Abstract

Connections between the principle of stationary action and optimal control, and between established notions of minimax and viscosity solutions, are combined to describe trajectories of energy conserving systems as solutions of corresponding Cauchy problems defined with respect to attendant systems of characteristics equations.

Key words: optimal control, stationary action, characteristics, Hamilton-Jacobi-Bellman equations.

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1 Introduction

In recent investigations [10, 8], connections between Hamilton’s action principle and optimal control have been exploited to synthesize fundamental solutions for conservative systems of differential equations, in finite and infinite dimensions, and their related two point boundary value problems (TPBVPs). In each case, an optimal control problem is identified whose cost is representative of the desired action, leading to a characteristic system corresponding to the desired conservative system. The tools of optimal control, including dynamic programming, semigroup theory, idempotent algebra, and convex analysis, subsequently provide a pathway for construction of its fundamental solution, for large classes of boundary conditions, see for example [10].

For short time horizons, convexity of the action functional with respect to the momentum trajectory is typically guaranteed (for finite dimensional dynamics). This ensures that an associated optimal control problem is well-defined, c.f. [10]. Consequently, stationary action is achieved as least action, as characterised by a corresponding value function, while the associated equations of motion are described by the characteristic system corresponding to a standard Hamilton-Jacobi-Bellman partial differential equation.

For longer or infinite time horizons, or for configurations with infinite dimensional dynamics, the equivalence of stationary action and optimal control breaks down, typically due to a loss of convexity of the action or to a presence of state constraints, in which further controllability assumptions are needed [8][4][5]. This leads to finite escape phenomena exhibited by the value function, and hence an inability to propagate solutions beyond these times. This limitation is particularly severe in the infinite dimensional setting [8], and motivates exploration of stationary control problems, as opposed to optimal control problems, whose value can propagate through these finite escape phenomena to longer horizons [8][11][12].
An optimal control problem can be relaxed to a stationary control problem by formally replacing the infimum (or supremum) operation in the definition of the attendant value function with a stat operation \([11, 12]\). As indicated, this stat operation requires only stationarity of its cost function argument, rather than optimality. In the stationary action problems considered to date, see for example \([11, 8]\), this has involved the characterization of open loop controls that render the cost stationary. However, motivated by the notion of minimax solutions considered in \([14, 15]\), it is also reasonable to consider initial adjoint or momentum variables that render an associated characteristics based cost stationary. An investigation in this direction forms the basis of this paper, building on the preliminary work of \([9]\). The main results document an equivalence between two stationary control problems, subject to uniqueness of solutions of an attendant TPBVP, and a verification result for stationary trajectories. An illustrative example is included.

In terms of organization, in Section 2 we recall some basic definitions and state the main assumptions of the present paper. The connection between least action and optimal control is reviewed in Section 3, along with the relevant notion of minimax solution \([14, 15]\). Optimality in the attendant minimax solution definition is then relaxed to stationarity in Section 4, and its relationship to the earlier work \([7]\) established. The paper concludes with a simple example in Section 5 and an Appendix.

2 Preliminaries and main assumptions

Throughout, \(\mathbb{R}, \mathbb{Z}, \mathbb{N}\) denote the real, integer, and natural numbers respectively, with extended reals defined as \(\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\}\). \(|\cdot|\) and \(\langle \cdot, \cdot \rangle\) stands for the Eucliden and the standard scalar product, respectively. The space of continuous mappings between Banach spaces \(\mathcal{X}\) and \(\mathcal{Y}\) is denoted by \(C(\mathcal{X}; \mathcal{Y})\). The set of bounded linear operators between the two spaces is denoted by \(\mathcal{L}(\mathcal{X}; \mathcal{Y})\), or \(\mathcal{L}(\mathcal{X})\) if \(\mathcal{X}\) and \(\mathcal{Y}\) coincide. Let \(I \subset \mathbb{R}\) a closed interval. If \(\mathcal{X}\) is a real Hilbert space, we denote by \(L^2(I; \mathcal{X})\) the space of square summable measurable functions on \(I\) endowed with the standard inner product.

Let \(\mathcal{X}, \mathcal{Y}\) two Banach spaces. A function \(f \in C(\mathcal{X}; \mathcal{Y})\) is said to be Fréchet differentiable at \(x \in \mathcal{X}\), with \(\mathcal{D}f(x) \in \mathcal{L}(\mathcal{X}; \mathcal{Y})\), if \(\lim_{\|h\|_X \to 0} \frac{\|f(h)\|_Y}{\|h\|_X} = 0\) in which \(df_x : \mathcal{X} \to \mathcal{Y}\) is defined by

\[
df_x(h) = \begin{cases} 
0 & \text{if } \|h\|_X = 0 \\
\frac{f(x+h) - f(x) - df_x(h)}{\|h\|_X} & \text{if } \|h\|_X > 0.
\end{cases}
\]  

By definition, the map \(h \mapsto df_x(h)\) is continuous at 0. The function \(f\) is said to be Fréchet differentiable if it is Fréchet differentiable at any \(x \in \mathcal{X}\).

A function \(f\) is continuously Fréchet differentiable, written \(f \in C^1(\mathcal{X}; \mathcal{Y})\), if \(Df\) is continuous on \(\mathcal{X}\). Higher order Fréchet derivatives are similarly defined, with \(f \in C^k(\mathcal{X}; \mathcal{Y})\) if \(f\) is \(k\)-times Fréchet differentiable.

Let \(\mathcal{X}\) denote a real Hilbert space of instantaneous generalized positions, and let \(T \in \mathbb{R}_{\geq 0}\) and \(t \in [0, T]\) denote the final and initial times of the desired motion. Denote the corresponding real Hilbert space of generalized momentum trajectories by \(\mathcal{U}[t, T] \cong L^2([t, T]; \mathcal{X})\).

Given an initial generalized position \(x \in \mathcal{X}\), a potential field \(V \in C^4(\mathcal{X}; \mathbb{R})\), and a coercive self-adjoint inertia operator \(\mathcal{M} \in \mathcal{L}(\mathcal{X})\), the action is defined as an integrated Lagrangian encapsulated by a cost function \(J_T(t, x, \cdot) : \mathcal{U}[t, T] \to \mathbb{R}\) using an artificial terminal cost \(\psi \in C^4(\mathcal{X}; \mathbb{R})\). In particular,

\[
J_T(t, x, u) \doteq \int_t^T \frac{1}{2} \langle u_s, \mathcal{M} u_s \rangle - V(\bar{x}_s) \, ds + \psi(\bar{x}_T),
\]

in which \(s \mapsto \bar{x}_s \in C([t, T]; \mathcal{X})\) is the generalized position trajectory satisfying

\[
\bar{x}_s \doteq x + \int_t^s u_\sigma \, d\sigma, \quad s \in [t, T],
\]

for all \(u \in \mathcal{U}[t, T]\).
It is assumed throughout that there exist constants \( m \in \mathbb{R}_{>0} \) and \( K \in \mathbb{R}_{>0} \) such that for all \( x, h \in \mathcal{X} \)
\[
\begin{align*}
    m \| h \|^2 - \langle h, M h \rangle & \leq 0, \\
    \max_{j=0,1,2} \| D^j \nabla^2 V(x) \|_{\mathcal{X}(\mathcal{X})} & \leq K/2, \\
    \| \nabla^2 \psi(x) \|_{\mathcal{X}(\mathcal{X})} & \leq \frac{K}{2},
\end{align*}
\]
i.e., \( M \) is coercive (and hence boundedly invertible), while second and third derivatives of the potential field and second derivative of the terminal cost are uniformly bounded.

### 3 Least action and optimal control

For sufficiently short time horizons, an optimal control problem can be formulated that encapsulates stationary action as least action. Given \( T \in \mathbb{R}_{>0} \) sufficiently small, this optimal control problem is defined via a value function \( W_T : [0, T] \times \mathcal{X} \to \mathbb{R} \) given by
\[
W_T(t, x) \doteq \inf_{u \in \mathcal{U}[t,T]} J_T(t, x, u)
\]
for all \( t \in [0, T], \ x \in \mathcal{X} \).

**Theorem 3.1.** Given \( T \in \mathbb{R}_{>0}, \ t \in [0, T] \) such that \( \max(T - t, 1) (T - t) < \frac{m}{K} \), c.f. \( \text{(6)} \), the value function \( W_T(t, \cdot) \) of \( \text{(5)} \) is well defined and real valued.

The proof of Theorem 3.1 uses the following lemma.

**Lemma 3.2.** Given arbitrary \( T \in \mathbb{R}_{>0}, \ t \in [0, T], \ x \in \mathcal{X}, \) and \( u \in \mathcal{U}[t,T], \) cost \( J_T(t, x, \cdot) : \mathcal{U}[t,T] \to \mathbb{R} \) is twice Fréchet differentiable, with second derivative \( D_u \nabla_u J_T(t, x, u) \in \mathcal{L}(\mathcal{U}[t,T]) \) given by
\[
D_u \nabla_u J_T(t, x, u) \delta_u = (M - \Delta_T(t, x, u)) \delta_u
\]
for all \( \delta_u \in \mathcal{U}[t,T] \), in which \( \nabla_u J_T(t, x, u) \in \mathcal{U}[t,T] \) denotes the Riesz representation of the first Fréchet derivative at \( u \), with \( \Delta_T(t, x, u) \in \mathcal{L}(\mathcal{U}[t,T]) \) given by
\[
[\Delta_T(t, x, u) \delta_u]_r = \int_t^T \left[ \int_{\sigma_r}^{\sigma_t} \nabla^2 V(\bar{x}_\sigma) d\sigma - \nabla^2 \psi(\bar{x}_T) \right] [\delta_u]_\rho \ d\rho
\]
for all \( r \in [t, T], \delta_u \in \mathcal{U}[t,T] \).

Moreover,
\[
\langle \delta_u, D_u \nabla_u J_T(t, x, u) \delta_u \rangle_{\mathcal{U}[t,T]} \geq K (\frac{m}{K} - \max(T - t, 1) (T - t)) \| \delta_u \|^2_{\mathcal{U}[t,T]}
\]
for all \( \delta_u \in \mathcal{U}[t,T] \), so that \( D_u \nabla_u J_T(t, x, u) \) is coercive and \( J_T(t, x, \cdot) \) is strictly convex and proper, provided that \( T - t \in \mathbb{R}_{>0} \) is sufficiently small.

**Proof.** Fix \( T \in \mathbb{R}_{>0}, \ t \in [0, T], \ x \in \mathcal{X}, \) and \( u \in \mathcal{U}[t,T] \). Hölder’s inequality and the second inequality in \( \text{(4)} \) yield \( \Delta_T(t, x, u) \in \mathcal{L}(\mathcal{U}[t,T]) \), with
\[
\| \Delta_T(t, x, u) \|_{\mathcal{X}(\mathcal{X})} \leq K \max(T - t, 1) (T - t).
\]
Twice Fréchet differentiability of \( J_T(t, x, \cdot) : \mathcal{U}[t,T] \to \mathbb{R} \), boundedness of the second derivative, i.e. \( D_u \nabla_u J_T(t, x, u) \in \mathcal{L}(\mathcal{U}[t,T]) \), and \( \text{(6)} \), subsequently follow by a minor generalization of \( \text{(7)} \) Theorem 3.6. Combining \( \text{(8)} \) with the first inequality in \( \text{(4)} \) via Cauchy-Schwartz and \( \text{(6)} \) yields
\[
\langle \delta_u, D_u \nabla_u J_T(t, x, u) \delta_u \rangle_{\mathcal{U}[t,T]} = \langle \delta_u, (M - \Delta_T(t, x, u)) \delta_u \rangle_{\mathcal{U}[t,T]}
\geq \langle \delta_u, (m - K \max(T - t, 1) (T - t)) \delta_u \rangle_{\mathcal{U}[t,T]}
\]
for all \( \delta_u \in \mathcal{U}[t, T] \), which is (7). By inspection, for sufficiently short time horizons, i.e. \( \max(T - t, 1) (T - t) < \frac{m}{K} \), it follows that \( D_u \nabla_u J_T(t, x, u) \in \mathcal{X}(\mathcal{U}[s, t]) \) is coercive, so that \( J_T(t, x, \cdot) \) is strictly convex and proper. Given an arbitrary \( \tilde{u} \in \mathcal{U}[t, T] \), and \( \tilde{u} = \hat{u} - u \in \mathcal{U}[t, T] \), Taylor’s theorem further implies that

\[
J_T(t, x, \hat{u}) = J_T(t, x, u) + \langle \hat{u}, \nabla_u J_T(t, x, u) \rangle_{\mathcal{U}[t, T]}
\]

\[
+ \left( \int_0^1 (1 - \sigma) D_u \nabla_u J_T(t, x, u + \sigma \tilde{u}) \, d\sigma \right) \langle \tilde{u}, \nabla_u J_T(t, x, u) \rangle_{\mathcal{U}[t, T]}
\]

\[
\geq J_T(t, x, u) + \langle \hat{u}, \nabla_u J_T(t, x, u) \rangle_{\mathcal{U}[t, T]} + \frac{1}{2} K \left( \frac{m}{K} - \max(T - t, 1) (T - t) \right) \| \tilde{u} \|^2_{\mathcal{U}[t, T]}
\]

Hence, \( \lim_{\|\hat{u}\|_{\mathcal{U}[t, T]} \to \infty} J_T(t, x, \hat{u}) = \infty \), and \( J_T(t, x, \cdot) \) is proper.

**Proof of Theorem 3.1.**  Fix \( T \in \mathbb{R}_0^+, t \in [0, T] \) as per the theorem statement, and any \( x \in \mathcal{X} \). Lemma 3.2 implies that the cost \( J_T(t, x, \cdot) : \mathcal{U}[t, T] \to \mathbb{R} \) of (2) is strictly convex and proper. Hence, the value function (5) is well defined, with the infimum achieved via a minimum, thereby yielding a real valued optimal cost.

