Practical optimal nonlinear filter with estimated noise probability density function

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Abstract: A theoretical nonlinear filtering method that can estimate a weak signal buried in strong non-Gaussian noise has been proposed in previous studies. This method is attractive because it maximizes the signal-to-noise ratio at the filter output. A mathematical expression for the probability density function (PDF) of the noise is necessary to determine the characteristics of the filter in this method. To enhance the usability of this filtering method, the present study proposes an estimation method for the PDF. Kernel density estimation is considered, and the design framework of two key parameters, the kernel function and bandwidth, is introduced. Employing the Epanechnikov kernel reduces the computational complexity, and the proposed bandwidth achieves a filtering performance close to the theoretical limit. A well-known optimal bandwidth, which minimizes the estimation error, is expected to achieve the best filtering performance, but our proposed method improves upon this performance. In a numerical evaluation, several typical examples of noise types are considered, and the effectiveness of the proposed method is confirmed.

Key Words: kernel density estimation, bandwidth, non-Gaussian noise, nonlinear filter, stochastic resonance

1. Introduction
The estimation of a weak signal buried in strong noise has been thoroughly discussed in the signal processing field. Gaussian noise is often considered, and one of the major methods of eliminating such noise is linear filtering; in particular, Wiener filtering is effective in terms of minimizing the mean squared error between the original and estimated signals [1, 2]. Nonlinear filtering is also an appealing method because its estimates are accurate even in situations with non-Gaussian and/or multiplicative noise [3]. Especially, a nonlinear method inspired by a nonlinear phenomena of stochastic resonance (SR) is attractive because it maximizes the signal-to-noise ratio (SNR) at the filter output [4–6]. Another advantage of the filter is that the output is obtained by calculating the simple filter function; for a given input signal \( x \), which contains the weak signal and the noise, the output is

\[
F_{opt}(x) = a - b \left\{ \log(\rho(n)) \right\}'_{n=x},
\]

(1)
where $\rho(n)$ is the PDF for the noise, the first derivative $\frac{\partial}{\partial n}$ is denoted by $'$, $a$ is a constant, and $b$ is a non-zero constant. Surprisingly, this filter has a strong relation to locally optimum detector: the filter function is similar to the test statistic in the detector [7, 8]. Owing to the statistic, the locally optimum detector realizes reliable detection for the weak signals.

Because Eq. (1) involves the derivative of the noise PDF, an estimated mathematical expression for the PDF is necessary. In the original work, the PDF was assumed to be known [4–6]. In practical situations, however, the noise characteristics depend on the surrounding environment and hence should be estimated from noise samples before filtering the noisy input. The aim of the present study is to build a practical optimum nonlinear filter in which the noise PDF is estimated according to the environment.

Though many nonparametric methods of PDF estimation have been discussed, a novel method must be developed for practical filtering use. One of the major conventional methods is based on histograms. This method is quite simple and can be easily implemented but has a critical issue in that the estimated PDF is not differentiable. The characteristic kernel method, which was recently developed by A. Gretton et al., is of great interest because its estimation performance is outstanding [9–11]. However, this method does not yield the mathematical expression of the PDF, which is useful in the discussion focusing on the independence of and consistency between the estimated PDFs.

Another option is employing the kernel density estimation (KDE) method [12, 13]. Using this method, the PDF is analytically estimated based on noise samples; the result takes the simple form of a summation of the kernel function. Of course, this is a nonparametric method, and thus it is effective in any type of noise. The problem with this method is the choice of the kernel function. Because the practical filter considered here has the two functions of estimating the PDF and filtering the noise, the computational cost tends to increase. In the KDE method, the cost depends primarily on the kernel, and the failure of the chosen kernel increases the cost. To obtain an accurate estimate, there is another problem: it is necessary to tune the key parameter of bandwidth. The optimal bandwidth has been analytically derived, but its value cannot be practically calculated [14]. Sub-optimal methods have also been proposed (e.g., the plug-in method), but in many cases, the Gaussian kernel is assumed and/or additional computational costs are necessary [15, 16].

