COMMUTATION FORMULAE WITH RESPECT TO NON-SYMMETRIC AFFINE CONNECTION

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Abstract. Commutation formulae with respect to a non-symmetric affine connection are obtained in this paper. The components of commutation formulae in this paper are covariant derivatives of tensors with respect to symmetric and non-symmetric affine connection.

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1. Introduction. Identities of Ricci Type [2, 3, 5, 4, 6, 7, 8, 9, 10, 11, 12, 13, 15] are important for different researches in the fields of differential geometry and the corresponding applications.

One curvature tensor of a symmetric affine connection space is obtained with respect to a symmetric affine connection [2, 12]. Many curvature tensors and curvature pseudotensors are founded with respect to a non-symmetric affine connection [3, 5, 4, 6, 7, 8, 9, 10, 11, 13, 15]. Curvature tensors and curvature pseudotensors are components of the curvature for the corresponding affine connection spaces.

Our purpose is to obtain all identities of Ricci Type with respect to a non-symmetric affine connection in this paper. In this research, we will try to simplify the previously obtained identities.

1.1. Affine connection space. An $N$-dimensional manifold $\mathcal{M}_N$ equipped with an affine connection with torsion $\nabla$ is the generalized affine connection space $\mathcal{G}A_N$ (see [1, 3, 5, 4, 6, 7, 8, 9, 10, 11, 13, 15]).

The affine connection coefficients with respect to the affine connection (with torsion) $\nabla$ are $L^i_{jk}$, $L^i_{jk} \neq L^i_{kj}$. The symmetric and anti-symmetric parts of the

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affine connection coefficients $L^i_{jk}$ are

$$L^i_{jk} = \frac{1}{2} (L^i_{jk} + L^i_{kj}) \quad \text{and} \quad L^i_{j,k} = \frac{1}{2} (L^i_{jk} - L^i_{kj}). \quad (1.1)$$

The double anti-symmetric parts $L^i_{j,k}$ are the components of the torsion tensor for the affine connection space $\mathbb{G}A_N$.

The symmetric parts $L^i_{jk}$ satisfy the transformation rule

$$L^i_{j',k'} = x^i_{i'} x^j_{j'} x^k_{k'} L^i_{jk} + x^i_{i'} x^j_{j'} x^k_{k'} L^i_{j'k}.$$  

(1.2)

For this reason, the manifold $\mathcal{M}_N$ equipped with the symmetric affine connection $^0\nabla$ whose coefficients are $L^i_{jk}$ is the associated (symmetric affine connection) space $A_N$ of the space $\mathbb{G}A_N$ (see [2, 12]).

 Covariant derivatives are defined with respect to torsion-free affine connections [2, 12] and affine connections with torsion [1, 3, 5, 4, 6, 7, 8, 9, 10, 11, 13, 15]. With respect to double covariant derivatives, corresponding commutation formulae are obtained. From the commutation formulae, the curvature tensors for the spaces $A_N$ and $\mathbb{G}A_N$ are founded.

1.2. About covariant derivatives. There exists one kind of covariant derivative with respect to the affine connection $^0\nabla$ (see [2, 12])

$$a^i_{j_1 \ldots j_q |k} = a^i_{j_1 \ldots j_q, k} + \sum_{u=1}^p L^i_{\alpha k} a^{i_1 \ldots i_u \alpha \alpha_{u+1} \ldots i_p}_{j_1 \ldots j_q} - \sum_{v=1}^q L^\alpha_{j v k} a^{i_1 \ldots i_p}_{j_1 \ldots j_{v-1} \alpha j_{v+1} \ldots j_q}, \quad (1.3)$$

for a tensor $\hat{a}$ of the type $(p, q)$ whose components are $a^i_{j_1 \ldots j_q}$ and the partial derivative $\partial / \partial x^k$ denoted by comma.

There exists one Ricci-Type identity with respect to the covariant derivative given by the equation (1.3)

$$a^i_{j_1 \ldots j_q |m |n} - a^i_{j_1 \ldots j_q |n |m} = \sum_{u=1}^p a^i_{j_1 \ldots j_{u-1} \alpha i_{u+1} \ldots i_p \alpha_{j_1 \ldots j_q} R^i_{\alpha mn} - \sum_{v=1}^q a^i_{j_1 \ldots j_{v-1} \alpha j_{v+1} \ldots j_q \alpha_{j_1 \ldots j_q} R^i_{\alpha mn},} \quad (1.4)$$

for the components

$$R^i_{jmn} = L^i_{jmn} - L^i_{jm,n} + L^\alpha_{jm} L^i_{\alpha mn} - L^\alpha_{jn} L^i_{\alpha mn}, \quad (1.5)$$

of the curvature tensor $\hat{R}$ of the type $(1, 3)$ for the associated space $A_N$.

There are four kinds of covariant derivatives with respect to the affine connect-
tion with torsion $\nabla$ (see [1, 3, 5, 4, 6, 7, 8, 9, 10, 11, 13, 15])

$$a^{i_1 \ldots i_p}_{j_1 \ldots j_q | k} = a^{i_1 \ldots i_p}_{j_1 \ldots j_q, k} + \sum_{u=1}^{p} L_{u\alpha}^{\alpha} a^{i_1 \ldots i_u-\alpha i_{u+1} \ldots i_p}_{j_1 \ldots j_q} - \sum_{v=1}^{q} L_{j_v k}^\alpha a^{i_1 \ldots i_p}_{j_1 \ldots j_{v-1} \alpha j_{v+1} \ldots j_q}, \quad (1.6)$$

$$a^{i_1 \ldots i_p}_{j_1 \ldots j_q | k} = a^{i_1 \ldots i_p}_{j_1 \ldots j_q, k} + \sum_{u=1}^{p} L_{u\alpha}^{\alpha} a^{i_1 \ldots i_u-\alpha i_{u+1} \ldots i_p}_{j_1 \ldots j_q} - \sum_{v=1}^{q} L_{j_v k}^\alpha a^{i_1 \ldots i_p}_{j_1 \ldots j_{v-1} \alpha j_{v+1} \ldots j_q}, \quad (1.7)$$

$$a^{i_1 \ldots i_p}_{j_1 \ldots j_q | k} = a^{i_1 \ldots i_p}_{j_1 \ldots j_q, k} + \sum_{u=1}^{p} L_{u\alpha}^{\alpha} a^{i_1 \ldots i_u-\alpha i_{u+1} \ldots i_p}_{j_1 \ldots j_q} - \sum_{v=1}^{q} L_{j_v k}^\alpha a^{i_1 \ldots i_p}_{j_1 \ldots j_{v-1} \alpha j_{v+1} \ldots j_q}, \quad (1.8)$$

$$a^{i_1 \ldots i_p}_{j_1 \ldots j_q | k} = a^{i_1 \ldots i_p}_{j_1 \ldots j_q, k} + \sum_{u=1}^{p} L_{u\alpha}^{\alpha} a^{i_1 \ldots i_u-\alpha i_{u+1} \ldots i_p}_{j_1 \ldots j_q} - \sum_{v=1}^{q} L_{j_v k}^\alpha a^{i_1 \ldots i_p}_{j_1 \ldots j_{v-1} \alpha j_{v+1} \ldots j_q}, \quad (1.9)$$

