Are Bundles Good Deals for FOML?

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Abstract

Bundled products are often offered as good deals to customers. When we bundle quantifiers and modalities together (as in $\exists x \square$, $\Diamond x \forall$ etc.) in first-order modal logic (FOML), we get new logical operators whose combinations produce interesting fragments of FOML without any restriction on the arity of predicates, the number of variables, or the modal scope. It is well-known that finding decidable fragments of FOML is hard, so we may ask: do bundled fragments that exploit the distinct expressivity of FOML constitute good deals in balancing the expressivity and complexity? There are a few positive earlier results on some particular fragments. In this paper, we try to fully map the terrain of bundled fragments of FOML in (un)decidability, and in the cases without a definite answer yet, we show that they lack the finite model property. Moreover, whether the logics are interpreted over constant domains (across states/worlds) or increasing domains presents another layer of complexity. We also present the loosely bundled fragment, which generalizes the bundles and yet retain decidability (over increasing domain models).

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1 Introduction

Among the gifts of logic to computer science, perhaps the most important are the use of first-order logic (FO) for descriptions of computational domains, and propositional modal logic (ML) for descriptions of state transitions. The former has led to striking results, not only in database theory, finite model theory, and descriptive complexity, but also for knowledge representation in artificial intelligence. The latter has led to industrial applications, especially in the use of propositional temporal logics and epistemic logics for system specification and verification.

A natural invitation then is to combine the best of both, using first-order modal logic (FOML) for the description of state transition systems where states are given by first-order descriptions of computational domains. Such an idea is implicit in the realm of database updates, in the control of infinite state systems, networks with unbounded parallelism (such as in the context of mobile processes), and in the study of cryptographic protocols. In such contexts, there are unboundedly many data elements or processes or, in the case of security protocols, multi-sessions, and we wish to study how these change on applying some transition.
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Unfortunately, first-order modal logic seems to combine the worst of both computationally, losing also some good properties of first-order logics and propositional modal logics.

One of the significant achievements of the last century was mapping the terrain of decidable fragments of first-order logic ([4]). Propositional modal logics are shown to be robustly decidable ([13, 1]), with many extensions going beyond first-order expressiveness (such as transitive closure) yielding decidability (cf. e.g., [7]). On the other hand, it is hard to obtain decidable fragments of FOML, and the situation seems quite hopeless: even the two-variable fragment with one single unary predicate is (robustly) undecidable over almost all useful model classes [16].

Despite such discouragement, a small band of researchers have managed to identify some decidable fragments of FOML. Notable among them are the monodic fragments [9, 23], which requires that there be at most one free variable in the scope of any modal subformula. Combining the monodic restriction with a decidable fragment of FO we often obtain decidable fragments of FOML even with extra frame conditions. The principal idea here is that two variables occurring free in modal scope allow us to code up a binary relation, which leads to undecidability in the presence of quantifiers, and hence we restrict the use of such variables. Interestingly, this idea originally came from the study on decidable description logics in knowledge representation (cf. [10]).

This line of research has led to applications in temporal logics for infinite state systems ([5]), branching time temporal logics ([11]), epistemic logics ([2, 3]) and logics with counting quantifiers ([8]), and so on.

In studying decidable syntactic fragments of FO, we typically study restrictions on quantifier alternation, vocabulary or the number of variables in the formula. The use of equality and constants often brings some complexity as well. When it comes to guarded fragments [1], fluted fragments [15] and such, there are no restrictions on variables, quantification or vocabulary, but on quantification scope: quantification of a variable is not free, but is subject to its position in the syntax tree of the formula. Essentially, this can limit the power of the quantifiers, which is also the ‘secret of success’ of propositional modal logic computationally. These are reminiscent of ‘quantifier phrases’ in linguistics, where quantifiers are combined with predicates or connectives (as in ‘both’ or ‘all up’).

A recent idea of this kind was explored by Wang ([20]) who showed that when the existential quantifier and a box modality were always bundled together to appear as a single quantifier-modality pair ($\exists x \Box$), the resulting fragment of FOML enjoys the attractive properties of propositional modal logic: finite tree model property, PSPACE decision procedure, simple axiom system and so on, without any restriction on predicates or the occurrences of variables. It does not build on a decidable FO fragment explicitly, but exploits the distinct feature of FOML in capturing the interaction between quantifiers and modalities, as in Barcan formulas. Interestingly, similar to the origin of monodic fragments, the motivation again came from a formal treatment of knowledge. Wang used $\exists x \Box$ as a new modality where $\Box$ is the know-that operator to capture the logic structure behind the knowing-uh expressions such as knowing what, knowing how, knowing why, and so on (cf. [21]), e.g., knowing how to achieve $\varphi$ is rendered as there exists $x$ such that the agent knows that $x$ can guarantee $\varphi$ [22].

Encouraged by this, [14] took the next step, by considering not only the combination $\exists x \Box$ but also its companion $\forall x \Box$. They found that the logic with both of these combinations continued to be decidable over increasing domain models, though it was later shown that there was a price to be paid in terms of complexity. Such modal-quantifier combinations were thus called bundled modalities or simply bundles. Clearly we can define more bundles
such as □∃, □∀ etc., with the obvious semantics.

Beyond the epistemic context, bundled fragments also offers many interesting possibilities for system specification:
- ¬∃x□(x < c): No element is guaranteed to be bounded by constant c (after update).
- ∃x□∃y(x > y): There is an element that dominates some element after every update.
- □∃x□∀y(x ≤ y): All updates admit a local minimum.
- ∃x□(∃y□(x > y) ∧ ∃y□(x < y)): There is an element that dominates another no matter the update and is dominated by another no matter the update.

The story in [14] came with a twist: the computational property of many bundled fragments depends on whether you assume the domain of each world is constant or only expanding along the relation. In the former, a single domain is fixed for the entire model, with only predicate interpretations changing during system evolution. This is natural for many database applications. In the latter, the domain itself may be state-dependent. This is natural in the context of systems of processes where new sub-processes are created during system evolution, and in security theory, where new parallel sessions of security protocols are spawned. [14] showed that while the ∃□ bundle cannot distinguish between the two interpretations, even the fragment of ∀□ with only unary predicates is undecidable over constant domain models. For other decidable fragments such as the monodic fragment, there is a natural translation showing that for any formula over a varying domain model, there is an equisatisfiable formula over a constant domain model. Unfortunately, such a translation does not preserve expressibility in bundled fragments.

All this opens up a range of questions: what about other bundles such as □∃x or □∀x and combinations thereof? Which of these distinguish constant domain and varying domain models? What about further bundles such as ∀x∃□? Can we identify the border line between decidability and undecidability in this terrain?

This is the project taken up in this paper, and what we present is a trichotomy on the decidability of bundled fragments: decidable ones, undecidable ones, and for those without definite answer yet, we show they lack the finite model property. Moreover, we present the loosely bundled fragment that generalizes the bundling idea to what we believe to be the strongest yet decidable bundled fragment.

The results are presented in Fig. 1 ordered by the number of various bundles. A * in the figure means that the result holds with or without the presence of the corresponding bundle. Note that not all of the cases are listed explicitly in the table: some cases are covered by others, e.g., the decidability of the fragment with ∀□ and □∀ follows from the case with an extra □∃. It is to be noted that constant domain and increasing domain interpretations make a significant difference. Where the logics are decidable, we present a tableau-based decision procedure. Proofs of undecidability involve coding of tiling problems, and lack of finite model property is shown by forcing infinite domains. The details are tricky, but technically interesting, combining many techniques that arise from the study of first-order logics and from modal logics. Specifically, as we will see in Section 5.2, for some combinations of bundles we can pull ∃ outside the scope of ∀ thereby allowing us to search for witnesses that work across worlds, using realised types.

The paper is structured as follows. After presenting the various fragments in the next section, we proceed from bad news to good news, presenting undecidability results, lack of finite model property and then decidability.

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1 Making the domain vary arbitrarily across the worlds causes problems defining proper semantics for quantifiers cf. [12]
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| Domain | ∀□ | ∃□ | □√ | □∃ | Upper/ Lower Bound |
|--------|-----|-----|-----|-----|-------------------|
| Constant | ✓ | * | ✓ | * | Undecidable |
| | * | * | ✓ | * | PSPACE-complete |
| | ✓ | ✓ | x | x | No FMP |
| | x | x | x | ✓ | |
| | x | ✓ | x | ✓ | |

| Increasing | ✓ | ✓ | ✓ | x | PSPACE-complete |
|            | x | ✓ | x | x | ExpSpace/ PSpace |
|            | x | x | ✓ | ✓ | ExpSpace/ NExpTime |
|            | ✓ | x | x | ✓ | Undecidable |
|            | x | ✓ | x | ✓ | No FMP |
|            | ✓ | ✓ | ✓ | ✓ | ExpSpace/ NExpTime |

Figure 1 Satisfiability problem classification for combinations of bundled fragments. The results are new in this paper, except those summarized in Theorem 1 below.

2 Syntax and Semantics

The syntax of First Order Modal logic is given by extending the first order logic with modal operators. Note that we exclude equality, constants and function symbols from the syntax.

► Definition 1 (FOML syntax). Given a countable set of predicates \( P \) and a countable set of variables \( \text{Var} \), the syntax of FOML is given by:

\[
\alpha ::= P(x_1, \ldots, x_n) \mid \neg \alpha \mid \alpha \land \alpha \mid \exists x \cdot \alpha \mid \Box \alpha
\]

where \( P \in \mathcal{P} \) has arity \( n \) and \( x, x_1, \ldots, x_n \in \text{Var} \).

The boolean connectives \( \lor, \rightarrow, \leftrightarrow \), and the modal operator \( \Diamond \) which is the dual of \( \Box \) and the quantifier \( \forall \) are all defined in the standard way. The notion of free variables, denoted by \( \text{FV}(\alpha) \) is similar to what we have for first order logic with \( \text{FV}(\Box \alpha) = \text{FV}(\alpha) \). We write \( \alpha(x) \) to mean that \( x \) occurs as a free variable of \( \alpha \). Also, \( \alpha[y/x] \) denotes the formula obtained from \( \alpha \) by replacing every free occurrence of \( x \) by \( y \).

► Definition 2 (FOML structure). An increasing domain model for FOML is a tuple \( \mathcal{M} = (W, D, \delta, \mathcal{R}, \rho) \) where, \( W \) is a non-empty countable set called worlds\(^2\), \( D \) is a non-empty countable set called domain; \( \mathcal{R} \subseteq (W \times W) \) is the accessibility relation. The map \( \delta : W \mapsto 2^D \) assigns to each \( w \in W \) a non-empty local domain set such that whenever \( (w, v) \in \mathcal{R} \) we have \( \delta(w) \subseteq \delta(v) \) and \( \rho : (W \times \mathcal{P}) \mapsto \bigcup 2^D \) is the valuation function which specifies the interpretation of predicates at every world over the local domain with appropriate arity.

\(^2\) Note that FOML can be translated into two-sorted FO, and due to the Löwenheim–Skolem theorem for countable languages, every model has an equivalent countable model, cf. [5].
The monotonicity condition is useful for evaluating the free variables present in the formula \([12]\). These models are called increasing domain models. The model \(\mathcal{M}\) is said to be a constant domain model if for all \(w \in W\) we have \(\delta(w) = D\).

For a given model \(\mathcal{M}\) we denote \(W^\mathcal{M}, R^\mathcal{M}\) etc. to refer to the corresponding components. We simply use \(W, R, \delta\) etc. when \(\mathcal{M}\) is clear from the context.

To evaluate formulas, we need an assignment function for variables. For a given model \(\mathcal{M}\), an assignment function \(\sigma : \text{Var} \rightarrow \mathcal{D}\) is relevant at \(w \in W\) if \(\sigma(x) \in \delta(w)\) for all \(x \in \text{Var}\).

\begin{definition}[FOML semantics] Given an FOML model \(\mathcal{M} = (W, D, \delta, R, \rho)\) and \(w \in W\), and \(\sigma\) relevant at \(w\), for all FOML formula \(\alpha\) define \(\mathcal{M}, w, \sigma \models \alpha\) inductively as follows:

\[
\begin{align*}
\mathcal{M}, w, \sigma \models P(x_1, \ldots, x_n) & \iff (\sigma(x_1), \ldots, \sigma(x_n)) \in \rho(w, P) \\
\mathcal{M}, w, \sigma \models \neg \alpha & \iff \mathcal{M}, w, \sigma \not\models \alpha \\
\mathcal{M}, w, \sigma \models (\alpha \land \beta) & \iff \mathcal{M}, w, \sigma \models \alpha \text{ and } \mathcal{M}, w, \sigma \models \beta \\
\mathcal{M}, w, \sigma \models \exists x \alpha & \iff \text{there is some } d \in \delta(w) \text{ such that } M, w, \sigma[d\rightarrow d] \models \alpha \\
\mathcal{M}, w, \sigma \models \Box \alpha & \iff \text{for every } u \in W if (w, u) \in R \text{ then } M, u, \sigma \models \alpha
\end{align*}
\]

We sometimes write \(\mathcal{M}, w \models \alpha(a)\) to mean \(\mathcal{M}, w, [x \mapsto a] \models \alpha(x)\).

A formula \(\alpha\) is satisfiable if there is some FOML structure \(\mathcal{M}\) and \(w \in W\) and some assignment \(\sigma\) relevant at \(w\) such that \(\mathcal{M}, w, \sigma \models \alpha\). In the sequel, we will only talk about the relevant \(\sigma\) given a pointed model. A formula \(\alpha\) is valid if \(\neg \alpha\) is not satisfiable.

\subsection{Bundled fragments}

The motivation of bundling is to restrict the occurrences of quantifiers using modalities. For instance allowing only formulas of the form \(\forall x \Box \alpha\) is one such bundling. We could also have \(\Diamond \exists y \alpha\). Thus, there are many ways to ‘bundle’ the quantifiers and modalities. We call these the ‘bundled operators’. The following syntax defines all possible bundled operators of one quantifier and one modality:

\begin{definition}[Bundled-FOML syntax] The bundled fragment of FOML is the set of all formulas constructed by the following syntax:

\[
\alpha ::= P(x_1, \ldots, x_n) | \neg \alpha | \alpha \land \beta | \Box \alpha | \exists x \Box \alpha | \forall x \forall \alpha | \forall \forall \alpha
\]

where \(P \in \mathcal{P}\) has arity \(n\) and \(x, x_1, \ldots, x_n \in \text{Var}\).
\end{definition}

Note that the dual of the bundled operators will give us the formulas of the form \(\forall x \Diamond \alpha\), \(\exists x \Diamond \alpha\), \(\Diamond \forall \alpha\), \(\Diamond \exists \alpha\). Also, note that \(\Box \exists \alpha\) can be defined using any one of the bundled operator where the quantifier is applied to a variable that does not occur in \(\alpha\). However, we retain \(\Box \alpha\) in the syntax for technical convenience.

The following constant domain models may help to get familiar with bundles.

\[
\begin{align*}
\mathcal{M}_1 & \quad \mathcal{M}_2 & \quad \mathcal{M}_3 \\
\begin{array}{c}
w_1 \quad v_1 : P a \\
u_1 : P b
\end{array} & \quad \begin{array}{c}
w_2 \quad v_2 : P c \\
u_2
\end{array} & \quad \begin{array}{c}
w_3 \quad v_3 : P c
\end{array}
\end{align*}
\]

where \(D^{\mathcal{M}_1} = \{a, b\}\), \(D^{\mathcal{M}_2} = D^{\mathcal{M}_3} = \{c\}\). \(\Box \exists x P x\) holds at \(w_1\) and \(w_3\) but not at \(w_2\); \(\exists x \Diamond P x\) holds only at \(w_3\); \(\neg \forall x \Box P x\) holds at \(w_1\) and \(w_2\); \(\neg \Box \forall x \neg P x\) holds at all the \(w_i\).

We denote \(\mathbb{AB}\) (to mean forAll-Box) to be the language that allows only atomic predicates, negation, conjunction, \(\Box \alpha\) and \(\forall x \Box \alpha\) (dually \(\exists x \Diamond \alpha\)) formulas. Similarly we have
Definition 6 (LBF syntax). The loosely bundled fragment of FOML is the set of all formulas constructed by the following syntax:

\[ \psi ::= P(z_1, \ldots, z_n) \mid \neg P(z_1, \ldots, z_n) \mid \psi \land \psi \mid \psi \lor \psi \mid \Box \alpha \mid \Diamond \alpha \]

\[ \alpha ::= \psi \mid \alpha \land \alpha \mid \alpha \lor \alpha \mid \exists x_1 \ldots \exists x_k \forall y_1 \ldots \forall y_l \psi \]

where \( k, l, n \geq 0 \) and \( P \in \mathcal{P} \) has arity \( n \) and \( x_1, \ldots, x_k, y_1, \ldots, y_l, z_1, \ldots, z_n \in \text{Var} \).

