SPECTRAL PROPERTIES OF STURM–LIOUVILLE EQUATIONS WITH SINGULAR ENERGY-DEPENDENT POTENTIALS

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Abstract. We study spectral properties of energy-dependent Sturm–Liouville equations, introduce the notion of norming constants and establish their interrelation with the spectra. One of the main tools is the linearization of the problem in a suitable Pontryagin space.

1. Introduction

The main aim of the present paper is to investigate spectral properties of Sturm–Liouville problems with energy-dependent potentials given by the differential equations

\[ -y'' + qy + 2\lambda py = \lambda^2 y \]

on \((0,1)\) and some boundary conditions. Here \(p\) is a real-valued function from \(L_2(0,1)\), \(q\) is a real-valued distribution from the Sobolev space \(W^{-1}_2(0,1)\), and \(\lambda \in \mathbb{C}\) is a spectral parameter. (A detailed definition will be given in the next section).

The spectral equation (1.1) is of importance in classical and quantum mechanics. For example, such problems arise in solving the Klein–Gordon equations, which describe the motion of massless particles such as photons (see [10, 22]). Sturm–Liouville energy-dependent equations are also used for modelling vibrations of mechanical systems in viscous media (see [31]). Note that in such models the spectral parameter \(\lambda\) is related to the energy of the system, and this motivates the terminology “energy-dependent” used for the spectral problem of the form (1.1).

The equations under study were also considered on the line and discussed in the context of the inverse scattering theory (see, e. g. [1,9,11,17,19,26,30], and [7] for a more extensive reference list). Some
of their spectral properties in this context were established in \cite{20}. The spectral problem (1.1) on an interval with \( p \in W^2_2[0, \pi] \) and \( q \in L^2_2[0, \pi] \) and with general boundary conditions was also studied by Gasymov and Nabiev in \cite{6,21}. An interesting approach to the spectral analysis of problems under consideration uses the theory of Krein spaces (i.e. spaces with indefinite scalar products). It was suggested by P. Jonas \cite{10} and H. Langer, B. Najman, and C. Tretter \cite{15,16,22}.

In the present paper, we consider (1.1) under minimal smoothness assumption on the real-valued potentials \( p \) and \( q \). As equation (1.1) contains terms depending on the spectral parameter \( \lambda \) and its square \( \lambda^2 \) as well, the spectral problem of interest is better understood as that for the corresponding quadratic operator pencil. And indeed, some of the spectral properties of the Sturm–Liouville energy-dependent equations (1.1) are derived in this paper from the general spectral theory of polynomial operator pencils (see \cite{18}) and some by the direct analysis of the corresponding quadratic operator pencil. We also prove equivalence of the spectral problem for (1.1) and that for its linearization \( \mathcal{L} \). The operator \( \mathcal{L} \) turns out to be self-adjoint in a suitably defined Pontryagin space, which provides some further properties of the operator pencil (1.1).

We also introduce the notion of the norming constants for the problem (1.1). For real and simple eigenvalues the definition of these quantities is analogous to that for the standard Sturm–Liouville operators. However, since the problem (1.1) can also have non-real and/or non-simple eigenvalues our definition is more general. These quantities are shown to be related to the spectra of (1.1) similarly to the classical Sturm–Liouville theory. We obtain an explicit formula determining norming constants via two spectra. We also derive sufficient conditions for simplicity of the spectra. The obtained results have their important applications in the inverse problems of reconstruction of the potentials \( p \) and \( q \) (or its primitive \( r \)) from two spectra or one spectrum and a set of norming constants (see \cite{7,25}).

The paper is organized as follows. In the next section we formulate the spectral problem under study as that for the corresponding operator pencil \( T \) and recall some notions from the operator pencil theory. In Section 3 we analyse the operator pencil \( T \) and obtain some of its spectral properties. We construct a linearization \( \mathcal{L} \) of the spectral problem for \( T \) in Section 4. The operator \( \mathcal{L} \) is considered in a specially defined Pontryagin space (i.e. in the space with indefinite inner product) and is shown to be self-adjoint therein. This gives more spectral properties of \( \mathcal{L} \) and so of \( T \). In Section 5 we introduce the notion of norming constants for the problem under study and derive some relations for
these quantities. Section 6 is devoted to the case when the spectra of the problems (1.1) under two types of boundary conditions are real and simple. We obtain sufficient conditions for this.

Notations. Throughout the paper, \( \rho(T) \), \( \sigma(T) \) and \( \sigma_p(T) \) denote the resolvent set, the spectrum and the point spectrum of a linear operator or a quadratic operator pencil \( T \). The superscript \( t \) will signify the transposition of vectors and matrices, e. g. \( (c_1, c_2)^t \) is the column vector \( (c_1, c_2) \).

2. Preliminaries

Consider equation (1.1) subject to the Dirichlet boundary conditions

\[ y(0) = y(1) = 0. \]

Notice that other separate boundary conditions can be treated similarly; in particular, in Sections 5 and 6 we shall consider (1.1) under the mixed conditions (5.11). We restrict our attention to (2.1) merely in order to enlighten the ideas and avoid unessential technicalities.

The spectral equation (1.1) depends on the parameter \( \lambda \) non-linearly. Thus to formulate the spectral problem of interest rigorously we should regard (1.1) as a spectral problem for some operator pencil. To start with, consider the differential expression

\[ \ell(y) := -y'' + qy. \]

As \( q \) is a real-valued distribution from \( W_2^{-1}(0, 1) \) we need to explain how \( \ell(y) \) is defined. The simplest and most convenient way uses the method of regularization by quasi-derivatives (see, e.g. [27, 28]) that proceeds as follows. Take a real-valued \( r \in L_2(0, 1) \) such that \( q = r' \) in the distributional sense and for every absolutely continuous function \( y \) denote by \( y^{[1]} := y' - ry \) its quasi-derivative. We then define \( \ell \) as

\[ \ell(y) = -\left(y^{[1]}\right)' - ry^{[1]} - r^2 y \]

on the domain

\[ \text{dom } \ell = \{ y \in AC'(0, 1) \mid y^{[1]} \in AC[0, 1], \ell(y) \in L_2(0, 1) \}. \]

Direct verification shows that with this definition \( \ell(y) = -y'' + qy \) in the distributional sense. Observe also that for every \( f \) from \( L_2(0, 1) \), every complex \( a, b \) and every \( x_0 \) from \( [0, 1] \) the equation \( \ell(y) = \mu y + f \) possesses a unique solution satisfying the initial conditions \( y(x_0) = a \) and \( y^{[1]}(x_0) = b \).

Denote by \( A \) the operator acting via

\[ Ay := \ell(y) \]
on the domain
\[ \text{dom } A := \{ y \in \text{dom } \ell \mid y(0) = y(1) = 0 \} . \]

For regular \( q \), the operator \( A \) is a standard Sturm–Liouville operator with potential \( q \) and the Dirichlet boundary conditions. It was shown in \([27, 28]\) that if \( q \in W_2^{-1}(0, 1) \) is real-valued, then the operator \( A \) is self-adjoint, bounded below and has a simple discrete spectrum.

**Remark 2.1.** Recall that an operator \( S \) is said to possess discrete spectrum if \( \sigma(S) \) consists of isolated points, each of which is an eigenvalue of finite algebraic multiplicity. By Theorem III.6.29 of \([12]\), \( S \) has discrete spectrum if its resolvent is compact for one (and then for all) \( \lambda \in \rho(S) \).

Next we denote by \( B \) the operator of multiplication by the function \( 2p \in L_2(0, 1) \), by \( I \) the identity operator and define the quadratic operator pencil \( T \) as
\[
(2.2) \quad T(\lambda) := \lambda^2 I - \lambda B - A, \quad \lambda \in \mathbb{C} .
\]
Then the spectral problem \([1.1], (2.1)\) can be regarded as the spectral problem for the operator pencil \( T \). Properties of the operators \( A \) and \( B \) guarantee that the pencil \( T \) is well defined on the \( \lambda \)-independent domain \( \text{dom } T := \text{dom } A \). More exactly, the following statement holds true.

**Proposition 2.2.** For every fixed \( \lambda_0 \in \mathbb{C} \) the operator \( T(\lambda_0) \) is closed on the domain \( \text{dom } T := \text{dom } A \) and has a discrete spectrum.

**Proof.** Since the domain of the operator \( A \) consists only of bounded functions we have that \( \text{dom } B \supset \text{dom } A \). This immediately gives that for every \( \lambda_0 \in \mathbb{C} \) the operator \( T(\lambda_0) \) is well defined.

