On analytic structure of weighted shifts on generalized directed semi-trees

Gargi Ghosh\textsuperscript{a} and Somnath Hazra\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Indian Institute of Science, Bangalore, India; \textsuperscript{b}Department of Mathematics & Statistics, Indian Institute of Science Education and Research Kolkata, Mohanpur, India

ABSTRACT
Inspired by natural classes of examples, we define generalized directed semi-trees and construct weighted shifts on them. Given an \( n \)-tuple of generalized directed semi-trees with certain properties, we associate an \( n \)-tuple of multiplication operators on a Hilbert space \( \mathcal{H}^{2}(\beta) \) of formal power series. Under certain conditions, \( \mathcal{H}^{2}(\beta) \) turns out to be a reproducing kernel Hilbert space consisting of holomorphic functions on some domain in \( \mathbb{C}^{n} \) and the \( n \)-tuple of multiplication operators on \( \mathcal{H}^{2}(\beta) \) is unitarily equivalent to an \( n \)-tuple of weighted shifts on generalized directed semi-trees. Finally, we exhibit two classes of examples of \( n \)-tuples of operators, which can be intrinsically identified as weighted shifts on generalized directed semi-trees.

ARTICLE HISTORY
Received 8 September 2021
Accepted 8 March 2022

COMMUNICATED BY
A. Khare

KEYWORDS
Weighted shift; generalized directed semi-tree; elementary symmetric polynomial; Schur polynomial

2020 MATHEMATICS SUBJECT CLASSIFICATIONS
Primary: 47B37; 47B38; Secondary: 05C63; 05C20

1. Introduction

The study of the adjacency operator for an infinite directed graph has been initiated in Ref. [1]. Later Jablonski, Stochel and Jung specialized in adjacency operators for weighted directed trees in Ref. [2]. They referred to these as weighted shifts on directed trees and obtained several significant results that enriched operator theory in several aspects. Inspired by generalized creation operators on Segal-Bargmann space, Majdak and Stochel generalized this to weighted shifts on directed semi-trees in Ref. [3].

Let \( \mathbb{D} \) denote the open unit disc in the complex plane \( \mathbb{C} \). For \( \lambda > 0 \), it is well known that

\[
K^{(\lambda)}(z, w) := \prod_{i=1}^{n} \frac{1}{1 - z_{i} \bar{w}_{i})^{\lambda}}, \quad z, w \in \mathbb{D}^{n}
\]

is a positive definite kernel on \( \mathbb{D}^{n} \). Let \( A^{(\lambda)}(\mathbb{D}^{n}) \) denote the Hilbert space consisting of holomorphic functions with reproducing kernel \( K^{(\lambda)} \). In particular, \( A^{(\lambda)}(\mathbb{D}^{n}) \) coincides with the Hardy space \( H^{2}(\mathbb{D}^{n}) \) and the Bergman space \( A^{2}(\mathbb{D}^{n}) \) for \( \lambda = 1 \) and \( \lambda = 2 \), respectively. Let
us denote the permutation group on \( n \) symbols by \( \mathfrak{S}_n \). The subspaces

\[
\mathbb{A}^{(\lambda)}_{\text{sym}}(\mathbb{D}^n) = \{ f \in \mathbb{A}^{(\lambda)}(\mathbb{D}^n) : f \circ \sigma^{-1} = f \text{ for } \sigma \in \mathfrak{S}_n \},
\]

and

\[
\mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n) = \{ f \in \mathbb{A}^{(\lambda)}(\mathbb{D}^n) : f \circ \sigma^{-1} = \text{sgn}(\sigma)f \text{ for } \sigma \in \mathfrak{S}_n \}
\]

are joint reducing subspaces of the \( n \)-tuple of multiplication operators \( M_s := (M_{s_1}, \ldots, M_{s_n}) \) on \( \mathbb{A}^{(\lambda)}(\mathbb{D}^n) \), where \( s_i \) denotes the elementary symmetric polynomial of degree \( i \) in \( n \) variables, see [4, p. 774]. In Ref. [5], Biswas et al. prove that each of the operators \( M_s|_{\mathbb{A}_{\text{anti}}^{(\lambda)}(\mathbb{D}^n)} \) and \( M_s|_{\mathbb{A}_{\text{sym}}^{(\lambda)}(\mathbb{D}^n)} \) is unitarily equivalent to the \( n \)-tuple of coordinate multiplication operators on some reproducing kernel Hilbert space containing holomorphic functions on the symmetrized polydisc. However, in [4, p. 771, Corollary 3.9], it is shown that neither \( M_s|_{\mathbb{A}_{\text{anti}}^{(\lambda)}(\mathbb{D}^n)} \) nor \( M_s|_{\mathbb{A}_{\text{sym}}^{(\lambda)}(\mathbb{D}^n)} \) is unitarily equivalent to any joint weighted shift. It has been observed that each of the operators \( M_s|_{\mathbb{A}_{\text{anti}}^{(\lambda)}(\mathbb{D}^n)} \) and \( M_s|_{\mathbb{A}_{\text{sym}}^{(\lambda)}(\mathbb{D}^n)} \) has a natural identification as an \( n \)-tuple of weighted shift operators on generalized structure of directed semi-trees. Motivated by these examples, we have defined generalized directed semi-trees and described weighted shifts on generalized directed semi-trees in this paper.

We have represented weighted shifts on generalized directed semi-trees as multiplication operators on Hilbert spaces consisting of analytic functions. Representing weighted shift operators as multiplication operators make various well-known operator theoretic tools accessible to analyse these operators. For example, understanding the unilateral shift as a multiplication operator by the coordinate function on the Hardy space of the unit disc yields a significant exposition. Shields exhibits an insightful association of weighted shift operators with analytic functions in Ref. [6]. Jewell and Lubin show a similar interplay between commuting weighted shifts and analytic functions in several variables, see [7]. Recently, in Ref. [8], Chavan et al. described an analytic model for left-invertible weighted shifts on directed trees, using Shimorin’s analytic model described in Ref. [9]. A different approach has been made to describe an analytic structure of weighted shifts on directed trees in Ref. [10]. However, we follow the framework of Ref. [7] in this paper.

Now we briefly outline the content of this paper. In Section 2, we have reproduced a few basic notions of graph theory. Given generalized directed semi-trees \( (\mathcal{G}_i, m_i), 1 \leq i \leq n \), an \( n \)-tuple of multiplication operators on a Hilbert space of formal power series \( \mathcal{H}^2(\beta) \) have been constructed in Section 3. A necessary and sufficient condition for the continuity of that \( n \)-tuple of multiplication operators on \( \mathcal{H}^2(\beta) \) is provided in Lemma 3.1. Under certain conditions, \( \mathcal{H}^2(\beta) \) turns out to be a reproducing kernel Hilbert space consisting of holomorphic functions on some domain in \( \mathbb{C}^n \). Moreover, we show in Theorem 3.1 that the \( n \)-tuple of multiplication operators on \( \mathcal{H}^2(\beta) \) is unitarily equivalent to an \( n \)-tuple of operators \( (\Lambda_1, \ldots, \Lambda_n) \) where each \( \Lambda_i \) is a weighted shift on the generalized directed semi-tree \( (\mathcal{G}_i, m_i) \).

In Sections 4 and 5, we provide the following natural classes of examples.

(1) For \( \lambda > 0 \), we denote the weighted Bergman space on the polydisc \( \mathbb{D}^n \) by \( \mathbb{A}^{(\lambda)}(\mathbb{D}^n) \). Moreover, the subspaces \( \mathbb{A}^{(\lambda)}_{\text{sym}}(\mathbb{D}^n) \) and \( \mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n) \) consist of all symmetric functions in \( \mathbb{A}^{(\lambda)}(\mathbb{D}^n) \) and all anti-symmetric functions in \( \mathbb{A}^{(\lambda)}(\mathbb{D}^n) \), respectively. The elementary symmetric polynomial of degree \( i \) in \( n \) variables is denoted by \( s_i \). For each
1 ≤ i ≤ n, the restrictions of the multiplication operator $M_{si}$ on the spaces $A_{\text{sym}}^{(k)}(D^n)$ and $A_{\text{anti}}^{(k)}(D^n)$ have a natural identification as weighted shifts on generalized directed semi-trees.

(2) Let $G$ be a finite pseudoreflection group and $D$ be a complete Reinhardt domain in $\mathbb{C}^n$ which is $G$-invariant. The Bergman space on $D$ is denoted by $A^2(D)$. For each $1 ≤ i ≤ n$, the operator $M_{\theta_i} : A^2(D) → A^2(D)$ is unitarily equivalent to a weighted shift on a generalized directed semi-tree, where $\{\theta_i : i = 1, \ldots, n\}$ is a set of basic polynomials associated with the group $G$.

## 2. Generalized directed semi-trees

### 2.1. Basic notions of graph theory

We begin by recalling a number of useful definitions from graph theory. For a nonempty set $V$ and a subset $E \subseteq (V \times V) \setminus \{(v, v) : v \in V\}$ of ordered pairs, we denote the directed graph $G$ by the pair $G = (V, E)$. An element of $V$ is called a vertex and an element of $E$ is called an edge. We enlist some requisite definitions related to a directed graph below, following the notations described in Ref. [3, p. 1429].

