HMLasso: Lasso for High Dimensional and Highly Missing Data

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Abstract
Sparse regression such as Lasso has achieved great success in dealing with high dimensional data for several decades. However, there are few methods applicable to missing data, which often occurs in high dimensional data. Recently, CoCoLasso was proposed to deal with high dimensional missing data, but it still suffers from highly missing data. In this paper, we propose a novel Lasso-type regression technique for Highly Missing data, called “HMLasso”. We use the mean imputed covariance matrix, which is notorious in general due to its estimation bias for missing data. However, we effectively incorporate it into Lasso, by using a useful connection with the pairwise covariance matrix. The resulting optimization problem can be seen as a weighted modification of CoCoLasso with the missing ratios, and is quite effective for highly missing data. We show that the proposed method is beneficial with regards to non-asymptotic properties of the covariance matrix. Numerical experiments show that the proposed method is highly advantageous in terms of estimation error and generalization error.

Introduction
Sparse regression has achieved great success for several decades in dealing with high dimensional data in a wide range of fields such as biology, economy and industry. There are a variety of methods including Lasso (Tibshirani 1996), Elastic Net (Zou and Hastie 2005), SCAD (Fan and Li 2001), MCP (Zhang 2010), Generalized Lasso (Tibshirani 2011), and so on. However, we often encounter missing values in high dimensional data, and most of these methods are not applicable to missing data.

Missing data analysis has a long history. Listwise deletion (complete case analysis) and pairwise deletion are widely used because of their simplicity. There are relatively modern methods such as expectation maximization (EM) algorithm (Dempster, Laird, and Rubin 1977), multiple imputation (MI) (Little and Rubin 1987, Little and Rubin 2002), and full information maximum likelihood (FIML) (Hartley and Hocking 1971, Enders 2001). However, these methods focus on low dimensional missing data and require a large computational cost for high dimensional missing data.

In order to deal with high dimensional missing data, another regression approach was proposed using the pairwise covariance matrix (Loh and Wainwright 2012). This approach had low computational cost and is possible to analyze statistical non-asymptotic properties. However, it requires a nonconvex optimization, and it depends heavily on some critical unknown parameters which must be determined in advance, so that there are some difficulties in practice. To overcome these problems, Convex Conditioned Lasso (CoCoLasso) was proposed (Datta and Zou 2017). Since CoCoLasso is a convex formulation using a positive semidefinite approximation of the pairwise covariance matrix, it does not suffer from local optima or critical parameters. Statistical non-asymptotic properties were also available for CoCoLasso. However, we have found that highly missing data can deteriorate the CoCoLasso estimate. CoCoLasso is quite sensitive to highly missing variables, so that the accuracy of the estimator can be significantly worse even if there exists one highly missing variable.

The highly missing problem is common in practice. For example, the missing ratio of 15% to 20% was common in educational and psychological studies (Enders 2003), typical industrial process maintenance data had over 75% missing values in over 50% of variables (Lakshminarayan, Harp, and Samad 1999), and in some extreme cases, the missing ratio could reach 90% in traffic data (Tan et al. 2013).

Motivated by the highly missing problem, we propose a novel regression method for high dimensional and highly missing data. We use the mean imputed covariance matrix, which is notorious in general due to its estimation bias for missing data. However, we effectively incorporate it into Lasso. The resulting optimization problem can be seen as a weighted modification of CoCoLasso with the missing ratios, and is quite effective for highly missing data. Our contributions are as follows:

- We propose a novel regression method for dealing with high dimensional and highly missing data. As far as we know, this is the first method that effectively overcomes both high dimensionality and highly missingness.
- We analyze non-asymptotic properties of our method. We show that our formulation balances uncertainty of the pairwise covariance estimators. Moreover, we obtain the optimal weight from a non-asymptotic property.
We demonstrate the effectiveness of our method by both numerical simulations and real-world data experiments. In almost all cases, our method outperforms other existing methods. In particular, our method delivers significant improvement for high dimensional and highly missing data.

The rest of this paper is organized as follows: First, we review existing methods and propose our method. Then, we describe some justifications of our formulation with theoretical analyses. Finally, we show the advantage of our method by numerical simulations and real-world data experiments.

### Notations

Let $v \in \mathbb{R}^p$, $\|v\|_q (q > 0)$ is the $\ell_q$ norm, i.e., $\|v\|_q = (|v_1|^q + \cdots + |v_p|^q)^{1/q}$. Let $M \in \mathbb{R}^{n \times p}$, $\|M\|_F$ is the Frobenius norm and $\|M\|_{\max}$ is the max norm, i.e., $\|M\|_F = (\sum_{j,k} M_{jk}^2)^{1/2}$ and $\|M\|_{\max} = \max_{j,k} |M_{jk}|$. Let $M_1, M_2 \in \mathbb{R}^{n \times p}$, $M_1 \odot M_2$ is the element-wise product (Hadamard product) of $M_1$ and $M_2$, $M_1 \odot M_2$ is the element-wise division of $M_1$ and $M_2$, $(M_1, M_2)$ is the element-wise product sum of $M_1$ and $M_2$, which is the matrix inner product inducing the Frobenius norm. Let $M \in \mathbb{R}^{p \times p}$ be a symmetric matrix. $M \succeq 0$ denotes that $M$ is positive semidefinite (PSD), i.e., $v^T M v \geq 0$ for any $v \in \mathbb{R}^p$.

### Methods

#### Problem Formulation

Consider a linear regression model:

$$ y = X \beta + \varepsilon, $$

where $X \in \mathbb{R}^{n \times p}$ is a complete design matrix, $y \in \mathbb{R}^n$ is a response, $\beta \in \mathbb{R}^p$ is a regression coefficient, and $\varepsilon \in \mathbb{R}^n$ is a noise. Suppose that each column of $X$ is centered without loss of generality. The ordinary problem is to estimate the regression coefficient $\beta$ given the complete data $X$ and $y$. Lasso is one of the most promising methods for high dimensional data (Tibshirani 1996). Lasso solves the following problem:

$$ \hat{\beta} = \arg\min_{\beta} \frac{1}{2n} \| y - X \beta \|_2^2 + \lambda \| \beta \|_1, $$

where $\lambda > 0$ is a regularization parameter. Since the objective function is regularized by $\ell_1$ norm of $\beta$, the solution is sparse and often has a small generalization error.

In this paper, we consider the Lasso problem with missing data. Some elements of $X$ are observed, and the other elements are missing. In the presence of missing values, we cannot apply Lasso directly.

