Intermediate dimension of images of sequences under fractional Brownian motion

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Abstract

We show that the almost sure $\theta$-intermediate dimension of the image of the set $F_p = \{0, 1, \frac{1}{2^p}, \frac{1}{3^p}, \ldots\}$ under index-$h$ fractional Brownian motion is $\frac{\theta}{p(h+\theta)}$, a value that is smaller than that given by directly applying the Hölder bound for fractional Brownian motion. In particular this establishes the box-counting dimension of these images.

1 Introduction

Intermediate dimensions were introduced in [10] to interpolate between the Hausdorff dimension and box-counting dimensions of sets where these differ, see the recent surveys [9,11] for surveys on intermediate dimensions and dimension interpolation. The lower and upper intermediate dimensions, $\overline{\dim_\theta}E$ and $\underline{\dim_\theta}E$ of a set $E \subseteq \mathbb{R}^n$ depend on a parameter $\theta \in [0, 1]$, with $\overline{\dim_\theta}E = \overline{\dim_0}E = \dim_H E$ and $\underline{\dim_\theta}E = \underline{\dim_1}E = \dim_B E$, where $\dim_H, \dim_B$ and $\dim_B$ denote Hausdorff, and lower and upper box-counting dimensions, respectively. Various properties of intermediate dimensions are established in [1,10] with the intermediate dimensions reflecting the range of diameters of sets needed to get coverings that are efficient for estimating dimensions. In particular $\overline{\dim_\theta}E$ and $\underline{\dim_\theta}E$ are monotonically increasing in $\theta \in [0, 1]$, are continuous except perhaps at $\theta = 0$, and are invariant under bi-Lipschitz mappings. Intermediate dimensions have been calculated for many sets which have differing Hausdorff and box-counting dimensions, including for sets of the form $F_p$ given in (1.5) below in [10], as well as for attractors of infinitely generated conformal iterated function systems [2], spirals [6], countable families of concentric spheres [16] and topologists’ sine curves [16], with non-trivial bounds obtained for self-affine carpets [10,13].

Specifically, for $E \subseteq \mathbb{R}^n$ and $0 \leq \theta \leq 1$, the lower intermediate dimension of $E$ may be defined as

$$\underline{\dim_\theta}E = \inf \{s \geq 0 : \text{ for all } \epsilon > 0 \text{ and all } 0 < r_0 < 1, \text{ there exists } 0 < r \leq r_0 \text{ and } \sum |U_i|^s \leq \epsilon \}$$

and a cover $\{U_i\}$ of $E$ such that $r^{1/\theta} \leq |U_i| \leq r$ and $\sum |U_i|^s \leq \epsilon$
and the corresponding upper intermediate dimension by
\[
\text{dim}_\theta E = \inf \left\{ s \geq 0 : \text{ for all } \epsilon > 0 \text{ there exists } 0 < r_0 < 1 \text{ such that for all } 0 < r \leq r_0, \right. \\
\left. \text{there is a cover } \{ U_i \} \text{ of } E \text{ such that } r^{1/\theta} \leq |U_i| \leq r \text{ and } \sum |U_i|^s \leq \epsilon \right\},
\]
(1.2)
where \(|U|\) denotes the diameter of a set \(U \subseteq \mathbb{R}^n\). When \(\theta = 0\) (1.1) and (1.2) reduce to Hausdorff dimension, since there are no lower bounds on the diameters of covering sets. When \(\theta = 1\) all covering sets are forced to have the same diameter and we recover the lower and upper box-counting dimensions.

