LINE, SPIRAL, DENSE

NEIL DOBBS

Abstract. Exponential of exponential of almost every line in the complex plane is dense in the plane. On the other hand, for lines through any point, for a set of angles of Hausdorff dimension one, exponential of exponential of a line with angle from that set is not dense in the plane.

In 1914, Harald Bohr and Richard Courant showed that for the Riemann zeta function, if $\sigma \in (\frac{1}{2}, 1]$, then $\zeta(\sigma + it) = C$, i.e. the image of any vertical line with real part in $(\frac{1}{2}, 1]$ is dense [1 §4, p.271]. One hundred years on, we ask what happens under the exponential map.

The exponential map, denoted $\exp : z \mapsto e^z$, maps vertical lines to circles centred on 0 and horizontal lines to rays emanating from 0. All other lines get mapped to logarithmic spirals. Circles are compact, so their images are compact. Rays are subsets of lines, so they get mapped into circles, rays or logarithmic spirals. The image of a logarithmic spiral is not immediately obvious and for good reason.

For $y \in \mathbb{C}, \alpha \in \mathbb{R}$, let $L_\alpha(y) := \{y + t(i+\alpha) : t \in \mathbb{R}\}$. Set $L(y) := \{L_\alpha(y) : \alpha \in \mathbb{R}\}$, the family of non-horizontal lines through a point $y \in \mathbb{C}$, parametrised by $\alpha \in \mathbb{R}$.

With this parametrisation, there is a natural one-dimensional Lebesgue measure on the set $L(y)$. It is equivalent to the measure obtained when parametrising the family by angle (points on the half-circle).

**Theorem 1.** For each $y \in \mathbb{C}$, for Lebesgue almost every $\alpha \in \mathbb{R}$,

$$\exp \circ \exp(L_\alpha(y)) = \mathbb{C}.$$ 

On the other hand, we have the following.

**Theorem 2.** For each $y \in \mathbb{C}$ and each open set $X \subset \mathbb{R}$, the set $\{\alpha \in X : \exp \circ \exp(L_\alpha(y)) \neq \mathbb{C}\}$ has Hausdorff dimension 1.

From the topological perspective, a property is generic in some space if it holds for all points in a residual set, that is, a set which can be written as a countable intersection of open, dense sets.

**Theorem 3.** For each $y \in \mathbb{C}$, the set $\{\alpha \in \mathbb{R} : \exp \circ \exp(L_\alpha(y)) = \mathbb{C}\}$ is residual.

We shall use $\Re(z)$ and $\Im(z)$ to denote the real and imaginary parts of a complex number $z$. We denote one-dimensional Lebesgue measure by $m$ and denote the length of an interval $I$ by $m(I)$ or by $|I|$.

Remark: Given a vertical line and its intersections with a logarithmic spiral, if one perturbs the spiral parameter, one gets intervals of intersections with the vertical line. Successive intervals are bigger (see Figure 2). If one quotients the vertical line

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by $2\pi i$, something resembling successive images under iteration by an expanding map of an interval (like $x \mapsto \beta x \pmod{1}$, for $\beta > 1$, say) appears. For expanding maps of the interval with certain properties, almost every orbit will be dense while the dimension of points with non-dense orbits will be 1. This is mostly just an analogy, there is no dynamical system in this paper. Nonetheless, the point of view underlies the proofs.

1. Proof of Theorem 1

Proof. Let $f$ denote $\exp \circ \exp$. Fix $y$ and write $L_\alpha$ for $L_\alpha(y)$. Let

$$X_U := \{ \alpha : f(L_\alpha) \cap U \neq \emptyset \}. \tag{1}$$

Given a sequence $(p_n)_{n=1}^\infty$ dense in $\mathbb{C}$ and a decreasing sequence of positive reals $(\delta_n)_{n=1}^\infty$ with $\delta_n \to 0^+$, let $\mathcal{U} := \{ B(p_n, \delta_n) : n \geq 1 \}$. Then a set is dense in $\mathbb{C}$ if and only if it has non-empty intersection with each $U \in \mathcal{U}$. Since $\mathcal{U}$ is countable, if for each $U \in \mathcal{U}$, $X_U$ has full measure, then $X_\infty := \bigcap_{U \in \mathcal{U}} X_U$ has full measure as a countable intersection of full-measure sets. Of course, for each $\alpha \in X_\infty$, $f(L_\alpha)$ is dense in $\mathbb{C}$.

