LENGTHS, AREAS AND LIPSCHITZ-TYPE SPACES OF PLANAR HARMONIC MAPPINGS

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Abstract. In this paper, we establish a three circles type theorem, involving the harmonic area function, for harmonic mappings. Also, we give bounds for length and area distortion for harmonic quasiconformal mappings. Finally, we will study certain Lipschitz-type spaces on harmonic mappings.

1. Introduction and main results

Let \( D \) be a simply connected subdomain of the complex plane \( \mathbb{C} \). A complex-valued function \( f \) defined in \( D \) is called a harmonic mapping in \( D \) if and only if both the real and the imaginary parts of \( f \) are real harmonic in \( D \). It is known that every harmonic mapping \( f \) defined in \( D \) admits a decomposition \( f = h + \overline{g} \), where \( h \) and \( g \) are analytic in \( D \). Since the Jacobian \( J_f \) of \( f \) is given by

\[
J_f = |f_z|^2 - |f_{\overline{z}}|^2 := |h'|^2 - |g'|^2,
\]

\( f \) is locally univalent and sense-preserving in \( D \) if and only if \( |g'(z)| < |h'(z)| \) in \( D \); or equivalently if \( h'(z) \neq 0 \) and the dilatation \( \omega = g'/h' \) has the property that \( |\omega(z)| < 1 \) in \( D \) (see [19]). Let \( \mathcal{H}(D) \) denote the class of all sense-preserving harmonic mappings in \( D \). We refer to [9, 11] for basic results in the theory of planar harmonic mappings.

For \( a \in \mathbb{C} \), let \( \mathbb{D}(a, r) = \{ z : |z - a| < r \} \). In particular, we use \( \mathbb{D}_r \) to denote the disk \( \mathbb{D}(0, r) \) and \( \mathbb{D}_1 \), the open unit disk \( \mathbb{D}_1 \).

The classical theorem of three circles [1, 28], also called Hadamard’s three circles theorem, states that if \( f \) is an analytic function in the annulus \( B(r_1, r_2) = \{ z : 0 < r_1 < |z| = r < r_2 < \infty \} \), continuous on \( \overline{B(r_1, r_2)} \), and \( M_1, M_2 \) and \( M \) are the maxima of \( f \) on the three circles corresponding to \( r_1, r_2 \) and \( r \), respectively, then

\[
M \log \frac{r}{r_1} \leq M_1 \log \frac{r_2}{r} M_2 \log \frac{r}{r_1}.
\]

Equivalently, we can reformulate this result into a simpler form. That is if \( f \) is analytic on the annulus \( B(r_1, 1) = \{ z : 0 < r_1 < |z| < 1 \} \), continuous on the closure, and

\[
|f(z)| \leq m = r_1^\alpha, \quad |z| = r_1 \quad \text{and} \quad |f(z)| \leq 1, \quad |z| = 1,
\]

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then Hadamard’s result states that, for \( r_1 \leq r \leq 1 \),
\[
|f(z)| \leq m_{\log r_1}^{\log r} = r^\alpha, \quad |z| = r,
\]
where \( \alpha \) is an integer.

The original three circles theorem was given by Hadamard without proof in 1896 [15], and comprehensive discussion about the history of this result can be found in [20, pp. 323–325] and [28]. It is a natural question, what results of this type can be proved for other classes of functions and, indeed, there are numerous generalizations of the thee circles theorem in the literature, see e.g. [3, 21, 26, 30]. In this paper, our first aim is to establish an area version of the three circles theorem (cf. area version of Schwarz’ lemma [4]).

For a harmonic mapping \( f \) in \( \mathbb{D} \) and \( r \in [0, 1) \), the harmonic area function \( S_f(r) \) of \( f \), counting multiplicity, is defined by
\[
S_f(r) = \int_{\mathbb{D}_r} J_f(z) \, d\sigma(z),
\]
where \( d\sigma \) denotes the normalized Lebesgue area measure on \( \mathbb{D} \) (cf. [8]). In particular, let
\[
S_f(1) = \sup_{0 < r < 1} S_f(r).
\]

**Theorem 1.** Let \( f = h + \overline{g} \) be harmonic in \( \mathbb{D} \), where \( h \) and \( g \) are analytic. If \( S_f(r_1) \leq m < 1 \), \( S_f(1) \leq 1 \) and for all \( n \in \{1, 2, \ldots\} \), \( |g^{(n)}(0)| \leq |h^{(n)}(0)| \), then for \( r_1 \leq r < 1 \),
\[
S_f(r) \leq m_{\log r_1}^{\log r}.
\]
The estimate of (1.1) is sharp and the extremal function is \( f(z) = \alpha z + \beta \overline{z} \), where \( \alpha \) and \( \beta \) are constant with \( |\alpha|^2 - |\beta|^2 = 1 \).