**Remark 3.3.** Theorem 3.1 ensures that, for sufficiently short time horizons, the principle of stationary action can be formulated as a least action principle, via the optimal control problem defined by the value function (5). Applying standard tools from optimal control [2,6], this value function may subsequently be characterized via the viscosity solution of a Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE).

**Remark 3.4.** With \( \mathcal{X} \) finite dimensional, \( V \) and \( \psi \) bounded, and \( T \in \mathbb{R}_0^+ \), [6] Theorems 5.2.12, 7.4.14 implies that the value function \( W_T \) of (9) is the unique viscosity solution of

\[
\begin{aligned}
- \frac{\partial W_T}{\partial t}(t, x) + H(x, \nabla_x W_T(t, x)) &= 0 \quad (t, x) \in [0, T] \times \mathcal{X} \\
W_T(t, x) &= \psi(x) \quad x \in \mathcal{X},
\end{aligned}
\]

(9) in which the Hamiltonian \( H : \mathcal{X} \times \mathcal{X} \to \mathbb{R} \) is given via completion of squares by

\[
H(x, p) = V(x) + \frac{1}{2} \langle p, \mathcal{M}^{-1} p \rangle = V(x) + \sup_{u \in \mathcal{X}} \left\{ -\langle p, u \rangle - \frac{1}{2} \langle u, \mathcal{M} u \rangle \right\}
\]

(10) for all \( x, p \in \mathcal{X} \). Alternatively, boundedness of \( V \) and \( \psi \) can be omitted for problems for which \( u \in \mathcal{U} \) where \( \mathcal{U} \) is a compact subset of \( \mathcal{X} \), see [6] Theorem 7.4.14.

The characteristic system associated with (9) is

\[
\begin{align*}
\dot{x}_s &= -\nabla_p H(x_s, \bar{p}_s) = \bar{u}_s = -\mathcal{M}^{-1} \bar{p}_s, \quad \bar{x}_T = y, \\
\dot{\bar{p}}_s &= \nabla_x H(x_s, \bar{p}_s) = \nabla V(x_s), \quad \bar{p}_T = \nabla \psi(y), \\
\dot{z}_s &= -\langle \bar{p}_s, \nabla_p H(x_s, \bar{p}_s) \rangle + H(x_s, \bar{p}_s) \\
&= V(x_s) - \frac{1}{2} \langle \bar{p}_s, \mathcal{M}^{-1} \bar{p}_s \rangle, \quad \bar{z}_T = \psi(y),
\end{align*}
\]

(11) for all \( s \in [t, T] \), \( y \in \mathcal{X} \), in which \( \nabla_x \) and \( \nabla_p \) denote the Riesz representations of the respective Fréchet derivatives. The first two equations in (11) correspond to the equations of motion defined by the action principle. As formalised later in Lemma 4.3, these equations coupled with either initial or terminal data exhibit a unique classical solution. In particular, the second derivative \( \ddot{x}_s \) is well defined, with

\[
\ddot{x}_s = -\mathcal{M}^{-1} \dot{p}_s = -\mathcal{M}^{-1} \nabla V(x_s)
\]

for all \( s \in [t, T] \), which is Newton’s second law. Observe also that the Hamiltonian \( H \) of (11) corresponds to the total energy, i.e. the sum of potential and kinetic energies. As expected, the chain rule implies that

\[
\begin{align*}
\dot{H}(x_s, \bar{p}_s) &= \langle \nabla_x H(x_s, \bar{p}_s), \dot{x}_s \rangle + \langle \nabla_p H(x_s, \bar{p}_s), \dot{\bar{p}}_s \rangle \\
&= -\langle \nabla_x H(x_s, \bar{p}_s), \nabla_p H(x_s, \bar{p}_s) \rangle + \langle \nabla_p H(x_s, \bar{p}_s), \nabla_x H(x_s, \bar{p}_s) \rangle \\
&= 0,
\end{align*}
\]

for all \( s \in [t, T] \), i.e. energy is conserved.
4 Stationary action

The connection between least action and optimal control is known to break down for longer time horizons, due typically to a loss of convexity of the action encapsulated by the cost \( \mathcal{J} \). This can be seen in Lemma 3.2 where the convexity guarantee provided for the cost \( \mathcal{J} \) is no longer valid, thereby rendering the optimal control interpretation of Theorem 3.1 inapplicable. In practice, the value function \( \mathcal{W} \) involved experiences finite escape phenomena as the horizon increases and convexity of the cost is lost.

On longer time horizons, it is well known that the stationary (rather than least) action principle continues to describe the motion of energy conserving systems. In order to encapsulate this description in a framework that is analogous to optimal control, the infimum operation appearing in (5) is relaxed to a \( \text{stat} \) operation \[11, 12\].

**Definition 4.1.** The \( \text{stat} \) operation, along with the corresponding \( \text{argstat} \) operation, with respect to a function \( F \in C^1(\mathcal{X};\mathbb{R}) \) is defined by

\[
\text{stat} F(\zeta) = \left\{ F(\tilde{\zeta}) \mid \tilde{\zeta} \in \text{arg\,stat} F(\zeta) \right\}, \quad \text{arg\,stat} F(\zeta) = \left\{ \zeta \in \mathcal{X} \mid 0 = \lim_{y \to \zeta} \frac{|F(y) - F(\zeta)|}{\|y - \zeta\|} \right\},
\]

in which \( \mathcal{X} \) is a Banach space. The elements in \( \text{arg\,stat} \) \( \zeta \in \mathcal{X} F(\zeta) \) are called stationary points for \( F \).

Relaxing the infimum in (5) to \( \text{stat} \) as indicated gives rise to the notion of a stationary control problem.

4.1 Stationary control problems

With \( \mathcal{X} = \mathcal{U}[t,T] \), the relaxed (and possibly set-valued) stationary value function \( \tilde{W}_T \) of interest is defined by \[11, 12, 8, 7\]

\[
\tilde{W}_T(t,x) = \text{stat}_{u \in \mathcal{U}[t,T]} J_T(t,x,u),
\]

for all \( t \in [0,T], x \in \mathcal{X} \), in which \( J_T \) is the same cost \( \mathcal{J} \). The utility of (13), relative to (5), in recovering the desired dynamics on arbitrary time horizons is illustrated via the following standard calculus of variations result.

**Theorem 4.2.** Given \( T \in \mathbb{R}_{\geq 0}, t \in [0,T], x \in \mathcal{X} \), a velocity trajectory \( \bar{u} \in \mathcal{U}[t,T] \) renders the cost \( \mathcal{J} \) stationary, i.e. \( \bar{u} \) satisfies

\[
\bar{u} \in \text{arg\,stat} J_T(t,x,u),
\]

if and only if there exists a mild solution \((\bar{x},\bar{p}) \in (\mathcal{U}[t,T])^2 \) of the TPBVP defined by (11) with \( \bar{x}_t = x \) fixed. Furthermore, \( \bar{u} \) in (14) satisfies

\[
\bar{u}_s = -M^{-1}\bar{p}_s \quad \text{a.e.} \quad s \in [t,T].
\]

**Proof.** See for example \[7, Theorem 3.9\].

Rather than focus immediately on a dynamic programming style approach to the synthesis of stationary trajectories \[11\], or to solving the TPBVP highlighted in Theorem 4.2, the aim is instead to consider an alternative to cost \( \mathcal{J} \) that appeals directly to the underlying characteristics system \[11\]. With this in mind, first observe that integration of the final equation of (11) yields

\[
\bar{z}_t = \psi(y) + \int_t^T \frac{1}{2} \langle \bar{u}_s, M \bar{u}_s \rangle - V(\bar{x}_s) \, ds,
\]
which is of the same form as the cost $J_T(t, x, \bar{u})$ of (2), provided that $\bar{x}_t = x$. For short horizons, this motivates an equivalent characterization of the optimal control value function [5] as the unique minimax solution (also called minimal selection) [13] of (1), given by

$$
\begin{align*}
W_T(t, x) &= \inf_{y \in Y_T(t, x)} \bar{z}_t, \\
Y_T(t, x) &= \{ y \in \mathcal{X} \mid \bar{x}_t = x, \bar{x}_T = y \}
\end{align*}
$$

for all $t \in [0, T]$, $x \in \mathcal{X}$. In view of (16), and as per [15], (17) may be recast as an optimization over an initial adjoint variable, i.e.

$$
W_T(t, x) \doteq \inf_{p \in \mathcal{X}} \tilde{J}_T(t, x, p)
$$

for all $t \in [0, T]$, $x \in \mathcal{X}$, in which $\tilde{J}_T(t, x, \cdot) : \mathcal{X} \to \mathbb{R}$ is the associated cost defined by

$$
\tilde{J}_T(t, x, p) \doteq \int_t^T \frac{1}{2} (\dot{x}_s, \mathcal{M}^{-1} \dot{p}_s) - V(\bar{x}_s) \, ds + \psi(\bar{x}_T),
$$

for all $t \in [0, T]$, with respect to the Cauchy problem

$$
\begin{align*}
\dot{x}_s &= -\mathcal{M}^{-1} \dot{p}_s, \quad \bar{x}_t = x, \\
\dot{p}_s &= \nabla V(\bar{x}_s), \quad \bar{p}_t = p,
\end{align*}
$$

for all $s \in [t, T]$, $x, p \in \mathcal{X}$, extracted from (11).

In view of (18), (19), (20), an alternative value function to (13) may be proposed by relaxing the infimum operation in (18) to the stat operation (12). In particular, define the value function $\bar{W}_T$ by

$$
\bar{W}_T(t, x) \doteq \text{stat} \tilde{J}_T(t, x, p)
$$

for all $T \in \mathbb{R}_{\geq 0}$, $t \in [0, T]$, $x \in \mathcal{X}$, with cost $\tilde{J}_T$ as per (19), and the stat involved possibly set-valued.

The subsequent analysis is concerned with (21), and in particular existence of the associated argstat and its relationship to corresponding argstat in Theorem 4.2. With this analysis in mind, it is convenient for brevity of notation to define $f : \mathcal{X}^2 \to \mathcal{X}^2$, $l : \mathcal{X}^2 \to \mathbb{R}$, and $\Psi : \mathcal{X}^2 \to \mathbb{R}$ by

$$
\begin{align*}
f(X) &\doteq \left( -\frac{1}{2} \mathcal{M}^{-1} p \right), \quad X \doteq Y_p(x) \doteq \begin{pmatrix} x \\ p \end{pmatrix} \in \mathcal{X}^2, \\
l(X) &\doteq \frac{1}{2} (p, \mathcal{M}^{-1} p) - V(x), \quad \Psi(X) \doteq \psi(x).
\end{align*}
$$

Note that (19), (20), (21) correspond to

$$
\begin{align*}
\bar{J}_T(t, Y) &= \tilde{J}_T(t, x, p) = \int_t^T l(X_s) \, ds + \Psi(X_T), \\
\dot{X}_s &= f(X_s), \quad s \in [t, T], \quad X_t = Y \doteq Y_p(x), \\
\bar{W}_T(t, x) &= \text{stat} \tilde{J}_T(t, Y_p(x)).
\end{align*}
$$

### 4.2 Fréchet differentiation of the cost functional

The objective now is to characterise the argstat in (21) via differentiation of (23) via (19). With this in mind, applying classical arguments as those in [13, Chapter 5], some intermediate lemmas are useful, whose proofs are referred to Appendix A.

**Lemma 4.3.** Given any $T \in \mathbb{R}_{\geq 0}$, $t \in [0, T]$, $Y \in \mathcal{X}^2$, the initial value problem (24) has a unique classical solution $\bar{Y}(Y) \in C([t, T]; \mathcal{X}^2) \cap C^1((t, T); \mathcal{X}^2)$. 


Lemma 4.4. The map \( Y \mapsto \overline{X}(Y) \) defined via the unique classical solution of Lemma 4.3 is continuous, i.e. \( \overline{X} \in C(\mathcal{X}^2; C([t,T]; \mathcal{X}^2)) \). In particular, there exists an \( \alpha \in \mathbb{R}_{\geq 0} \) such that
\[
\|\overline{X}(Y + h) - \overline{X}(Y)\|_\infty \leq \|h\| \exp(\alpha(T-t)) \tag{26}
\]
for all \( Y, h \in \mathcal{X}^2 \).

Lemma 4.5. The map \( Y \mapsto \overline{X}(Y) \in C(\mathcal{X}^2; C([t,T]; \mathcal{X}^2)) \) of Lemma 4.4 is Fréchet differentiable with derivative given by
\[
D\overline{X}(Y) \in \mathcal{L}(\mathcal{X}^2; C([t,T]; \mathcal{X}^2)), \quad [D\overline{X}(Y)]_s = U_{s,t}(Y) h, \quad s \in [t,T], \tag{27}
\]
for all \( Y, h \in \mathcal{X}^2 \), in which \( U_{s,t}(Y) \in \mathcal{L}(\mathcal{X}^2) \), \( r, s \in [t,T] \), and is an element of the two-parameter family of evolution operators generated by \( A_s \in \mathcal{L}(\mathcal{X}^2) \), i.e.
\[
U_{s,r} h = U_{s,t}(Y) h = h + \int_r^s A(Y) \sigma U_{\sigma,r}(Y) h d\sigma, \quad \forall \; r, s \in [t,T], \; h \in \mathcal{X}, \tag{28}
\]
in which \( s \mapsto A(Y) \sigma \) is defined uniquely, given \( Y \), by
\[
A(Y) \sigma \doteq \Lambda(\overline{X}(Y)) \doteq \begin{pmatrix} 0 & -\mathcal{M}^{-1} \\ \nabla^2 V(\overline{x}(Y)) & 0 \end{pmatrix}, \tag{29}
\]
for all \( s \in [t,T] \), and \( \overline{X}(Y) = \begin{pmatrix} \overline{x}(Y) \\ \overline{p}(Y) \end{pmatrix} \).

