Series expansion of the overlap reduction function for scalar and vector polarizations for gravitational wave search with pulsar timing arrays

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In our previous work [1] we calculated the overlap reduction function for the tensor polarization without employing the short wavelength approximation, this was done by obtaining a power series of nested sums which is valid for all gravitational wave frequencies and pulsar distances. In this work we generalize the power-series expansion method to vector and scalar polarizations. We have compared our expression for the breathing and vector modes with previous literature. We present for the first time analytic expressions for the overlap reduction function of the longitudinal mode for all angles between the pulsar pairs.

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I. INTRODUCTION

To clearly identify a signal picked up by pulsar timing arrays (PTA) as originating from a gravitational wave background, the cross correlations must follow a specific pattern, which for the tensor polarizations is known as the Hellings and Downs curve. Modified general relativity can also have vector and scalar polarizations and these would create different such patterns on the cross correlated signals. These patterns are called overlap reduction functions and are calculated as the sky average over all direction dependent terms in the cross correlation. The understanding of the methods for the detection of gravitational wave background from the non-Einsteinian scalar-transverse correlations in PTA data is becoming very relevant. Currently the 12.5-Year data set from the NANOGrav consortium [2] found a strong correlation between various pulsar data. This correlation is consistent across the pulsars with a common stochastic process which is a power-law of same spectral shape and amplitude [2]. Later it was found that although the non-Einsteinian scalar-transverse correlations are favoured more than the transverse-traceless correlation but including certain systematics (Solar System ephemeris) and removing just the pulsar J0030+0451 evidence in favour of non-Einsteinian scalar-transverse correlations are dramatically reduced [3, 4].

The integration over all sky directions presents some difficulties. The term in the denominator can become zero and thus produces a pole. If the pulsar term is neglected, which is justified in the literature (for example in [5, 7]) via short wavelength approximation, then this pole is in general not cancelled. For the purely transverse polarizations the polarization projections vanish as well in the same sky direction, which makes the calculation of this limiting integral possible [7]. This does not work for polarizations with longitudinal component. As discussed in [7] the pole is weakened sufficiently by polarization projections in the case of the vector mode such that the integral can be calculated analytically. This is not the case for the longitudinal polarization and only the limiting case for $\phi = 0$ could be calculated analytically.

We generalize the method we used in our previous work [1] to all possible polarizations in modified GR to calculate the analytic power series of nested sums for the overlap reduction functions for the vector, longitudinal and breathing modes. Since our method does not rely on the short wavelength approximation the poles from the denominator are always sufficiently cancelled by the pulsar term and thus the integral is always well defined. Moreover, our method yields the precise result for all gravitational wave frequencies $\omega$ and all pulsar distances $L$. (This is not required for the current pulsar timing array experiments but nice to have.)

Our method thus allows us to calculate the overlap reduction function for the longitudinal polarization analytically for all $\phi$-values for the first time. For the breathing and vector polarizations we find an agreement with Lee et. al. [7] but disagree with results presented in Chamberlin and Siemens [6] around the point $\phi = 0$. A disadvantage in our result comes from the fact, that it is a power series of nested sums, which is challenging to evaluate for high frequencies. Our generalization in this work allows to extract the main series terms, such that they can be pre-evaluated and then combined with different finite prefactors to form all overlap reduction functions.

We review the context in which the overlap reduction
functions appear shortly in section (II). In section (III) we present the generalized method from [1] in a summarized way and discuss the results, using plots which depict the most important features in section (IV). Finally, we present our conclusions in section (V).

\[ \Lambda = \frac{1}{\pi} \sum_A \int \frac{1}{2\mu^2} S_h^A(\nu) \delta^2(\nu - f) \frac{1}{\pi^2} \int_{S^2} F^A(\hat{\Omega}) F'^A(\hat{\Omega}) \frac{1 - e^{2\pi i \nu \tau}}{1 + \gamma} \frac{1 - e^{-2\pi i \nu \tau'}}{1 + \gamma'} d\Omega d\nu \tilde{Q}(f) df \]

II. OVERLAP REDUCTION FUNCTIONS

A general gravitational wave in modified general relativity can be written as:

\[ h = \sum_A h_A e^A, \quad A \in \{+, \times, x, y, b, l\}, \]

where \( h_A \) is the amplitude and \( e^A \) the polarization tensor of polarization \( A \).

