Some Operator and Trace Function Convexity Theorems

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Abstract

We consider trace functions \((A, B) \mapsto \text{Tr}[(A^{q/2}B^pA^{q/2})^s]\) where \(A\) and \(B\) are positive \(n \times n\) matrices and ask when these functions are convex or concave. We also consider operator convexity/concavity of \(A^{q/2}B^pA^{q/2}\) and convexity/concavity of the closely related trace functional \(\text{Tr}[A^{q/2}B^pA^{q/2}C]\). The concavity questions are completely resolved, thereby settling cases left open by Hiai; the convexity questions are settled in many cases. As a consequence, the Audenaert–Datta Rényi entropy conjectures are proved for some cases.

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1 Introduction

Let \(P_n\) denote the set of \(n \times n\) positive definite matrices. For \(p, q, s \in \mathbb{R}\), define

\[\Phi_{p,q,s}(A,B) = \text{Tr}[(A^{q/2}B^pA^{q/2})^s].\] (1.1)

We are mainly interested in the convexity or concavity of the map \((A, B) \mapsto \Phi_{p,q,s}(A,B)\), but we are also interested in the operator convexity/concavity of \(A^{q/2}B^pA^{q/2}\). When any of \(p, q\) or \(s\) is zero, the question of convexity is trivial, and we exclude these cases.

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Given any $n \times n$ matrix $K$, and with $p$, $q$, $s$ as above, define
\[ \Psi_{K,p,q,s}(A, B) = \text{Tr}[(A^{q/2}K^*B^pKA^{q/2})^s], \tag{1.2} \]
and note that
\[ \Phi_{p,q,s}(A, B) = \Psi_{K,p,q,s}(A, B) . \tag{1.3} \]

The main question to be addressed here is this: For which non-zero values of $p$, $q$ and $s$ is $\Psi_{K,p,q,s}(A, B)$ jointly convex or jointly concave on $P_n \times P_n$ for all $n$ and all $K$?

We begin with several simple reductions. Since invertible $K$ are dense, it suffices to consider all invertible operators $K$. Then, for $K$ invertible,
\[ \Psi_{K,p,q,s}(A, B) = \Phi_{K^{-1},-p,-q,-s}(A, B) , \]
and therefore it is no loss of generality to assume that $s > 0$. We always make this assumption in what follows.

Next, the convexity/concavity properties of $\Psi_{K,p,q,s}(A, B)$ are a consequence of those of $\Phi_{p,q,s}(A, B)$, and hence it suffices to study the special case $K = 1$. In fact, more is true as stated in the following Lemma 1.1.

These equivalences may be useful in other contexts. (For $s = 1$ the equivalence of (1) and (4) is in [11] and the equivalence of (1) and (3) is in [4]; the arguments in those papers extend to all $s$, but we repeat them here for completeness.)

1.1 LEMMA (Equivalent formulations). The following statements are equivalent for fixed $p, q, s$.

1. The map $(A, B) \mapsto \Psi_{K,p,q,s}(A, B)$ is convex for all $K$ and all $n$.
2. The map $(A, B) \mapsto \Psi_{K,p,q,s}(A, B)$ is convex for all unitary $K$ and all $n$.
3. The map $(A, B) \mapsto \Phi_{p,q,s}(A, B)$ is convex for all $n$.
4. The map $A \mapsto \Psi_{K,p,q,s}(A, A)$ is convex for all $K$ and all $n$.
5. The map $A \mapsto \Psi_{K,p,q,s}(A, A)$ is convex for all unitary $K$ and all $n$.

The same is true if convex is replaced by concave in all statements.

Proof. Trivially, (1) implies the other four items.

When $K$ is unitary, $K^*A^qK = (K^*AK)^q$, and hence (3) implies (2) (even for each fixed $n$). By taking $K = 1$, (2) implies (3) (again for each fixed $n$).

