Large-Small Equivalence in String Theory

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The simplest toroidally compactified string theories exhibit a duality between small and large radii: compactification on a circle, for example, is invariant under $R \rightarrow 1/(2R)$. Compactification on more general Lorentzian lattices (e.g. toroidal compactification in the presence of background metric, antisymmetric tensor, and gauge fields) yields theories for which a large-small equivalence is not so simple. Here an equivalence is demonstrated between large and small geometries for all toroidal compactifications. By repeatedly transforming the momentum mode corresponding to the smallest winding dimension to another mode on the lattice, it is possible to increase the volume to exceed a finite lower bound.
1. Introduction

It has been known for some time that the simplest toroidal string compactification exhibits an equivalence between small and large background geometries. That is, as pointed out in [1] and later expanded upon in [2] and [4], the presence of winding states leads to an \( R \to 1/(2R) \) invariance of a theory compactified on a circle without any background fields; this generalizes easily to a \( G \to 1/4G^{-1} \) invariance for a theory compactified on a torus given by the metric \( G \). \( (G_{ij} = e_i \cdot e_j \) where the \( e_i \) form a basis for the winding vectors.\)

This raises the question of how generally this phenomenon occurs in arbitrary toroidal compactifications of string theory. To answer this question we must consider the moduli space of Lorentzian lattice compactifications spanned by the background metric, antisymmetric tensor, and gauge field (Wilson lines). The worldsheet action is

\[
S = \int G_{ij} \partial^\alpha X^i \partial^\beta X^j + \epsilon^{\alpha\beta} B_{ij} \partial^\alpha X^i \partial^\beta X^j + \epsilon^{\alpha\beta} A_J^i \partial^\alpha X^i \partial^\beta X^J
\]

As described in [3], the left- and right-moving momenta are

\[
p_L = (P + A^T n, G^{-1} (m/2 - Bn - (1/4)AA^T n - (1/2)AP) + n)
\]

and

\[
p_R = (G^{-1} (m/2 - Bn - (1/4)AA^T n - (1/2)AP) - n)
\]

where \( n,m \in \mathbb{Z}^d \) are the winding and momentum vectors and \( P \) is a vector in the 16-dimensional root lattice of the gauge group. With general \( B \) and \( A \) values, interchanging \( G \) and \( 1/4G^{-1} \) does not always yield a new state on the same lattice or even preserve the spectrum of \( p_L^2 + p_R^2 \) values, as would be necessary for an invariance of this simple form. As discussed in [4], the moduli space of Lorentzian lattice compactifications contains discrete equivalences given by \( O(16+d,d,Z) \) transformations on \( (p_L,p_R) \), which leave the lattice invariant but which can be obtained equivalently by changing the background fields. (Here \( d \) is the number of compactified spatial dimensions.) That is, the moduli space of compactifications is \( O(16+d,d)/O(16+d)xO(16+d,d,Z) \). One example is the transformation taking \( B_{ij} \) to \( B_{ij} + 1 \); another is the well-known so-called duality transformation taking \( 2(B + G) \) to its inverse (discussed in [4] and [3]).

Thus we must investigate whether there exists some \( O(16+d,d,Z) \) transformation taking small \( G \) to large \( G \). First we must specify precisely what we mean by the size of the compactified space. One condition for small-large duality requires the volume \( \text{det} G \) to
have some finite effective lower bound. A stronger condition is that in addition the winding lengths \( n^T G n \) should themselves all exceed some lower bound. As discussed in section 3, it can be shown that the latter implies the former: in any dimension, if all winding lengths exceed some minimum, the volume must also have a lower bound.

In this paper we show that the compactified volume \( \sqrt{\det G} \) can be raised to exceed a finite volume; we can make the further statement that if the lengths of the windings cannot all be made to exceed a finite lower bound, then we can transform the volume to \( \infty \). In two dimensions with \( A=0 \) we will see explicitly that the winding lengths can be raised to exceed a lower bound. These results suggest that winding lengths can be raised in general; further work is necessary to show this for all cases. We will begin by discussing the \( d=2 \) case without Wilson lines in a way that we will then generalize to higher dimensions. Finally, we will demonstrate the result with Wilson lines included.