Lemma 4.6. Given \( T \in \mathbb{R}_{>0}, \; t \in [0,T] \), the map \( Y \mapsto U_{s,T}(Y) \) is continuously Fréchet differentiable, uniformly in \( r, s \in [t,T] \).

Next, Fréchet regularity of the cost functional is demonstrated.

Proposition 4.7. Given \( T \in \mathbb{R}_{\geq 0}, \; t \in [0,T] \), the map \( (t,Y) \mapsto \overline{J}_T(t,Y) \) of (24) is continuously Fréchet differentiable with derivative \( DJ_{T} \) given by
\[
D\overline{J}_T(t,Y) (\delta, h) = \left( D_t\overline{J}_T(t,Y), D_Y\overline{J}_T(t,Y) \right) \tag{30}
\]
where
\[
\begin{aligned}
D_t\overline{J}_T(t,Y) \doteq -l(\overline{X}(Y),t) \delta, \\
D_Y\overline{J}_T(t,Y) h = \langle \nabla_Y \overline{J}_T(t,Y), h \rangle_{\mathcal{X}^2},
\end{aligned}
\]
\( \forall t \in [0,T], \forall Y, h \in \mathcal{X}^2, \forall \delta \in (-T-t) \),
\( \nabla_Y \overline{J}_T(t,Y) \in \mathcal{X}^2 \) is the corresponding Riesz representation of \( D_Y\overline{J}_T(t,Y) \), given by
\[
\nabla_Y \overline{J}_T(t,Y) = U_{t,T}(Y)' \nabla \Psi(\overline{X}(Y)) + \int_t^T U_{s,t}(Y)' \nabla l(\overline{X}(Y)) ds. \tag{31}
\]
Moreover, the map \( (t,Y) \mapsto D\overline{J}_T(t,Y) \) is also continuously Fréchet differentiable.

Proof. Fix \( T \in \mathbb{R}_{\geq 0}, \; t \in [0,T], \; Y, h \in \mathcal{X}^2 \), and \( \delta \in (-T-t) \). Note immediately that \( (t,Y) \mapsto \overline{J}_T(t,Y) \) is Fréchet differentiable if and only if \( t \mapsto \overline{J}_T(t,Y) \) and \( Y \mapsto \overline{J}_T(t,Y) \) are Fréchet differentiable, using for example the norm \( ||(t,Y)||_2 = ||t|| + ||Y||_2 \). By inspection of (24), \( t \mapsto \overline{J}_T(t,Y) \) is Fréchet differentiable, with the derivative indicated in the left-hand equality in (30).

In order to demonstrate that the map \( Y \mapsto \overline{J}_T(t,Y) \) is Fréchet differentiable, with derivative as per the right-hand equality in (30), the chain rule for Fréchet differentiation \( \overline{I} \) may be applied. To this end, in view of (23), define \( \overline{I} : C([t,T]; \mathcal{X}^2) \to \mathbb{R} \) and \( \overline{I} : C([t,T]; \mathcal{X}^2) \to \mathcal{L}(C([t,T]; \mathcal{X}^2); \mathbb{R}) \) by
\[
\overline{I}(Z) \doteq \int_t^T l(Z_s) ds + \Psi(Z_T), \quad \overline{I}(Z) \delta \doteq \int_t^T \langle \nabla l(Z_s), \delta_s \rangle ds + \langle \nabla \Psi(Z_T), \delta_T \rangle_{\mathcal{X}^2} \tag{32}
\]
for all \( Z, \delta \in C([t, T]; \mathcal{X}) \). Note in particular that \( \bar{J}_T(t, Y) = \overline{I} \circ \overline{X}(Y) \), with \( \overline{X} \in C(\mathcal{X}; C([t, T]; \mathcal{X})) \) Fréchet differentiable by Lemma 4.3, and the candidate derivative of \( z \mapsto D\overline{I}(Z) \) is \( \iota(Z) \) in (32). Fix an arbitrary such \( Z, \delta \in C([t, T]; \mathcal{X}) \). By inspection,

\[
\overline{I}(Z + \delta) - \overline{I}(Z) - \iota(Z) \delta \leq \int_t^T |l(Z_s + \delta_s) - l(Z_s) - \langle \nabla l(Z_s), \delta_s \rangle_{\mathcal{X}}| \, ds \\
+ |\Psi(Z_T + \delta_T) - \Psi(Z_T) - \langle \nabla \Psi(Z_T), \delta_T \rangle_{\mathcal{X}}|.
\]

As \( l, \Psi \in C^3(\mathcal{X}; \mathbb{R}) \) by 41, 22, and \( Dl(Y) h = \langle \nabla l(Y), h \rangle \), Taylor’s theorem implies that

\[
|\overline{I}(Z + \delta) - \overline{I}(Z) - \iota(Z) \delta| \leq \int_t^T \left| \int_0^1 (1 - \eta) \langle \delta_s, \nabla^2 l(Z_s + \eta \delta_s) \rangle_{\mathcal{X}} \, d\eta \right| \, ds \\
+ \left| \int_0^1 (1 - \eta) \langle \delta_T, \nabla^2 \Psi(Z_T + \eta \delta_T) \rangle_{\mathcal{X}} \, d\eta \right| \\
\leq C \int_t^T \| \delta_s \|^2_{\mathcal{X}} \, ds + C \| \delta_T \|^2_{\mathcal{X}} \leq C \max(T - t, 1) \| \delta \|_{\mathcal{X}}^2,
\]

in which \( C < \infty \) is given by

\[
C := \frac{1}{2} \sup_{Y \in \mathcal{X}} \max(\| \nabla^2 l(Y) \|_{\mathcal{X}}, \| \nabla^2 \Psi(Y) \|_{\mathcal{X}}),
\]

and finiteness follows by 41, 22. Recalling 41 yields \( |d\overline{I}_\delta(\delta)| \leq C \max(T - t, 1) \| \delta \|_{\mathcal{X}}, \) i.e. \( \overline{I} \) is Fréchet differentiable with derivative \( D\overline{I} = \overline{I} \). Hence, \( \bar{J}_T(t, Y) = \overline{I} \circ \overline{X}(Y) \), in which \( \overline{I} : C([t, T]; \mathcal{X}) \rightarrow C([t, T]; \mathcal{X}) \) is Fréchet differentiable, as demonstrated above, and \( \overline{X} \in C(\mathcal{X}; C([t, T]; \mathcal{X})) \) is Fréchet differentiable by Lemma 4.5. The chain rule, along with (27), (32), thus yield

\[
D_Y \bar{J}_T(t, Y) h = D\overline{I}(\overline{X}(Y)) D\overline{X}(Y) h = \overline{\iota}(\overline{X}(Y)) U_{s,t}(Y) h
\]

\[
= \left\langle \int_t^T (\nabla l(\overline{X}(Y))_s, U_{s,t}(Y) h) \, ds + (\nabla \Psi(\overline{X}(Y))_T, U_{T,t}(Y) h) \right\rangle_{\mathcal{X}}
\]

\[
= \left\langle \int_t^T U_{s,t}(Y)' \nabla l(\overline{X}(Y))_s \, ds + U_{T,t}(Y)' \nabla \Psi(\overline{X}(Y))_T, h \right\rangle_{\mathcal{X}} = \langle \nabla_Y \bar{J}_T(t, Y), h \rangle_{\mathcal{X}},
\]

in which \( \nabla_Y \bar{J}_T(t, Y) \) is as per the lemma statement. Hence, the right-hand equality in (30) holds.

It may be verified that \( (t, Y) \mapsto D\overline{I}(t, Y) \) is continuous. In particular, by inspection of (30) and Lemma 4.4 that \( (t, Y) \mapsto D_t \bar{J}_T(t, Y) \) is continuous, i.e. \( l \in C^3(\mathcal{X}; \mathbb{R}) \) by 41, 22, and \( t \mapsto \overline{X}(Y)_t \) and \( Y \mapsto \overline{X}(Y)_t \) are both continuous. Similarly, by inspection of (30), (31), \( (t, Y) \mapsto D_Y \bar{J}_T(t, Y) \) is continuous if \( Y \mapsto U_{s,t}(Y) \) is continuous, uniformly in \( s \in [t, T] \). This uniform continuity property follows by Lemma 4.6

Twice continuous Fréchet differentiability follows similarly, via (4), (22), (30), and Lemma 4.6.

The next result describes Riesz representations of the cost functional, and an auxiliary statement may be found in Appendix B.

**Proposition 4.8.** Given \( T \in \mathbb{R}_{>0}, t \in [0, T) \), the maps \( x \mapsto \bar{J}_T(t, x, p) \) and \( p \mapsto \bar{J}_T(t, x, p) \) of (19) are Fréchet differentiable with derivatives given by

\[
D_x \bar{J}_T(t, x, p) \in \mathcal{L}(\mathcal{X}; \mathbb{R}), \quad D_p \bar{J}_T(t, x, p) \in \mathcal{L}(\mathcal{X}; \mathbb{R}),
\]

\[
D_x \bar{J}_T(t, x, p) h = \langle \nabla_x \bar{J}_T(t, x, p), h \rangle, \quad D_p \bar{J}_T(t, x, p) h = \langle \nabla_p \bar{J}_T(t, x, p), h \rangle,
\]

for all \( x, p, h \in \mathcal{X}^* \), in which \( \nabla_x \bar{J}_T(t, x, p) \) and \( \nabla_p \bar{J}_T(t, x, p) \) are the respective Riesz representations

\[
\left\{ \begin{array}{ll}
\nabla_x \bar{J}_T(t, x, p) \doteq \mathbb{I} & \nabla_x \bar{J}_T(t, Y_p(x)), \\
\nabla_p \bar{J}_T(t, x, p) \doteq \mathbb{I} & \nabla_Y \bar{J}_T(t, Y_p(x)),
\end{array} \right.
\]

(34)
and $\nabla \bar{J}_T(t, \cdot), Y_T(x)$ are as per \([31], [32]\).
Moreover, given

$$
\begin{aligned}
\begin{pmatrix}
\bar{p}_s \\
\bar{\beta}_s 
\end{pmatrix}
= X(Y_T(p))_s \in \mathcal{X},
\end{aligned}
$$

the maps $s \mapsto \zeta_s$ satisfies

$$
\zeta_s = U_{T,s}(Y_x(p))' \nabla \psi(X(Y_x(p))_T) + \int_s^T U_{\sigma,s}(Y_x(p))' \nabla l(X(Y_x(p))_\sigma) d\sigma,
$$

for all $s \in [t, T]$. Equivalently, $\zeta_s = \begin{pmatrix}
\bar{p}_s - \pi_s \\
\bar{\beta}_s 
\end{pmatrix}$ for all $s \in [t, T]$, where $s \mapsto \begin{pmatrix}
\pi_s \\
\bar{\beta}_s 
\end{pmatrix}$ satisfies the final value problem

$$
\begin{aligned}
\begin{pmatrix}
\dot{\xi}_s \\
\dot{\pi}_s 
\end{pmatrix}
&= \begin{pmatrix}
0 & -M^{-1} \\
\nabla^2 V(\bar{x}_s) & 0 
\end{pmatrix}
\begin{pmatrix}
\xi_s \\
\pi_s 
\end{pmatrix} = A_s \begin{pmatrix}
\xi_s \\
\pi_s 
\end{pmatrix}, \\
\xi_T &\equiv \begin{pmatrix}
0 \\
\bar{\beta}_T - \nabla \psi(\bar{x}_T) 
\end{pmatrix},
\end{aligned}
$$

Proof. Fix arbitrary $T \in \mathbb{R}_{\geq 0}, t \in [0, T], x, p, h \in \mathcal{X}$. By \([19], [23]\), $\bar{J}_T(t, x, p) = \tilde{J}_T(t, Y_x(p))$, and note that the maps $x \mapsto Y_x(p)$ and $p \mapsto Y_x(p)$ are Fréchet differentiable with respective derivatives given by $D_x Y_x(p) h = ( I \ 0 )^T h$ and $D_p Y_x(p) h = ( 0 I )^T h$, with $0, I \in \mathcal{L}(\mathcal{X})$ denoting the zero and identity maps. Applying the chain rule, and Proposition \([4.4]\)

$$
D_x \tilde{J}_T(t, x, p) h = D_Y \tilde{J}_T(t, Y_x(p)) D_x Y_x(p) h = (\nabla_Y \tilde{J}_T(t, Y_x(p)), D_p Y_x(p) h)_{\mathcal{X}^2}
$$

yields the first asserted derivative in \([34]\), with the other asserted derivative following similarly.

