MINIMAL PROBLEMS FOR THE
CALIBRATED TRIFOCAL VARIETY

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Abstract. We determine the algebraic degree of minimal problems for
the calibrated trifocal variety in computer vision. We rely on numerical
algebraic geometry and the homotopy continuation software Bertini.

1. Introduction

In computer vision, one fundamental task is 3D reconstruction: the re-
covery of three-dimensional scene geometry from two-dimensional images.
In 1981, Fischler and Bolles proposed a methodology for 3D reconstruc-
that is robust to outliers in image data [9]. This is known as Random
Sampling Consensus (RANSAC) and it is a paradigm in vision today [1].
RANSAC consists of three steps. To compute a piece of the 3D scene:

- Points, lines and other features that are images of the same source
  are detected in the photos. These matches are the image data.
- A minimal sample of image data is randomly selected. Minimal
  means that only a positive finite number of 3D geometries are exactly
  consistent with the sample. Those 3D geometries are computed.
- To each computed 3D geometry, the rest of the image data is com-
  pared. If one is approximately consistent with enough of the image
  data, it is kept. Else, the second step is repeated with a new sample.

Computing the finitely many 3D geometries in the second step is called
a minimal problem. Typically, it is done by solving a corresponding zero-
dimensional polynomial system, with coefficients that are functions of the
sampled image data [18]. Since this step is carried out thousands of times
in a full reconstruction, it is necessary to design efficient, specialized solvers.
One of the most used minimal solvers in vision is Nistère’s [25], based on
Gröbner bases, to recover the relative position of two calibrated cameras.

The concern of this paper is the recovery of the relative position of three
calibrated cameras from image data. To our knowledge, no satisfactory
solution to this basic problem exists in the literature. Our main result is the
determination of the algebraic degree of 66 minimal problems for the recovery
of three calibrated cameras; in other words, we find the generic number of
complex solutions (see Theorem 6). The solution sets for particular random
instances are available at this project’s computational webpage:

https://math.berkeley.edu/~jkileel/CalibratedMinimalProblems.html.

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braic geometry.
As a by-product, we can derive minimal solvers for each case. Our techniques come from numerical algebraic geometry \cite{28}, and we rely on the homotopy continuation software Bertini \cite{5}. This implies that our results are correct only with very high probability; in ideal arithmetic, with probability 1. Mathematically, the main object in this paper is a particular projective algebraic variety $T_{\text{cal}}$, which is a convenient moduli space for the relative position of three calibrated cameras. This variety is 11-dimensional, degree 4912 inside the projective space $\mathbb{P}^{26}$ of $3 \times 3 \times 3$ tensors (see Theorem 20). We call it the \textit{calibrated trifocal variety}. Theorem 21 formulates our minimal problems as slicing $T_{\text{cal}}$ by special linear subspaces of $\mathbb{P}^{26}$.

The rest of this paper is organized as follows. In Section 2, we make our minimal problems mathematically precise and we state Theorem 6. In Section 3, we examine image correspondences using multi-view varieties and then trifocal tensors \cite[Chapter 15]{13}. In Section 4, we prove that trifocal tensors and camera configurations are equivalent. In Section 5, we introduce the calibrated trifocal variety $T_{\text{cal}}$ and prove several useful facts. Finally, in Section 6, we present a computational proof of the main result Theorem 6.

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\section{Statement of Main Result}

We begin by giving several definitions. Throughout this paper, we work with the standard camera model of the projective camera \cite[Section 6.2]{13}.

\textbf{Definition 1.} A \textit{(projective) camera} is a full rank $3 \times 4$ matrix in $\mathbb{C}^{3 \times 4}$ defined up to multiplication by a nonzero scalar.

Thus, a camera corresponds to a linear projection $\mathbb{P}^3 \rightarrow \mathbb{P}^2$. The \textbf{center} of a camera $A$ is the point $\ker(A) \in \mathbb{P}^3$. A camera is \textbf{real} if $A \in \mathbb{R}^{3 \times 4}$.

\textbf{Definition 2.} A \textbf{calibrated camera} is a $3 \times 4$ matrix in $\mathbb{C}^{3 \times 4}$ whose left $3 \times 3$ submatrix is in the special orthogonal group $\text{SO}(3, \mathbb{C})$.

Real calibrated cameras have the interpretation of cameras with known and normalized \textit{internal parameters} (e.g. focal length) \cite[Subsection 6.2.4]{13}. In practical situations, this information can be available during 3D reconstruction. Note that calibration of a camera is preserved by right multiplication by elements of the following subgroup of $\text{GL}(4, \mathbb{C})$:

$$G := \{ g \in \mathbb{C}^{4 \times 4} \mid (g_{ij})_{1 \leq i, j \leq 3} \in \text{SO}(3, \mathbb{C}), g_{41} = g_{42} = g_{43} = 0 \text{ and } g_{44} \neq 0 \}.$$  

Elements in $G$ act on $A^3 \subset \mathbb{P}^3$ as composites of rotations, translations and central dilations. In the calibrated case of 3D reconstruction, one aims to recover camera positions (and afterwards the 3D scene) up to those motions, since recovery of absolute positions is not possible from image data alone.

\textbf{Definition 3.} A \textbf{configuration} of three calibrated cameras is an orbit of the action of the group $G$ above on the set:

$$\{(A, B, C) \mid A, B, C \text{ are calibrated cameras}\}$$

via simultaneous right multiplication.
By abuse of notation, we will call \((A, B, C)\) a calibrated camera configuration, instead of always denoting the orbit containing \((A, B, C)\).

As mentioned in Section 1, the image data used in 3D reconstruction typically are points and lines in the photos that match. This is made precise as follows. Call elements of \(\mathbb{P}^2\) image points, and elements of the dual projective plane \((\mathbb{P}^2)^\vee\) image lines. An element of \((\mathbb{P}^2 \sqcup (\mathbb{P}^2)^\vee)^{\times 3}\) is a point/line image correspondence. For example, an element of \(\mathbb{P}^2 \times \mathbb{P}^2 \times (\mathbb{P}^2)^\vee\) is called a point-point-line image correspondence, denoted \(PPL\).

**Definition 4.** A calibrated camera configuration \((A, B, C)\) is consistent with a given point/line image correspondence if there exist a point in \(\mathbb{P}^3\) and a line in \(\mathbb{P}^3\) containing it such that are such that \((A, B, C)\) respectively map these to the given points and lines in \(\mathbb{P}^2\).

For example, explicitly, a configuration \((A, B, C)\) is consistent with a given point-point-line image correspondence \((x, x', \ell') \in \mathbb{P}^2 \times \mathbb{P}^2 \times (\mathbb{P}^2)^\vee\) if there exist \((X, L) \in \mathbb{P}^3 \times \text{Gr}(\mathbb{P}^1, \mathbb{P}^3)\) with \(X \in L\) such that \(AX = x, BX = x',\) and \(CL = \ell'.\) In particular, this implies that \(X \neq \ker(A), \ker(B)\) and \(\ker(C) \notin L\). We say that a configuration \((A, B, C)\) is consistent with a set of point/line correspondences if it is consistent with each correspondence.

We give a numerical example to illustrate Theorem 6 on the next page:

**Example 5.** Given the following set of real, random correspondences:\(^1\)

\[
\begin{align*}
\text{PPP:} & \begin{bmatrix} 0.6132 & 0.4599 & 0.6863 \\ 0.5979 & 0.5713 & 0.4508 \\ 0.4970 & 0.5405 & 0.2692 \\ 0.8429 & 0.6734 & 0.1333 \\ 0.8933 & 0.7062 & 0.3328 \\ 0.3375 & 0.6669 & 0.8228 \\ 0.1054 & 0.7141 & 0.6781
\end{bmatrix} \\
\text{PPL:} & \begin{bmatrix} 0.6251 & 0.3232 & 0.3646 \\ 0.9248 & 0.5453 & 0.1497 \\ 0.2896 & 0.6898 & 0.6519 \\ 0.6909 & 0.9855 & 0.8469 \\ 0.4914 & 0.6777 & 0.6855 \
\end{bmatrix} \\
\end{align*}
\]

In the notation of Theorem 6, this is a generic instance of the minimal problem ‘\(1\text{PPP} + 4\text{PPL}\)’. Up to the action of \(G\), there are only a positive finite number of three calibrated cameras that are exactly consistent with this image data, namely 160 complex configurations. For this instance, it turns out that 18 of those configurations are real. For example, one is:

\[
A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -0.22 & 0.95 & -0.18 & 1 \\ 0.96 & 0.24 & 0.08 & 1.44 \\ -0.12 & 0.15 & 0.97 & 0.97 \\ -0.17 & 0.94 & -0.28 & 1.41 \end{bmatrix}, \quad C = \begin{bmatrix} -0.95 & 0.22 & 0.18 & -0.13 \\ -0.24 & -0.23 & -0.94 & -1.16 \end{bmatrix}.
\]

In a RANSAC run for 3D reconstruction, the image data above is identified by feature detection software such as SIFT [21]. Also, only the real configurations are compared for agreement with further image data.

In Example 5 above, 160 is the algebraic degree of the minimal problem ‘\(1\text{PPP} + 4\text{PPL}\)’. This means that for correspondences in a nonempty Zariski open (hence measure 1) subset of \((\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) \times (\mathbb{P}^2 \times \mathbb{P}^2 \times (\mathbb{P}^2)^\vee)^{\times 4}\), there are 160 consistent complex configurations. Given generic real correspondences, the number of real configurations varies, but 160 is an upper bound.

The cases in Theorem 6 admit a uniform treatment that we give below.

\(^1\)For ease of presentation, double precision floating point numbers are truncated here.
Theorem 6. The rows of the following table display the algebraic degree for 66 minimal problems across three calibrated views. Given generic point/line image correspondences in the amount specified by the entries in the first five columns, then the number of calibrated camera configurations over $\mathbb{C}$ that are consistent with those correspondences equals the entry in the sixth column.

