Complete Cyclic Proof Systems for Inductive Entailments

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Abstract. In this paper we develop cyclic proof systems for the problem of inclusion between the least sets of models of mutually recursive predicates, when the ground constraints in the inductive definitions belong to the quantifier-free fragments of (i) First Order Logic with the canonical Herbrand interpretation and (ii) Separation Logic, respectively. Inspired by classical automata-theoretic techniques of proving language inclusion between tree automata, we give a small set of inference rules, that are proved to be sound and complete, under certain semantic restrictions, involving the set of constraints in the inductive system. Moreover, we investigate the decidability and computational complexity of these restrictions for all the logical fragments considered and provide a proof search semi-algorithm that becomes a decision procedure for the entailment problem, for those systems that fulfill the restrictions.

1 Introduction

Inductive definitions play an important role in computing, being an essential component of the syntax and semantics of programming languages, databases, automated reasoning and program verification systems. The main advantage of using inductive definitions is the ability of reasoning about sets of logical objects, by means of recursion. The semantics of these definitions is defined in terms of least fixed points of higher-order functions on assignments of predicates to sets of models. A natural problem is the entailment, that asks whether the least solution of one predicate is included in the least solution of another. Examples of entailments are language inclusion between finite-state (tree) automata or context-free grammars, or verification conditions generated by shape analysis tools using specifications of recursive data structures as contracts of program correctness.

The principle of Infinite Descent [Bus18], formalized by Fermat, has become essential to reasoning about entailments between inductively defined predicates. A proof by infinite descent is a particular kind of proof by contradiction. We assume the existence of a counterexample from a well-founded domain and show that this leads to the existence of a strictly smaller counterexample for the same entailment problem. It is possible to infinitely repeat this step to obtain a chain of strictly smaller counterexamples. But, since the domain of counterexamples does not admit infinite descending chains, we reach a contradiction that indicates
there was actually no counterexample to start with. Cyclic proofs reflect this principle by inference rules of the sequent calculus \[BS11\].

The interest for automatic proof generation is two-fold. On one hand, machine-checkable proofs are certificates for the correctness of the answer given by an automated checker, that increase our trust in the reliability of a particular implementation \[SOR^+13\]. For instance, a language emptiness or inclusion checker for finite-state automata can either answer “no” and provide a counterexample (a finite word or tree) or it will answer “yes” and provide a proof certificate. Unfortunately, the generation of proofs for inclusion has received very little attention from the language-theoretic research community, unlike the generation of counterexamples \[WDHR06,HLSV11\].

On the other hand, the existence of a sound and complete proof system provides a (theoretical) decision procedure for the entailment problem, based on the following argument. Assuming that the sets of models and derivations are both recursively enumerable, one can interleave the enumeration of counter-models with the enumeration of derivations; if the entailment holds one finds a proof (provided that the proof system is complete), or a counterexample, otherwise. Proof search can be made effective by providing suitable strategies that limit the possibilities of applying the inference rules and guide the search towards finding a proof or a counterexample.

In this paper we give a set of inference rule types that are shown to be sound and complete for entailments in an inductive system \(S\), provided that the set of constraints of \(S\) meets several restrictions. We provide answers to the more general question: under which set of conditions \(C\) does there exist a complete set of inference rules for the entailment problems in a system \(S\), when the constraints of \(S\) comply with \(C\) ? In general, most authors define \(C\) by imposing syntactic restrictions on the logical fragment in which the constraints are written. In contrast, here we consider rather general logics and define \(C\) by a number of semantic conditions, which can be checked by existing decision procedures for the underlying logic.

**Tree Automata Inclusion as Cyclic Proof Search** We assume basic knowledge of tree automata \[CDG^+05\] and consider top-down nondeterministic tree automata (TA) that label trees with states, starting at the root and moving downwards. The actions of the TA are described by a finite number of transition rules of the form \(q \xrightarrow{f} (q_1, \ldots, q_n)\) with the following meaning: when the TA is at a node labeled with a function symbol \(f\) and its current state is \(q\), then it moves downwards and labels its children with states \(q_1, \ldots, q_n\), respectively. The TA accepts a tree when it can label all leaves with final states. The language inclusion problem \(L(p) \subseteq L(q)\) asks, given states \(p\) and \(q\) of the same TA, is every tree accepted starting with \(p\) also accepted starting with \(q\)?

A TA can be naturally viewed as a system of inductive definitions, associating states with predicates and transition rules with rules of the system. For instance, the transition rule \(f(q_1, \ldots, q_n) \rightarrow q\) is equivalent to the definition \(q(x) \leftarrow x \approx f(x_1, \ldots, x_n), q_1(x_1), \ldots, q_n(x_n)\) with the intended meaning that \(x\) denotes a tree
whose root is labeled by \( f \) and if the automaton has labeled \( x \) with state \( q \) then it can label its subtrees \( x_1, \ldots, x_n \) with \( q_1, \ldots, q_n \), respectively. The variables range over ground terms and the function symbols are interpreted in the canonical (Herbrand) sense. An entailment problem \( p \models q \) that asks whether every tree model of a predicate \( p \) is also a model of the predicate \( q \) can be seen as a language inclusion problem \( \mathcal{L}(p) \subseteq \mathcal{L}(q) \).

Since language inclusion is decidable for TA, we can obtain a complete set of inference rules and a proof search algorithm, by leveraging from an existing algorithm for language inclusion problem \( \mathcal{L}(p) \subseteq \mathcal{L}(q) \) \cite{HLSV11}. This algorithm enumerates pairs \( (p, \{q_1, \ldots, q_n\}) \), where \( p, q_1, \ldots, q_n \) are states of the given TA, with the following meaning: there exists a tree \( t \) and a node \( u \) of \( t \) which is labeled with \( p \) when the TA is started in state \( r \), and at the same time, \( q_1, \ldots, q_n \) is the set of possible labels of \( u \) in \( t \) when the TA starts in \( s \). Then \( \mathcal{L}(r) \nsubseteq \mathcal{L}(s) \) if and only if the search encounters a pair \( (p, \{q_1, \ldots, q_n\}) \) where \( p \) is a final state and none of \( q_1, \ldots, q_n \) are final.

The main idea of the proof systems in this paper is to view a (complete counterexample-free) search tree for language inclusion between top-down tree automata \cite{HLSV11} as a proof of the validity of an entailment between two predicates in a system \( S \). In a nutshell, we view the pairs \( (p, \{q_1, \ldots, q_n\}) \) explored by the language inclusion algorithm as sequents \( p(x) \vdash q_1(x), \ldots, q_n(x) \) in the proof and apply the principle of Infinite Descent to close those branches of the proof leading to an infinite sequence of strictly smaller counterexamples.

Related Work \cite{BS11} describe the cyclic proof system \( \text{CLKID}^\omega \), in which the inductive arguments are discovered during the construction of the proof, as opposed to the more traditional inductive proof system \( \text{LKID} \), in which the inductive invariant must be provided. The \( \text{CLKID}^\omega \) system is conjectured complete w.r.t. Henkin models. A sound variant of this system for Separation Logic is given in \cite{BDP11}, for symbolic heap constraints. Our approach builds upon their work, using a similar Infinite Descent rule type, relying on several semantic conditions delimiting the class of inductive systems for which our proof system is complete. Moreover, we consider more general Separation Logic formulae than symbolic heaps.

\cite{GM17} present a labeled sequent calculus that supports arbitrary inductive predicates, with Separation Logic constraints using the separating implication connective \cite{Rey02}. To our knowledge this is the first proof system that supports all connectives of Separation Logic, but unfortunately, no proof of completeness is provided. We chose not to include the separating implication (magic wand) in our systems because the Bernays-Schönfinkel-Ramsey fragment of Separation Logic with this connective appears to be undecidable \cite{RIS17}, which prevents us from effectively checking our sufficient conditions for completeness.

\cite{CJT15} propose a proof system for Separation Logic that extends the basic cyclic proof method with a cut rule type that uses previously encountered sequents as inductive hypothesis and applies them by matching and replacing the left with the right-hand side of such a hypothesis. This method can prove entail-
ments between predicates whose coverage trees differ, but again, only soundness is guaranteed. It remains an open question for which class of entailment problems this type of cut rules yields a complete proof system. An automata-based decision procedure that tackles such entailments is given in [IRV14]. This method translates the entailment problem to a language inclusion between tree automata and uses a closure operation on automata to match divergent predicates. Unlike proof search, this method uses existing tree automata inclusion algorithms, which do not produce proof witnesses.

In a different vein, [ESW15] give a proof system that uses automatic generation of concatenation lemmas for inductively defined Separation Logic predicates. Their system is sound and moreover, most concatenation lemmas can be shown to have a cyclic proof in our system, whereas completeness remains still an open question. Further, [TLKC16] describes mutual explicit induction proofs, an induction method based on a well-founded order on Separation Logic models. Akin to [ESW15], this method considers symbolic heaps extended with constraints on the data values stored within the heap structures. In a similar fashion with [CJT15], they keep a vault with hypotheses, which are marked as valid or unknown. Valid hypotheses can be freely applied, but unknown hypotheses are only applied if certain side conditions, which ensure a decrease in the size of the heap model, are satisfied. This approach allows the hypotheses to be used anywhere in the proof tree, and are not restricted to the branch from which they originated. This method is sound but no completeness arguments are given.

2 Preliminaries

For two integers \(0 \leq i \leq j\), we denote by \([i,j]\) the set \(\{i,i+1,\ldots,j\}\) and by \([i]\) the set \([1,i]\), where \([0]\) is the empty set. Given a finite set \(S\), \(|S|\) denotes its cardinality, \(\mathcal{P}(S)\) the powerset and \(\mathcal{P}_\leq(S)\) the set of finite subsets of \(S\).

A signature \(\Sigma = (\Sigma^s, \Sigma^f)\) consists of a set \(\Sigma^s\) of sort symbols and a set \(\Sigma^f\) of function symbols \(f^\sigma_1\cdots\sigma_n\sigma\), where \(n \geq 0\) is its arity, \(\sigma_1,\ldots,\sigma_n \in \Sigma^s\) are the sorts of its arguments and \(\sigma \in \Sigma^s\) is the sort of its result. If \(n = 0\), we call \(f^\sigma\) a constant symbol. We assume that every signature contains the boolean sort, and write \(\top\) and \(\bot\) for the boolean constants true and false. Let \(\text{Var}\) be a countable set of first-order variables, each variable \(x^\sigma \in \text{Var}\) having an associated sort \(\sigma\). We omit specifying the sorts of the function symbols and variables, whenever they are not important.

Terms are defined recursively: any constant symbol or variable is a term, and if \(t_1,\ldots,t_n\) are terms of sorts \(\sigma_1,\ldots,\sigma_n\), respectively, and \(f^\sigma_1\cdots\sigma_n\sigma \in \Sigma^f\), then \(t = f(t_1,\ldots,t_n)\) is a term of sort \(\sigma\), denoted \(t^\sigma\). We denote by \(T_\Sigma(x)\) the set of terms with function symbols in \(\Sigma^f\) and variables in \(x\), and we write \(T_\Sigma\) for the set \(T_\Sigma(\emptyset)\) of ground terms, in which no variable occurs.

Formulae are also defined recursively: a term of boolean sort, an equality \(t \approx u\), where \(t\) and \(u\) are terms of the same sort, are formulae, and each quantified boolean combination of formulae is a formula. For a formula \(\phi\) (set of formulae \(\mathcal{F}\)), we denote by \(\text{FV}(\phi)\) the set of variables not occurring under the scope
of a quantifier, and by writing \( \varphi(x) \) \((\mathcal{F}(x))\), we mean that \( x \subseteq \text{FV}(\varphi) \) \((x \subseteq \bigcup_{\phi \in \mathcal{F}} \text{FV}(\phi))\). The size of a formula \( \phi \) is the number of symbols occurring in it.

Given sets of variables \( x \) and \( y \), a substitution \( \theta : x \rightarrow T(x) \) is a mapping of the variables in \( x \) to terms in \( T(y) \). For a set of variables \( x \) we denote \( x\theta = \{ \theta(x) \mid x \in x \} \) its image under the substitution \( \theta \). A substitution \( \theta \) is flat if \( \text{Var} \theta \subseteq \text{Var} \), i.e. each variable is mapped to a variable. For a formula \( \phi(x) \), we denote by \( \phi\theta \) the formula obtained by replacing each occurrence of \( x \in x \) with the term \( \theta(x) \), and lift this notation to sets as \( \mathcal{F}\theta = \{ \phi\theta \mid \phi \in \mathcal{F} \} \).

An interpretation \( \mathcal{I} \) maps each sort symbol \( \sigma \in \Sigma^* \) to a non-empty set \( \sigma^\mathcal{I} \), each function symbol \( f^{\sigma_1 \cdots \sigma_n} \in \Sigma \) to a total function \( f^\mathcal{I} : \sigma_1^\mathcal{I} \times \cdots \times \sigma_n^\mathcal{I} \rightarrow \sigma^\mathcal{I} \) where \( n > 0 \), and to an element of \( \sigma^\mathcal{I} \) when \( n = 0 \). Given an interpretation \( \mathcal{I} \), a valuation \( \nu \) maps each variable \( x^\mathcal{I} \in \text{Var} \) to an element of \( \sigma^\mathcal{I} \). For a term \( t \), we denote by \( t^\mathcal{I} \) the value obtained by replacing each function symbol \( f \) by its interpretation \( f^\mathcal{I} \) and each variable \( x \) by its valuation \( \nu(x) \). For a quantifier-free formula \( \phi \), we write \( \mathcal{I}, \nu \models \phi \) if the formula obtained by replacing each term \( t \) in \( \phi \) by the value \( t^\mathcal{I} \) is equivalent to true. The semantics of first-order quantifiers is defined as: \( \mathcal{I}, \nu \models \exists x^\mathcal{I} . \phi(x) \) iff \( \mathcal{I}, \nu[x \leftarrow \alpha] \models \phi \), for some value \( \alpha \in \sigma^\mathcal{I} \), where \( \nu[x \leftarrow \alpha] \) is the same as \( \nu \), except for \( \nu[x \leftarrow \alpha](x) = \alpha \).

A formula \( \phi \) is satisfiable in the interpretation \( \mathcal{I} \) if there exists a valuation \( \nu \) such that \( \mathcal{I}, \nu \models \phi \). Given formulae \( \phi \) and \( \psi \), we say that \( \phi \) entails \( \psi \), denoted \( \phi \models \psi \) if \( \mathcal{I}, \nu \models \phi \) implies \( \mathcal{I}, \nu \models \psi \), for each valuation \( \nu \).

### 2.1 Systems of Inductive Definitions

Let \( \text{Pred} \) be a countable set of predicates, each \( p^{\sigma_1 \cdots \sigma_n} \in \text{Pred} \) having an associated tuple of argument sorts. Given a tuple of terms \( (t_1^\mathcal{I}, \ldots, t_n^\mathcal{I}) \), a predicate atom is \( p(t_1, \ldots, t_n) \). A rule is a pair \( \{ \{ \phi(x, x_1, \ldots, x_n), q_1(x_1), \ldots, q_n(x_n) \} \} \), \( p(x) \), where \( x, x_1, \ldots, x_n \) are pairwise disjoint sets of variables, \( \phi \) is a formula, called the constraint, \( p(x) \) is a predicate atom called the goal and \( q_1(x_1), \ldots, q_n(x_n) \) are predicate atoms called subgoals. The variables \( x \) are the goal variables, whereas the ones in \( \bigcup_{i=1}^n x_i \) are the subgoal variables of the rule. A system \( S \) is a finite set of rules. We assume w.l.o.g. that there are no goals with the same predicate and different goal variables, and write \( p(x) \leftarrow S R_1 \mid \ldots \mid R_m \) when \( \{ \langle R_1, p(x) \rangle, \ldots, \langle R_m, p(x) \rangle \} \) is the set of rules with goal \( p(x) \) in \( S \). The size of \( S \) is the sum of the sizes of all constraints occurring in \( S \).

**Example 1.** Consider the following inductive system of predicates:

\[
\begin{align*}
p(x) &\leftarrow_S x \approx f(x_1, x_2), p_1(x_1), p_2(x_2) \quad g(x) &\leftarrow_S x \approx f(x_1, x_2), q_1(x_1), q_2(x_2) \\
p_1(x) &\leftarrow_S x \approx g(x_1), p_1(x_1) \mid x \approx a &\quad q_1(x) &\leftarrow_S x \approx g(x_1), q_1(x_1) \mid x \approx a \\
p_2(x) &\leftarrow_S x \approx g(x_1), p_2(x_1) \mid x \approx b &\quad q_2(x) &\leftarrow_S x \approx g(x_1), q_2(x_1) \mid x \approx b
\end{align*}
\]

Intuitively, \( S \) models two tree automata with final states given by the predicates \( p \) and \( q \), where \( p \) accepts trees of the form \( f(g(\ldots g(a))), g(\ldots g(b))) \), while \( q \) accepts trees of the form \( f(g(\ldots g(a))), g(\ldots g(b))) \), and \( f(g(\ldots g(b))), g(\ldots g(a))) \).

\( \square \)
Given a system \( S \) and an interpretation \( I \) of the sorts and function symbols in \( S \), an assignment \( \mathcal{X} \) maps each predicate \( p^{\sigma_1\ldots\sigma_n} \in \text{Pred} \) to a set \( \mathcal{X}(p) \subseteq \sigma_1^T \times \ldots \times \sigma_n^T \). By a slight abuse of notation, we lift assignments from predicates to formulae, where for each predicate atom \( p(t_1,\ldots,t_n) \), we define \( \mathcal{X}(p(t_1,\ldots,t_n)) = \{ \nu | ⟨(t_1)^I,\ldots,(t_n)^I⟩ \in \mathcal{X}(p) \} \) and, for an arbitrary formula \( \phi \), we define \( \mathcal{X}(\phi) \) inductively on the structure of \( \phi \).

The system \( S \) and the interpretation \( I \) induce a function \( \mu_S^I(\mathcal{X}) \) on assignments, which maps each predicate \( p(x) \in \text{Pred} \) into the set \( \bigcup_{m=1}^{m} \{ \nu(x) | \nu \in \mathcal{X}(R_i) \} \), where \( p(x) \models_S R_1 | \ldots | R_m \) and, for a tuple of variables \( \mathbf{x} = (x_1,\ldots,x_k) \), we write \( \nu(x) \) for the tuple of values \( (\nu(x_1),\ldots,\nu(x_k)) \). A solution of \( S \) is an assignment \( \mathcal{X} \) such that \( \mu_S^I(\mathcal{X}) \subseteq \mathcal{X} \), where inclusion between assignments is defined pointwise. It can be easily shown that the set of all assignments, together with the \( \subseteq \) relation, is a complete lattice, since any powerset equipped with the subset relation is a complete lattice. Because \( \mu_S^I \) is provably monotone, it follows from Tarski’s theorem [Tar55] that \( \mu_S^I = \bigcap \{ \mathcal{X} | \mu_S^I(\mathcal{X}) \subseteq \mathcal{X} \} \) is the least fixpoint of \( \mu_S^I \) and the least solution of \( S \).

In this paper we are concerned with the following entailment problem: given a system \( S \), an interpretation \( I \), and two predicates \( p^{\sigma_1\ldots\sigma_n} \) and \( q^{\rho_1\ldots\rho_m} \), having the same tuple of argument sorts, does \( \mu_S^I(p) \subseteq \mu_S^I(q) \)? In the rest of the paper, we denote entailment problems as \( p \models_S^I q \). Moreover, we shall write \( \phi \models_S^I \psi \) for \( \mu_S^I(\phi) \subseteq \mu_S^I(\psi) \), when \( \phi \) and \( \psi \) are arbitrary formulae.

