Covariance-based Dissimilarity Measures Applied to Clustering Wide-sense Stationary Ergodic Processes

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Abstract

We introduce a new unsupervised learning problem: clustering wide-sense stationary ergodic stochastic processes. A covariance-based dissimilarity measure and consistent algorithms are designed for clustering offline and online data settings, respectively. We also suggest a formal criterion on the efficiency of dissimilarity measures, and discuss of some approach to improve the efficiency of clustering algorithms, when they are applied to cluster particular type of processes, such as self-similar processes with wide-sense stationary ergodic increments. Clustering synthetic data sampled from fractional Brownian motions is provided as an example of application.

Keywords: unsupervised clustering · wide-sense stationary ergodic processes · covariance-based dissimilarity measure · self-similar processes

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1 Introduction

Clustering, as a core subject in unsupervised learning, involves assigning a heterogeneous set of objects into non-overlapping clusters, in which any two objects are “close” enough in a particular sense, without a priori knowledge about the number of true clusters and which cluster the objects truly belong to. In practice, unsupervised clustering analysis is usually performed to detect hidden patterns of a given dataset. It has nowadays a number of applications in various fields of both industry and scientific research, such as biological and medical research (Damian et al., 2007; Zhao et al., 2014; Jäskinen et al., 2014), information technology (Slonim et al., 2005; Jain et al., 1999), signal and image processing (Rubinstein et al., 2013), geology (Juozapavičius and Rapsevicius, 2001) and finance (Pavlidis et al., 2006; Bastos and Caiado, 2014; Ieva et al., 2016). There is a rich literature of clustering analysis on random vectors, where the objects, waiting to be clustered, are often viewed as sampled from random vectors’ joint distributions. There is no shortage of such clustering algorithms (Xu and Wunsch, 2005). However, stochastic processes are a quite different setting in statistics, since their observations (sample paths) are sampled from processes distributions. While the clustering analysis on random vectors has developed aggressively, the clustering analysis on stochastic processes receives much less attention. Today clustering analysis on stochastic processes deserves increasingly intense study, thanks to their vital importance to many applied areas, where the collected information are often ordered with respect to time. Examples of these time-indexed information include biological data, financial data, marketing data, surface weather data, geological data and video/audio data, etc. In the setting of random vectors, a clustering process often consists of two steps:

**Step 1** One suggests a fitting dissimilarity measure, under which two objects are close to each other becomes meaningful.

**Step 2** One designs an enough accurate and computationally efficient clustering function based on the above dissimilarity measure.

Clustering stochastic processes is processed in a similar way but new challenges may arise in both Step 1 and Step 2. Intuitively, one can always apply existing random vectors clustering approaches to cluster arbitrary processes, such as non-hierarchical approaches (K-means clustering methods) and hierarchical approaches (agglomerative method, divisive method) (Hartigan, 1975), based on “naive” dissimilarity measures, for instance, Euclidean distance, Manhattan distance or Minkowski distance. However, there exist at least 2 potential risks when applying the above approaches to cluster processes:

1. These approaches might suffer from their huge complexities, due to the great length of a sample path. So classical clustering algorithms are often computationally forbidding (Ieva et al., 2016; Peng and Müller, 2008).

2. These approaches might suffer from over-fitting problem. For example, clustering stationary or periodic processes based on Euclidean distance
between the paths, without considering their path properties will result in "over fitting, bad clusters" situation.

In summary, classical dissimilarity measures or clustering strategies would fail in clustering stochastic processes.

Fortunately, the complexity and the over-fitting errors of clustering processes could be largely reduced, if one is aware of the fact that a stochastic process often possesses fine paths features (stationarity, Markov property, self-similarity, sparsity, seasons, etc.), which is unlike an arbitrary random vector. An appropriate dissimilarity measure then should be chosen to be able to capture these paths features. Clustering processes is then performed to group any two sample paths into one cluster, if they are relatively close to each other under that dissimilarity measure. Below are some examples provided in the literature.

Peng and Müller (2008) proposed a dissimilarity measure between two special ample paths of processes. In their setting it is supposed that, for each path only sparse and irregularly spaced measurements with additional measurement errors are available. Such features occur commonly in longitudinal studies and online trading data. Based on this particular dissimilarity measure, classification and clustering analysis could be made. Ieva et al (2016) developed a new algorithm to perform clustering of multivariate and functional data, based on a covariance-based dissimilarity measure. Their attention focused on the specific case of a set of observations from two populations, whose probability distributions have equal mean but differ in terms of covariances. Khaleghi et al (2016) designed consistent algorithms for clustering strict-sense stationary ergodic processes (see Eq. (1.3) for the definition of strict ergodicity), where the dissimilarity measure is proposed as distance of process distributions. It is worth noting that the consistency of their algorithms are guaranteed thanks to the strict-sense ergodicity.

In our setting, we aim to design consistent algorithms to cluster a general type of stochastic processes, i.e., wide-sense stationary ergodic stochastic processes (also called weakly stationary weakly ergodic process, see Definition 1.1 below). Consistent algorithms can be obtained in this setting, since stationarity and ergodicity allow the process to present some featured asymptotic behaviors with respect to their length, rather than to the total number of paths.

Definition 1.1 (Wide-sense stationary ergodic process). A stochastic process $X = \{X_t\}_{t \in T}$ (the time indexes set $T$ can be either discrete or continuous) is called weakly stationary (or stationary in the wide sense) if its mean and covariance structure are finite and time-invariant: $\mathbb{E}(X_t) = \mu < \infty$ for any $t \in T$, and for any subset $(X_{i_1}, \ldots, X_{i_r})$, its covariance structure remains invariant subject to time shift $h > 0$:

$$\text{Cov}(X_{i_1}, \ldots, X_{i_r}) = \text{Cov}(X_{i_1+h}, \ldots, X_{i_r+h}).$$

Denote by $\gamma$ the auto-covariance function of $X$. Then $X$ is further called weakly ergodic (or ergodic in the wide sense) if it is ergodic for the mean and the second-order moment:
• If \( X \) is a continuous-time process, then it satisfies for any \( s,t \in T, s < t \),

\[
\frac{1}{t-s} \int_s^t X_u \, du \xrightarrow{a.s., t-s \to \infty} \mu,
\]

and

\[
\frac{1}{t-s} \int_s^t (X_{u+\tau} - \mu)(X_u - \mu) \, du \xrightarrow{a.s., t-s \to \infty} \gamma(\tau), \text{ for all possible } \tau \in T,
\]

where \( \xrightarrow{a.s.} \) denotes the almost surely convergence.

• If \( X \) is a discrete-time process, then it satisfies for any \( s,t \in T, s < t \),

\[
\frac{X_s + X_{s+1} + \ldots + X_t}{t-s+1} \xrightarrow{a.s., t-s \to \infty} \mu,
\]

and

\[
\frac{\sum_{\tau=s}^{t-1} (X_{u+\tau} - \mu)(X_u - \mu)}{t-s+1} \xrightarrow{a.s., t-s \to \infty} \gamma(\tau), \text{ for all possible } \tau \in T.
\]

Wide-sense stationarity and ergodicity are believed to be a very general assumption, at least in the following senses:

1. The assumption that each process is generated by some covariance structure from a weakly stationary and weakly ergodic process is already sufficient for capturing any finite-dimensional characteristic of the process. Two processes in the same cluster have common means and covariance structures, but they may have different process distributions. In short, our algorithms intend for clustering covariance structures (invariant along the paths), not process distributions.

2. A second order (i.e. the variance of each member is finite) strictly stationary and ergodic process (see Eq. (1.3) for the definition of strict ergodicity) is also weakly stationary weakly ergodic. As a result strictly stationary ergodic stable processes are not wide-sense stationary ergodic, because their variances explode (Cambanis et al. [1987], Samorodnitsky [2004]).

3. A Gaussian process can be characterized by its means and covariance structure only. Then a weakly stationary weakly ergodic Gaussian process is also strictly stationary and ergodic.

4. The dependency among processes can be arbitrary.

There is a long list of processes which are wide-sense stationary and ergodic, but not necessarily in the strict sense. The examples of wide-sense stationary processes below are not exhausted.
Example 1 Non-independent White Noise.

Let $U$ be a random variable uniformly distributed over $(0, 2\pi)$ and define

$$Z(t) := \sqrt{2}\cos(tU), \text{ for } t \in \mathbb{N} := \{1, 2, \ldots\}.$$  

The process $Z = \{Z(t)\}_{t \in \mathbb{N}}$ then is a white noise because it verifies

$$E(Z(t)) = 0, \ Var(Z(t)) = 1 \text{ and } Cov(Z(s), Z(t)) = 0, \text{ for } s \neq t.$$  

We claim that $Z$ is wide-sense stationary and ergodic, which can be obtained by using the Kolmogorov’s strong law of large numbers, see e.g. Theorem 2.3.10 in [Sen and Singer (1993)]. However $Z$ is not strictly stationary since

$$(Z(1), Z(2)) \neq (Z(2), Z(3)) \text{ in law.}$$  

Indeed, it is easy to see that

$$0 < E(Z(1)^2Z(2)) \neq E(Z(2)^2Z(3)) = 0.$$  

Example 2 Autoregressive Models.

It is well-known that an auto-regressive model $AR(1)$ in the form:

$$Z(t) = a + bZ(t-1) + \varepsilon(t), \ a \in \mathbb{R}, \ |b| < 1,$$

is wide-sense stationary ergodic. However it is not necessarily strictly stationary ergodic, when the joint distributions of the white noise $\varepsilon(t)$ are not invariant with time-shifting (for example, take $\varepsilon(t)$ to be the white noise in Example 1).

Example 3 Increment Process of Fractional Brownian Motion.