Let us fix \( \lambda_0 \in \mathbb{C} \). Take an arbitrary \( \mu \in \rho(A) \) and denote by \( \varphi_- \) and \( \varphi_+ \) solutions of the equation \( \ell(y) = \mu y \) satisfying boundary conditions \( \varphi_-(0) = 0, \varphi_-^{[1]}(0) = 1 \) and \( \varphi_+(1) = 0, \varphi_+^{[1]}(1) = 1 \). Then the Green function of \( A - \mu \), i. e. the kernel of the operator \((A - \mu I)^{-1}\) is equal to
\[
k_0(x, s) := \begin{cases} 
\varphi_+(x)\varphi_-(y)/W, & \text{when } x > s \\
\varphi_-(x)\varphi_+(y)/W, & \text{when } x \leq s 
\end{cases},
\]
where \( W = \varphi_-(x)\varphi_+^{[1]}(x) - \varphi_+(x)\varphi_-^{[1]}(x) \) is the Wronskian of solutions \( \varphi_- \) and \( \varphi_+ \). In particular, the Green function is continuous on the square \( \Omega := [0, 1] \times [0, 1] \). It follows that the operator \((\lambda_0^2 I - \lambda_0 B)(A - \mu I)^{-1}\) is an integral one with the kernel \( k \) given by
\[
k(x, s) = (\lambda_0^2 - 2\lambda_0 p(x))k_0(x, s).
\]
As \( k \) is square integrable on \( \Omega \), the operator \((\lambda_0^2 I - \lambda_0 B)(A - \mu I)^{-1}\) is of the Hilbert–Schmidt class and thus \( \lambda_0^2 I - \lambda_0 B \) is \( A \)-compact (see [12, Ch. IV]). In view of Theorem IV.1.11 of [12] the operator \( T(\lambda_0) \) is closed on \( \text{dom} \ A \). Moreover, Theorem IV.5.35 of [12] implies the coincidence of the essential spectra of the operators \( A \) and \( T(\lambda_0) \). As \( A \) has discrete spectrum, we get that \( \sigma_{\text{ess}} T(\lambda_0) = \sigma_{\text{ess}}(A) = \emptyset \) and thus the spectrum of \( T(\lambda_0) \) is discrete. \( \square \)

Let us now recall some notions of the spectral theory of operator pencils, see [18].

An operator pencil \( T \) is an operator-valued function on \( \mathbb{C} \). The spectrum of an operator pencil \( T \) is the set \( \sigma(T) \) of all \( \lambda \in \mathbb{C} \) such that \( T(\lambda) \) is not boundedly invertible, i.e.,

\[ \sigma(T) = \{ \lambda \in \mathbb{C} \mid 0 \in \sigma(T(\lambda)) \}. \]

A number \( \lambda \in \mathbb{C} \) is called an eigenvalue of \( T \) if \( T(\lambda)y = 0 \) for some non-zero function \( y \in \text{dom} T \), which is then the corresponding eigenfunction. The eigenvalues of \( T \) constitute its point spectrum \( \sigma_p(T) \), i.e.,

\[ \sigma_p(T) = \{ \lambda \in \mathbb{C} \mid 0 \in \sigma_p(T(\lambda)) \}. \]

The set

\[ \rho(T) := \mathbb{C} \setminus \sigma(T) \]

is the resolvent set of an operator pencil \( T \).

Vectors \( y_1, \ldots, y_{m-1} \) are said to be associated with an eigenvector \( y_0 \) corresponding to an eigenvalue \( \lambda \) if

\[ \sum_{k=0}^{j} \frac{1}{k!} T^{(k)}(\lambda) y_{j-k} = 0, \quad j = 1, \ldots, m - 1. \]

Here \( T^{(k)} \) denotes the \( k \)-th derivative of \( T \) with respect to \( \lambda \). The number \( m \) is called the length of the chain \( y_0, \ldots, y_{m-1} \) of an eigen-and associated vectors. The maximal length of a chain starting with an eigenvector \( y_0 \) is called the algebraic multiplicity of an eigenvector \( y_0 \).

For an eigenvalue \( \lambda \) of \( T \) the dimension of the null-space of \( T(\lambda) \) is called the geometric multiplicity of \( \lambda \). The eigenvalue is said to be geometrically simple if its geometric multiplicity equals to one.

For the pencil \( T \) of (2.2) the operator \(-T(\lambda_0)\) is a Sturm–Liouville operator with potential \( q + 2\lambda_0 p - \lambda_0^2 \) and the Dirichlet boundary conditions, whence the dimension of its null-space is at most one. Therefore all the eigenvalues of the pencil \( T \) under study are geometrically simple, and then the algebraic multiplicity of an eigenvalue is the algebraic multiplicity of the corresponding eigenvector. (If the eigenvalue \( \lambda \) is not geometrically simple, its algebraic multiplicity is the number of vectors
in the corresponding canonical system, see [13][18]). An eigenvalue is said to be algebraically simple if its algebraic multiplicity is one.

3. Spectral properties of the operator pencil

In this section we discuss some basic spectral properties of the operator pencil $T$. We start with the following lemmas.

**Lemma 3.1.** The spectrum of the operator pencil $T$ consists only of eigenvalues.

**Proof.** By definition, $\lambda_0 \in \mathbb{C}$ belongs to the spectrum of the operator pencil $T$ if and only if $0 \in \sigma(T(\lambda_0))$. Since $\sigma(T(\lambda_0)) = \sigma_p(T(\lambda_0))$ (see Proposition 2.2), every $\lambda_0$ in the spectrum of $T$ is its eigenvalue. □

**Lemma 3.2.** The resolvent set of the operator pencil $T$ is not empty.

**Proof.** As the operator $A$ is lower semibounded, a number $\mu$ exists such that the operator $A + \mu^2 I$ is positive. Let us show that then the number $i\mu$ belongs to the resolvent set $\rho(T)$ of the operator pencil $T$. Suppose it does not; then, by the previous lemma, there exists a nonzero eigenfunction $y$ such that $T(i\mu)y = 0$ and so

$$(\mu^2 + A)y, y) + i\mu(By, y) = 0.$$ This contradicts positivity of $A + \mu^2 I$. Therefore $i\mu$ belongs to $\rho(T)$ and the lemma is proved. □

Using lemmas we shall prove discreteness of the spectrum of the operator pencil $T$.

**Lemma 3.3.** The spectrum of the operator pencil $T$ is a discrete subset of $\mathbb{C}$.

**Proof.** Let us take some $\lambda_0 \in \rho(T)$ and rewrite $T(\lambda)$ as

$$T(\lambda) = T(\lambda_0) + (\lambda - \lambda_0)[2\lambda_0 I - B] + (\lambda - \lambda_0)^2 I.$$ Set $\hat{B} := 2\lambda_0 I - B$, $\hat{A} := T(\lambda_0)$, and $\mu = \lambda - \lambda_0$ and consider the operator pencil $\hat{T}(\mu) := T(\lambda)^{-1}(\lambda_0)$, which can be written as

$$\hat{T}(\mu) := I + \mu^2 \hat{A}^{-1} + \mu \hat{B} \hat{A}^{-1}.$$ Using the arguments analogous to those used in the proof of Proposition 2.2 one can show that the operator $\mu^2 \hat{A}^{-1} + \mu \hat{B} \hat{A}^{-1}$ is from the Hilbert–Schmidt class and so is compact. Then applying the Gohberg theorem on analytic operator-valued functions [5, Ch.I] to the pencil $I - S(\mu)$ with $S(\mu) := -(\mu^2 \hat{A}^{-1} + \mu \hat{B} \hat{A}^{-1})$, we obtain that for all $\mu \in \mathbb{C}$ except possibly some isolated points the operator $\hat{T}(\mu)$ is...
boundedly invertible, while these isolated points are eigenvalues of $\hat{T}$ of finite algebraic multiplicity. This shows that the spectrum of $\hat{T}$ is a discrete subset of $\mathbb{C}$.

Assume $\lambda \in \sigma(T)$, which by Lemma 3.1 means that $\lambda \in \sigma_p(T)$, and let $x$ be the corresponding eigenfunction. Then $y = T^{-1}(\lambda_0)x$ is an eigenfunction of $\hat{T}$ corresponding to the eigenvalue $\mu = \lambda - \lambda_0$. Therefore,

$$\lambda \in \sigma(T) \Rightarrow \mu = \lambda - \lambda_0 \in \sigma(\hat{T}).$$

Observe also that if $\lambda \in \rho(T)$, i.e. if $T(\lambda)$ is boundedly invertible, then the operator $T(\lambda_0)T^{-1}(\lambda)$ is closable, defined on the whole space $L^2(0,1)$, and thus bounded by the closed graph theorem [12, Theorem III.5.20]. Direct verification shows that it is the inverse operator of $\hat{T}(\mu)$ with $\mu = \lambda - \lambda_0$. Therefore

$$\lambda \in \rho(T) \Rightarrow \mu = \lambda - \lambda_0 \in \rho(\hat{T}).$$

These two implications give the equivalence

$$\lambda \in \sigma(T) \iff \mu = \lambda - \lambda_0 \in \sigma(\hat{T});$$

thus the spectrum of the operator pencil $T$ is discrete in $\mathbb{C}$ along with the spectrum of $\hat{T}$.

