LONG TIME BEHAVIOUR FOR A CLASS
OF LOW-REGULARITY SOLUTIONS
OF THE CAMASSA-HOLM EQUATION

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Abstract. In this paper, we investigate the long time behaviour for a class of low-
regularity solutions of the Camassa-Holm equation given by the superposition of
infinitely many interacting traveling waves with corners at their peaks.

1. Introduction.

The Camassa-Holm (CH) equation

\[ u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx} \]  \hspace{1cm} (1.1)

was derived in [CH] as a model of long waves in shallow water. Since then, it
has received considerable attention. In this work, we will be concerned with the
non-dispersive case on the line, corresponding to \( \kappa = 0 \), that is,

\[ u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad x \in \mathbb{R}, \]  \hspace{1cm} (1.2)

and so from now on, we will refer to (1.2) as the CH equation. It is a fundamental
discovery of Camassa and Holm that (1.2) admits weak solutions, of the form

\[ u(x,t) = ce^{-|x-ct|} \]  \hspace{1cm} (1.3)

for any nonzero constant \( c \). Clearly, any such solution is a traveling wave with
a corner at its crest or trough, depending on the sign of \( c \). In [CH], apparently
motivated by such traveling wave solutions, the authors proposed an even more
general class of weak solutions of (1.2), namely,

\[ u(x,t) = \frac{1}{2} \sum_{j=1}^{N} e^{-|x-a_j(t)|}p_j(t), \]  \hspace{1cm} (1.4)
where $N$ is a positive integer. Clearly, when $N > 1$, this ansatz represents the superposition of $N$ interacting (peak-shaped) waves (called peakons if $p_j(t) > 0$, antipeakons if $p_j(t) < 0$) in which case the $p_j$’s are no longer constants and the $q_j$’s are no longer linear in $t$. Indeed, the location of the peaks and their signed amplitudes are now governed by the ordinary differential equations [CHH]:

$$\dot{q}_j = \frac{1}{2} \sum_{k=1}^{N} e^{-|q_j - q_k|} p_k, \quad j = 1, \ldots, N \tag{1.5}$$

which are the Hamiltonian equations of motion generated by

$$H = \frac{1}{4} \sum_{i,j=1}^{N} e^{-|q_i - q_j|} p_i p_j. \tag{1.6}$$

For $N = 2$, the equations in (1.5) were integrated explicitly and the long time behaviour was worked out in [CHH]. For $N > 2$, the explicit integration of (1.5) in the sector where $q_1 < \cdots < q_N$ was obtained by Beals, Sattinger and Szmigielski [BSS] using the inverse scattering method and a theorem of Stieltjes on continued fractions [S]. Furthermore, by making use of the explicit formulas for the $q_j$’s and $p_j$’s, these authors were able to analyze in detail the long time asymptotics of (1.5).

In this work, we will consider a class of weak solutions of (1.2) obtained by generalizing the ansatz in (1.4), namely, we take

$$u(x, t) = \frac{1}{2} \sum_{j=1}^{\infty} e^{-|x - q_j(t)|} p_j(t), \tag{1.7}$$

with $p_j(t) > 0$ for all $j \in \mathbb{N}$ and such that $p_j(t) \to 0$ sufficiently fast as $j \to \infty$. More precisely, we take $q(t) = (q_1(t), q_2(t), \cdots) \in l_\infty^+, p(t) = (p_1(t), p_2(t), \cdots) \in l_{1,2}^+$, where $l_\infty^+$ and $l_{1,2}^+$ are defined in (3.3). Thus the equations of motion for $(q(t), p(t))$ are given by

$$\begin{align*}
\dot{q}_j &= \frac{1}{2} \sum_{k=1}^{\infty} e^{-|q_j - q_k|} p_k, \\
\dot{p}_j &= \frac{1}{2} p_j \sum_{k=1}^{\infty} sgn(q_j - q_k) e^{-|q_j - q_k|} p_k, \quad j \in \mathbb{N}. \tag{1.8}
\end{align*}$$

Our main goal in this work is to investigate the long time behaviour for this class of solutions in the two sectors

$$\mathcal{S}_- = \{(q, p) \in l_\infty^+ \oplus l_{1,2}^+ \mid q_1 < q_2 < \cdots, p_j > 0 \text{ for all } j\}, \tag{1.9}$$
and
\[ S_+ = \{ (q,p) \in l_+^1 \oplus l_+^{1,2} \mid q_1 > q_2 > \cdots, p_j > 0 \text{ for all } j \}. \] (1.10)

Once this is achieved, the adaptation of our analysis to other sectors defined by a restricted class of permutations of N is straightforward. As the reader will see, our approach to this problem is based on the connection of (1.8) in the sectors \( S_\pm \) to the (\( \pm \) Toda flow) on some rather special Hilbert-Schmidt operators in \( l_2^+ \) whose kernels are parametrized by the \( q_j \)'s and the \( p_j \)'s.

The paper is organized as follows. In Section 2, we begin by introducing the Toda flows on Hilbert-Schmidt operators in \( l_2^+ \) using the r-matrix approach, then we discuss the associated Hilbert Lie groups and the factorization method to solve the so-called (\( \pm \) Toda flow) on some rather special Hilbert-Schmidt operators in \( l_2^+ \) whose kernels are parametrized by the \( q_j \)'s and the \( p_j \)'s.

In Section 3, we introduce the class of low-regularity solutions of the CH equation which we mentioned above and we establish the global existence of solutions of (1.8) in the sectors \( S_\pm \). Then we make the connection with the (\( \pm \) Toda flow) and we study the spectral properties of the Lax operators which are basic in our subsequent analysis. In Section 4, we study the long time behaviour of (1.7) and (1.8) in the sector \( S_- \). Here the main difficulty is in showing that the peaked waves separate out, i.e., \( \lim_{t \to \infty} |q_j(t) - q_k(t)| = \infty \) for \( j \neq k \).

Technically, this is due to the fact that the (positive) eigenvalues of the Lax operator \( L(t) = L(q(t), p(t)) \) accumulate at 0, which is in the essential spectrum of \( L(t) \). The fact that 0 is at the bottom of the spectrum is also responsible for the result that \( \lim_{t \to \infty} p_j(t) = 0 \) for all \( j \in \mathbb{N} \). Finally, we remark that although \( q_j(t) \) still approaches \( \infty \) as \( t \to \infty \), however, we only have \( q_j(t) = o(t) \) in the present case. Thus the long time behaviour of (1.7) in the sector \( S_- \) is quite different from the analogous one in [BSS] for the multipeakon solutions in (1.4).

In Section 5, the final section, we first analyze the long time behaviour in the sector \( S_+ \). Here our main tool consists of the induced derivations of the Lax operator, which are acting on the \( k \)-th exterior powers \( \wedge^k l_2^+ \), \( k \geq 1 \). Note that this has been also used in [DLT1], but in our context we can express all the relevant quantities explicitly in terms of eigenvalues and eigenvectors. An interesting feature here is that the long time behaviour of the (\( + \) Toda flow) \( L(q(t), p(t)) \) has the sorting property, as in the case of the Toda flow on \( N \times N \) Jacobi matrices [Mo]. In particular, if \( 0 < \cdots < \lambda_3 < \lambda_2 < \lambda_1 \) are the (simple) eigenvalues of \( L(q(0), p(0)) \), this means that \( \lim_{t \to \infty} p_j(t) = 2\lambda_j \) for all \( j \in \mathbb{N} \). In this case, the scattering behaviour
follows easily and we can show that $q_j(t) \sim \lambda_j t$ as $t \to \infty$. Finally, motivated by a convincing but heuristic discussion in [M], we end the paper by showing how our preceding analysis can be adapted to other sectors of the phase space defined by a restricted class of permutations of $\mathbb{N}$.

2. Toda flows on Hilbert-Schmidt operators and orbits of semiseparable operators.

Let $\mathcal{H}$ be the Hilbert space $l_2^+$ consisting of sequences $u = (u_1, u_2, \cdots)$ of real numbers which satisfy $||u|| = (\sum_{i=0}^{\infty} u_i^2)^{\frac{1}{2}} < \infty$. In this section, we begin by introducing a class of isospectral flows on the space $\mathfrak{g}$ of Hilbert-Schmidt operators on $\mathcal{H}$ which is in some sense a natural generalization of the Toda flows on $n \times n$ matrices [DLT1]. Then we will study some of the basic properties of these flows which are relevant in our study of the Camassa-Holm equation.

Throughout the paper, let $B(\mathcal{H})$ be the space of bounded operators on $\mathcal{H}$. If $A \in B(\mathcal{H})$, we shall denote its transpose by $A^T$, we shall also write $A = (A_{ij})_{i,j=1}^{\infty}$ if and only if

$$(Au)_i = \sum_{i=1}^{\infty} A_{ij} u_j, \quad i \geq 1, \; u \in l_2^+. \quad (2.1)$$

Thus $(A_{ij})_{i,j=1}^{\infty}$ is the matrix representation of $A$ with respect to the canonical basis $e_1, e_2, \cdots$ of $l_2^+$. With this notation, we have the following basic fact, namely, $A \in \mathfrak{g}$ if and only if the Hilbert-Schmidt norm $||A||_2 = \left(\sum_{i,j=1}^{\infty} A_{ij}^2\right)^{\frac{1}{2}} < \infty$. Hence $A \in \mathfrak{g}$ implies $A^T$ is also in $\mathfrak{g}$.

We start with the following proposition which is an easy consequence of the closure of $\mathfrak{g}$ under the operations of addition, subtraction, and composition in $B(\mathcal{H})$.

**Proposition 2.1.** $\mathfrak{g}$ is a Hilbert Lie algebra with Lie bracket $[\cdot, \cdot]$ defined by

$$[A, B] = AB - BA \quad (2.2)$$

and the inner product on $\mathfrak{g}$ is the usual Hilbert-Schmidt inner product $(\cdot, \cdot)_2$, i.e.,

$$(A, B)_2 = tr(A^T B) = \sum_{i,j=1}^{\infty} A_{ij} B_{ij}. \quad (2.3)$$

Moreover, $\mathfrak{g}$ is equipped with the non-degenerate ad-invariant pairing

$$(A, B) = \sum_{i,j=1}^{\infty} A_{ij} B_{ji}. \quad (2.4)$$
The Hilbert Lie algebra $\mathfrak{g}$ has two distinguished Lie subalgebras $\mathfrak{l}$ and $\mathfrak{k}$, where $\mathfrak{l}$ consists of lower triangular operators $A \in \mathfrak{g}$, i.e.,

$$A_{ij} = 0 \text{ for } i < j$$

(2.5)

and $\mathfrak{k}$ consists of operators $B \in \mathfrak{g}$ for which

$$B^T = -B.$$  

(2.6)

We will call $\mathfrak{l}$ the lower triangular subalgebra of $\mathfrak{g}$ and $\mathfrak{k}$ the skew-symmetric subalgebra. Our next result is obvious. (See also (2.10) below.)

