CYCLICITY IN THE HARMONIC DIRICHLET SPACE

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Abstract. The harmonic Dirichlet space is the Hilbert space of functions \( f \in L^2(\mathbb{T}) \) such that
\[
\|f\|_{D(\mathbb{T})}^2 := \sum_{n \in \mathbb{Z}} (1 + |n|)|\hat{f}(n)|^2 < \infty.
\]
We give sufficient conditions for \( f \) to be cyclic in \( D(\mathbb{T}) \), in other words, for \( \{\zeta^n f(\zeta) : n \geq 0\} \) to span a dense subspace of \( D(\mathbb{T}) \).

1. Introduction

Let \( \mathcal{X} \) be a topological linear space of complex functions on the unit circle \( \mathbb{T} \) such that the shift operator \( S \), given by
\[
S(f)(\zeta) := \zeta f(\zeta), \quad f \in \mathcal{X},
\]
is an isomorphism of \( \mathcal{X} \) onto itself.

A closed subspace \( \mathcal{M} \) of \( \mathcal{X} \) is called invariant if \( S(\mathcal{M}) \subset \mathcal{M} \). It is said to be 1-invariant (or simply invariant) for \( S \) if \( S(\mathcal{M}) \subsetneq \mathcal{M} \), and it is called 2-invariant (or doubly invariant) if \( S(\mathcal{M}) = \mathcal{M} \). The latter condition is equivalent to the invariance of \( \mathcal{M} \) under multiplication by both \( \zeta \) and \( \overline{\zeta} \).

Let \( \mathbb{Z} \) denote the integers and let \( \mathbb{N} := \{n \in \mathbb{Z} : n \geq 0\} \). Given \( f \in \mathcal{X} \), we write
\[
[f]_{\mathbb{N}} := \overline{\text{Span}^\mathcal{X}} \{z^n f : n \in \mathbb{N}\},
\]
\[
[f]_{\mathbb{Z}} := \overline{\text{Span}^\mathcal{X}} \{z^n f : n \in \mathbb{Z}\}.
\]
A function \( f \) is said to be 1-invariant if the space \( [f]_{\mathbb{N}} \) is 1-invariant for \( S \). We say that a function \( f \in \mathcal{X} \) is cyclic (resp. bicyclic) for \( \mathcal{X} \) if \( [f]_{\mathbb{N}} = \mathcal{X} \) (resp. \( [f]_{\mathbb{Z}} = \mathcal{X} \)).

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Let us begin with the classical case, namely $X = L^2(T)$. By a well-known theorem of Wiener, the 2-invariant subspaces have the form

$$ M = \{ f \in L^2(T) : f = 0 \text{ a.e. on } T \setminus \sigma \}, $$

where $\sigma$ is a Borel subset of $T$ (see e.g. [11, p.8, Theorem 1.2.1]). It follows from Szegő’s infimum theorem that a function $f$ is 1-invariant in $L^2(T)$ if and only if $\log |f| \in L^1(T)$ (see e.g. [10, p.12, Corollary 4]).

For $X = C^\infty(T)$, Makarov [7] gave a complete description of the invariant subspaces of $S$. He also obtained the following characterization of 1-invariant functions of $C^\infty(T)$.

**Theorem (Makarov [7, p.3]).** A function $f$ is 1-invariant in $C^\infty(T)$ if and only if $\log |f| \in L^1(T)$.

The case $X = C^n(T) \ (n \geq 1)$ is more complicated, and no characterization of 1-invariant functions is known ([7, p.3], see also [8, Theorem 1.3] or [9, Theorem 5]).

We shall focus our attention on the harmonic Dirichlet space $D(T)$. This is the set of functions $f \in L^2(T)$ whose Fourier coefficients satisfy

$$ D(f) := \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 |n| < \infty. $$

It becomes a Hilbert space if endowed with the norm $\| \cdot \|_{D(T)}$, given by

$$ \| f \|_{D(T)}^2 := \| f \|_{L^2(T)}^2 + D(f) = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 (1 + |n|). $$

According to Douglas’ formula [3], we have

$$ D(f) = \frac{1}{4\pi^2} \iint_{T^2} \frac{|f(\zeta) - f(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|. $$