We let LBF be the set of all formulas that can be obtained from the grammar of \( \alpha \) above.

Another advantage of the loosely bundled fragment is that some combination of bundled fragments can be embedded into LBF. Hence proving the decidability for LBF implies the decidability for these combinations as well.

Proposition 7. The fragments AEBE and BABE are subfragments of LBF.

The naming follows the convention that every pair of letters correspond to a bundled operator and we follow the precedence \( AB > EB > BA > BE \). That is, if the first two letters are not \( AB \) then the fragment does not include \( \forall x \Box \) as a bundle and so on.
2.2 Trichotomy

The goal of this paper is to classify the decidability status of the satisfiability problem for the various combinations of the bundled fragments. As discussed before, decidability depends on whether we are considering increasing domain models or constant domain models.

We consider all possible combinations and obtain a trichotomy classification. We prove for any combination of bundles, that it is undecidable, or that it admits a tableau based decision procedure or that it lacks the finite model property. The last cases do not give us any (un)decidability result, but demonstrate that the tableau method based on finite model property does not work.

The key idea is that in our setting a fragment can be proved to be undecidable if it can assert $\forall x \exists y \Box \alpha$ and $\forall x \Box \forall y \forall z \alpha$. As we will see, the first property is used in the tiling encoding to assert that every ‘grid point’ $x$ has a horizontal/vertical successor $y$. It is important for both quantifiers to be applicable over the same local domain. Moreover $\Box \alpha$ in $\forall x \exists y \Box \alpha$ ensures that the witness $y$ acts uniformly across all the descendants. The second formula $\forall x \Box \forall y \Box \forall z \alpha$ is used to verify the ‘diagonal property’ of the grid. These formulas can be asserted in the fragments $EBBA$ and $ABEBBE$ and consequently are shown to be undecidable over increasing domain models.

If a fragment can express $\forall x \exists y \Box \alpha$ but not $\forall x \Box \forall y \forall z \alpha$ then we will prove that such fragments do not have finite model property. For instance the fragment $EBBE$ has this property. Also note that over constant domain models, the fragment $BE$ can assert $\forall x \exists y \Box \alpha$ in the form $\Diamond \forall x \Box \exists y \Box \alpha$. In this case, even though the two quantifiers are applied at different worlds, since the local domain is same across all worlds, the formula will serve the same purpose as intended. We will prove that these fragments do not have the finite model property.

Finally, if a fragment cannot express $\forall x \exists y \Box \alpha$ then we will prove that it satisfies finite model property and give a tableau procedure. These fragments include $ABEBBA$ and $LBF$.

3 Undecidable fragments

In [14] the authors prove that the $AB$ fragment over constant domain models is undecidable. In [13] it is proved that a similar undecidability reduction holds for the $BA$ fragment over constant domain models. In this section, we consider the undecidable bundled fragments over increasing domain models. These are the fragments in which we can express both $\forall x \exists y \Box \alpha$ and $\forall x \Box \forall y \Box \forall z \alpha$ which include $EBBA$ and $ABEBBE$ fragments.

For the proof we use the tiling problem over the first quadrant. Given a tiling instance $\mathcal{T} = (T, H, V, t_0)$ where $T$ is a finite set of tiles and $t_0 \in T$ and $H, V \subseteq T \times T$ are the horizontal and vertical constraints respectively, a mapping $f : (\mathbb{N} \times \mathbb{N}) \to T$ is called a proper tiling if $f(0, 0) = t_0$ and for all $i, j \in \mathbb{N}$ if $f(i, j) = t$ and $f(i, j + 1) = t'$ then $(t, t') \in V$ and similarly if $f(i, j) = t$ and $f(i + 1, j) = t'$ then $(t, t') \in H$. The problem, to decide whether an input tiling instance $\mathcal{T}$ has a proper tiling is undecidable [17].

In the encoding, the idea is to interpret domain elements as grid points. We use two binary predicates $P$ and $Q$ to identify the horizontal (going right) and vertical successors (going up) respectively. Also, we abuse notation and consider every $t \in T$ as a unary predicate so that $t(x)$ means the ‘grid point’ $x$ is tiled with $t$. 
3.1 EBBA over increasing domain

This fragment allows formulas of the form $\exists x \Box \alpha$ and $\Box \forall x \alpha$ (and their duals). Hence we can express $\forall x \exists y \Box \alpha$ and $\Box \forall x \forall y \forall z \phi$ in the form of $\Box \forall x \exists y \Box \alpha$ and $\Box \forall x \forall y \forall z \phi$ respectively. For a given a tiling instance $T$ we construct an EBBA sentence $\varphi_T$ such that there exists a proper tiling of $T$ iff $\varphi_T$ is satisfiable in an increasing domain model. We first define some notations for short-hand reference:

| Formula | Definition |
|---------|------------|
| $\text{Only}_T(t, x)$ | $t(x) \land \bigwedge_{t' \neq t} \neg t'(x)$ |
| $H_{\text{suc}}(x, y)$ | $P(x, y) \rightarrow \bigvee_{(t, t') \in H} (t(x) \land t'(y))$ |
| $V_{\text{suc}}(x, y)$ | $Q(x, y) \rightarrow \bigvee_{(t, t') \in V} (t(x) \land t'(y))$ |
| $\Delta^0$ | $\top$ and $\Delta^n = \Diamond \top \land \Box (\Delta^{n-1})$ |

**Figure 2** Short hand formulas

The formula $\text{Only}_T(t, x)$ asserts that grid point $x$ is tiled with exactly $t$ and no other tile. The formulas $H_{\text{suc}}(x, y)$ and $V_{\text{suc}}(x, y)$ ensure that the horizontal and vertical successors satisfy the tiling constraints respectively. Also, $\Delta^n$ ensures that there is at least one path of length $n$ starting from the current world and all paths starting from the current world can be extended to length at least $n$.

For better readability, we sometimes drop the brackets in the predicates $P(x, y)$ and $Q(x, y)$ and write $P_{xy}$ and $Q_{xy}$ respectively. The tiling encoding formulas are described in Fig. 3.

| Formula | Definition |
|---------|------------|
| $\alpha_0$ | $\Diamond \exists x_0 \{ \Box \Box (\text{Only}_T(t_0, x_0)) \} \land \Box \forall x \{ \bigvee_{t \in T} \Box \Box \text{Only}_T(t, x) \}$ |
| $\alpha_H$ | $\Box \forall x \{ \exists x_1 \Box \Box P_{xx_1} \}$ |
| $\alpha_V$ | $\Box \forall x \{ \exists x_2 \Box \Box Q_{xx_2} \}$ |
| $\alpha_{H, s}$ | $\Box \forall x \{ \Box \forall y \Box H_{\text{suc}}(x, y) \}$ |
| $\alpha_{V, s}$ | $\Box \forall x \{ \Box \forall y \Box V_{\text{suc}}(x, y) \}$ |
| $\varphi_H$ | $\Box \forall x \{ \Box \forall y (\Diamond P_{xy} \leftrightarrow \Box P_{xy}) \}$ |
| $\varphi_V$ | $\Box \forall x \{ \Box \forall y (\Diamond Q_{xy} \leftrightarrow \Box Q_{xy}) \}$ |
| $\psi$ | $\Box \forall x \{ \Box \forall y (\exists z' (Q_{xz' \land P_{z'y}}) \rightarrow [\Box \forall z (P_{xz} \rightarrow Q_{zy})]) \}$ |

**Figure 3** EBBA formulas for encoding the tiling instance over increasing domain models

The formula $\alpha_0$ asserts that there is a grid point which is tiled with $t_0$ and every grid point has a unique tile (and $\Delta^3$ ensures that there are sufficiently many descendants). Further,
\( \alpha_H \) and \( \alpha_V \) assert that every grid point has a horizontal and vertical successor respectively. Formulas \( \alpha_{Hs} \) and \( \alpha_{Vs} \) ensure that the horizontal and vertical constraints of the input tiling instance are satisfied.

The formulas \( \varphi_H \) and \( \varphi_V \) ensure that the horizontal and vertical successor information is uniform across all descendants respectively. Finally \( \psi \) states the grid points satisfy the diagonal property.

For a given tiling instance \( T \), the corresponding formula \( \alpha_T \) is given by the conjunction of the above formulas.

**Theorem 8.** For a given tiling instance \( T \), there exists a proper tiling of \( T \) over \( \mathbb{N} \times \mathbb{N} \) iff \( \alpha_T \) is satisfiable in an increasing domain model.

**Proof.** (\( \Rightarrow \)) Suppose \( f : (\mathbb{N} \times \mathbb{N}) \rightarrow T \) is the tiling function. Then define the model \( M = (\mathcal{W}, D, \mathcal{R}, \rho) \) where

- \( \mathcal{W} = \{w_0, w_1, w_2, w_3\} \)
- \( D = \mathbb{N} \times \mathbb{N} \)
- \( \mathcal{R} = \{(w_k, w_{k+1}) \mid 0 \leq k < 3\} \)
- Define \( \rho \) as follows (for all \( k \leq 3 \)):
  \[
  \rho(w_k, P) = \left\{ i, j \in \mathbb{N} \mid (i, j), (i + 1, j) \right\}
  \]
  \[
  \rho(w_k, Q) = \left\{ i, j \in \mathbb{N} \mid (i, j), (i, j + 1) \right\}
  \]

For every \( t \in T \) define \( \rho(w_k, t) = \{(i, j) \mid f(i, j) = t\} \)

Valuation is important only at \( w_3 \) (valuations at other worlds are irrelevant). It can be verified that \( M, w_0 \models \alpha_T \).

(\( \Leftarrow \)) Suppose \( M, r \models \alpha_T \). We fix an enumeration of the elements in the domain of \( M \) since it is countable (cf. footnote\[2\]). Below, every time we pick a domain element satisfying a property, we pick the one that is least in the enumeration of domain elements satisfying that property.

Let \( \alpha_0, \alpha'_H, \alpha'_V, \alpha'_{Hs}, \alpha'_{Vs}, \varphi_H', \varphi_V' \) be the respective formulas where the outermost modality is removed. Then by \( \alpha_0 \) there is some \( r \rightarrow u \) such that \( M, u \models \alpha'_0 \wedge \alpha'_H \wedge \alpha'_V \wedge \alpha'_{Hs} \wedge \alpha'_{Vs} \wedge \varphi_H' \wedge \varphi_V' \wedge \psi' \).

Define a mapping \( g : (\mathbb{N} \times \mathbb{N}) \rightarrow \delta(u) \) by induction as follows: let \( d_0 \in \delta(u) \) such that \( M, u \models \Box \exists \text{Only}T(t_0, d_0) \). Define \( g(0, 0) = d_0 \).

For all \( j > 0 \) if \( g(0, j - 1) = c \) then let \( d \in \delta(u) \) be a witness for \( x_2 \) with the assignment \( x \mapsto c \) in \( \alpha_V \). So, we have \( M, u \models \Box \exists \text{Pac} \). Define \( g(0, j) = d \).

For all \( i, j > 0 \) if \( g(i - 1, j) = c \) then let \( d \in \delta(u) \) be a witness for \( x_1 \) with the assignment \( x \mapsto c \) in \( \alpha_H \). So, we have \( M, u \models \Box \exists \text{Pac} \). Define \( g(i, j) = d \).

Intuitively, we induce a grid over \( \delta(u) \), by first building the \( y \)-axis of the first quadrant and then build horizontal lines fixing each \( y \)-coordinate in such a way that the vertically adjacent elements are also connected to satisfy the ‘grid property’.

Note that for all \( i, j \) if \( g(i, j) = c \) and \( g(i + 1, j) = c' \) then by definition, we have \( M, u \models \Box \exists \text{Pac} \). First we prove that if \( g(i, j) = c \) and \( g(i, j + 1) = d \) then we have \( M, u \models \Box \exists \text{Pac} \). This is proved by induction on \( i \).

In the base case, \( i = 0 \) and if \( g(0, j) = c \) and \( g(0, j + 1) = d \) then by construction we have \( M, u \models \Box \exists \text{Qcd} \). Now, for the induction step, consider some \( i > 0 \) and let \( g(i, j) = c \) and \( g(i, j + 1) = d \). Let \( g(i - 1, j) = a \) and \( g(i - 1, j + 1) = b \). By induction hypothesis, \( M, u \models \Box \exists Qab \). By construction also have \( M, u \models \Box \exists Pac \) and \( M, u \models \Box \exists Pbd \).
We will prove that $\mathcal{M}, u \models □□Qcd$. For this, pick any successor $u \to v$. We will prove that $\mathcal{M}, v \models □Qcd$. Note that we have $\mathcal{M}, v \models □Pac$ and $\mathcal{M}, v \models □Pbd$ and $\mathcal{M}, v \models □Qab$.

Now since $\mathcal{M}, v \models △¹$, there is some $v \to w$ and we have $\mathcal{M}, w \models Pac \land Pbd \land Qab$. Further, by $ψ$ (assigning $x$ to $a$): $\mathcal{M}, v \models ∀y{[[∃z'(Qaz' ∧ Pz'y)]} → [□∀z(Paz \to Qzy)]\]$. By assigning $y$ to $d$ we have $\mathcal{M}, v \models [∃z'(Qaz' ∧ Pz'd)] → [□∀z(Paz \to Qzd)]$.

But note that we have $\mathcal{M}, v \models □∀z(Paz \to Qzd)$. Since $v \to w$, we have $\mathcal{M}, w \models ∀z(Paz \to Qzd)$. Now, by assigning $z$ to $c$ we have $\mathcal{M}, w \models Pac$ and hence we have $\mathcal{M}, w \models Qcd$. This implies that $\mathcal{M}, v \models □Qcd$. Now by $ϕ_V$ we have $\mathcal{M}, v \models □Qcd \iff □Qcd$. Hence $\mathcal{M}, v \models □Qcd$ as required.

The tiling function $f : (\mathbb{N} \times \mathbb{N}) \to T$ is defined as follows: For every $(i, j) \in (\mathbb{N} \times \mathbb{N})$, suppose $g(i, j) = d$ then since $\mathcal{M}, u \models ∀x{[[ ∨_{t \in T} □□OnlyT(t, x)]}$ there is some unique $t \in T$ such that $\mathcal{M}, u \models □□T(t)$. Define $f(i, j) = t$.

To prove that $f$ is a proper tiling function, first note that with $g(i, j) = d₀$ we have $f(0, 0) = t₀$ since $\mathcal{M}, u \models □□OnlyT(t₀, d₀)$.

Now suppose that $g(i, j) = c$ and $f(i, j) = t$. We verify that $f$ satisfies horizontal and vertical tiling constraints.

Let $f(i + 1, j) = t'$ and $g(i + 1, j) = d$ be the horizontal successor. By construction, $\mathcal{M}, u \models □□Pcd$ and $\mathcal{M}, u \models □□(t(c) \land t'(d))$. Since $\mathcal{M}, u \models △²$, there is some $u \to v \to w$, and hence we have $\mathcal{M}, w \models Pcd \land t(c) \land t'(d)$. But then by $α_H$, we have $\mathcal{M}, u \models □□Hsucc(c, d)$ and hence $(t, t') ∈ H$.

Similarly for vertical constraints, suppose $g(i, j + 1) = d$ and $f(i, j + 1) = t'$ then note that we have already proved that in this case $\mathcal{M}, u \models □□Qcd$. Again, we have $\mathcal{M}, u \models □□(t(c) \land t'(d))$. Since $\mathcal{M}, u \models △²$, there is some $u \to v \to w$ and hence we have $\mathcal{M}, w \models Qcd \land t(c) \land t'(d)$. But then by $α_V$, we have $\mathcal{M}, v \models □□Vsuc(c, d)$ and hence $(t, t') ∈ V$. ◀

**Corollary 9.** Let $\mathcal{L}$ be any fragment of FOML such that $\textit{EBBB} ⊆ \mathcal{L}$. Then the satisfiability problem for $\mathcal{L}$ over increasing domain models is undecidable.

### 3.2 ABEBBE over increasing domain

In this fragment we have the bundled operators $∀x □ α$, $∃x □ α$ and $∀x □α$ (and their duals). Hence, in this case also we can express $∀x ∃y □α$ and $∀x □∀y □∀zα$ as $□∀x ∃y □α$ and $∀x □∀y □∀zα$ respectively. Note that the former formula is a combination of BE and EB formulas and the later is an AB formula.

