Natanzon-Orlov model and refined superintegrability

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Abstract

We reconsider the simple matrix model description of Hurwitz numbers proposed by S. Natanzon and A. Orlov, which uses the superintegrability property of the complex matrix model, and discuss a way of its possible supersymmetric extension to approach spin Hurwitz numbers.

1 Introduction

In \cite{1}, it was suggested to reduce remarkable properties \cite{2} of matrix models \cite{3} to their superintegrability, that is, to the fact that averages of symmetric functions/characters are again expressed through the same symmetric functions/characters (see many examples in \cite{4}–\cite{18}, and also some preliminary results in \cite{19}–\cite{23}) depending only on the type of matrix model and on the matrix size.

In the simplest possible case of complex matrix model \cite{24}, where the integration goes over $N_1 \times N_2$ matrices $Z$, the claim was \cite{1}

\[
\langle S_R\{\text{Tr}(ZZ^\dagger)^k\}\rangle := \int S_R\{\text{Tr}(ZZ^\dagger)^k\} \cdot \exp(-\text{Tr} ZZ^\dagger) \, d^2Z = \frac{S_R\{N_1\} \cdot S_R\{N_2\}}{d_R}
\]  

(1)

Here $S_R\{p_k\}$ is the Schur polynomial (which is defined to be a symmetric function $S_R(z_i)$ of the variables $z_i$, or a graded polynomial $S_R\{p_k\}$ of the power sums $p_k := \sum z_i^k$). At the l.h.s., the role of $z_i$ is played by the eigenvalues of the matrix $ZZ^\dagger$, we will use the notation $S[ZZ^\dagger]$ in such cases. At the r.h.s., $p_k = N_1$ or $N_2$; $d_R := S_R\{\delta_{k,1}\}$. The integration measures is normalized so that $<1> := 1$.

It is natural to look for a further generalization (refinement) of this formula, where $N_1, N_2$ are substituted by arbitrary square matrices, and, indeed, such formula can be found \cite{6,16,25}:

\[
\langle S_R[AZBZ^\dagger] \rangle = \frac{S_R[A] \cdot S_R[B]}{d_R}
\]

(2)

and leads to a remarkable matrix model representation of Hurwitz numbers by S. Natanzon and A. Orlov \cite{25}. Here $A$ and $B$ are arbitrary square matrices of the sizes $N_1 \times N_1$ and $N_2 \times N_2$ respectively. In our opinion, this provides the complete form of the superintegrability relation for complex matrix model, and one should seek for its generalization to other matrix models, including Hermitian model and its fermionic counterpart, governing the spin Hurwitz numbers.

In this short note, we review the application of (2) to Hurwitz numbers in \cite{25}, and provide the combinatorial proof of this relation both in the bosonic and fermionic cases along the lines of \cite{16}.

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2 Application to Hurwitz numbers (NO model)

Generating function of the Hurwitz numbers is made out of the Schur polynomials:

\[ Z_m\{p^{(1)}, \ldots, p^{(m)}\} = \sum_{R} d_{R}^{2^{m}} \cdot S_{R}\{p^{(1)}\} \cdots S_{R}\{p^{(m)}\} \]  \hspace{1cm} (3) 

In [25], a matrix model realization of (3) was proposed,

\[ Z_m\{p^{(1)}, \ldots, p^{(m)}\} = \prod_{i=1}^{m} \int d^{2}Z_{i} \exp \left( \text{tr} \left( \prod_{i=1}^{m} A_{i} \right) \right) = \prod_{i=1}^{m} \int d^{2}Z_{i} e^{-\text{tr} Z_{i}^{\dagger} Z_{i}} = \prod_{i=1}^{m} \int d^{2}Z_{i} e^{-\text{tr} Z_{i}^{\dagger} Z_{i}} \exp \left( \text{tr} A_{1} Z_{2} A_{3} Z_{4} \ldots Z_{2}^{\dagger} Z_{1}^{\dagger} \right) \]  \hspace{1cm} (4)

with \( p_{k}^{(i)} = \text{Tr} A_{k}^{i} \). Evaluating of the integral at the r.h.s. of this expression is done using at the first step the elementary property

