Mapping Schrödinger equation into a Heun-type and identifying the corresponding potential function, energy and wavefunction

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Abstract: We transform the Schrödinger wave equation to a nine-parameter Heun-type differential equation. Using our solutions of the latter in [J. Math. Phys. 59 (2018) 113507], we are able to identify the associated potential function, energy parameter, and write the corresponding wave function. Some of the solutions obtained correspond to new integrable quantum systems.

Keywords: Heun-type equation, potential functions, orthogonal polynomials, recursion relation.

1. Introduction

Recently, we introduced the following nine-parameter Heun-type differential equation [1]

$$
\left[ \frac{d^2}{dy^2} + \left( \frac{a - b - c}{1 - y} - \frac{d}{d - y} \right) \frac{d}{dy} + \left( \frac{A}{y} - \frac{B}{1 - y} - \frac{C}{d - y} + yD - E \right) \frac{1}{y(1-y)(d-y)} \right] \chi(y) = 0, \quad (1)
$$

where \( \{a,b,c,d,A,B,C,D,E\} \) are real dimensionless parameters with \( d \neq 0,1 \) and positive. We can always take \( d > 1 \). In fact, even if \( d < 1 \) then we can rewrite the equation with the same exact form as (1) but with a redefined set of parameters \( \{a',b',c',d',A',B',C',D',E'\} \) such that \( d' > 1 \). This is accomplished by making the replacement \( y \to y' \) in (1) resulting in the following redefined parameters

$$
d' = d^{-1}, \quad (a',D') = (a,D), \quad (b',c') = (c,b), \quad (A',B',C') = (A,C,B)d^{-2}, \quad E' = Ed^{-1}. \quad (2)
$$

Then, the solution of Eq. (1) is obtained from the solution of the reparametrized equation as \( \chi(y) = \chi'(y/d) \). Now, equation (1) has four regular singularities at \( y = \{0,1,d,\infty\} \). The original Heun equation corresponds to \( A = B = C = 0 \) and \( D = \alpha \beta \) with the regularity condition at infinity that \( \alpha + \beta + 1 = a + b + c \) [2,3]. If the differential equation parameters \( A \) and \( B \) are below/above curtain critical values such that \( \frac{A}{d} \leq \frac{1}{4}(1-a)^2 \) and \( \frac{B}{d-1} \geq \frac{1}{4}(1-b)^2 \) then by using the Tridiagonal Representation Approach (TRA) [4], we were able to obtain four classes of solutions of this equation [1]. The differential equation parameters for each solution class must satisfy the respective constraints shown in Table 1. All of these solutions are written as convergent series of square integrable functions \( \{\phi_n(y)\} \) as follows

$$
\chi(y) = \sum_{n} f_n \phi_n(y). \quad (3)
$$

The expansion coefficients are written as \( f_n = f_n p_n \) and \( \{p_n\} \) turn out to be orthogonal polynomials, some of which are either new or modified versions of known polynomials [1]. The argument and parameters of these polynomials are related to the differential equation

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parameters whereas their positive weight function is $f_a^2$. The square integrable basis functions \( \{ \phi_n(y) \} \) are written in terms of the Jacobi polynomials $P_{n}^{(\mu,\nu)}(y)$ as follows

$$
\phi_n(y) = A_n y^n (1-y)^\beta (d-y)^\gamma P_{n}^{(\mu,\nu)}(y).
$$

The normalization constant is conveniently chosen as $A_n = \sqrt{(2n+\mu+\nu+1) \frac{\Gamma(n+1)\Gamma(n+\mu+\nu+1)}{\Gamma(n+\mu+1)\Gamma(n+\nu+1)}}$.

The basis parameters $\{ \alpha, \beta, \gamma, \mu, \nu \}$ are related to the differential equation parameters as shown in Table 2 for each of the four solution classes. The Jacobi polynomial is defined as

$$
P_{n}^{(\mu,\nu)}(y) = \frac{\Gamma(n+\mu+1)}{\Gamma(n+1)\Gamma(\mu+1)} P_n(-n,n+\mu+\nu+1;\mu+1;1-y),
$$

with $\mu > -1$, $\nu > -1$ and $y \in [0,1]$. This polynomial definition is obtained by the replacement $y \rightarrow 2y-1$ in the classical definition in which $y \in [-1,1]$.

In this work, we make a transformation that maps the Schrödinger equation into Eq. (1). Consequently, we will be able to identify the potential function, energy and corresponding solutions (wavefunctions) using the results found in [1]. The transformation used is a combination of an independent and dependent variable transformation (i.e., coordinate and wavefunction transformations). In the atomic units $h = m = 1$, the time-independent one-dimensional Schrödinger equation reads as follows

$$
\left[-\frac{1}{2} \frac{d^2}{dx^2} + V(x) - \mathcal{E}\right] \psi(x) = 0,
$$

where $V(x)$ is the potential function and $\mathcal{E}$ is the energy. In three dimensions with spherical symmetry, this equation could also be taken as the radial wave equation (with $x = r$) if we can write $V(r) = \frac{\ell(\ell+1)}{2r^2} + \tilde{V}(r)$, where $\ell$ is the angular momentum quantum number and $\tilde{V}(r)$ is the radial interaction potential. Now, we make the coordinate transformation $x \rightarrow y(x)$ and write $\psi(x) = h(y) \chi(y)$. If we define $\frac{dy}{dx} = \lambda g(y)$, where $\lambda$ is a positive scale parameter with inverse length dimension, then these transformations map Eq. (6) into the following second order differential equation in the dimensionless $y$ variable

$$
h g^2 \left[ \frac{d^2}{dy^2} + \left( \frac{g'}{g} + 2 \frac{h'}{h} \right) \frac{d}{dy} + \frac{h''}{h} + \frac{g'h'}{gh} - \frac{U}{hg^2} \right] \chi(y) = 0,
$$

where $U = \frac{2}{\lambda^2} (V - \mathcal{E})$ and the prime stands for the derivative with respect to $y$. Identifying this equation with (1) and assuming that $hg^2 \neq 0$ within the open interval $y \in (0,1)$ dictate that

$$
\frac{g'}{g} + 2 \frac{h'}{h} = \frac{a}{y} - \frac{b}{1-y} - \frac{c}{d-y},
$$

