BIRKHOFF SUMS AS DISTRIBUTIONS II: APPLICATIONS TO DEFORMATIONS OF DYNAMICAL SYSTEMS

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ABSTRACT. Often topological classes of one-dimensional dynamical systems are finite codimension smooth manifolds. We describe a method to prove this sort of statement that we believe can be applied in many settings. In this work we will implement it for piecewise expanding maps. The most important step will be the identification of infinitesimal deformations with primitives of Birkhoff sums (up to addition of a Lipschitz function), that allows us to use the ergodic properties of piecewise expanding maps to study the regularity of infinitesimal deformations.

CONTENTS

1. Introduction

Part 1. Heuristic of the method

2. Abstract nonsense results
3. Conditions for unique integrability
4. The heart of the paper: Derivatives of infinitesimal conjugacies
5. Perturbation of lyapunov exponents

Part 2. Piecewise expanding maps

6. The class of piecewise expanding maps
7. Regularity of infinitesimal conjugacies
8. Deformations
9. Flexibility of multipliers
10. Topological classes are Banach manifolds
11. Quasi-symmetric classes
12. Relation with partially hyperbolic framework and it nightmares
13. Pressure pseudo-metric on the topological class

References
1. Introduction

One of the most interesting features of one-dimensional dynamics, either real or complex, is that

*Often topological classes of one-dimensional dynamical systems are smooth manifolds with finite codimension.*

So we can consider topological classes as an (infinite-dimensional) Teichmüller space of those dynamical systems (see Gardiner and Lakic [22] for information on classical Teichmüller spaces). Given a dynamical system \( f \), there are many ways to represent this topological object, with distinct geometries. We can ask how the geometrical properties of these representations change when we deform them; that is when we consider a smooth curve inside the topological class. We can, for instance, study the impact of deformations on periodic orbits, the Hausdorff dimension of sets, and invariant measures (the so-called linear response problem. See, for example, Baladi [3]).

This fact is a quite useful tool to study these dynamical systems. This is a well-developed approach in the study of complex one-dimensional dynamical systems, as rational (and especially polynomial) maps and (quasi-)Fuchsian groups (Teichmüller theory). The introduction of holomorphic motions by Mañé, Sad and Sullivan [35] had a significant impact in this field. The nonexistence of rational maps with wandering domains established by Sullivan [53] relies on deformation methods. See also the Teichmüller space of rational maps by McMullen and Sullivan [42]. In both cases, the space of deformations is finite-dimensional.

One of the main results on (infinite-dimensional) deformations of dynamical systems is the study of topological classes (more precisely, hybrid classes) of quadratic-like maps by M. Lyubich [34] and real-analytic unimodal maps by Avila, Lyubich and de Melo [1], which were important to the study of renormalization and the typical behavior of such maps. There are also recent results for topological classes of real-analytic multimodal maps by Clark and van Strien [14]. All those results rely heavily on complex methods.

There are also related results for real analytic circle diffeomorphisms by E. Risyer [49] (see also Goucharuk and Yampolsky [25]), generalized interval exchange transformations by Marmi, Moussa and Yoccoz [37], dissipative gap mappings by Clark and Gouveia [13], piecewise expanding unimodal maps by Baladi and S. [5], and piecewise Möbius circle diffeomorphisms with a break by Khanin and Teplinsky [55], but one can expect that topological classes are finite codimension smooth manifolds in many settings in one-dimensional dynamics. Most of these works are deeply connected with renormalization theory, once in many settings the local stable manifolds of the omega-limit set of renormalization operator are topological classes.

We also refer to the universal Teichmüller space (see the survey by Gardiner and Harvey [21]), the study of the manifold structure of normalized potentials by Giulietti, Kloeckner, Lopes, and Marcon [23] and rigidity conjectures by Martens, Palisiano and Winckler [39] and Winckler [56].

One may ask if there is an *unified way* to study the smoothness and finite codimension of the topological classes in one-dimensional dynamics. Inspired by previous works by Lyubich [34] and Baladi and S. [5] we will describe a method
that we believe is quite general. In this work, we will implement it for the class of piecewise expanding maps.

Maps with discontinuities, non analytic critical points and/or Lorenz-like singularities have been quite resistant to complex dynamics methods. On the other hand, the ergodic theory of those maps had a massive development in the last decades. One of the main distinctions of this work is the use of purely real methods, where ergodic theory will play a surprisingly new and crucial role.

Part 1. Heuristic of the method

2. Abstract nonsense results

By abstract nonsense results one must understand that the arguments here can be carried out for every class of one-dimensional dynamical systems where assumptions A. and B. below holds. We only use soft arguments that one can adapt with minimal modifications in many settings. Suppose we have a smooth family of (piecewise) smooth dynamical systems $f_t: X \to X$, with $t$ in an open subset $O$ of some vector space, with $0 \in O$. The phase space $X$ is a one-dimensional manifold (perhaps with borders).

Suppose that $f_t$ belongs to the topological class of $f_0$ for every $t$. We call $f_t$ a smooth deformation of $f_0$. That is, there is a family of homeomorphisms $h_t$ such that

$$h_t \circ f_0 = f_t \circ h_t$$

Assume that this smooth deformation $f_t$ satisfies

Assumption A (smooth motions) For each $x \in X$ we have that

$$t \mapsto h_t(x)$$

is differentiable.

So we can derive (2.1) with respect to $t$ to obtain

$$\partial_t h_t \circ f_0 = \partial_t f_t \circ h_t + \partial_x f_t \circ h_t \cdot \partial_t h_t.$$  

that is, applying $h_t^{-1}$ on the right

$$v_t = \alpha_t \circ f_t - Df_t \cdot \alpha_t(x)$$

for every $t$. Here $v_t = \partial_t f_t$ and $\alpha_t = \partial_t h_t \circ h_t^{-1}$.

This suggest that if a function $v$ belongs to the tangent space of the topological class of a map $f$ the there is a solution $\alpha$ for the twisted cohomological equation

$$v = \alpha \circ f - Df \cdot \alpha$$

The function $\alpha$ will be called an infinitesimal deformation of $f$. Note also that $h_t$ satisfies the initial value problem

$$\begin{cases} 
\partial_t h_t(x) = \alpha_t(h_t(x)), \\
\end{cases}$$

$$h_0(x) = x.$$
It is remarkable that one can often solve (2.3) without knowing \( h_t \). Indeed note that the series
\[
\alpha(x) = -\sum_{i=0}^{\infty} \frac{v(f^i(x))}{Df^{i+1}(x)}
\]
is a formal solution of (2.4). Baladi and S. [4][5] proved that for piecewise expanding unimodal maps this series indeed converges to a Hölder solution. We are going to improve that showing that \( \alpha \) is indeed Log-Lipschitz in the piecewise expanding setting (see Theorem 8.1). When dealing with maps with critical points, as Collet-Eckmann maps, this formal series does not converge everywhere, but nevertheless one can implement an inducing scheme to obtain a continuous solution \( \alpha \) (See Baladi and S. [7]).

**Remark 2.1.** Assumption A seems to be quite strong. However, it is well-known in complex dynamics that for complex-analytic families of complex analytic maps with no bifurcations, the map \( t \mapsto h_t(x) \) is holomorphic for each \( x \), that is, a holomorphic motion, that has been an essential tool to study such dynamical systems since its introduction in the study of rational maps by Mané, Sad and Sullivan [35], and in particular in the study of deformations of quadratic-like maps in Lyubich [34]. The use of Beltrami paths to construction deformations of complex dynamical systems is a quite popular way to built deformations satisfying Assumption A. It is remarkable that Assumption A also holds for smooth deformations of real maps with finite smoothness. Indeed Baladi and S.[5] proved that Assumption A holds for smooth families of piecewise expanding unimodal maps, and we will prove that it holds for smooth families of piecewise expanding maps (See Theorem 8.3).

We started with a smooth family in the topological class of \( f_0 \) and verified that the twisted cohomological equation (2.4) for its tangent vector \( v = \partial_t f_t|_{t=0} \) has a solution \( \alpha \). We would like to do the reverse argument.

Suppose \( f_t \) is a smooth family, such that (2.3) has a solution \( \alpha_t \) for every \( t \). Then we can consider the initial value problem (2.5). If \( \alpha_t \) is regular enough, this problem is uniquely integrable and it defines a flow \( h_t \). One can see that due (2.3) that \( h_t \circ f_0 \) and \( f_t \circ h_t \) are both solution of the initial value problem
\[
\begin{align*}
\dot{y} &= \alpha_t(y), \\
y(0) &= f_0(x).
\end{align*}
\]
and consequently \( h_t \circ f_0 = f_t \circ h_t \), so \( h_t \) is a conjugacy between \( f_t \) and \( f_0 \) and \( f_t \) belongs to the topological class of \( f_0 \) for every \( t \).

So the reverse argument needs the following assumption on the smooth family \( f_t \)

**Assumption B (unique integrability)** If \( \alpha_t \) are solutions of (2.4) then the ordinary differential equation
\[
\dot{y} = \alpha_t(y)
\]
is uniquely integrable.

Note that we do not assume that the smooth family \( f_t \) is a deformation, but we conclude this from Assumption B.
Remark 2.2. For complex analytics dynamical systems we often have that $\alpha_t$ are quasiconformal vector fields, that implies the unique integrability (See McMullen [40]). Baladi and S. [5] showed that for families of piecewise expanding unimodal maps $f_t$ the unique integrability of this o.d.e. holds provided there are continuous solutions $\alpha_t$.

This heuristic suggests that

If the topological class of map $f$ is a smooth manifold, then the vectors $v$ in the tangent space of $f$ are those that admits solutions $\alpha$ of the twisted cohomological equation (2.4) that are regular enough to warranty that the ordinary differential equation (2.5) is uniquely integrable. Moreover, if the topological class has finite codimension $d$ then the subspace of vectors $v$ that admits such regular solutions has codimension $d$.

3. Conditions for unique integrability

In the setting of this work the unique integrability of (2.5) does not follow from the usual Picard-Lindelöf-Cauchy-Lipschitz theorem since typically the solutions $\alpha$ on (2.4) are not Lipchitz functions even when $f$ and $v$ are very smooth. For instance, if $f$ is a $C^\infty$ (or even analytic) expanding map of the circle and $v \in C^\infty$ (or even analytic), we have that the solution $\alpha$ is often nowhere differentiable and indeed it is not Lipchitz on any subset of the circle with positive one-dimensional Haar measure. See de Lima and S. [16].

There are two ways to obtain unique integrability. The first is to use a property similar to the sensibility of the initial conditions of $f_t$. This was done in Baladi and S. [5] for piecewise expanding unimodal maps. The downside of this approach is that it does not give us information about the regularity of the conjugacies $h_t$. The second approach, that we adopted here, is to prove that the solution $\alpha$ satisfies the Osgood condition [45], which implies unique integrability.

Indeed we are going to show that in the piecewise expanding setting $\alpha$ is Log-Lipschitz, which implies Osgood condition. See Theorem 8.1. Moreover, the Log-Lipschitz continuity of $\alpha$ implies that the conjugacy $h_t$ and its inverse are $1 - O(|t|)$-Hölder continuous by a result by Chemin [12], who was interested in the regularity of the flow generated by vectors that are weak solutions of the 2D Euler equations. Indeed there are even more recent results relating the modulus of continuity of $\alpha$ and the modulus of the continuity of the flow (See Kelliher [30]). If $\alpha_t$ is Zygmund, then the conjugacies are $1 + O(|t|)$-quasisymmetric by Riemann [47]. See Table 1.

Remark 3.1. For piecewise expanding maps, the conjugacies $h_t$ are typically not very regular. Indeed, Shub and Sullivan [51] show that if the conjugacy between two expanding maps on the circle is absolutely continuous, then it is indeed smooth. In particular, the conjugacy must preserve the multipliers of the periodic points. Since it is easy to see that there are deformations $f_t$ that do not preserve multipliers, $h_t$ is rarely absolutely continuous. A large class of unimodal maps have similar properties (See Martens and de Melo [38]). This, in particular, suggests that Lipschitz regularity of $\alpha_t$ is quite rare for one-dimensional maps with many periodic points.
4. THE HEART OF THE PAPER: DERIVATIVES OF INFINITESIMAL CONJUGACIES

So the crucial step in the above method demands the study of the existence and regularity of the solution \( \alpha \) for (2.4). Remark 3 tells us we can not expect \( \alpha \) to be very regular. However we can formally derive (2.4) to obtain

\[
Dv = D\alpha \circ f \cdot Df - D^2 f \cdot \alpha - Df \cdot D\alpha
\]

so \( D\alpha \) satisfies the Livsic cohomological equation

\[
\frac{Dv + D^2 f \cdot \alpha}{Df} = D\alpha \circ f - D\alpha.
\]

If we denote

\[
\phi = \frac{Dv + D^2 f \cdot \alpha}{Df}
\]

then we can formally solve (4.6) taking

\[
D\alpha = -\sum_{i=0}^{\infty} \phi \circ f^i.
\]

We are going to see that one can make this argument rigorous in the one-dimensional piecewise expanding setting. We will prove that the derivative of \( \alpha \) is indeed a Birkhoff sum in the sense of distributions (up to the addition of a bounded function) and this will allows us to study the regularity of \( \alpha \). See Theorem 7.1, it is heart of this work.

5. PERTURBATION OF LYAPUNOV EXPONENTS

The role of Birkhoff sums of the observable \( \phi \) in (4.7) is clarified when we study the perturbation of the lyapunov exponent along orbits in a deformation \( f_t \) of \( f_0 \).

If \( h_t \circ f_0 = f_t \circ h_t \) then for a given \( x \) we have that \( h_t(x) \) is a "smooth" continuation of \( x \) and we can see that

\[
\partial_t \ln |Df_t^k(h_t(x))|_{t=0} = \sum_{j<k} \partial_t \ln |Df_t(h_t(f_0^j(x)))|_{t=0} = \sum_{j<k} \phi(f_0^j(x)).
\]
Part 2. Piecewise expanding maps

6. The class of piecewise expanding maps

Let \( I = [a, b] \) and \( C = \{c_0, c_1, \ldots, c_n\} \subset [a, b] \) be such that \( a = c_0 < c_i < c_{i+1} < c_n = b \) for every \( i < n - 1 \). Denote by \( B^k(C) \), with \( k \in \mathbb{R}, k \geq 0 \), the space of all functions

\[
\begin{align*}
v: \bigcup_{i=0}^{n-1} (c_i, c_{i+1}) & \to \mathbb{R}
\end{align*}
\]
such that for each \( i < n - 1 \) we have that \( v: (c_i, c_{i+1}) \to \mathbb{R} \) extends to a \( C^m \) function in \( [c_i, c_{i+1}] \), where \( k = m + \beta \), with \( n \in \mathbb{N} \) and \( \beta \in [0, 1) \) and, in the case \( \beta \neq 0 \), a \( \beta \)-Hölder \( n \)th derivative. We can endowed \( B^k(C) \) with the norm

\[
|w|_k = \sup_{i < n} \sum_{j \geq m} |D^j w|_{L^\infty (c_i, c_{i+1})} + \sup_{x, y \in [c_i, c_{i+1}], x \neq y} \frac{|D^m f(x) - D^m f(y)|}{|x - y|^{\beta}}.
\]

We have that \( B^k(C) \) is a Banach space. Denote by \( B^k_{exp}(C) \), with \( k \geq 1 \), the set of all \( f \in B^k(C) \) such that the range of \( f \) is contained in \([a, b] \) and \( \inf |Df| > 1 \). This is a convex subset of \( B^k(C) \). Note that if \( f \in B^k_{exp}(C) \) then \( f^i \in B^k_{exp}(C) \) for some finite set \( C_i \) that depends on \( f \).

Let

\[
\hat{I} = (a, b) \times \{+ , -\} \cup \{(c_i, +), (c_i, -): 1 \leq i \leq n - 1\}.
\]

To simply the notation we will use \( x^+ \) instead of \((x, +) \) and \( x^- \) instead of \((x, -) \).

Suppose that \( f \in B^k(C) \) is piecewise monotone function on each interval \((c_i, c_{i+1}) \). This is the case for \( f \in B^k_{exp}(C) \). Then we can extend it to a function

\[
f: \hat{I} \to \hat{I}
\]

using the lateral limits of \( f \). For instance we define \( f(a^+) = y^- \) when

\[
\lim_{x \to a^+} f(x) = y^-.
\]

A function \( v \in B^k(C) \) can be extended to a function

\[
v: \hat{I} \to \mathbb{R}
\]
as \( v(a^+) = \lim_{x \to a^+} v(x) \) and \( v(a^-) = \lim_{x \to a^-} v(x) \). Note that if \( f \in B^k_{exp}(C) \) then \( D^i f \in B^k_{exp}(C) \) for every \( i \leq k \).

We say that \( f, g \in B^k_{exp}(C) \) are topologically conjugated by a homeomorphism \( h: \hat{I} \to \hat{I} \) if \( h \circ f = g \circ h \) on \( \hat{I} \). Note that the conjugacy \( h \) is always Hölder continuous (Buzzi [11] has an elegant proof of this for an even more general setting).

