Delaunay Hodge Star

Anil N. Hirani*, Kaushik Kalyanaramanb, and Evan B. VanderZee*c

aDepartment of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green St., Urbana, IL
bSCOREC, Rensselaer Polytechnic Institute, 110 8th St., Troy, NY
cArgonne National Laboratory, 9700 S. Cass Ave., Lemont, IL

Abstract

We define signed dual volumes at all dimensions for circumcentric dual meshes. We show that for pairwise Delaunay triangulations with mild boundary assumptions these signed dual volumes are positive. This allows the use of such Delaunay meshes for Discrete Exterior Calculus (DEC) because the discrete Hodge star operator can now be correctly defined for such meshes. This operator is crucial for DEC and is a diagonal matrix with the ratio of primal and dual volumes along the diagonal. A correct definition requires that all entries be positive. DEC is a framework for numerically solving differential equations on meshes and for geometry processing tasks and has had considerable impact in computer graphics and scientific computing. Our result allows the use of DEC with a much larger class of meshes than was previously considered possible.

Key words: discrete exterior calculus, primal mesh, circumcentric dual

NOTE: This version incorporates corrections to the previous arXiv version (v3). The previous version (minus the Appendix) appeared as the journal article [9]. The main correction in this version is replacement of Figure 1 of [9] by the Figure 1 here. The change is in columns 3 and 4 of the figure. This error in [9] (and hence in arXiv version v3 of this article) has been traced to a programming error resulting from the generation and processing of the meshes used in columns 3 and 4 of Figure 1. This error resulted in incorrect boundary conditions being used for the meshes in columns 3 and 4 and misidentification of some edges in column 4. We provide additional details in Section B in the Appendix of this arXiv version. The implication of this correction of Figure 1 is the possibility it raises that Discrete Exterior Calculus may be applicable in an even broader class of meshes than was believed possible when the previous version appeared. This version also fixes a formula error in Section 2 and error with an illustration in Figure 6 of [9]. There are no errors in the mathematical statements (lemmas and theorems) of [9].

1. Introduction

Discrete Exterior Calculus (DEC) is a framework for numerical solution of partial differential equations on simplicial meshes and for geometry processing tasks [4, 8]. DEC has had considerable impact in computer graphics and scientific computing. It is related to finite element exterior calculus and differs from it in how inner products are defined. The main objects in DEC are p-cochains, which for the purpose of this paper may be considered to be a vector of real values with one entry for each p-dimensional simplex in the mesh. For p-cochains a and b their inner product in DEC is \(a^T \ast_p b\) where \(\ast_p\) is a diagonal discrete Hodge star operator. This is a diagonal matrix of order equal to the number of p-simplices and with entries that are ratios of volumes of p-simplices and their \((n-p)\)-dimensional circumcentric dual cells.

For this to define a genuine inner product the entries have to be positive. Simply taking absolute values or considering all volumes to be unsigned does not lead to correct solutions of partial differential equations. See Figure 1 to see the spectacular failure when unsigned volumes are used in solving Poisson’s equation in mixed form.

When DEC was invented it was known that completely well-centered meshes were sufficient but perhaps not necessary for defining the Hodge star operator. (A completely well-centered mesh is one in which the circumcenters are contained within the corresponding simplices at all dimensions. Examples are acute-angled triangle meshes and tetrahedral meshes in which each triangle is acute and each tetrahedron contains its circumcenter.) For such meshes, the volumes of circumcentric dual cells are obviously positive. For some years now there has been numerical evidence that for codimension 1 duality some (but not all) pairwise Delaunay meshes yield positive dual volumes if volumes are given a sign based on some simple rules. Moreover, these seem to yield correct numerical solutions for a simple partial differential equation. A pairwise Delaunay mesh (of dimension n embedded in \(\mathbb{R}^N, N \geq n\)) is one in which each pair of adjacent n-simplices sharing a face of
dimension $n - 1$ is Delaunay when embedded in $\mathbb{R}^n$. (Imagine a pair of triangles with a hinge at the shared edge and lay the pair flat on a table.) This generalizes the Delaunay condition to triangle mesh surfaces embedded in three dimensions and analogous higher dimensional settings. For planar triangle meshes and for tetrahedral meshes in three dimensions pairwise Delaunay is same as Delaunay.

