Boltzmann Equation with Cutoff Rutherford Scattering Cross Section Near Maxwellian

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Abstract

The well-known Rutherford differential cross section, denoted by $d\Omega/d\sigma$, corresponds to a two body interaction with Coulomb potential. This leads to the logarithmic divergence of the momentum transfer (or the transport cross section), which is described by

$$\int_{S^2} (1 - \cos \theta) \frac{d\Omega}{d\sigma} d\sigma \sim \int_0^{\pi} \theta^{-1} d\theta.$$

Here $\theta$ is the deviation angle in the scattering event. Due to a screening effect, physically one can assume that $\theta_{\text{min}}$ is the order of magnitude of the smallest angles for which the scattering can still be regarded as Coulomb scattering. Under ad hoc cutoff $\theta \geq \theta_{\text{min}}$ on the deviation angle, L. D. Landau derived a new equation in [17] for the weakly interacting gas which is now referred to as the Fokker-Planck-Landau or Landau equation. In the present work, we establish a unified framework to justify Landau’s formal derivation in [17] and the so-called Landau approximation problem proposed in [5] in the close-to-equilibrium regime. Precisely, (i) we prove global well-posedness of the Boltzmann equation with cutoff Rutherford cross section which is perhaps the most singular kernel both in relative velocity and deviation angle; (ii) we prove a global-in-time error estimate between solutions to the Boltzmann and Landau equations with logarithm accuracy, which is consistent with the famous Coulomb logarithm. Key ingredients in the proofs of these results include a complete coercivity estimate of the linearized Boltzmann collision operator, a uniform spectral gap estimate and a novel linear-quasilinear energy method.
1. Introduction

The present work aims at the mathematical justification of Landau’s derivation of the Landau equation and the Landau’s approximation problem from the Boltzmann equation with angular cutoff Rutherford scattering cross section. These problems have a long history and we first recall the relevant physical background.

1.1. Review of Landau’s Derivation

In 1936, Landau [17] derived an effective equation for weakly interaction by Coulomb field in plasma physics. Loosely speaking, he derived the Landau equation from the Boltzmann equation with cutoff Rutherford scattering cross section.

1.1.1. Boltzmann Equation with Rutherford Cross Section  The typical Boltzmann equation is written as

\[ \partial_t f + v \cdot \nabla_x f = Q(f, f). \]

Here \( Q \) is the Boltzmann collision operator defined by

\[ Q(f, f)(v) := \int_{\mathbb{R}^3 \times S^2} |v - v_*| \frac{d\Omega}{d\sigma} \left( f(v'_*) f(v') - f(v'_*) f(v) \right) dv_* d\sigma, \]

where \( \frac{d\Omega}{d\sigma} \) is the differential scattering cross section determined by the potential function \( \phi \) for particles, and \( v', v'_* \) are given by

\[ v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma, \quad \sigma \in S^2. \]

In the scattering event between two electrons governed by the Coulomb potential,

\[ \phi(r) = \frac{e^2}{4\pi \varepsilon_0 r}, \]

the deviation angle \( \theta \) of relative velocity is determined through (see [21] for instance)

\[ \tan \frac{\theta}{2} = \frac{e^2}{2\pi \varepsilon_0 m |v - v_*|^2 b}, \quad (1.1) \]
where $\varepsilon_0$ is the permittivity of vacuum, $m$ is the mass of an electron with its charge $e$, $|v - v_*|$ is the relative velocity before the event and $b$ is the impact parameter which is defined as the distance of closest approach if the trajectory were undeflected. The well-known Rutherford differential cross section is computed as

$$\frac{d\Omega}{d\sigma} := \frac{b}{\sin \theta} \frac{db}{d\theta} = \frac{\left( \frac{e^2}{4\pi \varepsilon_0 m} \right)^2}{|v - v_*|^2 \sin^2(\theta/2)^4}.$$ 

Then the corresponding Boltzmann collision kernel $B$ reads as

$$B(v - v_*, \sigma) = |v - v_*| \frac{d\Omega}{d\sigma} = \frac{\left( \frac{e^2}{4\pi \varepsilon_0 m} \right)^2}{|v - v_*|^3 \sin^4(\theta/2)} = K |v - v_*|^{-3} b(\cos \theta),$$

where

$$K := \left( \frac{e^2}{4\pi \varepsilon_0 m} \right)^2, \quad b(\cos \theta) := \sin^{-4}(\theta/2), \quad \cos \theta := \frac{v - v_*}{|v - v_*|} \cdot \sigma. \quad (1.3)$$

### 1.1.2. Divergence of the Momentum Transfer

One may check that the momentum transfer defined by $\int_{S^2} b(\cos \theta) \sin^2(\theta/2) d\sigma$ is divergent in a logarithmic manner due to the singularity at $\theta = 0$. Indeed,

$$\frac{1}{8\pi} \int_{S^2} b(\cos \theta) \sin^2(\theta/2) d\sigma = \frac{1}{8\pi} \int_0^\pi \int_0^{2\pi} \sin^{-2}(\theta/2) \sin \theta d\theta d\varphi$$

$$= \frac{1}{2} \int_0^\pi \frac{\cos(\theta/2)}{\sin(\theta/2)} d\theta = \int_0^1 \frac{1}{t} dt = \infty,$$

where the change of variable $t = \sin(\theta/2)$ is used.

The reason for the divergence is due to the long-range interaction of Coulomb potential. As indicated in [18], the divergence at the lower limit has a physical cause: the slowness of the decrease of the Coulomb forces, which leads to a high probability of small-angle scattering. However, the phenomenon of screening effect implies that the role of collisions with a high impact parameter is not as important as the Coulomb potential suggests. Thanks to (1.1), a rough and artificial approximation is just to ignore grazing collisions. Such an argument can be found in [18]: “In reality, however, in an electrically neutral plasma the Coulomb field of a particle at sufficiently large distances is screened by other charges; let $\theta_{\text{min}}$ denote the order of magnitude of the smallest angles for which the scattering can still be regarded as Coulomb scattering”. In this way, one has

$$\frac{1}{8\pi} \int_{S^2} b(\cos \theta) \sin^2(\theta/2) 1_{\theta \geq \theta_{\text{min}}} d\sigma = -\ln(\sin(\theta_{\text{min}}/2)) \sim \ln(1/\theta_{\text{min}}),$$

which is relevant to the so-called “Coulomb logarithm” denoted by $\ln \Lambda$.

**Remark 1.1.** In most physical books (for instance, see [17–19]), $\ln \Lambda$ is derived through the integration with respect to the impact parameter $b$, that is,

$$\ln \Lambda := \int_{\lambda_L}^{\lambda_D} b^{-1} db = \ln \frac{\lambda_D}{\lambda_L},$$
where $\lambda_D$ is the Debye length which characterizes electrostatic screening and $\lambda_L$ is the Landau length which identifies strong interactions. In other words, the “weak interaction” is defined through the truncation of the impact parameter $b$ onto the interval $[\lambda_L, \lambda_D]$. Thanks to (1.1), by approximation (see [19]), it holds that

$$\Lambda = \frac{2}{\theta_{\text{min}}} = 24\pi n_0\lambda_D^3,$$

(1.4)

where $n_0$ is the density of the particle. Since $n_0\lambda_D^3 \gg 1$ (for instance, for electron-proton gas), one has

$$\theta_{\text{min}} \ll 1, \quad \Lambda \gg 1.$$  

(1.5)

1.1.3. Landau’s Strategy

Landau’s strategy in [17] can be summarized as follows:

**Step 1:** Based on (1.2), the Boltzmann kernel with cutoff Rutherford cross section is defined by

$$B^c(v - v_*, \sigma) = B(v - v_*, \sigma)1_{\sin(\theta/2) \geq \sin(\theta_{\text{min}}/2)}.$$ 

Landau’s first assumption is that the Boltzmann equation with cutoff Rutherford cross section

$$\begin{cases} 
\partial_t F + v \cdot \nabla_x F = Q^c(F, F), & t > 0, \; x \in \mathbb{T}^3, \; v \in \mathbb{R}^3, \\
F|_{t=0} = F_0, 
\end{cases}$$

(1.6)

admits a smooth solution. Here $\mathbb{T}^3 := [-\pi, \pi]^3$ is the torus. Here

$$Q^c(F, F):= \int_{\mathbb{R}^3 \times \mathbb{S}^2} B^c(v - v_*, \sigma) \left( F(v') F(v') - F(v_*) F(v) \right) dv_* d\sigma,$$

(1.7)

**Step 2:** To derive an effective equation for weakly coupling particles, Landau further assumed $v \sim v'$ and $v_* \sim v'_*$. In his language, if $q := v' - v$, then $|q| \ll 1$. Thanks to Taylor expansion,

$$F(v') = F(v + q) = F(v) + \nabla_v F(v) \cdot q + \frac{1}{2} \nabla^2_v F(v) : q \otimes q + O(|q|^3).$$

(1.8)

Plugging (1.8) into the Boltzmann collision operator (1.7) and taking the truncation of the impact parameter $b$ onto the interval $[\lambda_L, \lambda_D]$, he derived the leading equation, which is now named as the Landau equation

$$\partial_t F + v \cdot \nabla_x F = (\ln \Lambda) Q^L(F, F),$$

(1.9)

where the Landau collision operator $Q^L$ is defined by

$$Q^L(g, h)(v) := \nabla_v \cdot \left( \int_{\mathbb{R}^3} a(v - v_*) (g(v_*) \nabla_v h(v) - \nabla_{v_*} g(v_*) h(v)) dv_* \right).$$

Here the symmetric matrix $a$ is given by

$$a(z) = 2\pi K |z|^{-1} (I_3 - \frac{z \otimes z}{|z|^2}),$$

(1.10)

where $I_3$ is the $3 \times 3$ identity matrix.
1.2. Mathematical Problems on the Derivation

To set up mathematical problems, thanks to (1.5), we first introduce a small parameter \( \varepsilon \), which is related to the physical cutoff for the angle, that is,
\[
\varepsilon := \sin(\theta_{\text{min}}/2) \ll 1. \tag{1.11}
\]
For simplicity of presentation, we take \( K = 1 \) for the constant \( K \) in (1.3).

1.2.1. Mathematical Assumptions on the Kernel

Throughout the paper, we will consider the kernel \( B^\varepsilon \) verifying the following assumptions:

(A1). The kernel \( B^\varepsilon(v - v_*, \sigma) \) takes the form
\[
B^\varepsilon(v - v_*, \sigma) := |\ln \varepsilon|^{-1} B^c(v - v_*, \sigma) = |v - v_*|^{-3} b^\varepsilon(\cos \theta), \tag{1.12}
\]
\[
b^\varepsilon(\cos \theta) := |\ln \varepsilon|^{-1} \sin^{-4}(\theta/2) \sin(\theta/2) \geq \varepsilon. \tag{1.13}
\]

(A2). The kernel \( B^\varepsilon(v - v_*, \sigma) \) is supported in the set \( 0 \leq \theta \leq \pi/2 \), that is, \( \cos \theta \geq 0 \), for otherwise \( B^\varepsilon \) can be replaced by its symmetrized form
\[
B^\varepsilon(v - v_*, \sigma) = (B^\varepsilon(v - v_*, \sigma) + B^\varepsilon(v - v_*, -\sigma)) |\cos \theta > 0.
\]

Associated to \( B^\varepsilon \), the Boltzmann collision operator \( Q^\varepsilon \) is defined by
\[
Q^\varepsilon(g, h)(v) := \int_{\mathbb{R}^3 \times S^2} B^\varepsilon(v - v_*, \sigma) \left( g' h' - g h \right) dv_* d\sigma.
\]
Here and in what follows, we use the usual shorthand \( h = h(v), g_* = g(v_*) \), \( h' = h(v') \), \( g_*' = g(v_*') \).

1.2.2. Reformulation of the Equation

It is compulsory to rewrite equations (1.6) and (1.9) by taking into account the parameter \( \varepsilon \). To do this, we introduce the scaling
\[
\tilde{F}(t, x, v) = F(\ln \varepsilon)^{-1} t, x, |\ln \varepsilon|v). \tag{1.14}
\]

Thanks to the facts that \( Q^c(\tilde{F}, \tilde{F})(t, x, v) = Q^c(F, F)(|\ln \varepsilon|^{-1} t, x, |\ln \varepsilon|v) \) and \( Q^\varepsilon = |\ln \varepsilon|^{-1} Q^c \), (1.6) becomes
\[
\begin{cases}
\partial_t \tilde{F} + v \cdot \nabla_x \tilde{F} = Q^\varepsilon(\tilde{F}, \tilde{F}), & t > 0, x \in \mathbb{T}^3, v \in \mathbb{R}^3, \\
\tilde{F}|_{t=0} = F^\varepsilon_0 := F_0(x, |\ln \varepsilon|v).
\end{cases} \tag{1.15}
\]

Similarly, thanks to \( Q^L(\tilde{F}, \tilde{F})(t, x, v) = Q^L(F, F)(|\ln \varepsilon|^{-1} t, x, |\ln \varepsilon|v) \), (1.9) reduces to (thanks to (1.4) and (1.11), we choose \( \varepsilon \) satisfying \( |\ln \varepsilon| \sim \ln \Lambda \) and absorb some unimportant constant into the Landau operator)
\[
\begin{cases}
\partial_t \tilde{F} + v \cdot \nabla_x \tilde{F} = Q^L(\tilde{F}, \tilde{F}), & t > 0, x \in \mathbb{T}^3, v \in \mathbb{R}^3, \\
\tilde{F}|_{t=0} = F^\varepsilon_0 := F_0(x, |\ln \varepsilon|v).
\end{cases} \tag{1.16}
\]
Let us give some comments on the scaling (1.14). Roughly speaking, it enables us to put Landau’s derivation([17]) and the Landau approximation problem([5]) in a unified framework.

- **Validity of the scaling in physical sense.** In physical books and lectures, one may check that for typical weakly coupled plasmas, Coulomb logarithm $\ln \Lambda$ lies in the range: $[5, 20]$. Recalling (1.4) and (1.11), we have $|\ln \varepsilon| \sim \ln \Lambda \in [5, 20]$, which means $\varepsilon$ is sufficiently small (around $[e^{-20}, e^{-5}]$) but $|\ln \varepsilon|$ is relatively “normal” (around $[5, 20]$). In this range, the scaling (1.14) is harmless and thus (1.6) and (1.9) are equivalent to (1.15) and (1.16) respectively. For the same reason, it is mathematically equivalent to ignore the dependence of the initial data on the parameter $|\ln \varepsilon|$ in (1.15) and (1.16), as we will do in (1.20) and (1.23). This reduces the justification of Landau’s derivation to the study of (1.15).

- **Relation between the scaling and Landau approximation.** Landau approximation(or the grazing collisions limit) is a mathematical framework to derive Landau equation from Boltzmann equation with general potentials. The main idea is as follows: when the deviation angle is truncated up to the order $O(\varepsilon)$ with a proper scaling, the grazing collisions will dominate and then the Boltzmann equation will formally converge to the Landau equation. The convergence has drawn extensive attention from mathematicians(see [7,8,16,22]).

  In view of (1.13), only grazing collisions can survive in the limit in which $\varepsilon$ goes to zero. In the pioneering work [5], Alexandre and Villani derived Landau equation (1.16) from Boltzmann equation (1.15) in the inhomogeneous setting under physical assumptions of finite mass, energy, entropy and entropy production. We emphasize that the Coulomb logarithm $\ln \Lambda$ in front of the collision operator $Q^L$ in (1.9) has been normalized in (1.16). In other words, the derivation through Landau approximation will lose some information from the potential function $\phi$.

- **Effect of the scaling on the linearized Boltzmann collision operator.** The scaling factor $|\ln \varepsilon|^{-1}$ in (1.13) and (1.14) plays an essential role in getting the spectral gap estimates for the linearized operator $L^\varepsilon$ (see (1.21)) of $Q^\varepsilon$. We refer readers to Theorem 1.3 for details. For Maxwellian molecules, the Boltzmann kernel $B$ takes the special form

\[
B(v - v^*, \sigma) = b(\cos \theta).
\]  

(1.17)

In this case, Chang-Uhlenbeck in [23] proved that the first (smallest) positive eigenvalue $\lambda_1$ of the linearized Boltzmann collision operator can be computed explicitly by

\[
\lambda_1 \sim \int_0^\pi b(\cos \theta)(1 - \cos \theta) \sin \theta d\theta.
\]  

(1.18)

The scaling factor $|\ln \varepsilon|^{-1}$ in (1.13) ensures that in the limit process($\varepsilon \to 0$), it holds that (see Lemma 2.9)

\[
\lambda_1^\varepsilon := \int_0^\pi b^\varepsilon(\cos \theta)(1 - \cos \theta) \sin \theta d\theta \sim 1.
\]  

(1.19)

This motivates us to link the spectral gap estimate of $L^\varepsilon$ to Chang-Uhlenbeck’s work [23].
1.2.3. Mathematical Problems  Our setup enables us to handle the Landau’s derivation and the Landau approximation in a unified framework. We consider these problems near Maxwellian (a small perturbation around equilibrium: Maxwellian) since it is widely used in the study of kinetic equations (for instance, wave phenomena in plasma physics). Our work can be summarized as follows.

(1). We consider global well-posedness of (1.15) near Maxwellian. As a result, it shows that all the computation in Landau’s paper [17] is valid globally in time and thus we justify his derivation rigorously in mathematics.

(2). We revisit Landau approximation near Maxwellian from (1.15) to (1.16). Compared to [5], we work with classical solution rather than weak solution. Mathematically we need to establish a unified framework to solve Boltzmann and Landau equations simultaneously and obtain an explicit expansion formula for the approximation.

- Global wellposedness of (1.15) near Maxwellian. Recall the standard Maxwellian density function \( \mu(v) := (2\pi)^{-\frac{3}{2}} e^{-|v|^2/2} \). With the perturbation \( \bar{F} = \mu + \mu^{\frac{1}{2}} f \), (1.15) becomes
\[
\partial_t f + v \cdot \nabla_x f + \mathcal{L}^\varepsilon f = \Gamma^\varepsilon(f, f), \quad f|_{t=0} = f_0.
\] (1.20)

Here the linearized Boltzmann operator \( \mathcal{L}^\varepsilon \) and the nonlinear term \( \Gamma^\varepsilon \) are defined by
\[
\Gamma^\varepsilon(g, h) := \mu^{-\frac{1}{2}} Q^\varepsilon(\mu^{\frac{1}{2}} g, \mu^{\frac{1}{2}} h), \quad \mathcal{L}_1^\varepsilon g := -\Gamma^\varepsilon(\mu^{\frac{1}{2}}, g), \quad \mathcal{L}_2^\varepsilon g := -\Gamma^\varepsilon(g, \mu^{\frac{1}{2}}), \quad \mathcal{L}^\varepsilon g := \mathcal{L}_1^\varepsilon g + \mathcal{L}_2^\varepsilon g.
\] (1.21)

We aim at not only global well-posedness but also propagation of regularity which holds uniformly in \( \varepsilon \). These are crucial for the following reasons:

(1). Global well-posedness of (1.20) exactly corresponds to the first step of Landau’s original strategy. Propagation of regularity is necessary in order to apply Taylor expansion in the second step of Landau’s strategy.

(2). In order to find an asymptotic formula between solutions of the Boltzmann and Landau equations, some uniform estimates for propagation of regularity are essential.

- Asymptotics of (1.15) near Maxwellian. It is relevant to the second step of Landau’s strategy and establishes Landau approximation global-in-time in classical solution sense. As we reviewed before, Landau just used the Taylor expansion of the solution to get the desired equation. It is not formulated by a direct limit from Boltzmann equation to Landau equation while Landau approximation seeks to do so.

In [16], it was shown that at least locally in time,
\[
\bar{F}^\varepsilon = \bar{F}^L + O(|\ln \varepsilon|^{-1}),
\] (1.22)

where \( \bar{F}^\varepsilon \) and \( \bar{F}^L \) are solutions to (1.15) and (1.16) with the same initial data. The order \( O(|\ln \varepsilon|^{-1}) \) reflects the logarithmic accuracy as derived in [18].
In this paper, we will reconsider such approximation near Maxwellian. More precisely, if we set \( \tilde{F} = \mu + \mu^{1/2} f \), then (1.16) gives the linearized equation
\[
\partial_t f + v \cdot \nabla_x f + \mathcal{L}^L f = \Gamma^L(f, f), \quad f|_{t=0} = f_0.
\] (1.23)
Here the linearized Landau operator \( \mathcal{L}^L \) and the nonlinear term \( \Gamma^L \) are defined by
\[
\Gamma^L(g, h) := \mu^{-1/2} Q^L(\mu^{1/2} g, \mu^{1/2} h), \quad \mathcal{L}^L_1 g := -\Gamma^L(\mu^{1/2}, g),
\]
\[
\mathcal{L}^L_2 g := -\Gamma^L(g, \mu^{1/2}), \quad \mathcal{L}^L g := \mathcal{L}^L_1 g + \mathcal{L}^L_2 g.
\]
One may regard (1.23) as the limit case (\( \varepsilon = 0 \)) of (1.20). In another way, let \( \Gamma^0 := \Gamma^L, \mathcal{L}^0 := \mathcal{L}^L \). The Cauchy problem (1.20) contains two cases: the Boltzmann equation \( \varepsilon > 0 \) and the Landau equation \( \varepsilon = 0 \).

Our goal is to establish a global-in-time asymptotic formula with accuracy as in (1.22). More precisely, if \( f^\varepsilon \) and \( f^L \) are solutions to (1.20) and (1.23) with the same initial data, then the following formula holds
\[
f^\varepsilon = f^L + O(|\ln \varepsilon|^{-1}),
\] (1.24)
globally in time in some weighted Sobolev spaces.

1.2.4. Basic Properties  At the end of this subsection, we recall some properties of the Boltzmann equations. The solutions to (1.15) and (1.16) have the fundamental physical properties of conserving mass, momentum and kinetic energy, that is, for all \( t \geq 0 \),
\[
\int_{T^3 \times \mathbb{R}^3} F(t, x, v) \phi(v) dx dv = \int_{T^3 \times \mathbb{R}^3} F(0, x, v) \phi(v) dx dv,
\]
\[
\phi(v) = 1, v_j, |v|^2, \quad j = 1, 2, 3.
\] (1.25)
Without loss of generality, we assume that the initial data \( f_0 \) in (1.20) and (1.23) verifies
\[
\int_{T^3 \times \mathbb{R}^3} \mu^{1/2}(v) f_0(x, v) \phi(v) dx dv = 0, \quad \phi(v) = 1, v_j, |v|^2, \quad j = 1, 2, 3.
\] (1.26)
As a result of (1.25), the solutions to (1.20) and (1.23) verify for all \( t \geq 0 \),
\[
\int_{T^3 \times \mathbb{R}^3} \mu^{1/2}(v) f(t, x, v) \phi(v) dx dv = 0, \quad \phi(v) = 1, v_j, |v|^2, \quad j = 1, 2, 3.
\]
Recall that \( \mathcal{K}(\mathcal{L}^\varepsilon) \) and \( \mathcal{K}(\mathcal{L}^L) \), the kernel spaces of \( \mathcal{L}^\varepsilon \) and \( \mathcal{L}^L \) respectively, verify
\[
\mathcal{K}(\mathcal{L}^L) = \mathcal{K}(\mathcal{L}^\varepsilon) = \mathcal{K} := \text{span}\{\mu^{1/2}, \mu^{1/2} v_1, \mu^{1/2} v_2, \mu^{1/2} v_3, \mu^{1/2} |v|^2\}.\]
1.3. Main Results

Our main results are global well-posedness and propagation of regularity for the Boltzmann equation (1.20) with cutoff Rutherford cross section. Moreover, we derive the global-in-time asymptotic formula (1.24) for the Landau approximation from the equation (1.20) to the equation (1.23). We refer readers to subsection 1.5 to check details on the function spaces used throughout the paper.

Our results are based on the following energy functional

\[
\mathcal{E}^{N,l}(f):= \sum_{j=0}^{N} \| f \|^2_{H^{N-j}_x H^{j}_t H^{j}_y},
\]

(1.27)

where \( \gamma = -3, l \geq 3N + 2. \)

**Theorem 1.1.** Let \( 0 \leq \varepsilon \leq \varepsilon_0 \) where \( \varepsilon_0 > 0 \) is a small constant. Suppose \( f_0 \) verify (1.26). There is a universal constant \( \delta_0 > 0 \) such that, if

\[
\mu + \mu^\frac{1}{2} f_0 \geq 0, \quad \mathcal{E}^{4,14}(f_0) \leq \delta_0,
\]

then Cauchy problem (1.20) (interpreted as (1.23) if \( \varepsilon = 0 \)) admits a unique global strong solution \( f^\varepsilon \) verifying \( \mu + \mu^\frac{1}{2} f^\varepsilon (t) \geq 0 \) for any \( t \geq 0 \) and

\[
\sup_{t \geq 0} \mathcal{E}^{4,14}(f^\varepsilon (t)) \leq C \mathcal{E}^{4,14}(f_0),
\]

(1.28)

for some universal constant \( C \). Moreover, the family of solution \( \{ f^\varepsilon \}_{\varepsilon \geq 0} \) verifies

1. **(Propagation of regularity)** Fix \( N \geq 4, l \geq 3N + 2 \), if additionally \( \mathcal{E}^{N,l}(f_0) < \infty \) and \( \varepsilon \) is small enough, then

\[
\sup_{t \geq 0} \mathcal{E}^{N,l}(f^\varepsilon (t)) \leq P_{N,l} \left( \mathcal{E}^{N,l}(f_0) \right).
\]

(1.29)

Here \( P_{N,l}(\cdot) \) is a continuous and increasing function with \( P_{N,l}(0) = 0 \).

2. **(Global asymptotic formula)** Fix \( N \geq 4, l \geq 3N + 2 \), assume \( \mathcal{E}^{N+3,l+18}(f_0) < \infty \) and \( \varepsilon \) is small enough, then

\[
\sup_{t \geq 0} \mathcal{E}^{N,l}(f^\varepsilon (t) - f_0^0 (t)) \leq | \ln \varepsilon |^{-2} U_{N,l} (\mathcal{E}^{N+3,l+18}(f_0)).
\]

(1.30)

Here \( U_{N,l}(\cdot) \) is a continuous and increasing function with \( U_{N,l}(0) = 0 \).

Some remarks are in order.

**Remark 1.2.** The kernel studied in this work is the most singular one both in relative velocity and deviation angle, and is the borderline for the Boltzmann to be meaningful in the classical sense. To our best knowledge, Landau approximation in the inhomogeneous case has never been touched globally in time within classical solution setting. We manage to establish global-in-time asymptotic formula (1.30) with explicit accuracy order for the first time.
Remark 1.3. Our results are consistent with the results in [11] and [13] when $\varepsilon = 0$. In particular, the smallness assumption on initial data with finite regularity and finite weight is a universal constant, which is sufficient to prove propagation of regularity with arbitrary regularity and weight if $\varepsilon$ is sufficiently small.

Remark 1.4. To keep the paper in a reasonable length, we only consider one species of particle, which enables us to focus more on operator analysis and a so-called linear-quasilinear method to close energy estimate. In the future, we will consider a more physical model: two species Vlasov-Possion-Boltzmann system with cutoff Rutherford cross-section, to derive the Vlasov-Possion-Landau system.

Remark 1.5. Let us summarize the main difference between the Landau approximation proposed in [5] and Landau’s original strategy in [17] as follows:

1. Landau approximation is based on the assumption that $\ln \Lambda$ is sufficiently large. However it is invalid in many physical situations where $\ln \Lambda$ is a relatively normal constant.

2. The resulting equation derived from Landau is different from that by Landau approximation. We remind readers that the Coulomb logarithm $\ln \Lambda$ appears as a diffusive coefficient in (1.9) but it does not appear in (1.16).

3. The error estimate between the solutions to Boltzmann and Landau equations via Landau approximation is different from that by Landau’s strategy. One has logarithm accuracy while the other has a high order accuracy thanks to (1.8).

1.4. Main Difficulties

The Boltzmann and Landau equations are well studied near Maxwellian(see [2,3,9–13]). To explain the main difficulties and the new ideas of the present work, let us consider a typical kinetic equation near Maxwellian

$$\partial_t f + v \cdot \nabla_x f + \mathcal{L} f = \Gamma(f, f),$$

where $\mathcal{L}$ and $\Gamma$ denote the linearized operator and the nonlinear term. We focus on propagation of regularity (or a priori estimate). The general approach to prove propagation of regularity for the above equation can be divided into four steps:

Step 1: This step is to describe the behavior of the self-adjoint operator $\mathcal{L}$, including the spectral gap estimate and the coercivity estimate. Roughly speaking, they can be written as

$$\langle \mathcal{L} f, f \rangle \gtrsim ||| (I - P) f |||^2_{\text{gap}}, \quad \langle \mathcal{L} f, f \rangle \gtrsim ||| f |||^2_{\text{coercivity}} - ||| f |||^2_{\text{gap}},$$

where $P$ is a projection operator that maps a function into the null space of $\mathcal{L}$, $I$ is the identity operator. Here $||| \cdot |||^2_{\text{gap}}$ and $||| \cdot |||^2_{\text{coercivity}}$ are some explicit or implicit norms. From these two estimates, as $\mathcal{L}$ is self-adjoint, one has

$$\langle \mathcal{L} f, f \rangle = \langle \mathcal{L}(I - P) f, (I - P) f \rangle \gtrsim |||(I - P) f |||^2_{\text{coercivity}}.$$
Step 2: The second step is to use the norm \( ||| \cdot |||_{\text{coercivity}} \) to give the upper bound for the nonlinear term \( \Gamma(f, f) \). Roughly, the ideal estimate looks like
\[
|\langle \Gamma(f, f), f \rangle| \lesssim \| f \| \|||f|||_{\text{coercivity}}^2.
\]
Here \( \| \cdot \| \) is the usual \( L^2 \) norm.

Step 3: The third step is to derive an evolution equation for \( Pf \) and then to get some elliptic estimate for \( Pf \) under the control of \( |||(I - P)f|||_{\text{coercivity}} \). This is referred to as Micro-Macro decomposition.

Step 4: The final step is to construct a proper energy functional to close the energy estimates and then get propagation of regularity for the solution under smallness assumption.

Now we turn to our case (see (1.20)) to explain the main difficulties in each step. Due to the definition of the kernel (see (1.12)), all the difficulties result from the strong singularity: not only from the relative velocity \(|v - v_*|^{-3}\) and but also from the angular function \( b_\varepsilon(\cos \theta) = |\ln \varepsilon|^{-1} \sin^{-4}(\theta/2)1_{\sin(\theta/2) \geq \varepsilon} \). Loosely speaking, (i). on one hand, the high singularity from the relative velocity stops us to treat the equation like the cutoff Boltzmann equation; (ii). on the other hand, the high singularity from the deviation angle is the borderline case for non-cutoff Boltzmann equation in terms of finite momentum transfer.

Mathematically, we face the following essential difficulties:

(D1). The first one is concerned with the coercivity estimate of the linearized operator \( L^\varepsilon \). According to [16], for small \( \varepsilon > 0 \), there exists a characteristic function \( W^\varepsilon: \mathbb{R}^3 \to \mathbb{R}_+ \) defined by
\[
W^\varepsilon(y) := \langle y \rangle \phi(y) + \langle y \rangle \left(1 - \frac{|\ln |y||}{|\ln \varepsilon|} + \frac{1}{|\ln \varepsilon|} \right) \frac{1}{2} \left(\phi(\varepsilon y) - \phi(y)\right) + \frac{1}{|\ln \varepsilon|^{2\varepsilon}} (1 - \phi(\varepsilon y)),
\]
(1.31)
to catch the Sobolev regularity for the collision operator. More precisely, it holds that
\[
\langle -Q^\varepsilon(\mu, f), f \rangle \gtrsim |W^\varepsilon(D)f|_{L^2_{-3/2}}^2 - |f|_{L^2_{-3/2}}^2.
\]
Recall \( \langle y \rangle = (1 + |y|^2)^{-\frac{1}{2}} \). Here \( \phi \) is the smooth function appearing in (1.50). Note that \( \phi(y) \) has support in \( |y| \leq 4/3 \) and equals to 1 if \(|y| \leq 3/4 \). It is easy to see that the characteristic function \( W^\varepsilon \) behaves quite different when \(|y| \sim 1\) and \(|y| \sim \varepsilon^{-1} \). The function \( W^\varepsilon \) implies that the behavior of \( L^\varepsilon \) will be more complicated, in particular for the proof of gain of weight and gain of anisotropic regularity (see Theorem 1.2). Also, because of (1.31), we get stuck in the upper bound estimate of the operator.

(D2). Once the coercivity estimate is available, we still need to face the following spectral gap estimate. For any suitable function \( f \),
\[
\langle L^\varepsilon f, f \rangle \geq C \| (I - P)f \|_{L^2_{-3/2}}^2,
\]
where $C > 0$ is independent of $\varepsilon$. The main difficulty here results from the singular factor $|\ln \varepsilon|^{-1}$ in the angular function $b^\varepsilon$ because it goes to zero when $\varepsilon$ tends to zero. Thus to get the desired result we need a constructive proof. The easiest and also the clearest case is the Maxwellian molecules given in [23], where the authors proved that the constant $C$ only depends on the first positive eigenvalue which can be computed explicitly through (1.18). Fortunately, in our case, $b^\varepsilon$ satisfies the condition (1.19) which motivates us to utilize the result in [23].

(D3). Recall (1.32). Note that we gain one derivative only in the “low frequency part” (that is, $|\xi| \lesssim 1/\varepsilon$). Unfortunately because of the strong singularity of the relative velocity ($|v-v_*|^3$ is a borderline case in 3-dimension), we at least need one full derivative to give a uniformly upper bound for the collision operator with respect to $\varepsilon$. Roughly speaking, [16] provides

\[
|\langle Q^\varepsilon(g,h), f \rangle| \lesssim |g|_{H^2/2} |h|_{H^1} |W^\varepsilon(D)f|_{L^2_{-3/2}},
\]

which indicates that what we gain from the coercivity is not enough to control the upper bound of the nonlinear term $\langle \Gamma^\varepsilon(g,h), f \rangle$.

The above three difficulties stop us to implement energy method and prove propagation of regularity, which forces us to figure out new techniques.

1.5. Ideas and Strategies

Before explaining our strategy to overcome the above difficulties, we begin with basic facts on Micro-Macro decomposition and spherical harmonics.

- **Macro-micro decomposition**: Recall $\mathcal{K} = \text{span}\{ \mu_1^2, \mu_2^2 v_1, \mu_2^2 v_2, \mu_2^2 v_3, \mu_1^2 |v|^2 \}$, an orthonormal basis of which can be chosen as $\{ \mu_1^2, \mu_2^2 v_1, \mu_2^2 v_2, \mu_2^2 v_3, \mu_1^2 (|v|^2 - 3)/\sqrt{6} \} = \{ e_j \}_{1 \leq j \leq 5}$. The projection operator $\mathbb{P}$ on the null space $\mathcal{K}$ is given by

\[
\mathbb{P}f := \sum_{j=1}^{5} \langle f, e_j \rangle e_j = (a + b \cdot v + c |v|^2) \mu_2^1,
\]

where for $1 \leq i \leq 3$,

\[
a = \int_{\mathbb{R}^3} \left( \frac{5}{2} - \frac{|v|^2}{2} \right) \mu_1^2 f dv, \quad b_i = \int_{\mathbb{R}^3} v_i \mu_1^2 f dv,
\]

\[
c = \int_{\mathbb{R}^3} \left( \frac{|v|^2}{6} - \frac{1}{2} \right) \mu_2^1 f dv.
\]

As usual we call $\mathbb{P}f$ and $(\mathbb{I} - \mathbb{P})f$ the macroscopic part and microscopic part of $f$ respectively.
• **Spherical harmonics:** Let \( Y^m_l \) with \( l \in \mathbb{N}, m \in \mathbb{Z}, -l \leq m \leq l \) be real spherical harmonics. They are the eigenfunctions of the Laplace-Beltrami operator \(-\Delta_{S^2}\). Mathematically,

\[
(-\Delta_{S^2})Y^m_l = l(l + 1)Y^m_l.
\]

These functions are essential to help us to catch the anisotropic property of \( \mathcal{L}^\varepsilon \). We introduce the operator \( W^\varepsilon((-\Delta_{S^2})^\frac{1}{2}) \) defined by: if \( v = r\sigma \), then

\[
(W^\varepsilon((-\Delta_{S^2})^\frac{1}{2})f)(v) := \sum_{l=0}^{\infty} \sum_{m=-l}^{l} W^\varepsilon((l(l + 1))^\frac{1}{2}) Y^m_l(\sigma) f^m_l(r), \tag{1.35}
\]

where \( f^m_l(r) = \int_{S^2} Y^m_l(\sigma) f(r\sigma) d\sigma \). Note that a radial function maps to a function on \( \mathbb{R}_+ \). Since \( W^\varepsilon \) is a radial function, the coefficient \( W^\varepsilon((l(l + 1))^\frac{1}{2}) \) in (1.35) can be understood accordingly.

We are ready to present our main ideas and strategies.

### 1.5.1. Idea on the Coercivity Estimates

This part is related to (D1). The coercivity estimate of the linearized operator \( \mathcal{L}^\varepsilon \) plays an essential role in studying (1.20) and it reads

**Theorem 1.2.** There are two positive universal constants \( \varepsilon_0 > 0 \) and \( \nu_0 > 0 \) such that for \( 0 \leq \varepsilon \leq \varepsilon_0 \) and any suitable function \( f \), it holds that

\[
\langle \mathcal{L}^\varepsilon f, f \rangle + |f|^2_{L^2_{-3/2}} \geq \nu_0 |f|^2_{L^2_{-3/2}}, \tag{1.36}
\]

**Here for** \( l \in \mathbb{R} \) and \( W_l(v) := (1 + |v|^2)^{l/2} \), we denote

\[
|f|^2_{L^2_{l, l}} := |W^\varepsilon((-\Delta_{S^2})^\frac{1}{2}) W_l f|^2_{L^2_2} + |W^\varepsilon(D) W_l f|^2_{L^2_2} + |W^\varepsilon W f|^2_{L^2_2}, \tag{1.37}
\]

where \( W^\varepsilon \) is defined in (1.31) for \( \varepsilon > 0 \) and \( W^0(v) := (1 + |v|^2)^{\frac{1}{2}} \).

We have two remarks on Theorem 1.2.

**Remark 1.6.** The characteristic function \( W^\varepsilon \) indicates the same gain of weight simultaneously in phase space, frequency space and anisotropic space for \( \mathcal{L}^\varepsilon \). Moreover, it matches well the limiting operator \( \mathcal{L}^L \) as \( \varepsilon \) goes to zero. For this regard, note that \( W^\varepsilon(y) \to \langle y \rangle \) as \( \varepsilon \to 0 \). As \( \varepsilon \) goes to zero, (1.36) becomes to the following coercivity estimate of \( \mathcal{L}^L \)

\[
\langle \mathcal{L}^L f, f \rangle + |f|^2_{L^2_{-3/2}} \geq \nu_0 \left( |W((-\Delta_{S^2})^\frac{1}{2}) W_{-3/2} f|^2_{L^2_2} + |W(D) W_{-3/2} f|^2_{L^2_2} + |W W_{-3/2} f|^2_{L^2_2} \right).
\]

Here \( W(v) := (1 + |v|^2)^{\frac{1}{2}} \). In this sense, our estimate (1.36) is sharp.
Remark 1.7. We emphasize that gain of regularity only happens in the “low frequency part” \(|\xi| \lesssim 1/\varepsilon\) while gain of weight in the phase space only happens in the “big ball” \(|v| \lesssim 1/\varepsilon\). In other words, \(L^\varepsilon\) still keeps a hyperbolic structure due to the angular cutoff \(\theta \gtrsim \varepsilon\).

The intuition behind Theorem 1.2 comes from the knowledge that the linearized Boltzmann collision operator without angular cutoff corresponds to a unique characteristic function, which captures the key structure of the operator in phase, frequency and anisotropic spaces. One may check it in [14]. In [16], \(W^\varepsilon\) had been proved to be the symbol in the frequency space for \(Q^\varepsilon\) (see (1.32)). Therefore \(W^\varepsilon\) should be the characteristic function of \(L^\varepsilon\).

Note that the behavior of \(W^\varepsilon\) changes on the region \(|\cdot| \sim 1/\varepsilon\) with logarithm correction. The key idea to catch this behavior lies in the following two aspects:

- The first one is the “geometric decomposition” introduced in [14] resulting from the geometry inherent in an elastic collision event. The decomposition reads
  \[
  f(\xi) - f(\xi^+) = \left( f(\xi) - f\left(|\xi|\frac{\xi^+}{|\xi^+|}\right) \right) + \left( f\left(|\xi|\frac{\xi^+}{|\xi^+|}\right) - f(\xi^+) \right),
  \]
  where \(\xi^+ := \frac{|\xi|\sigma + \xi}{2}\). The first difference \(f(\xi) - f\left(|\xi|\frac{\xi^+}{|\xi^+|}\right)\) is referred as the “spherical” part since \(\xi\) and \(|\xi|\frac{\xi^+}{|\xi^+|}\) lie on the sphere centered at origin with radius \(|\xi|\). The second difference \(f\left(|\xi|\frac{\xi^+}{|\xi^+|}\right) - f(\xi^+)\) is referred as the “radial” part since \(|\xi|\frac{\xi^+}{|\xi^+|}, \xi^+\) and origin are on the same line. We can extract anisotropic information from the spherical part for both the lower and upper bounds of \(L^\varepsilon\). One can see section 2.3 for more details.

- The second one is the development of some localization technique: the dyadic decomposition in both phase and frequency spaces. The decomposition enables us to capture the leading term and the characteristic function. For more details, we refer readers to Section 2 and Section 4.

1.5.2. Idea on the Spectral Gap Estimates  This sequel aims to overcome the difficulty raised in (D2). Roughly speaking, the above Theorem 1.2 successfully catches the leading term \(|f|^2_{L^2_{L^{-3/2}}\varepsilon}\). To eliminate the lower order term \(|f|^2_{L^2_{L^{-3/2}}\varepsilon}\), we need the following uniform-in-\(\varepsilon\) spectral gap estimate of \(L^\varepsilon\).

Theorem 1.3. There exist two positive universal constants \(\varepsilon_0\) and \(\lambda\) such that for \(0 \leq \varepsilon \leq \varepsilon_0\) and any suitable function \(f\), it holds that
  \[
  \langle L^\varepsilon f, f \rangle \geq \lambda|\mathbb{1} - \mathbb{P}|f|^2_{L^2_{L^{-3/2}}\varepsilon},
  \]
  where \(\lambda\) depends only on \(\lambda_1^\varepsilon\) (see (1.19) for definition) and the constant \(\nu_0\) appearing in Theorem 1.2.

To give some illustration on the above theorem, in this sequel, we use \(L^\gamma\) to denote the linearized Boltzmann collision operator associated to the general kernel
  \[
  B(v - v_*, \sigma) := |v - v_*|^\gamma b(\cos \theta).
  \]
The first spectral gap estimate is attributed to [23]. In fact, thanks to the simple structure of $L^0$, which corresponds to the Maxwellian molecules (see (1.17)), the authors explicitly constructed its eigenvalues and their corresponding eigenfunctions. As $L^0$ is a self-adjoint operator, the spectral gap estimate reads

$$\langle L^0 f, f \rangle \geq \lambda_1 |(I - P)f|_{L^2}^2,$$  \hspace{1cm}  (1.39)

where $\lambda_1$ is the first positive eigenvalue of $L^0$ given by (1.18).

Based on a proper decomposition of the operator and also the dyadic decomposition on the modulus of the relative velocity, the authors in [6,20] extended the above result to the general case as follows,

$$\langle L^\gamma f, f \rangle \gtrsim C_\gamma C_b |(I - P)f|_{L^2_{\gamma/2}}^2,$$  \hspace{1cm}  (1.40)

where

$$C_b = \inf_{\sigma_1, \sigma_2 \in S^2} \int_{S^2} \min\{b(\sigma_1 \cdot \sigma_3), b(\sigma_2 \cdot \sigma_3)\} d\sigma_3.$$  

We remind readers that (1.40) is more general but the estimate depends on $C_b$. As a result, (1.40) cannot be applied directly if $C_b$ is not bounded from below, which unfortunately happens when the angular function $b$ concentrates on the grazing collisions. There are two typical examples

$$b(\cos \theta) = \varepsilon^2(\sin(\theta/2))^2 \sin(\theta/2) \geq 1,$$  \hspace{1cm}  \hspace{1cm} (1.41)

where $b^\varepsilon(\cos \theta)$ is defined in (1.13). It is not difficult to check that in both cases $C_b$ tends to zero when $\varepsilon$ goes to zero. We fail to get the desired result through (1.40). On the other hand, (1.39) works well for both cases in (1.41) since the following holds true uniformly in $\varepsilon$,

$$\int_0^{\pi} b(\cos \theta)(1 - \cos \theta) \sin \theta d\theta \sim 1.$$  

From the above short review on the spectral gap estimate, we need a new and constructive proof for Theorem 1.3, which can be regarded as one of our main contributions in this paper. The key step lies in finding a link between the desired result (1.38) and the well-known result (1.39) by noting that (1.39) is stable in the Landau approximation thanks to (1.19).

Because of some technical restriction (which will be seen soon), for $-3 \leq \gamma \leq 0$, we need to consider the general linearized collision operator $L^{\varepsilon,\gamma}$ associated to the Boltzmann kernel

$$B^{\varepsilon,\gamma}(v - v_*, \sigma) := |v - v_*|^\gamma b^\varepsilon(\cos \theta).$$

Our strategy consists of two parts:

- The first part utilizes the coercivity estimate in Theorem 1.2. More general than Theorem 1.2 (see Theorem 2.1), for $-3 \leq \gamma \leq 0$, we derive

$$\langle L^{\varepsilon,\gamma} f, f \rangle = \langle L^{\varepsilon,\gamma} (I - P)f, (I - P)f \rangle \geq v_0 |(I - P)f|_{L^2_{\gamma/2}}^2 - |(I - P)f|_{L^2_{\gamma/2}}^2,$$  \hspace{1cm}  (1.42)
The second part is to reduce the case of a general potential ($\gamma < 0$) to the case of Maxwellian molecules ($\gamma = 0$) with a small correction term. Roughly speaking, for any $0 < \delta \leq 1$, we derive
\[
\langle \mathcal{L}^{\varepsilon,\gamma} f, f \rangle \geq C_1 \delta^{-\gamma} \left( (1 - \mathbb{P}) f \right)^2_{L^{\gamma/2}} - C_2 \delta^{\gamma+2} \left( (1 - \mathbb{P}) f \right)^2_{L^{\gamma/2}}. \tag{1.43}
\]

When $-2 < \gamma \leq 0$, we can make a suitable combination of (1.42) and (1.43) to get (1.44) in Theorem 1.4. For $-3 \leq \gamma < -2$, we choose $-2 < \alpha, \beta < 0$ such that $\gamma = \alpha + \beta$ and thus $B^{\varepsilon,\gamma} = B^{\varepsilon,\alpha} |v - v_*|^\beta$. Then $B^{\varepsilon,\alpha}$ corresponds to $\mathcal{L}^{\varepsilon,\alpha}$ which has spectral gap estimate since $\alpha > -2$. We next reduce $|v - v_*|^\beta$ to the Maxwellian molecules $|v - v_*|^0$. In summary, we can deal with $\gamma \in (-2, 0)$ in the first stage, and $\gamma \in [-3, -2]$ in the second stage. For details, one can see the proof of Theorem 3.1 in Section 3.

We emphasize that the derivation of (1.43) is very tricky. To this end, we introduce a proper decomposition on the modulus of the relative velocity and a special weight function $U_\delta (v) = (1 + \delta^2 |v|^2)^{\frac{1}{2}}$ (see (3.9)) to keep the symmetric structure of $\langle \mathcal{L}^{\varepsilon,\gamma} f, f \rangle$.

More general than Theorem 1.3, we have

**Theorem 1.4.** Let $-3 \leq \gamma \leq 0$. There exist two positive universal constants $\varepsilon_0$ and $\lambda$ such that for $0 \leq \varepsilon \leq \varepsilon_0$ and any suitable function $f$, it holds that
\[
\langle \mathcal{L}^{\varepsilon,\gamma} f, f \rangle \geq \lambda |(1 - \mathbb{P}) f |^2_{L^{\gamma/2}}, \tag{1.44}
\]
where $\lambda$ depends only on $\gamma$, $\lambda_1^\varepsilon$ (see (1.19) for definition) and the constant $\varepsilon_0$ appearing in Theorem 1.2. Here for $\varepsilon = 0$, $\mathcal{L}^{0,\gamma}$ is the linearized Landau operator with the $|z|^{-1}$ in (1.10) replaced by $|z|^\gamma+2$.

### 1.5.3. Idea on the Linear-quasilinear Method

In this sequel, we deal with the difficulty raised in (D3). As explained before, for (1.20), it seems that the dissipation of $\mathcal{L}^\varepsilon$ cannot prevail the nonlinear term $\Gamma^\varepsilon (f, f)$, which is the biggest challenge for establishing global well-posedness. To overcome this obstacle, we introduce a so-called “linear-quasilinear method”.

In order to explain the method, we first introduce the truncation with respect to the modulus of the relative velocity. We associate $Q^{\varepsilon,\gamma,\eta}$ with kernel $B^{\varepsilon,\gamma,\eta} = B^{\varepsilon,\gamma} |v - v_*| \mathbb{1}_{|v - v_*| \geq \eta}$ and denote $\mathcal{L}^{\varepsilon,\gamma,\eta}$, $\mathcal{L}_1^{\varepsilon,\gamma,\eta}$, $\mathcal{L}_2^{\varepsilon,\gamma,\eta}$, $\Gamma^{\varepsilon,\gamma,\eta}$ correspondingly. To avoid ambiguity, we define explicitly the Boltzmann operator $Q^{\varepsilon,\gamma,\eta}$ as
\[
Q^{\varepsilon,\gamma,\eta} (g, h) (v) := \int \mathbb{R}^3 \times S^2 B^{\varepsilon,\gamma,\eta} (v - v_* , \sigma) (g_*^{\prime} h^{\prime} - g_* h) \, dv_* d\sigma,
\]
where
\[
B^{\varepsilon,\gamma,\eta} (v - v_* , \sigma) := |v - v_*|^\gamma \mathbb{1}_{|v - v_*| \geq \eta} b^\varepsilon (\cos \theta) = |v - v_*|^\gamma \mathbb{1}_{|v - v_*| \geq \eta} \ln \varepsilon^{-1} \sin^{-4} (\theta/2) |\sin (\theta/2)|^2 \varepsilon.
\]

In a fashion similar to (1.21), we define
\[
\Gamma^{\varepsilon,\gamma,\eta} (g, h) := \mu^{-\frac{1}{2}} Q^{\varepsilon,\gamma,\eta} (\mu^{\frac{1}{2}} g , \mu^{\frac{1}{2}} h), \tag{1.45}
\]
Then we set

\[ L_1^{\varepsilon, \gamma, \eta}g := - \Gamma^{\varepsilon, \gamma, \eta}(\mu \frac{1}{2}, g), \quad (1.46) \]

\[ L_2^{\varepsilon, \gamma, \eta}g := - \Gamma^{\varepsilon, \gamma, \eta}(g, \mu \frac{1}{2}), \quad (1.47) \]

\[ L^{\varepsilon, \gamma, \eta}g := L_1^{\varepsilon, \gamma, \eta}g + L_2^{\varepsilon, \gamma, \eta}g. \quad (1.48) \]

Then we set

\[ B^{\varepsilon, \gamma, \eta}_\eta := B^{\varepsilon, \gamma} - B^{\varepsilon, \gamma, \eta}, \quad Q^{\varepsilon, \gamma, \eta}_\eta := Q^{\varepsilon, \gamma} - Q^{\varepsilon, \gamma, \eta}, \]

\[ L^{\varepsilon, \gamma, \eta}_g := L^{\varepsilon, \gamma} - L^{\varepsilon, \gamma, \eta}, \quad \Gamma^{\varepsilon, \gamma, \eta}_\eta := \Gamma^{\varepsilon, \gamma} - \Gamma^{\varepsilon, \gamma, \eta}. \quad (1.49) \]

With these notations, we rewrite (1.20) as

\[ \partial_t f + v \cdot \nabla_x f + L^{\varepsilon, -3, \eta} f = \Gamma^{\varepsilon, -3, \eta}(f, f) + \left(- L^{\varepsilon, -3}_\eta f + \Gamma^{\varepsilon, -3, \eta}(f, f)\right). \]

Now we are in a position to illustrate the linear-quasilinear method in detail. Since the standard $L^2$ energy estimate is employed to establish global well-posedness, the method can be explained in terms of the integration domain of $L^2$ inner product. In fact, we separate the integration domain into two parts: $|v - v_u| \leq \eta$ and $|v - v_u| \geq \eta$, where $\eta$ is a parameter and principally it should be sufficiently small. We call them respectively the singular region and the regular region since $|v - v_u|^{-3}$ in the Boltzmann kernel $B$ is strongly singular near 0. The spirit of the new method can be summarized as follows: (i) In the regular region, we employ the standard linear method by showing that the dissipation of the linear term can dominate the nonlinear term; (ii) In the singular region, we use the quasi-linear method by utilizing the non-negativity of the solution to eliminate the dangerous strong singularity. More precisely,

- In the regular region, we follow the linear method to show that the dissipation of $L^{\varepsilon, -3, \eta} f$ dominates the nonlinear term $\Gamma^{\varepsilon, -3, \eta}(f, f)$ under smallness assumption on $f$. To this end, technically we need to show

\[ \langle L^{\varepsilon, -3, \eta} f, f \rangle + |f|^2_{L^2_{-\eta/2}} \geq v_0 |f|^2_{L^2_{-\eta/2}}, \]

\[ |\langle \Gamma^{\varepsilon, -3, \eta}(f, f), f \rangle| \lesssim C(\eta^{-1}, f) |f|^2_{L^2_{\eta/2}}. \]

That is, when $\eta$ is sufficiently small, $L^{\varepsilon, -3, \eta}$ yields the same dissipation as $L^\varepsilon$ in Theorem 1.2. Once $\eta > 0$ is fixed, we can assume $f$ is small enough such that $C(\eta^{-1}, f) \ll v_0$ to control $\langle \Gamma^{\varepsilon, -3, \eta}(f, f), f \rangle$ by $\langle L^{\varepsilon, -3, \eta} f, f \rangle$.

- In the singular region, we use the identity

\[ \langle - L^{\varepsilon, -3}_\eta f + \Gamma^{\varepsilon, -3}_\eta (f, f), f \rangle \]

\[ = \langle Q^{\varepsilon, -3}_\eta (\mu + \mu^\frac{1}{2} f, \mu^\frac{1}{2} f), f \rangle + \langle \Gamma^{\varepsilon, -3}_\eta (f, \mu^\frac{1}{2}), f \rangle \]

\[ = \langle Q^{\varepsilon, -3}_\eta (\mu + \mu^\frac{1}{2} f, f), f \rangle + \int_{\mathbb{S}^2 \times \mathbb{R}^3} B^{\varepsilon, -3}_\eta (\mu^\frac{1}{2} + f) \]

\[ \times ((\mu^\frac{1}{2})_s - \mu^\frac{1}{2}) d\sigma d\nu d\nu + \langle \Gamma^{\varepsilon, -3}_\eta (f, \mu^\frac{1}{2}), f \rangle. \]
In Theorem 4.5, we have
\[
\langle -L_{\eta}^{e,-3}f + \Gamma_{\eta}^{e,-3}(f, f), f \rangle \lesssim (\eta^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})(1 + |f|_{H^2})|f|^2_{L_{x,t}^{2,-3/2}}.
\]
Let us explain the idea. There are three terms in the previous identity. For the latter two terms, we make use of the regularity of \( \mu^{\frac{1}{2}} \) to cancel the singularity. For the first term we shall use the quasi-linear method to give the estimate. In fact, thanks to the fact that \( \mu + \mu^{\frac{1}{2}}f \) is a solution to the original Boltzmann equation, it holds that \( \mu + \mu^{\frac{1}{2}}f \geq 0 \) which implies that the coercivity type estimate (1.32) holds for \( \langle -Q_{\eta}^{e,-3}(\mu + \mu^{\frac{1}{2}}f, f), f \rangle \). Here we only use the good sign and cancellation Lemma 2.11 to get
\[
\langle Q_{\eta}^{e,-3}(\mu + \mu^{\frac{1}{2}}f, f), f \rangle \lesssim (\eta^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})(1 + |f|_{H^2})|f|^2_{L_{x,t}^{2,-3/2}}.
\]
One can see subsection 4.3 for details.

Using such a treatment, we can deal with the highest order derivative in the energy estimates and capture the highest order dissipation, which is crucial for the near Maxwellian framework. For other lower order derivatives, we have one order derivative to kill the strong singularity in \(|v - v_n|^{-3}\) near 0. In order to implement this plan, we need two types of upper bound estimates for the nonlinear term, see Table 1 at the beginning of Section 4 for a summary.

### 1.6. Notations, Function Spaces and Organization of the Paper

We list notations and state the organization of the paper in this subsection.

#### 1.6.1. Notations

For standard notations, we give the following list:

- For a multi-index \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3 \), \( |\alpha| := \alpha_1 + \alpha_2 + \alpha_3 \).
- We write \( a \lesssim b \) to indicate that there is a universal constant \( C \), which may be different on different lines, such that \( a \leq Cb \). We use the notation \( a \sim b \) whenever \( a \lesssim b \) and \( b \lesssim a \).
- The notation \( a^+ \) means the maximum value of \( a \) and 0 and \([a]\) denotes the maximum integer which does not exceed \( a \).
- The Japanese bracket \( \langle \cdot \rangle \) is defined by \( \langle v \rangle = (1 + |v|^2)^{\frac{1}{2}} \). Then the weight function \( W \) is defined by \( W_1(v) := \langle v \rangle^l \). When \( l = 1 \), we write \( W(v) := W_1(v) = \langle v \rangle \).
- We denote \( C(\lambda_1, \lambda_2, \ldots, \lambda_n) \) or \( C_{\lambda_1, \lambda_2, \ldots, \lambda_n} \) by a constant depending on parameters \( \lambda_1, \lambda_2, \ldots, \lambda_n \).
- For \( f, g \in L^2(\mathbb{R}^3) \), \( \langle f, g \rangle := \int_{\mathbb{R}^3} f(x)g(x)dx \) and \( |f|^2_{L^2} := \langle f, f \rangle \).
- For \( f, g \in L^2(\mathbb{T}^3) \), \( \langle f, g \rangle_x := \int_{\mathbb{T}^3} f(x)g(x)dx \) and \( |f|^2_{L^2} := \langle f, f \rangle_x \).
- For \( f, g \in L^2(\mathbb{T}^3 \times \mathbb{R}^3) \), \( \langle f, g \rangle := \int_{\mathbb{T}^3 \times \mathbb{R}^3} f(x, v)g(x, v)dx dv \) and \( \|f\|^2_{L^2_x L^2_v} := \langle f, f \rangle \).
- The translator operator \( T_u \) is defined by \( (T_u f)(v) := f(u + v) \), for any \( u, v \in \mathbb{R}^3 \).
- As usual, \( 1_A \) is the characteristic function of a set \( A \).
- If \( A, B \) are two operators, then their commutator \( [A, B] := AB - BA \).
1.6.2. Function Spaces  For simplicity, for \( \alpha, \beta \in \mathbb{N}^3 \), we set \( \partial^\alpha := \partial^\alpha_x \), \( \partial^\beta := \partial^\beta_v \), \( \partial^\alpha_v := \partial^\alpha_x \partial^\beta_v \). We will use the following spaces:

- For real number \( n, l \), we define the weighted Sobolev space on \( \mathbb{R}^3 \)
  
  \[ H^n_l := \left\{ f(v) \middle| f^2 |_{H^n_l} := \sum_{|\beta| \leq n} |\partial^\beta f|^2_{L^2_l} < \infty \right\}, \]

  Here \( a(D) \) is a pseudo-differential operator with the symbol \( a(\xi) \) defined by
  
  \[ (a(D) f)(v) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{i(v-y) \cdot \xi} a(\xi) f(y) dy d\xi. \]

- For \( n \in \mathbb{N}, l \in \mathbb{R} \), the weighted Sobolev space on \( \mathbb{R}^3 \) is defined by
  
  \[ H^n_l := \left\{ f(v) \middle| f^2 |_{H^n_l} := \sum_{|\beta| = n} |\partial^\beta f|^2_{L^2_l} < \infty \right\}. \]

- For \( m \in \mathbb{N} \), we denote the Sobolev space on \( \mathbb{T}^3 \) by
  
  \[ H^m_x := \left\{ f(x) \middle| f^2 |_{H^m_x} := \sum_{|\alpha| \leq m} |\partial^\alpha f|^2_{L^2_x} < \infty \right\}. \]

- For \( m, n \in \mathbb{N}, l \in \mathbb{R} \), the weighted Sobolev space on \( \mathbb{T}^3 \times \mathbb{R}^3 \) is defined by
  
  \[ H^n_l H^m_x := \left\{ f(x, v) \middle| f^2 |_{H^n_l H^m_x} := \sum_{|\alpha| \leq m, |\beta| \leq n} ||\partial^\alpha f||_{L^2_x} ||\partial^\beta f||_{L^2_l} < \infty \right\}. \]

We write \( ||f||_{H^n_l L^2_x} := ||f||_{H^n_l H^0_x} \) if \( n = 0 \) and \( ||f||_{L^2_x L^2_l} := ||f||_{H^0_x H^0_l} \) if \( m = n = 0 \). The space \( H^n_l H^m_x \) can be similarly defined.