Thus proving Theorem 1 reduces to showing that for any open ball $U$, $X_U$ has full measure. We say a point $x$ is an $\varepsilon$-density point for a set $X \subset \mathbb{R}$ if $\lim_{r \to 0^+} \frac{m(X \cap B(x, r))}{m(B(x, r))} \geq \varepsilon$. By the Lebesgue density point theorem, almost every point of $X$ is a 1-density point for $X$. On the other hand, if $\varepsilon > 0$ and almost every point in $\mathbb{R}$ is an $\varepsilon$-density point for a set $X \subset \mathbb{R}$, then the set of 1-density points for the complement of $X$ has zero measure, so the complement has zero measure, so $X$ must have full measure. It therefore suffices to prove that, given a ball $U$, there exists $\varepsilon > 0$ such that each $\alpha_0 \in \mathbb{R} \setminus \{0\}$ is an $\varepsilon$-density point for $X_U$. So let us do this.

Let $V := \exp^{-1}(U)$. Then $V$ is an open set. Let $H$ be a vertical line, with real part $h \neq 0$, which intersects $V$, see Figure 1. Since $\exp$ is $2\pi i$-periodic, $H \cap V$ contains an open interval $I$ and all $2\pi i$-translates of $I$. In particular, for any subinterval $T \subset H$ of length at least $2\pi$,

$$m(T \cap V)/m(T) \geq m(I)/4\pi. \tag{2}$$

Now consider $S = \exp^{-1}(H)$. If $h > 0$ then one connected component of $S$, $S_0$ say, can be parametrised by

$$\gamma_+ : t \mapsto \frac{1}{2} \log(t^2 + h^2) + i \arctan \frac{t}{h}$$

with $\gamma_+(\mathbb{R}) = S_0$. If $h < 0$ then $S_0$ can be parametrised by $\gamma_- : t \mapsto \pi i + \gamma_+(t)$. Taking the derivative of $\gamma_+$ and $\gamma_-,$

$$\gamma'_+(t) = \gamma'_-(t) = \frac{t}{t^2 + h^2} + i \frac{h}{t^2 + h^2},$$

so the slope of $\gamma_\pm$ tends to 0 as $|t| \to \infty$. For $k \in \mathbb{Z}$, if $\alpha_0 > 0$ let $S_k := S_0 + 2k\pi i$; otherwise let $S_k := S_0 - 2k\pi i$. Then $S_k$ for $k \in \mathbb{Z}$ are the connected components of $S$.

Let $W_k := S_k \cap \exp^{-1}(V)$. The absolute value of the derivative of $\exp$ on $S$ is bounded below by $|h| > 0$, so any segment of $S_k$ of length at least $2\pi/|h|$ gets mapped onto a segment of $H$ of length at least $2\pi$. The distortion of $\exp$
(by distortion, we mean the ratio of the absolute value of the derivative at any two points) is bounded by $e^{4\pi/|h|}$ on each vertical strip of width $4\pi/|h|$. By the distortion bound and (2), for any segment $B$ of $S_k$ of length between $2\pi/|h|$ and $4\pi/|h|$,  

$$
\frac{m(B \cap W_k)}{m(B)} \geq \frac{m(\exp(B) \cap V)}{m(\exp(B)) e^{4\pi/|h|}} \geq \frac{m(I)}{4\pi e^{4\pi/|h|}}.
$$

Any segment $B$ of $S_k$ of length at least $2\pi/|h|$ can be divided into segments of length between $2\pi/|h|$ and $4\pi/|h|$, so (3) continues to hold for all segments $B$ of $S_k$ of length at least $2\pi/|h|$.

