a consequence, we obtain the following result:

**Corollary 1.12** (Growth of high Sobolev norms). The Szegö equation admits solutions $u(t)$ whose high Sobolev norms $H^s$, for $s > 1/2$, grow to infinity:
\[
\|u(t)\|_{H^s} \to \infty \text{ as } t \to \pm \infty.
\]

More precisely, there exists a solution $u$ of the Szegö equation and a constant $C > 0$ such that $\|u(t)\|_{H^s} \geq C|t|^{2s-1}$ for sufficiently large $|t|$.

**Remark 1.13.** Corollary 1.12 presents an example of solutions whose high Sobolev norms grow to infinity. We could observe this phenomenon by considering non-generic initial data $u_0$ such that the operator $H^{s}_{u_0}$ has a double eigenvalue. We believe that the non-dispersive character of the Szegö equation plays an important role in the occurrence of this phenomenon. For example, consider the dispersionless NLS, $i u_t = |u|^2 u$. Then, $u(t, x) = \phi(x) \exp(-i |\phi(x)|^2 t)$ with smooth $\phi$ is a solution, satisfying $\|u(t)\|_{H^s} \sim |t|^s$ for $s \in \mathbb{N}$.

However, the situation is more subtle for the Szegö equation, due to the conservation of the $H^{1/2}$-norm. In particular, this explains why, for the Szegö equation, only the $H^s$-norms with $s > 1/2$ grow to infinity.

Corollary 1.12 shows that the energy is supported on higher frequencies while the mass is supported on lower frequencies. This phenomenon is called “forward cascade” and is consistent with some predictions in the weak turbulence theory.

Previously, Bourgain constructed, in [1, 5, 6], solutions with Sobolev norms growing to infinity. He considered, however, Hamiltonian PDEs involving a spectrally defined Laplacian. For general (dispersive) Hamiltonian PDEs, such a phenomenon is not known, but there are several partial results in this direction. In [16, Corollary 5], Gérard and Grellier noticed the growth of Sobolev norms for the Szegö equation on $\mathbb{T}$. However, their construction of a sequence of solutions $u^\varepsilon(t^\varepsilon)$ whose Sobolev norms become larger depends on the small parameter $\varepsilon$.

In the following theorem we introduce generalized action-angle coordinates for the Szegö equation in the case of generic rational functions.

\[\text{Note that this can be considered as a perturbation of the dispersionless NLS. See p.138 in [7]}.\]
Theorem 1.14 (Generalized action-angle coordinates). For $u \in \mathcal{M}(N)_{\text{gen}}$ denote by $0 < \lambda_1^2 < \lambda_2^2 < \ldots < \lambda_N^2$ the simple positive eigenvalues of $H_u^2$ and by $\{e_j\}_{j=1}^N$ an orthonormal basis of $\text{Ran}(H_u)$ such that $H_u e_j = \lambda_j e_j$. Denote $\nu_j = |(g, e_j)|$, $\phi_j = \arg(g, e_j)$, and $\gamma_j = \Re(T e_j, e_j)$.

Set $\Omega := (\mathbb{R}_+^*)^N \times \{0 < x_1 < x_2 < \ldots < x_N\} \times \mathbb{T}^N \times \mathbb{R}^N$. The application $\chi : \mathcal{M}(N)_{\text{gen}} \to \Omega$ defined by

\begin{equation}
\chi(u) = \left\{2\lambda_j^2 \nu_j^2 j_{j=1}^N, \{4\pi \lambda_j^2\}_{j=1}^N, \{2\phi_j\}_{j=1}^N, \{\gamma_j\}_{j=1}^N \right\},
\end{equation}

is a symplectic diffeomorphism. Moreover, $2\lambda_j^2 \nu_j^2, 4\pi \lambda_j^2, 2\phi_j, \gamma_j \in \mathbb{R}$, $j = 1, 2, \ldots, N$ are generalized action-angle coordinates for the Szegö equation on the real line.

As a corollary, we obtain that in the generic case, the trajectories of the Szegö equation spiral around toroidal-cylinders $\mathbb{T}^N \times \mathbb{R}^N$, $N \in \mathbb{N}^*$.

Corollary 1.15 (Lagrangian toroidal cylinders). Let $u_0 \in \mathcal{M}(N)_{\text{gen}}$. Consider

\begin{equation}
TC(u_0) := \left\{u \in \mathcal{M}(N)_{\text{gen}} | H_u^2, H_{u_0}^2 \text{ have same eigenvalues } \lambda_j^2 \text{ and same } \nu_j \right\}.
\end{equation}

Then, $u(t) \in TC(u_0)$ for all $t \in \mathbb{R}$, and the set $TC(u_0)$ is diffeomorphic to a toroidal cylinder $\mathbb{T}^N \times \mathbb{R}^N$ parameterized by the coordinates $(2\phi_j, \gamma_j)_{j=1}^N$, where $\gamma_j \in \mathbb{R}, 2 \phi_j \in \mathbb{T}$.

It seems difficult to extend Theorem 1.14 and Corollary 1.15 to arbitrary generic functions, which are not necessarily rational, as we did in Theorem 1.9. The main reasons are the lack of compactness and the fact that we are unable to characterize the conditions $u_0 \in H^s_+, s > 1/2$ and $x u_0(x) \in L^\infty(\mathbb{R})$ in terms of the spectral data.

The present paper was inspired by [15], where Gérard and Grellier introduced action-angle coordinates for the Szegö equation on $\mathbb{T}$. However, [15] does not treat the question of soliton resolution and growth of high Sobolev norms. Different difficulties are to be overcome in the two settings. In the case of $\mathbb{R}$, these difficulties are mostly related to the infinitesimal shift operator $T$ in (1.12), which does not appear in the case of $\mathbb{T}$.

1.3. Structure of the paper. We conclude this introduction by discussing the structure of the paper with some details. In Section 2, we prove Theorem 1.14 i.e. find an explicit formula for the solution of the Szegö equation with rational function initial condition. In the case of other completely integrable equations like KdV and one dimensional cubic NLS, an explicit formula for solutions was determined by the inverse scattering method [2, 10, 14]. Since in our case the operator $H_u$ is compact, we will not apply the inverse scattering method. We find a direct approach to solve the inverse spectral problem for the Hankel operator $H_u$, using the Lax pair structure and the commutation relation (1.13) between the operator $H_u$ and the infinitesimal shift $T$.

The inverse spectral problem for Hankel operators was considered in several papers, among which we cite [1, 25]. Our results are more precise than the previous ones and allow us to have a formula for the symbol $u$ of the Hankel operator $H_u$ only in terms of the spectral data.

Let us describe our strategy in Section 2. First we notice that $\hat{u}(\lambda) = (u, e^{i\lambda x} g)$, $\lambda > 0$. Then, we introduce the operators $S_\lambda(t) = P_{u_0} U^*(t) T_\lambda(t) U(t)$, $S(t) = U^*(t) T(t) U(t)$ acting on $\text{Ran}(H_{u_0})$. Exploiting the Lax pair structure, we obtain that

\begin{equation}
u(t, x) = \frac{i}{2\pi} \left( u_0, W(t)(S - xI)^{-1} W(t) g_0 \right).
\end{equation}
Since $S$ is defined using $U(t)$ and since the definition of $U(t)$ depends on $u(t)$ itself, the above formula is a vicious circle. To break it, we determine $S$ without using $U(t)$. The explicit expression for $S$ is obtained by computing the commutator $[H_{u_0}, S]$ and the derivative $\frac{d}{dt}S(t)$.

In Section 3, we prove Corollary 1.8 and Theorem 1.9. The proof of Theorem 1.9 uses an approximation argument, based on the remark that $u \in \text{Ran}(H_u)$ for all $u \in H_{+}^{1/2}$. The crucial step is to define the “adjoint of the infinitesimal shift operator”, $T^*$, for functions which are not necessarily rational functions (it seems more delicate to define the operator $T$ directly).

Notice that in Theorem 1.7, $S$ is a matrix whose eigenvalues are not real and thus the inverse $(S-xI)^{-1}$ can be explicitly computed. The result obtained in Theorem 1.9 is weaker. The operator $S^*$ acts between infinite dimensional spaces. Explicitly computing $(A-xI)^{-1}$ or the semi-group $S^*_t$ comes down to solving an infinite system of linear differential equations. Therefore, Theorem 1.9 actually states that we can transform our nonlinear infinite dimensional dynamical system into a linear one.

In Section 4, we prove Theorem 1.10. The soliton resolution conjecture is believed to be true for many dispersive equations for which the non-linearity is not strong enough to create finite-time blow-up. However, this was proved only for few equations like KdV [11] and one dimensional cubic NLS [27, 23], for which an explicit formula for the solution is available.

In Section 5, we prove Theorem 1.11 and Corollary 1.12. We show that soliton resolution still holds, even for non-generic solutions, but only in $H^s$, $0 \leq s < 1/2$. This is probably due to the fact that $H^{1/2}$ is the space of critical regularity.

The starting point in proving Theorems 1.10 and 1.11 is the explicit formula found in section 2, which we are able to write as a sum of simple fractions $\sum_{j=1}^{N} C_j(t) \overline{E_j(t)}$, $j = 1, 2, \ldots, N$.

The key remark is that the complex conjugates of the poles of $u(t)$, $E_j(t)$, are the eigenvalues of the operator $T(t)$ acting on $\text{Ran}(H_{u(t)})$. In the strongly generic case, the eigenvalues of $T(t)$ satisfy $E_j(t) = a_j t + b_j + O(\frac{1}{t})$ as $t \to \pm \infty$ with $a_j \neq 0$ and $\text{Im}(b_j) \neq 0$. This leads to the soliton resolution $u(t, x) = \sum_{j=1}^{N} C_j(t) \overline{E_j(t)} + \varepsilon(t, x)$ in $H^s$ for all $s \geq 0$. In the non-generic case, there is $j_0$ such that $E_{j_0}(t) = \text{Re}(b_{j_0}) + O(\frac{1}{t})$ as $t \to \pm \infty$. Then, $\text{Im}(E_{j_0}) = O(\frac{1}{t})$ and thus, one of the poles of the solution approaches the real line as $|t| \to \infty$. This causes $\|u(t)\|_{H^s}$ to grow to $\infty$ if $s > 1/2$.

In Section 6, we prove Theorem 1.14 and Corollary 1.15. The Szegö equation is an infinite dimensional, completely integrable system. The Lax pair structure establishes the existence of an infinite sequence of prime integrals $J_{2n} = (u, H_{2n-2}^2 u)$, $n \in \mathbb{N}$. Since the finite dimensional manifolds $\mathcal{M}(N)$ are invariant under the flow, by restricting the Szegö equation to $\mathcal{M}(N)$, we obtain a $4N$-dimensional, completely integrable system. The common level sets of the prime integrals $J_{2n}$ are not compact. Then, a generalization of the Liouville-Arnold theorem [13, 12] to the case of a $4N$-dimensional, completely integrable system, with non-compact
Lemma 2.1. Let \( u = \frac{A}{B} \in \mathcal{M}(N) \), where \( A \) and \( B \) are relatively prime, \( B(0) = 1 \), \( B(x) = (x-p_1)^{m_1} \cdots (x-p_k)^{m_k}, m_1 + \ldots + m_k = N \), and \( \text{Im}(p_j) < 0 \) for all \( j = 1, 2, \ldots, k \). Then \( \text{Ker}(H_u) = b_uL_2^2 \), where

\[
b_u = \prod_{j=1}^{k} \frac{(x-p_j)^{m_j}}{(x-p_j)^{m_j}}
\]

and

\[
g = 1 - b_u.
\]

Proof. Let \( f \in \text{Ker}(H_u) = (\text{Ran}(H_u))^\perp \). Then by equation \((1.8)\) we have that

\[
(f, \frac{1}{(x-p_j)^{l_j}}) = 0 \quad \text{for all } j = 1, 2, \ldots, k \text{ and } l_j = 1, 2, \ldots, m_j.
\]

By the residue theorem we have that

\[
(2.2) \quad \frac{1}{(x-p_j)^{l_j}}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} \frac{2\pi i l_j}{(l_j - 1)!} \xi^{l_j - 1} e^{-ip_j \xi}.
\]

Using the Plancherel formula, we obtain that

\[
0 = (f(\xi), \xi^{l_j - 1} e^{-ip_j \xi}) = \int e^{ip_j \xi} \xi^{l_j - 1} d\xi
\]

and thus \((D^{l_j-1}f)(\tilde{p}_j) = 0, \) for all \( j = 1, 2, \ldots, k \) and \( l_j = 1, 2, \ldots, m_j \). Then, the classical property \([26, \text{Corollary 3.7.4, p.38}]\) stating that if \( f \in L_2^2 \) is such that \( f(\tilde{p}) = 0, \text{Im}(p) < 0 \), then \( f(x) = \frac{x-p}{x-\bar{p}} f'(x) \) with \( f' \in L_2^2 \), applied recurrently to \( f \), \( Df, \ldots, D^{m_j-1}f \) yields the formula for \( b_u \). Using this formula we obtain

\[
(2.3) \quad \Pi(u\tilde{b}_u) = \Pi\left(\frac{A}{(x-\bar{p}_1)^{m_1} \cdots (x-\bar{p}_k)^{m_k}}\right) = 0.
\]

Moreover, equation \((1.8)\) yields that \( 1 - b_u \in \text{Ran}(H_u) \) and by \((1.9)\) we have that

\[
H_u(1 - b_u) = \Pi(u - u\tilde{b}_u) = u = H_u(g).
\]

Since \( H_u \) is one to one on its range, we conclude that \( 1 - b_u = g \). \( \square \)

Lemma 2.2. If \( u \in \mathcal{M}(N) \) and if \( g \) is such that \( u = H_u(g) \), then

\[
\hat{u}(\lambda) = (u, e^{i\lambda x} g) \quad \text{for all } \lambda > 0.
\]
Proof. Denoting by $\mathcal{F}$ the Fourier transform, we have that
\[
\hat{u}(\lambda) = \int e^{-i\lambda x} u dx = \int e^{-i\lambda x} u(1 - \tilde{g}) dx + \int e^{-i\lambda x} u \tilde{g} dx
\]
\[
= \mathcal{F}(u(1 - \tilde{g}))(\lambda) + (u, e^{i\lambda x} g) = \mathcal{F}(\bar{u}(1 - g))(-\lambda) + (u, e^{i\lambda x} g).
\]
By (1.10) we have that $\bar{u}(1 - g) \in L^2_{\pm}$. Thus, the first term is the Fourier transform at $\lambda < 0$ of a function in $L^2_{\pm}$, and hence it is zero.

**Lemma 2.3.** If $u(t)$ is the solution of the Szegö equation corresponding to the initial condition $u_0 \in \mathcal{M}(N)$ at time $t$ and $g(t) \in \text{Ran}(H_{u(t)})$ is such that $u(t) = H_{u(t)}(g(t))$, then we have:
\[
(2.4) \quad U^*(t)u(t) = e^{-i\frac{t}{2}H^2_{u_0}}u_0,
\]
\[
U^*(t)g(t) = e^{i\frac{t}{2}H^2_{u_0}}g_0.
\]

Proof. Differentiating with respect to $t$ and using equations (1.4), (2.2), and (1.6), we have
\[
\frac{d}{dt} U^* u = -U^* B_u u - iU^* T_{[1]} u = -U^*(-iT_{[1]} u + \frac{i}{2} H^2_{[1]} u) - iU^* T_{[1]} u
\]
\[
= -\frac{i}{2}U^* H^2_{u_0} u = -\frac{i}{2} H^2_{u_0} U^* u.
\]
Since $U^*(0) = U(0) = I$, this yields the first equality. By equation (1.6) and using the fact that the operator $H_u$ is skew-symmetric, we can rewrite (2.4) as
\[
H_{u_0} \left( U^*(t)g(t) - e^{i\frac{t}{2}H^2_{u_0}}g_0 \right) = 0.
\]
By (1.7) we have that $U^*(t)g(t) - e^{i\frac{t}{2}H^2_{u_0}}g_0 \in \text{Ran}(H_{u_0})$ and since $H_{u_0}$ is one to one on $\text{Ran}(H_{u_0})$, the second equality follows.