| #PPP | #PLL | #PLL | #LLL | #PLL | #configurations |
|------|------|------|------|------|-----------------|
| 3    | 1    | 0    | 0    | 0    | 272             |
| 4    | 0    | 0    | 2    | 0    | 248             |
| 2    | 2    | 0    | 0    | 1    | 424             |
| 2    | 1    | 1    | 0    | 1    | 528             |
| 2    | 1    | 0    | 0    | 0    | 424             |
| 2    | 0    | 0    | 2    | 1    | 304             |
| 2    | 0    | 0    | 0    | 0    | 448             |
| 2    | 0    | 0    | 0    | 5    | 1072            |
| 1    | 4    | 0    | 0    | 0    | 760             |
| 1    | 3    | 1    | 0    | 0    | 560             |
| 1    | 3    | 0    | 0    | 2    | 520             |
| 1    | 2    | 2    | 0    | 0    | 672             |
| 1    | 2    | 1    | 0    | 2    | 576             |
| 1    | 2    | 0    | 0    | 4    | 1048            |
| 1    | 1    | 1    | 0    | 1    | 456             |
| 1    | 1    | 1    | 1    | 2    | 896             |
| 1    | 1    | 0    | 0    | 4    | 1944            |
| 1    | 1    | 0    | 1    | 0    | 768             |
| 1    | 1    | 0    | 2    | 2    | 736             |
| 1    | 0    | 0    | 0    | 6    | 1672            |
| 0    | 0    | 0    | 2    | 0    | 596             |
| 0    | 0    | 0    | 0    | 4    | 360             |
| 0    | 0    | 0    | 0    | 6    | 1176            |
| 0    | 0    | 0    | 0    | 4    | 1680            |
| 0    | 0    | 0    | 0    | 8    | 2272            |
| 0    | 0    | 0    | 0    | 1    | 260             |
| 0    | 4    | 4    | 0    | 1    | 616             |
| 0    | 4    | 0    | 1    | 1    | 544             |
| 0    | 3    | 2    | 0    | 1    | 1152            |
| 0    | 3    | 1    | 1    | 1    | 480             |
| 0    | 3    | 1    | 0    | 3    | 1280            |
| 0    | 3    | 0    | 1    | 1    | 672             |
| 0    | 3    | 0    | 1    | 3    | 1008            |
| 0    | 3    | 0    | 0    | 5    | 1408            |
| 0    | 2    | 2    | 1    | 1    | 1152            |
| 0    | 2    | 2    | 0    | 3    | 1680            |
| 0    | 2    | 1    | 1    | 1    | 1520            |
| 0    | 2    | 0    | 1    | 0    | 2972            |
| 0    | 2    | 0    | 0    | 5    | 460             |
| 0    | 2    | 0    | 0    | 7    | 1296            |
| 0    | 2    | 0    | 0    | 9    | 1520            |
| 0    | 2    | 0    | 0    | 11   | 2444            |
| 0    | 1    | 1    | 2    | 1    | 1016            |
| 0    | 1    | 1    | 0    | 1    | 1056            |
| 0    | 1    | 0    | 7    | 0    | 2144            |
| 0    | 1    | 0    | 0    | 0    | 520             |
| 0    | 1    | 0    | 0    | 7    | 2800            |
| 0    | 1    | 0    | 0    | 2    | 712             |
| 0    | 1    | 0    | 0    | 4    | 1156            |
| 0    | 1    | 0    | 0    | 6    | 1456            |
| 0    | 1    | 0    | 0    | 8    | 1752            |
| 0    | 1    | 0    | 0    | 0    | 2444            |
| 0    | 0    | 0    | 1    | 1    | 1016            |
| 0    | 0    | 0    | 0    | 1    | 1056            |
| 0    | 0    | 0    | 0    | 11   | 2444            |
| 0    | 0    | 0    | 0    | 9    | 3936            |
| 0    | 0    | 0    | 0    | 7    | 4512            |
Remark. A calibrated camera configuration \((A, B, C)\) has 11 degrees of freedom (Theorem 20), and the first five columns in the table above represent conditions of codimension 3, 2, 2, 2, 1, respectively (Theorem 21).

Remark. The algebraic degrees in Theorem 6 are intrinsic to the underlying camera geometry. However, our method of proof uses a device from multi-view geometry called trifocal tensors, which breaks symmetry between \((A, B, C)\). There are other minimal problems for three calibrated views involving image correspondences of type \(‘LPP’, ‘LPL’, ‘LLP’\). These also possess intrinsic algebraic degrees; but they are not covered by the non-symmetric proof technique used here.

3. Correspondences

In this section, we examine point/line image correspondences. In the first part, we use multi-view varieties to describe correspondences. This approach furnishes exact polynomial systems for the minimal problems in Theorem 6. However, each parametrized system has a different structure (in terms of number and degrees of equations). This would force a direct analysis for Theorem 6 to proceed case-by-case, and moreover, each system so obtained is computationally unwieldy. In Subsection 3.2, we recall the construction of the trifocal tensor [13, Chapter 15]. This is a point \(T_{A,B,C} \in \mathbb{C}^{3 \times 3 \times 3}\) associated to cameras \((A, B, C)\). It encodes necessary conditions for \((A, B, C)\) to be consistent with different types of correspondences. Tractable relaxations to the minimal problems in Theorem 6 are thus obtained, each with similar structure. We emphasize that everything in Section 3 applies equally to calibrated cameras \((A, B, C)\) as well as to uncalibrated cameras.

3.1. Multi-view varieties. Let \(A, B, C \in \mathbb{C}^{3 \times 4}\) be three projective cameras, not necessarily calibrated. Denote by \(\alpha : \mathbb{P}^3 \rightarrow \mathbb{P}^2_A, \beta : \mathbb{P}^3 \rightarrow \mathbb{P}^2_B, \gamma : \mathbb{P}^3 \rightarrow \mathbb{P}^2_C\) the corresponding linear projections. We make:

Definition 7. Fix projective cameras \(A, B, C\) as above. Denote by \(\mathcal{F}_{\ell_{0,1}}\) the incidence variety \(\{(X, L) \in \mathbb{P}^3 \times \text{Gr}(\mathbb{P}^1, \mathbb{P}^3) \mid X \in L\}\). Then the:

- **PLL multi-view variety** denoted \(X_{A,B,C}^{PLL}\) is the closure of the image of \(\mathcal{F}_{\ell_{0,1}} \rightarrow \mathbb{P}^2_A \times (\mathbb{P}^2_B)^\vee \times (\mathbb{P}^2_C)^\vee, (X, L) \mapsto (\alpha(X), \beta(L), \gamma(L))\)

- **LLL multi-view variety** denoted \(X_{A,B,C}^{LLL}\) is the closure of the image of \(\text{Gr}(\mathbb{P}^1, \mathbb{P}^3) \rightarrow (\mathbb{P}^2_A)^\vee \times (\mathbb{P}^2_B)^\vee \times (\mathbb{P}^2_C)^\vee, L \mapsto (\alpha(L), \beta(L), \gamma(L))\)

- **PPL multi-view variety** denoted \(X_{A,B,C}^{PPL}\) is the closure of the image of \(\mathcal{F}_{\ell_{0,1}} \rightarrow \mathbb{P}^2_A \times \mathbb{P}^2_B \times (\mathbb{P}^2_C)^\vee, (X, L) \mapsto (\alpha(X), \beta(X), \gamma(L))\)

- **PLP multi-view variety** denoted \(X_{A,B,C}^{PLP}\) is the closure of the image of \(\mathcal{F}_{\ell_{0,1}} \rightarrow \mathbb{P}^2_A \times (\mathbb{P}^2_B)^\vee \times \mathbb{P}^2_C, (X, L) \mapsto (\alpha(X), \beta(L), \gamma(X))\)

- **PPP multi-view variety** denoted \(X_{A,B,C}^{PPP}\) is the closure of the image of \(\mathbb{P}^3 \rightarrow \mathbb{P}^2_A \times \mathbb{P}^2_B \times \mathbb{P}^2_C, X \mapsto (\alpha(X), \beta(X), \gamma(X))\).

Next, we give the dimension and equations for these multi-view varieties; the ‘PPP’ case has appeared in [3]. In the following, we notate \(x \in \mathbb{P}^2_A\).
are above, and let

\[ J \]

for image lines. Also, we postpone treatment of the \('PLL' case to Subsection 3.2. In particular, the trilinear form \( T_{AB,C}(x, \ell', \ell'') \) will be defined there.

**Theorem 8.** Fix \( A, B, C \). The multi-view varieties from Definition 7 are irreducible. If \( A, B, C \) have linearly independent centers in \( \mathbb{P}^3 \), then the varieties have the following dimensions and multi-homogeneous prime ideals.

- \( \dim(X^{PLL}_{A,B,C}) = 5 \) and \( I(X^{PLL}_{A,B,C}) = \langle T_{A,B,C}(x, \ell', \ell'') \rangle \subset \mathbb{C}[x_i, \ell'_j, \ell''_k] \)
- \( \dim(X^{LLL}_{A,B,C}) = 4 \) and \( I(X^{LLL}_{A,B,C}) \subset \mathbb{C}[\ell_i, \ell'_j, \ell''_k] \) is generated by the maximal minors of the matrix \( (A^T \ell \ B^T \ell' \ C^T \ell'')_{4 \times 3} \)
- \( \dim(X^{PLP}_{A,B,C}) = 4 \) and \( I(X^{PLP}_{A,B,C}) \subset \mathbb{C}[x_i, x'_j, \ell''_k] \) is generated by the maximal minors of the matrix \( \begin{pmatrix} A & x & 0 \\ B & 0 & x' \\ \ell''TC & 0 & 0 \end{pmatrix}_{7 \times 6} \)
- \( \dim(X^{PPL}_{A,B,C}) = 4 \) and \( I(X^{PPL}_{A,B,C}) \subset \mathbb{C}[x_i, \ell'_j, x''_k] \) is generated by the maximal minors of the matrix \( \begin{pmatrix} A & x & 0 \\ B & 0 & x' \\ C & 0 & x'' \end{pmatrix}_{7 \times 6} \) together with
  \[ \det \begin{pmatrix} A & x & 0 \\ B & 0 & x' \\ C & 0 & x'' \end{pmatrix}_{6 \times 6} \text{ and } \det \begin{pmatrix} A & x & 0 \\ B & 0 & x' \\ C & 0 & x'' \end{pmatrix}_{6 \times 6} \text{ and } \det \begin{pmatrix} A & x & 0 \\ B & 0 & x' \\ C & 0 & x'' \end{pmatrix}_{6 \times 6} \]

**Proof.** Irreducibility is clear from Definition 7. For the dimension and prime ideal statements, we may assume that:

\[
A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.
\]

This is without loss of generality in light of the following group symmetries. Let \( g, g', g'' \in \text{SL}(3, \mathbb{C}) \) and \( h \in \text{SL}(4, \mathbb{C}) \). To illustrate, consider the third case above, and let \( J^{PPL}_{Ah,Bh,Ch} \subset \mathbb{C}[x_i, x'_j, \ell''_k] \) be the ideal generated by the maximal minors mentioned there. It is straightforward to check that:

\[
I(X^{PPL}_{Ah,Bh,Ch}) = I(X^{PPL}_{A,B,C}) \quad \text{and} \quad J^{PPL}_{Ah,Bh,Ch} = J^{PPL}_{A,B,C}.
\]

Also, we can check that:

\[
I(X^{PPL}_{gAh,g'Bh,g''Ch}) = (g, g', (g'')^T) \cdot I(X^{PPL}_{A,B,C})
\]

and \( J^{PPL}_{gAh,g'Bh,g''Ch} = (g, g', (g'')^T) \cdot J^{PPL}_{A,B,C} \).

Here the left, linear action of \( \text{SL}(3, \mathbb{C}) \times \text{SL}(3, \mathbb{C}) \times \text{SL}(3, \mathbb{C}) \) on \( \mathbb{C}[x_i, x'_j, \ell''_k] \) is via \( (g, g', g'') \cdot f(x, x', \ell'') = f(g^{-1}x, g'^{-1}x', g''^{-1}\ell'') \) for \( f \in \mathbb{C}[x_i, x'_j, \ell''_k] \).
Also, $\wedge^2 g'' = (g''^T)^{-1} \in \mathbb{C}^{3 \times 3}$. So, for the 'PPL' case, $I$ and $J$ transform in the same way when $(A, B, C)$ is replaced by $(gA h, g' Bh, g'' Ch)$; in the other cases, this holds similarly. Assuming that $A, B, C$ have linearly independent centers, we may choose $g, g', g'', h$ to harmlessly move the cameras into the position above. Now using the computer algebra system Macaulay2 [10], we verify the dimension and prime ideal statements for this special position. □

Remark. In Theorem 8, if $A, B, C$ do not have linearly independent centers, then the minors described still vanish on the multi-view varieties, by continuity in $(A, B, C)$.

