Magnetic monopoles and cosmic inflation

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It is possible that the expansion of the universe began with an inflationary phase, in which the inflaton driving the process also was a Higgs field capable of stabilizing magnetic monopoles in a grand-unified gauge theory. If so, then the smallness of intensity fluctuations observed in the cosmic microwave background radiation implies that the self-coupling of the inflaton-Higgs field was exceedingly weak. It is argued here that the resulting broad, flat maximum in the Higgs potential makes the presence or absence of a topological zero in the field insignificant for inflation. There may be monopoles present in the universe, but the universe itself is not in the inflating core of a giant magnetic monopole.

I. INTRODUCTION – TOPOLOGY AS A POSSIBLE ALTERNATE SOURCE OF INFLATION

Two phenomena that should occur on totally different length scales nevertheless may be connected quite closely. Magnetic monopoles are classical-field solutions of grand-unified gauge theories, expected to survive quantum corrections and fluctuations because the poles are associated with long-range fields that would require infinite action to destroy. 't Hooft \(^1\) and Polyakov \(^2\) independently discovered the first such solution in SO(3) gauge theory with the apparent symmetry reduced to U(1) by the Higgs mechanism \(^3\). Kibble \(^4\) observed that in a universe without gravity and with the vacuum value of the Higgs field initially very small, quantum fluctuations of that field would produce structures with the topology of global monopoles, which could be deconfined by acquisition of magnetic monopole gauge fields. Preskill \(^5\) realized that this possible monopole production could create a crisis for cosmology, implying far more monopoles than observational limits allow. Because the expected energy scale of grand unification is quite high, the geometrical size of a monopole core must be quite small.

At the other extreme of length comes the visible universe. Guth \(^6\) introduced a concept which may underlie the origin of the universe, cosmic inflation. The essence is that a dynamical scalar field, the inflaton, initially may have in some region an expectation value which leaves the energy density of the field far from its minimum at zero energy density. This implies a dynamical (therefore variable) coefficient for Einstein's cosmological term, and hence exponential expansion, or inflation, of the region, coming to an end when the inflaton field reaches the value which minimizes the energy density.

At first sight it might seem that such a model would not be amenable to scientific investigation, because science exploits insights about structure to predict future observations. Nevertheless, as abundantly illustrated in a field like geology, it is possible to predict future observations related to events long ago, because we keep developing new methods to observe consequences of those events. In the case of inflationary cosmology, exactly such developments are occurring, in particular through improved sensitivity to angular fluctuations in the cosmic microwave background radiation, most recently with WMAP \(^7\).\(^1\)

How is all this connected to magnetic monopoles? Guth \(^6\) observed that one consequence of inflation is the disappearance of the monopole problem: Inflation sweeps apart all the monopoles implied by the Kibble mechanism, generating our entire visible universe from a tiny preinflation region, which might not contain even one pole. However, the possible connection might be even stronger, because the Higgs field stabilizing a monopole in a grand unified model is a candidate to play the role of the inflaton. Let us return to this possibility shortly.

The original Guth proposal had defects, some of which were addressed by ‘new inflation’, independently proposed by Albrecht and Steinhardt \(^8\) and Linde \(^9\). The next (and still viable) stage was the ‘chaotic eternal inflation’ of Linde \(^10\), and Steinhardt \(^11\), and Vilenkin \(^12\), where, the process of inflation never comes to a complete end. The transition from false to true vacuum occurs in some parts of the Universe, while the rest remains in the false vacuum state. In this way the global structure of the Universe is the following: thermalized phase surrounding islands of inflating space, or islands of thermalized phase surrounded by false vacuum, depending on the rate of formation of bubbles of the new phase. This eternal process is due to quantum fluctuations of the inflaton field, which in localized regions can inhibit the field from rolling down towards the minimum of its effective potential.\(^2\)

In 1994 Linde \(^13\) and Vilenkin \(^14\) independently proposed an alternative way to generate eternal inflation: In the center of a topological defect the scalar field is zero, and hence at a local maximum of its effective potential. This looks like a suitable condition for the onset of infla-

\(^1\) The scale-invariant spectrum of angular fluctuations appearing in many inflationary models was suggested even earlier on grounds of simplicity by Harrison and by Zeldovich \(^8\). Thus this phenomenon may be a consequence of inflation, but its confirmation would not necessarily single out inflationary cosmology as the only possible origin of such a spectrum. To accomplish that would require more detail, possibly forthcoming in future observations.

\(^2\) This description is drawn from the elegant formulation in \(^14\).
Numerical investigation of this proposal was performed in [10] for a global monopole which starts out as a static solution of classical field equations in the absence of gravity and Higgs-field self-coupling and in [17] for an ‘t Hooft-Polyakov magnetic monopole [1, 2] with the same assumptions for initial conditions. Such a calculation looks as if it might be relevant for a case in which the vacuum expectation value (vev) that minimizes the Higgs potential, initially small, slowly increases at all points in space to a value at or beyond the Planck scale, where the gravitational interactions become comparable with gauge and scalar field contributions: The monopole’s gauge and scalar fields rescale gradually, always solving the static Higgs-Yang-Mills-Einstein equations with the current vev, until a point is reached at which this structure becomes unstable.