6.1. Transfer operator and Lasota-Yorke inequality. Let \( f \in B^2_{exp}(C) \) and \( L \) be the transfer operator of \( f \) with respect to the Lebesgue measure \( m \) on \( I \). A well-known result by Lasota and Yorke [32] tell us that

- (Lasota-Yorke inequality in BV) There exist \( C_1 \) and \( \lambda_1 \) such that

\[
|L\gamma|_{BV} \leq \lambda_1|\gamma|_{BV} + C_1|\gamma|_{L^1},
\]

So

\[
|L^i\gamma|_{BV} \leq C_2 \lambda_1^i|\gamma|_{BV} + C_3|\gamma|_{L^1},
\]

for every \( i \). Denote

\[
\Lambda_f = \{\lambda \in S^1: \lambda \in \sigma(L)\},
\]

\[
\text{support}(\Lambda_f) = \{\lambda \in S^1: \lambda \in \sigma(L)\}.
\]
where $\sigma(L)$ denotes the spectrum of $L$ acting on $BV$. Then $\Lambda_1$ if finite, $1 \in \Lambda_1$ and

\begin{equation}
L = \sum_{\lambda \in \Lambda_1} \lambda \Phi_\lambda + K.
\end{equation}

Here $\Phi_\lambda^2 = \Phi_\lambda$, $\Phi_\lambda \Phi_{\lambda'} = 0$ if $j \neq j'$ and $K \Phi_\lambda = \Phi_\lambda K = 0$. Furthermore

i. $\Phi_\lambda$ is finite rank projection and it has an extension as a bounded linear transformation $\Phi_\lambda : L^1(m) \to BV$,

ii. $K$ is a bounded linear operator in $BV$ whose spectral radius is smaller than one, that is, there is $\lambda_2 \in (0, 1)$ and $C_4$ satisfying

$$|K^n(\phi)|_{BV} \leq C_4 \lambda_2^n |\phi|_{BV}.$$ 

iii. there exists $p = p(f) \in \mathbb{N}^*$ such that for all $\lambda \in \Lambda_1$ we have $\lambda^p = 1$.

Let $\lambda_3 = \sup_{x \in J} |Df(x)|^{-1}$. We will also need

**Lemma 6.1.** There exist $C_5 > 0$ and $C_6 > 0$ such that for all $n \in \mathbb{N}$ and $J \in \mathcal{P}^n$

\begin{equation}
\frac{1}{C_5} \leq \frac{Df^n(x)}{Df^n(y)} \leq C_5
\end{equation}

\begin{equation}
|\ln |Df^n(x)|| - |\ln |Df^n(y)|| \leq C_6 |f^n(x) - f^n(y)|.
\end{equation}

for every $x, y \in J$. Furthermore if $\gamma$ is a function with bounded variation and its support is contained in $J$ we have

\begin{equation}
v(L^n(F)(\gamma)) \leq C_5 \frac{1}{|DF^n(x)|} \left( v(\gamma) + C_6 ||\gamma||_{L^\infty(m)} \right),
\end{equation}

that for every $x \in J$, where $v(\gamma)$ is the variation of $\gamma$.

7. **Regularity of infinitesimal conjugacies**

**Theorem 7.1.** Let $f \in B_{exp}^{2+\beta}(C)$, with $\beta \in [0, 1)$, and $p = p(f)$. Let $\alpha : I \to \mathbb{C}$ be a continuous function. The following statements are equivalent

A. There is a function $v \in B^{1+\beta}(C)$ such that

$$v(x) = \alpha(f(x)) - Df(x)\alpha(x)$$

for every $x \notin Crit(f)$.

B. We have that

$$\alpha(x) = H(x) + G(x) + \int_{[a,x]} \left( \sum_{n=0}^{\infty} \sum_{i=0}^{p-1} \phi \circ f^{np+i} \right) dm,$$

where

- $\phi \in B^\beta(C)$ satisfies

\begin{equation}
\int \phi \Phi_1(\psi) \ dm = 0
\end{equation}

for every $\psi \in BV$,

- $H$ is a Lipschitz function such that $DH \circ f - DH$ belongs to $B^\beta(C)$ and
$G$ is given by

$$G(x) = \int \phi \left( \sum_{\lambda \in \Lambda \setminus \{1\}} \frac{1}{1 - \lambda} \Phi_{\lambda}(1_{[a,x]}) \right) \, dm$$

if $\Lambda_f \setminus \{1\} \neq \emptyset$, or zero otherwise. Here $\Phi_{\lambda}$ are the projections defined in Section 6.1. $G$ is also a Lipschitz function.

Indeed we can choose

$$\phi = \frac{Dv + D^2 f \alpha}{Df}.$$ 

Proof of Theorem 7.1 ($B \Rightarrow A$). Define

$$\hat{\alpha}_n(x) = \int 1_{[a,x]} \left( \sum_{k=0}^{n} \phi \circ f^k \right) \, dm$$

and

$$\hat{\alpha}_u(x) = \frac{1}{u} \sum_{n=0}^{u-1} \hat{\alpha}_n(x).$$

By G.R. and S. [27, Theorem 4.5] the following (uniform) limit

$$\beta(x) = \lim_{u} \hat{\alpha}_u(x) = \int 1_{[a,x]} \sum_{n=0}^{\infty} \sum_{j=0}^{p-1} \phi \circ f^{pn+j} \, dm + G(x),$$

exists for every $x \in I$ and $\beta$ is a Log-Lipschitz function, where

$$G(x) = \int \phi \left( \sum_{\lambda \in \Lambda \setminus \{1\}} \frac{1}{1 - \lambda} \Phi_{\lambda}(1_{[a,x]}) \right)$$

is a Lipschitz function. Define

$$w_n(x) = \hat{\alpha}_n(f(x)) - Df(x) \hat{\alpha}_n(x)$$

and

$$w(x) = \beta(f(x)) - Df(x) \beta(x).$$

Since $\hat{\alpha}_n$ is continuous and piecewise smooth, for every right continuous function $\psi \in BV$ such that

$$\text{supp } \psi \subset [c_i, c_{i+1}]$$

for some $i < n$, we have

$$- \int w_n \, d\psi = - \int (\hat{\alpha}_n \circ f - Df \hat{\alpha}_n) \, d\psi$$

$$= w_n(a^+) \psi(a) + \int D\hat{\alpha}_n \circ f Df \psi - \left( D^2 f \hat{\alpha}_n + Df D\hat{\alpha}_n \right) \psi \, dm$$

$$= w_n(a^+) \psi(a) + \int Df \psi \sum_{k=0}^{n} \phi \circ f^{k+1} - \left( D^2 f \hat{\alpha}_n + Df \sum_{k=0}^{n} \phi \circ f^k \right) \psi \, dm$$

$$= w_n(a^+) \psi(a) + \int \left( - D^2 f \hat{\alpha}_n - Df \phi + Df \phi \circ f^{n+1} \right) \psi \, dm,$$
(7.16) \[- \int (\hat{\alpha}_u \circ f - Df \hat{\alpha}_u) \, d\psi\]

\[= \psi(a) \frac{1}{u} \sum_{n=0}^{u-1} w_n(a^+) + \int \left(-D^2 f \hat{\alpha}_u - Df \phi \right) \psi \, dm + \frac{1}{u} \int \psi Df \sum_{n=0}^{u-1} \phi \circ f^{n+1} \, dm\]

Due (7.15) we have

\[\lim \int \frac{1}{u} \psi Df \sum_{n=0}^{u-1} \phi \circ f^{n+1} \, dm = 0,\]

so we conclude that

\[- \int w \, d\psi = w(a^+) \psi(a) + \int \left(-D^2 f \beta - Df \phi \right) \psi \, dm.\]

Taking \(\psi = 1_{[c_i, x]}\) we obtain

\[w(x) = w(c_i^+) + \int_{c_i}^{x} -D^2 f \beta - Df \phi \, dm.\]

for every \(x \in [c_i, c_{i+1}]\), so \(w\) is a piecewise \(C^{1+\beta}\) function. Define

\[s = H \circ f - Df \cdot H.\]

Then

\[Ds = Df(H' \circ f - H') - D^2 f H,\]

so \(s\) is a piecewise \(C^{1+\beta}\) function due the assumptions on \(H\). So \(\alpha = H + \beta\) satisfies

\[w + s = \alpha \circ f - Df \alpha,\]

where \(v = w + s\) is a piecewise \(C^{1+\beta}\) function. \(\square\)

**Lemma 7.2.** Let \(v \in B^{1+\beta}(C)\), with \(\beta \in [0, 1)\), such that there is a continuous function \(\alpha : [a, b] \to \mathbb{R}\) that satisfies the equation

(7.17) \[v(x) = \alpha(f(x)) - Df(x) \alpha(x).\]

for every \(x \notin C\). Then there exists \(C_7\) such that for every \(j\) there is a \(C^\infty\) function \(\theta_j : I \to \mathbb{C}\) satisfying

A. The function

\[\alpha_j(x) = \sum_{k=0}^{j-1} \frac{v(f^k(x))}{Df^{k+1}(x)} + \frac{\theta_j(f^j(x))}{Df^j(x)};\]

has a continuous extension to \(I\),

B. \(|D\theta_j|_{L^1(m)} \leq C_7 j\),

C. \(|\theta_j|_{L^\infty(m)} \leq C_7\),

D. We have that

\[\sup \sup \sup_{\lambda \in \Lambda_1} \sup_{j \in BV, \psi \neq 0} \frac{\left| \int D\theta_j \Phi_\lambda(\psi) \, dm \right|}{|\psi|_{L^1(m)}} \leq C_7.\]

E. \(\alpha_j(c) = \alpha(c)\) for every \(c \in C\).
Proof. Note that the continuity of $\alpha$ and $(7.17)$ implies that
\[
    \alpha(c) = -\frac{v(c^+) - \alpha(f(c^+))}{Df(c^+)}. \tag{7.18}
\]
for every $c \in C_f$. Let $h: \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function such that $h(0) = 1$ and $h(x) = 0$ for $x \notin (-1, 1)$. Let $j \in \mathbb{N}$. For $c \in C_f$ denote
\[
    \mathcal{O}^+(c, j) = \{ x \in I : x = f^i(c), \ i \leq j \}
\]
and
\[
    \mathcal{O}^+_f(j) = \bigcup_{c \in C} \mathcal{O}^+(c, j).
\]
Let
\[
    \theta_j(x) = \sum_{y \in \mathcal{O}^+_f(j)} \alpha(y) h(M(x - y)).
\]
We will choose $M$ later. Since $\alpha$ is bounded we have that $|D\theta_j|_{L^1} \leq C j$. Moreover if $M$ is large enough we have $\theta_j(x) = \alpha(x)$ for $x \in \mathcal{O}^+_f(j)$, and $|\theta_j|_{L^\infty} \leq C$. To prove $D$, consider $\lambda \in \Lambda_1$. Then the image $E_\lambda$ of $\Phi_\lambda$ has finite dimension. Let $w_1, \ldots, w_d \in BV$ be a basis of $E_\lambda$. Then
\[
    \Phi_\lambda(\psi) = \sum_{i=1}^d r_i(\psi) w_i, \tag{7.19}
\]
where $r_i \in L^1(I)^*$. Since $w_i$ has bounded variation we have
\[
    \sum_{x \in \mathcal{O}^+_f(j)} |w_i(x^+) - w_i(x^-)| \leq |w_i|_{BV}
\]
for every $j$. For every $j$ we can choose $M$ large enough such that
\[
    \sum_{y \in \mathcal{O}^+_f(j)} |w_i(z_y^+) - w_i(y^+)| + |w_i(z_y^-) - w_i(y^-)| \leq 1,
\]
for all $z_y^- \in [y - 1/M, y]$ and $z_y^+ \in [y, y + 1/M]$, and moreover
\[
    [y - 1/M, y + 1/M] \cap [z - 1/M, z + 1/M] = \emptyset
\]
for $y \neq z$ with $y, z \in \mathcal{O}^+_f(j)$. Since
\[
    \text{supp } \theta_j \subset \bigcup_{y \in \mathcal{O}^+_f(j)} [y - 1/M, y + 1/M],
\]
and furthermore
\[
    \int_{y-1/M}^y MDh(M(x - y)) \ dm(x) = 1, \quad \int_y^{y+1/M} MDh(M(x - y)) \ dm(x) = -1
\]
and
\[
    \int_{y-1/M}^{y+1/M} |MDh(M(x - y))| \ dm(x) = \int_{-1}^1 |Dh(x)| \ dm(x)
\]
we obtain
\[
\int D\theta_j w_i \, dm = \sum_{y \in \mathcal{O}^+_j} \int_{[y-1/M,y+1/M]} D\theta_j w_i \, dm \, dm
\]
\[
= \sum_{y \in \mathcal{O}^+_j} \alpha(y) \int_{y-1/M}^y MDh(M(x-y)) w_i(x) \, dm(x)
\]
\[
+ \sum_{y \in \mathcal{O}^+_j} \alpha(y) \int_{y}^{y+1/M} MDh(M(x-y)) w_i(x) \, dm(x)
\]
\[
= \sum_{y \in \mathcal{O}^+_j} \alpha(y) w_i(y^-) + \int_{y-1/M}^y MDh(M(x-y))(w_i(x) - w_i(y^-)) \, dm(x)
\]
\[
+ \sum_{y \in \mathcal{O}^+_j} -\alpha(y) w_i(y^+) + \int_{y}^{y+1/M} MDh(M(x-y))(w_i(x) - w_i(y^+)) \, dm(x)
\]
\[
= \sum_{y \in \mathcal{O}^+_j} \alpha(y)(w_i(y^-) - w_i(y^+)) + Q(j, i, M),
\]
with
\[
|Q(j, i, M)|
\]
\[
\leq |Dh|_{L^1(m)} \left( \sum_{y \in \mathcal{O}^+_j} \sup_{x \in [y,y+1/M]} |w_i(x) - w_i(y^+)| + \sup_{x \in [y-1/M,y]} |w_i(x) - w_i(y^-)| \right)
\]
\[
\leq |Dh|_{L^1(m)}.
\]
Moreover
\[
| \sum_{y \in \mathcal{O}^+_j} \alpha(y)(w_i(y^-) - w_i(y^+)) | \leq \sup_{x \in I} |\alpha(x)| |w_i|_{BV}.
\]
so we conclude that
\[
(7.20) \quad \left| \int D\theta_j w_i \, dm \right| \leq \sup_{x \in I} |\alpha(x)| |w_i|_{BV} + |Dh|_{L^1(m)}.
\]
By (7.19) and (7.20) we obtain $D$.
Define recursively the functions $\alpha_{j,k}$, $k \leq j$, as $\alpha_{j,0} = \theta_j$ and
\[
\alpha_{j,k+1}(x) = -\frac{v(x)}{Df(x)} + \frac{\alpha_{j,k}(f(x))}{Df(x)}
\]
for \( x \notin C_f \). One can easily see that due to (7.18) the function \( \alpha_{j,k} \) has a continuous extension to \( I \) and \( \alpha_{j,k}(x) = \alpha(x) \) for \( x \in \mathcal{O}_{j}^{f}(j - k) \). It follows that

\[
\alpha_{j,\ell}(x) = -\sum_{k=0}^{\ell-1} v(f^{k}(x)) \frac{\theta_{j}(f^{\ell}(x))}{Df^{\ell}(x)} + \sum_{n=0}^{\ell-1} D^{2}f(f^{n}(x)) L^{n}\psi dm.
\]

Take \( \alpha_{j} = \alpha_{j,j} \). \( \square \)

Let

\[
\hat{\alpha}_{j}(x) = -\sum_{k=0}^{j-1} v(f^{k}(x)) \frac{\theta_{j}(f^{j}(x))}{Df^{j}(x)},
\]

and

\[
\alpha_{j}(x) = \hat{\alpha}_{j}(x) + \frac{\theta_{j}(f^{j}(x))}{Df^{j}(x)},
\]

where \( \theta_{j} \) is given by Lemma 7.2. Note that for every \( x \notin \cup_{i=0}^{\infty} f^{-i}C_f \) we have

(7.21) \[
\alpha(x) - \alpha_{j}(x) = O(\lambda_{3}^{j}),
\]

and

(7.22) \[
\alpha(x) - \hat{\alpha}_{j}(x) = O(\lambda_{3}^{j}),
\]

Denote

\[
R_{j}(x) = D\left(\frac{\theta_{j}(f^{j}(x))}{Df^{j}(x)}\right) = D\theta_{j}(f^{j}(x)) - \frac{\theta_{j}(f^{j}(x))}{Df^{j}(x)} \sum_{n=0}^{j-1} \frac{D^{2}f(f^{n}(x))}{Df^{n}(x)}.
\]

In this work all Riemann-Stieltjes integrals of the form

\[
\int \eta \, d\gamma,
\]

where \( \eta \) and \( \gamma \) are right continuous \( BV \) functions on \( I = [a,b] \), must be considered as an Riemann-Stieltjes integral on \((a,b]\), that is

\[
\int \eta \, d\gamma = \int_{(a,b]} \eta \, d\nu,
\]

where \( \nu \) is the signed measure on \( I \) given by \( \nu([a,x]) = \gamma(x) \). See for instance Revuz and Yor [48] for information on integration by parts for Riemann-Stieltjes integrals of \( BV \) functions.

