Three-Body-Cluster Effects on $\Lambda$ Single-Particle Energies in $^{17}\Lambda O$ and $^{41}\Lambda Ca$

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A method for a microscopic description of $\Lambda$ hypernuclei is formulated in the framework of the unitary-model-operator approach. A unitarily transformed Hamiltonian is introduced and given in a cluster expansion form. The structure of three-body-cluster terms are discussed especially on the $\Lambda$ single-particle energy. The $\Lambda$ single-particle energies including the three-body-cluster contributions are calculated for the $0s_{1/2}$, $0p_{3/2}$ and $0p_{1/2}$ states in $^{17}\Lambda O$, and for the $0s_{1/2}$, $0p_{3/2}$, $0p_{1/2}$, $0d_{5/2}$, $0d_{3/2}$ and $1s_{1/2}$ states in $^{41}\Lambda Ca$, using the Nijmegen soft-core (NSC), NSC97a-f, the Jülich $\Lambda$ (J$\Lambda$) and JB hyperon-nucleon interactions. It is indicated that the three-body-cluster terms bring about sizable effects in the magnitudes of the $\Lambda$ single-particle energies, but hardly affect the $\Lambda$ spin-orbit splittings.

§1. Introduction

Much attention has been paid to the study of hypernuclear physics in recent years. One of the fundamental problems in hypernuclear physics is to understand properties of hypernuclei starting from the basic interaction between two kinds of constituent particles, nucleon ($N$) and hyperon ($Y$). Our knowledge on the $YN$ interaction is, however, quite inadequate because of the very limited experimental information on the $YN$ scattering. In such a situation many-body theoretical studies of the hypernuclear structure could provide alternative information on the $YN$ interaction. For this purpose, it would be desirable to have a theoretical method which enables us to make a high-precision calculation.

In many of microscopic structure calculations of hypernuclei, the $G$-matrix has been introduced as a $YN$ effective interaction. The $G$-matrix approach has been widely applied, and the use of this type of effective interaction has made important contributions to the description of hypernuclei.¹⁻⁷ However, we may say from a formal point of view that the $G$-matrix is energy-dependent and non-hermitian, and does not have the property of decoupling between a model space of low-lying two-particle states and its complement. The $G$-matrix itself is not considered to be an effective interaction in a formal sense of the effective interaction theory because it does not have the property of decoupling.⁸ In order to derive such an effective interaction we should add some higher-order corrections such as folded diagrams.⁹,¹⁰ It would be desirable if we could have a theory to describe many-body systems in

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terms of an energy-independent and hermitian effective interaction with the property of decoupling.

We have been proposed a many-body theory, the unitary-model-operator approach (UMOA),\(^{11}\) that was formulated on the basis of such an effective interaction. The UMOA was applied to finite nuclei \(^{16}\)O\(^{11}\) and \(^{40}\)Ca,\(^{12}\) and some encouraging results were obtained. Furthermore, in the previous work,\(^{13}\) we extended the formulation of the UMOA to a description of \(\Lambda\) hypernuclei and made calculations of properties of \(^{17}\)O which is a typical closed-shell nucleus plus one \(\Lambda\) system, using some realistic \(YN\) interactions.

The formulation of the UMOA is based essentially on a cluster expansion of a unitarily transformed hamiltonian in terms of a two-body correlation operator.\(^{14},\,^{15}\) In the previous work we gave a method of treating correlations within the two-body-cluster term, and the three-or-more-body-cluster terms remain to be evaluated. It has been known that the expansion of the unitarily transformed hamiltonian does not, in principle, terminate in finite order. Therefore, it needs to give a method of evaluating the three-or-more-body-cluster terms and confirm numerically convergence of the cluster expansion. It is of high interest to investigate to what extent the three-body-cluster term would affect the \(\Lambda\) single-particle energy, especially, the \(\Lambda\) spin-orbit splitting. The calculation of the \(\Lambda\) spin-orbit splitting is one of the important tasks in hypernuclear physics because it has a possibility of providing information to the determination of some parameters in the \(YN\) interaction such as the strengths of the spin-orbit and antisymmetric spin-orbit components.

The purposes of this work are to formulate a method of evaluating the three-body-cluster effect in the framework of the UMOA, and apply it to \(^{17}\)O and \(^{41}\)Ca by employing various realistic \(YN\) interactions. This paper is organized as follows: In \(\S\) 2 we present a formulation of the UMOA to treat the three-body-cluster term. In \(\S\) 3 we apply the UMOA to \(^{17}\)O and \(^{41}\)Ca by employing the Nijmegen soft-core (NSC),\(^{16}\) NSC97a-f,\(^{17}\) the Jülich \(\Lambda\) (JA) and JB\(^{18}\) \(YN\) potentials. In \(\S\) 4 we make some concluding remarks.

\(\S\) 2. Formulation

2.1. Cluster expansion of the unitarily transformed hamiltonian

We first consider a hamiltonian of a hypernuclear system consisting of nucleons and one \(\Lambda\) (or \(\Sigma\)) in the second-quantization form as

\[
H = \sum_{\alpha \beta} \langle \alpha | t_{N_1} | \beta \rangle c_\alpha^\dagger c_\beta + \frac{1}{4} \sum_{\alpha \beta \gamma \delta} \langle \alpha \beta | v_{N_1, N_2} | \gamma \delta \rangle c_\alpha^\dagger c_\beta^\dagger c_\delta c_\gamma + \sum_{\mu \nu} \langle \mu | t_Y + \Delta m_Y | \nu \rangle d_\mu^\dagger d_\nu + \sum_{\mu \nu} \langle \mu \alpha | v_{YN_1} | \nu \beta \rangle d_\mu^\dagger c_\alpha c_\beta d_\nu, \quad (2.1)
\]

where \(c^\dagger (c)\) is the creation (annihilation) operator for a nucleon in the usual notation, and \(d^\dagger (d)\) is the creation (annihilation) operator for a hyperon, \(\Lambda\) or \(\Sigma\). The kinetic energies of a nucleon and a hyperon are denoted by \(t_{N_1}\) and \(t_Y\), respectively.
The \( v_{N_1N_2} \) and \( v_{YN_1} \) represent the nucleon-nucleon (NN) and hyperon-nucleon (YN) interactions, respectively. The notations \( \alpha, \beta, \gamma \) and \( \delta \) are used for the sets of quantum numbers of nucleon states, and \( \mu \) and \( \nu \) for those of hyperon states. We note here that \( |\alpha\beta\rangle \) is the antisymmetrized and normalized two-body NN state. The \( \Delta m_Y \) denotes the difference between the rest masses of \( \Lambda \) and \( \Sigma \).

