CONTINUITY AND SCHATTEN PROPERTIES FOR PSEUDO-DIFFERENTIAL OPERATORS WITH SYMBOLS IN QUASI-BANACH MODULATION SPACES

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Abstract. We establish continuity and Schatten-von Neumann properties for matrix operators with matrices satisfying mixed quasi-norm estimates. These considerations also include the case when the Lebesgue and Schatten parameters are allowed to stay between 0 and 1. We use the results to deduce continuity and Schatten-von Neumann properties for pseudo-differential operators with symbols in a broad class of modulation spaces.

0. Introduction

The aim of the paper is to deduce continuity properties for pseudo-differential operators with symbols in modulation spaces, when acting on (other) modulation spaces. In particular we extend the results in [21,23,41,42,45] to broader classes of modulation spaces, in the sense of allowing the Lebesgue exponents to stay in the larger interval $[0, \infty]$ instead of $[1, \infty]$. In particular, in contrast to [21,23,41,42,45], our situations involve quasi-Banach spaces which are not Banach spaces. For this reason, our analysis is more involved compared to [21,23,41,42,45], since the absence of local-convexity cause several problems.

We also remark that our investigations also include Schatten-von Neumann properties (including cases where the Schatten parameters are allowed to be smaller than 1). For example, we prove that any pseudo-differential operator with symbol in the modulation space $M^{p,p}(\mathbb{R}^d)$, $0 < p \leq 2$, belongs to $\mathcal{S}_p$, the set of Schatten-von Neumann operators of order $p$ on $L^2(\mathbb{R}^d)$. (See [21,27] and Section 11 for definitions.) Furthermore we prove that this is sharp in the sense that any modulation space (with trivial weight) which is not covered by $M^{p,p}(\mathbb{R}^d)$, contains symbols, whose corresponding pseudo-differential operators fail to belong to $\mathcal{S}_p$.

The analysis behind the continuity and compactness results here is based on Gabor analysis for a broad family of modulation spaces, in combination of certain continuity results and factorization techniques for matrices and matrix operators. The Gabor analysis is deduced in [16], which are extensions of certain results in [16,21] in the sense of relaxed assumptions on involving weights and Lebesgue parameters, as
Theorem 0.1. Let $f \in \ell^p(J)$ such that $J \times J \ni (j,k) \mapsto a(j,j-k)\omega_0(j,j-k)$ belongs to $\ell^{p,q}(J \times J)$.

Evidently, if $A = (a(j,k))_{j,k \in J}$ is a matrix and $f$ belongs to $\ell_0(J)$, then $Af$ is uniquely defined in $\ell(J)$, i.e.

$$A : \ell_0(J) \mapsto \ell(J).$$

Here $\ell(J)$ is the set of all sequences on $J$, and $\ell_0(J)$ is the set of all $f \in \ell(J)$ such that $f(j)$ is non-zero for at most finite numbers of $j$.

For such classes of matrices we deduce Theorem 0.1 in Section 2 on mapping properties between appropriate mixed quasi-normed spaces. An important special case of the latter theorem is the following. Here the involved weight functions should satisfy

$$\frac{\omega_2(j)}{\omega_2(k)} \leq \omega_0(j,k).$$  \hspace{1cm} (0.2)

**Theorem 0.1.** Let $J = T\mathbb{Z}^d$ for some $T \in \text{GL}(d, \mathbb{R})$, $\omega_l$ be weights on $J$, $l = 1, 2$, and $\omega_0$ be a weight on $J \times J$ such that (0.2) holds. Also let $I = (I_1, \ldots, I_n)$ be a linear split of $J$, $q \in (0, 1]$ and $p \in (0, \infty]^n$ be such that

$$q \leq \nu(p),$$  \hspace{1cm} (0.3)

and let $A \in \mathbb{U}^{\infty,q}(\omega_0, J)$. Then $A$ in (0.1) is uniquely extendable to a continuous map from $\ell^{p_1}(I)$ to $\ell^{p_2}(I)$, and

$$\|A\|_{\ell^{p_1}(I) \rightarrow \ell^{p_2}(I)} \leq \|A\|_{\mathbb{U}^{\infty,q}(\omega_0, J)}. \hspace{1cm} (0.4)$$

By a combinations of the previous theorem with the Gabor expansion results in [16] or [46], the following special case of Theorem 0.2 is obtained.

**Theorem 0.2.** Let $\sigma \in S_{2d}$, $\omega_k \in \mathcal{P}_E(\mathbb{R}^{2d})$, $k = 1, 2$, and $\omega_0 \in \mathcal{P}_E(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$ be such that

$$\frac{\omega_2(x, \xi)}{\omega_1(y, \eta)} \lesssim \omega_0((1-t)x + ty, t\xi + (1-t)\eta, \xi - \eta, y - x)$$

Also let $t \in \mathbb{R}$, $q \in (0, 1]$ and $p \in (0, \infty]^n$ be such that (0.3) hold, and let $a \in \mathcal{M}^{\infty,q}(\omega_0)$ be such that $\mathcal{M}^{\infty,q}(\omega_0)$ extends uniquely to a continuous map from $\mathcal{M}^{p_1}(\omega_1)$ to $\mathcal{M}^{p_2}(\omega_2)$, and

$$\|a\|_{\mathcal{M}^{p_1}(\omega_1)} \lesssim \|a\|_{\mathcal{M}^{p_2}(\omega_2)}.$$  \hspace{1cm} (0.5)
By the previous result it follows that if $q_0 \in (0, 1]$, $p, q \in [q_0, \infty]$, $t \in \mathbb{R}$ and $a \in M^{\infty, q_0}(\mathbb{R}^{2d})$, then $O_{p}(a)$ is continuous on $M^{p, q}(\mathbb{R}^{d})$. In particular, Theorem 14.5.2 in [21] is obtained as a special case by choosing $q_0 = 1$.

The analysis behind obtaining Schatten-von Neumann properties of small orders is based on Theorem 2.1. The latter theorem implies that if

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_0},$$

then any matrix $A \in \mathbb{U}^{p_0}$ can be factorized as $A = A_1 \cdot A_2$, $A_j \in \mathbb{U}^{p_j}$.

Here we set $\mathbb{U}^{p} = \mathbb{U}^{p, p}(\mathbb{J}) = \mathbb{U}^{p, p}(\omega, \mathbb{J})$ when $\omega = 1$. Furthermore, the matrices $A_1$ and $A_2$ can be chosen such that

$$\|A_1\|_{\mathbb{U}^{p_1}} \|A_2\|_{\mathbb{U}^{p_2}} \leq \|A\|_{\mathbb{U}^{p_0}}.$$

The latter factorization property can be applied to deduce that

$$\mathbb{U}^{p} \subseteq \mathcal{I}_p,$$

when $p \in (0, 2]$, (0.6)

where $\mathcal{I}_p = \mathcal{I}_p(\ell^2(J))$ is the set of all Schatten-von Neumann operators on $\ell^2(J)$ of order $p \in (0, \infty]$.

In fact, the set of Hilbert-Schmidt operators on $\ell^2(J)$ can easily be identified with convenient square estimates on the involved matrix elements, giving that they agree with $\mathbb{U}^{2}$, with equality in norms. Furthermore, they agree with $\mathcal{I}_2$ (also in norms). Consequently, $\mathbb{U}^{2} = \mathcal{I}_2$, and Hölder’s inequality for Schatten-von Neumann in combination with the factorization property here above shows that for every $A \in \mathbb{U}^{2/N}$, where $N \geq 1$ is an integer, there are matrices $A_1, \ldots, A_N \in \mathbb{U}^{2}$ such that

$$A = A_1 \cdots A_N \in \mathbb{U}^{2} \cdots \mathbb{U}^{2} = \mathcal{I}_2 \cdots \mathcal{I}_2 = \mathcal{I}_{2/N}.$$

Hence $\mathbb{U}^{2/N} \subseteq \mathcal{I}_{2/N}$ for every integer $N \geq 1$. An interpolation argument between the cases

$$\mathbb{U}^{2/N} \subseteq \mathcal{I}_{2/N} \text{ and } \mathbb{U}^{2} = \mathcal{I}_2$$

now shows that that $\mathbb{U}^{p} \subseteq \mathcal{I}_p$ when $p \in [2/N, 2]$. Since $2/N$ can be chosen to stay arbitrarily close to 0, it follows that the latter inclusion relation holds for every $p \in (0, 2]$, and (0.6).

In Section 2 the recent arguments are used to deduce generalized version of (0.6), where $\mathbb{U}^{p}$ and $\mathcal{I}_p$ are replaced by

$$\mathbb{U}^{p, p}(\omega_0, J) \text{ and } \mathcal{I}_p(\ell^2(\omega_1), \ell^2(\omega_2)),$$

for some appropriate weights $\omega_j$. (Cf. Theorem 2.4.) In Section 3 we combine Theorem 2.4 with the Gabor results in [46] to prove Theorem 3.2. As a special case of the latter result one has

$$M^{p, p}(\mathbb{R}^{2d}) \subseteq st,p(\mathbb{R}^{2d}), \quad p \in (0, 2],$$

(0.7)
where \( s_{t,p}(\mathbb{R}^{2d}) \) is the set of all \( a \in \mathcal{S}'(\mathbb{R}^{2d}) \) such that the pseudo-differential operator \( \text{Op}_t(a) \) belongs to \( \mathcal{I}_p(L^2(\mathbb{R}^d)) \).

We note that (0.7) was proved in the case \( 1 \leq p \leq 2 \) by Gröchenig and Heil in [23], and that the case \( p = 1 \) was deduced already by Sjöstrand in [8, 39].

1. Preliminaries

In this section we explain some results available in the literature, which are needed later on, or clarify the subject. The proofs are in general omitted. Especially we recall some facts about weight functions, Gelfand-Shilov spaces, and modulation spaces.

1.1. Weight functions. We start by discussing general properties on the involved weight functions. A weight on \( \mathbb{R}^d \) is a positive function \( \omega \in L^\infty_{\text{loc}}(\mathbb{R}^d) \), and for each compact set \( K \subseteq \mathbb{R}^d \), there is a constant \( c > 0 \) such that

\[
\omega(x) \geq c \quad \text{when} \quad x \in K.
\]

A usual condition on \( \omega \) is that it should be moderate, or \( \nu \)-moderate for some positive function \( \nu \in L^\infty_{\text{loc}}(\mathbb{R}^d) \). This means that

\[
\omega(x + y) \leq C \omega(x) \nu(y), \quad x, y \in \mathbb{R}^d, \tag{1.1}
\]

for some constant \( C \) which is independent of \( x, y \in \mathbb{R}^d \). We note that (1.1) implies that \( \omega \) fulfills the estimates

\[
C^{-1} \nu(-x)^{-1} \leq \omega(x) \leq C \nu(x), \quad x \in \mathbb{R}^d. \tag{1.2}
\]

We let \( \mathcal{P}_E(\mathbb{R}^d) \) be the set of all moderate weights on \( \mathbb{R}^d \). Furthermore, if \( \nu \) in (1.1) can be chosen as a polynomial, then \( \omega \) is called a weight of polynomial type. We let \( \mathcal{P}(\mathbb{R}^d) \) be the set of all weights of polynomial type, and recall that an important class of weights in \( \mathcal{P}(\mathbb{R}^d) \) and \( \mathcal{P}(\mathbb{R}^{2d}) \) are

\[
\xi \mapsto \langle \xi \rangle^s, \quad (x, \xi) \mapsto \langle (x, \xi) \rangle^s
\]

and

\[
(x, \xi) \mapsto \langle x \rangle^t \langle \xi \rangle^s,
\]

when \( s, t \in \mathbb{R} \), and

\[
\langle \xi \rangle \equiv (1 + |\xi|^2)^{1/2} \quad \text{and} \quad \langle (x, \xi) \rangle \equiv (1 + |x|^2 + |\xi|^2)^{1/2}.
\]

It can be proved that if \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \), then \( \omega \) is \( \nu \)-moderate for some \( \nu(x) = e^{r|x|} \), provided the positive constant \( r \) is large enough. In particular, (1.2) shows that for any \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \), there are constants \( C, r > 0 \) such that

\[
C^{-1} e^{-r|x|} \leq \omega(x) \leq C e^{r|x|}, \quad x \in \mathbb{R}^d,
\]

or equivalently,

\[
v_{r,d}^{-1} \leq \omega \leq v_{r,d}.
\]
Here and in what follows we let
\[ v_{r,d}(x) \equiv e^{r|x|}, \quad x \in \mathbb{R}^d. \] (1.3)
Furthermore, \( A \lesssim B \) means that \( A \leq cB \) for a suitable constant \( c > 0 \).

We say that \( v \) is submultiplicative if \( v \) is even and (1.1) holds with \( \omega = v \). We note that \( v_{r,d} \) is submultiplicative and belongs to \( \mathcal{P}_E(\mathbb{R}^d) \), and that for any submultiplicative weight \( v \) on \( \mathbb{R}^d \), then
\[ 1 \lesssim v(x) \lesssim v_{r,d}(x), \quad x \in \mathbb{R}^d, \]
for some \( r > 0 \). In the sequel, \( v \) and \( v_j \) for \( j \geq 0 \), always stand for submultiplicative weights if nothing else is stated.

1.2. Gelfand-Shilov spaces. Next we recall the definition of Gelfand-Shilov spaces.

Let \( 0 < h, s, t \in \mathbb{R} \) be fixed. Then we let \( S_{t,h}^s(\mathbb{R}^d) \) be the set of all \( f \in C^\infty(\mathbb{R}^d) \) such that
\[ \| f \|_{S_{t,h}^s} \equiv \sup_{d} \frac{|x^\alpha \partial^\beta f(x)|}{h^{|\alpha|} + |\beta|} \]
is finite. Here the supremum should be taken over all \( \alpha, \beta \in \mathbb{N}^d \) and \( x \in \mathbb{R}^d \). For convenience we set \( S_{s,t} = S_{s,t,h}^s \).

Obviously \( S_{t,h}^s \subseteq \mathcal{S} \) is a Banach space which increases with \( h, s \) and \( t \). Furthermore, if \( s, t > 1/2 \), or \( s, t = 1/2 \) and \( h \) is sufficiently large, then \( S_{t,h}^s \) contains all finite linear combinations of Hermite functions. Since such linear combinations are dense in \( \mathcal{S} \), it follows that the dual \( (S_{t,h}^s)'(\mathbb{R}^d) \) of \( S_{t,h}^s(\mathbb{R}^d) \) is a Banach space which contains \( \mathcal{S}'(\mathbb{R}^d) \).

The Gelfand-Shilov spaces \( S_{t}^s(\mathbb{R}^d) \) and \( \Sigma_{t}^s(\mathbb{R}^d) \) are the inductive and projective limits respectively of \( S_{t,h}^s(\mathbb{R}^d) \) with respect to \( h \). This implies that
\[ S_{t}^s(\mathbb{R}^d) = \bigcup_{h>0} S_{t,h}^s(\mathbb{R}^d) \quad \text{and} \quad \Sigma_{t}^s(\mathbb{R}^d) = \bigcap_{h>0} S_{t,h}^s(\mathbb{R}^d), \] (1.4)
and that the topology for \( S_{t}^s(\mathbb{R}^d) \) is the strongest possible one such that each inclusion map from \( S_{t,h}^s(\mathbb{R}^d) \) to \( S_{t}^s(\mathbb{R}^d) \) is continuous. The space \( \Sigma_{t}^s(\mathbb{R}^d) \) is a Fréchet space with semi norms \( \| \cdot \|_{S_{t,h}^s} \), \( h > 0 \). Moreover, \( S_{t}^s(\mathbb{R}^d) \neq \{0\} \), and only if \( s, t > 0 \) satisfy \( s + t \geq 1 \), and \( \Sigma_{t}^s(\mathbb{R}^d) \neq \{0\} \), if and only if \( (s, t) \) belongs to
\[ J_{\text{GS}} \equiv \{ (s, t) \in \mathbb{R}^2 ; s, t > 0, \ s + t \geq 1 \} \setminus \{(1/2, 1/2)\}. \] (1.5)

For convenience we set \( S_s = S_s^s \) and \( \Sigma_s = \Sigma_s^s \), and remark that \( S_s(\mathbb{R}^d) \) is zero when \( s < 1/2 \), and that \( \Sigma_s(\mathbb{R}^d) \) is zero when \( s \leq 1/2 \). For each \( \varepsilon > 0 \) and \( s, t > 0 \) such that \( s + t \geq 1 \), we have
\[ \Sigma_s^t(\mathbb{R}^d) \subseteq S_s^t(\mathbb{R}^d) \subseteq S_{s+t}^{s+t}(\mathbb{R}^d). \]
On the other hand, in [30] there is an alternative elegant definition of \( \Sigma_{s_1}(\mathbb{R}^d) \) and \( S_{s_2}(\mathbb{R}^d) \) such that these spaces agrees with the definitions
above when \( s_1 > 1/2 \) and \( s_2 \geq 1/2 \), but \( \Sigma_{1/2}(\mathbb{R}^d) \) is non-trivial and contained in \( \Sigma_{1/2}(\mathbb{R}^d) \).

