GRAPH THEORY:
Modern developments

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Chapter 1

Number theory

Number theory is one of the most popular branches of mathematics. Graph theory is at the intersection of set theory and number theory and some of the sequences in graph theory can be predicted using concepts from number theory.

1.1 Integer sieves

The resulting closed-form expression for the number of primes in the interval \([0, N]\) is also a helpful tool that can be used with the Prime Number Theorem to resolve problems concerned with the distribution of primes on intervals of the reals.

**Theorem 1.1.1.** Let \(\beta = \prod_{r_k \leq N} (1 - 1/r_k)\) where the product is taken over composites \(r_k\) less than or equal to \(N\). Then \(\lfloor \beta N \rfloor = \pi(N)\).

**Theorem 1.1.2.** Let \(\alpha = \prod_{p_k \leq N^{1/2}} (1 - 1/p_k)\) where the product is taken over primes \(p_k\) less than or equal to \(N^{1/2}\). Then \(\lfloor \alpha N \rfloor = \pi(N) - \pi(N^{1/2})\).

**Theorem 1.1.3.** The following statement holds: If \(k\) is such that \(\lfloor n^{1/2} \rfloor = n_{k=1}\) and \(t = \min \{k: n_k < 3\}\), then
\[
\pi(n) - \pi(\lfloor n^{1/2} \rfloor) = n^{1-1/2+1/4-1/8+\cdots(-1)^{1/2^t}}.
\]

1.2 Representation theory

For a sufficiently large positive integer \(s_0 > S_t\), the congruence classes of the interval \([0, s_0]\) relative to a prime \(p_0 = t\) each contain at least one prime number. For a fixed prime \(p_0\), the largest difference between any two cardinalities of any two subsets of primes in any two congruence classes of the interval \([0, s_0]\) for all \(s_0 > S_t\) relative to \(p_0\) are bounded by some \(M_t\) where \(M_t = h(p_0)\) where \(S_t\) is a
function of \( p_0 \). The number of primes \( \pi(x) = f(x) \) is asymptotically \( x \cdot [\ln x]^{-1} \).
That is, the function \( \pi(x) \sim g(x) = x \cdot [\ln x]^{-1} \)
has \( f(x)/g(x) \to 1 \) as \( x \to \infty \).

The prime number theorem is considered to be a profound and important development in the theory of numbers. The prime number theorem was conjectured in various forms before the theorem was fully accepted. For all even numbers \( x \) greater than 6, there exist two distinct primes that sum to \( x \).

The number 6 is the only number we know of that cannot be represented as the sum of two distinct primes. That is, in this case, we disallow 1 as a prime under the required definitions and this is generally the standard definition in any event.

If the sequence \( \{a_0 + ib : 1 \leq i \leq k\} \subset [0, 2n] \)
is an arithmetic sequence such that \( \gcd(a_0, b) = 1 \) and \( k = \frac{2n - a_0}{b} \)
then substitute \( k \) for the value of \( n \). To see this inspect the previous derivation of
\[
\pi(N) = 1 + \sum_{i=1}^{r} \left( \left\lfloor \frac{N}{2^i} \right\rfloor - 1 \right)
\]
that holds when \( r \) is defined appropriately.

### 1.3 Representation theory II

For no integer \( t > 2 \) is there a solution in a natural number triple \( (x, y, z) \) that satisfies the equation \( x^t + y^t = z^t \).

Begin by moving the \( y^t \) term to the right-hand side of the equation and using the standard cyclotomic factorization of
\[
z^t - y^t = (z - y) \sum_{j=0}^{j=t-1} z^{t-j} y^j.
\]

By supposition, \( (x, y, z) \) is a coprime triple. Let \( x^t = [z - y] \sigma \). Then suppose \( \sigma | x^{t-j} \). Then \( [z - y] = \alpha \cdot x^t \). It follows \( [\alpha \cdot x^t] < x^t \). So \( z - y \mid x \). However, \( [z - y] \) cannot equal \( x \) or be less than \( x \), so we have a contradiction: Consider \( [z - y]^{-1} < \sigma \) from dividing both sides by \( x^{t-1} \).
**Definition 1.3.1.** An algebraic number is an element of a subset \( A \) of the reals which satisfy a polynomial \( p(x) \) with integer degrees and coefficients. Transcendental numbers are real numbers that form the complement of the algebraic numbers in the reals.

**Definition 1.3.2.** The algebraic degree of an algebraic number \( x_0 \) is the integer degree of a minimal degree polynomial which has \( f(x_0) = 0 \) where \( f \) is a polynomial function with integer coefficients and degrees.

**Proposition 1.3.3.** Every algebraic number has a well-defined algebraic degree.

**Proposition 1.3.4.** If the algebraic degree of \( x_0 \) and \( z_0 \) are not the same then the algebraic degree of a polynomial which gives \( f(x_0) = f(g_0) = 0 \) has \( \deg(f) \geq \max\{D_A(x_0), D_A(z_0)\} \).

