Finite-Time $H_{\infty}$ Static Output Feedback Control for Itô Stochastic Markovian Jump Systems

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Abstract: This paper focuses on the problem of finite-time $H_{\infty}$ static output feedback control for Itô stochastic systems with Markovian jumps (MJs). First of all, by introducing a new state vector and a novel signal, several sufficient conditions for the existence of static output feedback controllers are established for the considered systems with completely known transition rates (CKTRs) and partially known transition rates (PKTRs), respectively. Then the static output feedback controllers are designed via solving linear matrix inequalities (LMIs), which ensure the closed-loop systems are stochastic $H_{\infty}$ finite-time boundedness. The validity of the developed method was demonstrated through two examples.

Keywords: finite-time; static output feedback; markovian jump; stochastic system
It is worth noting that the above results are obtained through state feedback, while the system state cannot be directly measured in most cases. In addition, \( H_\infty \) control is an important robust control design applied to eliminate the effect of disturbance [21]. In order to overcome the difficulty of state measurement and attenuate the exogenous disturbance, the research on output feedback control has been developed. Among them, static output feedback control (SOFC) has drawn the attention of a number of investigators owing to its low maintenance cost and implementation in various output feedback control schemes [22–24]. To be more specific, for the finite-time SOFC problem of MJs, sufficient conditions for LMLs with fixed parameter constraints are given in [22]. In recent years, stochastic \( H_2/H_\infty \) control for MJs has become a popular area of research, in a context of partial information, the design of \( H_\infty \), and mixed \( H_2/H_\infty \) static output feedback controllers for Markov jump linear systems were studied [23]. Furthermore, the finite-time static output feedback \( H_\infty \) controller was designed for Takagi–Sugeno fuzzy nonlinear systems with time-varying transition rates [24]. In [25], by introducing two new signals, sufficient conditions that the closed-loop systems are stochastic finite-time boundness with the given \( H_\infty \) performance were obtained. From what has been discussed above, one can note that, at present, less research about static output feedback control focuses on Itô system. In this article, we will settle the issue of finite-time \( H_\infty \) static output feedback control for Itô systems with MJs.

The paper is outlined as follows. The problem is illustrated and some useful definitions are given in Section 2. Section 3 gives the sufficient conditions such that the closed-loop Itô stochastic systems with MJs is expressed as \( M^T \); \( A \geq 0(\lambda \leq 0) \): \( A \) is semi-positive (negative) definite symmetric matrix; \( \ast \) is the symmetric hidden matrix entries; The maximum and minimum eigenvalues of the matrix \( M \) are shown by \( \lambda_{\max}(M) \) and \( \lambda_{\min}(M) \). \( E \{ \cdot \} \) expresses the mathematical expectation; \( He(A + B) = (A + B) + (A + B)^T \). For matrices, unless otherwise specified, it is assumed that they have the appropriate dimensions.

2. Problem Statement and Preliminaries

Consider linear Itô stochastic system with Markovian jumps described by

\[
\begin{align*}
\dot{x}(t) &= [A(r_1)x(t) + B_1(r_1)z(t) + B_2(r_1)u(t)]dt + A_1(r_1)x(t)dW(t), \\
z(t) &= C(r_1)x(t) + D_1(r_1)z(t) + D_2(r_1)u(t), \\
y(t) &= E(r_1)x(t) + F(r_1)z(t), t \in [0, T^r]
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \), \( v(t) \in \mathbb{R}^l \), \( u(t) \in \mathbb{R}^m \), \( y(t) \in \mathbb{R}^r \) and \( z(t) \in \mathbb{R}^p \) denote the system state, the disturbance signal, control input, measured output and control output, respectively. \( W(t) \) is a one-dimensional standard Brownian motion which satisfies \( E\{dW(t)\} = 0 \) and \( E\{dW(t)^2\} = dt \); \( r_1 \) is a continuous homogeneous Markovian jump process taking values in a finite state space \( S = \{1, 2, \cdots, N\} \) with transition probability matrix \( \Pi = \{\pi_{ij}\}_{N \times N} \) given by

\[
Pr\{r_{t+1} = j | r_t = i\} = \begin{cases} 
\pi_{ij}h + o(h), & i \neq j, \\
1 + \pi_{ij}h + o(h), & i = j,
\end{cases}
\]
with $h > 0$, $\lim_{h \to 0} h = 0$, $\pi_{ij}$ represents the transition probability from mode $i$ to mode $j$, which satisfies $\pi_{ij} \geq 0 (i \neq j)$ and $\pi_{ii} = -\sum_{j=1,j \neq i}^{N} \pi_{ij}$. As a sequence, the corresponding transition probability matrix is

$$
\Pi_1 = \begin{bmatrix}
\pi_{11} & \pi_{12} & \cdots & \pi_{1N} \\
\pi_{21} & \pi_{22} & \cdots & \pi_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\pi_{N1} & \pi_{N1} & \cdots & \pi_{NN}
\end{bmatrix}.
$$

