GLOBAL CLASSICAL SOLUTIONS TO PARTIALLY DISSIPATIVE HYPERBOLIC SYSTEMS VIOLATING THE KAWASHIMA CONDITION

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ABSTRACT. This paper considers the Cauchy problem for the quasilinear hyperbolic system of balance laws in $\mathbb{R}^d$, $d \geq 2$. The system is partially dissipative in the sense that there is an eigen-family violating the Kawashima condition. By imposing certain supplementary degeneracy conditions with respect to the non-dissipative eigen-family, global unique smooth solutions near constant equilibria are constructed. The proof is based on the introduction of the partially normalized coordinates, a delicate structural analysis, a family of scaled energy estimates with minimum fractional derivative counts and a refined decay estimates of the dissipative components of the solution.

1. INTRODUCTION

Consider the following $n$-component quasilinear hyperbolic system of balance laws in $d$ space dimensions:

$$
\begin{cases}
\partial_t (G(u)) + \sum_{k=1}^{d} (F^k(u))_{x_k} = S(u), \\
u |_{t=0} = u_0.
\end{cases}
$$

(1.1)

Here $u = (u_1, \ldots, u_n)^T : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^n$ is the unknown, $G = (G_1, \ldots, G_n)^T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $F^k = (F^k_1, \ldots, F^k_n)^T : \mathbb{R} \rightarrow \mathbb{R}^n (k = 1, \ldots, d)$ and $S = (S_1, \ldots, S_n)^T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are given smooth functions. It is assumed that $\nabla_u G(u)$ is invertible, hence, by the chain rule, the system can be rewritten into

$$
\begin{cases}
\partial_t u + \sum_{k=1}^{d} A^k(u) u_{x_k} = Q(u), \\
u |_{t=0} = u_0.
\end{cases}
$$

(1.2)

Here the coefficient matrices $A^k(u) = (\nabla_u G(u))^{-1} \nabla_u F^k(u)$ ($k = 1, \ldots, d$) and the inhomogeneous term $Q(u) = (\nabla_u G(u))^{-1} S(u)$. By hyperbolicity, for any $u \in \mathbb{R}^n$ and any $\omega = (\omega_1, \ldots, \omega_d)^T \in \mathbb{S}^{d-1}$ the matrix

$$A(u, \omega) := \sum_{k=1}^{d} \omega_k A^k(u)$$

(1.3)

has $n$ real eigenvalues $\lambda_1(u, \omega), \ldots, \lambda_n(u, \omega)$ and a complete set of left (resp. right) eigenvectors $l_i(u, \omega)$ ($r_i(u, \omega)$) ($i = 1, \ldots, n$) are smooth with respect to $u$ and $\omega$, while $l_i(u, \omega)$, $r_i(u, \omega)$ ($i = 1, \ldots, n$) are all smooth functions of $u$ for any given $\omega \in \mathbb{S}^{d-1}$. One may normalize the eigenvectors so that

$$l_i(u, \omega) r_{i'}(u, \omega) \equiv \delta_{ii'}, \quad i, i' = 1, \ldots, n.$$  

(1.4)

It is well known that for any given suitably smooth initial data $u_0$, the system (1.1) has a unique local smooth solution [11] [19], but in general the solution develops singularities in finite time even for small and smooth initial data [4] [18] [19].

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It has been known that there are two structures that can prevent the finite-time singularity formation. One of them is the (weak) linear degeneracy for the homogeneous systems, which was firstly introduced in 1D study (see [3] and [10]). The other one is the dissipation effect induced by inhomogeneous terms. A well-known assumption in this aspect is given by the strict dissipation, which requires a damping term to enter into each of the equations of the system, see [10]. Notice that for some physical problems, this condition is too strong, while in many applications and as the interest in this paper, $Q(u)$ takes the form

$$Q(u) = (0, \ldots, 0, Q_{r+1}(u), \ldots, Q_n(u))^T,$$

which seems to produce the dissipation effect only in the last $n - r$ equations. In this case, the system (1.1) is referred to as a partially dissipative hyperbolic system in many literatures, such as [12, 23, 27, 25].

A constant vector $u_* \in \mathbb{R}^n$ is called an equilibrium state of the system (1.1) if $S(u_*) = 0$, or equivalently, $Q(u_*) = 0$. The main subject of this paper is to search for suitable structural formulations, and one of them reads as

$$(\text{A1}) \text{ The matrix } \left(\frac{\partial q_p}{\partial u_i}(0)\right)_{n, n} \text{ is invertible.}$$

$$(\text{A2}) \text{ There exist a strictly convex smooth entropy function } \eta(G) \text{ and } d \text{ smooth entropy flux functions } \psi^k(G) \text{ for the system (1.1) in terms of } G, \text{ such that for all } G,$$

$$(\nabla_G \psi^k(G))^T = (\nabla_G \eta(G))^T \nabla_G (F_k(u(G))).$$

This implies that if one defines $\eta(u) = \eta(G(u)) = \eta(0) - \nabla_G \eta(0) \cdot G(u)$ and $\psi^k(u) = \psi^k(G(u)) - \nabla_G \eta(0) \cdot F_k(u)$, then $\eta(0) = 0$, $\nabla_u \eta(0) = 0$, $\nabla_u^2 \eta(0)$ is positive definite and

$$(\nabla_u \psi^k(u))^T = (\nabla_u \eta(u))^T A^k(u), \quad k = 1, \ldots, d.$$

$$(\text{A3}) \text{ There exists a positive constant } c_e > 0 \text{ such that for all } u,$$

$$\nabla_u \eta(u) \cdot Q(u) \leq -c_e |Q(u)|^2.$$

It is clear that the entropy dissipation condition (A1)–(A3) is too weak to prevent the formation of singularities, for instance, a decoupled system composed of an inviscid Burgers’ equation and a damped linear equation satisfies (A1)–(A3), but generally would develop singularities in a finite time for non-degenerate initial data.

A natural supplementary coupling condition, the Shizuta–Kawashima stability condition, or “Kawashima condition” for short which was first formulated in [12, 23] for the hyperbolic-parabolic systems with a source term, was introduced in [27, 25]. This coupling condition implies that due to the wave propagation, the partial dissipation term indeed produces a kind of complete dissipation effect with respect to all the components of the solution, which yields the global existence of the small smooth solutions. The Kawashima condition has several equivalent formulations, and one of them reads as

$$(\text{A4}) \text{ The kernel of the Jacobian } \nabla_u Q(0) \text{ contains no eigenvector of } A(0, \omega) \text{ for any } \omega \in S^{d-1}.$$
flow (see \[26, 27, 28\]). Some works have discussed several aspects on the system violating the condition (A), such as \[20\] for special structures and \[18\] for partial strict dissipations in 1D.

In this paper we will continue the study on the system (1.1) with \(d \geq 2\) and \(r \geq 2\) (and hence \(n \geq 3\)) in (1.5) violating (A). More precisely, we assume

(A4) For each \(\omega \in S^{d-1}\) one and only one family of right eigenvectors of \(A(0, \omega)\) are not contained in the kernel of the Jacobian \(\nabla_u Q(0)\), that is, possibly after relabeling,
\[
\nabla_u Q(0)r_j(0, \omega) \neq 0, \quad j = 2, \ldots, n,
\]
but
\[
\nabla_u Q(0)r_1(0, \omega) = 0.
\]

One may refer to the assumption (A4) that the last \(n-1\) right eigenvectors satisfy the Kawashima condition and the first right eigenvector violates it. Note that (A1)–(A4) is still too weak to prevent the formation of singularities, and some additional structural conditions are needed in order to guarantee the global existence of the small smooth solutions. The goal of this paper is to identify one set of such conditions and prove the corresponding results.

For notational simplicity, \(L^p\) and \(H^s\) spaces are used to denote the usual \(L^p\) and Sobolev spaces on \(\mathbb{R}^d\). The notation \(\Lambda^s\) is defined by \(\Lambda^s f := (|\xi|^s \hat{f})^{\vee}\), where \(\hat{\cdot}\) is the Fourier transform and \((\cdot)^{\vee}\) is its inverse. \(C\) denotes the positive constants, whose value may change from line to line, and \(f \lesssim g\) is used for \(f \leq Cg\).

2. Main results

As mentioned in Section 1, the aim of this paper is to impose certain structural conditions in supplement with the assumptions (A1)–(A4) so that the system (1.1) admits a unique global smooth solution for the small initial data. First, for the technical point and as suggested by many physical applications, one may assume

(\(\mathcal{B}\)) The first family of eigenvectors satisfy an isotropic condition:

\[
l_1(u, \omega) \equiv l_1(u) \quad \text{and} \quad r_1(u, \omega) \equiv r_1(u), \quad \forall \omega \in S^{d-1}.
\]

In the spirit of the normalized coordinates method developed in \[16, 17\] for the one dimensional quasilinear hyperbolic systems, thanks to the isotropy of \(r_1\) in (2.1), the partially normalized coordinates can be introduced as follows. After the linear transformation performed in Appendix A let \(u^{(1)}(s, u^0)\) be the 1st characteristic trajectory passing through the point \(u^0\), which is defined by

\[
\begin{align*}
\frac{d}{ds} u^{(1)}(s, u^0) &= r_1(u^{(1)}(s, u^0)), \\
u^{(1)}(0, u^0) &= u^0.
\end{align*}
\]

(2.2)

For each point \(\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n)^T\), map it to \(u = (u_1, \ldots, u_n)^T\) by
\[
u = u(\tilde{u}) := u^{(1)}(\tilde{u}_1, 0, \tilde{u}_2, \ldots, \tilde{u}_n)^T.
\]

(2.3)

It will be verified in Section 3 that the mapping (2.3) is a local diffeomorphism near \(\tilde{u} = 0\). This then allows to define the partially normalized coordinates \(\tilde{u}\) by the inverse mapping \(\tilde{u} = \tilde{u}(u)\) for each point \(u\) near \(u(0) = u^{(1)}(0, 0) = 0\) in the phase space, and hence the system (1.1) can be equivalently reformulated in terms of \(\tilde{u}\). This reformulated system will be studied in details in Section 3. Throughout the paper the following notational convention is applied for vectors: for a vector \(f = (f_1, \ldots, f_n)^T \in \mathbb{R}^n\), denote
\[
\begin{align*}
f^0 &= (f_2, \ldots, f_n)^T \in \mathbb{R}^{n-1}, \\
f^c &= (f_2, \ldots, f_r)^T \in \mathbb{R}^{r-1} \quad \text{and} \\
f^p &= (f_{r+1}, \ldots, f_n)^T \in \mathbb{R}^{n-r}.
\end{align*}
\]

(2.4)

As one shall see, the decomposition (2.4) is closely related to the structure of the system (1.1) under assumptions (A1)–(A4) and (\(\mathcal{B}\)). Since the assumption (A4) implies that the first right eigenvector violates the Kawashima condition, the natural way is to supplement certain degeneracy conditions with respect to the first eigen-family. One may use the following weak degeneracy conditions:
(WΩ1) $Q(u)$ is weakly degenerate along the first characteristic trajectory that passes through 0:

$$Q(u^{(1)}(s, 0)) \equiv 0, \quad \forall |s| \text{ small},$$

which is equivalent to

$$(\nabla_u Q(u)r_1(u))|_{u=u^{(1)}(s, 0)} \equiv 0, \quad \forall |s| \text{ small}. \quad (2.5)$$

(2.7)

(2.12)

(WΩ2) The first eigen-family is weakly linearly degenerate (see [16, 17]):

$$\lambda_1(u^{(1)}(s, 0), \omega) \equiv \lambda_1(0, \omega), \quad \forall |s| \text{ small}, \quad \forall \omega \in S^{d-1},$$

which is equivalent to

$$(\nabla_u \lambda_1(u, \omega) \cdot r_1(u))|_{u=u^{(1)}(s, 0)} \equiv 0, \quad \forall |s| \text{ small}, \quad \forall \omega \in S^{d-1}. \quad (2.6)$$

It is easy to check that the conditions (WΩ1)–(WΩ2) are necessary since if any one of them is missing, the solution generally will blow up in finite time. The typical examples can be constructed in the spirit of the Riccati equation and the Burgers’ equation (see [16] and [18] for instance).

The first result of this paper is that under the assumptions (A1)–(A4), (B) and (WΩ1)–(WΩ2), the system (1.1) admits a unique global solution for each suitably smooth and small initial data.

**Theorem 1.** Suppose that the assumptions (A1)–(A4), (B) and (WΩ1)–(WΩ2) hold. Let $d \geq 3$, $\ell > d/2 + 1$ and $1 \leq p \leq 2$, $1 \leq p < d/2$. Denote $p^* = \min\{2, dp/(d - p)\}$ and $s^*_1 = d(1 - 1/p) + 1$.

There exists a sufficiently small $\varepsilon_0 > 0$ such that if

$$K_0 := \|u_0\|_H^\ell + \|\tilde{u}_0^B\|_{L^p} + \|\tilde{u}_0^P\|_{L^{p^*}} \leq \varepsilon_0, \quad (2.7)$$

then there exists a unique global solution $u(t, x)$ to the system (1.1) satisfying

$$\|u(t)\|^2_{H^\ell} + \int_0^t \left( \|\tilde{u}^P(\tau)\|^2_{H^\ell} + \|\Lambda \tilde{u}^C(\tau)\|^2_{H^{\ell-1}} \right) \, d\tau \leq C K_0^2. \quad (2.8)$$

Moreover, for all $s \in [0, s^*_1] \cap [0, d/2)$,

$$\|\Lambda^s \tilde{u}^C(t)\|_{H^{\ell-s}} \leq C K_0 (1 + t)^{-\frac{\ell}{2} \left( \frac{1}{2} - \frac{s}{d} \right) - \frac{1}{2}} \quad (2.9)$$

and for all $s \in [0, s^*_1 - 1] \cap [0, d/2 - 1)$,

$$\|\Lambda^s \tilde{u}^P(t)\|_{H^{\ell-s}} \leq C K_0 (1 + t)^{-\frac{\ell}{2} \left( \frac{1}{2} - \frac{s}{d} \right) - \frac{1}{2} - \frac{1}{2}}. \quad (2.10)$$

**Remark 2.1.** Note that if $d \geq 5$, then one can take $p = 2$ in Theorem 1 which yields the unique global solution to the system (1.1) for the small initial data in $H^\ell$ with $\ell > d/2 + 1$; but if $d = 3$ or 4, in order to obtain the global solutions one needs the low frequency assumption of the initial data as stated in (2.7) with $1 \leq p < d/2$ (if $p \geq 2d/(d + 2)$ and hence $p^* = 2$, then one needs only the low frequency assumption of $\tilde{u}_0^C$).

To improve the result of Theorem 1, one may use the following stronger degeneracy conditions:

(Ω1) $Q(u)$ is degenerate along each first characteristic trajectory:

$$\nabla_u Q(u)r_1(u) \equiv 0, \quad \forall u. \quad (2.11)$$

(Ω2) The first eigen-family is linearly degenerate (see [15]):

$$\nabla_u \lambda_1(u, \omega) \cdot r_1(u) \equiv 0, \quad \forall u, \quad \forall \omega \in S^{d-1}. \quad (2.12)$$

Then the second result of this paper can be stated as follows.
Theorem 2. Suppose that the assumptions (A1)-(A4), (B) and (D1)-(D2) hold. Let $d \geq 3$, $\ell > d/2 + 1$ and $1 \leq p \leq 2$, require further that $p < 2d/(2(3 - \ell) + d)$ if $\ell \leq 3$. Denote $s_2^* = \min\{d(1 - 1/p) + 1, \ell - 1\}$.

There exists a sufficiently small $\varepsilon_0 > 0$ such that if $K_0 \leq \varepsilon_0$, then there exists a unique global solution $u(t, x)$ to the system (1.1) satisfying
\[
\|u(t)\|_{H^1}^2 + \int_0^t \left(\|\tilde{u}_0^p(t)\|_{H^1}^2 + \|\Lambda u_0^C(\tau)\|_{H^{d-1}}^2\right) \, d\tau \leq CK_0^2.
\] (2.13)

Moreover, for all $s \in [0, s_2^*] \cap [0, d/2 + 1)$,
\[
\|\Lambda^s u_0^P(t)\|_{H^{d-s}} \leq CK_0(1 + t)^{-\frac{s}{2}\left(\frac{d}{p} - 1\right)} \left(\frac{d}{p} - 1\right)^{-\frac{1}{2}}
\] (2.14)

and for all $s \in [0, s_2^* - 1] \cap [0, d/2)$,
\[
\|\Lambda^s u_0^P(t)\|_{H^{d-s}} \leq CK_0(1 + t)^{-\frac{s}{2}\left(\frac{d}{p} - 1\right)} \left(\frac{d}{p} - 1\right)^{-\frac{1}{2}}.
\] (2.15)

Remark 2.2. Note that now one can take $p = 2$ in Theorem 2 for any $d \geq 3$ which yields the unique global solution to the system (1.1) for the small initial data in $H^2$ with $\ell > \max\{d/2 + 1, 3\}$ and without any low frequency assumption of the initial data.

Note that the result of Theorem 2 excludes the case $d = 2$. To obtain the global solution to the system (1.1) in two dimensions, one more assumption should be made on the structure of the system:

(D3) The first eigen-family gets no linear effect from the source term:
\[
l_1(0)\nabla uQ(0) = 0.
\] (2.16)

And the third result of this paper is stated as follows.

Theorem 3. Suppose that the assumptions (A1)-(A4), (B), (D1)-(D2) and (D3) hold. Let $d \geq 2$, $\ell > d/2 + 1$ and $1 \leq p < q \leq 2$, $1 \leq p \leq q < d$, require further that $p < 2d/(2(3 - \ell) + d)$ if $\ell \leq 3$. Denote $s_2^* = \min\{d(1/2 + 1/q - 1/p) + 1, d(1 - 1/p) + 2, \ell - 1\}$.

There exists a sufficiently small $\varepsilon_0 > 0$ such that if
\[
\tilde{K}_0 := \|u_0\|_{H^2} + \|\tilde{u}_0^P\|_{L_p} + \|\tilde{u}_0^P\|_{L_p^r} + \|\tilde{u}_0^C\|_{L^q} \leq \varepsilon_0,
\] (2.17)

then there exists a unique global solution $u(t, x)$ to the system (1.1) satisfying
\[
\|u(t)\|_{H^1}^2 + \int_0^t \left(\|\tilde{u}_0^p(t)\|_{H^1}^2 + \|\Lambda u_0^C(\tau)\|_{H^{d-1}}^2\right) \, d\tau \leq \tilde{C} \tilde{K}_0^2.
\] (2.18)

Moreover, for all $s \in [0, s_2^*] \cap [0, d/2 + 1)$,
\[
\|\Lambda^s u_0^C(t)\|_{H^{d-s}} \leq \tilde{C} \tilde{K}_0(1 + t)^{-\frac{s}{2}\left(\frac{d}{p} - 1\right)} \left(\frac{d}{p} - 1\right)^{-\frac{1}{2}}
\] (2.19)

and for all $s \in [0, s_2^* - 1] \cap [0, d/2)$,
\[
\|\Lambda^s u_0^P(t)\|_{H^{d-s}} \leq \tilde{C} \tilde{K}_0(1 + t)^{-\frac{s}{2}\left(\frac{d}{p} - 1\right)} \left(\frac{d}{p} - 1\right)^{-\frac{1}{2}}.
\] (2.20)

Remark 2.3. It is verified in Section 11 that the damping full Euler system of adiabatic flow satisfies all the assumptions (A1)-(A4), (B) and (D1)-(D3) and hence Theorem 3 can be applied. This improves the result of [24] in 3D and produces the first result of global unique solution in 2D which seems necessary to require the low frequency assumption of the initial data. However, our results could not cover the multidimensional thermal non-equilibrium flow [28].

Now let us illustrate the main difficulties encountered in proving these results of Theorems 1,2,3 and explain our strategies to overcome them. Since the components of the solution behave totally distinctively as already seen in the theorems, it is essential to divide the system, at least in the linear level, into the dissipative part and the non-dissipative one. Different from the specific models which usually have certain physical variables to give reasonable ways of doing this, for the general system considered in this paper these two parts are generally strongly coupled in an implicit manner. In [18], the methods of normalized coordinates and formulas...
of wave decomposition are used to identify the two parts for the one dimensional systems with partial strict dissipation. However, for our multidimensional systems, these methods, which are originally designed only for one dimensional problems, are just not applicable. Meanwhile, when discussing the dissipation structures given by the entropy dissipation condition together with Kawashima condition instead of the strict dissipation, the way of applying these methods would generally destroy the structure to identify the faster decaying variables from the slower ones. To get around this difficulty, the partially normalized coordinates \((2.3)\) is introduced under the assumption \((\mathcal{B})\). It is shown in Section 3 that by reformulating the system \((1.1)\) in terms of this new coordinates into the system \((2.5)\), the two parts are successfully identified, i.e., the dissipative part \((3.26)\) for \(\tilde{u}^3 = (\tilde{u}^\nu, \tilde{u}^p)\) and the rest part \((3.25)\) for the non-dissipative \(\tilde{u}_1\). Also, those structural conditions can be reformulated in terms of new variables in a more applicable way.

The main part of proving Theorems 1–3 is to derive the energy estimates. In the following, we explain the strategy by first dealing with the case under the assumptions of Theorem 1. Theorems 2, 3 are proved in the same fashion and we only need to explain the additional arguments for the proof. It is crucial to analyze the structures of the linearized system and the nonlinear terms. It is with the help of the partially normalized coordinates that the linear part \((3.26)\) is shown to be dissipative in the sense of satisfying the entropy dissipation condition and the Kawashima condition. By putting aside the nonlinear effect, one can follow the arguments of \([7, 25]\) to derive the basic \(L^2\) estimate by using the entropy dissipation condition \((\mathcal{A}_1)–(\mathcal{A}_3)\) as well as the high order energy estimates for \(\tilde{u}^3\) which include the damping dissipation of \(\tilde{u}^\nu\), the partial Kawashima condition \((\mathcal{A}_4)\) can be applied to recover the degenerate dissipation of \(\tilde{u}^\nu\). The remaining difficulty is to handle with the nonlinear terms. It is again with the help of the partially normalized coordinates that under the degenerate assumption \((\mathcal{D}_1, \mathcal{D}_2)\) there is no nonlinear term resulting from the source term composed solely by the non-dissipative part, that is, those nonlinear terms are of form \(O(1)\tilde{u}^3\), at least. This feature provides relatively nice structure so that we can complete the estimate of the dissipative part \(\tilde{u}^3\).

The next natural step is to derive the energy estimates for the remaining non-dissipative part \(\tilde{u}_1\). Under the weakly linearly degenerate assumption \((\mathcal{D}_2)\), there is no nonlinear term resulting from the convection term in \((3.25)\) composed solely by the non-dissipative part. However, there would be a loss of derivative if one directly performs the energy estimates by using \((2.24)\). To get around this difficulty, we return back to the original full system \((1.1)\) and resort to generalize the method of wave decomposition (see \([9]\)) to our multidimensional case. The isotropy assumption \((\mathcal{B})\) will be used again. After the delicate analysis of the nonlinear term under the degenerate conditions \((\mathcal{D}_1)–(\mathcal{D}_2)\), we can finish the full energy estimates as

\[
\|\tilde{u}(t)\|_{H^r} \lesssim \|\tilde{u}_0\|_{H^r} + \int_0^t \|\tilde{u}^\nu\|_{L^\infty} + \cdots d\tau. \tag{2.21}
\]

Note carefully that we cannot control the time integral in \((2.21)\) by the dissipations estimates. To bound this time integral, we will turn to derive a strong enough decay rate of \(\|\tilde{u}^\nu\|_{L^\infty} + \cdots\). This step is extremely delicate. We will need to highly explore a refined linear decay estimates which requires weaker low frequency assumption of \(\tilde{u}^\nu_0\) than \(\tilde{u}^\nu_0\), the stronger degeneracy of \(\tilde{Q}^\nu\) than \(\tilde{Q}^p\) and the faster decay of \(\tilde{u}^p\) than \(\tilde{u}^\nu\). The technical point thus lies in the way of balancing these despair. Taking into account these facts, by a bootstrap estimates, which implies that the decay of higher order norms can be deduced from the decay of lower order norms, and a use of the Duhamel formula, which implies that the decay of lower order norms is bounded in terms of the decay of higher order norms, we are able to derive a decay of the solution as stated in Theorem 1 in a recursive way by the smallness of the solution. It then follows from the Sobolev interpolation that \(\|\tilde{u}^\nu\|_{L^\infty}\) is integrable in time as long as the parameters \(p\) and \(d\) satisfy the required restriction given in the theorem. We remark here that the introduction of the fractional derivative in the study enlarges the available range of \(p\) and \(d\). This closes the energy estimates and then completes the proof of Theorem 1.
To prove Theorem 2, it is observed that under the stronger degenerate conditions (D1)–(D2), the nonlinear terms behave better. For example, under (D1), nonlinear terms resulting from the source term are of the form \( O(1)\tilde{u}^2 \), or so. Based on the stronger degeneracy, we can improve the full energy estimates as
\[
\|\tilde{u}(t)\|_{H^s} \lesssim \|\tilde{u}_0\|_{H^s} + \int_0^t \|\tilde{u}^2\|_{L^\infty} + \cdots \, dt.
\tag{2.22}
\]
Since \( \tilde{u}^2 \) decays at a faster 1/2 rate than \( \tilde{u}^4 \), this ultimately enlarges the range of \( p \) and \( d \) as stated in Theorem 2. Note that we can mostly deduce from (2.15) that \( \|\tilde{u}^2\|_{L^\infty} \) decays at the rate of \( (1 + t)^{-d/4-1/2+\varepsilon} \) for any \( \varepsilon > 0 \), which restricts Theorem 2 only hold for \( d \geq 3 \).

The main goal of Theorem 3 is to include the case \( d = 2 \). Note that even in view of the linear decay estimates \( \|\tilde{u}^2\|_{L^\infty} \) decays at the rate of \( (1 + t)^{-d/(2p)-1/2} \), and then we need to require the low frequency assumption of \( \tilde{u}_0^2 \) when \( d = 2 \). However, due to the nonlinear effect we cannot achieve this rate, and Theorem 2 Basically tells that \( \|\tilde{u}^2\|_{L^\infty} \) would decay at the same rate of \( (1 + t)^{-d/(2q)-1/2+\varepsilon} \) for all \( p \) if there is no further additional assumption. Thereby, the \( L^2 \) assumption of \( \tilde{u}_0^2 \) comes into the role. The idea is to derive the \( L^2 \) estimate of \( \tilde{u}_1 \) so that we can improve a bit some nonlinear estimates when using the Duhamel formula, for example the estimates of \( \|\tilde{u}_1\tilde{u}^2\|_{L^p} \). However, we cannot derive the \( L^2 \) estimate of \( \tilde{v}_1 \) by using (3.2), and we again need to return back to the original full system (3.8) and introduce the wave decomposition to consider instead the first wave \( \tilde{v}_1 \). It is the derivation of the \( L^2 \) estimate of \( \tilde{v}_1 \) that we need the assumption (D3). With the help of the \( L^2 \) estimate of \( \tilde{v}_1 \), we can improve the nonlinear estimates so that we can derive the decay rates of the solution as stated in Theorem 3. Note that now the decay rate of \( \|\tilde{u}^2\|_{L^\infty} \) is improved to be \( (1 + t)^{-d/(2q)-1/2+\varepsilon} \), which allows the case \( d = 2 \) for \( q < 2 \). One may then notice the necessity of the \( L^3 \) assumption of \( \tilde{u}_0^2 \).

The rest of the paper is organized as follows. In Section 3 partially normalized coordinates will be used to reformulate the system (1.1) and those structural conditions. In Section 4 the structure of the reformulated system is analyzed carefully as the foundation for the further analysis. The delicate energy estimates is performed in Sections 5–6, the low frequency estimates for two dimensional case is presented in Section 7, and the refined decay estimates is given in Section 8. Finally, those estimates are combined in Section 9 to prove Theorems 1–3.

3. Partially normalized coordinates

After a linear transformation if necessary (see Appendix A for details), one may assume
\[
\frac{\partial Q_i}{\partial u_q}(0) = 0, \quad q = 1, \ldots, r; \quad i = 1, \ldots, n,
\tag{3.1}
\]
and
\[
q_i(0) = e_i,
\tag{3.2}
\]
where \( e_i \) \((i = 1, \ldots, n)\) stands for the \( i \)th unit vector in \( \mathbb{R}^n \).