If one kept using the original vanishing Higgs self-coupling, then the instability would be towards collapse, thanks to the positive gravitational mass density of the gauge field, while if the Higgs self-coupling were increased, the Higgs field in the exterior region would approach its asymptotic vacuum value exponentially with increasing $r$, rather than simply as $1/r$. This implies that the starting points for the numerical calculations described above no longer should be appropriate for large self-coupling.

Given observational constraints on the self-coupling, even a correct numerical calculation with large self-coupling would not describe a realistic scenario. However, if it did yield inflation, then this would imply topological inflation in a theory with these parameters under almost any choice of initial conditions. On the other hand, for the extremely small self-coupling implied by observation, the main message of this paper is that nontrivial topology of the Higgs field is unimportant for inflation. Some of the considerations here were discussed briefly in earlier work, but with a more optimistic conclusion about the importance of topological inflation. If the present analysis is right, then the earlier work did not pursue these issues quite far enough.

The concept of topological inflation certainly is correct for at least some examples. It is quite straightforward in the case of a domain wall for large enough vev of an inflaton field with nonzero self-coupling [14, 15, 18]. Still, among all the possible cases, inflation of a magnetic monopole core holds special appeal. The attraction of such a scheme would only be increased by the economical notion that the Higgs field which stabilizes monopoles could also be the inflaton field.$^3$

Thus, inflation disconnects us from monopoles, but it remains possible that monopoles are connected to, and even generate, inflation. The main thesis of the present work, based on strong evidence from cosmic microwave background (CMB) fluctuations that there was very weak inflaton self-coupling during inflation preceding our present expanding universe, is that such inflation in no way was seeded or enhanced by the formation of zeros in an inflaton-Higgs field carrying the topological charge of a magnetic monopole. In other words, our universe might contain one or a few magnetic poles, but it does not constitute the inflated core of a magnetic pole.

Previous literature contains the observation that monopole topological inflation (with a rather large Higgs self-coupling) could have occurred in an earlier phase of inflation. This would get round the problem of inconsistency between naive expectations for a rather large Higgs self-coupling and the small inflaton self-coupling required for inflation directly initiating our current universe, but at the price of making the relationship between us and the inflating monopole core exceedingly remote! The main issue addressed in the present work, with a negative conclusion, is whether a small self-coupling for the Higgs field would lead to monopole topological inflation, as distinguished from chaotic inflation.

II. PREVIOUS ANALYSES

For the numerical calculations in [17] Sakai considered Yang-Mills and scalar fields minimally coupled to gravity. The action for this system is

$$S = \int d^4x \sqrt{-g} \left[ \frac{m^2_{\text{Pl}}}{16\pi} R - \frac{1}{4} (F_{\mu\nu}^a)^2 - \frac{1}{2} (D_\mu \Phi^a)^2 - V(\Phi) \right]$$

where $\Phi$ is a real triplet, and $F_{\mu\nu}^a$ is the field strength of the SU(2) gauge field $A_\mu^a$:

$$F_{\mu\nu}^a = \nabla_\mu A_\nu^a - \nabla_\nu A_\mu^a - e \epsilon^{abc} A_\mu^b A_\nu^c,$$  

$D_\mu$ is the fully covariant derivative:

$$D_\mu \Phi^a = \nabla_\mu \Phi^a + e \epsilon^{abc} A_\mu^b \Phi^c,$$

$^3$ This notion did not receive much consideration before the work of Linde and Vilenkin, probably because of the rather large expected scale for the Higgs self-coupling, suggested by perturbative quantum field theory. Phenomenology of the observed cosmic density variations suggests a much smaller coupling if inflation were the immediate precursor of our universe. These points are elaborated here a little later.
(\nabla_a$ being the spacetime covariant derivative), and the potential of the scalar field is:

$$V(\Phi) = \frac{1}{4} \lambda (\Phi^2 - \eta^2)^2.$$  \hspace{1cm} (4)

In [17] the field equations were solved numerically for different values of the parameters - the vev \(\eta\) of the scalar field and the ratio \(\frac{\lambda}{e}\) of the two coupling constants. The results were: Assuming a static initial configuration that solves the (first-order) Bogomol’nyi-Prasad-Sommerfield (BPS) equations \[14\], which hold for Higgs self-coupling \(\lambda = 0\), then for \(\eta\) order of the Planck mass \(m_{Pl}\) and \(\lambda/e^2\) order unity inflation does occur. The studies indicated that with \(\eta \approx m_{Pl}\) monopole inflation (that is, expansion of the core of an initially present monopole) is even possible for smaller values of \(\lambda/e^2\), but the smallest value considered was \(\lambda/e^2 = 0.1\). If the goal were to supply starting conditions corresponding to equilibrium without gravity, then the BPS ansatz would be questionable for \(\lambda \neq 0\) because the self-coupling suppresses the difference \(\Delta \Phi\) between the Higgs field and the vev, forcing exponential decay with radius \(r\), while the ansatz of course keeps \(\Delta \Phi\) independent of \(\lambda\), falling only as \(1/r\), and thus implies additional (spurious) contributions to the cosmological term which drives inflation.

