Cotorsion Pairs in Hopfological Algebra

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Abstract

In an intriguing paper [11] Khovanov proposed a generalization of homological algebra, called Hopfological algebra. Since then, several attempts have been made to import tools and techniques from homological algebra to Hopfological algebra. For example, Qi [15] introduced the notion of cofibrant objects in the category $C_{A,H}^H$ of $H$-equivariant modules over an $H$-module algebra $A$, which is a counterpart to the category of modules over a dg algebra, although he did not define a model structure on $C_{A,H}^H$.

In this paper, we show that there exists an Abelian model structure on $C_{A,H}^H$ in which cofibrant objects agree with Qi’s cofibrant objects under a slight modification. This is done by constructing cotorsion pairs in $C_{A,H}^H$ which form a Hovey triple in the sense of Gillespie [7]. This can be regarded as a Hopfological analogue of the work of Enochs, Jenda, and Xu [6] and Avramov, Foxby, and Halperin [1]. By restricting to compact cofibrant objects, we obtain a Waldhausen category $\text{Perf}_{A,H}$ of perfect objects. By taking invariants of this Waldhausen category, such as algebraic $K$-theory, Hochschild homology, cyclic homology, and so on, we obtain Hopfological analogues of these invariants.

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1 Introduction

Khovanov [11] proposed a generalization of homological algebra, called Hopfological algebra, based on finite dimensional Hopf algebras. An important observation of Khovanov is that the existence of an integral in a finite dimensional Hopf algebra $H$ allows us to define an analogue of chain homotopy and homology in the category $H$-$\text{Mod}$ of left $H$-modules, with which an analogue of homological algebra
can be developed, generalizing the fact that the category of chain complexes can be identified with the category of $\mathbb{Z}$-graded modules over the exterior Hopf algebra $\Lambda(d)$.

More generally, a differential graded algebra, dg algebra for short, $A$ is nothing else but a $\mathbb{Z}$-graded $\Lambda(d)$-module algebra. Given a finite dimensional Hopf algebra $H$ over a field $k$ and a left $H$-module algebra $A$, Khovanov proposed to study the category $C^H_{A,H}$ of $H$-equivariant left $A$-modules (see §2.2 for a precise definition) as a generalization of homological algebra over a dg algebra. Following Khovanov’s proposal, Qi [15] defined the derived category of compact objects $\mathcal{D}^c(A,H)$ and defined the Grothendieck group $K_0(A,H)$ of the pair $(A,H)$ as the Grothendieck group of $\mathcal{D}^c(A,H)$.

The analogy between homological and Hopfological algebra can be summarized in the following table.

| homological algebra | Hopfological algebra |
|---------------------|----------------------|
| chain complex       | $H$-module           |
| chain map           | $H$-module homomorphism |
| chain homotopy      | homotopy defined by an integral $\lambda$ |
| homology $H(M) = \ker d / \text{Im } d$ | homology $H(M) = M^H / \lambda \cdot M$ |
| dg algebra          | $H$-module algebra    |
| dg category         | $H$-module category   |
| left module over a dg category | $H$-equivariant left module over an $H$-module category |

Qi also proposed in Remark 7.17 of his paper to define and study the higher algebraic $K$-theory of $\mathcal{D}^c(A,H)$ by using the method introduced by Thomason and Trobaugh [21], in which the algebraic $K$-theory of derived categories (in the sense of usual homological algebra) is defined. However, the definition of $\mathcal{D}^c(A,H)$ is quite different from the usual definition of the derived category of a dg algebra or a dg category, since Qi did not use homology.

Recall that the algebraic $K$-theory of a dg category $A$ is defined as the Waldhausen $K$-theory of the category $\text{Perf}_A$ of compact cofibrant objects in the category $A\text{-Mod}$ of left $A$-modules by introducing a model structure on $A\text{-Mod}$. See Toën’s lecture note [22], for example. For an $H$-module algebra $A$, Qi introduced the notion of cofibrant objects in $C^H_{A,H}$ and proved the existence of a functorial cofibrant replacement functor without introducing a model structure.

This aim of this paper is to construct a model structure on $C^H_{A,H}$ in which cofibrant objects agree with Qi’s cofibrant objects under a slight modification. The modification is needed because of the difference of weak equivalences. We use isomorphisms of homology, while Qi used isomorphisms in the stable category of $H$-modules.

Since $C^H_{A,H}$ is an Abelian category, we should make use of Hovey’s theory of Abelian model structures [8]. By using the terminology of Gillespie [7], given an Abelian category $A$, Hovey found a one-to-one correspondence between Abelian model structures on $A$ and Hovey triples. Recall that a Hovey triple in $A$ is a triple $(\text{Cof, Triv, Fib})$ of subcategories such that both $(\text{Cof, Triv} \cap \text{Fib})$ and $(\text{Cof} \cap \text{Triv, Fib})$ are complete cotorsion pairs and $\text{Triv}$ is a thick subcategory. Recall also that cotorsion pairs are defined in terms of the orthogonality with respect to the biadditive functor $\text{Ext}^1_{A}(\cdot, \cdot)$. See §2.4 for details.

In the case of the category of chain complexes, and, more generally, in the category of left modules over a dg algebra, the corresponding orthogonality has been studied by Enochs, Jenda, and Xu [6] and Avramov, Foxby, and Halperin [1] in detail. As is stated in Hovey’s paper, their results lead to a Hovey triple which gives rise to the standard model structure on such categories, in which cofibrant objects are semiprojective modules.
We introduce the notions of $\Sigma$-semiprojective objects (Definition 3.1) and $\Sigma$-quasi-isomorphisms (Definition 2.31) in $CH$ that are analogues of semiprojective modules and quasi-isomorphisms in the dg context and show that they form a part of a Hovey triple.

**Theorem 1.1.** Let $H$ be a finite dimensional non-semisimple Hopf algebra over a field and $A$ a left $H$-module category. Denote the full subcategories of $CH$ consisting of $\Sigma$-semiprojective objects and of those objects that are $\Sigma$-quasi-isomorphic to 0 by $\text{SemiPrj}_A$ and $\text{Triv}_A$, respectively.

Then the triple $(\text{SemiPrj}_A, \text{Triv}_A, CH)$ is a Hovey triple and thus defines an Abelian model structure on $CH$ in which weak equivalences are $\Sigma$-quasi-isomorphisms, cofibrant objects are $\Sigma$-semiprojective modules, and all objects are fibrant.

By analogy, we call compact cofibrant objects in our model category perfect objects. The full subcategory $\text{Perf}_A$ of perfect objects has a structure of a Waldhausen category. We propose to call the algebraic $K$-theory of this Waldhausen category the algebraic $K$-theory of the pair $(A, H)$. Besides algebraic $K$-theory, the Waldhausen category $\text{Perf}_A$ allows us to extend invariants of dg categories, such as Hochschild homology, cyclic homology, and trace maps between them. Their properties will be studied in a sequel to this paper.

Recall that there is another approach to the algebraic $K$-theory of dg categories, as is described in §5.2 of Keller’s article [10]. Given a dg category $A$, Keller defines the algebraic $K$-theory of $A$ as the $K$-theory of the Waldhausen category of compact $A$-modules, whose cofibrations are morphisms $i : L \to M$ which admits a retraction in the category of $A$-modules, where $A$ is the underlying (graded) linear category of $A$.

We may also define a Hopfological analogue of this construction by using a cotorsion pair. For a left $H$-module category $A$, we define a structure of an exact category on $CH$ by declaring $A$-split extensions as exact sequences, where $A$ is the linear category obtained from $A$ by forgetting the $H$-action. Let us denote this exact category by $C^H_A$. We show that the pair $(C^H_A, \text{Cntr}_A)$ is a complete cotorsion pair, where $\text{Cntr}_A$ is the full subcategory consisting of objects that are homotopy equivalent to 0 in the category of left $H$-modules. See Proposition 3.18. Although this cotorsion pair is not part of a Hovey triple, a recent work of Sarazola’s [18] allows us to construct a Waldhausen category from this cotorsion pair and $\text{Triv}_A$. The Waldhausen subcategory of compact objects is another choice for defining algebraic $K$-theory of $(A, H)$. We note that this is closer to Qi’s approach to the Grothendieck group of $D^c(A, H)$.

Finally we remark that Kaygun and Khalkhali [9] introduced another kind of “projective” modules for an $H$-module algebra $A$, called $H$-equivariantly projective $A$-modules in their paper. Their purpose is to define the Hopf-cyclic homology of $A$ by using the exact category of $H$-equivariantly projective $A$-modules. We may use this exact category to define an algebraic $K$-theory. From this point of view, however, the action of $H$ is regarded as a generalization of group actions, while, in Hopfological algebra, the action of $H$ is a generalization of differentials. Thus the $K$-theory obtained from $H$-equivariant projective $A$-modules should be regarded as a generalization of Thomason’s equivariant $K$-theory [20] and is different from ours.

**Organization**

The rest of this paper consists of two sections.
• §2 is preliminary. Notations and conventions used in this paper are listed in §2.1. We recall basic properties of the category of $H$-equivariant $A$-modules $C_{A,H}^H$ in §2.2. Basics ideas in Hopfological algebra are recalled in §2.3. And §2.4 is a brief summary of Hovey’s theory of Abelian model structures used in this paper.

• §3 is the main part. In §3.1, the notion of $\Sigma$-semiprojective modules and related structures are introduced and studied, with which Theorem 1.1 is proved in §3.2 by studying the orthogonality in the category $C_{A,H}^H$.

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2 Preliminaries

2.1 Notations and conventions
In this paper, we fix a finite dimensional Hopf algebra $H$ over a field $k$. The coproduct, the counit, and the antipode are denoted by $\Delta$, $\varepsilon$, and $S$, respectively. We also fix a left integral $\lambda$ in $H$.

Other notations and conventions used in this paper are summarized in the following list.

• The tensor product over $k$ is denoted by $\otimes$.

• The category of $k$-modules is a symmetric monoidal category under $\otimes$. The morphism induced by a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ is denoted by

$$\sigma : M_1 \otimes \cdots \otimes M_n \longrightarrow M_{\sigma(1)} \otimes \cdots \otimes M_{\sigma(n)}.$$ 

For example, the symmetric monoidal structure $M_1 \otimes M_2 \rightarrow M_2 \otimes M_1$ is denoted by $(1, 2)$.

• The category of left $H$-modules is denoted by $H$-$\text{Mod}$. The full subcategory of finitely generated $H$-modules is denoted by $H$-$\text{mod}$. Given left $H$-modules $\mu_M : H \otimes M \rightarrow M$ and $\mu_N : H \otimes N \rightarrow N$, the left $H$-action on $M \otimes N$ is given by

$$H \otimes M \otimes N \xrightarrow{\Delta \otimes 1 \otimes 1} H \otimes H \otimes M \otimes N \xrightarrow{(2,3)} H \otimes M \otimes H \otimes N \xrightarrow{\mu_M \otimes \mu_N} M \otimes N.$$ 

The categories $H$-$\text{Mod}$ and $H$-$\text{mod}$ are regarded as monoidal categories under this tensor product.

• We regard $k$ as an $H$-module via the counit $\varepsilon : H \rightarrow k$ so that $\varepsilon$ is a morphism in $H$-$\text{Mod}$.