Now, certainly a point/line correspondence that is consistent with $(A, B, C)$ lies in the appropriate multi-view variety; consistency means that the correspondence is a point in the set-theoretic image of the appropriate rational map in Definition 7. Since the multi-view varieties are the Zariski closures of those set-theoretic images, care is needed to make a converse. We require:

Definition 9. Let $A, B, C$ be three projective cameras with distinct centers. The epipole denoted $e_{1\rightarrow 2}$ is the point $\alpha(\ker(B)) \in \mathbb{P}^2_A$. That is, $e_{1\rightarrow 2}$ is the image under $A$ of the center of $B$. Epipoles $e_{1\rightarrow 3}, e_{2\rightarrow 1}, e_{2\rightarrow 3}, e_{3\rightarrow 1}, e_{3\rightarrow 2}$ are defined similarly.

Lemma 10. Let $A, B, C$ be three projective cameras with distinct centers. Let $\pi \in (\mathbb{P}^2 \sqcup (\mathbb{P}^2)^\vee)^{\times 3}$. Assume this point/line correspondence avoids epipoles. For example, if $\pi = (x, x', \ell'') \in \mathbb{P}^2_A \times \mathbb{P}^2_B \times (\mathbb{P}^2_C)^\vee$, avoidance of epipoles means that $x \not\in e_{1\rightarrow 2}, e_{1\rightarrow 3}$; $x' \not\in e_{2\rightarrow 1}, e_{2\rightarrow 3}$; and $\ell'' \not\in e_{3\rightarrow 1}, e_{3\rightarrow 2}$. Then $\pi$ is consistent with $(A, B, C)$ if $\pi$ is in the suitable multi-view variety.

Proof. Assuming that $\pi$ is in the multi-view variety, then $\pi$ satisfies the equations from Theorem 8. This is equivalent to containment conditions on the back-projections of $\pi$, without any hypothesis on the centers of $A, B, C$.

We spell this out for the 'PPL' case, where $\pi = (x, x', \ell'') \in \mathbb{P}^2_A \times \mathbb{P}^2_B \times (\mathbb{P}^2_C)^\vee$. Here the back-projections are the lines $\alpha^{-1}(x), \beta^{-1}(x') \subseteq \mathbb{P}^3$ and the plane $\gamma^{-1}(\ell'') \subseteq \mathbb{P}^3$. The minors from Theorem 8 vanish if and only if there exists $(X, L) \in \mathcal{F}_{L, 0, 1}$ such that $X \in \alpha^{-1}(x), X \in \beta^{-1}(x')$ and $L \subseteq \gamma^{-1}(\ell'')$. To see this, note that the minors vanish only if:

$$
\begin{pmatrix}
A & x & 0 \\
B & 0 & x'
\end{pmatrix}
\begin{pmatrix}
X \\
\ell'' T C
\end{pmatrix} = 0 \quad \text{for some nonzero} \quad \begin{pmatrix}
X \\
-\lambda \\
-\lambda'
\end{pmatrix} \in \mathbb{C}^6,
$$

where $X \in \mathbb{C}^4, \lambda \in \mathbb{C}$ and $\lambda' \in \mathbb{C}$. Since $x, x' \in \mathbb{C}^3$ are nonzero, it follows that $X$ is nonzero, and so defines a point $X \in \mathbb{P}^3$. From $AX = \lambda x$, the line $\alpha^{-1}(x) \subseteq \mathbb{P}^3$ contains $X \in \mathbb{P}^3$. Similarly $AX = \lambda' x$ implies $X \in \beta^{-1}(x')$. Thirdly, $\ell'' T C X = 0$ says that $X$ lies on the plane $\gamma^{-1}(\ell'') \subseteq \mathbb{P}^3$. Now taking any line $L \subseteq \mathbb{P}^3$ with $X \in L \subseteq \gamma^{-1}(\ell'')$ produces a satisfactory point $(X, L) \in \mathcal{F}_{L, 0, 1}$, and reversing the argument gives the converse.

Returning to the lemma, since $\pi$ avoids epipoles, the back-projections of $\pi$ avoid the centers of $A, B, C$. In the 'PPL' case, this implies that $(X, L)$ avoids the centers of $A, B, C$. Thus $(X, L)$ witnesses consistency, because $\alpha(X) = x, \beta(X) = x', \gamma(L) = \ell''$. The other cases are finished similarly. □
The results of this subsection have provided tight equational formulations for a camera configuration and a point/line image correspondence to be consistent. This leads to a parametrized system of polynomial equations for each minimal problem in Theorem 6. For instance, for the minimal problem ‘1PPP + 4PPL’, the unknowns are the entries of \( A, B, C \), up to the action of the group \( G \). Due to Theorem 8, there are \( \binom{6}{y} + 3 + 4 \cdot \binom{4}{y} = 67 \) quartic equations. Their coefficients are parametrized cubically and quadratically by the image data in \( (\mathbb{P}^2)^{11} \times (\mathbb{P}^2)^3 \). Since this parameter space is irreducible, to find the generic number of solutions to the system, we may specialize to one random instance, such as in Example 5. Nonetheless, solving a single instance of this system – ‘as is’ – is computationally intractable, let alone solving systems for the other minimal problems present in Theorem 6.

The way out is to nontrivially replace the above systems with other systems, which enlarge the solution sets but amount to accessible computations. This key maneuver is based on trifocal tensors from multi-view geometry.

Before doing so, we justify calling the problems in Theorem 6 consistent. This leads to a parametrized system of polynomial equations for a camera configuration and a point/line image correspondence data, there is a finite number\(^2\) of solutions, i.e. calibrated camera configurations \( (A, B, C) \). Moreover, solutions have linearly independent centers.

**Proposition 11.** For each problem in Theorem 6, given generic correspondence data, there is a finite number\(^2\) of solutions, i.e. calibrated camera configurations \( (A, B, C) \). Moreover, solutions have linearly independent centers.

*Proof.* For calibrated \( A, B, C \), we may act by \( G \) so \( A = \begin{bmatrix} I_3 & 0 \end{bmatrix}, B = \begin{bmatrix} R_2 & t_2 \end{bmatrix} \) and \( C = \begin{bmatrix} R_3 & t_3 \end{bmatrix} \) where \( R_2, R_3 \in SO(3, \mathbb{C}) \) and \( t_2, t_3 \in \mathbb{C}^4 \). Furthermore, \( t_2 \) and \( t_3 \) may be jointly scaled. Thus, if \( A, B, C \) have non-identical centers, we get a point in \( SO(3, \mathbb{C})^2 \times \mathbb{P}^5 \). This point is unique and configurations with non-identical centers are in bijection with \( SO(3, \mathbb{C})^2 \times \mathbb{P}^5 \).

Now consider one of the minimal problems from Theorem 6, ‘\( w_1 PPP + w_2 PPL + w_3 PLL + w_5 PPL \)’. Notice that the problems in Theorem 6, are those for which the weights \( (w_1, w_2, w_3, w_4, w_5) \in \mathbb{Z}_{\geq 0} \) satisfy \( 3w_1 + 2w_2 + 2w_3 + 2w_4 + w_5 = 11 \) and \( w_2 \geq w_3 \). Image correspondence data is a point in the product \( \mathcal{D}_w := (\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)^{w_1} \times \mathbb{P}^{w_2} = (\mathbb{P}^2 \times (\mathbb{P}^2)^{\vee} \times (\mathbb{P}^2)^{\vee})^{w_5} \).

Consider the incidence diagram:

\[
\begin{align*}
SO(3, \mathbb{C})^2 \times \mathbb{P}^5 & \xleftarrow{\Gamma} \mathcal{D}_w \\
\end{align*}
\]

where \( \Gamma := \{(A, B, C), d \} \in (SO(3, \mathbb{C})^2 \times \mathbb{P}^5) \times \mathcal{D}_w \mid (A, B, C) \) and \( d \) are consistent\} and where the arrows are projections. The left map is surjective and a general fiber is a product of multi-view varieties described by Theorem 8. In particular, the fiber has dimension \( 3w_1 + 4w_2 + 4w_3 + 4w_4 + 5w_5 \). Therefore, by [8, Corollary 13.5], \( \Gamma \) has dimension \( 11 + 3w_1 + 4w_2 + 4w_3 + 4w_4 + 5w_5 \), as \( \dim(SO(3, \mathbb{C})^2 \times \mathbb{P}^5) = 11 \). Now, the second arrow is a regular map between varieties of the same dimension, because \( 11 + 3w_1 + 4w_2 + 4w_3 + 4w_4 + 5w_5 = 6(w_1 + w_2 + w_3 + w_4 + w_5) \). So, if it is dominant, then again by [8, Corollary 13.5], a general fiber has dimension 0; otherwise, a general fiber is empty. However, note that points in a general fiber of the second map correspond to solutions of a generic instance of the problem indexed by \( w \) from Theorem 6. This shows that those problems generically have finitely many solutions.

---

\(^2\) This number is shown to be positive in the proof of Theorem 6.
We can see that generically there are no solutions with non-identical but collinear centers, as follows. Let $C \subset \text{SO}(3, \mathbb{C})^{\times 2} \times \mathbb{P}^5$ be the closed variety of configurations $(A, B, C)$ with non-identical but collinear centers. Consider:

$$C \leftarrow \Gamma' \rightarrow D_w$$

where the definition of $\Gamma'$ is the definition of $\Gamma$ with $\text{SO}(3, \mathbb{C})^{\times 2} \times \mathbb{P}^5$ replaced by $C$, and where the arrows are projections. Here $\dim(C) = 10$. The left arrow is surjective, and a general fiber is a product of multi-view varieties, with the same dimension as in the above case. This dimension statement is seen by calculating the multi-view varieties as in the proof of Theorem 8, when $(A, B, C)$ have distinct, collinear centers. It follows that $\dim(\Gamma') = 10 + 3w_1 + 4w_2 + 4w_3 + 4w_4 + 5w_5 < 11 + 3w_1 + 4w_2 + 4w_3 + 4w_4 + 5w_5 = 6(w_1 + w_2 + w_3 + w_4 + w_5) = \dim(D_w)$ so that the right arrow is not dominant.

Finally, to see that generically there is no solution $(A, B, C)$ where the centers of $A, B, C$ are identical in $\mathbb{P}^3$, we may mimic the above argument with another dimension count. Calibrated configurations with identical centers are in bijection with $\text{SO}(3^\times 3)$, where the definition of $\Gamma$ with $\text{SO}(3^\times 3)$ replaced by $\text{SO}(3, \mathbb{C})^{\times 2}$, because each $G$-orbit has a unique representative of the form $A = [I_{3 \times 3} \ 0], B = [R_2 \ 0], C = [R_3 \ 0]$ where $R_2, R_3 \in \text{SO}(3, \mathbb{C})$. So, analogously to before, we consider the diagram:

$$\text{SO}(3, \mathbb{C})^{\times 2} \leftarrow \Gamma'' \rightarrow D_w$$

where the definition of $\Gamma''$ is the definition of $\Gamma$ with $\text{SO}(3, \mathbb{C})^{\times 2} \times \mathbb{P}^5$ replaced by $\text{SO}(3, \mathbb{C})^{\times 2}$, and where the arrows are projections. Again, the left arrow is surjective, and a general fiber is a product of multi-view varieties. Here, when $A, B, C$ have identical centers, a calculation as in the proof of Theorem 8 verifies that the dimensions of the multi-view varieties drop, as follows: $\dim(X_{A,B,C}^{PLL}) = 3, \dim(X_{A,B,C}^{LLL}) = 2, \dim(X_{A,B,C}^{PLP}) = 3, \dim(X_{A,B,C}^{PPP}) = 3, \dim(X_{A,B,C}^{PPP}) = 2$. So the dimension of a general fiber of the left arrow is $2w_1 + 3w_2 + 3w_3 + 2w_4 + 5w_5$. So $\dim(\Gamma'') = 6 + 2w_1 + 3w_2 + 3w_3 + 2w_4 + 5w_5 < 11 + 3w_1 + 4w_2 + 4w_3 + 4w_4 + 5w_5 = 6(w_1 + w_2 + w_3 + w_4 + w_5) = \dim(D_w)$, whence the right arrow is not dominant. This completes the proof. □

3.2. Trifocal tensors. In this subsection, we re-derive the trifocal tensor $T_{A,B,C} \in \mathbb{C}^{3 \times 3 \times 3}$ associated to cameras $(A, B, C)$, following the projective geometry approach of Hartley [11]. This explains the notation in the ‘PLL’ bullet of Theorem 8, and justifies the assertion made there. We also review how $T_{A,B,C}$ encodes other point/line images correspondences.