The tiling encoding formulas are described in Fig. 3 which are the modification of the formulas in Fig. 4 to suit the ABEBBE fragment.

For a given tiling instance $\mathcal{T}$, the corresponding formula $\hat{α}_\mathcal{T}$ is given by the conjunction of the above formulas.

**Theorem 10.** The satisfiability problem for $\textit{ABEBBE}$ fragment over increasing domain models is undecidable.

**Proof.** ($⇒$) It can be verified that $\mathcal{M}, w_0 \models \hat{α}_\mathcal{T}$ where $\mathcal{M}$ is the model described in the proof of Theorem 8.

($⇐$) Suppose $M, r \models \hat{α}_\mathcal{T}$. Then there is some $r \to u$ such that
we can argue that $\forall \exists x_0 \{ \square \square (\text{OnlyT}(t_0, x_0)) \} \land \Delta^2 T$

\begin{align*}
\hat{\alpha}_0 &:= \bigvee_{t \in T} \square \square \text{OnlyT}(t, x) \land \\
\hat{\alpha}_H &:= \exists x_1 \square \square Pxx_1 \land \\
\hat{\alpha}_V &:= \exists x_2 \square \square Qxx_2 \land \\
\hat{\phi}_H &:= \forall y \square \square \text{Hsuc}(x, y) \land \\
\hat{\phi}_V &:= \forall y \square \square \text{Vsuc}(x, y) \land \\
\psi &:= \forall y \square \square (\exists z \square (Qxxz \land Pz'_y) \rightarrow [\forall z \square (Pxxz \rightarrow Qyz)])
\end{align*}

\textbf{Figure 4} ABEBBE formulas to encoding the tiling instance over increasing domain models

Now define a mapping $g : (\mathbb{N} \times \mathbb{N}) \rightarrow \delta(u)$ by induction as follows: let $d_0 \in \delta(u)$ such that $\mathcal{M}, u \models \square \square \text{OnlyT}(t_0, d_0)$. Define $g(0, 0) = d_0$.

For all $j > 0$ if $g(0, j - 1) = c$ then let $d \in \delta(u)$ be a witness for $x_2$ with the assignment $x \mapsto c$ in $\hat{\alpha}_H$. So, we have $\mathcal{M}, u \models \square \square \text{Pcd}$. Define $g(0, j) = d$. For all $i, j > 0$ if $g(i - 1, j) = c$ then let $d \in \delta(u)$ be a witness for $x_1$ with the assignment $x \mapsto c$ in $\alpha_H$. So, we have $\mathcal{M}, u \models \square \square \text{Pcd}$. Define $g(i, j) = d$.

Note that for all $i, j$ if $g(i, j) = c$ and $g(i + 1, j) = c'$ then we have $\mathcal{M}, u \models \square \square \text{Pcc}'$. Now we claim that if $g(i, j + 1) = d$ then using the same reasoning as in the proof of Theorem we can argue that $\mathcal{M}, u \models \square \square \text{Pcd}$.

Now define the tiling function $f : (\mathbb{N} \times \mathbb{N}) \rightarrow T$ where for every $(i, j) \in (\mathbb{N} \times \mathbb{N})$, suppose $g(i, j) = d$ then since $\mathcal{M}, u \models \forall x \{ \bigvee_{t \in T} \square \square \text{OnlyT}(t, x) \}$ there is some unique $t \in T$ such that $\mathcal{M}, u \models \square \square t(d)$. Define $f(i, j) = t$.

Again, using the same arguments as in the proof of Theorem we can verify that $f$ is indeed a proper tiling function.
4 Fragments without Finite Model Property

In this section we consider the fragments that lack Finite Model Property, or rather, that they can force models with infinite domains. Of course, this does not necessarily mean that the satisfiability problem for these fragments is undecidable but any strategy to prove decidability based on finite model property will fail. Abstractly speaking, these are the fragments in which we can express $\forall x \exists y \square \alpha$ but we cannot express $\forall x \square \exists y \forall z \alpha$.

The inability to express $\forall x \square \exists y \forall z \alpha$ means that we cannot assert the ‘grid property’ needed for the tiling encoding and consequently we cannot prove undecidability. On the other hand, using formulas of the form $\forall x \exists y \square \forall z \alpha$, we can induce an irreflexive-transitive ordering over the local domain at the world where the formula is satisfied. Then, stating that such an ordering does not have a maximal element will imply that the domain of the model has to be infinite.

To encode the ordering we use a binary predicate $P$, with the intended meaning of $Pxy$ to mean $x < y$ in the ordering being defined.

4.1 EBBE over increasing domain

In this fragment we are allowed $\exists x \square \alpha$ and $\square \exists x \alpha$ (and their duals) in the syntax. Hence we can assert $\forall x \exists y \square \alpha$ in the form of $\square \forall x \exists y \square \alpha$. But we cannot express $\forall x \square \exists y \forall z \alpha$. This is because every $\forall$ operator needs to necessarily have a $\square$ either before or after its occurrence.

We prove that this fragment lacks the finite model property. The encoding formula is given as follows:

$$\varphi_1 := \diamond \forall x \left[ \exists y \square \square Pxy \land \square \square \lnot Pxx \land \diamond \forall y \left[ \left( \diamond Pxy \leftrightarrow \square \square Pxy \right) \land \diamond \forall z \left( \left( Pxy \land Pyz \right) \rightarrow \left( Pxz \right) \right) \right] \right]$$

The first conjunct asserts that every $x$ has a successor $y$ and the second conjunct asserts that the binary predicate $P$ is irreflexive. The last conjunct asserts that $P$ is transitive and the one before ensures that the relation $Pxy$ holds uniformly across all branches.

**Theorem 11.** The formula $\varphi_1$ is satisfiable in a model with infinite $D$. Moreover, for all increasing domain model $\mathcal{M}$ and $w \in W$ if $\mathcal{M}, w \models \varphi_1$ then there is some $w \rightarrow u$ such that $\delta(u)$ is infinite.

**Proof.** Consider the model $\mathcal{M} = (W, N, R, \delta, \rho)$ where:

$W = \{w_0, w_1, w_2\}$ and $R = \{(w_j, w_{j+1}) \mid j < 2\}$. For all $j \leq 2$ define $\delta(w_j) = N$ and $\rho(w_j, P) = \{(k, l) \mid k, l \in N$ and $k < l\}$. Note that only $\rho(w_2, P)$ is relevant. It can be verified that $\mathcal{M}, w_0 \models \varphi_1$.

Now assume that $\mathcal{M}, r \models \varphi_1$. Then there exists some $r \rightarrow u$ such that $\mathcal{M}, u \models \forall x \left[ \exists y \square \square Pxy \land \square \square \lnot Pxx \land \diamond \forall y \left[ \left( \diamond Pxy \leftrightarrow \square \square Pxy \right) \land \diamond \forall z \left( \left( Pxy \land Pyz \right) \rightarrow \left( Pxz \right) \right) \right] \right]$. We will prove that $\delta(u)$ is infinite. Towards this we will construct an infinite sequence of distinct domain elements $d_0, d_1, \ldots \rightarrow \delta(u)$. The sequence is constructed by induction such that for all $i$:

$\mathcal{M}, u, [x \rightarrow d_i, y \rightarrow d_{i+1}] \models \square \square Pxy$.

In the base case let since $\delta(u)$ is non-empty, pick some arbitrary $d_0 \in \delta(u)$.

For the induction step, suppose we have constructed $d_0, \ldots, d_i$. Let $d_{i+1}$ be such that $\mathcal{M}, u, [x \rightarrow d_i, y \rightarrow d_{i+1}] \models \square \square Pxy$ (such an element exists since $\mathcal{M}, u \models \forall x \exists y \square \square Pxy$).
We claim that \(d_{i+1}\) is distinct from \(d_0 \ldots d_i\). Suppose not; then let \(d_{i+1} = d_k\) for some \(k \leq i\). Let \(k + l = i\) for some \(l \geq 0\).

Now, with the assignment \(x \mapsto d_i\), we have
\[
\mathcal{M}, u \models \forall y \left( \left[ \exists P_{d_iy} \leftrightarrow \Box P_{d_iy} \right] \land \exists z \left( \left[ \left( \exists P_{d_iy} \land P_{g_{d_i}z} \right) \rightarrow \left( P_{d_iz} \right) \right] \right) \right)
\]

Hence there is some \(u \rightarrow w\) such that
\[
\mathcal{M}, w \models \forall y \left( \left[ \exists P_{d_iy} \leftrightarrow \Box P_{d_iy} \right] \land \exists z \left( \left[ \left( \exists P_{d_iy} \land P_{g_{d_i}z} \right) \rightarrow \left( P_{d_iz} \right) \right] \right) \right)
\]

By induction hypothesis, \(\mathcal{M}, w \models \Box P_{d_j d_{j+1}}\) for all \(j \leq i\) and by assumption we have \(\mathcal{M}, w \models \Box P_{d_j d_{k+1}}\) (since \(d_{i+1} = d_k\)).

Now with the assignment \(y \mapsto d_k\) there is some \(w \rightarrow w^k\) such that \(\mathcal{M}, w^k \models \forall z \left( P_{d^k d_{k+1}} \land P_{d^k z} \rightarrow P_{d^k z} \right)\). In particular with \(z \mapsto d_{k+1}\), we obtain \(\mathcal{M}, w^k \models P_{d^k d_{k+1}}\). This implies that \(\mathcal{M}, w \models \Box P_{d_j d_{k+1}}\).

Now since \(\mathcal{M}, w \models \forall y \left( \exists P_{d_j y} \leftrightarrow \Box P_{d_j y} \right)\), we have \(\mathcal{M}, w \models \Box P_{d_j d_{k+1}}\). But again, with \(y \mapsto d_{k+1}\) there is some \(w \rightarrow w^{k+1}\) be such that \(\mathcal{M}, w^{k+1} \models \forall z \left( P_{d^k d_{k+1}} \land P_{d^{k+1} z} \rightarrow P_{d^{k+1} z} \right)\). In particular with \(z \mapsto d_{k+2}\), we have \(\mathcal{M}, w^{(k+1)} \models P_{d^k d_{k+2}}\). This implies that \(\mathcal{M}, w \models \Box P_{d_j d_{k+2}}\) from which we can conclude that \(\mathcal{M}, w \models \Box P_{d_j d_{k+1}}\).

Applying this argument \(l\) times, we get \(\mathcal{M}, w \models \Box P_{d_j d_{k+1}}\). Hence we have \(\mathcal{M}, w \models \Box P_{d_j d_{k+1}}\) which contradicts to \(\mathcal{M}, u \models \forall x \Box \Box \Box P_{x x}\).

### 4.2 \(\forall \exists \Box\) fragment over increasing domain

Consider any fragment that can express \(\forall x \exists y \Box \alpha\) formulas. We prove that such a fragment does not have finite model property. For instance, any extension of the quantifier prefix of LBF fragment will allow us to express \(\forall x \exists y \Box \alpha\) formulas. Similarly the negation closed extension of LBF also allows us to express \(\forall x \exists y \Box \alpha\).

Note that the ability to express \(\forall x \exists y \Box \alpha\) implies that we can also write formulas of the form \(\forall \alpha \Box \alpha\) (by making \(y\) as a dummy variable) and also \(\Box \alpha\) (by making both \(x, y\) as dummy variables). The encoding formula is given as follows:

\[
\varphi_2 := \forall x \exists y \Box \exists \Box \Box P_{x y} \land \forall x \exists y \exists z \Box P_{x y} \land \Box \Box \Box \Box \Box \Box \Box
\]

The formula \(\varphi_2\) is essentially the same as \(\varphi_1\) only modified to ensure that it can be written using only \(\forall x \exists y \Box \alpha\) formulas.

\(\blacktriangleleft \) **Theorem 12.** The formula \(\varphi_2\) is satisfiable in a model with infinite \(D\). Moreover, for all increasing domain models \(\mathcal{M}\) and \(w \in W\), if \(\mathcal{M}, w \models \varphi_2\) then \(\delta(w)\) is infinite.

**Proof.** Consider the model \(\mathcal{M} = (W, \mathbb{N}, \mathcal{R}, \delta, \rho)\) where:
\[
W = \{w_0, w_1, w_2, w_3\} \text{ and } \mathcal{R} = \{(w_j, w_{j+1}) \mid j < 3\}.
\]
For all \(j \leq 3\) define \(\delta(w_j) = \mathbb{N}\) and \(\rho(w_j, P) = \{(k, l) \mid k, l \in \mathbb{N} \text{ and } k < l\}\). Note that only \(\rho(w_3, P)\) is relevant. It is easy to verify that \(\mathcal{M}, w_0 \models \varphi_2\).

Now assume that \(\mathcal{M}, w \models \varphi_2\). We will construct an infinite sequence of distinct domain elements \(d_0, d_1, \ldots\) over \(\delta(w)\) and this proves that \(\delta(w)\) is infinite. The sequence is constructed by induction. We maintain the invariant that for all \(i\) we have \(\mathcal{M}, w, [x \mapsto d_i, y \mapsto d_{i+1}] \models \Box \Box \Box P_{x y}\).

In the base case, since \(\delta(w)\) is non-empty, pick some arbitrary \(d_0 \in \delta(w)\). For the induction step, suppose we have constructed \(d_0, \ldots d_i\). Let \(d_{i+1}\) be such that \(\mathcal{M}, w, [x \mapsto d_i, y \mapsto d_{i+1}] \models \Box \Box \Box P_{x y}\) (such an element always exists since \(\mathcal{M}, w \models \forall x \exists y \Box \Box \Box P_{x y}\)).
Theorem 13. We can verify that $\square$ follows:

We have $\forall x | \Diamond x \Diamond x \top$. Hence, in $\Diamond$, we have $\forall x | \Diamond x \Diamond x \top$. The encoding formulas are given as

\[
\varphi_a := \Diamond \forall x \left[ \Box y \Box y \Box x y \land \Box y \Box x y \land \Box y \left( \Box x y \leftrightarrow \Box x y \right) \land \Box y \left( \Box x y \land \Box y z \rightarrow \Box x z \right) \right]
\]

The formula is similar to the formula $\varphi_1$, except that we need to insert one more $\Box$ as the padding due to the fact that we can only introduce an existential quantifier after a $\Box$ in $\Diamond$.

4.3 BE over constant domain

When we consider constant domain models, we can relax the condition that $\forall x \exists y \Box \alpha$ quantification has to be over the same domain. Since the local domain at every world is the same, all quantifiers at all worlds are evaluated over the same domain. Hence, in BE, using the $\Box \exists x \alpha$ operator we can write formula of the form $\Diamond \forall x \exists x \alpha$. Now we prove that this is sufficient to prove the lack of finite model property. The encoding formulas are given as follows:

\[
\varphi_3 := \Diamond \forall x \left[ \Box y \Box y \Box x y \land \Box y \Box x y \land \Diamond \left( \Diamond x y \leftrightarrow \Box x y \right) \land \Diamond \left( \Box x y \land \Box y z \rightarrow \Box x z \right) \right]
\]

Theorem 13. The formula $\varphi_3$ is satisfiable in a constant domain model with infinite $D$. Moreover, for all constant domain model $\mathcal{M}$ and $w \in W^M$ if $\mathcal{M}, w \models \varphi_3$ then $D$ is infinite.

Proof. We can verify that $\mathcal{M}, w_0 \models \varphi_3$ where $\mathcal{M}$ is the model described in the proof of Theorem 12 (Note that $\mathcal{M}$ is a constant domain model).

For the moreover part, let $\mathcal{M}$ be an arbitrary constant domain model such that $\mathcal{M}, r \models \varphi_3$. Then we will prove that $D$ is infinite. First note that there exists some $r \rightarrow r'$ such that $\mathcal{M}, r' \models \forall x \left[ \Box y \Box y \Box x y \land \Box y \Box x y \land \Diamond \left( \Diamond x y \leftrightarrow \Box x y \right) \land \Diamond \left( \Box x y \land \Box y z \rightarrow \Box x z \right) \right]$. Since $\mathcal{M}$ is a constant domain model, here is some successor $r' \rightarrow u$ such that:

\[
\mathcal{M}, u \models \forall x \left[ \exists y \Box y \Box x y \land \Box y \Box x y \land \Diamond \left( \Diamond x y \leftrightarrow \Box x y \right) \land \Diamond \left( \Box x y \land \Box y z \rightarrow \Box x z \right) \right]
\]

We will construct an infinite sequence of distinct domain elements $d_0, d_1 \ldots$ over $D$ and this will prove that $D$ is infinite. The sequence is constructed by induction. We maintain that the invariant that for all $i$: $\mathcal{M}, u \models \Box \Box P d_i d_{i+1}$.