However, it is hard to measure transition rates exactly. In this work, we also consider that the transition rates of $r_i$ is partially available. That is to say, some elements in $\Pi$ are unknown. It is assumed that the transition rate matrix $\Pi$ with two operation modes

$$
\Pi_2 = \begin{bmatrix}
\pi_{11} & ? \\
? & ?
\end{bmatrix},
$$

where '?' is unknown element and $\pi_{ij}$ is known. All above information was covered by two sets to formulate the accessibility of transition probability concisely,

$$
S_k = \{ j : \text{if } \pi_{ij} \text{ is known}\},
$$

(2)

$$
S_{uk} = \{ j : \text{if } \pi_{ij} \text{ is unknown}\}.
$$

(3)

If $S_k \neq \emptyset$, then

$$
S_k = \{k_1, k_2, \cdots, k_l\}, \ 1 \leq l \leq N,
$$

where $k_l \in \mathbb{N}^+$, expresses as the $m$th known element in the $m$th row of matrix $\Pi$. Then let’s assume that $r_i$ and $W(t)$ are independent, and the external disturbance signal $v(t)$ satisfies

$$
\int_0^1 v^T(s)v(s)ds < d^2, \ d > 0.
$$

To facilitate the following presentation, for $r_i = i \in S$, the $i$th mode system matrices can be simplified as $A_i, B_{1i}, B_{2i}, A_{1i}, C_i, D_{1i}, D_{2i}, E_i$ and $F_i$, which are with appropriate dimension.

Now, we consider the following static output feedback controller

$$
u(t) = K_{r_i}y(t),
$$

(4)

where $K_{r_i}$ is an output feedback gain matrix to be designed and is given by $K_i$ for $r_i = i$. Therefore, system (1) with controller (4) can be rewritten as follows:

$$
\begin{cases}
dx(t) = [(A_1 + B_2K_iE_i)dt + A_1dW(t)]x(t) + (B_1 dt + B_2K_iF_i dt)v(t), \\
z(t) = (C_i + D_2K_iE_i)x(t) + (D_{1i} + D_{2i}K_iF_i)v(t),
i \in S, t \in [0, T^*].
\end{cases}
$$

(5)

Before proceeding further, the following fundamental definitions and lemmas are introduced, which play a key role in this paper.

**Definition 1** (stochastic finite-time boundedness (SFTB)). The linear Itô stochastic system (5) is said to SFTB with respect to (w.r.t) $(c_1, c_2, T^*, R, d)$, if it has an impulse-free solution in the time interval $[0, T^*]$ and satisfies

$$
E\{x^T(0)Rx(0)\} \leq c_1 \Rightarrow E\{x^T(t)Rx(t)\} < c_2, \forall t \in [0, T^*].
$$

holds for $R > 0, c_1 > 0, c_2 > 0$ and $c_1 < c_2$. 
Remark 1. When \( v(t) = 0 \), the definition of SFTB can be reduced to stochastic finite-time stable with respect to \((c_1, c_2, T^*, R)\).

Remark 2. Definition 1 can be interpreted as: for a given initial condition a bound and a fixed time interval, during this time interval, if state is maintained in a region of ellipsoid shape in the mean square sense, then this linear stochastic system is stochastic finite-time stable.

Definition 2 (stochastic \( H_\infty \) finite-time boundedness (SH\( H_\infty \)FTB)). The linear Itô stochastic system (5) is said to SH\( H_\infty \)FTB w.r.t \((c_1, c_2, T^*, R, \gamma, d)\), if under the zero initial condition, it is SFTB w.r.t \((c_1, c_2, T^*, R, d)\) and satisfies the following inequality

\[
E\{\int_0^{T^*} z^T(t)z(t)dt\} < \gamma^2 E\{\int_0^{T^*} v^T(t)v(t)dt\}. \tag{6}
\]

In this article, our main target is to design a static output feedback controller (4) such that the closed-loop system (5) is SH\( H_\infty \)FTB.