We give a sign convention for dual cells and a mild assumption on boundary simplices. With these in hand, for pairwise Delaunay meshes it is easy to see that the codimension 1 dual lengths are positive in the most general case (dimension $n$ mesh embedded in $\mathbb{R}^N$). In addition, we prove that such triangle meshes embedded in two or three dimensions have positive vertex duals and that the duals of vertices and edges of tetrahedral meshes in three dimensions are positive. This settles the question of positivity for duals of vertices and edges of tetrahedral meshes in three dimensions have positive vertex duals and that the duals prove that such triangle meshes embedded in two or three dimensions pairwise Delaunay is same as Delaunay.

We will sometimes write the dimension of a

The dual mesh of a (primal) simplicial mesh is often
defined using a barycentric subdivision. For every $p$-
dimensional simplex, there is a dual $(n-p)$-dimensional

cell where $n$ is the dimension of the simplicial complex.
In DEC circumcenters are used instead of barycenters
because the resulting orthogonality between the primal
and dual is an integral part of the definition of some of

2. Signed Circumcentric Dual Cells

The dual mesh of a (primal) simplicial mesh is often
defined using a barycentric subdivision. For every $p$-
dimensional simplex, there is a dual $(n-p)$-dimensional


cell where $n$ is the dimension of the simplicial complex.
In DEC circumcenters are used instead of barycenters
because the resulting orthogonality between the primal
and dual is an integral part of the definition of some of

the operators of DEC [8]. The circumcentric dual cell of

a $p$-dimensional primal simplex $\tau$ is constructed from a
set of simplices incident to the circumcenter of $\tau$. These
are called elementary dual simplices with vertices being a
sequence of circumcenters of primal simplices incident to
$\tau$. The sequence begins with the circumcenter of $\tau$, moves
through circumcenters of higher-dimensional simplices $\sigma^i$
and ends with the circumcenter of a top-dimensional sim-
plex $\sigma^n$ such that $\sigma^1 \prec \sigma^2 \prec \cdots \prec \sigma^n$. Taking each of the
possibilities for $\sigma^i$ at each dimension $i$ yields the full dual


cell.

In the past in the software PyDEC the volume of the dual
cells has been taken to be the sum of unsigned volumes of
the elementary dual simplices. Instead we define the vol-
ume of the dual cells as the sum of signed volumes of its
elementary dual simplices. Our contribution is in defining
the sign convention and describing with proofs the class of
meshes for which the dual volumes and hence Hodge star
entries are positive. If the primal complex is completely well
centered every elementary dual has a positive volume. In
general, the sign of the volume of an elementary dual sim-
plex is defined as follows. Start from the circumcenter of $\tau$.
Let $v_p$ be the vertex such that $v_p \ast \tau$ is the simplex $\sigma^{p+1}$
formed by the vertices of $\tau$ together with $v_p$. Similarly, for
$p + 1 \leq i \leq n - 1$, let $v_i$ be the vertex such that $v_i \ast \sigma^i$
is the simplex $\sigma^{i+1}$. If the circumcenter of $\sigma^{p+1}$ is in the
same half space of $\sigma^{p+1}$ as $v_p$ relative to $\tau$, let $s_p = +1$, other-
wise, $s_p = -1$. Likewise, for $p + 1 \leq i \leq n - 1$, if the
 circumcenter of $\sigma^{i+1}$ is in the same half space as $v_i$ rela-
tive to $\sigma^i$, let $s_i = +1$, otherwise, $s_i = -1$. Then, the sign $s$
of the elementary dual simplex is the product of the signs at
each dimension, that is, $s = s_p s_{p+1} \cdots s_{n-1}$.

For illustration of this sign rule, we now consider various
cases in two and three dimensions. The first example is the
dual of an edge $ab$ in a triangle $abc$. From the midpoint of $ab$
– its circumcenter – we move to the circumcenter of triangle
$abc$. If this move is towards vertex $c$, then the sign is $s = s_1 = +1$, but if it is away from vertex $c$, as it will be if the
angle at vertex $c$ is obtuse, then the sign is $s = s_1 = -1$. The
next example is the dual of vertex $a$ in triangle $abc$. We will
consider the simplex formed from the circumcenter of $a$, the
 circumcenter of $ac$, and the circumcenter of $abc$. The move
from $a$ to the midpoint of $ac$ gives $s_0 = +1$, since vertex $c$
and the midpoint of $ac$ are in the same direction from $a$.