2. Small-Large Duality for \( d = 2 \) and \( A = 0 \)

As discussed in [4], the generalization of the interchange of momentum and winding vectors \( (n \leftrightarrow m) \) is \( 2(G + B) \rightarrow 1/2(G + B)^{-1} \). This transformation fails to transform all small metrics to large, as the following example demonstrates. When \( G = RI \), we have

\[
2 \begin{pmatrix} R^2 & -B \\ B & R^2 \end{pmatrix} \rightarrow \frac{1}{2(R^4 + B^2)} \begin{pmatrix} R^2 & B \\ -B & R^2 \end{pmatrix}
\]

under this transformation (here \( B = -B_{12} \)). Thus as \( R^2 \rightarrow 0 \) for \( B \neq 0 \) it is transformed to another nearly vanishing radius.

It is possible, taking into account other available \( O(2,2,Z) \) transformations, to satisfy both conditions for small-large duality mentioned in section 1. As described in [3], we can raise the volume \( \det G \) for \( d = 2 \) using \( SL(2, Z) \times SL(2, Z) = SO(2, 2, Z) \) transformations on the two complex parameters

\[
\rho = 2 \left( B + i \sqrt{\det G} \right)
\]

and

\[
\tau = \frac{G_{12}}{G_{11}} + \frac{i \sqrt{\det G}}{G_{11}}
\]

We simply need to transform \( \rho \) to its fundamental domain for which \( 2 \det G \geq \sqrt{3}/2 \) (see fig. 1). It is not too hard to see, however, that \( SL(2, Z) \times SL(2, Z) \) transformations do not suffice to raise the winding lengths of the two-dimensional torus: transforming \( \tau \)
by an SL(2,Z) transformation merely reparameterizes the torus (i.e. changes its basis); transforming ρ does not affect the shape of the torus, just its volume. Consequently, since the fundamental domain contains only one point in the orbit of each ρ, a torus with a small winding situated in ρ’s fundamental domain cannot be transformed to one with larger windings by SL(2,Z)xSL(2,Z). To illustrate this, consider a diagonal torus with $R_1 = \epsilon << 1$ and $R_2 = 1/\epsilon$. For this torus, det G = 1, putting it in ρ’s fundamental domain. Here Im τ = $R_1/R_2$; transforming ρ leaves this ratio fixed. Since ρ is already in its fundamental domain, the volume cannot be further increased. Thus $R_1$ will stay small as long as we confine ourselves to SL(2,Z)xSL(2,Z). If, however, we interchange ρ and τ (this is equivalent to exchanging the first component the momentum and winding modes, as will be seen in section 3), we find

$$G \rightarrow \begin{pmatrix} \frac{1}{4\epsilon^2} & \frac{B/(2\epsilon^2)}{B/(2\epsilon^2)(1+B^2)/\epsilon^2} \\ \end{pmatrix}$$

and

$$n^T G n \rightarrow (n_1 + (2B)n_2)^2/(4\epsilon^2) + n_2^2/\epsilon^2 \geq 1/(4\epsilon^2)$$

Thus we find that to raise winding lengths in $d = 2$ we must make use of the rest of O(2,2,Z), namely the $Z_2$ interchange of ρ and τ. First we show that this transformation provides a different way to raise the volume from the SL(2,Z)xSL(2,Z) transformations. Given a $d = 2$ torus G, reparameterize G if necessary to form a basis out of its two smallest winding lengths. The SL(2,Z) generators $\tau \rightarrow \tau + 1$, which takes $e_1 \rightarrow e_1 + e_2$, and $\tau \rightarrow -1/\tau$, which interchanges the 1 and 2 components, provide the necessary transformations for this reparameterization. Suppose the smallest squared winding, $G_{11}$, is less than 1/2. Then the interchange of ρ and τ, taking det G to det G/4$G_{11}$, increases the volume. We may repeat this procedure as long as $G_{11} < 1/2$. Then we have either detG $\rightarrow \infty$ or at least $G_{11}, G_{22} \geq 1/2$ (i.e. all squared winding lengths $\geq 1/2$, which is sufficient to put a lower bound on the volume, as discussed in section 3). This procedure will generalize to higher $d$.