(8a)
\[ U = \frac{hg^2}{y(1-y)(d-y)} \left[ -\frac{A}{y} + \frac{B}{1-y} + \frac{C}{d-y} - yD + E + y(1-y)(d-y) \left( \frac{h''}{h} + \frac{g'h'}{gh} \right) \right]. \quad (8b) \]

Compatibility of the wavefunction expansion (3) and basis ansatz (4) with Eq. (8a) suggest that we can take \( g(y) = y^a(1-y)^b \) and \( h(y) = (d-y)^c/2 \). Thus, the configuration space coordinate could be obtained in terms of \( y \) by evaluating the integral \( \lambda x = \int y^{-a}(1-y)^{-b} \, dy \). This is done in Appendix A where \( x(y) \) is written in terms of the incomplete beta function \([5,6]\). The inverse transform that gives the coordinate transformation \( y(x) \) is simple only for special values of the parameters \( a \) and \( b \). Now, substituting \( g(y) \) and \( h(y) \) in Eq. (8b) gives

\[ \frac{2}{\lambda^2} (V - \mathcal{E}) = y^{2a-1}(1-y)^{2b-1}(d-y)^{-1+c/2} \left\{ E + \frac{c}{2} \left[ \frac{c}{2} - 1 \right] (d-1) - a \right\} 
+ y \left[ -D + \frac{c}{2}(a+b+c-1) \right] + \frac{1}{d-y} \left[ C + \frac{cd}{2} \left( \frac{c}{2} - 1 \right) (1-d) \right] - A + \frac{B}{1-y} \]  

Moreover, the wavefunction series becomes

\[ \psi(x) = h(y)\chi(y) = f(z) y^a(1-y)^b(d-y)^{-c/2} \sum_n A_n p_n(z) P_n^{(a,b)}(y), \quad (10) \]

where \( z \) is some proper function of the differential equation parameters. In the ensuing analysis, we assume that the potential satisfies the following two conditions:

1. It is an energy independent function, and
2. It vanishes at infinity (i.e., at \( +\infty \) if \( x \geq 0 \), and at \( -\infty \) or at \( \pm\infty \) if \( -\infty < x < +\infty \)).

Thus, the parameters \( \{a,b,c\} \) must be chosen so that the right-hand side of Eq. (9) contains a constant to be identified with the energy parameter \(-2\mathcal{E}/\lambda^2\). We note that the individual terms inside the curly brackets in Eq. (9) are linear polynomials in \( y \) raised to the power zero or \( \pm1 \). Therefore, a necessary, but not sufficient, condition for the right-hand side of Eq. (9) to contain a constant is that the factor \( y^{2a-1}(1-y)^{2b-1}(d-y)^{-1+c/2} \) becomes a ratio of two polynomials in \( y \) each with a maximum degree of two and such that the difference between their two degrees is less than or equal to one. Consequently, it is necessary but not sufficient that the exponents of \( y \), \( 1-y \) and \( d-y \) in this factor be either zero or \( \pm1 \). That is, the parameters \( a \) and \( b \) should assume one of the values in the set \( \{0, \frac{1}{2}, 1\} \) whereas \( c \) must belong to the set \( \{0, 2, 4\} \). However, due to the exchange symmetry \( a \leftrightarrow b \) and \( y \leftrightarrow 1-y \) we end up with 18 independent choices for the parameter set \( \{a,b,c\} \) out of the possible 27. Moreover, the stated constraint on the degrees of the polynomial ratios in the factor \( y^{2a-1}(1-y)^{2b-1}(d-y)^{-1+c/2} \) leaves only 12 out of the 18 choices. For each of these 12 cases, Table 3 shows the corresponding coordinate transformation, potential function and energy parameter. Note that despite in 9 out of the 12 cases the energy parameter appears in the potential function, a simple parameter redefinition eliminates this superficial energy dependence of the potential. For example, in the case \( (a,b,c) = (0,1,2) \) the parameter redefinition \( E \rightarrow E - B \) will eliminate the energy parameter \( B \) from the potential function. It is interesting to observe from Table 3 an association of the energy parameter with the type of physical configuration space. For example, the parameter \( A \) is associated with the energy of the problem whose configuration space is the
whole real line. Whereas, the parameter $B$ is associated with the energy of five problems whose configuration space is half of the real line. On the other hand, the parameters $C$, $D$, and $E$ are associated with the energy of problems whose configuration space is a finite segment of the real line.

Table 2 gives the basis parameters $\{\alpha, \beta, \gamma, \mu, \nu\}$ in terms of the differential equation parameters $\{a, b, c, A, B\}$ making the basis functions (4) fully defined. Therefore, to completely determine the solution of the Schrödinger equation as given by the series expansion (10), we need only to figure out the polynomial coefficients $\{p_n(z)\}$ and their weight function $f_0^2(z)$.

In the following three sections, we do that for each of the solution classes obtained in [1]. The physical properties of any system corresponding to a given set of parameters $(a, b, c)$ in a given class is determined from the properties of the corresponding polynomials such as the weight function, generating function, asymptotics, spectrum formula, zeros, etc. [4,7]. However, these properties are known only for the Wilson polynomial, which is associated with the restricted solution class. The other two solution classes are either associated with a new polynomial or with a modified version of the Wilson polynomial, the properties of both are not yet known in the literature and need to be derived by experts in the field of orthogonal polynomials. Nonetheless, these new polynomials could be written explicitly to all degrees (albeit not in a closed form) using their respective recursion relation and initial value. Consequently, the series representation of the wavefunction (10) is fully determined and, in accordance with the postulates of quantum mechanics, the corresponding physical system is well defined.