\[ \int e^{\text{tr} A_{1} Z_{2} U_{2} Z_{2}^{\dagger} e^{-\text{tr} Z_{2}^{\dagger} Z_{2}}} d^{2}Z_{2} = \frac{1}{\text{Det}(I \otimes I - A_{1} \otimes U_{2})} = \exp \left( \sum_{k} \frac{\text{tr} A_{k} \text{tr} U_{k}^{2}}{k} \right) = \sum_{R} S_{R}\{p_{k}^{(2)}\} \cdot S_{R}[U_{2}] \]  \hspace{1cm} (5)

where \( U_{2} = A_{2} Z_{3} A_{3} \ldots Z_{2}^{\dagger} \), and further iterating with the superintegrability property [2]: at the second step, integrating (3) over \( Z_{3} \) makes out of \( S_{R}[U_{2}] \) the product \( \frac{S_{R}\{p_{k}^{(2)}\}}{d_{R}} \cdot S_{R}[U_{3}] \) with \( U_{3} = A_{3} Z_{4} A_{4} \ldots Z_{4}^{\dagger} \), etc. The final result is exactly (3).

3 Towards \( Q \)-Schur functions

An important goal is to find a generalization of the NO models to reproduce the spin Hurwitz numbers. The corresponding generating function is given, instead of (3) by [26]

\[ 3_{m}^{p}\{p^{(1)}, \ldots, p^{(m)}\} = \sum_{R \in SP} (-1)^{p_{R}} \cdot d_{R}^{2^{m}} \cdot Q_{R}\{p^{(1)}\} \cdots Q_{R}\{p^{(m)}\} \]  \hspace{1cm} (6)

where the generating function additionally depends on the parameter \( p = 0, 1 \) that encodes the spin structure, the sum goes over the strict partitions with \( l_{R} \) parts (i.e. over those with all parts distinct), \( q_{R} = \frac{1}{2} Q_{R}\{\delta_{k,1}\} \), and \( Q_{R} \) are the Schur \( Q \)-functions [26][27]. In other words, to deal with the spin Hurwitz numbers, one needs to find an extension of (2) from \( S_{R} \) to the \( Q \)-Schur functions \( Q_{R} \). To this end, one needs to add fermions already at the very first step. In order to see this, note that the counterpart of (2) is (see sections 4-5 below)

\[ \langle S_{R}[A \Psi B \Psi^{\dagger}] \rangle = (-1)^{|R|} \frac{S_{R}[\Psi] S_{R}[\Psi]}{d_{R}} \]  \hspace{1cm} (7)

It nicely reproduces the known result

\[ \int d^{2} \Psi e^{-\text{tr} \Psi \Psi^{\dagger} + \text{tr} A \Psi B \Psi^{\dagger}} = \sum_{R} (-1)^{|R|} S_{R}[\Psi] S_{R}[\Psi] = \exp \left( - \sum_{k} (-1)^{k} \frac{\text{tr} A_{k} \text{tr} B_{k}}{k} \right) = \text{det}(1 + A \otimes B) \]  \hspace{1cm} (8)

Now one easily gets for the superintegration:

\[ \int e^{\text{tr} A \Psi B \Psi^{\dagger} + \text{tr} \Psi^{\dagger} + \text{tr} A \Psi B \Psi^{\dagger}} e^{-\text{tr} Z \Psi^{\dagger} + \text{tr} Z \Psi^{\dagger}} d^{2}Z d^{2} \Psi \frac{\text{Det}(I \otimes I + A \otimes B)}{\text{Det}(I \otimes I - A \otimes B)} = \exp \left( 2 \sum_{k \text{ odd}} \frac{\text{tr} A_{k} \text{tr} B_{k}}{k} \right) = \sum_{R} Q_{R}\{\text{tr} A_{k}\} Q_{R}\{\text{tr} B_{k}\} \]  \hspace{1cm} (9)

Thus, the Schur \( Q \)-functions emerge when one adds fermions.
However, a $Q$-counterpart of (12) should not be found in this way, because the theory of $Q$-functions seems to generalize the Hermitian rather than the complex matrix model, i.e. the superintegrability relation is

$$\langle S_R(\text{Tr} X^k) \rangle := \int S_R(\text{Tr} X^k) \cdot \exp \left( -\frac{1}{2} \text{Tr} X^2 \right) dX = \frac{S_R(N)}{d_R} S_R(\delta_{k,2}) \tag{10}$$

instead of (11). It agrees with the fact that the $Q$-functions are characters of representations of the supergroup associated with a special reduction of $gl(n|m)$, of the queer algebra $q(n)$ [29], while the characters associated with $gl(n|m)$ are the supersymmetric Schur functions [30], which, in terms of the power sums $p_k$'s are reduced to the usual Schur functions. We address this story in a separate publication.

