STRUCTURE THEOREM FOR PROJECTIVE KLT PAIRS WITH NEF ANTI-CANONICAL DIVISOR

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Abstract. In this paper, we establish a structure theorem for a projective klt pair \((X, \Delta)\) with nef anti-log canonical divisor. Specifically, we prove that, up to replacing \(X\) with a finite quasi-étale cover, the variety \(X\) admits a locally trivial rationally connected fibration onto a projective klt variety with numerically trivial canonical divisor. As an application, we extend the Beauville-Bogomolov decomposition to projective klt Calabi-Yau pairs, by showing that klt Calabi-Yau pairs, which naturally appear as an outcome of the Log Minimal Model Program, are decomposed into building block varieties, namely rationally connected varieties and Calabi-Yau varieties.

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1. Introduction

1.1. Structure of varieties with ‘semi-positive’ curvature. We work over the complex number field throughout this paper. A central problem in birational geometry is revealing the structures of fibrations naturally associated with varieties so that the projective varieties can be decomposed into basic building blocks, namely Fano varieties, Calabi-Yau varieties, and (log) canonical models. The abundance conjecture predicts that varieties with ‘semi-negative’ curvature can be revealed by the Iitaka-Kodaira fibrations via the Minimal Model Program (MMP). Meanwhile, the pioneering studies \cite{Mor79, SY80} showed that varieties with ‘semi-positive’ curvature have a certain rigidity and are closely related to the geometry of rational curves. Based on this philosophy, several structure theorems have been established for Albanese maps or maximal rationally connected (MRC) fibrations of varieties with various semi-positivity conditions, including compact Kähler manifolds with semi-positive holomorphic bisectional curvature \cite{HSW81, MZ86, Mok88}; projective manifolds with semi-positive holomorphic sectional curvature \cite{Yan16, Yan18, HW20, Mat18}; compact Kähler manifolds with nef tangent bundle \cite{CP91, DPS94}; projective manifolds with pseudo-effective tangent bundle \cite{HIM19}; and projective manifolds with nef anti-canonical divisor \cite{Cao19, CH19, DLB20}. In particular, projective manifolds with nef anti-canonical divisor cover a large range of varieties with ‘semi-positive’ curvature, and the study of anti-canonical divisors is more natural than that of tangent bundles from the perspective of birational geometry.

In this paper, we study projective varieties with nef anti-canonical divisor. The class of such varieties includes Fano manifolds and Calabi-Yau manifolds as extreme cases, where Mori’s bend and break and the Beauville-Bogomolov decomposition have respectively played a decisive role in revealing their structures. Furthermore, on the basis of the previous works \cite{Pau97, Zha96, Zha05}, Cao-Höring established a structure theorem for smooth projective varieties with nef anti-canonical divisor, which interpolates between ‘Fano-like’ manifolds (rationally connected manifolds) and Calabi-Yau manifolds.

- (Mori’s bend and break, \cite{Mor79, KMM92a, Cam92}). Fano manifolds (i.e., smooth projective varieties with ample anti-canonical divisor) are rationally connected (i.e., any two points can be connected by a rational curve).
• (Beauville-Bogomolov decomposition, [Bea83]). Calabi-Yau manifolds (i.e., compact Kähler manifolds with numerically trivial canonical divisor) are decomposed, up to a finite étale cover, into the product of abelian varieties, strict Calabi-Yau manifolds, and holomorphic symplectic (hyperkähler) manifolds.

• (Cao-Höring’s structure theorem, [Cao19, CH19]). Projective manifolds with nef anti-canonical divisor are constructed by rationally connected manifolds and Calabi-Yau manifolds.

From the perspective of the MMP, it is more natural and of great importance to study the structure theorem not only for smooth varieties but also for varieties with klt singularities. In this direction, the works [Zha06, HM07] generalized the rational connectedness of Fano manifolds to varieties of weak Fano type; that is, if \((X, \Delta)\) is a weak Fano pair (i.e., a klt pair with nef and big anti-log canonical divisor), then \(X\) is rationally connected. Furthermore, the successive works [GKP16b, Dru18a, GGK19, HP19, CP19, Cam20] generalized the Beauville-Bogomolov decomposition to klt Calabi-Yau varieties, which we refer to as the singular Beauville-Bogomolov decomposition in this paper. The most important problem remaining in this field is to establish a structure theorem for klt pairs with nef anti-log canonical divisor, in the form of extending Cao-Höring’s structure theorem, Zhang-Hacon-McKernan’s result, and the singular Beauville-Bogomolov decomposition.

1.2. Projective klt pairs with nef anti-log canonical divisor. In this paper, we establish a structure theorem for projective klt pairs with nef anti-log canonical divisor (see Theorem 1.1), which naturally generalizes Cao-Höring’s structure theorem for smooth projective varieties and Hacon-McKernan’s result for varieties of weak Fano type. Moreover, our structure theorem reduces some problems on the structure of klt pairs with nef anti-log canonical divisor to the singular Beauville-Bogomolov decomposition for klt Calabi-Yau varieties.

**Theorem 1.1.** Let \((X, \Delta)\) be a projective klt pair with the nef anti-log canonical divisor \(-(K_X + \Delta)\). Then, there exists a finite quasi-étale cover \(\nu : X' \to X\) satisfying the following properties:

1. \(X'\) admits a holomorphic MRC fibration \(\psi : X' \to Y\).
2. \(Y\) is a projective klt variety with numerically trivial canonical divisor.
3. \(\psi : X' \to Y\) is a locally constant fibration with respect to the pair \((X', \Delta')\), where \(\Delta'\) is the Weil \(\mathbb{Q}\)-divisor defined by the pullback \(\Delta' := \nu^* \Delta\). In particular, the fibration \(\psi : X' \to Y\) is locally trivial with respect to \((X', \Delta')\), i.e., for a sufficiently small open set \(B \subset Y\), there exists an isomorphism

\[ (\psi^{-1}(B), \Delta') \simeq B \times (F, \Delta'_F) \]

over \(B \subset Y\), where \(F\) is the fiber of \(\psi\) and \(\Delta'_F := \Delta'|_F\).
Here locally constant fibrations are defined in Definition 2.3. The formulation of Theorem 1.1 using locally constant fibrations is essentially important. This viewpoint of locally constant fibrations has not explicitly appeared in previous studies yet, but is often important in applications. For the time being, this fibration can be regarded as a locally trivial fibration while keeping in mind that the local constancy is a much stronger condition than the local triviality. The structure theorems established in the previous studies do not require quasi-étale covers to be considered, but taking an appropriate quasi-étale cover is essential in the above theorem. Indeed, Theorem 1.1 does not hold without taking finite quasi-étale covers (see Remark 4.10 for such an example).

By the singular Beauville-Bogomolov decomposition, the base variety $Y$ in Theorem 1.1 has a finite quasi-étale cover $Y' \to Y$ such that $Y'$ is decomposed into the product of abelian varieties, strict Calabi-Yau varieties, and singular holomorphic symplectic varieties. Hence, by replacing $X'$ with the fiber product of $X' \times_Y Y'$, we deduce the following corollary:

**Corollary 1.2.** Let $(X, \Delta)$ be a projective klt pair with the nef anti-log canonical divisor $- (K_X + \Delta)$. Then, there exists a finite quasi-étale cover $\nu : X' \to X$ satisfying the properties listed in Theorem 1.1 such that the base variety $Y$ of $\psi : X' \to Y$ is decomposed into the product

$$Y \simeq A \times \prod_i Y_i \times \prod_j Z_j$$

of an abelian variety $A$, strict Calabi-Yau varieties $Y_i$, and singular holomorphic symplectic varieties $Z_j$.

Moreover, for klt Calabi-Yau pairs, we strengthen Theorem 1.1 to a splitting theorem (see Theorem 1.3) by combining Corollary 1.2 with [Amb05, Proposition 4.4, Theorem 4.7] and [Dru18a, Lemma 4.6]. Assuming the abundance conjecture, klt Calabi-Yau pairs naturally appear as an outcome of the Log MMP, as log minimal models of klt pairs of Kodaira dimension zero and as general fibers of the Iitaka-Kodaira fibration of log minimal models of positive Kodaira dimension. Theorem 1.3 shows that klt Calabi-Yau pairs can be further decomposed into building blocks comprising rationally connected varieties and Calabi-Yau varieties.

**Theorem 1.3.** Let $(X, \Delta)$ be a projective klt pair with numerically trivial log canonical divisor $K_X + \Delta$. Then, there exists a finite quasi-étale cover $\nu : X' \to X$ such that $(X', \Delta')$ is decomposed into the product

$$(X', \Delta') \simeq (F, \Delta'_F) \times A \times \prod_i Y_i \times \prod_j Z_j,$$

where $\Delta'_F := \nu^* \Delta$, $F$ is the fiber of the MRC fibration of $X'$ (in particular $F$ is rationally connected) with $\Delta'_F := \Delta' |_F$, $A$ is an abelian variety, $Y_i$ is a strictly Calabi-Yau variety, and $Z_i$ is a singular holomorphic symplectic variety as in Theorem 1.1 and Corollary 1.2.
Theorem 1.3 reveals the structure of some building blocks in the framework of the Log MMP, although Theorem 1.1 itself is not a part of the Log MMP. Note that Theorem 1.3 does not hold for lc pairs in general (see [EIM20, Example 6.2]).

1.3. Open problems on fundamental groups and slope rationally connected quotients. In this subsection, we discuss several topics relating to the uniformization, (topological) fundamental groups, and slope rationally connected (sRC) quotients of klt pairs with nef anti-log canonical divisor. This subsection does not directly relate to the proof of our main result, but illustrates the difficulty and interest in handling klt singularities.

We first review the proof of Theorem 1.1 in the case where $X$ is a smooth projective variety with nef anti-canonical divisor. The proof in this case is divided into the following steps:

Step (1) (Study of fundamental groups, [Pău97, Pău17]). The fundamental group $\pi_1(X)$ is shown to be virtually abelian by the theory of Cheeger-Colding [CC96].

Step (2) (Study of Albanese maps, [Cao19]). The Albanese map of $X$ is shown to satisfy the desired structure theorem. Then, together with Step (1), the problem is reduced to the case of $X$ being simply connected.

Step (3) (Study of MRC fibrations, [CH19]). The desired structure theorem for MRC fibrations is completed in the case of $X$ being simply connected (and thus in the general case by Step (2)).

Note that the previous works [Zha96, Zha05] played an important role in establishing Steps (2) and (3).

After Campana-Cao-Matsumura [CCM19] initiated the study of klt pairs $(X, \Delta)$, the second author [Wan22] revealed that the difficulty in adopting the above strategy arises in Step (1); precisely, Theorem 1.1 as well as the uniformization theorem can be deduced if the fundamental group $\pi_1(X_{\text{reg}})$ of the regular locus $X_{\text{reg}}$ has polynomial growth.

Conjecture 1.4 ([Wan22, Conjecture 1.2]). Let $(X, \Delta)$ be a projective klt pair with nef anti-log canonical divisor $-(K_X + \Delta)$. Then, the fundamental group $\pi_1(X_{\text{reg}})$ has polynomial growth.

Conjecture 1.4 is partially solved for the orbifold fundamental group $\pi_1^{\text{orb}}(X, \Delta)$ of an orbifold pair $(X, \Delta)$, but it seems quite difficult to solve Conjecture 1.4 in the general case. In fact, it is even unclear whether the argument in [Pău97, Pău17] works for klt pairs $(X, \Delta)$ with smooth $X$. Since Conjecture 1.4 for smooth projective varieties is a starting point of [CH19] as explained above, we need to take another strategy for the proof of Theorem 1.1. Our strategy displayed in this paper is independent of the results on fundamental groups and is new even in the smooth case.

Theorem 1.1 can be applied to prove the uniformization theorem of klt pairs $(X, \Delta)$ with nef anti-log canonical divisor. Indeed, Theorem 1.1 reduces the uniformization problem for $(X, \Delta)$ to the following conjecture, which is...
formulated only for the fundamental group of Calabi-Yau varieties (and not for the regular locus), and thus more straightforward than Conjecture 1.4.

**Conjecture 1.5.** Let $Y$ be a projective klt variety with numerically trivial canonical divisor. If $Y$ has vanishing augmented irregularity, then $\pi_1(Y)$ is finite.

Theorem 1.1 opens the study of sRC quotients. The MRC fibration of $X$ has the drawback of not containing any information of the boundary divisor $\Delta$, whereas the sRC quotient of $(X, \Delta)$, introduced in [Cam16], is a natural generalization of the MRC fibration that considers the boundary divisor $\Delta$. It would be an attractive problem to establish a structure theorem of sRC quotients by comparing MRC fibrations of $X$ to sRC quotients of $(X, \Delta)$. Based on [CCM19, Conjecture 1.5], we pose the following conjecture:

**Conjecture 1.6.** Let $(X, \Delta)$ be a klt pair with nef anti-log canonical divisor $-(K_X + \Delta)$. Then, up to a finite quasi-étale cover, the sRC quotient $(X, \Delta) \to (Z, \Delta_Z)$ is a locally constant fibration with numerically trivial $K_Z + \Delta_Z$. Moreover, the MRC fibration $X \to Y$ in Theorem 1.1 factorizes through $(X, \Delta) \to (Z, \Delta_Z)$.

**Organization of the paper.** This paper is organized as follows: In Section 2, we recall some definitions and preliminary results, especially the definitions of MRC fibrations, locally constant fibrations, singular Hermitian metrics, and finite quasi-étale covers. Unlike the previous works for smooth projective varieties, we need quasi-étale covers in studying klt pairs for Theorem 1.1 (see Remark 4.10). This requires non-trivial and technical discussions in Sections 3 and 4. In Section 3, we prove a technical result (see Theorem 3.2), which asserts that, up to replacing $X$ with a quasi-étale cover and a $\mathbb{Q}$-factorialization, appropriately defined ‘direct image sheaves $\psi_*(mL)$’ are flat for a positive integer $m \in \mathbb{Z}_{>0}$, where $\psi: X \to Y$ is an MRC fibration of $X$ and $L$ is a $\psi$-ample line bundle on $X$. The proof of this flatness clarifies the reason why we need a finite quasi-étale cover of $X$ in Theorem 1.1. In Section 4 we deduce the desired structure for $(X, \Delta)$. By combining the flatness result above with the results of (singular) foliations established by Stéphane Druel, we can prove the structure theorem in a birational sense, and the main difficulty and the major contribution of this section is to deduce the main theorem from such a birational structure result. Let us remark that in the discussion in Section 4, the formulation of Theorem 1.1 using locally constant fibrations plays a crucial role. This viewpoint is important not only for the proof but also for applications of the theorem.

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2. Preliminary results

2.1. Notation and conventions. Throughout this paper, all the varieties and morphisms are defined over the complex number field, and analytic varieties denote irreducible and reduced complex analytic spaces. We interchangeably use the terms ‘locally free sheaves’ and ‘vector bundles,’ and further the terms ‘Cartier divisors,’ ‘invertible sheaves,’ and ‘line bundles.’ For example, a Cartier divisor $D$ is regarded as a line bundle, and Hermitian metrics on $D$ denote those on the line bundle associated with $D$ (where this convention differs from \cite{Wan22}). We often use an additional notation for the tensor products of invertible sheaves $L$ and $M$ (e.g., $-L = L^*$ and $L + M = L \otimes M$).

2.2. MRC fibrations. We recall the definitions and properties of MRC fibrations and rationally (chain) connected varieties.

Definition 2.1. Let $X$ and $Y$ be normal projective varieties. A dominant rational map $\psi: X \to Y$ is called an RC fibration if it satisfies the following conditions:

- $\psi: X \to Y$ is almost holomorphic (i.e., the indeterminacy locus is not dominant over $Y$, and a general fiber is connected).
- A general fiber is rationally connected.

