Optimization of investment portfolio weight of stocks affected by market index

E. Azizah¹, E. Rusyaman² and S. Supian³
¹,²,³Department of Mathematics, Padjadjaran University, Indonesia.
E-mail: azizahelfa@gmail.com

Abstract. Stock price assessment, selection of optimum combination, and measure the risk of a portfolio investment is one important issue for investors. In this paper single index model used for the assessment of the stock price, and formulation optimization model developed using Lagrange multiplier technique to determine the proportion of assets to be invested. The level of risk is estimated by using variance. These models are used to analyze the stock price data Lippo Bank and Bumi Putera.

1. Introduction
In the face of risky investment opportunities, investment options cannot rely solely on the expected profit rate. Investors should be willing to bear the high risk if it expects to earn high rates of return [3]. Therefore, the stock price valuation analysis, the selection of a combination of stocks in a portfolio optimum, and measure the risk of investing in the capital market is very necessary. This is where the mathematical model has a very important role to analyze the formation of the portfolio [7].

The mathematical model, particularly single index model seeks to facilitate the analysis of investment. This is because in a single index model assumed that the correlations of returns between stocks occur due to the stock response to changes in the Composite Stock Price Index (general market index). When the market improves (indicated by the market index), the price of individual stocks also increased, and vice versa [4]. Single index model is used for valuation of stocks in an investment.

According to Dowd [2], in conditions of risky investments, a strategy that can be done to reduce the magnitude of the risk of investment is to build a portfolio. To establish the optimum portfolio can be done one of them with the help of optimization using the techniques of Lagrange Multiplier. Meanwhile, to measure the magnitude of risk can be used to estimate the variance (Var). Keep in mind, that the Var has been used widely and become a common standard for the calculation of risk in financial investments [5].

In this paper aims to analyze the stock price assessment with a single index model, determine the weight of the combined portfolio with Lagrange multiplier technique, and estimate the magnitude of the risk by using the size of the Var in portfolios that were formed.

2. Research methodology
In line with the stage of stock price valuation analysis, portfolio weight optimization, and estimation of the amount of investment risk, thus the first to do is to calculate stock returns and market returns as shown below.
2.1. Calculating the Stock Return and Market Return

To determine the expected return and variance of stock is required data of stock returns and market returns of each study period. According to Gruber et al. [4], stock returns for each period can be calculated using the following formula:

\[
R_{it} = \ln \left( \frac{P_{it+1}}{P_{it}} \right),
\]

(1)

where \( R_{it} \) is the stock returns in the period \( t \), and \( P_{it} \) is the stock price in the period \( t \). The market index is represented by the composed stock price index (CSPI). In the same way the market return is calculated by using the following formula:

\[
R_{mt} = \ln \left( \frac{IHSG_{t+1}}{IHSG_{t}} \right),
\]

(2)

where \( R_{mt} \) is the market return in the period \( t \), and \( IHSG_{t} \) is \( IHSG \) in the period \( t \). Then, the return data will be used for the establishment of a single index model below.

2.2. Establish a Single Index Model

Based Gruber et al. [4], the main characteristic of the single index model is the model is acceptable if and only if the underlying assumptions are met. Single index model has some characteristics as follows:

1. the basic equation \( R_i = \alpha_i + \beta_i R_m \) and \( \alpha_i = \alpha_i + e_i \), so that the single index model form:

\[
R_i = \alpha_i + \beta_i R_m + e_i, \quad \text{with} \ i = 1,2,\ldots,k,
\]

(3)

where \( k \) is the number of shares that were analysed.

2. Based on the formation of the equation, the average \( e_i \) is zero, or \( E(e_i) = 0 \)

3. Based on assumptions:
   a. Index unrelated (uncorrelated) with unique return.
      \( E[e_i(R_m - \bar{R}_m)] = 0 \)
   b. Stocks correlated simply because it provides a response to the market.
      \( E[e_i.e_j] = 0 \), for \( i \neq j \)

4. Based on the definition
   a. variance \( e_i \) is \( E[e_i^2] = \sigma_{ei}^2 \)
   b. variance \( R_m \) is \( E[(R_m - \bar{R}_m)^2] = \sigma_m^2 \)

Where \( \alpha_i \) is a unique stock return \( i \), \( \alpha_i \) is the expectations of unique return, \( \beta_i \) is a measure of sensitivity stock \( i \) to market, and \( e_i \) residual error from unique return. The above return data used to estimate the \( \alpha_i \) and \( \beta_i \).