Next we show that (2) implies (1), whence (1), (2) and (3) are equivalent. We may suppose, without loss of generality that $K$ is a contraction. Let $K = W|K|$ be its polar decomposition. Then
\[ U = \begin{bmatrix} K & W\sqrt{1 - |K|^2} \\ -W\sqrt{1 - |K|^2} & |K| \end{bmatrix} \]
is unitary. We consider the case $q < 0$ first. For arbitrary $t > 0$, let
\[ A_t = \begin{bmatrix} A & 0 \\ 0 & t\mathbf{1} \end{bmatrix}, \quad B = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} . \]
Then
\[
\begin{bmatrix}
A^{q/2}K^*B^pK^*A^{q/2} & 0 \\
0 & 0
\end{bmatrix} = \lim_{t \to \infty} A^{q/2}U^*B^pUA^{q/2}.
\]

Thus, recalling that we always assume \( s > 0 \),
\[
\text{Tr}[(A^{q/2}K^*B^pK^*A^{q/2})^s] = \lim_{t \to \infty} \Psi_{U,p,q,s}(A_t, B).
\]

Thus, (2) with \( 2n \) implies (1) with \( n \). The case \( q > 0 \) is treated analogously, letting \( t \to 0 \).

Trivially, (4) implies (5). To show that (5) (with \( 2n \)) implies (3) (with \( n \)), thereby completing the loop, replace \( A \) in (5) by \( \begin{bmatrix} A & 0 \\
0 & B \end{bmatrix} \), and replace \( K \) by the unitary \( \begin{bmatrix} 0 & 1 \\
1 & 0 \end{bmatrix} \). \( \square \)

2 Known results and our extension of them

Hiai has proved in [8] that if \( p, q \) are both non-zero, and \( s > 0 \), and \( \Phi_{p,q,s} \) is jointly convex in \( A \) and \( B \), then, necessarily, one of the following conditions holds:

(1.) \( 1 \leq p \leq 2 \) and \( -1 \leq q < 0 \) and \( s \geq 1/(p + q) \), or the same with \( p \) and \( q \) interchanged.

(2.) \( -1 \leq p, q < 0 \) and \( s > 0 \).

In the special case \( s = 1 \), condition (1.) was proved to be sufficient in [1, Corollary 6.3], and condition (2.) was proved to be sufficient in [11, Theorem 8]; see also [3] for \( s = 1 \) and one of \( p, q \) negative. Hiai [8] has also proved that \( \Phi_{p,q,s} \) is jointly convex in case \( -1 \leq p, q < 0 \) and \( 1/2 \leq s \leq -1/(p + q) \).

Our main focus is on (1.). The joint convexity in this case is known [7] when \( s = 1/(p + q) \), \( p = 1 \) and \( -1 \leq q < 0 \), and of course, with \( p \) and \( q \) interchanged.

Concerning concavity, Hiai has shown [8] that if \( p, q \) are both non-zero, and \( s > 0 \), and \( \Phi_{p,q,s} \) is jointly concave in \( A \) and \( B \), then, necessarily, the following condition holds:

(3.) \( 0 < p, q \leq 1 \) and \( 0 < s \leq 1/(p + q) \).

In the special case \( s = 1 \), this condition was proved to be sufficient in [11, Theorem 1]; Hiai [8] showed sufficiency for \( 1/2 \leq s \leq 1/(p + q) \).

Our contribution to the subject is to fill in parts of the table of sufficient/necessary conditions in the following manner. We were motivated in this endeavor by a recent paper of Audenaert and Datta [2], (and Datta’s Warwick lecture on it) and we prove some of their conjectures.

All the results mentioned above refer to trace inequalities. There are some operator convexity/concavity inequalities to be considered as well, and we will present some in the following.

\(^1\)After this work was submitted, Hiai posted the preprint arXiv:1507.00853 in which he extended our method to prove joint convexity under condition (2.).
As far as convexity of $\Phi_{p,q,s}$ is concerned we can summarize our results as follows. We are concerned with the region $p \in [1, 2]$, $q \in [-1, 0)$ and $s \geq 1/(p+q)$. (Clearly, $s$ cannot be smaller than $1/(p+q)$ by homogeneity.) We prove joint convexity for $s \geq \min \left\{ \frac{1}{p-1}, \frac{1}{1+q} \right\}$ (Thm. 4.1). Moreover, we prove joint convexity for $p = 1$ and $p = 2$ in the optimal range $s \geq 1/(p+q)$ (Thm. 4.2).