A fractional Brownian motion $\{B^H(t)\}_{t}$ (with Hurst index $H \in (0, 1)$, see [Mandelbrot and van Ness (1968)]) has increment process $\{Z^h(t) := B^H(t + h) - B^H(t)\}_{t}$, which is second order strictly stationary ergodic [Magdziarz and Weron (2011)]. As a result it is also wide-sense stationary ergodic. More detail will be discussed of in Section 4.

Example 4 Increment Process of More General Gaussian Processes.

Peng (2012) introduced a general class of Zero-mean Gaussian processes $X = \{X(t)\}_{t \in \mathbb{R}}$ having stationary increments. Its variogram $\nu(t) := 2^{-1}E(X(t)^2)$ satisfies:

1. There is a non-negative integer $d$ such that $\nu$ is $2d$-times continuously differentiable over $[-2, 2]$, but not $2(d + 1)$-times continuously differentiable over $[-2, 2]$.

2. There are 2 real numbers $c \neq 0$ and $s_0 \in (0, 2)$, such that for all $t \in [-2, 2]$, we have

$$\nu(t) = \nu^{(2d)}(0) + c|t|^{s_0} + r(t),$$

where the remainder $r(t)$ satisfies:
• \( r(t) = o(|t|^\omega) \), as \( t \to 0 \).

• There are two real numbers \( c > 0 \), \( \omega > s_0 \) and an integer \( q > \omega + 1/2 \) such that the remainder \( r \) is \( q \)-times continuously differentiable on \([-2, 2]\setminus\{0\}\) and for all \( t \in [-2, 2]\setminus\{0\} \), we have

\[
|r^{(q)}(t)| \leq c|t|^\omega q.
\]

It is shown that the process \( X \) extends fractional Brownian motion \cite{Peng2012}. It also has wide-sense (also strict-sense) stationary ergodic increments when \( d + s_0/2 \in (0, 1) \), with similar reasons to fractional Brownian motion’s increments (see Proposition 3.1 in Peng (2012)).

The problem of clustering processes via their covariance structures leads us to formulating our clustering targets in the following way.

**Definition 1.2** (Ground-truth \( G \) of covariance structures). Let

\[
G = \{G_1, \ldots, G_\kappa\},
\]

be a partitioning of \( \mathbb{N} \) into \( \kappa \) disjoint sets \( G_k, k = 1, \ldots, \kappa \), such that the means and covariance structures of \( x_i, i \in \mathbb{N} \) are identical, if and only if \( i \in G_k \) for some \( k = 1, \ldots, \kappa \). Such \( G \) is called ground-truth. We also denote by \( G|_N \) the restriction of \( G \) to the first \( N \) sequences:

\[
G|_N = \{G_k \cap \{1, \ldots, N\} : k = 1, \ldots, \kappa\}.
\]

Our clustering algorithms will aim to output the ground-truth partitioning \( G \), as the sample size grows. Before stating these algorithms, we introduce the inspiring framework done by Khaleghi et al (2016).

### 1.1 Preliminary Results: Clustering Strictly Stationary Ergodic Processes

Khaleghi et al (2016) considered the problem of clustering stationary and ergodic processes in the strict sense. The main fruit in Khaleghi et al (2016) is obtaining the so-called consistent algorithms to cluster processes of that type. We briefly state their work below. Depending on how the information is collected, the stochastic processes clustering problems consist of dealing with two models: offline setting and online setting.

**Offline setting:** The observations are assumed to be a finite number \( N \) of paths:

\[
x_1 = \left(X_1^{(1)}, \ldots, X_{n_1}^{(1)}\right), \ldots, x_N = \left(X_1^{(N)}, \ldots, X_{n_N}^{(N)}\right).
\]

Each sequence is generated by one of \( \kappa \) different unknown process distributions. In this case, a consistent clustering function should satisfy the following.
Definition 1.3 (Consistency: offline setting). A clustering function \( f \) is consistent for a set of sequences \( S \) if \( f(S, \kappa) = G \). Moreover, denoting \( n = \min\{n_1, \ldots, n_N\} \), \( f \) is called strongly asymptotically consistent in the offline sense if with probability 1 from some \( n \) on it is consistent on the set \( S \):

\[
P\left( \lim_{n \to \infty} f(S, \kappa) = G \right) = 1.
\]

It is weakly asymptotically consistent if \( \lim_{n \to \infty} P(f(S, \kappa) = G) = 1 \).

**Online setting:** The observations, having growing length and number of scenarios with respect to time \( t \), are denoted by

\[
x_1 = (X_1^{(1)}, \ldots, X_\kappa^{(1)}), \ldots, x_N(t) = (X_1^{(N(t))}, \ldots, X_\kappa^{(N(t))}).
\]

Then a consistent online clustering function is defined below:

**Definition 1.4** (Consistency: online setting). A clustering function is strongly (resp. weakly) asymptotically consistent in the online sense, if for every \( N \in \mathbb{N} \) the clustering \( f(S(t), \kappa)|_N \) is strongly (resp. weakly) asymptotically consistent in the offline sense, where \( f(S(t), \kappa)|_N \) is the clustering \( f(S(t), \kappa) \) restricted to the first \( N \) sequences:

\[
f(S(t), \kappa)|_N = \{f(S(t), \kappa) \cap \{1, \ldots, N\} : k = 1, \ldots, \kappa\}.
\]

There is a detailed discussion on the comparison of offline and online settings in Khaleghi et al (2016), stating that these two settings have significant differences, since using the offline algorithm in the online setting by simply applying it to the entire data observed at every time step, does not result in a consistent algorithm. Therefore an independent study on the consistent online clustering algorithms is meaningful.

As the main results in Khaleghi et al (2016), consistent clustering algorithms for both offline and online settings are designed. They are then successfully applied to clustering synthetic and real data sets.

Note that in the framework of Khaleghi et al (2016), a key step is to define the so-called distributional distance (Gray, 1988): the distributional distance between a pair of process distributions \( \rho_1, \rho_2 \) is defined to be

\[
d(\rho_1, \rho_2) = \sum_{m,l=1}^{\infty} w_m w_l \sum_{B \in B^{m,l}} |\rho_1(B) - \rho_2(B)|,
\]

where

- the weights \( \{w_j\}_{j \geq 1} \) are chosen to give precedence to chronologically earlier clusterings, protecting the clustering decisions from the presence of the newly formed paths, whose corresponding distance estimates may still be far from being accurate. For instance, it can be chosen that \( w_j = 1/j(j+1) \).
• The sets \( B^{m,l} \), \( m, l \geq 1 \) are obtained via the partitioning of \( \mathbb{R}^m \) into cubes of dimension \( m \) and volume \( 2^{-ml} \), starting at the origin.

Further, the distance between two sample paths \( x_1, x_2 \) of stochastic processes is defined to be
\[
\hat{d}(x_1, x_2) = \sum_{m=1}^{m_n} \sum_{l=1}^{l_n} w_m w_l \sum_{B \in B^{m,l}} |\nu(x_1, B) - \nu(x_2, B)|, \tag{1.2}
\]

where

• \( m_n, l_n (\leq n) \) can be arbitrary sequences of positive integers going to infinity.

• For a process path \( x = (X_1, \ldots, X_n) \), and an event \( B, \nu(x, B) \) denotes the average times that event \( B \) occurs over \( n - m_n + 1 \) steps. More precisely,
\[
\nu(x, B) := \frac{1}{n - m_n + 1} \sum_{i=1}^{n-m_n+1} \mathbb{1}\{(X_i, \ldots, X_{i+m_n-1}) \in B\}.
\]

The process distribution \( X \) from which \( x \) is sampled is called (strictly) ergodic if
\[
P \left( \lim_{n \to \infty} \nu(x, B) = \mathbb{P}(X \in B) \right) = 1, \text{ for all } B. \tag{1.3}
\]

The assumption that the processes are ergodic leads to that \( \hat{d} \) is a strongly consistent estimator of \( d \):
\[
P \left( \lim_{n \to \infty} \hat{d}(x_1, x_2) = d(\rho_1, \rho_2) \right) = 1,
\]

where \( \rho_1, \rho_2 \) are the process distributions corresponding to \( x_1, x_2 \), respectively.

Based on the distances \( d \) and \( \hat{d} \), the consistent algorithms to cluster stationary ergodic processes in each of the offline and online settings are provided (see Algorithms 1, 2 and Theorems 11, 12 in [Khaleghi et al 2016]). Khaleghi et al (2016) also show that their methods can be implemented efficiently: they are at most quadratic in each of their arguments, and are linear (up to log terms) in some formulations.

1.2 Statistical Setting: Clustering Wide-sense Stationary Ergodic Processes

Inspired by the framework of [Khaleghi et al 2016], we consider the problem of clustering wide-sense stationary ergodic processes. Without losing any generality, we assume that all the paths are sampled from zero-mean processes throughout the rest of the paper (if the observed path \( (X_1, \ldots, X_n) \) is not sampled from zero-mean distribution, we use the transformation \( \tilde{X}_i = X_i - \overline{X}_n \) for \( i = 1, \ldots, n \)). We first introduce the following covariance-based dissimilarity measure, which is one of the main contributions of this framework.
Definition 1.5. (Covariance-based dissimilarity measure) The covariance-based dissimilarity measure $d^*$ between a pair of processes $X^{(1)}$, $X^{(2)}$ (in fact $X^{(1)}, X^{(2)}$ denote two covariance structures, each may contain different process distributions) is defined as follows:

$$d^*(X^{(1)}, X^{(2)}) := \sum_{m,l=1}^{\infty} w_m w_l \rho^* \left( \text{Cov} \left( X^{(1)}_l, \ldots, X^{(1)}_{l+m-1} \right), \text{Cov} \left( X^{(2)}_l, \ldots, X^{(2)}_{l+m-1} \right) \right),$$

where:

- The sequence of positive weights $\{w_j\}$ is chosen such that $d^*(X^{(1)}, X^{(2)})$ is finite. We set $w_j = 1/j(j + 1)$ through this framework, but any summable (over $j$) sequence of positive weights may be used.