The present study proposes a practical nonlinear filter based on the KDE method. The Epanechnikov kernel is employed because the estimated PDF is differentiable, and the computational cost is thus reduced to half that of other kernels. Regarding the bandwidth, a novel criterion is analytically introduced: the key idea is that the desired bandwidth should maximize the output SNR. This is in contrast to the existing optimal bandwidth, which minimizes the estimation error. Unfortunately, maximizing the SNR is impossible because the SNR cannot be directly calculated. In the present study, the relation between the SNR and the in-out correlation is analytically derived, and the bandwidth is instead obtained by maximizing the correlation. It is worth mentioning that even if the existing optimal bandwidth is calculated, it does not give the maximum SNR. Conversely, of course, our proposed bandwidth does achieve the maximum SNR. This is a nontrivial result because it is generally expected that minimizing the estimation error improves the SNR, but in the focused filter, this statement is not valid. The numerical evaluation, which considers several noise scenarios, enhances the effectiveness of the proposed method.

### 2. Proposed filtering system with PDF estimation

A schematic of the proposed method is shown in Fig. 1. The nonlinear filter has an input of $x_i$, which contains a weak signal $s_i$ and white noise $n_i$. The subscript $i$ represents the time index. The filter extracts the weak signal component from the noisy input and outputs the signal $y_i$. The filter function $\hat{F}_{opt}(x)$ is calculated based on the estimated noise PDF $\hat{\rho}(n)$. For simplicity, it is assumed that a large number of noise samples $\bar{n}_j$ are measured in advance, and $n_i$ and $\bar{n}_j$ have the same PDF $\rho(n)$. In addition, to consider the situation in which a weak signal is buried in noise, the noise intensity is large compared to the signal power $P_s$.

The KDE method estimates the PDF by using a kernel function $K(u)$, as in [12]

\[
\hat{\rho}(n) = \frac{1}{n} \sum_{j=1}^{n} K\left( \frac{n - \bar{n}_j}{\sigma} \right)
\]

where $n$ is the number of samples, $\bar{n}_j$ is the $j$th sample, and $\sigma$ is the bandwidth. The optimal bandwidth $\hat{\sigma}$ is determined by maximizing the correlation between the input and output signals.

The effectiveness of the proposed method is demonstrated through numerical simulations, which show that the performance is comparable to that of the locally optimum detector. The proposed method is thus a practical solution for real-world applications.
\[ \hat{\rho}(n) = \frac{1}{Nh} \sum_{j=1}^{N} K\left( \frac{n - \tilde{n}_j}{h} \right), \]  

where \( N \) is the number of the noise samples \( \tilde{n}_j \) and \( h \) is the bandwidth. Substituting Eq. (1) into Eq. (2), we have

\[ \hat{F}_{\text{opt}}(x_i) = a - b \left( \frac{1}{N} \sum_{j=1}^{N} K\left( \frac{n - \tilde{n}_j}{h} \right) \right). \]  

This expression indicates that to calculate the function, a computation of order \( 2N \), which is denoted by \( O(2N) \) is required. Each of the summations in Eq. (3) requires \( N \) addition operations.

One of the contributions of the present study is a method of employing the Epanechnikov kernel, which is expressed as

\[ K(u) = \frac{3}{4} (1 - u^2) U(|u| \leq 1) \]  

and \( \alpha \) is a constant. From Eq. (5), the kernel is discontinuous at the point \( |u| = \alpha \). To obtain a differentiable kernel, a small parameter \( \epsilon \) is introduced. In the region \( \tilde{n}_j - h + \epsilon \leq x_i \leq \tilde{n}_j + h - \epsilon \), the kernel function \( K\left( \frac{n - \tilde{n}_j}{h} \right)_{n=x_i} \) is continuous and differentiable. Substituting Eq. (5) into Eq. (3) gives the proposed function,

\[ \hat{F}_{\text{opt}}(x_i) = a + b \frac{1}{h} \left( \frac{1}{1 - x_i - \mu x_i} - \frac{1}{1 + x_i - \mu x_i} \right). \]  

The variable \( \mu x_i \) denotes the averaged value of the noise samples \( \tilde{n}_j \) included in the region \( \tilde{n}_j - h + \epsilon \leq x_i \leq \tilde{n}_j + h - \epsilon \).

