Towards a classification of natural bi-hamiltonian systems.

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Abstract

For construction and classification of the natural integrable systems we propose to use a criterion of separability in Darboux–Nijenhuis coordinates, which can be tested without an a priori explicit knowledge of these coordinates.

1 Introduction

The search of integrable dynamical systems is one of the most fascinating branch of classical physics. Integrable systems are quite rare and still only a few examples are known. In this paper we will present a new direct method for construction of natural integrable systems on $\omega N$ bi-hamiltonian manifolds.

A bi-hamiltonian manifold is a smooth manifold $M$ endowed with a pair of compatible Poisson tensors $P$ and $P'$ associated with the Poisson brackets $\{.,.\}$ and $\{.,.\}'$ respectively. The class of manifolds we will consider are particular bi-Hamiltonian manifolds where one of the two Poisson tensors in non degenerate and thus defines a symplectic form $\omega = P^{-1}$ and a recursion operator $N = P'P^{-1}$.

Dynamical integrable systems on $M$ with functionally independent integrals of motion $H_1, \ldots, H_n$ in the bi-involution

$$\{H_i, H_j\} = 0, \quad \{H_i, H_j\}' = 0, \quad \text{for all } i, j$$

(1.1)

will be called bi-hamiltonian systems for brevity.

A Hamiltonian system is natural if its Hamiltonian is a sum of a positive-definite kinetic energy and a potential. For instance, on the symplectic manifold $M \simeq \mathbb{R}^{2n}$, i.e. in the Euclidean space, the natural Hamiltonian is equal to

$$H = \sum_{i=1}^{n} p_i^2 + V(q).$$

(1.2)

In the known direct method to find natural integrable systems [10] we substitute natural Hamiltonian and some ansatz for other integrals of motion into the equations (1.1) and try to solve these equations.

According to [6, 7] integrals of motion $H_1, \ldots, H_n$ are in bi-involution (1.1) if they are separable in Darboux-Nijenhuis coordinates which are canonical with respect to $\omega$.
and diagonalize $N$. So, instead of classification of bi-hamiltonian systems we can classify Hamiltonians whose associated Hamilton-Jacobi equations can be solved by separation of variables in Darboux-Nijenhuis coordinates.

A test for separability of a given Hamiltonian $H$ in a given coordinate system was proposed by Livi-Civita in 1904 [15]. Using Darboux–Nijenhuis coordinates associated with $N$ the Livi-Civita criteria may be rewritten in the following form [1]

$$d(N^*dH)(D_H, D_H) = 0.$$  \hspace{1cm} (1.3)

Here $D_H$ is distribution generated iteratively by the action of $N$ on the vector field $X_H$.

The relevance of a new form of the Livi-Civita criteria is that condition (1.3) can be checked in any coordinate system. So, for a given Hamiltonian $H$ we can look at (1.3) as one of the equations to determine torsion free tensor $N$ and hence the separated variables. This scheme, in its basic features, has already been considered in the literature and applied to various systems, see [1, 6, 7, 12, 17, 18, 19, 23] and references within.

In this paper we will take a different logical standpoint. We propose to solve equation (1.3) with respect to a torsionless tensor $N$ and a natural Hamiltonian $H$ simultaneously. We will prove that it is indeed possible and, actually, easy to solve such an equation by means of a couple of natural Ansätze. As an example, we reproduce a lot of known natural integrable systems on the plane with cubic and quartic integrals of motion [10]. The corresponding Poisson pencils and tensors $N$ are new.

## 2 Bihamiltonian geometry and separation of variables

In this section we sketch the main points of the separation of variables theory for bi-hamiltonian manifolds, referring to [1, 6, 7, 12] for complete proofs and more detailed discussions.

### 2.1 $\omega N$ manifolds

The phase spaces of integrable systems will be identified with $\omega N$-manifolds [6, 7] or Poisson-Nijenhuis manifolds [14, 17] endowed with a symplectic form $\omega$ and a torsion free tensor $N$ satisfying certain compatibility conditions.

**Definition 1** An $\omega N$ manifold is a bi-hamiltonian manifold in which one of the Poisson tensor (say, $P$) is non degenerate, i.e. $\det P \neq 0$.

Therefore, $M$ is endowed with a symplectic form $\omega = P^{-1}$ and with a tensor field

$$N = P'P^{-1},$$  \hspace{1cm} (2.1)

which is recursion operator (formally). The operator $N$ is called a Nijenhuis tensor [17] or a hereditary operator [8] as well.

In order to construct operator $N$ by the rule (2.1) we have to get a pair of compatible Poisson tensors $P$ and $P'$.

**Definition 2** The Poisson tensors $P$ and $P'$ are compatible if every linear combination of them is still a Poisson tensor. The corresponding linear combination $P^\lambda = P + \lambda P'$, $\lambda \in \mathbb{R}$, is a Poisson pencil.
Recall a bivector $P'$ on $M$ is said to be a Poisson tensor if the Poisson bracket associated with it
\[
\{f(z), g(z)\}' = \sum_{i,k=1}^{2n} P'_{ik}(z) \frac{\partial f(z)}{\partial z_i} \frac{\partial g(z)}{\partial z_k}
\]
satisfies the Jacobi identity, that is, if for all $f, g$ and $h \in C^\infty(M)$
\[
\{f, \{g, h\}'\}' + \{g, \{h, f\}'\}' + \{h, \{f, g\}'\}' = 0.
\] (2.2)

Here $z = (z_1, \ldots, z_{2n})$ is a point of $M$ in some coordinate system.

From the Jacobi identity we can get a system of equations on the entries $P'_{ik}(z)$ of the tensor $P'$. So, in order to construct the Poisson tensor $P'$ we have to solve algebraic equations
\[
P'_{ij} = -P'_{ji}, \quad i, j = 1, \ldots, 2n,
\]
and partial differential equations
\[
\sum_{m=1}^{2n} \left( P'_{im} \frac{\partial P'_{jk}}{\partial z_m} + P'_{jm} \frac{\partial P'_{ki}}{\partial z_m} + P'_{km} \frac{\partial P'_{ij}}{\partial z_m} \right) = 0, \quad i, j, k = 1, \ldots, 2n,
\] (2.3)

with respect to some unknown functions $P'_{ik}(z)$.

In order to construct the Poisson pencil on $\omega N$ manifold associated with a given tensor $P$ we have to solve equations (2.3-2.4) simultaneously with the equations
\[
\sum_{m=1}^{2n} \left( P'_{im} \frac{\partial P'_{jk}}{\partial z_m} + P'_{jm} \frac{\partial P'_{ki}}{\partial z_m} + P'_{km} \frac{\partial P'_{ij}}{\partial z_m} \right) = 0, \quad i, j, k = 1, \ldots, 2n,
\] (2.4)

which follows from the Jacobi identity for the Poisson bracket $\{\ldots\}_\lambda = \{\ldots\} + \lambda\{\ldots\}'$, where $\lambda$ is an arbitrary numerical parameter.

For any solution $P'$ of the equations (2.3-2.4) there is free torsion tensor $N$ (2.1).

**Theorem 1** [7] The Nijenhuis torsion of $N$,
\[
T_N(X,Y) = [NX, NY] - N\left([NX, Y] + [X, NY] - N[X, Y]\right) = 0,
\] (2.5)

vanishes as a consequence of the compatibility between $P$ and $P'$.

In a given coordinate system on $M$ entries of the Nijenhuis torsion of $N$ are equal to
\[
T'_{jk}(N) = \sum_{m=1}^{2n} \left( \frac{\partial N'_i}{\partial z_m} N'_j - \frac{\partial N'_i}{\partial z_m} N'_k + \frac{\partial N'_j}{\partial z_m} N'_i - \frac{\partial N'_k}{\partial z_m} N'_i \right).
\]

In general there are infinitely many solutions of the system of equations (2.3-2.4) or (2.5).

In order to identify $2n$ separated variables with Darboux-Nijenhuis coordinates we have to consider a special class of $\omega N$ manifolds.

