Integrable geometric flows of interacting curves/surfaces, multilayer spin systems and the vector nonlinear Schrödinger equation

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Abstract

In this paper, we study integrable multilayer spin systems, namely, the multilayer M-LIII equation. We investigate their relation with the geometric flows of interacting curves and surfaces in some space $\mathbb{R}^n$. Then we present their the Lakshmanan equivalent counterparts. We show that these equivalent counterparts are, in fact, the vector nonlinear Schrödinger equation (NLSE). It is well-known that the vector NLSE is equivalent to the $\Gamma$-spin system. Also, we have presented the transformations which give the relation between solutions of the $\Gamma$-spin system and the multilayer M-LIII equation. It is interesting to note that the integrable multilayer M-LIII equation contains constant magnetic field $H$. It seems that this constant magnetic vector plays an important role in theory of "integrable multilayer spin system" and in nonlinear dynamics of magnetic systems. Finally, we present some classes of integrable models of interacting vortices.

1 Introduction

There is an interesting geometric relationship between geometric flows of curves and surfaces and integrable nonlinear differential equations [1]-[38]. In the pioneering works of nineteenth-century geometers were already established main connections between the theory of ordinary and partial differential equations and differential geometry of curves and surfaces. In soliton theory, the well-known Darboux and Bäcklund transformations have origins in the relationship between solutions of the sine-Gordon equation and pseudospherical surfaces. In this and previous our papers, we launch the program to study integrable flows of curves and surfaces related with the vector (multi-component) nonlinear Schrödinger equation (NLSE) and with multilayer spin systems. Under "curve flows" we usually mean the following equations

$$e_i^x = C \wedge e_i, \quad e_{it} = D \wedge e_i,$$

(1.1)

where

$$C = \tau e_1 + \kappa e_3, \quad D = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3.$$

(1.2)

Similarly, we can write the geometric flows of immersed surfaces in some Euclidean space. They are given by the following Gauss-Weingarden equations

$$\chi_x = N \chi, \quad \chi_t = M \chi.$$

(1.3)
Integrability conditions of these equations are given by the following Gauss-Mainardi-Codazzi equation

\[ N_t - M_x + [N, M] = 0. \] (1.4)

The aim of the present work is to provide geometric formulations of multilayer spin systems. We will demonstrate that the multilayer M-LIII equation arises from the flow of interacting curves. The Lakshmanan equivalent counterparts of the multilayer M-LIII equations are derived from inelastic curve flows in the Euclidean space. Several new results are obtained by considering different classes of curves and surfaces.

The outline of this paper is as follows. In the next two sections we give a brief review of the vector NLSE and the M-LIII equation. In Sections 4-7, we consider the multilayer M-LIII equations and their relations with the vector NLSE. In Section 8, we provide a Hamilton structure of the multilayer M-LIII equation. The Γ-spin system which is the gauge equivalent counterpart of the vector NLSE is presented. The relation between solutions of the multilayer M-LIII equation and the Γ-spin system is considered in Section 10. The geometric flows of curves and surfaces were studied in Sections 11 and 12. In Section 13 we present an integrable filament equations of interacting vortices. At last, in Section 14, we give our conclusions.

2  Brief review of the vector NLSE

In this section we recall some general facts about the vector NLS equation. The vector NLS equation has many physical significant applications such as modeling crossing sea waves and in propagation of elliptically birefringent optical fibers.

2.1  The equation

The vector NLSE reads as

\[ iq_1 t + q_1 xx - v q_1 = 0, \] (2.1)

\[ iq_2 t + q_2 xx - v q_2 = 0, \] (2.2)

\[ \vdots \] (2.3)

\[ iq_N t + q_N xx - v q_N = 0, \] (2.4)

where

\[ v + 2(|q_1|^2 + |q_2|^2 + \cdots + |q_N|^2) = 0. \] (2.5)

The Manakov system is the particular case of the vector NLSE when \( N = 2 \). The Manakov system we studied in \([39]-[45]\).

2.2  Lax representation

The vector NLS equation (2.1)-(2.3) is integrable by the inverse scattering method. Its Lax representation (LR) reads as

\[ \Phi_x = U \Phi, \] (2.6)

\[ \Phi_t = V \Phi, \] (2.7)

where \( \Phi = (\phi_1, \phi_2, \phi_3) \) and

\[ U = -i\lambda \Sigma + U_0, \quad V = -2i\lambda^2 \Sigma + 2\lambda U_0 + V_0. \] (2.8)

Here

\[ \Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad U_0 = \begin{pmatrix} 0 & q_1 & q_2 & q_3 \\ -\bar{q}_1 & 0 & 0 & 0 \\ -\bar{q}_2 & 0 & 0 & 0 \\ -\bar{q}_3 & 0 & 0 & 0 \end{pmatrix}. \] (2.9)
\[ V_0 = i \begin{pmatrix} |q_1|^2 + |q_2|^2 + |q_3|^2 & q_{1x} & q_{2x} & q_{3x} \\ q_{1x} & -|q_1|^2 - q_1 q_2 - q_1 q_3 \\ q_{2x} & -q_2 q_1 - |q_2|^2 - q_2 q_3 \\ q_{3x} & -q_3 q_1 - q_3 q_2 - |q_3|^2 \end{pmatrix}. \] (2.10)

3 Brief review of the 1-layer M-LIII equation

3.1 The equation

3.1.1 The general form

For the pedagogical reason, in this section, we present some informations on the M-LIII equation for the 1-layer case. Consider the spin vector \( \mathbf{A} = (A_1, A_2, A_3) \), where \( A^2 = 1 \). Let this spin vector obey the 1-layer M-LIII equation which reads as

\[ A_t + A \wedge A_{xx} + u_1 A_x + F = 0, \] (3.1)

where \( u_1(x, t, A_j, A_{jx}) \) is the real function (potential), \( F \) is some vector function. The matrix form of the M-LIII equation looks like

\[ i A_t + \frac{1}{2} [A, A_{xx}] + i u_1 A_x + F = 0, \] (3.2)

where

\[ A = \begin{pmatrix} A_3 & A^- \\ A^+ & -A_3 \end{pmatrix}, \quad A^2 = I = diag(1, 1), \quad A^\pm = A_1 \pm i A_2. \] (3.3)

\[ F = \begin{pmatrix} F_3 & F^- \\ F^+ & -F_3 \end{pmatrix}, \quad F^\pm = F_1 \pm i F_2. \] (3.4)

3.1.2 The vector form

Sometime we consider the following particular case of the M-LIII equation

\[ A_t + A \wedge A_{xx} + u_1 A_x + v_1 H \wedge A = 0, \] (3.5)

where \( v_1(x, t, A_j, A_{jx}) \) is the real function (potential), \( H = (0, 0, 1) \) is the constant magnetic field.