1.6.3. Dyadic Decompositions  We now introduce the dyadic decomposition. Let \( B_{4/3} := \{ x \in \mathbb{R}^3 \mid |x| \leq 4/3 \} \) and \( C := \{ x \in \mathbb{R}^3 \mid 3/4 \leq |x| \leq 8/3 \} \). Then one may introduce two radial functions \( \phi \in C^\infty_0(B_{4/3}) \) and \( \psi \in C^\infty_0(C) \) which satisfy

\[
0 \leq \phi, \psi \leq 1, \text{ and } \phi(x) + \sum_{j \geq 0} \psi(2^{-j} x) = 1, \text{ for all } x \in \mathbb{R}^3. \quad (1.50)
\]

Now define \( \varphi_{-1}(x) := \phi(x) \) and \( \varphi_j(x) := \psi(2^{-j} x) \) for any \( x \in \mathbb{R}^3 \) and \( j \geq 0 \). Then one has the following dyadic decomposition

\[
f = \sum_{j=-1}^{\infty} \varphi_j f. \quad (1.51)
\]
for any function $f$ defined on $\mathbb{R}^3$. We also define

$$f^\varepsilon:=\phi(\varepsilon D)f, \quad f^\varepsilon:=\phi(\varepsilon D)f,$$

$$\Phi f := \phi(\varepsilon D)f, \quad \Phi f := (1-\phi(\varepsilon D))f.$$ (1.52)

### 1.6.4. Function Spaces Related to the Collision Operator

Now we introduce some spaces related to the coercivity estimate of $L^\varepsilon$.

- **Space $L^2_{\varepsilon,l}$.** For functions defined on $\mathbb{R}^3$, recalling (1.37), the space $L^2_{\varepsilon,l}$ with $l \in \mathbb{R}$ is defined by

$$L^2_{\varepsilon,l} := \left\{ f(v) \mid |f|^2_{L^2_{\varepsilon,l}} < \infty \right\}.$$

- **Space $H^m_x H^n_{\varepsilon,l}$.** For functions defined on $\mathbb{T}^3 \times \mathbb{R}^3$, the space $H^m_x H^n_{\varepsilon,l}$ with $m, n \in \mathbb{N}$ is defined by

$$H^m_x H^n_{\varepsilon,l} := \left\{ f(x,v) \mid \|f\|_{H^m_x H^n_{\varepsilon,l}} := \sum_{|\alpha| \leq m, |\beta| \leq n} \|\partial_\alpha \partial_\beta f\|_{L^2_{\varepsilon,l}}^2 < \infty \right\}.$$

We set $\|f\|_{H^m_x H^n_{\varepsilon,l}} := \|f\|_{H^m_x H^n_{\varepsilon,l}}$ if $n = 0$ and $\|f\|_{L^2_{\varepsilon,l}} := \|f\|_{H^0_x H^0_{\varepsilon,l}}$ if $m = n = 0$. Again, the space $H^m_x H^n_{\varepsilon,l}$ can be defined accordingly.

### 1.6.5. Organization of the Paper

In Section 2, we first give some elementary results and then endeavor to prove the coercivity estimate in Theorem 1.2. Section 3 is devoted to the spectral gap estimate in Theorem 1.3 and 1.4. In Section 4, the upper bound estimates of the nonlinear term $\Gamma^\varepsilon$ are provided. Some commutator estimates are given in Section 5. Based on the various estimates, Theorem 1.1 is proved in Section 6.

## 2. Coercivity Estimate

In this section, we will prove Theorem 1.2. In fact, we will capture the coercivity estimate of $L^\varepsilon_{\gamma,\eta}$ for $-3 \leq \gamma \leq 0$ and small $\eta \geq 0$, $\varepsilon > 0$ in Theorem 2.1, which is more general than Theorem 1.2. Throughout the article, we always assume that $0 < \varepsilon \leq \frac{1}{10}$ unless otherwise specified.

Our strategy relies on the following relation (see (2.76)). For $0 \leq \eta \leq r_0 = 6^{-1} 2^{-7/6} e^{-1/6}$ (first appearing in Proposition 2.4), it holds that

$$\langle L^\varepsilon_{\gamma,\eta} f, f \rangle + \|f\|_{L^2_{\varepsilon/2}}^2 \geq N^\varepsilon_{\gamma,\eta}(\mu^{1/4}, f) + N^\varepsilon_{\gamma,\eta}(\mu^{1/4}, f),$$ (2.1)

where for $\eta \geq 0$,

$$N^\varepsilon_{\gamma,\eta}(g, h) := \int_{\mathbb{S}^2 \times \mathbb{R}^3 \times \mathbb{R}^3} b^\varepsilon(\cos \theta)|v-v_*|^\gamma |v-v_*| |v_*| \geq \gamma \theta \cos \theta (h' - h)^2 d\sigma dv dv_*.$$

(2.2)
We remind readers that $\eta \geq 0$ is an additional parameter to deal with the high singularity $|v - v_\ast|^{-3}$. Thanks to (2.1), to get the coercivity estimate of $L_{\varepsilon, \gamma, \eta}$, it suffices to estimate from below the two functionals $N_{\varepsilon, \gamma, r_0}(\mu_1^2, f)$ and $N_{\varepsilon, \gamma, \eta}(f, \mu_1^1)$. We will study $N_{\varepsilon, \gamma, \eta}(f, \mu_1^1)$ in subsection 2.2 and $N_{\varepsilon, \gamma, \eta}(\mu_1^1, f)$ in subsection 2.3. The coercivity estimate is obtained in subsection 2.4 by utilizing (2.1).

In the rest of the article, we will omit the range of some frequently used variables in the integrals. Usually, $\sigma \in S^2, v, v_\ast, u, \xi \in \mathbb{R}^3$. For example, we set $\int (\cdots) d\sigma := \int_{S^2} (\cdots) d\sigma, \int (\cdots) d\sigma dv_\ast := \int_{S^2 \times \mathbb{R}^3} (\cdots) d\sigma dv_\ast$. Integration w.r.t. other variables should be understood in a similar way. Whenever a new variable appears, we will specify its range once and then omit it thereafter.

In the rest of the article, in the various estimates, the involved “suitable” functions($g, h, f,$ etc.) are assumed to be functions such that the corresponding norms of them are well-defined.

We begin with some elementary results which will be used frequently throughout the paper.

2.1. Elementary Results

We collect some properties of the function $W^\varepsilon$ defined in (1.31). Note that $W^\varepsilon$ is a radial function defined on $\mathbb{R}^3$.

Lemma 2.1. For any $x, y \in \mathbb{R}^3$, we have

\begin{align*}
W^\varepsilon(x) &\lesssim \min\{\langle x \rangle, |\ln \varepsilon|^{-\frac{1}{2}} \varepsilon^{-1}\}. \\
W^\varepsilon(x) &\gtrsim \phi(\varepsilon^\frac{1}{2}x)\langle x \rangle. \\
W^\varepsilon(x) &\gtrsim (1 - \phi(\varepsilon^\frac{1}{2}x))\varepsilon^{-\frac{1}{2}}. \\
W^\varepsilon(x) &\gtrsim \phi(\varepsilon x)|\ln \varepsilon|^{-\frac{1}{2}}(x). \\
W^\varepsilon(x) &\gtrsim (1 - \phi(\varepsilon x))|\ln \varepsilon|^{-\frac{1}{2}}\varepsilon^{-1}. \\
W^\varepsilon(x - y) &\lesssim W^\varepsilon(x)W^\varepsilon(y). 
\end{align*}

The following is an interpolation result.

Lemma 2.2. Let $m \geq 0$. For any $\eta > 0$, one has

$$
|f|_{H^m}^2 \lesssim (\eta + \varepsilon)|W^\varepsilon(D)f|_{H^m}^2 + C(\eta, m)|f|_{L_2}^2.
$$

Proof. Recall (1.52). Using $f = f^\varepsilon + f_\varepsilon$, by interpolation inequality, we have

$$
|f|_{H^m}^2 \lesssim |f^\varepsilon|_{H^m}^2 + |f_\varepsilon|_{H^m}^2 \lesssim |f^\varepsilon|_{H^m}^2 + \eta|f_\varepsilon|_{H^{m+1}}^2 + C(\eta, m)|f_\varepsilon|_{L_2}^2.
$$

Then the lemma follows from (2.4) and (2.5). \qed

Definition 2.1. A smooth function $a(v, \xi)$ is said to be a symbol of type $S^m_{1,0}$ if for any multi-indices $\alpha$ and $\beta$,

$$
|\partial^\alpha_\xi \partial^\beta_v a(v, \xi)| \leq C_{\alpha, \beta}(|\xi|^{m-|\alpha|},
$$

where $C_{\alpha, \beta}$ is a constant depending only on $\alpha$ and $\beta$. 


The following is a result on the commutator between multipliers in frequency and phase spaces.

**Lemma 2.3.** (Lemma 5.3 in [14]) Let $l, s, r \in \mathbb{R}$, $M(\xi) \in S^r_{1,0}$ and $\Phi(\xi) \in S^l_{1,0}$, then

$$|[M(D), \Phi]f|_{H^s} \lesssim C |f|_{H^{s-1}_{l+r}}.$$ 

We remark that $W^\varepsilon(\xi)$ is a symbol of type $S^1_{1,0}$. From which together with Lemma 2.3, we have

**Lemma 2.4.** Let $l, m \in \mathbb{R}$, then

$$|W^\varepsilon(D)W_l f|_{H^m} \sim |W_l W^\varepsilon(D) f|_{H^m}.$$ 

Thanks to Lemma 2.4, we interchangeably use $|W^\varepsilon(D)W_l f|_{H^m}$ and $|W_l W^\varepsilon(D) f|_{H^m}$ in the rest of the paper.

**Lemma 2.5.** (Theorem 3.1 in [16]) Let $A^\varepsilon(\xi):= \int b^\varepsilon(\frac{\xi}{|\xi|} \cdot \sigma) \min\{|\xi|^2 \sin^2(\theta/2), 1\}d\sigma$, then it holds that $A^\varepsilon(\xi) \sim |\xi|^2 1_{|\xi| \leq 2} + 1_{|\xi| \geq 2}(W^\varepsilon)^2(\xi)$.

**Lemma 2.6.** (Proposition 5.2 in [15]) Let $b : [0, 1] \to \mathbb{R}_+$ be a non-negative function. Let $h, f$ be real-valued functions and $\hat{h}, \hat{f}$ be their Fourier transform. It holds that

$$\int b\left(\frac{u}{|u|} \cdot \sigma\right)h(u)(f(u^+)-f\left(\frac{|u|}{|u^+|}u^+\right))d\sigma du$$

$$= \int b\left(\frac{\xi}{|\xi|} \cdot \sigma\right)(\hat{h}(\xi^+)-\hat{h}\left(\frac{|\xi|}{|\xi^+|}\xi^+\right))\hat{f}(\xi)d\sigma d\xi.$$ 

Here $u^+ = \frac{u+|u|\sigma}{2}$, $\xi^+ = \frac{\xi^+|\xi|}{2}$.

**Lemma 2.7.** (Lemma 5.8 in [14]) Let $\mathcal{F}$ be the Fourier transform, then $\mathcal{F}W^\varepsilon((-\Delta_{S^2})^{\frac{1}{2}}) = W^\varepsilon((-\Delta_{S^2})^{\frac{1}{2}})\mathcal{F}$.

We collect some properties of the translation operator $T_u$ defined through $(T_u f)(v):= f(u+v)$.

**Lemma 2.8.** Thanks to (2.8), for $u \in \mathbb{R}^3$, we have

$$|W^\varepsilon T_u f|_{L^2} \lesssim W^\varepsilon(u)|W^\varepsilon f|_{L^2}. \quad (2.9)$$

For $u \in \mathbb{R}^3$, $l \in \mathbb{R}$, one has $(T_u W_l)(v) = \langle v+u \rangle^l \lesssim C(l)|\langle u \rangle|^l |\langle v \rangle|^l$ and thus

$$|T_u f|_{L^2_l} \lesssim C(l)|\langle u \rangle|^l |f|_{L^2_l}. \quad (2.10)$$

Let us prepare some integrals regrading to the angular function $b^\varepsilon$ over the unit sphere $S^2$. 

---

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Lemma 2.9. If $0 < \varepsilon \leq 1/4$, the following estimates are valid.

\[
2\pi |\ln \varepsilon|^{-1}\varepsilon^{-2} \leq \int b^\varepsilon(\cos \theta) d\sigma \leq 4\pi |\ln \varepsilon|^{-1}\varepsilon^{-2}. \quad (2.11)
\]

\[
4\pi |\ln \varepsilon|^{-1}\varepsilon^{-1} \leq \int b^\varepsilon(\cos \theta) \sin(\theta/2) d\sigma \leq 8\pi |\ln \varepsilon|^{-1}\varepsilon^{-1}. \quad (2.12)
\]

\[
4\pi \leq \int b^\varepsilon(\cos \theta) \sin^2(\theta/2) d\sigma \leq 8\pi. \quad (2.13)
\]

**Proof.** By Assumption (A2), we can write $b^\varepsilon(\cos \theta) = |\ln \varepsilon|^{-1} \sin^{-4}(\theta/2) 1_{\sqrt{2}/2 \leq \sin(\theta/2) \leq \varepsilon}$. Note that $d\sigma = \sin \theta d\theta d\varphi = 4 \sin(\theta/2) d\sin(\theta/2) d\varphi$, we have

\[
\int b^\varepsilon(\cos \theta) \sin'(\theta/2) d\sigma = 4|\ln \varepsilon|^{-1} \int_0^\pi \int_0^{2\pi} 1_{\sqrt{2}/2 \leq \sin(\theta/2) \leq \varepsilon} \sin^{-3+i}(\theta/2) d\sin(\theta/2) d\varphi
\]

\[
= 8\pi |\ln \varepsilon|^{-1} \int_\varepsilon^{\sqrt{2}/2} t^{-3+i} dt.
\]

Then it is elementary to derive the desired results by using $0 < \varepsilon \leq 1/4$. □

Cancellation lemma is very important especially when we need to shift regularity between the three functions $g$, $h$, $f$ of the inner product $(Q(g,h),f)$. Depending on the parameter $\gamma$ in the kernel $B(v-v_\ast,\sigma) = |v-v_\ast|^\gamma b(\cos \theta)$, there are two types of cancellation lemma. For the case $\gamma > -3$, one may refer [1]. We mainly concern the case $\gamma = -3$ and recall the following cancellation lemma in [4]:

**Lemma 2.10.** (Cancellation Lemma) Recalling from Proposition 3 in [4], we define

\[
J^\varepsilon(z) := 1_{|z| \leq 1} \frac{2\pi}{|z|^3} \int_{2\cos^{-1}(|z|)}^{\pi/2} b^\varepsilon(\cos \theta) \sin \theta d\theta,
\]

whose $L^1$ norm is bounded uniformly in $\varepsilon$,

\[
|J^\varepsilon|_{L^1} = -8\pi^2 \int_0^{\pi/2} b^\varepsilon(\cos \theta) (\ln \cos(\theta/2)) \sin \theta d\theta \lesssim 1. \quad (2.14)
\]

Associated to $B_{\varepsilon;\gamma,\delta} = 1_{|v-v_\ast| \geq \delta}|v-v_\ast|^{-3} b^\varepsilon(\cos \theta)$, the convolution kernel is $S^\varepsilon_{\delta}(z) := \delta^{-3} J^\varepsilon(z/\delta)$. That is,

\[
\int B_{\varepsilon;\gamma,\delta} g_\ast(h' - h)d\sigma dv v_\ast = \int S^\varepsilon_{\delta}(v-v_\ast) g_\ast h dv v_\ast. \quad (2.15)
\]

For our purpose, we derive the following various types of cancellation effect:

**Lemma 2.11.** (Cancellation Lemma Continued) Let $p,q \geq 1$ satisfying $1/p + 1/q = 1$.

* Let $\delta = 0$, $a \in \mathbb{R}$, then

\[
|\int B_{\varepsilon;\gamma,0} g_\ast(h' - h)d\sigma dv v_\ast| \lesssim |\mu^{-a} g|_{L^p} |\mu^a h|_{L^q}. \quad (2.16)
\]
Let $1 \geq \delta \geq 0, a \geq 0$, then
\[
| \int B^{s, y, \delta} g_*(h' - h) d\sigma dv_* | \lesssim e^a | \mu^{-2a} g |_{L^p} | \mu^a h |_{L^q}.
\] (2.17)

Let $1 \geq \eta > \delta \geq 0, a \geq 0$, set $B^{s, y, \delta}_\eta := B^{s, y} 1_{\delta \leq |u - v_*| < \eta}$, then
\[
| \int B^{s, y, \delta}_\eta g_*(h' - h) d\sigma dv_* |
\lesssim (\eta + \varepsilon^{1/2} + a) e^{3a} |W^s(D)\mu^{-2a} g|_{L^2} |\mu^a h|_{L^2},
\] (2.18)
\[
| \int B^{s, y, \delta}_\eta (h' f' - h f) d\sigma dv_* |
\lesssim (\eta + \varepsilon^{1/2} + a) e^{3a} |\mu^{-2a} g|_{L^\infty} (|W^s(D)\mu^a h|_{L^2} |\mu^a f|_{L^2}
+ |\mu^a h|_{L^2} |W^s(D)\mu^a f|_{L^2}).
\] (2.19)

**Proof.** Recalling from (2.15) and $S^s_\phi(z) = \delta^{-3} J^s(z/\delta)$, we have
\[
\int B^{s, y, \delta} g_*(h' - h) d\sigma dv_* = \int J^s(u) g(\delta u + v) h(v) dv du.
\] (2.20)

We set to prove (2.16). By (2.20), (2.14) and Hölder’s inequality, we have
\[
| \int B^{s, y, 0} g_*(h' - h) d\sigma dv_* | = | \lim_{\delta \to 0} \int J^s(u) g(\delta u + v) h(v) dv du |
\leq |J^s|_{L^1} |gh|_{L^1} \lesssim |\mu^{-a} g|_{L^p} |\mu^a h|_{L^q}.
\]

We now go to prove (2.17). It is easy to check that for any $0 \leq \alpha < 1$, it holds that
\[
|x|^2 \geq \alpha |y|^2 - \frac{\alpha}{1 - \alpha} |x - y|^2.
\] (2.21)

Taking $\alpha = 1/2$ in (2.21), if $|x - y| \leq 1$, we have $|x|^2 \geq \frac{1}{2} |y|^2 - |x - y|^2 \geq \frac{1}{2} |y|^2 - 1$ and thus,
\[
|x - y| \leq 1 \Rightarrow \mu(x) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|x|^2}{2}} \leq e^{\frac{1}{2} |y|^2}.
\] (2.22)

From which, we get $\mu^a(v) \leq e^{a/2} \mu^\frac{a}{2}(v + \delta u)$ for $a \geq 0$. That is, $1 \leq e^a \mu^{-2a}(v) \mu^a(v + \delta u)$. By symmetry, $1 \leq e^a \mu^a(v) \mu^{-2a}(v + \delta u)$. Together with Hölder’s inequality, we get for any $|u| \leq 1, \delta \leq 1,$
\[
| \int_{\mathbb{R}^3} g(\delta u + v) h(v) dv | \leq e^{a} \int_{\mathbb{R}^3} \mu^a(v) \mu^{-2a}(v + \delta u) |g(\delta u + v)| |h(v)| dv
\leq e^a |\mu^{-2a} g|_{L^p} |\mu^a h|_{L^q}.
\] (2.23)

Recalling $J^s(u)$ has support in $|u| \leq 1$ and the parameter $\delta \leq 1$, plugging (2.23) into (2.20), together with (2.14), we get (2.17).
We now turn to (2.18). Since \( B_{n}^{\varepsilon, \gamma, \delta} = B_{n}^{\varepsilon, \gamma, \delta} - B_{n}^{\varepsilon, \gamma, \eta} \), we have

\[
| \int B_{n}^{\varepsilon, \gamma, \delta} g_{*}(h' - h) d\sigma dv d\nu | \leq | \int B_{n}^{\varepsilon, \gamma, \delta} g_{*}(h' - h) d\sigma dv d\nu |
\]

\[
+ | \int B_{n}^{\varepsilon, \gamma, \eta} g_{*}(h' - h) d\sigma dv d\nu |.
\]

Then by using (2.17), we get

\[
| \int B_{n}^{\varepsilon, \gamma, \delta} g_{*}(h' - h) d\sigma dv d\nu | \lesssim e^{a} | \mu^{-2a} g | L^{p} | \mu^{a} h | L^{q} \cdot (2.24)
\]

For notational brevity, we set \( G = \mu^{-2a} g, H = \mu^{a} h \) and then have

\[
\int B_{n}^{\varepsilon, \gamma, \delta} g_{*}(h' - h) d\sigma dv d\nu = \int J^{\varepsilon}(u) (g(\delta u + v) - g(\eta u + v)) h(v) dv du
\]

\[
= \int J^{\varepsilon}(u) \left( (\mu^{2a} G)(\delta u + v) - (\mu^{2a} G)(\eta u + v) \right)
\]

\[
(\mu^{-a} H)(v) dv du = \mathcal{I}(G, H).
\]

Recalling (1.52), we have \( G = G_{\varepsilon} + G_{\delta} \) and \( \mathcal{I}(G, H) = \mathcal{I}(G_{\varepsilon}, H) + \mathcal{I}(G_{\delta}, H) \). By Taylor expansion,

\[
(\mu^{2a} G_{\varepsilon})(\delta u + v) - (\mu^{2a} G_{\varepsilon})(\eta u + v) = (\delta - \eta) \int_{0}^{1} (\nabla (\mu^{2a} G_{\varepsilon}))(v(\kappa)) \cdot u d\kappa.
\]

where \( v(\kappa):= (\kappa \delta + (1 - \kappa) \eta) u + v. \) Noting that \((a |v| + 1) \mu^{a} \lesssim C(1 + a) \) for some universal constant \( C \), we have

\[
|\nabla (\mu^{2a} G_{\varepsilon})| \lesssim (a |v| + 1) \mu^{2a} (|G_{\varepsilon}| + |\nabla G_{\varepsilon}|) \lesssim C(1 + a) \mu^{\frac{3a}{2}} (|G_{\varepsilon}| + |\nabla G_{\varepsilon}|).
\]

(2.26)

Plugging (2.25) and (2.26) into the definition of \( \mathcal{I}(G_{\varepsilon}, H) \), we get

\[
|\mathcal{I}(G_{\varepsilon}, H)| \lesssim C(1 + a) (\eta - \delta) \int J^{\varepsilon}(u)(|G_{\varepsilon}(v(\kappa))| + |\nabla G_{\varepsilon}(v(\kappa))|)
\]

\[
\times \mu^{\frac{3a}{2}} (v(\kappa)) \mu^{-a}(\eta) |H(v)| du dv d\kappa.
\]

(2.27)

Taking \( \alpha = 3/4 \) in (2.21), if \( |x - y| \leq 1 \), we have \( |x|^{2} \geq \frac{3}{4} |y|^{2} - 3|x - y|^{2} \geq \frac{3}{4} |y|^{2} - 3 \) and thus

\[
|x - y| \leq 1 \Rightarrow \mu(x) \leq e^{3/2} \mu^{3/4}(y).
\]

(2.28)

Since \( |v(\kappa) - v| = |(\kappa \delta + (1 - \kappa) \eta) u| \leq \eta |u| \leq 1 \), by (2.28), we get \( \mu^{\frac{3a}{2}} (v(\kappa)) \mu^{-a}(\eta) \lesssim e^{3a} \) for \( a \geq 0 \). Plugging which into (2.27), we get

\[
|\mathcal{I}(G_{\varepsilon}, H)| \lesssim (1 + a) e^{3a} (\eta - \delta) \int J^{\varepsilon}(u)(|G_{\varepsilon}(v(\kappa))|)
\]
By the change of variable $v \rightarrow v(\kappa)$ and Cauchy-Schwartz inequality, we get

$$\left| \mathcal{I}(G_\varepsilon, H) \right| \lesssim (1 + a)e^{3\eta}(\eta - \delta)|J^\varepsilon|_{L^1} |G_\varepsilon|_{H^1} |H|_{L^2}.$$  

By (2.4), one has $|G_\varepsilon|_{H^1} \lesssim |W^\varepsilon(D)G|_{L^2}$. From which together with (2.14), we have

$$\left| \mathcal{I}(G_\varepsilon, H) \right| \lesssim (1 + a)e^{3\eta}|W^\varepsilon(D)G|_{L^2} |H|_{L^2}. \quad (2.30)$$

By (2.5), one has $|G^\varepsilon|_{L^2} \lesssim 1/\varepsilon^2|W^\varepsilon(D)G|_{L^2}$. From which together with (2.24), we get

$$\left| \mathcal{I}(G^\varepsilon, H) \right| \lesssim e^a|G^\varepsilon|_{L^2} |H|_{L^2} \lesssim 1/\varepsilon^2 e^a|W^\varepsilon(D)G|_{L^2} |H|_{L^2}. \quad (2.31)$$

Patching together (2.30) and (2.31), we get (2.18).

In the last, we go to prove (2.19). Thanks to the convolution structure in (2.20), we have

$$\int B^{\varepsilon, \gamma, \delta}_\eta g_\eta ((hf)' - hf) \, d\sigma dv \, d\nu = \int J^\varepsilon(u) ((hf)(v - \delta u) - (hf)(v - \eta u)) g(v) dv \, du.$$  

In this part, we set $G = \mu^{-2a}g$, $H = \mu^2 h$, $F = \mu^2 f$ and thus have

$$\int B^{\varepsilon, \gamma, \delta}_\eta g_\eta ((hf)' - hf) \, d\sigma dv \, d\nu = \int J^\varepsilon(u) ((\mu^{-a}HF)(v - \delta u) - (\mu^{-a}HF)(v - \eta u))(\mu^{-2a}G)(v) dv \, du : = \mathcal{J}(H, F).$$

Recalling from (1.52) for the definition of $H_\varepsilon, H^\varepsilon, F_\varepsilon, F^\varepsilon$. Decomposing $H = H_\varepsilon + H^\varepsilon$, $F = F_\varepsilon + F^\varepsilon$, we get

$$\mathcal{J}(H, F) = \mathcal{J}(H_\varepsilon, F_\varepsilon) + \mathcal{J}(H_\varepsilon, F^\varepsilon) + \mathcal{J}(H^\varepsilon, F_\varepsilon) + \mathcal{J}(H^\varepsilon, F^\varepsilon).$$

For $\mathcal{J}(H_\varepsilon, F_\varepsilon)$, we apply Taylor expansion to $\mu^{-a}H_\varepsilon F_\varepsilon$. Similar to (2.25), (2.26), (2.27) and (2.29), we get

$$|\mathcal{J}(H_\varepsilon, F_\varepsilon)| \lesssim C(|a| + 1)e^{3a}(\eta - \delta) \int J^\varepsilon(u) |(H_\varepsilon F_\varepsilon)(v(\kappa))|$$

$$+ |(F_\varepsilon \nabla H_\varepsilon)(v(\kappa))| + |(H_\varepsilon \nabla F_\varepsilon)(v(\kappa))|)$$

$$\times |G(v)|dv \, d\nu$$

$$\lesssim (|a| + 1)e^{3a}(\eta - \delta)\int J^\varepsilon |(H_\varepsilon H^\varepsilon)(v)|_{L^1} |G|_{L^\infty}(|H_\varepsilon|_{H^1} |F_\varepsilon|_{L^2} + |H_\varepsilon|_{L^2} |F_\varepsilon|_{H^1})$$

$$\lesssim (|a| + 1)e^{3a}|G|_{L^\infty}(|W^\varepsilon(D)H|_{L^2} |F|_{L^2} + |H|_{L^2} |W^\varepsilon(D)F|_{L^2}).$$

where we take out $|G|_{L^\infty}$ and use the change of variable $v \rightarrow v(\kappa)$, Cauchy-Schwartz inequality and (2.4). Using (2.22) to deal with the $\mu$-type weight, taking out $|G|_{L^\infty}$, applying Cauchy-Schwartz inequality and using (2.5), we get

$$|\mathcal{J}(H_\varepsilon, F^\varepsilon) + \mathcal{J}(H^\varepsilon, F_\varepsilon) + \mathcal{J}(H^\varepsilon, F^\varepsilon)|$$
Proof. If 0 < ε ≦ ε0 and 0 ≦ η ≦ 1, we have
\[ \left| f \right|_{L^2}^2 \geq C_{\gamma} \left| W^\varepsilon f \right|_{L^2}^2, \]
where \( C_{\gamma} \) is a positive constant depending only on \( \gamma \).

Proposition 2.1. Let \(-3 \leq \gamma \leq 0\). There exists \( \varepsilon_0 > 0 \) such that for \( 0 < \varepsilon \leq \varepsilon_0 \) and \( 0 \leq \eta \leq 1 \),
\[ \mathcal{N}^{\varepsilon, \eta}(f, \mu^1) \leq \frac{1}{\varepsilon} \mathcal{N}^{\varepsilon, \eta}(\mu^1, H^\varepsilon). \]

The functional \( \mathcal{N}^{\varepsilon, \eta}(f, \mu^1) \) produces gain of the weight \( W^\varepsilon \) in the phase space.

2.2. Gain of Weight from \( \mathcal{N}^{\varepsilon, \eta}(f, \mu^1) \)

The functional \( \mathcal{N}^{\varepsilon, \eta}(f, \mu^1) \) produces gain of the weight \( W^\varepsilon \) in the phase space.

Proposition 2.1. Let \(-3 \leq \gamma \leq 0\). There exists \( \varepsilon_0 > 0 \) such that for \( 0 < \varepsilon \leq \varepsilon_0 \) and \( 0 \leq \eta \leq 1 \),
\[ \mathcal{N}^{\varepsilon, \eta}(f, \mu^1) \leq \frac{1}{\varepsilon} \mathcal{N}^{\varepsilon, \eta}(\mu^1, H^\varepsilon). \]

The proof is divided into four steps. Let \( 0 < \delta < 1 \leq R \) be two constants which will be determined later.

Step 1: \( 16/\pi \leq |v_*| \leq \delta/\varepsilon \). One has \( \nabla \mu^1 = -\frac{\mu^1}{2} v \) and \( \nabla^2 \mu^1 = \frac{\mu^1}{4} (-2I_3 + v \otimes v) \). By Taylor expansion, we have
\[ \mu^1(v') - \mu^1(v) = -\frac{\mu^1}{2} v \cdot (v' - v) + \int_0^1 (1 - \kappa)(\nabla^2 \mu^1)(v(\kappa)) : (v' - v) \]
\[ \otimes (v' - v) d\kappa, \]
where \( v(\kappa) := v + \kappa(v' - v) \). Thanks to the fact that \( (a - b)^2 \geq \frac{a^2}{2} - b^2 \), we have
\[ (\mu^1(v') - \mu^1(v))^2 \geq \frac{\mu^1(v)}{8} |v \cdot (v' - v)|^2 \]
\[ - \int_0^1 (1 - \kappa)^2 |(\nabla^2 \mu^1)(v(\kappa))|^2 |v' - v|^4 d\kappa. \]

For \( \varepsilon \leq \pi \delta / 16 \), we set \( A(\varepsilon, \delta) := \{(v_*, v, \sigma) \in 16/\pi \leq |v_*| \leq \delta/\varepsilon, |v| \leq 8/\pi, \varepsilon \leq \sin(\theta/2) \leq 4\delta/|v_*|^{-1}\} \). It is easy to check that \( A(\varepsilon, \delta) \) is non-empty. Thus we have
\[ \mathcal{N}^{\varepsilon, \eta}(f, \mu^1) \geq \int B^{\varepsilon, \eta} A(\varepsilon, \delta) f^2_\ast \left| (\mu^1(v')) - \mu^1(v) \right|^2 d\sigma dv dv_\ast \]
\[ \geq \frac{1}{8} \int B^{\varepsilon, \eta} A(\varepsilon, \delta) f^2_\ast |v \cdot (v' - v)|^2 f^2_\ast d\sigma dv dv_\ast \]
\[ - \int B^{\varepsilon, \eta} A(\varepsilon, \delta) |(\nabla^2 \mu^1)(v(\kappa))|^2 |v' - v|^4 f^2_\ast d\sigma dv dv_\ast d\kappa. \]
\[ I_{1,1}^{\epsilon,\gamma,\delta}(\delta) := \frac{1}{8} \mathcal{I}_{1,1}^{\epsilon,\gamma,\delta}(\delta) - \mathcal{I}_{2,2}^{\epsilon,\gamma,\delta}(\delta). \] (2.32)

Estimate of \( I_{1,1}^{\epsilon,\gamma,\delta}(\delta) \). For fixed \( v, v_* \), we introduce an orthonormal basis \((h_{v,v_*}, h_{v,v_*}, v - v_*/|v - v_*|)\) such that \( d\sigma = \sin\theta d\theta d\varphi \). Then one has

\[
\begin{align*}
\frac{v' - v}{|v' - v|} &= \cos \frac{\theta}{2} \cos \varphi h_{v,v_*} + \cos \frac{\theta}{2} \sin \varphi h_{v,v_*} - \sin \frac{\theta}{2} \frac{v - v_*}{|v - v_*|}, \\
\frac{v}{|v|} &= a_1 h_{v,v_*} + a_2 h_{v,v_*} + a_3 \frac{v - v_*}{|v - v_*|},
\end{align*}
\]

where \( a_3 = \frac{v}{|v|} \cdot \frac{v - v_*}{|v - v_*|} \) and \( a_1, a_2 \) are constants independent of \( \theta \) and \( \varphi \). Then we have

\[
\frac{|v}{|v|} \cdot \frac{v' - v}{|v' - v|} = a_1^2 \cos^2 \frac{\theta}{2} \cos^2 \varphi + a_2^2 \cos^2 \frac{\theta}{2} \sin^2 \varphi + a_3^2 \sin^2 \frac{\theta}{2} + 2a_1a_2 \cos \theta \frac{\theta}{2} \cos \varphi \sin \varphi - 2a_3 \cos \frac{\theta}{2} \sin \frac{\theta}{2} (a_1 \cos \varphi + a_2 \sin \varphi).
\]

Integrating with respect to \( \sigma \), we have

\[
\int b^\delta (\cos \theta) 1_{A(\epsilon,\delta)} |v \cdot (v' - v)|^2 d\sigma
\]

\[
= \int_0^\pi \int_0^{2\pi} b^\delta (\cos \theta) \sin \theta 1_{A(\epsilon,\delta)} |v \cdot (v' - v)|^2 d\theta d\varphi
\]

\[
\geq \pi (a_1^2 + a_2^2) |v|^2 |v - v_*|^2 \int_0^\pi b^\delta (\cos \theta) \sin \theta \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} 1_{A(\epsilon,\delta)} d\theta.
\] (2.33)

Let \( B(\epsilon, \delta) := \{(v_*, v) \mid 16/\pi \leq |v_*| \leq \delta/\epsilon, |v| \leq 8/\pi\} \). If \((v, v_*) \in B(\epsilon, \delta)\), direct computation gives

\[
\int_0^\pi b^\delta (\cos \theta) \sin \theta \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} 1_{\epsilon \leq \sin(\theta/2) \leq 4\delta |v_*|^{-1}} d\theta
\]

\[
= 4 |\ln \epsilon|^{-1} \int_0^\pi \cos^2 \frac{\theta}{2} \sin^{-1} \frac{\theta}{2} 1_{\epsilon \leq \sin(\theta/2) \leq 4\delta |v_*|^{-1}} d\sin \frac{\theta}{2}
\]

\[
= 4 |\ln \epsilon|^{-1} \int_0^1 (1 - t^2) t^{-1} 1_{\epsilon \leq t \leq 4\delta |v_*|^{-1}} dt
\]

\[
\geq 2 |\ln \epsilon|^{-1} \int_{\epsilon}^{4\delta |v_*|^{-1}} t^{-1} dt = 2 |\ln \epsilon|^{-1} (\ln(4\delta) - \ln |v_*| - \ln \epsilon),
\]

where we use \( t \leq 4\delta |v_*|^{-1} \leq \sqrt{2}/2 \) and \( 1 - t^2 \geq 1/2 \). Back to (2.33), we get

\[
\int b^\delta (\cos \theta) 1_{A(\epsilon,\delta)} |v \cdot (v' - v)|^2 d\sigma
\]
\[ \geq 2\pi (a_1^2 + a_2^2) |v|^2 |v - v_*|^2 1_{B(\varepsilon, \delta)} |\ln \varepsilon|^{-1} (\ln (4\delta) - \ln |v_*| - \ln \varepsilon). \]

If \((v, v_*) \in B(\varepsilon, \delta)\), then \(|v - v_*| \geq \frac{8}{\pi} \geq 1\), which gives

\[
I_1^{\varepsilon, \gamma, 1}(\delta) \geq 2\pi |\ln \varepsilon|^{-1} \int (a_1^2 + a_2^2) |v - v_*|^\gamma + 2 |v|^2 1_{B(\varepsilon, \delta)} (\ln (4\delta) - \ln |v_*| - \ln \varepsilon) (\ln (4\delta) - \ln |v_*| - \ln \varepsilon) 1_{B(\varepsilon, \delta)} \mu(v) f_*^2 dv_* \]

where we use the fact \(a_1^2 + a_2^2 + a_3^2 = 1\) and the law of sines

\[ (1 - \left(\frac{v}{|v|} \cdot \frac{v_*}{|v_*|}\right)^2)^{-1}|v - v_*|^2 = (1 - a_3^2)^{-1}|v_*|^2. \]

If \((v, v_*) \in B(\varepsilon, \delta)\), one has \(|v_*|/2 \leq |v - v_*| \leq 3|v_*|/2\), which yields

\[
I_1^{\varepsilon, \gamma, 1}(\delta) \geq 2\pi (3/2)^\gamma |\ln \varepsilon|^{-1} \int (1 - \left(\frac{v}{|v|} \cdot \frac{v_*}{|v_*|}\right)^2) |v_*|^\gamma + 2 |v|^2 \]

\[
(\ln (4\delta) - \ln |v_*| - \ln \varepsilon) 1_{B(\varepsilon, \delta)} \mu(v) f_*^2 dv_* \]

\[ = 2\pi (3/2)^\gamma c_1 |\ln \varepsilon|^{-1} \int (\ln (4\delta) - \ln |v_*| - \ln \varepsilon) |v_*|^\gamma + 2 |v|^2 1_{6/\pi \leq |v_*| \leq \delta/\varepsilon} f_*^2 dv_* \tag{2.34} \]

where we denote \(c_1 := \int (1 - \left(\frac{v}{|v|} \cdot \frac{v_*}{|v_*|}\right)^2) |\mu(v)| |v|^2 1_{|v| \leq 8/\pi} dv\).

**Estimate** \(I_2^{\varepsilon, \gamma, 1}(\delta)\). Recalling \(B^{\varepsilon, \gamma, 1} = \{ |v - v_*| \geq 1, \sin(\theta/2) \geq \varepsilon \} |\ln \varepsilon|^{-1} |v - v_*|^\gamma \sin^{-4}(\theta/2), |v' - v| = |v - v_*| \sin(\theta/2)\), we have

\[
I_2^{\varepsilon, \gamma, 1}(\delta) = \int B^{\varepsilon, \gamma, 1} 1_{A(\varepsilon, \delta)} (|\nabla^2 \mu|^{1/2})(v(\kappa)) |v' - v|^4 f_*^2 d\sigma dv_* d\kappa
\]

\[ = |\ln \varepsilon|^{-1} \int 1_{A(\varepsilon, \delta)} (|\nabla^2 \mu|^{1/2})(v(\kappa)) |v - v_*|^\gamma + 4 f_*^2 d\sigma dv_* d\kappa \]

\[ \leq (3/2)^{\gamma+4} |\ln \varepsilon|^{-1} \int 1_{A(\varepsilon, \delta)} (|\nabla^2 \mu|^{1/2})(v(\kappa)) |v_*|^\gamma |v|^4 f_*^2 d\sigma dv_* d\kappa. \]

Given \(\kappa \in [0, 1]\), \(v_* \in \mathbb{R}^3\), we will use the following change of variables

\[ (\sigma = (\theta, \phi, v) \rightarrow (\sigma = (\theta(\kappa), \phi), v(\kappa)). \tag{2.35} \]

Here \(\theta(\kappa)\) is the angle between \(v(\kappa) - v_*\) and \(\sigma\). In the change, one has

\[
\frac{\theta}{2} \leq \theta(\kappa) \leq \theta, \quad \sin \theta \leq 2 \sin \theta(\kappa), \quad |v - v_*|/\sqrt{2} \leq |v(\kappa) - v_*| \leq |v - v_*|. \tag{2.36} \]

From which together with \(|\det \left(\frac{\partial(v(\kappa), \theta(\kappa))}{\partial(v, \theta)}\right)|^{-1} \leq (1 - \frac{\kappa}{2})^{-5} \leq 32\), we have

\[ d\sigma dv = \sin \theta d\theta d\phi dv \leq 2^6 \sin \theta(\kappa) d\theta(\kappa) d\phi(\kappa) dv(\kappa) = 2^6 d\sigma dv(\kappa), \tag{2.37} \]
and $I_{\epsilon, \delta} \leq 1_{16/\pi \leq |v_*| \leq \delta/\varepsilon, \sin(\theta(\kappa)/2) \leq 4\delta |v_*|^{-1}} =: I_{C(\epsilon, \delta)$. After the change, we get

\[
I_2^{\epsilon, \gamma, 1}(\delta) \leq 2^6 (3/2)^{\gamma+4} |\ln \varepsilon|^{-1}
+ \int 1_{C(\epsilon, \delta)} \left| (\nabla^2 \mu^{1/2}) (v(\kappa)) \right|^2 |v_*|^{\gamma+4} f_*^2 d\sigma d(\varepsilon, \delta)\ .
\]

(2.38)

In the region $C(\epsilon, \delta)$, one has $\sin(\theta(\kappa)/2) \leq 4\delta |v_*|^{-1}$, then we have

\[
\int 1_{A(\epsilon, \delta)} d\sigma \leq 2\pi \int_0^\pi 1_{16/\pi \leq |v_*| \leq \delta/\varepsilon} \sin(\theta(\kappa)/2) \leq 4\delta |v_*|^{-1} \sin \theta(\kappa) d\theta(\kappa) = 2^6 \pi \times 1_{16/\pi \leq |v_*| \leq \delta/\varepsilon} \delta^2 |v_*|^{-2}.
\]

Plugging this into (2.38), we get

\[
I_2^{\epsilon, \gamma, 1}(\delta) \leq 2^{12} \pi (3/2)^{\gamma+4} \delta^2 |\ln \varepsilon|^{-1}
+ \int 1_{16/\pi \leq |v_*| \leq \delta/\varepsilon} \left| (\nabla^2 \mu^{1/2}) (v(\kappa)) \right|^2 |v_*|^{\gamma+2} f_*^2 d(\varepsilon, \delta)\ .
\]

(2.39)

where $c_2 = \int |(\nabla^2 \mu^{1/2})(v)|^2 dv$. Plugging (2.34) and (2.39) into (2.32), thanks to $(\ln(4\delta) - |v_*| - |\ln \varepsilon|) \geq \ln 4$ when $|v_*| \leq \delta/\varepsilon$, we get

\[
\mathcal{N}^{\epsilon, \gamma, 1}(f, \mu^{1/2}) \geq \left( 2^{-2} \pi (3/2)^{\gamma} c_1 - 2^{12} \pi (3/2)^{\gamma+4} c_2 (4)^{-1} \delta^2 \right)
\times |\ln \varepsilon|^{-1} \int (\ln(4\delta) - |v_*| - |\ln \varepsilon|) 1_{16/\pi \leq |v_*| \leq \delta/\varepsilon} |v_*|^{\gamma+2} f_*^2 dv_*.
\]

For brevity, let $C_1 = 2^{-2} \pi (3/2)^{\gamma} c_1$, $C_2 = 2^{12} \pi (3/2)^{\gamma+4} c_2 (4)^{-1}$. We choose $\delta$ such that $C_2 \delta^2 = C_1/2$ and thus,

\[
\mathcal{N}^{\epsilon, \gamma, 1}(f, \mu^{1/2}) \geq \left( C_1/2 \right) |\ln \varepsilon|^{-1}
\int (\ln(4\delta) - |v_*| - |\ln \varepsilon|) 1_{16/\pi \leq |v_*| \leq \delta/\varepsilon} |v_*|^{\gamma+2} f_*^2 dv_*.
\]

(2.40)

**Step 2:** $|v_*| \geq R/\varepsilon$. Here $R \geq 1$, $\varepsilon \leq 1/2$. By direct computation, we have

\[
\mathcal{N}^{\epsilon, \gamma, 1}(f, \mu^{1/2}) = \int B^{\epsilon, \gamma, 1} f_*^2 ((\mu^{1/2})' - \mu^{1/2})^2 d\sigma dv dv_*
\geq \int b^{\epsilon}(\cos \theta) |v - v_*|^\gamma 1_{|v_*| \geq R/\varepsilon} |v| \leq 1 f_*^2 ((\mu^{1/2})' - \mu^{1/2})^2 d\sigma dv dv_*
\geq \int b^{\epsilon}(\cos \theta) |v - v_*|^\gamma 1_{|v_*| \geq R/\varepsilon} |v| \leq 1 f_*^2 \mu d\sigma dv dv_*
- 2 \int b^{\epsilon}(\cos \theta) |v - v_*|^\gamma 1_{|v_*| \geq R/\varepsilon} |v| \leq 1 f_*^2 (\mu^{1/2})' \mu^{1/2} d\sigma dv dv_*
:= \mathcal{J}^{\epsilon, \gamma}_1(R) - \mathcal{J}^{\epsilon, \gamma}_2(R).
\]

(2.41)
By (2.11), we have
\[
\mathcal{J}_1^{\epsilon, \gamma}(R) \geq 2\pi |\ln \epsilon|^{-1} e^{-2} \int |v - v_*|^{\gamma} 1_{|v_*| \geq R/\epsilon} 1_{|v| \leq 1} f_{\epsilon*}^2 \mu d\sigma dv_* \\
\geq 2\pi (3/2)^\gamma |\ln \epsilon|^{-1} e^{-2} \int |v_*|^{\gamma} 1_{|v_*| \geq R/\epsilon} 1_{|v| \leq 1} f_{\epsilon*}^2 \mu d\sigma dv_* \\
= 2\pi (3/2)^\gamma c_3 |\ln \epsilon|^{-1} e^{-2} \int |v_*|^{\gamma} 1_{|v_*| \geq R/\epsilon} f_{\epsilon*}^2 dv_*, \quad (2.42)
\]
where \(c_3 = \int 1_{|v| \leq 1} \mu dv\) and we use the following relation for \(|v_*| \geq R/\epsilon \geq 2, |v| \leq 1,\)
\[
|v_*|/2 \leq |v - v_*| \leq 3|v_*|/2. \quad (2.43)
\]
Since \(\sin(\theta/2) \geq \epsilon\), there holds \(|v'| + |v| \geq |v' - v| = \sin \theta/2 |v - v_*| \geq \epsilon |v - v_*| \geq \epsilon (|v_*| - |v|)\) and then \(|v' + (1 + \epsilon)|v| \geq \epsilon |v_*| \geq R\). From which we have \(R^2 \leq (|v'| + 2|v|)^2 \leq 8(|v'|^2 + |v|^2)\) and
\[
(\mu^{1/2})' \mu^{1/2} = (2\pi)^{-1/2} e^{-\frac{|v|^2 + |v_*|^2}{4}} \leq (2\pi)^{-1/2} e^{-\frac{|v|^2}{8}} e^{-\frac{R^2}{2\delta}}. \quad (2.44)
\]
Then by (2.44), the upper bound in (2.11), the lower bound in (2.43), we have
\[
\mathcal{J}_2^{\epsilon, \gamma}(R) = 2 \int b^\epsilon (\cos \theta)|v - v_*|^{\gamma} 1_{|v_*| \geq R/\epsilon} 1_{|v| \leq 1} f_{\epsilon*}^2 (\mu^{1/2})' \mu^{1/2} d\sigma dv_* \\
\leq 2(2\pi)^{-3/2} e^{-\frac{R^2}{2\delta}} \int b^\epsilon (\cos \theta)|v - v_*|^{\gamma} 1_{|v_*| \geq R/\epsilon} 1_{|v| \leq 1} f_{\epsilon*}^2 e^{-\frac{|v|^2}{8}} d\sigma dv_* \\
\leq 8\pi (2\pi)^{-1/2} e^{-\frac{R^2}{2\delta}} |\ln \epsilon|^{-1} e^{-2} \int |v - v_*|^{\gamma} 1_{|v_*| \geq R/\epsilon} 1_{|v| \leq 1} f_{\epsilon*}^2 dv_* \\
\leq 8\pi (2\pi)^{-1/2} e^{-\frac{R^2}{2\delta}} (1/2)^\gamma c_4 |\ln \epsilon|^{-1} e^{-2} \int |v_*|^{\gamma} 1_{|v_*| \geq R/\epsilon} f_{\epsilon*}^2 dv_*, \quad (2.45)
\]
where \(c_4 = \int 1_{|v| \leq 1} e^{-\frac{|v|^2}{8}} dv\). Plugging (2.42) and (2.45) into (2.41), we arrive at
for any \(\epsilon \leq 1/2, R \geq 1,\)
\[
\mathcal{N}^{\epsilon, \gamma, 1}(f, \mu^{1/2}) \geq \left( C_3 - C_4 e^{-\frac{R^2}{2\delta}} \right) |\ln \epsilon|^{-1} e^{-2} \int |v_*|^{\gamma} 1_{|v_*| \geq R/\epsilon} f_{\epsilon*}^2 dv_*, \quad (2.46)
\]
where \(C_3 = 2\pi (3/2)^\gamma c_3, C_4 = 8\pi (2\pi)^{-3/2} (1/2)^\gamma c_4.\)

Step 3: \(|v_*| \geq \delta/\epsilon\). Note that the above estimate (2.46) is valid for any \(R \geq 1\) and \(\epsilon \leq 1/2\). For the fixed \(\delta > 0\) in Step 1, we choose \(N\) large enough such that
\(N\delta \geq 1\) and \(C_3 - C_4 e^{-\frac{(N\delta)^2}{2\delta}} \geq C_3/2\). Taking \(R = N\delta\) in (2.46), when \(\epsilon\) is small such that \(N\epsilon \leq 1/2\), we have
\[
\mathcal{N}^{\epsilon, \gamma, 1}(f, \mu^{1/2}) \geq |\ln \epsilon|^{-1} |\ln(N\epsilon)| \mathcal{N}^{N\epsilon, \gamma, 1}(f, \mu^{1/2}) \\
\geq (C_3/2)N^{-2} |\ln \epsilon|^{-1} e^{-2} \int |v_*|^{\gamma} 1_{|v_*| \geq \delta/\epsilon} f_{\epsilon*}^2 dv_* \cdot (2.47)
\]
Step 4: gain of the weight $W^\varepsilon$. If $|v_*| \geq 16/\pi \geq 4$, then $|v_*|^2 \leq 1 + |v_*|^2 = \langle v_* \rangle^2 \leq \frac{17}{16} |v_*|^2$ and

$$|v_*|^\gamma + 2 \geq \min \{ 1, (17/16)^{-\gamma/2-1} \} \langle v_* \rangle^\gamma + 2, \quad |v_*|^\gamma \geq \langle v_* \rangle^\gamma.$$  

For simplicity, set $C_1(\gamma) = \min \{ 1, (17/16)^{-\gamma/2-1} \}$. It is obvious $|f|^2_{L^{2}\gamma/2} \geq \int 1_{|v_*| \leq 16/\pi} \langle v_* \rangle^\gamma f_*^2 dv_*$. From these facts together with (2.40), (2.47), we arrive at

$$N^{\varepsilon, \gamma, 1}(f, \mu \frac{1}{z}) + |f|^2_{L^{2}\gamma/2} \geq \int 1_{|v_*| \leq 16/\pi} \langle v_* \rangle^\gamma f_*^2 dv_* + (C_1/4) C_1(\gamma) \ln \varepsilon^{-1} \int (\ln(4\delta) - \ln |v_*| - \ln \varepsilon) \langle v_* \rangle^\gamma + 2 1_{16/\pi \leq |v_*| \leq \delta \varepsilon} f_*^2 dv_* + (C_3/4) N^{-2} |\ln \varepsilon|^{-1} \varepsilon^{-2} \int \langle v_* \rangle^\gamma 1_{|v_*| \geq \delta \varepsilon} f_*^2 dw_*.$$  

(2.48)

By (2.3), we have $1_{|v_*| \leq 16/\pi} W^\varepsilon(v_*) \leq C_5 (1 + 16^2/\pi^2)^{\frac{1}{2}}$ for some universal constant $C_5 > 0$, which gives

$$1_{|v_*| \leq 16/\pi} \geq C_5^{-2} (1 + 16^2/\pi^2)^{-1} 1_{|v_*| \leq 16/\pi} (W^\varepsilon)^2(v_*).$$  

(2.49)

In the middle region $16/\pi \leq |v_*| \leq \delta/\varepsilon$, we have

$$\langle v_* \rangle^2 |\ln \varepsilon|^{-1} (\ln(4\delta) - \ln |v_*| - \ln \varepsilon) \geq \frac{\ln 4}{1 - \ln \delta} \langle v_* \rangle^2 |\ln \varepsilon|^{-1} (1 - \ln |v_*| - \ln \varepsilon) \geq \frac{\ln 4}{1 - \ln \delta} (W^\varepsilon)^2(v_*).$$  

(2.50)

In the large velocity region $|v_*| \geq \delta/\varepsilon$, by (2.3), we have

$$|\ln \varepsilon|^{-1} \varepsilon^{-2} \geq C_5^{-2} (W^\varepsilon)^2(v_*).$$  

(2.51)

Plugging (2.49), (2.50) and (2.51) into (2.48), we get

$$N^{\varepsilon, \gamma, 1}(f, \mu \frac{1}{z}) + |f|^2_{L^{2}\gamma/2} \geq C_\gamma |W^\varepsilon f|^2_{L^{2}\gamma/2},$$  

where $C_\gamma = \min \{ C_5^{-2} (1 + 16^2/\pi^2)^{-1}, (C_1/4) C_1(\gamma) \ln 4 / (1 - \ln \delta), (C_3/4) N^{-2} C_5^{-2} \}$ is a positive constant depending only on $\gamma$. From the above proof, we can take $\varepsilon_0 = \min \{ \frac{\delta}{16}, \frac{1}{\pi N} \}$ where $\delta$ is the fixed constant in Step 1 and $N$ is the fixed constant in Step 3. The proof of the proposition is complete. $\Box$

We next show that the lower bound in Proposition 2.1 is sharp in the following sense:

**Proposition 2.2.** Let $-3 \leq \gamma \leq 0 \leq \eta$. Then $N^{\varepsilon, \gamma, \eta}(f, \mu \frac{1}{z}) \lesssim |W^\varepsilon f|^2_{L^{2}\gamma/2}$. 

Proof. It is obvious that $N^{ε,γ,η}(f, μ^{1/2}) \leq N^{ε,γ,0}(f, μ^{1/2})$. Therefore it suffices to consider the upper bound of $N^{ε,γ,0}(f, μ^{1/2})$. First we have

$$N^{ε,γ,0}(f, μ^{1/2}) \lesssim \int B^{ε,γ} f^2_*(|μ^{1/2})^2((μ^{1/2})^2(μ^{1/2})^2 + μ^{1/2})dσdv_*$

$$\lesssim \int B^{ε,γ} f^2_*(|μ^{1/2})^2(μ^{1/2})^2(μ^{1/2})^2dσdv_*

+ \int B^{ε,γ} f^2_*(|μ^{1/2})^2μ^{1/2}dσdv_*

:= I^1_{ε,γ}(f) + I^2_{ε,γ}(f).$$

By Taylor expansion, one has $((μ^{1/2})^2 - μ^{1/2})^2 \lesssim \min\{1, |v - v_*|^2\theta^2\} \sim \min\{1, |v' - v_*|^2\theta^2\}$. By Lemma 2.5, we have $\int b^ε(\cos θ)\min\{1, |v - v_*|^2\theta^2\}dσ \sim |v - v_*|^2|v - v_*|^2 + (W^ε)^2(v - v_*)|v - v_*|^2$. From which together with (2.8), we have

$$I^2_{ε,γ}(f) \lesssim \int f^2_*|v - v_*|^2|v - v_*|^2 + (W^ε)^2(v - v_*)μ^{1/2}dσdv_* \lesssim |W^ε f|^2_{L^2_{γ/2}}.$$ Here in the first integral $|v - v_*| \leq 2$, we used (2.21) to get $μ^{1/2} \lesssim μ^{1/2}_* μ^{1/2}_*$, and so,

$$\int f^2_*|v - v_*|^2|v - v_*|^2 + (W^ε)^2(v - v_*)μ^{1/2}dσdv_* \lesssim \int μ^2_* f^2_*(v - v_*)^2 + (W^ε)^2(v - v_*)μ^{1/2}dσdv_* \lesssim |f|^2_{L^2_{γ/2}}.$$ In the second integral $|v - v_*| \geq 2$, we used $|v - v_*|\lesssim (v - v_*)^γ \lesssim (v_*)^γ (v)$. The term $I^1_{ε,γ}(f)$ can be similarly estimated by the change of variable $v \rightarrow v'$ (take $κ = 1$ in (2.35)). We end the proof here. □

2.3. Gain of Regularity From $N^{ε,γ,η}(g, f)$

In what follows, we will focus on gain of regularity from $N^{ε,γ,η}(g, f)$. Our strategy can be concluded as follows:

(1) gain of regularity from $N^{ε,0,0}(g, f)$.
(2) gain of regularity from $N^{ε,0,η}(g, f)$ by reducing to $N^{ε,0,0}(g, f)$.
(3) gain of regularity from $N^{ε,γ,η}(g, f)$ by reducing to $N^{ε,0,η}(g, f)$.

2.3.1. Gain of Regularity From $N^{ε,0,0}(g, f)$ We first show gain of Sobolev regularity. Recalling from [1] that for $g \geq 0$ with $|g|_{L^1} \geq δ > 0$ and $|g|_{L^1_{γ} \cap L log L} \leq λ$ where $|g|_{L^1_{γ} \cap L log L} = \int |g(v)|v^2dv + \int |g(v)| log(1 + |g(v)|)dv$, it holds that

$$\int b(\cos θ)g_*(f' - f)^2dσdv_*dv + |f|^2_{L^2} \geq C(δ, λ)|a(D)f|^2_{L^2},$$

where $a(ξ) = \int b(\frac{ξ}{|ξ|} \cdot σ) min\{|ξ|^2 sin^2(θ/2), 1\}dσ + 1$. Therefore by Lemma 2.5, we get
Lemma 2.13. Let $g$ be a function such that $\|g^2\|_{L^1} \geq \delta > 0$, $\|g^2\|_{L^1\cap L\log L} \leq \lambda < \infty$, then there exists a constant $C(\delta, \lambda)$ such that

$$N^{r,0,0}(\xi,f) + |f|^2_{L^2} \geq C(\delta, \lambda)|W^r(D)f|^2_{L^2}.$$

Next we want to derive gain of anisotropic regularity from $N^{r,0,0}(\xi,f)$. In this part, we derive anisotropic regularity from $N^{r,\gamma,\eta}(\mu^{1/2}, f)$. To this end, our strategy is to apply the geometric decomposition in the frequency space. More precisely, we will use the following decomposition

$$\hat{f}(\xi) - \hat{f}(\xi^+) = (\hat{f}(\xi) - \hat{f}(\xi|\xi^+|)) + (\hat{f}(\xi|\xi^+|) - \hat{f}(\xi^+)).$$

(2.52)

Similar to Lemma 5.5 and Lemma 5.6 in [14], we can similarly derive the following gain of anisotropic regularity:

Lemma 2.12. Set $K^r(\xi) = |\ln \varepsilon|^{-r}1_{2\geq r \geq \varepsilon}$. For any suitable function $f$ defined on $\mathbb{S}^2$, we have

$$\int_{\mathbb{S}^2 \times \mathbb{S}^2} |f(\sigma) - f(\tau)|^2 K^r(|\sigma - \tau|)d\sigma d\tau$$

$$+ |f|^2_{L^2(\mathbb{S}^2)} \sim |W^r((-\Delta_{\mathbb{S}^2})^{1/2})f|^2_{L^2(\mathbb{S}^2)} + |f|^2_{L^2(\mathbb{S}^2)}.$$  

As a consequence, for any suitable function $f$ defined on $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{S}^2} |f(r\sigma) - f(r\tau)|^2 K^r(|\sigma - \tau|)r^2d\sigma d\tau$$

$$+ |f|^2_{L^2} \sim |W^r((-\Delta_{\mathbb{S}^2})^{1/2})f|^2_{L^2} + |f|^2_{L^2}.$$  

(2.53)

Remark 2.1. If the function $K^r(\xi)$ in Lemma 2.12 is changed to $K^r(\xi) = |\ln \varepsilon|^{-r}1_{2\geq r \geq \varepsilon}$, the results in Lemma 2.12 are still valid.