**Remark 3.4.** Without loss of generality we may and shall assume further in this paper that 0 is not in $\sigma(T)$ or, equivalently, that the operator $A$ is boundedly invertible. In view of the above lemma we can always achieve this by shifting of the spectral parameter by a real number.

As was noted in Section 2, every eigenvalue of $T$ is geometrically simple. However, in general the spectrum of the operator pencil $T$ is not necessarily real or algebraically simple as the following example demonstrates.

**Example 3.5.** Consider the operator pencil

$$T(\lambda) := \lambda^2 - 2\lambda \pi + \frac{d^2}{dx^2} + 5\pi^2 = (\lambda - \pi)^2 + 4\pi^2 + \frac{d^2}{dx^2},$$

i.e. the pencil $T$ with $p \equiv \pi$ and $q = r' \equiv -5\pi^2$. Then $\lambda_{\pm1} = (1 \pm i\sqrt{3})\pi$ are complex conjugate eigenvalues of this operator pencil, while $\lambda_2 = \pi$ is its eigenvalue of algebraic multiplicity at least 2, since $y_0 = \sin 2\pi x$ and $y_1 \equiv 0$ form the corresponding chain of eigen- and associated vectors.

We summarize the above considerations in the following theorem.
Theorem 3.6. The spectrum of the operator pencil $T$ of (2.2) is a discrete subset of $\mathbb{C}$ and consists of geometrically simple eigenvalues.

4. Linearization and its properties

In this section we shall recast the spectral problem for the operator pencil $T$ as a spectral problem for some linear operator $L$ and show equivalence of these problems. Considering $L$ in a specially defined Pontryagin space will then reveal some further spectral properties of the pencil $T$.

4.1. Linearization. Setting $u_1 := y$ and $u_2 := \lambda y$, we recast the problem (1.1)–(2.1) as the first order system

$$
\begin{align*}
    u_2 &= \lambda u_1 \\
    Au_1 + Bu_2 &= \lambda u_2.
\end{align*}
$$

The system (4.1) is the spectral problem for the operator

$$
L_0 := \begin{pmatrix} 0 & I \\ A & B \end{pmatrix}.
$$

Therefore the spectral properties of the operator pencil $T$ should be closely related to those of the operator $L_0$. The latter should be considered in the so called energy space $E$ which we next define. Recall that the operator $A$ is supposed to be boundedly invertible (see Remark 3.4). Denote by $H$ the space $L_2(0, 1)$ and by $H_\alpha$, $\alpha \in \mathbb{R}$, the scale of Hilbert spaces generated by the operator $A$. Thus the space $H_0$ coincides with $H$, for any $\alpha > 0$ the space $H_\alpha$ is the domain of the operator $|A|^\alpha$ endowed with the norm $\|x\|_\alpha := \|A^\alpha x\|$, and for $\alpha < 0$ the space $H_\alpha$ is the completion of $H$ by the norm $\|\cdot\|_\alpha$. Since the operator $A$ has compact resolvent for every $\beta < \alpha$, the embedding $H_\alpha \hookrightarrow H_\beta$ is compact. Note that for any $\alpha > \theta$ the restriction of the operator $A^\alpha : H_\theta \rightarrow H_{\theta-\alpha}$ is homeomorphism. Similarly, for $\alpha < \theta$ the extension of the operator $A^\alpha$ to $A^\alpha : H_\theta \rightarrow H_{\theta-\alpha}$ is homeomorphism.

Introduce the Hilbert space $(E, (\cdot, \cdot)_E)$, where $E := H_{1/2} \times H$ and the scalar product $(\cdot, \cdot)_E$ is given by

$$(x, y)_E = (|A|^{1/2}x_1, |A|^{1/2}y_1) + (x_2, y_2)$$

for every $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $E$. Then the operator $L_0$ of (4.2) is well defined on the domain

$$\text{dom } L_0 := \{(u_1, u_2)^t \mid u_1 \in H_1; u_2 \in H_{1/2} \cap \text{dom } B\}.$$

However $L_0$ is not closed on this domain. To describe its closure, we need the following auxiliary result.
Lemma 4.1. The operator $B$ extends by continuity to a compact mapping $\tilde{B}$ from $H_{1/2}$ to $H_{-1/2}$.

Proof. Using the arguments analogous to those in the proof of Proposition 2.2 one can show that the operator $BA^{-1} : H \to H$ is compact. This yields the compactness $B : H_1 \to H$ as a mapping from $H_1$ to $H$.

Observe that the space $H_{-1}$ is dual to $H_1$ with respect to the scalar product $(\cdot, \cdot)$. Denoting by $\langle \cdot, \cdot \rangle$ the pairing between $H_{-1}$ and $H_1$, we get for $x \in H_1$ that

$$\|Bx\|_{-1} = \sup_{y \in H_1 : \|y\| = 1} |\langle Bx, y \rangle| = \sup_{y \in H_1 : \|y\| = 1} |\langle (Bx, y) \rangle|$$

$$= \sup_{y \in H_1 : \|y\| = 1} |\langle x, By \rangle| \leq \|x\|_0 \|B\|_{H_1 \to H_0}.$$ 

Therefore $B$ extends by continuity to a bounded mapping from $H_0$ to $H_{-1}$. Now using the interpolation theorem for compact operators [23] we obtain compactness of $\tilde{B} : H_{1/2} \to H_{-1/2}$.

Next observe that the operator $A$ can be extended by continuity to a homeomorphism $\tilde{A} : H_{1/2} \to H_{-1/2}$.

Lemma 4.2 ([8]). The operator $L_0$ is closable and the closure $L$ is given by the formulae

$$(4.3) \quad L \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \tilde{A}x_1 + \tilde{B}x_2 \end{pmatrix},$$

$$\text{dom} L = \left\{ (x_1, x_2)^t \mid x_1, x_2 \in H_{1/2}, \tilde{A}x_1 + \tilde{B}x_2 \in H \right\}.$$ 

We are going to show the coincidence of the spectra of $T$ and $L$. To start with we prove that the spectrum of $L$ is discrete.

Lemma 4.3. The spectrum of the operator $L$ is discrete.

Proof. By Remark 2.1 to show discreteness of the spectrum of $L$ it is enough to establish that its inverse $L^{-1}$ is compact.

Consider the system

$$x_2 = y_1,$$

$$\tilde{A}x_1 + \tilde{B}x_2 = y_2,$$

with $x = (x_1, x_2)^t$ from dom $L$ and $y = (y_1, y_2)^t$ from $E$. Since the operator $\tilde{A}$ is boundedly invertible (see Remark 3.4) this system gives that the operator $L$ is boundedly invertible and its inverse $L^{-1}$ is given by the matrix

$$L^{-1} = \begin{pmatrix} -\tilde{A}^{-1} \tilde{B} & A^{-1} \\ I & 0 \end{pmatrix}.$$
on $E$. Next we show that $L^{-1} : E \to E$ is a compact operator. To do this we show compactness of all entries of the corresponding matrix.

The operator $I : H_{1/2} \to H$ is an embedding of the space $H_{1/2}$ into $H$ and thus it is compact.

Since the operator $A^{-1} : H \to H_1$ is bounded and the embedding $H_1 \hookrightarrow H_{1/2}$ is compact the operator $A^{-1} : H \to H_{1/2}$ is compact as the composition of a bounded operator and a compact operator.

It was established in Lemma 4.1 that the operator $\tilde{B} : H_{1/2} \to H^{-1/2}$ is compact. Since the operator $\tilde{A}^{-1} : H^{-1/2} \to H_1$ is bounded this gives compactness of $\tilde{A}^{-1} \tilde{B} : H_{1/2} \to H_{1/2}$.

These observations yield compactness of $L^{-1}$ and complete the proof. □

For $\lambda \in \mathbb{C}$ we set
\begin{equation}
(4.4) \quad \tilde{T}(\lambda) := \lambda^2 I - \lambda \tilde{B} - \tilde{A},
\end{equation}
and consider $\tilde{T}(\lambda)$ as an operator from $H_{1/2}$ to $H_{-1/2}$.

**Theorem 4.4** ([8]). The spectrum of the operator $L$ coincides with the spectrum $\sigma(\tilde{T})$ of the operator pencil $\tilde{T}$. For every nonzero $\lambda \in \rho(\tilde{T})$ the following representation holds:
\begin{equation}
(4.5) \quad (L - \lambda I)^{-1} = \begin{pmatrix}
-\lambda^{-1}(\tilde{T}^{-1}(\lambda) \tilde{A} + I) & \tilde{T}(\lambda)^{-1} \\
-\tilde{T}^{-1}(\lambda) \tilde{A} & -\lambda \tilde{T}(\lambda)^{-1}
\end{pmatrix}.
\end{equation}

Now we can show coincidence of the spectra of the operator $L$ and of the operator pencil $T$. In view of Lemmas 3.3 and 4.3, it is sufficient to show coincidence of the corresponding eigenvalues.