**Theorem 1.3.5.** [Weak abc-Conjecture] For no integers \( \min\{r, s, t\} > 2 \) is there a solution in a natural number triple \((x, y, z)\) that satisfies the equation

\[ x^r + y^s = z^t. \]

Assume the negation of the conjecture holds for some choice of an ordered 6-tuple \((r, s, t, x, y, z)\). Choose \( q = y^s/t \). Then for some 2-tuple \( \{p, q\} \) with \( p = x^{r/t} \), the negation holds. Then \( p^r + q^t = z^t \). There is no solution in rationals, \( \{p, q, z\} \), because otherwise there would be a solution to the Fermat equation in integers. So assume \( x \) is an integer. We have

\[ x^r = [z - q]\sigma \]

where both \([z - q]\) and \( \sigma \) have algebraic degree at least \( \min\{s, t\} \). Now choose \( r > \min\{s, t\} \). Otherwise, all three degrees are equal and the problem reduces to FLT.
Chapter 2

Graphs

2.1 Graph theory and parameters

A simple graph $G$ is an ordered pair of sets $V(G)$ and $E(G)$ such that the elements $uv \in E(G)$ are a sub-collection of the unordered pairs of elements of $V(G)$. The graph $K_n$ that has $n = V(G)$ is called a complete graph and is defined to be $E(G) := \{uv : u \in V(G) \land v \in V(G)\}$. All simple graphs are subgraphs of $K_n$ for some $n$ in the natural numbers. It is the case that vertices in a graph $G$ can be elements in $V(G)$ despite the fact that $v \notin uv$ for any $uv \in E(G)$. Such vertices are said to be isolated vertices or null vertices. The theory of graphs can begin from a variety of vantage points. Usually, the theorem of Erdos and Gallai, which is in the preceding section, is a good place to begin. Recall the definition $\deg v = |N(v)|$. An induced subgraph of a graph $G$, denoted $[U(G)]$, is the graph induced by the subset of vertices in $V(G)$, $U \subset V$. That is, $[U(G)] := \{uv \in E(G), v \in U(G) \land u \in V(G)\}$. The neighborhood of a vertex $v$ is the set of vertices $N(v) = U = \{u \in uv : u \neq v \in V(G)\}$. The closed neighborhood of $v$ is denoted $N[v] = N(v) \cup v$. That is, it is always the case that $|N(v)| + 1 = |N[v]|$, even in the case that $v$ is isolated.

The cardinality of $N(v)$, is said to be the degree of $v$, $|N(v)| = \deg v$.

Theorem 2.1.1. [Erdos-Gallai] A sequence $\pi$: $d_1 \geq d_2 \geq ... \geq d_p$ of non-negative integers whose sum (say $s$) is even, is graphic if and only if

$$\sum_{i=k}^{i=p} d_i \leq k(k - 1) + \sum_{j=k+1}^{j=p} \{d_j, k\}$$

for every $k$ such that $1 \leq k \leq p$.

Properties of Graphic Sequences. Any set of vertex degrees can be used to build a graph, but the same is not true for the list of vertex degrees. That is, some degree sequences are not graphic. A property of a degree sequence is either
potential or forcible. If the property $P$ is a forcible property of degree sequence $\pi$, then every graphic realization of the degree sequence $\pi$ has the property $P$. If the property $P$ is a potential property of degree sequence $\pi$, then there is at least one graphic realization of the degree sequence $\pi$ that has property $P$. Some properties, such as hamiltonicity, can be forcible or potential depending on the sequence $\pi$. In the case of hamiltonicity, an degree sequence $\pi$ with smallest element greater than $n/2$ is forcibly hamiltonian. However, the degree sequence $2^\pi$ is potentially hamiltonian and has a unique realization with a hamiltonian cycle. Graphs in general have a cycle spectrum, a list of lengths of cycles in the graph with may or may not compose the set, $3, 4, ..., n(G)$.

In general, maximal outerplanar graphs have the full cycle spectrum: Take the longest cycle, the cycle must be hamiltonian. Then we can iteratively reduce the length of this hamiltonian cycle by 1. The clique number of a graph is the size of the largest clique in the graph. In the case of perfect graphs, the clique number, $\omega(G) = \chi(G)$, where $\chi(G)$ is the chromatic number of the graph $G$. The chromatic number of a graph is the size of the smallest partition of the vertex sets into independent sets, that is, sets such that no two vertices in the set are incident by an edge. Let $\sigma(\pi) \geq 2n$ be graphic and $n \geq 6$. Then $\pi$ has a realization containing a $K_3$. Let $G$ be a realization of $\pi$ with smallest possible induced cycle length. We will show this cycle length is 3. (There exists a cycle because $G$ is too dense to be a tree or a forest.) If $G$ contains an induced cycle of length 6 or more, then the edges can be rearranged among the vertices of the induced cycle to form a $G'$ realizing $\pi$ and containing a cycle of length 3. Thus, we need only deal with $G$ containing a smallest induced cycle of length 4 or 5. Suppose the shortest cycle is $C_4$. If $x_1x_2x_3x_4$ form the $C_4$ and $yx_1 \in E(G)$ then $yx_1$ and $x_2x_3$ can be swapped for $x_1x_3$ and $yx_2$, forcing a $C_3$. So $G$ contains an isolated $C_4$ and another edge $yz$. Use a so-called bowtie swap to force a $C_5$ : Swap $x_1x_3 \cup yx_2z$ for $yz \cup x_1x_2x_3$. Thus, we assume the induced cycle is a $C_5$. No vertex off the cycle can be adjacent more than one vertex on the cycle; otherwise we get a $K_3$ or an induced $C_4$. If $xz$ is an isolated edge from the $C_5$, swap $x_5x_3 \cup x_1y \cup x_2z$ for $P_3 := x_5x_1x_2x_3$. Otherwise, swap $x_1x_4 \cup x_3y$ for $x_3x_4 \cup x_1y$.