This can also be written

$$ D(f) = \frac{1}{2\pi} \int_T D_\zeta(f) |d\zeta|, $$

where $D_\zeta(f)$ is the so-called local Dirichlet integral of $f$ at $\zeta$, given by

$$ D_\zeta(f) := \frac{1}{2\pi} \int_T \frac{|f(\zeta) - f(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'|, \quad \zeta \in T. $$

As $D(T)$ is dense in $L^2(T)$, if a function $f$ is cyclic for $D(T)$, then it is also cyclic for $L^2(T)$. So by Szegő’s theorem we have

$$ f \text{ is cyclic for } D(T) \Rightarrow \int_T \log |f(\zeta)| |d\zeta| = -\infty. $$

In [12], Ross, Richter and Sundberg gave a complete characterization of the 2-invariant subspaces $M$ of $D(T)$ in terms of their zero
sets. In order to state their result, we need to introduce the notion of logarithmic capacity (see for instance [6 §2.4]).

The energy of a Borel probability measure $\mu$ on $\mathbb{T}$ is defined by

$$I(\mu) := \int_{\mathbb{T}^2} \log \frac{1}{|\zeta - \zeta'|} \, d\mu(\zeta) d\mu(\zeta') = \sum_{n=1}^{\infty} \frac{|\hat{\mu}(n)|^2}{n}.$$ 

Then we define the logarithmic capacity of a Borel subset $E$ of $\mathbb{T}$ by

$$c(E) := \frac{1}{\inf \{ I(\mu) : \mu \in \mathcal{P}(E) \}},$$

where $\mathcal{P}(E)$ denotes the set of probability measures supported on a compact subset of $E$. We say that a property holds quasi–everywhere (q.e.) if it holds everywhere outside a set of logarithmic capacity zero.

It is known that, if $f \in \mathcal{D}(\mathbb{T})$, then the radial limit of the Poisson integral of $f$ exists q.e. and is equal to $f$ a.e. (for more details we refer to [12] and the references therein). In the sequel, $f$ will denote this limit, and will therefore be defined q.e. on $\mathbb{T}$. We shall write $Z(f)$ for the zero set of $f$, namely

$$Z(f) := \{ \zeta \in \mathbb{T} : f(\zeta) = 0 \}.$$ 

Note that this set is defined up to sets of logarithmic capacity zero.

**Theorem** (Richter–Ross–Sundberg [12]). $\mathcal{M}$ is a 2-invariant subspace of $\mathcal{D}(\mathbb{T})$ if and only if there exists a measurable set $E \subset \mathbb{T}$ such that $\mathcal{M} = \mathcal{D}_E := \{ f \in \mathcal{D}(\mathbb{T}) : f| E = 0 \text{ q.e.} \}$.

Note that the problem of characterization of 1-invariant subspaces of $\mathcal{D}(\mathbb{T})$ remains open. It was proved in [1] [13] that, for each $n \in \mathbb{N} \cup \{ \infty \}$, there exists an invariant subspace $\mathcal{M}$ of $\mathcal{D}(\mathbb{T})$ such that $\dim(\mathcal{M}/S(\mathcal{M})) = n$. This suggests that the lattice of 1-invariant subspaces has a very complicated structure.

As a direct consequence of the Richter–Ross–Sundberg theorem, we obtain the following necessary conditions for cyclicity in $\mathcal{D}(\mathbb{T})$.

**Theorem 1.** If $f$ is cyclic for $\mathcal{D}(\mathbb{T})$, then

$$\int_{\mathbb{T}} \log |f(\zeta)| \, d\zeta = -\infty \quad \text{and} \quad c(Z(f)) = 0.$$ 

Our goal in this paper is to give sufficient conditions for a function $f \in \mathcal{D}(\mathbb{T})$ to be cyclic.

For $\beta \in (0, 1]$, we shall denote by $\text{Lip}_\beta(\mathbb{T})$ the set of functions $f$ continuous on $\mathbb{T}$ such that

$$\|f\|_{\text{Lip}_\beta(\mathbb{T})} := \|f\|_{c(\mathbb{T})} + \sup_{\zeta, \zeta' \in \mathbb{T}} \frac{|f(\zeta) - f(\zeta')|}{|\zeta - \zeta'|^\beta} < \infty.$$
For $\alpha \in (0, 1)$, we set

$$C^{1+\alpha}(\mathbb{T}) := \{f \in C^1(\mathbb{T}) : f' \in \text{Lip}_\alpha(\mathbb{T})\}.$$  

Of course, if $f$ belongs to Lip$\beta(\mathbb{T})$ or $C^{1+\alpha}(\mathbb{T})$, then $Z(f)$ is closed in $\mathbb{T}$.