### Review of Existing Methods

A representative approach of regression for high dimensional missing data is to directly estimate the parameter $\beta$ without imputation. The Lasso objective function can be reformulated as

$$ \arg\min_{\beta} \frac{1}{2} \beta^T S \beta - \rho^T \beta + \lambda \| \beta \|_1, $$

where $S = \frac{1}{2} X^T X$ (i.e., sample covariance matrix of $X$) and $\rho = \frac{1}{2} X^T y$ (i.e., sample covariance vector of $X$ and $y$). Using this formulation, we can estimate $\beta$ via $S$ and $\rho$ instead of $X$ and $y$. If missing is completely at random, we can easily construct unbiased estimators of the covariance matrix and vector as $S_{\text{pair}} = (s_{jk})$ and $\rho_{\text{pair}} = (\rho_{jk})$, where

$$ S_{jk} := \frac{1}{n_{jk}} \sum_{i \in I_{jk}} X_{ij} X_{ik}, \quad \rho_{jk} := \frac{1}{n_{jk}} \sum_{i \in I_{ij}} X_{ij} y_i, $$

$I_{jk} := \{ i : X_{ij} \text{ and } X_{ik} \text{ are observed for } j \neq k, \text{ } X_{ij} \text{ are observed for } j = k \}$, $I_j := \{ i : X_{ij} \text{ is observed } \}$, and $n_{jk}$ and $n_j$ are the numbers of elements of $I_{jk}$ and $I_j$, respectively. We substitute $S_{\text{pair}}$ and $\rho_{\text{pair}}$ for $S$ and $\rho$, respectively.

The major problem here is that $S_{\text{pair}}$ may not be positive semidefinite (PSD), in other words, it may have negative eigenvalues. This is a critical problem because if it has negative eigenvalues, then the objective function can diverge to minus infinity, which means the optimization fails. To avoid this problem, the following two kinds of constraint optimization problems were proposed (Loh and Wainwright 2012):

$$ \hat{\beta} \in \arg\min_{\| \beta \| \leq R} \frac{1}{2} \beta^T S_{\text{pair}} \beta - \rho_{\text{pair}}^T \beta, $$

and

$$ \hat{\beta} \in \arg\min_{\| \beta \| \leq b_0 \sqrt{2}} \frac{1}{2} \beta^T S_{\text{pair}} \beta - \rho_{\text{pair}}^T \beta + \lambda \| \beta \|_1, $$

where $s$ is the number of true active variables, and $R$ and $b_0$ are some constants. The projected gradient descent algorithm was used to solve the problem. Careful analyses reveal that $\hat{\beta}$ is close to the true parameter with high probability. However, there are some difficulties in practice due to its nonconvexity. First, there may exist multiple global/local minimizers, so that different initial values output different solutions. Second, it depends on unknown parameters such as $R$ and $b_0$. These parameters must not be too large or too small, so that it is not trivial to appropriately determine such unknown parameter values in advance.

Recently, the following convex optimization problem, called CoCoLasso (Convex Conditioned Lasso), was proposed (Datta and Zou 2017):

$$ \hat{\beta} \in \arg\min_{\| \beta \|_1 \leq b_0 \sqrt{2}} \frac{1}{2} \beta^T S_{\text{pair}} \beta - \rho_{\text{pair}}^T \beta + \lambda \| \beta \|_1, $$

where $S$ is the PSD covariance matrix in the objective function via the Alternating Direction Method of Multipliers (ADMM), and then optimizes the Lasso-type objective function. This formulation overcomes the difficulties of Loh and Wainwright (2012). First, since $S$ is PSD by definition, the objective function is convex, and hence has no local minimizers. Second, it does not use any unknown parameters which must be determined in advance. For these reasons,
CoCoLasso is easy to use in practice. In addition, statistical non-asymptotic properties were also derived.

However, highly missing data can deteriorate the estimation of the covariance matrix in CoCoLasso. If $n_{jk}$ is very small, then $S_{\text{pair}}^{\text{imp}}$ is quite unreliable; it can be very large or small. Since (6) is based on the max norm, unreliable elements of $S_{\text{pair}}^{\text{imp}}$ will give a large effect on the estimator. As a result, other elements of the estimator can be far from corresponding elements of $S_{\text{pair}}$, even if their variables have few missing values. This consideration indicates that CoCoLasso can significantly worse, even if there is one highly missing variable. The problem is that $S_{\text{pair}}^{\text{imp}}$’s are used in the same manner regardless of different reliability. We will overcome this problem in the next subsection.

Here, we mention other approaches for regression with missing data. A simple approach is listwise deletion. This is very fast, but not appropriate when there are few complete samples, as is common with high dimensional data. Another typical approach is to impute missing values in $X$.

The mean imputation method is commonly used in practice.

**Proposed Method: HMLasso**

The mean imputation method is commonly used in practice. Let $Z$ be the mean imputed data of $X$. Because $X$ is centered, $Z_{jk} = X_{jk}$ for observed elements and $Z_{jk} = 0$ for missing elements. The covariance matrix of mean imputed data, $S_{\text{imp}} = (S_{\text{imp}}^{\text{imp}})_{jk}$, is defined as

$$S_{\text{imp}}^{\text{imp}} := \frac{1}{n} \sum_{i=1}^{n} Z_{ij} Z_{ik} = \frac{n_{jk}}{n} \frac{1}{n_{jk}} \sum_{i \in I_{jk}} X_{ij} X_{ik} = \frac{n_{jk}}{n} S_{\text{pair}}^{\text{imp}},$$

or equivalently,

$$S_{\text{imp}}^{\text{imp}} = R \odot S_{\text{pair}}^{\text{imp}},$$

where $R \in \mathbb{R}^{p \times p}$ is an observation ratio matrix whose $(j, k)$-th element is $R_{jk} = n_{jk}/n$. The mean imputed covariance matrix $S_{\text{imp}}^{\text{imp}}$ in the left-hand side of (8) is biased but PSD, while the pairwise covariance matrix $S_{\text{pair}}^{\text{imp}}$ in the right-hand side is unbiased but not PSD. To take the best of both, we came up with the idea to optimize the following problem to obtain a low biased PSD matrix:

$$\tilde{\Sigma} = \arg\min_{\Sigma \succeq 0} \| R \odot \Sigma - S_{\text{imp}}^{\text{imp}} \|_F^2.$$  

A direct use of mean imputation for covariance matrix estimation is notorious due to its estimation bias. However, we effectively incorporate $S_{\text{pair}}^{\text{imp}}$ and $S_{\text{pair}}^{\text{imp}}$ into the optimization problem (9) using the above relation between them.

