Bound states due to an accelerated mirror

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Abstract

In this Brief Report we discuss an effect of accelerated mirrors which remained hitherto unnoticed, the formation of a field condensate near its surface for massive fields. From the view point of an observer attached to the mirror, this effect is rather natural because a gravitational field is felt there. The novelty here is that since the effect is not observer dependent even inertial observers will detect the formation of this condensate. We further show that this localization is in agreement with Bekenstein’s entropy bound.

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The study of moving mirrors has been highly valuable to the understanding of accelerating system physics and of quantum field theory on curved spacetimes [1,2]. The case of a two dimensional massless scalar field in the presence of a moving mirror has enlightened the process of particle creation due to gravitational fields, especially the Hawking radiation. There is a vast literature on the subject and we indicate [1,2] for references.

The case of massless fields is particularly convenient because of the facility to impose the boundary condition on the mirror. The general solution of the massless Klein-Gordon equation $\Box \phi = 0$ has the form $\phi = f(v) + g(u)$, with $v = t+x$, $u = t-x$, and arbitrary $f$ and $g$. The reflection condition is that $\phi$ vanishes on the mirror. Assuming the mirror trajectory to be expressed by $v = z(u)$, the solution takes the simple form $\phi = f(v) - f(z(u))$, valid for any trajectory. From this solution, one can read off the Doppler shift associated with the mirror trajectory, and by exploiting Bogoliubov transformations determine the particle content of the vacuum state in the presence of the mirror as seen from the usual Minkowski vacuum.

In the present work we investigate the behavior of a massive scalar field $\phi$ in the presence of a uniformly accelerated mirror in two dimensions. Our conclusions are rather surprising: we show that the mirror acceleration implies on the existence of bound states for $\phi$, hence inducing spatial localization. We further discuss that quantum states in the presence of the accelerated mirror differ in an essential way from the case of an inertial mirror. This can be hinted by because in the first situation the energy spectrum is discrete while in the second one it is continuous, precluding the existence of a Bogoliubov transformation relating the corresponding modes. From a more mathematical standpoint, this is a consequence of the fact that the former set of modes are squared integrable while the latter are not.

Let us consider a real massive scalar field with mass $m$ confined to the right region of a uniformly accelerated mirror. The mirror’s trajectory corresponds to the hyperbola $x^2 - t^2 = \text{const, } x > 0$. This situation is described by the solutions of the massive Klein-Gordon equation
\[(\Box + m^2) \phi = 0, \tag{1}\]

such that \(\phi = 0\) on the mirror trajectory. We introduce Rindler coordinates \(t = y \sinh a\eta\) and \(x = y \cosh a\eta\), covering the region \(x \geq |t|\) of the Minkowski spacetime. In this coordinate system, the trajectory of a uniformly accelerated mirror is given by the lines of constant \(y = y_0\). In particular, for \(y = a^{-1}\) the proper acceleration is equal to \(a\). Equation (1) becomes

\[
\left( \frac{1}{a^2} \frac{\partial^2}{\partial \eta^2} - \frac{y}{\partial y} \frac{\partial}{\partial y} + m^2 y^2 \right) \phi = 0. \tag{2}\]

Writing \(\phi = e^{i\omega \eta} f_\omega(y)\) one has the following equation for \(f_\omega\)

\[
f''_\omega + \left( \frac{\omega^2}{a^2 y^2} - m^2 \right) f_\omega = 0. \tag{3}\]

Equation (3) is the Bessel’s equation with imaginary order and argument. Its solutions have the form \[f_\omega = A I_{i\omega a}(my) + B K_{i\omega a}(my).\] Since \(I_{i\omega a}(my)\) goes as \(e^{my}\) for \(y \to \infty\) we set \(A = 0\) and \(B\) can be determined by normalization. Imposing the boundary condition \(K_{i\omega a}(my_0) = 0\) one gets an eigenvalue equation for \(\omega\). We assume hereafter that \(y_0 = a^{-1}\).

The following representation for the Bessel functions \[K_{i\omega a}(ma^{-1}) = \int_0^\infty e^{-\frac{m}{a} \cosh t} \cos \frac{\omega}{a} t \, dt. \tag{4}\]

is convenient for our purposes. We can determine \(\omega\) numerically from (4). We could do it with good accuracy to the first eigenvalues. Figure 1 shows the aspect of the first eigenvectors \(f_{\omega n}\). For \(\frac{\omega}{a} \gg 1\), one can integrate this expression in the saddle point approximation and obtain the asymptotic localization of the zeroes of \(K_{i\omega a}(ma^{-1})\). Calling

\[
f(t) = -\frac{m}{a} \cosh t + i\frac{\omega}{a} t \tag{5}\]

we obtain from the stationary phase condition

\[
e^{i\theta_0} = i\frac{\omega}{m} \pm i \sqrt{\left(\frac{\omega}{m}\right)^2 - 1}. \tag{6}\]
In the $\omega/m \gg 1$ limit there is one root at $t_0 \to -\infty$, which is far away from the integration domain and a second one located approximately at $e^{\omega_0} \approx 2i\omega/m$. Superimposing a branch cut at the negative real axis, we obtain at this approximation,

$$f(t_0) = i\frac{\omega}{a} \left( \log \frac{2\omega}{me} + i\frac{\pi}{2} \right)$$  \hspace{1cm} (7)