**Lemma 7.3.** Let \( \psi \) and \( \alpha \) be as in Lemma 7.2. Then for every right continuous \( \psi \in BV \) such that \( \text{supp} \, \psi \subset I \) we have

\[
\int \alpha \, d\psi = -\alpha(a)\psi(a)
\]

\[+
\lim_{j} -\int R_{j}\psi \, dm + \sum_{n=0}^{j-1} \int \left( \frac{D\nu + D^{2}f \hat{\alpha}_{j-n}}{Df} \right) L^{n}\psi \, dm.
\]
Proof. For every $x \notin \bigcup_{i=0}^{j} f^{-i}C_f$

$D\alpha_j(x)$

\[
\begin{align*}
D\alpha_j(x) &= R_j(x) - \sum_{k=0}^{j-1} \left( \frac{Dv(f^k(x))}{Df(f^k(x))} - \frac{v(f^k(x))D^2 f^{k+1}(x)}{(Df^{k+1}(x))^2} \right) \\
&= R_j(x) - \sum_{k=0}^{j-1} \left( \frac{Dv(f^k(x))}{Df(f^k(x))} - \sum_{n=0}^{k} \frac{v(f^k(x))D^2 f^{n}(x)}{Df^{k-n+1}(f^n(x))Df(f^n(x))} \right) \\
&= R_j(x) - \sum_{n=0}^{j-1} \left( \frac{Dv(f^n(x))}{Df(f^n(x))} - \sum_{n=0}^{k} \frac{v(f^k(x))D^2 f^{n}(x)}{Df^{k-n+1}(f^n(x))Df(f^n(x))} \right) \\
&= R_j(x) - \sum_{n=0}^{j-1} \frac{Dv(f^n(x))}{Df(f^n(x))} + D^2 f^n(x)\hat{\alpha}_{j-n}(f^n(x)) \\
&= R_j(x) - \sum_{n=0}^{j-1} \frac{Dv(f^n(x))}{Df(f^n(x))} + D^2 f^n(x)\hat{\alpha}_{j-n}(f^n(x)).
\end{align*}
\]

Since $\alpha_j$ is continuous and piecewise smooth, and $\psi$ has bounded variation we have

\[-\int \alpha \, d\psi = -\lim_j \int \alpha_j \, d\psi = \lim_j \alpha_j(a) \psi(a) + \int Do_j \psi \, dm\]

\[= \alpha(a)\psi(a) + \lim_j \int R_j \psi \, dm - \sum_{n=0}^{j-1} \int \left( \frac{Dv \circ f^n + D^2 f \circ f^n \hat{\alpha}_{j-n} \circ f^n}{Df \circ f^n} \right) \psi \, dm\]

\[= \alpha(a)\psi(a) + \lim_j \int R_j \psi \, dm - \sum_{n=0}^{j-1} \int \left( \frac{Dv + D^2 f \hat{\alpha}_{j-n}}{Df} \right) L^n\psi \, dm.\]

\[\square\]

Lemma 7.4. Let $v$ and $\alpha$ be as in Lemma 7.2. For every $\psi \in BV$ we have

\[\sum_{n=0}^{j-1} \int \left[ \frac{(Dv + D^2 f \hat{\alpha}_{j-n})}{Df} - \frac{(Dv + D^2 f \alpha)}{Df} \right] L^n\psi \, dm\]

\[= \sum_{n=0}^{j-1} \int \left[ \frac{(Dv + D^2 f \hat{\alpha}_{j-n})}{Df} - \frac{(Dv + D^2 f \alpha)}{Df} \right] \left( \sum_{\lambda \in \Lambda_1} \lambda^n \Phi_\lambda(\psi) \right) \, dm + O(j\lambda^2_j)\]

\[\int (R_j - D\theta_j \circ f^j) \psi \, dm = O(j\lambda^2_j),\]

\[\int D\theta_j \circ f^j \psi \, dm = \int D\theta_j \left( \sum_{\lambda \in \Lambda_1} \lambda^j \Phi_\lambda(\psi) \right) \, dm + O(j\lambda^2_j).\]
Proof. By (7.21) we have
\[
\sum_{n=0}^{j-1} \int \left[ \frac{Dv + D^2 f \hat{\alpha}_{j-n}}{Df} - \left( \frac{Dv + D^2 f \alpha}{Df} \right) \right] L^n \psi \, dm \\
= \sum_{n=0}^{j-1} \int \left[ \left( \frac{Dv + D^2 f \hat{\alpha}_{j-n}}{Df} \right) - \left( \frac{Dv + D^2 f \alpha}{Df} \right) \right] \left( \sum_{\lambda \in \Lambda_1} \lambda^n \Phi_{\lambda}(\psi) \right) \, dm \\
+ O(\sum_{n=0}^{j-1} \lambda_2^{j-n} \lambda_2^n)
\]
\[
= \sum_{n=0}^{j-1} \int \left[ \left( \frac{Dv + D^2 f \hat{\alpha}_{j-n}}{Df} \right) - \left( \frac{Dv + D^2 f \alpha}{Df} \right) \right] \left( \sum_{\lambda \in \Lambda_1} \lambda^n \Phi_{\lambda}(\psi) \right) \, dm \\
+ O(j \max\{\lambda_2, \lambda_3\}^j).
\]

It is easy to see that (7.24) holds, since
\[
D\theta_j(f^j(x)) - R_j(x) = \frac{\theta_j(f^j(x))}{Df^j(x)} \sum_{n=0}^{j-1} \frac{D^2 f(f^n(x))}{Df(f^n(x))}.
\]

\(\theta_j\) and \(D^2 f/Df\) are uniformly bounded and \(f\) is expanding. Finally
\[
\int D\theta_j \circ f^j \psi \, dm = \int D\theta_j L^j \psi \, dm \\
= \int D\theta_j \left( \sum_{\lambda \in \Lambda_1} \lambda^j \Phi_{\lambda}(\psi) \right) \, dm + |D\theta_j|_{L^1(m)} O(\lambda_2^j) \\
= \int D\theta_j \left( \sum_{\lambda \in \Lambda_1} \lambda^j \Phi_{\lambda}(\psi) \right) \, dm + O(j \lambda_2^j).
\]

\(\square\)

Proof of Theorem 7.1(A \(\implies\) B). By Lemma 7.4 we have
\[
(7.26) \quad - \int R_j \psi \, dm + \sum_{n=0}^{j-1} \int \left( \frac{Dv + D^2 f \hat{\alpha}_{j-n}}{Df} \right) L^n \psi \, dm \\
= - \int D\theta_j \left( \sum_{\lambda \in \Lambda_1} \lambda^j \Phi_{\lambda}(\psi) \right) \, dm \\
+ \sum_{n=0}^{j-1} \int \left[ \left( \frac{Dv + D^2 f \hat{\alpha}_{j-n}}{Df} \right) - \left( \frac{Dv + D^2 f \alpha}{Df} \right) \right] \left( \sum_{\lambda \in \Lambda_1} \lambda^n \Phi_{\lambda}(\psi) \right) \, dm \\
+ \int \psi \sum_{n=0}^{j-1} \frac{Dv \circ f^n + D^2 f \circ f^n \alpha \circ f^n}{Df \circ f^n} \, dm + O(j \max\{\lambda_3, \lambda_2\}^j).
\]
Define the linear functional $T_j : L^1(m) \to \mathbb{C}$ as
\[
T_j(\psi) = -\int D\theta_j \left( \sum_{\lambda \in \Lambda_1} \lambda^j \Phi_\lambda(\psi) \right) dm \\
+ \sum_{n=0}^{j-1} \int \left( \frac{Dv + D^2f}{Df} \tilde{\alpha}_{j-n} - \frac{Dv + D^2f}{Df} \right) \left( \sum_{\lambda \in \Lambda_1} \lambda^n \Phi_\lambda(\psi) \right) dm.
\]

Due to Lemma 7.2.D and (7.22) we have \(\sup_j |T_j|_{(L^1(m))^*} < \infty\). In particular
\[
\text{(7.27)} \quad \sup_j |T_j(\psi)| < \infty
\]

By Theorem 7.3 we have that
\[
\int \alpha d\psi = -\alpha(a)\psi(a) + \lim_j - \int R_j \psi dm + \sum_{n=0}^{j} \int \frac{Dv + D^2f}{Df} \tilde{\alpha}_{j-n} L^n \psi dm.
\]

Let
\[
\phi = -\frac{Dv + D^2f}{Df} \alpha.
\]

Note that (7.26), (7.27) and (7.28) imply that
\[
\{ \int \psi \sum_{n=0}^{j-1} \phi \circ f^n \ dm : j \in \mathbb{N} \}
\]
is a bounded set for each $\psi \in BV$. Since
\[
\int \psi \sum_{n=0}^{j-1} \phi \circ f^n \ dm = \int \phi \left[ \sum_{n=0}^{j-1} \left( \sum_{\lambda \in \Lambda_1} \lambda^n \Phi_\lambda(\psi) + K^n(\psi) \right) \right] dm \\
= \int \phi \left[ j \Phi_1(\psi) + \sum_{\lambda \in \Lambda_1 \setminus \{1\}} \left( \frac{1 - \lambda^j}{1 - \lambda} \Phi_\lambda(\psi) \right) \right] dm + O(1) \\
= j \int \phi \Phi_1(\psi) dm + O(1),
\]

this only occurs if
\[
\int \phi \Phi_1(\psi) dm = 0
\]
for every $\psi \in BV$. So G.R. and S. [27, Lemma 4.3] implies that the limit
\[
\lim_j \int \psi \left( \sum_{n=0}^{p_j} \phi \circ f^n \right) dm
\]
exists. So the limit $T(\psi) = \lim_j T_{pj}(\psi)$ exists and it defines a bounded functional in $(L^1(m))^*$. In particular there is $A \in L^\infty(m)$ such that
\[
T(\psi) = \int A\psi dm
\]
for every $\psi \in BV$. We conclude that
\[
\text{(7.29)} \quad \int \alpha d\psi = -\alpha(a)\psi(a) + \int A\psi dm - \int \psi \left( \sum_{n=0}^{p-1} \sum_{i=0}^{p-1} \phi \circ f^{n+i} \right) dm
\]
for every $\psi \in BV$. For $\psi \in C^\infty(I)$ we have

\[(7.30) \quad - \int \alpha \, D\psi \, dm = \alpha(a)\psi(a) - \int A\psi \, dm + \int \psi \left( \sum_{n=0}^{\infty} \sum_{i=0}^{p-1} \phi \circ f^{np+i} \right) dm.\]

Taking $\psi = 1_{[a,x]}$ we get

\[
\alpha(x) = \alpha(a) + q(x) + \int 1_{[a,x]} \left( \sum_{n=0}^{\infty} \sum_{i=0}^{p-1} \phi \circ f^{np+i} \right) dm.
\]

where $q(x) = -T(1_{[a,x]})$ is a Lipschitz function, since

\[
|q(y) - q(x)| = |T(1_{[x,y]})| \leq |A|_{L^\infty(m)}|1_{[x,y]}|_{L^1(m)} = |A|_{L^\infty(m)}|y-x|.
\]

On the other hand, by Theorem 7.1 ($B \implies A$) we have that there is a piecewise $C^{1+\beta}$ function $w$ such that

\[
w = \beta(f(x)) - Df(x)\beta(x),
\]

where

\[
\beta(x) = G(x) + \int 1_{[a,x]} \left( \sum_{n=0}^{\infty} \sum_{i=0}^{p-1} \phi \circ f^{np+i} \right) dm
\]

and

\[
G(x) = \int \phi \left( \sum_{\lambda \in \Lambda_1 \setminus \{1\}} \frac{1}{1-\lambda} \Phi_\lambda(1_{[a,x]}) \right) dm
\]

when $\Lambda_1 \setminus \{1\} \neq \emptyset$, or $G(x) = 0$ otherwise. Consequently if $H = \alpha - \beta = \alpha(a) + q - G$ we have

\[
v - w = H \circ f - Df \, H.
\]

Since $H$ is a Lipschitz function one can derive this expression to obtain

\[
\frac{Dv - Dw + D^2 f \, H}{Df} = DH \circ f - DH.
\]

It follows that

\[
DH \circ f - DH
\]

is a piecewise $C^3$ function on $I$ and consequently $B$. holds.

\[\square\]

**Lemma 7.5.** Let $\psi \in L^1(m)$. If for some $n \in \mathbb{N}^*$ we have that

\[(7.31) \quad \int \Phi_{1,n}(\gamma) \sum_{i=0}^{n-1} \psi \circ f^i \, dm = 0 \text{ for every } \gamma \in BV\]

then for every $n \in \mathbb{N}^*$ we have that (7.31) holds.

**Proof.** We have

\[
\Phi^n = \sum_{\beta \in \Lambda_1, \beta^n=1} \Phi_\beta.
\]
so
\[
\int \Phi_1^n(\gamma) \sum_{i=0}^n \psi \circ f^i \, dm \\
= \int \left( \sum_{\beta \in \Lambda_1, \beta^n=1} \Phi_\beta(\gamma) \right) \left( \sum_{i=0}^{n-1} \psi \circ f^i \right) \, dm \\
= \int \psi \sum_{i=0}^{n-1} \sum_{\beta \in \Lambda_1, \beta^n=1} L^i(\Phi_\beta(\gamma)) \, dm \\
= \int \psi \left( n \Phi_1(\gamma) + \sum_{\beta \in \Lambda_1, \beta^n=1, \beta \neq 1} \frac{1 - \beta^n}{1 - \beta} \Phi_\beta(\gamma) \right) \, dm \\
= n \int \psi \Phi_1(\gamma) \, dm.
\]

This completes the proof. \(\square\)

8. Deformations

A family \(f_t \in B_{exp}^k(C)\), with \(t \in (a, b)\), is a \(C^j\) family if \(t \mapsto f_t\) is a \(C^j\) function from \((a, b)\) to \(B_k^k(C)\). We say that a \(C^j\) family is Lasota-Yorke stable if for every compact \(K \subset (a, b)\) there is \(C_1, \lambda_1\) such that
\[
(8.32) \quad |L_t \gamma|_{BV} \leq \lambda_1 |\gamma|_{BV} + C_1 |\gamma|_{L^1}.
\]
for every \(\gamma \in BV\) and for every \(t \in (a, b)\). Here \(L_t\) is the Ruelle-Perron-Frobenius operator of \(f_t\). We say that \(f_t \in B_{exp}^k(C)\) is locally Lasota-Yorke stable if there is a open neighbourhood of \(f_t\) in \(B_{exp}^k(C)\) such that all maps in there satisfies the Lasota-Yorke inequality on \(BV\) with the same constants.

A \(C^j\)-deformation of \(f_t \in B_{exp}^k(C)\) is a \(C^j\) family \(f_t \in B_{exp}^k(C)\) such that \(f_t\) is topologically conjugated to \(f\) for every \(t \in (a, b)\).

Let
\[
\hat{C} = \{(c_0, +), (c_n, -)\} \cup \{(c_1, \ldots, c_{n-1}) \times \{+, -\}\}. \quad (8.31)
\]
The set of critical relations \(R_f\) of \(f \in B_{exp}^k(C)\) is
\[
R_f = \{(x, y, k): \ x, y \in \hat{C}, k \in \mathbb{N}^*, \ and \ f^k(x) = y\}.
\]

For \(f \in B_{exp}^k(C)\) let
\[
\mathcal{O}^f = \cup_{i \geq 0} f^i(\hat{C})
\]
and
\[
\ell^\infty(\mathcal{O}^f) = \{v: \mathcal{O}^f \to \mathbb{R}: \sup_{a \in \mathcal{O}^f} |v(a)| < \infty\}.
\]

Then we can define the linear functional
\[
J(f, x, \cdot): \ell^\infty(\mathcal{O}^f) \to \mathbb{R},
\]
with \(x \in \hat{C}\) as
\[
J(f, x, v) = \sum_{i=0}^{k-1} \frac{v(f^i(x))}{Df^i(f(x))},
\]

\(a \in \mathcal{O}^f\).
if \( f^i(x) \notin \hat{C} \) for \( 1 \leq i < k \) and \( f^k(x) \in \hat{C} \), and
\[
J(f, x, v) = \sum_{i=0}^{\infty} \frac{v(f^i(x))}{Df^i(f(x))},
\]
if \( f^i(x) \notin \hat{C} \) for \( i \geq 1 \).
In particular \( J(f, x, v) \) is well defined for \( v \in B^k(\mathcal{C}) \).

**Theorem 8.1** (Characterization of infinitesimal deformations). Let \( f \in B_{exp}^k(\mathcal{C}) \) and \( v \in B^k(\mathcal{C}) \). The following statements are equivalent.

A. We have \( J(f, c, v) = 0 \) for every \( c \in \hat{C} \).

B. There is a continuous function \( \alpha: I \to \mathbb{R} \) such that
\[
v = \alpha \circ f - Df \cdot \alpha
\]
on \( I \setminus C \) and \( \alpha(c) = 0 \) for every \( c \in C \).

C. There is a continuous function \( \alpha: I \to \mathbb{R} \) such that
\[
\alpha(x) = g(x) + \int_{[-1, x]} \sum_{n=0}^{p-1} \sum_{i=0}^{\infty} \phi \circ f^{n+p+i} \, dm,
\]
where \( p = p(f) \) and \( g \) is a Lipchitz function. Indeed we can choose
\[
\phi(x) = \frac{Dv + D^2f \alpha}{Df}.
\]

D. There is a Log-Lipchitz function \( \alpha: I \to \mathbb{R} \) such that
\[
v = \alpha \circ f - Df \cdot \alpha
\]
on \( I \setminus C \) and \( \alpha(c) = 0 \) for every \( c \in C \).