We re-derived the correlated signal for pulsar timing arrays in our previous paper [1], where we found, that our result agrees with the literature:

\[ \mu = \frac{1}{24\pi^2} \sum_A \int \frac{1}{f^2} \left[ \sum_{A} S^A_h(f) \Gamma_A \right] \tilde{Q}(f) df, \]

where where we use the pattern functions without poles and discontinuities, which we derived in our first paper on this topic [8]:

\[ F^+ = \frac{\alpha^2 - \beta^2}{2}, \quad F^x = \alpha \gamma, \quad F^b = \frac{\alpha^2 + \beta^2}{2}, \]

\[ F^x = \alpha \beta, \quad F^y = \beta \gamma, \quad F^l = \frac{\gamma^2}{\sqrt{2}}, \]

and collected all the direction dependent terms in the overlap reduction functions:

\[ \Gamma_A = \Lambda \int_S F^A(\hat{\Omega}) F'^A(\hat{\Omega}) \frac{1 - e^{2\pi i \nu [1 + \gamma]}}{1 + \gamma} \frac{1 - e^{-2\pi i \nu [1 + \gamma']}}{1 + \gamma'} d\Omega, \]

where \( \Lambda = \frac{1}{2\pi} \) is a normalization constant which we use for better comparison with [9], where it is called \( \beta \).

The pattern function \( F^A(\hat{\Omega}) \) describes the response to GW polarizations of the detector formed with pulsar \( a \) which w.l.o.g. lies in direction \( \hat{x} = (1, 0, 0) \) from earth. The response function of pulsar \( a' \), which w.l.o.g. lies in direction \( \hat{x'} = (\cos \phi, \sin \phi, 0) \) is denoted with a prime \( F'^A(\hat{\Omega}) \).

The propagation direction of a gravitational wave introduces a special direction and breaks the rotation symmetry of the problem. We thus end up with an asymmetry. A gravitational wave background is assumed to be described by an isotropic power spectral density. The polarization pairs \( +, \times \) and \( x, y \) cannot be distinguished from each other in this case, since the asymmetry of the problem allows us to change the two polarizations into one another by rotating the basis. Therefore, we form the pairs of tensor \( T = \{+, \times\} \) and vector \( V = \{x, y\} \) polarizations. This cannot be done with the two scalar polarization, since the rotation around the propagation direction maps \( b \) and \( l \) into themselves. The longitudinal polarization \( l \) is, as the name says, purely longitudinal and the breathing polarization \( b \) is purely transverse, which are two different physical phenomena. Thus we can split the signal accordingly:

\[ \mu = \frac{T}{24\pi^2} \int \frac{1}{f^2} \left[ \sum_{A \in T} S^A_h(f) \Gamma_T + \sum_{A \in V} S^A_h(f) \Gamma_v + \sum_{A} S^A_h(f) \Gamma_b \right] \tilde{Q}(f) df, \]
overlap reduction functions of modified general relativity:

\[
\Gamma_V := \Lambda \int_S \left( F^x(\hat{\Omega})F'^x(\hat{\Omega}) + F^y(\hat{\Omega})F'^y(\hat{\Omega}) \right) \\
\cdot \left( 1 - e^{iL\omega[1+\gamma]} - e^{-iL\omega[1+\gamma']} \right) d\hat{\Omega}
\]

\[
\Gamma_b := \Lambda \int_S F^b(\hat{\Omega})F'^b(\hat{\Omega}) \left( 1 - e^{iL\omega[1+\gamma]} - e^{-iL\omega[1+\gamma']} \right) d\hat{\Omega},
\]

\[
\Gamma_l := \Lambda \int_S F^l(\hat{\Omega})F'^l(\hat{\Omega}) \left( 1 - e^{iL\omega[1+\gamma]} - e^{-iL\omega[1+\gamma']} \right) d\hat{\Omega},
\]

### III. Power Series Representation of the Overlap Reduction Functions for Scalar and Vector Polarization's

We follow the same methodology as described in our previous work [1] and use the residue theorem, to calculate the integral over the angle \( \varphi \in [0, 2\pi) \).

\[
\Gamma_M = \Lambda \sum_{A \in M} \int_0^{2\pi} \int_0^\pi F^A(\hat{\Omega})F'^A(\hat{\Omega}) h(\hat{\Omega}) h'(\hat{\Omega}) \sin \theta d\theta d\varphi = \Lambda \int_0^\pi \oint_{C_1} f_M(z) dz d\theta = 2\pi i \Lambda \int_0^\pi \text{Res}[f_M(z), 0] d\theta,
\]

\[
f_M(z) = \frac{1}{iz} \sum_{A \in M} \left( F^A F'^A h h' \right)(\theta, z) \sin \theta, \quad h = \frac{1 - e^{iL\omega[1+\gamma]}}{1+\gamma}, \quad z = e^{i\varphi}, \quad M \in \{V, \{b\}, \{l\}\}
\]

The prefactors of the series coming from the pulsar term contribute to the polynomial term \( P_M(z) \) such that it is related to \( F^A(z)F'^A(z) \) by:

\[
P_M(z) = \sum_{N' \in M} A_{N'} z^{N'},
\]

and then split the residue into two parts:

\[
\text{Res}[f_M(z), 0] = \text{Res} \left[ \frac{ie^{-i\phi}}{8 \sin \theta} P_M(z) G(z) \right]_{\text{Res}_1} + \text{Res} \left[ \frac{ie^{-i\phi}}{8 \sin \theta} P_M(z) E G(z) \right]_{\text{Res}_2}
\]

Thus the polarization term is entirely captured by the polynomial term and \( G(z) \) and \( E(z) \) are the same for every polarization.