**Theorem 4.5.** The eigenvalues of the operator pencil $T$ coincide with those of the operator $L$ counting multiplicities.

**Proof.** Observe firstly that for every $\lambda_0 \in \mathbb{C}$ the operator $\tilde{T}(\lambda_0)$ is an extension of $T(\lambda_0)$. Therefore if for some $\mu \in \rho(T(\lambda_0)) \cap \rho(\tilde{T}(\lambda_0))$ one has $\tilde{T}(\lambda_0) - \mu)u \in H$, then $u$ belongs to $H_1$, i.e. to $\text{dom} T$. We shall use this remark in our further discussions.

Assume that $\lambda_0 \in \mathbb{C}$ is an eigenvalue of $T$ with the corresponding chain of eigen- and associated vectors $y_0, y_1, \ldots, y_{m-1}$. By definition this means that

$$
(\lambda_0^2 - \lambda_0 B - A)y_k + (2\lambda_0 - B)y_{k-1} + y_{k-2} = 0
$$

for $k = 0, \ldots, m - 1$ with $y_{-1}, y_{-2}$ being zero. A direct verification shows that these equalities are equivalent to the following

$$(L - \lambda_0)Y_k = Y_{k-1}, \quad k = 0, \ldots, m - 1$$

and consider $\tilde{T}(\lambda)$ as an operator from $H_{1/2}$ to $H_{-1/2}$.
with \( Y_k = (y_k, \lambda y_k + y_{k-1})^\top \). In particular, \( \lambda_0 \) is an eigenvalue of \( \mathcal{L} \) and
the vectors \( Y_0, Y_1, \ldots, Y_{m-1} \) belong to the domain of \( \mathcal{L} \) and so form a chain of eigen- and associated vectors of \( \mathcal{L} \) corresponding to \( \lambda_0 \).

Next suppose that \( \lambda_0 \in \mathbb{C} \) is an eigenvalue of \( \mathcal{L} \) with the corresponding chain of eigen- and associated vectors \( Y_0, \ldots, Y_{m-1} \) of length \( m \). By definition, \((\mathcal{L} - \lambda_0)Y_k = Y_{k-1} \) (\( Y_{-1} \) is supposed to be zero) or, setting \( Y_k := (Y_{k,1}, Y_{k,2})^\top \),

\[-\lambda_0 Y_{k,1} + Y_{k,2} = Y_{k-1,1},\]
\[\tilde{A} Y_{k,1} + (\tilde{B} - \lambda_0) Y_{k,2} = Y_{k-1,2}.\]

This gives that \( Y_{k,2} = \lambda_0 Y_{k,1} + Y_{k-1,1} \) and

\[\left( \lambda_0^2 - \lambda_0 \tilde{B} - \tilde{A} \right) Y_{k,1} + (2 \lambda_0 - \tilde{B}) Y_{k-1,1} + Y_{k-2,1} = 0\]
for \( k = 0, \ldots, m - 1 \) and \( Y_{-1} = Y_{-2} = 0 \). Since \( Y_k = (Y_{k,1}, Y_{k,2})^\top \in \text{dom} \mathcal{L} \), we have that \( Y_{k,1} \) is from \( H_{1/2} \). Thus the last equality yields that \( Y_{0,1}, Y_{1,1}, \ldots, Y_{m-1,1} \) is a chain of eigen- and associated vectors of the operator pencil \( \tilde{T} \) corresponding to the eigenvalue \( \lambda_0 \).

Next we prove by induction that all the vectors \( Y_{0,1}, Y_{1,1}, \ldots, Y_{m-1,1} \) belong to \( H_1 \) and, therefore, form a chain of eigen- and associated vectors of \( T \) corresponding to \( \lambda_0 \). Consider firstly the vector \( Y_0 \). Take \( \mu \in \rho(T(\lambda_0)) \cap \rho(\tilde{T}(\lambda_0)) \) and observe that \((\tilde{T}(\lambda_0) - \mu)Y_{0,1} = -\mu Y_{0,1} \in H \). In view of remark made at the beginning of the proof, this gives that \( Y_{0,1} \) belongs to \( H_1 \) and so it is an eigenvector of \( T \) corresponding to \( \lambda_0 \). Now suppose that \( Y_{j,1} \) belongs to \( H_1 \) for every \( j < k \). Taking \( \mu \in \rho(\tilde{T}(\lambda_0)) \cap \rho(T(\lambda_0)) \) we obtain

\[(\tilde{T}(\lambda_0) - \mu)Y_{k,1} = -\mu Y_{k,1} - (2 \lambda_0 - \tilde{B}) Y_{k-1,1} + Y_{k-2,1} \in H.\]

By the same reason we deduce that \( Y_{k,1} \) is from \( H_1 \); thus the chain of eigen- and associated vectors of \( \mathcal{L} \) generates the corresponding chain for \( T \).

The above reasonings show that there is a one-to-one correspondence between the chains of eigen- and associated vectors of \( T \) and \( \mathcal{L} \) corresponding to the same eigenvalues, thus establishing the claim. \( \square \)

Coincidence of the eigenvalues of \( \mathcal{L} \) and \( T \) can be proved in another way. Observe that Lemma 4.3 and Theorem 4.4 imply that the spectrum of \( \tilde{T} \) is discrete. But it is known (see [29]) that the discrete parts of the spectra of \( T \) and \( \tilde{T} \) coincide. In view of Proposition 3.3 this gives that \( \sigma(T) = \sigma(\tilde{T}) \) and so \( \sigma(T) = \sigma(\mathcal{L}) \).
4.2. The Pontryagin space properties of $\mathcal{L}$. Now show that the linearization $\mathcal{L}$ is self-adjoint in some Pontryagin space. Consider the operator $J = P_+ - P_-$, where $P_+$ and $P_-$ are the $(\cdot, \cdot)$-orthogonal projectors onto the spectral subspaces of $A$ corresponding to the positive and negative parts of the spectrum respectively. Set $\mathcal{J} := \begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix}$

and define an inner product

$[x, y] := (J |A|^{1/2} x_1, |A|^{1/2} y_1) + (x_2, y_2)$

for every $x = (x_1, x_2)^t$ and $y = (y_1, y_2)^t$ from $\mathcal{E}$. The number of negative eigenvalues of the operator $\mathcal{J}$ equals that of $J$, which in turn is the number of negative eigenvalues of $A$. But the operator $A$ is lower semibounded and the number of its negative eigenvalues is finite, say $\kappa$. This gives that the product $[\cdot, \cdot]$ is indefinite and the space $\Pi := (\mathcal{E}, [\cdot, \cdot])$ is a Pontryagin space of negative index $\kappa$. Note that the topology of $\Pi$ coincide with that of $\mathcal{E}$.

Consider the operator $\mathcal{L}_0$ in the space $\Pi$. For every $x = (x_1, x_2)^t$ from $\text{dom} \mathcal{L}_0$ we have

$[\mathcal{L}_0 x, x] = (J |A|^{1/2} x_2, |A|^{1/2} x_1) + (Ax_1, x_2) + (Bx_2, x_2)$

$= (Ax_1, x_2) + (x_2, Ax_1) + (x_2, Bx_2) = 2 \text{Re}(Ax_1, x_2) + (x_2, Bx_2)$.

Clearly, this shows that $[\mathcal{L}_0 x, x]$ is real and thus the operator $\mathcal{L}_0$ is symmetric in $\Pi$. This together with the fact that the spectrum of $\mathcal{L}$ is discrete (see Lemma 4.3) gives the following proposition (see [2]).

**Proposition 4.6.** The operator $\mathcal{L}$ is self-adjoint in the Pontryagin space $\Pi$.

Using this result and properties of self-adjoint operators in Pontryagin spaces (see Proposition A.1) we obtain that the spectrum of the operator $\mathcal{L}$ is real with possible exception of at most $\kappa$ pairs of complex-conjugate eigenvalues $\lambda$ and $\bar{\lambda}$. The algebraic multiplicity of any eigenvalue of $\mathcal{L}$ can not exceed $2\kappa + 1$.

When the operator $A$ is positive the space $\Pi$ is a Hilbert space (the negative index $\kappa = 0$). Then $\mathcal{L}$ is a self-adjoint operator in a Hilbert space and thus its spectrum is real. Moreover, the algebraic and geometric multiplicities of the eigenvalues of $\mathcal{L}$ are equal. In view of geometric simplicity of the eigenvalues of $\mathcal{L}$, this yields that the spectrum of $\mathcal{L}$ is algebraically simple. Therefore, if the operator $A$ is positive, then the spectrum of the operator $\mathcal{L}$ is real and simple.

