Detecting relic gravitational waves in the CMB: A statistical bias

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Analyzing the imprint of relic gravitational waves (RGWs) on the cosmic microwave background (CMB) power spectra provides a way to determine the signal of RGWs. In this Letter, we discuss a statistical bias, which could exist in the data analysis and has the tendency to overlook the RGWs. We also explain why this bias exists, and how to avoid it.

I. INTRODUCTION

A stochastic background of relic gravitational waves was produced in the very early stage of the Universe due to the superadiabatic amplification of zero point quantum fluctuations of the gravitational field [1, 2]. The relic gravitational waves have a wide range of spreading frequency channels [14]. We adopt the ‘input’ cosmological model:

\[ P_l(k) = A_l(k_0) (k/k_0)^{n_t}, \]

where \( k_0 \) is the pivot wavenumber, which can be arbitrarily chosen. \( A_l(k_0) \) is the amplitude of RGWs, and \( n_t \) is the spectral index. The value of \( n_t \) is quite close to zero, predicted by the physical models of the early Universe. As usual, we can define the tensor-to-scalar ratio \( r = A_s(k_0)/A_\ell(k_0) \), where \( A_s(k_0) \) is the amplitude of the density perturbations. Obviously, assuming \( A_s(k_0) \) is known as in this Letter, \( r \) is just \( A_t(k_0) \) normalized by the constant \( A_s(k_0) \).

In order to discuss the statistical bias for the detection of RGWs in the data analysis, let us simulate the observable data for the Planck satellite, where we only consider the Planck instrumental noises at the 143GHz frequency channels [14]. We adopt the ‘input’ cosmological models as \( \Omega_b h^2 = 0.02267, \Omega_c h^2 = 0.1131, \Omega_\Lambda = 0.726, \tau_{\text{reion}} = 0.084, h = 0.705, A_s = 2.445 \times 10^{-9} \) and \( n_s = 1 \). The RGWs parameters are adopted as \( r = \hat{r} = 0.05 \), \( n_t = \hat{n}_t = 0 \). As we have discussed in the previous paper [13], this small \( r \) is expected to be detected at 2σ for the assumed noise level.

Based on this input cosmological model, and the assumed noise level, we simulate 500 data samples. For...
We expect the distribution of these 500 simulated samples. For different data samples, they have different values. Of course, their values do depend on the simulated data.

In the 1-dimensional posterior pdf for the parameters $r$ and $n_t$, and the flat prior of them. The red column shows the results by adopting the free parameters $r$ and $z$ and the flat prior of them.

In order to avoid this danger, the natural way is setting $r$ and $n_t$ as free parameters. We choose the flat priors of them in the range $r \in [0, 1]$ and $n_t \in [-3, 3]$. We adopt the best-pivot wavenumber, which is $k_0 = 0.0006$Mpc$^{-1}$ for the input model and the assumed noise level [18].

The most interesting final result is the maximum value in the 1-dimensional posterior pdf for the parameters $r$ and $n_t$. In this paper, we denote them by $r_{ML}$ and $n_{ML}$. Of course, their values do depend on the simulated data. For different data samples, they have different values. We expect the distribution of these 500 $r_{ML}$ and $n_{ML}$ are around their input values. However, it may be not the truth in the real analysis. In Fig 1 we plot the distribution of $r_{ML}$ and $n_{ML}$ with blue shadows. This figure shows that, the distribution of $r_{ML}$ is peaked at zero, the input value. However, the distribution of $r_{ML}$ obviously approaches to $r = 0$, and biased the input value at $r = 0.05$. This suggests that, if we deal with the data analysis in this way, the resulting conclusion has the tendency to deviate from the 'true' value of $r$, and to overlook the RGWs.

III. UNDERSTANDING THE STATISTICAL BIAS

It is important to understand why this statistical bias does exist. In order to realize it, let us proceed the following analytical approximation for the likelihood analysis.

The primordial power spectrum of RGW in (1) can be rewritten as,

$$ P_t(k) = A_t(k_0) (k/k_0)^{n_t} = A_z(k_0) r \exp [n_t \ln (k/k_0)] ,$$

which can be approximated as

$$ P_t(k) \simeq A_z(k_0) [r + r n_t \ln (k/k_0)].$$

In this approximation, we have used $|n_t| \ll 1$.