The formulation (9) has a good property. We see from (8) that (9) is equivalent to

$$\tilde{\Sigma} = \arg\min_{\Sigma \succeq 0} \| R \odot (\Sigma - S_{\text{pair}}) \|_F^2.$$  

This formulation can be seen as a weighted modification of CoCoLasso (6) using the observed ratio matrix $R$, when the max norm is replaced by the Frobenius norm. The weight is beneficial for highly missing data. When there are missing observations, the objective function downweights the corresponding term, $\Sigma_{jk} - S_{\text{pair}}^{\text{pair}}$, by the observed ratio $R_{jk} = n_{jk}/n$. In particular, when $n_{jk}$ is small, the downweighting will be reasonable, since the pairwise covariance $S_{\text{pair}}^{\text{pair}}$ is unreliable.

Based on the above idea, we extend the formulation and propose a novel optimization problem to estimate the regression model:

$$\tilde{\beta} = \arg\min_{\beta} \frac{1}{2} \beta^T \tilde{\Sigma} \beta - \rho_{\text{pair}}^T \beta + \lambda \| \beta \|_1,$$

$$\tilde{\Sigma} = \arg\min_{\Sigma \succeq 0} \| W \odot (\Sigma - S_{\text{pair}}) \|_F^2,$$

where $W$ is a weight matrix whose $(j, k)$-th element is $W_{jk} = R_{jk}^{\alpha}$ with a constant $\alpha > 0$. This Lasso formulation enables us to efficiently deal with Highly Missing data, so that we call our method “HMLasso”. HMLasso obtains the PSD matrix by minimizing the weighted Frobenius norm in (12), and then optimize the Lasso-type problem (11).

Several values of $\alpha$ can be considered. The case $\alpha = 0$ corresponds to the non-weighted case, which is just a projection of $S_{\text{pair}}^{\text{pair}}$ onto the PSD region. It is the same as CoCoLasso when the Frobenius norm in (12) is replaced by the max norm. The case $\alpha = 1$ relates to the mean imputation, as described above. In addition, as seen later, non-asymptotic analyses will support that $\alpha = 1$ is reasonable, and numerical experiments will show that it delivers the best performance. Therefore, we recommend $\alpha = 1$ in practice. The case $\alpha = 1/2$ can be roughly viewed as the maximum likelihood method from an asymptotic perspective. We discuss the case $\alpha = 1/2$ in the supplementary material.

Note that we use the Frobenius norm instead of the max norm. This is because the Frobenius norm delivers a better performance than the max norm in numerical experiments.

**Comparison through a Simple Example**

We give a simple example in which our weighted formulation (12) is better than the non-weighted formulation. Consider a three-dimensional data $X \in \mathbb{R}^{n \times 3}$. To derive simple analytical results, we suppose that the pairwise covariance matrix and observation ratio matrix are given by

$$S_{\text{pair}}^{\text{pair}} = \begin{bmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & s_2 \\ s_2 & s_2 & 1 \end{bmatrix},$$

and $R = \frac{1}{n} \begin{bmatrix} \cdot & n_1 & n_2 \\ n_1 & \cdot & n_2 \\ n_2 & n_2 & \cdot \end{bmatrix}$. 

which means that \(X_1\) and \(X_2\) are symmetric. In addition, we restrict the diagonal elements of the estimate to 1, for simplicity. Then, we can parameterize the covariance matrix \(\Sigma\) as
\[
\Sigma = \begin{bmatrix} 1 & \sigma_1 & \sigma_2 \\ \sigma_1 & 1 & \sigma_2 \\ \sigma_2 & \sigma_2 & 1 \end{bmatrix},
\]
because of the symmetry and convexity of the problem. In this setting, we can see \(S^{\text{pair}}\) and \(\Sigma\) on the same twodimensional space in Figure 1.

To show the difference among the methods, we focus on the case where \(S^{\text{pair}}\) is not PSD. By simple calculation, we see that the PSD condition of \(S^{\text{pair}}\) is \(2s_2^2 - 1 \leq s_1 \leq 1\), which is shown in gray in Figure 1. Hereafter, we suppose \(s_1 < 2s_2^2 - 1\) so that \(S^{\text{pair}}\) is not PSD, and also \(s_2 \geq 0\) without loss of generality. The point \(P\) in Figure 1 is an example of such a \(S^{\text{pair}}\).

We consider two highly missing scenarios as follows: 1) the pairwise observations of \((X_1, X_2)\) are rare, i.e., \(n_1\) is small enough and \(n_2\) is large enough, and 2) the pairwise observations of \((X_1, X_3)\) and \((X_2, X_3)\) are rare, i.e., \(n_1\) is large enough, and \(n_2\) is small enough. These scenarios are realized by simple missing patterns (see Figure 2). For the non-weighted formulations, missing scenarios do not affect their estimates; while do affect the estimates of the weighted formulations. We show that the weighted formulation is preferable for highly missing data.

**Non-weighted Max Norm Formulation (CoCoLasso)** The problem is \(\min_{\Sigma \succeq 0} \|\Sigma - S^{\text{pair}}\|_{\text{max}}\). The contour of \(\|\Sigma - S^{\text{pair}}\|_{\text{max}}\) is a square with the center point \(P\). The optimum is the tangent point of the gray region and the square contour, which is shown as the point \(A\) in Figure 1.

**Non-weighted Frobenius Norm Formulation** The problem is \(\min_{\Sigma \succeq 0} \|\Sigma - S^{\text{pair}}\|_F^2\). The contour of \(\|\Sigma - S^{\text{pair}}\|_F^2\) is an ellipse with the center point \(P\). The optimum is the tangent point of the gray region and the ellipse contour, which is shown as the point \(B\) in Figure 1. The analytical solution is given by \(\sigma_1 = s_1 + s_2 + \sqrt{8s_1 + 8s_2 + 9}/4\) and \(\sigma_2 = -1/4 + \sqrt{8s_1 + 8s_2 + 9}/4\).

**Weighted Norm Formulation** The problem is \(\min_{\Sigma \succeq 0} \|W \odot (\Sigma - S^{\text{pair}})\|_F\), where the norm \(\| \cdot \|_F\) is either the max norm or Frobenius norm.