2. The general solution class

Table 4 is a reproduction of Table 3 after imposing the class constraint $4C = (1 - c)^2 d(d - 1)$. The four cases corresponding to $c = 4$ has been eliminated because the corresponding potential functions are energy dependent. Aside from the scale parameter $\lambda$, all potentials in this class have four parameters. The orthogonal polynomials associated with this class of solutions satisfy the following symmetric three-term recursion relation [see Eq. (B10b) of Appendix B in Ref. 1]

\[
\left(\frac{1}{2} - R\right) p_n(z) = G_{n-1}(S_{n-1} + D)p_{n-1}(z) + G_n(S_n + D)p_{n+1}(z)
\]

\[
\left[-\frac{n(n + \mu)}{2n + \mu + \nu} + d\left(n + \frac{\nu + 1}{2}\right) + \frac{1}{2}(F_n + 1 - 2d)(S_n + D) - \frac{1}{4}(\nu + 1)^2\right] p_n(z)
\]

for $n = 0, 1, 2, \ldots$ with $p_0(z) = 1$, $p_1(z) := 0$ and where

\[
R = \frac{B}{d - 1} + Dd - E - \frac{c}{2}\left[d(a + b + c - 2) - a - \frac{c}{2} + 1\right] + \frac{1}{4}(1 - a)^2, \quad (12a)
\]

\[
S_n = \left(n + \frac{\mu + \nu}{2} + 1\right)^2 - \frac{1}{4}(a + b + c - 1)^2, \quad F_n = \frac{\nu^2 - \mu^2}{(2n + \mu + \nu)(2n + \mu + \nu + 2)}, \quad (12b)
\]

\[
G_n = \frac{1}{2n + \mu + \nu + 2} \sqrt{\frac{(n + 1)(n + \mu + 1)(n + \nu + 1)(n + \mu + \nu + 1)}{(2n + \mu + \nu + 1)(2n + \mu + \nu + 3)}}, \quad (12c)
\]
Comparing (11) to the recursion relation of the Racah-Heun polynomial given in Appendix B by (B6), we conclude that

$$ p_n(z) = (-1)^n W_n^\kappa(z^2; \sigma - \tau, \sigma + \tau, \eta, \eta) \text{ where} $$

$$ \kappa = d, \quad 2\sigma = \mu + 1, \quad 2\eta = \nu + 1, \quad 2\tau = \sqrt{(a + b + c - 1)^2 - 4D}, \quad z^2 = \frac{1}{2} - R. \quad (13) $$

In the comparison, we have used the following identity:

$$ \frac{(n + \nu + 1)(n + \mu + \nu + 1)(S_n + D)}{(2n + \mu + \nu + 1)(2n + \mu + \nu + 2)} + \frac{n(n + \mu)(S_{n-1} + D)}{(2n + \mu + \nu)(2n + \mu + \nu + 1)} = - \frac{n(n + \mu)}{2n + \mu + \nu} + \frac{1}{2}(1 + F_n)(S_n + D) \quad (14) $$

Finally, the exact series representation of the wavefunction in this general class is written as:

$$ \psi(x) = \sqrt{\rho^\kappa(z)} y^{v+1-a}(1 - y)^{v+1-b} (d - y) \sum_n (-1)^n A_n W_n^\kappa(z^2; \sigma - \tau, \sigma + \tau, \eta, \eta) P_n^{(\mu, \nu)}(y). \quad (15) $$

### 3. The special solution class

Table 5 is a reproduction of Table 3 after imposing the class constraint $4C = c(c - 2)d(d - 1)$. Similar to the general class, the four cases corresponding to $c = 4$ has been eliminated because the corresponding potential functions are energy dependent. Also, all potentials in this class have four parameters in addition to the scale parameter $\lambda$. The orthogonal polynomials associated with this class of solutions satisfy the following symmetric three-term recursion relation [see Eq. (B10a) of Appendix B in Ref. 1]

$$ (2d - 1)p_n(z) = \left[ \frac{2\bar{R}}{T_n + D} + F_n \right] p_n(z) + 2G_{n+1}p_{n-1}(z) + 2G_np_{n+1}(z), \quad (16) $$

for $n = 0, 1, 2, \ldots$ with $p_0(z) = 1$, $p_1(z) := 0$ and where

$$ \bar{R} = \frac{A}{d} + \frac{B}{d - 1} + Dd - E - \frac{c}{2} \left[ d(a + b + c - 2) - a - \frac{c}{2} + 1 \right], \quad (17a) $$

$$ T_n = \left( n + \frac{\mu + \nu + 1}{2} \right)^2 - \frac{1}{4}(a + b + c - 1)^2 = S_{\nu - \frac{1}{2}}. \quad (17b) $$

Comparing (16) to the recursion relation of the new orthogonal polynomial $V_n^{(\mu, \nu)}(z; \tau, \theta)$, which is given in Appendix C by (C1), we conclude that $p_n(z) = V_n^{(\mu, \nu)}(z; \tau, \theta)$ where

$$ \cosh \theta = 2d - 1, \quad 2\tau = \sqrt{(a + b + c - 1)^2 - 4D}, \quad z = \frac{\bar{R}}{2\sqrt{d^2 - d}}. \quad (18) $$

Therefore, the wavefunction associated with this special class of solutions is written as the following series:
\[
\psi(x) = \sqrt{p(z)} y^{v+1-u} (1-y)^{\mu+1-\nu} \sum_n A_n V_n^{(\mu,\nu)}(z; \tau, \theta) P_n^{(\mu,\nu)}(y). \tag{19}
\]

4. The restricted solution class

Due to the exchange symmetry between the two families of solution in this class, we consider only the first family corresponding to the third column of Table 2. Table 6 is a reproduction of Table 3 after imposing the class constraints on the parameter \( E \) as shown in the third column of Table 1 and setting \( 4C = c(c-2)d(d-1), \; D = 0 \). All entries with a fixed value of the energy are eliminated. For example, since \( D = 0 \) then all three cases with \( D \) in the energy parameter are deleted. Consequently, for a given choice of \( \{a, b, c\} \), the remaining free parameters in this class of solutions are four: \( \{d, A, B, \lambda\} \). Therefore, aside from the scale parameter \( \lambda \) and the energy parameter \( e \), we can only have a maximum of two potential parameters. This could also be understood by comparing the number of potential parameters with those in the above two classes of solution and noting that the additional constraints on the parameters \( D \) and \( E \) reduces that number by two, from four to two. For each member of this class that corresponds to a given set of parameters \( \{a, b, c\} \), Table 7 gives the potential function and the differential equation parameters \( \{d, A, B\} \) in terms of the potential parameters and the energy. The orthogonal polynomials associated with this class of solutions satisfy the following symmetric three-term recursion relation [see Eq. (B10c) of Appendix B in Ref. 1]

\[
\frac{1}{4}(\mu+1)^2 p_n(z) = \left[ \frac{n(n+1)}{2n+\mu+\nu} + \frac{1}{2}\left( F_n - 1 \right) S_n + \frac{1}{4}(\mu+1)^2 \right] p_n(z)
+ G_{n-1} S_{n-1} p_{n-1}(z) + G_n S_n p_{n+1}(z)
\tag{20}
\]

for \( n = 0, 1, 2, \ldots \) and with \( p_0(z) = 1, \; p_{-1}(z) = 0 \). Comparing this to the recursion relation of the normalized version of the Wilson polynomial given in Appendix B by Eq. (B2), we conclude that \( p_n(z) = W_n(z^2; \sigma - \tau, \sigma + \tau, \eta, \eta) \) where

\[
2\sigma = \nu + 1, \quad 2\eta = \mu + 1, \quad 2\tau = a + b + c - 1, \quad 4z^2 = -\left(\mu + 1\right)^2. \tag{21}
\]