3.1.3 The matrix form

The matrix form of this equation has the form

\[ i A_t + \frac{1}{2} [A, A_{xx}] + i u_1 A_x + v_1 [\sigma_3, A] = 0. \] (3.6)

3.1.4 The \( w \) - form

We now introduce a new complex function \( w \) as

\[ w = \frac{A^+}{1 + A_3}. \] (3.7)

Then the M-LIII equation takes the form

\[ i w_t - w_{xx} + \frac{2 \bar{w} w^2}{1 + |w|^2} = F', \] (3.8)

where \( F' \) is some complex function of the form

\[ F' = F'(w, w_x, \bar{w}, \bar{w}_x, u, v). \] (3.9)

It is the \( w \)-form of the M-LIII equation.
3.1.5 The r – form

Let us introduce the new function \( R \) as

\[
R = \partial^{-1}_x S. 
\] (3.10)

Then the r-form of the M-LIII equation reads as

\[
R_t = R_x \wedge R_{xx} + uR_x + 2vH \wedge R + L, 
\] (3.11)

where

\[
L = -\partial^{-1}_x [u_x R_x + 2v_x H \wedge R]. 
\] (3.12)

3.2 The Lakshmanan equivalent counterpart

Let us find the Lakshmanan equivalent counterpart of the 1-layer M-LIII equation (2.1)-(2.2). To do that, consider 3-dimensional curve in \( \mathbb{R}^3 \). This curve is given by the following vectors \( e_k \). These vectors satisfy the following equations

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}_x = C \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}, 
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}_t = D \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}. 
\] (3.13)

Here \( e_1, e_2 \) and \( e_3 \) are the unit tangent, normal and binormal vectors to the curve, \( x \) is its arclength parametrising the curve. The matrices \( C \) and \( G \) have the forms

\[
C = \begin{pmatrix}
0 & k_1 & 0 \\
-k_1 & 0 & \tau_1 \\
0 & -\tau_1 & 0
\end{pmatrix}, \quad G = \begin{pmatrix}
0 & \omega_3 & -\omega_2 \\
-\omega_3 & 0 & \omega_1 \\
\omega_2 & -\omega_1 & 0
\end{pmatrix}. 
\] (3.14)

The curvature and torsion of the curve are given by the following formulas

\[
k_1 = \sqrt{e_{1x}^2}, \quad \tau_1 = \frac{e_1 \cdot (e_{1x} \wedge e_{1xx})}{e_{1x}^2}. 
\] (3.15)

The compatibility condition of the equations (3.7) is given by

\[
C_t - G_x + [C, G] = 0, 
\] (3.16)

or in elements

\[
k_{1t} = \omega_3 x + \tau_1 \omega_2, 
\] (3.17)

\[
\tau_{1t} = \omega_1 x - k_1 \omega_2, 
\] (3.18)

\[
\omega_{2x} = \tau_1 \omega_1 - k_1 \omega_1. 
\] (3.19)

Now we do the following identifications:

\[
A \equiv e_1, \quad F = F_1 e_1 + F_2 e_2 + F_3 e_3. 
\] (3.20)

Then we have

\[
k_1^2 = A_x^2, 
\] (3.21)

\[
\tau_1 = \frac{A \cdot (A_x \wedge A_{xx})}{A_x^2}, 
\] (3.22)

and

\[
\omega_1 = -\frac{k_{1xx} + F_2 \tau_1 + F_3 x}{k_1} + (\tau_1 - u_1) \tau_1, 
\] (3.23)

\[
\omega_2 = k_{1x} x + F_3, 
\] (3.24)

\[
\omega_3 = k_1 (\tau_1 - u_1) - F_2. 
\] (3.25)
with $F_1 = E_1 = 0$. The equations for $k_1$ and $\tau_1$ reads as

$$k_{1t} = 2k_{1x}\tau_1 + k_1\tau_{1x} - (u_1k_1)x - F_2x + F_3\tau_1,$$

$$\tau_{1t} = -\left[\frac{k_{1xx} + F_2\tau_1 + F_3x}{k_1} + (\tau_1 - u_1)\tau_1 - \frac{1}{2}k_1^2\right]_x - F_3k_1. \tag{3.26}$$

Next we introduce a new complex function as

$$q_1 = \frac{k_1}{2}e^{-i\varphi_{11}^{-1}1_1} \tag{3.27}$$

This function satisfies the following equation

$$iq_{1t} + q_{1xx} + 2|q_1|^2q_1 + \ldots = 0. \tag{3.28}$$

It is the desired Lakshmanan equivalent counterpart of the M-LIII equation (3.5). If $u_1 = v_1 = 0$, it turns to the NLSE

$$iq_{1t} + q_{1xx} + 2|q_1|^2q_1 = 0. \tag{3.29}$$

## 4 The 2-layer M-LIII equation

In this section we consider the 2-layer M-LIII equation. As shown in [30]-[45], the 2-layer M-LIII equation is integrable by the inverse scattering transform (IST).

### 4.1 The equation

In this paper, we consider two spin vectors $\mathbf{A} = (A_1, A_2, A_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$, where $\mathbf{A}^2 = \mathbf{B}^2 = 1$. Let these spin vectors satisfy the following 2-layer M-LIII equation or the coupled M-LIII equation

$$A_t + A \wedge A_{xx} + u_1 A_x + 2v_1 H \wedge A = 0, \tag{4.1}$$

$$B_t + B \wedge B_{xx} + u_2 B_x + 2v_2 H \wedge B = 0, \tag{4.2}$$

or in matrix form

$$iA_t + \frac{1}{2}(A, A_{xx}) + iu_1 A_x + v_1 [\sigma_3, A] = 0, \tag{4.3}$$

$$iB_t + \frac{1}{2}(B, B_{xx}) + iu_2 B_x + v_2 [\sigma_3, B] = 0, \tag{4.4}$$

where $H = (0, 0, 1)^T$ is the constant magnetic field, $u_j$ and $v_j$ are coupling potentials. Here

$$u_1 = i[(q_2g_1g_3 - q_2g_1g_4) + (q_3g_1g_4 - q_3g_1g_4)], \tag{4.5}$$

$$v_1 = -\|[g_2]^2(\Delta_1 + [g_3]^2) + |g_3|^2(\Delta_1 + [g_4]^2) + q_2q_3g_3g_4 + q_2q_3g_3g_4, \tag{4.6}$$

$$u_2 = i[(q_2g_1g_3 - q_2g_1g_4) + (q_3g_1g_4 - q_3g_2g_4)], \tag{4.7}$$

$$v_2 = -\|[g_2]^2(\Delta_1 + [g_3]^2) + |g_3|^2(\Delta_1 + [g_4]^2) + q_2q_3g_3g_4 + q_2q_3g_3g_4. \tag{4.8}$$