For $p \in (1, 2)$, $q \in [-1, 0)$, the missing regions, where we believe joint convexity also holds, is $1/(p+q) \leq s < 1$ and $1 < s < \min \left\{ \frac{1}{p-1}, \frac{1}{1+q} \right\}$. (Ando’s theorem [1] covers the cases $1/(p+q) \leq s = 1$.)

On the other hand, our results completely close the gap between necessary and sufficient conditions for concavity to hold. The trace function $\Phi_{p,q,s}$ is jointly concave if and only if $0 < p, q \leq 1$ and $0 \leq s \leq 1/(p+q)$ (Thm. 4.4). This completes Hiai’s results discussed above.

As for joint operator convexity, we prove it for $(A, B) \mapsto BA^qB$ if $-1 \leq q < 0$, and show that it does not hold for $(A, B) \mapsto B^{p/2}A^qB^{p/2}$ for any $p < 2$ (Thm. 3.2). (Note that it cannot hold for $p > 2$ since $B \mapsto B^p$ is not operator convex when $p > 2$.)

3 Joint operator convexity

We investigate operator convexity and concavity of certain functions on $\mathcal{P}_n \times \mathcal{P}_n$. It is well known [10, 12] that

\[(A, B) \mapsto AB^{-1}A\]  

(3.1)

is jointly convex. In the scalar case ($n = 1$), $f(a, b) = a^pb^p$ is jointly convex on $(0, \infty) \times (0, \infty)$ if and only if $p \geq 1$, $q \leq 0$ and $p+q \geq 1$, or $q \geq 1$, $p \leq 0$ and $p+q \geq 1$, or $p, q \leq 0$. It is jointly concave if and only if $0 \leq p, q \leq 1$ and $p+q \leq 1$. It is natural to ask for which powers $p$ and $q$

\[(A, B) \mapsto A^{q/2}B^pA^{q/2}\]  

(3.2)

is jointly operator convex or concave.

This question is closely related to the question: For which values of $p, q, r$

is

\[(A, B, C) \mapsto \text{Tr} A^{q/2}B^pA^{q/2}C^r\]  

(3.3)

jointly convex or concave in the positive operators $A, B, C$?

3.1 LEMMA. When the function in (3.3) is convex (or concave) for some choice of $p, q$ and $r$ all non-zero, then the function in (3.2) is operator convex (or concave) for the same $p$ and $q$.

Proof. When $r$ is positive, simply take $C$ to be any rank-one projection. When $r$ is negative, let $P$ be any rank-one projection, $t > 0$. Take $C$ to be $P + tp^\perp$, so that $C^r = P + t^rP^\perp$ and let $t$ tend to $\infty$. \[\square\]

Thus, the operator convexity/concavity of the operator-valued function in (3.2) is a consequence of the seemingly weaker tracial convexity/concavity
of (3.3). In short, (3.3) is stronger than (3.2) for the same values of \( p, q \). The value of \( r \) is irrelevant as long as it is not zero, and the implication does not even require convexity/concavity in \( C \), only joint convexity/concavity in \( A \) and \( B \).

When \( p, r < 0 \), and \(-1 \leq p + r < 0\), then the map \((A, B, C) \mapsto \text{Tr}AB^pA^rC^r\) is jointly convex for \( B, C \) positive and \( A \) arbitrary. This was proved in [11, Corollary 2.1]. (This triple convexity theorem is deeper than the double convexity theorem [11, Theorem 8] referred to in the previous section because it uses [11, Theorem 2] in an essential way.) By restricting ourselves to \( A \) positive and taking \( q = 2 \) this function of \( A, B, C \) reduces to (3.3).

By Lemma 3.1, the function (3.2) is jointly convex when \( q = 2 \) and \(-1 \leq p < 0\). Our main result in this section is that there are no other cases in which this operator-valued function is either convex or concave!

### 3.2 Theorem

Let \( p, q \in \mathbb{R} \setminus \{0\} \) and consider the map

\[
(A, B) \mapsto A^{q/2}B^pA^{q/2}
\]  

(3.4)

from \( \mathcal{P}_n \times \mathcal{P}_n \) to \( \mathcal{P}_n \) for some fixed \( n \geq 2 \).