Let $\xi : \alpha \mapsto y + i + \alpha$. Let $\alpha_0 \in \mathbb{R} \setminus \{0\}$ and let $r_0 = |\alpha_0|/2$. For $r \in (0, r_0)$, let $J_r := \xi(B(\alpha_0, r))$ be the open line segment joining the points $y + i + \alpha_0 - r$ and $y + i + \alpha_0 + r$. For some $K \geq 1$, for each $k \geq K$, for each $\alpha \in B(\alpha_0, r_0)$, $L_\alpha$ intersects $S_k$ transversely (twice). For $k \geq K$, let $\phi_k$ denote the central projection with respect to $y$ from $J_{r_0}$ to $S_k$ (taking the first point of intersection). For some $K_0 > K$ and each $k \geq K_0$, $\phi_k(J_{r_0})$ is almost horizontal and the distortion of $\phi_k$ on $J_{r_0}$ is close to 1, in particular it is bounded by 2. Now simple geometry entails that
exp \frac{m(\phi_k(J_r))}{\pi kr} \to 1 as \ k \to \infty so, for each r \in (0, r_0), there exists k_r \geq K_0 with \ m(\phi_k(J_r)) > 2\pi |h|. Let \ X_r := J_r \cap \phi_k^{-1}(W_{k_r}). From (3) and the distortion bound of 2, we deduce that \ m(X_r)/m(J_r) \geq \varepsilon, for \ \varepsilon := \frac{m(I)}{8\pi e^{4\pi/|h|}}. For \ \alpha \in \xi^{-1}(X_r), L_\alpha \cap W_{k_r} \neq \emptyset so f(L_\alpha) \cap U \neq \emptyset. In particular, \ \xi^{-1}(X_r) \subset X_U and \ \frac{m(\xi^{-1}(X_r))}{m(f(L_\alpha \cap U))} \geq \varepsilon.

Noting that \ \varepsilon\ depends only on \ U \ and \ h, we have shown that \ \alpha_0 \ is an \ \varepsilon\-density point for \ X_U for each \ \alpha_0 \in \mathbb{R} \setminus \{0\}. □

2. Proof of Theorem 2

The Mass Distribution Principle is a standard source of lower bounds for the Hausdorff dimension. It is infused into the following lemma.

Lemma 4. Let \ J \ be a non-degenerate interval, let \ Y \subset J \ and let \ \mu \ be a measure with \ \mu(Y) > 0. For each \ n \geq 1, let \ \mathcal{P}_n \ be a finite partition of \ J \ into intervals, each of length at most \ 2^{-n}. Let \ \varepsilon \in (0,1), let \ \beta > 1 \ and suppose

\begin{equation}
\mu(P) \leq \beta(1+\varepsilon)^n|P|
\end{equation}

for every \ P \in \mathcal{P}_n. Then the Hausdorff dimension of \ Y \ is at least \ 1 - 2\varepsilon.

Proof. For \ r \in (0, 1), let \ n := \lfloor -\log_2 r \rfloor. Let \ x \in J. If \ P \in \mathcal{P}_n \ then \ |P| \leq 2^{-n} \leq r, so if \ P \cap B(x, r) \neq \emptyset \ then \ P \subset B(x, 2r). The total length of elements of \ \mathcal{P}_n \ intersecting \ B(x, r) \ is thus at most \ 4r. Summing (4) over such elements, we deduce that

\begin{equation}
\mu(B(x, r))/\beta \leq 4r(1+\varepsilon)^n \leq 4r(1+\varepsilon)e^{-\log(1+\varepsilon)\log r}/\log 2 \leq 8r^{1-\log(1+\varepsilon)}/\log 2.
\end{equation}

Now \ \log 2 > 1/2 \ and \ \log(1+\varepsilon) < \varepsilon, so \ \mu(B(x, r))/\beta \leq r^{1-2\varepsilon}. If \ U_1, U_2, \ldots \ is any countable cover of \ Y \ by balls of radius at most 1, then \ \sum_{j \geq 1} |U_j|^{1-2\varepsilon} \geq \sum_{j \geq 1} \mu(U_j)/\beta \geq \mu(Y)/\beta > 0. Since this positive lower bound does not depend on the cover, the Hausdorff dimension of \ Y \ is at least \ 1 - 2\varepsilon, as required. □

Together with the following lemma, one can glean an insight into the means of proving Theorem 2.

