On the teleportation of continuous variable

S.N. Molotkov and S.S. Nazin

Institute of Solid State Physics of Russian Academy of Sciences
Chernogolovka, Moscow district, 142432, Russia

Abstract

The measurement procedures used in quantum teleportation are analyzed from the viewpoint of the general theory of quantum-mechanical measurements. It is shown that to find the teleported state one should only know the identity resolution (positive operator-valued measure) generated by the corresponding instrument (quantum operation describing the system state change caused by the measurement) rather than the instrument itself. A quantum teleportation protocol based on a measurement associated with a non-orthogonal identity resolution is proposed for a system with non-degenerate continuous spectrum.

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1 Introduction

One of the major results of quantum information theory consists in the possibility of teleportation of an unknown quantum state by means of a classical and a distributed quantum communication channel, the latter being realized by a non-local entangled state chosen in a special way (e.g., an EPR-pair of particles). Quantum teleportation of an unknown state from user A to user B is performed in the following way: User A has the unknown state \( \rho_1 \) of quantum system 1 (e.g., spin-1/2 particle; Ref. contains also the more general case of a quantum system with arbitrary finite number of levels, i.e. with any finite-dimensional state space) which is to be teleported to user B. In addition, two other (also spin-1/2) particles labeled as systems 2 and 3 are employed which are in the spin-entangled EPR-state \( \rho_{23} \) such that the user A has access to particle 2 while user B has access to particle 3. User A performs a certain joint measurement \( m_{12} \) over the system 1 in the unknown state \( \rho_1 \) and particle 2 from the EPR-pair. As a result, the total system composed of particles 1, 2, and 3 changes its state from \( \rho_1 \otimes \rho_{23} \) to a new state \( \rho'_{123} \) which depends on the measurement result \( z \). It turns out that there exist such measurements \( m_{12} \) that the state \( \rho'_3 \) of particle 3 from the EPR-pair accessible to user B after the measurement (which is obtained from \( \rho'_{123} \) by performing trace of the state spaces of particles 2 and 3, \( \rho'_{3} = \text{Tr}_{1,2} \rho'_{123} \)) is related to the initial state \( \rho_1 \) of particle 1 through a certain unitary transformation \( U_z \) which does not depend on \( \rho_1 \) and is completely determined by the result \( z \) of the performed measurement \( m_{12} \):
so-called Bell measurement described by a certain self-adjoint operator with non-degenerate spectrum in the 4-dimensional state space (of particles 1 and 2) and the state $\rho'_{123}$ can be easily written explicitly.

The first algorithm for teleportation of quantum continuous variable (i.e. the wave function of a one-dimensional non-relativistic spinless particle whose state space is infinite-dimensional) was given by Vaidman [2]. Later this approach was extended to a more realistic algorithm of the teleportation of a single-mode electromagnetic field [3]. Both these algorithms actually assumed that in the case of an observable with a continuous spectrum the state of the system just after the measurement is described by the “eigenvector” belonging to the “eigenvalue” (of the corresponding self-adjoint operator) given by the measurement act.

However, for a continuous variable the correctly formulated question concerning the system state after the measurement turns out to be much more difficult than in the case of a discrete spectrum (e.g. see [4]). The problem here is not even only that for a continuous spectrum the Hilbert state space does not contain any correctly defined eigenvectors. Consider, for example, a self-adjoint operator $A$ with the continuous spectrum $\Lambda$. Let the point $z$ belong to this spectrum and the system state before the measurement be $\rho$. How sensible is then the question of what is the system state $\rho_z$ after the measurement which gave the result $r = z$?

The problem is that according to the statistical interpretation of quantum mechanics the very concept of “state” should only be associated with an ensemble of identical systems rather than single systems. In our case it is then natural to think that one should consider the subensemble of systems selected by the condition $r = z$ after the measurement. However, for a continuous spectrum the probability of obtaining any particular result $z$ is zero since any point has zero measure. Therefore, it is simply impossible to select the subensemble of systems by the condition $r = z$, since the probability of obtaining the same results in any two measurements is zero. Therefore, the problem of interpretation of the physical meaning ascribable to $\rho_z$ is not so straightforward. To analyze this problem we shall need some concepts of the general theory of quantum-mechanical measurements (e.g. see [4, 5, 6]). Basic ideas and some results of this theory are outlined in Section 2. In Section 3 the general theory is applied to a particular class of measurements used in quantum teleportation. Section 4 considers the teleportation protocol for a continuous variable presented in Ref.[2] within the framework of the results obtained in Section 3. Section 5 contains a new algorithm for the teleportation of the states of a model system with a continuous spectrum based on a measurement associated with a non-orthogonal identity resolution. Finally, the last Section 6 summarizes the results obtained in the paper.

2 Quantum-mechanical measurements

For a quantum system with a finite-dimensional Hilbert state space $\mathcal{H}$ (when any operator has a purely discrete spectrum), the canonical (von Neumann) measurement of the observable corresponding to a self-adjoint operator $A$ whose eigenvalues are $\lambda_i$, $i = 1 \ldots n$ results in the transformation of the system state from $\rho$ (density matrix just before the measurement) to $\rho_j$ if the measurement result is $\lambda_j$ (Lüders-von Neumann reduction postulate [4, 5]):

$$\rho \rightarrow \rho_j = \frac{E_j \rho E_j}{\text{Tr}\{E_j \rho\}}.$$  \hspace{1cm} (2)

Here $E_j$ is the orthogonal projector on the subspace associated with the eigenvalue $\lambda_j$ so that the following identity resolution takes place:

$$\sum_j E_j = I.$$  \hspace{1cm} (3)
where $I$ is the identity operator on $\mathcal{H}$, and the spectral representation of the operator $A$ is

$$A = \sum_j \lambda_j E_j.$$  \hspace{1cm} (4)

The probability of obtaining $j$-th results is

$$\text{Prob}(\lambda_j) = \text{Tr}\{\rho E_j\} = \text{Tr}\{E_j \rho E_j\}. \hspace{1cm} (5)$$