• We use Sweedler’s notation for coproducts, i.e.

$$\Delta(h) = h_{(1)} \otimes h_{(2)}$$

for $h \in H$. We also use an analogous notation for comodules.
For a category \( C \) and objects \( x, y \) in \( C \), the set of morphisms from \( x \) to \( y \) is denoted by \( C(x, y) \). When \( C \) is small, the sets of objects and morphisms are denoted by \( C_0 \) and \( C_1 \), respectively. The source, the target, and the unit maps are denoted by

\[
\begin{align*}
  s & : C_1 \rightarrow C_0 \\
  t & : C_1 \rightarrow C_0 \\
  \eta & : C_0 \rightarrow C_1,
\end{align*}
\]

respectively.

### 2.2 Module categories over a Hopf algebra and their modules

Khovanov and Qi studied \( H \)-module algebras and their modules. We would like to be slightly more general, since we are interested in Hopfological analogues of dg categories. We regard an \( H \)-module algebra as a one-object \( H \)-module category.

**Definition 2.1.** A left \( H \)-module category is a category enriched over the monoidal category \( H\text{-Mod} \). When it has a single object, it is called a left \( H \)-module algebra.

By forgetting the action of \( H \), we obtain the underlying \( k \)-linear category or \( k \)-algebra, which is denoted by \( A \).

We are interested in the category of left \( H \)-equivariant \( A \)-modules for a left \( H \)-module category \( A \).

In order to give a precise definition, we first need to fix notation and terminology for \( k \)-linear categories.

**Lemma 2.2.** For a \( k \)-module \( M \) and a set \( S \), there is a one-to-one correspondence between families of submodules \( \{M_s\}_{s \in S} \) indexed by \( S \) with \( M = \bigoplus_{s \in S} M_s \) and comodule structures on \( M \) over the free \( k \)-module \( kS \) spanned by \( S \), whose coalgebra structure is induced by the diagonal map on \( S \).

**Proof.** Given a comodule structure \( \delta : M \rightarrow kS \otimes M \), define

\[
M_s = \{m \in M \mid \exists m' \text{ s.t. } \delta(m) = s \otimes m' \}.
\]

Note that \( M_s \cap M_{s'} = 0 \) if \( s \neq s' \). Suppose \( \delta(m) = \sum_s s \otimes m_s \). Then the coassociativity implies that \( m_s \) belongs to \( M_s \) and the counit condition implies that \( \sum_{s \in S} m_s = m \). And we have \( M = \bigoplus_s M_s \).

Conversely a family of submodules with \( M = \bigoplus_{s \in S} M_s \) gives rise to a map \( \delta : M \rightarrow kS \otimes M \) by \( \delta(m) = s \otimes m \) for \( m \in M_s \). This is a comodule structure on \( M \).

**Remark 2.3.** This observation is due to Cohen and Montgomery [4]. The second author learned this fact from Hideto Asashiba.

Let \( A \) be a small \( k \)-linear category. Recall from section 1 that the set of objects and the modules of morphisms in \( A \) are denoted by \( A_0 \) and \( A_1 \), respectively. By Lemma 2.2, we may regard the total morphism space

\[
A_1 = \bigoplus_{x, y \in A_0} A(x, y).
\]

as a \( kA_0 \cdot kA_0 \)-bicomodule, where the right comodule structure is given by the source map and the left comodule structure is given by the target map. For simplicity, we use the following notations.
Convention 2.4.

- The free $k$-module $kA_0$ is denoted by $A_0$ and is regarded as a $k$-coalgebra by the diagonal of $A_0$.
- We identify $A$ with the module of all morphisms $A_1$ so that $A$ is an $A_0$-$A_0$-bicomodule.

Recall that given a right comodule $M$ and a left comodule $N$ over a coalgebra $C$, the cotensor product $M \square_C N$ is defined by the equalizer

$$M \square_C N \longrightarrow M \otimes N \xrightarrow{\delta_M \otimes 1_{\otimes_M \otimes N}} M \otimes C \otimes N ,$$

where $\delta_M$ and $\delta_N$ are comodule structure maps for $M$ and $N$, respectively. With this notation, we may identify

$$A \square_{A_0} A = \left( \bigoplus_{y,z \in A_0} \big( \bigoplus_{x, y \in A_0} A(x, y) \big) \right) \otimes A(y, z) \otimes A(x, y)$$

so that the composition of morphisms and the unit are given by bicomodule maps

$$\mu_A : A \square_{A_0} A \longrightarrow A$$

$$\eta_A : A_0 \longrightarrow A.$$

In other words, we regard a $k$-linear category as a monoid objects in the monoidal category of $A_0$-$A_0$-bicomodules whose monoidal structure is given by $\square_{A_0}$.

Definition 2.5. Let $A$ be a $k$-linear category. A left $A$-module consists of

- a left $A_0$-comodule $M$, and
- a morphism of left $A_0$-comodules $\mu_{A,M} : A \square_{A_0} M \rightarrow M$,

which satisfy the unit and the associativity conditions. For left $A$-modules $M$ and $N$, a morphism of left $A_0$-comodules $f : M \rightarrow N$ is called an $A$-module homomorphism if it commutes with the actions of $A$.

The category of left $A$-modules and $A$-module homomorphisms is denoted by $A$-$\text{Mod}$.

Remark 2.6. Thanks to Lemma 2.2, a left $A$-module $M$ can be regarded as a collection $\{M(x)\}$ of $k$-modules indexed by objects of $A$ equipped with a family of $k$-linear maps

$$A(x, y) \otimes M(x) \longrightarrow M(y)$$

satisfying the associativity and the unit conditions. In other words, a left $A$-module is nothing but a functor $A \rightarrow k$-$\text{Mod}$. Similarly, a right $A$-module is a contravariant functor from $A$ to $k$-$\text{Mod}$.

When $A$ is a left $H$-module category, we need to incorporate the action of $H$ as follows.

Definition 2.7. Let $A$ be a left $H$-module category. A left $H$-equivariant $A$-module consists of

- a left $A$-module $\mu_{A,M} : A \square_{A_0} M \rightarrow M$ and
- a left $H$-module structure $\mu_{H,M} : H \otimes M \rightarrow M$ on $M$

satisfying the following conditions:
(1) $\mu_{H,M}$ is a morphism of $A_0$-comodules.

(2) $\mu_{A,M}$ and $\mu_{H,M}$ are compatible in the sense that the following diagram is commutative

$$
\begin{array}{ccc}
H \otimes A \square A_0 M & \xrightarrow{1 \otimes \mu_{A,M}} & H \otimes M \\
\Delta \otimes 1 & \downarrow & \mu_{H,M} \\
H \otimes H \otimes A \square A_0 M & \xrightarrow{(2.3)} & (H \otimes A) \square A_0 (H \otimes M)
\end{array}
$$

Given two $H$-equivariant $A$-modules $M$ and $N$, an $H$-equivariant morphism from $M$ to $N$ is a morphism $f : M \to N$ of left $A_0$-comodules which commutes with both $A$-module structures and $H$-module structures.

The category of $H$-equivariant left $A$-modules and $A$-module homomorphisms is denoted by $C_{A,H}$. The wide subcategory of $H$-equivariant morphisms in $C_{A,H}$ is denoted by $C_{A,H}^H$.

Remark 2.8. When $A = k$ with the $H$-action given by $\varepsilon : H \to k$, we have an identification $C_{k,H}^H = H$-Mod.

The following fact plays a fundamental role in Hopfological algebra. See section 5.1 of Qi’s paper [15] for the case of $H$-module algebra. It is straightforward to obtain a generalization to the case of $H$-module category.

**Proposition 2.9.** For a left $H$-module category $A$ and left $H$-equivariant $A$-modules $M, N$, define a left $H$-action on $C_{A,H}(M, N)$ by

$$(hf)(m) = h_{(2)} f (S^{-1}(h_{(1)}) m)$$

for $h \in H$, $f \in C_{A,H}(M, N) = (A\text{-Mod})(M, N)$, and $m \in M$, where $S$ is the antipode of $H$. Then $C_{A,H}(M, N)$ becomes a left $H$-module with which the compositions of morphisms are $H$-module homomorphisms and the identity morphisms are $H$-invariant. In other words, $C_{A,H}$ becomes a left $H$-module category.

Recall that for an $H$-module $V$, the submodule of invariants is defined by

$$V^H = \{ x \in V \mid hx = \varepsilon(h)x \text{ for all } h \in H \}.$$

Our notation $C_{A,H}^H$ is designed to fit into the following identification.

**Corollary 2.10.** Let $f$ be a morphism in $C_{A,H}$. Then $hf = \varepsilon(h)f$ for all $h \in H$ if and only if $f$ is an $H$-module homomorphism. In other words, under the $H$-module structure on $C_{A,H}(M, N)$ in Proposition 2.9, we have

$$(C_{A,H}(M, N))^H = C_{A,H}^H(M, N).$$

A left $H$-module algebra $A$ gives rise to a new algebra $A \# H$ by the smash product construction. The construction can be extended to $H$-module categories.
Definition 2.11. Let $A$ be a left $H$-module category. Define a $k$-linear category $A\#H$ as follows. The objects are

$$(A\#H)_0 = A_0.$$ 

For $x, y \in (A\#H)_0$, the module of morphisms from $x$ to $y$ is

$$(A\#H)(x, y) = A(x, y) \otimes H$$

so that the module of morphisms is

$$A\#H = \bigoplus_{x, y \in A_0} A(x, y) \otimes H = A \otimes H.$$ 

The unit is given by

$$A_0 \cong A_0 \otimes k \xrightarrow{\eta_A \otimes \eta_H} A \otimes H,$$

where $\eta_H : k \to H$ is the unit of $H$. The composition of morphisms is given by

$$(A \otimes H) \square_{A_0} (A \otimes H) \xrightarrow{1 \otimes \mu_H \otimes 1} A \square_{A_0} (H \otimes A) \otimes (H \otimes H) \xrightarrow{\nu_A \otimes 1} A \otimes A_0 \otimes H \xrightarrow{\mu_A} A \otimes H.$$ 

The following description of $C_{A,H}^H$ is well known when $A$ is an $H$-module algebra. The proof is essentially the same as the case of $H$-module algebras and is omitted.

Proposition 2.12. For an $H$-module category $A$, the category $C_{A,H}^H$ is equivalent to $(A\#H)\text{-Mod}$.

Another important fact in Hopfological algebra is that $C_{A,H}^H$ is a module category over the monoidal category $(H\text{-Mod}, \otimes, k)$. Following Qi’s paper [15], we use a right action. Recall that a right action of a monoidal category $(V, \otimes, 1)$ on a category $C$ consists of a functor

$$\otimes : C \times V \to C,$$

a natural isomorphism

$$a : (X \otimes V) \otimes W \xrightarrow{\cong} X \otimes (V \otimes W)$$

for $X \in C_0$ and $V, W \in V_0$ satisfying the pentagon axiom, and a natural isomorphism

$$r : X \otimes 1 \xrightarrow{\cong} X$$

for $X \in C_0$ satisfying the unit axiom. A precise definition can be found, for example, in Ostrik’s paper [14], who refers to Bernstein’s lecture note [3] and a paper [5] by Crane and Frenkel for the first appearance in the literature.

In the case of $C_{A,H}^H$, the right action of $H\text{-Mod}$ is given by $M \otimes V$ for $M$ in $C_{A,H}^H$ and $V$ in $H\text{-Mod}$. The left $A$-module structure is given by that of $M$ and the left $H$-module structure is given by the composition

$$H \otimes M \otimes V \xrightarrow{\Delta \otimes 1 \otimes 1} H \otimes H \otimes M \otimes V \xrightarrow{(2,3)} H \otimes M \otimes H \otimes V \to M \otimes V,$$

should not be confused with an $H$-module category, which means a category enriched over $H\text{-Mod}$. 

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where the last map is given by the $H$-module structures of $M$ and $V$.