Let $a^{i_1 \ldots i_p}_{j_1 \ldots j_q | k} \equiv a^{i_1 \ldots i_p}_{j_1 \ldots j_q | k'}$. We will study the differences $a^{i_1 \ldots i_p}_{j_1 \ldots j_q | v_1} | m \rightarrow n - a^{i_1 \ldots i_p}_{j_1 \ldots j_q | v_2} | n \rightarrow m$, $v_1, v_2, w_1, w_2 \in \{0, 1, 2, 3, 4\}$, in this paper.

1.3. Motivation. It is obtained the Ricci-Type identity [1, 3, 5, 4, 6, 7, 8, 9, 10, 11, 13, 15]

$$a^i_{j|m} - a^i_{j|n} = a^\alpha_j A^i_{1 \alpha mn} - a^i_1 A^\alpha_{2 \alpha mn} + 4a^i_j <m \alpha n> + 4a^i_{j <m \alpha n>} + 2L^\alpha_{m \alpha} a^i_{j | \alpha}, \quad (1.10)$$

for

$$A^i_{1 \alpha mn} = R^i_{j mn} + L^i_{j \alpha m} - L^i_{j \alpha n} + L^\alpha_{j \alpha m} L^i_{j \alpha n} + 2L^\alpha_{j \alpha m} L^i_{j \alpha n}, \quad (1.11)$$

$$A^i_{2 \alpha mn} = R^i_{j mn} + L^i_{j \alpha m} - L^i_{j \alpha n} + L^\alpha_{j \alpha m} L^i_{j \alpha n} + 2L^\alpha_{j \alpha m} L^i_{j \alpha n}, \quad (1.12)$$

$$a^i_{j <m \alpha n>} = \frac{1}{2} L^\alpha_{j \alpha} a^i_{j, \alpha, n} - \frac{1}{2} L^\alpha_{j \alpha} a^i_{j, \alpha, n} + \frac{1}{2} L^\alpha_{j \alpha} a^i_{j, \alpha, m}, \quad (1.13)$$

$$a^i_{j <m \alpha n>} = \frac{1}{2} a^i_{\alpha \beta} \left( L^i_{j \alpha \beta} L^\alpha_{j \alpha \beta} - L^i_{j \alpha \beta} L^\alpha_{j \alpha \beta} - L^i_{j \alpha \beta} L^\alpha_{j \alpha \beta} + L^i_{j \alpha \beta} L^\alpha_{j \alpha \beta} \right). \quad (1.14)$$

The geometrical objects $A^i_{1 \alpha mn}$ and $A^i_{2 \alpha mn}$ are components of the curvature pseudotensors $\tilde{A}$ and $\tilde{A}$ of the type (1, 3). These objects are components of the curvature for the space $\mathcal{A}_N$.

In [14], and with respect to $L^i_{j \alpha k} = L^i_{jk} + L^i_{jk}$, the equation (1.10) is simplified to

$$a^i_{j|m} - a^i_{j|n} = 2L^i_{j \alpha m} a^j_{\alpha n} - 2L^i_{j \alpha n} a^j_{\alpha m} - 2L^\alpha_{j \alpha m} a^i_{\alpha n} + 2L^\alpha_{j \alpha n} a^i_{\alpha m} + 2L^\alpha_{m \alpha} a^i_{j | \alpha}$$

$$+ a^\alpha \left( R^i_{j mn} + L^i_{j \alpha m} - L^i_{j \alpha n} - L^\beta_{j \alpha m} L^\beta_{j \alpha n} + L^\beta_{j \alpha n} L^i_{j \alpha m} - 2L^\beta_{m \alpha} L^i_{j \alpha m} \right)$$

$$- a^\alpha \left( R^\alpha_{j mn} + L^\alpha_{j \alpha m} - L^\alpha_{j \alpha n} - L^\beta_{j \alpha m} L^\beta_{j \alpha n} + L^\beta_{j \alpha n} L^\alpha_{j \alpha m} - 2L^\beta_{m \alpha} L^\alpha_{j \alpha m} \right). \quad (1.10')$$
In [14], it is obtained the family of double covariant derivatives

\[
a^i_j|m|n = a^i_j|m|n + c_vL^i_\alpha m a^\alpha_j|m + c_wL^i_\beta n a^\beta_j|n + d_vL^i_j m a^\alpha_\alpha|n + d_wL^i_\alpha m a^\alpha_j|n
\]

\[
+ a^i_j \left( c_vL^i_\alpha m a^\alpha_j|m + c_wL^i_\beta n a^\beta_j|n + c_v(c_w + d_w)L^i_\alpha n L^i_\beta m - c_v d_wL^i_\beta m L^i_\alpha n \right)
\]

\[
- a^i_\alpha \left( - d_vL^i_\alpha n a^\alpha_j|m - d_v(c_w + d_w)L^i_\beta j m a^\alpha_\alpha|m - d_v d_wL^i_j m L^i_\alpha n + d_v d_wL^i_\alpha n L^i_\beta j \right)
\]

\[
+ a^i_\beta \left( c_w d_v L^i_\beta j m L^i_\alpha n + c_v d_w L^i_j m L^i_\alpha n \right),
\]

(1.15)

for \( v, w \in \{0, 1, 2, 3, 4 \} \).

When simplified the difference \( a^i_j|m|n - a^i_j|m|n \), we proved the next theorem.