There exist maximal rationally chain connected (MRCC) fibrations for normal projective varieties by \cite{Cam92, KMM92b}. Rationally chain connected varieties are not necessarily rationally connected, but the two notions coincide for potentially dlt projective varieties by \cite[Corollary 1.5]{HM07}. Hence, for a projective klt pair $(X, \Delta)$, the variety $X$ always admits an MRC fibration (i.e., an RC fibration $\psi: X \to Y$ such that a very general rational curve in $X$ is contained in the fiber $X_y := \psi^{-1}(y)$ at some point $y \in Y$. This definition of MRC fibrations can be rephrased as the condition that $K_Y$ is pseudo-effective when $Y$ is smooth by \cite{BDPP13, GHS03}.
Rationally connected varieties with mild singularities are simply connected by [Tak03, Theorem 1.1] and have no (non-trivial) holomorphic differential forms. These properties are used in the proof.

**Theorem 2.2.** Let $X$ be a rationally connected and potentially klt projective variety. Then $X$ is simply connected and $H^0(X, \mathcal{O}_X^p) = 0$ for any $p > 0$.

### 2.3. Locally constant fibrations

In this subsection, after we give the definition of locally constant fibrations, we explain the properties of the locally constant fibrations (e.g., Lemma 2.4) and a criterion for a fiber space to be locally constant using the flatness of certain direct image sheaves (see Proposition 2.5).

**Definition 2.3.** Let $\phi: X \to Y$ be a fiber space between normal analytic varieties (i.e., a proper surjective morphism with connected fibers), and let $\Delta$ be a Weil $\mathbb{Q}$-divisor on $X$.

1. $\phi: X \to Y$ is called a locally constant fibration with respect to the pair $(X, \Delta)$ if it satisfies the following conditions:
   - $\phi: X \to Y$ is an analytic fiber bundle with the fiber $F$.
   - Every component $\Delta_i$ of $\Delta$ is horizontal (i.e., $\phi(\Delta_i) = Y$).
   - There exists a Weil $\mathbb{Q}$-divisor $\Delta_F$ on $F$ and a representation $\rho: \pi_1(Y) \to \text{Aut}(F)$ of the fundamental group $\pi_1(Y)$ to the automorphism group $\text{Aut}(F)$ such that
     - $\Delta_F$ is invariant under the action of $\pi_1(Y)$;
     - $(X, \Delta)$ is isomorphic to the quotient $\left(Y^{\text{univ}} \times F; \text{pr}_2^* \Delta_F\right)/\pi_1(Y)$ over $Y$. Here $Y^{\text{univ}}$ is the universal cover of $Y$ and $\gamma \in \pi_1(Y)$ acts on $Y^{\text{univ}} \times F$ by the diagonal action
       $$\gamma \cdot (y, z) := (\gamma \cdot y, \rho(\gamma)(z))$$
       for $(y, z) \in Y^{\text{univ}} \times F$.

2. $\phi: X \to Y$ is simply said to be locally constant if it satisfies the above conditions for $\Delta = 0$.

Note that, when $\phi: X \to Y$ is a locally constant fibration, the fiber product $X \times_Y Y^{\text{univ}}$ is isomorphic to the product $Y^{\text{univ}} \times F$ and the induced morphism $X \times_Y Y^{\text{univ}} \to Y^{\text{univ}}$ is equal to the first projection $Y^{\text{univ}} \times F \to Y^{\text{univ}}$ under this identification. We thus have the following diagram:

$$
\begin{array}{c}
\phi \\
\downarrow \quad \downarrow \quad \downarrow \\
X & \xleftarrow{p_X} & X \times_Y Y^{\text{univ}} & \cong & Y^{\text{univ}} \times F & \xrightarrow{\text{pr}_2} & F \\
\text{pr}_1 & & \downarrow \quad \downarrow \quad \downarrow & & \downarrow \\
Y & \xleftarrow{p_Y} & Y^{\text{univ}}.
\end{array}
$$

(2.1)

A locally constant fibration $\phi: X \to Y$ with respect to a pair $(X, \Delta)$ is locally trivial, i.e., it is an analytic fiber bundle with fiber $F$, and there exists a Weil $\mathbb{Q}$-divisor $\Delta_F$ on $F$ such that

$$(\phi^{-1}(B), \Delta) \simeq B \times (F, \Delta_F)$$
over a sufficiently small open set \( B \subset Y \). However, the converse implication does not hold in general; e.g., the projective space bundle \( \mathbb{P}(E) \to Y \) of a locally free sheaf \( E \) on \( Y \) is always locally trivial, but not necessarily locally constant.

The projective space bundle \( \mathbb{P}(E) \to Y \) is a locally constant fibration if \( E \) is flat. Here, we recall that a vector bundle \( E \) on a smooth variety \( Y \) is said to be flat if \( E \) admits a flat connection or, equivalently, if \( E \) arises from a representation \( \pi_1(Y) \to \text{GL}_r(\mathbb{C}) \), where \( r := \text{rk} E \) (see [Kob87, §I.2, pp. 4-6]). Moreover, the category of flat vector bundles is equivalent to that of local systems (i.e., locally constant sheaves) via the Riemann-Hilbert correspondence. Hence, we interchangeably use them in the sequel.

The previous works [Cao19, CH19] formulated structure theorems using locally trivial fibrations. Nevertheless, it is important to formulate Theorem 1.1 with locally constant fibrations for several reasons; e.g., a locally constant fibration induces a splitting of tangent sheaves (see [Wan22, Remark 2.2] and the proof of Theorem 4.7); moreover, our proof needs the following lemma (see Subsection 4.4), which does not hold for locally trivial fibrations in general.

**Lemma 2.4.** Let \( \phi : X \to Y \) be a projective locally constant fibration such that the fiber \( F \) has vanishing irregularity. Let \( L \) be a line bundle \( X \). Then, there exist a line bundle \( L_F \) on \( F \) and a \( \mathbb{Q} \)-line bundle (\( \mathbb{Q} \)-Cartier divisor) \( L_Y \) on \( Y \) such that the pullback of \( p_X^*(L - \phi^*L_Y) \) is a flat vector bundle on \( Y \) (or the zero sheaf) for any sufficiently divisible integer \( m \in \mathbb{Z}_{> 0} \).

**Proof.** We use the notations in the diagram (2.1). Let \( \rho : \pi_1(Y) \to \text{Aut}(F) \) be a representation such that \( X \) is the quotient of \( Y^{\text{univ}} \times F \) by the action of \( \pi_1(Y) \). By assumption, the fiber \( F \) is a projective variety with vanishing irregularity. Hence we have the decomposition of the Picard group \( \text{Pic}(Y^{\text{univ}} \times F) \simeq \text{Pic}(Y^{\text{univ}}) \times \text{Pic}(F) \) by the analytic version of [Har77, §III.12, Exercise 12.6, p. 292]. In particular, there exists a line bundle \( L_Y^{\text{univ}} \) on \( Y^{\text{univ}} \) and a line bundle \( L_F \) on \( F \) such that \( p_X^*L \simeq \text{pr}_1^*L_Y^{\text{univ}} + \text{pr}_2^*L_F \). Note that \( L_F \) is a \( \rho \)-equivariant line bundle, but not necessarily \( \rho \)-linearizable.

By applying the above argument to a \( \phi \)-ample line bundle, we can find a \( \rho \)-equivariant ample line bundle on \( F \). Thus, the Zariski closure \( G \) of \( \text{Im} (\rho) \) is a linear algebraic group. By [KKLV89, Proposition 2.4], we can take an integer \( m_0 \in \mathbb{Z}_{> 0} \) such that \( mL_F \) is \( G \)-linearizable for any integer \( m \in \mathbb{Z}_{> 0} \) divisible by \( m_0 \). (Precisely speaking, the cited result [KKLV89, Proposition 2.4] assumes that \( G \) is connected, but this assumption can be removed since \( G \) has only finitely many connected components.) The line bundle \( L_0 \) defined by \( L_0 := (Y^{\text{univ}} \times m_0L_F)/\pi_1(Y) \) satisfies that \( p_X^*L_0 \simeq \text{pr}_2^*(m_0L_F) \) and \( (m_0L - L_0)|_F \simeq \mathcal{O}_F \). Hence, we can find a line bundle \( L_Y \) on \( Y \) such that \( m_0L - L_0 \sim \phi^*L_Y \).

We finally show that the line bundle \( L_F \) and the \( \mathbb{Q} \)-line bundle \( L_Y := (1/m_0)L_Y \) satisfy the desired properties. By construction, we can easily find
that $p_X^*(L - \phi^*L_Y) \sim Q \, \text{pr}_2^* L_F$. Furthermore, for any $m \in \mathbb{Z}$ divisible by $m_0$, we can see that

$$\text{pr}_1^*(p_X^*(m(L - \phi^*L_Y))) = \text{pr}_1^*(\text{pr}_2^*(mL_F)) = \mathcal{O}_{Y^{\text{univ}}} \otimes \mathbb{C} H^0(F, mL_F).$$

Hence, by the flat base change theorem, we obtain

$$(2.2) \quad p_Y^*(\phi_*(m(L - \phi^*L_Y))) \cong \mathcal{O}_{Y^{\text{univ}}} \otimes \mathbb{C} H^0(F, mL_F).$$

The representation $\rho: \pi_1(Y) \to \text{Aut}(F)$ linearly acts on $H^0(F, mL_F)$, and the quotient by this action coincides with $\phi_*(m(L - \phi^*L_Y))$. This implies that $\phi_*(m(L - \phi^*L_Y))$ is a flat vector bundle on $Y$. \qed

For a locally constant fibration $X \to Y$, every line bundle on $X$ can be modified (up to the tensoring of a line bundle on $Y$) so that its direct image is a flat vector bundle by Lemma 2.4. Conversely, if there exists a relatively very ample line bundle on $X$ whose direct image sheaf is flat, then $X \to Y$ is necessarily a locally constant fibration by the following proposition:

**Proposition 2.5.** Let $h: V \to W$ be a fiber space from a normal analytic variety $V$ to a complex manifold $W$, and let $D$ be an effective Weil $\mathbb{Q}$-divisor on $V$. Assume that $h$ is a flat projective morphism, and assume that there is an $h$-relatively very ample line bundle $L$ on $V$ such that

1. $E_m := h_*(mL)$ is a local system for every $m \in \mathbb{Z}_{>0}$;
2. for every $m \in \mathbb{Z}_{>0}$, the natural morphism $\text{Sym}^m E_1 \to E_m$ is a morphism between local systems (i.e., compatible with the flat connections);
3. for some $k \in \mathbb{Z}_{>0}$ rendering $kD$ a $\mathbb{Z}$-divisor, $F_m := h_*(mL - kD)$ is a sub-local system of $E_m$ for every $m \in \mathbb{Z}_{>0}$.

Then $h: V \to W$ is a locally constant fibration with respect to $(V, D)$.

**Proof.** By assumption, the line bundle $L$ induces a relative embedding $\iota: V \hookrightarrow \mathbb{P}(E_1)$ over $W$ such that $\iota^*\mathcal{O}_{\mathbb{P}(E_1)}(1) = L$. Let $p: W^{\text{univ}} \to W$ be the universal cover of $W$. Then $p^*E_1$ is a trivial vector bundle, and there are $(r + 1)$ global sections $e_0, e_1, \ldots, e_r \in H^0(W^{\text{univ}}, p^*E_1)$ that are parallel with respect to (the pullback of) the flat connection $\nabla_{E_1}$, where $r + 1 := \text{rk} E_1$.

Let $\mathcal{I}_V$ be the ideal sheaf of $V$ in $\mathbb{P}(E_1)$. By the relative Serre vanishing, for a sufficiently large $m$, we have the short exact sequence

$$0 \to S_m := g_*(\mathcal{I}_V \otimes \mathcal{O}_{\mathbb{P}(E_1)}(m)) \to g_*(\mathcal{O}_{\mathbb{P}(E_1)}(m)) = \text{Sym}^m E_1 \to E_m := h_*(mL) \to 0,$$

where $g$ denotes the natural morphism $\mathbb{P}(E_1) \to W$. Since $\mathcal{I}_V$ is flat over $W$ (by the flatness of $h$), the sheaf $S_m = g_*(\mathcal{I}_V \otimes \mathcal{O}_{\mathbb{P}(E_1)}(m))$ is a vector bundle for a sufficiently large $m$ by [ACG11, Proposition (3.3), p. 13, Vol. II]. The morphism $g_*(\mathcal{O}_{\mathbb{P}(E_1)}(m)) \to E_m$ is a morphism of local systems by condition (2). This indicates that $S_m$ is also a local system and the inclusion $S_m \to \text{Sym}^m E_1$ is also a morphism of local systems since the category of local systems is abelian. Hence, we get global sections
Proposition 2.5 differs from [Cao13, Cao19, DPS94, Den17, Wan22, CCM19] as follows.

We view a section of $p^*\text{Sym}^m E_1$ as a pullback of a section of $\nabla_{\text{Sym}^m E_1}$. Hence, for some constants $c_{i,\alpha} \in \mathbb{C}$.

\[
s_i = \sum_{\alpha = (\alpha_0, \ldots, \alpha_r) \in \mathbb{Z}_{\geq 0}^{r+1}} c_{i,\alpha} \cdot e_0^{\alpha_0} \cdots e_r^{\alpha_r}
\]

for some constants $c_{i,\alpha} \in \mathbb{C}$. This implies that the relative embedding $\bar{V} := V \times_W W^{\text{univ}}$ into $\mathbb{P}^r \times W^{\text{univ}}$ is defined by polynomials whose coefficients are independent of $w \in W^{\text{univ}}$. Hence $\bar{V}$ splits into a product $W^{\text{univ}} \times F$, where $F$ is the general fiber of $h$.

The vector bundle $E_1$ is induced by a representation $\pi_1(W) \to \text{GL}(r+1)$, which gives rise to a representation $\rho : \pi_1(W) \to \text{PGL}(r+1)$. For $\gamma \in \pi_1(W)$, the automorphism $\rho(\gamma) : \mathbb{P}^r \to \mathbb{P}^r$ sends the fiber $V_w$ at $w \in W$ to $V_{\rho(w)}$ viewed as subvarieties of $\mathbb{P}^r$. As seen before, the defining polynomial of $V_w$ in $\mathbb{P}^r$ is independent of $w$. Hence $\rho(\gamma)$ can be restricted to $F$ so that we get the representation $\rho : \pi_1(W) \to \text{Aut}(F)$. By construction, the variety $\bar{V}$ is isomorphic to the quotient of $\bar{V}$ by the action of $\pi_1(W)$. Hence $h$ is a locally constant fibration.

Moreover, since $F_m = h_* (mL - kD)$ is a sub-local system of $E_m$, the vector bundle $(h|_{kD})_* (mL|_{kD})$ is also a local system. Then, by the same argument as above, we see that the automorphism $\rho(\gamma) \in \text{Aut}(F)$ is also an automorphism of $DF := D|_F$ for every $\gamma \in \pi_1(W)$. Moreover $D$ is isomorphic to the quotient of $D \times W W^{\text{univ}}$ by the action of $\pi_1(W)$. Hence $h$ is a locally constant fibration with respect to $(V, D)$.

\[\square\]

Remark 2.6. (a) Proposition 2.5 differs from [Wan22, Proposition 2.1] at the point that we already assume that the $E_m$ denotes a local system so that we can tackle the non-compact case, and the result of Simpson [Sim92] is not required in the proof; however, we must assume the compatibility condition (2). Note that the result is not true without the compatibility condition (2), e.g., if $W$ is a non-compact Riemann surface, then every vector bundle on $W$ is trivial, but there are non-isotrivial flat families over $W$.