Additionally, the constants in the Log-Lipchitz condition on \( \alpha \) and in the Lipchitz constant of \( g \) depends only on the constants of the Lasota-Yorke inequality, \( p(f) \), \(|f|_2\) and \(|v|_1\).

**Proof.** We already proved that \( B, C \) and \( D \) are equivalent.

A \implies B. if \( x \in C \) define \( \alpha(x) = 0 \) and \( M(x) = 0 \). If \( f^k(x) \notin C \) for every \( k \in \mathbb{N} \) let \( M(x) = +\infty \). Otherwise \( x \notin C \) and there is \( k(x) \geq 1 \) such that \( f^{M(x)}(x) \in C \) and \( f^i(x) \notin C \) for \( 0 \leq i < M(x) \). Define
\[
\alpha(x) = -\sum_{j=0}^{M(x)-1} \frac{v(f^j(x))}{Df^{j+1}(x)}.
\]
Since the set of points \( Q \) that eventually arrive at \( C \) is countable, it is easy to verify that (8.33) holds for all points \( x \) except maybe those in \( Q \). So it is enough to show that \( \alpha \) is continuous to complete the proof. One can see that
\[
\lim_{x \to b^+} \alpha(x) = -\sum_{j=0}^{\infty} \frac{v(f^j(b^+))}{Df^{j+1}(b^+)}.
\]
Let \( b \in [-1, 1] \). If \( f^j(b^+) \notin \hat{C} \) for every \( j \) define \( n_0^+ = +\infty \), \( n_1^+ = +\infty \) and \( k_0^+ = 1 \). Otherwise let \( n_0^+ < n_1^+ < n_2^+ < \ldots \), with \( k < k_0^+ \), where \( k_0^+ \in \mathbb{N} \cup \{+\infty\} \), be the
sequence of all times \( j \) such that \( f^j(b^+) \in \hat{C} \). If \( k_0^+ \in \mathbb{N} \) define \( n_{k_0}^+ = +\infty \). Note that in all cases \( n_0^+ = M(b) \). Then

\[
\sum_{j=0}^{\infty} \frac{v(f^j(b^+))}{Df^{j+1}(b^+)} = \sum_{j=0}^{M(x)-1} \frac{v(f^j(b))}{Df^{j+1}(b)} + \sum_{k=0}^{k_0^+-1} \sum_{j=n_k^+}^{n_{k+1}^+-1} \frac{v(f^j(b^+))}{Df^{j+1}(b^+)}
\]

\[
= \sum_{j=0}^{M(x)-1} \frac{v(f^j(b))}{Df^{j+1}(b)} + \sum_{k=0}^{k_0^+-1} \frac{1}{Df^{n_k^++1}(b^+)} \sum_{j=0}^{n_{k+1}^+-n_k^+-1} \frac{v(f^j(f^{n_k^+}(b^+)))}{Df(f^{n_k^+}(b^+))}
\]

\[
= \sum_{j=0}^{M(x)-1} \frac{v(f^j(b))}{Df^{j+1}(b)} + \sum_{k=0}^{k_0^+-1} \frac{1}{Df^{n_k^++1}(b^+)} J(f,f^{n_k^+}(b^+),v)
\]

\[
= \sum_{j=0}^{M(x)-1} \frac{v(f^j(b))}{Df^{j+1}(b)} = \alpha(b).
\]

Here if no natural number satisfies the condition in the sum, consider its value to be zero.

In an analogous way, If \( f^j(b^-) \notin \hat{C} \) for every \( j \) define \( n_0^-=+\infty \), \( n_1^- =+\infty \) and \( k_0^- =1 \). Otherwise \( n_0^- < n_1^- < n_2^- < \ldots \), with \( k < k_0^- \), where \( k_0^- \in \mathbb{N} \cup \{+\infty\} \), be the sequence of all times \( j \) such that \( f^j(b^-) \in \hat{C} \). If \( k_0^- \in \mathbb{N} \) define \( n_{k_0^-}^- =+\infty \). Again, in all cases we have \( n_0^- = M(b) \). Then one can similarly conclude that

\[
\lim_{x \to b^-} \alpha(x) = -\sum_{j=0}^{M(x)-1} \frac{v(f^j(b))}{Df^{j+1}(b)} = \alpha(b).
\]

So we conclude that \( \alpha \) is continuous at \( b \). Since we concluded that \( \alpha \) is continuous, it follows from Theorem 7.1 that \( B \) holds.

\( D \implies A \). Note that for every \( k \) we have

\[
(8.35) \quad \sum_{j=0}^{k-1} Df^{k-j-1}(f^{j+1}(x)) \cdot v(f^j(x)) = \alpha \circ f^k(x) - Df^k(x) \cdot \alpha(x)
\]

whenever \( x \) is not a critical point of \( f^k \). Let \( c \in \hat{C} \) and suppose that there is \( c' \in \hat{C} \) and \( M \geq 1 \) such that \( f^M(c) = c' \) and \( f^k(c) \notin \hat{C} \) for every \( 1 \leq k < M \). Since \( \alpha(c) = \alpha(c') = 0 \), when \( x \) tends to \( c \) in (8.35) with \( k = M \) we obtain

\[
\sum_{j=0}^{M-1} Df^{M-j-1}(f^{j+1}(c)) \cdot v(f^j(c)) = 0.
\]

If we divide by \( Df^{M-1}(c) \) we get

\[
\frac{J(f,c,v)}{Df(c)} = \sum_{j=0}^{M-1} \frac{v(f^j(c))}{Df^{j+1}(c)} = 0,
\]
so \( J(f,c,v) = 0 \). On the other hand if \( f^k(c) \notin \hat{C} \) for every \( k \). When we divide (8.35) by \( Df^{k-1}(c) \) and \( x \) tends to \( c \) we get
\[
\sum_{j=0}^{k-1} \frac{v(f^j(c))}{Df^{j+1}(c)} = \frac{\alpha(f^k(c))}{Df^k(c)} - \alpha(c).
\]
Since \( \alpha(c) = 0 \) and \( \alpha \) is bounded we get
\[
\frac{J(f,c,v)}{Df(c)} = \sum_{j=0}^{\infty} \frac{v(f^j(c))}{Df^{j+1}(c)} = 0.
\]

The representation of \( \alpha \) in Theorem 8.1.B may not be unique. However

**Corollary 8.2.** Let \( \alpha \) and \( \phi \in B^\beta(C) \) be as in Theorem 8.1.C. Then there is \( \gamma \in L^\infty(I) \) such that
\[
\phi - \frac{Dv - D^2f \cdot \alpha}{Df} = \gamma \circ f - \gamma.
\]

**Proof.** By Theorem 8.1 we have
\[
\tilde{g}(x) + \int_{c_0,x} \sum_{n=0}^{p-1 n+1 dm} = \alpha(x) = g(x) + \int_{c_0,x} \sum_{n=0}^{\infty} \sum_{i=0}^{p-1} f^{n+p+i} dm,
\]
where \( g, \tilde{g} \) are Lipschitz functions and
\[
\tilde{\phi} = \frac{Dv + D^2f \cdot \alpha}{Df}.
\]
In particular
\[
b(x) = \int_{c_0,x} \sum_{n=0}^{\infty} \sum_{i=0}^{p-1} (\phi - \tilde{\phi}) \circ f^{n+p+i} dm
\]
is a Lipschitz function. By Theorem 8.1 there is a Lipschitz function \( \tilde{g} \) and \( w \in B^{1+\beta}(C) \) such that the Lipschitz function \( \tilde{\alpha} = \tilde{g} + b \) satisfies
\[
w = \tilde{\alpha} \circ f - Df \cdot \tilde{\alpha}.
\]
Deriving this expression with respect to \( x \) we obtain
\[
Dw = D\tilde{\alpha} \circ f \cdot Df - D^2f \cdot \tilde{\alpha} - Df \cdot D\tilde{\alpha},
\]
so \( D\tilde{\alpha} \in L^\infty(I) \) satisfies the Livsic cohomological equation
\[
\frac{Dw + D^2f \cdot \tilde{\alpha}}{Df} = D\tilde{\alpha} \circ f - D\tilde{\alpha}.
\]

The following theorem characterizes deformations

**Theorem 8.3** (Characterization of deformations). Let \( f_t \in B^k_{exp}(C) \), with \( t \in (c,d) \), be a \( C^j \) family. The following statements are equivalent.

A. The set of critical relations \( R_t \) does not depend on \( t \).

B. There exists a family of homeomorphisms \( h_t : I \to I \) such that \( h_t \circ f_0 = f_t \circ h_t \), that is, \( f_t \) is a deformation.
C. There is a family of conjugacies $h_t$ as in $B$, such that
\[(x, t) \mapsto h_t(x)\]
is a continuous function and for each $x \in I$ we have that $h_t(x)$ is $C^{k-1+\text{Lip}}$ on the variable $t$.

D. We have that $J(f_t, x, \partial_s f_s|_{s=t}) = 0$ for every $i \leq n$, $x \in \hat{C}$ and $t \in (c, d)$. Moreover the family $f_t$ is Lasota-Yorke stable.

E. For every $t \in (a, b)$ there is a continuous function $\alpha_t : [a, b] \to \mathbb{R}$ such that
\[(8.36) \quad \partial_s f_s|_{s=t} = \alpha_t \circ f_t - Df_t \cdot \alpha_t\]
on $\hat{I}$ satisfying $\alpha(c_i) = 0$ for every $0 \leq i \leq n$. Indeed $\alpha_t$ is Log-Lipschitz continuous and the constants in the Log-Lipschitz condition are such that
\[\sup_{t \in J}|\alpha_t|\text{LL} < \infty\]
for compact intervals $J$ of $(c, d)$. For every $x \in [a, b]$ there is the unique solution $h_t(x)$ of the initial value problem
\[(8.37) \quad \begin{cases} h_t(x) = \alpha_t(h_t(x)) \\ h_0(x) = x. \end{cases}\]
then we have
- $h_t$ is defined for every $(x, t) \in [a, b] \times (c, d)$ and $h([a, b]) = [a, b]$.
- For every $t \in (c, d)$ we have that $h_t : [a, b] \to [a, b]$ is a homeomorphism,
- We have $h_t \circ f_0 = f_t \circ h_t$ on $[a, b]$.
- For every compact interval $J \subset (c, d)$ there is $C_8 \geq 0$ such that
\[(8.38) \quad |h_t \circ h_t^{-1}(x) - h_t \circ h_t^{-1}(y)| \leq \exp(-C_8|t-t_0|)|x-y|\exp(-C_8|t-t_0|).
\]
and
\[(8.39) \quad |h_t \circ h_t^{-1}(x) - h_t \circ h_t^{-1}(y)| \leq \exp(-C_8|t-t_0|)|x-y|\exp(-C_8|t-t_0|).\]

**Lemma 8.4.** For every $f \in \mathcal{B}^e_{\exp}(C)$ the pre-periodic points are dense in $[a, b]$.

**Proof.** Let $\mu$ be an absolutely continuous ergodic probability of $f$. Then the support of $\mu$ is a finite union of intervals up to a zero measure set (Boyarsky and Góra [8]). Del Magno, Dias, Duarte and Galvão [36] proved that periodic points are dense on the support of $\mu$. Furthermore the union of the basins of these ergodic measures coincides with $[a, b]$. It is easy to see that for almost every point $x \in [a, b]$ we have that for every $n \in \mathbb{N}^+$ there is an open interval $I_n$ such that $f^n$ is a diffeomorphism on $I_n$ and $x \in I_n$. So for almost every point $x \in [a, b]$ we have that $f^n(x)$ belong to the interior of the support of one of those ergodic measures. So we can approximate $f^n(x)$ by periodic point and consequently we can approximate $x$ by preperiodic points. \ 

**Lemma 8.5.** Let $Q$ be a compact subset of $\mathcal{B}^e_{\exp}(C)$, $j \geq 0$ and $\ell \geq 2$. Let
\[C_k(f) = \bigcup_{i=0}^{k-1} f^{-i}C\]
Define
\[\lambda_Q = \inf_{f \in Q} \inf_{x \in I} |Df(x)| > 1.\]
Let \( k_Q \in \mathbb{N}^* \) such that \( \lambda^{k_Q} > 4 \). Suppose

\[
m_Q = \inf_{f \in Q} \inf_{x,y \in C_{k_Q}(f)} |f^k(x^+) - f^k(y^-)| > 0.
\]

Then we can choose constants in the Lasota-Yorke inequality for \( f \in Q \) that depends only on \( \lambda_Q, m_Q \) and \( N_Q \)

\[
N_Q = \sup_{f \in Q} |D^2 f^k|_{L^\infty} + |D f^k|_{L^\infty}.
\]

Moreover

\[
\sup_{f \in Q} p(f) < \infty.
\]

**Proof.** It is easy to check in the proof of Lasota-Yorke inequality in Broise [10] that constants in the Lasota-Yorke inequality can be chosen depending only on \( \lambda_Q, m_Q, N_Q \). To show the upper bound on \( p(f) \), suppose that there is \( f_n \in Q \) such that \( \lim_n p(f_n) = \infty \). That implies that \( \Lambda_{f_n} \) has a cyclic group with \( p(f_n) \) elements. So given a closed arc \( J \subset S^1 \), we have that for \( n \) large enough there is \( \lambda_n \in J \cap \Lambda_{f_n} \), so there is \( v_n \in BV \) with \( |v_n|_{L^1} = 1 \) such that \( L_{f_n} v_n = \lambda_n v_n \). Without loss of generality we can assume \( \lim_n f_n = f \in Q \). The uniformity in the constants of Lasota-Yorke inequality for \( f_n \) easily implies that \( \sup_n |v_n|_{BV} < \infty \), so we can assume that \( \lim_n v_n = v \) in \( BV \), with \( |v|_{L^1} = 1 \), \( \lim_n \lambda_n = \lambda \in J \), and consequently \( L_f v = \lambda v \), so \( \lambda \in \Lambda_f \cap J \). Since \( J \) is arbitrary, it follows that \( S^1 \) is contained in the spectrum of \( f \), which is not possible. So \( \sup_{f \in Q} p(f) < \infty \). \( \square \)

The only harmless distinction of the following statement with Theorem 5.2.1 in Chemin [12] is that \( \alpha_t \) is not defined for every \( t \in \mathbb{R} \).

**Proposition 8.6** (Theorem 5.2.1 in Chemin [12]). Consider the ordinary differential equation

\[
(8.40) \quad \begin{cases} h_t(x) = \alpha_t(h_t(x)) \\ h_0(x) = x. \end{cases}
\]

such that

- The function \( (x,t) \mapsto \alpha_t(x) \) is continuous,
- There are \( C_8 \) and \( C_9 \) such that functions \( \alpha_t : \mathbb{R} \to \mathbb{R} \) satisfy

\[
|\alpha_t(x) - \alpha_t(y)| \leq C_8 |x - y|(1 - \ln |x - y|)
\]

for every \( x, y \) such that \( |x - y| < 1 \), and

\[
|\alpha_t(x)| \leq C_9
\]

for every \( x \in \mathbb{R} \) and \( t \in (c, d) \).

Then (8.40) is uniquely integrable and it has a solution \( h_t(x) \) defined for every \( t \in (c, d) \) and moreover

\[
|h_t(x) - h_t(y)| \leq e^{1 - \exp(-C_8 t)} |x - y|^{\exp(-C_8 t)}
\]

provided \( |h_0(x) - h_0(y)| < 1 \) for every \( a \in [0, t] \) and \( t \geq 0 \).
Proof. Note that there are solutions $h_t(x)$ defined for every $t \in (c,d)$ since $(x,t) \mapsto \alpha_t(x)$ is a continuous and bounded function. Now apply the same methods of the proof of Theorem 5.2.1 in Chemin [12].

Proof of Theorem 8.3. Due Lemma 8.4 we can use the same argument in the proof of Theorem 1 in Baladi and S. [5] to show that $A$, $B$ and $C$ are equivalent. Note that $E \implies A$ is obvious.

$C \implies D$. The proof that $C$ implies $J(f_t, x, \partial_x f_s|_{s=t}) = 0$ for every $i \leq n$, $c \in \hat{C}$ and $t \in (a,b)$ is also quite similar to [5, Theorem 1], so we skip it.

It remains to show that $f_t$ is Lasota-Yorke stable. Indeed, note that $C$ implies that if $C_k(t)$ is the set of critical points of $f_t^k$ then $h_t(C_k(0)) = C_k(t)$ and consequently for every compact subset $Q \subset (a,b)$ we have
\[
\inf_{t \in Q} \sup_{x,y \in C_k(t)} |f_t^k(x^+) - f_t^k(y^-)| > 0,
\]
so by Lemma 8.5 we conclude that $f_t$ is Lasota-Yorke stable.