**Corollary 1.1.** Let \( f \) be analytic in \( \mathbb{D} \) satisfying \( S_f(r_1) \leq m \) and \( S_f(1) \leq 1 \), where \( 0 < r_1 < 1 \). Then for \( r_1 \leq r < 1 \),
\[
S_f(r) \leq m_{\log r_1}^{\log r}.
\]
The estimate of (1.2) is sharp and the extremal function is \( f(z) = \lambda z \), where \( |\lambda| = 1 \) are constant.

For \( p \in (0, \infty] \), the harmonic Hardy space \( h^p \) consists of all harmonic functions \( f \) such that \( \|f\|_p < \infty \), where
\[
\|f\|_p = \begin{cases} \sup_{0 < r < 1} M_p(r, f) & \text{if } p \in (0, \infty), \\ \sup_{z \in \mathbb{D}} |f(z)| & \text{if } p = \infty, \end{cases} \quad \text{and} \quad M_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta.
\]
If \( f \in h^p \) for some \( p > 0 \), then the radial limits
\[
f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})
\]
exist for almost every \( \theta \in [0, 2\pi) \) (cf. [11]).
We recall that a function \( f \in \mathcal{H}(\mathbb{D}) \) is said to be \( K \)-quasiregular, \( K \in [1, \infty) \), if for \( z \in \mathbb{D} \), \( \Lambda_f(z) \leq K \lambda_f(z) \). In addition, if \( f \) is univalent in \( \mathbb{D} \), then \( f \) is called a \( K \)-quasiconformal harmonic mapping \( \mathbb{D} \).

Let \( \Omega \) be a domain of \( \mathbb{C} \), with non-empty boundary. Let \( d_\Omega(z) \) be the Euclidean distance from \( z \) to the boundary \( \partial \Omega \) of \( \Omega \). In particular, we always use \( d(z) \) to denote the Euclidean distance from \( z \) to the boundary of \( \mathbb{D} \). The area of a set \( G \subset \mathbb{C} \) is denoted by \( A(G) \). The area problem of analytic functions has attracted much attention (see [2, 29, 31, 32]). We investigate the area problem of harmonic mappings and obtain the following result.

**Theorem 2.** Let \( \Omega_1 \) and \( \Omega_2 \) be two proper and simply connected subdomains of \( \mathbb{C} \) containing the point of origin. Then for a sense-preserving and \( K \)-quasiconformal harmonic mapping \( f \) defined in \( \Omega_1 \) with \( f(0) = 0 \),

\[
K A(f(\Omega_1) \cap \Omega_2) + A(f^{-1}(\Omega_2)) \geq \min \{ d_{\Omega_1}^2(0), d_{\Omega_2}^2(0) \}. 
\]

Moreover, if \( K = 1 \), then the estimate of (1.3) is sharp.

We remark that Theorem 2 is a generalization of [29, Theorem].

For a harmonic mapping \( f \) defined on \( \mathbb{D} \), we use the following standard notations:

\[
\Lambda_f(z) = \max_{0 \leq \theta \leq 2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = |f_z(z)| + |f_{\bar{z}}(z)| 
\]

and

\[
\lambda_f(z) = \min_{0 \leq \theta \leq 2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = \left| |f_z(z)| - |f_{\bar{z}}(z)| \right|. 
\]

Further, a planar harmonic mapping \( f \) defined on \( \mathbb{D} \) is called a harmonic Bloch mapping if

\[
\beta_f = \sup_{z, w \in \mathbb{D}, z \neq w} \frac{|f(z) - f(w)|}{\rho(z, w)} < \infty. 
\]

Here \( \beta_f \) is called the Lipschitz number of \( f \), and

\[
\rho(z, w) = \frac{1}{2} \log \left( \frac{1 + |(z - w)/(1 - \bar{z}w)|}{1 - |(z - w)/(1 - \bar{z}w)|} \right) = \arctanh \left| \frac{z - w}{1 - \bar{z}w} \right|. 
\]

denotes the hyperbolic distance between \( z \) and \( w \) in \( \mathbb{D} \). It is known that

\[
\beta_f = \sup_{z \in \mathbb{D}} \left\{ (1 - |z|^2) \Lambda_f(z) \right\}. 
\]

Clearly, a harmonic Bloch mapping \( f \) is uniformly continuous as a map between metric spaces,

\[
f : (\mathbb{D}, \rho) \to (\mathbb{C}, |\cdot|), 
\]

and for all \( z, w \in \mathbb{D} \) we have the Lipschitz inequality

\[
|f(z) - f(w)| \leq \beta_f \rho(z, w). 
\]