For the remaining assertions \([36], [37]\), given \([35]\), note that

$$
\zeta_s = \begin{pmatrix}
\nabla_x \tilde{J}_T(s, \bar{x}_s, \bar{p}_s) \\
\nabla_p \tilde{J}_T(s, \bar{x}_s, \bar{p}_s) 
\end{pmatrix} = \nabla_Y \tilde{J}_T(s, Y_{\bar{x}}(\bar{p}_s)) = \nabla_Y \tilde{J}_T(s, X(Y_x(p))_s),
$$

so that \([36]\) follows by Proposition \([4.4]\). As $U_{T,T} = I$, and $\psi$ is as per \([22]\), note by \([23]\) that

$$
\zeta_T = \nabla \psi(X(Y)_T) = \begin{pmatrix}
\nabla \psi(\bar{x}_T) \\
0 
\end{pmatrix},
$$

i.e. the terminal condition in \([37]\) holds. Note further that $s \mapsto \zeta_s$ of \([36]\) is differentiable, with the Leibniz integral rule implying that

$$
\begin{aligned}
\dot{\zeta}_s &= (\nabla X(Y)_T)' \nabla \psi(X(Y)_T) - \nabla l(X(Y)_s) + \int_s^T (\nabla U_{\sigma,s}(Y)_T)' \nabla l(X(Y)_\sigma) d\sigma \\
&= -A_s \zeta_s - \nabla l(X(Y)_s) = \begin{pmatrix}
0 & -\nabla^2 V(\bar{x}_s) \\
M^{-1} & -M^{-1} \bar{p}_s 
\end{pmatrix} \zeta_s + \begin{pmatrix}
\nabla V(\bar{x}_s) \\
\nabla V(\bar{x}_s) 
\end{pmatrix},
\end{aligned}
$$

for all $s \in (t, T)$. Define $s \mapsto \pi_s, \xi_s$ via

$$
\begin{aligned}
\bar{p}_s - \pi_s &\equiv \zeta_s = \begin{pmatrix}
\nabla_x \tilde{J}_T(s, \bar{x}_s, \bar{p}_s) \\
\nabla_p \tilde{J}_T(s, \bar{x}_s, \bar{p}_s) 
\end{pmatrix} \in \mathcal{X}^2, \\
\bar{\beta}_s = \bar{\beta}_s - \pi_s = \nabla V(\bar{x}_s) - \nabla V(\bar{x}_s) \xi_s + \nabla V(\bar{x}_s) \bar{\beta}_s,
\end{aligned}
$$

for all $s \in [t, T]$. That is, the ODE in \([37]\) also holds. \(\Box\)

**Remark 4.9.** An auxiliary statement of Proposition \([4.8]\).
4.3 Characterization of stationary trajectories

It may now be demonstrated, via the following lemma and theorem, that existence of the argstat in (21) corresponds to existence of the argstat in (13), under a condition of existence of an argstat along trajectories. An equivalent formulation, involving a TPBVP, is subsequently stated as a corollary.

**Lemma 4.10.** Let \( T \in \mathbb{R}_{>0}, t \in [0, T), x, p \in \mathcal{X} \), and \( (\bar{x}_s, \bar{p}_s) \doteq \overline{X}(Y_p(x))_s \) for all \( s \in [t, T] \). Then the following statements are equivalent:

(i) \( \bar{p}_s \in \text{arg stat}_{p \in \mathcal{X}} J_T(s, \bar{x}_s, p) \) for all \( s \in [t, T] \);

(ii) \( \bar{p}_s = \nabla_x J_T(s, \bar{x}_s, \bar{p}_s) \) for all \( s \in [t, T] \).

**Proof.** Fix \( T \in \mathbb{R}_{>0}, t \in [0, T), x, p \in \mathcal{X} \), and \( Y \doteq Y_p(x) \). Then, applying Proposition 4.8, i.e. (37), \( \bar{\xi}_s = -\mathcal{M}^{-1} \pi_s \), so that \( 0 = \bar{\pi}_s = \nabla_x J_T(s, \bar{x}_s, \bar{p}_s) \) for all \( s \in [t, T] \). That is, (ii) holds.

Alternatively, suppose (ii) holds true, i.e. \( \bar{p}_s = \nabla_x J_T(s, \bar{x}_s, \bar{p}_s) \) for all \( s \in [t, T] \). Then, \( 0 = \bar{p}_s - \nabla_x J_T(s, \bar{x}_s, \bar{p}_s) = \pi_s \) for all \( s \in [t, T] \), so that \( \bar{\xi}_s = -\mathcal{M}^{-1} \pi_s = 0 \) for all \( s \in [t, T] \). Moreover, the terminal condition \( \bar{J}_T = \nabla_p J_T(T, \bar{x}_T, \bar{p}_T) = \nabla_p \psi(\bar{x}_T) = 0 \) holds by definition of (19), so that \( 0 = \bar{\xi}_s = \nabla_p J_T(s, \bar{x}_s, \bar{p}_s) \) for all \( s \in [t, T] \), again by (37). That is, (i) holds.

**Theorem 4.11.** Let \( T \in \mathbb{R}_{>0}, t \in [0, T), x \in \mathcal{X} \). Then the following statements are equivalent:

(i) there exists \( \bar{u} \in \text{arg stat}_{u \in \mathcal{U}} J_T(t, x, u) \);

(ii) there exists \( \bar{p} \in \text{arg stat}_{p \in \mathcal{X}} J_T(t, x, p) \) such that \( (\bar{x}_s, \bar{p}_s) \doteq \overline{X}(Y_p(x))_s \) of (39) satisfies

\[
\bar{p}_s = \nabla_x J_T(s, \bar{x}_s, \bar{p}_s) \quad \forall \ s \in [t, T],
\]

in which \( J_T, \bar{J}_T \) are as per (2), (13), (17), (21). Moreover, if (i) holds true, then \( \bar{u} \) satisfies

\[
\bar{u}_s = -\mathcal{M}^{-1} \left( \frac{\partial}{\partial u} \right)^T \overline{X}(Y_p(x))_s = -\mathcal{M}^{-1} \bar{p}_s \quad \forall \ s \in [t, T].
\]

**Proof.** Fix arbitrary \( T \in \mathbb{R}_{>0}, t \in [0, T), x \in \mathcal{X} \).

(iii) \( \implies \) (i) Suppose \( \bar{p} \in \text{arg stat}_{p \in \mathcal{X}} J_T(t, x, p) \) exists such that \( (\bar{x}_s, \bar{p}_s) \doteq \overline{X}(Y_p(x))_s \) via (35) satisfies \( \bar{p}_s \in \text{arg stat}_{p \in \mathcal{X}} J_T(s, \bar{x}_s, \bar{p}_s) \) for all \( s \in [t, T] \). Applying Lemma 4.10, \( \bar{p}_s = \nabla_x J_T(s, \bar{x}_s, \bar{p}_s) \) for all \( s \in [t, T] \), and in particular, \( \bar{p}_T = \nabla_x J_T(T, \bar{x}_T, \bar{p}_T) = \nabla \psi(\bar{x}_T) \). Hence, \( s \mapsto \overline{X}(Y_p(x))_s \), \( s \in [t, T] \), solves the TPBVP (11) with \( \bar{x}_T = x \). Theorem 4.2 subsequently implies the existence of \( \text{arg stat}_{u \in \mathcal{U}} J_T(t, x, u) \), and its explicit form (39).

(iii) \( \implies \) (ii) Suppose there exists \( \bar{u} \in \text{arg stat}_{u \in \mathcal{U}} J_T(t, x, u) \). By Theorem 4.2 and subsequently Lemma 4.3 there exists a unique classical solution to TPBVP (11) with \( \bar{x}_T = x \) fixed. Denote \( \bar{p} = \bar{p}_T \) and \( (\bar{x}_s, \bar{p}_s) \doteq \overline{X}(Y_p(x))_s \) for all \( s \in [t, T] \), and note by (11) that \( \bar{p}_T = \nabla \psi(\bar{x}_T) \). Hence, FVP (37) has the trivial solution as its unique solution, i.e. \( (\xi_s, \pi_s) = 0 \) for all \( s \in [t, T] \). Hence, recalling (35), (36), (37),

\[
\xi_s = \left( \frac{\nabla_x J_T(s, \bar{x}_s, \bar{p}_s)}{\nabla_p J_T(s, \bar{x}_s, \bar{p}_s)} \right) \left( \frac{\bar{p}_s - \pi_s}{\xi_s} \right), \quad s \in [t, T],
\]

so that \( 0 = \nabla_p J_T(s, \bar{x}_s, \bar{p}_s) = \bar{p}_s \in \text{arg stat}_{p \in \mathcal{X}} J_T(s, \bar{x}_s, p) \) for all \( s \in [t, T] \).
Corollary 4.12. Let $T \in \mathbb{R}_{\geq 0}$, $t \in [0,T]$, and $x \in \mathcal{X}$. Then the following statements are equivalent:

(i) there exists $\bar{u} \in \arg\max_{u \in \mathcal{U}} J_T(t,x,u)$;

(ii) there exists $\bar{p} \in \arg\max_{p \in \mathcal{X}^\prime} J_T(t,x,p)$ such that TPBVP (40) has a unique solution, and it satisfies $\pi_T = 0$,
in which $J_T$, $\bar{J}_T$ are as per [2], [13], [19], [22].

Remark 4.13. As indicated, Corollary 4.12 uses a condition concerning the TPBVP (40) that is equivalent to the trajectory based argstat condition appearing in Theorem 4.11. This condition can be re-expressed via the operator

$$\bar{U}_{s,t} = \left( \begin{array}{c}
\bar{U}_{11}^{s,t} \\
\bar{U}_{21}^{s,t}
\end{array} \right) \in \mathcal{L}^{2}$$

which is defined as an element of the two-parameter family generated by $-A_s' \in \mathcal{L}^{2}$, $s \in [t,T]$, via (29). By definition of $\bar{U}_{T,t}$, and by inspection of (10),

$$\left( \begin{array}{c}
\xi_s \\
\pi_s
\end{array} \right) = \bar{U}_{T,t} \left( \begin{array}{c}
\xi_T \\
\pi_T
\end{array} \right), \quad s \in [t,T],$$

so that the boundary conditions in (40) require

$$0 = (\bar{U}_{11}^{21})' \pi_T.$$

Hence, the requirement that $\pi_T = 0$ in the statement of Corollary 4.12 amounts to an invertibility requirement for $(\bar{U}_{11}^{21})' \in \mathcal{L}^{2}$, i.e. perturbations in the terminal costate map bijectively to perturbations in the initial state. This type of condition arises in the application of the classical method of characteristics to optimal control problems, and so is unsurprising, see for example [14].

Lemma 4.14. Given any $T \in \mathbb{R}_{>0}$, $t \in [0,T)$, $x,p \in \mathcal{X}$, and $(\bar{x}, \bar{p}) \doteq \bar{X}(Y_p(x))_s$ for all $s \in [t,T]$,

$$0 = \nabla_p \bar{J}_T(t,\bar{x}_t,\bar{p}_t) \quad \iff \quad 0 = \nabla_p \bar{J}_T(s,\bar{x}_s,\bar{p}_s) \quad \forall \ s \in [t,T]$$

$$\bar{p}_t = \nabla_x \bar{J}_T(t,\bar{x}_t,\bar{p}_t) \quad \iff \quad \bar{p}_s = \nabla_x \bar{J}_T(s,\bar{x}_s,\bar{p}_s) \quad \forall \ s \in [t,T].$$

Proof. Fix arbitrary $T \in \mathbb{R}_{>0}$, $t \in [0,T)$, $x,p \in \mathcal{X}$. By Proposition 4.8 and in particular (35), (36), (37),

$$\left( \begin{array}{c}
\xi_s \\
\pi_s
\end{array} \right) = \left( \begin{array}{c}
\nabla_p \bar{J}_T(s,\bar{x}_s,\bar{p}_s) \\
\nabla_x \bar{J}_T(s,\bar{x}_s,\bar{p}_s)
\end{array} \right), \quad s \in [t,T],$$

satisfies the ODE in (37). Suppose that the left-hand condition in (11) holds, i.e. $0 = (\xi_t, \pi_t)$. Hence, by inspection of the ODE in (37), (12), $0 = (\xi_s, \pi_s)$ for all $s \in [t,T]$; so that the centre and right-hand conditions in (11) simultaneously hold. Conversely, suppose that either the centre or the right-hand condition in (11) holds. Then, by Lemma 4.10, both the centre and right-hand conditions in (11) simultaneously hold. Moreover, by inspection of (19), (35), selecting $s = t$ in the both the centre and right-hand conditions in (11) immediately yields the left-hand condition in (11).

Theorem 4.15. Let $T \in \mathbb{R}_{>0}$, $t \in [0,T)$, and $\bar{x}, \bar{p} \in \mathcal{X}$. Then the following statements are equivalent:

(i) $\bar{x} \in \arg\max_{x \in \mathcal{X}} \{ (x,\bar{p}) - \bar{J}_T(t,x,\bar{p}) \}$ and $\bar{p} \in \arg\max_{p \in \mathcal{X}^\prime} \bar{J}_T(t,\bar{x},p)$;

(ii) $\bar{u} \in \arg\max_{u \in \mathcal{U}[t,T]} J_T(t,\bar{x},u)$ where

$$\bar{u}_s \doteq -M^{-1} \left( \begin{array}{c}
0 \\
\bar{X}(Y_p(\bar{x}))_s
\end{array} \right), \quad \forall s \in [t,T].$$

Proof. Immediate by Theorem 4.11 and Lemma 4.14.
4.4 HJB equation and stationary trajectories

A verification theorem exists for the cost $\tilde{J}_T$ of (19), (21), formulated with respect to an extended Hamiltonian

$$\tilde{H}(x, p, \pi, \zeta) = -\frac{1}{2} \langle p, \mathcal{M}^{-1} p \rangle + V(x) + \langle \pi, \mathcal{M}^{-1} p \rangle - \langle \zeta, \nabla V(x) \rangle$$

(44)

for all $x, p, \pi, \zeta \in \mathcal{X}$.

**Theorem 4.16.** Given $T \in \mathbb{R}_{>0}$ suppose there exists a $W \in C([0, T] \times \mathcal{X}^2; \mathbb{R}) \cap C^1((0, T) \times \mathcal{X}^2; \mathbb{R})$ satisfying the partial differential equation and terminal condition

$$\begin{cases}
- \frac{\partial W}{\partial t}(t, x, p) + \tilde{H}(x, p, \nabla_x W(t, x, p), \nabla_p W(t, x, p)) = 0 & (t, x, p) \in (0, T) \times \mathcal{X}^2, \\
W(T, x, p) = \psi(x), & (x, p) \in \mathcal{X}^2,
\end{cases}$$

(45)

in which $\tilde{H}$ is as per (44), and $\psi$ is the terminal cost appearing in (2), (19). Then, $\tilde{J}_T(t, x, p) = W(t, x, p)$ for all $t \in (0, T), x, p \in \mathcal{X}$, where $\tilde{J}_T$ is as per (19).