(b) If we further assume that $W$ is a compact Kähler manifold, then the conditions (1), (2), and (3) can be replaced with the condition that $E_m$ and $F_m$ are both numerically flat vector bundles by [DPS94, Sim92, Den17, Cao13] (cf. [Cao19, Proposition 2.4], [CCM19, Proposition 2.8], and [Wan22, Proposition 2.1]). Indeed, the compatibility conditions are automatically satisfied by [Cao13, Lemma 4.3.3]).

(c) From the proof, we see that it is sufficient to check that the conditions (1), (2), and (3) hold for $m = 1$ and for some sufficiently large $m = m_0$. 

\[\square\]
2.4. **Singular Hermitian metrics on torsion-free sheaves.** In this subsection, we introduce singular Hermitian metrics on torsion-free sheaves on normal analytic varieties.

Let \( E \) be a torsion-free sheaf on a normal analytic variety \( X \). Throughout this paper, the notation \( X_E \) denotes the maximal locally free locus of \( E \), and \( X_{reg} \) (resp. \( X_{sing} \)) denotes the regular locus (resp. the singular locus) of \( X \). Note that \( \operatorname{codim} X_{sing} \geq 2 \) and \( \operatorname{codim}(X \setminus X_E) \geq 2 \) by the normality of \( X \) and the torsion-freeness of \( E \).

Let \( g \) be a singular Hermitian metric on \( E \) (which means a possibly singular Hermitian metric on the vector bundle \( E|_{X_{reg} \cap X_E} \)). See [Rau15, Pău18, PT18, HPS18] for singular Hermitian metrics on vector bundles and also [Wan20, §1.4, §2.2.4]. Let \( \theta \) be a smooth \((1,1)\)-form on \( X \) admitting a local potential function (i.e., \( \theta \) can be locally written as \( \theta = dd^c f \) for some smooth function \( f \)). Note that \( d \)-closed forms do not always admit local potential functions when \( X \) has singularities.

We write \( \sqrt{-1} \Theta_g \geq \theta \otimes \text{id} \) on \( X \) if the local function \( \log |e|_{g^*} - f \) is plurisubharmonic (psh) on \( X_{reg} \cap X_E \) for any (holomorphic) local section \( e \) of \( E \) for \( g^* \) being the induced metric on the dual sheaf \( E^* := \text{Hom}(E, \mathcal{O}_X) \). The notation \( \sqrt{-1} \Theta_g \), which resembles the curvature, does not make sense since an appropriate definition of curvature is not known in the higher-rank case, but the notation (2.3) does make sense. The function \( \log |e|_{g^*} - f \) is actually psh on \( X \) since any psh functions on a Zariski open set \( X_0 \) of \( \operatorname{codim}(X \setminus X_0) \geq 2 \) can be extended to \( X \) by [GR56, Satz 3, p.181].

**Definition 2.7.** Let \( E \) be a torsion-free sheaf and \( \omega \) be a positive \((1,1)\)-form on a compact normal analytic variety \( X \) admitting a local potential function.

1. \( E \) is said to be **weakly positively curved** if \( E \) admits singular Hermitian metrics \( \{g_\varepsilon\}_{\varepsilon > 0} \) such that \( \sqrt{-1} \Theta_{g_\varepsilon} \geq -\varepsilon \omega \otimes \text{id} \) on \( X \).

2. \( E \) is said to be **pseudo-effective** if, for any \( m \in \mathbb{Z}_{>0} \), there exists a singular Hermitian metric \( h_m \) on \((\text{Sym}^m E)^*\) such that \( \sqrt{-1} \Theta_{h_m} \geq -\omega \otimes \text{id} \) on \( X \).

By definition, weakly positively curved sheaves are always pseudo-effective. Note that there is an algebraic-geometric characterization of the pseudo-effectivity when \( E \) is a locally free sheaf on a smooth projective variety (see [HIM19, Proposition 2.2] and references therein). In this paper, we consider the pullback of weakly positively curved sheaves by surjective morphisms. If a locally free sheaf is weakly positively curved, its pullback is so (e.g., see [PT18]). However, the behavior of torsion-free sheaves under a pullback differs from that of locally free sheaves (see Remark 2.9 below). The following lemma gives a condition that guarantees that torsion-free sheaves satisfy this property.
Lemma 2.8. Let $\phi: M \to X$ be a surjective morphism between (not necessarily compact) normal analytic varieties $M$ and $X$. Let $\theta$ be a smooth $(1,1)$-form on $X$ admitting a local potential function, and let $(E, g)$ be a torsion-free sheaf on $X$ with a singular Hermitian metric $g$ satisfying $-\sqrt{-1}\Theta_g \succeq \theta \otimes \text{id}$ on $X$. Assume that the inverse image $\phi^{-1}(X \setminus X_E)$ is of codimension $\geq 2$ and that the pullback $\phi^*(E^*)$ is a reflexive sheaf on $M$.

Then, the induced metric $\phi^*g$ on $\phi^*E|_{\phi^{-1}(X_E)}$ (which is a vector bundle on $\phi^{-1}(X_E)$) can be extended to the singular Hermitian metric on the torsion-free sheaf $\phi^*E/\text{Tor}$ defined by the quotient by its torsion subsheaf satisfying that

$$\sqrt{-1}\Theta_{\phi^*g} \succeq \phi^*\theta \otimes \text{id \ on \ } M.$$  

Remark 2.9. (1) The assumptions in Lemma 2.8 are automatically satisfied if $\phi: M \to X$ is flat or if $E$ is locally free on $X$. Indeed, by $\text{codim}(X \setminus X_E) \geq 2$, the flatness implies $\text{codim}(\phi^{-1}(X \setminus X_E)) \geq 2$. Furthermore, the dual sheaf of any coherent sheaf is always reflexive, and the reflexivity is preserved under pullback by flat morphisms (see [Har80, Corollary 1.2, Proposition 1.8]).

(2) In general, the pullback $\phi^*E$ is not always a torsion-free sheaf. Even if we consider $\phi^*E/\text{Tor}$, the conclusion of the lemma does not hold without the assumptions. For example, the maximal ideal $m_p \subset O_{X,p}$ at a given point $p \in X$ admits a singular Hermitian metric $g$ such that $-\sqrt{-1}\Theta_g = 0$ on $X$, which is induced by the trivial metric on $O_X$. For the blow-up $\phi: M := \text{Bl}_p(X) \to X$ at the point $p \in X$, the quotient sheaf $\phi^*m_p/\text{Tor}$ of the pullback $\phi^*m_p$ is the ideal sheaf $O_M(-E)$ associated with the exceptional divisor $E$.

However, the sheaf $O_M(-E)$ obviously admits no singular Hermitian metric with semi-positive curvature.

Proof. Let $e$ be a section of $(\phi^*E)^*$ on an open set $B$ in $M$. Our purpose is to demonstrate that $\log |e|_{\phi^*g^*}$ is a $\theta$-psh function on $B$ (i.e., $dd^c \log |e|_{\phi^*g^*} \geq \phi^*\theta$).

The induced metric $\phi^*g^*$ is a priori defined on $\phi^{-1}(X_{\text{reg}} \cap X_E)$. The conclusion of the lemma follows from [PT18, Lemma 2.3.2] when $X$ is smooth and $E$ is locally free. Hence $\log |e|_{\phi^*g^*}$ is $\phi^*\theta$-psh on $B \cap \phi^{-1}(X_{\text{reg}} \cap X_E)$. In general, any $\theta$-psh functions on a Zariski open set can be extended on the ambient space if they are bounded from above. Therefore, it suffices to demonstrate that $\log |e|_{\phi^*g^*}$ is (locally) bounded from above.

The assumptions yield the following isomorphisms:

$$H^0(B, (\phi^*E)^*) \simeq H^0(B \cap \phi^{-1}(X_E), (\phi^*E)^*) \text{ by codim } \phi^{-1}(X \setminus X_E) \geq 2 \text{ and reflexivity},$$

$$\simeq H^0(B \cap \phi^{-1}(X_E), \phi^*E^*) \text{ by the local freeness of } E \text{ and } \phi^*E,$$

$$\simeq H^0(B, \phi^*(E^*)) \text{ by codim } \phi^{-1}(X \setminus X_E) \geq 2 \text{ and reflexivity},$$

$$= \lim_{\phi(B) \subset V \text{ open}} H^0(V, E^*) \otimes H^0(V, O_X) H^0(B, O_M) \text{ by the definition of the functor } \phi^*.$$
Therefore, the section $e$ can be written as

$$e = \sum_{i=1}^{m} s_i \otimes f_i$$

for some $s_i \in \mathcal{H}^0(V, \mathcal{E}^*)$ and $f_i \in \mathcal{H}^0(B, \mathcal{O}_M)$. Then, we have

$$|e|_{\phi^*g^*}^2 \leq \sum_{i=1}^{m} |f_i|^2 |s_i \otimes 1|_{\phi^*g^*}^2 = \sum_{i=1}^{m} |f_i|^2 |\phi^*s_i|^2_{g^*}$$

on $B \cap \phi^{-1}(X_{\text{reg}} \cap X_{\mathcal{E}})$ from the Cauchy-Schwarz inequality and the definition of $\phi^*g^*$. The function $\log |s_i|_{g^*}$ is $\theta$-psh on $V$ (in particular, bounded from above) by $\sqrt{-1} \Theta_{g^*} \geq \theta \otimes \text{id}$. Combining this fact with the above inequality, we see that $|e|_{\phi^*g^*}^2$ is bounded from above on $B$, and thus, it can be extended to the $\phi^*\theta$-psh function on $B$. □

At the end of this subsection, we investigate the push-forward of weakly positively curved sheaves by birational morphisms.

**Lemma 2.10.** Let $\pi: M \to X$ be a birational morphism from a smooth projective variety $M$ to a $\mathbb{Q}$-factorial projective variety $X$, and let $\mathcal{F}$ be a weakly positively curved sheaf on $M$. Then, the push-forward $\pi_* \mathcal{F}$ is weakly positively curved on $X$.

**Proof.** By assumption, there exists a singular Hermitian metric $g_\varepsilon$ on $\mathcal{F}$ such that $\sqrt{-1} \Theta_{g_\varepsilon} \geq -\varepsilon \omega_M \otimes \text{id}$ on $M$, where $\omega_M$ is a Kähler form on $M$. By comparing $\omega_M$ with a Kähler form on $X$, we investigate the singular Hermitian metrics $h_\varepsilon$ on $\pi_* \mathcal{F}$ defined by pushing-forward $g_\varepsilon$.

Let $\omega_X$ be a Kähler form on $X$ defined by $\omega_X := \Phi^* \omega_{FS}$, where $\Phi: X \to \mathbb{P}(|A|)$ is the embedding associated with a very ample line bundle $A$ on $X$ and $\omega_{FS}$ is the Fubini-Study form on $\mathbb{P}(|A|)$. Note that $\omega_X$ locally admits a smooth potential function by construction. By $\mathbb{Q}$-factoriality, we can take a $\pi$-exceptional effective $\mathbb{Q}$-divisor $G$ on $M$ such that $\pi^* A - G$ is ample on $M$ (cf. [Kol07, Lemma 2.9, Complement 2.10, pp. 73-74]). We may assume that $\omega_M$ is a Kähler form representing $c_1(\pi^* A - G)$. Then, we can take a smooth $(1,1)$-form $\theta$ on $M$ such that

$$\theta \in -c_1(G)$$

and $\omega_M = \pi^* \omega_X + \theta$.

We show that there exists a quasi-psh function $\varphi$ on $X$ such that

$$\omega_M = \pi^* \omega_X - [G] + dd^c \pi^* \varphi,$$

where $[G]$ is the integration current associated with the $\mathbb{Q}$-divisor $G$. This easily follows from the push-forward of currents when $X$ is smooth. For the reader’s convenience, we present a proof for a singular $X$. We first take a quasi-psh function $\psi$ on $M$ such that

$$\omega_M = \pi^* \omega_X - [G] + dd^c \psi.$$
Let $E$ be the $\pi$-exceptional locus. The function $\varphi$ on $X \setminus \pi(E)$ is defined by $\varphi = \pi_* \psi$ via the isomorphism $\pi: M \setminus E \simeq X \setminus \pi(E)$. Then, a local smooth function $f$ with $\omega_X = dd^c f$ satisfies that

$$dd^c(\pi^* f + \pi^* \varphi) = \pi^* \omega_X + dd^c \varphi = \omega_M$$

locally on $M \setminus E$ by $\text{Supp}(G) = E$. This indicates that $f + \varphi$ is a psh function on $X \setminus \pi(E)$. The function $\varphi$ can be extended to the quasi-psh function on $X$ since $f$ is smooth and $\pi(E)$ is of codimension $\geq 2$. Then, the extended function $\varphi$ satisfies that $\psi = \pi^* \varphi$ on $M$ since $\psi$ and $\pi^* \varphi$ are quasi-psh, which leads to the desired equality (2.4).

We define the singular Hermitian metric $h_\varepsilon$ on $\pi_* \mathcal{F}|_{X \setminus \pi(E)}$ by $h_\varepsilon := \pi_* g_\varepsilon$. We will construct weakly positively curved metrics on $\pi_* \mathcal{F}$ by modifying $h_\varepsilon$ with $\varphi$. For simplicity of the notation, we used the same notation $X$ (resp. $M$) to denote a sufficiently small open set in $X$ (resp. its inverse image by $\pi$). Let us consider the function

$$\varphi_\varepsilon := \log |e|_{h_\varepsilon}^2 \text{ on } X \setminus \pi(E)$$

for a local section $e$ of $(\pi_* \mathcal{F})^*$. We now have

$$H^0(X, (\pi_* \mathcal{F})^*) \simeq H^0(X \setminus \pi(E), (\pi_* \mathcal{F})^*) \text{ by codim } \pi(E) \geq 2 \text{ and reflexivity},$$

$$\simeq H^0(M \setminus E, \mathcal{F}^*) \text{ by the isomorphism } \pi: M \setminus E \simeq X \setminus \pi(E).$$

This indicates that the section $e$ of $(\pi_* \mathcal{F})^*$ can be identified with the section in $H^0(M \setminus E, \mathcal{F}^*)$, which we denote by $\pi^* e$. Then, by definition, we have

$$\pi^* \varphi_\varepsilon = \pi^*(\log |e|_{h_\varepsilon}^2) = \log |\pi^* e|_{\pi^* h_\varepsilon} = \log |\pi^* e|_{\varepsilon h_\varepsilon} \text{ on } M \setminus E.$$ 

Hence, we obtain

$$dd^c \pi^* \varphi_\varepsilon = dd^c \log |\pi^* e|_{\varepsilon h_\varepsilon} \geq -\varepsilon \omega_M = -\varepsilon (\pi^* \omega_X + dd^c \pi^* \varphi) \text{ on } M \setminus E$$

by the selection of $g_\varepsilon$ and equality (2.4). This equality indicates that

$$dd^c (\varphi_\varepsilon + \varepsilon \varphi) \geq -\varepsilon \omega_X \text{ on } X \setminus \pi(E).$$

Hence $\varphi_\varepsilon + \varepsilon \varphi$ can be extended to the $\varepsilon \omega_X$-psh function on $X$ by codim $\pi(E) \geq 2$. Note that $\varphi_\varepsilon + \varepsilon \varphi$ can be extended because of $\varphi$ (which may have the pole), but this extension is not expected for $\varphi_\varepsilon$ itself. Thus, the metric $H_\varepsilon := h_\varepsilon e^{-\varepsilon \varphi}$ on $\pi_* \mathcal{F}$ satisfies the definition of weakly positively curved metrics. \qed

2.5. **Finite quasi-étale covers.** In this subsection, following [GKP16a], we summarize the definition and basic properties of quasi-étale covers.