As in Subsection 3.1, let $A, B, C \in \mathbb{C}^{3 \times 4}$ be three projective cameras, not necessarily calibrated, and denote by $\alpha : \mathbb{P}^3 \rightarrow \mathbb{P}^2_A, \beta : \mathbb{P}^3 \rightarrow \mathbb{P}^2_B, \gamma : \mathbb{P}^3 \rightarrow \mathbb{P}^2_C$ the corresponding linear projections. Let the point and lines $x \in \mathbb{P}^2_A, \ell' \in (\mathbb{P}^2_B)^\vee, \ell'' \in (\mathbb{P}^2_C)^\vee$ be given as column vectors. The pre-image $\alpha^{-1}(x)$ is a line in $\mathbb{P}^3$, while $\beta^{-1}(\ell')$ and $\gamma^{-1}(\ell'')$ are planes in $\mathbb{P}^3$. We can characterize when these three have non-empty intersection as follows.

First, note that the plane $\beta^{-1}(\ell')$ is given by the column vector $B^T \ell'$, since $X \in \mathbb{P}^3$ satisfies $X \in \beta^{-1}(\ell')$ if and only if $0 = \ell'^T B X = (B^T \ell')^T X$. Similarly, the plane $\gamma^{-1}(\ell'')$ is given by $C^T \ell''$. For the line $\alpha^{-1}(x)$, note:

$$\alpha^{-1}(x) = \bigcap_{\ell' \in (\mathbb{P}^2_B)^\vee \atop \ell' x = 0} \alpha^{-1}(\ell') \subseteq \alpha^{-1}(x, [1 \ 1 \ 0]^T) \cap \alpha^{-1}(x, [1 \ 0 \ 1]^T).$$
Here \((\cdot)\) denotes span, and auxiliary points \([1 \ 1 \ 0]^T, [1 \ 0 \ 1]^T \in \mathbb{P}_A^2\) are simply convenient choices for this calculation. Unless those two points and \(x\) are collinear, the inclusion above is an equality, and the intersectands in the RHS are the planes given by the column vectors \(A^T[x] \in [1 \ 1 \ 0]^T\) and \(A^T[x] \in [1 \ 0 \ 1]^T\). The notation means \([x]_\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}\), and \([x]_\times y\) gives \(\langle x, y \rangle\) for \(x \neq y \in \mathbb{P}_A^2\). So, \(\alpha^{-1}(x) \cap \beta^{-1}(\ell') \cap \gamma^{-1}(\ell'') \neq \emptyset\) only if:

\[
\det \begin{pmatrix}
A^T[x]_\times & A^T[y]_\times & B^T[\ell'] & B^T[\ell'']
\end{pmatrix}_{4 \times 4} = 0. \tag{1}
\]

This determinant is divisible by \((x_1 - x_2 - x_3)\), since that vanishes if and only if \(x, [1 \ 1 \ 0]^T, [1 \ 0 \ 1]^T\) are collinear only if the first two columns above are linearly dependent. Hence, factoring out, we obtain a constraint that is trilinear in \(x, \ell', \ell''\), i.e., we get for some tensor \(T \in \mathbb{C}^{3 \times 3 \times 3}\):

\[
\sum_{1 \leq i, j, k \leq 3} T_{ijk} x_i \ell'_j \ell''_k = 0.
\]

The tensor entry \(T_{ijk}\) is computed by substituting into (1) the basis vectors \(x = e_i, \ell' = e_j, \ell'' = e_k\). Breaking into cases according to \(i\), this yields:

- \(T_{1ij} = \frac{1}{(i-1)} \det (a_3 \ b_j \ c_k) = \det (a_2 \ b_j \ c_k)\)
- \(T_{2ij} = \frac{1}{(i-1)} \det (-a_3 \ a_1-a_3 \ b_j \ c_k) = -\det (a_1 \ a_3 \ b_j \ c_k)\)
- \(T_{3ij} = \frac{1}{(i-1)} \det (-a_1+a_2 \ a_2 \ b_j \ c_k) = \det (a_1 \ a_2 \ b_j \ c_k)\)

where \(a_i\) denotes the transpose of the first row in \(A\), and so on.

At this point, we have derived formula (17.12) from [13, pg 415]:

**Definition 12.** Let \(A, B, C\) be cameras. Their trifocal tensor \(T_{A,B,C} \in \mathbb{C}^{3 \times 3 \times 3}\) is computed as follows. Form the \(4 \times 4\) matrix \((A^T|B^T|C^T)\). Then for \(1 \leq i, j, k \leq 3\), the entry \((T_{A,B,C})_{ijk}\) is \((-1)^{i+j} t_{ijk} \times \) the determinant of the \(4 \times 4\) submatrix gotten by omitting the \(i^{th}\) column from \(A^T\), while keeping the \(j^{th}\) and \(k^{th}\) columns from \(B^T\) and \(C^T\), respectively. If \(A, B, C\) are calibrated, then \(T_{A,B,C}\) is said to be a calibrated trifocal tensor.

**Remark.** Since \(A, B, C \in \mathbb{C}^{3 \times 4}\) are each defined only up to multiplication by a nonzero scalar, the same is true of \(T_{A,B,C} \in \mathbb{C}^{3 \times 3 \times 3}\).

**Remark.** By construction, \(T_{A,B,C}(x, \ell', \ell'') := \sum_{1 \leq i, j, k \leq 3} T_{ijk} x_i \ell'_j \ell''_k = 0\) is equivalent to \(\alpha^{-1}(x) \cap \beta^{-1}(\ell') \cap \gamma^{-1}(\ell'') \neq \emptyset\). In particular, \(T_{A,B,C} = 0\) if and only if the centers of \(A, B, C\) are all the same. Moreover, the ‘PLL’ cases in Theorem 8 and Lemma 10 postponed above are now immediate.

So far, we have constructed trifocal tensors so that they encode point-line-line image correspondences. Conveniently, the same tensors encode other point/line correspondences [11], up to extraneous components.
Proposition 13. Let \( A, B, C \) be projective cameras. Let \( x \in \mathbb{P}^2_A, x' \in \mathbb{P}^2_B, x'' \in \mathbb{P}^2_C \) and \( \ell \in (\mathbb{P}^2_A)^\vee, \ell' \in (\mathbb{P}^2_B)^\vee, \ell'' \in (\mathbb{P}^2_C)^\vee \). Putting \( T = T_{A,B,C} \), then \( (A, B, C) \) is consistent with:

- \((x, \ell', \ell'')\) only if \( T(x, \ell', \ell'') = 0 \) \([PLL]\)
- \((\ell, \ell', \ell'')\) only if \([\ell] \times T(-, \ell', \ell'') = 0 \) \([LLL]\)
- \((x, \ell', x'')\) only if \([x''] \times T(x, \ell', -) = 0 \) \([PLP]\)
- \((x, x', \ell'')\) only if \([x'] \times T(-, -; \ell'') = 0 \) \([PPL]\)
- \((x, x', x'')\) only if \([x'' \times T(x, -; -; x'') = 0 \) \([PPP]\)

In the middle bullets, each contraction of \( T \) with two vectors gives a column vector in \( \mathbb{C}^3 \). In the last bullet, \( T(x, -; -) = \sum_{i=1}^2 x_i (T_{ijk})_{1 \leq i,j,k \leq 3} \in \mathbb{C}^{3 \times 3} \).

Proof. This proposition matches Table 15.1 on [13, pg 372]. To be self-contained, we recall the proof. The first bullet is by construction of \( T \).

For the second bullet, assume that \((\ell, \ell', \ell'')\) is consistent with \((A, B, C)\), i.e. there exists \( L \in \text{Gr}(\mathbb{P}^2, \mathbb{P}^3) \) such that \( \alpha(L) = \ell, \beta(L) = \ell', \gamma(L) = \ell'' \). Now let \( y \in \ell \) be a point. So \( \alpha^{-1}(x) \) is a line in the plane \( \alpha^{-1}(\ell) \) and that plane contains the line \( L \). This implies \( \alpha^{-1}(x) \cap L \neq \emptyset \Rightarrow \alpha^{-1}(x) \cap \beta^{-1}(\ell') \cap \gamma^{-1}(\ell'') \neq \emptyset \Leftrightarrow T(y, \ell', \ell'') = 0 \). It follows that for \( y \in \mathbb{P}^2_A \), we have \( y^T \ell'' = 0 \Rightarrow y^T \ell'' = 0 \). This means that \( \ell \) and \( T(-, \ell', \ell'') \) are linearly independent, i.e. \([\ell] \times T(-, \ell', \ell'') = 0 \).

The third, fourth and fifth bullets are similar. They come from reasoning that the consistency implies, respectively:

- \( x'' \in k'' \Rightarrow T(x, \ell', k'') = 0 \)
- \( x' \in k' \Rightarrow T(x, k', \ell'') = 0 \)
- \( (x' \in k' \text{ and } x'' \in k'') \Rightarrow T(x, k', k'') = 0 \),

where \( k' \in (\mathbb{P}^2_B)^\vee \) and \( k'' \in (\mathbb{P}^2_C)^\vee \).

Remark. The constraints in Proposition 13 are linear in \( T \). We will exploit this in Section 6. Also, in fact, image correspondences of types ‘LPL’, ‘LLP’ and ‘LPP’ do not give linear constraints on \( T_{A,B,C} \). This is the reason that these types are not considered in Theorem 6. To get linear constraints nonetheless, one could permute \( A, B, C \) before forming the trifocal tensor.

In this subsection, we have presented a streamlined account of trifocal tensors, and the point/line image correspondences that they encode. Now, we sketch the relationship between the tight conditions in Theorem 8 and the necessary conditions in Proposition 13 for consistency.

Lemma 14. Fix projective cameras \( A, B, C \) with linearly independent centers. Then the trilinearities in Proposition 13 cut out subschemes of three-factor products of \( \mathbb{P}^2 \) and \( (\mathbb{P}^2)^\vee \). In all cases of Proposition 13, this subscheme is reduced and contains the corresponding multi-view variety as a top-dimensional component.

Proof. Without loss of generality, \( A, B, C \) are in the special position from the proof of Theorem 8. Then using Macaulay2, we form the ideal generated by the trilinearities of Proposition 13 and saturate with respect to the irrelevant ideal. This leaves a radical ideal; we compute its primary decomposition.
For example, in the case of ‘PPP’, the trilinearities from Proposition 13 generate a radical ideal in $\mathbb{C}[x_i, x'_j, x''_k]$ that is the intersection of:

- the 3 irrelevant ideals for each factor of $\mathbb{P}^2$
- 2 linear ideals of codimension 4
- the multi-view ideal $I(X_{A,B,C}^{PPP})$.