### 2.2 Distance

In order to understand the basic distance-metric on graphs, it is necessary to define a $uv$-path. A $uv$-path is a sequence of edges from $E(G)$ such that every two consecutive edges in the sequence of edges share an endpoint, the two distinct endpoints, the two distinct vertices which appear only once in the sequence of edges of the path, are $u$ and $v$, and no vertex, from the sequence of edges of the path, appears in more than two edges. The size of a path is the number of edges in the path. The distance between $u$ and $v$ is the size of the smallest $uv$-path, $P \subset E(G)$. If there is no $(u, v)$ path in $G$, then the distance between $u$ and $v$ is said to be infinite. Any graph such that there exist two vertices $\{u, v\} \subset V(G)$
such that \( d(u, v) > M_0 \) for all finite distances \( M_0 \) is said to be disconnected.

The distance \( d_G(u, v) \) is the length of the shortest path between the vertices \( u \) and \( v \) in the graph \( G \). The eccentricity of \( v \) is the value

\[
ecc(v) = \max_{u \in V(G)} d_G(u, v).
\]

The largest eccentricity of any vertex in \( G \) is the diameter of \( G \), \( \text{diam}(G) \), while the smallest eccentricity of any vertex in \( G \) is the radius of \( G \), \( \text{rad}(G) \).

**Theorem 2.2.1.** For all graphs \( G \), we have \( \text{rad}(G) \leq \text{diam}(G) \leq 2 \text{ rad}(G) \).

The center of \( G \), \( H = \text{cen}(G) \) is the subgraph in \( G \) induced by the set of vertices such that \( \text{ecc}(v) = \text{rad}(G) \). The periphery of \( G \), \( H' = \text{per}(G) \) is the subgraph in \( G \) induced by the set of vertices such that \( \text{ecc}(v) = \text{diam}(G) \).

### 2.3 Matchings

**Theorem 2.3.1.** [Tutte’s] For all simple graphs \( G \), the graph \( G \) has a 1-factor if and only if \( S \subset V(G) \) has the number of odd components in \( |G − S| \leq |S| \).

**Theorem 2.3.2.** [Konig-Egervary] The maximum cardinality of a matching in \( G \) is equal to the minimum cardinality of a vertex cover of its edges.

**Theorem 2.3.3.** [Hall’s] The graph \( G \) contains a matching of \( A \) if and only if \( |\mathcal{N}(S)| \geq |S| \) for all \( S \subset A \).

### 2.4 Connectivity

One of the most important theorems in the study of connectivity is Menger’s Theorem. This theorem’s statement relies on the definition of the parameter \( k \)-connectivity. In graph theory and digraph theory, the most sophisticated notions of strongly connected, hamiltonian-connected, and traceable for digraphs in particular have applications. A graph \( G \) is \( k \)-connected if \( G − S \) is connected for all \( S \subset V(G) \) where \( |S| \leq k − 1 \). A graph \( G \) is \( k \)-edge-connected if \( G − S \) is connected for all \( S \subset E(G) \) where \( |S| \leq k − 1 \).

If we consider the graph \( G \) to be a topological space, one of the only terms in modern topology that distinguishes \( k \)-connectivity from simple connectivity is the fundamental group of the space \( G \). In fact, the use of spanning trees for simplices is one of the beginning concepts used to introduce the set of groups that form the fundamental groups on topologies. The term \( k \)-linked implies that \( G \) has \( k \)-internally disjoint paths between any two disjoint sets that partition the vertex set. The term \( k \)-linked probably has a number of applications in topology that could distinguish spaces with a number of identical conditions.
For instance, the term connected implies that requires that in any 2-partition of the space into 2 disjoint sub-spaces one of the sub-spaces is not open. This definition allows for connected sets that are not path-connected. Consider finite sets with the strongest possible basis.

Notice any uniquely colorable graph $G$ with $\chi(G) = k$ is at least $k - 1$-connected.

**Theorem 2.4.1.** [Menger] If $x, y$ are vertices of a graph $G$ and $xy \notin E(G)$, then the minimum size of an $(x, y)$-cut equals the maximum number of pairwise internally disjoint $(x, y)$-paths.

**Theorem 2.4.2.** If $G$ is $k$-connected, then for all $S \subset V(G)$ with $|S| \leq k$, there is a cycle $C_t \subset G$ such that $S \subset V(C_t)$.

**Theorem 2.4.3.** If $\delta(G) \geq \lceil n/2 \rceil$, then the edge-connectivity of $G$ is exactly $\delta(G)$. 


Chapter 3

Multigraphs

3.1 \(m\)-edge graphs

The number of \((n, m)\)-graphs is partly determined by the fact that distribution on the number of edges per entry on the matrix \(A(K_n)\) is conditionally invariant. That is, the distribution of edges on the labelings of a connected \((n, m)\)-graph is uniform in \(A(K_n)\), regardless of a restriction to any original subset of entries.

There is exactly one graph with a single edge in standard graph theory, the singleton edge. There are two graphs up to isomorphism with two edges in standard graph theory, the double matching, \(2K_2\), and the path of size two: \(P_3\). One of these graphs is connected, \(P_3\), while the other graph, \(2K_2\), is disconnected. (This point is important in some contexts, for instance, if we wanted to calculate the number of connected graphs with \(m\) edges on \(n\) vertices.) Meanwhile, the number of labelings of \(2K_2\) with 4 labels is 3, while the number of labelings of \(P_3\) with 4 labels is 12. This follows by rote calculation, as well as the fact that \(\text{Aut}(P_3) = 2\) while \(\text{Aut}(2K_2) = 8\). Applying the fact that the number of labelings of \(G\) is equal to \(n!/\text{Aut}(G)\) gives the desired calculation.