We consider only sets of constraints with a decidable satisfiability problem. Unless mentioned otherwise, we consider that each constraint is a quantifier-free formula in which no disjunction occurs positively and no conjunction occurs negatively. Disjunctions can be eliminated w.l.o.g. from quantifier-free constraints, by splitting each rule \( ⟨\{\phi_1 \lor \ldots \lor \phi_m, q_1(x_1),\ldots,q_n(x_n)\},p(x)⟩ \) into \( m \) rules \( ⟨\{\phi_i, q_1(x_1),\ldots,q_n(x_n)\},p(x)⟩ \), one for each \( i \in [m] \). Finally, we assume that each predicate \( p \in \text{Pred} \) is the goal of at least one rule of \( S \). Predicates \( p \) that are not the goal of a rule will have empty least solutions, i.e. \( \mu_S^I(p) = \emptyset \), thus each rule containing a subgoal that is not the goal of a rule can be safely removed from the system.

Example 2. For the inductive system in Example 1:

\[
\begin{align*}
\mu_S^I(p) &= \{ f(g^n(a),g^m(b)) | n,m \geq 0 \} \\
\mu_S^I(p_1) &= \mu_S^I(q_1) = \{ g^n(a) | n \geq 0 \} \\
\mu_S^I(p_2) &= \mu_S^I(q_2) = \{ g^n(b) | n \geq 0 \} \\
\mu_S^I(q) &= \{ f(g^n(a),g^m(b)) | n,m \geq 0 \} \cup \{ f(g^n(b),g^m(a)) | n,m \geq 0 \}
\end{align*}
\]

Since \( \mu_S^I(p) \subseteq \mu_S^I(q) \), it follows that the entailment \( p \models_S^I q \) holds. In other words, the language accepted by the state represented as the predicate \( p \) is included in the language of the state represented by \( q \).

### 2.2 Well Quasi Orders

Given a set \( D \), a quasi-order (qo) is a reflexive and transitive relation \( \preceq \subseteq D \times D \). An infinite sequence \( d_1,d_2,\ldots \) from \( D \) is saturating if \( d_i \preceq d_j \) for some
$i < j$. A quasi-order $\preceq$ is a well-quasi-order (wqo) if every infinite sequence is saturating. A quasi-order $\preceq$ is well-founded (wfqo) iff there are no infinite decreasing sequences $d_1 > d_2 > \ldots$ Every wqo is well-founded, but not vice versa.

We extend any wqo $(D, \preceq)$ to the following order on the set of finite subsets of $D$. For all finite sets $S, T \in P_n(D)$, we have $S \preceq^D 3 T$ if and only if for all $a \in S$ there exists $b \in T$ such that $a \preceq b$. The following is a consequence of Higman’s Lemma [Hig52]:

**Lemma 1.** If $D$ is countable and $(D, \preceq)$ is a wqo, then $(P_n(D), \preceq^D 3 )$ is a wqo.

A multiset over $D$ is a mapping $M : D \to \N$. The multiset $M$ is finite if $M(d) > 0$ for a finite number of elements $d \in D$. We denote by $M(D)$ the set of finite multisets over $D$, and lift the operations of subset, union, intersection and difference to multisets, as usual. The multiset order induced by $\preceq$ is defined as in [DM79]. We write $N \preceq^ D M$ if and only if either $M = N$, or there exists a non-empty finite multiset $X \subseteq M$ and a (possibly empty) multiset $Y$, such that $Y \preceq X$ and $N = (M \setminus X) \cup Y$. Roughly, $N$ is obtained by replacing a non-empty submultiset of $M$ with a possibly empty multiset of strictly smaller elements. The following theorem was proved in [DM79]:

**Theorem 1.** $(M(D), \preceq^ D )$ is a wfqo if and only if $(D, \preceq)$ is a wfqo.

### 2.3 Canonical Interpretation

Let $\N^*$ be the set of sequences of natural numbers, $\varepsilon \in \N^*$ be the empty sequence, and $pq$ denote the concatenation of two sequences $p, q \in \N^*$. We say that $p$ is a prefix of $q$ iff $pr = q$, for some $r \in \N^*$. A set $X \subseteq \N^*$ is prefix-closed if $p \in X$ implies that every prefix of $p$ is in $X$. A tree over the signature $\Sigma = (\Sigma^t, \Sigma^f)$ is a ground term $t \in T_\Sigma$, viewed as a finite partial function $t : \N^* \to fin \Sigma^f$ such that $\text{dom}(t)$ is prefix-closed and, for all $p \in \text{dom}(t)$, such that $t(p) = f^{\sigma_1 \ldots \sigma_n}$, we have \{ $i \in \N \mid \pi \in \text{dom}(t)$ \} = $[n]$. We denote by $\text{fr}(t) = \{ p \in \text{dom}(t) \mid \pi \notin \text{dom}(t) \}$ the frontier of $t$. Given a tree $t$ and a position $p \in \text{dom}(t)$, we denote by $t_p$ the subtree of $t$ rooted at $p$, where, for each $q \in \N^*$, we have $t_{p_q}(q) = t(pq)$. The subtree order is defined by $u \subseteq t$ iff $u = t_p$, for some $p \in \text{dom}(t)$. It is easy to see that $(T_\Sigma, \subseteq)$ is a wfqo, because $T_\Sigma$ consists only of finite trees, making it impossible to build an infinite strictly decreasing sequence of subtrees.

A context is a tree $t_p$ that has a single position $p \in \text{fr}(t)$ labeled with a special symbol $\Box$, with no successors. When there is no risk for confusion, we denote by $t_p$ the context obtained from a tree $t$ by placing $\Box$ at the position $p \in \text{fr}(t)$. We sometimes write $t_{p_i}$ instead of $t_{p_i}$, when $p$ is not important, and $\langle \rangle$ for $t_{\varepsilon}$. The result of the concatenation $t_p \o u$ (or $t_{\langle \rangle} \o u$ when $p$ is not important) is the tree obtained from $t$, by replacing the $\Box$ symbol by $u$.

For a function symbol $f^{\sigma_1 \ldots \sigma_n} \in \Sigma^f$ and trees $t_1, \ldots, t_n \in T_\Sigma$, let $\tau_n(f, t_1, \ldots, t_n)$ be the tree $t$ such that $t(\varepsilon) = f$ and $t_{p_i} = t_i$, for all $i \in [n]$. The Herbrand
(canonical) interpretation $\mathcal{H}$ maps each sort $\sigma \in \Sigma^s$ into $T_\Sigma$, each constant symbol $c$ into the tree $e^c = \{(\varepsilon, c)\}$ consisting of a leaf which is also the root, and each function symbol $f^{\sigma_1,\ldots,\sigma_n}$ into the function $f^\mathcal{H}$ mapping each tuple of trees $t_1, \ldots, t_n$ into $\tau_n(f, t_1, \ldots, t_n)$. Even in this simple case, where function symbols do not have any equational properties (e.g. commutativity, associativity, etc.) entailment problems are undecidable, as stated by the following theorem:

**Theorem 2.** The entailment problem is undecidable in the Herbrand interpretation.

This negative result excludes the possibility of having a complete proof system for solving entailments between predicates of inductive systems using (unrestricted) first-order logic constraints, under the canonical interpretation. A possible workaround is to restrict the class of systems considered, by imposing several semantic restrictions on the set of constraints that occur within the rules of the system.

### 3 Cyclic Proofs for First Order Entailments

As usual, we denote *sequents* as $\Gamma \vdash \Delta$, where $\Gamma$ and $\Delta$ are sets of formulae. The comma in the sequents are read as set union, thus contraction rules are not necessary. A singleton $\{p(x)\}$ is denoted as $p(x)$, and a sequent of the form $p(x) \vdash q_1(x), \ldots, q_n(x)$ is *basic*. An inference rule is of the form:

\[
\frac{\begin{array}{c}
\Gamma_1 \vdash \Delta_1 \\
\vdots \\
\Gamma_n \vdash \Delta_n
\end{array}}{
\Gamma \vdash \Delta}
\]

(\text{R})

\[
\Gamma_p \vdash \Delta_p
\]

where $\Gamma_i \vdash \Delta_i$ are the antecedents, $\Gamma \vdash \Delta$ is the consequent and $\Gamma_p \vdash \Delta_p$ is the pivot of the rule and $C$ is a pivot constraint on the path between the pivot and the consequent. For conciseness reasons, we give rule types that are infinite sets of inference rules (instances) that share the same structure. Without loss of generality, we assume that, given sequents $\Gamma_p \vdash \Delta_p$ and $\Gamma \vdash \Delta$, there are only finitely many instances of a certain rule type, with pivot $\Gamma_p \vdash \Delta_p$ and consequent $\Gamma \vdash \Delta$. We denote by $\#(R)$ the number of antecedents of the rule type $R$ and write $\top$ for the antecedent list whenever $\#(R) = 0$.

A derivation is a (possibly infinite) tree in which the children of each node labeled by $\Gamma \vdash \Delta$ are antecedents of an inference rule. Given a path $\pi$ in a derivation, we denote by $A(\pi)$ the sequence of types of the inference rules that have been applied on $\pi$. The pivot constraint is a set of allowed sequences of rule types that label the path between the pivot and the consequent in the derivation.

A proof is a finite derivation whose leaves are all labeled with $\top$ (true). Given $\mathcal{S}$ and $\mathcal{I}$, a set of inference rules is sound if, for every sequent $p(x) \vdash q(x)$ labeling the root of a proof, the entailment $p \models_\mathcal{S} q$ holds, and complete if for every valid entailment $p \models_\mathcal{I} q$ there is a proof starting with $p(x) \vdash q(x)$. A strategy is a set
of sequences $S$ of inference rule types. A derivation (proof) $D$ is an $S$-derivation (S-proof) if the sequence of inference rules along each path in $D$ belongs to $S$.

Algorithm 1 Proof search semi-algorithm.

| data structure: Node$(\Gamma \vdash \Delta, CList, P, R)$, where $\Gamma \vdash \Delta$ is the sequent that labels the node, $CList$ is the list of children nodes, $P$ is a link to the parent of the node in the proof and $R$ is the type of the inference rule with consequent $\Gamma \vdash \Delta$. |
|---|
| input: a system $S$, a sequent $p(x) \vdash q(x)$, a set $R$ of inference rule types and a strategy $S$ |
| output: a proof of $p(x) \vdash q(x)$ |

1. $\text{Root} \leftarrow \text{Node}(p(x) \vdash q(x), [], \text{nil}, \text{nil})$
2. $\text{WorkList} \leftarrow \{\text{Root}\}$
3. while $\text{WorkList} \neq \emptyset$ do
4. remove $N$ from $\text{WorkList}$
5. let $\pi$ be the path between $\text{Root}$ and $N$
6. if $\Lambda(\pi) \in S$ then
7. match $N$ with $\text{Node}(\Gamma \vdash \Delta, CList, P, R)$
8. for each instance of $R \in R$ such that $\#(R) = 0$, whose consequent matches $\Gamma \vdash \Delta$ do
9. let $\Gamma_0 = \Delta_0$ be the pivot and $C$ the pivot constraint of $R$
10. match $\pi$ with $\rho \cdot \text{Node}(\Gamma_p \vdash \Delta_p, CList_p, P_p, R_p)$ do $\cdot \mu \cdot N$
11. if $\Lambda(\mu) \in C$ then
12. mark $N$ as closed
13. if $N$ not closed and the consequent of some instance $r$ of $R \in R$ matches $\Gamma \vdash \Delta$ then
14. for each antecedent $\Gamma' \vdash \Delta'$ of $r$ do
15. let $N' \leftarrow \text{Node}(\Gamma' \vdash \Delta', [], N, R)$
16. add $N'$ to $CList$ and $\text{WorkList}$

Given an input sequent $p(x) \vdash q(x)$, a set $R$ of inference rule types and a strategy $S$, the proof search semi-algorithm (1) uses a worklist iteration to build a derivation of $p(x) \vdash q(x)$. When a sequent is removed from the worklist, it chooses (non-deterministically) an inference rule and an instance whose consequent matches the current sequent, if one exists. In order to speed up termination, the nodes matching a rule with zero antecedent are considered eagerly (line 8). It is manifest that there exists a finite execution of the semi-algorithm (1) leading to a proof of $p(x) \vdash q(x)$ if such a proof exists. Moreover, we shall chose the strategy $S$ in a way that forbids infinite derivations, thus turning (1) into an algorithm. The algorithm is furthermore a decision procedure if the input entailment admits a complete set of inference rules.

3.1 Restricting the Set of Constraints

The following definitions introduce sufficient conditions that ensure the existence of a complete proof search algorithm using the inference rules given in §3.2. These definitions are not bound to a particular logic or interpretation, and will be extended to logics other than multisorted first-order logic, such as Separation Logic [Rey02] (§4).

Definition 1. Given an interpretation $\mathcal{I}$ and a system $S$, a rule $(\{\phi(x, x_1, \ldots, x_n), q_1(x_1), \ldots, q_n(x_n)\}, p(x)) \in S$ is non-filtering if and only if, for all $i \in [n]$ and $v_i \in \mu \mathcal{I}(q_i)$, there exists a valuation $\nu$ such that $\nu(x_i) = v_i$ and $\nu \models^\mathcal{I} \phi$. The system $S$ is non-filtering if and only if each rule in $S$ is non-filtering.
Example 3. The system in Example 1 is non-filtering. If we added the rule \( \{ x \approx f(x_1, x_2) \land x_1 \approx x_2, p_1(x_1), p_2(x_2) \} \), we would break this restriction, because it rejects all subgoals models assigning different values to \( x_1 \) and \( x_2 \).

Lifting this restriction leads, in general, to the undecidability of the entailment problem, as it is the case for tree automata with equality and disequality constraints\(^2\). Moreover, checking whether a given system is non-filtering is undecidable, as shown by the following lemma:

Lemma 2. It is undecidable if a system is non-filtering, in the Herbrand interpretation.

Due to this negative result, we adopt a stronger (sufficient) condition, which requires that \( \forall x_1 \ldots \forall x_n \exists x . \phi(x, x_1, \ldots, x_n) \) holds, for each constraint \( \phi \) in the system. For instance, in the canonical Herbrand interpretation, the latter problem becomes decidable, because each constraint \( \phi \) is a conjunction of equalities \( s \approx t \) and disequalities \( s \approx t \) between terms over the variables \( x \cup \bigcup_{i=1}^n x_i \). Establishing the validity of \( \forall x_1 \ldots \forall x_n \exists x . \phi(x, x_1, \ldots, x_n) \) reduces to checking the unsatisfiability of the equational problem \( \exists x_1 \ldots \exists x_n \forall x . \neg \phi(x, x_1, \ldots, x_n) \).

Because we assumed that constraints do not contain disjunctions, \( \neg \phi \) is a disjunction of equalities and disequalities, thus it is trivially in conjunctive normal form. Since satisfiability of the formulae \( \exists x \forall y . \phi(x, y) \), with \( \phi \) in conjunctive normal form, is \( \text{NP-complete} \)\(^3\) [Pic03], our validity problem is in \( \text{co-NP} \).

The second restriction guarantees that the principle of Infinite Descent [Bus18] can be applied to close a branch of the proof tree. To simplify the presentation, we fix an interpretation \( I \) and a wfqo \( (D, \preceq) \), such that \( \sigma^2 = D \) for every sort \( \sigma \in \Sigma^n \). The definition below can be easily extended for the case when each sort is interpreted as a separated wfqo.

Definition 2. A system \( S \) is ranked iff for every constraint \( \phi(x, x_1, \ldots, x_n) \) in \( S \), with goal variables \( x \) and subgoal variables \( \bigcup_{i=1}^n x_i \) and every valuation \( \nu \), such that \( I, \nu \models \phi \), for all \( y \in \bigcup_{i=1}^n x_i \), there exists \( x \in x \) such that \( \nu(y) \prec \nu(x) \).

Example 4. The system from Example 3 is ranked, because the only constraints involving subgoal variables are(i) \( x \approx f(x_1, x_2) \) and (ii) \( x \approx g(x_1) \) and for each valuation \( \nu \) we have \( \nu(x_1) \subseteq \nu(x) \) and \( \nu(x_2) \cup \nu(x) \), if \( \nu \) satisfies the constraint (i) and \( \nu(x_1) \cup \nu(x) \), if \( \nu \) satisfies the constraint (ii), where \( \subseteq \) is the subtree relation described in section 2.3.

Considering again systems whose constraints are interpreted in the canonical Herbrand interpretation, it is natural to ask whether a given system is ranked in the subtree order \( (T_\Sigma, \sqsubseteq) \). Since the satisfiability of the quantifier-free fragment of the first order logic with a binary relation symbol interpreted as the subterm relation is an \( \text{NP-complete} \) problem [Ven87], one can effectively decide if a given system is ranked. For each constraint \( \phi(x, x_1, \ldots, x_n) \) we check if the formula

\(^2\) See [CDG+05, Theorem 4.2.10].
\(^3\) See [Pic03, Theorem 5.2].
The third restriction guarantees that all constraints can be eliminated from a sequent, by instantiating the subgoal variables on the right hand side using finitely many substitutions, that map into the subgoal variables from the left hand side. Essentially, if the constraints \( \phi(x, x_1, \ldots, x_n) \) and \( \psi(x, y_1, \ldots, y_m) \) occur in a sequent \( \phi, p_1(x_1), \ldots, p_n(x_n) \vdash \exists y_1 \ldots \exists y_m . \psi \land q_1(y_1) \land \ldots \land q_m(y_m) \) and the entailment \( \phi \models^I \exists y_1 \ldots \exists y_m . \psi \) is valid, then we can replace the sequent with \( p_1(x_1), \ldots, p_n(x_n) \vdash \{ q_1\theta \land \ldots \land q_m\theta \mid \theta \in S \} \), where \( S \) is a finite set of substitutions witnessing the entailment, i.e. \( \phi \models^I \psi \), for each \( \theta \in S \). This elimination of constraints from sequents is generally sound but incomplete. For instance, the entailment \( \phi(x, x_1, \ldots, x_n) \models^I \exists y_1 \ldots \exists y_m . \psi(x, y_1, \ldots, y_m) \) is valid if and only if \( \phi(x, x_1, \ldots, x_n) \models^I \psi'(x, x_1, \ldots, x_m) \), where \( \psi' \) is obtained from \( \psi \) by replacing each \( y \in y_1 \cup \ldots \cup y_m \) with a Skolem function symbol \( f_y(x, x_1, \ldots, x_n) \) not occurring in \( \phi \) or \( \psi' \). A complete proof rule based on this replacement has to consider every possible interpretation of these Skolem witnesses. However, in general, this is impossible, because the definitions of these functions are not bound to any particular form. In order to achieve completeness, we require that these functions are always defined as flat substitutions ranging over the free variables of the entailment, i.e. \( x \cup \bigcup_{i=1}^n x_i \). This condition ensures moreover that there are finitely many possible interpretations of these Skolem witnesses.