It is convenient to work with equivalent definitions of these intermediate dimensions in terms of the exponential behaviour of sums over covers. For \(E \subseteq \mathbb{R}^n\) bounded and non-empty, \(\theta \in (0, 1]\), \(r > 0\) and \(s \in [0, n]\), define
\[
S_{r,\theta}^s(E) := \inf \left\{ \sum_i |U_i|^s : \{ U_i \}_i \text{ is a cover of } E \text{ such that } r \leq |U_i| \leq r^\theta \text{ for all } i \right\}.
\]
(1.3)
It is immediate that
\[
\text{dim}_\theta E = \text{ the unique } s \in [0, n] \text{ such that } \liminf_{r \to 0} \frac{\log S_{r,\theta}^s(E)}{-\log r} = 0
\]
(1.4)
and
\[
\text{dim}_\theta E = \text{ the unique } s \in [0, n] \text{ such that } \limsup_{r \to 0} \frac{\log S_{r,\theta}^s(E)}{-\log r} = 0.
\]

For \(p > 0\) let
\[
F_p = \left\{ 0, \frac{1}{1^p}, \frac{1}{2^p}, \ldots, \frac{1}{3^p}, \ldots \right\}.
\]
(1.5)

It is shown in \([10]\) that, for all \(\theta \in (0, 1]\),
\[
\text{dim}_\theta F_p = \frac{\theta}{p + \theta}.
\]

Index-\(h\) fractional Brownian motion \((0 < h < 1)\) is the stochastic process \(B_h : \mathbb{R}^{\geq 0} \to \mathbb{R}\) such that, almost surely, \(B_h\) is continuous with \(B_h(0) = 0\), and the increments \(B_h(x) - B_h(y)\) are stationary and Gaussian with mean 0 and variance \(|x - y|^{2h}\), see, for example, \([7, 8, 12, 14, 15]\). There is a considerable literature on the dimensions of images of sets under stochastic processes, see \([12]\) for Hausdorff dimensions, and \([17]\) where the packing dimensions of images are expressed in terms of dimension profiles.

Here we investigate the almost sure intermediate dimensions of \(B_h(F_p)\), the image of \(F_p\) under index-\(h\) fractional Brownian motion. By a simple estimate on the intermediate dimensions of Hölder images of sets, or see \([11] \text{ Section 4}\), since \(B_h\) has almost sure Hölder exponent \(h - \epsilon\) for all \(\epsilon > 0\),
\[
\text{dim}_\theta B_h(F_p) \leq \frac{1}{h} \text{dim}_\theta F_p = \frac{\theta}{h(p + \theta)};
\]
however the actual value is smaller than this.
Theorem 1.1. Let $B_h : \mathbb{R} \to \mathbb{R}$ be index-$h$ fractional Brownian motion. Then almost surely, for all $\theta \in [0, 1]$,
\[ \dim_\theta B_h(F_p) = \frac{\theta}{ph + \theta}, \]  
and in particular
\[ \dim_B B_h(F_p) = \frac{1}{ph + 1}. \]

We obtain the upper bound for $\dim_\theta B_h(F_p)$ by, for each $r$, covering the part of $B_h(F_p)$ near 0 by abutting intervals of lengths $r^\theta$ and the remaining points individually by intervals of length $r$. The lower bound uses a potential theoretic method, estimating an energy of the image under $B_h$ of the measure given by equal point masses on the points of $F_p$ between $1/(2M - 1)p$ and $1/Mp$.

'Intermediate dimension profiles' were introduced in [5] to develop a general theory of intermediate dimensions including their behaviour under projections and this was developed for random images in [4]. By [4, Theorem 3.4]
\[ \dim_\theta B_h(E) = \frac{1}{h} \dim^h_\theta E \]
where $\dim^h \theta E$ is the $\theta$-dimension profile of a general compact $E \subseteq \mathbb{R}$, but this does not give an explicit value of $\dim_\theta B_h(F_p)$. Whilst this might be found by an awkward calculation of dimension profiles, our proof here of (1.6) is self-contained.

2 Proofs

We recall from [5] the energy kernels $\tilde{\phi}^s_{r, \theta}$ on $\mathbb{R}^m$ defined for $0 < r < 1, \theta \in [0, 1]$ and $0 < s \leq m$ by
\[ \tilde{\phi}^s_{r, \theta}(x) = \begin{cases} 
1 & |x| < r \\
\left( \frac{r}{|x|} \right)^s & r \leq |x| < r^\theta \\
0 & r^\theta \leq |x| 
\end{cases} \]
(here we only need the case of $m = 1$).