The total CMB power spectra $C_{Y}^Y$ ($Y = T, C, E, B$) include the contributions of density perturbations and gravitational waves, i.e.

$$ C_{Y}^Y = C_{\ell,t}^Y + C_{\ell,t}^Y,$$

where $C_{\ell,t}^Y$ and $C_{\ell,t}^Y$ are the contributions of density perturbations and gravitational waves, separately. Note that $C_{Y}^Y = 0$. By considering $|n_t| \ll 1$, the spectra $C_{\ell,t}^Y$, as a function of $r$ and $n_t$, can be approximated as [18]

$$ C_{\ell,t}^Y \simeq C_{\ell,t}^Y [r + r n_t \ln (\ell/\ell_0)].$$

Here $C_{\ell,t}^Y \equiv C_{\ell,t}^Y (r = 1, n_t = 0)$, and best-pivot multipole $\ell_0 = k_0 \times 10^4$Mpc [18]. So $P_t(k)$ and $C_{\ell,t}^Y$ are all the linear combinations of the parameters $r$ and $n_t$.

Now, let us turn to the likelihood function. The exact form can be found in the previous works [19][15][14][18]. In the analytical approximation, it can be well approximated by [18]

$$ -2 \ln L = \sum_{\ell} \sum_Y \left( \frac{D_{Y}^Y - \bar{C}_{\ell}^Y}{\bar{\sigma}_{D_{Y}^Y}} \right)^2 .$$

where we have defined the quantities

$$ D_{Y}^Y = \bar{D}_{\ell,t}^Y - \bar{C}_{\ell,t}^Y , \quad a_{\ell}^Y = \bar{C}_{\ell,t}^Y , \quad b_{\ell}^Y = \ln (\ell/\ell_0) ,$$

which are all independent of the variables $r$ and $n_t$. Obviously, the value of $a_{\ell}^Y$ depends on the data. For a larger number of different sample, the average value of $D_{Y}^Y$ is $\langle D_{Y}^Y \rangle = a_{\ell}^Y (\bar{r} + \bar{r} n_t b_{\ell})$, due to the facts of $\langle D_{\ell,t}^Y \rangle = C_{\ell,t}^Y + C_{\ell,t}^Y (r = \bar{r}, n_t = \bar{n_t})$ and $C_{\ell,t}^Y (r = \bar{r}, n_t = \bar{n_t}) \simeq C_{\ell,t}^Y (\bar{r} + \bar{r} n_t b_{\ell}).$
Since we have adopted the best-pivot multipole \( \ell_0 \), which is defined by requiring \( \sum_{\ell} \sum_{Y} (a_Y^\ell)^2 b_\ell = 0 \), the likelihood (7) can be rewritten as

\[
-2 \ln \mathcal{L} = \left( \frac{r - r_p}{r_s} \right)^2 + \left( \frac{rn_t - z_p}{z_s} \right)^2 + C,
\]

where \( C \) is a constant, and the other quantities are defined by

\[
\begin{align*}
  r_p &= \frac{\sum_{\ell} \sum_{Y} Y a_Y^\ell b_\ell}{\sum_{\ell} \sum_{Y} (a_Y^\ell)^2}, & r_s &= \frac{1}{\sqrt{\sum_{\ell} \sum_{Y} (a_Y^\ell)^2}}, \\
  z_p &= \frac{\sum_{\ell} \sum_{Y} Y a_Y^\ell b_\ell}{\sum_{\ell} \sum_{Y} (a_Y^\ell)^2}, & z_s &= \frac{1}{\sqrt{\sum_{\ell} \sum_{Y} (a_Y^\ell)^2}}.
\end{align*}
\]

The posterior pdf relates to the likelihood by the prior. Here, let us adopt the flat prior for the parameters \( r \) and \( n_t \), the 2-dimensional posterior pdf for the variables is

\[
-2 \ln P(r, n_t) = \left( \frac{r - r_p}{r_s} \right)^2 + \left( \frac{rn_t - z_p}{z_s} \right)^2,
\]

which follows the 1-dimensional posterior pdf for \( r \) as follows,

\[
P(r) = \frac{1}{r} \exp \left[ -\frac{1}{2} \left( \frac{r - r_p}{r_s} \right)^2 \right] + C'.
\]