In the following we denote the unitary operator $e^{i\frac{t}{2}H^2_{u_0}}$ by $W(t)$. The skew-symmetry of the Hankel operator $H_{u_0}$ yields
\[
(2.5) \quad H_{u_0} W = W^* H_{u_0}.
\]
We also set
\[
(2.6) \quad \tilde{e}(t) := U^*(t)g(t) = e^{i\frac{t}{2}H^2_{u_0}}g_0 = W(t)g_0.
\]
With these notations we have, by equation (1.6), that
\[
(2.7) \quad u(t) = H_{u(t)}(g(t)) = U(t)H_{u_0}U^*(t)g(t) = U(t)(H_{u_0}\tilde{e}(t)).
\]

**Definition 2.1.** Let us denote by $P_u$ the orthogonal projection on $\text{Ran}(H_u)$. We also denote by $T_{\lambda}$, $\lambda > 0$, the compressed shift operators acting on $\text{Ran}(H_u)$ by
\[
T_{\lambda} f = P_u(e^{i\lambda x} f), \text{ for all } f \in \text{Ran}(H_u).
\]
If $u(t)$ is the solution of the Szegö equation with initial condition $u_0$ and $T_{\lambda}(t)$ acts on $\text{Ran}(H_{u(t)})$, then we define the operators $S_{\lambda}(t)$, $\lambda > 0$, $t \in \mathbb{R}$ on $\text{Ran}(H_{u_0})$ by
\[
S_{\lambda}(t)f = U^*(t)T_{\lambda}(t)U(t) = P_{u_0} U^*(t) e^{i\lambda x} U(t) f, \text{ for all } f \in \text{Ran}(H_{u_0}),
\]
Notice that using (1.7), we have
\[
(2.8) \quad P_{u(t)} e^{i\lambda x} g(t) = U(t)(P_{u_0} U^*(t) e^{i\lambda x} U(t))U^*(t)g(t) = U(t)(S_{\lambda}(t)\tilde{e}).
\]
Definition 2.2. Let \( u = \frac{A}{x^r} \in \mathcal{M}(N) \), where \( A \) and \( B \) are relatively prime, \( B(0) = 1 \), \( B(x) = (x - p_1)^{m_1} \cdots (x - p_k)^{m_k}, m_1 + \ldots + m_k = N \), and \( \text{Im}(p_j) < 0 \) for all \( j = 1, 2, \ldots, k \). For all \( f \in \text{Ran}(H_u) \),

\[
f = \sum_{j=1}^{k} \frac{\alpha_j}{x - p_j} + \sum_{j=1}^{k} m_j \sum_{l_j=2} \frac{\beta_j^l}{(x - p_j)^{l_j}},
\]

we define

\[
\Lambda(f) := \sum_{j=1}^{k} \alpha_j = \lim_{x \to \infty} xf(x).
\]

The infinitesimal shift operator is the linear operator \( T \) defined on \( \text{Ran}(H_u) \) by:

\[
T(f) = xf - \Lambda(f)b_u.
\]

Notice that by \([1,3]\), we have that \( T(f) \in \text{Ran}(H_u) \) for all \( f \in \text{Ran}(H_u) \).

If \( u(t) \) is the solution of the Szeg"{o} equation with initial condition \( u_0 \) and \( T(t) \) is the operator \( T \) acting on \( \text{Ran}(H_u(t)) \), we introduce the family of operators \( S(t) \) acting on \( \text{Ran}(H_u) \), by

\[
S(t) = U^*(t)T(t)U(t).
\]

Lemma 2.4. The eigenvalues of \( T \) and \( S \) are the complex conjugates of the poles of \( u \). In particular, the eigenvalues of \( T \) and \( S \) have strictly positive imaginary part.

Proof. Since \( T \) and \( S \) are conjugated, they have the same eigenvalues. If \( Tf = \lambda f \), then we have that \( (x - \lambda)f = \Lambda(f)b_u \). Taking \( x = \lambda \), we obtain that \( b_u(\lambda) = 0 \). Then, Lemma 2.1 yields that \( \lambda = \bar{\beta}_j \).

Remark 2.5. Notice that we can extend the definition of \( \Lambda \) to

\[
T_{[u]^2}(\text{Ran}(H_u)) = \left\{ \sum_{j=1}^{k} \sum_{l_j=1}^{2m_j} \frac{\beta_j^l}{(x - p_j)^{l_j}}; \beta_j^l \in \mathbb{C} \right\}.
\]

We then use formula (2.9) to extend the definition of \( T \) to \( T_{[u]^2}(\text{Ran}(H_u)) \).

Lemma 2.6. The operator \( iS \) is the infinitesimal generator of the semi-group \( S_\lambda \), i.e. \( S_\lambda = e^{i\lambda S} \) for all \( \lambda > 0 \).

Proof. Because of the definitions of \( S \) and \( S_\lambda \) in terms of \( T \) and \( T_\lambda \), it is enough to prove that

\[
-i \frac{d}{d\lambda|_{\lambda=0}} T_\lambda f = TT_{\lambda|_{\lambda=0}} f,
\]

where \( T \) and \( T_\lambda \) act on \( \text{Ran}(H_u) \).

Define the linear operator \( L : \text{Hol}(\mathbb{C}_+) \to \mathbb{C}^N \) by

\[
L : f \mapsto \{ \partial_x^m f(\bar{p}_j) | j \in \{ 1, 2, \ldots, k \}, m \in \{ 0, 2, \ldots, m_j - 1 \} \},
\]

where \( p_1, p_2, \ldots, p_k \) are the poles of \( u \) and \( m_j \) is the multiplicity of the pole \( p_j \). Then, we have that \( \text{Ker}L = b_u \text{Hol}(\mathbb{C}_+) \), where \( b_u = \prod_{j=1}^{k} \left( \frac{x - \bar{p}_j}{x - p_j} \right)^{m_j} \). In particular, \( L_{[\text{Ran}(H_u)]} : \text{Ran}(H_u) \to L(\text{Ran}(H_u)) \) is a isomorphism. Since \( T_\lambda f, Tf \in \text{Ran}(H_u) \) for all \( f \in \text{Ran}(H_u) \),
the lemma is proved once we show that \( L(-i\frac{d}{dx})\lambda=0 T\lambda f) = L(T f) \). This is indeed true since \( L(b_u) = 0, L(h) = L(P_u h) \) for all \( h \in L^2 \), and
\[
\frac{d}{d\lambda}|_{\lambda=0} L(T\lambda f) = \frac{d}{d\lambda}|_{\lambda=0} L(P_u(e^{i\lambda} f)) = \frac{d}{d\lambda}|_{\lambda=0} L(e^{i\lambda} f)
= iL(x f) = iL(x f - \Lambda(f)\lambda) = iL(T f).
\]

\[\square\]

**Proposition 2.7.** If \( u(t) \) is the solution of the Szegő equation corresponding to the initial data \( u_0 \in \mathcal{M}(N) \), then the following formula holds:

\[ (2.10) \quad u(t, x) = \frac{i}{2\pi} \left( u_0, W(t)(S - xI)^{-1} W(t)g_0 \right). \]

**Proof.** Using the Cauchy integral formula, Plancherel’s identity, equation (2.2), Lemma 2.2, equations (2.7) and (2.8), the fact that \( U(t) \) are unitary operators, equation (2.2), and Lemma 2.6, we have for \( \text{Im} z > \mu \)

\[ (2.11) \quad u(z, t) = \frac{1}{2\pi i} \int_0^\infty \frac{u(x)}{x - z} dx = \frac{1}{2\pi i} \int_0^\infty \hat{u}(t, \lambda) \frac{1}{\lambda - e^{iz}} d\lambda = \frac{1}{2\pi} \int_0^\infty e^{iz\lambda} \hat{u}(t, \lambda) d\lambda 
= \frac{1}{2\pi} \int_0^\infty e^{iz\lambda} (U(t)(H_{u_0} \hat{e}), U(t)(S\lambda(t) \hat{e})) d\lambda = \frac{1}{2\pi} \int_0^\infty e^{iz\lambda} (H_{u_0} \hat{e}, S\lambda(t) \hat{e}) d\lambda 
= \frac{1}{2\pi} \int_0^\infty e^{iz\lambda} (H_{u_0}(W(t)g_0), S\lambda(t)W(t)g_0) d\lambda = \frac{1}{2\pi} \int_0^\infty e^{iz\lambda} (W(t)^* H_{u_0}g_0, S\lambda(t)W(t)g_0) d\lambda 
= \frac{1}{2\pi} (W(t)^* u_0, \int_0^\infty e^{i(z(S - i\lambda I) - 1)} d\lambda W(t)g_0) = \frac{1}{2\pi} \left( u_0, W(t)(iS - i\hat{z}I)^{-1} W(t)g_0 \right). \]

The above formula also holds for \( x \in \mathbb{R} \) since, by Lemma 2.4, the eigenvalues of \( S \) are not real numbers. \[\square\]

Notice that in this formula for \( u(t) \), the operator \( S(t) \) is defined using \( U(t) \) whose definition depends on \( u(t) \). Our goal is to characterize \( S(t) \) without using \( U(t) \). In order to do that, we need to determine the derivative in time of \( S(t)h \), for any \( h \in \text{Ran}(H_{u_0}) \). This derivative is expressed in terms of commutators of \( T \) with Hankel and Toeplitz operators, that we compute in the following.

The below formula, that can be proved by passing into the Fourier space, will be useful:

\[ (2.11) \quad \Pi(x f) = x\Pi(f) + \frac{1}{2\pi i} \int f, \]

if \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) and \( xf \in L^2(\mathbb{R}) \).

**Lemma 2.8.** If \( u \in \mathcal{M}(N) \) and \( f \in \text{Ran}(H_u) \), then

\[ (2.12) \quad \Lambda(H_u f) = -\frac{1}{2\pi i} \int u \bar{f}, \]

\[ (2.13) \quad \Lambda(f) = -\frac{1}{2\pi i} \langle f, g \rangle \text{ for all } f \in \text{Ran}(H_u), \]

\[ (2.14) \quad \Lambda(T(u|f) = -\frac{1}{2\pi i} \int |u|^2 f. \]
Proof. The result follows once we prove that for all \( f_1, f_2 \in \mathcal{M}(N) \) that have the same poles, \( p_1, \ldots, p_k \), we have

\[
\Lambda(\Pi(f_1 f_2)) = \frac{1}{2\pi i} \int f_1 \tilde{f}_2.
\]

Indeed, (2.12) follows taking \( f_1 = u, f_2 = f \) and (2.14) follows taking \( f_1 = uf, f_2 = u \) Then, (2.13) is a direct consequence of (2.12). In order to prove (2.15) we decompose \( f_1 \tilde{f}_2 \) into simple rational fractions:

\[
f_1 \tilde{f}_2 = \frac{A_1}{x - p_1} + \cdots + \frac{A_k}{x - p_k} + \frac{B_1}{x - p_1} + \cdots + \frac{B_k}{x - p_k} + \frac{k}{\prod (x - p_j)^{l_j}} + \frac{k}{\prod (x - p_j)^{l_j}}.
\]

Since \( \text{Im}(p_j) < 0 \) for all \( j = 1, 2, \ldots, k \), the residue theorem yield:

\[
\int f_1 \tilde{f}_2 = -2\pi i (A_1 + \cdots + A_k) = -2\pi i \Lambda(\Pi(f_1 \tilde{f}_2)).
\]

\[\square\]

**Lemma 2.9.** For all \( h \in \text{Ran}(H_u) \) we have

\[
[T, T_{|u|^2}]h = -\frac{1}{2\pi i} \left( \int |u|^2 \right) g + \Lambda(h)|u|^2 b_u,
\]

\[
[T, H_u^2]h = -\frac{1}{2\pi i} \left( \int |u|^2 \right) g + \frac{1}{2\pi i} \left( \int \bar{u}h \right) u.
\]

Proof. Using equations (2.11), (2.14), and (2.3), we have

\[
[T, T_{|u|^2}]h = xT_{|u|^2}h - \Lambda(T_{|u|^2}h) b_u - T_{|u|^2} (xh - \Lambda(h)b_u)
\]

\[
= \left( x\Pi(|u|^2) - \Pi(x|u|^2)h \right) - \Lambda(T_{|u|^2}h) b_u + \Lambda(h)\Pi(|u|^2)b_u
\]

\[
= -\frac{1}{2\pi i} \left( \int |u|^2 \right) g + \frac{1}{2\pi i} \left( \int |u|^2 f \right) b_u + \Lambda(h)|u|^2 b_u.
\]

The first formula now follows using equation (2.1). Secondly, using equations (2.3), (2.11) twice, (2.12), and (2.1), we have

\[
[T, H_u^2]h = xH_u^2h - \Lambda(H_u^2h) b_u - H_u \left( \Pi(xu \tilde{h}) - \Lambda(h)\Pi(u \tilde{b}_u) \right)
\]

\[
= xH_u^2h - H_u \left( \Pi(xu \tilde{h}) \right) - \Lambda(H_u^2h) b_u
\]

\[
= \Pi(xu \tilde{H_u}h) - \frac{1}{2\pi i} \int u \tilde{H_u}h - H_u \left( x\Pi(u \tilde{h}) + \frac{1}{2\pi i} \int u \tilde{h} \right) - \Lambda(H_u^2h) b_u
\]

\[
= \Pi(xu \tilde{H_u}h) - \frac{1}{2\pi i} \int u \tilde{H_u}h - \Pi(u \tilde{H_u}h) + \frac{1}{2\pi i} \left( \int \bar{u}h \right) u - \Lambda(H_u^2h) b_u
\]

\[
= -\frac{1}{2\pi i} \left( \int u \tilde{H_u}h \right) (1 - b_u) + \frac{1}{2\pi i} \left( \int \bar{u}h \right) u
\]

\[
= -\frac{1}{2\pi i} (u, u \tilde{h}) g + \frac{1}{2\pi i} \left( \int \bar{u}h \right) u.
\]

\[\square\]
Lemma 2.10. For all \( h \in \text{Ran}(H_{u_0}) \), we have

\[
(2.18) \quad P_{u_0} \frac{d}{dt} S(t) h = \frac{1}{4\pi} \left( (h, H_{u_0}^2 \hat{e}) \hat{e} + (h, H_{u_0} \hat{e}) H_{u_0} \hat{e} \right).
\]

Proof. Using equation (1.5), we have that

\[
P_{u_0} i \frac{d}{dt} S(t) h = P_{u_0} U^* [T, T]|_u^2 - \frac{1}{2} H_{u_0}^2 U h + P_{u_0} U^* (i \frac{d}{dt} T(t)) U h.
\]

Then, by Lemma 2.9, equation (2.1), \( b_u = 1 - g \), equations (2.6) and (2.7), we have

\[
P_{u_0} i \frac{d}{dt} S(t) h = P_{u_0} U^* \left( -\frac{1}{2\pi i} \left( \int |u|^2 U h \right) g + \Lambda(U h)|u|^2 b_u \\
+ \frac{1}{4\pi i} \left( \int |u|^2 U h \right) \hat{e} - \frac{1}{4\pi i} \left( \int \bar{u} U h \right) + P_{u_0} U^* (ig' \Lambda(U h)) \\
+ \Lambda(U h) P_{u_0} U^* (|u|^2 b_u) + \Lambda(U h) P_{u_0} U^* (ig').
\]

In order to compute \( g'(t) \), we will differentiate the equality \( u = H_u g \). We obtain:

\[
-iT|_u^2 u = [B_u, H_u] g + H_u(g') = -iT|_u^2 H_u g - iH_u T|_u^2 g + iH_u^3 g + H_u(g').
\]

Then, \( H_u(g' + i\Pi(|u|^2 (g - 1))) = 0 \) and thus by (2.3) we have \( P_u (ig') = -P_u \Pi(|u|^2 b_u) = -P_u (|u|^2 b_u) \). Consequently, by (1.7) we have

\[
P_{u_0} U^* (ig') = U^* P_u (ig') = -U^* P_u (|u|^2 b_u) = -P_{u_0} U^* (|u|^2 b_u).
\]

Therefore we obtain

\[
P_{u_0} \frac{d}{dt} S(t) h = \frac{1}{4\pi} \left( \int |u|^2 U h \right) \hat{e} + \frac{1}{4\pi} \left( \int \bar{u} U h \right) H_{u_0} \hat{e}.
\]

To conclude, we only need to rewrite the two parenthesis so that they do not depend on \( U \). By equation (1.6), the definitions of \( g, \hat{e} \), and equation (1.3), we have:

\[
\int |u|^2 U h = (u, u \bar{U} h) = (u, \Pi(u \bar{U} h)) = (u, H_u(U h)) = (u, U H_{u_0} h) = (U^* H_u g, H_{u_0} h) \\
= (H_{u_0} U^* g, H_{u_0} h) = (H_{u_0} \hat{e}, H_{u_0} h) = (h, H_{u_0}^2 \hat{e}).
\]

(2.19) \[
\int \bar{u} U h = (U h, u) = (U h, H_u g) = (h, U^* H_u g) = (h, H_{u_0} U^* g) = (h, H_{u_0} \hat{e}).
\]

\( \square \)

In order to express \( S \) without using \( U(t) \), we also need to determine the adjoint \( S^* \) of the operator \( S \) and prove the commutation relation \( S^* H_{u_0} = H_{u_0} S \). We first determine \( T^* \).