The method is complicated by the fact that the automorphism group of a graph is not trivial to calculate. However, after observing these facts, it is good junc- ture to state that the number of labeled graphs on 2 edges and 4 vertices is 15 total and this number can be obtained with less effort by observing that \(\binom{4}{2} = 15\). Here, a displacement is a placement of an edge of the labeled graph \(G\) in the labeled complement of \(H\). An intersection is a placement of an edge of the labeled graph \(G\) in the labeled graph \(H\). Generally, two unlabeled graphs are not said to have intersections or displacements unless there is a function that maps them into the same vertex set and induces an unordered pair labeling on their edges; that is, a mutual placement is a labeling of two graphs on the same vertex set that has no intersections. An atlas of graphs is a directed multi-graph on the set of graphs of some range of \((n, m)\)-graphs. If we establish a 1:1
correspondence between a range of values for metrics on the adjacency matrices of graphs and the direction or color on edges we can build graphs on the vertex set with a 1:1 correspondence between graphs on simple \((n, m)\)-graphs and the vertex set of the atlas graph. Atlas graphs can be used to demonstrate that a glossary of graphs on \((n, m)\)-graphs is comprehensive. The theory of graphs also allows us to calculate the number of \((n, m)\)-graphs for all \((n, m)\) without actually drawing the atlas graphs or collecting the glossary of graphs for each pair of arguments in the \((n, m)\) pairs for \((n, m)\)-graphs.

A mutual placement of two graphs is a labeling of two graphs so that the edge labels induced by the vertex-labeling do not fix any two edges identically. By labeling, we mean an injection from the vertex set into a set of cardinality larger than \(n(G)\). A displacement is an edge covered in a labeled complete graph by a placement of two graphs which is overlapped (intersected) by precisely one of the two graphs. The \((0, 1)\)-adjacency matrix \(A(G)\) of a simple graph \(G\) is an \(n \times n\) array of zeros and ones. The diagonal of the matrix otherwise denotes loops at a vertex and is blank in the \((0, 1)\)-adjacency matrix of any simple graph. In fact, for any simple graph the adjacency matrix \(A(G)\) can be abbreviated by the upper triangular portion of the matrix; the lower triangular portion of the matrix is redundant. The number of \((u, v)\)-walks of arbitrary length can be determined by multiplying powers of the standard adjacency matrix and reading off the \((u, v)\)-entry. Generally, an atlas under our definition is a multi-digraph with arcs \(\vec{uv} \in M\) if \(u := G\) and \(v := H\) and the edge addition distance from \(G\) to \(H\) is one. Shortest paths in the atlas of one or more edge types represent the shortest number of operations required to form \(f(H) = G\), given the available operations. The method outlined here, and again later, for finding the number of simple graphs, can be applied to the question of finding the number of \((n, m)\)-graphs where the definition of an edge in \(G\) is extended to cover entries along the diagonal of \(A(G)\).

A connected \((n, m)\)-graph has one cycle when \(n = m\). However, the number of components of a graph \(k\) does not appear to be a sufficient third parameter such that the so called cyclotomic number \(cyc(G) \sim f(n, m, k)\).

### 3.2 Tournaments and oriented graphs

**Theorem 3.2.1.** A transitive tournament is an orientation of \(K_n\) such that the vertices have respectively degrees \((n-1, 0), (n-2, 1), ..., (0, n-1)\). Equivalently, a transitive tournament is acyclic, and has the property that if \(uv, vw \in A(T_n)\) then \(uw \in A(T_n)\).

**Theorem 3.2.2.** [Ford-Fulkerson] Let \(\vec{\pi} = \{a_i, b_i\}_{i=1}^{n}\). If for all subsets \(I \subset [n]\),
it follows that
\[ \sum_I b_i \leq \sum_I \{ a_i, |I| - 1 \} + \sum_{|I|=I} \{ a_i, |I| \}, \]
then \( \vec{\pi} \) is graphic.

**Proof.** The proof follows by induction on the number of terms in the sequence: Let the sequence \( \vec{\pi} \) be given and suppose it satisfies Fulkerson’s criteria. Then let \( d = (a, b) \) from the sequence be given and remove it from the sequence; at the same time subtract 1 from the first argument of \( b \) terms of the sequence and 1 from the second argument of \( a \) terms of the sequence. Let \( I \) be a subset of \( \vec{\pi} \), formed by the augmentation of \( \vec{\pi} \) just described. We are given that the original sequence satisfies Fulkerson’s criteria and thus, for all such \( I, I \cup d \) satisfies the criteria. When we remove \( d \), the left-hand side of the given inequality decreases by \( b \). But the right-hand side decreases by at most \( b \) as well since we augment at most \( b \) terms in the first coordinate and \( \{ a_i - 1, |I| - 2 \} = \{ a_i, |I| - 1 \} - 1 \).

**Theorem 3.2.3.** A graph \( G \) is \( k \)-colorable if and only if \( G \) has an orientation in which the length of every directed path is at most \( k - 1 \).

**Theorem 3.2.4.** The linear arboricity \( la(\vec{G}) \) of an arbitrary oriented graph \( \vec{G} \) is equal to the clique number \( \omega(G) \) of the graph \( G \) that underlies \( G \prec \vec{G} \) precisely when the achromatic number is equal to the clique number. That is, when
\[ \omega(G) = la(\vec{G}), \]
the graph \( G \cong K_{\omega(G)} \).