In the base case, since $D$ is non-empty, pick some arbitrary $d_0 \in D$. For the induction step, suppose we have constructed $d_0, \ldots, d_i$. Let $d_{i+1}$ be such that $\mathcal{M}, u \models \Box \Box P d_i d_{i+1}$ (such an element always exists since $\mathcal{M}, u \models \forall x \exists y \Box y P x y$).
Now we claim that $d_{i+1}$ is distinct from $d_0 \ldots d_i$. Suppose not then let $d_{i+1} = d_j$ for some $j \leq i$. Let $j + l = i$ for some $l \geq 0$. Let $u \rightarrow v^i$ such that with $x \mapsto d$, we have

$$M, v^i \models \forall y \left( \left[ \Box Pd_y \leftrightarrow \Box Pd_y \right] \land \forall z \left[ (Pd_y \land Pyz) \rightarrow Pd_z \right] \right)$$

By construction $M, v^i \models \Box Pd_y d_{k+1}$ for all $k \leq i$. Also, by assumption that $d_{i+1} = d_j$, we have $M, v^i \models \Box Pd_y d_j$. We also have $M, v^i \models \Box \neg Pd_y d_i$.

Let $v^i \rightarrow w^{ij}$ be such that with the assignment $y \mapsto d_j$ we have $M, w^{ij} \models \forall z \left( Pd_y d_j \land Pd_z \rightarrow Pd_y d_j \right)$. In particular with $z \mapsto d_{j+1}$, we obtain $M, w^{ij} \models Pd_y d_{j+1}$. This implies that $M, v^i \models \Box Pd_y d_{j+1}$.

Since $M, v^i \models \forall y \left( \left( \Box Pd_y \leftrightarrow \Box Pd_y \right) \right)$ we have $M, v^i \models \Box Pd_y d_{j+1}$.

Now, let $v^i \rightarrow w^{(j+1)}$ be such that $M, w^{(j+1)} \models \forall z \left( Pd_y d_{j+1} \land Pd_y d_{j+2} \rightarrow Pd_y d_{j+1} \right)$. In particular with $z \mapsto d_{j+2}$, we have $M, w^{(j+1)} \models Pd_y d_{j+2}$. This implies that $M, v^i \models \Box Pd_y d_{j+2}$ from this we can conclude that $M, v^i \models \Box Pd_y d_i$ and hence we obtain a contradiction.

### 5 Decidable fragments

Over constant domains, it is proved in [14] that the EB fragment is decidable and AB is undecidable. In [13] the undecidability of BA over constant domain is proved. Further, in Section 4 we have proved that the BE (and hence BABE) fragments do not have the finite model property. This completes the picture of bundled fragments over constant domain models.

Over increasing domain models, in [14] [13], the AEBE fragment and BABE fragments are respectively shown to be decidable. In Section 3 we have proved that the fragments EBBA and AEBBBA are undecidable and EBBB lacks the finite model property. The remaining cases are the fragments LBF and ABBBBA. In this section we take up these two fragments and prove them to be decidable. Note that in both of these fragments the formula $\forall x \exists y C(x)$ is not (syntactically) expressible.

We assume that the formulas are in negation normal form (where $\neg$ appears only in front of atomic predicates). We first define some useful terms and notations.

**Definition 14.** For any FOML formula $\varphi$:
- $\varphi$ is a literal if $\varphi$ is of the form $P(x_1, \ldots x_n)$ or of the form $\neg P(x_1, \ldots x_n)$
- $\varphi$ is a module if $\varphi$ is a literal or $\varphi$ is of the form $\Delta \alpha$ where $\Delta \in \{\Box, \Diamond\}$
- The component of $\varphi$ is defined inductively as follows:
  - If $\varphi$ is a module then $C(\varphi) = \{\varphi\}$
  - If $\varphi$ is of the form $\varphi_1 \land \varphi_2$ or $\varphi_1 \lor \varphi_2$ then $C(\varphi) = C(\varphi_1) \cup C(\varphi_2)$
  - If $\varphi$ is of the form $\forall x \varphi_1$ or $\exists x \varphi_1$ then $C(\varphi) = \{\varphi\} \cup C(\varphi_1)$
- A formula $\varphi$ is called Existential-safe if every $\psi \in C(\varphi)$ is a module or of the form $\forall x \psi'$.
- A finite set of formulas $\Gamma$ is Existential-safe if every $\varphi \in \Gamma$ is Existential-safe.

Intuitively, $C(\varphi)$ is the set of all subformulas of $\varphi$ that are ‘to be evaluated’ at the current world. An existential-safe formula $\varphi$ does need witnesses from the current local domain in order to make the formula true. The notions of components and existential-safeness will play a role in the tableau-based decision procedure to be introduced below.
Before going into the specific tableau rules for various bundled fragments, we first explain the general method. A tableau is a tree-like structure generated by repeatedly applying a few rules from a single formula \( \alpha \) with some auxiliary information as the root of the tree. Intuitively, a tableau for \( \alpha \) is a pseudo model which can be transformed into a real model of \( \alpha \) under some simple consistency conditions. We can then decide the satisfiability of a formula by trying to find a proper tableau. As in [20], a tableau \( T \) in our setting is a tree structure such that each node is a triple \((w, \Gamma, \sigma)\) where \( w \) is a symbol or a finite sequence of symbols intended as the name of a possible world in the real model, \( \Gamma \) is a finite set of FOML-formulas, and \( \sigma \) is an assignment function for variables. Since we intend to use the set of variables as the domain in the tableau-induced real model, \( \sigma \) is simply a partial identity function on \( \text{Var} \), i.e., \( \sigma(x) = x \) for all \( x \in \text{Dom}(\sigma) \subseteq \text{Var} \), where the domain of \( \sigma \), \( \text{Dom}(\sigma) \), is intended to be the local domain of the real model. The intended meaning of the node \((w, \Gamma, \sigma)\) is that all the formulas in \( \Gamma \) are satisfied on \( w \) with the assignment \( \sigma \), thus we also write \((w : \Gamma, \sigma)\) for the triple.

A tableau rule specifies how the node in the premise of the rule is transformed to or connected with one or more new nodes given by the conclusion of the rule. Applying the rules can generate a tree-like structure, a tableau, which is saturated if every leaf node contains only literals. For any formula \( \varphi \), we refer to a saturated tableau of \( \varphi \) simply as a tableau of \( \varphi \). Further, a saturated tableau is open if in every node \((w : \Gamma, \sigma)\) of the tableau, \( \Gamma \) does not contain both \( \alpha \) and \( \neg \alpha \) for any formula \( \alpha \).

We call a formula clean if no variable occurs both bound and free in it and every use of a quantifier quantifies a distinct variable. A finite set of formulas \( \Gamma \) is clean if \( \wedge \Gamma \), the conjunction of all formulas in \( \Gamma \), is clean. Note that every FOML-formula can be rewritten into an equivalent clean formula. For instance, the formulas \( \exists x \Box Px \vee \forall x \Diamond Qx \) and \( Px \land \Box \exists x Qx \) are not clean, whereas \( \exists x \Box Px \vee \forall y \Diamond Qy \) and \( Px \land \Box \exists y Qy \) are their clean equivalents respectively. Clean formulas help in handling the witnesses for existential formulas in the tableau in a syntactic way.

Consider a finite set of formulas \( \Gamma \) that is clean. Suppose we want to expand \( \Gamma \) to \( \Gamma \cup \{ \alpha_1, \ldots, \alpha_k \} \), then even if each of \( \alpha_i \) is clean, it is possible that a bounded variable of \( \alpha_i \) also occurs in some \( \varphi \in \Gamma \) or another \( \alpha_j \). To avoid this, first we rewrite the bound variables in each \( \alpha_i \) one by one by using the fresh variables that do not occur in \( \Gamma \) and other previously rewritten \( \alpha_j \). Such a rewriting can be fixed by always using the first fresh variable in a fixed enumeration of all the variables. When \( \Gamma \) and \( \{ \alpha_1, \ldots, \alpha_k \} \) are clear from the context, we denote \( \alpha_i^* \) to be such a fixed rewriting of \( \alpha_i \) into a clean formula. It is not hard to see that the resulting finite set \( \Gamma \cup \{ \alpha_1^*, \ldots, \alpha_k^* \} \) is clean.

### 5.1 LBF over increasing domain

Tableau rules for LBF fragment are described in Fig. 5. The \( (\land) \) and \( (\lor) \) rules are standard, where we make a non-deterministic choice of one of the branches for \( (\lor) \). The rule \((\text{END})\) says that if we are left with only modules and there are no \( \Diamond \) formulas, then the branch does not need to be explored further. The \( (\forall) \) rule creates one successor world for every \( \forall \) formula at the current node and includes all the \( \Box \) formulas that need to be satisfied along with the \( \Box \) formula and \( \sigma \) is inherited in the successor worlds to preserve increasing domain property. The \( (\exists) \) rule picks \( z \) itself as a witness to satisfy \( \exists x \varphi \) and \( (\forall) \) rule expands the set of formulas to include a clean version of \( \varphi[z/y] \) for every variable \( z \) in the current local

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4 Refer [20] for an illustration of a similar tableau construction.
\[
\begin{array}{c}
\text{Given } n \geq 1: \text{ and } m, s \geq 0 \\
\frac{w : \Diamond \varphi_1, \ldots, \Diamond \varphi_n, \Box \beta_1, \ldots, \Box \beta_m, l_1, \ldots, l_s, \sigma}{(\Diamond)} \\
\langle wv_i : \varphi_i, \{ \beta_j \mid j \in [1, m] \}, \sigma \rangle \text{ for all } i \in [1, n]
\end{array}
\]

\[
\begin{array}{c}
\text{Given } m \geq 1, s \geq 0: \\
\frac{w : \Box \beta_1, \ldots, \Box \beta_m, l_1, \ldots, l_s, \sigma}{(\text{END})}
\end{array}
\]

**Figure 5** Tableau rules for \( \text{LBF} \), here every \( l_i \) is a literal
Proposition 15. Suppose \( \Gamma \) contains at least one non-literal formula. If there is some \( \theta \) occurring in \( \Gamma \), we use \( \text{Dom}(t) \) to denote the domain of the domain of \( \sigma \).

Also, there is an implicit ordering on how rules are applied: \((\Diamond)\) rule can be applied at a node \((w, \Gamma, \sigma)\) only if all formulas of \( \Gamma \) are modules and hence may be applied only after the \((\land, \lor, \forall, \exists)\) rules have been applied as many times as necessary at \( w \). Similarly \((\forall)\) rule can be applied only when \( \Gamma \) is \text{Existential-safe} which means that the \((\exists)\) rule cannot be applied any more at the current node.

\[\text{Proposition 15. For every tableau } T \text{ and every node } v = (w, \Gamma, \sigma) \text{ in } T \text{ if } v \text{ is a leaf then either } \Gamma \text{ contains only literals or there is some rule that can be applied at } v.\]

\[\text{Proof. Suppose } \Gamma \text{ contains at least one non-literal formula. If there is some } \varphi \in \Gamma \text{ where } \varphi \text{ is of the form } \varphi_1 \land \varphi_2 \text{ or } \varphi_1 \lor \varphi_2 \text{ then we can apply } (\land) \text{ or } (\lor) \text{ rule respectively. So assume that the operators } \land, \lor \text{ does not occur as the outer most connective in any formula of } \Gamma.\]

Further, if every \( \varphi \in \Gamma \) is a module then we can apply \((\Diamond)\) rule if at least one formula of the form \( \Diamond \varphi_1 \in \Gamma \), otherwise we can apply \((\text{END})\) rule.

So the remaining case is that there is at least one formula in \( \Gamma \) which has a quantifier at the outermost level. Now if we have some \( \exists x \varphi_1 \in \Gamma \) then we can apply \((\exists)\) rule. Otherwise every quantified formula in \( \Gamma \) is of the form \( \forall y \varphi \). But then, from the syntax of LBF it follows that \( \Gamma \) is \text{Existential-safe} and hence we can pick some \( \forall y \varphi \in \Gamma \) and apply \((\forall)\) rule. \(\blacksquare\)

\[\text{Theorem 16. For any clean LBF formula } \theta, \text{ let } \text{Dom}(\sigma_r) = \text{FV}(\theta) \cup \{z\} \text{ where } z \text{ does not occur in } \theta. \text{ There is an open tableau } T \text{ with root } (r : \{\theta\}, \sigma_r) \text{ iff } \theta \text{ is satisfiable in an increasing domain model.}\]

\[\text{Proof. First we claim that the rules preserve cleanliness of the formulas. To see this, we verify that for every rule, if } \Gamma \text{ in the antecedent of the rule is clean then the } \Gamma' \text{ obtained after the application of the rules is also clean. This is obvious for } (\land), (\lor), (\Diamond) \text{ and } (\text{END}) \text{ rules. The } (\exists) \text{ rule preserves cleanliness because it frees variable } x \text{ which are not bound by any other quantifier in the antecedent. The } (\forall) \text{ rule preserves cleanliness by rewriting.}\]

\((\Rightarrow) \text{ : Let } T \text{ be an open tableau rooted at } (r : \{\theta\}, \sigma_r). \text{ Define a model } \mathcal{M} = (W, D, \delta, R, \rho) \text{ as follows:}\]

\[\begin{align*}
\text{\textbullet } W &= \{w \mid (w : \Gamma, \sigma) \text{ is a node in } T\} \\
\text{\textbullet } D &= \text{Var} \\
\text{\textbullet } R &= \{(w, v) \mid v \text{ is of the form } wv' \text{ for some } v'\} \\
\text{\textbullet } \text{For every } w \in W \text{ define } \delta(w) &= \text{Dom}(t_w) \\
&\text{where } t_w \text{ is the last node of } w \text{ in } T \\
\text{\textbullet } \text{For every } w \in W \text{ and } p \in T \text{ define } \\
\rho(w, P) &= \{\varphi \mid \overline{P} \varphi \in \Gamma \text{ where } t_w = (w, \Gamma, \sigma)\}
\end{align*}\]

Clearly, \( \mathcal{M} \) is an increasing domain model, and since \( z \in \text{Dom}(\sigma_r) \), there is no empty local domain. As \( T \) is an open tableau, \( \rho \) is well-defined.

\text{Claim. For every node } (w : \Gamma, \sigma) \text{ in } T \text{ and for every LBF formula } \varphi \text{ if } \varphi \in \Gamma \text{ then } \mathcal{M}, w, \sigma \models \varphi.\]

The proof of the claim is by induction on the nodes of \( T \) from leaf nodes to the root. For the base case, \((w : \Gamma, \sigma)\) is a leaf node and hence \( \Gamma \) contains only literals. Thus, by the definition of \( \rho(w) \), the claim holds as \( \sigma \) is an identity assignment.
For inductive step, \((w : \Gamma, \sigma)\) is non leaf node and hence some rule is applied at this node and we have one or more descendants depending on the rule. Also, the claim holds for all the descendants. Now we consider various cases depending on which rule was applied.

- If \(\land\) rule was applied then \(\Gamma\) is of the form \(\Gamma' \cup \{\varphi_1 \land \varphi_2\}\) and the node \((w : \Gamma, \sigma)\) has one descendant \((w : \Gamma' \cup \{\varphi_1, \varphi_2\}, \sigma)\). By induction hypothesis \(M, w, \sigma \models \varphi_1\) and \(M, w, \sigma \models \varphi_2\). Thus we have \(M, w, \sigma \models \varphi_1 \land \varphi_2\). Further, by induction for every \(\varphi' \in \Gamma'\) we have \(M, w, \sigma \models \varphi'\) and hence the claim holds.

- If \(\lor\) rule was applied then \(\Gamma\) is of the form \(\Gamma' \cup \{\varphi_1 \lor \varphi_2\}\) and the node \((w : \Gamma, \sigma)\) has one descendant \((w : \Gamma' \cup \{\varphi_1\}, \sigma)\) for some \(i \in \{1, 2\}\). By induction hypothesis \(M, w, \sigma \models \varphi_1\). Thus we have \(M, w, \sigma \models \varphi_1 \lor \varphi_2\). Further, by induction for every \(\varphi' \in \Gamma'\) we have \(M, w, \sigma \models \varphi'\) and hence the claim holds.