In nuclear many-body problems, it is important to treat properly the short-range two-body correlation because realistic NN and YN interactions have strongly repulsive cores at short distance. For this purpose, we introduce a unitary transformation of the hamiltonian as

\[
\hat{H} = e^{-S} H e^{S},
\]

where \( S \) is the sum of anti-hermitian two-body operators for NN and YN systems defined as

\[
S = S^{(NN)} + S^{(YN)}
\]

with

\[
S^{(NN)} = \frac{1}{4} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | S_{N_1N_2} | \gamma\delta \rangle c_\alpha \beta c_\gamma \delta c_\gamma \delta,
\]

\[
S^{(YN)} = \sum_{\mu\alpha\nu\beta} \langle \mu\alpha | S_{YN} | \nu\beta \rangle d_\mu \alpha c_\beta d_\nu
\]

According to Providência and Shakin,\(^{(14)}\) we make cluster expansion of the unitarily transformed hamiltonian as

\[
\hat{H} = \hat{H}^{(1)} + \hat{H}^{(2)} + \hat{H}^{(3)} + \cdots,
\]

where the first three terms are written explicitly as

\[
\hat{H}^{(1)} = \sum_{\alpha\beta} \langle \alpha | h_{N_1} | \beta \rangle c_\alpha \beta + \sum_{\mu\nu} \langle \mu | h_Y | \nu \rangle d_\mu \nu,
\]

\[
\hat{H}^{(2)} = \left( \frac{1}{2!} \right)^2 \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | \tilde{v}_{N_1N_2} | \gamma\delta \rangle c_\alpha \beta c_\gamma \delta c_\gamma \delta - \sum_{\alpha\beta} \langle \alpha | u_{N_1} | \beta \rangle c_\alpha \beta
\]

\[
+ \sum_{\mu\alpha\nu\beta} \langle \mu\alpha | \tilde{v}_{YN} | \nu\beta \rangle d_\mu \alpha c_\beta d_\nu - \sum_{\mu\nu} \langle \mu | u_Y | \nu \rangle d_\mu \nu,
\]

and

\[
\hat{H}^{(3)} = \left( \frac{1}{3!} \right)^2 \sum_{\alpha\beta\gamma\delta\epsilon\zeta} \langle \alpha\beta\gamma | \tilde{v}_{N_1N_2N_3} | \delta\epsilon\zeta \rangle c_\alpha \beta \gamma c_\delta \epsilon \zeta c_\epsilon \delta c_\delta
\]
The terms nucleon and a hyperon, respectively, and in this stage they are arbitrary. The \( 4 \) S. Fujii, R. Okamoto and K. Suzuki

\[
\begin{align*}
&\mathbb{H} = (\frac{1}{2!})^2 \sum_{\alpha \beta \gamma \delta} (\alpha \beta | \tilde{u}_{N_1 N_2} | \gamma \delta) c^\dagger_\alpha \beta c^\dagger_\gamma \delta c_\gamma \\
+& (\frac{1}{2!})^2 \sum_{\mu \alpha \beta \beta \gamma \delta} (\mu \alpha \beta | \tilde{v}_{Y N_1 N_2} | \nu \gamma \delta) d^\dagger_\mu \alpha c^\dagger_\beta c^\dagger_\gamma \delta d_\nu \\\n- & \sum_{\mu \alpha \beta \beta \gamma \delta} (\mu \alpha \beta | \tilde{u}_{Y N_1} | \nu \gamma \delta) d^\dagger_\mu \alpha c^\dagger_\beta c^\dagger_\gamma \delta d_\nu.
\end{align*}
\] (2.9)

Here \( |\alpha \beta \gamma\rangle \) is the antisymmetrized and normalized three-body NNN state, and the NN part of \( |\mu \alpha \beta\rangle \) is antisymmetrized and normalized. In the above, we have used the notations given as follows: Since the exponent \( S \) is a two-body operator, the one-body operators \( h_k \) for \( k = N_1 \) and \( Y \) in \( \mathbb{H} \) are unchanged and given by

\[
h_{N_1} = t_{N_1} + u_{N_1},
\] (2.10)

\[
h_{Y} = t_{Y} + u_{Y} + \Delta m_{Y}.
\] (2.11)

The terms \( u_k \) for \( k = N_1 \) and \( Y \) are the auxiliary single-particle potentials of a nucleon and a hyperon, respectively, and in this stage they are arbitrary. The \( \tilde{v}_{k l} \) for \( \{ k l \} = \{ N_1 N_2 \} \) and \( \{ Y N_1 \} \) are the transformed two-body interactions for NN andYN systems, and they are given by

\[
\tilde{v}_{N_1 N_2} = e^{-S_{N_1 N_2}} (h_{N_1} + h_{N_2} + v_{N_1 N_2}) e^{S_{N_1 N_2}} - (h_{N_1} + h_{N_2}),
\] (2.12)

\[
\tilde{v}_{Y N_1} = e^{-S_{Y N_1}} (h_{Y} + h_{N_1} + v_{Y N_1}) e^{S_{Y N_1}} - (h_{Y} + h_{N_1}).
\] (2.13)

The transformed hamiltonian \( \tilde{\mathbb{H}} \) contains, in general, three-ormore-body interactions even if the starting hamiltonian \( H \) in Eq. (2.1) does not include three-or-more-body interactions. The transformed three-body interactions \( \tilde{v}_{k l m} \) for \( \{ k l m \} = \{ N_1 N_2 N_3 \} \) and \( \{ Y N_1 N_2 \} \) are given, respectively, by

\[
\tilde{v}_{N_1 N_2 N_3} = e^{-S_{N_1 N_2 N_3}} (h_{N_1} + h_{N_2} + h_{N_3} + v_{N_1 N_2} + v_{N_2 N_3} + v_{N_3 N_1}) e^{S_{N_1 N_2 N_3}} \\
- (h_{N_1} + h_{N_2} + h_{N_3} + \tilde{v}_{N_1 N_2} + \tilde{v}_{N_2 N_3} + \tilde{v}_{N_3 N_1}),
\] (2.14)

\[
\tilde{v}_{Y N_1 N_2} = e^{-S_{Y N_1 N_2}} (h_{Y} + h_{N_1} + h_{N_2} + v_{Y N_1} + v_{N_1 N_2} + v_{N_2 Y}) e^{S_{Y N_1 N_2}} \\
- (h_{Y} + h_{N_1} + h_{N_2} + \tilde{v}_{Y N_1} + \tilde{v}_{N_1 N_2} + \tilde{v}_{N_2 Y}),
\] (2.15)

where the operators \( S_{k l m} \) for \( \{ k l m \} = \{ N_1 N_2 N_3 \} \) and \( \{ Y N_1 N_2 \} \) denote the sum of the two-body correlation operators defined as

\[
S_{N_1 N_2 N_3} = S_{N_1 N_2} + S_{N_2 N_3} + S_{N_3 N_1},
\] (2.16)

\[
S_{Y N_1 N_2} = S_{Y N_1} + S_{N_1 N_2} + S_{N_2 Y}.
\] (2.17)
We here note that the unitary transformation of the auxiliary potentials \( u_k \) for \( k = N_1 \) (\( N_2 \)) and \( Y \) generates two-body operators written as

\[
\tilde{u}_{N_1 N_2} = e^{-S_{N_1 N_2}}(u_{N_1} + u_{N_2}) e^{S_{N_1 N_2}} - (u_{N_1} + u_{N_2}),
\]