**Theorem 3.2.5.** [Sumner] Let \( A_n \) be any orientation of any tree \( T_n \) of order \( n \) : an arborescence. That is, \( A_n \) such that \( T_n \prec A_n \). Then \( A_n \) can be found as a subgraph of any tournament of order \( 2n - 1 \).
Chapter 4

Extremal graph theory

Complete Bipartite Graphs. Once the Turan number of a complete bipartite graph has been calculated closely, it is possible to give an upper bound for any subgraph of this complete bipartite graph. The Turan $r$-sphere of order $n$ is a graph consisting of ordered $d$-tuples which make up the vertex set, and edges between the $d$-tuples. That is, we have $d = \lceil \ln r \cdot n \rceil$ and

$V(rSPH_n) = \{(a_1, a_2, ..., a_{\lceil \ln n \rceil})\}$,

$E(rSPH_n) = \{(b_1, b_2, ..., b_{\lceil \ln n \rceil})(c_1, c_2, ..., c_{\lceil \ln n \rceil}) : b_i \neq c_i\}$.

Notice that the Turan $r$-sphere is $K_t,s$-free for all $t, s$ such that $\{t, s\} - 1 = r$. To see this, notice that if we want a $K_t,s$ it would require $t$ vertices adjacent to the same set of $s$ vertices.

If $\{t, s\} - 1 = r$, there must be two vertices, one vertex from the $t$-set and one vertex from the $s$-set that have a common element from the ordered $d$-tuples which describe those two vertices. Furthermore, if $n = t + s$, the Turan $r$-sphere of order $n$ associated with $K_t,s$ looks like a standard Turan graph with some cliques deleted. The $r$-sphere is not the largest $K_t,s$-free graph of order $t + s$ for all choices of $t, s$.

4.1 Potential

Swaps. Some degree sequences have only one realization; these sequences are said to be uniquely realizable. Any graph $G$ such that no Eulerian trail on the vertex set of $G$ has the property that each edge in $E(G)$ is followed by an edge that is not in $E(G)$ and vice versa has a uniquely realizable degree sequence $\pi(G)$. It is not the case that $C_4$ is the only possible alternating Eulerian trail that can induce an edge swap on the edge set of a graph that produces more than one realization of a graphic sequence. Furthermore, some edge-swaps on
realizations can produce a second realization that is isomorphic to the first realization.

Potential Number. The potential number of a graph, \( \sigma_G \), is the minimum sum of the degrees in a graphic degree sequence \( \pi \), such that every degree sequence with \( \sigma(\pi) = \sigma_G \) has a realization containing \( G \) as a subgraph.

Theorem 4.1.1. Any clique joined to any independent set forms a graph \( G \) with \( \pi(G) \) uniquely realizable.

Theorem 4.1.2. Any two realizations of a degree sequence can be obtained from one another by performing a series of \( C_4 \)-swaps, that is swaps of non-edges for edges in (edge, non-edge)-alternating \( C_4 \)s.

Theorem 4.1.3. The value \( \sigma_{C_4}(n) = 3n - 1 \) if \( n \) is odd and \( 3n - 2 \) if \( n \) is even.

If \( G \) contains a chordless path of size \( > 2 \) or cycle of length \( > 3 \), then there is, for all \( m, 0 \leq m \leq n(n-1) \) a uniquely realizable degree sequence with sum degree sum \( m \) and no induced \( G \). For all \( n + 1 \leq m \leq n(n-1) \), odd and all \( n + 2 \leq m \leq n(n-1) \), even, we have that if \( \pi \) is graphic, and the largest degree of \( \pi \) is \( \leq n - 2 \), then \( \pi \) is potentially induced-\( P_3 \) graphic.

Problems.

1. Given that \( \chi(G) = 3 \) and \( \chi(H) = 4 \), is the largest \( G, H \)-free graph of order \( n \) isomorphic to the largest \( G \)-free graph of order \( n \)?

2. Suppose \( G \) is connected and has \( t \) edges, what are the extremal graphs for bandwidth?

4.2 Turan numbers

The Strong Form of Turan’s Theorem states that the extremal Turan graph for \( K_n \) is unique up to order and that if \( m(G) > [p - 2][2(p - 1)]^{-1}[n^2] \), then \( G \) has a subgraph isomorphic to \( K_p \). Suppose there are two graphs, \( G_1 \), and \( G_2 \), that are \( H \)-free for \( H \subset K_r \), where \( m(H) < \left( \binom{r}{2} \right) \) and \( n = n(G_1) = n(G_2) \) and \( H \subset G_1 \cup e \) and \( H \subset G_2 \cup e \). It is not the case that \( m(G_1) = m(G_2) \) for all choices of \( H \). For instance, consider the case when \( H = C_{2n+1} \). Then \( G_1 = K_{2n} \cup e \) and \( G_2 = nK_1 \wedge K_{n+1} \) are both edge-maximal \( H \)-free graphs, but \( m(G_1) \neq m(G_2) \).

Problems.

1. Give the chromatic number of \( K_{2n} - nK_2 \).

2. Find the number of edges that forces a \( P_k \) in \( G \).
3. Suppose $G$ is maximal planar of order $n$. What is the maximum $n$ that excludes $P_k$.

4. State $R(3, 3)$ and $R(4, 4)$.

5. Prove the Strong Form of Turan’s Theorem.

6. The line graph of a graph $G$ is the graph formed by $G^L = E(G), \{u'v' : u' = e_1, v' = e_2; e_1 \sim e_2\}$ > . Prove or disprove: The edge chromatic number $\chi'(G) = \chi(G^L)$.

