Deformations of Lie algebras of type $D_l$

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Abstract

The description of global deformations of Lie algebras is important since it is related to unsolved problem of the classification of simple Lie algebras over a field of small characteristic. In this paper we study global deformations of Lie algebras of type $D_l$ over an algebraically closed field $K$ of characteristic 2. It is proved that Lie algebras of type $\bar{D}_l$ are rigid for odd $l > 3$. Some global deformations of Lie algebras of type $D_l$ are constructed for even $l \geq 4$.

Key words: modular Lie algebras, cohomology, deformations

A classical Lie algebra over a field of characteristic $p > 0$ is a Lie algebra obtained by reduction modulo $p$ of the Chevalley order of complex simple Lie algebra, or its quotient algebra by the centre. Let $L$ be Lie algebra over field $K$, $L_t$ denote the vector space obtained from $L$ by extending the coefficient domain from $K$ to $K((t))$, i.e., $L \otimes K((t))$. Suppose that bilinear function $f_t : L_t \times L_t \to L_t$ has the form

$$f_t(x, y) = [x, y] + tf_1(x, y) + t^2f_2(x, y) + \cdots$$

where $F_i$ – bilinear function over a field $K$. If $f_t$ satisfies condition of antisymmetry and Jacobi identity, then Lie algebras $L_t$ with multiplication $f_t$ are family of global deformations of Lie algebra $L$, and $F_1$ is called integrable. Conditions on $f_t$ mean, in particular, that $F_1$ is an element of the group $Z^2(L, L)$. For cohomologous cocycles corresponding Lie algebras are isomorphic. So we can choose representatives of cohomological classes as $F_1$. Second cohomology group with coefficients in adjoint module $H^2(L, L)$ is called space of local deformations. Lie algebra, which does not have non-trivial global deformations, is called rigid.
Classical Lie algebras over fields of characteristic 2 and 3 admit global deformations. Over a field of characteristic 3 there exist non-isomorphic Lie algebras with root system of type $C_2$. A. I. Kostrikin in [1] constructed parametric families of non-isomorphic simple Lie algebras over a field of characteristic 3, which are global deformations of Lie algebra of type $C_2$. A. S. Dzhumadildaev in [2] has proved that among Lie algebras of type $A_n$, $B_n$, $C_n$, $D_n$ over a field of characteristic 3 only $C_2$ admits non-trivial global deformations. A. I. Kostrikin and M. I. Kuznetsov in [3] fully described global deformations of Lie algebra of type $C_2$ over a field of characteristic 3. Their approach for study of global deformations, based on the study of orbits of the action of the automorphism group of the algebra on its cohomology space, is used in this paper. A.N. Rudakov [4] has proved that over a field of characteristic $p > 3$ all classical Lie algebras are rigid. In [5] and [6] M. I. Kuznetsov and N.G. Chebochko proposed a new scheme for studying rigidity and proved that over a field of characteristic $p > 2$ all classical Lie algebras are rigid except $C_2$ when $p = 3$. Spaces of local deformations of classical Lie algebras with homogeneous root system over a field of characteristic 2 are described in [7], in particular are described spaces of local deformations of classical Lie algebra of type $D_l$ over a field of characteristic 2.

Let $R$ be root system of type $D_l$, $\{\alpha_1, \ldots, \alpha_{l-2}, \alpha_{l-1}, \alpha_l\}$ — simple roots, where enumeration is chosen in such way that $\alpha_l$ is connected to $\alpha_{l-2}$ on Dynkin diagram. By $<\alpha, \beta>$ we denote Cartan number for roots $\alpha, \beta$. By $H_\alpha = H_{\alpha_i}$ and $E_\alpha, \alpha \in R$ we denote vectors of Chevalley basis in Lie algebra.