In the comparison, we used identity (14) after making the exchange \( \mu \leftrightarrow \nu \). Since \( z^2 < 0 \), then the spectrum is purely discrete and the spectrum formula (B4c) gives

\[
\frac{B}{d-1} = \left(k + 1 + \frac{\nu-a-b-c}{2} \right)^2 - \frac{1}{4}(1-b)^2, \tag{22}
\]

where \( k = 0, 1, 2, \ldots N \) and \( N \) is the largest integer less than or equal to \( \tau - \sigma = \frac{a+b+c-\nu}{2} - 1 \). The parameter values from Table 6 and Table 7 show that the maximum value of \( N \) is zero. Therefore, our restricted solution gives only the ground state. This observation could also be confirmed by comparing the recursion relation (20) to that of the Racah polynomial (which is the discrete version of the Wilson polynomial) given by Eq. (B5) in Appendix B. The comparison gives

\[
p_n(z) = R_n \left(k; \mu, \nu, \frac{\mu+\nu}{2} + \tau, \frac{\nu-\mu}{2} - \tau\right) \quad \text{with} \quad k = 0. \]

The ground state energy

\[
-6-
\]
could be obtained from (22) by setting \( k = 0 \) and substituting the values of the parameters \( \{a,b,c,d,A,B\} \) from Table 7. Moreover, the corresponding ground state wavefunction becomes the following series

\[
\psi(x) = \sqrt{\rho(z)} y^{x+1-w} (1-y)^{\mu+2-b} \sum_n A_n W_n(z^2; \sigma - \tau, \sigma + \tau, \eta, \eta) P_n^{(\mu, \nu)}(y).
\]  

(23)

5. Conclusion

We made a combined coordinate and wavefunction transformation of the one-dimensional time-independent Schrödinger equation. The transformed equation is identified with the nine-parameter Heun-type equation that we have already studied in an earlier publication [1]. Consequently, we were able to identify the potential function and energy parameter associated with a given solution of the Heun-type equation found in [1]. Moreover, the series solution of the equation obtained in our earlier publication for each of the three classes is identified with the wavefunction that solves the original Schrödinger equation. These are given by Eq. (15), Eq. (19) and Eq. (23) corresponding to each of the three classes. Some of the solutions obtained in this work correspond to new integrable physical systems.

Appendix A: Coordinate transformation

The coordinate transformation \( x \to y(x) \) is such that \( 0 \leq y(x) \leq 1 \) and \( \frac{dy}{dx} = \lambda y^a(1-y)^b \). Integration yields \( \lambda x = \int y^{-a}(1-y)^{-b} dy \), which is a match with the integral representation of the incomplete beta function [5,6]. Thus, we obtain two results depending on the physical space (values of the parameter \( a \) and \( b \)). The first is

\[
\int_0^x t^{-a}(1-t)^{-b} dt = B\left(y;1-a,1-b\right), \quad (A1)
\]

where \( B(z;\alpha,\beta) \) is the lower incomplete beta function, which is equal to \( \left(\frac{z}{\alpha}\right)_2 F_1 \left(\frac{\alpha-1}{1+\alpha},\frac{\alpha+1}{1+\alpha}\right|z\right) \). Therefore, we obtain (for \( a \neq 1 \))

\[
\lambda x = \int_0^x t^{-a}(1-t)^{-b} dt = B\left(y;1-a,1-b\right) = \frac{y^{1-a}}{1-a} {_2 F_1} \left(\frac{b-1,1-a}{2-a}\right.\left|y\right), \quad (A2)
\]

On the other hand, changing the integration limits and after some simple manipulations, we obtain the alternative result (the upper incomplete beta function)

\[
\int_y^1 t^{-a}(1-t)^{-b} dt = B\left(1-y;1-b,1-a\right), \quad (A3)
\]

where \( B\left(y;\alpha,\beta\right) + B\left(1-y;\beta,\alpha\right) = B(\alpha,\beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)} \) is the complete beta function. Therefore, we obtain the alternative result (for \( b \neq 1 \))
\[
\lambda x = \int_0^1 t^{-a}(1-t)^{-b} \, dt = B(1-y;1-b,1-a) = \frac{(1-y)^{1-b}}{1-b} _2 F_1 \left( \frac{1}{2}, 1-b \mid 1-y \right). \tag{A4}
\]

For the singular case \(a = b = 1\) direct integration gives \(y(x) = \frac{1}{1+e^{-x}}\) for \(-\infty \leq x \leq +\infty\) corresponding \(0 \leq y \leq 1\).

### Appendix B: The Wilson, Racah and Racah-Heun polynomials

Some symbols in this Appendix are local and not related to those in the rest of the paper. The normalized version of an orthogonal polynomial \(P_n(x)\) with \(P_0(x) = 1\) satisfies a symmetric three-term recursion relation of the form

\[
x P_n(x) = A_n P_{n-1}(x) + B_n P_{n+1}(x) \quad \text{with} \quad B_n > 0 \quad \text{for all} \quad n.
\]

The completely continuous version of this polynomial has an orthogonality that reads:

\[
\int_x \rho(x) P_n(x) P_m(x) dx = \delta_{n,m} \quad \text{where} \quad \rho(x) \text{ is the positive definite normalized weight function.}
\]

However, if the spectrum is a mix of continuous and discrete parts then this orthogonality is modified by the addition of a discrete (finite or infinite) sum. Now, the normalized version of the four-parameter Wilson polynomial is written as (see, Appendix A in Ref. [8])

\[
W_n(z^2; a,b,c,d) = 
\sqrt{\frac{(2n+a+b+c+d-1)n(n+a+b+c+d-2)}{(n+a+b+c+d)(n+a+b+c+d-1)}} \frac{(a+b)(a+c)(a+d)(a+b+c+d-1)}{(b+c)(b+d)(a+c+d)(a+b+c+d-1)}_4 F_3 \left( \begin{array}{c}
-n,n+a+b+c+d-1,1+i-z,1-i-z \\
1,1,1,1,1 
\end{array} \mid a+b+c+a+d \right) \tag{B1}
\]