Proposition 2.2. We have

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{k}.$$ 

Let $\Pi_\mathfrak{l}$ and $\Pi_\mathfrak{k}$ be the projection operators onto $\mathfrak{l}$ and $\mathfrak{k}$ respectively associated with the splitting $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{k}$. Then [STS]

$$R = \Pi_\mathfrak{l} - \Pi_\mathfrak{k}$$

(2.7)

is a classical $r$-matrix on $\mathfrak{g}$ satisfying the modified Yang-Baxter equation (mYBE)

$$\left[R(A), R(B)\right] - R(\left[R(A), B\right] + \left[A, R(B)\right]) = -[A, B]$$

(2.8)

for all $A, B \in \mathfrak{g}$. Consequently, the formula

$$[A, B]_R = \frac{1}{2}([R(A), B] + [A, R(B)]), \quad A, B \in \mathfrak{g}$$

(2.9)

defines a second Lie bracket on $\mathfrak{g}$ and we shall denote the associated Lie algebra by $\mathfrak{g}_R$. Note that explicitly, we have

$$\Pi_\mathfrak{k} A = A_+ - A_+^T,$$

$$\Pi_\mathfrak{l} A = A_- + A_0 + A_+^T$$

(2.10)

where for $A = (A_{ij})_{i,j=1}^\infty \in \mathfrak{g}$, the operators $A_+, A_-$ and $A_0$ are defined respectively by the strict upper triangular part, the strict lower triangular part and the diagonal part of $(A_{ij})_{i,j=1}^\infty$. In what follows, we will compute the dual maps of all linear operators on $\mathfrak{g}$ with respect to the pairing $(\cdot, \cdot)$ in (2.4).
Proposition 2.3. If $L \in \mathfrak{g}$, then
\begin{align*}
\Pi_t^* L &= L_- - L_+^T, \\
\Pi_t^* L &= L_+ + L_0 + L_+^T.
\end{align*}
(2.11)

Proof. For all $L, A \in \mathfrak{g}$, we have the obvious relation $(L, A_+) = (L_-, A_+) = (L_-, A)$. Similarly, $(L, A_+^T) = (L_+, A_+^T) = (L_+^T, A)$. Hence the formula for $\Pi_t^*$ follows. The formula for $\Pi_t^*$ is now obtained from $\Pi_t^* + \Pi_t^*$ = 1.

We will equip $\mathfrak{g}_R^* \simeq \mathfrak{g}$ with the Lie-Poisson structure
\begin{equation}
\{ F_1, F_2 \}_R(L) = (L, [dF_1(L), dF_2(L)]_R)
\end{equation}
where $F_1, F_2 \in C^\infty(\mathfrak{g}_R^*)$, and $dF_i(L) \in \mathfrak{g}$ is defined by the formula $\frac{d}{dt}|_{t=0} F_i(L + tL') = (dF_i(L), L')$, $i = 1, 2$.

The following result is a consequence of standard classical r-matrix theory. (See [STS] and [RSTS] for the general theory.)

Proposition 2.4. (a) The Hamiltonian equations of motion generated by $F \in C^\infty(\mathfrak{g}_R^*)$ is given by
\begin{equation}
\dot{L} = \frac{1}{2} [R(dF(L)), L] - \frac{1}{2} R^*[L, dF(L)].
\end{equation}
(2.13)

In particular, for the Hamiltonian $H_j(L) = \frac{1}{2(j+1)} tr(L^{j+1})$, $j = 1, 2, \ldots$, the corresponding equation is the Lax equation
\begin{equation}
\dot{L} = \frac{1}{2} [L, \Pi_t L^j].
\end{equation}
(2.14)

(b) The family of functions $H_j(L)$, $j = 1, 2, \ldots$ Poisson commute with respect to $\{\cdot, \cdot\}_R$.

Let
\begin{equation}
p = \{ L \in \mathfrak{g} \mid L = L^T \}.
\end{equation}
(2.15)

Corollary 2.5. (a) $p$ is a Poisson submanifold of $(\mathfrak{g}_R^*, \{\cdot, \cdot\}_R)$. Hence eqn. (2.14) with $L \in p$ is Hamiltonian with respect to the induced Poisson structure on $p$.

Usage. The flows defined by (2.14) will be called collectively the ($-$) Toda flows. On the other hand, the ($-$) Toda flow will refer to the $j = 1$ case in (2.14).
this work, we will also consider the (+) Toda flow which is defined by the equation
\[ \dot{L} = \frac{1}{2} [\Pi_t L, L]. \]

Our next goal in this section is to show how to solve the (±) Toda flow. In order to do so, we have to describe the Lie groups which integrates the Hilbert Lie algebras \( \mathfrak{g}, \mathfrak{l} \) and \( \mathfrak{k} \). For this purpose, let \( GL(\mathcal{H}) \) be the group of invertible operators in \( B(\mathcal{H}) \), and let \( I \in GL(\mathcal{H}) \) denote the identity operator. We begin by defining

\[ G = GL(\mathcal{H}) \cap (I + \mathfrak{g}). \quad (2.16) \]

Since \( \mathfrak{g} \) is a 2-sided ideal in \( B(\mathcal{H}) \) (see, for example [RS1]), it is clear that if \( I + A \in G \), then \( (I + A)^{-1} \in G \). Thus as in [L2], we can show that \( G \) is a Hilbert Lie group, it is indeed the Hilbert Lie group which integrates \( \mathfrak{g} \). We will call \( G \) the Hilbert-Schmidt group. On the other hand, the Lie subgroup of \( G \) which corresponds to the Lie algebra \( \mathfrak{k} \) is given by

\[ \mathcal{K} = \{ k \in G \mid kk^T = k^Tk = I \}. \quad (2.17) \]

In order to introduce the Lie subgroup of \( G \) which integrates \( \mathfrak{l} \), we need some preparation. Our next result is a discrete version of Lemma 1, Section 2.7 of [Sm]. We can prove it inductively as in [Sm]. For this reason, we will omit its details.

**Lemma 2.6.** Let \( A = (A_{ij})_{i,j=1}^{\infty} \in \mathfrak{l} \) and \( u = (u(1), u(2), \ldots) \in \mathcal{H} \). If

\[ u_n = A^n u, \quad n = 1, 2, \ldots \quad (2.18) \]

then

\[ |u_n(j)| \leq \frac{A_1(j)||u||}{[(n-1)!]^\frac{1}{2}} \left( \sum_{k=1}^{j} A_1(k)^2 \right)^{\frac{1}{2}(n-1)} \quad (2.19) \]

for all \( j \in \mathbb{N} \), where

\[ A_1(j) = \left( \sum_{k=1}^{j} a_{jk}^2 \right)^{\frac{1}{2}}, \quad j \in \mathbb{N}. \quad (2.20) \]

**Proposition 2.7.** The set

\[ \mathcal{L} = \{ g \in I + \mathfrak{l} \mid g_{ii} > 0 \text{ for all } i \} \quad (2.21) \]

with the induced group operation of \( G \) is a Lie subgroup of \( G \) which integrates \( \mathfrak{l} \).

**Proof.** It is clear that the group operation of \( G \) is closed on \( \mathcal{L} \). On the other hand, if \( A \in \mathfrak{l} \), we can show that the Neumann series \( \sum_{n=0}^{\infty} (-1)^n A^n \) converges. To see
this, take a nonzero vector \( u = (u(1), u(2), \cdots) \in \mathcal{H} \). Then from the inequality in (2.19), we have

\[
\|A^n u\|^2 \leq \sum_{j=1}^{\infty} \frac{A_1(j)^2\|u\|^2}{(n-1)!} \left( \sum_{k=1}^{j} A_1(k)^2 \right)^{n-1} \\
= \frac{\|u\|^2}{n!} \left( \sum_{j=1}^{\infty} A_1(j)^2 \right)^n \\
= \frac{\|u\|^2}{n!} \|A\|^2_n.
\]

from which we obtain the estimate

\[
\|A^n\| \leq \frac{\|A\|_2^2}{\sqrt{n!}}. \tag{2.23}
\]

The convergence of the Neumann series is now clear from (2.23). Thus \( I + A \) is invertible with \( (I + A)^{-1} = \sum_{n=0}^{\infty} (-1)^n A^n \in \mathcal{L} \). This shows that \( \mathcal{L} \) is a subgroup of \( G \). As \( \mathcal{L} \) is clearly a submanifold of \( G \), this completes the proof of the assertion. \( \square \)

Our next result is the global version of the direct sum decomposition in Proposition 2.2.

**Proposition 2.8.** Suppose \( I + A \in G \), then \( I + A \) has a unique factorization

\[
I + A = b_- b_+^{-1} \tag{2.24}
\]

where \( b_- \in \mathcal{L} \) and \( b_+ \in \mathcal{K} \).

**Proof.** The factorization problem in (2.24) is equivalent to \( (I + A)^T = b_+ b_-^T \), where \( b_- \in \mathcal{L} \) and \( b_+ \in \mathcal{K} \). Now we can certainly obtain a unique orthogonal \( b_+ \in GL(\mathcal{H}) \) and a unique lower triangular \( b_- \in GL(\mathcal{H}) \) with \( (b_-)_{ii} > 0 \) by applying the Gram-Schmidt orthogonalization process to the vectors \( (I + A)^T e_1, (I + A)^T e_2, \cdots \). To complete the proof, it suffices to show that \( b_- \in \mathcal{L} \). To this end, note that \( (I + A)(I + A)^T = b_- b_+^T \). Since \( \mathfrak{g} \) is a 2-sided ideal in \( B(\mathcal{H}) \) which is closed under the operation of taking the transpose, we can rewrite the above relation in the form

\[
K = b_- - (b_-^{-1})^T \tag{2.25}
\]

where

\[
K = ((I + A)(I + A)^T - I)(b_-^{-1})^T \in \mathfrak{g}. \tag{2.26}
\]
Now, from $||K||^2 < \infty$ and the relation in (2.25), we infer that

$$\sum_{j < i} (b_-)_{ij}^2 + \sum_{i < j} (b_-^{-1})_{ij}^2 + \sum_{i=0}^{\infty} \frac{((b_-)_{ii}^2 - 1)^2}{(b_-)_{ii}^2} < \infty. \quad (2.27)$$

But as

$$\frac{((b_-)_{ii}^2 - 1)^2}{(b_-)_{ii}^2} - ((b_-)_{ii} - 1)^2 = \frac{((b_-)_{ii} - 1)^2(2(b_-)_{ii}^2 + 1)}{(b_-)_{ii}^2} \geq 0, \quad (2.28)$$

it follows on using (2.28) in (2.27) that

$$||b_- - I||^2 = \sum_{j < i} (b_-)_{ij}^2 + \sum_{i=1}^{\infty} ((b_-)_{ii} - 1)^2 < \infty, \quad (2.29)$$

as desired.

\[\box]

We are now ready to give the solution to the $(\pm)$ Toda flow. As the proof is quite standard, we refer the reader to Theorem 3.2 and Remark 3.3 (b) in [L2]. (See [RSTS] for the general theory of the factorization method.)