In fact, this procedure in $d = 2$ gives us a two-step method for putting ρ in its fundamental domain. In the basis consisting of the two smallest windings $e_1 \leq e_2$, the angle between $e_1$ and $e_2$ ($\theta_{12}$) must be between π/3 and 2π/3; otherwise $|e_2 - e_1| < |e_2|$, contradicting the fact that $e_1$ and $e_2$ are the two smallest windings. Thus once τ has been transformed to this basis, it falls in the shaded region in figure 2. Then τ can be placed
in its fundamental region by $\tau \rightarrow \tau \pm 1$. Finally, $\rho \leftrightarrow \tau$ puts $\rho$ in its fundamental region, ensuring that $2\sqrt{\det G} > \sqrt{3/2}$.

Next we show that a $d = 2$ torus can be transformed via $\rho \leftrightarrow \tau$ into one with all windings bounded below. To see this we consider all possible windings $n^T G' n$ of the new metric. We find

$$G' = \begin{pmatrix} 1/(4G_{11}) & B/(2G_{11}) \\ B/(2G_{11}) & (\det G + B^2)/G_{11} \end{pmatrix}$$

This gives

$$n^T G' n = \frac{n_2^2 \det G}{G_{11}} + \frac{(n_1 + 2n_2 B)^2}{4G_{11}}$$

We see here that with $\det G > 3/16$ (i.e. with $\rho$ in its fundamental domain so that $2\sqrt{\det G} > \sqrt{3/2}$), interchanging $\tau$ and $\rho$ yields new windings whose squared lengths all exceed $3/8$: If $n_2 \neq 0$, the squared winding $e^2 \geq 3/(16G_{11}) \geq 3/8$ (for the smallest squared winding $G_{11} \leq 1/2$). If $n_2 = 0$, we have $e^2 > 1/(4G_{11}) \geq 1/2$. Thus it takes at most two interchanges of $\rho$ and $\tau$ to transform any torus to one with all windings bounded below. Unfortunately this argument for putting a lower bound on the winding lengths does not generalize to all higher $d$, although the above argument for the weaker form of small-large duality does survive in higher dimensions.

3. Volumes for General $d$ and $A=0$

As discussed in [3],

$$(1/2)(p^2_L + p^2_R) = (n^T m^T) \begin{pmatrix} G - BG^{-1}B & 1/2BG^{-1} \\ -1/2G^{-1}B & 1/4G^{-1} \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix}$$

Thus to obtain the effect of an $O(d,d,Z)$ transformation on the background fields, we simply transform the above matrix accordingly. That is,

$$\begin{pmatrix} n \\ m \end{pmatrix} \rightarrow S \begin{pmatrix} n \\ m \end{pmatrix}$$

where $S \in O(d,d,Z)$ is equivalent to

$$M(G, B) \rightarrow S^T M(G, B) S$$

where $M(G, B)$ is the matrix of background fields in the above expression for $p^2_L + p^2_R$. In particular, the lower right $dxd$ block gives the transformed dual torus, which determines
the new momentum modes $G^{-1}m$. The momentum modes characterize the volume much more directly than do the winding modes, since, as can be seen from the formulas for $p_L$ and $p_R$, turning on a winding mode generates nontrivial contributions to the momenta and thus to the energy which depend on $B$ as well as $G$. This makes it difficult to extract $G$ from the spectrum of winding modes.