1. A directed graph $G = (V, E)$ is said to be connected if for every two distinct vertices $u$ and $v$, there exists a finite sequence $v_1, v_2, \ldots, v_n \in V$, for $n ≥ 2$, such that $u = v_1$, either $(v_j, v_{j+1}) \in E$ or $(v_{j+1}, v_j) \in E$ for all $j = 1, 2, \ldots, n - 1$ and $v_n = v$.

2. A finite sequence $v_1, v_2, \ldots, v_n$ ($n ≥ 2$) of distinct vertices of $G$ is said to be a circuit if $(v_j, v_{j+1}) \in E$ for all $j = 1, 2, \ldots, n - 1$ and $(v_n, v_1) \in E$.

3. For any $u \in V$, the children of $u$ and the parents of $u$ are given by
   (a) $\text{Chi}(u) := \{v \in V : (v, u) \in E\}$ and
   (b) $\text{Par}(u) := \{v \in V : (v, u) \in E\}$, respectively.

4. A vertex $v$ is called a root of the graph $G$ if $\text{Par}(v)$ is empty. The set of all roots of $G$ is denoted by $\text{Root}(G)$. We also define $V^0 = V \setminus \text{Root}(G)$.

5. For a vertex $w \in V$, we fix $\text{Chi}^{(0)}(w) = \{w\}$ and $\text{Par}^{(0)}(w) = \{w\}$. For $n \in \mathbb{N}$, the $n$th children of $w$ and $n$th parents of $w$ are denoted by
   (a) $\text{Chi}^{(n)}(w) = \text{Chi}(\text{Chi}^{(n-1)}(w))$, and
   (b) $\text{Par}^{(n)}(w) = \text{Par}(\text{Par}^{(n-1)}(w))$, respectively.
   (c) In addition to it, the set of descendants of $w$ is defined as

   $$\text{Des}(w) = \bigcup_{n=0}^{\infty} \text{Chi}^{(n)}(w).$$

Now we are in a position to recall the definition of a directed tree from [2, p. 10].

**Definition 2.1:** A directed graph $G = (V, E)$ is called a directed tree if the following conditions are satisfied.

(i) $G$ has no circuit.
(ii) $G$ is connected.
(iii) For each vertex $v \in V^0$, the set $\text{Par}(v)$ has only one element.
Recently, Majdak and Stochel generalized the notion of a directed tree and defined a directed semi-tree in Ref. [3, p. 1430] as follows:

**Definition 2.2:** A directed graph \( G = (V, E) \) is called a directed semi-tree if the following conditions are satisfied.

(i) \( G \) has no circuit.
(ii) \( G \) is connected.
(iii) \( \text{card}(\text{Chi}(u) \cap \text{Chi}(v)) \leq 1 \) for distinct all \( u, v \in V \), where \( \text{card}(S) \) represents the cardinality of a set \( S \).
(iv) For all \( u, v \in V \), there exists \( w \in V \) such that \( u, v \in \text{Des}(w) \).

The motivation to generalize the notion of a directed semi-tree comes from a number of examples that occur naturally. A few such examples are discussed in Sections 4 and 5, and they emphasize that this generalization is not superficial.

**Definition 2.3:** A directed graph \( G = (V, E) \) is called a generalized directed semi-tree if the following conditions are satisfied.

(i) \( G \) has no circuit.
(ii) \( G \) has finite or countably infinite connected components.
(iii) There exists a fixed non-negative integer, say \( m \), such that for every distinct \( u, v \in V \);

\[
\text{card}(\text{Chi}(u) \cap \text{Chi}(v)) \leq m. \tag{1}
\]

The directed graphs (1) and (2) are examples of generalized directed semi-tree. Since each of the graphs (1) and (2) has two roots, they are not directed semi-trees.

Note that the cardinality of \( \text{Root}(G) \) can be countably infinite. If any non-negative integer \( m \) satisfies Equation (1), then for every natural number \( M > m \), we have \( \text{card}(\text{Chi}(u) \cap \text{Chi}(v)) \leq M \). So we take the convention of choosing the least non-negative integer that satisfies Equation (1). We denote a generalized directed semi-tree by \( (G, m) \) in the sequel, where \( m \) is the least non-negative integer satisfying Equation (1). The next proposition manifests a relation between the class of generalized directed semi-trees and the class of directed trees.
Proposition 2.1: A generalized directed semi-tree \((G, 0)\) is a directed tree if and only if it is connected.

Proof: The forward direction of the proposition follows immediately. Now we prove the backward direction. Assume that \(G\) is connected. Since \(G\) is a generalized directed semi-tree, it has no circuit. Arguing by contradiction, suppose there exists \(v_0 \in V^c\) such that \(\text{card(Par}(v_0)) > 1\). Let \(u_1, u_2 \in \text{Par}(v_0), u_1 \neq u_2\). Then \(\text{Chi}(u_1) \cap \text{Chi}(u_2) \supseteq \{v_0\}\). This implies that \(\text{card(Chi}(u_1) \cap \text{Chi}(u_2)) \geq 1\), which contradicts our hypothesis. Hence for all \(v \in V^c\), \(\text{Par}(v)\) has only one element. This shows that \(G\) is a directed tree. \(\blacksquare\)

Remark 2.1: Note that a generalized directed semi-tree \((G, 1)\) is a directed semi-tree if

1. \(\text{card(}\text{Root}(G)) \leq 1\),
2. For all \(u, v \in V\); there exists \(w \in V\) such that \(u, v \in \text{Des}(w)\).

3. Analytic structure for a weighted shift on a generalized directed semi-tree

3.1. Weighted shift on a generalized directed semi-tree

Let \((G, m) = (V, E)\) be a generalized directed semi-tree. We assign a complex number \(\lambda_{(u,v)}\) to each edge \((u, v) \in E\) such that the following holds for every \(v \in V\):

\[
\sum_{u \in \text{Par}(v)} |\lambda_{(u,v)}|^2 < \infty. \tag{2}
\]

We refer to \(\lambda_{(u,v)}\) as the weight on the edge \((u, v)\). The Hilbert space \(\ell^2(V)\) is the family of all square-summable complex-valued functions on \(V\) with the standard inner product \((f, g) := \sum_{v \in V} f(v)g(v)\) for \(f, g \in \ell^2(V)\). Let \(T_G\) be the operator defined on the set of all complex-valued functions on \(V\) by

\[
(T_G f)(v) = \begin{cases} 
\sum_{u \in \text{Par}(v)} \lambda_{(u,v)} f(u), & \text{if } v \in V^c, \\
0, & \text{if } v \in \text{Root}(G). 
\end{cases} \tag{3}
\]

We denote \(\mathcal{D}(\Lambda_G) := \{f \in \ell^2(V) : T_G f \in \ell^2(V)\}\). The operator \(\Lambda_G : \mathcal{D}(\Lambda_G) \rightarrow \ell^2(V)\), defined by

\[
\Lambda_G f = T_G f, \quad f \in \mathcal{D}(\Lambda_G), \tag{4}
\]

is called a weighted shift operator on the generalized directed semi-tree \(G\) with weights \(\{\lambda_{(u,v)} : (u, v) \in E\}\).

For an element \(v \in V\), if the set \(\text{Par}(v)\) is finite, \((\Lambda_G f)(v)\) is well-defined for every \(f \in \mathcal{D}(\Lambda_G)\). On the other hand, if \(\text{Par}(v) = \{u_i : i \in \mathbb{N}\}\) for some \(v \in V\), then by the Cauchy–Schwarz inequality we have

\[
|\sum_{i=1}^m \lambda_{(u_i,v)} f(u_i) - \sum_{i=1}^k \lambda_{(u_i,v)} f(u_i)| \leq \left(\sum_{i=k+1}^m |\lambda_{(u_i,v)}|^2\right)^{1/2} \left(\sum_{i=k+1}^m |f(u_i)|^2\right)^{1/2},
\]

for every \(m, k \in \mathbb{N}\) with \(k < m\). From Equation (2), we have \(\sum_{i=k+1}^m |\lambda_{(u_i,v)}|^2 \rightarrow 0\) as \(m, k \rightarrow \infty\). Moreover, since \(f \in \ell^2(V)\), it follows that \(\sum_{i=k+1}^m |f(u_i)|^2 \rightarrow 0\) as \(m, k \rightarrow \infty\). Therefore, \(\sum_{u \in \text{Par}(v)} \lambda_{(u,v)} f(u)\) is convergent and hence \((\Lambda_G f)(v)\) is well defined.
Remark 3.1: In the case that \( G \) is a directed tree or a directed semi-tree, the definition of \( \Lambda_G \) coincides with the definition of a weighted shift on a directed tree and a directed semi-tree, respectively, see [2, Definition 3.1.1] and [3, Equation (5.2), p. 1437].