- The distance $\rho^*$ between 2 equal-sized covariance matrices $M_1, M_2$ is defined to be

$$\rho^*(M_1, M_2) := \|M_1 - M_2\|_F,$$

with $\| \cdot \|_F$ being the Frobenius norm:

$$\|M\|_F := \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |M_{ij}|^2}.$$

Introduction to the matrices distance $\rho^*$ is inspired by Herdin et al. (2005). The matrices distance given in Herdin et al. (2005) is used to measure the distance between 2 correlation matrices. However, our distance $\rho^*$ is a modification of the one in the latter paper. Indeed, unlike Herdin et al. (2005), $\rho^*$ is a well-defined metric distance, as it satisfies the triangle inequalities.

For $1 \leq l \leq n$ and $m \leq n - l + 1$, define $\nu^*(x_{l \ldots n}, m)$ to be the empirical covariance matrix of a process $X$’s path $(X_l, \ldots, X_n)$:

$$\nu^*(x_{l \ldots n}, m) := \frac{1}{n - m - l + 2} \sum_{i=l}^{n-m+1} (x_{i \ldots X_{i+m-1}})^{T} (x_{i \ldots X_{i+m-1}}),$$

where $M^T$ denotes the transpose of the matrix $M$.

Recall that the notion of wide-sense ergodicity is given in Definition 1.1. The ergodicity theorem concerns what information can be derived from an average over time about the ensemble average at each point of time. For the weakly stationary weakly ergodic process $X$, being either continuous-time or discrete-time, the following statement holds: every empirical covariance matrix $\nu^*(x_{l \ldots n}, m)$
is a strongly consistent estimator of the covariance matrix $Cov(X_1, \ldots, X_{t+m-1})$ under the Frobenius norm: for all $m \geq 1$, we have
\[ P \left( \lim_{n \to \infty} \| \nu^* (X_{i..n}, m) - Cov(X_1, \ldots, X_{t+m-1}) \|_F = 0 \right) = 1. \]

Next we introduce empirical covariance-based dissimilarity measure $\hat{d}^*$, serving as a consistent estimator of the covariance-based distance $d^*$.

**Definition 1.6** (Empirical covariance-based dissimilarity measure). For two processes’ paths $\mathbf{x}_i = (X^{(j)}_1, \ldots, X^{(j)}_{n_i})$ for $j = 1, 2$, let $n = \min\{n_1, n_2\}$, then the empirical covariance-based dissimilarity measure between $\mathbf{x}_1$ and $\mathbf{x}_2$ is given by
\[
\hat{d}^*(\mathbf{x}_1, \mathbf{x}_2) := \sum_{m=1}^{m_n} \sum_{l=1}^{n-m+1} w_m w_l \rho^* \left( \nu^* (X^{(1)}_{l..n}, m), \nu^* (X^{(2)}_{l..n}, m) \right),
\]
where we usually take $m_n = \lfloor \log n \rfloor$, the floor number of $\log n$.

Remark that two observed paths have distinct lengths $n_1, n_2$, therefore in [1.7] we consider computing the distances between the parts of length $n = \min\{n_1, n_2\}$. It is easy to verify that both $d^*$ and $\hat{d}^*$ satisfy the triangle inequalities. This together with the fact that the processes are weakly ergodic, leads to the following result in Lemma 1.7, which is the key to demonstrate that our algorithms in the next section are consistent.

**Lemma 1.7.** For every pair of sequences
\[ \mathbf{x}_1 = (X^{(1)}_1, \ldots, X^{(1)}_{n_1}) \quad \text{and} \quad \mathbf{x}_2 = (X^{(2)}_1, \ldots, X^{(2)}_{n_2}), \]
which are sampled from weakly stationary and weakly ergodic processes, we have
\[
P \left( \lim_{n_1, n_2 \to \infty} \hat{d}^* (\mathbf{x}_1, \mathbf{x}_2) = d^* \left( X^{(1)}, X^{(2)} \right) \right) = 1, \tag{1.8} \]
and
\[
P \left( \lim_{n_i \to \infty} \hat{d}^* (\mathbf{x}_i, X^{(j)}) = d^* \left( X^{(1)}, X^{(2)} \right) \right) = 1, \quad \text{for } i, j \in \{1, 2\}. \tag{1.9} \]

**Proof.** Let $X^{(1)}$, $X^{(2)}$ be two arbitrary wide-sense stationary ergodic processes. To be convenient we denote the covariance matrix of $\left( X^{(j)}_1, \ldots, X^{(j)}_{l+m-1} \right)$ by
\[ V_{l,l+m-1}(X^{(j)}) := Cov \left( X^{(j)}_1, \ldots, X^{(j)}_{l+m-1} \right), \quad \text{for } j = 1, 2. \]
First, the weak ergodicity of the processes $X^{(1)}$ and $X^{(2)}$ tells that: for each $m \geq 1$, $\nu(X^{(j)}_{l..n_j}, m)$ ($j = 1, 2$) is a strongly consistent estimator of the covariance matrix $V_{l,l+m-1}(X^{(j)})$, in the sense of $\rho^*$, i.e.,
\[
P \left( \lim_{n_j \to \infty} \rho^* \left( \nu^* (X^{(j)}_{l..n_j}, m), V_{l,l+m-1}(X^{(j)}) \right) = 0 \right) = 1. \tag{1.10} \]
Observe that \( \sum_{m,l \in \mathbb{N}} w_m w_l < +\infty \). Then for fixed \( \varepsilon > 0 \), we can find an index \( J \) such that
\[
\sum_{(m,l) \in \mathbb{N}^2 \setminus \{1, \ldots, J\}^2} \omega_{m,l} \omega_{l,m} \leq \frac{\varepsilon}{3}. \tag{1.11}
\]
Moreover, thanks to (1.10), there exists some \( N \) (which depends on the realizations \( X_{1}^{(j)}, \ldots, X_{n_j}^{(j)}, \ j = 1, 2 \)) such that for all \( n_j \geq N \ (j = 1, 2) \), we have, with probability 1,
\[
\rho^* \left( \nu^* \left( X_{1}^{(j)}, m \right), \ V_{l,t+m-1}(X^{(j)}) \right) \leq \frac{\varepsilon}{6J^2 w_m w_l}, \ j = 1, 2. \tag{1.12}
\]
Denote by \( \mathbf{x}_j = (X_{1}^{(j)}, \ldots, X_{n_j}^{(j)}) \), \( j = 1, 2 \). For all \( n_j \geq N \), \( j = 1, 2 \), by using the triangle inequality, we obtain
\[
\left| d^* (\mathbf{x}_1, \mathbf{x}_2) - d^* (X^{(1)}, X^{(2)}) \right| = \left| \sum_{m,l=1}^{\infty} w_m w_l \left( \rho^* \left( \nu^* (X_{l}^{(1)}, m), \nu^* (X_{l}^{(2)}, m) \right) - \rho^* \left( V_{l,t+m-1}(X^{(1)}), V_{l,t+m-1}(X^{(2)}) \right) \right) \right|
\leq \sum_{m,l=1}^{\infty} w_m w_l \left| \rho^* \left( \nu^* (X_{l}^{(1)}, m), \nu^* (X_{l}^{(2)}, m) \right) - \rho^* \left( V_{l,t+m-1}(X^{(1)}), V_{l,t+m-1}(X^{(2)}) \right) \right|. \tag{1.13}
\]
Next we use the following two triangle inequalities
\[
\rho^* \left( V_{l,t+m-1}(X^{(1)}), V_{l,t+m-1}(X^{(2)}) \right) + \rho^* \left( \nu^* (X_{l}^{(1)}, m), V_{l,t+m-1}(X^{(1)}) \right) + \rho^* \left( \nu^* (X_{l}^{(2)}, m), V_{l,t+m-1}(X^{(2)}) \right)
\geq \rho^* \left( \nu^* (X_{l}^{(1)}, m), V_{l,t+m-1}(X^{(2)}) \right) + \rho^* \left( \nu^* (X_{l}^{(2)}, m), V_{l,t+m-1}(X^{(2)}) \right)
\geq \rho^* \left( \nu^* (X_{l}^{(1)}, m), \nu^* (X_{l}^{(2)}, m) \right),
\]
and
\[
\rho^* \left( \nu^* (X_{l}^{(1)}, m), \nu^* (X_{l}^{(2)}, m) \right) + \rho^* \left( \nu^* (X_{l}^{(1)}, m), V_{l,t+m-1}(X^{(1)}) \right) + \rho^* \left( \nu^* (X_{l}^{(2)}, m), V_{l,t+m-1}(X^{(2)}) \right)
\geq \rho^* \left( \nu^* (X_{l}^{(2)}, m), V_{l,t+m-1}(X^{(1)}) \right) + \rho^* \left( \nu^* (X_{l}^{(2)}, m), V_{l,t+m-1}(X^{(2)}) \right)
\geq \rho^* \left( V_{l,t+m-1}(X^{(1)}), V_{l,t+m-1}(X^{(2)}) \right),
\]
and
to obtain

\[
\left| \rho^* \left( \nu^*(X_{1,\ldots,n}^{(1)}, m), \nu^*(X_{1,\ldots,n}^{(2)}, m) \right) - \rho^* \left( V_{1,l+m-1}(X^{(1)}), V_{1,l+m-1}(X^{(2)}) \right) \right| \\
\leq \rho^* \left( \nu^*(x_{1,\ldots,n}^{(1)}, m), V_{1,l+m-1}(X^{(1)}) \right) + \rho^* \left( \nu^*(X_{1,\ldots,n}^{(2)}, m), V_{1,l+m-1}(X^{(2)}) \right).
\]

