Fischer type determinantal inequalities for accretive-dissipative matrices

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Abstract

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ be an $n \times n$ accretive-dissipative matrix, $k$ and $l$ be the orders of $A_{11}$ and $A_{22}$, respectively, and let $m = \min\{k, l\}$. Then

$$|\det A| \leq a |\det A_{11}| \cdot |\det A_{22}|,$$

where $a = \begin{cases} 2^{3m/2}, & \text{if } m \leq n/3; \\ 2m/2, & \text{if } n/3 < m \leq n/2. \end{cases}$ This improves a result of Ikramov.

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1 Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the set of $n \times n$ complex matrices. For any $A \in \mathbb{M}_n(\mathbb{C})$, $A^*$ stands for the conjugate transpose of $A$. $A \in \mathbb{M}_n(\mathbb{C})$ is accretive-dissipative if it can be written as

$$A = B + iC,$$

where $B = \frac{A + A^*}{2}$ and $C = \frac{A - A^*}{2i}$ are both (Hermitian) positive definite. Conformally partition $A, B, C$ as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21}^* & B_{22} \end{bmatrix} + i \begin{bmatrix} C_{11} & C_{12} \\ C_{21}^* & C_{22} \end{bmatrix}$$

such that all diagonal blocks are square. Say $k$ and $l$ $(k, l > 0$ and $k + l = n)$ the order of $A_{11}$ and $A_{22}$, respectively, and let $m = \min\{k, l\}$.

If $A$ is positive definite and partitioned as in (1.2), then the famous Fischer determinantal inequality (FDI) [3, p. 478] states that

$$\det A \leq \det A_{11} \cdot \det A_{22}. \quad (1.3)$$

Determinantal inequalities for accretive-dissipative matrices were first investigated by Ikramov [4], who obtained:
Theorem 1. Let $A \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative and partitioned as in (1.2). Then

$$|\det A| \leq 3^m |\det A_{11}| \cdot |\det A_{22}|.$$  

(1.4)

A reverse direction to that of Theorem 1 has been given in [5]. We call this kind of inequalities the Fischer type determinantal inequality for accretive-dissipative matrices. In this paper, we intend to give an improvement of (1.4). Our main result can be stated as

Theorem 2. Let $A \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative and partitioned as in (1.2). Then

$$|\det A| \leq a |\det A_{11}| \cdot |\det A_{22}|,$$  

(1.5)

where $a = \begin{cases} 2^{3m/2}, & \text{if } m \leq n/3; \\ 2^{n/2}, & \text{if } n/3 < m \leq n/2. \end{cases}$

As $a < 3^m$, it is clear that Theorem 2 improves Theorem 1. The proof of Theorem 2 is given in Section 3.

2 Auxiliary results

In this section, we present some lemmas that are needed in the proof of our main result.

Lemma 3. [2, Property 6] Let $A \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative and partitioned as in (1.2). Then \(A/A_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12},\) the Schur complement of $A_{11}$ in $A$, is also accretive-dissipative.

Lemma 4. [4, Lemma 1] Let $A \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative as in (1.1). Then $A^{-1} = E - iF$ with $E = (B + CB^{-1}C)^{-1}$ and $F = (C + BC^{-1}B)^{-1}$.

Lemma 5. [7, Lemma 3.2] Let $B, C \in \mathbb{M}_n(\mathbb{C})$ be Hermitian and assume $B$ is positive definite. Then

$$B + CB^{-1}C \geq 2C.$$  

(2.1)

Here we adopt the convention that, for two Hermitian matrices $X, Y$ of the same size, $X \geq Y$ means $X - Y$ is positive (semi)definite. Of course, we do not distinguish $Y \geq X$ from $X \geq Y$.