For each \( u \in V \), let \( \chi_u : V \rightarrow \mathbb{C} \) be defined by

\[
\chi_u(v) = \begin{cases} 
1, & \text{if } v = u, \\
0, & \text{otherwise.}
\end{cases}
\]  

(5)

From Equations (3) and (4), \( (\Lambda_G \chi_u)(v) = \sum_{w \in \text{Par}(v)} \lambda_{(w,v)} \chi_u(w) \). Therefore,

\[
(\Lambda_G \chi_u)(v) = \begin{cases} 
\lambda_{(u,v)}, & \text{if } u \in \text{Par}(v), \\
0, & \text{otherwise},
\end{cases}
\]

which can be rewritten as

\[
\Lambda_G \chi_u = \sum_{v \in \text{Chi}(u)} \lambda_{(u,v)} \chi_v.
\]  

(6)

If \( \text{Chi}(u) \) is finite for every \( u \in V \), then \{\( \chi_u \)| \( u \in V \} \subseteq \mathcal{D}(\Lambda_G) \). However, this does not ensure the boundedness of the operator \( \Lambda_G \). The following proposition provides a necessary and sufficient condition for the boundedness of \( \Lambda_G \).

Proposition 3.1: Let \( G = (V, E) \) be a generalized directed semi-tree with the property that \( \text{card}(\text{Par}(v)) \leq k \) for every \( v \in V \), and let \( \Lambda_G \) be the weighted shift operator on \( G \) with weights \( \{\lambda_{(u,v)} : (u,v) \in E\} \). The operator \( \Lambda_G \) is bounded on \( \ell^2(V) \) if and only if

\[
\sup_{v \in V} \sum_{u \in \text{Chi}(v)} |\lambda_{(v,u)}|^2 < \infty.
\]

Proof: Suppose that \( \Lambda_G \) is bounded on \( \ell^2(V) \), that is, the operator norm of \( \Lambda_G \) is finite and suppose that \( \|\Lambda_G\| = c \). For \( v \in V \), it follows from Equation (6) that the norm of \( \Lambda_G \chi_v \) in \( \ell^2(V) \) is given by \( \|\Lambda_G \chi_v\|^2 = \sum_{u \in \text{Chi}(v)} |\lambda_{(v,u)}|^2 \). Since \( \|\Lambda_G \chi_v\|^2 \leq c^2 \), the result follows.

Conversely, suppose that \( \sup_{v \in V} \sum_{u \in \text{Chi}(v)} |\lambda_{(v,u)}|^2 < c' \). Let \( f \in \ell^2(V) \) be such that \( f = \sum_{u \in W} f_u \chi_u \), where \( W \) is a finite subset of \( V \). Then we have

\[
\Lambda_G f = \sum_{u \in W} f_u (\Lambda_G \chi_u) = \sum_{u \in W} f_u \sum_{v \in \text{Chi}(u)} \lambda_{(u,v)} \chi_v = \sum_{v \in W} \left( \sum_{u \in \text{Par}(v)} \lambda_{(u,v)} f_u \right) \chi_v.
\]

Therefore,

\[
\|\Lambda_G f\|^2 = \sum_{v \in W} \left| \sum_{u \in \text{Par}(v)} f_u \lambda_{(u,v)} \right|^2 \leq k \sum_{v \in W} \sum_{u \in \text{Par}(v)} |f_u \lambda_{(u,v)}|^2
\]

\[
= k \sum_{u \in W} \left( \sum_{v \in \text{Chi}(u)} |\lambda_{(u,v)}|^2 \right) |f_u|^2 \leq kc' \|f\|^2.
\]

Note that \( \text{card}(\text{Par}(v)) \leq k \) for every \( v \in V \), so the first inequality follows by applying the Cauchy–Schwarz inequality on the constant function 1 and \( f_u \lambda_{(u,v)} \). Thus the operator \( \Lambda_G \) is bounded on a dense subset of \( \ell^2(V) \). Therefore, \( \Lambda_G \) is bounded on \( \ell^2(V) \).
3.2. Analytic structure

Given \( n \) generalized directed semi-trees, our aim is to determine an \( n \)-tuple of multiplication operators on a Hilbert space of formal power series, which is a reproducing kernel Hilbert space consisting of holomorphic functions under certain conditions. We restrict our attention to the generalized directed semi-trees with a few specific properties.

Suppose \( \mathbb{N} \) is the set of all natural numbers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Let \( V \subseteq \mathbb{N}_0^n \) and \((G, l) = (V, E)\) be a generalized directed semi-tree with the following properties:

1. If \((v, u) \in E\) then \( v_j \leq u_j \) for \( j \in \{1, 2, \ldots, n\} \).
2. There exists a natural number \( m \) such that \( \text{card}(\chi(u)) \leq m \) for every \( u \in V \) and \( V_m = \{v \in V : \text{card}(\chi(v)) = m\} \) is non-empty. Clearly, \( l \leq m \).

(a) For any two \( v, v' \in V_m \), if \( u \in \chi(v) \) there exists unique \( u' \in \chi(v') \) such that \( u_j = v_j \) for \( j = 1, \ldots, n \). Denote \( k_j = u_j - v_j \).

(b) Consider \( u \in V \). If \( \text{card}(\chi(u)) \leq m \), then for each \( w \in \chi(u) \), there exists a unique \( k \in \deg(G) \) such that \( w_j - u_j = k_j \) for all \( j \in \{1, 2, \ldots, n\} \).

Note that property (a) describes a condition on the set \( V_m = \{v \in V : \text{card}(\chi(v)) = m\} \). However, property (b) describes a condition for every \( u \in V \) with \( \text{card}(\chi(u)) \leq m \).

Let \( \mathcal{X}(V) \) be the set of all generalized directed semi-trees on \( V \) with Property (1) and Property (2). For \( V = \{(n, 0) : n \in \mathbb{N}_0\} \), the following graph yields an example of an element in \( \mathcal{X}(V) \).

For every \((G, l) \in \mathcal{X}(V)\), a unique polynomial

\[
\rho_G(z_1, \ldots, z_n) := \sum_{i=1}^{m} z_1^{k^{(i)}_1} \cdots z_n^{k^{(i)}_n},
\]

(7)

is associated to the graph \((G, l)\).

3.2.1. Construction

For \( 1 \leq i \leq n \), let \((G^{(i)}_n, m_i) = (V, E^{(i)}_n)\) be generalized directed semi-trees in \( \mathcal{X}(V) \). Let \( \chi^{(i)}_n(v) = \{u : (v, u) \in E^{(i)}_n\} \) and \( \text{Par}^{(i)}_n(u) = \{v : (v, u) \in E^{(i)}_n\} \).
For $1 \leq i \leq n$, denote the unique polynomial in Equation (7) associated to the graph $(G_n^{(i)}, m_i)$ by $p_i$. Let $\{S_v\}_{v \in V}$ be a collection of polynomials in $n$ variables $p_1, p_2, \ldots, p_n$ which satisfy the following relations:

(i) $S_v = \sum_{|a| \leq |v|, a \in \mathbb{N}_0^n} c_{a} p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$ for some $c_{a} \in \mathbb{C}$ and,

(ii) $p_i \cdot S_v = \sum_{u \in \text{Chi}_n^{(i)}(v)} S_u$,

where $a = (a_1, \ldots, a_n)$ is a tuple of non-negative integers and $|a| = \sum_{i=1}^{n} a_i$. Consider the space of formal power series $\mathcal{H}^2(\beta) = \{\sum_{v \in V} \hat{f}(v) S_v : \sum_{v \in V} |\hat{f}(v)|^2 \beta(v)^2 < \infty\}$ with the inner product

$$\langle f, g \rangle = \sum_{v \in V} \hat{f}(v) \overline{g(v)} \beta(v)^2$$

for $f, g \in \mathcal{H}^2(\beta)$,

where $f = \sum_{v \in V} \hat{f}(v) S_v$ and $g = \sum_{v \in V} \hat{g}(v) S_v$ are formal power series and $\{\beta(v) : v \in V\}$ is a net of positive real numbers. Note that we have

$$\langle S_u, S_v \rangle = \begin{cases} 0 & \text{if } u \not= v, \\ \beta(v)^2 & \text{if } u = v. \end{cases}$$

(8)

Hence, the set $\{\frac{1}{\beta(v)} S_v\}_{v \in V}$ forms an orthonormal basis of $\mathcal{H}^2(\beta)$. Consider the $n$-tuple of multiplication operators $T = (T_{p_1}, T_{p_2}, \ldots, T_{p_n})$ on the Hilbert space $\mathcal{H}^2(\beta)$ where $T_{p_i} : \mathcal{H}^2(\beta) \longrightarrow \mathcal{H}^2(\beta)$ is defined by

$$T_{p_i} f = p_i f, \quad f \in \mathcal{H}^2(\beta).$$

(9)

A priori it is not guaranteed that the operator $T_{p_i}$ is bounded. The following lemma describes a characterization of the boundedness of $T_{p_i}$.