where

$$\Delta_1 = |g_1|^2 + |g_2|^2, \tag{4.9}$$

$$\Delta_2 = |g_1|^2 + |g_4|^2, \tag{4.10}$$

$$\Delta = |g_1|^2 + |g_2|^2 + |g_3|^2. \tag{4.11}$$

In components, the coupled M-LIII equation (4.1)-(4.2) reads as

$$iA_t^+ + (A^+ A_{3xx} - A_{xx}^+ A_3) + iu_1 A_{xx}^+ - 2v_1 A^+ = 0, \tag{4.12}$$

$$iA_t^- - (A^- A_{3xx} - A_{xx}^- A_3) + iu_1 A_{xx}^- + 2v_1 A^- = 0, \tag{4.13}$$

$$iA_{3t} + \frac{1}{2}(A^+ A_{3xx}^+ - A_{xx}^+ A^+) + iu_1 A_{3xx} = 0, \tag{4.14}$$

$$iB_t^+ + (B^+ B_{3xx} - B_{xx}^+ B_3) + iu_2 B_{xx}^+ + 2v_2 B^+ = 0, \tag{4.15}$$

$$iB_t^- - (B^- B_{3xx} - B_{xx}^- B_3) + iu_2 B_{xx}^- + 2v_2 B^- = 0, \tag{4.16}$$

$$iB_{3t} + \frac{1}{2}(B^- B_{3xx}^+ - B_{xx}^- B^+) + iu_2 B_{3xx} = 0, \tag{4.17}$$

5
or

\[
\begin{align*}
A_{1t} + A_2A_{3xx} - A_{2xx}A_3 + u_1A_{1x} - 2v_1A_2 &= 0, \quad (4.18) \\
A_{2t} + A_3A_{1xx} - A_{3xx}A_1 + u_1A_{2x} - 2v_1A_1 &= 0, \quad (4.19) \\
A_{3t} + A_1A_{2xx} - A_{1xx}A_2 + u_1A_{3x} &= 0, \quad (4.20) \\
B_{1t} + B_2B_{3xx} - B_{2xx}B_3 + u_2B_{1x} - 2v_2B_2 &= 0, \quad (4.21) \\
B_{2t} + B_3B_{1xx} - B_{1xx}B_1 + u_2B_{2x} - 2v_2B_1 &= 0, \quad (4.22) \\
B_{3t} + B_1B_{2xx} - B_{1xx}B_2 + u_2B_{3x} &= 0. \quad (4.23)
\end{align*}
\]

4.2 The Lakshmanan equivalent counterpart

In this subsection we present the Lakshmanan equivalent counterpart of the 2-layer M-LIII equation (2.1)-(2.2). Now we consider two interacting 3-dimensional curves in $\mathbb{R}^n$. These curves are given by the following two basic vectors $e_k$ and $l_k$. The motion of these curves is defined by the following equations

\[
\begin{align*}
\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_x &= C \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \\
\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_t &= D \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix},
\end{align*}
\]

and

\[
\begin{align*}
\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}_x &= L \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}, \\
\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}_t &= N \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}.
\end{align*}
\]

Here $e_1$, $e_2$, and $e_3$ are the unit tangent, normal and binormal vectors respectively to the first curve, $l_1$, $l_2$, and $l_3$ are the unit tangent, normal and binormal vectors respectively to the second curve, $x$ is the arclength parametrising these both curves. The matrices $C, D, L, N$ are given by

\[
\begin{align*}
C &= \begin{pmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & \tau_1 \\ 0 & -\tau_1 & 0 \end{pmatrix}, \\
G &= \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}, \\
L &= \begin{pmatrix} 0 & k_2 & 0 \\ -k_2 & 0 & \tau_2 \\ 0 & -\tau_2 & 0 \end{pmatrix}, \\
N &= \begin{pmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{pmatrix}.
\end{align*}
\]

For the curvatures and torsions of curves we obtain

\[
\begin{align*}
k_1 &= \sqrt{k_{1xx}^2}, \quad \tau_1 = \frac{e_1 \cdot (e_{1x} \wedge e_{1xx})}{e_{1x}^2}, \\
k_2 &= \sqrt{l_{2xx}^2}, \quad \tau_2 = \frac{l_1 \cdot (l_{1x} \wedge l_{1xx})}{l_{1x}^2}.
\end{align*}
\]

The equations (3.9) and (3.11) are compatible if

\[
\begin{align*}
C_t - G_x + [C, G] &= 0, \quad (4.30) \\
L_t - N_x + [L, N] &= 0. \quad (4.31)
\end{align*}
\]

In elements these equations take the form

\[
\begin{align*}
k_{1tt} &= \omega_3 + \tau_1 \omega_2, \\
\tau_{1tt} &= \omega_{1x} - k_1 \omega_2, \\
\omega_{2xx} &= \tau_1 \omega_3 - k_1 \omega_1.
\end{align*}
\]
\[ k_{2t} = \theta_{3x} + \tau_{2}\theta_{2}, \quad (4.35) \]
\[ \tau_{2t} = \theta_{1x} - k_{2}\theta_{2}, \quad (4.36) \]
\[ \theta_{2x} = \tau_{2}\theta_{3} - k_{2}\theta_{1}. \quad (4.37) \]

Our next step is the following identifications:
\[ A \equiv e_{1}, \quad B \equiv l_{1}. \quad (4.38) \]

We also assume that
\[ F = F_{1}e_{1} + F_{2}e_{2} + F_{3}e_{3}, \quad E = E_{1}l_{1} + E_{2}l_{2} + E_{3}l_{3}, \quad (4.39) \]

where
\[ F = 2v_{1}H \wedge A, \quad E = 2v_{2}H \wedge B. \quad (4.40) \]

Then we obtain
\[ k_{1}^{2} = A_{x}^{2}, \quad (4.41) \]
\[ \tau_{1} = \frac{A \cdot (A_{x} \wedge A_{xx})}{A_{x}^{2}}, \quad (4.42) \]
\[ k_{2}^{2} = B_{x}^{2}, \quad (4.43) \]
\[ \tau_{2} = \frac{B \cdot (B_{x} \wedge B_{xx})}{B_{x}^{2}}, \quad (4.44) \]

and
\[ \omega_{1} = -\frac{k_{1xx} + F_{2}\tau_{1} + F_{3x}}{k_{1}} + (\tau_{1} - u_{1})\tau_{1}, \quad (4.45) \]
\[ \omega_{2} = k_{1x} + F_{3}, \quad (4.46) \]
\[ \omega_{3} = k_{1}(\tau_{1} - u_{1}) - F_{2}, \quad (4.47) \]
\[ \theta_{1} = -\frac{k_{2xx} + E_{2}\tau_{2} + E_{3x}}{k_{2}} + (\tau_{2} - u_{2})\tau_{2}, \quad (4.48) \]
\[ \theta_{2} = k_{2x} + E_{3}, \quad (4.49) \]
\[ \theta_{3} = k_{2}(\tau_{2} - u_{2}) - E_{2}. \quad (4.50) \]

with
\[ F_{1} = E_{1} = 0. \quad (4.51) \]