The proof of the lower bound for (1.6) uses the following three lemmas. The first is a slight variant of [5, Lemma 4.3] that relates the covering sums to energies with respect to the kernel $\tilde{\phi}^s_{r, \theta}$. We write $\mathcal{M}(F)$ for the set of Borel probability measures supported by $F$.

Lemma 2.1. Let $F \subseteq \mathbb{R}^m$ be compact, $\theta \in (0, 1]$, $0 < r < 1$ and $0 \leq s \leq m$ and let $\mu \in \mathcal{M}(F)$. Then
\[ S^s_{r, \theta}(F) \geq r^s \left[ \int \int \tilde{\phi}^s_{r, \theta}(x-y)d\mu(x)d\mu(y) \right]^{-1}. \]

Proof. Since $\tilde{\phi}^s_{r, \theta}$ is lower semicontinuous, by standard potential theory there is an equilibrium measure $\mu_0$ for which $\int \int \tilde{\phi}^s_{r, \theta}(x-y)d\mu(x)d\mu(y)$ attains its minimum, say,
\[ \int \int \tilde{\phi}^{s, n}_{r, \theta}(x-y)d\mu_0(x)d\mu_0(y) = \gamma. \]
Moreover,
\[ \int \tilde{\phi}^s_{r,\theta}(x-y) d\mu_0(y) \geq \gamma \]
for all \( x \in F \), with equality if \( x \in F_0 \) for a set \( F_0 \subseteq F \) with \( \mu_0(F_0) = 1 \).

If \( r \leq \delta < r^\theta \) and \( x \in F_0 \) then using (2.1),
\[ \gamma = \int \tilde{\phi}^s_{r,\theta}(x-y) d\mu(y) \geq \int \left( \frac{r}{\delta} \right)^s 1_{B(0,\delta)}(x-y) d\mu(y) \geq \left( \frac{r}{\delta} \right)^s \mu(B(x, \delta)). \] (2.3)

Let \( \{U_i\}_i \) be a finite cover of \( F \) by sets of diameters \( r \leq |U_i| < r^\theta \) and write \( I = \{ i : U_i \cap F_0 \neq \emptyset \} \), so for each \( i \in I \) we may choose \( x_i \in U_i \cap F_0 \) such that \( U_i \subseteq B(x_i, |U_i|) \). Then
\[ 1 = \mu(F_0) \leq \sum_{i \in I} \mu(U_i) \leq \sum_{i \in I} \mu(B(x_i, |U_i|)) \leq r^{-s} \gamma \sum_{i \in I} |U_i|^s \]
by (2.3), so
\[ \sum_{i} |U_i|^s \geq r^s \gamma^{-1}; \]

taking the infimum over all such covers gives (2.2). (Note that considering covers with \( r \leq |U_i| < r^\theta \) makes no difference in the definition (1.3).)

We next bound the expectation of \( \tilde{\phi}^s_{r,\theta} \) evaluated on increments of fractional Brownian motion in terms of another kernel:
\[ \psi^s_{r,\theta}(x) = \min \left\{ 1, \frac{x^{\theta(1-s)+s}}{|x|^h} \right\}. \] (2.4)

**Lemma 2.2.** Let \( B_h : \mathbb{R} \to \mathbb{R} \) be index-\( h \) fractional Brownian motion. Then there is a constant \( c \) depending only on \( s \) such that for \( 0 < r < 1 \), \( 0 < \theta \leq 1 \) and \( 0 < s < 1 \),
\[ \mathbb{E} \left( \tilde{\phi}^s_{r,\theta}(B_h(x) - B_h(y)) \right) \leq c \psi^s_{r,\theta}(x-y). \]