**Lemma 3.1:** For each $1 \leq i \leq n$, let $(G_n^{(i)}, m_i)$ be a generalized semi-tree with the property $\text{card}(\text{Par}_n^{(i)}(v)) \leq k$ for $v \in V$, and let the unique polynomial associated with the graph $(G_n^{(i)}, m_i)$ be $p_i$. The operator $T_{p_i}$ is bounded on $\mathcal{H}^2(\beta)$ if and only if

$$\sup_{v \in V} \sum_{u \in \text{Chi}_n^{(i)}(v)} \frac{\beta(u)^2}{\beta(v)^2} < \infty$$

for $i = 1, 2, \ldots, n$.

**Proof:** If each $T_{p_i}$ is bounded for $i = 1, \ldots, n$, then there exists $c > 0$ such that $\|T_{p_i} S_v\|^2 \leq c \|S_v\|^2$ for all $v \in V$. For $v \in V$,

$$\|T_{p_i} S_v\|^2 \leq c \|S_v\|^2 \Rightarrow \sum_{u \in \text{Chi}_n^{(i)}(v)} \|S_u\|^2 \leq c \beta(v)^2$$

$$\Rightarrow \sum_{u \in \text{Chi}_n^{(i)}(v)} \|S_u\|^2 \leq c \beta(v)^2$$

$$\Rightarrow \sum_{u \in \text{Chi}_n^{(i)}(v)} \frac{\beta(u)^2}{\beta(v)^2} \leq c.$$
Conversely, assume that \( \sup_{v \in V} \sum_{u \in \text{Chi}^{(i)}(v)} \frac{\beta(u)^2}{\beta(v)^2} = c < \infty \). Let \( f \in \mathcal{H}^2(\beta) \) be such that \( f = \sum_{u \in W} f_u S_u \), where \( W \) is a finite subset of \( V \). It follows from Property (i) (a few lines before Equations (8)) and (9) that

\[
T_p f = \sum_{v \in W} \left( \sum_{u \in \text{Par}^{(i)}(v)} \hat{f}(u) \right) S_v, \quad \text{(10)}
\]

for \( f \in \mathcal{H}^2(\beta) \). Hence by Cauchy–Schwarz inequality,

\[
\|T_p f\|^2 = \sum_{v \in W} |\sum_{u \in \text{Par}^{(i)}(v)} \hat{f}(u)|^2 \beta(v)^2 \leq \sum_{v \in W} \left( \sum_{u \in \text{Par}^{(i)}(v)} |\hat{f}(u)|^2 \right) \beta(v)^2 \leq k \sum_{v \in W} \left( \sum_{u \in \text{Chi}^{(i)}(v)} \beta(u)^2 \right) |\hat{f}(v)|^2 \beta(v)^2 \leq k c \|f\|^2.
\]

Since \( \text{card(Par}^{(i)}(v) \rangle \leq k \) for every \( v \in V \), the second inequality follows. Thus the operator \( T_p \) is bounded on a dense subset of \( \mathcal{H}^2(\beta) \) and therefore \( T_p \) is bounded. ■

A straightforward calculation yields the following lemma.

**Lemma 3.2:** Suppose that \( (T_{p_1}, \ldots, T_{p_n}) \) is an \( n \)-tuple of bounded operators on the Hilbert space \( \mathcal{H}^2(\beta) \). Then

\[
T^n_{p_i} S_v = \begin{cases} 
0, & \text{if } v \in \text{Root}(G_n^{(i)}), \\
\sum_{u \in \text{Par}^{(i)}(v)} \frac{\beta(u)^2}{\beta(v)^2} S_u, & \text{otherwise}.
\end{cases}
\]

**Definition 3.1:** For \( w \in \mathbb{C}^n \), let \( \lambda_w \) be the linear functional on the linear span \( \bigvee \{ S_v : v \in V \} \) of \( S_v \), defined by \( \lambda_w(f) = f(w) \), \( f \in \bigvee \{ S_v : v \in V \} \). The point \( w \) is said to be a bounded point evaluation on \( \mathcal{H}^2(\beta) \) if the functional \( \lambda_w \) on \( \bigvee \{ S_v : v \in V \} \) extends to a bounded linear functional on \( \mathcal{H}^2(\beta) \).

**Lemma 3.3:** \( w \in \mathbb{C}^n \) is a bounded point evaluation if and only if \( \sum_{v \in V} \frac{|S_v(w)|^2}{\beta(v)^2} < \infty \).
\textbf{Proof:} From Definition 3.1, it follows that if \( w \) is a bounded point evaluation, then \( \lambda_w(p) = p(w) \) for all polynomials \( p \) in the variables \( p_i \). By the Riesz representation theorem, if \( w \) is a bounded point evaluation, then there exists \( \gamma_w \in \mathcal{H}^2(\beta) \) such that \( \lambda_w(f) = \langle f, \gamma_w \rangle \), \( f \in \mathcal{H}^2(\beta) \). Therefore we have \( S_v(w) = \langle S_v, \gamma_w \rangle = \frac{\hat{\gamma}_w(v)}{\beta(v)} \beta(v)^2 \), that is, \( \gamma_w(v) = \frac{S_v(w)}{\beta(v)} \). Since \( \gamma_w \in \mathcal{H}^2(\beta) \), it follows that \( \|\gamma_w\|^2 = \sum_{v \in V} |\hat{\gamma}_w(v)|^2 \beta(v)^2 = \sum_{v \in V} \frac{|S_v(w)|^2}{\beta(v)^2} < \infty \).

Conversely, suppose that for \( w \in \mathbb{C}^n \), we have \( \sum_{v \in V} \frac{|S_v(w)|^2}{\beta(v)^2} = c \). Then, the power series \( \gamma_w = \sum_{v \in V} \frac{S_v(w)}{\beta(v)} \beta(v)^2 \) is in \( \mathcal{H}^2(\beta) \). Let us define \( \lambda_w : \mathcal{H}^2(\beta) \to \mathbb{C} \) such that \( \lambda_w(f) = \langle f, \gamma_w \rangle \). A direct computation gives us \( \lambda_w(S_v) = S_v(w) \) and thus \( \lambda_w(g) = g(w) \) for every \( g \in \sqrt{\{S_v : v \in V\}} \). Therefore, \( w \) is a bounded point evaluation. \( \blacksquare \)

\textbf{Proposition 3.2:} If \( w \) is a bounded point evaluation and \( f = \sum_{v \in V} \hat{f}(v)S_v \in \mathcal{H}^2(\beta) \), then the series \( \sum_{v \in V} \hat{f}(v)S_v(w) \) converges absolutely to \( \lambda_w(f) \).

\textbf{Proof:} Since \( w \) is a bounded point evaluation, Lemma 3.3 yields that \( \sum_{v \in V} \frac{|S_v(w)|^2}{\beta(v)^2} < \infty \). Let \( \sum_{v \in V} \frac{|S_v(w)|^2}{\beta(v)^2} = c \). Suppose \( U \) is a finite subset of \( V \). We have

\[
\left( \sum_{v \in U} |\hat{f}(v)S_v(w)| \right)^2 = \left( \sum_{v \in U} |\hat{f}(v)\beta(v)S_v(w)\beta(v)| \right)^2 \leq \sum_{v \in U} |\hat{f}(v)|^2 \beta(v)^2 \sum_{v \in U} \frac{|S_v(w)|^2}{\beta(v)^2} \leq \sum_{v \in V} |\hat{f}(v)|^2 \beta(v)^2 \sum_{v \in V} \frac{|S_v(w)|^2}{\beta(v)^2} = \|f\|^2 \sum_{v \in V} \frac{|S_v(w)|^2}{\beta(v)^2} = \|f\|^2 c.
\]

This proves that the series \( \sum_{v \in V} \hat{f}(v)S_v(w) \) converges absolutely. Since we have \( \lambda_w(\sum_{v \in U} \hat{f}(v)S_v) = \sum_{v \in U} \hat{f}(v)S_v(w) \) for every finite subset \( U \) of \( V \) and \( \lambda_w \) is continuous, it follows that the series \( \sum_{v \in V} \hat{f}(v)S_v(w) \) converges absolutely to \( \lambda_w(f) \). \( \blacksquare \)

\textbf{Remark 3.2:} In view of Proposition 3.2, there is no ambiguity in writing \( \lambda_w(f) = f(w) \), whenever \( w \) is a bounded point evaluation.

The set of all bounded point evaluations on \( \mathcal{H}^2(\beta) \) is denoted by \( \Omega_{bpe} \). Then the following corollary follows from Proposition 3.2.
Corollary 3.1: The Hilbert space \( \mathcal{H}^2(\beta) \) is a reproducing kernel Hilbert space with reproducing kernel

\[
\kappa(z, w) = \sum_{v \in V} \frac{S_v(z)S_v(w)}{\beta(v)^2}, \quad z, w \in \Omega_{bpe},
\]

consisting of holomorphic functions on \( \Omega_{bpe} \) if the interior of \( \Omega_{bpe} \) is non-empty.

There are natural examples where the interior of \( \Omega_{bpe} \) is non-empty. We have provided two such classes of examples in Section 4.

For \( w \in \mathbb{C}^n \), let \( p_w = (p_1(w), p_2(w), \ldots, p_n(w)) \). Then the following result holds.