We are ready to show that the “spherical part” in (2.52) produces anisotropic regularity.

Lemma 2.13. Let $\mathcal{A}^r(f) := \int b^r(\frac{\xi}{|\xi|} \cdot \sigma) |\hat{f}(\xi) - \hat{f}(\xi|\xi^+|))d\sigma d\xi$ where $\xi^+ = \frac{\xi + \xi|\sigma|}{2}$, then

$$\mathcal{A}^r(f) + |f|^2_{L^2} \sim |W^r((-\Delta_{\mathbb{S}^2})^{1/2})f|^2_{L^2} + |f|^2_{L^2}.$$  

(2.54)

Proof. Let $r = |\xi|$, $\tau = \xi/|\xi|$ and $\varsigma = \frac{\tau + \sigma}{|\tau + \sigma|}$, then $\frac{\xi}{|\xi|} \cdot \sigma = 2(\tau \cdot \varsigma)^2 - 1$ and $|\xi|\frac{\xi^+}{|\xi^+|} = r \varsigma$. For the change of variable $(\xi, \sigma) \to (r, \tau, \varsigma)$, one has $d\xi d\sigma = 4(\tau \cdot \varsigma)r^2 dr d\tau d\varsigma$. Let $\theta$ be the angle between $\tau$ and $\sigma$, then

$$\cos \frac{\theta}{2} = \tau \cdot \varsigma, \quad 2\sin \frac{\theta}{2} = |\tau - \varsigma|, \quad |\tau - \varsigma| = 2 \left(1 - \cos \frac{\theta}{2}\right).$$
and thus \( \sin \frac{\theta}{2} = \frac{1}{2} |\tau - \sigma| \leq |\tau - \varsigma| \leq |\tau - \sigma| = 2 \sin \frac{\theta}{2} \). Therefore

\[
| \ln \varepsilon |^{-1} |\tau - \varsigma|^{-4} 1_{|\tau - \varsigma| \geq 2 \varepsilon} \leq b^\varepsilon(\cos \theta) \leq | \ln \varepsilon |^{-1} 2^4 |\tau - \varsigma|^{-4} 1_{|\tau - \varsigma| \geq \varepsilon}.
\]

By the \( \lesssim \) direction of (2.53), we have

\[
A^\varepsilon(f) + |f|_{L^2}^2 = 4 \int_{\mathbb{R}^+ \times S^2 \times S^2} b^\varepsilon(\cos \theta)|\hat{f}(r \tau) - \hat{f}(r \varsigma)|^2(\tau \cdot \varsigma)r^2drd\tau d\varsigma + |f|_{L^2}^2 \lesssim | \ln \varepsilon |^{-1} \int |\hat{f}(r \tau) - \hat{f}(r \varsigma)|^2 |\tau - \varsigma|^{-4} 1_{|\tau - \varsigma| \geq \varepsilon}r^2drd\tau d\varsigma + |f|_{L^2}^2 \lesssim |W^\varepsilon((-\Delta_{S^2})^{\frac{1}{2}})\hat{f}|_{L^2}^2 + \hat{f}^2_{L^2} \sim |W^\varepsilon((-\Delta_{S^2})^{\frac{1}{2}})f|_{L^2}^2 + |f|_{L^2}^2,
\]

where we use Lemma 2.7 and Plancherel’s theorem in the last line. Thanks to the \( \gtrsim \) direction of (2.53) and Remark 2.1, we similarly get the \( \gtrsim \) direction of (2.54).

The “radial part” in (2.52) is controllable by gain of \( W^\varepsilon \) in the phase and frequency space. Namely,

**Lemma 2.14.** Let \( Z^{\varepsilon, \gamma}(f) := \int b^\varepsilon(\frac{\varsigma}{|\xi|} \cdot \sigma)\langle \xi \rangle^\gamma |f(\frac{\varsigma}{|\xi|} + \sigma)|^2d\sigma d\xi \) with \( \xi^+ = \varsigma + \frac{\varepsilon + \tau}{|\sigma + \tau|} \). Then

\[
Z^{\varepsilon, \gamma}(f) \lesssim |W^\varepsilon(D)W_{\gamma/2}f|_{L^2}^2 + |W^\varepsilon W_{\gamma/2}f|_{L^2}^2.
\]

**Proof.** We divide the proof into two steps.

**Step 1:** \( \gamma = 0 \). By the change of variable \( (\xi, \sigma) \rightarrow (r, \tau, \varsigma) \) with \( \xi = r \tau \) and \( \varsigma = \frac{\sigma + \tau}{|\sigma + \tau|} \), we have

\[
Z^{\varepsilon, 0}(f) = 4 \int b^\varepsilon(2(\tau \cdot \varsigma)^2 - 1)|f(r \varsigma) - f(\tau \cdot \varsigma r \varsigma)|^2(\tau \cdot \varsigma)r^2drd\tau d\varsigma.
\]

Let \( u = r \varsigma \), and \( \theta \) be the angle between \( \tau \) and \( \varsigma \). Since \( b^\varepsilon(2(\tau \cdot \varsigma)^2 - 1) = b^\varepsilon(\cos 2\theta) \lesssim | \ln \varepsilon |^{-1} \theta^{-4} \), and \( r^2drd\varsigma = \sin \theta dud\theta dS \), we have

\[
Z^{\varepsilon, 0}(f) \lesssim | \ln \varepsilon |^{-1} \int_{\mathbb{R}^3} \int_{\mathbb{S}/2}^{\pi/2} \theta^{-3}|f(u) - f(u \cos \theta)|^2dud\theta \lesssim |W^\varepsilon(D)f|_{L^2}^2 + |W^\varepsilon f|_{L^2}^2,
\]

where we use the change of variable \( u \rightarrow u/\cos \theta \) and Lemma 4.2 in the last inequality.

**Step 2:** \( \gamma \neq 0 \). We reduce the general case \( \gamma \neq 0 \) to the special case \( \gamma = 0 \). For simplicity, we denote \( w = |\xi|\frac{\varsigma}{|\xi| + \sigma} \), then \( W_{\gamma}(\xi) = W_{\gamma}(w) \). From which we have

\[
\langle \xi \rangle^\gamma(f(w) - f(\xi^+))^2 = \left( (W_{\gamma/2}f)(w) - (W_{\gamma/2}f)(\xi^+) + (W_{\gamma/2}f)(\xi^+) \right) \left( 1 - W_{\gamma/2}(w)W_{-\gamma/2}(\xi^+) \right) \leq 2|(W_{\gamma/2}f)(\xi^+) - (W_{\gamma/2}f)(w)|^2 + 2|(W_{\gamma/2}f)(\xi^+)|^2 |1 - W_{\gamma/2}(w)W_{-\gamma/2}(\xi^+)|^2.
\]
Thus we have
\[
Z^{e, \gamma}(f) \lesssim Z^{e, 0}(W_{\gamma/2} f) + \int b^e \left( \frac{\xi}{|\xi|} \cdot \sigma \right) |(W_{\gamma/2} f)(\xi^+)|^2 \\
|1 - W_{\gamma/2}(w) W_{\gamma/2}(\xi^+)|^2 d\sigma d\xi
:= Z^{e, 0}(W_{\gamma/2} f) + A.
\]

By noticing that \(|W_{\gamma/2}(w) W_{\gamma/2}(\xi^+)| - 1| \lesssim \theta^2\), using the change of variable \(\xi \to \xi^+\) and (2.13), we have |A| \lesssim |W_{\gamma/2} f|^2. By (2.55), we get \(Z^{e, 0}(W_{\gamma/2} f) \lesssim |W^e(D) W_{\gamma/2} f|^2_{L^2} + |W^e W_{\gamma/2} f|^2_{L^2}\). Patching together, we get the desired result.

\(\Box\)

Now we are in a position to get \(|W^e((-\Delta_{S^2})^{1/2}) f|^2_{L^2_{\gamma/2}}\) from \(N^{e, 0, 0}(g, f)\).

**Lemma 2.15.** For any suitable functions \(g, f\), the follow two estimates hold true:

\[
N^{e, 0, 0}(g, f) + |g|^2_{L^2_1} |W^e(D) f|^2_{L^2_1} + |g|^2_{L^2_2} |W^e f|^2_{L^2_2} \gtrsim |g|^2_{L^2_2} |W^e((-\Delta_{S^2})^{1/2}) f|^2_{L^2_1},
\]

(2.56)

\[
N^{e, 0, 0}(g, f) \lesssim |g|^2_{L^2_2} |W^e((-\Delta_{S^2})^{1/2}) f|^2_{L^2_1} + |g|^2_{L^1_1} |W^e(D) f|^2_{L^2_1} + |g|^2_{L^2_2} |W^e f|^2_{L^2_1}.
\]

(2.57)

**Proof.** By Bobylev’s formula, we have

\[
N^{e, 0, 0}(g, f) = \frac{1}{(2\pi)^3} \int b^e \left( \frac{\xi}{|\xi|} \cdot \sigma \right) |\hat{g}^2(0)\|\hat{f}(\xi) - \hat{f}(\xi^+)||^2 \\
+ 2\Re((\hat{g}^2(0) - \hat{g}^2(\xi^-))\hat{f}(\xi^+)\hat{\xi}(\xi)) d\sigma d\xi
:= \frac{|g|^2_{L^2_2}}{(2\pi)^3} I_1 + \frac{2}{(2\pi)^3} I_2,
\]

(2.58)

where \(\xi^+ = \frac{\xi + |\xi|\sigma}{2}\) and \(\xi^- = \frac{\xi - |\xi|\sigma}{2}\). Thanks to the fact \(\hat{g}^2(0) - \hat{g}^2(\xi^-) = \hat{f}(1 - \cos(v \cdot \xi^-))g^2(v)dv\), we have

\[
|I_2| = |\int b^e \left( \frac{\xi}{|\xi|} \cdot \sigma \right)(1 - \cos(v \cdot \xi^-))g^2(v)\Re(\hat{f}(\xi^+)\hat{\xi}(\xi)) d\sigma d\xi dv|
\leq (\int b^e \left( \frac{\xi}{|\xi|} \cdot \sigma \right)(1 - \cos(v \cdot \xi^-))g^2(v)|\hat{f}(\xi^+)|^2 d\sigma d\xi d\nu)
\frac{1}{2}
\times (\int b^e \left( \frac{\xi}{|\xi|} \cdot \sigma \right)(1 - \cos(v \cdot \xi^-))g^2(v)|\hat{f}(\xi)|^2 d\sigma d\xi d\nu)^{1/2}.
\]

Observe that \(1 - \cos(v \cdot \xi^-) \lesssim |v|^2|\xi^-|^2 = \frac{1}{4} |v|^2|\xi|^2 |\frac{\xi^+}{|\xi^+|} - \sigma|^2 \sim |v|^2|\xi|^2 |\frac{\xi^+}{|\xi^+|} - \sigma|^2\), thus \(1 - \cos(v \cdot \xi^-) \lesssim \min\{|v|^2|\xi|^2 |\frac{\xi^+}{|\xi^+|} - \sigma|^2, 1\} \sim \min\{|v|^2|\xi|^2 |\frac{\xi^+}{|\xi^+|} - \sigma|^2, 1\}. By the fact \(\frac{\xi^+}{|\xi^+|} \cdot \sigma = 2(\frac{\xi^+}{|\xi^+|} \cdot \sigma)^2 - 1\), and the change of variable \(\xi \to \xi^+\), and the property \(W^e(|v||\xi|) \lesssim W^e(|v|) W^e(|\xi|)\), we have

\[
|I_2| \lesssim \int (W^e)^2(|v||\xi|)|\hat{f}(\xi)|^2 g^2(v) dv d\xi \lesssim |W^e g|^2_{L^2_2} |W^e(D) f|^2_{L^2_1}
\]
Now we set to investigate $\mathcal{I}_1$. By the decomposition (2.52) and the inequality $\\frac{1}{2}a^2 - b^2 \leq (a + b)^2 \leq 2a^2 + 2b^2$, we have

$$
\frac{1}{2} A^e(f) - Z^{e,0}(\hat{f}) \leq \mathcal{I}_1 = \int b^e(\\frac{\xi}{|\xi|} \cdot \sigma) |\hat{f}(\xi) - \hat{f}(\xi^+)|^2 d\sigma d\xi \\
\leq 2A^e(f) + 2Z^{e,0}(\hat{f}).
$$

(2.60)

where $A^e(\cdot)$ and $Z^{e,\gamma}(\cdot)$ are defined in Lemma 2.13 and Lemma 2.14. By Lemma 2.14, the fact $\hat{f}(v) = f(-v)$ and Plancherel’s theorem, we have

$$
Z^{e,0}(\hat{f}) \lesssim |W^e(D)\hat{f}|^2_{L^2} + |W^e\hat{f}|^2_{L^2} \lesssim |W^e f|^2_{L^2} + |W^e(D)f|^2_{L^2}.
$$

(2.61)

Recalling (2.58) and (2.60), patching together (2.59), (2.61) and (2.4), we get (2.56) and (2.57).

2.3.2. Gain of Regularity From $\mathcal{N}^{e,0,\eta}(g, f)$

We first introduce some notations. Let $\chi$ be a smooth function such that $0 \leq \chi \leq 1$, $\chi = 1$ on $B_1$ and $\text{Supp}\chi \subset B_2$. Here $B_1$ is the ball centered at origin with radius $1$. $B_2$ is interpreted in a similar way. Let $\chi_R(v) := \chi(v/R)$, $\chi_{r,u}(v) := \chi_r(v - u)$ and $\phi_{R,r,u} := \chi_{7R} - \chi_{3R, u}$ for some $r, R > 0$ and $u \in \mathbb{R}^3$. The following lemma bounds $\mathcal{N}^{e,0,\eta}(g, f)$ by $\mathcal{N}^{e,0,0}(g, f)$ from below once the distance between supports of $g$ and $f$ is larger than some multiples of $\eta$.

**Lemma 2.16.** For $0 \leq \eta \leq r \leq 1 \leq R$, $u \in B_{7R}$, the following two estimates hold true:

$$
\mathcal{N}^{e,0,\eta}(g, f) + |g|^2_{L^2} |f|^2_{L^2} \gtrsim \mathcal{N}^{e,0,0}(\chi_{Rg}, (1 - \chi_{3R})f).
$$

(2.62)

$$
\mathcal{N}^{e,0,\eta}(g, f) + r^{-2} R^2 |g|^2_{L^2} |f|^2_{L^2} \gtrsim \mathcal{N}^{e,0,0}(\phi_{R,r,u} g, \chi_{r,u} f).
$$

(2.63)

**Proof.** We proceed in the spirit of [16]. If $|v_s| \leq 2R$ and $|v| \geq 3R$, then $|v - v_s| \geq R \geq \eta$. Then we have

$$
\mathcal{N}^{e,0,\eta}(g, f) = \int b^e(\cos \theta) 1_{|v - v_s| \geq \eta} \hat{g}^e(v' - f)^2 d\sigma dv dv_s \\
\gtrsim \int b^e(\cos \theta)(\chi_{Rg})^2_\eta(f' - f)^2(1 - \chi_{3R})^2 d\sigma dv dv_s \\
\gtrsim \frac{1}{2} \int b^e(\cos \theta)(\chi_{Rg})^2_\eta((1 - \chi_{3R})f)' - (1 - \chi_{3R})f)^2 d\sigma dv dv_s \\
- \int b^e(\cos \theta)(\chi_{Rg})^2_\eta(f^2)'(\chi_{3R} - \chi_{3R})^2 d\sigma dv dv_s := \frac{1}{2} \mathcal{I}_1 - \mathcal{I}_2.
$$

Observe that $\mathcal{I}_1 = \mathcal{N}^{e,0,0}(\chi_{Rg}, (1 - \chi_{3R})f)$. Since $|\nabla \chi_{3R}|_{L^\infty} \lesssim R^{-1} |\nabla \chi|_{L^\infty} \lesssim R^{-1}$, we get $(\chi_{Rg})^2_\eta(\chi_{3R} - \chi_{3R})^2 \lesssim R^{-2} |v' - v|^2 = R^{-2} |v - v_s|^2 \sin^2(\theta/2)$. If $|v_s| \leq 2R$, $|v| \geq 20R$, $\theta \leq \pi/2$, we have

$$
|v' - v_s| = \cos(\theta/2)|v - v_s| \geq \cos(\theta/2)(|v| - |v_s|) \geq 9\sqrt{2}R.
$$
Then we have $|v'| \geq |v - v_*| - |v_*| \geq 9\sqrt{2}R - 2R \geq 6R$, which gives $\chi_{3R}' = 0$. As a result, we have

$$(\chi_R')^2(\chi_{3R} - \chi_{3R})^2 \leq 1_{|v| \leq 20R, |v_*| \leq 2R} R^{-2} |v - v_*|^2 \sin^2(\theta/2) \lesssim \sin^2(\theta/2).$$

By the change of variable $(v, \theta) \rightarrow (v', \theta/2)$ and using (2.13), we get

$$I_2 \lesssim \int g_*^2 f^2 \, dv \, dv_* \lesssim |g|^2_{L^2} |f|_{L^2}^2.$$  

From which together with the fact $I_1 = N^{r,0,0}(\chi_{R} g, (1 - \chi_{3R}) f)$, we get (2.62).

If $v \in \text{supp} \chi_{R, u}, v_* \in \text{supp} \phi_{R, r, u}$, then $|v - v_*| \geq r \geq \eta$, which gives

$$N^{r,0,0}(g, f) = \int b^\ell(\cos \theta) 1_{|v-v_*| \geq \eta} g_*^2 (f' - f)^2 \, d\sigma \, dv \, dv_*$$

$$\geq \int b^\ell(\cos \theta)(\phi_{R, r, u} g)^2 (f' - f)^2 \chi_{r,u} \, d\sigma \, dv \, dv_*$$

$$\geq \frac{1}{2} \int b^\ell(\cos \theta)(\phi_{R, r, u} g)^2 ((\chi_{r,u} f')' - \chi_{r,u} f)^2 \, d\sigma \, dv \, dv_*$$

$$- \int b^\ell(\cos \theta)(\phi_{R, r, u} g)^2 (f' - f)^2 \chi_{r,u} - \chi_{r,u}) \, d\sigma \, dv \, dv_* := \frac{1}{2} J_1 - J_2.$$  

Observe that $J_1 = N^{r,0,0}(\phi_{R, r, u} g, \chi_{r,u} f)$. Since $|\nabla \chi_{r,u}(v)| \lesssim r^{-1} |\nabla \chi|_{L^\infty}$, $1_r \leq |v - u| \leq 2r$, together with Taylor expansion, we get

$$|\chi_{r,u} - \chi_{r,u}|^2 = |\int_0^1 \nabla \chi_{r,u} (v(\kappa)) \cdot (v' - v) \, d\kappa|^2 \lesssim r^{-2} |v - v_*|^2 \sin^2(\theta/2)$$

$$\int_0^1 1_{r \leq |v(\kappa) - u| \leq 2r} \, d\kappa.$$  

For $u \in B_{7R}, |v_*| \leq 14R, r \leq |v(\kappa) - u| \leq 2r$, we have

$$|v - v_*| \leq \sqrt{2}|v(\kappa) - v_*| \leq \sqrt{2}|v(\kappa) - u|$$

$$+ \sqrt{2}|u - v_*| \leq 2\sqrt{2}r + \sqrt{2}(7R + 14R) \leq 23\sqrt{2}R,$$

and thus

$$(\phi_{R, r, u} g)^2 (\chi'_{r,u} - \chi_{r,u})^2 \lesssim r^{-2} |v - v_*|^2 1_{|v-v_*| \leq 23\sqrt{2}R} \sin^2(\theta/2) \lesssim r^{-2} R^2 \sin^2(\theta/2).$$

By the change of variable $v \rightarrow v'$ and (2.13), we get

$$J_2 \lesssim r^{-2} R^2 \int g_*^2 f^2 \, dv \, dv_*$$

$$\lesssim r^{-2} R^2 |g|^2_{L^2} |f|_{L^2}^2.$$  

From which together with the fact $J_1 = N^{r,0,0}(\phi_{R, r, u} g, \chi_{r,u} f)$, we get (2.63). □
2.3.3. Gain of Regularity From $\mathcal{N}^{e,\gamma,\eta}(g, f)$ To reduce $\mathcal{N}^{e,\gamma,\eta}$ to $\mathcal{N}^{e,0,\eta}$, we introduce an intermediate quantity
\[
\tilde{\mathcal{N}}^{e,\gamma,\eta}(g, h) := \int b^\epsilon (\cos \theta) 1_{|v-v_*| \geq \eta} \langle v-v_* \rangle^\gamma g_*^2 (h' - h)^2 d\sigma d\nu_*.
\]

The next lemma reduces $\tilde{\mathcal{N}}^{e,\gamma,\eta}$ to $\mathcal{N}^{e,0,\eta}$.

**Lemma 2.17.** For $\gamma \leq 0 \leq \eta$, it holds that
\[
\frac{1}{2} C_1 \mathcal{N}^{e,0,\eta}(W_{\gamma/2} g, W_{\gamma/2} f) - C_3 |g|_{L^2_{\gamma/2+1}} |f|_{L^2_{\gamma/2}}^2 
\leq \tilde{\mathcal{N}}^{e,\gamma,\eta}(g, f) \leq 2 C_2 \mathcal{N}^{e,0,\eta}(W_{-\gamma/2} g, W_{\gamma/2} f) + 2 C_3 |g|_{L^2_{\gamma/2+1}} |f|_{L^2_{\gamma/2}}^2,
\]
where $C_1, C_2, C_3$ are constants depending only on $\gamma$. They are universally bounded if $-3 \leq \gamma \leq 0$.

**Proof.** Set $F = W_{\gamma/2} f$. By definition, we have
\[
\tilde{\mathcal{N}}^{e,\gamma,\eta}(g, f) = \int b^\epsilon (\cos \theta) 1_{|v-v_*| \geq \eta} \langle v-v_* \rangle^\gamma g_*^2 ((W_{-\gamma/2} F)' - W_{-\gamma/2} F)^2 d\sigma d\nu_*.
\]

We make the following decomposition
\[
(W_{-\gamma/2} F)' - W_{-\gamma/2} F = (W_{-\gamma/2})'(F' - F) + F(W_{-\gamma/2} - W_{-\gamma/2}) := A + B.
\]
By $\frac{1}{2} a^2 - b^2 \leq (a + b)^2 \leq 2a^2 + 2b^2$, we get $\frac{1}{2} I_1 - I_2 \leq \mathcal{N}^{e,\gamma,\eta}(g, f) \leq 2(I_1 + I_2)$, where
\[
I_1 := \int b^\epsilon (\cos \theta) 1_{|v-v_*| \geq \eta} \langle v-v_* \rangle^\gamma g_*^2 W_{-\gamma}(F' - F)^2 d\sigma d\nu_*,
\]
\[
I_2 := \int b^\epsilon (\cos \theta) 1_{|v-v_*| \geq \eta} \langle v-v_* \rangle^\gamma g_*^2 F^2 (W_{-\gamma/2} - W_{-\gamma/2})^2 d\sigma d\nu_*.
\]

Since $\langle v_* \rangle^\gamma \lesssim \langle v_* - v \rangle^\gamma \langle v' \rangle^{-\gamma} \lesssim \langle v_* \rangle^{-\gamma}$, we get
\[
\mathcal{N}^{e,0,\eta}(W_{\gamma/2} g, W_{\gamma/2} f) \lesssim I_1 \lesssim \mathcal{N}^{e,0,\eta}(W_{-\gamma/2} g, W_{\gamma/2} f).
\]

By Taylor expansion, one has $(W_{-\gamma/2} - W_{-\gamma/2})^2 \lesssim \int \langle v(\kappa) \rangle^{-\gamma-2} |v - v_*|^2 \sin^2 (\theta/2) d\kappa$. Note that
\[
\langle v - v_* \rangle^\gamma |v - v_*|^2 \langle v(\kappa) \rangle^{-\gamma-2} \lesssim \langle v - v_* \rangle^{\gamma+2} \langle v(\kappa) \rangle^{-\gamma-2} \lesssim \langle v(\kappa) - v_* \rangle^{\gamma+2} \langle v(\kappa) \rangle^{-\gamma-2} \lesssim \langle v_* \rangle^{\gamma+2}.
\]

From which together with (2.13), we get
\[
I_2 \lesssim \int g_*^2 \langle v_* \rangle^{\gamma+2} F^2 d\nu_* \lesssim |g|_{L^2_{\gamma/2+1}} |F|_{L^2}^2.
\]

Patching together the above estimates of $I_1$ and $I_2$, we finish the proof. \qed
If $\gamma \leq 0$, then $|v - v_a|^\gamma \geq (v - v_a)^\gamma$, and thus $\mathcal{N}^{\epsilon,\gamma,\eta}(g, f) \geq \tilde{\mathcal{N}}^{\epsilon,\gamma,\eta}(g, f)$. From which together with Lemma 2.17, we have

**Lemma 2.18.** For $\gamma \leq 0 \leq \eta$, it holds that

$$\mathcal{N}^{\epsilon,\gamma,\eta}(g, f) + |g|_{L_{\gamma/2+1}^{\gamma}}^2 |f|_{L_{\gamma/2}^{\gamma}}^2 \geq C \mathcal{N}^{\epsilon,0,\eta}(W_{\gamma/2}g, W_{\gamma/2}f),$$

where $C$ is a constant depending only on $\gamma$. The constant is universally bounded if $-3 \leq \gamma \leq 0$.

We are ready to derive gain of regularity from $\mathcal{N}^{\epsilon,\gamma,\eta}(\mu^\frac{1}{2}, f)$.

**Proposition 2.4.** For $-3 \leq \gamma \leq 0 \leq \eta \leq r_0 := 6^{-1}2^{-7/6}e^{-1/6}$, the following estimates are valid:

$$\mathcal{N}^{\epsilon,\gamma,\eta}(\mu^\frac{1}{2}, f) + |f|_{L_{\gamma/2}^{\gamma}}^2 \gtrsim |W^\epsilon(D)W_{\gamma/2}f|_{L^2}^2.$$  \hspace{1cm} (2.64)

$$\mathcal{N}^{\epsilon,\gamma,\eta}(\mu^\frac{1}{2}, f) + |W^\epsilon(D)W_{\gamma/2}f|_{L^2}^2 + |W^\epsilon((-\Delta^2)^\frac{1}{2})W_{\gamma/2}f|_{L^2}^2.$$ \hspace{1cm} (2.65)

Making suitable combination, we have

$$\mathcal{N}^{\epsilon,\gamma,\eta}(\mu^\frac{1}{2}, f) + |W^\epsilon W_{\gamma/2}f|_{L^2}^2 \gtrsim |W^\epsilon((-\Delta^2)^\frac{1}{2})W_{\gamma/2}f|_{L^2}^2 + |W^\epsilon(D)W_{\gamma/2}f|_{L^2}^2.$$ \hspace{1cm} (2.66)

**Proof.** Set $F = W_{\gamma/2}f$. Taking $g = \mu^\frac{1}{2}$ in Lemma 2.18, we have

$$\mathcal{N}^{\epsilon,\gamma,\eta}(\mu^\frac{1}{2}, f) + |F|_{L^2}^2 \gtrsim \mathcal{N}^{\epsilon,0,\eta}(W_{\gamma/2}\mu^\frac{1}{2}, f) \geq \mathcal{N}^{\epsilon,0,\eta}(W_{-3/2}\mu^\frac{1}{2}, f).$$

Taking $g = W_{-3/2}\mu^\frac{1}{2}, f = F$ in Lemma 2.16, we have for $\eta \leq r \leq 1 \leq R, u \in B_{7R}$,

$$\mathcal{N}^{\epsilon,0,\eta}(W_{-3/2}\mu^\frac{1}{2}, f) + |F|_{L^2}^2 \gtrsim \mathcal{N}^{\epsilon,0,0}(\chiRW_{-3/2}\mu^\frac{1}{2}, (1 - \chi_3R)F).$$ \hspace{1cm} (2.67)

$$\mathcal{N}^{\epsilon,0,\eta}(W_{-3/2}\mu^\frac{1}{2}, f) + r^{-2}R^2 |F|_{L^2}^2 \gtrsim \mathcal{N}^{\epsilon,0,0}(\phi_{R,r,u}W_{-3/2}\mu^\frac{1}{2}, \chi_{r,u}F).$$ \hspace{1cm} (2.68)

From now on in the proof, we take $R = 1$. Then $\chi_R = \chi$, we get

$$|\chiRW_{-3/2}\mu^\frac{1}{2}|_{L^1}^2 = |\chiRW_{-3/2}\mu^\frac{1}{2}|_{L^2}^2 = \int \chi^2 W_{-3}\mu \, dv \geq \frac{4\pi}{3}2^{-3/2}(2\pi)^{-\frac{3}{2}}e^{-1/2} := 4\delta_. $$

Recalling $\phi_{R,r,u} = \chi_{7R} - \chi_{3r,u}$ and $\chi_{7R} \geq \chi_R$, we have

$$\int \phi_{R,r,u}^2 W_{-3}\mu \, dv \geq \frac{1}{2} \int \chi_{7R}^2 W_{-3}\mu \, dv - \int \chi_{3r,u}^2 W_{-3}\mu \, dv$$
\[ \geq 2\delta_\ast - \int \chi_{3r,u}^2 W^{-3}\mu dv. \]

Note that \[ \int \chi_{3r,u}^2 W^{-3}\mu dv \leq \frac{4\pi}{3}(6r)^3(2\pi)^{-\frac{3}{2}} := Cr^3. \] By choosing \( r \) such that \( Cr^3 = \delta_\ast \), we get

\[ |\phi_{R,r,u} W^{-3/2}\mu^\frac{1}{2}|_{L^2}^2 \geq \delta_\ast. \]

It is easy to check \( r = 6^{-1}2^{-7/6}e^{-1/6} = r_0 \). Therefore we have

\[ \min\{|(\chi_{R} W^{-3/2}\mu^\frac{1}{2})^2|_{L^1}, |(\phi_{R,r,u} W^{-3/2}\mu^\frac{1}{2})^2|_{L^1}\} \geq \delta_\ast. \tag{2.69} \]

On the other hand, it is obvious to see that

\[ \max\{|(\chi_{R} W^{-3/2}\mu^\frac{1}{2})^2|_{L^1 \cap L \log L}, |(\phi_{R,r,u} W^{-3/2}\mu^\frac{1}{2})^2|_{L^1 \cap L \log L}\} \leq |\mu|_{L^1 \cap L \log L} = \lambda_\ast. \tag{2.70} \]

Thanks to (2.69) and (2.70), by Proposition 2.3, we get

\[ \mathcal{N}^{e,0,0}(\chi_{R} W^{-3/2}\mu^\frac{1}{2}, (1 - \chi_{3R})F + |(1 - \chi_{3R})F|_{L^2}^2 \]
\[ \geq C(\delta_\ast, \lambda_\ast)|W^e(D)(1 - \chi_{3R})F|_{L^2}^2. \tag{2.71} \]

\[ \mathcal{N}^{e,0,0}(\phi_{R,r,u} W^{-3/2}\mu^\frac{1}{2}, \chi_{r,u} F) + |\chi_{r,u} F|_{L^2}^2 \]
\[ \geq C(\delta_\ast, \lambda_\ast)|W^e(D)\chi_{r,u} F|_{L^2}^2. \tag{2.72} \]

There is a finite cover of \( B_{6R} \) with open ball \( B_r(u_j) \) for \( u_j \in B_{6R} \). More precisely, there exists \( \{u_j\}_{j=1}^N \subset B_{6R} \) such that \( B_{6R} \subset \bigcup_{j=1}^N B_r(u_j) \), where \( N \sim \frac{1}{r^3} \) is a universal constant. We then have \( \chi_{3R} \leq \sum_{j=1}^N \chi_{r,u_j} \) and thus \( |W^e(D)\chi_{3R} F|_{L^2}^2 \leq N \sum_{j=1}^N |W^e(D)\chi_{r,u_j} F|_{L^2}^2 \). From which together with (2.67), (2.68), (2.71), (2.72), we get for any \( 0 \leq \eta \leq r \),

\[ \mathcal{N}^{e,\eta}(\mu^\frac{1}{2}, f) + |f|_{L^2}^2 \gtrsim r^8 |W^e(D)W_{\gamma/2}f|_{L^2}^2. \]

Since \( r \) is a universal constant, we get (2.64).

Thanks to (2.69) and (2.70), by (2.56) in Lemma 2.15, we get

\[ \mathcal{N}^{e,0,0}(\chi_{R} W^{-3/2}\mu^\frac{1}{2}, (1 - \chi_{3R})F) \]
\[ + \lambda_\ast(|W^e(D)(1 - \chi_{3R})F|_{L^2}^2 + |W^e(1 - \chi_{3R})F|_{L^2}^2) \]
\[ \gtrsim \delta_\ast|W^e((-\Delta_{D2})^\frac{1}{2})(1 - \chi_{3R})F|_{L^2}^2, \]

\[ \mathcal{N}^{e,0,0}(\phi_{R,r,u} W^{-3/2}\mu^\frac{1}{2}, \chi_{r,u} F) + \lambda_\ast(|W^e(D)\chi_{r,u} F|_{L^2}^2 + |W^e\chi_{r,u} F|_{L^2}^2) \]
\[ \gtrsim \delta_\ast|W^e((-\Delta_{D2})^\frac{1}{2})\chi_{r,u} F|_{L^2}^2. \]

Then a similar finite cover argument yields (2.65). \( \square \)
2.4. Lower Bound of $\langle \mathcal{L}^{\varepsilon, \gamma, \eta} f, f \rangle$

By Proposition 2.1 and the estimate (2.66), recalling (1.37), we get

**Proposition 2.5.** Let $-3 \leq \gamma \leq 0 \leq \eta \leq r_0$, $0 < \varepsilon \leq \varepsilon_0$, then

$$
\mathcal{N}^{\varepsilon, \gamma, \eta}(\mu_1^{\frac{1}{2}}, f) + \mathcal{N}^{\varepsilon, \gamma, \eta}(f, \mu_1^{\frac{1}{2}}), \|f\|_{L^{r/2}_{\gamma/2}}^2 \gtrsim |f|_{L^2_{r/2}}^2.
$$

Now we are ready to prove the following coercivity estimate.

**Theorem 2.1.** Let $-3 \leq \gamma \leq 0 \leq \eta \leq r_0$, $0 < \varepsilon \leq \varepsilon_0$, then

$$
\langle \mathcal{L}^{\varepsilon, \gamma, \eta} f, f \rangle + |f|_{L^{r/2}_{\gamma/2}}^2 \gtrsim |f|_{L^2_{r/2}}^2.
$$

**Proof.** We recall that $\mathcal{N}^{\varepsilon, \gamma, \eta}(\mu_1^{\frac{1}{2}}, f) + \mathcal{N}^{\varepsilon, \gamma, \eta}(f, \mu_1^{\frac{1}{2}})$ corresponds to the anisotropic norm $||| \cdot |||$ introduced in [3]. By the proof of Proposition 2.16 in [3], it holds that

$$
\langle \mathcal{L}_1^{\varepsilon, \gamma, \eta} f, f \rangle \geq \frac{1}{10} \left( \mathcal{N}^{\varepsilon, \gamma, \eta}(\mu_1^{\frac{1}{2}}, f) + \mathcal{N}^{\varepsilon, \gamma, \eta}(f, \mu_1^{\frac{1}{2}}) \right) - \frac{3}{10} \int B^{\varepsilon, \gamma, \eta} \mu_*(f^2 - (f^2'))d\sigma dv dv_*.
$$

If $\gamma = -3$, by (2.17) with $a = 1/2, p = \infty, q = 1$, we have

$$
\int B^{\varepsilon, \gamma, \eta} \mu_*(f^2 - (f^2'))d\sigma dv dv_* \leq C|\mu_1^{\frac{1}{2}} f|_{L^1} \leq C|f|_{L^{r/2}_{\gamma/2}}^2.
$$

If $\gamma > -3$, referring to the cancellation Lemma 1 of [1], by (2.13), we have

$$
\int B^{\varepsilon, \gamma, r_0} \mu_*(f^2 - (f^2'))d\sigma dv dv_* \lesssim \int 1_{|v - v_*| \geq r_0}|v - v_*|^\gamma \mu_* f^2 dv dv_* \lesssim |f|_{L^{r/2}_{\gamma/2}}^2.
$$

Therefore for some universal constant $C > 0$, we have

$$
\langle \mathcal{L}_1^{\varepsilon, \gamma, \eta} f, f \rangle \geq \frac{1}{10} \left( \mathcal{N}^{\varepsilon, \gamma, \eta}(\mu_1^{\frac{1}{2}}, f) + \mathcal{N}^{\varepsilon, \gamma, \eta}(f, \mu_1^{\frac{1}{2}}) \right) - C|f|_{L^{r/2}_{\gamma/2}}^2. \tag{2.74}
$$

By Lemma 4.3, we have

$$
|\langle \mathcal{L}_2^{\varepsilon, \gamma, \eta} f, f \rangle| \lesssim |\mu_1^{\frac{1}{2}} f|_{L^2} \lesssim |f|_{L^{r/2}_{\gamma/2}}^2. \tag{2.75}
$$

Recalling (1.48), patching (2.74) and (2.75), we arrive at

$$
\langle \mathcal{L}^{\varepsilon, \gamma, \eta} f, f \rangle + |f|_{L^{r/2}_{\gamma/2}}^2 \gtrsim \langle \mathcal{L}_1^{\varepsilon, \gamma, \eta} f, f \rangle + |f|_{L^{r/2}_{\gamma/2}}^2 \gtrsim \mathcal{N}^{\varepsilon, \gamma, r_0}(\mu_1^{\frac{1}{2}}, f) + \mathcal{N}^{\varepsilon, \gamma, r_0}(f, \mu_1^{\frac{1}{2}}). \tag{2.76}
$$

From this, together with Proposition 2.5, we finish the proof. $\square$

Note that for $0 < \varepsilon \leq \varepsilon_0$, by taking $\gamma = -3, \eta = 0$ in Theorem 1.2, we get (1.36). As $\varepsilon \to 0$, we get (1.36) for the linearized Landau operator.
3. Spectral Gap Estimate

In this section, we will consider the spectral gap estimate of $L_{\eta}^{\varepsilon,\gamma,\eta}$. As we explained in the introduction, it will yield Theorem 1.3 and Theorem 1.4. Recalling (1.49), $L_{\eta}^{\varepsilon,0}$ is the linearized Boltzmann operator with kernel $B_{\eta}^{\varepsilon,0} = b^{\varepsilon}(\cos \theta)1_{|v-v_*|<\eta}$. We first prove the smallness of $\langle L_{\eta}^{\varepsilon,0} f, f \rangle$ in terms of smallness of $\eta$.

Lemma 3.1. Let $0 \leq \eta \leq 1$. Then

$$\langle L_{\eta}^{\varepsilon,0} f, f \rangle \lesssim \eta^3|W^{\varepsilon}(D)(\mathbb{I}-\mathbb{P})f|^2_{L^2}.$$

Proof. The kernel space of $L_{\eta}^{\varepsilon,0}$ is $\mathcal{K}$ and $L_{\eta}^{\varepsilon,0}$ is a self-joint operator. Therefore it suffices to prove $\langle L_{\eta}^{\varepsilon,0} f, f \rangle \lesssim \eta^3|W^{\varepsilon}(D) f|^2_{L^2}$ for $f \in \mathcal{K}^\perp$. Similar to (2.2), we define $N_{\eta}^{\varepsilon,0}(g, h)$ with the restriction $|v-v_*| < \eta$. It is easy to check $\langle L_{\eta}^{\varepsilon,0} f, f \rangle \lesssim 2N_{\eta}^{\varepsilon,0}(\mu^{\frac{1}{2}}, f) + 2N_{\eta}^{\varepsilon,0}(f, \mu^{\frac{1}{2}})$. We divide the proof into two steps.

Step 1: Estimate of $N_{\eta}^{\varepsilon,0}(f, \mu^{\frac{1}{2}})$. Recall

$$N_{\eta}^{\varepsilon,0}(f, \mu^{\frac{1}{2}}) = \int b^{\varepsilon}(\cos \theta)1_{|v-v_*|<\eta} f^{2}_{\varepsilon}((\mu^{\frac{1}{2}}) - \mu^{\frac{1}{2}})^2 d\sigma dv dv_*.$$

By Taylor expansion, $| (\mu^{\frac{1}{2}}) - \mu^{\frac{1}{2}} | \lesssim | \nabla \mu^{\frac{1}{2}} |_{L^\infty} | v' - v | \lesssim | v - v_* | \theta$. From which together with (2.13), we have

$$N_{\eta}^{\varepsilon,0}(f, \mu^{\frac{1}{2}}) \lesssim \int b^{\varepsilon}(\cos \theta)1_{|v-v_*|<\eta} f^{2}_{\varepsilon} | v - v_* |^2 \theta^2 d\sigma dv dv_*$$

$$\lesssim \int 1_{|v-v_*|<\eta} f^{2}_{\varepsilon} | v - v_* |^2 dv dv_* \lesssim \eta^5 | f |^2_{L^2}.$$

Step 2: Estimate of $N_{\eta}^{\varepsilon,0}(\mu^{\frac{1}{2}}, f)$. Recalling the decomposition $f = \tilde{f}_\phi f + \tilde{f}^f f$ in (1.52), we have

$$N_{\eta}^{\varepsilon,0}(\mu^{\frac{1}{2}}, f) = \int b^{\varepsilon}(\cos \theta)1_{|v-v_*|<\eta} \mu^{\star}(f'-f)^2 d\sigma dv dv_*$$

$$\leq 2 \int b^{\varepsilon}(\cos \theta)1_{|v-v_*|<\eta} \mu^{\star}((\tilde{f}_\phi f)' - \tilde{f}^f f)^2 d\sigma dv dv_*$$

$$+ 2 \int b^{\varepsilon}(\cos \theta)1_{|v-v_*|<\eta} \mu^{\star}((\tilde{f}_\phi f)' - \tilde{f}^f f)^2 d\sigma dv dv_*$$

$$:= 2I_{\text{low}} + 2I_{\text{high}}.$$

Step 2.1: Estimate of $I_{\text{high}}$. By $((\tilde{f}_\phi f)' - \tilde{f}^f f)^2 \leq 2 ((\tilde{f}_\phi f)^2 + (\tilde{f}^f f)^2)$, the change of variable $v \rightarrow v'$, the estimate (2.11) and the fact (2.7), we have

$$I_{\text{high}} \lesssim \int b^{\varepsilon}(\cos \theta)1_{|v-v_*|<\eta} \mu^{\star} (\tilde{f}^{f} f)^2 d\sigma dv dv_*$$

$$\lesssim |\ln \varepsilon|^{-1} \varepsilon^{-2} \int 1_{|v-v_*|<\eta} \mu^{\star} (\tilde{f}^{f} f)^2 dv dv_* \lesssim \eta^3 |\ln \varepsilon|^{-1} \varepsilon^{-2} |\tilde{f}^{f} f|^2_{L^2}.$$
\[ \lesssim \eta^3 |W^\varepsilon(D) f|^2_{L^2}. \]

**Step 2.2: Estimate of \( I_{\text{low}} \).** Observe \((\tilde{\phi} f)' - \tilde{\phi} f)^2 = (\tilde{\phi} f)'^2 - (\tilde{\phi} f)^2 + 2\tilde{\phi} f (\tilde{\phi} f - (\tilde{\phi} f)')\). We get

\[
I_{\text{low}} = \int b^\varepsilon(\cos \theta) 1_{|v - v_*| < \eta \mu_*} ((\tilde{\phi} f)'^2 - (\tilde{\phi} f)^2) \, d\sigma dv_*
+ 2 \int b^\varepsilon(\cos \theta) 1_{|v - v_*| < \eta \mu_*} \tilde{\phi} f (\tilde{\phi} f - (\tilde{\phi} f)') \, d\sigma dv_*
\]

:= \( I_{\text{low, cancell}} + 2 I_{\text{low, dyadic}} \).

**Estimate of \( I_{\text{low, cancell}} \).** Thanks to (2.13), by the cancellation Lemma 1 of [1], we get

\[
I_{\text{low, cancell}} \lesssim \int 1_{|v - v_*| < \eta \mu_*} (\tilde{\phi} f)^2 dv_* \lesssim \eta^3 |\tilde{\phi} f|^2_{L^2} \lesssim \eta^3 |f|^2_{L^2}.
\tag{3.1}
\]

**Estimate of \( I_{\text{low, dyadic}} \).** For simplicity, we define

\[
\mathcal{Y}(g, h) = \int b^\varepsilon(\cos \theta) 1_{|v - v_*| < \eta \mu_*} g (h' - h) \, d\sigma dv_*.
\]

Recalling (1.50) and (1.51), the dyadic decomposition in frequency space reads

\[
f = \sum_{j=-1}^{\infty} \tilde{\phi}_j f,
\tag{3.2}
\]

where \( \tilde{\phi}_j f = \varphi_j(D) f \). By the dyadic decomposition (3.2), we have

\[
I_{\text{low, dyadic}} = -\mathcal{Y}(\tilde{\phi}_j f, \tilde{\phi}_j f) = - \sum_{j,k=-1}^{\infty} \mathcal{Y}(\tilde{\phi}_j \tilde{\phi}_k f, \tilde{\phi}_k \tilde{\phi}_j f)
\tag{3.3}
\]

\[
= - \sum_{-1 \leq j \leq k \leq |\ln \varepsilon|} \mathcal{Y}(\tilde{\phi}_j \tilde{\phi}_k f, \tilde{\phi}_k \tilde{\phi}_j f)
- \sum_{-1 \leq k < j \leq |\ln \varepsilon|} \mathcal{Y}(\tilde{\phi}_j \tilde{\phi}_k f, \tilde{\phi}_k \tilde{\phi}_j f).
\]

For simplicity, we set \( F_k = \tilde{\phi}_k \tilde{\phi}_j f \).

**Case 1: \( k < j \).** Note that

\[
\mathcal{Y}(F_j, F_k) = \int B^\varepsilon_\eta 0 (\phi(\sin(\theta/2)/2^k)) \mu_* F_j (F_k' - F_k) d\sigma dv_*
+ \int B^\varepsilon_\eta 0 (1 - \phi(\sin(\theta/2)/2^k)) \mu_* F_j (F_k' - F_k) d\sigma dv_*
:= \mathcal{Y}_1(F_j, F_k) + \mathcal{Y}_2(F_j, F_k).
\]
Let us first consider $\mathcal{Y}_i(F_j, F_k)$ in which $\varepsilon \leq \sin(\theta/2) \leq \frac{1}{2} \times 2^{-k} \leq 2^{-k+1}$.
By Taylor expansion to $F'_k - F_k$ up to second order, we get $\mathcal{Y}_1(F_j, F_k) = \mathcal{Y}_{1,1}(F_j, F_k) + \mathcal{Y}_{1,2}(F_j, F_k)$ where

$$\mathcal{Y}_{1,1}(F_j, F_k) := \int B_{\eta}^{\varepsilon, 0} \phi(\sin(\theta/2)/2^k) \mu_* F_j(\nabla F_k)(v) \cdot (v - v) d\sigma dv u_*,$$

$$\mathcal{Y}_{1,2}(F_j, F_k) := \int \int_0^1 B_{\eta}^{\varepsilon, 0} \phi(\sin(\theta/2)/2^k) \mu_* F_j(1 - \kappa) (\nabla^2 F_k)(v(\kappa)) : (v - v) \otimes (v - v)) d\sigma dv u_* d\kappa.$$

We first estimate $\mathcal{Y}_{1,1}(F_j, F_k)$. Note that

$$|\int b^\varepsilon \phi(\sin(\theta/2)/2^k)(v - v) d\sigma| = \int b^\varepsilon \phi(\sin(\theta/2)/2^k) \sin^2 \frac{\theta}{2} d\sigma (v_* - v) | \ (3.4)$$

$$\lesssim |\ln \varepsilon|^{-1} (|\ln \varepsilon| - k \ln 2 + \ln 2) |v_* - v|,$$

which yields

$$|\mathcal{Y}_{1,1}(F_j, F_k)| \lesssim |\ln \varepsilon|^{-1} (|\ln \varepsilon| - k \ln 2 + \ln 2) \int \int_0^1 |v_* - v| |\mu_* F_j(\nabla F_k)| dv u_*$$

$$\lesssim \eta^4 |\ln \varepsilon|^{-1} (|\ln \varepsilon| - k \ln 2 + \ln 2) 2^k |F_j|_{L^2} |F_k|_{L^2}.$$

We go to estimate $\mathcal{Y}_{1,2}(F_j, F_k)$. By Cauchy-Schwartz inequality and the change (2.35), using (2.36) and (2.37), we get

$$|\mathcal{Y}_{1,2}(F_j, F_k)| \lesssim |\ln \varepsilon|^{-1} (|\ln \varepsilon| - k \ln 2 + \ln 2) \int \int_0^1 |v_* - v| |\mu_* F_j(\nabla^2 F_k)(v(\kappa))| dv u_*$$

$$\lesssim \left( \int b^\varepsilon \phi(\sin(\theta/2)/2^k) \theta^2 1_{|v_* - v| \leq \eta} |v - v| \mu_* F_j |\nabla^2 F_k|^2 dv u_* \right)^{\frac{1}{2}}$$

$$\times \left( \int b^\varepsilon \phi(\sin(\theta/2)/2^k) \theta^2 1_{|v_* - v| \leq \eta} |v - v| \mu_* |\nabla^2 F_k|^2 dv u_* \right)^{\frac{1}{2}}$$

$$\lesssim |\ln \varepsilon|^{-1} (|\ln \varepsilon| - k \ln 2 + \ln 2) \eta^5 |F_j|_{L^2} |F_k|_{H^2}$$

$$\lesssim \eta^5 |\ln \varepsilon|^{-1} (|\ln \varepsilon| - k \ln 2 + \ln 2) 2^{2k} |F_j|_{L^2} |F_k|_{L^2}.$$

Patching together the estimates of $\mathcal{Y}_{1,1}(F_j, F_k)$ and $\mathcal{Y}_{1,2}(F_j, F_k)$, we have

$$|\mathcal{Y}_1(F_j, F_k)| \lesssim \eta^4 |\ln \varepsilon|^{-1} (|\ln \varepsilon| - k \ln 2 + \ln 2) 2^{2k} |F_j|_{L^2} |F_k|_{L^2}.$$

We now turn to $\mathcal{Y}_2(F_j, F_k)$ in which $\theta \gtrsim 2^{-k}$. By Taylor expansion up to order 1, we have

$$|F'_k - F_k| = |\int_0^1 (\nabla F_k)(v(\kappa)) \cdot (v - v) d\kappa| \lesssim |\theta| |v - v| \int_0^1 |(\nabla F_k)(v(\kappa))| d\kappa.$$
Plugging the above inequality into the definition of $\mathcal{Y}_2(F_j, F_k)$, by Cauchy-Schwartz inequality and the change (2.35), using (2.36), (2.37) and the fact $\int_{2^{-k}}^{\pi/2} \theta^{-2} d\theta \lesssim 2^k$, we get

$$|\mathcal{Y}_2(F_j, F_k)| \lesssim \int B_{\eta}^{\epsilon, 0} \mu_* F_j(F_k' - F_k) d\sigma dv_*,$$

$$= \int B_{\eta}^{\epsilon, 0} \mu_* ((F_j F_k)' - F_j F_k) d\sigma dv_*$$

$$+ \int B_{\eta}^{\epsilon, 0} \mu_* (F_j - F_j') F_k' d\sigma dv_*$$

$$:= \mathcal{X}_1(F_j, F_k) + \mathcal{X}_2(F_j, F_k).$$

Case 2: $j \leq k$. We have

$$|\mathcal{Y}(F_j, F_k)| \lesssim 2^k |F_j|_{L^2} |F_k|_{L^2}.$$

Similar to (3.1), using the cancellation Lemma 1 of [1], we get

$$|\mathcal{X}_1(F_j, F_k)| \lesssim \eta^3 |F_j|_{L^2} |F_k|_{L^2}.\quad (3.6)$$

Similar to the estimate of $\mathcal{Y}(F_j, F_k)$ in Case 1 where $k < j$, here we apply Taylor expansion to $F_j$, similar to (3.5), we can get

$$|\mathcal{X}_2(F_j, F_k)| \lesssim \eta^4 |\ln \epsilon|^{-1}(1 + |\ln \epsilon| - j \ln 2) 2^j |F_j|_{L^2} |F_k|_{L^2}.\quad (3.7)$$

Patching together (3.6) and (3.7), we get for $j \leq k$,

$$|\mathcal{Y}(F_j, F_k)| \lesssim \eta^4 |\ln \epsilon|^{-1}(1 + |\ln \epsilon| - j \ln 2) 2^j |F_j|_{L^2} |F_k|_{L^2}$$

$$+ \eta^3 |F_j|_{L^2} |F_k|_{L^2}.\quad (3.8)$$

By (3.5) and (3.8), recalling (3.3) and (1.31), we have

$$|I_{\text{low, dyadic}}| \lesssim \eta^4 \sum_{-1 \leq k < j \leq \ln \epsilon} 2^k |\ln \epsilon|^{-1}(1 + |\ln \epsilon| - k \ln 2 + 1) |F_j|_{L^2} |F_k|_{L^2}.$$
\[ + \eta^4 \sum_{1 \leq j \leq k \leq \ln \varepsilon} 2^{2j} \frac{\ln \varepsilon - j \ln 2 + 1}{\ln \varepsilon} |F_j|_{L^2} |F_k|_{L^2} \]
\[ + \eta^3 \sum_{1 \leq j \leq k \leq \ln \varepsilon} |F_j|_{L^2} |F_k|_{L^2} \lesssim \eta^3 |W^\varepsilon(D) f|_{L^2}^2. \]

The lemma follows by patching together all the estimates. \( \square \)

Before giving the spectral gap result, we first introduce a special weight function \( U_\delta \) defined by
\[ U_\delta(v) := (1 + \delta^2 |v|^2)^{1/2} \geq \max \{ \delta |v|, 1 \}. \] (3.9)

We remark that \( U_\delta \) plays an important role in deriving (1.43) and here \( \delta \) is a sufficiently small parameter. We recall the function \( \chi \) and its dilation \( \chi_R \) at the beginning of section 2.3.2 (right before Lemma 2.16).

**Lemma 3.2.** Set
\[ X(\gamma, R, \delta) := \delta^{-\gamma} (\chi_R)'(\chi_R)'_*(U_\delta^{\gamma/2})'(U_\delta^{\gamma/2})_* - \chi_R(\chi_R)' \]
\[ U_\delta^{\gamma/2}(U_\delta^{\gamma/2})_* \]
with \( \gamma \leq 0 < \delta \leq 1 \leq R \), then
\[ X(\gamma, R, \delta) \lesssim (\delta^2 + R^{-2}) \theta^2 \langle v \rangle^{\gamma + 2} \langle v_* \rangle^2 1_{|v| \leq 4R}. \] (3.10)

**Proof.** Recall that \( \chi_R \) has support in \( |v| \leq 2R \). If \( |v|^2 + |v_*|^2 \geq 8R^2 \), then either \( |v| \geq 2R \) or \( |v_*| \geq 2R \), which implies \( (\chi_R)_*^2 \chi_R^2 = 0 \). Note that \( |v|^2 + |v_*|^2 = |v'|^2 + |v_*|^2 \), then \( |v|^2 + |v_*|^2 \geq 8R^2 \) also implies \( (\chi_R)'(\chi_R)'_* = 0 \). Therefore, we have
\[ X(\gamma, R, \delta) = X(\gamma, R, \delta) 1_{|v|^2 + |v_*|^2 \leq 8R^2} \leq X(\gamma, R, \delta) 1_{|v| \leq 4R}. \] (3.11)

By adding and subtracting terms, we get
\[ X(\gamma, R, \delta) \lesssim \delta^{-\gamma} ((\chi_R)' - \chi_R)^2 (\chi_R^2)_* (U_\delta^{\gamma/2})'(U_\delta^{\gamma/2})_* + \delta^{-\gamma} ((\chi_R)_*)^2 \chi_R^2 (U_\delta^{\gamma/2})'(U_\delta^{\gamma/2})_* \]
\[ (\chi_R)_*^2 (U_\delta^{\gamma/2})'(U_\delta^{\gamma/2})_* \]
\[ + \delta^{-\gamma} \chi_R^2 (\chi_R)_* (U_\delta^{\gamma/2})'(U_\delta^{\gamma/2})_* \]
\[ + \delta^{-\gamma} \chi_R^2 (\chi_R)_* (U_\delta^{\gamma/2})'(U_\delta^{\gamma/2})_* \]
\[ := A_1 + A_2 + A_3 + A_4. \]

**Estimate of** \( A_1 and A_2. **Since** \( \gamma \leq 0 \) and \( |v'|^2 + |v_*|^2 = |v_*|^2 + |v|^2 \), we derive \( (U_\delta^{\gamma/2})'(U_\delta^{\gamma/2})_* = (1 + \delta^2 |v'|^2 + \delta^2 |v_*|^2 + \delta^4 |v'|^2 |v_*|^2)^{\gamma/2} \leq (1 + \delta^2 |v|^2)^{\gamma/2} \), which yields \( \delta^{-\gamma} (U_\delta^{\gamma/2})'(U_\delta^{\gamma/2})_* \leq (\delta^2 + |v|^2)^{\gamma/2} \leq \langle v \rangle^{\gamma} \).**

Since \( |\nabla \chi_R| \leq R^{-1} \), \( |v' - v| = |v_* - v_*| = |v - v_*| \sin(\theta/2) \), we get
\[ ((\chi_R)' - \chi_R)^2 + ((\chi_R)_* - (\chi_R)_*)^2 \leq R^{-2} \theta^2 |v - v_*|^2 \leq R^{-2} \theta^2 \langle v \rangle^2 \langle v_* \rangle^2. \]
Therefore we deduce that $A_1 + A_2 \lesssim \epsilon^{-2}(v)^{\gamma+2}(v_*)^2$.

**Estimate of $A_3$ and $A_4$.** We first consider $A_3$. Noting that $|\nabla U_\delta^{\gamma/2}| \lesssim \delta U_\delta^{\gamma/2}$, we get

$$
\left( (U_\delta^{\gamma/2})' - U_\delta^{\gamma/2} \right)^2 = \left| \int_0^1 (\nabla U_\delta^{\gamma/2})'(v(\kappa)) \cdot (v' - v) d\kappa \right|^2 
\lesssim \delta^2 \theta^2 |v - v_*|^2 \int_0^1 U_\delta^{\gamma/2}(v(\kappa)) d\kappa.
$$

Thanks to $|v_*|^2 + |v(\kappa)|^2 \sim |v|^2 + |v_*|^2$, we have $\delta^{-\gamma}(U_\delta^{\gamma/2})'_* U_\delta^{\gamma/2}(v(\kappa)) \lesssim \langle v \rangle^\gamma$, which gives $A_3 \lesssim \delta^2 \theta^2 \langle v \rangle^{\gamma+2}(v_*)^2$. Similarly, we have $A_4 \lesssim \delta^2 \theta^2 \langle v \rangle^{\gamma+2}(v_*)^2$.

Patching together the above estimates of $A_1$, $A_2$, $A_3$, $A_4$ and (3.11), we arrive at (3.10).

Now we are in a position to prove the following spectral gap result.

**Theorem 3.1.** Let $-3 \leq \gamma \leq 0$. There are three universal constants $\epsilon_0$, $\eta_0$, $\lambda_0 > 0$ ($\lambda_0$ is related to $\lambda_1$ in (1.19)), such that for any $0 < \epsilon \leq \epsilon_0$, $0 \leq \eta \leq \eta_0$ and suitable function $g$, the following estimate holds true.

$$
(\mathcal{L}^{\epsilon, \gamma, \eta} g, g) \geq \lambda_0 |(I - \mathbb{P})g|^2_{L^{2, \gamma/2}_c}.
$$

(3.12)

Note that for $0 < \epsilon \leq \epsilon_0$, by taking $\eta = 0$ in Theorem 3.1, we get (1.44). As $\epsilon \to 0$, we get (1.44) for the linearized Landau operator.