This action of $H$-$\text{Mod}$ allowed Khovanov to introduce functors

$$C, \Sigma : C^H_{A,H} \longrightarrow C^H_{A,H}$$

by

$$C(M) = M \otimes H$$

$$\Sigma(M) = M \otimes (H/(\lambda)),$$

respectively. These are called the cone and the suspension functors, respectively.

The cone functor allows us to define an analogue of chain homotopy.

**Definition 2.13.** Two morphisms $f, g : M \rightarrow N$ in $C^H_{A,H}$ are called homotopic if there exists a morphism $\varphi : C(M) \rightarrow N$ making the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{f-g} & N \\
\downarrow^\varphi & & \downarrow^f \\
M \otimes k & \xrightarrow{1 \otimes \lambda} & M \otimes H & \xrightarrow{C(M)} \\
\end{array}
\]

commutative, in which case, we denote $f \simeq g$. The morphism $\varphi$ is called a homotopy from $f$ to $g$.

The homotopy category or the stable category of $C^H_{A,H}$, denoted by $\mathcal{T}^H_{A,H}$, is the category having the same objects as $C^H_{A,H}$ whose set of morphisms from $M$ to $N$ is given by

$$\mathcal{T}^H_{A,H}(M, N) = C^H_{A,H}(M, N)/\sim.$$

When $A = k$, under the identification $C^H_{k,H} = H$-$\text{Mod}$, we denote

$$\text{ho}(H$-$\text{Mod}) = \mathcal{T}^H_{k,H}.$$

Khovanov noticed that the homotopy category $\mathcal{T}^H_{A,H}$ has a structure of triangulated category with $\Sigma$ a shift functor. An inverse to $\Sigma$ on $\mathcal{T}^H_{A,H}$ is given by

$$\Sigma^{-1}(M) = M \otimes (\text{Ker}\, \varepsilon),$$

which is called the desuspension functor.

**Lemma 2.14.** The functors $\Sigma$ and $\Sigma^{-1}$ induce functors on $\mathcal{T}^H_{A,H}$ that are inverse to each other.

**Proof.** Khovanov showed that there exists a projective $H$-module $Q$ such that

$$\text{Ker}\,(\varepsilon) \otimes (H/(\lambda)) \cong k \oplus Q.$$

Thus

$$\Sigma(\Sigma^{-1}(M)) \cong M \oplus M \otimes Q$$

for any $M$ in $C^H_{A,H}$. By Lemma 2.15 below, we have $M \otimes Q \cong 0$ in $\mathcal{T}^H_{A,H}$.

Similarly we see that $\Sigma^{-1}(\Sigma(M)) \cong M$ in $\mathcal{T}^H_{A,H}$. 

\[\square\]
The following fact used in the above proof is stated as Proposition 2 in Khovanov’s paper [11]. Khovanov refers to Montgomery’s book [13] for a proof.

**Lemma 2.15.** For any $H$-module $M$, $H \otimes M$ is a free $H$-module. If $M$ is of finite dimensional over $k$. Then $H \otimes M$ is a free $H$-module of rank $\dim_k M$.

**Corollary 2.16.** Let $P$ be a projective $H$-module. Then, for any $H$-module $M$, both $P \otimes M$ and $M \otimes P$ are projective $H$-modules.

In order to describe distinguished triangles, let us recall the definition of mapping cones from Khovanov’s paper, which is essentially the same as the definition of mapping cones in the category of chain complexes.

**Definition 2.17.** Given a morphism $f : M \to N$ in $C^H_{A,H}$, the pushout of the extension

$$0 \to M \xrightarrow{i_M} C(M) \xrightarrow{P_M} \Sigma(M) \to 0$$

along $f$ is denoted by

$$0 \to N \xrightarrow{j_f} C_f \xrightarrow{\delta_f} \Sigma(M) \to 0.$$ 

Here $i_M$ is the composition $M \cong M \otimes k \xrightarrow{1 \otimes \lambda} M \otimes H = C(M)$. The object $C_f$ is called the mapping cone of $f$.

A sequence of the form

$$M \xrightarrow{f} N \xrightarrow{j_f} C_f \xrightarrow{\delta_f} \Sigma(M)$$

for a morphism $f : M \to N$ is called a standard triangle in $T^H_{A,H}$.

**Theorem 2.18** (Khovanov). The category $T^H_{A,H}$ becomes a triangulated category with $\Sigma$ a shift functor, by declaring a sequence $X \to Y \to Z$ in $T^H_{A,H}$ to be a distinguished triangle if it is isomorphic to a standard triangle.

The following fact can be verified immediately.

**Lemma 2.19.** Let $M$ and $N$ be $H$-equivariant $A$-modules. For any $H$-module $V$, the canonical isomorphism of $k$-modules

$$C_{A,H}(M, N) \otimes V \to C_{A,H}(M, N \otimes V)$$

is an isomorphism of $H$-modules. In particular, we have isomorphisms of $H$-modules

$$\Sigma C_{A,H}(M, N) \cong C_{A,H}(M, \Sigma N)$$

$$\Sigma^{-1} C_{A,H}(M, N) \cong C_{A,H}(M, \Sigma^{-1} N).$$

Recall that the underlying $k$-linear category of an $H$-module category $A$ is denoted by $\underline{A}$.

**Definition 2.20.** Let $U : C^H_{A,H} \to \underline{A}$-Mod be the forgetful functor. We say a short exact sequence

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

$C^H_{A,H}$ is $A$-split if $0 \to U(L) \to U(M) \to U(N) \to 0$ is a split short exact sequence in $\underline{A}$-Mod.

Qi found a characterization of distinguished triangles in $T^H_{A,H}$ in terms of $A$-split sequences. See Lemma 4.3 of [15].
Lemma 2.21. Let
\[ 0 \longrightarrow L \overset{f}{\longrightarrow} M \overset{g}{\longrightarrow} N \longrightarrow 0 \]
be an $A$-split short exact sequence in $\mathcal{C}^H_{A,H}$. Then there exists a distinguished triangle in $\mathcal{T}^H_{A,H}$ of the form
\[ L \overset{[f]}{\longrightarrow} M \overset{[g]}{\longrightarrow} N \overset{\delta}{\longrightarrow} \Sigma(L). \]
Conversely, any distinguished triangle in $\mathcal{T}^H_{A,H}$ is isomorphic to the one that arises from an $A$-split short exact sequence in $C^H_{A,H}$.

The following useful fact is proved as Lemma 4.4 in Qi’s paper and used in the proof of Lemma 2.21.

Lemma 2.22. Any $A$-split extension of the form
\[ 0 \longrightarrow L \longrightarrow M \longrightarrow Z \otimes H \longrightarrow 0 \]
splits.

Definition 2.23. We say a short exact sequence
\[ 0 \longrightarrow L \overset{f}{\longrightarrow} M \overset{g}{\longrightarrow} N \longrightarrow 0 \]
in $\mathcal{C}^H_{A,H}$ homotopically splits if there exists a morphism $s : N \to M$ with $g \circ s \simeq 1_N$. The morphism $s$ is called a homotopy section of $g$.

Corollary 2.24. Let
\[ 0 \longrightarrow L \overset{f}{\longrightarrow} M \overset{g}{\longrightarrow} N \longrightarrow 0 \]
be an $A$-split short exact sequence in $\mathcal{C}^H_{A,H}$. Then it homotopically splits if and only if there exists a morphism $t : M \to L$ with $f \circ t \simeq 1_L$.

Proof. Since the sequence splits in $\mathcal{A}$-$\text{Mod}$, it defines a triangle
\[ L \overset{[f]}{\longrightarrow} M \overset{[g]}{\longrightarrow} N \overset{\delta}{\longrightarrow} \Sigma(L) \]
in $\mathcal{T}^H_{A,H}$. Then $f$ has a homotopy section if and only if this triangle is isomorphic to the trivial triangle, which in turn is equivalent to saying that $[f]$ is a section, or there exists a morphism $t : M \to L$ with $f \circ t \simeq 1_L$.

We have the following closely related fact.

Lemma 2.25. A morphism $f : X \to Y$ in $\mathcal{C}^H_{A,H}$ is homotopic to 0, if and only if there exists an isomorphism of extensions
\[ 0 \longrightarrow Y \overset{1_Y \cdot 0}{\longrightarrow} Y \oplus \Sigma(X) \overset{\text{pr}_2}{\longrightarrow} \Sigma(X) \longrightarrow 0. \]
Proof. Suppose \( f \simeq 0 \). By definition, there exists \( h : C(X) \to Y \) such that \( f = h \circ i_X \), which gives rise to a morphism \( r : Cf \to Y \) making the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{i_X} & & \downarrow{r} \\
C(X) & \xrightarrow{q_f} & Cf \\
& \downarrow{h} & \downarrow{i_Y} \\
& Y &
\end{array}
\]

commutative. Then we have

\[
(1 - j_f \circ r) \circ j_f = j_f - j_f = 0.
\]

Since \( \delta_f \) is a cokernel of \( j_f \), and we obtain a morphism \( s : \Sigma(X) \to Cf \) with

\[
1 - j_f \circ r = s \circ \delta_f.
\]

Then the maps

\[
(r, \delta_f) : Cf \to Y \oplus \Sigma(X) \\
j_f + s : Y \oplus \Sigma(X) \to Cf
\]

are inverse to each other and we obtain an isomorphism of extensions that we wanted.

Conversely, suppose we have a map \( \varphi : Cf \to Y \oplus \Sigma(X) \) which defines an isomorphism of extensions. Then in the pushout diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{i_X} & & \downarrow{j_f} \\
C(X) & \xrightarrow{q_f} & Cf \\
& \downarrow{h} & \downarrow{\Sigma(C)} \\
& Y &
\end{array}
\]

the composition \( \text{pr}_2 \circ \varphi : Cf \to Y \) defines a left inverse to \( j_f \) and thus \( f \simeq 0 \).

The mapping cone construction has the following nice property.

**Lemma 2.26.** Let \( f : M \to N \) be a morphism in \( CH_{A,H} \) and \( P \) be an object of \( CH_{A,H} \) which is projective as an \( A \)-module. Let \( CA_{A,H}(P, f) : CA_{A,H}(P, M) \to CA_{A,H}(P, N) \) be the morphism induced by \( f \), then we have a natural isomorphism

\[
CA_{A,H}(P, f) \simeq CA_{A,H}(P, CF).
\]

**Proof.** Since \( P \) is projective as an \( A \)-module, \( CA_{A,H}(P, -) \) is an exact functor and we obtain a diagram of short exact sequences

\[
\begin{array}{ccccccc}
0 & \longrightarrow & CA_{A,H}(P, N) & \longrightarrow & CA_{A,H}(P, f) & \longrightarrow & \Sigma CA_{A,H}(P, M) & \longrightarrow & 0 \\
& & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} & & \\
0 & \longrightarrow & CA_{A,H}(P, N) & \longrightarrow & CA_{A,H}(P, CF) & \longrightarrow & CA_{A,H}(P, \Sigma(M)) & \longrightarrow & 0,
\end{array}
\]

where the middle vertical arrow is the morphism obtained by the universality of pushout. Since the right vertical arrow is an isomorphism by Lemma 2.19, so is the middle vertical arrow.
2.3 Homological algebra in Hopfological algebra

In order to perform homological algebra in $C_{A,H}^H$, $C_{A,H}$, and $\mathcal{T}^H_{A,H}$, we need homology. Let us recall the definition from Qi’s paper.