1 Deformations of Lie algebra of type $\bar{D}_l$ for odd $l > 3$.

For odd $l > 3$ Lie algebra of type over a field $K$ of characteristic 2 has one-dimensional centre $H_l + H_{l-1}$. Let $L$ be Lie algebra of type $\bar{D}_l$ — quotient algebra of by the centre. According to [8], the automorphism group of Lie algebra $L$ of type $\bar{D}_l$ is isomorphic to Chevalley group of type $B_l$. Let $V$ be vector space over $K$ of dimension $2l$ with a symplectic form $(\ , \ )$. The automorphism group of $L$ contains group $G = Sp(2l)$ — the symplectic group, associated with the form $(\ , \ )$. We choose a symplectic basis $\{e_1, \ldots, e_l, e_{-l}, \ldots, e_{-1}\}$ in $V$, consisting of eigenvectors under the action of a maximal torus $T$ of the group $G$. Let $e_1$ have weight $\varepsilon_1$, $e_{-1}$ have
weight \(-\varepsilon_i, i = 1, \ldots, l\). The quotient algebra of the exterior algebra \(\Lambda^2 V\) by the ideal \(I = \langle e_1 e_{-1} + \cdots + e_i e_{-i} \rangle\) is isomorphic to the Lie algebra \(L\) (this is true only for odd \(l\)). The basic complex of cohomologies can be decomposed into the direct sum of weight subcomplexes using the natural action of a maximal torus of the Chevalley group \(G(L)\). The corresponding cohomology groups are weight subspaces in the cohomology group of the basic complex. We denote by \(C^n_\mu(L, L), Z^n_\mu(L, L), B^n_\mu(L, L), H^n_\mu(L, L)\) the weight subspaces of cochains, cocycles, coboundaries and cohomologies of weight \(\mu\), respectively. In [7] it is proved that \(H^n_\mu(L, L) = 0\), \(H^2(L, L)\) is the direct sum of non-zero subspaces \(H^2_\mu(L, L)\) where \(\mu\) has the form \(\gamma + \delta\) for \(\gamma, \delta \in R\) satisfying condition \(<\gamma, \delta> = 0\). Any weight \(\mu\) is conjugate under the action of the Weyl group with \(\alpha_i + \alpha_{i-1}\). Total number of such weights is \(2l\): \(\{\pm(\alpha_i - \alpha_{i-1}), \pm(\alpha_i + \alpha_{i-1}), \pm(\alpha_i + \alpha_{i-1} + 2\alpha_{i-2}), \ldots, \pm(\alpha_l + \alpha_{l-1} + 2\alpha_{l-2} + \cdots + 2\alpha_1)\}\) and \(\dim H^2(L, L) = 2l\). Also natural description of cohomology group \(H^2(L, L)\) as a module over the group \(Sp(2l) \subset Aut(L)\) is presented in [7]. It is proved that for Lie algebra of type \(\bar{D}_l\) (\(l > 3\) - odd) \(H^2(L, L)\) as a module over \(Sp(2l)\) is, up to the Frobenius morphism, isomorphic to \(V\), where \(V\) is the standard module over \(Sp(2l)\). Below we describe in details this implementation of \(H^2(L, L)\). Let \(v \in V, w_1 w_2, w_3 w_4 \in \Lambda^2 V/I \cong L\). Set \(\Phi(v) = \varphi\), where
\[
\varphi(v(w_1 w_2, w_3 w_4) = (v, w_1)(v, w_3)w_2w_4 + (v, w_2)(v, w_3)w_1w_4 + (v, w_3)(v, w_4)w_1w_2 + (v, w_4)(v, w_1)w_3w_2 + (v, w_2)(v, w_4)w_3w_4 + (v, w_3)(v, w_1)w_2w_4 + (v, w_1)(v, w_2)w_3w_4.
\]
Using the Poisson brackets: \(\{v v_1 v_2, v\} = (v_1, v)v_2 + (v_2, v)v_1\), \(\Phi(v)\) can be defined as follows:
\[
\Phi(v)(w_1 w_2, w_3 w_4) = \{ w_1, w_2 \} \{ w_3, w_4 \} + (w_3, w_4)w_1w_2 + (w_1, w_2)w_3w_4.
\]
\(\varphi\) is correctly defined on elements of \(\Lambda^2 V\), also \(\varphi\) is linear and skew-symmetric, and, consequently, is an element of \(C^2(L, L)\). Also \(\Phi\) commutes with the action of \(Sp(2l)\):
\[
\Phi(gv)(w_1 w_2, w_3 w_4) = g(\Phi(v)(g^{-1}w_1 g^{-1}w_2, g^{-1}w_3 g^{-1}w_4))
\]
for any \(g \in Sp(2l)\). Using the Poisson brackets in \(\Lambda^2 V\):
\[
[v_1 v_2, v_3 v_4] = \{ v_1 v_2, v_3 v_4 \} = (v_1, v_3)v_2v_4 + (v_1, v_4)v_2v_3 + (v_2, v_3)v_1v_4 + (v_2, v_4)v_1v_3,\]
calculation of \(d\varphi\) on arbitrary elements \(w_1 w_2, w_3 w_4, w_5 w_6\) shows that \(\varphi \in Z^2(L, L)\). Mapping \(\Phi\) induces mapping of \(V\) into cohomology group \(H^2(L, L)\), which we will also denote as \(\Phi\). So, \(\Phi(\varepsilon_{\pm 1}), \ldots, \Phi(\varepsilon_{\pm l})\) define \(2l\) non-trivial cocycles of different weights. Since \(\Phi(kv) = k^2\Phi(v)\) for any
$k \in K$, the action of symplectic group on $H^2(L, L)$ is the action on $V$ up to the Frobenius morphism.

