HIGHER ORDER RELATIONS FOR ADE-TYPE GENERALIZED $q$–ONSAGER ALGEBRAS

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Abstract. For the family of generalized $q$–Onsager algebras associated with simply-laced affine Lie algebras $\hat{g}$, new relations between certain monomials of the fundamental generators are proposed. These relations can be seen as deformed analogues of Lusztig’s higher order $q$–Serre relations associated with $U_q(\hat{g})$, which are recovered as special cases in relation with certain coideal subalgebras of $U_q(\hat{g})$.

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1. INTRODUCTION

Introduced in [BB], the generalized $q$–Onsager algebra $O_q(\hat{g})$ associated with the affine Lie algebra $\hat{g}$ is a higher rank generalization of the so-called $q$–Onsager algebra [Ter [B]]. For $\hat{g} = a_1^{(1)}$, it can be understood as a $q$–deformation of the $sl_{n+1}$–Onsager algebra introduced by Uglow and Ivanov [IT]. By analogy with the $sl_2$ case [BB, IT], let us mention that an algebra homomorphism from $O_q(\hat{g})$ to a coideal subalgebra of the Drinfeld-Jimbo quantum universal enveloping algebra $U_q(g)$ is known [BB] (see also [Ko]). Realizations in terms of finite dimensional quantum algebras may be also considered: for instance, the coideal subalgebras of $U_q(\hat{g})$ studied by Letzter [Le] or the non-standard $U_q'(so_n)$ introduced by Klimyk, Gavrilik and Iorgov [GI, Klim]. Part of the motivation for the present letter comes from the fact that generalized $q$–Onsager algebras already find applications in the study of quantum integrable systems with boundaries [4] and so deserve further investigation.

Besides the definition of the generalized $q$–Onsager algebra in terms of generators and relations [BB Definition 2.1], most of its properties remain to be studied. In view of its relation with coideal subalgebras of $U_q(\hat{g})$ [BB, Ko] and its application to the theory of quantum integrable systems, an important problem is to identify those properties of $U_q(\hat{g})$ which could be somehow extended to $O_q(\hat{g})$. For instance, define the extended Cartan matrix $\{a_{ij}\}$ of $\hat{g}$. The quantum universal enveloping algebra $U_q(\hat{g})$ is generated by the elements $\{h_j, e_j, f_j\}$, $j = 0, 1, ..., \text{rank}(\hat{g})$. As shown by Lusztig [L], besides the fundamental defining relations the basic generators satisfy the so-called $r$–th higher order $q$–Serre relations:

$$\sum_{k=0}^{r-1} (-1)^k \left(\begin{array}{c} r+1 \\ k \end{array}\right)_q x_i^{r-k} x_j^k x_i^k = 0, \quad x \in \{e, f\}, \quad (i \neq j).$$

For the family of generalized $q$–Onsager algebras $O_q(\hat{g})$, by analogy with (1.1) linear relations between monomials of the fundamental generators - denoted $A_i$ below - are expected too. For the case $\hat{g} = sl_2$, this problem has been considered in details in [BV]: higher order relations satisfied by the fundamental generators $A_0, A_1$ of the $q$–Onsager algebra were proposed. In the present letter, we extend the construction to the family of simply-laced affine Lie algebras (of the so-called ADE-type): the $r$–th higher order relations associated with the generalized $q$–Onsager algebras $O_q(\hat{g})$ are proposed, given by (2.3) with (3.4). There are three main motivations for doing so:

1. Integrable scalar [GZ, VGG] or dynamical [BK3] boundary conditions of boundary affine Toda field theories are classified according to the representation theory of $O_q(\hat{g})$ [BB]. Also, several known solutions of the reflection equations are intertwiners of $O_q(\hat{g})$. For explicit examples, see e.g. [DeG, DeM, BF].

2. For $\hat{g} = sl_2$, recall that $a_{ii} = 2$, $a_{ij} = -2$. For the family of simply-laced affine Lie algebras, $a_{ii} = 2$, $a_{ij} = -1$ for $i, j$ simply linked and $a_{ij} = 0$ otherwise [Ko]. For non-simply laced cases, fix coprime integers $d_i$ such that $d_i a_{ij}$ is symmetric. We define $q_i = q^{d_i}$.

3. As usual, we denote: $n \frac{[n]}{m}_q = \frac{[n]_q!}{[m]_q! [n-m]_q!}$, $[n]_q! = \prod_{i=1}^{n} [i]_q$, $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, $[0]_q = 1$. 