3. Main Results

3.1. MJS with CKTR

In this part, we give the main results through two theorems. First of all, we solve the problem of SH\( H_\infty \)FTB for MJS (5) with completely known transition rate (CKTR), by introducing two new signals under the output feedback gain \( K_i = J_i P_i^{-1} \).

**Theorem 1.** For constants \( \alpha, \phi > 0 \), system (5) with CKTR is SH\( H_\infty \)FTB w.r.t \((c_1, c_2, T^*, R, \gamma, d)\), if there exist matrices \( J_i, P_i \), positive definite matrix \( Y_i \), scalars \( \rho_1, \rho_2 > 0 \), such that the following matrix inequalities hold

\[
\rho_1 I < \hat{\gamma}^{-1}_i < \rho_2 I \tag{7}
\]

\[
c^{\alpha T^*} c_1 \rho_2 + \gamma^2 d^2 - c_2 \rho_1 < 0 \tag{8}
\]

\[
\begin{bmatrix}
\Theta_{11} & \Theta_{12} & 0 & -\phi F_i & J_i^T D_{2i}^T \\
* & \Theta_{22} & A_i Y_i & B_{1i} & \Theta_{24} \\
* & * & -Y_i & 0 & 0 \\
* & * & * & -\gamma^2 e^{-\alpha T^*} I & D_{1i}^T \\
* & * & * & * & -I
\end{bmatrix} < 0, \tag{9}
\]

where

\[
\Theta_{11} = -\phi (P_i^T + P_i),
\]

\[
\Theta_{12} = \phi (P_i E_i - E_i Y_i) + J_i^T B_{2i},
\]

\[
\Theta_{22} = He(A_i Y_i + B_{2i} J_i E_i) - \alpha Y_i + Y_i^T \sum_{j=1}^{N} \pi_{ij} Y_j^{-1} Y_i,
\]

\[
\Theta_{24} = Y_i C_i^T + E_i J_i^T D_{2i}^T,
\]

\[
Y_i^{-1} = R_i \hat{\gamma}^{-1}_i R_i^T.
\]

Moreover, a expected static output feedback controller is given in the form of (4) with \( K_i = J_i P_i^{-1} \).

**Proof.** First, define a new state vector \( \mu(t) \) and a novel signal \( \varphi(t) \): \[
\mu(t) = Y_i^{-1} x(t), \tag{10}
\]

\[
\varphi(t) = \hat{\gamma}^{-1}_i \mu(t) + \gamma^2 d^2 - c_2 \rho_1 < 0.
\]


\( \mathcal{q}(t) = E_i \mu(t) - P_i^{-1} y(t). \) \hspace{1cm} (11)

Then,
\[ V(x(t), i) = x^T(t) Y_i^{-1} x(t), \] \hspace{1cm} (12)

where \( Y_i > 0. \) Pre-multiplying (11) by \( I_i \) results in
\[ K_i y(t) = I_i E_i \mu(t) - I_i \mathcal{q}(t). \] \hspace{1cm} (13)

Using Schur complement, (9) is equivalent to
\[ \begin{bmatrix}
-\Phi(P_i^T + P_i) & \Phi(P_i E_i - E_i Y_i) + J_i^T B_{2i}^T & -\Phi F_i \\
* & \Delta_{22} & B_{1i} \\
* & * & -\gamma^2 e^{-\alpha T} I
\end{bmatrix} < 0, \] \hspace{1cm} (14)

where \( \Delta_{22} = H e(A_i Y_i + B_{2i} I_i E_i) + (A_i Y_i)^T Y_i^{-1} A_i Y_i - \alpha Y_i + Y_i^T \sum_{j=1}^N \pi_{ij} Y_j^{-1} Y_i. \) Together with (10)–(13), the closed-loop system (5) can be rewritten as
\[ \begin{cases}
\dot{x}(t) = (A_i Y_i d t + B_{2i} I_i E_i d t + A_i Y_i d \omega) \mu(t) + B_{1i} d t \upsilon(t) - B_{2i} I_i \mathcal{q}(t) d t, \\
z(t) = (C_i Y_i + D_{2i} I_i E_i) \mu(t) + D_{1i} \upsilon(t) - D_{2i} I_i \mathcal{q}(t).
\end{cases} \]