(2.18)

\[
\tilde{u}_{YN_1} = e^{-S_{YN_1}}(u_Y + u_{N_1}) e^{S_{YN_1}} - (u_Y + u_{N_1}).
\]

(2.19)

In nuclear many-body problems, it is important how to choose the auxiliary potential \( u_k \). In general, it is useful to introduce \( u_{N_1} \) (\( u_{N_2} \)) and \( u_Y \) as self-consistent potentials defined with the transformed two-body interactions \( \tilde{v}_{N_1 N_2} \) and \( \tilde{v}_{YN_1} \) as

\[
\langle \alpha | u_{N_1} | \beta \rangle = \sum_{\xi \leq \rho_F} \langle \alpha \xi | \tilde{v}_{N_1 N_2} | \beta \xi \rangle,
\]

(2.20)

\[
\langle \mu | u_{Y} | \nu \rangle = \sum_{\xi \leq \rho_F} \langle \mu \xi | \tilde{v}_{YN_1} | \nu \xi \rangle,
\]

(2.21)

where \( \rho_F \) is the uppermost occupied level, and the notation \( \xi \) means an occupied state for nucleons. If \( u_{N_1} \) and \( u_Y \) are defined as given in Eqs. (2.20) and (2.21), one can prove that the second and fourth terms on the right-hand side of Eq. (2.8) are cancelled by the contributions of the bubble diagrams of the first and third terms, respectively. In the same way, the second and fourth terms on the right-hand side of Eq. (2.9) are also cancelled by the bubble-diagram contributions of the first and third terms, respectively. Therefore, the cluster terms \( \bar{H}^{(1)} \), \( \bar{H}^{(2)} \) and \( \bar{H}^{(3)} \) include only the one-, two- and three-body operators, respectively, if we write them in the normal-product form with respect to particles and holes as discussed in Ref. 11). As has been well known, this cancellation mechanism is advantageous in perturbative calculations, because there is no need of evaluating \( u \)-insertion diagrams and bubble diagrams.

An important problem in the present approach is how to determine the two-body correlation operators \( S_{kl} \) for \( \{kl\} = \{N_1 N_2\} \) and \( \{YN_1\} \). These operators are given as solutions to equations of decoupling as

\[
Q_{kl} e^{-S_{kl}} (h_k + h_l + \tilde{v}_{kl}) e^{S_{kl}} P_{kl} = 0,
\]

(2.22)

where \( P_{kl} \) and \( Q_{kl} \) for \( \{kl\} = \{N_1 N_2\} \) and \( \{YN_1\} \) are projection operators which act in the space of two-body states and project a two-body state onto the low-momentum model space and the high-momentum space, respectively. It has been proved that Eq. (2.22) for \( S_{kl} \) can be solved in a nonperturbative way\(^{11},^{13} \) under the restrictive conditions

\[
P_{kl} S_{kl} P_{kl} = Q_{kl} S_{kl} Q_{kl} = 0.
\]

(2.23)

In the previous work\(^{13} \) the structure of the two-body-cluster term for the \( \Lambda \) single-particle energy was studied, and three-or-more-body cluster terms were not considered. After the evaluation of the two-body-cluster term, the next important
term will be the three-body-cluster (TBC) term. The significance of studying the TBC term may be understood from the following two reasons: The first is to examine whether the TBC effect is large or not in comparison with the one- and two-body-cluster effects, in other words, whether the cluster expansion of $\bar{H}$ is convergent so fast or not as to be useful in a practical application. The second is that, when we investigate effects of the real three-body force, we must be careful of the possibility that such three-body-force effects may interfere with effects of the effective three-body force.

2.2. Structure of the three-body-cluster term

We here discuss general structure of the TBC term $\bar{H}^{(3)}$ in Eq. (2.9). The TBC term $\bar{H}^{(3)}$ contains the transformed three-body $NNN$ and $YN$ interactions. These interactions include the bare two-body $NN$ and $YN$ interactions which have strongly repulsive cores. Therefore, we represent the transformed three-body interactions in terms of the transformed two-body $NN$ and $YN$ interactions $\tilde{v}_{ijk}$ which are well-behaved in the actual calculation.

Using a factorization formula of Campbell and Hausdorf, we have for $\exp(S_{ijk})$

$$e^{S_{ijk}} = e^{S_{ij}}e^{S_{(ij)k}} = e^{S_{jk}}e^{S_{(jk)i}} = e^{S_{ki}}e^{S_{(ki)j}}$$

(2.24)

with

$$S_{(ij)k} = S_{jk} + S_{ki} - \frac{1}{2}[S_{ij}, S_{jk} + S_{ki}]$$

$$- \frac{1}{12}[S_{ij}, S_{jk} + S_{ki}, S_{ijk} + S_{ij} + S_{ji}] + \cdots.$$  

(2.25)

Hereafter, a set $\{ijk\}$ denotes that of $\{N_1N_2N_3\}$ or $\{YN_N\}$. The correlation operators $S_{(jk)i}$ and $S_{(ki)j}$ are also written in a similar way. Substituting $\exp(S_{ijk})$ in Eq. (2.24) into Eqs. (2.14) and (2.15), we have

$$\tilde{v}_{ijk} = \tilde{v}_{ijk}^{(v)} + \tilde{v}_{ijk}^{(h)}$$

(2.26)

with

$$\tilde{v}_{ijk}^{(v)} = \sum_{(ijk)}(e^{-S_{(ij)k}}\tilde{v}_{ij}e^{S_{(ij)k}})$$

(2.27)

and

$$\tilde{v}_{ijk}^{(h)} = \sum_{(ijk)}[e^{-S_{(ij)k}}(h_i + h_j)e^{S_{(ij)k}} - (h_i + h_j)]$$

$$+ (h_i + h_j + h_k - e^{-S_{(ij)k}}(h_i + h_j + h_k)),$$

(2.28)

where $\sum_{(ijk)}$ denotes the summation of exchange terms defined for an arbitrary function $f(ijk)$ as

$$\sum_{(ijk)} f(ijk) = f(ijk) + f(jki) + f(kij).$$

(2.29)
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The terms $\tilde{v}_{ijk}^{(v)}$ and $\tilde{v}_{ijk}^{(h)}$ can be expanded into series in powers of $S_{ij}$, $S_{jk}$ and $S_{ki}$. The term $\tilde{v}_{ijk}^{(v)}$ is given by

$$
\tilde{v}_{ijk}^{(v)} = \sum_{(ijk)} \left\{ [\tilde{v}_{ij}, S_{jk} + S_{ki}] - \frac{1}{2} [\tilde{v}_{ij}, [S_{ij}, S_{jk} + S_{ki}]] 
+ \frac{1}{2} [[\tilde{v}_{ij}, S_{jk} + S_{ki}], S_{jk} + S_{ki}]] + \cdots \right\}.
$$

(2.30)

Similarly, $\tilde{v}_{ijk}^{(h)}$ is given by

$$
\tilde{v}_{ijk}^{(h)} = \sum_{(ijk)} \left\{ -\frac{1}{2} [[h_i + h_j, S_{ij}], S_{jk} + S_{ki}]
+ \frac{1}{6} [[h_i + h_j, S_{ij}], S_{ij}, S_{jk} + S_{ki}]
+ \frac{1}{3} [[h_i + h_j, S_{ij}], S_{jk} + S_{ki}, S_{ij}]] + \cdots \right\}.
$$

(2.31)

The expressions of $\tilde{v}_{ijk}^{(v)}$ and $\tilde{v}_{ijk}^{(h)}$ give us desirable forms for the calculation of the TBC terms because they do not include the bare interaction $\tilde{v}_{ij}$ and are written in terms of only the terms $h_i$, $\tilde{v}_{ij}$ and $S_{ij}$. The TBC effect on the ground-state and one-body energies for ordinary nuclei such as $^{16}O$ and $^{40}Ca$ has already been studied.\cite{11,12,15} In the present work we study the effect of the TBC term on the energy of $\Lambda$.