**Definition 3.** A system \( S \) has the finite instantiation property if and only if for any two constraints \( \phi(x, x_1, \ldots, x_n) \) and \( \psi(x, y_1, \ldots, y_m) \) from \( S \), with goal variables \( x \) and subgoal variables \( \bigcup_{i=1}^n x_i \) and \( \bigcup_{j=1}^m y_j \), respectively, the set \( \text{Sk}(\phi, \psi) = \{ \theta : \bigcup_{i=1}^n y_i \rightarrow T_\Sigma(x, x_1, \ldots, x_n) \mid \phi \models^I \psi \theta \} \) is finite. Moreover, \( S \) has the finite variable instantiation (fvi) property if for all \( i \in [n] \) there exists \( j \in [m] \) such that \( y_i \theta = x_j \), for each \( \theta \in \text{Sk}(\phi, \psi) \).

**Example 5.** Consider the constraints \( \phi \equiv x \approx f(x_1, x_2) \) and \( \psi \equiv x \approx f(y_1, y_2) \). Then \( \phi \models^\approx \exists y_1 \exists y_2 . \psi \iff \phi \models^\approx \psi \theta \), where \( \theta(x_1) = y_1 \) and \( \theta(x_2) = y_2 \), i.e. \( \text{Sk}(\phi, \psi) = \{ \theta \} \).

**Remark** Whenever \( S \) has the fvi property, a constraint with no subgoal variables may not entail a constraint with more than one subgoal variables. If \( S \) has the fvi property, \( \phi(x) \models^I \exists y_1 \ldots \exists y_m . \psi(x, y_1, \ldots, y_m) \) and \( m > 0 \) imply \( \text{Sk}(\phi, \psi) \neq \emptyset \). But then each flat substitution \( \theta \in \text{Sk}(\phi, \psi) \) would have an empty range, which is not possible.

Below we give an upper bound for the complexity of the problem whether a given system has the fvi property, in the canonical Herbrand interpretation of constraints. It is unclear, for now, whether the bound below can be tightened.

---

4 We assume w.l.o.g. that these function symbols belong to the signature, i.e. \( f_y \in \Sigma^d \).
because the exact complexity of the satisfiability of equational problems is still unknown, in general\(^5\).

**Lemma 3.** The fvi problem is in NEXPTIME in the Herbrand interpretation. If there exists a constant \(K > 0\), independent of the input, such that for each constraint \(\phi(x, x_1, \ldots, x_n)\), with goal variables \(x\) and subgoal variables \(\bigcup_{i=1}^n x_i\), respectively, we have \(|x_i| \leq K\), the fvi problem is in NP.

The last condition required for completeness is also related to the elimination of constraints from sequents. Intuitively, we do not allow two constraints to overlap, having at least one model in common, without one entailing the other.

**Definition 4.** Given an interpretation \(I\), a system \(S\) is non-overlapping if and only if for any two constraints \(\phi(x, x_1, \ldots, x_n)\) and \(\psi(x, y_1, \ldots, y_m)\) in \(S\), with goal variables \(x\) and subgoal variables \(\bigcup_{i=1}^n x_i\) and \(\bigcup_{j=1}^m y_j\) respectively, \(\phi \land \psi\) is satisfiable only if \(\phi \models^I \exists y_1 \ldots \exists y_m \cdot \psi\).

**Example 6.** The system from Example 3 is non-overlapping, because for instance \(x \approx f(x_1, x_2) \land x \approx f(y_1, y_2)\) is satisfiable and \(x \approx f(x_1, x_2) \models^h \exists y_1 \exists y_2 \cdot x \approx f(y_1, y_2)\), whereas \(x \approx f(x_1, x_2) \land x \approx g(y_1)\) is unsatisfiable. \(\square\)

**Remark** For a non-overlapping system, if \(\phi(x, x_1, \ldots, x_n) \land \psi(x, y_1, \ldots, y_m)\) is a satisfiable conjunction of constraints, then the formulae \(\exists x_1 \ldots \exists x_n \cdot \phi\) and \(\exists y_1 \ldots \exists y_m \cdot \psi\) are equivalent. \(\square\)

For constraints interpreted in the Herbrand interpretation this condition is decidable in nondeterministic polynomial time, because it suffices to check, for each pair of constraints such that (i) \(\phi \land \psi\) is satisfiable, the satisfiability of (ii) \(\forall y_1 \ldots \forall y_m \cdot \phi \land \neg \psi\). Since \(\phi\) and \(\psi\) are both conjunctions of literals, \(\phi \land \psi\) and \(\phi \land \neg \psi\) is in conjunctive normal form, both problems are in NP [Pic03].

### 3.2 Inference Rules

Figure 1 gives a set \(\mathcal{R}\) of inference rule types for proving entailments. To shorten the presentation, we write \((\Gamma_1 \vdash \Delta_1 \ldots \Delta_n)_{i=1}^n\) for \(\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_n \vdash \Delta_n\).

The inference rules of type (LU) and (RU) correspond to the unfolding of a predicate atom \(p(x)\) occurring on the left and right hand side of a sequent \(\Gamma \vdash \Delta\), respectively. By unfolding, we mean essentially the replacement of \(p(x)\) with the set of rules in \(p(x) \leftarrow R_1 \mid \ldots \mid R_n\). Observe that the left unfolding yields a set of sequents that must be all proved, one for each \(R_i\), whereas the right unfolding simply replaces \(p(x)\), on the right hand side of the sequent, with a set of formulae in which the subgoal variables of \(R_1, \ldots, R_n\) are existentially quantified.

The rules of type (RD) are used to simplify sequents by eliminating the constraints from both the left and right hand sides. In the process, we eliminate

\(^5\) Converting a formula into CNF requires exponential time at most, thus NEXPTIME is an upper bound.
the existentially quantified variables on the right hand side of the sequent, using a finite set of substitutions that witness the entailments between the (unique) constraint of the left and the ones on the right. Observe that, by applying the (RD) rule, one can obtain conjunctions of predicates sharing the same set of arguments. These cases are eagerly eliminated using a weakening rule of type (RU). In fact we shall assume, from now on, that every application of an (RD) contains it by default. If, after the cleanup done by applying (RU), there are conjunctions of predicates sharing the same number of predicates on the left hand side and in every set of predicate conjunctions on the right hand side, thus enabling the application of (SP).

We introduce a set of universal predicate rules \( \mathcal{S}_{\text{uni}} = \{ \{ \top \}, p_k^k(x_1, \ldots, x_k) \mid k \geq 0 \} \) and assume that any system \( S \) contains it by default. If, after the cleanup done by applying (RU), there are conjunctions of predicates \( Q \) on the right hand side such that there exist sets of subgoals \( x_i \) for which \( \exists p \in S . p(x_i) \in \Gamma \), but \( \forall q \in S . q(x_i) \notin Q \), then we add \( p_{\Delta}^{x_i} \) to \( Q \). While not changing the semantics of the entailment, this makes sure that, after every application of (RD), there always are the same number of predicates on the left hand side and in every set of predicate conjunctions on the right hand side, thus enabling the application of (SP).

The rule type (AX) closes the current branch of the proof, if the sequent from its consequent can be proved using a decision procedure for the following.
constraint logic, treating all predicate symbols as uninterpreted function symbols of boolean sort.

The rules of type Infinite Descent (ID) work as follows. A sequent \( \Gamma \theta \vdash \Delta' \theta \) denotes a valid entailment whenever the pivot \( \Gamma \vdash \Delta \), encountered earlier in the proof is a weaker sequent \( \Delta' \subseteq \Delta \), up to the renaming of variables by a flat substitution. Further, the pivot condition \( R^* \cdot LU \cdot R^* \) asks that a rule of type LU occurs on the path between the pivot \( \Gamma \vdash \Delta \) and the consequent \( \Gamma \theta \vdash \Delta' \theta \) in the proof. ID rules are sound under the assumption that the system is ranked (Definition 2), by an application of Fermat’s Infinite Descent principle [Bus18]. Consider, by contradiction, that \( \Gamma \vdash \Delta \) does not denote a valid entailment, meaning that there exists a valuation \( \nu \in \mu S^T(\bigwedge \Gamma) \setminus \bigcup \mu S^T(\bigvee \Delta) \), for the given interpretation \( \mathcal{I} \). Because the system is ranked and (LU) was used at least once on the path from \( \Gamma \vdash \Delta \) to \( \Gamma \theta \vdash \Delta' \theta \), we can deduce that a strictly smaller\(^6\) counterexample \( \nu' \) has been discovered such that \( \nu' \in \mu S^T(\bigwedge \Gamma \theta) \setminus \bigcup \mu S^T(\bigvee \Delta \theta) \). Since \( \Delta' \subseteq \Delta \), we can obtain a from \( \nu' \) a strictly smaller counterexample for \( \Gamma \vdash \Delta \). But since the path between \( \Gamma \vdash \Delta \) and \( \Gamma \theta \vdash \Delta' \theta \) can be repeated indefinitely, this would result in a strictly decreasing sequence of tuples of trees, contradicting the well-foundedness of the multiset order (Theorem 1). Thus \( \bigwedge \Gamma \models^T \bigvee \Delta \) must hold and this branch of the proof can be closed.

Finally, the rules of type (SP) split a sequent \( p_1(x_1), \ldots, p_n(x_n) \vdash \bigwedge_{j=1}^n q_j^1(x_j), \ldots, \bigwedge_{j=1}^n q_j^n(x_j) \), without constraints, into \( n \) basic sequents, with left hand sides \( p_1, \ldots, p_n \), respectively. Given a set of tuples \( \{ \overline{\mathcal{Q}}_1, \ldots, \overline{\mathcal{Q}}_k \} \subseteq \text{Pred}^n \), for some \( n \geq 1 \), a choice function \( f \) maps each tuple \( \overline{\mathcal{Q}}_i \) into an index \( f(\overline{\mathcal{Q}}_i) \in [n] \) corresponding to a given coordinate in the tuple. Let \( \mathcal{F}(\overline{\mathcal{Q}}_1, \ldots, \overline{\mathcal{Q}}_k) \) be the set of such choice functions. This set has cardinality \( n^k \leq n^{[\text{Pred}]}^n \), for any set of \( n \)-tuples of predicates. Observe that (SP) is applied to each tuple \( \overline{i} \in [n]^k \) of choices, indexed by the set of choice functions. The following lemma states an important property for the soundness of the (SP) rules.

**Lemma 4.** Given a system \( \mathcal{S} \), with predicates \( p_1(x), \ldots, p_n(x) \) and tuples of predicates \( \overline{\mathcal{Q}}_i = (q_1^i(x), \ldots, q_n^i(x)) \) in \( \mathcal{S} \), for all \( i \in [k] \). Then

\[
\mu S^T(p_1) \times \ldots \times \mu S^T(p_n) \subseteq \bigcup_{i=1}^{k} \mu S^T(q_1^i) \times \ldots \times \mu S^T(q_n^i)
\]

if and only if there exists a tuple \( \overline{i} \in [n]^k \), such that:

\[
\mu S^T(p_{\overline{i}}) \subseteq \bigcup \{ \mu S^T(q_{\ell}^{\ell_1}) | \ell \in [k], f_j(\overline{\mathcal{Q}}_\ell) = \overline{i}_j \}
\]

for all \( j \in [n^k] \), where \( \mathcal{F}(\overline{\mathcal{Q}}_1, \ldots, \overline{\mathcal{Q}}_k) = \{ f_1, \ldots, f_{n^k} \} \).

**Example 7.** In the system of Example 3, the sequent \( p(x) \vdash q(x) \) has the following proof:

\(^6\) In the multiset order applied to the multisets \( [\nu(x) | x \in \text{Var}] \).
The dashed arrow indicates the pivot of the ID rule. For space reasons, some branches following the application of (SP) are omitted. The full proof is provided as additional material.

The soundness of the set $\mathcal{R}$ of inference rule types from Figure 1 follows from the local soundness of each inference rule type, as proved by the following lemma. Observe that the only condition required for soundness is that the system be ranked.

**Lemma 5.** Given a ranked system $\mathcal{S}$, if a sequent $\Gamma \vdash \Delta$ has a proof using the inference rules in Figure 1, then the entailment $\Delta \vdash_{\mathcal{S}} \Gamma$ holds.

### 3.3.3 Completeness

We prove that the set of inference rules in Figure 1 is complete for entailments between predicates in non-filtering and non-overlapping systems with the fvi property (§3.1). As discussed earlier, the non-overlapping restriction (Definition 4) can be removed at the cost of introducing quantifiers and an exponential blowup in the size of the system. We chose to maintain it mostly for easing the technical developments in this section.

A derivation is said to be **maximal** if it cannot be extended by an application of an inference rule, and **irreducible** if it cannot be rewritten into a smaller derivation of the same sequent by an application of a rule of type (ID). Observe that the proof search semi-algorithm 1 presented in the previous, generates only irreducible derivations, because the (ID) type rules are always applied before other inference rules.
A derivation $D$ is structured if, on each path of $D$, between any two consecutive applications of a rule of type (LU) there exists an application of a (RD) rule. Intuitively, unstructured derivations constitute poor candidates for proofs. For instance, a derivation consisting only of applications of (LU) rules will only grow the size of the left hand sides of the sequents, without making progress towards $\top$ or a counterexample. Each subtree of a structured derivation is also structured. We denote by $\mathcal{D}(\Gamma \vdash \Delta)$ the set of irreducible, maximal and structured derivations rooted in $\Gamma \vdash \Delta$.

**Lemma 6.** If $S$ has the fvi property, then the following hold:
1. any irreducible and structured derivation is finite, and
2. for any sequent $\Gamma \vdash \Delta$, the set $\mathcal{D}(\Gamma \vdash \Delta)$ is finite.

**Definition 5.** A set $F = \{\phi_1, \ldots, \phi_n, q_1(x_1), \ldots, q_m(x_m)\}$ is tree-shaped if and only if $\phi_1, \ldots, \phi_n$ are constraints, $q_1(x_1), \ldots, q_m(x_m)$ are predicate atoms, and there exist trees $t_1, \ldots, t_k$ such that:

- each node labeled with a constraint $\phi_i(y, y_1, \ldots, y_n)$ in some tree $t_\ell$, $\ell \in [k]$ has exactly $n$ children and for all $j \in [n]$, the $j$-th child is labeled either (i) with a constraint whose goal variables are $y_j$, or (ii) with a predicate atom $q_k(y_j)$, and
- a predicate atom $q_i(x_i)$ may occur only on the frontier of a tree $t_j$, for some $j \in [k]$.

If $k = 1$ we say that $F$ is singly-tree shaped.

Tree-shaped sets can be uniquely represented by trees labeled with formulae, thus we use sets of trees instead of sets of formulae interchangeably. We write $\Gamma \vdash \Delta \Rightarrow \Gamma' \vdash \Delta'$ if $\Gamma' \vdash \Delta'$ occurs in a derivation from $\mathcal{D}(\Gamma \vdash \Delta)$. Next, we prove an invariant on the shape of the sequents occurring in a proof of a basic sequent $p(x) \vdash q(x)$.

**Lemma 7.** Given a system $S$ and two predicate atoms $p(x)$ and $q(x)$, in every sequent $\Gamma \vdash \Delta$ such that $p(x) \vdash q(x) \Rightarrow \Gamma \vdash \Delta$, $\Gamma$ is a tree-shaped set and $\Delta$ consists of finite conjunctions of tree-shaped sets, with all subgoal variables existentially quantified.

The following lemma gives a characterization of the cases in which the root of a derivation denotes an invalid entailment, and from which a counterexample can be extracted. This characterization of invalid entailments in terms of derivations, that produce sequents with empty right hand sides, is crucial in establishing our main completeness result (Theorem 3).

**Lemma 8.** Given an interpretation $I$, a non-filtering and non-overlapping system $S$ with the fvi property, two predicate atoms $p(x)$ and $q(x)$, and a sequent $\Gamma \vdash \Delta$ such that $p(x) \vdash q(x) \Rightarrow \Gamma \vdash \Delta$, if every derivation $D \in \mathcal{D}(\Gamma \vdash \Delta)$ contains a leaf $\Gamma' \vdash \emptyset$ then there exists a valuation $\nu \in \mu S^I(\bigwedge \Gamma) \setminus \mu S^I(\bigvee \Delta)$.

The following theorem proves that the set of inference rule types $\mathcal{R}$, given in Figure 1, is complete under the canonical interpretation, and provides a proof search strategy.
Theorem 3. Given an interpretation $I$ and a non-filtering, non-overlapping system $S$, with the fvi property, let $p(x)$ and $q(x)$ be predicates occurring in $S$. Then the entailment $p \models^I_S q$ holds only if the sequent $p(x) \vdash q(x)$ has an $S$-proof with the inference rule types $R$, where $S$ is defined by the regular expression $(LU \cdot RU^* \cdot RD \cdot R^* \cdot SP?)^* \cdot LU? \cdot RU^* \cdot (AX \mid ID)$.

Since the proof search semi-algorithm 1 only explores irreducible derivations, if it is executed with the strategy $S$ from Theorem 3, then every derivation it generates is, moreover, structured. By Lemma 6 (1), each irreducible and structured derivation is finite, thus every execution of the semi-algorithm 1 is guaranteed to terminate. If, moreover the system $S$ given in input is ranked, non-filtering, non-overlapping and has the fvi property, the set of inference rule types in Figure 1 is complete, thus algorithm 1 is a decision procedure for this class of entailment problems.

4 Separation Logic

In this section we apply the method described in §3 to deciding entailments between predicates whose defining rules use constraints from a fragment of Separation Logic [Rey02]. These predicates are common for specifications of recursive data structures implemented using pointers, thus having complete sets of proof rules for these systems is important for obtaining decision procedures that solve verification conditions generated by program analysis tools. Using a similar approach as for first order logic, we give a set of inference rules and prove completeness under a number of (decidable) restrictions on the set of constraints that occur in the system.

Throughout this section, we consider a signature $\Sigma$, such that $\Sigma^s = \{\text{Loc, Bool}\}$ and $\Sigma^f = \emptyset$, i.e. the only sorts are the boolean and location sort, with no function symbols defined on it, other than equality. Observe that, in this case $T_\Sigma(x) = x$, for any $x \subseteq \text{Var}$, i.e. the only terms occurring in a formula are variables of sort Loc. In the rest of this section we consider systems whose constraints are Separation Logic (SL) formulae, generated by the following syntax:

$$\varphi ::= \bot \mid x \approx y \mid \text{emp} \mid x \mapsto (y_1, \ldots, y_k) \mid \varphi_1 \ast \varphi_2 \mid \neg \varphi_1 \mid \varphi_1 \wedge \varphi_2 \mid \exists x . \varphi_1$$

where $k > 0$ is a fixed constant. As usual, we consider that the constraints of an inductive system are quantifier-free SL formulae in the above fragment. For a set of formulae $F = \{\varphi_1, \ldots, \varphi_n\}$, we write $\ast F$ for $\varphi_1 \ast \ldots \ast \varphi_n$ if $F \neq \emptyset$, and $\text{emp}$ if $F = \emptyset$. The size of a formula is the number of variables and connectives occurring in it. The size of a system is the sum of the sizes of its constraints.