The first step is to verify that the nonlinear mapping \( u = u(\tilde{u}) \) in the phase space, defined by (2.3) and (2.2), is invertible near \( \tilde{u} = 0 \). Denote the Jacobian matrix of the transformation \( u = u(\tilde{u}) \) by
\[
J(\tilde{u}) := \frac{\partial u}{\partial \tilde{u}}(\tilde{u}).
\tag{3.3}
\]
By the definition, it holds
\[
u((\tilde{u})) = u^{(1)}(0, (\tilde{u})) = (\tilde{u})
\tag{3.4}
\]
and
\[
\frac{\partial u}{\partial \tilde{u}_1}(\tilde{u}) = q_1(u(\tilde{u})).
\tag{3.5}
\]
Hence,
\[
J((\tilde{u})) = (q_1((\tilde{u})), e_2, \ldots, e_n).
\tag{3.6}
\]
This together with (3.2) yields
\[
J(0) = (q_1(0), e_2, \ldots, e_n) = I_n,
\tag{3.7}
\]
which implies that the transformation \( u = u(\tilde{u}) \) is a local diffeomorphism near \( \tilde{u} = 0 \). In the partially normalized coordinates \( \tilde{u} \) of the phase space, the system (1.1) (and (1.2)) is transformed into

\[
\begin{aligned}
\partial_t \tilde{u} + \sum_{k=1}^{d} \hat{A}^k(\tilde{u}) \tilde{u}_x^k &= \tilde{Q}(\tilde{u}), \\
\tilde{u} \big|_{t=0} &= \tilde{u}_0,
\end{aligned}
\]  

(3.8)

Here the coefficient matrices

\[
\hat{A}^k(\tilde{u}) = J^{-1}(\tilde{u})A(u(\tilde{u}))J(\tilde{u}), \quad k = 1, \ldots, d
\]  

(3.9)

and the inhomogeneous term

\[
\tilde{Q}(\tilde{u}) = J^{-1}(\tilde{u})Q(u(\tilde{u})).
\]  

(3.10)

Now it is important to study the structure of \( \hat{A}^k(\tilde{u}) \), which will exhibit the advantage of our partially normalized coordinates \( \tilde{u} \). First of all, it is direct to check that \( \hat{A}^k(\tilde{u}) \) possesses the following expression

\[
\hat{A}^k(\tilde{u}) = \tilde{A}(\tilde{u}, e_k), \quad k = 1, \ldots, d
\]  

(3.11)

with \( e_k \) the kth unit vector in \( \mathbb{R}^d \) and

\[
\tilde{A}(\tilde{u}, \omega) := J^{-1}(\tilde{u})A(u(\tilde{u})), \quad \omega \in S^{d-1},
\]  

(3.12)

where the matrix \( A(u, \omega) \) is defined by (1.3). Therefore, it suffices to study the structure of \( \tilde{A}(\tilde{u}, \omega) \). Recalling that \( l_i(u, \omega) \) (resp. \( r_i(u, \omega) \)) is the ith left (resp. right) eigenvector of \( A(u, \omega) \) with the corresponding eigenvalue \( \lambda_i(u, \omega) \), \( i = 1, \ldots, n \), one could then choose

\[
\tilde{l}_i(\tilde{u}, \omega) := l_i(u(\tilde{u})), \omega, \quad i = 1, \ldots, n,
\]  

(3.13)

\[
\tilde{r}_i(\tilde{u}, \omega) := (\tilde{r}_{11}(\tilde{u}), \ldots, \tilde{r}_{n1}(\tilde{u}))^T := J^{-1}(\tilde{u})r_i(u(\tilde{u})), \omega,
\]  

(3.14)

\[
\tilde{l}_i(\tilde{u}, \omega) = \left( \tilde{l}_{11}(\tilde{u}), \ldots, \tilde{l}_{n1}(\tilde{u}) \right) := l_i(u(\tilde{u})), J(\tilde{u})
\]  

(3.15)

as the eigenvalues and a complete set of left (resp. right) eigenvectors of \( \tilde{A}(\tilde{u}, \omega) \), that is,

\[
\tilde{l}_i(\tilde{u}, \omega)\tilde{A}(\tilde{u}, \omega) = \tilde{\lambda}_i(\tilde{u}, \omega)\tilde{l}_i(\tilde{u}, \omega), \quad i = 1, \ldots, n,
\]  

(3.16)

\[
\tilde{A}(\tilde{u}, \omega)\tilde{r}_i(\tilde{u}, \omega) = \tilde{\lambda}_i(\tilde{u}, \omega)\tilde{r}_i(\tilde{u}, \omega), \quad i = 1, \ldots, n.
\]  

(3.17)

From the normalization (1.4), it follows

\[
\tilde{l}_i(\tilde{u}, \omega)\tilde{r}_{i'}(\tilde{u}, \omega) \equiv \delta_{ii'}, \quad i, i' = 1, \ldots, n.
\]  

(3.18)

Moreover, by the assumption (2B) and (3.3), respectively, for any \( \omega \in S^{d-1} \),

\[
\tilde{l}_i(\tilde{u}, \omega) = l_i(u(\tilde{u})), J(\tilde{u}) \equiv \tilde{l}_i(\tilde{u})
\]  

(3.19)

and

\[
\tilde{r}_i(\tilde{u}, \omega) = J^{-1}(\tilde{u})r_i(u(\tilde{u})) \equiv e_1.
\]  

(3.20)

Combining (3.18)–(3.20), it follows

\[
\tilde{l}_{11}(\tilde{u}) \equiv 1 \quad \text{and} \quad \tilde{l}_{j1}(\tilde{u}, \omega) \equiv 0, \quad j = 2, \ldots, n, \quad \forall \omega \in S^{d-1}.
\]  

(3.21)

More importantly, taking \( i = 1 \) in (3.17) and noting (3.20), leads to

\[
\tilde{A}_{11}(\tilde{u}, \omega) = \tilde{\lambda}_1(\tilde{u}, \omega) \quad \text{and} \quad \tilde{A}_{ji}(\tilde{u}, \omega) \equiv 0, \quad j = 2, \ldots, n, \quad \forall \omega \in S^{d-1}.
\]  

(3.22)

Throughout the paper the following notational convention are used for matrices: for an \( n \times n \) matrix \( M = (M_{ii'})_{i,i'=1}^n \), denote

\[
M^2 = (M_{12}, \ldots, M_{1n}), \quad M^3 = (M_{jj'})_{j,j'=2}^n \quad \text{and} \quad M^D = \left( M_{pp'} \right)_{p,p'=r+1}^n.
\]  

(3.23)

Then by (3.22), one has the following structure of \( \tilde{A}(\tilde{u}, \omega) \):

\[
\tilde{A}(\tilde{u}, \omega) = \begin{pmatrix} \tilde{\lambda}_1(\tilde{u}, \omega) & \tilde{A}^2(\tilde{u}, \omega) \\ 0 & \tilde{A}^D(\tilde{u}, \omega) \end{pmatrix}.
\]  

(3.24)
Accordingly, the system (3.5) indeed takes the form
\[ \partial_t \tilde{u}_1 + \sum_{k=1}^{d} \tilde{\lambda}_1(\tilde{u}, \varrho_k)(\tilde{u}_k)_{x_k} + \sum_{k=1}^{d} \tilde{A}^{k,b} (\tilde{u}) \tilde{u}_{x_k}^b = \tilde{Q}_1(\tilde{u}), \]
\[ \partial_t \tilde{u}_b^b + \sum_{k=1}^{d} \tilde{A}^{k,b}(\tilde{u}) \tilde{u}_{x_k}^b = \tilde{Q}^b(\tilde{u}). \] (3.26)

Note that the system (3.5) is split into two subsystems in some sense. We will see that the subsystem (3.26) of \( \tilde{u}_b^b \) is dissipative in the sense of satisfying the entropy dissipation condition and the Kawashima condition and the main part of the subsystem (3.25) of the non-dissipative \( \tilde{u}_1 \) is indeed (weakly) linearly degenerate. This structure property is crucial for the whole analysis in this paper.

Recall that from the assumption (B), the isotropy (3.13)–(3.20) hold in the partially normalized coordinates, and we restate them as follows

(B) The first family of eigenvectors satisfy an isotropic condition:
\[ \tilde{l}_1(\tilde{u}, \omega) = \tilde{l}_1(\tilde{u}) \text{ and } \tilde{r}_1(\tilde{u}, \omega) = e_1, \quad \forall \omega \in S^{d-1}. \] (3.27)

The next task is to derive the corresponding forms of the assumptions (A1)–(A4), (B1D1)–(D2), (D1)–(D2) and (D3). First, recalling (3.10) and (1.5) it trivially holds that \( \tilde{Q}(0) = 0 \) and
\[ \sum_{i=1}^{n} J_{q_i}(\tilde{u}) \tilde{Q}_i(\tilde{u}) \equiv 0, \quad q = 1, \ldots, r. \] (3.28)

Moreover, by (3.10) and (3.7), one has
\[ \frac{\partial \tilde{Q}(0)}{\partial \tilde{u}_i} = \left( \frac{\partial J^{-1}_i(\tilde{u})}{\partial \tilde{u}_i}(\tilde{u}) \right) \bigg|_{\tilde{u}=0} \tilde{Q}(0) + J^{-1}_i(0) \left( \sum_{i'=1}^{n} \frac{\partial u_{i'}}{\partial \tilde{u}_i} \frac{\partial \tilde{Q}(u(\tilde{u}))}{\partial u_{i'}} \bigg|_{\tilde{u}=0} \right) = \frac{\partial \tilde{Q}(0)}{\partial \tilde{u}_i}. \] (3.29)

Hence the assumption (A1) is translated to be

(A1) The matrix \( \left( \nabla_{\tilde{u}} \tilde{Q} \right)_B(0) \) is invertible.

Next, set
\[ \tilde{\eta}(\tilde{u}) = \eta(u(\tilde{u})), \] (3.30)
then
\[ \tilde{\eta}(0) = 0 \text{ and } \nabla_{\tilde{u}} \tilde{\eta}(0) = 0. \] (3.31)

Moreover, since
\[ \frac{\partial^2 \tilde{\eta}}{\partial \tilde{u}_i \partial \tilde{u}_r}(\tilde{u}) = \sum_{r',r''=1}^{n} \frac{\partial^2 \eta}{\partial u_{i'} \partial u_{r'}}(u(\tilde{u})) \frac{\partial u_{r'}}{\partial \tilde{u}_i}(\tilde{u}) \frac{\partial u_{r''}}{\partial \tilde{u}_r}(\tilde{u}) + \sum_{r=1}^{n} \left( \frac{\partial \eta}{\partial u_r}(u(\tilde{u})) - \frac{\partial \eta}{\partial u_r}(0) \right) \frac{\partial^2 u_r}{\partial \tilde{u}_i \partial \tilde{u}_r}(\tilde{u}), \] (3.32)

by (3.7) it holds
\[ \nabla_{\tilde{u}}^2 \tilde{\eta}(0) = J^T(0) \nabla_{\tilde{u}}^2 \eta(0) J(0) = \nabla_{\tilde{u}}^2 \eta(0). \] (3.33)

Set \( \tilde{\psi}^k(\tilde{u}) = \psi^k(u(\tilde{u})) \), then the assumption (A2) leads to

(A2) The strictly convex smooth entropy function \( \tilde{\eta}(\tilde{u}) \) and the \( d \) smooth entropy flux functions \( \tilde{\psi}^k(\tilde{u}) \) (\( k = 1, \ldots, d \)) satisfy that for all \( \tilde{u} \),
\[ (\nabla_{\tilde{u}} \tilde{\psi}^k(\tilde{u}))^T = (\nabla_{\tilde{u}} \tilde{\eta}(\tilde{u}))^T \tilde{A}^k(\tilde{u}). \] (3.34)

Now, noting (3.7), (3.10), (3.14) and (3.20), it follows from the assumptions (A3) and (A4) that

(A3) There exists a positive constant \( \tilde{c}_e > 0 \) such that for all \( \tilde{u} \),
\[ \nabla_{\tilde{u}} \tilde{\eta}(\tilde{u}) \cdot \tilde{Q}(\tilde{u}) \leq -\tilde{c}_e |\tilde{Q}(\tilde{u})|^2. \] (3.35)
The last \( n - 1 \) right eigenvectors of \( \tilde{A}(0, \omega) \) are not contained in the kernel of the Jacobian \( \nabla u \tilde{Q}(0) \) for any \( \omega \in S^{d-1} \), that is,

\[
\nabla u \tilde{Q}(0) \tilde{r}_j(0, \omega) \neq 0, \quad j = 2, \ldots, n,
\]

but

\[
\nabla u \tilde{Q}(0) \tilde{r}_1(0) = 0.
\]

For the assumptions (\( WD_1 \))–(\( WD_2 \)), we notice that the definition (\( 3.36 \)) directly leads to

\[
u(\tilde{u}_1 e_1) = u(\tilde{u}_1, 0).
\]

Then by (\( 3.10 \)), (\( 3.13 \)) and (\( 3.27 \)), one can express them as

(\( WD_1 \)) \( \tilde{Q}(\tilde{u}) \) is degenerate along the first characteristic trajectory passing through 0:

\[
\frac{\partial(J \tilde{Q})}{\partial \tilde{u}_1}(\tilde{u}_1 e_1) \equiv 0, \quad \forall |	ilde{u}_1| \text{ small}.
\]

(\( WD_2 \)) The first eigen-family is weakly linearly degenerate:

\[
\frac{\partial \tilde{\lambda}_1}{\partial \tilde{u}_1}(\tilde{u}_1 e_1, \omega) = 0, \quad \forall |	ilde{u}_1| \text{ small}, \quad \forall \omega \in S^{d-1}.
\]

Similarly, the assumptions (\( D_1 \))–(\( D_2 \)) are translated to be

(\( D_1 \)) \( \tilde{Q}(\tilde{u}) \) is degenerate along each first characteristic trajectory:

\[
\frac{\partial(J \tilde{Q})}{\partial \tilde{u}_1}(\tilde{u}) \equiv 0, \quad \forall \tilde{u}.
\]

(\( D_2 \)) The first eigen-family is linearly degenerate:

\[
\frac{\partial \tilde{\lambda}_1}{\partial \tilde{u}_1}(\tilde{u}, \omega) \equiv 0, \quad \forall \tilde{u}, \quad \forall \omega \in S^{d-1}.
\]

At last, by (\( 3.17 \)), (\( 3.19 \)) and (\( 3.29 \)), the assumption (\( D_3 \)) can be expressed as

(\( D_3 \)) The first eigen-family gets no linear effect from the source term

\[
\tilde{I}_1(0) \nabla u \tilde{Q}(0) = 0.
\]

**Remark 3.1.** Since the assumptions (\( A_1 \))–(\( A_4 \)) and (\( B \)) are made in all the three Theorems (\( A_1 \))–(\( A_4 \)) and (\( B \)) are always supposed to be hold in the rest of the paper.

4. Structural analysis

In this section, useful algebraic consequences are established for the structural assumptions (\( A_1 \))–(\( A_4 \)), (\( WD_1 \)) and (\( D_1 \)).

4.1. Structure of the source term \( \tilde{Q} \). In this subsection the source term \( \tilde{Q}(\tilde{u}) \) is analyzed under the assumption (\( WD_1 \)) or (\( D_1 \)) by the Taylor expansion in an appropriate way.

4.1.1. The expansion of \( \tilde{Q}(\tilde{u}) \) under (\( WD_1 \)): First, assume (\( WD_1 \)), then it holds that \( \tilde{Q}(\tilde{u}_1 e_1) \equiv J^{-1}(\tilde{u}_1 e_1).J(0)\tilde{Q}(0) = 0 \). Using this fact, one may first expand \( \tilde{Q}(\tilde{u}) \) around \( \tilde{u}_1 e_1 \), that is, fixing \( \tilde{u}_1 \) and viewing \( \tilde{Q}(\tilde{u}) \) as a function of \( \tilde{u} = (\tilde{u}_2, \ldots, \tilde{u}_n) \) and then expanding it around \( \tilde{u}^0 = 0 \), to have

\[
\tilde{Q}_i(\tilde{u}) = \sum_{j' = 2}^{n} \frac{\partial \tilde{Q}_i}{\partial \tilde{u}_{j'}}(\tilde{u}_1 e_1)\tilde{u}_{j'} + \sum_{j', j'' = 2}^{n} \tilde{Y}_{ij'j''}(\tilde{u})\tilde{u}_{j'}\tilde{u}_{j''}, \quad i = 1, \ldots, n.
\]
Expanding further the functions of $\tilde{u}_1$ inside the first summation in (4.1) around $\tilde{u}_1 = 0$, one can obtain

$$
\tilde{Q}_i(\tilde{u}) = \sum_{j' = 2}^{n} \frac{\partial \tilde{Q}_i}{\partial \tilde{u}_{j'}}(0) \tilde{u}_{j'} + \sum_{j' = 2}^{n} \frac{\partial^2 \tilde{Q}_i}{\partial \tilde{u}_{j'} \partial \tilde{u}_1}(0) \tilde{u}_{j'} \tilde{u}_1 + \sum_{j', j'' = 2}^{n} \tilde{Y}_{ij', j''}(\tilde{u}) \tilde{u}_{j'} \tilde{u}_{j''}
+ \sum_{j' = 2}^{n} \tilde{Z}_{ij'11}(\tilde{u}) \tilde{u}_{j'} \tilde{u}_1^2, \quad i = 1, \ldots, n.
$$

(4.2)

Now examine the first two summations in (4.2). First, it follows from (3.29), (3.11) and (1.5) that

$$
\frac{\partial \tilde{Q}_i}{\partial \tilde{u}_q}(0) = 0, \quad q = 1, \ldots, r; \quad i = 1, \ldots, n.
$$

(4.3)

On the other hand, by (3.10), it holds

$$
\frac{\partial^2 \tilde{Q}}{\partial \tilde{u}_i \partial \tilde{u}_i}(\tilde{u}) = \frac{\partial^2 J^{-1}}{\partial \tilde{u}_i' \partial \tilde{u}_i}(\tilde{u}) \tilde{J}(\tilde{u}) \tilde{Q}(\tilde{u}) + \frac{\partial J^{-1}}{\partial \tilde{u}_i'}(\tilde{u}) \tilde{J}(\tilde{u}) \frac{\partial \tilde{Q}}{\partial \tilde{u}_i}(\tilde{u})
+ \frac{\partial J^{-1}}{\partial \tilde{u}_i}(\tilde{u}) \tilde{J}(\tilde{u}) \frac{\partial \tilde{Q}}{\partial \tilde{u}_i}(\tilde{u})
+ J^{-1}(\tilde{u}) \frac{\partial^2 (Q(u(\tilde{u})))}{\partial \tilde{u}_i \partial \tilde{u}_i'}(\tilde{u})
.$$

(4.4)

Hence, by the fact $\tilde{Q}(0) = 0$, (4.3), (4.7) and (1.5), one has

$$
\frac{\partial^2 \tilde{Q}_i}{\partial \tilde{u}_i \partial \tilde{u}_{q'}}(0) = \left. \frac{\partial^2 (Q(u(\tilde{u})))}{\partial \tilde{u}_i \partial \tilde{u}_{q'}} \right|_{\tilde{u}=0} = 0, \quad q, q', q'' = 1, \ldots, r.
$$

(4.5)

Denote the constant matrix

$$
\Theta = \nabla_\tilde{u} \tilde{Q}(0)
$$

(4.6)

and for $i, i', i'' = 1, \ldots, n,$

$$
\Upsilon_{ii'ii''} = \frac{\partial^2 \tilde{Q}_i}{\partial \tilde{u}_{i'} \partial \tilde{u}_{i''}}(0).
$$

(4.7)

Then (4.3) and (4.6) imply that

$$
\Theta_{qi} = \Theta_{iq} = 0, \quad i = 1, \ldots, n; \quad q = 1, \ldots, r,
$$

(4.8)

$$
\Upsilon_{qq'q''} = 0, \quad q, q', q'' = 1, \ldots, r.
$$

(4.9)

Accordingly, it follows from (4.2) that

$$
\tilde{Q}_q(\tilde{u}) = \sum_{p' = r + 1}^{n} \Upsilon_{qp'1} \tilde{u}_{p'} \tilde{u}_1 + \sum_{j' = 2}^{n} \tilde{Y}_{qj'1}(\tilde{u}) \tilde{u}_{j'} \tilde{u}_1 + \sum_{j = 2}^{n} \tilde{Z}_{qj'11}(\tilde{u}) \tilde{u}_{j'} \tilde{u}_1^2, \quad q = 1, \ldots, r
$$

(4.10)

and

$$
\tilde{Q}_p(\tilde{u}) = \sum_{p' = r + 1}^{n} \Theta_{pp'} \tilde{u}_{p'} + \sum_{j'' = 2}^{n} \sum_{j' = 2}^{n} \tilde{Y}_{pj'1'}(\tilde{u}) \tilde{u}_{j'} \tilde{u}_{j''}, \quad p = r + 1, \ldots, n,
$$

(4.11)

where $\tilde{Y}_{pj'1}(\tilde{u}) = \Upsilon_{p1j'} + \tilde{Z}_{pj'11}(\tilde{u}) \tilde{u}_1$.

Remark 4.1. By the matrix $\Theta$ defined in (4.6), (4.1) is equivalent to

$$
\det \Theta^p \neq 0.
$$

(4.12)
4.1.2. The expansion of $\tilde{Q}(\tilde{u})$ under $(\mathfrak{D}1)$. Since $(\mathfrak{D}1)$ implies $(\mathfrak{D}1\mathfrak{D}1)$, the expansions (4.10)–(4.14) and all the computations in the previous subsection hold in this case. But under the stronger assumption $(\tilde{\mathfrak{D}}1)$, it is reasonable to expect a better expansion.

Again, using the fact $\tilde{Q}(\tilde{u}_1e_1) \equiv 0$, one may first expand $\tilde{Q}(\tilde{u})$ around $\tilde{u}_1e_1$ to have

$$
\tilde{Q}_1(\tilde{u}) = \sum_{j' = 2}^{n} \frac{\partial \tilde{Q}_1}{\partial \tilde{u}_{j'}}(\tilde{u}_1e_1)\tilde{u}_{j'} + \sum_{j', j'' = 2}^{n} \frac{1}{2} \frac{\partial^2 \tilde{Q}_1}{\partial \tilde{u}_{j'} \partial \tilde{u}_{j''}}(\tilde{u}_1e_1)\tilde{u}_{j'}\tilde{u}_{j''} + \sum_{j', j'' = 2}^{n} \tilde{Z}_{ij'j''}(\tilde{u})\tilde{u}_{j'}\tilde{u}_{j''}, \quad i = 1, \ldots, n.
$$

(4.13)

Due to the assumption $(\tilde{\mathfrak{D}}1)$ and the fact $\tilde{Q}(\tilde{u}_1e_1) = \tilde{Q}(0) = 0$, it holds

$$
\tilde{Q}(\tilde{u}_1e_1 + \tilde{u}_q e_q) - \tilde{Q}(\tilde{u}_1e_1) = J^{-1}(\tilde{u}_1e_1 + \tilde{u}_q e_q)J(\tilde{u}_q e_q) \tilde{Q}(\tilde{u}_q e_q) = J^{-1}(\tilde{u}_1e_1 + \tilde{u}_q e_q)J(\tilde{u}_q e_q) \left( \tilde{Q}(\tilde{u}_q e_q) - \tilde{Q}(0) \right).
$$

(4.14)

This together with (4.3) yields

$$
\frac{\partial \tilde{Q}}{\partial \tilde{u}_q}(\tilde{u}_1e_1) = J^{-1}(\tilde{u}_1e_1)J(0)\frac{\partial \tilde{Q}}{\partial \tilde{u}_q}(0) = 0, \quad q = 1, \ldots, r.
$$

(4.15)

Thus, from (4.13), one can get

$$
\tilde{Q}_1(\tilde{u}) = \sum_{p' = r+1}^{n} \frac{\partial \tilde{Q}_1}{\partial \tilde{u}_{p'}}(\tilde{u}_1e_1)\tilde{u}_{p'} + \sum_{p' = r+1}^{n} \frac{1}{2} \frac{\partial^2 \tilde{Q}_1}{\partial \tilde{u}_{p'} \partial \tilde{u}_{p''}}(\tilde{u}_1e_1)\tilde{u}_{p'}\tilde{u}_{p''} + \sum_{j', j'' = 2}^{n} \tilde{Z}_{ij'j''}(\tilde{u})\tilde{u}_{j'}\tilde{u}_{j''}, \quad i = 1, \ldots, n.
$$

(4.16)

Expanding further the functions of $\tilde{u}_1$ inside the first two summations in (4.16) around $\tilde{u}_1 = 0$, one has

$$
\tilde{Q}_1(\tilde{u}) = \sum_{p' = r+1}^{n} \frac{\partial \tilde{Q}_1}{\partial \tilde{u}_{p'}}(0)\tilde{u}_{p'} + \sum_{p' = r+1}^{n} \tilde{Y}_{zp'1}(0)\tilde{u}_{p'}\tilde{u}_1 + \sum_{j', j'' = 2}^{n} \tilde{Z}_{ij'j''}(0)\tilde{u}_{j'}\tilde{u}_{j''} + \sum_{j', j'' = 2}^{n} \tilde{Z}_{ij'j''}(\tilde{u})\tilde{u}_{j'}\tilde{u}_{j''}, \quad i = 1, \ldots, n.
$$

(4.17)

Recall $\Theta$ from (1.6) and $\Upsilon$ from (1.7), and note (1.8) and (1.9), then it holds

$$
\tilde{Q}_q(\tilde{u}) = \sum_{p' = r+1}^{n} \sum_{i' = 1}^{n} \tilde{Y}_{qp'1}(\tilde{u})\tilde{u}_{p'}\tilde{u}_1 + \sum_{j', j'' = 2}^{n} \tilde{Z}_{ij'j''}(\tilde{u})\tilde{u}_{j'}\tilde{u}_{j''} + \sum_{j', j'' = 2}^{n} \tilde{Z}_{ij'j''}(\tilde{u})\tilde{u}_{j'}\tilde{u}_{j''} + \sum_{j', j'' = 2}^{n} \tilde{Z}_{ij'j''}(\tilde{u})\tilde{u}_{j'}\tilde{u}_{j''}, \quad q = 1, \ldots, r.
$$

(4.18)

and

$$
\tilde{Q}_p(\tilde{u}) = \sum_{p' = r+1}^{n} \Theta_{pp'}\tilde{u}_{p'} + \sum_{p' = r+1}^{n} \tilde{Y}_{zp'1}(\tilde{u})\tilde{u}_{p'}\tilde{u}_1 + \sum_{j', j'' = 2}^{n} \tilde{Y}_{zp'j''}(\tilde{u})\tilde{u}_{j'}\tilde{u}_{j''}, \quad p = r + 1, \ldots, n.
$$

(4.19)

where $\tilde{Y}_{zp'j''}(\tilde{u}) = \frac{1}{2} \tilde{\Upsilon}_{zp'j''} + \sum_{i = 1}^{n} \tilde{Z}_{zp'j''i}(\tilde{u})\tilde{u}_{i}$.

4.2. Structure of the subsystem (4.20). In this subsection the subsystem (4.20) is analyzed under the assumptions (3\mathfrak{A}1)–(3\mathfrak{A}4).
4.2.1. Symmetrizer. The first task is to search for a positive definite \((n - 1) \times (n - 1)\) symmetric matrix to symmetrize the subsystem (3.20). Indeed,

\[
\tilde{A}^{0,\beta} (\tilde{u}) := \left( \frac{\partial^2 \tilde{\eta}}{\partial \tilde{u}_j \partial \tilde{u}_{j'}} (\tilde{u}) - \sum_{i,i'=1}^{n} \frac{\partial \tilde{\eta}}{\partial \tilde{u}_i} (\tilde{u}) \left( \frac{\partial G}{\partial \tilde{u}} \right)_{ii'}^{-1} \frac{\partial^2 G_{i'}^{\beta}}{\partial \tilde{u}_j \partial \tilde{u}_{j'}} (\tilde{u}) \right)_{j,j'=2}^{n} \tag{4.20}
\]

is just the desired symmetrizer.