The move from the midpoint of $ac$ to the circumcenter of
$abc$ gives $s_1 = +1$ if we go towards $b$ and $s_1 = -1$ if we move
away from $b$. The sign of the volume of this contribution to
the dual of vertex $a$ is $s = s_0 s_1 = s_1$.

For a tetrahedron $abcd$ we can expand on the cases for
triangle $abc$. For the dual to face $abc$, we move from the
 circumcenter of $abc$ to the circumcenter of $abcd$. If the cir-

circumcenter of $abcd$ is in the same half space as vertex $d$
relative to $abc$, this move is towards $d$, the sign is $s = s_2 = +1$,
and the signed length (volume) is positive; otherwise, it is negative. Of the two contributions to the dual of edge $ab$, we focus on the simplex formed from the circumcenter of $ab$, the circumcenter of $abc$ and the circumcenter of $abcd$. The sign $s_1$ is determined as it was for the dual of edge $ab$ in triangle $abc$. The sign $s_2$ is $+1$ if vertex $d$ and the circumcenter of $abcd$ are in the same half space relative to $abc$. Thus for the dual of edge $ab$, the sign of the volume is $s = s_1 s_2$, and both $s_1$ and $s_2$ can be either positive or negative. As a final example, consider the simplex formed from vertex $a$, the circumcenter of $ac$, the circumcenter of $abc$, and the circumcenter of $abcd$. This simplex contributes to the dual of vertex $a$. Signs $s_0$ and $s_1$ are the same as they were for the dual of vertex $a$ in triangle $abc$. Sign $s_2$ is $-1$ if triangle $abc$ separates vertex $d$ from the circumcenter of tetrahedron $abcd$. The sign of this elementary volume then is $s = s_0 s_1 s_2$.

The significance of the sign rule defined above is that it orients the elementary dual simplices in a particular way with respect to the dual orientation for a completely well-centered simplex. Consider two $n$-dimensional simplices $\sigma$ and $\sigma_w$ which have the same orientation but such that $\sigma_w$ is well-centered. We are given a bijection between the vertices of these two simplices such that the resulting simplicial map is orientation preserving. This vertex map induces a bijection between faces of the two simplices and between their elementary duals. Let $\tau$ and $\tau_w$ be two corresponding $p$-dimensional faces in the two simplices and consider their duals $\ast \tau$ and $\ast \tau_w$. If we consider two corresponding elementary duals in $\ast \tau$ and $\ast \tau_w$ we can affinely map these such that the first vertex (the circumcenter of $\tau$ or $\tau_w$) is mapped to the origin and the others are mapped to $+1$ or $-1$ along a coordinate axis. For the elementary dual in $\ast \tau_w$ we always choose $+1$ for all $n-p$ coordinate axes. For $\ast \tau$ we choose $+1$ if the sign along that direction of the

Fig. 1. Solution of Poisson’s equation $-\Delta u = f$ in mixed form. In mixed form this equation is the system $\sigma = -\text{grad} u$ and $\text{div} \sigma = f$. The boundary condition is constant influx on left and outflux on right. The correct solution is linear $u$ which varies only along $x$-direction and a constant horizontal $\sigma$. The top row shows $u$ and bottom row shows $\sigma$. The first column shows the correct solution using the results of this paper on a Delaunay mesh with correct boundary simplices. The second column is for unsigned duals using the same mesh as first column. This fails to produce the correct solution. The next two columns are instances in which the $1$-Hodge star mass matrix is not positive definite.