For the three modes of interest, the polynomial part reads:
\[ P_1(z) = \frac{e^{2i\phi} \sin^4 \theta}{z^3} + \frac{2(1 + e^{2i\phi}) \sin^4 \theta}{z} + 2(2 + \cos(2\phi)) \sin^4 \theta z + 2(1 + e^{-2i\phi}) \sin^4 \theta z^3 + e^{-2i\phi} \sin^4 \theta z^5 \] (14)

\[ P_2(z) = \frac{e^{2i\phi} \sin^4 \theta}{2z^3} + \frac{(1 + e^{2i\phi})(\cos^4 \theta - 1)}{z} + (2 + \cos^2 \theta)^2 + \cos(2\phi) \sin^4 \theta z \\
+ (1 + e^{-2i\phi})(\cos^4 \theta - 1) \sin^4 \theta z^3 + \frac{1}{2} e^{-2i\phi} \sin^4 \theta z^5 \] (15)

\[ P_3(z) = -\frac{2e^{2i\phi} \sin^4 \theta}{z^3} + \frac{(1 + e^{2i\phi}) \sin^2(2\theta)}{z} + 4(\cos(2\phi)(1 + \cos^2 \theta) + 2 \cos^2 \theta) \sin^2 \theta z \\
+ (1 + e^{-2i\phi}) \sin^2(2\theta) \sin^4 \theta z^3 - 2e^{-2i\phi} \sin^4 \theta z^5 \] (16, 17)

As it can be seen, the polynomial part in the three different modes contain the same powers of \( z \) as the one for the tensor mode did [1], (supplemental material) (7). Since \( G(z) \) only contains positive powers (see [1] eq. (25)), the first residue is a finite sum:

\[
\text{Res}_1^M = \text{Res} \left[ -\frac{ie^{-i\phi}}{8 \sin \theta} P_M(z) G(z), 0 \right] \\
= \text{Res} \left[ \sum_{n \in \mathbb{Z}} b_n z^n, 0 \right] = b_{-1} \tag{18}
\]

For a polynomial starting at \( N' = -3 \) we get the following residue:

\[ \text{Res}_1^M = \frac{ie^{-i\phi}}{8 \sin \theta} \left( 2e^{-i\phi} \left[ \cos \phi - 2 \frac{1 + 2 \cos\phi}{\sin^2 \theta} \right] A_{-3}^M + 2 \frac{1 + e^{-i\phi}}{\sin \theta} A_{-2}^M - A_{-1}^M \right) \] (19)

The contribution of this residue to the overlap reduction function for those three cases then read:

\[ 2\pi \int_0^\pi \text{Res}_1^M d\theta = \frac{2}{3} \pi (3 + 7 \cos \phi) \] (20)
\[ 2\pi \int_0^\pi \text{Res}_1^L d\theta = \frac{\pi}{3} (3 + \cos \phi) \] (21)
\[ 2\pi \int_0^\pi \text{Res}_1^V d\theta = -\frac{4}{3} \pi (3 + 4 \cos \phi) \] (22)

To calculate the second residues we can use the fact, that Res is a linear map and pull the prefactors out front:

\[ \text{Res}_2 = \text{Res} \left[ -\frac{ie^{-i\phi}}{8 \sin \theta} P_M(z) EG(z), 0 \right] \]
\[ = -\frac{ie^{-i\phi}}{8 \sin \theta} \sum_{N'} A_{N'}^M S_{N'}, \tag{23} \]
\[ S_{N'} = \text{Res} \left[ z^{N'} EG(z), 0 \right] \tag{24} \]