**Proof.** (Proof of Theorem 3.1.) Suppose $\mathbb{P}g = 0$ and then it suffices to prove $\langle \mathcal{L}^{\epsilon, \gamma, \eta} g, g \rangle \gtrsim |g|^2_{L^{2, \gamma/2}_c}$. For brevity, we set

$$
J^{\epsilon, \gamma, \eta}(g) := 4(\mathcal{L}^{\epsilon, \gamma, \eta} g, g),
$$

$$
\mathcal{H}(f, g) := (f_* g + f g_* - f'_* g' - f' g'_*), \quad \mathcal{F}(f, g) := \mathcal{H}^2(f, g).
$$

With these notations, we have $J^{\epsilon, \gamma, \eta}(g) = \int B^{\epsilon, \gamma, \eta}(\mu^{1/2}_* \mu^{1/2}, g) d\sigma d\nu_*$. Our proof is divided into four steps.

**Step 1: Localization of $J^{\epsilon, \gamma, \eta}(g)$.** Due to (3.9) and the condition $\gamma \leq 0$, we get

$$
|v - v_*|^{-\gamma} \leq C_\gamma \delta^{-\gamma} ((\delta|v|)^{-\gamma} + (\delta|v_*|)^{-\gamma}) \lesssim 2C_\gamma \delta^{-\gamma} U_\delta^{\gamma/2} (v) U_\delta^{\gamma/2} (v_*),
$$

which gives $|v - v_*|^{-\gamma} \gtrsim \delta^{-\gamma} U_\delta^{\gamma/2} (v) U_\delta^{\gamma/2} (v_*)$ and thus

$$
J^{\epsilon, \gamma, \eta}(g) \gtrsim \delta^{-\gamma} \int b^{\epsilon} 1_{|v - v_*| \lesssim \eta} \chi_R^2 (\mathcal{X}_R^2 U_\delta^{\gamma/2} (U_\delta^{\gamma/2})_*) \mathcal{F}(\mu^{1/2}_*, g) d\sigma d\nu_*.
$$

We move the function $\chi_R^2 (\mathcal{X}_R^2 U_\delta^{\gamma/2} (U_\delta^{\gamma/2})_*)$ inside $\mathcal{F}(\mu^{1/2}_*, g)$, which leads to $\mathcal{F}(\mathcal{X}_R U_\delta^{\gamma/2} \mu^{1/2}_*, \mathcal{X}_R U_\delta^{\gamma/2} g)$ with some correction terms. For simplicity, set $h = \mathcal{X}_R U_\delta^{\gamma/2}$, $f = \mu^{1/2}_*$, then

$$
\chi_R^2 (\mathcal{X}_R^2 U_\delta^{\gamma/2} (U_\delta^{\gamma/2})_*) \mathcal{F}(\mu^{1/2}_*, g) = h^2 h^2 \mathcal{F}(f, g) = (h_* (f_* g + f g_* - h_* (f'_* g' + f' g'_*)))^2
$$

$$
- (h_* (f_* g + f g_* - h'_* (f'_* g' + f' g'_*))^2.
$$
\[
\frac{1}{2} \left( h h_\ast (f_\ast g + f g_\ast) - h' h'_\ast (f'_\ast g' + f' g'_\ast) \right)^2
\]
\[
\geq \frac{1}{2} \left( h h_\ast (f_\ast g + f g_\ast) - h' h'_\ast (f'_\ast g' + f' g'_\ast) \right)^2
\]
\[
- (h' h'_\ast - h h_\ast)^2 (f'_\ast g' + f' g'_\ast)^2
\]
\[
= \frac{1}{2} F(h f, h g) - (h' h'_\ast - h h_\ast)^2 (f'_\ast g' + f' g'_\ast)^2. \quad (3.13)
\]

By (3.13), we get
\[
J^{\varepsilon, \gamma, \eta}(g) \geq \frac{1}{2} \delta^{-\gamma} \int b^\varepsilon 1_{|v-v_\ast| \leq \eta} F(\chi_R U^{\gamma/2}_\delta, \chi_R U^{\gamma/2}_\delta) d\sigma d v d v_\ast
\]
\[
- \delta^{-\gamma} \int b^\varepsilon (h' h'_\ast - h h_\ast)^2 (f'_\ast g' + f' g'_\ast)^2 d\sigma d v d v_\ast. \quad (3.14)
\]

We now move \( \chi_R U^{\gamma/2}_\delta \) before \( \mu^{1/2} \) out of \( F(\chi_R U^{\gamma/2}_\delta, \chi_R U^{\gamma/2}_\delta) \), which leads to
\[
F(\mu^{1/2}, \chi_R U^{\gamma/2}_\delta) \) with some correction terms. That is,
\[
F(\chi_R U^{\gamma/2}_\delta, \chi_R U^{\gamma/2}_\delta) = \int b^\varepsilon (h' h'_\ast - h h_\ast)^2 (f'_\ast g' + f' g'_\ast)^2 d\sigma d v d v_\ast. \quad (3.15)
\]

By symmetry, we have
\[
\int b^\varepsilon (h' h'_\ast - h h_\ast)^2 (f'_\ast g' + f' g'_\ast)^2 d\sigma d v d v_\ast
\]
\[
\leq 4 \int b^\varepsilon (h' h'_\ast - h h_\ast)^2 f'_\ast g'^2 d\sigma d v d v_\ast. \quad (3.16)
\]

Thanks to (3.14), (3.15) and (3.16), we get
\[
J^{\varepsilon, \gamma, \eta}(g) \geq \frac{1}{4} \delta^{-\gamma} \int b^\varepsilon 1_{|v-v_\ast| \leq \eta} F(\mu^{1/2}, \chi_R U^{\gamma/2}_\delta) d\sigma d v d v_\ast
\]
\[
- \frac{1}{2} \delta^{-\gamma} \int b^\varepsilon F((1 - \chi_R U^{\gamma/2}_\delta) \mu^{1/2}, \chi_R U^{\gamma/2}_\delta) d\sigma d v d v_\ast
\]
\[
- 4 \delta^{-\gamma} \int b^\varepsilon (h' h'_\ast - h h_\ast)^2 f'_\ast g'^2 d\sigma d v d v_\ast := \frac{1}{4} J_1 - \frac{1}{2} J_2 - 4 J_3. \quad (3.17)
\]

**Step 2: Estimates of \( J_i(i = 1, 2, 3) \).** We will give the estimates term by term.

*Lowerboundof \( J_1 \).* We claim that for \( \varepsilon \leq 64^{-1} R^{-2} \) and some universal constant \( C \),
\[
J_1 \geq \delta^{-\gamma} |g|^2_{L^2_{\gamma/2}} - C(\eta^3 + \delta^2 + R^{-2})|g|^2_{L^2_{\gamma/2}}. \quad (3.18)
\]
Thanks to (1.39), (1.19) and (2.13), for any suitable function $F$, we have
\[
\langle L^e,0,0, F,F\rangle \geq \lambda_{l}^e \langle (I-P) F\rangle_{L^2}^2 \gtrsim \langle (I-P) F\rangle_{L^2}^2.
\]
From this, together with Lemma 3.1, for some universal constant $C$, we have
\[
\langle L^{e,0,\eta}, F,F\rangle = \langle L^{e,0,0}, F,F\rangle - \langle L^{e,0}, F,F\rangle \\
\gtrsim \langle (I-P) F\rangle_{L^2}^2 - C\eta^3 |W^e(D)\langle (I-P) F\rangle_{L^2}^2. (3.19)
\]
Applying (3.19) with $F = \chi_{R}U_{\delta}^{\gamma/2}g$, and using $(a-b)^2 \geq a^2/2 - b^2$, we have
\[
J_1 = \delta^{-\gamma} \int b^*1_{|v-u| \geq \eta\sqrt{\mu}^2, \chi_{R}U_{\delta}^{\gamma/2}g})d\sigma dv = 4\delta^{-\gamma} \langle L^{e,0,\eta}, \chi_{R}U_{\delta}^{\gamma/2}g\rangle_{L^2}^2 \\
\gtrsim \delta^{-\gamma} \langle (I-P)(\chi_{R}U_{\delta}^{\gamma/2}g)\rangle_{L^2}^2 - C\eta^3 \delta^{-\gamma} |W^e(D)\langle (I-P)(\chi_{R}U_{\delta}^{\gamma/2}g)\rangle_{L^2}^2 \\
\gtrsim \frac{1}{2} \delta^{-\gamma} \langle \chi_{R}U_{\delta}^{\gamma/2}g\rangle_{L^2}^2 \delta\eta^3 \delta^{-\gamma} |W^e(D)\langle (I-P)(\chi_{R}U_{\delta}^{\gamma/2}g)\rangle_{L^2}^2 \\
\gtrsim \frac{1}{4} \delta^{-\gamma} \langle U_{\delta}^{\gamma/2}g\rangle_{L^2}^2 - \frac{1}{2} \delta^{-\gamma} \langle (1 - \chi_{R})U_{\delta}^{\gamma/2}g\rangle_{L^2}^2 \delta\eta^3 \delta^{-\gamma} |W^e(D)\langle (I-P)(\chi_{R}U_{\delta}^{\gamma/2}g)\rangle_{L^2}^2 \\
- C\eta^3 \delta^{-\gamma} |W^e(D)\langle (I-P)(\chi_{R}U_{\delta}^{\gamma/2}g)\rangle_{L^2}^2 := J_{1,1} - J_{1,2} - J_{1,3} - J_{1,4}. (3.20)
\]
• Since $\delta \leq 1$ and $\gamma \leq 0$, then $U_{\delta}^{\gamma/2} \geq W_{\gamma/2}$, which enables us to get the leading term
\[
J_{1,1} \gtrsim \delta^{-\gamma} |g|_{L_{\gamma/2}^2}^2. (3.21)
\]
• Thanks to the fact $\delta^{-\gamma} U_{\delta}^{\gamma} \leq W_{\gamma}$ and $1 - \chi_{R}(v) = 0$ when $|v| \leq R$, we have
\[
J_{1,2} = \frac{1}{2} \delta^{-\gamma} |(1 - \chi_{R})U_{\delta}^{\gamma/2}g|_{L^2}^2 \lesssim |(1 - \chi_{R})W_{\gamma/2}g|_{L^2}^2 \\
\lesssim |1_{|v| \geq \eta\sqrt{\mu}^2}W_{\gamma/2}g|_{L^2}^2 + |(1 - \phi(\frac{v}{\mu}^2))W_{\gamma/2}g|_{L^2}^2 \\
\lesssim R^{-2} |\phi(\frac{1}{\mu}^2)W_{\gamma/2+1}g|_{L^2}^2 + |(1 - \phi(\frac{v}{\mu}^2))e^{-\frac{1}{2}}W_{\gamma/2}g|_{L^2}^2 \\
\lesssim (R^{-2} + \varepsilon) |W^eW_{\gamma/2}g|_{L^2}^2. (3.22)
\]
where we use (2.4) and (2.5) in the last inequality. By the assumption $\varepsilon \leq 64^{-1} R^{-2}$, we have
\[
J_{1,2} \lesssim R^{-2} |W^eW_{\gamma/2}g|_{L^2}^2. (3.23)
\]
• We now estimate $J_{1,3}$. Recalling (1.33) for the definition of $P$ and by the condition $P g = 0$, we have
\[
P(\chi_{R}U_{\delta}^{\gamma/2}g) = \sum_{i=1}^{5} e_i \int e_i \chi_{R}U_{\delta}^{\gamma/2}g dv = \sum_{i=1}^{5} e_i \int (\chi_{R}U_{\delta}^{\gamma/2} - 1) g dv.
\]
Observing
\[
1 - \chi_{R}U_{\delta}^{\gamma/2} \lesssim 1 - \chi_{R} + \delta |v| \chi_{R}, (3.24)
\]
and thus \( e_i(1 - \chi_R U_\delta^{\gamma/2}) \lesssim (\delta + R^{-1})\mu_\delta^2 \), we have \( \int e_i(\chi_R U_\delta^{\gamma/2} - 1)g dv \lesssim (\delta + R^{-1})|\mu_\delta^1 g|_{L^2} \), which gives

\[
J_{1.3} = \delta^{-\gamma} |\mathbb{P}(\chi_R U_\delta^{\gamma/2} g)|^2_{L^2} \lesssim (\delta^2 + R^{-2})|\mu_\delta^1 g|^2_{L^2} \lesssim (\delta^2 + R^{-2})|g|_{L^{\gamma/2}}^2 .
\]

(3.25)

- Using Lemma 2.3 with \( \Phi = \delta^{-\gamma/2} \chi_R U_\delta^{\gamma/2} \in S_{1,0}^{\gamma/2} \) and \( M = W^\varepsilon \in S_{1,0}^{1} \), we have

\[
J_{1.4} \leq C \eta^3 \delta^{-\gamma} |W^\varepsilon(D)(\chi_R U_\delta^{\gamma/2} g)|^2_{L^2} \lesssim \eta^3 |W^\varepsilon(D)W_{\gamma/2}^\varepsilon |_{L^2}^2 .
\]

(3.26)

Recalling (3.20), patching together the estimates (3.21), (3.23), (3.25), (3.26), we get (3.18).

**Upper bound of \( J_2 \).** For simplicity, setting \( f_\gamma = (1 - \chi_R U_\delta^{\gamma/2})\mu_\delta^1 \), \( g_\gamma = \chi_R U_\delta^{\gamma/2} g \), we get

\[
J_2 = \delta^{-\gamma} \int b^\varepsilon \mathbb{P}((1 - \chi_R U_\delta^{\gamma/2})\mu_\delta^1, \chi_R U_\delta^{\gamma/2} g) d\sigma dv dv_*
\]

\[
= \delta^{-\gamma} \int b^\varepsilon(f_\gamma, g_\gamma) d\sigma dv dv_*
\]

\[
\lesssim \delta^{-\gamma} \int b^\varepsilon(f_\gamma^2) (g_\gamma' - g_\gamma)^2 d\sigma dv dv_*
\]

\[
+ \delta^{-\gamma} \int b^\varepsilon(g_\gamma^2) (f_\gamma' - f_\gamma)^2 d\sigma dv dv_* := J_{2.1} + J_{2.2} .
\]

(3.27)

Thanks to (3.24), we have

\[
(f_\gamma^2)_* = ((1 - \chi_R U_\delta^{\gamma/2})\mu_\delta^1)_*^2 \lesssim (\delta^2 + R^{-2})\mu_\delta^1.
\]

(3.28)

Plugging (3.28) into \( J_{2.1} \), we have

\[
J_{2.1} \lesssim (\delta^2 + R^{-2})\delta^{-\gamma} \int b^\varepsilon\mu_\delta^1 (g_\gamma' - g_\gamma)^2 d\sigma dv dv_*
\]

\[
= (\delta^2 + R^{-2})\delta^{-\gamma} \mathcal{A}^{0,0,0}(\mu_\delta^1, g_\gamma)
\]

\[
\lesssim (\delta^2 + R^{-2})\delta^{-\gamma} |\chi_R U_\delta^{\gamma/2} g|^2_{L^{\gamma/2}} \lesssim (\delta^2 + R^{-2})|g|^2_{L^{\gamma/2}} .
\]

(3.29)

where we use (2.57) and Lemma 2.3 with \( \Phi = \delta^{-\gamma/2} \chi_R U_\delta^{\gamma/2} \in S_{1,0}^{\gamma/2} \) and \( M = W^\varepsilon \in S_{1,0}^{1} \).

By Taylor expansion up to order 1, \( f_\gamma' - f_\gamma = \int_0^1 (\nabla f_\gamma)(v(\kappa)) \cdot (v' - v) d\kappa \).

From which together with

\[
|\nabla f_\gamma| = |\nabla((1 - \chi_R U_\delta^{\gamma/2})\mu_\delta^1)|
\]

\[
= |(1 - \chi_R U_\delta^{\gamma/2})\nabla \mu_\delta^1 - U_\delta^{\gamma/2} \mu_\delta^1 \nabla \chi_R - \chi_R \mu_\delta^1 \nabla U_\delta^{\gamma/2}|
\]

\[
\lesssim \mu_\delta^1 (\delta + R^{-1}),
\]
we get

$$|f'_\nu - f_\nu|^2 \lesssim (\delta^2 + R^{-2})\theta^2 \int_0^1 \mu^{\frac{1}{2}}(\nu(\kappa))|\nu(\kappa) - v_*|^2d\kappa. \quad (3.30)$$

Since $R \leq 8^{-1} \varepsilon^{-\frac{1}{2}}$, by the change (2.35), using (2.36) and (2.37), recalling (2.4), we have

$$J_{2,2} \lesssim (\delta^2 + R^{-2})\varepsilon^{-\gamma} \int b^\varepsilon \theta^2(\chi R U_\delta^{\nu/2}g)^2\mu^{\frac{1}{2}}(\nu(\kappa))|\nu(\kappa) - v_*|^2d\sigma dv(\kappa)dv_*d\kappa$$

$$\lesssim (\delta^2 + R^{-2})|\chi R W_{\nu/2+1}|_2^2 \lesssim (\delta^2 + R^{-2})|W_{\nu/2}W^\varepsilon g|_2^2$$

$$\lesssim (\delta^2 + R^{-2})|g|^2_{L^2_{\nu,y/2}}. \quad (3.31)$$

Plugging the estimates (3.29) and (3.31) into (3.27), we get

$$J_2 \lesssim (\delta^2 + R^{-2})|g|^2_{L^2_{\nu,y/2}}. \quad (3.32)$$

**Upper bound of $J_3$.** By Lemma 3.2, we have $\delta^{-\gamma}(h' h_* - hh_*)^2 \lesssim (\delta^2 + R^{-2})\theta^2(v)^{\nu/2}(v)|v|^2_{|v| \leq 4R}$. Since $8R \leq \varepsilon^{-\frac{1}{2}}$, by (2.4), we have,

$$J_3 = \delta^{-\gamma} \int b^\varepsilon (h' h_* - hh_*)^2 \mu_* g^2 d\sigma dv_*$$

$$\lesssim (\delta^2 + R^{-2}) \int b^\varepsilon \theta^2(\nu(\kappa)|\nu(\kappa) - v_*|^2\mu_*|v|_{|v| \leq 4R}g^2 d\sigma dv_*$$

$$\lesssim (\delta^2 + R^{-2})|1|_{|v| \leq 4R} W_{\nu/2}g|_{L^2_{\nu,y/2}}^2 \lesssim (\delta^2 + R^{-2})|W_{\nu/2}W^\varepsilon g|_{L^2_{\nu,y/2}}^2$$

$$\lesssim (\delta^2 + R^{-2})|g|^2_{L^2_{\nu,y/2}}. \quad (3.33)$$

**Step 3: Case $-2 < \gamma < 0$.** Plugging the estimates of $J_1$ in (3.18), $J_2$ in (3.32), $J_3$ in (3.33) into (3.17), for $\varepsilon \leq 64^{-1} R^{-2}$, $0 < \delta < 1$, we get

$$J^{\varepsilon,\gamma,\eta}(g) \gtrsim \delta^{-\gamma}|g|^2_{L^2_{\nu,y/2}} - C(\eta^3 + \delta^2 + R^{-2})|g|^2_{L^2_{\nu,y/2}}.$$

Choosing $R = \delta^{-1}$, $\eta = \delta^{2/3}$, for some universal constants $C_1$, $C_2$, we have

$$J^{\varepsilon,\gamma,\eta}(g) \geq C_1 \delta^{-\gamma}|g|^2_{L^2_{\nu,y/2}} - C_2 \delta^2 |g|^2_{L^2_{\nu,y/2}}. \quad (3.34)$$

By Theorem 2.1, for any $0 \leq \eta \leq r_0$, for some universal constants $C_3$, $C_4$, we have

$$J^{\varepsilon,\gamma,\eta}(g) \geq C_3 |g|^2_{L^2_{\nu,y/2}} - C_4 |g|^2_{L^2_{\nu,y/2}}. \quad (3.35)$$

Multiplying (3.35) by $C_5 \delta^2$ and adding the resulting inequality to (3.34), we get

$$(1 + C_5 \delta^2) J^{\varepsilon,\gamma,\eta}(g) \geq (C_1 \delta^{-\gamma} - C_4 C_5 \delta^2)|g|^2_{L^2_{\nu,y/2}} + (C_3 C_5 - C_2) \delta^2 |g|^2_{L^2_{\nu,y/2}}.$$
First take $C_5$ large enough such that $C_3 C_5 - C_2 \geq C_2$, for example let $C_5 = 2C_2 / C_3$. Then take $\delta$ small enough such that $C_4 \delta^{-\gamma} - C_4 C_5 \delta^2 \geq 0$, for example, let $\delta = \left( \frac{C_1 C_5}{2C_4 C_2} \right)^{1/(2+\gamma)} = \left( C_1 C_3 \right)^{1/(2+\gamma)}$. Then we get

$$J^{\epsilon,\gamma,\eta}(g) \geq C_2^{2/3} \left| g \right|^2_{L_{\epsilon,\gamma}^2} = C_2 \left( \frac{C_1 C_3}{2C_4 C_2} \right)^{2/(2+\gamma)} \left| g \right|^2_{L_{\epsilon,\gamma}^2}, \quad (3.36)$$

for any $0 < \epsilon \leq 64^{-1} R^{-2} = 64^{-1} \left( \frac{C_1 C_3}{2C_4 C_2} \right)^{2/(2+\gamma)}$ and $0 \leq \eta \leq \min\{r_0, \delta^{2/3}\} = \min\{r_0, \left( \frac{C_1 C_3}{2C_4 C_2} \right)^{2/(6+3\gamma)} \}$.

**Step 4: Case $-3 \leq \gamma \leq -2$.** In this case, we take $-2 < \alpha, \beta < 0$ such that $\alpha + \beta = \gamma$. Replacing $b^\epsilon$ by $b^\epsilon |v - v_\ast| \alpha^\epsilon$ and $\gamma$ by $\beta$, similar to (3.17), we get

$$J^{\epsilon,\gamma,\eta}(g) \geq \frac{1}{4} \delta^{-\beta} \int b^\epsilon |v - v_\ast| \alpha^\epsilon |v - v_\ast|^2_{v_\ast} \eta^\alpha (\mu_2^2, X R U_\delta^{\beta/2}) d\sigma dv dv_\ast - \frac{1}{2} \delta^{-\beta} \int b^\epsilon |v - v_\ast| \alpha^\epsilon F((1 - X R U_\delta^{\beta/2}) \mu_2^2, X R U_\delta^{\beta/2}) d\sigma dv dv_\ast - 4\delta^{-\beta} \int b^\epsilon |v - v_\ast| \alpha^\epsilon (h'h' - hh_\ast)^2 \mu_2^2 g^2 d\sigma dv dv_\ast$$

$$:= \frac{1}{4} J_1^{\alpha,\beta} - \frac{1}{2} J_2^{\alpha,\beta} - 4 J_3^{\alpha,\beta}, \quad (3.37)$$

where $h: = X R U_\delta^{\beta/2}$.

**Lower bound of $J_1^{\alpha,\beta}$**. Since $-2 < \alpha < 0$, we can use previous estimate (3.36) to get

$$J_1^{\alpha,\beta} = \delta^{-\beta} J^{\epsilon,\gamma,\eta}(X R U_\delta^{\beta/2} g) \geq \delta^{-\beta} \left| W_{\alpha/2} (1 - F) (X R U_\delta^{\beta/2} g) \right|_{L_{\epsilon,\gamma}^{2}},$$

for any $0 < \epsilon \leq 64^{-1} \left( \frac{C_1 C_3}{2C_4 C_2} \right)^{2/(2+\alpha)}$ and $0 \leq \eta \leq \min\{r_0, \left( \frac{C_1 C_3}{2C_4 C_2} \right)^{2/(6+3\alpha)} \}$.

Using $(a - b)^2 \geq a^2 - b^2$, we get

$$J_1^{\alpha,\beta} \geq \frac{1}{4} \delta^{-\beta} \left| W_{\alpha/2} U_\delta^{\beta/2} g \right|_{L_{\epsilon,\gamma}^{2}}^2 - \frac{1}{2} \delta^{-\beta} \left| W_{\alpha/2} (1 - X R) U_\delta^{\beta/2} g \right|_{L_{\epsilon,\gamma}^{2}}^2$$

$$- \delta^{-\beta} \left| W_{\alpha/2} F (X R U_\delta^{\beta/2} g) \right|_{L_{\epsilon,\gamma}^{2}}^2$$

$$:= J_{1,1}^{\alpha,\beta} - J_{1,2}^{\alpha,\beta} - J_{1,3}^{\alpha,\beta}. \quad (3.38)$$

Note that $U_\delta \leq W$ gives $U_\delta^{\beta/2} \geq W/2$ and thus

$$J_{1,1}^{\alpha,\beta} \geq \delta^{-\beta} \left| W_{\alpha/2} W_{\beta/2} g \right|_{L_{\epsilon,\gamma}^{2}}^2 = \delta^{-\beta} \left| g \right|_{L_{\epsilon,\gamma}^{2}}^2. \quad (3.39)$$

Thanks to $\delta^{-\beta} U_\delta^{\beta/2} \leq W_{\beta}$, similar to (3.22) and (3.23), we have

$$J_{1,2}^{\alpha,\beta} \leq \left| W_{\alpha/2} (1 - X R) W_{\beta/2} g \right|_{L_{\epsilon,\gamma}^{2}}^2 \leq R^{-2} \left| W_{\epsilon} W_{\gamma/2} g \right|_{L_{\epsilon,\gamma}^{2}}^2. \quad (3.40)$$
Similarly to (3.25), we get

\[
J_{1,3}^{\alpha,\beta} = \delta^{-\beta}|W_{\alpha/2}\tilde{w}(\chi R U_\delta^{\beta/2} g)|^2_{L_2} \lesssim (\delta^2 + R^{-2})|\mu^{1/2} g|^2_{L_2} \lesssim (\delta^2 + R^{-2})|g|^2_{L_{r,y/2}}.
\]

(3.41)

Plugging (3.39), (3.40), (3.41) into (3.38), we get

\[
J_1^{\alpha,\beta} \gtrsim \delta^{-\beta}|g|^2_{L_{r,y/2}} - C(\delta^2 + R^{-2})|g|^2_{L_{r,y/2}}.
\]

(3.42)

**Upper bound of** \( J_2^{\alpha,\beta} \). Now we analyze

\[
J_2^{\alpha,\beta} = \delta^{-\beta}\int b^e|v - v_\ast|^\alpha \mathbb{F}((1 - \chi R U_\delta^{\beta/2})\mu^{1/2}, \chi R U_\delta^{\beta/2} g)\,d\sigma \,dv \ast.
\]

For simplicity, set \( f_\beta = (1 - \chi R U_\delta^{\beta/2})\mu^{1/2}, g_\beta = \chi R U_\delta^{\beta/2} g \), we get

\[
J_2^{\alpha,\beta} \lesssim \delta^{-\beta}\int b^e|v - v_\ast|^\alpha (f_\beta^2 g_\beta - g_\beta)^2 \,d\sigma \,dv \ast
\]

\[
+ \delta^{-\beta}\int b^e|v - v_\ast|^\alpha (g_\beta^2 (f_\beta' - f_\beta)^2 \,d\sigma \,dv \ast
\]

\[
:= J_{2,1}^{\alpha,\beta} + J_{2,2}^{\alpha,\beta}.
\]

(3.43)

Similar to (3.28), we get \( f_\beta^2 = (1 - \chi R U_\delta^{\beta/2})\mu^{1/2} \lesssim (\delta^2 + R^{-2})\mu^{1/2} \). From this, together with \( \delta^{-\beta}U_\delta^{\beta/2} \leq W_{\beta/2} \), we get

\[
J_{2,1}^{\alpha,\beta} \lesssim (\delta^2 + R^{-2})\delta^{-\beta}\int b^e|v - v_\ast|^\alpha (g_\beta' - g_\beta)^2 \,d\sigma \,dv \ast
\]

\[
= (\delta^2 + R^{-2})\delta^{-\beta}N^{e,\alpha,0}(\mu^{1/2}, g_\beta)
\]

\[
\lesssim (\delta^2 + R^{-2})\delta^{-\beta}|\chi R U_\delta^{\beta/2} g|^2_{L_{r,\alpha/2}} \lesssim (\delta^2 + R^{-2})|g|^2_{L_{r,y/2}}.
\]

(3.44)

where we use Lemma 2.3 with \( \Phi = W_{\alpha/2}\delta^{-\beta/2}U_\delta^{\beta/2} \chi R \) and \( M = W^e \). Similarly to (3.30), we have

\[
|f_\beta' - f_\beta|^2 \lesssim (\delta^2 + R^{-2})\theta^2 \int_0^1 \mu^{1/2}(\nu(\kappa))|\nu(\kappa) - v_\ast|^2 d\kappa.
\]

Thanks to \( |v - v_\ast| \sim |\nu(\kappa) - v_\ast| \), since \( 8R \leq e^{-1/2} \), by the change (2.35), using (2.36) and (2.37), we have

\[
J_{2,2}^{\alpha,\beta} \lesssim (\delta^2 + R^{-2})\delta^{-\beta}\int b^e\theta^2(\chi R U_\delta^{\beta/2} g)^2_{L_2} \mu^{1/2}(\nu(\kappa))|\nu(\kappa)) - v_\ast|^2 \,d\nu(\kappa)d\nu d\kappa
\]

\[
\lesssim (\delta^2 + R^{-2})|\chi R W_{r/2} g|^2_{L_2} \lesssim (\delta^2 + R^{-2})|g|^2_{L_{r,y/2}}.
\]

(3.45)

Plugging (3.44) and (3.45) into (3.43), we get

\[
J_2^{\alpha,\beta} \lesssim (\delta^2 + R^{-2})|g|^2_{L_{r,y/2}}.
\]

(3.46)
Upper bound of $J^{\alpha, \beta}_3$. Recall $J^{\alpha, \beta}_3 = \delta^{-\beta} \int b^\varepsilon |v - v_*|^\alpha (h'h_* - hh_*)^2 \mu_* g^2 \, d\sigma d\nu v_*$. By Lemma 3.2, we have

$$
\delta^{-\beta} (h'h_* - hh_*)^2 = X (\beta, R, \delta) \lesssim (\delta^2 + R^{-2}) \theta^2 (v_*)^2 (v)^{\beta+2} |v| \leq 4R.
$$

Thanks to $\int |v - v_*|^\alpha (v_*)^2 \mu_* d\nu v_* \lesssim (v)^\alpha$, since $8R \leq \varepsilon^{-\frac{1}{2}}$, we get

$$
J^{\alpha, \beta}_3 \lesssim (\delta^2 + R^{-2}) \int b^\varepsilon |v - v_*|^\alpha \theta^2 (v_*)^2 (v)^{\beta+2} \mu_* 1_{|v| \leq 4R} g^2 \, d\sigma d\nu v_* \\
\lesssim (\delta^2 + R^{-2}) |1|_{L^2} \lesssim (\delta^2 + R^{-2}) |g|_{L^2_{r, \gamma/2}}^2. \tag{3.47}
$$

Plugging the estimates of $J^{\alpha, \beta}_1$ in (3.42), $J^{\alpha, \beta}_2$ in (3.46), $J^{\alpha, \beta}_3$ in (3.47) into (3.37), we get

$$
J^{\varepsilon, \gamma, \eta}(g) \gtrsim \delta^{-\beta} |g|_{L^2_{r, \gamma/2}}^2 - C (\delta^2 + R^{-2}) |g|_{L^2_{r, \gamma/2}}^2.
$$

Choosing $R = \delta^{-1}$, for some universal constants $C_6, C_7$, we get

$$
J^{\varepsilon, \gamma, \eta}(g) \gtrsim C_6 \delta^{-\beta} |g|_{L^2_{r, \gamma/2}}^2 - C_7 \delta^2 |g|_{L^2_{r, \gamma/2}}^2. \tag{3.48}
$$

By (3.48) and the coercivity estimate (3.35), since $-2 < \beta < 0$, by a similar argument as in Step 3, similar to (3.36), we get for $-3 \leq \gamma \leq -2$,

$$
J^{\varepsilon, \gamma, \eta}(g) \gtrsim C_7 \left( \frac{C_6 C_3}{2 C_4 C_7} \right)^{2/(2+\beta)} |g|_{L^2_{r, \gamma/2}}^2, \tag{3.49}
$$

for any $0 < \varepsilon \leq \min \left\{ 64^{-1} \left( \frac{C_1 C_3}{2 C_4 C_2} \right)^{2/(2+\alpha)}, 64^{-1} \left( \frac{C_6 C_3}{2 C_4 C_7} \right)^{2/(2+\beta)} \right\}$ and $0 \leq \eta \leq \min \{ r_0, \left( \frac{C_1 C_3}{2 C_4 C_2} \right)^{2/(6+3\alpha)} \}$.

Patching together (3.36) and (3.49), we get (3.12). One can trace the above proof to settle down two universal constants $\varepsilon_0, \eta_0 > 0$ such that (3.12) holds true for any $0 < \varepsilon \leq \varepsilon_0$ and $0 \leq \eta \leq \eta_0$. One should not worry that the above constants could blow up if $\alpha, \beta \to -2$. Indeed, in Step 3, we can deal with $-7/4 \leq \gamma < 0$. Then in Step 4, we deal with $-3 \leq \gamma < -7/4$, where we can choose $-7/4 \leq \alpha, \beta < 0$ such that $\alpha + \beta = \gamma$. In this way, all the constants are universally bounded. \[ \Box \]

4. Upper Bound Estimate

In this section, we will provide various upper bounds on the nonlinear operator $\Gamma^\varepsilon$ and linear operator $L^\varepsilon$. We recall the definition of $\Gamma^{\varepsilon, \gamma, \eta}(g, h)$ from (1.45) and get
\[ \Gamma^{\varepsilon, \gamma, \eta}(g, h) = \mu^{-\frac{1}{2}} Q^{\varepsilon, \gamma, \eta}(\mu^\frac{1}{2} g, \mu^\frac{1}{2} h) \]
\[ = \int B^{\varepsilon, \gamma, \eta}(v - v_\ast, \sigma) \mu^\frac{1}{2}_\ast (g_\ast h' - g_\ast h) d\sigma dv_\ast \]
\[ = \int B^{\varepsilon, \gamma, \eta}(v - v_\ast, \sigma) \left( (\mu^\frac{1}{2}_\ast g)_\ast h' - (\mu^\frac{1}{2}_\ast g)_\ast h \right) d\sigma dv_\ast \]
\[ + \int B^{\varepsilon, \gamma, \eta}(v - v_\ast, \sigma) \left( \mu^\frac{1}{2}_\ast - (\mu^\frac{1}{2}_\ast)_\ast \right) g_\ast h' d\sigma dv_\ast \]
\[ = Q^{\varepsilon, \gamma, \eta}(\mu^\frac{1}{2} g, h) + I^{\varepsilon, \gamma, \eta}(g, h), \]

where for notational brevity, we set
\[ I^{\varepsilon, \gamma, \eta}(g, h) := \int B^{\varepsilon, \gamma, \eta}(v - v_\ast, \sigma) \left( \mu^\frac{1}{2}_\ast - (\mu^\frac{1}{2}_\ast)_\ast \right) g_\ast h' d\sigma dv_\ast. \quad (4.1) \]

We recall from (1.49) the operators \( \Gamma^{\varepsilon, \gamma, \eta}, Q^{\varepsilon, \gamma, \eta}, \mathcal{L}^{\varepsilon, \gamma, \eta} \) containing subscript \( \eta \). Similar to (1.46) and (1.47), we can define \( \mathcal{L}^{\varepsilon, \gamma, \eta}_1, \mathcal{L}^{\varepsilon, \gamma, \eta}_2 \) through \( \Gamma^{\varepsilon, \gamma, \eta}_\eta \). We define \( I^{\varepsilon, \gamma}(g, h) \) using kernel \( B^{\varepsilon, \gamma}(v - v_\ast, \sigma) \) in (1.49) in the way as in (4.1). When \( \eta = 0 \), we drop the superscript \( \eta \) for brevity. That is, \( Q^{\varepsilon, \gamma} := Q^{\varepsilon, \gamma, 0}, \Gamma^{\varepsilon, \gamma} := \Gamma^{\varepsilon, \gamma, 0}, I^{\varepsilon, \gamma} := I^{\varepsilon, \gamma, 0} \).

With these notations in hand, we have the following identities:
\[ \Gamma^{\varepsilon, \gamma}(g, h) = Q^{\varepsilon, \gamma}(\mu^\frac{1}{2} g, h) + I^{\varepsilon, \gamma}(g, h). \]
\[ \Gamma^{\varepsilon, \gamma, \eta}(g, h) = Q^{\varepsilon, \gamma, \eta}(\mu^\frac{1}{2} g, h) + I^{\varepsilon, \gamma, \eta}(g, h). \]
\[ \Gamma^{\varepsilon, \gamma}(g, h) = Q^{\varepsilon, \gamma, \eta}(g, h). \]
\[ \Gamma^{\varepsilon, \gamma, \eta}(g, h) = Q^{\varepsilon, \gamma, \eta}(g, h) + I^{\varepsilon, \gamma, \eta}(g, h). \]

Throughout this section, we assume \(-3 \leq \gamma \leq 0, 0 < \varepsilon \leq 1\) unless otherwise specified. Our results on the upper bounds can be summarized in Table 1.

It is easy to see that \( (Q^{\varepsilon, \gamma, \eta}(g, h), f) \) and \( (I^{\varepsilon, \gamma, \eta}(g, h), f) \) involve the regular region \(|v - v_\ast| \geq \eta\), while \( (Q^{\varepsilon, \gamma}(g, h), f), (I^{\varepsilon, \gamma}(g, h), f), (-\mathcal{L}^{\varepsilon, \gamma, \eta}_1 h + \Gamma^{\varepsilon, \gamma}(f, h), h) \) and \( (-\mathcal{L}^{\varepsilon, \gamma, \eta}_2 f, f) \) focus on the singular region \(|v - v_\ast| \leq \eta\). We provide two types of estimates on these functionals because we will meet two cases for the nonlinear term \( \Gamma^{\varepsilon} \) when the standard energy method is applied. These two cases can be clarified as follows: \( (\Gamma^{\varepsilon}(f, \partial^\alpha f), \partial^\alpha f) \) and \( (\Gamma^{\varepsilon}(\partial^\alpha f, \partial^\alpha f), \partial^\alpha f) \), where \( \alpha_1 + \alpha_2 = \alpha \) and \(|\alpha_2| < |\alpha|\).

- The first case corresponds to the highest order estimate of the solution. As explained in section 1.4.3, the linear-quasilinear method will be employed. Technically we need to separate the integration domain into two regions: singular region and regular region. In this situation, all the upper bounds will depend on the parameter \( \eta \).
- For the second case, since \(|\alpha_2| < |\alpha|\), we have one more derivative freedom on the function \( \partial^\alpha f \). In this situation, all the upper bounds are independent of the parameter \( \eta \) and allow more regularity.
Table 1. Results summary

| Functionals | Proposition or Theorem |
|-------------|------------------------|
| $\langle Q^{\varepsilon,\gamma,\eta}(g, h), f \rangle$ | Proposition 4.1 |
| $\langle I^{\varepsilon,\gamma,\eta}(g, h), f \rangle$ | Proposition 4.2 |
| $\langle Q^{\varepsilon,\gamma}_{\eta}(g, h), f \rangle$ | Proposition 4.3 |
| $\langle I^{\varepsilon,\gamma}_{\eta}(g, h), f \rangle$ | Proposition 4.4 |
| $\langle -L^{\varepsilon,\gamma}_{1,\eta}(h + \Gamma^{\varepsilon,\gamma}_{\eta}(f, h), h) \rangle$ | Proposition 4.5 |
| $\langle L^{\varepsilon,\gamma}_{2,\eta}(f, h) \rangle$ | Proposition 4.6 |
| $\langle \Gamma^{\varepsilon,\gamma,\eta}(g, h, f) \rangle$ | Theorem 4.1 |
| $\langle Q^{\varepsilon,\gamma}(g, h), f \rangle$ | Theorem 4.2 |
| $\langle I^{\varepsilon,\gamma}(g, h), f \rangle$ | Theorem 4.3 |
| $\langle \Gamma^{\varepsilon,\gamma}(g, h, f) \rangle$ | Theorem 4.4 |
| $\langle I^{\varepsilon,\gamma}_{\eta}(f, h) - L^{\varepsilon,\gamma}_{\eta}(h, h) \rangle$ | Theorem 4.5 |

4.1. Upper Bounds of $(Q^{\varepsilon,\gamma,\eta}(g, h), f)$ and $(I^{\varepsilon,\gamma,\eta}(g, h), f)$

By (4.2), we need to consider $Q^{\varepsilon,\gamma,\eta}$ and $I^{\varepsilon,\gamma}.$

4.1.1. Upper Bound of $(Q^{\varepsilon,\gamma,\eta}(g, h), f)$ We begin with two technical lemmas, which rely on some localization techniques in phase and frequency space.

Lemma 4.1. Let $0 < \eta \leq 1,$ $\Upsilon^{\varepsilon,\gamma}(h, f) := \int b^\varepsilon (\frac{u}{|u|} \cdot \sigma) |u|^{\gamma} 1_{|u| \geq \eta} h(u)(f(u^+) - f(|u| \frac{u^+}{|u^+|}))d\sigma du,$ then

$$\Upsilon^{\varepsilon,\gamma}(h, f) \lesssim \eta^{\gamma - 3}(|W^{\varepsilon} W_{\gamma/2}h|_{L^2} + |W^{\varepsilon}(D)W_{\gamma/2}h|_{L^2})$$

$$\quad (|W^{\varepsilon} W_{\gamma/2}f|_{L^2} + |W^{\varepsilon}(D)W_{\gamma/2}f|_{L^2}).$$

Proof. We divide the proof into two steps.

Step 1: without the term $|u|^\gamma 1_{|u| \geq \eta}.$ For ease of notation, we denote

$$\mathcal{X}(h, f) := \int b^\varepsilon (\frac{u}{|u|} \cdot \sigma) h(u)(f(u^+) - f(|u| \frac{u^+}{|u^+|}))d\sigma du.$$ First applying dyadic decomposition in the phase space, as $\sqrt{2} |u| \leq |u^+| \leq |u|,$ we have

$$\mathcal{X}(h, f) = \sum_{k = -\infty}^{\infty} \int b^\varepsilon (\frac{u}{|u|} \cdot \sigma) (\tilde{\varphi}_k h(u)((\varphi_k f)(u^+) - (\varphi_k f)(|u| \frac{u^+}{|u^+|}))d\sigma du$$

$$:= \sum_{k = -\infty}^{\infty} \mathcal{X}_k.$$ where $\tilde{\varphi}_k = \sum_{|l - k| \leq 3} \varphi_l.$ We split the proof into two cases: $2^k \geq 1/\varepsilon$ and $2^k \leq 1/\varepsilon.$

Case 1: $2^k \geq 1/\varepsilon.$ We first have

$$|\mathcal{X}_k| \leq \left( \int b^\varepsilon (\frac{u}{|u|} \cdot \sigma)(\tilde{\varphi}_k h(u))^2 d\sigma du \right)^{\frac{1}{2}}$$

$$\quad \left( \int b^\varepsilon (\frac{u}{|u|} \cdot \sigma)(|(\varphi_k f)(u^+)|^2 + |(\varphi_k f)(|u| \frac{u^+}{|u^+|})|^2) d\sigma du \right)^{\frac{1}{2}}.$$
By the changes $u \to u^+$ and $u \to |u| u^+$, the estimate (2.11), we have $|\mathcal{X}_k| \lesssim |\ln \varepsilon|^{-1} \varepsilon^{-2} |\phi_k h|_{L^2} |\varphi_k f|_{L^2}$, which gives

$$
| \sum_{2^k \geq 1/\varepsilon} \mathcal{X}_k | \lesssim \sum_{2^k \geq 1/\varepsilon} | \ln \varepsilon |^{-1} \varepsilon^{-2} |\phi_k h|_{L^2} |\varphi_k f|_{L^2} \lesssim |W^\varepsilon h|_{L^2} |W^\varepsilon f|_{L^2}.
$$

**Case 2**: $2^k \leq 1/\varepsilon$. By Lemma 2.6 and the dyadic decomposition in the frequency space, we have

$$
\mathcal{X}_k = \int b^\varepsilon \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \left( \overline{\phi_k h}(\xi^+) - \overline{\phi_k h}(|\xi| \frac{\xi^+}{|\xi^+|}) \right) \overline{\varphi_k f}(\xi) d\sigma d\xi
$$

$$
= \sum_{l=1}^\infty \int b^\varepsilon \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \left( (\psi_l \overline{\phi_k h})(\xi^+) - (\psi_l \overline{\phi_k h})(|\xi| \frac{\xi^+}{|\xi^+|}) \right) (\overline{\psi_l \varphi_k f})(\xi) d\sigma d\xi
$$

$$
:= \sum_{l=1}^\infty \mathcal{X}_{k,l}.
$$

**Subcase 1**: $2^l \geq 1/\varepsilon$. In this case, we have $|\mathcal{X}_{k,l}| \lesssim |\ln \varepsilon|^{-1} \varepsilon^{-2} |\psi_l \overline{\phi_k h}|_{L^2} |\overline{\varphi_k f}|_{L^2}$, which yields

$$
\sum_{2^k \geq 1/\varepsilon} |\mathcal{X}_{k,l}| \lesssim \sum_{2^k \geq 1/\varepsilon} |\ln \varepsilon|^{-1} \varepsilon^{-2} |\psi_l \overline{\phi_k h}|_{L^2} |\overline{\varphi_k f}|_{L^2} \lesssim |W^\varepsilon (D) \phi_k h|_{L^2} |W^\varepsilon (D) \varphi_k f|_{L^2}.
$$

By Lemma 2.3, we have $||W^\varepsilon (D), 2^k \varphi_k f||_{L^2} \lesssim |f|_{L^2}$, which gives

$$
\sum_{k \geq -1} |W^\varepsilon (D) \varphi_k f|_{L^2}^2 \lesssim \sum_{k \geq -1} 2^{-2k} (|2^k \varphi_k W^\varepsilon (D) f|_{L^2}^2 + |f|_{L^2}^2) \lesssim |W^\varepsilon (D) f|_{L^2}^2.
$$

Together with Cauchy-Schwartz inequality, we have $\sum_{2^k \leq 1/\varepsilon, 2^l \geq 1/\varepsilon} |\mathcal{X}_{k,l}| \lesssim |W^\varepsilon (D) h|_{L^2} |W^\varepsilon (D) f|_{L^2}$.

**Subcase 2**: $2^l \leq 1/\varepsilon$. We have

$$
\mathcal{X}_{k,l} = \int b^\varepsilon \left( \frac{\xi}{|\xi|} \cdot \sigma \right) 1_{\sin(\theta/2) \geq 2^{-\frac{k+1}{2}}} \left[ (\psi_l \overline{\phi_k h})(\xi^+) - (\psi_l \overline{\phi_k h})(|\xi| \frac{\xi^+}{|\xi^+|}) \right] (\overline{\psi_l \varphi_k f})(\xi) d\sigma d\xi
$$

$$
- (\psi_l \overline{\phi_k h})(|\xi| \frac{\xi^+}{|\xi^+|}) (\overline{\psi_l \varphi_k f})(\xi) d\sigma d\xi := \mathcal{X}_{k,l,1} + \mathcal{X}_{k,l,2}.
$$

As $\int_2^{1/\varepsilon} t^{-3} dt \lesssim 2^{k+l}$, we have $|\mathcal{X}_{k,l,1}| \lesssim |\ln \varepsilon|^{-1} 2^{k+l} |\psi_l \overline{\phi_k h}|_{L^2} |\overline{\varphi_k f}|_{L^2}$.

Recalling (2.6), we have
\[
\sum_{2^k \leq 1/\varepsilon, 2^l \leq 1/\varepsilon} |X_{k,l,1}| \leq \left( \sum_{2^k \leq 1/\varepsilon, 2^l \leq 1/\varepsilon} |\ln \varepsilon|^{-1} 2^l |\varphi_l \varphi_k h|^2_{L^2} \right)^{1/2} \\
\times \left( \sum_{2^k \leq 1/\varepsilon, 2^l \leq 1/\varepsilon} |\ln \varepsilon|^{-1} 2^{2k} |\varphi_l \varphi_k f|^2_{L^2} \right)^{1/2} \\
\lesssim |W^\varepsilon(D)h|_{L^2} |W^\varepsilon f|_{L^2}.
\]

By Taylor expansion, \((\varphi_l \varphi_k h)(\xi^+) - (\varphi_l \varphi_k h)(|\xi| \xi^+) = (1 - \frac{1}{\cos \theta}) \int_0^1 (\nabla (\varphi_l \varphi_k h))(\xi^+) \cdot \xi^+ d\kappa\), where \(\xi^+(\kappa) = (1 - \kappa) |\xi| \xi^+ + \kappa \xi^+\). From which we obtain

\[
|X_{k,l,2}| = \left| \int b^\varepsilon \left( \frac{\xi}{|\xi|} \right) \cdot \sigma (1 - \frac{1}{\cos \theta}) 1_{\varepsilon \leq \sin(\theta/2) \leq 2} \frac{k+l}{2} (\varphi_l \varphi_k f)(\xi) (\nabla (\varphi_l \varphi_k h)(\xi^+(\kappa))) \right| \\
\lesssim |\ln \varepsilon|^{-1} \left( \int_{\varepsilon}^{2^2} \int r^{-1} |\varphi_l \varphi_k f(\xi)|^2 \, dr \, d\xi \right)^{1/2} \\
\left( \int_{\varepsilon}^{2^{k+l}} \int r^{-1} |u|^2 |\nabla (\varphi_l \varphi_k h)(u)|^2 \, dr \, du \right)^{1/2} \\
\lesssim |\ln \varepsilon|^{-1} \left( |\ln \varepsilon| - \frac{k + l}{2} \ln 2 \right) |\varphi_l \varphi_k f|_{L^2} (\| \varphi_l \varphi_k h \|^2_{L^2}) \\
\lesssim |\ln \varepsilon|^{-1} \left( |\ln \varepsilon| - \frac{k + l}{2} \ln 2 \right) |\varphi_l \varphi_k h|_{L^2}^2 + 2^l |\varphi_l \varphi_k f|_{L^2}^2.
\]

where we use the change of variable \(\xi \to u = \xi^+(\kappa)\) and the fact

\[
\int |u|^2 |\nabla (\varphi_l \varphi_k h)(u)|^2 \, du \lesssim |\varphi_l \varphi_k h|_{L^2}^2 + 2^l |\varphi_l \varphi_k f|_{L^2}^2.
\]

Since \(|\ln \varepsilon|^{-1} (|\ln \varepsilon| - \frac{k + l}{2} \ln 2) \lesssim 1\), we have

\[
\sum_{2^k \leq 1/\varepsilon, 2^l \leq 1/\varepsilon} |\ln \varepsilon|^{-1} (|\ln \varepsilon| - \frac{k + l}{2} \ln 2) |\varphi_l \varphi_k f|_{L^2} |\varphi_l \varphi_k h|_{L^2} \lesssim |f|_{L^2} |h|_{L^2}.
\]

It is easy to check \(|\ln \varepsilon| - \frac{k + l}{2} \ln 2 \leq (|\ln \varepsilon| - k \ln 2 + 2) \frac{1}{2} (|\ln \varepsilon| - l \ln 2 + 2) \frac{1}{2} \) and thus

\[
\sum_{2^k \leq 1/\varepsilon, 2^l \leq 1/\varepsilon} |\ln \varepsilon|^{-1} (|\ln \varepsilon| - \frac{k + l}{2} \ln 2) 2^l |\varphi_l \varphi_k h|_{L^2}^2 |\varphi_l \varphi_k f|_{L^2} \\
\lesssim \left( \sum_{2^k \leq 1/\varepsilon, 2^l \leq 1/\varepsilon} |\ln \varepsilon|^{-1} (|\ln \varepsilon| - k \ln 2 + 2) 2^l |\varphi_l \varphi_k h|^2_{L^2} \right)^{1/2} \\
\times \left( \sum_{2^k \leq 1/\varepsilon, 2^l \leq 1/\varepsilon} |\ln \varepsilon|^{-1} (|\ln \varepsilon| - l \ln 2 + 2) 2^l |\varphi_l \varphi_k f|^2_{L^2} \right)^{1/2}.
\]
\[ \lesssim \left( \sum_{2^k \leq 1/\varepsilon} |W^e(D)\varphi_k f|_{L^2}^2 \right)^{1/2} \left( \sum_{2^k \leq 1/\varepsilon} |\ln \varepsilon|^{-1} (|\ln \varepsilon| - k \ln 2 + 2)|\varphi_k h|_{L^2}^2 \right)^{1/2} \]
\[ \lesssim |W^e(D) f|_{L^2} |W^e h|_{L^2}. \]

By the previous two results, we have \[ \sum_{2^k \leq 1/\varepsilon, 2^l \leq 1/\varepsilon} |X_{k,l,2}| \lesssim |W^e(D) f|_{L^2} |W^e h|_{L^2}. \]

Patch together all the above results, we conclude that \[ |X(h, f)| \lesssim (|W^e h|_{L^2} + |W^e(D) h|_{L^2})(|W^e f|_{L^2} + |W^e(D) f|_{L^2}). \] (4.4)

**Step 2: with the term** \(|u|^\gamma 1_{|u| \geq \eta} \). \[ \text{Observe that } |u|^\gamma 1_{|u| \geq \eta} = |u|^\gamma (1 - \phi(u)) + |u|^\gamma (\phi(u) - 1_{|u| < \eta}). \]

From which, we separate \( \mathcal{J}^e,\gamma (h, f) \) into two parts \( \mathcal{J}_1^{e,\gamma} (h, f) \) and \( \mathcal{J}_2^{e,\gamma} (h, f) \) which correspond to \( |u|^\gamma (1 - \phi(u)) \) and \( |u|^\gamma (\phi(u) - 1_{|u| < \eta}) \) respectively.

**Estimate of** \( \mathcal{J}_1^{e,\gamma} (h, f) \). \[ \text{Set } H(u) := h(u)|u|^{-\gamma}|u|^\gamma (1 - \phi(u)) \text{ and } w = |u| u^+_{|u^+|}, \]

then \( W_{\gamma/2} W_{\gamma/2} = W_{\gamma/2} (w) \) and \[ \langle u \rangle^\gamma H(u) (f(u^+) - f(w)) = (W_{\gamma/2} H(u) ((W_{\gamma/2} f)(u^+) - (W_{\gamma/2} f)(w)) \]

\[ + (W_{\gamma/2} H(u) (W_{\gamma/2} f)(u^+))(W_{\gamma/2}(w) W_{-\gamma/2}(u^+) - 1). \]

From which we have \[ \mathcal{J}_1^{e,\gamma} (h, f) = \mathcal{J}(W_{\gamma/2} H, W_{\gamma/2} f) + A \]

\[ A := \int b^e \left( \frac{u}{|u|} \cdot \sigma \right) (W_{\gamma/2} H)(u)(W_{\gamma/2} f)(u^+) \]

\[ (W_{\gamma/2}(w) W_{-\gamma/2}(u^+) - 1) \, d\sigma \, du. \]

Observing that \( |W_{\gamma/2}(u) W_{\gamma/2}(u^+) - 1| \lesssim \theta^2 \), we have \[ |A| = \left( \int b^e \left( \frac{u}{|u|} \cdot \sigma \right) |(W_{\gamma/2} H)(u)|^2 |W_{\gamma/2}(w) W_{-\gamma/2}(u^+) - 1| \, d\sigma \, du \right)^{1/2} \]

\[ \times \left( \int b^e \left( \frac{u}{|u|} \cdot \sigma \right) |(W_{\gamma/2} f)(u^+)|^2 |W_{\gamma/2}(w) W_{-\gamma/2}(u^+) - 1| \, d\sigma \, du \right)^{1/2} \]

\[ \lesssim |W_{\gamma/2} H|_{L^2} |W_{\gamma/2} f|_{L^2}, \]

where the change of variable \( u \to u^+ \) is used. Thanks to the result (4.4) in **Step 1** and Lemma 2.3 with \( M = W^e \in S^1_{1,0} \) and \( \Phi = \langle \cdot \rangle^{-\gamma} \cdot |\gamma (1 - \phi) (\cdot) \in S^0_{1,0}, \)

we have \[ |\mathcal{J}_1^{e,\gamma} (h, f)| \lesssim (|W^e W_{\gamma/2} h|_{L^2} \]

\[ + |W^e(D) W_{\gamma/2} h|_{L^2})(|W^e W_{\gamma/2} f|_{L^2} + |W^e(D) W_{\gamma/2} f|_{L^2}). \]

**Estimate of** \( \mathcal{J}_2^{e,\gamma} (h, f) \). Since the support of \( |u|^\gamma (\phi(u) - 1_{|u| < \eta}) \) belongs to \( \eta \lesssim u \lesssim 1 \), we notice that \[ \mathcal{J}_2^{e,\gamma} (h, f) = \int b^e \left( \frac{u}{|u|} \cdot \sigma \right) \tilde{W}(u) \tilde{H}(u) (\tilde{F}(u^+) - \tilde{F}(\langle |u| u^+_{|u^+|} \rangle) \, d\sigma \, du, \]
where \( \tilde{W}(u):= |u|^\gamma (\phi(u) - 1_{|u|<\eta}), \tilde{\phi}(u):= \phi(u/4), \tilde{H}: = \tilde{\phi}h, \tilde{F}: = \tilde{\phi}f \). By the result (4.4) in Step 1, we derive that

\[
|Y_2^{\varepsilon, \gamma}(h, f)| \lesssim (|W^\varepsilon \tilde{W}\tilde{H}|_{L^2} + |W^\varepsilon(D)(\tilde{W}\tilde{H})|_{L^2})(|W^\varepsilon \tilde{F}|_{L^2} + |W^\varepsilon(D)\tilde{F}|_{L^2}).
\]

First by \( |\tilde{W}| \lesssim \phi(u)\eta^\gamma \), we have \( |W^\varepsilon \tilde{W}\tilde{H}|_{L^2} \lesssim \eta^\gamma |W^\varepsilon W_{\gamma/2}h|_{L^2} \). Next, let us focus on \( W^\varepsilon(D)(\tilde{W}\tilde{H}) \). Note that for any \( \Psi \in L^2 \), there holds

\[
\int \Psi W^\varepsilon(D)(\tilde{W}\tilde{H})dv = \int \tilde{W}(\xi)W^\varepsilon(\xi)\tilde{W}(v)\tilde{H}(\xi - v)dvd\xi.
\]

By (2.8), Fubini’s theorem, \( |\cdot|_{L^1} \lesssim |\cdot|_{L^2}^2 \) and (2.3), we have

\[
|\int \Psi W^\varepsilon(D)(\tilde{W}\tilde{H})dv| \lesssim |W^\varepsilon \tilde{W}|_{L^1}|W^\varepsilon \tilde{H}|_{L^2}|\Psi|_{L^2} \lesssim |\tilde{W}|_{\dot{H}^3}|W^\varepsilon(D)\tilde{H}|_{L^2}|\Psi|_{L^2} \lesssim \eta^{-3} |W^\varepsilon(D)\tilde{H}|_{L^2}|\Psi|_{L^2}.
\]

From which we infer that \( |W^\varepsilon(D)(\tilde{W}\tilde{H})|_{L^2} \lesssim \eta^{-3} |W^\varepsilon(D)\tilde{H}|_{L^2} \). From which together with the support of \( \tilde{H} \) and \( \tilde{F} \), and Lemma 2.3, we finally have

\[
|Y_2^{\varepsilon, \gamma}(h, f)| \lesssim \eta^{-3} (|W^\varepsilon W_{\gamma/2}h|_{L^2} + |W^\varepsilon(D)W_{\gamma/2}h|_{L^2}) (|W^\varepsilon W_{\gamma/2}f|_{L^2} + |W^\varepsilon(D)W_{\gamma/2}f|_{L^2}).
\]

We conclude the desired result by patching together all the estimates. \( \square \)

Similar to Lemma 4.1, we can derive

**Lemma 4.2.** There holds

\[
\mathcal{A} := \ln |\varepsilon|^{-1} \int_{\mathbb{R}^3} \int_{\varepsilon/2}^{\pi/4} \theta^{-3} |f(v) - f(v/\cos \theta)|^2 dv d\theta \lesssim |W^\varepsilon(D)f|_{L^2}^2 + |W^\varepsilon f|_{L^2}^2.
\]

We omit the proof of Lemma 4.2 since it can be proved in a similar way as Lemma 4.1.

Now we are in a position to prove the following upper bound of \( Q^{\varepsilon, \gamma, \eta} \).