Lemma 4.2. \(\tilde{A}^{0,\beta}(\tilde{u})\) is symmetric and positive definite near \(\tilde{u} = 0\). Moreover,

\[
\tilde{A}^{0,\beta}(\tilde{u}) \tilde{A}^{k,\beta}(\tilde{u}) = \tilde{A}^{k,\beta}(\tilde{u})^T \tilde{A}^{0,\beta}(\tilde{u}), \quad k = 1, \ldots, d. \tag{4.21}
\]

Proof. It is direct to check that \(\tilde{A}^{0,\beta}(\tilde{u})\) is symmetric. Meanwhile, by (3.31), it holds

\[
\tilde{A}^{0,\beta}(0) = (\nabla^2 \tilde{\eta}(0))^\beta. \tag{4.22}
\]

On the other hand, according to the assumption (A2), \(\tilde{\eta}\) is strictly convex, then it follows from (4.22) that \(\tilde{A}^{0,\beta}(\tilde{u})\) is positive definite near \(\tilde{u} = 0\). Moreover, it follows from (3.9) and (3.31) that, setting \(F^k(\tilde{u}) = F^k(u(\tilde{u}))\),

\[
0 = \frac{\partial}{\partial \tilde{u}_{i'}} \left( \frac{\partial \psi^k}{\partial \tilde{u}_i} \right) - \frac{\partial}{\partial \tilde{u}_i} \left( \frac{\partial \psi^k}{\partial \tilde{u}_{i'}} \right) = \sum_{r,l=1}^{n} \left( \frac{\partial^2 \tilde{\eta}}{\partial \tilde{u}_r \partial \tilde{u}_l} \left( \frac{\partial G}{\partial \tilde{u}} \right)_{rl}^{-1} \frac{\partial F^k}{\partial \tilde{u}_l} + \frac{\partial \tilde{\eta}}{\partial \tilde{u}_r} \frac{\partial}{\partial \tilde{u}_l} \left( \left( \frac{\partial G}{\partial \tilde{u}} \right)_{rl}^{-1} \frac{\partial \tilde{F}^k}{\partial \tilde{u}_l} + \frac{\partial \tilde{\eta}}{\partial \tilde{u}_r} \frac{\partial}{\partial \tilde{u}_l} \left( \left( \frac{\partial G}{\partial \tilde{u}} \right)_{rl}^{-1} \frac{\partial \tilde{F}^k}{\partial \tilde{u}_l} \right) \right) \right)
\]

This together with (4.22) gives rise to (4.21). \(\square\)

4.2.2. Dissipative structure. In this part the linear dissipative structure is investigated for the subsystem (3.20), which yields the damping effect on the component \(\tilde{u}^\beta\) of the system (5.8).

Lemma 4.3. Recall \(\tilde{A}^{0,\beta}(0)\) from (4.20) and \(\Theta\) from (4.6). Then it holds

\[
\tilde{A}^{0,\beta}(0) \begin{pmatrix} 0 & 0 \\ 0 & \Theta^\beta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (\Theta^\beta)^T \end{pmatrix} \tilde{A}^{0,\beta}(0) = - \begin{pmatrix} 0 & 0 \\ 0 & \tilde{M}^\beta \end{pmatrix} \tag{4.23}
\]

for a constant \((n - r) \times (n - r)\) symmetric matrix \(\tilde{M}^\beta\), which is positive definite

\[
\tilde{M}^\beta \geq \tilde{c}_m I_{n-r}, \tag{4.24}
\]

for a constant \(\tilde{c}_m > 0\).
Proof. We employ a modification of the proof of Theorem 2.1 of [25]. Recall that \( \nabla_u \tilde{Q}(0) = \begin{pmatrix} 0 & 0 \\ 0 & \Theta^p \end{pmatrix} \), and block accordingly \( \nabla_u^2 \tilde{q}(0) = \begin{pmatrix} B_1 & B_2^T \\ B_2 & B_4 \end{pmatrix} \). It follows that
\[
\nabla_u^2 \tilde{q}(0) \nabla_u \tilde{Q}(0) + (\nabla_u \tilde{Q}(0))^T \nabla_u^2 \tilde{q}(0) = \begin{pmatrix} B_1 & B_2^T \\ B_2 & B_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \Theta^p \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (\Theta^p)^T \end{pmatrix} \begin{pmatrix} B_1 & B_2^T \\ B_2 & B_4 \end{pmatrix} = \begin{pmatrix} B_1^TB_1 & B_1^TB_2^T + B_2(\Theta^p)^TB_4 \\ B_2^TB_1 & B_4(\Theta^p)^TB_4 + (\Theta^p)^TB_4 \end{pmatrix}.
\]

(4.25)

Since \( (\nabla_u \tilde{q}(\tilde{u}))^T \tilde{Q}(\tilde{u}) \) achieves its local maximum at \( \tilde{u} = 0 \) by (A3), one gets that \( \nabla_u^2 \tilde{q}(0) \nabla_u \tilde{Q}(0) + (\nabla_u \tilde{Q}(0))^T \nabla_u^2 \tilde{q}(0) \) is semi-negative definite. It then follows from (4.25) that
\[
\nabla_u^2 \tilde{q}(0) \nabla_u \tilde{Q}(0) + (\nabla_u \tilde{Q}(0))^T \nabla_u^2 \tilde{q}(0) = - \begin{pmatrix} 0 & 0 \\ 0 & \tilde{M}^p \end{pmatrix},
\]

(4.26)

where \( \tilde{M}^p = -B_4\Theta^p - (\Theta^p)^TB_4 \) is symmetric and semi-positive definite.

Note that (4.23) follows by (4.20) and (4.22). It remains to prove that \( \tilde{M}^p \) is indeed positive definite. Note that it is equivalent to prove that \( \tilde{M}^p \) is invertible. Suppose that \( \tilde{M}^p \) has a zero eigenvalue, one may take a corresponding right eigenvector \( \tilde{u}_p \neq 0 \) and set \( \tilde{u}_* = \frac{0}{\tilde{u}_p^2} \). Then by a Taylor expansion, since \( \tilde{Q}(0) = 0 = \nabla_u \tilde{q}(0) \),
\[
0 \geq (\nabla_u \tilde{q}(\tilde{u}_*))^T \tilde{Q}(\tilde{u}_*) + c_*|\tilde{Q}(\tilde{u}_*)|^2
\]
\[
= u_*^T (\nabla_u \tilde{q}(0) \nabla_u \tilde{Q}(0) + (\nabla_u \tilde{Q}(0))^T \nabla_u^2 \tilde{q}(0)) \tilde{u}_* + \tilde{c}_e|\nabla_u \tilde{Q}(0) \tilde{u}_*|^2 + o(|\tilde{u}_*|^2)
\]
\[
= -(\tilde{u}_p^2)2\tilde{M}^p \tilde{u}_p \tilde{u}_p + \tilde{c}_e|\Theta^p \tilde{u}_p|^2 + o(|\tilde{u}_p|^2)
\]
\[
= \tilde{c}_e|\Theta^p \tilde{u}_p|^2 + o(|\tilde{u}_p|^2),
\]

which yields the contradiction since \( \Theta^p \) is invertible by (A1), if \( \tilde{u}_p^2 \) has been taken sufficiently small. \( \square \)

4.2.3. Partial Kawashima condition. Finally, the partial Kawashima condition inside the assumption (A4) for the subsystem (3.26) is explored, which implies that the component \( \tilde{u}_c \) of the system (3.28) is indeed affected by the dissipation linearly.

Lemma 4.4. For any \( \omega \in S^{d-1}, \) recall \( \tilde{A}^\omega(0, \omega) \) from (3.24). Then there exists a constant \( c_k > 0 \) and an \((n-1) \times (n-1)\) skew-symmetric real matrix \( K = K(\omega) \in C^\infty(S^{d-1}) \) with
\[
K(-\omega) = -K(\omega)
\]

(4.27)

such that
\[
K(\omega) \tilde{A}^\omega(0, \omega) - (\tilde{A}^\omega(0, \omega))^T K(\omega) \geq 2c_k I_{n-1} - 2\text{diag}(0, I_{r-1}), \quad \forall \omega \in S^{d-1}.
\]

(4.28)

Proof. Consider the following linear system
\[
\tilde{A}^{0,\omega}(0)V^\omega_x + \sum_{k=1}^{d} \tilde{A}^{0,\omega}(0) \tilde{A}^{k,\omega}(0)V^\omega_x = - \begin{pmatrix} 0 & 0 \\ 0 & \tilde{M}^p \end{pmatrix} V^\omega.
\]

(4.29)

Due to (3.3) and (3.36), it holds that \( \tilde{r}^\omega_j(0, \omega) \neq 0 \) and hence
\[
\begin{pmatrix} 0 & 0 \\ 0 & \tilde{M}^p \end{pmatrix} \tilde{r}^\omega_j(0, \omega) \neq 0, \quad j = 2, \ldots, n.
\]

(4.30)

On the other hand, by (3.17) and (3.24), \( \{\tilde{r}^\omega_j(0, \omega)\}_{j=2}^n \) is a set of right eigenvectors of the matrix \( \tilde{A}^\omega(0, \omega) \) with \( \tilde{\lambda}_j(0, \omega) \) \( (j = 2, \ldots, n) \) as the corresponding eigenvalues. Moreover, by (3.18) and (3.21), \( \{\tilde{r}^\omega_j(0, \omega)\}_{j=2}^n \) are also linearly independent for each given \( \omega \in S^{d-1} \). This
verifies the Kawashima condition (akhir) for the system (akhir). Noting (akhir), one can conclude the lemma from Theorem 1.1 of Shizuta–Kawashima [23].

5. ENERGY ESTIMATES WITH FULL DISSIPATION

This section is devoted to derive the energy estimates for the system (akhir) which includes the full dissipative estimates of the component \( \tilde{u}^k \), under the assumption (akhir) or (akhir). Throughout Sections 5–8 it will be assumed a priori that for sufficiently small \( \delta > 0 \),

\[
\| \tilde{u}(t) \|_{H^t} \lesssim \delta. \tag{5.1}
\]

5.1. Entropy dissipation estimates. One may first derive the \( L^2 \) energy estimates for the solution \( \tilde{u} \) by using the entropy \( \tilde{\eta} = \tilde{\eta}(\tilde{u}) \), defined by \( (akhir) \).

**Lemma 5.1.** The following estimates hold.

(i) Under (akhir),

\[
\frac{d}{dt} \int \tilde{\eta}(\tilde{u}) + \| \tilde{u}^p \|_{L^2}^2 \lesssim \delta^2 \| \tilde{u}^p \|_{L^\infty}^2. \tag{5.2}
\]

(ii) Under (akhir),

\[
\frac{d}{dt} \int \tilde{\eta}(\tilde{u}) + \| \tilde{u}^p \|_{L^2}^2 \lesssim \| \tilde{u}^b \|_{L^2} \| \tilde{u}^b \|_{L^\infty}^2. \tag{5.3}
\]

**Proof.** Applying \( (\nabla \tilde{u} \tilde{\eta}(\tilde{u}))^T \) to \( (akhir) \), the entropy assumption (akhir) implies

\[
\partial_t \tilde{\eta}(\tilde{u}) + \sum_{k=1}^{d} \left( \tilde{\psi}^k(\tilde{u}) \right) = \nabla \tilde{u} \tilde{\eta}(\tilde{u}) \cdot \tilde{Q}(\tilde{u}). \tag{5.4}
\]

Integrating \( (5.4) \) over \( \mathbb{R}^d \), by the entropy dissipation assumption (akhir), one has

\[
\frac{d}{dt} \int \tilde{\eta}(\tilde{u}) + \| \tilde{u}^p \|_{L^2}^2 \leq 0. \tag{5.5}
\]

Since trivially \( \| \tilde{Q}(\tilde{u}) \|_{L^2} \) and recalling \( \Theta \) defined in \( (akhir) \), it follows from \( (5.5) \) that

\[
\frac{d}{dt} \int \tilde{\eta}(\tilde{u}) + \| \tilde{u}^p \|_{L^2}^2 \lesssim \| \tilde{Q}(\tilde{u}) - \Theta \tilde{u}^p \|_{L^2}^2. \tag{5.6}
\]

Now one may prove (i) under (akhir). By the expansion \( (akhir) \) and the a priori assumption (akhir), using Sobolev’s inequality, it holds that

\[
\| \tilde{Q}(\tilde{u}) - \Theta \tilde{u}^p \|_{L^2}^2 \lesssim \| \tilde{Y}(\tilde{u}) \tilde{u}^2 \tilde{u} \|_{L^2}^2 \lesssim \| \tilde{u} \|_{L^2} \| \tilde{u}^p \|_{L^\infty}^2 \lesssim \delta^2 \| \tilde{u}^p \|_{L^\infty}^2. \tag{5.7}
\]

Plugging the estimates \( (5.7) \) into \( (5.6) \), by the assumption (akhir) (or equivalently, \( (akhir) \)), one deduces \( (5.2) \).

One then turns to prove (ii) under (akhir). By the expansion \( (akhir) \), it holds that

\[
\| \tilde{Q}(\tilde{u}) - \Theta \tilde{u}^p \|_{L^2}^2 \lesssim \| \tilde{Y}(\tilde{u}) \tilde{u}^2 \tilde{u} \|_{L^2}^2 + \| \tilde{Y}(\tilde{u}) \tilde{u}^2 \tilde{u} \|_{L^2}^2 \lesssim \delta^2 \| \tilde{u}^p \|_{L^2}^2 + \| \tilde{u}^p \|_{L^2}^2 \| \tilde{u}^p \|_{L^\infty}^2. \tag{5.8}
\]

Plugging the estimates \( (5.8) \) into \( (5.6) \), since \( \delta \) is small, one deduces \( (5.3) \). \( \square \)

5.2. Energy estimates for \( \tilde{u}^b \). One may next derive the higher-order energy estimates for the component \( \tilde{u}^b \) by using the structure of the subsystem (akhir).

**Lemma 5.2.** For \( 1 \leq s \leq \ell \), the following estimates hold.

(i) Under (akhir),

\[
\frac{d}{dt} \int (\Lambda^s \tilde{u}^b)^T \tilde{A}^{0b}(\tilde{u}) \Lambda^s \tilde{u}^b + \| \Lambda^s \tilde{u}^p \|_{L^2}^2 \lesssim \delta \left( \| \Lambda^s \tilde{u}^b \|_{L^2}^2 + \| \tilde{u}^b \|_{L^\infty}^2 + \| \nabla \tilde{u}^b \|_{L^\infty}^2 \right). \tag{5.9}
\]
(ii) Under $(D_1)$,
\[
\frac{d}{dt} \int (\Lambda^s \tilde{\varphi}^T \tilde{A}^{0,b}(\tilde{u}) \Lambda^s \tilde{\varphi} + \|\Lambda^s \tilde{u}\|^2_{L^2}) \lesssim \delta \left( \|\Lambda^s \tilde{u}\|^2_{L^2} + \|\tilde{u}\|^2_{L^\infty} + \|\nabla \tilde{u}\|^2_{L^\infty} \right) + \|\Lambda^s \tilde{u}\|^2_{L^\infty} \|\tilde{u}\|^2_{L^\infty}.
\]  
(5.10)

**Proof.** For $1 \leq s \leq \ell$, applying $\Lambda^s$ to (3.26) and employing the commutator notation (B.7) yields
\[
\partial_t (\Lambda^s \tilde{u}^b) + \sum_{k=1}^d \tilde{A}^{k,b}(\tilde{u})(\Lambda^s \tilde{u}^b)_{x_k} = - \sum_{k=1}^d \left[ \Lambda^s, \tilde{A}^{k,b}(\tilde{u}) \right] \tilde{u}^b_{x_k} + \Lambda^s \tilde{Q}^b(\tilde{u}).
\]  
(5.11)

Applying $(\Lambda^s \tilde{u}^b)^T \tilde{A}^{0,b}(\tilde{u})$ to (5.11), since by Lemma 4.2 $\tilde{A}^{0,b}(\tilde{u})$ and $\tilde{A}^{0,b}(\tilde{u}) \tilde{A}^{k,b}(\tilde{u})$ are symmetric, one has
\[
\partial_t \left( (\Lambda^s \tilde{u}^b)^T \tilde{A}^{0,b}(\tilde{u}) \Lambda^s \tilde{u}^b \right) + \sum_{k=1}^d \left( (\Lambda^s \tilde{u}^b)^T \tilde{A}^{0,b}(\tilde{u}) \tilde{A}^{k,b}(\tilde{u}) \Lambda^s \tilde{u}^b \right)_{x_k}
\]
\[
= (\Lambda^s \tilde{u}^b)^T \left( \partial_t (\tilde{A}^{0,b}(\tilde{u})) + \sum_{k=1}^d \left( \tilde{A}^{0,b}(\tilde{u}) \tilde{A}^{k,b}(\tilde{u}) \right)_{x_k} \right) \Lambda^s \tilde{u}^b
\]
\[ - 2Re \left( \sum_{k=1}^d (\Lambda^s \tilde{u}^b)^T \tilde{A}^{0,b}(\tilde{u}) \left[ \Lambda^s, \tilde{A}^{k,b}(\tilde{u}) \right] \tilde{u}^b_{x_k} \right) + 2Re \left( (\Lambda^s \tilde{u}^b)^T \tilde{A}^{0,b}(\tilde{u}) \Lambda^s \tilde{Q}^b(\tilde{u}) \right).
\]  
(5.12)

Here $R e f = (f + f^T)/2$.

Now one may estimate the integration of the right hand side of (5.12) over $\mathbb{R}^d$. Replacing $\partial_t \tilde{u}$ by using the system (3.8), since $\ell > d/2 + 1$ and by Sobolev’s inequality, the first term is easily bounded by
\[
\left( \|\partial_t \tilde{u}\|_{L^\infty} + \|\nabla \tilde{u}\|_{L^\infty} \right) \left( \|\Lambda^s \tilde{u}\|^2_{L^2} \right) \lesssim \delta \left( \|\Lambda^s \tilde{u}\|^2_{L^2} \right).
\]  
(5.13)

By the commutator estimates (B.8) of Lemma B.3 using Cauchy’s inequality, one may bound the second term by
\[
\left\| \Lambda^s \tilde{u}^b \right\|_{L^2} \sum_{k=1}^d \left( \left\| \Lambda^s, \tilde{A}^{k,b}(\tilde{u}) \right\| \tilde{u}^b_{x_k} \right\|_{L^2}.
\]
\[
\lesssim \left\| \Lambda^s \tilde{u}^b \right\|_{L^2} \sum_{k=1}^d \left( \|\nabla (\tilde{A}^{k,b}(\tilde{u}))\|_{L^\infty} \left\| \Lambda^s \tilde{u}^b \right\|_{L^2} + \left\| \Lambda^s (\tilde{A}^{k,b}(\tilde{u})) \right\|_{L^2} \|\nabla \tilde{u}\|_{L^\infty} \right) \]
\[ \lesssim \delta \left( \left\| \Lambda^s \tilde{u}^b \right\|_{L^2}^2 + \|\nabla \tilde{u}\|_{L^\infty}^2 \right).
\]  
(5.14)

For the last term, one can rewrite
\[
(\Lambda^s \tilde{u}^b)^T \tilde{A}^{0,b}(\tilde{u}) \Lambda^s \tilde{Q}^b(\tilde{u})
\]
\[
= (\Lambda^s \tilde{u}^b)^T \tilde{A}^{0,b}(\tilde{u}) \Theta^s \Lambda^s \tilde{u}^b + (\Lambda^s \tilde{u}^b)^T \tilde{A}^{0,b}(\tilde{u}) \Lambda^s \left( \tilde{Q}^b(\tilde{u}) - \Theta^s \tilde{u}^b \right)
\]
\[
= (\Lambda^s \tilde{u}^b)^T \tilde{A}^{0,b}(0) \Theta^s \Lambda^s \tilde{u}^b + (\Lambda^s \tilde{u}^b)^T \left( \tilde{A}^{0,b}(\tilde{u}) - \tilde{A}^{0,b}(0) \right) \Theta^s \Lambda^s \tilde{u}^b
\]
\[ + (\Lambda^s \tilde{u}^b)^T \tilde{A}^{0,b}(\tilde{u}) \Lambda^s \left( \tilde{Q}^b(\tilde{u}) - \Theta^s \tilde{u}^b \right).\]
\[ \]
(5.15)

By (3.26)–(3.34) from Lemma 3.3, it holds that
\[
\int \! 2Re \left( (\Lambda^s \tilde{u}^b)^T \tilde{A}^{0,b}(0) \Theta^s \Lambda^s \tilde{u}^b \right) = - \int (\Lambda^s \tilde{u}^b)^T M^b \Lambda^s \tilde{u}^b \leq - \tilde{c}_m \left\| \Lambda^s \tilde{u}^b \right\|_{L^2}^2.
\]  
(5.16)

On the other hand, it is easy to bound
\[
\int \! 2Re \left( (\Lambda^s \tilde{u}^b)^T \left( \tilde{A}^{0,b}(\tilde{u}) - \tilde{A}^{0,b}(0) \right) \Theta^s \Lambda^s \tilde{u}^b \right) \lesssim \|\tilde{u}\|_{L^\infty} \left\| \Lambda^s \tilde{u}^b \right\|_{L^2}^2 \lesssim \delta \left\| \Lambda^s \tilde{u}^b \right\|_{L^2}^2.
\]  
(5.17)
Hence, integrating (5.12) over $\mathbb{R}^d$, it follows from (5.13)–(5.17) that

$$
\frac{d}{dt} \int (\Lambda^s \tilde{w}^b)^T \tilde{A}^{b, b}(\tilde{u}) \Lambda^s \tilde{w}^b + \| \Lambda^s \tilde{w}^b \|_{L^2}^2 \\
\lesssim \delta \left( \| \Lambda^s \tilde{w}^b \|_{L^2}^2 + \| \nabla \tilde{w}^b \|_{L^\infty}^2 \right) + \| \Lambda^s \tilde{w}^b \|_{L^2} \| \Lambda^s \left( \tilde{Q}^b(\tilde{u}) - \Theta^b \tilde{w}^b \right) \|_{L^2}.
$$

(5.18)

Now one may prove (i) under (\ref{MD3}). By the expansions \ref{4.10}–\ref{4.11} and using the product estimates (B.9) of Lemma B.3, one may estimate

$$
\| \Lambda^s \left( \tilde{Q}^b(\tilde{u}) - \Theta^b \tilde{w}^b \right) \|_{L^2} \lesssim \| \Lambda^s \left( \tilde{Y}(\tilde{u}) \tilde{w}^b \right) \|_{L^2} \\
\lesssim \| \tilde{Y}(\tilde{u}) \tilde{w}^b \|_{L^\infty} \| \Lambda^s \tilde{w}^b \|_{L^2} + \| \Lambda^s \left( \tilde{Y}'(\tilde{u}) \tilde{w}^b \right) \|_{L^2} \| \tilde{w}^b \|_{L^\infty} \\
\lesssim \delta \left( \| \Lambda^s \tilde{w}^b \|_{L^2} + \| \tilde{w}^b \|_{L^\infty} \right).
$$

(5.19)

Plugging the estimates (5.19) into (5.18) yields (5.9).

One then turns to prove (ii) under (\ref{D1}). By the expansions \ref{4.18}–\ref{4.19} and using the product estimates (B.9) of Lemma B.3, one may estimate

$$
\| \Lambda^s \left( \tilde{Q}^b(\tilde{u}) - \Theta^b \tilde{w}^b \right) \|_{L^2} \lesssim \| \Lambda^s \left( \tilde{Y}'(\tilde{u}) \tilde{w}^b \right) \|_{L^2} + \| \Lambda^s \left( \tilde{Y}''(\tilde{u}) \tilde{w}^b \right) \|_{L^2} \\
\lesssim \| \tilde{Y}'(\tilde{u}) \tilde{w}^b \|_{L^\infty} \| \Lambda^s \tilde{w}^b \|_{L^2} + \| \Lambda^s \left( \tilde{Y}'(\tilde{u}) \tilde{w}^b \right) \|_{L^2} \| \tilde{w}^b \|_{L^\infty} \\
\lesssim \delta \left( \| \Lambda^s \tilde{w}^b \|_{L^2} + \| \tilde{w}^b \|_{L^\infty} \right) + \| \tilde{w}^b \|_{L^\infty}^2.
$$

(5.20)

Plugging the estimates (5.20) into (5.18) yields (5.10). \hfill \Box

5.3. Recovering the full dissipation. Noting that Lemmas \ref{5.4}–\ref{5.5} only contain the dissipation estimates for the component $\tilde{w}^b$, one may then use the partial Kawashima condition of the subsystem (3.26) to recover the dissipation estimates for the component $\tilde{w}^c$ (and hence $\tilde{w}^y$).

Lemma 5.3. For $0 \leq s \leq \ell - 1$, the following estimates hold.

(i) Under (\ref{MD3}) and that $d \geq 3$,

$$
\frac{d}{dt} \int \left( |\xi|^s \tilde{w}^b \right)^T K \left( |\xi|^{s+1} \tilde{w}^b \right) d\xi + \| \Lambda^{s+1} \tilde{w}^b \|_{L^2}^2 \\
\lesssim \| \Lambda^s \tilde{w}^b \|_{L^2}^2 + \| \Lambda^{s+1} \tilde{w}^b \|_{L^2}^2 + \delta \left( \| \tilde{w}^b \|_{L^\infty}^2 + \| \nabla \tilde{w}^b \|_{L^\infty}^2 \right).
$$

(5.21)

(ii) Under (\ref{D1}),

$$
\frac{d}{dt} \int \left( |\xi|^s \tilde{w}^b \right)^T K \left( |\xi|^{s+1} \tilde{w}^b \right) d\xi + \| \Lambda^{s+1} \tilde{w}^b \|_{L^2}^2 \\
\lesssim \| \Lambda^s \tilde{w}^b \|_{L^2}^2 + \| \Lambda^{s+1} \tilde{w}^b \|_{L^2}^2 + \delta \left( \| \tilde{w}^b \|_{L^\infty}^2 + \| \nabla \tilde{w}^b \|_{L^\infty}^2 \right) + \| \tilde{w}^b \|_{L^\infty}^4
$$

(5.22)

Here $K = K(|\xi|/|\xi|)$ is defined in Lemma 4.4.