1) For the first missing scenario, we can see that the optimal point of the weighted norm will be close to the point \(C\), i.e., \(s_1 \approx 2s_2^2 - 1\) and \(s_2 \approx s_2\). The reasons are that i) we have \(\sigma_2 \approx s_2\), because the weights \(R_{13}\) and \(R_{23}\) are much larger than \(R_{12}\), and ii) the optimum of \(s_1\) must be close to the tangent point of the gray PSD region and the line \(CD\). On the other hand, the sample covariance matrix \(S\) with complete observations satisfies \(s_2 \approx s_2\) and \(2s_2^2 - 1 \leq s_1 \leq 1\), which represents a line segment \(CD\) in Figure 1. Since the point \(C\) is closer to the any points on the line segment \(CD\) than \(A\) and \(B\), the optimum of the weighted norm is always closer to \(S\) than the non-weighted norms.

2) For the second missing scenario, we can see that the optimal point will be close to the point \(E\), i.e., \(s_1 \approx s_1\) and \(s_2 \approx \sqrt{(s_1 + 1)/2}\). The reasons are similar to the first scenario; i) \(s_1 \approx s_1\), because the weight \(R_{12}\) is much larger than \(R_{13}\) and \(R_{23}\), and ii) the optimum of \(s_2\) must be close to the tangent point of the gray region and the line \(EF\). On the other hand, the sample covariance matrix \(S\) with complete observations satisfies \(s_1 \approx s_1\) and \(2s_2^2 - 1 \leq s_1 \leq 1\), which represents a line segment \(EF\) in Figure 1. Since the point \(E\) is closer to any points on the line segment \(DF\) than \(A\) and \(B\), the optimum of the weighted norm is always closer to \(S\) than the non-weighted norms.
Theorem 1. Assume that $M_{ij}$’s are independent and identically distributed with mean $\mu_M$, covariance $\Sigma_M$, and sub-Gaussian parameter $\tau^2$.

Proposition 1. Let $\hat{\Sigma} := \Sigma_M + \mu_M \mu_M^\top$. Then, the rows of $\hat{\Sigma}$ are sub-Gaussian with $\tau^2 = \max_j (1 - \mu_j)$.

Non-asymptotic Properties

We first derive a refined non-asymptotic property of $\hat{\Sigma}$.

Theorem 2. Under Assumption 7, we have, for all $\epsilon \leq c \tau^2 X_{\text{max}}^2 / \pi_{jk}$,

$$\Pr \left( |\hat{\Sigma}_{jk} - S_{jk}| \leq \epsilon \right) \geq 1 - C \exp \left( -cn\pi_{jk}^2 \frac{\tau^2}{\zeta} \right),$$

(14)

where $\zeta = \tau^2 X_{\text{max}}^4 \max \{ \tau^2, \mu_j^2, \mu_k^2 \}$, $X_{\text{max}} = \max_{i,j} |X_{ij}|$, and $C$ and $c$ are some universal constants.

Preparations

Let $M = (M_{ij}) \in \mathbb{R}^{n \times p}$ be the observation pattern matrix whose elements are one/zero when it is observed/missing, so that $Z = M \odot X$. We suppose a sub-Gaussian assumption on $M$, which plays a key role in non-asymptotic properties. This assumption often holds, as seen in the following proposition.

Definition 1. A random variable $X$ is said to be sub-Gaussian with $\tau^2$ if $\mathbb{E}[\exp(s(X - \mathbb{E}[X])]) \leq \exp(\frac{\tau^2 s^2}{2})$, for all $s \in \mathbb{R}$. A random vector $X \in \mathbb{R}^p$ is said to be sub-Gaussian with $\tau^2$ if $v^\top X$ is a sub-Gaussian variable with $\tau^2$ for all $v \in \mathbb{R}^p$ satisfying $\|v\|_2 = 1$.

Assumption 1. The rows of $M$ are independent and identically distributed with mean $\mu_M$, covariance $\Sigma_M$, and sub-Gaussian parameter $\tau^2$.

The Lasso-type optimization problem (11) can be solved by various algorithms for Lasso, such as the coordinate descent algorithm (Friedman, Hastie, and Tibshirani 2010), and the least angle regression algorithm (Efron et al. 2004). In our implementation, we use the coordinate descent algorithm because it is efficient for high dimensional data. The algorithm is described in the supplementary materials. Note that we use the warm-start and safe screening techniques to speed up the algorithm.

To derive the covariance matrix optimization algorithm for (12), we use the ADMM algorithm (Boyd et al. 2011). The problem (12) can be written as

$$f(A, B) = \arg\min_{A \succeq 0, B = A - S_{\text{pair}}} \|W \odot B\|_F^2,$$

Therefore, the augmented Lagrangian function is

$$f(A, B, \Lambda) = \frac{1}{2} \|W \odot B\|_F^2 - \langle \Lambda, A - B - S_{\text{pair}} \rangle + \frac{1}{2\mu} \|A - B - S_{\text{pair}}\|_F^2,$$

(13)

where $\Lambda$ is the Lagrangian matrix, and $\mu$ is an augmented Lagrangian parameter. We iteratively update $A$, $B$ and $\Lambda$ subject to $A \succeq 0$ by minimizing (13) in terms of each variable. The resulting algorithm is similar to the CoCoLasso algorithm except for the update rule of $B$ due to the weight matrix $W$. To derive the $B$-step update equation, differentiating $f(A, B, \Lambda)$ with respect to $B$ yields

$$\partial_B f(A, B, \Lambda) = W \odot W \odot B + \Lambda - \frac{1}{\mu} (A - B - S_{\text{pair}}).$$

Solving $\partial_B f(A, B, \Lambda) = 0$, we obtain the update rule as

$$B \leftarrow (A - S_{\text{pair}} - \mu \Lambda) \odot (\mu W \odot W + I).$$

The whole algorithm for solving (12) is described in Algorithm 1.

The difference of the max norm and Frobenius norm is trivial when we use the ADMM. We describe the algorithm for the weighted max norm in the supplementary material.

Theoretical Properties

We investigate non-asymptotic properties of our estimator. Specifically, we obtain a refined non-asymptotic property for the pairwise covariance matrix, which explicitly includes the effect of missing ratio. Moreover, we derive a non-asymptotic property of our weighted estimator. These results show that the weighted formulation with $\alpha = 1$ is superior in terms of non-asymptotic properties. Since the essential difference from CoCoLasso is the weighted norm in (12), we focus on non-asymptotic properties of the covariance matrix. We can easily see that it holds the similar non-asymptotic properties of the coefficient in Datta and Zou (2017). The proofs of the proposition and theorems are given in the supplementary material.
In Theorem\textsuperscript{2} the missing rate appears explicitly and the non-asymptotic property is stricter than Definition\textsuperscript{1} and Lemma\textsuperscript{2} in Datta and Zou (2017). To see an effect of missing rate clearly, let us replace $\varepsilon$ by $\varepsilon/\pi_{jk}$. Then, we have, for all $\varepsilon \leq c\tau^2 X_{\max}^2$, $\Pr \left( \pi_{jk} \left| \hat{\Sigma}_{jk} - S_{jk} \right| \leq \varepsilon \right) \geq 1 - C \exp \left( -cn\varepsilon^2 \zeta^{-1} \right)$.