Because other kernels, such as Gaussians, are differentiable, the merit of Eq. (6) may not be immediately apparent. However, one of the advantages of using the Epanechnikov kernel is that the computational complexity can be reduced by a factor of two. As previously mentioned, KDE-based functions generally require \( O(2N) \) computations. For example, in the case of a Gaussian kernel, the derivative is an exponential function, and thus Eq. (3) involves two summations of exponential functions. Each requires \( O(N) \) computations, and unlike Eq. (6), the resulting function does not have a simple form. The function in Eq. (6) is derived using a second-order kernel, and the filter output can be obtained simply by calculating the average of the noise samples \( \mu x_i \). Such an operation requires \( O(N) \) computations, which means that the computational complexity of the proposed method is half that with other kernels.
3. Optimization of the bandwidth

The estimation performance of the KDE method depends on the bandwidth. The function in Eq. (6) has an implicit bandwidth dependence indicated by the presence of $\mu_x$. In this section, a method of setting the optimal bandwidth to achieve a filtering performance close to the theoretical performance is proposed.

The optimal bandwidth $h_{opt}$ has been derived by minimizing the asymptotic mean squared error (AMSE) [12]:

$$h_{opt} = \arg\min_h \left[ E \left[ \int_{-\infty}^{+\infty} \{\hat{\rho}(n) - \rho(n)\}^2 \, dn \right] \right] = \left( \frac{R(k)}{\beta_2(k)R(\mu^2)} \right)^{\frac{1}{2}} N^{-\frac{1}{2}}, \quad (7)$$

where $R(z) = \int z^2(v) dv$ and $\beta_2(k)$ is the second moment of the kernel function. In practice, the value of $h_{opt}$ cannot be calculated because the original PDF $\rho(n)$ must be known.

A method optimized for the function in Eq. (1) is proposed in this study. Considering the fact that the optimum function $F_{opt}(x)$ yields the maximum output SNR, the proposed bandwidth $\hat{h}_{opt}$ can be derived using the following criterion:

$$\hat{h}_{opt} = \arg \max_h \left[ \gamma \right], \quad (8)$$

where $\gamma = \frac{(F_{opt}(x))^2}{(F_{opt}(x) - \langle F_{opt}(x) \rangle)^2}$ denotes the normalized output SNR and $\langle z \rangle = \frac{1}{N} \sum_{i=1}^{N} z_i$ [5].

Equation (8) is unfortunately not solvable because it is impossible to take the derivative of the estimated function $\hat{F}_{opt}(x)$ (more precisely, the variable $\mu_x$ is not differentiable). An alternative method is now introduced using the relation between the output SNR and the in-out correlation. The correlation $C$ between the weak signal and the filter output is defined as

$$C = \frac{\frac{1}{N} \sum_{i=1}^{N}(s_i - \langle s_i \rangle)(y_i - \langle y_i \rangle)}{\sqrt{\frac{1}{N} \sum_{i=1}^{N}(s_i - \langle s_i \rangle)^2} \sqrt{\frac{1}{N} \sum_{i=1}^{N}(y_i - \langle y_i \rangle)^2}}. \quad (9)$$

Again, the signal $s_i$ is sufficiently small compared to the noise $n_i$ in this work. Under this condition, the filter output can be rewritten as a Taylor expansion:

$$y_i = \hat{F}_{opt}(s_i + n_i) \approx \hat{F}_{opt}(n_i) + s_i \hat{F}_{opt}(n_i)', \quad (10)$$

Because the input signal and the noise are uncorrelated, $\langle y_i \rangle = \langle \hat{F}_{opt}(n_i) \rangle + \langle s_i \rangle \langle \hat{F}_{opt}(n_i)' \rangle$ and $\langle y_i^2 \rangle = \langle \hat{F}_{opt}^2(n_i) \rangle + 2 \langle s_i \rangle \langle \hat{F}_{opt}(n_i) \rangle \langle \hat{F}_{opt}(n_i)' \rangle + \langle s_i^2 \rangle \langle \hat{F}_{opt}(n_i)' \rangle$. The value of the bias term $\langle s_i \rangle$ does not affect the SNR because the information in signals is defined with respect to the bias. In this sense, $\langle s_i \rangle = 0$ is assumed. Substituting these results into Eq. (9), an alternative expression of the correlation is now obtained as

$$C = \left[ \frac{(\hat{F}_{opt}(n_i))^2}{(\hat{F}_{opt}(n_i)')^2 + \frac{1}{P_s}} \right]^{\frac{1}{2}}. \quad (11)$$