Proposition 3.3: Suppose that \( (T_{p_1}, \ldots, T_{p_n}) \) is an \( n \)-tuple of bounded operators on \( \mathcal{H}^2(\beta) \). Then \( P_w \) is in the point spectrum \( \sigma(T_{p_1}^*, \ldots, T_{p_n}^*) \) with common eigenvector \( K_w \) if \( w \) is a bounded point evaluation on \( \mathcal{H}^2(\beta) \).

Proof: Suppose \( w \) is a bounded point evaluation on \( \mathcal{H}^2(\beta) \). By the Riesz representation theorem, there exists \( K_w \in \mathcal{H}^2(\beta) \) such that \( \lambda_w(f) = \langle f, K_w \rangle \) for \( f \in \mathcal{H}^2(\beta) \). Then we have,

\[
\langle f, T_{p_i}^*K_w \rangle = \langle T_{p_i}f, K_w \rangle = \langle pf, K_w \rangle = p_i(w)\langle f, K_w \rangle = \langle f, p_i(w)K_w \rangle
\]

Therefore \( T_{p_i}^*K_w = p_i(w)K_w \) and hence, \( P_w \in \sigma(T_{p_1}^*, T_{p_2}^*, \ldots, T_{p_n}^*) \) corresponding to the common eigenvector \( K_w \). \( \blacksquare \)

3.2.2. Relation between the analytic structure and discrete structure

Assume that \( \{(G^{(i)}_n, m_i) = (V, E^{(i)}_n) : i = 1, \ldots, n\} \subset \mathcal{P}(V) \) and the unique polynomial associated to the graph \( (G^{(i)}_n, m_i) \) is \( p_i \). Suppose that there exists \( k > 0 \) such that \( \text{card}(\text{Par}^{(i)}_n(v)) \leq k \) for all \( v \in V \) and \( i = 1, \ldots, n \). For each \( 1 \leq i \leq n \), the weighted shift \( \Lambda_i \) on \( (G^{(i)}_n, m_i) \) is defined as follows: for \( f \in D(\Lambda_i) \subseteq \ell^2(V) \)

\[
(\Lambda_if)(v) = \begin{cases} 
0, & \text{if } v = \text{Root}(G^{(i)}_n), \\
\sum_{u \in \text{Par}^{(i)}_n(v)} \frac{\beta(u)}{\beta(v)}f(u), & \text{otherwise},
\end{cases}
\]

(11)

where \( \beta(v) \) is as in Equation (8).

Theorem 3.1: Let \( \Lambda_i \) be the weighted shift on the generalized directed semi-tree \( (G_i, m_i) \), as described in Equation (11), for \( i = 1, \ldots, n \) such that each \( \Lambda_i \) extends to a bounded linear operator to \( \ell^2(V) \) and the \( n \)-tuple \( (\Lambda_1, \ldots, \Lambda_n) \) is commuting. Then there exists a unitary operator \( U : \ell^2(V) \rightarrow \mathcal{H}^2(\beta) \) such that \( U^*T_{p_i}U = \Lambda_i \) for each \( 1 \leq i \leq n \).
Proof: For \( u \in V \), let \( \chi_u \in \ell^2(V) \) be given by Equation (5). Note that the sets \( \{ \chi_u : u \in V \} \) and \( \{ \frac{S_u}{\beta(u)} : u \in V \} \) are orthonormal bases of \( \ell^2(V) \) and \( \mathcal{H}^2(\beta) \), respectively. Let \( U : \ell^2(V) \to \mathcal{H}^2(\beta) \) be defined by \( U \chi_u = \frac{S_u}{\beta(u)} \). It follows from Equations (6) and (11) that

\[
\Lambda_i \chi_v = \sum_{u \in \text{Chi}^{(i)}(v)} \frac{\beta(u)}{\beta(v)} \chi_u, \quad v \in V,
\]

that is,

\[
(\Lambda_i \chi_v)(u) = \begin{cases} 
\frac{\beta(u)}{\beta(v)}, & \text{if } u \in \text{Chi}^{(i)}(v), \\
0, & \text{otherwise}.
\end{cases}
\]

From Property (ii) of the collection of polynomials \( \{ S_v(z) \}_{v \in V} \), we have

\[
T_{pi} \frac{S_v}{\beta(v)} = \sum_{u \in \text{Chi}^{(i)}(v)} \frac{\beta(u)}{\beta(v)} \frac{S_u}{\beta(u)}, \quad v \in V.
\]

Therefore, from Equations (12) and (13), it follows that \( U^* T_{pi} U = \Lambda_i \).

4. Example I

Fix \( n > 1 \). Recall that the weighted Bergman space \( A^{(\lambda)}(\mathbb{D}^n), \lambda > 0 \), consisting of holomorphic functions on \( \mathbb{D}^n \), is determined by the reproducing kernel \( K^{(\lambda)} : \mathbb{D}^n \times \mathbb{D}^n \to \mathbb{C} \), which is given by the formula

\[
K^{(\lambda)}(z, w) = \prod_{j=1}^n (1 - z_j \bar{w}_j)^{-\lambda}, \quad z, w \in \mathbb{D}^n.
\]

Let \( s_i \) denote the elementary symmetric polynomial of degree \( i \) in \( n \) variables, that is,

\[
s_i(z_1, \ldots, z_n) = \sum_{1 \leq a_1 < \cdots < a_i \leq n} z_{a_1} \cdots z_{a_i}.
\]

The map \( s = (s_1, \ldots, s_n) : \mathbb{C}^n \to \mathbb{C}^n \) is called the symmetrization map. For \( 1 \leq i \leq n \), \( M_{s_i} \) denotes the multiplication operator by the elementary symmetric polynomial \( s_i \) on \( A^{(\lambda)}(\mathbb{D}^n) \) and \( M_s := (M_{s_1}, \ldots, M_{s_n}) \). The permutation group on \( n \) symbols is denoted by \( \mathfrak{S}_n \). The subspaces

\[
A^{(\lambda)}_{\text{sym}}(\mathbb{D}^n) = \{ f \in A^{(\lambda)}(\mathbb{D}^n) : f \circ \sigma^{-1} = f \text{ for } \sigma \in \mathfrak{S}_n \},
\]

and

\[
A^{(\lambda)}_{\text{anti}}(\mathbb{D}^n) = \{ f \in A^{(\lambda)}(\mathbb{D}^n) : f \circ \sigma^{-1} = \text{sgn}(\sigma)f \text{ for } \sigma \in \mathfrak{S}_n \}
\]

are two joint reducing subspaces of the \( n \)-tuple of multiplication operators \( M_s \) on \( A^{(\lambda)}(\mathbb{D}^n) \) [4, p. 774]. In the following discussion, the restriction operators \( M_{s_i}|_{A^{(\lambda)}_{\text{sym}}(\mathbb{D}^n)} \) and \( M_{s_i}|_{A^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)} \) have been identified with two \( n \)-tuples of weighted shift operators on generalized directed semi-trees.
The collection of all elements \( u = (u_1, u_2, \ldots, u_n) \in \mathbb{N}_0^n \) with \( u_1 \geq u_2 \geq \cdots \geq u_n \geq 0 \) is denoted by \( \mathcal{Q}_n \). For \( u \in \mathcal{Q}_n \), let \( (\lambda)_{u} = \prod_{j=1}^{n} (\lambda)_{u_j} \), where \( (\lambda)_{u_j} = \lambda(\lambda + 1) \cdots (\lambda + u_j - 1) \) is the Pochhammer symbol.

For \( u \in \mathcal{Q}_n \), suppose that \( a_u(z) = \det((z_{ij}^{u_j}))_{i,j=1}^{n} \). It is given by

\[
P_n^{(\text{anti})} = \{ v + \delta : v \in \mathcal{Q}_n \text{ and } \delta = (n - 1, n - 2, \ldots, 1, 0) \}. \]

For \( u \in \mathcal{P}_n^{(\text{anti})} \), the norm of \( a_u \) in \( \mathbb{A}^{(\lambda)}(\mathbb{D}^n) \) is calculated in Ref. [4, p. 775]. It is given by

\[
\| a_u \| = \sqrt{\prod_{j=1}^{n} (\lambda)_{u_j}} = \frac{1}{\gamma_u} \quad \text{(say),} \tag{14}
\]

where \( u! = \prod_{j=1}^{n} u_j! \). Moreover, the following holds [11, p. 2365]:

**Lemma 4.1:** The set \( \{ \gamma_ua_u : u \in \mathcal{P}_n^{(\text{anti})} \} \) forms an orthonormal basis of \( \mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n) \).

The action of \( \mathfrak{S}_n \) on \( \mathbb{N}_0^n \) is given by \( (\sigma, u) \mapsto \sigma \cdot u = (u_{\sigma^{-1}(1)}, \ldots, u_{\sigma^{-1}(n)}) \). Let \( \mathfrak{S}_n u \) denote the orbit of \( u \in \mathbb{N}_0^n \). We say \( u_1 \sim u_2 \) if \( u_1 = \sigma \cdot u_2 \) for some \( \sigma \in \mathfrak{S}_n \). For an element \( u \in \mathcal{Q}_n \), let \( [u] = \{ v \in \mathcal{Q}_n : v \sim u \} \) be the equivalence class and \( \mathcal{P}_n^{(\text{sym})} = \{ [u] : u \in \mathcal{Q}_n \} \).