It satisfies the following symmetric three-term recursion relation

\[
z^2 W_n = \left[ \frac{(n+a+b)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(n+a+b+c+d-2)} - a^2 \right] W_n - \frac{1}{2n+a+b+c+d-2} \frac{(n+a+b)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(n+a+b+c+d-2)} W_{n-1} - \frac{1}{2n+a+b+c+d} \frac{(n+a+b)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d+1)} W_{n+1} \tag{B2}
\]

If \(\Re(a,b,c,d) > 0\) and non-real parameters occur in conjugate pairs, then the polynomial has only a continuous spectrum with the following normalized weight function

\[
\rho(z) = \frac{1}{2\pi} \frac{\Gamma(a+b+c+d)\Gamma(a+i\zeta)\Gamma(b+i\zeta)\Gamma(c+i\zeta)\Gamma(d+i\zeta)/\Gamma(2i\zeta)^2}{\Gamma(a+b)\Gamma(c+d)\Gamma(a+c)\Gamma(a+d)\Gamma(a+b+c)\Gamma(b+d)}. \tag{B3}
\]

On the other hand, if the parameters are such that \(a < 0\) and \(a+b, a+c, a+d\) are positive or a pair of complex conjugates with positive real parts, then the polynomial will have a mix of continuous positive spectrum and a finite-size negative discrete spectrum and the polynomial satisfies a generalized orthogonality relation [see, Eq. (C3) in Ref. 7 and Eq. (9.1.3) in Ref. 9]. The asymptotics \((n \to \infty)\) of the Wilson polynomial gives the following scattering amplitude, phase shift and spectrum formula (see Appendix B in Ref. [8])

\[
A(z) = 2\sqrt{\rho(z)}, \tag{B4a}
\]

\[
\delta(z) = \arg \Gamma(2i\zeta) - \arg \left[ \Gamma(a+i\zeta)\Gamma(b+i\zeta)\Gamma(c+i\zeta)\Gamma(d+i\zeta) \right], \tag{B4b}
\]

\[\text{Page 8}\]
\[ z_k^2 = -(k + a)^2, \]  
\[ \text{(B4c)} \]

where \( k = 0,1,2,\ldots, N \) and \( N \) is the largest integer less than or equal to \(-a\). The discrete version of the Wilson polynomial is the four-parameter Racah polynomial \( R_n(k;\alpha,\beta,\gamma,\delta) \) whose orthonormal version satisfies the following symmetric three-term recursion relation (see, Appendix A in Ref. [8] or Appendix C in Ref. [7])

\[
\begin{align*}
R_n & = \left( k + \frac{\gamma + \delta + 1}{2} \right)^2 R_n - \left[ 
\frac{(n+1)(n+\alpha+1)(n+\beta+1)}{2n+\alpha+\beta+1}(2n+\alpha+\beta+2) + \frac{n(n+\beta)(n+\alpha-\delta)(n+\alpha+\beta-\gamma)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} - \frac{1}{4}(\gamma + \delta + 1)^2 \right] R_{n-1} \\
& + \frac{1}{2n+\alpha+\beta+2} \left[ \frac{n(n+\alpha)(n+\gamma)(n+\beta+\delta)(n+\alpha-\delta)(n+\alpha+\beta-\gamma)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+3)} \right] R_{n+1},
\end{align*}
\]
\[ \text{(B5)} \]

where \( k = 0,1,2,\ldots, N \) and either \( \alpha + 1 = -N \) or \( \gamma + 1 = -N \) or \( \beta + \delta + 1 = -N \).

The modified version of the Wilson polynomial (named as the “Racah-Heun polynomial” by the authors of Ref. [10]) is written as \( W_n^\lambda(z^2;\alpha,\beta,\gamma,\delta) \), where \( \lambda \) is a deformation (modification) parameter. It satisfies a modified version of the three-term recursion relation (B2) that reads

\[
\begin{align*}
z^2 W_n^\lambda & = A_n W_n^\lambda + B_{n-1} W_n^\lambda + B_n W_{n+1}^\lambda \\
& \quad - \lambda \left[ (n+a+c)(n+b+d) - \frac{1}{2} (2n+a+b+c+d-1) \right] W_n^\lambda, 
\end{align*}
\]
\[ \text{(B6)} \]

where \( \{A_n,B_n\} \) are the recursion coefficients in (B2). All properties of the polynomial \( W_n^\lambda(z^2;\alpha,\beta,\gamma,\delta) \) are yet to be derived analytically. This is still an open problem in orthogonal polynomials [11].

**Appendix C: The new orthogonal polynomial associated with the special solution class**

This polynomial, referred to as \( V_n^{(\mu,\nu)}(z;\tau,\theta) \), was introduced in an open problem in orthogonal polynomials [11]. It is defined, up to now, by its three-term recursion relation and initial value \( V_0^{(\mu,\nu)}(z;\tau,\theta) = 1 \). Its other properties (weight function, generating function, orthogonality, asymptotics, zeros, etc.) are yet to be derived analytically. Its normalized version satisfies the following symmetric three-term recursion relation

\[
\begin{align*}
(cosh \theta)V_n^{(\mu,\nu)}(z;\tau,\theta) & = \left\{ z \left( \sinh \theta \right) \left[ \left( n + \frac{\mu + \nu + 1}{2} \right)^2 - \tau^2 \right] \right\}^{-1} V_n^{(\mu,\nu)}(z;\tau,\theta) \\
& \quad + 2G_{n-1}^{(\mu,\nu)}(z;\tau,\theta) + 2G_n^{(\mu,\nu)}(z;\tau,\theta) 
\end{align*}
\]
\[ \text{(C1)} \]
where $\theta \geq 0$ and $\{F_n, G_n\}$ are defined by Eq. (12) in the text above. It was conjectured that if $\tau$ is pure imaginary, then the spectrum is purely continuous and positive. However, if $\tau$ is real then the spectrum is a mix of a continuous positive spectrum and a discrete negative spectrum of finite size $N+1$, where $N$ is the largest integer less than or equal to $|\tau| - \frac{\mu+\nu+1}{2}$. In that case, the polynomial satisfies the following generalized orthogonality