We now can write the equations for \( k_{j} \) and \( \tau_{j} \). They look like
\[ k_{1t} = 2k_{1x}\tau_{1} + k_{1}\tau_{1x} - (u_{1}k_{1})_{x} - F_{2x} + F_{3}\tau_{1}, \quad (4.52) \]
\[ \tau_{1t} = \frac{-k_{1xx} + F_{2}\tau_{1} + F_{3x}}{k_{1}} + (\tau_{1} - u_{1})\tau_{1} - \frac{1}{2}k_{1}^{2}, \quad (4.53) \]
\[ k_{2t} = 2k_{2x}\tau_{2} + k_{2}\tau_{2x} - (u_{2}k_{2})_{x} - E_{2x} + E_{3}\tau_{2}, \quad (4.54) \]
\[ \tau_{2t} = \frac{-k_{2xx} + E_{2}\tau_{2} + E_{3x}}{k_{2}} + (\tau_{2} - u_{2})\tau_{2} - \frac{1}{2}k_{2}^{2}, \quad (4.55) \]

Let us now introduce new four real functions \( \alpha_{j} \) and \( \beta_{j} \) as
\[ \alpha_{1} = 0.5k_{1}\sqrt{1 + \zeta_{1}}, \quad (4.56) \]
\[ \beta_{1} = \tau_{1}(1 + \xi_{1}), \quad (4.57) \]
\[ \alpha_{2} = 0.5k_{2}\sqrt{1 + \zeta_{2}}, \quad (4.58) \]
\[ \beta_{2} = \tau_{2}(1 + \xi_{2}), \quad (4.59) \]
In this paper, we consider three spin vectors $\mathbf{A} = (A_1, A_2, A_3)$, $\mathbf{B} = (B_1, B_2, B_3)$ and $\mathbf{C} = (C_1, C_2, C_3)$, where $\mathbf{A}^2 = \mathbf{B}^2 = \mathbf{C}^2 = 1$. Let these spin vectors satisfy the 3-layer M-LIII equation

\[ \mathbf{A} = (A_1, A_2, A_3), \mathbf{B} = (B_1, B_2, B_3) \text{ and } \mathbf{C} = (C_1, C_2, C_3), \text{ where } \mathbf{A}^2 = \mathbf{B}^2 = \mathbf{C}^2 = 1. \]
of the form

\[ \begin{align*}
A_t + A \wedge A_{xx} + u_1 A_x + 2v_1 H \wedge A &= 0, \\
B_t + B \wedge B_{xx} + u_2 B_x + 2v_2 H \wedge B &= 0, \\
C_t + C \wedge C_{xx} + u_3 C_x + 2v_3 H \wedge C &= 0,
\end{align*} \]

or in matrix form

\[ \begin{align*}
iA_t + \frac{1}{2}[A, A_{xx}] + iu_1 A_x + v_1[\sigma_3, A] &= 0, \\
iB_t + \frac{1}{2}[B, B_{xx}] + iu_2 B_x + v_2[\sigma_3, B] &= 0, \\
iC_t + \frac{1}{2}[C, C_{xx}] + iu_3 C_x + v_3[\sigma_3, C] &= 0.
\end{align*} \]

Here \( u_j \) and \( v_j \) are coupling potentials and have the forms

\[ \begin{align*}
u_1 &= i[(\bar{g}_2 g_1 g_3 - q_2 g_1 g_4) + (q_3 g_1 g_4 - q_3 g_1 g_4)], \\
v_2 &= -[|q_2|^2(\Delta_1 + |g_3|^2) + |g_3|^2(\Delta_2 + |g_4|^2) + q_2 q_3 g_3 g_4 + q_2 q_3 g_3 g_4], \\
u_2 &= i[(\bar{g}_1 g_2 + q_3 g_4 g_1) - (q_1 g_2 + q_2 g_3) g_1], \\
v_3 &= \frac{2i}{\Delta_1}[(\bar{g}_1 g_2 + q_3 g_4) g_1 - (q_1 g_2 + q_2 g_3) g_1], \\
v_3 &= \frac{2i}{\Delta_2}[(\bar{g}_1 g_2 + q_3 g_4 g_1) - (q_1 g_2 + q_2 g_3) g_1].
\end{align*} \]

where

\[ \begin{align*}
\Delta_1 &= |g_1|^2 + |g_2|^2, \\
\Delta_2 &= |g_1|^2 + |g_3|^2, \\
\Delta_3 &= |g_1|^2 + |g_4|^2, \\
\Delta &= |g_1|^2 + |g_2|^2 + |g_3|^2 + |g_4|^2.
\end{align*} \]

### 5.2 The Lakshmanan equivalent counterpart

In this subsection we obtain the Lakshmanan equivalent counterpart of the 3-layer M-LIII equation (2.1)-(2.2). For this purpose we consider three interacting 3-dimensional curves in some Euclidean space. These curves are given by the following three vectors \( \mathbf{e}_k, \mathbf{l}_k \) and \( \mathbf{n}_k \). These vectors are governed by the following equations

\[ \begin{align*}
\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_x &= C \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_t, \\
\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}_x &= L \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}_t, \\
\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}_x &= M \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}_t.
\end{align*} \]

The matrices \( C, D, L, N \) are given by

\[ \begin{align*}
C &= \begin{pmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & \tau_1 \\ 0 & -\tau_1 & 0 \end{pmatrix}, \\
G &= \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}.
\end{align*} \]
We also assume that

\[
L = \begin{pmatrix}
0 & k_2 & 0 \\
-k_2 & 0 & \tau_2 \\
0 & -\tau_2 & 0
\end{pmatrix}, \quad N = \begin{pmatrix}
0 & \theta_3 & -\theta_2 \\
-\theta_3 & 0 & \theta_1 \\
\theta_2 & -\theta_1 & 0
\end{pmatrix}, \quad (5.21)
\]

\[
M = \begin{pmatrix}
0 & k_3 & 0 \\
-k_3 & 0 & \tau_3 \\
0 & -\tau_3 & 0
\end{pmatrix}, \quad J = \begin{pmatrix}
0 & \delta_3 & -\delta_2 \\
-\delta_3 & 0 & \delta_1 \\
\delta_2 & -\delta_1 & 0
\end{pmatrix}. \quad (5.22)
\]