Consider now the most general case when the complete set of all possible measurement results constitute a measurable space $\mathcal{Z}$ with measure $dz$ and the quantum system $S$ is described by the (generally speaking, infinite-dimensional) Hilbert space $\mathcal{H}$, i.e. the set of all its states can be identified with the set $K(\mathcal{H})$ of all positive operators on $\mathcal{H}$ with trace 1 (i.e., density matrices; an operator $A$ on $\mathcal{H}$ is called positive if $\langle \psi | A | \psi \rangle \geq 0$ for all $\psi \in \mathcal{H}$). The set $K(\mathcal{H})$ is a subset of the space $B_1(\mathcal{H})$ of all finite trace operators on $\mathcal{H}$. In that case the adequate mathematical entity completely characterizing any particular measurement procedure with the space $\mathcal{Z}$ of all possible results which can be applied to $S$ is the instrument [5] (or, in a different terminology, operation [5]) $T$, which is a map $\Delta \to T(\Delta)$ of the set $\Gamma$ of all measurable (with respect to the measure $dz$) subsets $\Delta \subset \mathcal{Z}$ into the set of all trace decreasing (more strictly, non-increasing) completely positive operators $P(B_1(\mathcal{H}))$ which map $B_1(\mathcal{H})$ into itself and satisfies the following two requirements:

1) $T(\Delta) = \sum_j T(\Delta_j)$, if $\Delta = \cup_j \Delta_j$, $\Delta_j \cap \Delta_i = \emptyset$ for $i \neq j$ (additivity) and
2) $\text{Tr}\{T(\mathcal{Z})\rho\} = \text{Tr}\rho$ for any $\rho \in B_1(\mathcal{H})$ (normalization).

Remember that the linear map $F$ from $B_1(\mathcal{H})$ into itself is called completely positive if $F(L) > 0$ for any $L > 0$ from $B_1(\mathcal{H})$, i.e. if it maps any positive operator from $B_1(\mathcal{H})$ into a positive operator, and in addition, possesses a property that if $\mathcal{H}_0$ is another Hilbert space then the map $F \otimes I$: $B_1(\mathcal{H} \otimes \mathcal{H}_0) \to B_1(\mathcal{H} \otimes \mathcal{H}_0)$, defined on the elements of the type $W \otimes W_0 \in B_1(\mathcal{H} \otimes \mathcal{H}_0)$ by a formula $F \otimes I(W \otimes W_0) = F(W) \otimes W_0$ and extended to the entire space $B_1(\mathcal{H} \otimes \mathcal{H}_0)$ by the linearity, where $I$ is the identity operator on $B_1(\mathcal{H}_0)$, is also a positive map for any $\mathcal{H}_0$. The essence of the instrument $T$ is that for any measurable subset $\Delta \subset \mathcal{Z}$, $\Delta \in \Gamma$ the state $\rho_\Delta$ of the subensemble of the systems initially prepared in the state $\rho \in K(\mathcal{H})$ and then selected after a repeated application of the specified measurement procedure by the condition that the measurement result $r = z$ belongs to the set $\Delta$, is (for brevity we write $T(\Delta)\rho$ instead of $[T(\Delta)](\rho)$)

$$\rho_\Delta = \frac{\tilde{\rho}(\Delta)}{\text{Tr}\{\tilde{\rho}(\Delta)\}} = \frac{T(\Delta)\rho}{\text{Tr}\{T(\Delta)\rho\}} \in K(\mathcal{H}), \quad \tilde{\rho}(\Delta) = T(\Delta)\rho$$  \hspace{1cm} (6)

while the probability of obtaining result $r = z \in \Delta$ is

$$\text{Prob}(z \in \Delta) = \text{Tr}\{T(\Delta)\rho\} = \text{Tr}\{\tilde{\rho}(\Delta)\};$$  \hspace{1cm} (7)

here and later we label by the tilde symbol the “unnormalized density matrices” (positive operators with trace $\leq 1$) which arise after the application of the operator $T(\Delta)$ corresponding to the considered instrument to the initial density matrix $\rho$. We shall also apply the term “density matrices” to these “unnormalized density matrices” in the cases where it cannot cause confusion.

It is easily checked that for a fixed $T$ the formula (7) generates an affine map of the convex set $K(\mathcal{H})$ of all possible states $\rho$ of the system $S$ into the set of probability measures $\nu_{\text{Prob}}$ defined on $\mathcal{Z}$: each state $\rho \in K(\mathcal{H})$ corresponds to the measure $\mu_\rho$ on $\mathcal{Z}$ such that for every set $\Delta \in \Gamma$ its measure $\mu_\rho(\Delta)$ is exactly $\text{Prob}(z \in \Delta)$. It is known [6] that the set of all such maps $\rho \to \mu_\rho$ from $K(\mathcal{H})$ into $\nu_{\text{Prob}}$ is in one-to-one correspondence with the families of
Hermitian operators $M(\Delta)$, $\Delta \in \Gamma$ defined on the Hilbert state $\mathcal{H}$ and satisfying the following requirements:

1') $M(\emptyset) = 0$, $M(\mathcal{Z}) = I$, (normalization)

2') $M(\Delta) \geq 0$, (positivity) and

3') $M(\Delta) = \sum_j M(\Delta_j)$, if $\Delta = \cup_j \Delta_j$, $\Delta_j \cap \Delta_i = \emptyset$ for $i \neq j$ (additivity),
i.e. with the identity resolutions on $\mathcal{Z}$ with the values in the set of positive operators on $\mathcal{H}$.

The measure $\mu_\rho$ of the set $\Delta$ is given by

$$\mu_\rho(\Delta) = \text{Prob}(z \in \Delta) = \text{Tr}\{\rho M(\Delta)\}. \tag{8}$$

In other words, $M(\Delta)$ defines a positive operator-valued measure. A special case of the measures of that kind is given by the identity resolutions corresponding to the families of spectral projectors associated with the self-adjoint operators in $\mathcal{H}$ which, in addition, possess the property

$$M(\Delta_1)M(\Delta_2) = 0, \text{ if } \Delta_1 \cap \Delta_2 = \emptyset;$$
it is natural to call the measurements described by these identity resolutions the “orthogonal measurements”.