Lemma 2.11. The adjoint of the operator \( T \) on \( \text{Ran}(H_u) \) is the operator \( T^* \) defined by

\[
T^* f = xf - \Lambda(f), \text{ for all } f \in \text{Ran}(H_u).
\]
Lemma 2.12.

\[ S^* H_{u_0} = H_{u_0} S \]

and

\[(2.20) \quad S = S^* - \frac{1}{2\pi i} (\cdot, \tilde{e})\tilde{e}. \]

Proof. By projecting equation (2.11) on \( \text{Ran}(H_u) \), we obtain \( T^*_\lambda H_u = H_u T_\lambda \). Then, by Lemma 2.6 it follows that \( T^*_H = H_u T \). This and equation (1.6) yield for all \( h \in \text{Ran}(H_{u_0}) \) that

\[ H_{u_0} S h = H_{u_0} U^* T U h = U^* H U T h = U^* T^*_H U h = U^* T^*_H U h = S^* H_{u_0} h. \]

Notice that (2.13) yields that

\[(2.21) \quad T = T^* - \frac{1}{2\pi i} (\cdot, g)g. \]

Then, (2.20) follows immediately by conjugating the above relation with \( U(t) \).

\[ \square \]

Proof of Theorem 1.7 By conjugating equation (2.17) by \( U(t) \), we obtain:

\[ [H_{u_0}^2, S] h = \frac{1}{2\pi i} \left( (h, H_{u_0}^2)\tilde{e} - (h, H_{u_0} \tilde{e})H_{u_0} \tilde{e} \right), \]

for all \( h \in \text{Ran}(H_{u_0}) \). Applying this to \( h = e_j \) we have

\[ (H_{u_0}^2 - \lambda_j^2) Se_j = \frac{\lambda_j}{2\pi i} \left( \lambda_j (e_j, \tilde{e})\tilde{e} - (e_j, \tilde{e})H_{u_0} \tilde{e} \right). \]

Suppose that \( \lambda_j \) is an eigenvalue of multiplicity \( m_j \) and that \( M_j \) is the set of all indices \( k \) such that \( H_{u_0} e_k = \lambda_j e_k \). Plugging \( \tilde{e} = \sum_{k=1}^N (\tilde{e}, e_k)e_k \) in the above formula we have:

\[ (H_{u_0}^2 - \lambda_j^2) Se_j = \frac{\lambda_j}{2\pi i} \sum_{k \in M_j} \left( \lambda_j (e_j, \tilde{e})\tilde{e} - \lambda_k (e_j, e_k)\tilde{e} \right) e_k \]

\[ + \frac{\lambda_j^2}{2\pi i} \sum_{k \in M_j} \left( (e_j, \tilde{e})\tilde{e} - (e_j, \tilde{e})\tilde{e} \right) e_k. \]

Since

\[ (\tilde{e}, e_j) = (e^{i\frac{\lambda_j}{2} H_{u_0}} g_0, e_j) = e^{i\frac{\lambda_j}{2} \lambda_j} (g_0, e_j) = e^{i\frac{\lambda_j}{2} \lambda_j} \beta_j, \]

we have

\[ (H_{u_0}^2 - \lambda_j^2) Se_j = \frac{\lambda_j}{2\pi i} \sum_{k \in M_j} \left( \lambda_j (e_j, \tilde{e})\tilde{e} - \lambda_k (e_j, e_k)\tilde{e} \right) e_k \]

\[ + \frac{\lambda_j^2}{2\pi i} \sum_{k \in M_j} \left( (e_j, \tilde{e})\tilde{e} - (e_j, \tilde{e})\tilde{e} \right) e_k. \]
we obtain
\[(H_{u_0}^2 - \lambda_j^2)S e_j = \frac{\lambda_j}{2\pi i} \sum_{k \notin M_j} \left( \lambda_j e^{i\frac{\lambda_k^2 - \lambda_j^2}{2\pi i} \beta_j \beta_k} - \lambda_k e^{i\frac{\lambda_k^2 - \lambda_j^2}{2\pi i} \beta_j \beta_k} \right) e_k \]
\[+ \frac{\lambda_j^2}{2\pi i} \sum_{k \in M_j} (\beta_j \beta_k - \beta_j \overline{\beta_k}) e_k \]
Writing
\[S(t)e_j = \sum_{k=1}^{N} c_j^k(t) e_k,\]
we have
\[(H_{u_0}^2 - \lambda_j^2)S(t)e_j = \sum_{k \notin M_j} (\lambda_k^2 - \lambda_j^2) c_j^k(t) e_k.\]
Identifying the coefficients of \((H_{u_0}^2 - \lambda_j^2)S(t)e_j\) in the basis \(\{e_k\}_{k=1}^{N}\), we obtain that
\[(2.22) \quad \overline{\beta_j} \beta_k \in \mathbb{R}, \text{ for all } k \in M_j \]
and
\[c_j^k(t) = \lambda_j \frac{\lambda_j}{2\pi i(\lambda_k^2 - \lambda_j^2)} \left( \lambda_j e^{i\frac{\lambda_k^2 - \lambda_j^2}{2\pi i} \beta_j \beta_k} - \lambda_k e^{i\frac{\lambda_k^2 - \lambda_j^2}{2\pi i} \beta_j \beta_k} \right) \]
for all \(k \notin M_j\). Finally, we determine \(c_j^k(t)\) for \(k \in M_j\) using Lemma 2.10:
\[\frac{d}{dt} c_j^k(t) = \frac{d}{dt} (S(t)e_j, e_k) = (P_{u_0} \frac{d}{dt} S(t)e_j, e_k) = \frac{\lambda_j}{4\pi} \left( (e_j, \dot{e})(e_j, e_k) + (\dot{e}, e_j)(e_k, e) \right) \]
\[= \frac{\lambda_j^2}{4\pi} (\overline{\beta_j} \beta_k + \beta_j \overline{\beta_k}) = \frac{\lambda_j^2}{2\pi i} \beta_j \beta_k. \]
Therefore, for \(k \in M_j\) we have
\[c_j^k(t) = \frac{\lambda_j^2}{2\pi i} \beta_j \beta_k t + c_j^k(0), \]
where \(c_j^k(0) = (S(0)e_j, e_k) = (Te_j, e_k). \]

3. Extension of the formula to general initial data

Proof of Corollary 2.8. The proof of Theorem 1.7 can be adapted to the case of a general initial data, as long as \(u_0 \in \text{Ran}(H_{u_0})\), i.e. there exists \(g_0 \in \text{Ran}(H_{u_0})\) such that \(u_0 = H_{u_0}(g_0)\). Writing \(g_0 = \sum_{j=1}^{\infty} (g_0, e_j) e_j\) in the basis \(\{e_j\}_{j=1}^{\infty}\), the fact that \(g_0 \in L^2(\mathbb{R})\) is equivalent to \(\sum_{j=1}^{\infty} |(g_0, e_j)|^2 < \infty\). Since \(u_0 = H_{u_0}(g_0)\) yields \((u_0, e_j) = \lambda_j (e_j, g)\) for all \(j \in \mathbb{N}^*\), it follows that \(\{\beta_j\}_{j=1}^{\infty} = \{\lambda_j (u_0, e_j)\}_{j=1}^{\infty} \in \ell^2\).

The main difference with the case of rational functions data is that \(S\) is no longer a matrix, but an operator acting between infinite dimensional spaces. Then, the infinitesimal generator of the semi-group \(S_t\) is not \(iS\), but its closure \(i\tilde{S}\) (like in Proposition 3.4). This explains the operator \(\tilde{S}\) appearing in the explicit formula. \(\square\)

Proposition 3.1. Let \(s \geq 1\). If \(u_0 \in H_+^s\) and \(xu_0 \in L^\infty(\mathbb{R})\), then the corresponding solution of the Szegö equation satisfies \(xu(t,x) \in L^\infty(\mathbb{R})\) for all \(t \in \mathbb{R}\).
Proof. The local well-posedness follows using a fixed point argument in the space \((L^\infty_t, X)\), where

\[
X := H^s_+(\mathbb{R}) \cap \{f \mid xf(x) \in L^\infty(\mathbb{R})\}.
\]

By equation (2.11), the Hölder inequality, and Sobolev embedding, we have:

\[
\left\| x \int_0^T \Pi(|u(t)|^2u(t)) \right\|_{L^\infty_t} \leq T \left\| x \Pi(|u(t)|^2u(t)) \right\|_{L^\infty_t} \leq T \left\| \Pi(x|u(t)|^2u(t)) \right\|_{L^\infty_t} + \frac{T}{2\pi} \left\| \int |u(t)|^2u(t) dt \right\|_{L^\infty_t} \leq T \left\| \Pi(x|u(t)|^2u(t)) \right\|_{L^\infty_t} + \frac{T}{2\pi} \left\| \int |u(t)|^2u(t) dt \right\|_{L^\infty_t} \leq T \left(2 \|u\|_{L^\infty_t} + \|u\|_{L^\infty_t} \right) \leq T \left(4 \|u\|_{L^\infty_t} + \|u\|_{L^\infty_t} \right) \leq T \left(4 \|u\|_{L^\infty_t} + \|u\|_{L^\infty_t} \right).
\]

The global well-posedness is a consequence of the Brezis-Galouet estimate

\[
\|u\|_{L^\infty(\mathbb{R})} \leq C \|u\|_{H^{1/2}(\mathbb{R})} \left(\log \left(2 + \frac{\|u\|_{H^s}}{\|u\|_{H^{1/2}}}\right)\right)^{1/2},
\]

and of Gronwall’s inequality. \(\Box\)

Lemma 3.2. For all \(u \in H^{1/2}_+(\mathbb{R})\), we have that \(u \in \text{Ran}(H_u)\).

Moreover, if \(u \in H^s(\mathbb{R})\), \(s > \frac{1}{2}\) and \(xu(x) \in L^\infty(\mathbb{R})\), we have that \(u = \lim_{\varepsilon \to 0} H_u(\frac{1}{1-i\varepsilon x})\).

Proof. For \(h \in L^2_+\), we have that

\[
(u, h) = \lim_{\varepsilon \to 0} \left(u, \frac{h}{1-i\varepsilon x}\right) = \lim_{\varepsilon \to 0} \left(u\overline{h}, \frac{1}{1-i\varepsilon x}\right) = \lim_{\varepsilon \to 0} \left(H_u h, \frac{1}{1-i\varepsilon x}\right).
\]

Taking \(h \in \text{Ker}(H_u)\), it follows that \((u, h) = 0\) and \(u \in (\text{Ker}(H_u))^\perp = \text{Ran}(H_u)\).

By (L3), the above equation also yields that for all \(h \in L^2_+\), we have that

\[
(u, h) = \lim_{\varepsilon \to 0} \left(H_u(\frac{1}{1-i\varepsilon x}), h\right).
\]

Then, \(H_u(\frac{1}{1-i\varepsilon x})\) converges weakly to \(u\) in \(L^2_+\). We now intend to prove that, if \(u \in H^s(\mathbb{R})\) and \(xu(x) \in L^\infty(\mathbb{R})\), then \(\|H_u(\frac{1}{1-i\varepsilon x})\|_{L^2} \to \|u\|_{L^2}\). This yields that the convergence is strong in \(L^2_+\).

Computing the Fourier transform with the residue theorem, we have that

\[
H_u(\frac{1}{1-i\varepsilon x}) = \Pi\left(\frac{u(x)}{1+i\varepsilon x}\right) = \frac{1}{i\varepsilon} \Pi\left(\frac{u(x)}{x-i\varepsilon}\right) = \frac{u(x) - u(\frac{x}{\varepsilon})}{1+i\varepsilon x} = \frac{u(x) - u(\frac{x}{\varepsilon})}{1+i\varepsilon x}.
\]

By the Sobolev embedding \(H^s(\mathbb{R}) \subset L^\infty(\mathbb{R})\) for \(s > \frac{1}{2}\), we have that there exists \(C_0 > 0\) such that \(|u(x)| \leq C_0\) for all \(x \in \mathbb{R}\). Since \(u\) is a holomorphic function in \(\mathbb{C}_+\), we can write
using the Poisson integral
\[ u(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \text{Im} z \frac{u(x)}{|z - x|^2} dx, \]
for all \( z \in \mathbb{C}_+. \) Then
\[ |u(z)| \leq \frac{C_0}{\pi} \text{Im} z \int_{-\infty}^{\infty} \frac{1}{|z - x|^2} dx = C_0, \]
for all \( z \in \mathbb{C}_+. \) Thus, \( u \) is bounded in \( \mathbb{C}_+ \cup \mathbb{R}. \) Similarly, since \( xu(x) \in L^\infty(\mathbb{R}) \), we have that \( zu(z) \) is bounded in \( \mathbb{C}_+ \cup \mathbb{R} \) by a constant \( C_1. \) In particular, we have that \( \frac{1}{\varepsilon}u(\frac{1}{\varepsilon}) \leq C_1 \) and thus \( \lim_{\varepsilon \to 0} u(\frac{1}{\varepsilon}) = 0. \) Then, by (3.2), we have that \( H_u \left( \frac{1}{1 - i\varepsilon x} \right) \) converges pointwise to \( u(x). \) Furthermore,
\[ \left| H_u \left( \frac{1}{1 - i\varepsilon x} \right) \right|^2 = \left| \frac{u(x) - u(\frac{1}{\varepsilon})}{1 + i\varepsilon x} \right|^2 \leq \left| u(x) - u(\frac{1}{\varepsilon}) \right|^2 \leq 2(\left| u(x) \right|^2 + \left| u(\frac{1}{\varepsilon}) \right|^2) \leq 4C_0 \]
and
\[ \left| H_u \left( \frac{1}{1 - i\varepsilon x} \right) \right|^2 \leq \frac{2(\left| u(x) \right|^2 + \left| u(\frac{1}{\varepsilon}) \right|^2)}{1 + \varepsilon^2 x^2} \leq \frac{C_0 + C_1 \varepsilon^2}{1 + \varepsilon^2 x^2} = C_1 \frac{1}{x^2}. \]
Then, the functions \( \left| H_u \left( \frac{1}{1 - i\varepsilon x} \right) \right|^2 \) are bounded by an integrable function. By the dominated convergence theorem, it follows that \( \|H_u \left( \frac{1}{1 - i\varepsilon x} \right)\|_{L^2} \to \|u\|_{L^2}. \) Hence, \( H_u \left( \frac{1}{1 - i\varepsilon x} \right) \to u \) in \( L^2_+. \)

The key point in extending the explicit formula for the solution to the case of general initial data is the below definition of the operator \( T^*: \text{Ran}(H_u) \to L^2_+. \)

(3.3) \[ T^*(H_u f) = xH_u(f) + \frac{1}{2\pi i}(u, f). \]
If \( xu \in L^\infty(\mathbb{R}), \) by (2.11) we have that
\[ T^*(H_u f) = \Pi(xu \tilde{f}). \]

**Remark 3.3.** If \( u \in H^s_+ \) for \( s > \frac{1}{2} \) and \( xu \in L^\infty(\mathbb{R}), \) then the operator \( T^* \) takes values in \( \text{Ran}(H_u). \)

**Proof.** For all \( f \in \text{Ran}(H_u) \) and \( h \in \text{Ker}(H_u), \) we have that
\[ (T^* f, h) = (\Pi(xu \tilde{f}), h) = (xu \tilde{f}, h) = \lim_{\varepsilon \to 0} (xu \tilde{f}, \frac{h}{1 - i\varepsilon x}) = \lim_{\varepsilon \to 0} (uh, x \frac{f}{1 - i\varepsilon x}) \]
\[ = \lim_{\varepsilon \to 0} (H_u h, x \frac{f}{1 - i\varepsilon x}) = 0. \]
Then, \( T^* f \in (\text{Ker}(H_u))^\perp = \text{Ran}(H_u). \)

For \( \lambda > 0, \) we introduce the operators \( T^*_\lambda : L^2_+ \to L^2_+ \) by
\[ T^*_\lambda h(x) = Pu e^{-i\lambda x} \mathcal{F}^{-1}(\hat{h}(\xi) \chi_{[\lambda, +\infty)}(\xi)) = Pu \left( e^{-i\lambda x} h(x) - \frac{e^{-i\lambda x}}{2\pi} \int_{\mathbb{R}} h(x - y) \frac{e^{i\lambda y} - 1}{iy} dy \right). \]
Then
\[ \lim_{\lambda \to 0} \frac{T^*_\lambda h(x) - h(x)}{\lambda} = Pu \left( -ixh(x) - \frac{1}{2\pi} \int_{\mathbb{R}} h(x) dx \right). \]
Let us now conjugate $T^*$ and $T^*_\lambda$ using the operators $U(t)$. We obtain $S^*(t)$ and $S^*_\lambda(t)$:

$$S^*(t) = U^*(t)T^*U(t), \quad S^*_\lambda(t) = U^*(t)T^*_\lambda U(t).$$

**Proposition 3.4.** The closure of the operator $-iS^*$ is the infinitesimal generator of the semi-group $S^*_\lambda$. Moreover, $\text{Ran}(H_{u_0})$ is a core for the infinitesimal generator of the semi-group $S^*_\lambda$.