Most definitions of common recursive data structures employed by programmers (e.g. lists, trees, etc.) use a restricted fragment of quantifier-free SL, consisting of formulae $\Pi \wedge \Theta$, called symbolic heaps, in the following syntax, for pure ($\Pi$) and spatial ($\Theta$) formulae defined as follows:

$$\Pi ::= x \approx y \mid \neg x \approx y \mid \Pi_1 \wedge \Pi_2 \quad \Theta ::= \text{emp} \mid x \mapsto (y_1, \ldots, y_k) \mid \Theta_1 \ast \Theta_2$$
In the rest of this section, we fix an interpretation $\mathcal{I}$ such that $\mathcal{I}(\text{Loc}) = L$ is a countably infinite set and omit to specify $\mathcal{I}$ any further. A heap is a finite partial mapping $h : L \rightarrow \text{fin} L^k$ associating locations with $k$-tuples of locations. We denote by $\text{dom}(h)$ the set of locations on which $h$ is defined, by $\text{img}(h)$ the set of locations occurring in the range of $h$, and by Heaps the set of heaps. Two heaps $h_1$ and $h_2$ are disjoint if $\text{dom}(h_1) \cap \text{dom}(h_2) = \emptyset$. In this case $h_1 \uplus h_2$ denotes their union, which is undefined if $h_1$ and $h_2$ are not disjoint. Given a valuation $\nu : \text{Var} \rightarrow L$ and a heap $h$, the semantics of SL formulae is defined as:

\[
\begin{align*}
\nu, h \models ^e x & \iff \nu(x) = e \\
\nu, h \models ^{\text{emp}} & \iff h = \emptyset \\
\nu, h \models ^{x \mapsto (y_1, \ldots, y_k)} & \iff h = \{(\nu(x), (\nu(y_1), \ldots, \nu(y_k)))\}
\end{align*}
\]

\[
\begin{align*}
\nu, h \models \phi_1 \land \phi_2 & \iff \exists h_1, h_2 \in \text{Heaps} : h = h_1 \uplus h_2 \And \mathcal{I}, h_1 \models \phi_1, \mathcal{I}, h_2 \models \phi_2, i \in [2]
\end{align*}
\]

The semantics of boolean connectives is the usual one, omitted for brevity.

An assignment $\mathcal{X}$ maps each predicate $p(x_1, \ldots, x_n)$ to a set $\mathcal{X}(p) \subseteq L^n \times \text{Heaps}$. For a set $\mathcal{F} = \{\phi, p_1(x_1), \ldots, p_m(x_m)\}$, where $\phi$ is an SL formula and $p_1(x_1), \ldots, p_m(x_m) \in \text{Pred}$, we define $\mathcal{X}(\mathcal{F}) = \{\nu, h_0 \uplus \bigcup_{i=1}^m h_i \mid \nu, h_0 \models ^e \phi, (\nu(x_j), h_j) \in \mathcal{X}(p_j), j \in [m]\}$. The least solution $\mu S^e$ of a system $S$ is the least fixpoint of the function $\mathbb{F}_S$, where $\mathbb{F}_S(\mathcal{X})$ maps each predicate $p(x) \in \text{Pred}$, such that $p(x) \leftarrow S R_1 \mid \ldots \mid R_m$, into the set $\bigcup_{i=1}^m \{\nu(x), h \mid \nu, h \models \mathcal{X}(\mathcal{F})\}$. Observe that the heaps from the subgoal assignments are separately joined with a heap satisfying the constraint of the rule to obtain a heap for the goal. In this section we consider entailments between predicates $p \models ^e q$ if and only if $\mu S^e(p) \subseteq \mu S^e(q)$. As before, we extend this notation to SL formulae and write $\phi \models ^e \psi$ for $\mu S^e(\phi) \subseteq \mu S^e(\psi)$, where, for an arbitrary SL formula $\phi$, $\mu S^e(\phi)$ is defined recursively on its structure.

**Example 8.** Consider the following system, with symbolic heap constraints:

\[
\begin{align*}
ls^e(x, y) & \leftarrow S x \mapsto y \mid y \approx y' \land x \mapsto z, \ls^e(z, y') \\
ls^o(x, y) & \leftarrow S x \approx y \land \text{emp} \mid y \approx y' \land x \mapsto z, \ls^o(z, y') \\
\ls^e(x, y) & \leftarrow S x \mapsto y \mid y \approx y' \land x \mapsto z, \ls^e(z, y') \\
\ls^o(x, y) & \leftarrow S x \approx y \land \text{emp} \mid y \approx y' \land x \mapsto z, \ls^o(z, y')
\end{align*}
\]

Intuitively, $\ls^e(x, y)$ defines the set of finite list segments of at least one element between $x$ and $y$, $\ls^e$ and $\ls^o$ are list segments of even and odd length, respectively, and $\ls^e(x, y)$ is the definition of a list segment consisting of one element followed by an even or an odd list segment. It is immediate to see that both entailments $\ls^e \models ^e \ls^e$ and $\ls^o \models ^e \ls^o$ hold. \ □

The following negative result [IRV14,AGH+14] justifies a number of restrictions on the set of SL constraints occurring in a system\footnote{See e.g. [IRV14, Theorem 2] and [AGH+14, Theorem 3]}.\[\text{Theorem 4.} \text{ The entailment problem is undecidable for systems with symbolic heap constraints.}\]
4.1 Restricting the Set of Constraints

Before giving a set of inference rules for the entailment problem in SL and analyzing its proof-theoretic properties (§4.2), we state the counterparts of the semantic restrictions introduced in §3.1, necessary for soundness and completeness. Moreover, we give complexity bounds for the problem of deciding whether a certain system, with quantifier-free SL and symbolic heap constraints, respectively, complies with these restrictions.

**Definition 6.** Given a system \( S \), a rule \( \langle \{ \phi, q_1(x_1), \ldots, q_n(x_n) \}, p(x) \rangle \in S \) is non-filtering if and only if, for all \( i \in [n] \) and \( \langle \ell_i, h_i \rangle \in \mu S^*(q_i) \), where \( h_i \) are pairwise disjoint, there exists a valuation \( \nu \) and a heap \( h \), disjoint from \( \bigcup_{i=1}^{n} h_i \), such that \( \nu, h \models \phi \) and \( \nu(x_i) = \ell_i \), for all \( i \in [n] \). The system \( S \) is non-filtering if and only if each rule in \( S \) is non-filtering.

**Example 9.** The system from Example 8 is non-filtering because there exists a model \( \nu, h \models y \models y' \land x \mapsto z \), such that \( \nu(y') = \nu'(y') \), \( \nu(z) = \nu'(z) \) and \( \text{dom}(h) \cap \text{dom}(h') = \emptyset \), for each given pair \( (\nu', h') \). Since the system \( L \) is infinite, it is always possible to find a value \( \nu(x) \not\in \text{dom}(h') \). \( \square \)

As opposed to the case of systems with first-order constraints, under the Herbrand interpretation (Lemma 2), the non-filtering property is decidable for systems with SL constraints. This is because one can build an over-approximation of the least solution, that is both necessary and sufficient to characterize the satisfiability of a quantifier-free SL formula using predicate atoms [BFPG14].

Formally, the abstraction is defined as the least fixpoint of an operator \( \mathcal{F}_S^\mathcal{E} \), denoted \( \mu S^\mathcal{E} \). An abstract assignment \( \nu \) is a mapping of predicates \( p_1, \ldots, p_n \) into sets of pairs \((A, E)\), where \( A \in \mathcal{P}(\{[n]\}) \) is a set of allocated arguments and \( E \subseteq \{[n] \times [n] \) is a set of equality constraints such that, for each model \( \langle \ell_1, \ldots, \ell_n \rangle \), \( h \in \mu S^\mathcal{E}(p) \) there exists a pair \((A, E) \in \mu S^\mathcal{E}(p) \) such that \( A = \{i \in [n] \mid \ell_i \in \text{dom}(h)\} \) and \((i, j) \in E \) if and only if \( \ell_i = \ell_j \). Given a quantifier-free SL formula \( \varphi(x) \), we define the following sets:

\[
\text{alloc}^+ (\varphi) = \{ x \in \text{FV}(\varphi) \mid \varphi \models x \models z_1 \ldots z_k \quad x \mapsto (z_1, \ldots, z_k) \land \text{T is satisfiable} \}
\]

\[
\text{alloc}^- (\varphi) = \{ x \in \text{FV}(\varphi) \mid \varphi \models \neg \exists z_1 \ldots z_k \quad x \mapsto (z_1, \ldots, z_k) \land \text{T} \}
\]

\[
\text{eq}(\varphi) = \{ (x, y) \in \text{FV}(\varphi) \times \text{FV}(\varphi) \mid \varphi \models x \models y \}
\]

Computing the above sets can be done in polynomial space, in general, using the decision procedures for quantifier-free [CYO01] and Bernays-Schoenfinkel-Ramsey SL formulae [RIS17], and in polynomial time for symbolic heaps, respectively. Dually, we consider the formulae \( A(X) = \bigwedge_{x \in X} \exists z_1 \ldots z_k \cdot x \mapsto (z_1, \ldots, z_k) \) and \( E(R) = \bigwedge_{(x,y) \in R} x \models y \land \bigwedge_{(x,y) \notin R} \neg x \models y \), for any set \( X \subseteq \text{Var} \) and relation \( R \subseteq \text{Var} \times \text{Var} \) on variables.

Given a rule \( R = \langle \{ \phi(x, y_1, \ldots, y_m), q_1(y_1), \ldots, q_m(y_m) \}, p(x) \rangle \in S \), let \( y_i = (y_{i1}, \ldots, y_{ik_i}) \) for all \( i \in [m] \). For a tuple \( P = (A_1, E_1), \ldots, (A_m, E_m) \in \mu S^\mathcal{E}(q_1) \times \cdots \times \mu S^\mathcal{E}(q_m) \) and a relation \( C \) on the free variables of \( \phi \), we define:

\[
\omega_R(P, C) = \bigwedge_{i \in [m]} A_i \{ (x_j \mid j \in A_i) \} \land E_1 \{ (y_{i1}, y_{i1}') \mid (r, s) \in E_i, \quad i \in [m] \} \land E(C)
\]

\[
\eta_R(P, C) = \exists y_1 \ldots y_m : \phi(x, y_1, \ldots, y_m) \land \omega_R(P, C)
\]
Then an abstract assignment $F_S^F(Y)$ maps each predicate $p$ into the set $\bigcup_{i=1}^m Y(R_i)$, where $p(x_1, \ldots, x_n) \leftrightarrow R_1 | \ldots | R_m$ and $Y(R)$ is the set of pairs $(A, E)$ for which there exists a tuple of pairs $P \in \mu S^q(q_1) \times \ldots \times \mu S^q(q_m)$ and a relation $C \subseteq \text{FV}(\phi) \times \text{FV}(\phi)$ such that:

$$A = \{ i \in [n] | x_i \in A, \text{alloc}^- (\eta_R(P, C)) \subseteq A \subseteq \text{alloc}^+ (\eta_R(P, C)) \}$$

$$E = \{ (i, j) \in [n] \times [n] | (x_i, x_j) \in \text{eq}(\eta_R(P, C)) \}$$

Remark If $\eta_R(P, C)$ is unsatisfiable, $\text{alloc}^- (\eta_R(P, C)) = x$ and $\text{alloc}^+ (\eta_R(P, C)) = \emptyset$, and there is no choice for the set $A$, thus no corresponding pair $(A, E)$. □

Since there are finitely many variables in the system, the set of pairs $(A, E)$ is finite, of cardinality at most $2^{n+n^2}$, and can be enumerated in exponential time in the size of the system. Then the least fixpoint of $F_S^F$ can be computed in an exponential number of steps\(^8\), each step requiring polynomial space. These pairs can be stored in a table that requires $O(n)$ pairs can be stored in a table that requires $2^{O(n^2)}$ space, indexed by $O(n^2)$ bits. Then, we can check if a rule in $S$ is non-filtering, by checking the satisfiability of an exponential number of SL formulae, where each satisfiability check can be done in polynomial space. The lemma below adds up to the upper bound for the complexity of deciding whether a given system is non-filtering.

**Lemma 9.** The problem “given a system $S$ with SL constraints, is $S$ non-filtering?” is in EXPSPACE.

Next, we turn to the ranking condition, that ensures the soundness of applying the principle of Infinite Descent to a system with SL constraints. In the absence of a natural wqo on the set of locations $L$ (since there are no relations other than equality defined on it), we consider the following wqo on heaps. For any $h_1, h_2 \in \text{Heaps}$, we have $h_1 \preceq h_2$ if there exists $h \in \text{Heaps}$ such that $h_2 = h_1 \uplus h$. We write $h_1 \prec h_2$ if, moreover, $h \neq \emptyset$.

**Definition 7.** A system $S$ is ranked if and only if for each rule $\langle \{ \phi, q_1(x_1), \ldots, q_n(x_n) \}, p(x) \rangle \in S$, for each heap $(\overline{t}, h_1) \in \mu S(q_1)$ there exists a heap $(\overline{t}, h) \in \mu S(p)$ such that $h_1 < h$.

**Example 10.** The system of Example 8 is ranked because each rule with at least one subgoal has a constraint $y \approx y' \wedge x \mapsto z$, which does not admit an empty heap model. □

Deciding whether a given system is ranked is possible in polynomial space when all constraints are quantifier-free SL formulae, by checking the validity of $\phi \models^{\text{aemp}} \neg \text{emp}$, i.e. the satisfiability of $\phi \wedge \text{emp}$ for each constraint $\phi$, which is in PSPACE \cite{CY001}. This bound drops to polynomial time for systems with symbolic heap constraints, because each model of a symbolic heap $\Pi \wedge \Theta$ is empty iff $\Theta$ does not contain atoms of the form $x \mapsto (y_1, \ldots, y_k)$.

We continue with the finite variable instantiation (fvi) property (cf. Definition 3) for quantifier-free SL constraints. We show that this problem is decidable and provide several upper bounds.

---

\(^8\) See \cite[Lemma 4.6]{BFPG14} for an analogous construction for systems with symbolic heap constraints.
The fvi problem is in PSPACE for systems with quantifier-free SL constraints and in $\Sigma_2^P$ for systems with symbolic heap constraints, respectively.

Example 11. The system from Example 8 has the fvi property, because the entailment $y \equiv y_1 \land x \rightarrow z_1 \models \exists y_2 \exists z_2 . y \equiv y_2 \land x \rightarrow z_2$ is witnessed by a single substitution $\theta(y_2) = y_1$ and $\theta(z_2) = z_1$.

A related question is whether the system is non-overlapping (cf. Definition 4). Since satisfiability of formulae $\forall y . \varphi_1(x) \land \neg \varphi_2(x, y)$ is in PSPACE [RIS17], where $\varphi_1$ and $\varphi_2$ are quantifier-free SL formulae, deciding whether a given system with quantifier-free constraints is non-overlapping is in PSPACE. Moreover, since the satisfiability problem for symbolic heaps is in NP [CY001] and the entailment between existentially quantified symbolic heaps is $\Pi^2_2$-complete [AGH+14], checking whether a system with symbolic heap constraints is non-overlapping is in $\Pi^2_2$.

Example 12. The system from Example 8 is non-overlapping because the only constraints with a satisfiable conjunction are $x \rightarrow y$ and $y \equiv y' \land x \rightarrow z$ and both entailments $x \rightarrow y \models \exists y' \exists z . y \equiv y' \land x \rightarrow z$ and $y \equiv y' \land x \rightarrow z \models \neg y' \rightarrow y$ are valid.

4.2 Inference Rules for Entailments in Separation Logic

The inference rule types from Figure 2 are the SL counterparts of the rules from Figure 1. These rules are obtained from the ones in Figure 1 by systematically replacing boolean with spatial conjunctions, in order to match the semantics of the rules in a system, which separately join the constraint and subgoal heaps into a goal heap. The rule types $LU$, $\land R_d$ and $ID_d$ are identical to $LU$, $\land R$ and ID, respectively, and are omitted from Figure 2.

\[
\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta} \quad \text{(AX)}
\]

\[
\frac{\Gamma \vdash \{ \exists y_i . \ast R_i(x, y_i) \}^n_{i=1} : \Delta \setminus p(x)}{\Gamma \vdash \Delta}
\quad \text{(RU$_a$)}
\]

\[
\frac{p_i(x_1), \ldots, p_n(x_n) \vdash \{ Q_j \theta \mid \theta \in S_j \}^{j=1}_j}{\phi(x, x_1, \ldots, x_n), p_i(x_1), \ldots, p_n(x_n) \vdash \{ \exists y_j . \psi_j \ast Q_j(y_j) \}^{j=1}_j}
\quad \text{(RD$_a$)}
\]

\[
\frac{\{ p_i(x_1) \vdash \{ q^i_j(x) \mid \ell \in [k], f_j(\overline{\psi}_\ell) = \overline{r}_j \}^{k=1}_1 \}^n_{1} \quad x_i \cap x_j = \emptyset, [x_i] \cup [x_j], i, j \in [n]}{p_1(x_1), \ldots, p_n(x_n) \vdash Q_1(x_1, \ldots, x_n), \ldots, Q_k(x_1, \ldots, x_n)}
\quad \text{(SP$_a$)}
\]

\[
\quad \text{(AX$_a$)}
\]

\[
\quad \text{(RU$_a$)}
\]

\[
\quad \text{(RD$_a$)}
\]

\[
\quad \text{(SP$_a$)}
\]

\textbf{Fig. 2.} Proof rules for SL entailments.
Example 13. Below we give a proof for the sequent $ls^+(x, y) \vdash u \widehat{ls}^+(x, y)$, using the rules in Figure 2:

\[
\begin{array}{c|c}
RD_d & \text{id} \\
\hline
ls^+(u_1, y) & ls^+(u_1, y), ls^+(u_1, y) \\
\hline
y \approx u_2 \land z_1 \rightarrow u_1, ls^+(u_1, u_2) & z_1 \approx y \land \text{emp}, \exists v_1 \exists v_2, y \approx v_2 \land z_1 \rightarrow v_1 \ast ls^+(v_1, v_2), \\
\hline
\text{ru}_d & \text{lu} \\
\hline
y \approx u_2 \land z_1 \rightarrow u_1, ls^+(u_1, u_2) & z_1 \approx y \land \text{emp}, \exists v_1 \exists v_2, y \approx v_2 \land z_1 \rightarrow v_1 \ast ls^+(v_1, v_2), \exists v_1 \exists v_2, y \approx v_2 \land z_1 \rightarrow v_1 \ast ls^+(v_1, v_2), \\
\end{array}
\]

For space reasons, several simple branches of the proof are omitted. The full proof is provided as additional material.

The following lemma is the counterpart of Lemma 4 for rules of type $SP_d$:

Lemma 11. Given a system $S$, with predicates $p_1(x_1), \ldots, p_n(x_n)$, such that $x_i \cap x_j = \emptyset$, for all $1 \leq i < j \leq n$, and let $\overline{Q}_i = \langle q_1^i(x_1), \ldots, q_n^i(x_n) \rangle$ in $S$, for all $i \in [k]$, be tuples of predicates. Then

$$\mu_S(p_1(x_1) \ast \ldots \ast p_n(x_n)) \subseteq \bigcup_{i=1}^k \mu_S(q_1^i(x_1) \ast \ldots \ast q_n^i(x_n))$$

if and only if there exists a tuple $\overline{i} \in [n]^n$, such that:

$$\mu_S(p_{\overline{i}}) \subseteq \bigcup \{\mu_S(q_{i_j}^\ell) \mid \ell \in [k], f_j(\overline{Q}_\ell) = i_j\}$$

for all $\ell \in [n^k]$, where $F(\overline{Q}_1, \ldots, \overline{Q}_k) = \{f_1, \ldots, f_{n^k}\}$.

