A lower bound for the error term in Weyl’s law
for certain Heisenberg manifolds

Werner Georg Nowak (*) (Vienna)

Abstract. This article provides an Omega-result for the remainder term in Weyl’s law for the spectral counting function of certain rational \((2\ell + 1)\)-dimensional Heisenberg manifolds.

Introduction. For \(M\) a closed \(n\)-dimensional Riemannian manifold with a metric \(g\) and Laplace-Beltrami operator \(\Delta\), let \(N(t)\) denote the spectral counting function

\[
N(t) := \sum_{\lambda \text{ eigenvalue of } \Delta} d(\lambda) \quad \lambda \leq t
\]

where \(d(\lambda)\) is the dimension of the eigenspace corresponding to \(\lambda\), and \(t\) is a large real variable. Then a deep and very general theorem due to L. Hörmander [6] tells us that

\[
N(t) = \frac{\text{vol}(M)}{(4\pi)^{n/2} \Gamma(\frac{1}{2}n + 1)} t^{n/2} + O\left(t^{(n-1)/2}\right)
\]

(“Weyl’s law”), and that the error term in general cannot be improved. Nevertheless, it is of interest to study the order of magnitude and the asymptotic behavior of the remainder

\[
R(t) = N(t) - \frac{\text{vol}(M)}{(4\pi)^{n/2} \Gamma(\frac{1}{2}n + 1)} t^{n/2}
\]

for special manifolds \(M\).

The most classic example, namely the case that \(M = \mathbb{R}^n / \mathbb{Z}^n\), the \(n\)-dimensional torus, is (equivalent to) a central problem in the theory of lattice points in large domains, namely to provide asymptotic results for the number \(A_n(x)\) of integer points in an origin-centered \(n\)-dimensional ball of radius \(x\), for any dimension \(n \geq 2\). There exists a vast literature on this particular subject: We only refer to the works of Huxley [7], [8], Hafner [4], and Soundararajan [17] for the planar case, for the papers by Chamizo & Iwaniec [1], Heath-Brown [5], and Tsang [18] for dimension \(n = 3\), and to the monographs of Walfisz [20], and Krätzel [14], [15], as well as to the recent, quite comprehensive, survey article [9].

In fact, for \(M = \mathbb{R}^n / \mathbb{Z}^n\), we see that\(^{(1)}\) \(\{u \mapsto e(m \cdot u) : m \in \mathbb{Z}^n\}\) is a basis for the eigenfunctions of the Laplace operator \(\Delta = -\sum_{j=1}^{n} \partial_{jj}\), acting on functions from \(\mathbb{R}^n / \mathbb{Z}^n\).

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\(^{(1)}\) Bold face letters will denote throughout elements of \(\mathbb{R}^n\), resp., \(\mathbb{Z}^n\), which may be viewed also as \((1 \times n)\)-matrices (“row vectors”) where applicable. Further, \(|| \cdot ||\) stands for the Euclidean norm.
into $\mathbb{C}$. The corresponding eigenvalues are $4\pi^2|m|^2$, $m \in \mathbb{Z}^n$. For any integer $k \geq 0$, let us denote the number of ways to write $k$ as the sum of $n$ squares. Then, for each $k$ with $r_n(k) > 0$, $4\pi^2k$ is an eigenvalue of $\Delta$ whose eigenspace consists of all functions

$$u \mapsto \sum_{|m|^2=k} c(m) e(m \cdot u),$$

where $c(m)$ are any complex coefficients. Its dimension obviously equals $r_n(k)$, hence

$$N(t) = \sum_{k \geq 0: 4\pi^2k \leq t} r_n(k) = A_n \left(\frac{\sqrt{t}}{2\pi}\right).$$