**Proof.** If $h = H_{u_0}f \in \text{Ran}(H_{u_0})$, then we have

$$\lim_{\lambda \to 0} \frac{T^*_{\lambda}h(x) - h(x)}{\lambda} = -iP_u(xh(x) + \frac{1}{2\pi i}(u, f)) = -iT^*h.$$

Conjugating with $U(t)$, we obtain that the restriction of the infinitesimal generator of $S^*_\lambda$ to $\text{Ran}(H_{u_0})$ is $-iS^*$.

Moreover, by conjugating $T^*_\lambda H_u H_{u_0} = H_u T^*_\lambda$ with $U(t)$, we obtain $S^*_\lambda H_u = H_u S^*_\lambda$. This yields

$$S^*_\lambda(\text{Ran}(H_{u_0})) \subset \text{Ran}(H_{u_0}).$$

By Theorem X.49, vol. II in [30], we have that $\text{Ran}(H_{u_0})$ is a core of the infinitesimal generator of $S^*_\lambda$. Then, the infinitesimal generator of $S^*_\lambda$ is the closure $-iA$ of $-iS^*$.

**Proof of Theorem 4.4.** According to Proposition 3.4, we have that $u(t) \in H^s$ and $xu(t, x) \in L^\infty(\mathbb{R})$ for all $t \in \mathbb{R}$. Then, by Lemma 3.2, we obtain that

$$u(t) = \lim_{\varepsilon \to 0} H_{u(t)}(\frac{1}{1 - i\varepsilon x}) \in L^2_{+}.$$

By Plancherel’s identity, this is equivalent to

$$\lim_{\varepsilon \to 0} \mathcal{F}\left(u(t)(1 - \frac{1}{1 + i\varepsilon x})\right) = 0 \in L^2(\mathbb{R}_{+}).$$

Since,

$$\hat{u}(t, \lambda) = \int_{\mathbb{R}} e^{-i\lambda x} u(t, x)dx = \int_{\mathbb{R}} e^{-i\lambda x} u(t)(1 - \frac{1}{1 + i\varepsilon x})dx + \int_{\mathbb{R}} e^{-i\lambda x} u(t) \frac{1}{1 + i\varepsilon x} dx$$

$$= \mathcal{F}\left(u(t)(1 - \frac{1}{1 + i\varepsilon x})\right)(\lambda) + \int_{\mathbb{R}} e^{-i\lambda x} u(t) \frac{1}{1 + i\varepsilon x} dx,$$

we obtain that

$$\hat{u}(t, \lambda) = \lim_{\varepsilon \to 0} \left(u(t), e^{i\lambda x} \frac{1}{1 - i\varepsilon x}\right) dx.$$

The rest of the proof follows the same lines as the proof of Theorem 4.7 but uses $T^*$ and $S^*$ instead of $T$ and $S$. Special attention should be given to the fact that the infinitesimal generator of the semi-group $S^*_\lambda$ is not $-iS^*$, but its closure $-iA$.

**4. Soliton resolution in the case of strongly generic, rational function data**

We prove that all the solutions with strongly generic, rational function initial data $u_0 \in \mathcal{M}(N)_{sgen}$ resolve into N solitons and a remainder which tends to zero in all the $H^s$-norms for $s \geq 0$, when $t \to \pm \infty$. 
Proof of Theorem 1.10. The strategy is to write all the vectors in \( \text{Ran}(H_{u_0}) \) in the basis \( \{ e_j \}_{j=1}^N \) and make formula (2.10) more explicit.

According to Theorem 1.7, we have

\[
S(t)e_j = \left( \frac{\lambda_j^2 \nu_j^2}{2\pi} t + (S(0)e_j, e_j) \right)e_j + \sum_{i=1, i \neq j}^N a_{ji}(t)e_i.
\]

Since \( a_{ji}(t) \) are linear combinations of \( e^{\pm \frac{i}{2}(\lambda_j^2 - \lambda_i^2)} \) with constant coefficients, there exists \( M > 0 \) such that

\[
|a_{ji}(t)| \leq M,
\]

for all \( j \neq i \) and all \( t \in \mathbb{R} \). Denoting \( A_j = \frac{\lambda_j^2 \nu_j^2}{2\pi} t + (S(0)e_j, e_j) \), the operator \( S \) in the basis \( \{ e_j \}_{j=1}^N \) can be written as the following matrix:

\[
S = \begin{pmatrix}
A_1 & a_{12} & \cdots & a_{1N} \\
a_{21} & A_2 & \cdots & a_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
a_{N1} & a_{N2} & \cdots & A_N
\end{pmatrix}
\]

Let us first compute \( \text{Im}(A_j) = \text{Im}(S(0)e_j, e_j) \) for \( t \) large enough. By equation (2.20) and noting that \( \tilde{e}(0) = g_0 \), we have that

\[
2i\text{Im}(S(0)e_j, e_j) = (S(0)e_j, e_j) - (e_j, S(0)e_j) = (S(0) - S(0)^*)e_j, e_j
\]

\[
= \left( -\frac{1}{2\pi i} (e_j, g_0)g_0, e_j \right) = i\frac{2\pi}{2\pi} |(g_0, e_j)|^2.
\]

Therefore

\[
\text{Im}(S(0)e_j, e_j) = \frac{\nu_j^2}{4\pi}.
\]

Then, we notice that

\[
\int_{-\infty}^{\infty} \frac{dx}{|x - at + id|^2|x - ct + id|^2} = O\left( \frac{1}{t^2} \right) \quad \text{as} \quad t \to \pm\infty,
\]

if \( 0 < a < c \) and \( b, d \neq 0 \). This can be proved by estimating the integral on each of the intervals \( (-\infty, at - 1], [at - 1, at + 1], [at - 1, at + 1], [ct - 1, ct + 1], [ct + 1, \infty) \) if \( t > 0 \) large enough, and similarly for \( t < 0 \). Since \( \text{Im}(A_j) = \frac{\nu_j^2}{4\pi} > 0 \) and by the strong genericity hypothesis \( \frac{\lambda_j^2 \nu_j^2}{2\pi} \neq \frac{\lambda_k^2 \nu_k^2}{2\pi} \) for \( j \neq k \), this yields that

\[
\frac{1}{(x - A_j)(x - A_k)} = O\left( \frac{1}{|t|} \right) \quad \text{in} \quad L^2(\mathbb{R}) \quad \text{as} \quad t \to \pm\infty,
\]
Moreover, using \( \| \frac{1}{x-A_j} \|_{L^\infty} = \frac{1}{\text{Im} A_j} = \frac{4\pi}{\nu_j} \), we have that \( \frac{1}{(x-A_j)(x-A_k)} = O(\frac{1}{t}) \) in \( H^s(\mathbb{R}) \) for all \( s \geq 0 \). Furthermore, we have

\[
\left\| \frac{1}{(x-A_j)(x-A_k)} \right\|_{L^\infty} = \left\| \frac{1}{A_k - A_j} \left( \frac{1}{x-A_j} - \frac{1}{x-A_k} \right) \right\|_{L^\infty} \\
\leq \left\| \frac{1}{A_k - A_j} \right\| \left( \| \frac{1}{x-A_j} \|_{L^\infty} + \| \frac{1}{x-A_k} \|_{L^\infty} \right) \\
= \frac{1}{|A_k - A_j|} \left( \frac{4\pi}{\nu_j} + \frac{4\pi}{\nu_k} \right) = O(\frac{1}{t}).
\]

Therefore, \( \frac{\det(S-xI)}{(A_1-x)...(A_N-x)} - 1 \rightarrow 0 \) in \( L^\infty(\mathbb{R}) \) and in \( H^s, s \geq 0 \), as \( t \rightarrow \pm \infty \), since it is equal to a linear combination of \( \frac{1}{(x-A_1)...(x-A_N)} \) in \( H^s \). We notice that, using the definition of the determinant, the terms \( \frac{1}{x-A_j} \) do not appear in the above linear combination.

Then,

\[
(S - xI)^{-1} = \frac{1}{\det(S-xI)} \begin{pmatrix}
C_{11} & C_{12} & \cdots & C_{1N} \\
C_{21} & C_{22} & \cdots & C_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
C_{N1} & C_{N2} & \cdots & C_{NN}
\end{pmatrix}
\]

\[
= \frac{(A_1-x)...(A_N-x)}{\det(S-xI)} \begin{pmatrix}
\frac{1}{A_1-x} + O(\frac{1}{t}) & O(\frac{1}{t}) & \cdots & O(\frac{1}{t}) \\
O(\frac{1}{t}) & \frac{1}{A_2-x} + O(\frac{1}{t}) & \cdots & O(\frac{1}{t}) \\
\vdots & \vdots & \ddots & \vdots \\
O(\frac{1}{t}) & O(\frac{1}{t}) & \cdots & \frac{1}{A_N-x} + O(\frac{1}{t})
\end{pmatrix}
\]

as \( t \rightarrow \pm \infty \) in \( H^s(\mathbb{R}) \).

Therefore,

\[
W(S - xI)^{-1}Wg_0 = W(S - xI)^{-1}(\beta_1e^{i\frac{\lambda_1^2}{2}t}, \ldots, \beta_Ne^{i\frac{\lambda_N^2}{2}t})^t
\]

\[
= \left( \frac{e^{it\lambda_1^2} \beta_1}{x-A_1} + O(\frac{1}{t}), \ldots, \frac{e^{it\lambda_N^2} \beta_N}{x-A_N} + O(\frac{1}{t}) \right)^t.
\]

Since \( u_0 = \sum_{j=1}^N (u_0, e_j) e_j \) and by (1.3),

\[
(4.3) \quad (u_0, e_j) = (H_{u_0}g_0, e_j) = (H_{u_0}e_j, g_0) = \lambda_j(g_0, e_j) = \lambda_j\overline{\beta_j},
\]

we have

\[
u(t) = \frac{1}{2\pi}(u_0, W(S - xI)^{-1}Wg_0) = \frac{1}{2\pi} \cdot \frac{e^{-it\lambda_1^2} \lambda_1^2}{x-A_1} + \cdots + \frac{1}{2\pi} \cdot \frac{e^{-it\lambda_N^2} \lambda_N^2}{x-A_N} + O(\frac{1}{t})
\]
in $H^s$, $s \geq 0$. Since $\text{Im}(\tilde{A}_j) = -\frac{c^2}{4\pi} < 0$, we have that $u \in H^s_+$. Moreover, by (1.14), we have that each of the functions 

$$\frac{1}{2\pi} e^{-i\lambda t^2} \frac{\lambda^2}{x-A_j}$$

is a soliton of speed $c = \frac{\lambda^2}{2\pi}$ and frequency $\omega = \lambda^2_j$.

Let us notice that the result in Theorem 1.10 can also be restated in terms of $N$-solitons.

**Definition 4.1.** A $N$-soliton is a solution of the Szegő equation $u(t)$, such that there exist $N$ solitons $c_j(t)$, $j=1,2,\ldots,N$ satisfying

$$\|u(t) - \sum_{j=1}^{N} \frac{c_j(t)}{x-q_j(t)}\|_{H^{1/2}} \to 0 \text{ as } t \to -\infty.$$ 

If, moreover, there exist $\delta_j \in \mathbb{R}$, $j=1,2,\ldots,N$ such that

$$\|u(t) - \sum_{j=1}^{N} \frac{c_j(t)}{x-\delta_j-q_j(t)}\|_{H^{1/2}} \to 0 \text{ as } t \to +\infty,$$

we say that the $N$-soliton is pure or that the collision of the $N$ solitons is elastic, in the sense that there is no loss of energy in the collision.

Theorem 1.10 states for $s=1/2$ that if $u_0 \in \mathcal{M}(N\text{sgen})$, then the corresponding solution is a pure $N$-soliton. Moreover, there is no shift in the trajectories of the $N$ solitons, i.e. $\delta_j = 0$ for all $j=1,2,\ldots,N$. This situation is characteristic to completely integrable equations. For the one dimensional cubic NLS, KdV and mKdV, which are all completely integrable, it is known that $N$-solitons exist and are pure [32, 18]. For the gKdV equation with fourth order nonlinearity, which is not completely integrable, it was proved in [24] that the collision of solitons fails to be elastic by loss of a small quantity of energy.

5. **ASYMPTOTIC BEHAVIOR OF THE SOLUTION IN THE CASE OF NON-GENERIC, RATIONAL FUNCTION DATA**

We show that when $u_0 \in \mathcal{M}(2)$ is such that $H_{u_0}^2$ has a double eigenvalue, then the solution $u$ behaves as the sum of a soliton and a remainder, which tends to zero in the $H^s$-norms, $0 \leq s < 1/2$. However, $\|u(t)\|_{H^s} \to \infty$ if $s > 1/2$. An example of such an initial condition is $u_0 = \frac{2}{x+1} - \frac{4}{x+2}$. The operator $H_{u_0}^2$ has the double eigenvalue $(\frac{4}{3})^2$ in this case.

Let us consider an orthonormal basis $\{\tilde{e}_1, \tilde{e}_2\}$ of $\text{Ran}(H_{u_0})$ such that $H_{u_0} \tilde{e}_j = \lambda_j \tilde{e}_j$. Denoting $\tilde{\beta}_j = (g_0, \tilde{e}_j)$ and $\tilde{\nu}_j = |\tilde{\beta}_j|$ we have $g_0 = \beta_1 \tilde{e}_1 + \beta_2 \tilde{e}_2$ and $\|g_0\|_{L^2} = \tilde{\nu}_1^2 + \tilde{\nu}_2^2$. By (2.22), we have that $\beta_1 \beta_2 \in \mathbb{R}$. We assume that $\beta_1 \beta_2 = \tilde{\nu}_1 \tilde{\nu}_2$, and thus $\tilde{\beta}_j = e^{i\theta} \tilde{\nu}_j$ for $j=1,2$.

We make the following change of basis

$$e_1 := \frac{1}{\|g_0\|_{L^2}} (\tilde{\nu}_1 \tilde{e}_1 + \tilde{\nu}_2 \tilde{e}_2),$$

$$e_2 := \frac{1}{\|g_0\|_{L^2}} (\tilde{\nu}_2 \tilde{e}_1 - \tilde{\nu}_1 \tilde{e}_2).$$

Notice that this is also an orthonormal basis of $\text{Ran}(H_{u_0})$ and $H_{u_0} e_j = \lambda e_j$. Moreover, setting $\beta_j := (g_0, e_j)$ and $\nu_j = |\beta_j|$, we have

$$\beta_2 := (g_0, e_2) = 0,$$

(5.1)
and \( \nu_2 := |\beta_2| = 0 \). In the case when \( \beta_1 \bar{\beta}_2 = -\bar{\nu}_1 \nu_2 \), we can similarly choose an orthonormal basis for which \( \beta_2 = \nu_2 = 0 \).

**Lemma 5.1.** With the notations in Theorem 1.11 we set \( c_j(0) = (S(0)e_1, e_j), \) \( d_j(0) = (S(0)e_2, e_j) \) for \( j = 1, 2 \) and

\[
(5.2) \quad A := \frac{\lambda^2 \nu_1^2}{2\pi},
B := \frac{\lambda^2 \nu_1^2}{\pi} (c_1(0) - d_2(0)),
C := (c_1(0) - d_2(0))^2 + 4e_2(0)d_1(0).
\]

Then, \( 4A^2 C - B^2 > 0 \) and \( \text{Im}(B) = \frac{\lambda^2 \nu_1^2}{4\pi^2} > 0 \).