We used this definition of the series terms \( S_{N'} \) in the same manner in [1], (supplemental material), (35). Picking the coefficient to the power \( z^{-1} \) leads to the equation:

\[ z^{-n+2j+l+m+2i+N'} = z^{-1} \]
\[ \iff n = 2j + l + m + 2i + N', \quad N = N' + 1 \] (25) (26)
\[
S_{N-1} = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \sum_{m}^{j-k} \sum_{l=0}^{j-k} (-1)^j e^{-i(k+m)\phi} \binom{k}{m} \binom{j-k}{l} \frac{2}{\sin \theta} \left( \sum_{m}^{j-l-m} i^{j-l+m+N} g^+_{j+l+m+N}(\theta) \right),
\]

and for the special case \(N' = -3\) from [1], (supplemental material), (39):

\[
S_{-3} = \sum_{j=2}^{\infty} \sum_{k=0}^{j} \sum_{m}^{j-k} \sum_{l=0}^{j-k} (-1)^j e^{-i(k+m)\phi} \binom{k}{m} \binom{j-k}{l} \frac{2}{\sin \theta} \left( \sum_{m}^{j-l-m} i^{j-l+m-2} g^+_{j+l+m-2}(\theta) \right) - (1 + e^{-2i\phi}) g_0^+(\theta) - \frac{2i}{\sin \theta} (1 + e^{-i\phi}) g_1^-(\theta) - g_2^-(\theta)
\]

where we used the \(g^\pm_n(\theta)\) as defined in [1], (supplemental material), (33):

\[
g^\pm_n(\theta) = e^{i(L-L')\omega} \left( \frac{L - e^{\pm i\phi} L'}{\sqrt{L^2 + L'^2 - 2LL' \cos \phi}} \right)^n J_n \left( \sqrt{L^2 + L'^2 - 2LL' \cos \phi} \omega \sin \theta \right) - e^{iL\omega} J_n(L \omega \sin \theta) - e^{i(\pm n\phi - L'\omega)} J_n(-L' \omega \sin \theta)
\]

with the Bessel functions \(J_n\).

We can calculate the contribution of the second residue for all polarizations in an efficient manner, by introducing the functions \(H^M_{N'}\):

\[
2\pi i \int_0^\pi \operatorname{Res}_2 d\theta = \frac{\pi}{4} e^{-i\phi} \sum_{N'} H^M_{N'}(\phi),
\]

\[
H^M_{N'}(\phi) = \int_0^\pi \sin^{-1} \theta A^M_{N',S_{N'}} d\theta
\]

They can be calculated, by expanding the coefficients \(A^M_{N',S_{N'}}\) of the polynomial \(P_M\) in powers \(s\) of \(\sin \theta\) and \(t\) of \(\cos \theta\):

\[
A^M_{N',S_{N'}} = \sum_{s,t} A^M_{s,t,N',S_{N'}} \sin^{s+1} \theta \cos^{t} \theta,
\]

so that we can apply the identity:

\[
\int_0^\pi \cos^m(\theta) \sin^n(\theta) d\theta = \frac{(1 + (-1)^m) \Gamma \left( \frac{1+m}{2} \right) \Gamma \left( \frac{1+n}{2} \right)}{2\Gamma \left( \frac{1}{2}(2 + m + n) \right)},
\]

\(\text{Re}(m), \text{Re}(n) > -1\)

in the same way as in our previous work [1], eq. (28).

The coefficients of the extended multi variate polynomials

\[
Q_M = \sum_{N'} \sum_{s,t} A^M_{s,t,N'} \sin^{s} \theta \cos^{t} \theta z^{N'}
\]

in \(z, \sin \theta\) and \(\cos \theta\) for the polarization modes \(l, b\) and \(V\) are given by:
Note that instead of collecting all series terms for the special case.

We can write the

\[ A_{3,0,-3}^t = 2(1 + e^{2i\phi}) \]

\[ A_{3,0,-1}^t = 2((2 + \cos(2\phi)) \]

\[ A_{3,0,1}^t = 2(1 + e^{-2i\phi}) \]

\[ A_{3,0,3}^t = e^{-2i\phi} \]

This lead us to the definition in eq. (30) [1]:

\[ h_{j,b,s,t,N}^\pm(\phi) := \int_0^\pi \sin^{-j+b+s} \theta \cos^l \theta \, g_{j+b+N}(\theta) \, d\theta \quad (35) \]

The special case for which we used \( f_{a,b} \) does not occur in the cases of the vector and breathing modes here.

When we introduce the \( \theta \)-integrated series term

\[ \mathcal{H}_{j,s,t,N}(\phi) := \sum_{j=J}^{\infty} \sum_{k=0}^{j} \sum_{m=0}^{k} \sum_{l=0}^{j-k} \left( \frac{1}{2} \right)^{m+l} \binom{k}{m} \binom{j-k}{l} e^{-i(k+m)\phi} h_{j,m+l,s,t,N}^\pm(\phi) \quad (36) \]

we can write the \( H_{N'}^M \) explicitly:

\[ H_{N'}^M = \sum_{s,t} A_{s,t}(\phi) \mathcal{H}_{0,s,t,N}(\phi), \quad N' > -1 \quad (37) \]

and

\[ H_{-3}^M = \sum_{s,t} A_{s,t} \left( \mathcal{H}_{2,s,t,N}(\phi) + \left[ (1 + e^{-2i\phi}) h_{0,0,0,s,t,0}(\phi) \right. \right. \]

\[ + 2i(1 + e^{-i\phi}) h_{0,0,0,-s+1}(\phi) + h_{0,0,0,s+2}(\phi) \left. \right]) \quad (38) \]

for the special case.