If the self-coupling of the scalar field arose as a perturbative quantum effect, as in the Coleman-Weinberg calculation [20], one would have \(\lambda \approx e^4/4\pi\), and hence \(\frac{\lambda}{e} \approx e^2/4\pi \ll 1\), because for the perturbative expansion to make sense one has to have \(e < 1\). Beyond this theoretical argument, phenomenological considerations give a far stronger version of the condition. Comparison of inflation-model predictions with astronomical observations (data on the CMB fluctuations) yield an upper limit \(\lambda \lesssim 10^{-10}\) [21, 22, 23, 24]. At the same time, the slow (logarithmic) running of gauge couplings with scale assures that a grand unified magnetic monopole would be associated with a value \(e^2 \approx 0.1 - 1.0\). This implies \(\frac{\lambda}{e} \lesssim 10^{-9}\), and is commonly taken to show that the inflation is not the Higgs field of a grand unified theory, and therefore has no direct connection to the monopoles in such a theory. As we shall discuss later, supersymmetry, if present, could greatly suppress the Higgs self-coupling, perhaps making it consistent with the mentioned CMB limit. It’s worth noting that mysteriously small couplings abound in the current scheme of physics. Whether or not these are explained by supersymmetry, one might take an empirical view that a small self-coupling of the Higgs field in grand-unified gauge theory would be no more or less surprising than the small masses of quarks and leptons, and the really small magnitude of an apparent cosmological term driving accelerated expansion of our universe.

We see that accepted values of the second parameter are restricted to the range \(\frac{\lambda}{e} \ll 1\). We shall encounter below analytic evidence that rules out monopole core inflation for small \(\lambda/e^2\). All of this depends on choosing initial conditions for a monopole in equilibrium without gravity, and following the evolution when gravity is included (as was attempted (using a simplified ansatz) in the numerical calculations mentioned above). We shall need to consider later a very different and arguably more appropriate choice of initial conditions, where the expectation value of the Higgs field everywhere starts out close to zero, but with a very small, but nonzero, Higgs self-coupling calling for a nonzero vev. Under these circumstances, topological zeros must arise in the Higgs field, and one may study the influence of their presence on the evolution.

The existence of a solution of the Yang-Mills-Higgs system coupled to gravity (an ‘t Hooft-Polyakov monopole in curved background) was shown by van Nieuwenhuizen, Wilkinson, and Perry [25]. As in the flat-space case with nonzero \(\lambda\), this solution cannot be written in closed analytic form. Some time after that analysis, there were several (mostly numerical) investigations of a different aspect of large \(\eta\): Ignoring the possibility of inflation, what happens in a theory with both classical gravity and a Higgs-Yang-Mills action, so that two types of monopole, a Reissner-Nordstrom (extremal black hole) and an ‘t Hooft-Polyakov hedgehog, both are possible? In general, the naive value of the mass of the first or the second will be greater, depending on whether \(m_{Pl}\) is greater or smaller than \(\eta\). An obvious expectation is that the more massive one (as determined by naive estimates) will be unstable. Instability of the Reissner-Nordstrom solution for \(\eta < m_{Pl}\) was noted by Goldhaber [26], Lee, Nair, and Weinberg [27], and Breitenlohner, Forgacs, and Maison [28]. Instability of the ‘t Hooft-Polyakov monopole for \(\eta \gtrsim O(m_{Pl})\) was noted by Frieman and Hill [29], Lee et al. [30], and Ortiz [31], as well as Breitenlohner et al. [28]. Even though this was not discussed explicitly in the works just cited, it is quite possible that for sufficiently large \(\lambda/e^2\) the instability is associated with monopole inflation.

In [28] new static solutions of the spherically symmetric Einstein-Yang-Mills [EYM]-Higgs system were found, which disappear when gravity is decoupled. These solutions are a discrete family of radial excitations with an increasing number of zeros of the gauge field. In the limit \(\eta \to 0\) they join smoothly to the (again numerical) Bartnik-McKinnon solutions of the EYM system [32], which may be viewed as gravitationally bound classical ‘glueballs’, and of which all but the two lowest are known to be unstable. Arguably these two are unstable as well (though quite possibly metastable), because there is no obvious lower bound for the mass of such an object.

Let us concentrate for our analytic estimates of ‘soliton explosion’ on the ‘t Hooft-Polyakov monopole, coupled to gravity. As mentioned, the method to be used is designed for solutions having a zero-gravity limit. The same is true also for the numerical work [17]. The solutions of [28], which are intrinsically nonperturbative in the gravitational coupling, in principle could give a new result for inflation conditions, even though they do not carry net topological charge. However, this seems unlikely be-
cause the gauge-field energy, more than the Higgs-field self-coupling energy, evidently increases for these solutions – that generates positive gravitational mass, and hence gravitational binding, precisely the opposite phenomenon to the inflation we are seeking. Further, as already mentioned, these excitations may be unstable against ordinary disintegration, as opposed to explosive inflation.

There is an interesting further consideration here: Much discussion in the literature points to the necessity of introducing some physics besides classical inflation to describe the past, including the case of a universe generated from a bubble of false vacuum \[27, 28, 30\], and in particular, as discussed by Borde, Trodden and Vachaspati, a universe generated by monopole inflation \[31\]. Indeed, the inevitability of a past singularity for any region of a classical inflating spacetime was indicated by Borde and Vilenkin \[32\]. A recent theorem based on very weak assumptions appears to make this constraint inescapable \[36\]. The required additional physics might be supplied by a quantum fluctuation at or later than the point in the past where purely classical backward evolution would have become incomplete \[37\].