1
The higher order $q$–Serre relations (1.1) arise in the construction of a basis of $\mathcal{U}_q(\widehat{g})$. They also appear in the discussion of the quantum Frobenius homomorphism [4]. For the family of generalized $q$–Onsager algebras and corresponding coideal subalgebras of $\mathcal{U}_q(\widehat{g})$ [Ko], the higher order relations (2.3) with (3.4) here proposed will play a similar role. See also [IT1, Problem 3.4] for the $sl_2$ case.

The generalized $q$–Onsager algebras provide a representation theoretic framework for a large class of quantum integrable systems (see e.g. [UI, BK1, BK2, BB, BF, FK]). The higher order relations here proposed will allow to study the hidden symmetry of boundary quantum integrable models for $q$ a root of unity, extending the analysis conducted for $\mathcal{U}_q(\widehat{g})$–related models [DFM, KM].

In [BV], it was shown that the higher order relations for the $q$–Onsager algebra - the case $\widehat{g} = sl_2$ - can be determined from the properties of $A_0, A_1$ acting on a finite dimensional irreducible vector space $(A_0, A_1$ is called a tridiagonal pair [Ter]). By analogy, the concept of tridiagonal pair can be extended to higher rank affine Lie algebras. Based on it, the two-variable polynomial generating function associated with any simply-laced $\widehat{g}$ here proposed - see Definition 3.1 - can be derived. Details will be considered elsewhere.

This letter is organized as follows. In the next Section, extending the analysis of [BV, Section 3], the $r−th$ higher order relations satisfied by the fundamental generators of the generalized $q$–Onsager algebra $\mathcal{O}_q(\widehat{g})$ are derived for $r ≤ 5$. They take the form (2.2), which coefficients $c^{r,p}_k$ are explicitly obtained. Then, for generic values of $r$ and using an inductive argument, recursive formulae for the coefficients are derived. In Section 3, a polynomial generating function for the coefficients is independently proposed, which leads to the closed formula (3.4) for the coefficients. Several checks are done, which support the proposal. In Appendix A, useful recursion relations are reported.

For simplicity, here we focus on the simply-laced affine Lie algebras $\widehat{g}$. Although technically more involved, non-simply laced cases can be treated along the same line. The basic defining relations of the generalized $q$–Onsager algebra associated with non-simply laced cases are given in [BB].

Notations: Here \( \{x\} \) denotes the integer part of \( x \). \( q \) is assumed not to be a root of unity.

2. THE HIGHER ORDER RELATIONS FOR ADE−TYPE AFFINE LIE ALGEBRAS

Generalized $q$–Onsager algebras are extensions of the $q$–Onsager algebra to higher rank affine Lie algebras [BB]. Extending the analysis of [BV], analogues of Lusztig’s higher order relations for $\mathcal{O}_q(\widehat{g})$ can be constructed explicitly, which is the purpose of this Section. First, we recall some basic definitions.

Definition 2.1. (see [BB]) Let \( \{a_{ij}\} \) be the extended Cartan matrix of the simply-laced affine Lie algebra $\widehat{g}$. The generalized $q$–Onsager algebra $\mathcal{O}_q(\widehat{g})$ is an associative algebra with unit 1, elements $A_i$ and scalars $\rho_i$. The defining relations are:

\[
\sum_{k=0}^{2} (-1)^{k} \left[ \frac{2}{k} \right]_{q} A^2_{k} A^k_{j} - \rho_i A_j = 0 \quad \text{if } i, j \text{ are linked},
\]

(2.1) \[
[A_i, A_j] = 0 \quad \text{otherwise}.
\]

Remark 1. For $\rho_i = 0$ the relations (2.1) reduce to the $q$–Serre relations (1.1) of $U_q(\widehat{g})$.

Remark 2. For $q = 1$, the relations (2.1), (2.2) coincide with the defining relations of the so-called $sl_{n+1}$–Onsager’s algebra for $n > 1$ introduced by Uglow and Ivanov [UI].