Therefore, if we are only interested in the probability distribution of obtaining a particular result and do not touch a much more difficult problem of the system state after the measurement, it is sufficient to restrict ourselves to the analysis of the positive identity resolutions rather than the families of operators $T(\Delta) \in \mathcal{P}(B_1(\mathcal{H}))$. The relationship between them is established by the requirement that the probability of obtaining the result $z \in \Delta$ after the measurement act is performed on the system $S$ in any initial state $\rho$ originally defined by Eq. (7) can be calculated with the operator $M(\Delta)$ employing Eq. (8). Comparing Eqs. (7) and (8), one can easily see that they are compatible if and only if

$$M(\Delta) = [T(\Delta)]^* I, \tag{9}$$

where asterisk means the dual map from the space $B(\mathcal{H})$ into itself and $I \in B(\mathcal{H})$ is the identity operator on $\mathcal{H}$ (remember that the linear space of all bounded operators $B(\mathcal{H})$ on $\mathcal{H}$ is isomorphic to the dual space of $B_1(\mathcal{H})$, and the corresponding isomorphism is generated by the bilinear mapping $B(\mathcal{H}) \times B_1(\mathcal{H}) \to \mathbb{C}$: $a \in B(\mathcal{H}), b \in B_1(\mathcal{H}) \to \text{Tr}\{a \cdot b\} \in \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers).

For a canonical measurement of an observable $A$ on a finite-dimensional space state $\mathcal{H}$ (i.e., the discrete spectrum) described by Eqs. (2,3,4), the space $\mathcal{Z}$ coincides with the finite set of all eigenvalues $\lambda_i, i = 1 \ldots n$ of the operator $A$ while the set $\Gamma$ consists of all the subsets of the set $\mathcal{Z}$ and for all the sets consisting of a single point $\{\lambda_j\}$ the operators $T(\{\lambda_j\})$ and $M(\{\lambda_j\})$ are given by the formulas

$$T(\{\lambda_j\})\rho = E_j\rho E_j, \quad M(\{\lambda_j\}) = E_j. \tag{10}$$

It is clear that the family of operators $T(\Delta)$ provides a much more comprehensive description of the measurement process than the corresponding identity resolution $M(\Delta)$ since the former allows one not only to calculate the statistics of obtaining various measurement outcomes, but also determines the state of the system after the measurement is performed (6); generally, the same identity resolution can be generated by different instruments $T_1 \neq T_2$.

Further, it turns out [4] that for the case $\mathcal{Z} = \mathbb{R}$ (real numbers) for any fixed $\rho$ the density matrix $\tilde{\rho}(\Delta) = T(\Delta)\rho$ allows the following integral representation:

$$\tilde{\rho}(\Delta) = T(\Delta)\rho = \int_\Delta \rho_z \text{Tr}\{\rho M(\Delta)\}, \tag{11}$$
where \( \rho_z \) is a certain function from the space of all possible measurement results \( Z \) into the normalized density matrices \( K(H) \), and \( \text{Tr}\{\rho M(dz)\} \) is the “density” of the measure \( \mu_\rho \) on \( Z \), i.e.

\[
\mu_\rho(\Delta) = \text{Prob}(z \in \Delta) = \int_\Delta \text{Tr}\{\rho M(dz)\},
\]

\[
\mu_\rho(\Delta) = \int_\Delta d\mu_\rho(z), \quad d\mu_\rho(z) = \text{Tr}\{\rho M(dz)\}.
\]

The function \( \rho_z \) defined in this way can already be interpreted as the “state of the system after the measurement which gave the outcome \( z \)”. This does not contradict to the statistical interpretation of quantum mechanics since actually \( \rho_z \) is only a convenient auxiliary tool which allows to calculate the final state of the system after the measurement. Physical interpretation of Eq. (11) is absolutely transparent since \( \text{Tr}\{\rho M(dz)\} \) is the probability of obtaining after the measurement a result in the neighborhood \( dz \) of point \( z \).

The reason why the representation of the type (11) is important for us is that in the teleportation algorithms the system state after the measurement is corrected with a unitary transformation \( U_z \) which depends on the measurement outcome \( z \). Obviously, in this case the subensemble of systems selected by the condition \( z \in \Delta \) after the unitary correction is described by the density matrix

\[
\tilde{\rho}_{U,\Delta} = \int_\Delta U_z \rho_z U_z^\dagger \text{Tr}\{\rho M(dz)\};
\]

therefore, introduction of the function \( \rho_z \) is a natural step in the attempt to extend the algorithm of the teleportation of the state of a finite-dimensional quantum system proposed in Ref. [1] to the case of continuous variable.

### 3 Measurements used in quantum teleportation

Consider now the measurements used in quantum teleportation from the viewpoint of the general theory of quantum mechanical measurements outlines in the preceding section. Let the particles 1 and 2 be subjected to the measurement corresponding to the instrument \( T_{12} \). Then for the whole system including particle 3 this measurement is described by the instrument \( T_{123}(\Delta) = T_{12}(\Delta) \otimes I_3 \), where \( I_3 \) is the identity operator on \( B_1(H_3) \). Hence, after the joint measurement performed on the first and second particles the subensemble of systems selected by the condition \( z \in \Delta, \Delta \subset Z, \Delta \in \Gamma \) (at this moment we do not specify the space of possible results \( Z \)) is described by the density matrix

\[
\rho'_{123,\Delta} = \frac{T_{123}(\Delta)\rho}{\text{Tr}_{1,2,3}\{T_{123}(\Delta)\rho\}},
\]

and the probability of the event \( z \in \Delta \) is \( \text{Tr}_{1,2,3}\{T_{123}(\Delta)\rho\} \); the reduced density matrix representing the state of particle 3 is

\[
\rho'_{3,\Delta} = \frac{\text{Tr}_{1,2}\{T_{123}(\Delta)\rho\}}{\text{Tr}_{1,2,3}\{T_{123}(\Delta)\rho\}}.
\]
described by the instrument $T_A$ and we wish to find the state $\rho'_{B,\Delta}$ of the system $B$ after the measurement (from here on the prime is used to label the state of a quantum system just after the measurement). It is obvious that the instrument $T_{AB}$ describing the change of the state of the entire system $A + B$ is $T_A \otimes I_B$ so that

$$\rho'_{B,\Delta} = \frac{\text{Tr}_A\{T_{AB}(\Delta)\rho_{AB}\}}{\text{Tr}_{AB}\{T_{AB}(\Delta)\rho_{AB}\}}.$$  \hspace{1cm} (17)