This convincing case for something in inflation beyond classical field theory suggests a thought experiment for numerical calculations of the type carried out by Sakai \[17\]: After verifying that inflation occurs starting from an initial condition for a monopole coupled to gravity, one could run the equations backwards, and look for the past singularity. Clearly the time symmetry of the system is such that if at \(t = 0\) one had a static solution, its past history would simply be the reflection of its future evolution. If so, indeed in this model inflation would not be eternal from the past – instead there would be inflation looking backwards, equivalent to past contraction terminating at the equilibrium point. This also would violate singularity theorems about future singularities of collapsing systems. The obvious resolution is that there must be a quantum fluctuation leading to any such solution, and moreover there must be time-dependence at every stage (no static solution at any time), increasing the conventional matter energy and so further raising the threshold for inflation.

The next two sections are devoted to examining the influence of the two dimensionless parameters in the EYM-Higgs system, \(\lambda/e^2\) and \(\eta/mp_i\), on the possibility of inflation starting from an initial monopole field configuration in equilibrium without gravity. It is worth repeating that the numerical studies \[27, 28, 31\] already give significant constraints, as of course does the work of \[17\].

III. RELATION OF \(\lambda/e^2\) TO NEGATIVE GRAVITATIONAL MASS DENSITY

Let us now examine the question of possible monopole core inflation for specified \(\lambda/e^2\), assuming the solution in the absence of gravity as an initial state, with gravity then added. A prerequisite to produce inflation, i.e., an exponential growth with time in the volume of some region of space, is the presence of negative gravitational mass density. For a Friedmann-Robertson-Walker spacetime, the driving force for inflation is well known to be a negative value for the combination \(\mu = \rho + 3p\), where \(\rho\) is the energy density, \(p\) the pressure, and this expression defines \(\mu\) as the gravitational mass density. For such an isotropic system this is equivalent to

\[
\mu = T_{00} + \sum_{i=1}^{3} T_{ii} ,
\]

where \(T_{\mu\nu}\) represents the non-gravitational (i.e., matter) contributions to the energy-momentum tensor. Sakai \[17\] emphasizes the form \(\mu\) for \(\mu\) and utilizes it as a way of determining when and where inflation (or collapse) should be expected. Indeed, \(\mu\) applies quite widely; for example, it holds in the case of a photon gas in the vicinity of a large mass such as the sun or the earth \[35\]. It is useful for our problem, because except at the exact center of the monopole spatial isotropy is lost.

We want to determine the conditions for \(\mu\) to be negative, so that self-gravitational effects might cause inflation in the monopole core. Consider the same scalar and gauge fields as above, only omitting the gravitational field. The energy-momentum tensor of the system is:

\[
T_{\mu\nu} = D_{\mu} \Phi^a D_{\nu} \Phi^a - \frac{1}{2} g_{\mu\nu} D_{\sigma} \Phi^a D^\sigma \Phi^a - \frac{1}{4} g_{\mu\nu} F^2 - F_{\mu\rho} F^\rho_{\nu} - g_{\mu\nu} V(\Phi) ,
\]

where \(g_{\mu\nu} = \text{diag}(-,+,+,+).\) As we are interested in the parameter range \(\frac{\lambda}{e^2} \ll 1\), let us consider as a starting point the static solution in the BPS limit \(\frac{\lambda}{e^2} \rightarrow 0\). One can easily check, using the BPS equation \(F^2 = \epsilon_{ijk} D_k \Phi^a\) (the indices \(i,j,k\) run only over the spatial components), that the sum of the principal pressures vanishes. Hence the gravitational mass density \(\mu = \rho + \Sigma T_{ii}\) is strictly positive. Therefore, in this limiting case inflation is not possible. In fact one may notice that in the BPS limit every component of the space-space part of the stress tensor is zero:

\[
T_{ij} = \frac{\delta S}{\delta g_{ij}} = 0 \quad \forall i, j = 1, 2, 3 ,
\]

a consequence of the topological nature of the solution in this case (metric-independent form of the energy). Furthermore, as the solution is static in the gauge \(A_0 = 0\), we have \(T_{0i} = T_{i0} = 0 \quad \forall i = 1, 2, 3\), which, along with \(\mu\), implies that the only nonvanishing component of the energy-momentum tensor is the energy density \(T_{00}\).
To study the case of nonzero $\lambda$, we need some inequalities on two contributions, from the gauge field and from the scalar field. As we have

$$\mu = \frac{1}{2} F^2 - 2V,$$

(8)

to show that $\mu$ is everywhere greater than zero we require a function which is a lower bound on the gauge field contribution $F^2/2$, and a function which is an upper bound on the scalar field potential $V$. In the BPS limit the gauge field contribution $F^2/2$ for the density in terms of the dimensionless Cartesian coordinates $\tilde{x} = e\eta \tilde{r}$ may be written

$$F^2/2 = \frac{\eta}{e \sqrt{2}} \frac{d}{dx} \left[ (\coth x - 1) \left( 1 - \frac{x}{\sinh x} \right)^2 \right] = \frac{\eta}{e \sqrt{2}} \times \left[ (1 - (x/\sinh x))^2 + 2(x \coth x - 1)^2 \left( \frac{x}{\sinh x} \right)^2 \right].$$

(9)

A simple function which never is greater than this (but equal for $x = 0$, and discrepant only by $O(1/x^6)$ for $x \to \infty$) is

$$\left( F^2/2 \right)_{\text{inf}} = \frac{\eta}{e (3 + x^2)(1 + x^2)}.$$ 

(10)

Note that this expression is valid for $\lambda = 0$. For nonzero $\lambda$, one expects a compression of the monopole core. Consequently, any change in $\lambda$ will only be an increase, so we indeed have a robust lower bound.