**Definition 3** A $2n$-dimensional $\omega N$ manifold $M$ is said to be semisimple if its recursion operator $N$ has, at every point, $n$ distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. It is called regular if eigenvalues of $N$ are functionally independent on $M$.

Of course, equations (2.3-2.4) also have infinitely many solutions on semisimple regular $\omega N$ manifolds. In order to get a bounded set of the solutions we have to fix form of the admissible Poisson tensors $P'$.
Example 1 Let us consider canonical Poisson tensor in the Euclidean space

$$ P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad I = \text{diag}(1, \ldots, 1) \quad (2.6) $$

associated with the usual Poisson bracket for arbitrary functions $f, g \in C^\infty(M)$

$$ \{f, g\} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) . \quad (2.7) $$

At $n = 2$ there are two nontrivial solutions of the system of equations (2.3-2.4)

$$ P' = \begin{pmatrix} 0 & 0 & 0 & q_1 \\ * & 0 & q_1 & 2q_2 \\ * & * & 0 & -p_1 \\ * & * & * & 0 \end{pmatrix} \quad \text{and} \quad P' = \begin{pmatrix} 0 & 0 & a_1 - q_1^2 & -q_1q_2 \\ * & 0 & -q_1q_2 & a_2 - q_2^2 \\ * & * & 0 & -q_1p_2 + q_2p_1 \\ * & * & * & 0 \end{pmatrix}, \quad (2.8) $$

which are matrix polynomials in variables $z = (p, q)$. The Nijenhuis torsion of the corresponding recursion operator $N$ (2.1) vanishes and its eigenvalues are distinct in the both cases.

### 2.2 The Darboux–Nijenhuis coordinates

According to [3] the recursion operator $N$ exists for overwhelming majority of integrable by Liouville systems. Therefore classification of the Poisson pencils has to coincide with the classification of integrable systems, which has a more rich history [10]. This connection is based on the separation of variables method in which separated variables for the Hamilton-Jacobi equation are identified with Darboux–Nijenhuis coordinates [16].

**Definition 4** A set of local coordinates $(x_i, y_i)$ on an $\omega N$ manifold is a set of Darboux–Nijenhuis coordinates if they are canonical with respect to the symplectic form

$$ \omega = P^{-1} = \sum_{i=1}^{n} dy_i \wedge dx_i , $$

and put the recursion operator $N$ in diagonal form

$$ N = \sum_{i=1}^{n} \lambda_i \left( \frac{\partial}{\partial x_i} \otimes dx_i + \frac{\partial}{\partial y_i} \otimes dy_i \right) , \quad (2.9) $$

This means that the only nonzero Poisson brackets are $\{x_i, y_j\} = \delta_{ij}$ and $\{x_i, y_j\}' = \lambda_i \delta_{ij}$. According to (2.9) differentials of the Darboux–Nijenhuis coordinates span an eigenspace of the adjoint operator $N^*$

$$ N^* dx_i = \lambda_i dx_i, \quad N^* dy_i = \lambda_i dy_i . \quad (2.10) $$

As a consequence of the vanishing of the Nijenhuis torsion of $N$, the eigenvalues $\lambda_i$ always satisfy

$$ N^* d\lambda_i = \lambda_i d\lambda_i . $$

It allows us to find very easy one special family of Darboux–Nijenhuis coordinates.
Theorem 2 \cite{6, 7} In a neighborhood of any point of a regular \(\omega N\) manifold where the eigenvalues of \(N\) are distinct it is possible to find by quadratures \(n\) functions \(\mu_1, \ldots, \mu_n\) that, along with the eigenvalues \(\lambda_1, \ldots, \lambda_n\), are Darboux–Nijenhuis coordinates.

In this case the coordinates \(\lambda_j\) can be computed algebraically as the roots of the minimal polynomial of \(N\)

\[
\Delta(\lambda) = \left(\det(N - \lambda I)\right)^{1/2} = \prod_{j=1}^{n} (\lambda - \lambda_j).
\]

On the contrary, the conjugated momenta \(\mu_j\) must be computed (in general) by a method involving quadratures \cite{6, 7}.

Example 2 For the first Poisson tensor in (2.8) the minimal characteristic polynomial of \(N\) is equal to

\[
\Delta(\lambda) = \lambda^2 - q_2\lambda - \frac{q_1}{4} = (\lambda - \lambda_1)(\lambda - \lambda_2).
\]

The corresponding special Darboux–Nijenhuis coordinates

\[
\lambda_{1,2} = \frac{1}{2} \left( q_2 \pm \sqrt{q_1^2 + q_2^2} \right), \quad \mu_{1,2} = p_2 - \frac{p_1}{q_1} \left( q_2 \pm \sqrt{q_1^2 + q_2^2} \right)
\]

coincide with the parabolic coordinates on the plane and their conjugated momenta.

For the second Poisson tensor in (2.8) the corresponding minimal polynomial

\[
\Delta(\lambda) = \lambda^2 + (q_1^2 + q_2^2 - a_1 - a_2)\lambda + a_1a_2 - a_2q_1^2 - a_1q_2^2 = (\lambda - \lambda_1)(\lambda - \lambda_2)
\]

is a generating function of the elliptic coordinates \(\lambda_{1,2}\) on the plane \cite{2}.

2.3 Integrable systems in the Jacobi method

The Darboux–Nijenhuis coordinates \((x_i, y_i)\) or \((\lambda_i, \mu_i)\) may be identified with the separated variables for a huge family of integrable systems. In order to construct these integrable systems in framework of the Jacobi method we can use the following theorem.

Theorem 3 Let \((\lambda_i, \mu_i)\) are canonical coordinates \(\{\lambda_i, \mu_j\} = \delta_{ij}\). The product of \(n\) one-dimensional Lagrangian submanifolds

\[
C_i : \Phi_i(\lambda_i, \mu_i, \alpha_1, \ldots, \alpha_n) = 0 \quad \text{with} \quad \det \left| \frac{\partial \Phi_i(\lambda_i, \mu_i)}{\partial \alpha_j} \right| \neq 0 (2.11)
\]

is an \(n\)-dimensional Lagrangian submanifold \(\mathcal{F}(\alpha) = C_1 \times \cdots \times C_n\). The solutions of the separated equations (2.11) with respect to \(n\) parameters \(\alpha_k\) are functionally independent integrals of motion \(H_k = \alpha_k(\lambda, \mu)\) in involution.

The main problem in the Jacobi approach is to choose separated equations \(\Phi_i = 0\) for which the Hamilton function \(H = H_1\) has a natural form (1.2) in the initial physical variables \((p, q)\).

We can reformulate this problem by using the separability condition (1.3) which implies that the distribution \(\mathcal{D}_H\) is integrable. So, there exist \(n\) independent local functions \(H_1, \ldots, H_n\) that are constant on the leaves of \(\mathcal{D}_H\).
Theorem 4 [6, 7] Let $M$ be a semisimple regular $\omega N$ manifold. The functions $H_1, \ldots, H_n$ on $M$ are separable in Darboux–Nijenhuis coordinates if and only if they are in bi-involution [17]. The distribution $\mathcal{D}$ tangent to the foliation defined by $H_1, \ldots, H_n$ is Lagrangian with respect to $\omega$ and invariant with respect to $N$.

The invariance of the Lagrangian distribution $\mathcal{D}$ means that there exists a matrix $F$ with eigenvalues $(\lambda_1, \ldots, \lambda_n)$ such that

$$N^*dH_j = \sum_{k=1}^n F_{jk}dH_k, \quad j = 1, \ldots, n. \quad (2.12)$$

Here $F$ is the control matrix with respect to integrals of motion $H_1, \ldots, H_n$, which form the Nijenhuis chain [6, 7].

Using this theorem we can formulate a new method to find natural integrable systems. Namely, substituting a torsionless tensor $N$, some control matrix $F$ and a natural Hamiltonian $H = H_1$ [12] into the $n$ relations (2.12), one gets a system of $n$ differential equations on the potential $V(q)$ and on the remaining integrals of motion $H_2, \ldots, H_n$. Solving these equations one gets a natural integrable system separable in Darboux–Nijenhuis coordinates. In order to avoid a lot of intermediate calculations we can solve equations (2.12) together with equations (2.13, 2.14).