Since the right-hand side does not depend on $\pi_{jk}$, we can see that the concentration probability of $\pi_{jk} \left| \hat{\Sigma}_{jk} - S_{jk} \right|$ is equally bounded regardless of missing rate. This implies that our weighted formulation balances uncertainty of each element of $\hat{\Sigma}$ induced by different missing rates. On the other hand, non-weighted formulation such as CoCoLasso suffers from this unbalancedness.

Next, we derive a non-asymptotic property of our weighted estimator $\hat{\Sigma}$.

**Theorem 3.** Under Assumption\textsuperscript{7} we have, for all $\varepsilon \leq c\tau^2 X_{\max}^2 (\min_{j,k} W_{jk} / \pi_{jk}) / W_{\min}$,

$$\Pr \left( \frac{1}{p^2} \left\| \hat{\Sigma} - S \right\|_F^2 \leq \varepsilon^2 \right) \geq 1 - p^2 C \exp \left( -cn\varepsilon^2 W_{\min}^2 \left( \min_{j,k} \frac{\pi_{jk}}{W_{jk}} \right)^2 \zeta^{-1} \right),$$

(15)

where $\zeta = \tau^2 X_{\max}^4 \max \{ \tau^2, \mu_2, \ldots, \mu_p \}$, $W_{\min} = \min_{j,k} W_{jk}$, $X_{\max} = \max_{i,j} |X_{ij}|$, and $C$ and $\epsilon$ are some universal constants.

Based on Theorem\textsuperscript{3} we show that the weighted norm with $\alpha = 1$ is superior in terms of the non-asymptotic property. Let $\pi_{\min} = \min_{j,k} \pi_{jk}$, and $\pi_{\max} = \max_{j,k} \pi_{jk}$. Substituting $W_{jk} = \pi_{jk}^\alpha$, we have the concentration probability (15) as

$$1 - p^2 C \exp \left( -cn\varepsilon^2 \pi_{\min}^{2\alpha} \left( \min_{j,k} \pi_{jk}^{2-\alpha} \right) \zeta^{-1} \right),$$

with the constraint $\varepsilon \leq c\tau^2 X_{\max}^2 (\min_{j,k} \pi_{jk}^{-1}) / \pi_{\min}$. i) For $0 \leq \alpha \leq 1$, the concentration probability turns into $1 - p^2 C \exp \left( -cn\varepsilon^2 \pi_{\min}^{2\alpha} \zeta^{-1} \right)$, and the constraint of $\varepsilon$ turns into $\varepsilon \leq c\tau^2 X_{\max}^2 (\pi_{\max} / \pi_{\min})^\alpha / \pi_{\max}$. Since $\pi_{\max} / \pi_{\min} \geq 1$, the constraint region of $\varepsilon$ is maximized at $\alpha = 1$. ii) For $\alpha \geq 1$, the concentration probability turns into $1 - p^2 C \exp \left( -cn\varepsilon^2 \pi_{\min}^{2(\pi_{\max} / \pi_{\min})^\alpha} \zeta^{-1} \right)$ and the constraint of $\varepsilon$ turns into $\varepsilon \leq c\tau^2 X_{\max}^2 / \pi_{\min}$. Since $\pi_{\min} / \pi_{\max} \leq 1$, the concentration is maximized at $\alpha = 1$. These considerations imply that the case $\alpha = 1$ is superior in the weighted norm formulation from the viewpoint of the non-asymptotic theory.

Note that a similar non-asymptotic property for the weighted max norm can be obtained in the same manner. We can see that the weight with $\alpha = 1$ is superior for the max norm as well, although CoCoLasso uses the non-weighted norm ($\alpha = 0$).

**Numerical Experiments**

We conducted various experiments for both synthetic and real-world data. In particular, we investigated the performances of several methods extensively in many simulation settings.

**Numerical Simulations with Several Methods**

We compared our proposed method with the following formulations of the covariance matrix estimation, including two representative methods; the mean imputation method and CoCoLasso.

- **Mean imputation method:** $\hat{\Sigma} = \hat{S}_{\text{imp}}$.
- **Max norm formulation:** formulation (12) using the max norm instead of the Frobenius norm with $\alpha = 0, 1/2, 1, 2$. The case $\alpha = 0$ corresponds to CoCoLasso formulation.
- **Frobenius norm formulation:** formulation (12) with $\alpha = 0, 1/2, 1, 2$. The case $\alpha = 0$ is just a projection of $S_{\text{max}}$ onto the PSD region.

Although we examined missing imputation methods such as mice (Buuren and Groothuis-Oudshoorn 2011) and missForest (Stekhoven and Buehlmann 2012), their computational costs were over 100 times larger than the above methods, so that we excluded these methods in our experiments.

The training data were generated as follows: We generated a design matrix $X \in \mathbb{R}^{n \times p}$ with $n = 10,000$ and $p = 100$ from $\mathcal{N}(0, \Sigma^*)$ with $\Sigma_{jk} = 0.5$ for $j \neq k$ and $\Sigma_{jk} = 1$ for $j = k$. Then, we generated an objective variable $y$ as $y = X\beta + \varepsilon$ with $\beta_1 = 10, \beta_{31} = -9, \beta_{21} = 8, \beta_{33} = -7, \ldots, \beta_{99} = -1$, and $\beta_j = 0$ otherwise, and $\varepsilon \sim \mathcal{N}(0, 1)$. We introduced missing values completely at random; for each column, we set a missing rate sampled from a uniform distribution $U(0, 1)$. To avoid columns with no observations, we modified the missing ratio such that at least 10 observations existed for each variable.

The test data were also generated in the same manner except that we used different random seeds and did not introduce missing values for evaluation. We used the following three performance measures:

- **Frobenius error of covariance matrix:** $\frac{1}{p^2} \left\| \hat{\Sigma} - \Sigma^* \right\|_F$.
- **$L_2$ error of regression coefficients:** $\| \hat{\beta} - \beta^* \|_2$.
- **Root MSE (RMSE) of prediction:** $\sqrt{\left\| y - X\hat{\beta} \right\|_2^2 / n}$.