**Theorem 2.9.** Let $\mathbf{L}_0 \in \mathfrak{g}$, and let $b_-(t) \in \mathcal{L}$, $b_+(t) \in \mathcal{K}$ be the unique solution of the factorization problem

$$\exp \left( \pm \frac{1}{2} t \mathbf{L}_0 \right) = b_-(t)b_+(t)^{-1}. \quad (2.30)$$

Then for all $t$,

$$\mathbf{L}(t) = b_+^{-1}(t)\mathbf{L}_0b_+(t) = b_-^{-1}(t)\mathbf{L}_0b_-(t) \quad (2.31)$$

solves the initial value problem

$$\dot{\mathbf{L}} = \pm \frac{1}{2} [\Pi_t \mathbf{L}, \mathbf{L}], \quad \mathbf{L}(0) = \mathbf{L}_0. \quad (2.32)$$

We next give the first result on the long time behaviour of the $(\pm)$ Toda flow when the initial data $\mathbf{L}_0 \in \mathfrak{p}$. It is in fact just a special case of Proposition 5 in Section 2 of [DLT1]. (The proof is a modification of Moser’s argument in [Mo].)
Proposition 2.10. Let $L(t)$ be the solution of $\dot{L} = \pm \frac{1}{2} [\Pi_t L, L]$, $L(0) = L_0 \in p$. Then $L(t)$ converges strongly to a diagonal operator $L^\pm(\infty) = \text{diag}(\alpha_1^\pm, \alpha_2^\pm, \cdots)$ with $\alpha_i^\pm$ belonging to the spectrum $\sigma(L_0)$ of $L_0$ as $t \to \infty$.

Remark 2.11 Recall that $L(t)$ converges strongly to $L^\pm(\infty)$ as $t \to \infty$ means $\|L(t)u - L^\pm(\infty)u\| \to 0$ for each $u \in \mathcal{H}$ (see [RS1]). Since $\mathcal{H}$ is infinite dimensional, this notion of convergence is weaker than norm convergence, so in general the spectrum of $L^\pm(\infty)$ can shrink. (See VIII.7 of [RS1] for a discussion of such matters.) As the reader will see, this is indeed what happens in Section 4 below.

In the rest of the section, we will describe the symplectic leaves of the Lie-Poisson structure $\{\cdot, \cdot\}_R$ which are given by the coadjoint orbits of the infinite dimensional Lie group $G_R$ which integrates $g_R$. In particular, we will consider the coadjoint action of $G_R$ on the class $p_*$ of semiseparable operators $L \in g$. By definition, a Hilbert-Schmidt operator $L = (L_{ij})_{i,j=1}^\infty \in p_*$ if and only if

\[
L_{ij} = \begin{cases}
u_i v_j, & i \leq j \\
u_j v_i, & i > j,
\end{cases}
\quad (2.33)
\]

where $u = (u_1, u_2, \cdots)$, $v = (v_1, v_2, \cdots)$ are sequences of real numbers, which are not necessarily in $l_2^+$. The Lie group $G_R$ can be described in the following way (cf. [DLT2]): the underlying manifold is $G$, but now the group operation is defined by

\[
g * h \equiv g_- h g_-^{-1}
\quad (2.34)
\]

where $g = g_- g_+^{-1}$ is the unique factorization into $g_- \in L$ and $g_+ \in K$. Moreover, the coadjoint action of $G_R$ on $g_R^*$ is given by

\[
\text{Ad}_{G_R}^*(g) L = \Pi_t^*(g_- L g_-^{-1}) + \Pi_t^*(g_+ L g_+^{-1})
\quad (2.35)
\]

and the orbits of this action are the symplectic leaves of $\{\cdot, \cdot\}_R$.

Proposition 2.12. The class $p_* \subset p$ of Hilbert-Schmidt operators in $\mathcal{H}$ which are semiseparable is invariant under $\text{Ad}_{G_R}^*$.

Proof. From (2.35), we have $\text{Ad}_{G_R}^*(g) L = \Pi_t^*(g_- L g_-)$ for $L \in p_*$. If

\[
L_{ij} = \begin{cases}
u_i v_j, & i \leq j \\
u_j v_i, & i > j,
\end{cases}
\]

for some sequences of real numbers $u = (u_1, u_2, \cdots)$ and $v = (v_1, v_2, \cdots)$, a straightforward computation shows that

\[
(g_-^{-1} L g_-)_{ij} = (g_-^{-1} u_i)(g^T v)_{ij}, \quad i \leq j
\]
where we have used the fact that \( g_\nu \) is lower triangular. Therefore the assertion follows from the formula for \( \Pi^*_1 \) in (2.11)

\[ \square \]

From this result, it follows that if the initial data \( L_0 \) of (2.32) is in \( p_* \), then \( L(t) \in p_* \) for all \( t \). In the next section, the reader will see that we will be dealing with the \((\pm)\) Toda flow on some rather special semiseparable operators which are related to the CH equation.

3. A class of low-regularity solutions of the Camassa-Holm equation.

In this section, we will consider a class of weak solutions of the CH equation

\[
\begin{align*}
u_t - u_{xxt} + 3uu_x &= 2u_x u_{xx} + uu_{xxx}, & (3.1) \\
\end{align*}
\]

of the form

\[
\begin{align*}
u(x, t) &= \frac{1}{2} \sum_{j=1}^{\infty} e^{-|x-q_j(t)|} p_j(t), & (3.2) \\
\end{align*}
\]

where \( p_j(t) \neq 0 \) for all \( j \in \mathbb{N} \) and such that \( p_j(t) \to 0 \) sufficiently fast as \( j \to 0 \). To be more precise, we assume (for small values of \( t \)) that \( q(t) = (q_1(t), q_2(t), \cdots) \in l^+_\infty \), while \( p(t) = (p_1(t), p_2(t), \cdots) \in l^+_{1,2} \). Here \( l^+_\infty \) and \( l^+_1,2 \) are Banach spaces defined as follows:

\[
\begin{align*}
l^+_\infty &= \{q = (q_1, q_2, \cdots) \mid ||q||_\infty = \sup_j |q_j| < \infty \}, \\
l^+_1,2 &= \{p = (p_1, p_2, \cdots) \mid ||p||_{1,2} = \sum_{j=1}^{\infty} j^2 |p_j| < \infty \}. & (3.3) \\
\end{align*}
\]

Following [BSS], we rewrite the CH equation (3.1) in the form

\[
m_t + (mu)_x + mu_x = 0, \quad m = u - u_{xx}. \quad (3.4)
\]

Then the solution in (3.2) corresponds to the measure

\[
m(x, t) = \sum_{j=1}^{\infty} e^{-|x-q_j(t)|} p_j(t) \delta(x - q_j(t)). \quad (3.5)
\]

Therefore, if we mimic the calculation in [BSS], we find that \( u(x, t) \) and \( m(x, t) \) satisfy the equation in (3.4) in a weak sense if and only if

\[
\begin{align*}
\dot{q}_j &= \frac{1}{2} \sum_{k=1}^{\infty} e^{-|q_j - q_k|} p_k, \\
\dot{p}_j &= \frac{1}{2} p_j \sum_{k=1}^{\infty} sgn(q_j - q_k) e^{-|q_j - q_k|} p_k, \quad j \in \mathbb{N} \quad (3.6)
\end{align*}
\]
where we adopt the convention that $\text{sgn} \, 0 = 0$. Note that our assumptions above means that we are considering these equations in the Banach space direct sum $l^+_{\infty} \oplus l^+_{1,2}$, equipped with the norm $||(q, p)|| = ||q||_\infty + ||p||_{1,2}$. Clearly, the signs of the $p_j$’s are preserved as long as no blowup occurs. In this work, we will focus on the case where $p_j > 0$ for all $j$. Indeed, we will restrict our attention to the two sectors

$$S_+ = \{(q, p) \in l^+_{\infty} \oplus l^+_{1,2} \mid q_1 < q_2 < \cdots, p_j > 0 \text{ for all } j\},$$

and

$$S_- = \{(q, p) \in l^+_{\infty} \oplus l^+_{1,2} \mid q_1 < q_2 < \cdots, p_j > 0 \text{ for all } j\}$$

for the most part. At the end of the paper, we will show how to adapt our analysis to other sectors which are defined by a restricted class of permutations of $\mathbb{N}$.

**Proposition 3.1.** Suppose $(q^0, p^0) \in S_\pm$. Then the initial value problem

$$\dot{q}_j = \frac{1}{2} \sum_{k=1}^{\infty} e^{-|q_j - q_k|} p_k,$$

$$\dot{p}_j = \frac{1}{2} p_j \sum_{k=1}^{\infty} \text{sgn}(q_j - q_k) e^{-|q_j - q_k|} p_k$$

$$= \pm \frac{1}{2} p_j \sum_{k=1}^{\infty} \text{sgn}(k - j) e^{-|q_j - q_k|} p_k,$$

$$q_j(0) = q_j^0, \quad p_j(0) = p_j^0, \quad j \in \mathbb{N}$$

has a unique global solution in $S_\pm$.