$$\int_{0}^{\infty} \rho(z)V^{(\mu,\nu)}_{n}(z;\tau,\theta)V^{(\mu,\nu)}_{m}(z;\tau,\theta)dz + \sum_{k=0}^{N} \omega(k)V^{(\mu,\nu)}_{n}(z_{k};\tau,\theta)V^{(\mu,\nu)}_{m}(z_{k};\tau,\theta) = \delta_{n,m}, \quad (C2)$$

where $\rho(z)$ and $\omega(k)$ are the positive definite continuous and discrete weight functions, respectively. The finite discrete spectrum $\{z_{k}\}_{k=0}^{N}$ could be determined from the condition that forces the asymptotics ($n \to \infty$) of $V^{(\mu,\nu)}_{n}(z;\tau,\theta)$ to vanish. It remains an open problem to determine the discrete spectrum and weight functions analytically.

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Tables Caption:

Table 1: The conditions on the parameters of the differential equation (1) in each of its four solution classes.

Table 2: The parameters \( \{\alpha, \beta, \gamma, \mu, \nu\} \) of the basis (4) in terms of the differential equation parameters for each of the four solution classes.

Table 3: The twelve choices for the parameter set \((a,b,c)\) along with the corresponding coordinate transformation, energy parameter and potential function.

Table 4: The potential function and energy for each of the eight cases corresponding to a given set of parameters \((a,b,c)\) in the class of general solutions.

Table 5: The potential function and energy for each of the eight cases corresponding to a given set of parameters \((a,b,c)\) in the class of special solutions.

Table 6: The potential function and energy for each of the five cases corresponding to a given set of parameters \((a,b,c)\) in the class of restricted solutions.

Table 7: The potential function and the differential equation parameters \(\{d,A,B\}\) in terms of the potential parameters and the energy for each member in the class of restricted solutions.
### Table 1

| General Solution | Special Solution | Two Restricted Solutions |
|------------------|------------------|--------------------------|
| \( \frac{4A}{d} \leq (1-a)^2 \) | \( \frac{4A}{d} \leq (1-a)^2 \) | \( \frac{4A}{d} \leq (1-a)^2 \) |
| \( \frac{4B}{d-1} \geq -(1-b)^2 \) | \( \frac{4B}{d-1} \geq -(1-b)^2 \) | \( \frac{4B}{d-1} \geq -(1-b)^2 \) |
| \( \frac{4C}{d(d-1)} = (1-c)^2 \) | \( \frac{4C}{d(d-1)} = (1-c)^2 - 1 \) | \( \frac{4C}{d(d-1)} = (1-c)^2 - 1 \) |
| \( D = 0 \) | \( E = \frac{A}{d} + \frac{B}{d-1} - \frac{c}{2} \left[ d(a+b+c-2) - a - \frac{c}{2} + 1 \right] \) |

### Table 2

| General Solution | Special Solution | First Restricted Solution | Second Restricted Solution |
|------------------|------------------|---------------------------|-----------------------------|
| \( 2\alpha = \nu + 1 - a \) | \( 2\alpha = \nu + 1 - a \) | \( 2\alpha = \nu + 1 - a \) | \( 2\alpha = \nu + 2 - a \) |
| \( 2\beta = \mu + 1 - b \) | \( 2\beta = \mu + 1 - b \) | \( 2\beta = \mu + 2 - b \) | \( 2\beta = \mu + 1 - b \) |
| \( 2\gamma = 1 - c \) | \( 2\gamma = -c \) | \( 2\gamma = -c \) | \( 2\gamma = -c \) |
| \( \nu^2 = (1-a)^2 - 4 \frac{A}{d} \) | \( \nu^2 = (1-a)^2 - 4 \frac{A}{d} \) | \( \nu^2 = (1-a)^2 - 4 \frac{A}{d} \) | \( (\nu+1)^2 = (1-a)^2 - 4 \frac{A}{d} \) |
| \( \mu^2 = (1-b)^2 + \frac{4B}{d-1} \) | \( \mu^2 = (1-b)^2 + \frac{4B}{d-1} \) | \( (\mu+1)^2 = (1-b)^2 + \frac{4B}{d-1} \) | \( \mu^2 = (1-b)^2 + \frac{4B}{d-1} \) |
Table 3