**Proof.** Since \( B_h(x) - B_h(y) \) has Gaussian density with mean 0 and variance \( |x-y|^{2h} \),
\[ \mathbb{E} \left( \tilde{\phi}^s_{r,\theta}(B_h(x) - B_h(y)) \right) = \frac{1}{\sqrt{2\pi}} \frac{1}{|x-y|^h} \int_0^\infty \tilde{\phi}^s_{r,\theta}(t) \exp \left( \frac{-t^2}{2|x-y|^{2h}} \right) dt. \]

This is bounded above by 1 since \( \tilde{\phi}^s_{r,\theta}(t) \leq 1 \) and the Gaussian has integral 1. With \( c_1 = 2/\sqrt{2\pi} \),
\[ \mathbb{E} \left( \tilde{\phi}^s_{r,\theta}(B_h(x) - B_h(y)) \right) = \frac{c_1}{|x-y|^h} \int_0^\infty \tilde{\phi}^s_{r,\theta}(t) \exp \left( \frac{-t^2}{2|x-y|^{2h}} \right) dt \]
\[ = \frac{c_1}{|x-y|^h} \left[ \int_0^r \exp \left( \frac{-t^2}{2|x-y|^{2h}} \right) dt + \int_r^\infty \frac{r^s}{t^s} \exp \left( \frac{-t^2}{2|x-y|^{2h}} \right) dt \right] \]
\[ \leq \frac{c_1}{|x-y|^h} \left[ \int_0^r dt + \int_r^\infty \frac{r^s}{t^s} dt \right] \]
\[ \leq \frac{c_1}{|x-y|^h} \left[ r + \frac{1}{1-s} r^{s+\theta(1-s)} \right] \]
\[ \leq \frac{c_2}{|x-y|^h} r^{s+\theta(1-s)} \]
since \( r < 1 \) and \( s + \theta(1-s) \leq 1 \). \( \square \)
The next lemma estimates the energy of a measure on $F_p$ under the kernel $\psi_{r,\theta}^s$.

**Lemma 2.3.** Given $\theta, s \in (0, 1)$ and $h \in (0, 1)$ there is a number $c > 0$ such that for all $0 < r < 1$ there is a measure $\mu_r \in \mathcal{M}(F_p)$ such that

$$E(\mu_r) := \iint \psi_{r,\theta}^s(x - y)d\mu_r(x)d\mu_r(y) \leq cr^{(\theta(1-s)+s)/(1+ph)}.$$ 

**Proof.** Given $r$ we will find $M \equiv M(r) \in \mathbb{N}$ such that the measure $\mu_r$ formed by placing a point mass of $1/M$ on each of the $M$ points of

$$F_p^M = \left\{ \frac{1}{(2M-1)^p}, \frac{1}{(2M-2)^p}, \ldots, \frac{1}{M^p} \right\} \subseteq F_p$$

satisfies the conclusion. Note that if $M \leq k \leq 2M-1$, then

$$\frac{p}{(2M)^{p+1}} \leq \frac{p}{(k+1)^{p+1}} < \left| \frac{1}{k^p} - \frac{1}{(k+1)^p} \right| < \frac{p}{k^{p+1}} \leq \frac{p}{M^{p+1}},$$

by a mean value theorem estimate. In particular, if $g$ is the minimum gap length between any pair of points of $F_p^M$, then $g \geq p/(2M)^{p+1}$ so any interval of length $R \geq g$ intersects at most $aM^{p+1}$ points of $F_p^M$, where $a = 2^{p+2}/p$. We estimate the energy

$$E(\mu_r) = \iint \psi_{r,\theta}^s(x - y)d\mu_r(x)d\mu_r(y) = \frac{1}{M^2} \sum_{M \leq i,j \leq 2M-1} \psi_{r,\theta}^s(x_i - x_j)$$

where for convenience we write $x_i = 1/i^p$. Let $m$ be the greatest integer such that $2^mg \leq M^{-p}$. Using (2.4),

$$M^2 E(\mu_r) = \sum_{x_i = x_j} \psi_{r,\theta}^s(x_i - x_j) + \sum_{g \leq |x_i - x_j| \leq 1/M^p} \psi_{r,\theta}^s(x_i - x_j)$$