### 2. Heisenberg manifolds.

In recent times, presumably motivated by quite different areas like quantum physics and the abstract theory of PDE's, a lot of work has been done on another special case, namely that of so-called Heisenberg manifolds. To recall basics and to fix notions, let $\ell \geq 1$ be a given integer, and

$$\gamma(x, y, z) = \begin{pmatrix} 1 & x & z \\ t_o & I_\ell & t_y \\ 0 & o_\ell & 1 \end{pmatrix},$$

where $x, y \in \mathbb{R}^\ell$, $z \in \mathbb{R}$, $o_\ell = (0, \ldots, 0) \in \mathbb{R}^\ell$, $I_\ell$ is the $(\ell \times \ell)$-unit matrix, and $t$. denotes transposition. Then the $(2\ell + 1)$-dimensional Heisenberg group $H_\ell$ is defined by

$$(2.1) \quad H_\ell = \{\gamma(x, y, z) : x, y \in \mathbb{R}^\ell, z \in \mathbb{R}\},$$

with the usual matrix product. Further, for any $\ell$-tuple $r = (r_1, \ldots, r_\ell) \in \mathbb{Z}_+^\ell$ with the property that $r_j \mid r_{j+1}$ for all $j = 1, \ldots, \ell - 1$, we put $r * \mathbb{Z}^\ell := r_1 \mathbb{Z} \times \ldots \times r_\ell \mathbb{Z}$ and define

$$(2.2) \quad \Gamma_r = \{\gamma(x, y, z) : x \in r * \mathbb{Z}^\ell, y \in \mathbb{Z}^\ell, z \in \mathbb{Z}\}.$$

$\Gamma_r$ is a uniform discrete subgroup of $H_\ell$, i.e., the Heisenberg manifold $H_\ell/\Gamma_r$ is compact. Fortunately, according to a deep work by Gordon and Wilson [2], Theorem 2.4, this seemingly quite special choice of $\Gamma_r$ is in fact fairly general. The subgroups $\Gamma_r$ classify all uniform discrete subgroups of $H_\ell$ up to automorphisms: For every uniform discrete subgroup $\Gamma$ of $H_\ell$ there exists a unique $\ell$-tuple $r$ and an automorphism of $H_\ell$ which maps $\Gamma$ to $\Gamma_r$.

However, to get a "rational" or "arithmetic" Heisenberg manifold - borrowing an expression due to Petridis & Toth [16] - we have to make a quite particular choice of the metric involved\(^{(2)}\). Following the example of [16], Theorem 1.1, and also Zhai [21], we pick

$$(2.3) \quad g_\ell = \begin{pmatrix} I_2 & t_o \ell \\ o_\ell & 2\pi \end{pmatrix}.$$

\(^{(2)}\) Compare the discussion below concerning the bound (3.3) valid for "almost all" metrics $g$. 
The spectrum of the Laplace-Beltrami operator on $H_{\ell}/\Gamma_{r}$ has been analyzed in Gordon and Wilson [2], p. 259, and also in Khosravi and Petridis [12], p. 3564. It consists of two different classes $S_I$ and $S_{II}$, where $S_I$ is the spectrum of the Laplacian on the $2\ell$-dimensional torus, and

$$S_{II} = \{2\pi (n_0^2 + n_0(2n_1 + \ell)) : n_0 \in \mathbb{Z}^+, n_1 \in \mathbb{Z}_0^+ \},$$

with multiplicities ( = dimensions of corresponding eigenspaces) $2n_0^\ell r_1 \cdots r_\ell (n_1^{\ell-1} + \ell)$. 

3. Statement of problem and results. In this article, we shall be concerned with the size of the error term in (1.1) for the special case that $M = (H_{\ell}/\Gamma_{r}, g_{\ell})$ as described above, i.e., with the asymptotic behavior of

$$R(t) = N(t) - \frac{\text{vol}(M)}{(4\pi)^{\ell+1/2} \Gamma(\ell + \frac{3}{2})} t^{\ell+1/2} = N(t) - \frac{r_1 \cdots r_\ell}{2^{2\ell+1/2} \pi^{\ell} \Gamma(\ell + \frac{3}{2})} t^{\ell+1/2}.$$