The third column has a single bad (i.e., not one-sided) boundary triangle shown shaded in a Delaunay mesh. The fourth column is a non-Delaunay mesh. It appears that Discrete Exterior Calculus produces correct solution even for these cases for meshes used in columns 3 and 4.

|              | Unsigned | Bad Boundary | Not Delaunay |
|--------------|----------|--------------|--------------|
| Analytical   | DEC      | DEC          | DEC          |
elementary dual is positive according to the sign rule described above and \(-1\) otherwise. It is clear (and is easy to show using determinants) that the orientation of the corresponding elementary duals will be same if an even number of \(-1\) directions are used for the elementary dual in \(\star \tau\) and the orientations will be opposite otherwise. Thus we have shown the following result.

**Theorem 3 (Codimension 1)** Let \(\tau\) be a codimension 1 shared face of two \(n\)-dimensional simplices embedded in \(\mathbb{R}^N\),

The above lemma can now be used to show easily that the codimension 1 duals always have positive net length. This is the content of the next result.

**Theorem 3 (Codimension 1)** Let \(\tau\) be a codimension 1 shared face of two \(n\)-dimensional simplices embedded in \(\mathbb{R}^N\),

3. **Signed Dual of a Delaunay Triangulation**

We first consider the codimension 1 case in the most general setting of a simplicial complex of arbitrary dimension \(n\) embedded in dimension \(N \geq n\). After that we consider cases other than codimension 1 but in more restricted settings. For these latter cases we restrict ourselves to the physically most useful cases of triangle meshes embedded in two or three dimensions \((n = 2\) and \(N = 2\) or \(3\)) and tetrahedral meshes embedded in three dimensions \((n = N = 3\) ). We conjecture that these results can be extended to the more general setting of arbitrary \(n\) and \(N \geq n\) but those cases are not as important for physical applications and we leave those for future work. For the general codimension 1 case we first prove the following basic fact about circumcenter ordering for Delaunay pairs.

**Lemma 2 (Circumcenter Order)** Let \(\tau\) be an \((n - 1)\)-dimensional simplex in \(\mathbb{R}^n\). Let \(L\) and \(R\) be points such that \(\lambda = L \star \tau\) and \(\rho = R \star \tau\) form a non-degenerate Delaunay pair of \(n\)-dimensional simplices with circumcenters \(c_\lambda\) and \(c_\rho\), respectively. Then, \(c_\lambda\) and \(c_\rho\) have the same relative ordering with respect to \(\tau\) as \(L\) and \(R\).

**Proof.** Consider the collection of \((n - 1)\)-dimensional spheres containing the vertices of \(\tau\). Since \(\lambda\) and \(\rho\) are a non-degenerate Delaunay pair, their circum spheres are empty and belong to this collection. It is then easy to see that \(c_\lambda\) and \(c_\rho\) will be in the same order as \(\lambda\) and \(\rho\). See Figure 3. For an alternative, more algebraic and detailed proof, see Appendix A. (In fact there we show the stronger result that the correctness of the circumcenter ordering is equivalent to the simplices being a non-degenerate Delaunay pair.)

Theorem 3 shows that the signed area of the dual of a Delaunay triangulation is always positive. We prove this below by showing that the net dual area corresponding to a pair of triangles is positive.

**Theorem 4** Let \(\tau\) be an internal vertex in a pairwise Delaunay triangle mesh embedded in \(\mathbb{R}^N\), \(N = 2, 3\). Then the signed area of \(\star \tau\) is a positive.

**Proof.** \(\star \tau\) is the Voronoi cell of vertex \(\tau\) in the pairwise Delaunay mesh. Consider a pair of triangles sharing a common edge incident to \(\tau\) and if they are embedded in \(\mathbb{R}^3\), isometrically project to \(\mathbb{R}^2\) (i.e., treat the shared edge as a hinge, and flatten the pair.) The circumcenters of these two triangles are in correct order by Lemma 2 and there are three possible cases as shown in Figure 4. Thus the net area of the two elementary dual simplices is positive when the signs are assigned using the rule described in Section 2.
Summing over all edges containing $\tau$ yields the full $\star \tau$ as a positive area.

Fig. 4. Elementary dual simplices of a vertex in a pair of triangles sharing an edge. The cases shown correspond to various positions of the circumcenters of the shared edge and the two triangles.

3.2. Dual of an Edge in Tetrahedral Mesh

**Theorem 5** Let $\tau$ be an internal edge in a tetrahedral Delaunay triangulation embedded in $\mathbb{R}^3$. Then $\star \tau$ is a simple, planar, convex polygon whose signed area is positive.