**Theorem 1.1.** (First Ricci-type identities theorem, [14]) The family of identities of the Ricci type with respect to a non-symmetric affine connection \( \nabla \) is

\[
a^i_j|m|n - a^i_j|m|n - (c_v - c_w) L^i_\alpha m a^\alpha_j|m + (c_w - c_v) L^i_\alpha n a^\alpha_j|m + (d_v - d_w) L^i_\alpha m a^\alpha_j|m
\]

\[
+ (d_v - d_w) L^i_\alpha n a^\alpha_j|m + (d_w + d_w) L^i_\alpha m a^\alpha_j|m + a^i_j \left\{ R^i_\alpha m n - (c_v L^i_\alpha m a^\alpha_j|m + c_w L^i_\alpha n a^\alpha_j|m
\]

\[
+ [c_v c_w - c_v (c_w + d_w)] L^i_\alpha n L^i_\beta m + c_v (c_w + d_w) L^i_\alpha n L^i_\beta m - (c_v c_w + c_v d_w) L^i_\beta m L^i_\alpha n \right\}
\]

\[
- a^i_\alpha \left\{ R^i_\beta j m n - (d_v L^i_\beta j m a^\beta_j|m + d_w L^i_\beta j m a^\beta_j|m
\]

\[
- (d_v c_w - d_v d_w) L^i_\beta j m a^\beta_j|m + (d_v + d_w) c_w d_w L^i_\beta j m a^\beta_j|m
\]

\[
+ (d_v + d_w) c_w d_w L^i_\beta j m a^\beta_j|m \right\}
\]

\[
+ a^i_\beta \left\{ (c_v c_w - c_v d_w) L^i_\beta j m a^\beta_j|m + (c_v d_v - c_w d_w) L^i_\beta j m a^\beta_j|m \right\},
\]

(1.16)

for \( v, w, w_1, w_2 \in \{0, 1, 2, 3, 4 \} \).

It is obtained [14] that the geometrical objects \( a^i_j|k \), \( a^i_j|k \), \( a^i_j|k \) are linearly independent and that the geometrical objects \( a^i_j|k \) and \( a^i_j|k \) may be uniquely expressed in the terms of the first three kinds of covariant derivative.
The purpose of this paper is to generalize the first Ricci-type identities theorem in the sense of changing the summands $L^i_{jk}a^l_{s|r}$ with linear combinations of the geometrical objects $L^i_{jk}a^l_{s|r} = L^i_{jk}a^l_{s|r}, L^i_{jk}a^l_{s|1}, L^i_{jk}a^l_{s|2}, L^i_{jk}a^l_{s|3}, L^i_{jk}a^l_{s|4}.$

At the start of the research, we will prove that three of covariant derivatives $a^i_{j|k}, a^i_{j|k}, a^i_{j|k}, a^i_{j|k}$ are enough for all commutation formulae to be obtained.

The next result of our research will be the commutation formulae with respect to double covariant derivatives of a tensor $\hat{a}$ of a type $(p, q), p, q \in \mathbb{N}$.

2. Four plus one kinds of covariant derivatives. For the research in this paper, we need the next propositions.

**Proposition 2.1.** The covariant derivatives given by the equations (1.6, 1.7, 1.8, 1.9) and the covariant derivative given by the equation (1.3) satisfy the equations

\[
a^i_{j_1...j_p|k_1} = a^i_{j_1...j_p|k_1} + \sum_{u=1}^p L^i_{\alpha u}a^{i_{\alpha i_{u+1}}...i_p}_{j_1...j_q|k_2} - \sum_{v=1}^q L^i_{\alpha v}a^{i_{\alpha i_{v+1}}...i_p}_{j_1...j_q|k_2}, \tag{2.1}
\]

\[
a^i_{j_1...j_p|k_2} = a^i_{j_1...j_p|k_2} - \sum_{u=1}^p L^i_{\alpha u}a^{i_{\alpha i_{u+1}}...i_p}_{j_1...j_q|k_2} + \sum_{v=1}^q L^i_{\alpha v}a^{i_{\alpha i_{v+1}}...i_p}_{j_1...j_q|k_2}, \tag{2.2}
\]

\[
a^i_{j_1...j_p|k_3} = a^i_{j_1...j_p|k_3} + \sum_{u=1}^p L^i_{\alpha u}a^{i_{\alpha i_{u+1}}...i_p}_{j_1...j_q|k_3} + \sum_{v=1}^q L^i_{\alpha v}a^{i_{\alpha i_{v+1}}...i_p}_{j_1...j_q|k_3}, \tag{2.3}
\]

\[
a^i_{j_1...j_p|k_4} = a^i_{j_1...j_p|k_4} - \sum_{u=1}^p L^i_{\alpha u}a^{i_{\alpha i_{u+1}}...i_p}_{j_1...j_q|k_4} - \sum_{v=1}^q L^i_{\alpha v}a^{i_{\alpha i_{v+1}}...i_p}_{j_1...j_q|k_4}. \tag{2.4}
\]

**Remark 2.1.** With respect to the equations (1.3, 2.1–2.4), we obtain

\[
a^i_{j_1...j_q|k} = a^i_{j_1...j_q|k} + c_0 \sum_{u=1}^p L^i_{\alpha u}a^{i_{\alpha i_{u+1}}...i_p}_{j_1...j_q|k} + d_0 \sum_{v=1}^q L^i_{\alpha v}a^{i_{\alpha i_{v+1}}...i_p}_{j_1...j_q|k}, \tag{2.5}
\]

for $z = 0, \ldots, 4$, and the corresponding coefficients $c_0 = d_0 = 0, c_1 = 1, c_2 = 1, c_3 = 1, c_4 = 1, d_1 = 1, d_2 = 1, d_3 = 1, d_4 = 1$.

Let us obtain the commutation formulae with respect to covariant derivatives of tensors $\hat{a}$ of the type $(1, 1), \hat{u}$ of the type $(1, 0)$ and $\hat{v}$ of the type $(0, 1)$.

**Proposition 2.2.** For a tensor $\hat{a}$ of the type $(1, 1)$, three of the geometrical objects $a^i_{j|k}, a^i_{j|k}, a^i_{j|k}, a^i_{j|k}$ are linearly independent.

**Proof.** With respect to the equation (2.5), the number of linearly independent geometrical objects $a^i_{j|k}, a^i_{j|k}, a^i_{j|k}, a^i_{j|k}$ is equal to the rank of the matrix

\[
M = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & -1 \\
1 & -1 & 1 \\
1 & 1 & 1 \\
1 & -1 & -1
\end{bmatrix}.
\]
Because $\text{Rank}(M) = 3$, three of the geometrical objects $a_{j|k}^i$, $a_{j|k}^i$, $a_{j|k}^i$, $a_{j|k}^i$ are linearly independent.

**Corollary 2.1.** For a tensor $\hat{u}$ of the type $(1, 0)$, two of the geometrical objects $u_{k|1}^i$, $u_{k|3}^i$, $u_{k|4}^i$ are linearly independent.

For a tensor $\hat{v}$ of the type $(0, 1)$, two of the geometrical objects $v_{j|k}$, $v_{j|k}$, $v_{j|k}$, $v_{j|k}$ are linearly independent.