Conversely, $\tilde{J}_T$ of (19) always satisfies (45), and it is consequently the unique solution.

**Proof.** Fix $T \in \mathbb{R}_{>0}, t \in (0, T)$, and let $W$ be as per the theorem statement. Fix any $x, p \in \mathcal{X}$. With $X$ as per Lemmas 1.3, 1.4, 1.5 let $(\tilde{x}_s, \tilde{p}_s) \equiv X(Y_p(x))_s$ for all $s \in [t, T]$. Note in particular that $s \mapsto (\tilde{x}_s, \tilde{p}_s)$ is a classical solution of the Cauchy problem (20), (24). Hence, by the asserted smoothness of $W$, $s \mapsto W(s, \tilde{x}_s, \tilde{p}_s)$ is differentiable, so that the chain rule and (45) yield

$$\frac{d}{ds} W(s, \tilde{x}_s, \tilde{p}_s) = \frac{\partial}{\partial s} W(s, \tilde{x}_s, \tilde{p}_s) + \langle \nabla_x W(s, \tilde{x}_s, \tilde{p}_s), \dot{\tilde{x}}_s \rangle + \langle \nabla_p W(s, \tilde{x}_s, \tilde{p}_s), \dot{\tilde{p}}_s \rangle$$

$$= -\left[ -\frac{\partial}{\partial s} W(s, \tilde{x}_s, \tilde{p}_s) + \tilde{H}(x, p, \nabla_x W(s, \tilde{x}_s, \tilde{p}_s), \nabla_p W(s, \tilde{x}_s, \tilde{p}_s)) \right]$$

$$+ \langle \nabla_x W(s, \tilde{x}_s, \tilde{p}_s), -\mathcal{M}^{-1} \dot{\tilde{p}}_s \rangle + \langle \nabla_p W(s, \tilde{x}_s, \tilde{p}_s), \nabla V(\tilde{x}_s) \rangle$$

$$= -\frac{1}{2} \langle \tilde{p}_s, \mathcal{M}^{-1} \dot{\tilde{p}}_s \rangle + V(\tilde{x}_s) + \langle \nabla_x W(s, \tilde{x}_s, \tilde{p}_s), \mathcal{M}^{-1} \dot{\tilde{p}}_s \rangle - \langle \nabla_p W(s, \tilde{x}_s, \tilde{p}_s), \nabla V(\tilde{x}_s) \rangle$$

$$+ \langle \nabla_x W(s, \tilde{x}_s, \tilde{p}_s), -\mathcal{M}^{-1} \dot{\tilde{p}}_s \rangle + \langle \nabla_p W(s, \tilde{x}_s, \tilde{p}_s), \nabla V(\tilde{x}_s) \rangle$$

$$= -\frac{1}{2} \langle \tilde{p}_s, \mathcal{M}^{-1} \dot{\tilde{p}}_s \rangle + V(\tilde{x}_s),$$

for all $s \in (t, T)$. Integrating with respect to $s \in (t, T)$, and recalling the boundary condition in (45), subsequently yields

$$\psi(\tilde{x}_T) - W(t, x, p) = W(T, \tilde{x}_T, \tilde{p}_T) - W(t, x, p) = \int_t^T -\frac{1}{2} \langle \tilde{p}_s, \mathcal{M}^{-1} \dot{\tilde{p}}_s \rangle + V(\tilde{x}_s) \, ds.$$

Rearranging, and recalling (19), yields the asserted equality $\tilde{J}_T(t, x, p) = W(t, x, p)$. Recalling that $t \in (0, T)$, $x, p \in \mathcal{X}$ are arbitrary, the first assertion follows trivially holds. Fix $t \in (0, T)$, and let $(\tilde{x}_s, \tilde{p}_s) \equiv X(Y_p(x))_s$ for all $s \in [t, T]$. Fix $r \in (t, T)$. By (19),

$$\tilde{J}_T(t, x, p) = \int_t^r -\frac{1}{2} \langle \tilde{p}_s, \mathcal{M}^{-1} \dot{\tilde{p}}_s \rangle + V(\tilde{x}_s) \, ds + \int_r^T -\frac{1}{2} \langle \tilde{p}_s, \mathcal{M}^{-1} \dot{\tilde{p}}_s \rangle + V(\tilde{x}_s) \, ds + \psi(\tilde{x}_T)$$

$$= \int_t^r -\frac{1}{2} \langle \tilde{p}_s, \mathcal{M}^{-1} \dot{\tilde{p}}_s \rangle + V(\tilde{x}_s) \, ds + \tilde{J}_T(r, \tilde{x}_r, \tilde{p}_r).$$

Dividing through by $r - t$ and sending $r \to t^+$, yields

$$-\frac{1}{2} \langle p, \mathcal{M}^{-1} p \rangle + V(x) = \frac{d}{dt} \tilde{J}_T(t, \tilde{x}_t, \tilde{p}_t)$$

$$= \frac{d}{dt} \tilde{J}_T(t, x, p) + \langle \nabla_x \tilde{J}_T(t, x, p), -\mathcal{M}^{-1} p \rangle + \langle \nabla_p \tilde{J}_T(t, x, p), \nabla V(x) \rangle,$$

i.e. $\tilde{J}_T$ satisfies (45), and uniqueness follows by the first assertion. \qed
Theorem 4.17. Given $T \in \mathbb{R}_{>0}$, suppose there exists a $W \in C([0,T] \times \mathcal{X};\mathbb{R}) \cap C^1((0,T) \times \mathcal{X};\mathbb{R})$ satisfying the partial differential equation and terminal condition (46). Suppose further that, given $t \in [0,T]$ and $\bar{x} \in \mathcal{X}$, there exists a $\bar{p} \in \mathcal{X}$ such that

$$\bar{p} \in \arg \sup_{x \in \mathcal{X}} W(t,\bar{x},p), \quad \bar{x} \in \arg \sup_{x \in \mathcal{X}} \{ \langle x, \bar{p} \rangle - W(t, x, \bar{p}) \}.$$  

(46)

Then, there exists a $u \in \mathcal{U}[t,T]$ such that

$$J_T(t,\bar{x},u) = \sup_{x \in \mathcal{X}} W(t,\bar{x},\bar{p}) = -\mathcal{M}^{-1} \bar{p},$$

(47)

for all $s \in [t,T]$.

Proof. Fix $T \in \mathbb{R}_{>0}$, $W \in C([0,T] \times \mathcal{X};\mathbb{R}) \cap C^1((0,T) \times \mathcal{X};\mathbb{R})$, $t \in [0,T]$, $x \in \mathcal{X}$, as per the theorem statement. Suppose that $\bar{p} \in \mathcal{X}$ exists such that (46) holds. Observe by Theorem 4.15 that $J_T \equiv W$. Hence, by (40) and Theorem 4.15, there exists $\bar{u} \in \arg \sup_{x \in \mathcal{X}} J_T(t,\bar{x},u)$ satisfying $\bar{u}_s = -\mathcal{M}^{-1} \bar{p}_s$ for all $s \in [t,T]$. Moreover, by Lemma 4.14 $\bar{p}_s = \nabla_x J_T(s,\bar{x},\bar{p}_s) = \nabla_x W(s,\bar{x},\bar{p}_s)$, and $0 = \nabla_p J_T(s,\bar{x},\bar{p}_s)$ for all $s \in [t,T]$, so that (47) holds.

Remark 4.18.

(i) The characteristic system associated with (44) is

$$\begin{aligned}
\dot{x}_t &= -\nabla_x H(x, x_p, \pi_s, \zeta_s) = -\mathcal{M}^{-1} p_s, \quad x_t = x, \\
\dot{p}_s &= -\nabla_x H(x, x_p, \pi_s, \zeta_s) = \nabla V(x), \quad p_t = p, \\
\dot{\pi}_s &= \nabla_x \hat{H}(x, x_p, \pi_s, \zeta_s) = \nabla V(x) - \nabla^2 V(x) \zeta_s, \\
\dot{\zeta}_s &= \nabla_p \hat{H}(x, x_p, \pi_s, \zeta_s) = -\mathcal{M}^{-1} (p_s - \pi_s),
\end{aligned}$$

(48)

for all $s \in [0,T]$, in which $\pi_s = \nabla_x W(s, x, x_p)$ and $\zeta_s = \nabla_p W(s, x, x_p)$. By adopting the condition (46), it may be noted that $p_s = \pi_s$, $\zeta_s = 0$, and $\hat{H}(x, x_p, \pi_s, \zeta_s) = H(x, x_p)$ for all $s \in [t,T]$.

(ii) It may be noted that under the stated assumptions, if $\arg \sup_{x \in \mathcal{X}} J_T(t, x, p)$ is convex for a.e. $t \in (0,T)$ and all $x \in \mathcal{X}$, then from the $C^1$ regularity of $J_T$ (c.f. Proposition 4.7) it follows that $\arg \sup_{x \in \mathcal{X}} J_T(t, x, p)$ is single-valued for a.e. $t \in (0,T)$ and all $x \in \mathcal{X}$ and $\mathcal{W}_T(\cdot, \cdot)$ is the (viscosity) solution of the HJB equation

$$\begin{aligned}
\frac{\partial W}{\partial t}(t, x) + H(x, \nabla_x W(t, x)) &= 0 \quad (t, x) \in [0,T] \times \mathcal{X}, \\
W(T, x) &= \psi(x),
\end{aligned}$$

(49)

where $H$ is the Hamiltonian defined in (10). Indeed, since it is known that the minimal selection $W_T$ is the unique viscosity solution of (49) (c.f. [14]), it is sufficient to show that

$$W_T(t, x) = \mathcal{W}_T(t, x) \quad \forall t \in [0,T], \forall x \in \mathcal{X}. $$

(50)

Fix $t \in [0,T]$ and all $x \in \mathcal{X}$. Then, letting $L(y, u) = \frac{1}{2} \langle u, M u \rangle - V(y)$ and using that $H(y, p) = p \nabla_p H(y, p) - L(y, \nabla_p H(y, p))$ for any $p, y \in \mathcal{X}$, applying Theorem 4.2 and keeping the same notation, we have that there exists $p \in \mathcal{X}$ satisfying

$$J_T(t, x, \bar{u}) = \psi(\bar{x}_T) + \int^T_t L(\bar{x}_s, \bar{u}_s) \, ds = \psi(\bar{x}_T) + \int^T_t L(\bar{x}_s, \nabla_p H(\bar{x}_s, \bar{p}_s)) \, ds$$

$$= \psi(\bar{x}_T) + \int^T_t (-H(\bar{x}_s, \bar{p}_s) + \langle \bar{p}_s, \nabla_p H(\bar{x}_s, \bar{p}_s) \rangle) \, ds = \bar{J}_T(t, x, p),$$

where $(\bar{x}, \bar{p})$ is the Hamiltonian flow (20) with initial condition $(x, p)$. Hence, the minimal selection coincide with $\mathcal{W}_T$, and (50) follows. □
5 A one-dimensional example

A one-dimensional linear mass-spring system consists of a mass $M \doteq m \in \mathbb{R}_{>0}$ located at position $x \in \mathbb{R}$ whose motion is a consequence of a quadratic potential field $V: \mathbb{R} \to \mathbb{R}_{\geq 0}$, $V(x) \doteq \frac{1}{2} \kappa x^2$, $x \in \mathbb{R}$. For simplicity, supposes that the terminal velocity $v \in \mathbb{R}$ of this mass is of interest. The terminal cost $\psi: \mathbb{R} \to \mathbb{R}$ in (19), (23) is consequently defined by $\psi(x) \doteq -m v x$ for all $x \in \mathbb{R}$. Observe by inspection that (11) holds, with $K = 2 \kappa$. With a view to demonstrating the existence of an explicit solution to (4.16), fix $t \in [0, T]$, and define

$$\Sigma \doteq \begin{pmatrix} \kappa & 0 \\ 0 & -\frac{m}{\kappa} \end{pmatrix}, \quad \Gamma \doteq \begin{pmatrix} 0 & -\frac{m}{\kappa} \\ \kappa & 0 \end{pmatrix}.$$

(51)

Recalling (44), (51), note that the PDE (45) may be compactly written as

$$0 = -\frac{\partial W}{\partial s}(s,x,p) + \frac{1}{2} \left( \begin{pmatrix} x \\ p \end{pmatrix} , \Sigma \begin{pmatrix} x \\ p \end{pmatrix} \right) - \left( \begin{pmatrix} \nabla_x W(s,x,p) \\ \nabla_p W(s,x,p) \end{pmatrix} , \Gamma \begin{pmatrix} x \\ p \end{pmatrix} \right)$$

for all $s \in (t,T)$, $Y \doteq Y_s(p) \in \mathcal{X}^2$. Define $\tilde{W}: [t, T] \times \mathcal{X}^2 \to \mathbb{R}$ by

$$\tilde{W}(s,Y) \doteq \frac{1}{2} \langle Y, P_s Y \rangle + \langle Q_s, Y \rangle,$n
(53)

$$P_s \doteq -\int_t^T \exp(\Gamma' (\sigma - s)) \Sigma \exp(\Gamma (\sigma - s)) d\sigma, \quad Q_s \doteq \exp(\Gamma' (T - s)) \left( \begin{pmatrix} -m v \\ 0 \end{pmatrix} \right),$$

(54)

for all $s \in (t,T)$, $Y \in \mathcal{X}^2$. Applying Leibniz, note that

$$\dot{P}_s = \Sigma - \Gamma' P_s - P_s \Gamma, \quad \dot{Q}_s = -\Gamma' Q_s,$$

(55)

for all $s \in (t,T)$. Hence, differentiating (53) yields

$$\frac{\partial}{\partial s} \tilde{W}(s,Y) = \frac{1}{2} \langle Y, \dot{P}_s Y \rangle + \langle \dot{Q}_s, Y \rangle, \quad \nabla_Y \tilde{W}(s,Y) = P_s Y + Q_s.$$