**Proof.** For $(q, p) \in S_\pm$, put

$$f(q, p) = (f_1(q, p), f_2(q, p))$$

where

$$f_1(q, p) = \left( \frac{1}{2} \sum_{j=1}^{\infty} e^{-|q_j - q_k|} p_k \right)^{\infty}_{j=1},$$

$$f_2(q, p) = \left( \frac{1}{2} p_j \sum_{j=1}^{\infty} \text{sgn}(q_j - q_k) e^{-|q_j - q_k|} p_k \right)^{\infty}_{j=1}. \tag{3.11}$$

We first show $f : S_\pm \rightarrow l^+_{\infty} \oplus l^+_{1,2}$. For this purpose (and for later usage), we put

$$||p||_1 = \sum_{k=1}^{\infty} p_k, \quad p = (p_k)_{k=1}^{\infty} \in l^+_{1,2}, \quad p_k > 0. \tag{3.12}$$
Then from the expressions for \( f_1 \) and \( f_2 \) above, we find that

\[
\|f(q, p)\| \leq \frac{1}{2}\|p\|_1(1 + \|p\|_{1,2}),
\]  

(3.13)
as desired. We next show \( f \) is locally Lipschitz. To do this, take \((q, p), (\tilde{q}, \tilde{p})\) in an open ball centered at \((q^0, p^0)\) which is contained in \( S_{\pm} \). Then by making use of the inequality \(|e^{-\xi} - e^{-\eta}| \leq |\xi - \eta|\) for \( \xi, \eta > 0 \) and the triangle inequalities, we have

\[
\left| \sum_{k=1}^{\infty} e^{-|q_j - q_k|} p_k - \sum_{k=1}^{\infty} e^{-|\tilde{q}_j - \tilde{q}_k|} \tilde{p}_k \right| 
\leq \sum_{k=1}^{\infty} e^{-|q_j - q_k|} |p_k - \tilde{p}_k| + \sum_{k=1}^{\infty} |\tilde{p}_k||e^{-|q_j - q_k|} - e^{-|\tilde{q}_j - \tilde{q}_k|}| 
\leq \sum_{k=1}^{\infty} |p_k - \tilde{p}_k| + \sum_{k=1}^{\infty} |\tilde{p}_k|(|q_j - \tilde{q}_j| + |q_k - \tilde{q}_k|) 
\leq \sum_{k=1}^{\infty} |p_k - \tilde{p}_k| + ||\tilde{p}||_1|q_j - \tilde{q}_j| + ||\tilde{p}||_1||q - \tilde{q}||_\infty.
\]  

(3.14)

Consequently,

\[
\|f_1(q, p) - f_1(\tilde{q}, \tilde{p})\|_\infty 
\leq \frac{1}{2}\|p - \tilde{p}\|_{1,2} + ||\tilde{p}||_1||q - \tilde{q}||_\infty
\]  

(3.15)

On the other hand, by using the fact that \( sgn(q_j - q_k) = sgn(\tilde{q}_j - \tilde{q}_k) = \pm sgn(k - j) \) and the estimate in (3.14), we find

\[
\|f_2(q, p) - f_2(\tilde{q}, \tilde{p})\|_{1,2} 
\leq \frac{1}{2}\sum_{j=1}^{\infty} j^2 \sum_{k=1}^{\infty} |e^{-|q_j - q_k|} p_k - e^{-|\tilde{q}_j - \tilde{q}_k|} \tilde{p}_k| p_j 
+ \frac{1}{2}\sum_{j=1}^{\infty} j^2 \sum_{k=1}^{\infty} |e^{-|\tilde{q}_j - \tilde{q}_k|} \tilde{p}_k| p_j - \tilde{p}_j|
\leq \|p\|_{1,2} \left( \frac{1}{2}\|p - \tilde{p}\|_{1,2} + ||\tilde{p}||_1||q - \tilde{q}||_\infty \right) + \frac{1}{2}||\tilde{p}||_1||p - \tilde{p}||_{1,2}.
\]  

(3.16)

Therefore, upon combining (3.15) and (3.16), we obtain

\[
\|f(q, p) - f(\tilde{q}, \tilde{p})\| 
\leq C(||p||_{1,2}, ||\tilde{p}||_1)||q(p) - (\tilde{q}, \tilde{p})||.
\]  

(3.17)

Finally, to establish global existence, let us suppose the solution \((q(t), p(t))\) exists for \( 0 \leq t \leq T \) for some \( T > 0 \). We will establish an a priori estimate for \(||(q(t), p(t))||\).
To do so, observe that \( P = \sum_{j=1}^{\infty} p_j(t) \) is a conserved quantity. Hence the equations for \( q_j(t) \) gives \( |\dot{q}_j(t)| \leq \frac{1}{2} P \) from which we obtain the estimate

\[
\|q(t)\|_\infty \leq \|q(0)\|_\infty + \frac{1}{2} P t. \tag{3.18}
\]

Similarly, the equations for \( p_j(t) \) gives \( |\dot{p}_j(t)| \leq \frac{1}{2} p_j(t) P \) from which we find \( p_j(t) \leq p_j(0) e^{\frac{1}{2} P t} \). Therefore,

\[
\|p(t)\|_{1,2} \leq \|p(0)\|_{1,2} e^{\frac{1}{2} P t}. \tag{3.19}
\]

Consequently, on combining (3.18) and (3.19), we conclude that

\[
\|\mathbf{(q(t), p(t))}\| \leq \|q(0)\|_\infty + \frac{1}{2} P t + \|p(0)\|_{1,2} e^{\frac{1}{2} P t} \tag{3.20}
\]

for \( 0 \leq t \leq T \). This completes the proof. \( \square \)

Our next result relates the equations in (3.6) with \( (q, p) \in S_\pm \) to the \((\pm)\) Toda flow and gives the spectral properties of the Lax operator.

**Proposition 3.2.** For \((q, p) \in S_\pm\), define the operator \( L(q, p) = (L_{ij}(q, p))_{i,j=1}^{\infty} \) on \( l_2^+ \) by

\[
L_{ij}(q, p) = \frac{1}{2} e^{-\frac{1}{2}|q_i - q_j|} \sqrt{p_i p_j}, \quad i, j \geq 1. \tag{3.21}
\]

Then

(a) \( L(q, p) \) is a positive, semiseparable trace-class operator. Indeed,

\[
L_{ij}(q, p) = \begin{cases} u_i(q, p)v_j(q, p), & i \leq j \\ u_j(q, p)v_i(q, p), & i > j, \end{cases} \tag{3.22a}
\]

for \((q, p) \in S_-\), while

\[
L_{ij}(q, p) = \begin{cases} u_j(q, p)v_i(q, p), & i \leq j \\ u_i(q, p)v_j(q, p), & i > j, \end{cases} \tag{3.22b}
\]

for \((q, p) \in S_+\), where

\[
u_i(q, p) = \frac{1}{\sqrt{2}} e^{\frac{1}{2} q_i} \sqrt{p_i}, \quad v_i(q, p) = \frac{1}{\sqrt{2}} e^{-\frac{1}{2} q_i} \sqrt{p_i}, \quad i \geq 1. \tag{3.23}\]

(b) If \( f = (f(1), f(2), \ldots) \) is an eigenvector of \( L(q, p) \), then \( f(1) \neq 0 \).

(c) The eigenvalues of \( L(q, p) \) are simple and \( \ker(L(q, p)) = \{0\} \).

(d) If \((q, p) \) evolves under (3.6), then

\[
\dot{L}(q, p) = \begin{cases} \frac{1}{2} [L(q, p), \Pi_L L(q, p)], & (q, p) \in S_- \\ \frac{1}{2} [\Pi_L L(q, p) L(q, p)], & (q, p) \in S_+. \end{cases} \tag{3.24}
\]
Proof. We will give the proof for \((q, p) \in \mathcal{S}_-\), the arguments for the other case are similar.

(a) It is clear from the definition of \(L(q, p)\) that \(L_{ij}(q, p)\) is of the form given in (3.22a) with \(u_i(q, p), v_i(q, p)\) defined in (3.23). Moreover, it follows from the natural ordering of the \(q_i\)’s that

\[
0 < \frac{u_1(q, p)}{v_1(q, p)} < \frac{u_2(q, p)}{v_2(q, p)} < \cdots . \tag{3.25}
\]

To show that \(L(q, p)\) is positive, let us denote by \(L^{(n)}(q, p)\) the \(n \times n\) matrix in which \(L_{ij}^{(n)}(q, p) = L_{ij}(q, p), i, j = 1 \cdots , n\). Clearly, \(\det(L^{(1)}(q, p)) = p_1 > 0\). For \(n > 1\), it follows from [GK] [eqn.(29) on p.78] that

\[
\det(L^{(n)}(q, p)) = u_1(q, p)v_n(q, p) \prod_{j=1}^{n-1} \det \left( \begin{array}{cc} v_i(q, p) & v_{i+1}(q, p) \\ u_i(q, p) & u_{i+1}(q, p) \end{array} \right) = v_1^2(q, p) \cdots v_n^2(q, p) \prod_{j=1}^{n} \left( \frac{u_j(q, p)}{v_j(q, p)} - \frac{u_{j-1}(q, p)}{v_{j-1}(q, p)} \right) \tag{3.26}\]

where we formally set \(\frac{u_0(q, p)}{v_0(q, p)} = 0\). Hence we conclude from (3.25) that \(\det(L^{(n)}(q, p)) > 0\). Consequently, \(L(q, p)\) is positive. Finally, the fact that \(L(q, p)\) is trace-class follows as we have \(\sum_{j=1}^{\infty} (e_j, L(q, p)e_j) = \sum_{j=1}^{\infty} p_j < \infty\).

(b) Suppose \(L(q, p)f = \lambda f\). Writing this out in terms of components, we have

\[
\begin{aligned}
&u_1(q, p)(v_1(q, p)f(1) + v_2(q, p)f(2) + v_3(q, p)f(3) + \cdots ) = \lambda f(1), \\
u_1(q, p)v_2(q, p)f(1) + u_2(q, p)(v_2(q, p)f(2) + v_3(q, p)f(3) + \cdots ) = \lambda f(2), \\
u_1(q, p)v_3(q, p)f(1) + u_2(q, p)v_3(q, p)f(2) + u_3(q, p)(v_3(q, p)f(3) + \cdots ) = \lambda f(3), \\
&\vdots
\end{aligned} \tag{3.27}
\]

If \(f(1) = 0\), then it follows from the first equation of (3.27) that

\[v_2(q, p)f(2) + v_3(q, p)f(3) + \cdots = 0.\]

Substitute this into the second equation of (3.27), we find \(f(2) = 0\). Hence \(v_3(q, p)f(3) + \cdots = 0\). When we substitute this into the third equation of (3.27), we obtain \(f(3) = 0\). Proceeding inductively, it is easy to see that \(f = 0\), a contradiction to the assumption that \(f\) is an eigenvector.

(c) Suppose there exist independent eigenvectors \(f\) and \(g\) of \(L(q, p)\) corresponding to the eigenvalue \(\lambda\). Since \(f(1), g(1) \neq 0\), we can find \(c_1, c_2 \in \mathbb{R} \setminus \{0\}\) such that
\( c_1 f(1) + c_2 g(1) = 0 \). But on the other hand, \( c_1 f + c_2 g \) is obviously an eigenvector of \( L(q, p) \), which is impossible as \( c_1 f(1) + c_2 g(1) = 0 \). To show that \( \ker(L(q, p)) = \{0\} \), suppose \( L(q, p) h = 0 \). Writing this out in terms of components, we get

\[
\begin{align*}
u_1(q, p)v_1(q, p)h(1) + v_2(q, p)h(2) + v_3(q, p)h(3) + \cdots & = 0, \\
u_1(q, p)v_2(q, p)h(1) + u_2(q, p)v_2(q, p)h(2) + v_3(q, p)h(3) + \cdots & = 0, \\
u_1(q, p)v_3(q, p)h(1) + u_2(q, p)v_3(q, p)h(2) + u_3(q, p)(v_3(q, p)h(3) + \cdots) & = 0,
\end{align*}
\]

(3.28)

Next, we multiply the second equation of (3.28) by \( -\frac{u_1(q, p)}{u_2(q, p)} \) and add it to the first equation, this gives

\[
v_1(q, p)v_2(q, p) \left( \frac{u_2(q, p)}{v_2(q, p)} - \frac{u_1(q, p)}{v_1(q, p)} \right) u_1(q, p)h(1) = 0.
\]

(3.29)

Thus it follows from (3.29) and (3.25) that \( h(1) = 0 \). Clearly, if we proceed inductively and make use of (3.25), the conclusion is that \( h = 0 \), as required.