**Definition 2.11.** Let $\nu: X \to Y$ be a surjective morphism between normal analytic varieties $X$ and $Y$.

1. $\nu: X \to Y$ is called a cover (resp. finite cover) of $Y$ if every fiber of $\nu$ is a finite set of discrete points (resp. $\nu: X \to Y$ is a finite morphism).
(2) $\nu: X \to Y$ is said to be quasi-étale if there exists a Zariski closed set $Z \subset X$ of codimension $\geq 2$ such that the induced morphism $\nu|_{X\setminus Z}: X \setminus Z \to Y$ is étale.

(3) $\nu: X \to Y$ is said to be maximally quasi-étale if it is a finite quasi-étale cover such that any finite étale cover of $X_{\text{reg}}$ extends to a finite étale cover of $X$, i.e., the natural morphism

$$i_*: \hat{\pi}_1(X_{\text{reg}}) \to \hat{\pi}_1(X)$$

between the étale fundamental groups (which are the profinite completion of the topological fundamental groups) is isomorphic. We simply say that $X$ is maximally quasi-étale when $X$ itself satisfies the above condition.

Note that the category of finite quasi-étale covers of $X$ is equivalent to that of finite index subgroups in the étale fundamental group $\hat{\pi}_1(X_{\text{reg}})$ of $X_{\text{reg}}$. For potentially klt projective varieties, a maximally quasi-étale cover always exists by [GKP16b, Theorem 1.5] and this property is preserved under birational morphisms (see Theorem 2.12 (2)), which plays an important role in Section 3.

**Theorem 2.12.** Let $X$ be a potentially klt projective variety. Then, we have:

1. There exists a maximally quasi-étale cover Galois cover $\nu: \bar{X} \to X$.
2. Let $X'$ be a potentially klt projective variety and $\pi: X' \to X$ be a birational morphism. If $X$ is a maximally quasi-étale cover, then so is $X'$.

**Proof.** Conclusion (1) is a direct consequence of [GKP16b, Theorem 1.5]. We check conclusion (2). Take a Zariski open set $U \subset X_{\text{reg}}$ such that $\text{codim}(X \setminus U) \geq 2$ holds and $\pi: X' \to X$ is isomorphic over $U$. Set $V := \pi^{-1}(U) \subset X'$ and consider the following commutative diagram:

$$
\begin{array}{ccc}
\pi_1(V) & \xrightarrow{j_*} & \pi_1(X'_{\text{reg}}) & \xrightarrow{\pi_*} & \pi_1(X') \\
\pi_* \cong & & & & \pi_* \cong \\
\pi_1(U) & \xrightarrow{i_*} & \pi_1(X_{\text{reg}}) & \xrightarrow{\cong} & \pi_1(X).
\end{array}
$$

The vertical arrows are isomorphic by [Tak03]. All the horizontal arrows are surjective since $X$ and $X'$ are normal, and further, the morphism $i_*: \pi_1(U) \to \pi_1(X_{\text{reg}})$ induced by the natural inclusion $i: U \hookrightarrow X_{\text{reg}}$ is isomorphic by $\text{codim}(X_{\text{reg}} \setminus U) \geq 2$. Then, by taking the profinite completion (which is a right-exact functor), we obtain the following diagram:

$$
\begin{array}{ccc}
\hat{\pi}_1(V) & \xrightarrow{j_*} & \hat{\pi}_1(X'_{\text{reg}}) & \xrightarrow{\hat{\pi}_1(X')} \\
\hat{\pi}_1(U) & \xrightarrow{i_*} & \hat{\pi}_1(X_{\text{reg}}) & \xrightarrow{\hat{\pi}_1(X)}.
\end{array}
$$
Note that $\hat{\pi}_1(X_{\text{reg}}) \to \hat{\pi}_1(X)$ is isomorphic since $X$ is a maximally quasi-étale cover. This implies that $\hat{\pi}_1(X'_{\text{reg}}) \to \hat{\pi}_1(X')$ is also isomorphic. □

The following lemma is an elementary result obtained from the Stein factorization, and we thus omit the proof.

**Lemma 2.13.** Let $g: X' \to X$ be a birational morphism and $\eta': X'_1 \to X'$ be a finite quasi-étale cover between normal projective varieties. Then, there exists a projective variety $X_1$ with a finite quasi-étale cover $\eta_1: X_1 \to X$ and a birational morphism $g_1: X'_1 \to X_1$ such that $\eta_1 \circ g_1 = g \circ \eta'$.

### 3. Flat connections on direct image sheaves

In this section, we study projective klt pairs with nef anti-log canonical divisor and direct image sheaves appropriately defined by their MRC fibrations. This section is devoted to the proof of Theorem 3.2, which states that the direct image sheaves satisfy a certain flatness up to $\mathbb{Q}$-factorializations and finite quasi-étale covers.

For a given klt pair $(X, \Delta)$, we take a maximally quasi-étale cover $\nu: \bar{X} \to X$ by Theorem 2.12 (1) and a $\mathbb{Q}$-factorial log terminal model $\pi: (\bar{X}_{\text{qf}}, \bar{\Delta}_{\text{qf}}) \to (\bar{X}, \bar{\Delta})$ by [BCHM10, Corollary 1.4.3], where $\bar{\Delta}_{\text{qf}}$ and $\bar{\Delta}$ are $\mathbb{Q}$-divisors defined by the pullbacks. Then, the pair $(\bar{X}_{\text{qf}}, \bar{\Delta}_{\text{qf}})$ is a $\mathbb{Q}$-factorial terminal pair and $\bar{X}_{\text{qf}}$ is a maximally quasi-étale cover by Theorem 2.12 (2). We will apply Theorem 3.2 to $(\bar{X}_{\text{qf}}, \bar{\Delta}_{\text{qf}})$ to derive the holomorphicity and local constancy of MRC fibrations of $\bar{X}_{\text{qf}}$, and prove that the MRC fibration of $\bar{X}_{\text{qf}}$ induces the desired MRC fibration for $(X, \Delta)$, which is discussed in Section 4.

The proof of Theorem 3.2 clarifies why we need a finite quasi-étale cover of $X$ in Theorem 1.1, whereas the quasi-étale cover never appears when $X_{\text{reg}}$ is simply connected (see [Wan22]) or when $X$ is smooth (see [CCM19]). Theorem 3.2 requires a deeper insight into nef anti-log canonical divisors and a more involved argument than what has been presented in the previous works [Cao19, CH19, CCM19, Wan22].

#### 3.1. Setting and goal for this section

In this subsection, we give a precise formulation of Theorem 3.2 after we describe our setting in this section.

**Setting 3.1.** Let $(X, \Delta)$ be a projective $\mathbb{Q}$-factorial terminal pair such that $X$ is a maximally quasi-étale cover and the anti-log canonical divisor $-(K_X + \Delta)$ is nef. Let $\psi: X \dashrightarrow Y$ be an MRC fibration of $X$ to a smooth projective variety $Y$. Let $\pi: M \to X$ be a resolution of singularities of $X$ and indeterminacies of $\psi: X \dashrightarrow Y$ with a morphism $\phi: M \to Y$ in the following diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{\pi} & X \\
\phi \downarrow & & \downarrow \psi \\
& & Y \\
\end{array}
\]
Let $A$ be a (sufficiently) ample line bundle on $X$ and set $E := \text{Exc}(\pi)$. (In fact, Theorem 3.2 is proved for any effective divisor $E$ whose support coincides with $\text{Exc}(\pi)$, but in this paper we set $E := \text{Exc}(\pi)$ for simplicity.) For a fixed integer $c \in \mathbb{Z}_{>0}$ (which we take to be large enough later), we consider the determinant sheaf
det \phi_* \mathcal{O}_M(\pi^* A + cE) := \left( \bigwedge^r \phi_* \mathcal{O}_M(\pi^* A + cE) \right)^{**},
where $r$ is the rank of the direct image sheaf $\phi_* \mathcal{O}_M(\pi^* A + cE)$. Note that this
determinant sheaf is an invertible sheaf (a line bundle) since it is reflexive
and $Y$ is smooth. We define the $\phi$-big line bundle $L_m$ on $M$ by
$L_m := \mathcal{O}_M(m(\pi^* A + cE)) - \frac{m}{r} \phi^* \text{det} \phi_* \mathcal{O}_M(\pi^* A + cE),$
where $m$ is a positive integer with $m/r \in \mathbb{Z}$. Here, we used the additive
notation for tensor products and we can regard $L_m$ as a line bundle by
$m/r \in \mathbb{Z}$ (see Subsection 2.1). Furthermore, we define the direct image
sheaf $V_m$ on $Y$ by
$V_m := \phi_* \mathcal{O}_M(L_m).
$The subscript $m$ in $V_m$ and $L_m$ is not important in most of this section, and
we thus often omit the subscript $m$ to simplify the notation. Let $Y_0 \subset Y$ be
the maximal Zariski open set in $Y$ satisfying the following properties:
* $\phi: M \to Y$ is a flat morphism over $Y_0$.
* $\phi^* P$ is not $\pi$-exceptional for any prime divisor $P$ on $Y_0$.

This section aims to prove that $V_m$ satisfies a certain flatness on $Y_0$.

**Theorem 3.2.** Consider the same situation as in Setting 3.1. Then, for
some fixed $c \in \mathbb{Z}_{>0}$ and for every $m \in \mathbb{Z}_{>0}$ with $m/r \in \mathbb{Z}$, there exists a
Zariski closed set $C_m \subset Y$ such that
* $C_m \subset Y$ is of codimension $\geq 2$;
* $V_m$ is locally free on $Y_0 \setminus C_m$;
* $V_m$ admits a flat connection on $Y_0 \setminus C_m$.

The proof of Theorem 3.2 is divided into three subsections. Throughout
this section, we keep Setting 3.1 and promise that $m$ always satisfies $m/r \in \mathbb{Z}$.

**3.2. Birational semi-stability for MRC fibrations.** In this subsection,
we confirm a certain birational semi-stability for the MRC fibration $\psi: X \to Y$. Such a semi-stability result, which essentially follows from [Zha05, Main
Theorem], has been explicitly formulated in [CH19], [CCM19, Theorem 3.2],
and [Wan22, Proposition 3.1]. Proposition 3.3 is a slight refinement of the
above results.

**Proposition 3.3.** The following statements hold:
* $\phi^* N_Y$ is $\pi$-exceptional for any effective divisor $N_Y$ on $Y$ with $N_Y \sim_{\mathbb{Q}} K_Y$. 

(b) The Kodaira dimension of $Y$ is zero (i.e., $\kappa(K_Y) = 0$).
(c) $\pi(\phi^{-1}(Y \setminus Y_0))$ is of codimension $\geq 2$.
(d) $Y_0$ has the generalized Liouville property in the following sense: For a flat vector bundle $(\mathcal{H}_0, \nabla_0)$ on $Y_0$ with the following condition ($\cdot$), every global section of $\mathcal{H}_0$ is parallel with respect to $\nabla_0$.
($\cdot$) There exists a numerically flat vector bundle $\mathcal{H}$ on $X$ such that
\[(\phi^*\mathcal{H}_0, \phi^*\nabla_0) \simeq (\pi^*\mathcal{H}, \nabla)\] on $M_0 := \phi^{-1}(Y_0)$,
where $\phi^*\nabla_0$ is the connection on $\phi^*\mathcal{H}_0$ defined by the pullback and $\nabla$ is the (unique) flat connection on $\pi^*\mathcal{H}$ defined by [Sim92, Corollary 3.10 and the discussion thereafter] (which is compatible with the filtration given by [DPS94, Theorem 1.18]).
(e) $\psi$ is semi-stable in codimension one in the following sense: Let $P$ be a prime divisor on $Y_0$ and $\phi^*P = \sum c_iP_i$ be the irreducible decomposition of $\phi^*P$. Then, any non-reduced component $P_i$ (i.e., a component $P_i$ with $c_i > 1$) is $\pi$-exceptional.
(f) $\Delta$ is horizontal with respect to $\psi$ (i.e., $Y = \phi(\pi^{-1}_*\Delta_i)$ for any component $\Delta_i$ of $\Delta$).

Remark 3.4. In the proof, we assume that $\psi: X \to Y$ is an MRC fibration only to deduce that $Y$ is not uniruled. In fact, we can obtain all the conclusions for an almost holomorphic map $\psi: X \to Y$ with a non-uniruled $Y$ under the weaker assumption that $(X, \Delta)$ is log canonical. The case of $\mathcal{H} = \mathcal{O}_{Y_0}$ in property (d) is nothing but [Wan22, Proposition 3.1(d)].

Proof. Note that the base variety $Y$ of MRC fibrations is not uniruled by [GHS03]. Hence, all the conclusions except for (d) follow from [Wan22, Proposition 3.1].

The generalized Liouville property (i.e. conclusion (d)) essentially follows from property (c). Let $s \in H^0(Y_0, \mathcal{H}_0)$. Then $s$ is parallel with respect to $\nabla_0$ if and only if $\phi^*s$ is parallel with respect to $\phi^*\nabla_0 = \nabla|_{\phi^{-1}(Y_0)}$. Since the complement of $\pi(\phi^{-1}(Y_0) \setminus E)$ in $X$ is of codimension 2 by property (c),
\[\text{the section } \phi^*s|_{\phi^{-1}(Y_0) \setminus E} \text{ on } \phi^{-1}(Y_0) \setminus E \simeq \pi(\phi^{-1}(Y_0) \setminus E)\]
induces a section $\sigma \in H^0(X, \mathcal{H})$ by reflexivity (via the isomorphism $\pi|_{\phi^{-1}(Y_0) \setminus E}$). Then $\pi^*\sigma$ is parallel with respect to $\nabla$ by [Cao13, Theorem 4.3.3], and hence so is $\phi^*s$. \qed

From the next subsection, following [DPS94], we recall the definition of numerically flat vector bundles. Note that we do not define the numerical flatness for non-locally free sheaves.
Definition 3.5 (Numerically flat locally free sheaves). Let \( \mathcal{E} \) be a locally free sheaf on a projective variety. The sheaf \( \mathcal{E} \) is said to be \textit{numerically effective} (\( \text{nef} \)) if the hyperplane bundle \( \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \) on the projective space bundle \( \mathbb{P}(\mathcal{E}) \) is a nef line bundle. Furthermore, the sheaf \( \mathcal{E} \) is said to be \textit{numerically flat} if \( \mathcal{E} \) is nef and \( c_1(\mathcal{E}) = 0 \).

3.3. Positivity of singular Hermitian metrics on direct image sheaves.

The purpose of this subsection is to prove Proposition 3.9, which states that \( \mathcal{E}_m \) is a numerically flat vector bundle on \( X \). In the proof of Proposition 3.9, we use the assumption that \( X \) is a maximally quasi-étale cover and smooth in codimension two.

We first prove Proposition 3.6 by applying Lemma 2.10 and the positivity of direct image sheaves. In the proof of Proposition 3.6, we use the assumption that \( X \) is \( \mathbb{Q} \)-factorial.

Proposition 3.6. There exists a sufficiently large integer \( c \in \mathbb{Z}_{>0} \) (recalling that \( c \) is an integer appearing in the definition of \( \mathcal{V}_m \)) such that every \( m \in \mathbb{Z}_{>0} \) the following properties holds:

1. \( \mathcal{E}_m \) satisfies that \( c_1(\mathcal{E}_m) = 0 \).
2. \( \mathcal{E}_m \) is weakly positively curved on \( X \).

Proof of Proposition 3.6. We deduce (1) and (2) from [Wan22, Proposition 3.5, Lemma 3.1] and Lemmas 2.8 and 2.10. Note that the assumption of [Wan22, Proposition 3.5, Lemma 3.1] is satisfied thanks to the \( \mathbb{Q} \)-factoriality of \( X \).