2.3. Derivation of the three-body-cluster contribution to the $\Lambda$ single-particle energy

The contribution of the TBC term to a $\Lambda$ single-particle energy in a state $\lambda_{\Lambda}$ is derived from Eq. (2.9) by making the summation of diagonal occupied states of nucleons $\mu_N$ and $\nu_N$ as

$$
\Delta E^{(TBC)}_{\lambda_{\Lambda}} = \frac{1}{2} \sum_{\mu_N \nu_N \leq PF} \langle \lambda_{\Lambda} \mu_N \nu_N | \tilde{v}_{\Lambda N_1 N_2} | \lambda_{\Lambda} \mu_N \nu_N \rangle
- \sum_{\mu_N \nu_N} \langle \lambda_{\Lambda} \mu_N | \tilde{u}_{\Lambda N_1} | \lambda_{\Lambda} \mu_N \rangle,
$$

(2.32)

where $\tilde{v}_{\Lambda N_1 N_2}$ is the transformed three-body $\Lambda NN$ interaction $\tilde{v}_{ijk}$ for $\{ijk\} = \{\Lambda N_1 N_2\}$ in Eq. (2.26), and $\tilde{u}_{\Lambda N_1}$ is the transformed operator of the one-body potential defined for $Y = \Lambda$ in Eq. (2.19).

In order to obtain the TBC effect in Eq. (2.32), we need to calculate the matrix element of the two-body correlation operator $S_{ij}$, the two-body effective interaction $\tilde{v}_{ij}$ and the single-particle potential $u_k$. The calculation procedure for these quantities has already been given in Refs. 11) and 13) in detail. Therefore, we here present only the outline of the calculation procedure. In an actual calculation for the $YN$ system, we employ a two-step procedure by introducing two kinds of model spaces, namely, large and small model spaces as shown in Figs. 1 and 2. We here use the
harmonic oscillator wave functions as basis states. The numbers $\rho_F$, $\rho_X$, $\rho_1$ and $\rho_2$ in Figs. 1 and 2 are defined as

$$\rho_F = 2n_N + l_N,$$

$$\rho_2 = 2n_A + l_A,$$

$$\rho_X = 2n_Y + l_Y + 2n_N + l_N, \quad (Y = \Lambda, \Sigma)$$

and

$$\rho_1 = 2n_A + l_A + 2n_N + l_N,$$

where $n_k$ and $l_k$ for $k = \Lambda, \Sigma$ and $N$ are the harmonic oscillator quantum numbers. The number $\rho_2$ is introduced so as to specify the uppermost bound state of $\Lambda$.

In the first-step calculation, we calculate the $\Lambda N$ effective interaction $\tilde{v}_{\Lambda N}^{(1)}$ acting in the large model space $P_{\Lambda N}^{(1)}$ by solving the equation of decoupling between the $P_{\Lambda N}^{(1)}$ and the $\{(Q_{\Lambda N}^{(1)} - Q_{\Lambda N}^{(X)}) + (Q_{\Sigma N}^{(1)} - Q_{\Sigma N}^{(X)})\}$ spaces in Eq. (2.22). It is noted here that the $Q_{\Lambda N}^{(X)}$ and $Q_{\Sigma N}^{(X)}$ spaces are the excluded spaces due to the Pauli principle.
for nucleons. The $\Lambda$ potential energy $u^{(1)}_{\Lambda}$ in the first step is calculated as given in Eq. (2.21), using the $\Lambda N$ effective interaction $\tilde{v}^{(1)}_{\Lambda N}$.

In the second-step calculation, the $\Lambda N$ effective interaction $\tilde{v}^{(2)}_{\Lambda N}$ acting in the $P^{(2)}_{\Lambda N}$ space and the correlation operator $S^{(2)}_{\Lambda N}$ are obtained by solving the equation of decoupling between the $P^{(2)}_{\Lambda N}$ and the $Q^{(2)}_{\Lambda N}$ spaces, using $\tilde{v}^{(1)}_{\Lambda N}$ and $u^{(1)}_{\Lambda}$. The $\Lambda$ potential energy $u^{(2)}_{\Lambda}$ is finally calculated in Eq. (2.21), using $\tilde{v}^{(2)}_{\Lambda N}$. However, at this stage, we consider only the diagonal part of $h^{(2)}_{\Lambda} = t_{\Lambda} + u^{(2)}_{\Lambda}$ as the unperturbed one-body hamiltonian. Therefore, we evaluate perturbation corrections arising from the non-diagonal terms of $h^{(2)}_{\Lambda}$ as shown in diagram (a) of Fig. 3. Furthermore, the effective interaction between $\Lambda N$ states in the $P^{(2)}_{\Lambda N}$ and $P^{(Y)}_{\Lambda N}$ spaces induces core polarization in the closed-shell nucleus. We take account of this effect by calculating diagrams (b) and (c) of Fig. 3, using the $\Lambda N$ effective interaction $\tilde{v}^{(1)}_{\Lambda N}$ determined in the first step. The validity of this two-step calculation procedure is confirmed numerically in the previous work.\(^{13}\)

In order to calculate the TBC effect on the $\Lambda$ single-particle energy, we need the effective interaction and the correlation operator for the $NN$ system which are determined by solving the equation of decoupling in Eq. (2.22). Using the two-body effective interactions and the two-body correlation operators for the $\Lambda N$ and $NN$ systems, one can derive that an approximate expression of the TBC terms from Eq. (2.32) as

$$\Delta E^{(TBC)}_{\Lambda \Lambda} \simeq \frac{1}{2} \sum_{\mu \nu \mu' \nu' \leq \rho \rho' \rho \rho'} \langle \lambda_{\Lambda \mu N \nu N} \mid \frac{1}{2} \left[ \tilde{v}_{\Lambda N}^{(1)} \right. \left. S_{\Lambda N}^{(1)} + S_{\Lambda N}^{(2)} \right] + S_{\Lambda N}^{(2)} \right] \mid \lambda_{\Lambda \mu N \nu N} \rangle = (a_T) + (b_T) + (c_T) + (d_T) + (e_T) + (f_T),$$