A well-known fact is that the set of all harmonic Bloch mappings, denoted by the symbol \( \mathcal{HB} \), forms a complex Banach space with the norm \( \| \cdot \| \) given by

\[
\|f\|_{\mathcal{HB}} = |f(0)| + \sup_{z \in \mathbb{D}} \left\{ (1 - |z|^2) \Lambda_f(z) \right\}. 
\]
Specially, we use $\mathcal{B}$ to denote the set of all analytic functions defined in $\mathbb{D}$ which forms a complex Banach space with the norm
\[ \|f\|_\mathcal{B} = |f(0)| + \sup_{z \in \mathbb{D}} \{(1 - |z|^2)|f'(z)|\}. \]

The reader is referred to [10, Theorem 2] (or [5, 6]) for a detailed discussion.

For $r \in [0, 1)$, the length of the curve $C(r) = \{ w = f(re^{i\theta}) : \theta \in [0, 2\pi] \}$, counting multiplicity, is defined by
\[ l_f(r) = \int_0^{2\pi} |df(re^{i\theta})| = r \int_0^{2\pi} |f_z(re^{i\theta}) - e^{-2i\theta}f_z(re^{i\theta})| \, d\theta, \]
where $f$ is a harmonic mapping defined in $\mathbb{D}$. In particular, let $l_f(1) = \sup_{0 < r < 1} l_f(r)$.

**Theorem 3.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n$ be a sense-preserving $K$-quasiconformal harmonic mapping. If $l_f(1) < \infty$, then for $n \geq 1$,
\[ |a_n| + |b_n| \leq \frac{Kl_f(1)}{2n\pi} \]
and
\[ \Lambda_f(z) \leq \frac{l_f(1)\sqrt{K}}{2\pi(1 - |z|)}. \]

Moreover, $f \in \mathcal{HB}$ and $\beta_f \leq \frac{l_f(1)\sqrt{K}}{\pi}$. In particular, if $K = 1$, the estimates of (1.4) and (1.5) are sharp, and the extremal function is $f(z) = z$.

A continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ is called a **majorant** if $\omega(t)/t$ is non-increasing for $t > 0$. Given a subset $\Omega$ of $\mathbb{C}$, a function $f : \Omega \to \mathbb{C}$ is said to belong to the **Lipschitz space** $L_\omega(\Omega)$ if there is a positive constant $C$ such that
\[ |f(z) - f(w)| \leq C\omega(|z - w|) \quad \text{for all } z, w \in \Omega. \]

For $\delta_0 > 0$, let
\[ \int_0^\delta \frac{\omega(t)}{t} \, dt \leq C \cdot \omega(\delta), \quad 0 < \delta < \delta_0, \]
and
\[ \delta \int_{\delta}^{+\infty} \frac{\omega(t)}{t^2} \, dt \leq C \cdot \omega(\delta), \quad 0 < \delta < \delta_0, \]
where $\omega$ is a majorant and $C$ is a positive constant.

A majorant $\omega$ is said to be **regular** if it satisfies the conditions (1.7) and (1.8) (see [12, 13, 22, 23, 24]).

Let $G$ be a proper subdomain of $\mathbb{C}$. We say that a function $f$ belongs to the **local Lipschitz space** $\text{loc}L_\omega(G)$ if (1.6) holds, with a fixed positive constant $C$, whenever $z \in G$ and $|z - w| < \frac{1}{2}d_G(z)$ (cf. [14, 18]). Moreover, $G$ is said to be a $L_\omega$-extension domain if $L_\omega(G) = \text{loc}L_\omega(G)$. The geometric characterization of $L_\omega$-extension domains was first given by Gehring and Martio [14]. Then Lappalainen [18] generalized their characterization, and proved that $G$ is a $L_\omega$-extension domain.
if and only if each pair of points $z, w \in G$ can be joined by a rectifiable curve $\gamma \subset G$ satisfying
\begin{equation}
\int_{\gamma} \frac{\omega(d_{G}(z))}{d_{G}(z)} \, ds(z) \leq C \omega(|z - w|)
\end{equation}
with some fixed positive constant $C = C(G, \omega)$, where $ds$ stands for the arc length measure on $\gamma$. Furthermore, Lappalainen [18, Theorem 4.12] proved that $L_{\omega}$-extension domains exist only for majorants $\omega$ satisfying (1.7).

Theorem A. ([16, Theorem 3]) $f \in B$ if and only if
\[ \sup_{z, w \in \mathbb{D}, z \neq w} \left\{ \frac{\sqrt{(1 - |z|^2)(1 - |w|^2)}|f(z) - f(w)|}{|z - w|} \right\} < \infty. \]

The following result is a generalization of Theorem A. For the related studies of this topic for real functions, we refer to [25, 27].