$D \implies E$. By Theorem 8.1 there are log-Lipchitzian solutions $\alpha_t$ for (8.36) such that $\alpha_t(c) = 0$ for every $c \in \hat{C}$ and $\sup_{i \in J} |\alpha_t|_{\log-Lip} < \infty$ and $\sup_{i \in J} |\alpha_t|_{L^\infty(m)} < \infty$ on any compact interval $J$. We can extend $\alpha_t$ to be zero outside $I$ in such way that $\alpha_t$ became log-Lipchitzian on $\mathbb{R}$ with a similar uniform bound on the log-Lipchitzian norm and the sup norm. Note that
\[
(x, t) \mapsto \alpha_t(x)
\]
is continuous. By Osgood we have that (8.37) has a unique solution $h_t(x)$ defined for every $t \in (a,b)$. Note that $h_t$ is defined for every $t \in (a,b)$ since $\alpha_t(x)$ are uniformly bounded on compact intervals in $(a,b)$. Note also that since $h_t(0) = 0$ and $h_t(1) = 1$ for every $t$ we conclude that $h_t(x) \in (0,1)$ for every $x \in (0,1)$ and $t \in (a,b)$.

Let $x \in [0,1]$. It follows from (8.36) that $h_t(f_0(x))$ and $f_t(h_t(x_0))$ are both solutions of $\dot{y} = \alpha_t(y)$ with initial condition $y(0) = f_0(x) \in [0,1]$ so the unique integrability implies that $h_t \circ f_0 = f_t \circ h_t$ (the verification of this when solutions cross critical points is a slightly more delicate, see Baladi and S. [5] for details).

Let $J \subset (a,b)$ be a compact interval and
\[
C_8 = \sup_{t \in J} |\alpha_t|_{LL}.
\]
By Proposition 8.6 we have that (8.37) defines a flow that satisfies
\[
|h_t(x) - h_t(y)| \leq e^{1 - \exp(-C_8 t)}|x - y|\exp(-C_8 t)
\]
for $t \geq 0$. This proves (8.38) for $t_0 = 0$ and $t \geq 0$. For a general $t_0$, note that it is enough to show (8.38) and (8.39) when $t \geq t_0$. To show (8.38) for $t \geq t_0$ consider the smooth family $\tilde{f}_t = f_{t_0+t}$, apply the same argument to $\tilde{f}_t$ and note that $\tilde{h}_t \circ f_0 = \tilde{f}_t \circ \tilde{h}_t$, where $\tilde{h}_t = h_{t_0+t} \circ h_{t_0}$. Then we obtain
\[
|h_{t_0+t} \circ h_{t_0}^{-1}(x) - h_{t_0+t} \circ h_{t_0}^{-1}(y)| \leq e^{1 - \exp(-C_8 t)}|x - y|\exp(-C_8 t)
\]
for every $t \geq 0$ such that $t_0, t_0 + t \in J$. 

To show (8.38), fix \( t \geq t_0 \). One can apply a similar argument to the family \( \hat{f}_s = f_{t-s} \) since in this case \( h_s \circ f_0 = \hat{f}_s \circ h_s \), where \( h_s = h_{t-s} \circ h_t^{-1} \), and we obtain

\[
|h_t \circ h_t^{-1}(x) - h_{t-s} \circ h_t^{-1}(y)| \leq e^{1-\exp(-Cs)}|x-y|\exp(-Cs)
\]

for \( s \geq 0 \) such that \( t, t-s \in J \). Choosing \( s = (t-t_0) \) we obtain

\[
|h_{t_0} \circ h_t^{-1}(x) - h_{t_0} \circ h_t^{-1}(y)| \leq e^{1-\exp(-Cs(t-t_0))}|x-y|\exp(-Cs(t-t_0)).
\]

which implies (8.39) for \( t \geq t_0, t, t_0 \in J \). \( \square \)

9. Flexibility of multipliers

**Proposition 9.1.** Let \( f \in \mathcal{B}_c^2(\hat{C}) \). Consider the set indexed family \( \mathcal{F} \) containing all functionals of the following types

A. Functionals of the form \( \Psi_{c,0}(v) = J(f, c, v) \), with \( c \in \hat{C} \).

B. Functionals of the form

\[
\Psi_{\Omega_i}(v) = \sum_{j=0}^{m-1} \frac{Dv(f^j(p)) + D^2f(f^j(p)) \cdot \alpha_{v, j, p}}{Df(f^j(p))}
\]

where \( p \in \hat{I} \) is a periodic point with period \( m \) and

\[
\alpha_{v, p, j} = -\sum_{k=j}^{\infty} \frac{v(f^k(p))}{Df^{k+1}(f^j(p))}.
\]

Then \( \mathcal{F} \) is a linear independent indexed family in \( \mathcal{B}^\ell(\hat{C})^* \), for every \( \ell \geq 1 \).

**Proof.** It is enough to show the following statement. Given a finite sequence of periodic points \( p_1, \ldots, p_k \) in \( \text{distinct} \) orbits, and \( c_1, \ldots, c_{2n-2} \) be an enumeration of the elements of \( \hat{C} \), the linear transformation

\[
T: \mathcal{B}^\ell(\hat{C}) \to \mathbb{R}^{k+2n-2}
\]

given by

\[
T(v) = (\Psi_{\Omega(p_1), 1}(v), \ldots, \Psi_{\Omega(p_k), 1}(v), J(f, c_1, v), \ldots, J(f, c_1, v))
\]

has \( \mathbb{R}^{k+q} \) as its image. Of course it is enough to show that \( T(\mathcal{B}^\ell(\hat{C})) \) is dense in \( \mathbb{R}^{k+q} \). Let \( m_i \) be the period of \( p_i \).

Let \( u_0 \) and \( u_1 \) be \( C^\infty \) functions defined in \( \mathbb{R} \) with support in \([-1, 1]\), such that \( u_0(0) = Du_1(0) = 1, Du_0(0) = u_1(0) = 0 \) and \( |u_i|_\infty \leq 1 \) for \( i = 0, 1 \). For every \( x_0 \in \hat{I} \setminus \hat{C} \) and \( \delta > 0 \) define

\[
u(x_0, a, b, \delta, x) = au_0((x-x_0)/\delta) + b\delta u_1((x-x_0)/\delta).
\]

Note that the support of this function is in \([-\delta, \delta] \), and

\[
u(x_0, a, b, \delta, x_0) = a,
D_x \nu(x_0, a, b, \delta, x_0) = b.
\]

If \( x_0 = c^+ (c^-) \), with \( c \in C \), define \( u(x_0, a, b, \delta, x) \) as before for \( x \geq 0 (x \leq 0) \), and 0, otherwise.

Given \( \epsilon > 0 \), choose \( N \) such that

\[
\sum_{i=N}^{\infty} \frac{2n-2}{|D^2f^i(x)|} < \epsilon/2
\]
for every \( x \in \hat{I} \). Let
\[
(e_1, \ldots, e_k, w_1, \ldots, w_{2n-2}) \in \mathbb{R}^{k+2n-2}.
\]
For \( c \in \hat{C} \) define
\[
N_c = \min \{N \cup \{k : f^k(c) \in \hat{C}, \ k \geq 1\}\}.
\]
Consider the set
\[
\Omega = \{f^i(c), \ c \in \hat{C}, \ i \leq N_c\} \cup \bigcup_i \mathcal{O}(p_i)
\]
and
\[
\delta_0 = \min_{x,y \in \Omega, x \neq y} |x - y|.
\]
We are going to define \( a_{x_0}, b_{x_0} \) for every \( x_0 \in \Omega \). Define \( a_{c_i} = w_i \) for every \( i \leq 2n-2 \).
And for all \( x_0 \in \Omega \) define \( a_{x_0} = 0 \). For each \( x_0 \in \mathcal{O}(p_i) \) define
\[
\beta_{x_0} = -\sum_{k=0}^{\infty} \frac{a_{f^k(x_0)}}{Df^{k+1}(x_0)}.
\]
If \( x_0 \in \mathcal{O}(p_i) \) we define \( b_{x_0} = -D^2f(x_0)\beta_{x_0} \) for \( x_0 \neq p_i \), and \( b_{p_i} = e_i Df(p_i) - D^2f(p_i)\beta_{p_i} \). For every \( x_0 \in \Omega \) \( \setminus \cup_i \mathcal{O}(p_i) \)
define \( b_{x_0} = 0 \).

Given \( \delta \in (0, \delta_0/2) \) define
\[
v_\delta(x) = \sum_{x_0 \in \Omega} u(x_0, a_{x_0}, b_{x_0}, \delta, x).
\]
One can see that \( v_\delta(x) = a_x \) and \( Dv_\delta(x) = b_x \) for every \( x \in \Omega \). In particular
\[
\alpha_{v_\delta, p} = \beta_{f(p_i)}
\]
for every \( x \in \mathcal{O}(p_i) \), \( i \leq k \), and consequently
\[
\Psi_{\mathcal{O}(p_i), 1}(v) = e_i.
\]
Since
\[
\lim_{\delta \to 0^+} |v_\delta|_\infty = 1
\]
we have that (9.41) implies that for \( \delta \) small enough
\[
|J(f, c_i, v_\delta) - w_i| < \epsilon
\]
for every \( i \leq 2n - 2 \). This shows that \( T(B'(C)) \) is dense in \( \mathbb{R}^{k+q} \). \( \square \)

**Remark 9.2.** Suppose that \( f_t \) is a smooth family in the topological class of \( f_0 \) such that \( f_t = f_0 + tv + o(t) \) and \( p \) is a \( f_0 \)-periodic point. Let \( p_t \) be the smooth continuation of \( p \), that is, \( t \mapsto p_t \) is smooth and \( f^m_t(p_t) = p_t \). Then
\[
\partial_t f^j_t(p_t)|_{t=0} = \alpha_{v_t, p, j}
\]
and
\[
\partial_t \log |Df^m_t(p_t)|_{t=0} = \Psi_{\mathcal{O}(p_t), 1}(v).
\]
So Proposition 9.1 (and Theorem 10.2) tell us that in the topological class of \( f_0 \) we can perturb the multipliers of periodic points in the independent way.
10. Topological classes are Banach manifolds

Lemma 10.1. Let $f \in B^k_{\exp}(C)$ and $\{w_c\}_{c \in \hat{C}} \subset B^k(C)$ be such that $J(f, w_c, c) = 1$ and $J(f, d, w_c) = 0$ for $d \neq c$. Then for every $v \in \ell^\infty(\mathcal{O})$ there is an unique vector $(t_c)_{c \in \hat{C}} \in \mathbb{R}^{\hat{C}}$ such that

$$J(f, c, (v(a) + \sum_{c \in \hat{C}} t_c w_c(a))_{a \in \mathcal{O}}) = 0$$

for every $c \in \hat{C}$ and there is an unique $(\alpha(a))_{a \in \mathcal{O}} \in \ell^\infty(\mathcal{O})$ such that $\alpha(c) = 0$ for every $c \in \hat{C}$ and

$$v(a) + \sum_{c \in \hat{C}} t_c w_c(a) = \alpha(f(a)) - Df(a)\alpha(a).$$

Proof. The existence of the set $\{w_c\}_{c \in \hat{C}}$ follows from Proposition 9.1. It is easy to see that the unique solution of (10.42) is $t_c = -J(f, v, c)$. Let

$$w = v + \sum_{c \in \hat{C}} t_c w_c.$$

For every $a \in \mathcal{O}$ define $\alpha(c) = 0$ for $c \in \hat{C}$,

$$\alpha(a) = -\sum_{i=0}^\infty \frac{w(f^i(a))}{Df^i(f(a))}$$

if $f^i(a) \notin \hat{C}$ for every $i \geq 0$, and

$$\alpha(a) = -\sum_{i=0}^{N-1} \frac{w(f^i(a))}{Df^i(f(a))}$$

if $f^N(a) \in \hat{C}$ and $f^i(a) \notin \hat{C}$ for every $i < N$. Then $\alpha$ satisfies (10.43). If $\alpha$ is also a solution of (10.43) then $\beta = \alpha - \hat{\alpha}$ satisfies $\beta(c) = 0$ for every $c \in \hat{C}$ and $Df(b)\beta(b) = \alpha(f(b))$ for every $b \notin \hat{C}$. It easily follows that $\beta = 0$, so $\alpha = \alpha$. \qed

Theorem 10.2 (Topological classes are Banach manifolds with finite codimension). Let $f \in B^k_{\exp}(C)$, with $k \geq 3$, be locally Lasota-Yorke stable. Then the topological class $\mathcal{T}(f) \subset B^k_{\exp}(C)$ is a $C^\infty$ Banach manifold modelled on the horizontal direction $E^h(f)$, where $\bar{r} = [k/2]$, and with codimension $2n - 2$.

Proof. Our approach is to use the Implicit Function Theorem. To this end, choose $f_0 \in \mathcal{T}(f)$. We will show that there is a neighbourhood $W$ of $f_0$ such that $\mathcal{T}(f) \cap W$ is defined implicitly by an equation involving certain function $G$ we now define.

Let $\{w_c\}_{c \in \hat{C}} \subset B^k(C)$ be such that $J(f_0, w_c, c) = 1$ and $J(f_0, d, w_c) = 0$ for $d \neq c$. Let $U \subset E^h(f_0) \times \mathbb{R}^{\hat{C}}$ be an open neighbourhood of 0 in $E^h_{\bar{r}} \times \mathbb{R}^{\hat{C}}$ such that for every

$$(v, \{\beta_c\}_{c \in \hat{C}}) \in U$$

we have that

$$f(v, \{\beta_c\}_{c \in \hat{C}}) = f_0 + v + \sum_{c \in \hat{C}} \beta_c w_c \in B^k(C)$$
satisfies
\[ \theta = \inf_{(v, \{ \beta_i \}_{i \in C}) \in U} |Df_{(v, \{ \beta_i \}_{i \in C})}|_{\infty} > 1. \]

Moreover note that
\[ (v, \{ \beta_i \}_{i \in C}) \in E^k(f_0) \times \mathbb{R}^\hat{C} \rightarrow f_{(v, \{ \beta_i \}_{i \in C})} \in B^k(C) \]
parametrizes \( B^k(C) \), so from now on we use \((v, \{ \beta_i \}_{i \in C})\) to represent \( f_{(v, \{ \beta_i \}_{i \in C})}\) without further notice. Let
\[ I_i = (c_i, c_{i+1}) \]
and
\[ \hat{I}_i = \{ c_i^+, c_{i+1}^- \} \cup \{ x^\pm : x \in (c_i, c_{i+1}) \}. \]

By Merrien [43] (see also Fefferman [20] for a historical account of related results) one can find a bounded linear transformation
\[ T_i : C^k[c_i, c_{i+1}] \rightarrow C^k(\mathbb{R}) \]
that such that \( T_i(g) \) is an extension of \( g \). Let
\[ g_{(i,v,\{ \beta_i \}_{i \in C})} : \mathbb{R} \rightarrow \mathbb{R} \]
be defined by
\[ g_{(i,v,\{ \beta_i \}_{i \in C})} = T_i(f_{(v,\{ \beta_i \}_{i \in C})}/[c_i, c_{i+1}]). \]

\((v, \{ \beta_i \}_{i \in C}) \rightarrow g_{(i,v,\{ \beta_i \}_{i \in C})}\)
is Fréchet \( C^\infty\)-differentiable (indeed, an affine map) considering the product norm in \( B^k(C) \times \mathbb{R}^\hat{C} \) and the \( C^k(\mathbb{R})\)-norm in its image. Reducing \( U \) if necessary, there are \( \delta > 0 \) and \( \epsilon > 0 \) such that \( g_{(i,v,\{ \beta_i \}_{i \in C})} \) is a diffeomorphism on \([c_i - \delta, c_{i+1} + \delta]\)
for every \( i \) and \((v, \{ \beta_i \}_{i \in C}) \in U \) and
\[ \theta_1 = \inf_{(v, \{ \beta_i \}_{i \in C}) \in U} \inf_{x \in [c_i - \delta, c_{i+1} + \delta]} |Dg_{(i,v,\{ \beta_i \}_{i \in C})}(x)|_{\infty} > 1, \]
and moreover
\[ \{ y : \text{dist}(y, [f_0(c_i^+, f_0(c_{i+1}^-))] < \epsilon \} \subset g_{(i,v,\{ \beta_i \}_{i \in C})}([c_i - \delta, c_{i+1} + \delta]). \]
for every \( i \) and \((v, \{ \beta_i \}_{i \in C}) \in U \). Let
\[ J_i = \{ y : \text{dist}(y, [f_0(c_i^+, f_0(c_{i+1}^-))] \leq \epsilon/2 \}. \]

So we can consider the “inverse branches”
\[ \phi_{i,v,\{ \beta_i \}_{i \in C}} : J_i \rightarrow \mathbb{R} \]
defined by \( g_{(i,v,\{ \beta_i \}_{i \in C})} \circ \phi_{i,v,\{ \beta_i \}_{i \in C}}(x) = x \). By Farkas and Garay [18][19] for every \( r < k \) the map
\[(v, \{ \beta_i \}_{i \in C}) \rightarrow \phi_{i,v,\{ \beta_i \}_{i \in C}} \]
is \( C^r \) Fréchet differentiable considering the product norm in \( B^k(C) \times \mathbb{R}^\hat{C} \) and the \( C^{k-r}(J_i)\)-norm in its image. Let
\[ \mathcal{O} = \bigcup_{i \geq 0} f_0(J_i). \]
Define \( s : \hat{I} \rightarrow \mathbb{N} \) by \( s(a) = i \) if \( a \in \hat{I}_i \). Consider
\[ \ell^\infty(\mathcal{O}) = \{(x_a)_{a \in \mathcal{O}} \in \mathbb{R}^{\mathcal{O}} : \sup_{a \in \mathcal{O}} |x_a| < \infty \} \]
endowed with the supremum norm, its affine subspace
\[ \ell_c^\infty(\mathcal{O}) = \{(x_a)_{a \in \mathcal{O}} \in \ell^\infty(\mathcal{O}) : x_{c+} = c \text{ for every } c \in C\}. \]
and its corresponding tangent space
\[ \ell_0^\infty(\mathcal{O}) = \{(x_a)_{a \in \mathcal{O}} \in \ell_0^\infty(\mathcal{O}) : x_{c^z} = 0 \text{ for every } c \in C\}. \]