**Definition 2.27.** For a left $H$-module $M$, we denote

\[
Z(M) = M^H = \{ m \in M \mid hm = \varepsilon(h)m \text{ for all } h \in H \}
\]

\[
B(M) = \lambda M
\]

\[
H(M) = Z(M) / B(M).
\]

The functor $H : H\text{-Mod} \rightarrow k\text{-Mod}$ is called the **canonical homological functor**. The composition with the forgetful functor $C_{A,H}^H \rightarrow H\text{-Mod}$ is also denoted by

\[
H : C_{A,H}^H \rightarrow k\text{-Mod}.
\]

**Example 2.28.** Suppose $P$ is projective as an $H$-module. There exists a free $H$-module $F$ with $F \cong P \oplus Q$ as $H$-modules. The homology of $H$ is trivial, since $Z(H)$ is the submodule of integrals, which is known to be of 1-dimensional over $k$ generated by a fixed integral $\lambda$. It implies that $H(F) = 0$, and we have $H(P) = 0$. In particular, $H(C(M)) = 0$ for any $H$-equivariant $A$-module $M$ and we see that the canonical homological functor descends to $H : \mathcal{T}^H_{A,H} \rightarrow k\text{-Mod}$.

**Example 2.29.** Let $M$ and $N$ be $H$-equivariant $A$-modules. Then by Corollary 2.10, we have

\[
Z(C_{A,H}(M,N)) = C_{A,H}^H(M,N).
\]

For morphisms $f, g : M \rightarrow N$ in $C_{A,H}^H$, $f \simeq g$ if and only if $f - g \in B(C_{A,H}(M,N))$ by definition.

Thus $H(C_{A,H}(M,N))$ can be identified with the set of $H$-homotopy classes of $H$-equivariant $A$-module maps from $M$ to $N$. In other words,

\[
H(C_{A,H}(M,N)) = \mathcal{T}^H_{A,H}(M,N).
\]

In particular, we have an isomorphism

\[
\text{Ext}^1_{\mathcal{T}^H_{A,H}}(M,N) \cong \mathcal{T}^H_{A,H}(M,\Sigma(N)) = H(C_{A,H}(M,\Sigma(N))).
\]

**Proposition 2.30.** The canonical homological functor is homological, i.e. any triangle in $\mathcal{T}^H_{A,H}$ induces a long exact sequence by the canonical homological functor $H$.

**Proof.** For a left $H$-module $M$, we have an identification

\[
M \cong (H\text{-Mod})(k,M) = C_{k,H}^H(k,M).
\]

in $H\text{-Mod} = C_{k,H}^H$ and we have

\[
H(M) \cong \mathcal{T}^H_{k,H}(k,M)
\]

by the previous example. Since $\mathcal{T}^H_{k,H}$ is a triangulated category, this is a homological functor. The functor $\mathcal{T}^H_{A,H} \rightarrow \mathcal{T}^H_{k,H}$ which forgets $A$-module structures preserves triangles and thus

\[
H : \mathcal{T}^H_{A,H} \rightarrow k\text{-Mod}
\]

is also homological. □
Definition 2.31. A morphism \( f : M \to N \) in \( \mathcal{C}^H_{A,H} \) is called a quasi-isomorphism or a quism if the induced map \( H(f) : H(M) \to H(N) \) is an isomorphism in \( k\text{-Mod} \). It is called a \( \Sigma \)-quism if \( H(\Sigma^n(f)) : H(\Sigma^n(M)) \to H(\Sigma^n(N)) \) is an isomorphism for all \( n \in \mathbb{Z} \). The wide subcategory of \( \Sigma \)-quisms is denoted by \( \text{Quism}^\Sigma \).

An object \( M \) of \( \mathcal{C}^H_{A,H} \) is called acyclic if \( H(M) = 0 \). The class of acyclic objects is denoted by \( \text{Triv}^A_{A,H} \). If \( H(\Sigma^n M) = 0 \) for all \( n \in \mathbb{Z} \), \( M \) is called \( \Sigma \)-acyclic. The class of \( \Sigma \)-acyclic objects is denoted by \( \text{Triv}^{\Sigma}_{A,H} \). We regard them as full subcategories of \( \mathcal{C}^H_{A,H} \) or \( \mathcal{T}^H_{A,H} \).

Remark 2.32. Khovanov [11] and Qi [15] used a different notion of quasi-isomorphisms. A morphism \( f : M \to N \) in \( \mathcal{C}^H_{A,H} \) is a quasi-isomorphism in their sense if it is a homotopy equivalence in \( H\text{-Mod} \).

Hence their acyclic objects are different from ours.

Lemma 2.33. The category \( \text{Triv}^\Sigma_{A,H} \) is a thick subcategory of both \( \mathcal{C}^H_{A,H} \) and \( \mathcal{T}^H_{A,H} \).

Proof. Since the canonical homological functor commutes with direct sums, \( \text{Triv}^\Sigma_{A,H} \) is closed under taking direct summands. The two-out-of-three property follows from the fact that \( H \) is a homological functor and the fact that \( \text{Triv}^\Sigma_{A,H} \) is closed under \( \Sigma \).

Example 2.34. For any \( M \) and \( n \in \mathbb{Z} \), \( \Sigma^n C(M) \) is acyclic, i.e. \( C(M) \) is \( \Sigma \)-acyclic. This can be verified as follows.

Suppose \( n \geq 0 \). By Lemma 2.35, we have an isomorphism of \( H \)-modules

\[
\Sigma^n C(M) = M \otimes H \otimes (H/(\lambda))^\otimes n \cong H \otimes M \otimes (H/(\lambda))^\otimes n.
\]

This is a free left \( H \)-module by Lemma 2.15 and hence is acyclic by Example 2.28. By replacing \( H/(\lambda) \) by Ker \( \varepsilon \), we see that \( \Sigma^n C(M) \) is acyclic for \( n < 0 \).

The following fact, used in the above argument, appears as Lemma 2 in Khovanov’s paper [11].

Lemma 2.35. In the category of \( H \)-modules, there exists an isomorphism \( r : V \otimes H \to H \otimes V \) which is natural in \( V \) and makes the following diagram commutative.

\[
\begin{array}{ccc}
V \otimes k & \cong & V \\
\downarrow \text{id}_V \otimes \lambda & & \downarrow \lambda \otimes \text{id}_V \\
V \otimes H & \underset{r}{\longrightarrow} & H \otimes V.
\end{array}
\]

The following is an analogue of Proposition 2.3.5 (1) in [1].

Lemma 2.36. Let \( M, N \) be objects of \( \mathcal{C}^H_{A,H} \). If \( f : M \to N \) is a surjective quism, then both \( B(f) : B(M) \to B(N) \) and \( Z(f) : Z(M) \to Z(N) \) are surjective.

Proof. The morphism \( f \) gives rise to a commutative diagram

\[
\begin{array}{ccc}
M & \overset{f}{\longrightarrow} & B(M) \\
\downarrow & & \downarrow B(f) \\
N & \longrightarrow & B(N).
\end{array}
\]

Since horizontal arrows are surjective, if \( f \) is surjective, then so is \( B(f) \).
By assumption, \( H(f) \) is an isomorphism. The commutativity of the diagram of extensions

\[
\begin{array}{c}
0 & \rightarrow & B(M) & \rightarrow & Z(M) & \rightarrow & H(M) & \rightarrow & 0 \\
\downarrow{B(f)} & & \downarrow{Z(f)} & & \downarrow{H(f)} & & \\
0 & \rightarrow & B(N) & \rightarrow & Z(N) & \rightarrow & H(N) & \rightarrow & 0
\end{array}
\]

implies that \( Z(f) \) is surjective.

Recall that the forgetful functor from \( C_{A,H}^H \) to \( A\text{-Mod} \) is denoted by \( U \). Left and right adjoints to this functor are useful in studying the orthogonality in \( C_{A,H}^H \). By regarding an \( A \)-module \( M \) as a trivial \( H \)-module, the cone functor \( C: C_{A,H}^H \rightarrow C_{A,H}^H \) gives us a functor

\[
C: A\text{-Mod} \rightarrow C_{A,H}^H.
\]

The following fact is obvious.

**Lemma 2.37.** The cone functor \( C \) is an exact functor which is left adjoint to \( U \).

The functor \( U \) also has a right adjoint.

**Definition 2.38.** Define a functor

\[
E : A\text{-Mod} \rightarrow A\text{-Mod}
\]

by \( E(M) = (k\text{-Mod})(H,M) \). The \( A \)-module structure is defined by \( (a\varphi)(g) = a\varphi(g) \) for \( a \in A, \varphi \in E(M) \), and \( g \in H \).

**Lemma 2.39.** For \( \varphi \in E(M) \), define an action of \( h \in H \) on \( \varphi \) by

\[
(h \cdot \varphi)(g) = \varphi(S(h)g)
\]

Then it defines a left \( H \)-module structure on \( E(M) \). It is compatible with the action of \( A \) and we obtain an exact functor

\[
E : A\text{-Mod} \rightarrow C_{A,H}^H.
\]

**Proof.** Let us verify the associativity. For \( h, h', g \in H \),

\[
(h \cdot (h' \cdot \varphi))(g) = (h \cdot \varphi)(S(h')g)
\]

\[
= \varphi(S(h')S(h)g)
\]

\[
= \varphi(S(hh')g)
\]

\[
= ((hh') \cdot \varphi)(g).
\]

We also have \( 1 \cdot \varphi = \varphi \), since \( S(1) = 1 \).

In order to verify that \( E(M) \) is an \( H \)-equivariant \( A \)-module, let \( a \in A, \varphi \in E(M) \), and \( g, h \in H \).

Then

\[
((h(1)a)(h(2)\varphi))(g) = (h(1)a)((h(2)\varphi)(g))
\]

\[
= (h(1)a)\varphi(S(h(2))g)
\]

\[
= (\varepsilon(h(1))a)\varphi(S(h(2))g)
\]

\[
= a\varphi(S(\varepsilon(h(1))h(2))g)
\]

\[
= a\varphi(S(h)g)
\]

\[
= (h \cdot (a\varphi))(g),
\]

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which means that the $A$-module structure on $E(M)$ is compatible with the $H$-module structure.

Since $k$ is a field, $E$ is an exact functor.

It is a fundamental fact that if $H$ is finite dimensional, the antipode $S$ is bijective. See [12] or Corollary 5.1.6 of Sweedler’s book [19], for example.

**Proposition 2.40.** The functor $E$ is right adjoint to the forgetful functor $U : C_{A,H}^H \to \underline{A}\text{-Mod}$.

**Proof.** For an $H$-equivariant $A$-module $M$ and an $A$-module $N$, define

$$\Phi : (\underline{A}\text{-Mod})(U(M), N) \to C_{A,H}^H (M, E(N))$$

by

$$\Phi(\varphi)(m)(h) = \varphi(S^{-1}(h)m)$$

for $\varphi \in (\underline{A}\text{-Mod})(U(M), N)$, $m \in M$ and $h \in H$. Then $\Phi(\varphi)$ is a $H$-module homomorphism, since

$$\Phi(\varphi)(h'm)(h) = \varphi(S^{-1}(h)h'm)$$

$$= \varphi(S^{-1}(S(h')h)m)$$

$$= \varphi(S^{-1}(h)S^{-1}(h')m)$$

Define

$$\Psi : C_{A,H}^H (M, E(N)) \to (\underline{A}\text{-Mod})(U(M), N)$$

by

$$\Psi(\psi)(m) = \psi(m)(1).$$

Then $\Phi$ and $\Psi$ are inverse to each other, since

$$((\Psi \circ \Phi)(\varphi))(m) = \Psi(\Phi(\varphi))(m)$$

$$= \Phi(\varphi(m))(1)$$

$$= \varphi(S^{-1}(1)m)$$

$$= \varphi(m)$$

$$= ((\Phi \circ \Psi)(\psi))(m)(h) = (\Phi(\Psi(\psi))(m))(h)$$

$$= \Psi(\psi)(S^{-1}(h)m)$$

$$= \psi(S^{-1}(h)m)(1)$$

$$= (S^{-1}(h) \cdot \psi(m))(1)$$

$$= \psi(m)(S^{-1}(h)1)$$

$$= \psi(m)(h).$$

**Corollary 2.41.** Under the assumption of Proposition 2.40, if $P$ is projective in $C_{A,H}^H$, then $U(P)$ is projective in $\underline{A}\text{-Mod}$. 