The proposed construction inherits the main disadvantage of the Jacobi method because we have to choose control matrix $F$ for which equations (2.12) have non-trivial solutions.

Example 3 If $H_k = \frac{1}{2k}\text{tr}N^k$ then the control matrix in (2.12) is equal to

$$F = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \cdots & 0 & 1 \\ \sigma_n & \sigma_{n+1} & \cdots & \sigma_1 \end{pmatrix}. \quad (2.13)$$

Here $\sigma_k$ are the elementary symmetric polynomials of degree $k$ on the eigenvalues of $N$.

In this case relations (2.12) are the Lenard-Magri recurrence relations

$$N^*dH_i = dH_{i+1} \quad \text{or} \quad P'dH_i = PdH_{i+1}, \quad (2.14)$$

such that the corresponding vector fields $X_{H_i}$ are bi-hamiltonian [12, 17].

Remark 1 It is known that for finite-dimensional systems the bi-hamiltonian property of the vector fields is a very strong condition and too restrictive for construction of natural integrable systems [7, 18]. Of course, we could easy solve equations (2.12) with such matrix $F$ (2.13), but for the majority of the known Poisson pencils system of equations (2.12) or (2.14) is an inconsistent system if $H_1 = H$ is a natural Hamiltonian [12].

2.4 The uniform Stäckel systems

The equations (2.12) do not give explicit information on the form of the separated equations (2.11). In order to restrict a set of the possible separated equations $\Phi_i = 0$ we will consider the following family of the separated equations

$$\Phi_i(\mu_i, \lambda_i, \alpha_1, \ldots, \alpha_n) = \sum_{j=1}^n S_{ij}(\lambda_i) \alpha_j - s_i(\mu_i, \lambda_i) = 0, \quad H_i = \alpha_i, \quad (2.15)$$
where $S$ is a non-degenerate Stäckel matrix and $s$ is a Stäckel vector. The entries $S_{ij}$ and $s_i$ depend on a pair of the separated variables $\lambda_i$ and $\mu_i$ only.

**Theorem 5** [6, 7] Let $H_1, \ldots, H_n$ are independent integrals of motion in bi-involution (1.1) defining a bi-Lagrangian foliation on a regular semisimple $\omega N$ manifold. If the control matrix $F$ satisfies

$$N^*dF = FdF,$$

that is

$$N^*dF_{ij} = \sum_{k=1}^{n} F_{ik}dF_{kj}, \quad i, j = 1, \ldots, n, \quad (2.16)$$

then the functions $H_1, \ldots, H_n$ are Stäckel separable in the Darboux–Nijenhuis coordinates.

In this case the Stäckel matrix $S$ (2.15) is defined by

$$F = S^{-1}\Lambda S, \quad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n).$$

Construction of the corresponding Stäckel vector $s$ is not so algorithmic.

This theorem could be very useful for a construction of natural integrable systems of the Stäckel type. Namely, substituting a $(1, 1)$ tensor $N$ and a natural Hamiltonian $H = H_1$ (1.2) into the relations (2.5), (2.12) and (2.16) one gets the closed system of algebro-differential equations

$$T_N = 0, \quad N^*dH = FdH, \quad N^*dF = FdF, \quad (2.17)$$

on potential $V(q)$, integrals of motion $H_2(p, q), \ldots, H_n(p, q)$, entries $N_{ij}(p, q)$ of the recursion operator and entries $F_{ij}(p, q)$ of the control matrix.

Unfortunately, we can not solve equations (2.17) in general. More precisely, usually equations (2.17) have infinitely many solutions. The main reason is that we fix form of the Hamiltonian $H$ (1.2), but an admissible set of the Stäckel vectors $s$ in (2.15) remains undefined and, therefore, unbounded set. So, in order to solve equations (2.17) in practice we have to narrow a class of the separated equations (2.15) once more.

Below we will determine a class of the separated equations by using the Stäckel matrix. Recall the Stäckel matrix is a $n \times n$ block of the transpose Brill-Noether matrix, which is a differential of the Abel-Jacobi map associated with a product $\mathcal{F}(\alpha)$ of the algebraic curves $C_j$ (2.11), see [21] and references within. There is one to one correspondence between $j$-th rows of the Stäckel matrix $S$ and a basis of the differentials on the $j$-th curve $C_j$.

Let us consider uniform Stäckel systems [21] for which the Lagrangian submanifold $\mathcal{F}(\alpha)$ is the $n$-th symmetric product of a hyperelliptic curve $C$. For the standard basis of the holomorphic differentials on $C$ associated with the Veronese map the corresponding Stäckel matrix is equal to

$$S = \begin{pmatrix} \lambda_1^{n-1} & \ldots & \lambda_1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \lambda_n^{n-1} & \ldots & \lambda_n & 1 \end{pmatrix}, \quad \Leftrightarrow \quad \Phi_i = \sum_{k=0}^{n-1} \alpha_k\lambda_i^k = s_i(\mu_i, \lambda_i). \quad (2.18)$$

It is easy to prove that the control matrix

$$F = S^{-1}\Lambda S = \begin{pmatrix} \sigma_1 & 1 & 0 & \ldots & 0 \\ \sigma_2 & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \sigma_n & 0 & \ldots & 1 & 0 \end{pmatrix}, \quad (2.19)$$
fulfills (2.16) \[6, 7\]. Here \(\sigma_j\) are elementary symmetric polynomials on eigenvalues \(\lambda_j\) of tensor \(N\):

\[
\Delta(\lambda) = \lambda^n - \sigma_1\lambda^{n-1} - \sigma_2\lambda^{n-2} - \cdots - \sigma_n.
\]

(2.20)

Solving the separated equations (2.15) with respect to \(\alpha_1, \ldots, \alpha_n\) one gets the integrals of motion \(H_i\) as functions on the separated variables

\[
H_i = \sum_{j=1}^{n} \frac{\partial \sigma_j}{\partial \lambda_j} s_i(\mu_i, \lambda_i) \prod_{i \neq k}(\lambda_i - \lambda_k).
\]

(2.21)

The corresponding vector fields \(X_{H_i}\) are tangent to a bi-Lagrangian foliation, but they are not, in general, bi-Hamiltonian. These vector fields are Pfaffian quasi-bi-hamiltonian fields in the terminology of [18].

**Remark 2** The Stäckel matrix \(S\) (2.18) is one of the most studied matrices, which appears very often in various applications [2, 6, 7, 12, 21]. But we have to keep in mind that there are many other Stäckel matrices associated with hyperelliptic or non-hyperelliptic curves.

**Example 4** Let us consider natural integrable systems on the plane separable in the polar coordinates \((r, \phi, p_r, p_\phi)\). The corresponding second Poisson tensor \(P'\) is degenerate

\[
P' = \begin{pmatrix}
0 & 0 & r^2 & 0 \\
* & 0 & 0 & 0 \\
* & * & 0 & 0 \\
* & * & * & 0
\end{pmatrix}.
\]

(2.22)

Operator \(N\) (2.1) has two different eigenvalues \(\lambda_1 = r^2\) and \(\lambda_2 = 0\). Integrals of motion

\[
H_1 = p_r^2 + \frac{p_\phi^2}{r^2} + V(r), \quad H_2 = p_\phi
\]

(2.23)

give rise to the following control matrix \(F\) and the separated equations \(\Phi_{1,2}\)

\[
F = \begin{pmatrix}
r^2 & 0 \\
0 & 0
\end{pmatrix}, \quad \begin{cases}
\Phi_1 = \ p_r^2 + \frac{\alpha_2^2}{r^2} + V(r) - \alpha_1 = 0, \\
\Phi_2 = \ p_\phi - \alpha_2 = 0.
\end{cases}
\]

The corresponding Stäckel matrix \(S\) is non-homogeneous in contrast with (2.18). If \(V(r)\) is polynomial then the separated equations \(\Phi_1 = 0\) determines hyperelliptic curve, whereas curve associated with \(\Phi_2 = 0\) is non-hyperelliptic.