We shall establish the following result.

**Theorem 2.** Let $f \in D(\mathbb{T})$ such that $|f| \in C^{1+\alpha}(\mathbb{T})$, where $\alpha \in (0, 1)$. Suppose further that $\log |f| \notin L^1(\mathbb{T})$. Then $[f^2]_N = D_{Z(f)}$.

Combining Theorems 1 and 2 we deduce

**Corollary.** Let $f \in D(\mathbb{T})$ such that $|f| \in C^{1+\alpha}(\mathbb{T})$, where $\alpha \in (0, 1)$. Then the following assertions are equivalent:

1. $f^2$ is cyclic for $D(\mathbb{T})$;
2. $\log |f| \notin L^1(\mathbb{T})$ and $c(Z(f)) = 0$.

A closed set $E \subset \mathbb{T}$ is said to be a *Carleson set* (and we write $E \in (C)$) if

$$\int_\mathbb{T} \log \frac{1}{d(\zeta, E)} |d\zeta| < \infty.$$

For background information on Carleson sets, see e.g. [6, §4.4]. Note that, if $f \in \text{Lip}_\beta(\mathbb{T})$ and $Z(f) \notin (C)$, then $\log |f| \notin L^1(\mathbb{T})$.

It is known that Lip$\beta(\mathbb{T}) \subset D(\mathbb{T})$ if and only if $\beta > 1/2$. The inclusion Lip$\beta(\mathbb{T}) \subset D(\mathbb{T})$ for $\beta > 1/2$ can easily be obtained from Douglas’ formula.

We shall establish the following theorem.

**Theorem 3.** Let $f \in \text{Lip}_\beta(\mathbb{T})$, where $\beta \in (\frac{1}{2}, 1]$. If $Z(f) \notin (C)$, then $[f^2]_N = D_{Z(f)}$. If furthermore $c(Z(f)) = 0$, then $f$ is cyclic for $D(\mathbb{T})$.

### 2. Proof of Theorem 2

For the proof of Theorem 2 we shall need the following standard result.

**Lemma 4.** Let $f \in D(\mathbb{T})$. The following assertions are equivalent:

1. $[f]_N = [f]_Z$;
2. $f \in [Sf]_N$;
3. $\inf \{\|pf\|_{D(\mathbb{T})} : p \in H^\infty, pf \in D(\mathbb{T}) \text{ and } p(0) = 1\} = 0$.

**Proof.** Since $S$ is invertible, (1) and (2) are equivalent.

If $f \in [Sf]_N$, then there is a sequence $(p_n)$ of polynomials such that $p_n(0) = 1$ and $\|(1 - p_n)f\|_{D(\mathbb{T})} \to 0$. This proves that (2) implies (3).

Finally, suppose that (3) holds. Let $(p_n) \subset H^\infty$ be a sequence such that $p_n(0) = 1$, $p_n f \in D(\mathbb{T})$ and $\|p_n f\|_{D(\mathbb{T})} \to 0$. Writing $p_n = 1 - z_{q_n}$, by [12, Proposition 3.4] we have $z_{q_n} f \in [Sf]_N$. Since $z_{q_n} f$ converges to $f$, it follows that $f \in [Sf]_N$, so that (2) holds. $\square$
We shall also need the following result, which is a special case of a theorem due to Carleson–Jacobs–Havin–Shamoyan [2, Theorem 6.1].

**Lemma 5.** Let $F$ be an outer function on $\mathbb{D}$ that is continuous on $\mathbb{D}$.
If $|F| \in C^{1+\alpha}(\mathbb{T})$, where $\alpha \in (0, 1)$, then $F \in \text{Lip}_{(1+\alpha)/2}(\mathbb{D})$. Furthermore, the Lipschitz constant associated to $F$ on $\mathbb{D}$ depends only on the Lipschitz constants and bounds for the derivatives of $|F|$ on $\mathbb{T}$.