Lemma 6. Let $B, C \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite. Then

$$|\det(B + iC)| \leq \det(B + C) \leq 2^{n/2}|\det(B + iC)|.$$  

(2.2)

Proof. The first inequality follows from [6, Theorem 2.2] while the second one follows from [1, Theorem 1.1]. Here we provide a direct proof of (2.2) for the convenience of readers. We may assume $B$ is positive definite, the general case is by a continuity
argument. Let $\lambda_j$, $j = 1, \ldots, n$, be the eigenvalues of $B^{-1/2}CB^{-1/2}$, where $B^{1/2}$ means the unique positive definite square root of $B$. Then

$$|1 + i\lambda_j| \leq 1 + \lambda_j \leq \sqrt{2}|1 + i\lambda_j|, \quad j = 1, \ldots, n.$$ 

Also, we denote the identity matrix by $I$.

Compute

$$|\det(B + iC)| = \det B \cdot |\det(I + iB^{-1/2}CB^{-1/2})|$$

$$= \det B \cdot \prod_{j=1}^{n} |1 + i\lambda_j|$$

$$\leq \det B \cdot \prod_{j=1}^{n} (1 + \lambda_j)$$

$$= \det B \cdot \det(I + B^{-1/2}CB^{-1/2})$$

$$= \det(B + C).$$

This proves the first inequality. To show the other, compute

$$\det(B + C) = \det B \cdot \det(I + B^{-1/2}CB^{-1/2})$$

$$= \det B \cdot \prod_{j=1}^{n} (1 + \lambda_j)$$

$$\leq \det B \cdot \prod_{j=1}^{n} \sqrt{2}|1 + i\lambda_j|$$

$$= 2^{n/2} \det B \cdot |\det(I + iB^{-1/2}CB^{-1/2})|$$

$$= 2^{n/2} |\det(B + iC)|.$$

$$\square$$

3 Main results

Theorem 2 follows from the next two theorems.

**Theorem 7.** Let $A \in \mathbb{M}_n(C)$ be accretive-dissipative and partitioned as in (1.2). Then

$$|\det A| \leq 2^{n/2} |\det A_{11}| \cdot |\det A_{22}|.$$  \hspace{1cm} (3.1)

**Proof.** Compute

$$|\det A| = |\det(B + iC)|$$

$$\leq \det(B + C) \quad \text{(By Lemma 6)}$$

$$\leq \det(B_{11} + C_{11}) \cdot \det(B_{22} + C_{22}) \quad \text{(By FDI)}$$

$$\leq 2^{k/2} |\det(B_{11} + iC_{11})| \cdot 2^{l/2} |\det(B_{22} + iC_{22})| \quad \text{(By Lemma 6)}$$

$$= 2^{n/2} |\det A_{11}| \cdot |\det A_{22}|.$$

$$\square$$
\textbf{Theorem 8.} Let $A \in \mathbb{M}_n(C)$ be accretive-dissipative and partitioned as in \((1.2)\). Then
\[|\det A| \leq 2^{m/2}|\det A_{11}| \cdot |\det A_{22}|.\] (3.2)

\textit{Proof.} We have, by Lemma 4, that
\[A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}\]
\[= B_{22} + iC_{22} - (B_{12}^* + iC_{12}^*)^{-1}(B_{11} + iC_{11})^{-1}(B_{12} + iC_{12})\]
\[= B_{22} + iC_{22} - (B_{12}^* + iC_{12}^*)(E_k - iF_k)(B_{12} + iC_{12})\]

with
\[E_k = (B_{11} + C_{11}B_{11}^{-1}C_{11})^{-1},\]
\[F_k = (C_{11} + B_{11}C_{11}^{-1}B_{11})^{-1}.\]

By Lemma 5 and the operator reverse monotonicity of the inverse, we get
\[E_k \leq \frac{1}{2}C_{11}^{-1},\quad F_k \leq \frac{1}{2}B_{11}^{-1}.\] (3.3)

Setting $A/A_{11} = R + iS$ with $R = R^*$ and $S = S^*$. By Lemma 3 we know $R$ and $S$ are positive definite. A calculation shows
\[R = B_{22} - B_{12}^*E_kB_{12} + C_{12}^*E_kC_{12} - B_{12}^*F_kC_{12} - C_{12}^*F_kB_{12};\]
\[S = C_{22} - B_{12}^*F_kB_{12} - C_{12}^*F_kC_{12} - C_{12}^*E_kB_{12} - B_{12}^*E_kC_{12}.\]