For \( u \in \mathcal{Q}_n \), consider the monomial symmetric polynomials (as in Ref. [12, p. 454])

\[
m_u(z) = \sum_{\beta \in \mathfrak{S}_n u} z^\beta,
\]

where the sum is over all the distinct permutations \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \) of \( u \) and \( z^\beta = z_1^{\beta_1} z_2^{\beta_2} \cdots z_n^{\beta_n} \). Observe that \( m_u(z) = \sum_{\beta} z^\beta = m_{u'}(z) \) for \( u' \in [u] \) and the elements of the set \( \{ m_u : u \in \mathcal{P}_n^{(\text{sym})} \} \) are mutually orthogonal in \( \mathbb{A}^{(\lambda)}(\mathbb{D}^n) \).

If \( u \in \mathcal{Q}_n \) has \( k(\leq n) \) distinct components, that is, there are \( k \) distinct non-negative integers \( u_1 > \cdots > u_k \) such that

\[
u = (u_1, u_1, u_2, \ldots, u_2, \ldots, u_k, \ldots, u_k),
\]

where each \( u_i \) is repeated \( \alpha_i \) times, for \( i = 1, \ldots, k \), then \( \alpha = (\alpha_1, \ldots, \alpha_k) \) is said to be the multiplicity of \( u \). For a fixed \( u \in \mathcal{Q}_n \) with multiplicity \( \alpha \), we have \( \text{card}(\mathfrak{S}_n u) = \frac{n!}{\alpha!} \). For an element \( u \in \mathcal{P}_n^{(\text{sym})} \) with multiplicity \( \alpha \), the norm of \( m_u \) in \( \mathbb{A}^{(\lambda)}(\mathbb{D}^n) \) can be calculated by the orthogonality of the distinct monomials in \( \mathbb{A}^{(\lambda)}(\mathbb{D}^n) \), and it is given by

\[
\| m_u \| = \sqrt{\frac{u! u!}{(\lambda)_u \alpha!}} = \frac{1}{k_u} \quad \text{(say).} \tag{15}
\]

Moreover, from the discussion in Ref. [4, Section 4], we get the following lemma.

**Lemma 4.2:** The set \( \{ k_u m_u : u \in \mathcal{P}_n^{(\text{sym})} \} \) forms an orthonormal basis of \( \mathbb{A}^{(\lambda)}_{\text{sym}}(\mathbb{D}^n) \).
For each partition $u \in Q_n$, the associated Schur polynomial in $n$ variables is defined by

$$S_u(z_1, z_2, \ldots, z_n) = \frac{\det((z_j^{u_i+n-i})_{i,j=1}^n)}{\det((z_j^{n-i})_{i,j=1}^n)} = a_{u+\delta}(z) / a_\delta(z),$$

(16)

[12, p. 454, A.4]. For $1 \leq k \leq n$, the partition with the first $k$ components as 1 and the rest $(n-k)$ components as 0, is denoted by $(1^k)$. Observe from the second of Giambelli’s formulas, stated in Ref. [12, p. 455], that $S_{(1^k)} = s_k$. By the Pieri rule in Ref. [13, p. 25], we have

$$s_k s_u = \sum_{v \in \mathcal{I}_u^{(k)}} S_v,$$

(17)

where $\mathcal{I}_u^{(k)} = \{ v = (v_1, \ldots, v_n) \in Q_n : \sum_{j=1}^n (v_j - u_j) = k \text{ and } 0 \leq v_j - u_j \leq 1 \text{ for all } j \}$. Equivalently, $s_k a_{u+\delta} = \sum_{v \in \mathcal{I}_u^{(k)}} a_{\nu+\delta}$. Then it follows that for $u \in \mathcal{P}_n^{(\text{anti})}$,

$$s_k a_u = \sum_{v \in \mathcal{I}_u^{(k)} \cap \mathcal{P}_n^{(\text{anti})}} a_v.$$

(18)

A straightforward calculation shows that the monomial symmetric polynomials satisfy a similar summation formula as the Schur polynomials. That is, for $u \in \mathcal{P}_n^{(\text{sym})}$,

$$s_k m_u = \sum_{v \in (\mathcal{I}_u^{(k)} / \sim) \cap \mathcal{P}_n^{(\text{sym})}} c_v m_v,$$

(19)

where $c_v$ are positive rational numbers.

**Remark 4.1:** Let $C_{n,k}$ be the collection of all $(t_1, \ldots, t_n) \in \mathbb{N}_0^n$ such that exactly $k$ many $t_i$’s are equal to 1 and the rest of the $t_i$’s are 0. Thus the cardinality of $C_{n,k}$ is $\binom{n}{k}$. We note that if $v \in \mathcal{I}_u^{(k)}$, then $v = u + t$ for some $t \in C_{n,k}$. Therefore, for any $u \in Q_n$, $\text{card}(\mathcal{I}_u^{(k)}) \leq \binom{n}{k}$.

For each $1 \leq k \leq n$, we define the directed graph $G^{(k)}_{\text{anti}} = (\mathcal{P}_n^{(\text{anti})}, E^{(k)}_{\text{anti}})$, where the set of all edges is given by $E^{(k)}_{\text{anti}} = \bigcup_{u \in \mathcal{P}_n^{(\text{anti})}} \{ (u, v) : v \in \mathcal{I}_u^{(k)} \cap \mathcal{P}_n^{(\text{anti})} \}$.

**Proposition 4.1:** For each $1 \leq k \leq n$, there exists a non-negative integer $m_k \leq \binom{n}{k}$ such that the directed graph $(G^{(k)}_{\text{anti}}, m_k)$ is a generalized directed semi-tree.

**Proof:** Fix $1 \leq k \leq n$. Since $\mathcal{P}_n^{(\text{anti})}$ is countable, the graph $G^{(k)}_{\text{anti}}$ can have at most countably many connected components. Since $\text{Root}(G^{(k)}_{\text{anti}}) \subseteq \mathcal{P}_n^{(\text{anti})}$, the graph can have at most countable roots.

Moreover, for every $u \in \mathcal{P}_n^{(\text{anti})}$ we have $\text{Chi}^{(k)}_n(u) = \mathcal{I}_u^{(k)} \cap \mathcal{P}_n^{(\text{anti})}$, where the children of $u$ in $G^{(k)}_{\text{anti}}$ are denoted by $\text{Chi}^{(k)}_n(u)$. By Remark 4.1, $\text{card}(\mathcal{I}_u^{(k)}) \leq \binom{n}{k}$ and thus $\text{card}(\mathcal{I}_u^{(k)} \cap \mathcal{P}_n^{\text{anti}}) \leq \text{card}(\mathcal{I}_u^{(k)}) \leq \binom{n}{k}$. Therefore, for any two distinct $u, v \in \mathcal{P}_n^{(\text{anti})}$, $\text{card}(\text{Chi}^{(k)}_n(u) \cap \text{Chi}^{(k)}_n(v)) \leq \binom{n}{k}$. Consequently, we have $m_k = \sup_{u, v \in \mathcal{P}_n^{\text{anti}}, u \neq v} \text{card}(\text{Chi}^{(k)}_n(u) \cap \text{Chi}^{(k)}_n(v))$. 

To prove that \( G^{(k)}_{\text{anti}} \) has no circuit, we argue by contradiction. Suppose not, then there exists a sequence \( \{v_1, \ldots, v_n\}, n > 1 \) such that \((u, v_1), (v_n, u)\) and \((v_j, v_{j+1})\) \( E^{(k)}_{\text{anti}} \) for all \( j = 1, \ldots, n - 1 \). Since \((u, v_1) \in E^{(k)}_{\text{anti}}\), by construction there exists at least one \( i = 1, \ldots, n \) such that \( u_i < v_1 \), where \( u = (u_1, \ldots, u_n) \) and \( v_1 = (v_{11}, \ldots, v_{1n}) \). This implies \( u_i < v_1 \leq v_2 \leq v_n \leq u_i \), which is a contradiction.

In a similar manner, we define the directed graph \( G^{(k)}_{\text{sym}} = (P^{(sym)}_n, E^{(k)}_{\text{sym}}) \), where

\[
E^{(k)}_{\text{sym}} = \bigcup_{u \in P^{(sym)}_n} \{(u, v) : v \in (T^{(k)}_u / \sim) \cap P^{(sym)}_n\},
\]

for each \( 1 \leq k \leq n \). The children of \( u \) in \( E^{(k)}_{\text{sym}} \) are denoted by \( \text{Chi}^{(k)}(u) \). Suppose that \( r_k = \sup_{u,v \in P^{(sym)}_n, u \neq v} \text{card} \left( \text{Chi}^{(k)}(u) \cap \text{Chi}^{(k)}(v) \right) \). Then similar arguments as above lead us to the following proposition.

**Proposition 4.2:** The directed graph \( (G^{(k)}_n, r_k) \) is a generalized directed semi-tree, where \( r_k \leq \binom{n}{k} \) for each \( 1 \leq k \leq n \).