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Proof. As a right adjoint to an exact functor, $U$ maps projectives to projectives.

**Lemma 2.42.** For any $\Delta$-module $M$, $E(M)$ is $\Sigma$-acyclic.

**Proof.** By Lemma 2.19, we have

$$\Sigma^n E(M) = E(\Sigma^n M)$$

and it suffices to prove the case when $n = 0$. Let us show that both $Z(E(M))$ and $B(E(M))$ are isomorphic to $k(\varepsilon) \otimes M$.

Suppose $\varphi \in Z(E(M))$, which means that

$$(h \cdot \varphi)(h') = \varepsilon(h) \varphi(h')$$

or

$$\varphi(S(h)h') = \varepsilon(h) \varphi(h')$$

for all $h, h' \in H$. Take $h' = 1$. Then we have

$$\varphi(S(h)) = \varepsilon(h) \varphi(1)$$

or

$$\varphi(h) = \varepsilon(S^{-1}(h)) \varphi(1) = \varepsilon(h) \varphi(1).$$

And we have $Z(E(M)) \cong k(\varepsilon) \otimes M$.

Suppose $\varphi \in B(E(M))$. Then there exists $\psi \in E(M)$ such that

$$\varphi(h) = \psi(S(\lambda)h)$$

for all $h \in H$. It is immediate to verify that $S(\lambda)$ is a right integral and thus the right hand side is $\varepsilon(h) \psi(S(\lambda))$. Therefore $B(E(M)) \cong k(\varepsilon) \otimes M$. 

\Box

**2.4 Model structures on Abelian categories**

This is a summary of Hovey’s theory of Abelian model categories and cotorsion pairs used in this paper. Our main reference is Gillespie’s survey [7].

Let us first recall the definition of cotorsion pairs introduced by Salce in [16].

**Definition 2.43.** Let $\mathbf{A}$ be an Abelian category. For objects $X, Y \in \mathbf{A}$, define

$$X \perp Y \iff \text{Ext}^1_{\mathbf{A}}(X, Y) = 0.$$  

More generally for a class $\mathbf{C}$ of objects, define

$$\mathbf{C}^\perp = \{ Y \in \mathbf{A}_0 \mid C \perp Y \text{ for all } C \in \mathbf{C} \}$$

$$\perp \mathbf{C} = \{ X \in \mathbf{A}_0 \mid X \perp C \text{ for all } C \in \mathbf{C} \}.$$  

These classes are also regarded as full subcategories.

**Definition 2.44.** Let $\mathbf{A}$ be as above. A cotorsion pair on $\mathbf{A}$ is a pair $(\mathbf{P}, \mathbf{I})$ of classes of objects of $\mathbf{A}$ satisfying
(1) $P = \perp I$.

(2) $I = P^\perp$.

**Definition 2.45.** A cotorsion pair $(P, I)$ is said to have enough projectives if, for any $X$ in $A$, there exists a short exact sequence of the form

$$0 \rightarrow A \rightarrow B \rightarrow X \rightarrow 0$$

with $B \in P$ and $A \in I$. It is said to have enough injectives if, for any $X$ in $A$, there exists a short exact sequence of the form

$$0 \rightarrow X \rightarrow A' \rightarrow B' \rightarrow 0$$

with $A' \in I$ and $B' \in P$.

It is called complete if it has both enough projectives and injectives.

**Example 2.46.** Let $\text{Prj}(A)$ and $\text{Inj}(A)$ be the classes of projectives and injectives in an Abelian category $A$. Then $(A, \text{Inj}(A))$ and $(\text{Prj}(A), A)$ are cotorsion pairs. The former is complete precisely when $A$ has enough injectives and the latter is complete precisely when $A$ has enough projectives.

These are called categorical cotorsion pairs by Hovey. Let us call the former the projective cotorsion pair and the latter the injective cotorsion pair.

In the case of Grothendieck Abelian categories, the completeness of cotorsion pairs is closely related to the notion of generation of cotorsion pairs. The following terminology is used in [17].

**Definition 2.47.** Let $G$ be a set of objects in an Abelian category $A$. Then the pair $(\perp(G^\perp), G^\perp)$ is called the cotorsion pair generated by $G$.

**Remark 2.48.** The pair $(\perp(G^\perp), G^\perp)$ is always a cotorsion pair by definition. Some authors say that the cotorsion pair is cogenerated by $G$ in the above situation. For example, this terminology is used by Hanno Becker in [2].

The following fact can be found as Proposition 1.2.1 in [2] and is attributed to [17].

**Proposition 2.49.** Let $A$ be a Grothendieck Abelian category. If $(D, E)$ is a cotorsion pair generated by a set $X$, then the following hold:

1. The pair $(D, E)$ has enough injectives.
2. The pair $(D, E)$ has enough projectives if and only if $D$ is generating.

The following terminology is used in [2].

**Definition 2.50.** A cotorsion pair $(D, E)$ is called small if $D$ is generated by a set and $D$ is generating.

**Corollary 2.51.** If $A$ is a Grothendieck Abelian category with enough projectives, then any cotorsion pair generated by a set is small. Thus it is complete.

**Definition 2.52.** Let $A$ be a bicomplete Abelian category. A model structure on $A$ is called Abelian if

(1) a morphism is a cofibration if and only if it is a monomorphism with cofibrant cokernel,
(2) a morphism is a trivial cofibration if and only if it is a monomorphism with trivially cofibrant cokernel,

(3) a morphism is a fibration if and only if it is an epimorphism with fibrant kernel, and

(4) a morphism is a trivial fibration if and only if it is an epimorphism with trivially fibrant kernel.

The following terminology is introduced by Gillespie [7].

**Definition 2.53.** Let $A$ be an Abelian category. A triple of subcategories $(\text{Cof}, \text{Triv}, \text{Fib})$ is called a **Hovey triple** if

1. $\text{Triv}$ is a thick subcategory.
2. $(\text{Cof}, \text{Fib} \cap \text{Triv})$ is a complete cotorsion pair.
3. $(\text{Cof} \cap \text{Triv}, \text{Fib})$ is a complete cotorsion pair.

**Theorem 2.54 (Hovey).** Let $A$ be a bicomplete Abelian category. Suppose $A$ is equipped with an Abelian model structure. Denote the full subcategories of trivial, cofibrant, and fibrant objects by $\text{Triv}$, $\text{Cof}$, and $\text{Fib}$, respectively. Then $(\text{Cof}, \text{Triv}, \text{Fib})$ is a Hovey triple

Conversely, given a Hovey triple $(\text{Cof}, \text{Triv}, \text{Fib})$ there exists a unique Abelian model structure on $A$ such that $\text{Triv}$, $\text{Cof}$, and $\text{Fib}$ are subcategories of trivial, cofibrant, and fibrant objects, respectively.

By definition, in the Abelian model structure defined by a Hovey triple $(\text{Cof}, \text{Triv}, \text{Fib})$, a morphism $f : X \to Y$ is

- a cofibration if and only if $f$ is a monomorphism and $\text{Coker } f \in \text{Cof}$,
- a trivial cofibration if and only if $f$ is a monomorphism and $\text{Coker } f \in \text{Cof} \cap \text{Triv}$,
- a fibration if and only if $f$ is an epimorphism and $\text{Ker } f \in \text{Fib}$, and
- a trivial fibration if and only if $f$ is an epimorphism and $\text{Ker } f \in \text{Fib} \cap \text{Triv}$.

Furthermore the following characterization of weak equivalences is obtained by Hovey.

**Lemma 2.55.** In the Abelian model structure defined by a Hovey triple $(\text{Cof}, \text{Triv}, \text{Fib})$, a morphism $f : X \to Y$ is a weak equivalence if and only if there exist an epimorphism $p$ with $\text{Ker } p \in \text{Triv}$ and a monomorphism $i$ with $\text{Coker } i \in \text{Triv}$ such that $f = p \circ i$.

By the projective cotorsion pair (Example 2.46) and Corollary 2.51, we obtain the following.

**Lemma 2.56.** Let $A$ be a Grothendieck Abelian category having enough projectives. Given a thick subcategory $\text{Triv}$, $(\perp \text{Triv}, \text{Triv}, A)$ is a Hovey triple if and only if

1. $\perp \text{Triv} \cap \text{Triv} = \text{Prj}(A)$, and
2. $(\perp \text{Triv})^\perp = \text{Triv}$,

where $\text{Prj}(A)$ is the full subcategory of projective objects in $A$. 

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3 Cotorsion pairs in the category of equivariant modules

3.1 Equivariant projective modules

In [1], Avramov, Foxby, and Halperin compared various notions of projectives in the category of dg modules over a dg algebra. In this section, we study Hopfological counterparts. Throughout this section, we fix a left $H$-module category $A$. Given an $H$-equivariant $A$-module $P$, we have two functors

\[ C_{A,H}(P, -) : C_{A,H} \to H\text{-Mod} \]

\[ C^H_{A,H}(P, -) : C^H_{A,H} \to k\text{-Mod}. \]

Various notions of projectivities are defined by the degrees of preservation of surjectivities by these functors.

**Definition 3.1.** Let $P$ be an $H$-equivariant left $A$-module.

1. $P$ is called $\Sigma$-linearly projective, if, for any surjective morphism $f : M \to N$,

\[ C_{A,H}(P, \Sigma^n f) : C_{A,H}(P, \Sigma^n M) \to C_{A,H}(P, \Sigma^n N) \]

is surjective for all $n \in \mathbb{Z}$.

2. $P$ is called $\Sigma$-homotopically projective, if, for any $\Sigma$-quism $f : M \to N$,

\[ C_{A,H}(P, \Sigma^n f) : C_{A,H}(P, \Sigma^n M) \to C_{A,H}(P, \Sigma^n N) \]

is a quism for all $n \in \mathbb{Z}$.

3. $P$ is called $\Sigma$-semiprojective, if, for any surjective $\Sigma$-quism $f : M \to N$,

\[ C_{A,H}(P, \Sigma^n f) : C_{A,H}(P, \Sigma^n M) \to C_{A,H}(P, \Sigma^n N) \]

is a surjective quism for all $n \in \mathbb{Z}$.

4. $P$ is called $\Sigma$-Qi-projective, if for any surjective $\Sigma$-quism $f : M \to N$,

\[ C^H_{A,H}(P, \Sigma^n f) : C^H_{A,H}(P, \Sigma^n M) \to C^H_{A,H}(P, \Sigma^n N) \]

is surjective for all $n \in \mathbb{Z}$.

The full subcategories of $C^H_{A,H}$ consisting of $\Sigma$-homotopically projectives, $\Sigma$-semiprojectives, and $\Sigma$-Qi-projectives are denoted by $\text{HoPrj}_\Sigma$, $\text{SemiPrj}_\Sigma$, and $\text{QiPrj}_\Sigma$ respectively. Corresponding full subcategories of $\mathcal{T}^H_{A,H}$ are denoted by the same symbols.