- If \(\exists\) rule was applied then \(\Gamma\) is of the form \(\Gamma' \cup \{\exists x \varphi_1\}\) and the node \((w : \Gamma, \sigma)\) has one descendant \((w : \Gamma' \cup \{\varphi_1\}, \sigma)\). By induction hypothesis, \(M, w, \sigma \models \exists x \varphi_1\). Also note that since we applied \((w : \Gamma, \sigma)\) rule, suppose \(\Gamma\) is of the form \(\Gamma' \cup \{\exists x \varphi_1\}\) and \(\Gamma\) is of the form \(\Gamma' \cup \{\varphi_1\}\) for some \(i \in \{1, 2\}\). Further, by induction for every \(\varphi' \in \Gamma'\) we have \(M, w, \sigma \models \varphi'\) and hence the claim holds.

- If \(\forall\) rule is applied then \(\Gamma\) is of the form \(\Gamma' \cup \{\forall y \varphi_2\}\) and we have a descendant \((w : \Gamma' \cup \{\varphi_2\}, \sigma)\) where \(\sigma' = \sigma \cup \{x, y\}\). This implies that \(M, w, \sigma' \models \forall \varphi_2\). Also since \(\Gamma\) is clean, \(x\) does not occur in any \(\varphi' \in \Gamma'\). Thus for every \(\varphi' \in \Gamma'\) by induction hypothesis we have \(M, w, \sigma' \models \varphi'\) and hence \(M, w, \sigma \models \varphi\).

- If \(\emptyset\) rule is applied at \((w : \Gamma, \sigma)\) then \(\Gamma\) is of the form \(\{l_1 \ldots l_s\} \cup \{\lnot \varphi_1 \ldots \lnot \varphi_n\} \cup \{\Box \beta_1 \ldots \Box \beta_m\}\) for some \(s, m \geq 0\) and \(n \geq 1\). Consequently there are \(n\) children each of the form \((w^i : \Gamma^i, \sigma)\) where \(\Gamma^i = \{\varphi_i\}\) and \(\beta_i \ldots \beta_m\). Further, we have \((w, w^i) \in R\) for every \(i \leq n\) and there are the only worlds accessible from \(w\) in \(M\). Thus, by induction hypothesis and semantics, we have \(M, w, \sigma \models \Box \beta_i\) for every \(i \leq m\) and \(M, w, \sigma \models \lnot \varphi_i\) for every \(i \leq n\). Also note that \((w, \Gamma, \sigma)\) is the last node of \(w\) and hence, by definition of \(\rho\) at \(w\) we have \(M, w, \sigma \models l_i\) for every \(i \leq s\).

- If \(\text{END}\) rule is applied at \((w : \Gamma, \sigma)\) then \(\Gamma\) is of the form \(\{l_1 \ldots l_s\} \cup \{\Box \beta_1 \ldots \Box \beta_m\}\) for some \(s, m \geq 0\). There is one descendant \((w : \{l_1 \ldots l_s\}, \sigma)\) which is a leaf node in \(T\). Thus, there are no accessible worlds from \(w\) in \(M\) and hence \(M, w, \sigma \models \Box \beta_j\) for all \(j \leq m\) vacuously. Finally since \((w : \{l_1 \ldots l_s\}, \sigma)\) is the last node of \(w\), by definition of \(\rho\), we have \(M, w, \sigma \models l_j\) for every \(j \leq s\).
element \( d \in \delta(w) \) such that \( \mathcal{M}, w, \pi_{[x \rightarrow d]} \models \varphi \). By cleanliness, \( x \) is not free in \( \Gamma \) and hence \( \mathcal{M}, w, \pi_{[x \rightarrow d]} \models \bigwedge \Gamma \). Thus, \( \{ \varphi \} \cup \Gamma \) is also satisfiable.

For \( (\forall) \) rule, suppose \( \mathcal{M}, w, \pi \models \forall y \varphi \land \bigwedge \Gamma \). By semantics, for every \( d \in \delta(w) \) we have \( \mathcal{M}, w, \pi_{[y \rightarrow d]} \models \varphi \).

Let \( \text{Dom}(\sigma) = \{ z_0, z_1, \ldots, z_k \} \) and for all \( i \leq k \) let \( \pi(z_i) = d_i \). Now since every \( \varphi^*[z/y] \) is a clean rewriting of \( \varphi[z/y] \), we have \( \mathcal{M}, w, \pi_{[z_0 \rightarrow d_0, \ldots, z_k \rightarrow d_k]} \models \varphi^*[z_0/y] \land \ldots \land \varphi^*[z_k/y] \). Also, by cleanliness for every \( i \leq k \) the variable \( z_i \) does not occur in \( \bigwedge \Gamma \). Hence we also have \( \mathcal{M}, w, \pi_{[z_0 \rightarrow d_0, \ldots, z_k \rightarrow d_k]} \models \bigwedge \Gamma \).

Hence the set of formulas \( \{ \varphi^*[z/y] \mid z \in \text{Dom}(\sigma) \} \cup \Gamma \) is satisfiable.

Note that the depth of the tableau is linear in the size of the formula. However, as we have to rewrite formulas using new variables when applying \( (\forall) \) rule, the size of the domain is exponential in the size of the formula. Therefore, the tableau procedure can be implemented by an \textsc{ExpSpace} algorithm.

From Proposition \ref{prop:abebe} \textsc{ABEB} and \textsc{BABE} are subfragments of \textsc{LBF}. As a corollary, these fragments are also decidable. Also note that the quantifier prefix in \textsc{LBF} is of the form \( \exists^* \forall^* \) and hence any extension of this quantifier prefix or extending \textsc{LBF} with negation closure will result in a fragment that will be able to express \( \forall x \exists y \square \alpha \) (and hence will not have the finite model property, cf Theorem \ref{thm:fm}). In this sense \textsc{LBF} is the largest fragment in which \( \forall x \exists y \square \alpha \) is not (syntactically) expressible.

### 5.2 \textbf{ABBBABE over increasing domain}

In this fragment we are allowed \( \forall x \alpha, \Box \forall x \alpha \) and \( \Box \exists x \alpha \) and their duals. Note that \textsc{ABBBABE} fragment is not closed under subformulas. For instance, \( \varphi := \forall x \left( \exists y \alpha \lor \forall z \Box \beta \right) \) is a subformula of \( \varphi' := \Diamond \forall x \left( \exists y \alpha \lor \forall z \Box \beta \right) \). But \( \varphi' \) is in the fragment and \( \varphi \) is not in the fragment.

We say that \( \varphi \) is a subformula of \textsc{ABBBABE} if there is some formula \( \varphi' \in \textsc{ABBBABE} \) such that \( \varphi \) is a subformula of \( \varphi' \).

\begin{proposition}
Let \( \varphi \) be a subformula of \textsc{ABBBABE} such that \( \varphi \) is of the form \( Q x \psi \) where \( Q \in \{ \forall, \exists \} \). Then for every \( \beta \in C(\psi) \), \( \beta \) is a module or \( \beta \) is of the form \( \forall x \Box \beta' \) or \( \exists x \beta' \).
\end{proposition}

From Proposition \ref{prop:module} it follows that we cannot (syntactically) express formulas of the form \( \forall x \exists y \Box \varphi \) in the fragment. However, \( \forall x \exists y \Box \varphi \) is still allowed. For instance, \( \Diamond \forall x \left( \exists y \Box \alpha \right) \) formula is in the fragment.

Thus, we can have \( \forall x \exists y \Box \varphi \) but not \( \forall x \exists y \Box \varphi \). Intuitively this means that the different witnesses \( y \) for each \( x \) can work on different successor worlds. The fragment cannot enforce the interaction between \( x \) and \( y \) at all successors. This property can be used to prove that we can reuse the witnesses by creating new successor subtrees as required.

To get the decidability for \textsc{ABBBABE} fragment, the main idea is to prove that the formulas of the form \( \forall x \exists y \Box \varphi \) can be satisfied by picking some boundedly many witnesses \( y \) that will work for all \( x \). This is the same as proving that if \( \forall x \exists y \Box \varphi \) is satisfiable then \( \exists y_1, \ldots, \exists y_l \forall x (\Diamond \varphi[y/y_l]) \) is satisfiable (where \( l \) is bounded). We illustrate the proof idea with an example.

\begin{example}
Consider the formula \( \alpha := \forall x \left( \Box \neg P x x \land \exists y \Box P x y \right) \) which is a subformula of the fragment. Let \( \mathcal{T}, r \models \alpha \) where \( \mathcal{T} \) is a tree model rooted at \( r \). Now we will modify \( \mathcal{T} \)
to obtain $\mathcal{M}$ which is also a tree model rooted at $r$ such that $\mathcal{M}, r \models \exists y_1 \exists y_2 \forall x \left( \lozenge \neg Pxx \land (\lozenge Px \lor \lozenge Px_2) \right)$.

The model $\mathcal{M}$ is obtained by extending $\mathcal{T}$ in the following way. Let $\delta^T(r) = D_r$. To obtain $\mathcal{M}$, first we extend the local domain of $r$ by adding a fresh element $a$. The idea is that for every $d \in D_r$ (when assigned to $x$) we will ensure that the new element $a$ can be picked as the $y$-witness. To achieve this, we do the following: For every $d \in D_r$ let $d' \in D_r$ and $(r, s^d) \in R^T$ such that $\mathcal{T}, s^d \models Pd'$. Let $\mathcal{T}^d$ be the subtree of $\mathcal{T}$ rooted at $s^d$. We will create a new copy of $\mathcal{T}^d$ and call its root as $a^d$. Now, in the new subtree rooted at $a^d$, we make the new element $a$ ‘behave’ like $d'$ and we add an edge from $r$ to $a^d$. So, in particular, $\mathcal{M}$ will have $(r, a^d) \in R^M$ such that $\mathcal{M}, a^d \models Pda$. Since we do this construction for every $d \in D_r$ we obtain that for all $d \in D_r$ we have $\mathcal{M}, r \models \lozenge Pda$.

Now note that while evaluating $\alpha$ at $(\mathcal{M}, r)$ the $\forall x$ quantification will now also apply to $a$ (since $a$ is added to the local domain at $r$ in $\mathcal{M}$). But then, we cannot use $a$ itself as the witness for $a$ since we also need to ensure that $\mathcal{M}, r \models \forall x \lozenge \neg Pxx$. Hence we will add another element $b$ that acts as a witness for $a$. Further, $b$ also needs a witness. But now we can choose $a$ to be the witness for $b$ since that does not violate the formula $\forall x \lozenge \neg Pxx$.

So to complete the construction, we pick some arbitrary $d \in D_r$ for which we have some $d' \in D_r$ and $(r, s^d) \in R^T$ such that $\mathcal{T}, s^d \models Pd'$. We create two copies of $\mathcal{T}^d$ (subtree rooted at $s^d$) and call their roots as $v^d$ and $w^d$ respectively. In the subtree rooted at $v^d$ we ensure that $a$ and $b$ ‘behave’ like $d', d''$ respectively and in the subtree rooted at $w^d$ we ensure that $a$ and $b$ ‘behave’ like $d', d$ respectively. In particular we have $\mathcal{M}, v^d \models Pab$ and $\mathcal{M}, w^d \models Pba$. Finally we add edges from $r$ to $v^d$ and from $r$ to $w^d$ in $\mathcal{M}$.

Thus, we have: $\delta^M(r) = \delta^T(r) \cup \{a, b\}$ and $\mathcal{M}, r \models \exists y_1 \exists y_2 \forall x \left( \lozenge \neg Pxx \land (\lozenge Px \lor \lozenge Px_2) \right)$. With the above construction, this assertion can be verified by assigning $y_1$ and $y_2$ to $a$ and $b$ respectively.

Note that in principle, it is possible for a $\exists$ quantified formula to occur in the scope of a $\forall$ quantifier as a boolean combination with other $\exists$ quantified formulas and modules. Moreover these additional formulas can assert some ‘type’ information that may force us to pick additional witnesses. For example if the formula is $\forall x \left[ (\lozenge \neg Pxx \land R x) \lor \lozenge (\neg Pxx \land \neg R x) \right] \land \exists y \lozenche (Rx \rightarrow (Px \lor \neg Ry)) \land \neg Rx \rightarrow (Px \lor Ry)$, then we need two initial $y$-witnesses $a_1, a_2$ where one is used for witness whose ‘type’ is $\lozenche \neg Pyy \land Ry$ and other for witness whose ‘type’ is $\lozenche \neg Pyy \land \neg Ry$ and we also need the corresponding additional witnesses $b_1, b_2$. In general the formula can force us to pick witness of a particular ‘1-type’ which means we might need exponentially many witnesses.

Thus, we need to replace one $\exists$ inside the scope of $\forall$ by $2l$ many $\exists$ quantifiers outside the scope of $\forall$ where $l$ is bounded exponentially in the size of the given formulas. We now prove this formally.

For any formula $\varphi$ if $\alpha \in C(\varphi)$ we denote this by $\varphi[\alpha]$. This means that $\alpha$ does not occur inside the scope of any modality in $\varphi$. For any formula $\beta$ we denote $\varphi[\beta/\alpha]$ obtained by rewriting $\varphi$ where $\alpha$ is replaced by $\beta$. In particular we are interested in the case where $\alpha$ is of the form $\exists y \lozenche \psi$. Thus we always consider $\varphi[\exists y \lozenche \psi]$.

For every $l \geq 0$ if $y = y_1, y'_1 \ldots y_l, y'_l$ are fresh variables, we denote $\overline{\varphi} \lozenche \psi$ to be the formula $\bigvee_{i \leq l} \left( \varphi[\psi[y_i/y] \lor \varphi[\psi[y'_i/y]] \right)$ which is a big disjunction where each disjunct replaces $y$ in $\psi$ with one of $y_i$ or $y'_i$. Further, we denote $\varphi[\overline{\varphi} \lozenche \psi/\exists y \lozenche \psi]$ as simply $\varphi[\overline{\varphi} \lozenche \psi]$.

For instance, for the formula $\varphi := \left( Px \lor \exists y \lozenche Qxy \right)$ where $\psi := \exists y \lozenche Qxy$, for $l = 2$
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where:

The size of a formula denoted by $|\varphi|$ is the number of symbols occurring in $\varphi$ and for a finite set of formulas $\Gamma$, let $|\Gamma| = \sum_{\varphi \in \Gamma} |\varphi|$.

Lemma 19. Let $\Gamma'$ be a clean finite set of formulas such that every $\alpha \in \Gamma$ is a subformula of $ABBABE$ where $\Gamma' = \Gamma \cup \{\exists y \varphi[\exists y \varphi]\}$. If $\Gamma \land \exists x \varphi[\exists y \varphi]$ is satisfiable then there exists $l \leq 2^{|\Gamma'|}$ such that $\Gamma \land \exists y_1 \exists y_2 \exists y_3 \ldots \exists y_l \exists y' \forall x \varphi[\exists y \varphi]$ is satisfiable, where $\mathcal{F} = y_1, y_2, \ldots, y_l, y'$ are variables.

Note that the $\exists y$ quantifier is pulled outside the scope of the $\forall x \varphi$ quantifier and replaced with a disjunction each replacing $y$ with one of $y_i$ or $y_i'$ for every $i \leq l$.

To prove the lemma first we formally define the tree editing operation described in the example. Given a tree model $T$ rooted at $r$, let $d \notin D_T$. To add the new domain element $d$ to a local domain of $r$, we also need to specify the "type" of the new element $d$ at $r$ and its descendants. Towards this, we pick some domain element $c$ that is already present in $\delta(r)$ and assign the type of $d$ to the type of $c$ at every world.

Definition 20. Given a tree model $T = (W, D, R, \delta, \rho)$ rooted at $r$, let $d \notin D$ and $c \in \delta(r)$. Define the operation of "adding $d$ to $\delta(r)$ by mimicking $c'$", denoted by $T_{d \rightarrow c} = (W, D', R, \delta', \rho')$ where:

- $D' = D \cup \{d\}$
- For all $w \in W$ we have $\delta'(w) = \delta(w) \cup \{d\}$
- For every $w \in W$ and predicate $P$ define

$\rho'(w, P) = \{\bar{c}' | $ there is some $\bar{c} \in \delta(w, P)$ and $\bar{c}'$ is obtained from $\bar{c}$ by replacing zero or more occurrences of $c$ in $\bar{c}$ by $d\}$

Suppose that we want to extend the domain with $\bar{d} = d_1, \ldots, d_n$ which are fresh. Let $\omega : \bar{d} \rightarrow D'$ where $D' \subseteq \delta'(r)$ and we want each $d_i$ to mimic $\omega(d_i)$. Then we denote $T_{\omega}$ to be the tree obtained by $\left((T_{d_1 \rightarrow \omega(d_1)})_{d_2 \rightarrow \omega(d_2)} \ldots \right)_{d_n \rightarrow \omega(d_n)}$.