By analogy with the $\widehat{sl}_2$ case discussed in details in [BV], we expect the following form for the higher order relations

\[
\sum_{p=0}^{\lfloor \frac{r-2p+1}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{r-p}{2} \rfloor} (-1)^{k+p} \rho_i^{p} c^{[r,p]}_{k} A^2_{k} A^{r-2p+1-k}_{j} A^{k}_{j} = 0 \quad \text{if } i, j \text{ are linked}.
\]

(2.3)

In the first part of this Section, we only assume $\{A_i\}_{i=0,1,\ldots, \text{rank}(g)}$ are the fundamental generators of $\mathcal{O}_q(\widehat{g})$. Explicit examples of higher order relations of the form (2.3) are then derived, for which the coefficients $c^{[r,p]}_{k}$ are explicitly obtained. In a second part, the general structure of the $r−th$ higher order relations is studied.
It leads to explicit recursive formulae for the coefficients $c^{[r,p]}_k$, thus extending the results of [BV Section 3] - the $\tilde{s}l_2$ case - to all simply-laced cases.

2.1. Proof of the higher order relations for $r \leq 5$. For $r = 1$, the relations (2.3) are the defining relations of the generalized $q$–Onsager algebra $O_q(\hat{g})$. Assume $A_i$ are the fundamental generators of $O_q(\hat{g})$. To derive the simplest example of higher order relations, we are looking for a linear relation between monomials of the type

$$A_i^n A_j^2 A_k^m \quad \text{with} \quad n + m = 3, 1.$$  

Suppose it is of the form (2.3) for $r = 2$ with yet unknown coefficients $c^{[r,p]}_k$. We show $c^{[r,p]}_k$ are uniquely determined. First, according to the defining relations (2.1) the monomial $A_i^2 A_j$ can be ordered as:

$$A_i^2 A_j = [2]_q A_i A_j - A_i A_j + r_i A_j .$$

Multiplying from the left and/or right by $A_i, A_j (i \neq j)$, the corresponding monomials can be ordered as follows: each time a monomial of the form $A_i^n A_j^2 A_k^m$ with $n \geq 2$ arise, it is reduced using (2.5). For instance, one has:

$$A_i^3 A_j = ([2]_q - 1)A_i A_j A_i^2 - [2]_q A_j A_i^3 + r_i ([2]_q A_j A_i + A_i A_j) .$$

Now, observe that the two monomials in (2.3) for $r = 2$ and $k = 0, 1, p = 0$ can be written as $A_i^3 A_j^2 \equiv (A_i^3 A_j) A_j$ and $A_i^2 A_j A_k^2 \equiv (A_i^2 A_j) A_k$. Following the ordering prescription, each of these monomials can be reduced as a combination of monomials of the type:

$$A_i^n A_j^2 A_k^m \quad \text{with} \quad n \leq 1, n + m = 3, 1,$$

$$A_i^n A_j A_k^s A_l^t \quad \text{with} \quad p \leq 1, s \geq 1, p + s + t = 3, 1 .$$

Plugging the reduced expressions of $A_i^3 A_j^2$ and $A_i^2 A_j A_k^2$ in (2.3) for $r = 2$, one finds that all monomials of the form (2.3) cancel provided a simple system of equations for the coefficients $c^{[r,p]}_k$ is satisfied. The solution of this system is unique, given by:

$$c^{[2,0]}_k = \begin{bmatrix} 3 \\ k \end{bmatrix}_q \quad \text{for} \quad k = 0, 1, 2, 3 , \quad \text{and} \quad c^{[2,1]}_0 = c^{[1,1]}_1 = q^2 + q^{-2} + 2 .$$

For $r = 3, 4, 5$, we proceed similarly: the monomials entering in the relations (2.3) are ordered according to the prescription described above. Given $r$, the reduced expression of the corresponding relation (2.3) holds provided the coefficients $c^{[r,p]}_k$ satisfy a system of equation which solution is unique. In each case, one finds:

$$c^{[r,p]}_k = \begin{bmatrix} r + 1 \\ k \end{bmatrix}_q \quad \text{for} \quad k = 0, \ldots, r + 1 , \quad r = 3, 4, 5 ,$$

whereas for $p \geq 1$, the other coefficients are such that $c^{[r,p]}_k = c^{[r,p]}_{r-2p+1-k}$, given by:

**Case $r = 3$:**

$$c^{[3,1]}_0 = q^4 + 2q^2 + 4 + 2q^{-2} + q^{-4} , \quad c^{[3,1]}_1 = [4]_q (q^2 + q^{-2} + 3) ,$$

$$c^{[3,2]}_0 = (q^2 + q^{-2} + 1)^2 ;$$

**Case $r = 4$:**

$$c^{[4,1]}_0 = (q^4 + 3 + q^{-4}) [2]_q^2 , \quad c^{[4,1]}_1 = [5]_q [3]_q [2]_q^2 ,$$

$$c^{[4,2]}_0 = (q^2 + q^{-2})^2 [2]_q^4 ;$$

**Case $r = 5$:**

$$c^{[5,1]}_0 = q^8 + 2q^6 + 4q^4 + 6q^2 + 6 + 6q^{-2} + 4q^{-4} + 2q^{-6} + q^{-8} ,$$

$$c^{[5,1]}_1 = [6]_q [3]_q^{-1} (q^8 + 4q^6 + 8q^4 + 14q^2 + 16 + 14q^{-2} + 8q^{-4} + 4q^{-6} + q^{-8}) ,$$

$$c^{[5,1]}_2 = [6]_q [2]_q^{-1} [5]_q (q^4 + 3q^2 + 6 + 3q^{-2} + q^{-4}) ,$$

$$c^{[5,2]}_0 = q^{12} + 4q^{10} + 11q^8 + 20q^6 + 31q^4 + 40q^2 + 45 + 40q^{-2} + 31q^{-4} + 20q^{-6} + 11q^{-8} + 4q^{-10} + q^{-12} ,$$

$$c^{[5,2]}_1 = [6]_q [3]_q^{-1} (q^{10} + 6q^8 + 17q^6 + 32q^4 + 47q^2 + 53 + 47q^{-2} + 32q^{-4} + 17q^{-6} + 6q^{-8} + q^{-10}) ,$$

$$c^{[5,3]}_0 = [3]_q [5]_q^2 .$$
2.2. Generic case \( r \). Above examples suggest that higher order relations of the form (2.3) exist for generic values of \( r \). To derive the coefficients recursively, one first assumes that given \( r \), the relation (2.3) exists and that all coefficients \( c_{k}^{[r,p]} \) are already known in terms of \( q \). The relation (2.3) for \( r \to r + 1 \) is then considered. In this case, the combination

\[
(2.10) \quad f_{r}^{ADE}(A_i, A_j) = A_i^{r+2} A_j^{r+1} - c_{1}^{[r+1,0]} A_i^{r+1} A_j^{r+1} A_i
\]

is introduced. By analogy with the procedure described in [BV], the monomials \( A_i^{r+2} A_j^{r+1} \) and \( A_i^{r+1} A_j^{r+1} A_i \) are reduced using (2.1) and (2.3). The ordered expression of the first monomial follows:

\[
A_i^{r+2} A_j^{r+1} = \sum_{k=2}^{r+2} (-1)^{k+1} M_k^{(r,0)} A_i^{r+2-k} A_j^{k} A_j
\]

+ \[ \sum_{p=1}^{r+2} \sum_{k=0}^{r+2} (-1)^{p+k+1} \rho^{p} M_k^{(r,p)} A_i^{r+2-2p-k} A_j^{k} A_j , \]

where the recursive relations for the coefficients \( M_k^{(r,p)} \) are reported in Appendix A. Obviously, an ordered expression for the second monomial immediately follows from (2.3). As a consequence, the whole combination \( f_{r}^{ADE}(A_i, A_j) \) can be further reduced. As an intermediate step, note that one uses (2.1) to obtain:

\[
A_i^{2n+1} A_j = \sum_{p=0}^{n} \rho_{1}^{p} (\eta_{0}^{(2n+1,p)} A_i A_j A_i^{2n-2p} + \eta_{1}^{(2n+1,p)} A_j A_i^{2n-2p+1}) ,
\]

\[
A_i^{2n+2} A_j = \sum_{p=0}^{n} \rho_{1}^{p} (\eta_{0}^{(2n+2,p)} A_i A_j A_i^{2n+1-2p} + \eta_{1}^{(2n+2,p)} A_j A_i^{2n+2-2p} + \rho^{p+1} A_j ,
\]

where the coefficients \( \eta_{j}^{(r,p)} \) are given in Appendix A. The ordered expression of \( f_{r}^{ADE}(A_i, A_j) \) is then studied. A detailed analysis shows that all coefficients of monomials of the type \( A_i^{p} A_j^{r} A_i A_j \) (with \( p + s + t = r, r-2, ..., 3, 1 \) if \( r \) is odd, and \( p + s + t = r, r-2, ..., 4, 2 \) if \( r \) is even) vanish provided the coefficients \( c_{k}^{[r+1,0]} \) satisfy a system of equations. If such a system is satisfied, it implies that the coefficients \( c_{k}^{[r+1,0]} \) are uniquely determined in terms of \( c_{k}^{[r,0]} \). Firstly, the coefficient of the monomial \( A_i^{r+1} A_j^{r+1} A_i \) is found, given by:

\[
c_{1}^{[r+1,0]} = \left[ \begin{array}{c} r+1 \\ 1 \end{array} \right]_{q}.
\]