From now on we assume that \( s, t > 1/2 \) when considering \( \Sigma_t^s(\mathbb{R}^d) \).

The Gelfand-Shilov distribution spaces \((\mathcal{S}_t^s)'(\mathbb{R}^d)\) and \((\Sigma_t^s)'(\mathbb{R}^d)\) are the projective and inductive limit respectively of \((\mathcal{S}_{t,h}^s)'(\mathbb{R}^d)\). This means that
\[
(\mathcal{S}_t^s)'(\mathbb{R}^d) = \bigcap_{h > 0} (\mathcal{S}_{t,h}^s)'(\mathbb{R}^d) \quad \text{and} \quad (\Sigma_t^s)'(\mathbb{R}^d) = \bigcup_{h > 0} (\mathcal{S}_{t,h}^s)'(\mathbb{R}^d). \tag{1.4}
\]

We remark that already in \([17]\) it is proved that \((\mathcal{S}_t^s)'(\mathbb{R}^d)\) is the dual of \(\mathcal{S}_t^s(\mathbb{R}^d)\), and if \( s > 1/2 \), then \((\Sigma_t^s)'(\mathbb{R}^d)\) is the dual of \(\Sigma_t^s(\mathbb{R}^d)\) (also in topological sense).

The Gelfand-Shilov spaces are invariant or possess convenient mapping properties under several basic transformations. For example they are invariant under translations, dilations, tensor product, and to some extent under Fourier transformation. Here tensor products of elements in Gelfand-Shilov distribution spaces are defined in similar ways as for tensor products for distributions (cf. Chapter V in \([27]\)). If \( s, s_0, t, t_0 > 0 \) satisfy
\[ s_0 + t_0 \geq 1, \quad s \geq s_0 \quad \text{and} \quad t \geq t_0, \]
and \( f, g \in (\mathcal{S}_{s_0}^s)'(\mathbb{R}^d) \setminus \{0\} \) and then \( f \otimes g \in (\mathcal{S}_t^s)'(\mathbb{R}^{2d}) \), if and only if \( f, g \in (\mathcal{S}_t^s)'(\mathbb{R}^d) \). Similar facts hold for any other choice of Gelfand-Shilov spaces of functions or distributions.

From now on we let \( \mathcal{F} \) be the Fourier transform which takes the form
\[
(\mathcal{F} f)(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i(x, \xi)} \, dx
\]
when \( f \in L^1(\mathbb{R}^d) \). Here \( (\cdot, \cdot) \) denotes the usual scalar product on \( \mathbb{R}^d \). The map \( \mathcal{F} \) extends uniquely to homeomorphisms on \( \mathcal{S}'(\mathbb{R}^d) \), \( \mathcal{S}_t'(\mathbb{R}^d) \) and \( \Sigma_t'(\mathbb{R}^d) \), and restricts to homeomorphisms on \( \mathcal{F}(\mathbb{R}^d) \), \( \mathcal{S}(\mathbb{R}^d) \) and \( \Sigma(\mathbb{R}^d) \), and to a unitary operator on \( L^2(\mathbb{R}^d) \). More generally, \( \mathcal{F} \) extends uniquely to homeomorphisms from \( (\mathcal{S}_t^s)'(\mathbb{R}^d) \) and \( (\Sigma_t^s)'(\mathbb{R}^d) \) to \( (\mathcal{S}_t^s)'(\mathbb{R}^d) \) and \( (\Sigma_t^s)'(\mathbb{R}^d) \) respectively, and restricts to homeomorphisms from \( \mathcal{S}_t(\mathbb{R}^d) \) and \( \Sigma_t(\mathbb{R}^d) \) to \( \mathcal{S}_t(\mathbb{R}^d) \) and \( \Sigma_t(\mathbb{R}^d) \) respectively.

The following lemma shows that functions in Gelfand-Shilov spaces can be characterized by estimates on the functions and their Fourier transform of the form
\[
|f(x)| \lesssim e^{-\varepsilon|x|^{1/t}} \quad \text{and} \quad |\hat{f}(\xi)| \lesssim e^{-\varepsilon|\xi|^{1/s}}. \tag{1.6}
\]
The proof is omitted, since the result can be found in e.g. \([4, 17]\).

**Lemma 1.1.** Let \( f \in \Sigma_{1/2}^1(\mathbb{R}^d) \). Then the following is true:

1. if \((s, t) \in J_{\text{GS}} \) or \( s = t = 1/2 \), then \( f \in \Sigma_t^s(\mathbb{R}^d) \), if and only if \((1.6)\) holds for some \( \varepsilon > 0 \);
(2) if \((s,t) \in J_{CS}\), then \(f \in \Sigma_s^t(\mathbb{R}^d)\), if and only if \((1.6)\) holds for every  \(\varepsilon > 0\).

The estimates \((1.6)\) are equivalent to
\[
|f(x)| \leq Ce^{-\varepsilon |x|^{1/t}} \quad \text{and} \quad |\hat{f}(\xi)| \leq Ce^{-\varepsilon |\xi|^{1/s}}.
\]

In (2) in Lemma \([11.1]\) it is understood that the (hidden) constant \(C > 0\) depends on \(\varepsilon > 0\).

Next we recall some mapping properties of Gelfand-Shilov spaces under short-time Fourier transforms.

Let \(\phi \in \mathscr{S}(\mathbb{R}^d) \setminus \{0\}\) be fixed. For every \(f \in \mathscr{S}(\mathbb{R}^d)\), the short-time Fourier transform \(V_\phi f\) is the distribution on \(\mathbb{R}^{2d}\) defined by the formula
\[
(V_\phi f)(x,\xi) = \mathcal{F}(f \cdot \phi(\cdot - x))(\xi).
\]
We note that the right-hand side defines an element in \(\mathscr{S}'(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d})\), and that \(V_\phi f\) takes the form
\[
V_\phi f(x,\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(y)\phi(y-x)e^{-i(y,\xi)} \, dy
\]
when \(f \in L_{(\omega)}^1\) for some \(\omega \in \mathscr{P}(\mathbb{R}^d)\).

In order to extend the definition of the short-time Fourier transform we reformulate \((1.7)\) in terms of partial Fourier transforms and tensor products (cf. \([15]\)). More precisely, let \(\mathscr{F}_2 F\) be the partial Fourier transform of \(F(x, y) \in \mathscr{S}'(\mathbb{R}^{2d})\) with respect to the \(y\)-variable, and let \(U\) be the map which takes \(F(x,y)\) into \(F(y,y-x)\). Then it follows that
\[
V_\phi f = (\mathscr{F}_2 \circ U)(f \otimes \overline{\phi})
\]
when \(f \in \mathscr{S}'(\mathbb{R}^d)\) and \(\phi \in \mathscr{S}(\mathbb{R}^d)\).

The following result concerns the map
\[
(f, \phi) \mapsto V_\phi f.
\]

\textbf{Proposition 1.2.} The map \((1.9)\) from \(\mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d)\) to \(\mathscr{S}'(\mathbb{R}^{2d})\) is uniquely extendable to a continuous map from \(S'_{1/2}(\mathbb{R}^d) \times S'_{1/2}(\mathbb{R}^d)\) to \(S'_{1/2}(\mathbb{R}^{2d})\). Furthermore, if \(s \geq 1/2\) and \(f, \phi \in S'_{1/2}(\mathbb{R}^d) \setminus \{0\}\), then the following is true:

1. the map \((1.9)\) restricts to a continuous map from \(S_s(\mathbb{R}^d) \times S'_s(\mathbb{R}^d)\) to \(S'_s(\mathbb{R}^{2d})\). Moreover, \(V_\phi f \in S'_s(\mathbb{R}^{2d})\), if and only if \(f, \phi \in S_s(\mathbb{R}^d)\);

2. the map \((1.9)\) restricts to a continuous map from \(S'_s(\mathbb{R}^d) \times S_s(\mathbb{R}^d)\) to \(S'_s(\mathbb{R}^{2d})\). Moreover, \(V_\phi f \in S'_s(\mathbb{R}^{2d})\), if and only if \(f, \phi \in S'_s(\mathbb{R}^d)\).

Similar facts hold after the spaces \(S_s\) and \(S'_s\) have been replaced by \(\Sigma_s\) and \(\Sigma'_s\) respectively.
Proof. The first part of the proposition, (1) and (3) follow immediately from \(1.3\), and the facts that tensor products, \(\mathcal{F}_2\) and \(U\) are continuous on \(S_*, \Sigma_*\) and their duals. See also \(7\) for details. \(\square\)

There are several other ways to characterize Gelfand-Shilov spaces (cf. e.g. \([18, 25, 44]\)).

1.3. **Mixed quasi-normed space of Lebesgue types.** Next we discuss mixed quasi-norm spaces. Let \(p, q \in (0, \infty]\), and let \(\omega \in \mathcal{P}_E(\mathbb{R}^{2d})\). A common type of mixed quasi-norm space on \(\mathbb{R}^{2d}\) is \(L_{p,q}^\omega(\mathbb{R}^{2d})\), which consists of all measurable functions \(F\) on \(\mathbb{R}^{2d}\) such that

\[
\|g\|_{L_{p,q}^\omega(\mathbb{R}^{2d})} < \infty, \quad \text{where} \quad g(\xi) \equiv \|F(\cdot, \xi)\omega(\cdot, \xi)\|_{L_q(\mathbb{R}^d)}.
\]

Next we introduce a broader family of mixed quasi-norm spaces on \(\mathbb{R}^d\), where the pair \((p, q)\) above is replaced by a vector in \((0, \infty]^d\) of Lebesgue exponents. If

\[
p = (p_1, \ldots, p_d) \in (0, \infty]^d \quad \text{and} \quad q = (q_1, \ldots, q_d) \in (0, \infty]^d
\]

are two such vectors, then we use the conventions \(p \leq q\) when \(p_j \leq q_j\) for every \(j = 1, \ldots, d\), and \(p < q\) when \(p_j < q_j\) for every \(j = 1, \ldots, d\).

Let \(S_d\) be the set of permutations on \(\{1, \ldots, d\}\). For every \(p \in (0, \infty]^d, \omega \in \mathcal{P}_E(\mathbb{R}^d), \sigma \in S_d\) and measurable \(f\) on \(\mathbb{R}^d\), let

\[
\|f\|_{L_p^\omega(\mathbb{R}^d)} \equiv g_{d,\sigma},
\]

where \(g_{j,\omega}, j = 1, \ldots, d,\) are defined inductively by the formulas

\[
g_{0,\omega}(x_{\sigma(1)}, \ldots, x_{\sigma(d)}) = |f(x_1, \ldots, x_d)\omega(x_1, \ldots, x_d)|,
\]

\[
g_{1,\omega}(x_2, \ldots, x_d) = \|g_{0,\omega}(\cdot, x_2, \ldots, x_d)\|_{L_{p_1}(\mathbb{R})},
\]

\[
g_{k,\omega}(x_{k+1}, \ldots, x_d) = \|g_{k-1,\omega}(\cdot, x_{k+1}, \ldots, x_d)\|_{L_{p_k}(\mathbb{R})}, \quad k = 2, \ldots, d-1.
\]

and

\[
g_{d,\omega} = \|g_{d-1,\omega}\|_{L_{p_d}(\mathbb{R})}.
\]

The mixed norm space \(L_{p,\sigma,\omega}(\mathbb{R}^d)\) of Lebesgue type is defined as the set of all measurable functions \(f\) on \(\mathbb{R}^d\) such that \(\|f\|_{L_{p,\sigma,\omega}(\mathbb{R}^d)} < \infty\).

The set of sequences \(\ell_{p,\sigma,\omega}(\Lambda)\), for an appropriate lattice \(\Lambda\) is defined in an analogous way. More precisely, let \(\theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}_+^d\), where \(\mathbb{R}_+ = \mathbb{R} \setminus 0\), and let \(T_\theta\) denote the diagonal matrix with diagonal elements \(\theta_1, \ldots, \theta_d\). Also let

\[
\Lambda = T_\theta \mathbb{Z}^d = \{(\theta_1j_1, \ldots, \theta_dj_d); (j_1, \ldots, j_d) \in \mathbb{Z}^d\}.
\]

For any sequence \(a\) on \(\mathbb{Z}^d\), let

\[
\|a\|_{\ell_{p,\sigma,\omega}(\Lambda)} \equiv b_{d,\omega}(0),
\]
where $b_{j,\omega}$, $j = 1, \ldots, d$, are defined inductively by the formulas

$$b_{\omega}(j_1, \ldots, j_d) = |a(j_1, \ldots, j_d)\omega(j_1, \ldots, j_d)|,$$

$$b_{1,\omega}(j_2, \ldots, j_d) = \|b_{\omega}(\cdot, x_2, \ldots, x_d)\|_{\ell^p(\theta)}(\Phi),$$

$$b_{k,\omega}(j_{k+1}, \ldots, j_d) = \|b_{k-1,\omega}(\cdot, (j_{k+1}, \ldots, j_d))\|_{\ell^p_k(\theta)}(\Phi), \quad k = 2, \ldots, d - 1$$

and

$$b_{d,\omega} = \|b_{d-1,\omega}\|_{\ell^p_d(\theta)}.$$

We also write $L^p_\omega$ and $\ell^p_\omega$ instead of $L^p_{\sigma,\omega}$ and $\ell^p_{\sigma,\omega}$ respectively when $\sigma$ is the identity map. Furthermore, if $\omega$ is equal to 1, then we write

$$L^p_\sigma, \ell^p_\sigma, L^p \text{ and } \ell^p$$

instead of

$$L^p_{\sigma,\omega}, \ell^p_{\sigma,\omega}, L^p_\omega \text{ and } \ell^p_\omega,$$

respectively.

For any $p \in (0, \infty]^d$, let

$$\max p \equiv \max(p_1, \ldots, p_d) \quad \text{and} \quad \min p \equiv \min(p_1, \ldots, p_d).$$

We note that if $\max p < \infty$, then $\ell^p(\Lambda)$ is dense in $\ell^p_{\sigma,\omega}(\Lambda)$. Here $\ell^p(\Lambda)$ is the set of all sequences $\{a(j)\}_{j \in \Lambda}$ on $\Lambda$ such that $a(j) \neq 0$ for at most finite numbers of $j$.

1.4. Modulation spaces. Next we define modulation spaces. First we modify the definition of invariant Banach spaces of measurable functions on the phase space (cf. e.g. [43, 44]).