Under the $O(d,d,Z)$ transformation which exchanges $n_1$ and $m_1$ taking the first momentum mode to the first winding mode (leaving all other momentum modes fixed), $G^{-1}$ becomes (generalizing $\rho \leftrightarrow \tau$ from 2 dimensions)

$$1/4G' = \begin{pmatrix} G_{11} - (BG^{-1}B)_{11} & 1/2(BG^{-1})_{12} & \cdots & 1/2(BG^{-1})_{1d} \\ 1/2(BG^{-1})_{12} & 1/4G^{-1}_{22} & \cdots & 1/4G^{-1}_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ 1/2(BG^{-1})_{1d} & 1/4G^{-1}_{2d} & \cdots & 1/4G^{-1}_{dd} \end{pmatrix}$$

Note that the squared volume $\text{det} G'^{-1}$ is independent of $B$: without the $G_{11}$ term, the first column is a linear combination of the other columns so that the contributions of $B_{ij}$ to the determinant cancel. That is,

$$(BG^{-1}B)_{11} = B_{1k}G_{kl}^{-1}B_{l1} = (1/2B_{1k}G_{kl}^{-1})(2B_{l1})$$

and

$$1/2(BG^{-1})_{1j} = 1/2B_{1l}G_{lj}^{-1} = (1/4G_{jl}^{-1})(2B_{l1})$$

so that

$$\text{column1} = (\text{columnl}) (2B_{l1})$$

We are left with

$$\text{det}G'^{-1} = 4G_{11}V^*_{2d}$$

where $V^*_{2d}$ is the squared dual volume restricted to the $2,3,\ldots,d$ directions (i.e. the determinant of $G^{-1}$ restricted to these components). Now $G_{11} = e^2_1$ and $e^2_1 = 1/cos^2\theta e^*_{1i} e^*_{1k}$ where $\theta$ is the angle between $e_1$ and its dual $e^*_1$. So

$$\text{det}G'^{-1} = \text{det}G^{-1} (4e^4_1) \left( \frac{V^*_{2d} e^*_{1i} cos^2\theta_1}{V^*_{1d}} \right)$$

But the last factor is 1, since the volume of the dual lattice is just the volume of the sublattice spanned by $e^*_{2i}, \ldots, e^*_{di}$ times the component of the remaining dual basis vector $e^*_1$ orthogonal to this sublattice. Thus exchanging $n_1$ and $m_1$ decreases $\text{det}G^{-1}$ if $e^2_1 < 1/4$. By
repeating this procedure, reparameterizing if necessary to render \( e_1 \) the smallest winding, we can continue increasing \( \det G = 1/\det G^{-1} \) as long as the smallest squared winding is less than \( 1/2 \). Then either \( \det G \) increases ad infinitum or eventually all windings exceed \( 1/\sqrt{2} \).

In the latter case, it follows from a theorem in the geometry of numbers (a generalization of Blichfeldt’s theorem \([6]\)) that

\[
\sqrt{\det G} > V_d
\]

where

\[
V_d = \frac{\pi^d (1/2\sqrt{2})^d}{\Gamma(1 + d/2)}
\]

is the volume of a sphere of diameter \( 1/\sqrt{2} \) in \( R^d \). According to this theorem, any region in \( R^d \) of volume \( V \) greater than \( \sqrt{\det G} \) must contain two points whose difference is in the lattice determined by \( G \). Consider a \((d-1)\)-sphere whose diameter is \( 1/\sqrt{2} \). After the above procedure we must have \( \sqrt{\det G} \) greater than the volume of the sphere since otherwise the sphere would contain a winding shorter than \( 1/\sqrt{2} \). Thus for any \( d \), every theory is equivalent to one compactified on a volume exceeding \( V_d \).

4. Small-large duality for general toroidal compactifications

With \( A \neq 0 \), the zero-mode mass spectrum is given by

\[
(1/2)(p_L^2 + p_R^2) = (n^T \quad m^T \quad P^T) M(G, B, A) \begin{pmatrix} n \\ m \\ P \end{pmatrix}
\]

where \( M \) can be read off the expressions for \( p_L \) and \( p_R \).