**Corollary 2.2.** The triples

\[
\mathcal{A}: \begin{cases}
  a_{j|k}^i, \\
  a_{j|k}^i, \\
  a_{j|k}^i
\end{cases}
  \quad
\mathcal{A}: \begin{cases}
  a_{j|k}^i, \\
  a_{j|k}^i, \\
  a_{j|k}^i
\end{cases}
  \quad
\mathcal{A}: \begin{cases}
  a_{j|k}^i, \\
  a_{j|k}^i, \\
  a_{j|k}^i
\end{cases}
  \quad
\mathcal{A}: \begin{cases}
  a_{j|k}^i, \\
  a_{j|k}^i, \\
  a_{j|k}^i
\end{cases}
\]

are triples of linearly independent geometrical objects $a_{j|k}^i$, $z = 0, \ldots, 4$.

The pairs

\[
\mathcal{U}: \begin{cases}
  u_{k|1}^i, \\
  u_{k|1}^i
\end{cases}
  \quad
\mathcal{U}: \begin{cases}
  u_{k|2}^i, \\
  u_{k|2}^i
\end{cases}
  \quad
\mathcal{U}: \begin{cases}
  u_{k|3}^i, \\
  u_{k|3}^i
\end{cases}
  \quad
\mathcal{V}: \begin{cases}
  v_{j|k}, \\
  v_{j|k}
\end{cases}
  \quad
\mathcal{V}: \begin{cases}
  v_{j|k}, \\
  v_{j|k}
\end{cases}
  \quad
\mathcal{V}: \begin{cases}
  v_{j|k}, \\
  v_{j|k}
\end{cases}
\]

are pairs of linearly independent geometrical objects $u_{k|z}^i$, $v_{j|z}$, for $z = 0, \ldots, 4$.

3. **Identities of Ricci type with respect to tensor $\hat{a}$ of type $(1, 1)$.** Let us generalize the first Ricci-type identities theorem.

**Theorem 3.1.** (Second Ricci-type identities theorem) Suppose that

\[
X_{jk}^i = \rho_0^1 a_{j|k}^i + \rho_1^1 a_{j|k}^i + \rho_2^1 a_{j|k}^i + \rho_3^1 a_{j|k}^i + \rho_4^1 a_{j|k}^i, \]

\[
Y_{jk}^i = \rho_0^2 a_{j|k}^i + \rho_1^2 a_{j|k}^i + \rho_2^2 a_{j|k}^i + \rho_3^2 a_{j|k}^i + \rho_4^2 a_{j|k}^i, \]

\[
Z_{jk}^i = \rho_0^3 a_{j|k}^i + \rho_1^3 a_{j|k}^i + \rho_2^3 a_{j|k}^i + \rho_3^3 a_{j|k}^i + \rho_4^3 a_{j|k}^i, \]

\[
U_{jk}^i = \rho_0^4 a_{j|k}^i + \rho_1^4 a_{j|k}^i + \rho_2^4 a_{j|k}^i + \rho_3^4 a_{j|k}^i + \rho_4^4 a_{j|k}^i, \]

(3.1) (3.2) (3.3) (3.4)
\[ V_{jk}^i = \rho_0^i a_j^i | k + \rho_1^i a_j^i | k + \rho_2^i a_j^i | k + \rho_3^i a_j^i | k + \rho_4^i a_j^i | k, \]  

(3.5)

for a tensor \( \hat{a} \) of the type \( (1,1) \) and scalars \( \rho_0^z, \rho_1^z, \rho_2^z, \rho_3^z, \rho_4^z, \ z \in \{1, \ldots, 5\}, \rho_0^z + \rho_1^z + \rho_2^z + \rho_3^z + \rho_4^z = 1. \)

The following equation holds:

\[
\begin{align*}
  a_j^i & \mid m \mid n - a_j^i \mid n \mid m \\
  &= (c_{v_1} - c_{w_2}) L^i_{\alpha \gamma} X_{\gamma j}^\alpha + (c_{w_1} - c_{v_2}) L^i_{\alpha \gamma} Y_{\gamma j}^\alpha + (d_{v_1} - d_{w_2}) L^i_{\gamma \alpha} Z_{\gamma j}^\alpha \\
  &+ (d_{w_1} - d_{v_2}) L^i_{\gamma j} U_{\alpha m}^\alpha + (d_{v_1} + d_{w_2}) L^i_{\alpha m} V_{\gamma j}^\alpha \\
  &+ a_j^i \{ R_{\alpha mn}^i + c_{v_1} L^i_{\alpha \gamma} n - c_{w_2} L^i_{\alpha \gamma} m \\
  &+ p_1 L^i_{\gamma \alpha} L^i_{\alpha \beta} + p_2 L^i_{\alpha \gamma} L^i_{\alpha \beta} + p_3 L^i_{\alpha \gamma} L^i_{\alpha \beta} \} \\
  &- a_j^i \{ R_{\gamma jm}^i - d_{v_1} L^i_{\gamma j} m + d_{w_2} L^i_{\gamma j} m \\
  &+ q_1 L^i_{\gamma j} L^i_{\alpha \gamma} + q_2 L^i_{\gamma j} L^i_{\alpha \beta} + q_3 L^i_{\alpha \gamma} L^i_{\alpha \beta} \} \\
  &+ a_j^i \{ r_1 L^i_{\alpha \gamma} L^i_{\alpha \beta} + r_2 L^i_{\gamma j} L^i_{\alpha \beta} \},
\end{align*}
\]

(3.6)

where

\[
\begin{align*}
  p_1 &= c_{v_1} c_{w_1} - c_{v_2} (c_{w_1} + d_{w_2}) - (c_{w_1} - c_{v_2}) (\rho_1^2 - \rho_2^2 + \rho_3^2 - \rho_4^2), \tag{3.7} \\
  p_2 &= c_{v_1} (c_{w_1} + d_{w_1}) - c_{v_2} c_{w_2} - (c_{v_1} - c_{w_2}) (\rho_1^2 - \rho_2^2 - \rho_3^2 - \rho_4^2), \tag{3.8} \\
  p_3 &= -c_{v_1} d_{w_1} - c_{v_2} d_{w_2} + (d_{w_1} + d_{w_2}) (\rho_1^2 - \rho_2^2 - \rho_3^2 - \rho_4^2), \tag{3.9} \\
  q_1 &= -d_{v_1} (c_{w_1} + d_{w_1}) + d_{w_2} (d_{v_1} + d_{w_2}) (\rho_1^3 - \rho_2^3 - \rho_3^3 + \rho_4^3), \tag{3.10} \\
  q_2 &= -d_{v_1} d_{w_1} - d_{w_2} (c_{w_1} + d_{w_2}) - (d_{w_1} + d_{w_2}) (\rho_1^4 - \rho_2^4 - \rho_3^4 + \rho_4^4), \tag{3.11} \\
  q_3 &= d_{v_1} d_{w_1} + d_{w_2} (c_{w_1} + d_{w_2}) - (d_{w_1} + d_{w_2}) (\rho_1^5 - \rho_2^5 - \rho_3^5 + \rho_4^5), \tag{3.12} \\
  r_1 &= c_{w_1} d_{v_1} - c_{w_2} d_{w_2} + (c_{w_1} - c_{w_2}) (\rho_1^2 - \rho_2^2 - \rho_3^2 + \rho_4^2) \\
  &- (d_{v_1} - d_{w_2}) (\rho_1^3 - \rho_2^3 + \rho_3^3 - \rho_4^3), \\  r_2 &= c_{v_1} d_{v_1} - c_{v_2} d_{v_2} + (c_{v_1} - c_{v_2}) (\rho_1^4 - \rho_2^4 - \rho_3^4 + \rho_4^4) \\
  &- (d_{v_1} - d_{v_2}) (\rho_1^5 - \rho_2^5 + \rho_3^5 - \rho_4^5). \tag{3.14}
\end{align*}
\]