Proposition 21. Let $T = (W, D, R, \delta, \rho)$ be a tree model rooted at $r$ with $T_{d \rightarrow c}$ being an extended tree where $d \notin D$ and $c \in \delta(r)$. Then for all interpretation $\sigma$ and for all FOML formula $\varphi$ and all $w \in W$:

$T, w, \sigma_{[x \rightarrow c]} \models \varphi$ if $T_{d \rightarrow c}, w, \sigma_{[x \rightarrow d]} \models \varphi$ and $T_{d \rightarrow c}, w, \sigma_{[x \rightarrow c]} \models \varphi$

Proof. Proof is by induction on the structure of $\varphi$. The atomic case follows by the definition of $\delta'$.

- For the $\exists y \varphi$ case, if $T, w, \sigma_{[x \rightarrow c]} \models \exists y \varphi$ then there is some $w \rightarrow v$ such that $T, v, \sigma_{[y \rightarrow v]} \models \varphi$. By induction, $T_{d \rightarrow c}, v, \sigma_{[y \rightarrow v]} \models \varphi$ and $T_{d \rightarrow c}, v, \sigma_{[x \rightarrow d]} \models \varphi$. Hence $T_{d \rightarrow c}, w, \sigma_{[x \rightarrow c]} \models \exists y \varphi$. Conversely, if $T_{d \rightarrow c}, w, \sigma_{[x \rightarrow d]} \models \exists y \varphi$ then there is some $w \rightarrow v$ such that $T_{d \rightarrow c}, v, \sigma_{[x \rightarrow d]} \models \varphi$. By induction $T, v, \sigma_{[x \rightarrow c]} \models \varphi$ and $T_{d \rightarrow c}, v, \sigma_{[x \rightarrow c]} \models \varphi$. Hence $T, w, \sigma_{x \rightarrow c} \models \exists y \varphi$.

- For the case $\exists y \varphi$, if $T, w, \sigma_{[x \rightarrow c]} \models \exists y \varphi$ then let $c' \in \delta'(r)$ such that $T, w, \sigma_{[x \rightarrow cc']} \models \varphi$. By induction $T_{d \rightarrow c'}, w, \sigma_{[y \rightarrow cc']} \models \varphi$ and $T_{d \rightarrow c'}, w, \sigma_{[x \rightarrow d]} \models \varphi$. Thus, $T_{d \rightarrow c}, w, \sigma_{[x \rightarrow c]} \models \exists y \varphi$ and $T_{d \rightarrow c}, w, \sigma_{[x \rightarrow c]} \models \exists y \varphi$. 
Conversely, suppose $T_{d\rightarrow c}, w, \sigma_{[x\rightarrow d]} \models \exists y \varphi$ then there exists $d' \in \delta T_{d\rightarrow c}(r)$ such that $T_{d\rightarrow c}, w, \sigma_{[xy\rightarrow d']} \models \varphi$. Now, if $d' \neq d$ then $d' \in \delta T(r)$ and by induction we have $T, w, \sigma_{[xy\rightarrow c]} \models \varphi$ and $T_{d\rightarrow c}, w, \sigma_{[xy\rightarrow c]} \models \varphi$. Otherwise if $d' = c$ then also by induction we have $T, w, \sigma_{[xy\rightarrow c]} \models \varphi$ and $T_{d\rightarrow c}, w, \sigma_{[xy\rightarrow c]} \models \varphi$. So we have both $T, w, \sigma_{[x\rightarrow c]} \models \exists y \varphi$ and $T_{d\rightarrow c}, w, \sigma_{[x\rightarrow c]} \models \exists y \varphi$.

Now we are ready to prove Lemma 11.

Proof of Lemma 11 Let $T$ be a tree model rooted at $r$ such that $T, r, \sigma \models \bigwedge \Gamma$.
For every domain element $a \in \delta T(r)$ define:

$$\Pi(r,a) = \bigcup_{\forall \varphi(a) \in T'} \{ \lambda | \lambda \in C(a) \text{ and } T, r, \sigma_{[x'=a]} \models \lambda \}$$

Let $\Pi(r) = \{ \Pi(r,a) | a \in \delta T(r) \}$. Let $|\Pi(r)| = l$ (note that $l \leq 2|\Gamma|^l$). Enumerate $\Pi(r) = \{ \Lambda_1, \ldots, \Lambda_l \}$ and for every $i \leq l$ pick $a_i \in \delta T(r)$ such that $\Pi(r, a_i) = \Lambda_i$.

Now let $\overline{d} = d_1, d'_1, d_2, d'_2, \ldots, d_i, d'_i$ be fresh domain elements and let $\omega: \overline{d} \mapsto \{a_1, a_2, \ldots, a_l \}$ where for all $i \leq l$ we have $\omega(d_i) = \omega(d'_i) = a_i$. We define the required model $M = (W', T', \mathcal{R}', \delta', \rho')$ as follows:

Let $M_0 = T_{\omega}$ be the new tree model rooted at $r$ obtained by adding $d_1, d'_1, \ldots, d_i, d'_i$ to $\delta T(r)$ where each $d_i$ and $d'_i$ mimics $a_i$. Now $M_0$ is obtained by extending $M_0$ as follows:

For every $c \in \delta(r)$ such that $T, r, [x \mapsto c] \models \exists y \varphi$ we pick $c' \in \delta T(r)$ and $r \rightarrow s^c$ be such that $T, s^c, [xy \rightarrow cc'] \models \psi$. Let $T^{c'}$ be the sub-tree of $T$ rooted at $s^c$ and $\Pi(r, c') = \Lambda_j$.

Then:

Create a new subtree $T^{c'}_0 = T_{\omega'}$ where for all $h \neq j$ we have $\omega'(d_h) = \omega(d_h) = a_h$ and $\omega'(d_j) = \omega(d'_j) = c'$. Let $u^c$ be the root of $T^{c'}_0$. Add an edge from $r$ to $u^c$ in $M$.

Further, for every $i \leq l$ if $T, r, [x \mapsto a_i] \models \exists y \varphi$, then let $b \in \delta T(r)$ and $r \rightarrow s^i \in \mathcal{R}^T$ be such that $T, s^i, [xy \rightarrow a_i, b] \models \psi$. Let $\Pi(r, b) = \Lambda_j$. Then:

Create $T^i_1 = T^{c'}_0$ and $T^i_2 = T_{\omega'}$ where $\omega_1$ and $\omega_2$ are defined as follows:

- For all $h \neq j$,
  - $\omega_1(d_h) = \omega_2(d_h) = a_h$
  - $\omega_1(d_j) = \omega_2(d'_j) = a_j$
- $\omega_1(d_j) = b$ and $\omega_2(d'_j) = a_j$

Let $u^i$ and $u^{i'}$ be the root of $T^i_1$ and $T^i_2$ respectively. Add the edges from $r$ to $u^i$ and from $r$ to $u^{i'}$ in $M$. The two copies of subtrees are intended to provide witnesses for $\exists y \varphi$ for $d_j$ and $d'_j$ respectively. We need the two copies to ensure that $\omega_1$ and $\omega_2$ are well defined in the case when $i = j$ and $a_j \neq b$.

Note that $T^{c'}_0$ rooted at $u^c$ is created for every $c \in \delta T(r)$ such that $T, r, \sigma_{[x\rightarrow c]} \models \exists y \varphi$. If $c'$ is the picked witness for $c$ with $(r, s^c) \in \mathcal{R}^T$ such that $T, s^c, \sigma_{[xy\rightarrow cc']} \models \psi$ and $\Pi(r, c') = \Lambda_j$ then by construction $T^{c'}_0$ rooted at $u^c$ is obtained where $d_j$ mimics $c'$ in the subtree rooted at $u^c$. All these together indicate that we can use $d_j$ and the subtree rooted at $u^c$ in $M$ to verify that $M, u^c, \sigma_{[xy\rightarrow c]} \models \psi$. Also note that for all $h \neq j$ the fresh elements $d_h$ and $d'_h$ mimic $a_h$ in the subtree $T^{c'}_0$ (i.e., we have not added any extra ‘types’).

Further, we want the type of $d_j$ and $d'_j$ at $r$ in $M$ to mimic the type of $a_j$ at $r$ in $T$. All type information is taken care in $M_0$ where both $d_j$ and $d'_j$ mimic $a_j$ except the formula $\exists y \varphi$. So if $T, r, \sigma_{[x\rightarrow a_j]} \models \exists y \varphi$ then we need witness to verify $M, r, \sigma_{[xy\rightarrow d_j]} \models \exists y \varphi$ and $M, r, \sigma_{[xy\rightarrow d'_j]} \models \exists y \varphi$. If the witness for $y$ for $a_j$ is $b$ and $\Pi(r, b) = \Lambda_j$ then we want $d'_j$ to be the witness for $d_j$ and $d_j$ to be the witness for $d'_j$. 

- For all $h \neq j$,
  - $\omega_1(d_h) = \omega_2(d_h) = a_h$
  - $\omega_1(d_j) = \omega_2(d'_j) = a_j$
- $\omega_1(d_j) = b$ and $\omega_2(d'_j) = a_j$
Consequently if \( s^i \) is the world such that \( a \rightarrow s^i \in R^T \) and \( T, s', [xy \mapsto ab] \models \psi \) then we create two new copies of subtree \( T' \) rooted at \( s^i \) and call it \( T'_1 \) and \( T'_2 \). By construction, in particular, the new element \( d_i \) mimics \( a_i \) and \( d'_j \) mimics \( b_i \) in \( T'_1 \). Similarly \( d'_j \) mimics \( a_i \) and \( d'_j \) mimics \( b_i \) in \( T'_2 \). Thus, we can pick \( d'_j \) to be the witness for \( d_i \) (and consider \( T'_1 \)) and pick \( d'_j \) to be the witness for \( d_i \) (and consider \( T'_2 \)).

Also, it is important to note that for every \( r \rightarrow v \in R^M \), if \( d_i \) mimics \( c \) and \( d'_j \) mimics \( c' \) at \( v \) then we will always have \( \Pi(r, c) = \Pi(r, c') = \Lambda_i = \Pi(r, a_i) \). Now it can be verified that \( M, r, c \models \bigwedge \Gamma \land \exists y_1 \exists y_1' \ldots \exists y_l \exists y'_l \forall x \phi \psi \).

Towards this, first we prove some useful claims.

**Claim 1.** For every interpretation \( \pi : Var \rightarrow \delta^T(r) \) and every module \( \alpha \), if \( T, r, \pi \models \alpha \) then \( M, r, \pi \models \alpha \).

**Proof.** Let \( \alpha \) be a module. We consider all possible cases. If \( \alpha \) is a literal then the claim follows since \( \rho^M(r) \) is an extension of \( \rho^T(r) \).

For the case \( \square \alpha' \) let \( T, r, \pi \models \square \alpha' \). To verify \( M, r, \pi \models \square \alpha' \) pick any arbitrary successor \( r \rightarrow v \in R^M \). Let \( v \) be a copy of \( w \in W^T \) extended with \( d_1, d'_1, \ldots, d_l, d'_l \). Since \( T, r, \pi \models \square \alpha' \), we have \( T, w, \pi \models \alpha' \). By Proposition 21 we have \( M, v, \pi \models \alpha' \).

Since we picked \( v \) arbitrarily, we have \( M, r, \pi \models \square \alpha' \).

For the case \( \diamond \alpha' \) let \( T, r, \pi \models \diamond \alpha' \). By semantics, there exists \( r \rightarrow w \in R^T \) such that \( T, w, \pi \models \alpha' \). Then by construction the subtree of \( T \) rooted at \( w \) continues to be a part of \( M_0 \) (and hence \( M \)) extended with \( d_1, d'_1, \ldots, d_l, d'_l \). So by Proposition 21 we have \( M, w, \pi \models \alpha' \) and \( (r, w) \in R^M \). Thus, \( M, r, \pi \models \diamond \alpha' \).

**Claim 2.** Let \( \pi : Var \rightarrow \delta^T(r) \) be any interpretation. Let \( \alpha \) be a module or of the form \( \forall z \square \beta \) or \( \exists z \diamond \beta \). If \( T, r, \pi \models \alpha \) then \( M, r, \pi \models \alpha \).

**Proof.** We consider all possible cases of \( \alpha \):

- If \( \alpha \) is a module then Claim 2 follows from Claim 1.
- If \( \alpha = \forall z \square \beta \) then by semantics, for every \( d \in \delta^T(r) \) and every \( r \rightarrow w \in R^T \) we have \( T, w, \pi[\{z \mapsto d\}] \models \beta \).

Suppose \( M, r, \pi \not\models \forall z \square \beta \) then by semantics we have \( M, r, \pi \models \exists z \diamond (\neg \beta) \). Let \( c \in \delta^M(r) \) and \( r \rightarrow v \in R^M \) such that \( M, v, \pi[\{z \mapsto c\}] \models \neg \beta \).

Let the subtree rooted at \( v \) in \( M \) be a copy of the subtree rooted at \( w \in W^T \) in \( T \). Now if \( c \in \delta^T(r) \) then by Proposition 21 we have \( T, w, \pi[\{z \mapsto c\}] \models \neg \beta' \), which is a contradiction. Otherwise if \( c \) mimics some \( c' \) at the world \( v \) then again by Proposition 21 we have \( T, w, \pi[\{z \mapsto c'\}] \models \neg \beta' \) which is a contradiction.

- If \( \alpha = \exists z \diamond \beta \) then let \( d \in \delta^T(r) \) and \( r \rightarrow w \in R^T \) such that \( T, w, \pi[\{z \mapsto d\}] \models \beta \). By construction, the subtree rooted at \( w \) in \( T \) continues to be a part of \( M_0 \) (and hence \( M \)) extended with \( d_1, d'_1, \ldots, d_l, d'_l \).

So by Proposition 21 we have \( M, w, \pi[\{z \mapsto d\}] \models \beta \) and \( (r, w) \in R^M \). Thus, \( M, r, \pi[\{z \mapsto d\}] \models \exists z \diamond \beta \) which is a contradiction.

**Claim 3.** Let \( d \in \{d_k, d'_k\} \) for some \( k \leq l \). Then for every \( \forall z \alpha \in \Gamma' \) and for every \( \beta \in C(\alpha) \): If \( T, r, \sigma[\{z \mapsto d\}] \models \beta' \) then \( M, r, \sigma[\{z \mapsto d\}] \models \beta' \).

**Proof.** From Proposition 17 it follows that every \( \beta \in C(\alpha) \) is either a module or of the form \( \forall z' \square \beta' \) or \( \exists z' \diamond \beta' \). We consider various possible cases for \( \beta \):
If $\beta$ is a literal then the claim follows since $d$ mimics $a_k$ at $r$ and $\rho^M$ is defined accordingly.

If $\beta$ is of the form $\Box \beta'$ then by definition, $\Box \beta' \in \Pi(r, a_k)$ and hence for all $c \in \delta^T(r)$ if $\Pi(r, c) = \Pi(r, a_k)$ then $T, r, \sigma[z \mapsto c] \models \Box \beta'$. Thus, for every $c$ such that $\Pi(r, a_k) = \Pi(r, c)$ and for every $r \to w \in R^T$ we have $T, w, \sigma[z \mapsto c] \models \beta'$. Now to verify that $M, r, \sigma[z \mapsto d] \models \Box \beta'$, pick some arbitrary world $v$ such that $r \to v \in R^M$. Let the subtree rooted at $v$ in $M$ be a copy of the subtree rooted at $w \in W^T$ in $T$. Let $d$ mimic some $c$ in the subtree rooted at $v$ in $M$. By construction it follows that $\Pi(r, c) = \Pi(r, a_k)$. So we have $T, w, \sigma[z \mapsto c] \models \beta'$ and by Proposition 21 we have $T, w, \sigma[z \mapsto d] \models \beta'$.

If $\beta$ is of the form $\Diamond \beta'$ then there is some $r \to w \in R^T$ such that $T, w, \sigma[z \mapsto d] \models \beta'$. Then by construction, the subtree rooted at $w$ is a subree of $M_0$ (and hence subtree of $M$) where $d$ mimics $a_k$. Hence by proposition 21 we have $M, w, \sigma[z \mapsto d] \models \beta'$ which implies $M, r, \sigma[z \mapsto d] \models \Diamond \beta'$.