We iterated each experiment 30 times, and plotted their averages with plus/minus standard errors. The regularization parameter $\lambda$ was selected by calibrated 5-fold cross validation (Datta and Zou 2017).

The results are shown in Figure\textsuperscript{3}. The weighted norms were effective for both the Frobenius and max norm formulations. The case $\alpha = 1$ was better than $\alpha = 0, 1/2, 2$, and 2, as suggested by the non-asymptotic theoretical analyses. In addition, the Frobenius norm with $\alpha = 1$ was the best in terms of most of the performance measures. Therefore, we use this formulation as the proposed method HMLasso in the subsequent experiments. In contrast, CoCoLasso (the
max norm with \( \alpha = 0 \) was apparently inferior to HMLasso. The simple mean imputation method was the worst because of its estimation bias.

**Numerical Simulations under Various Conditions**

We compared HMLasso (the Frobenius norm with \( \alpha = 1 \)) with mean imputation method and CoCoLasso (the max norm with \( \alpha = 0 \)) under various conditions. Based on the simulation setting in the previous subsection, we changed various conditions including missing pattern, missing rate, covariance pattern, covariance level, sample size, and the number of variables. We evaluated the \( L_2 \) error of regression coefficients, since other performance measures had similar tendencies.

**Missing Patterns and Missing Rates** We examined three missing patterns and three missing rates, resulting in nine conditions. The missing rates were set to \( \mu = 0.1, 0.5, 0.9 \). We introduced missing values according to the following missing patterns, which are thought to be common in real-world data. (1) Random pattern: Missing elements were selected with the same probability for all the elements. (2) Column pattern: Missing rates differ for each column. The \( j \)-th column missing rate \( \mu_j \) was sampled from the uniform distribution so that the overall missing rate was \( \mu \). \( \mu_j \) was sampled from \( U(0, 0.2) \) for \( \mu = 0.1 \), from \( U(0, 1) \) for \( \mu = 0.5 \), and from \( U(0.8, 1) \) for \( \mu = 0.9 \). (3) Row column pattern: Missing rates differ for each row and each column. The \((i, j)\)-th element missing rate \( \mu_{ij} \) was set so that the overall missing rate was \( \mu \). Specifically, we defined \( \mu_{ij} = \mu_i \mu_j \) where \( \mu_i \) and \( \mu_j \) were sampled from \( U(0, 0.632) \) for \( \mu = 0.1 \), \( \mu_{ij} = 1 - (1 - \mu_i)(1 - \mu_j) \) where \( \mu_i \) and \( \mu_j \) were sampled from \( U(0.586, 1) \) for \( \mu = 0.5 \), and \( \mu_{ij} = 1 - (1 - \mu_i)(1 - \mu_j) \) where \( \mu_i \) and \( \mu_j \) were sampled from \( U(0.368, 1) \) for \( \mu = 0.9 \).

The results are shown in Figure 4. HMLasso outperformed other methods, when the missing rate was moderate or high. In particular, in the cases of the column pattern and row column pattern, HMLasso delivered significant improvements. This might be because the number of pairwise observations were very small for these missing patterns. The mean imputation and CoCoLasso suffered from highly missing variables, while HMLasso suppressed the effects of them.

Note that the column and row missing patterns often appear in practice. The column missing pattern appears when some variables are frequently observed and others are rarely observed. This is typically caused by different data collection cost for each variable. The row missing pattern appears when some samples are filled and other samples are highly missing. This happens when some samples are considered to be important and they are frequently measured.

**Covariance Patterns and Covariance Levels** We examined three covariance patterns and three covariance levels, resulting in nine conditions. The covariance levels were set to \( r = 0.1, 0.5, 0.9 \). The covariance matrix was generated according to the following covariance matrix patterns.
(1) Uniform pattern: Covariances were uniform among all variables, where $\Sigma_{jk}^* = r$ for $j \neq k$ and $\Sigma_{jk}^* = 1$ for $j = k$. (2) Autoregressive pattern: Covariances among neighbors were strong, such that $\Sigma_{jk}^* = r^{|j-k|}$ for $j \neq k$ and $\Sigma_{jk}^* = 1$ for $j = k$. (3) Block pattern: All of the variables were divided into some blocks. The intra-block covariances were strong and inter-block covariances are zeros. We set $\Sigma^* = \text{diag}(\Sigma_{11}^*, \ldots, \Sigma_{qq}^*)$ with $q = 10$, where $\Sigma_{jj}^*$ was a 10-dimensional square matrix with the above uniform pattern.

The results are shown in Figure 5. HMLasso outperformed the other methods for almost all covariance patterns and covariance levels. The mean imputation method was comparable to HMLasso under low covariance conditions, because the shrinkage estimator such as the mean imputation tends to show a good performance when the true covariance is close to zero. However, the mean imputation deteriorated its estimation under a moderate or high covariance condition.

**Data Size** We examined three kinds of row size and four kinds of column size; $n = 10^3, 10^4, 10^5$ and $p = 50, 100, 150, 200$. We fixed either one as the base condition ($n = 10^3$ or $p = 100$) while changing the other parameter.

The results are shown in Figure 6. HMLasso outperformed the other methods under all the conditions. Remarkably, HMLasso did not suffer from high dimensionality, although the other methods showed gradually worse performances.

**Residential Building Dataset**

We used the residential building dataset [Rafiei and Adeli 2015], which is available in UCI datasets repository\footnote{https://archive.ics.uci.edu/ml/datasets.html}. The residential building dataset includes the construction cost, sale prices, project variables, and economic variables corresponding to real estate single-family residential apartments in Tehran, Iran. The objective is to predict the sale prices and construction costs from project physical/financial variables and economic variables. The data consists of $n = 372$ samples and $p = 105$ variables including two output variables. Because the original data does not contain missing values, we introduced missing values in our experiments. We examined two missing patterns (random pattern and column pattern) with various missing rates ($\mu = 0, 0.2, 0.4, 0.6, 0.8$). We examined the performances of several methods in terms of prediction error by complete samples, since we cannot know the true coefficient. We randomly split data into 300 samples for training, 36 samples for validation, and 36 samples for testing, and iterated the experiments 30 times.

The results are shown in Figure 7. HMLasso outperformed the other methods for almost all the conditions. HMLasso delivered significant improvements especially in highly missing cases with the column missing pattern.