7. Prove the Erdos-Gallai criteria for graphic sequences.

8. Prove the Handshake Lemma: No simple graph is fully irregular.

9. Prove: If $\pi$ has a realization $G$ with $H$ of order $n$ having $H \subset G$, then $\pi$ has a realization with $H \subset G$ such that $V(H)$ has the $\deg_G(v) \geq \deg_G(w)$ for all $v \in V(H)$ and all $w \in V(G) - V(H)$.

4.3 Graphic structures: Ramsey theory

The following discussion is taken identically from various versions of [16]. Any subgraph $G$ of $K_{n,n}$ with $m(G) \geq n^2/2$ has some $K_{p,q}$ as a subgraph where $p + q \geq n + 1$. If a graph spanned by $tK_t$ red cliques is red $K_{t+1}$-free, then every parallel class of $K_{t,t}$ that spans a pair of red cliques is $K_{p,q}$-free. Any graph that is spanned by $tK_t$ red cliques and which is also red-blue $K_{t+1}$-free has a spanning $t$-sphere. The definition of a $t$-sphere of 2 dimensions reoccurs several times in the given argument. Any red $t$-sphere joined to a monochromatic red $K_t$ by an arbitrary set of edges contains a monochromatic $K_{t+1}$. Consider a $t$-sphere and a $t$-clique joined by a dichromatic $K_{t,t}$2. Then there are not enough edges in the graph formed by this join to cover the desired complete graph. That is, there is either a $K_{t+1} \subset G$, or we have the contradiction that $G$ is tri-chromatic. The remainder of the section goes on to prove the opposite inclusive inequality.

If the clique number of $G$ is $\omega(G) = t$, then the bigraphic split of $G$, called $S(G)$, has no proper subgraph $K_{p,q}$ where $p + q > t$. For a graph $G$ isomorphic to a $t$-sphere, both $S(G)$ and $S(G^c)$ have no $K_{p,q}$ where $p + q > t$. The extremal graph for the lower bound on $R(t)$ for $t = r$, the graph $SF(r)$, is defined explicitly next: Let $G_z = G_x + G_y$. Let

$$G = [(r-1)K_1 \times G_z],$$

and

$$H = [(r-1)K_1 \times G_z]^c.$$
vertices $V(G) \to V(H)$ wherein $V$ and $H$ decompose the complete graph. The construction of the edge set of the graph $F(r)$ is outlined next: Let

$$E(F(r)) = E(G) \cup E(H) \cup \{vw : [g(v) = g(w) + 1]^{2}, v \in G, w \in H\}.$$  

The graph $SF(r)$ is defined to be $SF(t) := SF(t - 1) \cup E(G) \cup E(H) \cup \{vw : [g(v) = g(w) + 1]^{2}, v \in G, w \in H\}$.

The largest monochromatic clique under $g$ in $F(r)$ has size $\lceil r/2 \rceil$.

There are no graphs isomorphic to $K_{r+1}$ or $(r+1)K_1$ contained in the extremal graph $SF(r+1)$. Then $SF(r)$ is $K_{r/2}$-free of cliques composed of vertex labels that are monochromatic under $g$. Observably, $F(r+1)$ is $K_{(r+1)/2}$-free of cliques composed of edge labels monochromatic under $g$. Assume the following holds for $r > 3$. Let $G_x$ be a clique $\lceil r/2 \rceil$ vertices labeled 1. Next, let $G_y$ be a clique $\lceil r/2 \rceil$ vertices labeled 2. In the case that $r = 3$, the graph $F(r) = \{vu, uw, wx, xy, yt\}$. The vertices $\{v, w, x\} \to [1]$ by $g$. The vertices $\{u, t\} \to [2]$ by $g$.

**Theorem 4.3.1.** The value of $R(t + 1) = \frac{2}{3}t^{3} - t + 2$.

**Problems.**

1. Let $G$ be a graph with $\chi(G) > k$ and $V_1 \cup V_2 = V(G)$ a partition of the vertex set of $G$. If both the graphs induced by the partition of $V(G)$ are $k$-colorable, then the edge cut between $V_1[G]$ and $V_2[G]$ has cardinality at least $k$.

2. Let $H = K[G]$ be an induced subgraph of $G$. If we orient $\tilde{G}$ arbitrarily and claim $E(H)/V(H) = P(H)$ for all graphs $H$, and subsequently let

$$Q([\tilde{G} - \tilde{H}, \tilde{H}]) = \int_{0}^{t} |\text{deg}^+ - \text{deg}^-| |\tilde{G} - \tilde{H}, \tilde{H}| dt,$$

find an expression for max $P(H)$ if $t_0 = 1$.

3. Show that if $G$ has a total coloring of index $t + 1$, then the vertices can be labeled in $t + 1$ labels so that the vertices have difference basis $[t]$.

4. The Erdos-Gyarfas conjecture claims no graph with $\delta(G) \geq 3$ and all vertices of odd valence is free of cycles that have power of 2 length. Disprove the Erdos-Gyarfas conjecture.
Chapter 5

Algorithmic graph theory

The following solution to the irregularity strength question is not a probabilistic solution, though it does have some elements from analysis.