**Proposition 4.1.** Fix \( 0 < \eta \leq 1 \). For suitable functions \( g, h \) and \( f \), there holds

\[
|\langle Q^{\varepsilon, \gamma, \eta}(g, h), f \rangle| \lesssim \eta^{-3} |g|_{L^1_{\varepsilon/2, \gamma/2}} |h|_{L^2_{\varepsilon/2, \gamma/2}} |f|_{L^2_{\varepsilon/2, \gamma/2}}.
\]

**Proof.** Recalling the translation operator \( T_{v_\varepsilon} \) defined by \((T_{v_\varepsilon} f)(v) = f(v_\varepsilon + v)\). By geometric decomposition in the phase space, we have \( \langle Q^{\varepsilon, \gamma, \eta}(g, h), f \rangle = D_1 + D_2 \), where

\[
D_1 := \int b^\varepsilon(\frac{u}{|u|} \cdot \sigma)|u|^\gamma 1_{|u| \geq \eta} g^*(T_{v_\varepsilon} h)(u)(T_{v_\varepsilon} f)(u^+) - (T_{v_\varepsilon} f)(|u| \frac{u^+}{|u^+|})d\sigma dv_\varepsilon du,
\]

\[
D_2 := \int b^\varepsilon(\frac{u}{|u|} \cdot \sigma)|u|^\gamma 1_{|u| \geq \eta} g^*(T_{v_\varepsilon} h)(u)(T_{v_\varepsilon} f)(|u| \frac{u^+}{|u^+|}) - (T_{v_\varepsilon} f)(u)d\sigma dv_\varepsilon du.
\]

We remark that \( D_1 \) represents the “radial” part and \( D_2 \) stands for the “spherical” part.
**Step 1: Estimate of $D_1$.** By Lemma 4.1, we have

$$|D_1| \lesssim \eta^{r-3} \int |g_v|(|W^\varepsilon W_{\gamma/2}T_{v*,h}|_{L^2} + |W^\varepsilon (D)W_{\gamma/2}T_{v*,h}|_{L^2})$$

$$\times (|W^\varepsilon W_{\gamma/2}T_{v*,f}|_{L^2} + |W^\varepsilon (D)W_{\gamma/2}T_{v*,f}|_{L^2}) dv_*.$$

By (2.9) and (2.10), we have

$$|W^\varepsilon W_{\gamma/2}T_{v*,h}|_{L^2} \lesssim |W^\varepsilon (v_*) W_{|\gamma/2|}(v_*)| W^\varepsilon W_{\gamma/2}h|_{L^2}$$

$$\lesssim W_{|\gamma/2|+1}(v_*) |W^\varepsilon W_{\gamma/2}h|_{L^2}. \quad (4.5)$$

Since $W^\varepsilon \in S^1_{1,0}$, $W_{\gamma/2} \in S^1_{1,0}$, by (2.10) and Lemma 2.3, we have

$$|W^\varepsilon (D)W_{\gamma/2}T_{v*,h}|_{L^2} \lesssim |W^\varepsilon (D)W_{\gamma/2}h|_{L^2} + |W^\varepsilon W_{\gamma/2}h|_{L^2}$$

$$= |W^\varepsilon (D)W_{\gamma/2}h|_{L^2} + |W^\varepsilon (D)h|_{L^2} + |W_{\gamma/2}h|_{L^2}$$

$$\lesssim W_{|\gamma/2|+1}(v_*) (|W^\varepsilon (D)h|_{L^2} + |h|_{L^2})$$

$$\lesssim W_{|\gamma/2|+1}(v_*) |W^\varepsilon (D)h|_{L^2}. \quad (4.6)$$

By (4.5) and (4.6), we get

$$|D_1| \lesssim \eta^{r-3} |g|_{(\eta^1)} (|W^\varepsilon (D)W_{\gamma/2}h|_{L^2} + |W^\varepsilon W_{\gamma/2}h|_{L^2})$$

$$= (|W^\varepsilon (D)W_{\gamma/2}h|_{L^2} + |W^\varepsilon W_{\gamma/2}h|_{L^2}) dv_*.$$

**Step 2: Estimate of $D_2$.** Let $u = r \tau$ and $\varsigma = \frac{\tau+\sigma}{|\tau+\sigma|^2}$, then $\frac{u+\sigma}{|u+\sigma|^2} = r \varsigma$. By the change of variable $(u, \sigma) \to (r, \tau, \varsigma)$, one has $d\sigma du = 4(\tau \cdot \varsigma) r^2 d\tau d\varsigma$. Then

$$D_2 = 4 \int \tau^r (1 - \phi)(r) b^\varepsilon (2(\tau \cdot \varsigma)^2 - 1) (T_{v*,h})(r \tau)$$

$$\times ((T_{v*,f})(r \varsigma) - (T_{v*,f})(r \tau))(\tau \cdot \varsigma) r^2 d\tau d\varsigma dv_*$$

$$= 2 \int \tau^r (1 - \phi)(r) b^\varepsilon (2(\tau \cdot \varsigma)^2 - 1) ((T_{v*,h})(r \tau) - (T_{v*,h})(r \varsigma))$$

$$\times ((T_{v*,f})(r \varsigma) - (T_{v*,f})(r \tau))(\tau \cdot \varsigma) r^2 d\tau d\varsigma dv_*$$

$$= -\frac{1}{2} \int b^\varepsilon (\frac{u+\sigma}{|u+\sigma|^2} - (T_{v*,f})(u)) d\sigma dv_* du.$$
Note that $D_{2,1}$ and $D_{2,2}$ have exactly the same structure. It suffices to focus on $D_{2,2}$. Since
\[
((T_{v_2} f)(|u| \frac{u^+}{|u^+|}) - (T_{v_2} f)(u))^2 \leq 2((T_{v_2} f)(|u| \frac{u^+}{|u^+|}) - (T_{v_2} f)(u^+))^2 + 2((T_{v_2} f)(u^+) - (T_{v_2} f)(u))^2,
\]
we have
\[
D_{2,2} \lesssim \int b^\varepsilon \left( \frac{u}{|u|} \cdot \sigma \right) (u') |g_\ast|((T_{v_2} f)(|u| \frac{u^+}{|u^+|}) - (T_{v_2} f)(u^+))^2 \, d\sigma \, d\nu_\ast du
\]
\[+ \int b^\varepsilon \left( \frac{u}{|u|} \cdot \sigma \right) (u') |g_\ast|((T_{v_2} f)(u^+) - (T_{v_2} f)(u))^2 \, d\sigma \, d\nu_\ast du \]
\[= D_{2,2,1} + D_{2,2,2}.
\]

- By Lemma 2.14, and the facts (4.5) and (4.6), we have
\[
D_{2,2,1} \lesssim \int |g_\ast| Z^{\varepsilon, \gamma}(T_{v_2} f) d\nu_\ast \lesssim |g|_{L^1_{|\gamma|+2}} |(|W^\varepsilon(D)W_{\gamma/2} f|^2_{L^2_{\gamma}} + |W^\varepsilon W_{\gamma/2} f|^2_{L^2_{\gamma}}).
\]

- Observe that $D_{2,2,2} = \tilde{N}^{\varepsilon, \gamma}_{\ast, \eta} (\sqrt{|g|}, f)$. By Lemma 2.17, we have
\[
\tilde{N}^{\varepsilon, \gamma}_{\ast, \eta} (\sqrt{|g|}, f) \lesssim N^{\varepsilon, \gamma}_{\ast, \eta} (W_{-\gamma/2} \sqrt{|g|}, W_{\gamma/2} f) + |g|_{L^1_{|\gamma|+2}} |f|^2_{L^2_{\gamma}}.
\]

By (2.57) in Lemma 2.15, we get
\[
N^{\varepsilon, \gamma}_{\ast, \eta} (W_{-\gamma/2} \sqrt{|g|}, W_{\gamma/2} f) \lesssim |W_{-\gamma/2} \sqrt{|g|}|^2_{L^1_{\varepsilon}} |f|^2_{L^2_{\varepsilon, \gamma/2}} \lesssim |g|_{L^1_{|\gamma|+2}} |f|^2_{L^2_{\varepsilon, \gamma/2}}.
\]

So we have $D_{2,2,2} \lesssim |g|_{L^1_{|\gamma|+2}} |f|^2_{L^2_{\varepsilon, \gamma/2}}$.

Patching together the estimates for $D_{2,2,1}$ and $D_{2,2,2}$, we get $D_{2,2} \lesssim |g|_{L^1_{|\gamma|+2}} |f|^2_{L^2_{\varepsilon, \gamma/2}}$, which yields $|D_2| \lesssim \eta^\varepsilon (D_{2,1}) \frac{1}{2} (D_{2,2}) \frac{1}{2} \lesssim \eta^\varepsilon |g|_{L^1_{|\gamma|+2}} |h|_{L^2_{\varepsilon, \gamma/2}} |f|_{L^2_{\varepsilon, \gamma/2}}$.

We complete the proof by patching together the estimates of $D_1$ and $D_2$. \(\square\)

### 4.1.2. Upper Bound of \((I^{\varepsilon, \gamma}_{\ast, \eta} (g, h), f)\) To implement the energy estimates for the nonlinear equations, in this subsection, we will give the upper bound of \((I^{\varepsilon, \gamma}_{\ast, \eta} (g, h; \beta), f)\) where
\[
I^{\varepsilon, \gamma}_{\ast, \eta} (g, h; \beta) := \int B^{\varepsilon, \gamma}_{\ast, \eta} (v - u_\ast, \sigma) ((\partial_\beta \mu^{\frac{1}{2}})_{\ast} - (\partial_\beta \mu^{\frac{1}{2}})'_{\ast}) g_\ast h' \, d\sigma \, d\nu_\ast. \quad (4.7)
\]

Let us deviate to explain why we consider the additional differential operator $\partial_\beta$. By binomial expansion, we have
\[
\partial_\beta^{\alpha} \Gamma^\varepsilon (g, h) = \sum_{\beta_0 + \beta_1 + \beta_2 = \beta, \alpha_1 + \alpha_2 = \alpha} C_{\beta_0, \beta_1, \beta_2}^\varepsilon C_\alpha^{\varepsilon} \Gamma^\varepsilon (\partial_\beta^{\alpha_1} g, \partial_\beta^{\alpha_2} h; \beta_0). \quad (4.8)
\]
where
\[ \Gamma^\varepsilon(g, h; \beta)(v) := \int_{\mathbb{S}^2 \times \mathbb{R}^3} B^\varepsilon(v - v_*, \sigma)(\partial_\beta \mu^{1/2})_s(g_*' h' - g_* h) d\sigma dv_. \]  
(4.9)

Note that
\[ \Gamma^{\varepsilon, \gamma, \eta}(g, h; \beta) = Q^{\varepsilon, \gamma, \eta}(g \partial_\beta \mu^{1/2}, h) + I^{\varepsilon, \gamma, \eta}(g, h; \beta). \]  
(4.10)

This explains why we consider the general version \( I^{\varepsilon, \gamma, \eta}(g, h; \beta) \).

By writing \( \partial_\beta \mu^{1/2} = \mu^{1/2} P_\beta \) where \( P_\beta \) is a polynomial, we observe that
\[
(\mu^{1/2} P_\beta)'_s - (\mu^{1/2} P_\beta)_s = ((\mu^{1/2})'_s - (\mu^{1/2})_s)((\mu^{1/2} P_\beta)'_s - (\mu^{1/2} P_\beta)_s) \\
\quad + (\mu^{1/2})_s((\mu^{1/2} P_\beta)'_s - (\mu^{1/2} P_\beta)_s) \\
\quad + (\mu^{1/2} P_\beta)_s((\mu^{1/2})'_s - (\mu^{1/2})_s). 
\]  
(4.11)

Then we have
\[
\langle I^{\varepsilon, \gamma, \eta}(g, h; \beta), f \rangle = \int B^{\varepsilon, \gamma, \eta}((\mu^{1/2})'_s + \mu^{1/2}_s) \\
\quad - ((\mu^{1/2})'_s - \mu^{1/2})((\mu^{1/2} P_\beta)'_s - (\mu^{1/2} P_\beta)_s)g\, h f' d\sigma dv_\sigma dv \\
\quad + \int B^{\varepsilon, \gamma, \eta}(((\mu^{1/2} P_\beta)'_s - (\mu^{1/2} P_\beta)_s)(\mu^{1/2} g)_s \\
\quad + ((\mu^{1/2})'_s - (\mu^{1/2})_s)(\mu^{1/2} P_\beta g)_s)h - h') f' d\sigma dv_\sigma dv \\
\quad + \int B^{\varepsilon, \gamma, \eta}(((\mu^{1/2} P_\beta)'_s - (\mu^{1/2} P_\beta)_s)(\mu^{1/2} g)_s + ((\mu^{1/2})'_s) \\
\quad - (\mu^{1/2})_s)(\mu^{1/2} P_\beta g)_s)h' f' d\sigma dv_\sigma dv \\
= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\]

**Proposition 4.2.** Let \( 0 < \eta \leq 1, 0 < \delta \leq 1/2, s_1, s_2 \geq 0 \) and \( s_1 + s_2 = 3/2 + \delta \). It holds that
\[
|\langle I^{\varepsilon, \gamma, \eta}(g, h; \beta), f \rangle| \lesssim \delta^{-1/2} |\mu^{1/2} g|_{H_1} |\mu^{1/2} h|_{H_2} |W^\varepsilon f|_{L^{2}_{\gamma/2}} \\
+ \eta^{\gamma/2} |g|_{L^2} |h|_{L^{2}_{\epsilon, \gamma/2}} |W^\varepsilon f|_{L^{2}_{\gamma/2}}.
\]

The \( \lesssim \) could bring a constant depending on \( |\beta| \).

**Proof.** Let us consider the \( \beta = 0 \) case since the following arguments also work when \( |\beta| > 0 \). There are three steps in the proof. We will indicate the main difference at the end of each step.

In the proof, we will frequently use the following fact:
\[
((\mu^{1/2})'_s - \mu^{1/2}_s)^2 \lesssim \min\{1, |v - v_*|^2 \theta^2\} \sim \min\{1, |v' - v_*|^2 \theta^2\} \\
\quad \sim \min\{1, |v - v_*'|^2 \theta^2\}. 
\]  
(4.12)
Step 1: Estimate of $I_1$. When $|\beta| = 0$, recall

$$I_1 = \int B^{(\gamma, \eta)}_{\epsilon, \frac{1}{2}, \frac{1}{2}} ((\mu_*)^\frac{1}{2} + \mu_*)^2 ((\mu_*)^\frac{1}{2} - \mu_*)^2 g_s h f' d\sigma dv_* dv.$$ 

Since $|v - v_*| \geq \eta$, we have

$$|v - v_*|^\gamma 1_{|v - v_*| \geq \eta} \lesssim \eta^\gamma |v - v_*|^\gamma.$$ \hspace{1cm} (4.13)

By (4.13) and Cauchy-Schwarz inequality, we have

$$|I_1| \lesssim \eta^\gamma \left( \int b^\epsilon (\cos \theta) \langle v - v_* \rangle^\gamma ((\mu_*)^\frac{1}{2} + \mu_*)^2 ((\mu_*)^\frac{1}{2} - \mu_*)^2 g_s^2 h^2 d\sigma dv_* dv) \right)^{\frac{1}{2}} \times \left( \int b^\epsilon (\cos \theta) \langle v - v_* \rangle^\gamma ((\mu_*)^\frac{1}{2} + \mu_*)^2 ((\mu_*)^\frac{1}{2} - \mu_*)^2 (f^2)^\gamma d\sigma dv_* dv) \right)^{\frac{1}{2}}$$

$$=: \eta^\gamma (I_{1,1})^\frac{1}{2} (I_{1,2})^\frac{1}{2}.$$

Estimate of $I_{1,1}$. We claim that

$$\mathcal{A} := \int b^\epsilon (\cos \theta) \langle v - v_* \rangle^\gamma ((\mu_*)^\frac{1}{2} + \mu_*)^2 ((\mu_*)^\frac{1}{2} - \mu_*)^2 d\sigma \lesssim (W^\epsilon)^2 (v) \langle v \rangle^\gamma,$$ \hspace{1cm} (4.14)

which immediately gives $I_{1,1} \lesssim |g|_{L^2}^2 |W^\epsilon h|_{L^2}^2$. It remains to prove (4.14). Since

$$(\mu_*)^\frac{1}{2} + \mu_* \lesssim (\mu_*)^\frac{1}{2} + \frac{1}{2},$$

we have $\mathcal{A} \lesssim \mathcal{A}_1 + \mathcal{A}_2$, where

$$\mathcal{A}_1 := \int b^\epsilon (\cos \theta) \langle v - v_* \rangle^\gamma \mu_*^\frac{1}{2} ((\mu_*)^\frac{1}{2} - \mu_*)^2 d\sigma,$$

$$\mathcal{A}_2 := \int b^\epsilon (\cos \theta) \langle v - v_* \rangle^\gamma (\mu_*)^\frac{1}{2} ((\mu_*)^\frac{1}{2} - \mu_*)^2 d\sigma.$$

Thanks to (4.12), Lemma 2.5 and property (2.8), one has

$$\mathcal{A}_1 \lesssim \langle v - v_* \rangle^\gamma \mu_*^\frac{1}{2} (W^\epsilon)^2 (v - v_*) \lesssim (W^\epsilon)^2 (v) \langle v \rangle^\gamma.$$ \hspace{1cm} (4.15)

As for $\mathcal{A}_2$, thanks to $|v - v_*| \sim |v' - v_*|$ and thus $\langle v - v_* \rangle^\gamma \lesssim \langle v' - v_* \rangle^\gamma \lesssim \langle v \rangle^\gamma \langle v' \rangle^\gamma$, we have

$$\mathcal{A}_2 \lesssim \langle v \rangle^\gamma \int b^\epsilon (\cos \theta) (\mu_*)^\frac{1}{2} \min \{1, |v - v_*|^2 \} d\sigma.$$

- If $|v - v_*| \geq 10|v|$, then there holds $|v' - v + v| \geq |v' - v| - |v| \geq (1/\sqrt{2} - 1/10)|v - v_*| \geq \frac{1}{5}|v - v_*|$, and thus $(\mu_*)^\frac{1}{2} \lesssim \mu_\frac{1}{2} (v - v_*)$, which yields

$$\mathcal{A}_2 \lesssim \langle v \rangle^\gamma \mu_\frac{1}{2} (v - v_*) (W^\epsilon)^2 (v - v_*) \lesssim \langle v \rangle^\gamma.$$
By the change of variable 

$$A_2 \lesssim \langle v \rangle^\gamma \int b^\varepsilon (\cos \theta) \min \{1, |v|^2 \theta^2 \} d\sigma \lesssim (W^\varepsilon)^2(v) \langle v \rangle^\gamma.$$  

Patching together the estimates of $A_1$ and $A_2$, we get the claim (4.14).

**Estimate of $I_{1,2}$.** By the change of variable $(v, v_{*}, \sigma) \rightarrow (v^*, v_{*}', \tau = (v - v_{*})/|v - v_{*}|)$, we have

$$I_{1,2} = \int b^\varepsilon (\cos \theta) \langle v - v_{*} \rangle^\gamma ((\mu_{\varepsilon})_{*}^{1/2} + \mu_{\varepsilon}^{1/2})^2 ((\mu_{\varepsilon})_{*}^{1/2} - \mu_{\varepsilon}^{1/2})^2 f^2 d\sigma d v_{*} dv$$

$$\lesssim 2 \int b^\varepsilon (\cos \theta) \langle v - v_{*} \rangle^\gamma \mu_{\varepsilon}^{1/2} (\mu_{\varepsilon})_{*}^{1/2} - \mu_{\varepsilon}^{1/2})^2 f^2 d\sigma d v_{*} dv$$

$$+ 2 \int b^\varepsilon (\cos \theta) \langle v - v_{*} \rangle^\gamma (\mu_{\varepsilon})_{*}^{1/2} ((\mu_{\varepsilon})_{*}^{1/2} - \mu_{\varepsilon}^{1/2})^2 f^2 d\sigma d v_{*} dv$$

$$:= I_{1,2,1} + I_{1,2,2}.$$  

By (4.12), Lemma 2.5 and property (2.8), we have

$$I_{1,2,1} \lesssim \int b^\varepsilon (\cos \theta) \langle v - v_{*} \rangle^\gamma (\mu_{\varepsilon})_{*}^{1/2} \min \{1, |v - v_{*}|^2 \theta^2 \} f^2 d\sigma d v_{*} dv$$

$$\lesssim \int \langle v - v_{*} \rangle^\gamma \mu_{\varepsilon}^{1/2} (W^\varepsilon)^2(v) f^2 d v_{*} dv \lesssim |W^\varepsilon f|_{L_{4/2}^2}^2.$$  

By the fact $|v - v_{*}| \sim |v - v_{*}'|$, the change of variable $v_{*} \rightarrow v_{*}'$, we have

$$I_{1,2,2} \lesssim \int b^\varepsilon (\cos \theta) \langle v - v_{*}' \rangle^\gamma (\mu_{\varepsilon})_{*}^{1/2} \min \{1, |v - v_{*}'|^2 \theta^2 \} f^2 d\sigma d v_{*}' dv \lesssim |W^\varepsilon f|_{L_{4/2}^2}^2.$$  

Therefore we have $I_{1,2} \lesssim |W^\varepsilon f|_{L_{4/2}^2}^2$. Patching together the estimates of $I_{1,1}$ and $I_{1,2}$, we have

$$I_1 \lesssim \eta^\gamma |g|_{L^2} |W^\varepsilon h|_{L_{4/2}^2} |W^\varepsilon f|_{L_{4/2}^2}.$$  

Since (4.12) also holds for $(\mu_{\varepsilon}^{1/2} P_{\beta})_{*}' - (\mu_{\varepsilon}^{1/2} P_{\beta})_{*}$, the above estimates in this step are also valid for the $|\beta| > 0$ case.

**Step 2: Estimate of $I_2$.** When $|\beta| = 0$, by Cauchy-Schwartz inequality, we have

$$I_2 = 2 \int B_{\varepsilon, \gamma, \eta} ((\mu_{\varepsilon})_{*}^{1/2} - \mu_{\varepsilon}^{1/2}) (\mu_{\varepsilon})_{*}^{1/2} g_{*}(h - h') f' d\sigma dv_{*} dv$$

$$\lesssim ( \int B_{\varepsilon, \gamma, \eta} (\mu_{\varepsilon})_{*}^{1/2} g_{*}(h - h')^2 d\sigma dv_{*} dv)^{1/2}$$

$$\times ( \int B_{\varepsilon, \gamma, \eta} ((\mu_{\varepsilon})_{*}^{1/2} - \mu_{\varepsilon}^{1/2})^2 (\mu_{\varepsilon})_{*}^{1/2} f^2 d\sigma dv_{*} dv)^{1/2}$$

$$:= (I_{2,1})^{1/2} (I_{2,2})^{1/2}. \tag{4.16}$$
Estimate of $I_{2.1}$. Since $(h - h')^2 = (h^2)' - h^2 - 2h(h' - h)$, we have

$$I_{2.1} = \int B^{e, \gamma, \eta}(\mu^{1/2} g)_*((h^2)' - h^2) d\sigma dv_\sigma dv = 2(Q^e(\mu^{1/2} g, h), h)$$

For $\gamma = -3$, we apply (2.17) with $a = 1/6, 1/p + 1/q = 1$ to get

$$|\int B^{e, \gamma, \eta}(\mu^{1/2} g)_*((h^2)' - h^2) d\sigma dv_\sigma dv| \lesssim |\mu^{1/2} g|_{L^p} |\mu^{1/2} h|_{L^q}^2 \lesssim \delta^{-1/2} |\mu^{1/2} g|_{H^{1/2}} |\mu^{1/2} h|_{H^{3/2}},$$

where $s_1 + s_2 = 3/2 + \delta$ and we use the Sobolev imbedding $|\cdot|_{L^p} \lesssim \delta^{-\frac{1}{2}} |\cdot|_{H^{\frac{3}{2} + \delta}}$ with $\delta > 0$ and $|\cdot|_{L^p} \lesssim |\cdot|_{H^s}$ with $s/3 = 1/2 - 1/p$. For For $\gamma > -3$, by the cancellation Lemma 1 of [1] and (2.13), similar to (2.73), we have

$$|\int B^{e, \gamma, \eta}(\mu^{1/2} g)_*((h^2)' - h^2) d\sigma dv_\sigma dv| \lesssim \eta^3 |\mu^{1/2} g|_{L^2} |h|_{L^2_{2, \gamma/2}}^2.$$ 

By Proposition 4.1, we have

$$|(Q^{e, \gamma, \eta}(\mu^{1/2} g, h), h)| \lesssim \eta^{-3} |\mu^{1/2} g|_{L^2} |h|_{L^2_{2, \gamma/2}}^2 \lesssim \eta^{-3} |\mu^{1/2} g|_{L^2} |h|_{L^2_{2, \gamma/2}}^2.$$ 

Patching together the previous results, we get

$$|I_{2.1}| \lesssim \delta^{-1/2} |\mu^{1/2} g|_{H^{1/2}} |\mu^{1/2} h|_{H^{3/2} \gamma/2} + \eta^{-3} |\mu^{1/2} g|_{L^2} |h|_{L^2_{2, \gamma/2}}^2.$$ 

(4.17)

Estimate of $I_{2.2}$. By the change of variable $v \rightarrow \nu'$, and the estimate (4.15) of $A_1$, we have $I_{2.2} \lesssim \eta^3 |\mu^{1/2} g|_{L^2} |W^e f|_{L^2_{2, \gamma/2}}^2$.

Patching the estimates for $I_{2.1}$ and $I_{2.2}$, we get

$$|I_{2}| \lesssim \delta^{-1/2} |\mu^{1/2} g|_{H^{1/2}} |\mu^{1/2} h|_{H^{3/2}} |W^e f|_{L^2_{2, \gamma/2}}^2 + \eta^{-3} |\mu^{1/2} g|_{L^2} |h|_{L^2_{2, \gamma/2}} |W^e f|_{L^2_{2, \gamma/2}}^2.$$ 

We remark that the $|\beta| > 0$ case can be dealt with in a similar way and there is no essential difference.

Step 3: Estimate of $I_3$. By the change of variables $(v, v_\sigma) \rightarrow (v', v'_\sigma)$ and $(v, v_\sigma, \sigma) \rightarrow (v_\sigma, v, -\sigma)$,

$$I_3 = \int b^*(\cos \theta)|v - v_\sigma|^\gamma (\mu^{1/2} - (\mu^{1/2}))'(\mu^{1/2} g)' h_\sigma f_\sigma d\sigma dv_\sigma dv.$$ 

We separate the integration domain into three parts, $S^2 \times \mathbb{R}^3 \times \mathbb{R}^3 = E_1 \cup E_2 \cup E_3$, where $E_1 = \{(\sigma, v, \sigma_1) : |v - v_\sigma| \geq 1/\varepsilon\}, E_2 = \{(\sigma, v, \sigma) : |v - v_\sigma| \leq 1/\varepsilon, |v - v_\sigma| \leq \sin(\theta/2) \leq \sqrt{2}/2\}, E_3 = \{(\sigma, v, \sigma) : |v - v_\sigma| \leq 1/\varepsilon, \varepsilon \leq |\sin(\theta/2)| \leq |v - v_\sigma| \leq |v - v_\sigma|\}$. Then $I_3 = I_{3.1} + I_{3.2} + I_{3.3}$ where $I_{3.3} = \int_{E_3} (\cdots) d\sigma dv_\sigma dv$.

Estimate of $I_{3.1}$. By the change of variable $v \rightarrow v'$, the fact $|v' - v_\sigma| \geq |v_\sigma|/\sqrt{2}$ and (2.11), we have

$$|I_{3.1}| \lesssim \int b^*(\cos \theta)|v' - v_\sigma|^\gamma |v' - v_\sigma| (\mu^{1/2} g)' h_\sigma f_\sigma d\sigma dv_\sigma dv.'$$
On one hand, by Cauchy-Schwartz inequality, we have

\[
\ln |\varepsilon|^{-1} e^{-2} \int |v' - v_\ast|^{\gamma} 1_{|v' - v_\ast| \geq (\sqrt{2} \varepsilon)^{-1}} \left( (\mu^{\frac{1}{2}} g)^{\prime} h \ast f \ast |dv_\ast dv'.
\]

On the other hand, if \( |v' - v_\ast|^{2} \geq (\sqrt{2} \varepsilon)^{-1} \), we have

\[
\ln |\varepsilon|^{-1} e^{-2} \int |v' - v_\ast|^{\gamma} 1_{|v' - v_\ast| \geq (\sqrt{2} \varepsilon)^{-1}} \left( (\mu^{\frac{1}{2}} g)^{\prime} |dv'.
\]

With estimates (4.18) and (4.19) in hand, by (2.6) and (2.7),

\[
\min \{ \ln |\varepsilon|^{-1} e^{-2}, |\varepsilon|^{-1} v_\ast \} \lesssim (W^\varepsilon)^2 (v_\ast),
\]

which gives \( |T_{3,1}| \lesssim |\mu^{\frac{1}{2}} g|_{L^2} |W^\varepsilon h|_{L^2_{\gamma/2}} |W^\varepsilon f|_{L^2_{\gamma/2}}. \)

Estimate of \( T_{3,2} \). By the change of variable \( v \to v' \) and the fact \( |v' - v_\ast| \leq |v - v_\ast| \), we get

\[
T_{3,2} \lesssim \int b^{\varepsilon} (\cos \theta)_{1} \sin(\theta/2)^{\varepsilon} (\sqrt{2} b_{v' - v_\ast})^{-1} |v' - v_\ast|^{\gamma} 1_{|v' - v_\ast| \leq 1/\varepsilon} \left( (\mu^{\frac{1}{2}} g)^{\prime} h \ast f \ast |dv_\ast dv'.
\]

On one hand, similar to the argument in (4.19), we have

\[
\ln |\varepsilon|^{-1} \int |v' - v_\ast|^{\gamma+2} 1_{|v' - v_\ast| \leq 1/\varepsilon} \left( (\mu^{\frac{1}{2}} g)^{\prime} |dv'.
\]

On the other hand, if \( |v_\ast| \geq 2/\varepsilon \), then \( |v'| \geq |v_\ast| - |v' - v_\ast| \geq |v_\ast|/2 \geq 1/\varepsilon \), which gives \( \mu' \lesssim \mu^{\frac{1}{2}} \lesssim e^{-\frac{1}{10}} \). Then we deduce that

\[
\ln |\varepsilon|^{-1} 1_{|v_\ast| \geq 2/\varepsilon} \int |v' - v_\ast|^{\gamma+2} 1_{|v' - v_\ast| \leq 1/\varepsilon} \left( (\mu^{\frac{1}{2}} g)^{\prime} |dv'.
\]

\[
\lesssim \ln |\varepsilon|^{-1} 1_{|v_\ast| \geq 2/\varepsilon} |\mu^{\frac{1}{2}} g|_{L^2} \left( \int |v' - v_\ast|^{2\gamma+4} 1_{|v' - v_\ast| \leq 1/\varepsilon} (\mu^{\frac{1}{2}} g^{\prime})^{\prime} |dv'.
\]
\[ \lesssim |\ln \varepsilon|^{-1} |v_\ast|^{2/\varepsilon} |\mu^{\frac{1}{2}} g| L^2 \mu_*^{\frac{1}{44}} (\varepsilon^{-1})^{\nu+2+3/2} e^{-\frac{1}{32\pi^2}} \]
\[ \lesssim |\ln \varepsilon|^{-1} |v_\ast|^{2/\varepsilon} |\mu^{\frac{1}{2}} g| L^2 \mu_*^{\frac{1}{44}}. \]  

(4.22)

With estimates (4.21) and (4.22) in hand, recalling (4.20), we have \( |\mathcal{I}_{3,2}| \lesssim |\mu^{\frac{3}{2}} g| L^2 |W^g h| L^2_{\gamma/2} |W^g f| L^2_{\gamma/2} \).

Estimate of \( \mathcal{I}_{3,3} \). By Taylor expansion, one has
\[ \mu^{\frac{1}{2}} - (\mu^{\frac{1}{2}})' = (\nabla \mu^{\frac{1}{2}})(v') \cdot (v - v') + \int_0^1 (1 - \kappa) \left( (\nabla^2 \mu^{\frac{1}{2}})(v(\kappa)) : (v - v') \otimes (v - v') \right) d\kappa, \]
where \( v(\kappa) = v' + \kappa(v - v') \). For any fixed \( v_\ast \), there holds
\[ \int b^g(\cos \theta)|v - v_\ast|^\gamma 1_{|v - v_\ast| \leq 1/\varepsilon, \varepsilon \leq \sin(\theta/2) \leq |v - v_\ast|^{-1}} (\nabla \mu^{\frac{1}{2}})(v') \cdot (v - v')(\mu^{\frac{1}{2}} g)'\, d\sigma\, dv = 0. \]

By the change of variable \( v \to v' \), the fact \( |v' - v_\ast| \geq |v - v_\ast|/\sqrt{2} \) and \( |\nabla^2 \mu^{\frac{1}{2}}| L^\infty \lesssim 1 \), we have
\[ |\mathcal{I}_{3,3}| = |\int b^g(\cos \theta)|v - v_\ast|^\gamma 1_{|v - v_\ast| \leq 1/\varepsilon, \varepsilon \leq \sin(\theta/2) \leq |v - v_\ast|^{-1}} \]
\[ \times (1 - \kappa) \left( (\nabla^2 \mu^{\frac{1}{2}})(v(\kappa)) : (v - v') \otimes (v - v') \right)(\mu^{\frac{1}{2}} g)' h_\ast f_\ast \, d\kappa \, d\sigma \, dv_\ast \, dv \]
\[ \lesssim |\int b^g(\cos(\theta))|v' - v_\ast|^\gamma+2\theta^2 1_{|v' - v_\ast| \leq 1/\varepsilon, \varepsilon \leq \sin(\theta/2) \leq |v' - v_\ast|^{-1}} \]
\[ |(\mu^{\frac{1}{2}} g)' h_\ast f_\ast| \, d\sigma \, dv_\ast \, dv' \]
\[ \lesssim |\ln \varepsilon|^{-1} \int (|\ln \varepsilon| - \ln |v' - v_\ast|) |v' - v_\ast|^\gamma+2 1_{|v' - v_\ast| \leq 1/\varepsilon} \]
\[ |(\mu^{\frac{1}{2}} g)' h_\ast f_\ast| \, dv_\ast \, dv'. \]

We claim that
\[ |\ln \varepsilon|^{-1} \int (|\ln \varepsilon| - \ln |v - v_\ast|) |v - v_\ast|^\gamma+2 1_{|v - v_\ast| \leq 1/\varepsilon} |\mu^{\frac{1}{2}} g| \, dv \]
\[ \lesssim (W^g_\ast)^2(v_\ast)^\gamma |\mu^{\frac{3}{2}} g| L^2, \]

which immediately gives \( |\mathcal{I}_{3,3}| \lesssim |\mu^{\frac{3}{2}} g| L^2 |W^g h| L^2_{\gamma/2} |W^g f| L^2_{\gamma/2} \). By Cauchy-Schwartz inequality, it suffices to prove
\[ K(v_\ast) := |\ln \varepsilon|^{-1} \left( \int (|\ln \varepsilon| - \ln |v - v_\ast|)^2 |v - v_\ast|^{2\gamma+4} 1_{|v - v_\ast| \leq 1/\varepsilon} |\mu^{\frac{1}{2}} g| \, dv \right)^{\frac{1}{2}} \]
\[ \lesssim (W^g_\ast)^2(v_\ast)^\gamma. \]  

(4.23)
Case 1: $|v_*| \geq 2/\varepsilon$. Since $|v - v_*| \leq 1/\varepsilon$, we have $|v - v_*| \leq |v_*|/2$ and thus $|v| \geq |v_*| - |v - v_*| \geq |v_*|/2$. Then we get $\mu \lesssim \mu_*^{1/4}$. On the other hand, since $|v|^2 \geq \frac{1}{2} |v - v_*|^2 - |v_*|^2$, we have

$$
\mu \lesssim \mu_*^{1/2} (v - v_*) \mu_*^{-1}, \quad (4.24)
$$

which gives $\mu_*^{1/4} 1_{|v_*| \geq 2/\varepsilon, |v - v_*| \leq |v_*|/2} \lesssim \mu_*^{1/128} (v - v_*) \mu_*^{1/4} 1_{|v_*| \geq 2/\varepsilon, |v - v_*| \leq |v_*|/2}$. Plugging which into $K(v_*)$, we get

$$
K(v_*) \lesssim \mu_*^{1/128} |\ln \varepsilon|^{-1} \left( \int (|\ln \varepsilon| - |\ln |u||)^2 |u|^{2\gamma + 4} 1_{|u| \leq 1} \mu_*^{1/128} (v - v_*) du \right)^{1/2} \\
= \mu_*^{1/128} |\ln \varepsilon|^{-1} \left( \int (|\ln \varepsilon| - |\ln |u||)^2 |u|^{2\gamma + 4} 1_{|u| \leq 1} \mu_*^{1/128} (v - v_*) du \right)^{1/2}.
$$

We separate the integration domain into two regions: $|u| \leq 1$ and $|u| \geq 1$.

- For the part $|u| \leq 1$, we have

$$
\int (|\ln \varepsilon| - |\ln |u||)^2 |u|^{2\gamma + 4} 1_{|u| \leq 1} \mu_*^{1/128} (v - v_*) du \leq 2 \int (|\ln \varepsilon|^2 + (\ln |u|^2) |u|^{2\gamma + 4} 1_{|u| \leq 1} du \leq 2(C_1(\gamma) |\ln \varepsilon|^2 + C_2(\gamma)) \lesssim |\ln \varepsilon|^2.
$$

where $C_1(\gamma) := \int |u|^{2\gamma + 4} 1_{|u| \leq 1} du$ and $\int (\ln |u|^2) |u|^{2\gamma + 4} 1_{|u| \leq 1} du \leq C \int |u|^{2\gamma + 7/2} 1_{|u| \leq 1} du := C_2(\gamma)$.

- For the part $|u| \geq 1$, since $0 \leq \ln |u| \leq |\ln \varepsilon|$ and $\gamma < 0$ we have

$$
\int (|\ln \varepsilon| - |\ln |u||)^2 |u|^{2\gamma + 4} 1_{|u| \geq 1} \mu_*^{1/128} (v - v_*) du \leq |\ln \varepsilon|^2 \int |u|^{4} 1_{|u| \geq 1} \mu_*^{1/128} (v - v_*) du \lesssim |\ln \varepsilon|^2.
$$

By these two estimates, we get $1_{|v_*| \geq 2/\varepsilon} K(v_*) \lesssim \mu_*^{1/128} \lesssim (W^\varepsilon)_*(v_*)^\gamma$.

Case 2: $1 \leq |v_*| \leq 2/\varepsilon$. We separate the integration domain into two regions: $|v - v_*| \leq |v_*|/2 \leq 1/\varepsilon$ and $|v - v_*| \geq |v_*|/2$. Using $\sqrt{A + B} \leq \sqrt{A} + \sqrt{B}$, we get

$$
K(v_*) \leq |\ln \varepsilon|^{-1} \left( \int (|\ln \varepsilon| - |\ln |v - v_*||)^2 |v - v_*|^{2\gamma + 4} 1_{|v - v_*| \leq |v_*|/2} \mu_*^{1/2} dv \right)^{1/2} \\
+ |\ln \varepsilon|^{-1} \left( \int (|\ln \varepsilon| - |\ln |v - v_*||)^2 |v - v_*|^{2\gamma + 4} 1_{|v_*|/2 \leq |v - v_*| \leq |v_*|/2} \mu_*^{1/4} dv \right)^{1/2} \\
:= K_1(v_*) + K_2(v_*).
$$

When $|v - v_*| \leq |v_*|/2$, we can follow the computation in Case 1 to get $K_1(v_*) \lesssim \mu_*^{1/128}$. When $|v - v_*| \geq |v_*|/2$, then $|\ln \varepsilon| - |v - v_*| \leq |\ln \varepsilon| - \ln |v_*| + \ln 2$, we get

$$
K_2(v_*) \leq |\ln \varepsilon|^{-1} (|\ln \varepsilon| - \ln |v_*| + \ln 2) \left( \int |v - v_*|^{2\gamma + 4} 1_{|v_*|/2 \leq |v - v_*| \leq |v_*|/2} \mu_*^{1/4} dv \right)^{1/2}
$$
By \( \mu_{\ast}^{\frac{1}{2}} \approx \langle v_{\ast} \rangle^{\gamma} \) and \( 1 \leq |v_{\ast}| \leq 2/\varepsilon \), we have
\[
\begin{aligned}
\ln|\varepsilon|^{-1} \left( |\ln|\varepsilon| - \ln|v_{\ast}| + \ln 2 \right) \langle v_{\ast} \rangle^{\gamma + 2}. \\
\end{aligned}
\]

By the computation in Case 1, we get
\[
K(v_{\ast}) \approx \langle W^\varepsilon \rangle_{v_{\ast}}^{2}(v_{\ast})^\gamma.
\]

Case 3: \( |v_{\ast}| \leq \frac{1}{\varepsilon} \). By (4.24), we have \( \mu \approx \mu_{\ast}^{\frac{1}{2}}(v - v_{\ast}) \). Plugging which into \( K(v_{\ast}) \), we have
\[
\begin{aligned}
1_{|v_{\ast}| \leq \frac{1}{\varepsilon}} K(v_{\ast}) \\
\ll |\ln|\varepsilon|^{-1} \left( |\ln|\varepsilon| - \ln|v - v_{\ast}| + \ln 2 \right) (v - v_{\ast})^{2} 1_{|v - v_{\ast}| \leq 1/\varepsilon} K(v - v_{\ast})dv \frac{1}{2} \\
= |\ln|\varepsilon|^{-1} \left( |\ln|\varepsilon| - \ln|\mu| + \ln 2 \right) |\mu|^{2} 1_{|\mu| \leq 1/\varepsilon} K(v - v_{\ast})dv \frac{1}{2}.
\end{aligned}
\]

By the computation in Case 1, we get
\[
1_{|v_{\ast}| \leq \frac{1}{\varepsilon}} K(v_{\ast}) \ll 1_{|v_{\ast}| \leq \frac{1}{\varepsilon}} (W^\varepsilon)_{v_{\ast}}^{2}(v_{\ast})^{\gamma}.
\]

By the above upper bounds of \( I_{3,1}, I_{3,2} \) and \( I_{3,3} \), we have
\[
|I_{3}| \approx |\mu_{\ast}^{\frac{1}{2}} g_{L}^{2} |W^\varepsilon h|_{L_{r/2}^{2}}^{2} |W^\varepsilon f|_{L_{r/2}^{2}}^{2}.
\]

In the \( |\beta| \geq 1 \) case, \( I_{3} \) contains two parts. The first part involving \((\mu_{\ast}^{\frac{1}{2}} P_{\beta}^{\ast})(\mu_{\ast}^{\frac{1}{2}} g)_{\ast} \), by the change of variables \((v, v_{\ast}) \rightarrow (v', v_{\ast}')\) and \((v, v_{\ast}, \sigma) \rightarrow (v_{\ast}, v, -\sigma)\), gives
\[
I_{3} = \int b_{\varepsilon}(\cos \theta)|v - v_{\ast}|^{\gamma'}(\mu_{\ast}^{\frac{1}{2}} P_{\beta}^{\ast} - (\mu_{\ast}^{\frac{1}{2}} P_{\beta})')(\mu_{\ast}^{\frac{1}{2}} g_{\ast})h_{\ast} f_{\ast} \mathrm{d}\sigma dv_{\ast} dv.
\]

With the same decomposition as above according to \( E_{1}, E_{2}, E_{3} \), we have \( I_{3} = I_{3,1} + I_{3,2} + I_{3,3} \). In \( I_{3,1}, I_{3,2} \), we can use \( |\mu_{\ast}^{\frac{1}{2}} P_{\beta}| \ll 1 \). In \( I_{3,3} \), we can use \( |\nabla^{2} \mu_{\ast}^{\frac{1}{2}} P_{\beta}| \ll 1 \). The second part involving \( P_{\beta} \mu_{\ast}^{\frac{1}{2}} g \) can be dealt with in the same way as the above for the case \( |\beta| = 0 \).

We end the proof by patching together the above estimates of \( I_{1}, I_{2} \) and \( I_{3} \).

\[\square\]

4.1.3. Upper Bound of \( \Gamma^{\varepsilon, \gamma, \eta}(g, h) \) We are ready to give the upper bound of \( \langle \Gamma^{\varepsilon, \gamma, \eta}(g, h; \beta, f) \rangle \).

Theorem 4.1. Let \( 0 < \eta \leq 1, 0 < \delta \leq 1/2, s_{1}, s_{2} \geq 0 \) and \( s_{1} + s_{2} = \frac{3}{2} + \delta \). It holds that
\[
|\langle \Gamma^{\varepsilon, \gamma, \eta}(g, h; \beta, f) \rangle| \lesssim \delta \frac{1}{2} |\mu_{\ast}^{1/2} g|_{H^{s_{1}}} |\mu_{\ast}^{1/2} h|_{H^{s_{2}}} |W^{\varepsilon} f|_{L^{2}_{r/2}}^{2} + \eta^{\gamma - 3} |g|_{L^{2}} |h|_{L^{2}_{r, \gamma/2}}^{2} |f|_{L^{2}_{r, \gamma/2}}^{2}.
\]

Proof. Recalling from (4.10), we have
\[
\langle \Gamma^{\varepsilon, \gamma, \eta}(g, h; \beta, f) \rangle = \langle Q^{\varepsilon, \gamma, \eta}(P_{\beta} \mu_{\ast}^{1/2} g, h, f) \rangle + \langle I^{\varepsilon, \gamma, \eta}(g, h; \beta, f) \rangle.
\]

The theorem follows directly from Proposition 4.1 and Proposition 4.2. \[\square\]
4.2. Upper Bound of \( Q_{\eta}^{\varepsilon, \gamma}(g, h), f) \) and \( f_{\eta}^{\varepsilon, \gamma}(g, h), f) \)

We will provide two estimates for each functional. One allows us to make use of the smallness of \( \eta \) later, and the other is independent of \( \eta \).

### 4.2.1. Upper Bound of \( Q_{\eta}^{\varepsilon, \gamma} \)

We give the upper bound of \( Q_{\eta}^{\varepsilon, \gamma} \) in the following proposition.

**Proposition 4.3.** Let \( \delta \in (0, 1/2], \eta \in (0, 1], a \in [0, 1] \) and \((s_3, s_4) = \left( \frac{1}{2} + \frac{\delta}{2}, 0 \right) \) or \((0, \frac{1}{2} + \delta) \). Then for any suitable functions \( g, h \) and \( f \), the following estimates are valid.

\[
|\langle Q_{\eta}^{\varepsilon, \gamma}(g, h), f \rangle| \lesssim \delta^{-\frac{1}{2}}(\eta^{\delta} + \varepsilon^{\delta})|\mu^{-2a} g|^2_{H^{\frac{3}{2} + \delta}} |\mu^{\frac{a}{2}} h|^2_{H^{1}} |W^\varepsilon(D)\mu^{\frac{a}{2}} f|_{L^2}. \quad (4.25)
\]

\[
|\langle Q_{\eta}^{\varepsilon, \gamma}(g, h), f \rangle| \lesssim \delta^{-\frac{1}{2}}|\mu^{-2a} g|^2_{H^{1}} |\mu^{\frac{a}{2}} h|^2_{H^{1} + s_4} |W^\varepsilon(D)\mu^{\frac{a}{2}} f|_{L^2}. \quad (4.26)
\]

**Proof.** Set \( G = \mu^{-2a} g, H = \mu^\frac{a}{2} h, F = \mu^\frac{a}{2} f \). By the definition of \( Q_{\eta}^{\varepsilon, \gamma} \), we have

\[
\langle Q_{\eta}^{\varepsilon, \gamma}(g, h), f \rangle = \int B_{\eta}^{\varepsilon, \gamma}(\mu^{2a} G_s)\mu^{-\frac{a}{2}} H((\mu^{-\frac{a}{2}} F) - \mu^\frac{a}{2} F) d\sigma dv_d v^*_s.
\]

Recalling (1.52), by the decomposition \( F = \mathcal{F}_\phi F + \mathcal{F}_\phi F \) and \( H = \mathcal{F}_\phi H + \mathcal{F}_\phi H \), we have

\[
\mathcal{Y}(G, H, F) = \mathcal{Y}(G, \mathcal{F}_\phi H, \mathcal{F}_\phi F) + \mathcal{Y}(G, \mathcal{F}_\phi H, \mathcal{F}_\phi F) + \mathcal{Y}(G, H, \mathcal{F}_\phi F).
\]

**Step 1:** \( \mathcal{Y}(G, H, \mathcal{F}_\phi F) \). In order to transfer regularity from \( \mu^{-\frac{a}{2}} \mathcal{F}_\phi F \) to \( \mu^{-\frac{a}{2}} H \), we rearrange

\[
\mathcal{Y}(G, H, \mathcal{F}_\phi F) = \int B_{\eta}^{\varepsilon, \gamma}(\mu^{2a} G_s)\mu^{-\frac{a}{2}} H((\mu^{-\frac{a}{2}} \mathcal{F}_\phi F) - \mu^{-\frac{a}{2}} \mathcal{F}_\phi F) d\sigma dv_d v^*_s
\]

\[= \int B_{\eta}^{\varepsilon, \gamma}(\mu^{2a} G_s)(\mu^{-a} H \mathcal{F}_\phi F) - \mu^{-a} H \mathcal{F}_\phi F) d\sigma dv_d v^*_s
\]

\[+ \int B_{\eta}^{\varepsilon, \gamma}(\mu^{2a} G_s)(\mu^{-\frac{a}{2}} H - (\mu^{-\frac{a}{2}} H)')(\mu^{-\frac{a}{2}} \mathcal{F}_\phi F)' d\sigma dv_d v^*_s
\]

\[:= \mathcal{Y}_1(G, H, \mathcal{F}_\phi F) + \mathcal{Y}_2(G, H, \mathcal{F}_\phi F).
\]

Estimate of \( \mathcal{Y}_1(G, H, \mathcal{F}_\phi F) \). We will give two results on it. For the case \( \gamma = -3 \), we have

(1) Using (2.19), we have

\[
|\mathcal{Y}_1(G, H, \mathcal{F}_\phi F)| \lesssim (\eta + \varepsilon^{\frac{1}{2}})|G|_{L^\infty}(|W^\varepsilon(D)H|_{L^2} |\mathcal{F}_\phi F|_{L^2}
\]

\[+ |H|_{L^2} |W^\varepsilon(D)\mathcal{F}_\phi F|_{L^2})
\]

\[\lesssim \delta^{-\frac{1}{2}}(\eta + \varepsilon^{\frac{1}{2}})|G|_{H^{\frac{3}{2} + \delta}} |W^\varepsilon(D)H|_{L^2} |W^\varepsilon(D)\mathcal{F}_\phi F|_{L^2}.
\]

(4.27)
(2) Using (2.24), we have
\[ |Y_1(G, H, \Theta F)| \lesssim |G|_{L^p} |H| \Theta F |_{L^q} \lesssim |G|_{L^p} |H| |\Theta F|_{L^2}, \]
where \( \frac{1}{p} + \frac{1}{r} = \frac{1}{2} \). Taking \( p, r = 2, \infty \) or \( p, r = 3, 6 \), by Sobolev imbedding one has
\[ |G|_{L^2} |H|_{L^\infty} \lesssim \delta^{-\frac{1}{2}} |G|_{H^0} |H|_{H^{\frac{3}{2}}} \]
or
\[ |G|_{L^3} |H|_{L^6} \lesssim |G|_{H^1} |H|^1. \]
Therefore we have for \((s_3, s_4) = (\frac{1}{2} + \delta, 0)\) or \((s_3, s_4) = (0, \frac{1}{2} + \delta)\),
\[ |Y_1(G, H, \Theta F)| \lesssim \delta^{-\frac{1}{2}} |G|_{H^{s_3}} |H|_{H^{1+s_4}} |\Theta F|_{L^2}. \tag{4.28} \]
For the case \( \gamma > -3 \), we can use the cancellation Lemma 1 of [1] to get (4.27) and (4.28). As \( \gamma = -3 \) is the most singular case, we omit the details of the case \( \gamma > -3 \).

Estimate of \( Y_2(G, H, \Theta F) \). By Taylor expansion up to order 1,
\[ |(\mu^{-\frac{a}{2}} H)' - \mu^{-\frac{a}{2}} H| = \left| \int_0^1 (\nabla (\mu^{-\frac{a}{2}} H))(v(\kappa)) \cdot (v' - v) d\kappa \right| \lesssim \theta |v - v_s| \int_0^1 |(\nabla (\mu^{-\frac{a}{2}} H))(v(\kappa))| d\kappa, \]
and the fact
\[ |\nabla (\mu^{-\frac{a}{2}} H)| = |\mu^{-\frac{a}{2}} \nabla H + H \nabla \mu^{-\frac{a}{2}}| \lesssim (1 + a|v|) \mu^{-\frac{a}{2}} (|\nabla H| + |H|) \lesssim (1 + a) \mu^{-a} (|\nabla H| + |H|), \tag{4.29} \]
we get
\[ Y_2(G, H, \Theta F) \lesssim \int B_{\eta, \gamma}^{\Theta} \theta |v - v_s| |(\mu^{2a} G_s)\mu^{-a} (v(\kappa))| \times (|\nabla H(v(\kappa))| + |H(v(\kappa))|) |(\mu^{-\frac{a}{2}} \Theta F)'| d\sigma dv d v_s d \kappa. \]
Thanks to \( |v_s - v(\kappa)| \leq 1 \) and \( |v_s - v'| \leq 1 \), we can apply (2.28) to get \( \mu^{2a}(v_s) \mu^{-a}(v(\kappa)) \mu^{-\frac{a}{2}} (v') \leq e^{3a} \), which gives
\[ Y_2(G, H, \Theta F) \lesssim \int B_{\eta, \gamma}^{\Theta} \theta |v - v_s| |G_s||(\nabla H(v(\kappa)))| + |H(v(\kappa))| |(\Theta F)'| d\sigma dv d v_s d \kappa. \]
By Cauchy-Schwartz inequality and the change (2.35), using (2.36), (2.37), (2.12) and \( |v - v_s|^{\epsilon + 1} \leq |v - v_s|^{-2} \), we get
\[ |Y_2(G, H, \Theta F)| \lesssim |\ln \varepsilon|^{-1} e^{-1} \left( \int 1_{|v - v_s| \leq \eta} |v - v_s|^{-1 - 2\theta} G_s^2 (|\nabla H|^2 + H^2) dv d v_s \right)^{\frac{1}{2}} \times \left( \int 1_{|v - v_s| \leq \eta} |v - v_s|^{-3 + 2\theta} |\Theta F|^2 dv d v_s \right)^{\frac{1}{2}}. \tag{4.30} \]
It is easy to check that for \( \delta \in (0, 1/2] \), \((s_3, s_4) = (1/2 + \delta, 0)\) or \((s_3, s_4) = (0, 1/2 + \delta)\) and suitable functions \( G, H, F \), by Hardy inequality and Hardy-Littlewood-Sobolev inequality,

\[
\int_{|v-v_*| \leq \eta} |v - v_*|^{-1-2\delta} G^2 H^2 d\nu_* v_* \lesssim |G|^2_{H^{s_3}} |H|^2_{H^{s_4}}.
\]

(4.31)

\[
\int 1_{|v-v_*| \leq \eta} |v - v_*|^{-3+2\delta} F^2 d\nu_* v_* \lesssim \delta^{-1} \eta^{2\delta} |F|_{L^2}^2.
\]

(4.32)

By (4.31) and (4.32), back to (4.30), using (2.7), we have

\[
|\mathcal{Y}_2(G, H, \tilde{\phi} F)| \lesssim \delta^{-\frac{1}{2}} \eta^{\delta} |\ln \varepsilon|^{-\frac{1}{2}} |G|_{H^{s_3}} |H|_{H^{1+s_4}} |W^\varepsilon(D) F|_{L^2}.
\]

(4.33)

Patching together (4.27) and (4.33), we get

\[
|\mathcal{Y}(G, H, \tilde{\phi} F)| \lesssim \delta^{-\frac{1}{2}} \eta^{\delta} |G|_{H^{s_3}} |H|_{H^{1+s_4}} |W^\varepsilon(D) F|_{L^2} + \delta^{-\frac{1}{2}} (\eta + \varepsilon^\frac{1}{2}) |G| _{H^{\frac{3}{2}+\delta}} |W^\varepsilon(D) H|_{L^2} |W^\varepsilon(D) F|_{L^2}.
\]

(4.34)

Patching together (4.28) and (4.33), we get

\[
|\mathcal{Y}(G, H, \tilde{\phi} F)| \lesssim \delta^{-\frac{1}{2}} |G|_{H^{s_3}} |H|_{H^{1+s_4}} |W^\varepsilon(D) F|_{L^2}.
\]

(4.35)

**Step 2:** \(\mathcal{Y}(G, \tilde{\phi} F, \tilde{\phi} F)\). Recall

\[
\mathcal{Y}(G, \tilde{\phi} F, \tilde{\phi} F) = \int B^\varepsilon(\mu^{2\alpha} G_*)((\mu^{\frac{\beta}{2}} \tilde{\phi} F)' - \mu^{\frac{\beta}{2}} \tilde{\phi} F) d\sigma d\nu_* v_*.
\]

The analysis of term \(\mathcal{Y}(G, \tilde{\phi} F, \tilde{\phi} F)\) is similar to that of \(\mathcal{Y}_2(G, H, \tilde{\phi} F)\) in **Step 1**. In this step, we can apply Taylor expansion to function \(\mu^{-\frac{\beta}{2}} \tilde{\phi} F\). Then similar to (4.30), we will get

\[
|\mathcal{Y}(G, \tilde{\phi} F, \tilde{\phi} F)| \lesssim |\ln \varepsilon|^{-1} \varepsilon^{-1} \left( \int 1_{|v-v_*| \leq \eta} |v - v_*|^{-1-2\delta} G^2 H^2 d\nu_* v_* \right)^{\frac{1}{2}}
\]

\[
\times \left( \int 1_{|v-v_*| \leq \eta} |v - v_*|^{-3+2\delta} (|\nabla \tilde{\phi} F|^2 + |\tilde{\phi} F|^2) d\nu_* v_* \right)^{\frac{1}{2}}.
\]

Thanks to (4.31) and (4.32), for \((s_3, s_4) = (1/2 + \delta, 0)\) or \((s_3, s_4) = (0, 1/2 + \delta)\), we have

\[
|\mathcal{Y}(G, \tilde{\phi} F, \tilde{\phi} F)| \lesssim \delta^{-\frac{1}{2}} \eta^{\delta} |\ln \varepsilon|^{-1} \varepsilon^{-1} |G|_{H^{s_3}} |\tilde{\phi} F|_{H^{s_4}} |\tilde{\phi} F|_{H^1}
\]

\[
\lesssim \delta^{-\frac{1}{2}} \eta^{\delta} |G|_{H^{s_3}} |H|_{H^{1+s_4}} |W^\varepsilon(D) F|_{L^2},
\]

(4.36)

where we use (2.7), (2.6) and (2.3) in the last line.