Pre-multiplying (11) by \( P_i \), we can get the following equation associated with \( \mathcal{q}(t), \mu(t) \) and \( \upsilon(t) \)
\[ \xi_i = (P_i E_i - E_i Y_i) \mu(t) - F_i \upsilon(t) - P_i \mathcal{q}(t) = 0. \]

Defining \( \xi \) as the weak infinitesimal operator, and using Itô’s formula, (6) will satisfy if
\[ \dot{\xi} V(x(t), i) - \alpha V(x(t), i) - \gamma^2 e^{-\alpha T} \upsilon^T(t) \upsilon(t) + z^T(t) z(t) < 0. \] \hspace{1cm} (15)

To this end, pre and post-multiplying (14) by \( \Phi \) and \( \Phi^T \), we can obtain
\[ \Phi \begin{bmatrix}
-\Phi(P_i^T + P_i) & \Phi(P_i E_i - E_i Y_i) + J_i^T B_{2i}^T & -\Phi F_i \\
* & \Delta_{22} & B_{1i} \\
* & * & -\gamma^2 e^{-\alpha T} I
\end{bmatrix} \Phi^T < 0, \] \hspace{1cm} (16)

where
\[ \Phi = \begin{bmatrix} \mathcal{q}(t) & \mu(t) & \upsilon(t) & z(t) \end{bmatrix}, \]
\[ \Delta_{22} = H e(A_i Y_i + B_{2i} I_i E_i) + (A_i Y_i)^T Y_i^{-1} A_i Y_i - \alpha Y_i + Y_i^T \sum_{j=1}^N \pi_{ij} Y_j^{-1} Y_i, \]
\[ Y_i^{-1} = R_i^2 \gamma_i^{-1} R_i^2. \]

We can see that (16) is equivalent to
\[ \dot{\xi} V(x(t), i) - \alpha V(x(t), i) - \gamma^2 e^{-\alpha T} \upsilon^T(t) \upsilon(t) + 2 \Phi \mathcal{q}^T(t) \xi_i + z^T(t) z(t) < 0. \]

Therefore, (15) holds.

By (15), we can get
\[ \dot{\xi} V(x(t), i) - \alpha V(x(t), i) - \gamma^2 e^{-\alpha T} \upsilon^T(t) \upsilon(t) < 0, \] \hspace{1cm} (17)
Theorem 2. Given positive scalars $\beta$ and $\varphi$, system (5) with CKTR is $S_{\infty}$FTB w.r.t. $(c_1, c_2, T^*, R, \gamma, d)$, if there exist matrices $Q_i, L_i$, and positive definite matrices $X_i (i \in S)$, scalars $\rho_1, \rho_2 > 0$ satisfying

$$\rho_1 R < X_i < \rho_2 R,$$

then, taking the mathematical expectation of (17), we have

$$E\{\xi e^{-at}V(x(t), i)\} < \gamma^2 e^{-a(t-T^*)}E\{v^T(\tau)v(\tau)\},$$

integrating both sides of (18) from 0 to $t$, with $t \in [0, T^*]$, it can be deduced that

$$e^{-at}E\{V(x(t), i)\} < E\{V(x(0), r_0)\} + \gamma^2 E\{\int_0^{t} e^{-a(\tau-T^*)}v^T(\tau)v(\tau)d\tau\},$$

pre-multiplying (19) by $e^{at}$, it is easy to see

$$E\{V(x(t), i)\} < e^{at}E\{V(x(0), r_0)\} + \gamma^2 e^{at}\rho_{\max}(\hat{Y}_i^{-1})x^T(0)Rx(0) + \gamma^2 e^{at}d^2 \leq e^{at}\rho_{\max}(\hat{Y}_i^{-1})c_1 + \gamma^2 e^{at}d^2.$$

Since

$$E\{V(x(t), i)\} \geq \rho_{\min}(\hat{Y}_i^{-1})E\{x^T(t)Rx(t)\},$$

it can be verified that

$$E\{x^T(t)Rx(t)\} < \frac{e^{at}\rho_{\max}(\hat{Y}_i^{-1})c_1 + \gamma^2 e^{at}d^2}{\rho_{\min}(\hat{Y}_i^{-1})} < c_2.$$

From Definition 1, system (5) with CKTR is SFTB.