Lemma 5. Let \ Y \ be a compact subset of \ \mathbb{R}. Let \ I \ be a subinterval of the imaginary axis with \ |I| < 1 \ and let \ \tilde{I} := \bigcup_{k \in \mathbb{Z}} (2k\pi i + I) \ be the union of all \ 2\pi i\-translates of \ \tilde{I}. Suppose \ \tilde{I} \ is disjoint from \ B(0, 1). Let \ y \in \mathbb{C}. Suppose that \ \exp(L_\alpha(y)) \cap \tilde{I} = \emptyset \ for every \ \alpha \in Y. Then there is an open set \ U \ with \ \exp \circ \exp(L_\alpha(y)) \cap U = \emptyset \ for each \ \alpha \in Y.

Proof. Deriving \ t \mapsto \exp(y+it(i+\alpha)) \ gives \ (i+\alpha) \exp(y+it(i+\alpha)). Thus \ \exp(L_\alpha(y)) has slope \ -\alpha \ at each intersection with the imaginary axis. Moreover, since \ Y \ is bounded, there is a constant \ C > 1 \ such that the slope of \ \exp(L_\alpha(y)) \ is bounded in absolute value by \ C \ in the region \ \{z: |\Re(z)| < 1/2, |\Im(z)| > 1/2\}. Let \ D \ denote the body of the rhombus with diagonal \ I \ and sides of slope \ \pm C, and \ \tilde{D} \ the union of all \ 2\pi i\-translates of \ D. Then \ \exp(L_\alpha(y)) \cap \tilde{D} = \emptyset \ for each \ \alpha \in Y. Let \ x \ be the midpoint of \ I \ and denote by \ U \ the open set \ \exp(B(x, |I|/4C)). By construction, \ B(x, |I|/4C) \subset D \ so \ \exp^{-1}(U) \subset \tilde{D}. Thus \ \exp \circ \exp(L_\alpha(y)) \cap U = \emptyset \ for each \ \alpha \in Y, as required. □

Now we can prove Theorem 2, which states that for each \ y \in \mathbb{C} \ and each open set \ X \subset \mathbb{R}, the set \ \{\alpha \in X: \exp \circ \exp(L_\alpha(y)) \neq \mathbb{C}\} \ has Hausdorff dimension 1. 
\[ \psi_k(J), \psi_{k+1}(J), \psi_{k+2}(J) \text{ of the imaginary axis.} \]

**Proof.** We can assume \( 0 \notin X \). Writing \( \sigma \) for the map sending points to their complex conjugates, \( \exp \circ \sigma = \sigma \circ \exp \) and \( \sigma(L_\alpha(y)) = L_{-\alpha}(\sigma(y)) \) so, without loss of generality (replacing \( y \) by \( \sigma(y) \) and \( X \) by \( -X \), if necessary), one can assume \( X \subset \mathbb{R}^+ \).