Theorem 4. Let $f$ be a harmonic mapping in $\mathbb{D}$ and $\omega$ be a majorant. Then the following are equivalent:
(a) There exists a constant $C_1 > 0$ such that for all $z \in \mathbb{D},$
\[ \Lambda f(z) \leq C_1 \omega \left( \frac{1}{d(z)} \right); \]
(b) There exists a constant $C_2 > 0$ such that for all $z, w \in \mathbb{D}$ with $z \neq w,$
\[ \frac{|f(z) - f(w)|}{|z - w|} \leq C_2 \omega \left( \frac{1}{\sqrt{d(z)d(w)}} \right); \]
(c) There exists a constant $C_3 > 0$ such that for all $r \in (0, d(z)],$
\[ \frac{1}{|D(z, r)|} \int_{B^{n}(z, r)} |f(\zeta) - f(z)| \, dA(\zeta) \leq C_3 r \omega \left( \frac{1}{r} \right), \]
where $dA$ denotes the Lebesgue area measure in $\mathbb{D}$.

Note that if $\omega(t) = t$ and $f$ is analytic, then (a)$\iff$(b) in Theorem 4 implies that Theorem A.

Krantz [17] proved the following Hardy-Littlewood-type theorem for harmonic functions with respect to the majorant $\omega(t) = \omega_{n}(t) = t^{\alpha}$ ($0 < \alpha \leq 1$).

Theorem B. ([17, Theorem 15.8]) Let $u$ be a real harmonic function in $\mathbb{D}$ and $0 < \alpha \leq 1$. Then $u$ satisfies
\[ \left| \nabla u(z) \right| \leq C \frac{\omega_{n}(d(z))}{d(z)} \quad \text{for all} \quad z \in \mathbb{D} \]
if and only if
\[ |u(z) - u(w)| \leq C \omega_{n}(|z - w|) \quad \text{for all} \quad z, w \in \mathbb{D}. \]

We generalize Theorem B to the following form.
Theorem 5. Let \( \omega \) be a majorant satisfying (1.7), \( \Omega \) be a \( L_\omega \)-extension domain and \( f \) be a harmonic mapping in \( \Omega \). Then there exists a constant \( C_4 > 0 \) such that

\[
\Lambda_f(z) \leq C_4 \frac{\omega(d_\Omega(z))}{d_\Omega(z)} \quad \text{for all } z \in \Omega
\]

if and only if, for some \( C_5 > 0 \),

\[
|f(z) - f(w)| \leq C_5 \omega(|z - w|) \quad \text{for all } z, w \in \Omega.
\]

The proofs of Theorems 1, 2 and 3 will be given in Section 2. We will show Theorems 4 and 5 in the last part of this paper.

2. LENGTH AND AREA OF HARMONIC MAPPINGS

Proof of Theorem 1. Let \( f = h + \overline{g} \) be harmonic in \( \mathbb{D} \) with the following expansion

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n \overline{z}^n,
\]

where \( a_n = \frac{h^{(n)}(0)}{n!} \) and \( b_n = \frac{g^{(n)}(0)}{n!} \). By hypothesis, \( |a_n| \geq |b_n| \). Then

\[
S_f(r) = \int_{\mathbb{D}_r} J_f(\zeta) \, d\sigma(\zeta) = \int_{\mathbb{D}_r} (|h'(\zeta)|^2 - |g'(\zeta)|^2) \, d\sigma(\zeta) = \sum_{n=1}^{\infty} n \left( |a_n|^2 - |b_n|^2 \right) r^{2n}.
\]

For \( z \in \mathbb{D} \), let \( F(z) = \sum_{n=1}^{\infty} A_n z^{2n} \), where \( A_n = n (|a_n|^2 - |b_n|^2) \). Since \( A_n \geq 0 \), we see that the maximum of \( F \) on \( \partial \mathbb{D}_r \) is obtained on the real axis, that is

\[
S_f(r) = F(r) = \max_{|z|=r} |F(z)|,
\]

where \( r_1 \leq r < 1 \). Hence the result follows from Hadamard’s theorem.