\hspace{1cm} (2.37)

where

$$(a_T) = - \sum_{\mu \nu \nu' \leq \rho \rho' \rho \rho'} \sum_{\rho' \rho > \rho \rho} \tilde{v}_{\Lambda \nu \nu N}^{(AN)} S_{\nu N}^{(AN)} S_{\nu N}^{(AN)} S_{\rho N}^{(AN)} S_{\rho N}^{(AN)} S_{\rho N}^{(AN)}.$$
(b_T) = \frac{1}{2} \sum_{\mu \rho \nu \rho' \lambda \rho' \lambda' \rho' \lambda'} v_{\lambda \mu \rho \lambda' \rho' \lambda'}^{(AN)} \sum_{\alpha N \beta N > \rho' \alpha > \rho} S^{(NN)}_{\rho \nu \rho' \alpha N \beta N} S^{(NN)}_{\alpha \beta N \mu \nu N},

(c_T) = -2 \sum_{\mu \rho \nu \rho' \alpha N \beta N > \rho' \alpha > \rho} v_{\lambda \mu \rho \lambda' \rho' \beta N}^{(AN)} S^{(NN)}_{\mu \nu \rho' \alpha N \beta N} S^{(NN)}_{\alpha \beta N \lambda \nu N},

(d_T) = \sum_{\mu \rho \nu \rho' \alpha N \beta N > \rho' \alpha > \rho} \tilde{v}_{\lambda \mu \rho \lambda' \rho' \beta N}^{(NN)} S^{(AN)}_{\lambda \mu \alpha N \alpha \mu N} S^{(AN)}_{\alpha \beta N \nu \beta N},

(e_T) = \sum_{\mu \rho \nu \rho' \alpha N \beta N > \rho' \alpha > \rho} \tilde{v}_{\alpha \nu \beta N \gamma \nu N}^{(AN)} S^{(AN)}_{\alpha \beta N \lambda \mu N} S^{(AN)}_{\lambda \mu \gamma \nu N},

(f_T) = -\frac{1}{2} \sum_{\mu \rho \nu \rho' \alpha N \beta N > \rho' \alpha > \rho} v_{\lambda \mu \rho \lambda' \rho' \gamma N}^{(AN)} \sum_{\alpha N \beta N > \rho' \alpha > \rho} S^{(NN)}_{\rho \nu \rho' \alpha N \beta N} S^{(NN)}_{\alpha \beta N \mu \nu N}. \quad (2.38)

In the above, we have used the notations defined by

\tilde{v}_{\alpha \beta N \gamma \delta N}^{(AN)} = \langle \alpha A \beta N | \tilde{v}_{AN} | \gamma \delta N \rangle,

\tilde{v}_{\alpha N \beta N \gamma \delta N}^{(NN)} = \langle \alpha N | \tilde{v}_{i N 2} | \gamma \delta N \rangle \quad (2.39)

and

S_{\alpha \beta N \gamma \delta N}^{(AN)} = \langle \alpha A \beta N | S_{AN} | \gamma \delta N \rangle,

S_{\alpha N \beta N \gamma \delta N}^{(NN)} = \langle \alpha N | S_{i N 2} | \gamma \delta N \rangle. \quad (2.40)

The contributions of the TBC terms to \( \Delta E_{\Lambda}^{(TBC)} \) come from the effective three-body interactions \( \tilde{v}_{ijk}^{(v)} \) and \( \tilde{v}_{ijk}^{(h)} \). We here note that the first and second terms of \( \tilde{v}_{ijk}^{(v)} \) in Eq. (2.30) do not bring about non-zero contributions to the \( \Lambda \) single-particle potential. On the other hand, the contributions of \( \tilde{v}_{ijk}^{(h)} \) in Eq. (2.31) vanish up to second order in \( S_{ij} \). Therefore, the contributions of \( \tilde{v}_{ijk}^{(v)} \) and \( \tilde{v}_{ijk}^{(h)} \) start with the second and third order terms in \( S_{ij} \), respectively. These facts concerning the TBC terms can be proved by using the property of decoupling of \( \tilde{v}_{ij} \) and the restrictive conditions for \( S_{ij} \) in Eq. (2.23) as discussed in Ref. 15). On the assumption that the matrix elements of \( S_{ij} \) are sufficiently small, we neglect the contribution of \( \tilde{v}_{ijk}^{(h)} \). The validity of this assumption will be confirmed numerically in §3.

The \( \Lambda \) single-particle energy is given finally by

\[ E_{\Lambda} = \langle \lambda_A | t_A | \lambda_A \rangle + \langle \lambda_A | u_A^{(2)} | \lambda_A \rangle + \Delta E_{\Lambda}^{(p)} + \Delta E_{\Lambda}^{(TBC)}, \quad (2.41) \]

where the first term is the kinetic energy, the second the first-order potential energy, the third the sum of the perturbation corrections given in Fig. 3 and the last the sum of the TBC contributions given in Eq. (2.37).

2.4. Diagrammatical expression of the three-body-cluster term

We here make some discussions on the relation between the UMOA and the \( G \)-matrix theory in the calculation of the \( \Lambda \) single-particle energy in order to clarify
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Fig. 4. Diagrammatical expression of the first-order $\Lambda$ potential energy in terms of the $\Lambda N$ effective interaction $\tilde{v}_{\Lambda N}$ obtained in the UMOA and the $G$-matrix $G_{\Lambda N}$.

Fig. 5. Diagrams corresponding to the leading order terms of the three-body-cluster terms in terms of the $G$-matrix shown by the wavy line.

the physical meaning of the TBC effect. The formal relation between the effective interaction in the UMOA and the $G$-matrix has been discussed in Refs. 11) and 15). The $\Lambda N$ effective interaction $\tilde{v}_{\Lambda N}$ in the UMOA in the calculation of the first-order $\Lambda$ potential energy $u_\Lambda$ can be given in an expansion form in terms of the $G$-matrix as shown diagrammatically in Fig. 4. The second diagram on the right-hand side in Fig. 4 is a kind of folded diagram. The leading contributions of the TBC terms in Eq. (2.38) can also be given diagrammatically in terms of the $G$-matrix as shown in Fig. 5. It is noted that the second diagram on the right-hand side in Fig. 4 has a minus sign which is against the diagram rule. However, this diagram is just cancelled by diagram (aT) in Fig. 5. It should be noted that such a folded diagram emerges again as one of higher-order corrections with the vertices of the transformed interaction $\tilde{v}_{ij}$. 

\[
\tilde{v}_{\Lambda N} = G_{\Lambda N} + (-) G_{\Lambda N} - \ldots
\]
§3. Numerical results for $^{17}_{A}$O and $^{41}_{A}$Ca using the Nijmegen and the Jülich $YN$ potentials

We performed calculations employing the Nijmegen soft-core (NSC), \(^{16}\) NSC97a-f, \(^{17}\) the Jülich $\tilde{A}$ ($\tilde{J}\tilde{A}$) and JB \(^{18}\) potentials for the $YN$ interaction. We used the harmonic oscillator basis states and took a common $\hbar\Omega$ to single-particle states of $A$, $\Sigma$ and $N$. We used the values $\hbar\Omega = 14$MeV for $^{17}_{A}$O and 12MeV for $^{41}_{A}$Ca. The dependence of the calculated effective interaction and the single-particle energy on the value $\hbar\Omega$ has already been examined, and we confirmed that the dependence was quite small around $\hbar\Omega = 14$MeV in $^{16}$O \(^{15}\) if perturbation corrections were included. As for the nucleon single-particle potential $u_N$, we used the fixed data, \(^{11}\), \(^{12}\) calculated using the Paris potential, \(^{19}\) and we have assumed that $u_S = 0$. The angular momentum $l$ of the partial wave of the $NN$ and $YN$ interactions is taken into account up to $l = 4$. The numbers $\rho_2$, $\rho_F$, $\rho_1$ and $\rho_X$ are taken as $\rho_2 = 1$, $\rho_F = 1$, $\rho_1 = 8$ and $\rho_X = 12$ for $^{17}_{A}$O, and $\rho_2 = 2$, $\rho_F = 2$, $\rho_1 = 10$ and $\rho_X = 12$ for $^{41}_{A}$Ca.