It can be verified that
\[\pm(B_{12}^*F_kC_{12} + C_{12}^*F_kB_{12}) \leq B_{12}^*F_kB_{12} + C_{12}^*F_kC_{12};\]
\[\pm(C_{12}^*E_kB_{12} + B_{12}^*E_kC_{12}) \leq B_{12}^*E_kB_{12} + C_{12}^*E_kC_{12}.\]

Thus,
\[R + S \leq B_{22} + 2B_{12}^*F_kB_{12} + C_{22} + 2C_{12}^*E_kC_{12}.\] (3.4)

As $B, C$ are positive definite, we also have
\[B_{22} > B_{12}^*B_{11}^{-1}B_{12}, \text{ and } C_{22} > C_{12}^*C_{11}^{-1}C_{12}.\] (3.5)

Without loss of generality, we assume $m = l$. Compute
\[|\det(A/A_{11})| = |\det(R + iS)|\]
\[\leq \det(R + S) \quad \text{(by Lemma 6)}\]
\[\leq \det(B_{22} + 2B_{12}^*F_kB_{12} + C_{22} + 2C_{12}^*E_kC_{12}) \quad \text{(by (3.4))}\]
\[\leq \det(B_{22} + B_{12}^*B_{11}^{-1}B_{12} + C_{22} + C_{12}^*C_{11}^{-1}C_{12}) \quad \text{(by (3.3))}\]
\[< \det(2(B_{22} + C_{22})) \quad \text{(by (3.5))}\]
\[= 2^m \det(B_{22} + C_{22})\]
\[\leq 2^m \cdot 2^{m/2} |\det(B_{22} + iC_{22})| \quad \text{(by Lemma 6)}\]
\[= 2^{3m/2} |\det A_{22}|.\]

The proof is complete by noting $\det(A/A_{11}) = \frac{\det A}{\det A_{11}}$. \qed
It is natural to ask whether $a$ in (1.3) can be replaced by a smaller number? There is evidence that the following could hold:

**Conjecture 9.** Let $A \in \mathbb{M}_n(C)$ be accretive-dissipative and partitioned as in (1.3). Then

$$|\det A| \leq 2^m |\det A_{11}| \cdot |\det A_{22}|.$$

We end the paper by an example showing that if the above conjecture is true, then the factor $2^m$ is optimal.

**Example 10.** Let $A = \begin{bmatrix} (1 + \epsilon)(1 + i) & i - 1 \\ i - 1 & (1 + \epsilon)(1 + i) \end{bmatrix} = \begin{bmatrix} 1 + \epsilon & -1 \\ -1 & 1 + \epsilon \end{bmatrix} + i \begin{bmatrix} 1 + \epsilon & 1 \\ 1 & 1 + \epsilon \end{bmatrix}$ with $\epsilon > 0$. Then $A$ is accretive-dissipative. As $\epsilon \to 0^+$, we have

$$\frac{|\det A|}{|\det A_{11}| \cdot |\det A_{22}|} \to 2.$$

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## References

[1] R. Bhatia, F. Kittaneh, The singular values of $A + B$ and $A + iB$, Linear Algebra Appl. 431 (2009) 1502-1508.

[2] A. George, Kh. D. Ikramov, On the properties of accretive-dissipative matrices, Math. Notes 77 (2005) 767-776.

[3] R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge University Press, London, 1985.

[4] Kh. D. Ikramov, Determinantal inequalities for accretive-dissipative matrices, J. Math. Sci. (N. Y.) 121 (2004) 2458–2464.

[5] M. Lin, Reversed determinantal inequalities for accretive-dissipative matrices, Math. Inequal. Appl. 12 (2012) 955-958.

[6] X. Zhan, Singular values of differences of positive semidefinite matrices, SIAM J. Matrix Anal. Appl. 22 (2000) 819-823.

[7] X. Zhan, Computing the extremal positive definite solutions of a matrix equation, SIAM J. Sci. Comput. 17 (1996) 1167-1174.
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