(d) See Remark 3.3 (a) below.

\textbf{Remark 3.3} (a) If we replace \( \infty \) by a positive integer \( N \) in (3.2), we obtain the multipeakon solutions of the CH equation [CH], [BSS] if \( p_j(t) > 0 \) for \( j = 1, \ldots, N \). In that case, the Lax pair of the Hamiltonian equations for \( (q, p) \in \mathbb{R}^{2N} \) with \( p_j > 0 \) for all \( j \) was discovered in [CF] in a remarkable calculation. On the other hand, in the sector where the \( q_j \)’s satisfy the natural ordering \( q_1 < q_2 < \cdots < q_N \), the realization that the Lax equation is just a special case of the Toda flows on \( N \times N \) matrices was pointed out in [RB]. In this regard, the calculation which leads to (3.24) is just an extension of the one in [CF]. We should mention, however, that the \( r \)-matrix in [RB] is not the appropriate one to use from the point of view of Poisson geometry. Indeed, it is easy to check that the set of \( N \times N \) symmetric matrices is not even a Poisson submanifold of the Lie-Poisson structure associated with the corresponding \( R \)-bracket.

(b) In the case of the peakons lattice in [RB], the Lax operator is invertible. (This also follows from (3.26) above.) In our case, although 0 is not an eigenvalue of \( L(q, p) \) by Proposition 3.2 (c) above, however, it is in the essential spectrum \( \sigma_{\text{ess}}(L(q, p)) \) (indeed, \( \sigma_{\text{ess}}(L(q, p)) = \{0\} \)). Hence \( L(q, p) \) is not invertible.

We close this section with the following result which is a consequence of Proposition 2.10, Proposition 3.2 and the equation for \( q_j \) in (3.6).
**Proposition 3.4.** Let \((q(t), p(t))\) be a solution of (3.6) in the sector \(S_\pm\). Then

(a) \(L(q(t), p(t)) \rightarrow \text{diag}(\alpha_1^\pm, \alpha_2^\pm, \cdots)\) strongly as \(t \rightarrow \infty\) where \(\alpha_i^\pm \in \sigma(L(q(0), p(0))\) and hence \(\lim_{t \rightarrow \infty} p_j(t) = 2\alpha_j^\pm\) for each \(j \in \mathbb{N}\),

(b) \(q_j(t) > 0\) for each \(j \in \mathbb{N}\) and \(\lim_{t \rightarrow \infty} e^{-\frac{1}{2}|q_j(t)-q_k(t)|} \sqrt{p_j(t)p_k(t)} = 0\) whenever \(j \neq k\).

**Remark 3.5** (a) From Proposition 3.4 (b) above, we conclude that the peakons are traveling to the right.

(b) In Sections 4 and 5, we will show that the peakons separate out, i.e., the \(q_j(t)\)'s have the scattering behaviour. However, in contrast to the semi-infinite Toda lattice [L1], this does not follow immediately from the long time behaviour of \(L(q(t), p(t))\). This is clear from Proposition 3.4 (b) above as \(\lim_{t \rightarrow \infty} e^{-\frac{1}{2}|q_j(t)-q_k(t)|} \sqrt{p_j(t)p_k(t)} = 0\) does not follow automatically from \(\lim_{t \rightarrow \infty} e^{-\frac{1}{2}|q_j(t)-q_k(t)|} \sqrt{p_j(t)p_k(t)} = 0\). Indeed, for \((q(t), p(t)) \in S_\pm\), we will show that \(\lim_{t \rightarrow \infty} p_j(t) = 0\) for all \(j \in \mathbb{N}\), so even the explicit values of the \(\alpha_j^\pm\)'s are of no help in this case in establishing the scattering behaviour.

### 4. Long time behaviour in the sector \(S_-\).

Let \((q(t), p(t))\) be the solution of (3.6) with \((q(0), p(0)) = (q^0, p^0) \in S_-\) and let \(\sigma(L(q^0, p^0)) \setminus \{0\} = \{\lambda_i\}_{i=1}^\infty\). In view of Proposition 3.2, we will order the eigenvalues as follows:

\[
0 < \cdots < \lambda_3 < \lambda_2 < \lambda_1. \tag{4.1}
\]

We will also take the normalized eigenvectors \(\phi_1(t), \phi_2(t), \cdots\) of \(L(q(t), p(t))\) to be such that \(\phi_k(1, t) > 0\), \(k = 1, 2, \cdots\).

Now it follows from the same proposition that \(L(q(t), p(t))\) is one-to-one. Hence \(L(q(t), p(t))\) has a left inverse \(J(q(t), p(t))\) which is an unbounded operator defined on the dense linear subspace \(\text{Ran} \ L(q(t), p(t))\) of \(\mathcal{H}\). In order to give the formula for \(J(q(t), p(t))\), set

\[
e_j(t) = e^{-\frac{1}{2}(q_{j+1}(t)-q_j(t))}, \quad j \in \mathbb{N}. \tag{4.2}
\]

**Proposition 4.1.** The matrix of \(J(q(t), p(t))\) is tridiagonal:

\[
J(q(t), p(t)) = \begin{pmatrix}
a_1(t) & -b_1(t) & 0 & \cdots \\
-b_1(t) & a_2(t) & -b_2(t) & \ddots \\
0 & -b_2(t) & a_3(t) & \ddots \\
& \ddots & \ddots & \ddots 
\end{pmatrix} \tag{4.3}
\]
Theorem 2.9, for any \( j \)

Proof. Let

\[
\begin{align*}
a_j(t) &= \frac{2}{p_j(t)} \frac{1 - e_j^2(t)}{(1 - e_j^2(t))(1 - e_j^2(t))}, \\
b_j(t) &= \frac{2}{\sqrt{p_j(t)p_{j+1}(t)}} \frac{e_j(t)}{1 - e_j^2(t)}, \quad j \in \mathbb{N}
\end{align*}
\]

where we formally set \( e_0(t) = 0 \).

Moreover,

\[
e_j(t) \phi_k(j + 1, t) = - \frac{e_{j-1}(t)(1 - e_j^2(t))}{1 - e_j^2(t)} \phi_k(j - 1, t) + \frac{1}{2}(1 - e_j^2(t))p_j(t) \left( a_j(t) - \frac{1}{\lambda_k} \right) \phi_k(j, t) \quad (4.5)
\]

for all \( j, k \in \mathbb{N} \).

Proof. To obtain the formula for \( J(q(t), p(t)) \), we solve \( L(q(t), p(t))f = g \) recursively for \( f(1), f(2), \ldots \) in terms of the components of \( g \). On the other hand, from \( L(q(t), p(t))\phi_k(t) = \lambda_k\phi_k(t) \), we find \( J(q(t), p(t))\phi_k(t) = \frac{1}{\lambda_k}\phi_k(t) \). Since \( J(q(t), p(t)) \) is given by (4.3), we obtain the recurrence relation

\[
b_j(t)\phi_k(j + 1, t) = -b_{j-1}(t)\phi_k(j - 1, t) + \left( a_j(t) - \frac{1}{\lambda_k} \right) \phi_k(j, t). \quad (4.6)
\]

Therefore, the formula in (4.5) follows upon multiplying both sides of (4.6) by \( \frac{1}{2} \sqrt{p_j(t)(1 - e_j^2(t))} \) and making use of the formulas in (4.4).

Our next result shows \( \phi_k(1, t) \) can be solved explicitly.

Lemma 4.2. For each \( k \in \mathbb{N} \),

\[
\phi_k(1, t) = \frac{e^{-\frac{1}{2}\lambda_k t}\phi_k(1, 0)}{\left( \sum_{j=1}^{\infty} e^{-\lambda_j t}\phi_j^2(1, 0) \right)^{\frac{1}{2}}}. \quad (4.7)
\]

Proof. Let \( L(t) = L(q(t), p(t)) \). Then \( L(t) \) evolves under the \((-\) Toda flow. By Theorem 2.9, for any \( j \in \mathbb{N} \),

\[
\begin{align*}
(e_1, \phi_j(t)) &= (e_1, b_+ (t)^{-1}\phi_j(0)) \\
&= ((b_- (t)^{-1})^T e_1, e^{-\frac{1}{2}t L(0)}\phi_j(0)) \\
&= e^{-\frac{1}{2}\lambda_j t}(e_1, \phi_j(0)) \\
&= (b_- (t))_{11}.
\end{align*}
\]

The assertion therefore follows from (4.8) and the relation \( \sum_{j=1}^{\infty} \phi_j^2(1, t) = 1 \). \( \square \)

As the \( \lambda_j \)'s accumulate at 0, it is a difficult problem to get the asymptotics of \( \phi_k(1, t) \) as \( t \to \infty \) from (4.7) above. In the following, we will bypass this difficulty.
Theorem 4.3. Let \((q(t), p(t))\) be the solution of (3.6) with \((q(0), p(0)) = (q^0, p^0) \in S_-\), then

(a) \(\lim_{t \to \infty} p_j(t) = 0\) for all \(j \in \mathbb{N}\),
(b) \(\lim_{t \to \infty} |q_j(t) - q_k(t)| = \infty\) for all \(j \neq k\),
(c) \(q_j(t) \to \infty\) as \(t \to \infty\) for all \(j \in \mathbb{N}\),
(d) \(q_j(t) = o(t)\) as \(t \to \infty\) for all \(j \in \mathbb{N}\).