(1.14)

It follows from (1.13), (1.14), (1.11) and (1.12) that

\[
\left| \hat{d}^* (x_{n_1}, x_{n_2}) - d^* (X^{(1)}, X^{(2)}) \right| \\
\leq \sum_{m,l=1}^{\infty} w_m w_l \left( \rho^* \left( \nu^*(X_{1,\ldots,n}^{(1)}, m), V_{1,l+m-1}(X^{(1)}) \right) \\
+ \rho^* \left( \nu^*(X_{1,\ldots,n}^{(2)}, m), V_{1,l+m-1}(X^{(2)}) \right) \right) \\
\leq \sum_{(m,l)\in[N^2(1,\ldots,J)^2]} w_m w_l \left( \rho^* \left( \nu^*(X_{1,\ldots,n}^{(1)}, m), V_{1,l+m-1}(X^{(1)}) \right) \\
+ \rho^* \left( \nu^*(X_{1,\ldots,n}^{(2)}, m), V_{1,l+m-1}(X^{(2)}) \right) \right) + 2 \sum_{(m,l)\in[N^2(1,\ldots,J)^2]} w_m w_l \\
\leq \sum_{(m,l)\in[N^2(1,\ldots,J)^2]} w_m w_l \left( \frac{\varepsilon}{6J^2 w_m w_l} + \frac{\varepsilon}{6J^2 w_m w_l} + \frac{2\varepsilon}{3} \right) \\
= \varepsilon.
\]

(1.15)

which proves (1.8). The statement (1.9) can be proven analogously. \qed

Remark 1.8. Thanks to (1.9) and the definitions of $d^*$ (1.4) and $\hat{d}^*$ (1.7), we see that the triangle inequality holds for the covariance-based dissimilarity measure $d^*$, as well as for its empirical estimates $\hat{d}^*$. Therefore for arbitrary processes $X^{(i)}$, $i = 1, 2, 3$ and arbitrary random vectors $x_i$, $i = 1, 2, 3$ we have

\[
\begin{align*}
    d^* (X^{(1)}, X^{(2)}) &\leq d^* (X^{(1)}, X^{(3)}) + d^* (X^{(3)}, X^{(2)}), \\
    \hat{d}^* (x_1, x_2) &\leq \hat{d}^* (x_1, x_3) + \hat{d}^* (x_2, x_3), \\
    \hat{d}^* (x_1, X^{(1)}) &\leq \hat{d}^* (x_1, X^{(2)}) + d^* (X^{(1)}, X^{(2)}).
\end{align*}
\]

2 Consistent Clustering Algorithms

2.1 Offline and Online Algorithms

In this section we state two consistent algorithms for clustering offline and online datasets respectively. We explain how the algorithms work, and proofs of consistency are also provided. It is worth noting that the consistency of our algorithms below relies on the assumption of knowing the number of clusters.
The more general case for \( \kappa \) being unknown has been studied in Khaleghi et al. (2016) in the problem of clustering strictly stationary ergodic processes. However, in the setting of wide-sense stationary ergodic processes, the problem remains open.

Algorithm 1 below first initializes the first two clusters using the two farthest observations among all observations, then assigns each remaining observation to the nearest cluster, under the empirical dissimilarity measure \( \hat{d}^* \). More precisely, the sample \( x_1, x_2 \) are assigned as the first and the second cluster center, respectively. Then for each \( k = 3, \ldots, \kappa \) the \( k^{th} \) cluster center is sought as the path with the largest minimum distance from the already assigned cluster centers for \( 1, \ldots, k - 1 \). By the last iteration we have \( \kappa \) cluster centers. Next, the remaining samples are each assigned to the closest cluster.

Algorithm 1: Offline clustering, with known \( \kappa \)

**Input:** sample sequences \( S = \{x_1, \ldots, x_N\} \); number \( \kappa \) of clusters.
1. \((c_1, c_2) \leftarrow \arg\max_{(i,j) \in \{1, \ldots, N\}^2, i \neq j} \hat{d}^*(x_i, x_j)\);
2. \( C_1 \leftarrow \{c_1\}\);
3. \( C_2 \leftarrow \{c_2\}\);
4. for \( k = 3, \ldots, \kappa \) do
   5. \( c_k \leftarrow \arg\max_{i=1, \ldots, N} \min_{j=1, \ldots, k-1} \hat{d}^*(x_i, x_j) \);
6. end

**Assign the remaining points to closest centers:**
7. for \( i = 1, \ldots, N \) do
6. \( k \leftarrow \arg\min_{k \in \{1, \ldots, \kappa\}} \{ \hat{d}^*(x_i, x_j) : j \in C_k \} \);
9. \( C_k \leftarrow C_k \cup \{i\} \);
10. end

**Output:** The \( \kappa \) clusters \( \{C_1, C_2, \ldots, C_\kappa\} \).

We point out that Algorithm 1 is different from Algorithm 1 in Khaleghi et al. (2016) at two points:

1. As mentioned previously, our algorithm relies on the covariance-based dissimilarity \( \hat{d}^* \), in lieu of the process distributional distances.
2. Our algorithm suggests 2-point initialization, which is different from Algorithm 1 in Khaleghi et al. (2016). In our algorithm, we pick the two farthest observations and assign each to a different cluster at the initial stage. However, the initialization in Khaleghi et al. (2016) is picking (randomly) one sample as cluster center. The latter initialization was proposed for use with \( k \)-means clustering by Katsavounidis et al. (1994). Algorithm 1 in Khaleghi et al. (2016) requires \( \kappa N \) distance calculations, while our algorithm requires \( N(N-1)/2 \) distances calculations. It is very important to point out that, to reduce the computational complexity of our algorithm, it is fine to replace our 2-points initialization with the one in
However there are two reasons based on which we recommend using our approach of initialization:

**Reason 1** For fixed number $N$, our empirical comparison of the two initializations show that the 2-points initialization turns out to be more accurate in clustering than the 1-point initialization.

**Reason 2** Concerning the complexity, we have the following loss and earn: one one hand, the 2-points initialization requires $N(N - 1)/2$ steps of calculations, against $\kappa N$ steps for the 1-point initialization; on the other hand, in our covariance-based dissimilarity measure $\hat{d}^*$ defined in (1.7), the matrices distance $\rho^*$ requires $m_n^2$ computations of Euclidean distances, while the distance

$$\sum_{B\in\mathcal{B}^n, i} |\nu(x_1, B) - \nu(x_2, B)|,$$

given in (1.2) requires at least $n_1 + n_2 - 2m_n + 2$ computations of Euclidean distances (see Eq. (33) in Khaleghi et al. (2016)). Recall that we take $m_n = \log n$ in this setting. Therefore the computational complexity of the covariance-based dissimilarity $\hat{d}^*$ makes the overall complexity of Algorithm 1 quite competitive to the algorithm in Khaleghi et al. (2016), especially when the paths lengths $n_i$, $i = 1, \ldots, n$ are relatively large, or when the database of all distance values are at hand.

Next we present the clustering algorithm for online setting. As mentioned in Khaleghi et al. (2016), one regards recently-observed paths as unreliable observations, for which sufficient information has not yet been collected, and for which the estimators of the covariance-based dissimilarity measures are not accurate enough. Consequently, farthest-point initialization would not work in this case; and using any algorithm for offline setting on all available data at every time step results in not only mis-clustering unreliable paths, but also in clustering incorrectly those for which sufficient data are already available. The strategy is presented in Algorithm 2 below: clustering based on a weighted combination of several clusterings, each obtained by running the offline algorithm (Algorithm 1) on different portions of data.

More precisely, Algorithm 2 works as follows. Suppose the number of clusters $\kappa$ is known. At time $t$, assume a set $S(t)$ of $N(t)$ samples is observed, the algorithm iterates over $j = \kappa, \ldots, N$ where at each iteration Algorithm 1 is utilized to cluster the first $j$ paths $\{x'_1, \ldots, x'_j\}$ into $\kappa$ clusters. In each cluster the path with the smallest index is assigned as the candidate cluster center. A performance score $\gamma_j$ is calculated as the minimum distance $\hat{d}^*$ between the $\kappa$ candidate cluster centers obtained at iteration $j$. Indeed, $\gamma_j$ is an estimate of the minimum inter-cluster distance (see Cesa-Bianchi and Lugosi (2006)). At this point we have $N(t) - \kappa + 1$ sets of $\kappa$ cluster centers $c'_1, \ldots, c'_j, j = 1, \ldots, N(t) - \kappa + 1$. Next, every sample $x'_i$, $i = 1, \ldots, N(t)$ is assigned to a
cluster, according to the weighted combination of the distances between \( x_t^j \) and the candidate cluster centers obtained at each iteration on \( j \).

Algorithm 2: Online clustering, with known \( \kappa \)

Input: number \( \kappa \) of target clusters.