Summing up all these results and using the equivalence of spectral problems for the operator pencil $T$ and for the operator $\mathcal{L}$ we obtain more spectral properties of $T$ given in the following proposition.
Theorem 4.7. Let $\kappa$ be the number of negative eigenvalues of the operator $A$. Then

(i) the spectrum of the operator pencil $T$ is real with possible exception of at most $\kappa$ pairs of complex-conjugate eigenvalues $\lambda$ and $\bar{\lambda}$;

(ii) algebraic multiplicity of every eigenvalue of $T$ does not exceed $2\kappa + 1$.

If the operator $A$ is positive, then the spectrum of the operator pencil $T$ is real and simple.

5. Norming constants

5.1. Notion of norming constants. In this section we introduce the notion of norming constants for the operator pencil $T$ and establish some of their properties. In the case where the pencil $T$ has only real and simple eigenvalues the norming constants were used in [7] to solve the inverse spectral problem of determination of the potentials $p$ and $q$ of the pencil.

We say that a matrix is upper (lower) anti-triangular if all its elements under (above) anti-diagonal are zero. Denote by $M^+[\gamma_1, \gamma_2, \ldots, \gamma_p]$, respectively by $M^-[\gamma_1, \gamma_2, \ldots, \gamma_p]$, a Hankel upper, respectively lower, anti-triangular matrices given by

$$
M^+[\gamma_1, \gamma_2, \ldots, \gamma_p] = 
\begin{pmatrix}
\gamma_1 & \gamma_2 & \cdots & \gamma_p \\
\gamma_2 & \ddots & 0 \\
\vdots & \ddots & \ddots \\
\gamma_p & 0 & \cdots & 0
\end{pmatrix}
$$

and

$$
M^-[\gamma_1, \gamma_2, \ldots, \gamma_p] = 
\begin{pmatrix}
0 & \cdots & 0 & \gamma_1 \\
\vdots & \ddots & \ddots \\
0 & \cdots & \gamma_2 \\
\gamma_1 & \gamma_2 & \cdots & \gamma_p
\end{pmatrix}
$$

If some matrix $M = M^+[\gamma_1, \ldots, \gamma_p]$ we say that the sequence $\gamma_1, \ldots, \gamma_p$ is associated with the matrix $M$.

In this section we shall often work with infinite block-diagonal matrices with upper (lower) anti-triangular blocks of two types. The first type blocks are just upper (lower) anti-triangular Hankel matrices. The second type ones have the form

$$
(5.1) \quad \begin{pmatrix}
0 & B_1 \\
B_2 & 0
\end{pmatrix}
$$

where $B_1$ is upper (lower) anti-triangular Hankel matrix and $B_2$ is its complex conjugate. Denote the blocks of such a matrix $M$ by $M_n$, $n \in \mathbb{Z}$. To every block $M_n$ of size $m$ there is associated a number sequence of length $m$; these finite sequences together form an infinite sequence $(\gamma_k)_{k \in \mathbb{Z}}$ associated with $M$.

List the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, of the operator pencil $T$ so that

(i) each eigenvalue is repeated according to its multiplicity;
(ii) the real parts of eigenvalues do not decrease, i.e. $\text{Re}\lambda_i \leq \text{Re}\lambda_j$ for $i < j$;
(iii) the moduli of the imaginary parts of the eigenvalues with equal real parts non-decrease, i.e. if $\text{Re}\lambda_i = \text{Re}\lambda_j$ for some $i < j$ then $|\text{Im}\lambda_i| \leq |\text{Im}\lambda_j|$. Moreover, if $|\text{Im}\lambda_i| = |\text{Im}\lambda_j|$ for some $i \leq j$ then $\text{Im}\lambda_i \geq \text{Im}\lambda_j$.

This enumeration is such that if some $\lambda$ is an eigenvalue of $T$ of multiplicity $m$, then there is $n \in \mathbb{Z}$ such that $\lambda = \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m-1}$. If, moreover, $\lambda$ is non-real, then $\overline{\lambda} = \lambda_{n+m} = \lambda_{n+m+1} = \cdots = \lambda_{n+2m-1}$. Along with the eigenvalue sequence $(\lambda_k)_{k \in \mathbb{Z}}$ we introduce the sequence $(y_k)_{k \in \mathbb{Z}}$ of vectors from $\text{dom} A$ in the following way: if $\lambda = \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m-1}$ is an eigenvalue of $T$ of multiplicity $m$, then $y_n, y_{n+1}, \ldots, y_{n+m-1}$ is a chain of eigen- and associated vectors of $T$ corresponding to $\lambda$ and such that $y_n(x, \lambda)$ satisfies the initial conditions $y_n(0, \lambda) = 0$, $y_n^{[1]}(0, \lambda) = \lambda$ and the functions $y_k, y_{k+1}, \ldots, y_{k+m-1}$ are defined via

\begin{equation}
 y_{n+j}(x, \lambda) := \frac{1}{j!} \frac{\partial^j y_n(x, z)}{\partial z^j} \bigg|_{z=\lambda}, \quad j = 0, 1, \ldots, m - 1.
\end{equation}

Note that for complex conjugate eigenvalues $\lambda = \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m-1} = \overline{\lambda_{n+m}} = \cdots = \overline{\lambda_{n+2m-1}}$, the vectors $y_{j+m} = \overline{y_j}$ for $j = n, \ldots, n + m - 1$.

Next define the norming constants $\alpha_n$, $n \in \mathbb{Z}$, for the operator pencil $T$ as follows. For an eigenvalue $\lambda = \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m-1}$ we put

\begin{equation}
 \alpha_n = \left( \frac{T'(\lambda)}{\lambda} y_n, y_n \right),
\end{equation}

\begin{equation}
 \alpha_{n+j} = \left( \frac{T'(\lambda)}{\lambda} y_n, y_{n+j} \right) + \frac{1}{\lambda} (y_n, y_{n+j-1}), \quad \text{for } j = 1\ldots m - 1.
\end{equation}

Defining the norming constants for the operator pencil $T$ in the described way is quite reasonable. Firstly, note that for real and simple
eigenvalues so defined norming constants determine the type of eigenvalues (see [18]) as

\[ (T'(\lambda_n)y_n, y_n) = \lambda_n \alpha_n, \]

where \( T' \) is the \( \lambda \)-derivative of \( T \). Secondly, if the potential \( p \) is identically zero, (1.1) is the spectral equation for Sturm–Liouville operator \( A \) and the given definition of the norming constants for the operator pencil \( T \) coincides with the standard definition of norming constants, [4]. Further we shall see that so defined norming constants are closely related to the norming constants for the linearization \( L' \).

Recall (see Theorem 4.5) that the sequence (\( \lambda_k \)\)\( k \in \mathbb{Z} \)) is also an eigenvalues sequence for the operator \( L \). Consider the sequence of vectors \( (Y_k)_{k \in \mathbb{Z}} \), such that for the eigenvalue \( \lambda = \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m-1} \) of multiplicity \( m \) the vectors \( Y_k, k = n, \ldots, n+m-1 \) are defined by formulae \( Y_n = (y_n, \lambda y_n)^t \) and \( Y_j = (y_j, \lambda y_j + y_{j-1})^t \), \( j = n+1, \ldots, n+m-1 \). From the proof of Theorem 4.5 we know that \( Y_n, Y_{n+1}, \ldots, Y_{n+m-1} \) is the chain of eigen- and associated vectors of \( L \) corresponding to the eigenvalue \( \lambda \) and so \( (Y_n)_{n \in \mathbb{Z}} \) is the sequence of all eigen- and associated vectors of \( L \).

Let us put \( g_{kl} := [Y_k, Y_l] \) and associate with the operator \( L \) the Gramm matrix \( G = (g_{kl}) \). In view of the equalities

\[ [Y_{k-1}, Y_l] = [ (L - \lambda I) Y_k, Y_l] = [Y_k, (L - \overline{\lambda} I) Y_l], \]

for every real \( \lambda = \lambda_n = \cdots = \lambda_{n+m-1} \) we have \( g_{ij} = g_{kl} \) for \( n \leq i, j, k, l \leq n + m - 1 \) such that \( i + j = k + l \). Moreover, \( g_{kl} = 0 \) for \( k + l < 2n + m - 1 \). Similarly, for complex \( \lambda = \lambda_n = \cdots = \lambda_{n+m-1} = \lambda_{n+1} = \cdots = \lambda_{n+2m-1} \) we obtain \( g_{ij} = g_{kl} \) for \( n \leq i, j, k, l \leq n + 2m - 1 \) such that \( 2n + m \leq i + j = k + l \leq 2n + 3m - 2 \) and \( g_{kl} = 0 \) for \( 2n + m \leq k + l < 2n + 2m - 1 \). Therefore the matrix \( G \) is of block-diagonal form with blocks of two types: blocks corresponding to real eigenvalues and those corresponding to pairs of complex conjugate ones. The block corresponding to a real eigenvalue \( \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m-1} \) of multiplicity \( m \) is a Hankel lower anti-triangular matrix \( M^- [\beta_n, \ldots, \beta_{n+m-1}] \). The sub-matrix of \( G \) corresponding to the pair of complex conjugate eigenvalues \( \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m-1} = \lambda_{n+m} = \lambda_{n+m+1} = \cdots = \lambda_{n+2m-1} \) is of the form (5.1) with \( B_1 = M^- [\beta_n, \ldots, \beta_{n+m-1}] \) and its complex conjugate \( B_2 \) (see Proposition A.2). We call the number \( \beta_k, k \in \mathbb{Z} \), the norming constant of \( L \) corresponding to the eigenvalue \( \lambda_k \).