| $(a,b,c)$ | $y(x)$ | $2E/\lambda^2$ | $2V(x)/\lambda^2$ | Parameter Redefinition |
|----------|--------|----------------|-------------------|----------------------|
| $(\frac{1}{2}, \frac{1}{2}, 2)$ | $\frac{1}{2}[1+\sin(\lambda x)],$ | $\frac{1}{2} - E$ | $(1-D)y - A + \frac{B}{y} + \frac{C}{d-y}$ | $x x x x x$ |
| | $-\frac{\pi}{2} \leq \lambda x \leq +\frac{\pi}{2}$ | | | |
| $(\frac{1}{2}, \frac{1}{2}, 4)$ | $\frac{1}{2}[1+\sin(\lambda x)],$ | $2d(d-1) - C$ | $(d-y)\left[E + 2d - 3 + (4-D)y - A + \frac{B}{y} + \frac{C}{d-y}\right]$ | $x x x x x$ |
| | $-\frac{\pi}{2} \leq \lambda x \leq +\frac{\pi}{2}$ | | | |
| $(\frac{1}{2}, \frac{1}{2}, 0)$ | $\frac{1}{2}[1+\sin(\lambda x)],$ | $-D$ | $\frac{1}{d-y}\left[E - Dd - A + \frac{B}{y} + \frac{C}{d-y}\right]$ | $E \rightarrow E + Dd$ |
| | $-\frac{\pi}{2} \leq \lambda x \leq +\frac{\pi}{2}$ | | | |
| $(0,0,4)$ | $\lambda x,$ | $D - 2$ | $\frac{d-y}{y(1-y)}\left[E + D(d-1) - A + \frac{B}{y} + \frac{C}{d-y}\right]$ | $E \rightarrow E + (1-d)D$ |
| | $0 \leq \lambda x \leq 1$ | | | $C \rightarrow C + d(d-1)D$ |
| $(1,1,0)$ | $\frac{1}{d}e^{-\lambda x}^{-1},$ | $A$ | $\frac{y(1-y)}{d-y}\left[E - Dy + A + \frac{B}{1-d} + \frac{(d-1)C}{d-y}\right]$ | $B \rightarrow B + A(1-d)/d$ |
| | $-\infty \leq \lambda x \leq +\infty$ | | | |
| $(\frac{1}{2}, 1,2)$ | $\tanh^2(\lambda x),$ | $-B$ | $(1-y)\left[E - \frac{1}{2} + \left(\frac{3}{2} - D\right)y - A + \frac{C}{d-y}\right]$ | $x x x x x$ |
| | $x \geq 0$ | | | |
| $(\frac{1}{2}, 1,0)$ | $\tanh^2(\lambda x),$ | $B$ | $\frac{1-y}{d-y}\left[E - D + A + \frac{B}{1-d} - Dy + \frac{C}{d-y}\right]$ | $E \rightarrow E + \frac{B}{1-d}$ |
| | $x = 0$ | | | |
| $(\frac{1}{2}, 0,2)$ | $\frac{(\lambda x/2)^2}{2},$ | $\frac{1}{2} - D$ | $\frac{1}{1-y}\left[E - D - A + \frac{B}{d-y} + \frac{C}{d-y}\right]$ | $E \rightarrow E + D$ |
| | $0 \leq \lambda x \leq 2$ | | | |
| $(\frac{1}{2}, 0,4)$ | $\frac{(\lambda x/2)^2}{2},$ | $3 - 2d - E$ | $\frac{d-y}{1-y}\left[(3-D)y - A + \frac{B}{1-y} + \frac{C}{d-y}\right]$ | $C \rightarrow C + (1-d)(E-3)$ |
| | $0 \leq \lambda x \leq 2$ | | | |
| $(0,1,2)$ | $1-e^{-\lambda x},$ | $-B$ | $\frac{1-y}{y}\left[B + E + (1-D)y - A + \frac{C}{d-y}\right]$ | $E \rightarrow E - B$ |
| | $x \geq 0$ | | | |
| $(0,1,0)$ | $1-e^{-\lambda x},$ | $\frac{B}{1-d}$ | $\frac{1-y}{y(d-y)}\left[B + E - \left(D + \frac{B}{d-1}\right)y - A + \frac{C}{d-y}\right]$ | $E \rightarrow E - B$ |
| | $x \geq 0$ | | | $D \rightarrow D + \frac{B}{1-d}$ |
| $(0,1,4)$ | $1-e^{-\lambda x},$ | $B(1-d)$ | $\frac{1-y}{y}\left[(1-y)(d-y)\left[E - 2d(1-D) + (4-D)y + \frac{A}{y} + \frac{C}{d-y}\right]\right]$ | $C \rightarrow C - Bd$ |
| | $x \geq 0$ | | | |
| $(a,b,c)$ | $y(x)$ | $2\xi/\lambda^2$ | $2\nu(x)/\lambda^2$ | Parameters |
|---------|--------|-----------------|-------------------|------------|
| $(\frac{1}{2}, \frac{1}{2}, 2)$ | $\frac{1}{2}[1 + \sin(\lambda x)]$, $-\frac{\pi}{2} \leq \lambda x \leq +\frac{\pi}{2}$ | $\frac{1}{2} - E$ | $(1-D)y - \frac{A}{y} + \frac{B}{1-y} + \frac{d(d-1)/4}{d-y}$ | $u_0 = 2\nu_0/\lambda^2$, $u_1 = 2\nu_1/\lambda^2$ |
| $(\frac{1}{2}, \frac{1}{2}, 0)$ | $\frac{1}{2}[1 + \sin(\lambda x)]$, $-\frac{\pi}{2} \leq \lambda x \leq +\frac{\pi}{2}$ | $-D$ | $\frac{1}{d-y}\left[u_0 - \frac{A}{y} + \frac{B}{1-y} + \frac{d(d-1)/4}{d-y}\right]$ | $E = u_0 + Dd$ |
| $(1,0)$ | $(1 + e^{-\lambda x})^{-1}$, $-\infty \leq \lambda x \leq +\infty$ | $\frac{A}{d}$ | $\frac{y(1-y)}{d-y}\left[E - Dy + \frac{u_0}{1-y} + \frac{d(d-1)/4}{d-y}\right]$ | $B = u_0 + \frac{A}{d}(d-1)$ |
| $(\frac{1}{2}, 1, 2)$ | $\tanh^2(\lambda x)$, $x \geq 0$ | $-B$ | $E - \frac{1}{2} + \left(\frac{3}{2} - D\right)y - \frac{A}{y} + \frac{d(d-1)/4}{d-y}$ | $u_0 + Dd$ |
| $(\frac{1}{2}, 1, 0)$ | $\tanh^2(\lambda x)$, $x \geq 0$ | $-B$ | $\frac{1-y}{d-y}\left[u_0 - Dy - \frac{A}{y} + \frac{d(d-1)/4}{d-y}\right]$ | $E = u_0 + \frac{B}{d-1}$ |
| $(\frac{1}{2}, 0, 2)$ | $(\lambda x/2)^2$, $0 \leq \lambda x \leq 2$ | $\frac{1}{2} - D$ | $\frac{1-y}{1-y}\left[u_0 - \frac{A}{y} + \frac{B}{1-y} + \frac{d(d-1)/4}{d-y}\right]$ | $E = u_0 + D$ |
| $(0,1,2)$ | $1 - e^{-\lambda x}$, $x \geq 0$ | $-B$ | $\frac{1-y}{y}\left[u_0 + (1-D)y - \frac{A}{y} + \frac{d(d-1)/4}{d-y}\right]$ | $E = u_0 - B$ |
| $(0,1,0)$ | $1 - e^{-\lambda x}$, $x \geq 0$ | $-B$ | $\frac{1-y}{y(d-y)}\left[u_0 - u_1y - \frac{A}{y} + \frac{d(d-1)/4}{d-y}\right]$ | $E = u_0 - B$ |
Table 5