Proof. We get

\[
\begin{align*}
  L^i_{\alpha \gamma} X_{\gamma j}^\alpha &= L^i_{\alpha \gamma} a_j^\alpha | n + (\rho_1^2 - \rho_2^2 + \rho_3 - \rho_4) a_j^\alpha | n - (\rho_1^2 - \rho_2^2 + \rho_3 + \rho_4) a_j^\alpha | n,
  \\
  L^i_{\gamma \alpha} Y_{\gamma j}^\alpha &= L^i_{\gamma \alpha} a_j^\alpha | n + (\rho_1^2 - \rho_2^2 + \rho_3 - \rho_4) a_j^\alpha | n - (\rho_1^2 - \rho_2^2 + \rho_3 + \rho_4) a_j^\alpha | n,
  \\
  L^i_{\gamma j} Z_{\gamma j}^\alpha &= L^i_{\gamma j} a_j^\alpha | n + (\rho_1^2 - \rho_2^2 + \rho_3 - \rho_4) a_j^\alpha | n - (\rho_1^2 - \rho_2^2 + \rho_3 + \rho_4) a_j^\alpha | n.
\end{align*}
\]

(3.15)
Theorem 3.2. \((3.20, 3.21)\) substituted into the equation \((3.33)\), the next theorem holds.

\[ L^\alpha_{\nu \nu} U_{\alpha m} = L^\alpha_{\nu \nu} a^i_{\alpha | m} + (\rho_1^2 - \rho_2^2 + \rho_3^2 - \rho_4^2) L^\beta_{\nu \nu} L^i_{\alpha m} a^\beta_{\alpha} - (\rho_1^2 - \rho_2^2 + \rho_3^2 - \rho_4^2) L^\beta_{\nu \nu} L^i_{\alpha m} a^\beta_{\alpha}, \]
\[ (3.18) \]

\[ L^\alpha_{\nu \nu} V^i_{\alpha} = L^\alpha_{\nu \nu} a^i_{\alpha | \alpha} + (\rho_1^2 - \rho_2^2 + \rho_3^2 - \rho_4^2) L^\beta_{\nu \nu} L^i_{\alpha \beta} a^\beta_{\alpha} - (\rho_1^2 - \rho_2^2 + \rho_3^2 - \rho_4^2) L^\beta_{\nu \nu} L^i_{\alpha \beta} a^\beta_{\alpha}, \]
\[ (3.19) \]

After expressing the terms
\[ (c_v - c_w) L^i_{\nu \mu} a^\mu_{\alpha | \alpha} \]
\[ (c_v - c_w) L^i_{\nu \mu} a^\mu_{\alpha | m}, \]
\[ (c_v - c_w) L^i_{\nu \mu} a^\mu_{\alpha | m}, \]
\[ (d_v - d_w) L^\alpha_{\nu \nu} a^i_{\alpha | \alpha}, \]
\[ (d_v - d_w) L^\alpha_{\nu \nu} a^i_{\alpha | m}, \]
\[ (d_v + d_w) L^\alpha_{\nu \nu} a^i_{\alpha | \alpha}, \]
with respect to the equalities \((3.15–3.19)\) and substituting them into the equation \((1.16)\), one confirms the validity of the equation \((3.33)\). 

The next equalities are satisfied
\[ \rho_1^2 - \rho_2^2 + \rho_3^2 - \rho_4^2 = (-1)^{1-1} \rho_1^2 + (-1)^{2-1} \rho_2^2 + (-1)^{3-1} \rho_3^2 + (-1)^{4-1} \rho_4^2, \]
\[ (3.20) \]
\[ \rho_1^2 - \rho_2^2 - \rho_3^2 + \rho_4^2 = (-1)^{1-1} \rho_1^2 + (-1)^{2-1} \rho_2^2 + (-1)^{3-1} \rho_3^2 + (-1)^{4-1} \rho_4^2, \]
\[ (3.21) \]

for \(z = 1, \ldots, 5\), for the floor function \(|x|\) (the function that takes as input a real number \(x\) and gives the greatest integer less than or equal to \(x\) as output).

Let \(\{n_1, n_2, n_3, n_4\} = \{1, 2, 3, 4\}\). With respect to the Proposition 2.2, we conclude that it is enough to consider the case of \(\rho_0^5 = \rho_4^5 = 0\), \(n_4 \in \{1, 2, 3, 4\}\) \(\setminus \{n_1, n_2, n_3\}\).

For integers \(n_1, n_2, n_3, 1 \leq n_1 < n_2 < n_3 \leq 4\), and with respect to the equations \((3.20, 3.21)\) substituted into the equation \((3.33)\), the next theorem holds.

Theorem 3.2. \((n_1 - n_2 - n_3)\) second Ricci-type identities theorem) Suppose that

\[ \tilde{A}^i_{jk} = \rho_{n_1} a^i_{n_1 | k} + \rho_{n_2} a^i_{n_2 | k} + \rho_{n_3} a^i_{n_3 | k}, \]
\[ \tilde{B}^i_{jk} = \rho_{n_1} a^i_{n_1 | k} + \rho_{n_2} a^i_{n_2 | k} + \rho_{n_3} a^i_{n_3 | k}, \]
\[ \tilde{C}^i_{jk} = \rho_{n_1} a^i_{n_1 | k} + \rho_{n_2} a^i_{n_2 | k} + \rho_{n_3} a^i_{n_3 | k}, \]
\[ \tilde{D}^i_{jk} = \rho_{n_1} a^i_{n_1 | k} + \rho_{n_2} a^i_{n_2 | k} + \rho_{n_3} a^i_{n_3 | k}, \]
\[ (3.22) \]

for the tensor \(\tilde{a}\) of the type \((1, 1)\).