This discrepancy between the trifocal and multi-view conditions for ‘PPP’ correspondences was studied in [29]. To demonstrate our main result, in Section 6 we shall relax the tight multi-view equations in Theorem 8 to the merely necessary trilinearities in Proposition 13. The ‘top-dimensional’ clause in Lemma 14, as well as Theorem 16 in Section 4 below, indicate that this gives ‘good’ approximations to the minimal problems in Theorem 6.

4. Configurations

In this section, it is proven that trifocal tensors, in both the uncalibrated and calibrated case, are in bijection with camera triples up to the appropriate group action, i.e. with camera configurations. Statements tantamount to Proposition 15 are made throughout [13, Chapter 15] and are well-known in the vision community, however, we could not find any proof in the literature. As far as our main result Theorem 6 is concerned, Theorem 16 below enables us to compute consistent calibrated trifocal tensors in exchange for consistent calibrated camera configurations. To our knowledge, this theorem is new; subtly, the analog for two calibrated is false [13, Result 9.19].

**Proposition 15.** Let $A, B, C$ be three projective cameras, with linearly independent centers in $\mathbb{P}^3$. Let $\tilde{A}, \tilde{B}, \tilde{C}$ be another three projective cameras. Then $T_{A,B,C} = T_{\tilde{A}, \tilde{B}, \tilde{C}} \in \mathbb{P}(\mathbb{C}^{3 \times 3 \times 3})$ if and only if there exists $h \in \text{SL}(4, \mathbb{C})$ such that $Ah = \tilde{A}, Bh = \tilde{B}, Ch = \tilde{C} \in \mathbb{P}(\mathbb{C}^{3 \times 4})$.

**Proof.** As in the proof of Theorem 8, for $g, g', g'' \in \text{SL}(3, \mathbb{C})$, $h \in \text{SL}(4, \mathbb{C})$:

$$T_{gA, g'B, g''C} = (g, \wedge^2 g', \wedge^2 g'') \cdot T_{A,B,C} \quad \text{and} \quad T_{Ah, Bh, Ch} = T_{A,B,C}. \quad (2)$$

The second equality gives the ‘if’ direction. Conversely, for ‘only if’, for any $g, g', g'' \in \text{SL}(3, \mathbb{C})$, $h_1, h_2 \in \text{SL}(4, \mathbb{C})$, we are free to replace $(A, B, C)$ by $(gAh_1, g'Bh_1, g''Ch_1)$ and to replace $(\tilde{A}, \tilde{B}, \tilde{C})$ by $(gAh_2, g'Bh_2, g''\tilde{Ch}_2)$, and then to exhibit an $h$ as in the proposition. Hence we may assume that:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

where each ‘*’ denotes an indeterminate. Now consider the nine equations:

$$(T_{A,B,C})_{i3k} = (T_{\tilde{A}, \tilde{B}, \tilde{C}})_{i3k}$$
where $1 \leq i, k \leq 3$. Under the above assumptions, these are linear and in the nine unknowns $\tilde{a}_m$ for $1 \leq l, m \leq 3$. Here we have fixed the nonzero scale on $\tilde{C}$ so that these are indeed equalities, on the nose. It follows that:

$$\tilde{C} = \begin{bmatrix} 1 & 0 & 0 & \ast \\ 0 & 0 & 1 & \ast \\ 0 & 0 & 0 & \ast \end{bmatrix}.$$

At this point, we have reduced to solving 18 equations in 11 unknowns:

$$(T_{A,B,C})_{i,j,k} = (T_{\tilde{A},\tilde{B},\tilde{C}})_{i,j,k}$$

where $1 \leq i, k \leq 3$ and $1 \leq j \leq 2$. These equations are quadratic monomials and binomials. The system is simple to solve by hand or with Macaulay2:

$$\tilde{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{bmatrix}$$

for $\lambda \in \mathbb{C}^*$. Now taking $h = \lambda^{-3/4} \text{diag}(\lambda, \lambda, \lambda, 1) \in \text{SL}(4, \mathbb{C})$ gives $Ah = \tilde{A}, Bh = \tilde{B}, Ch = \tilde{C} \in \mathbb{P}(\mathbb{C}^{3 \times 4})$, as desired. This completes the proof. \hfill \Box

With a bit of work, we can promote Proposition 15 to the calibrated case.

**Theorem 16.** Let $A, B, C$ be three calibrated cameras, with linearly independent centers in $\mathbb{P}^3$. Let $\tilde{A}, \tilde{B}, \tilde{C}$ be another three calibrated cameras. Then $T_{A,B,C} = T_{\tilde{A},\tilde{B},\tilde{C}} \in \mathbb{P}(\mathbb{C}^{3 \times 3 \times 3})$ if and only if there exists $h \in G$ (where $G$ is defined on page 2) such that $Ah = \tilde{A}, Bh = \tilde{B}, Ch = \tilde{C} \in \mathbb{P}(\mathbb{C}^{3 \times 4})$.

**Proof.** The ‘if’ direction is from Proposition 15. For ‘only if’, here for any $g, g', g'' \in \text{SO}(3, \mathbb{C}), h_1, h_2 \in G$, we are free to replace $(A, B, C)$ by $(gAh_1, g'Bh_1, g''Ch_1)$ and to replace $(\tilde{A}, \tilde{B}, \tilde{C})$ by $(g\tilde{A}h_2, g'\tilde{B}h_2, g''\tilde{C}h_2)$, and then to exhibit an $h \in G$ as above. In this way, we may assume that:

$$A = [I_{3 \times 3} \ 0], \quad B = [I_{3 \times 3} \ s_1], \quad C = [I_{3 \times 3} \ s_2]$$

$$\tilde{A} = [I_{3 \times 3} \ 0], \quad \tilde{B} = [R_1 \ t_1], \quad \tilde{C} = [R_2 \ t_2]$$

where $R_1, R_2 \in \text{SO}(3, \mathbb{C})$ and $s_1, s_2, t_1, t_2 \in \mathbb{C}^3$. Now from Proposition 15, there exists $h' \in \text{SL}(4, \mathbb{C})$ such that $Ah' = \tilde{A}, Bh' = \tilde{B}, Ch' = \tilde{C} \in \mathbb{P}(\mathbb{C}^{3 \times 3})$.

From the first equality, it follows that $h' = \begin{bmatrix} I_{3 \times 3} \ 0 \\ u^T \ \lambda \end{bmatrix} \in \mathbb{P}(\mathbb{C}^{4 \times 4})$ for some $u \in \mathbb{C}^3, \lambda \in \mathbb{C}^*$. It suffices to show that $u = 0$, so $h' \in G$. By way of contradiction, let us assume that $u \neq 0$. Substituting into $Bh' = \tilde{B}$ gives:

$$[I_{3 \times 3} \ s_1] \begin{bmatrix} I_{3 \times 3} \\ u^T \\ \lambda \end{bmatrix} = [I_{3 \times 3} + s_1 u^T \ \lambda s_1] = [R_1 \ t_1] \in \mathbb{P}(\mathbb{C}^{3 \times 4}).$$

In particular, there is $\mu_1 \in \mathbb{C}^*$ so that $\mu_1 (I_{3 \times 3} + s_1 u^T) = R_1$. In particular, $R_1 - \mu_1 I_{3 \times 3}$ is rank at most 1. Equivalently, $\mu_1$ is an eigenvalue of the rotation $R_1 \in \text{SO}(3, \mathbb{C})$ of geometric multiplicity at least 2. The only possibilities are $\mu_1 = 1$ or $\mu_1 = -1$. If $\mu_1 = 1$, then $R_1 = I$ and $s_1 u^T = 0$. 
From \( u \neq 0 \), we get that \( s_1 = 0 \); but then \( A = B \), contradicting linear independence of the centers of \( A, B, C \). So in fact \( \mu_1 = -1 \). Now \( R_1 \) is a \( 180^\circ \) rotation. From \( R_1 + I_{3 \times 3} = s_1 u^T \in \mathbb{C}^{3 \times 3} \), it follows that the axis of rotation is the line through \( u \), and \( s_1 = \frac{2u}{u^T u} \). The exact same analysis holds starting from \( C'h' = \tilde{C} \). So in particular, \( s_2 = \frac{2u}{u^T u} \). But now \( B = C \), contradicting linear independence of the centers of \( A, B, C \). We conclude that \( u = 0 \). \( \square \)

5. Varieties

So far in Subsection 3.2 and Section 4, we have worked with individual trifocal tensors, uncalibrated or calibrated. This is possible once a camera configuration \((A, B, C)\) is given. To determine an unknown camera configuration from image data, we need to work with the set of all trifocal tensors.

**Definition 17.** The trifocal variety, denoted \( \mathcal{T} \subset \mathbb{P}(\mathbb{C}^{3 \times 3 \times 3}) \), is defined to be the Zariski closure of the image of the following rational map:

\[
\mathbb{P}(\mathbb{C}^{3 \times 4}) \times \mathbb{P}(\mathbb{C}^{3 \times 4}) \times \mathbb{P}(\mathbb{C}^{3 \times 4}) \dashrightarrow \mathbb{P}(\mathbb{C}^{3 \times 3 \times 3}), \quad (A, B, C) \mapsto T_{A,B,C}
\]

where \((T_{A,B,C})_{ijk} := (-1)^{i+j} \det \begin{bmatrix} a_i \\ b_j \\ c_k \end{bmatrix}_{4 \times 4}\) for \( 1 \leq i, j, k \leq 3 \).

Here \( \sim a_i \) is gotten from \( A \) by omitting the \( i \)th row, and \( b_j, c_k \) are the \( j \)th, \( k \)th rows of \( B, C \) respectively. So, \( \mathcal{T} \) is the closure of the set of all trifocal tensors.

**Definition 18.** The calibrated trifocal variety, denoted \( \mathcal{T}_{\text{cal}} \subset \mathbb{P}(\mathbb{C}^{3 \times 3 \times 3}) \), is defined to be the Zariski closure of the image of the following rational map:

\[
(\text{SO}(3, \mathbb{C}) \times \mathbb{C}^3) \times (\text{SO}(3, \mathbb{C}) \times \mathbb{C}^3) \times (\text{SO}(3, \mathbb{C}) \times \mathbb{C}^3) \dashrightarrow \mathbb{P}(\mathbb{C}^{3 \times 3 \times 3}),
\]

\[
((R_1, t_1), (R_2, t_2), (R_3, t_3)) \mapsto T_{[R_1|t_1], [R_2|t_2], [R_3|t_3]}
\]

where the formula for \( T \) is as in Definitions 12 and 17. So, \( \mathcal{T}_{\text{cal}} \) is the closure of the set of all calibrated trifocal tensors.

In the remainder of this paper, the calibrated trifocal variety \( \mathcal{T}_{\text{cal}} \) is the main actor. It has recently been studied independently by Martynushev [22] and Matthews [23]. They both obtain implicit quartic equations for \( \mathcal{T}_{\text{cal}} \). However, a full set of ideal generators for \( I(\mathcal{T}_{\text{cal}}) \subset \mathbb{C}[I_{ijk}] \) is currently not known. We summarize the state of knowledge on implicit equations for \( \mathcal{T}_{\text{cal}} \):

**Proposition 19.** The prime ideal of the calibrated trifocal variety \( I(\mathcal{T}_{\text{cal}}) \subset \mathbb{C}[I_{ijk}] \) contains the ideal of the trifocal variety \( I(\mathcal{T}) \), and \( I(\mathcal{T}) \) is minimally generated by 10 cubics, 81 quintics and 1980 sextics. Additionally, \( I(\mathcal{T}_{\text{cal}}) \) contains 15 linearly independent quartics that do not lie in \( I(\mathcal{T}) \).