Consider now the numerator of this fraction which, according to the adopted conventions, will be written as $\tilde{\rho}'_{B,\Delta} : \tilde{\rho}'_{B,\Delta} = \text{Tr}_A\{T_{AB}(\Delta)\rho_{AB}\}$ (so that probability of the event $z \in \Delta$ for the measurement result $z$ is $\text{Tr}_B\tilde{\rho}'_{B,\Delta}$). Let $u_B$ be an arbitrary operator from $B(\mathcal{H})$. Let us compute the trace $\text{Tr}_B\{u_B\tilde{\rho}'_B\}$ (for brevity we omit everywhere the subscript $\Delta$):

$$\text{Tr}_B\{u_B\tilde{\rho}'_B\} =$$

$$\text{Tr}_B\{u_B\text{Tr}_A\{T_A \otimes I_B \rho_{AB}\}\} =$$

$$\text{Tr}_B\{\text{Tr}_A\{I_A \otimes u_B \cdot T_A \otimes I_B \rho_{AB}\}\} =$$

$$\text{Tr}_{AB}\{I_A \otimes u_B \cdot T_A \otimes I_B \rho_{AB}\} =$$

$$\text{Tr}_{AB}\{((T_A \otimes I_B)^*I_A \otimes u_B) \cdot \rho_{AB}\} =$$

$$\text{Tr}_{AB}\{((T_A I_A) \otimes I_B u_B) \cdot \rho_{AB}\} =$$

$$\text{Tr}_{AB}\{[M_A \otimes u_B] \cdot \rho_{AB}\} =$$

$$\text{Tr}_{AB}\{[(M_A \otimes I_B) \cdot (I_A \otimes u_B)] \cdot \rho_{AB}\} =$$

$$\text{Tr}_{AB}\{[(I_A \otimes u_B) \cdot (M_A \otimes I_B)] \cdot \rho_{AB}\} =$$

$$\text{Tr}_B\{\text{Tr}_A\{[(I_A \otimes u_B) \cdot (M_A \otimes I_B)] \cdot \rho_{AB}\}\} =$$

$$\text{Tr}_B\{u_B\text{Tr}_A\{(M_A \otimes I_B) \cdot \rho_{AB}\}\}.$$  \hspace{1cm} (18)

Therefore,

$$\tilde{\rho}'_{B,\Delta} = \text{Tr}_A\{T_{AB}(\Delta)\rho_{AB}\} = \text{Tr}_A\{(M_A(\Delta) \otimes I_B) \cdot \rho_{AB}\}.$$  \hspace{1cm} (19)

Thus, if we wish to find the state of the system $B$ just after the measurement performed over the system $A$, it is sufficient for us to know only the identity resolution in $\mathcal{H}_A$ on $Z$ generated by the instrument $T_A$ rather than the instrument $T_A$ itself.

It should be noted that the technique of quantum operations seems first to have been applied to the problem of teleportation in the work [7] where the simplest case of “ideal” teleportation with the discrete space of possible measurement results $Z$ was considered when the change of the system state caused by the measurement is described by the instrument of the type

$$\rho \rightarrow A_i \rho A_i^+,$$  \hspace{1cm} (20)

where $A_i$ is a positive operator and the subscript $i = 1, 2, \ldots$ labels different measurement results, i.e., points of $Z$. However, in that work the teleported state was expressed through the operators $A_i$ which completely characterize the entire instrument.

We are interested in the possibility of the representation of $\tilde{\rho}'_{B,\Delta}$ in the form

$$\tilde{\rho}'_{B,\Delta} = \int_{\Delta} \rho_{z,B} d\mu_{\rho_{AB}}(z),$$  \hspace{1cm} (21)

where $\rho_{z,B} \in K(\mathcal{H}_B)$, and the measure $d\mu_{\rho_{AB}}(z)$ is the probability density of obtaining measurement result in the neighbourhood of point $z$, i.e. satisfies the condition

$$\text{Tr}_B\tilde{\rho}'_{B,\Delta} = \int_{\Delta} d\mu_{\rho_{AB}}(z).$$  \hspace{1cm} (22)
Formally, such a representation can be easily found if the measure \( \mu_{\rho AB} \) is absolutely continuous with respect to the initial measure \( dz \) on \( Z \) and the matrix elements of the operators \( M_A(\Delta) \) calculated for some orthogonal basis \( |\varphi_{nA}\rangle \) of the system \( A \) can be written as

\[
\langle \varphi_{mA}|M_A(\Delta)|\varphi_{nA}\rangle = \int_\Delta dz F_{mn}(z),
\]

where \( F_{mn}(z) \) are the c-numbers valued functions on \( Z \) (for example, if the measurement \( M \) corresponds to a simultaneous measurement of the complete set of commuting observables with the continuous spectrum, since in that case \( \mathcal{H}_A = L^2(Z) \), while the space \( Z \) is a direct product of the spectra of the operators comprising this set so that \( F_{mn}(z) = \varphi_{mA}(z)^* \psi_{nA}(z) \)). Indeed, in this case

\[
\tilde{\rho}'_{B,\Delta} = \text{Tr}_A\{(M_A(\Delta) \otimes I_B) \cdot \rho_{AB}\} = \sum_{mn} \langle \varphi_{mA}|M_A(\Delta)|\varphi_{nA}\rangle \rho_{nm,B}
\]

\[
= \sum_{mn} \int_\Delta dz F_{mn}(z) \rho_{nm,B} = \int_\Delta dz \left[ \sum_{mn} F_{mn}(z) \rho_{nm,B} \right] = \int_\Delta dz \tilde{\rho}_{z,B},
\]

where the operator \( \rho_{nm,B} \) on \( \mathcal{H}_B \) is obtained from the operator \( \rho_{AB} \) by taking a “partial matrix element” over the vectors \( \varphi_{nA} \) and \( \varphi_{mA} \) from \( \mathcal{H}_A \),

\[
\rho_{nm,B} = \langle \varphi_{mA}|\rho_{AB}|\varphi_{nA}\rangle,
\]

and

\[
\tilde{\rho}_{z,B} = \sum_{mn} F_{mn}(z) \rho_{nm,B}.
\]