Summing up, the separation of variables theory for \(\omega N\) manifolds outlined above provides intrinsic and algorithmic recipe to construction of the natural integrable systems of the Stäckel type.

The algorithm consists of solution of the extended system of equations (2.12) and (2.14) with some fixed Stäckel matrix \(S\). These equations contain the following unknown functions \(P'_{ij}, V\) and \(H_2, \ldots, H_n\). Solving this overdetermined system of equations in physical variables we get a Poisson pencil, recursion operator, Darboux-Nijenhuis coordinates and the corresponding natural integrable system simultaneously.

**Remark 3** In the proposed algorithm we have to substitute some concrete Stäckel matrix \(S\) into the equations (2.12) and then to solve these equations. In general matrix \(S\) defines only first part of the separated equations (2.15) whereas the Stäckel vector \(s\) remains arbitrary. However if we use the Stäckel matrices (2.18) associated with the Abel-Jacoby map on the hyperelliptic curves, we implicitly fix the Stäckel vector \(s\) as well, because the separated equation has to generate namely hyperelliptic curve.
3 Bi-hamiltonian systems on the plane

In this section we substitute natural Hamiltonian $H$ and some polynomial anzats for the Poisson tensor $P'$ and describe nontrivial physical solution of these equations by $n = 2$.

**Definition 5** Solution of the equations is nontrivial, if the functions $H_i$ are functionally independent integrals of motion, $dH_i \neq 0$ for any $i$, and potential $V(q)$ is some real function, $V(q) \neq \text{const}$.

Moreover, throughout the rest of the section we consider only physical solutions for which all the integrals of motion are polynomials in momenta.

**Example 5** Substituting an independent on momenta tensor $P'$ and the following Hamiltonian $H_1 = p_1 p_2 + V(q_1, q_2)$ into (2.12) we can get the following solution

$$P' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & f(q_1) & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$$

$$H_1 = p_1 p_2 + \frac{q_2}{4C_1 q_1} + f(q_1),$$

$$H_2 = \sqrt{q_1} \exp(C_1 p_2^2).$$

It is example of the non physical solutions which will not considered below.

Of course, integrable systems with two-degrees of freedom are one of the well studied integrable systems and, therefore, we can only obtain some new tensors $N$ for known integrable systems listed in [10]. Nevertheless below we present two new integrable systems on the plane with pseudo-Euclidean metric as well.

### 3.1 Linear Poisson tensors and the L-systems

Let $Q$ be an $n$-dimensional Riemannian manifold endowed with a conformal Killing tensor $L$ of gradient type with vanishing Nijenhuis torsion [2]. According to [12] its cotangent bundle $M = T^*Q$ is an $\omega N$ manifold, whose recursion operator

$$N \frac{\partial}{\partial q^k} = \sum_{i=1}^n L^i_k \frac{\partial}{\partial q^i} + \sum_{ij} p_j \left( \frac{\partial L^i_j}{\partial q^k} - \frac{\partial L^j_i}{\partial q^k} \right) \frac{\partial}{\partial p_i}, \quad N \frac{\partial}{\partial p_k} = \sum_{i=1}^n L^i_k \frac{\partial}{\partial p_i}$$

(3.1)

is a complete lifting of operator $L$. In this case the Poisson tensor

$$P' = N \ P = \begin{pmatrix} 0 & -L^i_i \\ L^j_i & \left( \frac{\partial L^k_j}{\partial q_i} - \frac{\partial L^k_i}{\partial q_j} \right) p_k \end{pmatrix}$$

(3.2)

is a linear matrix polynomial in momenta.

Substituting a Poisson tensor $P'$ and a natural Hamiltonian $H_1 = T + V$ into the system of equations (2.12) and expanding it in powers of the momenta we can see that (2.17) splits into the equations on $L$ and on $V$

$$d(\mathcal{L}_{X_T} \theta' - T d \sigma_1) = 0, \quad d(\mathcal{L}_{X_V} \theta' - V d \sigma_1) = 0,$$

(3.3)

and the cyclic recurrence relations for other integrals of motion

$$N^* dH_i = dH_{i+1} + \sigma_i dH_1, \quad i = 2, \ldots, n,$$

(3.4)
which we have to solve starting with $i = n$ and $H_{n+1} = 0$, see e.g. [12, 23, 24].

Here $\mathcal{L}$ is a Lie derivative along the vector field $X_T$ or $X_V$ and $\theta' = \sum_{i,j=1}^{n} L^i_j p_i dq^j$ is a deformation of the Liouville form $\theta = \sum p_i dq^i$ for any set of fibered coordinates $(p, q)$.

In [23, 24] the computer program for finding of the $L$-tensors and the associated integrable systems was constructed. This software allows to solve equations (3.3) and (3.4) on the Riemannian manifolds with a positive-definite metric.

The eigenvalues on $N$ (3.1) coincide with the eigenvalues $\lambda_i(q_1, \ldots, q_n)$ of the operator $L$ defined on the configuration space $Q$. As a consequence, functions $\sigma_i$ depend on coordinates $q$ only and, according to (3.4), for the natural integrable systems with $H_1 = T + V$ (1.2) all the integrals of motion $H_2, \ldots, H_n$ are quadratic polynomials in the momenta $p_i$.

The associated with $N$ (3.1) Darboux-Nijenhuis coordinates are related with initial physical variables $(p, q)$ by the point canonical transformations. Indeed in this case Darboux-Nijenhuis coordinates coincide with well-studied orthogonal curvilinear coordinates on the Riemannian manifolds.

**Example 6** On the plane $Q = \mathbb{R}^2$ there are four $L$-tensors, which give rise to the natural integrable systems [2]. The corresponding Poisson tensors $P'$ (2.8) generate elliptic and parabolic coordinates, tensor $P'$ (2.22) gives rise to the polar coordinates and numerical skewsymmetric tensor generates cartesian web on the plane [2].

**Remark 4** On the plane $Q = \mathbb{R}^2$ there are many other solutions of the equations (2.3,2.4) in the form (3.2)

$$P' = \begin{pmatrix} A_p & B \\ -B & C_p \end{pmatrix},$$

where entries of the matrices $A, B, C$ depend on coordinates $q$. However if $H_1 = T + V$ is a natural Hamiltonian (1.2) there are only four nontrivial solutions of the extended system of equations (2.3,2.4,2.12) associated with the four known coordinate systems.

### 3.2 Linear Poisson tensors and systems with cubic integrals of motion

In this section we consider the following ansatz for the Poisson tensor

$$P' = \begin{pmatrix} 0 & h(q) & c_1 p_1 + c_2 p_2 & c_3 p_3 + c_4 p_4 \\ * & 0 & c_5 p_1 + c_6 p_2 & c_7 p_1 + c_8 p_2 \\ * & * & 0 & f(q) \\ * & * & * & 0 \end{pmatrix}, \quad c_k \in \mathbb{R}. \quad (3.6)$$

Here $c_k$ are undetermined coefficients and $f, h$ are some functions on coordinates $q$.

Substituting $P'$ (3.6) into the Jacobi equations (2.3,2.4) we can extract equations for the function $h$. Solving these equations it is easy to prove that either function $h(q)$ is a constant $h(q) = \text{const}$, or it is a linear function of coordinates $h(q) = aq_1 + bq_2$.

**Proposition 1** If the Poisson tensor $P'$ has the form (3.6) and integrals of motion

$$H_1 = p_1^2 + p_2^2 + V(q_1, q_2), \quad H_2 = H_2(p_1, p_2, q_1, q_2),$$

satisfy to the Lenard-Magri recurrence relations (2.14) then the extended system of equations (2.3,2.4,2.12) with control matrix $F$ (2.13) has only one nontrivial solution up to
canonical transformations

\[ P' = \begin{pmatrix} 0 & a & p_1 & 0 \\ * & 0 & 0 & p_2 \\ * & 0 & 0 & b e^{\frac{q_2 - q_1}{a}} \\ * & * & * & 0 \end{pmatrix}, \quad H_1 = p_1^2 + p_2^2 + 2a b e^{\frac{q_2 - q_1}{a}}. \quad (3.7) \]

The corresponding second integral of motion \( H_2 = (p_1 + p_2)H_1 - \frac{1}{3}(p_1 + p_2)^3 \) is formally a third order polynomial in the momenta.