The ideal containment follows from \( \mathcal{T}_{\text{cal}} \subset \mathcal{T} \), and the statement about minimal generators of \( I(\mathcal{T}) \) was proven by Aholt and Oeding [2]. For the additional quartics, see [22, Theorems 8, 11] and [23, Corollary 51].

In the rest of this paper, using numerical algebraic geometry, we always interact with the calibrated trifocal variety \( \mathcal{T}_{\text{cal}} \) directly via (a restriction of) its defining parametrization. Therefore, we do not need the ideal of implicit equations \( I(\mathcal{T}_{\text{cal}}) \), nor do we use the known equations from Proposition 19.

At this point, we discuss properties of the rational map in Definition 18. First, since the source \((\text{SO}(3, \mathbb{C}) \times \mathbb{C}^3)^{\times 3}\) is irreducible, the closure
of the image $\mathcal{T}_{\text{cal}}$ is irreducible. Second, the base locus of the map consists of triples of calibrated cameras $([R_1 | t_1], [R_2 | t_2], [R_3 | t_3])$ all with the same center in $\mathbb{P}^3$, by the remarks following Definition 12. Third, the two equations in (2), the second line of the proof of Proposition 15, mean that the rational map in Definition 18 satisfies group symmetries. Namely, the parametrization of $\mathcal{T}_{\text{cal}}$ is equivariant with respect to $SO(3, \mathbb{C})^3$, and each of its fibers carry a $G$ action. In vision, these two group actions are interpreted as changing image coordinates and changing world coordinates. Here, by the equivariance, it follows that $\mathcal{T}_{\text{cal}}$ is an $SO(3, \mathbb{C})^3$-variety. Also, we can use the $G$ action on fibers to pick out one point per fiber, and thus restrict the map in Definition 18 so that the restriction is generically injective and dominant onto $\mathcal{T}_{\text{cal}}$. Explicitly, we restrict to the domain where $[R_1 | t_1] = [I_{3 \times 3} \ 0]$, $t_2 = [\ast \ast \ 1]^T$. This restriction $(SO(3, \mathbb{C}) \times \mathbb{C}^2) \times (SO(3, \mathbb{C}) \times \mathbb{C}^3) \rightarrow \mathcal{T}_{\text{cal}}$ is generically injective by Theorem 16. Generic injectivity makes the restricted map particularly amenable to numerical algebraic geometry, where computations regarding a parametrized variety are pulled back to the source of the parametrization. We now obtain the major theorem of this section using that technique:

**Theorem 20.** The calibrated trifocal variety $\mathcal{T}_{\text{cal}} \subset \mathbb{P}(\mathbb{C}^{3 \times 3 \times 3})$ is irreducible, dimension 11 and degree 4912. It equals the $SO(3, \mathbb{C})^3$-orbit closure generated by the following projective plane, parametrized by $[\lambda_1 \ \lambda_2 \ \lambda_3]^T \in \mathbb{P}^2$:

$$ T_{1**} = \begin{bmatrix} 0 & \lambda_1 & \lambda_2 \\ 0 & 0 & 0 \\ \lambda_1 & 0 & 0 \end{bmatrix}, \quad T_{2**} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_1 & \lambda_2 \\ 0 & \lambda_3 & 0 \end{bmatrix}, \quad T_{3**} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \lambda_1 & \lambda_2 + \lambda_3 \end{bmatrix}. $$

**Computational Proof.** Dimension 11 follows from the generically injective parametrization given above. The $SO(3, \mathbb{C})^3$ statement follows from (2). In more detail, given a calibrated camera configuration $(A, B, C)$ with linearly independent centers, we may act by $G$ so that the centers of $A, B, C$ are:

$$ [0 \ 0 \ 0 \ 1]^T, \ [0 \ 0 \ 1 \ 1]^T, \ [0 \ \ast \ \ast \ 1]^T, $$

respectively. Then we may act by $SO(3, \mathbb{C})^3$ so that the left submatrices of $A, B, C$ equal $I_{3 \times 3}$. The calibrated trifocal tensor $T_{A,B,C}$ now lands in the stated $\mathbb{P}^2$. Hence, $\mathcal{T}_{\text{cal}}$ is that orbit closure due to transformation laws (2).

To compute the degree of $\mathcal{T}_{\text{cal}}$, we use the open-source homotopy continuation software *Bertini*. We fix a random linear subspace $L \subset \mathbb{P}(\mathbb{C}^{3 \times 3 \times 3})$ of complementary dimension to $\mathcal{T}_{\text{cal}}$, i.e. $\dim(L) = 15$. This is expressed in floating-point as the vanishing of 11 random linear forms $\ell_m(T_{ijk}) = 0$ (3), where $m = 1, \ldots, 11$. Our goal is to compute $\#(\mathcal{T}_{\text{cal}} \cap L)$. As homotopy continuation calculations are sensitive to the formulation used, we carefully explain our own formulation to calculate $\mathcal{T}_{\text{cal}} \cap L$. Our formulation starts with the parametrization of $\mathcal{T}_{\text{cal}}$ above, and with its two copies of $SO(3, \mathbb{C})$.

Recall that unit norm quaternions double-cover $SO(3, \mathbb{R})$. Complexifying:

$$ R_2 = \begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2(bc - ad) & 2(bd + ac) \\ 2(bc + ad) & a^2 + c^2 - b^2 - d^2 & 2(cd - ab) \\ 2(bd - ac) & 2(cd + ab) & a^2 + d^2 - b^2 - c^2 \end{pmatrix} $$
where \(a, b, c, d \in \mathbb{C}\) and \(a^2 + b^2 + c^2 + d^2 = 1\) (4). Similarly for \(R_3\) with \(e, f, g, h \in \mathbb{C}\) subject to \(e^2 + f^2 + g^2 + h^2 = 1\) (5). For our purposes, it is computationally advantageous to replace (4) by a random patch \(\alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d = 1\) (6), where \(\alpha_i \in \mathbb{C}\) are random floating-point numbers fixed once and for all. Similarly, we replace (5) by a random patch \(\beta_1 e + \beta_2 f + \beta_3 g + \beta_4 h = 1\) (7). The patches (6) and (7) leave us with injective parameterizations of two subvarieties of \(\mathbb{C}^{3 \times 3}\), that we denote by \(SO(3, \mathbb{C})^\alpha, SO(3, \mathbb{C})^\beta\). These two varieties have the same closed affine cone as the closed affine cone of \(SO(3, \mathbb{C})\). This affine cone is:
\[SO(3, \mathbb{C}) := \{ R \in \mathbb{C}^{3 \times 3} : \exists \lambda \in \mathbb{C} \text{ s.t. } RR^T = R^T R = \lambda I_{3 \times 3}\}\]
and it is parametrized by \(a, b, c, d\) as above, but with no restriction on \(a, b, c, d\). In the definition of the cone \(SO(3, \mathbb{C})\), note \(\lambda = 0\) is possible; it corresponds to \(a^2 + b^2 + c^2 + d^2 = 0\), or to \(e^2 + f^2 + g^2 + h^2 = 0\). By the first remark after Definition 12, we are free to scale cameras \(B\) and \(C\) so that their left \(3 \times 3\) submatrices satisfy \(R_2 \in SO(3, \mathbb{C})^\alpha\) and \(R_3 \in SO(3, \mathbb{C})^\beta\), and for our formulation here we do so. Finally, for \(\mathbb{C}^5\) in the source of the parametrization of \(T_{\text{cal}}\), write \(t_2 = \begin{bmatrix} t_{2,1} & t_{2,2} & 1 \end{bmatrix}^T\) and \(t_3 = \begin{bmatrix} t_{3,1} & t_{3,2} & t_{3,3} \end{bmatrix}^T\).

At this point, we have replaced the dominant, generically injective map \(SO(3, \mathbb{C})^{\times 2} \times \mathbb{C}^5 \rightarrow T_{\text{cal}}\) by the dominant, generically injective parametrization \(SO(3, \mathbb{C})^\alpha \times SO(3, \mathbb{C})^\beta \times \mathbb{C}^5 \rightarrow T_{\text{cal}}\). Also, we have injective, dominant maps \(V(\alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d - 1) \rightarrow SO(3, \mathbb{C})^\alpha\) and \(V(\beta_1 e + \beta_2 f + \beta_3 g + \beta_4 h - 1) \rightarrow SO(3, \mathbb{C})^\beta\). Composing gives the generically 1-to-1, dominant \(V(\alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d - 1) \times V(\beta_1 e + \beta_2 f + \beta_3 g + \beta_4 h - 1) \times \mathbb{C}^5 \rightarrow T_{\text{cal}}\).

With exactly this parametrization of \(T_{\text{cal}}\), it will be most convenient to perform numerical algebraic geometry calculations. Hence, here to compute \(\deg(T_{\text{cal}}) = \#(T_{\text{cal}} \cap L)\), we consider the square polynomial system:

- in 13 variables: \(a, b, c, d, e, f, g, h, t_{2,1}, t_{2,2}, t_{3,1}, t_{3,2}, t_{3,3} \in \mathbb{C}\);
- with 13 equations: the 11 cubics (3) and 2 linear equations (6), (7).

The solution set equals the preimage of \(T_{\text{cal}} \cap L\). This system is expected to have \(\deg(T_{\text{cal}})\) many solutions. We can solve zero-dimensional square systems of this size (in floating-point) using the \texttt{UseRegeneration:1} setting in \texttt{Bertini}. That employs the \textit{regeneration} solving technique from [17]. For the present system, overall, \texttt{Bertini} tracks 74,667 paths in 1.5 hours on a standard laptop computer to find 4912 solutions. Numerical path-tracking in \texttt{Bertini} is based on a \textit{predictor-corrector} approach. Prediction by default is done by the Runge-Kutta 4th order method; correction is by Newton steps. For more information, see [6, Section 2.2]. Here, this provides strong numerical evidence for the conclusion that \(\deg(T_{\text{cal}}) = 4912\). Up to the numerical accuracy of \texttt{Bertini} and the reliability of our random number generator used to choose \(L\), this computation is correct with probability 1. Practically speaking, 4912 is correct only with very high probability.