Hence

\[
\text{Tr}_B\{\tilde{\rho}'_{B,\Delta}\} = \int_\Delta d\mu_{\rho AB}(z) = \int_\Delta dz \text{Tr}_B\{\tilde{\rho}_{z,B}\}.
\]

Therefore, multiplying and dividing the integrand in the last integral in Eq. (24) by \( H(z) = \text{Tr}\{\tilde{\rho}_{z,B}\} > 0 \), we obtain Eq. (21) where

\[
\rho_{z,B} = \frac{\tilde{\rho}_{z,B}}{\text{Tr}\{\tilde{\rho}_{z,B}\}} = \frac{\tilde{\rho}_{z,B}}{H(z)},
\]

so that \( \text{Tr}\{\rho_{z,B}\} = 1 \) and \( d\mu_{\rho AB}(z) = H(z)dz \), i.e. \( H(z) \) is the Radon-Nikodem derivative of the measure \( d\mu_{\rho AB}(z) \) with respect to measure \( dz \). We shall not dwell on the correctness of the procedure of changing the order of summation of an infinite series and integration in Eq. (24) and other similar operations since in the particular cases considered in the rest of the paper the integral representation of the form (24) directly follows from the specific form of the operators \( M(\Delta) \).

4 Teleportation with an orthogonal measurement

To illustrate the outlined general scheme, we shall first consider the teleportation of an unknown quantum state \( |\psi\rangle \) of a one-dimensional non-relativistic spinless particle. To avoid the complications associated with the particle permutation symmetry we shall assume that all three particles are different. It is sufficient to consider the case where the initial state of particle 1 is a pure state

\[
\rho_1 = \rho_\psi = |\psi; 1\rangle\langle \psi; 1|, \quad |\psi; 1\rangle = \int_{-\infty}^{+\infty} dx \psi(x)|x; 1\rangle.
\]
The entangled state of particles 2 and 3 will be chosen in the form of an EPR-state (with an infinite norm)
\[
\rho_{23} = |\psi_{23}\rangle \langle \psi_{23}|, \quad |\psi_{23}\rangle = \int_{-\infty}^{\infty} dx |x; 2\rangle |x; 3\rangle, \quad (30)
\]
which can be represented as a limit of a normalized state
\[
|\Psi_{23}\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \Psi(x, y)|x; 2\rangle |y; 3\rangle, \quad (31)
\]
where \(\Psi(x, y) \rightarrow \delta(x - y)\) (in the momentum representation \(\Psi_{23}(p_1, p_2) \rightarrow \delta(p_1 + p_2)\)); formally, the state (31) is an eigenvector of the operator of the difference of the positions of the second and the third particle: \((X_2 - X_3)|\psi_{23}\rangle = 0\).

Consider now the joint measurement performed over one of the particles from the EPR-pair (particle 2) and the system in the unknown state to be teleported (particle 1) defined by the following identity resolution:
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_{12}(dXdP) = I, \quad (32)
\]
\[
E_{12}(dXdP) = |\Phi_{XP}\rangle \langle \Phi_{XP}| \frac{dXdP}{2\pi} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' e^{iP(x-x')} |x + X; 1\rangle |x'; 2\rangle \langle x' + X; 1| \langle x; 2| dXdP, \quad (33)
\]
where
\[
|\Phi_{XP}\rangle = \int_{-\infty}^{\infty} dx e^{iPx} |x + X; 1\rangle |x; 2\rangle; \quad (35)
\]

note that formally the state (35) is a common eigenvector for a pair of the commuting observables \(X_1 - X_2 \) and \(P_1 + P_2\) (position difference and the total momentum) which form a complete set of commuting observables on the state space of two particles: \((X_1 - X_2)|\Phi_{XP}\rangle = X|\Phi_{XP}\rangle\), \((P_1 + P_2)|\Phi_{XP}\rangle = P|\Phi_{XP}\rangle\); therefore, the teleportation procedure with \(\rho_{23}\) taken in the form (34) and the measurement (33) is exactly coincides with the algorithm [2]. In that case the space of all possible measurement results \(Z\) is the set of ordered pairs \((X, P)\), \((-\infty < X < \infty, -\infty < P < \infty)\) constituting a plane \(\mathbb{R}^2\) which is actually a direct product of two copies of the real line \(\mathbb{R}_X\) and \(\mathbb{R}_P\) corresponding to the position \(X\) and momentum \(P\): \(Z = \mathbb{R}_X \times \mathbb{R}_P\).

The exact meaning of Eq. (33) is that the matrix elements of the positive operator \(E(\Delta)\) associated with the set \(\Delta\) can be calculated as
\[
\langle \Phi | E_{12}(\Delta) | \Psi \rangle = \int_{\Delta} \frac{dXdP}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' e^{iP(x-x')} \Phi^*(x + X, x) \Psi(x' + X, x'), \quad (36)
\]
similar to Eq. (23).

Simple calculations reveal that the teleported density matrix in channel 3 becomes
\[
\tilde{\rho}_{3,\Delta} = \text{Tr}_{1,2}\{ (\rho_1 \otimes \rho_{23}) E_{12}(\Delta) \} = \int_{\Delta} \rho_{XP} \frac{dXdP}{2\pi}, \quad (37)
\]
where
\[
\rho_{XP} = |\psi_{XP}; 3\rangle \langle \psi_{XP}; 3|, \quad \psi_{XP}(x) = e^{iPx} \psi(x + X). \quad (38)
\]
Since
\[
\text{Tr}_3\{ \rho_{XP} \} = \int_{-\infty}^{+\infty} dx |\psi(x + X)|^2 = 1, \quad (39)
\]
it is clear that the probability density of obtaining after the measurement a result in the interval \((dX, dP)\) in the neighbourhood of point \((X, P)\) is \(1/2\pi\) and does not depend on \(|\psi; 1\rangle\), so that
the measurement does not provide any information on the teleported state. Total probability of obtaining any pair \((X, P)\) turns out to be infinite because of the unnormalizability of state \((30)\).