Given \( X \) and \( y \), let \( X' = (\alpha_0, \alpha_1) \) be a non-degenerate subinterval of \( X \) with \( 0 < \alpha_0 < \alpha_1 \). Let \( \xi : \alpha \mapsto y + i + \alpha \) and let \( J \) be the line segment \( \xi(X') \). For \( k \in \mathbb{Z} \), let \( S_k := (k + \frac{1}{2})\pi i + \mathbb{R} \). Then \( \exp(S_k) \) is a vertical ray leaving 0, heading up if \( k \) is even and down if \( k \) is odd. Let \( \phi_k \) be the central projection from \( y \) onto \( S_k \), so

\[
\phi_k(y + i + \alpha) = \Re(y) + \left( (k + \frac{1}{2})\pi - \Im(y) \right) \alpha + i \left( (k + \frac{1}{2})\pi - \Im(y) \right). \]

In particular, as a map from \( J \) to \( S_k \), \( \phi_k \) is affine with derivative \( D\phi_k(z) = (k + \frac{1}{2})\pi - \Im(y) \) for every \( z \in J \). There exists a \( k_0 \in \mathbb{Z} \) such that, for all \( k \leq k_0 \), \( \phi_k(J) \subset \{ z : \Re(z) < 0 \} \), and thus, for \( k \leq k_0 \), \( \exp \circ \phi_k(J) \subset B(0,1) \). Writing \( \psi_k := \exp \circ \phi_k \) on \( J \), \( \psi_k \) maps \( J \) onto a subinterval of the imaginary axis, see Figure 2. We have \( |D\psi_k| = |D \exp(\phi_k)||D\phi_k| = |D\phi_k|\exp(\Re(\phi_k)) \), so

\[
|D\psi_k(y + i + \alpha)| = ( (k + \frac{1}{2})\pi - \Im(y) ) e^{\Re(y)} e^{\left( \frac{\pi}{2} - \Im(y) \right) \alpha} e^{k\pi \alpha}. \]

**Figure 2.** Two logarithmic spirals \( \exp(L_{\alpha_0}(y)) \) and \( \exp(L_{\alpha_1}(y)) \), drawn with \( y = 0 \) and the increasing (in length) subintervals \( \psi_k(J), \psi_{k+1}(J), \psi_{k+2}(J) \) of the imaginary axis.
Thus for \( k > |3(y)|/\pi, \)
\[
|D\psi_{k+1}(y + i + \alpha)/D\psi_k(y + i + \alpha)| > e^{\alpha\pi}.
\]

Moreover, there exists \( C \in (0, 1) \) such that, for each \( k \geq k_0 \) with \( y \notin S_k, \)
\[
|D\psi_k| > C.
\]

Let an integer \( N > 2|k_0| + 8\pi \) be large enough that
\[
\begin{align*}
& N\pi > 2|3(y)|; \\
& e^{N\pi\alpha_0/2} > N^6; \\
& N e^{R(y)} > 1; \\
& 1/N^2 < |J|.
\end{align*}
\]

By \( \textcircled{5} \) and choice of \( N, \) for all \( z \in J, \)
\[
|D\psi_N(z)| > (N\pi/2)e^{R(y)}e^{N\pi\alpha_0/2} > N^6.
\]

From \( \textcircled{6} \) and choice of \( N, \) we obtain
\[
|D\psi_{n+1,N}(z)/D\psi_{n,N}(z)| > N^6
\]
for each \( n \geq 1 \) and \( z \in J. \)

Let \( M := \sup_{z \in J} |D\psi_N(z)|, \) so for any subinterval \( J' \subset J, |\psi_N(J')| \leq M|J'|. \)

Let \( I \) be an open subinterval of the imaginary axis of length \( N^{-8}C/M \) whose \( 2\pi i \)-translates are disjoint from \( \exp(y) \) and from \( B(0, 1). \)

Let \( \hat{I} := \bigcup_{k \in \mathbb{Z}} (2k\pi i + I). \)

For \( k \leq k_0, \psi_k(J) \subset B(0, 1), \) so \( \psi_k(J) \cap \hat{I} = \emptyset. \)

Let \( J' \) be a subinterval of length \( 1/N^3. \)

For \( k = k_0 + 1, \ldots, N, |\phi_k(J')| < |J'|(N + 1/2)\pi - 3(y) < N^{-3}(N+1)\pi/2 < 1/N. \)

Hence the distortion of \( \psi_k \) is bounded by \( e^{1/N} < 2. \)