**Definition 1.3.** Let $v \in \mathcal{P}_E(\mathbb{R}^{2d})$ be submultiplicative

1. A quasi-Banach space $\mathcal{B}$ of measurable functions on $\mathbb{R}^{2d}$ is called **invariant** (with respect to $v$), if there is a constant $C > 0$ such that

$$\|F\|_{\mathcal{B}} \leq C\|G\|_{\mathcal{B}} \quad \text{and} \quad \|F(\cdot - X)\|_{\mathcal{B}} \leq C\|F\|_{\mathcal{B}}v(X),$$

when

$$F, G \in \mathcal{B}, \quad |F| \leq |G|, \quad \Phi \in \mathcal{S}_{1/2}(\mathbb{R}^{2d}), \quad \text{and} \quad X \in \mathbb{R}^{2d}.$$

2. A Banach space $\mathcal{B}$ of measurable functions on $\mathbb{R}^{2d}$ is called **strongly invariant** (with respect to $v$), if $\mathcal{B}$ is invariant, the convolution map $(F, \Phi) \mapsto F * \Phi$ is continuous from $\mathcal{B} \times \mathcal{S}_{1/2}(\mathbb{R}^{2d})$ to $\mathcal{B}$, and there is a constant $C > 0$ such that

$$\|F * \Phi\|_{\mathcal{B}} \leq C\|F\|_{\mathcal{B}}\|\Phi\|_{L^1_v}.$$
Let \( \phi \in S_{1/2}(\mathbb{R}^d) \setminus 0 \), \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \) be such that \( \omega \) is \( v \)-moderate, and let \( \mathcal{B} \) be an invariant quasi-Banach space on \( \mathbb{R}^{2d} \) with respect to \( v \). Then the modulation space \( M(\omega, \mathcal{B}) \) consists of all \( f \in S_{1/2}(\mathbb{R}^d) \) such that

\[
\| f \|_{M(\omega, \mathcal{B})} \equiv \| V_\omega f \|_{\mathcal{B}}
\]

if finite. It follows from Proposition One of our goal is to prove that \( M(\omega, \mathcal{B}) \) is independent of \( \phi \), and that different \( \phi \) gives rise to equivalent quasi-norms \( \| \cdot \|_{M(\omega, \mathcal{B})} \).

For any \( p, q \in (0, \infty] \) and \( \omega \mathcal{P}_E(\mathbb{R}^{2d}) \), the standard modulation space \( M^p_q(\mathbb{R}^d) \) is defined as \( M(\omega, L^p(\mathbb{R}^{2d})) \). We remark that \( M^p_q(\mathbb{R}^d) \) is one of the most common types of modulation spaces.

More generally, for any \( \sigma \in S_{2d} \), \( p \in (0, \infty]^d \) and \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \), the modulation space \( M^p_q(\omega, \sigma, \mathbb{R}^d) \) is defined as \( M(\omega, L^p(\mathbb{R}^{2d})) \).

The following results are essentially restatements of Propositions 1.3, 1.4 and 3.5 in [46]; we list some properties for modulation. The proofs are therefore omitted.

**Proposition 1.4.** Let \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \), and let \( \mathcal{B} \) be an invariant space on \( \mathbb{R}^{2d} \). Then the following is true:

1. \( \Sigma_1(\mathbb{R}^d) \subseteq M(\omega, \mathcal{B}) \subseteq \Sigma_1'(\mathbb{R}^d) \);
2. if in addition

\[
e^{-\varepsilon |.|} \lesssim \omega \lesssim e^{\varepsilon |.|},
\]

holds for every \( \varepsilon > 0 \), then \( \Sigma_1(\mathbb{R}^d) \subseteq M(\omega, \mathcal{B}) \subseteq \Sigma_1'(\mathbb{R}^d) \);
3. if in addition \( \omega \in \mathcal{P}(\mathbb{R}^{2d}) \), then \( \mathcal{S}(\mathbb{R}^d) \subseteq M(\omega, \mathcal{B}) \subseteq \mathcal{S}'(\mathbb{R}^d) \).

**Proposition 1.5.** Let

\[
p, q \in [1, \infty], \quad p_j \in (0, \infty)^{2d} \quad \text{and} \quad \omega, \omega_j, v, v_0 \in \mathcal{P}_E(\mathbb{R}^{2d}), \quad j = 1, 2,
\]

be such that \( p_1 \leq p_2 \), \( \omega_2 \lesssim \omega_1 \), \( v \) and \( v_0 \) are submultiplicative, and \( \omega \) is \( v \)-moderate. Also let \( \sigma \in S_{2d} \), and let \( \mathcal{B} \) be a strongly invariant Banach space on \( \mathbb{R}^{2d} \) with respect to \( v_0 \). Then the following is true:

1. if \( \phi \in M^1_{(\omega v_0)}(\mathbb{R}^d) \setminus 0 \), then \( f \in M(\omega, \mathcal{B}) \) if and only if

\[
\| V_\phi f \|_{\mathcal{B}} < \infty.
\]

That is \( M(\omega, \mathcal{B}) \) is independent of the choice of \( \phi \in M^1_{(\omega v_0)}(\mathbb{R}^d) \setminus 0 \). Moreover, \( M(\omega, \mathcal{B}) \) is a Banach space under the norm in (1.10), and different choices of \( \phi \) give rise to equivalent norms;

2. \( M^p_{\sigma, (\omega v_0)}(\mathbb{R}^d) \subseteq M^p_{\sigma, \omega}(\mathbb{R}^d) \);
3. the \( L^2 \)-form on \( \mathcal{S}_{1/2}(\mathbb{R}^d) \) extends uniquely to a dual form between \( M^p_q(\mathbb{R}^d) \) and \( M^p_{1/q}(\mathbb{R}^d) \). Furthermore, if in addition \( p, q < \infty \), then the dual of \( M^p_{\sigma}(\mathbb{R}^d) \) can be identified with \( M^p_{1/q}(\mathbb{R}^d) \) through this form.
Next we recall the notion of Gabor expansions. First we recall some facts on sequences and lattices. In what follows we let $\Lambda_1$ and $\Lambda_2$ be the lattices

$$\Lambda_1 \equiv \{ x_j \}_{j \in J} \equiv T_\theta \mathbb{Z}^d \quad \text{and} \quad \Lambda_2 \equiv \{ \xi_k \}_{k \in J} \equiv T_\vartheta \mathbb{Z}^d,$$

where $\theta, \vartheta \in \mathbb{R}^d$ and some index set $J$.

**Definition 1.6.** Let $\Lambda = \Lambda_1 \times \Lambda_2$, where $\Lambda_1$ and $\Lambda_2$ are given by (1.11). Let $p, r \in (0, \infty]$ be such that $r_m \in \min(1, p_1, \ldots, p_m)$, $m = 1, \ldots, 2d$, $\omega, v \in \mathcal{P}_E(\mathbb{R}^{2d})$ be such that $v$ is submultiplicative and $\omega$ is $v$-moderate, and let $\phi, \psi \in M^r_v(\mathbb{R}^d)$.

1. The analysis operator $C^\Lambda_\phi$ is the operator from $\Sigma_1(\mathbb{R}^d)$ to $\ell_\infty(\omega)(\Lambda)$, given by

   $$C^\Lambda_\phi f \equiv \{ V_\phi f(x_j, \xi_k) \}_{j, k \in J};$$

2. The synthesis operator $D^\Lambda_\psi$ is the operator from $\ell_0(\Lambda)$ to $M^p_\omega(\mathbb{R}^d)$, given by

   $$D^\Lambda_\psi c \equiv \sum_{j, k \in J} c_{j, k} e^{i \langle \cdot, \xi_k \rangle} \phi(\cdot - x_j);$$

3. The Gabor frame operator $S^{\Lambda}_{\phi, \psi}$ is the operator from $\Sigma_1(\mathbb{R}^d)$ to $M^p_\omega(\mathbb{R}^d)$, given by $D^\Lambda_\psi \circ C^\Lambda_\phi$, i.e.

   $$S^{\Lambda}_{\phi, \psi} f \equiv \sum_{j, k \in J} V_\phi f(x_j, \xi_k) e^{i \langle \cdot, \xi_k \rangle} \phi(\cdot - x_j).$$

It is easily seen that the operators in Definition 1.6 are well-defined and continuous.

We finish the section by some reflections concerning Theorem 13.1.1 in [21], which can be considered as a special case of Theorem S in [20]. The following result follows from Theorem 13.1.1 in [21].

**Proposition 1.7.** Let $v \in \mathcal{P}_E(\mathbb{R}^{2d})$ be submultiplicative, and $\phi \in M^1_v(\mathbb{R}^d) \setminus \{0\}$. Then there is a constant $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$, the frame operator $S^{\Lambda}_{\phi, \phi}$, with $\Lambda = \varepsilon \mathbb{Z}^{2d}$ is a homeomorphism on $M^1_v(\mathbb{R}^d)$.

Let $v, \phi$ and $\Lambda$ be as in Proposition 1.7. Then

$$(S^{\Lambda}_{\phi, \phi})^{-1} \phi$$

is called the canonical dual window to $\phi$, with respect to $\Lambda$. By duality, it follows that $S^{\Lambda}_{\phi, \phi}$ extends to a continuous operator on $M^\infty_\omega(\mathbb{R}^d)$, and

$$S^{\Lambda}_{\phi, \phi}(e^{i \langle \cdot, \xi_k \rangle} f(\cdot - x_j)) = e^{i \langle \cdot, \xi_k \rangle} (S^{\Lambda}_{\phi, \phi} f)(\cdot - x_j),$$

when $f \in M^\infty_\omega(\mathbb{R}^d)$ and $(x_j, \xi_k) \in \Lambda$. 

11
Remark 1.8. Let \( r \in (0, 1), v \in \mathcal{P}_E(\mathbb{R}^d) \) be submultiplicative, and set

\[
(\Theta, v)(x, \xi) = v(x, \xi)(x, \xi)^\rho, \quad \text{where} \quad \rho > 2d(1 - r)/r.
\]

(1.12)

Then \( L^1(\Theta, v)(\mathbb{R}^{2d}) \) is continuously embedded in \( L^r_v(\mathbb{R}^{2d}) \), giving that \( M^1(\Theta, v)(\mathbb{R}^d) \subseteq M^r_v(\mathbb{R}^d) \). Hence if \( \phi \in M^1(\Theta, v) \), \( \varepsilon_0 \) is chosen such that \( S^{\Lambda}_{\phi, \varepsilon} \) is invertible on \( M^1(\Theta, v)(\mathbb{R}^d) \) for every \( \Lambda = \varepsilon Z^d, \varepsilon \in (0, \varepsilon_0] \), it follows that both \( \phi \) and its canonical dual with respect to \( \Lambda \) belong to \( M^r_v(\mathbb{R}^d) \).

The following extensions of results on Gabor expansions in [16, 21] agree with Proposition 3.6 and Theorem 3.7 in [16]. The proofs are therefore omitted.

Proposition 1.9. Let \( \Lambda = T_\theta Z^{2d} \) for some \( \theta_1, \ldots, \theta_{2d} > 0, p \in (0, \infty]^{2d}, r \in \min(1, p) \), and let \( \omega, v \in \mathcal{P}_E(\mathbb{R}^{2d}) \) be such that \( \omega \) is \( v \)-moderate. Also let \( \phi, \psi \in M^1(\Theta, v)(\mathbb{R}^d) \), and let \( C_\phi \) and \( D_\psi \) be as in Definition 1.6. Then the following is true:

1. \( C_\phi \) is uniquely extendable to continuous mappings from \( M^p_{\sigma, (\omega)}(\mathbb{R}^d) \) to \( \ell^p_{\sigma, (\omega)}(\Lambda) \);
2. \( D_\psi \) is uniquely extendable to continuous mappings from \( \ell^p_{\sigma, (\omega)}(\Lambda) \) to \( M^p_{\sigma, (\omega)}(\mathbb{R}^d) \).

Furthermore, if \( \max p < \infty, f \in M^p_{\sigma, (\omega)}(\mathbb{R}^d) \) and \( c \in \ell^p_{\sigma, (\omega)}(\Lambda) \), then \( C_\phi f \) and \( D_\psi c \) converge unconditionally and in norms. If instead \( \max p = \infty \), then \( C_\phi f \) and \( D_\psi c \) converge in the weak* topology in \( \ell^\infty_{\sigma, (\omega)}(\Lambda) \) and \( M^\infty_{\sigma, (\omega)}(\mathbb{R}^d) \), respectively.

Theorem 1.10. Let \( \Lambda = T_\theta Z^{2d} = \{(x_j, \xi_k)\}_{j,k \in J} \), where \( \theta \in \mathbb{R}_+^{2d}, p, r \in (0, \infty]^{2d}, \sigma \in S_{2d}, \) and let \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \) be the same as in Proposition 1.9. Also let \( \phi, \psi \in M^1(\sigma, (\omega)) \) be such that

\[
\{e^{i\langle \cdot, \xi_k \rangle} \phi(\cdot - x_j)\}_{j,k \in J} \quad \text{and} \quad \{e^{i\langle \cdot, \xi_k \rangle} \psi(\cdot - x_j)\}_{j,k \in J}
\]

are dual frame to each other. Then the following is true:

1. The operators \( S_{\phi, \psi} = D_\psi \circ C_\phi \) and \( S_{\psi, \phi} = D_\phi \circ C_\psi \) are both the identity map on \( M^p_{\sigma, (\omega)}(\mathbb{R}^d) \), and

\[
f = \sum_{j,k \in J} \langle V_\phi f(x_j, \xi_k), e^{i\langle \cdot, \xi_k \rangle} \psi(\cdot - x_j) \rangle
\]

\[
= \sum_{j,k \in \mathbb{Z}^d} (V_\psi f(x_j, \xi_k)) e^{i\langle \cdot, \xi_k \rangle} \phi(\cdot - x_j),
\]

with unconditional norm-convergence in \( M^p_{\sigma, (\omega)}(\mathbb{R}^d) \) when \( \max p < \infty \), and with convergence in \( M^\infty_{\sigma, (\omega)}(\mathbb{R}^d) \) with respect to the weak* topology otherwise;

2. \( \|f\|_{M^p_{\sigma, (\omega)}} \asymp \|C_\phi f\|_{\ell^p_{\sigma, (\omega) \Theta}} \asymp \|D_\psi f\|_{\ell^p_{\sigma, (\omega) \Theta}} \asymp \|(V_\phi f) \circ T_\theta\|_{\ell^p_{\sigma, (\omega) \Theta}} \asymp \|(V_\psi f) \circ T_\theta\|_{\ell^p_{\sigma, (\omega) \Theta}} \).
1.5. **Classes of matrices.** Next we define the classes of matrices. In the most general setting, the matrices are given by $A = (a(j, k))_{j,k \in J}$, for some index set $J$. In some situations, the index set $J$ is given by $T \mathbb{Z}^d$, where $T \in \text{GL}(d, \mathbb{R})$. For such $J$, let $A$ be the matrix $(a(j, k))_{j,k \in J}$, $p \in (0, \infty]$, and let $\omega$ be a map from $J \times J$ to $\mathbb{R}_+$. Then it is convenient to let the function $h_{A,p,\omega}$ from $J$ to $\mathbb{R}$ be defined as

$$h_{A,p,\omega}(k) \equiv \|H_{A,\omega}(:, k)\|_{\ell^p},$$

where $H_{A,\omega}(j, k) = a(j, j - k)\omega(j, j - k)$. (1.13)

**Definition 1.11.** Let $0 < p \leq \infty$, $J$ be an index set and let $\omega$ be a map from $J \times J$ to $\mathbb{R}_+$.

1. The set $U_0(J)$ is the set of matrices $(a(j, k))_{j,k \in J}$ such that at most finite numbers of $a(j, k)$ are non-zero;
2. Assume in addition $J = T \mathbb{Z}^d$ for some $T \in \text{GL}(d, \mathbb{R})$ and $p, q \in (0, \infty]$. Then the set $\mathbb{U}^{p,q}(\omega, J)$ consists of all matrices $A = (a(j, k))_{j,k \in J}$ such that $\|A\|_{\mathbb{U}^{p,q}(\omega, J)}$ is finite. Here

$$\|A\|_{\mathbb{U}^{p,q}(\omega, J)} \equiv \|h_{A,p,\omega}\|_{\ell^q},$$

where $h_{A,p,\omega}$ is given by (1.13). Furthermore, $\mathbb{U}_0^{p,q}(\omega, J)$ is the completion of $U_0(J)$ under the norm $\| \cdot \|_{\mathbb{U}^{p,q}(\omega, J)}$.

For convinieny we also set $\mathbb{U}^p = \mathbb{U}^{p,p}$ and $\mathbb{U}_0^p = \mathbb{U}_0^{p,p}$.

Later on it will be convenient to use the following notation. Let $J$ be an index set and let $f = \{f(j)\}_{j \in J}$ be a sequence. Then $|f|$ denotes the sequence given by

$$|f| = \{|f(j)|\}_{j \in J}.$$  

Furthermore, $\text{Re}(f)$ and $\text{Im}(f)$ are the real and imaginary parts of $f$, and if $f$ is real, then $f_+$ and $f_-$ denote the sequences

$$f_+ = \{(f(j))_+\}_{j \in J} \text{ and } f_- = \{(f(j))_-\}_{j \in J}.$$  

Here $t_+ = \max(t, 0)$ and $t_- = -\min(t, 0)$ when $t \in \mathbb{R}$.