**Step 3:** \(\mathcal{Y}(G, \tilde{\phi} F, \tilde{\phi} F)\). We make dyadic decomposition in the frequency space and get

\[
\mathcal{Y}(G, \tilde{\phi} F, \tilde{\phi} F) = \sum_{j,k=-1}^{\infty} \mathcal{Y}(G, \tilde{\phi} j \tilde{\phi} F, \tilde{\phi} k \tilde{\phi} F)
\]
We will apply Taylor expansion to \( \mathcal{Y}(G, \mathbb{F}_j \mathbb{F} H, \mathbb{F}_k \mathbb{F} F) \) or 
where we have (4.29) and (2.28). Thanks to (4.31) and (4.32), for 
\( (s_3, s_4) = (\frac{1}{2} + \delta, 0) \) or 
\( (s_3, s_4) = (0, \frac{1}{2} + \delta) \), we have 
\[
\mathcal{X}_{11}(G, H_j, F_k) \lesssim | \ln \varepsilon |^{-1} | \ln \varepsilon | - k \ln 2 + 1 \big| \\
\int 1_{|v-v_*| \leq \eta} |v-v_*|^r \big( \mu^{2a} G \big)_* \mu^{-\frac{a}{2}} H_j (\nabla \mu^{-\frac{a}{2}} F_k) | d\sigma d\nu_* \\
\lesssim | \ln \varepsilon |^{-1} | \ln \varepsilon | - k \ln 2 + 1 \\
\int 1_{|v-v_*| \leq \eta} |v-v_*|^{-2} G_* H_j (|F_k| + |\nabla F_k|) d\sigma d\nu_* ,
\]
where we use (4.29) and (2.28). Thanks to (4.31) and (4.32), for 
\( (s_3, s_4) = (\frac{1}{2} + \delta, 0) \) or 
\( (s_3, s_4) = (0, \frac{1}{2} + \delta) \), we have 
\[
\mathcal{X}_{11}(G, H_j, F_k) \lesssim | \ln \varepsilon |^{-1} | \ln \varepsilon | - k \ln 2 + 1 + 2^k \delta^{-\frac{1}{2}} \eta ^{\delta} | G |_{H^{s_3}} | H_j |_{H^{s_4}} | F_k |_{L^2} .
\]

**Estimate of \( \mathcal{X}_{12}(G, H_j, F_k) \).** By Cauchy-Schwarz inequality and the change 
(2.35), using (2.36) and (2.37), we get 
\[
\mathcal{X}_{12}(G, H_j, F_k) \lesssim \int b^\varepsilon \phi \left( \sin(\theta/2)/2^k \right) \theta^2 1_{|v-v_*| \leq \eta} |v-v_*|^r + 2
\]
In this case, one has
\[ \int f^\phi (\sin(\theta/2)/2^k) \theta^2 1_{|v-v_s|\leq \eta}|v-v_s|^{-1-2\delta}|(\mu^{-\frac{d}{2}} H_j (\nabla^2 \mu^{-\frac{d}{2}} F_k))^2 dv dv_s \delta_0 \]
\[ \times \left( \int f^\phi (\sin(\theta/2)/2^k) \theta^2 1_{|v-v_s|\leq \eta}|v-v_s|^{-1+2\delta}|(\mu^{-\frac{d}{2}} H_j)^2 dv dv_s \delta_0 \right) \]
\[ \times \left( \int 1_{|v-v_s|\leq \eta}|v-v_s|^{-1+2\delta} (|\nabla^2 F_k|^2 + |\nabla F_k|^2 + |F_k|^2) dv dv_s \right)^\frac{1}{2}, \]
where we use the fact \(|\nabla^2 (\mu^{-\frac{d}{2}} F)| \lesssim (1 + a^2) \mu^{-a} (|\nabla^2 F| + |\nabla F| + |F|)\) and (2.28). Thanks to (4.31) and (4.32), we get
\[ \mathcal{X}_{1,2}(G, H_j, F_k) \lesssim |\ln \varepsilon|^{-1} (|\ln \varepsilon| - k \ln 2 + 1) 2^{2k} \delta^{-\frac{1}{2}} \eta^2 |G| \| H_j \|_{H^3} \| H_j \|_{H^4} \| F_k \|_{L^2}. \]

Patching together the estimates of \( \mathcal{X}_{1,1}(G, H_j, F_k) \) and \( \mathcal{X}_{1,2}(G, H_j, F_k) \), we get
\[ \mathcal{X}_{1}(G, H_j, F_k) \lesssim |\ln \varepsilon|^{-1} (|\ln \varepsilon| - k \ln 2 + 1) 2^{2k} \delta^{-\frac{1}{2}} \eta^2 |G| \| H_j \|_{H^3} \| H_j \|_{H^4} \| F_k \|_{L^2}. \]

Estimate of \( \mathcal{X}_{2}(G, H_j, F_k) \). In this case, one has \( \theta \gtrsim 2^{-k} \). By Taylor expansion up to order 1,
\[ |(\mu^{-\frac{d}{2}} F_k)' - \mu^{-\frac{d}{2}} F_k| \lesssim \theta |v - v_s| \int_0^1 \left| (\nabla (\mu^{-\frac{d}{2}} F_k))(v(\kappa)) \right| d\kappa. \]
Plugging which into \( \mathcal{X}_{2}(G, H_j, F_k) \), since \( \int_0^1 t^{-2} dt \lesssim 2^k \), by similar computation as in (4.30), using (4.31) and (4.32), we get
\[ \mathcal{X}_{2}(G, H_j, F_k) \lesssim |\ln \varepsilon|^{-1} 2^k \left( \int 1_{|v-v_s|\leq \eta}|v-v_s|^{-1-2\delta} G^2 H_j^2 dv dv_s \right)^\frac{1}{2} \]
\[ \times \left( \int 1_{|v-v_s|\leq \eta}|v-v_s|^{-3+2\delta} (|\nabla F_k|^2 + |F_k|^2) dv dv_s \right)^\frac{1}{2} \]
\[ \lesssim |\ln \varepsilon|^{-1} 2^k \delta^{-\frac{1}{2}} \eta^2 |G| \| H_j \|_{H^3} \| H_j \|_{H^4} \| F_k \|_{L^2}. \]

Patching together the estimates of \( \mathcal{X}_{1}(G, H_j, F_k) \) and \( \mathcal{X}_{2}(G, H_j, F_k) \), for \( k < j \), we have
\[ \mathcal{Y}(G, H_j, F_k) \lesssim |\ln \varepsilon|^{-1} (1 + |\ln \varepsilon| - k \ln 2) 2^{2k} \delta^{-\frac{1}{2}} \eta^2 |G| \| H_j \|_{H^3} \| H_j \|_{H^4} \| F_k \|_{L^2}. \quad (4.37) \]

**Case 2**: \( j \leq k \). Note that
\[ \mathcal{Y}(G, H_j, F_k) = \int B^\xi \eta (\mu^{2a} G)_\ast \mu^{-\frac{d}{2}} H_j ((\mu^{-\frac{d}{2}} F_k)' - \mu^{-\frac{d}{2}} F_k) dv dv_s \]
\[ = \int B^\xi \eta (\mu^{2a} G)_\ast ((\mu^{-a} H_j F_k)' - \mu^{-a} H_j F_k) dv dv_s \]
\[ + \int B^\xi \eta (\mu^{2a} G)_\ast (\mu^{-\frac{d}{2}} H_j - (\mu^{-\frac{d}{2}} H_j)')(\mu^{-\frac{d}{2}} F_k)' dv dv_s \]
where \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) are defined in Step 1. Since \( \mathcal{Y}_1(G, H_j, F_k) \) is handled in Step 1 and \( \mathcal{Y}_2(G, H_j, F_k) \) enjoys almost the same argument as that for \( \mathcal{Y}(G, H_j, F_k) \) in the Case I where \( k < j \), we conclude from (4.27), (4.28) and (4.37) that for \( j \leq k \),

\[
|\mathcal{Y}(G, H_j, F_k)| \lesssim |\ln \varepsilon| |\ln \varepsilon| - j \ln 2 |\mathcal{Y}^{\frac{1}{2}}(G) H_j | H^{\frac{1}{2}} |F_k| L^2
+ \delta^{-\frac{1}{2}} (\eta + \varepsilon^2) |G| H^{\frac{1}{2}} |W^\varepsilon(D) H_j | L^2 |W^\varepsilon(D) F_k| L^2,
\]

(4.38)

where we refer to [16](Page 4132) for the last inequality. Similarly, by (4.37) and (4.38), we have

\[
|\mathcal{Y}(G, \mathcal{H}, F)| \lesssim \delta^{-\frac{1}{2}} \eta^\delta |G| H^{\frac{1}{2}} \sum_{-1 \leq k \leq j \leq |\ln \varepsilon|} 2^{2k} |\ln \varepsilon| - k \ln 2 + 1 |H_j | H^{\frac{1}{2}} |F_k| L^2
+ \delta^{-\frac{1}{2}} (\eta + \varepsilon^2) |G| H^{\frac{1}{2}} \sum_{-1 \leq j \leq k \leq |\ln \varepsilon|} 2^{2j} |\ln \varepsilon| - j \ln 2 + 1 |H_j | H^{\frac{1}{2}} |F_k| L^2
+ \delta^{-\frac{1}{2}} (\eta + \varepsilon^2) |G| H^{\frac{1}{2}} \sum_{-1 \leq j \leq k \leq |\ln \varepsilon|} |W^\varepsilon(D) H_j | L^2 |W^\varepsilon(D) F_k| L^2,
\]

(4.39)

where we refer to [16](Page 4132) for the last inequality. Similarly, by (4.37) and (4.39), we have

\[
|\mathcal{Y}(G, \mathcal{H}, F)| \lesssim \delta^{-\frac{1}{2}} \eta^\delta |G| H^{\frac{1}{2}} |W^\varepsilon(D) H | H^{\frac{1}{2}} |W^\varepsilon(D) F| L^2
+ \delta^{-\frac{1}{2}} |G| H^{\frac{1}{2}} |H^{\frac{1}{2}} |F| L^2
\lesssim \delta^{-\frac{1}{2}} |G| H^{\frac{1}{2}} |H^{\frac{1}{2}} |F| L^2.
\]

(4.40)

Patching all the estimates, we get the proposition. Indeed, patching together (4.34), (4.36) and (4.40), taking \( (s_3, s_4) = (\frac{1}{2} + \delta, 0) \), we get (4.25). Patching together (4.35), (4.36) and (4.41), we get (4.26). \( \square \)

**4.2.2. Upper Bound of** \( Q^{\varepsilon, \gamma}(g, h) \) **Recalling** (4.3), as a result of Proposition 4.1 and Proposition 4.3, we get

**Theorem 4.2.** Let \( \delta \in (0, 1/2] \), \( \eta \in (0, 1] \), \( a \in [0, 1] \) and \( (s_3, s_4) = (\frac{1}{2} + \delta, 0) \) or \( (0, \frac{1}{2} + \delta) \). Then for any suitable functions \( g, h \) and \( f \), we have

\[
|\langle Q^{\varepsilon, \gamma}(g, h), f \rangle| \lesssim \delta^{-\frac{1}{2}} (\eta^\delta + \varepsilon^2) |\mu^{-2a} g| H^{\frac{1}{2}} |h| H^{\frac{1}{2}} |W^\varepsilon(D) H^{\frac{1}{2}} |W^\varepsilon(D) \mu^{\frac{a}{2}} f| L^2
+ \eta^\gamma^{-3} |g| L^{1}_{\varepsilon, \gamma/2} |h| L^{2}_{\varepsilon, \gamma/2} |f| L^{2}_{\varepsilon, \gamma/2},
\]

\[
|\langle Q^{\varepsilon, \gamma}(g, h), f \rangle| \lesssim \delta^{-\frac{1}{2}} |\mu^{-2a} g| H^{\frac{1}{2}} |\mu^{\frac{a}{2}} h| H^{\frac{1}{2}} |W^\varepsilon(D) \mu^{\frac{a}{2}} f| L^2
+ |g| L^{1}_{\varepsilon, \gamma/2} |h| L^{2}_{\varepsilon, \gamma/2} |f| L^{2}_{\varepsilon, \gamma/2}.
\]
4.2.3. Upper Bound of $\langle I^{E,Y}_\eta (g, h), f \rangle$  Recalling (4.7), we can define $I^{E,Y}_\eta (g, h; \beta)$ by replacing $B^{E,Y,\eta}$ with $B^{E,Y}_\eta$. We have the following result for upper bound of $\langle I^{E,Y}_\eta (g, h; \beta), f \rangle$.

**Proposition 4.4.** Let $\delta \in (0, 1/2]$, $\eta \in (0, 1]$ and $(s_3, s_4) = (1/2 + \delta, 0)$ or $(0, 1 + \delta)$. Then for any suitable functions $g, h$ and $f$, there holds

$$|\langle I^{E,Y}_\eta (g, h; \beta), f \rangle| \lesssim \delta^{-1/2} \eta |\mu_1^{\frac{1}{16}} g|_{H^{s_3}} |W^E(D) \mu_1^{\frac{1}{16}} h|_{H^{s_4}} |W^E(D) \mu_1^{\frac{1}{16}} f|_{L^2}.$$  

**Proof.** Let us consider the $\beta = 0$ case since the following arguments also work when we replace $\mu_1^{\frac{1}{16}}$ with $\partial_\mu_1^{\frac{1}{16}}$. Recall

$$\langle I^{E,Y}_\eta (g, h), f \rangle = \int B^{E,Y}_\eta ((\mu_1^{\frac{1}{16}})_a' - \mu_2^\frac{1}{16}) g_a h df \sigma dv_a dv.$$  

(4.42)

By setting $G = \mu_1^{\frac{1}{16}} g$, $H = \mu_1^{\frac{1}{16}} h$, $F = \mu_1^{\frac{1}{16}} f$, we have

$$\langle I^{E,Y}_\eta (g, h), f \rangle = \int B^{E,Y}_\eta ((\mu_1^{\frac{1}{16}})_a' - \mu_2^\frac{1}{16})(\mu_1^{\frac{1}{16}} G)_a \mu_1^{\frac{1}{16}} H(\mu_1^{\frac{1}{16}} F)' \sigma dv_a dv = \int B^{E,Y}_\eta ((\mu_1^{\frac{1}{16}})_a' - \mu_2^\frac{1}{16})(\mu_1^{\frac{1}{16}} H - (\mu_1^{\frac{1}{16}} H'))(\mu_1^{\frac{1}{16}} F)' \sigma dv_a dv + \int B^{E,Y}_\eta ((\mu_1^{\frac{1}{16}})_a' - \mu_2^\frac{1}{16})(\mu_1^{\frac{1}{16}} G)_a (\mu_1^{\frac{1}{16}} H F)' \sigma dv_a dv =: A(G, H, F) + B(G, H, F).$$

**Step 1:** $A(G, H, F)$. By the decomposition $F = \emptyset F$ and $H = \emptyset H + \emptyset F$, we have

$$A(G, H, F) = A(G, \emptyset F, \emptyset H) + A(G, \emptyset H, \emptyset F) + A(G, \emptyset F, \emptyset F).$$

**Step 1.1:** low $-$ high, high $-$ low, high $-$ high. By Taylor expansion,

$$|(\mu_1^{\frac{1}{16}})_a' - \mu_2^\frac{1}{16}| = |\int_0^1 (\nabla \mu_1^{\frac{1}{16}})(v_a(i)) \cdot (v' - v) dt| \lesssim \theta |v - v_*| \int \mu_1^{\frac{1}{16}} (v_a(i)) dt,$$

where $v_a(i) = v_* + \iota(v' - v_*).$ Then

$$|A(G, H, F)| \lesssim \int B^{E,Y}_\eta \theta |v - v_*| \mu_1^{\frac{1}{16}} (v_a(i)) (\mu_1^{\frac{1}{16}} G)_a (\mu_1^{\frac{1}{16}} H - (\mu_1^{\frac{1}{16}} H')) (\mu_1^{\frac{1}{16}} F)' \sigma dv_a dv + \int B^{E,Y}_\eta \theta |v - v_*| \mu_1^{\frac{1}{16}} (v_a(i)) \mu_1^{\frac{1}{16}} (v) \mu_1^{\frac{1}{16}} (v') \mu_1^{\frac{1}{16}} (v_*).$$

$$\int G_1 H F' \sigma dv_a dv + \int B^{E,Y}_\eta \theta |v - v_*| \mu_1^{\frac{1}{16}} (v_a(i)) \mu_1^{\frac{1}{16}} (v) \mu_1^{\frac{1}{16}} (v_*).$$

$$\mu_1^{\frac{1}{16}} (v) |G_2 H F'| \sigma dv_a dv + \int B^{E,Y}_\eta \theta |v - v_*| \mu_1^{\frac{1}{16}} (v_a(i)) \mu_1^{\frac{1}{16}} (v) \mu_1^{\frac{1}{16}} (v_*).$$

$$\mu_1^{\frac{1}{16}} (v) |G_3 H F'| \sigma dv_a dv.$$
Note that $|v - v_*| \leq \eta \leq 1$ yields $|v_* - v_*(t)| \leq 1, |v - v_*(t)| \leq 1, |v' - v_*(t)| \leq 1$.
Then by (2.28), one has
\[ \mu^{\frac{1}{2}}(v_*(t)) \mu^{-\frac{1}{16}}(v_*) \mu^{-\frac{1}{16}}(v') \leq 1, \mu^{\frac{1}{2}}(v_*(t)) \mu^{-\frac{1}{16}}(v_*) \mu^{-\frac{1}{16}}(v') \leq 1, \]
which yields
\[
|A(G, H, F)| \lesssim \int B^{\varepsilon, \gamma}_1 \theta |v - v_*||G_*|(|H| + |H'|)|F'|d\sigma dv_*
\]
\[
\lesssim \left( \int B^{\varepsilon, \gamma}_1 \theta |v - v_*|^2-2\delta G^2_*H^2d\sigma dv_* \right)^{\frac{1}{2}}
\]
\[
(\int B^{\varepsilon, \gamma}_1 \theta |v - v_*|^{2\delta}F^2d\sigma dv_* \right)^{\frac{1}{2}},
\]
where we use Cauchy-Schwartz inequality and the change of variable $v \to v'$.
By the estimate (2.12), thanks to (4.31) and (4.32), for $(s_3, s_4) = (\frac{1}{2} + \delta, 0)$ or $(s_3, s_4) = (0, \frac{1}{2} + \delta)$, we derive
\[
|A(G, H, F)| \lesssim |\ln \varepsilon|^{-1}e^{-1}\delta^{-\frac{1}{2}}\eta^\delta |G|_{H^2} |H|_{H^4} |F|_{L^2}.
\]
Taking $(H, F) = (\tilde{\Phi}H, \tilde{\Phi}F)$, or $(H, F) = (\tilde{\Phi}H, \Phi F)$, or $(H, F) = (\Phi H, \Phi F)$, by (2.6) and (2.7), we have
\[
|A(G, \tilde{\Phi}H, \tilde{\Phi}F) + A(G, \tilde{\Phi}H, \Phi F) + A(G, \Phi H, \Phi F)|
\]
\[
\lesssim \delta^{-\frac{1}{2}}\eta^\delta |G|_{H^2} |W^\varepsilon(D)H|_{H^4} |W^\varepsilon(D)F|_{L^2}.
\]
**Step 1.2: low-low.** We make dyadic decomposition in the frequency space and get
\[
A(G, \tilde{\Phi}H, \tilde{\Phi}F) = \sum_{j, k = -1}^{\infty} A(G, \tilde{\Phi}j \tilde{\Phi}H, \tilde{\Phi}k \tilde{\Phi}F)
\]
\[= \sum_{-1 \leq j \leq k \leq |\ln \varepsilon|} A(G, \tilde{\Phi}j \tilde{\Phi}H, \tilde{\Phi}k \tilde{\Phi}F)
\]
\[+ \sum_{-1 \leq k < j \leq |\ln \varepsilon|} A(G, \tilde{\Phi}j \tilde{\Phi}H, \tilde{\Phi}k \tilde{\Phi}F).
\]
For simplicity, let $H_j = \tilde{\Phi}j \tilde{\Phi}H, F_k = \tilde{\Phi}k \tilde{\Phi}F$. We set to estimate $A(G, H_j, F_k)$.
**Case 1:** $j \leq k$. Let us first consider $A(G, H_j, F_k)$ for $-1 \leq j \leq k \leq |\ln \varepsilon|$. Recall
\[
A(G, H_j, F_k) = \int B^{\varepsilon, \gamma}_1 ((\mu^{\frac{1}{2}})_* - \mu^2_*)(\mu^{-\frac{1}{16}} G_*)(\mu^{-\frac{1}{16}} H_j - (\mu^{-\frac{1}{16}} H_j'))
\]
\[(\mu^{-\frac{1}{16}} F_k')d\sigma dv_*dv.
\]
By Taylor expansion up to order 1,
\[ |\mu^{-\frac{1}{16}} H_j - (\mu^{-\frac{1}{16}} H_j)'| = \int (\nabla \mu^{-\frac{1}{16}} H_j)(v(\kappa)) \cdot (v' - v) \, dk | \]
\[ \lesssim |v - v_*| \int |(\nabla \mu^{-\frac{1}{16}} H_j)(v(\kappa))| \, dk, \]
\[ |(\mu^{\frac{1}{2}})_* - \mu^{\frac{1}{2}}_*| = \int (\nabla \mu^{\frac{1}{2}})(v_*(i)) \cdot (v' - v) \, dt | \]
\[ \lesssim |v - v_*| \int |(\nabla \mu^{\frac{1}{2}})(v_*(i))| \, dt. \]

From which together with \( |\nabla \mu^{-\frac{1}{16}} H_j| \lesssim \mu^{-\frac{1}{8}} (|H_j| + |\nabla H_j|) \) and \( |\nabla \mu^{\frac{1}{2}}| \lesssim \mu \frac{1}{3}, \)
we get
\[ |\mathcal{A}(G, H_j, F_k)| \lesssim \int B^{\varepsilon, \gamma}_{\eta} \theta^2 |v - v_*|^2 \mu \frac{1}{3} (v_*(i)) \mu^{-\frac{1}{8}} H_j \mu^{-\frac{1}{16}} (v') \]
\[ \times |G_*| (|\nabla H_j|) (v(\kappa)) | + |H_j (v(\kappa))|) |F_k'| \, d\sigma \, dv_* \, dv \, dk \, dt. \]

By (2.28), we have \( \mu \frac{1}{3} (v_*(i)) \mu^{-\frac{1}{8}} (v(\kappa)) \mu^{-\frac{1}{16}} (v') \lesssim 1. \) From which together with Cauchy-Schwartz inequality and the change (2.35), using (2.36) and (2.37), we have
\[ |\mathcal{A}(G, H_j, F_k)| \lesssim \int B^{\varepsilon, \gamma}_{\eta} \theta^2 |v - v_*|^2 |G_*| (|\nabla H_j|)^2 + |H_j|^2) \, d\sigma \, dv_* \, dv \]
\[ \lesssim (\int B^{\varepsilon, \gamma}_{\eta} \theta^2 |v - v_*|^2 |G_*| (|\nabla H_j|^2 + |H_j|^2) \, d\sigma \, dv_* \, dv)^{\frac{1}{2}} \]
\[ \times \left( \int B^{\varepsilon, \gamma}_{\eta} \theta^2 |v - v_*|^2 |G_*| |F_k|^2 \, d\sigma \, dv_* \, dv \right)^{\frac{1}{2}} \]
\[ \lesssim \left( \int 1_{|v - v_*| \leq \eta} |v - v_*|^{-1} |G_*| (|\nabla H_j|^2 + |H_j|^2) \, d\sigma \, dv_* \, dv \right)^{\frac{1}{2}} \]
\[ \times \left( \int 1_{|v - v_*| \leq \eta} |v - v_*|^{-1} |G_*| |F_k|^2 \, d\sigma \, dv_* \, dv \right)^{\frac{1}{2}} \]
\[ \lesssim \eta^{\frac{1}{2}} 2^j |G|_{L^2} |H_j|_{L^2} |F_k|_{L^2}, \]

where we use the inequality \( \int 1_{|v - v_*| \leq \eta} |v - v_*|^{-2} \, dv_* \lesssim \eta. \) Note that \( 2^j \lesssim 2^j \frac{|\ln \varepsilon| - j \ln 2 + 1}{|\ln \varepsilon|} \), we arrive at
\[ |\mathcal{A}(G, H_j, F_k)| \lesssim 2^j \frac{|\ln \varepsilon| - j \ln 2 + 1}{|\ln \varepsilon|} \eta^n |G|_{H^{\frac{1}{2}}} |H_j|_{H^{\frac{1}{2}}} |F_k|_{L^2}. \quad (4.43) \]

**Case 2 :** \( k < j. \) Let us now consider \( \mathcal{A}(G, H_j, F_k) \) for \(-1 \leq k < j \lesssim |\ln \varepsilon|. \)

Note that
\[ \mathcal{A}(G, H_j, F_k) = \int B^{\varepsilon, \gamma}_{\eta} ((\mu^{\frac{1}{2}})_* - \mu^{\frac{1}{2}}_*)(\mu^{-\frac{1}{16}} G)_* (\mu^{-\frac{1}{16}} H_j - (\mu^{-\frac{1}{16}} H_j)') \]
\[ \times \mu^{-\frac{1}{16}} F_k' \, d\sigma \, dv_* \, dv \]
\[ = \int B^{\varepsilon, \gamma}_{\eta} (\mu^{-\frac{1}{16}} G)_* (\mu^{-\frac{1}{8}} H_j F_k - (\mu^{-\frac{1}{8}} H_j F_k)'). \]
\[ ((\mu^\frac{1}{2})_* - \mu_\ast^\frac{1}{2}) d\sigma dv_n dv + \int B_{\eta}^{E,Y} (\mu^{\frac{1}{2}} G) (\mu^{\frac{1}{2}} H_j) \]
\[ ((\mu^{\frac{1}{2}} F_k)_* - \mu_\ast^{\frac{1}{2}} F_k)((\mu^\frac{1}{2})_* - \mu_\ast^\frac{1}{2}) d\sigma dv_n dv \]
\[ := A_1(g, H_j, F_k) + A_2(g, H_j, F_k). \]

- **Estimate of** \( A_1(g, H_j, F_k). \) By Taylor expansion, one has

\[
(\mu^\frac{1}{2})_* - \mu_\ast^\frac{1}{2} = (\nabla \mu^\frac{1}{2})_* \cdot (v'_* - v_*) \quad \text{and} \quad \int_0^1 (1 - \kappa) ((\nabla^2 \mu^\frac{1}{2})(v_*(\kappa)) : (v'_* - v_*) \wedge (v'_* - v_*) d\kappa. \quad (4.44)
\]

where \( v_*(\kappa) = v_\ast + \kappa (v'_\ast - v_\ast). \) Then \( A_1(g, H_j, F_k) = A_{1,1}(g, H_j, F_k) + A_{1,2}(g, H_j, F_k), \) where

\[ A_{1,1}(g, H_j, F_k) := \int B_{\eta}^{E,Y} (\mu^{\frac{1}{2}} G)_* (\mu^{\frac{1}{2}} H_j F_k) - (\mu^{\frac{1}{2}} H_j F_k) (\nabla \mu^\frac{1}{2})_* \cdot (v'_* - v_*) d\sigma dv_n dv, \]

\[ A_{1,2}(g, H_j, F_k) := \int B_{\eta}^{E,Y} (\mu^{\frac{1}{2}} G)_* (\mu^{\frac{1}{2}} H_j F_k) - (\mu^{\frac{1}{2}} H_j F_k) \times (1 - \kappa) ((\nabla^2 \mu^\frac{1}{2})(\mu^\frac{1}{2})(v_*(\kappa)) : (v'_* - v_*) \wedge (v'_* - v_*) d\sigma dv_n dv d\kappa. \]

Using the that \( v'_* - v_\ast = v - v' \) and the identities (see [4])

\[ \int B_{\eta}^{E,Y} f'(v' - v) d\sigma dv = 0, \quad (4.45) \]

\[ \int B_{\eta}^{E,Y} (v' - v) d\sigma = \left( \int B_{\eta}^{E,Y} \sin^2 (\theta/2) d\sigma \right) (v_\ast - v), \quad (4.46) \]

we have

\[ |A_{1,1}(g, H_j, F_k)| = | \int B_{\eta}^{E,Y} (\mu^{\frac{1}{2}} G)_* (\mu^{\frac{1}{2}} H_j F_k) (\nabla \mu^\frac{1}{2})_* \cdot (v' - v) d\sigma dv_n dv | \]

\[ = | \int B_{\eta}^{E,Y} \sin^2 (\theta/2) (\mu^{\frac{1}{2}} G)_* (\mu^{\frac{1}{2}} H_j F_k) (\nabla \mu^\frac{1}{2})_* \cdot (v_\ast - v) d\sigma dv_n dv | \]

\[ \lesssim \int_{|v_\ast - v| \leq \eta} |v - v_\ast|^{-2} |G_s H_j F_k| dv_n dv \]

\[ \lesssim \delta^{-\frac{1}{2}} \eta^2 |G|_{H^2} |H_j|_{H^4} |F_k|_{L^2}, \]

where we use (4.31) and (4.32). Similar to the estimate of \( A(G, F_k, H_j) \) in Case I, we get that

\[ |A_{1,2}(g, H_j, F_k)| \lesssim \int B_{\eta}^{E,Y} \theta^2 |v - v_\ast|^2 (|\mu^{\frac{1}{2}} G|_s (|\mu^{\frac{1}{2}} H_j F_k| + |(\mu^{\frac{1}{2}} H_j F_k)'|)) \]

\[ \times |(\nabla^2 \mu^\frac{1}{2})(\mu^\frac{1}{2})(v_*(\kappa))| d\sigma dv_n dv d\kappa \]

\[ \lesssim \int B_{\eta}^{E,Y} \theta^2 |v - v_\ast|^2 |G_s| (|H_j F_k| + |(H_j F_k)'|) d\sigma dv_n dv \]

\[ \lesssim \int_{|v_\ast - v| \leq \eta} |v - v_\ast|^{-1} |G_s H_j F_k| dv_n dv \lesssim \eta^\frac{1}{2} |G|_{L^2} |H_j|_{L^2} |F_k|_{L^2}. \]
Patching together the previous two estimates, we have $|A_1(g, H_j, F_k)| \lesssim \delta^{-\frac{1}{2}} \eta^8 |G|_{H^3} |H_j|_{H^4} |F_k|_{L^2}$.

- **Estimate of $A_2(g, H_j, F_k)$**: Similar to the idea used to estimate $A(G, F_k, H_j)$ in Case 1, we apply Taylor expansion to $\mu^{-\frac{j}{16}} F_k$ and get

  $$|A_2(g, H_j, F_k)| \lesssim 2^k \frac{|\ln \varepsilon| - j \ln 2 + 1}{|\ln \varepsilon|} \eta^j |G|_{H^3} |H_j|_{H^4} |F_k|_{L^2}.$$  

  Patching together the estimates of $A_1(g, H_j, F_k)$ and $A_2(g, H_j, F_k)$, we have for $-1 \leq k < j \leq |\ln \varepsilon|,

  $$|A(G, H_j, F_k)| \lesssim 2^k \frac{|\ln \varepsilon| - j \ln 2 + 1}{|\ln \varepsilon|} \eta^j |G|_{H^3} |H_j|_{H^4} |F_k|_{L^2} + \delta^{-\frac{1}{2}} \eta^j |G|_{H^3} |H_j|_{H^4} |F_k|_{L^2}.$$  

  (4.47)

  Patching together (4.43) and (4.47), we have

  $$|A(G, \delta_\phi H, \delta_\phi F)| \lesssim \eta^j \sum_{-1 \leq j \leq k \leq |\ln \varepsilon|} 2^j \frac{|\ln \varepsilon| - j \ln 2 + 1}{|\ln \varepsilon|} |G|_{H^3} |H_j|_{H^4} |F_k|_{L^2}$$

  $$+ \eta^j \sum_{-1 \leq k < j \leq |\ln \varepsilon|} 2^k \frac{|\ln \varepsilon| - j \ln 2 + 1}{|\ln \varepsilon|} |G|_{H^3} |H_j|_{H^4} |F_k|_{L^2}$$

  $$+ \delta^{-\frac{1}{2}} \eta^j \sum_{-1 \leq k < j \leq |\ln \varepsilon|} |G|_{H^3} |H_j|_{H^4} |F_k|_{L^2}$$

  $$\lesssim \delta^{-\frac{1}{2}} \eta^j |G|_{H^3} |W^0(D) H|_{H^4} |W^0(D) F|_{L^2}.$$  

  **Step 2: Estimate of $B(G, H, F)$**: Recalling (4.44) and (4.45), similar to the estimate of $A(G, F_k, H_j)$ in Case 1, by the change of variable $v \to v'$, we get

  $$|B(G, H, F)| \leq |\int B^{e,\gamma}_{\eta} (\mu^{-\frac{j}{16}} G)_* (\mu^{-\frac{j}{8}} H F)' ((\nabla^2 \mu^\frac{1}{2})(v_* (\kappa)))$$

  $$: (v' - v) \otimes (v' - v)) d\sigma dv_* dv dk|$$

  $$\lesssim |\int B^{e,\gamma}_{\eta} |v - v_*|^2 \theta^2 |G_* (HF)'| d\sigma dv_* dv \lesssim \eta^j |G|_{L^2} |H|_{L^2} |F|_{L^2}.$$  

  Patching together the estimates in Step 1 and Step 2, we finish the proof.  

  □

**4.2.4. Upper Bound of $\langle I^{e,\gamma}(g, h), f \rangle$**  As $I^{e,\gamma}(g, h; \beta) = I^{e,\gamma,\eta}(g, h; \beta) + I^{e,\gamma}_{\eta}(g, h; \beta)$, by Proposition 4.2 and Proposition 4.4, we get

**Theorem 4.3**: Let $\delta \in (0, 1/2]$, $\eta \in (0, 1]$ and $(s_3, s_4) = (\frac{1}{2} + \delta, 0)$ or $(0, \frac{1}{2} + \delta)$. Then for any suitable functions $g$, $h$ and $f$, the following two estimates are valid.

$$|\langle I^{e,\gamma}(g, h; \beta), f \rangle| \lesssim \eta^{\gamma - 3} |g|_{L^2} |h|_{L^2} |H|_{L^2} |W^0 f|_{L^2} + \delta^{-\frac{1}{2}} \eta^8 |\mu^{\frac{1}{16}} g|_{H^\frac{1}{2}}$$

$$|\mu^{\frac{1}{16}} H|_{H^1} |f|_{L^2} + \delta^{-\frac{1}{2}} |\mu^{\frac{1}{16}} g|_{H^\frac{1}{2}} |\mu^{\frac{1}{16}} H|_{L^2} |f|_{L^2}$$

$$|\langle I^{e,\gamma}(g, h; \beta), f \rangle| \lesssim |g|_{L^2} |h|_{L^2} |W^0 f|_{L^2} + \delta^{-\frac{1}{2}} |\mu^{\frac{1}{16}} g|_{H^3} |\mu^{\frac{1}{16}} H|_{H^1} |f|_{L^2}.$$
4.2.5. Upper Bound of $\Gamma^{\varepsilon,\gamma}$  
Recalling (4.10), as a result of Theorem 4.2 with $a = \frac{1}{2}$ and Theorem 4.3, we get

**Theorem 4.4.** Let $\delta \in (0, 1/2]$, $\eta \in (0, 1]$ and $(s_3, s_4) = (\frac{1}{2} + \delta, 0)$ or $(0, \frac{1}{2} + \delta)$. Then for any suitable functions $g, h$ and $f$, the following two estimates are valid.

$$
\left| \langle \Gamma^{\varepsilon,\gamma}(g, h; \beta), f \rangle \right| \lesssim \eta^{-\frac{3}{2}} |g|_{L^2} |h|_{L^2_{\varepsilon,\gamma}} \left| f \right|_{L^2_{\varepsilon,\gamma}'} + \delta^{-\frac{1}{2}} \left( \eta^\delta + \varepsilon^2 \right) |\mu^{\frac{1}{6}} g|_{H^{\frac{3}{2} + \delta}}^2 \\
|\mu^{\frac{1}{6}} | h | H^1 | f |_{L^2_{\varepsilon,\gamma}'} + \delta^{-\frac{1}{2}} |\mu^{\frac{1}{6}} g|_{H^{\frac{3}{2} + \delta}} | f |_{L^2_{\varepsilon,\gamma}'}^{1} \\
|\langle \Gamma^{\varepsilon,\gamma}(g, h; \beta), f \rangle \rangle | \lesssim |g|_{L^2} |h|_{L^2_{\varepsilon,\gamma}} \left| f \right|_{L^2_{\varepsilon,\gamma}'} + \delta^{-\frac{1}{2}} |\mu^{\frac{1}{6}} g|_{H^{\frac{3}{2} + \delta}} \left| f \right|_{L^2_{\varepsilon,\gamma}'}^{1}.
$$

We remark that Theorem 4.4 also holds for the Landau case $\varepsilon = 0$. Taking $\delta = \frac{1}{2}$ in Theorem 4.4, we have

**Corollary 4.1.** For any $\eta \in (0, 1]$ and any $\beta_0, \beta_1 \in \mathbb{N}^3$, there holds

$$
\left| \langle \Gamma^{\varepsilon,\gamma}(\partial_{\beta_1} \mu^\frac{1}{2}, h), f \rangle \right| \lesssim (\eta^{-\frac{3}{2}} |h|_{L^2_{\varepsilon,\gamma}} + (\eta^\frac{1}{2} + \varepsilon^2) |\mu^{\frac{1}{6}} h|_{H^1}) | f |_{L^2_{\varepsilon,\gamma}'}^{1}.
$$

4.3. Upper Bound of $\langle \Gamma^{\varepsilon,\gamma}_\eta(f, h) - \mathcal{L}^{\varepsilon,\gamma}_\eta h, h \rangle$

This is the core part of the linear-quasilinear method. Observe

$$
\langle \Gamma^{\varepsilon,\gamma}_\eta(f, h) - \mathcal{L}^{\varepsilon,\gamma}_\eta h, h \rangle = \langle -\mathcal{L}^{\varepsilon,\gamma}_\eta h + \Gamma^{\varepsilon,\gamma}_\eta(f, h), h \rangle + \langle -\mathcal{L}^{\varepsilon,\gamma}_2 h, h \rangle.
$$

4.3.1. Upper Bound of $\langle -\mathcal{L}^{\varepsilon,\gamma}_1 h + \Gamma^{\varepsilon,\gamma}_\eta(f, h), h \rangle$

**Proposition 4.5.** Let $\delta \in (0, 1/2]$, $\eta \in (0, 1]$. For any suitable functions $h$ and $f$ with $\mu^\frac{1}{2} + f \geq 0$, it holds that

$$
\langle -\mathcal{L}^{\varepsilon,\gamma}_1 h + \Gamma^{\varepsilon,\gamma}_\eta(f, h), h \rangle \lesssim \delta^{-\frac{1}{2}} \eta^\delta |\mu^{\frac{1}{6}} (\mu^{\frac{1}{2} + f})|_{H^{\frac{3}{2} + \delta}} |f|_{L^2_{\varepsilon,\gamma}'}^{1} \\
+ \delta^{-\frac{1}{2}} (\eta + \varepsilon^2) |\mu^{\frac{1}{6}} (\mu^{\frac{1}{2} + f})|_{H^{\frac{3}{2} + \delta}} |f|_{L^2_{\varepsilon,\gamma}'}^{1}.
$$

**Proof.** Set $F = \mu + \mu^{\frac{1}{2}} g$, $g = \mu^{\frac{1}{2}} + f$, then $\langle -\mathcal{L}^{\varepsilon,\gamma}_1 h + \Gamma^{\varepsilon,\gamma}_\eta(f, h), h \rangle = \langle \mu^{\frac{1}{2}} Q^{\varepsilon,\gamma}_\eta(F, \mu^{\frac{1}{2}} h), h \rangle$ and $F = \mu^{\frac{1}{2}} g \geq 0$. We make the decomposition

$$
\langle \mu^{\frac{1}{2}} Q^{\varepsilon,\gamma}_\eta(F, \mu^{\frac{1}{2}} h), h \rangle = \int B^{\varepsilon,\gamma}_\eta F_\star (\mu^{\frac{1}{2}} h') - \mu^{\frac{1}{2}} h) d\sigma d\nu_d d\nu_v \\
= \int \int B^{\varepsilon,\gamma}_\eta F_\star h(h' - h) d\sigma d\nu_d d\nu_v \\
+ \int \int B^{\varepsilon,\gamma}_\eta g_\star h h' (\mu^{\frac{1}{2}} h' - \mu^{\frac{1}{2}} h) d\sigma d\nu_d d\nu_v.
$$

By the inequality $2h(h' - h) \leq ((h^2)' - h^2)$ and the condition $F \geq 0$, we have

$$
I_1 \leq \frac{1}{2} \int B^{\varepsilon,\gamma}_\eta F_\star ((h^2)' - h^2) d\sigma d\nu_d d\nu_v.
$$
We only deal with the most singular case $\gamma = -3$. By taking $a = \frac{1}{8}$ in (2.19), we have
\[
|\int B^{e,\gamma}_{\eta} F_* \left( (h^2)' - h^2 \right) d\sigma dv_* dv | \lesssim (\eta + \varepsilon \frac{1}{2}) |\mu_-^{\frac{1}{2}} F|_{L^\infty} \| W^\varepsilon(D) \mu^{\frac{1}{2}} h \|_{L^2} |\mu^{\frac{1}{2}} h|_{L^2}.
\]
Recalling $F = \mu^\frac{1}{2} (\mu^\frac{1}{2} + f)$, one has $\mu_-^{\frac{1}{2}} F = \mu^\frac{1}{2} (\mu^\frac{1}{2} + f)$ and thus
\[
I_1 \lesssim \delta^{-\frac{1}{2}} (\eta + \varepsilon \frac{1}{2}) |\mu^\frac{1}{2} (\mu^\frac{1}{2} + f)|_{H^\frac{1}{2} + \delta} \| W^\varepsilon(D) \mu^{\frac{1}{2}} h \|_{L^2} |\mu^{\frac{1}{2}} h|_{L^2}.
\]
Observe that $I_2 = \int B^{e,\gamma}_{\eta} g_* hh'( (\mu^\frac{1}{2})_* - \mu^\frac{1}{2} ) d\sigma dv_* dv = \langle I^{e,\gamma}_{\eta}(g, h), h \rangle$. Then by Proposition 4.4, we have
\[
|I_2| \lesssim \delta^{-\frac{1}{2}} \eta^\delta |\mu^{\frac{1}{2}} g|_{H^\frac{1}{2} + \delta} \| W^\varepsilon(D) \mu^{\frac{1}{2}} h \|_{L^2}^2.
\]
Patching together the previous two inequalities, we finish the proof. \(\square\)

4.3.2. Upper Bound of \(\langle -L^{e,\gamma}_{2,\eta} f, h \rangle\) We have

**Proposition 4.6.** Fix $0 < \eta \leq 1$. For any suitable functions $f$ and $h$, there holds
\[
\langle -L^{e,\gamma}_{2,\eta} f, h \rangle \lesssim (\eta^\frac{1}{2} + \varepsilon \frac{1}{2}) \| W^\varepsilon(D) \mu^{\frac{1}{2}} f \|_{L^2} |\mu^{\frac{1}{2}} h|_{L^2}.
\]

**Proof.** Note that
\[
\langle -L^{e,\gamma}_{2,\eta} f, h \rangle = \langle \mu^{-\frac{1}{2}} Q^{e,\gamma}_{\eta} (\mu^\frac{1}{2} f), h \rangle = \int B^{e,\gamma}_{\eta} (\mu^\frac{1}{2} f)_* \mu ((\mu^\frac{1}{2} h)' - \mu^{-\frac{1}{2}} h) d\sigma dv_* dv
\]
\[
= \int B^{e,\gamma}_{\eta} (\mu^\frac{1}{2} f)_* \mu^\frac{1}{2} (h' - h) d\sigma dv_* dv + \int B^{e,\gamma}_{\eta} f_* \mu^\frac{1}{2} h' ((\mu^\frac{1}{2})_* - \mu^\frac{1}{2}) d\sigma dv_* dv := Y_1 + Y_2.
\]
We first estimate $Y_1$. Observe that
\[
Y_1 = \int B^{e,\gamma}_{\eta} (\mu^\frac{1}{2} f)_* \mu^\frac{1}{2} (h' - h) d\sigma dv_* dv
\]
\[
= \int B^{e,\gamma}_{\eta} (\mu^\frac{1}{2} f)_* ((\mu^\frac{1}{2} h)' - \mu^\frac{1}{2} h) d\sigma dv_* dv + \int B^{e,\gamma}_{\eta} (\mu^\frac{1}{2} f)_* (\mu^\frac{1}{2} - (\mu^\frac{1}{2})') h' d\sigma dv_* dv
\]
\[
:= Y_{1,1} + Y_{1,2}.
\]
For $Y_{1,1}$, we only deal with the most singular case $\gamma = -3$. Use (2.18) with $\delta = a = 0$ to get
\[
|Y_{1,1}| \lesssim (\eta + \varepsilon \frac{1}{2}) \| W^\varepsilon(D) \mu^{\frac{1}{2}} f \|_{L^2} |\mu^{\frac{1}{2}} h|_{L^2}.
\]
Similarly to the estimate of $B(G, H, F)$ in the proof of Proposition 4.4, we have

$$|Y_{1,2}| \lesssim \eta^{1/2} |\mu^1 f|_{L^2} |\mu^1 h|_{L^2}.$$  

We turn to $Y_2$. Note that

$$Y_2 = \int B_\eta^{p,q} f_* \mu^{1/2} h'((\mu^{1/2})' - \mu^{1/2}) d\sigma dv* dv$$

$$= \int B_\eta^{p,q} f_* (\mu^{1/2} - (\mu^{1/2})') h'((\mu^{1/2})' - \mu^{1/2}) d\sigma dv* dv 
+ \int B_\eta^{p,q} f_* (\mu^{1/2})' h'((\mu^{1/2})' - \mu^{1/2}) d\sigma dv* dv 
:= Y_{2,1} + Y_{2,2}.$$  

Similarly to the estimate of $B(G, H, F)$ in the proof of Proposition 4.4, we get

$$|Y_{2,2}| \lesssim \eta^{1/2} |\mu^1 f|_{L^2} |\mu^1 h|_{L^2}.$$  

By Cauchy-Schwartz inequality, we get

$$|Y_{2,1}| \lesssim \left( \int B_\eta^{p,q} f_*^2 (\mu^{1/2} - (\mu^{1/2})')^2 d\sigma dv* dv \right)^{1/2} \left( \int B_\eta^{p,q} (h^{2})'((\mu^{1/2})' - \mu^{1/2})^2 d\sigma dv* dv \right)^{1/2}$$

$$= \left( \int B_\eta^{p,q} f_*^2 (\mu^{1/2} - (\mu^{1/2})')^2 d\sigma dv* dv \right)^{1/2} \left( \int B_\eta^{p,q} h^2 ((\mu^{1/2})' - (\mu^{1/2})')^2 d\sigma dv* dv \right)^{1/2}.$$  

By Taylor expansion up to order 1, $(\mu^{1/2})' - \mu^{1/2} = \int_0^1 (\nabla \mu^{1/2})(v(\kappa)) \cdot (v' - v) d\kappa.$

By the change (2.35), using (2.36), (2.37) and (2.28), we get

$$\int B_\eta^{p,q} f_*^2 (\mu^{1/2} - (\mu^{1/2})')^2 d\sigma dv* dv$$

$$\lesssim \int f_*^2 \mu^{1/2} |v - v_*|^{-1} 1_{|v - v_*| < \eta} dv* dv_* \lesssim \eta^{2} |\mu^1 f|_{L^2}^2,$$

which gives $|Y_{2,1}| \lesssim \eta^{2} |\mu^1 f|_{L^2} |\mu^1 h|.$

Patching together the above estimates, we finish the proof. \hfill \Box

### 4.3.3. Quasilinear Estimate

As a result of Proposition 4.5 and Proposition 4.6, we have

**Theorem 4.5.** Let $\delta \in (0, 1/2], \eta \in (0, 1]$. For any suitable functions $h$ and $f$ with $\mu^{1/2} + f \geq 0$, if holds that

$$\langle \Gamma_\eta^{p,q} (f, h) - L_\eta^{p,q} h, h \rangle \lesssim \delta^{-1/2} (\eta^{\delta} + e^{1/2}) (1 + |\mu^{1/2} f|_{H^{1/2+\delta}}^2) |W^q(D) \mu^{1/2} h|_{L^2}^2.$$
4.3.4. Byproducts  In this part, we give some byproducts of previous results. We define

\[
\mathcal{L}^{\varepsilon, \gamma, \eta, \beta_{0}, \beta_{1}} g = \mathcal{L}_{1}^{\varepsilon, \gamma, \eta, \beta_{0}, \beta_{1}} g + \mathcal{L}_{2}^{\varepsilon, \gamma, \eta, \beta_{0}, \beta_{1}} g, \tag{4.48}
\]

where

\[
\begin{align*}
\mathcal{L}_{1}^{\varepsilon, \gamma, \eta, \beta_{0}, \beta_{1}} g &:= -\Gamma^{\varepsilon, \gamma, \eta}(\partial_{\beta_{1}}^{1/2} g; \beta_{0}), \\
\mathcal{L}_{2}^{\varepsilon, \gamma, \eta, \beta_{0}, \beta_{1}} g &:= -\Gamma^{\varepsilon, \gamma, \eta}(g, \partial_{\beta_{1}}^{1/2} \beta_{0}). \tag{4.49}
\end{align*}
\]

**Lemma 4.3.** For any \( \eta \geq 0 \), if holds that

\[
|\langle \mathcal{L}_{2}^{\varepsilon, \gamma, \eta, \beta_{0}, \beta_{1}} f, h \rangle| \lesssim |\mu^{1/2} f|_{L^{2}} |\mu^{1/2} h|_{L^{2}}. 
\]

**Proof.** The proof is similar to Proposition 4.6, so we omit the details. \( \square \)

By Corollary 4.1 and Lemma 4.3, we have the following lemma for upper bound of \( \mathcal{L}^{\varepsilon, \gamma, \beta_{0}, \beta_{1}} = \mathcal{L}^{\varepsilon, \gamma, 0, \beta_{0}, \beta_{1}} \).

**Lemma 4.4.** For any \( 0 < \eta \leq 1 \), if holds that

\[
|\langle \mathcal{L}_{2}^{\varepsilon, \gamma, \beta_{0}, \beta_{1}} f, h \rangle| \lesssim \eta^{\gamma-3} |f|_{L^{2}_{\varepsilon, \gamma/2}} |h|_{L^{2}_{\varepsilon, \gamma/2}} + ( \eta^{2} + \varepsilon^{2}) |\mu^{1/2} f|_{H^{1}} |h|_{L^{2}_{\varepsilon, \gamma/2}}. 
\]

5. Commutator Estimate

This section is devoted to the estimate of the commutator estimates between \( \Gamma^{\varepsilon}(g, \cdot) \) and \( W_{l} \), which are necessary for energy estimates in weighted Sobolev space. In this section, unless otherwise specified, \(-3 \leq \gamma \leq 0\) and \( g, h, f \) are suitable functions.

5.1. Commutator Estimates for \( Q^{\varepsilon, \gamma} \)

For the operator \( Q^{\varepsilon, \gamma, \eta} \), we have We first have

**Proposition 5.1.** Let \( 0 < \eta \leq 1, l \geq 2 \), if holds that

\[
|\langle Q^{\varepsilon, \gamma, \eta}(\mu^{1/2} g, W_{l} h) - W_{l} Q^{\varepsilon, \gamma, \eta}(\mu^{1/2} g, h), f \rangle| \lesssim \eta^{\gamma-3} C_{l} |\mu^{1/2} g|_{L^{2}} |h|_{L^{2}_{\varepsilon, \gamma/2}} |f|_{L^{2}_{\varepsilon, \gamma/2}}. 
\]

**Proof.** We observe that

\[
\begin{align*}
\langle Q^{\varepsilon, \gamma, \eta}(\mu^{1/2} g, W_{l} h) - W_{l} Q^{\varepsilon, \gamma, \eta}(\mu^{1/2} g, h), f \rangle &= \int B^{\varepsilon, \gamma, \eta}(W_{l} - W_{l}')(\mu^{1/2} g) h f' d\sigma d\nu_* dv \\
&= \int B^{\varepsilon, \gamma, \eta}(W_{l} - W_{l}')(\mu^{1/2} g) h (f' - f) d\sigma d\nu_* dv \\
&\quad + \int B^{\varepsilon, \gamma, \eta}(W_{l} - W_{l}')(\mu^{1/2} g) h f d\sigma d\nu_* dv =: A_{1} + A_{2}.
\end{align*}
\]
Step 1: Estimate of $A_1$. By Cauchy-Schwarz inequality, we have
\[ |A_1| \lesssim \left( \int B^{\varepsilon,\gamma}(f')^2 \, d\sigma \, dv \right)^{1/2} \left( \int B^{\varepsilon,\gamma}(W_I - W_I')^2 \mu_{1,2}^2 \sigma^2 \, d\sigma \, dv \right)^{1/2} \]
\[ := (A_{1,1})^{1/2} (A_{1,2})^{1/2}. \]
Note that $A_{1,1}$ has the same structure as $T_{2,1}$ in (4.16). Taking $\delta = 1/2$, $s_1 = 2$, $s_2 = 0$ in (4.17), we have $A_{1,1} \lesssim \gamma^{\gamma-3} |f|_{L^2_{\varepsilon,\gamma/2}}^2$. It is easy to derive $\int b^{\varepsilon} (W_I - W_I')^2 \, d\sigma \lesssim |v - v_*|^2 |v|^{2l-2} |v_*|^{2l-2}$, which gives
\[ A_{1,2} \lesssim \int 1_{|v - v_*| \leq \eta} |v - v_*|^{\gamma+2} (v) |v|^{2l-2} |v_*|^{2l-2} \mu_{1,2}^2 \sigma^2 \, d\sigma \, dv. \]
If $\gamma + 2 \geq 0$, there holds $A_{1,2} \lesssim |\mu_{1,2} g|_{L^2}^2 |h|_{L^2_{\varepsilon,\gamma/2}}^2$. If $\gamma + 2 \leq 0$, we get
\[ A_{1,2} \lesssim \gamma^{\gamma+2} \int (v - v_*)^2 |v|^{2l-2} |v_*|^{2l-2} \mu_{1,2}^2 \sigma^2 \, d\sigma \, dv \lesssim \gamma^{\gamma+2} |\mu_{1,2} g|_{L^2}^2 |h|_{L^2_{\varepsilon,\gamma/2}}^2. \]

Patch together the estimates of $A_{1,1}$ and $A_{1,2}$, we get $|A_1| \lesssim \gamma^{\gamma-3} |\mu_{1,2} g|_{L^2} |h|_{L^2_{\varepsilon,\gamma/2}} |f|_{L^2_{\varepsilon,\gamma/2}}^2$.

Step 2: Estimate of $A_2$. By Taylor expansion to $W_I' - W_I$ up to order 2, we have
\[ A_2 = -\int B^{\varepsilon,\gamma}(\nabla W_I)(v) \cdot (v' - v) - \int B^{\varepsilon,\gamma}(1 - \kappa) (\nabla^2 W_I)(v) : (v' - v) \]
\[ \otimes (v' - v) \mu_{1,2} g h f \, d\kappa \, d\sigma \, dv \]
\[ = A_{2,1} + A_{2,2}. \]

Estimate of $A_{2,1}$. Thanks to the fact (4.46), using $|\nabla W_I(v)| \lesssim |v|^{l-1}$, we have
\[ |A_{2,1}| \lesssim \int 1_{|v - v_*| \leq \eta} |v - v_*| \gamma^{\gamma+1} |v|^{l-1} \mu_{1,2} g h f \, d\sigma \, dv. \]
If $\gamma + 1 \geq 0$, there holds $|A_{2,1}| \lesssim |\mu_{1,2} g|_{L^2} |h|_{L^2_{\varepsilon,\gamma/2}} |f|_{L^2_{\varepsilon,\gamma/2}}^2$. If $\gamma + 1 \leq 0$, we have
\[ |A_{2,1}| \lesssim \gamma^{\gamma+1} \int (v - v_*)^2 \gamma^{\gamma+1} |v|^{l-1} \mu_{1,2} g h f \, d\sigma \, dv \]
\[ \lesssim \gamma^{\gamma+1} |\mu_{1,2} g|_{L^2} |h|_{L^2_{\varepsilon,\gamma/2}} |f|_{L^2_{\varepsilon,\gamma/2}}^2. \]

Patch together the two cases, we get $|A_{2,1}| \lesssim \gamma^{(\gamma+1)^0} |\mu_{1,2} g|_{L^2} |h|_{L^2_{\varepsilon,\gamma/2}} |f|_{L^2_{\varepsilon,\gamma/2}}^2$.

Estimate of $A_{2,2}$. Since $|\nabla^2 W_I(v)(\kappa)| \lesssim |\kappa|^{l-2} \lesssim |v|^{l-2} |v_*|^{l-2}$ and $|v' - v|^2 \lesssim \theta^2 |v - v_*|^2$, we have
\[ |A_{2,2}| \lesssim \int b^{\varepsilon} (\cos \theta)^2 \theta^2 1_{|v - v_*| \leq \eta} |v - v_*|^2 |v|^{2l-2} |v_*|^{2l-2} \mu_{1,2} g h f \, d\sigma \, dv \]
\[ \lesssim \int 1_{|v-v_*| \geq \eta}|v-v_*|^\gamma+2 (v)^f - \frac{1}{\mu_*} \sum \left| g(h) \right| dv d\nu. \]

Similar as in the estimate of \( A_{2,1} \), we have \( |A_{2,2}| \lesssim \eta^{(\gamma+2)^\wedge 0} \mu^\frac{1}{\gamma} g L^2 h L^2_{t+\gamma/2} |f| L^2_{t+\gamma/2}. \)

Patching together the estimates of \( A_{2,1} \) and \( A_{2,2} \), we have \( |A_2| \lesssim \eta^{(\gamma+1)^\wedge 0} \mu^\frac{1}{\gamma} g L^2 h L^2_{t+\gamma/2} |f| L^2_{t+\gamma/2}. \)

The proposition follows by patching together the estimates of \( A_1 \) and \( A_2 \).

Observe that
\[ (Q^\epsilon,\gamma (\mu^\frac{1}{2} g, W_l h) - W_l Q^\epsilon,\gamma (\mu^\frac{1}{2} g, h), f) = \int B^\epsilon,\gamma (W_l - W_l') \mu^\frac{1}{2} g h f^\prime d\sigma dv d\nu. \] (5.1)

Comparing (5.1) with (4.42), we find that they enjoy almost the same structure. Thus, following the argument there and using the fact that \( \mu^\frac{1}{2} \mu^\frac{3}{2} |v-v_*| \lesssim \mu^\frac{1}{2} \mu^\frac{3}{2} |v-v_*| \), we get

Proposition 5.2. Let \( 0 < \eta \leq 1, l \geq 2, 0 < \delta \leq 1/2, (s_3, s_4) = (1/2 + \delta, 0) \) or \( (s_3, s_4) = (0, 1/2 + \delta) \), there holds
\[ |(Q^\epsilon,\gamma (\mu^\frac{1}{2} g, W_l h) - W_l Q^\epsilon,\gamma (\mu^\frac{1}{2} g, h), f)| \lesssim C_l \delta^{-\frac{1}{2}} \eta \mu^\frac{1}{\gamma} g |H^3| W^\epsilon(D) \mu^\frac{1}{\gamma} h |H^{s_4}| W^\epsilon(D) \mu^\frac{1}{\gamma} f |L^2_{t+\gamma/2}. \]

5.2. Commutator Estimates for \( I^\epsilon,\gamma,\eta \)

We have

Proposition 5.3. Let \( \eta \geq 0, l \geq 1 \), it holds that
\[ |(I^\epsilon,\gamma,\eta (g, W_l h; \beta) - W_l I^\epsilon,\gamma,\eta (g, h; \beta), f)| \lesssim C_l |g| L^2 h L^2_{t+\gamma/2} |W^\epsilon f| L^2_{t+\gamma/2}. \]

Proof. Let us consider the \( \beta = 0 \) case since the following arguments also work when we replace \( \mu^\frac{1}{2} \) with \( P_\beta \mu^\frac{1}{2} \) by using the decomposition (4.11). There are two steps in the proof. We will indicate the main difference at the end of each step.

By the definition (4.1) of \( I^\epsilon,\gamma,\eta (g, h) \) and the fact \( ((\mu^\frac{1}{2})' - \mu^\frac{1}{2}) = ((\mu^\frac{1}{2})'_* - \mu^\frac{1}{2})^2 + 2 \mu^\frac{1}{2} ((\mu^\frac{1}{2})'_* - \mu^\frac{1}{2}) \), we have
\[ (I^\epsilon,\gamma,\eta (g, W_l h) - W_l I^\epsilon,\gamma,\eta (g, h), f) \]
\[ = \int B^\epsilon,\gamma,\eta ((\mu^\frac{1}{2})'_* - \mu^\frac{1}{2}) (W_l - W_l') g h f^\prime d\sigma dv d\nu \]
\[ = \int B^\epsilon,\gamma,\eta ((\mu^\frac{1}{2})'_* - \mu^\frac{1}{2})^2 (W_l - W_l') g h f^\prime d\sigma dv d\nu \]
\[ + 2 \int B^\epsilon,\gamma,\eta \mu^\frac{1}{2} ((\mu^\frac{1}{2})'_* - \mu^\frac{1}{2}) (W_l - W_l') g h f^\prime d\sigma dv d\nu \]
\[ := A_1 + 2 A_2. \]
By (5.2) and (5.3), we have
\[ |A_1| \leq \left( \int B^{ε,γ}((μ^{1/4})_* - μ^{1/2})^2 (f^2)'dσ dv_u dv \right)^{1/2} \]
\[ \times \left( \int B^{ε,γ}((μ^{1/4})_* - μ^{1/2})^2 (W_l - W'_l)^2 g_{8^2} h^2 dσ dv_u dv \right)^{1/2} = (A_{1,1})^{1/2} (A_{1,2})^{1/2}. \]

By the change of variables \((v, v_u) → (v', v')\) and Proposition 2.2 (the result still holds with \(μ^{1/2}\) replaced by \(μ^{1/4}\)), we have
\[ A_{1,1} = \int B^{ε,γ}((μ^{1/4})'_* - μ^{1/2})^2 f_{8^2}^2 dσ dv_u dv = N^{ε,γ}(f, μ^{1/2}) \lesssim |W^ε f|^2 \gamma/2. \]

Thanks to \(((μ^{1/4})'_* - μ^{1/2})^2 = ((μ^{1/4})'_* + μ^{1/2})^2 (μ^{1/2})'_* - μ^{1/2})^2 \lesssim 2((μ^{1/4})'_* + μ^{1/2})(μ^{1/4})'_* - μ^{1/2})^2\), we have
\[ A_{1,2} \lesssim \int B^{ε,γ} μ^{1/8}_u ((μ^{1/4})'_* - μ^{1/2})^2 (W_l - W'_l)^2 g_{8^2} h^2 dσ dv_u dv \]
\[ + \int B^{ε,γ} ((μ^{1/4})'_* - μ^{1/2})^2 (W_l - W'_l)^2 g_{8^2} h^2 dσ dv_u dv = A_{1,2,1} + A_{1,2,2}. \]

We first estimate \(A_{1,2,2}\). We recall that \(|v - v'_u| \sim |v - v_u|\), and thus,
\[ (W_l - W'_l)^2 \lesssim \min\{θ^2 |v - v'_u|^2 (v')^{2l-2} (v_u')^{2l-2}, θ^2 (v)^{2l/(v'_u)^{2l}}\}, \quad (5.2) \]
\[ ((μ^{1/4})'_* - μ^{1/2})^2 \lesssim \min\{θ^2 |v - v'_u|^2, 1\}. \quad (5.3) \]

We set to prove that
\[ B := \int B^{ε,γ}((μ^{1/4})'_* - μ^{1/2})^2 (W_l - W'_l)^2 dσ \lesssim \langle v \rangle^{2l+γ}, \quad (5.4) \]
which immediately gives \(A_{1,2,2} \lesssim \|g\|_{L^2}^2 \|h\|_{L^{2,γ/2}}^2\).