**Proof.** By equation (4.11) we have that \( \text{Im} c_1(0) = \frac{\nu_1^2}{4\pi} \) and \( \text{Im} d_2(0) = \frac{\nu_2^2}{4\pi} = 0 \). Then, we obtain

\[
(5.3) \quad \text{Im}(B) = \frac{\lambda^2 \nu_1^2}{\pi} \text{Im} c_1(0) = \frac{\lambda^2 \nu_1^4}{4\pi^2}.
\]

Let us notice that

\[
(5.4) \quad \beta_j = 2\pi i \Lambda(e_j).
\]

Indeed, since \( e_j \in \text{Ran}(H_{u_0}) \), there exists \( f_j \in L^2_+ \) such that \( e_j = H_{u_0}(f_j) \) and by equation (2.12) we have

\[
\Lambda(e_j) = \Lambda(H_{u_0}(f_j)) = -\frac{1}{2\pi i} (u_0, f_j) = -\frac{1}{2\pi i} (H_{u_0} g_0, f_j) = -\frac{1}{2\pi i} (H_{u_0} f_j, g_0) = -\frac{1}{2\pi i} \beta_j.
\]

Then,

\[
4A^2 C - B^2 = 4 \left( \frac{\lambda^2 \nu_1^2}{2\pi} \right)^2 \left( (c_1(0) - d_2(0))^2 + 4e_2(0)d_1(0) \right) - 4 \left( \frac{\lambda^2 \nu_1^2}{2\pi} \right)^2 (c_1(0) - d_2(0))^2
= 16 \left( \frac{\lambda^2 \nu_1^2}{2\pi} \right)^2 c_2(0)d_1(0).
\]

By equation (2.20) and noticing that \( \tilde{c}(0) = g_0 \), we have

\[
d_1(0) = (S(0)e_2, e_1) = (S^*(0)e_2, e_1) - \frac{1}{2\pi i} (e_2, g_0)(g_0, e_1) = (S^*(0)e_2, e_1) = (e_2, S(0)e_1) = \overline{c_2(0)}.
\]

Thus,

\[
4A^2 C - B^2 = 16 \left( \frac{\lambda^2 \nu_1^2}{2\pi} \right)^2 |d_2(0)|^2.
\]

Suppose by absurd that \( d_2(0) = (S(0)e_2, e_1) = 0 \). Since \( (e_2, e_1) = 0 \), and since \( e_1, e_2, S(0)e_2 \) belong to the two dimensional complex vector space \( \text{Ran}(H_{u_0}) \), it results that there exists \( a \in \mathbb{C} \) such that \( S(0)e_2 = ae_2 \). Using the fact that \( S(0) = T \) and the definition of \( T \), we obtain that \( e_2(x - a) = \Lambda(e_2)b_{u_0} \). Then, by equation (5.1) and (5.4), we obtain that \( \Lambda(e_2) = 0 \) and therefore \( e_2 = 0 \), which is a contradiction. Hence, \( 4A^2 C - B^2 > 0 \). \( \Box \)

**Proof of Theorem 1.11.** Let us first express \( S \) in the basis \{\( e_1, e_2 \)\} of \( \text{Ran}(H_{u_0}) \). By Theorem 1.7 we have

\[
S(t)e_1 = c_1(t)e_1 + c_2(t)e_2,
\]
with \( c_1(t) = \frac{\lambda^2 \nu^2}{2\pi} t + c_1(0) \) and \( c_2(t) = \frac{\lambda^2 \nu^2}{2\pi} \beta_2 t + c_2(0) = c_2(0) \), and
\[
S(t)e_2 = d_1(t)e_1 + d_2(t)e_2,
\]

with \( d_1(t) = \frac{\lambda^2 \nu^2}{2\pi} \beta_2 t + d_1(0) = d_1(0) \) and \( d_2(t) = \frac{\lambda^2 \nu^2}{2\pi} t + d_2(0) = d_2(0) \). We denoted \( c_j(0) = (S(0)e_1, e_j) \) and \( d_j(0) = (S(0)e_2, e_j) \), \( j = 1, 2 \). Moreover, by equation \((2.22)\), we have that \( \beta_1 \beta_2 \in \mathbb{R} \).

Therefore, in the basis \( \{e_1, e_2\} \), the operator \( S \) is the matrix
\[
S = \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix},
\]

and its characteristic equation is \( x^2 - (c_1 + d_2) x + c_1 d_2 - d_1 c_2 = 0 \). Since \( (\beta_1 \beta_2)^2 = \nu_1^2 \nu_2^2 \), we obtain that the discriminant of this equation writes
\[
\Delta = (c_1 - d_2)^2 + 4d_1 c_2 = \left( \frac{\lambda^2 \nu^2}{2\pi} t + c_1(0) - d_2(0) \right)^2 + 4c_2(0)d_1(0)
= \left( \frac{\lambda^2 \nu^2}{2\pi} \right)^2 t^2 + \frac{\lambda^2 \nu^2}{\pi} (c_1(0) - d_2(0)t) + (c_1(0) - d_2(0))^2 + 4c_2(0)d_1(0)
= A^2 t^2 + Bt + C,
\]

where \( A, B, C \) are defined in \((5.2)\). The eigenvalues of \( S \) will be written in terms of \( \sqrt{\Delta} \), where we use the principal determination of the square root. In order to do so, we have to make sure that \( \Delta \) is not negative. We will show that when \(|t|\) is large enough, \( \Delta \) cannot be a real number. In what follows we suppose that \( t > 0 \). The case \( t < 0 \) can be treated similarly.

Using equations \((4.1)\) and \((\beta_1 \beta_2)^2 = \nu_1^2 \nu_2^2 \), we obtain
\[
\text{Im}(\Delta) = \frac{\lambda^2 \nu^2}{\pi} \left( \text{Im}(c_1(0)) - \text{Im}(d_2(0)) \right) t + \text{Im} \left( \left( c_1(0) - d_2(0) \right)^2 + 4c_2(0)d_1(0) \right)
= \frac{\lambda^2 \nu^4}{4\pi^2} t + \text{Im} \left( \left( c_1(0) - d_2(0) \right)^2 + 4c_2(0)d_1(0) \right)
\]

and thus \( \text{Im}(\Delta) \neq 0 \) for \(|t|\) large enough. Using the Taylor approximation \((1 + x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{3!} + \cdots \) if \(|x| < 1\), we have by \((5.5)\) that
\[
\sqrt{\Delta} = At \left( 1 + \frac{B}{A^2 t} + \frac{C}{A^2 t^2} \right)^{1/2}
= At \left( 1 + \frac{B}{A^2 t} + \frac{C}{A^2 t^2} - \frac{1}{8} \left( \frac{B}{A^2 t} + \frac{C}{A^2 t^2} \right)^2 + \frac{1}{16} \left( \frac{B}{A^2 t} + \frac{C}{A^2 t^2} \right)^3 + \left( \frac{B}{A^2 t} + \frac{C}{A^2 t^2} \right)^4 \right)
= At \left( 1 + \frac{B}{A^2 t} + \frac{1}{t^2} \left( \frac{C}{A^2 t^2} - \frac{B^2}{8A^4} \right) + \frac{1}{t^3} \left( - \frac{BC}{4A^4} + \frac{B^3}{16A^6} \right) + O \left( \frac{1}{t^4} \right) \right)
= At + \frac{B}{2A} + \frac{4A^2 C - B^2}{8A^3} \cdot \frac{1}{t} + \frac{B(B^2 - 4A^2 C)}{16A^6} \cdot \frac{1}{t^2} + O \left( \frac{1}{t^3} \right)
\]

We set
\[
F(t) := \frac{4A^2 C - B^2}{8A^3} \cdot \frac{1}{t} + \frac{B(B^2 - 4A^2 C)}{16A^6} \cdot \frac{1}{t^2} + O \left( \frac{1}{t^3} \right).
\]
By Lemma 5.1 we have that
\begin{align}
|F(t)| &= \frac{4A^2C - B^2}{8A^3} \cdot \frac{1}{t} + O\left(\frac{1}{t^2}\right), \\
\text{Im} F(t) &= -\frac{\lambda^2 \nu_1^4}{4 \pi^2} \cdot \frac{(4A^2C - B^2)}{16A^6} \cdot \frac{1}{t^2} + O\left(\frac{1}{t^3}\right)
\end{align}
with $4A^2C - B^2 > 0$. Then, we have
\begin{equation}
\sqrt{\Delta} = At + \frac{B}{2A} + F(t) = At + c_1(0) - d_2(0) + F(t).
\end{equation}
and the eigenvalues of $S$ are
\begin{align}
E_1 &= \frac{c_1 + d_2 + \sqrt{\Delta}}{2} = \frac{\lambda^2 \nu_1^2}{2\pi} t + c_1(0) + \frac{F(t)}{2} \\
E_2 &= \frac{c_1 + d_2 - \sqrt{\Delta}}{2} = d_2(0) - \frac{F(t)}{2}.
\end{align}
Therefore,
\begin{align}
(S - xI)^{-1} &= \frac{1}{\det(S - xI)} \begin{pmatrix} d_2 - x & -d_1 \\ -c_2 & c_1 - x \end{pmatrix} = \frac{1}{(x - E_1)(x - E_2)} \begin{pmatrix} d_2 - x & -d_1 \\ -c_2 & c_1 - x \end{pmatrix}
\end{align}
and
\begin{equation}
(S - xI)^{-1} W g_0 = (S - xI)^{-1} (e^{i\frac{\lambda}{2} x_2} \beta_1, 0)^t = \left(e^{i\frac{\lambda}{2} x_2} \frac{(d_2(0) - x) \beta_1}{(x - E_1)(x - E_2)}, -e^{i\frac{\lambda}{2} x_2} \frac{c_2(0) \beta_1}{(x - E_1)(x - E_2)} e_2\right).
\end{equation}
Since $u_0 = \lambda \beta_1 e_1 + \lambda \beta_2 e_2 = \lambda \beta_1 e_1$, we obtain
\begin{equation}
u(t) = \frac{1}{2\pi} (u_0, W(S - xI)^{-1} W g_0) = \frac{\lambda e^{-i\lambda x^2}}{2\pi} \cdot \frac{(d_2(0) - x) \beta_1^2}{(x - E_1)(x - E_2)}.
\end{equation}
Using (5.11), we obtain that
\begin{align}
u(t) &= -\frac{\lambda}{2\pi} e^{-i\lambda x^2} \frac{\beta_1^2}{x - E_1} + \bar{F}(t) \frac{\lambda}{4\pi} e^{-i\lambda x^2} \frac{\beta_1^2}{(x - E_1)(x - E_2)}.
\end{align}
Let us denote
\begin{align}
R(t, x) &= \bar{F}(t) \frac{\lambda}{4\pi} e^{-i\lambda x^2} \frac{\beta_1^2}{(x - E_1)(x - E_2)}.
\end{align}
We will study the $H^s$-norms of $R$, for $s \geq 0$. First, we determine $\text{Im}(E_1)$ and $\text{Im}(E_2)$. By equations (5.11), (4.1), and (5.1), we have
\begin{align}
\text{Im}(E_2) &= \text{Im}(d_2(0)) - \frac{\text{Im} F(t)}{2} = \frac{\nu_1^2}{4\pi} - \frac{\text{Im} F(t)}{2} = -\frac{\text{Im} F(t)}{2},
\end{align}
and similarly, we obtain that
\begin{align}
\text{Im}(E_1) &= \frac{\nu_1^2}{4\pi} + \frac{\text{Im} F(t)}{2}.
\end{align}
Let us now estimate $\|R(t, x)\|_{H^s}$. First, we write
\begin{equation}
R(t, x) = \frac{\bar{F}(t)}{E_1 - E_2} \cdot \frac{\lambda}{4\pi} e^{-i\lambda x^2} \frac{\beta_1^2}{(x - E_1)(x - E_2)}.
\end{equation}
We compute the $\dot{H}^s$-norm, $s \geq 0$, of each of the two terms in $R$. Let $p \in \mathbb{C}$, $\text{Im} p < 0$. By (2.2), we have that
\[
\left\| \frac{1}{x - p} \right\|_{\dot{H}^s}^2 = \int_0^\infty \xi^{2s} |F(\frac{1}{x - p})| (\xi) d\xi = c \int_0^\infty \xi^{2s} e^{-i\text{Im}(p)\xi} d\xi = c \int_0^\infty \xi^{2s} e^{2\text{Im}(p)\xi} d\xi.
\]
Integrating by parts, we can explicitly compute the last integral. If $p = \tilde{E}_2$, then $\text{Im}(p) = \text{Im}(\tilde{E}_2)$ and we obtain
\[
\left\| \frac{1}{x - \tilde{E}_2} \right\|_{\dot{H}^s} = O\left(\frac{1}{|\text{Im}(\tilde{E}_2)|^{(2s+1)/2}}\right) = O\left(\frac{1}{|\text{Im}(F(t))|^{(2s+1)/2}}\right).
\]
More precisely, by (5.8) there exist $c, C > 0$ such that
\[
|t|^{2s+1} \leq \left\| \frac{1}{x - \tilde{E}_2} \right\|_{\dot{H}^s} \leq C|t|^{2s+1}.
\]
Similarly, for $p = \tilde{E}_1$, we have $\text{Im}(p) = \text{Im}(\tilde{E}_1) = -\frac{v^2 + \nu^2}{4\pi} - \frac{\text{Im}(F(t))}{2}$ and thus
\[
\left\| \frac{1}{x - \tilde{E}_1} \right\|_{\dot{H}^s} = O(1).
\]
Consequently, by (5.13), (5.10), (5.11), (5.7), (5.8) we obtain for $0 \leq s < \frac{1}{2}$ that
\[
\|R(t, x)\|_{\dot{H}^s} \leq C \frac{|F(t)|}{|E_1 - E_2|} \left( \left\| \frac{1}{x - E_1} \right\|_{L^2} + \left\| \frac{1}{x - E_2} \right\|_{L^2} + C \frac{|F(t)|}{|E_1 - E_2|} \left( \left\| \frac{1}{x - E_1} \right\|_{H^s} + \left\| \frac{1}{x - E_2} \right\|_{H^s} \right) \right) \leq C \frac{1}{|t|^2} (|t| + |t|^{2s+1}),
\]
and thus $\|R(t, x)\|_{\dot{H}^s} \to 0$ for $0 \leq s < \frac{1}{2}$. For $s > \frac{1}{2}$ we have that
\[
\|R(t, x)\|_{\dot{H}^s} \geq C \frac{|F(t)|}{|E_1 - E_2|} \left( \left\| \frac{1}{x - E_1} \right\|_{H^s} - \left\| \frac{1}{x - E_1} \right\|_{H^s} \right) \geq C \frac{1}{|t|^2} (|t|^{2s+1} - |t|).
\]
Therefore, $\|R(t, x)\|_{\dot{H}^s} \to +\infty$ if $s > \frac{1}{2}$.

Moreover, for $s = 1/2$ we have
\[
c \frac{|F(t)|}{|E_1 - E_2|} \left( \left\| \frac{1}{x - E_2} \right\|_{L^2} - \left\| \frac{1}{x - E_1} \right\|_{L^2} + \left\| \frac{1}{x - E_2} \right\|_{H^{1/2}} - \left\| \frac{1}{x - E_1} \right\|_{H^{1/2}} \right) \leq \|R(t, x)\|_{H^{1/2}} \leq C \frac{|F(t)|}{|E_1 - E_2|} \left( \left\| \frac{1}{x - E_1} \right\|_{L^2} + \left\| \frac{1}{x - E_2} \right\|_{L^2} + \left\| \frac{1}{x - E_1} \right\|_{H^{1/2}} + \left\| \frac{1}{x - E_2} \right\|_{H^{1/2}} \right),
\]
and thus there exist $0 < c \leq C$ such that
\[
c \leq \|R(t, x)\|_{H^{1/2}} \leq C
\]
for $|t|$ large enough. We proceed similarly for $\|R(t, x)\|_{L^\infty}$.
\[
c \frac{|F(t)|}{|E_1 - E_2|} \left( \left\| \frac{1}{x - E_2} \right\|_{L^\infty} - \left\| \frac{1}{x - E_1} \right\|_{L^\infty} \right) \leq \|R(t, x)\|_{L^\infty} \leq C \frac{|F(t)|}{|E_1 - E_2|} \left( \left\| \frac{1}{x - E_1} \right\|_{L^\infty} + \left\| \frac{1}{x - E_2} \right\|_{L^\infty} \right).
\]
Since $\left\| \frac{1}{x - E_j} \right\|_{L^\infty} = \frac{1}{|\text{Im} E_j|}$ for $j = 1, 2$, there exist $0 < c \leq C$ such that
\[
c < c \frac{1}{t^2} (t^2 - 1) \leq \|R(t, x)\|_{L^\infty} \leq C \frac{1}{t^2} (t^2 + 1) < C.
\]
Hence, $R(t, x)$ stays away from zero in the $H^{1/2}$-norm and $L^\infty$-norm.
Proof of Corollary 1.12. Notice that the Sobolev norms of solitons are constant in time. By (5.14)
\[ u(t, x) = -\frac{\lambda^2 \nu_1^2 e^{-it\lambda^2}}{2\pi} + R(t, x) + \tilde{\varepsilon}(t, x). \]
where \( \tilde{\varepsilon}(t, x) = -\frac{\lambda^2 \nu_1^2 e^{-it\lambda^2}}{x - p(t)} \)
and
\[ \tilde{\varepsilon}(t, x) = C \frac{E_1 - p(t)}{x - E_1(x - p(t))} = o\left(\frac{1}{t}\right) \]
in all \( H^s \), \( s \geq 0 \).