$$= M + \sum_{g \leq |x_i - x_j| \leq 1/M^p} r^{\theta(1-s)+s}\frac{1}{|x_i - x_j|^h}$$

$$\leq M + \sum_{k=0}^{m} \sum_{2^kg \leq |x_i - x_j| \leq 2^{k+1}g} r^{\theta(1-s)+s}\frac{1}{|x_i - x_j|^h}$$

$$\leq M + c_1 M^{p+2} r^{\theta(1-s)+s} g^{1-h} 2^{m(1-h)}$$

$$\leq M + c_1 M^{p+2} r^{\theta(1-s)+s} g^{1-h} (M-p g^{-1})^{(1-h)}$$

$$= M + c_1 M^{2+ph} r^{\theta(1-s)+s}$$

where $c_1$ is a constant depending only on $h$ and we have taken the dominant term of the geometric sum. Thus

$$E(\mu_r) \leq M^{-1} + c_1 M^{ph} r^{\theta(1-s)+s}$$

$$= (2 + c_1 2^{ph}) r^{(\theta(1-s)+s)/(1+ph)}$$

on setting $M = \lceil r^{-(\theta(1-s)+s)/(1+ph)} \rceil \leq 2r^{-(\theta(1-s)+s)/(1+ph)}$ as $r < 1$. 

\hfill \Box
Proof of Theorem 1.1. The conclusion is clear when \( \theta = 0 \), as \( B_h(F_p) \) is countable so has Hausdorff dimension 0, so assume \( \theta \in (0, 1] \).

**Upper bound:** Let \( 0 < \epsilon < h \). Index-\( h \) fractional Brownian motion satisfies

\[
|B_h(x)| \leq K t^{h-\epsilon} \quad (0 \leq x \leq 1)
\]

almost surely for some \( K < \infty \) see, for example, [3].

Let \( 0 < r < 1 \) and \( M \) be an integer and take a cover of \( B_h(F_p) \) by intervals \( \{U_i\} \) with \( r \leq |U_i| \leq r^\theta \) by covering each point \( B_h(1/k^p) \) \((1 \leq k \leq M)\) by an interval of length \( r \) and covering \( B_h([0,1/M^p]) \subseteq [-KM^{-p(h-\epsilon)}, KM^{-p(h-\epsilon)}] \) by abutting intervals of length \( r^\theta \). Then

\[
\sum_i |U_i|^s \leq Mr^s + \left[ \frac{2KM^{-p(h-\epsilon)}}{r^\theta} + 1 \right] (r^\theta)^s = Mr^s + 2Kr^\theta(s-1)M^{-p(h-\epsilon)} + r^{s\theta}.
\]

Setting \( M = \lceil r^{(\theta(s-1)-(1/p(h-\epsilon))} \rceil \geq 2 \) gives

\[
\sum_i |U_i|^s \leq 2(1 + K)r^{s(p(h-\epsilon)+\theta)/(1+p(h-\epsilon))} + r^{s\theta} \to 0
\]

as \( r \to 0 \) provided that \( s > \theta/(p(h-\epsilon)+\theta) \). Taking \( \epsilon \) arbitrarily small we conclude that

\[
\bar{\dim}_\theta B_h(F_p) \leq \frac{\theta}{ph+\theta} \quad (2.5)
\]

almost surely, by the definition [1,2] of \( \bar{\dim}_\theta \).

**Lower bound:** From Lemmas 2.2 and 2.3 and using Fubini’s theorem, there is a number \( c \) independent of \( r \) such that for all \( 0 < r < 1 \) there is a measure \( \mu_r \) on \( F_p \) such that

\[
\mathbb{E} \left( \int \int \widetilde{\phi}_{r,\theta}(B_h(x) - B_h(y)) \, d\mu_r(x) \, d\mu_r(y) \right) \leq c \, r^{(\theta(1-s)+s)/(1+ph)}.
\]