For the curvatures and torsions of curves we obtain

\[
k_1 = \sqrt{e_{1x}^2}, \quad \tau_1 = \frac{e_1 \cdot (e_{1x} \wedge e_{1xx})}{e_{1x}^3},
\]

\[
k_2 = \sqrt{l_{1x}^2}, \quad \tau_2 = \frac{l_1 \cdot (l_{1x} \wedge l_{1xx})}{l_{1x}^3},
\]

\[
k_3 = \sqrt{n_{1x}^2}, \quad \tau_3 = \frac{n_1 \cdot (n_{1x} \wedge n_{1xx})}{n_{1x}^3}.
\]

The equations (3.9) and (3.11) are compatible if

\[
C - G + [C, G] = 0,
\]

\[
L - N + [L, N] = 0,
\]

\[
M - J + [M, J] = 0.
\]

In elements these equations take the form

\[
k_{1t} = \omega_{1z} + \tau_1 \omega_2,
\]

\[
\tau_{1t} = \omega_{1x} - k_1 \omega_2,
\]

\[
\omega_{2x} = \tau_1 \omega_3 - k_1 \omega_1,
\]

\[
k_{2t} = \theta_{3x} + \tau_2 \theta_2,
\]

\[
\tau_{2t} = \theta_{1x} - k_2 \theta_2,
\]

\[
\theta_{2x} = \tau_2 \theta_3 - k_2 \theta_1.
\]

and

\[
k_{3t} = \delta_{3x} + \tau_2 \delta_2,
\]

\[
\tau_{3t} = \delta_{1x} - k_3 \delta_2,
\]

\[
\delta_{2x} = \tau_3 \delta_3 - k_3 \delta_1.
\]

As in the previous section we assume the following identifications:

\[
A \equiv e_1, \quad B \equiv l_1, \quad C \equiv n_1.
\]

We also assume that

\[
F = F_1 e_1 + F_2 e_2 + F_3 e_3, \quad E = E_1 l_1 + E_2 l_2 + E_3 l_3, \quad P = P_1 n_1 + P_2 n_2 + P_3 n_3.
\]

where

\[
F = 2v_1 H \wedge A, \quad E = 2v_2 H \wedge B, \quad P = 2v_3 H \wedge C.
\]

Then we obtain

\[
k_1^2 = A_x^2,
\]

\[
\tau_1 = \frac{A \cdot (A_x \wedge A_{xx})}{A_x^2},
\]

\[
k_2^2 = B_z^2,
\]

\[
\tau_2 = \frac{B \cdot (B_x \wedge B_{xx})}{B_z^2},
\]

\[
k_3^2 = C_x^2,
\]

\[
\tau_3 = \frac{C \cdot (C_x \wedge C_{xx})}{C_x^2}.
\]

\[
(5.40)
\]

(5.41)
and

\[
\omega_1 = -\frac{k_{1xx} + F_2 \tau_1 + F_3}{k_1} + (\tau_1 - u_1)\tau_1, \quad (5.47)
\]

\[
\omega_2 = k_{1x} + F_3, \quad (5.48)
\]

\[
\omega_3 = k_1(\tau_1 - u_1) - F_2, \quad (5.49)
\]

\[
\theta_1 = -\frac{k_{2xx} + E_2 \tau_2 + E_3}{k_2} + (\tau_2 - u_2)\tau_2, \quad (5.50)
\]

\[
\theta_2 = k_{2x} + E_3, \quad (5.51)
\]

\[
\theta_3 = k_2(\tau_2 - u_2) - E_2, \quad (5.52)
\]

\[
\delta_1 = -\frac{k_{3xx} + P_2 \tau_3 + P_3}{k_3} + (\tau_3 - u_3)\tau_3, \quad (5.53)
\]

\[
\delta_2 = k_{3x} + P_3, \quad (5.54)
\]

\[
\delta_3 = k_3(\tau_3 - u_3) - P_2, \quad (5.55)
\]

with

\[
F_1 = E_1 = P_1 = 0. \quad (5.56)
\]

As a result, we obtain the following equations for \(k_j\) and \(\tau_j\). They read as

\[
k_{1t} = 2k_{1x}\tau_1 + k_1\tau_1_x - (u_1k_1)_x - F_{2x} + F_3\tau_1, \quad (5.57)
\]

\[
\tau_{1t} = \left[ -\frac{k_{1xx} + F_2 \tau_1 + F_3}{k_1} + (\tau_1 - u_1)\tau_1 - \frac{1}{2}k_1^2 \right]_x - F_3k_1, \quad (5.58)
\]

\[
k_{2t} = 2k_{2x}\tau_2 + k_2\tau_2_x - (u_2k_2)_x - E_{2x} + E_3\tau_2, \quad (5.59)
\]

\[
\tau_{2t} = \left[ -\frac{k_{2xx} + E_2 \tau_2 + E_3}{k_2} + (\tau_2 - u_2)\tau_2 - \frac{1}{2}k_2^2 \right]_x - E_3k_2, \quad (5.60)
\]

\[
k_{3t} = 2k_{3x}\tau_3 + k_3\tau_3_x - (u_3k_3)_x - P_{2x} + P_3\tau_3, \quad (5.61)
\]

\[
\tau_{3t} = \left[ -\frac{k_{3xx} + P_2 \tau_3 + P_3}{k_3} + (\tau_3 - u_3)\tau_3 - \frac{1}{2}k_3^2 \right]_x - P_3k_3. \quad (5.62)
\]

According to our approach (see e.g. the refs. [39]-[45]), we now introduce the following new functions \(\alpha_j\) and \(\beta_j\) as

\[
\alpha_1 = 0.5k_1\sqrt{1 + \xi_1}, \quad (5.63)
\]

\[
\beta_1 = \tau_1(1 + \xi_1), \quad (5.64)
\]

\[
\alpha_2 = 0.5k_2\sqrt{1 + \xi_2}, \quad (5.65)
\]

\[
\beta_2 = \tau_2(1 + \xi_2), \quad (5.66)
\]

\[
\alpha_3 = 0.5k_3\sqrt{1 + \xi_3}, \quad (5.67)
\]

\[
\beta_3 = \tau_3(1 + \xi_3). \quad (5.68)
\]

where \(\xi_j\) and \(\xi_j\) are some real functions. We now can show that the equations for the functions \(\alpha_j\) and \(\beta_j\) read as

\[
\alpha_{1t} - 2\alpha_{1x}\beta_1 - \alpha_1\beta_{1x} = 0, \quad (5.69)
\]

\[
\beta_{1t} + \left[ \frac{\alpha_{1xx}}{\alpha_1} - \beta_1^2 + 2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) \right]_x = 0, \quad (5.70)
\]