For \( k = k_0 + 1, \ldots, N, \) \( |\psi_k(J')| \leq |\psi_N(J')| \) and by \( \textcircled{8} \), \( |\psi_N(J')| > N^6/N^3 > 1. \)

The number of connected components of \( \hat{I} \) intersecting \( \psi_k(J') \) is bounded by \( |\psi_N(J')| \) for \( k \leq N; \) consequently \( m(\hat{I} \cap \psi_k(J')) \leq |\psi_N(J')||I|. \)

Let \( J_n \) be the set of points \( z \in J' \) for which \( \psi_k(z) \notin \hat{I} \) for all \( k \leq n. \)

Note that \( J_{k_0} = \emptyset. \)

Using \( \textcircled{7} \) and then choice of \( M \) and \( I, \)
\[
m(J_N) = |J'| - m\left( J' \cap \bigcup_{k=k_0+1}^N \psi_k^{-1}(\hat{I}) \right)
\]
\[
\geq |J'| - (N - k_0)|\psi_N(J')||I|/C
\]
\[
> |J'| - (N - k_0)|J'|N^{-8}
\]
\[
> |J'|/2,
\]
say. Meanwhile, \( J_N \) has at most \( (N - k_0)|\psi_N(J')|+1 \) connected components. Therefore, at least one connected component \( Z \) of \( J_N \) must satisfy \( |Z| > |J'|/3(N - k_0)|\psi_N(J')| \) and, more importantly (by the distortion bound of 2), \( |\psi_N(Z)| > 1/6(N - k_0) > 1/N^3. \)

We claim there is a finite collection of pairwise-disjoint subintervals \( \{A_i\}_I \) of \( V \) with the following properties: For each \( i, \)
\[
\begin{align*}
& |A_i| \leq |V|/2; \\
& \psi_k(A_i) \cap \hat{I} = \emptyset \text{ for } k = nN + 1, \ldots, (n + 1)N; \\
& |\psi_{(n+1)N}(A_i)| > 1/N^3; \\
& m(\bigcup_i A_i)/m(V) > 1 - 1/N.
\end{align*}
\]
Let us show the claim. Note that since $\psi_{n,N}(V) \cap \hat{I} = 0$, $|\psi_{n,N}(V)| \leq 2\pi$. Thus $\phi_{n,N}(V)$ has length (trivially) bounded by $\log 2$ and the distortion of $\psi_{n,N}$ on $V$ is bounded by 2. Divide $V$ into $N^2$ subintervals $\{W_j\}_{j=1}^{N^2}$ of equal length. From the distortion bound for $\psi_{n,N}$, it follows that $|\psi_{n,N}(W_j)| > |\psi_{n,N}(V)|/2N^2 > 1/2N^3$. By \( \square \), each $\psi_k(W_j)$, for $k \geq nN$, has length at least 2. Hence

\[ m(\hat{I} \cap \psi_k(W_j))/|\psi_k(W_j)| \leq 2N^5/|I| \]

for $k \geq nN$. Now $|D\phi_k|/|D\phi_{n,N}| = ((k + \frac{1}{2})\pi - 3(y))/((nN + \frac{1}{2})\pi - 3(y)) < 4$ for $k = nN + 1, \ldots, (n + 1)N$, and so $|\phi_k(W_j)|$ is bounded by $4|\phi_{n,N}(V)|/N^2 \leq 8\pi/N < 1/N$. Therefore the distortion of $\psi_k$ on $W_j$ is bounded by $e^{\pi/N} < 2$ for $k = nN + 1, \ldots, (n + 1)N$. We deduce from this and \( \square \) that the set $Z_j$ of those $z \in W_j$ for which $\psi_k(z) \notin \hat{I}$ for any $k = nN+1, \ldots, (n+1)N$ satisfies $m(Z_j)/|W_j| \geq 1 - 4N_6^0/|I|$. Meanwhile by \( \square \), $|\psi_{n,N}(W_j)| \geq N_6^0|W_j|/2N_6^0 = N/2$. The set $Z_j$ has at most $N(|\psi_{n,N}(W_j)| + 1)$ connected components, and those of length bounded by $2|W_j|/\psi_{n,N}(W_j)|N^3$ have measure bounded by $2|W_j|/N^2 < |W_j|/2N$, see Figure \( \square \). Let $\{A_j^I\}_I$ denote the collection of connected components of $Z_j$ of length at least $2|W_j|/\psi_{n,N}(W_j)|N^3$. These have combined measure at least $m(Z_j) - |W_j|/2N$. Each $A_j^I$ gets mapped by $\psi_{n,N}(W_j)$ onto an interval of length at least $1/3N^3$, by bounded distortion. Combining our estimates,