According to the parity of \( r \) and replacing \( r + 1 \to r \), for the other coefficients one finds:

**Case r odd:** For \( r = 2t + 1 \) and \( p = 0 \):

\[
c_{2}^{[2t+1,0]} = M_{2}^{(2t,0)} \eta_{1}^{(2,0)}, \quad c_{2h}^{[2t+1,0]} = M_{2h}^{(2t,0)} \eta_{1}^{(2h,0)} + c_{1}^{[2t+1,0]} c_{2h-1}^{(2h-1,0)} \eta_{1}^{(2h-1,0)}, \quad h = \frac{2, t + 1}{}, \]

\[
c_{2h+1}^{[2t+1,0]} = M_{2h+1}^{(2t,0)} \eta_{1}^{(2h+1,0)} + c_{1}^{[2t+1,0]} c_{2h}^{(2h,0)} \eta_{1}^{(2h,0)}, \quad h = \frac{1, t}{1, t}.
\]

Using the recursion relations given in Appendix A, it is possible to show that these coefficients can be simply written in terms of \( q \)-binomials:

\[
(2.11) \quad c_{k}^{[r,0]} = \left[ \begin{array}{c} r+1 \\ k \end{array} \right]_{q}.
\]

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4Let \( j, m, n \) be integers, we write \( j = m/n \) for \( j = m, m + 1, ..., n - 1, n \).
Other coefficients $c_0^{[2t+1,p]}$ for $p \geq 1$ are determined by the following recursion relations:

$$c_0^{[2t+1,t+1]} = \sum_{p=0}^{t} (-1)^{p+t+1} M_{2(t+1-p)}^{(2t,p)},$$

$$c_0^{[2t+1,1]} = -M_2^{(2t,0)} + M_0^{(2t,1)},$$

$$c_0^{[2t+1,h]} = \sum_{p=0}^{h} (-1)^{p+h} M_{2(h-p)}^{(2t,p)}, \quad h = 2t,$$

$$c_1^{[2t+1,1]} = -\left( M_3^{(2t,0)} \eta_1^{(3,1)} - c_1^{[2t+1,0]} (-c_2^{[2t,0]} + c_0^{[2t,1]}) \right),$$

$$c_1^{[2t+1,h]} = \sum_{p=0}^{h-1} (-1)^{p+h} c_1^{[2t+1,p]} M_{2(h-p)+1}^{(2t,p)} \eta_1^{(2(h-p)+1, h-p)},$$

$$+ c_1^{[2t+1,0]} \sum_{p=0}^{h} (-1)^{p+h} c_2^{[2t,p]} \eta_1^{(2(h-p), h-p)}, \quad h = 2t,$$

$$c_2^{[2t+1,1]} = -M_4^{(2t,0)} \eta_1^{(4,1)} + M_2^{(2t,1)} \eta_1^{(2,0)} - c_1^{[2t+1,0]} c_3^{[2t,0]} \eta_1^{(2,0)},$$

$$c_0^{[2t+1,t+1]} = \sum_{p=0}^{t} (-1)^{p+t+1} M_{2(h-t+1-p)}^{(2t,p)},$$

$$c_0^{[2t+1,1]} = -M_2^{(2t,0)} + M_0^{(2t,1)},$$

$$c_0^{[2t+1,h]} = \sum_{p=0}^{h} (-1)^{p+h} M_{2(h-p)}^{(2t,p)}, \quad h = 2t.$$
\[
[2t+2,t]_c_{2h-2l} = \sum_{p=0}^{l} (-1)^{p+l} M_{2(h-p)}^p (2(h-p), l-p) \\
\left. + c_{1}^{[2t+2,0]} \sum_{p=0}^{\min(l,h-2)} (-1)^{p+l} c_{1}^{[2t+1,1]} (2(h-p)-1, l-p), \quad h = 3, l + 1, l = 1, h - 1, \right]
\]
\[
[2t+2,t]_c_{2h-2l+1} = \sum_{p=0}^{l} (-1)^{p+l} M_{2(h-p)+1} (2(h-p)+1, l-p) \\
\left. + c_{1}^{[2t+2,0]} \sum_{p=0}^{l} (-1)^{p+l} c_{1}^{[2t+1,1]} (2(h-p), l-p), \quad h = 2, l + 1, l = 1, h - 1. \right]
\]