**Proof.** $\star \tau$ of an internal edge $\tau$ in a Delaunay triangulation may or may not intersect $\tau$. The vertices of $\star \tau$ are circumcenters of tetrahedra incident to $\tau$ and the boundary edges of $\star \tau$ are dual edges of triangles incident to $\tau$. Note that $\star \tau$ is the interface between the Voronoi cells corresponding to the two vertices of $\tau$ and thus is a bounding face of both Voronoi cells. Since the Voronoi cell of a vertex is a convex polyhedron [6], $\star \tau$ is simple, planar and convex.

Suppose $\tau$ intersects $\star \tau$. Then the tetrahedra incident to $\tau$ and the edges of $\star \tau$ have to be in a configuration shown in left part of Figure 5. A configuration in which the triangles incident to $\tau$ are reflected about $\tau$ is impossible due to Lemma 2.

Now, to see that the signed area of $\star \tau$ is positive, consider two elementary dual simplices of $\star \tau$ incident to a shared face $\sigma$ of two tetrahedra in the fan of tetrahedra incident to $\tau$. These two elementary dual simplices can be in one of the two configurations as shown in Figure 6. In both cases, $c_\tau$ is the circumcenter of the edge $\tau$, $c_\sigma$ is the circumcenter of the shared face $\sigma$, and $c_\rho$ and $c_\lambda$ are the circumcenters of the two tetrahedra. Also, in both cases, using the sign rule of Section 2 the sum of the signed areas of the elementary dual simplices is positive, and hence, the signed area of $\star \tau$ composed of these elementary dual simplices is positive.

Next consider the case in which $\tau$ does not intersect $\star \tau$ as shown in right part of Figure 5. A boundary edge of $\star \tau$ is called near side if it is visible from the midpoint of $\tau$, otherwise, it is called a far side edge. Figure 6 shows the net dual simplices of a near side and far side boundary edge of $\star \tau$. By the sign rule of Section 2, far side elementary dual simplices have a net positive signed area while near side elementary dual simplices have a net negative signed area. The negative areas of the near side dual simplices are covered by the positive areas of the far side dual simplices. Thus, the sum of all these elementary dual simplices which is the signed area of $\star \tau$ is positive.
3.3. Dual of a Vertex in Tetrahedral Mesh

Theorem 6 Let \( \tau \) be an internal vertex of a tetrahedra Delaunay mesh embedded in \( \mathbb{R}^3 \). Then the volume of \( \star \tau \) is positive.

**PROOF.** \( \star \tau \) of a vertex \( \tau \) in a Delaunay tetrahedral mesh is a convex polyhedron that is the Voronoi dual cell of \( \tau \) \([6]\) and thus \( \tau \) is inside \( \star \tau \). The faces of \( \star \tau \) are duals of edges incident to \( \tau \). By Theorem 5 all these faces have a positive signed area. The direction corresponding to traversal from \( \tau \) to an edge center always has a positive sign. Thus each pyramid formed by \( \tau \) and a boundary face of \( \star \tau \) has positive volume. Thus, the volume of \( \star \tau \) is positive.

4. Requirements on Boundary Simplices

In the previous section we have only considered internal simplices in a pairwise Delaunay mesh. For simplices lying in the boundary of a domain we require an assumption to ensure positive duals. We call a simplex \( \sigma \) one-sided with respect to a codimension 1 face \( \tau \) if its circumcenter \( c_\sigma \) lies in the same half space as the apex with respect to \( \tau \) in the affine space of \( \sigma \).

We show below that the only assumption then needed is that a top dimensional simplex with a codimension 1 face in the domain boundary should be one-sided with respect to the boundary face.

Consider a pairwise Delaunay mesh of dimension \( n \) embedded in \( \mathbb{R}^N \), \( N \geq n \). Assume that \( \tau \) is an \( (n-1) \)-dimensional face appearing in domain boundary and \( \tau \prec \sigma^n \) such that \( \sigma^n \) is one-sided with respect to \( \tau \).