**Concluding Remarks**

In this paper, we proposed a novel regression method for high dimensional and highly missing data, and showed the advantages of our method by theoretical analyses and numerical simulations. Unfortunately, we could not find an appropriate high dimensional and highly missing real-world open data. This may be because, e.g., people feel ashamed of opening highly missing data to the public, people feel confident that highly missing data is not useful, or high dimensional industrial data is hard to open since it contains large amount of information. However, the authors often encountered highly missing data especially in industrial data. Our method will enable us to utilize data which people have not yet been utilized.
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Supplementary Materials

Discussion from an Asymptotic Perspectives
In this section, we describe another view of our weighted formulation. This is a rough result, but we intuitively interpret why weighted norm with $\alpha = 1/2$ performs well from an asymptotic perspective in the case $n \gg p$. The standard asymptotic theory shows that we have, for a large pairwise observation number $n_{jk}$,

$$\sqrt{n_{jk}}(S^\text{pair}_{jk} - \Sigma^*_j) \sim \mathcal{N}(0, \tau^2_{jk}),$$

where $\Sigma^*$ is a population covariance matrix and $\tau_{jk}$ is a constant. Here we assume that $S^\text{pair}_{jk}$ are independent and $\tau_{jk} = \tau$ for all $j, k$. The likelihood of $S^\text{pair}$ can be approximated to

$$\prod_{j,k} \frac{1}{\sqrt{2\pi \tau^2}} \exp \left( \frac{-1}{2\tau^2} \left( \frac{1}{n_{jk}}(S^\text{pair}_{jk} - \Sigma^*_j)^2 \right) \right).$$

Then, the maximum likelihood estimator of $\Sigma^*$ under the PSD constraint can be approximated to

$$\arg\min_{\Sigma \geq 0} \sum_{j,k} n_{jk}(S^\text{pair}_{jk} - \Sigma_{jk})^2,$$

which is equivalent to our method with $\alpha = 1/2$.

Algorithms

Cordinate Descent Algorithm with the Covariance Matrix
Let $\mathcal{L}(\beta)$ be the objective function of (11). To derive the update equation, when $\beta_j \neq 0$, differentiating $\mathcal{L}(\beta)$ with respect to $\beta_j$ yields

$$\partial_{\beta_j} \mathcal{L}(\beta) = \Sigma_{j,-j} \beta_j - \rho_{j,j}^\text{pair} + \lambda \text{sgn}(\beta_j),$$

where $\Sigma_{j,-j}$ denotes $\Sigma$ without the $j$-th component, and $X_{j,-j}$ denotes the $j$-th row of $X$ without the $j$-th column. Solving $\partial_{\beta_j} \mathcal{L}(\beta) = 0$, we obtain the update rule as

$$\beta_j \leftarrow \frac{1}{\Sigma_{jj}} S \left( (\rho_{j,j}^\text{pair} - \tilde{\Sigma}_{j,-j} \beta_j) , \lambda \right),$$

where $S(z, \gamma)$ is a soft thresholding function

$$S(z, \gamma) := \text{sgn}(z)(|z| - \gamma)_+$$

$$= \begin{cases} 
  z - \gamma & \text{if } z > 0 \text{ and } \gamma < |z|, \\
  z + \gamma & \text{if } z < 0 \text{ and } \gamma < |z|, \\
  0 & \text{if } |z| \leq \gamma.
\end{cases}$$

The whole algorithm for the Lasso-type optimization problem (11) using the covariance matrix is described in Algorithm 2.

ADMM for the Weighted Max Norm Formulation
We describe the ADMM algorithm for the weighted max norm formulation. This is a natural extension of the CoCoLasso algorithm.

Proofs

Proof of Proposition 1

Proof. First, we prove that random variables with Bernoulli distribution is sub-Gaussian. By $m_{ij} \sim \text{Bernoulli}(\mu_j)$, we have $E[m_{ij}] = \mu_j$ and

$$E[\exp(s(m_{ij} - \mu_j))] = \exp(1) = 1.$$

Then, we prove the following proposition. For a random vector $M_i$, the $i$-th row of $M$, we have

$$E[\exp(s(v^T M_i - E[v^T M_i])]| = E[\exp(s(v^T M_i - E[v^T M]))] = E[\exp(sv_j(M_{ij} - \mu_j))].$$

The standard distribution is sub-Gaussian. By

$$E \left[ \exp \left( \frac{\lambda s^2 v_j^2}{2} \right) \right] = \exp \left( \frac{\lambda s^2 v_j^2}{2} \right).$$

We can see $\tau^2_j = \mu_j(1 - \mu_j)$.

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for any unit vector $v$. \hfill \Box

Algorithm 2 Lasso with Covariance Matrix

Input: $\Sigma$, $\rho^\text{pair}$, $\lambda$
initialize $\beta$
while until convergence do
  for $j = 1, \cdots, p$ do
    $\beta_j \leftarrow \frac{1}{\Sigma_{jj}} S \left( (\rho_{j,j}^\text{pair} - \tilde{\Sigma}_{j,-j} \beta_j) , \lambda \right)$
  end for
end while
Output: $\beta$
Algorithm 3 B-step Update for max norm in ADMM

Input: $A_{k+1}, A_k, \Sigma, \mu, W$

define $c = \text{vec} \left( A_{k+1} - \hat{\Sigma} - \mu A_k \right)$, $w = \text{vec}(W)$

sort $c$ as $w_1 |c_1| \geq w_2 |c_2| \geq \ldots$

find $l = \max_{l'} \left\{ l' : w_i |c_{l'}| = \frac{(\Sigma_i + |c_i|) - \frac{\mu}{2}}{\sum_{j=1}^{j<n} w_j} > 0 \right\}$

define $d = \frac{(\Sigma_i + |c_i|) - \frac{\mu}{2}}{\sum_{j=1}^{j<n} w_j}$

define $B_{k+1} = \text{mat}(b)$ such that $b_j = c_j$ for $|c_j| \leq \frac{d}{w_j}$, and $b_j = \frac{\text{sgn}(c_j)}{w_j}$ for $|c_j| > \frac{d}{w_j}$

Output: $B_{k+1}$

Proof of Theorem 2

Proof. We see that

$$|\hat{\Sigma}_{jk} - S_{jk}| = \left| \frac{1}{n} \sum_{i=1}^{n} m_{ij} m_{ik} x_{ij} x_{ik}/r_{jk} - \frac{1}{n} \sum_{i=1}^{n} x_{ij} x_{ik} \right|$$

$$\leq \frac{1}{r_{jk}} \frac{1}{n} \sum_{i=1}^{n} x_{ij} x_{ik} (m_{ij} - \mu_j) (m_{ik} - \mu_k)$$

$$+ \frac{\mu_j}{r_{jk}} \frac{1}{n} \sum_{i=1}^{n} x_{ij} x_{ik} (m_{ik} - \mu_k)$$

$$+ \frac{\mu_k}{r_{jk}} \frac{1}{n} \sum_{i=1}^{n} x_{ij} x_{ik} (m_{ij} - \mu_j).$$

We denote the three terms on the right-hand side by $T_1, T_2$ and $T_3$, respectively.