Straightforward verification shows that the norming constants of \( L \) are related with those of \( T \) as follows

\[ \beta_k = \lambda_k^2 \alpha_k \]

for every \( k \in \mathbb{Z} \).
Theorem 5.1. All the norming constants of $T$ are positive if and only if the operator $A$ is positive.

Proof. Observe firstly that by (5.4) for nonzero eigenvalues the norming constants of $T$ are positive if and only if those of $L$ are also positive.

Sufficiency. Obviously, if $A$ is positive then the space $\Pi$ is a Hilbert space and so $L$ is a self-adjoint operator in a Hilbert space and all its norming constants are positive as norms of eigenvectors in Hilbert space.

Necessity. Suppose that $A$ is not positive. Then $\Pi$ is a Pontryagin space of finite negativity index, say $\kappa$, and so by the Pontryagin theorem it has a maximal non-positive subspace of dimension $\kappa$ invariant with respect to $L$. Therefore $L$ possesses an eigenvector in this subspace, and the norming constant generated by this eigenvector is non-positive. Thus not all norming constants of $T$ are positive, and the proof is complete. □

5.2. Relations for norming constants. Let us now consider the vector $X = (0, x)^t$ and compute the residues of $(L - z)^{-1}X$ at an eigenvalue $\lambda$ in two different ways. Equating the results, we shall obtain some relations for norming constants $\alpha_k$.

Denote by $D = (d_{kl})$ the matrix inverse to $G$, i.e. $D = G^{-1}$. Observe that $D$ as well as $G$ has a block-diagonal structure but with upper anti-triangular matrices in blocks. Associate with $D$ the sequence $(\delta_k)_{k \in \mathbb{Z}}$.

Since the system of eigen- and associated vectors $Y_j$ of $L$ forms a Riesz basis in the Pontryagin space $\Pi$ (see Proposition A.1) we can write the resolution of identity in $\Pi$ as follows

$$I = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} d_{kl}[\cdot, Y_l]Y_k,$$

where the sum is convergent in the strong operator topology. Using this it is straightforward to compute the residue of $(L - z)^{-1}X$ at the real eigenvalue $\lambda = \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m-1}$ as

$$\text{res}_{z = \lambda} (L - z)^{-1}X = - \sum_{k=n}^{n+m-1} \sum_{l=n}^{n+m-1} d_{kl}[X, Y_l]Y_k = - \sum_{k=n}^{n+m-1} \sum_{l=n}^{n+m-1} d_{kl}(x, Y_{l,2})Y_k$$

and for complex $\lambda = \lambda_n = \cdots = \lambda_{n+m-1} = \lambda_{n+2m-1} = \cdots = \lambda_{n+2m-1}$

$$\text{res}_{z = \lambda} (L - z)^{-1}X = - \sum_{k=n}^{n+m-1} \sum_{l=n+m}^{n+2m-1} d_{kl}[X, Y_l]Y_k = - \sum_{k=n}^{n+m-1} \sum_{l=n+m}^{n+2m-1} d_{kl}(x, Y_{l,2})Y_k.$$
Next using the representation \([4.5]\) we obtain
\[
(\mathcal{L} - zI)^{-1} X = \begin{pmatrix}
-T(z)^{-1}x \\
-zT(z)^{-1}x
\end{pmatrix}.
\]

By Green’s Formula
\[
T(z)^{-1}f(x) = \frac{1}{W(z)} \left[ \varphi(x, z) \int_x^1 f(t)\psi(t, z)dt + \psi(x, z) \int_0^x f(t)\varphi(t, z)dt \right],
\]
where \(\varphi(\cdot, z)\) is a solution of the equation \(\ell(y) = (z^2 - 2zp)y\) satisfying the initial conditions \(y(0) = 0, y[1](0) = z\), \(\psi(\cdot, z)\) is a solution of the same equation satisfying the conditions \(y(1) = 0, y[1](1) = z\) and \(W(z)\) is the Wronskian of the solutions \(\varphi(\cdot, z)\) and \(\psi(\cdot, z)\), i.e. \(W(z) = \varphi(x, z)\psi[1](x, z) - \psi(x, z)\varphi[1](x, z)\). Set \(s(z) := \varphi(1, z)\) and \(c(z) := \varphi[1](1, z)\). Since the Wronskian \(W\) does not depend on \(x\) we observe that \(W(z) = z\varphi(1, z) = zs(z)\). Next, note that for an eigenvalue \(\lambda\) of \([1.1], [2.1]\) the functions \(\psi(x, \lambda)\) and \(\varphi(x, \lambda)\) are related as follows
\[
\psi(x, \lambda) = \frac{\varphi[1](1, \lambda)}{\varphi[1](1, \lambda)} \varphi(x, \lambda) = \frac{\lambda}{c(\lambda)} \varphi(x, \lambda).
\]
Taking these remarks into account, we compute the residues of \(T(z)^{-1}\) and \(zT(z)^{-1}\) at the real eigenvalue \(\lambda_1 = \lambda_2 = \ldots = \lambda_{n+m-1}\) of multiplicity \(m\)
\[
\text{res}_{z=\lambda} T(z)^{-1}x = \sum_{k=n}^{n+m-1} \sum_{l=n}^{n+m-1} h_{kl}(x, y_l) y_k,
\]
\[
\text{res}_{z=\lambda} zT(z)^{-1}x = \sum_{k=n}^{n+m-1} \sum_{l=n}^{n+m-1} h_{kl}(x, y_l)(\lambda y_k + y_{k-1}),
\]
and at the complex eigenvalue \(\lambda_1 = \lambda_2 = \ldots = \lambda_{n+2m-1}\)
\[
\text{res}_{z=\lambda} T(z)^{-1}x = \sum_{k=n}^{n+2m-1} \sum_{l=n+m}^{n+2m-1} h_{kl}(x, y_l) y_k,
\]
\[
\text{res}_{z=\lambda} zT(z)^{-1}x = \sum_{k=n}^{n+2m-1} \sum_{l=n+m}^{n+2m-1} h_{kl}(x, y_l)(\lambda y_k + y_{k-1})
\]
where \(h_{kl}\) form a matrix \(H = (h_{kl})\) of block diagonal form with upper anti-triangular blocks of two types. The elements of the sequence \((\eta_k)_{k \in \mathbb{Z}}\) associated with \(H\) are given explicitly in the following
way. For real eigenvalue \( \lambda = \lambda_n = \ldots = \lambda_{n+m-1} \) of multiplicity \( m \)

\[
\eta_j = \frac{1}{(m+n-1-j)!(z-\lambda)^{m-1}} \left[ \frac{(z-\lambda)^{m-1}}{s(z)c(z)} \right] \bigg|_{z=\lambda} \quad \text{for} \quad j = n, \ldots, n+m-1.
\]

And for complex \( \lambda = \lambda_n = \lambda_{n+1} = \ldots = \lambda_{n+2m-1} \) the element \( \eta_j, j = n, \ldots, n+m-1 \), is defined by (5.7) and \( \eta_j = \eta_{j-m} \) for \( j = n+m, n+m+1, \ldots, n+2m-1 \). Therefore,

\[
\text{res}(\mathcal{L} - z)^{-1}X = - \sum_{k=n}^{n+m-1} \sum_{l=n}^{n+m-1} h_{kl}(x, y_l)Y_k.
\]

for real eigenvalue \( \lambda = \lambda_n = \ldots = \lambda_{n+m-1} \) and

\[
\text{res}(\mathcal{L} - z)^{-1}X = - \sum_{k=n}^{n+2m-1} \sum_{l=n}^{n+2m-1} h_{kl}(x, y_l)Y_k.
\]

for complex \( \lambda = \lambda_n = \lambda_{n+1} = \ldots = \lambda_{n+m-1} = \lambda_{n+2m-1} \). Now equating (5.5) to (5.8) and (5.6) to (5.9) and taking linear independence of \( Y_j \) (see Proposition A.1) into account we derive the relation

\[
\delta_{n+m-1} = \eta_{n+m-1}/\lambda, \quad \delta_k = (\delta_{k-1})/\lambda, \quad \text{for} \quad k = n, \ldots, n+m-2.
\]

for eigenvalue \( \lambda = \lambda_n = \lambda_{n+1} = \ldots = \lambda_{n+m-1} \) of multiplicity \( m \).