1. The map (3.4) is jointly operator convex if and only if \( q = 2 \) and \(-1 \leq p < 0\).

2. The map (3.4) is not jointly operator concave.

### 3.3 Corollary

Let \( p, q \in \mathbb{R} \setminus \{0\} \). The function \((A, B, C) \mapsto \text{Tr}A^{q/2}B^pA^{q/2}C^r\) is never concave, and it is convex if and only if \( q = 2 \), \( p, r < 0 \) and \(-1 \leq p + r < 0\).

**Proof.** By Lemma 3.1, any triple convexity/concavity would imply the corresponding operator convexity/concavity, which is ruled out by the previous Theorem 3.2, except when \( q = 2 \), \( p, r < 0 \) and \(-1 \leq p + r < 0\). In this case convexity is provided by [11, Corollary 2.1]. \(\square\)

Our counterexamples to operator convexity and concavity given in Theorem 3.2 will be based on the following lemma.

### 3.4 Lemma

Let \( r \in (-\infty, 0) \cup (0, 1) \), let \( Y \geq 0 \) be rank one and \( n \geq 2 \). Then the map \( X \mapsto X^rYX^r \) from \( \mathcal{P}_n \) to \( \mathcal{P}_n \) is not operator convex.

**Proof of Lemma 3.4.** First assume that \( r \in (0, 1/2) \). Then for any non-trivial \( Y \geq 0 \) (not necessarily rank one) the map \( X \mapsto X^rYX^r \) from \( \mathcal{P}_n \) to \( \mathcal{P}_n \) is not operator convex. This follows simply from the fact that the map \( x \mapsto x^{2r}Y \) from \( (0, \infty) \) to \( \mathcal{P}_n \) is not operator convex for \( 0 < r < 1/2 \). It is, in fact, strictly concave in this region.

Now let \( r \in (-\infty, 0) \). (The proof actually also works for \( r \in (0, 1/2) \), which is hardly surprising in light of the concavity mentioned above.) Clearly, we may assume \( n = 2 \). Let \( Y = \langle v | v \rangle \). If the convexity were true, then for all \( X_1, X_2 \in \mathcal{P}_2 \), with \( X = (X_1 + X_2)/2 \), we would have

\[
X^r|v\rangle\langle v|X^r \leq \frac{1}{2}X_1^r|v\rangle\langle v|X_1^r + \frac{1}{2}X_2^r|v\rangle\langle v|X_2^r.
\]

(3.5)
Without loss of generality, let $|v⟩ = (1, 1)$. If we take $X_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $X_2 = t \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$, with $t > 0$, then (3.5) becomes

$$\begin{bmatrix} (1 + t)^{2r} & (1 + t)^r(1 + 2t)^r \\ (1 + t)^r(1 + 2t)^r & (1 + 2t)^{2r} \end{bmatrix} \leq (2^{2r-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + t^{2r}2^{2r-1} \begin{bmatrix} 1 & 2^r \\ 2^r & 2^r \end{bmatrix}) \quad (3.6)$$

The vector $|w⟩ = (2^r, -1)$ is in the null space of the second matrix on the right in (3.6), and taking the trace of both sides against $|w⟩⟨w|$ yields

$$\langle w, \begin{bmatrix} (1 + t)^{2r} & (1 + t)^r(1 + 2t)^r \\ (1 + t)^r(1 + 2t)^r & (1 + 2t)^{2r} \end{bmatrix} w \rangle \leq 2^{2r-1}(2^r - 1)^2,$$

which, in the limit $t \to 0$, becomes $(2^r - 1)^2 \leq 2^{2r-1}(2^r - 1)^2$, so that for $r \neq 0$, we would have $1 \leq 2^{2r-1}$. This is false for all $r < 1/2$, which shows that (3.5) leads to a contradiction for nonzero $r \in (-\infty, 0) \cup (0, 1/2)$.