Eqs. (37-38) imply that subjecting the particle 3 to a unitary transformation

\[
U_{XP}: \psi(x) \rightarrow e^{iP(x-X)}\psi(x - X)
\]

which only depends on the result of the measurement performed over particles 1 and 2, one obtains in the channel 3 the state identical to the initial state of particle 1, i.e. achieves the teleportation of the state of particle 1. It should be noted that in the present example the unitary correction (which does not depend on \(\rho_1\)) of the state of the third particle to the initial state of particle 1 proves to be possible for any input state \(\rho_1\) and any measurement outcome, i.e. any pair \((X, P)\). However, it is generally reasonable to consider also the teleportation algorithms which allow the teleportation of only a subset \(K'(\mathcal{H}_1)\) of all possible states, e.g. only the states belonging to a certain subspace \(\mathcal{H}'_1 \subset \mathcal{H}_1\) rather than the total space \(\mathcal{H}_1\) [9] (an example of that kind of algorithm is presented in the next section). In addition, the requirement that the necessary unitary correction \(U_z\) exists for all measurement outcomes is also unnecessary. Indeed, the entire space of all possible measurement outcomes \(Z\) can always be divided into two disjoint subsets \(Z_1\) and \(Z_2\), \(Z = Z_1 \cap Z_2 = \emptyset\), \(Z = Z_1 \cup Z_2\), in the following way: an arbitrary point \(z \in Z\) belongs to the subset \(Z_1\) if and only if the unitary transformation \(U_z\) with the required properties exists. A sufficient condition for the possibility of the teleportation will then be a non-zero measure \(\mu_\rho(Z_1)\) for all \(\rho \in K'(\mathcal{H}_1)\). In that case the teleportation algorithm looks as follows: the ensemble of systems representing the initial state \(\rho_1\) is subjected to the joint (together with the particle 2) to the measurement \(m_{12}\). If the outcome \(z \in Z_2\), then the particular copy of the system 3 is discarded. On the other hand, if \(z \in Z_1\), then the system 3 is subjected to the unitary correction \(U_z\). Under these conditions the subensemble of particles 3 selected and corrected in the above outlined way will be found in the state identical to the initial state \(\rho_1\) of particle 1.

5 Teleportation with a non-orthogonal measurement

We shall now consider an example of the teleportation of an unknown state based on the measurement associated with a non-orthogonal identity resolution. Consider a model quantum system whose Hamiltonian has a continuous non-degenerate spectrum coinciding with the positive part of the real line \((0, +\infty)\) (e.g. a free non-relativistic one-dimensional spinless particle whose allowed states are restricted by the condition of only positive momentum components occurring in their momentum representation). Thus we shall assume that an arbitrary pure state of the system 1 can be described by a wave function defined on the positive part of the real line:

\[
|\psi; 1\rangle = \int_0^{+\infty} \psi(E)|E; 1\rangle dE, \quad \langle E|E'\rangle = \delta(E - E').
\]

The EPR-state in the energy representation can be chosen, e.g. in the form

\[
|\psi_{23}\rangle = \int_0^{\varepsilon_0} d\varepsilon |\varepsilon; 2\rangle|\varepsilon_0 - \varepsilon; 3\rangle.
\]

Such an EPR-pair can be considered as the limit of a normalized state

\[
|\Psi_{23}\rangle = \int_0^{\varepsilon_0} \int_0^{\varepsilon_0} d\varepsilon_1 d\varepsilon_2 \psi(\varepsilon_1, \varepsilon_2)|\varepsilon_1; 1\rangle|\varepsilon_2; 2\rangle,
\]

(43)
where \( \psi(\varepsilon_1, \varepsilon_2) \rightarrow \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_0) \). The states of that kind are produced in the parametric down-conversion if the pump frequency is \( \varepsilon_0 \). Formally, the EPR-state can also be chosen in the form \( \psi(\varepsilon_1, \varepsilon_2) \rightarrow \delta(\varepsilon_1 - \varepsilon_2) \); however, it is not clear how this state can be realized experimentally.

Consider now a joint measurement \( M_{12}(d\Omega dT) \) on the particles 1 and 2 defined by the following non-orthogonal identity resolution

\[
M_{12}(d\Omega dT) = \frac{1}{\pi} \left( \int_{-\Omega}^{\Omega} d\omega e^{i\omega T} |\Omega + \omega; 1\rangle \langle \Omega - \omega; 2| \right) \left( \int_{-\Omega}^{\Omega} d\omega' e^{-i\omega' T} |\Omega + \omega'; 1\rangle \langle \Omega - \omega'; 2| \right) d\Omega dT
\]

(44)

where \( \Omega \) and \( T \) vary in the ranges \( \mathbb{R}_\Omega^+ = (0; +\infty) \) and \( \mathbb{R}_T = (-\infty; +\infty) \), respectively, so that the space of all possible measurement results is \( \mathcal{Z} = \mathbb{R}_\Omega^+ \times \mathbb{R}_T \). The quantities \( \Omega \) and \( \omega \) have the meaning of the half-sum and half-difference of the energies of two particles (we do not distinguish energy and frequency), e.g., two photons in the biphoton. This measurement which in some sense is an intermediate measurement between the frequency and time parameter measurement for two-particle states can be realized, at least in principle, for the photons experimentally employing the parametric up-conversion phenomenon [11].

It is easily checked that \( M_{12}(d\Omega dT) \) is actually an identity resolution:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} M_{12}(d\Omega dT) =
\]

\[
\frac{1}{\pi} \int_{0}^{\infty} d\Omega \int_{-\infty}^{\infty} dT \int_{-\Omega}^{\Omega} d\omega \int_{-\Omega}^{\Omega} d\omega' e^{i(\omega - \omega')T} |\Omega + \omega; 1\rangle \langle \Omega - \omega; 2| =
\]

\[
2 \int_{0}^{\infty} d\Omega \int_{-\Omega}^{\Omega} d\omega \int_{-\Omega}^{\Omega} d\omega' \delta(\omega - \omega') |\Omega + \omega; 1\rangle \langle \Omega - \omega; 2| =
\]

\[
\int_{0}^{\infty} d\omega_1 \int_{0}^{\infty} d\omega_2 |\omega_1; 1\rangle |\omega_2; 2\rangle = I_{12}, \]

where \( \omega_1 = \Omega + \omega \) and \( \omega_2 = \Omega - \omega \).