For $\ell = 1$, Petridis and Toth [16] proved that $R(t) \ll t^{5/6} \log t$. They were the first to realize that the question is related to a certain planar lattice point problem which can be dealt with usual tools for the estimation of fractional part sums. This result was sharpened and generalized to arbitrary $\ell \geq 1$ by Khosravi and Petridis [12] who obtained $R(t) \ll t^{\ell-7/41}$. Zhai [21] noticed that Huxley’s deep method [7], [8] can be used to derive, for any $\ell \geq 1$,

$$R(t) \ll t^{\ell-77/416} (\log t)^{26957/8320}.$$

In fact, the difficulty in these estimations comes from the "rational" nature of the metric $g_{\ell}$. As Khosravi and Petridis [12] showed, for "almost all" metrics $g$ the much sharper bound

$$R_g(t) \ll_g t^{\ell-1/4} \log t$$

holds true. Returning to the rational case (2.3), a result of Khosravi [11] and Khosravi & Toth [13] tells us that

$$\int_0^T (R(t))^2 \, dt = C_\ell T^{2\ell+1/2} + O \left( T^{2\ell+1/4+\varepsilon} \right)$$

where $C_\ell > 0$ is an explicit constant. A recent paper of Zhai [21] provides estimates and asymptotics for higher power moments of $R(t)$. In fact, (3.3) and (3.4) may suggest the conjecture that

$$R(t) \ll t^{\ell-1/4+\varepsilon}$$

for every $\varepsilon > 0$. The situation has a good deal in common with the Dirichlet divisor and the Gaussian circle problems.
The objective of the present article is to provide a lower bound which shows that the \( \varepsilon \) in (3.5) cannot be removed. We shall prove that

\[
R(t) = \Omega \left( t^{\ell - 1/4} (\log t)^{1/4} \right).
\]

Together with (3.4), we may say that " \( R(t) \ll t^{\ell - 1/4} \) in mean-square, with an unbounded sequence of exceptionally large values \( t \)".

Unfortunately, we have to impose the restriction that \( \ell \) is an even integer. We will comment on this condition at the end of the paper.

**Theorem.** For a fixed even positive integer \( \ell \), let \((H_{\ell}/\Gamma_r, g_{\ell})\) be a rational \((2\ell + 1)\)-dimensional Heisenberg manifold with metric \( g_{\ell} \), as described above. Then the error term \( R(t) \) for the associated spectral counting function, defined in (3.1), satisfies

\[
\limsup_{t \to \infty} \frac{R(t)}{t^{\ell - 1/4} (\log t)^{1/4}} > 0.
\]

4. Some Lemmas.

**Lemma 1.** (Vaaler’s approximation of fractional parts by trigonometric polynomials.) For arbitrary \( w \in \mathbb{R} \) and \( H \in \mathbb{Z}^+ \), let \( \psi(w) := w - \lfloor w \rfloor - \frac{1}{2} \),

\[
\Sigma_H(w) := \sum_{h=1}^{H} \alpha_{h,H} \sin(2\pi hw), \quad \Sigma^*_H(w) := \sum_{h=1}^{H} \beta_{h,H} \cos(2\pi hw) + \frac{1}{2H + 2},
\]

where, for \( h = 1, \ldots, H \),

\[
\alpha_{h,H} := \frac{1}{\pi h} \rho \left( \frac{h}{H + 1} \right), \quad \beta_{h,H} := \frac{1}{H + 1} \left( 1 - \frac{h}{H + 1} \right),
\]

and

\[
\rho(\xi) = \pi \xi (1 - \xi) \cot(\pi \xi) + \xi \quad (0 < \xi < 1).
\]

Then the following inequality holds true:

\[
|\psi(w) + \Sigma_H(w)| \leq \Sigma^*_H(w).
\]