### 5.3. Determining norming constants from two spectra.

Let us now consider the problem (1.1) with so-called mixed boundary conditions

\[
y(0) = y^{[1]}(1) = 0
\]

and denote by \( A_M \) the operator acting via

\[
A_My := \ell(y)
\]

on the domain

\[
\text{dom} A_M := \{ y \in \text{dom} \ell \mid y(0) = y^{[1]}(1) = 0 \}.
\]

Define the operator pencil \( T_M \) by (2.2) with \( A_M \) instead of \( A \). Then the spectral problem (1.1), (5.11) can be regarded as that for \( T_M \).

We can analyse the pencil \( T_M \) in the same way as \( T \). Moreover, by means of the operator \( A_M \) we can construct an energy space \( \mathcal{E}_M \), the corresponding Pontryagin space \( \Pi_M \) and consider the corresponding linearization \( \mathcal{L}_M \) therein as it was done for \( T \). This will give that all the results of sections 3 and 4 concerning the pencil \( T \) hold for \( T_M \).
Note that the function \( s(\lambda) \) is a characteristic function for the problem \((1.1), (2.1)\) and \( c(z) \) is that for the problem \((1.1), (5.11)\). This means that some \( \lambda \in \mathbb{C} \) is an eigenvalue of the pencil \( T \) (\( T_M \) respectively) of algebraic multiplicity \( m \) if and only if it is a zero of \( s(z) \) (of \( c(z) \) respectively) of order \( m \). The functions \( s(z) \) and \( c(z) \) are of exponential type one and are determined uniquely by their zeros by means of a canonical product (see [32]). Therefore, the spectra of \( T \) and \( T_M \) determine the functions \( s(z) \) and \( c(z) \) which by \((5.7)\) and \((5.10)\) determine the elements \( \delta_n \) of the matrix \( D \) inverse to the Gramm matrix \( G \). Having \( D \) we compute \( \beta_n \) and then the norming constants \( \alpha_n \) of \( T \) by \((5.4)\).

For instance, let the spectra of the pencils \( T \) and \( T_M \) consist of simple eigenvalues \( (\lambda_n)_{n \in \mathbb{Z}^*}, \mathbb{Z}^*: = \mathbb{Z} \setminus \{0\} \), and \( (\mu_n)_{n \in \mathbb{Z}} \). Then for the eigenvalue \( \lambda_n \) of \( T \) we have (see \((5.7)\))

\[
\eta_n = \frac{1}{s(\lambda_n)c(\lambda_n)}
\]

and, by definition,

\[
\delta_n = \frac{1}{\beta_n},
\]

where \( \beta_n \) is a norming constant of \( \mathcal{L} \) corresponding to \( \lambda_n \). These equalities together with \((5.10)\) yield

\[
\frac{\lambda_n}{\beta_n} = \frac{1}{s(\lambda_n)c(\lambda_n)}
\]

which, by \((5.4)\), gives the equality for the norming constant \( \alpha_n \) of \( T \) corresponding the eigenvalue \( \lambda_n \)

\[
(5.12) \quad \alpha_n = \frac{s(\lambda_n)c(\lambda_n)}{\lambda_n}.
\]

Therefore, the norming constant \( \alpha_n \) corresponding to the eigenvalue \( \lambda_n, n \in \mathbb{Z} \) of \( T \) is determined by the formula

\[
(5.13) \quad \alpha_n = \frac{1}{\lambda_n} \prod_{k=\infty}^{\infty} \frac{\lambda_n - \lambda_k}{\pi k} \prod_{k=-\infty}^{\infty} \frac{\lambda_n - \mu_k}{\pi(k - 1/2)}.
\]

For the operator pencil \( T_M \) we can also define the norming constants \( \alpha_n^M \) by \((5.3)\) with \( T_M \) instead of \( T \) and obtain for these norming constants analogous results as for those of \( T \). In particular, the following theorem holds.

**Theorem 5.2.** All the norming constants of \( T_M \) are positive if and only if the operator \( A_M \) is positive.
For the pencil $T_M$ and the corresponding linearization $L_M$ we can define matrices $G_M$, $D_M$, $H_M$ in the same way as $G$, $D$, $H$ were defined for $T$ and obtain analogous relations. It can be shown by direct analysis that for real eigenvalue $\mu = \mu_n = \cdots = \mu_{n+m-1}$ of $T_M$ of algebraic multiplicity $m$ the elements of the matrix $H_M$ are

$$\eta_j^M = \frac{1}{(m+n-1-j)!} \frac{\partial^{m+n-1-j} \left[ (z - \mu)^{m-1} \right]}{s(z)c(z)} \bigg|_{z=\mu}$$

for $j = n, \ldots, n+m-1$. For complex $\mu = \mu_n = \mu_{n+1} = \cdots = \mu_{n+m-1} = \bar{\mu}_{n+m} = \bar{\mu}_{n+m+1} = \cdots = \bar{\mu}_{n+2m-1}$ the elements $\eta_j^M$, $j = n, \ldots, n+m-1$, are defined by the same formula and $\eta_j^M = \overline{\eta_{j-m}^M}$ for $j = n+m, n+m+1, \ldots, n+2m-1$. Therefore, in the case of simple eigenvalues we obtain the formulas for norming constants $\alpha_n^M$ corresponding to the eigenvalues $\mu_n$ of $T_M$

$$\alpha_n^M = -\frac{s(\mu_n)\dot{c}(\mu_n)}{\mu_n}$$

and so

$$\alpha_n^M = -\frac{1}{\mu_n} \prod_{k=-\infty}^{\infty} \frac{\mu_n - \lambda_k}{\pi k} \prod_{k=-\infty}^{\infty} \frac{\mu_n - \mu_k}{\pi (k - 1/2)}.$$

6. The case of real and simple eigenvalues

This section is devoted to the special case when the spectra of $T$ and $T_M$ are real and simple. We shall establish some conditions which guarantee that these spectra are real and simple.

We say that the spectra of the operator pencils $T$ and $T_M$ almost interlace if they consist only of real and simple eigenvalues, which can be labeled in increasing order as $\lambda_n$, $n \in \mathbb{Z}^*$, and $\mu_n$, $n \in \mathbb{Z}$, respectively so that they satisfy the condition

$$\mu_k < \lambda_k < \mu_{k+1} \text{ for every } k \in \mathbb{Z}^*.$$

**Theorem 6.1.** The following statements are equivalent

(i) The spectra of $T$ and $T_M$ almost interlace.

(ii) A real number $\mu_*$ exists such that the operator $T_M(\mu_*)$ is negative.

**Proof.** ((i) $\Rightarrow$ (ii)) Let us firstly prove this implication for the case when $0 \in (\mu_0, \mu_1)$. We shall show that then the operator $A_M = -T_M(0)$ is positive, which means that $T_M(0)$ is negative.
In view of (5.14) and the fact that the product
\[
(\mu_n - \lambda_n) \prod_{k=-\infty}^{\infty} \frac{\mu_n - \lambda_k}{\pi k} \frac{\mu_n - \mu_k}{\pi(k - 1/2)}
\]
is negative, the sign of the norming constant \(\alpha_n^M\) is defined by the sign of \((\mu_n - \mu_0)/\mu_n\). Thus under our assumption all the norming constants \(\alpha_n^M\) of \(T_M\) are positive. By Theorem 5.2, this gives positivity of \(A_M\).

If 0 does not belong to \((\mu_0, \mu_1)\) we take any point \(\mu_*\) from this interval and shift the spectral parameter of \(T\) and \(T_M\) by \(\mu_*\) to obtain the pencils
\[
\hat{T}(\lambda) = T(\lambda + \mu_*) = \lambda^2 I - \lambda(B - 2\mu_* I) + T(\mu_*) = \lambda^2 I - 2\lambda \hat{B} - \hat{A}
\]
(6.2)
\[
\hat{T}_M(\lambda) = T_M(\lambda + \mu_*) = \lambda^2 I - \lambda(B - 2\mu_* I) + T_M(\mu_*) = \lambda^2 I - 2\lambda \hat{B} - \hat{A}_M
\]
with \(\hat{B} := B - 2\mu_* I, \hat{A} := -T(\mu_*)\) and \(\hat{A}_M := -T_M(\mu_*)\). Clearly, the spectra of \(\hat{T}\) and \(\hat{T}_M\) almost interlace with 0 \(\in (\mu_0, \mu_1)\). In view of the first part of this proof the operator \(\hat{A}_M\) is positive. Therefore, the operator \(T_M(\mu_*)\) with this \(\mu_*\) is negative.