Now turn to the potential term. In the vicinity of $x = 0$, we have

$$-2V \geq -\frac{\eta}{2e} \left( \frac{\lambda}{e^2} \right),$$

(11)

with the value decreasing monotonically in absolute magnitude as $x$ increases. This gives at $x = 0$ the condition for $\mu \leq 0$

$$\frac{\lambda}{e^2} \geq 2/3,$$

(12)

as noted already by Sakai 17. Because at this point $2V$ attains its maximum, one might think that this is an absolute condition. However, it is easy to see that for the BPS ansatz $2V$ falls more slowly with $r$ than $F^2$, opening the counterintuitive possibility that $\mu$ may be negative away from the center of the monopole for an even smaller value of $\lambda/e^2$. To address this question, we need to obtain an improved bound on $-2V$ away from the origin (where it is fixed by the topological condition $\Phi(0) = 0$).

At large $x$, with the BPS solution substituted into $V$, we have

$$-2V \approx -\frac{2\eta}{e} \left( \frac{\lambda}{e^2 x^2} \right).$$

(13)

This is the form used as a starting ansatz by Sakai. However, for nonzero $\lambda$ small fluctuations about the vacuum have a mass $m = \sqrt{2\lambda \eta}$. A conservative lower bound on the asymptotic behavior therefore is

$$-2V_{\text{bound}} \approx -\frac{2\eta}{e} \left( \frac{\lambda}{e^2 x^2} \right) e^{-2\sqrt{2(\lambda/e^2) x}}.$$

(14)

This Yukawa falloff actually gives an upper bound on the magnitude of $V$ because the effective source at the geometric center of the Yukawa field $\Phi(x)$ is weakened by the same quartic potential which gives mass to the long-distance small fluctuations. The exponential falloff is important in principle, because otherwise at sufficiently long distances $-2V$ clearly would overcome $F^2/2$. Let us combine the expressions for small and large $x$ to get an infimum for $-2V$. Let us take at each $x$ the less negative of our two forms, matching them at the crossover point, for which infinitesimal $\lambda$ occurs at $x = 2$:

$$-2V_{\text{inf}} = -\frac{\eta}{2e} \left( \frac{\lambda}{e^2} \right), \ x \leq 2$$

$$-2V_{\text{inf}} = -\frac{2\eta}{e} \left( \frac{\lambda}{e^2 x^2} \right) e^{-2\sqrt{2(\lambda/e^2) x}}, \ x > 2.$$

(15)

For the interval $x \leq 2$ the strongest condition for positivity of $\mu$ comes at the end ($x = 2$):

$$\frac{\lambda}{e^2} \leq 2/35.$$ 

(16)

At larger $x$, we may simplify initially by using the asymptotic form for $F^2/2$, yielding the requirement

$$\sqrt{2\frac{\lambda}{e^2} x} e^{-\sqrt{2\frac{\lambda}{e^2} x}} \leq 1.$$ 

(17)

As the maximum of this expression is $1/e$ for $\sqrt{2\lambda/e^2} = 1$ or $x \approx 3$ with the previously obtained value for $\lambda/e^2$, and at that value of $x$ the infimum of $F^2/2$ is smaller than the asymptotic form by a factor $81/120 \approx 0.68 > 1/e = 0.36$, we see that with ample assurance, for

$$\frac{\lambda}{e^2} \leq 2/35 \approx 0.06,$$

(18)

the gravitational mass density $\mu$ of the ‘t Hooft-Polyakov monopole is nowhere negative, so that the most elementary criterion for inflation is not satisfied.

In particular, if gravitational coupling were to be introduced at this point, and then increased, it could produce compression, even collapse, but never inflation. While we have a rigorous lower bound on the critical value of $\lambda/e^2$, it would be helpful to increase the bound. The following semi-quantitative argument assures that the actual bound must be larger by an order of magnitude. For the BPS solution, the gauge field energy and the gradient energy of the Higgs field are identically distributed, but as $\lambda/e^2$ increases the Higgs gradient energy is suppressed in the external region. Consequently, a $1/R$ term in the total energy, where $R$ is the radius below which $F^2$ stops increasing as $1/r^4$, has its coefficient reduced.
by as much as $\frac{1}{4}$, while the coefficient of a term proportional to $R$, associated with the integral of the gradient energy inside $R$, is roughly doubled. This means that the equilibrium $R$ is reduced by a factor of about two, so that the central density $F^2/2$ of the gauge field contribution to $\mu$ is increased by at least an order of magnitude. When this is inserted into the previous estimates, it leads immediately to the claimed order of magnitude increase in the critical value of $\lambda/e^2$, which therefore surely is at least 0.5 - the same order of magnitude as the smallest values considered in [17]. This claim could be tested numerically if Sakai’s calculation were repeated using as a starting point the static solution to the Yang-Mills-Higgs equations including the $\lambda$ term (a solution which itself is determined numerically [1]).