If
\[ P = (p_a)_{a \in \mathcal{O}}, \]
where \( p_a = a \) for every \( a \in \mathcal{O} \), then \( P \in \ell_0^\infty(\mathcal{O}) \) and there is \( \eta > 0 \) such that if \( X = (x_a)_{a \in \mathcal{O}} \in \ell_0^\infty(\mathcal{O}) \) satisfies
\[ |X - P|_{\ell_0^\infty(\mathcal{O})} < \eta \]
then \( x_{f_0(a)} \in \mathcal{J}_{s(a)} \). Consequently if we denote
\[ W = \{ X \in \ell_0^\infty(\mathcal{O}) : |X - P|_{\ell_0^\infty(\mathcal{O})} < \eta \} \]
we can define
\[ G: U \times W \to \ell_0^\infty(\mathcal{O}) \]
by
\[ G(v, \{ y_\beta \}_{c \in \check{C}}, (x_a)_{a \in \mathcal{O}}) = (y_a)_{a \in \mathcal{O}}, \]
where
\[ y_a = \phi_{s(a), v, \{ y_\beta \}_{c \in \check{C}}} (x_{f_0(a)}) - x_a, \]
for \( a \in \mathcal{O} \setminus \check{C} \), and
\[ y_{a^\pm} = \phi_{s(a^\pm), v, \{ y_\beta \}_{c \in \check{C}}} (x_{f_0(a^\pm)}) - a, \]
if \( a \in C \). For \( 1 \leq r < k \) we have that \( G \) is \( \check{r} \)-Fréchet differentiable, where \( \check{r} = \min\{k - r, r\} \). Choosing \( r = \lfloor k/2 \rfloor \) we have \( \check{r} = \lfloor k/2 \rfloor \). Note that
\[ G(0, 0, P) = 0. \]

For \( c \in \check{C} \) we have
\[ \partial_{\beta_c} G(0, 0, P) = \left( \frac{-w_c(a)}{D f_0(a)} \right)_{a \in \mathcal{O}}, \]
and for \( b \in \mathcal{O} \setminus \check{C} \)
\[ \partial_{x_b} G(0, 0, P) = \left( y_{a^b} \right)_{a \in \mathcal{O}}, \]
where
\[ y_{a^b} = \begin{cases} \frac{1}{D f_0(a)} & \text{if } f_0(a) = b \text{ and } a \neq b, \\ \frac{1}{D f_0(a)} & \text{if } f_0(a) = b \text{ and } a \in \check{C}, \\ \frac{1}{D f_0(a)} - 1 & \text{if } a = f_0(a) = b \notin \check{C}, \\ -1 & \text{if } f_0(a) \neq b \text{ and } a = b \notin \check{C}, \\ 0 & \text{in other cases}. \end{cases} \]

So the directional derivative of \( G \) with respect to the subspace \( \{0\} \times \mathbb{R}^{\check{C}} \times \ell_0^\infty(\mathcal{O}) \)
with \( 0 \in E^h(f_0) \), is the linear transformation
\[ \mathcal{D}: \mathbb{R}^{\check{C}} \times \ell_0^\infty(\mathcal{O}) \to \ell_0^\infty(\mathcal{O}) \]
given by
\[ \mathcal{D}( (t_c)_{c \in \check{C}}, (\alpha(b))_{b \in \mathcal{O}} ) = \sum_{c \in \check{C}} \partial_{\beta_c} G(0, 0, P) \cdot t_c + \sum_{b \in \mathcal{O} \setminus \check{C}} \partial_{x_b} G(0, 0, P) \cdot \alpha(b) = (z_a)_{a \in \mathcal{O}}, \]
where for every \((t_c)_{c \in \hat{C}}\) and \((\alpha(b))_{b \in \mathcal{O}} \in \ell_0^\infty(\mathcal{O})\) the \(a\)-th component \(z_a\) is

\[
(10.44) \quad z_a = -\frac{1}{Df_0(a)} \sum_{c \in \hat{C}} t_c w_c(a) + \frac{\alpha(f_0(a))}{Df_0(a)} - \alpha(a).
\]

We claim that \(\mathcal{D}\) is invertible. To prove that \(T\) is a subjective, let \((z_a)_{a \in \mathcal{O}} \in \ell^\infty(\mathcal{O})\).

Then there is an unique vector \((t_c)_{c \in \hat{C}} \in \mathbb{R}^\hat{C}\) such that

\[
J(f_0, c, (Df_0(a)z_a + \sum_{c \in \hat{C}} t_c w_c(a))_{a \in \mathcal{O}}) = 0
\]

for every \(c \in \hat{C}\). By Lemma 10.1 there is an unique \((\alpha(a))_{a \in \mathcal{O}} \in \ell_0^\infty(\mathcal{O})\) such that

\[
Df_0(a)z_a + \sum_{c \in \hat{C}} t_c w_c(a) = \alpha(f_0(a)) - Df_0(a)\alpha(a)
\]

for every \(a \notin \hat{C}\) and \(\alpha(c) = 0\) for every \(c \in \hat{C}\). So \(((t_c)_{c \in \hat{C}}, (\alpha(b))_{b \in \mathcal{O}})\) satisfies (10.44). To prove the injectivity of \(\mathcal{D}\), suppose that

\[
\mathcal{D}((t_c)_{c \in \hat{C}}, (\alpha(a))_{a \in \mathcal{O}}) = \mathcal{D}((q_c)_{c \in \hat{C}}, (\beta(a))_{a \in \mathcal{O}}).
\]

Then

\[
\frac{1}{Df_0(a)} \sum_{c \in \hat{C}} (t_c - q_c) w_c(a) = \frac{(\alpha - \beta)(f_0(a))}{Df_0(a)} - (\alpha - \beta)(a).
\]

for every \(a \in \mathcal{O}\). That implies

\[
J(f_0, c, (\sum_{c \in \hat{C}} (t_c - q_c) w_c(a))_{a \in \mathcal{O}}) = 0
\]

for every \(c \in \hat{C}\) and consequently \(t_c = q_c\) for every \(c \in \hat{C}\). By Lemma 10.1 we have \(\alpha - \beta = 0\).

So \(\mathcal{D}\) is invertible. By the Implicit Function Theorem there is an open neighbourhood \(\mathcal{O}\) of \((0, 0, P)\) and a \(C^{k/2}\)-function

\[
\theta: \mathcal{U} \to \mathbb{R}^\hat{C} \times \ell_0^\infty(\mathcal{O})
\]

where \(\mathcal{U}\) is an open neighbourhood of 0 in \(E^h(f_0)\) such that

\[
(10.45) \quad \{(v, (\beta_c)_{c \in \hat{C}}, (x_a)_{a \in \mathcal{O}}) \in \mathcal{O}: G(v, (\beta_c)_{c \in \hat{C}}, (x_a)_{a \in \mathcal{O}}) = 0\}
\]

\[
= \{(v, \theta(v)): v \in \mathcal{U}\}.
\]

It remains to show that this subset is indeed \(\mathcal{T}(f)\) near to \(f_0\). This is not obvious since the definition of \(G\) involves the extension operators \(T_i\).

We claim that for every \(\epsilon > 0\) there is a neighbourhood \(\mathcal{V} \subset \mathcal{U}\) of 0 in \(E^h(f_0)\) such that for each \(v \in \mathcal{V}\) there is \((\beta_c(v))_{c \in \hat{C}}\) such that \(|\beta_c(v)_{c \in \hat{C}}|_{\mathbb{R}^C} < \epsilon\) such that if \(f\) is defined by

\[
(10.46) \quad f_v = f_0 + v + \sum_{c \in \hat{C}} \beta_c(v) w_c,
\]

and we define recursively \(x_v^{f_C} = x_c(v_c)\) for \(c \in C\) and \(x_v^{f_0(a)} = x_a^{f_0(a)}\) we have

- \(f_v\) belongs in the topological class of \(f_0\),
- \((v, (\beta_c(v))_{c \in \hat{C}}, (x_v^{f_0})_{a \in \mathcal{O}})\) belongs to \(\mathcal{O}\),
The claim and (10.45) imply that there is an open neighborhood $\mathcal{W}$ of $f_0$ such that
\[ \{(v,\theta(v)) : v \in \mathcal{V}\} = \{(v, (\beta_c(v)))_{c \in \hat{C}}, (x_a^{f_0})_{a \in \mathcal{O}}) : v \in \mathcal{V}\} = \mathcal{T}(f) \cap \mathcal{W}, \]
so $\mathcal{T}(f) \cap \mathcal{W}$ is a Banach manifold modelled on $E^h(f_0)$. Since $E^h(f_0)$ and $E^h(f_0)$ are isomorphic spaces (once there are subspaces with the same finite codimension) this completes the proof of the theorem.

To prove the claim, note that, reducing $\mathcal{U}$ if necessary, there is an open neighbourhood $\mathcal{U}$ of 0 in $\mathbb{R}^\hat{C}$ such that for every $v \in \mathcal{U}$ and $(\beta_c)_{c \in \hat{C}} \in \mathcal{U}$ and $t \in (-1, 1)$ there is a unique $u = u(t, (\beta_c)_{c \in \hat{C}}) \in \mathcal{U}$ such that
\[ u = w + \sum_{c \in \hat{C}} \zeta_c w_c, \]
\[ J(f, u, c) = 0 \text{ for every } c \in \hat{C}, \]
\[ f = f_0 + tw + \sum_{c \in \hat{C}} \beta_c w_c. \]
So $u$ is a vector field in the open set
\[ \mathcal{A} = \{f_0 + tw + \sum_{c \in \hat{C}} \beta_c w_c, \text{ with } t \in (-1, 1) \text{ and } (\beta_c)_{c \in \hat{C}} \in \mathcal{U}\} \]
of an affine subspace. This vector field is continuous due to Theorem 8.1, so we can find a $C^1$ integral curve $g_t$, $|t| < \epsilon$, with $g_0 = f_0$. Here we can choose $\epsilon > 0$ that does not depends on $v$. By Theorem 8.3 we have that $g_t$ belongs to the topological class of $f_0$ for every $t$. Reducing $\mathcal{U}$ and $\mathcal{U}$ again we can assume that $g_t \in O$ for every $v \in \mathcal{U}$, $|t| < \epsilon$. Notice that $g_t = f_0 + tw + \sum_{c \in \hat{C}} \beta'_c w_c$ for some $\beta'_c \in \mathbb{R}$. Define $\mathcal{V} = \epsilon/2 \mathcal{U}$. For every $u \in \mathcal{V}$ choose $w = 2u/\epsilon$ and $f = g_{t/2}$. If $h$ is the homeomorphism that satisfies $h \circ f_0 = f \circ h$ and $h(c) = c$ for every $c \in C$ then
\[ G(v, (\beta'_c)_{c \in \hat{C}}, (h(a))_{a \in \mathcal{O}}) = 0 \]
since $h(a) \in (c_{s(a)}, c_{s(a)+1})$ for every $a \in \mathcal{O} \setminus \hat{C}$. By Theorem 8.3 once can choose $\mathcal{U}$ small enough such that
\[ \|(h(a))_{a \in \mathcal{O}} - P|_{\mathcal{U} \cap \mathcal{O}}| < \eta. \]
This proves the claim. \qed

11. QUASI-SYMMETRIC CLASSES

One of the main tools in one-dimensional dynamics is quasi-symmetric rigidity. Often the conjugacy $h$ between two real-analytic maps is a quasi-symmetric map; that is, there is $C$ such that
\[ \frac{1}{C} \leq \frac{|h(x + \delta) - h(x)|}{|h(x) - h(x - \delta)|} \leq C \]
for every $x, x + \delta, x - \delta$ in the phase space. The proof that a conjugacy is quasi-symmetric typically uses quite sophisticated methods, including complex analytic extensions of the original real dynamics and quasiconformal maps. In the setting of real-analytic dynamics, a quasi-symmetric conjugacy can be extended to a quasiconformal conjugacy between the extended complex dynamics (a method pioneered
We have
\[ F = \text{quasisymmetric classes, which are also submanifolds of finite codimension.} \]

Beltrami paths, and quasiconformal surgeries. See Sullivan [32, 33], Graczyk and Światek [26], Kozlovski, Shen and van Strien [31], and Clark, van Strien and Trejo [15] for more information.

Piecewise expanding maps, however, have discontinuities (either in the map itself or its derivative) that are an additional difficulty for (and maybe even precludes) the use of complex extension methods. *One may wonder if the topological classes of such maps coincide with their quasisymmetric classes.* We are going to see that this is not always the case. More unexpectedly, some topological classes are *laminated* by quasisymmetric classes, which are also submanifolds of finite codimension.

In this section, we have strong assumptions on the dynamics of the maps. However, we obtain a quite complete description in this setting.

**Proposition 11.1** (Obstruction for quasisymmetric conjugacies). Let \( f \in B_{exp}^k(C) \) be a piecewise expanding map. Suppose there is \( c \in C \setminus \partial I \) such that
\[
\inf_{x \in O^+(f,c^+) \setminus C} \text{dist}(x, \hat{C}) > 0
\]
and there is \( N_{c^+}, M_{c^+} \in \mathbb{N} \) such that
\[
f^{M_{c^+}}(f^{N_{c^+}}(c^+)) = f^{N_{c^+}}(c^+)
\]
and
\[
f^{M_{c^-}}(f^{N_{c^-}}(c^-)) = f^{N_{c^-}}(c^-).
\]
Let \( g \in B_{exp}^k(C) \) be a map such that there is an orientation preserving homeomorphism \( h : I \to I \) such that \( h \circ f = g \circ h \) and \( h(c^\pm) = c^\pm \). If
\[
\frac{\ln |Df^{M_{c^+}}(f^{N_{c^+}}(c^+))|}{\ln |Df^{M_{c^-}}(f^{N_{c^-}}(c^-))|} \neq \frac{\ln |Dg^{M_{c^+}}(g^{N_{c^+}}(c^+))|}{\ln |Dg^{M_{c^-}}(g^{N_{c^-}}(c^-))|}
\]
then \( h \) is not quasisymmetric.

**Proof.** Let \( T \) be a multiple of \( p(f) \), \( N_{c^\pm} \) and \( M_{c^\pm} \) and \( F(x) = f^T(x), G(x) = g^T(x) \). We have \( F^2(c^\pm) = F(c^\pm) \), \( h \circ F = G \circ h \), and
\[
(11.47) \quad \frac{\ln |DF(F(c^+))|}{\ln |DF(F(c^-))|} \neq \frac{\ln |DG(G(c^+))|}{\ln |DG(G(c^-))|}.
\]
Define
\[
d = \frac{1}{2} \inf_{x \in O^+(f,c) \setminus \hat{C}_F} \text{dist}(x, \hat{C}_F) > 0.
\]
Let \( J^1 = [c, c+\delta] \) and \( J^2 = [c-\delta, c] \) and \( q_k = q_k(\delta) \), with \( k = 1, 2 \) be the smallest integers such that
\[
|DF^{q_k}(c_k)||J^k| > \frac{d}{C_5}.
\]
Then \( F^{q_k} \) is a diffeomorphism on \( J^k \) and
\[
|I| \geq |F^{q_k}J^k| \geq \frac{d}{C_5}.
\]
Since \( F \) and \( G \) are conjugated by \( h \) there is \( C_{10}, C_{11} > 0 \) such that
\[
|I| \geq |G^{q_k}h(J^k)| \geq C_{10},
\]
and (due an usual bounded distortion argument)
\[
|DG^{q_k}(c_k)||h(J^k)| > C_{11},
\]
where \( C_{10}, C_{11} \) does not depend on \( h \). As a consequence we have
\[
- \ln |J^k| - q_k \ln |DF(F(c_k))| = O(1)
\]
so since \( |J^1| = |J^2| = \delta \) and \( q_k(\delta) \to +\infty \) when \( \delta \) tends to zero we obtain
\[
\frac{\ln |DF(F(c^+))|}{\ln |DF(F(c^-))|} = \lim_{\delta \to 0^+} \frac{q_1(\delta)}{q_2(\delta)}.
\]
But in other hand
\[
- \ln |h(J^k)| - q_k \ln |DG(G(c_k))| = O(1),
\]
so if \( h \) is quasisymmetric we have \( \ln |h(J^1)| - \ln |h(J^1)| = O(1) \) and
\[
\frac{\ln |DG(G(c^+))|}{\ln |DG(G(c^-))|} = \lim_{\delta \to 0^+} \frac{q_1(\delta)}{q_2(\delta)},
\]
we conclude that
\[
\frac{\ln |DF(F(c^+))|}{\ln |DF(F(c^-))|} = \frac{\ln |DG(G(c^+))|}{\ln |DG(G(c^-))|}
\]
which is impossible due (11.47). \( \square \)

**Remark 11.2.** We may wonder if there are others obstructions to quasisymmetric conjugacy than the one described by Proposition 11.1. Note that this obstruction does not occur in piecewise expanding unimodal maps. Is the conjugacy always quasisymmetric in this case?