**Case 1:** \(|v - v_u| \leq 1\). By (5.2) and (5.3), we have
\[ B \lesssim \int b^ε (cos θ) θ^4 |v - v'_u|^{γ+4} (μ^{1/4})'_* (v)^{2l-2} (v'_u)^{2l-2} dσ. \]

Since \(|v - v_u| \leq 1\), there holds \(|v - v'_u| \leq 1\), \(|v - v'_u|^{γ+4} \leq 1\) and \(\langle v \rangle \sim \langle v'_u \rangle\), thus \(\langle v \rangle^{2l-2} \lesssim \langle v \rangle^{2l+γ} \langle v'_u \rangle^{-2-γ}\), which implies
\[ B \lesssim \int b^ε (cos θ) θ^4 (μ^{1/4})'_* \langle v \rangle^{2l+γ} \langle v'_u \rangle^{2l-4-γ} dσ \]
\[ \lesssim \int b^ε (cos θ) \theta^4 \langle v \rangle^{2l+γ} dσ \lesssim \langle v \rangle^{2l+γ}. \]
Case 2: $|v - v_*| \geq 1$. By (5.2) and (5.3), we have $B \lesssim \int b^c(\cos \theta)\theta^2 |v - v_*|^\gamma (\mu^{\frac{1}{2}})^\gamma (v_*)^2\|d\sigma$. Since $|v - v_*| \geq 1$, there holds $|v - v_*|^\gamma \sim (v - v_*)^\gamma \lesssim \langle v \rangle^\gamma (v_*)^\gamma$, which implies

$$B \lesssim \int b^c(\cos \theta)\theta^2 (\mu^{\frac{1}{2}})^\gamma (v_*)^2 |v - v_*|^\gamma |d\sigma$$

$$\lesssim \int b^c(\cos \theta)\theta^2 (v)^2 |d\sigma | \lesssim \langle v \rangle^{2+\gamma}.$$  

We get (5.4) by patching together the two cases.

We then go to estimate $\mathcal{A}_{1,2.1}$. Thanks to $(W_t - W_t')^2 \lesssim \min\{\theta^2 |v - v_*|^2 (v_*)^{2l-2}, \theta^2 (v)^2 (v_*)^{2l}\}$, and $((\mu^{\frac{1}{2}})^\gamma - \mu^{\frac{1}{2}})^2 \lesssim \min\{\theta^2 |v - v_*|^2, 1\}$, similar to (5.4), we can prove

$$\int B^{c,\gamma,\eta}|\mu|^{\frac{1}{2}} (\mu^{\frac{1}{2}})^\gamma (v_*)^2 (W_t - W_t')^2 |d\sigma | \lesssim \langle v \rangle^{2+\gamma} |\mu|^{\frac{1}{2}}.$$  

(5.5)

Plugging (5.5) into $\mathcal{A}_{1,2.1}$, we get $\mathcal{A}_{1,2.1} \lesssim |\mu|^{\frac{1}{2}} |g|^2 |L^2| |h|^{2} |L^2_{1+\gamma/2}$. Patching together the upper bound estimates of $\mathcal{A}_{1,2.1}$ and $\mathcal{A}_{1,2.2}$, we arrive at $\mathcal{A}_{1.2} \lesssim |g|^2 |L^2| |h|^{2} |L^2_{1+\gamma/2}$. Patching together the estimates of $\mathcal{A}_{1.1}$ and $\mathcal{A}_{1.2}$, we conclude $|A_1| \lesssim |g| |L^2| |h|^{2} |L^2_{1+\gamma/2}$.

In the $|\beta| > 0$ case, by recalling (4.8) and (4.11), changes only happen in $\mathcal{A}_{1.1}$, in which $((\mu^{\frac{1}{2}})^\gamma - \mu^{\frac{1}{2}})^2$ is replaced with $(P_{\beta} \mu^{\frac{1}{2}})^\gamma - P_{\beta} \mu^{\frac{1}{2}})^2$. Then Proposition 2.2 also holds since it only utilizes the condition (4.12).

Step 2: Estimate of $\mathcal{A}_2$. By Cauchy-Schwartz inequality, we have

$$|A_2| \leq \left( \int B^{c,\gamma,\eta}|\mu|^{1}\mu^{\frac{1}{2}} ((\mu^{\frac{1}{2}})^\gamma - \mu^{\frac{1}{2}})^2 g_\ast (f)^2 |d\sigma | dv_\ast dv \right)^{\frac{1}{2}}$$

$$\times \left( \int B^{c,\gamma,\eta}|\mu|^{1}\mu^{\frac{1}{2}} (W_t - W_t')^2 g_\ast h^2 |d\sigma | dv_\ast dv \right)^{\frac{1}{2}} := (A_{2.1})^{\frac{1}{2}} (A_{2.2})^{\frac{1}{2}}.$$  

Estimate of $A_{2.1}$. By the change of variable $v \rightarrow v'$, we have

$$A_{2.1} \lesssim \int b^c(\cos \theta)|v' - v_*|^{\gamma} \mu^{\frac{1}{2}} ((\mu^{\frac{1}{2}})^\gamma - \mu^{\frac{1}{2}})^2 g_\ast (f)^2 |d\sigma | dv_\ast dv'.$$

By Lemma 2.5, one has $\int b^c(\cos \theta)((\mu^{\frac{1}{2}})^\gamma - \mu^{\frac{1}{2}})^2 |d\sigma | \lesssim |v' - v_*|^2 |v' - v_*|_2 + (W^\epsilon)^2 |v' - v_*|_2 |v' - v_*|_2$, which gives

$$A_{2.1} \lesssim \int |v' - v_*|^{\gamma} \mu^{\frac{1}{2}} ((\mu^{\frac{1}{2}})^\gamma - \mu^{\frac{1}{2}})^2 g_\ast (f)^2 |dv_\ast dv'|$$

$$\lesssim |\mu|^{\frac{1}{2}} g_\ast |W^\epsilon|^{2} |L^2_{1+\gamma/2}.$$
Estimate of $A_{2.2}$. Using $(W_l - W_{l'})^2 \lesssim \theta^2 |v - v_*|^2 (v)^{2l-2} (v_*)^{2l-2}$ to get
\[
A_{2.2} \lesssim \int b^e (\cos \theta)^2 |v - v_*|^{\gamma + 2} (v)^{2l-2} (v_*)^{2l-2} \frac{1}{\mu_*} g_* h^2 d\sigma dv_* dv
\]
\[
\lesssim \int |v - v_*|^{\gamma + 2} (v)^{2l-2} (v_*)^{2l-2} \frac{1}{\mu_*} g_* h^2 dv_* dv.
\]
Noting that
\[
\int |v - v_*|^{\gamma + 2} (v)^{2l-2} \frac{1}{\mu_*} g_* dv_*
\]
\[
\leq \left( \int |v - v_*|^{2\gamma + 4} \frac{1}{\mu_*} dv_* \right)^{\frac{1}{2}} \left( \int (v)^{4l-4} \frac{1}{\mu_*} g_*^2 dv_* \right)^{\frac{1}{2}}
\]
\[
\lesssim (v)^{\gamma + 2} |\mu_*^{\frac{1}{4}} g|_{L^2},
\]
which gives $A_{2.2} \lesssim |\mu_*^{\frac{1}{4}} g|_{L^2} |h|_{L^2_{i+\gamma/2}}^2$. Putting together the estimates of $A_{2.1}$ and $A_{2.2}$, we arrive at
\[
|A_2| \lesssim |\mu_*^{\frac{1}{4}} g|_{L^2} |h|_{L^2_{i+\gamma/2}} |W^e f|_{L^2_{i+\gamma/2}}.
\]

In the $|\beta| > 0$ case, by recalling (4.8) and (4.11), $\mu_*^{\frac{1}{4}} ((\mu_1^{\frac{1}{4}})_a - \mu_2^{\frac{1}{4}})^2$ is replaced by $(P_\beta \mu_1^{\frac{1}{4}})_a ((\mu_1^{\frac{1}{4}})_a - \mu_2^{\frac{1}{4}})^2$ or $\mu_*^{\frac{1}{4}} ((P_\beta \mu_1^{\frac{1}{4}})_a - (P_\beta \mu_1^{\frac{1}{4}})_a)^2$. The above arguments also work. In the former, just use $|(P_\beta \mu_1^{\frac{1}{4}})_a| \lesssim \mu_*^{\frac{1}{4}}$. In the latter, $((P_\beta \mu_1^{\frac{1}{4}})_a - (P_\beta \mu_1^{\frac{1}{4}})_a)^2$ enjoys the condition (4.12).

The proposition follows the estimates of $A_1$ and $A_2$. □

5.3. Applications of Previous Results

We first have

**Theorem 5.1.** Let $0 < \eta \leq 1$, $l \geq 2$. It holds that
\[
|\langle \Gamma^{e, \gamma, \eta} (g, W_l h; \beta) - W_l \Gamma^{e, \gamma, \eta} (g, h; \beta), f \rangle| \lesssim \eta^{-3} C_l |g|_{L^2} |h|_{L^2_{i+\gamma/2}} |f|_{L^2_{i+\gamma/2}}. \tag{5.6}
\]

Let $0 < \delta \leq 1/2$, $(s_3, s_4) = (\frac{1}{2} + \delta, 0)$ or $(s_3, s_4) = (0, \frac{1}{2} + \delta)$, there holds
\[
|\langle \Gamma^{e, \gamma} (g, W_l h; \beta) - W_l \Gamma^{e, \gamma} (g, h; \beta), f \rangle| \lesssim \eta^{-3} C_l |g|_{L^2} |h|_{L^2_{i+\gamma/2}} |f|_{L^2_{i+\gamma/2}} \tag{5.7}
\]
\[
+ C_l \delta^{-\frac{1}{2}} \eta^{\frac{1}{2}} |\mu_1^{\frac{1}{4}} g|_{H^{s_3}} |W^e (D) \mu_1^{\frac{1}{4}} h|_{H^{s_4}} |W^e (D) \mu_1^{\frac{1}{4}} f|_{L^2}.
\]

**Proof.** By Proposition 5.1 and Proposition 5.3, we have (5.6). By Proposition 5.1 and Proposition 5.2 and Proposition 5.3, we get (5.7). □
Theorem 4.4 and Theorem 5.1 together give the following upper bound estimate with weight.

**Corollary 5.1.** Let $0 < \delta \leq 1/2$, $(s_3, s_4) = \left(\frac{1}{2} + \delta, 0\right)$ or $(s_3, s_4) = (0, \frac{1}{2} + \delta)$, it holds that

\[
|\langle W_1 \Gamma^{e,\gamma}(g, h; \beta), f \rangle| \lesssim |g|_{L^2} |h|_{L_{\gamma, \gamma/2}^{2}} |f|_{L_{\gamma, \gamma/2}^{2}} + C_l |g|_{L^2} |h|_{L_{\gamma, \gamma/2}^{2}} |f|_{L_{\gamma, \gamma/2}^{2}}
\]

\[
+ C_l \delta^{-\frac{1}{2}} |\mu|_{H^3} g |\mu|_{H^{1+\delta}} h |f|_{L_{\gamma, \gamma/2}^{2}}.
\]

(5.8)

We remark that Corollary 5.1 also holds for the Landau case $\varepsilon = 0$.

As an application of Theorem 5.1, we have

**Corollary 5.2.** Let $0 < \eta \leq 1$, $l \geq 2$, it holds that

\[
\langle [L^{e,\gamma, \beta_0, \beta_1}_1 g, W_l f] \rangle \lesssim \eta^{-3} C_l |g|_{L_{\gamma_1+\eta/2}^{2}} |f|_{L_{\gamma_1+\eta/2}^{2}} + C_l \eta^{\frac{1}{2}} |W^{e}(D)\mu^{\frac{1}{2}} g|_{L^2} |f|_{L_{\gamma_1+\eta/2}^{2}}.
\]

(5.9)

**Proof.** Recall from (4.49), $L^{e,\gamma, \beta_0, \beta_1}_1 g = -\Gamma^{e,\gamma}(\partial_{\beta_1} \mu^{\frac{1}{2}}; g; \beta_0)$. Taking $\delta = 1/2$, $s_3 = 1$, $s_4 = 0$ in (5.7), we get

\[
\langle [L^{e,\gamma, \beta_0, \beta_1}_1, W_l] g, W_l f \rangle \lesssim \eta^{-3} C_l |g|_{L_{\gamma_1+\eta/2}^{2}} |f|_{L_{\gamma_1+\eta/2}^{2}} + C_l \eta^{\frac{1}{2}} |W^{e}(D)\mu^{\frac{1}{2}} g|_{L^2} |f|_{L_{\gamma_1+\eta/2}^{2}}.
\]

Recall from (4.49), $L^{e,\gamma, \beta_0, \beta_1}_2 g := -\Gamma^{e,\gamma}(g, \partial_{\beta_1} \mu^{\frac{1}{2}}; \beta_0)$. Taking $\delta = 1/2$, $s_3 = 0$, $s_4 = 1$ in (5.7), we get

\[
\langle [L^{e,\gamma, \beta_0, \beta_1}_2, W_l] g, W_l f \rangle \lesssim \eta^{-3} C_l |g|_{L^2} |f|_{L^2} + C_l \eta^{\frac{1}{2}} |W^{e}(D)\mu^{\frac{1}{2}} g|_{L^2} |f|_{L^2}.
\]

Patching together the above two estimates, recalling (4.48), thanks to the fact $|W^{e}(D)\mu^{\frac{1}{2}} W_l f|_{L^2} \lesssim C_l |f|_{L_{\gamma_1+\eta/2}^{2}}$, we finish the proof. \quad \Box

When $\gamma = -3$, recall the notation $L^{e, \beta_0, \beta_1} = L^{e, -3, \beta_0, \beta_1}$. As a special case of Corollary 5.2, we have

**Corollary 5.3.** Let $0 < \eta \leq 1$, $l \geq 2$, it holds that

\[
\langle [L^{e, \beta_0, \beta_1}, W_l] g, W_l f \rangle \lesssim \eta^{-6} C_l |g|_{L_{\gamma_1+\eta/2}^{2}} |f|_{L_{\gamma_1+\eta/2}^{2}} + \eta \frac{5}{2} C_l |g|_{L_{\gamma_1+\eta/2}^{2}} |f|_{L_{\gamma_1+\eta/2}^{2}}.
\]

6. Energy Estimate and Asymptotic Formula

In this section, we will give the proof to Theorem 1.1. We divide the proof into three subsections. The first subsection is devoted to the a priori estimates for the linear equation (6.1). In subsection 4.2, we consider the global well-posedness (1.28) and regularity propagation (1.29) of the linearized Boltzmann equation (1.20). In subsection 4.3, we derive the global asymptotic formula (1.30). Throughout this section, we set $\gamma = -3$. 
Given $g$, we consider the linear equation
\[ \partial_t f + v \cdot \nabla_x f + \mathcal{L}_f f = g. \quad (6.1) \]

Let us set up some notations which will be used throughout this section.

- We set $f_1 := \mathbb{P} f$ and $f_2 := f - \mathbb{P} f$, where $\mathbb{P}$ is the projection operator defined in (1.33). By (1.33) and (1.34),
\[ f_1(t, x, v) = (a(t, x) + b(t, x) \cdot v + c(t, x)|v|^2)\mu_1^{\frac{1}{2}}, \]
which solves
\[ \partial_t f_1 + v \cdot \nabla_x f_1 = -\partial_t f_2 - v \cdot \nabla_x f_2 - \mathcal{L}_f f_2 + g. \quad (6.3) \]

- The column vector $e = \{e_j\}_{1 \leq j \leq 13}$ is defined explicitly by
\[ e_1 = \mu_1^{\frac{1}{2}}, e_2 = v_1 \mu_1^{\frac{1}{2}}, e_3 = v_2 \mu_1^{\frac{1}{2}}, e_4 = v_3 \mu_1^{\frac{1}{2}}, \]
\[ e_5 = v_1^2 \mu_1^{\frac{1}{2}}, e_6 = v_2^2 \mu_1^{\frac{1}{2}}, e_7 = v_3^2 \mu_1^{\frac{1}{2}}, \]
\[ e_8 = v_1 v_2 \mu_1^{\frac{1}{2}}, e_9 = v_2 v_3 \mu_1^{\frac{1}{2}}, e_{10} = v_3 v_1 \mu_1^{\frac{1}{2}}, \]
\[ e_{11} = |v|^2 v_1 \mu_1^{\frac{1}{2}}, e_{12} = |v|^2 v_2 \mu_1^{\frac{1}{2}}, e_{13} = |v|^2 v_3 \mu_1^{\frac{1}{2}}. \]

- Let $A = (a_{ij})_{1 \leq i \leq 13, 1 \leq j \leq 13}$ be the matrix defined by $a_{ij} = \langle e_i, e_j \rangle$ and $y$ be the 13-dimensional column vector defined by
\[ y = (\partial_t a, \partial_t b_i + \partial_i a)_{1 \leq i \leq 3}, (\partial_t c + \partial_i b_1)_{1 \leq i \leq 3}, \]
\[ (\partial_t b_j + \partial_j b_1)_{1 \leq i < j \leq 3}, (\partial_i c)_{1 \leq i \leq 3} \) \(T\). \]

Set $z = (z_i)_{1 \leq i \leq 13} = (-\partial_t f_2 - v \cdot \nabla_x f_2 - \mathcal{L}_f f_2 + g, e)$. By (6.2) and taking inner product between (6.3) and $e$ in the space $L^2(\mathbb{R}^3)$ for variable $v$, one has $Ay = z$, which implies $y = A^{-1}z$. Denote
\[ \tilde{f} = (\tilde{f}^{(0)}, \{\tilde{f}_i^{(1)}\}_{1 \leq i \leq 3}, \{\tilde{f}_i^{(2)}\}_{1 \leq i \leq 3}, \]
\[ \{\tilde{f}_{ij}^{(2)}\}_{1 \leq i < j \leq 3}, \{\tilde{f}_i^{(3)}\}_{1 \leq i \leq 3} \) \(T\) = A^{-1}(f_2, e). \]

Then
\[ y = -\partial_t \tilde{f} + A^{-1}(-v \cdot \nabla_x f_2 - \mathcal{L}_f f_2 + g, e). \quad (6.4) \]

- Define the temporal energy functional $\mathcal{I}^N(f)$ as (see [9])
\[ \mathcal{I}^N(f) := \sum_{|\alpha| \leq N-1} \sum_{i=1}^{3} (\mathcal{I}^a_{\alpha,i}(f) + \mathcal{I}^{b}_{\alpha,i}(f) + \mathcal{I}^{c}_{\alpha,i}(f) + \mathcal{I}^{ab}_{\alpha,i}(f)), \]
\[ \mathcal{I}^{a}_{\alpha,i}(f) := (\partial^\alpha f_i^{(1)}), \partial_t \partial^\alpha a \rangle_x, \quad \mathcal{I}^{c}_{\alpha,i}(f) := (\partial^\alpha f_i^{(3)}), \partial_t \partial^\alpha c \rangle_x, \]
\[ \partial^\alpha := \partial_t^\alpha, \partial^\alpha_x := \partial_t^\alpha \partial_x^\alpha. \]
\[ T^{ab}_{\alpha,i}(f) := \langle \partial_i \partial^\alpha a, \partial^\alpha b_i \rangle_x, \quad T^{b}_{\alpha,i}(f) := - \sum_{j \neq i} \langle \partial^\alpha f_j^{(2)}, \partial_i \partial^\alpha b_i \rangle_x \]

\[ + \sum_{j \neq i} \langle \partial^\alpha f_j^{(2)}, \partial_j \partial^\alpha b_i \rangle_x + 2 \langle \partial^\alpha f_i^{(2)}, \partial_i \partial^\alpha b_i \rangle_x. \]

Note that there is some universal constant \( C_1 \) such that

\[ |T^N(f)| \leq C_1 \| f \|_{H^N_x L^2}. \tag{6.5} \]

The above set-up is standard for the near Maxwellian framework. The temporal energy functional \( T^N(f) \) appears naturally in deriving the dissipation of \((a, b, c)\). Based on (6.4), one can study the evolution of the macroscopic quantities \((a, b, c)\) in terms of the microscopic part \( f_2 \). That is, we have

**Lemma 6.1.** Let \( N \geq 1 \). There exists are two universal constants \( C, c_0 > 0 \) such that

\[
\frac{d}{dt} T^N(f) + c_0 |(a, b, c)|^2_{H^N} \leq C \| f_2 \|^2_{H^N_x L^2_{t,\gamma/2}} + \sum_{|\alpha| \leq N-1} \sum_{|\beta| = 1}^{13} \int_{T^3} \left| \langle \partial^\alpha g, e_j \rangle \right|^2 dx. \tag{6.6}
\]

We refer readers to [9,12] for more details of Lemma 6.1, and for brevity we omit the proof.

Before giving the estimates for (6.1), we prepare some technical lemmas to deal with the inner products that will appear in energy estimates. The dissipation of the microscopic part \( f_2 \) is produced by the linearized Boltzmann operator.

**Lemma 6.2.** Let \( 0 < \epsilon \leq \epsilon_0, 0 \leq \eta \leq \eta_0. \) Let \( \alpha, \beta \in \mathbb{N}^3 \) with \(|\alpha| + |\beta| \leq N \) and \( q \geq 0 \), then

\[
(\mathcal{L}^{\epsilon,\eta} W_q \partial^\alpha f, W_q \partial^\alpha f) \geq \frac{7}{8} \lambda_0 \| W_q \partial^\alpha f_2 \|_{L^2_{t,\gamma/2}}^2 - C_{q,N} (\| \partial^\alpha f_2 \|_{L^2_{t,\gamma/2}} + |\partial^\alpha (a, b, c)|_{L^2_x}).
\]

**Proof.** By Theorem 3.1, we have

\[
(\mathcal{L}^{\epsilon,\eta} W_q \partial^\alpha f, W_q \partial^\alpha f) \geq \lambda_0 (\| - \mathbb{P} \| W_q \partial^\alpha f \|_{L^2_{t,\gamma/2}}^2.
\]

By the macro–micro decomposition \( f = f_1 + f_2 \), we deduce that

\[
(\mathcal{L}^{\epsilon,\eta} W_q \partial^\alpha f, W_q \partial^\alpha f) \geq \lambda_0 (\| - \mathbb{P} \| W_q \partial^\alpha (f_1 + f_2) \|_{L^2_{t,\gamma/2}}^2
\]

\[
\geq \frac{7}{8} \lambda_0 \| W_q \partial^\alpha f_2 \|_{L^2_{t,\gamma/2}}^2 - C_{q,N} (\| \partial^\alpha f_2 \|_{L^2_{t,\gamma/2}} + |\partial^\alpha (a, b, c)|_{L^2_x}),
\]

where we use (2.21) to take out \( W_q \partial^\alpha f_2 \) as the leading term and integration by parts formula to deal with the operator \( \partial_\beta \). In addition, all polynomial weights can be controlled by the factor \( \mu^{1/2} \) in \( f_1 \). \( \square \)
Lemma 6.3. Let $|\alpha| + |\beta| \leq N$, $\beta_0 + \beta_1 + \beta_2 = \beta$, $q \geq 2$, then for any $0 < \delta \leq 1$, we have

$$
|\{ [W_q, \mathcal{L}^e, \beta_0, \beta_1] \partial_{\beta_2} f, W_q \partial_{\beta} f \}| \leq \delta (\| \partial_{\beta_2} f \|_{L^2_{x,q+y/2}}^2 + \| \partial_{\beta_2} f \|_{L^2_{x,q+y/2}}^2 ) + C_{\delta,q} \| \partial_{\beta_2} f \|_{L^2_{x,q+y/2}}^2 + C_{\delta,q,N} |\partial^a (a, b, c)|_{L^2_{x,y}}^2.
$$

Proof. By Corollary 5.3, we get for any $0 < \eta < 1$,

$$
|\{ [W_q, \mathcal{L}^e, \beta_0, \beta_1] \partial_{\beta_2} f, W_q \partial_{\beta} f \}| \leq \eta^{-6} C_q \| \partial_{\beta_2} f \|_{L^2_{x,q+y/2}} \| \partial_{\beta} f \|_{L^2_{x,y/2}}^2 + \eta^{-1/2} C_q \| \partial_{\beta_2} f \|_{L^2_{x,y/2}}^2 + \eta^{-1/2} C_q \| \partial_{\beta} f \|_{L^2_{x,y/2}}^2
$$

where we set $\delta = \eta^{1/2} C_q$, and the constant $C_q$ may change across different lines. From which together with the decomposition $f = f_1 + f_2 = \mathbb{P} f + f_2$, we get the lemma. \(\Box\)

Lemma 6.4. Let $|\alpha| + |\beta| \leq N$, $|\beta| \geq 1$, $q \geq 2$ or $q = 0$, then for any $0 < \delta \leq 1$, we have

$$
|\{ [W_q[\partial_{\beta}, \mathcal{L}^e] \partial^a f, W_q \partial_{\beta} f \}| \leq (\delta + C_{N, \varepsilon}) \| \partial_{\beta} f \|_{L^2_{x,q+y/2}}^2 + C_{\delta,q,N} |\partial^a (a, b, c)|_{L^2_{x,y}}^2 + C_{\delta,q,N} \sum_{\beta_2 < \beta} (\| \partial_{\beta_2} f \|_{L^2_{x,q+y/2}}^2 + \| \partial_{\beta_2} f \|_{L^2_{x,y/2}}^2 )^2.
$$

Proof. Recalling $\mathcal{L}^e g = -\Gamma^e (\mu_{1/2}, g) - \Gamma^e (g, \mu_{1/2})$, (4.9), (4.8), and (4.48), we have

$$
\partial_{\beta} \mathcal{L}^e g = \mathcal{L}^e \partial_{\beta} g - \sum_{\beta_0 + \beta_1 + \beta_2 = \beta, \beta_2 < \beta} C_{\beta}^{\beta_0, \beta_1, \beta_2} [\Gamma^e (\partial_{\beta_1} \mu_{1/2}, \partial_{\beta_2} g ; \beta_0) + \Gamma^e (\partial_{\beta_1} g, \partial_{\beta_2} \mu_{1/2} ; \beta_0)]
$$

$$
= \mathcal{L}^e \partial_{\beta} g + \sum_{\beta_0 + \beta_1 + \beta_2 = \beta, \beta_2 < \beta} C_{\beta}^{\beta_0, \beta_1, \beta_2} \mathcal{L}^e \partial_{\beta_1} g.
$$

In this way, we have $[\partial_{\beta}, \mathcal{L}^e] = \sum_{\beta_2 < \beta} C_{\beta}^{\beta_0, \beta_1, \beta_2} \mathcal{L}^e \partial_{\beta_1} g$, and thus,

$$
W_q [\partial_{\beta}, \mathcal{L}^e] \partial^a f = W_q \sum_{\beta_2 < \beta} C_{\beta}^{\beta_0, \beta_1, \beta_2} \mathcal{L}^e \partial_{\beta_1} g \partial_{\beta_2} f
$$

$$
= \sum_{\beta_2 < \beta} C_{\beta}^{\beta_0, \beta_1, \beta_2} \mathcal{L}^e \partial_{\beta_1} g W_q \partial_{\beta_2} f + \sum_{\beta_2 < \beta} C_{\beta}^{\beta_0, \beta_1, \beta_2} [W_q, \mathcal{L}^e \partial_{\beta_1} g] \partial_{\beta_2} f.
$$

By upper bound estimate in Lemma 4.4, we get
\[ \| (L^{e, \beta_0, \beta_1} W_q \partial^\alpha_{\beta_2} f, W_q \partial^\alpha_{\beta} f) \| \lesssim \delta^{-6} \| \partial^\alpha_{\beta_2} f \|_{L^2_{x, \tau, \gamma+y/2}} \| \partial^\alpha_{\beta} f \|_{L^2_{x, \tau, \gamma+y/2}} \\
+ (\delta^2 + \varepsilon) \| \mu \partial^\alpha_{\beta_2} f \|_{L^2_{x, \tau}} \| \partial^\alpha_{\beta} f \|_{L^2_{x, \tau, \gamma+y/2}} \\
\lesssim (\delta + \varepsilon) \| \partial^\alpha_{\beta_2} f \|_{L^2_{x, \tau, \gamma+y/2}}^2 + C_{\delta, q} N |\partial^\alpha (a, b, c)|_{L^2_{x}}^2 \\
+ C_{\delta, q} \| \partial^\alpha_{\beta_2} f \|_{L^2_{x, \tau, \gamma+y/2}}^2 + C_{q} \| \partial^\alpha_{\beta} f \|_{L^2_{x, \tau}}^2 \]

where we use \( f = f_1 + f_2 \) and the definition of \( a, b, c \).

If \( q \geq 2 \), we use Lemma 6.3 to deal with \( ([W_q, L^{e, \beta_0, \beta_1}] \partial^\alpha_{\beta_2} f, W_q \partial^\alpha_{\beta} f) \). If \( q = 0 \), the commutator \([W_q, L^{e, \beta_0, \beta_1}] = 0 \). Taking sum over \( \beta_2 < \beta \), we get the result. \( \square \)

For non-negative integers \( n, m \), we recall that

\[ \| f \|_{H^N_{x, \tau}}^2 = \sum_{|\alpha| \leq n, |\beta| = m} \| \partial^\alpha_{\beta} f \|_{L^2_{x}}^2, \quad \| f \|_{H^m_{x, \tau}}^2 = \sum_{|\alpha| \leq n, |\beta| = m} \| \partial^\alpha_{\beta} f \|_{L^2_{x}}^2. \]

Let \( N \geq 4, l \geq 3N + 2 \). For some universal constants \( M, L, K_j, 0 \leq j \leq N \) (which could depend on \( N, l \) and will be explicitly determined later), we define

\[ \Xi^{N,l}(f) = M T^N(f) + L \| f \|_{H^N_{x, \tau}}^2 + \sum_{j=0}^{N} K_j \| f \|_{H^{N-j}_{x, \tau}}^2 H_{l+j}^j, \quad (6.7) \]

\[ D_{\varepsilon}^{N,l}(f) = c_0 M |(a, b, c)|_{H^N_{x}}^2 + \lambda_0 L \| f \|_{H^{N}_{x, \tau}}^2 + \lambda_0 \sum_{j=0}^{N} K_j \| f \|_{H^{N-j}_{x, \tau}}^2 H_{l+j}^j. \quad (6.8) \]

Here \( \lambda_0 \) is the constant in (3.12) and \( c_0 \) is the constant in (6.6).

Now we are in a position to prove the following \textit{a priori} estimate for (6.1):

**Proposition 6.1.** Let \( N \geq 4, l \geq 3N + 2 \). Suppose \( f \) is a suitable solution to (6.1), then there exist a constant \( \varepsilon_1 \) verifying \( 0 < \varepsilon_1 \leq \varepsilon_0 \), such that for any \( 0 \leq \eta \leq \eta_0, 0 < \varepsilon \leq \varepsilon_1 \) or \( \eta = 0 = \varepsilon \), it holds that

\[ \frac{d}{dt} \Xi^{N,l}(f) + \frac{1}{4} D_{\varepsilon}^{N,l}(f) \leq MC \sum_{|\alpha| \leq N-1} \sum_{j=1}^{13} \int |(\partial^\alpha g, e_j)|^2 dx \\
+ 2L \sum_{|\alpha| \leq N} (\partial^\alpha g - L^{e, \gamma} \partial^\alpha \bar{f}, \partial^\alpha f) \\
+ \sum_{j=0}^{N} 2K_j \sum_{|\alpha| \leq N-j, |\beta| = j} (W_{l+j} \partial^\alpha_{\beta} g - L^{e, \gamma} W_{l+j} \partial^\alpha_{\beta} \bar{f}, W_{l+j} \partial^\alpha_{\beta} f). (6.9) \]

The constant \( \varepsilon_1 > 0 \) could depend on \( N, l \). Here \( \eta_0, \varepsilon_0 \) are the universal constants in Theorem 3.1 and \( C \) is the constant in (6.6). Moreover, \( M \sim L \).
**Proof.** We divide the proof into three steps to construct the energy functional $\Xi_{N,L}(f)$ in (6.7).

**Step 1: Propagation of $\|f\|_{H^N L^2}^2$.** Applying $\partial^\alpha$ to equation (6.1), taking inner product with $\partial^\alpha f$, taking sum over $|\alpha| \leq N$, we have

$$\frac{1}{2} \frac{d}{dt} \|f\|_{H^N L^2}^2 + \sum_{|\alpha| \leq N} (L^\varepsilon \partial^\alpha f, \partial^\alpha f) = \sum_{|\alpha| \leq N} (\partial^\alpha g, \partial^\alpha f). \quad (6.10)$$

Split $L^\varepsilon \gamma = L^\varepsilon \gamma \eta + L^\varepsilon \gamma$. Thanks to $\partial^\alpha f_2 = (\partial^\alpha f)_2$, by Theorem 3.1, for $\eta \leq \eta_0, \varepsilon \leq \varepsilon_0$, we have

$$(L^\varepsilon \gamma \partial^\alpha f, \partial^\alpha f) \geq \lambda_0 \|\partial^\alpha f_2\|_{L^2_{x,y/2}}^2.$$ 

Plugging which into (6.10), we have

$$\frac{1}{2} \frac{d}{dt} \|f\|_{H^N L^2}^2 + \lambda_0 \|f_2\|_{H^N L^2_{x,y/2}}^2 \leq \sum_{|\alpha| \leq N} (\partial^\alpha g - L^\varepsilon \gamma \partial^\alpha f, \partial^\alpha f). \quad (6.11)$$

Multiplying (6.11) by a large constant $2M_1$ and adding it to (6.6), we get

$$\frac{d}{dt} (M_1 \|f\|_{H^N L^2}^2 + \mathcal{I}^N (f)) + (c_0 |(a, b, c)|^2 H^N + M_1 \lambda_0 \|f_2\|_{H^N L^2_{x,y/2}}^2) \quad (6.12)$$

$$\leq 2M_1 \sum_{|\alpha| \leq N} (\partial^\alpha g - L^\varepsilon \gamma \partial^\alpha f, \partial^\alpha f) + C \sum_{|\alpha| \leq N-1} \int \int |\langle \partial^\alpha g, e_j \rangle|^2 dx.$$ 

Here $M_1$ is large enough such that $M_1 \geq 2C_1$ and $M_1 \lambda_0 \geq C$ to insure $M_1 \|f\|_{H^N L^2}^2 + \mathcal{I}^N (f) \sim \|f\|_{H^N L^2}^2$ by (6.5) and cancel the term $C \|f_2\|_{H^N L^2_{x,y/2}}^2$ on the right hand side of (6.6).

**Step 2: Propagation of $\|f\|_{H^N L^2}^2$.** Applying $W_i \partial^\alpha$ to equation (6.1), taking inner product with $W_i \partial^\alpha f$, taking sum over $|\alpha| \leq N$, we have

$$\frac{1}{2} \frac{d}{dt} \|f\|_{H^N L^2}^2 + \sum_{|\alpha| \leq N} (W_i L^\varepsilon \partial^\alpha f, W_i \partial^\alpha f) = \sum_{|\alpha| \leq N} (W_i \partial^\alpha g, W_i \partial^\alpha f).$$

Using commutator to transfer weight and splitting $L^\varepsilon \gamma = L^\varepsilon \gamma \eta + L^\varepsilon \gamma$, we get

$$W_i L^\varepsilon \partial^\alpha f = L^\varepsilon W_i \partial^\alpha f + [W_i, L^\varepsilon] \partial^\alpha f$$

$$= L^\varepsilon \gamma \eta W_i \partial^\alpha f + L^\varepsilon \gamma W_i \partial^\alpha f + [W_i, L^\varepsilon] \partial^\alpha f.$$ 

By Lemma 6.2, we get

$$(L^\varepsilon \gamma \eta W_i \partial^\alpha f, W_i \partial^\alpha f) \geq \frac{7}{8} \lambda_0 \|\partial^\alpha f_2\|_{L^2_{x,y/2}}^2 - C_1 \lambda (\|\partial^\alpha f_2\|_{L^2_{x,y/2}}^2 + |\partial^\alpha (a, b, c)|_{L^2_x}^2).$$
Thanks to Lemma 6.3, we have

$$|\langle W_l, \mathcal{L}^\varepsilon \partial^\alpha f, W_l \partial^\alpha f \rangle| \leq \delta \|\partial^\alpha f_2\|_{L_t^2 L_{t,1+y/2}^2}^2 + C_{\delta, l} \|\partial^\alpha f_{2}\|_{L_t^2 L_{t,1+y/2}^2}^2 + C_{\delta, l, N} \|\partial^\alpha (a, b, c)\|_{L_t^2}^2.$$  

By (2.4), (2.5) and interpolation, for any $\delta_2 > 0$, it holds that $|h|_{L^2_\theta} \lesssim (\delta_2^\frac{1}{2} + \varepsilon \frac{1}{2}) |W^\varepsilon h|_{L^2_\theta} + C(\delta_2, q)|h|_{L^2_\theta}$, which yields

$$\|\partial^\alpha f_2\|_{L_t^2 L_{t,1+y/2}^2}^2 \leq (\delta_2 + \varepsilon) \|\partial^\alpha f_2\|_{L_t^2 L_{t,1+y/2}^2}^2 + C(\delta_2, l) \|\partial^\alpha f_2\|_{L_t^2 L_{t,1+y/2}^2}^2.$$  

First taking $\delta = \lambda_0/8$, then taking $\delta_2$ such that $\delta_2 C_{\delta, l} = \lambda_0/8$, when $\varepsilon C_{\delta, l} \leq \lambda_0/8$, we get

$$\frac{d}{dt} \|f\|^2_{H_x^N L_t^2} + \frac{\lambda_0}{2} \|f_2\|^2_{H_x^N L_{t,1+y/2}^2} \leq C_{l, N} \|f_2\|^2_{H_x^N L_{t,1+y/2}^2} + C_{l, N} \|f_1\|^2_{H_x^N L_{t,1+y/2}^2}$$

$$+ 2 \sum_{|\alpha| \leq N} (W_l \partial^\alpha g - \mathcal{L}^\varepsilon \partial^\alpha f, W_l \partial^\alpha f).$$

(6.13)

There is a constant $C_l$ such that $\|f_1\|^2_{H_x^N L_{t,1+y/2}^2} \leq C_l \|f_1\|^2_{H_x^N L_{t,1+y/2}^2}$. We choose a constant $M_2$ large enough such that $c_0 M_2/4 \geq C_{l, N}, c_0 M_2/4 \geq C_l \lambda_0/2, M_2 M_1 \lambda_0/2 \geq C_{l, N}$. Multiplying (6.12) by the constant $M_2$ and adding the resulting inequality to (6.13), we get

$$\frac{d}{dt} \left( M_2 \mathcal{T}^N (f) + M_1 M_2 \|f\|^2_{H_x^N L_{t,1+y/2}^2} + \|f_1\|^2_{H_x^N L_{t,1+y/2}^2} \right) + \frac{1}{2} (M_2 c_0 \|f_2\|^2_{H_x^N L_{t,1+y/2}^2} + \lambda_0 \|f_1\|^2_{H_x^N L_{t,1+y/2}^2} + \lambda_0 \|f_2\|^2_{H_x^N L_{t,1+y/2}^2})$$

$$\leq M_2 C \sum_{|\alpha| \leq N-1} \sum_{j=1}^{13} \int |\langle \partial^\alpha g, e_j \rangle|^2 dx + 2 M_2 M_1 \sum_{|\alpha| \leq N} (\partial^\alpha g - \mathcal{L}^\varepsilon \partial^\alpha f, \partial^\alpha f),$$

$$+ 2 \sum_{|\alpha| \leq N} (W_l \partial^\alpha g - \mathcal{L}^\varepsilon \partial^\alpha f, W_l \partial^\alpha f).$$

(6.14)

**Step 3: Propagation of $\sum_{j=1}^{N} K_j \|f_{H_x^N L_{t,1+y/2}^2} \|_{L_t^2}^2$**. We shall use mathematical induction to prove that for any $0 \leq i \leq N$, there are some constants $M^i, L^i, K^i, 0 \leq j \leq i$, such that

$$\frac{d}{dt} \left( M^i \mathcal{T}^N (f) + L^i \|f\|^2_{H_x^N L_{t,1+y/2}^2} + \sum_{0 \leq j \leq i} K^i \|f\|^2_{H_x^{N-i} H_{t+j}^j} \right)$$

$$+ 2^{-i-N} (c_0 M^i \|f_2\|^2_{H_x^N L_{t,1+y/2}^2} + \lambda_0 L^i \|f_2\|^2_{H_x^N L_{t,1+y/2}^2})$$

(6.15)
By Cauchy-Schwartz inequality and using \(L^2\) splitting
Taking inner product with that our final goal (6.9) is actually (6.15) with (6.15) is also valid for
By Lemma 6.2, Lemma 6.3 and Lemma 6.4, we have

\[
\frac{1}{2} \frac{d}{dt} \| \partial_\beta^g f \|_{L^2_q L^2}^2 + \sum_{\beta_1 \leq \beta, |\beta_1| = 1} (W_q \partial_{\beta - \beta_1}^\alpha f, W_q \partial_{\beta_1}^\alpha f) + (W_q \partial_\beta^g, W_q \partial_\beta^g f) = (W_q \partial_\beta^g, W_q \partial_\beta^g f).
\]

Estimate of \((W_q \partial_{\beta - \beta_1}^\alpha f, W_q \partial_{\beta_1}^\alpha f)\). By Cauchy-Schwartz inequality and using \(f = f_1 + f_2\), we get

\[
| (W_q \partial_{\beta - \beta_1}^\alpha f, W_q \partial_{\beta_1}^\alpha f) | \leq \| \partial_{\beta - \beta_1}^\alpha f \|_{L^2_q L^2} \| \partial_{\beta_1}^\alpha f \|_{L^2_q L^2} \leq \| f_2 \|_{H^{N-k} L^{2-q+y/2}}^2 + \| f_2 \|_{H^{N-k+1} L^{2-q+y/2}}^2 + C_l(a, b, c)^2_{H^{N-k}}.
\]

Estimate of \((W_q \partial_\beta^g \mathcal{L}^g f, W_q \partial_\beta^g f)\). Using commutator to transfer weight and splitting \(\mathcal{L}^g = \mathcal{L}^{g-\eta} + \mathcal{L}^{g, \eta}\), we get

\[
W_q \partial_\beta^g \mathcal{L}^g f = \mathcal{L}^{g-\eta} W_q \partial_\beta^g f + \mathcal{L}^{g, \eta} W_q \partial_\beta^g f + [W_q, \mathcal{L}^g] \partial_\beta^g f + W_q [\partial_\beta, \mathcal{L}^g] \partial_\beta^g f.
\]

By Lemma 6.2, Lemma 6.3 and Lemma 6.4, we have

\[
(\mathcal{L}^{g-\eta} W_q \partial_\beta^g f + [W_q, \mathcal{L}^g] \partial_\beta^g f + W_q [\partial_\beta, \mathcal{L}^g] \partial_\beta^g f, W_q \partial_\beta^g f) \geq \left( \frac{7}{8} \lambda_0 - 3 \delta - C N \epsilon \right) \| W_q \partial_\beta^g f \|_{L^2_q L^2}^2 - C \delta, q \| \partial_\beta^g f \|_{L^2_q L^2}^2.
\]
\[-C_{\delta,q,N} |\partial^q(a, b, c)|^2_{L^2} - C_{\delta,q,N} \sum_{\rho_2 < \beta} (\|\partial_{\rho_2}^q f_2\|^2_{L^2_{*_{r,q+y/2}}} + \|\partial_{\beta}^q f_2\|^2_{L^2_{\partial_*^{q+y/2}}}).\]

Taking $\delta$ such that $3\delta = \lambda_0/16$, when $\epsilon$ is small such that $C_N \epsilon \leq \lambda_0/16$, we get

\[
(\mathcal{L}^{\epsilon,\gamma_0} W_q \partial_{\beta}^q g + [W_q, \mathcal{L}^\epsilon] \partial_{\beta}^q f + W_q [\partial_{\beta}, \mathcal{L}^\epsilon] \partial^q f, W_q \partial_{\beta}^q f) \geq (3\lambda_0/4) \|\partial_{\beta}^q f_2\|^2_{L^2_{*_{r,q+y/2}}}
\]

\[
-C_{q,N} \|f_2\|^2_{H^{N-k-1}_{*_{r,q+y/2}}} - C_{q,N} \|f_2\|^2_{H^{N-k-1}_{*_{r,q+y/2}}} - C_{q,N} |\partial^q(a, b, c)|^2_{L^2}.
\]

Plugging (6.18), (6.19) and (6.20) into (6.17), taking sum over $|\alpha| \leq N - (k + 1)$, $|\beta| = k + 1$, we have

\[
\frac{d}{dt} \|f\|^2_{H^{N-k-1}_{*_{r,q+y/2}}} + 3 \lambda_0 \|f_2\|^2_{H^{N-k-1}_{*_{r,q+y/2}}} + C(\delta', k) \|f_2\|^2_{H^{N-k-1}_{*_{r,q+y/2}}}.
\]

Taking $\delta'$ such that $\delta'C_{l,N} = \lambda_0/4$, when $\epsilon$ satisfies $\epsilon C_{l,N} \leq \lambda_0/4$, recalling $q = l + (k + 1)\gamma$, we get

\[
\frac{d}{dt} \|f\|^2_{H^{N-k-1}_{*_{r,q+y/2}}} + 3 \lambda_0 \|f_2\|^2_{H^{N-k-1}_{*_{r,q+y/2}}} + C(\delta', k) \|f_2\|^2_{H^{N-k-1}_{q+y/2}}.
\]

By Lemma 2.2, for any $0 < \delta' < 1$, we have

\[
\|f_2\|^2_{H^{N-k-1}_{*_{r,q+y/2}}} \leq (\delta' + \epsilon) \|f_2\|^2_{H^{N-k-1}_{*_{r,q+y/2}}} + C(\delta', k) \|f_2\|^2_{H^{N-k-1}_{q+y/2}}.
\]

For notational convenience, set $\mathcal{X}^k(f) := M^k \mathcal{I}^N(f) + L^k \|f\|^2_{H^k_{*_{r,y}}} + \sum_{0 \leq j \leq k} K_j^k \|f\|^2_{H^{N-j}_{*_{r,y}}} \text{ and }$

\[
\mathcal{Y}^k_\epsilon(f) := c_0 M^k |(a, b, c)|^2_{H^{N}_{*_{r,y}}} + \lambda_0 L^k \|f_2\|^2_{H^k_{*_{r,y}}} + \lambda_0 \sum_{j=0}^{k} K_j^k \|f_1\|^2_{H^{N-j}_{*_{r,y}}} \text{ and }$

\[
\mathcal{Y}^k_\epsilon(f) \geq \lambda_0 \sum_{j=0}^{k} K_j^k \|f_1\|^2_{H^{N-j}_{*_{r,y}}} + \|f_2\|^2_{H^k_{*_{r,y}}}.
\]

By our induction assumption, (6.15) is true when $i = k$, that is,

\[
\frac{d}{dt} \mathcal{X}^k(f) + 2^{1-k/N} \mathcal{Y}^k_\epsilon(f)
\]
Theorem 6.1. There exists a universal constant $\delta_0 > 0$ such that the following statement is valid. Let $T > 0$, $N \geq 4$, $l \geq 3N + 2$, there is a constant $\epsilon_0$ which may depend on $N$, $l$, such that if $0 \leq \epsilon \leq \epsilon_0$ and $f^\epsilon$ is a solution of the Cauchy problem (1.20) satisfying $\mu + \mu^\epsilon f^\epsilon(t) \geq 0$ and $\mathcal{E}^4.14(f^\epsilon(t)) \leq \delta_0$ for any $0 \leq t \leq T$, then for any $t \in [0, T]$, the solution $f^\epsilon$ verifies the following estimates.

\begin{align*}
\sum_{|\alpha| \leq N-1} \sum_{j=1}^{13} |\langle \partial^\alpha g, e_j \rangle|^2 dx &+ 2L_k^k \sum_{|\alpha| \leq N} (\partial^\alpha g - \mathcal{L}^\epsilon_{\eta}\partial^\alpha f, \partial^\alpha f) \\
+ \sum_{j=0}^{k} 2K_j^k \sum_{|\alpha| \leq N-j, |\beta| = j} (W_l+jY \partial^\beta g - \mathcal{L}^\epsilon_{\eta} W_l+jY \partial^\beta f, W_l+jY \partial^\beta f).
\end{align*}

(6.22) There is a constant $C_{l,N}$ such that $\|f_1\|^2_{H^{N-k-1}_x H^{k+1}_{x,q+y/2}} \leq C_{l,N} |(a, b, c)|^2_{H^N_x}$. We choose a constant $M_3$ large enough such that

\begin{align*}
M_3(1 - 2^{1/N})c_0 M^k/2 &\geq C_{l,N}, \\
M_3(1 - 2^{1/N})\lambda_0 \min_{0 \leq j \leq k} \{ K_j^k \} &\geq C_{l,N}.
\end{align*}

Multiplying (6.22) by the constant $M_3$, and adding the resulting inequality to (6.21), we get

\begin{align*}
\frac{d}{dt} (M_3 \lambda^k (f) + \| f \|^2_{H^{N-k-1}_x H^{k+1}_{x,q+y/2}}) &+ M_3 2^{-1/N} 2^{1-k/N} \mathcal{Y}^k (f) \\
&+ \lambda_0 \| f_1 \|^2_{H^{N-k-1}_x H^{k+1}_{x,q+y/2}} + \lambda_0 \| f_2 \|^2_{H^{N-k-1}_x H^{k+1}_{x,q+y/2}} \\
&\leq M_3 M_k^k C \sum_{|\alpha| \leq N-1} \sum_{j=1}^{13} |\langle \partial^\alpha g, e_j \rangle|^2 dx + 2M_3 L_k^k \sum_{|\alpha| \leq N} (\partial^\alpha g - \mathcal{L}^\epsilon_{\eta} \partial^\alpha f, \partial^\alpha f) \\
&+ \sum_{j=0}^{k} 2M_3 K_j^k \sum_{|\alpha| \leq N-j, |\beta| = j} (W_l+jY \partial^\beta g - \mathcal{L}^\epsilon_{\eta} W_l+jY \partial^\beta f, W_l+jY \partial^\beta f) \\
&+ 2 \sum_{|\alpha| \leq N-k-1, |\beta| = k+1} (W_q \partial^\alpha g - \mathcal{L}^\epsilon_{\eta} W_q \partial^\alpha f, W_q \partial^\alpha f).
\end{align*}

So we get (6.15) for $i = k + 1$. In detail, we set $M^{k+1} = M_3 M^k$, $L^{k+1} = M_3 L^k$, $K^{k+1}_j = M_3 K^k_j$ for $0 \leq j \leq k$ and $K^{k+1}_{k+1} = 1$. \(\Box\)

6.2. Global Well-posedness and Propagation of Regularity

The local well-posedness and the non-negativity of the solution to (1.15) were well established in [16]. Thus to prove global well-posedness, we only need to provide the a priori estimates for the equation, which is Theorem 6.1.

6.2.1. A Priori Estimate of Boltzmann Equation (1.20). In this subsection, we derive the following a priori estimate for solutions to the Cauchy problem (1.20):

**Theorem 6.1.** There exists a universal constant $\delta_0 > 0$ such that the following statement is valid. Let $T > 0$, $N \geq 4$, $l \geq 3N + 2$, there is a constant $\epsilon_0$ which may depend on $N$, $l$, such that if $0 \leq \epsilon \leq \epsilon_0$ and $f^\epsilon$ is a solution of the Cauchy problem (1.20) satisfying $\mu + \mu^\epsilon f^\epsilon(t) \geq 0$ and $\mathcal{E}^4.14(f^\epsilon(t)) \leq \delta_0$ for any $0 \leq t \leq T$, then for any $t \in [0, T]$, the solution $f^\epsilon$ verifies the following estimates.
(1) If \( N = 4, l = 14 \), then the solution \( f^\varepsilon \) verifies
\[
\mathcal{E}^{4,14}(f^\varepsilon(t)) + \int_0^t \mathcal{D}^{4,14}_\varepsilon(f^\varepsilon(s))ds \leq C\mathcal{E}^{4,14}(f_0).
\] (6.23)

(2) If \( N = 4, l > 14 \), then the solution \( f^\varepsilon \) verifies
\[
\mathcal{E}^{4,l}(f^\varepsilon(t)) + \int_0^t \mathcal{D}^{4,l}_\varepsilon(f^\varepsilon(s))ds \leq C_l \mathcal{E}^{4,l}(f_0).\] (6.24)

(3) If \( N \geq 5, l \geq 3N + 2 \), then the solution \( f^\varepsilon \) verifies
\[
\mathcal{E}^{N,l}(f^\varepsilon(t)) + \int_0^t \mathcal{D}^{N,l}_\varepsilon(f^\varepsilon(s))ds \leq P_{N,l}(\mathcal{E}^{N,l}(f_0)).\] (6.25)

Here \( C \) is a universal constant, \( C_l \) is a constant depending on \( l \) and \( P_{N,l}(\cdot) \) is a continuous and increasing function with \( P_{N,l}(0) = 0 \).

Recall from (1.27) the energy functional \( \mathcal{E}^{N,l} = \sum_{j=0}^N \| f \|_H^2 (\hat{h}_l^j \hat{h}_l^{j+\gamma}) \). For some constants \( C_0 \) depending only on \( N, l \), we have
\[
\mathcal{E}^{N,l}(f) \leq \Xi^{N,l}(f) \leq C_0(N, l)\mathcal{E}^{N,l}(f), \quad (6.26)
\]

In order to prove Theorem 6.1, we employ Proposition 6.1 by taking \( g = \Gamma^\varepsilon(f^\varepsilon, f^\varepsilon) \) and get
\[
\frac{d}{dt} \Xi^{N,l}(f^\varepsilon) + \frac{1}{4} \mathcal{D}^{N,l}_\varepsilon(f^\varepsilon) \leq \sum_{j=0}^N 2K_j \mathcal{A}^{N,j,l}_\varepsilon(f^\varepsilon, f^\varepsilon) + 2LB^N_\varepsilon(f^\varepsilon, f^\varepsilon) + MCC^N_\varepsilon(f^\varepsilon, f^\varepsilon),
\] (6.27)

where
\[
\mathcal{A}^{N,j,l}_\varepsilon(g, f) := \sum_{|\omega| \leq N-j, |\beta| = j} (W_{l+j\beta} \partial_\beta^\varepsilon \Gamma^\varepsilon(g, f) - \mathcal{L}_\varepsilon^{j,\gamma} W_{l+j\beta} \partial_\beta^\varepsilon f, W_{l+j\beta} \partial_\beta^\varepsilon f),
\] (6.28)

\[
B^N_\varepsilon(g, f) := \sum_{|\omega| \leq N} (\partial_\omega^\varepsilon \Gamma^\varepsilon(g, f) - \mathcal{L}_\varepsilon^{1,\gamma} \partial_\omega^\varepsilon f, \partial_\omega^\varepsilon f),
\] (6.29)

\[
C^N_\varepsilon(g, f) := \sum_{|\omega| \leq N-1} \sum_{j=1}^{13} \int \| (\partial_\omega^\varepsilon \Gamma^\varepsilon(g, f), e_j) \|^2 dx.
\] (6.30)

To move forward based on (6.27), we need to estimate \( \mathcal{A}^{N,j,l}_\varepsilon(f^\varepsilon, f^\varepsilon) \), \( B^N_\varepsilon(f^\varepsilon, f^\varepsilon) \), \( C^N_\varepsilon(f^\varepsilon, f^\varepsilon) \). This to end, we will give estimates of functionals \( \mathcal{A}^{N,j,l}_\varepsilon \) and \( B^N_\varepsilon \) in Lemma 6.6, functional \( C^N_\varepsilon \) in Lemma 6.7. To keep the proof of Lemma 6.6 in a reasonable length, we prepare a commutator estimate as Lemma 6.5.