As a check for 4912, we apply the \textit{trace test} from [27], [14] and [20]. A random linear form \(l'\) on \(\mathbb{P}(\mathbb{C}^{3 \times 3})\) is fixed. For \(s \in \mathbb{C}\), we set \(L_s := V(l_1 + sl', \ldots, l_{11} + sl')\), so \(L_0 = L\). Varying \(s \in \mathbb{C}\), the intersection \(T_{\text{cal}} \cap L_s\) consists of \(\deg(T_{\text{cal}})\) many complex paths. Let \(T_s \subset T_{\text{cal}} \cap L_s\) be a subset of paths. Then the trace test implies (for generic \(l', \ell_i\) that \(T_s = T_{\text{cal}} \cap L_s\) if and only if the centroid of \(T_s\) computed in a consistent affine chart \(\mathbb{C}^{26}\), i.e.
is an affine linear function of $s$. Here, we set $T_0$ to be the $4912$ intersection points found above. Then we calculate $T_1$ with the UserHomotopy : 1 setting in Bertini, where the variables are $a, \ldots, t_{3,3}$, and the start points are the preimages of $T_0$. After this homotopy in parameter space, $T_1$ is obtained by evaluating the endpoints of the track via TrackType : -4. Similarly, $T_{-1}$ is computed. Then we calculate that the following quantity in $\mathbb{C}^{26}$:

$$(\text{cen}(T_1) - \text{cen}(T_0)) - (\text{cen}(T_0) - \text{cen}(T_{-1}))$$

is indeed numerically 0. This trace test is a further verification of $4912$. □

Remark. In the proof of Theorem 20, when we select one point per fiber per member of $T_{\text{cal}} \cap L$, we obtain a pseudo-witness set $W$ for $T_{\text{cal}}$. This is the fundamental data structure in numerical algebraic geometry for computing with parameterized varieties (see [16]). Precisely, here it is the quadruple:

- the parameter space $\mathcal{P} \subset \mathbb{C}^{13}$, where $\mathbb{C}^{13}$ has coordinates $a, \ldots, t_{3,3}$ and $\mathcal{P} = V(\alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d - 1, \beta_1 e + \beta_2 f + \beta_3 g + \beta_4 h - 1)$
- the dominant map $\Phi : \mathcal{P} \rightarrow T_{\text{cal}}$ in the proof of Theorem 20, e.g.
  $\Phi_{1,1,1} = -2bct_{2,1} - 2adt_{2,1} + a^2 t_{2,2} + b^2 t_{2,2} - c^2 t_{2,2} - d^2 t_{2,2}$
- the generic complimentary linear space $L = V(\ell_1, \ldots, \ell_{11}) \subset \mathbb{P}(\mathbb{C}^{3 \times 3 \times 3})$
- the finite set $W \subseteq \mathcal{P} \subset \mathbb{C}^{13}$, mapping bijectively to $T_{\text{cal}} \cap L$

We heavily use this representation of $T_{\text{cal}}$ for the computations in Section 6.

Now, we re-visit Proposition 13. When $T_{A,B,C}$ is unknown but the point/line correspondence is known, the constraints there amount to special linear slices of $\mathcal{T}$ and of the subvariety $T_{\text{cal}}$. The next theorem may help the reader appreciate the specialness of these linear sections of $T_{\text{cal}}$; in general, the intersections are not irreducible, equidimensional, nor dimensionally transverse.

**Theorem 21.** Fix generic points $x, x', x'' \in \mathbb{P}^2$ and generic lines $\ell, \ell', \ell'' \in (\mathbb{P}^2)^\vee$. In the cases of Proposition 13, we have the following codimensions:

- $L = \{T \in \mathbb{P}(\mathbb{C}^{3 \times 3 \times 3}) : T(x, \ell', \ell'') = 0\}$ is a hyperplane and $T_{\text{cal}} \cap L$ consists of one irreducible component of codimension 1 in $T_{\text{cal}}$ [PLL]
- $L = \{T \in \mathbb{P}(\mathbb{C}^{3 \times 3 \times 3}) : [\ell] \cap T(-, \ell', \ell'') = 0\}$ is a codimension 2 subspace and $T_{\text{cal}} \cap L$ consists of two irreducible components both of codimension 2 in $T_{\text{cal}}$ [LLL]
- $L = \{T \in \mathbb{P}(\mathbb{C}^{3 \times 3 \times 3}) : [x''] \cap T(x, \ell', -) = 0\}$ is a codimension 2 subspace and $T_{\text{cal}} \cap L$ consists of two irreducible components both of codimension 2 in $T_{\text{cal}}$ [PLP]
- $L = \{T \in \mathbb{P}(\mathbb{C}^{3 \times 3 \times 3}) : [x'] \cap T(x, -, \ell'') = 0\}$ is a codimension 2 subspace and $T_{\text{cal}} \cap L$ consists of two irreducible components both of codimension 2 in $T_{\text{cal}}$ [PPL]
\[ L = \{ T \in \mathbb{P}(\mathbb{C}^{3 \times 3}) : [x'] \times T(x, -,-)[x'] = 0 \} \] is a codimension 4 subspace and \( \tau_{\text{cal}} \cap L \) consists of five irreducible components, one of codimension 3 and four of codimension 4 in \( \tau_{\text{cal}} \).

**Computational Proof.** The statements about the subspaces may shown symbolically. In the case of ‘LLL’, e.g., work in the ring \( \mathbb{Q}[\ell_0, \ldots, \ell_2] \) with 8 variables, and write the constraint on \( T \in \mathbb{P}(\mathbb{C}^{3 \times 3}) \) as the vanishing of a \( 3 \times 27 \) matrix times a vectorization of \( T \). Now we check that all of the \( 3 \times 3 \) minors of that long matrix are identically 0, but not so for \( 2 \times 2 \) minors.

For the statements about \( \tau_{\text{cal}} \cap L \), we offer a probability 1, numerical argument. By [28, Theorem A.14.10] and the discussion on page 348 about generic irreducible decompositions, we can fix random floating-point coordinates for \( x, x', x'', \ell, \ell', \ell'' \). With the parametrization \( \Phi \) of \( \tau_{\text{cal}} \) from the proof of Theorem 20, the \texttt{TrackType:1} setting in \texttt{Bertini} is used to compute a numerical irreducible decomposition for the preimage of \( \tau_{\text{cal}} \cap L \) per each case. That outputs a witness set, i.e. general linear section, per irreducible component. \texttt{Bertini’s TrackType:1} is based on regeneration, monodromy and the trace test; see [28, Chapter 15] or [6, Chapter 8] for a description.

Here, the ‘PPP’ case is most subtle since the subspace \( L \subseteq \mathbb{P}(\mathbb{C}^{3 \times 3}) \) is codimension 4, but the linear section \( \tau_{\text{cal}} \cap L \subseteq \tau_{\text{cal}} \) includes a codimension 3 component. The numerical irreducible decomposition above consists of five components of dimensions \( 8, 7, 7, 7, 7 \) in \( a, \ldots, t_{3,3} \)-parameter space. Thus, it suffices to verify that the map to \( \tau_{\text{cal}} \) is generically injective restricted to the union of these components. For that, we take one general point on each component from the witness sets, and test whether that point satisfies \( a^2 + b^2 + c^2 + d^2 \neq 0 \) and \( e^2 + f^2 + g^2 + h^2 \neq 0 \). This indeed holds for all components. Then, we test using singular value decomposition (see [7, Theorem 3.2]) whether the point maps to a camera triple with linearly independent centers. Linear independence indeed holds for all components. From Theorem 16, the above parametrization is generically injective on this locus. Hence the image \( \tau_{\text{cal}} \cap L \) consists of distinct components with the same dimensions \( 8, 7, 7, 7, 7 \). This finishes ‘PPP’. The other cases are similar. \( \square \)

Mimicking the proof of Proposition 11, and using the ‘top-dimensional’ clause in Lemma 14, we can establish the following finiteness result for \( \tau_{\text{cal}} \):

**Lemma 22.** For each problem in Theorem 6, given generic image correspondence data, there are only finitely many tensors \( T \in \tau_{\text{cal}} \) that satisfy all of the linear conditions from Proposition 13.

We have arrived at a relaxation for each minimal problem in Theorem 6, as promised. Namely, for a problem there we can fix a random instance of image data, and we seek those calibrated trifocal tensors that satisfy the merely necessary – linear conditions in 13. Geometrically, this is equivalent to intersect the special linear sections of \( \tau_{\text{cal}} \) from Theorem 21. In Section 6, we will use the pseudo-witness set representation \( (\mathcal{P}, \Phi, \mathbf{L}, \mathcal{W}) \) of \( \tau_{\text{cal}} \) from Theorem 20 to compute these special slices of \( \tau_{\text{cal}} \) in \texttt{Bertini}. Conveniently, \texttt{Bertini} outputs a calibrated camera triple per calibrated trifocal tensor in the intersection; this is because all solving is done in the parameter space \( \mathcal{P} \), or in other words, camera space. To solve the original minimal problem, we then test these configurations against the tight conditions of Theorem 8.
6. Proof of Main Result

In this section, we put all the pieces together and we determine the algebraic degrees of the minimal problems in Theorem 6. Mathematically, these degrees represent interesting enumerative geometry problems; in vision, related work for three uncalibrated views appeared in [26]. The authors considered correspondences ‘PPP’ and ‘LLL’ and they determined 3 degrees for projective (uncalibrated) views, using the larger group actions present in that case. Here, all 66 degrees for calibrated views in Theorem 6 are new.

Now, recall from Proposition 11 that solutions \((A, B, C)\) to the problems in Theorem 6 in particular must have non-identical centers. So, by the second remark after Definition 12, they associate to nonzero tensors \(T_{A,B,C}\), and thus to well-defined points in the projective variety \(\mathcal{T}_{\text{cal}}\). Conversely, however, there are special subloci of \(\mathcal{T}_{\text{cal}}\) that are not physical. Points in these subvarieties (introduced next) are extraneous to Theorem 6, because they correspond to configurations with a \(3 \times 4\) matrix whose left \(3 \times 3\) submatrix \(R\) is not a rotation, but instead satisfies \(RR^T = R^T R = 0\).

**Definition/Proposition 23.** Recall the parametrization of \(\mathcal{T}_{\text{cal}}\) by \(a, \ldots, t_{3,3}\) from Theorem 20. Let \(T_{\text{cal}}^{0,1} \subset \mathcal{T}_{\text{cal}}\) be the closure of the image of the rational map restricted to the locus \(a^2 + b^2 + c^2 + d^2 = 0\). Let \(T_{\text{cal}}^{1,0} \subset \mathcal{T}_{\text{cal}}\) be the closure of the image of the rational map restricted to the locus \(e^2 + f^2 + g^2 + h^2 = 0\). Let \(T_{\text{cal}}^{0,0} \subset \mathcal{T}_{\text{cal}}\) be the closure of the image of the rational map restricted to the locus \(a^2 + b^2 + c^2 + d^2 = 0\) and \(e^2 + f^2 + g^2 + h^2 = 0\). Then these subvarieties are irreducible with: \(\dim(T_{\text{cal}}^{0,0}) = 9\) and \(\deg(T_{\text{cal}}^{0,0}) = 1296\); \(\dim(T_{\text{cal}}^{1,0}) = 10\) and \(\deg(T_{\text{cal}}^{1,0}) = 2616\); \(\dim(T_{\text{cal}}^{1,0}) = 10\) and \(\deg(T_{\text{cal}}^{1,0}) = 2616\).