The following lemma proves the soundness of the set of inference rule types from Figure 2, together with (LU) and (ID) (Figure 1), denoted $\mathcal{R}_d$ in the following.

Lemma 12. Given a ranked system $S$, if a sequent $\Gamma \vdash \Delta$ has a proof using the set of inference rule types $\mathcal{R}_d$, then $\ast \Gamma \vdash_S \Delta$ holds.

4.3 Completeness

The set of inference rules from Figure 2 is not complete for $SL$ entailments, even for those systems which comply with the conditions of 4.1. This section proves
first the completeness of the set of rule types $R_d$ (Figure 2) for a more restricted class of entailment problems. The existence of a complete set of inference rules for the general entailment problem of $SL$ is, to our knowledge, still open.

We consider assignments $X$ mapping a predicate $p$ of arity $n$ into a subset of $L^n \times Heaps \times Cover$. For a singly-tree shaped set (Definition 5) represented as a tree $T$, we define $X(\ast T)$ to be the set of tuples $(\nu, h, t)$, where $\nu : \bigcup_{p \in \text{dom}(T)} \text{FV}(T(p)) \rightarrow L$ is a valuation, $h$ is a heap and $t$ is a coverage tree for $h$, where:

- for each $p \in \text{dom}(T) \setminus \text{fr}(T)$, we have $\nu, t(p) \models^a T(p)$,
- for each $p \in \text{fr}(T)$ where $T(p) = q(x)$, there exists $(\nu(x), t(p), t_{pq}) \in X(q)$.

The above definition extends to tree-shaped sets $\{T_1, \ldots, T_k\}$ as $X(\ast \{T_1, \ldots, T_k\}) = \{ (\nu, h_1 \uplus \ldots \uplus h_k, \{t_1, \ldots, t_k\}) | (\nu, h_i, t_i) \in X(\ast T_i), \ i \in [k] \}$. With this interpretation of predicates and formulae, the least solution $\mu S^\nu$ of a system $S$ is the least fixpoint of the function $\mathbb{P}_S^\nu(X)$, mapping each predicate $p \in \text{Pred}$ into $\bigcup_{m=1}^\infty \{ (\nu(x), h, t) | (\nu, h, t) \in X(\ast R_i) \}$, where $p(x) \leftarrow S R_1 | \ldots | R_m$.

For each $(\nu, h, t) \in \mu S^\nu(p)$, for some $p \in \text{Pred}$, we say that $t$ is an unfolding tree for the singly-tree shaped set $T$. Then the entailment problem becomes $\nu(x) \models^a q(x)$ iff $\mu S^\nu(p) \subseteq \mu S^\nu(q)$, given predicates $p(x), q(x) \in \text{Pred}$. It is not difficult to prove that $p(x) \models^a q(x)$ implies $\nu(x) \models^a q(x)$, but not vice versa.

Akin to the derivations using the rules in Figure 1, we denote by $D^u(\Gamma \vdash \Delta)$ the set of irreducible, maximal and structured\(^9\) derivations of $\Gamma \vdash \Delta$, with respect to the rules in Figure 2. Also, we write $\Gamma \vdash \Delta \rightarrow^u \Gamma' \vdash \Delta'$ if $\Gamma' \vdash \Delta'$ occurs inside a derivation from $D^u(\Gamma \vdash \Delta)$. It can be shown, as in Lemma 6, that any irreducible, maximal and structured derivation rooted in $\Gamma \vdash \Delta$ is finite, and the set $D^u(\Gamma \vdash \Delta)$ is finite.

The following lemma proves an invariant that relates tree-shaped sets with their corresponding unfolding trees, and is the counterpart of Lemma 8, needed to prove the completeness of the set of inference rules from Figure 2, for systems with symbolic heap constraints, with the above definition of entailments.

**Lemma 13.** Given a non-filtering and non-overlapping system $S$ with quantifier-free $SL$ constraints, having the fvi property, two predicate atoms $p(x), q(x)$ and a sequent $\Gamma \vdash \Delta$ such that $p(x) \vdash q(x) \sim^a \Gamma \vdash \Delta$, if every derivation $D \in D^u(\Gamma \vdash \Delta)$ contains a leaf $\Gamma' \vdash \emptyset$ then there exists a valuation $\nu$, a heap $h$ and a set of unfolding trees $U$ such that $(\nu, h, U) \in \mu S^\nu(\ast \Gamma) \setminus \mu S^\nu(\emptyset)$.

Since the structure of the inference rule types $R_d$ (Figure 2) mirrors the one of $R$ (Figure 1), the completeness proof for $SL$ systems mirrors closely the proof of Theorem 3 (§3.3).

**Theorem 5.** Let $S$ be a system with quantifier-free $SL$ constraints and $p(x)$ and $q(x)$ be predicates occurring in $S$, such that $p \models^a q$ holds. If $S$ is non-filtering, non-overlapping and has the fvi property, then the sequent $p(x) \vdash q(x)$ has an $S$-proof with the inference rule types $R_d$, where $S$ is defined by the regular expression $\text{RU}_d^a \cdot \text{RD}_d \cdot \land R^a \cdot \text{SP}_d ?^* \cdot \text{LU} ? \cdot \text{RU}_d^a \cdot (\text{AX}_d | \text{ID})$.

\(^9\) A derivation is structured if and only if there is an occurrence of $(\text{RD}_d)$ between any two consecutive applications of $(\text{LU})$.
Because each $S$-derivation is structured, the proof search semi-algorithm 1 terminates on all inputs, when given $S$ as strategy. A direct consequence of Theorem 5 is that algorithm 1 is a decision procedure for entailments $p \models q$, when $p$ and $q$ are defined by an system $S$ with quantifier-free $SL$ constraints, that is ranked, non-filtering, non-overlapping and has the fvi property. As discussed in §4.1, the problem whether a given system enjoys there properties is decidable.

5 Conclusions

We present a cyclic proof system for entailments between inductively defined predicates written using (multisorted) First Order Logic, based on Fermat’s principle of Infinite Descent. The advantage of this principle over classical induction is that the inductive invariants are produced during proof search, whereas induction requires them to be provided. The soundness of this principle is coined by a semantic restriction on the constraints of the inductive system, that asks that models generated by unfoldings decrease in a well-founded domain. On the other hand, completeness relies on an argument inspired by the theory of tree automata, that is applicable under three semantic restrictions on the set of constraints. In general all these restrictions are decidable, with computational complexities that depend on the logical fragment in which the constraints of the inductive system are written.

Moreover, we extend the proof system for First Order Logic to Separation Logic and analyze its proof-theoretic properties. While soundness is maintained by a similar ranking property as in First Order Logic, completeness is lost, in general. We recover completeness partially by restricting the semantics of entailments with a notion of (matching) unfolding trees. Extending the proof system to handle limited cases of entailments between divergent predicates, whose unfolding trees do not match, but are related by reversal and rotation relations is possible. The completeness and algorithmic properties of such extensions, such as the decidability of the entailment problem, are considered for future work.

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A Additional Material

A.1 Proof of Lemma 1

Proof. Let $D^*$ be the set of finite sequences of elements from $D$, where $u_i$ denotes the $i$-th element of $u \in D^*$ and $|u|$ is the length of $u$. The subword order $\leq_{sw}$ on $D^*$ is defined as $u \leq_{sw} v$ iff there exists a strictly increasing mapping $f : [|u|] \to [|v|]$ such that $u_i = v_{f(i)}$ for all $i \in [|u|]$. A qo $\preceq$ on $D$ induces the following order on the set $D^*$: for all $u, v \in D^*$, $u \preceq v$ if there exists $v' \leq_{sw} v$ such that $|u| = |v'|$ and $u_i \preceq v'_i$, for all $i = 1, \ldots, |u|$. Because $D$ is countable, there is an indexing of its elements. Then each finite set $S \in \mathcal{P}_{fin}(D)$ is uniquely represented as a finite word and the result follows from Higman’s Lemma [Hig52], which states that $(D, \preceq)$ is a wqo only if $(D^*, \preceq^*)$ is a wqo. \hfill $\Box$
A.2 Proof of Theorem 2

Proof. By reduction from the inclusion problem for context-free languages, a known undecidable problem [Sip97, Theorem 5.10]. Let \( G = (\Xi, \Sigma, \Delta) \) be a context-free grammar, where \( \Xi \) is the set of nonterminals, \( \Sigma \) is the alphabet of terminals, and \( \Delta \) is a set of productions \( (X, w) \in \Xi \times (\Xi \cup \Sigma)^* \). For a nonterminal \( X \in \Xi \), we denote by \( L_X(G) \subseteq \Sigma^* \) the language produced by \( G \) starting with \( X \) as axiom. The problem ”given \( X, Y \in \Xi \), does \( L_X(G) \subseteq L_Y(G) \)?” is undecidable. Given a context-free grammar \( G = (\Xi, \Sigma, \Delta) \), we define a system \( S_G \) as follows:

- each nonterminal \( X \in \Xi \) corresponds to a predicate \( X(x^\sigma, y^\sigma) \), where \( \sigma \) is the only sort used in the reduction,
- each alphabet symbol \( a \in \Sigma \) corresponds to a function symbol \( \pi^a \), and a word \( w = a_1 \ldots a_n \in \Sigma^* \) is encoded by the context (i.e. the term with a hole) \( \pi = \pi_1(\ldots \pi_n(().) \),
- each grammar rule \( (X, u_1X_1 \ldots u_nX_n u_{n+1}) \in \Delta \) corresponds to a rule:

\[
\{(\phi(x, y, x_1, y_1, \ldots, x_n, y_n), X_1(x_1, y_1), \ldots, X_n(x_n, y_n); X(x, y))\}
\]

of \( S_G \), where \( \phi = x \equiv (\pi_1(x_1) \land (\bigwedge_{i=1}^{n-1} y_i \equiv (\pi_{i+1}(x_{i+1}) \land y_n \equiv (\pi_{n+1}(y)). In particular, a rule \( (\epsilon, X) \in \Delta \) maps into a rule \( \{(x \equiv y); X(x, y)\} \) of \( S_G \).

We must check that, for any nonterminals \( X, Y \in \Xi \), we have \( L_X(G) \subseteq L_Y(G) \) if and only if \( X \models_{S_G} Y \). This is proved using the following invariant:

\[ \forall w \in \Sigma^* . \ (\forall t \sigma^w . [x \leftarrow \pi(t), y \leftarrow t] \in \mu S_G^w(X)) \iff w \in L_X(G) \]

where \([x \leftarrow t, y \leftarrow u] \) denotes the valuation mapping \( x \) to \( t \) and \( y \) to \( u \). \( \square \)

A.3 Proof of Lemma 2

Proof. By reduction from the following undecidable problem: given a context-free grammar \( G = (\Xi, \Sigma, \Delta) \), where \( \Xi \) is the set of nonterminals, \( \Sigma \) is the alphabet of terminals, and \( \Delta \) is a set of productions \( (X, w) \in \Xi \times (\Xi \cup \Sigma)^* \), and two nonterminals \( X, Y \in \Xi \), is it the case that \( L_X(G) \cap L_Y(G) \neq \emptyset \)?

We encode \( G \) as a system, in the same way as done in the proof of Theorem 2, each nonterminal \( Z \in \Xi \) corresponding to a predicate \( Z(x, y) \).

Then we encode the problem \( L_X(G) \cap L_Y(G) \neq \emptyset \) using an additional rule \( \{(x_1 \approx x_2 \land y_1 \approx y_2, X(x_1, y_1), Y(x_2, y_2); P()) \). It is easy to check that the system is non-filtering iff \( L_X(G) \cap L_Y(G) \neq \emptyset \). \( \square \)

A.4 Proof of Lemma 3

Proof. Let \( \mathcal{S} \) be a system and \( \phi(x, x_1, \ldots, x_n), \psi(x, y_1, \ldots, y_m) \) be two arbitrary constraints, with goal variables \( x \) and subgoal variables \( \bigcup_{i=1}^n x_i \) and \( \bigcup_{j=1}^m y_j \), respectively. Then \( \mathcal{S} \) has the fvi property if and only if the following entailment does not hold:

\[
\phi(x, x_1, \ldots, x_n) \models^\mathcal{S} \exists y_1 \ldots \exists y_m . \psi(x, y_1, \ldots, y_m) \land \bigwedge_{j=1}^m \neg (y_j \approx x_i)
\]
where $y_j \equiv x_i$ is a shorthand for $\left( \bigwedge_{y \in y_j} \bigvee_{x \in x_i} y \approx x \right) \land \left( \bigwedge_{x \in x_i} \bigvee_{y \in y_j} y \approx x \right)$. In other words, $S$ has the fvi property if and only if the following equational problem has a solution:

$$\exists x \exists x_1 \ldots \exists x_n \forall y_1 \ldots \forall y_m . \phi(x, x_1, \ldots, x_n) \land \bigwedge_{j=1}^{m} \neg \psi(x, y_1, \ldots, y_m) \lor \bigvee_{i=1}^{n} y_j \equiv x_i$$

The last formula is not in CNF and expanding the formulae $y_j \equiv x_i$ to obtain a CNF form causes a simply exponential blowup. Since checking the satisfiability of an equational problem in CNF is $\text{NP}$-complete, the above check can be performed in $\text{NEXPTIME}$. If the size of each set of subgoal variables is bound to a constant, not part of the input, the size of each clause in the CNF expansion of the above formula is constants, thus there are at most polynomially many such constants and we apply [Pic03, Theorem 5.2] to obtain the $\text{NP}$ upper bound. \hfill \Box

**A.5 Proof of Lemma 4**

**Proof.** By [HLSV11, Theorem 1], we have:

$$\mu S^\pi(p_1) \times \ldots \times \mu S^\pi(p_n) \subseteq \bigcup_{i=1}^{k} \mu S^\pi(q_i) \times \ldots \times \mu S^\pi(q_i^n) \iff \bigwedge_{i=1}^{k} \bigvee_{j=1}^{n} (\mu S^\pi(p_i) \subseteq \bigcup \{ \mu S^\pi(q_i) \mid \ell \in [k], f_j(\overline{Q}_i) = i \}) \iff \bigvee_{i=1}^{n} (\mu S^\pi(p_i) \subseteq \bigcup \{ \mu S^\pi(q_i) \mid \ell \in [k], f_j(\overline{Q}_i) = i \})$$

The last step is the expansion of the second last formula in disjunctive normal form. \hfill \Box

**A.6 Proof of Lemma 5**

**Proof.** Assume that there is a proof for a sequent $\Gamma \vdash \Delta$ using the rule types in Figure 1. We prove the entailment $\bigwedge \Gamma \models_{S} \Delta$ by induction on the structure of the proof, using a case split, on the type of the inference rule used at the root. For the case (AX) soundness follows by the side condition of the rule. The other cases are:

- (LU) Let $p(x) \in \Gamma$ be a predicate atom, where $p(x) \leftarrow s R_1(x) \mid \ldots \mid R_n(x)$:

$$\mu S^\pi(\bigwedge \Gamma) = \mu S^\pi(p(x)) \cap \mu S^\pi(\bigwedge (\Gamma \setminus \{p(x)\}))$$

$$= (\bigcup_{i=1}^{n} \mu S^\pi(\bigwedge R_i(x))) \cap \mu S^\pi(\bigwedge (\Gamma \setminus \{p(x)\}))$$

$$= \bigcup_{i=1}^{n} \mu S^\pi(\bigwedge R_i(x) \land \bigwedge (\Gamma \setminus \{p(x)\}))$$

The second equivalence holds because $\mu S^\pi$ is the least solution of $S$, thus $\mu S^\pi(p(x)) = \bigcup_{i=1}^{n} \mu S^\pi(\bigwedge R_i(x))$. Since each sequent $R_i, \Gamma \setminus \{p(x)\} \vdash \Delta$, $i \in [n]$, has a proof which is a subtree of the proof for $\Gamma \vdash \Delta$, by the induction hypothesis we have:

$$\mu S^\pi(\bigwedge R_i(x) \land \bigwedge (\Gamma \setminus \{p(x)\})) \subseteq \mu S^\pi(\bigvee \Delta), \text{ for all } i \in [n]$$

which leads to $\bigwedge \Gamma \models_{S} \Delta$, as required.
(RU) Let \( p(x) \in \Delta \) be a predicate atom, such that \( p(x) \leftarrow R_1(x, y_1) \mid \ldots \mid R_n(x, y_n) \). We have that \( \mu^{S^2}(p(x)) = \bigcup_{i=1}^n \mu^{S^2}(\exists y_i \land R_i(x, y_i)) \), because \( \mu^{S^2} \) is the least solution of \( S \). Since \( \Gamma \vdash \exists y_1 \land R_1(x, y_1), \ldots, \exists y_n \land R_n(x, y_n), \Delta \vdash p(x) \) has a proof which is a subtree of the proof for \( \Gamma \vdash \Delta \), by the induction hypothesis, we have:

\[
\mu^{S^2}(\bigwedge \Gamma) \subseteq \bigcup_{i=1}^n \mu^{S^2}(\exists y_i \land R_i(x, y_i)) \cup \mu^{S^2}(\Delta \setminus p(x)) = \mu^{S^2}(\bigvee \Delta)
\]

and thus \( \bigwedge \Gamma \models^T \bigvee \Delta \), as required.

(RD) Assume that \( x = \{x_1, \ldots, x_n\} \) and

\[
\Gamma = \{\phi(x, x_1, \ldots, x_n), p_1(x_1), \ldots, p_n(x_n)\}
\]

\[
\Delta = \{\exists y_1 \cdot \phi_1(x, y_1) \land Q_1(y_1), \ldots, \exists y_n \cdot \phi_k(x, y_k) \land Q_k(y_k)\}.
\]

Since the sequent \( p_1(x_1), \ldots, p_n(x_n) \vdash \{Q_j \mid \theta \in S_j\}_{j=1}^i \) has a proof which is a subtree of the proof for \( \Gamma \vdash \Delta \), by the induction hypothesis, we have:

\[
\mu^{S^2}(p_1(x_1) \land \ldots \land p_n(x_n)) \subseteq \bigcup_{\theta \in S_j} \mu^{S^2}(Q_j) \subseteq \bigcup_{\theta \in S_j} \mu^{S^2}(Q_j)
\]

because for all \( j \in [i] \), we have \( S_j \subseteq Sk(\phi, \psi_j) \). Furthermore, by Definition 3, for each \( \theta \in Sk(\phi, \psi_j) \), for all \( j \in [i] \) we have \( \mu^{S^2}(\phi) \subseteq \mu^{S^2}(\psi_j) \). We compute:

\[
\mu^{S^2}(\phi \land p_1(x_1) \land \ldots \land p_n(x_n)) \subseteq \bigcup_{\theta \in S_j} \mu^{S^2}(\psi_j \land Q_j)
\]

because for all \( j \in [i] \), we have \( S_j \subseteq Sk(\phi, \psi_j) \). Furthermore, by Definition 3, for each \( \theta \in Sk(\phi, \psi_j) \), for all \( j \in [i] \) we have \( \mu^{S^2}(\phi) \subseteq \mu^{S^2}(\psi_j) \). We compute:

- (SP) is a direct consequence of Lemma 4.