We first prove conclusion (1). By [Wan22, Proposition 3.5] (or [CCM19, Proposition 3.9]), we have already known that \( \mathcal{V}_m \) satisfies that \( c_1(\pi_*\phi^*\mathcal{V}_m) = 0 \). Strictly speaking, for some sufficiently large integer \( c \in \mathbb{Z}_{>0} \), we obtain \( c_1(\pi_*\phi^*\mathcal{V}_m) = 0 \). The sheaf \( \phi^*\mathcal{V}_m \) is reflexive on \( M_0 = \phi^{-1}(Y_0) \) since \( \phi: M \to Y \) is flat over \( Y_0 \) by [Har80, Proposition 1.8]. Furthermore, the Zariski closed subset \( \pi(M \setminus M_0) \) is of codimension \( \geq 2 \) by property (c) of Proposition 3.3. Hence, we obtain

\[
\pi_*\phi^*\mathcal{V}_m = (\pi_*\phi^*\mathcal{V}_m)^{**} = \mathcal{E}_m \text{ on } X \setminus \pi(\text{Exc}(\pi))
\]

from the isomorphism \( \pi: M \setminus \text{Exc}(\pi) \simeq X \setminus \pi(\text{Exc}(\pi)) \). Then, we see that 
\[ 0 = c_1(\pi_*\phi^*\mathcal{V}_m) = c_1(\mathcal{E}_m) \text{ by codim } \pi(\text{Exc}(\pi)) \geq 2. \]

We finally prove conclusion (2). By [Wan22, Lemma 3.1], the sheaf \( \mathcal{V}_m \) is weakly positively curved on \( Y_0 \) in the following sense: For any \( \varepsilon > 0 \), there exists a singular Hermitian metric \( b_\varepsilon \) on \( \mathcal{V}_m|_{Y_0} \) such that

\[
\sqrt{-1} \Theta_{b_\varepsilon} \succeq -\varepsilon \omega_Y \otimes \text{id} \text{ on } Y_0
\]

for some Kähler form \( \omega_Y \) on \( Y \). The key point here is that \( \omega_Y \) in the above inequality is a Kähler form on \( Y \) (not only on \( Y_0 \)). Then, we deduce that \( \phi^*\mathcal{V}_m \) is weakly positively curved on \( M \setminus E \) in the sense that

\[
\sqrt{-1} \Theta_{b_\varepsilon} \succeq -\varepsilon \omega_M \otimes \text{id} \text{ on } M \setminus E,
\]

(3.2)
for some singular Hermitian metric $g_\varepsilon$ on $\phi^*V_m|_{M\setminus E}$; where $\omega_M$ is a fixed Kähler form on $M$. Indeed, the induced morphism $\phi: M_0 = \phi^{-1}(Y_0) \to Y_0$ satisfies the assumptions of Lemma 2.8 (see also Remark 2.9). By applying Lemma 2.8 to $V_m$ (which is weakly positively curved on $Y_0$), we see that (3.2) holds on $M_0$. Here, we use the fact that $\phi^*\omega_Y \leq k\omega_M$ holds on $M$ for a sufficiently large $k$. Any $\varepsilon\omega_M$-psh functions on the complement of a Zariski closed set of codimension $\geq 2$ can be automatically extended to the ambient variety by [GR56, Satz 3, p. 181]. By applying this fact to the non-divisorial part of $M \setminus M_0$, we see that (3.2) holds outside the divisorial components of $M \setminus M_0$. The divisorial components of $M \setminus M_0$ are contained in $\text{Exc}(\pi)$ by property (c) of Proposition 3.3. We therefore conclude that $\pi_\ast\phi^*V_m$ (and thus $E_m$) is weakly positively curved on $X \setminus \pi(\text{Exc}(\pi))$. Then conclusion (2) follows from $\text{codim} \pi(\text{Exc}(\pi)) \geq 2$. □

Hereafter, we fix $c \in \mathbb{Z}_{>0}$ satisfying Proposition 3.6. We now discuss the numerical flatness of pseudo-effective sheaves with vanishing first Chern class. If $X$ is smooth, then a pseudo-effective reflexive sheaf with vanishing first Chern class is a fortiori locally free and numerically flat by [HIM19, Wu20]. This is nevertheless not true if $X$ has singularities as in the following example:

**Example 3.7.** Let $X$ be a rationally connected $Q$-abelian variety, i.e., there exists a finite quasi-étale cover $\nu: A \to X$ by an abelian variety $A$ (see [Cam10], [KL09, §2, Corollary 24, p. 193] for details of $Q$-abelian varieties). Then, the tangent sheaf $T_X$ admits a flat Hermitian metric on $X_{\text{reg}}$ (in particular, $T_X$ is pseudo-effective) since $T_{X_{\text{reg}}}$ is an étale quotient of the trivial tangent bundle on the abelian variety. Furthermore, the variety $X$ has klt singularities by [KM98, Proposition 5.20, p.160] and the canonical divisor $K_X$ is $Q$-linearly trivial by $p^*K_X \sim_Q K_A = O_A$. Nevertheless, the tangent sheaf $T_X$ never is locally free. Indeed, if $T_X$ is locally free, the variety $X$ is smooth, which contradicts the rational connectedness.

This example also shows that the fiber dimension of the MRC fibration is not preserved under taking quasi-étale covers. Indeed, the constant map $X \to \{\ast\}$ is an MRC fibration of $X$, but the identity map $\text{id}: A \to A$ is an MRC fibration of the quasi-étale cover $A$.

Taking a finite quasi-étale cover, we can obtain the local freeness and numerical flatness as in [HIM19, Wu20].

**Theorem 3.8 ([HP19, Theorem 1.8], [GKP16a, Theorem 1.20]).** Let $\mathcal{F}$ be a reflexive sheaf on a normal projective variety $Z$. Assume that $Z$ is potentially klt and smooth in codimension two, and that $\mathcal{F}$ is pseudo-effective (which is satisfied if it is weakly positively curved) and satisfies $c_1(\mathcal{F}) = 0$. Then, there exists a finite quasi-étale Galois cover $\nu: \tilde{Z} \to Z$ with normal $\tilde{Z}$ such that $(\nu^*\mathcal{F})^{**}$ is locally free and numerically flat on $\tilde{Z}$.

**Proof.** The same conclusion has been proved in [HP19, Theorem 1.8] when $\mathcal{F}$ is almost nef. We can easily see that the same proof works when $F$ is
pseudo-effective. (Strictly speaking, the cited result [HP19, Theorem 1.8]
assumes that $X$ is klt, but this assumption can be relaxed to potentially klt
(see [GKP16a, Theorem 1.20]). □

**Proposition 3.9.** The sheaf $E_m$ is locally free and numerically flat on $X$ for
every $m \in \mathbb{Z}_{>0}$.

**Proof.** When $X$ is smooth, the sheaf $E_m$ is locally free and numerically flat on
$X$ by [HIM19, Theorem 1.2]. When $X$ is singular, we need to take a quasi-
étale cover to obtain the same conclusion for the pullback of $E_m$, as explained
in Example 3.7 and Theorem 3.8. Nevertheless, since we are assuming that
$X$ is a maximally quasi-étale cover, we obtain the desired conclusion without
taking any quasi-étale cover.

We first show that it suffices to prove that $E_m$ is locally free on $X$ for
every $m \in \mathbb{Z}_{>0}$. For the resolution of singularities $\pi: M \to X$, the pullback
$\pi^*E_m$ also satisfies the conclusions of Proposition 3.6. Here, we essentially
used the local freeness. Hence $\pi^*E_m$ is numerically flat by [HIM19, Theorem
1.2] (which corresponds the smooth case of Theorem 3.8) and so is $E_m$.

Since $(X, \Delta)$ is a $\mathbb{Q}$-factorial terminal pair, the variety $X$ has terminal
singularities. Hence $X$ is smooth in codimension two. This indicates that $X$
and $E_m$ satisfy the assumptions of Theorem 3.8. By Theorem 3.8, for each
$m \in \mathbb{Z}_{>0}$, we can take a finite quasi-étale Galois cover $\nu_m: X_m \to X$
such that $(\nu_m^*E_m)^{**}$ is locally free. Since $X$ is a maximally quasi-étale cover, the
cover $\nu_m: X_m \to X$ is actually étale. This implies that $E_m$ is locally free on
$X$. □

**Remark 3.10.** Hereafter, the dependence of $E_m$ and $V_m$ on $m$ is negligible,
and we thus omit the subscript $m$.

### 3.4. Rationally connected fibers and flat connections.

In this subsection, we prove Theorem 3.2 by applying Proposition 3.9.

**Proof of Theorem 3.2.** We omit the subscript $m$ in $V_m$ and $E_m$ in the proof.
The variety $X$ in our setting may have singularities. Thus, the $\pi$-exceptional
locus $\text{Exc}(\pi)$ may be dominant over $Y$, which causes a problem in proving
the theorem. To overcome this difficulty, we consider the normalization $\Gamma$ of the
graph of $\psi: X \dashrightarrow Y$. Both $\phi: M \to Y$ and $\pi: M \to X$ factorize through
$\Gamma$, equipped with the morphisms $\varphi: \Gamma \to Y$, $\mu: \Gamma \to X$, and $\gamma: M \to \Gamma$ in
the following diagram:

\[
\begin{array}{c}
M \\
\downarrow \phi \\
\gamma \\
\downarrow \\
\Gamma \\
\downarrow \mu \\
X \\
\downarrow \psi \\
Y
\end{array}
\]
The \( \pi \)-exceptional locus \( \text{Exc}(\pi) \) may be dominant over \( Y \). However, the \( \mu \)-exceptional locus \( E_\mu := \text{Exc}(\mu) \) is not dominant over \( Y \) since \( \psi : X \longrightarrow Y \) is almost holomorphic.

Let \( B \subset Y \) be the union of the non-locally free locus of \( \mathcal{V} \) and non-flat locus of \( \varphi : \Gamma \rightarrow Y \). Note that \( B \) is of codimension \( \geq 2 \) by the torsion-freeness of \( \mathcal{V} \) and the normality of \( Y \). We first confirm the following claim:

**Claim 3.11.** We have

\[
\pi^* \mathcal{E} = \phi^* \mathcal{V} \quad \text{on} \quad M \setminus (\gamma^{-1}(E_\mu) \cup \phi^{-1}(B)).
\]

**Proof.** The sheaf \( \varphi^* \mathcal{V} \) is locally free on \( \Gamma \setminus \varphi^{-1}(B) \). Hence, we obtain

\[
\gamma_* \gamma^* \varphi^* \mathcal{V} = \varphi^* \mathcal{V} \quad \text{on} \quad \Gamma \setminus \varphi^{-1}(B)
\]

by applying the projection formula to the algebraic fiber space

\[
\gamma : M \setminus \phi^{-1}(B) \rightarrow \Gamma \setminus \varphi^{-1}(B).
\]

Then, we see that

\[
\mu^* \pi_* \phi^* \mathcal{V} = \mu^* \mu_* \gamma_* \gamma^* \varphi^* \mathcal{V} = \mu^* \mu_* \phi^* \mathcal{V} = \varphi^* \mathcal{V} \quad \text{on} \quad \Gamma \setminus (E_\mu \cup \varphi^{-1}(B))
\]

by using the diagram (3.3) and the isomorphism \( \mu : \Gamma \setminus E_\mu \simeq X \setminus \mu(E_\mu) \).

This implies that

\[
\pi^* \mathcal{E} = \gamma^* \mu^* ((\pi_* \phi^* \mathcal{V})^{**}) = \gamma^* ((\mu^* \pi_* \phi^* \mathcal{V})^{**}) = \phi^* \mathcal{V}
\]

on \( M \setminus (\gamma^{-1}(E_\mu) \cup \phi^{-1}(B)) \). Here, we used the definition of \( \mathcal{E} \) and the local freeness (reflexivity) of \( \varphi^* \mathcal{V} \) on \( \Gamma \setminus \varphi^{-1}(B) \). \( \square \)

Let us return to the proof of Theorem 3.2. By assumption, we can easily see that the pullback \( \pi^* \mathcal{E} \) is a numerically flat locally free sheaf on \( M \). Hence \( \pi^* \mathcal{E} \) admits a unique flat connection \( \nabla \) by [Sim92, Corollary 3.10 and the discussion after] and [DPS94, Theorem 1.18]. By the definition of \( B \) and codim \( B \geq 2 \), it is sufficient to show that \( \mathcal{V} \) admits a flat connection on \( Y_0 \setminus (B \cup C) \) for some Zariski closed set \( C \) of codimension \( \geq 2 \). For simplicity, we assume that \( B = \emptyset \) by replacing \( Y \) with \( Y \setminus B \) (i.e., \( \mathcal{V} \) is locally free and \( \varphi : \Gamma \rightarrow Y \) is flat). We construct the desired flat connection of \( \mathcal{V} \) on \( Y_0 \setminus C \) adopting the following strategy:

1. We demonstrate that the connection \( \nabla \) of \( \pi^* \mathcal{E} \) descends to a flat connection of \( \mathcal{V} \) on a Zariski open set in \( Y \), by using \( \pi^* \mathcal{E} = \phi^* \mathcal{V} \) over \( Y \setminus \varphi(E_\mu) \).
2. We demonstrate that the flat connection of \( \mathcal{V} \) constructed in (1) can be extended on \( Y_0 \setminus C \) for some Zariski closed set \( C \) of codim \( \geq 2 \).

(1) We first consider only a neighborhood (in the analytic topology) of a given point in \( Y \). Therefore, assuming that \( Y \) is a (sufficiently small) open ball, we take a local frame of \( \mathcal{V} \) on \( Y \). This frame determines the frame of \( \pi^* \mathcal{E} \) on \( M \setminus \gamma^{-1}(E_\mu) \) by Claim 3.11 (recalling that \( B = \emptyset \)). The flat connection \( \nabla \) of \( \pi^* \mathcal{E} \) can be written as \( \nabla = d + \Xi \). Here \( d \) is the exterior derivative and \( \Xi \), which is the connection form, is a \((1,0)\)-form valued in \( \text{End}(\pi^* \mathcal{E}) \). We
regard $\Xi$ as a matrix of $(1,0)$-forms with respect to the fixed frame. By the flatness of $\nabla$, the matrix $\Xi$ satisfies the equation
\[ d\Xi + \Xi \wedge \Xi = 0. \]
The $(1,1)$-part of the left-hand side is $\overline{\partial}\Xi$ (which is zero). Hence $\Xi$ can be considered as a matrix of holomorphic one-forms on $M \setminus \gamma^{-1}(E_\Gamma)$.

We will confirm that $\Xi$ descends to a holomorphic $(1,0)$-form $\Xi'$ valued in $\text{End}(\mathcal{V})$ on $Y_1$. Here $Y_1$ is defined by
\[ Y_1 := \left\{ y \in Y \setminus \varphi(E_\Gamma) \mid \begin{array}{c}
\text{The fiber } M_y := \phi^{-1}(y) \text{ is } \\
\text{smooth and rationally connected.}
\end{array} \right\} \]
Since $\psi: X \to Y$ is an MRC fibration, the fiber $X_y := \psi^{-1}(y)$ of a general point $y \in Y$ is rationally connected, and so is the fiber $M_y$. This indicates that $Y_1$ is a non-empty Zariski open set in $Y$. It is possible that $\varphi(E) = Y$, but we have $\varphi(E_\Gamma) \subset Y$, which is the reason why we consider $E_\Gamma$ rather than $E$. The fiber $M_y$ at $y \in Y_1$ has no non-trivial holomorphic one-form. Hence, the components of $\Xi$, which are holomorphic one-forms, vanish along the fiber $M_y$ at $y \in Y_1$. Furthermore, we have $\text{End}(\pi^*E) = \phi^* \text{End}(\mathcal{V})$ over $Y \setminus \varphi(E_\Gamma)$ by Claim 3.11. Therefore, we find a holomorphic section
\[ \Xi' \in H^0(Y_1, \Omega^1_\mathcal{E} \otimes \text{End}(\mathcal{V})) \]such that $\Xi = \phi^* \Xi'$.