For any matrix $A$, the matrices $|A|$, $\text{Re}(A)$, $\text{Im}(A)$, $A_+$ and $A_-$ are defined analogously. Here we note that for a matrix $A$ in the literature usually stands for the matrix $(A^*A)^{1/2}$, and not the matrix with whose matrix elements are the modulus of corresponding elements for $A$.

1.6. **Pseudo-differential operators.** Next we recall some properties in pseudo-differential calculus. Let $s \geq 1/2$, $a \in \mathcal{S}_s(\mathbb{R}^{2d})$, and $t \in \mathbb{R}$ be fixed. Then the pseudo-differential operator $\text{Op}(a)$ is the linear and continuous operator on $\mathcal{S}_s(\mathbb{R}^d)$, given by

$$(\text{Op}(a)f)(x) = (2\pi)^{-d} \int \int a((1-t)x + ty, \xi)f(y)e^{i(x-y, \xi)}
\text{d}y \text{d}\xi. \quad (1.14)$$
For general $a \in \mathcal{S}'(\mathbb{R}^{2d})$, the pseudo-differential operator $\text{Op}_t(a)$ is defined as the continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ with distribution kernel given by

$$K_{a,t}(x, y) = (2\pi)^{-d/2}(\mathcal{F}_2^{-1} a)((1-t)x + ty, x-y).$$

Here $\mathcal{F}_2 F$ is the partial Fourier transform of $F(x, y) \in \mathcal{S}'(\mathbb{R}^{2d})$ with respect to the $y$ variable. This definition makes sense, since the mappings $\mathcal{F}_2$ and $F(x, y) \mapsto F((1-t)x + ty, y-x)$

are homeomorphisms on $\mathcal{S}'(\mathbb{R}^{2d})$. In particular, the map $a \mapsto K_{a,t}$ is a homeomorphism on $\mathcal{S}'(\mathbb{R}^{2d})$.

The standard or Kohn-Nirenberg representation $\text{Op}(a)$ of $a$ is given by $\text{Op}_0(a)$ with $t = 0$, and the Weyl quantization $\text{Op}^w(a)$ is obtained by choosing $t = 1/2$ in (1.14) and (1.15). Since especially the former is important to us, we set $K_a = K_{a,t}$ when $t = 0$.

For any $K \in \mathcal{S}'(\mathbb{R}^{d_1+d_2})$, we let $T_K$ be the linear and continuous mapping from $\mathcal{S}(\mathbb{R}^{d_1})$ to $\mathcal{S}'(\mathbb{R}^{d_2})$, defined by the formula

$$(T_K f, g)_{L^2(\mathbb{R}^{d_2})} = (K, g \otimes \overline{f})_{L^2(\mathbb{R}^{d_1+d_2})}.$$  

(1.17)

It is well-known that if $t \in \mathbb{R}$, then it follows from Schwartz kernel theorem that $K \mapsto T_K$ and $a \mapsto \text{Op}_t(a)$ are bijective mappings from $\mathcal{S}'(\mathbb{R}^{2d})$ to the set of linear and continuous mappings from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ (cf. e.g. [27]).

In this context we remark that the maps $K \mapsto T_K$ and $a \mapsto \text{Op}_t(a)$ are uniquely extendable to bijective mappings from $\mathcal{S}'(\mathbb{R}^{2d})$ to the set of linear and continuous mappings from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. In fact, the asserted bijectivity for the map $K \mapsto T_K$ follows from the kernel theorem [29, Theorem 2.2], by Lozanov–Crvenković, Perišić and Taskovic. This kernel theorem corresponds to Schwartz kernel theorem in the usual distribution theory. The other assertion follows from the fact that $a \mapsto K_{a,t}$ is a homeomorphism on $\mathcal{S}'(\mathbb{R}^{2d})$.

In particular, for each $a_1 \in \mathcal{S}'(\mathbb{R}^{2d})$ and $t_1, t_2 \in \mathbb{R}$, there is a unique $a_2 \in \mathcal{S}'(\mathbb{R}^{2d})$ such that $\text{Op}_{t_1}(a_1) = \text{Op}_{t_2}(a_2)$. The relation between $a_1$ and $a_2$ is given by

$$\text{Op}_{t_1}(a_1) = \text{Op}_{t_2}(a_2) \iff a_2(x, \xi) = e^{i((t_2-t_1)(D_x, D_\xi))} a_1(x, \xi).$$

(1.18)

(Cf. [27].) Note here that the right-hand side makes sense, since it is equivalent to $\widehat{a_2} = e^{i((t_2-t_1)(x, \xi))} \widehat{a_1}$, and that the map $a \mapsto e^{i(t(x, \xi))} a$ is continuous on $\mathcal{S}'(\mathbb{R}^{2d})$.

Next we recall the links between pseudo-differential operators and $t$-Wigner distributions. Let $t \in \mathbb{R}$ and $a \in \mathcal{S}'(\mathbb{R}^{2d})$ be fixed. Then $a$ is called a rank-one element with respect to $t$, if the corresponding
pseudo-differential operator is of rank-one, i.e.

$$\text{Op}_t(a) f = (f, f_2) f_1, \quad f \in S_s(\mathbb{R}^d),$$  \hspace{1cm} (1.19)

for some $f_1, f_2 \in S'_s(\mathbb{R}^d)$. By straight-forward computations it follows that (1.19) is fulfilled, if and only if $a = (2\pi)^{d/2} W_{f_1, f_2}$, where $W_{f_1, f_2}$ is the $t$-Wigner distribution, defined by the formula

$$W_{f_1, f_2}(x, \xi) \equiv \mathcal{F}(f_1(x + t \cdot) f_2(x - (1 - t) \cdot))(\xi),$$  \hspace{1cm} (1.20)

which takes the form

$$W_{f_1, f_2}(x, \xi) = (2\pi)^{-d/2} \int f_1(x + ty) f_2(x - (1 - t)y) e^{-i(y, \xi)} dy,$$

when $f_1, f_2 \in S_s(\mathbb{R}^d)$. By combining these facts with (1.18), it follows that

$$W_{f_1, f_2}(x, \xi) = e^{i(t_2 - t_1)(D_x, D_\xi)} W_{f_1, f_2},$$  \hspace{1cm} (1.21)

for each $f_1, f_2 \in S'_s(\mathbb{R}^d)$ and $t_1, t_2 \in \mathbb{R}$. Since the Weyl case is particularly important, we set $W_{f_1, f_2} = W_{f_1, f_2}$ when $t = 1/2$, i.e. $W_{f_1, f_2}$ is the usual (cross-)Wigner distribution of $f_1$ and $f_2$.

1.7. Links between symbols and operator kernels. Next we discuss links between symbols of pseudo-differential operators and their operator kernels, on levels of short-time Fourier transforms and Gabor expansions. For simplicity we only consider the Kohn-Nirenberg case for pseudo-differential operators. That is, $t = 0$ in (1.14) and (1.15).

The following lemma explains the links between the short-time Fourier transforms of symbols in pseudo-differential calculus and their kernels, and follows by straight-forward applications of Fourier’s inversion formula. The details are left for the reader.

**Lemma 1.12.** Let $a, \Phi, \Phi_1, \Phi_2 \in S'_{1/2}(\mathbb{R}^{2d})$ be such that

$$\Phi_1(x, y) = \Phi(x, x - y) \quad \text{and} \quad \Phi_2(x, \xi) = (\mathcal{F}_2 \Phi_1)(x, \xi) = e^{-i(x, \xi)} (\mathcal{F}_2 \Phi)(x, -\xi).$$  \hspace{1cm} (1.22)

Then

$$\left( V_a K_a \right)(x, y, \xi, \eta) = \left( V_{\Phi_1}(\mathcal{F}_2^{-1} a)(x, x - y, \xi + \eta, -\eta) \right)$$

$$= e^{i(x, y, \eta)} (V_{\Phi_2} a)(x, -\eta, \xi + \eta, y - x),$$

or equivalently,

$$\left( V_{\Phi_2} a \right)(x, \xi, \eta, y) = e^{-i(y, \xi)} (V_{\Phi_1}(\mathcal{F}_2^{-1} a)(x, -y, \eta, \xi)$$

$$= e^{-i(y, \xi)} (V_{\Phi} K_a)(x, x + y, -\xi, \xi + \eta).$$
The next results are important when carrying over continuity properties for matrices into corresponding continuity properties for pseudo-differential operators with symbols in modulation spaces.

First we recall that if $\Phi, \Phi_1, \Phi_2 \in M^1_{(v)}(\mathbb{R}^{2d})$ satisfy (1.22), then corresponding Gabor systems with respect to the positive parameters $\alpha$ and $\beta$ are given by

$$
\{ e^{i\langle x, \kappa_1 \rangle + \langle y, \kappa_2 \rangle} \Phi(x - \alpha j_1, y - \alpha j_2) \}_{j_1, j_2, \kappa_1, \kappa_2 \in \mathbb{Z}^d},
$$

$$
\{ e^{i\beta \langle x, \kappa_1 \rangle + \langle y, \kappa_2 \rangle} \Phi_1(x - \alpha j_1, y - \alpha j_2) \}_{j_1, j_2, \kappa_1, \kappa_2 \in \mathbb{Z}^d},
$$

$$
\{ e^{i(\alpha \langle x, \kappa_2 \rangle + \beta \langle j_2, \xi \rangle)} \Phi_2(x - \alpha j_1, \xi - \beta \kappa_1) \}_{j_1, j_2, \kappa_1, \kappa_2 \in \mathbb{Z}^d}.
$$

The following proposition shows that if these Gabor systems are frames for $L^2$, then the dual Gabor atoms $\Psi, \Psi_1, \Psi_2$ satisfy similar boundary properties as $\Psi_1, \Psi_2$, and can be related to each other by the formulas

$$
\Psi_1(x, y) = \Psi(x, x - y) \quad \text{and} \quad \Psi_2(x, \xi) = (\mathcal{F}_2 \Psi_1)(x, -\xi).
$$

In particular, the dual frames of the frames in (1.23) can be expressed as

$$
\{ e^{i\langle x, \kappa_1 \rangle + \langle y, \kappa_2 \rangle} \Psi(x - \alpha j_1, y - \alpha j_2) \}_{j_1, j_2, \kappa_1, \kappa_2 \in \mathbb{Z}^d},
$$

$$
\{ e^{i\beta \langle x, \kappa_1 \rangle + \langle y, \kappa_2 \rangle} \Psi_1(x - \alpha j_1, y - \alpha j_2) \}_{j_1, j_2, \kappa_1, \kappa_2 \in \mathbb{Z}^d},
$$

$$
\{ e^{i(\alpha \langle x, \kappa_2 \rangle + \beta \langle j_2, \xi \rangle)} \Psi_2(x - \alpha j_1, \xi - \beta \kappa_1) \}_{j_1, j_2, \kappa_1, \kappa_2 \in \mathbb{Z}^d}.
$$

**Proposition 1.13.** Let $v \in \mathcal{P}_E(\mathbb{R}^{4d})$, and let $\Phi, \Phi_1, \Phi_2 \in M^1_{(v)}(\mathbb{R}^{2d}) \setminus \{0\}$ satisfy (1.22). Then for some constants $\alpha_0, \beta_0 > 0$, the sets in (1.23) are frames for every $\alpha \in [0, \alpha_0]$ and $\beta \in (0, \beta_0]$, with dual frames of the form (1.23), where $\Psi, \Psi_1, \Psi_2 \in M^1_{(v)}(\mathbb{R}^{2d})$ satisfy (1.24).

Next we consider Gabor frames for operator kernels and symbols to pseudo-differential operators. Let $v \in \mathcal{P}_E(\mathbb{R}^{4d})$ be submultiplicative, and assume that

$$
\{ \Phi_{1, j_2} \}_{j_1, j_2 \in \mathbb{Z}^{2d}}, \quad \{ \Phi_{1, j_1, j_2} \}_{j_1, j_2 \in \mathbb{Z}^{2d}} \quad \text{and} \quad \{ \Phi_{2, j_1, j_2} \}_{j_1, j_2 \in \mathbb{Z}^{2d}},
$$

are Gabor frames, where

$$
\Phi_{j_1, j_2}(x, y) = e^{i\langle x, \kappa_1 \rangle + \langle y, \kappa_2 \rangle} \Phi(x - \alpha j_1, y - \alpha j_2),
$$

$$
\Phi_{1, j_1, j_2}(x, y) = e^{i\langle x, \kappa_1 \rangle + \langle y, \kappa_2 \rangle} \Phi_1(x - \alpha j_1, y - \alpha j_2),
$$

$$
\Phi_{2, j_1, j_2}(x, \xi) = e^{i\langle \alpha \langle x, \kappa_2 \rangle + \beta \langle j_2, \xi \rangle \rangle} \Phi_2(x - \alpha j_1, \xi - \beta \kappa_1),
$$

$$
\mathbf{j}_1 = (j_1, \kappa_1) \in \mathbb{Z}^{2d}, \quad \mathbf{j}_2 = (j_2, \kappa_2) \in \mathbb{Z}^{2d},
$$

(1.27)
and the Gabor atoms $\Phi$, $\Phi_1$ and $\Phi_2$ belonging to $M_{(v)}^1(\mathbb{R}^{2d})$. Also let
\[
\{\Psi_{j_1,j_2}\}_{j_1,j_2 \in \mathbb{Z}^{2d}}, \quad \{\Psi_{1,j_1,j_2}\}_{j_1,j_2 \in \mathbb{Z}^{2d}} \quad \text{and} \quad \{\Psi_{2,j_1,j_2}\}_{j_1,j_2 \in \mathbb{Z}^{2d}},
\]
be the dual frames, defined analogously, with Gabor atoms $\Psi$, $\Psi_1$ and $\Psi_2$ belonging to $M_{(v)}^1(\mathbb{R}^{2d})$.

We shall recall some facts concerning the expansions
\[
K_a(x, y) = \sum_{j,k \in \mathbb{Z}^{2d}} b(j, k) \Phi_{j,k}(x, y)
= \sum_{j,k \in \mathbb{Z}^{2d}} c(j, k) \Phi_{1,j,k}(x, y)
= \sum_{j,k \in \mathbb{Z}^{2d}} b_1(j, k) \Phi_{1,j,k}(x, y)
\]
\[
(\mathcal{F}^{-1}_2 a)(x, y) = \sum_{j,k \in \mathbb{Z}^{2d}} b(j, k) \Phi_{j,k}(x, y)
= \sum_{j,k \in \mathbb{Z}^{2d}} b_1(j, k) \Phi_{1,j,k}(x, y)
\]
and
\[
a(x, \xi) = e^{-i(x, \xi)} \sum_{j,k \in \mathbb{Z}^{2d}} b(j, k) (\mathcal{F}_2 \Phi_{j,k})(x, -\xi)
= \sum_{j,k \in \mathbb{Z}^{2d}} b_2(j, k) \Phi_{2,j,k}(x, \xi),
\]
where $b(j, k)$, $b_1(j, k)$ and $b_2(j, k)$ are given by
\[
b(j_1,j_2) = (V_{\Psi}K_a)(\alpha j_1, \alpha j_2, \beta \kappa_1, \beta \kappa_2),
b_1(j_1,j_2) = (V_{\Psi_1} (\mathcal{F}^{-1}_2 a))(\alpha j_1, \alpha j_2, \beta \kappa_1, \beta \kappa_2),
b_2(j_1,j_2) = (V_{\Psi_2} a)(\alpha j_1, \beta \kappa_1, \beta \kappa_2, \alpha j_2),
\]
and are related to each others by the formulas
\[
b_1(j, k) = b(j, j - \tau(k)), \quad \tau(j, \kappa) = (j, -\kappa),
\]
and
\[
b_2(j_1,j_2) = e^{-i\alpha \beta(j_2, \kappa_1)} b(j_1, \kappa_1 + \kappa_2, j_1 + j_2, -\kappa_1),
\]
\[
j_1 = (j_1, \kappa_1), \quad j_2 = (j_2, \kappa_2).
\]
The following proposition shows how the coefficients $b(j, k)$, $b_1(j, k)$ and $b_2(j, k)$ can be estimated in terms of modulation space norms of $a$. 

\[17\]
Here the involved weight functions should satisfy
\[ \omega_0(x, \xi, \eta, y) \asymp \omega(x, x + y, \xi + \eta, -\xi) \]
\[ \vartheta_0(j_1, j_2) = \omega_0(\alpha j_1, \beta \kappa_1, \beta \kappa_2, j_2) \]
\[ \vartheta(j_1, j_2) = \omega(\alpha j_1, \alpha j_2, \beta \kappa_1, \beta \kappa_2), \quad j_1 = (j_1, \kappa_1), \quad j_2 = (j_2, \kappa_2), \]
and we let
\[ v_{r,d}(x) = e^{r|x|}, \quad x \in \mathbb{R}^d. \]

**Proposition 1.14.** Let \( p, q \in (0, \infty], 0 \leq r_0 < r, \omega_0, \omega, \omega_0, \omega \in \mathcal{P}_E(\mathbb{R}^d) \) and \( \vartheta_0 \) be such that \( \omega \) and \( \omega_0 \) are \( v_{r_0,d} \)-moderate and \((1.35)\) is fulfilled. Let \( \alpha > 0, \beta > 0 \) and \( \Phi, \Phi_j, \Psi, \Psi_j \in M_{(1/r_0, d)}(\mathbb{R}^{2d}) \setminus 0, j = 1, 2 \) be chosen such that \((1.26)\) are frames with dual frames \((1.28)\).