The teleported density matrix is now

\[
\tilde{\rho}_{3,\Delta} = \text{Tr}_{1,2}\{(\rho_1 \otimes \rho_{23})M_{12}(\Delta)\} = \int_{\Delta} \rho_{\Omega T} \frac{d\Omega dT}{\pi}, \quad \rho_{\Omega T} = |\psi_{\Omega T}; 3\rangle \langle \psi_{\Omega T}; 3|,
\]

(46)

where (for brevity we write \( |\psi_3\rangle \) instead of \( |\psi_{\Omega T}; 3\rangle \))

\[
|\psi_{\Omega T}; 3\rangle = |\psi_3\rangle = \int_{\varepsilon_0 - \min(\varepsilon_0, 2\Omega)}^{\varepsilon_0} d\varepsilon e^{-i(2\Omega - \varepsilon_0 + \varepsilon)T} \psi(2\Omega - \varepsilon_0 + \varepsilon)|\varepsilon; 3\rangle.
\]

(47)

The probability of obtaining the measurement result in the interval \( (\Omega, \Omega + d\Omega; T, T + dT) \) is

\[
\text{Tr}\{\tilde{\rho}_{3dT}\} = \int_{1,2,3}\{(\rho_1 \otimes \rho_{23})M_{12}(d\Omega dT)\} = \frac{d\Omega dT}{\pi} \int_{\varepsilon_0 - \min(\varepsilon_0, 2\Omega)}^{\varepsilon_0} |\psi(2\Omega - \varepsilon_0 + \varepsilon)|^2 d\varepsilon.
\]

(48)

Note that the corresponding probability density does not depend on \( T \). Since \( T \) varies in the infinite interval, the total probability, just as in the preceding section, proves to be infinite. Formally, this is related to the fact that the state \( |\psi_{3}\rangle \) has an infinite norm. However, this
circumstance does not create any problems since all the physically meaningful results can be calculated on the basis of the relative probabilities of different events.

Suppose now that the support of wave function \( \psi \) of the system 1 is known to lie in a certain segment \([E_{\text{min}}, E_{\text{max}}]\), i.e. \( \psi(E) = 0 \) at \( E > E_{\text{max}} \) and \( E < E_{\text{min}} \). In that case the probability density \( |\psi|^2 \) does depend on \( \Omega \); e.g. it vanishes for \( 2\Omega > E_{\text{max}} + \varepsilon_0 \), since then the function \( \psi \) is zero in the entire integration interval. Clearly, the condition for the exact teleportation is that the support of function \( \psi \) should belong to the integration interval in Eq. (17); in that case the probability of obtaining a particular result \( \Omega \) does not depend on \( |\psi; 1\rangle \) since the integral in Eq. (18) is identically equal to 1 due to the normalization of \( |\psi; 1\rangle \).

It is convenient to perform the further analysis for the cases \( \varepsilon_0 > E_{\text{max}} \) and \( \varepsilon_0 < E_{\text{max}} \) separately. Consider first the case \( \varepsilon_0 > E_{\text{max}} \). If the measurement gave the result \( 2\Omega < \varepsilon_0 \) (case 1a), the state of the system 3 will be \( |\psi_3\rangle\langle\psi_3| \), where

\[
|\psi_3\rangle = \int_\gamma^{\varepsilon_0} d\varepsilon e^{-i(\varepsilon-\gamma)T} \psi(\varepsilon - \gamma)|3\rangle, \quad \gamma = \varepsilon_0 - 2\Omega. \tag{49}
\]

Here the argument of the function \( \psi \) in the integrand ranges from 0 to \( 2\Omega \). Therefore, the state \( \psi \) can only be teleported if its support \([E_{\text{min}}, E_{\text{max}}]\subset [0, 2\Omega] \), i.e. if \( E_{\text{max}} < 2\Omega \). Thus, \([E_{\text{max}}, \varepsilon_0]\subset Z_1 \) (we omit the trivial factor \( R_T \) in \( Z_1 \), since the value of \( T \) does not matter).

On the other hand, if the measurement gave the result \( 2\Omega > \varepsilon_0 \) (case 1b), the state of the system 3 will be \( |\psi_3\rangle\langle\psi_3| \), where

\[
|\psi_3\rangle = \int_0^{\varepsilon_0} d\varepsilon e^{-i(\varepsilon+\gamma)T} \psi(\varepsilon + \gamma)|3\rangle, \quad \gamma = 2\Omega - \varepsilon_0. \tag{50}
\]

Now the argument of the function \( \psi \) in the integrand ranges from \( \gamma \) to \( 2\Omega \) and the state \( \psi \) can only be teleported if its support \([E_{\text{min}}, E_{\text{max}}]\subset [\gamma, 2\Omega] \), i.e. if \( \gamma < E_{\text{min}} \) or, in other words, \( 2\Omega < \varepsilon_0 + E_{\text{min}} \) (the condition \( E_{\text{max}} < 2\Omega \) is fulfilled automatically since \( 2\Omega > \varepsilon_0 > E_{\text{max}} \)). Thus, \([\varepsilon_0, \varepsilon_0 + E_{\text{min}}]\subset Z_1 \). Bringing the case 1a and 1b together we obtain \( Z_1 = [E_{\text{max}}, \varepsilon_0 + E_{\text{min}}] \).