For practical purpose, for any positive integer \(r\) all coefficients \(c_{k}^{[r,p]}\) entering in the higher order relation \([2,3]\) can be computed recursively. Using a computer program, note that we have checked that the relation \([2.3]\) holds for a large number of values \(r \geq 6\) provided the coefficients satisfy above recursive formulæ. Finally, let us point out that for any setting \(\rho_1 = 0\), the relations \([2.3]\) reproduce the higher order \(q\)-Serre relations \([1.1]\) with \(x \rightarrow A\).

### 3. A two-variable polynomial generating function

For the \(q\)-Onsager algebra, in \([BV\) Section 2\] it was shown that the coefficients \(c_{k}^{[r,p]}\) entering in the \(r-th\) higher order relations could be generated from a two-variable generating function. Here, for any simply-laced affine Lie algebras we propose a two-variable generating function for the coefficients too. Various checks are done at the end of this Section, which support the proposal.

**Definition 3.1.** Let \(r \in \mathbb{Z}^+\). Let \(x, y\) be commuting indeterminates and \(\rho\) a scalar. To any simply-laced affine Lie algebra \(\hat{\mathfrak{g}}\), we associate the polynomial generating function \(p_{r}^{ADE}(x, y)\) such that:

\[
(3.1) \quad p_{2t+1}(x, y) = \prod_{l=1}^{t+1} \left( x^2 - \frac{[4l - 2]}{[2l - 1]} q xy + y^2 - \rho [2l - 1] q^2 \right),
\]
\[
(3.2) \quad p_{2t+2}(x, y) = \left( x - y \right) \prod_{l=1}^{t+1} \left( x^2 - \frac{[4l]}{[2l]} q xy + y^2 - \rho [2l] q^2 \right).
\]

**Lemma 3.1.** The polynomial \(p_{r}^{ADE}(x, y)\) can be expanded as:

\[
(3.3) \quad p_{r}^{ADE}(x, y) = \sum_{p=0}^{r-2p+1} \sum_{k=0}^{\frac{k}{2}} (-1)^{k+p} c_{k}^{[r,p]} x^{r-2p+1-k} y^{k},
\]

where the coefficients are given by:

\[
(3.4) \quad c_{k}^{[r,p]} = \sum_{i=0}^{\frac{k}{2}} \frac{\left( \frac{r+1}{2} \right)! - k + \alpha l - p)!}{\left( \frac{r+1}{2} \right)! - k + \alpha l - p - \left( \frac{\alpha l}{2} \right)!} \sum_{p} \frac{s_1 q^2 \cdots s_p q^2 [s_{p+1}] q \cdots [s_{p+k-\alpha l}] q}{[s_{p+1}] q \cdots [s_{p+k-\alpha l}] q},
\]

with

\[
\left\{ \begin{array}{ll}
k = 0, \left( \frac{r+1}{2} \right), & s_i \in \{ r - 2 \left( \frac{r+1}{2} \right), \ldots, r - 2, r \}, \\
\mathcal{P}, & s_1 \leq \cdots \leq s_p \leq s_{p+1} \leq \cdots \leq s_{p+k-\alpha l} \land \{ s_1, \ldots, s_p \} \cap \{ s_{p+1}, \ldots, s_{p+k-\alpha l} \} = \emptyset, \\
\alpha = \begin{cases} 1 & \text{if } r \text{ is even}, \\
2 & \text{if } r \text{ is odd} \end{cases}
\end{array} \right.
\]

**Proof.** By induction. See \([BV\) for the \(\hat{\mathfrak{s}}_2\) case. \)

We claim that the two-variable polynomial given by \([3.1]+[3.2]\) is the generating function for the coefficients \(c_{k}^{[r,p]}\) entering in the higher order relations \([2.3]\) in view of the following observations:

- For \(r \leq 5\), it is an exercise to check that the coefficients \(c_{k}^{[r,p]}\) given by \([3.4]\) coincide exactly with the ones derived in the previous Section (see cases \(r = 2, 3, 4, 5\)).
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\[ \text{HIGHER ORDER RELATIONS FOR ADE-TYPE } q-\text{ONSAGER ALGEBRAS} \]

- Using the \( q \)-binomial theorem, for \( r \) generic it is easy to check that the coefficients \( c_k^{[r,0]} \) obtained from (3.4) are the \( q \)-binomials (2.11):
- For \( r \geq 6 \) and \( p \geq 1 \), the comparison is more involved. However, for a large number of values \( r \geq 6 \) using a computer program we have checked that the coefficients derived using the recursive formulae coincide exactly with the ones given by (3.4):
- Let \( \{c_i, \tau_i, w_i\} \in \mathbb{C} \). Let \( \hat{g} = a_n^{(1)}(n > 1), d_i^{(1)}, c_i^{(1)}, e_i^{(1)} \). There exists an algebra homomorphism: \( \mathcal{O}_q(\hat{g}) \to \mathcal{U}_q(\hat{g}) \) given by [BB]:

\[
A_i \mapsto \mathcal{A}_i = c_i e_i q_i^{h_i} + \tau_i f_i q_i^{h_i} + w_i q_i^{k_i}
\]

iff the parameters \( w_i \) are subject to the constraints: \( w_i \left( w_i^2 + \frac{c_i \tau_i}{q+q^{-1}} \right) = 0 \), \( w_j \left( w_j^2 + \frac{c_j \tau_j}{q+q^{-1}} \right) = 0 \) where \( i, j \) are simply linked and \( \rho_i \to c_i \tau_i \). Let \( V \) be the so-called evaluation representation of \( \mathcal{U}_q(\hat{g}) \) on which \( \mathcal{A}_i \) act (see e.g. [J2] Proposition 1) for \( \hat{g} = a_n^{(1)} \). For generic parameters \( c_i, \tau_i, q, V \) is irreducible and \( \mathcal{A}_i \) is diagonalizable on \( V \). Let \( \theta_k^{(i)}, k = 0, 1, \ldots \) denote the (possibly degenerate) corresponding eigenvalues of \( \mathcal{A}_i \). For instance, for the fundamental representation \( \mathbb{F} \) of \( \mathcal{U}_q(a_n^{(1)}) \), the eigenvalues take the simple form:

\[
\theta_k^{(i)} = C^{(i)}(v q^k + v^{-1} q^{-k}),
\]

where \( v, C^{(i)} \) are scalars and \( C^{(i)} \) depend on \( c_i, \tau_i, q \). Let \( E^{(i)}_k \) be the projector on the eigenspace associated with the eigenvalue \( \theta_k^{(i)} \). Denote \( \Delta^{(i)} \) as the l.h.s of the first equation in (2.1). The relation (2.1) implies that it must exist integers \( k, l \) such that:

\[
E^{(i)}_k \Delta^{(i)}_l E^{(i)}_l = 0 \Rightarrow p^{ADE}_{l}(\theta_k^{(i)}, \theta_l^{(i)}) E^{(i)}_k \mathcal{A}_j E^{(i)}_l = 0 \quad \text{with} \quad \rho \equiv \rho_i.
\]

For generic parameters \( c_i, \tau_i, q \), \( E^{(i)}_k \mathcal{A}_j E^{(i)}_l \neq 0 \). It implies \( p^{ADE}_{l}(\theta_k^{(i)}, \theta_l^{(i)}) = 0 \) which, according to (3.6), is consistent with the structure (3.4) for \( t = 0 \) provided \( l = k \pm 1 \). The same observation about the structure of the two-variable polynomial can be generalized as follows. Denote \( \Delta^{(i)} \) as the l.h.s of (2.3). If the relation (2.3) with (3.4) holds, then it must exist integers \( k, l \) such that:

\[
E^{(i)}_k \Delta^{(i)}_l E^{(i)}_l = 0 \Rightarrow p^{ADE}_{l}(\theta_k^{(i)}, \theta_l^{(i)}) E^{(i)}_k \mathcal{A}_j E^{(i)}_l = 0 \quad \text{with} \quad \rho \equiv \rho_i.
\]

For generic \( c_i, \tau_i, q \), \( E^{(i)}_k \mathcal{A}_j E^{(i)}_l \neq 0 \). It implies \( p^{ADE}_{l}(\theta_k^{(i)}, \theta_l^{(i)}) = 0 \) which, according to (3.6), leads to the following constraints one the integers \( k, l \):

\[
k = l \pm 1, \quad l \pm 3, \quad l \pm 5, \ldots, \quad l \pm r \quad \text{for} \quad r \text{ odd} \, ;
\]

\[
k = l, \quad l \pm 2, \quad l \pm 4, \ldots, \quad l \pm r \quad \text{for} \quad r \text{ even} \, .
\]

Again, this is in perfect agreement with the factorized form (3.1), (3.2). Thus, for \( \hat{g} = a_n^{(1)} \) the structure of the two-variable polynomial (3.1) is consistent with the spectral properties of \( \mathcal{A}_i \).