Proof. The subsystem (3.26) can be rewritten in the following form

$$
\partial_t \tilde{w}^b + \sum_{k=1}^{d} \tilde{A}^{k,b}(0) \tilde{w}_{z_k}^b = - \sum_{k=1}^{d} \left( \tilde{A}^{k,b}(\tilde{u}) - \tilde{A}^{k,b}(0) \right) \tilde{w}_{z_k}^b + \tilde{Q}^b(\tilde{u}) := h^b.
$$

(5.23)
Taking the Fourier transform yields

$$\partial_t \hat{u}^b + i \sum_{k=1}^d \xi_k \hat{A}^{k,b} (0) \hat{u}^b = \hat{h}^b. \tag{5.24}$$

Let \( K = K(\xi/|\xi|) \) be the skew-symmetric real matrix from Lemma 4.1, then

$$K \sum_{k=1}^d \xi_k \hat{A}^{k,b} (0) - \sum_{k=1}^d \xi_k \left( \hat{A}^{k,b} (0) \right)^T K \geq 2 |\xi| (c_k I_{n-1} - \text{diag}(0, I_{n-1})). \tag{5.25}$$

Applying \(-i(\hat{u}^b)^T K\) to \(5.24\) yields

$$- i(\hat{u}^b)^T K \partial_t \hat{u}^b + (\hat{u}^b)^T K \sum_{k=1}^d \xi_k \hat{A}^{k,b} (0) \hat{u}^b = - i(\hat{u}^b)^T K \hat{h}^b. \tag{5.26}$$

It then follows from \(5.25\) that

$$- i \partial_t \left( (\hat{u}^b)^T K \hat{u}^b \right) + 2 c_k |\xi| \left| \hat{u}^b \right|^2 \leq 2 |\xi| \left| \hat{u}^b \right|^2 + 2 \left| (\hat{u}^b)^T K \hat{h}^b \right|. \tag{5.27}$$

For \(0 \leq s \leq \ell - 1\), multiplying \(5.27\) by \(|\xi|^{2s+1}\) and then integrating over the frequency space \(\mathbb{R}^d_\xi\), yields, by the Plancherel theorem,

$$\frac{d}{dt} \int -i |\xi|^{2s+1} (\hat{u}^b)^T K \hat{u}^b \, d\xi + \left\| \Lambda^{s+1} \hat{u}^b \right\|_{L^2}^2 \lesssim \left\| \Lambda^{s+1} \hat{u}^b \right\|_{L^2}^2 + \left\| \Lambda^{s+1} \hat{u}^b \right\|_{L^2} \left\| \Lambda^s \hat{h}^b \right\|_{L^2}. \tag{5.28}$$

To estimate \(\left\| \Lambda^s \hat{h}^b \right\|_{L^2}\), one may use the product estimates \((B.9)\) of Lemma \(B.3\) to get

$$\left\| \Lambda^s \left( \hat{A}^{k,b} (\hat{u}) - \hat{A}^{k,b} (0) \right) \hat{u}^b \right\|_{L^2} \lesssim \left\| \hat{A}^{k,b} (\hat{u}) - \hat{A}^{k,b} (0) \right\|_{L^\infty} \left\| \Lambda^s \nabla \hat{u}^b \right\|_{L^2} + \left\| \Lambda^s \left( \hat{A}^{k,b} (\hat{u}) - \hat{A}^{k,b} (0) \right) \right\|_{L^2} \left\| \nabla \hat{u}^b \right\|_{L^\infty} \tag{5.29}.$$

On the other hand

$$\left\| \Lambda^s \left( \hat{Q}^b (\hat{u}) \right) \right\|_{L^2} \leq \left\| \Theta^b \Lambda^s \hat{u}^b \right\|_{L^2} + \left\| \Lambda^s \left( \hat{Q}^b (\hat{u}) - \Theta^b \hat{u}^b \right) \right\|_{L^2}. \tag{5.30}$$

Hence, by \((5.29)-(5.30)\), it follows from \(5.28\) that

$$\frac{d}{dt} \int -i |\xi|^{s+1} (\hat{u}^b)^T K (|\xi|^{s+1} \hat{u}^b) \, d\xi + \left\| \Lambda^{s+1} \hat{u}^b \right\|_{L^2}^2 \lesssim \left\| \Lambda^s \hat{u}^b \right\|_{L^2}^2 + \left\| \Lambda^{s+1} \hat{u}^b \right\|_{L^2}^2 + \delta \left\| \nabla \hat{u}^b \right\|_{L^\infty}^2 + \left\| \Lambda^{s+1} \hat{u}^b \right\|_{L^2} \left\| \Lambda^s \left( \hat{Q}^b (\hat{u}) - \Theta^b \hat{u}^b \right) \right\|_{L^2}. \tag{5.31}$$

Now one may prove \((i)\) under \((5.1)\) and that \(d \geq 3\). By the expansions \((1.10)-(1.11)\) and the product estimates \((B.9)\) of Lemma \(B.3\) it holds that

$$\left\| \Lambda^s \left( \hat{Q}^b (\hat{u}) - \Theta^b \hat{u}^b \right) \right\|_{L^2} \lesssim \left\| \Lambda^s \left( \hat{Y} (\hat{u}) \hat{u}^b \right) \right\|_{L^2} \tag{5.32} \lesssim \left\| \hat{Y} (\hat{u}) \hat{u}^b \right\|_{L^s} \left\| \Lambda^s \hat{u}^b \right\|_{L^2} + \left\| \Lambda^s \left( \hat{Y} (\hat{u}) \hat{u}^b \right) \right\|_{L^2} \left\| \hat{u}^b \right\|_{L^\infty} \tag{5.33} \lesssim \delta \left( \left\| \Lambda^{s+1} \hat{u}^b \right\|_{L^2} + \left\| \hat{u}^b \right\|_{L^\infty} \right).$$

Plugging the estimates \((5.32)\) into \((5.31)\), one deduces \((5.21)\).
One then turns to prove (ii) under \( \mathfrak{D}_1 \). By the expansions (4.18)–(4.19) and the product estimates (5.2) of Lemma 5.3, it holds that
\[
\left\| \Lambda^s \left( \tilde{Q}^b(\tilde{u}) - \Theta^b\tilde{u}^2 \right) \right\|_{L^2} \lesssim \left\| \Lambda^s \left( \tilde{Y}'(\tilde{u})\tilde{u}^2\tilde{u} \right) \right\|_{L^2} + \left\| \Lambda^s \left( \tilde{Y}'(\tilde{u})\tilde{u}^2 \right) \right\|_{L^2} + \left\| \tilde{Y}'(\tilde{u})\tilde{u}^2 \right\|_{L^\infty} + \left\| \Lambda^s \tilde{u}^2 \right\|_{L^2}
\]
(5.33)
Plugging the estimates (5.33) into (5.31), one deduces (5.22).

5.4. **Synthesis.** Now one can combine Lemmas 5.1–5.3 to arrive the following propositions.

**Proposition 5.4.** For \( 1 \leq s \leq \ell - 1 \), there exists an energy functional \( E_s(\tilde{u}^2) \) equivalent to \( \|\Lambda^s \tilde{u}^2\|_{H^{\ell-s}} \) such that the following estimates hold.

(i) **Under (\( \mathfrak{M}_1 \))** and that \( d \geq 3 \),
\[
\frac{d}{dt} E_s(\tilde{u}^2) + \left\| \Lambda^s \tilde{u}^2 \right\|_{H^{\ell-s}}^2 + \left\| \Lambda^{s+1} \tilde{u}^2 \right\|_{H^{\ell-s-1}}^2 \lesssim \delta \left( \left\| \Lambda^s \tilde{u}^2 \right\|_{L^2}^2 + \left\| \tilde{u}^2 \right\|_{L^\infty}^2 + \left\| \nabla \tilde{u}^2 \right\|_{L^\infty}^4 \right)
\]
(5.34)

(ii) **Under (\( \mathfrak{D}_1 \)),**
\[
\frac{d}{dt} E_s(\tilde{u}^2) + \left\| \Lambda^s \tilde{u}^2 \right\|_{H^{\ell-s}}^2 + \left\| \Lambda^{s+1} \tilde{u}^2 \right\|_{H^{\ell-s-1}}^2 \lesssim \delta \left( \left\| \Lambda^s \tilde{u}^2 \right\|_{L^2}^2 + \left\| \tilde{u}^2 \right\|_{L^\infty}^2 + \left\| \nabla \tilde{u}^2 \right\|_{L^\infty}^2 \right)
\]
(5.35)

**Proof.** One may first prove (i) under (\( \mathfrak{M}_1 \)) and that \( d \geq 3 \) by using the (i) assertions of Lemmas 5.2–5.3. Taking first \( 1 \leq s \leq \ell - 1 \) and then \( s = \ell \) in the estimates (5.9) of Lemma 5.2 noting that \( f \|_{H^k} \) is equivalent to the sum \( \|f\|_{L^2} + \|\Lambda^k f\|_{L^2} \), one has
\[
\frac{d}{dt} \int \left( (\Lambda^s \tilde{u}^2)^T \tilde{A}^{0,3}(\tilde{u})\Lambda^s \tilde{u}^2 + (\Lambda^s \tilde{u}^2)^T \tilde{A}^{0,3}(\tilde{u})\Lambda^s \tilde{u}^2 \right) + \left\| \Lambda^s \tilde{u}^2 \right\|_{H^{\ell-s}}^2 \lesssim \delta \left( \left\| \Lambda^s \tilde{u}^2 \right\|_{H^{\ell-s}}^2 + \left\| \tilde{u}^2 \right\|_{L^\infty}^2 + \left\| \nabla \tilde{u}^2 \right\|_{L^\infty}^2 \right).
\]
(5.36)
Next, taking first \( 1 \leq s \leq \ell - 1 \) and then \( s = \ell - 1 \) in the estimates (5.21) of Lemma 5.3, one obtains
\[
\frac{d}{dt} \int -i \left( (|\xi|^s \tilde{\omega})^T K (|\xi|^{s+1} \tilde{\omega}) + (|\xi|^{\ell-1} \tilde{\omega})^T K (|\xi|^{\ell} \tilde{\omega}) \right) d\xi + \left\| \Lambda^{s+1} \tilde{u}^2 \right\|_{H^{\ell-s-1}}^2 \lesssim \left\| \Lambda^s \tilde{u}^2 \right\|_{H^{\ell-s}}^2 + \delta \left( \left\| \tilde{u}^2 \right\|_{L^\infty}^2 + \left\| \nabla \tilde{u}^2 \right\|_{L^\infty}^2 \right).
\]
(5.37)
For sufficiently small constant \( \epsilon > 0 \), if one defines the instant energy constant
\[
E_s(\tilde{u}^2) = \int \left( (\Lambda^s \tilde{u}^2)^T \tilde{A}^{0,3}(\tilde{u})\Lambda^s \tilde{u}^2 + (\Lambda^s \tilde{u}^2)^T \tilde{A}^{0,3}(\tilde{u})\Lambda^s \tilde{u}^2 \right)
\]
\[
+ \epsilon \int -i \left( (|\xi|^s \tilde{\omega})^T K (|\xi|^{s+1} \tilde{\omega}) + (|\xi|^{\ell-1} \tilde{\omega})^T K (|\xi|^{\ell} \tilde{\omega}) \right) d\xi
\]
(5.38)
for \( 1 \leq s \leq \ell - 1 \), since \( \tilde{A}^{0,3}(\tilde{u}) \) is positive definite near \( \tilde{u} = 0 \) from Lemma 5.2, then it follows that \( E_s(\tilde{u}^2) \) is equivalent to \( \|\Lambda^s \tilde{u}^2\|_{H^{\ell-s}}^2 \). Moreover, since both \( \epsilon \) and \( \delta \) are small, one can deduce (5.34) from (5.36)–(5.37).

Similarly, to prove (ii) under \( \mathfrak{D}_1 \), one needs only to use instead the (ii) assertions of Lemmas 5.2–5.3 in the arguments above to deduce (5.35).

\[\square\]
Proposition 5.5. The following estimates hold.

(i) Under (W1) and that $d \geq 3$,
\[
\|\tilde{u}(t)\|_{L^2}^2 + \left\|\Lambda \tilde{u}^\delta(t)\right\|_{H^{d-1}}^2 + \int_0^t \left(\left\|\tilde{u}_0^\delta(\tau)\right\|_{H^d}^2 + \left\|\Lambda \tilde{u}^\delta(\tau)\right\|_{H^{d-1}}^2\right) \, d\tau \\
\lesssim \|\tilde{u}_0\|_{L^2}^2 + \left\|\Lambda \tilde{u}_0^\delta\right\|_{H^{d-1}}^2.
\]
(5.39)

(ii) Under (D1),
\[
\|\tilde{u}(t)\|_{L^2}^2 + \left\|\Lambda \tilde{u}^\delta(t)\right\|_{H^{d-1}}^2 + \int_0^t \left(\left\|\tilde{u}_0^\delta(\tau)\right\|_{H^d}^2 + \left\|\Lambda \tilde{u}^\delta(\tau)\right\|_{H^{d-1}}^2\right) \, d\tau \\
\lesssim \|\tilde{u}_0\|_{L^2}^2 + \left\|\Lambda \tilde{u}_0^\delta\right\|_{H^{d-1}}^2.
\]
(5.40)

Proof. One may first prove (i) under (W1) and that $d \geq 3$. Taking $s = 1$ in the estimates (5.34) of Proposition 5.4, the estimates (5.22) of Lemma 5.3 and the estimates (5.3) of Lemma 5.1 to

Taking $\delta(5.2)$ of Lemma 5.1, since

By this, integrating in time the sum of the estimates (5.41)–(5.42) and the entropy estimates (5.39)
once

Since $\ell > d/2 + 1$ and $d \geq 3$, by the interpolation estimates of Lemma 5.2, it holds that
\[
\left\|\tilde{u}\right\|_{L^\infty}^2 + \left\|\nabla \tilde{u}\right\|_{L^\infty}^2 \lesssim \left\|\Lambda \tilde{u}\right\|_{H^{d-1}}^2.
\]
(5.43)

By this, integrating in time the sum of the estimates (5.41)–(5.42) and the entropy estimates (5.2) of Lemma 5.1, since $\delta$ is small, one can deduce (5.39).

Similarly, to prove (ii) under (D1), one needs only to use instead the estimates (5.34) of Proposition 5.4, the estimates (5.22) of Lemma 5.3 and the estimates (5.3) of Lemma 5.1 to deduce (5.40).

6. Full energy estimates

In view of the energy estimates (5.39)–(5.40) of Proposition 5.5 in Section 5, it appears natural to derive the energy estimates for the derivatives of the remaining component $\tilde{u}_1$. However, the evolution equation (5.22) of $\tilde{u}_1$ is apparently not suitable for this goal since there would be a loss of derivatives if one directly performs the energy estimate. To get around this obstacle, despite that the component $\tilde{u}_2$ has already been controlled, one should still return back to derive the estimates for the derivatives of the whole $\tilde{u}$ by using the original system (3.3).

Proposition 6.1. The following estimates hold.

(i) Under (W1)–(W2) and that $d \geq 3$,
\[
\|\tilde{u}(t)\|_{H^d} \lesssim \|\tilde{u}_0\|_{H^d} + \int_0^t \left(\|\tilde{u}\|_{L^\infty}^2 + \|\nabla \tilde{u}\|_{L^\infty}^2 + \|\Lambda \tilde{u}^\delta\|_{L^2}^2 + \|\Lambda^\ell \tilde{u}^\delta\|_{L^2}^2\right) \, d\tau.
\]
(6.1)

(ii) Under (D1)–(D2) and that $d \geq 2$,
\[
\|\tilde{u}(t)\|_{H^d} \lesssim \|\tilde{u}_0\|_{H^d} + \int_0^t \left(\|\tilde{u}\|_{L^\infty}^2 + \|\tilde{u}_0\|_{H^d} + \|\nabla \tilde{u}\|_{L^\infty}^2 + \|\Lambda \tilde{u}^\delta\|_{L^2}^2 + \|\Lambda^\ell \tilde{u}^\delta\|_{L^2}^2\right) \, d\tau.
\]
(6.2)
Thanks to the entropy estimates in Lemma 5.1, by the Sobolev interpolation, it suffices to derive the energy estimates for \( \Lambda^{\ell-1} \tilde{u} \). Note that this is equivalent to derive the energy estimates of \( \Lambda^{\ell-1} \nabla \tilde{u} \). In the spirit of one dimensional study [9, 16, 17], one may do the wave decomposition

\[
\tilde{u}_{x_k} = \sum_{i=1}^{n} \tilde{w}_{i,k} \tilde{r}_i(\tilde{u}, \e_k), \quad k = 1, \ldots, d,
\]

where

\[
\tilde{w}_{i,k} = \tilde{l}_i(\tilde{u}, \e_k) \tilde{u}_{x_k}, \quad i = 1, \ldots, n; \quad k = 1, \ldots, d.
\]

Using the product estimate (3.9) of Lemma 3.3 one has

\[
\left\| \Lambda^{\ell-1} \tilde{u}_{x_k} \right\|_{L^2} \lesssim \sum_{i=1}^{n} \left\| \tilde{w}_{i,k} \right\|_{L^\infty} \left\| \Lambda^{\ell-1}(\tilde{r}_i(\tilde{u}, \e_k) - \tilde{r}_i(0, \e_k)) \right\|_{L^2} + \sum_{i=1}^{n} \left\| \Lambda^{\ell-1} \tilde{w}_{i,k} \right\|_{L^2} \left\| \tilde{r}_i(\tilde{u}, \e_k) \right\|_{L^\infty}
\]

\[
\lesssim \delta \| \tilde{u} \|_{H^\ell} + \sum_{i=1}^{n} \left\| \Lambda^{\ell-1} \tilde{w}_{i,k} \right\|_{L^2}, \quad k = 1, \ldots, d.
\]

This together with the estimates (5.39)–(5.40) of Proposition 5.5 implies that it suffices to derive the energy estimates for \( \Lambda^{\ell-1} \tilde{w}_{i,k} \) for each \( k' = 1, \ldots, d \). Note that for the notational convenience the index notation \( k \) is changed into \( k' \).

Now it is crucial to derive the equation of \( \tilde{w}_{1,k'} \). Taking \( \partial_{x_{k'}} \) to the system (3.8) yields

\[
\partial_t \tilde{u}_{x_{k'}} + \sum_{k=1}^{d} \left( \tilde{A}^k(\tilde{u}) \tilde{u}_{x_k} \right)_{x_{k'}} = \nabla \tilde{Q}(\tilde{u}) \tilde{u}_{x_{k'}}.
\]

Substituting the decomposition (6.3) into (6.7), by (3.17) and the system (3.8), one can deduce

\[
\sum_{i=1}^{n} \partial_t \tilde{w}_{i,k'} \tilde{r}_i(\tilde{u}, \e_{k'}) + \sum_{i=1}^{n} \sum_{k=1}^{d} \tilde{\lambda}_i(\tilde{u}, \e_k) (\tilde{w}_{i,k})_{x_{k'}} \tilde{r}_i(\tilde{u}, \e_k) = \frac{d}{2} \tilde{A}^k(\tilde{u}) \tilde{u}_{x_{k'}} - \tilde{Q}(\tilde{u})
\]

\[
+ \sum_{i=1}^{n} \tilde{w}_{i,k'} \nabla \tilde{Q}(\tilde{u}) \tilde{r}_i(\tilde{u}, \e_{k'}) \left( \sum_{i'=1}^{d} \sum_{k=1}^{d} \tilde{\lambda}_{i'}(\tilde{u}, \e_k) \tilde{w}_{i',k'} \tilde{r}_{i'}(\tilde{u}, \e_k) - \tilde{Q}(\tilde{u}) \right)
\]

\[
- \sum_{i=1}^{n} \sum_{k=1}^{d} \tilde{w}_{i,k} \tilde{\lambda}_i(\tilde{u}, \e_k) \nabla \tilde{r}_i(\tilde{u}, \e_k) \left( \sum_{i'=1}^{d} \tilde{w}_{i',k'} \tilde{r}_{i'}(\tilde{u}, \e_{k'}) \right)
\]

\[
- \sum_{i=1}^{n} \sum_{k=1}^{d} \tilde{w}_{i,k} \left( \nabla \tilde{\lambda}_i(\tilde{u}, \e_k) \cdot \tilde{r}_{i'}(\tilde{u}, \e_{k'}) \right) \tilde{\lambda}_{i'}(\tilde{u}, \e_k) \tilde{w}_{i',k'} \tilde{r}_{i'}(\tilde{u}, \e_{k'}). 
\]

Multiplying (6.8) by \( \tilde{l}_1(\tilde{u}) \) (owing to (2B)) and using (3.15), one has

\[
\partial_t \tilde{w}_{1,k'} + \sum_{k=1}^{d} \tilde{\lambda}_1(\tilde{u}, \e_k) (\tilde{w}_{1,k})_{x_{k'}}
\]

\[
= \sum_{i=1}^{n} \tilde{l}_1(\tilde{u}) \nabla \tilde{Q}(\tilde{u}) \tilde{r}_i(\tilde{u}, \e_{k'}) \tilde{w}_{i,k'} - \sum_{i=1}^{n} \tilde{l}_1(\tilde{u}) \nabla \tilde{r}_i(\tilde{u}, \e_{k'}) \tilde{Q}(\tilde{u}) \tilde{w}_{i,k'}
\]

\[
+ \sum_{i,i'=1}^{n} \tilde{l}_1(\tilde{u}) \nabla \tilde{r}_i(\tilde{u}, \e_{k'}) \tilde{r}_{i'}(\tilde{u}, \e_k) \tilde{\lambda}_{i'}(\tilde{u}, \e_k) \tilde{w}_{i,k'} \tilde{w}_{i',k'}
\]
and inconvenience in doing the energy estimates. One may artificially recover this with the following:

\[ \tilde{I}_1(\tilde{u}) = \tilde{I}_1(\tilde{u}) \frac{\partial \tilde{Q}}{\partial \tilde{u}_1}(\tilde{u}) \tilde{w}_{1,k'} + \sum_{j=2}^{n} \tilde{I}_1(\tilde{u}) \nabla_{\tilde{u}} \tilde{Q}(\tilde{u}) \tilde{r}_{j}(\tilde{u}, \epsilon_{k'}) \tilde{w}_{j,k'} , \]  

(6.10)

\[ I_2 = - \sum_{j=2}^{n} \tilde{I}_1(\tilde{u}) \nabla_{\tilde{u}} \tilde{r}_{j}(\tilde{u}, \epsilon_{k'}) \tilde{Q}(\tilde{u}) \tilde{w}_{j,k'}, \]  

(6.11)

\[ I_3 = \sum_{j=2}^{n} \sum_{j'=1}^{d} \tilde{I}_1(\tilde{u}) \nabla_{\tilde{u}} \tilde{r}_{j}(\tilde{u}, \epsilon_{k'}) \tilde{r}_{j'}(\tilde{u}, \epsilon_{k}) \tilde{\lambda}_{j'}(\tilde{u}, \epsilon_{k}) \tilde{w}_{j,k'} \tilde{w}_{i',k'}, \]  

(6.12)

\[ I_4 = - \sum_{j=2}^{n} \sum_{j'=1}^{d} \tilde{I}_1(\tilde{u}) \nabla_{\tilde{u}} \tilde{r}_{j}(\tilde{u}, \epsilon_{k'}) \tilde{r}_{j'}(\tilde{u}, \epsilon_{k}) \tilde{\lambda}_{j}(\tilde{u}, \epsilon_{k}) \tilde{w}_{j,k} \tilde{w}_{i',k'} \]  

(6.13)

and

\[ I_5 = - \sum_{k=1}^{d} \frac{\partial \tilde{\lambda}_1}{\partial \tilde{u}_1}(\tilde{u}, \epsilon_{k}) \tilde{w}_{1,k} \tilde{w}_{1,k'} - \sum_{j=2}^{n} \sum_{k=1}^{d} \left( \nabla_{\tilde{u}} \tilde{\lambda}_1(\tilde{u}, \epsilon_{k}) \cdot \tilde{r}_{j}(\tilde{u}, \epsilon_{k'}) \right) \tilde{w}_{1,k} \tilde{w}_{j,k'} . \]  

(6.14)

One next needs to reexamine each term in (6.9). Indeed, by (2.8), it holds that

Lastly, notice that the left hand side of (6.9) is not a transported one, and it causes the inconvenience in doing the energy estimates. One may artificially recover this with the following:

\[ (\tilde{w}_{1,k'})_{x_k} - (\tilde{w}_{1,k'})_{x_{k'}} = \left( \tilde{I}_1(\tilde{u}) \tilde{u}_{x_k} \right)_{x_k} - \left( \tilde{I}_1(\tilde{u}) \tilde{u}_{x_k} \right)_{x_{k'}} \]  

\[ = \sum_{i,i'=1}^{n} \tilde{r}_1(\tilde{u}, \epsilon_{k}) \left( \nabla_{\tilde{u}} \tilde{I}_1(\tilde{u}) - \left( \nabla_{\tilde{u}} \tilde{I}_1(\tilde{u}) \right)^T \right) \tilde{r}_{i'}(\tilde{u}, \epsilon_{k'}) \tilde{w}_{i,k} \tilde{w}_{i',k'}. \]  

(6.15)

Moreover, for \( i = i' = 1 \) in the summation, by (2.8),

\[ \tilde{r}_1(\tilde{u}, \epsilon_{k}) \left( \nabla_{\tilde{u}} \tilde{I}_1(\tilde{u}) - \left( \nabla_{\tilde{u}} \tilde{I}_1(\tilde{u}) \right)^T \right) \tilde{r}_1(\tilde{u}, \epsilon_{k'}) = e_1^T \left( \nabla_{\tilde{u}} \tilde{I}_1(\tilde{u}) - \left( \nabla_{\tilde{u}} \tilde{I}_1(\tilde{u}) \right)^T \right) e_1 = 0. \]  

(6.16)

Hence,

\[ (\tilde{w}_{1,k'})_{x_k} = (\tilde{w}_{1,k'})_{x_k} - \sum_{i+i' \geq 3} \tilde{r}_1(\tilde{u}, \epsilon_{k}) \left( \nabla_{\tilde{u}} \tilde{I}_1(\tilde{u}) - \left( \nabla_{\tilde{u}} \tilde{I}_1(\tilde{u}) \right)^T \right) \tilde{r}_{i'}(\tilde{u}, \epsilon_{k'}) \tilde{w}_{i,k} \tilde{w}_{i',k'}. \]  

(6.17)

Collecting these facts (6.10)-(6.14) and (6.17) together with (2.8), one may thus deduce from (6.9) that

\[ \partial t \tilde{w}_{1,k'} + \sum_{k=1}^{d} \hat{\lambda}_1(\tilde{u}, \epsilon_{k}) \left( \tilde{w}_{1,k'} \right)_{x_k} \]  

\[ = \tilde{I}_1(\tilde{u}) \frac{\partial \tilde{Q}}{\partial \tilde{u}_1}(\tilde{u}) \tilde{w}_{1,k'} - \sum_{k=1}^{d} \frac{\partial \hat{\lambda}_1}{\partial \tilde{u}_1}(\tilde{u}, \epsilon_{k}) \tilde{w}_{1,k} \tilde{w}_{1,k'} + \sum_{j=2}^{n} \tilde{S}_{j,k'}(\tilde{u}) \tilde{u}_{j,k'} \]  

(6.18)

\[ + \sum_{j=2}^{n} \sum_{j'=1}^{d} \sum_{k_k=1}^{d} \tilde{T}_{j,j',k_k}(\tilde{u}) \tilde{u}_{j,k} \tilde{u}_{j',k_k}. \]