**Proof.** This is one of the main results in Vaaler [19]. A very well readable exposition can also be found in the monograph by Graham and Kolesnik [3].
Lemma 2. Let $F \in C^4[A, B]$, $G \in C^2[A, B]$, and suppose that, for positive parameters $X, Y, Z$, we have $1 \ll B - A \ll X$ and
\[ F^{(j)} \ll X^{2-j}Y^{-1} \quad \text{for } j = 2, 3, 4, \quad |F''| \geq c_0Y^{-1}, \quad G^{(j)} \ll X^{-j}Z \quad \text{for } j = 0, 1, 2, \]
throughout the interval $[A, B]$, with some constant $c_0 > 0$. Let $J'$ denote the image of $[A, B]$ under $F'$, and $F^*$ the inverse function of $F'$. Then, with $e(w) = e^{2\pi iw}$ as usual,
\[
\sum_{A < m \leq B} G(m) e(F(m)) = e\left( \frac{\text{sgn}(F'')}{8} \sum_{k \in J'} \frac{G(F^*(k))}{\sqrt{|F''(F^*(k))|}} e(F(F^*(k)) - kF^*(k)) \right) + O\left( Z\left( \sqrt{Y} + \log(2 + \text{length}(J')) \right) \right).
\]

Proof. Transformation formulas of this kind are quite common, though often with worse error terms. This very sharp version can be found as f. (8.47) in the recent monograph [10] of H. Iwaniec and E. Kowalski.

Lemma 3. For a real parameter $T \geq 1$, let $F_T$ denote the Fejér kernel
\[
F_T(v) = T \left( \frac{\sin(\pi Tv)}{\pi Tv} \right)^2.
\]
Then for arbitrary real numbers $Q > 0$ and $\delta$, it follows that
\[
\int_{-1}^{1} F_T(v) e(Qv + \delta) dv = \max\left( 1 - \frac{Q}{T}, 0 \right) e(\delta) + O\left( \frac{1}{Q} \right),
\]
where the $O$-constant is independent of $T$ and $\delta$.

Proof. This useful result is due to Hafner [4]. It follows from the classic Fourier transform formula
\[
\int_{\mathbb{R}} F_T(v) e(Qv) dv = \int_{\mathbb{R}} \left( \frac{\sin(\pi v)}{\pi v} \right)^2 e\left( \frac{Q}{T} v \right) dv = \max\left( 1 - \frac{Q}{T}, 0 \right).
\]
Since $F_T(\pm 1) \ll T^{-1}$ and $F_T'(v) \ll v^{-2}$ for $|v| \geq 1$, uniformly in $T \geq 1$, integration by parts readily shows that the intervals $]-\infty, -1]$ and $[1, \infty[$ contribute only $O(Q^{-1})$.

5. Proof of the Theorem. We start from Lemma 3.1 in Zhai [21] which approximates the error term involved by a fractional part sum. Let $U$ be a large real parameter, $u \in [U - 1, U + 1]$, and put
\[
E(u) := \frac{2^{\ell-2}(\ell-1)!}{r_1 \cdots r_\ell} R(2\pi u^2).
\]
In fact, Zhai in his notation tacitly assumes that $r_1 = \ldots = r_{\ell} = 1$, which means no actual loss of generality. We have supplemented the factor $r_1 \cdot \ldots \cdot r_{\ell}$ in (5.1).

We apply Lemma 1 in the form $-\psi \geq \Sigma_H - \Sigma_H^*$, choosing $H = [U]$. Thus we get

$$E^*(u) = E^*(u) + O \left( u^{2\ell-1} \right),$$

$$E^*(u) := - \sum_{1 \leq m \leq u} m(u^2 - m^2)^{\ell-1} \psi \left( \frac{u^2}{2m} - \frac{m}{2} \right).$$

(5.2)

We split up the range $1 \leq m \leq u$ into dyadic subintervals $M_j := [M_{j+1}, M_j]$, $M_j = u 2^{-j}$ for $j = 0, \ldots, J$, where $J$ is minimal such that $(U - 1)2^{-j-1} < 1$. We thus have to deal with exponential sums

$$E_j(h, u) := \sum_{m \in M_j} m(u^2 - m^2)^{\ell-1} e \left( -h \left( \frac{u^2}{2m} - \frac{m}{2} \right) \right).$$