Also let \( a \in M_{(1/r_0, d)}(\mathbb{R}^{2d}) \) be a symbol such that corresponding distribution kernel satisfies \( K_a \in M_{(1/r_0, d)}(\mathbb{R}^{2d}) \), and let \( b(j, k), c(j, k) \) and \( b_2(j, k) \) be given by \((1.32)\). If \( \tau(x, \xi, y, \eta) = (x, \xi, y, -\eta) \), then the following conditions are equivalent:

1. \( a \in M_{p,q}(\mathbb{R}^{2d}); \)
2. \( b \circ \tau \in \mathcal{U}_{p,q}(\vartheta \circ \tau, \mathbb{Z}^{2d}); \)
3. \( b_2 \in \ell^{p,q}_{(\vartheta_0)}(\mathbb{Z}^{2d}). \)

Furthermore,
\[ \|a\|_{M_{p,q}(\omega_0)} \asymp \|b \circ \tau\|_{\mathcal{U}_{p,q}(\vartheta \circ \tau, \mathbb{Z}^{2d})} \asymp \|b_2\|_{\ell^{p,q}_{(\vartheta_0)}}, \quad a \in M_{(1/r_0, d)}(\mathbb{R}^{2d}). \]

**Proof.** It suffices to prove \((1.36)\). The relation \( \|a\|_{M_{p,q}(\omega_0)} \asymp \|b_2\|_{\ell^{p,q}_{(\vartheta_0)}} \) follows from \([45] \) Theorem 3.7, and the relation \( \|b \circ \tau\|_{\mathcal{U}_{p,q}(\vartheta \circ \tau, \mathbb{Z}^{2d})} \asymp \|b_2\|_{\ell^{p,q}_{(\vartheta_0)}} \) follows from \((1.34)\). The proof is complete.

### 1.8. Schatten-von Neumann classes

Next we recall some Schatten-von Neumann properties of operators from a Hilbert space \( \mathcal{H}_1 \) to another Hilbert space \( \mathcal{H}_2 \). (Cf. e.g. \([136, 38, 47]\).)

Let \( \text{ON}(\mathcal{H}_j) \), \( j = 1, 2 \), denote the family of orthonormal sequences in \( \mathcal{H}_j \), and assume that \( T : \mathcal{H}_1 \to \mathcal{H}_2 \) is linear, and that \( p \in (0, \infty] \). Then we set
\[ \|T\|_{\mathcal{S}_p} = \|T\|_{\mathcal{S}_p(\mathcal{H}_1, \mathcal{H}_2)} \equiv \sup \left( \sum \left| \langle Tf_j, g_j \rangle_{\mathcal{H}_2} \right|^p \right)^{1/p} \]
(with obvious modifications when \( p = \infty \)). Here the supremum is taken over all \( \{f_j\}_{j \in J} \in \text{ON}(\mathcal{H}_1) \) and \( \{g_j\}_{j \in J} \in \text{ON}(\mathcal{H}_2) \). Then \( \mathcal{S}_p = \mathcal{S}_p(\mathcal{H}_1, \mathcal{H}_2) \), the set of Schatten-von Neumann operators from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \) of order \( p \), consists of all linear and continuous operators \( T \) from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \) such that \( \|T\|_{\mathcal{S}_p(\mathcal{H}_1, \mathcal{H}_2)} \) is finite.
We note that
\[ \mathcal{I}_1(\mathcal{H}_1, \mathcal{H}_2) = \text{Tr}(\mathcal{H}_1, \mathcal{H}_2), \quad \mathcal{I}_2(\mathcal{H}_1, \mathcal{H}_2) = \text{HS}(\mathcal{H}_1, \mathcal{H}_2) \]

and \( \mathcal{I}_\infty(\mathcal{H}_1, \mathcal{H}_2) = \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \)

with equality in norms, where \( \text{Tr}(\mathcal{H}_1, \mathcal{H}_2) \), \( \text{HS}(\mathcal{H}_1, \mathcal{H}_2) \) and \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \), are the sets of trace-class, Hilbert Schmidt and continuous operators, respectively, from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \). Furthermore, if \( p < \infty \), then \( \mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2) \subseteq \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2) \), the set of compact operators from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \).

For future references we recall that \( \| T \|_{\mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2)} = \| T \|_{\text{HS}(\mathcal{H}_1, \mathcal{H}_2)} = \| T \|_{\mathcal{I}_\infty(\mathcal{H}_1, \mathcal{H}_2)} \), (1.37) and that for \( p_0, p_1, p_2 \in (0, \infty], T_1 \in \mathcal{I}_{p_1}(\mathcal{H}_1, \mathcal{H}_2) \) and \( T_2 \in \mathcal{I}_{p_2}(\mathcal{H}_2, \mathcal{H}_3) \) we have

\[ \| T_2 \circ T_1 \|_{\mathcal{I}_{p_0}(\mathcal{H}_1, \mathcal{H}_3)} \leq \| T_1 \|_{\mathcal{I}_{p_1}(\mathcal{H}_1, \mathcal{H}_2)} \| T_2 \|_{\mathcal{I}_{p_2}(\mathcal{H}_2, \mathcal{H}_3)} \]

when \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_0} \). (1.38)

For convenience we set \( \mathcal{I}_p(\mathcal{H}) = \mathcal{I}_p(\mathcal{H}, \mathcal{H}) \), and similarly for \( \mathcal{B}(\mathcal{H}) \) and \( \mathcal{K}(\mathcal{H}) \).

Let \( \{ e_j \}_{j \in J} \) be an orthonormal basis in \( \mathcal{H}_1 \), and let \( S \in \mathcal{I}_1(\mathcal{H}_1) \). Then the trace of \( S \) is defined as

\[ \text{tr}_{\mathcal{H}_1} S = \sum (Se_j, e_j)_{\mathcal{H}_1}. \]

For each pairs of operators \( T_1, T_2 \in \mathcal{I}_\infty(\mathcal{H}_1, \mathcal{H}_2) \) such that \( T_2^* \circ T_1 \in \mathcal{I}_1(\mathcal{H}_1) \), the sesqui-linear form

\[ (T_1, T_2) = (T_1, T_2)_{\mathcal{H}_1, \mathcal{H}_2} \equiv \text{tr}_{\mathcal{H}_1}(T_2^* \circ T_1) \]

of \( T_1 \) and \( T_2 \) is well-defined. Here we note that \( T \in \mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2) \) if and only if \( T^* \in \mathcal{I}_p(\mathcal{H}_2, \mathcal{H}_1) \). We refer to \[1,35,47\] for more facts about Schatten-von Neumann classes.

We are especially interested of deducing Schatten-von Neumann properties for pseudo-differential operators acting between Hilbert modulation spaces. For any \( p \in (0, \infty] \) and \( \omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^d) \), we therefore let \( s_{t,p}(\omega_1, \omega_2) \) be the set of all \( a \in \mathcal{S}_t(\mathbb{R}^d) \) such that \( \text{Op}_t(a) \in \mathcal{I}_p(M^2_{(\omega_1)}, M^2_{(\omega_2)}) \). We note that \( s_{t,p}(\omega_1, \omega_2) \) is a quasi-Banach space under the quasi-norm

\[ \| a \|_{s_{t,p}(\omega_1, \omega_2)} \equiv \| \text{Op}_t(a) \|_{\mathcal{I}_p(M^2_{(\omega_1)}, M^2_{(\omega_2)})}. \]

Furthermore, if in addition \( p \geq 1 \), then \( \| \cdot \|_{s_{t,p}(\omega_1, \omega_2)} \) is a norm and \( s_{t,p}(\omega_1, \omega_2) \) is a Banach space.

We also note that

\[ \Sigma_1(\mathbb{R}^d) \subseteq s_{t,p}(\omega_1, \omega_2) \subseteq \Sigma_1'(\mathbb{R}^d), \]

where the notations \( \Sigma_p \) denotes the Sobolev spaces on \( \mathbb{R}^d \).
and if addition $\omega_1, \omega_2 \in \mathcal{P}(\mathbb{R}^{2d})$, then
\[
\mathcal{P}(\mathbb{R}^{2d}) \subseteq s_{t,p}(\omega_1, \omega_2) \subseteq \mathcal{P}'(\mathbb{R}^{2d}).
\]

2. ESTIMATES FOR MATRICES

In the first part of the section we establish convenient factorization results for matrices in the class $U_p(\omega, J)$, where the most general result is Theorem 2.1. Thereafter we prove certain continuity results for matrix operators.

The involved weights
\[
\omega_l : J \times J \to \mathbb{R}_+,
\]
should fulfill
\[
\omega_1(j,j)\omega_2(j,k) \leq \omega_0(j,k)
\]
or
\[
\omega_1(j,k)\omega_2(k,k) \leq \omega_0(j,k),
\]
and the involved Lebesgue exponents should satisfy the Hölder condition
\[
\frac{1}{p_0} \leq \frac{1}{p_1} + \frac{1}{p_2},
\]

Theorem 2.1. Let $p_l \in (0, \infty]$ and $\omega_l, l = 0, 1, 2,$ be such that (2.1) and (2.4) hold, and let $A_0 \in U_{p_0}(\omega_0, J)$. Then the following is true:

(1) if (2.2) holds, then $A_0 = A_1 \cdot A_2$ for some $A_l \in U_{p_l}(\omega_l, J), l = 1, 2$. Furthermore, $A_1$ can be chosen as a diagonal matrix;

(2) if (2.3) holds, then $A_0 = A_1 \cdot A_2$ for some $A_l \in U_{p_l}(\omega_l, J), l = 1, 2$. Furthermore, $A_2$ can be chosen as a diagonal matrix.

Moreover, the matrices in (1) and (2) can be chosen such that
\[
\|A_1\|_{U_{p_1}(\omega_1,J)}\|A_2\|_{U_{p_2}(\omega_2,J)} \leq \|A_0\|_{U_{p_0}(\omega_0,J)}.
\]

Proof. It is no restrictions to assume that equality is attained in (2.4), and by transposition it also suffices to prove (1).

We only prove the result for $p_0 < \infty$. The small modifications to the case when $p_0 = \infty$ are left for the reader. Let $a(j,k)$ be the matrix elements for $A_0$, and let $A_1 = (b(j,k))$ and $A_2 = (c(j,k))$ be the matrices such that

\[
b(j,k) = \begin{cases} 
(\omega_1(j,j))^{-1} \left( \sum_m |a(j,m)\omega_0(j,m)|^{p_0} \right)^{1/p_1}, & j = k \\
0, & j \neq k
\end{cases}
\]

and $c(j,k) = a(j,k)/b(j,k)$ when $b(j,j) \neq 0$, and $c(j,k) = 0$ otherwise.

Since
\[
b(j,j) \geq (\omega_1(j,j))^{-1}|a(j,k)\omega_0(j,k)|^{p_0/p_1}, \quad \text{and} \quad \frac{1}{p_0} - \frac{1}{p_1} = \frac{1}{p_2},
\]

20
(2.2) gives
\[ |c(j, k)\omega_2(j, k)| \leq |a(j, k)|^{p_0/p_2}\omega_1(j, j)\omega_2(j, k)/\omega_0(j, k)^{p_0/p_1} \]
\[ \leq |a(j, k)|^{p_0/p_2}\omega_0(j, k)^{p_0/p_2}. \]
This in turn gives
\[
\|A_1\|_{\mathbb{U}_p(\omega_1, J)} = \left( \sum_{j,k} |b(j, k)\omega_1(j, k)|^{p_1} \right)^{1/p_1} \\
= \left( \left( \sum_j \left( \sum_m |a(j, m)\omega_0(j, m)|^{p_0} \right)^{1/p_1} \right)^{p_1} \right)^{1/p_1} = \|A_0\|^{p_0/p_1}_{\mathbb{U}_p(\omega_0, J)},
\]
and
\[
\|A_2\|_{\mathbb{U}_p(\omega_2, J)} = \left( \sum_{j,k} |c(j, k)\omega_2(j, k)|^{p_2} \right)^{1/p_2} \\
\leq \left( \sum_{j,k} |a(j, k)\omega_0(j, k)|^{p_0} \right)^{1/p_2} = \|A_0\|^{p_0/p_2}_{\mathbb{U}_p(\omega_0, J)}.
\]
Hence \( A_l \in \mathbb{U}_p(\omega_l, J), l = 1, 2. \) Since \( A_0 = A_1 \cdot A_2 \) and \( p_0/p_1 + p_0/p_2 = 1, \) the result follows.

If \( \omega_l, l = 0, 1, 2, \) in (2.1) fulfill
\[ \omega_1(j, m)\omega_2(m, k) \leq \omega_0(j, k), \quad \text{for every } j, k, m \in J, \quad (2.6) \]
then it is obvious that both (2.2) and (2.3) are fulfilled. Hence the following result is a special case of Theorem 2.1.

**Proposition 2.2.** Let \( p_l \in (0, \infty] \) and \( \omega_l, l = 0, 1, 2, \) be such that (2.1), (2.4) and (2.6) hold, and let \( A_0 \in \mathbb{U}_p(\omega_0, J). \) Then \( A_0 = A_1 \cdot A_2 \) for some \( A_l \in \mathbb{U}_p(\omega_l, J), l = 1, 2. \) Moreover, the matrices \( A_1 \) and \( A_2 \) can be chosen such that (2.5) holds.

Next we prove certain continuity results for matrix operators. We recall that if \( A = (a(j, k))_{j,k \in J} \) is a matrix, then \( Af \) is uniquely defined in \( \ell(J) \) when \( f \in \ell_0(J), \) i.e.
\[
A : \ell_0(J) \ni \ell(J). \quad (0.1)
\]
Furthermore, if in addition \( A \) belongs to \( \mathbb{U}_0(J), \) then \( Af \) is uniquely defined as an element in \( \ell_0(J) \) when \( f \in \ell(J), \) i.e.
\[
A : \ell(J) \ni \ell_0(J) \quad \text{when } A \in \mathbb{U}_0(J). \quad (2.7)
\]
Proposition 2.3. Let $\omega_1$, $\ell = 1, 2$ be weights on $J$ and $\omega_0$ be a weight on $J \times J$ such that (1.2) holds. Also let $p \in (0, \infty]$, $q = \infty$ when $p \leq 1$ and $q = p'$ otherwise, and let $A \in \mathbb{U}^p(\omega_0, J)$. Then $A$ in (1.1) is uniquely extendable to a continuous map from $\ell^p(\omega_1)(J)$ to $\ell^p(\omega_2)(J)$, and

$$\|A\|_{\ell^p(\omega_1)(J) \to \ell^p(\omega_2)(J)} \leq \|A\|_{\mathbb{U}^p(\omega_0, J)}. \quad (2.8)$$

Proof. First we assume that $p \leq 1$, and we let $A \in \mathbb{U}_0(J)$. Let $f = (f(k))_{k \in J} \in \ell^\infty(J)$, and let $g = (g(j))_{j \in J}$ be a weight $A$. Let $f = A \in \mathbb{U}_0(J)$, and we let $g = \omega_0(j,k) \omega_1(k)$. Then

$$|g(j)\omega_2(j)|^p = \left|\sum_k a(j,k)f(k)\omega_2(j)\right|^p \leq \left(\sum_k |a(j,k)f(k)|\omega_0(j,k)\omega_1(k)\right)^p \leq \|f\|_{\ell^p(\omega_1)} \|\omega_0(j,k)\|_p. \quad (2.9)$$

In the second inequality we have used the fact that $p \leq 1$. This gives

$$\|g\|_{\ell^p(\omega_2)} = \left(\sum_j |g(j)\omega_2(j)|^p\right)^{1/p} \leq \|A\|_{\mathbb{U}^p(\omega_0, J)} \|f\|_{\ell^p(\omega_1)},$$

and the continuity extension follows in this case, and the fact that $\mathbb{U}_0(J)$ is dense in $\mathbb{U}^p(\omega_0, J)$ when $p < \infty$.