If $\beta$ is of the form $\exists z' \Box \beta'$ then let $c' \in \delta^T(r)$ and $r \to w \in R^T$ such that $T, w, \sigma[z' \mapsto a_k, c'] \models \beta'$. By construction, the subtree of $T$ rooted at $w$ continues to be a part of $M_0$ (and hence of $M$) where $d$ mimics $a_k$. Hence, by Proposition 21 we have $M, w, \sigma[z' \mapsto d, c'] \models \beta'$. Thus, $M, r, \sigma[z \mapsto d] \models \exists z' \Box \beta'$.

If $\beta$ is of the form $\forall z' \Box \beta'$ then then pick some arbitrary $c' \in \delta^M(r)$ and some arbitrary $r \to v \in R^M$. We will verify that $\exists r, v, \sigma[z' \mapsto v, c'] \models \beta'$. Let $v$ be a copy of $w \in W^T$ and let $d$ mimic some $e$ at $v$. Then by construction we have $\Pi(r, a_k) = \Pi(r, c)$ and hence $\forall r, \sigma[z \mapsto v] \models \forall z' \Box \beta'$. Now if $c' \in \delta^T(r)$ then we have $T, w, \sigma[z' \mapsto e, c'] \models \beta'$ and since $d$ mimics $e$ at $v$, by Proposition 21 we have $M, w, \sigma[z' \mapsto d, c'] \models \beta'$.

Since we picked $c'$ and $v$ arbitrarily, we have $M, r, \sigma[z \mapsto d] \models \forall z' \Box \beta'$.

Now we are ready to prove that $M, r, \sigma \models \Lambda \Gamma$. First pick some $\alpha \in \Gamma$. We have $T, r, \sigma \models \alpha$ and we need to verify that $M, r, \sigma \models \alpha$. Suppose not then there is some $\chi \in C(\alpha)$ such that $T, r, \sigma \models \chi$ and $M, r, \sigma \models \neg \chi$ where $\chi$ is a module of or the form $Qz \alpha'$.

Note that $\sigma$ is a mapping from Var to $\delta^T(r)$. Assume that $\chi$ is a module then the assumption is a contradiction to Claim 1. Otherwise $\chi$ is of the form $Qz \alpha'$. Then by Proposition 17 every $\beta \in C(\alpha')$ is either a module or of the form $\forall z' \Box \beta'$ or $\exists z' \Box \beta'$. Now we have two cases:

If $\chi$ is of the form $\exists z \alpha'$ then let $c \in \delta^T(r)$ such that $T, r, \sigma[z \mapsto c] \models \alpha'$. But now if $M, r, \sigma[z \mapsto c] \not\models \alpha'$ then there is some $\beta' \in C(\alpha')$ such that $T, r, \sigma[z \mapsto c] \models \beta'$ and $M, r, \sigma[z \mapsto c] \models \neg \beta'$. This is a contradiction to Claim 2.

If $\chi$ is of the form $\forall z \alpha'$ then pick any arbitrary $d \in \delta^M(r)$ and we verify that $M, r, \sigma[z \mapsto c] \models \alpha'$ (this implies $M, r, \sigma \models \forall z \alpha'$ which contradicts the assumption) Suppose $d \in \delta^T(r)$ then by Claim 2, for every $\beta' \in C(\alpha')$ if $T, r, \sigma[z \mapsto d] \models \beta'$ then $M, r, \sigma[z \mapsto d] \models \beta'$. Hence, $M, r, \sigma[z \mapsto d] \models \alpha'$.

Suppose $d \in \{d_1, d_1', \ldots, d_l, d_l'\}$ then let $d \in \{d_k, d_k'\}$ for some $k \leq l$. Note that we have $\forall r, \sigma[z \mapsto a_k] \models \alpha'$. By Claim 3, for every $\beta' \in C(\alpha')$ if $T, r, \sigma[z' \mapsto a_k] \models \beta'$ then $M, r, \sigma[z' \mapsto a_k] \models \beta'$. Hence, $M, r, \sigma[z \mapsto d_k] \models \alpha'$. Thus we have proved that $M, r, \sigma \models \Lambda \Gamma$. 

\[ \sqcap \]
Now we verify $\mathcal{M}, r, \sigma \models \exists y_1 \exists y'_1 \ldots \exists y_l \exists y'_l \forall x \varphi[\exists y \psi]$. Let $d_1, d'_1, \ldots, d_l, d'_l$ be the witness for $y_1, y'_1, \ldots, y_l, y'_l$ respectively. Let $\sigma'$ be an extension of $\sigma$ where for all $i \leq l$ we have $\sigma'(y_i) = d_i$ and $\sigma'(y'_i) = d'_i$. Thus, we will verify that $\mathcal{M}, r, \sigma' \models \forall x \varphi[\exists y \psi]$. Pick any arbitrary $c \in \delta^\mathcal{M}(r)$. We claim that $\mathcal{M}, r, \sigma'_{[x=c]} \models \varphi[\exists y \psi]$. Again we have two cases:

- Suppose $c \in \delta^M(r)$ then note we have $\mathcal{T}, r, \sigma_{[x=c]} \models \varphi$. So, if the claim is false then there is some $\chi \in C(\varphi)$ such that $\mathcal{T}, r, \sigma_{[x=c]} \models \chi$ and $\mathcal{M}, r, \sigma_{[x=c]} \not\models \chi$ where $\bar{\chi} = \chi$ if $\chi \not\models \exists y \psi$ and otherwise $\bar{\chi} = \exists y \psi$.

  Also by Proposition 17 $\chi$ is either a module or of the form $\forall x \beta$ or $\exists x \beta$. Now, if $\chi \not\models \exists y \psi$ then $\bar{\chi} = \chi$ and by Claim 2, since $\mathcal{T}, r, \sigma_{[y=c]} \models \chi$ we have $\mathcal{M}, r, \sigma_{[y=c]} \models \chi$ which is a contradiction to the assumption.

  So let $\chi = \exists y \psi$. Then let $c' \in \delta^T(r)$ and $r \rightarrow s^c \in \mathcal{R}^T$ be such that $\mathcal{T}, s^c, \sigma_{[x=c]} \models \psi$. Let $\Pi(r, c') = \Pi(r, a_k)$ for some $k \leq l$. By construction we have a subtree $\mathcal{T}_{d'}^c$ of $\mathcal{M}$ rooted at $u^c$ which is a copy of $c$ (the subtree rooted at $s^c$ in $\mathcal{T}$) extended with $d_1, d'_1, \ldots, d_l, d'_l$ where $d_k$ mimics $c'$. So by Proposition 21 it follows that $\mathcal{M}, u^c, \sigma_{[x=c],d_k} \models \psi$ which implies $\mathcal{M}, u^c, \sigma'_{[x=c]} \models \psi[y_k/y]$. Hence $\mathcal{M}, r, \sigma'_{[x=c]} \models \exists y \psi$.

- Otherwise $c \in \{d_1, d'_1, \ldots, d_l, d'_l\}$. Let $c \in \{d_j, d'_j\}$ for some $j \leq l$. Note that we have $\mathcal{T}, r, \sigma_{[x=a_j]} \models \varphi$. So, if the claim is false then there is some $\chi \in C(\varphi)$ such that $\mathcal{T}, r, \sigma_{[x=a_j]} \models \chi$ and $\mathcal{M}, r, \sigma'_{[x=a_j]} \not\models \bar{\chi}$ where $\bar{\chi} = \chi$ if $\chi \not\models \exists y \psi$ and otherwise $\bar{\chi} = \exists y \psi$.

  Also by Proposition 17 $\chi$ is either a module or of the form $\forall x \beta$ or $\exists x \beta$. Now, if $\chi \not\models \exists y \psi$ then $\bar{\chi} = \chi$ and by Claim 3, since $\mathcal{T}, r, \sigma_{[y=a_j]} \models \chi$ we have $\mathcal{M}, r, \sigma_{[y=a_j]} \models \chi$ which is a contradiction to the assumption (note that $\sigma$ and $\sigma'$ differ only on the valuation of $y_1, y'_1, \ldots, y_l, y'_l$ and these variables do not occur in $\chi$ in this case).

  So let $\chi = \exists x \psi$. Then let $b \in \delta^T(r)$ and $r \rightarrow s^j \in \mathcal{R}^T$ be such that $\mathcal{T}, s^j, \sigma_{[x=a_j,b]} \models \psi$. Let $\Pi(r, b) = \Pi(r, a_k)$ for some $k \leq l$.

  Now by construction, we have the subtree $\mathcal{T}_{d_j}^j$ rooted at $v^j$ in $\mathcal{M}$ which is a copy of $\mathcal{T}^j$ (subtree rooted at $s^j$ in $\mathcal{T}$) such that $d_j$ and $d'_j$ mimic $a_k$ and $b$ respectively in $\mathcal{T}_{d_j}^j$. Hence by Proposition 21 we have $\mathcal{M}, v^j, \sigma_{[x=a_j,d_j]} \models \psi$ which implies $\mathcal{M}, v^j, \sigma'_{[x=d_j]} \models \psi[y_k/y]$. Hence, we have $\mathcal{M}, r, \sigma'_{[x=d_j]} \models \exists y \psi$.

  Similarly, we have the subtree $\mathcal{T}_{d'_j}^j$ rooted at $w^j$ and $w^j$ in $\mathcal{M}$ which is a copy of $\mathcal{T}^j$ (subtree rooted at $s^j$ in $\mathcal{T}$) such that $d'_j$ and $d_k$ mimic $a_k$ and $b$ respectively in $\mathcal{T}_{d'_j}^j$. Hence by Proposition 21 we have $\mathcal{M}, v^j, \sigma_{[x=a_j,d'_j]} \models \psi$ which implies $\mathcal{M}, v^j, \sigma'_{[x=d'_j]} \models \psi[y_k/y]$. Hence, we have $\mathcal{M}, r, \sigma'_{[x=d'_j]} \models \exists y \psi$.

  Thus, in both cases of $c \in \{d_j, d'_j\}$ we have $\mathcal{M}, r, \sigma'_{[x=c]} \models \exists y \psi$ (which is a contradiction to the assumption).

**Corollary 22.** Let $\Gamma'$ be a clean finite set of formulas such that every $\alpha \in \Gamma'$ is a subformula of $\text{ABBABE}$ where $\Gamma' = \Gamma \cup \{\forall x \varphi[\exists y \psi]\}$. If $\bigwedge \Gamma' \land \forall x \varphi[\exists y \psi]$ is satisfiable then $\bigwedge \Gamma \land \exists y_1 \exists y'_1 \exists y_2 \exists y'_2 \ldots \exists y_l \exists y'_l \forall x \varphi[\exists y \psi]$ is satisfiable, where $l = 2^{\# \Gamma'}$ and $\Gamma = y_1, y'_1, \ldots, y_l, y'_l$ are fresh variables.

To see why the corollary is true, by Lemma 19 we get some $l \leq 2^{\# \Gamma'}$, and we can pad sufficiently many dummy variables to get a strict equality. This gives us a useful tableau rule which we call $(\forall \exists \downarrow)$ rule for ABBAE fragment, described in figure 6. The full tableau rules for ABBAE are given by the tableau rules of LBF (Figure 5) along with the $(\forall \exists \downarrow)$-rule.
\[ w : \forall x \varphi[\exists y \psi], \; \Gamma, \; \sigma \vdash (\forall \exists) \]
\[ w : \forall x \varphi[\exists y \psi], \; \Gamma, \; \sigma' \vdash (\forall \exists) \]

where \( l = 2^{l_1 + | \sigma |} \) and \( \overline{y} = y_1, y'_1, \ldots, y_l, y'_l \)
are fresh variables and \( \sigma' = \sigma \cup \{(y_i, y_i'), (y'_i, y'_i) \mid i \leq l\} \)

\[ \text{Figure 6 (}\forall \exists\text{) rule for ABBABE fragment} \]

\[ \text{Theorem 23. For any clean ABBABE formula } \theta, \text{ let } \text{Dom}(\sigma_r) = \text{Free}(\theta) \cup \{z\} \text{ where } z \text{ does not occur in } \theta. \text{ There is an open tableau with } (r : \{\theta\}, \sigma_r) \text{ as the root iff } \theta \text{ is satisfiable in an increasing domain model.} \]

\[ \text{Proof. First note that } (\forall \exists) \text{ preserves cleanliness because the new free variables } y_1, y'_1, \ldots, y_l, y'_l \text{ are fresh and hence not bound by any other quantifier in the antecedent.} \]

\[ \text{Let } \overline{d} \in \text{Free}(\theta) \cup \{z\}. \text{ We only prove this for } (\exists \exists) \text{ and for every LBF formula } \varphi \text{ if } \varphi \in \Gamma \text{ then } M, w, \sigma \models \varphi. \]

The proof of the claim is by induction on the nodes of \( T \) from leaf nodes to the root which same as in the proof of Theorem 16. The only additional part is to prove the inductive claim for the application of (\forall \exists) rule at a node (w : \Gamma, \sigma).

In this case \( \Gamma \) is of the form \( \Gamma' \cup \{\forall x \varphi[\exists y \psi]\} \) and we have a descendant (w : \Gamma' \cup \{\forall x \varphi[\exists y \psi]\}, \sigma') where \( \sigma' = \sigma \cup \{(y_i, y_i'), (y'_i, y'_i) \mid i \leq l\} \).

First note that since \( y_1, y'_1, \ldots, y_l, y'_l \) are fresh, by induction hypothesis, for every \( \varphi' \in \Gamma' \) we have \( M, w, \sigma \models \varphi' \). Now to prove that \( M, w, \sigma \models \forall x \varphi[\exists y \psi] \), pick any arbitrary \( z \in \text{Dom}(\sigma) \). We claim that \( M, w, \sigma_{[x \leftarrow z]} \models \varphi[\exists y \psi] \).

Suppose \( M, w, \sigma_{[x \leftarrow z]} \not\models \varphi[\exists y \psi] \) then we can evaluate the conjunction and disjunctions and obtain some subformula \( \chi \) of \( \varphi[\exists y \psi] \) such that \( M, w, \sigma_{[x \leftarrow z]} \models \chi \) and \( M, w, \sigma_{[x \leftarrow z]} \not\models \chi \).

So suppose \( M, w, \sigma_{[x \leftarrow z]} \not\models \varphi[\exists y \psi] \) then we can evaluate the conjunction and disjunctions and obtain some subformula \( \chi \) of \( \varphi[\exists y \psi] \) such that \( M, w, \sigma_{[x \leftarrow z]} \models \chi \) and \( M, w, \sigma_{[x \leftarrow z]} \not\models \chi \).

If \( \chi = \exists y \psi \) then \( M, w, \sigma_{[x \leftarrow y]} \models \bigvee_{i \leq l} \left( \Diamond \psi_{[y_i]} \lor \Diamond \psi_{[y'_i]} \right) \). So there is some \( j \leq l \) and \( \psi \in \{\psi_{[y_j]}, \psi_{[y'_j]}\} \) and \( y_j \in \{y_j, y'_j\} \) such that \( M, w, \sigma_{[x \leftarrow y]} \models \Diamond \psi \).

Since \( \text{FV}(\psi) = \text{FV}(\psi) \cup \{y_j\} \), we have \( M, w, \sigma_{[x \leftarrow y]} \models \Diamond \psi \). Thus \( M, w, \sigma_{[x \leftarrow z]} \models \exists y \psi \).

If \( \chi \) is a contradiction.

Otherwise \( \chi = \varphi[\exists y \psi] \) which is a contradiction.

Hence \( M, w, \sigma_{[x \leftarrow y]} \) do not occur in \( \chi \). Hence \( M, w, \sigma_{[x \leftarrow z]} \models \chi \) iff \( M, w, \sigma_{[x \leftarrow y]} \) which is a contradiction.

First note that we can prove a proposition analogous to Proposition 15 since if \( \Gamma \) is not Existential-safe then either (\exists) rule can be applied or (\forall \exists) rule can be applied. Thus, we can always get a saturated tableau and to show that such a tableau is open, it is sufficient to show that all rules preserve satisfiability. We only prove this for (\forall \exists) rule (Other rules are already proved in Theorem 16).

Let \( M, w, \pi \models \forall x \varphi[\exists y \psi] \land \Lambda \Gamma \). By lemma 19 there exists \( M' \) and \( w' \in \forall' \) and \( \pi' = \Lambda \Gamma \land \exists y_1 \exists y'_1 \ldots \exists y_{l'} \exists y'_l \forall x \varphi[\exists y \psi] \).