(5.3)

We transform them by means of Lemma 2, with

$$G(\xi) = \xi(u^2 - \xi^2)^{\ell-1}, \quad F(\xi) = -h \left( \frac{u^2}{2\xi} - \frac{\xi}{2} \right).$$

By straightforward computations, on each interval $M_j$ the conditions of Lemma 2 are fulfilled with the parameters $X = M_j$, $Y = \frac{M_j^3}{hu^2}$, $Z = M_j u^{2\ell-2}$. We obtain

$$E_j(h, u) = h^{3/4} u^{2\ell-1/2} \sum_{k \in P^{\prime}(M_j)} \frac{(2k - 2h)^{\ell-1}}{(2k - h)^{\ell+1/4}} e \left( -u \sqrt{h} \sqrt{2k - h} - \frac{1}{8} \right)$$

$$+ O \left( u^{2\ell-3} \frac{M_j^{5/2}}{h^{1/2}} + u^{2\ell-1} \log u \right).$$

(5.4)

We first bound the overall contribution of the error terms, summing over $j$ and $h$. Let $\gamma_{h, [U]}$ denote $\alpha_{h, [U]}$ or $\beta_{h, [U]}$, thus $\gamma_{h, [U]} \ll h^{-1}$ in any case, then

$$\sum_{j=0}^{[U]} \sum_{h=1}^{[U]} \gamma_{h, [U]} \left( u^{2\ell-3} \frac{M_j^{5/2}}{h^{1/2}} + u^{2\ell-1} \log u \right) \ll u^{2\ell-1/2} + u^{2\ell-1} (\log u)^3 \ll u^{2\ell-1/2}.$$

(5.5)
Summing up the main terms in (5.4), we notice that the total range of $k$ becomes $h = F'(M_0) < k \leq F'(M_1) = \frac{1}{2}h + 2^{2j+1}h =: K_{h,U}$, and we obtain
\[
\sum_{1 \leq h \leq U} \sum_{1 \leq m \leq u} m(u^2 - m^2)^{\ell-1} \gamma_{h,[U]} e\left(-h\left(\frac{u^2}{2m} - \frac{m}{2}\right)\right) = \\
= u^{2\ell-1/2} \sum_{h=1}^{[U]} \gamma_{h,[U]} h^{3/4} \sum_{h < k \leq K_{h,U}} \frac{(2k - 2h)^{\ell-1}}{(2k - h)^{\ell+1/4}} e\left(-u\sqrt{h}\sqrt{2k - h} - \frac{1}{8}\right) + \\
+ O\left(u^{2\ell - 1/2}\right).
\]

Using the real and imaginary part of this result in (5.3), we arrive at
\[
E^*(u) \geq u^{2\ell - 1/2} S(u, U) - c_1 u^{2\ell-1/2},
\]
where
\[
S(u, U) := \sum_{(h,k) \in \mathcal{D}(U)} h^{3/4} \frac{(2k - 2h)^{\ell-1}}{(2k - h)^{\ell+1/4}} \times \\
\times \left(\alpha_{h,[U]} \sin\left(2\pi u\sqrt{h}\sqrt{2k - h} + \frac{\pi}{4}\right) - \beta_{h,[U]} \cos\left(2\pi u\sqrt{h}\sqrt{2k - h} + \frac{\pi}{4}\right)\right),
\]
\[
\mathcal{D}(U) := \{(h,k) \in \mathbb{Z}^2 : 1 \leq h \leq U, h < k \leq K_{h,U}\},
\]
and $c_1$ is an appropriate positive constant. Our next step is to get rid of ”most” of the terms of the last double sum. To this end we use Lemma 3, multiplying $S(u, U)$ by a Fejér kernel $\mathcal{F}_T(u - U)$, where $T$ is a new large parameter, and integrating over $U - 1 \leq u \leq U + 1$. We obtain
\[
I(T, U) := \int_{U-1}^{U+1} S(u, U) \mathcal{F}_T(u - U) \, du = \int_{-1}^{1} S(U + v, U) \mathcal{F}_T(v) \, dv = \\
= \sum_{(h,k) \in \mathcal{D}(U), h(2k-h) \leq T^2} h^{3/4} \frac{(2k - 2h)^{\ell-1}}{(2k - h)^{\ell+1/4}} \left(1 - \frac{\sqrt{h}(2k-h)}{T}\right) \times \\
\times \left(\alpha_{h,[U]} \sin\left(2\pi U\sqrt{h}(2k - h) + \frac{\pi}{4}\right) - \beta_{h,[U]} \cos\left(2\pi U\sqrt{h}(2k - h) + \frac{\pi}{4}\right)\right) + \\
+ O\left(\sum_{(h,k) \in \mathcal{D}(U)} h^{-3/4} \frac{(2k - 2h)^{\ell-1}}{(2k - h)^{\ell+3/4}}\right).
\]