Multiplying (15) with $e^{-at}$ and taking mathematical expectation

$$E\{\xi [e^{-at}V(x(t), i)]\} < E\{e^{-at}[\gamma^2 e^{-at}v^T(t)v(t) - z^T(t)z(t)]\},$$

integrating (20) from 0 to $t$, the result follows

$$E\{\int_0^{t} e^{-a(h)}[z^T(h)z(h) - \gamma^2 e^{-a(t)}v^T(h)v(h)]dh\} < 0.$$

For all $t \in [0, T^*]$, it implies that

$$E\{\int_0^{T^*} z^T(t)z(t)dt\} \leq e^{-at}E\{\int_0^{T^*} e^{-a(t)}z^T(t)z(t)dt\} < e^{at}E\{\int_0^{T^*} \gamma^2 e^{-a(t+T^*)}v^T(t)v(t)dt\} < \gamma^2 E\{\int_0^{T^*} v^T(t)v(t)dt\}.$$

From the above, the closed-loop system (5) with CKTR is $S_{\infty}$FTB with the proposed static output feedback controller.  

**Remark 3.** By the introducing $\mu(t)$ and $\varphi(t)$, we can decouple the connection between input matrix and output matrix. We use a zero term $\zeta_i$ to make the conditions given in Theorem 1 solvable via linear matrix inequalities.

Considering another form of the output feedback gain matrix $K_i = Q_i^{-1}L_i$, and the corresponding theorem is given as follows.

**Theorem 2.** Given positive scalars $\beta$ and $\varphi$, system (5) with CKTR is $S_{\infty}$FTB w.r.t. $(c_1, c_2, T^*, R, \gamma, d)$, if there exist matrices $Q_i, L_i$, and positive definite matrices $X_i (i \in S)$, scalars $\rho_1, \rho_2 > 0$ satisfying

$$\rho_1 R < X_i < \rho_2 R,$$
\[ e^{\beta T} c_1 \rho_2 + \gamma^2 \rho_1^2 - c_2 \rho_1 < 0, \]

\[
\begin{bmatrix}
\Xi_{11} & \Xi_{12} & L_i F_i & \varphi D_{2i}^T \\
* & \Xi_{22} & \Xi_{23} & C_i^T \\
* & * & -\gamma^2 e^{-\beta T} I & D_i^T \\
* & * & * & -I
\end{bmatrix} < 0,
\]  \hspace{1cm} (21)

where

\[ \Xi_{11} = -\varphi (Q_i^T + Q_i), \]
\[ \Xi_{12} = \varphi (X_i B_{2i} - B_{2i} Q_i)^T + L_i E_i, \]
\[ \Xi_{22} = H e(A_i X_i + B_{2i} L_i E_i) + A_i^T X_i A_{1i} - \beta X_i + \sum_{j=1}^N \pi_{ij} X_j, \]
\[ \Xi_{23} = X_i B_{1i} + B_{2i} L_i F_i. \]

Beyond that,

\[ K_i = Q_i^{-1} L_i. \]

**Proof.** Set \( \mu(t) = x(t) \) and \( \varphi(t) = \varphi^{-1} u(t) \) and construct the following function

\[ V(x(t), i) = x^T(t) X_i x(t), \]

where \( X_i > 0. \)

By \( K_i = Q_i^{-1} L_i, \) multiply \( u(t) = K_i \varphi(t) \) from the left by \( Q_i, \) one has

\[ \zeta_i = L_i E_i x(t) + L_i F_i \varphi(t) - Q_i u(t) = 0. \]

Then, (1) is described

\[
\begin{align*}
\frac{dx(t)}{dt} &= A_i x(t) dt + B_{1i} \varphi(t) dt + (B_{2i} - X_i^{-1} B_{2i} Q_i) u(t) dt + A_{1i} x(t) d\omega \\
&\quad + X_i^{-1} (B_{2i} L_i E_i x(t) + B_{2i} L_i F_i \varphi(t)) dt.
\end{align*}
\]  \hspace{1cm} (22)

Pre-multiplying (22) by \( X_i, \) we can get

\[
\begin{align*}
X_i \frac{dx(t)}{dt} &= (X_i A_i dt + X_i A_{1i} d\omega + B_{2i} L_i E_i dt) x(t) + (X_i B_{1i} dt + B_{2i} L_i F_i dt) \varphi(t) \\
&\quad + \varphi(X_i B_{2i} dt - B_{2i} Q_i dt) \varphi(t).
\end{align*}
\]