Let \( d_1, d'_1, \ldots, d_l, d'_l \in \delta'(w') \) be the witness for \( y_1, y'_1, \ldots, y_l, y'_l \) respectively and let \( \pi'' \) be the extension of the assignment \( \pi \) where \( y_1, y'_1, \ldots, y_l, y'_l \) are assigned to \( d_1 d'_1, \ldots, d_l d'_l \) respectively. So we have \( M', w', \pi'' = \Lambda \Gamma \land \forall x \varphi[\exists y \psi] \) as required.
Note that at if we start with a formula of length \( n \) then the application of \((\forall \exists \cdot)\) rule will blow up the formula to size \( 2^n \). Thus the size of the tableau is \( 2^{O(n^2)} \). So we have a tableau procedure entails an \( \text{ExpSpace} \) algorithm.

5.3 Lower bound

The \( \text{PSpace} \) upper bound for the fragments \( \text{AB}, \text{EB} \) and \( \text{BA} \) follows for these fragments since we do not need ‘clean rewriting’ of formulas in the tableau rules specialized for these fragments from propositional modal logic. So the tableau size remains polynomial and hence satisfiability problem for these fragments are \( \text{PSpace}\)-complete. The \( \text{PSpace} \) lower bound follows from that of propositional modal logic.

We can prove a \( \text{NexpTime} \) lower bound for \( \text{AEBE} \) and \( \text{BAEB} \) fragments over increasing domain models which implies the same lower bound for \( \text{LBF} \) and other fragments that contains these as sub fragments.

The \( \text{NexpTime} \) complete version of the tiling problem is that given a tiling instance \( T = (T, H, V, t_0) \) and a natural number \( n \) (in unary representation), does there exists a tiling function \( f: [0, 2^n]^2 \rightarrow T \) that can tile the \( 2^n \times 2^n \) grid \( [17] \).

Given a natural number \( k \in [0, 2^n] \), it has a binary representation with \( n \) bits. We use unary predicates \( P_0, P_1, \ldots, P_{n-1} \) using which \( k \) can be encoded. For example, given \( n = 3 \) and \( k = 5 \), the binary representation of 5 is 101. It can be encoded by a domain element \( d \) at \( w \) such that \( M, w \models P_2(d) \land \neg P_1(d) \land P_0(d) \).

First, we need a set of formulas to force exponential size domain with respect to \( n \), and then use this domain to encode the tiling problem. The encoding formulas for \( \text{AEBE} \) fragment are given in Figure 7.

Formula \( \varphi_0 \) says natural number 0 exists at the root. Formula \( \varphi_1 \) ensures that the encoding propagates to successors. Formula \( \varphi_2 \) asserts that for every position \( i \) and every element \( x \) if \( i^{th} \) bit of \( x \) is 0 then there exists \( y \) such that in all the successor worlds, \( i^{th} \) bit of \( y \) is 1, and for all \( j \neq i, j^{th} \) bit of \( x \) and \( y \) is identical. Formula \( \alpha_n \) propagates \( \varphi_1 \) and \( \varphi_2 \) to \( n \) depth.

In a given model \( M \) for all \( u, v \in W \) we say that \( v \) is at a distance \( n \) from \( u \) if there is some path from \( u \) to \( v \) of length \( n \) (note that \( v \) could be at distance \( m \) and also at distance \( n \) from \( u \)).

Let \( \Sigma^n_j \) be the set of all \( n \)-length binary strings in which there are at most \( j \) many 1-bit. Clearly, \( \Sigma^n_j \) covers all binary strings of length \( n \). Also for every string \( s \) of length \( n \) let \( s(j) \) denote the \( j^{th} \) position of \( s \).

Lemma 24. The formula \( \alpha^n \) is satisfiable. Moreover, for all models \( M \) and \( r \in W \) if \( M, r \models \alpha^n \) then for every \( 0 \leq j < n \) there exists some world \( w \) at distance \( j \) from \( r \) and for every world \( w \) at a distance \( j \) from \( r \), there exists a one-one function \( f_w : \Sigma^n_j \rightarrow \delta(w) \) such that if \( f_w(s) = d \) then for every \( 0 \leq k < n \), \( s(k) = 1 \) iff for every \( u \) which is a child of \( w \), we have \( d \in \rho(u, P_k) \).

Proof. To prove that \( \alpha^n \) is satisfiable, consider the model \( M = (W, R, D, \delta, \rho) \) where

\(^5\) \( \text{BE} \) fragment needs clean rewriting and hence we get an \( \text{ExpSpace} \) upper bound.
\(^6\) The authors of [12] claim \( \text{PSpace} \) upper bound for \( \text{ABEB} \) fragment which is rectified here. The bug is that they do not consider clean rewriting in the intermediate steps and hence calculate the tableau size to be polynomial.
The invariant holds since $\varphi \leq \phi$ then for every position $0 < f$ define $w$ by induction hypothesis there is $\rho$ and $\varphi_0 := \exists x \Box( \bigwedge_{0 \leq i \leq n} \neg P_i(x) )$

$\varphi_1 := \forall x \Box \left( \bigwedge_{0 \leq i \leq n} \left( P_i(x) \rightarrow \Box P_i(x) \land \neg P_i(x) \rightarrow \Box \neg P_i(x) \right) \right)$

$\varphi_2 := \forall x \Box \left( \bigwedge_{0 \leq i \leq n} \left( \neg P_i(x) \rightarrow \exists y \Box \left( P_i(y) \land \bigwedge_{j \leq i} P_j(x) \leftrightarrow P_j(y) \right) \right) \right)$

$\alpha^n := \varphi_0 \land \Box \Box \left( \varphi_1 \land \varphi_2 \land \diamond \top \right)$

where $\Box \Box \varphi = \psi \land \Box \Box \neg \psi$ and $\Box \Box \varphi = \top$

$\psi_0 := \exists x \Box \left( \bigwedge_{0 \leq i \leq n} \left( \neg P_i(x) \right) \rightarrow Q_{0n}(x,x) \right)$

$\psi_1 := \forall x \Box \forall y \Box \left( \bigvee_{i \leq i} \left( Q_i(x,y) \land \bigwedge_{j \neq i} \neg Q_{ij}(x,y) \right) \right)$

$\psi_2 := \forall x \Box \forall y \Box \forall z \Box \left( \text{succ}(x,y) \rightarrow \bigvee_{(t,t') \in H} \left( Q_i(x,z) \land Q_{ij}(y,z) \right) \right)$

$\psi_3 := \forall x \Box \forall y \Box \forall z \Box \left( \text{succ}(x,y) \rightarrow \bigvee_{(t,t') \in V} \left( Q_i(x,z) \land Q_{ij}(z,y) \right) \right)$

where $\text{succ}(x,y) := \bigvee_{0 \leq i \leq n} \left( \neg P_i(x) \land P_i(y) \land \bigwedge_{j < i} \left( P_j(x) \land \neg P_j(y) \right) \land \bigwedge_{j < i} \left( P_j(x) \leftrightarrow P_j(y) \right) \right)$

$\beta^n := \alpha^n \land \Box^n \left( \varphi_1 \land \Box \varphi_1 \land \Box \Box \varphi_1 \land \Box \top \land \psi_0 \land \psi_1 \land \psi_2 \land \psi_3 \right)$

![Figure 7 ABEB formulas for encoding NEXP-time complete tiling problem over increasing domain models](image)

- $W = \{w_0, w_1, \ldots, w_n, w_{n+1}\}$
- $\mathcal{R} = \{(w_i, w_{i+1}) \mid i \leq n\}$
- $\mathcal{D} = \bigcup_{j=0}^{n} \Sigma^n_j$ and $\delta(w_i) = \mathcal{D}$ for all $i \leq n + 1$
- For every $P_j$ where $j \leq n$ and every $w_i \in W$ define $\rho(w_i, P_j) = \{s \mid s \in \mathcal{D} \text{ and } s(j) = 1\}$

It can be verified that $\mathcal{M}, w_0 \models \alpha^n$.

For the second part of the lemma, let $\mathcal{M}, r \models \alpha^n$. The proof is by induction on $j$.

In the base case, $j = 0$. Let $d_0 \in \delta(r)$ be the witness for $x$ in $\varphi_0$. Define $f_0(0^n) = d_0$. The invariant holds since $\varphi_0$ ensures that for all $0 \leq k < n$ and every $r \rightarrow u \in \mathcal{R}$ we have $d_0 \notin \rho(u, P_k)$.

In the induction step, let $u'$ be any world at distance $j < n$ from $r$. By $\alpha^n$ there is some $w' \rightarrow w \in \mathcal{R}^M$ which implies $w$ is at a distance $j + 1$ from $r$.

Now pick any arbitrary $w$ be at a distance $j + 1$ from $r$ and let $w'$ be the parent of $w$. By induction hypothesis there is $f_{w'} : \Sigma^n_j \rightarrow \delta(w')$ such that for all $s \in \Sigma^n_j$ if $f_{w'}(s) = d$ then for every position $0 \leq k < n$ (since $w$ is the child of $w'$), $s^k = 1$ iff $d \in \rho(w, P_k)$. Now, define $f_w : \Sigma^n_{j+1} \rightarrow \delta(w)$ such that for every $t \in \Sigma^n_{j+1}$:

- If $t \in \Sigma^n_j$ then $f_w(t) = f_{w'}(t)$.
- Otherwise, let $k$ be an arbitrary position such that $k^{th}$ bit of $t$ is 1. Consider $t'$ such that for all $l \neq k$ we have $t'(l) = t(l)$ and $t'(k) = 0$. Then, $t' \in \Sigma^n_j$ and let $f_w(t') = d'$. By induction hypothesis (since $w$ is child of $w'$), for every position $j$ we have $t'(j) = 1$ iff $d' \in \rho(w, P_j)$. 
Now, by $\alpha^n$ we have $\mathcal{M}, w \models \Box \top$. Hence there is at least one child for $w$ and moreover, since $\mathcal{M}, w \models \varphi_1$, for every child $u$ of $w$ and for every position $j$ we have $\ell'(j) = 1$ iff $d \in \rho(u, P_j)$.

Now since $\mathcal{M}, w \models \neg P_k(d')$ and $\mathcal{M}, w \models \varphi_2$, there is some $d$ (witness for $y_k$) such that for every child $u$ of $w$, we have $\mathcal{M}, u \models P_k(d) \land \bigwedge_{k' \neq k} P_{k'}(d') \leftrightarrow P_{k'}(d)$.

Define $f_w(t) = d$.

To see that the claim holds, if $t \in \Sigma^n_j$ then the claim holds since $\mathcal{M}, w' \models \varphi_1 \land \Box \varphi_1$. Otherwise, if $t \in \Sigma^n_{j+1}$ then we have $t' \in \Sigma^n_j$ (as defined above) and $t$ and $t'$ differ exactly in the position $k$. Hence by $\mathcal{M}, w \models \varphi_2$, the defined $f_w(t) = d$ satisfies the required condition of the invariant.

Thus, the exponential tiling problem is encoded in the scope of $n$ modal depth as described in Figure 7. Note that in this encoding we have exponential domain instead of the grid. Thus, we use the binary predicate $Q$ described in Figure 7. Note that in this encoding we have exponential domain instead of the grid.

The formula $\psi_0$ says $(0,0)$ has tile $t_0$; $\psi_1$ ensures every grid point has exactly one tile; $\psi_2$ and $\psi_3$ encode horizontal and vertical constraints respectively. Also, $\varphi_1 \land \Box \varphi_1 \land \Box \Box \varphi_1$ ensures that the ‘bit encoding’ continues for the next 3 modal depth. The formula $\Diamond^3 \top$ is to ensure that there is some world after 3 modal depth. Finally, $\beta^n_T$ is the ABEB formula that encodes the exponential tiling problem. Note that the size of $\beta$ is polynomial in the size of the input instance (since $n$ is given in unary representation).

**Theorem 25.** For all tiling instance $T = (T, H, V, t_0)$ and all $n \in \mathbb{N}$, there is a proper tiling $f : (2^n \times 2^n) \to T$ if and only if $\beta^n_T$ is satisfiable.

**Proof.** Suppose there is a tiling, then consider the model $\mathcal{M} = (W, R, D, \delta, \rho)$ where

- $W = \{w_0, \ldots, w_n, w_{n+1}\} \cup \{u_0, u_1, u_2\}$
- $R = \{\langle w_i, w_{i+1} \rangle \mid i \leq n\} \cup \{\langle w_{n+1}, u_0\rangle, \langle u_0, u_1\rangle, \langle u_1, u_2\rangle\}$
- $D = \{0, 1, \ldots, 2^n\}$ and for all $v \in W$, $\delta(v) = D$
- For all $v \in W$ and for all $0 \leq k < n$
  - $\rho(v, P_k) = \{d \mid k^{th} \text{ bit of } d \text{ in binary representation is } 1\}$ and for $u_2$ and every tile $t \in T$ we have
  - $\rho(u_2, Q_3) = \{(c, d) \mid (c, d) \text{ has the tile } t \text{ in the tiling}\}$

It can be verified that $\mathcal{M}, w_0 \models \beta^n_T$.

For the other direction, let $\mathcal{M}, r \models \beta^n_T$. First note that since $\mathcal{M}, r \models \alpha^n$ there exists a $w$ at distance $n$ from $r$ and by Lemma 23 we assume that $\{0, 1, \ldots, 2^n\} \subseteq \delta(w)$ such that and for all $u$ that is a child of $w$ and every $d \in \{0, 1 \ldots 2^n\}$ we have $\mathcal{M}, u \models P_h(d)$ iff $k^{th}$ bit in the binary representation of $d$ in 1.

Define the tiling $g : \{0, 1, \ldots, 2^n\}^2 \to T$ where $g(c, d) = t$ iff $\mathcal{M}, w \models \Box \Box \Box \psi_1(c, d)$. To prove that $g$ is well defined, note that $\mathcal{M}, w \models \Diamond^3 \top$ and also since $\mathcal{M}, w \models \psi_1$ there is a unique tile for every grid point $(c, d)$.

Further, since $\mathcal{M}, w \models \psi_0$, we have $g(0,0) = t_0$. Finally, $\psi_2$ and $\psi_3$ ensure that the horizontal and vertical tiling constraints are satisfied.

**Corollary 26.** The satisfiability problem for ABEB over increasing domain is NExpTime-hard.
The encoding formulas of BABE fragment are very similar. We only need to swap positions of □ and quantifiers to suit the fragment in the encoding formulas (Fig. 7). The details are omitted.

**Corollary 27.** The satisfiability problem for BABE over increasing domain is NExpTime-hard.

6 Conclusion

We began with the question: are bundles good deals? Now we see the answer is: *it depends*, on the combinations of bundles you have chosen, and whether you work with varying domain interpretations or insist on constant domains.

In this paper, we have studied the decidability of bundled fragments of FOML, where we have no restrictions on the use of variables or arity of relations. Over constant domain interpretations, it is only the ∃□ bundle that is well-behaved, but over increasing domain interpretations, we get a trichotomy of decidability, lack of finite model property and undecidability. The obvious question is to settle the issue of decidability for cases where we have only shown lack of finite model property. This work is one step in the programme of mapping the terrain of decidable bundled fragments and identifying the borderline between decidability and undecidability. We identify some strands that constitute immediate next steps in the programme.

In the context of verification of infinite state systems, security theory and database theory, we are often more interested in the model checking problem. If the domain is finite, this is no different from model checking of first order modal logic. However we are usually interested in the specification being checked against a finitely specified (potentially infinite) model, e.g. when the domain elements form a regular infinite set. This is a direction to be pursued in the context of bundled fragments.

The applications we have referred to often call for reachability analysis rather than the study of single updates. Moreover, the richness of modal logic stems from its extensions over various classes of frames, and hence the study of bundles over models with various frame conditions is relevant. Unfortunately, while it is clear that equivalence frames seem to lead to undecidability [20], even with transitive frames the situation is unclear. Obtaining good decidable fragments over linear frames is an important challenge.

We consider only the “pure” fragments, without constants, function symbols, or equality. The addition of constants is by itself simple, but equality complicates things considerably. Since equality is extensively used in specifications, mapping fragments with equality is an important direction.

While the decision procedure for the ∃□ bundle is tight in terms of complexity, the other results involve only an ExpSpace upper bound, leaving a gap between the upper bound and known lower bound. We need sharper technical tools for investigating lower bounds for bundled fragments.

We have presented tableau-based decision procedures that are easily implementable, but inference systems for reasoning in these logics require further study.

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