Now we are ready to prove the sharpness part. It is not difficult to see that \( S_f(r_1) = r_1^2 \) and \( S_f(1) = 1 \), where \( f(z) = \alpha z + \beta \overline{z} \) with \( |\alpha|^2 - |\beta|^2 = 1 \). Then for \( 0 < r_1 \leq r < 1 \), \( S_f(r) = r^2 = m \log r, \) where \( m = \frac{\log r}{\log r_1} \). The proof of the theorem is complete. \( \square \)

Lemma 1. Let \( f \) be a sense-preserving and \( K \)-quasiconformal harmonic mapping in \( \mathbb{D} \) with \( f(0) = 0 \). If \( A(f(\mathbb{D})) < \infty \), then \( f \in h^2 \) and

\[
\|f\|_2^2 \leq KA(f(\mathbb{D})). \tag{2.1}
\]

Moreover, if \( K = 1 \), then the estimate of (2.1) is sharp and the extremal function is \( f(z) = z \).
Proof. Let $f$ be a sense-preserving and $K$-quasiconformal harmonic mapping in $\mathbb{D}$ with $f(0) = 0$. Then, by the definition of $S_f(r)$, we see that

$$S_f(1) = A(f(\mathbb{D})) = \int_{\mathbb{D}} J_f(z) \, d\sigma(z) = \int_{\mathbb{D}} \Lambda_f(z) \lambda_f(z) \, d\sigma(z)$$

$$\geq \frac{1}{K} \int_{\mathbb{D}} \Lambda_f^2(z) \, d\sigma(z)$$

$$\geq \frac{1}{K} \int_{\mathbb{D}} (|z'(z)|^2 + |f'(z)|^2) \, d\sigma(z)$$

$$= \frac{1}{K} \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2)$$

$$\geq \frac{1}{K} \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2)$$

$$= \frac{1}{K} \|f\|_2^2.$$ 

Then $\|f\|_2^2 \leq KA(f(\mathbb{D}))$. Furthermore, if $K = 1$, then the function $f(z) = z$ shows the estimate of (2.1) is sharp. The proof of this lemma is complete. \qed

Proof of Theorem 2. Let $D_1 = f(\Omega_1) \cap \Omega_2$ and $D_2 = f^{-1}(\Omega_2)$. It is not difficult to see that $D_2 = f^{-1}(D_1)$. Without loss of generality, we assume that for $k = 1, 2$, $A(D_k) < \infty$. Let $E$ be the component of the open set $D_2$ containing the original point. Then there is a universal covering mapping $\varphi$ such that $\varphi : \mathbb{D} \to E$ with $\varphi(0) = 0$. Let $F = f \circ \varphi$. It is easy to see that $F$ is also a $K$-quasiconformal harmonic mapping. By using Lemma 1, we have

(2.2) \quad \|\varphi\|_2^2 \leq A(E) \leq A(D_2)

and

(2.3) \quad \|F\|_2^2 \leq A(f(E)) \leq KA(f(D_2)) = KA(D_1),

which imply that

(2.4) \quad \|\varphi\|_2^2 + \|F\|_2^2 \leq A(D_2) + KA(D_1).

Since $\varphi$ and $F$ belong to $h^2$, we conclude that the linear measure $m(\gamma)$ of

$$\gamma = \{\xi \in \partial \mathbb{D} : \text{both } |\varphi(\xi)| \text{ and } |F(\xi)| \text{ are finite}\}$$

is $2\pi$. Let

$$\gamma_\varphi = \{\xi \in \gamma : |\varphi(\xi)| \geq d_{\Omega_1}(0)\}$$

and, similarly,

$$\gamma_F = \{\xi \in \gamma : |F(\xi)| \geq d_{\Omega_2}(0)\}.$$

Then

(2.5) \quad \|\varphi\|_2^2 \geq \frac{1}{2\pi} \int_{\gamma_\varphi} d_{\Omega_1}^2(0) \, d\xi \geq \frac{d_{\Omega_1}^2(0)m(\gamma_\varphi)}{2\pi}$$. 

and

\begin{equation}
\|F\|_2^2 \geq \frac{1}{2\pi} \int_{\gamma} d\Omega_2(0) \|d\xi\| \geq \frac{d^2(0)m(\gamma) + d^2(0)m(\gamma_F)}{2\pi}.
\end{equation}

**Claim.** \( \gamma = \gamma_\varphi \cup \gamma_F \).

Suppose \( \gamma \neq \gamma_\varphi \cup \gamma_F \). Then there is a \( \xi_0 \in \gamma \setminus (\gamma_\varphi \cup \gamma_F) \) such that \( \varphi(\xi) \in \Omega_1 \) with \( |\varphi(\xi)| < d\Omega_1(0) \), and \( F(\xi) \in \Omega_2 \) with \( |F(\xi)| < d\Omega_2(0) \). Since \( f \) is continuous at \( \varphi(\xi_0) \), we know that

\[ F(\xi) = \lim_{r \to 1} F(r\xi) = \lim_{r \to 1} f(\varphi(r\xi)) = f(\varphi(\xi)). \]

On the other hand, \( l = \{ \varphi(r\xi) : r \in [0, 1] \} \) is a curve joining 0 and \( \varphi(\xi) \) in \( \Omega_1 \). Hence \( \varphi(\xi) \in \Omega \) which is a contradiction with the covering property of \( \mathbb{D} \) induced by \( \varphi \) over \( \Omega \).