- (ID) Suppose, by contradiction, that \( \bigwedge \Gamma \not\models^T \bigvee \Delta \), i.e. there exists a valuation \( \nu \in \mu^{S^2}(\bigwedge \Gamma) \) such that \( \nu \not\models \mu^{S^2}(\bigvee \Delta) \). Let \( \Gamma \vdash \Delta = \Gamma_1 \cup \Delta_1, \ldots, \Gamma_k \cup \Delta_k \) be a path in the proof and \( R_1, \ldots, R_k \) be the sequence of inference rules applied on this path. Clearly, none of these rules are (AX), or (ID), and at least one of them is of the type (LU), as required by the side condition of (ID). By the previous points, the existence of a counterexample for the consequent of a rule \( R_i \) implies the existence of a counterexample for at least one of its antecedents, if \( R_i \) is of the type (LU), (RU), (RD) or (SP). If the antecedent on the path has no counterexample, then it is safe to close the path using (ID), so we are left with the case when each consequent on the path has a counterexample. Let \( x = x_1, \ldots, x_{k+1} = x' \) be the sets of free variables in \( \bigwedge \Gamma_1, \ldots, \bigwedge \Gamma_k \). Since \( S \) is ranked, we have that \( \nu(x_i) \geq 1 \nu(x_{i+1}) \) for all \( i \in [k] \), by a case split on \( R_i \):

  - (LU) if \( \{\phi(y, y_1, \ldots, y_n), p_1(y_1), \ldots, p_n(y_n)\} \), \( p(y) \in S \) is the rule that replaces a predicate atom \( p(y) \in \Gamma_i \), in each valuation \( \nu : y \cup y_1 \cup \cdots \cup y_n \) satisfy...
... \cup y_n \rightarrow D$, for each $y \in y_1 \cup \ldots \cup y_n$ there exists $z \in y$ such that
\[ \nu(z) > \nu(y). \]
Thus, since $y \subseteq x_i$ and $x_{i+1} = x_i \cup y_1 \cup \ldots \cup y_n$, then
\[ \nu(x_i) > \nu(x_{i+1}). \]

- (RU) Since $\text{FV}(\Gamma_i) = \text{FV}(\Gamma_{i+1})$, we have $\nu(x_i) = \nu(x_{i+1}).$
- (RD), (SP) Since $\text{FV}(\Gamma_i) \supseteq \text{FV}(\Gamma_{i+1})$, we have $\nu(x_i) \geq \nu(x_{i+1}).$

Because (LU) is applied at least once on the path between the pivot and the consequent, we obtain that $\nu(x) > \nu(x').$ Since, however, $\Delta' \subseteq \Delta$, one can obtain an infinite path by repeating the rules $R_1, \ldots, R_k$ any number of times. But this would result in an infinite decreasing sequence, in contradiction with the fact that $\geq^1$ is a wfqo (Theorem 1).

\[ \square \]

### A.7 Proof of Lemma 6

**Proof.** Let $p^\#$ be the number of predicates in the inductive system $S$, $r^\#$ the number of rules in $S$ and $s^\#$ the maximum number of subgoals occurring in the rules of $S$. Consider a structured derivation starting from a basic sequent $p(x) \vdash q_1(x), \ldots, q_n(x)$ and a path $\pi$ in this derivation that leads to another basic sequent $r(x) \vdash s_1(x), \ldots, s_m(x)$ without containing any other basic sequents. Clearly, only LU, RU, $\land R$, RD and SP can be applied on $\pi$, otherwise $\text{AX}$ and $\text{ID}$ would not allow us to reach the second basic sequent. The left-hand side of the sequents along the path $\pi$ allow us to apply LU at most once (on $p(x)$), RD at most once (after LU, on the rule that replaces $p(x)$) and SP at most once (after RD, if we are left with multiple predicates on the left-hand side).

The right-hand side of the sequents along $\pi$ allows for $n$ applications of RU, where $n$ is at most $p^\#$, and $r^\# \ast (s^\# - 1)$ applications of $\land R$ (there can be at most $s^\# - 1$ for every rule resulted from RU and reduced by RD, and there can be at most $r^\#$ rules). Thus, on any path between two consecutive basic sequents is of length at most $b^\# = 3 + p^\# + r^\# \ast (s^\# - 1)$, which is a constant determined by the system $S$.

(1) Let $D$ be an irreducible structured derivation and suppose, by contradiction, that $\pi$ is an infinite path in $D$. Let $\rho$ is any subsequence of $\pi$ on which no (LU) rule has been applied. The case in which $\rho$ reaches its maximum possible length is when it starts right after the application of LU on a basic sequent and it extends until the next possible application of LU, while encountering the next basic sequent and containing the results of applying all possible rules before LU. The only rule that can be applied on a basic sequent before LU is RU and it can occur a maximum of $p^\#$ times. Thus, $\rho$ must be finite, with a maximum length of $b^\# - 1 + p^\#$. Then (LU) is applied infinitely often on $\pi$, and since $D$ is structured, also (RD) must be applied infinitely often. But, since the antecedent of each application of an (RD) rule contains no constraints, and moreover, the consequents thereof are of the form $\phi(x, x_1, \ldots, x_n), p_1(x_1), \ldots, p_n(x_n) \vdash \Delta$, the left hand side of such sequents must have been produced by a (LU) rule with consequent of the form $p(x) \vdash \Delta$, because (LU) are the only rules introducing constraints on the left hand side of a sequent. But such sequents can only be the antecedents of (SP) or (RD) rules, with $n = 1$ in the latter case. In the case of (SP) rules, the right hand side $\Delta$ is a set consisting of predicates only,
thus $p(x) \vdash \Delta$ is a basic sequent. But this must be the case also for (RD) rules, because $S$ has the fvi property and the assumption that $(\land R)$ is used eagerly after an application of (RD) to rule out conjunctions of predicates with the same argument list. Since there are finitely many predicates in $S$, the number of basic sequents is bounded thus some basic sequent must occur twice on $\pi$ and (ID) is applicable, which contradicts the assumption that $D$ is irreducible. Then $\pi$ must be finite, and since it was chosen arbitrarily, we obtain that $D$ is finite, by an application of König’s Lemma.

(2) Suppose, by contradiction, that $D(\Gamma \vdash \Delta)$ is infinite and let $D_1, D_2, \ldots$ be an infinite sequence of finite, maximal derivations of $\Gamma \vdash \Delta$. By point (1), each derivation in $D(\Gamma \vdash \Delta)$ is finite. W.l.o.g. we assume that all $D_i$ are obtained by applying more than one rule — in the opposite case, one can extract an infinite subsequence that satisfies this condition. The number of all possible basic sequents is $p^\pi \ast (2^{p^\pi} - 1)$, as there are $p^\pi$ predicates that can be on the left-hand side and $2^{p^\pi} - 1$ possible non-empty subsets of predicates on the right-hand side. Because we have a finite number of base sequents and the length of a path between two consecutive basic sequents is of finite length at most $b^\pi$, then there must exist a derivation $D_i$ in the infinite sequence chosen above that contains a path in which the same basic sequent appears at least twice. But then this means that $D_i$ is irreducible and, thus, that $D_i \notin D(\Gamma \vdash \Delta)$, which contradicts our initial assumption.

A.8 Proof of Lemma 7

Proof. By induction on the length of the path $p(x) \vdash q(x) = \Gamma_1 \vdash \Delta_1, \ldots, \Gamma_N \vdash \Delta_N = \Gamma \vdash \Delta$ from the derivation in which $\Gamma \vdash \Delta$ occurs. The case $N = 1$ is trivial. Assuming that $\Gamma_{N-1}$ and $\Delta_{N-1}$ are of the required form, we prove that $\Gamma_N$ is tree-shaped and $\Delta_N$ consists of finite conjunctions of tree-shaped sets, in which all subgoal variables occur existentially quantified. We make a case split, based on the type of the last inference rule on the path:

- (LU) in this case $\Gamma_{N-1}$ is tree-shaped and there exists a tree $t$ associated with $\Gamma_{N-1}$ such that $t(q) = p(y)$, for some frontier position $q \in \text{fr}(t)$ and $p(x) \in \text{Pred}$. Then there exists a rule $R = (\phi(x, x_1, \ldots, x_h), p_1(x_1), \ldots, p_h(x_h)) \in S$ such that $\Gamma_N = R, \Gamma_{N-1}\setminus p(y)$ and $t[q] \circ \tau_h(\phi(y, x_1, \ldots, x_h), p_1(x_1), \ldots, p_h(x_h))$ replaces $t$ in the set of trees that represents $\Gamma_N$.
- (RU) in this case there exists $p(x) \in \Delta_{N-1}$ and a tree consisting of a single node labeled with $p(x)$. This tree is replaced in $\Delta_N$ by trees $t_1, \ldots, t_h$ corresponding to the tree-shaped sets $R_1(x, x_1^1, \ldots, x_{n_1}^1), \ldots, R_h(x, x_1^h, \ldots, x_{n_h}^h)$, where $p(x) \leftarrow S R_1 | \ldots | R_h$ are the rules from the definition of $p(x)$.
- the cases (RD) and (SP) are trivial, because the consequents of these rules consist of sequents of the form $p_1(x_1), \ldots, p_n(x_n) \vdash Q_1(x_1, \ldots, x_n), \ldots, Q_m(x_1, \ldots, x_n)$, where each $Q_i$ is a conjunction of predicates. \qed
A.9 Proof of Lemma 8

Proof. By induction on $\mathcal{D}(\Gamma \vdash \Delta)$, ordered by $\subseteq^\forall$. By the fact that $\mathcal{D}(\Gamma \vdash \Delta)$ is finite, by Lemma 6 (2), and Lemma 1, we have that $\mathcal{D}(\Gamma \vdash \Delta), \subseteq^\forall$ is a wqo, thus we can apply the induction principle to it. We have $\mathcal{D}(\Gamma \vdash \Delta) = \bigcup_R \mathcal{D}_R(\Gamma \vdash \Delta)$, where $\mathcal{D}_R(\Gamma \vdash \Delta)$ denotes the subset of $\mathcal{D}(\Gamma \vdash \Delta)$ consisting of derivations starting with an inference rule of type $R$. Observe that $R$ cannot be of type (AX) because then no derivation in $\mathcal{D}_R(\Gamma \vdash \Delta)$ may contain a leaf $\Gamma' \vdash \emptyset$, and $R$ cannot be of type (ID), because no derivation may start with an application of (ID). We distinguish the following remaining cases for $R$:

- (LU) Let $p(x) \in \Gamma$ be the predicate atom chosen for replacement and $R_1, \Gamma \setminus p(x) \vdash \Delta$, $\ldots, R_n, \Gamma \setminus p(x) \vdash \Delta$ be the antecedents of $R$, where $p(x) \leftarrow \mu R \vdash \emptyset \setminus R_n$. It is sufficient to prove that there exists $i \in [n]$ such that every derivation in $\mathcal{D}(R_i, \Gamma \setminus p(x) \vdash \Delta)$ contains a leaf $\Gamma' \vdash \emptyset$ with no subgoals, and conclude by an application of the induction hypothesis. Suppose, by contradiction, that for each $i \in [n]$, there exists $D_i \in \mathcal{D}(R_i, \Gamma \setminus p(x) \vdash \Delta)$ not containing such a leaf. Then there exists a derivation for $\Gamma \vdash \Delta$ with the same property, which contradicts the hypothesis of the lemma. Thus, there must exist $i \in [n]$ such that every derivation $D \in \mathcal{D}(R_i, \Gamma \setminus p(x) \vdash \Delta)$ contains a leaf $\Gamma' \vdash \emptyset$. By the induction hypothesis, there exists $\nu \in \mu \mathcal{S}(\bigwedge R_i \setminus p(x)) \setminus \mu \mathcal{S}(\bigvee \Delta)$, because $\mu \mathcal{S}(\bigwedge R_i \setminus p(x)) \subseteq \mu \mathcal{S}(\bigvee \Delta)$.

- (RU) Let $p(x) \leftarrow \mu R \vdash \emptyset$. Every $D \in \mathcal{D}(\Gamma \vdash \emptyset) \setminus p(x) \vdash \Delta$ contains a leaf $\Gamma' \vdash \emptyset$, therefore, by the induction hypothesis, there exists $\nu \in \mu \mathcal{S}(\bigwedge \Gamma) \setminus \mu \mathcal{S}(\bigvee \Delta)$

because

$$
\begin{align*}
\mu \mathcal{S}(\bigwedge \Gamma) \setminus \mu \mathcal{S}(\bigvee \Delta) &= \bigcup_{i=1}^n \exists y_i \cdot R_i(x, y_i) \setminus p(x) \bigvee \Delta \bigwedge (\forall x \setminus \bigwedge \nu) \\
&= \bigcup_{i=1}^n \mu \mathcal{S}(\exists y_i \cdot R_i(x, y_i)) \setminus \mu \mathcal{S}(\bigvee \Delta) \bigwedge (\forall x \setminus \bigwedge \nu)
\end{align*}
$$

- (RD) Let $\Gamma = \{ \phi(x, x_1, \ldots, x_m), p_1(x_1), \ldots, p_m(x_m) \}$ and $\Delta = \{ \exists y_1 \cdot \psi_1(x, y_1) \}$.

\ldots, where $\phi, \psi_1, \ldots, \psi_k$ are constraints, $p_1, \ldots, p_m$ are predicates, and $Q_1, \ldots, Q_k$ are conjunctions of predicates. W.l.o.g. we assume that $S_j = S_k(\phi, \psi_j)$, for all $j \in [k]$. Since $\mathcal{S}$ has the fvi property, each $S_j$ is finite. We distinguish the following cases:

- if $m = 0$, i.e. $\Gamma$ contains no predicate atoms. Since $p(x) \setminus q(x) \sim 0$, by Lemma 7, $\Gamma$ is tree-shaped, meaning that $\Gamma = \phi(x)$, and $\phi$ has no subgoal variables. Again, we distinguish two cases:
  - if $k = 0$ then $\Delta = \emptyset$ and, since $\phi$ is a constraint from $\mathcal{S}$, it must be satisfiable. Thus any model of $\phi(x)$ contradicts the entailment $\phi(x) \models T \bot$, and moreover, such a model exists, because $\mathcal{S}$ is non-filtering.
such that for all $j \in [k]$, because $S$ has the fvi property. In this case we can trivially find a counterexample for the entailment $\phi(x) \models^\Delta \Delta$.

• else, if $m > 0$ the antecedent of the rule (RD) is $p_1(x_1), \ldots, p_m(x_m) \vdash \{ Q_j \mid j \in \text{Sk}(\phi, \psi_j) \}_{j=1}^k$. Moreover, by the side condition of the rule, we have $\phi \models^\Delta \bigwedge_{j=1}^k \exists y_j \cdot \psi_j$ and $\phi \not\models^\Delta \bigwedge_{j=i+1}^k \exists y_j \cdot \psi_j$, via a possible reordering of $\Delta$. Since every derivation $D \in \mathcal{D}(p_1(x_1), \ldots, p_m(x_m) \vdash \{ Q_j \mid \theta \in \text{Sk}(\phi, \psi_j) \}_{j=1}^i)$ must contain a leaf $I' \vdash \emptyset$, by the induction hypothesis there must exist a counterexample $\nu \in \mu_S^\Delta(\bigwedge_{\ell=1}^m p_\ell(x_\ell)) \setminus \mu_S^\Delta(Q_j, \theta)$, for all $j \in [i]$ and all $\theta \in \text{Sk}(\phi, \psi_j)$. Because we assumed that $S$ is non-filtering, there exists $\nu'$ such that $\mathcal{I},\nu' |= \phi$ and $\nu$ and $\nu'$ agree on $x_1, \ldots, x_m$. Furthermore, because $S$ is assumed to be non-overlapping and $\phi \not\models^\Delta \exists y_j \cdot \psi_j$, for all $j \in [i+1, k]$, we obtain that $\phi \land \exists y_j \cdot \psi_j$ is unsatisfiable, hence $\nu'$ is also a counterexample for the entailment $\phi \models^\Delta \exists y_j \cdot \psi_j$, for each $j \in [i+1, k]$ and thus for the entailment $\phi \models^\Delta \bigwedge_{j=i+1}^k \exists y_j \cdot \psi_j \land Q_j$. Suppose now, by contradiction, that $\nu'$ is a model of $\exists y_j \cdot \psi_j(x, y_j) \land Q_j(y_j)$, for some $j \in [i]$. Then $\nu'$ is a model of $\exists y_j \cdot \psi_j(x, y_j)$ also. Since $\phi \models^\Delta \exists y_j \cdot \psi_j$ and $S$ has the fvi property, it must be the case that $\phi \models^\Delta \exists y_j \psi_j \theta$ for all $\theta \in \text{Sk}(\phi, \psi_j)$ and, moreover, there is no other Skolem function that witnesses this entailment, besides the ones in $\text{Sk}(\phi, \psi_j)$. But then it must be that $\nu'$ is a model of $\psi_j \theta$, for all $\theta \in \text{Sk}(\phi, \psi_j)$, and only for those substitutions. Since the range of each $\theta \in \text{Sk}(\phi, \psi_j)$ is $x_1 \cup \ldots \cup x_m$, we have that $\nu' \in \mu_S^\Delta(Q_j, \theta)$ for some $\theta \in \text{Sk}(\phi, \psi_j)$, which contradicts the assumption that $\nu'$ is a counterexample of the antecedent. Then $\nu'$ cannot be a model of $\exists y_j \cdot \psi_j(x, y_j) \land Q_j(y_j)$, for some $j \in [i]$, and since it cannot be a model of the right hand side for $j \in [i+1, k]$ either, it is a counterexample for the entailment $\bigwedge I \models^\Delta \bigwedge_{j=i+1}^k \exists y_j \cdot \psi_j \land Q_j$ as required.

• (SP) Every derivation for $I \vdash \Delta$ starts with the following inference rule, for some tuple of indices $(i_1, \ldots, i_m) \in [m]^{m^k}$:

\[
\begin{array}{c}
p_{i_1}(x_1) \vdash \{ q_{i_1}^\ell(x_1) \mid \ell \in [k], f_{1}(\overline{\ell}) = i_1 \} \ldots p_{i_m}(x_m) \vdash \{ q_{i_m}^\ell(x_m) \mid \ell \in [k], f_{m}(\overline{\ell}) = i_m \} \\
p_{1}(x_1), \ldots, p_{m}(x_m) \vdash Q_1(x_1, \ldots, x_m) \ldots, Q_k(x_1, \ldots, x_m)
\end{array}
\]

By our assumption, each derivation $D \in \mathcal{D}(I \vdash \Delta)$ contains a leaf $I' \vdash \emptyset$. Suppose, by contradiction, that there exists a tuple $(i_1, \ldots, i_m) \in [m]^{m^k}$ such that for all $j \in [m^k]$ there exists $D \in \mathcal{D}(p_{i_j}(x) \vdash \{ q_{i_j}^\ell(x) \mid \ell \in [k], f_{j}(\overline{\ell}) = i_j \})$ does not have a leaf $I' \vdash \emptyset$. Then we can build a derivation for $I \vdash \Delta$ that does not contain any such leaves, which contradicts our assumption. Thus it must be the case that for all tuples $(i_1, \ldots, i_m) \in [m]^{m^k}$ there exists $j \in [m^k]$ such that for all derivations $D \in \mathcal{D}(p_{i_j}(x) \vdash \{ q_{i_j}^\ell(x) \mid \ell \in [k], f_{j}(\overline{\ell}) = i_j \})$ contains a leaf $I' \vdash \emptyset$. By the inductive hypothesis, for all tuples $(i_1, \ldots, i_m) \in [m]^{m^k}$ there exists $j \in [m^k]$ such that $\mu_S(p_{i_j}(x)) \not\subseteq \mu_S^\Delta(\{ q_{i_j}^\ell(x) \mid \ell \in [k], f_{j}(\overline{\ell}) = i_j \})$. Then, by Lemma 4, we
have \(\mu S^\mathcal{T}(p_1) \times \ldots \times \mu S^\mathcal{T}(p_m) \not\subseteq \bigcup_{i=1}^r \mu S^\mathcal{T}(q_i) \times \ldots \times \mu S^\mathcal{T}(q_{m_i})\), proving the claim. \(\square\)

### A.10 Proof of Theorem 3

**Proof.** Since \(\mathcal{S}\) is non-filtering and non-overlapping, and moreover, it has the fvi property, since \(p \models^2 q\), by Lemma 8, there exists a finite maximal, structured and irreducible derivation \(D \in \mathcal{D}(p(x) \vdash q(x))\) which does not contain a leaf \(\Gamma \vdash \emptyset\). But then, no node in \(D\) is of the form \(\Gamma \vdash \emptyset\), because all descendants of such a node must have empty right-hand sides as well.