\[
\alpha_{2t} - 2\alpha_{2x}\beta_2 - \alpha_2\beta_{2x} = 0, \quad (5.71)
\]

\[
\beta_{2t} + \left[ \frac{\alpha_{2xx}}{\alpha_2} - \beta_2^2 + 2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) \right]_x = 0, \quad (5.72)
\]

\[
\alpha_{3t} - 2\alpha_{3x}\beta_3 - \alpha_3\beta_{3x} = 0, \quad (5.73)
\]

\[
\beta_{3t} + \left[ \frac{\alpha_{3xx}}{\alpha_3} - \beta_3^2 + 2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) \right]_x = 0. \quad (5.74)
\]
Next we introduce new three complex functions \( q_j \) as
\[
q_1 = \alpha_1 e^{-i\delta_1 z^3}, \quad (5.75)
q_2 = \alpha_2 e^{-i\delta_2 z^4}, \quad (5.76)
q_3 = \alpha_3 e^{-i\delta_3 z^5}, \quad (5.77)
\]
The straight calculation shows that these functions obey the following vector NLS equation
\[
iq_{1t} + q_{1xx} + 2(|q_1|^2 + |q_2|^2 + |q_3|^2)q_1 = 0, \quad (5.78)
iq_{2t} + q_{2xx} + 2(|q_1|^2 + |q_2|^2 + |q_3|^2)q_2 = 0, \quad (5.79)
iq_{3t} + q_{3xx} + 2(|q_1|^2 + |q_2|^2 + |q_3|^2)q_3 = 0. \quad (5.80)
\]
Thus we have proved that the 3-component NLS equation (5.78)-(5.80) is the Lakshmanan equivalent counterpart of the 3-layer M-LIII equation (2.1)-(2.2).

6 The 4-layer M-LIII equation

6.1 The equation

In this section we study the 4-layer M-LIII equation. It is given as
\[
A_t + A \wedge A_{xx} + u_1 A_x + 2v_1 H \wedge A = 0, \quad (6.1)
B_t + B \wedge B_{xx} + u_2 B_x + 2v_2 H \wedge B = 0, \quad (6.2)
C_t + C \wedge C_{xx} + u_3 C_x + 2v_3 H \wedge C = 0, \quad (6.3)
D_t + D \wedge D_{xx} + u_4 D_x + 2v_4 H \wedge D = 0. \quad (6.4)
\]
In matrix form, this equation takes the form
\[
iA_t + \frac{1}{2}[A, A_{xx}] + iu_1 A_x + v_1 [\sigma_3, A] = 0, \quad (6.6)
iB_t + \frac{1}{2}[B, B_{xx}] + iu_2 B_x + v_2 [\sigma_3, B] = 0, \quad (6.7)
iC_t + \frac{1}{2}[C, C_{xx}] + iu_3 C_x + v_3 [\sigma_3, C] = 0, \quad (6.8)
iD_t + \frac{1}{2}[D, D_{xx}] + iu_4 D_x + v_4 [\sigma_3, D] = 0, \quad (6.9)
\]
where \( A = (A_1, A_2, A_3) \), \( B = (B_1, B_2, B_3) \), \( C = (C_1, C_2, C_3) \), \( D = (D_1, D_2, D_3) \) and \( A^2 = B^2 = C^2 = D^2 = 1 \). Here \( u_j \) and \( v_j \) are coupling potentials.

6.2 The Lakshmanan equivalent counterpart

The similar algebra as in the previous sections shows that the Lakshmanan equivalent counterpart of the 4-layer M-LIII equation (2.1)-(2.2) reads as
\[
iq_{1t} + q_{1xx} + 2(|q_1|^2 + |q_2|^2 + |q_3|^2 + |q_4|^2)q_1 = 0, \quad (6.10)
iq_{2t} + q_{2xx} + 2(|q_1|^2 + |q_2|^2 + |q_3|^2 + |q_4|^2)q_2 = 0. \quad (6.11)
iq_{3t} + q_{3xx} + 2(|q_1|^2 + |q_2|^2 + |q_3|^2 + |q_4|^2)q_3 = 0. \quad (6.12)
iq_{4t} + q_{4xx} + 2(|q_1|^2 + |q_2|^2 + |q_3|^2 + |q_4|^2)q_4 = 0. \quad (6.13)
\]
7 The N-layer M-LIII equation

7.1 The equation

In this section we present the N-layer M-LIII equation. This equation can be written as

\[ \begin{align*}
A_t^{(1)} + A^{(1)} \wedge A_{xx}^{(1)} + u_1 A_x^{(1)} + 2v_1 H \wedge A^{(1)} &= 0, \\
A_t^{(2)} + A^{(2)} \wedge A_{xx}^{(2)} + u_2 A_x^{(2)} + 2v_2 H \wedge A^{(2)} &= 0, \\
&\vdots \\
A_t^{(N)} + A^{(N)} \wedge A_{xx}^{(N)} + u_N A_x^{(N)} + 2v_N H \wedge A^{(N)} &= 0.
\end{align*} \] (7.1)-(7.4)

Let us rewrite this equation in the matrix form as

\[ iA_t^{(1)} + \frac{1}{2}[A^{(1)}, A_x^{(1)}] + iu_1 A_{xx}^{(1)} + v_1[\sigma_3, A^{(1)}] = 0, \] (7.5)
\[ iA_t^{(2)} + \frac{1}{2}[A^{(2)}, A_x^{(2)}] + iu_2 A_{xx}^{(2)} + v_2[\sigma_3, A^{(2)}] = 0, \] (7.6)
\[ \vdots \]
\[ iA_t^{(N)} + \frac{1}{2}[A^{(N)}, A_x^{(N)}] + iu_N A_{xx}^{(N)} + v_N[\sigma_3, A^{(N)}] = 0. \] (7.7)

where \( \mathbf{A}^{(j)} = (A_1^{(j)}, A_2^{(j)}, A_3^{(j)}) \) and \( \mathbf{A}^{(j)2} = 1 \) with

\[ A^{(j)} = \begin{pmatrix} A_3^{(j)} \\ A_1^{(j)} + iA_2^{(j)} \\ A_2^{(j)} - iA_1^{(j)} \end{pmatrix}, \quad A^{(j)2} = I, \quad A^{(j)\pm} = A_1^{(j)} \pm iA_2^{(j)}, \] (7.9)

and \( u_j \) and \( v_j \) are some coupling potentials.