1. for \( t = 1, \ldots, \infty \) do

2. Obtain new sequences: \( S(t) \leftarrow \{x_1^t, \ldots, x_N^t(t)\} \);

3. Initialize the normalization factor: \( \eta \leftarrow 0 \);

4. Initialize the final clusters: \( C_k(t) \leftarrow \emptyset, \ k = 1, \ldots, \kappa; \)

5. Generate \( N(t) - \kappa + 1 \) candidate cluster centers:

6. for \( j = \kappa, \ldots, N(t) \) do

7. \( \{C_1^j, \ldots, C_\kappa^j\} \leftarrow \text{Alg1}(\{x_1^t, \ldots, x_t^j\}, \kappa); \)

8. \( c_k^j \leftarrow \min \{i \in C_k^j\}, k = 1, \ldots, \kappa; \)

9. \( \gamma_j \leftarrow \min_{k, k' \in \{1, \ldots, \kappa\}, k \neq k'} \hat{d}^2(x_{c_k^j}^t, x_{c_{k'}^j}^t); \)

10. \( \omega_j \leftarrow 1/j(j+1); \)

11. \( \eta \leftarrow \eta + \omega_j \gamma_j \)

end

Assign each point to a cluster:

13. for \( i = 1, \ldots, N(t) \) do

14. \( k \leftarrow \arg\min_{k \in \{1, \ldots, \kappa\}} \frac{1}{\eta} \sum_{j=\kappa}^{N(t)} w_j \gamma_j \hat{d}^2(x_{c_k^j}^t, x_i^t); \)

15. \( C_k(t) \leftarrow C_k(t) \cup \{i\} \)

end

Output: The \( \kappa \) clusters \( \{C_1(t), \ldots, C_\kappa(t)\}, t = 1, 2, \ldots, \infty. \)

Remark that for online setting, our algorithm requires the same number of distance calculations as in Algorithm 2 in [Khaleghi et al. (2016)]. They are both bounded by \( O(N(t)^2) \). Our algorithm then presents advantage in overall computational complexity. Finally we note that both Algorithm 1 and Algorithm 2 require \( \kappa \geq 2 \). When \( \kappa \) is known, this condition is not a practical issue.

2.2 Consistency and Computational Complexity of the Algorithms

Theorem 2.1. Algorithm \( \text{Alg2} \) is strongly asymptotically consistent (in the offline sense), provided that the correct number \( \kappa \) of clusters is known, and the marginal distribution of each sequence \( x_i^t, i = 1, \ldots, N \) is wide-sense stationary ergodic.

Proof. To prove the consistency statement we use Lemma 1.7 to show that if the samples in \( S \) are long enough, the samples that are generated by the same process covariance structure are closer to each other than to the rest of the samples. Therefore, the samples chosen as cluster centers are each generated by a different covariance structure, and since the algorithm assigns the rest of the samples to the closest clusters, the statement follows. More formally, let \( n_{\text{min}} \)
denote the shortest path length in $S$:
\[
n_{\text{min}} := \min \{ n_i : i = 1, \ldots, N \}.
\]
For $k, k' \in \{1, \ldots, \kappa\}$, denote by $\delta(k, k')$ the covariance-based dissimilarity measure between the processes $X^{(k)}$ and $X^{(k')}$:
\[
\delta(k, k') = d^* \left( X^{(k)}, X^{(k')} \right).
\] (2.1)

Also denote by $\delta$ the minimum non-zero distance between the processes with different covariance structures:
\[
\delta := \min \{ \delta(k, k') : k, k' \in \{1, \ldots, \kappa\}, k \neq k' \}.
\]
For $k \neq k'$, fix $\varepsilon(k, k') \in (0, \delta(k, k')/4)$. Since there are a finite number $N$ of samples, by Lemma 1.7 there is $n(k, k')$ such that for $n_{\text{min}} > n(k, k')$ we have
\[
\max_{i \in \{k, k'\}} \min_{i \in G_k \cap \{1, \ldots, N\}} \frac{\widehat{d}^*(x_i, X^{(l)})}{\delta(k, k')} \leq \varepsilon(k, k'),
\] (2.2)

where $G_i, l = 1, \ldots, \kappa$ denote the ground-truth partitions given by Definition 1.2. By (2.2) and applying the triangle inequality we obtain
\[
\max_{i \in \{k, k'\}} \min_{i \in G_k \cap \{1, \ldots, N\}} \frac{\widehat{d}^*(x_i, x_j)}{\delta(k, k')} \leq \frac{2\varepsilon(k, k')}{\delta(k, k')} < \frac{\delta(k, k')}{2}.
\] (2.3)

Thus, for $n_{\text{min}} > n(k, k')$ we have
\[
\min_{i \in G_k \cap \{1, \ldots, N\}} \frac{d^*(x_i, x_j)}{\varepsilon(k, k')} \geq \min_{i \in G_k \cap \{1, \ldots, N\}} \left\{ d^*(X^{(k)}, X^{(k')}) - \frac{\widehat{d}^*(x_i, X^{(k)})}{\delta(k, k')} - \frac{\widehat{d}^*(x_i, X^{(k')})}{\delta(k, k')} \right\}
\[
\geq \frac{\delta(k, k')}{2}.
\] (2.4)

In words, (2.3) and (2.4) mean that the samples in $S$ that are generated by the same process covariance structure (corresponding to $G_k$ and $G_{k'}$) are closer to each other than to the rest of the samples. Since the number of clusters $\kappa$ is known and finite, then for $n_{\text{min}}$ large enough (i.e. $n_{\text{min}} > \max\{n(k, k') : k, k' \in \{1, \ldots, \kappa\}, k \neq k'\}$) to have (2.3) and (2.4) for all $k, k' \in \{1, \ldots, \kappa\}$, $k \neq k'$, we obtain
\[
\max_{i \in \{1, \ldots, \kappa\}} \min_{i \in G_k \cap \{1, \ldots, N\}} \frac{\widehat{d}^*(x_i, x_j)}{\varepsilon(k, k')} < \frac{\delta}{2} \quad \text{and} \quad \min_{i \in G_{k'} \cap \{1, \ldots, N\}} \max_{i \in G_k \cap \{1, \ldots, N\}} \frac{\widehat{d}^*(x_i, x_j)}{\varepsilon(k, k')} > \frac{\delta}{2}.
\] (2.5)
This tells that the sample paths in $S$ that are generated by the same covariance structures are closer to each other than to the rest of sample paths. Finally for $n_{\text{min}}$ large enough to have (2.5), we obtain
\[
\max_{i=1,\ldots,N} \min_{k=1,\ldots,\kappa-1} \hat{d}^s(x_i, x_{c_k}) > \frac{\delta}{2},
\]
where as specified by Algorithm 1
\[c_1 := 1 \text{ and } c_k := \arg\max_{i=1,\ldots,N} \min_{j=1,\ldots,k-1} \hat{d}^s(x_i, x_{c_j}), \quad k = 2, \ldots, \kappa.
\]
Hence, $c_1, \ldots, c_\kappa$ will be chosen to index sequences generated by different process covariance structures. To derive the consistency statement, we note that, by (2.5), each remaining sequence will be assigned to the cluster center corresponding to the sequence generated by the same process covariance structure.

**Theorem 2.2.** Algorithm 2 is strongly asymptotically consistent (in the online sense), provided the correct number of clusters $\kappa$ is known, and each sequence $x_i, i \in \mathbb{N}$ is weakly stationary weakly ergodic.

**Proof.** First, we show that for every $k \in \{1, \ldots, \kappa\},$
\[
P \left( \lim_{N(t) \to \infty} \frac{1}{N(t)} \sum_{j=1}^{N(t)} w_j \gamma_j \hat{d}^s \left( x_{s_k, c_k}^{(j)}, x^{(k)} \right) \right) = 1. \tag{2.6}
\]
For each $\delta(k, k')$ defined in (2.1), $k \neq k'$, fix $\varepsilon(k, k') \in (0, \delta(k, k')/4)$. We can find an index $J(k, k')$ such that $\sum_{j=J(k, k')}^{\infty} w_j \leq \varepsilon(k, k')$. Let $S(t)|_j = \{x_1^{(j)}, \ldots, x_j^{(j)}\}$ denote the subset of $S(t)$ consisting of the first $j$ sequences for $j \in \{1, \ldots, N(t)\}$. For $k = 1, \ldots, \kappa$ let
\[
s_k := \min \{ i : G_k \cap \{1, \ldots, N(t)\} \}, \tag{2.7}
\]
index the first sequence in $S(t)$ that is generated by $X^{(k)}$. Define
\[
m := \max_{k \in \{1, \ldots, \kappa\}} s_k. \tag{2.8}
\]
Recall that the sequence lengths $n_i(t)$ grow with time. Therefore, by Lemma 1.7 for every $j \in \{1, \ldots, J(k, k')\}$ there exists some $T_1(j)$ such that for all $t \geq T_1(j)$ we have
\[
\sup_{l \in \{k, k'\} \cap \{1, \ldots, j\}} \hat{d}^s(x_l^{(j)}, x^{(l)}) \leq \varepsilon(k, k'). \tag{2.9}
\]
Moreover, by Theorem 2.1 for every $j \in \{m, \ldots, J\}$ there exists some $T_2(j)$ such that $\text{Alg1}(S(t)|_j, \kappa)$ is consistent for all $t \geq T_2(j)$. Let
\[
T(k, k') := \max_{i=1,2} \max_{j \in \{1, \ldots, J(k, k')\}} T_i(j).
\]

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Recall that, by definition of \( m \), \( S(t) \mid \gamma \) contains sample paths from all \( \kappa \) covariance structures. Therefore, for all \( t \geq T(k, k') \) we have

\[
\hat{d}^\alpha \left(x_{c_{m}^k}, x_{c_{m}^{k'}}\right) \geq d^\alpha \left(X^{(k)}, X^{(k')}\right) - \left(\hat{d}^\alpha \left(x_{c_{m}^{k}}, X^{(k)}\right) + \hat{d}^\alpha \left(x_{c_{m}^{k'}}, X^{(k')}\right)\right) \\
\geq \delta(k, k') - 2\varepsilon(k, k') \geq \frac{\delta(k, k')}{2}.
\]  
(2.10)

Recall that (as specified in Algorithm \( 2 \)) we have \( \eta := \sum_{j=1}^{N(t)} w_j \gamma_j^t \). Hence, by (2.10) for all \( t \geq T(k, k') \) we have

\[
\eta \geq \frac{w_m \delta(k, k')}{2}.
\]  
(2.11)