4 A proof

In this section we provide a formal proof of (12) and (7). It is based on the Wick theorem [12 Eq.(7)]

$$\left\langle \prod_{i=1}^{n} Z_{a_i(\alpha_i, Z_{b_i(\beta_i)} \right\rangle = \sum_{\gamma \in S_n} \prod_{i=1}^{n} \delta_{\alpha_i(\gamma)}^{\beta_i(\gamma)} \tag{11}$$

let $|R| = |\Delta| = n$, then

$$\langle S_R[AZBZ]\rangle = \sum_{\Delta \vdash n} \frac{\psi_R(\Delta)}{z_{\Delta}} \cdot \langle p_{\Delta} \rangle \tag{12}$$

where

$$\langle p_{\Delta} \rangle = \left( \prod_{i=1}^{l(\Delta)} \text{tr} (AZBZ)^{\Delta_i} \right) = \prod_{i=1}^{n} A_{a_i(\alpha_i, B_{b_i(\beta_i)}} \cdot \left\langle \prod_{i=1}^{n} Z_{a_i(\alpha_i, Z_{b_i(\beta_i)} \right\rangle \tag{13}$$

here $\sigma$ is any element from the conjugacy class $\Delta$. Then

$$\langle p_{\Delta} \rangle = \prod_{i=1}^{n} A_{a_i(\alpha_i, B_{b_i(\beta_i)}} \prod_{\gamma \in S_n} \prod_{i=1}^{n} \delta_{a_i(\gamma)}^{\beta_i(\gamma)} \cdot \sum_{\gamma \in S_n} \prod_{i=1}^{n} A_{a_i(\gamma(\sigma_i(\gamma)) \beta_i(\gamma(\sigma_i(\gamma)) \sum_{\gamma \in S_n} A_{\gamma \sigma} B_{\gamma} \tag{14}$$

where the last equation means that if $\gamma \circ \sigma$ and $\gamma$ are in the conjugacy classes $\lambda$ and $\mu$ respectively, then

$$\lambda_{\gamma \sigma} = \lambda = \prod_{i=1}^{l(\lambda)} \text{tr} A^\lambda, \quad B_{\gamma} = B_{\mu} = \prod_{i=1}^{l(\mu)} \text{tr} B^{\mu} \tag{15}$$

and (12) has the form

$$\langle S_R[AZBZ]\rangle = \sum_{\Delta \vdash n} \frac{\psi_R(\Delta)}{z_{\Delta}} \cdot \sum_{\gamma \in S_n} A_{\gamma \sigma} B_{\gamma} \tag{16}$$

Now we use the formula

$$p_{\Delta} = \sum_{R \vdash n} \psi_R(\Delta) S_{R}(p) \tag{17}$$

to rewrite (10):

$$\langle S_R[AZBZ]\rangle = \sum_{\Delta \vdash n} \frac{\psi_R(\Delta)}{z_{\Delta}} \cdot \sum_{\gamma \in S_n} \left( \sum_{R \vdash n} \psi_{R_1(\gamma \circ \sigma) S_{R_1}(A)} \right) \left( \sum_{R_2 \vdash n} \psi_{R_2(\gamma) S_{R_2}(B)} \right) \tag{18}$$

The generalized orthogonality relations [12 Eq.(9)]

$$\sum_{\gamma \in S_n} \psi_{R_1(\gamma \circ \sigma) \psi_{R_2}(\gamma)} = \frac{\psi_{R_1(\sigma)}}{d_{R_1}} \delta_{R_1, R_2} \tag{19}$$
\[ \langle S_R[AZBZ^\dagger] \rangle = \sum_{R_1} \frac{S_{R_1}[A]S_{R_1}[B]}{d_{R_1}} \left( \sum_{\Delta} \frac{\psi_R(\Delta)\psi_{R_1}(\Delta)}{z_\Delta} \right) = \frac{S_{R}[A]S_{R}[B]}{d_{R}} \] \hspace{1cm} (20)

This completes the proof of (2).