According to the law of large numbers, if the number of noise samples is sufficiently large, the term $(\hat{F}_{opt}(n_i))^2$ equals unity. This is the case in which the KDE method is effective because it is well known that the estimation error is reduced when there is a large number of samples [12]. In this case, the SNR is simply expressed as $\gamma = \frac{P_s}{\beta_2}$. Substituting this $\gamma$ into Eq. (8), the proposed bandwidth is now obtained as

$$\hat{h}_{opt} = \arg \max_h \left[ \frac{P_s}{\beta_2} \right] = \arg \max_h \left[ C^2 \right].$$

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signal is considered. Let the frequency of the sinusoidal signal be $\omega$. The filter function is estimated based on the proposed bandwidth $h_{\text{opt}}$. It is confirmed that our previously proposed method, given in Eq. (1), improves the SNR. Even when the power of the weak signal is larger than that with $(\alpha, \beta) = (10.0, 1.0)$, it is observed that the filtering performance in the proposed method is approximately the same as that in the ideal case. Comparing the output SNR for the pulse signal with that for the sinusoidal signal, it is confirmed that the filtering performance in the proposed method does not depend on the shape of the weak signal. It is worth mentioning that a large difference is observed between the case using the original PDF and that using the estimated PDF with the well-known optimal bandwidth $h_{\text{opt}}$. Although the estimation with $h_{\text{opt}}$ minimizes the AMSE, i.e., the estimated PDF is close to the original, the resulting SNR is degraded compared to both the ideal

**Table I.** Parameter settings related to noise.

| Parameter          | Model / value |
|--------------------|---------------|
| Mixed Gaussian     | $\rho(n) = w\mathcal{N}(n; \nu_1, \sigma_1^2) + (1-w)\mathcal{N}(n; \nu_2, \sigma_2^2)$ |
| Mean               | $\nu_1 = 2.0, \nu_2 = -1.0$ |
| Variance           | $\sigma_1^2 = 2.0, \sigma_2^2 = 1.0$ |
| Weight parameter   | $w = 0.5$ |
| Generalized Gaussian | $\rho(n) = \beta \{2\alpha \Gamma(1/\beta)\}^{-1} \exp(-|n-\mu|/\alpha)^\beta$ |
| Mean               | $\mu = 1.0$ |
| (Scale, Shape)     | $(\alpha, \beta) = (10.0, 1.0), (10.0, 3.0)$ |
| Num. of noise samples | $N = 30000$ |

$$
= \arg \max_h \left\{ \frac{1}{N} \sum_{i=1}^{N} (\hat{s}_i - \langle \hat{s}_i \rangle)(y_i - \langle y_i \rangle) \right\}^2
$$

(12)

Second equation is valid since the signal power of the received weak signal $P_s$ does not depend on the bandwidth $h$. The correlation $C$ can be calculated using a dummy signal $\tilde{s}_i$, which is introduced to the filter as a substitution for the original weak signal $s_i$. This is a reasonable result because as shown in Eq. (1), the filter function does not depend on the shape of the weak signal. The merit of using the dummy signal is twofold; the correlation can be calculated without the knowledge of the shape of the original weak signal, and the estimated filter function (i.e., the filtering performance) is guaranteed not to depend on the shape.

4. Numerical examples and discussion

The performance of the proposed method is numerically evaluated and discussed in this section. To demonstrate the effectiveness of the proposed method in various applications and/or situations, two types of non-Gaussian noise, mixed Gaussian [17–19] and generalized Gaussian [20–22], are considered. The former noise is often observed in communication systems and image processing, whereas the latter is in Acoustic signal processing in addition to communication. The details of the noise are given in Table I. Note that $\mathcal{N}(n; \nu, \sigma^2)$ represents a Gaussian PDF of mean $\nu$ and variance $\sigma^2$, and $\Gamma(\cdot)$ denotes the gamma function. In the numerical simulation, the noise samples $\tilde{n}_i$ are generated, and the noisy input samples $x_i$ are then filtered to obtain the output. The values of $a$ and $b$ are set as 0 and 1, respectively.

This work focuses on two types of unbiased signals for the weak input signal: periodical sinusoidal and single pulsed signals. The peak-to-peak amplitudes of both signal types are set to be $A = 0.20$ to consider the situation in which the weak signal is buried in the noise. The frequency of the sinusoidal signal is $f_c = 100$ kHz, and the time interval of the pulse is equivalent to the inverse of this frequency, i.e., 10 $\mu$s. The duty ratio is 0.20.