Denote by \( M(t) = \max_{j \geq 1} \gamma_j^t \). By (2.11), for every \( k \in \{1, \ldots, \kappa\} \) we obtain

\[
\frac{1}{\eta} \sum_{j=1}^{N(t)} w_j \gamma_j^t \hat{d}^\alpha \left(x_{c_{j}^k}, X^{(k)}\right) \leq \frac{1}{\eta} \sum_{j=1}^{J(k, k')} w_j \gamma_j^t \hat{d}^\alpha \left(x_{c_{j}^k}, X^{(k)}\right) + \frac{2M(t)\varepsilon(k, k')}{w_m \delta(k, k')}.
\]  
(2.12)

On the other hand, by the definition of \( m \), the sequences in \( S(t) \mid j \) for \( j = 1, \ldots, m - 1 \) are generated by at most \( \kappa - 1 \) out of the \( \kappa \) process covariance structures. Therefore, at every iteration on \( j \in \{1, \ldots, m - 1\} \) there exists at least one pair of distinct cluster centers that are generated by the same process covariance structure. Therefore, by (2.9) and (2.11), for all \( t \geq T(k, k') \) and \( l = k, k' \) we have

\[
\frac{1}{\eta} \sum_{j=1}^{m-1} w_j \gamma_j^t \hat{d}^\alpha \left(x_{c_{j}^k}, X^{(i)}\right) \leq \frac{1}{\eta} \sum_{j=1}^{m-1} w_j \gamma_j^t \leq \frac{2M(t)\varepsilon(k, k')}{w_m \delta(k, k')}.
\]  
(2.13)

Noting that the clusters are ordered in the order of appearance of the different covariance structures, we have \( x_{c_{j}^l} = x_{c_{j}^m} \) for all \( j = m, \ldots, J \) and \( l = k, k' \), where the index \( s_1 \) is defined by (2.7). Therefore, by (2.9) for all \( t \geq T(k, k') \) and every \( l = k, k' \) we have

\[
\frac{1}{\eta} \sum_{j=m}^{J(k, k')} w_j \gamma_j^t \hat{d}^\alpha \left(x_{c_{j}^l}, X^{(i)}\right) = \frac{1}{\eta} \hat{d}^\alpha \left(x_{c_{s_1}^l}, X^{(i)}\right) \sum_{j=m}^{J(k, k')} w_j \gamma_j^t \leq \varepsilon(k, k').
\]  
(2.14)

Combining (2.12), (2.13), and (2.14) we obtain

\[
\frac{1}{\eta} \sum_{j=1}^{N(t)} w_j \gamma_j^t \hat{d}^\alpha \left(x_{c_{j}^l}, X^{(i)}\right) \leq \varepsilon(k, k') \left(1 + \frac{4M(t)}{w_m \delta(k, k')}\right),
\]  
(2.15)

for all \( l = k, k' \) and all \( t \geq T(k, k') \), establishing (2.6).

To finish the proof of the consistency, consider an index \( i \in G_{k'} \) for some \( k' \in \{1, \ldots, \kappa\} \). By Lemma 1.7 increasing \( T(k, k') \) if necessary, for all \( t \geq T(k, k') \)
we have
\[
\max_{l \in \{k,k'\}} \sum_{j \in G_l \cap \{1,\ldots,N\}} \tilde{d}^\varepsilon \left( x_j^l, X^{(l)} \right) \leq \varepsilon(k,k').
\] (2.16)

For all \( t \geq T(k,k') \),
\[
\frac{1}{\eta} \sum_{j=1}^{N(t)} w_j \gamma_j^t \tilde{d}^\varepsilon \left( x_j^t, x_{e_k}^t \right) \\
\geq \frac{1}{\eta} \sum_{j=1}^{N(t)} w_j \gamma_j^t \tilde{d}^\varepsilon \left( x_j^t, X^{(k)} \right) - \frac{1}{\eta} \sum_{j=1}^{N(t)} w_j \gamma_j^t \tilde{d}^\varepsilon \left( x_j^t, X^{(k')} \right) \\
\geq \frac{1}{\eta} \sum_{j=1}^{N(t)} w_j \gamma_j^t \left( \tilde{d}^\varepsilon \left( X^{(k)}, X^{(k')} \right) - \tilde{d}^\varepsilon \left( x_j^t, X^{(k')} \right) \right) \\
\geq \delta(k,k') - 2\varepsilon(k,k') \left( 1 + \frac{2M(t)}{w_m \delta(k,k')} \right) .
\] (2.17)

Since the choice of \( \varepsilon(k,k') \) is arbitrary, from (2.16) and (2.17) we obtain
\[
\arg\min_{k \in \{1,\ldots,\kappa\}} \frac{1}{\eta} \sum_{j=1}^{N(t)} w_j \gamma_j^t \tilde{d}^\varepsilon \left( x_j^t, x_{e_k}^t \right) = k'.
\] (2.18)

It remains to note that for any fixed \( N \in \mathbb{N} \) from some \( t \) on (2.18) holds for all \( i = 1,\ldots,N \), and the consistency statement follows by the fact that \( k, k' \in \{1,\ldots,\kappa\} \) in the above argument can be arbitrarily chosen.

The next part involves discussion of the complexities of the above two algorithms.

1. For offline setting, our Algorithm 1 requires \( N(N-1)/2 \) calculations of \( \tilde{d}^\varepsilon \), against \( \kappa N \) calculations of \( \hat{d} \) in the offline algorithm in Khaleghi et al (2016). In each \( \tilde{d}^\varepsilon \), the matrices distance \( \rho^\varepsilon \) consists of \( m^2 n^2 \) calculations of Euclidean distances. Then iterating over \( m, l \) in \( \tilde{d}^\varepsilon \) we see that at most \( O(nm^3) \) computations of Euclidean distances, against \( O(nm^3/|\log s|) \) computations of \( \hat{d} \) for the offline algorithm in Khaleghi et al (2016), where
\[
s = \min_{X^{(1)} \neq X^{(2)}} \min_{i=1,\ldots,n_1, j=1,\ldots,n_2} |X^{(1)}_i - X^{(2)}_j|.
\]

It is known that efficient searching algorithm can be utilized to determine \( s \), with at most \( O(n \log(n)) \) (with \( n = \min\{n_1, n_2\} \)) computations. Therefore our Algorithm 1 is computationally competitive to the one in Khaleghi et al (2016).
2. For online setting, we can hold a similar discussion as in Khaleghi et al. (2016), Section 5.1. There it shows the computational complexity of updates of $\hat{d}^\circ$ for both our Algorithm 2 and the online algorithm in Khaleghi et al. (2016) is at most $O(N(t)^2 + N(t)\log^3 n(t))$ (here we take $m_n(t) = \log n(t)$). Therefore the overall difference of computational complexities between the 2 algorithms are reflected by the complexity of computing $\hat{d}^\circ$ and $\hat{d}$ (see Point 1).

2.3 Efficient Dissimilarity Measure

Kleinberg (2003) presented a set of three simple properties that a good clustering function should have: scale-invariance, richness and consistency. Further, Kleinberg (2003) demonstrated that there is no clustering function that satisfies these properties. As one particular example, he pointed out that the centroid-based Clustering basically does not satisfy the above consistency property. In this section we show that, although the consistency property is not satisfied, there exists some other criterion of efficiency of dissimilarity measure in a particular setting. It is the so-called efficient dissimilarity measure.

**Definition 2.3 (Efficient dissimilarity measure).** Assume that the samples $S = \{x(\xi) : \xi \in H \} \ (H \subset \mathbb{R}^q$ for some $q \in \mathbb{N})$, meaning that all the paths $x(\xi)$ are indexed by a set of real-valued parameters $\xi$. Then a clustering function is called efficient if its dissimilarity measure $d$ satisfies that, there exists $c > 0$ so that for any $x(\xi_1), x(\xi_2) \in S$,

$$d(x(\xi_1), x(\xi_2)) = c\|\xi_1 - \xi_2\|,$$

where $\| \cdot \|$ denotes some norm defined over $\mathbb{R}^q$.

As in Definition 2.3, clustering processes based on efficient dissimilarity measure will possibly be equivalent to clustering via classical distances in $\mathbb{R}^q$, such as Euclidean distance, Manhattan distance, or Minkowski distance. The latter setting has well-known advantages in clustering analysis. For example, Euclidean distance performs well when deployed to datasets that include compact or isolated clusters (Jain and Mao, 1996; Jain et al., 1999); when the shape of clusters is hyper-rectangular (Xu and Wunsch, 2005), then Manhattan distance can be used; Minkowski distance, including Euclidean and Manhattan distances as its particular cases, can be used to solve clustering obstacles (Wilson and Martinez, 1997). There is a rich literature on comparing the above three distances and discussing of their advantages and inconveniences. We refer to Hirkhorshidi et al. (2015) and the references therein.

In the next section we present one excellent example, to show how to improve the efficiency of our consistent algorithms, for clustering self-similar processes with wide-sense stationary ergodic increments.
3 Self-similar Processes and Logarithmic Transformation

In this section we apply the consistent algorithms to cluster self-similar processes with stationary (in wide sense) increments.