5.1 Irregularity strength

An irregular labeling of a graph $G$ is a not necessarily proper edge-labeling of the edge set of $G$ such that
\[ \sum_{uv \in E(G)} f(uv) = \sum_{uw \in E(G)} f(uw) \]
if and only if $v = w$. Consider an arc weighting of a digraph $D$, $f : A(D) \rightarrow \mathbb{Z}^+$. Define a vertex labeling induced by our arc weighting to be $g : V(D) \rightarrow \mathbb{Z}^+ \cup \{0\}$, where $g(v) = (\sigma_f(vx), \sigma_f(xv))$. If the arc labeling such that every vertex label is distinct, then the labeling is irregular. Let $I(D)$ denote the set of irregular labelings of a digraph $D$. The irregularity strength of a digraph $D$ is defined as $\bar{s}(D) = \min_{f \in I(D)} \max_{e \in A(D)} f(e)$. Let
\[ \text{Ran}(v) = \{(a, b) : \text{deg}^+(v) \leq a \leq t, \text{deg}^-(v) \leq b \leq t'\} \]
where $t = s \cdot \text{deg}^+(v)$ and $t' = s \cdot \text{deg}^-(v)$. Let
\[ S_x = \{v : \text{Ran}(v) \subset \text{Ran}(x)\}. \]
Let
\[ D_f(x) = \min\{|\text{Ran}(x)| - |S_x|\}. \]
If $D_f(v) \geq 0$, for all $v \in D$, then $[D^x]_f(v) \geq 0$, for all $v \in D^x$ where $D^x$ is some weighting of the arcs incident $x$. No matter how we weight the arcs of $x$, the relation $|S_v| \leq |\text{Ran}(v)|$ holds for $v$ not equal to $x$. If $s$ is given, $D$ can be weighted irregularly by recursively weighting the arcs incident the vertex $v$ (the
vertex that accomplishes the minimum value for $D_f(v)$) with a distinct vertex label at the given stage of the algorithm. Furthermore, $\lambda_f = s$ is given by the value that defines $D_f(v) > 0$ for all $v \in V(D)$.

5.2 Edge-labelings

Define an $(|A|, |B|)$-tree to be an acyclic spanning graph with opposing partite sets of cardinalities $|A|$ and $|B|$. Let $f$ be a graceful labeling of $T$ with $f(v) = n(T)$, so that $v \in B$. Then the largest label assigned to a vertex in partite set $A$ is called the primary interior label while similarly, $f(v)$ is said to be the primary label of the graceful labeling $f$. The term upper imbeddable is already in use in some branches of topological graph theory. For our purposes, we define upper imbeddable as follows. An $(|A|, |B|)$-tree is $k$-upper imbeddable if $f$ labels the vertices in partite set $A$ with labels from $[|A|]$, further, $f$ labels the vertices in partite set $B$ with labels from $[k] - [k - |B| - 1]$, and finally all the induced edge labels are distinct and from the set $[k] - [k - |B|]$.

Algorithm $\beta$. Suppose $T$ is an arbitrary simple tree. Out-direct the tree from an arbitrary leaf. Start at the pendant vertex with degree $(1, 0)$ and follow the up-down labeling procedure for a path, always traversing the shortest directed path available to a vertex with no out-degree, and labeling the 1-distant vertices incident the path if those 1-distant vertices have no neighbors. Next, return to the labeled vertices in the order they were encountered if they are incident unlabeled vertices. Use the up-down procedure to proceed along a caterpillar against the directed orientation until the algorithm reaches a vertex that is already labeled.

Certificate of Proof for Algorithm $\beta$. There should be a set of vertices whose labels can be permuted so that the branch has consecutive edge-difference labels and so that the labeled edge-differences on the tree are consecutive as a whole. The up-down procedure always starts in the correct partite set so that the labeling is locally up-down. Any difference value that we skip on a branch of the labeling scheme is immediately recovered on the next branch. To see that the additional caterpillars can be attached to the tree without missing any edge-difference labels, consider that the zero label can be rotated to any vertex in a caterpillar, preserving consecutive differences. (This result is due to Cahit.)

To see this, draw a caterpillar with an up-down labeling, remove a bridge, and then insert an edge between the zero label and the tail of the caterpillar. Appeal to the inequality in the size of the two branches, if necessary, and upper imbeddability. If two caterpillars are appended to the same branch of the labeling, neglect the structure of the pre-labeled branch. The missing edge-difference label gets smaller at each juncture and it follows no difference value can be skipped on the final branch of the labeling scheme.
Problems.

1. A ρ-labeling of a graph is a vertex-labeling of an \( n(G) = n \)-graph with labels from \([2n + 1]\). Find the lowest complexity proof and lowest complexity proof-certificate that lobsters (2-distant trees) have a ρ-labeling.

2. Show that complete \( m \)-ary trees \( T \) have ρ-labelings such that the center-vertex of the tree \( T \) is labeled with 0 and vertices are ordered according to distance from the center of the tree \( T \).

5.3 Packings

Consider the problem of packing \( P_5 \) or \( C_4 \) in subgraphs of \( K_{32} \). It is easy to verify that fairly dense graphs in \( K_{32} \) admit dense packings of either of the two graphs. There are generalized approaches to packing graphs, but perhaps the most eloquent phrasing of any packing problem in graph theory belongs to Bollobas.

P. Catlin is infrequently credited with contributing to the following conjecture. Erdos, Bollobas, Catlin contributed to the proof that the Hadwiger conjecture and Hajos conjecture almost always hold given the assumption of the probabilistic method.

**Theorem 5.3.1.** [Bollobas-Eldridge] If \( n \geq 6 \), \( (\Delta(G) + 1)(\Delta(H) + 1) \leq n + 1 \), \( n(G) \leq n \), and \( n(H) \leq n \), then there is an \( \{H, G\} \)-placement in \( K_n \).