Next we consider the case $1 < p < \infty$. Again we let $A \in \mathbb{U}_0(J)$. By the first inequality in (2.9) and Hölder’s inequality we get

$$|g(j)\omega_2(j)|^p \leq \left(\sum_k |a(j,k)f(k)|\omega_0(j,k)\omega_1(k)\right)^p \leq \|f\|_{\ell^p(\omega_1)} \|\omega_0(j,k)\|_p,$$

which gives

$$\|g\|_{\ell^p(\omega_2)} \leq \|A\|_{\mathbb{U}^p(\omega_0, J)} \|f\|_{\ell^p(\omega_1)},$$

by using similar arguments as in the first part of the proof. This proves the continuity extension follows in this case.

Finally, if $p = \infty$, then it suffices to prove (2.8) when $f \in \ell_0(J)$ and $A \in \mathbb{U}_0^\infty(\omega_0, J)$, since $\ell_0(J)$ is dense in $\ell^1(\omega_1)(J)$. The norm estimate (2.8) follows by similar arguments as in the first part of the proof, and is left for the reader. The proof is complete. \qed

By Theorem 2.1, Proposition 2.3, and the fact that $\ell^p(\omega)(J)$ increases with $p \in (0, \infty]$, we get the following.
Theorem 2.4. Let \( \omega_l, \ l = 1, 2 \) be weights on \( J \) and \( \omega_0 \) be a weight on \( J \times J \) such that (0.2) holds. Also let \( p \in (0, 2] \), and let \( A \in \mathbb{U}^p(\omega_0, J) \). Then \( A \in \mathcal{J}_p(\ell^2(\omega_1)(J), \ell^2(\omega_2)(J)) \), and

\[
\|A\|_{\mathcal{J}_p(\ell^2(\omega_1)(J), \ell^2(\omega_2)(J))} \leq \|A\|_{\mathbb{U}^p(\omega_0, J)}. \tag{2.10}
\]

Proof. We may assume that equality is attained in (0.2), and that \( p > j \) respectively, for every \( \vartheta \) and \( p \) between the cases with equality in norms.

First assume that \( p = 2/N \) for some integer \( N \geq 3 \), and let \( A \in \mathbb{U}^{2/N}(\omega_0, J) \). Also let \( \vartheta_1(j, k) = \omega_2(j) \), \( \vartheta_2(j, k) = 1 \), \( j = 2, \ldots, N-1 \) and \( \vartheta_N(j, k) = \omega_1(k) \). By Theorem 2.1 we have

\[
A = A_1 \circ \cdots \circ A_N,
\]

where \( A_j \in \mathbb{U}^2(\vartheta_j, J) \) satisfies \( \|A_m\|_{\mathbb{U}^2(\vartheta_m, J)} \leq 1 \), for every \( m = 1, \ldots, N \).

By (1.33) we get

\[
\|A\|_{\mathcal{J}_p(\ell^2(\omega_1)(J), \ell^2(\omega_2)(J))} \leq \|A_1\|_{\mathcal{J}_p(\ell^2(\omega_1)(J))} \|A_N\|_{\mathcal{J}_p(\ell^2(\omega_1)(J))} \prod_{m=2}^{N-1} \|A_m\|_{\mathcal{J}_p(\ell^2(\omega_1)(J))}^N = \prod_{m=2}^{N-1} \|A_m\|_{\mathbb{U}^2(\vartheta_m, J)} \leq 1,
\]

and the result follows in the case \( p = 2/N \) where \( N \geq 3 \) is an integer.

The result is therefore true when \( p = 2/N \) for some integer \( N \geq 3 \), and when \( p = 2 \). For general \( p \), the result now follows by interpolation between the cases \( p = 2 \) and \( p = 2/N \), where \( N \geq 3 \) is chosen such that \( p > 2/N \). The proof is complete. \( \square \)

In what follows it is convenient to use the following convention. Let \( p = (p_1, \ldots, p_n) \in [0, \infty]^n \), \( q = (q_1, \ldots, q_n) \in [0, \infty]^n \) and \( t \in [-\infty, \infty] \). Then we set

\[
p \leq q \quad \text{and} \quad p \leq t \quad \text{when} \quad p_j \leq q_j \quad \text{and} \quad p_j \leq t,
\]

respectively, for every \( j = 1, \ldots, n \), and

\[
p = q \quad \text{and} \quad p = t \quad \text{when} \quad p_j = q_j \quad \text{and} \quad p_j = t,
\]

respectively, for every \( j = 1, \ldots, n \). We also let

\[
p \pm q = (p_1 \pm q_1, \ldots, p_n \pm q_n) \quad \text{and} \quad p \pm t = (p_1 \pm t, \ldots, p_n \pm t),
\]

provided the right-hand sides are well-defined and belongs to \([-\infty, \infty]^n\). Finally we set \( 1/0 = \infty \), \( 1/\infty = 0 \) and \( 1/p = 1/p_1, \ldots, 1/p_n \).

**Theorem 0.1.** Let \( J = \mathbb{H}^d \) for some \( T \in \text{GL}(d, \mathbb{R}) \), \( \omega_1 \) be weights on \( J, \ l = 1, 2 \), and \( \omega_0 \) be a weight on \( J \times J \) such that (0.2) holds.
Also let \( I = (I_1, \ldots, I_n) \) be a linear split of \( J \), \( p_1, p_2 \in (0, \infty]^n \), and \( p, q \in (0, \infty] \) be such that
\[
\frac{1}{p_2} - \frac{1}{p_1} = \frac{1}{p} + \min \left( 0, \frac{1}{q} - 1 \right), \quad q \leq p \quad \text{and} \quad q \leq \nu(p_2), \quad (0.3)
\]
and let \( A \in \mathbb{U}^p_q(\omega_0, J) \). Then \( A \) in \( (0.1) \) is uniquely extendable to a continuous map from \( \ell^p_{(\omega_1)}(I) \) to \( \ell^p_{(\omega_2)}(I) \), and
\[
\|A\|_{\ell^p_{(\omega_1)}(I) \rightarrow \ell^p_{(\omega_2)}(I)} \leq \|A\|_{\mathbb{U}^p_q(\omega_0, J)}. \quad (0.4)
\]

We note that \((0.4)\) is the same as
\[
\|Af\|_{\ell^p_{(\omega_2)}(I)} \leq \|A\|_{\mathbb{U}^p_q(\omega_0, J)} \|f\|_{\ell^p_{(\omega_1)}(I)} \quad (2.11)
\]
when \( f \in \ell^p_{(\omega_1)}(I) \).

**Proof.** The case \( p = q \) follows immediately from Proposition 2.3 and then the case \( q \leq p \leq 1 \) follows from the fact that \( \mathbb{U}^p_q(J) \subseteq \mathbb{U}^p_p(J) \).

In the other cases we first prove the result when \( \mathbb{U}^p_q(J) \) and \( \ell^p_{(\omega_1)}(J) \) are replaced by \( \mathbb{U}^p_q(\omega_0, J) \) and \( \ell^p_{(\omega_1)}(\omega_0, J) \), respectively.

First we consider the case when \( p = \infty \) and \( q \leq 1 \), and then we only prove the result when \( n = 2 \) and \( T \) is the identity operator, which implies that \( I_1 = \mathbb{Z}^{d_1} \) and \( I_2 = \mathbb{Z}^{d_2} \), where \( d_1 + d_2 = d \). The general situation follows by similar arguments and induction, and is left for the reader.

Let \( j = (j_1, j_2) \) and \( k = (k_1, k_2) \), \( j_l, k_l \in \mathbb{Z}^{d_l} \), \( f \in \ell^p_{(\omega_1)} \) and let \( A \in \mathbb{U}_0(J) \). By \((0.3)\) it follows that \( p_1 = p_2 = (r, s) \), where \( r, s \geq q \).

We set
\[
c(k) = |x(k)\omega_1(k)|, \quad g = Af, \quad c_0(k_2) = \left( \sum_{k_1} c(k_1, k_2)^r \right)^{1/r},
\]
and
\[
h_0(j_2) = \begin{cases} \|h(\cdot, j_2)\|_{\nu(I_1)}, & r \leq 1, \\ \|h(\cdot, j_2)\|_{\nu(I_1)}, & r > 1, \end{cases}
\]
where \( h = h_{A,\infty,\omega} \) is the same as in Definition 1.11 with \( \omega = \omega_0 \) and \( f \in \ell^p_{(\omega_1)}(I) \). Furthermore,
\[
|g(j)\omega_2(j)| \leq \sum_k |a(j, k)\omega_0(j, k)| c(k) = \sum_k |a(j, j - k)\omega_0(j, j - k)| c(j - k) \leq \sum_k h(k)c(j - k) = (h * c)(j).
\]

We need to consider separate cases.
First we consider the case when \( r = s \leq 1 \). Then \( r = s \geq q \), and the fact that \( q \leq r \leq 1 \) gives

\[
\|Af\|_{\ell^r_{(\omega_2)}(J)} = \left( \sum_j |g(j)\omega_2(j)|^r \right)^{1/r} \leq \left( \sum_j (h \ast c(j))^r \right)^{1/r} \leq \|h\|_{\ell^r} \|c\|_{\ell^r} \leq \|h\|_{\ell^r} \|c\|_{\ell^r} = \|A\|_{U_{(\omega_0)}} \|f\|_{\ell^r_{(\omega_2)}(J)},
\]

which is the desired estimate in this case.

Next we consider the case when \( r \leq 1 \) and \( s = \infty \). We have

\[
\left( \sum_j |g(j)\omega_2(j)|^r \right)^{1/r} \leq \left( \sum_j (h \ast c(j_1, j_2))^r \right)^{1/r} \leq (h_0 \ast c_0)(j_2). \tag{2.12}
\]

By applying the \( \ell^\infty(I_2) \) norm we obtain

\[
\|Af\|_{\ell^r_{(\omega_2)}(J)} = \text{ess sup}_{j_2} \left( \sum_j |f(j)\omega_2(j)|^r \right) \leq \|h_0 \ast c_0\|_{\ell^\infty} \leq \|h\|_{\ell^r} \|c_0\|_{\ell^\infty} = \|A\|_{U_{(\omega_0)}} \|f\|_{\ell^{p_1}_{(\omega_1)}(J)}.
\]

This gives (2.11) in this case.

In the next case we let \( r = \infty \) and \( s = q \). By (2.12) and the fact that \( s \leq 1 \) we get

\[
\|Af\|_{\ell^r_{(\omega_2)}(J)} = \left( \sum_{j_2} \text{ess sup}_{j_1} (|g(j)\omega_2(j)|)^s \right)^{1/s} \leq \|h_0\|_{\ell^1} \|c_0\|_{\ell^s} \leq \|h\|_{\ell^1} \|c_0\|_{\ell^s} = \|A\|_{U_{(\omega_0)}} \|f\|_{\ell^{p_1}_{(\omega_1)}(J)}.
\]

In the next case we let \( r = s = \infty \). Then

\[
\|Af\|_{\ell^\infty_{(\omega_2)}(J)} = \text{ess sup}_{j} |g(j)\omega_2(j)| \leq \|h \ast c\|_{\ell^\infty} \leq \|h\|_{\ell^\infty} \|c\|_{\ell^\infty} = \|A\|_{U_{(\omega_0)}} \|f\|_{\ell^\infty_{(\omega_2)}(J)},
\]

which is desired estimate in this case.

The estimate (2.11) with \( A \in U_{0,\omega}^{p,q}(\omega, J) \) and \( f \in \ell^{p_1}_{(\omega_1)} \) now follows in the case when \( p = \infty \) and \( q \leq 1 \) from these four cases and interpolation, and then the case \( p = \infty \) and \( 1 < q < \infty \) follows by multi-linear interpolation between the cases \( (p, q) = (\infty, 1) \) and \( (p, q) = (\infty, \infty) \).

Finally for general \( p \) and \( q \), (2.11) now follows by interpolation between the cases \( p = 1 \) and \( p = \infty \).
It remains to prove that similar facts hold after the assumptions $A \in \mathbb{U}^p_0(\omega, J)$ and $f \in \ell^p_{(\omega_1)}$ have been replaced by $A \in \mathbb{U}^p(\omega, J)$ and $f \in \ell^p_{(\omega_1)}$.

First assume that $A \in \mathbb{U}_0(J)$, and let $f \in \ell^p_{(\omega_1)}$. Then $Af$ is uniquely defined, and (2.11) must hold due to the first part of the proof, since $Af = Af_0$, for some $f_0 \in \ell_0(J)$ such that
\[
\|Af_0\|_{\ell^p_{(\omega_1)}} \leq \|Af\|_{\ell^p_{(\omega_1)}}.
\]
The result now follows in the case $p < \infty$ by the fact that $\mathbb{U}_0(J)$ is dense in $\mathbb{U}^p(\omega, J)$.

It remains to consider the case $p = \infty$. First assume that $A = A_+ \in \mathbb{U}^p(\omega_0, J)$ with matrix elements $a(j, k)$, and $f = f_+ \in \ell^p_{(\omega_1)}(I)$. Let $\Omega_N$, $N \geq 1$, be an increasing family of finite subsets of $J \times J$ such that their union equals $J \times J$, and let $A_N$ let $A_N = (a_N(j, k))_{j,k \in J} \in \mathbb{U}_0(J)$ be such that $a_N(j, k) = a(j, k)$ when $(j, k) \in \Omega_N$, and $a_N(j, k) = 0$ otherwise. Then $A_N$ is a sequence of matrices such that $0 \leq a_N(j, k)$ increases to $a(j, k)$ as $N \to \infty$, for every fixed $(j, k)$. Also let
\[
g_N(j) \equiv \sum_{k \in J} a(j, k)f(k).
\]
By (2.11) and the fact that we have
\[
\|g_N\|_{\ell^p_{(\omega_2)}(I)} \leq \|A_N\|_{\mathbb{U}^p(\omega_0, J)}\|f\|_{\ell^p_{(\omega_1)}(I)} \leq \|A\|_{\mathbb{U}^p(\omega_0, J)}\|f\|_{\ell^p_{(\omega_1)}(I)}.
\]
Since $g_N(j)$ is increasing, it follows from the latter inequality and Beppo-Levis theorem that $g_N(j)$ has a limit $g(j)$ as $N$ tends to the infinity, and
\[
\|g\|_{\ell^p_{(\omega_2)}(I)} \leq \|A\|_{\mathbb{U}^p(\omega_0, J)}\|f\|_{\ell^p_{(\omega_1)}(I)}.
\]
Furthermore, since
\[
0 \leq \sum_{k \in J} a(j, k)f(k) \leq \sum_{k \in J} b(j, k)f(k), \quad \text{when} \quad 0 \leq a(j, k) \leq b(j, k), \quad j, k \in J,
\]
it follows by straight-forward computations that $g(j)$ is independent of the choice of sequences $A_N \mathbb{U}_0(J)$, $N \geq 1$ such that $0 \leq a_N(j, k)$ increases to $a(j, k)$ as $N \to \infty$. This proves the result when in addition $A = A_+$ and $f = f_+$.