**Definition 3.1** (Self-similar process, see [Samorodnitsky and Taqqu (1994)]) A process \(X^{(H)} = \{X_i^{(H)}\}_{i \in T}\) is self-similar with index \(H > 0\) if, for all \(n \in \mathbb{N}\), all \(t_1, \ldots, t_n \in T\), and all \(c \in \mathbb{R}\) such that \(ct_i \in T\) (\(i = 1, \ldots, n\)),

\[
\left(X_{t_1}^{(H)}, \ldots, X_{t_n}^{(H)}\right) \overset{\text{law}}{=} \left(|c|^{-H} X_{ct_1}^{(H)}, \ldots, |c|^{-H} X_{ct_n}^{(H)}\right).
\]

It can be shown that a self-similar process’ covariance structure is indexed by its self-similarity index \(H\), in the following way [Embrechts and Maejima 2000]:

**Theorem 3.2.** Let \(\{X_i^{(H)}\}_{i \in T}\) be a zero-mean self-similar process with index \(H\) and wide-sense stationary ergodic increments. Assume \(\mathbb{E}|X_1^{(H)}|^2 < +\infty\), then for any \(s, t \in T\),

\[
\text{Cov} \left(X_s^{(H)}, X_t^{(H)}\right) = \frac{\mathbb{E}|X_1^{(H)}|^2}{2} \left(|s|^{2H} + |t|^{2H} - |s - t|^{2H}\right).
\]

The corollary below follows.

**Corollary 3.3.** Let \(\{X_i^{(H)}\}_{i \in T}\) be a zero-mean self-similar process with index \(H\) and weakly stationary increments. Assume \(\mathbb{E}|X_1^{(H)}|^2 < +\infty\). For \(h > 0\) small enough, define the increment process \(Z_h^{(H)}(s) = X_{s+h}^{(H)} - X_s^{(H)}\), then for \(s, t \in T\) such that \(s - t \geq h\), we have

\[
\text{Cov} \left(Z_h^{(H)}(s), Z_h^{(H)}(t)\right) = \frac{\mathbb{E}|X_1^{(H)}|^2}{2} \left((s - t - h)^{2H} + (s - t + h)^{2H} - 2(s - t)^{2H}\right).
\]

Applying three times the mean value theorem to (3.1) leads to

\[
\text{Cov} \left(Z_h^{(H)}(s), Z_h^{(H)}(t)\right) = H\mathbb{E}|X_1^{(H)}|^2 \left((v_1^{(H)})^{2H-1} - (v_2^{(H)})^{2H-1}\right) h
\]

\[
= H(2H - 1)\mathbb{E}|X_1^{(H)}|^2 (v^{(H)})^{2H-2} h,
\]

(3.2)

for some \(v_1^{(H)} \in (s - t, s - t + h)\), \(v_2^{(H)} \in (s - t - h, s - t)\) and \(v^{(H)} \in (v_2^{(H)}, v_1^{(H)})\).

We see that the items \(\text{Cov} \left(Z_h^{(H)}(s), Z_h^{(H)}(t)\right)\) is a non-linear function of \(H\). We introduce the following \(\log^*\)-transformation: for \(x \in \mathbb{R}\), define

\[
\log^*(x) := \text{sgn}(x) \log |x| = \begin{cases} \log(x) & \text{if } x > 0; \\ -\log(-x) & \text{if } x < 0; \\ 0 & \text{if } x = 0. \end{cases}
\]

With the computation below we show the major 2 motivations of introduction to \(\log^*\)-transformation:
**Motivation 1** The log\(^*\) function transforms the current dissimilarity measure to the one which “linearly” depends on its variable \(H\).

**Motivation 2** The value log\(^*\)(\(x\)) preserves the sign of \(x\), which leads to the consequence that large distance between \(x, y\) implies large distance between log\(^*\)(\(x\)) and log\(^*\)(\(y\)).

Applying the log\(^*\)-transformation the covariances of \(Z^{(H)}_h\) given in (3.2), we obtain

\[
\log^* \left( \text{Cov} \left( Z^{(H)}_h(s), Z^{(H)}_h(t) \right) \right) = \text{sgn}(2H - 1) \left( (2H - 2) \log v^{(H)} + \log h + \log^* (H|1 - 2H|\text{Var}(X^{(H)}_1)) \right).
\]

When \(v^{(H)}\) and \(h\) are small the items \(\log v^{(H)}\) and \(\log h\) are significantly large so \(\log^* (H|1 - 2H|\text{Var}(X^{(H)}_1))\) becomes negligible compared to it. Thus we can write

\[
\log^* \left( \text{Cov} \left( Z^{(H)}_h(s), Z^{(H)}_h(t) \right) \right) \approx \text{sgn}(2H - 1) \left( (2H - 2) \log v^{(H)} + \log h \right).
\]

In conclusion,

- the item \(\log^* \left( \text{Cov} \left( Z^{(H)}_h(s), Z^{(H)}_h(t) \right) \right)\) is “approximately linear” on \(H \in (0,1/2]\) or on \(H \in [1/2,1)\). Use the approximation \(\log v^{(H_1)} \approx \log v^{(H_2)}\) for \(H_1, H_2 \in (0,1/2]\) or \(H_1, H_2 \in [1/2,1)\), we have

\[
\log^* \left( \text{Cov} \left( Z^{(H_1)}_h(s), Z^{(H_1)}_h(t) \right) \right) - \log^* \left( \text{Cov} \left( Z^{(H_2)}_h(s), Z^{(H_2)}_h(t) \right) \right) \approx 2 \text{sgn}(2H_1 - 1)(H_1 - H_2) \log v^{(H_1)}.
\]

- When \(H_1 \in (0,1/2]\) and \(H_2 \in (1/2,1)\), \(\log^* \left( \text{Cov} \left( Z^{(H)}_h(s), Z^{(H)}_h(t) \right) \right)\) turns out to be large, because we have

\[
\log^* \left( \text{Cov} \left( Z^{(H_1)}_h(s), Z^{(H_1)}_h(t) \right) \right) - \log^* \left( \text{Cov} \left( Z^{(H_2)}_h(s), Z^{(H_2)}_h(t) \right) \right) \approx -(2H_1 - 2) \log v^{(H_1)} - (2H_2 - 2) \log v^{(H_2)} \\
\geq 2(2 - H_1 - H_2) \min \{\log v^{(H_1)}, \log v^{(H_2)}\}.
\]

Taking advantage of the above facts we define the new empirical covariance-based dissimilarity measure to be

\[
\widehat{\delta}^{**}(z_1, z_2) := \sum_{m=1}^{m_n} \sum_{l=1}^{n-m+1} w_m w_l p^* \left( \nu^{**}(Z^{(H_1)}_{l \ldots n}, m), \nu^{**}(Z^{(H_2)}_{l \ldots n}, m) \right),
\]

where \(\nu^{**}(Z^{(H_1)}_{l \ldots n}, m)\) is the empirical covariance matrix of \(Z^{(H_1)}_{l \ldots n}\), \(\nu^{**}(Z^{(H_1)}_{l \ldots n}, m)\), with each of its coefficients transformed by log\(^*\); let \(M = \{M_{i,j}\}_{i=1,\ldots,m; j=1,\ldots,n}\) be an arbitrary real-valued matrix, define

\[
\log^* M := \{\log^* (M_{i,j})\}_{i=1,\ldots,m; j=1,\ldots,n}.
\]
Then we have
\[ \nu^*(Z_i^{(H_1)}, m) := \log^* \left( \nu^*(Z_i^{(H_1)}, m) \right). \]

Now for 2 wide-sense stationary ergodic processes \( X^{(1)}, X^{(2)} \), define
\[ d^*(X^{(1)}, X^{(2)}) := \sum_{m,l=1}^{\infty} w_m w_l \times \rho^* \left( \log^* \left( \text{Cov}(X^{(1)}_l, \ldots, X^{(1)}_{l+m-1}) \right), \log^* \left( \text{Cov}(X^{(2)}_l, \ldots, X^{(2)}_{l+m-1}) \right) \right). \]

Using the fact that \( \log^* \) is continuous over \( \mathbb{R} \setminus \{0\} \) and the weak ergodicity of \( Z_h^{(H)} \), we have the following version of ergodicity:
\[ \hat{d}^*(z_1, z_2) \xrightarrow{a.s.} d^*(Z_h^{(H_1)}, Z_h^{(H_2)}). \]

Unlike \( \hat{d}^* \), the dissimilarity measure \( \hat{d}^* \) is approximately linear with respect to the self-similarity index \( H \). Indeed, it is easy to see that
\[ \hat{d}^*(z_1, z_2) \sim \begin{cases} |H_1 - H_2| < 1, & \text{for } H_1, H_2 \in (0, 1/2] \text{ or } H_1, H_2 \in [1/2, 1); \\ 2(2 - H_1 - H_2) > 1, & \text{for } H_1 \in (0, 1/2) \text{ and } H_2 \in [1/2, 1); \end{cases} \]
where \( H_1, H_2 \) correspond to the self-similarity indexes of \( X^{(H_1)}, X^{(H_2)} \). In fact, from (3.3) we can believe that \( \hat{d}^* \) satisfies Definition 2.3 in the wide sense. This allows our consistent algorithms to be more efficient when clustering self-similar processes with weakly stationary increments, having different values of \( H \). In the next section we provide an example of clustering using our consistent algorithms, when the observed paths are from a well-known self-similar processes – fractional Brownian motions.

4 Experimental Results

In this section, we present the performance of proposed offline (Algorithm 1) and online (Algorithm 2) methods, on a synthetic data set. The wide-sense stationary ergodic processes that we choose is the first order increment process of fractional Brownian motions. Denote by \( \{B^H(t)\}_{t \geq 0} \) a fractional Brownian motion with Hurst index \( H \in (0, 1) \). It is well-known that \( B^H \) is a zero-mean self-similar Gaussian process with self-similarity index \( H \) and covariance function
\[ \text{Cov} \left( B^H(s), B^H(t) \right) = \frac{1}{2} \left( s^{2H} + t^{2H} - |s - t|^{2H} \right), \text{ for } s, t \geq 0. \]

Fix \( h > 0 \), define its increment process (with time variation \( h \)) to be
\[ Z_h^{(H)}(t) = B^H(t + h) - B^H(t). \]
Z_{h}^{(H)} is also called fractional Gaussian noise. Using the covariance function \( (4.1) \) we obtain the autocovariance function of \( Z_{h}^{(H)} \) below:

\[
\gamma(\tau) = \text{Cov} \left( Z_{h}^{(H)}(s), Z_{h}^{(H)}(s + \tau) \right) = \frac{1}{2} \left( |\tau + h|^{2H} + |\tau - h|^{2H} - 2|\tau|^{2H} \right).
\]

(4.2)

For stationary Gaussian processes such as \( Z_{h}^{(H)} \), the strict ergodicity can be expressed in the language of its autocovariance function \( \gamma \). The following result is obtained in [Maruyama (1970)] and summarized in [Slezak (2017)]. It provides a sufficient and necessary condition for a stationary Gaussian process to be strictly ergodic.