**Remark 3.2.** Under the isomorphism $C_{A,H}(P, \Sigma^n M) \cong \Sigma^n C_{A,H}(P, M)$, $P$ is $\Sigma$-homotopically projective if and only if $C_{A,H}(P, -)$ transforms $\Sigma$-quisms to $\Sigma$-quisms.

**Lemma 3.3.** If $P$ is either $\Sigma$-semiprojective or $\Sigma$-Qi-projective, then $U(P)$ is projective in $A\text{-Mod}$. 

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Proof. Suppose we have a diagram of $A$-modules

\[
\begin{array}{ccc}
U(P) & \rightarrow & 0 \\
\downarrow g & & \\
M & \rightarrow & N \\
\downarrow f & & \\
0 & & 
\end{array}
\]

in which the bottom row is exact. By taking the right adjoint, we obtain a diagram

\[
\begin{array}{ccc}
P & \rightarrow & E(N) \\
\downarrow \tilde{g} & & \\
E(M) & \rightarrow & 0 \\
\downarrow E(f) & & \\
\end{array}
\]

by Proposition 2.40. Since $E$ is an exact functor, the bottom row is exact. By Lemma 2.42, both $E(M)$ and $E(N)$ are $\Sigma$-acyclic. Hence $E(f)$ is a surjective $\Sigma$-quism.

When $P$ is $\Sigma$-Qi-projective, the induced map

\[C^H_{A,H}(P, E(f)) : C^H_{A,H}(P, E(M)) \rightarrow C^H_{A,H}(P, E(N))\]

is surjective by definition.

When $P$ is $\Sigma$-semiprojective, the induced map

\[C_{A,H}(P, E(f)) : C_{A,H}(P, E(M)) \rightarrow C_{A,H}(P, E(N))\]

is a surjective quism. By Lemma 2.36, $C^H_{A,H}(P, f)$ is surjective.

Thus $\tilde{g}$ has a lift $P \rightarrow E(M)$ in $C^H_{A,H}$ in both cases. The left adjoint to this morphism is a lift of $g$. Hence $U(P)$ is projective as an $A$-module in both cases.

The following is a modification of Lemma 6.2 in Qi’s paper.

**Lemma 3.4.** If $P$ is $\Sigma$-Qi-projective, then, for any $\Sigma$-acyclic object $T$, $C_{A,H}(P, T)$ is $\Sigma$-acyclic. In other words, $\tau^H_{A,H}(P, \Sigma^n T) = 0$ for all $n$.

**Proof.** If $T$ is $\Sigma$-acyclic, the surjective map

\[1 \otimes \varepsilon : T \otimes H \rightarrow T\]

is a $\Sigma$-quism, since $T \otimes H = C(T)$ is $\Sigma$-acyclic by Corollary 2.34. Since $P$ is $\Sigma$-Qi-projective, the induced map

\[C^H_{A,H}(P, \Sigma^n (T \otimes H)) \rightarrow C^H_{A,H}(P, \Sigma^n (T))\]

is surjective for all $n \in \mathbb{Z}$. For any $\varphi \in C^H_{A,H}(P, \Sigma^n T)$, there exists $\psi \in C^H_{A,H}(P, \Sigma^n (T \otimes H))$ such that $(1 \otimes \varepsilon) \circ \psi = \varphi$. By Lemma 2.19, we have an isomorphism of $H$-modules

\[C_{A,H}(P, \Sigma^n (T \otimes H)) \cong \Sigma^n (C_{A,H}(P, T) \otimes H).\]

By Corollary 2.34, this is acyclic, which implies that there exists $\rho \in C_{A,H}(P, \Sigma^n (T \otimes H))$ such that $\psi = \lambda \rho$. By the same calculation as in the proof of Lemma 6.2 of Qi’s paper, we have

\[\varphi = \lambda (1 \otimes \varepsilon) \circ \rho.\]

Thus $\tau_{A,H}(P, \Sigma^n T) = 0$ for all $n$. \qed
Avramov, Foxby, and Halperin [1] found many equivalent descriptions of homotopical projectivity and semiprojectivity. Here we prove analogues of some of them.

**Lemma 3.5.** For an $H$-equivariant $A$-module $P$, the following conditions are equivalent:

1. $P$ is $\Sigma$-semiprojective.
2. $P$ is $\Sigma$-homotopically projective and $U(P)$ is projective as an $A$-module.
3. $P$ is $\Sigma$-Qi-projective.

**Proof.** Let us first show that (1) and (2) are equivalent.

Suppose $P$ is $\Sigma$-semiprojective. By Lemma 3.3, $P$ is projective as an $A$-module. In order to show that $P$ is $\Sigma$-homotopically projective, let $f : M \to N$ be a $\Sigma$-quism. We have a short exact sequence

$$0 \to \Sigma^n(N) \to C_{\Sigma^n f} \to \Sigma^{n+1}(M) \to 0,$$

which defines a triangle

$$\Sigma^n(M) \xrightarrow{\Sigma^n f} \Sigma^n(N) \to C_{\Sigma^n f} \to \Sigma^n+1(M).$$

Since $H$ is a homological functor, we see from the long exact sequence associated with this triangle that $H(C_{\Sigma^n f}) = 0$ for all $n$. In other words, $C_f \to 0$ is a surjective $\Sigma$-quism. Since $P$ is $\Sigma$-semiprojective,

$$C_{A,H}(P,C_f) \to C_{A,H}(P,0) = 0$$

is a surjective $\Sigma$-quism, i.e. $C_{A,H}(P,C_f)$ is $\Sigma$-acyclic.

On the other hand, since $P$ is projective as an $A$-module, $C_{A,H}(P,−) = (A-\text{Mod})(P,−)$ is an exact functor and we obtain an extension

$$0 \to C_{A,H}(P,\Sigma^n(N)) \to C_{A,H}(P,\Sigma^n f) \to C_{A,H}(P,\Sigma^{n+1}(M)) \to 0$$

in $H-\text{Mod}$. By Lemma 2.26 and 2.19, this is isomorphic to

$$0 \to \Sigma^n C_{A,H}(P,\Sigma^n(N)) \to C_{A,H}(P,\Sigma^n f) \to \Sigma^{n+1}(C_{A,H}(P,M)) \to 0,$$

and thus we obtain a triangle

$$C_{A,H}(P,\Sigma^n(M)) \xrightarrow{C_{A,H}(P,\Sigma^n f)} C_{A,H}(P,\Sigma^n(N)) \to C_{A,H}(P,\Sigma^n f) \to \Sigma^{n+1}(C_{A,H}(P,M))$$

in the homotopy category $\text{ho}(H-\text{Mod})$.

By the associated long exact sequence of homology, we see that

$$H(C_{A,H}(P,\Sigma^n(f))) : H(C_{A,H}(P,\Sigma^n(M))) \to H(C_{A,H}(P,\Sigma^n(N)))$$

is an isomorphism for all $n$, since $C_{A,H}(P,C_f)$ is $\Sigma$-acyclic. And thus $P$ is homotopically $\Sigma$-semiprojective.

Conversely suppose that $U(P)$ is projective as an $A$-module and $P$ is $\Sigma$-homotopically semiprojective. Let $f : M \to N$ be a surjective $\Sigma$-quism. Since $P$ is $\Sigma$-homotopically projective, $C_{A,H}(P,f)$ is a $\Sigma$-quism. Furthermore, since $U(P)$ is projective in $A-\text{Mod}$, $C_{A,H}(P,−)$ is an exact functor. In particular, $C_{A,H}(P,f)$ is surjective. Hence $P$ is $\Sigma$-semiprojective.
We next show that (1) and (3) are equivalent. Suppose \( P \) is \( \Sigma \)-semiprojective and let \( f : M \to N \) be a surjective \( \Sigma \)-quism. By definition,
\[
C_{A,H}(P, \Sigma^n f) : C_{A,H}(P, \Sigma^n M) \longrightarrow C_{A,H}(P, \Sigma^n N)
\]
is a surjective quism for all \( n \in \mathbb{Z} \). By Lemma 2.36, the induced map
\[
C^H_{A,H}(P, \Sigma^n f) : C^H_{A,H}(P, \Sigma^n M) = Z(C_{A,H}(P, \Sigma^n M)) \xrightarrow{Z(C_{A,H}(P, \Sigma^n f))} Z(C_{A,H}(P, \Sigma^n N)) = C^H_{A,H}(P, \Sigma^n N)
\]
is surjective. Hence \( P \) is \( \Sigma \)-Qi-projective.

Conversely, suppose that \( P \) is \( \Sigma \)-Qi-projective. Let \( f : M \to N \) be a surjective \( \Sigma \)-quism. We need to show that the induce morphism
\[
C_{A,H}(P, \Sigma^n f) : C_{A,H}(P, \Sigma^n M) \longrightarrow C_{A,H}(P, \Sigma^n N)
\]
is a surjective quism for all \( n \in \mathbb{Z} \). By Lemma 3.3, \( U(P) \) is a projective \( \mathcal{A} \)-module and thus \( C_{A,H}(P, -) \) is an exact functor, which implies that \( C_{A,H}(P, \Sigma^n f) \) is surjective.

Let us show that \( C_{A,H}(P, \Sigma^n f) \) is a quism for all \( n \), i.e. the induced map
\[
H(C_{A,H}(P, \Sigma^n M)) \cong \mathcal{I}_{A,H}^H(P, \Sigma^n M) \xrightarrow{\mathcal{I}_{A,H}^H(P, \Sigma^n f)} \mathcal{I}_{A,H}^H(P, \Sigma^n N) \cong H(C_{A,H}(P, \Sigma^n N))
\]
is an isomorphism for all \( n \in \mathbb{Z} \).

Under the identification in Lemma 2.26, the short exact sequence
\[
0 \longrightarrow \Sigma^n N \longrightarrow \Sigma^n C_f \longrightarrow \Sigma^{n+1} M \longrightarrow 0
\]
defines a triangle in \( \mathcal{I}_{A,H}^H \) and thus we obtain a long exact sequence
\[
\cdots \longrightarrow \mathcal{I}_{A,H}^H(P, \Sigma^{n-1} C_f) \longrightarrow \mathcal{I}_{A,H}^H(P, \Sigma^n M) \longrightarrow \mathcal{I}_{A,H}^H(P, \Sigma^n N) \longrightarrow \mathcal{I}_{A,H}^H(P, \Sigma^n C_f) \longrightarrow \cdots .
\]
Since \( f : M \to N \) is a \( \Sigma \)-quism, \( C_f \) is \( \Sigma \)-acyclic. Since \( P \) is \( \Sigma \)-Qi-projective, by Lemma 3.4, we have \( \mathcal{I}_{A,H}^H(P, \Sigma^n C_f) = 0 \). Therefore \( \mathcal{I}_{A,H}^H(P, \Sigma^n f) \) is an isomorphism for all \( n \).

**Corollary 3.6.** An object \( P \) in \( C_{A,H}^H \) is \( \Sigma \)-semiprojective if and only if \( U(P) \) is projective in \( \mathcal{A} \)-Mod and \( C_{A,H}(P, T) \in \text{Triv}_{A,H}^\Sigma \) for all \( T \in \text{Triv}_{A,H}^\Sigma \).

**Proof.** Suppose \( P \) is \( \Sigma \)-semiprojective. For \( T \in \text{Triv}_{A,H}^\Sigma \), \( T \to 0 \) is a surjective \( \Sigma \)-quism and hence induces a \( \Sigma \)-quism \( C_{A,H}(P, T) \to 0 \).