Theorem 7 For a mesh such as above, a dual of codimension 1 faces has positive length. For \( n = 2 \) and \( N = 2 \) or 3, and for \( n = N = 3 \), duals of all simplices at all dimensions have positive areas or volumes.

**PROOF.** The codimension 1 dual of \( \tau \) in all cases has positive length using our sign rule since \( \sigma^n \) is one-sided with respect to \( \tau \). As a result, for a surface triangle mesh, that is \( n = 2 \) and \( N = 2 \) or 3, it easily follows from our sign rule in Section 2 that the dual of a vertex on the boundary also has a positive area.

For \( n = N = 3 \), one configuration for the dual of an edge \( \tau \) incident to the boundary is shown in Figure 7. In this figure, the plane containing the codimension 1 faces incident to \( \tau \) are shown as short line segments, and the coloring of boundary edges of \( \star \tau \) show the corresponding codimension 1 face they are dual to. The other configuration in which the planes containing the faces incident to \( \tau \) are mirror images of ones shown is not possible since then the circumcenters of tetrahedra will not be in the correct order as in Lemma 2. Thus, by our sign rule, all elementary dual simplices of \( \star \tau \) are positive and hence the signed area of \( \star \tau \) is positive. Finally, it follows from our sign rule that the dual of a vertex on the boundary is also positive since each of the elementary dual pyramids will have a positive volume.

Fig. 7. Dual of an edge \( \tau \) lying in the boundary of a Delaunay tetrahedral mesh. The meaning of colors and small lines is as in Figure 5.

5. Conclusions and Outlook

For planar triangle meshes and for tetrahedral meshes in three dimensional space the condition of being pairwise Delaunay is equivalent to being Delaunay. Thus most commercial and freely available meshing software can generate such meshes. In our experience, several codes for planar meshing also generate meshes for which the one-sidedness condition on the boundary is satisfied. For example, the popular meshing code called Triangle has an option for conforming Delaunay triangulation which generates Delaunay meshes with one-sided boundary triangles. For tetrahedral meshes with acute input angles this property may be harder to achieve. In general however, algorithms for creating tetrahedral meshes with one-sided boundary tetrahedra do exist \([3,5,7]\). Note that one-sidedness is equivalent to an “oriented” Gabriel property (using diametral half-balls) for the boundary faces.

The pairwise Delaunay condition also appears to be more natural for DEC than other conditions that are used in place of Delaunay in the case of surfaces. For example, some researchers require that the equatorial balls of triangles not contain another vertex. This disqualifies surfaces with many folds or sharp turns. Another alternative is to define intrinsic Delaunay condition based on geodesics on the triangle mesh but algorithms for such surfaces can be complicated to implement. Yet another alternative is to use Hodge-optimized triangulations \([10]\). But creation of these requires an additional optimization step. On the other hand Hodge-optimized triangulation is a very interesting generalization of Voronoi-Delaunay duality with many applications.

The invention of algorithms that generate pairwise Delaunay surface meshes is left for future work. So is the proof of our conjecture that the case of codimension other than 1 has positive volume for general dimension and embedding.
space for pairwise Delaunay meshes with one-sided boundary simplices. Nevertheless, the practically important cases have all been settled by this paper.

Acknowledgement: ANH and KK were supported by NSF Grant DMS-0645604. We thank the anonymous referees for pointing out some important references and for their other suggestions.