Substituting these derivatives in the right-hand side of (52), and applying (53), subsequently yields

$$-\frac{\partial W}{\partial s}(s,x,p) + \frac{1}{2} \langle Y, \Sigma Y \rangle - \langle \nabla_Y \tilde{W}(s,Y), \Gamma Y \rangle$$

$$= -\frac{1}{2} \langle Y, \dot{P}_s Y \rangle - \langle \dot{Q}_s, Y \rangle + \frac{1}{2} \langle Y, \Sigma Y \rangle - \langle P_s Y + Q_s, \Gamma Y \rangle$$

$$= \frac{1}{2} \langle Y, [-P_s + \Sigma - P_s \Gamma - \Gamma' P_s] Y \rangle + \langle -Q_s - \Gamma' Q_s, Y \rangle$$

$$= 0,$$

for all $s \in (t,T)$, $Y \in \mathcal{X}^2$. Note further that $\tilde{W}(T, x, p) = \tilde{W}(T, Y) = \langle Q_T, Y \rangle = -m v x = \psi(x)$. Hence, $\tilde{W}$ of (50) is a solution of the PDE and terminal condition of (40). Hence, by Theorem 4.14, the cost $\tilde{J}_T(s, x, p)$ of (19), (23) is given explicitly by $\tilde{J}_T(s, x, p) = \tilde{W}(s, Y_s(p))$ for all $s \in [t,T], x, p \in \mathcal{X}$. By diagonalizing $\Gamma$, direct integration of (54) yields

$$P_s = \frac{1}{2} \begin{pmatrix} -\kappa \sin(2 \omega (T - s)) & 1 - \cos(2 \omega (T - s)) \\ 1 - \cos(2 \omega (T - s)) & \frac{m}{\kappa} \sin(2 \omega (T - s)) \end{pmatrix}, \quad \omega = \sqrt{\frac{\kappa}{m}}.$$n

$$Q_s = -m v \begin{pmatrix} \cos(\omega (T - s)) \\ -\frac{\kappa}{m} \sin(\omega (T - s)) \end{pmatrix}.$$

(56)

With a view to illustrating Theorem 4.17, fix $x \in \mathbb{R}$, and note that

$$\nabla_x \tilde{W}(t, x, p) = \begin{pmatrix} 1 & 0 \end{pmatrix} \nabla_Y \tilde{W}(t, Y_s(p)) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} P_t \left( \begin{pmatrix} x \\ p \end{pmatrix} \right) + Q_t \end{pmatrix}.$$
(a) Time evolution.

(b) Phase portrait.

(c) Time evolution of phase portrait.

Figure 1: State and costate trajectories.

\[
\begin{align*}
\n\bar{p}(t,x,p) &= (0 \quad 1) \nabla_{\bar{p}} \bar{W}(t,x,p) = (0 \quad 1) \left( P_t \left( \begin{array}{c} x \\ p \end{array} \right) + Q_t \right) \\
&= \frac{1}{2} \left[ 1 - \cos(2\omega (T - t)) \right] x + \frac{1}{2m \omega} \sin(2\omega (T - t)) p + m v \left( \frac{\omega}{\kappa} \right) \sin(\omega (T - t)).
\end{align*}
\]

(56)

(57)

Note that \( m \omega = \sqrt{\kappa m} = \frac{\omega}{\kappa}. \) Motivated by (46), let \( \bar{p} \in \mathbb{R} \) be such that \( 0 = \nabla_{\bar{p}} \bar{W}(t,x,\bar{p}) \) and \( \bar{p} = \nabla_x \bar{W}(t,x,\bar{p}) \) via (56) and (57). Applying double angle formulae subsequently yields the system of linear equations

\[
\Omega_t \left( \frac{x\sqrt{\kappa m}}{\bar{p}} \right) = \Theta_t,
\]

(58)

in which

\[
\begin{align*}
\Omega_t &= \begin{pmatrix}
\sin^2(\omega (T - t)) & \sin(\omega (T - t)) \cos(\omega (T - t)) \\
\sin(\omega (T - t)) \cos(\omega (T - t)) & \cos^2(\omega (T - t))
\end{pmatrix},
\quad \Theta_t = -m v \begin{pmatrix}
\sin(\omega (T - t)) \\
\cos(\omega (T - t))
\end{pmatrix}.
\end{align*}
\]
By inspection, the matrix \( (\Omega | \Theta_i ) \in \mathbb{R}^{2 \times 3} \) is rank one, i.e. the two equations in (60) are linearly dependent. Moreover, some minor manipulations yield
\[
\begin{cases}
p = -\sqrt{\kappa} m \tan(\omega (T-t)) x - m v \sec(\omega (T-t)), & \omega (T-t) \notin \{ n \pi, (n+\frac{1}{2}) \pi : n \in \mathbb{Z} \}, \\
p = (-1)^{n+1} m v, & \omega (T-t) \in \{ n \pi : n \in \mathbb{Z} \}, x \in \mathbb{R}, \\
p \text{arbitrary,} & \omega (T-t) \in \{ (n+\frac{1}{2}) \pi : n \in \mathbb{Z} \}, x = (-1)^{n+1} \frac{p}{\omega}, \\
p \text{does not exist,} & \omega (T-t) \in \{ (n+\frac{1}{2}) \pi : n \in \mathbb{Z} \}, x \neq (-1)^{n+1} \frac{p}{\omega}.
\end{cases}
\]

Note in the second case that \( \bar{p} \) must correspond to the desired terminal momentum, with sign determined by whether \( T - t \) is a period or half-period of the mass-spring oscillation. In the third and fourth cases, \( T - t \) corresponds to a quarter or three quarter period of the mass-spring oscillation, and \( \bar{p} \) is either arbitrary, or does not exist, depending on the specific choice of \( x \). An example of the third case, where \( \bar{p} \) is arbitrary, is illustrated in Figures 1a 1b and 1c for \( v = -2 \) and \( x = (-1)^{4} \frac{p}{\omega} \approx -4.47 \).

Appendix

A Proofs of lemmas 4.3, 4.4, 4.5, and 4.6

Proof of Lemma 4.3. The proof employs a standard fixed point argument, exploiting global Lipschitz continuity of \( f \) of (22), see for example [13, Theorem 5.1, p.127]. We notice that the global Lipschitz continuity of \( Df(x) \) directly follows from the second inequality in (11) for \( j = 0 \).

Proof of Lemma 4.4. Fix \( T \in \mathbb{R}_{>0}, t \in [0,T] \), and \( Y, h \in \mathcal{X}^{2} \). Applying Lemma 4.3, there exist unique classical solutions \( \bar{X}(Y) \) and \( \bar{X}(Y+h) \) to (23) satisfying respectively \( \bar{X}(Y)_t = Y \) and \( \bar{X}(Y+h)_t = Y + h \). In integral form,
\[
\bar{X}(Y)_s = Y + \int_t^s f(\bar{X}(Y)_{\sigma}) d\sigma, \\
\bar{X}(Y+h)_s = Y + h + \int_t^s f(\bar{X}(Y+h)_{\sigma}) d\sigma,
\]
so that
\[
\bar{X}(Y+h)_s - \bar{X}(Y)_s = h + \int_t^s f(\bar{X}(Y+h)_{\sigma}) - f(\bar{X}(Y)_{\sigma}) d\sigma
\]
for all \( s \in [t,T] \). Consequently, as \( f \) is globally Lipschitz by inspection of (22),
\[
\| \bar{X}(Y+h)_s - \bar{X}(Y)_s \| \leq \| h \| + \int_t^s \| f(\bar{X}(Y+h)_{\sigma}) - f(\bar{X}(Y)_{\sigma}) \| d\sigma
\]
\[
\leq \| h \| + \alpha \int_t^s \| \bar{X}(Y+h)_\sigma - \bar{X}(Y)_\sigma \| d\sigma
\]
in which \( \alpha \in \mathbb{R}_{>0} \) is the associated Lipschitz constant. Applying Gronwall’s inequality, and recalling the definition of \( \| \cdot \|_{\infty} \), yields
\[
\| \bar{X}(Y+h) - \bar{X}(Y) \|_{\infty} \leq \| h \| \exp(\alpha (T-t)),
\]
so that (20) holds. As \( Y, h \in \mathcal{X}^{2} \) are arbitrary, continuity is immediate.

Proof of Lemma 4.5. Fix \( T \in \mathbb{R}_{>0}, t \in [0,T] \), and \( \bar{X} \in C(\mathcal{X}^{2}; C([t,T];\mathcal{X}^{2}) as per Lemma 4.4. Fix \( Y \in \mathcal{X}^{2} \) and \( s \mapsto A(Y)_s \) as per (20), and note that (28) follows by [13, Theorem 5.2, p.128]. Fix any \( h \in \mathcal{X}^{2} \), \( s \in [t,T] \), and note by inspection of (22) that \( A(Y)_s = Df(\bar{X}(Y)_s) \). Hence, recalling (59),
\[
\bar{X}(Y+h)_s - \bar{X}(Y)_s - U_{s,t}(Y) h = \int_t^s f(\bar{X}(Y+h)_{\sigma}) - f(\bar{X}(Y)_{\sigma}) - A(Y)_{\sigma} U_{s,t}(Y) h d\sigma
\]
Hence, by Taylor’s theorem, given \( L \).

\[
\| f(x) - f(a) - Df(a)(x-a) \| \leq \frac{L}{2} \| x-a \|^2
\]

for all \( x \in C([t,T]; \mathcal{X}^2) \). Note that \( Y \mapsto f(Y) \) is twice Fréchet differentiable by \([1]\), with \( D^2f(Y) \in \mathcal{L}(\mathcal{X}^2; \mathcal{L}(\mathcal{X}^2)) = \mathcal{L}(\mathcal{X}^2 \times \mathcal{X}^2; \mathcal{X}^2) \) for all \( Y \in \mathcal{X}^2 \). Again by \([1]\), there exists an \( M \in \mathbb{R}_{>0} \) such that

\[
\sup_{Y \in \mathcal{X}^2} \| D^2f(Y) \|_{\mathcal{L}(\mathcal{X}^2 \times \mathcal{X}^2; \mathcal{X}^2)} \leq M < \infty.
\]

Hence, by Taylor’s theorem, given \( X, \delta \in C([t,T]; \mathcal{X}^2) \),

\[
\left\| \tilde{I}_f(X + \delta) - \tilde{I}_f(X) - \int_t^s Df(X_\sigma) \delta_\sigma d\sigma \right\| \leq \int_t^s \| f(X_\sigma + \delta_\sigma) - f(X_\sigma) - Df(X_\sigma) \delta_\sigma \| d\sigma
\]

\[
= \int_t^s \left\| \left( \int_0^1 (1-\eta) D^2f(X_\sigma + \eta \delta_\sigma) d\eta \right)(\delta_\sigma, \delta_\sigma) \right\| d\sigma
\]

\[
\leq \int_t^s \int_0^1 (1-\eta) \| D^2f(X_\sigma + \eta \delta_\sigma) \|_{\mathcal{L}(\mathcal{X}^2 \times \mathcal{X}^2; \mathcal{X}^2)} \| \delta_\sigma \|^2 \quad \leq \frac{M}{2} \int_t^s \| \delta_\sigma \|^2 \quad \leq \frac{M}{2} (s-t) \| \delta \|^2_{\infty}.
\]

That is,

\[
\left\| \tilde{I}_f(X + \delta) - \tilde{I}_f(X) - \int_t^s Df(X_\sigma) \delta_\sigma d\sigma \right\| \leq \frac{M}{2} (T-t) \| \delta \|^2_{\infty},
\]

so that \( \tilde{I}_f \) is Fréchet differentiable with derivative

\[
[D\tilde{I}_f(X)] \delta_\sigma = \int_t^s Df(X_\sigma) \delta_\sigma d\sigma
\]

for all \( X, \delta \in C([t,T]; \mathcal{X}^2), s \in [t,T] \). So, recalling \([60]\), and \([1]\),

\[
\mathcal{X}(Y+h)_s - \mathcal{X}(Y)_s - U_{s,t}(Y) h = \int_t^s [\tilde{I}_f]_{\mathcal{X}(Y)}(\mathcal{X}(Y+h) - \mathcal{X}(Y))_s \| \mathcal{X}(Y+h) - \mathcal{X}(Y) \|^\infty
\]

\[
+ \int_t^s Df(\mathcal{X}(\mathcal{X})) (\mathcal{X}(Y+h)_s - \mathcal{X}(Y)_s - U_{s,t}(Y) h) d\sigma.
\]

Noting that \( L = \sup_{\sigma \in [t,T]} \| Df(\mathcal{X}(\mathcal{X})) \|_{\mathcal{X}(\mathcal{X})} < \infty \), taking the norm of both sides yields

\[
\| \mathcal{X}(Y+h)_s - \mathcal{X}(Y)_s - U_{s,t}(Y) h \| \leq \| d[\tilde{I}_f]_{\mathcal{X}(Y)}(\mathcal{X}(Y+h) - \mathcal{X}(Y))_s \| \| \mathcal{X}(Y+h) - \mathcal{X}(Y) \|_{\infty}
\]

\[
+ \int_t^s L \| \mathcal{X}(Y+h)_s - \mathcal{X}(Y)_s - U_{s,t}(Y) h \| d\sigma.
\]

Hence, by Gronwall’s inequality,

\[
\| \mathcal{X}(Y+h)_s - \mathcal{X}(Y)_s - U_{s,t}(Y) h \|
\leq \| d[\tilde{I}_f]_{\mathcal{X}(Y)}(\mathcal{X}(Y+h) - \mathcal{X}(Y))_s \| \infty \| \mathcal{X}(Y+h) - \mathcal{X}(Y) \|_{\infty} \exp(L(T-t)),
\]

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or, with \( \theta_Y(h) \triangleq \|d[T]\bar{X}(Y)\|_{L^\infty} \),

\[
\|\bar{X}(Y + h) - \bar{X}(Y) - U_{r,t}(Y) h\|_\infty \leq \theta_Y(h) \|\bar{X}(Y + h) - \bar{X}(Y)\|_{L^\infty} \exp(L(T - t))
\]

\[
\leq \theta_Y(h) \|\bar{X}(Y + h) - \bar{X}(Y) - U_{r,t}(Y) h\|_\infty \exp(L(T - t))
\]

\[
+ \theta_Y(h) \sup_{s \in [t,T]} \|U_{s,t}(Y)\|_{L^2} \|h\| \exp(L(T - t)).
\]

As \( \theta_Y \) is continuous at 0, there exists an \( r > 0 \) sufficiently small such that \( h \in B_0(r) \) implies that \( \theta_Y(h) \exp(L(T - t)) < \frac{1}{4} \). Hence, with \( h \in B_0(r) \),

\[
\|\bar{X}(Y + h) - \bar{X}(Y) - U_{r,t}(Y) h\|_\infty < 2 \theta_Y(h) \sup_{s \in [t,T]} \|U_{s,t}(Y)\|_{L^2} \|h\| \exp(L(T - t))
\]

\[
= Q \theta_Y(h) \|h\|,
\]

in which \( Q \triangleq 2 \sup_{s \in [t,T]} \|U_{s,t}(Y)\|_{L^2} \exp(L(T - t)) \). Consequently, taking a limit,

\[
\lim_{\|h\| \to 0} \frac{\|\bar{X}(Y + h) - \bar{X}(Y) - U_{r,t}(Y) h\|_\infty}{\|h\|} \leq \lim_{\|h\| \to 0} Q \theta_Y(h) = 0.
\]

That is, \( Y \mapsto \bar{X}(Y) \) is Fréchet differentiable, with the indicated derivative.