Our proof for $1/2 < r < 1$ is different; this proof actually works in the range $0 < r < 1$. Let $|v⟩$ be a unit vector in $\mathbb{C}^n$. Then we will show that there is another vector $|w⟩$ in $\mathbb{C}^n$ such that

$$X \mapsto |⟨w|X^r|v⟩|^2$$

is not convex. Again, we may assume that $n = 2$ and that $|v⟩ = (0, 1)$. Take

$$X_1 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$  

Let $|w⟩ = (1, -1)$, so that $X_1^r|w⟩ = 0$ and $X_2^r|v⟩ = 0$. Evidently,

$$\frac{1}{2}|⟨w|X_1^r|v⟩|^2 + \frac{1}{2}|⟨w|X_2^r|v⟩|^2 = 0.$$  

However, the eigenvalues of $X = \frac{1}{2}(X_1 + X_2)$ are easily computed to be $\lambda_\pm = (3 \pm \sqrt{5})/2$, and then a further simple computation yields

$$⟨w|X^r|v⟩ = \frac{1}{\sqrt{5}}(\lambda_+^{r-1} - \lambda_-^{r-1}),$$

and this is strictly negative for all $0 < r < 1$.  

**Proof of Theorem 3.2.** As explained above, the convexity assertion in (1.) is a consequence of [11, Corollary 2.1]. Our goal now is to prove that there are no other cases of convexity or concavity.

A number of exponents can be excluded by considering the scalar case. Moreover, since $X \mapsto X^r$ is operator convex on $\mathcal{P}_n$ if and only if $r \in [-1, 0] \cup [1, 2]$, and is operator concave on $\mathcal{P}_n$ if and only if $r \in [0, 1]$, the only cases in which convexity cannot be immediately ruled out are $p \in [1, 2]$, $q \in [-1, 0]$ and $p + q \geq 1$ (or the same with $p$ and $q$ interchanged). Likewise, the only
cases of in which concavity cannot be immediately ruled out are \( p, q \in [0, 1] \), 
\( p + q \leq 1 \).

For part (1.), it remains for us to show that (3.4) is not jointly operator 
convex in the following three cases,

(a) \( p \in [-1, 0) \), \( q \in [1, 2) \) and \( p + q \geq 1 \).

(b) \( p \in [1, 2) \), \( q \in [-1, 0) \) and \( p + q \geq 1 \).

(c) \( p \in (-1, 0) \) and \( p + q \geq -1 \).

Let us prove failure of convexity in case (a). Let \( |v\rangle \) be any unit vector 
in \( \mathbb{C}^n \). Let \( P \) be the orthogonal projection onto the span of \( v \), and let \( P^\perp \) 
denote the complementary projection. Fix \( t > 0 \), and define \( B_t = P + tP^\perp \).

Then \( B_t^p = P + tpP^\perp \). If convexity would hold, then for any \( |w\rangle \) the map \( A \mapsto 
\langle w|A^{q/2}B_t^pA^{q/2}|w\rangle \) would be convex. Since \( \lim_{t \to \infty} B_t^p = |v\rangle\langle v| \), and since 
limits of convex functions are convex, it would follow that \( A \mapsto |\langle v|A^{q/2}|w\rangle|^2 \) would be convex on \( P_n \) for any \( |w\rangle \). This contradicts Lemma 3.4 with 
\( r = q/2 \in [1/2, 1) \). The proof for (c) is almost exactly the same, except one 
uses Lemma 3.4 with \( r = q/2 < 0 \).

The proof in case (b) is similar. Again, we let \( |v\rangle \) be a unit vector in 
\( \mathbb{C}^n \) and set \( B = |v\rangle\langle v| \). Then \( B^p = |v\rangle\langle v| \) and, if convexity would hold, 
then for any \( |w\rangle \) the map \( A \mapsto |\langle v|A^{q/2}|w\rangle|^2 \) would be convex on \( P_n \). This 
contradicts Lemma 3.4 with \( r = q/2 \in [-1/2, 0) \).