Therefore,

$$K_{i\omega a}(ma^{-1}) \approx \text{Re} \left( e^{-\frac{\omega a}{2e} e^{i\omega a \log \left( \frac{2\omega}{me} \right)} + \phi} \exp \left( -i\frac{\omega a}{2e} s^2 e^{2i\phi} \right) ds \right),$$  \hspace{1cm} (8)

where $se^{i\phi} = t - t_0$. The appropriate deformation of the contour integration yields for $\phi = -\pi/4$. Putting all these pieces together,

$$K_{i\omega a}(ma^{-1}) \approx \sqrt{\frac{2a\pi}{\omega}} e^{-\omega\pi/2a} \cos \left[ \frac{\omega a}{2e} \log \left( \frac{2\omega}{me} \right) - \frac{\pi}{4} \right].$$  \hspace{1cm} (9)

Defining $z = 2\omega/me$, we obtain the asymptotic expression for the field eigenvalues,

$$z \log z = \frac{a}{em} \left( 2n + \frac{1}{2} \right) \pi,$$  \hspace{1cm} (10)

where $z = \frac{2\omega}{em} \gg 1$.

Returning to the usual coordinates, we have finally that the solutions of the massive Klein-Gordon equation in the presence of a uniformly accelerated mirror with proper acceleration $a$ has the form

$$\phi(t, x) = \sum_{n=0}^{\infty} a_n e^{-i\frac{\omega_n}{a} \text{arctanh} \frac{1}{2} K_{i\omega a} \left( m\sqrt{x^2 - t^2} \right)} + \text{hermitean conjugated},$$  \hspace{1cm} (11)

where $\omega_n$ are the zeroes of $K_{i\omega a}(ma^{-1})$. Note that we have a discrete number of modes, in contrast to the case of a mirror at rest in $x = a^{-1}$, for which we have

$$\phi(t, x) = \int dk b_k e^{-i\sqrt{k^2 + m^2}t} \sin k(x - a^{-1}) + \text{hermitean conjugated},$$  \hspace{1cm} (12)

and the modes are continuous. As we already mentioned, the usual Bogoliubov transformation connecting (11) and (12) cannot be constructed.

Let us now consider two physically inequivalent situations. First, suppose that an uniformly accelerated mirror with proper acceleration $a$ ceases its motion at $t = 0$ and remains
at rest for \( x = a^{-1} \). For \( t < 0 \), the scalar field \( \phi \) is described by the superposition of modes given in eq. (11) and is totally determined by \( m \) and \( a \). As long as the mirror starts its inertial trajectory for \( t > 0 \) the field is described by another superposition of modes, namely, that one given by eq. (12). The coefficients \( b_k \) and \( a_n \) are related by a singular (non-invertible) Bogoliubov transformation. Such a transformation can be determined by equaling (11) and (12) and their first time derivatives for \( t = 0 \) and using that both sets of modes are orthonormal with respect to the usual time independent inner product of the Klein-Gordon equation. We have \( b_k = \alpha_{kn}a_n + \beta_{kn}a_n^\dagger \), where

\[
\alpha_{kn} = \frac{1}{2\pi} \int_0^\infty dy K_i\frac{\omega_n}{\sqrt{k^2 + m^2}} \sin ky \left( 1 + \frac{\omega_n}{\sqrt{k^2 + m^2}} \frac{1}{ay+1} \right),
\]

\[
\beta_{kn} = \frac{1}{2\pi} \int_0^\infty dy K_i\frac{\omega_n}{\sqrt{k^2 + m^2}} \sin ky \left( 1 - \frac{\omega_n}{\sqrt{k^2 + m^2}} \frac{1}{ay+1} \right). \tag{13}
\]

We have that an originally localized state at \( t = 0 \) starts a spreading phase and for \( t > 0 \) it becomes totally unlocalized.

The other relevant situation is that one for which the mirror originally at rest in \( x = a^{-1} \) starts a uniformly accelerated motion with proper acceleration \( a \) at \( t = 0 \). In this case, the scalar field is in a superposition of plane wave states for \( t < 0 \), and we know that for \( t > 0 \) it will be certainly localized. However, we cannot determine the coefficients \( a_n \) from \( b_k \) easily as in the previous case: the situation depends on the derivatives of the mirror acceleration at \( t = 0 \). Nevertheless, we know that for large \( t \), when the transient effects are no longer appreciable the field will relax to a superposition of modes given eq. (11). As the the mirror starts its motion the originally unlocalized scalar field will be dragged to near the mirror. Clearly, transients will be present which depend very much on the way the mirror is set into accelerated motion.