The $O$-term here is harmless: In fact,
\[
\sum_{h=1}^{\infty} h^{-3/4} \sum_{k > h} \frac{(2k - 2h)^{\ell-1}}{(2k - h)^{\ell+3/4}} \ll \sum_{h=1}^{\infty} h^{-3/4} \sum_{m > h/2} m^{-7/4} \ll \sum_{h=1}^{\infty} h^{-3/2} \ll 1.
\]
We proceed to derive a lower bound for the main term in (5.8). To this end we relate the two parameters $U$ and $T$ to each other: For arbitrary $T$ sufficiently large, we choose $U$ according to the Dirichlet approximation theorem, such that

(i) \[ T^2 \leq U \leq T^2 16T^2 \]

and

(ii) \[ \left\| U \sqrt{h(2k-h)} \right\| \leq \frac{1}{16} \]

for all $(h,k) \in \mathcal{D}(U)$, $h(2k-h) \leq T^2$, where $\| \cdot \|$ denotes the distance from the nearest integer. Now $h(2k-h) \leq T^2$, $k > h$, together with (i) implies that

(5.10) \[ h \leq T \leq \sqrt{U} \]

By the definitions in Lemma 1, for all of the $h$ occurring in (5.8), $\alpha_{h,[U]} \asymp h^{-1}$, and $\beta_{h,[U]} \asymp U^{-1}$, hence

\[
\alpha_{h,[U]} \sin \left( 2\pi U \sqrt{h(2k-h)} + \frac{\pi}{4} \right) - \beta_{h,[U]} \cos \left( 2\pi U \sqrt{h(2k-h)} + \frac{\pi}{4} \right) \geq
\]

\[
\geq c_1 \sin \left( \frac{\pi}{8} \right) \frac{1}{h} - c_2 \frac{c_4}{U} \geq \frac{c_4}{h},
\]

with suitable positive constants $c_1, c_2, c_3$. Putting for short

\[
\theta_{U,T,\ell}(n) := \sum_{(h,k) \in \mathcal{D}(U), h(2k-h)=n} \frac{h^{1/2}}{(2k-h)^{1/2}} \left( 1 - \frac{h}{2k-h} \right)^{\ell-1},
\]

we thus readily infer from (5.8) and (5.9) that, for some $c_4, c_5 > 0$,

(5.11) \[
I(T,U) \geq c_3 \sum_{1 \leq n \leq T^2} \frac{\theta_{U,T,\ell}(n)}{n^{3/4}} \left( 1 - \frac{\sqrt{n}}{T} \right) - c_4
\]

\[
\geq c_5 \sum_{1 \leq n \leq T^2/2} \frac{\theta_{U,T,\ell}(n)}{n^{3/4}} - c_4.
\]