Since, as discussed above, the momentum modes characterize \( G^{-1} \), we would like to transform the momentum modes orthogonally (i.e. via \( O(16+d,d,Z) \)) to other modes on the lattice with small values of \( p_L^2 + p_R^2 \). If we transform only one of the momentum modes, that corresponding to the smallest winding \( e_1 \), leaving all other momentum modes fixed, the new dual volume will be the old dual subvolume for the \( 2 \ldots d \) directions multiplied by the component of the transformed first momentum mode orthogonal to this subvolume. If this new orthogonal component is smaller than the original orthogonal component, the transformation will reduce the dual volume, as occurred for the \( n_1 \leftrightarrow m_1 \) transformation.
with $A = 0$. Knowing the transformed mode $(n, m, P)$ (with $2n \cdot m + P^2 = 0$ so that the new mode is still 0-norm) suffices to compute the new inverse metric $G'^{-1}$, as long as such an $O(16+d,d,Z)$ transformation exists. We choose

$$n = \begin{pmatrix} n_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

so that our new mode will still be orthogonal to the other momentum modes.

As pointed out in [7], it is always possible to find a dual $x^*$ to $x = (n,m,P)$ such that the Lorentzian inner products $x^* \cdot x = 1$ and $x^* \cdot x = 0$ hold and that $x^*$ is still orthogonal to the winding modes corresponding to the other momentum modes $(2...d)$. (Here we are taking $x$ to be indivisible since we are interested in small values of $p^2_L + p^2_R$; we can always divide out any common factor if necessary.) The $2...d$ momentum modes still have the corresponding winding modes as their duals. The space orthogonal to the new momentum and winding modes is a 16-dimensional even self-dual positive definite lattice, either $\Gamma_8 + \Gamma_8$, the root lattice for $E_8 \times E_8$, or $\Gamma_{16}$, the root lattice for Spin$(32)/Z_2$.

If upon transforming the first momentum mode, the orthogonal space switches from one root lattice to the other, our transformation is not quite Lorentzian and therefore cannot be absorbed into a background field transformation. It is, however, an $O(16+d,d)$ transformation from a theory with the same $G,B$, and $A$ and with our original momentum modes but with the orthogonal space the switched root lattice. This Lorentz transformation can be absorbed into the background fields, giving $G'^{-1}$ as described above, since this is determined by the momentum-mode part of the transformation. This is simply a restatement of the isomorphism between $\Gamma_8 + \Gamma_8$ with one compactified dimension and $\Gamma_{16}$ with one compactified dimension noted in [3] and [7]. Thus we can always transform the first momentum mode to any other 0-norm vector on the lattice in a way that can be obtained by an $O(16+d,d)$ transformation on the background fields.

We find

$$G'^{-1} = \begin{pmatrix} (1/2)(P + A^T n)^2 + n_1^2 G_{11} + (1/4)z_i G_{ij}^{-1} z_j & (1/4)G_{2k}^{-1} z_k & \cdots & (1/4)G_{dk}^{-1} z_k \\ (1/4)G_{2k}^{-1} z_k & (1/4)G_{22}^{-1} & \cdots & (1/4)G_{2d}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ (1/4)G_{dk}^{-1} z_k & (1/4)G_{d2}^{-1} & \cdots & (1/4)G_{dd}^{-1} \end{pmatrix}$$

where

$$z = m - (2B + (1/2)AA^T)n - AP$$
The $(1,1)$ component of the new dual metric is of course the $(1/2)(p_L^2 + p_R^2)$ value for our new momentum mode; the $2 \ldots d$ components are those from the original dual metric, since we left the corresponding momentum modes alone. The others were obtained by transforming $M(G,B,A) \rightarrow O^TMO$, where $O$ transforms the first momentum mode to the mode $(n,m,P)$, leaving the others fixed.