(2) We will prove that $\Xi'$ can be extended to a section on $Y_0 \setminus C$. Let $P \subset Y_0$ be a divisorial component of $Y \setminus Y_1$. It is sufficient to extend $\Xi'$ through a general point $y \in P$. The pullback $\phi^*P$ is not $\pi$-exceptional by the definition of $Y_0$. Hence, we find a reduced component $P_0$ of $\phi^*P$ by property (e) in Proposition 3.3. Note that $P_0 \not\subset \gamma^{-1}(E_\Gamma)$ since $\gamma^{-1}(E_\Gamma)$ is $\pi$-exceptional. Furthermore, the induced morphism $\phi: P_0 \to P$ is surjective since $\phi: M \to Y$ is flat over $Y_0$ (recalling that $P \subset Y_0$). Therefore, for a general point $y \in P$, we find a point $x \in P_0 \cap M_y$ such that
\[ x \not\in \gamma^{-1}(E_\Gamma) \text{ and } x \text{ is a smooth point of } \phi. \]
The first property shows that all the components of the matrix $\Xi = \phi^* \Xi'$ are bounded on a neighborhood of $x$, since $\Xi$ is defined not only on $\phi^{-1}(Y_1)$ but also on $M \setminus \gamma^{-1}(E_\Gamma)$. The second property yields a local section of $\phi: M \to Y$ from $y$ to $x$. By pulling back $\Xi = \phi^* \Xi'$ by this local section, we see that all the components of the matrix $\Xi'$ are also bounded on a neighborhood of $y$. Therefore, the section $\Xi'$ can be extended through a general point $y \in P$ by the Riemann extension theorem.

The extended connection of $\mathcal{V}$ (defined locally on $Y_0 \setminus C$) satisfies the gluing condition as a connection and is flat from the flatness of $\nabla$. Hence, the glued connection determines the flat connection of $\mathcal{V}$ on $Y_0 \setminus C$. \hfill \Box

Remark 3.12. (a) The key point in this proof is that a general fiber of $\phi: M \to Y$ is rationally connected (in particular, it has no holomorphic one-form). Furthermore, we implicitly used the fact that $\phi: M \to Y$ has connected fibers.
(b) By the proof of Theorem 3.2 we see that the flat vector bundle $V_m|_{Y_m}\cdot C_m$ satisfies the condition (●) in Proposition 3.3(d) for every $m$.

4. MRC fibrations of klt pairs with nef anti-log canonical divisor

This section is devoted to the proof of Theorem 1.1. To this end, we study MRC fibrations of projective klt pairs with nef anti-log canonical divisor by applying Theorem 3.2 and the theory of foliations.

4.1. Local constancy of algebraic fiber spaces. In this subsection, we study the local constancy and relative anti-canonical divisors of algebraic fiber spaces. The content of this subsection is not only motivated by the proof of Theorem 1.1, but also independently interesting. The first result (see Theorem 4.1) gives conditions on singularities and the positivity of relative anti-canonical divisors to guarantee that algebraic fiber spaces are locally constant, which generalizes [Wan22, Theorem A] (see Corollary 4.2).

**Theorem 4.1.** Let $h: V \to W$ be an algebraic fiber space between normal projective varieties with a smooth $W$. Let $D$ be an effective $\mathbb{Q}$-divisor on $V$ such that $(V, D)$ is klt and $-(K_{V/W} + D)$ is nef. Then $h: V \to W$ is a locally constant fibration with respect to $(V, D)$.

**Corollary 4.2.** Let $(X, \Delta)$ be a projective klt pair with the nef anti-canonical divisor $-(K_X + \Delta)$. Then, the Albanese map of $X$ is a locally constant fibration with respect to $(X, \Delta)$.

To prove Theorem 4.1, we need the following preliminary result:

**Proposition 4.3.** In the same setting as in Theorem 4.1, the following statements hold:

(a) $h$ is flat.
(b) $D$ is horizontal with respect to $h$.
(c) For every pseudo-effective and $\phi$-big divisor $G$ on $V$, the direct image sheaf $h_* \mathcal{O}_V(q(K_{V/W} + D) + G)$ is weakly positively curved for every $q \in \mathbb{Z}$.
(d) For every $\phi$-big divisor $G$ on $V$ and every $m \in \mathbb{Z}_{>0}$, we define the Cartier divisor by $D_{G, m} := \frac{1}{r_m} \cdot$ the Cartier divisor associated with $\det h_* \mathcal{O}_V(mG)$, where $r_m := \text{rk } h_* \mathcal{O}_V(mG)$. Then $G - h^* D_{G, 1}$ is pseudo-effective.
(e) Let $A$ be a sufficiently ample divisor on $V$ such that $\text{Sym}^k \mathcal{H}^0(V_w, \mathcal{O}_{V_w}(A)) \to \mathcal{H}^0(V_w, \mathcal{O}_{V_w}(kA))$
for a general fiber $V_w$. Then $D_{A,m} \equiv_{\text{num}} mD_{A,1}$ holds for every $m \in \mathbb{Z}_{>0}$.

**Proof.** The proof is obtained from considering a resolution of the singularities of $V$ and applying the same argument as in [CCM19, Wan22]. Specifically, (a) is a direct consequence of [Wan22, Lemma 43.14]; (b) follows from the proof of [Wan22, Proposition 3.1(b)] or [CCM19, Theorem 1.3]; (c) follows from the proof of [CCM19, Lemma 3.4] or [Wan22, Proposition 3.2]; (d) follows from the proof of [CCM19, Lemma 3.5] or [Wan22, Proposition 3.3]; and (e) follows from [CH19, Proposition 3.6] or [Wan22, Proposition 3.5]. □

**Proof of Theorem 4.1.** Let $A$ be a sufficiently ample divisor on $V$ satisfying Proposition 4.3(e). By Proposition 4.3(e), we may assume that $D_{A,1}$ is a $\mathbb{Z}$-divisor after replacing $A$ with its multiple. By replacing $A$ with $A - \phi^*D_{A,1}$, we have the following:

- $A$ is pseudo-effective on $V$ by Proposition 4.3(d).
- $A$ is $h$-very ample.
- $\text{Sym}^2H^0(V_w, \mathcal{O}_{V_w}(A)) \to H^0(V_w, \mathcal{O}_{V_w}(kA))$.
- $D_{A,1} \sim 0$.

Then, the sheaf $h_*\mathcal{O}_V(mA)$ is reflexive since $h$ is flat by Proposition 4.3(a) and weakly positively curved for any $m \in \mathbb{Z}_{>0}$ by Proposition 4.3(c). Moreover, since $D_{A,m} \equiv_{\text{num}} mD_{A,1} \sim 0$ holds, we can conclude that $h_*\mathcal{O}_V(mA)$ is a numerically flat vector bundle on $W$ by [CCM19, Proposition 2.7] or [Wu20, §1, Corollary of Main Theorem].

It sufficient to prove that $h_*\mathcal{O}_V(mA - pD)$ is a numerically flat vector bundle for every $m \in \mathbb{Z}_{>0}$ and for some $p \in \mathbb{Z}_{>0}$ rendering $pD$ a $\mathbb{Z}$-divisor, which follows from Proposition 2.5 and Remark 2.6(b). To prove this, for a log resolution $\mu: V' \to V$ of $(V, D)$, we write as

$$K_{V'} + D' \sim_{\mathbb{Q}} \mu^*(K_V + D) + E$$

with $(V',D')$ being a klt pair and $E$ being an effective $\mu$-exceptional $\mathbb{Q}$-divisor. We fix $p \in \mathbb{Z}_{>0}$ such that both $pD$ and $pD'$ are $\mathbb{Z}$-divisors. For a sufficiently large $q$ (noting that $q$ may depend on $m$), we write as

$$qE + m\mu^*A - pD' \sim qK_{V'/W} - q\mu^*(K_{V'/W} + D) + (q - p)D' + m\mu^*A.$$ 

By the klt condition, we have

$$\mathcal{J}(h_D^* \cdot \mu^* h_A^* \cdot \mu^* \mathbb{F}) \simeq \mathcal{O}_{V'}$$

for $q \gg 1$, where $h_A$ is a singular Hermitian metric on $A$ with semi-positive curvature and $h_D'$ is the singular Hermitian metric associated with the effective $\mathbb{Q}$-divisor $D'$. Hence, by [Wan22, Corollary 2.1.3], the direct image sheaf $(\mu \circ h)_*\mathcal{O}_{V'}(qE + m\mu^*A - pD)$ is weakly positively curved. Moreover, by $\mu_*\mathcal{O}_{V'}(m\mu^*A - pD') = mA - pD$, we obtain

$$\mu_*\mathcal{O}_{V'}(qE + m\mu^*A - pD') \simeq (\mu_*\mathcal{O}_{V'}(m\mu^*A - pD'))^{**} \simeq \mathcal{O}_V(mA - pD)$$
for $q \gg 1$ (cf. [Wan20, Theorem 1.3.1 and Lemma 1.3.2]). Consequently, we see that $h_*O_V(mA - pD)$ is weakly positively curved and $D_{mA - pD,1}$ is pseudo-effective. Meanwhile, we have $D_{mA - pD,1} \equiv_{\text{num}} 0$ since $D$ is effective and $D_{A,m} \equiv_{\text{num}} 0$. Furthermore, the sheaf $h_*O_V(mA - pD)$ is reflexive since $h$ is flat and $O_V(mA - pD)$ is reflexive. Thus, by [CCM19, Proposition 2.7] or [Wu20, §1, Corollary of Main Theorem], we see that $h_*O_V(mA - pD)$ is a numerically flat vector bundle for every $m \in \mathbb{Z}_{>0}$.

We next prove the three lemmas (Lemmas 4.4, 4.5, and 4.6), which are needed for comparing MRC fibrations of a given variety $X$ to those of quasi-étale covers or birational models of $X$.

**Lemma 4.4.** Let $h: V \to W$ be a locally constant fibration between normal projective varieties such that the fiber $F$ has vanishing irregularity. Assume that there is an effective $Q$-divisor $D$ on $V$ such that $(V, D)$ is klt and $-(K_{V/W} + D)$ is nef. Then, any nef and big divisor $B$ on $V$ can be written as $B = B_0 + h^*B_W$, where $B_0$ is a nef $Q$-divisor on $V$ and $B_W$ is a nef and big $Q$-divisor on $W$.

**Proof.** Let $m$ be a sufficiently divisible integer. By Lemma 2.4, we can write as $B = B_0 + h^*B_W$ such that $h_*O_V(mB_0)$ is a flat vector bundle for every $m$ sufficiently divisible. We will prove that $B_0$ is nef and $B_W$ is nef and big. To this end, we may assume that $W$ is smooth after $W$ is replaced with a resolution of singularities and $(V, D)$ is replaced with the induced fiber product. Indeed, after the base change, the klt condition is preserved, the fibration $h$ remains a locally constant fibration, both $B_0$ and $B_W$ are obtained from the pullbacks.

We can apply Theorem 4.1 since $W$ is smooth. Hence $h$ is a locally constant fibration also with respect to the pair $(V, D)$. Moreover, we see that $B_0$ is pseudo-effective by Proposition 4.3(d) and $E_m := h_*O_V(mB_0)$ is weakly positively curved by Proposition 4.3(b). Therefore $E_m = h_*O_V(mB_0)$ is a numerically flat vector bundle since $h_*O_V(mB_0)$ is a flat vector bundle.

We next show that $h_*O_V(mB - mD)$ is a nef vector bundle. The tangent sheaf $T_V$ is decomposed into $T_V \simeq T_{V/W} \oplus \mathcal{E}$ with $\mathcal{E} \simeq h^*T_W$ (e.g., see [Wan22, Remark 21.27]) since $h$ is a locally constant fibration, which implies that $\mathcal{E}$ is locally free. Then, since $(V, D)$ is a log canonical pair, a functorial resolution $\mu: V' \to V$ satisfies $T_{V'} \simeq \mu^*\mathcal{E} \oplus T_{V'/W}$ by [Dru18a, Lemma 5.10]. (Note that [Dru18a, Lemma 5.10] holds for any klt pairs since it depends only on [GKKP11]). Consequently, the fibration $h \circ \mu: V' \to W$ is a locally constant fibration by the classical Ehresmann theorem (cf. [Här07, 3.17.Theorem] and [CLN85, §V.3, Theorem 1 and Theorem 3, pp. 91-95]). Meanwhile, since $B - (K_{V/W} + D)$ is nef and big by assumption, the direct image sheaf

$$(h \circ \mu)_*O_V(mK_{V'/W} + m\mu^*(B - K_{V/W} - D))$$

is a nef vector bundle for $m \in \mathbb{Z}_{>0}$ sufficiently divisible, which essentially follows from a variant of [Mou97, Théorème 1] or [Fuj16, Theorem 1.4 and...
§5. Indeed, we obtain the nefness by replacing [Fuj16, Theorem 5.1] or [PS14, Theorem 1.4] with [PS14, Variant 1.5] in [Fuj16, §5, Proof of Theorem 1.4]. Moreover, by [Wan20, Lemma 1.3.2], we obtain

\[(h \circ \mu)_* \mathcal{O}_V (mK_V + m\mu^*(B - K'_V - D)) \simeq h_* \mathcal{O}_V (mB - mD)\]

since the left-hand side is locally free (and thus reflexive). Hence \(h_* \mathcal{O}_V (mB - mD)\) is a nef vector bundle.

The decomposition of \(B - D\) given by Lemma 2.4 should be \(B - D = (B_0 - D) + h^*B_W\) since \(h: V \to W\) is a locally constant fibration with respect to the pair \((V, D)\). Therefore \(E_{m,D} := h_* \mathcal{O}_V (mB_0 - mD)\) is a flat vector bundle. Furthermore, by the projection formula, we have

\[h_* \mathcal{O}_V (mB - mD) \simeq E_{m,D} \otimes \mathcal{O}_W (mB_W),\]

which indicates that \(B_W\) is nef.

Since \(B_W\) is nef, by [Dem01, (7.5) Corollary, p. 52] or [Dem10, (7.5) Corollary, p. 66], there is an ample divisor \(H\) on \(W\) (independent of \(m\)) such that \(H + mB_W\) is very ample for every \(m\). For a general hypersurface \(H_{m,1} \in |H + mB_W|\), we have the exact sequence

\[0 \to E_m \otimes \mathcal{O}_W (-H) \to E_m \otimes \mathcal{O}_W (mB_W) \to E_m \otimes \mathcal{O}_W (mB_W)|_{H_{m,1}} \to 0.\]

Since \(E_m\) is numerically flat, we have \(H^0(W, E_m \otimes \mathcal{O}_W (-H)) = 0\). Hence, we see that

\[h^0(W, E_m \otimes \mathcal{O}_W (mB_W)) \leq h^0(H_{m,1}, (E_m \otimes \mathcal{O}_W (mB_W))|_{H_{m,1}}).\]

By repeating this process, we obtain

\[h^0(V, \mathcal{O}_V (mB)) = h^0(W, E_m \otimes \mathcal{O}_W (mB_W))\]
\[\leq \sharp(H_{m,1} \cap H_{m,2} \cap H_{m,\dim Y}) \cdot \rk E_m\]
\[\leq (mB_W + H)^{\dim Y} \cdot h^0(F, mB_0|_F).\]

By considering \(\lim_{m \to +\infty} (\bullet / m^{\dim X})\) (cf. [Laz04, §1.2.B, Example 1.2.36, p. 38-39, §2.2.C, Definition 2.2.31 and (2.9), p. 148, vol.I]), we obtain

\[0 < \vol_V (B) \leq C \cdot \vol_W (B_W) \cdot \vol_F (B_0|_F)\]

for some constant \(C\) (depending only on the dimensions of \(X\) and \(Y\)), which implies that \(B_W\) is big.