By (1.14), the first term in the sum in (5.14) is a soliton. Using equation (5.1), we have
\[ \|u_0\|_{L^2}^2 = (u_0, u_0) = (H_{u_0} g_0, H_{u_0} g_0) = \lambda^2 \nu_1^2. \]
In [29, Lemma 3.5] it was shown that \( H_{u_0} \) is a Hilbert-Schmidt operator of Hilbert-Schmidt norm \( \|u_0\|_{H^{1/2}} \). Then, \( 2\lambda^2 = \text{Tr}(H_{u_0}^2) = \|u_0\|_{H^{1/2}}^2 \).
Therefore, the soliton satisfies
\[ \left| \frac{\lambda^2 \nu_1^2 e^{-it\lambda^2}}{2\pi} \right| = \frac{\lambda^2 \nu_1^2}{2\pi} = \sqrt{\pi} \|u_0\|_{H^{1/2}}, \]
\[ \text{Im}(p) = \frac{\nu_1^2}{4\pi} = -\frac{\lambda^2 \nu_1^2}{\lambda^2} = -\frac{\|u_0\|_{H^{1/2}}}{\|u_0\|_{H^{1/2}}^2}. \]
We set \( \varepsilon(t, x) = R(t, x) + \tilde{\varepsilon}(t, x) \). Then, \( \varepsilon(t, x) \to 0 \) as \( t \to \pm\infty \) in all the \( H^s \)-norms, \( 0 \leq s < 1/2 \). However, \( \lim_{t \to \infty} \|\varepsilon(t, x)\|_{H^s} = \infty \) if \( s > 1/2 \) and \( t \to \pm\infty \). □

**Proof of Corollary 1.12.** Notice that the Sobolev norms of solitons are constant in time. Then, the solution in Theorem 1.11 having a non-generic initial data \( u_0 \in \mathcal{M}(2) \) such that \( H_{u_0} \) has a double eigenvalue, provides an example of a solution whose \( H^s \)-norms, with \( s > 1/2 \) grow
\[ \|u(t)\|_{H^s} \geq C|t|^{2s-1} \text{ if } s > 1/2 \]
and \( |t| \) is big enough.

This does not contradict the complete integrability of the Szegő equation, since the conservation laws \( J_{2n} = (u, H^{2n-2}_{u}(u)) \) can all be controlled by the \( H^{1/2}_{+} \)-norm, as it was noticed in Remark 1.1. □

6. **Generalized action-angle coordinates**

On \( L^2_2(\mathbb{R}) \) we introduce the symplectic form
\[ \omega(u, v) = 4\text{Im} \int_{\mathbb{R}} u\bar{v}. \]
A function \( F : L^2_2(\mathbb{R}) \to \mathbb{R} \) admits a Hamiltonian vector field \( X_F \) if
\[ d_u F(h) = \omega(h, X_F(u)), \]
for all \( u, h \in L^2_2(\mathbb{R}) \). If the functions \( F, G : L^2_2(\mathbb{R}) \to \mathbb{R} \) admit the Hamiltonian vector fields \( X_F, X_G \), then we define the Poisson bracket of \( F \) and \( G \) by:
\[ \{ F, G \}(u) = \omega(X_F(u), X_G(u)) = d_u G(X_F(u)). \]
A consequence of the Lax pair is the existence of an infinite sequence of conservation laws as we noticed in Corollary 1.3.

We now introduce the Szegő hierarchy, i.e. the evolution equations associated to the Hamiltonian vector fields of $J_{2n}$ for all $n \in \mathbb{N}^*$, and prove that each of these equations possesses a Lax pair. We will need the following lemma:

**Lemma 6.1.** For all $f \in L^2(\mathbb{R})$ we have

$$ (I - \Pi)f = \overline{\Pi(f)}. $$

As a consequence, the following identity holds:

$$ H_{aH_u(a)}(h) = H_{ua}(h) + H_u(a\Pi(\bar{a}h)). $$

**Proof.** The first equation is equivalent to $f = \Pi(f) + \overline{\Pi(f)}$ and it follows by passing into the Fourier space. Then

$$ H_{aH_u(a)}(h) = \Pi(aH_u(a)\bar{h}) = \Pi\left( H_{ua}(\Pi(\bar{a}h) + (I - \Pi)(\bar{a}h)) \right) $$

$$ = H_{ua}(aH_u(h)) + \Pi(H_{ua}(a\Pi(\bar{a}h))) = H_{ua}(aH_u(h)) + \Pi(ua\Pi(\bar{a}h)) $$

$$ = H_{ua}(aH_u(h)) + H_u(a\Pi(\bar{a}h)). $$

$\square$

**Proposition 6.2.** Let $u \in H^s_+$, $s > \frac{1}{2}$. The Hamiltonian vector field associated to $J_{2n}(u)$ is

$$ X_{J_{2n}}(u) = \frac{1}{2i} \sum_{k=0}^{n-1} H_{u}^{2n-2k-1}(g) H_{u}^{2k}(g) $$

Moreover,

$$ H_{X_{J_{2n}}}(u) = [B_{u,n}, H_u], $$

where

$$ B_{u,n}(h) = -\frac{i}{4} \sum_{j=0}^{2n-2} H_{u}^{4j}(g) \Pi(H_{u}^{2n-2-j}(g)h). $$

**Proof.** The proof follows using the above lemma and similar computations as in the proof of Theorem 8.1 in [16]. Denote

$$ w(x) := (1 - xH_u^2)^{-1}(u) = \sum_{n=0}^{\infty} x^n H_{u}^{2n} u $$

$$ J(x, u) := (u, w(x, u)) = \sum_{n=0}^{\infty} x^n J_{2n+2}(u). $$

A computation shows that

$$ d_u J(x, u)(h) = \omega(h, X(x)), \text{ where} $$

$$ X(x) = \frac{1}{2i}(w(x) + xw(x)H_u w(x)). $$

Identifying the coefficients of $x^n$, we obtain the desired formula for $X_{J_{2n}}(u)$. For the second part of the proposition, we use

$$ w(x) = u + xH_u^2(w) $$
and the above lemma to obtain
\[
H_iX_{j(x,u)}(h) = \frac{1}{2} H_w(h) + \frac{x}{2} H_{wH_u(w)}(h) = \frac{1}{2} H_w(h) + \frac{x}{2} H_{uH_u(w)}(h) + \frac{x^2}{2} H_{2H_u(w)H_u(w)}(h)
\]
\[= G_u H_u(h) + H_u D_u(h), \]
where
\[
G_u(h) = \frac{x}{2} w \Pi(\bar{h} h)
\]
\[
D_u(h) = \frac{1}{2} \sum_{n=0}^{\infty} x^n h^{2n-1}(u) + \frac{x}{2} H_h H_u(w) + \frac{x^2}{2} H_u(w) \Pi(H_u(w))
\]
Since, by (1.3), \( H_iX_{j(x,u)}(h) = \frac{1}{2} H_w(h) + \frac{x}{2} H_{wH_u(w)}(h) \) satisfies
\[
(H_iX_{j(x,u)}(h_1), h_2) = (H_iX_{j(x,u)}(h_2), h_1)
\]
for all \( h_1, h_2 \in L^2_{\text{loc}}, \) we have that
\[
H_iX_{j(x,u)}(h) = G_u H_u(h) + H_u D_u(h) = H_u G_u(h) + D_u H_u(h) = C_u H_u + H_u C_u,
\]
where \( C_u = \frac{G_u + D_u}{2}. \) Identifying once more the coefficients of \( x^n \) and using the fact that \( H_u \) is a skew-symmetric operator, we obtain the formula for \( H_{X_{j2n}}(u). \)

As in \[16\], the following result holds:

**Theorem 6.3.** For every \( u_0 \in H_+^s, s > 1, \) there exists a unique solution \( u \in C(\mathbb{R}, H_+^s) \) of the Cauchy problem
\[
(6.6) \begin{cases}
\partial_t u = X_{J_{2n}}(u) \\
u(0) = u_0.
\end{cases}
\]
Moreover, \( u \) satisfies
\[
(6.7) \quad \partial_t H_u = [B_{u,n}, H_u]
\]
and
\[
(6.8) \quad \{J_{2n}, J_{2k}\} = 0,
\]
for all \( k \in \mathbb{N}^*. \)

In what follows we compute \( g'(t) \) and the commutator \([T, B_{u,n}]\) and use this result to determine the evolution of the angles and generalized angles along the flow of \( X_{j2n}. \)

**Lemma 6.4.** Let \( u \in C(\mathbb{R}, H_+^s), s > 1 \) be a solution of (6.6) and for all \( t \in \mathbb{R} \) let \( g(t) \in \text{Ran}(H_u(t)) \) be such that \( H_u(t)g(t) = u(t). \) Then
\[
(6.9) \quad g'(t) = \frac{i}{4} H_u^{2n-2}(g) + P_u B_{u,n}(g).
\]
Moreover,
\[
[T, B_{u,n}]h = \frac{1}{8\pi} \sum_{j=1}^{2n-2} H_u^j(g)(h, H_u^{2n-2-j}(g)) + \frac{1}{8\pi} (h, H_u^{2n-2}(g)) g - \Lambda(h) B_{u,n}(g).
\]
Proof. In order to compute $g'(t)$, we differentiate with respect to time the equality $H_u(g) = u$:

$$[B_{u,n}, H_u]g + H_u(g') = X_{J_{2n}}(u).$$

Thus

$$H_u(g') = X_{J_{2n}}(u) - [B_{u,n}, H_u]g$$

$$= -i \frac{n-1}{2} \sum_{k=0}^{n-1} H_u^{2n-2k-1}(g) H_u^{2k}(g) + \frac{i}{4} \sum_{j=0}^{2n-2} H_u^{j}(g) \Pi(H_u^{2n-2-j}(g)u) + H_u B_{u,n}(g)$$

$$= -i \frac{n-1}{2} \sum_{k=0}^{n-1} H_u^{2n-2k-1}(g) H_u^{2k}(g) + \frac{i}{4} \sum_{j=0}^{n-1} H_u^{2j}(g) H_u^{2n-1-2j}(g)$$

$$+ \frac{i}{4} \sum_{j=1}^{n-1} H_u^{2n-2j-1}(g) H_u^{2j}(g) + H_u B_{u,n}(g)$$

$$= -i \frac{n-1}{2} H_u^{2n-1}(g) + H_u B_{u,n}(g).$$

Using the fact that $H_u$ is a skew-symmetric operator and is onto on its range, we obtain (6.9).

Since the product of two rational functions has $\Lambda$ equal to zero, we notice that

$$\Lambda(B_{u,n}h) = -i \frac{n-1}{4} \sum_{j=0}^{2n-2} \Lambda(H_u^{j}(g) \Pi(H_u^{2n-2-j}(g)h)) = -i \frac{n}{4} \Lambda(\Pi(H_u^{2n-2}(g)h)).$$

A similar computation yields the second equation in the statement.

Proposition 6.5. If $u_0 \in H_s^k$, $s > 1$ and $u_0 \in \mathcal{M}(N)_{\text{gen}}$, then the solution $u(t)$ of the equation (6.6) is contained in the toroidal cylinder $TC(u_0)$ defined by (1.17), for all $t \in \mathbb{R}$. Moreover, the angles $\phi_j$ and the generalized angles $\gamma_j$ evolve along the flow of this equation as follows:

$$\{J_{2n}, \phi_j\} = \frac{d}{dt} \phi_j = \frac{\lambda_j^{2n-2}}{4}$$

$$\{J_{2n}, \gamma_j\} = \frac{d}{dt} \gamma_j = \frac{n-1}{4\pi} \lambda_j^{2n-2} \nu_j^2.$$

Proof. Since the evolution equation (6.6) admits the Lax pair (6.7), the classical theory yields that if $u(t)$ is a solution of (6.6), then

$$H_u(0) = U_n(t)^* H_u(t) U_n(t),$$

where $U_n(t)$ is a unitary operator on $L_+^2$ satisfying

$$\frac{d}{dt} U_n(t) = B_{u,n} U_n, \quad U(0) = I.$$

Therefore, the eigenvalues $\lambda_j^2, j = 1, 2, \ldots, N$ of $H_u(t)$ are conserved in time. Moreover, if we denote by $\{e_j(0)\}_{j=0}^{N}$ an orthonormal basis of Ran($H_u$) such that $H_u e_j(0) = \lambda_j e_j(0)$, then $e_j(t) = U_n(t)e_j(0)$ form a basis of Ran($H_u(t)$) such that $H_u(t)e_j(t) = \lambda_j e_j(t)$. Then,
by (6.9) and using the fact that $B_{u,n}$ is a skew-symmetric operator, we have

$$
\frac{d}{dt}(e_j(t), g(t)) = (B_{u,n}e_j(t), g(t)) - \frac{i}{4}(e_j(t), H_u^{2n-2}(g(t))) + (e_j(t), B_{u,n}(g(t)))
$$

$$
= -\frac{i}{4} \lambda_j^{2n-2} (e_j(t), g(t)).
$$

Therefore $(e_j(t), g(t)) = e^{-i\phi_j(t)}(e_j(0), g_0)$, with $\frac{d}{dt}\phi_j = \frac{\lambda_j^{2n-2}}{4}$. Thus $|(e_j(t), g(t))| = |(e_j(0), g_0)|$ and $u(t) \in TC(u_0)$ for all $t \in \mathbb{R}$.