The formulae for calculating the TBC contributions to the $A$ single-particle energy have been derived in Eq. (2.38). The TBC terms are given in powers of the two-body correlation operator $S_{ij}$ as given in Eqs. (2.30) and (2.31). In the UMOA, it is necessary that the matrix element of the correlation operator is sufficiently small in the sense of cluster expansion. In Table I typical matrix elements of $S_{ij}$ for the $NN$ and $AN$ systems are given. We present only the matrix elements of $S_{ij}$ of comparatively large magnitudes among all the matrix elements of $S_{ij}$. The magnitudes of the matrix elements are at most of the order of $10^{-2}$. Besides, the TBC effect yields the contributions of the order of the square of $S_{ij}$ as shown in Eq. (2.38). Thus, we expect that the TBC contribution is considerably small compared with the contribution from the one- and two-body-cluster terms.

We calculated the $A$ single-particle energies with the correction terms (a), (b) and (c) given in Fig. 3 and the TBC terms ($a_T$)-($f_T$) in Eq. (2.38) for $^{17}_{A}$O using the NSC97f potential as shown in Table II.\(^{\text{10}}\) One see that the TBC contributions

\(^{\text{10}}\) The mark *** in Tables II and III means that the calculated absolute value is less than 0.01MeV.

| $a$ | $b$ | $J$ | $T$ | $\langle aa|S_{NN}^{(2)}(bb)_{J,T}\rangle_T$ | $\langle aa|S_{NN}^{(2)}(bb)_{J,T=1/2}\rangle_T$ |
|-----|-----|-----|-----|--------------------------------|---------------------------------|
| $1s_{1/2}$ | $0s_{1/2}$ | 1 | 0 | $2.32 \times 10^{-2}$ | $5.38 \times 10^{-3}$ |
| $1s_{1/2}$ | $0s_{1/2}$ | 0 | 1 | $1.69 \times 10^{-2}$ | $9.42 \times 10^{-3}$ |
| $1s_{1/2}$ | $0p_{1/2}$ | 1 | 0 | $-2.43 \times 10^{-2}$ | $2.49 \times 10^{-3}$ |
| $1s_{1/2}$ | $0p_{1/2}$ | 0 | 0 | $-5.75 \times 10^{-3}$ | $9.36 \times 10^{-4}$ |
| $0d_{3/2}$ | $0s_{1/2}$ | 1 | 0 | $8.31 \times 10^{-3}$ | $1.51 \times 10^{-4}$ |
| $0d_{3/2}$ | $0s_{1/2}$ | 0 | 0 | $7.82 \times 10^{-3}$ | $4.58 \times 10^{-3}$ |
| $0d_{3/2}$ | $0p_{1/2}$ | 0 | 1 | $-6.48 \times 10^{-3}$ | $3.67 \times 10^{-3}$ |
| $0d_{3/2}$ | $0p_{1/2}$ | 1 | 0 | $-8.85 \times 10^{-3}$ | $1.29 \times 10^{-3}$ |
are considerably small in comparison with the contributions from the one- and two-body-cluster terms. This ensures that the cluster expansion method in powers of the correlation operator $S_{ij}$ may be justified in the calculation of the $\Lambda$ single-particle energy.

One also see that two of the TBC terms, $(b_T)$ and $(f_T)$, have dominant contributions among the six TBC terms. It may be clear from Fig. 5 that these two TBC terms include the vertices inducing typical two-particle two-hole excitations of the core nucleus in intermediate states. On the other hand, the TBC terms $(a_T)$ and $(c_T)-(e_T)$ do not include this type of excitations. The two terms $(b_T)$ and $(f_T)$, however, give the contributions of the opposite sign. Therefore, the sum of the TBC terms becomes rather small. The effect of the TBC terms is, on the whole, repulsive and at most about 3-4% of the $\Lambda$ one-body potential energies of the one- and two-body-cluster terms for the $0p$ and $0s$ states in $^{17}_A O$ as given in Table II.

The calculated value of the spin-orbit splitting of the $\Lambda$ single-particle energies for the $0p$ orbits in $^{17}_A O$ is 1.11MeV for the first-order potential energy. Corrections $(a)$, $(b)$ and $(c)$ in Fig. 3 yield the contribution of $-0.11$MeV to the $\Lambda$ spin-orbit splitting. The sum of the TBC terms for the $\Lambda$ spin-orbit splitting is only $-0.06$MeV. We may conclude from the present calculation that the magnitude of the $\Lambda$ spin-orbit splitting is determined dominantly by the contribution of the one- and two-body-cluster terms, and the polarization of the core nucleus does not play an important role. This trend is also confirmed for the other $YN$ interactions used in this study.

The calculated result in $^{41}_A Ca$ for the NSC97f is shown in Table III. The effects of the second-order and the TBC correction terms on the $\Lambda$ single-particle energy are not so different from those in $^{17}_A O$. The magnitude of the TBC terms is at most 3% of the $\Lambda$ one-body potential energy of the one- and two-body-cluster terms for the $0s$ state in $^{41}_A Ca$. We note here that the $\Lambda$ spin-orbit splittings are calculated for