\[ m\left(\bigcup_i A_j^I\right)/|W_j| \geq m(Z_j) - |W_j|/2N \geq 1 - 4N^6|I| - 1/2N > 1 - 1/N, \]

noting $|I| \leq 1/N^8$. The collection $\{A_j^I\}_{j,I}$ satisfies the claim.

From the claim, we deduce, by induction, the existence of a sequence $(W'_n)_{n \geq 1}$ of finite collections of pairwise-disjoint subintervals of $J'$, with $W'_1 = \{Z\}$ as before, with the following properties: For each $W \in W_n$, denote by $A_W$ the elements of $W'_{n+1}$ which are subintervals of $W$; set $W^+ := \bigcup_{A \in A_W} A$ and $\theta_W := m(W^+)/m(W)$; then
\[ \mathcal{W}_{n+1} = \bigcup_{W \in \mathcal{W}_n} A_W; \]
\[ \theta_W > 1 - 1/N; \]
\[ \text{if } A \in A_W, \text{ then } |A| \leq |W|/2; \]
\[ \text{for } k \leq nN \text{ and } W \in \mathcal{W}_n, \psi_k(W) \cap \hat{I} = \emptyset. \]

Since \(|J| = 1/N^3, |Z| < 1/2\) for (the unique interval) \(Z \in \mathcal{W}_1\). It follows that for each \(W \in \mathcal{W}_n\), \(|W| \leq 2^{-n}\). Define the closed set
\[ \Lambda := \bigcap_{n \geq 1} \bigcup_{W \in \mathcal{W}_n} W. \]
For each \(n\) and each \(z \in W \in \mathcal{W}_n\), \(\psi_k(z) \notin \hat{I}\) for \(k \leq nN\). Since \(\hat{I}\) is open, the same holds for \(z \in W\). Thus for \(z \in \Lambda\), \(\psi_k(z) \notin \hat{I}\) for all \(k \in \mathbb{Z}\).

Now we wish to construct a measure on \(\Lambda\), in order to estimate its dimension using Lemma \[\] For each \(n \geq 1\), let us introduce a measure \(\mu_n\) on \(\bigcup_{W \in \mathcal{W}_n} W\). Let \(\mu_1\) be Lebesgue measure restricted to the unique interval \(Z \in \mathcal{W}_1\). Define inductively \(\mu_n\), for \(n \geq 2\), as follows. For each \(W \in \mathcal{W}_{n-1}\), set
\[ \mu_n := \mu_{n-1}/\theta_W \]
on \(W^+\), and \(\mu_n := 0\) on \(W \setminus W^+\). As defined, \(\mu_n(W^+) = \mu_{n-1}(W)\) for each \(W \in \mathcal{W}_{n-1}\), whence \(\mu_k(W) = \mu_n(W)\) for all \(k \geq n\) and each \(W \in \mathcal{W}_n\). Since also \(\max_{W \in \mathcal{W}_n} |W| \leq 2^{-n}\), there exists a unique (weak) limit measure \(\mu := \lim_{n \to \infty} \mu_n\) and \(\mu\) is supported on \(\Lambda\) with \(\mu(\Lambda) = \mu_n(J') = |Z|\). By induction using (11), \(\mu_n(W) \leq |W|(1 - 1/N)^{-n+1}\) for \(W \in \mathcal{W}_n\). Thus for \(z \in \Lambda\) and \(n \geq 1\), there are at most two elements \(W_1, W_2 \in \mathcal{W}_n\) intersecting all tiny neighbourhoods of \(z\), with \(\mu_n(W_i) \leq |W_i|(1 - 1/N)^{-n+1} \leq 2^{-n/2+1}\) for \(i = 1, 2\). Hence \(\mu_k(W_i) \leq 2^{-n/2+1}\) for all \(k \geq n\), and so \(\mu(z) \leq 2^{-n/2+2}\) for each \(n\); therefore \(\mu\) is continuous (i.e. it has no atoms). Since \(\mu\) is continuous, \(\mu(W) = \mu_k(W)\) for each \(W \in \mathcal{W}_n\) and \(k \geq n\).