Proof. From (4.7), we have

\[
\frac{\phi_r(1, t)}{\phi_s(1, t)} = e^{-\frac{1}{2}(\lambda_r - \lambda_s)t} \frac{\phi_r(1, 0)}{\phi_s(1, 0)}
\]  

for all \(r, s \in \mathbb{N}\). Therefore,

\[
\frac{\phi_k^2(1, t)}{p_1(t)} = \frac{\phi_k^2(1, t)}{2 \sum_{j=1}^{\infty} \lambda_j \phi_j^2(1, t)} < \frac{1}{2\lambda_{k+1}} \phi_{k+1}^2(1, t)
\]

\[
= c_{k1} e^{-(\lambda_k - \lambda_{k+1})t}
\]

for each \(k \geq 1\) where \(c_{k1}\) depends only \(L(q^0, p^0)\). Note that as \(\lim_{t \to \infty} p_1(t) = 2\alpha^- < \infty\), the above inequality not only gives

\[
\lim_{t \to \infty} \frac{\phi_k(1, t)}{\sqrt{p_1(t)}} = 0,
\]

but also

\[
\lim_{t \to \infty} \phi_k(1, t) = 0.
\]

Consequently, we conclude that

\[
\lim_{t \to \infty} p_1(t) = 2 \lim_{t \to \infty} \sum_{j=1}^{\infty} \lambda_j \phi_j^2(1, t) = 0
\]

as the series on the right hand side is uniformly convergent. Now, by using (4.13) and the formula for \(a_1(t)\) in (4.4), we find

\[
(1 - e_1^2(t))p_1(t) \left( a_1(t) - \frac{1}{\lambda_k} \right) = \frac{2\lambda_k - p_1(t)(1 - e_1^2(t))}{\lambda_k} \sim 2 \quad \text{as} \quad t \to \infty.
\]

As a result, it follows from the recurrence relation (4.5) for \(j = 1\) and (4.11) that

\[
\lim_{t \to \infty} e_1(t) \frac{\phi_k(2, t)}{\sqrt{p_2(t)}} = 0.
\]
Hence
\[
\lim_{t \to \infty} e_1(t) = \lim_{t \to \infty} \left( 2 \sum_{k=1}^{\infty} \lambda_k \frac{\phi_k(1, t)}{\sqrt{p_1(t)}} \frac{\phi_k(2, t)}{\sqrt{p_2(t)}} \right)^{1/2} = 0 \tag{4.16}
\]
by (4.11), (4.15) and so we can make the following improvement to (4.14):
\[
p_1(t) \left( a_1(t) - \frac{1}{\lambda_k} \right) \sim 2 \quad \text{as } t \to \infty. \tag{4.17}
\]

Next, we claim that
\[
p_2(t) b_1(t) \frac{\phi_k(1, t)}{\phi_k(2, t)} \sim 2e_1(t) \quad \text{as } t \to \infty. \tag{4.18}
\]
To see that this is true, use the recurrence relation (4.6) for \( j = 1 \) and the formula for \( b_1(t) \) in (4.4), we find that
\[
p_2(t) b_1(t) \frac{\phi_k(1, t)}{\phi_k(2, t)} = \frac{4e_1^2(t)}{(1 - e_1^2(t))^2} \frac{1}{p_1(t)(a_1(t) - \frac{1}{\lambda_k})}.
\tag{4.19}
\]
Therefore the claim follows from (4.16) and (4.17). Note that in particular, the relation in (4.18) implies that
\[
\phi_k(2, t) > 0 \quad \text{for } t \text{ sufficiently large}. \tag{4.20}
\]

We next use the recurrence relation (4.6) for \( j = 1 \), (4.9) and (4.17) to obtain
\[
\frac{\phi_r(2, t)}{\phi_s(2, t)} = \frac{p_1(t)(a_1(t) - \frac{1}{\lambda_k})\phi_r(1, t)}{p_1(t)(a_1(t) - \frac{1}{\lambda_k})\phi_s(1, t)}
= e^{-\frac{t}{2}(\lambda_r - \lambda_s)} \frac{\phi_r(1, 0)}{\phi_s(1, 0)} \quad \text{as } t \to \infty. \tag{4.21}
\]
From this, it follows that
\[
\frac{\phi_k^2(2, t)}{p_2(t)} = \frac{\phi_k^2(2, t)}{2 \sum_{j=1}^{\infty} \lambda_j \phi_j^2(2, t)} < \frac{1}{2\lambda_{k+1}} \frac{\phi_k^2(2, t)}{\phi_{k+1}^2(2, t)}
\sim \frac{c_k}{e^{-t(\lambda_k - \lambda_{k+1})}} \frac{\phi_k(1, 0)}{\phi_{k+1}(1, 0)} \quad \text{as } t \to \infty. \tag{4.22}
\]
Consequently, from (4.22) and (4.20), we have
\[
\lim_{t \to \infty} \frac{\phi_k(2, t)}{\sqrt{p_2(t)}} = 0. \tag{4.23}
\]
Therefore, as \( \lim_{t \to \infty} p_2(t) = \alpha_2^- < \infty \), we also obtain from (4.22) that

\[
\lim_{t \to \infty} \phi_k(2, t) = 0
\]

and so

\[
\lim_{t \to \infty} p_2(t) = \sum_{j=1}^{\infty} \lambda_j \phi_j^2(2, t) = 0.
\]

Hence it follows from the formula for \( a_2(t) \) together with (4.16) and (4.25) that

\[
(1 - e_2^2(t)) p_2(t) \left( a_2(t) - \frac{1}{\lambda_k} \right)
= \frac{2 \lambda_k (1 - e_1^2(t) e_2^2(t)) - p_2(t) (1 - e_1^2(t))(1 - e_2^2(t))}{\lambda_k (1 - e_1^2(t))}
\sim 2 \quad \text{as} \quad t \to \infty.
\]

Using the recurrence relation in (4.5) for \( j = 2 \), (4.11), (4.16), (4.23) and (4.26), we now conclude that

\[
\lim_{t \to \infty} e_2(t) \phi_k(3, t) = 0.
\]

As a consequence of (4.27) and (4.23), we find

\[
\lim_{t \to \infty} e_2(t) = \lim_{t \to \infty} \left( 2 \sum_{k=1}^{\infty} \lambda_k \frac{\phi_k(2, t)}{\sqrt{p_2(t)}} \frac{\phi_k(3, t)}{\sqrt{p_3(t)}} \right)^{\frac{1}{2}} = 0
\]

and so we can improve (4.26) to

\[
p_2(t) \left( a_2(t) - \frac{1}{\lambda_k} \right) \sim 2 \quad \text{as} \quad t \to \infty.
\]

We next show

\[
p_3(t) b_2(t) \frac{\phi_k(2, t)}{\phi_k(3, t)} \sim 2 e_2^2(t) \quad \text{as} \quad t \to \infty.
\]

To establish this, we make use of the recurrence relation (4.6) for \( j = 2 \) and the formula for \( b_2(t) \) in (4.4), this yields

\[
p_3(t) b_2(t) \frac{\phi_k(2, t)}{\phi_k(3, t)} = \frac{p_3(t) b_2^2(t)}{(a_2(t) - \frac{1}{\lambda_k} - b_1(t) \phi_k(1, t))} \frac{\phi_k(1, t)}{\phi_k(2, t)}
= \frac{4 e_2^2(t)}{(1 - e_2^2(t))^2} p_2(t) (a_2(t) - \frac{1}{\lambda_k} - p_2(t) b_1(t) \phi_k(1, t)).
\]
from which we obtain the assertion by using (4.18), (4.28) and (4.29). Note that if we combine (4.30) with (4.20), we conclude that
\[ \phi_k(3, t) > 0 \quad \text{for } t \text{ sufficiently large}. \] (4.32)

Now let \( n \geq 3 \) and assume by induction that the following sequence of assertions holds for \( j \leq n - 1 \) for each \( k, r, s \in \mathbb{N} \):

1. \[ \frac{\phi_r(j, t)}{\phi_s(j, t)} \sim e^{-\frac{1}{2}(\lambda_r + \lambda_s)t} \frac{\phi_r(1, 0)}{\phi_s(1, 0)} \quad \text{as } t \to \infty, \]

2. \[ \frac{\phi_k^2(j, t)}{p_j(t)} < \frac{1}{2\lambda_{k+1}} \frac{\phi_k^2(j, t)}{\phi_{k+1}^2(j, t)} \sim c_{k1} e^{-(\lambda_k - \lambda_{k+1})t} \quad \text{as } t \to \infty, \]

3. \[ \lim_{t \to \infty} \frac{\phi_k(j, t)}{\sqrt{p_j(t)}} = 0, \]

4. \[ \lim_{t \to \infty} \phi_k(j, t) = 0, \]

5. \[ \lim_{t \to \infty} p_j(t) = 0, \]

6. \[ (1 - c_j^2(t))p_j(t) \left( a_j(t) - \frac{1}{\lambda_k} \right) \sim 2 \quad \text{as } t \to \infty, \] (4.33)

7. \[ \lim_{t \to \infty} e_j(t) \frac{\phi_k(j + 1, t)}{\sqrt{p_{j+1}(t)}} = 0, \]

8. \[ \lim_{t \to \infty} e_j(t) = 0, \]

9. \[ p_j(t) \left( a_j(t) - \frac{1}{\lambda_k} \right) \sim 2 \quad \text{as } t \to \infty, \]

10. \[ p_{j+1}(t) b_j(t) \frac{\phi_k(j, t)}{\phi_k(j + 1, t)} \sim 2c_j^2(t) \quad \text{as } t \to \infty. \]

We shall prove the sequence of assertions holds for \( j = n \). We begin by invoking the recurrence relation (4.6) for \( j = n - 1 \), this gives

\[
\frac{\phi_r(n, t)}{\phi_s(n, t)} = \frac{\left( a_{n-1}(t) - \frac{1}{\lambda_r} \right) \phi_r(n - 1, t) - b_{n-2}(t) \phi_r(n - 2, t)}{\left( a_{n-1}(t) - \frac{1}{\lambda_s} \right) \phi_s(n - 1, t) - b_{n-2}(t) \phi_s(n - 2, t)}
\]

\[= \frac{\left( a_{n-1}(t) - \frac{1}{\lambda_r} \right) \phi_r(n - 1, t) - b_{n-2}(t) \phi_r(n - 2, t)}{\phi_s(n - 1, t) \left( a_{n-1}(t) - \frac{1}{\lambda_s} \right) - \phi_s(n - 2, t) \phi_r(n - 1, t)} \frac{\phi_r(n - 2, t)}{\phi_s(n - 1, t)} \]

\[= e^{-\frac{1}{2}(\lambda_r - \lambda_s)t} \frac{\phi_r(1, 0)}{\phi_s(1, 0)} \quad \text{as } t \to \infty \] (4.34)

by the induction assumptions (1)_{n-1}, (9)_{n-1}, (10)_{n-2} and (8)_{n-2}. Thus as before, we have