References

[1] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus: from Hodge theory to numerical stability. Bull. Amer. Math. Soc. (N.S.), 47(2):281–354, 2010.
[2] Nathan Bell and Anil N. Hirani. PyDEC: Software and algorithms for Discretization of Exterior Calculus. ACM Transactions on Mathematical Software, 39(1):3:1 – 3:41, 2012.
[3] R. Chaine. A geometric convection approach of 3-d reassembly. In Proc. Sympos. Geometry Processing, 2003.
[4] M. Desbrun, E. Kanso, and Y. Tong. Discrete differential forms for computational modeling. In Discrete Differential Geometry, volume 38, pages 287–324. Birkhäuser Basel, 2008.
[5] H. Edelsbrunner. Surface reconstruction by wrapping finite sets in space. In Discrete and computational geometry, Algorithms Combin. Springer, 2003.
[6] H. Edelsbrunner. Geometry and topology for mesh generation. Cambridge University Press, 2006.
[7] J. Giesen and M. John. The flow complex: a data structure for computational modeling. Comput. Geom., 39(3):178–190, 2008.
[8] A. N. Hirani. Discrete Exterior Calculus. PhD thesis, California Institute of Technology, May 2003.
[9] Anil N. Hirani, Kaushik Kalyanaraman, and Evan B. VanderZee. Delaunay Hodge star. Computer-Aided Design, 45(2):540–544, 2013.
[10] P. Mullen, P. Memari, F. de Goes, and M. Desbrun. HOT: Hodge-optimized triangulations. ACM Trans. Graph., 30:103:1–103:12, 2011.
[11] J. R. Shewchuck. What is a good linear element? Interpolation, conditioning, and quality measures. In Eleventh International Meshing Roundtable, 2002.
[12] V. Zobel. Spectral analysis of the hodge laplacian on discrete manifolds. Master’s thesis, Technische Universität Berlin, 2010.

Appendix A. Circumcenter Order

In fact here we prove a stronger result than Lemma 2. We will show that the circumcenters are in the correct order if and only if the pair of simplices is a non-degenerate Delaunay pair.

Lemma 8 Let $\tau$ be an $(n−1)$-dimensional simplex in $\mathbb{R}^n$. Let $L$ and $R$ be points separated by $\tau$. Let $c_\lambda$ and $c_\rho$ be the circumcenters of the $n$-dimensional simplices $\lambda = L \star \tau$ and $\rho = R \star \tau$, respectively. Then, $c_\lambda$ and $c_\rho$ have the same relative ordering with respect to $\tau$ as $L$ and $R$ if and only if $\lambda$ and $\rho$ are a pair of non-degenerate Delaunay simplices.

**Proof.** Since $\lambda$ and $\rho$ are a Delaunay pair, the affine space of $\tau$ separates $L$ and $R$. See Figure A.1. Let $c_\tau$ and $r_\tau$ be the circumcenter and the circumradius of $\tau$, respectively. Now, $c_\lambda$ and $c_\rho$ lie on a line $\ell$ that passes through $c_\tau$ and is orthogonal to the affine space of $\tau$. Let $h_\lambda$ be the signed distance along $\ell$ from $c_\lambda$ to $c_\tau$. Similarly, let $h_\rho$ be the signed distance from $c_\rho$ to $c_\tau$. For now, it is sufficient that these distances be signed and whether the positive direction is along $L$ or $R$ is not important. Next, orthogonally project $R$ onto $\ell$, and let $r_R$ be the (positive) distance from $R$ to its projection onto $\ell$. Finally, let $h_R$ be the signed distance (along $\ell$) from the projection of $R$ onto $\ell$ to $c_\tau$. Notice that $h_R$ is necessarily either negative or positive depending on the choice of positive direction to be either along $L$ or $R$, respectively.

By elementary geometry, the squared circumradius of $\lambda$ is $h_\lambda^2 + r_\tau^2$ and the squared circumradius of $\rho$ is $h_\rho^2 + r_\tau^2$. Similarly, the squared distance from $c_\lambda$ to $R$ is $(h_\lambda - h_R)^2 + r_R^2$. Since $\lambda$ and $\rho$ form a Delaunay pair, $R$ lies outside the circumsphere of $\lambda$. Thus, the squared circumradius of $\lambda$ is less than the squared distance from $c_\lambda$ to $R$:

$$h_\lambda^2 + r_\tau^2 < (h_\lambda - h_R)^2 + r_R^2,$$

$$r_\tau^2 < r_R^2 + h_\lambda^2 - 2 h_R h_\lambda.$$

Also, since $R$ lies on the circumsphere of $\rho$, the distance from $c_\rho$ to $R$ is the same as the distance from $c_\rho$ to a vertex of $\tau$. Thus, we have:

$$h_\rho^2 + r_\tau^2 = (h_\rho - h_R)^2 + r_R^2,$$

$$r_\tau^2 = r_R^2 + h_\rho^2 - 2 h_R h_\rho.$$

Using this in the previous inequality, we obtain:

$$r_R^2 + h_\lambda^2 - 2 h_R h_\rho < r_R^2 + h_\rho^2 - 2 h_R h_\lambda,$$

$$h_R h_\rho > h_\lambda h_R.$$

Finally, we choose a coordinate direction along $\ell$ to fix signs in the signed distances along $\ell$. If we choose the direction towards the half space containing $R$ to be positive, $h_R$ is positive. (We will call this the positive $R$-direction.) As a result, the last inequality above simplifies to $h_\rho > h_\lambda$. This
means that \( h_\rho \) is larger along the positive \( R \)-direction. If we choose the direction along \( L \) to be positive, \( h_R \) is negative and we obtain \( h_\rho < h_\lambda \). In this case, \( h_\lambda \) is larger along the positive \( L \)-direction.

Conversely, if \( \lambda \) and \( \rho \) are not a Delaunay pair, then the distance from \( c_\lambda \) to \( R \) is less than the circumradius of \( \lambda \). Thus, all inequalities will reverse directions and therefore the circumradii will be in the wrong order.

Appendix B. Corrigendum

In this work (which appeared, minus the Appendix, in \([9]\)), we established positivity of entries of the diagonal Hodge star matrix assembled from signed elementary duals for meshes that are pairwise Delaunay, non-degenerate, and with a one-sidedness property for boundary simplices. There are no errors in these mathematical statements (lemmas and theorems) in \([9]\). Further, in \([9]\) we did not provide any mathematical statements for meshes that violate the pairwise Delaunay condition and/or meshes that violate the one-sidedness condition. However in \([9]\), a numerical computation of solution of scalar Poisson’s equation was shown for such meshes. These were shown in columns 3 and 4 of Figure 1 in \([9]\).

We recently discovered errors in some computer programs that were used for the numerical experiments corresponding to columns 3 and 4 of Figure 1 in \([9]\). Those figures in the published paper seemed to suggest that meshes which violate the pairwise Delaunay condition and/or meshes that violate the one-sidedness condition may need to be avoided. After correcting the errors in the programs we noticed that DEC appears to produce the correct solution in the cases shown even for such meshes. In light of this, we have deleted the following sentence from the Introduction in this version as in \([9]\): “That figure also shows the importance of the Delaunay property and of our boundary assumptions and the success of the signed dual volumes for such meshes that we describe in this paper.”. We have also updated Figure 1, in particular columns 3 and 4. The figures in columns 1 and 2 of Figure 1 in \([9]\) are unchanged by this correction – those figures in \([9]\) (as well as the previous arXiv v3) are correct.

We now describe our programming error. We traced the error to the generation and processing of mesh used for Figure 1, column 3 of \([9]\). The vertex and edge numberings for this mesh are shown in Figure B.1. (For clarity, only numbering of vertices and edges on the boundary are shown.) Notice that vertices numbered 0 and 19 overlap (bottom left corner) and vertices 14 and 15 overlap (top left corner). This led to an incorrect assignment of boundary conditions. (The boundary condition for the problem is inflow through left boundary, outflow through right, and no flow across the top and bottom boundaries.) As a result, the edge numbered 22 on the right was assigned an outflow velocity of 0 and edge 27 was assigned an outflow velocity of 1 pointing to the right. Due to normal to edge 27 pointing upwards this resulted in 0 flux through edge 27 (which is correct). However, edge 22 was assigned an incorrect flux of 0. All outgoing flux was thus assigned to edge 24 on the right.

For column 4 in Figure 1 of \([9]\), the non-Delaunay mesh was obtained by starting from the erroneous mesh of column 3. In particular, one of the triangles of the mesh from column 3 was subdivided to yield a non-Delaunay pair. Consequently, the non-Delaunay mesh inherited the edge indexing problems of the mesh from column 3. In addition, due to modification of edge markers during subdivision, some internal edges were misidentified as being boundary edges. This again led to an incorrect assignment of boundary conditions.

![Fig. B.1. The cause of the programming error was an incorrect mesh. Notice the overlapping vertices on top left and bottom. This led to an incorrect assignment of boundary conditions for columns 3 and 4 of Figure 1 in \([9]\) as explained in the text.](image)