The following proof relies on the tools we have used in the previous two sections. We introduce a definition for the purpose of the proof. (The definition originally appeared in Journal of Graph Theory.) A star-resolution of a tree \( T_n \) is the shortest possible list of trees \( S_n = A_0, A_1, ..., A_k = T_n \) where \( m(S_n) = m(T_n) = m(A_t) \) for \( 0 \leq t \leq k \), and where \( A_{t+1} \) is formed by replacing an edge \( e_x = xz \), where \( x = \Delta(A_t) \) is fixed throughout, with the edge \( yz \).

**Lemma 5.3.2.**

\[ E(\sigma(\{S_i\}) | \sigma(\{S_i\}) \geq 1) > E(\sigma(\{S_i\})) \]

under the condition that the \( L_i \) are formed from the \( S_i \) element-wise by star-resolutions.

The proof is by induction on the largest \( i \) we resolve and the greatest \( k \) in the star-resolution. Consider that if we have an edge-disjoint placement of the list \( \{L_i\} - L_j(= A_t) \) in \( K_n \) and we take the expectation of the number of edge-overlaps in the various placements of \( L_j \) in the packing, we get that there is a greater conditional expectation once there is a first edge-overlap by induction.
5.4 Topological graph theory

**Theorem 5.4.1.** If $G$ is Kuratowski subgraph free, and the $G$ parameter-tuple

$$\langle n(G), m(G), r(G) \rangle = \langle n, 3n - 6, 2n - 4 \rangle,$$

then $G^d$ is 1-factorable.

**Proof.** Take the smallest homeomorph $G'$ of any graph $G$ and take the dual $G^d$. If the dual has a circumference 2 graph with $n = n(G)$ edges, then the graph $G$ is hamiltonian. If $G$ is Kuratowski subgraph free, and the $G$ parameter-tuple

$$\langle n(G), m(G), r(G) \rangle = \langle n, 3n - 6, 2n - 4 \rangle,$$

then $G^d$ is 1-factorable. If the upper imbeddable image of $G$ contains a circumference 2 multisubgraph of size $n(G^d)$, then $G^d$ is Hamiltonian. The upper imbeddable image of $G$ has $r_0 = r_{im}(G) = 1$, $m_0 = m_{im}(G) = 3n - 6$, because maximal planar graphs have splitting trees \[31\]. Then $m_0 \geq (1/2)[2n - 3][r_0]$ edges. Therefore, $G$ has a circumference 2 multigraph of size $n(G) - 1$ as a subgraph of $G^d$. 

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Chapter 6

Probabilistic methods

Let $\mu = [m^2](\binom{n}{2})^{-1}$. Let

$$\Delta(\mu) = \mu - \mu_C = m \binom{n}{2}^{-1}.$$  

Finally, let $\phi(I)$ be the cardinality of the isomorphism class $I$ and let $||I||$ be the cardinality of the labeled isomorphism class $L(I)$. The strict isomorphism class of $(n, m)$-graphs is the set of $m$-edge graphs that span the vertex set $V(G) \cong [n]$. The proper isomorphism class of $(n, m)$-graphs is the set of $m$-edge graphs that have $n(G) \leq n$.

Let $\tau_{AV} = E[X]$ where the random variable $X = \tau(G)$ over the class of graphs in the set of connected graphs in the proper $(n, m)$-isomorphism class $CG_{n,m}$. If $m_F + n - 1$ is $m(G)$ for all $G \in CG_{n,m}$, then the number of connected labeled $(n, m)$-graphs is given by

$$||CG_{n,m}|| = \tau_{AV}^{-1} \cdot \frac{[n^{n-2}]}{m_F} \cdot \binom{n}{2} - n + 1.$$  

It is reasonable to assume that the expectation $X$ has $|\tau_{AV}| = |E[X]| = 3$ in the case of very sparse graphs.

Continuing, we conclude that almost all graphs are connected, noting that this conclusion is somewhat misleading: (1) the proportion of measure in $I = \{G_{n,m}\}$ does not have $\alpha \to 1$, and, (2) we assume the result that we are in essence attempting to prove, that is that the conditional expectation

$$\mu_C \longrightarrow (m^2 - m) \binom{n}{2}^{-1}.$$  

I.e., in any event, it is clear from our set of assumptions and deductions that the connected graphs do form a set of positive measure in the space of $(n, m)$-graphs.
as \((n, m)\) diverges in \(\mathbb{R} \times \mathbb{R}\). Use the given recurrence relation to find \(\tau(C_{n,m})\):

\[
\tau(C_{n,m}) = \tau(C_{n-1,m-1}) + \tau(C_{n,m-1}),
\]

for appropriate values of \((n, m)\). Here, \(C_{n,m}\) is the set of labeled connected \((n, m)\)-graphs and \(\tau(C_{n,m}) = \binom{m}{n}\) is the number of labeled spanning trees of connected labeled \((n, m)\)-graphs.

**Conjecture 6.1.** The value

\[
\frac{n!\mu|I|^2 - 2m|I|}{|I| \cdot |L(I)|} \sim \mu_C.
\]

**Conjecture 6.2.** [Tree conjecture] If \(T(n+1)\) is the cardinality of the strict \((n+1, n)\) isomorphism class, then

\[
n^{n-2}[(n-1)!]^{-1} \longrightarrow T(n+1).
\]

**Conjecture 6.3.** The cardinality of the labeled isomorphism class for strict \((n, n)\)-graphs has \(L(I)/(n-1)! \longrightarrow e\) as \((n, n)\) has \(n \rightarrow \infty\).

**Conjecture 6.4.** The cardinality of the strict \((n, m)\)-graph isomorphism class \(I\) is given by

\[
o|I| = |L(I)||m - \mu_C||\mu - \mu_C|^{-1}
\]

where \(o \sim n!\).
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