As in the $A=0$ case, we find that the first column contains a linear combination of the others (multiply column $j$ by $z_j$). When we subtract this from the first column (since we are taking the determinant), the first row of the resulting matrix contains the same linear combination of the other rows as we had above for columns. Subtracting this out, the resulting dual volume is

$$\det(1/4)G'^{-1} = [(1/2)(P_1 + A_1^1 n_1)^2 + n_1^2 G_{11}](1/4)V_{2 \ldots d}^*(1/4)^{d-1} + z_1^2 \det(1/4)G^{-1}$$

So

$$\frac{\det(1/4)G'^{-1}}{\det(1/4)G^{-1}} = e_1^2[(1/2)(P + A_1^1 n_1)^2 + n_1^2 e_1^2] + (m_1 - (1/2)A_1^I A_1^J n_1 - A_1^I P_J)^2$$

We need to find $n_1$, $m_1$, and $P$ to render this ratio smaller than 1 as long as $e_1$ remains smaller than some lower bound. We can enforce the $0$-(Lorentzian) norm condition by taking $n_1 = 2y^2$, $P_I = 2S_I y$ and $m_1 = -S^2$ where $y$ is an integer and $S$ is a vector in the root lattice. This ensures that $2n_1 m_1 + P^2 = 0$, with $m_1$ integral. With this choice we have

$$(P + A_1^1 n_1)^2 = (2Sy + A_1^1 (2y^2))^2 = 4y^2 (S + A_1^1 y)^2$$
$$= 4y^2 (D_I + A_I^J y) \Gamma^IJ (D_J + A_J^I y)$$

and

$$(m_1 - (1/2)A_1^I A_1^J n_1 - A_1^I P)^2 = ((S + A_1^1 y)^2)^2$$
$$= ((D_I + A_I^J y) \Gamma^IJ (D_J + A_J^I y))^2$$

where the $D_I \in \mathbb{Z}$ give $S$ in the integer basis: $S = D_I E_I$ where the $E_I$ are the basis vectors for the root lattice. Here $\Gamma$ is the metric for the root lattice: $E_I \cdot E_J = \Gamma_{IJ}$. Similarly $A_1^1 = A_I^J E_I$.

By a result of diophantine approximation [8], it is possible to choose integers $D_I$ and $y$ such that
\[ |D_I + A_I^1 y| < (\sqrt{e_1})^{1/16} \]

with

\[ y < (1/\sqrt{e_1}) \]

Then

\[ e_1 y |D_I + A_I^1 y| < (\sqrt{e_1})^{17/16} \]

and

\[ n_1^2 e_1^4 < 4e_1^2 \]

So

\[ \frac{\det G'^{-1}}{\det G^{-1}} < f(e_1) \]

where \( f(e_1) \) is a positive polynomial in \( e_1^{1/16} \) since the root lattice metric \( \Gamma \) is positive definite.

Setting this polynomial equal to 1 gives a bound on \( e_1 \) below which the transformation lowers the dual volume, and thus raises the volume. Once again, if the smallest winding \( e_1 \) never exceeds this finite lower bound, the volume diverges by repeated transformations of the corresponding momentum mode.

5. Conclusion

We have seen that by repeatedly transforming the momentum mode on the smallest circle, we can prevent the volume of the compactified dimensions from getting too small. In two compactified dimensions without Wilson lines, this transformation is sufficient in fact to raise all winding lengths. For any \( d \), if it turns out that the winding lengths cannot be raised by \( O(16+d,d,Z) \), then we can at least say that the corresponding volume becomes unbounded. If this is the whole story, the fact that \( \sqrt{\det G} > V_d \) will imply that some sort of uncertainty principle is operating on the winding lengths, requiring at least one of them to blow up if any approach zero. Since the theory has infinitely many chances to escape this volume divergence as we iteratively perform our transformation on the first momentum mode, and since in two dimensions without Wilson lines windings are bounded below, we expect windings to have a lower bound in all cases.

This equivalence between large and small spaces is often taken to account for the necessary imprecision in measurements made by fundamental strings of nonzero (Planckian)
size, since it keeps strings insensitive to arbitrarily small distances. This does not completely resolve the problem, however, since duality does not prevent one from measuring exactly the volumes of the large and small spaces. With a Planck-sized ruler one could certainly distinguish a tiny space from a huge one; what one could not do is measure precisely the size of either. The fact that they cannot be distinguished results directly from the presence of the winding modes.

Further work is necessary to determine whether windings can be raised in all cases and the relation of small-large duality to the modular invariance of the worldsheet.

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