According to Definition 2, we prove that the following inequality holds

\[
E \{ \xi V(x(t), i) - \beta V(x(t), i) - \gamma^2 e^{-\beta T} \varphi^T(t) \varphi(t) + z^T(t) z(t) \} < 0.
\]  \hspace{1cm} (23)

By taking a similar approach to Theorem 1, (21) can be rewritten as

\[
E \{ \xi V(x(t), i) - \beta V(x(t), i) + 2 \xi^T(t) \xi_i - \gamma^2 e^{-\beta T} \varphi^T(t) \varphi(t) + z^T(t) z(t) \} < 0.
\]  \hspace{1cm} (24)

It is easy to see that (24) is correct from (23).

Then, using the same method as Theorem 1, we can easily prove that the closed-loop system (5) with CKTR is \( SH_{\infty} \) FTB with the output feedback gain matrix \( K_i = Q_i^{-1} L_i. \)

3.2. MJS with PKTR

The following theorems will give a sufficient condition of \( SH_{\infty} \) FTB for system (5) with PKTR.
**Theorem 3.** Given positive scalars $\alpha$ and $\phi$, system (5) with PKTR is $SH_{\omega FTB}$ w.r.t. $(c_1, c_2, T^*, R, \gamma, d)$, if there exist matrices $\hat{Y}_i$, positive definite matrix $Y_i$, two positive scalars $\rho_1, \rho_2$, satisfying

$$\rho_1 I < \hat{Y}_i^{-1} < \rho_2 I,$$

$$\alpha^T c_1 \rho_2 + \gamma^2 d^2 - c_2 \rho_1 < 0,$$

$$\begin{bmatrix}
-\phi(\hat{P}_i^T + \hat{P}_i) & \phi(\hat{P}_i E_i - E_i Y_i) + \hat{F}_i^T B_i^T & 0 & -\phi F_i & \hat{F}_i^T D_i^T \\
* & \Lambda_{22} & A_{1i} Y_i & B_{1i} & Y_i C_i^T + \hat{E}_i^T \hat{F}_i^T D_i^T \\
* & * & -Y_i & 0 & 0 \\
* & * & * & -\gamma^2 \alpha^T I & D_i^T \\
* & * & * & * & -I
\end{bmatrix} < 0,$$

where

$$\Omega_j = He(A_i Y_i + B_2 \hat{Y}_i),$$

$$\Lambda_{22} = (1 + Y_i^T \sum_{j \in S_i} \pi_{ij}) \Omega_j + Y_i^T \sum_{j \in S_i} \pi_{ij} Y_j^{-1} Y_i + Y_i^T \sum_{j \in S_{ik}} \pi_{ij} (\Omega_j + Y_j^{-1}) - \alpha Y_i.$$

The static output feedback controller gain is given by $K_i = \hat{Y}_i^{-1}$.

**Proof.** Using Schur complement, (25) is equivalent to

$$\begin{bmatrix}
-\phi(\hat{P}_i^T + \hat{P}_i) & \phi(\hat{P}_i E_i - E_i Y_i) + \hat{F}_i^T B_i^T & 0 & -\phi F_i & \hat{F}_i^T D_i^T \\
* & \Lambda_{22} & A_{1i} Y_i & B_{1i} & Y_i C_i^T + \hat{E}_i^T \hat{F}_i^T D_i^T \\
* & * & -Y_i & 0 & 0 \\
* & * & * & -\gamma^2 \alpha^T I & D_i^T \\
* & * & * & * & -I
\end{bmatrix} < 0,$$

where

$$Y_{22} = (1 + Y_i^T \sum_{j \in S_i} \pi_{ij}) \Omega_j + (A_{1i} Y_i)^T Y_i^{-1} A_{1i} Y_i + Y_i^T \sum_{j \in S_i} \pi_{ij} Y_j^{-1} Y_i + Y_i^T \sum_{j \in S_{ik}} \pi_{ij} (\Omega_j + Y_j^{-1}) - \alpha Y_i.$$