**Proof of Theorem 2.** Let $p_\epsilon$ be the outer function such that

$$|p_\epsilon(\zeta)| = \frac{e^{-M_\epsilon}}{|f(\zeta)| + \epsilon} \text{ a.e. on } \mathbb{T},$$

where the constant $M_\epsilon$ is chosen so that $p_\epsilon(0) = 1$. Thus

$$M_\epsilon = \int_\mathbb{T} \log \left( \frac{1}{|f(\zeta)| + \epsilon} \right) \frac{|d\zeta|}{2\pi},$$

and since $\log |f| \notin L^1(\mathbb{T})$, it follows that $M_\epsilon \to \infty$ as $\epsilon \to 0^+$. We are going to prove that

$$\lim_{\epsilon \to 0^+} \|p_\epsilon f^2\|_{D(\mathbb{D})} = 0.$$ 

If this holds, then by Lemma 4 we have $[f^2]_N = [f^2]_Z$, and since clearly $Z(f^2) = Z(f)$, we can apply the Richter–Ross–Sundberg theorem to obtain the desired result.

We have

$$\|p_\epsilon f^2\|_{D(\mathbb{D})}^2 = \|p_\epsilon f^2\|_{L^2(\mathbb{T})}^2 + D(p_\epsilon f^2).$$

For the first term, we have

$$\|p_\epsilon f^2\|_{L^2(\mathbb{T})}^2 = \int_\mathbb{T} \frac{e^{-2M_\epsilon} |f|^4}{(|f| + \epsilon)^2} \frac{|d\zeta|}{2\pi} \leq e^{-2M_\epsilon} \|f^2\|_{L^2(\mathbb{T})}^2 \to 0 \text{ as } \epsilon \to 0^+.$$ 

The second term we estimate using Douglas’ formula, namely

$$D(p_\epsilon f^2) = \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} \frac{|(p_\epsilon f^2)(\zeta) - (p_\epsilon f^2)(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|. $$

Let

$$\Gamma := \{(\zeta, \zeta') \in \mathbb{T}^2 : |f(\zeta')| \leq |f(\zeta)|\}. $$

Then, by symmetry,

$$D(p_\epsilon f) = 2 \frac{1}{4\pi^2} \iint_{\Gamma} \frac{|(p_\epsilon f)(\zeta) - (p_\epsilon f)(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|. $$

Now, for all $\zeta, \zeta' \in \mathbb{T}$, we have

$$|(p_\epsilon f^2)(\zeta) - (p_\epsilon f^2)(\zeta')|^2$$

$$= |p_\epsilon(\zeta)(f^2(\zeta) - f^2(\zeta')) + f^2(\zeta')(p_\epsilon(\zeta) - p_\epsilon(\zeta'))|^2$$

$$\leq 2|p_\epsilon(\zeta)|^2|f^2(\zeta) - f^2(\zeta')|^2 + 2|f^2(\zeta')|^2|p_\epsilon(\zeta) - p_\epsilon(\zeta')|^2.$$
Hence
\[
\iint_{\Gamma} \frac{|(p_\epsilon f^2)(\zeta) - (p_\epsilon f^2)(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| \leq 2A_\epsilon + 2B_\epsilon,
\]
where
\[
A_\epsilon := \iint_{\Gamma} |p_\epsilon(\zeta)|^2 \frac{|f^2(\zeta) - f^2(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|
\]
and
\[
B_\epsilon := \iint_{\Gamma} |f^2(\zeta')|^2 \frac{|p_\epsilon(\zeta) - p_\epsilon(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|.
\]
We estimate \(A_\epsilon\) directly as follows:
\[
A_\epsilon = e^{-2M_\epsilon} \iint_{\Gamma} \frac{|f(\zeta) + f(\zeta')|^2 |f(\zeta) - f(\zeta')|^2}{(|f(\zeta)| + \epsilon)^2 |\zeta - \zeta'|^2} |d\zeta'| |d\zeta|
\leq 4e^{-2M_\epsilon} \iint_{\Gamma} \frac{|f(\zeta) - f(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|
\leq 4e^{-2M_\epsilon} 4\pi^2 D(f).
\]
Hence \(A_\epsilon \to 0\) as \(\epsilon \to 0^+\).