Note that instead of collecting all series terms \( H_{N'}^M \) in one series as we did last time [1] we leave the series terms separate. This way we can let the cases \( N' > -1 \) start from \( j = 0 \) and there is no need to extract a sum-term to make the series mach with the lower bound of the \( H_{-3}^M \)-series at \( j = 2 \).

As explained in Section IV in our previous paper [1] we reorder the series to increase the efficiency of the numerical evaluation. We transform our indices according to [1], (supplemental material), eq. (62):

\[ j, k, m, l \rightarrow a = j + b, \quad b = m + l, \quad k, m \quad (39) \]

From [1], (supplemental material), eq. (63), (65), (68) and (70) we see, that \( \mathcal{H} \) transforms as:
The exceptional term in $H_{2,s,t,N}$ for $a = 2$ comes from the lower bound of the sum over $a$. When we start from $a = 0$ this term falls away. The $h^+$-functions are the redefined ones from \[\Pi\], (supplemental material), eq. (66).

Finally the overlap reduction function for the polarization mode $M$ can be written as:

$$
\Gamma_M = 2\pi i \Lambda \left( \int_0^\pi \text{Res}_1 \, d\theta + \int_0^\pi \text{Res}_2 \, d\theta \right) \\
= 2\pi i \Lambda \int_0^\pi \text{Res}_1 \, d\theta + \frac{\pi}{4} \Lambda e^{-i\phi} \sum_{N'} H_{N'}^M, \quad \text{(42)}
$$

\[\Gamma_l = \frac{1}{2}(3 + 7 \cos \phi) - \frac{3}{16} e^{i\phi} \left[ (1 + e^{-2i\phi}) h^+_{0,0,3,0,0}(\phi) + 2i(1 + e^{-i\phi}) h^-_{0,0,2,0,1}(\phi) + h^-_{0,0,3,0,2}(\phi) \right]
\]

\begin{align*}
&+ \frac{3}{16} e^{-i\phi} \left[ \sum_{j=0}^{1} (-2)^j \sum_{k=0}^{j} \sum_{m=0}^{j-k} \left( \frac{i}{2} \right)^{m+l} \binom{k}{m} \binom{j-k}{l} e^{-i(k+m)\phi} \right. \\
&\cdot \left\{ -e^{-2i\phi} h^+_{j,m+l,3,0,0}(\phi) + 2(1+e^{2i\phi}) h^+_{j,m+l,3,0,0}(\phi) + 2(1+e^{-2i\phi}) h^+_{j,m+l,3,0,0}(\phi) - 2(2+\cos(2\phi)) h^+_{j,m+l,3,0,2}(\phi) \right\} \\
&+ \sum_{a=2}^{\infty} \sum_{b=0}^{\infty} (-2)^a \sum_{b=0}^{\infty} \left( -\frac{1}{4} \right)^b \\
&\cdot \left\{ -e^{2i\phi} h^+_{a,b,3,0,0}(\phi) + e^{-2i\phi} h^+_{a,b,3,0,0}(\phi) + 2(1+e^{2i\phi}) h^+_{a,b,3,0,0}(\phi) + 2(1+e^{-2i\phi}) h^+_{a,b,3,0,0}(\phi) - 2(2+\cos(2\phi)) h^+_{a,b,3,0,2}(\phi) \right\}
\end{align*}

\begin{align*}
&+ \sum_{a=2}^{\infty} \sum_{b=0}^{\infty} \sum_{k=0}^{\infty} \left( -\frac{1}{4} \right)^b \\
&\cdot \left\{ e^{-ik\phi} \binom{a-b-k}{b} \binom{2}{-b} F_1(-b, -k; 1 + a - 2b - k; e^{-i\phi}) \quad a \geq 2b + k \right. \\
&\left. \cdot \sum_{k=0}^{\infty} \left( e^{-i\phi} \binom{a-b-k}{b} \binom{2}{-b} F_1(-a + 2b, a + b + k; 1 - a + 2b + k; e^{-i\phi}) \quad a < 2b + k \right) \right]
\end{align*}