To conclude, let us point out that for \( c_i = 0 \) or \( \tau_i = 0 \), one has \( \rho_i = 0 \): in this special case the higher order relations associated with the coideal subalgebra generated by (3.5) reduce to the Lusztig’s higher order \( q \)-Serre relations (1.1) [L]. As an additional support for the proposal (3.4), note that extending the arguments presented in [BV] Section 2 and based on [Ter] (see the related literature on tridiagonal pairs), it is independently possible to derive the polynomial \( p^{ADE}_{l}(x, y) \) within the framework of tridiagonal algebras associated with higher rank affine Lie algebras. This will be discussed elsewhere.

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\[ ^5 \text{For } \hat{g} = a_n^{(1)}, \quad n \geq 2, \text{ see e.g. [DeM]. For } \hat{g} = a_n^{(1)}, \quad n \geq 4, \text{ see e.g. [DeG].} \]
APPENDIX A: Coefficients $\eta_{k,j}^{(m)}$, $M_j^{(r,p)}$

The initial values of $\eta_{k,j}^{(m)}$ are given by:

$$
\eta_{0,0}^{(2)} = [2]^2, \quad \eta_{0,1}^{(2)} = -1.
$$

The recursion relations for $\eta_{k,j}^{(m)}$ and $M_j^{(r,p)}$ read:

$$
\eta_{p,0}^{(2n+1)} = [2]^2 \eta_{p,0}^{(2n)} + \eta_{p,0}^{(2n)}, \quad p = 0, n-1, \\
\eta_{p,1}^{(2n+1)} = 1, \\
\eta_{p,0}^{(2n+1)} = -\eta_{p,0}^{(2n)} + \eta_{p-1,0}^{(2n)}, \quad p = 1, n-1, \\
\eta_{0,1}^{(2n+1)} = \eta_{0,0}^{(2n)}, \\
\eta_{p,1}^{(2n+1)} = \eta_{n-1,0}^{(2n)}, \\
\eta_{p,0}^{(2n+2)} = [2]^2 \eta_{p,0}^{(2n+1)} + \eta_{p,1}^{(2n+1)}, \quad p = 0, n, \\
\eta_{p,1}^{(2n+2)} = \eta_{p-1,0}^{(2n+1)} - \eta_{p,0}^{(2n+1)}, \quad p = 1, n, \\
\eta_{0,1}^{(2n+2)} = -\eta_{0,0}^{(2n+1)}.
$$

$$
M_0^{(2t,p)} = \frac{[2t,p]}{c_0}, \\
M_0^{(2t+1,2-2p)} = -\frac{[2t,0]}{c_0} \frac{[2t,p]}{c_{2t+1-2p}}, \\
M_k^{(2t,p)} = \frac{[2t,p]}{c_k} - \frac{[2t,0]}{c_k} \frac{[2t,p]}{c_{k-1}}, \quad k = 1, 2t + 1 - 2p, \\
M_0^{(2t+1,1+t+1)} = \frac{[2t+1,1+t+1]}{c_0}, \\
M_k^{(2t+1,1+t+1)} = -\frac{[2t+1,1+t+1]}{c_k} \frac{[2t+1,0]}{c_{k-1}}, \\
M_k^{(2t+1,1+t+1)} = -\frac{[2t+1,0]}{c_k} \frac{[2t+1,1]}{c_{k-1}} + \frac{[2t+1,1]}{c_k}, \quad k = 1, 2t + 2 - 2p, \\
M_0^{(2t+1,1+t+1)} = \frac{[2t+1,1+t+1]}{c_0}, \\
M_k^{(2t+1,1+t+1)} = -\frac{[2t+1,0]}{c_k} \frac{[2t+1,1+p]}{c_{2t+2-2p}}.
$$

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