We are nearly at a stage where we can apply Lemma \[\]. For each \(n\), let \(Q_n\) denote a finite partition of \(J' \setminus \bigcup_{W \in \mathcal{W}_n} W\) into intervals such that each \(Q \in Q_n\) has \(|Q| < 2^{-n}\). For each \(Q \in Q_n\), \(\mu_k(Q) = 0\) for all \(k \geq n\), hence \(\mu(Q) = 0\) (using continuity of \(\mu\)). Let \(\mathcal{P}_n := Q_n \cup \mathcal{W}_n\), so \(\mathcal{P}_n\) is a partition of \(J'\). From the construction, \(\mu(P) \leq |P|(1 - 1/N)^{-n} \leq |P|(1 + 2/N)^n\) for each \(n \geq 1\) and \(P \in \mathcal{P}_n\). By Lemma \[\] the Hausdorff dimension of \(\Lambda\) is at least \(1 - 4/N\). Recalling \(\Lambda \subset J'\), set \(Y := \xi^{-1}(\Lambda) \subset X'\). Applying Lemma \[\] to \(Y\), we obtain that for each \(\alpha \in Y\), \(\exp \circ \exp(L_\alpha(y))\) is not dense. As \(\xi\) is a translation it preserves Hausdorff dimension, and the dimension of \(Y\) is at least \(1 - 4/N\). But \(N\) could be taken arbitrarily large (of course, \(I\) and therefore \(Y\) depend on choice of \(N\)). Noting that any set with subsets of dimension arbitrarily close to \(1\) has dimension at least \(1\), the proof of Theorem \[\] is complete. \[\]

3. Proof of Theorem \[\]

It remains to provide the straightforward proof of Theorem \[\].

Proof. Fix \(y \in \mathcal{C}\). Let \((p_n)_{n=1}^\infty\) be a dense sequence in \(\mathcal{C}\) and let \((\delta_n)_{n=1}^\infty\) be a decreasing sequence of positive reals with \(\delta_n \to 0^+\). Let \(U := \{B(p_n, \delta_n) : n \geq 1\}\). As per (11), given an open set \(U\), let \(X_U := \{\alpha : \exp \circ \exp(L_\alpha(y)) \cap U \neq \emptyset\}\). Since \(\exp\) is continuous (so \(\exp^{-2}(U)\) is open) and the central projection is an open map, \(X_U\) is open. By Theorem (11), \(X_U\) has full measure and thus is dense and open for each open set \(U\). Consequently, \(X_\infty := \bigcap_{U \in \mathcal{U}} U\) is countable intersection of open, dense
sets. As in the proof of Theorem 1, each point $\alpha \in X_{\infty}$ satisfies $\exp \circ \exp(L_\alpha(y))$ is dense.

References

[1] Harald Bohr and Richard Courant. Neue Anwendungen der Theorie der Diophantischen Approximationen auf die Riemannische Zetafunktion. Journal für die reine und angewandte Mathematik, 144:249–274, 1914.

DEPARTMENT OF MATHEMATICS AND STATISTICS, PL 68, UNIVERSITY OF HELSINKI, FINLAND.