2.3. Estimating \( \alpha_i \) and \( \beta_i \)

In the assessment of stocks, the market risk that is worth considering is the systematic risk. Therefore, investors need to estimate the amount of beta as a measure of market risk. To estimate the beta can be done with the least squares method. From equation (3) can be obtained by the number of squares residual error [6]:

\[
e_i = \sum_{t=1}^{n} e_{it}^2 = \sum_{t=1}^{n} (R_i - \alpha_i - \beta_i R_m)^2
\]

(4)
Derivate equation (4) respects to parameter $\alpha_i$ and $\beta_i$ will be obtained system of equations as follows:

$$\frac{\partial e_i}{\partial \alpha_i} = 0 \text{ and } \frac{\partial e_i}{\partial \beta_i} = 0.$$  

(5)

By completing the system of equations (5) for the parameters $\beta_i$ will be obtained:

$$\hat{\beta}_i = \frac{n \sum_{t=1}^{n} R_{mt} R_{it} - \sum_{t=1}^{n} R_{mt} \sum_{t=1}^{n} R_{it}}{n \sum_{t=1}^{n} R_{mt}^2 - (\sum_{t=1}^{n} R_{mt})^2}.$$  

(6)

While for parameter $\alpha_i$ can be calculated by using equation:

$$\hat{\alpha}_i = \overline{R}_i - \hat{\beta}_i \overline{R}_m.$$  

(7)

The next step is to determine the expected return and variance stock as follows.

2.4. Estimating Expected Return and Stock Variance

Based on the characteristics of single index model, expectations of individual stock returns are:

$$E[R_i] = E[\alpha_i + \beta_i R_m + e_i]$$

$$= E[\alpha_i] + E[\beta_i R_m] + E[e_i]$$

$$\overline{R}_i = \alpha_i + \beta_i \overline{R}_m.$$  

(8)

Equation (2.8) shows that the expectations of individual stock returns consisting of a unique return (not affected the market) and the level of benefits associated with the market (market related return).

Variance individual stock returns using a single index can be calculated by the equation [6]:

$$E[(R_i - \overline{R}_i)^2] = \beta_i^2 E[(R_m - \overline{R}_m)^2] + 2 \beta_i E[e_i (R_m - \overline{R}_m)] + E[e_i^2]$$  

(9)

Because of the characteristics based on Index unrelated (uncorrelated) with unique return and the definition of variance, then the equation (9) becomes:

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{e_i}^2.$$  

(10)

Equation (10) shows that the variance of individual stock returns is composed of two parts, namely the unique risk ($\sigma_{e_i}^2$) and risks associated with the market ($\beta_i^2 \sigma_m^2$). Furthermore, it also estimated covarince and correlation between stocks [6], [1].

2.5. Estimating Covariance and Correlation Between Stocks

To determine the amount of variance between the stock model single indexes can be done as follows:

$$\text{Cov}(R_i, R_j) = E[(R_i - \overline{R}_i)(R_j - \overline{R}_j)]$$
Based on the characteristics (3a) and (3b), equation (11) is written as:

\[ \text{Cov}(R_i, R_j) = \beta_i \beta_j \sigma_m^2 \]  

(12)

From equation (12) it appears that the covariance between the stocks is affected by market risk (this is consistent with the assumption). This is what is known as the single index model. Furthermore, based on the equation (12), the correlation coefficient between stocks, can be expressed as [6], [1]:

\[ \rho_{ij} = \frac{\beta_i \beta_j \sigma_m^2}{\sigma_i \sigma_j} \]  

(13)

After each individually assessed and selected, then determine the model of expectation and variance of portfolio return follows.