(T1): Let $v_i := x_{ij} x_{ik}$. Then we have $\|v\|_{\infty} \leq X_{\max}^2$. Remember that $m_{ij} - \mu_j$ and $m_{ik} - \mu_k$ are sub-Gaussian with parameter $\tau^2$. Then, by applying Lemma B.1 in CoCoLasso, we have

$$\Pr(T_1 > \varepsilon) = \Pr \left( \frac{1}{n} \left| \sum_{i=1}^{n} x_{ij} x_{ik} (m_{ij} - \mu_j) (m_{ik} - \mu_k) \right| > r_{jk} \varepsilon \right)$$

$$\leq C \exp \left( -\frac{c n \varepsilon r_{jk}^2}{\tau^2 X_{\max}^4} \right)$$

for all $r_{jk} \varepsilon \leq c \tau^2 X_{\max}^2$, i.e., $\varepsilon \leq c \tau^2 X_{\max}^2/r_{jk}$.

(T2) and (T3): By property (B.2) in CoCoLasso, we can see that for any vector $v$ and independent sub-Gaussian vector $w_i$ with parameter $\tau^2$, we have

$$\Pr \left( \frac{1}{n} \left| \sum_{i=1}^{n} v_i w_i \right| > \varepsilon \right) \leq C \exp \left( -\frac{c n \varepsilon^2}{\|v\|^2_{2^2}} \right).$$

If we define $v_i := x_{ij} x_{ik}$, we have $\|v\|^2_{2^2} \leq n^2 X_{\max}^4$. Remember that $m_{ij} - \mu_j$ and $m_{ik} - \mu_k$ are sub-Gaussian with parameter $\tau^2$. Hence, we have

$$\Pr(T_2 > \varepsilon) = \Pr \left( \frac{1}{n} \sum_{i=1}^{n} x_{ij} x_{ik} (m_{ij} - \mu_k) > r_{jk} \varepsilon \right)$$

$$\leq C \exp \left( -\frac{c n \varepsilon^2 r_{jk}^2}{\tau^2 X_{\max}^4} \right).$$

Similarly, we have

$$\Pr(T_3 > \varepsilon) = \Pr \left( \frac{1}{n} \sum_{i=1}^{n} x_{ij} x_{ik} (m_{ij} - \mu_j) > r_{jk} \varepsilon \right)$$

$$\leq C \exp \left( -\frac{c n \varepsilon^2 r_{jk}^2}{\tau^2 X_{\max}^4} \right).$$

Putting all together, we obtain that for all $\varepsilon \leq c \tau^2 X_{\max}^2/r_{jk}$,

$$\Pr \left( |\hat{\Sigma}_{jk} - S_{jk}| > \varepsilon \right) \leq C \exp \left( -\frac{c n \varepsilon^2 r_{jk}^2}{\tau^2 X_{\max}^4} \right).$$

Proof of Theorem 3

Proof. Since $\hat{\Sigma} = \arg \min_{\Sigma \geq 0} \|W \odot (\Sigma - \hat{\Sigma})\|_F^2$, we have, using the triangular equation,

$$\|W \odot (\Sigma - S)\|_F^2$$

$$\leq \|W \odot (\hat{\Sigma} - \Sigma)\|_F^2 + \|W \odot (\hat{\Sigma} - S)\|_F^2$$

$$\leq 2\|W \odot (\hat{\Sigma} - S)\|_F.$$

From Theorem 2, we have

$$\Pr \left( \|W \odot (\hat{\Sigma} - S)\|_F > \varepsilon \right)$$

$$= \Pr \left( \sum_{j,k} W_{jk}^2 \left( \hat{\Sigma}_{jk} - S_{jk} \right)^2 > \varepsilon^2 \right)$$

$$\leq \sum_{j,k} \Pr \left( W_{jk}^2 \left( \hat{\Sigma}_{jk} - S_{jk} \right)^2 > \varepsilon^2 p^{-2} \right)$$

$$\leq p^2 \max_{j,k} \Pr \left( W_{jk} \left| \hat{\Sigma}_{jk} - S_{jk} \right| > \varepsilon p^{-1} \right)$$

$$\leq p^2 C \exp \left( -c n \varepsilon^2 p^{-2} \left( \min_{j,k} \frac{r_{jk}}{w_{jk}} \right)^2 \right).$$
for all $\varepsilon \leq c p r^2 X^2_{\max} \min_{j,k} \left( \frac{w_{jk}}{r_{jk}} \right)$, where $\zeta = \max \{ \tau^2, \mu_1^2, \ldots, \mu_p^2 \}$. Hence, we have

$$
\Pr \left( \| W \odot (\tilde{\Sigma} - S) \|_F > \varepsilon \right) \\
\leq \Pr \left( \| W \odot (\hat{\Sigma} - S) \|_F > \varepsilon/2 \right) \\
\leq p^2 C \exp \left( -cn \varepsilon^2 p^{-2} \left( \min_{j,k} \frac{r_{jk}}{w_{jk}} \right)^2 \zeta^{-1} \right),
$$

for all $\varepsilon \leq c p r^2 X^2_{\max} \min_{j,k} \left( \frac{w_{jk}}{r_{jk}} \right)$. This is equivalent to

$$
\Pr \left( \frac{1}{p^2} \| W \odot (\tilde{\Sigma} - S) \|_F^2 > \varepsilon^2 \right) \\
\leq p^2 C \exp \left( -cn \varepsilon^2 \left( \min_{j,k} \frac{r_{jk}}{W_{jk}} \right)^2 \zeta^{-1} \right).
$$

Using the inequality

$$W_{\min}^2 \| \tilde{\Sigma} - S \|_F^2 \leq \| W \odot (\tilde{\Sigma} - S) \|_F^2,$$

we have

$$
\Pr \left( \frac{1}{p^2} \| \tilde{\Sigma} - S \|_F^2 > \varepsilon^2 \right) \\
\leq p^2 C \exp \left( -cn \varepsilon^2 W_{\min}^2 \left( \min_{j,k} \frac{r_{jk}}{W_{jk}} \right)^2 \zeta^{-1} \right),
$$

for $\varepsilon \leq c r^2 X^2_{\max} \left( \min_{j,k} W_{jk}/r_{jk} \right)/W_{\min}$.  

$\square$