A simple, still useful corollary is

\[ \langle p_{1^n} \rangle = \langle (\text{tr} AZBZ^\dagger)^n \rangle = \sum_{\gamma \in S_n} A_{\gamma \circ \text{id}} B_{\gamma} = \sum_{\Delta \vdash n} |\Delta| A_\Delta B_\Delta \] \hspace{1cm} (21)

where \( |\Delta| = \frac{n!}{z_\Delta} \) is the number of elements in the conjugacy class \( \Delta \).

\[ \sum_{R \vdash n} S_{R}[A]S_{R}[B] = \sum_{R \vdash n} \frac{\psi_R(\Delta_1)}{z_{\Delta_1}} A_{\Delta_1} \sum_{\Delta_2} \frac{\psi_R(\Delta_2)}{z_{\Delta_2}} B_{\Delta_2} = \sum_{\Delta_1 \Delta_2} \frac{A_{\Delta_1}B_{\Delta_2}}{z_{\Delta_1}} \left( \sum_{R \vdash n} \frac{\psi_R(\Delta_1)\psi_R(\Delta_2)}{z_{\Delta_2}} \right) \hspace{1cm} (22) \]

Thus we can rewrite (21) as

\[ \langle p_{1^n} \rangle = \langle (\text{tr} AZBZ^\dagger)^n \rangle = \sum_{\Delta \vdash n} |\Delta| A_\Delta B_\Delta = n! \sum_{R} S_{R}[A]S_{R}[B] \] \hspace{1cm} (23)

### 5 Implications for fermionic averages

Now in order to get (7), a fermionic counterpart of (2), one needs the fermionic Wick theorem:

\[ \langle \prod_{i=1}^{n} \Psi_{a_i \alpha_i} \Psi_{b_i \beta_i} \rangle = \sum_{\gamma \in S_n} \text{sgn}(\gamma) \prod_{i=1}^{n} \delta_{a_i}^{\gamma_{(i)}} \delta_{b_i}^{\beta_{(i)}} \] \hspace{1cm} (24)

where \( \text{sgn}(\gamma) = \pm 1 \) depends on the parity of permutation:

\[
\begin{array}{c|c}
\gamma & \text{sgn}(\gamma) \\
\hline
[1] & 1 \\
[2] & -1 \\
[1, 1] & 1 \\
[3] & 1 \\
[2, 1] & -1 \\
[1, 1, 1] & 1 \\
[4] & -1 \\
[3, 1] & 1 \\
[2, 2] & 1 \\
[2, 1, 1] & -1 \\
[1, 1, 1, 1] & 1 \\
\end{array}
\] \hspace{1cm} (25)

An explicit formula for \( \text{sgn}(\gamma) \) is

\[ \text{sgn}(\gamma) = (-1)^{\sum_i (\gamma_i - 1)} = (-1)^{\gamma_1 \vdash 1(\gamma) + (\gamma_1 \vdash 1(\gamma))} = (-1)^{\gamma_1 \vdash 1(\gamma) + \ell(\gamma)} \] \hspace{1cm} (26)

Now literally repeating the derivation of (2), we arrive at (7).

Again, the simplest check is

\[ \langle p_{1^n} \rangle = \langle (\text{tr} A\Psi B\Psi^\dagger)^n \rangle = \sum_{\gamma \in S_n} \text{sgn}(\gamma) A_{\gamma \circ \text{id}} B_{\gamma} = \sum_{\Delta \vdash n} (-1)^{|\Delta| + \ell(\Delta)} |\Delta| A_\Delta B_\Delta = (-1)^{n} \sum_{\Delta} (-A)_{\Delta} B_{\Delta} = n! \sum_{R \vdash n} S_{R}[-A]S_{R}[B] \] \hspace{1cm} (27)
6 Conclusion

In this note, we revised the extension (2) of the superintegrability property for the simplest and archetypical complex matrix model [6,7,16,25], which gives rise to the matrix model description of the Hurwitz numbers by S. Natanzon and A. Orlov [25]. We came across this description by reexamining the legacy of Sergey Natanzon. This remarkable result adds to the variety of his contributions to modern mathematical physics. We discussed a possible extension to fermionic models in order to approach to the spin Hurwitz numbers.

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