\[
\frac{\phi_k^2(n, t)}{p_n(t)} < \frac{1}{2\lambda_{k+1}} \frac{\phi_k^2(n, t)}{\phi_{k+1}^2(n, t)} \sim c_{k1} e^{-(\lambda_k - \lambda_{k+1})t} \quad \text{as } t \to \infty \] (4.35)
from which it follows that
\[
\lim_{t \to \infty} \frac{\phi_k(n, t)}{\sqrt{p_n(t)}} = 0 \quad (4.36)
\]
since \(\phi_k(n, t) > 0\) for \(t\) sufficiently large by the induction assumptions. Consequently,
\[
\lim_{t \to \infty} \phi_k(n, t) = 0 \quad (4.37)
\]
and so
\[
\lim_{t \to \infty} p_n(t) = \lim_{t \to \infty} \frac{\sqrt{2}}{\infty} \sum_{j=1}^{\infty} \lambda_j \phi_j^2(n, t) = 0. \quad (4.38)
\]
To establish \((6)_n\), we make use of the formula for \(a_n(t)\) in \((4.4)\), the induction assumptions and \((4.38)\), thus
\[
(1 - e_{n-1}^2(t))p_n(t) \left( a_n(t) - \frac{1}{\lambda_k} \right) = 2 \lambda_k (1 - e_{n-1}^2(t)) (1 - e_n^2(t)) \frac{\phi_k(n + 1, t)}{\phi_k(n, t)} \approx 2 \quad \text{as } t \to \infty. \quad (4.39)
\]
Therefore, on using the recurrence relation \((4.5)\) for \(j = n\), the induction assumptions \((3)_{n-1}, (8)_{n-1}\) together with \((4.36)\) and \((4.39)\), we obtain
\[
\lim_{t \to \infty} e_n(t) \frac{\phi_k(n + 1, t)}{\sqrt{p_{n+1}(t)}} = 0. \quad (4.40)
\]
Hence
\[
\lim_{t \to \infty} e_n(t) = \lim_{t \to \infty} \left( 2 \sum_{k=1}^{\infty} \lambda_k \frac{\phi_k(n, t)}{\sqrt{p_n(t)}} e_n(t) \frac{\phi_k(n + 1, t)}{\sqrt{p_{n+1}(t)}} \right)^{\frac{1}{2}} = 0 \quad (4.41)
\]
by \((4.36)\) and \((4.40)\). Using this result in \((4.39)\), we obtain \((9)_n\). Finally, the assertion \((10)_n\) follows from
\[
\frac{p_{n+1}(t) b_n(t) \phi_k(n, t)}{\phi_k(n + 1, t)} = \frac{p_{n+1}(t) b_n^2(t)}{(a_n(t) - \frac{1}{\lambda_k}) - b_{n-1}(t) \frac{\phi_k(n - 1, t)}{\phi_k(n, t)}} = \frac{4 e_n^2(t)}{(1 - e_n^2(t))^2 p_n(t) (a_n(t) - \frac{1}{\lambda_k}) - p_n(t) b_{n-1}(t) \frac{\phi_k(n - 1, t)}{\phi_k(n, t)}} \quad (4.42)
\]
upon using (9), the induction assumptions (8), (10) and (4.41). This completes the proof of the sequence of assertions (1), (10) by induction. Hence we have established parts (a) and (b) of the theorem as the relation \( \lim_{t \to \infty} e_j(t) = 0 \) is equivalent to \( \lim_{t \to \infty} (q_{j+1}(t) - q_j(t)) = \infty \). To prove part (c), we begin with the assertion for \( j = 1 \).

For this case, note that from the equation for \( p_1(t) \), we have

\[
p_1(t) = p_1(0) \exp \left( -\frac{1}{2} \int_0^t \sum_{k \neq 1} e^{-(q_k(s) - q_1(s))} p_k(s) \, ds \right). \tag{4.43}
\]

Since \( \lim_{t \to \infty} p_1(t) = 0 \), we conclude from (4.43) that

\[
\int_0^t \sum_{k \neq 1} e^{-(q_k(s) - q_1(s))} p_k(s) \, ds \to \infty \quad \text{as} \quad t \to \infty. \tag{4.44}
\]

Meanwhile, from the equation for \( q_1(t) \) in (3.6), we find

\[
q_1(t) = q_1(0) + \frac{1}{2} \int_0^t \sum_{k=1}^{\infty} e^{-|q_k(s) - q_1(s)|} p_k(s) \, ds
> q_1(0) + \frac{1}{2} \int_0^t \sum_{k \neq 1} e^{-(q_k(s) - q_1(s))} p_k(s) \, ds, \quad t > 0. \tag{4.45}
\]

Therefore, on taking the limit as \( t \) tends to infinity in (4.45) and making use of (4.44), we conclude that \( q_1(t) \to \infty \) as \( t \to \infty \). To show that \( q_j(t) \to \infty \) as \( t \to \infty \) for \( j > 1 \), note that \( \dot{q}_{j-1}(t) > 0 \) by Proposition 3.4 (b). Since \( (q(t), p(t)) \in S_- \), it follows from this property that

\[
0 < q_j(t) - q_{j-1}(t) < q_j(t) - q_{j-1}(0) \quad \text{for all} \quad t > 0. \tag{4.46}
\]

Hence the assertion follows from (4.46) as we have \( \lim_{t \to \infty} (q_j(t) - q_{j-1}(t)) = \infty \).

To establish part (d), note that by parts (a) and (b) above, and the equation for \( q_j(t) \) in (3.6), we have \( \lim_{t \to \infty} \dot{q}_j(t) = 0 \). Since \( q_j(t) \to \infty \) as \( t \to \infty \), it follows by the L'Hôpital's rule that

\[
\lim_{t \to \infty} \frac{q_j(t)}{t} = \lim_{t \to \infty} \dot{q}_j(t) = 0. \tag{4.47}
\]

Hence \( q_j(t) = o(t) \) as \( t \to \infty \), as asserted. \( \square \)
Corollary 4.4. If \( u_0(x) = \frac{1}{2} \sum_{j=1}^{\infty} e^{-|x-q_j^0| p_j^0} \) where \((q^0, p^0) \in S_+\), then the solution \( u(x, t) \) of the CH equation (3.1) with \( u(x, 0) = u_0(x) \) is such that
\[
u(x, t) \simeq 0 \quad \text{as} \quad t \to \infty. \tag{4.48}
\]

Remark 4.5 (a) Theorem 4.3 (a) can also be proved using the method in Section 5 below.
(b) From the fact that \( \text{Ran} L(q, p) = H \), it is straightforward to show that the equation \( L(q, p) = \frac{1}{2}[L(q, p), \Pi_t L(q, p)] \) implies
\[
\dot{J}(q, p) = \frac{1}{2}[J(q, p), \Pi_t L(q, p)].
\]
However, as \( L(q, p) \) is not invertible, it is not possible to express this equation in terms of \( J(q, p) \) alone.
(c) From the definition of \( a_j(t) \) in (4.4) and the proof of Theorem 4.3 above, we see that \( a_j(t) \to \infty \) as \( t \to \infty \).

5. Long time behaviour in \( S_+ \) and other sectors.

Let \((q(t), p(t))\) be the solution of (3.6) with \((q(0), p(0)) = (q^0, p^0) \in S_+\). Then \( L(t) = L(q(t), p(t)) \) evolves under the (+) Toda flow with initial condition \( L(0) = L(q^0, p^0) \). As in Section 4, we order the eigenvalues of \( L(q^0, p^0) \) in such a way that
\[0 < \cdots < \lambda_3 < \lambda_2 < \lambda_1.\]
Also, we let \( \phi_k(t) \) be the normalized eigenvector of \( L(t) \) corresponding to \( \lambda_k \) with \( \phi_k(1, t) > 0, k = 1, 2, \ldots \).

Denote by \( \wedge^k H \) the \( k \)-th exterior power of \( H \), \( k \geq 1 \). (See [LS] and [RS2] for more details.) Then the operator \( L(t) \) gives rise to the induced derivations
\[
(L(t))_k \equiv \sum_{r=0}^{k-1} \mathbf{I}^r \otimes L(t) \otimes I^{k-1-r} : \wedge^r H \longrightarrow \wedge^k H, \quad k \geq 1 \tag{5.1}
\]
where \( \mathbf{I} \) and \( I^{k-1-r} \) are the identity operators on \( \wedge^r H \) and \( \wedge^{k-1-r} H \) respectively. Since \( \ker L(t) = \{0\}, \{\phi_k(t)\}_k^{\infty} \) is an orthonormal basis of \( H \). Thus
\[
\{\phi_{i_1}(t) \wedge \cdots \wedge \phi_{i_k}(t) | 1 \leq i_1 < \cdots < i_k \} \tag{5.2}
\]
is an orthonormal basis of \( \wedge^k H \) with respect to the natural inner product on \( \wedge^k H \) defined by
\[
(\xi_1 \wedge \cdots \wedge \xi_k, \eta_1 \wedge \cdots \wedge \eta_k) = \det((\xi_i, \eta_j)). \tag{5.3}
\]
Moreover, the elements \( \phi_{i_1}(t) \wedge \cdots \wedge \phi_{i_k}(t) \) \((i_1 < \cdots < i_k)\) are eigenvectors of \( (L(t))_k \) as we have
\[
(L(t))_k(\phi_{i_1}(t) \wedge \cdots \wedge \phi_{i_k}(t)) = (\lambda_{i_1} + \cdots + \lambda_{i_k})(\phi_{i_1}(t) \wedge \cdots \wedge \phi_{i_k}(t)). \tag{5.4}
\]
Lemma 5.1. For any increasing sequence \(1 \leq i_1 < \cdots < i_k\) of numbers from \(\mathbb{N}\),

\[
(e_1 \land \cdots \land e_k, \phi_{i_1}(t) \land \cdots \land \phi_{i_k}(t))^2 \\
= e^{(\lambda_{i_1} + \cdots + \lambda_{i_k}) t}(e_1 \land \cdots \land e_k, \phi_{i_1}(0) \land \cdots \land \phi_{i_k}(0))^2 \\
= \sum_{1 \leq j_1 < \cdots < j_k} e^{(\lambda_{i_1} + \cdots + \lambda_{i_k}) t}(e_1 \land \cdots \land e_k, \phi_{j_1}(0) \land \cdots \land \phi_{j_k}(0))^2.
\]  

(5.5)

Hence

\[
\lim_{t \to \infty} (e_1 \land \cdots \land e_k, \phi_{i_1}(t) \land \cdots \land \phi_{i_k}(t))^2 = \begin{cases} 
1, & \text{for } (i_1, \ldots, i_k) = (1, \ldots, k) \\
0, & \text{otherwise.}
\end{cases}
\]

(5.6)