**Theorem 4.1** (Strict ergodicity of Gaussian processes). A Gaussian stationary process \( X \) is strictly ergodic if and only if

\[
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} |\gamma_X(\tau)| d\tau = 0,
\]

where \( \gamma_X \) denotes the autocovariance function of \( X \).

From the covariance function \( \gamma \) in (4.2) we know that

\[
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} |\gamma(\tau)| d\tau = 0.
\]

This together with Theorem 4.1 yields that \( Z_{h}^{b} \) is second-order strictly stationary ergodic, so it is also wide-sense stationary ergodic.

To test our algorithms we simulate \( \kappa = 5 \) groups of independent fractional Brownian paths, with the \( i \)th group containing 10 paths as

\[
\left\{ B^{H_i} \left( \frac{1}{n} \right), \ldots, B^{H_i} \left( \frac{n-1}{n} \right), B^{H_i}(1) \right\},
\]

for the self-similarity indexes

\[
H_1 = 0.3, \quad H_2 = 0.4, \ldots, H_5 = 0.7.
\]

We intentionally choose the indexes \( H_1, \ldots, H_5 \) to be different but close to each other. Remark that clustering a zero-mean fractional Brownian motion \( B^{H} \) is equivalent to clustering its increments \( Z_{1/n}^{(H)}(t) = B^{H}(t + 1/n) - B^{H}(t) \). These total number of \( N = 50 \) observed paths of \( Z_{1/n}^{(H)}(t) \), each of length \( n = 50 \), compose an offline data set. But online data set can be also constructed by varying the number of observed paths as time increases. To be specific, we construct our online data set in the following way: there are 6 observed paths when \( k = 1 \). Each time one more newly simulated path is added to the data set as \( k \) increases by 10. We keep increasing \( N(t) \) till \( k \) reaches \( n = 50 \).
After clustering the observed stochastic processes in datasets using the proposed algorithms, we calculate the misclassification rates to measure each algorithm’s performance: dividing the number of misclassified paths (without labeling the group) by the total number of paths:

\[ p := \frac{\# \text{ of mis-clustered paths}}{\# \text{ of total paths collected}}. \]

Figure 4 presents the comparisons of 2 algorithms: one is using the dissimilarity measure \( d^*_d \), the other one is using the dissimilarity measure \( d^{**}_d \), based on the behavior of misclassification rates as time increases. We conclude that, in both offline and online settings, the covariance-based dissimilarity measure algorithms with \( \log^* \)-transformation (dashed red lines) have 30% lower misclassification rates on average than that of algorithms without \( \log^* \)-transformation (solid blue lines). This simulation study proves the necessity of utilizing \( \log^* \)-transformed covariance-based dissimilarity measure when the underlying observations have nonlinear, especially power based, covariance dissimilarity measure.

We finally provide the easily readable and editable Matlab codes of proposed algorithms and simulation study replications. All the codes used in this section can be found publicly online\(^{1}\).

5 Conclusion

Inspired by Khaleghi et al (2016), we introduce the problem of clustering wide-sense stationary ergodic processes. A new covariance-based dissimilarity measure is proposed to obtain consistent clustering algorithms for both offline and online settings. We have shown that the recommended algorithms are competitive for at least two reasons:

1. Our algorithms are applicable to clustering any strict-sense stationary ergodic processes whose covariance structures are finite.
2. Our algorithms are efficient enough in terms of their computational complexity.

As one application, we studied the problem of clustering a class of self-similar processes. For this particular type of processes, we suggest the so-called \( \log^* \)-transformation on the process’ nonlinear covariance-based dissimilarity measure to improve the clustering performance. Empirical study is made on simulated paths of fractional Brownian motions. The results show that \( \log^* \)-transformations largely increased the efficiency of the clustering algorithms.

Finally, the clustering framework proposed in our paper focuses on the cases where the true number of clusters \( \kappa \) is known. The case for which \( \kappa \) is unknown is left to future research.

\(^{1}\)https://github.com/researchcoding/clustering_stochastic_processes/
Figure 1: The top graph illustrates the misclassification rates of (log) covariance-based dissimilarity measure along with the increase of time using offline algorithm and offline dataset. The bottom graph plots misclassification rates using online algorithm and online dataset, where one more observed time series in each cluster is added as time step increases by 10.
References

Bastos J. A., Caiado J. (2014) Clustering financial time series with variance ratio statistics. Quantitative Finance 14(12):2121–2133.

Cambanis S., Hardin C. J., Weron A. (1987) Ergodic properties of stationary stable processes. Stochastic Processes and their Applications 24(1):1–18.

Cesa-Bianchi N., Lugosi G. (2006) Prediction, Learning, and Games. Cambridge university press.

Damian D., Orešič M., Verheij E. e. a. (2007) Applications of a new subspace clustering algorithm (COSA) in medical systems biology. Metabolomics 3(1):69–77.

Embrechts P., Maejima M. (2000) An introduction to the theory of self-similar stochastic processes. International Journal of Modern Physics B 14(12):1399–1420.

Gray R. M. (1988) Probability, Random Processes, and Ergodic Properties. Springer.

Hartigan J. A. (1975) Clustering Algorithms. John Wiley & Sons, Inc.

Herdin M., Czink N., Ozcelik H., Bonek E. (2005) Correlation matrix distance, a meaningful measure for evaluation of non-stationary MIMO channels. In: IEEE 61st Vehicular Technology Conference, 2005., vol 1, pp. 136–140.

Hirkhorshidi A. S., Aghabozorgi S., Wah T. Y. (2015) A comparison study on similarity and dissimilarity measures in clustering continuous data. PLoS ONE 10(12).

Ieva F., Paganoni A. M., Tarabelloni N. (2016) Covariance-based clustering in multivariate and functional data analysis. Journal of Machine Learning Research 17:1–21.

Jääskinen V., Parkkinen V., Cheng L., Corander J. (2014) Bayesian clustering of DNA sequences using markov chains and a stochastic partition model. Stat Appl Genet Mol Biol 13(1):105–121.

Jain A. K., Mao J. (1996) A self-organizing network for hyperellipsoidal clustering (HEC). IEEE Trans Neural Networks 7:16–29.

Jain A. K., Murty M. N., Flynn P. J. (1999) Data clustering: a review. ACM Computing Surveys (CSUR) 31(3):264–323.

Juozapavičius A., Rapsvevičius V. (2001) Clustering through decision tree construction in geology. Nonlinear Analysis: Modelling and Control 6(2):29–41.

Katsavounidis I., Kuo C. J., Zhang Z. (1994) A new initialization technique for generalized Lloyd iteration. IEEE Signal processing letters 1(10):144–146.

Khaleghi A., Ryabko D., Mari J., Preux P. (2016) Consistent algorithms for clustering time series. Journal of Machine Learning Research 17(3):1–32.

Kleinberg J. M. (2003) An impossibility theorem for clustering. In Neural In-
Magdziarz M., Weron A. (2011) Ergodic properties of anomalous diffusion processes. Annals of Physics 326:2431–2443.

Mandelbrot B., van Ness J. W. (1968) Fractional Brownian motions, fractional noises and applications. SIAM Review 10(4):422–437.

Maruyama G. (1970) Infinitely divisible processes. Theory of Probability and its Applications 15(1):1–22.

Pavlidis N. G., P P. V., Tasoulis D. K., Vrahatis M. N. (2006) Financial forecasting through unsupervised clustering and neural networks. Operational Research 6(2):103–127.

Peng J., Müller H.-G. (2008) Distance-based clustering of sparsely observed stochastic processes, with applications to online auctions. The Annals of Applied Statistics 2(3):1056–1077.

Peng Q. (2012) Uniform Hölder exponent of a stationary increments Gaussian process: estimation starting from average values. Statistics & Probability Letters 81(8):1326–1335.

Rubinstein M., Joulin A., Kopf J., Liu C. (2013) Unsupervised joint object discovery and segmentation in internet images. In: The IEEE Conference on Computer Vision and Pattern Recognition (CVPR), pp. 1939–1946.

Samorodnitsky G. (2004) Extreme value theory, ergodic theory and the boundary between short memory and long memory for stationary stable processes. The Annals of Probability 32(2):1438–1468.

Samorodnitsky G., Taqqu M. S. (1994) Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance. Chapman & Hall, New York.

Sen P. K., Singer J. M. (1993) Large Sample Methods in Statistics. Chapman & Hall, Inc.

Ślezak J. (2017) Asymptotic behaviour of time averages for non-ergodic Gaussian processes. Annals of Physics 383:285–311.

Slonim N., Atwal G. S., Tkačik G., Bialek W. (2005) Information-based clustering. PNAS 102(51):18297–18302.

Wilson D. R., Martinez T. R. (1997) Improved heterogeneous distance functions. JAIR 6:1–34.

Xu R., Wunsch D. (2005) Survey of clustering algorithms. IEEE Transactions on Neural Networks 16(3):645–678.

Zhao W., Zou W., Chen J. J. (2014) Topic modeling for cluster analysis of large biological and medical datasets. BMC Bioinformatics 15:S11.