Now one may prove (i) under (2.8.1)-(2.8.2) and that \( d \geq 3 \). Assume (2.8.1)-(2.8.2), then it holds that \( \frac{\partial \tilde{Q}}{\partial \tilde{u}_1}(\tilde{u}_1 e_1) \equiv 0 \) and \( \frac{\partial \hat{\lambda}_1}{\partial \tilde{u}_1}(\tilde{u}_1 e_1, \epsilon_{k}) \equiv 0 \). Using this fact, one may fix \( \tilde{u}_1 \) and view
\[
\frac{\partial \tilde{Q}_1}{\partial \tilde{u}_1}(\tilde{u}) \quad \text{and} \quad \frac{\partial \lambda_1}{\partial \tilde{u}_1}(\tilde{u}, e_k) \quad \text{as functions of} \quad \tilde{u}^k = (\tilde{u}_2, \ldots, \tilde{u}_n)^T. \]
By the mean value theorem, it holds that
\[
\frac{\partial \tilde{Q}_1}{\partial \tilde{u}_1}(\tilde{u}) = \frac{\partial \tilde{Q}_1}{\partial \tilde{u}_1}(\tilde{u}_1 e_1) + \sum_{j=2}^n \tilde{V}_{ij}(\tilde{u})\tilde{u}_j \equiv \sum_{j=2}^n \tilde{V}_{ij}(\tilde{u})\tilde{u}_j \quad (6.19)
\]
and
\[
\frac{\partial \lambda_1}{\partial \tilde{u}_1}(\tilde{u}, e_k) = \frac{\partial \lambda_1}{\partial \tilde{u}_1}(\tilde{u}_1 e_1, e_k) + \sum_{j=2}^n \tilde{W}_{j,k}(\tilde{u})\tilde{u}_j \equiv \sum_{j=2}^n \tilde{W}_{j,k}(\tilde{u})\tilde{u}_j. \quad (6.20)
\]
Also,
\[
\tilde{\lambda}_1(\tilde{u}, e_k) = \tilde{\lambda}_1(\tilde{u}_1 e_1, e_k) + \sum_{j=2}^n \tilde{W}_{j,k}^*(\tilde{u})\tilde{u}_j \equiv \tilde{\lambda}_1(0, e_k) + \sum_{j=2}^n \tilde{W}_{j,k}^*(\tilde{u})\tilde{u}_j. \quad (6.21)
\]
Hence, under (\(\mathcal{W}_1\))–(\(\mathcal{W}_2\)), it holds that
\[
\tilde{l}_1(\tilde{u}) \frac{\partial \tilde{Q}_1}{\partial \tilde{u}_1}(\tilde{u})\tilde{w}_{1,k'} = \sum_{i=1}^n \tilde{l}_1(\tilde{u})\tilde{V}_{ij}(\tilde{u})\tilde{u}_j\tilde{w}_{1,k'}, \quad (6.22)
\]
and
\[
\sum_{k=1}^d \frac{\partial \tilde{\lambda}_1}{\partial \tilde{u}_1}(\tilde{u}, e_k)\tilde{w}_{1,k}\tilde{w}_{1,k'} = \sum_{k=1}^d \sum_{j=2}^n \tilde{W}_{j,k}(\tilde{u})\tilde{u}_j\tilde{w}_{1,k}\tilde{w}_{1,k'}. \quad (6.23)
\]
Substituting (6.22)–(6.23) into (6.18), one can get
\[
\partial_t \tilde{w}_{1,k'} + \sum_{k=1}^d \tilde{\lambda}_1(\tilde{u}, e_k) (\tilde{w}_{1,k'}) = \sum_{j=2}^n \tilde{X}_j(\tilde{u}, \nabla \tilde{u})\tilde{u}_j + \sum_{j=2}^n \tilde{S}_j(\tilde{u})(\tilde{u}_j, \tilde{w}_{1,k'}) + \sum_{j=2}^n \sum_{j'=2}^n \sum_{k'=1}^d \tilde{T}_{j,j',k,k'}(\tilde{u})\tilde{u}_j\tilde{w}_{1,k}\tilde{w}_{1,k'} := \tilde{\mathcal{K}}^1. \quad (6.24)
\]
The \(\Lambda^{\ell-1}\) energy estimates on (6.21) yields
\[
\frac{1}{2} \frac{d}{dt} \|\Lambda^{\ell-1}\tilde{w}_{1,k'}\|^2_{L^2} + \sum_{k=1}^d \int \tilde{\lambda}_1(\tilde{u}, e_k) \left( \Lambda^{\ell-1}\tilde{w}_{1,k'} \right)_{x_k} \Lambda^{\ell-1}\tilde{w}_{1,k'} \quad (6.25)
\]
Integrating by parts, by (6.21), one has
\[
\left| \int \tilde{\lambda}_1(\tilde{u}, e_k) \left( \Lambda^{\ell-1}\tilde{w}_{1,k'} \right)_{x_k} \Lambda^{\ell-1}\tilde{w}_{1,k'} \right| = \frac{1}{2} \int \tilde{\lambda}_1(\tilde{u}, e_k) \left( \left| \Lambda^{\ell-1}\tilde{w}_{1,k'} \right|^2 \right)_{x_k} \quad (6.26)
\]
By the commutator estimates (B.3), the product estimates (B.9) from Lemma (B.3), the expansion (6.21) and since \(d \geq 3, \ell > d/2 + 1 > 2\), it yields
\[
\left\| \left[ \Lambda^{\ell-1}, \tilde{\lambda}_1(\tilde{u}, e_k) \right] \right\|_{L^2} \leq \left\| \nabla \tilde{\lambda}_1(\tilde{u}, e_k) \right\|_{L^\infty} \left\| \Lambda^{\ell-2} \tilde{w}_{1,k'} \right\|_{L^2} + \left\| \Lambda^{\ell-1} \left( \tilde{\lambda}_1(\tilde{u}, e_k) \right) \right\|_{L^{\frac{2d}{d-2}}} \left\| \tilde{w}_{1,k'} \right\|_{L^d} \quad (6.27)
\]
Lastly, one may apply the product estimates \([3.9]\) from Lemma \([3.3]\) to obtain, by the expression of \(\tilde{N}^1\) in \([6.24]\),
\[
\left\| \Lambda^{\ell-1} \tilde{N}^1 \right\|_{L^2} \lesssim \left\| \tilde{u}^b \right\|_{L^\infty} + \left\| \nabla \tilde{u}^b \right\|_{L^\infty} + \left\| \Lambda^{\ell-1} \tilde{u}^b \right\|_{L^2} + \left\| \Lambda^\ell \tilde{u}^b \right\|_{L^2}.
\] (6.28)

In view of these estimates \([6.26]-[6.28]\), it follows from \([6.26]\) that
\[
\frac{d}{dt} \left\| \Lambda^{\ell-1} \tilde{u}_{1,k'} \right\|_{L^2}^2 \lesssim \left( \left\| \tilde{u}^b \right\|_{L^\infty} + \left\| \nabla \tilde{u}^b \right\|_{L^\infty} + \left\| \Lambda^{\ell-1} \tilde{u}^b \right\|_{L^2} + \left\| \Lambda^\ell \tilde{u}^b \right\|_{L^2} \right) \left\| \tilde{u} \right\|_{H^\ell}.
\] (6.29)

Finally, integrating the inequality \([6.29]\) directly in time, by \([6.3]-[6.6]\) together with the estimates \([5.39]\), one has
\[
\left\| \tilde{u}(t) \right\|_{H^\ell}^2 \lesssim \left\| \tilde{u}_0 \right\|_{H^\ell}^2 + \int_0^t \left( \left\| \tilde{u}^b \right\|_{L^\infty} + \left\| \nabla \tilde{u}^b \right\|_{L^\infty} + \left\| \Lambda^{\ell-1} \tilde{u}^b \right\|_{L^2} + \left\| \Lambda^\ell \tilde{u}^b \right\|_{L^2} \right) \left\| \tilde{u} \right\|_{H^\ell} dt. \] (6.30)

By Cauchy’s inequality, this implies \([6.1]\).

One then turns to prove \((ii)\) under \((\mathfrak{D}1)-(\mathfrak{D}2)\) and that \(d \geq 2\). Assume \((\mathfrak{D}1)-(\mathfrak{D}2)\), then it holds that \(\frac{\partial (JQ)}{\partial u_1}(\tilde{u}) \equiv 0\) and \(\frac{\partial \tilde{\lambda}_1}{\partial u_1}(\tilde{u}, \phi_k) \equiv 0\). By this fact, it follows from \([6.18]\) that
\[
\partial_{t} \tilde{w}_{1,k'} + \sum_{k=1}^d \tilde{\lambda}_1(\tilde{u}, \phi_k) \left( \tilde{w}_{1,k'} \right) x_k
\]
\[
= \tilde{l}_1(\tilde{u}) \frac{\partial (J^{-1})}{\partial u_1}(\tilde{u}) \tilde{Q}(\tilde{u}) \tilde{w}_{1,k'} + \sum_{j=2}^n \tilde{S}_{j,k'}(\tilde{u}) \tilde{u}_{j,k'}
\]
\[
+ \sum_{j=2}^n \sum_{k'=1}^d \tilde{T}_{j',kk'}(\tilde{u}) \tilde{u}_{j,k'} \tilde{u}_{j',k'} := \tilde{N}^2.
\] (6.31)

The \(\Lambda^{\ell-1}\) energy estimates on \([6.31]\) yields
\[
\frac{1}{2} \frac{d}{dt} \left\| \Lambda^{\ell-1} \tilde{w}_{1,k'} \right\|_{L^2}^2 + \sum_{k=1}^d \tilde{\lambda}_1(\tilde{u}, \phi_k) \left( \Lambda^{\ell-1} \tilde{w}_{1,k'} \right) x_k \Lambda^{\ell-1} \tilde{w}_{1,k'}
\]
\[
= \int \Lambda^{\ell-1} \tilde{N}^2 \Lambda^{\ell-1} \tilde{w}_{1,k'} - \sum_{k=1}^d \int \left[ \Lambda^{\ell-1} \tilde{\lambda}_1(\tilde{u}, \phi_k) \right] (\tilde{w}_{1,k'}) x_k \Lambda^{\ell-1} \tilde{w}_{1,k'}.
\] (6.32)

Integrating by parts, by the linear degeneracy \((\mathfrak{D}2)\), one has
\[
\left| \int \tilde{\lambda}_1(\tilde{u}, \phi_k) \left( \Lambda^{\ell-1} \tilde{w}_{1,k'} \right) x_k \Lambda^{\ell-1} \tilde{w}_{1,k'} \right| = \frac{1}{2} \left| \int \tilde{\lambda}_1(\tilde{u}, \phi_k) \left( \left| \Lambda^{\ell-1} \tilde{w}_{1,k'} \right|^2 \right) x_k \right|
\]
\[
= \frac{1}{2} \sum_{j=2}^n \int \left\| \nabla \tilde{u}_j \right\|_{L^\infty} \left\| \Lambda^{\ell-1} \tilde{w}_{1,k'} \right\|_{L^2}^2 \] (6.33)
By the commutator estimates (6.38) from Lemma (B.3), by again (5.2), it holds that
\[
\left\| \Lambda^{\ell-1} \tilde{\lambda}_1 (\tilde{u}, \mathbf{e}_k) (\tilde{w}_{1,k}^j)_{,k} \right\|_{L^2} 
\lesssim \left\| \nabla \tilde{\lambda}_1 (\tilde{u}, \mathbf{e}_k) \right\|_{L^\infty} \left\| \Lambda^\ell \left( \tilde{w}_{1,k}^j \right)_{,k} \right\|_{L^2} + \left\| \Lambda^{\ell-1} \left( \tilde{\lambda}_1 (\tilde{u}, \mathbf{e}_k) \right) \right\|_{L^{n/2}} \left\| (\tilde{w}_{1,k}^j)_{,k} \right\|_{L^2} 
\lesssim \sum_{j=2}^n \left\| \nabla_{\tilde{u}} \tilde{\lambda}_1 (\tilde{u}, \mathbf{e}_k) \nabla \tilde{u}_j \right\|_{L^\infty} \left\| \Lambda^{\ell-1} \tilde{w}_{1,k}^j \right\|_{L^2} 
+ \sum_{j=2}^n \left\| \Lambda^{\ell-2} \left( \nabla_{\tilde{u}} \tilde{\lambda}_1 (\tilde{u}, \mathbf{e}_k) \nabla \tilde{u}_j \right) \right\|_{L^\infty} \left\| \nabla \tilde{w}_{1,k}^j \right\|_{L^2} 
\lesssim \left( \left\| \nabla \tilde{u}_1^b \right\|_{L^\infty} + \left\| \Lambda^{\ell-1} \tilde{u}_1^b \right\|_{L^2} + \left\| \Lambda^\ell \tilde{u}_1^b \right\|_{L^2} \right\| \tilde{u}_1 \right\|_{H^\ell}, \tag{6.34}
\]
where \(p_d = d\) for \(d \geq 3\); \(p_d = 4\) for \(d = 2\) and \(l = 5/2\); \(p_2 = 2/(3-l)\) for \(d = 2\) and \(2 < l < 5/2\).

Lastly, one may apply the product estimates (B.9) from Lemma (B.3) to obtain, by the expressions (1.18)–(1.19) of \(Q\),
\[
\left\| \Lambda^{\ell-1} \tilde{\nabla}^2 \right\|_{L^2} \lesssim \| \tilde{\nabla}^b \|_{L^2}^2 + \| \tilde{\nabla}^b \|_{L^\infty} + \| \nabla \tilde{\nabla}^b \|_{L^\infty} + \left\| \Lambda^{\ell-1} \tilde{\nabla}^b \right\|_{L^2} + \left\| \Lambda^\ell \tilde{\nabla}^b \right\|_{L^2} \| \tilde{u}_1 \|_{H^\ell}. \tag{6.35}
\]
In view of these estimates (6.33)–(6.35), it follows from (6.32) that
\[
\frac{d}{dt} \left\| \Lambda^{\ell-1} \tilde{w}_{1,k}^j \right\|_{L^2}^2 \lesssim \left( \left\| \tilde{\nabla}^b \right\|_{L^\infty} + \| \tilde{\nabla}^b \|_{L^\infty} + \| \nabla \tilde{\nabla}^b \|_{L^\infty} + \left\| \Lambda^{\ell-1} \tilde{\nabla}^b \right\|_{L^2} + \left\| \Lambda^\ell \tilde{\nabla}^b \right\|_{L^2} \right\| \tilde{u}_1 \right\|_{H^\ell}. \tag{6.36}
\]
This is an improvement of (B.24) which then, together with the estimates (5.40), leads to (6.2) in this case.

7. \(L^q\) estimates under (\(\tilde{D}_3\))

This section is devoted to derive an \(L^q\) estimates of the solution in preparation to prove Theorem 3 under (\(\tilde{D}_1\))–(\(\tilde{D}_3\)) and the initial condition \(\tilde{u}_{0,1} \in L^q\) with \(1 \leq q \leq 2\). However, one may not be able to derive the \(L^q\) estimates of \(\tilde{u}_1\) by directly using the subsystem (3.25). The essential difficulty lies in the presence of the second summation involving \(\tilde{u}^b\) on the left hand side. One feasible way is going back to the original system (3.8) of \(\tilde{u}\) and then resorting to the wave decomposition.

In the spirit of the one dimensional study (3) [10], one may consider, owing to (\(\tilde{D}_3\)),
\[
\tilde{v}_1 = \tilde{l}_1 (\tilde{u}) \tilde{u}. \tag{7.1}
\]
But by (5.24), it holds that
\[
\tilde{v}_1 = \tilde{u}_1 + \sum_{j=2}^n \tilde{l}_{1,j} (\tilde{u}) \tilde{u}_j. \tag{7.2}
\]
The key point is to derive the \(L^q\) estimates of \(\tilde{v}_1\), and the result is stated as the following.

Proposition 7.1. Under (\(\tilde{D}_1\))–(\(\tilde{D}_3\)) and that \(d \geq 2\) and \(\tilde{v}_{1,0} \in L^q\) with \(1 \leq q \leq 2\), it holds that
\[
\| \tilde{v}_1 \|_{L^q} \lesssim \exp \left( \int_0^t \left( \| \tilde{\nabla} \tilde{u}^b \|_{L^\infty} + \| \nabla \tilde{\nabla} \tilde{u}^b \|_{L^\infty} + \| \tilde{u}^b \|_{L^\infty} \right) d\tau \right) \times \left( \| \tilde{v}_{1,0} \|_{L^q} + \int_0^t \| \tilde{u}^b \|_{L^{2q}} \left( \| \tilde{\nabla} \tilde{u}^b \|_{L^{2q}} + \| \tilde{u}^b \|_{L^{2q}} \right) d\tau \right). \tag{7.3}
\]

Proof. First of all, one needs to derive the equation of \(\tilde{v}_1\). Multiplying the system (3.8) by \(\tilde{l}_1 (\tilde{u})\), by (3.19), (3.24) and an use of (3.8) to substitute \(\partial_t \tilde{u}\), one can get
\[
\partial_t \tilde{v}_1 + \sum_{k=1}^d \tilde{\lambda}_1 (\tilde{u}, \mathbf{e}_k) (\tilde{v}_1)_{,k} \]
By Hölder’s inequality, the second integral in the right hand side of (7.8) is easily bounded by

\[
= \tilde{l}_1(\tilde{u})\tilde{Q}(\tilde{u}) + \partial_k\tilde{u}^T\nabla \tilde{u}_k \tilde{l}_1(\tilde{u})\tilde{u} + \sum_{k=1}^{d} \tilde{\lambda}_1(\tilde{u}, e_k)\tilde{u}_k^T \nabla \tilde{u}_k \tilde{l}_1(\tilde{u})\tilde{u}
\]

\[
= \tilde{l}_1(\tilde{u})\tilde{Q}(\tilde{u}) + \sum_{j=2}^{n} \sum_{i=1}^{n} \tilde{Q}_j(\tilde{u})\frac{\partial \tilde{l}_1(\tilde{u})}{\partial u_j} \tilde{u}_j
\]

\[
+ \sum_{k=1}^{d} \sum_{j,j'=2}^{n} \sum_{i=1}^{n} \left( \tilde{\lambda}_1(\tilde{u}, e_k) - \tilde{\lambda}_{j'}(\tilde{u}, e_k) \right) \tilde{w}_{j',k} \tilde{r}_{ij'}(\tilde{u}) \frac{\partial \tilde{l}_1(\tilde{u})}{\partial u_i} \tilde{u}_{ij}.
\] (7.4)

Here in the second equality the wave composition \((\tilde{u}, \tilde{w}_{j',k})\) for \(\tilde{u}_x\) is used.

By \((\tilde{D}_3), (1.0), (1.8)\) and \((1.12)\), it holds that

\[
\tilde{l}_{1p}(0) = 0, \quad p = r + 1, \ldots, n,
\] (7.5)

which implies that

\[
\tilde{l}_1(\tilde{u})\tilde{Q}(\tilde{u}) = \sum_{q=1}^{r} \tilde{l}_{1q}(\tilde{u})\tilde{Q}_q(\tilde{u}) + \sum_{p=r+1}^{n} \left( \tilde{l}_{1p}(\tilde{u}) - \tilde{l}_{1p}(0) \right) \tilde{Q}_p(\tilde{u}).
\] (7.6)

Substituting \((7.6)\) and the expansions \((4.18)-(4.19)\) (due to \((\tilde{D}_1)\)) into \((7.4)\), using the expression \((6.6)\) for \(\tilde{w}_{j',k}\) and \((7.2)\) for \(\tilde{v}_1\), one may deduce

\[
\partial_t \tilde{v}_1 + \sum_{k=1}^{d} \tilde{\lambda}_1(\tilde{u}, e_k)(\tilde{v}_1)_{x_k}
\]

\[
= \sum_{p=r+1}^{n} \tilde{R}_{1p}(\tilde{u})\tilde{v}_1 \tilde{u}_p + \sum_{p=r+1}^{n} \sum_{j=2}^{n} \tilde{R}_{jp}(\tilde{u})\tilde{u}_j \tilde{u}_p + \sum_{k=1}^{d} \sum_{j,j'=2}^{n} \tilde{H}_{j,j',k}(\tilde{u})\tilde{u}_j \tilde{u}_j \tilde{u}_x (\tilde{v}_1(\tilde{u})x_k)
\]

\[
+ \sum_{j,j'=2}^{n} \tilde{R}_{1j}(\tilde{u})\tilde{v}_1 \tilde{u}_j \tilde{u}_j - \sum_{j,j'=2}^{n} \tilde{R}_{1j}(\tilde{u})\tilde{v}_1 \tilde{u}_j \tilde{u}_j
\] (7.7)

The key point lying in \((7.7)\) is that unlike \((3.25)\), there is no term involving \(\tilde{w}^p\) on the left hand side, which makes it possible to derive the \(L^q\) estimate for \(\tilde{v}_1\).

Now multiplying the equation by \(|\tilde{v}_1|^q - 2 \tilde{v}_1\) and then integrating over \(\mathbb{R}^d\), one has

\[
\frac{d}{dt} \left\| \tilde{v}_1 \right\|_{L^q} \leq \sum_{k=1}^{d} \left| \int \tilde{\lambda}_1(\tilde{u}, e_k)(|\tilde{v}_1|^q)_{x_k} \right|
\]

\[
+ \int \left( |\tilde{v}_1|^q |\tilde{w}^p| + |\tilde{v}_1|^{q-1} |\tilde{w}^p| (|\tilde{w}^p| + |\nabla \tilde{w}^p|) + |\tilde{v}_1|^q |\tilde{w}^p|^2 + |\tilde{v}_1|^{q-1} |\tilde{w}^p|^3 \right).
\] (7.8)

By the linear degeneracy \((\tilde{D}_2)\), it holds that

\[
\left| \int \tilde{\lambda}_1(\tilde{u}, e_k)(|\tilde{v}_1|^q)_{x_k} \right| = \left| \sum_{j=2}^{n} \frac{\partial \tilde{\lambda}_1(\tilde{u}, e_k)}{\partial u_j}(|\tilde{v}_1|^q)_{x_k} \right| \leq \left\| |\tilde{v}_1|^q \right\|_{L^q} \left\| \nabla \tilde{w}^p \right\|_{L^\infty}
\] (7.9)

By Hölder’s inequality, the second integral in the right hand side of \((7.8)\) is easily bounded by

\[
\left\| \tilde{v}_1 \right\|^q_{L^q} \left\| \tilde{w}^p \right\|_{L^\infty} + \left\| \tilde{v}_1 \right\|^q_{L^q} \left\| \nabla \tilde{w}^p \right\|_{L^2}^2 \left( \left\| \tilde{w}^p \right\|_{L^q} + \left\| \nabla \tilde{w}^p \right\|_{L^q} \right)
\]

\[
+ \left\| \tilde{v}_1 \right\|^q_{L^q} \left\| \tilde{w}^p \right\|^2_{L^q} + \left\| \tilde{v}_1 \right\|^q_{L^q} \left\| \nabla \tilde{w}^p \right\|^2_{L^q}
\] (7.10)

Plugging the estimates \((5.3)\), \((6.8)\) into \((7.8)\) yields

\[
\frac{d}{dt} \left\| \tilde{v}_1 \right\|_{L^q} \leq \left\| \tilde{v}_1 \right\|_{L^q} \left( \left\| \tilde{w}^p \right\|_{L^\infty} + \left\| \nabla \tilde{w}^p \right\|_{L^\infty} + \left\| \tilde{w}^p \right\|^2_{L^q} \right)
\]

\[
+ \left\| \tilde{w}^p \right\|_{L^2}^2 \left( \left\| \tilde{w}^p \right\|_{L^q} + \left\| \nabla \tilde{w}^p \right\|_{L^q} + \left\| \tilde{w}^p \right\|^2_{L^q} \right),
\] (7.11)
Applying the Gronwall lemma to (7.11), one can deduce (7.10).

**Remark 7.2.** In the statement of Proposition 7.1, it is required that \( \tilde{v}_{1,0} \in L^q \) with \( 1 \leq q \leq 2 \). However, this can be guaranteed under the assumptions of Theorem 3. Indeed, by (7.2) and (7.3), it holds that

\[
\tilde{v}_1 = \tilde{u}_1 + \sum_{q=2}^{r} \tilde{l}_1(\tilde{u}) \tilde{u}_q + \sum_{p=r+1}^{n} \left( \tilde{l}_1 p(\tilde{u}) - \tilde{l}_1 p(0) \right) \tilde{u}_p,
\]

which implies, since \( 1 \leq p \leq q \leq 2 \),

\[
\|\tilde{v}_{1,0}\|_{L^q} \lesssim \|\tilde{u}_{1,0}\|_{L^q} + \|\tilde{u}_0^C\|_{L^q} + \|\tilde{v}_0^D\|_{L^{2q}} \lesssim \|\tilde{u}_{1,0}\|_{L^q} + \|\tilde{u}_0^C\|_{L^q} + \|\tilde{u}_0\|_{H^\ell}^2.
\]
It thus follows from (8.6) that
\[
\frac{d}{dt} \mathcal{E}_s(\tilde{u}^\alpha) + \left\| \Lambda^s \tilde{u}^\alpha \right\|^2_{H^{s/2}} + \left\| \Lambda^{s+1} \tilde{u}^\alpha \right\|^2_{H^{s-1}} \\
\lesssim \delta \left( \left\| \Lambda^s \tilde{u}^\alpha \right\|^2_{L^2} + \left\| \Lambda^{s-1} \tilde{u}^\beta \right\|^2_{L^2} + \left\| \tilde{u}^\beta \right\|^4_{L^\infty} \right) + \left\| \Lambda^s \tilde{u}^\alpha \right\|^2_{L^2} \left\| \tilde{u}^\beta \right\|^2_{L^\infty}. \tag{8.7}
\]

Adding \( \left\| \Lambda^s \tilde{u}^\alpha \right\|^2_{L^2} \) to both sides of (8.7), by Cauchy’s inequality, one can obtain
\[
\frac{d}{dt} \mathcal{E}_s(\tilde{u}^\alpha) + \left\| \Lambda^s \tilde{u}^\alpha \right\|^2_{H^{s/2}} \lesssim \left\| \Lambda^s \tilde{u}^\alpha \right\|^2_{L^2} + \delta \left\| \Lambda^{s-1} \tilde{u}^\beta \right\|^2_{L^2} + \delta \left\| \tilde{u}^\beta \right\|^4_{L^\infty}, \tag{8.8}
\]
which implies
\[
\frac{d}{dt} \mathcal{E}_s(\tilde{u}^\alpha) + \lambda \mathcal{E}_s(\tilde{u}^\alpha) \lesssim \left\| \Lambda^s \tilde{u}^\alpha \right\|^2_{L^2} + \delta \left\| \Lambda^{s-1} \tilde{u}^\beta \right\|^2_{L^2} + \delta \left\| \tilde{u}^\beta \right\|^4_{L^\infty}. \tag{8.9}
\]
Applying the Gronwall lemma to (8.9) yields (8.2).

8.2. Linear decay estimates. In order to derive the decay of the lower-order norms of \( \tilde{u}^\beta(t) \) appearing in Lemma 8.1 in this subsection one may consider the linear decay estimates for the linearized system of (8.20):
\[
\begin{cases}
\partial_t \tilde{u}^\beta + \tilde{L}^\beta \tilde{u}^\beta = 0, \\
\tilde{u}^\beta |_{t=0} = \tilde{u}_0^\beta,
\end{cases} \tag{8.10}
\]
where the linear operator \( \tilde{L}^\beta \) is defined by
\[
\tilde{L}^\beta \tilde{u}^\beta = \sum_{k=1}^d A^{k,\beta}(0) \tilde{u}_k + \Theta^\beta \tilde{u}^\beta. \tag{8.11}
\]

The solution operator \( e^{-t\tilde{L}^\beta} \) of the system (8.10) has the following time-decay property:

**Lemma 8.2.** Let \( d \geq 2 \). For \( 1 \leq p \leq 2 \), \( \alpha \geq 0 \) and denote \( p^* = \min\{2, dp/(d-p)\} \). There exists a constant \( \lambda > 0 \) such that
\[
\left\| \Lambda^\alpha \Pi^c e^{-t\tilde{L}^\beta} \tilde{u}_0^\beta \right\|_{L^2} \lesssim (1 + t)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{\alpha}{2}} \left( \left\| \Pi^c \tilde{u}_0^\beta \right\|_{L^p} + \left\| \Pi^p \tilde{u}_0^\beta \right\|_{L^{p^*}} \right) + e^{-\lambda t} \left\| \Lambda^\alpha \tilde{u}_0^\beta \right\|_{L^2}, \tag{8.12}
\]
\[
\left\| \Lambda^\alpha \Pi^c e^{-t\tilde{L}^\beta} \tilde{u}_0^\beta \right\|_{L^2} \lesssim (1 + t)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{\alpha}{2}} \left( \left\| \Pi^c \tilde{u}_0^\beta \right\|_{L^p} + \left\| \Pi^p \tilde{u}_0^\beta \right\|_{L^{p^*}} \right) + e^{-\lambda t} \left\| \Lambda^\alpha \tilde{u}_0^\beta \right\|_{L^2}, \tag{8.13}
\]
and
\[
\left\| \Lambda^\alpha \Pi^p e^{-t\tilde{L}^\beta} \tilde{u}_0^\beta \right\|_{L^2} \lesssim (1 + t)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{\alpha}{2}} \left( \left\| \Pi^c \tilde{u}_0^\beta \right\|_{L^p} + \left\| \Pi^p \tilde{u}_0^\beta \right\|_{L^{p^*}} \right) + e^{-\lambda t} \left\| \Lambda^\alpha \tilde{u}_0^\beta \right\|_{L^2}, \tag{8.14}
\]
\[
\left\| \Lambda^\alpha \Pi^p e^{-t\tilde{L}^\beta} \tilde{u}_0^\beta \right\|_{L^2} \lesssim (1 + t)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{\alpha}{2}} \left( \left\| \Pi^c \tilde{u}_0^\beta \right\|_{L^p} + \left\| \Pi^p \tilde{u}_0^\beta \right\|_{L^{p^*}} \right) + e^{-\lambda t} \left\| \Lambda^\alpha \tilde{u}_0^\beta \right\|_{L^2}. \tag{8.15}
\]
where \( \Pi^c \) and \( \Pi^p \) are the projection operators defined by
\[
\Pi^c \tilde{u}^\beta = (a_1, \ldots, a_r)^T, \quad \Pi^p \tilde{u}^\beta = (a_{r+1}, \ldots, a_n)^T, \quad \text{for } a^\beta = (a_2, \ldots, a_n)^T \in \mathbb{R}^{n-1}. \tag{8.16}
\]
**Proof.** These decay estimates follow by refining the initial condition in (2). Indeed, since similar as in Lemma 8.4 the linear system (8.10) satisfies the Kawashima condition, from Section 4.2 of (2), one can decompose the solution \( \tilde{u}^\beta(t) \) of (8.10) as
\[
\tilde{u}^\beta(t) = e^{-t\tilde{L}^\beta} \tilde{u}_0^\beta = K(t) \tilde{u}_0^\beta + \mathcal{K}(t) \tilde{u}_0^\beta, \tag{8.17}
\]
and the two components satisfy the following bounds:
\[
\left\| \Pi^c \mathcal{K}(t) \tilde{u}_0^\beta \right\| \leq C e^{-c|\xi|^2 t} \left( \left\| \Pi^c \tilde{u}_0^\beta \right\| + |\xi| \left\| \Pi^p \tilde{u}_0^\beta \right\| \right), \tag{8.18}
\]
\[
\left\| \Pi^p \mathcal{K}(t) \tilde{u}_0^\beta \right\| \leq C e^{-c|\xi|^2 t} \left( |\xi| \left\| \Pi^c \tilde{u}_0^\beta \right\| + |\xi|^2 \left\| \Pi^p \tilde{u}_0^\beta \right\| \right). \tag{8.19}
\]
for some constants $C, c > 0$. Then the estimates \(8.12 - 8.15\) follow in the standard way. \(\square\)

8.3. **Nonlinear estimates.** In order to use the linear decay estimates of Lemma 8.2 one may rewrite the subsystem \(8.20\) in the following perturbed operator form

$$
\partial_t \tilde{u}^\theta + \tilde{L} \tilde{u}^\theta = \tilde{N}^\theta, 
$$

where the linear operator $\tilde{L}$ is defined by \(8.11\) and the nonlinear term

$$
\tilde{N}^\theta = - \sum_{k=1}^d \left( \tilde{A}^{k,\theta}(\tilde{u}) - \tilde{A}^{k,\theta}(0) \right) \tilde{u}^\theta_{x_k} + \tilde{Q}^\theta(\tilde{u}) - \Theta^\theta \tilde{u}^\theta.
$$

For the convenience of the later analysis, one may record the following estimates of $\tilde{N}^\theta$.