To estimate \(B_\epsilon\), we consider the outer function \(F_\epsilon\) such that
\[
|F_\epsilon(\zeta)| = |f(\zeta)| + \epsilon \quad \text{a.e. on } \mathbb{T}.
\]
By Lemma 5, since \(|F_\epsilon| \in C^{1+\alpha}(\mathbb{T})\), we have \(F_\epsilon \in Lip(1+\alpha)/2(\mathbb{T}) \subset D(\mathbb{T})\) and there exists a positive constant \(D\), depending only on \(|f|\), such that \(D(F_\epsilon) \leq D\) for all \(\epsilon \in (0, 1)\). We then have
\[
B_\epsilon = \iint_{\Gamma} \frac{e^{-2M_\epsilon} |f^2(\zeta')|^2}{(|f(\zeta)| + \epsilon)^2 (|f(\zeta')| + \epsilon)^2} \frac{|1/p_\epsilon(\zeta) - 1/p_\epsilon(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|
\leq e^{-2M_\epsilon} \iint_{\Gamma} \frac{|F_\epsilon(\zeta) - F_\epsilon(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|
\leq e^{-2M_\epsilon} 4\pi^2 D(F_\epsilon).
\]
Thus \(B_\epsilon \to 0\) as \(\epsilon \to 0^+\). This completes the proof of Theorem 2.  

3. Proof of Theorem 3

To prove Theorem 3 we shall need the following additional lemma.

**Lemma 6.** Let \(f \in Lip_\beta(\mathbb{T})\), where \(\beta > 1/2\). Then, for \(\eta \in (0, \frac{2\beta - 1}{2\beta})\), we have
\[
\iint_{\mathbb{T}^2} \frac{|f(\zeta) - f(\zeta')|^{2-2\eta}}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| < +\infty.
\]
Proof. Since \( \beta > 1/2(1-\eta) \), we get

\[
\iint_{\mathbb{T}^2} \frac{|f(\zeta) - f(\zeta')|^{2-2\eta}}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| \lesssim \iint_{\mathbb{T}^2} \frac{|d\zeta'| |d\zeta|}{|\zeta - \zeta'|^{2(1-(1-\eta)\beta)}} < \infty. \quad \square
\]

Note that here, and in what follows, we write \( A \lesssim B \) to mean that there is an absolute constant \( C \) such that \( A \leq CB \).

Proof of Theorem 3. By (12), it suffices to prove that \( [f]_N \) is \( 2 \)-invariant, which is equivalent to proving that \( f \in [Sf]_N \).

Let \( \epsilon, \gamma > 0 \), where \( \gamma \) will be taken small. Let \( E \) be a closed subset of \( \mathcal{Z}(f) \) such that \( |E| = 0 \) and \( E \notin (C) \). Let \( p_\epsilon \) be the outer function satisfying

\[
|p_\epsilon(\zeta)| = e^{-M_\epsilon} \frac{1}{(d(\zeta, E)^\gamma + \epsilon)^{1/2}} \quad \text{a.e. on } \mathbb{T},
\]

where the constant \( M_\epsilon \) is chosen so that \( p_\epsilon(0) = 1 \). Thus

\[
M_\epsilon := \frac{1}{2} \int_\mathbb{T} \log \left( \frac{1}{d(\zeta, E)^\gamma + \epsilon} \right) \frac{|d\zeta|}{2\pi},
\]

and since \( E \notin (C) \), it follows that \( M_\epsilon \to \infty \) as \( \epsilon \to 0^+ \). By Lemma 4, it suffices to prove that

\[
\lim_{\epsilon \to 0^+} \|p_\epsilon f\|_{\mathcal{D}(\mathbb{T})} = 0.
\]

Now

\[
\|p_\epsilon f\|_{\mathcal{D}(\mathbb{T})}^2 = \|p_\epsilon f\|_{L^2(\mathbb{T})}^2 + \mathcal{D}(p_\epsilon f).
\]

For the first term, we have

\[
\|p_\epsilon f\|_{L^2(\mathbb{T})}^2 \lesssim e^{-2M_\epsilon} \int_\mathbb{T} \frac{d(\zeta, E)^{2\beta}}{d(\zeta, E)^\gamma} |d\zeta|.
\]

Thus \( \|p_\epsilon f\|_{L^2(\mathbb{T})} \to 0 \) as \( \epsilon \to 0^+ \), provided that \( \gamma < 2\beta \).