| $(a,b,c)$ | $y(x)$ | $2\xi/\lambda^2$ | $2V(x)/\lambda^2$ | Parameters Definition |
|-----------|--------|-------------------|-------------------|------------------------|
| $(\frac{1}{2},\frac{1}{2},2)$ | $\frac{1}{2}[1+\sin(\lambda x)],$ $-\frac{\pi}{2} \leq \lambda x \leq +\frac{\pi}{2}$ | $\frac{1}{2}E$ | $(1-D)y - \frac{A}{y} + \frac{B}{1-y}$ | $u_0 = 2V_0/\lambda^2$, $u_1 = 2V_1/\lambda^2$ |
| $(\frac{1}{2},\frac{1}{2},0)$ | $\frac{1}{2}[1+\sin(\lambda x)],$ $-\frac{\pi}{2} \leq \lambda x \leq +\frac{\pi}{2}$ | $-D$ | $\frac{1}{d-y}\left(u_0 - \frac{A}{y} + \frac{B}{1-y}\right)$ | $E = u_0 + Dd$ |
| $(1,1,0)$ | $(1+e^{-\lambda x})^{-1},$ $-\infty \leq \lambda x \leq +\infty$ | $\frac{A}{d}$ | $\frac{y(1-y)}{d-y}\left(E-Dy + \frac{u_0}{1-y}\right)$ | $B = u_0 - \frac{A}{d} (d-1)$ |
| $(\frac{1}{2},1,2)$ | $\tanh^2(\lambda x),$ $x \geq 0$ | $-B$ | $(1-y)\left[E - \frac{1}{2} + \left(\frac{3}{2} - D\right)y - \frac{A}{y}\right]$ | $	imes\times\times$ |
| $(\frac{1}{2},1,0)$ | $\tanh^2(\lambda x),$ $x \geq 0$ | $-B$ | $\frac{1-y}{d-y}\left(u_0 - Dy - \frac{A}{y}\right)$ | $E = u_0 + \frac{B}{d-1}$ |
| $(\frac{1}{2},0,2)$ | $(\lambda x/2)^2,$ $0 \leq \lambda x \leq 2$ | $\frac{1}{2} - D$ | $\frac{1-y}{d-y}\left(u_0 - \frac{A}{y} + \frac{B}{1-y}\right)$ | $E = u_0 + D$ |
| $(0,1,2)$ | $1-e^{-\lambda x},$ $x \geq 0$ | $-B$ | $\frac{1-y}{y}\left[u_0 + (1-D)y - \frac{A}{y}\right]$ | $E = u_0 - B$ |
| $(0,1,0)$ | $1-e^{-\lambda x},$ $x \geq 0$ | $-B$ | $\frac{1-y}{y(d-y)}\left(u_0 - u_1y - \frac{A}{y}\right)$ | $E = u_0 - B$, $D = u_1 - \frac{B}{d-1}$ |
Table 6

| (a,b,c) | y(x) | $2\mathcal{E}/\lambda^2$ | $2V(x)/\lambda^2$ | Parameters Definition |
|--------|------|----------------|----------------|----------------------|
| $(\frac{1}{2}, \frac{1}{2}, 2)$ | $\frac{1}{2}[1+\sin(\lambda x)], \ -\frac{\pi}{2} \leq \lambda x \leq +\frac{\pi}{2}$ | $d - \frac{A}{d} - \frac{B}{d-1}$ | $y = -\frac{A}{y} + \frac{B}{1-y}$ | $u_0 = 2V_0/\lambda^2, \ u_1 = 2V_1/\lambda^2$ |
| (1,1,0) | $(1+e^{-\beta x})^{-1}, \ -\infty \leq \lambda x \leq +\infty$ | $\frac{A}{d}$ | $u_0y$ | $\frac{B}{d-1} = u_0 - \frac{A}{d}$ |
| $(\frac{1}{2}, 1, 2)$ | $\tanh^2(\lambda x), \ x \geq 0$ | $-B$ | $(1-y)\left(u_0 + \frac{3}{2}y - \frac{A}{y}\right)$ | $\frac{3}{2d} = \frac{A}{d} + \frac{B}{d-1} - u_0$ |
| $(\frac{1}{2}, 1, 0)$ | $\tanh^2(\lambda x), \ x \geq 0$ | $-B$ | $\frac{A}{d}\left(1 - \frac{1}{y}\right)$ | $u_0 - \frac{A}{d}$ |
| (0,1,2) | $1 - e^{-\beta x}, \ x \geq 0$ | $-B$ | $\frac{1-y}{y}\left(u_0 + \frac{A}{y}\right)$ | $d = \frac{A}{d} + \frac{Bd}{d-1} - u_0$ |

Table 7

| (a,b,c) | x | $V(x)$ | Parameters Definition |
|--------|---|-------|----------------------|
| $(\frac{1}{2}, \frac{1}{2}, 2)$ | $-\frac{\pi}{2} \leq \lambda x \leq +\frac{\pi}{2}$ | $\frac{V_0 - V_\sin(\lambda x)}{\cos^2(\lambda x)} + \frac{\lambda^2}{4}[1+\sin(\lambda x)]$ | $A = -(V_+ + V_-)/2\lambda^2, \ B = (V_+ - V_-)/2\lambda^2$ |
| (1,1,0) | $-\infty \leq \lambda x \leq +\infty$ | $\frac{V_0}{1+e^{-\beta x}}$ | $\frac{A}{d} = 2\mathcal{E}/\lambda^2, \ B = \frac{2}{\lambda^2}(V_0 - \mathcal{E})$ |
| $(\frac{1}{2}, 1, 2)$ | $x \geq 0$ | $\frac{V_0}{\cosh^2(\lambda x)} - \frac{V_1}{\sinh^2(\lambda x)} + \frac{3\lambda^2}{4}/\cosh^4(\lambda x)$ | $A = 2V_1/\lambda^2, \ B = -2\mathcal{E}/\lambda^2$ |
| $(\frac{1}{2}, 1, 0)$ | $x \geq 0$ | $-\frac{V_0}{\sinh^2(\lambda x)}$ | $\frac{A}{d} = 2V_0/\lambda^2, \ B = \frac{2}{\lambda^2}(V_0 + \mathcal{E})$ |
| (0,1,2) | $x \geq 0$ | $\frac{1}{e^{\lambda^2x} - 1}\left(V_0 - \frac{V_1}{1-e^{-\beta x}} - \frac{\lambda^2}{2}e^{-\beta x}\right)$ | $A = 2V_1/\lambda^2, \ B = -2\mathcal{E}/\lambda^2$ |

$\mathcal{E} = \sqrt{V_+ V_-}$