This integrable system coincides with open Toda lattice of \( A_2 \) type.

Remark 5 From the Lenard-Magri relations (2.14-2.12) with control matrix \( F \) (2.13) it follows that

\[
\begin{align*}
N^* dH_1 &= dH_2, \\
N^* dH_2 &= \sigma_2 dH_1 + \sigma_1 dH_2, \\
N^*^2 dH_1 &= dH_2^{(2)}, \\
N^*^2 dH_2 &= \sigma_2^{(2)} dH_1 + \sigma_1^{(2)} dH_2^{(2)}, \\
&\vdots & \vdots
\end{align*}
\]

where \( dH_2^{(2)} = \sigma_2 dH_1 + \sigma_2 dH_2 \) and \( \sigma_i^{(2)} \) are symmetric polynomials associated with \( N^*^2 \). Similar relations (3.8) may be constructed only for the \( N^*^{2m} \).

In our case (3.7) \( H_2^{(2)} = (p_1 p_2 - ab \exp((q_2 - q_1)/a))^2 - \frac{1}{2} H_1^2 \) is a fourth order polynomial in the momenta. Below we will use deformations of the operator \( N^2 \) in order to construct open Toda lattices of \( BC_2 \) and \( D_2 \) type. To get the Toda lattice of \( G_2 \) type we can use deformations of the operator \( N^4 \).

Remark 6 The Poisson tensor \( P' \) (3.7) admits a natural multi-dimensional generalization

\[ P' = \sum_{i=1}^{n-1} 2e^{2(q_i - q_{i+1})} \frac{\partial}{\partial p_{i+1}} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i<j}^{n} \frac{\partial}{\partial q_j} \wedge \frac{\partial}{\partial q_i} \quad (3.9) \]

which was considered in [9]. The corresponding open Toda lattice of \( A_n \) type are one of the well-studied bi-hamiltonian systems.

Throughout the rest of the section we will use the matrix \( F \) (2.19) associated with the homogeneous Stäckel matrix \( S \) (2.18) of the Veronese type.

Proposition 2 If the Poisson tensor \( P' \) has the form (3.6) and

\[ H_1 = p_1^2 + p_2^2 + V(q_1, q_2), \quad H_2 = H_2(p_1, p_2, q_1, q_2), \]

then equations (2.3, 2.4, 2.12) with matrix \( F \) (2.19) has only four nontrivial solutions up
to canonical transformations

\begin{equation}
P' = \begin{pmatrix}
0 & 2 & p_2 & 3p_1 \\
* & 0 & p_1 & p_2 \\
* & * & 0 & be_{q_i} \\
* & * & * & 0
\end{pmatrix}
, \quad V = -\frac{2b}{3} e^{q_1} + C_1 e^{(-\frac{q_1}{\sqrt{3}})} + C_2 e^{(-\frac{\sqrt{3}q_2}{3})},
\end{equation}

\begin{equation}
P' = \begin{pmatrix}
0 & q_1 & -\frac{2}{3}p_2 & 0 \\
* & 0 & \frac{4}{3}p_1 & -\frac{2}{3}p_2 \\
* & * & 0 & bq_1 \\
* & * & * & 0
\end{pmatrix}
, \quad V = \frac{b(3q_1^2 + 4q_2^2) + C_1q_2 + C_2}{4q_1^{2/3}},
\end{equation}

\begin{equation}
P' = \begin{pmatrix}
0 & q_1 & 0 & 0 \\
* & 0 & p_1 & -2p_2 \\
* & * & 0 & bq_1 \\
* & * & * & 0
\end{pmatrix}
, \quad V = b\left(\frac{q_1^2}{4} + q_2^2\right) + C_1q_2 + \frac{C_2}{q_1^2},
\end{equation}

\begin{equation}
P' = \begin{pmatrix}
0 & q_1 & 0 & 4p_1 \\
* & 0 & p_1 & 2p_2 \\
* & * & 0 & \frac{b}{q_1^3} \\
* & * & * & 0
\end{pmatrix}
, \quad V = \frac{b}{4q_1^2} + C_1q_2^2 + C_2q_2 - C_3q_1^2.
\end{equation}

Here $C_j$ are arbitrary constants, which appear by integration of differential equations.

The corresponding integrals of motion are the third order polynomials in the momenta $p_i$ which may be found in [10]. The first solution is a periodic Toda lattice of $A_3$ type in the center of mass system [10]. The second solution is a Holt system [10, 11]. Two remaining solutions may be considered as integrable systems of Calogero type.

In fact we proved that all these systems are the uniform Stäckel systems with the homogeneous Stäckel matrix $S$ (2.18). It is easy to check that the corresponding separated equations give rise to the hyperelliptic curves.

Remark 7 There are many other solutions of the equations (2.3,2.4) if we suppose that coefficients $c_k$ in $P'$ (3.6) are some functions of coordinates $q$. However if $H_1 = T + V$ is a natural Hamiltonian $[12]$ there are only four nontrivial solutions of the extended system of equations (2.3,2.4,2.12).

We fix the form of the desired Poisson tensor $P'$ (3.6) and, therefore, in the equations (2.12) we could use the natural Hamilton function defined up to canonical transforms

\begin{equation}
H = \tilde{T} + V(q), \quad \tilde{T} = \sum_{i,j=1}^{n} b_{ij}(q)p_i p_j + \sum_{k=1}^{n} b_k(q)p_k.
\end{equation}

Here $b_{ij}(q)$ and $b_k(q)$ are functions of coordinates $q$, such that kinetic energy $\tilde{T}$ may be reduced to the canonical form $[12]$ by point transformations $q \to q' = Aq + b$ and canonical shifts of the momenta $p_i \to p_i + b_i(q_i)$.

As an example we consider natural Hamiltonians with pseudo-Euclidean metric $[5]$ and reproduce one Drach system which is not superintegrable Stäckel system with quadratic integrals of motion $[22]$. Only for this Drach system the separated variables was unknown.
Proposition 3 If the Poisson tensor $P'$ has the form (3.10) and $H_1 = p_1 p_2 + V(q_1, q_2)$ then one of the solutions of the equations (2.3, 2.4, 2.12) reads

$$P' = \begin{pmatrix} 0 & q_1 & 0 & 0 \\ * & 0 & p_1 + a p_2 & - \frac{p_2}{\sqrt{b}} \\ * & * & 0 & \frac{b}{\sqrt{b}} \\ * & * & * & 0 \end{pmatrix}, \quad H_1 = p_1 p_2 - 2b \frac{2aq_1 - q_2}{\sqrt{q_1}} + \frac{C_1}{\sqrt{q_1}} + C_2 \left( q_2 - \frac{2a}{3} q_1 \right).$$

The second integral of motion $H_2$ is a third order polynomial in momenta. The corresponding separated variables are the special Darboux-Nijenhuis variables

$$\lambda_{1,2} = -\frac{p_2}{4} \pm \sqrt{\frac{p_2^2 + 16 b \sqrt{q_1}}{4}}$$

and the separated equations give rise to hyperelliptic curve.

Remark 8 In this section we identify the first integral $H_1$ from the Nijenhuis chain (2.12) with the natural Hamiltonian $H = T + V$. Of course, we can choose any integral $H_k$ from the chain as the natural Hamilton function.