It remains to show that \(B_0\) is nef. Let \(A\) be a very ample divisor on \(W\) so that it induces an embedding \(W \hookrightarrow \mathbb{P} H^0(W, \mathcal{O}_W (A))\). Since \(E_m\) is a flat vector bundle, \(E_m \otimes \mathcal{O}_W (kA)\) is Nakano positive for every \(m, k \in \mathbb{Z}_{\geq 0}\) (cf. [Dem01, (3.9) Definition, p. 24] or [Dem10, (3.9) Definition, p. 28]). Hence, by the Nakano vanishing theorem (cf. [Dem01, (4.9), p. 30] or [Dem10, (4.9), p. 35]), we obtain

\[H^i(W, E_m \otimes \mathcal{O}_W ((d + 1 - i)A)) = 0\]
for every $i > 0$, where $d = \dim Y$. Consequently, the vector bundle $E_m$ is $(d+1)$-regular with respect to $A$. By [Laz04, Theorem 1.8.3, pp. 99-100], we see that

$$E_m \otimes \mathcal{O}_W((d+1)A) \simeq h_*\mathcal{O}_V(mB_0 + (d+1)h^*A)$$

is globally generated on $W$. Moreover, since $-K_{V'/W'}$ is nef and $B_0$ is $h$-relatively big and nef, the relative base point freeness shows that $B_0$ is $h$-globally generated. Hence $mB_0 + (d+1)h^*A$ is globally generated. By letting $m \to +\infty$, we can conclude that $B_0$ is nef. □

Lemma 4.5. Let $V, W, V'$ and $W'$ be normal projective varieties satisfying following commutative diagram:

$$\begin{array}{ccc}
V' & \xrightarrow{\beta_V} & V \\
\downarrow f' & & \downarrow f \\
W' & \xrightarrow{\beta_W} & W.
\end{array}$$

Here $\beta_V$ and $\beta_W$ are birational morphisms and $f$ and $f'$ are fiber spaces. Assume that $f'$ is a locally constant fibration whose fiber $F'$ is simply connected. Assume $W$ has klt singularities and that there is an effective divisor $\Delta'$ on $V'$ such that $(V', \Delta')$ is klt and that $-(K_{V'/W'} + \Delta')$ is nef. Then $f$ is a locally trivial fibration.

Proof. Let $p_W : W^{\text{univ}} \to W$ (resp. $p_{W'} : W'^{\text{univ}} \to W'$) be the universal cover of $W$ (resp. of $W'$), we will prove the stronger statement that the pullback of $V$ over $W^{\text{univ}}$ splits into a product $W^{\text{univ}} \times F$, where $F$ is the general fiber of $f$ (which implies that $f$ is a locally trivial fibration).

Since $(V', \Delta')$ is klt and since $f'$ is a locally constant fibration, we see that $W'$ has klt singularities. Moreover, since $W$ has also klt singularities, we have $\pi_1(W_1) \simeq \pi_1(W)$ from [Tak03, Theorem 1.1]. Then, the morphism $\beta_W \circ p_{W'}$ lifts to a projective morphism $\tilde{\beta}_W : W'^{\text{univ}} \to W^{\text{univ}}$ by the map lifting lemma, and the bottom square of the commutative diagram below is Cartesian. In addition, we have $(\tilde{\beta}_W)_*\mathcal{O}_{W'^{\text{univ}}} \simeq \mathcal{O}_{W^{\text{univ}}}$. 
Let us take the fiber product $V \times_W W^{\text{univ}}$ equipped with the natural morphisms $p_V: V \times_W W^{\text{univ}} \to V$ and $\tilde{f}: V \times_W W^{\text{univ}} \to W^{\text{univ}}$. Note that $p_V'$ is the universal cover of $V'$ since $F'$ is simply connected. Hence $\beta_V \circ p_V'$ lifts to the morphism $\tilde{\beta}_V: W^{\text{univ}} \times F' \to V \times_W W^{\text{univ}}$.

Furthermore, the morphism $\tilde{\beta}_V$ is projective and we have $(\tilde{\beta}_V)_* O_{W^{\text{univ}} \times F'} \simeq O_{V \times W W^{\text{univ}}}$ since both $\tilde{f}'$ and $\tilde{\beta}_W$ are projective and have connected fibers.

Take an ample divisor $A$ on $V$. Then $B := \beta_0^* A$ is nef and big over $V'$, we can apply Lemma 4.4, and write $B = B_0 + (f')^* B_W$, such that $B_0$ is nef and $B_W$ is nef and big. Moreover $B_W$ is $\beta_W$-relatively numerically trivial. To see this, we take an integral curve $C_0$ on $W'$ that is contracted by $\beta_W$, and let $C_1$ be a curve on $V'$ such that $g(C_1) = C_0$. Then $C_1$ must be contracted by $\beta_V$. Indeed, otherwise $\beta_V(C_1)$ would be contained in some fiber of $f$ and thus $C_1$ would be contracted by $f'$, which is absurd. Therefore we have

$$B_0 \cdot C_1 + B_W \cdot C_0 = B \cdot C_1 = \beta_0^* A \cdot C_1 = 0.$$ 

Since $B_0$ and $B_W$ are nef, we must have $B_W \cdot C_0 = 0$, i.e., $B_W$ is $\beta_W$-relatively numerically trivial. Hence, we can find a big and nef divisor $A_W$ on $W$ such that $B_W \equiv \beta_W^* A_W$.

By applying [Wan20, Lemma 4.2.4] we see that $\tilde{\beta}_V$ factorizes through $\tilde{\beta}_W \times \text{id}_{F'}$, i.e., there is a projective morphism $\beta: W^{\text{univ}} \times F' \to V \times_W W^{\text{univ}}$ such that $\tilde{\beta}_V = \beta \circ (\tilde{\beta}_W \times \text{id}_{F'})$. Furthermore, we have $\beta_* O_{W^{\text{univ}} \times F'} \simeq O_{V \times W W^{\text{univ}}}$. Hence $\beta$ must be birational. Then, from [Wan20, Lemma 4.2.3], we conclude that $\tilde{f}$ induces a decomposition of $V \times_W W^{\text{univ}}$ into a product $W^{\text{univ}} \times F$. In particular, the fibration $f$ is locally trivial. $\Box$
Lemma 4.6. Let $h: V \rightarrow W$ be an algebraic fiber space between projective varieties, and $D$ be an effective $\mathbb{Q}$-divisor on $V$. Assume that $W$ has klt singularities and that there is a projective birational morphism $\mu: W' \rightarrow W$ such that the base change morphism $h': V':= V \times_W W' \rightarrow W'$ is a locally constant fibration with respect to $(V', D')$ where $D'$ is the pullback of $D$. Then $h$ is also a locally constant fibration with respect to $(V, D)$.

Proof. By [Tak03, Theorem 1.1], we have

$$\pi_1(W') \cong \pi_1(W).$$

Then, by considering the pullback family over the universal cover of $W$ and $W'$ respectively and applying [Wan20, Lemma 4.2.3, Lemma 4.2.4], we can easily deduce that $h$ is a locally constant fibration with respect to $(V, D)$. The argument is quite similar (and in fact much simpler than) to that of Lemma 4.5.

4.2. Splitting of tangent sheaves. In this subsection, we deduce the splitting of tangent sheaves for projective klt pairs with anti-log canonical divisor from Theorem 3.2.

Theorem 4.7. Let $(X, \Delta)$ be a projective klt pair with nef anti-canonical divisor $-(K_X + \Delta)$. Assume that $X$ is maximally quasi-étale. Then, the MRC fibration of $X$ induces a splitting of the tangent sheaf $T_X$ of $X$:

$$T_X \cong \mathcal{F} \oplus \mathcal{G}$$

such that

- $\mathcal{F}$ is an algebraically integrable foliation whose general leaves are rationally connected fibers of MRC fibrations of $X$;
- $\mathcal{G}$ is a (possibly singular) foliation whose canonical divisor $K_\mathcal{G} \sim \mathbb{Q}$ 0.

Here a (singular) foliation $\mathcal{H}$ denotes a saturated subsheaf $\mathcal{H} \subset T_X$ that is closed under the Lie bracket. By the normality of $X$, the restriction $\mathcal{H}|_{X_{\text{reg}}}$ is a genuine foliation in the usual sense, and the leaves of $\mathcal{H}$ are thus defined. A foliation is said to be algebraically integrable if its general leaf is Zariski open in its closure (cf. [GM89]). (See [Wan20, §2.4] for a general account.)

For the proof of Theorem 4.7, we first confirm the following key lemma, which states that the desired splitting holds in the $\mathbb{Q}$-factorial terminal case.

Lemma 4.8. Let $X$ be as in Theorem 4.7, and assume in addition that $(X, \Delta)$ is a $\mathbb{Q}$-factorial terminal pair. Then, the MRC fibration of $X$ induces a splitting of the tangent sheaf of $X$ with the same properties as those in the statement of Theorem 4.7.

Proof. This proof is based on [CH19, §3.C, Proof of Theorem 1.2] and [Wan22, Step 3 & Step 4 of the Proof of Theorem 54.1]. For the reader’s convenience, we briefly recall the proof. Let the conditions be the same as those in Setting 3.1. For a very ample divisor $A$ on $X$, we set $G := \pi^* A + cE$ and

$$D_{A,c,m} := \frac{1}{r_m} \cdot \text{the Cartier divisor associated with } \det \phi_* \mathcal{O}_M(mG),$$
where $r_m := \text{rk}_* \mathcal{O}_M(mG)$. By replacing $M$ with its successive blow-ups, we assume that the $\phi$-relative base locus of $G$ is a divisor. Then $G$ can be written as follows:

$$G = G_t + G_b,$$

where $G_b$ is the $\phi$-relative fixed part of the linear system $|G|$ and $G_t := G - G_b$ is the $\phi$-relatively generated part. The adjunction morphism now admits a factorization

$$\phi^* \mathcal{O}_M(G) \to \mathcal{O}_M(G_t) \hookrightarrow \mathcal{O}_M(G),$$

which pushes down to $Y$ to give the morphisms

$$\phi_* \mathcal{O}_M(G) \to \phi_* \mathcal{O}_M(G_t) \to \phi_* \mathcal{O}_M(G).$$

By construction, the composition morphism is the identity, and the inclusion $\phi_* \mathcal{O}_M(G_t) \to \phi_* \mathcal{O}_M(G)$ is thus an isomorphism. Then, the surjection $\phi^* \phi_* \mathcal{O}_M(G_t) \twoheadrightarrow \mathcal{O}_M(G_t)$ induces the morphism

$$\pi_G: M \to \mathbb{P}(\phi_* \mathcal{O}_M(G_t))$$

such that $\mathcal{O}_M(G_t) = \pi_G^* \mathcal{O}_{\mathbb{P}(\phi_* \mathcal{O}_M(G_t))}(1)$.

We define $X_G$ to be the image $\pi_G(X)$ with the induced morphism $\psi_G: X_G \to Y$, then we obtain the following commutative diagram:

Since $(X, \Delta)$ is a $\mathbb{Q}$-factorial terminal pair and $X$ is maximally quasi-étale, the sheaf $\mathcal{V}_m := \phi_* \mathcal{O}_X(mG) \otimes \mathcal{O}_Y(-mD_{A,c,1})$ is a flat vector bundle over $Y_0 \setminus C_m$ for every $m$ by Theorem 3.2, where $\text{codim} C_m \geq 2$. We set

$$W_m := \psi_{G*} \mathcal{O}_{X_G}(m) \otimes \mathcal{O}_Y(-mD_{A,c,1}).$$

Then, by Remark 3.12(b) and by applying the same argument as [Wan22, Lemma 5.3], we see that $\mathcal{W}_m \simeq \mathcal{V}_m$ over $Y_0$. As a consequence, for every $m$, the sheaf $W_m$ is a flat vector bundle on $Y_0 \setminus C_m$ and satisfies the condition (●) in Proposition 3.3(d). Therefore $\mathcal{W}(\text{Sym}^n \mathcal{W}_1, \mathcal{W}_m)$ is a flat vector bundle on $Y_0 \setminus C_m$ satisfying the condition (●) in Proposition 3.3(d). Thus, each of the global sections is parallel. In particular, this implies that the compatibility condition (2) in Proposition 2.5 is satisfied. Combining this result with Remark 2.6(c), we conclude that the morphism $\psi_G: X_G \to Y$ is a locally constant fibration over $Y_0$ with the typical fiber $F$ (up to replacing $Y_0$ with a Zariski open set whose complement is of codimension $\geq 2$). In particular, the tangent sheaf $T_{\phi^{-1}_G(Y_0)}$ splits into two foliations. Indeed, by the definition of locally constant fibrations, the fundamental group $\pi_1(Y_0)$
diagonally acts on $Y_0^{\text{univ}} \times F$. In addition to the normality of $Y$, the natural splitting
\[ T_{Y_0^{\text{univ}} \times F} = \text{pr}_1^* T_{Y_0^{\text{univ}}} \oplus \text{pr}_2^* T_F \]
induces a splitting of $T_{\phi_G^{-1}(Y_0)}$. Meanwhile, by [Wan22, Lemma 5.5] (noting that the assumption $\pi_1(X_{\text{reg}}) = \{1\}$ is not needed), any divisorial component of the exceptional locus
\[ \psi_G|_{\phi_0^{-1}(Y_0)} : \phi_0^{-1}(Y_0) \to \psi_1^{-1}(Y_0) \]
is contained in the $\pi$-exceptional locus $E$. Hence, the splitting of $T_{\phi_G^{-1}(Y_0)}$ induces a splitting $T_X \simeq F \oplus G$ of $T_X$. By construction, general leaves of $F$ are fibers of the MRC fibration $\psi : X \to Y$ (in particular, they are rationally connected). Thus, by Proposition 3.3(a), we find
\[ K_G|_{\pi(\phi^{-1}(Y_0) \setminus E)} \sim_Q 0. \]
This fact indicates that $K_G \sim_Q 0$ since both $\pi(E)$ and $\pi(\phi^{-1}(Y \setminus Y_0))$ are of codimension $\geq 2$ in $X$ according to Proposition 3.3(c).

We now prove Theorem 4.7.

**Proof of Theorem 4.7.** Considering a $\mathbb{Q}$-factorial terminal model, we will reduce the general case to the case of Lemma 4.8. Applying [BCHM10, Corollary 1.4.3], we take a $\mathbb{Q}$-factorial terminal model $g : X^\text{term} \to X$ of $(X, \Delta)$. By construction, there is an effective $\mathbb{Q}$-divisor $\Delta^\text{term}$ on $X^\text{term}$ such that $K_{X^\text{term}} + \Delta^\text{term} \sim g^*(K_X + \Delta)$.

Hence, the anti-log canonical divisor $-(K_{X^\text{term}} + \Delta^\text{term})$ is nef. By Theorem 2.12 (2), the variety $X^\text{term}$ is still maximally quasi-étale, and we thus apply Lemma 4.8 to get the splitting $T_{X^\text{term}} \simeq F^\text{term} \oplus G^\text{term}$ with the properties in Theorem 4.7. Since $g : X^\text{term} \to X$ is birational, the splitting $T_{X^\text{term}} \simeq F^\text{term} \oplus G^\text{term}$ induces a splitting $T_X \simeq F \oplus G$ with the properties in Theorem 4.7. $\square$

4.3. **Case of $\mathbb{Q}$-factorial terminal pairs with splitting tangent sheaf.**

The purpose of this subsection is to confirm the following theorem, which asserts that the conclusion of Theorem 1.1 holds for $\mathbb{Q}$-factorial terminal pairs with the splitting tangent sheaf as in Theorem 4.7.