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**Table II. Contributions to the $\Lambda$ single-particle energies in $^{17}_A O$ for the NSC97f potential. KE and PE stand for the kinetic and first-order potential energies, respectively. Rows $(a)$, $(b)$ and $(c)$ are the contributions of diagrams $(a)$, $(b)$ and $(c)$ in Fig. 3, respectively. Rows $(a_T)$-$(-f_T)$ mean the contributions given in Eq. (2.38). All entries are in MeV.**

|          | $0s_{1/2}$ | $0p_{3/2}$ | $0p_{1/2}$ | $\Delta\varepsilon_{ls}$ |
|----------|------------|------------|------------|---------------------------|
| KE       | 10.50      | 17.50      | 17.50      | ***                       |
| PE       | -23.13     | -17.20     | -16.09     | 1.11                      |
| $(a)$    | -1.84      | -2.35      | -2.42      | -0.08                     |
| $(b)$    | -1.20      | -0.37      | -0.13      | 0.24                      |
| $(c)$    | -0.55      | -0.98      | -1.25      | -0.28                     |
| Subtotal | -16.22     | -3.40      | -2.40      | 1.00                      |
| $(a_T)$  | -0.01      | -0.01      | -0.01      | ***                       |
| $(b_T)$  | 1.49       | 1.22       | 1.15       | -0.07                     |
| $(c_T)$  | -0.09      | -0.16      | -0.16      | ***                       |
| $(d_T)$  | ***        | ***        | -0.02      | -0.02                     |
| $(e_T)$  | 0.01       | 0.02       | 0.02       | ***                       |
| $(f_T)$  | -0.43      | -0.46      | -0.44      | 0.02                      |
| Sum of TBC | 0.97    | 0.60       | 0.54       | -0.06                     |
| Total    | -15.25     | -2.80      | -1.86      | 0.95                      |
both the $0p$ and $0d$ states in $^{41}\Lambda\text{Ca}$. The values of the $\Lambda$ spin-orbit splittings for the $0p$ and $0d$ states are 0.79MeV and 1.11MeV, respectively. The net contribution of core polarizations to the $\Lambda$ spin-orbit splittings is very small for both the $0p$ and $0d$ states. This situation is quite different from that in the ordinary nucleus $^{40}\text{Ca}$, as well as in $^{16}\text{O}$.\textsuperscript{11}) It has been observed that many-body correlations bring about corrections of more than 30% for the spin-orbit splittings of nucleon states.

It might be of interest to compare the results obtained in this study with those in other many-body methods. Recently, the $\Lambda$ spin-orbit splittings in $^{17}\Lambda\text{O}$ were calculated on another method by Motoba\textsuperscript{20}) in which configuration mixing of the $\Lambda$ single-particle states is taken into account, using the YNG interaction\textsuperscript{21}) derived from the NSC97 potentials. The result for the NSC97f was almost 1MeV for the $0p$ states. This value is consistent with our result of 0.95MeV for the NSC97f as given in Table II.

We now discuss the results obtained for various $YN$ interactions. The partial-wave contributions of the $\Lambda N$ effective interactions to the first-order $\Lambda$ potential energy are given for various $YN$ potentials, namely, the $\text{JA}$, $\text{JB}$, the NSC\textsuperscript{(*)} and

\textsuperscript{*}) The mark (*) for the NSC in Table V and Fig. 6 means that the antisymmetric spin-orbit force is not included. This potential was also used in the calculation of properties in $^{17}\Lambda\text{O}$.

Table III. Results of $^{41}\Lambda\text{Ca}$ for the NSC97f. The notation is the same as in Table II. All entries are in MeV.

|          | $0s_{1/2}$ | $0p_{3/2}$ | $0p_{1/2}$ | $0d_{5/2}$ | $0d_{3/2}$ | $1s_{1/2}$ |
|----------|------------|------------|------------|------------|------------|------------|
| KE       | 9.00       | 15.00      | 15.00      | 21.00      | 21.00      | 21.00      |
| PE       | -28.01     | -22.75     | -21.76     | -18.08     | -16.63     | -16.36     |
| (a)      | -2.16      | -3.19      | -3.25      | -3.69      | -3.83      | -2.96      |
| (b)      | -1.44      | -0.99      | -0.87      | -0.39      | -0.15      | 0.16       |
| (c)      | -0.58      | -1.02      | -1.25      | -1.32      | -1.73      | -1.62      |
| Subtotal | -23.19     | -12.96     | -12.14     | -2.48      | -1.33      | 0.22       |

Table IV. Partial-wave contributions to the first-order $\Lambda$ potential energy for the $0s_{1/2}$ state in $^{17}\Lambda\text{O}$. All entries are in MeV.

|          | $0s_{0}$ | $0s_{1}$ | $0p_{0}$ | $0p_{1}$ | $0p_{1}$ | $0p_{2}$ | $0p_{2}$ | Total    |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| NSC97a   | -2.37    | -22.93   | -0.03    | 0.84     | 0.80     | -0.98    | -24.68   |
| NSC97b   | -3.52    | -22.35   | 0.04     | 0.90     | 0.98     | -0.88    | -24.82   |
| NSC97c   | -5.06    | -22.00   | 0.18     | 0.95     | 1.12     | -0.78    | -25.60   |
| NSC97d   | -7.16    | -20.55   | 0.30     | 1.06     | 1.40     | -0.56    | -25.52   |
| NSC97e   | -8.42    | -19.28   | 0.35     | 1.14     | 1.62     | -0.41    | -24.99   |
| NSC97f   | -9.46    | -17.11   | 0.32     | 1.29     | 2.00     | -0.14    | -23.10   |
NSC97a-f potentials for the $0s_{1/2}$ state, in Table IV for $^{17}_A\Lambda O$, and in Table V for $^{41}_A\Lambda Ca$. It is seen that the partial-wave contributions are very different, especially in the spin singlet and triplet states, dependently on the $YN$ potentials employed. It is noted, however, that the ratios of the contributions of the spin singlet and triplet states vary smoothly from the NSC97a to the NSC97f. This variation is caused mainly by the difference in the strength of the spin-spin interaction of the NSC97 potentials. The similar tendency was observed in the calculation of the potential energy of $\Lambda$ contained in nuclear matter.\textsuperscript{17}

One of the features of the NSC97 model for the $YN$ interaction is that the NSC97a-f potentials are given by varying smoothly the magnetic $F/(F + D)$ ratio $\alpha_V^{(m)}$ for the vector mesons in the NSC97a-f potentials. This change in the magnetic ratio yields the variation of the strength of not only the spin-spin interaction but also other components including the spin-orbit interaction. According to the variation of the $YN$ interactions, the calculated result of the $\Lambda$ single-particle potential energies vary dependently on the $YN$ potentials employed as seen in Tables IV and V. It is important to investigate the difference among the NSC97 potentials quantitatively in the calculation of properties in finite $\Lambda$ hypernuclei. This quantitative study would help to improve the $YN$ interaction, complementarily to the progress in relevant previous work.\textsuperscript{13}

On the other hand, the NSC newly used in the present study is the complete version of the NSC potential code which includes the antisymmetric spin-orbit force.

### Table V. Partial-wave contributions to the first-order $\Lambda$ potential energy for the $0s_{1/2}$ state in $^{41}_A\Lambda Ca$.