For the second term we again use Douglas’ formula, namely

\[
\mathcal{D}(p_\epsilon f) = \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} \frac{||(p_\epsilon f)(\zeta) - (p_\epsilon f)(\zeta')||^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|.
\]

Let

\[
\Gamma := \{(\zeta, \zeta') \in \mathbb{T}^2 : d(\zeta', E) \leq d(\zeta, E)\}.
\]

Arguing as in the proof of Theorem 2, we have

\[
\mathcal{D}(p_\epsilon f) \lesssim \iint_\Gamma |p_\epsilon(\zeta)|^2 \frac{|f(\zeta) - f(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta||d\zeta'| + \iint_\Gamma |f(\zeta')|^2 \frac{|p(\zeta) - p(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta||d\zeta'| = A_\epsilon + B_\epsilon, \quad \text{say}.
\]
Thus where \( \delta \) depends only on \( f \), and the double integral is finite, thanks to Lemma \( \text{[5]} \). Hence \( A_\epsilon \to 0 \) as \( \epsilon \to 0^+ \).

To estimate \( B_\epsilon \), we introduce the outer function \( F_\epsilon \) satisfying

\[
|F_\epsilon(\zeta)| = d(\zeta, E)^\gamma + \epsilon \quad \text{a.e. on } \mathbb{T}.
\]

By the Carleson–Richter–Sundberg formula [6, Theorem 7.4.2], we have

\[
\mathcal{D}_\zeta(F_\epsilon) = \int_{\mathbb{T}} \frac{|F_\epsilon(\zeta)|^2 - |F_\epsilon(\zeta')|^2 - 2|F_\epsilon(\zeta')| \log |F_\epsilon(\zeta)/F_\epsilon(\zeta')|}{|\zeta - \zeta'|^2} \, d\zeta' \quad |d\zeta'| = \frac{2\pi}{\log \pi}.
\]

Therefore

\[
B_\epsilon \lesssim e^{-2M_\epsilon} \int_{\mathbb{T}} \frac{d(\zeta', E)^{2\beta}}{(d(\zeta, E)\gamma + \epsilon)(d(\zeta, E)\gamma + \epsilon)} \frac{|F_\epsilon(\zeta) - F_\epsilon(\zeta')|^2}{|\zeta - \zeta'|^2} \, |d\zeta| \, |d\zeta'|.
\]

\[
\lesssim e^{-2M_\epsilon} \int_{\mathbb{T}} \frac{|F_\epsilon(\zeta) - F_\epsilon(\zeta')|^2}{|\zeta - \zeta'|^2} \, d(\zeta', E)^{2(\beta - \gamma)} \, |d\zeta| \, |d\zeta'|.
\]

\[
\lesssim e^{-2M_\epsilon} \int_{\mathbb{T}} \mathcal{D}_\zeta(F_\epsilon) \, d(\zeta, E)^{2(\beta - \gamma)} \, |d\zeta|.
\]

\[
\lesssim e^{-2M_\epsilon} \int_{\mathbb{T}} \frac{|F_\epsilon(\zeta)|^2 - |F_\epsilon(\zeta')|^2 - 2|F_\epsilon(\zeta')| \log |F_\epsilon(\zeta)/F_\epsilon(\zeta')|}{|\zeta - \zeta'|^2}
\]

\[
\times \left( d(\zeta, E)^{2(\beta - \gamma)} + d(\zeta', E)^{2(\beta - \gamma)} \right) \, |d\zeta| \, |d\zeta'|.
\]

Exchanging the roles of \( \zeta \) and \( \zeta' \), and taking the average, we obtain

\[
B_\epsilon \lesssim e^{-2M_\epsilon} \int_{\mathbb{T}} \frac{(|F_\epsilon(\zeta)|^2 - |F_\epsilon(\zeta')|^2) \log |F_\epsilon(\zeta)/F_\epsilon(\zeta')|}{|\zeta - \zeta'|^2}
\]

\[
\times \left( d(\zeta, E)^{2(\beta - \gamma)} + d(\zeta', E)^{2(\beta - \gamma)} \right) \, |d\zeta| \, |d\zeta'|.
\]