**Theorem 4.9.** Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial terminal pair with nef anti-log canonical divisor $-(K_X + \Delta)$. Assume that $X$ is maximally quasi-étale. Then $X$ admits a holomorphic MRC fibration $f : X \to Y$ such that $f$ is a locally constant fibration with respect to $(X, \Delta)$ and $Y$ is a projective variety with terminal singularities and $K_Y \sim_Q 0$.

**Remark 4.10.** • This theorem does not directly imply Theorem 1.1 even in the $\mathbb{Q}$-factorial terminal case. The point is that to make $X$ maximally quasi-étale, we need to take a quasi-étale cover (see Theorem 2.12(1)), but taking quasi-étale covers destroys $\mathbb{Q}$-factoriality,
and thus prevents us from applying Theorem 4.9. Therefore, the argument in Subsection 4.4 is necessary for the proof of Theorem 1.1 even in the $\mathbb{Q}$-factorial terminal case.

- In the above theorem, the condition that $X$ is maximally quasi-étale is indispensable. Indeed, the theorem does not hold without this condition, even when we assume the splitting of the tangent sheaf. Consider the following example: Let $A$ be an abelian variety of dimension $\geq 2$, and $\sigma$ be an involution of $A$, consider the group $\langle \sigma \rangle \cong \mathbb{Z}/2\mathbb{Z}$ acting on $A \times \mathbb{P}^1$ by $\sigma \cdot (a, [z, w]) \mapsto (\sigma(a), [z, -w])$, and let $X$ be the quotient of $A \times \mathbb{P}^1$ by this action. Then $X$ has $\mathbb{Q}$-factorial terminal singularities with $-K_X$ nef, and we get a fiber space $f: X \to Y$ where $Y := A/\langle \sigma \rangle$. Moreover, the natural splitting of $T_{A \times \mathbb{P}^1}$ induces a splitting of $T_X$. Nevertheless $f$ is not a locally constant fibration. (In fact, $f$ is not even semistable.)

Proof of Theorem 4.9. The proof can be divided into three steps:

Step 1 (Holomorphicity of MRC fibrations). A key observation in this step is that $F$ and $G$ are weakly regular foliations (cf. [Dru18b, Definition 5.4, Lemma 5.8]) and $F \subset T_X$ is an algebraically integrable foliation. Therefore we can apply [DGP20, Theorem 4.6] to confirm that $F$ is induced by an equidimensional fiber space:

Lemma 4.11. Let the conditions be the same as those in Theorem 4.9. Then, there exists a holomorphic MRC fibration $f: X \to Y$ to a normal projective variety $Y$ (which may differ from the original $Y$ chosen in Setting 3.1) such that

- $f: X \to Y$ has equidimensional fibers;
- $F$ coincides with the foliation induced by $f: X \to Y$.

Since $K_G \sim_{\mathbb{Q}} 0$, we have $K_Y \sim_{\mathbb{Q}} 0$. Thus we establish the first conclusion of Theorem 4.9. It remains to show the second conclusion.

Step 2 (Semi-stable reduction in codimension one and local constancy of the pullback fibration). From the previous step, we have a holomorphic MRC fibration $f: X \to Y$. We now intend to apply Theorem 4.1 to show that $f$ is a locally constant fibration (up to a quasi-étale cover). To this end, we take a resolution of $Y$ and consider the pullback fibration over it, but this new fibration is not necessarily semi-stable in codimension one. This means that the relative anti-canonical divisor is not necessarily nef, which prevents us from applying Theorem 4.1. To overcome this obstruction, we need to take a further finite cover as we precise in the sequel.

The morphism $f: X \to Y$ is semi-stable in codimension one by Proposition 3.3(e); hence the ramification divisor of $f$ is zero (cf. [CKT16, Definition 2.16]). Then, from [CKT16, Lemma 2.31] and $K_G \sim_{\mathbb{Q}} 0$, we have

$$K_{X/Y} \sim K_F \sim_{\mathbb{Q}} K_X,$$
which indicates that $K_Y \sim_{\mathbb{Q}} 0$. Meanwhile, since $(X, \Delta)$ is a $\mathbb{Q}$-factorial terminal pair, we see that $(\mathcal{F}, \Delta)$ has canonical singularities by writing

$$-K_{\mathcal{F}} \sim_{\mathbb{Q}} -(K_X + \Delta) + \Delta$$

and by [Dru17, Proposition 5.5]. Then, we consider the following diagram:

$$
\begin{array}{c}
X_2 \xrightarrow{\sigma_X} X_1 \xrightarrow{\mu_X} X \\
\downarrow f_2 \hspace{1cm} \downarrow f_1 \hspace{1cm} \downarrow f \\
Y_2 \xrightarrow{\sigma} Y_1 \xrightarrow{\mu} Y.
\end{array}
$$

Here $\mu : Y_1 \to Y$ is a resolution of singularities of $Y$ and $\sigma : Y_2 \to Y_1$ is a finite cover (of Kawamata) that renders $f$ semi-stable in codimension one, and $X_1$ (resp. $X_2$) is the normalization of the fiber product $X \times_Y Y_1$ (resp. $X \times_Y Y_2$). Note that, since $f$ is equidimensional, the fiber product $X \times_Y Y_i$ has only one irreducible component for $i = 1, 2$. Then, by [Wan20, Proposition 2.4.20], we see that $\mathcal{F}_1 := \mu_X^{-1}\mathcal{F} = T_{X_1/Y_1}$ and $K_{\mathcal{F}_1} \sim_{\mathbb{Q}} \mu_X^* K_{\mathcal{F}}$ since $\mathcal{F}$ has canonical singularities. Moreover, the branch locus of $\sigma$ is invariant under $\mathcal{F}_1$ (cf. [Dru18b, §3.4]), and hence we obtain

$$K_{\mathcal{F}_2} \sim_{\mathbb{Q}} \sigma_X^* K_{\mathcal{F}_1} \sim_{\mathbb{Q}} (\mu_X \circ \sigma_X)^* K_{\mathcal{F}}$$

by [Dru18b, Lemma 3.4] (cf. [Dru18b, Proof of Lemma 4.3]). Meanwhile, since $f_2$ is semi-stable in codimension one, we have $K_{X_2/Y_2} \sim K_{\mathcal{F}_2}$. Set $\Delta_2 := (\mu_X \circ \sigma_X)^* \Delta$ (noting that $X$ is $\mathbb{Q}$-factorial). Then, by [Dru17, Proposition 5.6], we see that $(X_2, \Delta_2)$ is canonical, and thus conclude that $f_2$ is a locally constant fibration by Theorem 4.1.

**Step 3 (Conclusion).** In this last step, we will deduce the latter conclusion of Theorem 4.9 from the local constancy of $f_2$. By taking the Stein factorization of $\mu \circ \sigma : Y_2 \to Y$, we get a finite cover $\gamma : Y' \to Y$, which is ramified along the non-semi-stable locus of $f$, and thus it is quasi-étale. Let $X'$ be the fiber product $X \times_Y Y'$ with the induced morphisms $f' : X' \to Y'$ and $\gamma_X : X' \to X$. Then, we have the following commutative diagram:

$$
\begin{array}{c}
X_2 \xrightarrow{\rho_X} X' \xrightarrow{\gamma_X} X \\
\downarrow f_2 \hspace{1cm} \downarrow f' \hspace{1cm} \downarrow f \\
Y_2 \xrightarrow{\rho} Y' \xrightarrow{\gamma} Y.
\end{array}
$$

Here, the morphisms $\rho$ and $\rho_X$ are birational by construction. Since $\gamma$ is quasi-étale, so is $\gamma_X$. Moreover, set $\Delta' := \gamma_X^* \Delta$, then $(X', \Delta')$ is terminal.
and \(-(K_X + \Delta')\) is nef. Moreover, since \(Y\) is non-uniruled, so is \(Y'\). Hence \(f'\)
coincides with the MRC fibration of \(X'\). Since \(f'\) is a locally trivial fibration
by Lemma 4.5, the left square in the diagram must be Cartesian. Then, by
Lemma 4.6, the fibration \(f'\) is locally constant with respect to \((X', \Delta')\).
To conclude, we use the fact that \(X\) is maximally quasi-étale. This indicates
that the cover \(\gamma_X\) is in fact étale, and thus so is \(\gamma\). As a consequence, the
fibration \(f\) is a locally constant fibration with respect to \((X, \Delta)\). The proof
of Theorem 4.9 is thus completed.

\[\square\]

### 4.4. Case of klt pairs with nef anti-log canonical divisor

In this subsection, we deduce Theorem 1.1 by considering the case treated in Subsection 4.3.

**Proof of Theorem 1.1.** Let \((X, \Delta)\) be a projective klt pair with nef anti-log
canonical divisor \(-(K_X + \Delta)\). Up to replacing \((X, \Delta)\) with a maximally
quasi-étale cover of \(X\) with the boundary divisor defined by pullback, we may
assume that \(X\) is maximally quasi-étale (see Theorem 2.12(1)). Take a \(\mathbb{Q}\)-
factorial terminal model \(g: (X', \Delta') \to (X, \Delta)\) of \((X, \Delta)\). Then, the variety
\(X'\) is also maximally quasi-étale by Theorem 2.12(2). By Theorem 4.9, the
MRC fibration \(f': X' \to Y'\) of \(X'\) is a locally constant fibration with respect to
\((X', \Delta')\) with \(Y'\) having terminal singularities and \(K_{Y'} \sim_{\mathbb{Q}} 0\).
Moreover, the splitting of \(T_{X'}\) induces a splitting of the tangent sheaf of \(X: T_X \simeq F \oplus G\)
such that \(F\) (resp. \(G\)) coincides with \(F'\) (resp. \(G'\)) over \(X \setminus g(\text{Exc}(g))\).
Moreover since \(K_{Y'} \sim_{\mathbb{Q}} 0\), we have \(K_{G} \sim_{\mathbb{Q}} 0\).

This subsection aims to show that \(f': X' \to Y'\) induces an MRC fibration
fibration \(f: X \to Y\) satisfying the conclusions of Theorem 1.1, namely, that
\(f: X \to Y\) is a locally constant fibration with respect to the pair \((X, \Delta)\) and
\(Y\) is a projective klt variety with the numerically trivial \(K_{Y}\). The proof is
divided into two steps:

**Step 1 (Holomorphicity of MRC fibrations).** First, we prove that \(f'\)
gives rise to an MRC fibration \(f: X \to Y\) of \(X\) defined everywhere.

**Theorem 4.12.** Let the conditions remain the same as above. Then, there
exists a normal projective variety \(Y\) with canonical singularities whose canonical
divisor \(K_Y \sim_{\mathbb{Q}} 0\), a birational morphism \(g_Y: Y' \to Y\), and an equidi-
men-tional fiber space \(f: X \to Y\) such that \(f \circ g = g_Y \circ f'\) and \(f\) coincides
with the MRC fibration of \(X\).

**Proof of Theorem 4.12.** The proof of this theorem makes use of Lemma 4.4.
Let us take a very ample divisor \(A\) on \(X\) and set \(B = g^*A\). Then \(B\) is a nef
and big divisor on \(X'\). The fiber of \(f'\) has vanishing irregularity since it has
terminal (and thus rational) singularities and is rationally connected. Hence,
from Lemma 4.4, we can write \(B = B_0 + (f')^*B_{Y'}\), where \(B_0\) is nef and \(B_{Y'}\) is
nef and big. By \(K_{Y'} \sim_{\mathbb{Q}} 0\), the basepoint-free theorem [KM98, §3.1, Theorem
3.3, p. 75] implies that \(B_{Y'}\) is semi-ample. Up to multiplying by a sufficiently
large and divisible integer, the divisor $B_{Y'}$ induces a birational morphism $g_Y: Y' \to Y$ (cf. [Laz04, §2.1.B, Theorem 2.1.27, pp. 129-131, Vol.I]) and there is a very ample line bundle $A_Y$ on $Y$ such that $B_{Y'} \simeq g_Y^* A_Y$. Up to replacing $Y$ with its normalization, we can assume that $Y$ is normal.

We next prove that $g_Y \circ f'$ factorizes through $g: X' \to X$. To this end, let $C$ be a curve contracted by $g$. Then, we have $g^* A \cdot C = B \cdot C = 0$ by the choice of $C$. Since both $B_{Y'}$ and $B_0$ are nef, then we have

$$(f')^* B_{Y'} \cdot C = B_0 \cdot C = 0,$$

which implies that $(g_Y \circ f')^* A_Y \cdot C = 0$. Hence $C$ is contracted by $g_Y \circ f'$.

By [Deb01, Lemma 1.15, pp.12-13], there is a morphism $f: X \to Y$ such that $g_Y \circ f' = f \circ g$. We claim that $f$ is equidimensional. To see this, let $X''$ be the normalization of $X \times_Y Y' \to Y'$. Then, by considering everything over the universal cover of $Y'$ and applying [Wan20, Lemma 4.2.3], we find that $X'' \to Y'$ is locally trivial and in particular equidimensional, and thus so is $f$. Moreover, since $g_Y$ is birational, $Y$ is not uniruled. Hence $f$ must coincide with the MRC fibration of $X$ and $T_{X/Y} \simeq \mathcal{F}$.

By Proposition 3.3(e), the morphism $f$ is semi-stable in codimension one. Hence, the ramification divisor of $f$ is zero, and $K_{X/Y} \sim K_{\mathcal{F}} \sim_{\mathbb{Q}} K_X$. Therefore, we obtain that $K_Y$ is $\mathbb{Q}$-Cartier and $K_Y \sim_{\mathbb{Q}} 0$. Consequently, we have $K_{Y'} \sim_{\mathbb{Q}} g_Y^* K_Y$, but $Y'$ has terminal singularities, and thus $Y$ has canonical singularities.

**Step 2 (Conclusion).** To prove that $f$ is a locally constant fibration, we intend to apply Theorem 4.1 and Lemma 4.6. Nevertheless, in order to apply Theorem 4.1, we need to consider the base change fibration over a resolution of singularities of $Y$, but we do not know whether this still satisfies the assumptions of Theorem 4.1 (i.e., the klt condition and nefness).

To overcome these difficulties, we apply Lemma 4.5 (noting that $Y$ has canonical singularities, in particular klt singularities) to show that $f$ is a locally trivial fibration. Let $\mu: Y_1 \to Y$ be a resolution of singularities of $Y$. Since $f$ is locally trivial, the base change morphism $f_1: X_1 \to Y_1$ is still locally trivial and $(X_1, \Delta_1)$ is still klt, where $X_1 := X \times_Y Y_1$ and $\Delta_1$ is the base change of $\Delta$. Note that $\Delta_1$ is locally trivial over $Y_1$ since $\Delta$ is locally trivial over $Y$. The relative anti-canonical divisor $-(K_{X_1/Y_1} + \Delta_1)$, which is equal to the pullback of $-(K_X + \Delta)$, is nef. We now can apply Theorem 4.1 to $f_1: (X_1, \Delta_1) \to Y_1$, and conclude that $f_1$ is a locally constant fibration with respect to $(X_1, \Delta_1)$. Then, since $Y$ has canonical singularities, the same holds for $f$ and $(X, \Delta)$ by Lemma 4.6. This shows that $f$ is a locally constant fibration with respect to $(X, \Delta)$.

Our main result Theorem 1.1 has thus been proved.

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