This consideration suggests a very attractive scenario: the eventuality of having Bose-condensation for a system which is not confined inside a box, in which confinement is produced quantum mechanically. Thus, let us consider a situation where prior to the acceleration of the mirror, the field is in a thermal state at some fixed temperature. As the
mirror accelerates, the state becomes localized in the vicinity of the mirror’s face. By the time transients get over and relaxation takes place the quantum state relaxes to a maximum entropy configuration, a thermal state localized in a region whose typical length scale is $m^{-1}$ (this scale arises from the exponential factor in the asymptotic expansion of the $K_{i\omega}^\nu$). As well know in condensed matter physics, the excited states has a finite storage capacity. The total number of quanta that can be accommodated in the system is

$$N = \sum_\omega \frac{1}{\tau^{-1} e^{\beta \omega} - 1}$$

where $\beta = \frac{1}{kT}$ and $\tau = \exp\left(\frac{\mu}{kT}\right)$, where $\mu = \mu(T)$ stands for the chemical potential and $T$ for the system’s temperature after relaxation.  In order to sum this series we have to recall our dispersion relation for the large eigenvalues. One can see that there is a no large integer $l$ such that

$$N \approx \frac{em}{2a} \int_l^\infty \frac{d(z \log z)}{\tau^{-1} e^{\gamma z} - 1} + \sum_{n=0}^l \frac{1}{\tau^{-1} e^{\beta \omega_n} - 1}$$

where $\gamma = \frac{em}{2kT}$. Calling $N_l$ the second term in the previous equation, we obtain the maximum storage capacity in the high excited states

$$N_e \equiv N - N_l = \frac{em}{2a} \int_l^\infty \frac{\tau d(z \log z)}{e^{\gamma z} - \tau}.$$  

Recalling that $0 \leq \tau \leq 1$, the r.h.s. of this equation is clearly a monotonic function of $\tau$, and thus

$$N_e \leq \frac{em}{2a} \int_l^\infty \frac{d(z \log z)}{e^{\gamma z} - 1}.$$  

This shows the finite storage capacity in the excited states. For $N \geq N_e$, boson condensation takes place and a thin film is formed on the face of the accelerated mirror.

In our model, the scalar field is selfconfined into a region of typical length $R = m^{-1}$ near to the mirror surface. We could ask about a possible violation of the Bekenstein’s bound on the entropy of localized systems

$$S \leq 2\pi \frac{ER}{\hbar}.$$  

(18)
where $S, E$ and $R$ are the system’s entropy, energy and largest linear dimension, owing to the large density of states of our system in the high frequency region. In our case, $E$ and $R$ are measured in the accelerated frame. A heuristic argumentation suggests that one does not have a violation of (18). Consider a massive scalar field confined into a box of size $R$ under a given acceleration. The spectrum of high frequency modes (frequencies much bigger than both $R^{-1}$ and $m^{-1}$) should be very close to the spectrum of a massless field in similar conditions. But for the massless case, one should not expect any violation of Bekenstein’s bound since one has thermal photons in an external field. We will show below that this is indeed the case.

An upper bound on $N^*$, the number of possible configurations with a given energy, is provided by [5,6]

$$N^*(E) = \sum_{i=0}^{\infty} g_i N^*(E - \omega_i),$$

where $g_i$ is $i$-th’s level degeneracy ($\omega_i$ is the corresponding frequency). With the ansatz $N^*(E) = e^{\frac{E}{\Gamma}}$, $\Gamma > 0$,

$$S \leq \frac{\Gamma E}{\hbar},$$

with $\Gamma$ given by the solution of the equation

$$\sum_{j=0}^{\infty} g_j e^{-\frac{\omega_j}{\Gamma \hbar}} = 1. \quad (21)$$

Because

$$z^2 \geq z \log z = \frac{a}{em} \left(2n - \frac{1}{2}\right) \pi \quad (22)$$

and $g_i = 1$ for our case, an estimative for an upper bound on $\Gamma$ follows from

$$\sum_{j=0}^{\infty} \exp \left(-\frac{\Gamma}{\hbar} \sqrt{\frac{ema}{2} \left(2j - \frac{1}{2}\right) \pi}\right) = 1, \quad (23)$$

which in the continuous limit provides

$$\Gamma \approx (ema\pi)^{-1/2}\hbar. \quad (24)$$
Consequently,

\[ S \leq m^{-1} \left( \frac{m}{e\pi a} \right) \frac{E}{\hbar} \]  

(25)

Recalling that the typical linear dimension of the system is \( R \approx m^{-1} \), violations of the bound would occur at scales \( m \gg a \). Nevertheless, at this mass scale the asymptotic expansion obtained for the eigenvalues is no longer valid and the above expression is not valid anymore.

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FIG. 1. The first eigenvectors $f_{\omega_n}$. The corresponding eigenvalues are $\omega_0 \approx 2.96m$, $\omega_1 \approx 4.53m$, and $\omega_2 \approx 5.88m$ (assumed that $a = m$).
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