It is seen from Eqs. (49) and (50) that the state of the system 3 can be made identical to the initial state of the system 1 just before the measurement if the system 3 just after the measurement is subjected to the unitary transformation

\[
\begin{align*}
\psi(\varepsilon) &\rightarrow \tilde{\psi}(\varepsilon) = \\
&= \begin{cases} 
\psi(\varepsilon), & \text{if } \varepsilon > \varepsilon_0 \\
\psi(\varepsilon + \gamma)e^{i\varepsilon T}, & \text{if } 0 < \varepsilon < 2\Omega \\
\psi(\varepsilon - 2\Omega), & \text{if } 2\Omega < \varepsilon < \varepsilon_0
\end{cases}
\tag{51}
\end{align*}
\]

or

\[
\begin{align*}
\psi(\varepsilon) &\rightarrow \tilde{\psi}(\varepsilon) = \\
&= \begin{cases} 
\psi(\varepsilon), & \text{if } \varepsilon > 2\Omega \\
\psi(\varepsilon + \gamma)e^{i\varepsilon T}, & \text{if } \gamma < \varepsilon < 2\Omega \\
\psi(\varepsilon - 2\Omega), & \text{if } 0 < \varepsilon < \gamma
\end{cases}
\tag{52}
\end{align*}
\]

in the cases 1a and 1b, respectively.

Consider now the situation where \( \varepsilon_0 < E_{\text{max}} \). In that case the measurements yielding the results \( 2\Omega < \varepsilon_0 \) should certainly be discarded since the range of the variation of the argument of function \( \psi \) in Eq. (19) does not cover the support of function \( \psi \). However, if the measurement gave the result \( 2\Omega > \varepsilon_0 \) then, similar to the case 1b, the teleportation is still possible (using the unitary transformation (51), if \([E_{\text{min}}, E_{\text{max}}]\subset [\gamma, 2\Omega], \) i.e. if the conditions \( \gamma < E_{\text{min}} \) (i.e. \( 2\Omega < \varepsilon_0 + E_{\text{min}} \)) and \( E_{\text{max}} < 2\Omega \) are simultaneously satisfied (the latter inequality now imposes an additional constraint rather than being satisfied automatically). The existence of an interval of the values of \( \Omega \) where the inequalities \( 2\Omega < \varepsilon_0 + E_{\text{min}} \) and \( E_{\text{max}} < 2\Omega \) are simultaneously satisfied is only possible if the inequality \( E_{\text{max}} < E_{\text{min}} + \varepsilon_0 \) hold, or, in other
words if $\varepsilon_0 > E_{\text{max}} - E_{\text{min}}$. In that case again $Z_1 = [E_{\text{max}}, \varepsilon_0 + E_{\text{min}}]$. Thus in the proposed scheme the teleportation is possible if and only if the spectrum width of the EPR-pair exceeds the spectral width of the support of function $\psi$.

It should be noted the teleportation of a broadband single-photon wave packet was first considered in [10, 11]. Recently, the algorithm for the teleportation of a single-mode electromagnetic field based on the squeezed states was extended to the case of a broadband input state whose spectral density is restricted to the vicinity of the half-frequency of the pump field producing the indicated squeezed state. In contrast to the algorithm proposed in the present paper, the scheme of Ref.[12] employs the orthogonal measurements. Physically, the non-orthogonal measurement is naturally arising when considering the system states in the energy representation: just as the originally proposed teleportation scheme formulated in the position representation actually employs the simultaneous measurement of position and momentum, it is natural to suppose that a similar procedure can be implemented measuring the energy and the conjugated quantity, i.e. time. However, since in quantum mechanics the time observable is not associated with any self-adjoint operator, the resulting measurement turns out to be non-orthogonal (and, of course, the involved EPR-pair is entangled in energy rather than position).

It should also be noted that Ref.[3] addressed the teleportation of a quantum state described by dynamic variables $(x,p)$ (the unknown state in Ref.[3] corresponds to the single-mode photon state) for the case of a non-ideal EPR-pair (squeezed state). Non-ideality of the EPR-correlation reduces the accuracy (fidelity) of the teleportation. The example based on an orthogonal measurement shows that the singular EPR-states allows the achievement of an unconditional exact (fidelity = 1) teleportation. The word “unconditional” here means that any measurement outcome leads to an exact teleportation. In the case of the proposed non-orthogonal measurement the unconditional exact teleportation is impossible even with a singular EPR-pair since for some measurement outcomes there exist no unitary transformations recovering the exact copy of the initial input state in the channel; these outcomes should be discarded. All the left outcomes provide an exact teleportation.

The experiments on teleportation may involve the situation when actually realized instead of a theoretically unconditional measurement (leading to the exact teleportation for any outcome) is its certain approximation and the teleportation becomes conditional even if one assumes that the experiment uses an ideal EPR-pair. Formally, any measurement is described by an identity resolution; the experimental implementation of a particular identity resolution requires the selection of a suitable interaction between the quantum system and the measuring device reproducing the necessary space of all possible measurement outcomes an the probability density distribution on that space specified by the given identity resolution. Usually this a very difficult task even for the systems with the discrete degree of freedom (e.g. spin or polarization). Therefore, the surplus outcomes arise which should be discarded. For example, the non-orthogonal identity resolution can be realized through the coalescence of a pair of photons in a non-linear crystal (parametric up-conversion) and subsequent detection of the arising photon. However, because of the small non-linear perceptibility, a lot of idle outcomes (when the photodetector does not fire) occur which should be discarded.

6 Conclusions

In summary, we have analyzed the measurements used in quantum teleportation from the viewpoint of the general quantum mechanical theory of measurements. It is shown that the teleported state is completely determined by the identity resolution (positive operator-valued...
measure) in the system state space generated by the corresponding instrument (quantum operation describing the change of the system state caused by the act of measurement) rather than the instrument itself, so that it is not actually necessary to specify the instrument itself providing the most complete description of the measuring procedure allowed by general laws of quantum mechanics. An algorithm for the teleportation of the state of a quantum system with a continuous non-degenerate spectrum is proposed based on a non-orthogonal measurement. Similar to all other available protocols providing an exact teleportation, our protocol employs an ideal EPR-pair with the singular correlations which corresponds to an unnormalizable wave function$^1$. It should be noted that the question of the possibility of achieving an exact teleportation of continuous quantum variable with the physically realizable (normalized) states is still open; for example, no algorithms of exact quantum teleportation for continuous variable have yet been proposed with non-singular EPR-states.

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$^1$Strictly speaking, the correct analysis of such states should be based on the rigged Hilbert state approach; in our case the problem of the infinite norm of employed EPR-states is avoided since we are only interested in the relative probabilities of different events.