Proof. By Theorem 2.9, if \(b_\pm(t)\) are the solutions of the factorization problem \(e^{-\frac{1}{2}L(0)} = b_+(t)b_-(t)^{-1}\) with \(b_+(t) \in K\) and \(b_-(t) \in L\), then \(\phi_j(t) = b_+(t)^{-1}\phi_j(0)\) for all \(j\). Therefore,

\[
(e_1 \land \cdots \land e_k, \phi_{j_1}(t) \land \cdots \land \phi_{j_k}(t)) \\
= (e_1 \land \cdots \land e_k, (b_+(t)^{-1})^\land k \phi_{j_1}(0) \land \cdots \land \phi_{j_k}(0)) \\
= (((b_-(t)^{-1})^T)^\land k e_1 \land \cdots \land e_k, (e^{\frac{1}{2}L(0)} - e^L(0))^{k \phi_{j_1}(0)} \land \cdots \land \phi_{j_k}(0)) \\
= e^{\frac{1}{2}(\lambda_{j_1} + \cdots + \lambda_{j_k}) t}(e_1 \land \cdots \land e_k, \phi_{j_1}(0) \land \cdots \land \phi_{j_k}(0)) \\
\in K^{(0,k)} (b_-(t))_{11} \cdots (b_-(t))_{kk}
\]

for any increasing sequence \(1 \leq j_1 < \cdots < j_k\). Consequently, upon substituting the above expression into

\[
1 = \sum_{1 \leq j_1 < \cdots < j_k} (e_1 \land \cdots \land e_k, \phi_{j_1}(t) \land \cdots \land \phi_{j_k}(t))^2,
\]

(5.8)

we obtain

\[
((b_-(t))_{11} \cdots (b_-(t))_{kk})^2 \\
= \sum_{1 \leq j_1 < \cdots < j_k} e^{(\lambda_{j_1} + \cdots + \lambda_{j_k}) t}(e_1 \land \cdots \land e_k, \phi_{j_1}(0) \land \cdots \land \phi_{j_k}(0))^2.
\]

(5.9)

Hence (5.5) follows from (5.7) and (5.9). The assertion in (5.6) is now seen to be a corollary of the fact that

\[
\sum_{1 \leq j_1 < \cdots < j_k} e^{(\lambda_{j_1} + \cdots + \lambda_{j_k}) t}(e_1 \land \cdots \land e_k, \phi_{j_1}(0) \land \cdots \land \phi_{j_k}(0))^2 \\
\sim e^{(\lambda_1 + \cdots + \lambda_k) t}(e_1 \land \cdots \land e_k, \phi_1(0) \land \cdots \land \phi_k(0))^2 \quad \text{as } t \to \infty.
\]

\(\square\)
Theorem 5.2. Let \((q(t), p(t))\) be the solution of (3.6) with \((q(0), p(0)) = (q^0, p^0) \in S_+\). Then

(a) \(\lim_{t \to \infty} p_j(t) = 2\lambda_j\) for all \(j \in \mathbb{N}\),
(b) \(\lim_{t \to \infty} |q_j(t) - q_k(t)| = \infty\) for all \(j \neq k\),
(c) \(q_j(t) \sim \lambda_j t\) as \(t \to \infty\) for all \(j \in \mathbb{N}\).

Proof. For all \(k \in \mathbb{N}\),

\[
\sum_{i=1}^{k} L_{ii}(t) = \langle e_1 \wedge \cdots \wedge e_k, (L(t))_k e_1 \wedge \cdots \wedge e_k \rangle = \sum_{1 \leq i_1 < \cdots < i_k} (\lambda_{i_1} + \cdots + \lambda_{i_k}) (e_1 \wedge \cdots \wedge e_k, \phi_{i_1}(t) \wedge \cdots \wedge \phi_{i_k}(t))^2 \tag{5.11}
\]

where we have used (5.4) and the fact that \(\{\phi_{i_1}(t) \wedge \cdots \wedge \phi_{i_k}(t) \mid 1 \leq i_1 < \cdots < i_k\}\) is an orthonormal basis of \(\wedge^k \mathcal{H}\). Taking the limit as \(t \to \infty\) in (5.11), and using Proposition 3.4 (a) and (5.6), we find

\[
\alpha_1^+ + \cdots + \alpha_k^+ = \lambda_1 + \cdots + \lambda_k. \tag{5.12}
\]

Consequently we have

\[
\alpha_k^+ = \lambda_k \tag{5.13}
\]

for all \(k\) and this proves part (a). The assertion in part (b) is now a consequence of part (a) and Proposition 3.4 (b). To establish part (c), note that all the terms on the right hand side of the equation for \(\dot{q}_j(t)\) in (3.6) tend to zero as \(t \to \infty\) except for the term \(\frac{1}{2} p_j(t)\). Hence \(\dot{q}_j(t) \sim \lambda_j t\) as \(t \to \infty\) from which the assertion follows. \(\Box\)

Corollary 5.3. If \(u_0(x) = \frac{1}{2} \sum_{j=1}^{\infty} e^{-|x-q_0^j|} p_0^j\) where \((q_0^0, p_0^0) \in S_+\), then the solution \(u(x, t)\) of the CH equation (3.1) with \(u(x, 0) = u_0(x)\) is such that

\[
u(x, t) \simeq \sum_{j=1}^{\infty} \lambda_j e^{-|x-\lambda_j t|} \text{ as } t \to \infty. \tag{5.14}\]

Remark 5.4 (a) In [DLT1], the authors study the long time behaviour of the Toda flows on general bounded symmetric operators in \(l_2^+\). It is in this context that the induced derivations of the Lax operator was first introduced. Indeed, the quantity corresponding to the one which appears on the second line of (5.11) (with \(L(t)\)
replaced by the solution of the Toda flow on a general symmetric operator on $l^+_2$) was also considered in Lemma 1, Section 4 of [DLT1]. The difference here is that there is no need for us to introduce spectral measures as we can do everything explicitly in terms of eigenvalues and eigenvectors.

(b) The proof of Theorem 5.2 above shows that the long time behaviour of $L(t)$ has the sorting property, as in the case of the Toda flow on $N \times N$ Jacobi matrices [Mo]. Note that this is not true for the Toda flow on general symmetric operators on $l^+_2$ [DLT1].

(c) In their analysis of the multipeakon solutions in the sector where $q_1 < \cdots < q_N$, the authors in [BSS] found that $\lim_{t \to -\infty} p_j(t) = \lim_{t \to \infty} p_{N+1-j}(t)$ for $j = 1, \cdots, N$. However, from our investigation in Section 4 and in the present section, we see that there is no analog of this relation in our case.

Note that the behaviour in (5.14) is given in a convincing but heuristic discussion in [M] with no explicit assumptions on the ordering of the $q_j$'s. In view of this, we ask if there are other sectors in the phase space besides $S_+$ for which this behaviour is valid. We give an answer to this question as follows. Let $\text{Aut}(\mathbb{N})$ be the group of all permutations of $\mathbb{N}$ and let

$$\Sigma = \{p = (p_1, p_2, \cdots) \in l^+_{1,2} \mid p_j > 0 \text{ for all } j\}. \quad (5.15)$$

If $\pi \in \text{Aut}(\mathbb{N})$, $p = (p_1, p_2, \cdots) \in l^+_{1,2}$ and $q = (q_1, q_2, \cdots) \in l^+_{\infty}$, define $\pi \cdot p = (p_{\pi(1)}, p_{\pi(2)}, \cdots)$ and similarly for $\pi \cdot q$. With this notation, we set

$$\text{Aut}(\mathbb{N})_\Sigma = \{\pi \in \text{Aut}(\mathbb{N}) \mid \pi \cdot p \in \Sigma \text{ for all } p \in \Sigma\}. \quad (5.16)$$

Clearly, $\text{Aut}(\mathbb{N})_\Sigma$ contains the finite permutations of $\mathbb{N}$, it also contains those permutations that satisfy the condition $\lim_{j \to \infty} \pi(j)/j = 1$, for example. For $\pi \in \text{Aut}(\mathbb{N})_\Sigma$, consider the sector

$$S_+(\pi) = \{(q, p) \in l^+_{\infty} \oplus l^+_{1,2} \mid q_{\pi(1)} > q_{\pi(2)} > \cdots, p_j > 0 \text{ for all } j\}. \quad (5.17)$$

Note that for $(q, p) \in S_+(\pi)$, we can rewrite the equations of motion (3.6) in the form

$$\dot{q}_{\pi(j)} = \frac{1}{2} \sum_{k=1}^{\infty} e^{-|q_{\pi(j)} - q_{\pi(k)}|} p_{\pi(k)} \quad (5.18)$$

$$\dot{p}_{\pi(j)} = \frac{1}{2} p_{\pi(j)} \sum_{k=1}^{\infty} \text{sgn}(q_{\pi(j)} - q_{\pi(k)}) e^{-|q_{\pi(j)} - q_{\pi(k)}|} p_{\pi(k)},$$

$$= \frac{1}{2} p_{\pi(j)} \sum_{k=1}^{\infty} \text{sgn}(k - j) e^{-|q_{\pi(j)} - q_{\pi(k)}|} p_{\pi(k)}, \quad j \in \mathbb{N}$$
as the series on the right hand side of (3.6) are absolutely convergent and rearrangements of such series have the same sum. Therefore, when we compare (5.18) with (3.9), we see that the calculation which leads to Proposition 3.2 (d) also gives the Lax pair for (5.18). Namely,

\[ \dot{L}_\pi(q, p) = \frac{1}{2} [\Pi_t L_\pi(q, p) L_\pi(q, p) ] \]  

(5.19)

where \( L_\pi(q, p) = \left( \frac{1}{2} e^{-\frac{1}{2} [q_\pi(i) - q_\pi(j)]} \sqrt{p_\pi(i)p_\pi(j)} \right)_{i,j=1}^\infty \) is a positive, semiseparable trace class operator in \( l_2^+ \) with simple eigenvalues \( 0 < \cdots < \lambda_3 < \lambda_2 < \lambda_1 \). Hence Proposition 5.2 can be applied to \( (\pi \cdot q(t), \pi \cdot p(t)) \) and we conclude that for all \( j \in \mathbb{N}, \)

\[ \lim_{t \to \infty} p_j(t) = 2\lambda_{\pi^{-1}(j)}, \quad q_j(t) \sim \lambda_{\pi^{-1}(j)} t \quad \text{as} \quad t \to \infty. \]  

(5.20)

Consequently the long time asymptotics for \( u(x, t) \) is given by (5.14), as before.

Clearly, similar consideration and Theorem 4.3 can be applied to obtain the long time behaviour in sectors of the form

\[ \mathcal{S}_-(\pi) = \{(q, p) \in l_\infty^+ \oplus l_1^+ \mid q_{\pi(1)} < q_{\pi(2)} < \cdots, p_j > 0 \text{ for all } j \} \]  

(5.21)

for \( \pi \in \text{Aut}(\mathbb{N})_{\Sigma} \) with the following result:

\[ \lim_{t \to \infty} p_j(t) = 0, \quad q_j(t) \to \infty, \quad q_j(t) = o(t) \quad \text{as} \quad t \to \infty \]  

(5.22)

for all \( j \in \mathbb{N} \). Hence (4.48) remains valid for \( (q^0, p^0) \in \mathcal{S}_-(\pi) \).

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