**Lemma 8.3.** The following estimates hold.

(i) Under (2D1) and that $d \geq 3$, for $1 \leq s \leq \ell - 1$,

$$
\left\| \Lambda^s \tilde{N}^\theta \right\|_{L^2} + \left\| \Lambda^{s-1} \tilde{N}^\theta \right\|_{L^2} \lesssim \delta \left( \left\| \Lambda^{s+1} \tilde{u}^\theta \right\|_{L^2} + \left\| \Lambda^s \tilde{u}^\theta \right\|_{L^2} + \left\| \nabla \tilde{u}^\theta \right\|_{L^\infty} + \left\| \tilde{u}^\theta \right\|_{L^\infty} \right),
$$

for $1 \leq p < 2$,

$$
\left\| \tilde{N}^\theta \right\|_{L^p} \lesssim \delta \left( \left\| \Lambda^d \left( 1 - \frac{1}{r} \right) + 1 \tilde{u}^\theta \right\|_{L^2} + \left\| \Lambda^d \left( 1 - \frac{1}{r} \right) \tilde{u}^\theta \right\|_{L^2} + \left\| \nabla \tilde{u}^\theta \right\|_{L^\infty} + \left\| \tilde{u}^\theta \right\|_{L^\infty} \right)^2,
$$

for $1 \leq p < 2d/(d+2)$,

$$
\left\| \tilde{N}^\theta \right\|_{L^\frac{dp}{d-p}} \lesssim \delta \left( \left\| \Lambda^d \left( 1 - \frac{1}{r} \right) + 2 \tilde{u}^\theta \right\|_{L^2} + \left\| \Lambda^d \left( 1 - \frac{1}{r} \right) + 1 \tilde{u}^\theta \right\|_{L^2} \right),
$$

and

$$
\left\| \tilde{N}^\theta \right\|_{L^2} \lesssim \delta \left( \left\| \nabla \tilde{u}^\theta \right\|_{L^\infty} + \left\| \tilde{u}^\theta \right\|_{L^\infty} \right).
$$

(ii) Under (2D1), for $0 \leq s \leq \ell - 1$,

$$
\left\| \Lambda^s \tilde{N}^\theta \right\|_{L^2} \lesssim \delta \left( \left\| \Lambda^{s+1} \tilde{u}^\theta \right\|_{L^2} + \left\| \Lambda^s \tilde{u}^\theta \right\|_{L^2} + \left\| \nabla \tilde{u}^\theta \right\|_{L^\infty} + \left\| \tilde{u}^\theta \right\|_{L^\infty} \right)
+ \left\| \tilde{u}^\theta \right\|_{L^\infty} \left( \left\| \Lambda^s \tilde{u}^\theta \right\|_{L^2} + \left\| \tilde{u}^\theta \right\|_{L^\infty} \right),
$$

for $1 \leq p < 2$,

$$
\left\| \tilde{N}^\theta \right\|_{L^p} \lesssim \delta \left( \left\| \Lambda^d \left( 1 - \frac{1}{r} \right) + 1 \tilde{u}^\theta \right\|_{L^2} + \left\| \Lambda^d \left( 1 - \frac{1}{r} \right) \tilde{u}^\theta \right\|_{L^2} \right) + \left\| \Lambda^d \left( 1 - \frac{1}{r} \right) \tilde{u}^\theta \right\|_{L^2} \left\| \tilde{u}^\theta \right\|_{L^\infty},
$$

and

$$
\left\| \tilde{N}^\theta \right\|_{L^2} \lesssim \delta \left( \left\| \nabla \tilde{u}^\theta \right\|_{L^\infty} + \left\| \tilde{u}^\theta \right\|_{L^\infty} \right) + \left\| \tilde{u}^\theta \right\|_{L^\infty}^2.
$$

for $1 \leq p < 2d/(d+2)$,

$$
\left\| \tilde{N}^\theta \right\|_{L^\frac{dp}{d-p}} \lesssim \delta \left( \left\| \Lambda^d \left( 1 - \frac{1}{r} \right) + 2 \tilde{u}^\theta \right\|_{L^2} + \left\| \Lambda^d \left( 1 - \frac{1}{r} \right) + 1 \tilde{u}^\theta \right\|_{L^2} \right) + \left\| \Lambda^d \left( 1 - \frac{1}{r} \right) + \frac{1}{2} \tilde{u}^\theta \right\|_{L^2}^2,
$$

and

$$
\left\| \tilde{N}^\theta \right\|_{L^2} \lesssim \delta \left( \left\| \nabla \tilde{u}^\theta \right\|_{L^\infty} + \left\| \tilde{u}^\theta \right\|_{L^\infty} \right) + \left\| \tilde{u}^\theta \right\|_{L^2} \left\| \tilde{u}^\theta \right\|_{L^\infty},
$$

for $2d/(d+2) < p < 2$,

$$
\left\| \Lambda \tilde{N}^\theta \right\|_{L^p} \lesssim \delta \left( \left\| \Lambda^{\frac{1}{2} + 1} \tilde{u}^\theta \right\|_{L^2} + \left\| \Lambda^{\frac{1}{2} - \frac{1}{r} + 1} \tilde{u}^\theta \right\|_{L^2} + \left\| \Lambda^{\frac{1}{2} - \frac{1}{r} + \frac{1}{2}} \tilde{u}^\theta \right\|_{L^2} \right) + \left\| \Lambda^{\frac{1}{2} - \frac{1}{r} + \frac{1}{2}} \tilde{u}^\theta \right\|_{L^2} \left\| \tilde{u}^\theta \right\|_{L^\infty}.
$$
and
\[ \left\| \Lambda \tilde{N}^\nu \right\|_{L^2} \lesssim \delta \left( \left\| \Lambda^{\frac{d}{2} + 1} \tilde{v}^1 \right\|_{L^2} + \left\| \nabla \tilde{u}^2 \right\|_{L^\infty} + \left\| \tilde{u}^p \right\|_{L^\infty} \right) + \left\| \tilde{u}^p \right\|_{L^\infty}^2. \] (8.33)

(iii) Moreover, under (\( \tilde{\Omega}^1 \)) and that \( \| v_1 \|_{L^q} \lesssim \delta \) with \( 1 \leq p \leq q \leq 2 \), for \( 1 \leq p < 2 \),
\[ \left\| \tilde{N}^\nu \right\|_{L^p} \lesssim \delta \left( \left\| \Lambda^{d(\frac{1}{2} + \frac{1}{q} - \frac{1}{p}) + 1} \tilde{u}^1 \right\|_{L^2} + \left\| \Lambda^{d(\frac{1}{2} + \frac{1}{q} - \frac{1}{p})} \tilde{v}^p \right\|_{L^2} \right) \]
\[ + \left\| \Lambda^{d(\frac{1}{2} - \frac{1}{q})} \tilde{u}^1 \right\|_{L^2} \left( \left\| \tilde{u}^1 \right\|_{L^\infty} + \left\| \tilde{u}^p \right\|_{L^2} + \left\| \tilde{u}^p \right\|_{L^2} \right), \] if \( p < q \); (8.34)
\[ \left\| \tilde{N}^\nu \right\|_{L^p} \lesssim \delta \left( \left\| \nabla \tilde{u}^2 \right\|_{L^\infty} + \left\| \tilde{u}^p \right\|_{L^\infty} \right) + \left\| \tilde{u}^p \right\|_{L^2} + \left\| \tilde{u}^p \right\|_{L^2} \right), \] if \( p = q \), (8.35)
and for \( 2d/(d+2) < p < 2 \),
\[ \left\| \Lambda \tilde{N}^\nu \right\|_{L^p} \lesssim \delta \left( \left\| \Lambda^{d(\frac{1}{2} + 1)} \tilde{u}^1 \right\|_{L^2} + \left\| \Lambda^{d(\frac{1}{2} + \frac{1}{q} - \frac{1}{p}) + 1} \tilde{u}^1 \right\|_{L^2} + \left\| \Lambda^{d(\frac{1}{2} + \frac{1}{q} - \frac{1}{p})} \tilde{v}^p \right\|_{L^2} + \left\| \Lambda^{d(\frac{1}{2} - 1)} \tilde{u}^1 \right\|_{L^2} \right) \]
\[ + \left\| \Lambda^{d(\frac{1}{2} - \frac{1}{q})} \tilde{u}^1 \right\|_{L^2} \left( \left\| \tilde{u}^2 \right\|_{L^\infty} + \left\| \tilde{u}^p \right\|_{L^2} + \left\| \tilde{u}^p \right\|_{L^2} \right), \] if \( p < q \); (8.36)
\[ \left\| \Lambda \tilde{N}^\nu \right\|_{L^p} \lesssim \delta \left( \left\| \Lambda^{d(\frac{1}{2} + 1)} \tilde{u}^1 \right\|_{L^2} + \left\| \nabla \tilde{u}^2 \right\|_{L^\infty} + \left\| \tilde{u}^p \right\|_{L^2} + \left\| \tilde{u}^p \right\|_{L^2} \right) \]
\[ + \left\| \Lambda^{d(\frac{1}{2} - \frac{1}{q})} \tilde{u}^1 \right\|_{L^2} \left( \left\| \tilde{u}^2 \right\|_{L^\infty} + \left\| \tilde{u}^p \right\|_{L^2} + \left\| \tilde{u}^p \right\|_{L^2} \right), \] if \( p = q \). (8.37)

Proof. One may first prove (i) under (\( \tilde{\Omega}^3 \)) and that \( d \geq 3 \). In this case, recalling from the definition (8.22) of \( \tilde{N}^\nu \) and the expansion (1.10)–(1.11), one has
\[ \tilde{N}^\nu = \tilde{Y}(\tilde{u})(\tilde{u}\nabla \tilde{v}^1 + \tilde{u}^p \tilde{u}^1 + \tilde{u}^1 \tilde{u}^2) + \tilde{Z}(\tilde{u}) \tilde{v}^1 \tilde{u}^1, \] (8.38)
and
\[ \tilde{N}^\nu = \tilde{Y}(\tilde{u})(\tilde{u}\nabla \tilde{v}^1 + \tilde{u}^1 \tilde{u}). \] (8.39)

By the product estimates (13.9) of Lemma (13.3) it holds that for \( 0 \leq s \leq \ell - 1 \),
\[ \left\| \Lambda^s \tilde{N}^\nu \right\|_{L^2} \lesssim \left\| \tilde{Y}(\tilde{u}) \tilde{u}^1 \right\|_{L^\infty} \left( \left\| \Lambda^{s+1} \tilde{u}^1 \right\|_{L^2} + \left\| \Lambda^s \tilde{u}^1 \right\|_{L^2} \right) \]
\[ + \left\| \Lambda^s (\tilde{Y}(\tilde{u}) \tilde{u}^1) \right\|_{L^2} \left( \left\| \nabla \tilde{u}^2 \right\|_{L^\infty} + \left\| \tilde{u}^1 \right\|_{L^\infty} \right) \]
\[ \lesssim \delta \left( \left\| \Lambda^{s+1} \tilde{u}^1 \right\|_{L^2} + \left\| \Lambda^s \tilde{u}^1 \right\|_{L^2} + \left\| \nabla \tilde{u}^2 \right\|_{L^\infty} + \left\| \tilde{u}^1 \right\|_{L^\infty} \right) \] (8.40)
and that for \( 1 \leq s \leq \ell - 1 \), since \( d \geq 3 \),
\[ \left\| \Lambda^{s-1} \tilde{N}^\nu \right\|_{L^2} \lesssim \left\| \tilde{Y}(\tilde{u}) \tilde{u}^1 \right\|_{L^d} \left( \left\| \Lambda^{s} \tilde{u}^1 \right\|_{L^\frac{2d}{d-2}} + \left\| \Lambda^{s-1} \tilde{u}^1 \right\|_{L^\frac{2d}{d-2}} \right) \]
\[ + \left\| \Lambda^{s-1} (\tilde{Y}(\tilde{u}) \tilde{u}^1) \right\|_{L^2} \left( \left\| \nabla \tilde{u}^2 \right\|_{L^\infty} + \left\| \tilde{u}^1 \right\|_{L^\infty} \right) \]
\[ \lesssim \delta \left( \left\| \Lambda^{s+1} \tilde{u}^1 \right\|_{L^2} + \left\| \Lambda^s \tilde{u}^1 \right\|_{L^2} + \left\| \nabla \tilde{u}^2 \right\|_{L^\infty} + \left\| \tilde{u}^1 \right\|_{L^\infty} \right). \] (8.41)

For \( 1 \leq p < 2 \), by Hölder’s and Sobolev’s inequalities, one can get
\[ \left\| \tilde{N}^\nu \right\|_{L^p} \lesssim \left\| \tilde{Y}(\tilde{u}) \tilde{u}^1 \right\|_{L^2} \left( \left\| \nabla \tilde{u}^2 \right\|_{L^\frac{2p}{d-2}} + \left\| \tilde{u}^1 \right\|_{L^\frac{2p}{d-2}} \right) \]
\[ + \left\| \tilde{Y}(\tilde{u}) \right\|_{L^\frac{2p}{d-2}} \left\| \tilde{u}^1 \right\|_{L^\frac{2p}{d-2}} + \tilde{Z}(\tilde{u}) \tilde{u}^1 \left\| \tilde{u}^1 \right\|_{L^\frac{2p}{d-2}} \]
\[ \lesssim \left\| \tilde{u}^1 \right\|_{L^2} + \delta \left( \left\| \nabla \tilde{u}^2 \right\|_{L^\frac{2p}{d-2}} + \left\| \tilde{u}^1 \right\|_{L^\frac{2p}{d-2}} \right) \]
\[ \lesssim \left\| \Lambda^{\frac{d}{2} - \frac{1}{q}} \tilde{u}^1 \right\|_{L^2} + \delta \left( \left\| \Lambda^{\frac{d}{2} - \frac{1}{q}} \tilde{u}^1 \right\|_{L^2} + \left\| \tilde{u}^1 \right\|_{L^\infty} \right). \] (8.42)
For \(1 \leq p < 2d/(d + 2)\) and hence \(dp/(d - p) < 2\), \((1 - 1/p) + 2 < d/2 + 1\), it holds that
\[
\left\| \tilde{N}^c \right\|_{L^p_{\tau = \tau_0}} \lesssim \left\| \tilde{Y} (\tilde{u}) \tilde{u} \right\|_{L^2} \left( \left\| \nabla \tilde{u} \right\|_{L^p_{\tau = \tau_0}} + \left\| \tilde{u} \right\|_{L^p_{\tau = \tau_0}} \right) \\
\lesssim \delta \left( \left\| \Lambda^{d(1 - \frac{1}{2})_p + 2} \tilde{u} \right\|_{L^2} + \left\| \Lambda^{d(1 - \frac{1}{2})_p + 1} \tilde{u} \right\|_{L^2} \right).
\] (8.43)

Thus, (8.40)–(8.43) yield (8.24)–(8.25) correspondingly. On the other hand, (8.24) can be proved with the simple modification in (8.43).

Next, one may turn to prove \((ii)\) under \(\tilde{\mathcal{D}}_1\). In this case, by the expansions (1.18)–(1.19), the nonlinear terms can be expressed as
\[
\tilde{N}^c = \tilde{Y} (\tilde{u}) (\tilde{u} \nabla \tilde{u}^b + \tilde{u}^b \tilde{u}) + \tilde{Z} (\tilde{u}) \tilde{u} \tilde{u}^b \tilde{u}
\] (8.44)
and
\[
\tilde{N}^b = \tilde{Y} (\tilde{u}) (\tilde{u} \nabla \tilde{u}^b + \tilde{u}^b \tilde{u}) + \tilde{W} (\tilde{u}) \tilde{u} \tilde{u}^b \tilde{u}.
\] (8.45)

By the product estimates (13.9) of Lemma 13.3 it holds that for \(0 \leq s \leq \ell - 1\),
\[
\left\| \Lambda^s \tilde{N}^b \right\|_{L^2} \lesssim \left\| \tilde{Y} (\tilde{u}) \tilde{u} \right\|_{L^\infty} \left( \left\| \Lambda^{s+1} \tilde{u}^b \right\|_{L^2} + \left\| \Lambda^s \tilde{u}^b \right\|_{L^2} \right) \\
+ \left\| \Lambda^s \tilde{Y} (\tilde{u}) \tilde{u} \right\|_{L^\infty} \left( \left\| \nabla \tilde{u} \right\|_{L^\infty} + \left\| \tilde{u} \right\|_{L^\infty} \right) \\
+ \left\| \tilde{Y} (\tilde{u}) \right\|_{L^\infty} \left( \left\| \Lambda^s \tilde{u} \right\|_{L^2} + \left\| \nabla \tilde{u} \right\|_{L^\infty} + \left\| \tilde{u} \right\|_{L^\infty} \right) \\
\lesssim \delta \left( \left\| \Lambda^{s+1} \tilde{u}^b \right\|_{L^2} + \left\| \Lambda^s \tilde{u}^b \right\|_{L^2} + \left\| \nabla \tilde{u} \right\|_{L^\infty} + \left\| \tilde{u} \right\|_{L^\infty} \right) \\
+ \left\| \Lambda^s \tilde{u} \right\|_{L^2} \left\| \tilde{u} \right\|_{L^\infty} \right. + \left. \delta \left\| \tilde{u} \right\|_{L^\infty}^2.
\] (8.46)

For \(1 \leq p < 2\), by Hölder’s and Sobolev’s inequalities, one can get
\[
\left\| \tilde{N}^c \right\|_{L^p} \lesssim \left\| \tilde{Y} (\tilde{u}) \tilde{u} \right\|_{L^2} \left( \left\| \nabla \tilde{u} \right\|_{L^\infty} \left\| \Lambda^{s+1} \tilde{u}^b \right\|_{L^2} + \left\| \Lambda^s \tilde{u}^b \right\|_{L^2} \right) \\
+ \left\| \tilde{Y} (\tilde{u}) \tilde{u} \right\|_{L^\infty} \left( \left\| \nabla \tilde{u} \right\|_{L^\infty} + \left\| \tilde{u} \right\|_{L^\infty} \right) \\
\lesssim \delta \left( \left\| \nabla \tilde{u} \right\|_{L^\infty} \left\| \Lambda^s \tilde{u} \right\|_{L^2} + \left\| \nabla \tilde{u} \right\|_{L^\infty} \right. + \left. \left\| \tilde{u} \right\|_{L^\infty} \right) \\
\lesssim \delta \left( \left\| \Lambda^{d(1 - \frac{1}{p})_p + 1} \tilde{u} \right\|_{L^2} + \left\| \Lambda^{d(1 - \frac{1}{p})_p + 1} \tilde{u} \right\|_{L^2} \right). \] (8.47)

For \(1 \leq p < 2d/(d + 2)\), it holds that
\[
\left\| \tilde{N}^b \right\|_{L^p_{\tau = \tau_0}} \lesssim \left\| \tilde{Y} (\tilde{u}) \tilde{u} \right\|_{L^2} \left( \left\| \nabla \tilde{u} \right\|_{L^\infty} \left\| \Lambda^{d(1 - \frac{1}{p})_p + 2} \tilde{u} \right\|_{L^2} + \left\| \tilde{u} \right\|_{L^\infty} \right) \\
\lesssim \delta \left( \left\| \Lambda^{d(1 - \frac{1}{p})_p + 2} \tilde{u} \right\|_{L^2} + \left\| \Lambda^{d(1 - \frac{1}{p})_p + 1} \tilde{u} \right\|_{L^2} \right). \] (8.48)

For \(2d/(d + 2) \leq p < 2\), by the product estimates (13.9) of Lemma 13.3, it holds that
\[
\left\| \Lambda \tilde{N}^b \right\|_{L^p} \lesssim \left\| \tilde{Y} (\tilde{u}) \tilde{u} \right\|_{L^\infty} \left\| \nabla^2 \tilde{u} \right\|_{L^d} + \left\| \tilde{Y} (\tilde{u}) \tilde{u} \right\|_{L^2} \left\| \nabla \tilde{u} \right\|_{L^{2p}_{\tau = \tau_0}} \\
+ \left\| \nabla \tilde{u} \right\|_{L^{2p}_{\tau = \tau_0}} \right. + \left. \left\| \tilde{u} \right\|_{L^{2p}_{\tau = \tau_0}} \right) \\
+ \left\| \nabla \tilde{u} \right\|_{L^{2p}_{\tau = \tau_0}} \left\| \tilde{u} \right\|_{L^2} + \left\| \nabla \tilde{u} \right\|_{L^\infty} \left\| \tilde{u} \right\|_{L^2} \\
\lesssim \delta \left( \left\| \Lambda^{d(1 - \frac{1}{p})_p + 1} \tilde{u} \right\|_{L^2} + \left\| \Lambda^{d(1 - \frac{1}{p})_p + 1} \tilde{u} \right\|_{L^2} + \left\| \Lambda^{d(1 - \frac{1}{p})_p + 1} \tilde{u} \right\|_{L^2} \right). \] (8.49)

Thus, (8.46)–(8.49) yields (8.24)–(8.25), (8.30), (8.32), respectively. On the other hand, (8.29), (8.31), (8.33) can be proved with the simple modifications in (8.47)–(8.49), respectively.
Finally, one may first assume $p < q$. For $1 \leq p \leq q \leq 2$ by (8.42), the expansions (8.44)–(8.45) can be rewritten as
\[
\tilde{N}^c = \tilde{Y}(\tilde{u})\tilde{v}_1(\nabla \tilde{u}^b + \tilde{u}^b) + \tilde{Y}(\tilde{u}) \tilde{u}^b(\nabla \tilde{u}^b + \tilde{u}^b) + \tilde{Z}(\tilde{u}) \tilde{u}^b \tilde{u}^b \tilde{u}
\] (8.50)
and
\[
\tilde{N}^p = \tilde{Y}(\tilde{u})\tilde{v}_1(\nabla \tilde{u}^b + \tilde{u}^b) + \tilde{Y}(\tilde{u}) \tilde{u}^b(\nabla \tilde{u}^b + \tilde{u}^b).
\] (8.51)

One may first assume $p < q$. For $1 \leq p < q$, it holds that
\[
\left\| \tilde{N}^c \right\|_{L^p} \lesssim \left\| \tilde{Y}(\tilde{u})\tilde{v}_1 \right\|_{L^q} \left( \left\| \nabla \tilde{u}^b \right\|_{L^{\infty,p}} + \left\| \tilde{u}^b \right\|_{L^{\infty,p}} \right)
\]
\[
+ \left\| \tilde{Y}(\tilde{u}) \tilde{u}^b \right\|_{L^{2,p}} \left( \left\| \nabla \tilde{u}^b \right\|_{L^2} + \left\| \tilde{u}^b \right\|_{L^2} \right) + \left\| \tilde{Z}(\tilde{u}) \tilde{u} \right\|_{L^2} \left\| \tilde{u}^b \right\|_{L^{2,p}} \left\| \tilde{u} \right\|_{L^{\infty}}
\]
\[
\lesssim \delta \left( \left\| \Lambda^{d(\frac{1}{2} + \frac{1}{\ell} - \frac{1}{p})} \tilde{u}^b \right\|_{L^2} + \left\| \Lambda^{d(\frac{1}{2} + \frac{1}{\ell} - \frac{1}{p})} \tilde{u}^b \right\|_{L^2} \right)
\]
\[
+ \left\| \Lambda^{d(1 - \frac{1}{p})} \tilde{u}^b \right\|_{L^2} \left( \left\| \nabla \tilde{u}^b \right\|_{L^2} + \left\| \tilde{u}^b \right\|_{L^2} + \delta \left\| \tilde{u}^b \right\|_{L^\infty} \right).
\] (8.52)

For $2d/(d + 2) \leq p < 2$, it holds that
\[
\left\| \Lambda \tilde{N}^p \right\|_{L^p} \lesssim \left\| \nabla (\tilde{Y}(\tilde{u})\tilde{v}_1) \right\|_{L^q} \left( \left\| \nabla \tilde{u}^b \right\|_{L^{\infty,p}} + \left\| \tilde{u}^b \right\|_{L^{\infty,p}} \right)
\]
\[
+ \left\| \tilde{Y}(\tilde{u})\tilde{u} \right\|_{L^{2,p}} \left\| \nabla \tilde{u} \right\|_{L^2} + \left\| \tilde{Y}(\tilde{u})\tilde{u} \right\|_{L^{2,p}} \left\| \nabla \tilde{u} \right\|_{L^2} \left\| \tilde{u}^b \right\|_{L^{2,p}}
\]
\[
+ \left\| \Lambda^{d(\frac{1}{2} + 1)} \tilde{u}^b \right\|_{L^2} + \left\| \Lambda^{d(\frac{1}{2} + 1)} \tilde{u}^b \right\|_{L^2} \left\| \tilde{u}^b \right\|_{L^2} \left\| \tilde{u}^b \right\|_{L^{2,p}} \left\| \tilde{u}^b \right\|_{L^\infty}
\]
\[
\lesssim \delta \left( \left\| \Lambda^{\frac{d}{2} + 1} \tilde{u}^b \right\|_{L^2} + \left\| \Lambda^{d(\frac{1}{2} + \frac{1}{\ell} - \frac{1}{p})} \tilde{u}^b \right\|_{L^2} + \left\| \Lambda^{d(\frac{1}{2} + \frac{1}{\ell} - \frac{1}{p})} \tilde{u}^b \right\|_{L^2} \right)
\]
\[
+ \left\| \Lambda^{d(1 - \frac{1}{p})} \tilde{u}^b \right\|_{L^2} \left( \left\| \nabla \tilde{u}^b \right\|_{L^2} + \left\| \tilde{u}^b \right\|_{L^2} + \delta \left\| \tilde{u}^b \right\|_{L^\infty} \right).
\] (8.53)

Here the following bound is used
\[
\left\| \nabla \tilde{v}_1 \right\|_{L^q} \lesssim \left\| \tilde{v}_1 \right\|_{L^q} + \left\| \nabla \tilde{v}_1 \right\|_{L^2} \lesssim \delta,
\] (8.54)
which follows from an use of Lemma B.1. Thus, (8.52)–(8.53) yield (8.54), (8.55), respectively. On the other hand, (8.55), (8.57) can be proved with the simple modifications in (8.52)–(8.53), respectively.