2.6. Forming Expectations Return and Variance Portfolio Model

Suppose \( w_i \) represents the proportion (weight) of assets invested in stocks \( i \) in the portfolio formation, thus the portfolio return is:

Based on the equation (13) expected portfolio return is obtained as follows [8]:

\[ E[R_p] = \sum_{i=1}^{k} w_i R_i = E[\sum_{i=1}^{k} w_i (\alpha_i + \beta_i R_m)] \]

Or

\[ \bar{R}_p = \sum_{i=1}^{k} w_i \alpha_i + \bar{R}_m \sum_{i=1}^{k} w_i \beta_i = \alpha_p + \beta_p \bar{R}_m \]  

(14)

With \( \alpha_p = \sum_{i=1}^{k} w_i \alpha_i \) and \( \beta_p = \sum_{i=1}^{k} w_i \beta_i \)

While, the variance of portfolio return can be determined as follows [8], [3]:

\[ \sigma_p^2 = \text{Var}(R_p) = \text{Var}(\sum_{i=1}^{k} w_i (\alpha_i + \beta_i R_m)) \]

\[ = \sum_{i=1}^{k} w_i^2 \sigma_i^2 + \sum_{i=1}^{k} \sum_{j=1}^{k} w_i w_j \text{Cov}(R_i, R_j); i \neq j \]

\[ = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{k} w_i^2 \sigma_{ei}^2 \]  

(15)

To get the proportions right assets, portfolio optimization carried out as follows.

2.7. Doing weight optimization of combination portfolio

Suppose an investor has an aversion factor (risk aversion) \( \theta \), it can be estimated by comparison of the two portfolios. If an investor with risk aversion \( \theta \), with two portfolios that have average rate of return \( \bar{R}_p \) and \( \tilde{R}_p \), along with the variance \( \sigma_p^2 \) and \( \tilde{\sigma}_p^2 \), the comparison between the two is by equating portfolio \([7]\):
\[ R_p - \theta \sigma_p^2 = \bar{R}_p^2 - \theta \bar{\sigma}_p^2 \]  

(16)

Resolving equation (16) for \( \theta \) will be obtained:

\[ \theta = \frac{\bar{R}_p - \bar{R}_p^2}{\sigma_p^2 - \bar{\sigma}_p^2} \]  

(17)

Optimization aims to maximize \( \bar{R}_p - \theta \sigma_p^2 \), so the portfolio optimization problems can be formulated as follows:

**The objective function:**

\[ f(w_1, w_2, ..., w_k) = \left( \sum_{i=1}^{k} w_i \alpha_i + \bar{R}_m \sum_{i=1}^{k} w_i \beta_i \right) - \theta\left( \beta_p^2 \sigma_m^2 + \sum_{i=1}^{k} w_i^2 \sigma_{ei}^2 \right) \]  

(18)

**Constraint functions:**

\[ g(w_1, w_2, ..., w_k) = 1 - \sum_{i=1}^{k} w_i = 0 \]  

(19)

From equation (18) and (19) subsequently formed as a summation function Lagrange, objective function and constraint functions, the Lagrange multiplier factor as follows:

\[ L(w_1, w_2, ..., w_k; \lambda) = f(w_1, w_2, ..., w_k) + \lambda g(w_1, w_2, ..., w_k) \]  

(20)

Derivative partially respect to \( w_1, w_2, ..., w_k \) and \( \lambda \), it will be obtained [7]:

\[ \frac{\partial L}{\partial w_1} = 0 \Rightarrow \lambda = \alpha_1 + (\bar{R}_m - \theta \sigma_m^2) \beta_1 + 2w_1 \sigma_{e1}^2 \]  

(21)

\[ \frac{\partial L}{\partial w_2} = 0 \Rightarrow \lambda = \alpha_2 + (\bar{R}_m - \theta \sigma_m^2) \beta_2 + 2w_2 \sigma_{e2}^2 \]  

(22)

And so on up:

\[ \frac{\partial L}{\partial w_k} = 0 \Rightarrow \lambda = \alpha_k + (\bar{R}_m - \theta \sigma_m^2) \beta_k + 2w_k \sigma_{ek}^2 \]  

(23)

\[ \frac{\partial L}{\partial \lambda} = 0 \Rightarrow 1 - \sum_{i=1}^{k} w_i = 0 \]  

(24)

Resolving the equation system, will be obtained \( w_1, w_2, ..., w_k \) . Match the equation (23) by (21) and the finish will be obtained:

\[ w_k = \frac{(\alpha_k - \alpha_1) + (\beta_k - \beta_1)(\bar{R}_m - \theta \sigma_m^2) + 2w_1 \sigma_{e1}^2}{2e_{1}^2} \]  

(25)

And so on until the equalization equation (22) by (21) and the finish is obtained:

\[ w_2 = \frac{(\alpha_2 - \alpha_1) + (\beta_2 - \beta_1)(\bar{R}_m - \theta \sigma_m^2) + 2w_1 \sigma_{e1}^2}{2e_{1}^2} \]  

(26)

and

\[ w_1 = 1 - \sum_{i=2}^{k} w_i \]  

(27)
The values of this proportion \( w_1, w_2, \ldots, w_k \) is showing the composition of asset allocation (funding) will be invested in each stock in the portfolio formation, so that the rate of return on the maximum investment portfolio [8]. Finally, the estimated variance will do the following.