We first show that this derivation is actually a proof (i.e. all its leaves are \(\top\)). Suppose there exists a leaf that is not \(\top\). Let \(\pi\) be the path in \(D\) leading to this leaf. Since \(D\) is a maximal derivation, \(\pi\) cannot be extended any further by the application of an inference rule. Assume that the last inference rule applied on \(\pi\) is of type \(R\) and has the consequent \(\Gamma' \vdash \Delta'\). Since \(p(x) \vdash q(x) \leadsto \Gamma' \vdash \Delta'\), by Lemma 7, \(\Gamma'\) is a tree-shaped set and \(\Delta'\) consists of existentially quantified finite conjunctions of tree-shaped sets. Also, \(R\) cannot be AX or ID, because then the leaf would be \(\top\). We do a case split based on \(R\):

1. (LU) Then \(\Gamma \vdash \Delta\) is of the form \(\Gamma'' \vdash \Delta'\), where \(\Delta \neq \emptyset\). Let \(\pi\) be the path in \(D\) leading to this leaf. Since \(D\) is a maximal derivation, \(\pi\) cannot be extended any further by the application of an inference rule. Assume that the last inference rule applied on \(\pi\) is of type \(R\) and has the consequent \(\Gamma' \vdash \Delta'\). Since \(p(x) \vdash q(x) \leadsto \Gamma' \vdash \Delta'\), by Lemma 7, \(\Gamma'\) is a tree-shaped set and \(\Delta'\) consists of existentially quantified finite conjunctions of tree-shaped sets. Also, \(\Delta'\) cannot be AX or ID, because then the leaf would be \(\top\). We do a case split based on \(R\):
   - \(\Gamma'' \vdash \Delta'\) then we can apply (RD) to the sequent \(\Gamma'' \vdash \Delta'\) and extend \(D\), which results in a contradiction.
   - \(\Gamma'' \vdash \Delta'\) then we can apply (RD) to the sequent \(\Gamma'' \vdash \Delta'\) and extend \(D\), which results in a contradiction.

2. (RU) Then \(\Gamma \vdash \Delta\) is of the form \(\Gamma' \vdash \exists y_i \cdot (\bigwedge_{i=1}^n R_i(x,y_i))\), where \(p'(x)\) is a predicate atom and \(p'(x) \leftarrow S R_1 \mid \ldots \mid R_q\). If \(\Delta' \vdash \Delta'\) contains at least one predicate, we can apply (RU) and extend \(D\), contradiction. Otherwise, because \(\Delta' \neq \emptyset\) consists of existentially quantified finite conjunctions of tree-shaped sets and it does not contain any predicates, then it must be the case that \(\Delta'\) consists of existentially quantified conjunctions over rules from \(\mathcal{S}\), obtained from previous applications of (RU), or predicate conjunctions that are not singleton, obtained from previous applications of (RD). We distinguish the following cases:
– $\Gamma'$ contains a predicate atom. Then we can apply (LU) and extend $D$, contradiction.
– $\Gamma'$ does not contain predicate atoms. Because $\Gamma'$ is tree-shaped and $D$ is structured, $\Gamma'$ can only contain a constraint with no subgoal variables.

Then we can apply (RD) and extend $D$, contradiction.

3. (RD) Then $\Gamma \vdash \Delta$ is of the form $p_1(x_1), \ldots, p_n(x_n) \vdash Q_1(x_1, \ldots, x_n), \ldots, Q_m(x_1, \ldots, x_n)$ and we can apply (LU) – or even $(\land R)$, (RU) or (SP) if possible – to $\Gamma \vdash \Delta$, which means that we can still extend $\pi$, leading to a contradiction.

4. $(\land R)$ $\Gamma \vdash \Delta$ is of the form $\Gamma \vdash p(x) \land Q$. Since we only apply $(\land R)$ as cleanup after (RD), $\Gamma$ only contains predicates and $Q$ is a conjunction of predicates. Then we can continue to apply $(\land R)$ if $Q$ contains $p'(x') \land q'(x')$, or apply (LU), (RU), or (SP), leading to a contradiction.

5. (SP) Then $\Gamma \vdash \Delta$ is of the form $p(x) \vdash q_1(x), \ldots, q_n(x)$ and we can apply (LU) or (RU) to $\Gamma \vdash \Delta$, which means that we can still extend $\pi$, contradiction.

We will now show that the sequence of inference rules fired on each path in $D$ is captured by the strategy $S$. Let $\pi$ be an arbitrary path in $D$. Since $D$ is a maximal derivation, $\pi$ cannot be extended any further by the application of an inference rule. W.l.o.g. we assume that the first application of (LU) is not immediately preceded by an application of (RU) — otherwise, one can obtain the same sequent by swapping the first applications of (RU) and (LU), respectively. The proof goes by induction on the number $N \geq 1$ of basic sequents that occur on $\pi$.

If $N = 1$, i.e. the only basic sequent $p(x) \vdash q(x)$ occurs on the first position of $\pi$. In this case (SP) is never applied on $\pi$, because its antecedent is a basic sequent, and thus $N > 1$, contradiction. We distinguish two cases:

1. If (LU) is not applied on $\pi$, the only possibility is to apply directly $AX \in S$ to $p(x) \vdash q(x)$, thus ending the path. Otherwise, (LU) is enabled, which contradicts the maximality of $\pi$.

2. Otherwise, (LU) is applied on $\pi$, and it must be applied in the beginning, because only (LU) and (RU) are applicable on $p(x) \vdash q(x)$ and we assumed that (RU) does not immediately precede (LU). Assume that the first rule application on $\pi$ is:

$$
\begin{array}{c}
\text{LU} \\
\hline
\Gamma' \vdash \Delta' \\
p(x) \vdash q(x)
\end{array}
$$

Then $\langle \Gamma', p(x) \rangle \in S$ and (LU) cannot be applied again without applying (RD) first, due to the assumption that $D$ is structured. Since $\Delta' = \{q(x)\}$ after the first application of (LU), we can now apply (RU). Because (SP) is never applied on $\pi$, either (AX) or (RD) can be applied next. In the first case, we obtain $\Lambda(\pi) \in LU \cdot RU \cdot AX \in S$. In the second case, if $n = 1$ in the antecedent of RD, since $S$ has the fvi property, we obtain that the antecedent of RD is a basic sequent, contradicting our assumption. Then it must be the case that $n > 1$, and in this case RD is not applicable any longer, because
the number of predicates will always be bigger than the number of subgoal variables in the constraint, on the left hand side. The only possibilities for continuation are then (AX), (ID), (∧R), (LU) and (RU). However, (∧R) is applicable only a finite number of times, equal to the number of predicate conjunctions with the same arguments on the right and side, (RU) can also only be applied a finite number of times, equal to the number of predicates on the right hand side, and (LU) is applicable at most once, because (RD) is no longer applicable. In both cases, π is not maximal, because (LU) is enabled.

Then the only possibility is to end the path by (AX) or (ID), obtaining $LU \cdot RU \cdot RD \cdot AR^* \cdot LU? \cdot RU^* \cdot (AX \mid ID) \subseteq S$.

If $N > 1$, let $π = τρ$, where $ρ$ starts with the second occurrence of a basic sequent in $π$. As before, the first occurrence is the initial sequent $p(x) \models q(x)$. Then the head of $ρ$ is the antecedent of either a (SP) rule or a (RD) rule with $n = 1$. In the first case, the last rule on $τ$ is preceded by a (RD) rule (and, optionally, several applications of (RD) for cleanup), and let $τ'$ denote the prefix of $τ$ up to the last application of an (RD) rule. As argued before, the consequent of this rule is a rule of $S$ with goal $p(x)$, and there must have been a previous application of (LU). Since between the antecedent of (LU) and the consequent of (RD) the left hand side of the sequents is unchanged, the only possibility is that (RU) has been used all along, thus $Λ(τ) ∈ LU \cdot RU^* \cdot RD \cdot SP?$. By the inductive hypothesis, the sequence of rules on $Λ(ρ) ∈ S$, thus $Λ(π) ∈ (LU \cdot RU^* \cdot RD \cdot \land R^* \cdot SP?) \cdot S \subseteq S$. □

A.11 Proof of Lemma 9

Proof. Given a rule $R = \{\{φ(x, y_1, \ldots, y_m), q_1(y_1), \ldots, q_m(y_m)\}, p(x)\} ∈ S$, let $P = \{(A_1, E_1), \ldots, (A_m, E_m)\} ∈ µS^1(q_1) × \ldots × µS^1(q_m)$ be a tuple of pairs and $C ⊆ FV(φ) \times FV(φ)$ be a relation on variables, such that the formula $ω_R(P, C)$ is satisfiable. We claim that, if $φ * ω_R(P, C)$ is satisfiable then for each tuple of models $\{⟨T_1, h_1⟩, \ldots, ⟨T_m, h_m⟩\} ∈ µS^1(q_1) × \ldots × µS^1(q_m)$, such that $\bigcup_{j=1}^m h_j$ satisfies $ω_R(P, C)$, there exists $u, h \models φ$, such that $\text{dom}(h) \cap (\bigcup_{i=1}^m \text{dom}(h_i)) = \emptyset$ and $ν(y_i) = ν(y_j) = T_j$, for all $j ∈ [m]$. The proof idea for this claim is that, because $ω_R(P, C)$ specifies exactly those variables which are allocated and those which are not, and the pairs of variables which are equal, as well as the ones which are not, the truth value of $φ * ω_R(P, C)$ is invariant under the renaming of the values of $x ∪ \bigcup_{i=1}^m y_i$, as long as the allocations and equalities are preserved. Moreover, for each tuple of models $\{⟨T_1, h_1⟩, \ldots, ⟨T_m, h_m⟩\} ∈ µS^n(q_1) × \ldots × µS^n(q_m)$ is a model of some formula $ω_R(P, C)$, for some $P = \{(A_1, E_1), \ldots, (A_m, E_m)\} ∈ µS^1(q_1) × \ldots × µS^1(q_m)$ and $C ⊆ FV(φ) × FV(φ)$. Then, for each rule $R = \{\{φ(x, y_1, \ldots, y_m), q_1(y_1), \ldots, q_m(y_m)\}, p(x)\} ∈ S$ we need to check satisfiability of $φ * ω_R(P, C)$, for each $P ∈ \times_{i=1}^m µS^1(q_i)$ and $C ∈ FV(φ) × FV(φ)$, such that $ω_R(P, C)$ is satisfiable. But this requires storing at most $2^n + n^2$ pairs, where each pair can be stored using $O(n)$ bits. Moreover, checking the satisfiability of the formula $φ * ω_R(P, C)$ is possible in polynomial space and the generation of the set $µS^1(p)$, for each $p ∈ \text{Pred}$, requires polynomial space as well. □
A.12 Proof of Lemma 10

Proof. Consider two constraints \( \phi_1(x, x_1, \ldots, x_n) \) and \( \phi_2(x, y_1, \ldots, y_m) \), with goal variables \( x \) and subgoal variables \( \bigcup_{i=1}^{n} x_i \) and \( \bigcup_{j=1}^{m} y_j \), respectively, and the entailment:

\[
\phi_1(x, x_1, \ldots, x_n) \models \exists y_1 \ldots \exists y_m \cdot \phi_2(x, y_1, \ldots, y_m) \land \bigvee_{j=1}^{m} \bigwedge_{i=1}^{n} (x_i \not \equiv y_j) \lor (y_j \not \equiv x_i)
\]

where \( y_j \equiv x_i \) is a shorthand for \( \left( \bigwedge_{y \in y_j} \bigvee_{x \in x_i} y \approx x \right) \land \left( \bigwedge_{x \in x_i} \bigvee_{y \in y_j} y \approx x \right) \).

Then the system does not meet the fvi condition iff the entailment above holds for some constraints \( \phi_1 \) and \( \phi_2 \). Since the entailment problem is in \( \text{PSPACE} \) for formulae of the form \( \forall \theta \cdot \phi_1(x, \theta) \land \phi_2(x, \theta) \), where \( \phi_1 \) and \( \phi_2 \) are quantifier-free SL formulae [RIS17], the fvi problem is in \( \text{PSPACE} \). If \( \phi_1 \) and \( \phi_2 \) are symbolic heaps, the above entailment problem is in \( \Pi^2_2 \) [AGH+14, Theorem 6], thus the fvi problem is in \( \Sigma^P_2 \). \( \square \)

A.13 Proof of Lemma 11

Proof. Let \( \bigotimes_{i=1}^{n} \mu S(r_i) \overset{\text{def}}{=} \{(u_1, \ldots, u_n), h_1 \uplus \ldots \uplus h_n) \mid (u_i, h_i) \in \mu S(r_i), i \in [n]\} \), for some predicates \( r_1(x_1), \ldots, r_n(x_n) \) in \( S \), and \( U_k = L^k \times \text{Heaps} \). Then the following property holds:

\[
\bigotimes_{i=1}^{n} \mu S(r_i) = \bigcap_{i=1}^{n} \left( \bigotimes_{j=1}^{i-1} U_{|x_i|} \otimes \mu S(r_i) \otimes \bigotimes_{j=i+1}^{n} U_{|x_i|} \right)
\]

Using this property, the inclusion we need to prove can be rewritten as

\[
\mu S(p_1(x_1) \ast \ldots \ast p_n(x_n)) \subseteq \bigcup_{i=1}^{n} \bigotimes_{j=1}^{n} \mu S(q_j(x_j)) \iff \\
\bigotimes_{i=1}^{n} \mu S(p_i) \subseteq \bigcup_{i=1}^{n} \bigotimes_{j=1}^{n} \mu S(q_j(x_j)) \iff \\
\bigotimes_{i=1}^{n} \mu S(p_i) \subseteq \bigcap_{i=1}^{n} \bigotimes_{j=1}^{n} \mu S(q_j(x_j)) \otimes \bigotimes_{j=i+1}^{n} U_{|x_i|}
\]

As in the proof of [HLSV11, Theorem 1], because the power set lattice \( (2^V, \subseteq) \) of any set \( V \) is a completely distributive lattice, for any doubly indexed family \( \{S_{j,k} \mid j \in J, k \in K_j\} \) it holds that \( \bigcap_{i \in J} \bigcup_{k \in K_i} S_{j,k} = \bigcup_{f \in F} \bigcap_{j \in J} S_{j,f(j)} \), where \( F \) is the set of choice functions \( f \) choosing for each index \( j \) in \( J \) some index \( f(j) \) in \( K_j \). In our case, let \( F = F(\overline{Q}_1, \ldots, \overline{Q}_k) \). Then,

\[
\bigotimes_{i=1}^{n} \mu S(p_i) \subseteq \bigcup_{i=1}^{n} \bigotimes_{j=1}^{n} \left( \bigotimes_{f \in F} (\bigotimes_{i=1}^{n} U_{|x_i|} \otimes \mu S(q_j(x_j)) \otimes \bigotimes_{j=i+1}^{n} U_{|x_i|}) \right) \iff \\
\bigotimes_{i=1}^{n} \mu S(p_i) \subseteq \bigcap_{f \in F} \left( \bigotimes_{i=1}^{n} U_{|x_i|} \otimes \mu S(q_j(x_j)) \otimes \bigotimes_{j=i+1}^{n} U_{|x_i|} \right) (\ast)
\]

For a fixed \( f \), we can rewrite the right hand-side of the inclusion as
\[ \bigcup_{i=1}^{k} \left( \bigotimes_{j=1}^{f(i)} \mathcal{U}[x_i] \otimes \mu S(q_i^j) \right) \otimes \bigotimes_{l=j+1}^{n} \mathcal{U}[x_l] = \]
\[ \bigcup_{i=1}^{n} \left( \bigotimes_{j=1}^{f(i)} \mathcal{U}[x_i] \otimes \left( \bigcup_{l \in [k], f(l) = j} \mu S(q_l^j) \right) \otimes \bigotimes_{l=j+1}^{n} \mathcal{U}[x_l] \right) = \]
\[ \bigcup_{i=1}^{n} \left( \bigotimes_{j=1}^{f(i)} \mathcal{U}[x_i] \otimes \left( \bigcup_{l \in [k], f(l) = j} \mu S(q_l^j) \right) \otimes \bigotimes_{l=j+1}^{n} \mathcal{U}[x_l] \right) \]

Then the inclusion query \( \ast \) becomes

\[ \forall f \in F. \bigotimes_{i=1}^{n} \mu S(p_i) \subseteq \bigcup_{j=1}^{n} \left( \bigotimes_{j=1}^{f(i)} \mathcal{U}[x_i] \otimes \left( \bigcup_{l \in [k], f(l) = j} \mu S(q_l^j) \right) \otimes \bigotimes_{l=j+1}^{n} \mathcal{U}[x_l] \right) \iff \]
\[ \forall f \in F \exists j \in [n]. \mu S(p_j) \subseteq \bigcup_{i \in [k], f(i) = j} \mu S(q_i^j) \iff \]
\[ \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} \mu S(p_i) \subseteq \bigcup \{ \mu S(q_i^j) \mid l \in [k], f_j(l) = j \} \iff \]
\[ \bigvee_{i \in [n], k} \bigwedge_{j=1}^{n} \mu S(p_i) \subseteq \bigcup \{ \mu S(q_i^j) \mid l \in [k], f_j(l) = j \} \]

\[ \square \]

### A.14 Proof of Lemma 12

**Proof.** Assume that there is a proof for a sequent \( \Gamma \vdash \Delta \) using the rules in Figure 2, together with (LU) and (ID) from Figure 1. We prove the entailment \( \ast \) by induction on the structure of the proof tree, using a case split, on the type of the inference rule used at the root of the proof tree. For the case (AXₐ) soundness follows by the side condition of the rule. The other cases are:

- (LU) Let \( p(x) \in \Gamma \) be a predicate atom, such that \( p(x) \leftarrow_s R_1 \ldots R_n \). We have that \( \mu S^a(p(x)) = \bigcup_{i=1}^{n} \mu S^a(R_i) \), because \( \mu S^a \) is the least solution of \( S \). We compute:

\[
\mu S^a(\ast R_i) = \{ (\nu, h_1 \uplus h_2) \mid (\nu, h_1) \in \mu S^a(p(x),_i), (\nu, h_2) \in \mu S^a(\ast (\Gamma \setminus p(x))) \}
\]
\[
= \{ (\nu, h_1 \uplus h_2) \mid (\nu, h_1) \in \bigcup_{i=1}^{n} \mu S^a(\ast R_i), (\nu, h_2) \in \mu S^a(\ast (\Gamma \setminus p(x))) \}
\]
\[
= \bigcup_{i=1}^{n} \{ (\nu, h_1 \uplus h_2) \mid (\nu, h_1) \in \mu S^a(\ast R_i), (\nu, h_2) \in \mu S^a(\ast (\Gamma \setminus p(x))) \}
\]
\[
= \bigcup_{i=1}^{n} \mu S^a(\ast (R_i \cup (\Gamma \setminus p(x))))
\]

Since each sequent \( R_i, \Gamma \setminus p(x) \vdash \Delta \), \( i \in [n] \), has a proof which is a subtree of the proof for \( \Gamma \vdash \Delta \), by the induction hypothesis we have:

\[
\mu S^a(\ast (R_i \cup (\Gamma \setminus p(x)))) \subseteq \mu S^a(\bigvee \Delta), \text{ for all } i \in [n]
\]

which leads to \( \ast \mu S^{\ast} \vdash \bigvee \Delta \), as required.