We say that \( f \in B_{\exp}^k(C) \) satisfies the Assumption FOorMC if

**Finite orbit or Misiurewicz condition (FOorMC).** We have
\[
\inf_{c \in C} \inf_{x \in \mathcal{O}^+(f,c^\pm) \cap \hat{C}} \text{dist}(x, \hat{C}) > 0
\]
and all \( c \in C \setminus \partial I \) one of the following conditions holds

- **Type I (Finite orbit).** there is \( N_c^+, M_c^+ \in \mathbb{N} \) such that
  \[
  f^{M_c^+} f^{N_c^+} c^\pm = f^{N_c^+} c^\pm.
  \]

- **Type II (Misiurewicz and Continuous).** \( f^i \) is continuous at \( c \) for every \( i \geq 0 \) and there is \( N_c \) \( N_{c^+} = N_{c^-} \) such that \( f^i(c) \notin C \) for every \( i \geq N_c \).

**Proposition 11.3** (Quasisymmetric conjugacies). Let \( f \in B_{\exp}^k(C) \) be a piecewise expanding map satisfying Assumption FOorMC. Let \( g \in B_{\exp}^k(C) \) and suppose there is a homeomorphism \( h \) such that \( h \circ f = g \circ h \) and \( h(c^\pm) = c^\pm \). Suppose that for every Type I critical point \( c \in C \)
\[
\frac{\ln |Df^{M_c^+}(f^{N_c^+}(c^+))|}{\ln |Df^{M_c^-}(f^{N_c^-}(c^-))|} = \frac{\ln |Dg^{M_c^+}(g^{N_c^+}(c^+))|}{\ln |Dg^{M_c^-}(g^{N_c^-}(c^-))|}
\]
Then \( h \) is a quasisymmetric map.

**Proof.** Let \( T \) be a common multiple of \( p(f) \), \( p(g) \), \( N_c \), and \( M_c \) for every \( c \in C \). Let \( F(x) = f^T(x) \). Then \( F \in B_{\exp}^k(C_F) \) and \( G \in B_{\exp}^k(h(C_F)) \), for a finite set \( C_F \), and for every \( c \in C_F \)
- **Type I.** either \( F^2(c^\pm) = F(c^\pm) \) and

\[
\ln\left| \frac{\mathrm{D}F(F(c^+))}{\mathrm{D}F(F(c^-))} \right| = \ln\left| \frac{\mathrm{D}G(h(c^+))}{\mathrm{D}G(h(c^-))} \right|
\]

(11.48)

- **Type II.** or \( F^i \) is continuous at \( c \) for every \( i \geq 0 \) and \( F^i(c) \notin C_F \) for every \( i \geq 1 \).

Furthermore \( F \circ h = h \circ G \),

\[
d_F = \frac{1}{2} \inf_{c \in C_F} \inf_{x \in O^+(F,c) \setminus C_F} \text{dist}(x, \hat{C}_F) > 0
\]

and

\[
d_G = \frac{1}{2} \inf_{c \in C_G} \inf_{x \in O^+(G,c) \setminus C_G} \text{dist}(x, \hat{C}_G) > 0.
\]

Let \( x \in I \) and \( \delta > 0 \) be such that \([x - \delta, x + \delta] \subset I \). We may assume \(|\delta| < d_F \).

Note that there is \( C_{12} > 0 \) and \( C_{13} > 0 \) such that for every interval \( R \subset I \) and \( n \in \mathbb{N} \) where \( F^i(R) \cap C_F = \emptyset \) for \( i < n \) we have

\[
\frac{1}{C_5} \leq \frac{D^n_F(x)}{D^n_F(y)} \leq C_{12}
\]

(11.49)

\[
|| \ln |D^n_F(x)| - \ln |D^n_F(y)|| \leq C_{13} |F^n(x) - F^n(y)|.
\]

(11.50)

for all \( x, y \in R \). Moreover \( G \) has analogous properties. Let \( q \) be the smallest integer satisfying

\[ F^q((x - \delta, x + \delta)) \cap C_F \neq \emptyset. \]

We have

\[ |F^i([x - \delta, x + \delta])| \leq (\min_{z \in I} |DF(z)|)^{-i} \]

Define \( J = F^q([x - \delta, x + \delta]) \). Let \( c \in C_F \) be defined by

\[
\{ c \} = F^q((x - \delta, x + \delta)) \cap C_F.
\]

Denote by \( J^1 \) and \( J^2 \) the right and left connected components of \( F^q([x - \delta, x + \delta]) \setminus \{ c \} \). Due (11.49) there is \( C_{14} > 1 \) such that

\[
\frac{1}{C_{14}} \leq \frac{|J^1|}{|J^2|} \leq C_{14}.
\]

Fix \( y \in [x - \delta, x + \delta] \) such that \( F^q(y) = c \). It follows from (11.49) that

\[
\frac{1}{|DF^q(y)|} \leq 2C_5 \frac{|h|}{|J|}
\]

Define \( c_1 = c^+ \) and \( c_2 = c^- \).

Given an interval \( S = [z_1, z_2] \subset I \) such that \( z_k \) is between \( z_{3-k} \) and \( c_k \), and

\[
\text{dist}(S, c_k) \leq C_{12} |S|,
\]

let \( q(S) \) be the smallest integer satisfying

\[
|DF^{q(S)}(c_k)| |S| > \frac{d_F}{C_{12}(C_{12} + 1)}.
\]

This implies

\[
\ln |DF^{q(S)}(c_k)| + \ln |S| = O(1).
\]

(11.51)
Moreover \( F^q(S) \) is a diffeomorphism on \([z_{3-k}, c_k]\) and

\[
|F^q(S)| \geq \frac{d_F}{C_{12}^2(C_{12} + 1)},
\]

that implies that \( G^q(S) \) is a diffeomorphism on \( h([z_{3-k}, c_k]) \) and there is \( C_{15} > 0 \) such that

\[
|G^q(S)h(S)| > C_{15}
\]

and

\[
|DG^q(S)(c_k)||h(S)| > C_{15},
\]

so

\[
(11.52) \quad \ln |DG^q(S)(c_k)| + \ln |h(S)| = O(1).
\]

Note that

\[
(11.53) \quad |F^i(S)| \leq (\max_{z \in \tilde{l}} |DF(F(z))|)^{-i}.
\]

for every \( i \leq q(S) \).

Let \( J = [y_1, y_2] \). If \( Q_1 \) and \( Q_2 \) are the right and left connected components of \( J \setminus \{F^q(x)\} \) then

\[
(11.54) \quad \frac{1}{C_{12}} \leq \frac{|Q_1|}{|Q_2|} \leq C_{12},
\]

so

\[
\frac{1}{1 + C_{12}} \leq \frac{|Q_1|}{|J|} \leq \frac{C_{12}}{1 + C_{12}}.
\]

Suppose \( c \in Q_1 \) (the case \( c \in Q_2 \) is analogous). Then \( J^1 \subset Q_1 \) and \( Q_2 \subset J^2 \) and

\[
dist(z_2, Q_2) \leq C_{12}|Q_2|.
\]

We consider two cases.

\textit{First case. \( c \) is a type I point.} Then (11.51) and (11.52) imply

\[
q(S) = -\frac{\ln |S|}{\ln |DF(F(c_k))|} + O(1) = -\frac{\ln |h(S)|}{\ln |DG(G(c_k))|} + O(1).
\]

so there is a \( C_{16} \) such that

\[
\frac{1}{C_{16}} |S|^{\frac{\ln |DG(G(c_k))|}{\ln |DF(F(c_k))|}} \leq |h(S)| \leq C_{16}|S|^{\frac{\ln |DG(G(c_k))|}{\ln |DF(F(c_k))|}}.
\]

By (11.48) we can define

\[
r = \frac{\ln |DG(G(c^+))|}{\ln |DF(F(c^+))|} = \frac{\ln |DG(G(c^-))|}{\ln |DF(F(c^-))|},
\]

and consequently

\[
(11.55) \quad \frac{1}{C_{16}} |S|^r \leq |h(S)| \leq C_{16}|S|^r
\]

for every interval \( S \) satisfying the conditions we imposed on \( S \).

So (11.55) implies

\[
\frac{1}{C_{16}} |Q_2|^r \leq |h(Q_2)| \leq C_{16}|Q_2|^r.
\]
Since $Q_1 = [y_1, c] \cup [c, F^q(x)]$ by (11.55) we have

$$|h(Q_1)| = |h(J)| + |h([c, F^q(x)])| \leq C_{16}(|J| + ||c, F^q(x)|| \leq 2C_{16} \max\{|J|, ||c, F^q(x)||\}

\leq 2C_{16}|Q_1|^r.$$

and

$$|h(Q_1)| = |h(J)| + |h([c, F^q(x)])| \geq \frac{1}{C_{16}}(|J| + ||c, F^q(x)|| \geq \frac{1}{C_{16}} \max\{|J|, ||c, F^q(x)||\}

\leq \frac{1}{2C_{16}}|Q_1|^r.$$

so there is $C_{17} \geq 1$ such that

(11.56) \quad \frac{1}{C_{17}} \leq \frac{|h(Q_1)|}{|h(Q_2)|} \leq C_{17},

and the bounded distortion of $G$ implies

(11.57) \quad \frac{1}{C_{12}C_{17}} \leq \frac{|h([x, x + h])|}{|h([x - h, x])|} \leq C_{12}C_{17}.$$

Second case. $c$ is a type II point. In this case

$$\ln |DF^q(c^+)| - \ln |DF^q(c^-)| = O(1)$$

and

$$\ln |DG^q(c^+)| - \ln |DG^q(c^-)| = O(1)$$

for every $q$. So

$$\ln |DF^q(S)(c_1)| + \ln |S| = O(1),$$

$$\ln |DG^q(S)(c_1)| + \ln |h(S)| = O(1).$$

For $S = Q_2$ we obtain

(11.58) \quad \ln |DF^q(Q_2)(c_1)| + \ln |Q_2| = O(1),

$$\ln |DG^q(Q_2)(c_1)| + \ln |h(Q_2)| = O(1).$$

Note if $\tilde{S} = [s_1, c] \cup [c, s_2]$ then $2||s_i, c|| \geq |\tilde{S}|$, for some $i \in \{1, 2\}$ we have

$$\ln |\tilde{S}| = \ln ||s_i, c|| + O(1)$$

$$\ln |\tilde{S}| > \ln ||s_{3-i}, c|| + \ln 2,$$

so

$$\ln |DF^q([s_i, c])(c_1)| + \ln |\tilde{S}| = O(1),$$

and there is $C_{18}$ such that

$$\ln |DF^q([s_{3-i}, c])(c_1)| + \ln |\tilde{S}| > C_{18}.$$

Since $F$ is uniformly expanding we conclude that

$$\ln |\tilde{S}| + \ln |DF^{\min(q([s_i, c]), q([s_{3-i}, c]))}(c_1)| = O(1).$$

We can use a similar argument with $G$ and $h(\tilde{S})$ and obtain

$$\ln |h(\tilde{S})| + \ln |DG^{\min(q([s_i, c]), q([s_{3-i}, c]))}(c_1)| = O(1).$$
Take \( S = Q_1 \). Then
\[
(11.59) \quad \ln |Q_1| + \ln |DF_{\min}(q(J^1), q([c, F^q(x)])) (c_1)| = O(1).
\]
We can use a similar argument with \( \text{Theorem } 11.4 \) (Quasisymmetric deformations) and finally (11.58) to obtain
\[
(11.60) \quad \ln |h(Q_1)| + \ln |DG_{\min}(q(J^1), q([c, F^q(x)])) (c_1)| = O(1).
\]
Since (11.54), (11.59) and (11.58) imply
\[
\ln |DG^q(Q_2) (c_1)| - \ln |DF_{\min}(q(J^1), q([c, F^q(x)])) (c_1)| = O(1)
\]
so the uniform expansion of \( F \) gives us
\[
q(Q_2) - \min q(J^1), q([c, F^q(x)]) = O(1),
\]
and finally (11.58) and (11.60) imply (11.56) and (11.57). This completes the proof.

**Theorem 11.4** (Quasisymmetric deformations). Let \( f_0 \in \mathcal{B}_{exp}^k (C) \) be a piecewise expanding map satisfying Assumption FOorMC. Let \( f_t \in \mathcal{B}_{exp}^k (C) \) be a smooth family, \( t \in (c, d) \). The following statements are equivalent

A. For every \( t \) there is a quasisymmetric map \( h_t \) such that \( h_t (c) = c \) for every \( c \in C \) and \( f_t \circ h_t = h_t \circ f_0 \).

B. For every Type I critical point \( c \in C \setminus \partial I \) and every \( t \) we have
\[
(11.61) \quad \frac{1}{\ln |DF_t^{N_e^+} (f_{N_e^+} (c^+))|} \sum_{i=N_e^+}^{N_e^+ + M_{e^+} - 1} \phi_t (f_t^i (c^+)) \quad \frac{1}{\ln |DF_t^{N_e^-} (f_{N_e^-} (c^-))|} \sum_{i=N_e^-}^{N_e^- + M_{e^-} - 1} \phi_t (f_t^i (c^-)),
\]
where
\[
\phi_t = \frac{Dv_t + D_f f_t \cdot \alpha_t}{Df_t},
\]
\( v_t = \partial_t f_t \) and \( \alpha_t \) is the unique continuous solution of
\[
(11.62) \quad v_t = \alpha_t \circ f_t - D f_t \circ \alpha_t.
\]
Moreover the family \( f_t \) is Lasota-Yorke stable.

C. For every \( t_0 \) and \( x, x + \delta, x - \delta \in I \) we have that
\[
(11.63) \quad \frac{|h_t \circ h_{t_0}^{-1} (x + \delta) - h_t \circ h_{t_0}^{-1} (x)|}{|h_t \circ h_{t_0}^{-1} (x - \delta) - h_t \circ h_{t_0}^{-1} (x)|} \leq (1 + O(|t - t_0|))
\]

**Proof.** Of course \( C \implies A \).

A \implies B. Since \( f_t \in \mathcal{T}(f_0) \) for every \( t \), Theorem 8.3 implies that (11.62) as a unique continuous solution \( \alpha_t \). Proposition 11.1 and A implies
\[
\frac{\ln |DF_t^{M_e^+} (h_t (f_0^{N_e^+} (c^+)))|}{\ln |DF_t^{M_e^-} (h_t (f_0^{N_e^-} (c^-)))|} = \frac{\ln |DF_0^{M_e^+} (f_0^{N_e^+} (c^+))|}{\ln |DF_0^{M_e^-} (f_0^{N_e^-} (c^-))|}
\]
for every $t$ and Type II critical point $c$. Deriving with respect to $t$ we obtain

$$
\frac{1}{\ln |Df_t^{M_+}(h_t(f_0^{N_+}(c^+)))|} \partial_t \ln |Df_t^{M_+}(h_t(f_0^{N_+}(c^+)))| = \frac{1}{\ln |Df_t^{M_-}(h_t(f_0^{N_-}(c^-)))|} \partial_t \ln |Df_t^{M_-}(h_t(f_0^{N_-}(c^-)))|.
$$

Since for every $M$ and $N$ and $c \in C \setminus \partial I$

$$
\partial_t \ln |Df_t^M(h_t(f_0^N(c^\pm)))| = \partial_t \left( \sum_{i=0}^{N-1} \ln |Df_t(h_t(f_0^{N+i}(c^\pm)))| \right)
$$

we have that (11.61) holds. The family $f_t$ is Lasota-Yorke stable due Theorem 8.3. $B \implies C$. $B$ and Theorem 8.3 imply that for every compact interval $K$ we have that $\alpha_t$ are uniformly Log-Lipschitz for $t \in K$. In particular $\alpha_t$ are uniformly $\beta$-Hölder for every $\beta \in (0, 1)$, for $t \in K$. In particular $\phi_t$ are uniformly piecewise $\beta$-Hölder for every $\beta \in (0, 1)$, for $t \in K$. Theorem 7.1 implies that

$$
\alpha_t(x) = H_t(x) + G_t(x) + \int_{[a,b]} \left( \sum_{n=0}^{\infty} \sum_{i=0}^{p-1} \phi_t \circ f^{np+i} \right) dm,
$$

where $H_t$ and $G_t$ are uniformly Lipschitz functions for $t \in K$. G.R. and S. [27, Theorem 4.3] implies that there is $C_{19}$ such that

$$
|\alpha_t(x + \delta) + \alpha_t(x - \delta) - 2\alpha_t(x)| \leq C_{19}|\delta|
$$

for every $t \in K$, $x, x + \delta, x - \delta \in I$. Since due Theorem 8.3 we have that $h_t$ are the solutions of the differential equations $\dot{h}_t = \alpha_t \circ h_t$ with initial condition $h_0(x) = x$, we have by Remann [47, Proposition 8] that $h_t$ is $C_{19}|t|$-quasisymmetric and (11.63) holds for $t_0 = 0$. The case for general $t_0$ follows for an argument similar to those used in the the proof of Theorem 8.3.