For \( 1 \leq k \leq n \), let \( \Lambda^{(\text{anti})}_k : \ell^2(P^{(\text{anti})}_n) \to \ell^2(P^{(\text{anti})}_n) \) be such that

\[
\Lambda^{(\text{anti})}_k \chi_u = \sum_{v \in T^{(k)}_u \cap P^{(\text{anti})}_n} \frac{\gamma_v}{\gamma_u} \chi_v,
\]

where \( \gamma_u \) is given by Equation (14). Therefore, \( \Lambda^{(\text{anti})} := (\Lambda^{(\text{anti})}_1, \ldots, \Lambda^{(\text{anti})}_n) \) is an \( n \)-tuple of weighted shifts, where each \( \Lambda^{(\text{anti})}_k \) is a weighted shift on the generalized directed semi-tree \( G^{(k)}_n \).

Similarly, we define \( \Lambda^{(\text{sym})}_k : \ell^2(P^{(\text{sym})}_n) \to \ell^2(P^{(\text{sym})}_n) \) by

\[
\Lambda^{(\text{sym})}_k \chi_u = \sum_{v \in (T^{(k)}_u / \sim) \cap P^{(\text{sym})}_n} \frac{k_v}{k_u} \chi_v,
\]

where \( k_u \) is given by Equation (15). The \( n \)-tuple of operators is denoted by \( \Lambda^{(\text{sym})} := (\Lambda^{(\text{sym})}_1, \ldots, \Lambda^{(\text{sym})}_n) \), where each \( \Lambda^{(\text{sym})}_k \) is a weighted shift on the generalized directed semi-tree \( G^{(k)}_n \).

The operators \( U^{(\text{sym})} : \ell^2(P^{(\text{sym})}_n) \to A^{(\lambda)}_{\text{sym}}(\mathbb{D}^n) \) and \( U^{(\text{anti})} : \ell^2(P^{(\text{anti})}_n) \to A^{(\lambda)}_{\text{anti}}(\mathbb{D}^n) \), defined by

\[
U^{(\text{sym})} \chi_u = m_u, \text{ for } u \in P^{(\text{sym})}_n \text{ and,}
\]

\[
U^{(\text{anti})} \chi_u = a_u, \text{ for } u \in P^{(\text{anti})}_n,
\]

are unitary. From the definition, it follows that for every \( k = 1, \ldots, n \), the unitary \( U^{(\text{anti})} \) intertwines the operators \( \Lambda^{(\text{anti})}_k \) on \( \ell^2(P^{(\text{anti})}_n) \) and \( M_{sk} \) on \( A^{(\lambda)}_{\text{anti}}(\mathbb{D}^n) \) and \( U^{(\text{sym})} \) intertwines the operators \( \Lambda^{(\text{sym})}_k \) on \( \ell^2(P^{(\text{sym})}_n) \) and \( M_{sk} \) on \( A^{(\lambda)}_{\text{sym}}(\mathbb{D}^n) \). Therefore, we have the following results:
**Theorem 4.1:** Suppose \( \lambda > 0 \). We have the following:

1. The \( n \)-tuple of operators \( (M_1, \ldots, M_n) \) on \( \mathcal{A}_{\text{sym}}^{(\lambda)}(\mathbb{D}^n) \) is unitarily equivalent to the \( n \)-tuple of operators \( (\Lambda_{1}^{(\text{sym})}, \ldots, \Lambda_{n}^{(\text{sym})}) \) on \( \ell^2(P_{n}^{(\text{sym})}) \), where \( \Lambda_{k}^{(\text{sym})} \) is a weighted shift on the generalized directed semi-tree \( (G_{\text{sym}}, r_{k}) \), defined in Equation (21).

2. The \( n \)-tuple \( (M_1, \ldots, M_n) \) on \( \mathcal{A}_{\text{anti}}^{(\lambda)}(\mathbb{D}^n) \) is unitarily equivalent to the \( n \)-tuple of operators \( (\Lambda_{1}^{(\text{anti})}, \ldots, \Lambda_{n}^{(\text{anti})}) \) on \( \ell^2(P_{n}^{(\text{anti})}) \), where \( \Lambda_{k}^{(\text{anti})} \) is a weighted shift on the generalized directed semi-tree \( (G_{\text{anti}}, m_{k}) \), defined in Equation (20).

In addition to this, note that the point spectra of \( M_{s}^{n} |_{\mathcal{A}_{\text{anti}}^{(\lambda)}(\mathbb{D}^n)} \) and \( M_{s}^{n} |_{\mathcal{A}_{\text{sym}}^{(\lambda)}(\mathbb{D}^n)} \) contain the symmetrized polydisc \( \mathbb{G}_{n}^{(s)}(= s(\mathbb{D}^n)) \). This provides two classes of examples that satisfy the hypothesis in Corollary 3.1.

### 4.1. Pictorial representation for \( G_{\text{sym}}^{(k)} \)

The generalized directed semi-trees \( G_{\text{sym}}^{(1)} \), \( G_{\text{sym}}^{(2)} \) and \( G_{\text{sym}}^{(3)} \) (cf. Proposition 4.2) represent the multiplication operators \( M_{s1}, M_{s2} \) and \( M_{s3} \) on \( \mathcal{A}_{\text{sym}}^{(\lambda)}(\mathbb{D}^3) \), respectively.
5. Example II

A pseudoreflection on \( \mathbb{C}^n \) is a linear homomorphism \( \sigma : \mathbb{C}^n \to \mathbb{C}^n \) such that \( \sigma \) has finite order in \( GL(n, \mathbb{C}) \) and the rank of \( id - \sigma \) is 1. A group generated by pseudoreflections is called a pseudoreflection group.

Let \( G \) be a finite pseudoreflection group. The group action of \( G \) on \( \mathbb{C}^n \) is defined by \( (\sigma, z) \mapsto \sigma \cdot z = \sigma^{-1} \cdot z, \quad z \in \mathbb{C}^n, \quad \sigma \in G \). Let \( G \) act on the set of functions on \( \mathbb{C}^n \) by \( \sigma(f)(z) = f(\sigma^{-1} \cdot z) \). A function is said to be \( G \)-invariant if \( \sigma(f) = f \), for all \( \sigma \in G \). The ring of all \( G \)-invariant polynomials in \( n \) variables is denoted by \( \mathbb{C}[z_1, \ldots, z_n]_G \). Moreover, the set of all \( G \)-invariant polynomials, denoted by \( \mathbb{C}[z_1, \ldots, z_n]^G \), forms a ring. Chevalley, Shephard and Todd characterized finite pseudoreflection groups in the following theorem.

\[ \text{Theorem 5.1 (see [14, Theorem 3, p.112]): The ring } \mathbb{C}[z_1, \ldots, z_n]^G \text{ consisting of all } G \text{-invariant polynomials is equal to } \mathbb{C}[\theta_1, \ldots, \theta_n], \text{ where } \theta_i \text{ are algebraically independent homogeneous polynomials if and only if } G \text{ is a finite pseudoreflection group.} \]

The map \( \theta : \mathbb{C}^n \to \mathbb{C}^n \), defined by

\[ \theta(z) = (\theta_1(z), \ldots, \theta_n(z)) \quad z \in \mathbb{C}^n \]

(22)
is called a basic polynomial map associated to the group \( G \). The set of the degrees of the homogeneous polynomials \( \theta_i \), \( \{ \deg(\theta_i) = \eta_i : i = 1, \ldots, n \} \), is unique for the group \( G \). Let \( \mathbb{N}_0^n \) denote the set of all non-negative integers and \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^n \) be a multi-index. For \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), denote \( z^\alpha := \prod_{j=1}^{n} z_j^{\alpha_j} \). We consider \( \theta_i(z) = \sum_{u \in \mathbb{N}_0^n} a_u^{(i)} z^u \), where \( N_i \subseteq \{ u = (u_1, \ldots, u_n) \in \mathbb{N}_0^n : \sum_{j=1}^{n} u_j = \eta_i \} \) and the \( a_u^{(i)} \)'s are positive real numbers. Let \( \text{card}(N_i) = c_i \).