Conversely, suppose \( U(P) \) is projective and that \( C_{A,H}(P, T) \) is \( \Sigma \)-acyclic for all \( T \in \text{Triv}_{A,H}^\Sigma \). By the previous Lemma, it suffices to show that \( P \) is \( \Sigma \)-homotopically projective. For a \( \Sigma \)-quism \( f : M \to N \), \( C_f \) is \( \Sigma \)-acyclic. Since \( U(P) \) is projective, we have a triangle
\[
C_{A,H}(P, M) \xrightarrow{C_{A,H}(P, f)} C_{A,H}(P, N) \xrightarrow{C_{A,H}(P, j_f)} C_{A,H}(P, C_f) \xrightarrow{C_{A,H}(P, f_j)} C_{A,H}(P, \Sigma(M)).
\]
By assumption \( C_{A,H}(P, C_f) \) is \( \Sigma \)-acyclic. The long exact sequence of homology implies that \( C_{A,H}(P, f) \) is a \( \Sigma \)-quism.
3.2 The orthogonality in the category of equivariant modules

In this section, we study the orthogonality in $\mathcal{C}^{H}_{A,H}$ with respect to the biadditive functor $\text{Ext}^{1}_{\mathcal{C}^{H}_{A,H}}(-,-)$. For objects $X,Y \in \mathcal{C}^{H}_{A,H}$, we denote $X \perp Y$ when $\text{Ext}^{1}_{\mathcal{C}^{H}_{A,H}}(X,Y) = 0$. For a class $\mathcal{T}$ of objects in $\mathcal{C}^{H}_{A,H}$, define

$$\mathcal{T}^{\perp} = \{ Y \mid C \perp Y \text{ for all } C \in \mathcal{T} \}$$

$$\perp \mathcal{T} = \{ X \mid X \perp C \text{ for all } C \in \mathcal{T} \}. $$

We begin with the following general fact.

**Lemma 3.7.** If $\mathcal{T}$ is a class of objects in $\mathcal{C}^{H}_{A,H}$ closed under $\Sigma^{n}$ for all $n \in \mathbb{Z}$, then so are both $\perp \mathcal{T}$ and $\mathcal{T}^{\perp}$.

**Proof.** We first prove that $\perp \mathcal{T}$ is closed under $\Sigma$. Suppose $M \in \perp \mathcal{T}$. We need to prove $\text{Ext}^{1}_{\mathcal{C}^{H}_{A,H}}(\Sigma M, N) = 0$ for all $N \in \mathcal{T}$. Let

$$0 \longrightarrow N \longrightarrow E \longrightarrow \Sigma M \longrightarrow 0$$

be an extension in $\mathcal{C}^{H}_{A,H}$. Since $\Sigma^{-1}$ is an exact functor, we obtain an extension

$$0 \longrightarrow \Sigma^{-1} N \longrightarrow \Sigma^{-1} E \longrightarrow \Sigma^{-1} \Sigma M \longrightarrow 0.$$

Let

$$i : M \rightarrow M \oplus M \otimes Q \cong \Sigma^{-1} \Sigma M$$

be the inclusion under the isomorphism used in the proof of Lemma 2.14. By taking the pullback along $i$, we obtain an extension

$$0 \longrightarrow \Sigma^{-1} N \longrightarrow i^{*} \Sigma^{-1} E \longrightarrow M \longrightarrow 0.$$ 

Since $\mathcal{T}$ is closed under $\Sigma^{-1}$, this sequence splits in $\mathcal{C}^{H}_{A,H}$. Let $s : M \rightarrow i^{*} \Sigma^{-1} E$ be a splitting. The composition

$$\Sigma M \xrightarrow{\Sigma s} \Sigma i^{*} \Sigma^{-1} E \longrightarrow \Sigma \Sigma^{-1} E \cong E \oplus E \oplus Q \longrightarrow E$$

gives us a splitting we wanted by the commutativity of the following diagram

```
\begin{array}{cccccc}
0 & \longrightarrow & \Sigma \Sigma^{-1} N & \longrightarrow & \Sigma i^{*} \Sigma^{-1} E & \longrightarrow & \Sigma M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma \Sigma^{-1} N & \longrightarrow & \Sigma \Sigma^{-1} E & \longrightarrow & \Sigma \Sigma^{-1} \Sigma M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & \Sigma M & \longrightarrow & 0.
\end{array}
```

The same argument can be used to show that $\perp \mathcal{T}$ is closed under $\Sigma^{-1}$ by switching $\Sigma$ and $\Sigma^{-1}$. The detail is omitted.

We next show that $\mathcal{T}^{\perp}$ is closed under $\Sigma$. The argument is essentially dual to the above case. Suppose $N \in \mathcal{T}^{\perp}$. We need to show that $\text{Ext}^{1}_{\mathcal{C}^{H}_{A,H}}(M, \Sigma N) = 0$ for all $M \in \mathcal{T}$. Let

$$0 \longrightarrow \Sigma N \longrightarrow E \longrightarrow M \longrightarrow 0$$

(3.1)
be an extension. Apply $\Sigma^{-1}$ to obtain an extension

$$0 \rightarrow \Sigma^{-1}N \rightarrow \Sigma^{-1}E \rightarrow \Sigma^{-1}M \rightarrow 0.$$ 

Let $p : \Sigma^{-1}N \cong N \oplus N \otimes Q \rightarrow N$ be the projection. By taking the pushout along $p$, we obtain an extension

$$0 \rightarrow N \rightarrow p_*\Sigma^{-1}E \rightarrow \Sigma^{-1}M \rightarrow 0.$$ 

Since $T$ is closed under $\Sigma^{-1}$, this extension splits. Let $q : p_*\Sigma^{-1}E \rightarrow N$ be a splitting. Then the composition

$$E \rightarrow \Sigma\Sigma^{-1}E \rightarrow \Sigma p_*\Sigma^{-1}E \xrightarrow{\Sigma q} \Sigma N$$

is a splitting of (3.1). And we have $\Ext^1_C(M, \Sigma N) = 0$ for all $M \in T$. The case of $\Sigma^{-1}$ on $T^\perp$ is analogous and is omitted.

**Corollary 3.8.** Let $T$ be a class of objects closed under $\Sigma^n$ for all $n \in \mathbb{Z}$. Suppose $P \in \perp T$ and $T \in T$. Then $T^H_{A,H}(P, \Sigma^n T) = 0$ for all $n \in \mathbb{Z}$.

**Proof.** Let $T$ be an object in $T$. Under the identification $T^H_{A,H}(P, \Sigma^n T) = C^H_{A,H}(P, \Sigma^n T)/\simeq$, it suffices to show that any morphism $\varphi : P \rightarrow \Sigma^n T$ is null homotopic. By Lemma 3.7, $\Sigma P \in \perp T$. In particular, the extension

$$0 \rightarrow \Sigma^n T \rightarrow C \varphi \rightarrow \Sigma P \rightarrow 0$$

splits. By Corollary 2.25, $\varphi \simeq 0$.

Recall from Lemma 2.33 that the subcategory $\text{Triv}_{A,H}^\Sigma$ of $\Sigma$-acyclic objects is a thick subcategory of $C^H_{A,H}$. By the identification $C^H_{A,H} \cong (A\# H)$-$\text{Mod}$ of Proposition 2.12, $C^H_{A,H}$ is an Abelian category with enough projectives. Thus, by Lemma 2.56, the triple $(\perp \text{Triv}_{A,H}^\Sigma, \text{Triv}_{A,H}^\Sigma, C^H_{A,H})$ is a Hovey triple if

$$\perp \text{Triv}_{A,H}^\Sigma \cap \text{Triv}_{A,H}^\Sigma = \text{Prj}(C^H_{A,H})$$

$$\left(\perp \text{Triv}_{A,H}^\Sigma\right)^{\perp} = \text{Triv}_{A,H}^\Sigma.$$

In order to prove these statements, we first need to understand $\perp \text{Triv}_{A,H}^\Sigma$.

**Proposition 3.9.** We have

$$\perp \text{Triv}_{A,H}^\Sigma = \text{SemiPrj}_{A,H}.$$ 

**Proof.** Let $P$ be an object of $\perp \text{Triv}_{A,H}^\Sigma$. Let us first show that $U(P)$ is projective as an $A$-module. Consider a diagram of $A$-modules

$$\begin{array}{ccc}
U(P) & \rightarrow & 0 \\
\downarrow f & & \\
M & \xrightarrow{\varphi} & N \\
\end{array}$$

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By taking the right adjoint, we have a diagram of $H$-modules

$$
\begin{array}{ccc}
P & \xrightarrow{f} & E(N) \\
E(M) & \xrightarrow{E(\varphi)} & 0
\end{array}
$$

by Lemma 2.40. Since $E$ is an exact functor, the bottom sequence is exact. By taking the pullback, we obtain a map of short exact sequences in $C^H_{A,H}$

$$
\begin{array}{cccc}
0 & \longrightarrow & E(\text{Ker } \varphi) & \longrightarrow & E(M) \times_{E(N)} P \\
& & \downarrow & & \downarrow \\
0 & \longrightarrow & E(\text{Ker } \varphi) & \longrightarrow & E(M) \xrightarrow{E(\varphi)} E(N) \longrightarrow 0.
\end{array}
$$

By Lemma 2.42, $E(\text{Ker } \varphi)$ belongs to $\text{Triv}_{A,H}^\Sigma$, which implies

$$
\text{Ext}^1_{C^H_{A,H}}(P, E(\text{Ker } \varphi)) = 0
$$

by assumption. In other words, the top row splits in $C^H_{A,H}$. Let $s : P \to E(M) \times_{E(N)} P$ be a splitting. Then the adjoint to the composition

$$
P \xrightarrow{s} E(M) \times_{E(N)} P \longrightarrow E(M)
$$

gives us a lift $U(P) \to M$ of $f$. Hence $U(P)$ is projective as an $A$-module.

By Lemma 3.5, it remains to show that $P$ is $\Sigma$-homotopically projective. Let $f : M \to N$ be a $\Sigma$-quism in $C^H_{A,H}$. By Lemma 2.26, It suffices to show that $C_{A,H}(P, \Sigma^n C_f)$ is acyclic for any $n \in \mathbb{Z}$. Under the identification

$$
H(C_{A,H}(P, \Sigma^n C_f)) \cong \mathcal{T}^H_{A,H}(P, \Sigma^n C_f) = C^H_{A,H}(P, \Sigma^n C_f)/\sim,
$$

we are going to show that any morphism $\varphi : P \to \Sigma^n C_f$ is null homotopic. Consider the extension associated with the mapping cone of $\varphi$

$$
0 \longrightarrow \Sigma^n C_f \longrightarrow C_\varphi \longrightarrow \Sigma(P) \longrightarrow 0.
$$

By assumption, $\Sigma^n C_f$ is acyclic for all $n$. Since $P$ belongs to $\text{Triv}_{A,H}^\Sigma$, so does $\Sigma P$ by Lemma 3.7, which implies that this extension splits. By Lemma 2.25, we have $f \simeq 0$.

Conversely, suppose that $P$ is $\Sigma$-semiprojective. For an extension

$$
0 \longrightarrow T \xrightarrow{\gamma} E \xrightarrow{p} P \longrightarrow 0
$$

with $T \in \text{Triv}_{A,H}^\Sigma$, since $P$ is projective as an $A$-module, it defines a distinguished triangle

$$
T \xrightarrow{|\gamma|} E \xrightarrow{|p|} P \xrightarrow{\delta} \Sigma T
$$

in $\mathcal{T}^H_{A,H}$ by Lemma 2.21. The induced long exact sequence of homology

$$
\cdots \longrightarrow H(\Sigma^n T) \xrightarrow{\delta^n} H(\Sigma^n E) \xrightarrow{p_*} H(\Sigma^n P) \xrightarrow{\delta^n} H(\Sigma^{n+1} T) \longrightarrow H(\Sigma^{n+1} E) \longrightarrow \cdots
$$
and the assumption that $T \in \text{Triv}_{A,H}^\Sigma$ implies that $p : E \to P$ is a surjective $\Sigma$-quism.