For \( \epsilon > 0 \), setting \( r = 2^{-k}, k \in \mathbb{N} \), and summing,

\[
\mathbb{E} \left( \sum_{k=1}^{\infty} 2^{k[(\theta(1-s)+s)/(1+ph)-\epsilon]} \int \int \widetilde{\phi}_{2^{-k},\theta}(B_h(x) - B_h(y)) \, d\mu_{2^{-k}}(x) \, d\mu_{2^{-k}}(y) \right) \leq c \sum_{k=1}^{\infty} 2^{-ke} < \infty.
\]

Hence, almost surely there exists a random \( K < \infty \) such that

\[
\int \int \widetilde{\phi}_{r,\theta}(B_h(x) - B_h(y)) \, d\mu_r(x) \, d\mu_r(y) \leq Kr^{\theta(1-s)+s)/(1+ph)-\epsilon}
\]

for \( r = 2^{-k} \) for all \( k \in \mathbb{N} \) and thus for all \( 0 < r < 1 \) with a modified \( K \), noting that the two sides change only by a bounded ratio on replacing \( r \) by \( 2^{-k} \) for the least \( k \) such that \( 2^{-k} \leq r \). Writing \( \tilde{\mu}_r \) for the image measure of \( \mu_r \) under \( B_h \), so \( \tilde{\mu}_r \) is supported by \( B_h(F_p) \) (in the notation of Lemma 2.3 \( \tilde{\mu}_r \) consists of a mass of \( 1/M \) on each point \( B_r(x_i) \)), this becomes

\[
\int \int \widetilde{\phi}_{r,\theta}(u - v) \, d\tilde{\mu}_r(u) \, d\tilde{\mu}_r(v) \leq Kr^{\theta(1-s)+s)/(1+ph)-\epsilon}.
\]
Thus, by Lemma 2.1 almost surely there is a $K < \infty$ such that for all $0 < r < 1$,

$$S_{r,\theta}^s(B_h(F_p)) \geq K^{-1} r^{s-(\theta(1-s)+s)/(1+ph)+\epsilon}$$

so

$$\liminf_{r \to 0} \frac{\log S_{r,\theta}^s(B_h(F_p))}{-\log r} \geq -s + \frac{\theta(1-s) + s}{(1+ph)} - \epsilon = \frac{\theta(1-s) - sph}{(1+ph)} - \epsilon$$

and this remains true almost surely on setting $\epsilon = 0$. Hence $\liminf_{r \to 0} \log S_{r,\theta}^s(B_h(F_p))/ -\log r \geq 0$ if $\theta(1-s) + sph = 0$, that is if $s = \theta/(ph + \theta)$, so by (1.4)

$$\dim_\theta B_h(F_p) \geq \frac{\theta}{ph + \theta}. \quad (2.6)$$

Combined with (2.5) this gives (1.6) almost surely for each $\theta \in [0, 1]$. Thus, almost surely, (1.6) holds for all rational $\theta \in [0, 1]$ simultaneously, and so, since box dimensions are continuous for $\theta \in (0, 1]$, for all $\theta \in [0, 1]$ simultaneously. \qed

Finally we remark that a similar approach can be used to find or estimate the intermediate and box-counting dimensions of fractional Brownian images of sets defined by other sequences tending to 0. For example, let $f : [1, \infty) \to \mathbb{R}^+$ be a decreasing function and let $F = \{0, f(1), f(2), \ldots\}$. Then if $f(x) = O(x^{-p})$ the upper bound argument gives

$$\dim_\theta B_h(F) \geq \frac{\theta}{ph + \theta}. \quad (2.7)$$

almost surely. If we also assume that $f$ is differentiable with non-increasing absolute derivative $|f'|$ then $F$ has ‘decreasing gaps’, that is $f(k) - f(k+1)$ is non-increasing, and if

$$\frac{f(x)^{1-h}}{|f'(2x)|} = O(x^{1+ph})$$

a similar energy argument gives that almost surely, for all $\theta \in [0, 1]$,

$$\dim_\theta B_h(F) \geq \frac{\theta}{ph + \theta}. \quad (2.8)$$

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