By Lemma 2.8 and equation (3.1), we have that $\Lambda(H_u h) = -\frac{1}{2\pi i}(u, h) = -\frac{1}{2\pi i} \lim_{\epsilon \to 0} (H_u h, \frac{1}{1+i\epsilon x})$. Then

$$
\frac{d}{dt} \gamma_j(t) = \frac{d}{dt} \left( T(t)e_j(t), e_j(t) \right) = \lim_{\epsilon \to 0} \left( xB_{u,n}e_j(t) - \frac{1}{2\pi i} (B_{u,n}e_j(t), \frac{1}{1-i\epsilon x})(g(t) - 1), e_j(t) \right)
$$

$$
= \lim_{\epsilon \to 0} \left( \frac{1}{2\pi i} \lim_{\epsilon \to 0} (e_j(t), \frac{1}{1-i\epsilon x}) \left( \frac{i}{4} H_u^{2n-2}(g) + B_{u,n}(g), e_j(t) \right) \right)
$$

$$
= \left( [T, B_{u,n}]e_j(t), e_j(t) \right) + \Lambda(e_j(t)) \left( \frac{i}{4} H_u^{2n-2}(g) + B_{u,n}(g), e_j(t) \right)
$$

$$
= \left( \frac{1}{8\pi} \sum_{k=1}^{2n-2} \left( e_j(t), H_u^{2n-2-k}(g) \right) H_u^{k}(g), e_j(t) \right) + \frac{1}{8\pi} \left( e_j(t), H_u^{2n-2}(g) e_j(t) \right) + \Lambda(e_j(t)) \left( B_{u,n}(g), e_j(t) \right).
$$

Writing $e_j(t) = H_u(t)f_j(t) \in \text{Ran}(H_u(t))$, we have

$$
\Lambda(e_j(t)) = \Lambda(H_u(t)f_j(t)) = -\frac{1}{2\pi i} (u(t), f_j(t)) = -\frac{1}{2\pi i} (H_u(t)g(t), f_j(t))
$$

$$
= -\frac{1}{2\pi i} (H_u(t)f_j(t), g(t)) = -\frac{1}{2\pi i} (e_j(t), g(t))
$$

$$
= -\frac{1}{2\pi i} e^{-i\phi_j(t)}(e_j(0), g_0) = -\frac{1}{2\pi i} (e_j(t), g(t)).
$$

Then

$$
\frac{d}{dt} \gamma_j(t) = \frac{1}{8\pi} \sum_{k=1}^{2n-2} \lambda_j^{2n-2} \nu_j^2 + \frac{1}{8\pi} \lambda_j^{2n-2} \nu_j^2 - \frac{1}{8\pi} \lambda_j^{2n-2} \nu_j^2 = \frac{n-1}{4\pi} \lambda_j^{2n-2} \nu_j^2.
$$

\[ \square \]

**Proposition 6.6.** If $u \in \mathcal{M}(N)_{\text{gen}}$, then

$$
u(x) = \frac{i}{2\pi} \sum_{j,k=1}^N \lambda_j \nu_j \nu_k e^{2i\phi_j} (T - xI)^{-1}_{jk},
$$

where

$$
Te_j = \sum_{k \neq j} \frac{\lambda_j \nu_j \nu_k}{2\pi i} - \frac{\lambda_j - \lambda_k e^{i(2\phi_j - 2\phi_k)}}{\lambda_k^2 - \lambda_j^2} e_k + (\gamma_j + \frac{\nu_j^2}{4\pi}) e_j,
$$

for all $j \in \{1, 2, \ldots, N\}$. In particular, $\chi$ is a one to one map.
Proof. The proof follows the same lines as the proof of Theorem 1.14. The only difference is that we work with the orthonormal basis \( e_j = e^{i\phi_j}e_j \). Since \( H_u \) is anti-linear, the orthonormal basis \( \{ e_j \}_{j=1}^N \) satisfying \( H_u e_j = \lambda_j e_j \) is determined only modulo the sign of \( e_j \). Therefore, \( \phi_j = \text{arg}(e_j, u_0) \) is determined modulo \( \pi \). We intend to introduce generalized action-angle coordinates, and the angles should be defined modulo \( 2\pi \). Considering the basis \( \hat{e}_j \), the formulas we obtain only depend on \( 2\phi_j \), which are therefore good candidates for the angles. \( \square \)

Proof of Theorem 1.14. Let us first notice that, if we prove that \( \chi \) is a symplectic diffeomorphism, then the coordinates \( (2\lambda_j^2 \nu_j^2, 4\pi \lambda_j^2, 2\phi_j, \gamma_j) \) are canonical. Denote \( I_j = 2\lambda_j^2 \nu_j^2 \). By equation \( E = 2J_4 = 2 \sum_{j=1}^N \lambda_j^2 \nu_j^2 = \sum_{j=1}^N \lambda_j^2 I_j \) and using Proposition 6.5 we obtain that for the flow of the Szegő equation we have:

\[
\frac{d}{dt}(2\phi_j(t)) = \{E, 2\phi_j \} = 4\{J_4, \phi_j \} = \lambda_j^2 \\
\frac{d}{dt}\gamma_j(t) = \{E, \gamma_j \} = 2\{J_4, \gamma_j \} = \frac{\lambda_j^2 \nu_j^2}{2\pi}.
\]

Thus, the Szegő equation can be indeed rewritten as

\[
\begin{cases}
\frac{d}{dt}I_j = 0 \\
\frac{d}{dt}\phi_j(t) = \frac{\partial E}{\partial I_j} \\
\frac{d}{dt}(4\pi \lambda_j^2) = 0 \\
\frac{d}{dt}\gamma_j(t) = \frac{\partial E}{\partial (4\pi \lambda_j^2)}.
\end{cases}
\]

The first step in proving that \( \chi \) is a symplectic diffeomorphism is to compute the Poisson brackets between actions and (generalized) angles. This will lead to \( \chi \) being a local diffeomorphism.

6.1. Poisson brackets between actions and (generalized) angles. First notice that

\[
(6.11) \quad J_{2n}(u) = (u, H_u^{2n-2}u) = \sum_{k=1}^N \lambda_k^{2n-2} \nu_k^2.
\]

Fix \( j \in \{1, 2, \ldots, N\} \). Writing

\[
\{ J_{2n}, 2\phi_j \} = \sum_{k=1}^N \lambda_k^{2n-2} \{ \lambda_k^2 \nu_k^2, 2\phi_j \} + \sum_{k=1}^N (n-1)\lambda_k^{2n-2} \nu_k^2 \{ \lambda_k^2, 2\phi_j \}
\]

for all \( n = 1, 2, \ldots, 2N \) we obtain the following linear system of equations:

\[
\sum_{k=1}^N \lambda_k^{2n-2} \{ \lambda_k^2 \nu_k^2, 2\phi_j \} + \sum_{k=1}^N (n-1)\lambda_k^{2n-2} \nu_k^2 \{ \lambda_k^2, 2\phi_j \} = \frac{\lambda_j^{2n-2}}{2}
\]

with \( 2N \) unknowns, \( \{ \lambda_k^2 \nu_k^2, \phi_j \} \) and \( \lambda_k^2 \nu_k^2 \{ \lambda_k^2, \phi_j \} \). The matrix of this system is invertible. Indeed, supposing by absurd that the matrix is not invertible, it results that the columns are linearly dependent. Therefore, there exist numbers \( a_n, n = 1, 2, \ldots, 2N \), not all zero, such that

\[
\sum_{n=1}^{2N} a_n (\lambda_k^2)^{n-1} = 0, \quad \sum_{n=1}^{2N} a_n (n-1)(\lambda_k^2)^{n-2} = 0.
\]
Considering the polynomial $P(x) = \sum_{n=1}^{2N} a_n x^{n-1}$, this yields $P(\lambda_k^2) = 0$, $P'(\lambda_k^2) = 0$. Thus, each $\lambda_k^2$ is a double root of the polynomial. Since a polynomial of degree $2N - 1$ with $2N$ roots is identically zero, we obtain a contradiction. Therefore, the system is a Cramer system and one can easily verify that its solutions are

\begin{equation}
\{\lambda_k^2 \nu_k^2, 2\phi_j\} = \frac{\delta_{kj}}{2}
\end{equation}

\begin{equation}
\{\lambda_k^2, 2\phi_j\} = 0.
\end{equation}

Similarly, computing $\{J_{2n}, \gamma_j\}$, we obtain the Cramer system

\begin{equation}
\sum_{k=1}^{N} \lambda_k^{2n-2}\{\lambda_k^2 \nu_k^2, \gamma_j\} + \sum_{k=1}^{N} (n-1)\lambda_k^{2n-2}\nu_k^2 \{\lambda_k^2, \gamma_j\} = \frac{n-1}{4} \lambda_j^{2n-2} \nu_j^2
\end{equation}

with solutions

\begin{equation}
\{\lambda_k^2 \nu_k^2, \gamma_j\} = 0
\end{equation}

\begin{equation}
\{\lambda_k^2, \gamma_j\} = \frac{\delta_{kj}}{4\pi}.
\end{equation}

Since $\lambda_j^2$ and $\nu_j^2$ are conserved by the flow of any of the equations in the Szegő hierarchy, we have that

\begin{equation}
\{J_{2n}, \lambda_j^2\} = \frac{d}{dt} \lambda_j^2 = 0, \quad \{J_{2n}, \nu_j^2\} = \frac{d}{dt} \nu_j^2 = 0.
\end{equation}

Proceeding as above, we have two homogeneous Cramer systems, whose solutions must be null. Thus, we obtain

\begin{equation}
\{\lambda_k^2 \nu_k^2, \lambda_j^2\} = 0
\end{equation}

\begin{equation}
\{\lambda_k^2, \lambda_j^2\} = 0
\end{equation}

\begin{equation}
\{\lambda_k^2 \nu_k^2, \lambda_j^2 \nu_j^2\} = 0.
\end{equation}

6.2. $\chi$ is a local diffeomorphism. The fact that $\chi$ is a local diffeomorphism is equivalent to proving that the differentials $d\lambda_j^2$, $d(\lambda_j^2 \nu_j^2)$, $d\phi_j$, $d\gamma_j$, $j = 1, 2, \ldots, N$, are linearly independent. Suppose

\begin{equation}
\sum_{j=1}^{N} \alpha_j d(\lambda_j^2) + \beta_j d(\lambda_j^2 \nu_j^2) + \theta_j d\phi_j + \eta_j d\gamma_j = 0.
\end{equation}

Applying this differential to the vector field $X_{\lambda_k^2}$, using $df(X_g) = \{g, f\}$, (6.17), (6.16), (6.13), and (6.15), we obtain

\begin{equation}
\sum_{j=1}^{N} \eta_j \frac{\delta_{kj}}{4\pi} = 0
\end{equation}

and thus $\eta_k = 0$, for all $k = 1, 2, \ldots, N$. Applying the same differential to $X_{\lambda_k^2 \nu_k^2}$ and using (6.16), (6.18), and (6.12), we obtain $\theta_k = 0$ for all $k = 1, 2, \ldots, N$. Applying the differential to $X_{\phi_k}$ and using (6.13) and (6.12) we have $\beta_k = 0$ for all $k = 1, 2, \ldots, N$. Finally, applying the differential to $X_{\gamma_k}$ and using (6.15) we obtain $\alpha_k = 0$ for all $k = 1, 2, \ldots, N$. Therefore, $d\lambda_j^2$, $d(\lambda_j^2 \nu_j^2)$, $d\phi_j$, $d\gamma_j$, $j = 1, 2, \ldots, N$, are linearly independent and $\chi$ is a local diffeomorphism.
Since a bijective local diffeomorphism is a diffeomorphism, and we have by Proposition 6.6 that $\chi$ is one to one, we only need to show that $\chi$ is onto. A proper local diffeomorphism taking values in a connected manifold is onto. Thus, it is enough to show that $\chi$ is proper.

### 6.3. $\chi$ is a proper mapping

Let $K \subset \Omega$ be a compact set. Set

$$(I, \tilde{I}, 2\phi, \gamma) := \left(2\left(\lambda_j^{(p)}\right)^2, 4\pi \left(\lambda_j^{(p)}\right)^2, 2\phi_j^{(p)}, \gamma_j^{(p)}\right)_{j=1}^N.$$

Let $(I^{(p)}, \tilde{I}^{(p)}, 2\phi^{(p)}, \gamma^{(p)}), (I, \tilde{I}, 2\phi, \gamma) \in K$ such that

$$\left(\tilde{I}^{(p)}, \tilde{I}^{(p)}_{\tilde{j}}^{(p)}, 2\phi_j^{(p)}, \gamma_j^{(p)}\right) \to (I, \tilde{I}, 2\phi, \gamma) \quad \text{as} \quad p \to \infty.$$

Consider $u_p \in \chi^{-1}(I^{(p)}, \tilde{I}^{(p)}, 2\phi^{(p)}, \gamma^{(p)})$. Then $\left(\lambda_j^{(p)}\right)^2$ are the eigenvalues of $H_{u_p}^2$. By Lemma 3.5 in [29], which states that $H_u$ is a Hilbert-Schmidt operator of norm $\|H_u\|_{HS} = \frac{1}{\sqrt{2\pi}}\|u\|_{H^{1/2}}$, we have that

$$\|u_p\|_{L^2}^2 = J_2(u_p) = \sum_{j=1}^N \left(\lambda_j^{(p)}\right)^2 = \frac{1}{2} \sum_{j=1}^N I_j^{(p)},$$

$$\|u_p\|_{H^{1/2}}^2 = 2\pi \|H_{u_p}\|_{HS} = 2\pi \text{Tr}(H_{u_p}^2) = 2\pi \sum_{j=1}^N \left(\lambda_j^{(p)}\right)^2 = \frac{1}{2} \sum_{j=1}^N \tilde{I}_j^{(p)}.$$

Since $I^{(p)} \to I$ and $\tilde{I}^{(p)} \to \tilde{I}$ as $p \to \infty$, this yields that $\|u_p\|_{H^{1/2}}$ is bounded. Consequently, there exists $u \in H^{1/2}_r$ such that $u_p \to u$ in $H^{1/2}_r$. It follows in particular that $u_p \to u$ in $L^2_{loc}$. We denote by $\lambda_j(u), \nu_j(u), \phi_j(u)$, and $\gamma_j(u)$ the spectral data for $u$.

By Proposition 6.6 we have that

$$u_p(x) = \frac{i}{2\pi} \sum_{j,k=1}^N \lambda_j^{(p)} \nu_j^{(p)} \nu_k^{(p)} e^{2i\phi_j^{(p)}} \frac{(T^{(p)} - xI)^{-1}_{jk}}{2\pi i},$$

where $(T^{(p)} - xI)^{-1}_{jk}$ is a component of the matrix $(T^{(p)} - xI)^{-1}$ in the basis $\{e_j^{(p)}\}_{j=1}^N$, and

$$T^{(p)} e_j^{(p)} = \sum_{k \neq j} \frac{\lambda_j^{(p)} \nu_j^{(p)} \nu_k^{(p)}}{2\pi i} \cdot \frac{\lambda_j^{(p)} - \lambda_k^{(p)} e^{(2\phi_j^{(p)} - 2\phi_k^{(p)})}}{(\lambda_j^{(p)})^2 - (\lambda_k^{(p)})^2} e_k + \left(\gamma_j^{(p)} + i \frac{\nu_j^{(p)} e^{(2\phi_j^{(p)})}}{4\pi}\right) e_j,$$

for all $j \in \{1, 2, \ldots, N\}$. By equation [6.19], there exists $R > 0$ such that $\|T^{(p)}\| \leq \frac{R}{2}$ for all $p \in \mathbb{N}$. Using the Neumann series, we have that if $|x| \geq R$, then there exists $A > 0$ such that

$$\|(T^{(p)} - xI)^{-1}\| \leq \frac{A}{|x|},$$

for all $p \in \mathbb{N}$. This yields that

$$\lim_{R \to \infty} \sup_p \int_{|x| > R} |u_p(x)|^2 dx \leq \lim_{R \to \infty} \int_{|x| > R} \frac{A^2}{|x|^2} dx = 0.$$

Since $u_p \to u$ in $L^2_{loc}$, this triggers $u_p \to u$ in $L^2_\infty(\mathbb{R})$. 
Let us now prove that $H_{u_p}(h) \to H_u(h)$ in $L^2_+$. First, notice that there exists $C > 0$ such that

$$\|H_{u_p} - H_u\| = \|H_{u_p-u}\| \leq \|H_{u_p-u}\|H^{-S} \leq \frac{1}{\sqrt{2\pi}}\|u_p - u\|_{H^{1/2}} \leq C.$$ 

In particular, it follows that it suffices to prove that $H_{u_p}(h) \to H_u(h)$ for $h$ in a dense subset of $L^2_+$, for example $h \in L^\infty \cap L^2_+$. For such $h$, we have that

$$\|H_{u_p}(h) - H_u(h)\|_{L^2_+} = \|H_{u_p-u}(h)\|_{L^2_+} = \|\Pi((u_p - u)h)\|_{L^2} \leq \|u_p - u\|_{L^2_+} \|h\|_{L^\infty}.$$ 

Thus, $u_p \to u$ in $L^2(\mathbb{R})$ yields $H_{u_p}(h) \to H_u(h)$ in $L^2_+$.