For general $A \in \mathbb{U}^p(\omega_0, J)$ and $f \in \ell^p_{(\omega_2)}(I)$, we write
\[
A = (A_1 - A_2) + i(A_3 - A_4) \quad \text{and} \quad f = (f_1 - f_2) + i(f_3 - f_4),
\]
where $A_1 = (\text{Re}(A))_+$, $A_2 = (\text{Re}(A))_-$, $A_3 = (\text{Im}(A))_+$ and $A_4 = (\text{Im}(A))_-$, and similarly for $f_k$, $k = 1, \ldots, 4$. Then $Af$ is a linear combination of elements on the form $A_j f_k$, and the estimate
\[
\|A_j f_k\|_{\ell^p_{(\omega_2)}(I)} \leq \|A_j\|\|f_k\|_{\ell^p_{(\omega_2)}(I)} \leq \|A\|\|f_k\|_{\ell^p_{(\omega_2)}(I)} \leq \|A\|\|f_k\|_{\ell^p_{(\omega_1)}(I)} \leq \|A\|\|1\|\|f_k\|_{\ell^p_{(\omega_1)}(I)} = \|A\|\|f_k\|_{\ell^p_{(\omega_1)}(I)}
\]
for each $j, k$.
It now follows from the latter estimate and using sequences $A_N$ as in the previous case that $Af$ is uniquely defined and belongs to $ℓ^2(ω_2)(I)$. Furthermore, since (2.11) is true when $A ∈ U_0(J)$, it also follows that this estimate remains valid when $A ∈ U^{p,q}(ω, J)$. This completes the proof. □

Remark 2.5. Let $I$, $J$, $p$, $q$, $p_1$, $ω_0$ and $ω_I$, $l = 1, 2$ be the same as in Theorem 0.1. Furthermore, let $L = (L_1, \ldots, L_m)$ be an other linear split of $J$, and let $T$ be a linear map on $R^d × R^d$ such that

$$T(j_1, \ldots, j_m, k_1, \ldots, k_m) = (α_1j_1, \ldots, α_mj_m, β_1k_1 + γ_1k_1, \ldots, β_mk_m + γ_mk_m),$$

for some choices of $α_l, β_l, γ_l ∈ \{-1, 1\}$ which are independent of $j_1, \ldots, j_m, k_1, \ldots, k_m$. Then let $U^{p,q}(ω_0, J)$ be the set of all matrices $(a(j, k))_{j,k∈J}$ such that

$$∥A∥_{U^{p,q}(ω_0, J)} ≲ \left(\sum_{k∈J} \left(\sum_{j∈J} |a(T(j, k))ω_0(T(j, k))|^p \right)^{q/p} \right)^{1/q}$$

is finite. Then it follows from Theorem 0.1 and its proof that if $A ∈ U^{p,q}(ω_0, J)$, then $A$ in (2.11) is uniquely extendable to a continuous map from $ℓ^{p_1}(ω_1)(I)$ to $ℓ^{p_2}(ω_2)(I)$, and

$$∥A∥_{ℓ^{p_1}(ω_1)(I) → ℓ^{p_2}(ω_2)(I)} \leq ∥A∥_{U^{p,q}(ω_0, J)}. \quad (0.4)$$

3. CONTINUITY AND SCHATTEN-VO 
NEUMANN PROPERTIES FOR 
PSEUDO-DIFFERENTIAL OPERATORS

In this section we deduce continuity and Schatten-von Neumann results for pseudo-differential operators with symbols in modulation spaces. In particular we extend results in [21, 23, 41, 42, 45] to include Schatten parameters less than one. Furthermore, in contrast to [21, 23, 41, 42, 45], the continuity results also include cases where the involved Lebesgue parameters in the definition of modulation spaces are allowed to be smaller than one. We start with the following result on continuity, and which extends Theorem 0.2 in the introduction.

Theorem 0.2. Let $σ ∈ S_{2d}$, $ω_k ∈ \mathcal{P}_E(R^{2d})$, $k = 1, 2$, and $ω_0 ∈ \mathcal{P}_E(R^{2d} ⨁ R^{2d})$ be such that

$$\frac{ω_2(x, ξ)}{ω_1(y, η)} ≲ ω_0((1 - t)x + ty, tξ + (1 - t)η, ξ - η, y - x)$$

Also let $t ∈ R$, $p_1, p_2 ∈ (0, ∞)^{2d}$, $p, q ∈ (0, ∞]$ be such that (0.3)′ hold, and let $a ∈ M^{p,q}(ω_0)(R^{2d})$. Then $Op_l(a)$ from $S_{1/2}(R^d)$ to $S_{1/2}′(R^d)$
extends uniquely to a continuous map from $M_{\sigma, (\omega_1)}^{p_1}(\mathbb{R}^d)$ to $M_{\sigma, (\omega_2)}^{p_2}(\mathbb{R}^d)$, and

\[
\| \text{Op}_t(a) \|_{M_{\sigma, (\omega_1)}^{p_1}} \leq \| a \|_{M_{\sigma, (\omega_0)}^{p_2}}.
\]

(0.5)

We need the following lemma for the proof.

**Lemma 3.1.** Let $\sigma, \omega_0$ and $\omega_k$, $p_k$, $k = 1, 2$, be the same as in Theorem 7.2 with $t = 0$, and let $\omega$, $\vartheta$ and $\vartheta_0$ be chosen such that (1.35) holds. Also let $r > 0$ be chosen such that $\omega$ and $\omega_0$ are moderate with respect to $\nu_{r/2,4}$, and let $\Phi_{1,k}$ and $\Phi_{2,k}$ be as in Proposition 1.13 and such that $\Phi = \phi \otimes \phi$, where $\phi$ is the standard Gaussian on $\mathbb{R}^d$. If $p, q \in (0, \infty]$, $a \in M_{\omega_0}^{p,q}(\mathbb{R}^d)$, then $\text{Op}(a)$ as a map from $S_{1/2}(\mathbb{R}^d)$ to $S'_{1/2}(\mathbb{R}^d)$, is given by

\[
\text{Op}(a) = D_\phi \circ A \circ C_\phi,
\]

where $b(j, \tau(k))$ is given by (1.32) and $\tau(j, k) = (j, -k)$. If instead $a \in S_{1/2}(\mathbb{R}^d)$, then (3.1) still holds as an operator from $M_{\sigma, (\omega_1)}^{p_1}(\mathbb{R}^d)$ to $S_{1/2}(\mathbb{R}^d)$.

**Proof.** Let $a \in M_{\omega_0}^{p,q}(\mathbb{R}^d)$ and $f \in S_{1/2}(\mathbb{R}^d)$, or let $a \in S_{1/2}(\mathbb{R}^d)$ and $f \in M_{\sigma, (\omega_1)}^{p_1}(\mathbb{R}^d)$. We note that $\phi \in M_{\omega_0}^{1}$ is invariant under Fourier transform and positive everywhere.

By letting $J = \mathbb{Z}^{2d}$, and using the same notations as in Propositions 1.13 and 1.14 and their proofs, we obtain

\[
(\text{Op}(a)f)(x) = \sum_{j_1, j_2 \in J} b(j_1, j_2) \int \Phi_{j_1, j_2}(x, y) f(y) \, dy
\]

\[
= \sum_{j_1, j_2 \in J} b(j_1, j_2)(\nu_{x, \vartheta})(\nu_{y, \vartheta_0})\phi_{j_1}(x)
\]

\[
= \sum_{j_1, j_2 \in J} b(\tau(j_1, j_2))(\nu_{x, \vartheta})(\nu_{y, \vartheta_0})\phi_{j_1}(x).
\]

By straightforward computations, it follows that the right-hand side is equal to $((D_\phi \circ A \circ C_\phi)f)(x)$, and the result follows.

**Proof of Theorem 7.2** By Proposition 1.7 in [46] and its proof, it suffices to prove the result for $t = 0$ or for $t = 1/2$. In the first cases we shall prove the result for $t = 0$, and in the last cases for $t = 1/2$.

Let $a \in M_{\omega_0}^{p,q}(\mathbb{R}^d)$ and $f \in S_{1/2}(\mathbb{R}^d)$, or when $a \in S_{1/2}(\mathbb{R}^d)$ and $f \in M_{\sigma, (\omega_1)}^{p_1}(\mathbb{R}^d)$. By Theorems 1.10 and 0.1 Proposition 1.14 and Lemma 3.1 we obtain

\[
\| \text{Op}_t(a) \|_{M_{\sigma, (\omega_2)}^{p_2}} \leq \| a \|_{M_{\sigma, (\omega_0)}^{p_1}} \| f \|_{M_{\sigma, (\omega_1)}^{p_2}}.
\]

28
This gives
\[ \| \text{Op}_t(a)f \|_{M^{p_2}_{\sigma,(\omega_t)}} \lesssim \| a \|_{M^{p,q}_{(\omega_0)}} \| f \|_{M^{p_1}_{\sigma,(\omega_1)}}. \]  
and the result when \( a \in \mathcal{S}_{1/2}(\mathbb{R}^d) \) or \( f \in \mathcal{S}_{1/2}(\mathbb{R}^d) \).

If \( p < \infty \), then \( q < \infty \), and the result follows from the fact that \((0.5)'\) holds when \( a \in \mathcal{S}_{1/2}(\mathbb{R}^d) \), and \( f \in M^{p_1}_{\sigma,(\omega_1)}(\mathbb{R}^d) \), and the fact \( \mathcal{S}_{1/2}(\mathbb{R}^d) \) is dense in \( M^{p,q}_{(\omega_0)}(\mathbb{R}^d) \).

Next assume that \( p = \infty \) and \( \max p_1 < \infty \). Then the result now follows from the fact that \((0.5)'\) holds when \( a \in M^{p,q}_{(\omega_0)}(\mathbb{R}^d) \), and \( f \in \mathcal{S}_{1/2}(\mathbb{R}^d) \), and the fact \( \mathcal{S}_{1/2}(\mathbb{R}^d) \) is dense in \( M^{p_1}_{\sigma,(\omega_1)}(\mathbb{R}^d) \).

If instead \( p = \infty \) and \( \max p_1 = \infty \), then \( q \leq 1 \), in view of \((0.3)\). Let \( t = 1/2 \), and assume that \( q = 1 \) and \( \min p_1 \geq 1 \). Then \( \min p_2 = p_1 \), and \((0.5)'\) shows that the definition of \( \text{Op}^w(a) \) when \( a \in M^{(1,\omega_1)}_\sigma(\mathbb{R}^d) \) extends to a continuous map from \( M^{p_1}_{\sigma,(\omega_1)}(\mathbb{R}^d) \) to \( M^{p_2}_{\sigma,(\omega_2)}(\mathbb{R}^d) \).

Now let \( f \in M^{p_1}_{\sigma,(\omega_1)}(\mathbb{R}^d) \), \( g \in \mathcal{S}_{1/2}(\mathbb{R}^d) \), and let \( a_j \in \mathcal{S}_{1/2}(\mathbb{R}^d) \), \( j = 1, 2, \ldots \), be such that \( a_j \) converges narrowly to \( a \) when \( j \) tends to \( \infty \). Then
\[
(\text{Op}^w(a)f, g) = (2\pi)^{-d/2}(a, W_{g,f}) = \lim_{j \to \infty} (2\pi)^{-d/2}(a_j, W_{g,f}) = \lim_{j \to \infty} (\text{Op}^w(a_j)f, g).
\]
Since the right-hand side is uniquely defined, it follows that \( \text{Op}^w(a)f \) is uniquely defined as an element in \( \mathcal{S}'_{1/2}(\mathbb{R}^d) \), and the result follows in this case.

Finally, assume that \( p = \infty \), \( q \leq 1 \) and \( \min p_1 \leq 1 \). Then Proposition \((1.5)\) and the latter case show that \( \text{Op}^w(a)f \) is uniquely defined as an element in \( M^{\infty}_{(\omega_2)}(\mathbb{R}^d) \) when \( f \in M^{p}_{\sigma,(\omega_1)}(\mathbb{R}^d) \), \( p = p_1 \) and \( r = (r_1, \ldots, r_{2d}) \) with \( r_j = \max(1, p_j) \). By Proposition \((1.5)\) \((0.5)'\), and the first part of the proof, it follows that \( \text{Op}^w(a)f \in M^{p}_{\sigma,(\omega_2)}(\mathbb{R}^d) \), which proves the result. \( \square \)

We have also the following result on Schatten-von Neumann properties for pseudo-differential operators.

**Theorem 3.2.** Let \( t \in \mathbb{R} \), \( \omega_k \in \mathcal{P}_E(\mathbb{R}^{2d}) \), \( k = 1, 2 \), and \( \omega_0 \in \mathcal{P}_E(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}) \) be such that
\[
\frac{\omega_2(x, \xi)}{\omega_1(y, \eta)} \asymp \omega_0((1-t)x + ty, t\xi + (1-t)\eta, \xi - \eta, y - x)
\]
Also let \( p, p_j, q, q_j \in (0, \infty] \) be such that
\[
p_1 \leq p, \quad q_1 \min(p, p'), \quad p_2 \geq \max(p, 1), \quad q_2 \geq \max(p, p').
\]
Then
\[
M^{p_1,q_1}_{(\omega_0)}(\mathbb{R}^d) \subseteq s_{t,p}(\omega_1, \omega_2) \subseteq M^{p_2,q_2}_{(\omega_0)}(\mathbb{R}^d) \]  
(3.2)
and
\[ \|a\|_{M^{p_1,q_1}} \lesssim \|a\|_{M^{p_2,q_2}} \lesssim \|a\|_{M^{p_3,q_3}}. \]  
(3.3)

**Proof.** The result is true for \( p \in [1, \infty] \) in view of Theorem A.3 in [15] and Proposition 1.4. Hence it suffices to prove the assertion for \( p \in (0,1). \)

By Proposition 1.7 in [16] and its proof, it again suffices to prove the result for \( t = 0. \)

Let \( t \) and \( t_j, j = 1, 2, \) be the same as in the proof of Theorem 0.2.

By (1.38) and (3.1) we get
\[ \|a\|_{M^{p_1,q_1}} \lesssim \|a\|_{M^{p_2,q_2}} \lesssim \|a\|_{M^{p_3,q_3}}. \]

and the result follows. \( \square \)

Next we show that Theorem 3.2 is optimal with respect to \( p. \) More precisely, we have the following result

**Theorem 3.3.** Let \( t \in \mathbb{R}, \omega_k \in \mathcal{P}_E(\mathbb{R}^{2d}), k = 1, 2, \) and \( \omega_0 \in \mathcal{P}_E(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}) \) be such that
\[ \frac{\omega_2(x, \xi)}{\omega_1(y, \eta)} \lesssim \omega_0((1-t)x + ty, t\xi + (1-t)\eta, \xi - \eta, y - x) \]

Also let \( r \in (0, \infty], p, q \in (0, \infty], \) and assume that
\[ \mathrm{Op}(r(M^{p,q}_{(\omega_0)}(\mathbb{R}^{2d}))) \subseteq J_r(M^{2}_{(\omega_1)}(\mathbb{R}^{d}), M^{2}_{(\omega_2)}(\mathbb{R}^{d})). \]  
(3.4)

Then the following is true:

(1) \( p \leq r \) and \( q \leq \min(2, r); \)

(2) if in addition \( \omega_1, \omega_2 \in \mathcal{P}(\mathbb{R}^{2d}) \) and \( r \geq 2, \) then \( q \leq \min(2, p'). \)

We need some preparations for the proof, and start by considering certain quasi-norm estimates for Wigner distributions. More precisely, certain estimates for Wigner distributions with respect to modulation space norms can be found in [7,11,12,15]. The next result extends [15] Proposition A.4 in the sense of replacing the interval \([1, \infty]\) for the involved Lebesgue exponents by the larger interval \((0, \infty]\). We omit the proof since the arguments are the same as in the proof of [15] Proposition A.4.