Figure 2 describes four types of SNR results: at the input of the filter and at the output based on the original PDF $\rho(n)$, the estimated $\hat{\rho}(n)$ with the optimum bandwidth $h_{\text{opt}}$, and the estimated $\hat{\rho}(n)$ with the proposed bandwidth $\hat{h}_{\text{opt}}$. The Generalized Gaussian noise with $(\alpha, \beta) = (10.0, 1.0)$ has larger power than that with $(\alpha, \beta) = (10.0, 3.0)$. From the ideal case (using the original PDF), it is confirmed that our previously proposed method, given in Eq. (1), improves the SNR. Even when the filter function is estimated based on the proposed bandwidth $\hat{h}_{\text{opt}}$, the SNR is improved and is approximately the same as that in the ideal case. Comparing the output SNR for the pulse signal with that for the sinusoidal signal, it is confirmed that the filtering performance in the proposed method does not depend on the shape of the weak signal. It is worth mentioning that a large difference is observed between the case using the original PDF and that using the estimated PDF with the well-known optimal bandwidth $h_{\text{opt}}$. Although the estimation with $h_{\text{opt}}$ minimizes the AMSE, i.e., the estimated PDF is close to the original, the resulting SNR is degraded compared to both the ideal
Fig. 2. SNR performances in (a) mixed Gaussian noise, (b) generalized Gaussian noise with $(\alpha, \beta) = (1.0, 1.0)$, and (c) generalized Gaussian noise with $(\alpha, \beta) = (10.0, 3.0)$.

Table II shows the values of the bandwidth obtained by the proposed and the optimal methods. As shown in Table II, the proposed method yields a different value than the other methods. The reason is that the proposed method maximizes the SNR, whereas the other methods minimize the AMSE. As described in Eq. (7), using the optimal bandwidth ensures accurate estimates for all regions of the noise amplitude in the average sense. However, focusing on a specific region, another solution exists when the SNR is maximized. If a noise included in a specific region significantly affects the deterioration of the SNR, a method of improving the SNR by minimizing the error in this region should be applied. Figure 3 shows two examples of the estimated PDF result in addition to the original PDF. The AMSE for the optimal method (dotted line) in Fig. 3(a) is $5.8177 \times 10^{-5}$, which is smaller than that for the proposed method (solid line), $1.4252 \times 10^{-4}$. In this sense, the optimal bandwidth yields an accurate estimation. However, it is observed that near a noise amplitude of $-1.5$ V, our
Table II. Numerical example of the bandwidth.

| Noise                        | Proposed $h_{opt}$ | Optimum $h_{opt}$ |
|------------------------------|--------------------|-------------------|
| Mixed Gaussian               | 0.140              | 0.406             |
| Generalized Gaussian $(\alpha, \beta) = (10.0, 1.0)$ | 0.113              | 2.885             |
| Generalized Gaussian $(\alpha, \beta) = (10.0, 3.0)$ | 0.102              | 1.843             |

Fig. 3. Example of estimated noise PDF.

The number of the noise samples $N$ affects the estimation of the noise PDF. To evaluate this point, Fig. 4 describes the output SNR as a function of the number of the noise samples $N$. For the comparison, the output SNR with original PDF, i.e., in the case of ideal estimation, is also described. It is observed that as the number of the noise samples increases, the accuracy of the estimation is improved so that the output SNR is close to the one in the case with the original PDF. The amount of the improvement in the output SNR is decreased when $N \geq 10000$, which numerically indicates that the minimum required number of the noise samples should be $N = 10000$. This claim is confirmed in other cases including generalized Gaussian noise and pulsed weak signal. Note that as shown in Fig. 4, the above claim is valid only when the bandwidth is optimized for the given number of the noise samples with the proposed manner (Eq. (12)).

5. Conclusions

The present paper proposed a method of estimating the noise PDF and the corresponding in–out function of the nonlinear filter. The proposed method is based on the KDE method, which is a nonparametric estimation method, thus allowing it to be applied to any type of white noise. Due
to the use of a second-order Epanechnikov kernel, the computational complexity was reduced to half that for other kernels. The bandwidth for achieving the maximum output SNR was also analytically derived. Since such maximization is apparently impossible, an alternative method focusing on the in–out correlation was proposed. It has been made clear that the existing optimal bandwidth, which is given as Eq. (7), does not effectively maximize the SNR in the focused nonlinear filter. The effectiveness of the proposed method was confirmed by numerical evaluation, which implies that it is likely to function as a practical filtering system.

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