The following equation holds:

\[ a^i_{\nu_1 \nu_1} - a^i_{\nu_2 \nu_2} = (c_{v_1} - c_{w_2}) L^i_{\nu \nu} \tilde{A}^{\alpha}_{\nu m} + (c_{v_1} - c_{w_2}) L^i_{\nu \nu} \tilde{B}^{\alpha}_{\nu m} + (d_v - d_{w_2}) L^i_{\nu \nu} \tilde{C}^{\alpha}_{\nu m} + (d_v + d_{w_2}) L^i_{\nu \nu} \tilde{D}^{\alpha}_{\nu m} \]
\[a_j^i \{ R^i_{\alpha mn} + c_v L^i_{\alpha |m} - c_v L^i_{\alpha |n} + \tilde{p}_1 L^\beta_{\alpha m} L^i_{\beta n} + \tilde{p}_2 L^\beta_{\alpha n} L^i_{\beta m} + \tilde{p}_3 L^\beta_{mn} L^i_{\beta \alpha}\}
\]
\[-a^i_\alpha \{ R^\alpha_{j mn} - d_v L^\alpha_{jm |n} + d_v L^\alpha_{jn |m} + \tilde{q}_1 L^\beta_{jm} L^\alpha_{\beta n} + \tilde{q}_2 L^\beta_{jn} L^\alpha_{\beta m} + \tilde{q}_3 L^\beta_{mn} L^\alpha_{\beta j}\}
\]
\[+ a^\alpha_\beta \{ \tilde{r}_1 L^\beta_{jm} L^i_{\alpha v} + \tilde{r}_2 L^\beta_{jn} L^i_{\alpha v}\}, \tag{3.23}\]

where

\[\tilde{p}_1 = c_v c_w - c_v (c_{w_2} + d_{w_2})\]
\[-(c_{w_1} - c_{w_2})((-1)^{n_1-1} \rho^2_{n_1} + (-1)^{n_2-1} \rho^2_{n_2} + (-1)^{n_3-1} \rho^2_{n_3}), \tag{3.24}\]
\[\tilde{p}_2 = c_v (c_w + d_w)
- c_v c_{w_2} - (c_v - c_{w_2})((-1)^{n_1-1} \rho^1_{n_1} + (-1)^{n_2-1} \rho^1_{n_2} + (-1)^{n_3-1} \rho^1_{n_3}), \tag{3.25}\]
\[\tilde{p}_3 = -c_v d_{w_1} - c_v d_{w_2} + (d_{w_1} + d_{w_2})((-1)^{n_1-1} \rho^5_{n_1} + (-1)^{n_2-1} \rho^5_{n_2} + (-1)^{n_3-1} \rho^5_{n_3}), \tag{3.26}\]
\[\tilde{q}_1 = -d_v (c_{w_1} + d_{w_1}) + d_{w_2} d_{w_2}\]
\[-(d_{w_1} - d_{w_2})((-1)^{n_1} \rho^3_{n_1} + (-1)^{n_2} \rho^3_{n_2} + (-1)^{n_3} \rho^3_{n_3}), \tag{3.27}\]
\[\tilde{q}_2 = -d_v d_{w_1} + d_v (c_{w_2} + d_{w_2})\]
\[-(d_{w_1} - d_{w_2})((-1)^{n_1} \rho^4_{n_1} + (-1)^{n_2} \rho^4_{n_2} + (-1)^{n_3} \rho^4_{n_3}), \tag{3.28}\]
\[\tilde{q}_3 = d_v d_{w_1} + d_v d_{w_2} - (d_{w_1} + d_{w_2})((-1)^{n_1} \rho^5_{n_1} + (-1)^{n_2} \rho^5_{n_2} + (-1)^{n_3} \rho^5_{n_3}), \tag{3.29}\]
\[\tilde{r}_1 = c_v d_{w_1} - c_v d_{w_2} + (c_v - c_{w_2})((-1)^{n_1} \rho^2_{n_1} + (-1)^{n_2} \rho^2_{n_2} + (-1)^{n_3} \rho^2_{n_3})\]
\[-(d_{w_1} - d_{w_2})((-1)^{n_1-1} \rho^3_{n_1} + (-1)^{n_2-1} \rho^3_{n_2} + (-1)^{n_3-1} \rho^3_{n_3}), \tag{3.30}\]
\[\tilde{r}_2 = c_v d_{w_1} - c_v d_{w_2} + (c_v - c_{w_2})((-1)^{n_1} \rho^1_{n_1} + (-1)^{n_2} \rho^1_{n_2} + (-1)^{n_3} \rho^1_{n_3})\]
\[-(d_{w_1} - d_{w_2})((-1)^{n_1-1} \rho^4_{n_1} + (-1)^{n_2-1} \rho^4_{n_2} + (-1)^{n_3-1} \rho^4_{n_3}), \tag{3.31}\]
\[\rho^z_{n_1} + \rho^z_{n_2} + \rho^z_{n_3} = 1, z \in \{1, 2, 3, 4, 5\}.
\]

With respect to the Proposition 2.2, we conclude that it is enough to consider the case of $\rho^z_{n_3} = 0, \rho^z_{n_4} = 0, 1 \leq n_3 < n_4 \leq 4, \{n_1, n_2\} = \{1, 2, 3, 4\}\}\{n_3, n_4\},
\quad (n_1, n_2) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\} as in the next theorem.

**Theorem 3.3. (n_1 - n_2-second Ricci-type identities theorem)** Suppose that

\[\tilde{X}^i_{jk} = \rho^0_{a^i_{j |k}} + \rho^1_{a^i_{j |n_1}} + \rho^2_{a^i_{j |n_2}} + \rho^3_{a^i_{j |n_3}} + \rho^4_{a^i_{j |n_4}} + \rho^5_{a^i_{j |n_5}}, \quad \tilde{Y}^i_{jk} = \rho^0_{a^i_{j |k}} + \rho^1_{a^i_{j |n_1}} + \rho^2_{a^i_{j |n_2}} + \rho^3_{a^i_{j |n_3}} + \rho^4_{a^i_{j |n_4}} + \rho^5_{a^i_{j |n_5}},\]
\[\tilde{Z}^i_{jk} = \rho^3_{a^i_{j |k}} + \rho^4_{a^i_{j |n_1}} + \rho^5_{a^i_{j |n_2}} + \rho^0_{a^i_{j |n_3}} + \rho^1_{a^i_{j |n_4}} + \rho^2_{a^i_{j |n_5}}, \quad \tilde{U}^i_{jk} = \rho^0_{a^i_{j |k}} + \rho^1_{a^i_{j |n_1}} + \rho^2_{a^i_{j |n_2}} + \rho^3_{a^i_{j |n_3}} + \rho^4_{a^i_{j |n_4}} + \rho^5_{a^i_{j |n_5}},\]
\[\tilde{V}^i_{jk} = \rho^0_{a^i_{j |k}} + \rho^1_{a^i_{j |n_1}} + \rho^2_{a^i_{j |n_2}} + \rho^3_{a^i_{j |n_3}} + \rho^4_{a^i_{j |n_4}} + \rho^5_{a^i_{j |n_5}}, \tag{3.32}\]
for the tensor $\tilde{a}$ of the type $(1,1)$.