### 3.3 The Poisson tensors second order in the momenta

In this section we consider the following ansatz for the Poisson tensor

$$P' = \begin{pmatrix} 0 & h_1(q)p_1 + h_2(q)p_2 & \sum c_{ij}^1 p_i p_j + g_1(q) & \sum c_{ij}^2 p_i p_j + g_2(q) \\ * & 0 & \sum c_{ij}^3 p_i p_j + g_3(q) & \sum c_{ij}^4 p_i p_j + g_4(q) \\ * & * & 0 & f_1(q)p_1 + f_2(q)p_2 \\ * & * & * & 0 \end{pmatrix}, \quad c_k \in \mathbb{R}.$$  \hspace{1cm} (3.11)

Here $f_i, g_i, h_i$ are some functions on coordinates $q$ and $c_{ij}^k$ are undetermined numerical coefficients.

As above there are solutions with potentials $V$ separable in physical coordinates $q_{1,2}$, $\frac{\partial^2 V}{\partial q_1 \partial q_2} = 0$, and solutions with formally fourth order integrals of motion for which $H_2$ consists of $H_1$ and another independent quadratic polynomial $\widetilde{H}_2$, for instance $H_2 = \alpha H_1^2 + \widetilde{H}_2$ or $H_2 = H_1 \widetilde{H}_2$. In contrast with the previous section we will not present such solutions below.

Example 7 By using tensor $P'$ (3.11) we can obtain integrable systems with quadratic integrals of motion. For instance one of the solutions of the equations (2.3, 2.4, 2.12) with the control matrix $F$ (2.19) is equal to

$$H_1 = p_1^2 + p_2^2 + \frac{C_3}{2} \left( q_1^2 + q_2^2 \right) - \frac{C_1 - C_2}{2(q_1 + q_2)^2} - \frac{C_1 + C_2}{2(q_1 - q_2)^2}$$

and

$$P' = \begin{pmatrix} 0 & p_1 q_2 - p_2 q_1 & -p_1^2 + g_1(q) & -p_2 p_1 + g_2(q) \\ * & 0 & -p_2 p_1 + g_3(q) & -p_2^2 + g_4(q) \\ * & * & 0 & p_1 f_1(q) + p_2 f_2(q) \\ * & * & * & 0 \end{pmatrix}$$  \hspace{1cm} (3.12)
where
\[ g_1 = \frac{q_1(C_1 q_1^3 + 3 C_2 q_1 q_2 + 3 C_1 q_2^3 + C_2 q_2^3)}{(q_1 - q_2)^3(q_1 + q_2)}, \quad g_3 = q_2 q_1^{-1} g_1(q), \quad f_2 = q_1^{-1} g_1(q) \]
\[ g_2 = \frac{q_1(C_2 q_1^3 + 3 C_2 q_1 q_2 + 3 C_2 q_2^3 + C_1 q_1^3)}{(q_1 - q_2)^3(q_1 + q_2)}, \quad g_4 = q_2 q_1^{-1} g_2(q), \quad f_1 = q_1^{-1} g_2(q). \]

The corresponding second integral of motion \( H_2 = \tilde{H}_2 - \frac{1}{4} H_1^2 \) is a fourth order polynomial in momenta, which may be reduced to a quadratic integral.

The associated with \( P' \) \( (3.12) \) solution of the equations \((2.3, 2.4, 2.12)\) with another control matrix \( F \) \( (2.13) \) has a form
\[
H_1 = p_1^2 + p_2^2 - \frac{C_1(q_1^2 + q_2^2)}{(q_1 + q_2)^2(q_1 - q_2)^2} - \frac{2 C_2 q_1 q_2}{(q_1 + q_2)^2(q_1 - q_2)^2}.
\]

As above second integral of motion is reduced to a quadratic polynomial in the momenta.

**Integrals of motion fourth order in the momenta.**

Below we consider integrable systems for which second integrals of motion are the fourth order polynomials in the momenta, which can not be reduced to quadratic polynomials.

Substituting \( P' \) \( (3.11) \) into the equations \( (2.3, 2.4) \) we can extract equations for the functions \( h_i \). Solving these equations we get that either functions \( h_i \) are constant \( h_i = \text{const} \), or they are linear functions of coordinates \( h_i = a_i q_1 + b_i q_2 \).

At \( h_i = \text{const} \) we consider only the Lenard-Magri recurrence relations. Recall that in this case the control matrix \( F \) in \( (2.12) \) has a special form \( (2.13) \).

**Proposition 4** If the Poisson tensor \( P' \) has the form \( (3.11) \) there are only two solutions of the extended system of equations \((2.3, 2.4, 2.12)\) with a control matrix \( F \) \( (2.13) \) and a natural Hamilton function \( H_1 = p_1^2 + p_2^2 + V(q) \).

The corresponding Poisson tensors are distinguished by functions \( f_i \) and \( g_i \) only
\[
P' = \begin{pmatrix}
0 & 2 a p_1 + 2 b p_2 & p_1^2 + g_1(q) & g_2(q) \\
* & 0 & g_3(q) & p_2^2 + g_4(q) \\
* & * & 0 & f_1(q)p_1 + f_2(q)p_2 \\
* & * & * & 0
\end{pmatrix}.
\] (3.13)

If \( a \neq 0 \) and \( b \neq 0 \) then the Hamiltonian reads as
\[
H_1 = p_1^2 + p_2^2 + (a + b) C_2 \exp \left( \frac{q_2 - q_1}{a + b} \right) - (a - b) C_1 \exp \left( \frac{q_2 + q_1}{a - b} \right),
\]
whereas functions \( f_i \) and \( g_i \) in \( (3.13) \) are equal to
\[
f_1 = C_2 e^{\frac{q_2 - q_1}{a + b}} - C_1 e^{\frac{q_1 + q_2}{a + b}}, \quad g_1 = 2 b f_2, \quad g_2 = b f_1 - a f_2
\]
\[
f_2 = C_2 e^{\frac{q_2 - q_1}{a + b}} + C_1 e^{\frac{q_1 + q_2}{a - b}}, \quad g_3 = -g_2, \quad g_4 = 2 a f_1.
\]

If \( a = 0 \) or \( b = 0 \), then this is another nontrivial solution. At \( a = 0 \) and \( b = 1 \) the corresponding Hamiltonian reads as
\[
H_1 = p_1^2 + p_2^2 + 2 C_1 e^{-q_1 - q_2} + 2 C_2 e^{q_2 - q_1} - \frac{C_3 e^{2 q_2}}{2 (C_2 e^{2 q_2} - C_1)} - \frac{C_4 e^{2 q_2} (C_2 e^{2 q_2} + C_1)}{(C_2 e^{2 q_2} - C_1)^2} \quad (3.14)
\]
Proposition 5
If the Poisson tensor nontrivial solutions of the equations (2.3, 2.4, 2.12) with control matrix 
H Toda lattices at C the following product of linear Poisson tensor (3.7) (see Remark 5) 
P natural multi-dimensional generalization of K where 
Remark 9 which was constructed by Inozemtsev [13].

Now let us consider the second case at C. If = (1) = (2) = (3) = (4) = 0, the second 
Hamiltonian H1 (3.14) describes open Toda lattice of D2 type. If = 0 or = 0 the 
Hamiltonian H1 (3.14) describes also an open Toda lattice associated with a root system 
BC2. If = = we obtain the Hamiltonian H1 (3.14) of the generalized Toda lattice, which was constructed by Inozemtsev [13].

Remark 9 The quadratic matrix polynomial P′ (3.13) may be considered as deformation of the following product of linear Poisson tensor (3.7) (see Remark 5)
P′ = P′linP−1P′lin + Klin, (3.15)
where Klin is a very simple linear matrix polynomial. Therefore, substituting the multi-
dimensional linear Poisson tensor (3.9) into (3.15) and adding the similar term Klin one gets a natural multi-dimensional generalization of P′ (3.13), which was proposed in [4] for the open 
Toda lattices at C = C = 0 and at C = 0 (C = 0).

Now let us consider the second case at hi = aiq1 + biq2.