8.4. Nonlinear decay estimates. Now one can prove the decay estimates of $\tilde{u}^b$:

**Proposition 8.4.** The following decay estimates hold.

(i) Under (OD1) and that $d \geq 3$ and $1 \leq p \leq 2$, $p < d/2$, then for all $s \in [0,s^*_1] \cap [0,d/2)$ with $s^*_1 = d(1 - 1/p) + 1$,
\[
\left\| \Lambda^s \tilde{u}^c(t) \right\|_{H^{1-s}} \lesssim K_0(1 + t)^{-\frac{d}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{d}{2}},
\] (8.55)
and for all $s \in [0,s^*_1 - 1] \cap [0,d/2 - 1]$,
\[
\left\| \Lambda^s \tilde{u}^p(t) \right\|_{H^{1-s}} \lesssim K_0(1 + t)^{-\frac{d}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{d}{2} - \frac{1}{2}}.
\] (8.56)

(ii) Under (OD1) and that $d \geq 3$ and $1 \leq p \leq 2$, $p < d$; requiring further that $p < 2d/(2(3 - \ell) + d)$ if $\ell \leq 3$, then for all $s \in [0,s^*_2] \cap [0,d/2 + 1)$ with $s^*_2 = \min\{d(1 - 1/p) + 1, \ell - 1\}$,
\[
\left\| \Lambda^s \tilde{u}^c(t) \right\|_{H^{1-s}} \lesssim K_0(1 + t)^{-\frac{d}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{d}{2}},
\] (8.57)
and for all $s \in [0,s^*_2 - 1] \cap [0,d/2)$,
\[
\left\| \Lambda^s \tilde{u}^p(t) \right\|_{H^{1-s}} \lesssim K_0(1 + t)^{-\frac{d}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{d}{2} - \frac{1}{2}}.
\] (8.58)
Then one may apply the linear decay estimates (8.12) and (8.14) of Lemma 8.2 to the formula

\[ \| \Lambda^s \tilde{u}^C(t) \|_{H^{t-s}} \lesssim K_0(1 + t)^{-\frac{d}{2} \frac{1}{p} - \frac{s}{2}} ; \]  

(8.59)

and for all \( s \in [0, s^*_3 - 1] \cap [0, d/2) , \)

\[ \| \Lambda^s \tilde{u}^D(t) \|_{H^{t-s}} \lesssim K_0(1 + t)^{-\frac{d}{2} \frac{1}{p} - \frac{s}{2}} . \]  

(8.60)

Here \( K_0 = \| u_0 \|_{H^t} + \| \nabla^C u_0 \|_{L^p} + \| \nabla^D u_0 \|_{L^{p^*}} \) is sufficiently small.

**Proof.** One may first prove (i) under (\( \mathfrak{D}_1 \), (\( \mathfrak{D}_3 \)) and that \( d \geq 2 \) and \( \| \tilde{u} \|_{L^p} \lesssim \delta \) with \( 1 \leq p \leq q \leq 2 , q < d ; \) requiring further that \( p < 2d/(2(3 - \ell) + 1) \) if \( \ell \leq 3 \), then for all \( s \in [0, s_3^*] \cap [0, d/2 + 1) \) with \( s_3^* = \min \{ d(1/2 + 1/q - 1/p) + 1, d(1 - 1/p) + 2, \ell - 1 \} \),

\[ \| \Lambda^s \tilde{u}^C(t) \|_{H^{t-s}} \lesssim K_0(1 + t)^{-\frac{\tilde{d}}{2} \frac{1}{p} - \frac{s}{2}} ; \]  

(8.59)

where \( s \) is sufficiently small.

First, taking \( s = \tilde{s}_1 \) in (8.1) yields

\[ \| \Lambda^{\tilde{s}_1} \tilde{u}^\tau(t) \|_{H^{t-\tilde{s}_1}}^2 \lesssim e^{-\lambda t} K_0^2 + \int_0^t e^{-\lambda (t - \tau)} \| \Lambda^{\tilde{s}_1} \tilde{u}^C(\tau) \|_{L^2}^2 d\tau . \]  

(8.62)

On the other hand, by the Duhamel principle, it follows from (8.21) that

\[ \tilde{u}^\tau(t) = e^{-t\tilde{\mathcal{L}}^\tau} \tilde{u}^\tau_0 + \int_0^t e^{-(t - \tau)\tilde{\mathcal{L}}^\tau} \tilde{\mathcal{G}}^\tau(\tau) d\tau . \]  

(8.63)

Then one may apply the linear decay estimates (8.12) and (8.14) of Lemma 8.2 to the formula (8.63) to obtain

\[ \| \Lambda^{\tilde{s}_1} \tilde{u}^C(t) \|_{L^2} + \| \Lambda^{\tilde{s}_1 - 1} \tilde{u}^D(t) \|_{L^2} \]

\[ \lesssim (1 + t)^{-\sigma_{p^*} - \frac{d}{2}} \left( \| \tilde{u}_0^C \|_{L^p} + \| \tilde{u}_0^D \|_{L^{p^*}} + \| \Lambda^{\tilde{s}_1} \tilde{u}^\tau_0 \|_{L^2} + \| \Lambda^{\tilde{s}_1 - 1} \tilde{u}^\tau_0 \|_{L^2} \right) \]

\[ + \int_0^t (1 + t - \tau)^{-\sigma_{p^*} - \frac{d}{2}} \left( \| \tilde{\mathcal{G}}^\tau(\tau) \|_{L^p} + \| \tilde{\mathcal{G}}^\tau(\tau) \|_{L^{p^*}} + \| \Lambda^{\tilde{s}_1} \tilde{\mathcal{G}}^\tau(\tau) \|_{L^2} + \| \Lambda^{\tilde{s}_1 - 1} \tilde{\mathcal{G}}^\tau(\tau) \|_{L^2} \right) d\tau . \]

(8.64)

By the estimates (8.23)–(8.26) from Lemma 8.3, it holds that

\[ \| \tilde{\mathcal{G}}^\tau \|_{L^p} + \| \tilde{\mathcal{G}}^\tau \|_{L^{p^*}} + \| \Lambda^{\tilde{s}_1} \tilde{\mathcal{G}}^\tau \|_{L^2} + \| \Lambda^{\tilde{s}_1 - 1} \tilde{\mathcal{G}}^\tau \|_{L^2} \]

\[ \lesssim \delta \left( \| \Lambda^{\tilde{s}_1} \tilde{u}^\tau \|_{H^{t-\tilde{s}_1}} + \| \Lambda^{\tilde{s}_1 - 1} \tilde{u}^D \|_{L^2} + \| \nabla \tilde{u}^\tau \|_{L^\infty} + \| \tilde{u}^\tau \|_{L^\infty} \right) + \| \Lambda^\frac{d}{2} (1 - \frac{1}{p}) \tilde{u}^\tau \|_{L^2}^2 \]

\[ \lesssim \delta \left( \| \Lambda^{\tilde{s}_1} \tilde{u}^\tau \|_{H^{t-\tilde{s}_1}} + \| \Lambda^{\tilde{s}_1 - 1} \tilde{u}^D \|_{L^2} \right) + \| \Lambda^\frac{d}{2} (1 - \frac{1}{p}) \tilde{u}^\tau \|_{L^2}^2 . \]  

(8.65)

Here in the first inequality it has been used that \( \tilde{s}_1 \leq d(1 - 1/p) + 1 \) and that \( d(1 - 1/p) + 2 < d/2 + 1 < \ell \) for \( 1 \leq p < 2d/(d + 2) \), and in the second inequality it has been used that, since \( \tilde{s}_1 < d/2 \) and \( \ell > d/2 + 1 \) and by the Sobolev interpolation,

\[ \| \nabla \tilde{u}^\tau \|_{L^\infty} + \| \tilde{u}^\tau \|_{L^\infty} \lesssim \| \Lambda^{\tilde{s}_1} \tilde{u}^\tau \|_{H^{t-\tilde{s}_1}} . \]  

(8.66)
Then plugging the estimates (8.65) into (8.64) yields
\[
\left\| \Lambda^{\tilde{s}_1} \tilde{u}^c(t) \right\|_{L^2} + \left\| \Lambda^{\tilde{s}_1 - 1} \tilde{u}^b(t) \right\|_{L^2} \\
\lesssim (1 + t)^{-\sigma_p - \frac{d}{2}} K_0 + \int_0^t (1 + t - \tau)^{-\sigma_p - \frac{d}{2}} \delta \left( \left\| \Lambda^{\tilde{s}_1} \tilde{u}^b \right\|_{H^{t-\tilde{s}_1}} + \left\| \Lambda^{\tilde{s}_1 - 1} \tilde{u}^b \right\|_{L^2} \right) d\tau \\
+ \int_0^t (1 + t - \tau)^{-\sigma_p - \frac{d}{2}} \left( \Lambda^{\tilde{s}_1} \tilde{u}^b(\tau) \right)^2 \right\|_{L^2} d\tau. 
\] (8.67)

Now one may estimate the two time integrals in the right hand side of (8.67). In view of the definition of \( M_1(t) \), the first time integral is bounded by
\[
\delta \int_0^t (1 + t - \tau)^{-\sigma_p - \frac{d}{2}} (1 + t)^{-2\sigma_p - \frac{d}{2}} (1 + \frac{1}{2}) M_1(\tau) d\tau \lesssim \delta (1 + t)^{-\sigma_p - \frac{d}{2}} \sqrt{M_1(t)}. 
\] (8.68)

Here it has been used that \( \sigma_p + \tilde{s}_1/2 > 1 \): if \( \tilde{s}_1 = d(1 - 1/p) + 1 \), then \( \sigma_p + \tilde{s}_1/2 = d/4 + 1/2 > 1 \) since \( d \geq 3 \); if \( \tilde{s}_1 = d/2 - \epsilon \), then \( \sigma_p + \tilde{s}_1/2 = d/(2p) - \epsilon/2 > 1 \) since \( 1 \leq p < d/2 \) and one can take \( \epsilon \) sufficiently small. On the other hand, since \( d/2(1 - 1/p) < \tilde{s}_1 \), by the interpolation the second time integral is bounded by
\[
\int_0^t (1 + t - \tau)^{-\sigma_p - \frac{d}{2}} (1 + t)^{-2\sigma_p - \frac{d}{2}} (1 + \frac{1}{2}) M_1(\tau) d\tau \lesssim (1 + t)^{-\sigma_p - \frac{d}{2}} \sqrt{M_1(t)}. 
\] (8.69)

since \( \tilde{s}_1 < d/2 \). Thus substituting the estimates (8.68)–(8.69) into (8.67) yields
\[
\left\| \Lambda^{\tilde{s}_1} \tilde{u}^c(t) \right\|_{L^2} + \left\| \Lambda^{\tilde{s}_1 - 1} \tilde{u}^b(t) \right\|_{L^2} \lesssim (1 + t)^{-\sigma_p - \frac{d}{2}} \left( K_0 + \delta \sqrt{M_1(t) + M_1(t)} \right). 
\] (8.70)

Then by (8.70), it follows from (8.62) that
\[
\left\| \Lambda^{\tilde{s}_1} \tilde{u}^c(t) \right\|_{H^{t-\tilde{s}_1}} \lesssim e^{-\lambda t} K_0^2 + \int_0^t e^{-\lambda(t-\tau)} (1 + \tau)^{-2\sigma_p - \tilde{s}_1} \left( K_0^2 + \delta^2 M_1(\tau) + M_1^2(\tau) \right) d\tau \\
\lesssim (1 + t)^{-2\sigma_p - \tilde{s}_1} \left( K_0^2 + \delta^2 M_1(t) + M_1^2(t) \right). 
\] (8.71)

Lastly and similarly, by using in addition the estimates (8.26), one can get
\[
\left\| \tilde{u}^c(t) \right\|_{L^2} \lesssim (1 + t)^{-\sigma_p} \left( \left\| \tilde{u}^c_0 \right\|_{L^p} + \left\| \tilde{u}^b_0 \right\|_{L^p} + \left\| \tilde{u}^c_{0} \right\|_{L^2} \right) \\
+ \int_0^t (1 + t - \tau)^{-\sigma_p} \left( \left\| \tilde{N}^c(\tau) \right\|_{L^p} + \left\| \tilde{N}^b(\tau) \right\|_{L^p} \right) d\tau \\
\lesssim (1 + t)^{-\sigma_p} K_0 + \int_0^t (1 + t - \tau)^{-\sigma_p (1 + \tau)^{-\sigma_p - \frac{d}{2}}} \left( \delta \sqrt{M_1(t)} + M_1(\tau) \right) d\tau \\
\lesssim (1 + t)^{-\sigma_p} \left( K_0 + \delta \sqrt{M_1(t) + M_1(t)} \right) 
\] (8.72)

and
\[
\left\| \tilde{u}^b(t) \right\|_{L^2} \lesssim (1 + t)^{-\sigma_p - \frac{1}{2}} K_0 + \int_0^t (1 + t - \tau)^{-\sigma_p - \frac{1}{2}} (1 + \tau)^{-\sigma_p - \frac{d}{2}} \left( \delta \sqrt{M_1(t)} + M_1(t) \right) d\tau \\
\lesssim (1 + t)^{-\sigma_p - \frac{1}{2}} \left( K_0 + \delta \sqrt{M_1(t) + M_1(t)} \right). 
\] (8.73)

since \( \tilde{s}_1 \geq 1 \). Now combining the estimates (8.70)–(8.73) and recalling the definition of \( M_1(t) \) yields
\[
M_1(t) \lesssim K_0^2 + \delta^2 M_1(t) + M_1^2(t), 
\] (8.74)

which implies, since both \( \delta \) and \( M_1(0) \leq K_0 \) are small,
\[
M_1(t) \lesssim K_0^2. 
\] (8.75)

By again the definition of \( M_1(t) \) and by the interpolation, one can deduce (8.55)–(8.56) from (8.75) since \( \epsilon > 0 \) can be taken arbitrarily small.
One then turns to prove \((ii)\) under \((\hat{Q}1)\) and that \(d \geq 3, 1 \leq p \leq 2, p < d\) and further that \(p < 2d/(2(3 - \ell + d))\) if \(\ell \leq 3\). Take \(\bar{s} = \min\{s^2_2, d/2 + 1 - \epsilon\}\) for a sufficiently small positive constant \(\epsilon\). Then define
\[
\mathcal{M}_2(t) = \sup_{0 \leq \tau \leq t} \left\{ (1 + \tau)^{2\sigma_p + \bar{s}} \left( \left\| \Lambda^\bar{s}_2 \bar{u}^p(\tau) \right\|^2_{H^\ell} + \left\| \Lambda^\bar{s}_2 - 1 \bar{u}^D(\tau) \right\|^2_{L^2} \right) + (1 + \tau)^{2\sigma_p} \left\| \bar{u}^p(\tau) \right\|^2_{L^2} + (1 + \tau)^{2\sigma_p + 1} \left\| \bar{u}^D(\tau) \right\|^2_{L^2} \right\}.
\]

(8.76)

First, taking \(s = \bar{s}_2\) in \((8.2)\) yields
\[
\left\| \Lambda^\bar{s}_2 \bar{u}^p(t) \right\|^2_{H^\ell} \lesim e^{-\lambda t} K_0^2 + \int_0^t e^{-\lambda(t - \tau)} \left( \left\| \Lambda^\bar{s}_2 \bar{u}^C(\tau) \right\|^2_{L^2} + \left\| \Lambda^\bar{s}_2 - 1 \bar{u}^D(\tau) \right\|^2_{L^2} + \left\| \bar{u}^p(\tau) \right\|^4_{L^\infty} \right) \mathrm{d}\tau.
\]

(8.77)

On the other hand, if \(1 \leq p \leq 2d/(d + 2)\) and hence \(p^* \equiv dp/(d - p) \leq 2\), one may apply the linear decay estimates \((8.12)\) and \((8.14)\) of Lemma \(8.2\) to the formula \((8.8)\) to deduce
\[
\left\| \Lambda^\bar{s}_2 \bar{u}^C(t) \right\|_{L^2} + \left\| \Lambda^\bar{s}_2 - 1 \bar{u}^D(t) \right\|_{L^2} \lesim (1 + t)^{-\sigma_p - \frac{\bar{s}}{2}} \left( \left\| \bar{u}^C_0 \right\|_{L^p} + \left\| \bar{u}^D_0 \right\|_{L^\frac{dp}{d-p}} + \left\| \Lambda^\bar{s}_2 \bar{u}^p_0 \right\|_{L^2} + \left\| \Lambda^\bar{s}_2 - 1 \bar{u}^p_0 \right\|_{L^2} \right) + \int_0^t (1 + t - \tau)^{-\sigma_p - \frac{\bar{s}}{2}} \left( \left\| \bar{N}^C(\tau) \right\|_{L^p} + \left\| \bar{N}^D(\tau) \right\|_{L^\frac{dp}{d-p}} + \left\| \Lambda^\bar{s}_2 \bar{N}^p(\tau) \right\|_{L^2} + \left\| \Lambda^\bar{s}_2 - 1 \bar{N}^p(\tau) \right\|_{L^2} \right) \mathrm{d}\tau.
\]

(8.78)

If \(2d/(d + 2) < p \leq 2\), one may apply instead the linear decay estimates \((8.13)\) and \((8.15)\) to deduce
\[
\left\| \Lambda^\bar{s}_2 \bar{u}^C(t) \right\|_{L^2} + \left\| \Lambda^\bar{s}_2 - 1 \bar{u}^D(t) \right\|_{L^2} \lesim (1 + t)^{-\sigma_p - \frac{\bar{s}}{2}} \left( \left\| \bar{u}^C_0 \right\|_{L^p} + \left\| \Lambda \bar{u}^D_0 \right\|_{L^2} + \left\| \Lambda^\bar{s}_2 \bar{u}^p_0 \right\|_{L^2} + \left\| \Lambda^\bar{s}_2 - 1 \bar{u}^p_0 \right\|_{L^2} \right) + \int_0^t (1 + t - \tau)^{-\sigma_p - \frac{\bar{s}}{2}} \left( \left\| \bar{N}^C(\tau) \right\|_{L^p} + \left\| \Lambda \bar{N}^D(\tau) \right\|_{L^2} + \left\| \Lambda^\bar{s}_2 \bar{N}^p(\tau) \right\|_{L^2} + \left\| \Lambda^\bar{s}_2 - 1 \bar{N}^p(\tau) \right\|_{L^2} \right) \mathrm{d}\tau.
\]

(8.79)

By the estimates \((8.30)\) from Lemma \(8.3\) it holds that for \(1 \leq p < 2d/(d + 2)\),
\[
\left\| \bar{N}^p \right\|_{L^\frac{dp}{d-p}} \lesim \delta \left\| \Lambda^\bar{s}_2 \bar{u}^p \right\|^2_{H^\ell} + \left\| \Lambda^\bar{s}_2 - 1 \bar{u}^p \right\|^2_{H^\ell}.
\]

(8.80)

where it has been used that \(\bar{s}_2 \leq d(1 - 1/p) + 1\) and that \(d(1 - 1/p) + 2 < d/2 + 1 < \ell\) for \(1 \leq p < 2d/(d + 2)\), and by the estimates \((8.32) - (8.33)\) that for \(2d/(d + 2) < p < 2\),
\[
\left\| \Lambda \bar{N}^p \right\|_{L^p} \lesim \delta \left( \left\| \Lambda^\bar{s}_2 \bar{u}^p \right\|_{H^\ell} + \left\| \Lambda^\bar{s}_2 - 1 \bar{u}^D \right\|_{L^2} \right) + \left\| \Lambda^\bar{s}_2 \bar{u}^p \right\|_{H^\ell} \right\|_{L^\infty}.
\]

(8.81)

These two estimates together with the estimates \((8.27) - (8.29)\) and \((8.33)\) imply that the two sums inside the two time integrals in the right hand side of \((8.8)\) and \((8.79)\) are bounded by
\[
\delta \left( \left\| \Lambda^\bar{s}_2 \bar{u}^p \right\|_{H^\ell} + \left\| \Lambda^\bar{s}_2 - 1 \bar{u}^D \right\|_{L^2} + \left\| \nabla \bar{u}^p \right\|_{L^\infty} + \left\| \bar{u}^D \right\|_{L^\infty} \right) + \left\| \Lambda^\bar{s}_2 \bar{u}^p \right\|^2_{H^\ell} + \left\| \bar{u}^p \right\|^2_{L^\infty} \lesim \delta \left( \left\| \Lambda^\bar{s}_2 \bar{u}^p \right\|_{H^\ell} + \left\| \Lambda^\bar{s}_2 - 1 \bar{u}^D \right\|_{L^2} + \left\| \bar{u}^p \right\|^2_{H^\ell} + \left\| \bar{u}^D \right\|^2_{H^\ell} \right) \lesim \delta \left( \left\| \Lambda^\bar{s}_2 \bar{u}^p \right\|_{H^\ell} + \left\| \Lambda^\bar{s}_2 - 1 \bar{u}^D \right\|_{L^2} + \left\| \bar{u}^p \right\|^2_{H^\ell} + \left\| \bar{u}^D \right\|^2_{H^\ell} \right).\]

(8.82)

Here in the first bound it has been used that \(\bar{s}_2 \leq d(1 - 1/p) + 1, \bar{s}_2 < d/2 + 1\) and in the second bound it has been used that, since \(\bar{s}_2 < d/2 + 1 < \ell\) and by Sobolev interpolation,
\[
\left\| \nabla \bar{u}^p \right\|_{L^\infty} + \left\| \bar{u}^D \right\|_{L^\infty} \lesim \left\| \Lambda^\bar{s}_2 \bar{u}^p \right\|_{H^\ell} + \left\| \Lambda^\bar{s}_2 - 1 \bar{u}^D \right\|_{L^2}.
\]

(8.83)
Plugging the estimates (8.82) into (8.78) or (8.79), in each case one can get
\[
\left\| \Lambda^{\delta_2} \hat{u}^c(t) \right\|_{L^2} + \left\| \Lambda^{\delta_2-1} \hat{u}^D(t) \right\|_{L^2} 
\lesssim (1 + t)^{-\sigma_p - \frac{\delta_2}{2}} \int_0^t (1 + t - \tau)^{-\sigma_p - \frac{\delta_2}{2}} \delta \left( \left\| \Lambda^{\delta_2} \hat{u}^b \right\|_{H^{t-\delta_2}} + \left\| \Lambda^{\delta_2-1} \hat{u}^D \right\|_{L^2} \right) d\tau
\]
(8.84)
\[
+ \int_0^t (1 + t - \tau)^{-\sigma_p - \frac{\delta_2}{2}} \left\| \Lambda^{\delta_2} \hat{u}^b \right\|_{H^{t-\delta_2/2}}^2 d\tau
\]
\[
+ \int_0^t (1 + t - \tau)^{-\sigma_p - \frac{\delta_2}{2}} \left\| \hat{u}^b \right\|_{L^\infty} \left\| \Lambda^{\delta_2-1} \hat{u}^b \right\|_{H^{t-\delta_2+1}} d\tau.
\]

Now one may estimate the three time integrals in the right hand side of (8.84). In view of the definition of $M_2(t)$, the first time integral is bounded by
\[
\delta \int_0^t (1 + t - \tau)^{-\sigma_p - \frac{\delta_2}{2}} (1 + \tau)^{-2\sigma_p - \frac{\delta_2}{2}} \sqrt{M_2(\tau)} d\tau \lesssim (1 + t)^{-\sigma_p - \frac{\delta_2}{2}} \sqrt{M_2(t)}.
\]
(8.85)

To bound the third time integral one may use the interpolation to estimate
\[
\left\| \hat{u}^b \right\|_{L^\infty} \lesssim \left\| \Lambda^{\min\left(\frac{d}{2}, \delta_2\right)} \hat{u}^b \right\|_{H^{\min\left(\frac{d}{2}, \delta_2\right)}} \lesssim (1 + t)^{-\sigma_p - \min\left(\frac{d}{2}, \delta_2\right)} \sqrt{M_2(t)}.
\]
(8.87)

Hence, the third time integral is bounded by
\[
\int_0^t (1 + t - \tau)^{-\sigma_p - \frac{\delta_2}{2}} (1 + \tau)^{-2\sigma_p - \frac{\delta_2}{2}} \sqrt{M_2(\tau)} d\tau \lesssim (1 + t)^{-\sigma_p - \frac{\delta_2}{2}} M_2(t).
\]
(8.88)

Here it has been used that $\sigma_p + \delta_2 > 2$: if $\delta_2 = d(1 - 1/p) + 1$, then $\sigma_p + \delta_2 = d/4 + 1/2 > 1$ since $d \geq 3$; if $\delta_2 = d/2 + 1 - \epsilon$, then $\sigma_p + \delta_2 = d/(2p) + 1/2 - \epsilon/2 > 1$ since $1 \leq p < d$ and $\epsilon$ can be taken sufficiently small; if $\delta_2 = \ell - 1$, then $\sigma_p + \delta_2 = d/2(1/p - 1/2) + (\ell - 1)/2 > 1$, which trivially holds when $\ell > 3$, and follows from $1 \leq p < 2d/(2(3 - \ell) + d)$ when $\ell \leq 3$. On the other hand, by the interpolation the second time integral is bounded by
\[
\int_0^t (1 + t - \tau)^{-\sigma_p - \frac{\delta_2}{2}} (1 + \tau)^{-2\sigma_p - \frac{\delta_2}{2}} M_2(\tau) d\tau \lesssim (1 + t)^{-\sigma_p - \frac{\delta_2}{2}} M_2(t).
\]
(8.86)

Then by (8.89) together with an use of the estimates (8.88), the bound of the third time integral in the right hand side of (8.84), one may deduce from (8.77) that
\[
\left\| \Lambda^{\delta_2} \hat{u}^c(t) \right\|_{L^2} + \left\| \Lambda^{\delta_2-1} \hat{u}^D(t) \right\|_{L^2} \lesssim (1 + t)^{-\sigma_p - \delta_2} \left( K_0 + \delta \sqrt{M_2(t)} + M_2(t) \right).
\]
(8.89)

Finally, note that one needs to revise the estimates (8.72) in the case (i) since now $\sigma_p - \delta_1 > 1$ may not be true for $d/2 \leq p < d$. Then it is crucial to exploit the exponential decay property of the high frequency part in the linear decay estimates of Lemma 8.2. Indeed, applying the linear decay estimates (8.12) and (8.15) and using in addition the estimates (8.31),
one can deduce
\[
\|\bar{u}^c(t)\|_{L^2} \lesssim (1 + t)^{-\sigma_p} \left( \|\bar{u}_0^c\|_{L^p} + \|\bar{u}_0^D\|_{L^p} + \|\bar{u}_0^\flat\|_{L^2} \right) \\
+ \int_0^t (1 + t - \tau)^{-\sigma_p} \left( \|\bar{N}^c(\tau)\|_{L^p} + \|\bar{N}^D(\tau)\|_{L^p} + e^{-\lambda(t-\tau)} \|\bar{N}^h(\tau)\|_{L^2} \right) d\tau \\
\lesssim (1 + t)^{-\sigma_p} K_0 + \int_0^t (1 + t - \tau)^{-\sigma_p} (1 + \tau)^{-\sigma_p} \frac{\tilde{s}_2}{2} \left( \delta \sqrt{M_2(\tau)} + M_2(\tau) \right) d\tau \\
\lesssim (1 + t)^{-\sigma_p} \left( K_0 + \delta \sqrt{M_2(t)} + M_2(t) \right), \\
\text{for } 1 \leq p \leq \frac{2d}{d+2},
\]
\[
\|\bar{u}^c(t)\|_{L^2} \lesssim (1 + t)^{-\sigma_p} \left( \|\bar{u}_0^c\|_{L^p} + \|\Lambda\bar{u}_0^D\|_{L^p} + \|\bar{u}_0^\flat\|_{L^2} \right) \\
+ \int_0^t (1 + t - \tau)^{-\sigma_p} \left( \|\bar{N}^c(\tau)\|_{L^p} + \|\Lambda\bar{N}^D(\tau)\|_{L^p} + e^{-\lambda(t-\tau)} \|\bar{N}^h(\tau)\|_{L^2} \right) d\tau \\
\lesssim (1 + t)^{-\sigma_p} \left( K_0 + \delta \sqrt{M_2(t)} + M_2(t) \right), \\
\text{for } \frac{2d}{d+2} < p \leq 2.
\]
and
\[
\|\bar{u}^D(t)\|_{L^2} \lesssim (1 + t)^{-\sigma_p - \frac{1}{2}} K_0 + \int_0^t (1 + t - \tau)^{-\sigma_p - \frac{1}{2}} (1 + \tau)^{-\sigma_p - \frac{1}{2}} \left( \delta \sqrt{M_2(\tau)} + M_2(\tau) \right) d\tau \\
+ \int_0^t e^{-\lambda(t-\tau)} (1 + \tau)^{-2\sigma_p - \frac{\min(p-p_1)}{2}} M_2(\tau) d\tau \\
\lesssim (1 + t)^{-\sigma_p - \frac{1}{2}} \left( K_0 + \delta \sqrt{M_2(t)} + M_2(t) \right).
\]
Here it has been used that \(\tilde{s}_2 \geq 1\) and that \(\sigma_p + d/4 = d/(2p) > 1/2\) since \(1 \leq p < d\). Based on these estimates (8.89)–(8.93), one can get that for \(\delta\) and \(M_2(0) \leq K_0\) small,
\[
M_2(t) \lesssim K_0^2.
\]
This implies (8.57)–(8.58).