By Lemma 3.5, the induced map

$$C^H_{A,H}(P, p) : C^H_{A,H}(P, E) \to C^H_{A,H}(P, P)$$

is surjective. Any $s \in C^H_{A,H}(P, E)$ with $C^H_{A,H}(P, p)(s) = 1_P$ is a splitting of (3.2), which implies that $P \in \perp \text{Triv}_{A,H}^\Sigma$.

**Corollary 3.10.** The subcategory $\text{SemiPrj}_\Sigma$ is closed under $\Sigma^n$ for all $n \in \mathbb{Z}$.

**Proposition 3.11.** We have

$$\perp \text{Triv}_{A,H}^\Sigma \cap \text{Triv}_{A,H}^\Sigma = \text{Prj}(C^H_{A,H}),$$

where the right hand side is the full subcategory of projective objects in $C^H_{A,H}$.

**Proof.** Since $\text{Triv}_{A,H}^\Sigma \subset C^H_{A,H}$, we have

$$\perp \text{Triv}_{A,H}^\Sigma \supset \perp C^H_{A,H} = \text{Prj}(C^H_{A,H}).$$

Let $P$ be a projective object in $C^H_{A,H}$. We show that $P$ is $\Sigma$-acyclic. Under the identification

$$C^H_{A,H} \cong A#H\text{-Mod},$$

$P$ is a direct summand of a free $A#H$-module, i.e. there exists a free $A$-module $F$ such that $P$ is a direct summand of $C(F)$. By Corollary 2.34, $C(F)$ belongs to $\text{Triv}_{A,H}^\Sigma$, and so does $P$. And we have

$$\text{Prj}(C^H_{A,H}) \subset \perp \text{Triv}_{A,H}^\Sigma \cap \text{Triv}_{A,H}^\Sigma.$$

Conversely suppose that

$$P \in \perp \text{Triv}_{A,H}^\Sigma \cap \text{Triv}_{A,H}^\Sigma = \text{SemiPrj}_\Sigma \cap \text{Triv}_{A,H}^\Sigma.$$

Since $C^H_{A,H} \cong A#H\text{-Mod}$ has enough projectives, we may take a projective object $Q$ in $C^H_{A,H}$ and a surjection

$$Q \xrightarrow{\phi} P \to 0.$$

Since $P$ is $\Sigma$-acyclic, this is a surjective $\Sigma$-quism. By Lemma 3.5, $P$ is $\Sigma$-Qi-projective and the induced map

$$C^H_{A,H}(P, f) : C^H_{A,H}(P, Q) \to C^H_{A,H}(P, P)$$

is surjective, which implies that $Q \xrightarrow{\phi^P} P$ has a section. As a direct summand of a projective object, $P$ is projective.

**Proposition 3.12.** We have

$$\perp \text{Triv}_{A,H}^\Sigma \perp = \text{Triv}_{A,H}^\Sigma.$$

**Proof.** By definition, we have an inclusion

$$\text{Triv}_{A,H}^\Sigma \subset \perp \left( \perp \text{Triv}_{A,H}^\Sigma \right).$$

We prove

$$\left( \text{SemiPrj}_\Sigma \right) \perp = \left( \perp \text{Triv}_{A,H}^\Sigma \right) \perp \subset \text{Triv}_{A,H}^\Sigma.$$
Suppose $T \in (\text{SemiPrj}_\Sigma)$. Then for any $P \in \text{SemiPrj}_\Sigma$, $\text{Ext}^1_{C_{A,H}}(P, T) = 0$. Let $P = A \otimes k$, where $k$ is regarded as a trivial $H$-module. The $A$-module structure is given by the composition of morphisms of $A$. Note that $A \otimes k$ is $\Sigma$-semiprojective, since for any $\Sigma$-quism $f : M \to N$, the commutativity of the diagram

$$
\begin{array}{ccc}
C_{A,H}(P, \Sigma^n M) & \xrightarrow{C_{A,H}(P, \Sigma^n f)} & C_{A,H}(P, \Sigma^n N) \\
\cong & & \cong \\
(k\text{-Mod})(k, \Sigma^n M) & \xrightarrow{\cong} & (k\text{-Mod})(k, \Sigma^n N) \\
\cong & & \cong \\
\Sigma^n M & \xrightarrow{\Sigma^n f} & \Sigma^n N
\end{array}
$$

allows us to identify $C_{A,H}(P, \Sigma^n f)$ with $\Sigma^n f$.

Since $\text{SemiPrj}_\Sigma$ is closed under $\Sigma^n$ for all $n \in \mathbb{Z}$, so does $(\text{SemiPrj}_\Sigma)^\perp$ by Lemma 3.7 and we have $\text{Ext}^1_{C_{A,H}}(P, \Sigma^n T) = 0$ for all $n \in \mathbb{Z}$. By Corollary 3.8, we have $\mathcal{T}^H_{A,H}(P, \Sigma^n T) = 0$ for all $n \in \mathbb{Z}$. Under the identification

$$
\mathcal{T}^H_{A,H}(P, \Sigma^n T) = H(C_{A,H}(P, \Sigma^n T)) \cong H((k\text{-Mod})(k, \Sigma^n T)) \cong H(\Sigma^n T),
$$

this implies that $T$ belongs to $\text{Triv}_{A,H}$.

\begin{proof}
\end{proof}

Corollary 3.13. The triple $\left(\text{SemiPrj}_\Sigma, \text{Triv}_{A,H}^\Sigma, C_{A,H}^H\right)$ is a Hovey triple in $C_{A,H}^H$.

Now Theorem 1.1 is a corollary to this fact. By Hovey’s correspondence this model structure is described as follows.

Corollary 3.14. There exists an Abelian model structure on $C_{A,H}^H$ with the following properties:

1. A morphism $f : M \to N$ is a cofibration if and only if it is a monomorphism and $\text{Coker } f$ is $\Sigma$-semiprojective.

2. A morphism $f : M \to N$ is a fibration if and only if it is an epimorphism.

3. A morphism $f : M \to N$ is a weak equivalence if and only if it is a $\Sigma$-quism.

In particular, all objects are fibrant and cofibrant objects are $\Sigma$-semiprojectives.

\begin{proof}
It only remains to verify that weak equivalences agree with $\Sigma$-quisms. Suppose $f$ is a weak equivalence. By Lemma 2.55, it factors as $f = p \circ i$ with $\text{Ker } p, \text{Coker } i \in \text{Triv}_{A,H}^\Sigma$. In particular both $p$ and $i$ are $\Sigma$-quisms and thus so is $f$. Conversely suppose $f : M \to N$ is a $\Sigma$-quism. By the model structure, it factors as $f = p \circ i$ with $i : M \to E$ a cofibration and $p : E \to N$ a trivial fibration. By the characterization of trivial fibrations in an Abelian model category, $p$ is an epimorphism with $\text{Ker } p \in \text{Triv}_{A,H}^\Sigma$. And $i$ is a monomorphism with $\text{Coker } i \in \text{SemiPrj}_\Sigma$. By Lemma 3.5, the sequence

$$
0 \to M \xrightarrow{i} E \to \text{Coker } i \to 0
$$

is an $A$-split sequence. Hence defines a triangle in $\mathcal{T}^H_{A,H}$. Since both $f$ and $p$ are $\Sigma$-quisms, so is $i$. The long exact sequence of homology induced by this triangle implies that $\text{Coker } i \in \text{Triv}_{A,H}^\Sigma$.
\end{proof}

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Definition 3.15. Under this model structure, the full subcategory of compact cofibrant objects in $C_{A,H}^H$ is denoted by $\text{Perf}_{A,H}^H$. Objects of this category are called perfect $H$-equivariant $A$-modules. Cofibrations and weak equivalences in this model structure make $\text{Perf}_{A,H}^H$ into a Waldhausen category, whose algebraic $K$-theory is denoted by $K(A,H)$ and is called the Hopfological algebraic $K$-theory of $A$.

Remark 3.16. Kaygun and Khalkhali [9] introduced the notion of $H$-equivariantly projective $A$-modules and used the exact category of $H$-equivariantly projective $A$-modules to define the Hopf-cyclic homology of $A$. We may use this exact category to define an algebraic $K$-theory, which should be regarded as a generalization of Thomason’s equivariant $K$-theory [20] and should be called the $H$-equivariant $K$-theory of $A$. We note, however, that this $K$-theory is different from ours.

It should be noted that our Waldhausen category can be also obtained from the cotorsion pair $(\text{SemiPrj}_H \Sigma, C_{A,H})$ by using a recent work of Sarazola’s [18]. Sarazola’s work also suggests the existence of another interesting cotorsion pair in $C_{A,H}^H$ if we replace the structure of exact category as follows.

Definition 3.17. The class of extensions in $C_{A,H}^H$ that are split as $A$-modules is denoted by $E_{\text{split}}$. We obviously obtain an exact category $(C_{A,H}^H, E_{\text{split}})$. Let us denote this exact category by $C_{A,H}^H, \text{split}$.

Proposition 3.18. Let $\text{Cntr}_{A,H}^H$ be the class of $H$-equivariant $A$-modules that are contractible in $H$-Mod. Then the pair $(C_{A,H}^H, \text{Cntr}_{A,H}^H)$ is a complete cotorsion pair in $C_{A,H}^H, \text{split}$.

Proof. The completeness is obvious from the definition.

Let us verify that $(C_{A,H}^H, \text{Cntr}_{A,H}^H)$ is a cotorsion pair. Suppose $\text{Ext}_{C_{A,H}^H, \text{split}}^1(M,T) = 0$ for all $M$. We show that $T$ is contractible. Consider the extension

$$0 \to T \to C(T) \to \Sigma(T) \to 0.$$ 

This is an $A$-split extension. Apply $\Sigma^{-1}$ and take the pullback along the map $T \to \Sigma^{-1}\Sigma(T)$ which induces an isomorphism in $T_{A,H}^H$ by Lemma 2.14 to obtain

$$0 \to \Sigma^{-1}(T) \to \Sigma^{-1}C(T) \to \Sigma^{-1}\Sigma(T) \to 0.$$

By assumption, the bottom sequence splits and thus $T$ is a retract of $E$. Since these are $A$-split exact sequences, they define triangles in $T_{A,H}^H$. Since $T \to \Sigma^{-1}\Sigma(T)$ is an isomorphism in $T_{A,H}^H$, so is $E \to \Sigma^{-1}C(T)$. Since $\Sigma^{-1}C(T)$ is contractible, so is $E$. As a retract of a contractible object, $T$ is contractible.

Conversely, suppose $T$ is contractible. We have a commutative diagram

$$
\begin{array}{ccc}
T & \xrightarrow{1_T} & T \\
\downarrow & & \downarrow \\
C(T). & \xrightarrow{\varphi} & \\
\end{array}
$$

For an $A$-split sequence

$$0 \to F \to E \to T \to 0$$

take the pullback along $\varphi$ to obtain

$$0 \to F \to \varphi^*(E) \to C(T) \to 0.$$
By Lemma 2.22 this sequence splits. Let \( s : C(T) \to \varphi^*(E) \) be section. Then the composition

\[
T \xrightarrow{s_T} C(T) \xrightarrow{s} \varphi^*(E) \rightarrow E
\]

is a section of \( E \to T \).

Remark 3.19. This is an analogue of Example 8.3 in Sarazola’s paper [18].

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