IV. INFLUENCE OF $\eta$ ON THE POSSIBILITY OF INFLATION

Let us turn now to consider the other (dimensionful) parameter in [1]. The fact that the vev $\eta$ of the scalar field must be at least of the order of the Planck mass to have inflation is commonly accepted. It was identified in [14] and [15], for example, as required by the slow-roll condition for new inflation ($|\dot{\Phi}| \ll 3H\Phi$ and $\dot{\Phi}^2 \ll V(\Phi)$). However, the usual inflationary considerations assume that the scalar field is constant throughout the whole space, or at least within some big region. Here is a simple proof that $\eta \approx m_{pl}$ is a necessary condition for inflation in the case under consideration (in which the negative gravitational mass density is restricted to a region in the core of the monopole), based on a paper by Blau, Guendelman, and Guth [39]. Because from the result above negative $\mu$ implies $\frac{\eta}{\lambda} \approx 1$, in the following let us assume that this condition is met.

In [39] the authors consider a spherically symmetric bubble of false vacuum separated by a domain wall from an infinite region of true vacuum. They obtain a critical mass $M_{cr}$ such that for mass $M$ of the bubble bigger than $M_{cr}$ the bubble will inflate, while for $M < M_{cr}$ it will collapse. If one neglects the surface energy density on the bubble wall, then from (5.14) in [39] one sees that the critical mass is

$$M_{cr} = \frac{4\pi}{3} \chi^{-2} \rho,$$

(19)

where $\chi$ (in the notation of [39]) is the Hubble parameter of the de Sitter region inside the bubble and $\rho$ is the energy density of the false vacuum. This simply means that the radius of the false vacuum region should be bigger than the distance to the horizon in the de Sitter space for inflation to be possible. Let us now consider the core of the ’t Hooft-Polyakov monopole as a spherical region of false vacuum with average energy density $\bar{\rho}$. As the mass of the monopole is $M = \frac{4\pi}{3} f(\frac{\lambda}{\rho})$, where $f(\frac{\lambda}{\rho})$ depends very little on its argument and is $f(\frac{\lambda}{\rho}) \approx 1$, we get for the average energy density:

$$\bar{\rho} = 3e^2 \eta^4,$$

(20)

using the fact that the equilibrium radius of the monopole is $R = 1/e\eta$. As the Hubble parameter $H$ is given by

$$H^2 = \frac{8\pi G \bar{\rho}}{3},$$

(21)

from the condition for inflation $R > H^{-1}$ follows:

$$\eta > \sqrt{\frac{1}{8\pi G}},$$

(22)

using (20). In units $\hbar = c = 1$:

$$\eta > \sqrt{\frac{1}{8\pi} m_{pl}} \approx 0.2 m_{pl}.$$  

(23)

This rough estimate agrees pretty well with the numerical calculations in [17], which give $\eta > 0.3 m_{pl}$. The idea that a topological defect can inflate if its size is bigger than the cosmological horizon is suggested already in [17], but without derivation.

The above estimate of the critical value of $\eta$ would be accurate if as in [39] the entire mass of the bubble were associated with the Higgs self-coupling, but that is not true here because of the positive gauge field contributions to $\mu$, which surely will increase the threshold further. Again, this could be checked by numerical work with the flat-space solution for nonzero $\lambda$ as a starting point.

V. INFLUENCE OF PRIMORDIAL TOPOLOGICAL ZEROS ON INFLATION

We have seen that starting out with a monopole in equilibrium and then adding gravity will not yield inflation unless strict conditions are satisfied. However, as gravity presumably is always present, we should consider the early stage of a process which would have generated monopoles in the absence of gravity, but now with gravity included. Let us imagine an initial condition introduced at some stage in which the Higgs field everywhere has a very small expectation value, but the minimum of the Higgs self-coupling, i.e., the equilibrium vev, has a substantial nonzero value. From the continuity of the small yet fluctuating expectation value, we expect topological zeros. The canonical view [14] [15] is that because this gives an initial condition for inflation, the gradient of the Higgs field in the neighborhood of each zero will rapidly decrease in magnitude, so that the field will be pinned near zero indefinitely by the topology, thus generating eternal inflation.

This issue has been addressed in numerical calculations for models in 2+1 dimensions by Linde and Linde [LL]
Using quite large values for the parameters, so that one would be at least near the classical threshold for inflation of a soliton (vortex in this case), initially in equilibrium without gravity, when gravitational coupling is introduced. Even then, when counting inflationary zones associated with maxima of the Higgs potential, LL found only a minority of those zones carried topological zeros, so that even under these optimal circumstances (large $\lambda$) topological zeros at most make a minor quantitative enhancement in the rate of inflation.

Let us approach the question here in a different way, which should be complementary but may serve to highlight the circumstances in which topological inflation does or does not take place. The process which eventually can produce normal vacuum invokes quantum fluctuations to accomplish this task. This implies that for consistency we should treat quantum-mechanically all degrees of freedom of the Higgs field configuration, including the spatial coordinates of each topological zero. The same inflationary effect which flattens the gradients of the field treated classically also spreads the probability distribution associated with what classically would be a localized zero.