7.2 The Lakshmanan equivalent counterpart

We now can write the Lakshmanan equivalent counterpart of the N-layer M-LIII equation (7.1)-(7.4). It is the following vector NLSE

\[ \begin{align*}
\mathbf{i}q_{1t} + q_{1xx} + 2(|q_1|^2 + |q_2|^2 + \cdots + |q_N|^2)q_1 &= 0, \\
\mathbf{i}q_{2t} + q_{2xx} + 2(|q_1|^2 + |q_2|^2 + \cdots + |q_N|^2)q_2 &= 0, \\
&\vdots \\
\mathbf{i}q_{Nt} + q_{Nxx} + 2(|q_1|^2 + |q_2|^2 + \cdots + |q_N|^2)q_N &= 0.
\end{align*} \] (7.10)-(7.13)

8 Hamiltonian structure

In this section we briefly present main elements of the Hamiltonian structure of the multilayer M-LIII equation. As example, here we consider only \( N = 3 \) case that is the 3-layer M-LIII equation (5.1)-(5.3). Let us start from quantum Heisenberg magnetic model for the 3-layer case. We write it as

\[ \mathbf{H} = J \sum (\mathbf{\hat{A}}_i \mathbf{\hat{A}}_{i+1} + \mathbf{\hat{B}}_i \mathbf{\hat{B}}_{i} + \mathbf{\hat{C}}_i \mathbf{\hat{C}}_{i}) + H_{\text{int}}. \] (8.1)

In classical limit, this model takes the form

\[ \mathbf{H} = J \sum (A_i A_{i+1} + B_i B_{i} + C_i C_{i}) + H_{\text{int}}. \] (8.2)

The continuum limit of this model looks like

\[ H = H_a + H_b + H_c + H_{\text{int}}, \] (8.3)
where

\[ H_a = 0.5 \int A_a^2 dx, \quad H_b = 0.5 \int B_b^2 dx, \quad H_c = 0.5 \int C_c^2 dx, H_{int} = 0.5 \int h_{int} dx, \]   \hspace{1cm} (8.4)

with \( h_{int} = h_{int}(A_j, B_j, C_j, A_{jx}, B_{jx}, C_{jx}) \). The Hamilton form of the 3-layer M-LIII equation (5.1)-(5.3) is given by

\[ A_{it} = \{ H, A_i \} = \epsilon_{ijk} \delta H/\delta A_j A_k, \]  \hspace{1cm} (8.5)
\[ B_{it} = \{ H, B_i \} = \epsilon_{ijk} \delta H/\delta B_j B_k, \]  \hspace{1cm} (8.6)
\[ C_{it} = \{ H, C_i \} = \epsilon_{ijk} \delta H/\delta C_j C_k. \]  \hspace{1cm} (8.7)

Here the Poisson bracket is given by

\[ \{ P, Q \} = \epsilon_{ijk} \int \left[ \delta P/\delta A_i \delta Q/\delta A_j A_k + \delta P/\delta B_i \delta Q/\delta B_j B_k + \delta P/\delta C_i \delta Q/\delta C_j C_k \right] dx. \]   \hspace{1cm} (8.8)

Hence for the components of the spin vectors we obtain

\[ \{ A_i, A_j \} = \epsilon_{ijk} A_k \delta(x - y), \] \hspace{1cm} (8.9)
\[ \{ B_i, B_j \} = \epsilon_{ijk} B_k \delta(x - y), \] \hspace{1cm} (8.10)
\[ \{ C_i, C_j \} = \epsilon_{ijk} C_k \delta(x - y). \] \hspace{1cm} (8.11)

and

\[ \{ A_i, B_j \} = \{ A_i, C_j \} = \{ A_i, C_j \} = 0. \] \hspace{1cm} (8.12)

9 The \( \Gamma \)-spin system

In the previous section we have shown that between the multilayer M-LIII equation and the vector NLSE takes place the Lakshmanan equivalence. But it is well known that there exists another "spin system", namely, the \( \Gamma \)-spin system which is also (gauge) equivalent to the vector NLSE. The \( \Gamma \)-spin system reads as

\[ i\Gamma_t + \frac{1}{2} [\Gamma, \Gamma_{xx}] = 0, \] \hspace{1cm} (9.1)

where

\[ \Gamma = (\Gamma_{ij}) \in \text{su}(N + 1). \] \hspace{1cm} (9.2)

10 The relation between solutions of the multilayer M-LIII equation and the \( \Gamma \)-spin system

It is natural that the solutions of the multilayer M-LIII equation (4.3)-(4.4) and the \( \Gamma \)-spin system (9.1) is related to each other by some exact transformations. As example, here we present these transformations for the 2-layer M-LIII equation that is for \( N = 2 \) case.

10.1 Direct M-transformation

Let \( A \) and \( B \) be the solution of the 2-layer M-LIII equation (4.3)-(4.4). Then, according to the direct M-transformation \([39]-[45]\), the solutions of the \( \Gamma \)-spin system (9.1) are expressed as

\[ \Gamma = \frac{1}{2 + K} \begin{pmatrix} 2A_3 - K & 2A^- \qquad 2(1 + A_3)B^- \qquad \frac{2(1 + A_3)B^-}{1 + B_3} \\ 2A^+ & -2(A_3 + K) \qquad \frac{2A^+ B^-}{1 + B_3} \qquad \frac{2A^+ B^-}{1 + B_3} \\ 2(1 + A_3)B^- & \frac{2A^- B^-}{1 + B_3} \qquad K - 2 \end{pmatrix}, \] \hspace{1cm} (10.1)
where
\[ K = \frac{(1 + A_3)(1 - B_3)}{1 + B_3}. \] (10.2)

### 10.2 Inverse M-transformation

We now consider the inverse M-transformation. Let \( \Gamma_{ij} \) be solutions of the \( \Gamma \) – spin system (9.1). Then the solutions of the coupled or (that same) 2-layer M-LIII equation (4.3)-(4.4) are defined as
\[ A = \frac{1}{1 - \Gamma_{33}} \begin{pmatrix} \Gamma_{11} - \Gamma_{22} & 2\Gamma_{12} \\ 2\Gamma_{21} & \Gamma_{22} - \Gamma_{11} \end{pmatrix}, \] (10.3)
\[ B = \frac{1}{1 - \Gamma_{22}} \begin{pmatrix} \Gamma_{11} - \Gamma_{33} & 2\Gamma_{13} \\ 2\Gamma_{31} & \Gamma_{33} - \Gamma_{11} \end{pmatrix}, \] (10.4)

where \( \Gamma_{ij} \) are given by the formulas (9.2). Similarly, we find the direct and inverse M-transformations for \( N \neq 2 \) cases.