Recalling from (6.8) the dissipation functional \( \mathcal{D}^{N,l}_\varepsilon \), we have
\[
\mathcal{D}^{N,l}_\varepsilon(f) \geq (c_0 M/2) \|(a, b, c)\|_{H^2} + c_1 \lambda_0 L \| f \|_{H^2}^2 + c_2 \| f \|_{H^2}^2 \quad (6.25)
\]
6.1. For simplicity, we define $c_1 = (M c_0 / 4L \lambda_0) \wedge 1$ since $M \sim L$ by Proposition 6.1. For simplicity, we define

$$
\| f \|_{H^{N,\omega}_x}^2 := \sum_{|\alpha|+|\beta| \leq m} \| \partial^\alpha_{\beta} f \|_{L^2_x}^2, \quad D^m_{\beta} (f) := \sum_{|\alpha|+|\beta| \leq m} \| \partial^\alpha_{\beta} f \|_{L^2_x L^{2,\gamma/2}_y}^2.
$$

Lemma 6.5. Let $N \geq 4, l \geq 3N + 2, 0 \leq j \leq N$. Let $\alpha, \beta$ satisfy $|\alpha| \leq N - j, |\beta| = j$. The following three statements hold true:

1) If $N = 4$ and $l = 14$, then

$$
|\langle W_{l+jY} \partial^\alpha_{\beta}, \Gamma^\epsilon (g, \cdot) \rangle h, W_{l+jY} \partial^\alpha_{\beta} f \rangle| \lesssim \| g \|_{H^4_x, v} \left( D_{\beta}^{4,14} (h) \right)^{\frac{1}{2}} \left( D_{\beta}^{4,14} (f) \right)^{\frac{1}{2}}. 
$$

(6.32)

2) If $N = 4$ and $l > 14$, then for any $\delta > 0$,

$$
|\langle W_{l+jY} \partial^\alpha_{\beta}, \Gamma^\epsilon (g, \cdot) \rangle h, W_{l+jY} \partial^\alpha_{\beta} f \rangle| \lesssim \| g \|_{H^4_x, v} \| h \|_{H^{N-1}_x, \hat{H}_y^l} \| \partial^\alpha_{\beta} f \|_{H^0_{x,y} L^0_{l+jY+\gamma/2}} + \nabla \| \partial^\alpha_{\beta} f \|_{H^0_{x,y} L^0_{l+jY+\gamma/2}}^2 + \delta^{-1} C_l \| g \|_{H^4_x, v}^2 \mathcal{C}_{4,l}^{4,l} (h). 
$$

(6.33)

3) If $N \geq 5$ and $l \geq 3N + 2$, then for any $\delta > 0$,

$$
|\langle W_{l+jY} \partial^\alpha_{\beta}, \Gamma^\epsilon (g, \cdot) \rangle h, W_{l+jY} \partial^\alpha_{\beta} f \rangle| \lesssim \delta \| \partial^\alpha_{\beta} f \|_{H^0_{x,y} L^0_{l+jY+\gamma/2}}^2 + \delta^{-1} C_{N,l} \| g \|_{H^N_x, v} D^{N-1,l}_{\beta} (h) + \delta^{-1} C_{N,l} \| g \|_{H^N_x, v}^2 \mathcal{C}_{N,l}^{N,l} (h). 
$$

(6.34)

Let $N \geq 4$. Let $\alpha$ satisfy $|\alpha| \leq N$. The following two statements hold true:

1) If $N = 4$, then

$$
|\langle \partial^\alpha, \Gamma^\epsilon (g, \cdot) \rangle h, \partial^\alpha f \rangle| \lesssim \| g \|_{H^4_x, v} \left( D_{\beta}^{4} (h) \right)^{\frac{1}{2}} \left( D_{\beta}^{4} (f) \right)^{\frac{1}{2}}. 
$$

(6.35)

2) If $N \geq 5$, then for any $\delta > 0$,

$$
|\langle \partial^\alpha, \Gamma^\epsilon (g, \cdot) \rangle h, \partial^\alpha f \rangle| \lesssim \delta \| \partial^\alpha f \|_{H^0_{x,y} L^{0,\gamma/2}_y}^2 + \delta^{-1} C_N \| g \|_{H^N_x, v} D^{N-1}_{\beta} (h) + \delta^{-1} C_N \| g \|_{H^N_x, v}^2 \mathcal{C}_{N,l}^{N,l} (h). 
$$

(6.36)
Proof. Set $q = l + j\gamma$. By the binomial expansion (4.8), we have
\[
W_q[\partial_{\beta}^{\alpha} \Gamma^e(g, \cdot)]h = W_q[\partial_{\beta}^{\alpha} \Gamma^e(g, h) - W_q \Gamma^e(g, \partial_{\beta}^{\alpha} h)]
\]
where the sum is over $\alpha_1 + \alpha_2 = \alpha, \beta_1 + \beta_2 \leq \beta, |\alpha_2 + \beta_2| \leq |\alpha| + |\beta| - 1$. By (5.8) in Corollary 5.1, we have for $b_1 \geq 0, b_2 \geq 1$ with $b_1 + b_2 = 2$,
\[
|\langle W_q \Gamma^e(g, h), W_q f \rangle| \lesssim |g|_{H_{\frac{q}{2}}^{1+\gamma}} |h|_{H_{\frac{q}{2}}^{1+\gamma}} + C_q |g|_{H_{\frac{q}{2}}^{1+\gamma}} |f|_{H_{\frac{q}{2}}^{1+\gamma}} + C_q |\mu^{\frac{1}{2}} g|_{H_{\frac{q}{2}}^{1+\gamma}} |f|_{H_{\frac{q}{2}}^{1+\gamma}}.
\]
(6.38)

If we denote the Fourier transform of $f$ with respect to $x$ variable by $\widehat{f}$, then we have
\[
(\Gamma^e(g, h), f) = \sum_{k, m \in \mathbb{Z}^3} \langle \Gamma^e(\widehat{g}(k), \widehat{h}(m-k); \beta_0), \widehat{f}(m) \rangle.
\]
From this together with (6.38), we get
\[
|\langle W_q \Gamma^e(\partial_{\beta_1}^{\alpha_1} g, \partial_{\beta_2}^{\alpha_2} h; \beta_0), W_q f \rangle| \lesssim \sum_{k, m \in \mathbb{Z}^3} |k|^{\alpha_1} |m-k|^{\alpha_2} |\partial_{\beta_1}^{\alpha_1} g(k)|_{H_{\frac{q}{2}}^{1+\gamma}} |\partial_{\beta_2}^{\alpha_2} h(m-k)|_{H_{\frac{q}{2}}^{1+\gamma}} + C_q |\mu^{\frac{1}{2}} g|_{H_{\frac{q}{2}}^{1+\gamma}} |f|_{H_{\frac{q}{2}}^{1+\gamma}}.
\]
(6.39)

From this we derive that for $a_1 + a_2 \geq 0$ with $a_1 + a_2 = 2$,
\[
|\langle W_q \Gamma^e(\partial_{\beta_1}^{\alpha_1} g, \partial_{\beta_2}^{\alpha_2} h, W_q \partial_{\beta}^{\alpha} f \rangle| \lesssim \|g\|_{H_{x}^{1+\gamma}} \|h\|_{H_{x}^{1+\gamma}} \|\partial_{\beta}^{\alpha} f\|_{H_{x}^{1+\gamma}} + C_q \|\mu^{\frac{1}{2}} g\|_{H_{x}^{1+\gamma}} \|\partial_{\beta}^{\alpha} f\|_{H_{x}^{1+\gamma}}.
\]
(6.40)

In what follows, we choose $a_1, a_2, b_1, b_2 \in \{0, 1, 2\}$ with $a_1 + a_2 = b_1 + b_2 = 2$ and $b_2 \geq 1$. For $N \geq 4$ and multi-indices $\alpha, \beta$ with $|\alpha| + |\beta| \leq N$, we consider all the combinations of $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $\alpha_1 + \alpha_2 = \alpha, \beta_1 + \beta_2 \leq \beta, |\alpha_2 + \beta_2| \leq |\alpha| + |\beta| - 1$ in Table 2 for the choice of $a_1, a_2, b_1, b_2$.

To summarize, if $|\alpha_1| + |\beta_1| \leq 4$, with the choice of $(a_1, a_2, b_1, b_2)$ (except the last line) in Table 2, we have $|\alpha_1| + a_1 + |\beta_1| + b_1 \leq 4, |\alpha_2| + a_2 + |\beta_2| + b_2 \leq |\alpha| + |\beta|, |\beta_2| \leq |\beta|$.

Case 1: $N = 4, l = 14$. We recall that $q = l + j\gamma \leq 14$ which implies $C_q \lesssim 1$.
By the lower bound of $D^e_{\gamma, l}(f)$ in (6.31) and the estimate (6.39), we have
\[
\langle W_q \Gamma^e(\partial_{\beta_1}^{\alpha_1} g, \partial_{\beta_2}^{\alpha_2} h, W_q \partial_{\beta}^{\alpha} f \rangle \lesssim \|g\|_{H_{x}^{1+\gamma}} \left( D^e_{\gamma, 14}^4(h) \right)^{\frac{1}{2}} \left( D^e_{\gamma, 14}^4(f) \right)^{\frac{1}{2}}.
\]
Table 2. Parameter choice

| $(|\alpha_1|, |\beta_1|)$ | $(|\alpha_2|, |\beta_2|)$ | $(a_1, a_2, b_1, b_2)$ | $|\alpha_1|+a_1+|\beta_1|+b_1$ | $|\alpha_2|+a_2+|\beta_2|+b_2$ |
|-------------------------|-------------------------|------------------------|-----------------------------|-----------------------------|
| (0,0)                   | $(|\alpha|, \leq |\beta| - 1)$ | (2,0,1,1)              | 3                           | $|\alpha| + |\beta|$          |
| (0,1)                   | $(|\alpha|, \leq |\beta| - 1)$ | (2,0,1,1)              | 4                           | $|\alpha| + |\beta|$          |
| (1,0)                   | $(|\alpha| - 1, \leq |\beta|)$ | (2,0,1,1)              | 4                           | $|\alpha| + |\beta|$          |
| (0,2)                   | $(|\alpha|, \leq |\beta| - 2)$ | (2,0,0,2)              | 4                           | $|\alpha| + |\beta|$          |
| (1,1)                   | $(|\alpha| - 1, \leq |\beta| - 1)$ | (1,1,1,1)              | 4                           | $|\alpha| + |\beta|$          |
| (2,0)                   | $(|\alpha| - 2, \leq |\beta|)$ | (1,1,1,1)              | 4                           | $|\alpha| + |\beta|$          |
| (0,3)                   | $(|\alpha|, \leq |\beta| - 3)$ | (1,1,0,2)              | 4                           | $|\alpha| + |\beta|$          |
| (1,2)                   | $(|\alpha| - 1, \leq |\beta| - 2)$ | (1,1,0,2)              | 4                           | $|\alpha| + |\beta|$          |
| (2,1)                   | $(|\alpha| - 2, \leq |\beta| - 1)$ | (0,2,1,1)              | 4                           | $|\alpha| + |\beta|$          |
| (3,0)                   | $(|\alpha| - 3, \leq |\beta|)$ | (0,2,1,1)              | 4                           | $|\alpha| + |\beta|$          |
| $|\alpha_1| + |\beta_1| = 4$ | $(|\alpha| - |\alpha_1|, \leq |\beta| - |\beta_1|)$ | (0,2,0,2)              | 4                           | $|\alpha| + |\beta|$          |
| $|\alpha_1| + |\beta_1| \geq 5$ | $(|\alpha| - |\alpha_1|, \leq |\beta| - |\beta_1|)$ | (0,2,0,2)              | N                           | $|\alpha| + |\beta| - 1$         |
Since $N = 4$, the constants $C(\alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2)$ in (6.37) are universally bounded. Taking sum over $\alpha_1 + \alpha_2 = \alpha, \beta_0 + \beta_1 + \beta_2 = \beta$, $|\alpha_2 + \beta_2| \leq |\alpha| + |\beta| - 1$, we get (6.32).

**Case 2 : $N = 4, l \geq 14$.** If $|\beta_2| = |\beta|$, we use $\|h\|_{H^{N-j}_x H^j_t r,q+y/2} \leq \|h\|_{H^{N-j}_x H^j_t r,q+y/2} \leq \|h\|_{H^{N-j}_x H^j_t r,q+y/2} \leq \|h\|_{H^{N-j}_x H^j_t r,q+y/2}$.

Plugging these facts into (6.39), we get

$$
|(W_q \Gamma^e (\partial^{\alpha_1}_{\beta_1} g, \partial^{\alpha_2}_{\beta_2} h), W_q \partial^g f)| \leq 1 |\beta_2| = |\beta| \|g\|_{H^4_{x,v}} \|h\|_{H^{N-j}_x H^j_t r,q+y/2} \|\partial^g f\|_{H^0_{r,q+y/2}}
$$

$$
+ (1 + \|g\|_{H^4_{x,v}} \|h\|_{H^{N-j}_x H^j_t r,q+y/2} \|\partial^g f\|_{H^0_{r,q+y/2}} + C_q \|g\|_{H^4_{x,v}} \|h\|_{H^{N-j}_x H^j_t r,q+y/2} \|\partial^g f\|_{H^0_{r,q+y/2}}
$$

$$
\leq \|g\|_{H^4_{x,v}} \|h\|_{H^{N-j}_x H^j_t r,q+y/2} \|\partial^g f\|_{H^0_{r,q+y/2}} + \delta \|\partial^g f\|_{H^0_{r,q+y/2}}^2 + \delta^{-1} C_l \|g\|_{H^4_{x,v}}^2 \|h\|_{H^0_{r,q+y/2}}^2,
$$

where the facts $\|h\|_{H^{N-j}_x H^j_t r,q+y/2} \leq 4(h)$ and $\|h\|_{H^0_{r,q+y/2}} \leq 4(h)$ are used. Since $N = 4$, the constants $C(\alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2)$ in (6.37) are universally bounded. Taking sum over according to (6.37), we get (6.33).

**Case 3 : $N \geq 5, l \geq 3N + 2$.** When $N \geq 5$, since $|\alpha_2| + a_2 + |\beta_2| \leq |\alpha| + |\beta| - b_2 \leq N - 1$, we have

$$
|\\(W_q \Gamma^e (\partial^{\alpha_1}_{\beta_1} g, \partial^{\alpha_2}_{\beta_2} h), W_q \partial^g f)\| \leq C_q \|g\|_{H^4_{x,v}} (D^N_{e_{H^0_{r,q+y/2}}})\|\partial^g f\|_{H^0_{r,q+y/2}}
$$

$$
+ C_q \|g\|_{H^4_{x,v}} \|h\|_{H^{N-j}_x H^j_t r,q+y/2} \|\partial^g f\|_{H^0_{r,q+y/2}}
$$

$$
\leq \delta \|\partial^g f\|_{H^0_{r,q+y/2}}^2 + \delta^{-1} C_l \|g\|_{H^4_{x,v}} D^N_{e_{H^0_{r,q+y/2}}}(h).
$$

(6.40)

If $|\alpha_1| + |\beta_1| \geq 5$, which occurs only when $N \geq 5$, with the choice of $(a_1, a_2, b_1, b_2)$ in the last line of Table 2, we have $|\alpha_1| + a_1 + |\beta_1| + b_1 \leq N, |\alpha_2| + a_2 + |\beta_2| + b_2 \leq |\alpha| + |\beta| - 1 \leq N - 1, |\beta_2| \leq |\beta|$ and thus

$$
|\\(W_q \Gamma^e (\partial^{\alpha_1}_{\beta_1} g, \partial^{\alpha_2}_{\beta_2} h), W_q \partial^g f)\| \leq C_q \|g\|_{H^4_{x,v}} (D^N_{e_{H^0_{r,q+y/2}}})\|\partial^g f\|_{H^0_{r,q+y/2}}
$$

$$
\leq \delta \|\partial^g f\|_{H^0_{r,q+y/2}}^2 + \delta^{-1} C_l \|g\|_{H^4_{x,v}}^2 D^N_{e_{H^0_{r,q+y/2}}}(h).
$$

(6.41)

Taking sum over according to (6.37), by (6.40) for the case of $|\alpha_1| + |\beta_1| \leq 4$ and (6.41) for the case of $|\alpha_1| + |\beta_1| \geq 5$, we get (6.34). We remark that the sum will bring a constant depending on $N$ due to the constants $C(\alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2)$.
However, thanks to the arbitrariness of $\delta$, only the latter two terms in (6.34) depend on $N$.

We turn to the case when $q = 0$. By Theorem 4.4, a counterpart to (6.39) is
\[
[(\Gamma^{(\delta)} g, \partial^{\alpha_2} h) + (\partial^{\alpha} f)] \lesssim \|g\|_{H^{[\alpha_1]^{+} c_1}} \|h\|_{H^{[\alpha_2]^{+} \delta_2}} \|\partial^{\alpha} f\|_{H^{0}_{c, y/2}}
\]
\[
+ \|\mu^{1/8} g\|_{H^{[\alpha_1]^{+} \delta_1}} \|h\|_{H^{[\alpha_2]^{+} \delta_2}} \|\partial^{\alpha} f\|_{H^{0}_{c, y/2}},
\]
which gives (6.35) and (6.36) in a similar way. □

**Lemma 6.6.** Let $0 < \eta, \epsilon < 1$ and $g, f$ be suitable functions with $\mu^{1/2} + g \geq 0$. Recall the definition of $A_{N, j, l}^{N_{e}}$ in (6.28). The following three statements hold true:

1. If $N = 4, l = 14$, then
\[
A_{N, j, l}^{N_{e}} (g, f) \leq C (\eta^{1/2} + \epsilon^{1/2} + \eta^{-6} \|g\|_{H^{2}_{e, y}}) D_{e}^{4,14} (f).
\]

2. If $N = 4, l > 14$, then for any $\delta > 0$,
\[
A_{N, j, l}^{N_{e}} (g, f) \leq C (\delta + \eta^{1/2} + \epsilon^{1/2} + \eta^{-6} \|g\|_{H^{2}_{e, y}}) \|f\|_{H^{2}_{N_{e} - \delta_{1}^{+} \delta_{2}^{+} y/2}}^{2} + \delta^{-1} C_{l} \|g\|_{H^{2}_{e, y}} \|f\|_{H^{2}_{N_{e} - \delta_{1}^{+} \delta_{2}^{+} y/2}}^{2} \|f\|_{H^{2}_{N_{e}}}.
\]

3. If $N \geq 5, l \geq 3N + 2$, then for any $\delta > 0$,
\[
A_{N, j, l}^{N_{e}} (g, f) \leq C (\delta + \eta^{1/2} + \epsilon^{1/2} + \eta^{-6} \|g\|_{H^{2}_{e, y}}) \|f\|_{H^{2}_{N_{e} - \delta_{1}^{+} \delta_{2}^{+} y/2}}^{2} + \delta^{-1} C_{N, j, l} \|g\|_{H^{2}_{e, y}} \|f\|_{H^{2}_{N_{e}}}.
\]

Recall the definition of $B_{N_{e}}$ in (6.29). The following two statements hold true:

1. If $N = 4$, then
\[
B_{N_{e}} (g, f) \leq C (\eta^{1/2} + \epsilon^{1/2} + \eta^{-6} \|g\|_{H^{2}_{e, y}}) D_{e}^{4} (f).
\]

2. If $N \geq 5$, then for any $\delta > 0$,
\[
B_{N_{e}} (g, f) \leq C (\delta + \eta^{1/2} + \epsilon^{1/2} + \eta^{-6} \|g\|_{H^{2}_{e, y}}) \|f\|_{H^{2}_{N_{e}}} + C_{N} \delta^{-1} \|g\|_{H^{2}_{e, y}} D_{e}^{N - 1}_{N_{e}} (f) + C_{N} \delta^{-1} \|g\|_{H^{2}_{e, y}} \|f\|_{H^{2}_{N_{e}}}.
\]

We emphasize that $C$ is a universal constant independent of $N, l$.

**Proof.** Let $q = l + j y$. Note that a typical term in $A_{N, j, l}^{N_{e}} (g, f)$ is $(W_{q} \partial^{\alpha} \Gamma^{(\delta)} (g, f) - L_{(N_{e}, y)}^{N_{e}, y} W_{q} \partial^{\alpha} f, W_{q} \partial^{\alpha} f)$ for some fixed $\alpha, \beta$ such that $|\alpha| \leq N - j, |\beta| = j$. We make the following decomposition
\[
W_{q} \partial^{\alpha} \Gamma^{(\delta)} (g, f) = W_{q} \Gamma^{(\delta)} (g, \partial^{\alpha} f) + W_{q} [\partial^{\alpha}, \Gamma^{(\delta)} (g, \cdot)] f
\]
\[
\begin{align*}
&= \Gamma^\varepsilon (g, W_q \partial^\alpha \beta f) + [W_q, \Gamma^\varepsilon (g, \cdot)] \partial^\alpha \beta f + W_q [\partial^\alpha \beta, \Gamma^\varepsilon (g, \cdot)] f \\
&= \Gamma^\varepsilon (g, W_q \partial^\alpha \beta f) + \Gamma^\varepsilon (g, W_q \partial^\alpha \beta f) + [W_q, \Gamma^\varepsilon (g, \cdot)] \partial^\alpha \beta f \\
&\quad + W_q [\partial^\alpha \beta, \Gamma^\varepsilon (g, \cdot)] f.
\end{align*}
\] (6.47)

Estimate of \(\Gamma^\varepsilon (g, W_q \partial^\alpha \beta f) - L^\varepsilon (g, W_q \partial^\alpha \beta f)\). Since \(\mu^2 + g \geq 0\), taking \(\delta = \frac{1}{2}\) in Theorem 4.5, by the embedding \(H^2 \rightarrow L^\infty\), we have
\[
\begin{align*}
(\Gamma^\varepsilon (g, W_q \partial^\alpha \beta f) - L^\varepsilon (g, W_q \partial^\alpha \beta f)) \\
&\leq C(\eta^2 + \varepsilon^2) \int (1 + |\mu| g |H^2|) |W_q \partial^\alpha \beta f|^2 L^2_{\varepsilon, \gamma/2} \ dx \\
&\leq C(\eta^2 + \varepsilon^2) \| \partial^\alpha \beta f \|^2 L^2_{\varepsilon, \gamma/2} + C(\eta^2 + \varepsilon^2) \| \mu \ |H^2| \ |H^2| \ |\partial^\alpha \beta f \|^2 L^2_{\varepsilon, \gamma/2} \\
&\leq C(\eta^2 + \varepsilon^2) \| \partial^\alpha \beta f \|^2 L^2_{\varepsilon, \gamma/2}.
\end{align*}
\] (6.48)

Estimate of \(W_q, \Gamma^\varepsilon (g, \cdot) \partial^\alpha \beta f\). Taking \(\delta = \frac{1}{2}, s_1 = 2, s_2 = 0\) in Theorem 4.1, by the embedding \(H^2 \rightarrow L^\infty\), we have
\[
\begin{align*}
|((\Gamma^\varepsilon (g, W_q \partial^\alpha \beta f), W_q \partial^\alpha \beta f))| &\leq C \eta^{-6} \int |g| |H^2| |W_q \partial^\alpha \beta f|^2 L^2_{\varepsilon, \gamma/2} \ dx \\
&\leq C \eta^{-6} \| g \|_{H^4} \| \partial^\alpha \beta f \|^2 L^2_{\varepsilon, \gamma/2}.
\end{align*}
\] (6.49)

Estimate of \([W_q, \Gamma^\varepsilon (g, \cdot) \partial^\alpha \beta f]\). Taking \(\delta = \frac{1}{2}, s_3 = 1, s_4 = 0\) in (5.7), by the embedding \(H^2 \rightarrow L^\infty\), we have
\[
\begin{align*}
|([W_q, \Gamma^\varepsilon (g, \cdot) \partial^\alpha \beta f])| \\
&\leq \eta^{-6} C_l \int |g| \|H^2| \| \partial^\alpha \beta f \|^2 L^2_{\varepsilon, \gamma/2} \ dx \\
&\quad + \eta^2 C_l \int |g| |H^1| \| \partial^\alpha \beta f \|^2 L^2_{\varepsilon, \gamma/2} \ dx \\
&\leq C_l \| g \|_{H^4} \| \partial^\alpha \beta f \|_{L^2_{\varepsilon, \gamma/2}} \| \partial^\alpha \beta f \|^2 L^2_{\varepsilon, \gamma/2} + \| g \|_{H^4} \| \partial^\alpha \beta f \|^2 L^2_{\varepsilon, \gamma/2},
\end{align*}
\]
where we choose \(\eta\) such that \(\eta^2 C_l = 1\) and then \(\eta^{-6} C_l\) is a constant depending only on \(l\). When \(N = 4, l = 14\), the constant \(C_l\) is a universal constant, which gives
\[
|([W_q, \Gamma^\varepsilon (g, \cdot) \partial^\alpha \beta f])| \lesssim \| g \|_{H^4} \| \partial^\alpha \beta f \|^2 L^2_{\varepsilon, \gamma/2}.
\] (6.50)

When \(N = 4, l > 14\) or \(N \geq 5, l \geq 3N + 2\), we get
\[
|([W_q, \Gamma^\varepsilon (g, \cdot) \partial^\alpha \beta f])| \lesssim \delta \| g \|_{H^4} \| \partial^\alpha \beta f \|^2 L^2_{\varepsilon, \gamma/2} + \delta^{-1} C_l \| g \|_{H^4} \| \partial^\alpha \beta f \|^2 L^2_{\varepsilon, \gamma/2}.
\] (6.51)
The last term \( W_q[\partial^q_{\mu}, \Gamma^q(g, \cdot)]f \) in (6.47) is handled in Lemma 6.5 by (6.32), (6.33) and (6.34). From which together with (6.48), (6.49), (6.50) and (6.51), we get the desired results (6.42), (6.43) and (6.44). The estimates (6.45) and (6.46) of \( B^N_\varepsilon(g, f) \) can be derived similarly, so we omit the details and end the proof of the lemma.

**Lemma 6.7.** Let \( \varepsilon \geq 0 \) be small enough. Recall the definition of \( C^N_\varepsilon \) in (6.30). The following two statements hold true:

1. If \( N = 4 \), then
   \[
   C^N_\varepsilon(g, f) \leq C\|g\|_{H^4_{x, v}}^2 D^4_\varepsilon(f).
   \] (6.52)

2. If \( N \geq 5 \), then
   \[
   \sum_{|\alpha| \leq N-1} \sum_{j=1}^{13} \int |\partial^\alpha \Gamma^\varepsilon(g, f, e_j)|^2 dx \leq C_N(\|g\|_{H^N_{x, v}}^2 D^{N-1}_\varepsilon(f) + D^{N-1}_\varepsilon(g)\|f\|_{H^N_{x, v}}^2). \] (6.53)

**Proof.** Thanks to Theorem 4.4, for \( a_1, a_2, b_1, b_2 \in \{0, 1, 2\} \) with \( a_1 + a_2 = b_1 + b_2 = 2 \) and \( b_2 \geq 1 \), when \( N = 4 \), we have

\[
\int |\langle \partial^{a_1} \Gamma^\varepsilon, \partial^{a_2} f, e_j \rangle|^2 dx \lesssim \|g\|_{H^{|a_1|+a_1}_x H^{|a_2|+a_2}_v}^2 \|f\|_{H^{|a_1|+a_1}_x H^{|a_2|+a_2}_v}^2 + \|\mu^{1/2} g\|_{H^{|a_1|+b_1}_x H^{|a_2|+b_2}_v}^2 \|\mu^{1/2} f\|_{H^{|a_1|+b_1}_x H^{|a_2|+b_2}_v}^2 \lesssim \|g\|_{H^4_{x, v}}^2 D^4_\varepsilon(f).
\]

Similarly, when \( N \geq 5 \), we have

\[
\int |\langle \partial^{a_1} \Gamma^\varepsilon, \partial^{a_2} f, e_j \rangle|^2 dx \lesssim \|g\|_{H^{|a_1|+a_1}_x H^{|a_2|+a_2}_v}^2 \|f\|_{H^{|a_1|+a_1}_x H^{|a_2|+a_2}_v}^2 + \|\mu^{1/2} g\|_{H^{|a_1|+b_1}_x H^{|a_2|+b_2}_v}^2 \|\mu^{1/2} f\|_{H^{|a_1|+b_1}_x H^{|a_2|+b_2}_v}^2 \lesssim \|g\|_{H^N_{x, v}}^2 D^{N-1}_\varepsilon(f) + D^{N-1}_\varepsilon(g)\|f\|_{H^N_{x, v}}^2.
\]

Here in both cases, we use Table 2 for the choice of \( a_1, a_2, b_1, b_2 \). □

Now we are ready to prove Theorem 6.1.

**Proof of Theorem 6.1.** Taking \( g = \Gamma^\varepsilon(f^\varepsilon, f^\varepsilon) \) in Proposition 6.1, recalling (6.27), for \( 0 < \eta \leq \eta_0, 0 \leq \varepsilon \leq \varepsilon_1 \), we have

\[
\frac{d}{dt} \Xi^{N, l}(f^\varepsilon) + \frac{1}{4} D^{N, l}_\varepsilon(f^\varepsilon) \leq \sum_{j=0}^{N} 2K_j A^{N, j, l}_\varepsilon(f^\varepsilon, f^\varepsilon) + 2LB^N_\varepsilon(f^\varepsilon, f^\varepsilon) + MCC^N_\varepsilon(f^\varepsilon, f^\varepsilon),
\]
Case 1: $N = 4, l = 14$. In this case, the constants $M, L, K_j$ are universal. Then by (6.42) and (6.45) in Lemma 6.6, and (6.52) in Lemma 6.7, and the natural inequality $D_e^4(f^\varepsilon) \lesssim D_e^{4,14}(f^\varepsilon)$, we have
\[
\frac{d}{dt} \Xi^{4,14}(f^\varepsilon) + \frac{1}{4} D_e^{4,14}(f^\varepsilon) \leq C (\eta^2 + \varepsilon^\frac{1}{2} + \eta^{-6} \|f^\varepsilon\|_{H_{x,v}^4}^2 + \|f^\varepsilon\|_{H_{x,v}^4}^2) D_e^{4,14}(f^\varepsilon).
\]
Let $\eta_1$ verify $C\eta_1^2 = \frac{1}{32}$. Let $\varepsilon_2$ verify $C\varepsilon_2^2 = \frac{1}{32}$. Let $\delta_1$ be the largest number satisfying $C\eta_1^{-6} \varepsilon_1^2 \leq \frac{1}{32}$ and $C\delta_1 \leq \frac{1}{32}$. We choose $\eta = \min\{\eta_0, \eta_1\}$. When $\varepsilon \leq \min\{\varepsilon_1, \varepsilon_2\}$, under the assumption $\sup_{0 \leq t \leq T} \mathcal{E}^{4,14}(f^\varepsilon(t)) \leq \delta_1$, since $\|f^\varepsilon\|_{H_{x,v}^4}^2 \leq \mathcal{E}^{4,14}(f^\varepsilon)$, we have
\[
\frac{d}{dt} \Xi^{4,14}(f^\varepsilon) + \frac{1}{8} D_e^{4,14}(f^\varepsilon) \leq 0.
\] (6.54)
We emphasize that when $N = 4, l = 14$, the constants $C_0(N, l)$ in (6.26) is universal. Therefore we get (6.23) from (6.54).

Case 2: $N = 4, l > 14$. In this case, the constants $M, L, K_j$ could depend on $l$. Then by (6.43) and (6.45) in Lemma 6.6 and (6.52) in Lemma 6.7, and the natural inequality $\|f^\varepsilon\|_{H_{x,v}^4}^2 \leq D_e^{4,14}(f^\varepsilon)$, we have
\[
\frac{d}{dt} \Xi^{4,l}(f^\varepsilon) + \frac{1}{4} D_e^{4,l}(f^\varepsilon)
\leq MC \|f^\varepsilon\|_{H_{x,v}^4}^2 D_e^{4,l}(f^\varepsilon) + C_l (\eta^2 + \varepsilon^\frac{1}{2} + \eta^{-6} \|f^\varepsilon\|_{H_{x,v}^4}^2) D_e^{4,l}(f^\varepsilon)
+ 2C (\delta + \eta^2 + \varepsilon^\frac{1}{2} + \eta^{-6} \|f^\varepsilon\|_{H_{x,v}^4}^2) \sum_{j=0}^{N} K_j \|f^\varepsilon\|_{H_{x,v}^4}^2 \leq \mathcal{E}^{4,14}(f^\varepsilon)
+ \delta^{-1} C_l D_e^{4,14}(f^\varepsilon) \mathcal{E}^{4,l}(f^\varepsilon).
\]
We take $\delta, \eta_2, \varepsilon_2, \delta_2$ such that $2C\delta = \frac{\lambda_0}{32}, 2C\eta_2^2 = \frac{\lambda_0}{32}, 2C\varepsilon_2^\frac{1}{2} = \frac{\lambda_0}{32}, 2C\eta^{-6} \delta_2^2 = \frac{\lambda_0}{32}$. We choose $\eta = \min\{\eta_0, \eta_1, \eta_2\}$. When $\varepsilon \leq \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, under the assumption $\sup_{0 \leq t \leq T} \mathcal{E}^{4,14}(f^\varepsilon(t)) \leq \min\{\delta_1, \delta_2\}$, since $\|f^\varepsilon\|_{H_{x,v}^4}^2 \leq \mathcal{E}^{4,14}(f^\varepsilon)$, we have $2C (\delta + \eta^2 + \varepsilon^\frac{1}{2} + \eta^{-6} \|f^\varepsilon\|_{H_{x,v}^4}^2) \leq \frac{\lambda_0}{4}$. Recalling the definition of $D_e^{N,l}$ in (6.31), we get
\[
\frac{d}{dt} \Xi^{4,l}(f^\varepsilon) + \frac{1}{8} D_e^{4,l}(f^\varepsilon) \leq C_l D_e^4(f^\varepsilon) + C_l D_e^{4,14}(f^\varepsilon) \mathcal{E}^{4,l}(f^\varepsilon).
\]
By Gronwall inequality, we arrive at
\[
\Xi^{4,l}(f^\varepsilon(t)) \leq (\Xi^{4,l}(f_0) + C_l \int_0^t D_e^{4,l}(f^\varepsilon(s)) ds) \exp \left( C_l \int_0^t D_e^{4,14}(f^\varepsilon(s)) ds \right)
\leq (\Xi^{4,l}(f_0) + C_l \mathcal{E}^{4,14}(f_0)) \exp \left( C_l \mathcal{E}^{4,14}(f_0) \right) \leq C_l \Xi^{4,l}(f_0).
\]
where we use $D_{f}^{2}(f^{e}) \leq D_{f}^{2,14}(f^{e})$, and $\int_{0}^{t} D_{f}^{2,14}(f^{e})ds \leq C_{f}^{2,14}(f_{0})$ by the proved result (6.23), and $C^{2,14}(f_{0}) \leq \Xi^{2,1}(f_{0})$, and the assumption $C^{2,14}(f_{0}) \leq \delta_{1} \leq 1$. Then by (6.26), we get (6.24).

**Case 3 :** $N \geq 5, l \geq 3N + 2$. In this case, the constants $M, L, K_{j}$ could depend on $N, l$. Then by (6.44) and (6.46) in Lemma 6.6 and (6.53) in Lemma 6.7, and the inequalities $\|f^{e}\|^{2}_{H_{x,v}^{0}} \leq D_{f}^{N-1,1}(f^{e}), D_{f}^{N-1}(f^{e}) \leq D_{f}^{N-1,1}(f^{e})$, $\|f^{e}\|^{2}_{H_{x,v}^{0}} \leq \Xi^{N,l}(f^{e})$, we have

$$
\frac{d}{dt} \Xi^{N,l}(f^{e}) + \frac{1}{8} D_{f}^{N,l}(f^{e}) \leq 2C(\delta + \eta + \epsilon^{2} + \epsilon^{6} \|f^{e}\|_{H_{x,v}^{0}})L \|f^{e}\|^{2}_{H_{x,v}^{0}} + 2C(\delta + \eta + \epsilon^{2} + \epsilon^{6} \|f^{e}\|_{H_{x,v}^{0}}) \sum_{j=0}^{N} K_{j} \|f^{e}\|_{H_{x,v}^{0}}^{2} \Xi^{N-1,l}(f^{e}) + \delta C_{N,l} D_{f}^{N-1,1}(f^{e}) \Xi^{N,l}(f^{e}).
$$

We take $\delta, \eta_{3}, \epsilon_{4}, \epsilon_{3}$ such that $2C \delta = \frac{\epsilon_{0}}{32}, 2C \eta_{3}^{2} = \frac{\epsilon_{0}}{32}, 2C \epsilon_{4}^{2} = \frac{\epsilon_{0}}{32}, 2C \eta_{3}^{6} \delta_{3}^{2} = \frac{\epsilon_{0}^{2}}{32}$. We choose $\eta = \min_{0 \leq \eta \leq \eta_{3}} \eta_{i}$. Let $\delta_{0} = \min_{1 \leq i \leq 3} \delta_{i}$ and $\epsilon_{0} = \min_{2 \leq i \leq 3} \epsilon_{i}$.

When $\epsilon \leq \epsilon_{0}$, if $\sup_{0 \leq t \leq T} E^{4,14}(f^{e}(t)) \leq \delta_{0}$, since $\|f^{e}\|_{H_{x,v}^{0}}^{2} \Xi^{4,14}(f^{e})$, we have $2C(\delta + \eta + \epsilon^{2} + \epsilon^{6} \|f^{e}\|_{H_{x,v}^{0}}) \leq \frac{\epsilon_{0}^{2}}{8}$. Recalling the definition of $D_{f}^{N,l}$ in (6.31), we conclude that for any $N \geq 5, l \geq 3N + 2$, it holds that

$$
\frac{d}{dt} \Xi^{N,l}(f^{e}) + \frac{1}{8} D_{f}^{N,l}(f^{e}) \leq C_{N,l} D_{f}^{N-1,l}(f^{e}) \Xi^{N,l}(f^{e}).
$$

(6.55)

In the following we use mathematical induction to finish the proof. Suppose for some $k \geq 4$, (6.25) is valid for $N = k$, that is,

$$
E^{k,l}(f^{e}(t)) + \int_{0}^{t} D_{f}^{k,l}(f^{e}(s))ds \leq P_{k,l}(E^{k,l}(f_{0})).
$$

(6.56)

Then for $N = k + 1 \geq 5, l \geq 3N + 2$, by (6.55), we get

$$
\frac{d}{dt} \Xi^{k+1,l}(f^{e}) + \frac{1}{8} D_{f}^{k+1,l}(f^{e}) \leq C_{k+1,l} D_{f}^{k+1,l}(f^{e}) \Xi^{k+1,l}(f^{e}).
$$

Now since $\int_{0}^{t} D_{f}^{k+1,l}(f^{e}(s))ds \leq P_{k,l}(E^{k,l}(f_{0}))$ by (6.56) and $E^{k+1,l}(f^{e}) \leq \Xi^{k+1,l}(f^{e})$, by Gronwall’s inequality, we arrive at

$$
\Xi^{k+1,l}(f^{e}(t)) + \frac{1}{8} \int_{0}^{t} D_{f}^{k+1,l}(f^{e}(s))ds \leq \Xi^{k+1,l}(f_{0}) \exp \{C_{k+1,l} \int_{0}^{t} D_{f}^{k+1,l}(f^{e}(s))ds\} \leq \Xi^{k+1,l}(f_{0}) \exp \{C_{k+1,l} P_{k,l}(E^{k,l}(f_{0}))\}.
$$

Then by the equivalence relation (6.26), we have

$$
E^{k+1,l}(f^{e}(t)) + \int_{0}^{t} D_{f}^{k+1,l}(f^{e}(s))ds \leq C_{k+1,l} E^{k+1,l}(f_{0}) \exp \{C_{k+1,l} P_{k,l}(E^{k+1,l}(f_{0}))\}
$$
That is, we get (6.25) for the case \( N = k + 1, l \geq 3N + 2 \). Starting from \( P_{4,l}(x) = C_l x \), we can define \( P_{N,l}(x) := C_{N,l} x \exp \left( C_{N,l} P_{N-1,l}(x) \right) \) in an iterating manner for \( N \geq 5 \). □

Proof of Theorem 1.1 (global well-posedness and regularity propagation) We remind readers that local well-posedness of the equation (1.20) and the non-negativity \( \mu + \mu^{\frac{1}{2}} f^\varepsilon \geq 0 \) were proved in [16]. Thanks to Theorem 6.1, the standard continuity argument yields the global well-posedness result (1.28). The propagation of regularity result (1.29) follows Theorem 6.1. □

6.3. Asymptotic Formula for the Limit

We prove (1.30) in this subsection. Let \( f^\varepsilon \) and \( f \) be the solutions to (1.20) and (1.23) respectively with the initial data \( f_0 \). Setting \( F^\varepsilon_R := \ln \varepsilon |(f^\varepsilon - f)| \), this solves

\[
\partial_t F^\varepsilon_R + v \cdot \nabla_x F^\varepsilon_R + \mathcal{L}^L F^\varepsilon_R = |\ln \varepsilon|((\mathcal{L}^L - \mathcal{L}^\varepsilon) f^\varepsilon + (\Gamma^\varepsilon - \Gamma^L)(f^\varepsilon, f))
\]

\[
+ \Gamma^\varepsilon (f^\varepsilon, F^\varepsilon_R) + \Gamma^L (F^\varepsilon_R, f).
\]

We will apply Proposition 6.1 to the above equation for \( F^\varepsilon_R \). For notational brevity, we set

\[
G_1 = |\ln \varepsilon|((\mathcal{L}^L - \mathcal{L}^\varepsilon) f^\varepsilon + (\Gamma^\varepsilon - \Gamma^L)(f^\varepsilon, f)), \quad G_2 = \Gamma^\varepsilon (f^\varepsilon, F^\varepsilon_R),
\]

\[
G_3 = \Gamma^L (F^\varepsilon_R, f).
\]

When \( N \geq 4, \eta = \varepsilon = 0 \), by applying Proposition 6.1 with \( g = G_1 + G_2 + G_3 \), since \( |(\partial^\alpha g, e_j)|^2 \leq 3(|(\partial^\alpha G_1, e_j)|^2 + |(\partial^\alpha G_2, e_j)|^2 + |(\partial^\alpha G_3, e_j)|^2) \), we have

\[
\frac{d}{dt} \Delta^{N,l}(F^\varepsilon_R) + \frac{1}{4} \mathcal{D}_0^{N,l}(F^\varepsilon_R)
\]

\[
\leq 3MC \sum_{i=1}^3 \sum_{|\alpha| \leq N-1} \sum_{j=1}^{13} \int |(\partial^\alpha G_1, e_j)|^2 dx + \sum_{i=1}^3 2L \sum_{|\alpha| \leq N} (\partial^\alpha G_1, \partial^\alpha F^\varepsilon_R)
\]

\[
+ \sum_{i=1}^3 \sum_{j=0}^{N} 2K_j \sum_{|\alpha| \leq N-j, |\beta| = j} (W_{i+j, \beta} \partial^\alpha G_1, W_{i+j, \beta} \partial^\alpha F^\varepsilon_R)
\]

\[
= 3MC \left( \mathcal{X}^N(G_1) + C_\varepsilon^N(f^\varepsilon, F^\varepsilon_R) + C_0^N(F^\varepsilon_R, f) \right)
\]

\[
+ 2L \left( \mathcal{Y}^N(G_1) + \mathcal{Z}_\varepsilon^N(f^\varepsilon, F^\varepsilon_R) + \mathcal{Z}_0^N(F^\varepsilon_R, f, F^\varepsilon_R) \right)
\]

\[
+ \sum_{j=0}^N 2K_j \left( \mathcal{W}^{N,j,l}(G_1) + \mathcal{Y}^{N,j,l}(f^\varepsilon, F^\varepsilon_R, F^\varepsilon_R) + \mathcal{Y}_0^{N,j,l}(F^\varepsilon_R, f, F^\varepsilon_R) \right),
\]

(6.58)
where we recall (6.30) and define

\[ \mathcal{X}^N(h) := \sum_{|\alpha| \leq N-1} \sum_{j=1}^{13} \int |\langle \partial^\alpha h, e_j \rangle|^2 dx, \]  

(6.59)

\[ \Upsilon^N(h) := \sum_{|\alpha| \leq N} (\partial^\alpha h, \partial^\alpha F_R^\varepsilon), \]  

(6.60)

\[ Z^N_\varepsilon(g, h, f) := \sum_{|\alpha| \leq N} (\partial^\alpha \Gamma^\varepsilon(g, h), \partial^\alpha f), \]  

(6.61)

\[ \mathcal{W}^{N,j,l}(h) := \sum_{|\alpha| \leq N-j, |\beta| = j} (W_{l+j} \partial^\alpha \beta^l h, W_{l+j} \partial^\alpha \beta^l F_R^\varepsilon), \]  

(6.62)

\[ \Upsilon^{N,j,l}(g, h, f) := \sum_{|\alpha| \leq N-j, |\beta| = j} (W_{l+j} \partial^\alpha \beta^l \Gamma^\varepsilon(g, h), W_{l+j} \partial^\alpha \beta^l f). \]  

(6.63)

In order to further analyze (6.58), we need to estimate the nine terms on the right hand side. Note that the functional \( C^N_\varepsilon \) is already handled in Lemma 6.7. We will deal with the functionals \( Z^N_\varepsilon \) and \( \Upsilon^{N,j,l}_\varepsilon \) in Lemma 6.8, functional \( \mathcal{X}^N \) in Lemma 6.10, functionals \( \Upsilon^N \) and \( \mathcal{W}^{N,j,l} \) in Lemma 6.11.

**Lemma 6.8.** Let \( \varepsilon \geq 0. \) Let \( g, h, f \) be suitable functions. Recall the definition of \( \Upsilon^{N,j,l}_\varepsilon \) in (6.63). The following three statements hold true:

1. If \( N = 4, l = 14, \) we have

\[ \Upsilon^{N,j,l}(g, h, f) \leq C \|g\|_{H^4_{x,v}} (D^{4,14}_0(h))^\frac{1}{2} (D^{4,14}_0(f))^\frac{1}{2}. \]  

(6.64)

2. If \( N = 4, l > 14, \) we have for any \( \delta > 0, \)

\[ \Upsilon^{N,j,l}(g, h, f) \leq C \|g\|_{H^4_{x,v}} \|h\|_{H^{4-j}_{x,v} H^{j+\gamma+y/2}_{0,l+j+y/2}} \|f\|_{H^{4-j}_{x,v} H^{j+\gamma+y/2}_{0,l+j+y/2}} + \delta \|f\|^2_{H^{4-j}_{x,v} H^{j+\gamma+y/2}_{0,l+j+y/2}} + \delta^{-1} C \|g\|^2_{H^4_{x,v}} E^{4,l}(h). \]  

(6.65)

3. If \( N \geq 5, l \geq 3N + 2, \) we have for any \( \delta > 0, \)

\[ \Upsilon^{N,j,l}(g, h, f) \leq C \|g\|_{H^4_{x,v}} \|h\|_{H^{4-j}_{x,v} H^{j+\gamma+y/2}_{0,l+j+y/2}} \|f\|_{H^{4-j}_{x,v} H^{j+\gamma+y/2}_{0,l+j+y/2}} + \delta \|f\|^2_{H^{4-j}_{x,v} H^{j+\gamma+y/2}_{0,l+j+y/2}} + \delta^{-1} C \|g\|^2_{H^4_{x,v}} D^{4,1-l}_N(h) + \delta^{-1} C \|g\|^2_{H^4_{x,v}} \|h\|^2_{H^4_{x,v}}. \]  

(6.66)

Recall the definition of \( Z^N_\varepsilon \) in (6.61). The following two statements hold true:

1. If \( N = 4, \) then

\[ |Z^N_\varepsilon(g, h, f)| \leq C \|g\|_{H^4_{x,v}} (D^{4}_0(h))^\frac{1}{2} (D^{4}_0(f))^\frac{1}{2}. \]  

(6.67)
(2) If \( N \geq 5 \), then for any \( \delta > 0 \),

\[
| \mathcal{Z}_\varepsilon^N (g, h, f) | \leq C \| g \|_{H^4_\varepsilon} \| h \|_{H^N_\varepsilon H^0_{0,q,y/2}} \| f \|_{L^2_{\varepsilon,y/2}^N} + \delta \| f \|_{L^2_{\varepsilon,y/2}^N}^2 + \delta^{-1} C_N \| g \|_{H^4_\varepsilon}^2 \| h \|_{H^N_\varepsilon}^2.
\]

(6.68)

We emphasize that \( C \) is a universal constant independent of \( N, l \).

Proof. A typical term in \( \mathcal{Y}_\varepsilon^{N,j,l} (g, h, f) \) is \((W_{l+j,y} \partial_\alpha^\varepsilon \Gamma^\varepsilon (g, h), W_{l+j,y} \partial_\beta^\varepsilon f)\) for some fixed \( \alpha, \beta \) such that \( |\alpha| \leq N - j, |\beta| = j \). For simplicity, let \( q = l + jy \).

We use

\[
W_q \partial_\beta^\varepsilon \Gamma^\varepsilon (g, h) = W_q \Gamma^\varepsilon (g, \partial_\beta^\varepsilon h) + W_q [\partial_\beta^\varepsilon, \Gamma^\varepsilon (g, \cdot)] h.
\]

Since \((W_q [\partial_\beta^\varepsilon, \Gamma^\varepsilon (g, \cdot)] h, W_q \partial_\beta^\varepsilon f)\) is handled in Lemma 6.5, we only need to focus on the first term.

By (5.9) in Corollary 5.1, we have

\[
| \langle W_q \Gamma^\varepsilon (g, \partial_\beta^\varepsilon h), W_q \partial_\beta^\varepsilon f \rangle | \lesssim \| g \|_{H^2_\varepsilon} \| \partial_\beta^\varepsilon h \|_{L^2_{\varepsilon,0,q+y/2}} \| \partial_\beta^\varepsilon f \|_{L^2_{\varepsilon,q+y/2}}^2 + C_l \| g \|_{H^2_\varepsilon} \| \partial_\beta^\varepsilon h \|_{H^0_{\varepsilon,q+y/2}} \| \partial_\beta^\varepsilon f \|_{H^0_{\varepsilon,q+y/2}}^2.
\]

Then, by the imbedding \( H^2_\varepsilon \rightarrow L^\infty_\varepsilon \), we get

\[
| \langle W_q \Gamma^\varepsilon (g, \partial_\beta^\varepsilon h), W_q \partial_\beta^\varepsilon f \rangle | \lesssim \| g \|_{H^4_\varepsilon} \| \partial_\beta^\varepsilon h \|_{H^0_{\varepsilon,0,q+y/2}} \| \partial_\beta^\varepsilon f \|_{H^0_{\varepsilon,q+y/2}}^2 + C_l \| g \|_{H^4_\varepsilon} \| \partial_\beta^\varepsilon h \|_{H^0_{\varepsilon,q+y/2}} \| \partial_\beta^\varepsilon f \|_{H^0_{\varepsilon,q+y/2}}^2.
\]

When \( N = 4, l = 14 \), since \( C_l \) is a universal constant, we have

\[
| \langle W_q \Gamma^\varepsilon (g, \partial_\beta^\varepsilon h), W_q \partial_\beta^\varepsilon f \rangle | \lesssim \| g \|_{H^4_\varepsilon} \left( D^4_{1,14} (h) \right)^2 \left( D^4_{1,14} (f) \right)^2.
\]

From this, together with (6.32) in Lemma 6.5, we get (6.64). When \( N = 4, l > 14 \) or \( N \geq 5, l \geq 3N + 2 \), one has

\[
| \langle W_q \Gamma^\varepsilon (g, \partial_\beta^\varepsilon h), W_q \partial_\beta^\varepsilon f \rangle | \lesssim \| g \|_{H^4_\varepsilon} \| \partial_\beta^\varepsilon h \|_{H^0_{\varepsilon,0,q+y/2}} \| \partial_\beta^\varepsilon f \|_{H^0_{\varepsilon,q+y/2}}^2 + \delta \| \partial_\beta^\varepsilon f \|_{H^0_{\varepsilon,q+y/2}}^2 + \delta^{-1} C_l \| g \|_{H^4_\varepsilon} \| \partial_\beta^\varepsilon h \|_{H^0_{\varepsilon,q+y/2}}^2.
\]

From this, together with (6.33) and (6.34) in Lemma 6.5, we get (6.65) and (6.66).

For \( \mathcal{Z}_\varepsilon^N (g, h, f) \), it is not difficult to copy the above argument to get the desired result. \( \square \)

We recall an estimate on the operator \( \Gamma^L - \Gamma^\varepsilon \), which can be derived similarly as in [24].

Lemma 6.9. It holds that

\[
| \langle W_q (\Gamma^L - \Gamma^\varepsilon) (g, h), f \rangle | \lesssim C_q | \ln \varepsilon |^{-1} | \mu \gamma^{3/2} g |_{H^3} |h|_{H^4_{4,15/2}} \| f \|_{L^2_{\varepsilon,3/2}^2}.
\]

As an application of Lemma 6.9, we have
Lemma 6.10. Let $N \geq 4$. Recall the function $G_1$ in (6.57) and the functional $\mathcal{X}^N$ in (6.59). The following estimate is valid:

$$\mathcal{X}^N(G_1) \leq CN D^N_{e} + 3, N + 11 (f^\varepsilon) + C_N \| f^\varepsilon \|_{H^{N+3, N+11}}^2 D_0^{N+3, N+11} (f).$$

Proof. By Lemma 6.9 with $q = 0$, we have

$$|\{\partial^\alpha G_1, e_j\} \lesssim |\partial^\alpha f^\varepsilon|_{H^{3, N+11}$ /2} + C_N \sum_{\alpha_1 + \alpha_2 = \alpha} |\partial^{\alpha_1} f^\varepsilon|_{H^{3, N+11}_x} |\partial^{\alpha_2} f|_{H^{3, N+11}_x}.$$

Since $N \geq 4$, by the embedding $H^2 \to L^\infty_x$, we get

$$\mathcal{X}^N(G_1) \leq CN \| f^\varepsilon \|_{H^{N+3, N+11}}^2 H^3_{x} H^{3, N+11}_x + C_N \| f^\varepsilon \|_{H^{N+3, N+11}_x} \| f^\varepsilon \|_{H^{N+3, N+11}_x} \leq CN D_{e}^{N+3, N+11} (f^\varepsilon) + C_N \| f^\varepsilon \|_{H^{N+3, N+11}_x}^2 D_0^{N+3, N+11} (f),$$

thanks to $D_{e}^{N+3, N+11} (f^\varepsilon) \geq \| f^\varepsilon \|_{H^{N+3-j, 3, N+11-3j+\gamma/2}_x} \geq \| f^\varepsilon \|_{H^{N+3-j, 3, N+11-3j+\gamma/2}_x} \geq \| f^\varepsilon \|_{H^{N+3-j, 3, N+11-3j+\gamma/2}_x}$

for any $0 \leq j \leq 3$. \hfill \Box

As another application of Lemma 6.9, we have

Lemma 6.11. Let $N \geq 4, l \geq 3N + 2$. Recall the function $G_1$ in (6.57) and the functional $\mathcal{V}^N$ in (6.60) and $\mathcal{W}^{N, j, l}$ in (6.62). For any $\delta > 0$, it holds that

$$\mathcal{V}^N(G_1) + \mathcal{W}^{N, j, l}(G_1) \leq \delta D_{0}^{N, l} (F^\varepsilon_1) + \delta^{-1} C_N, l (D_{e}^{N+3, j, l+18} (f^\varepsilon)) + \| f^\varepsilon \|_{H^{N+3, j, l+18}_x}^2 D_0^{N+3, j, l+18} (f)).$$

Proof. It suffices to only consider $\mathcal{W}^{N, j, l}(G_1)$. Set $q = l + j, j$. By Lemma 6.9, we have

$$|\{W_q \partial^\alpha G_1, W_q \partial^\alpha F^\varepsilon\}| \lesssim \sum_{\beta \leq \beta} C_{N, q} |\partial^\alpha \partial^\beta f^\varepsilon|_{H^{N+3, N+11}_x} |\partial^\alpha \partial^\beta F^\varepsilon|_{L^{2, N+11}_x}$$

$$+ \sum_{\alpha_1 + \alpha_2 = \alpha, \beta_1 + \beta_2 = \beta} C_{N, q} |\partial^{\alpha_1} f^\varepsilon|_{H^{N+3, N+11}_x} |\partial^{\alpha_2} f|_{H^{N+3, N+11}_x} |\partial^{\alpha_1} \partial^\beta F^\varepsilon|_{L^{2, N+11}_x}.$$

Since $N \geq 4$, by embedding $H^2 \to L^\infty_x$, we get

$$\mathcal{W}^{N, j, l}(G_1) = \sum_{|\alpha| + |\beta| \leq N} |\{W_{l+|\beta|} \partial^\alpha G_1, W_{l+|\beta|} \partial^\alpha F^\varepsilon\}|$$

$$\lesssim C_{N, l} (D_{e}^{N+3, j, l+18} (f^\varepsilon))^{1/2} (D_{0}^{N, l} (F^\varepsilon_1))^{1/2}$$

$$+ C_{N, l} \| f^\varepsilon \|_{H^{N+3, j, l+18}_x} (D_{e}^{N+3, j, l+18} (f))^{1/2} (D_{0}^{N, l} (F^\varepsilon_1))^{1/2},$$

since $D_{e}^{N+3, j, l+18} (f^\varepsilon) \geq \| f^\varepsilon \|_{H^{N+3-j, 3, N+11-3j+\gamma/2}_x}$ for any $0 \leq j \leq N + 3$. Then by the basic inequality $2ab \leq \delta a^2 + \delta^{-1} b^2$, we get the result. \hfill \Box

We are ready to prove (1.30).
Proof of Theorem 1.1 (Asymptotic formula). We give a detailed proof to the case \( N = 4, l = 14 \). For the other two cases, we only illustrate the main differences.

Case 1: \( N = 4, l = 14 \). In this case the constants \( M, L, K_j \) in (6.58) are universal. By (6.52) in Lemma 6.7 for \( C^N_k (f^e, F^e_R) \) and \( C^N_0 (F^e_R, f) \), (6.64) in Lemma 6.8 for \( Y^{N,j,l} (f^e, F^e_R, F^e_R) \) and \( Y^{N,j,0} (F^e_R, f, F^e_R) \), (6.67) in Lemma 6.8 for \( Z^N_k (f^e, F^e_R, F^e_R) \) and \( Z^N_0 (F^e_R, f, F^e_R) \), Lemma 6.10 for \( A^N (G_1) \), Lemma 6.11 for \( V^{N,j,l} (G_1) \) and \( V^{N,j,0} (G_1) \), we get

\[
\frac{d}{dt} \Xi^{4,14} (F^e_R) + \frac{1}{4} D_0^{4,14} (F^e_R) \leq C \| f^e \|_{H^4_{x,v}}^2 + \| f^e \|_{H^4_{x,v}}^\delta D_0^{4,14} (F^e_R)
+ \delta^{-1} C \| F^e_R \|_{H^4_{x,v}}^2 D_0^{4,14} (f)
+ \delta^{-1} C (D_7^{7,32} (f^e) + \| f^e \|_{H^4_{x,v}}^2 D_0^{7,32} (f)).
\]

For the moment, let \( \delta_0 \) be the universal constant such that Theorem 6.1, global well-posedness (1.28) and propagation of regularity (1.29) in Theorem 1.1 are valid.

Let \( \delta \) verify \( C \delta = \frac{1}{16} \). Let \( \delta_4 \) be the largest number verifying \( C (\delta_4 + \delta_4^\frac{1}{2}) \leq \frac{1}{16} \), and \( \delta_4 \leq \delta_0 \). Let \( \delta_5 \) be the largest number verifying \( C \delta_5 \leq \delta_4 \) and \( \delta_5 \leq \delta_0 \). Then by (1.28), if \( \mathcal{E}^{4,14} (f_0) \leq \delta_5 \), we have \( \sup_{t \geq 0} \mathcal{E}^{4,14} (f^e (t)) \leq C \delta_5 \leq \delta_4 \), which gives \( \sup_{t \geq 0} \| f^e (t) \|^2_{H^4_{x,v}} \leq \delta_4 \) since \( \| f^e \|^2_{H^4_{x,v}} \leq \mathcal{E}^{4,14} (f^e) \). Therefore

\[
C (\| f^e \|^2_{H^4_{x,v}} + \| f^e \|_{H^4_{x,v}} + \delta) \leq \frac{1}{8}
\]

and thus

\[
\frac{d}{dt} \Xi^{4,14} (F^e_R) + \frac{1}{8} D_0^{4,14} (F^e_R) \leq C \| F^e_R \|^2_{H^4_{x,v}} D_0^{4,14} (f) + C (D_7^{7,32} (f^e)
+ \| f^e \|^2_{H^4_{x,v}} D_0^{7,32} (f)).
\]

By the propagation result (6.25) in Theorem 6.1, we have for \( \varepsilon \geq 0 \) small enough, that

\[
\mathcal{E}^{7,32} (f^e (t)) + \int_0^t D_\varepsilon^{7,32} (f^e (s)) ds \leq P_{7,32} (\mathcal{E}^{7,32} (f_0)).
\]

Recall the natural relation \( \| F^e_R \|^2_{H^4_{x,v}} \leq \Xi^{4,14} (F^e_R) \), \( \| f^e \|^2_{H^4_{x,v}} \leq \mathcal{E}^{7,32} (f^e) \). By Gronwall’s inequality and the initial condition \( F^e_R (0) = 0 \), we have

\[
\Xi^{4,14} (F^e_R (t)) + \frac{1}{8} \int_0^t D_0^{4,14} (F^e_R (s)) ds \leq \exp(C \int_0^\infty D_0^{4,14} (f (s)) ds) (C \int_0^\infty D_\varepsilon^{7,32} (f^e (s)) ds
+ C \sup_{t \geq 0} \| f^e (t) \|^2_{H^4_{x,v}} \int_0^\infty D_0^{7,32} (f (s)) ds)
\leq C \exp \left( C P_{7,32} (\mathcal{E}^{7,32} (f_0)) \right) P_{7,32} (\mathcal{E}^{7,32} (f_0)) (1 + P_{7,32} (\mathcal{E}^{7,32} (f_0))).
\]
By the equivalence relation (6.26) and recalling \( F_R^ε(t) = |\ln ε|(f^ε(t) - f(t)) \), we get

\[
\mathcal{E}^{4,14}(f^ε(t) - f(t)) + \int_0^t \mathcal{D}^{4,14}_0(f^ε(s) - f(s))ds \\
\leq |\ln ε|^{-2}C \exp \left( CP_{7,32}(\mathcal{E}^{7,32}(f_0)) \right) P_{7,32}(\mathcal{E}^{7,32}(f_0)) (1 + P_{7,32}(\mathcal{E}^{7,32}(f_0))) \\
:= |\ln ε|^{-2}U_{4,14}(\mathcal{E}^{7,32}(f_0)).
\]

**Case 2:** \( N = 4, l > 14 \). We use (6.65) in Lemma 6.8 to deal with \( \mathcal{Y}^N_{j,l}(f^ε, F^ε_R, F^ε_R) \) and \( \mathcal{Y}^N_{0,j,l}(F^ε_R, f, F^ε_R) \). The other terms can be handled in the same way as in **Case 1**. We skip the details here.

**Case 3:** \( N \geq 5, l \geq 3N + 2 \). We use (6.53) in Lemma 6.7 to handle \( \mathcal{C}^N_{j,l}(f^ε, F^ε_R, F^ε_R) \) and \( \mathcal{C}^N_{0,j,l}(F^ε_R, f, F^ε_R) \), (6.66) in Lemma 6.8 to handle \( \mathcal{Y}^N_{j,l}(f^ε, F^ε_R, F^ε_R) \) and \( \mathcal{Y}^N_{0,j,l}(F^ε_R, f, F^ε_R) \), (6.68) in Lemma 6.8 to handle \( Z^N_{j,l}(f^ε, F^ε_R, F^ε_R) \) and \( Z^N_{0,j,l}(F^ε_R, f, F^ε_R) \). As in the **Proof of Theorem 6.1**, we can apply mathematical induction on \( N \) and a sequence of functions \( U_{N,l} \) can be defined in an iterating manner. We skip the details here. However, the smallness assumption on \( \mathcal{E}^{4,14}(f_0) \) (bounded by a universal constant) is not affected in the process. □

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