Thus

\[
B_\epsilon \lesssim e^{-2M_\epsilon} \int_{\mathbb{T}} \frac{\delta^{\gamma} - \delta'^{\gamma}}{|\zeta - \zeta'|^2} \log \left( \frac{\delta^{\gamma} + \epsilon}{\delta'^{\gamma} + \epsilon} \right) (\delta^{2(\beta - \gamma)} + \delta'^{2(\beta - \gamma)}) \, |d\zeta| \, |d\zeta'|,
\]

where \( \delta := d(\zeta, E) \) and \( \delta' := d(\zeta', E) \).
Let \((I_j)\) be the connected components of \(\mathbb{T} \setminus E\), and set
\[
N_E(t) := 2 \sum_j 1_{\{|I_j| > 2t\}}, \quad 0 < t < 1.
\]

Then, for every measurable function \(\Omega : [0, \pi] \to \mathbb{R}^+\), we have
\[
\int_{\mathbb{T}} \Omega(d(\zeta, E)) |d\zeta| = \int_0^\pi \Omega(t) N_E(t) \, dt.
\]

Using similar ideas to those in [4, 5], we obtain
\[
J := \iint_{\mathbb{T}^2} \frac{\delta^{\gamma} - \delta^{\gamma'}}{|\zeta - \zeta'|^2} \log\left(\frac{\delta^{\gamma} + \epsilon}{\delta^{\gamma} + \epsilon'}\right) (\delta^{2(\beta - \gamma)} + \delta'^{2(\beta - \gamma)}) |d\zeta| |d\zeta'|
\leq \int_0^\pi \int_0^t \frac{(s+t)^{\gamma} - t^{\gamma}}{s^2} \log\left(\frac{(s+t)^{\gamma} + \epsilon}{t^{\gamma} + \epsilon}\right) (t+s)^{2(\beta - \gamma)} N_E(t) \, ds \, dt
\leq \int_0^\pi \int_0^t \frac{(s+t)^{\gamma} - t^{\gamma}}{s^2} \log[(s+t)^{\gamma}/t^{\gamma}] (t+s)^{2(\beta - \gamma)} N_E(t) \, ds \, dt
+ \int_0^\pi \int_t^\pi \frac{(s+t)^{\gamma} - t^{\gamma}}{s^2} \log[1/(t^{\gamma} + \epsilon)] (t+s)^{2(\beta - \gamma)} \, ds \, N_E(t) \, dt
= J_1 + J_2,
\]
where
\[
J_1 \lesssim \int_0^\pi i^{2\beta - \gamma - 1} \int_0^1 \frac{(1+x)^{\gamma} - 1}{x^2} \log(1+x) \, dx \, N_E(t) \, dt
\lesssim \int_0^\pi i^{2\beta - \gamma - 1} N_E(t) \, dt = O(1),
\]
and
\[
J_2 \lesssim \int_0^\pi i^{2\beta - \gamma - 1} \log[1/(t^{\gamma} + \epsilon)] \int_1^{\pi/t} \frac{(1+x)^{\gamma} - 1}{s^2} (1+x)^{2(\beta - \gamma)} \, dx \, N_E(t) \, dt
\lesssim \int_0^\pi \log[1/(t^{\gamma} + \epsilon)] N_E(t) \, dt
\lesssim \int_\pi \log(d(\zeta, E)^{\gamma} + \epsilon) |d\zeta| = 2M_e.
\]
Thus \(J = O(M_e)\). Combining this with the estimate (3.1), we get
\[
B_\epsilon \lesssim M_e \epsilon^{-2M_e}.
\]
Hence \(B_\epsilon \to 0\) as \(\epsilon \to 0^+\). This completes the proof of Theorem 3. \(\square\)
4. Concluding remarks

1. In order to produce cyclic functions for $\mathcal{D}(\mathbb{T})$ using Theorem 3, we need to construct closed subsets $E \subset \mathbb{T}$ such that $E \notin (C)$ and $c(E) = 0$. An easy example can be given by countable sets. Indeed, taking $E_\beta := \{e^{i/(\log n)^\beta} : n \geq 2\}$ with $\beta \leq 1$ provides such an example. Using Cantor-type sets, it is also possible to construct perfect sets $E$ such that $E \notin (C)$ and $c(E) = 0$.