We notice that, when \( r_p \gg r_s \), corresponding to \( S/N \gg 1 \) (see 18 for details), this pdf can be reduced that

\[
P(r) \simeq \frac{1}{r_p} \exp \left[ -\frac{1}{2} \left( \frac{r - r_p}{r_s} \right)^2 \right] + C'.
\]

This is gaussian function for \( r \), and peaks at \( r = r_p \) with spread \( r_s \). From the expression of \( r_p \), we know that, the value of \( r_p \) depends on the data \( D_\ell^Y \) by the quantity \( d_Y^\ell \). However, the average value of \( r_p \) for a larger number of sample is \( \bar{r}_p = \hat{r} \), i.e. \( r_p \) is an unbiased estimator for \( \hat{r} \). This has been mentioned in the previous paper 18.

But here, we want to emphasize that, when \( r_p \) is not much larger than \( r_s \), the peak of the posterior pdf in 18 is smaller than \( r_p \), due to the term \( 1/r \). Especially when \( r_p < 3r_s \), the peak of the pdf is very close to zero, which is never an unbiased estimator for the input value \( \hat{r} \). This explains what we have found in the left panel of Fig. 1.

IV. AVOIDING THE STATISTICAL BIAS

Now, let us consider the possible way to avoid this bias in the data analysis. Let us return to the likelihood function in 19. We find that, if considering \( r \) and \( z \equiv rn_t \) as two independent parameters, this likelihood is a simple gaussian function for the uncorrected parameters \( r \) and \( z \).

Now, we adopt the flat prior for the variables \( r \) and \( z \), and the posterior pdf for \( r \) and \( z \) becomes

\[
-2 \ln P(r, z) = \left( \frac{r - r_p}{r_s} \right)^2 + \left( \frac{z - z_p}{z_s} \right)^2,
\]

from which follows that the 1-dimensional posterior pdf for \( r \) is

\[
P(r) = \exp \left[ -\frac{1}{2} \left( \frac{r - r_p}{r_s} \right)^2 \right] + C'.
\]

This pdf peaks at \( r = r_p \), which is an unbiased estimator for the input value \( \hat{r} \). Similarly, we can also find that, the 1-dimensional posterior pdf for \( z \) peaks at \( z = z_p \), which is also an unbiased estimator for \( \hat{z} \equiv \hat{r}n_t \). So the statistical bias in data analysis is elegantly avoided.

In order to clearly show this result, we have analyzed the same 500 samples, by adopting the flat prior on \( r \) and \( z \). In Fig. 1 we plot the distribution of the \( r_{ML} \) and \( z_{ML} \) with the solid columns. As expected, we find that, these \( r_{ML} \) and \( z_{ML} \) are all distributed around at their input values \( \hat{r} = 0.05 \) and \( \hat{z} = 0 \), and the bias for the tensor-to-scalar ratio is naturally avoided. In this figure, we also plot the distribution of \( n_{ML} \), which also unbiased distributed around its input value \( \hat{n}_t = 0 \).

It is interesting to compare the difference between the prior \( f(r, z) \) and the general prior \( f(r, n_t) \). They can be related by the Jacobi, i.e.

\[
f(r, n_t) = \left| \frac{\partial(r, z)}{\partial(r, n_t)} \right| f(r, z) = rf(r, z).
\]

This relation shows that, the flat prior \( f(r, z) = 1 \) exactly corresponds to \( f(r, n_t) = r \). So, comparing with the analysis with flat prior \( f(r, n_t) = 1 \), the new flat prior \( f(r, z) \) induces a larger value of the variable \( r \).

V. CONCLUSION

In this Letter, we find a statistical bias in the CMB data analysis for the detection of RGWs, when the signal-to-noise ratio is not very high. This could overlook the signal of RGWs in the CMB data analysis. We explain why this bias does exist by the analytical approximation of the likelihood function, and also find this bias can be elegantly avoided by adopting the orthogonalized parameters \( r \) and \( z \equiv rn_t \), instead of the general parameters \( r \) and \( n_t \).

In the end, we should emphasize that a similar statistical bias might exist for any data analysis 20, which should be carefully treated.

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