**Proof of Lemma 4.2**. Fix \( T \in \mathbb{R}_{\geq 0}, \ t \in [0,T] \) as per the lemma statement. It is first demonstrated that \( Y \mapsto U_{s,r}(Y) \) is continuous, uniformly in \( r, s \in [t,T] \), as this motivates the subsequent proof of continuous differentiability. Fix \( r, s \in [t,T] \), \( h, \tilde{h} \in \mathcal{X}^2 \). As \( U_{s,r}(Y) \in \mathcal{L}(\mathcal{X}^2) \) is an element of the two-parameter family of evolution operators generated by \( A(Y)_s \in \mathcal{L}(\mathcal{X}^2) \), see (29),

\[
U_{s,r}(Y) h = h + \int_r^s A(Y)_{\sigma} U_{\sigma,r}(Y) h \, d\sigma,
\]

\[
U_{s,r}(Y + \tilde{h}) h = h + \int_r^s A(Y + \tilde{h})_{\sigma} U_{\sigma,r}(Y + \tilde{h}) h \, d\sigma,
\]

so that

\[
[U_{s,r}(Y + \tilde{h}) - U_{s,r}(Y)] h = \int_r^s [A(Y + \tilde{h})_{\sigma} U_{\sigma,r}(Y + \tilde{h}) - A(Y)_{\sigma} U_{\sigma,r}(Y)] h \, d\sigma
\]

\[
= \int_r^s [A(Y + \tilde{h})_{\sigma} - A(Y)_{\sigma}] [U_{\sigma,r}(Y + \tilde{h}) - U_{\sigma,r}(Y)] h \, d\sigma
\]

\[
+ \int_r^s A(Y)_{\sigma} [U_{\sigma,r}(Y + \tilde{h}) - U_{\sigma,r}(Y)] h \, d\sigma + \int_r^s [A(Y + \tilde{h})_{\sigma} - A(Y)_{\sigma}] U_{\sigma,r}(Y) h \, d\sigma. \tag{63}
\]

Hence, by the triangle inequality,

\[
\|[U_{s,r}(Y + \tilde{h}) - U_{s,r}(Y)] h\| \leq \int_r^s \|A(Y + \tilde{h})_{\sigma} - A(Y)_{\sigma}\|_{L^2} \|[U_{\sigma,r}(Y + \tilde{h}) - U_{\sigma,r}(Y)] h\| \, d\sigma
\]

\[
+ \int_r^s \|A(Y)_{\sigma}\|_{L^2} \|[U_{\sigma,r}(Y + \tilde{h}) - U_{\sigma,r}(Y)] h\| \, d\sigma
\]

\[
+ \int_r^s \|A(Y + \tilde{h})_{\sigma} - A(Y)_{\sigma}\|_{L^2} \|U_{\sigma,r}(Y) h\| \, d\sigma. \tag{64}
\]

Recall (4), and in particular the uniform bound on \( x \mapsto D\nabla^2 V(x) \). Given \( x, \bar{x} \in \mathcal{X}^2 \), the mean value theorem implies that \( \nabla^2 V(x) - \nabla^2 V(\bar{x}) = (\int_0^1 D\nabla^2 V(\eta(x - \bar{x})) \, d\eta) (x - \bar{x}) \), so that \( \|\nabla^2 V(x) - \nabla^2 V(\bar{x})\|_{L^2(\mathcal{X}^2)} \leq \frac{d}{2} \|x - \bar{x}\| \) by (4). Hence, by (29), there exists an \( \alpha_1 \in \mathbb{R}_{\geq 0} \) such that \( \Lambda : \mathcal{X}^2 \to \mathcal{L}(\mathcal{X}^2) \) satisfies \( \|\Lambda(Z) - \Lambda(\bar{Z})\|_{L^2(\mathcal{X}^2)} \leq \alpha_1 \|Z - \bar{Z}\|_{L^2(\mathcal{X}^2)} \).
\[ \alpha_1 \| Z - \bar{Z} \| \text{ for all } Z, \bar{Z} \in \mathcal{X}^2. \] So, applying Lemma \[\text{[4]}\] there exists an \( \alpha \in \mathbb{R}_{\geq 0} \), \( L_0 = \sup_{\sigma \in [t, T]} \| A(0)\| \mathcal{X}(\mathcal{X}^2), L_1 = \alpha_1 \exp(\alpha (T - t)) \), such that

\[
\sup_{\sigma \in [t, T]} \| A(Y + \hat{h})\| \mathcal{X}(\mathcal{X}^2) \leq \alpha_1 \sup_{\sigma \in [t, T]} \| \bar{X}(Y + \hat{h}) - \bar{X}(Y)\| \leq L_1 \| \hat{h} \|, \tag{65}
\]

in which the second inequality follows from the first, via the triangle inequality, by selecting \( \hat{h} = -Y \). Note further that as \( \sigma \rightarrow A(Y) \) is continuous, \( L_2 = \sup_{\sigma \in [t, T]} \| U_{\sigma,t}(Y)\| \mathcal{X}(\mathcal{X}^2) < \infty \), see \[\text{[13]}\] Theorem 5.2, p.128. Hence, substituting these inequalities in (64) yields

\[
\| U_{s,r}(Y + \hat{h}) - U_{s,r}(Y)\| h \leq (L_0 + 2L_1 \| \hat{h} \|) \int_r^s \| U_{\sigma,r}(Y + \hat{h}) - U_{\sigma,r}(Y)\| h \| d\sigma + (T - t)L_1 \| L_2 \| \| \hat{h} \| \| h \|. \tag{66}
\]

Gronwall's inequality subsequently implies that

\[
\| U_{s,r}(Y + \hat{h}) - U_{s,r}(Y)\| h \leq (T - t)L_1 \| L_2 \| \| \hat{h} \| \| \hat{h} \| \exp((L_0 + 2L_1 \| \hat{h} \|)(T - t)) \implies \sup_{r,s \in [t, T]} \| U_{s,r}(Y + \hat{h}) - U_{s,r}(Y)\| \mathcal{X}(\mathcal{X}^2) \leq (T - t)L_1 \| L_2 \| \| \hat{h} \| \exp((L_0 + 2L_1 \| \hat{h} \|)(T - t)). \tag{67}
\]

Continuity of \( Y \rightarrow U_{s,r}(Y) \), uniformly in \( r, s \in [t, T] \), thus follows.

Now, we show that \( Y \rightarrow U_{s,r}(Y) \) is Fréchet differentiable, uniformly in \( r, s \in [t, T] \). Appealing to the contraction theorem and Picard's principle, for any \( t \leq r < s \leq T \) and \( Y \in \mathcal{X}^2 \) we consider the two-parameter family of operators \( V_{s,r}(y) \in \mathcal{L}(\mathcal{X}^2; \mathcal{X}(\mathcal{X}^2)) \) solving

\[
V_{s,r}(Y) \hat{h} h = \int_r^s A(Y) \sigma V_{s,r}(Y) \hat{h} h d\sigma + \int_r^s D Y A(Y) \sigma \hat{h} U_{\sigma,r}(Y) \hat{h} h d\sigma, \tag{67}
\]

for all \( h, \hat{h} \in \mathcal{X}^2, r, s \in [t, T] \), in which \( D Y A(Y) = DA(\bar{X}(Y)) \sigma U_{\sigma,t}(Y) \in \mathcal{L}(\mathcal{X}^2; \mathcal{X}(\mathcal{X}^2)) \) by the chain rule and Lemma \[\text{[14]}\]. Note in particular by \[\text{[4]}, \text{[29]}, \text{and Lemma } \text{[4]}\] that

\[
L_3 = \sup_{\sigma \in [t, T]} \| DA(\bar{X}(Y)) \| \mathcal{X}(\mathcal{X}^2; \mathcal{X}(\mathcal{X}^2)) < \infty.
\]

Applying the triangle inequality to (67), and recalling the definitions of \( L_0, L_1, L_2 \), yields

\[
\| V_{s,r}(Y) \hat{h} h \| \leq \int_r^s (L_0 + L_1 \| \hat{h} \|) \| V_{s,r}(Y) \hat{h} h \| d\sigma + \int_r^s L_3 \| h \| L_2 \| h \| d\sigma
\]

\[
\leq (T - t)L_2 L_3 \| \hat{h} \| \| h \| + (L_0 + L_1 \| \hat{h} \|) \int_r^s \| V_{s,t}(Y) \hat{h} h \| d\sigma,
\]

so that by Gronwall's inequality,

\[
\| V_{s,r}(Y) \hat{h} h \| \leq (T - t)L_2 L_3 \| \hat{h} \| \| h \| \exp\left((L_0 + L_1 \| \hat{h} \|) (T - t)\right).
\]

As \( \hat{h}, \hat{h} \in \mathcal{X}^2 \) are arbitrary, it follows immediately that \( V_{s,r}(Y) \in \mathcal{L}(\mathcal{X}^2; \mathcal{X}(\mathcal{X}^2)) \) for all \( r, s \in [t, T] \). Recalling (67), observe by adding and subtracting terms that

\[
[ U_{s,r}(Y + \hat{h}) - U_{s,r}(Y) - V_{s,r}(Y) \hat{h} ] h
\]

\[
= \int_r^s [ A(Y + \hat{h}) \sigma U_{\sigma,r}(Y + \hat{h}) - A(Y) \sigma U_{\sigma,r}(Y) ] h d\sigma - V_{s,r}(Y) \hat{h} h
\]

\[
= \int_r^s A(Y) \sigma [ U_{\sigma,r}(Y + \hat{h}) - U_{\sigma,r}(Y) - V_{\sigma,r}(Y) \hat{h} ] h d\sigma
\]

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\begin{align*}
&\quad + \int_r^s [A(Y + \hat{h}) - A(Y)] [U_{\sigma,r}(Y + \hat{h}) - U_{\sigma,r}(Y)] h \, d\sigma \\
&\quad + \int_r^s [A(Y + \hat{h}) - A(Y) - D_Y A(Y)\hat{h}] U_{\sigma,r}(Y) h \, d\sigma \\
&\quad - \left[ V_{s,r}(Y) \hat{h} h - \int_r^s A(Y) V_{s,r}(Y) \hat{h} \, h \, d\sigma - \int_r^s D_Y A(Y)\hat{h} U_{\sigma,r}(Y) h \, d\sigma \right],
\end{align*}
\number{68}

and the last term in square brackets is zero by definition \number{67} of \( V_{s,r}(Y) \). Define \( \hat{A} : C([t,T];\mathcal{X}^2) \to C([t,T];\mathcal{L}(\mathcal{X}^2)) \) by \( \hat{A}(X)_{\sigma} = A(X_{\sigma}) \) for all \( X \in C([t,T];\mathcal{X}^2) \), and note that the range of \( \hat{A} \) follows by \number{14}, \number{29}.

Combining \number{14}, \number{29} with the mean value theorem, there exists \( \hat{a} \in \mathbb{R}_{\geq 0} \) such that

\begin{align*}
\| \hat{A}(X + \delta) - \hat{A}(X) - DA(X) \delta \|_{C([t,T];\mathcal{L}(\mathcal{X}^2))} &= \sup_{\sigma \in [t,T]} \| A(X_{\sigma} + \delta_{\sigma}) - A(X_{\sigma}) - DA(X_{\sigma})\delta_{\sigma} \|_{\mathcal{L}(\mathcal{X}^2)} \\
&= \sup_{\sigma \in [t,T]} \left\| \left( \nabla^2 V([X_{\sigma} + \delta_{\sigma}]_1) - \nabla^2 V([X_{\sigma}]_1) - D\nabla^2 V([X_{\sigma}]_1) \delta_{\sigma} \right) \right\|_{\mathcal{L}(\mathcal{X}^2)} \\
&\leq \hat{a} \sup_{\sigma \in [t,T]} \left\| \nabla^2 V([X_{\sigma} + \delta_{\sigma}]_1) - \nabla^2 V([X_{\sigma}]_1) - D\nabla^2 V([X_{\sigma}]_1) \delta_{\sigma} \right\| \\
&\leq \hat{a} \sup_{\sigma \in [t,T]} \int_0^1 \| D^2 \nabla^2 V([X_{\sigma}]_1 + \eta \delta_{\sigma}]_1) \|_{\mathcal{L}(\mathcal{X}^2 \times \mathcal{X}^2;\mathcal{L}(\mathcal{X}^2))} d\eta \sup_{\sigma \in [t,T]} \| \delta_{\sigma} \|^2 \\
&\leq \frac{\hat{a} \hat{K}}{2} \| \delta \|_{\infty}^2,
\end{align*}
\number{69}

for all \( X, \delta \in \mathcal{X}^2 \). Dividing both sides by \( \| \delta \|_{\infty} \) and taking the limit as \( \| \delta \|_{\infty} \to 0 \) subsequently yields that \( \hat{A} \) is Fréchet differentiable with derivative \( DA(X) \in \mathcal{L}(C([t,T];\mathcal{X}^2); C([t,T];\mathcal{L}(\mathcal{X}^2))) \).