Consequently, even though the zero exists, it becomes possible that it has little influence on the fluctuation process which can lead to regions of normal vacuum. When that is correct, then sooner or later a zero might find itself in such a normal region, clothed with the gauge field configuration required to make its total mass finite. In that case the considerations above become relevant, because a monopole in approximate equilibrium will only be able to inflate if the values of the parameters $\lambda/e^2$ and $\eta$ are appropriately large. Thus, the zero will be eternal, but inflation in the vicinity of the monopole need not be. One may argue that this reasoning is robust, because classical physics always is an approximation to quantum physics. If treating a degree of freedom as quantum-mechanical gives different results from treating it classically, that immediately implies the classical approximation is invalid, and the quantum description is necessary.

The earlier discussions in this paper appear not to be found in the literature, but nevertheless are quite straightforward. On the other hand the argument just given about quantum behavior of topological zero coordinates may be less obvious. Here are some questions it raises: In the standard approach leading to the appearance and significance of topological zeros [14, 15], it is quantum fluctuations which produce the zeros. Indeed (as pointed out in [14, 15]), continuing quantum fluctuations during the epoch of very small Higgs field will be generating additional zeros. Further, for a zero at the center of a soliton in equilibrium – an object which is well-described by a stable, classical field configuration – quantum fluctuations of the position are unimportant because the Compton wavelength of the soliton is small compared to the internal dimensions of the object. Thus one could ask both “Don’t we already have a fully quantum description?” and “Isn’t the location of the zero anyway a classical degree of freedom?”

To address both questions, we need to look at the early evolution of a topological zero – before it has `nucleated' a stable configuration. Of course, if the couplings are such that a monopole stable without gravity would inflate with gravity taken into account, then there isn’t a stable configuration, but we now are investigating the opposite case, so that only early development is possibly relevant. It is obvious that the zero appears during a period when inflation is occurring, precisely because the necessary fluctuations depend on the Higgs field being small, and therefore close to a maximum in the Higgs potential. Thus the issue is whether the presence of topological zeros will keep inflation going longer than it otherwise would, e.g., in a theory with a single Higgs field constrained to be nonnegative, with maximum potential at zero value of that field.

To simplify the problem take an Ansatz for the Higgs field of the form

$$\Phi(\vec{r}, t) = \Phi_0 \sin(|\vec{r} - r_0(t)\|/\ell), |\vec{r} - r_0| \leq \ell \pi/2$$

$$\Phi(\vec{r}, t) = \Phi_0, |\vec{r} - r_0| \geq \ell \pi/2. \quad (24)$$

In the light of the comment about the classical character of the zero location for a stable soliton, to address this issue we need to examine the behavior of the coordinate for a zero in the presence of a very small Higgs field. Let us do this initially while neglecting gravity, adopting the following method: Assume the Higgs field profile has the same shape and size it would have in the static limit, except that the magnitude of the field at asymptotic distance from the zero is much smaller. Under rigid motion of this structure (neglecting the gauge field contribution which presumably only is important for large Higgs field and stable monopole), what is the kinetic energy, and hence the inertial mass corresponding to motion of the coordinate $r_0$. This quantity is evaluated by substituting into the energy density expression given by (10) the form (24), and evaluating the integrated density as $m_0 (d\Phi_0/d\ell)^2/2$:

$$m_0 = \frac{\pi^2}{6} \left( \frac{\pi^2}{6} - 1 \right) \Phi_0^2 \ell. \quad (25)$$

Evidently for $\Phi_0 \ell \ll 1$ this gives rise to an uncertainty in position of the zero much greater than the size $\ell$ of the region in which the field departs from its temporarily small `vacuum' value. Even if one made the extreme assumption that there were only one scale in the problem, given by the Hubble length, still the uncertainty would be at least comparable with the size of the causally connected region.

A consequence of this proposed uncertainty in position is that the expectation value of $\Phi^2(x)$ will be nearly constant, and an appreciable fraction of $\Phi_0^2$ throughout the region, rather than vanishing at some point. Thus, if this view were correct, the evolution of $\langle \Phi^2(x) \rangle$ would be
It is well-known that in inflationary models based on supersymmetric (susy) extensions of the Standard Model the quartic self-coupling of the inflaton field can be strongly suppressed. (For an extensive recent review of susy inflationary models the reader is referred to [45].) With exact global supersymmetry, the potential is typically independent of some of the fields i.e., it has flat directions in field space. Due to the slow-roll condition for inflation, those fields are exactly the candidates for the inflaton field, and because of the flatness of the potential one has $\lambda = 0$, independently of whether supersymmetry is broken or unbroken and if broken whether spontaneously or softly. If susy is present in nature, it is expected to be local so as to accommodate gravity. That is, supersymmetry and gravity only can be reconciled in the form of supergravity. It is known that in supergravity the flatness of global susy is lifted: $\lambda$ is of the order of $M^2_s/m_{Pl}^2$, where $M_s$ is the supersymmetry breaking scale. If $M_s \ll m_{Pl}$, then $\lambda$ is strongly suppressed, though not identically zero. Hence in global or local susy inflation of the monopole core must depend on a different type of effective potential for the Higgs field from the ‘conventional’ one described here. The considerations above do not apply directly, but one would expect them to give a correct qualitative analysis of the phenomenon, most probably meaning no inflation.