Hence by (2.5), (2.6) and the Claim, we get

\[ A(D_2) + KA(D_1) \geq \|\varphi\|_2^2 + \|F\|_2^2 \geq \frac{m(\gamma) + m(\gamma_F)}{2\pi} \min\{d^2(0), d^2(0)\} \]

Next we prove the sharpness part. We consider the case that \( K = 1 \). Let \( \Omega_1 = \Omega_2 = \mathbb{D} \) and for \( z \in \mathbb{D} \), let \( f(z) = tz \), where \( 0 < t < 1 \). Then

\[ A(D_1) + A(D_2) = (1 + t^2) \]

and \( d(0) = 1 \). The arbitrariness of \( t \) shows the estimate of (1.3) is sharp. The proof of the theorem is complete. \( \Box \)

The following result is well-known.

**Lemma C.** Among all rectifiable Jordan curves of a given length, the circle has the maximum interior area.

**Proof of Theorem 3.** We first prove (1.4). By elementary computations, we have

\[ l_f(r) = r \int_0^{2\pi} \left| f_z(re^{i\theta}) - e^{-2i\theta} f_z(re^{i\theta}) \right| d\theta \]

\[ \geq r \int_0^{2\pi} \left( |f_z(re^{i\theta})| - |f_z(re^{i\theta})| \right) d\theta \]

\begin{equation}
(2.7) \quad \geq \frac{r}{K} \int_0^{2\pi} \Lambda_f(re^{i\theta}) d\theta.
\end{equation}

Cauchy’s integral formula applied to \( h'(z) = f_z(z) \) and \( g'(z) = \overline{f_z(z)} \) shows for \( n \geq 1 \),

\begin{equation}
na_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f_z(z)}{z^n} dz \quad \text{and} \quad nb_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{\overline{f_z(z)}}{z^n} dz,
\end{equation}

and

\begin{equation}
(2.6) \quad \|F\|_2^2 \geq \frac{1}{2\pi} \int_{\gamma} d\Omega_2(0) |d\xi| \geq \frac{d^2(0)m(\gamma) + d^2(0)m(\gamma_F)}{2\pi}.
\end{equation}
respectively. By (2.7) and (2.8), we get
\begin{align*}
n(|a_n| + |b_n|) &= \frac{1}{2\pi} \left( \left| \int_{|z|=r} \frac{f_z(z)}{z^n} \, dz \right| + \left| \int_{|z|=r} \frac{\overline{f_z(z)}}{z^n} \, dz \right| \right) \\
&\leq \frac{1}{2\pi r^n} \int_0^{2\pi} r \Lambda_f(re^{i\theta}) \, d\theta \\
&\leq \frac{K l_f(r)}{2\pi r^n} \leq \frac{K l_f(1)}{2\pi r^n},
\end{align*}
which implies that
\[ |a_n| + |b_n| \leq \frac{K l_f(1)}{2n\pi}. \]

Now we are ready to prove the inequality (1.5). First we observe that
\begin{equation}
S_f(r) = \int_{D_r} J_f(z) \, d\sigma(z) \geq \frac{1}{K} \int_{D_r} \Lambda_f^2(z) \, d\sigma(z). \tag{2.9}
\end{equation}
For \( \theta \in [0, 2\pi) \) and \( z \in \mathbb{D} \), let \( P_\theta(z) = (f_z(z) + e^{i\theta} \overline{f_z(z)})^2 \). By (2.9) and subharmonicity of \( |P_\theta| \), we have
\begin{align*}
|P_\theta(z)| &\leq \frac{1}{\pi(1-|z|)^2} \int_0^{1-|z|} \int_0^{2\pi} |P_\theta(z + \rho e^{i\beta})| \rho \, d\beta \, d\rho \\
&\leq \frac{1}{(1-|z|)^2} \int_{D_{1-|z|}} \Lambda_f^2(z) \, d\sigma(z) \\
&\leq \frac{S_f(1)K}{(1-|z|)^2},
\end{align*}
and the arbitrariness of \( \theta \in [0, 2\pi) \) gives the inequality
\begin{equation}
\Lambda_f^2(z) \leq \frac{S_f(1)K}{(1-|z|)^2}. \tag{2.10}
\end{equation}
By Lemma C, we get
\begin{equation}
S_f(r) \leq \frac{l_f^2(r)}{4\pi^2}. \tag{2.11}
\end{equation}
By (2.10) and (2.11), we have
\[ \Lambda_f^2(z) \leq \frac{l_f^2(1)K}{4\pi^2(1-|z|)^2}, \]
which gives
\begin{equation}
\Lambda_f(z) \leq \frac{l_f(1)\sqrt{K}}{2\pi(1-|z|)}. \tag{2.12}
\end{equation}
Finally, \( f \in \mathcal{HB} \) easily follows from (2.12). The proof of this theorem is complete. \( \square \)
3. Bloch and Lipschitz spaces on harmonic mappings