2. One can consider weighted harmonic Dirichlet spaces instead of the classical harmonic Dirichlet space. More precisely, given $\alpha \in [0, 1)$, the weighted harmonic Dirichlet space $\mathcal{D}_\alpha(\mathbb{T})$ is the space of functions $f \in L^2(\mathbb{T})$ such that
   $$\|f\|_{\mathcal{D}_\alpha(\mathbb{T})}^2 := \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2(1 + |n|)^{1-\alpha} < \infty.$$  
   We define the $\alpha$-capacity of a Borel subset $E \subset \mathbb{T}$ by
   $$c_\alpha(E) = 1/\inf\{I_\alpha(\mu) : \mu \in \mathcal{P}(E)\},$$
   where $\mathcal{P}(E)$ is the set of all probability measures supported on a compact subset of $E$ and $I_\alpha(\mu) := \sum_{n \geq 1} |\hat{\mu}(n)|^2/n^{1-\alpha}$ is the $\alpha$-energy of $\mu$. We say that a property holds $c_\alpha$-quasi-everywhere if it holds everywhere outside a set of $c_\alpha$-capacity zero.

   It is well known that $\text{Lip}_\beta(\mathbb{T}) \subset \mathcal{D}_\alpha(\mathbb{T})$ if and only $\beta > (1 - \alpha)/2$. Theorem 3 may be extended to show that, if $f \in \text{Lip}_\beta(\mathbb{T})$, where $\beta \in ((1 - \alpha)/2, 1]$, and if $Z(f) \notin (C)$, then
   $$[f]_N = \{g \in \mathcal{D}(\mathbb{T}) : g|_{Z(f)} = 0\} c_\alpha\text{-quasi-everywhere}.$$  

3. One can equally well consider the holomorphic Dirichlet space, namely $\mathcal{D} := \{f \in \mathcal{D}(\mathbb{T}) : \hat{f}(n) = 0 (n < 0)\}$. Here too the problem of characterizing the cyclic functions is still open. For more on this topic, see e.g. [6, Chapter 9].

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References

[1] A. Aleman, S. Richter, W. Ross, Bergman spaces on disconnected domains, *Canad. J. Math.* 48 (1996), no. 2, 225–243.
[2] J. E. Brennan, Approximation in the mean by polynomials on non-Carathéodory domains, *Ark. Mat.* 15 (1977), no. 1, 117–168.

[3] J. Douglas, Solution of the problem of Plateau, *Trans. Amer. Math. Soc.* 33 (1931), no. 1, 263–321.

[4] O. El-Fallah, K. Kellay, T. Ransford, On the Brown–Shields conjecture for cyclicity in the Dirichlet space, *Adv. Math.* 222 (2009), no. 6, 2196–2214.

[5] O. El-Fallah, K. Kellay, T. Ransford, Cantor sets and cyclicity in weighted Dirichlet spaces, *J. Math. Anal. Appl.* 372 (2010), no. 2, 565–573.

[6] O. El-Fallah, K. Kellay, J. Mashreghi, T. Ransford, *A Primer on the Dirichlet Space*, Cambridge University Press, Cambridge, 2014.

[7] N. G. Makarov, Invariant subspaces of the space $C^\infty$, *Mat. Sb. (N.S.)* 119(161) (1982), no. 1, 1–31, 160.

[8] N. G. Makarov, Sets of simple invariance. Investigations on linear operators and the theory of functions, *X, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 107 (1982), 104–135, 230.

[9] N. G. Makarov, Sets of 1-invariance and 1-invariant subspaces (smooth functions), *Dokl. Akad. Nauk SSSR* 262 (1982), no. 5, 1072–1075.

[10] N. K. Nikolskii, *Treatise on the Shift Operator*, Springer, Berlin, 1986.

[11] N. K. Nikolski. *Operators, Functions, and Systems: An Easy Reading, Vol. 1: Hardy, Hankel, and Toeplitz*, Amer. Math. Soc., Providence RI, 2002.

[12] S. Richter, W. Ross, C. Sundberg, Hyperinvariant subspaces of the harmonic Dirichlet space, *J. Reine Angew. Math.* 448 (1994), 1–26.

[13] W. Ross, Invariant subspaces of the harmonic Dirichlet space with large codimension, *Proc. Amer. Math. Soc.* 124 (1996), no. 6, 1841–1846.

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