((ii) ⇒ (i)) Let the operator \(T_M(\mu_*)\) be negative. Consider the operator pencil \(\hat{T}_M\) of (6.3) obtained from \(T_M\) by the shift of the spectral parameter by \(\mu_*\). Then the operator \(\hat{A}_M = -T_M(\mu_*)\), and the operator \(\hat{A} = -T(\mu_*)\) is also negative. By Theorem 4.1 the spectra of \(T\) and \(T_M\) are real and simple. The eigenvalues \(\lambda_n\) and \(\mu_n\) can be enumerated so that \(\lambda_n = \pi n + p_0 + \hat{\lambda}_n\) and \(\mu_n = \pi \left(n - \frac{1}{2}\right) + p_0 + \hat{\mu}_n\) with \(p_0 := \int_0^1 p(x)\, dx\) and \(\ell_2\)-sequences \((\bar{\mu}_n), (\bar{\lambda}_n)\).

Next define the norming constants \(\hat{\alpha}_j^M, j \in \mathbb{Z}^*\), for \(\hat{T}_M\). In view of Theorem 5.2 all these norming constants are positive and by (5.14) they are determined by the formula
\[
\hat{\alpha}_n^M = \frac{1}{\mu_n - \mu_*} \prod_{k=-\infty}^{\infty} \frac{\mu_n - \lambda_k}{\pi k} \prod_{k=-\infty \atop k \neq n}^{\infty} \frac{\mu_n - \mu_k}{\pi(k - 1/2)},
\]
where \(\lambda_n, n \in \mathbb{Z}^*\), and \(\mu_n, n \in \mathbb{Z}\), are the eigenvalues of \(T\) and \(T_M\) respectively. Therefore, the expression
\[
\frac{\hat{\alpha}_{n+1}^M}{\hat{\alpha}_n^M} = \frac{\mu_{n+1} - \mu_*}{\mu_{n+1} - \mu_n} \prod_{k=-\infty \atop k \neq n+1}^{\infty} \frac{\mu_{n+1} - \lambda_k}{\mu_n - \lambda_k} \prod_{k=-\infty \atop k \neq n, n+1}^{\infty} \frac{\mu_{n+1} - \mu_k}{\mu_n - \mu_k}
\]
is positive. This gives that if $\mu_n < \mu_* < \mu_{n+1}$, then there is an even number of $\lambda_k$ between $\mu_n$ and $\mu_{n+1}$ and otherwise there is an odd number of $\lambda_k$ between $\mu_n$ and $\mu_{n+1}$. But the asymptotics of $\mu_n$ and $\lambda_n$ (see [24]) implies that the number of elements of $(\lambda_k)$ between $\mu_n$ and $\mu_{n+1}$ can not exceed 1 and that there is no $\lambda_k$ between $\mu_0$ and $\mu_1$. This gives that $(\lambda_n)$ and $(\mu_n)$ almost interlace and $\mu_* \in (0, \mu_1)$. This completes the proof. □

From the proof of the first implication in the above theorem we immediately obtain the following corollaries.

**Corollary 6.2.** If the spectra $(\lambda_n)_{n \in \mathbb{Z}^*}$ of $T$ and $(\mu_n)_{n \in \mathbb{Z}}$ of $T_M$ almost interlace, then for every number $\mu_*$ from the interval $(0, \mu_1)$ the operator $T_M(\mu_*)$ is negative.

**Corollary 6.3.** If for some $\mu_* \in \mathbb{R}$ the operator $T_M(\mu_*)$ is negative, then the spectra $(\lambda_n)_{n \in \mathbb{Z}^*}$ of $T$ and $(\mu_n)_{n \in \mathbb{Z}}$ of $T_M$ almost interlace with $\mu_* \in (0, \mu_1)$. Moreover, for every $\mu$ from $(0, \mu_1)$ the operator $T_M(\mu)$ is negative.

**Appendix A. Basics of Pontryagin spaces theory**

In this appendix we recall some facts from the Pontryagin space theory, which we use in the paper. The details of the theory, more spectral properties of self-adjoint operators in Pontryagin spaces and the proofs of the propositions given here can be found in [2, 3, 14].

A linear space $\Pi$ is called an **inner product space** if there is a complex-valued function $\langle \cdot, \cdot \rangle$ defined on $\Pi \times \Pi$ so that the conditions

\[
\begin{align*}
\langle \alpha_1 u_1 + \alpha_2 u_2, v \rangle &= \alpha_1 \langle u_1, v \rangle + \alpha_2 \langle u_2, v \rangle \\
\langle u, v \rangle &= \overline{\langle v, u \rangle}
\end{align*}
\]

hold for every $\alpha_1, \alpha_2 \in \mathbb{C}$ and $u_1, u_2, u, v \in \Pi$. The function $\langle \cdot, \cdot \rangle$ is then called an **inner product**. An inner product space $(\Pi, \langle \cdot, \cdot \rangle)$ is a *Pontryagin space of negative index* $\kappa$ if $\Pi$ can be written as

(A.1) \[ \Pi = \Pi_+ \oplus \Pi_-, \]

where $\oplus$ denotes the direct $\langle \cdot, \cdot \rangle$-orthogonal sum, $(\Pi_{\pm}, \pm \langle \cdot, \cdot \rangle)$ are Hilbert spaces and the component $\Pi_-$ is of finite dimension $\kappa$.

An element $x \in \Pi$ is said to be positive (negative, non-positive, non-negative, neutral resp.) if $\langle x, x \rangle > 0$ ($\langle x, x \rangle < 0$, $\langle x, x \rangle \leq 0$, $\langle x, x \rangle \geq 0$, $\langle x, x \rangle = 0$ resp.). A subspace $\mathcal{M}$ of $\Pi$ is called positive (negative, non-positive, non-negative, neutral resp.) if all its non-zero vectors are positive (negative, non-positive, non-negative, neutral resp.)
In Pontryagin space of negative index $\kappa$ the dimension of any non-positive subspace can not exceed $\kappa$. Moreover, a non-positive subspace of Pontryagin space is maximal (i.e. such that it is not properly included in any other non-positive subspace) if and only if it is of dimension $\kappa$.

Pontryagin spaces often arise from Hilbert spaces in the following way. Suppose we have a Hilbert space $(H, (\cdot, \cdot))$ and a bounded self-adjoint operator $G$ in $H$ with $0 \in \rho(G)$ which has exactly $\kappa$ negative eigenvalues counted according to their multiplicities. Then with an inner product $[x, y] := (Gx, y)$, $x, y \in H$ the space $(H, [\cdot, \cdot])$ is a Pontryagin space of negative index $\kappa$ for which the decomposition (A.1) can be given with $\Pi_+$ and $\Pi_-$ being the spectral subspaces of $G$ corresponding to the positive and negative spectrum of $G$ respectively.

Consider a Pontryagin space $\Pi := (\Pi, [\cdot, \cdot])$ and a closed operator $A$ densely defined on $\Pi$. An adjoint $A^\ast$ of $A$ in $\Pi$ is defined on the domain $\text{dom} A^\ast := \{ y \in \Pi \mid [A^\ast y, x] \text{ is a continuous linear functional on } \text{dom} A \}$ by the relation $[Ax, y] = [x, A^\ast y]$, $x \in \text{dom} A$, $y \in \text{dom} A^\ast$.

The operator $A$ is symmetric if $A \subset A^\ast$ and self-adjoint if $A = A^\ast$. In contrast to the case of Hilbert space, the spectrum of self-adjoint operator in Pontryagin space is not necessarily real, but it is always symmetric with respect to the real axis.

If for some eigenvalue $\lambda_0$ of a self-adjoint operator in a Pontryagin space all eigenvectors are positive (negative resp.) then $\lambda_0$ is called of positive (negative resp.) type.

Proposition A.1. Assume $A$ is a self-adjoint operator in a Pontryagin space $\Pi$. Then

(i) The spectrum of $A$ is real with possible exception of at most $\kappa$ pairs of eigenvalues $\lambda$ and $\bar{\lambda}$ of finite multiplicities.

(ii) If the spectrum of the operator $A$ is discrete, then the set of all eigenvectors and the corresponding associated vectors of $A$ forms a basis in $\Pi$.

Denote by $M_\lambda(A)$ the root space of the operator $A$ corresponding to eigenvalue $\lambda$.

Proposition A.2. Suppose $A$ is a self-adjoint operator in a Pontryagin space. Then
(1) For eigenvalues $\lambda$ and $\mu$ of $A$ such that $\lambda \neq \mu$ the root spaces $M_\lambda(A)$ and $M_\mu(A)$ are orthogonal;
(2) The linear span of all the algebraic root spaces corresponding to the eigenvalues of $A$ in the upper (or lower) half plane is a neutral subspace of $\Pi$;
(3) The root spaces $M_\lambda(A)$ and $M_\bar{\lambda}(A)$ corresponding to complex conjugate eigenvalues $\lambda$ and $\bar{\lambda}$ are isomorphic. The spaces $M_\lambda(A)$ and $M_\bar{\lambda}(A)$ are of the same dimension and have the same Jordan structure;
(4) The length of a chain of eigen- and associated vectors does not exceed $2\kappa + 1$.

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