\[\text{(43)}\]
\[
\Gamma_b = \frac{1}{4} (3 + \cos \phi) - \frac{3}{32} e^{i \phi} \left[ (1 + e^{-2i \phi}) h^+_{0,0,3,0,0}(\phi) + 2i(1 + e^{-i \phi}) h^-_{0,0,2,0,1}(\phi) + h^-_{0,0,3,0,2}(\phi) \right]
+ \frac{3}{16} e^{-i \phi} \left[ \sum_{j=0}^{\infty} (-2i)^j \sum_{k=0}^{j} \sum_{m=0}^{j-k} \sum_{l=0}^{(j-k) m+l} \left( \frac{i}{2} \right) \binom{k}{m} \binom{j-k}{l} e^{-i(k+m) \phi} \right]
\cdot \left\{ -\frac{e^{-2i \phi}}{2} h^+_{j,m+l,3,0,6}(\phi) + (1 + e^{2i \phi}) \left( -2 h^+_{j,m+l,1,0,0}(\phi) + h^+_{j,m+l,3,0,0}(\phi) \right) + (1 + e^{-2i \phi}) \left( -2 h^+_{j,m+l,1,1,0,4}(\phi) \right. \right.
+ h^+_{j,m+l,1,0,2}(\phi) \left. \right) - 8 h^+_{j,m+l,-1,0,2}(\phi) + 8 h^+_{j,m+l,1,0,2}(\phi) - 2(2 + \cos(2 \phi)) h^+_{j,m+l,3,0,2}(\phi) \right\}
+ \sum_{a=2}^{\infty} \sum_{b=0}^{\left\lfloor \frac{a}{2} \right\rfloor} \left( \frac{1}{4} \right)^b
\cdot \left\{ -\frac{e^{2i \phi}}{2} h^+_{a,b,3,0,6}(\phi) - \frac{e^{-2i \phi}}{2} h^+_{a,b,3,0,6}(\phi) + (1 + e^{2i \phi}) \left( -2 h^+_{a,b,1,0,0}(\phi) + h^+_{a,b,3,0,0}(\phi) \right) + (1 + e^{-2i \phi}) \right.
\cdot \left( -2 h^+_{a,b,1,0,4}(\phi) + h^+_{a,b,3,0,4}(\phi) \right) - 8 h^+_{a,b,-1,0,2}(\phi) + 8 h^+_{a,b,1,0,2}(\phi) - 2(2 + \cos(2 \phi)) h^+_{a,b,3,0,2}(\phi) \left\} \right.
\left. \sum_{k=0}^{a-b} e^{-ik \phi(\frac{a-b}{2} - k)} 2F_1(-b, -k; 1 + a - 2b - k; e^{-i \phi}) \quad a \geq 2b + k \right.
\left. \sum_{k=0}^{a-b} e^{i(2b+k) \phi(\frac{a-b}{2} + k)} 2F_1(-a + 2b, -a + b + k; 1 - a + 2b + k; e^{-i \phi}) \quad a < 2b + k \right\}
\]

\[
\Gamma_V = -(3 + 4 \cos \phi) + \frac{3}{8} e^{i \phi} \left[ (1 + e^{-2i \phi}) h^+_{0,0,3,0,0}(\phi) + 2i(1 + e^{-i \phi}) h^-_{0,0,2,0,1}(\phi) + h^-_{0,0,3,0,2}(\phi) \right]
+ \frac{3}{16} e^{-i \phi} \left[ \sum_{j=0}^{\infty} (-2i)^j \sum_{k=0}^{j} \sum_{m=0}^{j-k} \sum_{l=0}^{(j-k) m+l} \left( \frac{i}{2} \right) \binom{k}{m} \binom{j-k}{l} e^{-i(k+m) \phi} \right]
\cdot \left\{ 2e^{-2i \phi} h^+_{j,m+l,3,0,6}(\theta) + 4 \left( 1 + e^{2i \phi} h^+_{j,m+l,1,0,0}(\theta) + 4(1 + e^{-2i \phi}) h^+_{j,m+l,1,2,4}(\theta) \right.ight.
+ 4 \cos(2 \phi) h^+_{j,m+l,1,0,2}(\theta) - 4(\cos(2 \phi) + 2) h^+_{j,m+l,1,2,2}(\theta) \left. \right\}
+ \sum_{a=2}^{\infty} \sum_{b=0}^{\left\lfloor \frac{a}{2} \right\rfloor} \left( \frac{1}{4} \right)^b
\cdot \left\{ 2e^{2i \phi} h^+_{a,b,3,0,6}(\theta) + 2e^{-2i \phi} h^+_{a,b,3,0,0}(\theta) + 4 \left( 1 + e^{2i \phi} h^+_{a,b,1,0,0}(\theta) + 4 \right. \right.
+ 4(1 + e^{-2i \phi}) h^+_{a,b,1,2,4}(\theta) - 4 \cos(2 \phi) h^+_{a,b,1,0,2}(\theta) - 4 \cos(2 \phi) + 2) h^+_{a,b,1,2,2}(\theta) \left. \right\}
\cdot \sum_{k=0}^{a-b} e^{i(2b+k) \phi(\frac{a-b}{2} - k)} 2F_1(-b, -k; 1 + a - 2b - k; e^{-i \phi}) \quad a \geq 2b + k
\cdot \sum_{k=0}^{a-b} e^{-i(2b+k) \phi(\frac{a-b}{2} + k)} 2F_1(-a + 2b, -a + b + k; 1 - a + 2b + k; e^{-i \phi}) \quad a < 2b + k \right\}
\]
where we use the reordered version from [1], (supplemental material), (66) in the series over $a$:

$$h_{a,b,s,t,N}^{\pm}(\phi) := \Gamma\left(\frac{t+1}{2}\right) \Gamma\left(b + \frac{s+N+1}{2}\right)
\cdot e^{i(L-L')\omega}\left[\frac{(L-e^{\pm i\varphi}L')\omega}{2}\right]^{a+N} \tilde{F}_2\left(b + \frac{s+N+1}{2} ; a + N + 1, b + \frac{s+t+N}{2} + 1 ; -\left(L^2 + L'^2 - 2LL'\cos\phi\right)\omega^2\right)
- e^{i\omega}\left(\frac{L'\omega}{2}\right)^{a+N} \tilde{F}_2\left(b + \frac{s+N+1}{2} ; a + N + 1, b + \frac{s+t+N}{2} + 1 ; -\left(\frac{L\omega}{2}\right)^2\right)
- e^{i(a+N)\varphi-L'\omega}\left(\frac{-L'\omega}{2}\right)^{a+N} \tilde{F}_2\left(b + \frac{s+N+1}{2} ; a + N + 1, b + \frac{s+t+N}{2} + 1 ; -\left(L'\omega\right)^2\right)$$

(46)

**IV. DISCUSSION OF THE RESULTS**

To check the validity of our calculations we compare the truncated series to a direct numerical integration in Fig. 1. Since the evaluation the series becomes numerically expensive with growing $L\omega$, we plot the $\phi$ dependent overlap reduction functions for a sample of $L\omega$-parameters to visualize its tendency with increasing $L\omega$ up to 75. We discussed the behaviour of the series for different pulsar distances $L \neq L'$ in detail in our previous paper [1] and focus on the different properties of the overlap reduction functions for the vector, longitudinal and breathing modes in the case of equidistant pulsars $L = L'$ here. In this case the imaginary part vanishes.

The limiting overlap reduction functions of the vector and breathing mode for $L\omega \rightarrow \infty$ were calculated in [7], (A44) and (A33), using the short wavelength approximation. We include the normalization constant $\Lambda$ here for better comparison:

$$\Gamma_{V,\text{Lee}}(\phi) = 3\left(-1 - \frac{4\cos\phi}{3} + \log\left(\frac{2}{1 - \cos\phi}\right)\right)$$

(47)

$$\Gamma_{b,\text{Lee}}(\phi) = \frac{1}{4}(\cos\phi + 3 + 4\delta(\phi))$$

(48)

Since taking the limit of $L\omega \rightarrow \infty$ of the power series expression is a daunting if not impossible task, we compare it to the limits obtained by [7] given in (47) and (48) in the middle and lower plot of Fig. 1. The singularity in the integrand for the longitudinal mode is not sufficiently cancelled by the polarization projections and thus the short wavelength approximation was not applicable. Here we calculate it (48) analytically for the first time and show its graph in the upper plot of Fig. 1.

As for the tensor polarization mode [1], the overlap reduction function for the breathing mode converges to 2 for large $L\omega$ at $\phi = 0$, which agrees with the intuition.

![FIG. 1: A comparison between the truncated series and numerical integration of the real part of the overlap reduction functions for different $L\omega$ values for the longitudinal (top), breathing (middle) and vector (bottom) polarization. The imaginary parts vanish. For the breathing and vectorial modes, an analytical curve from [7] is displayed for comparison.](image-url)
that two aligned detector arms would double the signal. As depicted in Fig. 2, the function shows the same behaviour like the one for the tensor mode. The upper plot shows, how it converges rapidly to the value obtained using the short wavelength approximation away from $\phi = 0$. In the lower plot we show that it converges slower as we approach $\phi = 0$ until it assumes the value 2 at $\phi = 0$, in agreement with the $\delta$-term in [7].