Proposition 5 If the Poisson tensor P′ has the form (3.11) there are only two explicit 
nontrivial solutions of the equations (2.3, 2.4, 2.12) with control matrix F (2.12) and a 
natural Hamilton function H1 = p12 + p22 + V(q).
The first solution is the Holt-type system for which
H1 = p12 + p22 − C1q12 + C2q22) + C3q23 − C4q22 (3.16)
and
P′ = (0 3p1q2 p12 + C2 q2 q1 0 p1p2 + C1q1q21/3
* 0 − 3C2 q2 q1 p12 − 3C1q24/3 + C2 q2 q1 + 2C3q24/3
* * 0
* * * −C1p1q21/3 − C2q2q1/3
* * * * 0).
(3.17)
The second solution is the system with fourth order potential for which
H1 = p12 + p22 − C1 4q14 + 3q2 q2 q2 + q2 2 + C2 q12 + q22) + C3 q12 + q22 − C4 q22 (3.18)
and
P′ = (0 p1q2 p12 + q1(q) p1p2 + g2(q)
* 0 g3(q) p12 + 2p22 + g4(q)
* * 0 p1f1(q) + p2f2(q)
* * * 0).
(3.19)
where

\[ f_1 = -C_1q_2^3, \quad f_2 = 2C_1q_1(4q_1^3 + 3q_2^3) - C_2q_1 - \frac{C_3}{q_1}, \]
\[ g_2 = C_1q_1q_2^3, \quad g_1 = -4C_1q_1^4 + C_2q_1^2 - \frac{C_3}{q_1}, \]
\[ g_3 = 4g_2 + q_2f_2, \quad g_4 = -C_1(4q_1^4 + 6q_1^2q_2^2 + q_2^4) + C_2\left(q_1^2 + \frac{q_2^2}{2}\right) - \frac{C_3}{q_1} \]

The second integrals of motion are the fourth order polynomials in the momenta \( H \).

According to [20], in the second case we can make a special contraction of variables and obtain the Henon-Heiles system with the cubic potential

\[ H_1 = p_1^2 + p_2^2 + \frac{1}{3}q_1(16q_1^2 + 3q_2^2) + \frac{C_1}{q_2^2}. \]  \( (3.20) \)

The corresponding Poisson tensor \( P' \) is equal to

\[
P' = \begin{pmatrix}
0 & q_2p_1 & p_2^2 + \frac{16}{3}q_1^3 & p_1p_2 - \frac{1}{6}q_2^3 \\
* & 0 & 8q_2p_1 + \frac{16}{3}q_1^3 & p_1^2 + 2p_2^2 + 2q_1q_2^2 + \frac{16}{3}q_1^3 \\
* & * & 0 & -p_2(8q_2^2 + q_1^3) \\
* & * & * & 0
\end{pmatrix} \]  \( (3.21) \)

**Remark 10** The eigenvalues of the operators \( N \) associated with the Poisson tensors \( (3.19) \) and \( (3.21) \) are integrals of motion \( \lambda_{1,2} = 0 \). So, the special Darboux-Nijenhuis coordinates can not be the separated variables for the system with fourth order potential \( (3.18) \) and for the Henon-Heiles system \( (3.20) \). The corresponding separated variables are generic Darboux-Nijenhuis coordinates \( (x,y) \) \( (2.39) \).

If the Poisson tensor \( P' \) has a form \( (3.11) \) then there are some implicit solutions of the equations \( (2.3,2.4,2.12) \) with control matrix \( F \) \( (2.19) \). In these cases functions \( f_i(q), g_i(q) \) and integral of motion \( H_2(p,q) \) are determined by potential \( V(q_1, q_2) \), which satisfies some special partial differential equations of the second and of the third order. For instance, one of the obtained equations has the form

\[
(q_1q_2 + C_1)\left(\frac{\partial^2}{\partial q_1^2} - \frac{\partial^2}{\partial q_2^2}\right)V + 3\left(q_2\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial q_2}\right)V = (q_1^2 - q_2^2 + C_2)\frac{\partial^2}{\partial q_1\partial q_2}V.
\]

Similar second order equations were considered in [10, 25]. The third order partial differential equations obtained in this method we could not find in the literature. These implicit systems will be studied separately.

As above we can substitute into the equations \( (2.12) \) the Hamilton function \( H_1 = T + V \) in the form \( (3.10) \). Here we present two new integrable systems with pseudo-Euclidean metric.

**Proposition 6** If the Poisson tensor \( P' \) has a form \( (3.11) \) there are only two explicit solutions of the equations \( (2.3,2.4,2.12) \). The first solution reads as

\[
H_1 = p_1p_2 + C_1(4q_1^2 + 5\alpha q_1q_2 + \alpha^2q_2^2) + \frac{C_2(q_1 + \alpha q_2)}{q_2^{1/3}} + \frac{C_3}{q_2^{2/3}} + \frac{C_4}{q_2^{4/3}}, \quad \alpha \in \mathbb{R},
\]
In this section we consider the normalized Poisson tensors

\[ P' = \begin{pmatrix} 0 & 3q_2p_1 & p_1^2 - 6C_1q_2(q_1 + \alpha q_2) & \alpha p_1^2 - 3p_1p_2 - 3C_1(q_1 + \alpha q_2)(2q_1 + \alpha q_2) - \frac{C_2(q_1 + \alpha q_2)}{q_2^{3/2}} + \frac{2C_3}{q_2^2} \\ * & 0 & 18C_1q_2^2 & 9C_1q_2(2q_1 + \alpha q_2) + 3C_2q_2^{2/3} \\ * & * & 0 & \left(3C_1(2q_1 - \alpha q_2) + \frac{C_2}{q_2^{1/2}}\right) + p_1 + 18C_1q_2p_2 \\ * & * & * & \end{pmatrix} \]

3.4 The normalized Poisson tensors

In this section we consider the normalized Poisson tensors

\[ P' = \det(\tilde{P}')^{-1} \tilde{P}', \tag{3.22} \]

where \( \tilde{P}' \) has the form (3.11). Such tensors induce formal \( \omega N \) structure on \( M \), but their eigenvalues \( \lambda_i \) will be a priori functionally dependent functions.

**Proposition 7** For the Poisson tensor \( P' \) (3.22) there are only two nontrivial solutions of the equations (2.3, 2.4, 2.12) with the control matrix \( F' \) (2.19).

The corresponding Poisson tensors are distinguished by functions \( f_i \) and \( g_i \) only

\[ \tilde{P}' = \begin{pmatrix} 0 & p_1q_2 & -p_2^2 + g_1(q) & p_1p_2 + g_2(q) \\ * & 0 & g_3(q) & p_2^2 + g_4(q) \\ * & * & 0 & f_1(q)p_1 + f_2(q)p_2 \\ * & * & * & 0 \end{pmatrix}. \tag{3.23} \]

First solution corresponds to the Henon-Heiles system for which

\[ H_1 = p_1^2 + p_2^2 - C_1q_1(16q_1^2 + 3q_2^2) - 2C_2q_1, \]

whereas functions \( f_i \) and \( g_i \) in (3.23) are equal to

\[ f_1 = 0, \quad g_1 = 3C_1q_1q_2^2, \quad g_2 = \frac{C_2}{2}q_2^3, \]

\[ f_2 = 3C_1(8q_1^2 + q_2^2) + C_2, \quad g_3 = 4g_2 - f_2q_2, \quad g_4 = -g_1. \]

Second solution corresponds to the system with the fourth order potential for which

\[ H_1 = p_1^2 + p_2^2 + C_1(8q_1^4 + 6q_1^2q_2^2 + q_2^4) + C_2(4q_1^2 + q_2^2) + \frac{C_3}{q_1^2}, \]
whereas functions \( f_i \) and \( g_i \) in (3.23) are equal to

\[
\begin{align*}
  f_1 &= 2C_1q_2^3, \\
  g_1 &= -(C_1(q_2^2 + 6q_1^2) + C_2)q_2^2, \\
  g_2 &= -2C_1q_1q_2^3, \\
  f_2 &= -4C_1q_1(3q_2^2 + 4q_1^2) - 4C_2q_1 + \frac{C_3}{q_1^2}, \\
  g_3 &= 4g_2 - f_2q_2, \\
  g_4 &= -g_1.
\end{align*}
\]

The corresponding second integrals of motion are equal to

\[
H_2 = \det(\hat{P}') = \sqrt{p_2^4 + q_1f_1p_1^2 + (q_2f_2 + g_3)p_1p_2 - 2g_1p_2^2 - g_1^2 + g_2g_3}.
\]

In the both cases eigenvalues of the recursion operator \( N \) are trivial \( \lambda_{1,2} = \pm 1 \). In order to get the separated variables we have to put operator \( N \) in diagonal form (2.9) by using canonical transformation \( z = (q, p) \to \zeta = (x, y) \).