### 11 Geometric flows of immersed surfaces

Let \( \mathbf{r}_j = \mathbf{r}_j(x, t) \) be the position vector of the immersed \( j \)-th surface in the some Euclidean space. Such surfaces are given by the following set of first fundamental forms
\[ I_1 = dx^2 + 2\mathbf{r}_{1x} \cdot \mathbf{r}_{1t} dx dt + r_{1t}^2 dt^2, \] (11.1)
\[ I_2 = dx^2 + 2\mathbf{r}_{2x} \cdot \mathbf{r}_{2t} dx dt + r_{2t}^2 dt^2, \] (11.2)
\[ \vdots \] (11.3)
\[ I_N = dx^2 + 2\mathbf{r}_{Nx} \cdot \mathbf{r}_{Nt} dx dt + r_{Nt}^2 dt^2, \] (11.4)

where we assumed that \( r_x^2 = 1 \). We write the set of the second fundamental forms as
\[ II_1 = L_1 dx^2 + 2M_1 dx dt + N_1 dt^2, \] (11.5)
\[ II_2 = L_2 dx^2 + 2M_2 dx dt + N_2 dt^2, \] (11.6)
\[ \vdots \] (11.7)
\[ II_N = L_N dx^2 + 2M_N dx dt + N_N dt^2. \] (11.8)

Finally, we can also write the third fundamental forms. As it is well known, the third fundamental forms can be written in terms of the first and second forms as
\[ III_1 = 2H_1 II_1 - K_1 I_1, \] (11.9)
\[ III_2 = 2H_2 II_2 - K_2 I_2, \] (11.10)
\[ \vdots \] (11.11)
\[ III_N = 2H_N II_N - K_N I_N, \] (11.12)

where \( H_j \) and \( K_j \) are the mean curvature and the Gaussian curvature of the \( j \)-th surface respectively.

### 12 Geometric flows of curves

In this section we want to present another but the equivalent approach to derive the Lakshmanan equivalent of the 1-layer M-LIII equation (3.5). For this purpose we consider the curve which is given by
\[ \mathbf{r}_t = a \mathbf{e}_1 + b \mathbf{e}_2 + c \mathbf{e}_3, \] (12.1)
where \( \mathbf{e}_1, \mathbf{e}_2 \) and \( \mathbf{e}_3 \) denote the tangent, normal and binormal vectors of the curve, respectively. The velocities \( a, b \) and \( c \) depend on the \( \kappa \) and \( \tau \) as well as their derivatives with respect to arclength parameter \( x \). The arclength parameter \( x \) is defined implicitly by \( ds = h dp, h = |r(p)| \), where \( p \) is a free parameter and is independent of time. From the flow (12.1), the time evolutions of the vectors \( \mathbf{e}_j \) are given by

\[
\begin{align*}
\dot{\mathbf{e}}_3 &= (a_x - \tau b + \kappa c) \mathbf{e}_1 + (b_x + \tau a) \mathbf{e}_2, \\
\dot{\mathbf{e}}_1 &= - \left( a_x - \tau b + \kappa c \right) \mathbf{e}_3 + \frac{1}{\kappa} \left( (b_x + \tau a)_{x} + \tau (a_x - \tau b + \kappa c) \right) \mathbf{e}_2, \\
\dot{\mathbf{e}}_2 &= - (b_x + \tau a) \mathbf{e}_3 - \frac{1}{\kappa} \left( (b_x + \tau a)_{x} + \tau (a_x - \tau b + \kappa c) \right) \mathbf{e}_1, \\
\dot{h} &= 2h(c_x - \kappa a) \
\end{align*}
\]  

(12.2) 

(12.3) 

(12.4) 

(12.5) 

and

\[
\begin{align*}
\tau_t &= \left[ \frac{1}{\kappa} \left( b_x + \tau a \right) + \frac{\tau}{\kappa} (a_x - \tau b) + \tau \int x \kappa a dx' \right] + \kappa \tau a + \kappa b_x, \\
\kappa_t &= a_{xx} + (\kappa - \tau^2)a + \kappa_x \int x \kappa a dx' - 2\tau b_x - \tau_x b. 
\end{align*}
\]  

(12.6) 

(12.7) 

Assuming that the flow is intrinsic, namely that the arclength does not depend on time, it implies from (12.5) that

\[
ce_s = \kappa a. 
\]

(12.8) 

Let \( a = 0, b = \kappa \), where \( \kappa \) is a real function, then (12.8) implies that \( c = c_1 \), where \( c_1 \) is a constant. Let \( c_1 = 0 \), then the set of equations (12.2)-(12.4) takes the form

\[
\begin{align*}
\dot{\mathbf{e}}_3 &= -\tau \kappa \mathbf{e}_1 + \kappa_x \mathbf{e}_2, \\
\dot{\mathbf{e}}_1 &= \tau \kappa \mathbf{e}_3 + \left[ \frac{\kappa_{xx}}{\kappa} - \tau^2 \right] \mathbf{e}_2, \\
\dot{\mathbf{e}}_2 &= -\kappa_x \mathbf{e}_3 - \left[ \frac{\kappa_{xx}}{\kappa} - \tau^2 \right] \mathbf{e}_1. 
\end{align*}
\]  

(12.9) 

13 Integrable filament equations of interacting vortices

Let us consider the following r-form of the multilayer M-LIII equation, shortly, the multilayer r-M-LIII equation

\[
\mathbf{r}_{jt} = \mathbf{r}_{jx} \wedge \mathbf{r}_{jxx} + u_j \mathbf{r}_{jx} + 2v_j \mathbf{H} \wedge \mathbf{r}_j + \mathbf{L}_j, 
\]

(13.1) 

where \( j = 1, 2, \cdots, N \) and

\[
\mathbf{L}_j = -\partial_j^{-1} \left[ u_j \mathbf{r}_{jx} + 2v_j \mathbf{H} \wedge \mathbf{r}_j \right].
\]

(13.2) 

This closed set of equations is integrable. It describes the (integrable) interaction of \( N \) vortices. Indeed, this system is the closed set of the filament equations for interacting \( N \) vortices. In the case \( u_j = v_j = 0 \), we obtain the following uncoupled (noninteracting) \( N \) vortices filament equations

\[
\mathbf{r}_{jt} = \mathbf{r}_{jx} \wedge \mathbf{r}_{jxx}. 
\]

(13.3) 

14 Conclusion

In this paper, we have shown that the \( N \)-layer M-LIII equation can be related with the geometric flows of interacting curves and surfaces in some space \( R^6 \). Then we have found the Lakshmanan equivalent counterparts of the \( N \)-layer M-LIII equations. After some algebra we have proved that these counterparts in fact are the vector NLSE. On the other hand, it is well-known that the vector NLSE is equivalent to the \( \Gamma \)-spin system. Also, we have presented the transformations which give the relation between solutions of the \( \Gamma \)-spin system and the multilayer M-LIII equation. It is interesting to understand the role of the constant magnetic field \( \mathbf{H} \). It seems that this constant magnetic vector plays an important role in our construction of integrable multilayer spin systems and in nonlinear dynamics of multilayer magnetic systems.
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