The necessary condition of integrability of arbitrary cocycle $\psi$ to global deformation is the triviality of cocycle $\psi \cup \psi$ from $Z^3(L, L)$, where

$$\psi \cup \psi(x, y, z) = \psi(\psi(x, y), z) + \psi(\psi(y, z), x) + \psi(\psi(z, x), y)$$

for any $x, y, z$ from $L$. In Lie algebra of type $D_l$ ($l > 3$ - odd) exists cocycle $\psi$, such that $\psi \cup \psi \neq 0$ and $\psi \cup \psi$ has weight $\mu$, for which $C^2_\mu(L, L) = 0$. Lets prove it. Since $l > 3$ is odd, in basis $\{e_1, \ldots, e_l, e_{-l}, \ldots, e_{-1}\}$ exist vectors $e_4, e_5, e_5, e_4$. Consider the cocycle $\psi = \Phi(e_4)$. We have

$$\psi \cup \psi(e_{-4}e_5, e_{-4}e_5, e_3e_4) = \psi(\psi(e_{-4}e_5, e_{-4}e_5), e_3e_4) + \psi(\psi(e_3e_4, e_{-4}e_5), e_3e_4) = \psi(\psi(e_{-4}e_5, e_{-4}e_5), e_3e_4) = \psi(e_5e_5, e_3e_4) = e_3e_4.$$  

It is seen that $\psi \cup \psi$ is nonzero cocycle from $Z^3(L, L)$. Since space $C^2_4(L, L)$ is zero, $B^3_4(L, L) = 0$. Therefore, $\psi \cup \psi$ is not a coboundary. So, $\psi$ is nonintegrable cocycle. The group $G$ acts transitively on $V$, so cocycle from $H^2(L, L)$ is not integrable and Lie algebras of type $D_l$ ($l > 3$ - odd) are rigid.

2 Deformations of Lie algebra of type $D_l$ for even $l \geq 4$.

Let $L$ be Lie algebra of type $D_l$ for even $l \geq 4$. Then $L$ has two-dimensional centre: $< H_l + H_{l-1}, H_{l-1} + H_{l-3} + \cdots + H_3 + H_1 >$. The space $H^2(L, L)$ is the direct sum of non-zero subspaces $H^2_\mu(L, L)$, where $\mu$ has the form $\gamma + \delta$ for $\gamma, \delta \in R$, satisfying condition $< \gamma, \delta > = 0$. For $l > 4$ any of such weights $\mu$ is conjugate under the action of the Weyl group with $\alpha_l + \alpha_{l-1}$. Total number of such weights is $2l$: $\{\pm(\alpha_l - \alpha_{l-1}), \pm(\alpha_l + \alpha_{l-2}), \ldots, \pm(\alpha_l + \alpha_{l-1} + 2\alpha_{l-2} + \cdots + 2\alpha_l)\}$. Corresponding weight subspaces of cohomologies are one-dimensional and $\dim H^2(L, L) = 2l$. For $l = 4$ there are 24 weights and they are conjugate with $\alpha_1 + \alpha_3, \alpha_1 + \alpha_4, \alpha_3 + \alpha_4$, $\dim H^2(L, L) = 24$. We will use isomorphism $C^n(L, L) \cong L^* \wedge \ldots \wedge L^* \otimes L^*$.

Since all weights are conjugate (in case of $D_4$ conjugate with $\alpha_1 + \alpha_3, \alpha_1 + \alpha_4, \alpha_3 + \alpha_4$), it is sufficient to describe space $H^2_{\alpha_1 + \alpha_{l-1}}(L, L)$. Cocycle

$$\psi = \sum_{\gamma + \delta = \alpha_1 + \alpha_{l-1}} E^*_\gamma \wedge E^*_\delta \otimes z,$$
where $z = H_l-1 + H_l-3 + \cdots + H_3 + H_1$, generates subspace $H^2_{\alpha_l+\alpha_{l-1}}(L, L)$. Condition integrability for cocycle $\psi$ has the form:

$\psi \cup \psi(x, y, z) = 0$ for any $x, y, z$ from $L$. Therefore, cocycle $\psi$ is integrable. Mapping $f_t(x, y) = [x, y] + t\psi(x, y)$ gives global deformation of Lie algebra $L$. Since for any cocycle $\psi \in H^2(L, L)$ $\psi(x, y)$ is contained in the centre of $L$ for any $x, y$ from $L$ and $\psi(x, y) = 0$ if $x, y \in Z(L)$, condition $\psi \cup \psi(x, y, z) = 0$ holds true for all elements from $H^2(L, L)$. So any cocycle $H^2(L, L)$ for Lie algebra of type $D_l$ for even $l$ are integrable.

Conclusions.

Main results of this paper are formulated in the following theorem.

**Theorem 1** Let $L$ be Lie algebra over an algebraically closed field of characteristic 2.

(1) If $L$ has type $\bar{D}_l$ for odd $l > 3$, then $L$ is rigid Lie algebra.

(2) If $L$ has type $D_l$ for even $l \geq 4$, then any cocycle from $H^2(L, L)$ is integrable.

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