As a consequence, we have that $J_{2n}(u_p) \to J_{2n}(u)$ as $p \to \infty$. Indeed, we write

$$J_{2n}(u_p) - J_{2n}(u) = (H_{u_p}^{2n-2}u_p, u_p) - (H_u^{2n-2}u, u)$$

$$= (H_{u_p}^{2n-2}u_p, u_p - u) + (H_u^{2n-2}(u_p - u), u) + \sum_{j=1}^{2n-2} (H_{u_p}^{2n-2-j}H_{u_p-u}H_u^{j-1}u, u).$$

For the first term we notice that

$$|(H_{u_p}^{2n-2}u_p, u_p - u)| \leq \|H_{u_p}^{2n-2}u_p\|_{L^2} \|u_p - u\|_{L^2} \leq \|H_{u_p}\|^{2n-2} \|u_p\|_{L^4_+} \|u_p - u\|_{L^2_+}$$

$$\leq C\|u_p\|_{H^{1/2}}^{2n-2} \|u_p\|_{L^4_+} \|u_p - u\|_{L^2_+} \to 0 \text{ as } p \to \infty.$$ 

For the second term we have that

$$|(H_u^{2n-2}(u_p - u), u)| \leq \|H_u^{2n-2}(u_p - u)\|_{L^4_+} \|u\|_{L^2_+} \leq \|H_u\|^{2n-2} \|u_p - u\|_{L^4_+} \|u\|_{L^2_+}$$

$$\leq \|u_p\|_{H^{1/2}}^{2n-2} \|u_p - u\|_{L^4_+} \|u\|_{L^2_+} \to 0 \text{ as } p \to \infty.$$ 

For the other terms, in the case when $j$ is even, we use the self-adjointness of the operator $H_u^2$. We then obtain:

$$|(H_{u_p}^{2n-2-j}H_{u_p-u}H_u^{j-1}u, u)| = |(H_{u_p-u}H_u^{j-1}u, H_{u_p}^{2n-2-j}u)|$$

$$\leq \|H_{u_p-u}H_u^{j-1}u\|_{L^2_+} \|H_{u_p}^{2n-2-j}u\|_{L^2_+}$$

$$\leq \|H_{u_p-u}H_u^{j-1}u\|_{L^2_+} \|u_p\|_{H^{1/2}} \|u\|_{L^2_+} \to 0$$

and the first factor tends to zero since $H_{u_p-u}(h) \to 0$ in $L^2_+$ for all $h \in L^2_+$. For the case when $j$ is odd, we use equation (1.3) and then proceed similarly.

We prove in the following that $\lambda_j(u) = \lambda_j$ and $\nu_j(u) = \nu_j$. Since, by equation (6.11), $J_{2(n+1)}(u_p) = \sum_{j=1}^{N} (\lambda_j^{(p)})^{2(n+1)}(\nu_j^{(p)})^2$, we have that

$$\sum_{n=0}^{\infty} x^n J_{2(n+1)}(u_p) = \sum_{n=0}^{\infty} x^n \sum_{j=1}^{N} (\lambda_j^{(p)})^{2(n+1)}(\nu_j^{(p)})^2$$

$$= \sum_{j=1}^{N} (\lambda_j^{(p)})^2 (\nu_j^{(p)})^2 \sum_{n=0}^{\infty} x^n (\lambda_j^{(p)})^{2n} = \sum_{j=1}^{N} (\lambda_j^{(p)})^2 (\nu_j^{(p)})^2 \frac{1}{1 - x(\lambda_j^{(p)})^2}.$$
for $|x| < 1/\lambda_j^2$, and thus for every $x$ distinct from the poles. Then, using $\lambda_j^{(p)} \to \lambda_j$ and $\nu_j^{(p)} \to \nu_j$, we obtain

$$\sum_{n=0}^{\infty} x^n J_{2(n+1)}(u_p) \to \sum_{j=1}^{N} \frac{\lambda_j^2 \nu_j^2}{1 - x\lambda_j^2}.$$ 

On the other hand, we have that

$$\sum_{n=0}^{\infty} x^n J_{2(n+1)}(u_p) \to \sum_{n=0}^{\infty} x^n J_{2(n+1)}(u) = \sum_{j=1}^{N} \frac{(\lambda_j(u))^2 (\nu_j(u))^2}{1 - x(\lambda_j(u))^2}.$$ 

Therefore,

$$\sum_{j=1}^{N} \frac{\lambda_j^2 \nu_j^2}{1 - x\lambda_j^2} = \sum_{j=1}^{N} \frac{(\lambda_j(u))^2 (\nu_j(u))^2}{1 - x(\lambda_j(u))^2},$$

which yields, by identification, $\lambda_j(u) = \lambda_j$ and $\nu_j(u) = \nu_j$.

At last, we show that $e_j(u_p) \to \pm e_j(u)$. It then follows that

$$2\phi_j^{(p)} = 2\phi_j(u_p) = \arg(u_p, e_j(u_p))^2 \to \arg(u, e_j(u))^2 = 2\phi_j(u)$$

$$\gamma_j^{(p)} = \gamma_j(u_p) = \Re(T^{(p)} e_j(u_p), e_j(u_p)) \to \Re(T e_j(u), e_j(u)) = \gamma_j(u).$$

Since, by (6.19), we also have $2\phi_j^{(p)} \to 2\phi_j$ and $\gamma_j^{(p)} \to \gamma_j$, we obtain that $2\phi_j(u) = 2\phi_j$ and $\gamma_j(u) = \gamma_j$. Hence $\chi(u) = (I, I, 2\phi, \gamma) \in K$, and $u \in \chi^{-1}(K)$. Thus $\chi^{-1}(K)$ is compact, which proves that $\chi$ is proper.

We still need to show that $e_j(u_p) \to \pm e_j(u)$. Using $\lambda_j^{(p)} \to \lambda_j = \lambda_j(u)$, $\nu_j^{(p)} \to \nu_j = \nu_j(u)$, we have that

$$\|u_p\|_{H^1_+} = \sum_{j=1}^{N} \frac{(\lambda_j^{(p)})^2 (\nu_j^{(p)})^2}{1 - \lambda_j^{(p)}} \to \sum_{j=1}^{N} \frac{(\lambda_j(u))^2 (\nu_j(u))^2}{1 - \lambda_j(u)} = \|u\|_{H^1_+}.$$ 

Since $u_p \to u$ in $H^{1/2}_+$ and $u \to u$ in $L^2_+$, it follows that $u_p \to u$ in $H^{1/2}_+$. This yields that $H_{u_p} \to H_u$ in the sense of the norm. As a consequence, setting

$$P_j^{(p)} h := (h, e_j(u_p)) e_j(u_p).$$

to be the orthogonal projection onto the eigenspace of $H^2_{u_p}$, corresponding to the eigenvalue $(\lambda_j^{(p)})^2$ and similarly,

$$P_j(u) h := (h, e_j(u)) e_j(u)$$

to be the orthogonal projection onto the eigenspace of $H^2_u$, corresponding to the eigenvalue $\lambda_j^2(u)$, we have by Theorem VIII.23 in [30], that $P_j^{(p)} \to P_j(u)$.

Therefore, $(h, e_j(u_p)) e_j(u_p) \to (h, e_j(u)) e_j(u)$ as $p \to \infty$, for all $h \in L^2_+$. Taking $h = e_j(u)$, we have that $(e_j(u), e_j(u_p)) e_j(u_p) \to e_j(u)$. Since $e_j(u_p)$ and $e_j(u)$ are unitary
Writing which gives by (6.12), (6.13), that
\[
\lim_{p \to \infty} (e_j(u), e_j(u_p)) = \lim_{p \to \infty} (e_j(u), e_j(u)) = \lim_{p \to \infty} (e_j(u), e_j(u_p)).
\]

Since the above limit is of absolute value 1, we obtain that
\[
\chi = \text{symplectic, we only need to prove that the Poisson brackets involving only angles and generalized angles, } \{\phi_j, \phi_k\}, \{\gamma_j, \phi_k\}, \text{ and } \{\gamma_j, \gamma_k\}, \text{ are zero.}
\]

We first remark that the Jacobi identity yield that
\[
\{\phi_j, \phi_k\}, \{\gamma_j, \phi_k\}, \text{ and } \{\gamma_j, \gamma_k\}, \text{ are only functions of } \lambda^2_\ell \text{ and } \lambda^2_\ell \nu^2_\ell \text{ for } \ell = 1, 2, \ldots, N. \text{ Indeed, for the first one we take } f = \lambda^2_\ell \text{ and then } f = \lambda^2_\ell \nu^2_\ell \text{ in}
\]

\[
\{f, \{\phi_j, \phi_k\}\} + \{\phi_k, \{f, \phi_j\}\} + \{\phi_j, \{\phi_k, f\}\} = 0,
\]

which gives by (6.12), (6.13), that
\[
\{\lambda^2_\ell, \{\phi_j, \phi_k\}\} = \{\lambda^2_\ell \nu^2_\ell, \phi_j, \phi_k\} = 0.
\]

Writing \(\{\phi_j, \phi_k\} = h(\lambda^2_\ell \cdot \lambda^2_\ell \nu^2_\ell, \phi_\ell, \gamma_\ell)\) for \(\ell = 1, 2, \ldots, N\), we obtain
\[
\frac{\partial h}{\partial \phi_\ell} = \frac{\partial h}{\partial \gamma_\ell} = 0.
\]

Define now \(J_1(u) = (u, g)\) and \(J_3(u) = (H_u^2 u, g)\). We will compute \(\{J_1, J_3\}\) to prove that \(\{\phi_j, \phi_k\} = 0\). We have that
\[
d_u J_1(h) = \lim_{t \to 0} \left( \frac{u + th, g(u + th) - (u, g(u))}{t} \right) = (h, g(u)) + \lim_{t \to 0} \left( u, \frac{g(u + th) - g(u)}{t} \right) = (h, g(u)) + (H_u g, d_u g(h)) = (h, g(u)) + (H_u(d_u g(h)), g).
\]

In order to compute \(H_u(d_u g(h))\), we differentiate the equation \(u = H_u g\):
\[
h = \lim_{t \to 0} \frac{H_u u + u g(u + th) - H_u g(u)}{t} = \lim_{t \to 0} \left( H_u \left( \frac{g(u + th) - g(u)}{t} \right) + H_h g(u + th) \right) = H_u(d_u g(h)) + H_h g(h).
\]

Thus, \(H_u(d_u g(h)) = h - H_h g(1 - g)\) and \(d_u J_1(h) = (h, g) + (h, g(1 - g))\). Therefore, the vector fields corresponding to the real and imaginary part of \(J_1\) are:
\[
X_{\text{Re} J_1} = -\frac{i}{4}(g + g(1 - g)), \quad X_{\text{Im} J_1} = \frac{1}{4}(g + g(1 - g)).
\]
Similarly we have that
\[
d_uJ_3(h) = (H^2_u, g) + (H_uH_uu + H_uH_uu, g) + (H^2_u, d_u g(h))
\]
\[
= (h, H^2 g) + (H_uH_uu, g) + (H_uH_uu, g) + (H_u(d_u g(h)), H_u) = (h, H_uu) + (u^2, h) + (h, gH_uu) + (h, (1 - g)H_uu) = 2(h, H_uu) + (u^2, h).
\]

Then
\[
\{J_1, J_3\} = d_u J_3 \cdot X_{Re J_1}(u) + i d_u J_3 \cdot X_{Im J_1}(u)
\]
\[
= 2\left( -\frac{i}{4}(g + g(1 - g)), H_uu \right) + \left( u^2, -\frac{i}{4}(g + g(1 - g)) \right)
\]
\[
+ 2i\left( \frac{1}{4}(g + g(1 - g)), H_uu \right) + i\left( u^2, \frac{1}{4}(g + g(1 - g)) \right)
\]
\[
= \frac{i}{2}(u^2, g + g(1 - g)).
\]

Using equations (1.10) and (5.1), we have
\[
(u^2, g + (1 - g)g) = (u^2, g) + (u^2, (1 - g)g) = (u^2, g) + (u(1 - \bar{g}), \bar{u}g)
\]
\[
= (u^2, g) + (u(1 - \bar{g}), (I - \Pi)(\bar{u}g)) = (u^2, g) + (u(1 - \bar{g}), \Pi(\bar{u}g))
\]
\[
= (u^2, g) + (u(1 - \bar{g}), \bar{u}) = (u^2, g) + \int_{-\infty}^{\infty} u^2 - \int_{-\infty}^{\infty} u^2 = 0.
\]

Thus, we obtain \(\{J_1, J_3\} = 0\). On the other hand, we have
\[
\{J_1, J_3\} = \left\{ \sum_{j=1}^{N} \lambda_j \nu_j^2 e^{-2i\phi_j}, \sum_{k=1}^{N} \lambda_k \nu_k^2 e^{-2i\phi_k} \right\}
\]
\[
= \sum_{j,k=1}^{N} e^{-2i(\phi_j + \phi_k)} \left( -i\{\lambda_j \nu_j^2, \phi_j\} + i\{\lambda_k \nu_k^2, \phi_j\} + \{\phi_j, \phi_k\} \lambda_j \lambda_k \nu_j^2 \nu_k^2 \right).
\]

Since \(\{\phi_j, \phi_k\}\) only depends on \(\lambda_j^2\) and \(\lambda_k^2\), \(\nu_j^2\), \(\ell = 1, 2, \ldots, N\), we have that the coefficient of \(e^{-2i(\phi_j + \phi_k)}\) in the above expression is
\[-i\{\lambda_j \nu_j^2, \phi_k\} - i\{\lambda_k \nu_k^2, \phi_j\} + i\{\lambda_k \nu_k^2, \phi_j\} + i\{\phi_j, \phi_k\} \lambda_j \lambda_k \nu_j^2 \nu_k^2 (\lambda_j^2 - \lambda_k^2)\].

Comparing the two expressions for \(\{J_1, J_3\}\), we have that \(\{J_1, J_3\}\) is a trigonometric polynomial which is equal to zero. Therefore all its coefficients are zero, which triggers, by taking the real part, that \(\{\phi_j, \phi_k\} = 0\).

In order to compute \(\{\gamma_j, \gamma_k\}\) and \(\{2\phi_j, \gamma_k\}\) we denote \(A := (Tu, u), C := (Tu, g)\) and compute \(\{A, C\}\) in two different ways. First, we use \(\{A, C\}(u) = d_u C \cdot X_{ReA} + id_u C \cdot X_{ImA}\).

Since
\[
d_u A(h) = 2\Re(h, Tu) + \Lambda(u)(g(1 - g), h) + (h, \frac{1}{2\pi i}(u, g)g),
\]
for all \(h\) rational function (notice that we extend the definition of \(T\) to \(\bigcup_{N \in \mathbb{N}} M_N\)), we obtain the following Hamiltonian vector fields:
\[
X_{ReA} = -\frac{i}{2} Tu - \frac{i}{4} \Lambda(u)g(1 - g) - \frac{1}{8\pi}(u, g)g,
\]
\[
X_{ImA} = -\frac{i}{4} \Lambda(u)g(1 - g) - \frac{i}{8\pi}(u, g)g.
\]
Similarly we have
\[ d_u C(h) = \Lambda(u) (f(1-g), h) - \frac{1}{2\pi i} (u, g) (h, f(1-g)) + (Th, g) + (h, (1-g)Tg), \]
where \( f \) is the unique element in \( \text{Ran}(H_u) \) such that \( H_u f = g \). By Lemma 2.11 we have that \( \text{Ker}(H_u) = (1-g)L^2_+ \) and using the orthogonality of \( \text{Ker}(H_u) \) and \( \text{Ran}(H_u) \) we obtain
\[ \{ A, C \} = -\frac{i}{2} \left( T^2 u, g \right) + \frac{1}{4\pi} \Lambda(u)(u, g)(g, f) - \frac{i}{2} \Lambda(u)(Tg(1-g), g) - \frac{i}{2} \Lambda(u)(g, Tg). \]
Notice also that, by Lemmas 2.1 and 2.11, we have
\[ Tg(1-g) = \chi(1-g) - \Lambda(g(1-g))(1-g) = \chi(1-g) - \Lambda(g)(1-g) \]
and thus by orthogonality of \( \text{Ker}(H_u) \) and \( \text{Ran}(H_u) \), the third term vanishes. By (2.21), we rewrite the first term as
\[ -\frac{i}{2} \left( T^2 u, g \right) = -\frac{i}{2} \left( T^* Tu - \frac{1}{2\pi i} (Tu, g)g, g \right) = -\frac{i}{2} (Tu, Tg) + \frac{1}{4\pi} (Tu, g)(g, g). \]
Hence, we have
\[ \{ A, C \} = -\frac{i}{2} (Tu, Tg) + \frac{1}{4\pi} (Tu, g)(g, g) + \frac{1}{4\pi} \Lambda(u)(u, g)(g, f) - \frac{i}{2} \Lambda(u)(g, Tg). \]
Proceeding as in the case of \( \{ J_1, J_2 \} \), we obtain after tedious computations that \( \{ \gamma_j, 2\phi_k \} = 0 \) and \( \{ \gamma_j, \gamma_k \} = 0 \) for all \( j, k \in \{1, 2, \ldots, N\} \), which proves that the coordinates we defined are symplectic.

\[ \Box \]

**Proof of Corollary 1.15.** Fixing \( \lambda_j \) and \( \nu_j \) the application \( \chi \) in Theorem 1.14 yields a diffeomorphism between \( TC(u_0) \) and \( \mathbb{T}^N \times \mathbb{R}^N \).

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