Part of this qualitative reasoning is the following: In exact susy, one expects the BPS bound on the mass to be saturated, hence no inflation of the monopole core, indeed, no inflation at all. If there is some breaking, then there will be some potential for the Higgs field other than the $\phi^4$ term. In a homogeneous space this would be enough for inflation. However in the nontrivial background provided by an ’t Hooft-Polyakov monopole the phenomenological requirement that the couplings in this potential be small for compatibility with CMB data [44] implies that the $F^2$ term in $\lambda$ would dominate over the potential term, and hence inflation of the monopole core would be forbidden.

VII. CONCLUSIONS

We have seen that for observationally indicated values of $\lambda/e^2$ the gravitational mass density $\mu$ of an ’t Hooft-Polyakov monopole is everywhere positive, so that without some new factor there could not be monopole inflation. As mentioned, introducing gravity would only strengthen the result by spatially compressing the system, further increasing $\mu$, and eventually leading towards a Reissner-Nordstrom black hole, rather than inflation (again modulo the solutions of [28] which have no flat space-time limit). Having confirmed a simple-minded and intuitive expectation, we still should explore possible ways of resuscitating the notion. A first possibility is...
quantum fluctuations leading to an expanded monopole in which the gauge field energy is diluted. If indeed $\lambda/e^2$ is only somewhat smaller than unity, such fluctuations surely become plausible. A second possibility is two-stage inflation, where the first stage involves the Higgs field as inflaton field (with large $\lambda$, meaning no supersymmetry), possibly producing monopole core inflation, while the second involves a separate, non-gauge-coupled inflaton field with small $\lambda$ [14]. Clearly, this would make our connection to the core region of the monopole exceedingly indirect. It would mean that the center of the monopole would be exponentially large compared the size of our visible universe.

In summary, for a monopole in equilibrium without gravity, the slow or fast introduction of gravitational couplings could not precipitate inflation through the classical equations of motion unless the parameters were above the threshold values $(\lambda/e^2)_{\text{thresh}} \simeq 1$ and $\eta_{\text{thresh}} \simeq m_{\text{Pl}}$. These values are a bit higher than suggested by Sakai [17], and a numerical test has been proposed to verify the higher values. For parameters only a bit below the threshold values, quantum fluctuations still could initiate inflation. The singularity theorems mentioned earlier imply that, regardless of the values of the parameters, quantum fluctuations are essential to monopole core inflation. The main result of the classical computations is that not even quantum fluctuations could produce such a phenomenon unless the parameters $\lambda$ and $\eta$ both were sufficiently large.

For the starting conditions associated with a topological zero of a field which initially has very small magnitude everywhere in some region, it has been argued that quantizing the degree of freedom associated with the position of the zero implies effective decoupling of the zero from the spatial evolution. If so, by the time the zero is treatable as a classical coordinate, it likely will be the center of a monopole field configuration moderately close to equilibrium, and so could not inflate unless the model parameters were at least close to the threshold values. If they were close, then the eternal inflation associated with the zero would be in a sawtooth pattern, where inflation would start, the zero would eventually emerge in a noninflating region, inflation would reignite producing more topological zeros, and so on.

At first sight, the claim about the influence of quantum uncertainty sounds quite different from earlier work, including the numerical lattice studies of quantum fluctuations by LL [14], but a more helpful view of these very low-mass topological defect ‘embryos’ may be to say that not only their positions but also their numbers show large quantum fluctuations, as found in [14]. Thus the position uncertainty may be viewed as only one aspect of the rapid pair creation. This has interesting implications, including the ability of net topological charge to shift position faster than the speed of light. It makes a problem of which comes first: Small magnitude of $\Phi_0$ allowing copious production of topological zeros, or topological zeros plus inflation enforcing small $\Phi_0$ and hence continued inflation. Based on the quantum uncertainties, one can argue for the former as the dynamically significant statement. The results of LL quoted earlier indicate that the topological zeros at most give a modest quantitative enhancement to the maintenance of inflation. Even those results were for a large value of the self-coupling $\lambda$, where as noted earlier one already would expect a monopole in equilibrium to inflate once coupling to gravity is included. A numerical test has been proposed to check whether, even for the configurations with topological zeros, the topology is more than an accidental aspect of the fact that the Higgs field comes close to a local maximum in the Higgs potential. Boubekeur and Lyth [11] have argued that almost all viable models of inflation involve starting from a ‘hilltop’, that is, the near neighborhood of a rather flat maximum in the inflaton potential. For this purpose it shouldn’t matter whether at some point the potential actually achieved its maximum, which for a locally analytic solution would mean a topological zero.

If supersymmetry were broken only at low scales, then monopole inflation from equilibrium would be strongly suppressed, and by the above argument a topological zero of the Higgs field no longer should be expected to generate inflation. Thus, if this understanding of the implications of quantum physics for zeros of a Higgs field is correct, and if phenomenologically indicated parameters of inflationary cosmology should be accepted at face value, then monopole topological inflation, while conceptually instructive and appealing, does not provide a compelling alternative or even a significant enhancement to chaotic eternal inflation [11]. However, domain wall inflation in a theory with a single inflaton field remains a viable and important mechanism for eternal inflation. In this case there is no static-gradient positive contribution to the gravitational mass density to compensate for the negative density coming from the scalar field potential. Further, if the domain wall achieves large area, it also achieves large inertial mass, so that the argument about quantum uncertainty in its position disappears. By the same token it becomes harder to envision copious pair creation of walls and antiwalls.

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