**Proposition 8** Let \( J = \frac{\partial \zeta}{\partial z} \) is the Jacobi matrix associated with some change of variables \( \zeta = \zeta(z) \). Solving overdetermined system of the partial differential equations

\[
JPJ^T = P, \quad JNJ^{-1} = \text{diag}(1, -1, 1, -1)
\]

with respect to \( \zeta(p, q) \) we reproduce the separated variables for the Henon-Heiles system and the separated variables for the system with fourth order potential, which were constructed in [27].

The first equation in (3.24) provides canonicity of transformation \( z \to \zeta \), the second one provides diagonability of \( N \) in \( \zeta \) variables. Equations (3.24) were easy solved in the symbolic computational system Maple.

Relations of such transformations, which put operator \( N \) in diagonal form, with the Bäcklund transformations for these systems will be discussed in forthcoming publications.

## 4 Appendix: a programm for the search of natural integrable systems

This paper as well as [23, 24] belongs mostly to computational mathematical physics and, therefore, in this section we present an implementation of the discussed algorithm made in the symbolic computational system Maple.

At the first step we need to determine dimension of the phase space \( M \) and canonical variables \( q = (q_1, \ldots, q_n) \) and \( p = (p_1, \ldots, p_n) \):

\[
> \text{restart: with(linalg):}
> \text{n:=2:}
> \text{q:=seq(q||i,i=1.. n): p:=seq(p||i,i=1.. n):}
> \text{var:=q,p;}
\]

Now we can introduce canonical Poisson tensor \( P \) (2.6)

\[
> \text{ed:=array(identity, 1.. n,1.. n): z:=array(sparse,1.. n,1.. n):}
> \text{P1:=blockmatrix(2,2,[z,ed,-ed,z]):}
\]

and a linear ansatz for the second Poisson tensor \( P' \) (3.6) at \( h(q) = aq_1 \)

\[
> \text{P2 := array(antisymmetric,1.. 2*n,1.. 2*n):}
> \text{P2[1,2]:=a*q1: P2[1,3]:=c1*p1+c2*p2: P2[1,4]:=c3*p1+c4*p2:}
> \text{P2[2,3]:}=-c5*p1+c6*p2: P2[2,4]:=c7*p1+c8*p2: P2[3,4]:=f1(q):
\]

The Poisson brackets associated with the Poisson tensor \( P' \) and with the Poisson pencil
$P + \lambda P'$ are defined by the standard rule

```maple
> P3 := evalm(P1 + lambda * P2);
> PB2 := proc (f, g) options operator, arrow:
    add(add(P2[i,j] * diff(f, var[i]) * diff(g, var[j]), i=1.. 2*n), j=1.. 2*n)
end:
> PB3 := proc (f, g) options operator, arrow:
    add(add(P3[i,j] * diff(f, var[i]) * diff(g, var[j]), i=1.. 2*n), j=1.. 2*n)
end:
```

The corresponding Jacobi identities (2.2) look like

```maple
> f := F(var); g := G(var); h := H(var):
> Y2 := simplify(PB2(PB2(f, g), h) + PB2(PB2(g, h), f) + PB2(PB2(h, f), g)):
> Y3 := simplify(PB3(PB3(f, g), h) + PB3(PB3(g, h), f) + PB3(PB3(h, f), g)):
```

It allows us to build up the system of equations (2.3, 2.4)

```maple
> ListEq := NULL:
> for i to 2*n do
    for j to 2*n do
        for k to 2*n do
            Eq2 := coeff(coeff(coeff(Y2, diff(f, var[i]), 1), diff(h, var[j]), 1), diff(g, var[k]), 1):
            Eq3 := coeff(coeff(coeff(Y3, diff(f, var[i]), 1), diff(h, var[j]), 1), diff(g, var[k]), 1):
            ListEq := ListEq, coeffs(collect(Eq2, {p}, distributed), {p}),
            coeffs(collect(Eq3, {p, lambda}, distributed), {p, lambda});
        end do;
    end do;
end do:
```

On the next step we can construct the operator $N^*$ and its minimal polynomial $\Delta(\lambda)$

```maple
> N := evalm(inverse(P1) &* P2);
> ed2 := array(identity, 1.. 2*n, 1.. 2*n):
> Delta := collect(simplify(sqrt(factor(det(N - lambda * ed2)))), lambda, factor);
> if coeff(Delta, lambda, n) < 0 then Delta := -Delta end if:
```

The coefficients of the minimal polynomial $\Delta(\lambda)$ are entries of the control matrix $F$ (2.19)

```maple
> sigma := vector(n):
> for k to n do sigma[k] := -coeff(Delta, lambda, n-k) end do:
> F := delcols(augment(sigma, ed), n+1.. n+1):
```

Now we introduce natural Hamilton function $H = H_1$ and unknown integrals of motion $H_2, \ldots, H_n$ and calculate the corresponding differentials $dH_k$

```maple
> H1 := add(p[k] * k-2, k=1.. n) + V(q);
> for k from 2 to n do H||k := h||k(var); dH||k := vector(2*n): end do;
> for k to n do
    for i to 2*n do dH||k[i] := simplify(diff(H||k, var[i]), symbolic):
    end do:
```

On the final step after building of the equations (2.12)

```maple
> ListEqH := NULL:
> for i to n do
    Z||i := simplify(evalm(N &* dH||i - add(F[i,j] * dH||j, j=1.. n))):
    ListEqH := ListEqH, seq(Z||i[k], k=1.. 2*n):
end do:
```

we solve the complete overdetermined system of algebro-differential equations

```maple
> Ans := pdsolve({ListEq, ListEqH}, {f1(q), V(q), seq(H||k, k=2.. n)});
```
This program allows us to construct three integrable systems listed in the Proposition 2 in a few seconds on the standard personal computer. A little larger program allows us to construct all the systems listed in this paper in a few minutes.

**Remark 11** In order to cut the computer time and to exclude the trivial solutions we have to add some simple equations to our system. For instance we added the following inequalities $\frac{\partial}{\partial p_i} H_k \neq 0$ and $\frac{\partial^2}{\partial q_1 \partial q_2} V(q_1, q_2) \neq 0$.

## 5 Conclusion

Using some recent results on the separation of variables for bi-hamiltonian manifolds we propose a new method for construction on the natural integrable systems. In this method the desired integrals of motion are solutions of the overdetermined system of algebro-differential equations, which arise from the Jacobi identities for the Poisson pencil (2.3, 2.4) and from invariance of the corresponding bi-Lagrangian distribution with respect to recurrence operator $N$ (2.12). The algorithm may be easy realized in any modern symbolic computational system. One of the possible implementations may be found in the Appendix. So the search for natural integrable systems is reduced to simple computer calculations.

The main advantage of the proposed method is that we construct integrals of motion of a natural integrable system, the corresponding Poisson pencil and a part of the separated variables simultaneously.

The main disadvantage is that we have to use undefined polynomial anzats for the second Poisson tensor $P'$ and to choose some control matrix $F$. So, we construct special families of the integrable systems associated with the a given $P'$ and $F$.

Of course, by using very simple substitutions (3.6, 3.11) we reproduce a majority of the known natural integrable systems on the plane with cubic and quartic integrals of motion. Nevertheless we did not get all the known integrable systems [10]. Moreover we do not know how to construct similar polynomials $P'$ on the sphere or on the ellipsoid.

The second disadvantage is an exponential growth of the computer time with the growth of $n = \text{rank} P'$ and with the growth of the higher power of the matrix polynomial $P'$.

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