**Computational Proof.** The restricted parameter spaces:
\[
\mathcal{P} \cap V(a^2 + b^2 + c^2 + d^2), \quad \mathcal{P} \cap V(e^2 + f^2 + g^2 + h^2),
\]
\[
\mathcal{P} \cap V(a^2 + b^2 + c^2 + d^2, e^2 + f^2 + g^2 + h^2) \subset \mathbb{C}^{13},
\]
where \(\mathcal{P} = V(\alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d - 1, \beta_1 e + \beta_2 f + \beta_3 g + \beta_4 h - 1)\), are irreducible, therefore their images \(T_{\text{cal}}^{0,1}, T_{\text{cal}}^{1,0}, T_{\text{cal}}^{0,0} \subset \mathbf{P}(\mathbb{C}^{3 \times 3 \times 3})\) are irreducible. The dimension statements are verified by picking a random point in the restricted parameter spaces, and then by computing the rank of the derivative of the restricted rational map \(\Phi\) at that point. This rank equals the dimension of the image with probability 1, by generic smoothness over \(\mathbb{C}\) [12, III.10.5] and the preceding [12, III.10.4]. For the degree statements, the approach from Theorem 20 may be used. For \(T_{\text{cal}}^{0,1}\) we fix a random linear subspace \(M \subset \mathbf{P}(\mathbb{C}^{3 \times 3 \times 3})\) of complementary dimension, i.e. \(\dim(M) = 16\), so \(\deg(T_{\text{cal}}^{0,1}) = \#(T_{\text{cal}}^{0,1} \cap M)\). We pull back to \(\mathcal{P} \cap V(a^2 + b^2 + c^2 + d^2) \cap \Phi^{-1}(M)\), and use the UseRegegeneation:1 setting in Bertini to solve for this. This run outputs 2616 floating-point tuples in \(a, \ldots, t_{3,3}\) coordinates. Then, we apply the parametrization \(\Phi\) and check that the image of these are 2616 numerically distinct tensors, i.e. the restriction \(\Phi|_{\mathcal{P} \cap V(a^2 + b^2 + c^2 + d^2)}\) is generically injective. It follows that \(\deg(T_{\text{cal}}^{0,1}) = 2616\), up to numerical accuracy and random choices. To verify this degree further, we apply the trace test as in Theorem 20, and this finishes the computation for \(\deg(T_{\text{cal}}^{0,1})\). Since \(T_{\text{cal}}^{0,1}\)
and $T^0\mathcal{L}$ are linearly isomorphic under the permutation $T_{ijk} \mapsto T_{ikj}$, this implies $\text{deg}(T^1\mathcal{L}) = 2616$. The computation for $\text{deg}(T^0\mathcal{L})$ is similar. \hfill $\square$

Now, we come to the proof of Theorem 6, at last. The outline was given in the last paragraph of Section 5: for computations, solving the polynomial systems of multi-view equations (see Theorem 8) is relaxed to taking a special linear section of the calibrated trifocal variety $T_{\mathcal{L}}$ (see Theorem 21). Then, to take this slice, we use the numerical algebraic geometry technique of coefficient-parameter homotopy [28, Theorems 7.1.1, A.13.1], i.e. a general linear section is moved in a homotopy to the special linear section.

**Computational Proof of Theorem 6.** Consider one of the problems $'w_1 PPP + w_2 PPL + w_3 PLP + w_4 LLL + w_5 PLL'$ in Theorem 6, so that the weights $(w_1, w_2, w_3, w_4, w_5) \in \mathbb{Z}_2 \geq 0$ satisfy $3w_1 + 2w_2 + 2w_3 + 2w_4 + w_5 = 11$ and $w_2 \geq w_3$. Fix one general instance of this problem, by taking image data with random floating-point coordinates. Each point/line image correspondence in this instance defines a special linear subspace of $\mathbb{P}(\mathbb{C}^{3 \times 3 \times 3})$, as in Theorem 21. The intersection of these is one subspace $L_{\mathcal{L}}$ expressed in floating-point; using singular value decomposition, we verify that its codimension in $\mathbb{P}(\mathbb{C}^{3 \times 3 \times 3})$ is the expected $4w_1 + 2w_2 + 2w_3 + 2w_4 + w_5 = 11 + w_1$. By Proposition 13, $L_{\mathcal{L}}$ represents necessary conditions for consistency, so we seek $T_{\mathcal{L}} \cap L_{\mathcal{L}}$. If $w_1 > 0$, then this intersection is not dimensionally transverse by the ‘PPP’ clause of Theorem 21. To deal with a square polynomial system, we fix a general linear space $L_{\mathcal{L}} \supseteq L_{\mathcal{L}}$ of codimension 11 in $\mathbb{P}(\mathbb{C}^{3 \times 3 \times 3})$ and now seek $T_{\mathcal{L}} \cap L_{\mathcal{L}}$. This step is known as randomization [28, Section 13.5] in numerical algebraic geometry, and it is needed to apply the parameter homotopy result [28, Theorem 7.1.1].

The linear section $T_{\mathcal{L}} \cap L_{\mathcal{L}}$ is found numerically by a degeneration. In the proof of Theorem 20, we computed a pseudo-witness set for $T_{\mathcal{L}}$. This includes a general complimentary linear section $T_{\mathcal{L}} \cap L$, and the preimage $\Phi^{-1}(T_{\mathcal{L}} \cap L)$ of $\text{deg}(T_{\mathcal{L}}) = 4912$ points in $a, \ldots, t_{3,3}$ space. Writing $L = V(\ell_1, \ldots, \ell_{11})$ and $L_{\mathcal{L}} = V(\ell'_1, \ldots, \ell'_{11})$ for linear forms $\ell_i$ and $\ell'_i$ on $\mathbb{P}(\mathbb{C}^{3 \times 3 \times 3})$, consider the following homotopy function $H : \mathbb{C}^{13} \times \mathbb{R} \rightarrow \mathbb{C}^{13}$:

$$H(a, \ldots, t_{3,3}, s) := \begin{bmatrix} s \cdot \ell_1(\Phi(a, \ldots, t_{3,3})) + (1-s) \cdot \ell'_1(\Phi(a, \ldots, t_{3,3})) \\ \vdots \\ s \cdot \ell_{11}(\Phi(a, \ldots, t_{3,3})) + (1-s) \cdot \ell'_{11}(\Phi(a, \ldots, t_{3,3})) \\ \alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d - 1 \\ \beta_1 e + \beta_2 f + \beta_3 g + \beta_4 h - 1 \end{bmatrix}$$

Here $s \in \mathbb{R}$ is the path variable. As $s$ moves from 1 to 0, $H$ defines a family of square polynomial systems in the 13 variables $a, \ldots, t_{3,3}$. The start system $H(a, \ldots, t_{3,3}, 1) = 0$ has solution set $\Phi^{-1}(T_{\mathcal{L}} \cap L)$ and the target system $H(a, \ldots, t_{3,3}, 0) = 0$ has solution set $\Phi^{-1}(T_{\mathcal{L}} \cap L')$. With the UserHomotopy1 setting in Bertini, we track the 4912 solution paths from the start to target system. By genericity of $L$ in the start system, these solution paths are smooth [28, Theorem 7.1.1(4), Lemma 7.1.2]. The finite
endpoints of this track consist of solutions to the target system. By the principle of coefficient-parameter homotopy [28, Theorem A.13.1], every isolated point in $\Phi^{-1}(T_{\text{cal}} \cap L'_\text{special})$ is an endpoint, with probability 1. Note that in general, coefficient-parameter homotopy – i.e., the tracking of solutions of a general instance of a parametric system of equations to solutions of a special instance – may be used to find all isolated solutions to square polynomial systems. Here, by Lemma 22, $T_{\text{cal}} \cap L'_\text{special}$ is a scheme with finitely many points. By Bertini’s theorem [28, Theorem 13.5.1(1)], $T_{\text{cal}} \cap L'_\text{special}$ also consists of finitely many points, using genericity of $L'_\text{special}$. On the other hand, by Proposition 11, all solutions $(A, B, C)$ to the instance of the original minimal problem indexed by $w \in \mathbb{Z}_{\geq 0}$ have linearly independent centers in $\mathbb{P}^3$. Moreover, a configuration $(A, B, C)$ with linearly independent centers is an isolated point in $\Phi^{-1}(T_{A,B,C})$, thanks to Theorem 16. Therefore, it follows that all solutions to the problem from Theorem 6 are among the isolated points in $\Phi^{-1}(T_{\text{cal}} \cap L'_\text{special})$, and so the endpoints of the above homotopy.

For each minimal problem in Theorem 6, after the above homotopy, Bertini returns 4912 finite endpoints in $a, \ldots, t_{3,3}$ space. We pick out which of these endpoints are solutions to the original minimal problem by performing a sequence of checks, as explained next. First of all, of these endpoints, let us keep only those that lie in $\Phi^{-1}(T_{\text{cal}} \cap L'_\text{special})$, as opposed to those that lie just in the squared-up target solution set $\Phi^{-1}(T_{\text{cal}} \cap L'_\text{special})$. Second, we remove points that satisfy $a^2 + b^2 + c^2 + d^2 \approx 0$ or $e^2 + f^2 + g^2 + h^2 \approx 0$, because they are non-physical (see Definition/Proposition 23). Third, we verify that, in fact, all remaining points correspond to camera configurations $(A, B, C)$ with linearly independent centers. This means that the equations in Theorem 8 generate the multi-view ideals (recall Definition 7). Fourth, we check which remaining points satisfy those tight multi-view equations. To test this robustly in floating-point, note that the equations in Theorem 8 are equivalent to rank drops of the concatenated matrices there, hence we test for those rank drops using singular value decomposition. If the ratio of two consecutive singular values exceeds $10^5$, then this is taken as an indication that all singular values below are numerically 0, thus the matrix drops rank. Fifth, and conversely, we verify that all remaining configurations $(A, B, C)$ avoid epipoles (recall Definition 9) for the fixed random instance of image correspondence data, so the converse Lemma 10 applies to prove consistency. Lastly, we verify that all solutions are numerically distinct. Ultimately, the output of this procedure is a list of all calibrated camera configurations over $\mathbb{C}$ that are solutions to the fixed random instances of the minimal problems, where these solutions are expressed in floating-point and $a, \ldots, t_{3,3}$ coordinates. The numbers of solutions are the algebraic degrees from Theorem 6.

As a check for this numerical computation, we repeat the entire calculation for other random instances of correspondence data. For each minimal problem, we obtain the same algebraic degree each time. One instance per problem solved to high precision is provided on this paper’s webpage. □

Example 24. We illustrate the proof of Theorem 6 by returning to the instance of ‘$1PPP + 4PPL$’ in Example 5. Here $L_{\text{special}} \subset \mathbb{P}(\mathbb{C}^{3 \times 3 \times 3})$ formed by intersecting subspaces from Theorem 21 is codimension 12, hence
Tracking $\deg(T_{cal})$ many points in the pseudo-witness set $\Phi^{-1}(T_{cal} \cap L')$ to the target $\Phi^{-1}(T_{cal} \cap L_{\text{special}})$, we get 4912 finite endpoints. Testing membership in $L_{\text{special}}$, we get 2552 points in $\Phi^{-1}(T_{cal} \cap L_{\text{special}})$. Among these, 888 points satisfy $a^2 + b^2 + c^2 + d^2 \approx 0$, so they are non-physical (corresponding to $3 \times 4$ matrices with left submatrices that are not rotations). The remaining 1664 points turn out to correspond to calibrated camera configurations with linearly independent centers. Checking satisfaction of the equations from Theorem 8, we end up with 160 solutions.

**Remark.** The proof of Theorem 6 is constructive. From the solved random instances, one may build solvers for each minimal problem, using coefficient-parameter homotopy. Here the start system is the solved instance of the minimal problem and the target system is another given instance. Such a solver is optimal in the sense that the number of paths tracked equals the true algebraic degree of the problem. Implementation is left to future work.

**Remark.** All degrees in Theorem 6 are divisible by 8. We would like to understand why. What are the Galois groups [15] for these minimal problems?

**Remark.** Practically speaking, given image correspondence data defined over $\mathbb{R}$, only real solutions $(A, B, C)$ to the minimal problems in Theorem 6 are of interest to RANSAC-style 3D reconstruction algorithms. Does there exist image data such that all solutions are real? Also, for the image data observed in practice, what is the distribution of the number of real solutions?

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