Finally, one turns to prove (iii) under (\(\tilde{D}_1\), (\(\tilde{D}_3\)) and that \(d \geq 2\) and \(\|\bar{v}_1\|_{L^p} \lesssim \delta\) with \(1 \leq p \leq q \leq 2\), \(q < d\) and further that \(p < 2d/(2(3 - \ell) + d)\) if \(\ell \leq 3\). Take \(\tilde{s}_3 = \min\{s_3^*, d/2 + 1 - \epsilon\}\) for a sufficiently small positive constant \(\epsilon\). Then define
\[
M_3(t) = \sup_{0 \leq \tau \leq t} \left\{ (1 + \tau)^{2\sigma_p + \tilde{s}_3} \left( \|\Lambda^\tilde{s}_3 \bar{u}^c(\tau)\|_{H^{d-\tilde{s}_3}}^2 + \|\Lambda^\tilde{s}_3 - 1 \bar{u}^D(\tau)\|_{L^2}^2 \right) \\
+ (1 + \tau)^{2\sigma_p} \|\bar{u}^\flat(\tau)\|_{L^2}^2 + (1 + \tau)^{2\sigma_p + 1} \|\bar{u}^D(\tau)\|_{L^2}^2 \right\}
\]
\[
(8.95)
\]
One can still use the same estimates (8.77)–(8.79) in the case (ii), however, the nonlinear estimates can be improved by using the estimates (8.34)–(8.37) from Lemma 8.3. Indeed, by the estimates (8.34)–(8.35), it holds that for \(1 \leq p < 2\),
\[
\|\bar{N}^c\|_{L^p} \lesssim \delta \left( \|\Lambda^\tilde{s}_3 \bar{u}^c\|_{H^{d-\tilde{s}_3}} + \|\Lambda^\tilde{s}_3 - 1 \bar{u}^D\|_{L^2} \right) \\
+ \|\Lambda^{d(1 - \frac{1}{2})} \bar{u}^\flat\|_{L^2} \left( \|\bar{u}^\flat\|_{L^\infty} + \|\Lambda \bar{u}^\flat\|_{L^2} + \|\bar{u}^D\|_{L^2} \right),
\]
and by the estimates (8.36)–(8.37), it holds that for \(2d/(d + 2) < p < 2\),
\[
\|\Lambda \bar{N}^c\|_{L^p} \lesssim \delta \left( \|\Lambda^\tilde{s}_3 \bar{u}^c\|_{H^{d-\tilde{s}_3}} + \|\Lambda^\tilde{s}_3 - 1 \bar{u}^D\|_{L^2} \right) \\
+ \|\Lambda^{d(1 - \frac{1}{2})} \bar{u}^\flat\|_{L^2} \left( \|\bar{u}^\flat\|_{L^\infty} + \|\Lambda \bar{u}^\flat\|_{L^2} + \|\bar{u}^D\|_{L^2} \right),
\]
\[
(8.97)
\]
Hence, noting further (8.249) and (8.354), one may improve the estimates (8.82) to be
\[
\delta \left( \left\| \Delta^{\tilde{s}} \tilde{u}^h \right\|_{H^{\ell-\tilde{s}}} + \left\| \Delta^{\tilde{s}-1} \tilde{u}^D \right\|_{L^2} \right) \\
+ \left\| \tilde{u}^h \right\|_{L^\infty} \left( \left\| \Delta^{\tilde{s}-1} \tilde{u}^h \right\|_{H^{\ell-\tilde{s}+1}} + \left\| \tilde{u}^h \right\|_{L^\infty} \left\| \Delta^{d(1-\frac{1}{p})} \tilde{u}^h \right\|_{L^2} \right) \\
+ \left\| \Delta^{d(1-\frac{1}{p})+\frac{1}{2}} \tilde{u}^h (\tau) \right\|_{L^2}^2 + \left\| \Delta^{d(1-\frac{1}{p})} \tilde{u}^h \right\|_{H^1} \left( \left\| \Delta \tilde{u}^h \right\|_{L^2} + \left\| \tilde{u}^D \right\|_{L^2} \right).
\]  
(8.98)

Plugging the estimates (8.98) to (8.78) or (8.79), in each case it holds that
\[
\left\| \Delta^{\tilde{s}} \tilde{u}^C (t) \right\|_{L^2} + \left\| \Delta^{\tilde{s}-1} \tilde{u}^D (\tau) \right\|_{L^2} \\
\lesssim (1 + t)^{-\sigma_p - \frac{\tilde{s}}{2}} K_0 + \int_0^t (1 + t - \tau)^{-\sigma_p - \frac{\tilde{s}}{2}} \delta \left( \left\| \Delta^{\tilde{s}} \tilde{u}^h (\tau) \right\|_{H^{\ell-\tilde{s}}} + \left\| \Delta^{\tilde{s}-1} \tilde{u}^h (\tau) \right\|_{L^2} \right) d\tau \\
+ \int_0^t (1 + t - \tau)^{-\sigma_p - \frac{\tilde{s}}{2}} \left( \left\| \tilde{u}^h (\tau) \right\|_{L^\infty} \left( \left\| \Delta^{\tilde{s}-1} \tilde{u}^h (\tau) \right\|_{H^{\ell-\tilde{s}+1}} + \left\| \Delta^{d(1-\frac{1}{p})} \tilde{u}^h (\tau) \right\|_{L^2} \right) \\
+ \left\| \Delta^{d(1-\frac{1}{p})+\frac{1}{2}} \tilde{u}^h (\tau) \right\|_{L^2}^2 + \left\| \Delta^{d(1-\frac{1}{p})} \tilde{u}^h \right\|_{H^1} \left( \left\| \Delta \tilde{u}^h \right\|_{L^2} + \left\| \tilde{u}^D \right\|_{L^2} \right) \right) d\tau.
\]  
(8.99)

Now one may estimate the two time integrals in the right hand side of (8.99). In view of the definition of \(\mathcal{M}_3(t)\), the first time integral is bounded by
\[
\delta \int_0^t (1 + t - \tau)^{-\sigma_p - \frac{\tilde{s}}{2}} \left( 1 + t \right)^{-\sigma_p - \frac{\tilde{s}}{2}} \sqrt{\mathcal{M}_3(\tau)} d\tau \lesssim \delta (1 + t)^{-\sigma_p - \frac{\tilde{s}}{2}} \sqrt{\mathcal{M}_3(t)}.
\]  
(8.100)

Here it has been used that \(\sigma_p + \tilde{s}_3 > 2 > 1\): if \(\tilde{s}_3 = d(1/2 + 1/2 - 1/p) + 1\), then \(\sigma_p + \tilde{s}_3/2 = d/(2q) + 1/2 > 1\) since \(1 \leq q < d\); if \(\tilde{s}_3 = d(1 - 1/p) + 2\), then \(\sigma_p + \tilde{s}_3/2 = d/4 + 1 > 1\); if \(\tilde{s}_3 = d/2 + 1 - \epsilon\), then \(\sigma_p + \tilde{s}_3/2 = d/(2p) + 1/2 - \epsilon/2 > 1\) since \(1 \leq p < d\) and one can take \(\epsilon\) sufficiently small; if \(\tilde{s}_3 = \ell - 1\), then \(\sigma_p + \tilde{s}_3/2 = d/(1 - 1/p) + (\ell - 1)/2 > 1\) which trivially holds when \(\ell > 3\), and follows from \(1 \leq p < 2d/(3 - \ell) + d\) when \(\ell \leq 3\). To estimate the second time integral, it is noted that, similar to (8.88), the first term is bounded by
\[
(1 + t)^{-\sigma_p - \frac{\tilde{s}}{2}} \mathcal{M}_3(t).
\]  
(8.101)

Since \(d(1 - 1/p) \leq \tilde{s}_3\) one may use the interpolation, together with the estimates similar to (8.87), to bound the second term by
\[
\int_0^t (1 + t - \tau)^{-\sigma_p - \frac{\tilde{s}}{2}} (1 + t)^{-\sigma_p - \frac{\tilde{s}}{2}} (1 - \frac{1}{2})^{-\frac{\tilde{s}}{2}} \mathcal{M}_3(\tau) d\tau \lesssim (1 + t)^{-\sigma_p - \frac{\tilde{s}}{2}} \mathcal{M}_3(t).
\]  
(8.102)

Here it has been used that \(2\sigma_p + d/2(1-1/p) + d/4 = \sigma_p + d/2 \geq \sigma_p + (d/2 + 1)/2 > \sigma_p + \tilde{s}_3/2\) since \(d \geq 2\) and \(\tilde{s}_3 < d/2 + 1\). Similarly, the last three terms are bounded by
\[
\int_0^t (1 + t - \tau)^{-\sigma_p - \frac{\tilde{s}}{2}} (1 + t)^{-\sigma_p - \frac{\tilde{s}}{2}} (1 - \frac{1}{2})^{-\frac{\tilde{s}}{2}} \mathcal{M}_3(t) \left( \omega_k + \delta \sqrt{\mathcal{M}_3(t)} + \mathcal{M}_3(t) \right).
\]  
(8.103)

since \(\tilde{s}_3 < d/2 + 1\). Thus substituting the estimates (8.100)–(8.103) into (8.99) yields
\[
\left\| \Delta^{\tilde{s}} \tilde{u}^C (t) \right\|_{L^2} + \left\| \Delta^{\tilde{s}-1} \tilde{u}^D (t) \right\|_{L^2} \lesssim (1 + t)^{-\sigma_p - \frac{\tilde{s}}{2}} \left( K_0 + \delta \sqrt{\mathcal{M}_3(t)} + \mathcal{M}_3(t) \right).
\]  
(8.104)

Then by (8.104), similarly as (8.90), it follows from (8.77) that
\[
\left\| \Delta^{\tilde{s}} \tilde{u}^h (t) \right\|_{H^{\ell-\tilde{s}}} \lesssim (1 + t)^{-2\sigma_p - \tilde{s}_3} \left( K_0^2 + \delta^2 \mathcal{M}_3(t) + \mathcal{M}_3^2(t) \right).
\]  
(8.105)

Finally, similarly as (8.91)–(8.93) in the case (ii), one can get
\[
\left\| \tilde{u}^C (t) \right\|_{L^2} \lesssim (1 + t)^{-\sigma_p} \left( K_0 + \delta \sqrt{\mathcal{M}_3(t)} + \mathcal{M}_3(t) \right).
\]  
(8.106)
and
\[ \| \tilde{u}^D(t) \|_{L^2} \lesssim (1 + t)^{-\frac{1}{2}} \left( K_0 + \delta \sqrt{M_3(t)} + M_3(t) \right). \] (8.107)

Thus combining (8.104)–(8.107), one can get that for \( \delta \) and \( M_3(0) \leq K_0 \) small,
\[ M_3(t) \lesssim K_0^2. \] (8.108)

This implies (8.59)–(8.60).

**Remark 8.5.** In the statement of the assertions (ii) and (iii) of Proposition 8.4, it has been used implicitly that for \( 2d/(d + 2) < p < 2 \),
\[ \| \Lambda \tilde{u}^D \|_{L^p} \lesssim \| \tilde{u}^0 \|_{L^2} + \| \Lambda \tilde{u}^0 \|_{L^2} \leq K_0, \] (8.109)
which follows from an use of Lemma [BL7].

### 9. Proof of the Theorems

This section is devoted to prove the main theorems. By the standard local existence theory [11] [19] and a continuity argument one needs only to show how to close the *a priori* estimates.

**Proof of Theorem 1.** Under the assumptions of Theorem 1 one assumes a priori that \( \| \tilde{u}(t) \|_{H^\ell} \leq \delta \). Then by the (i) assertions of Propositions 6.1 and 8.1, taking \( \tilde{s}_1 \) be the one in the proof of the assertion (i) of Proposition 8.4, it holds that
\[ \| \tilde{u}(t) \|_{H^\ell} \lesssim \| \tilde{u}_0 \|_{H^\ell} + \int_0^t \left( \left\| \Lambda^{s_1} \tilde{u}^\ell(\tau) \right\|_{H^{\ell - s_1}} + \left\| \Lambda^{s_1} \tilde{u}^\ell(\tau) \right\|_{L^2} + \left\| \tilde{u}^\ell(\tau) \right\|_{L^2} \right) \, d\tau \]
\[ \lesssim \| \tilde{u}_0 \|_{H^\ell} + K_0 \int_0^t (1 + \tau)^{-\alpha p - \frac{d}{p} + \epsilon} \, d\tau \lesssim K_0. \] (9.1)

This closes the a priori estimates by taking \( K_0 \) sufficiently small. Consequently, by the estimates (9.1) and the (i) assertions of Propositions 6.1 and 8.1, one can obtain (2.8)–(2.10) and thus conclude the proof of Theorem 1.

**Proof of Theorem 2.** Under the assumptions of Theorem 2 one assumes a priori that \( \| \tilde{u}(t) \|_{H^\ell} \leq \delta \). Then by the (ii) assertions of Propositions 6.1 and 8.1, taking \( \tilde{s}_2 \) be the one in the proof of the assertion (ii) of Proposition 8.4, it holds that
\[ \| \tilde{u}(t) \|_{H^\ell} \lesssim \| \tilde{u}_0 \|_{H^\ell} + \int_0^t \left( \left\| \Lambda^{s_2} \tilde{u}^\ell(\tau) \right\|_{H^{\ell - s_2}} + \left\| \Lambda^{s_2 - 1} \tilde{u}^\ell(\tau) \right\|_{L^2} + \left\| \tilde{u}^\ell(\tau) \right\|_{L^2} \right) \, d\tau \]
\[ \lesssim \| \tilde{u}_0 \|_{H^\ell} + K_0 \int_0^t (1 + \tau)^{-\alpha p - \frac{d}{p} + \epsilon} \, d\tau \lesssim K_0 \] (9.2)
since \( d \geq 3 \). This closes the a priori estimates by taking \( K_0 \) sufficiently small. Consequently, by the estimates (9.2), the assertion (i) of Proposition 5.5 and the assertion (ii) of Proposition 8.4, one can obtain (2.13)–(2.15) and thus conclude the proof of Theorem 2.

**Proof of Theorem 3.** Under the assumptions of Theorem 3 one assumes a priori that \( \| \tilde{u}(t) \|_{H^\ell} + \| \tilde{v}_1(t) \|_{L^q} \leq \delta \). Then by the assertion (ii) of Proposition 6.1 and the assertion (iii) of Proposition 8.4, taking \( \tilde{s}_3 \) be the one in the proof of the assertion (iii) of Proposition 8.4, it holds that
\[ \| \tilde{u}(t) \|_{H^\ell} \lesssim \| \tilde{u}_0 \|_{H^\ell} + \int_0^t \left( \left\| \Lambda^{s_3} \tilde{u}^\ell(\tau) \right\|_{H^{\ell - s_3}} + \left\| \Lambda^{s_3 - 1} \tilde{u}^\ell(\tau) \right\|_{L^2} + \left\| \tilde{u}^\ell(\tau) \right\|_{L^2} \right) \, d\tau \]
\[ \lesssim \| \tilde{u}_0 \|_{H^\ell} + \tilde{K}_0 \int_0^t (1 + \tau)^{-\alpha p - \frac{d}{p} + \epsilon} \, d\tau \lesssim \tilde{K}_0 \] (9.3)
since \( 2\alpha p + d/2 = d/p > 1 \) for \( 1 \leq p < d \) and one can take \( \epsilon \) sufficiently small. Moreover, by the estimates (7.3) of Proposition 7.1 and Remark 7.2, it holds that
\[ \| \tilde{v}_1(t) \|_{L^q} \lesssim \exp(CK_0) \left( \tilde{K}_0 + \int_0^t \left\| \tilde{u}^\ell \right\|_{L^2} \left( \left\| \tilde{u}^D \right\|_{L^2} + \left\| \nabla \tilde{u}^\ell \right\|_{L^2} + \left\| \tilde{u}^\ell \right\|_{L^2} \right) \, d\tau \right) \]
\[ \lesssim \tilde{K}_0 + \tilde{K}_0 \int_0^t (1 + \tau)^{-\frac{d}{2} \left( \frac{3}{p} - \frac{1}{2} \right)} \, d\tau \lesssim \tilde{K}_0 \] (9.4)
since \(d/2(2/p - 1/q) + 1/2 \geq d/(2p) + 1/2 > 1\) and \(d/2(3/p - 1/q) \geq d/p > 1\) for \(1 \leq p \leq q < d\). These close the a priori estimates by taking \(\tilde{K}_0\) sufficiently small. Consequently, by the estimates \(\ref{4.3}\), the assertion (ii) of Proposition \(\ref{8.4}\) and the assertion (iii) of Proposition \(\ref{8.4}\) one can obtain \(\ref{2.11}\) and \(\ref{2.12}\) and thus conclude the proof of Theorem \(\ref{3.1}\). \(\square\)

10. Application to Damping Full Euler System

In this section, we verify that the damping full Euler system of adiabatic flow satisfies all the assumptions (\(\mathfrak{A}_1\)–(\(\mathfrak{A}_4\), (\(\mathfrak{B}\) and (\(\mathfrak{D}_1\)–(\(\mathfrak{D}_3\). The system can be written in terms of the density \(\rho\), the velocity \(v\) and the entropy \(S\) of the flow as

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho v) &= 0, \\
\partial_t(pv) + \text{div}(pv \otimes v + pI_d) &= -\rho v, \\
\partial_t(\rho S) + \text{div}(\rho v S) &= 0.
\end{align*}
\]

(10.1)

It is well known that the thermodynamics variables \(\rho, p, S\), the temperature \(\theta\) and the internal energy \(e\) satisfy the relation

\[
de = \theta \, dS - p \, d\left(1/\rho\right),
\]

(10.2)

and they can be determined by knowing any two of them; we may view \(p, \theta, \) and \(e\) as functions of \(\rho\) and \(S\). It is assumed that \(e\) is strictly convex with respect to \(\rho\) and \(S\).

The system (10.1) can be rewritten into

\[
\begin{align*}
\partial_t S + \sum_{k=1}^{d} v_k S_{x_k} &= 0, \\
\partial_t \rho + \sum_{k=1}^{d} v_k \rho_{x_k} + \sum_{k=1}^{d} \rho(v_k)_{x_k} &= 0, \\
\partial_t v_i + \frac{p_i}{\rho} \rho_{x_i} + \sum_{k=1}^{d} v_k (v_i)_{x_k} + \frac{p_i}{\rho} S_{x_i} &= -v_i, \quad i = 1, \ldots, d.
\end{align*}
\]

(10.3)

Denote \(u = (S, \rho, v^T)^T\). By a direct calculation, the matrix

\[
A(u, \omega) = \begin{pmatrix}
v \cdot \omega & 0 & 0 \\
0 & v \cdot \omega & \rho \omega^T \\
\frac{p_s}{\rho} \omega & \frac{p_s}{\rho} \omega & v \cdot \omega I_d
\end{pmatrix}
\]

possesses \(d + 2\) real eigenvalues

\[
\lambda_1(u, \omega) = \cdots = \lambda_d(u, \omega) = v \cdot \omega, \quad \lambda_{d+1}(u, \omega) = v \cdot \omega + \sqrt{p}, \quad \lambda_{d+2}(u, \omega) = v \cdot \omega - \sqrt{p}
\]

and a complete set of left eigenvectors

\[
l_1(u, \omega) = (1, 0, 0)^T, \quad l_k(u, \omega) = (0, 0, \chi_k^T) \quad (k = 2, \ldots, d),
\]

\[
l_{d+1}(u, \omega) = \left(\frac{p_s}{\sqrt{p}}, \sqrt{p}, \rho \omega\right), \quad l_{d+2}(u, \omega) = \left(\frac{p_s}{\sqrt{p}}, \sqrt{p}, -\rho \omega\right)
\]

and right eigenvectors

\[
r_1(u, \omega) = \left(1, \frac{p_s}{\rho}, 0\right), \quad r_k(u, \omega) = \left(0, 0, \chi_k\right) \quad (k = 2, \ldots, d),
\]

\[
r_{d+1}(u, \omega) = \left(0, 1, \frac{1}{2\sqrt{p}}\right), \quad r_{d+2}(u, \omega) = \left(0, 1, \frac{1}{2\sqrt{p}}\right).
\]

Here \(\chi_k \quad (k = 2, \ldots, d)\) are \(d - 1\) orthogonal unit vectors that are perpendicular to \(\omega\).

It is clear that the assumption (\(\mathfrak{B}\)) is satisfied. Noting that \(Q(u) = (0, 0, -v^T)^T\), we check that around the equilibrium state \(u_* = (S_*, \rho_*, 0^T)^T\) with \(S_*\) and \(\rho_* > 0\) given constants,
\[ \nabla_u Q(u) = \begin{pmatrix} 0 & 0 \\ 0 & -I_d \end{pmatrix}. \]

- Take the entropy \( \eta = \frac{1}{2} \rho |v|^2 + \rho e(\rho, S) \) and the entropy flux \( \psi^k = v^k(\frac{1}{2} \rho |v|^2 + \rho e(\rho, S) + p(\rho, S)) \), and then define

\[ \eta(u) = \frac{1}{2} \rho |v|^2 + \rho e(\rho, S) - \rho e(\rho_*, S_*) - \rho S \theta(\rho_*, S_*) + \rho_*=S \theta(\rho_*, S_*) - \frac{\partial}{\partial \rho} p(\rho_*, S_*) + p(\rho_*, S_*). \]

- \( \nabla_u \eta(u) \cdot Q(u) = -\rho |v|^2 \leq -\frac{\rho}{2} |v|^2. \)
- \( \nabla_u Q(u) r_j(u_*, \omega) \neq 0, \ j = 2, \ldots, d + 2. \)
- \( \nabla_u Q(u) r_1(u) = 0 \text{ and } \nabla_u \lambda_1(u, \omega) \cdot r_1(u) = 0. \)
- \( I_1(u) \nabla_u Q(u) = 0. \)

These verify the assumptions (A1)–(A4) and (B1)–(B3). Thus, Theorem 3 produces the unique global solution to the damping full Euler system in \( \mathbb{R}^d \), \( d \geq 2. \)

**Appendix A. Linear transformation**

In this appendix it will be showed that for the system (1.1) under the assumptions (A1)–(A4) and (B), the conditions (3.1) and (3.2) can be always satisfied after a proper linear transformation.

By (A1), one can apply to the system (1.1) the following linear transformation

\[ u = u(u^*) := \begin{pmatrix} \frac{\partial Q_r}{\partial u_r}(0) \end{pmatrix}^{-1} I_r \begin{pmatrix} \frac{\partial Q_p}{\partial u_p}(0) \end{pmatrix}^{-1} u^*. \]  

(A.1)

It is easy to check that the system after this transformation keeps the same form as (1.1), and all the assumptions (A1)–(A4) and (B) still hold. Moreover, one has

\[ \frac{\partial Q^*_r}{\partial u^*_r}(0) = 0, \ q = 1, \ldots, r, \ i = 1, \ldots, n. \]  

(A.2)

This verifies the condition (3.1).

Now, by (A1), (A4) and (3.1), it is direct to get

\[ r^*_p(0) \in \text{span}\{e_1, \ldots, e_r\}, \]  

(A.3)

which implies that \( r^*_p(0) = 0 \) (\( p = r + 1, \ldots, n \)) and for some \( i_1 \) \( \in \{1, \ldots, r\}, \)

\[ r^*_i(0) \neq 0. \]  

(A.4)

Then one may choose \( \{i_2, \ldots, i_r\} = \{1, \ldots, r\} \setminus \{i_1\} \) and have that

\[ \text{span}\{r^*_i(0), e_{i_2}, \ldots, e_{i_r}, e_{r+1}, \ldots, e_n\} = \mathbb{R}^n. \]  

(A.5)

One can further apply to the system (1.1) the linear transformation

\[ u^* = u^*(u^{**}) := (r^*_i(0), e_{i_2}, \ldots, e_{i_r}, e_{r+1}, \ldots, e_n) u^{**}. \]  

(A.6)

Again, it is easy to check that the transformed system keeps the same form as (1.1), and all the assumptions (A1)–(A4), (B) and the condition (3.1) still hold. Moreover, one has

\[ r^*_i(0) = e_1. \]  

(A.7)

This verifies the condition (3.2).
Appendix B. Analytic tools

First recall the classical Sobolev interpolation of the Gagliardo-Nirenberg inequality.

Lemma B.1. Let \(1 \leq q, r \leq \infty\) and \(0 \leq \alpha < \beta\). Then it holds that
\[
\|\Lambda^\alpha f\|_{L^p} \lesssim \left\|\Lambda^\beta f\right\|_{L^q}^{\theta} \left\|f\right\|_{L^r}^{1-\theta},
\]
where
\[
\frac{1}{p} - \frac{\alpha}{d} = \theta \left(\frac{1}{r} - \frac{\beta}{d}\right) + (1 - \theta) \frac{1}{q},
\]
for all \(\theta\) in the interval
\[
\frac{i}{m} \leq \theta \leq 1
\]
with the following exceptional cases.

1. If \(\alpha = 0\), \(r\beta < d\), \(q = \infty\), then one needs an additional assumption that either \(f\) tends to zero at infinity or \(f \in L^\delta\) for some finite \(\delta > 0\).

2. If \(1 < r < \infty\), meanwhile \(\beta - \alpha - d/r\) is a non negative integer, then \((B.1)\) holds only for \(\theta\) satisfying \(\alpha/\beta < \theta < 1\).

Proof. See THEOREM of Lecture II in [21].

In the lemma above the restriction \(0 \leq \alpha < \beta\) is inconvenient for the use in this paper, one may employ the following variant.

Lemma B.2. Let \(2 \leq p \leq \infty\) and \(\alpha, \beta, \gamma \in \mathbb{R}\). Then it holds that
\[
\|\Lambda^\alpha f\|_{L^p} \lesssim \left\|\Lambda^\beta f\right\|_{L^2}^{\theta} \left\|\Lambda^\gamma f\right\|_{L^2}^{1-\theta}.
\]
Here \(0 \leq \theta \leq 1\) (if \(p = \infty\), then it is required that \(0 < \theta < 1\)) and \(\alpha\) satisfies
\[
\alpha + d \left(\frac{1}{2} - \frac{1}{p}\right) = \beta \theta + \gamma (1 - \theta).
\]

Proof. For the case \(2 \leq p < \infty\), it follows from the classical Sobolev inequality that
\[
\|\Lambda^\alpha f\|_p \lesssim \|\Lambda^\zeta f\|_2 \text{ with } \zeta = \alpha + d \left(\frac{1}{2} - \frac{1}{p}\right).
\]
By the Parseval theorem and Hölder’s inequality, we have
\[
\|\Lambda^\zeta f\|_2 \lesssim \left\|\Lambda^\beta f\right\|_{L^2}^{\theta} \left\|\Lambda^\gamma f\right\|_{L^2}^{1-\theta},
\]
where \(0 \leq \theta \leq 1\) is defined by \((B.4)\). Hence, \((B.3)\) follows by \((B.5)-(B.6)\).

For the case \(p = \infty\), one may refer to Exercise 6.1.2 in [6].

Recall lastly the following commutator and product estimates:

Lemma B.3. Let \(l \geq 0\) and define the commutator
\[
\left[\Lambda^l, g\right] h = \Lambda^l (gh) - g \Lambda^l h.
\]
Then it holds that
\[
\|\left[\Lambda^l, g\right] h\|_{L^p} \lesssim \|\nabla g\|_{L^p} \left\|\Lambda^{l-1} h\right\|_{L^p} + \left\|\Lambda^l g\right\|_{L^p} \|h\|_{L^p}.
\]
In addition, for \(l \geq 0\),
\[
\left\|\Lambda^l (gh)\right\|_{L^p} \lesssim \|g\|_{L^p} \left\|\Lambda^l h\right\|_{L^p} + \left\|\Lambda^l g\right\|_{L^p} \|h\|_{L^p}.
\]
Here \(p_0, p_2, p_3 \in (1, \infty)\) and
\[
\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
\]
Proof. One may refer to Lemma 3.1 in [10].
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