The following equation holds:

\[
\begin{align*}
  a^i_{j, \nu_1 \nu_2} | m | n - a^i_{j, \nu_1 \nu_2} | n | m \\
  = & \quad (c_{v_1} - c_{v_2})L^i_{\alpha \nu_1} \tilde{X}^\alpha_{j, \nu_2} + (c_{w_1} - c_{w_2})L^i_{\alpha \nu_1} \tilde{Y}^\alpha_{j, m} + (d_{v_1} - d_{w_2})L^i_{\alpha \nu_1} \tilde{Z}^i_{j, \alpha m} \\
  & + (d_{w_1} - d_{v_2})L^i_{\beta \nu_1} \tilde{U}^\alpha_{j, \nu_2} + (d_{w_1} + d_{w_2})L^i_{\beta \nu_1} \tilde{V}^\alpha_{j, \alpha} \\
  + & a^\alpha_j \{ R^i_{\alpha \nu_1 \nu_2} + c_{v_1} L^i_{\alpha \nu_1 | n} - c_{v_2} L^i_{\alpha \nu_1 | m} \\
  & + \tilde{p}_1 L^\beta_{\alpha \nu_1} L^i_{\beta \nu_1} + \tilde{p}_2 L^\beta_{\alpha \nu_1} L^i_{\beta \nu_2} + \tilde{p}_3 L^\beta_{\nu_1 \nu_2} L^i_{\beta \nu_2} \} \\
  - & a^\alpha_j \{ R^i_{\beta \nu_1 \nu_2} - d_{v_1} L^i_{\alpha \nu_1 | n} + d_{v_2} L^i_{\alpha \nu_1 | m} \\
  & + \tilde{q}_1 L^\beta_{\alpha \nu_1} L^i_{\beta \nu_1} + \tilde{q}_2 L^\beta_{\beta \nu_1} L^i_{\beta \nu_2} + \tilde{q}_3 L^\beta_{\nu_1 \nu_2} L^i_{\beta \nu_2} \} \\
  + & a^\alpha_j \{ \tilde{r}_1 L^\beta_{\alpha \nu_1} L^i_{\beta \nu_1} + \tilde{r}_2 L^\beta_{\nu_1 \nu_2} L^i_{\alpha \nu_2} \},
\end{align*}
\]

(3.33)

where

\[
\begin{align*}
  \tilde{p}_1 &= c_{v_1} c_{w_1} - c_{v_2} (c_{v_2} + d_{w_2}) - (c_{w_1} - c_{v_2}) ((-1)^{n_1 - 1} \rho_{n_1}^2 + (-1)^{n_2} \rho_{n_2}^2), \\
  \tilde{p}_2 &= c_{v_1} (c_{w_1} + d_{w_1}) - c_{v_2} c_{w_2} - (c_{v_1} - c_{w_2}) ((-1)^{n_1 - 1} \rho_{n_1}^1 + (-1)^{n_2} \rho_{n_2}^1), \\
  \tilde{p}_3 &= -c_{v_1} d_{w_1} - c_{v_2} d_{w_2} + (d_{w_1} + d_{w_2})((-1)^{n_1} \rho_{n_1} + (-1)^{n_2} \rho_{n_2}), \\
  \tilde{q}_1 &= -d_{v_1} (c_{w_1} + d_{w_1}) + d_{v_2} d_{w_2} - (d_{v_1} - d_{w_2})((-1)^{n_1} \rho_{n_1} + (-1)^{n_2} \rho_{n_2}), \\
  \tilde{q}_2 &= -d_{v_1} d_{w_1} - d_{v_2} (c_{w_1} + d_{w_2}) - (d_{w_1} - d_{w_2})((-1)^{n_1} \rho_{n_1} + (-1)^{n_2} \rho_{n_2}), \\
  \tilde{q}_3 &= d_{v_1} d_{v_1} + d_{v_2} d_{v_2} - (d_{w_1} - d_{w_2})((-1)^{n_1} \rho_{n_1} + (-1)^{n_2} \rho_{n_2}), \\
  \tilde{r}_1 &= c_{w_1} d_{v_1} - c_{w_2} d_{w_2} + (c_{v_1} - c_{v_2})((-1)^{n_1} \rho_{n_1}^2 + (-1)^{n_2} \rho_{n_2}^2) \\
  & - (d_{v_1} - d_{v_2})((-1)^{n_1} \rho_{n_1}^1 + (-1)^{n_2} \rho_{n_2}^1), \\
  \tilde{r}_2 &= c_{w_1} d_{v_1} - c_{w_2} d_{v_2} + (c_{v_1} - c_{w_2})((-1)^{n_1} \rho_{n_1}^1 + (-1)^{n_2} \rho_{n_2}^1) \\
  & - (d_{v_1} - d_{v_2})((-1)^{n_1} \rho_{n_1}^2 + (-1)^{n_2} \rho_{n_2}^2),
\end{align*}
\]

(3.39)

(3.40)

(3.41)

(3.42)

(3.43)

(3.44)

\[
\begin{align*}
  \rho_{n_1}^z + \rho_{n_2}^z + \rho_{n_2}^z = 1, \ z \in \{1, 2, 3, 4, 5\}.
\end{align*}
\]

**Theorem 3.4.** (Commutation formulae theorem) Fifteen of the geometrical objects $a^i_{j, \nu_1 \nu_2} | m | n - a^i_{j, \nu_1 \nu_2} | n | m$, $v_1, v_2, w_1, w_2 \in \{0, 1, 2, 3, 4\}$, are linearly independent.