**Lemma 2.** Let $\omega$ be a majorant. For $t > 0$, if $\lambda \geq 1$, then

$$(3.1) \quad \omega(\lambda t) \leq \lambda \omega(t).$$

**Proof.** The inequality (3.1) easily follows from the monotonicity of $\omega(t)/t$ for $t > 0$. The proof of this lemma is complete. □

**Lemma D.** ([10, Lemma 1]) Let $z$, $w$ be complex numbers. Then

$$\max_{\theta \in [0,2\pi]} |w \cos \theta + z \sin \theta| = \frac{1}{2}(|w + iz| + |w - iz|).$$

**Proof of Theorem 4.** (a)$\iff$(c) easily follows from [7, Theorem 1.1]. We only need to prove (a)$\iff$(b). We first prove that (a)$\implies$(b). Let $z, w \in \mathbb{D}$ with $z \neq w$, and let $\varphi(t) = zt + (1-t)w$, where $t \in [0,1]$. Since $|\varphi(t)| \leq t|z| + (1-t)|w|$, we see that

$$1 - |\varphi(t)| = 1 - t|z| - |w| + tw \geq 1 - t + |w|(t - 1) = (1-t)d(w)$$

and

$$1 - |\varphi(t)| = 1 - t|z| - |w| + tw \geq 1 - t|z| - (1-t) = td(z).$$

Using the last two inequalities, one has

$$(1 - |\varphi(t)|)^2 \geq (1-t)td(w)d(z)$$

and therefore, we get

$$(3.2) \quad \frac{1}{1-|\varphi(t)|} \leq \frac{1}{\sqrt{(1-t)td(w)d(z)}}.$$

By Lemma 2 and the inequality (3.2), for any $z, w \in \mathbb{D}$ with $z \neq w$, we have

$$|f(z) - f(w)| = \left| \int_0^1 \frac{df}{dt}(\varphi(t)) \, dt \right| \quad \text{for } \zeta = \varphi(t) = w + t(z-w)$$

$$= \left| (z-w) \int_0^1 f'_\zeta(\varphi(t)) \, dt + (\overline{z} - \overline{w}) \int_0^1 f''_\zeta(\varphi(t)) \, dt \right|$$

$$\leq |z-w| \int_0^1 (|f'_\zeta(\varphi(t))| + |f''_\zeta(\varphi(t))|) \, dt$$

$$\leq |z-w| \int_0^1 \Lambda_f(\varphi(t)) \omega \left( \frac{1}{1-|\varphi(t)|} \right) dt.$$
\[ \begin{align*}
&\leq C|z - w| \int_0^1 \omega \left( \frac{1}{1 - |\phi(t)|} \right) dt \\
&\leq C|z - w| \omega \left( \frac{1}{\sqrt{d(w) d(z)}} \right) \int_0^1 \frac{1}{\sqrt{1 - t}} dt \\
&\leq C|z - w| \omega \left( \frac{1}{\sqrt{d(w) d(z)}} \right) \int_0^1 \frac{1}{\sqrt{1 - t}} dt \\
&= C|z - w| \omega \left( \frac{1}{\sqrt{d(w) d(z)}} \right) \int_0^\pi \frac{2 \sin \theta \cos \theta}{\sqrt{\sin^2 \theta \cos^2 \theta}} d\theta \\
&= C\pi|z - w| \omega \left( \frac{1}{\sqrt{d(w) d(z)}} \right),
\end{align*} \]

for some constant \( C > 0 \), which gives

\[ \frac{|f(z) - f(w)|}{|z - w|} \leq \pi C \omega \left( \frac{1}{\sqrt{d(w) d(z)}} \right). \]

Now we prove that \((b)\implies(a)\). Let \( f = h + g \), where \( h \) and \( g \) are analytic in \( \mathbb{D} \). By Lemma D, we obtain

\[ \max_{\theta \in [0, 2\pi]} |f_x(z) \cos \theta + f_y(z) \sin \theta| = \frac{1}{2} (|f_x(z) + if_y(z)| + |f_x(z) - if_y(z)|) \]

\[ = \frac{1}{2} (|g_x(z)| + |2h_x(z)|) \]

\[ = |h'(z)| + |g'(z)| \]

\[ = \Lambda_f(z). \]