**Proposition 3.4.** Let \( t \in \mathbb{R}, \) and let \( p_j, q_j, p, q \in (0, \infty] \) be such that \( p \leq p_j, q_j \leq q, \) for \( j = 1, 2, \) and
\[ \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p} + \frac{1}{q}. \]  
(3.5)

Also let \( \omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^{2d}) \) and \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}) \) be such that
\[ \omega_0((1-t)x + ty, t\xi + (1-t)\eta, \xi - \eta, y - x) \lesssim \omega_1(x, \xi) \omega_2(y, \eta). \]  
(3.6)
Then the map \((f_1, f_2) \mapsto W_{f_1, f_2}\) from \(S_{1/2}'(\mathbb{R}^d) \times S_{1/2}'(\mathbb{R}^d)\) to \(S_{1/2}'(\mathbb{R}^{2d})\) restricts to a continuous mapping from \(M_{\omega_1}^{p_1,q_1}(\mathbb{R}^d) \times M_{\omega_2}^{p_2,q_2}(\mathbb{R}^d)\) to \(M_{\omega_1}^{p,q}(\mathbb{R}^{2d})\), and
\[
\|W_{f_1, f_2}\|_{M_{\omega_1}^{p,q}} \lesssim \|f_1\|_{M_{\omega_1}^{p_1,q_1}} \|f_2\|_{M_{\omega_2}^{p_2,q_2}}
\] (3.7)
when \(f_1, f_2 \in S_{1/2}'(\mathbb{R}^d)\).

We have now the following extension of Corollary 4.2 (1) in [46].

**Corollary 3.5.** Let \(p \in (0, \infty], q \in (2, \infty], t \in \mathbb{R}\), and let \(\omega_2 \in \mathcal{P}_E(\mathbb{R}^{2d})\) and \(\omega_0 \in \mathcal{P}_E(\mathbb{R}^d)\) be such that
\[
\omega_0((1-t)x, t\xi, \xi-x) \lesssim \omega_2(x, \xi).
\]
Then there is an element \(a\) in \(M_{\omega_0}^{p,q}(\mathbb{R}^{2d})\) such that \(\text{Op}_t(a)\) is not continuous from \(S_{1/2}(\mathbb{R}^d)\) to \(M_{\omega_2}^{2,2}(\mathbb{R}^d)\).

**Proof.** Let \(a = W_{f_2, f_1}\), where \(f_1 \in \Sigma_1(\mathbb{R}^d) - 0\) and \(f_2 \in M_{\omega_2}^{q_p}(\mathbb{R}^d) \setminus M_{\omega_2}^{2,2}(\mathbb{R}^{2d})\). Such choices of \(f_2\) are possible in view of Proposition [3.4].

By using the fact that \(\omega_0, \omega_2\) are moderate weights, it follows that (3.6) holds when \(\omega_1(x, \xi) = e^{c(|x|+|\xi|)}\), and the constant \(c > 0\) is chosen large enough. By Proposition [3.4] it follows that \(f_1 \in M_{\omega_1}^{p,q}(\mathbb{R}^d)\). Hence \(a \in M_{\omega_0}^{p,q}(\mathbb{R}^{2d})\) in view of Proposition [3.4].

On the other hand, if \(f \in S_{1/2}(\mathbb{R}^d) - 0\) is chosen such that \(f\) and \(f_1\) are not orthogonal, then
\[
\text{Op}_t(a)f = (f, f_1) \cdot f_2 \in M_{\omega_2}^{q_p}(\mathbb{R}^d) \setminus M_{\omega_2}^{2,2}(\mathbb{R}^{2d}),
\]
and the result follows. \(\square\)

We also need the following lemma. We omit the proof since the result is a special case of Proposition 4.3 in [46]. Here \(\hat{f}\) is defined as \(\hat{f}(x) = f(-x)\) for every function or distribution \(f\) on \(\mathbb{R}^d\).

**Lemma 3.6.** Let \(\omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^{2d})\), \(a \in \mathcal{S}'(\mathbb{R}^{2d})\), and that \(p \in (0, \infty]\). Then
\[
\mathcal{F}_\sigma(s^w_p(\omega_1, \omega_2)) = s^w_p(\omega_1, \omega_2).
\]

**Proof.** We may assume that \(t = 1/2\), and consider first the case when \(1 \leq r\). Let \(\sigma\) be the canonical symplectic form on \(\mathbb{R}^{2d}\), \(\{X_j\}_{j \in J}\) and \(\{Y_k\}_{k \in J}\) be lattices in \(\mathbb{R}^{2d}\) such that \(X_0 = 0\), and let \(\phi \in \Sigma(\mathbb{R}^{2d})\) and \(\alpha, \beta > 0\) be chosen such that \(\{\phi(\cdot - \alpha X_j)e^{-2i\sigma(\cdot, Y_k)}\}_{j,k \in J}\) is a Gabor frame. Also let \(\vartheta(j, k) = \omega_0(\alpha X_j, \beta X_k)\), \(\vartheta(k) = \vartheta(0, k)\) and \(\omega(Y) = \omega(0, Y)\). Furthermore, let \(c \in \ell_j^\infty(J)\) and let
\[
a(X) = \sum_{k \in J} c(k)\phi(X)e^{-2i\sigma(X, Y_k)} = \sum_{j, k \in J} c_0(j, k)\phi(X - X_j)e^{-2i\sigma(X, Y_k)},
\]
when \( c_0(0, k) = c(k) \) and \( c_0(j, k) = 0 \) when \( j \neq 0 \). Then \( a \in M_{(\omega_0)}^{p, \infty} \) for every \( p \in (0, \infty] \). Furthermore, for every \( p \in (0, \infty] \), the equivalences

\[
a \in M_{(\omega_0)}^{p, q} \iff a \in W_{(\omega_0)}^{p, q} \iff c \in \ell_{(\omega)}^q
\]

holds.

Now if \( q > r \), then choose \( c \in \ell_{(\omega)}^q \setminus \ell_{(\omega)}^r \), and it follows from (3.2) and (3.8) that \( a \in M_{(\omega_0)}^{p, q} \setminus s^w_r(\omega_1, \omega_2) \). This shows that \( q \leq r \) when (3.4) holds.

Assume instead that \( p > r \), let \( q \in (0, \infty] \) be arbitrary, choose \( c \in \ell_{(\omega)}^p \setminus \ell_{(\omega)}^r \), and consider

\[
b = \mathcal{F}_\sigma a \in \mathcal{F}_\sigma W_{(\omega_0)}^{q, p} = M_{(\omega_0, \omega)}^{p, q},
\]

where \( \omega_{T, 0}(X, Y) = \omega_0(Y, X) \). Furthermore, by Lemma 3.6 (3.2) and (3.8) it follows that \( b \notin s^w_r(\omega_1, \omega_2) \), where \( \hat{\omega}_2(X) = \omega_2(-X) \). This shows that \( p \leq r \) when (3.4) holds, and the result follows in the case \( r \geq 1 \).

Next assume that \( r < 1 \). If (3.4) holds for some \( q > r \), then it follows by interpolation between the cases (3.4) and

\[
\text{Op}_t(M_{(\omega)}^{2, 2}(R^{2d})) = \mathcal{F}_2(M_{(\omega_1)}^2(R^d), M_{(\omega_2)}^2(R^d))
\]

that (3.4) holds for \( r = 1 \) and some \( q > 1 \). This contradicts the first part of the proof. If instead (3.4) holds for some \( p > r \), then it again follows by interpolation that (3.4) holds for \( r = 1 \) and some \( p > 1 \), which contradicts the first part of the proof. This shows that \( p, q \leq r \) if (3.4) should hold. Furthermore, by Corollary 3.5 it follows that \( q \leq 2 \) when eqrefSchattenModIncl holds, and (1) follows.

It remains to prove (2). By [23, Corollary 3.5] it follows that the result is true for trivial weights in the modulation space norms, and the result is carried over to the case by using lifting properties, established in [24]. The proof is complete.

\[\Box\]

REFERENCES

[1] Birman, Solomyak Estimates for the singular numbers of integral operators (Russian), Usbehi Mat. Nauk. 32, (1977), 17–84.
[2] L. Borup, M. Nielsen Banach frames for multivariate \( \alpha \)-modulation spaces, J. Math. Anal. Appl. 321 (2006), 880–895.
[3] L. Borup, M. Nielsen Frame decomposition of decomposition spaces, J. Fourier Anal. Appl. 13 (2007), 39–70.
[4] J. Chung, S.-Y. Chung, D. Kim Characterizations of the Gelfand-Shilov spaces via Fourier transforms, Proc. Amer. Math. Soc. 124 (1996), 2101–2108.
[5] E. Cordero, K. Gröchenig Time-Frequency Analysis of Localization Operators, J. Funct. Anal. (1) 205 (2003), 107–131.
[6] E. Cordero, S. Pilipović, L. Rodino, N. Teofanov Localization operators and exponential weights for modulation spaces, Mediterr. J. Math. 2 (2005), 381–394.
[7] E. Cordero, S. Pilipović, L. Rodino, N. Teofanov *Quasianalytic Gelfand-Shilov spaces with applications to localization operators*, Rocky Mt. J. Math. 40 (2010), 1123–1147.

[8] M. Dimassi, J. Sjöstrand *Spectral Asymptotics in the Semi-Classical Limit*, vol 268, London Math. Soc. Lecture Note Series, Cambridge University Press, Cambridge, New York, Melbourne, Madrid, 1999.

[9] H. G. Feichtinger *Modulation spaces on locally compact abelian groups. Technical report*, University of Vienna, Vienna, 1983; also in: M. Krishna, R. Radha, S. Thangavelu (Eds) Wavelets and their applications, Allied Publishers Private Limited, New Delhi Mumbai Kolkata Chennai Hapur Ahmedabad Bangalore Hyderabad Lucknow, 2003, pp. 99–140.

[10] H. G. Feichtinger *Gabor frames and time-frequency analysis of distributions*, J. Funct. Anal. (2) 146 (1997), 464–495.

[11] H. G. Feichtinger *Modulation spaces: Looking back and ahead*, Sampl. Theory Signal Image Process. 5 (2006), 109–140.

[12] H. G. Feichtinger, K. H. Gröchenig *Banach spaces related to integrable group representations and their atomic decompositions, I*, J. Funct. Anal., 86 (1989), 307–340.

[13] H. G. Feichtinger, K. H. Gröchenig *Banach spaces related to integrable group representations and their atomic decompositions, II*, Monatsh. Math., 108 (1989), 129–148.

[14] H. G. Feichtinger, K. H. Gröchenig, D. Walnut *Wilson bases and modulation spaces*, Math. Nach. 155 (1992), 7–17.

[15] G. B. Folland *Harmonic analysis in phase space*, Princeton U. P., Princeton, 1989.

[16] Y. V. Galperin, S. Samarah *Time-frequency analysis on modulation spaces $M^p_q$, $0 < p, q \leq \infty$*, Appl. Comput. Harmon. Anal. 16 (2004), 1–18.

[17] I. M. Gelfand, G. E. Shilov *Generalized functions, I–III*, Academic Press, New York London, 1968.

[18] T. Gramchev, S. Pilipović, L. Rodino *Classes of degenerate elliptic operators in Gelfand-Shilov spaces in: L. Rodino, M. W. Wong (Eds) New developments in pseudo-differential operators, Operator Theory: Advances and Applications 189*, Birkhäuser Verlag, Basel 2009, pp. 15–31.

[19] P. Gröbner Banachräume Glatter Funktionen und Zerlegungsmethoden, Thesis, University of Vienna, Vienna, 1992.

[20] K. H. Gröchenig *Describing functions: atomic decompositions versus frames*, Monatsh. Math.,112 (1991), 1–42.

[21] K. H. Gröchenig *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

[22] K. Gröchenig *Weight functions in time-frequency analysis* in: L. Rodino, M. W. Wong (Eds) Pseudodifferential Operators: Partial Differential Equations and Time-Frequency Analysis, Fields Institute Comm., 52 2007, pp. 343–366.

[23] K. H. Gröchenig and C. Heil *Modulation spaces and pseudo-differential operators*, Integral Equations Operator Theory (4) 34 (1999), 439–457.

[24] K. H. Gröchenig and J. Toft, *Isomorphism properties of Toeplitz operators and pseudo-differential operators between modulation spaces*, J. Anal. Math. 114 (2011), 255–283.

[25] K. Gröchenig, G. Zimmermann *Spaces of test functions via the STFT* J. Funct. Spaces Appl. 2 (2004), 25–53.

[26] J. Han, B. Wang *α-modulation spaces (I)*, preprint, arXiv:1108.0460v2.

[27] L. Hörmander *The Analysis of Linear Partial Differential Operators*, vol I–III, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1983, 1985.
28] K. Johansson, S. Pilipovic, N. Teofanov, J. Toft Micro-local analysis in some spaces of ultradistributions, Publ. Inst. Math. (Beograd) 92 (2012), 1–24.
29] Z. Lozanov-Crvenković, D. Perišić, M. Tasković Gelfand-Shilov spaces structural and kernel theorems, (preprint), arXiv:0706.2268v2.
30] S. Pilipovic Generalization of Zemanian spaces of generalized functions which have orthonormal series expansions, SIAM J. Math. Anal. 17 (1986), 477D484.
31] S. Pilipović, N. Teofanov Wilson Bases and Ultramodulation Spaces, Math. Nachr. 242 (2002), 179–196.
32] S. Pilipović, N. Teofanov On a symbol class of Elliptic Pseudodifferential Operators, Bull. Acad. Serbe Sci. Arts 27 (2002), 57–68.
33] H. Rauhut Wiener amalgam spaces with respect to quasi-Banach spaces, Colloq. Math. 109 (2007), 345–362.
34] H. Rauhut Coorbit space theory for quasi-Banach spaces, Studia Math. 180 (2007), 237–253.
35] M. Ruzhansky, M. Sugimoto, N. Tominaga, J. Toft Changes of variables in modulation and Wiener amalgam spaces, Math. Nachr. 284 (2011), 2078–2092.
36] R. Schatten Norm ideals of completely continuous operators, Springer, Berlin, 1960.
37] B. W. Schulze, N. N. Tarkhanov Pseudodifferential operators with operator-valued symbols. Israel Math. Conf. Proc. 16, 2003.
38] B. Simon Trace ideals and their applications, I, London Math. Soc. Lecture Note Series, Cambridge University Press, Cambridge London New York Mel bourne, 1979.
39] J. Sjöstrand An algebra of pseudodifferential operators, Math. Res. L. 1 (1994), 185–192.
40] M. Sugimoto, N. Tomita The dilation property of modulation spaces and their inclusion relation with Besov Spaces, J. Funct. Anal. (1), 248 (2007), 79–106.
41] J. Toft Continuity properties for modulation spaces with applications to pseudo-differential calculus, I, J. Funct. Anal. (2), 207 (2004), 399–429.
42] J. Toft Continuity and Schatten properties for pseudo-differential operators on modulation spaces in: J. Toft, M. W. Wong, H. Zhu (eds) Modern Trends in Pseudo-Differential Operators, Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, 2007, 173–206.
43] J. Toft Pseudo-differential operators with smooth symbols on modulation spaces, Cubo, 11 (2009), 87–107.
44] J. Toft The Bargmann transform on modulation and Gelfand-Shilov spaces, with applications to Toeplitz and pseudo-differential operators, J. Pseudo-Differ. Oper. Appl. 3 (2012), 145–227.
45] J. Toft Multiplication properties in Gelfand-Shilov pseudo-differential calculus in: S. Molahajlo, S. Pilipović, J. Toft, M. W. Wong (eds) Pseudo-Differential Operators, Generalized Functions and Asymptotics, Operator Theory: Advances and Applications Vol 231, Birkhäuser, Basel Heidelberg NewYork Dordrecht London, 2013, pp. 117–172.
46] J. Toft Gabor analysis for a broad class of quasi-Banach modulation spaces, (preprint), arxiv1404.0758.
47] J. Toft, P. Boggiatto Schatten classes for Toeplitz operators with Hilbert space windows on modulation spaces, Adv. Math. 217 (2008), 305–333.
48] J. Toft, A. Khrennikov, B. Nilsson and S. Nordebo Decompositions of Gelfand-Shilov kernels into kernels of similar class, J. Math. Anal. Appl. 396 (2012), 315–322.
[49] J. Toft, P. Wahlberg *Embeddings of α-modulation spaces*, Pliska Stud. Math. Bulgar. 21 (2012), 25–46.

[50] B. Wang, C. Huang *Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations*, J. Differential Equations, 239, 2007, 213–250.

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