**Proof.** With respect to the Corollary 2.2 and the equation (2.6) in this corollary, we get
\[ a^i_{j_1 \ldots j_q} \mid m | n = a^i_{j_1 \ldots j_q} \mid m | n + \sum_{k=1}^p L^i_{\alpha m} L^i_{\beta n} a^i_{j_1 \ldots j_q} \mid m | n + \sum_{k=1}^p L^i_{\alpha m} L^i_{\beta n} a^i_{j_1 \ldots j_q} \mid m | n \]

for the corresponding scalars \( x_1, x_2, x_3, y_1, y_2, y_3 \).

After substituting the expression (3.42) into the equation (3.43), one gets that the double covariant derivative \( a^i_{j_1 \ldots j_q} \mid m | n \) is a linear combination of the geometrical objects \( a^i_{j_1 \ldots j_q} \mid m | n \), for \( v'_1, w'_1 \in \{1, 2, 3\} \).

In 1–2–3-commutation formulae theorem [14], it is proved that sixteen of the geometrical objects \( a^i_{j_1 \ldots j_q} \mid m | n - a^i_{j_1 \ldots j_q} \mid m | n \), for \( v'_1, v'_2, w'_1, w'_2 \in \{1, 2, 3\} \), are linearly independent, which completes the proof for this theorem. \( \square \)

4. Identities of Ricci type with respect to tensor \( \hat{a} \) of type \((p, q)\). The next equation holds.

\[ a^i_{j_1 \ldots j_q} \mid m | n = a^i_{j_1 \ldots j_q} \mid m | n + \sum_{k=1}^p L^i_{\alpha m} L^i_{\beta n} a^i_{j_1 \ldots j_q} \mid m | n + \sum_{k=1}^p L^i_{\alpha m} L^i_{\beta n} a^i_{j_1 \ldots j_q} \mid m | n \]

for the corresponding scalars \( x_1, x_2, x_3, y_1, y_2, y_3 \).
\[- \sum_{l=1}^{q} a_{j_1 \ldots j_l-1 \alpha j_l+1 \ldots j_q}^{i_1 \ldots i_p} \left( -d_v L_{j_lm}^\alpha \nabla_{j_m} - d_v (c_w + d_w) L_{j_{m+1}m}^\beta L_{\beta n}^\alpha - d_v d_w L_{j_{m+1}m}^\beta L_{\beta n}^\alpha \right. \\
\quad \left. + d_v d_w L_{mn}^\beta L_{\beta j}^i \right) \]
\[+ \sum_{k=1}^{p} \sum_{l=1}^{q} a_{j_1 \ldots j_l-1 \alpha j_l+1 \ldots j_q}^{i_1 \ldots i_p} \left( c_w d_v L_{j_{m+1}m}^\beta L_{\beta j}^{i_k} + c_v d_w L_{j_{m+1}m}^\beta L_{\beta j}^{i_k} \right). \]  
(4.1)

With respect to the equation (2.5), one generalizes the results obtained in the previous section with the next theorems.

**Theorem 4.1.** (General first Ricci-type identities theorem) The family of identities of the Ricci type with respect to a non-symmetric affine connection \( \nabla \) and a tensor \( \tilde{a} \) of the type \((p, q), p, q \in \mathbb{N}\), is

\[
a_{j_1 \ldots j_q | m | n}^{i_1 \ldots i_p} - a_{j_1 \ldots j_q | v_1 w_1}^{i_1 \ldots i_p} \]
\[= \sum_{k=1}^{p} \sum_{l=1}^{q} \left( c_v - c_w \right) L_{\alpha v}^{i_k} a_{j_1 \ldots j_q | m}^{i_1 \ldots i_k \ldots i_{k+1} \ldots i_p} \]
\[+ \left( c_w - c_v \right) L_{\alpha v}^{i_k} a_{j_1 \ldots j_q | n}^{i_1 \ldots i_k \ldots i_{k+1} \ldots i_p}, \]  
(4.3)

\[
a_{j_1 \ldots j_q | m | n}^{i_1 \ldots i_p} = \sum_{k=1}^{p} \sum_{l=1}^{q} \left( d_v - d_w \right) L_{\alpha v}^{i_k} a_{j_1 \ldots j_q | m}^{i_1 \ldots i_{k-1} \alpha i_{k+1} \ldots i_p} \]
\[+ \left( d_w - d_v \right) L_{\alpha v}^{i_k} a_{j_1 \ldots j_q | n}^{i_1 \ldots i_{k-1} \alpha i_{k+1} \ldots i_p}, \]  
(4.4)

\[
1 R_{jlmn}^\alpha = R_{jlmn}^\alpha + c_v L_{j_{m+1}m}^\alpha - c_v L_{j_{m+1}m}^\alpha + \left[ c_v - c_w (c_w + d_w) \right] L_{j_{m+1}m}^\alpha L_{\alpha v}^i \]
\[+ \left[ c_v (c_w + d_w) - c_v c_w \right] L_{j_{m+1}m}^\alpha L_{\alpha v}^i - (c_v d_v + c_v d_w) L_{j_{m+1}m}^\alpha L_{\alpha v}^i, \]  
(4.5)

\[
2 R_{jlmn}^\alpha = R_{jlmn}^\alpha - d_v L_{j_{m+1}m}^\alpha + d_v L_{j_{m+1}m}^\alpha - \left[ d_v (c_w + d_w) - d_v d_w \right] L_{j_{m+1}m}^\alpha L_{\alpha v}^i \]
\[+ \left[ d_v (c_w + d_w) - d_v c_w \right] L_{j_{m+1}m}^\alpha L_{\alpha v}^i + (c_v d_v + c_v d_w) L_{j_{m+1}m}^\alpha L_{\alpha v}^i, \]  
(4.6)

\[
3 R_{jlmn}^\alpha = c_v d_v - c_v d_w \right) L_{j_{m+1}m}^\alpha L_{\alpha v}^i + \left( c_v d_{w_1} - c_v d_{w_2} \right) L_{j_{m+1}m}^\alpha L_{\alpha v}^i, \]  
(4.7)
and \( v_1, v_2, w_1, w_2 \in \{0, 1, 2, 3, 4\} \).

5. **Conclusion.** In this article, we generalized the first Ricci-type identities theorem. It was proved that three of geometrical objects \( a_{j[k}^i, a_{j[k}^i_1, a_{j[k}^i_2, a_{j[k}^i_3, a_{j[k}^i_4 \) are linearly independent here.

After that, we generalized the first Ricci-type identities theorem with respect to a tensor \( \hat{a} \) of the type \((p, q), p, q \in \mathbb{N}\).

In the future work, we will generalize the commutation formulae theorem, \( n_1 - n_2 - n_3 \)-second Ricci-type identities theorem and the \( n_1 - n_2 \)-second Ricci-type identities theorem with respect to tensors of the types \((p, q), (p, 0), (0, q)\).

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