Because of $\sum_{j=1}^N \pi_{ij} = 0$, then

$$\Delta_{22} = \Omega_j + (A_{1i} Y_i)^T Y_i^{-1} (A_{1i} Y_i) - \alpha Y_i + Y_i^T \sum_{j=1}^N \pi_{ij} Y_j^{-1} Y_i + Y_i^T \sum_{j=1}^N \pi_{ij} (\Omega_j + Y_j^{-1}).$$

Together with (2) and (3),

$$\Delta_{22} = \Omega_j + (A_{1i} Y_i)^T Y_i^{-1} (A_{1i} Y_i) + Y_i^T \sum_{j \in S_i} \pi_{ij} Y_j^{-1} Y_i + Y_i^T \sum_{j \in S_{ik}} \pi_{ij} (\Omega_j + Y_j^{-1}) - \alpha Y_i + \gamma^2 d^2.$$ 

The rest of the proof is similar to Theorem 1, so it is omitted. □

Considering another form of the output feedback gain matrix $K_i = \hat{Q}_i^{-1} \hat{L}_i$, and alternative theorem is presented below.

**Theorem 4.** The closed-loop system (5) with PKTR is $SH_{\omega FTB}$ w.r.t. $(c_1, c_2, T^*, R, \gamma, d)$, if there exist matrices $\hat{Q}_i, \hat{L}_i$, and positive definite matrix $X_i (i \in S)$, two positive scalars $\rho_1, \rho_2$ and given positive scalars $\beta$ and $\varphi$, satisfying

$$\rho_1 R < X_i < \rho_2 R,$$
The static output feedback controller gain is given by 

\[
K = \text{Shen M et al. [25]},
\]

and this paper investigates the static output feedback control problem of Itô stochastic system. It is not easy to deal with the condition (7) and (8) by applying LMIs. To address this problem, set \( \hat{Y}_i^{-1} \) with \( 1 < \hat{Y}_i^{-1} < \delta^{-1}I \) and \( Y_i^{-1} = R \hat{Y}_i^{-1} R^T \), then (7) and (8) can be replaced with following constraints

\[
\delta R^{-1} < Y_i^{-1} < R^{-1}, \tag{26}
\]

\[
\begin{bmatrix}
    e^{-\alpha T}(-c_2 + \gamma^2 d^2) & \sqrt{\gamma^2} \\
    \sqrt{\gamma^2} & -\delta I
\end{bmatrix} < 0. \tag{27}
\]

Through this transformation, the static output controller can be calculated by LMIs (26), (27) and (9) or (25).

Remark 4. It is not easy to deal with the condition (7) and (8) by applying LMIs. To address this problem, set \( \hat{Y}_i^{-1} \) with \( 1 < \hat{Y}_i^{-1} < \delta^{-1}I \) and \( Y_i^{-1} = R \hat{Y}_i^{-1} R^T \), then (7) and (8) can be replaced with following constraints

\[
\delta R^{-1} < Y_i^{-1} < R^{-1}, \tag{26}
\]

\[
\begin{bmatrix}
    e^{-\alpha T}(-c_2 + \gamma^2 d^2) & \sqrt{\gamma^2} \\
    \sqrt{\gamma^2} & -\delta I
\end{bmatrix} < 0. \tag{27}
\]

Remark 5. The conditions given in Theorem 1 can be transformed into the problem as follows

\[
\min_c c_2^2 + \gamma^2 \\
\text{s.t. } \tag{28}
\]

Remark 6. The work of static output feedback control problem in deterministic systems has been studied by Shen M et al. [25], and this paper investigates the static output feedback control problem of Itô stochastic system through Theorems 1 and 2. In addition, compared with [25], this paper also studies the SH∞FTB for system (5) with partially known transition rates in Theorems 3 and 4.

4. Numerical Examples

Simulations of two examples are presented in this section to verify the effectiveness and applicability of the proposed method in this paper.

Example 1. Consider system (5) with two modes and the transition probability is completely known.