Let $f \in \mathcal{B}_{exp}^k(C)$ satisfying assumption FOorMC. Define

$$
\Omega_f = \{ \mathcal{O}^+(c^\pm), \ c \text{ is a Type I critical point} \}.
$$

Let $D_f$ be the dimension of the linear space

$$
\{ f: \Omega \to \mathbb{R}: f(\mathcal{O}^+(c^+)) = f(\mathcal{O}^+(c^-)), \text{ for all } c \in C \text{ that is a Type I critical point} \}.
$$

Note that $D_f$ is a topological invariant.

**Theorem 11.5** (Lamination by quasisymmetric classes). Let $f \in \mathcal{B}_{exp}^k(C)$ be a map satisfying assumption FOorMC. Then

A. Given $g_0$ in the topological class $T$ of $f$ the quasisymmetric class of $g$ is an embedded submanifold $M_g$ of codimension $D_f$ in $T$.  

B. Moreover \( v \in \mathcal{B}^k(C) \) belongs to the tangent space of \( M_g \) at \( g \) if and only if

\[
\frac{1}{\ln |Dg^{M_+}(g^{N_+}(c^+))|} \sum_{i=N_+}^{N_++M_+ -1} \phi(g^i(c^+)) = \frac{1}{\ln |Dg^{M_-}(g^{N_-}(c^-))|} \sum_{i=N_-}^{N_-+M_- -1} \phi(g^i(c^-))
\]

for every \( c \in C \) of Type I and \( \mathcal{O}^+(f, f^{N_+}(c^+)) \neq \mathcal{O}^+(f, f^{N_-}(c^-)) \). Here

\[
\phi = \frac{Dv + D^2 f \alpha}{Dg}
\]

and \( \alpha \) is the only continuous solution of \( v = \alpha \circ g - Dg \alpha \).

Proof. By Theorem 10.2 we have that in a neighborhood \( U \) to \( f \) its topological class \( T \cap U \) is a \( \mathcal{C}^{[k/2]} \) Banach manifold with codimension \( 2n-2 \) modelled over the Banach space \( E^h(f) \). Given \( g_0 \in T \cap U \) let \( M_{g_0} \) be the set of all \( g \in T \cap U \) such that

\[
\frac{\ln |Dg^{M_+}(g^{N_+}(c^+))|}{\ln |Dg^{M_-}(g^{N_-}(c^-))|} = \frac{\ln |Dg_0^{M_+}(g_0^{N_+}(c^+))|}{\ln |Dg_0^{M_-}(g_0^{N_-}(c^-))|}
\]

for every Type I critical point \( c \in C \setminus \partial I \). Let \( e^{Y_c} \) be the right hand side of this expression. By Theorem 11.1 and Proposition 11.3 we have that \( g \in T \cap U \) is conjugate to \( g_0 \) by a quasisymmetric map if and only if \( g \in M_{g_0} \). Consider the function

\[
Q: T \cap U \rightarrow \mathbb{R}^{\Omega_{g_0}}
\]

defined by

\[
Q(g)(\mathcal{O}^+_{g_0}(x)) = \ln |Dg^n(h_g(x))|
\]

where \( \mathcal{O}^+_{g_0}(x) \in \Omega_{g_0} \) is an \( g_0 \)-orbit with period \( n \). Here \( h_g \) is the unique homeomorphism such that \( h_g \circ g_0 = g \circ h_g \). Note that \( Q \) is a \( \mathcal{C}^{[k/2]} \) function. Consider the affine subspace \( S \subset \mathbb{R}^{\Omega_{g_0}} \) given by

\[
\{(s\mathcal{O}^+_{g_0}(x))\mathcal{O}^+_{g_0}(x)\in\Omega_{g_0} : s\mathcal{O}^+_{g_0}(g^{N_+}(c^+)) = s\mathcal{O}^+_{g_0}(g_0^{N_-}(c^-)) = Y_c \text{ for all } c \text{ that is Type I}\}
\]

Note that the tangent space of \( S \) is

\[
\{(s\mathcal{O}^+_{g_0}(x))\mathcal{O}^+_{g_0}(x)\in\Omega_{g_0} : s\mathcal{O}^+_{g_0}(g^{N_+}(c^+)) = s\mathcal{O}^+_{g_0}(g_0^{N_-}(c^-)) \text{ for all } c \text{ that is Type I}\}
\]

which has dimension \( Df \). We have

\[
M_{g_0} = Q^{-1}S.
\]

We will apply Submersion Theorem to prove that \( M_{g_0} \) is a submanifold of codimension \( Df \). If we define \( Q \) at \( g \) in the direction \( E^h_g \) we obtain

\[
DgQ \cdot v = \left( \frac{\sum_{i=0}^{p(x)-1} \phi(g^i(h_g(x)))}{\ln |Dg^p(x)(h_g(x))|} \right)_{\mathcal{O}^+_{g_0}(x)\in\Omega_{g_0}},
\]

where \( p(x) \) is the period of the \( g_0 \)-orbit of \( x \) and

\[
\phi = \frac{Dv + D^2 g_0 \alpha}{Dg_0}.
\]
and \(\alpha\) is the solution of the equation \(v = \alpha \circ g_0 - Dg_0 \circ \alpha\). We need to show that the image of

\[
D_{g_0}Q : T_{g_0} \mathcal{T} \to \mathbb{R}^{\Omega_{g_0}}
\]

is \(\mathbb{R}^{\Omega_{g_0}}\). This follows from Proposition 9.1. The description of the tangent space of \(M_{g_0}\) follows from Theorem 11.4. \(\square\)

12. Relation with partially hyperbolic framework and it nightmares

We can interpret most of the results wherein the framework of 2-dimensional piecewise smooth partially hyperbolic endomorphisms.

**Proposition 12.1.** Let \(f_t \in \mathcal{B}_{\text{exp}}(C), t \in [0, 1]\), be a smooth family, with \(t \in [0, 1]\).

Define

\[
F : \mathcal{I} \times [0, 1] \to \mathcal{I} \times [0, 1].
\]

as \(F(x, t) = (f_t(x), t)\). Then \(F\) is a partially hyperbolic piecewise smooth endomorphism in the following sense

A. The unstable manifolds are horizontal lines. More precisely

\(W^u(x, t) = \mathcal{I} \times \{t\}\), \(E^u(x, t) = \mathbb{R} \times \{0\}\),

\(W^u(x, t)\) is invariant and there is \(\theta > 1\) such that

\[|D_x F(x, t)| = |Df_t(x)| \geq \theta\]

for all \((x, t)\) where \(Df_t(x)\) is defined.

B. There exists a measurable subset \(S\) such that

\(S^c \cap (\mathcal{I} \times \{t\})\)

is countable for every \(t\), and a continuous and bounded function

\(\hat{\alpha} : S \to \mathbb{R}^2\)

such that \(E^c_{(x, t)} = \langle \hat{\alpha} \rangle.\) Indeed

\[
DF(x, t) \cdot \hat{\alpha}(x, t) = \hat{\alpha}(F(x, t)).
\]

**Proof.** Statement A is obvious. Let

\(S = \{(x, t) : f_t^i(x)\text{ is well-defined and } f^i(x) \notin C \text{ for every } i \geq 0\}\).

Let

\[
\alpha_t(x) = -\sum_{i=0}^{\infty} \frac{v_i(f_t^i(x))}{Df_t^{i+1}(x)}
\]

for \((x, t) \in S\), where \(v_i = \partial_i f_t\). It is easy to see that

\[(x, t) \mapsto \alpha_t(x)\]

is continuous and bounded function. Note that

\[
v_t(x) = \alpha_t(f_t(x)) - Df_t(x)\alpha_t(x)
\]

for \((x, t) \in S\). Define

\[
\hat{\alpha}(x) = \begin{bmatrix} \alpha_t(x) \\ 1 \end{bmatrix}.
\]

We have

\[
DF(x, t) \cdot \hat{\alpha}(x, t) = \begin{bmatrix} Df_t(x) & v_t(x) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_t(f_t(x)) \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_t(x) \\ 1 \end{bmatrix} = \hat{\alpha}(x, t).
\]
Proposition 12.2. Let $F$ be as in Proposition 12.1, and assume that $f_t$ is Lasota-Yorke stable. Then $E^c$ extends to a continuous distribution on $I \times [0,1]$ if and only if $f_t$ is a deformation, that is, $f_t$ is topologically conjugate with $f_0$ for every $t$.

Proof. $E^c$ has as continuous extension to $I \times [0,1]$ if and only if $\alpha_t$ has a continuous extension $\alpha_t: I \to I$ for every $t$. But it is equivalent to $f_t$ be a deformation by the characterization of deformations by Theorem 8.3. \qed

Proposition 12.3. Let $F$ be as in Proposition 12.1, and assume that $f_t$ is a deformation. Then for every $t$ the function

$$x \mapsto \alpha_t(x, t) = (\alpha_t(x), 1)$$

is Log-Lipschitz continuous. The central direction is uniquely integrable and the holonomy between unstable manifolds trough the central lamination are Hölder. Moreover there are examples such that for every $t$ the function

$$x \mapsto \alpha_t(x)$$

satisfies a Central Limit Theorem for its modulus of continuity.

Proof. This follows from Theorems 8.1 and 8.3 and G.R. and S. [27, Theorem 4.10]. \qed

Proposition 12.4. Choose a deformation $f_t$ such that for every support of an ergodic absolutely continuous $f_0$-invariant probability $\mu_0$ there is a $f_0$-periodic point $q$, $f_0^M(q) = q$ in the support $\mu_0$ such that

$$t \mapsto |Df_t^M(\gamma(q))|$$

is injective where $\gamma_t$ is the conjugacy between $f_0$ and $f_t$. Then the center foliation $W^c$ of $F$ is a Fubini’s nightmare. Indeed, there is a subset $A$ of positive Lebesgue measure in $I \times [0,1]$ such that $A \cap I \times \{t\}$ is an atomic set for every $t \in [0,1]$.

Proof. This is quite similar to the Katok’s example (see Milnor [44]). Let $A$ defined in the following way. A point $(x,t)$ belongs to $A$ if and only if $x$ is typical with respect to some ergodic absolutely continuous invariant probability $\mu_{x,t}$ of $f_t$. $A$ has full Lebesgue measure on $I \times [0,1]$.

Note first that $S$ is the support of an ergodic absolutely continuous $f_0$-invariant probability $\mu_0$ if and only if $h_t(S)$ is the support of an ergodic absolutely continuous $f_t$-invariant probability $\mu_t$. Indeed by Boyarsky and Góra [8] the set $S$ is a $f_0$-invariant finite union of intervals, so $h_t(S)$ is a $f_t$-invariant finite union of intervals. That implies that $h_t(S)$ contain the support of an ergodic absolutely continuous $f_t$-invariant probability $\mu_t$. If we exchange the roles of 0 and $t$ we conclude that $h_t^{-1}(\text{supp } \mu_t) \subset S$ is $f_0$-invariant a finite union of intervals and the ergodicity of $\mu_0$ implies $h_t^{-1}(\text{supp } \mu_t) = S$.

Suppose that $(x_1, t_1), (x_2, t_2)$, with $t_0 \neq t_1$, belongs to $A \cap \gamma$, where $\gamma$ is a center leave. That means that there is $(x_0, 0)$ such that $x_i = h_i(x_0)$, with $i = 1, 2$. Let $h = h_1 \circ h_0^{-1}$ be the conjugacy between $f_{t_0}$ and $f_{t_1}$. Then $h(x_0) = x_1$. Note that the typically of $x_i$ imply

$$\mathcal{O}^{-1}(x_i) = \text{supp } \mu_{x_i,t_i} = S_i$$
so \( h(\text{supp } \mu_{x_1, t_1}) = \text{supp } \mu_{x_2, t_2} \) and
\[
\mu_{x_2, t_2}(h(B)) = \mu_{x_1, t_1}(B)
\]
for every borelian set \( B \). Indeed the support \( S_i \) is a finite union of intervals and, since \( \mu_{x_i, t_i} = \rho_i m \), where \( \rho_i \) has a positive upper and lower bound on \( S_i \) we conclude that \( h \) is absolutely continuous with respect to the Lebesgue measure in \( S_1 \) and in fact a bi-Lipschitz function. In particular
\[
\log |Df_{t_2}(h(q))| - \log |Df_{t_1}(h(q))| = \log |Dh(f_{t_1}(q))| - \log |Dh(f_{t_2}(q))|.
\]
The left hand side is a piecewise Lipschitz function and \( \log |Dh| \in L^\infty(S_1) \). G.R. and S. [27, Theorem 4.13] implies
\[
\sum_{j=0}^{M-1} \log |Df_{t_2}(f_{t_2}^j(h(q)))| = \sum_{j=0}^{M-1} \log |Df_{t_1}(f_{t_1}^j(h(q)))|
\]
for every \( f_0 \) periodic point \( q \) in the support of an absolutely continuous ergodic probability of \( f_0 \). This is not possible. \( \square \)

The following is an immediate consequence of the results on deformation of piecewise expanding maps

**Proposition 12.5.** We have

A. For every \( f_0 \) there are examples of deformations \( f_t \) for which the holonomies between unstable leaves through the central lamination are not absolutely continuous, and the central lamination is a Fabini’s nightmare.

B. For every \( f_0 \) there are examples of deformations \( f_t \) for which the holonomies between unstable leaves through the central lamination are quasisymmetric. However they are not absolutely continuous, and the central lamination is a Fabini’s nightmare.

**Remark 12.6.** Pathological invariant foliations, as foliations with atomic decomposition, seem to be ubiquitous in partially hyperbolic dynamics, and have been intensively studied by many authors. See Shub and Wilkinson [52], Ruelle and Wilkinson [50], Hirayama and Pesin [28], Homburg [29], Gogolev and Tahzibi [24] and Avila, Viana and Wilkinson [2].

Examples similar to Katok’s example as those in this section are quite special cases. However statistical properties of the distribution of the central direction similar to Proposition 12.3 does not seem to appear in the previous literature. One may wonder if similar statistical properties hold for more general classes of partially hyperbolic maps, and if they can help to understand their dynamics.

13. **Pressure pseudo-metric on the topological class**

Once we know that the topological class \( T(f_0) \) of a piecewise expanding map \( f_0 \) is a Banach manifold, one may ask if there is an interesting, dynamically defined riemannian (pseudo-)metric on \( T(f_0) \). The work of McMullen [41] on the characterisation via thermodynamical formalism of the Weil–Petersson metric on the Teichmüller space (and its generalisations for Blaschke products) suggest that a ”nice” dynamically-defined pseudo-metric would be the pressure pseudo-metric
\[
\langle v_1, v_2 \rangle_{E^h(f_0)} = \sigma(Dv_1 + D^2f \cdot \alpha_1, Dv_2 + D^2f \cdot \alpha_2).
\]
where \( \sigma \) is the hermitian form

\[
\sigma(\phi_1, \phi_2) = \lim_{N \to \infty} \int \left( \sum_{i=0}^{N-1} \frac{\phi_1 \circ f^i}{\sqrt{N}} \right) \left( \sum_{i=0}^{N-1} \frac{\phi_2 \circ f^i}{\sqrt{N}} \right) \, dm,
\]

that due G.R. and S. [27] is well defined for every pair \( (\phi_1, \phi_2) \in B^3(C) \) such that

\[
\int \phi_i \Phi_1(\gamma) \, dm = 0
\]

for every \( \gamma \in BV, \, i = 1, 2 \), and \( v_i = \alpha_i \circ f - Df \cdot \alpha_i \), where \( \alpha_i \) are Log-Lipschitz. Note that \( m \) does not need to be \( f \)-invariant. One must compare this with Giulietti, Kloekner, Lopes, and Marcon [23], a study of thermodynamical formalism in a geometric framework. See also Pollicott and Sharp [46] and Bridgeman, Canary and Sambarino [9] and the Weil-Petersson metric in the infinite-dimensional Teichmüller space in Takhtajan and Teo [55].

There are many interesting questions one can ask on this pseudo-metric. We give a result that follows immediately from our results on Birkhoff sums as distributions and deformations.

**Proposition 13.1.** Let \( w \in E^h(f) \) and define

\[
\Theta(v) = \langle v, w \rangle_{E^h(f)}.
\]

Let

\[
\phi = \frac{Dw + D^2f \cdot \alpha}{Df}.
\]

The following statements are equivalent

A. \( \Theta \) is a signed measure.
B. \( \phi = \psi \circ f - \psi \), where \( \psi \in L^2(m) \) and \( \psi \in L^\infty(S_\ell) \) for every \( \ell \leq E \).
C. \( \Theta = 0 \).

Moreover A. – C. implies

D. We have that

\[
\sum_{j=0}^{M-1} \phi(f^j(q)) = 0
\]

holds for every \( M \) and \( q \in \hat{S}_\ell \), with \( \ell \leq E \), such that \( f^M(q) = q \).

Furthermore if \( f \) is markovian, \( p(f) = 1 \) and it has an absolutely continuous ergodic invariant probability whose support is \( I \) then D. is equivalent to A. – C.

**Proof.** This follows from G.R. and S. [27, Theorem 4.17].

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