3. Case Analysis

The data is analysed daily stock price of Lippo Bank and Bumi Putera and Stock Price Index (CSPI) performance of 129 working days in 2015. Then the data is each determined the value of log returns using equation (2.1) and (2.2). Based on descriptive statistics, each return log data has a mean and variance as given in Table 1 below.

Table 1 Mean and Variance of Log Return Data

| Log Return Data   | Mean     | Variance   |
|-------------------|----------|------------|
| Log Return Lippo  | 0.000718 | 0.00010588 |
| Log Return Bumi Putera | 0.001261 | 0.01575025 |
| Log Return IHSG   | 0.017440 | 0.05112120 |

Regression between log return Lippo stock and ISHG obtained equation \( \text{LIPPO} = 0.000664 + 0.00310 \text{IHSG} \), while regression between log returns of Bumi Putera obtained equation \( \text{BUMPU} = 0.0019 + 0.0378 \text{IHSG} \). Squared Residual of regression LIPPO is \( \sigma_{\epsilon L}^2 = 0.0001063 \) and squared residuals of the regression Bumi Putera is \( \sigma_{\epsilon B}^2 = 0.0158000 \). Often, to determine the weight of the investment portfolio is determined based on the comparison of the value of stock returns last log data, where the value of last log data from Lippo stock return is 0.0057907 and last log data value of stock returns from Bumi Putera is 0.028029; thereby weighting the portfolio’s stake in Lippo is 0.17 and the portfolio weights for shares in Bumi Putera is 0.83. Using these values variance portfolio weights are calculated using the formula (2.16), the result is \( \sigma_p^2 = 0.104 \).

It will be different results if used Lagrange Multiplier optimization approach. If the equation (2.18) was approached by several levels challenger risk (risk taker): \( 0 < a \leq 1 \) and several levels of risk-averse: \( -1 \leq a < 0 \); then weighting the portfolio for investors for the risk taker and risk aversion investor can be shown in Figure 1 below.

![Figure 1. Weight portfolio for investors risk taker and risk aversive](image)
It appears that for investors who have a level of risk taker tends to halve its capital allocation with equal weights, which are 0.5 to stock in Lippo and 0.5 for stock in Bumi Putera. As for investors who have a level of risk aversion one would halve its capital allocation with a weight of 0.67 for the Lippo stock and 0.33 for the Bumi Putera stock.

4. Conclusions
Based on results and discussions above can finally be concluded and suggestions for other researchers who are interested in further researching this study. The result of the weight optimization portfolio for investors risk taker with a level of risk taker/avers high tends to divide two equal allocations of investment funds in stocks in its portfolio. As for investors with a level of risk taker / avers higher proportionally while maintaining different weighting the allocation of funds to the investment shares in its portfolio.

Acknowledgment
We thank the program of the Academic Leadership Grant (ALG), Faculty of Mathematics and Natural Sciences, Universitas Padjadjaran (Indonesia), which has provided facilities to conduct a research and publication.

References
[1] Alexader, C. (Editor). (1999). Risk Management and Analysis. Volume 1: Measuring and Modelling Financial Risk. New York : John Wiley & Sons Inc.
[2] Dowd, K. (2002). An Introduction to Market Risk Measurement. United State American: John Wiley & Sons Inc.
[3] Bodie et al. (1999). Investment. Fourth Edition. Singapore : Irwin / McGraw-Hill.
[4] Gruber et al. (1991). Modern Portfolio Theory and Investment Analysis. Singapore: John Wiley & Sons, Inc.
[5] Jorion, P. (2004). Bank Trading Risk and Systemic Risk. Third draft: December 2004.
[6] Ruppert, David. 2004. Statistics and Finance an Introduction. New York: Springer.
[7] Sukono. (2002). Model Matematika Pembentukan Portofolio Investasi Optimum Dengan Menggunakan Indeks Tunggal. Jurnal Matematika Integratif. Vol. 1, No. 1, April 2002.
[8] Tadelilin, E. (2001). Analisis Investasi. Yogyakarta: BPFE.