All entries are in MeV.

| $l$ | $^{1}S_0$ | $^{3}S_1$ | $^{3}D_1$ | $^{1}D_2$ | $^{3}D_2$ | $^{3}D_3$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|
| JA  | -3.55     | -23.76    | -0.12     | -0.13     | -0.14     | -0.15     |
| JB  | 0.10      | -31.71    | -0.09     | -0.13     | -0.14     | -0.11     |
| NSC97a | -2.80    | -29.42    | -0.09     | -0.17     | -0.22     | -0.21     |
| NSC97b | -4.40    | -28.69    | -0.08     | -0.19     | -0.23     | -0.22     |
| NSC97c | -6.57    | -28.25    | -0.08     | -0.20     | -0.23     | -0.23     |
| NSC97d | -9.55    | -26.27    | -0.08     | -0.23     | -0.24     | -0.24     |
| NSC97e | -11.33   | -24.52    | -0.07     | -0.24     | -0.25     | -0.24     |
| NSC97f | -12.82   | -21.55    | -0.05     | -0.26     | -0.27     | -0.24     |
| NSC  | -13.83   | -8.92     | -0.06     | -0.25     | -0.23     | -0.19     |
| NSC(*) | -13.12   | -9.50     | -0.05     | -0.24     | -0.23     | -0.19     |

| $l$ : odd |
|-----------|
| $^{3}P_0$ | $^{1}P_1$ | $^{3}P_1$ | $^{3}P_2$ | Total  |
| JA  | 0.76      | 0.82      | 1.41      | -0.31   | -25.17  |
| JB  | 0.70      | 1.16      | 2.00      | 0.52    | -27.71  |
| NSC97a | -0.05    | 1.70      | 1.75      | -2.02   | -31.53  |
| NSC97b | 0.09     | 1.83      | 2.14      | -1.83   | -31.57  |
| NSC97c | 0.38     | 1.93      | 2.42      | -1.63   | -32.48  |
| NSC97d | 0.63     | 2.16      | 3.02      | -1.21   | -32.01  |
| NSC97e | 0.72     | 2.34      | 3.47      | -0.90   | -31.03  |
| NSC97f | 0.67     | 2.67      | 4.26      | -0.36   | -27.97  |
| NSC  | 0.49      | 1.79      | 2.19      | -3.67   | -22.69  |
| NSC(*) | 0.51     | 1.87      | 2.25      | -3.71   | -22.42  |
In Figs. 6 and 7 we show the calculated \( \Lambda \) single-particle energies including the second-order perturbation corrections and the TBC terms for the JA, JB, the NSC and NSC97 interactions. It is noted here that the calculated results for the NSC97f potential are, on the whole, fairly good in comparison with the empirical values for \(^{16}\text{O}\) and \(^{40}\text{Ca}\). Our result for the NSC97f is consistent with the fact that the calculations of few-body \( \Lambda \) hypernuclei such as \(^{3}\Lambda\text{H}\) and \(^{4}\Lambda\text{He}\) using the NSC97f can reproduce successfully experimental energy levels.

The calculated \( \Lambda \) spin-orbit splittings of the 0\( p \) states vary in the range of 0.49-1.31MeV from the NSC97a to the NSC in \(^{17}\text{O}\) as shown in Fig. 6. Similarly, the \( \Lambda \) spin-orbit splittings of the 0\( p \) and 0\( d \) states vary in the range of 0.35-1.08MeV and 0.61-1.55MeV, respectively, in \(^{41}\text{Ca}\) as shown in Fig. 7. The experimental values of the spin-orbit splitting of a nucleon are known as 6.20MeV for the 0\( p \) states in \(^{16}\text{O}\) and 6.75MeV for the 0\( d \) states in \(^{40}\text{Ca}\). Therefore, the obtained values of the \( \Lambda \) spin-orbit splitting are considerably smaller than those of a nucleon. Recently, some
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Fig. 7. Dependence of the calculated \( ^\Lambda \) single-particle energies in \( ^{41}_\Lambda Ca \) on various \( YN \) interactions. All entries are in MeV.

Experiments in connection with the \( ^\Lambda \) spin-orbit splittings have been performed at KEK for \( ^{89}_\Lambda Y \) and \( ^{51}_\Lambda V \) \(^{23} \) and at BNL for \( ^{13}_\Lambda C \) \(^{24} \) and \( ^{9}_\Lambda Be \) \(^{25} \). Higher resolution has been attained in these experiments than that in old experiments though the analysis of these new experiments are still being in progress. Therefore, it will be highly desirable that the magnitude of the \( \Lambda \) spin-orbit splitting is established experimentally.

§4. Concluding remarks

A method of evaluating the three-body-cluster (TBC) effect was presented for the calculation of \( \Lambda \) single-particle energies. The TBC terms were generated as the transformed three-body interactions among the \( YNN \) and \( NNN \) systems. The transformed three-body interactions were given in the cluster expansion form in powers of two-body correlation operators of the \( YN \) and \( NN \) systems. Therefore, it is necessary for fast convergence of the cluster expansion that the matrix elements of
the correlation operator should be sufficiently small. In our calculations, it was made sure that the matrix elements of the correlation operator were quite small, and thus the TBC contributions to the \( \Lambda \) single-particle energies were considerably small in comparison with the contributions from the one- and two-body-cluster terms. This ensured that the cluster expansion method in terms of the correlation operator was acceptable in the calculation of \( \Lambda \) single-particle energies.

The present results showed that \( \Lambda \) in the \( 0p_{1/2} \) state in \(^{17}_{\Lambda}{\text{O}}\) was not bound for the J\( \tilde{A} \) and the NSC potentials but bound for the J\( \tilde{B} \) and the NSC97a-f potentials. The spin-orbit splittings of the \( \Lambda \) single-particle orbits in the \( 0p \) states, including the TBC contributions, were given by 1.35, 1.49, 1.31 and 0.49-0.95MeV for the J\( \tilde{A} \), J\( \tilde{B} \), the NSC and NSC97a-f potentials, respectively. The value of 1.49MeV for the J\( \tilde{B} \) potential is very large compared with that given in Ref. 7). On the other hand, the value 0.95MeV for the NSC97f potential is consistent with that given in Ref. 20). The calculated \( \Lambda \) spin-orbit splittings depend largely on the \( YN \) interactions employed. The dependence of the calculated results on the \( YN \) interactions was also examined for \(^{41}_{\Lambda}{\text{Ca}}\). It seemed that the NSC97f potential yielded better results for \(^{17}_{\Lambda}{\text{O}}\) and \(^{41}_{\Lambda}{\text{Ca}}\) in comparison with the relevant experimental values. The present results were consistent with those obtained in the calculation of few-body \( \Lambda \) hypernuclei.

In the application of the UMOA to \( \Lambda \) hypernuclei, some problems still remain. First, in the present calculation, single-particle energies of \( \Sigma \) in intermediate states were neglected. However, the single-particle spectrum of \( \Sigma \) has a possibility of bringing about some contributions to the \( \Lambda N \) effective interaction. Although single-particle energies of \( \Sigma \) has not yet been known well, it would be important to investigate how much the spectrum of \( \Sigma \) affects the determination of the \( \Lambda N \) effective interaction. Second, there would be some relativistic effect for the \( \Lambda \) single-particle energy as well as in calculations of ordinary nuclei. In the present approach, the relativistic effect has not yet been considered in studies of ordinary nuclei and hypernuclei. On the other hand, the relativistic many-body approach such as the Dirac-Brueckner-Hartree-Fock approach\(^{26}\) does not fully take account of the effect of three- or more-body correlations. Therefore, it would be of great importance for the understanding of nuclear properties from the many-body theoretical point of view that the relativistic effect is incorporated in the framework of the UMOA. This problem, which would be closely related to the treatment of the three-body force,\(^{27},^{28}\) is also a remaining task in the UMOA.

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