- (RUₐ) Let \( p(x) \in \Delta \) be a predicate atom, such that \( p(x) \leftarrow_s R_1(x, y_1) \ldots R_n(x, y_n) \). We have that \( \mu S^a(p(x)) = \bigcup_{i=1}^{n} \{ (\nu(x), h) \mid (\nu, h) \in \mu S^a(\exists y_i \ast R_i(x, y_i)) \} = \{ (\nu(x), h) \mid (\nu, h) \in \bigcup_{i=1}^{n} \mu S^a(\exists y_i \ast R_i(x, y_i)) \} \), because \( \mu S^a \) is the least solution of \( S \). Since the sequent \( \Gamma \vdash \exists y_1 \ast R_1(x, y_1) \ldots \exists y_n \ast R_n(x, y_n), \Delta \vdash p(x) \) has a proof which is a subtree of the proof for \( \Gamma \vdash \Delta \), by the induction
hypothesis, we have:

\[ \mu S^i(\star \Gamma) \subseteq \bigcup_{i=1}^{n} \mu S^i(\exists y_i \ast R_i(x, y_i)) \cup \mu S^i(\forall (\Delta \setminus p(x))) \]

\[ = \bigcup_{i=1}^{n} \mu S^i(\exists y_i \ast R_i(x, y_i)) \cup (\mu S^i(\forall \Delta) \setminus \mu S^i(p(x))) \]

\[ = \bigcup_{i=1}^{n} \mu S^i(\exists y_i \ast R_i(x, y_i)) \cup (\mu S^i(\forall \Delta) \setminus \bigcup_{i=1}^{n} \mu S^i(\exists y_i \ast R_i(x, y_i))) \]

\[ = \mu S^i(\forall \Delta) \]

and thus \( \Gamma \models^s \forall \Delta \) follows.

– (RD\(_d\)) Assume that

\[ \Gamma = \{ \phi(x, x_1, \ldots, x_n), p_1(x_1), \ldots, p_n(x_n) \} \]

\[ \Delta = \{ \exists y_1 \cdot \psi_1(x, y_1) \land Q_1(y_1), \ldots, \exists y_k \cdot \psi_k(x, y_k) \land Q_k(y_k) \} . \]

Since the sequent \( p_1(x_1), \ldots, p_n(x_n) \vdash \{ Q_j \theta \mid \theta \in S_j \}_{j=1}^{i} \) has a proof which is a subtree of the proof for \( \Gamma \vdash \Delta \), by the induction hypothesis, we have:

\[ \mu S^i(p_1(x_1) \ast \ldots \ast p_n(x_n)) \subseteq \bigcup_{j=1}^{i} \bigcup_{\theta \in S_j} \mu S^i(Q_j \theta) \subseteq \bigcup_{j=1}^{i} \bigcup_{\theta \in Sk(\phi, \psi_j)} \mu S^i(Q_j \theta) \]

because for all \( j \in [i] \), we have \( S_j \subseteq Sk(\phi, \psi_j) \). Furthermore, by Definition 3, for each \( \theta \in Sk(\phi, \psi_j) \), for all \( j \in [i] \) we have \( \mu S^i(\phi) \subseteq \mu S^i(\psi_j) \). We compute:

\[ \mu S^i(\phi \ast p_1(x_1) \ast \ldots \ast p_n(x_n)) \subseteq \bigcup_{j=1}^{i} \bigcup_{\theta \in Sk(\phi, \psi_j)} \mu S^i(\psi_j \theta \ast Q_j \theta) \]

\[ \subseteq \bigcup_{j=1}^{i} \bigcup_{\theta \in Sk(\phi, \psi_j)} \mu S^i((\psi_j \ast Q_j) \theta) \]

\[ \subseteq \bigcup_{j=1}^{i} \mu S^i(\exists y_j \cdot \psi_j \ast Q_j) \]

\[ \subseteq \bigcup_{j=1}^{i} \mu S^i(\exists y_j \cdot \psi_j \ast Q_j) . \]

– (SP\(_d\)) is a direct consequence of Lemma 11.

– (ID) Suppose, by contradiction, that \( \star \Gamma \not\models \forall \Delta \). Then there exists a tuple \( (\nu, h) \in \mu S^i(\star \Gamma) \) such that \( (\nu, h) \not\in \mu S^i(\forall \Delta) \). Let \( \Gamma \vdash \Delta = \Gamma_1 \vdash \Delta_1, \ldots, \Gamma_{k+1} \vdash \Delta_{k+1} = \Gamma' \vdash \Delta' \) be a path in the proof and \( \mathcal{R}_1, \ldots, \mathcal{R}_k \) be the sequence of inference rules applied on this path. Clearly, none of these rules are \( (AX_i) \), or \( (ID) \), and at least one of them is of the type \( (LU) \), as required by the side condition of \( (ID) \). By the previous points, the existence of a counterexample for the consequent of \( \mathcal{R}_i \) implies the existence of a counterexample for its antecedent, if \( \mathcal{R}_i \) is of the type \( (LU) \), \( (RU_\mu) \), \( (RD_\mu) \) or \( (SP_\mu) \). If the antecedent on the path has no counterexample, then it is safe to close the path using \( (ID) \), so we are left with the case when each sequent on the path has a counterexample. Since \( S \) is ranked, we exhibit a sequence of counterexamples \((\nu, h) = (\nu_1, h_1), \ldots, (\nu_{k+1}, h_{k+1}) = (\nu', h') \) such that \( h_i \geq h_{i+1} \) for all \( i \in [k-1] \). By a case split on the type of \( \mathcal{R}_i \):

– (LU) Let \( p(x) \not\models S R \) be the rule that replaces a predicate atom \( p(x) \in \Gamma_i \). Then \( \Gamma_{i+1} = \{ R, \Gamma_i \setminus p(x) \} \). Because \( \mu S^i(\star (R \cup (\Gamma_i \setminus p(x))) \setminus \mu S^i(\ast \Gamma_i) \), we have \( (\nu_{i+1}, h_{i+1}) \in \mu S^i(\star (R \cup (\Gamma_i \setminus p(x))) \setminus \mu S^i(\forall \Delta_i) \subseteq \mu S^i(\ast \Gamma_i) \setminus \mu S^i(\forall \Delta_i) \), thus \( (\nu_i, h_i) = (\nu_{i+1}, h_{i+1}) \).
• (RU_a) Since \( \Gamma_i = \Gamma_{i+1} \), we have \((\nu_i, h_i) = (\nu_{i+1}, h_{i+1})\).

• (RD_a) Let \( \Gamma_i = \{ \phi(x_1, \ldots, x_n), p_1(x_1), \ldots, p_n(x_n) \} \). Then \( \Gamma_{i+1} = \{ p_1(x_1), \ldots, p_n(x_n) \} \) and \( h_i = h_i \cup h_{i+1} \), where \((\nu_i, h) \in \mu \mathcal{S}^d(\phi(x_1, \ldots, x_n))\). Moreover, we must have \( n > 0 \), or else the path would end before the consequent of (ID). Because the constraint \( \phi \) can only be introduced by an application of (LU) and because \( n > 0 \), we obtain that \( h \neq \emptyset \) and \( h_i \triangleright h_{i+1} \), due to the fact that \( \mathcal{S} \) is ranked.

• (SP_a) Let \( h_i = h_i^1 \cup \ldots \cup h_i^n \) and let \( i \) and \( j \) be fixed for the application of \( \mathcal{R}_i \) on this path. Then \( h_{i+1} = h_i^j \) and \( h_i \triangleright h_{i+1} \).

Because the system is ranked and (LU) is applied at least once on the path, it follows that (RD_a) must also be applied at least once in the path. Otherwise, the application of (LU) introduces at least one additional spatial constraint and it would be impossible to find the same \( \Gamma \) later on the path. Since \( \Delta' \subseteq \Delta \), one can obtain an infinite path by repeating the rules \( \mathcal{R}_1, \ldots, \mathcal{R}_k \) any number of times. But this would result in an infinite decreasing sequence of heaps, in contradiction with \( \triangleright \) being a wfqo. \( \square \)

A.15 Proof of Lemma 13

Proof. By induction on \( \mathcal{D}^\forall(\Gamma \vdash \Delta) \), ordered by \( \sqsubseteq \forall^\exists \). We make a case split according to the type \( R \) of the rule applied at the root of the derivation:

− (LU) Let \( p(x) \in \Gamma \) be the predicate atom chosen for replacement and \( R_1, \Gamma \vdash p(x) \vdash \Gamma \vdash \Delta, \ldots, R_m, \Gamma \vdash p(x) \vdash \Delta \) be the antecedents of \( R \), where \( p(x) \vdash \langle R_1 \mid \ldots \mid R_m \rangle \). Using a similar reasoning as in the (LU) case of Lemma 8, there must exist \( i \in [m] \) such that every derivation \( D \in \mathcal{D}^\forall(R_i, \Gamma \vdash p(x) \vdash \Delta) \) contains a leaf \( \Gamma_i \vdash \emptyset \). Because \( p(x) \vdash q(x) \vdash \mu \mathcal{S}^d(\phi(x_1, \ldots, x_n)) \), \( \Gamma_i \) is a tree-shaped set and \( \Gamma_1, \ldots, \Gamma_n \) be the singly-tree shaped sets, represented as trees labeled with formulae, such that \( \Gamma = \bigcup_{i=1}^n \Gamma_i \). Then there exists a tree \( T_j, j \in [n] \) and a frontier position \( q \in \text{fr}(T_j) \) such that \( T_j(q) = p(x) \). Then the tree-shaped set \( R_i, \Gamma \vdash p(x) \vdash \Delta \) is represented by the singly-tree shaped sets \( T_1', \ldots, T_n' \), where \( T'_l = T_l \) for all \( l \in [n] \setminus \{ j \} \) and \( T'_j = T_{j[q]} \circ R_i \). By the induction hypothesis, there exists a counterexample \( (\nu, h_1 \cup \ldots \cup h_n, \{ t_1, \ldots, t_n \}) \in \mu \mathcal{S}^d(\psi(R_i \cup (\Gamma \vdash p(x)))) \) \( \mu \mathcal{S}^d(\forall \Delta) \). Then \( \nu, h_1, t_i \in \mu \mathcal{S}^d(\forall T'_j) \), for all \( \nu \in [n] \). Because \( T'_j = T_j \) for all \( l \in [n] \setminus \{ j \} \), it is sufficient to prove that \( (\nu, h_j, t_j) \in \mu \mathcal{S}^d(\forall T'_j) \), in order to obtain that \( (\nu, h_1 \cup \ldots \cup h_n, \{ t_1, \ldots, t_n \}) \in \mu \mathcal{S}^d(\forall \Delta) \). Since \( (\nu, h_j, t_j) \in \mu \mathcal{S}^d(\forall T_{j[q]} \circ R_i) \), there are disjoint heaps \( h_j' \) and \( h_j'' \) a context \( t'_j [q] \) and a cover \( t''_j [q] \) such that:

• \( h_j = h_j' \cup h_j'' \) and \( t_j = t'_j [q] \circ t''_j \),

• \( t'_j [q] \) covers \( h_j' \) and \( (\nu, t''_j [q], h_j'') \in \mu \mathcal{S}^d(\forall R_i) \).

Since \( (R_i, p(x)) \in \mathcal{S} \) is a rule, we have \( \mu \mathcal{S}^d(\forall R_i) \subseteq \mu \mathcal{S}^d(p) \) and thus \( (\nu, h_1, t_j) \in \mu \mathcal{S}^d(\forall T_j) \), as required.

− (RU_a) Let \( p(x) \in \Delta \) be the predicate chosen for replacement, defined by \( p(x) \vdash \langle R_1(x, y_1), \ldots, R_m(x, y_m) \rangle \) and \( \Gamma \vdash \exists y_1 \ldots \exists y_m. \ast * R_m(x, y_m), \Delta \vdash p(x) \) be the antecedent of \( R \). Clearly, every \( D \in \mathcal{D}^\forall(\Gamma \vdash \exists y_1 \ldots \exists y_m. \ast * R_1(x, y_1), \ldots, \exists y_m. \ast * R_m(x, y_m), \Delta \vdash p(x) \) contains a leaf \( \Gamma' \vdash \emptyset \),
thus, by the induction hypothesis, there exists a counterexample \((\nu, h, \Gamma) \in \mu \mathcal{S}^*(\Gamma) \setminus \mu \mathcal{S}^*(\emptyset)\) as required. The last step is proved as follows:

\[
\mu \mathcal{S}^*(\emptyset) \not\models \exists y_1 : * R_i(x, y_i) \lor (\Delta \setminus \{p(x)\}) = \mu \mathcal{S}^*(\emptyset) \setminus \mu \mathcal{S}^*(\emptyset)\]

\(\cup_{i=1}^n \mu \mathcal{S}^*(\emptyset) \not\models R_i(x, y_i) \lor \mu \mathcal{S}^*(\emptyset) \setminus \mu \mathcal{S}^*(\emptyset)\)

\(\mu \mathcal{S}^*(p(x)) = \mu \mathcal{S}^*(\emptyset)\).

- (RD) Let \(\Gamma = \{\phi(x, x_1, \ldots, x_n), p_1(x_1), \ldots, p_m(x_m)\}\) and \(\Delta = \{\exists y_1 \cdot \psi_1(x, y_1) \mid Q_1(y_1), \ldots, \exists y_n \cdot \psi_k(x, y_k) \mid Q_k(y_k)\}\), where \(\phi, \psi_1, \ldots, \psi_k\) are symbolic heap constraints, \(p_1, \ldots, p_m\) are predicates, and \(Q_1, \ldots, Q_k\) are separated conjunctions of predicates. The case \(m = 0\) is similar to the one in the proof of Lemma 8, so we are left with the case \(m > 0\). In this case, the antecedent of the rule (RD) is \(p_1(x_1), \ldots, p_m(x_m) \models \{Q_j \mid \theta \in \text{Sk}(\phi, \psi_j)\}\) and, by the side condition of the rule, we have \(\phi \equiv \exists y \land \exists y_j \land \psi_j\) and \(\phi \not\models \exists y \land \exists y_j \land \psi_j\), via a possible reordering of \(\Delta\). Since every derivation \(D \in \mathcal{D}^*(\Gamma) \models \{Q_j \mid \theta \in \text{Sk}(\phi, \psi_j)\}\) must contain a leaf \(\Gamma' \not\models \emptyset\), by the induction hypothesis there must exist a counterexample \((\nu, h, t_1, \ldots, t_m) \in \mu \mathcal{S}^*(\emptyset) \setminus \mu \mathcal{S}^*(\emptyset)\) for all \(j \in [i]\) and all \(\theta \in \text{Sk}(\phi, \psi_j)\). Then \(h = \bigcup_{i=1}^m \theta_i\), where \(\theta_i(x, t_i, t) \in \mu \mathcal{S}^*(\emptyset)\), for all \(i \in [m]\). Because we assumed that \(\mathcal{S}\) is non-filtering, there exists a pair \((\ell, h_0) \in L^{[k]} \times \text{Heaps}\) such that \(h_0\) is disjoint from \(h\) and \(\nu[x \leftarrow t_j], h_0 \models \phi\). We prove that the tuple \((\nu[x \leftarrow t_j], h_0 \cup h, \tau_m(h_0, t_1, \ldots, t_m)\) is a counterexample for \(\Gamma \models \emptyset \lor \Delta\). Because of the assumption that \(p(x) \not\models q(x) \rightarrow \Gamma \models \Delta\), by Lemma 7 applied for the set of rules \(\mathcal{R_e}\), the set \(\Gamma = \{\phi(x, x_1, \ldots, x_n), p_1(x_1), \ldots, p_m(x_m)\}\) is tree-shaped, and because the number of predicate atoms equals the number of tuples of subgoal variables, it must be a singly-tree shaped set. But then we have \(\nu[x \leftarrow t_j], h_0 \cup h, \tau_m(h_0, t_1, \ldots, t_m) \in \mu \mathcal{S}^*(\emptyset)\) and, by the definition of models of single-tree shaped sets, \(\nu[x \leftarrow t_j], h_0 \models \phi\). Now suppose, for a contradiction, that \((\nu[x \leftarrow t_j], h_0 \cup h, \tau_m(h_0, t_1, \ldots, t_m) \in \mu \mathcal{S}^*(\emptyset) \lor \psi_j(x, y_j) \mid Q_j(y_j))\) for some \(j \in [k]\). But then we also have \(\nu[x \leftarrow t_j], h_0 \models \exists y_j \land \psi_j(x, y_j)\). We distinguish two cases:

(a) if \(j \in [i+1, k]\), then \(\nu[x \leftarrow t_j, y \leftarrow t_j], h_0 \models \psi_j\) and since \(\nu[x \leftarrow t_j], h_0 \models \phi\), it must be that \(\phi \equiv \exists y \land \psi_j\), because \(\mathcal{S}\) is non-overlapping. But this contradicts the side condition \(\phi \not\equiv \exists y \land \psi_j\).

(b) otherwise, \(j \in [i]\) and \(\nu[x \leftarrow t_j], h_0 \models \exists y_j \land \psi_j(x, y_j)\), therefore \(\nu[x \leftarrow t_j], h_0 \models \psi_j \theta\) for some \(\theta \in \text{Sk}(\phi, \psi_j)\). By the definition of the set \(\text{Sk}(\phi, \psi_j)\) it must be the case that \(\nu[x \leftarrow t_j], \bigcup_{i=1}^m h_0 \models Q_j \theta\), which contradicts the assumption that \((\nu, h, \{t_1, \ldots, t_m\} \not\in \mu \mathcal{S}^*(Q_j \theta)\) for all \(j \in [m]\) and \(\theta \in \text{Sk}(\phi, \psi_j)\).

- (SP) similar to the (SP) case in the proof of Lemma 8, using a variation of Lemma 11, which we prove by defining \(\bigotimes_{i=0}^n \mu \mathcal{S}^*(p_i) \equiv \{(u_1, \ldots, u_n), h_1 \cup h_0 \cup h_2 \cup \ldots \cup h_m\}\).
\[
\ldots \psi h_n, \{t_1, \ldots, t_n\} \mid (u_i, h_i, t_i) \in \mu \mathcal{S}(p_i), \ i \in [n]\}, \text{ for some predicates } p_1(x_1), \ldots, p_n(x_n) \text{ in } \mathcal{S}, \text{ and } \mathcal{U}_k = \mathbb{L}^k \times \text{Heaps} \times \text{Cover}.
\]