(3.3)

For \( r \in (0, 1) \) and \( \theta \in [0, 2\pi] \), let \( w = z + re^{i\theta} \). Then

\[ \lim_{r \to 0^+} \frac{|f(z) - f(w)|}{z - w} = \lim_{r \to 0^+} \frac{|f(z) - f(z + re^{i\theta})|}{r} \]

\[ = |f_x(z) \cos \theta + f_y(z) \sin \theta| \]

\[ \leq C \lim_{r \to 0^+} \omega \left( \frac{1}{\sqrt{d(z) d(z + re^{i\theta})}} \right) \]

\[ = C \omega \left( \frac{1}{d(z)} \right) \]

(3.4)

for some constant \( C > 0 \). By (3.3) and (3.4), we conclude that

\[ \Lambda_f(z) = \max_{\theta \in [0, 2\pi]} |f_x(z) \cos \theta + f_y(z) \sin \theta| \leq \max_{\theta \in [0, 2\pi]} C \omega \left( \frac{1}{d(z)} \right) = C \omega \left( \frac{1}{d(z)} \right). \]

Hence \((a)\iff(b)\iff(c)\). The proof of the theorem is complete. \qed
Proof of Theorem 5. We first prove the necessity. Since $\Omega$ is a $L_w$-extension domain, we see that for any $z, w \in \Omega$, by using (1.9), there is a rectifiable curve $\gamma \subset \Omega$ joining $z$ to $w$ such that

$$|f(z) - f(w)| \leq \int_{\gamma} \Lambda_f(\zeta) \, ds(\zeta) \leq C \int_{\gamma} \omega\left(\frac{d\Omega(\zeta)}{d\Omega(\zeta)}\right) \, ds(\zeta) \leq C\omega(|z - w|)$$

for some constant $C > 0$.

Now we prove the sufficiency. Let $z \in \Omega$ and $r = d_{\Omega}(z)/2$. For all $w \in D(z, r)$, using (1.3), we get

$$f(w) = \frac{1}{2\pi} \int_{0}^{2\pi} P(w, re^{i\theta}) f(re^{i\theta} + z) \, d\theta,$$

where

$$P(w, re^{i\theta}) = \frac{r^2 - |w - z|^2}{|w - z - re^{i\theta}|^2}.$$

By elementary calculations, we have

$$\frac{\partial}{\partial w} P(w, re^{i\theta}) = -\left(\frac{w - z}{w - z - re^{i\theta}}\right)^2 - \frac{r^2 - |w - z|^2}{(w - z - re^{i\theta})^3},$$

and

$$\frac{\partial}{\partial \overline{w}} P(w, re^{i\theta}) = -\left(\frac{w - z}{w - z - re^{i\theta}}\right)^2 - \frac{r^2 - |w - z|^2}{(w - z - re^{i\theta})^3}.$$

Then for all $w \in D(z, r/2)$,

$$\left| \frac{\partial}{\partial w} P(w, re^{i\theta}) \right| \leq \frac{|w - z||w - z - re^{i\theta}|^2 + (r^2 - |w - z|^2)|w - z - re^{i\theta}|}{|w - z - re^{i\theta}|^4} \leq \frac{r^2 + r^2}{r^4/4} = \frac{21}{2r},$$

and

$$\left| \frac{\partial}{\partial \overline{w}} P(w, re^{i\theta}) \right| \leq \frac{21}{2r},$$

which implies that

$$\Lambda_f(w) = \frac{1}{2\pi} \left( \left| \int_{0}^{2\pi} \frac{\partial}{\partial w} P(w, re^{i\theta}) (f(z + re^{i\theta}) - f(z)) \, d\theta \right| + \left| \int_{0}^{2\pi} \frac{\partial}{\partial \overline{w}} P(w, re^{i\theta}) (f(z + re^{i\theta}) - f(z)) \, d\theta \right| \right) \leq \frac{21C}{2\pi} \int_{0}^{2\pi} \left| f(z + re^{i\theta}) - f(z) \right| \, d\theta \leq \frac{21C}{2\pi} \int_{0}^{2\pi} \left( \frac{d\Omega(\zeta)}{2\pi} \right) \, d\Omega(\zeta).$$
Since $\omega(t)$ is increasing on $t \in (0, \infty)$, we conclude that

$$
\Lambda_f(z) \leq \frac{21C}{\pi} \frac{\omega\left(\frac{d\Omega(z)}{2}\right)}{d\Omega(z)} \leq \frac{21C}{\pi} \omega\left(d\Omega(z)\right),
$$

for some constant $C > 0$. The proof of the theorem is complete. □

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