Mode 1:

\[
A_1 = \begin{bmatrix}
-3 & 1 \\
-2 & -5
\end{bmatrix}, \ A_{11} = \begin{bmatrix}
2.1 & 0.9 \\
1.5 & 1.8
\end{bmatrix}, \ B_{11} = \begin{bmatrix}
2 \\
1
\end{bmatrix}, \ B_{21} = \begin{bmatrix}
1 \\
1
\end{bmatrix}, \ C_1 = \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix},
\]

\[
D_{11} = \begin{bmatrix}
0.2 \\
0.2
\end{bmatrix}, \ D_{21} = \begin{bmatrix}
0.4 \\
0.2
\end{bmatrix}, \ E_1 = \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}, \ F_1 = \begin{bmatrix}
0.1 \\
0.1
\end{bmatrix}.
\]

Mode 2:

\[
A_2 = \begin{bmatrix}
-5 & -2 \\
4 & -1
\end{bmatrix}, \ A_{12} = \begin{bmatrix}
2.1 & 1.1 \\
1 & 1.2
\end{bmatrix}, \ B_{12} = \begin{bmatrix}
1 \\
3
\end{bmatrix}, \ B_{22} = \begin{bmatrix}
1 \\
1
\end{bmatrix}, \ C_2 = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]
\[
D_{12} = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \quad D_{22} = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix}, \quad E_2 = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.
\]

The transiting between the two modes is
\[
\Pi_1 = \begin{bmatrix} -0.3 & 0.3 \\ 0.35 & -0.35 \end{bmatrix}.
\]

The external disturbance \(v(t) = \frac{0.1}{1+7t}\). One possible mode evolution is given in Figure 1.

![Figure 1. One possible Markov mode evolution.](image)

Given \(\alpha = 1, c_1 = 0.6, T^* = 1, d = 2, R = I, x_0 = [0 \ 0]^T\). With the help of MATLAB, from Theorem 1, it yields that \(c_2 = 11.064, \gamma = 3.820(\epsilon_1 = 1, \epsilon_2 = 1)\). The static output feedback gains are calculated as follows:

\[
K_1 = \begin{bmatrix} -0.3238 \\ -0.5388 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.0322 \\ -0.0287 \end{bmatrix}.
\]

With these conditions, we can obtain the state response curves of closed-loop system (5) in Figure 2 and the simulated curves of \(E[x^T(t)Rx(t)]\) in Figure 3. From Figures 2 and 3, we can see that (5) with CKTR is \(SH_\infty\) FTB.

It should note that the optimization problem (28) relies on the parameter \(\alpha\), once the value of \(\alpha\) is determined, we can treat (28) as a linear matrix inequality. By using the linear search algorithm, we can find feasible solution when \(0 \leq \alpha \leq 3\). Figures 4 and 5 show the optimal value with different value of \(\alpha\).
Figure 2. The response of system (5).

Figure 3. The trajectory of $E[x^T(t)Rx(t)]$. 
Figure 4. The local optimal bound of $c_2$ when $\alpha \in [0,3]$.

Figure 5. The local optimal bound of $\gamma$ when $\alpha \in [0,3]$. 
Example 2. Consider system (5) with two modes, and the transition probability is partially known. In this case, the system parameters of Example 1 are used. The transition probability matrix between the two modes is

$$\Pi_2 = \begin{bmatrix} -1.2 & 1.2 \\ . & . \end{bmatrix}.$$ 

By using the relevant data from Example 1: $a = 1, c_1 = 0.6, T^* = 1, d = 2, R = I, x_0 = [0, 0]^T$, from Theorem 3, we have $c_2 = 11.064, \gamma = 3.820(e_1 = 1, e_2 = 1)$, the static output feedback gains

$$K_1 = \begin{bmatrix} -0.3238 \\ -0.5388 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.0322 \\ -0.0287 \end{bmatrix}.$$ 

For the simulation purpose, it is assumed that the actual partially unknown transition rate matrix is

$$\Pi_1 = \begin{bmatrix} -1.2 & 1.2 \\ 0.5 & -0.5 \end{bmatrix}.$$ 

In this case, we can get another possible Markov mode evolution Figure 6. Meanwhile, Figures 7 and 8 are the simulated curves of $E[x^T(t)Rx(t)]$ and the state response curves of system (5), which show that MJS (5) with partially known transition rates is $SH_{\infty}FTB$.

Figure 6. Another possible Markov mode evolution.
Figure 7. State response curves of closed-loop system.

Figure 8. The trajectory of $E[x^T(t)Rx(t)]$. 
5. Conclusions

In this paper, the problem of finite-time $H_\infty$ static output feedback control for Itô stochastic systems with Markovian jump has been studied. By introducing a new state vector and a novel signal, several new sufficient conditions are presented for the considered system with completely known or partially unknown transition rates to be $SH_\infty$-FTB. At last, the validity of proposed methods is verified by two numerical examples.

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