Recall that a domain \( U \subset \mathbb{C}^n \) (containing the origin \( 0 \)) is said to be a complete Reinhardt domain with centre \( 0 \) if for any \( z = (z_1, \ldots, z_n) \in U \), the domain \( U \) contains the closure of the polydisc \( D(0; r) \), where \( |z_j| = r_j \) for \( j = 1, \ldots, n \) and \( r = (r_1, \ldots, r_n) \) [15, p. 21]. Let \( D \) be a bounded complete Reinhardt domain in \( \mathbb{C}^n \). The Bergman space on \( D \), denoted by \( \mathbb{A}^2(D) \), is the subspace of holomorphic functions in \( L^2(D) \) with respect to the Lebesgue measure on \( D \). The Bergman space \( \mathbb{A}^2(D) \) is a Hilbert space with a reproducing kernel called the Bergman kernel. The set \( \{ \frac{z^\alpha}{\|z^\alpha\|} : \alpha \in \mathbb{N}_0^n \} \) forms an orthonormal basis of \( \mathbb{A}^2(D) \). The multiplication operator \( M_{\alpha} : \mathbb{A}^2(D) \to \mathbb{A}^2(D) \) is bounded for each \( i = 1, \ldots, n \). Note that

\[ M_{\alpha} z^\alpha = \sum_{u \in \mathbb{N}_0^n} a_u^{(i)} z^{u+\alpha} \]

\[ M_{\alpha} \frac{z^\alpha}{\|z^\alpha\|} = \sum_{u \in \mathbb{N}_0^n} a_u^{(i)} \frac{z^{u+\alpha}}{\|z^\alpha\|} \| z^{u+\alpha} \| \frac{z^{u+\alpha}}{\|z^u\|}. \]

(23)

For each \( i = 1, \ldots, n \), we fix the notation \( E_n^{(i)} = (\mathbb{N}_0^n, E_n^{(i)}) \) is a generalized directed semi-tree, for some \( c_i' \leq c_i \).

\[ \text{Theorem 5.2: For every } i = 1, \ldots, n, \text{ the graph } (G_i, c_i') = (\mathbb{N}_0^n, E_n^{(i)}) \text{ is a generalized directed semi-tree, for some } c_i' \leq c_i. \]

\[ \text{Proof: We fix } i \in \{1, \ldots, n\} \text{ and set } \text{Chi}_i(u) = \{ v : (u, v) \in E_n^{(i)} \}. \text{ Then for every } u, v \in \mathbb{N}_0^n, \text{ card}(\text{Chi}_i(u) \cap \text{Chi}_i(v)) \leq \text{card}(\text{Chi}_i(u)) \leq c_i. \text{ We suppose that sup}_{u, v \in \mathbb{N}_0^n, u \neq v} \text{card}(\text{Chi}_i(u)) \]
Moreover, it is clear that the supremum is attained for some \( u, v \) and \( \text{card}(\text{Chi}_i(u) \cap \text{Chi}_i(v)) \leq c_i^1 \leq c_i \) for every \( u, v \in \mathbb{N}_0^n \). Similar arguments as in the proof of the Proposition 4.1 prove that these graphs can have at most countable components and they have no circuit. Hence, the result follows.

Let \( \lambda_{(\alpha,\alpha+u)}^{(i)} := a_u \frac{\|z^{\alpha+u}\|}{\|z^\alpha\|} \). For each \( i = 1, \ldots, n \), we define a weighted shift \( \Lambda_i : \ell^2(\mathbb{N}_0^n) \to \ell^2(\mathbb{N}_0^n) \) on \((G_i, c_i')\) by

\[
\Lambda_i \chi_{\alpha} = \sum_{u \in \mathbb{N}_i} \lambda_{(\alpha,\alpha+u)}^{(i)} \chi_{\alpha+u}, \quad \alpha \in \mathbb{N}_0^n,
\]

where \( \chi_{\alpha} \) is the characteristic function at \( \alpha \). The set \( \{\chi_{\alpha}\}_{\alpha \in \mathbb{N}_0^n} \) forms an orthonormal basis of the Hilbert space \( \ell^2(\mathbb{N}_0^n) \).

**Proposition 5.1:** For each \( i = 1, \ldots, n \), the weighted shift \( \Lambda_i \) on the generalized directed semi-tree \((G_i, c_i')\) is unitarily equivalent to \( M_{\theta_i} \) on \( A^2(D) \).

**Proof:** The unitary operator \( U : \ell^2(\mathbb{N}_0^n) \to A^2(D) \), defined by

\[
U(\chi_{\alpha}) = \frac{z^\alpha}{\|z^\alpha\|}, \quad \alpha \in \mathbb{N}_0^n,
\]

intertwines the operator \( \Lambda_i \) on \( \ell^2(\mathbb{N}_0^n) \) and \( M_{\theta_i} \) on \( A^2(D) \), for all \( i = 1, \ldots, n \). □

### 5.1. On the Bergman space of the bidisc

We consider the bounded Reinhardt domain \( \mathbb{D}^2 \), the bidisc. The domain \( \mathbb{D}^2 \) is closed under the action of the symmetric group \( \mathcal{S}_2 \) on two symbols. The symmetrization map \( s : (s_1, s_2) : \mathbb{D}^2 \to s(\mathbb{D}^2) \) is a basic polynomial map associated with the group \( \mathcal{S}_2 \), where

\[
s_1(z_1, z_2) = z_1 + z_2 \quad \text{and} \quad s_2(z_1, z_2) = z_1 z_2, \quad \text{for } (z_1, z_2) \in \mathbb{D}^2.
\]

From the above discussion, we get that the multiplication operators \( M_{s_i} : A^2(\mathbb{D}^2) \to A^2(\mathbb{D}^2) \), \( i = 1, 2 \), yield two generalized directed semi-trees \((G_1, c_1')\) and \((G_2, c_2')\) which are generated with respect to the orthonormal basis \( \{a_{(n_1,n_2)} = \frac{z_1^{n_1} z_2^{n_2}}{(n_1+1)(n_2+1)} : (n_1, n_2) \in \mathbb{N}_0^2\} \) for \( A^2(\mathbb{D}^2) \), that is,

\[
M_{z_1+z_2}a_{(n_1,n_2)} = \frac{\sqrt{(n_1+2)}}{\sqrt{(n_1+1)}} a_{(n_1+1,n_2)} + \frac{\sqrt{(n_2+2)}}{\sqrt{(n_2+1)}} a_{(n_1,n_2+1)},
\]

\[
M_{z_1 z_2}a_{(n_1,n_2)} = \frac{\sqrt{(n_1+2)(n_2+2)}}{\sqrt{(n_1+1)(n_2+1)}} a_{(n_1+1,n_2+1)}.
\]

Moreover, it is clear from the above expression that \( c_1' \) can be at most 1 and \( c_2' = 0 \). Note that both \((n_1, n_2 + 1)\) and \((n_1 + 1, n_2)\) have \((n_1 + 1, n_2 + 1)\) as one of their children, so \( \text{card}(\text{Chi}((n_1, n_2 + 1)) \cap \text{Chi}((n_1 + 1, n_2))) = 1 \) for \( n_1, n_2 \in \mathbb{N}_0 \). Thus \( c_1' = 1 \). We provide pictorial representations for \((G_1, 1)\) and \((G_2, 0)\) below.
Acknowledgments

The authors would like to express their sincere gratitude to Subrata Shyam Roy for several comments and suggestions in the preparation of this paper. The authors are grateful to the anonymous referee for many useful comments and suggestions.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

The research of the first-named author was supported by IoE-IISc Fellowship. The research of the second-named author was supported by a post-doctoral fellowship of the NBHM.

References

[1] Fujii M, Sasaoka H, Watatani Y. Adjacency operators of infinite directed graphs. Math Jpn. 1989;34:727–735.
[2] Jablonski ZJ, Jung IB, Stochel J. Weighted shifts on directed trees. Mem Am Math Soc. 2012;216:viii+106.
[3] Majdak W, Stochel JB. Weighted shifts on directed semi-trees: an application to creation operators on Segal–Bargmann spaces. Complex Anal Oper Theory. 2016;10:1427–1452.
[4] Biswas S, Ghosh G, Misra G, et al. On reducing submodules of Hilbert modules with $\mathfrak{S}_n$-invariant kernels. J Funct Anal. 2019;276:751–784.
[5] Biswas S, Datta S, Ghosh G, et al. A Chevalley–Shephard–Todd theorem for analytic Hilbert module. https://arxiv.org/abs/1811.06205. 2018.
[6] Shields AL. Weighted shift operators and analytic function theory. In: Topics in operator theory. 1974. p. 49–128. Math. Surveys, No. 13.
[7] Jewell NP, Lubin AR. Commuting weighted shifts and analytic function theory in several variables. J Oper Theory. 1979;1:207–223.
[8] Chavan S, Trivedi S. An analytic model for left-invertible weighted shifts on directed trees. J Lond Math Soc (2). 2016;94:253–279.
[9] Shimorin S. Wold-type decompositions and wandering subspaces for operators close to isometries. J Reine Angew Math. 2001;531:147–189.
[10] Budzyński P, Dymek P, Ptak M. Analytic structure of weighted shifts on directed trees. Math Nachr. 2017;290:1612–1629.
[11] Misra G, Shyam Roy S, Zhang G. Reproducing kernel for a class of weighted Bergman spaces on the symmetrized polydisc. Proc Am Math Soc. 2013;141:2361–2370.
[12] Fulton W, Harris J. Representation theory. New York: Springer-Verlag; 1981. (vol. 129 of Graduate Texts in Mathematics). A first course, Readings in Mathematics
[13] Fulton W. Young tableaux. Cambridge: Cambridge University Press; 1997. (vol. 35 of London Mathematical Society Student Texts). With applications to representation theory and geometry
[14] Bourbaki N. Lie groups and Lie algebras, Chapters 4–6. Berlin: Springer-Verlag; 2002 (Elements of mathematics (Berlin)). Translated from the 1968 French original by Andrew Pressley.
[15] Lebl J. Tasty bits of several complex variables. https://www.jirka.org/scv/. 2020.