If the polarization has a longitudinal component, the sensitivity of the detector seems to diverge for co-located pulsars when $L \rightarrow \infty$. In appendix A of [8] we found that photons traveling in the same direction as a longitudinal gravitational wave appear to be travelling slightly faster or slower than light speed as seen in the Earth reference frame. This happens, because the gravitational wave is constantly decrease or increase the distance the photons cover by squeezing or stretching the space-time in travel direction. Therefore, two subsequent pulses can increase their time of arrival difference with increasing pulsar difference.

This feature is more obvious in Fig 3 and Fig. 2 where we display the overlap reduction functions dependence on $L\omega$ for different $\phi$ near 0. In agreement with [6, 7] we find that for large $L\omega$ the overlap reduction function for the longitudinal polarization increases linearly for $\phi$-values close enough to 0. The overlap reduction function for the vector mode converges to increasingly large values with decreasing $\phi$ for $L\omega \rightarrow \infty$. Thus, it diverges at $\phi = 0$ for $L\omega \rightarrow \infty$. This is confirmed by the logarithmic growth in $\Phi \propto L\omega$ shown in [7], eq. (A42) at $\phi = 0$. In [6], Fig. 6 however they claim, that $\Gamma_V$ converges to a finite value with $L\omega \rightarrow \infty$ at $\phi = 0$.

![FIG. 2: The plot on the top shows the fast convergence of the overlap reduction function for the breathing mode at $\phi$ sufficiently far away from 0. The bottom plot shows how the function converges slower for $\phi$ close to 0. The limiting values are calculated from (48).](image1)

![FIG. 3: We plot the $L\omega$ dependence of the overlap reduction functions for the longitudinal polarization (top) and the vector polarization (bottom) for $\phi$-values around 0. We compare our results with the expressions found in [7] (dashed lines).](image2)

**V. CONCLUSIONS**

We find an analytical expression for the overlap reduction functions of the breathing, longitudinal and vector mode, in form of a power series of nested sums, which is valid for all $L\omega$ since the method does not make any prior assumptions on the gravitational wave frequency or pulsar distance. This is done by generalizing the method we derived in [1], which does not use the short wavelength approximation which made it possible to calculate the overlap reduction function for the longitudinal polarization for all $\phi$-values for the first time.

We formulate the method in a more compact way, which allows us to extract the series terms. This allows to pre-evaluate the series terms and then combine them into the overlap reduction functions for the different modes. This saves computation time since the evaluation of power series of nested sums is numerically costly.

Our results are in agreement with prior literature [6, 7]. We confirm the divergent behaviour of the longitudinal...
components at $\phi = 0$ when $L\omega$ tends to infinity. This should not be interpreted as an infinitely strong signal. Two subsequent pulses acquire a larger time of arrival difference during their journey to earth. Thus, for finite distances the signal will be finite. Also, since the time of arrival increases with $L\omega$ one also requires a longer observation time to detect the two subsequent pulses, which naturally increases the $SNR$ of a persistent signal.

The breathing mode behaves similar to the tensor mode and we conclude that the transverse and longitudinal components show a very different behaviour and in the superposition of the two, which is the case for the vector mode the longitudinal feature takes over (since the transverse is finite at $\phi = 0$, while the longitudinal grows monotonically with increasing $L\omega$.)

The alignment or orthogonality of the polarization with the direction of travel of a wave is a geometrically well-defined (independent of choice of basis) and thus physically relevant property. With that in mind it is not surprising, that for a GWB the two scalar modes, breathing and longitudinal, can be distinguished and the vector $x$, $y$ and tensor $+, \times$ cannot.

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[1] A. Boitier, S. Tiwari, and P. Jetzer, Phys. Rev. D 103, 064044 (2021).
[2] Z. Arzoumanian et al. (NANOGrav), Astrophys. J. Lett. 905, L34 (2020), arXiv:2009.04496 [astro-ph.HE].
[3] Z. Arzoumanian et al. (NANOGrav), (2021), arXiv:2109.14706 [gr-qc].
[4] Z.-C. Chen, Y.-M. Wu, and Q.-G. Huang, (2021), arXiv:2109.00296 [astro-ph.CO].
[5] M. Anholm, S. Ballmer, J. D. E. Creighton, L. R. Price, and X. Siemens, Phys. Rev. D 79, 084030 (2009).
[6] S. J. Chamberlin and X. Siemens, Phys. Rev. D 85, 082001 (2012), arXiv:1111.5661 [astro-ph.HE].
[7] K. J. Lee, F. A. Jenet, and R. H. Price, Astrophys. J. 685, 1304 (2008).
[8] A. Boitier, S. Tiwari, L. Philippoz, and P. Jetzer, Phys. Rev. D 102, 064051 (2020).
[9] W. R. Inc., “Mathematica, Version 12.1,” Champaign, IL, 2019.