We notice further that $(h,k) \in \mathcal{D}(U)$ explicitly means that

\[
1 \leq h \leq U, \quad h < k \leq \frac{1}{2}h + 2^{2J+1}h \asymp U^2 h,
\]

while $h(2k-h) \leq T^2$, $k > h$ implies (5.10) and

\[
k < 2k-h \leq \frac{T^2}{h} \leq T^2 \leq U.
\]
Hence, for $1 \leq n \leq T^2$,

$$\theta_{U,T,\ell}(n) = \theta_{\ell}(n) := \sum_{h(2k-h) = n \atop k>h} \frac{h^{1/2}}{(2k-h)^{1/2}} \left(1 - \frac{h}{2k-h}\right)^{\ell-1} \gg \sum_{h(2k-h) = n \atop k>2h} \frac{h^{1/2}}{(2k-h)^{1/2}} \gg \sum_{h,m=n, \ m \geq 3h \atop h \equiv m \mod 2} \frac{\sqrt{h}}{\sqrt{m}} \gg \sum_{h,m=n, \ 3h < m \leq 4h \atop h \equiv m \mod 2} 1.$$ 

Therefore, by (5.11),

$$I(T, U) + c_4 \gg \sum_{T^2/4 \leq n \leq T^2/2 \atop n \ odd} n^{-3/4} \sum_{h,m=n, \ 3h < m \leq 4h} 1 \gg T^{-3/2} \sum_{T^2/4 \leq h,m \leq T^2/2 \atop 3h < m \leq 4h, \ h, m \ odd} 1 \gg T^{1/2}.$$ 

Thus,

$$I(T, U) \gg T^{1/2} \gg (\log U)^{1/4},$$

since (i) readily implies that $T \gg (\log U)^{1/2}$. On the other hand, it follows from the definition of $I(T, U)$ that

$$I(T, U) \leq \left(\sup_{U-1 \leq u \leq U+1} S(u, U)\right) \int_{-1}^{1} F_T(v) \, dv.$$ 

Since

$$\int_{-1}^{1} F_T(v) \, dv = \int_{-T}^{T} \left(\frac{\sin(\pi v)}{\pi v}\right)^2 \, dv \leq 1,$$

this implies that there exists a value $u^* \in [U - 1, U + 1]$ for which

(5.12) \quad $S(u^*, U) \geq c_6 (\log u^*)^{1/4},$

$c_6$ a suitable positive constant. It remains to recall that if $T$ runs through an unbounded sequence of positive reals, by construction so do $U$ and $u^*$. Therefore, (5.1), (5.2), (5.7) and (5.12) together complete the proof of our theorem.

6. Concluding remarks. 1. It is appropriate to comment on the somewhat disturbing restriction that $\ell$ must be even. In general, in (5.2) the argument of the function $\psi$ contains an additional term $-\frac{\ell}{2}$, according to Zhai’s [21] Lemma 3.1. This gives an additional factor $e(\frac{1}{2}h\ell)$ at the right-hand side of (5.6), and as a consequence, additional terms $\pi h\ell$ in the arguments of the sine and cosine in (5.8). For $\ell$ odd, the definitions of $\theta_{U,T,\ell}(n)$ and $\theta_{\ell}(n)$ ultimately contain alternating factors $(-1)^h$ which fatally affect our argument, at least in its present form.
2. It is a natural question whether the sophisticated new methods due to Hafner [4] and Soundararajan [17] can be applied, in order to improve the result by a loglog-factor, as it is the case for the divisor and circle problems. However, both arguments are based on the fact that for \( d(n) \) and \( r(n) \), the average order is accomplished by a "thin" set of integers on which \( d(n) \), resp., \( r(n) \) attain exceptionally large values. The improvement is effected by restricting the application of the Dirichlet approximation theorem to such a thin set.

In our present case, a similar observation concerning the arithmetic function \( \theta_\ell(n) \) is at least not at all straightforward, \textit{inter alia} because \( \theta_\ell(n) \) fails to be multiplicative.

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Institute of Mathematics
Department of Integrative Biology
Universität für Bodenkultur Wien
Gregor Mendel-Straße 33
1180 Wien, Österreich
E-mail: nowak@boku.ac.at