Hence, taking the norm of both sides of \number{68}, applying the triangle inequality, \number{66}, \number{69}, and recalling the definitions of \( L_1, L_2, L_3, \)

\begin{align*}
&\| U_{s,r}(Y + \hat{h}) - U_{s,r}(Y) - V_{s,r}(Y) \hat{h} \| \\
&\quad \leq (L_0 + L_1 \| \hat{h} \|) \int_r^s \| U_{s,r}(Y + \hat{h}) - U_{s,r}(Y) - V_{s,r}(Y) \hat{h} \| \, d\sigma \\
&\quad + (T - t)^2 L_2^2 L_2 \| \hat{h} \|^2 \exp((L_0 + 2 L_1 \| \hat{h} \|)(T - t)) \| \hat{h} \| \\
&\quad + (T - t) L_2 \| A \circ \overline{X}(Y + \hat{h}) - A \circ \overline{X}(Y) - D\overline{A}(Y) \|_{C([t,T];\mathcal{L}(\mathcal{X}^2))} \| \hat{h} \| \\
&\quad = (L_0 + L_1 \| \hat{h} \|) \int_r^s \| U_{s,r}(Y + \hat{h}) - U_{s,r}(Y) - V_{s,r}(Y) \hat{h} \| \, d\sigma \\
&\quad + (T - t)^2 L_2^2 L_2 \| \hat{h} \|^2 \exp((L_0 + 2 L_1 \| \hat{h} \|)(T - t)) \| \hat{h} \| \\
&\quad + (T - t) L_2 \| d(\hat{A} \circ \overline{X})_Y(\hat{h}) \|_{C([t,T];\mathcal{L}(\mathcal{X}^2))} \| \hat{h} \| \| \hat{h} \|,
\end{align*}

in which \( d(\hat{A} \circ \overline{X})_Y(\cdot) \) is defined via \number{14}. Hence, by Gronwall’s inequality,

\begin{align*}
&\| U_{s,r}(Y + \hat{h}) - U_{s,r}(Y) - V_{s,r}(Y) \hat{h} \| \\
&\quad \leq (T - t) L_2 \left[ (T - t) L_2 \| \hat{h} \| \exp((L_0 + 2 L_1 \| \hat{h} \|)(T - t)) \right. \\
&\quad \left. + \| d(\hat{A} \circ \overline{X})_Y(\hat{h}) \|_{C([t,T];\mathcal{L}(\mathcal{X}^2))} \| \hat{h} \| \right] \| \hat{h} \| \\
&\quad \times \exp((L_0 + L_1 \| \hat{h} \|)(T - t)).
\end{align*}

As \( \hat{h}, \hat{h} \in \mathcal{X}^2 \) are arbitrary,

\begin{align*}
\lim_{\| \hat{h} \| \to 0} \frac{\sup_{r,s \in [t,T]} \| U_{s,r}(Y + \hat{h}) - U_{s,r}(Y) - V_{s,r}(Y) \hat{h} \|_{\mathcal{L}(\mathcal{X}^2)}}{\| \hat{h} \|}.
\end{align*}
\number{70}
Hence, $Y \mapsto U_{s,r}(Y)$ is Fréchet differentiable, uniformly in $r,s \in [t,T]$, with derivative $V_{s,r}(Y)$. An analogous argument to (63), (64) applied to (67) can be applied to show that $Y \mapsto V_{s,r}(Y)$ is continuous, uniformly in $r,s \in [t,T]$, and the details are omitted.

**Remark A.1.** We remark that, from the smoothness of $V$, we can compute the second order Fréchet derivative of $Y \mapsto U_{s,r}(Y)$. Indeed, using a first order expansion, we have that $A(z + \delta) = A(z) + D_A(z)(\delta) + o(\|\delta\|)$, where $O(.)$ stands for the second order integral rest. Now, fix $r > 0$. In the same way as in (63), we have, putting $U_Y(s) := U_{s,r}(Y)$, for all $s \in [r,T]$

$$
\dot{U}_{Y+h} (s) - \dot{U}_Y (s) = A(Y)(s)(U_{Y+h} (s) - U_Y (s)) + DA(Y)(s)(\dot{h})U_{Y+h} (s) + o(|\dot{h}|)(s)U_{Y+h} (s).
$$

So, letting $\xi(.) = U_{Y+h} (s) - U_Y (s)$, $A(.) = A(Y)(.)$, $Q(.) = DA(Y)(.)(\dot{h})$, $v(.) = U_{Y+h} (s)$, and $R(.) = o(|\dot{h}|)(.)U_{Y+h} (s)$, the previous equation reduce to the ODE $\dot{\xi} = A\xi + Qv + R$, solvable via standard tools. Then, we have that $\xi(s) = X(r) \dot{h} + \int_r^s X(s)X(t)^{-1}Q(t)v(t)dt + \int_r^s X(s)X(t)^{-1}R(t)dt$ where $X(.)$ is the fundamental solution of $\dot{X} = AX$ a.e. on $[r,T]$, $X(r) = I$. We have that $\int_r^s X(s)X(t)^{-1}R(t)dt$ is closed to zero for $\dot{h} \to 0$ (uniformly), and hence $X(r)\dot{h} + \int_r^s X(s)X(t)^{-1}Q(t)v(t)dt$ provide the derivative in (67). Moreover, it is possible to iterate such argument, in order to compute high order Fréchet derivatives of $Y \mapsto U_{s,r}(Y)$, as times as the potential $V$ is differentiable.

## B Auxiliary statement of Proposition 4.8.

**Proposition B.1.** Given $T \in \mathbb{R}_{>0}$, $t \in [0,T)$, $x,p \in \mathcal{X}$, and $(\tilde{x}_s, \tilde{p}_s) = \overline{X}(Y_p(x))_s$ for all $s \in [t,T]$, the maps $s \mapsto \nabla_p \overline{J}_T(s, \tilde{x}_s, \tilde{p}_s)$ and $s \mapsto \nabla_{\tilde{x}} \overline{J}_T(s, \tilde{x}_s, \tilde{p}_s)$ are continuously differentiable, with derivatives given by

\[
\frac{d}{ds} \left[ \nabla_p \overline{J}_T(s, \tilde{x}_s, \tilde{p}_s) \right] = -\mathcal{M}^{-1} \left( \tilde{p}_s - \nabla_{\tilde{x}} \overline{J}_T(s, \tilde{x}_s, \tilde{p}_s) \right),
\]

\[
\frac{d}{ds} \left[ \nabla_{\tilde{x}} \overline{J}_T(s, \tilde{x}_s, \tilde{p}_s) \right] = \nabla V(\tilde{x}_s) - \nabla^2 V(\tilde{x}_s) \nabla_p \overline{J}_T(s, \tilde{x}_s, \tilde{p}_s),
\]

for all $s \in (t,T)$. Moreover, $s \mapsto \nabla_p \overline{J}_T(s, \tilde{x}_s, \tilde{p}_s)$ is twice continuously differentiable, and satisfies

\[
0 = \frac{d^2}{ds^2} \left[ \nabla_p \overline{J}_T(s, \tilde{x}_s, \tilde{p}_s) \right] + \mathcal{M}^{-1} \nabla^2 V(\tilde{x}_s) \nabla_p \overline{J}_T(s, \tilde{x}_s, \tilde{p}_s),
\]

for all $s \in (t,T)$.

**Proof.** Fix $T \in \mathbb{R}_{>0}$, $x,p \in \mathcal{X}$, and let $(\tilde{x}_s, \tilde{p}_s) \in \mathcal{X}^2$, $s \in [t,T]$, be as per the lemma statement. Fix $h \in \mathcal{X}$. Applying Proposition 4.7 $(x,p) \mapsto \overline{J}_T(s,x,p)$ is twice continuously differentiable, and the order of differentiation may be swaped. In particular,

\[
\frac{d}{ds} \left[ D_p \overline{J}_T(s, \tilde{x}_s, \tilde{p}_s) h \right] = \frac{\partial}{\partial s} D_p \overline{J}_T(s, \tilde{x}_s, \tilde{p}_s) h + \frac{\partial}{\partial \tilde{x}} D_p \left[ \overline{J}_T(s, \tilde{x}_s, \tilde{p}_s) h \right] h_{\tilde{x}} + \frac{\partial}{\partial \tilde{p}} \left[ D_p \overline{J}_T(s, \tilde{x}_s, \tilde{p}_s) h \right] h_{\tilde{p}}
\]

\[
= \frac{\partial}{\partial s} D_p \overline{J}_T(s, \tilde{x}_s, \tilde{p}_s) h + \frac{\partial}{\partial \tilde{x}} D_p \left[ \overline{J}_T(s, \tilde{x}_s, \tilde{p}_s) \right] h_{\tilde{x}} + \frac{\partial}{\partial \tilde{p}} \left[ D_p \overline{J}_T(s, \tilde{x}_s, \tilde{p}_s) \right] h_{\tilde{p}} h.
\]

Meanwhile, $\overline{J}_T$ satisfies (15) by Theorem 4.10 i.e.

\[
0 = -\frac{\partial}{\partial \tilde{x}} \overline{J}_T(s, x, p) - \frac{1}{2} \langle p, \mathcal{M}^{-1} p \rangle + V(x) + \frac{\partial}{\partial \tilde{x}} \overline{J}_T(s, x, p) \mathcal{M}^{-1} p - D_p \overline{J}_T(s, x, p) \nabla V(x),
\]

for all $s \in (t,T)$, $x, p \in \mathcal{X}$. Differentiating (75) with respect to $p$,

\[
0 = -D_p \frac{\partial}{\partial \tilde{x}} \overline{J}_T(s, x, p) \mathcal{M}^{-1} p + D_p \frac{\partial}{\partial \tilde{x}} \overline{J}_T(s, x, p) \mathcal{M}^{-1} p + D_p \overline{J}_T(s, x, p) \mathcal{M}^{-1} h - D_p h \overline{J}_T(s, x, p) \nabla V(x) h
\]
Recalling that $\bar{h}$ is arbitrary immediately yields (71).

Similarly, for (72), observe that

\[
\frac{d}{ds}[D_x \bar{J}(s, \bar{x}, \bar{p})] = \frac{d}{ds} D_x \bar{J}(s, \bar{x}, \bar{p}) h + D_x [D_x \bar{J}(s, \bar{x}, \bar{p}) h] \hat{x} + D_p [D_x \bar{J}(s, \bar{x}, \bar{p}) h] \hat{p} \\
= \frac{d}{ds} D_x \bar{J}(s, \bar{x}, \bar{p}) h + (D_x [D_x \bar{J}(s, \bar{x}, \bar{p})]) \hat{x} + (D_p [D_x \bar{J}(s, \bar{x}, \bar{p})]) \hat{p} h \\
= [D_x \frac{d}{ds} \bar{J}(s, \bar{x}, \bar{p})] + D_x D_x \bar{J}(s, \bar{x}, \bar{p}) \hat{x} + D_p D_x \bar{J}(s, \bar{x}, \bar{p}) \hat{p} h.
\]

Evaluating along the trajectory $s \mapsto (\bar{x}, \bar{p})$ yields

\[
[D_x \frac{d}{ds} \bar{J}(s, \bar{x}, \bar{p})] \hat{x} + D_x D_x \bar{J}(s, \bar{x}, \bar{p}) \hat{x} + D_p D_x \bar{J}(s, \bar{x}, \bar{p}) \hat{p} h = \langle \nabla V(\bar{x}) - \nabla^2 V(\bar{x}) \nabla p \bar{J}(s, \bar{x}, \bar{p}), h \rangle.
\]

Hence, substitution in (74) yields

\[
\langle \frac{d}{ds}[\nabla p \bar{J}(s, \bar{x}, \bar{p})], h \rangle = \langle \nabla V(\bar{x}) - \nabla^2 V(\bar{x}) \nabla p \bar{J}(s, \bar{x}, \bar{p}), h \rangle.
\]

Recalling that $h \in \mathcal{B}^\ast$ is arbitrary immediately yields (71).

The remaining assertion regarding twice differentiability is immediate by inspection of (71), (72), with

\[
\frac{d^2}{ds^2}[\nabla p \bar{J}(s, \bar{x}, \bar{p})] = -\mathcal{M}^{-1} \left( \hat{p} - \frac{d}{ds}[\nabla p \bar{J}(s, \bar{x}, \bar{p})] \right)
= -\mathcal{M}^{-1} \left( \nabla V(\bar{x}) - [\nabla V(\bar{x}) - \nabla^2 V(\bar{x}) \nabla p \bar{J}(s, \bar{x}, \bar{p})] \right)
= -\mathcal{M}^{-1} \nabla V(\bar{x}) \nabla^2 V(\bar{x}) \nabla p \bar{J}(s, \bar{x}, \bar{p}),
\]

as required.
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