Finally, we prove (2.), the failure of concavity. According to the discussion above, it remains for us to show that (3.4) is not jointly operator 
concave for \( p, q \in (0, 1] \) and \( p + q \leq 1 \). Suppose \( (A, B) \mapsto A^{q/2}B^pA^{q/2} \) were 
concave for some \( p, q \) in this range. Then for all non-negative \( A \) and \( B \) we have

\[
\frac{1}{2}A^{q/2}B^pA^{q/2} + \frac{1}{2}B^{q/2}A^pB^{q/2} \leq \left( \frac{A + B}{2} \right)^{q/2} \left( \frac{B + A}{2} \right)^{p} \left( \frac{A + B}{2} \right)^{q/2} = 2^{-p-q}(A + B)^{p+q} .
\]

Suppose that \( A \) has a non-trivial null space (here we use the assumption 
\( n \geq 2 \)), and \( |v\rangle \) is a unit vector with \( A|v\rangle = 0 \). By Jensen’s inequality, since 
\( p + q \leq 1 \),

\[
\langle v|(A + B)^{p+q}|v\rangle \leq \langle v|(A + B)|v\rangle^{p+q} = \langle v|B|v\rangle^{p+q} .
\]

Thus we would have

\[
\langle v|B^{q/2}A^pB^{q/2}|v\rangle \leq 2^{1-p-q}\langle v|B|v\rangle^{p+q} .
\]

The left side is homogeneous of degree \( q \) in \( B \), while the right side is ho-
maneogeneous of degree \( p + q \), and hence the inequality cannot be generally 
valid. (The positivity of the powers is essential here; the argument of course 
cannot be adapted to yield a counterexample to the convexity proved in the 
first part of the theorem.) \( \square \)

3.5 Remark. There is another way to prove the convexity in (3.4) for \( q = 2 \) 
and \(-1 \leq p < 0 \). For \( p = -1 \) one can use the Schwarz type inequality in [12,
In this section we prove, among other things, two cases of a conjecture of Audenaert and Datta [2]. Much of our analysis is based on the formulas
\[ C \Phi \]
with \( C \) by any Herglotz function \( B \), we have by (4.1),

Now define \( D \) and \( s \) is jointly convex for all \( p \) \( \in [1,2] \), \( q \in [-1,0) \), \( \Phi_{p,q,s}(A,B) \) is jointly convex.

\[ \Phi_{p,q,s}(A,B) = s \sup_{Z \geq 0} \left\{ \text{Tr}[A^{q/2}B^pA^{q/2}Z^{1-1/s}] + \left( \frac{1}{s} - 1 \right) \text{Tr}[Z] \right\} \ . \]

For \( 1 \leq p \leq 2 \), the map \( B \mapsto B^p \) is operator convex and therefore \( B \mapsto \text{Tr}[B^pD] \) is convex. Moreover, by Hiai’s extension of Epstein’s Theorem [8, Thm. 4.1] the map \( A \mapsto \text{Tr}((DA^{-q}D)^{s/(s-1)}) \) is concave as long as \( s/(s-1) \leq -1/q \), which is the same as \( s \geq 1/(1+q) \). Thus, (4.3) represents \( \Phi_{p,q,s}(A,B) \) as a supremum of jointly convex functions and so \( \Phi_{p,q,s}(A,B) \) is jointly convex for \( s \geq 1/(1+q) \). This proves the first part of the theorem.
We now prove convexity if $s \geq 1/(p - 1)$. Let us first consider the case $p = 2$ and $s = 1$, where $\Phi_{2,q,1}(A, B) = \text{Tr}[A^{q/2}B^2 A^{q/2}] = \text{Tr}[BA^qB]$. For $-1 \leq q < 0$, the map $(A, B) \mapsto BA^qB$ is operator convex by Theorem 3.2 and therefore $(A, B) \mapsto \text{Tr}[BA^qB]$ is convex, as claimed. We now assume that $s > 1$ (and still $s \geq 1/(p - 1)$). Then by (4.1), making use of 
\[ \Phi_{p,q,s}(A, B) = s \sup_{Z \geq 0} \left\{ \text{Tr}[B^{p/2}A^qB^{p/2}Z^{1-1/s}] + \left(\frac{1}{s} - 1\right) \text{Tr}[Z] \right\}. \]
Note that
\[ \text{Tr}[B^{p/2}A^qB^{p/2}Z^{1-1/s}] = \text{Tr}[BA^qB(B^{p/2-1}Z^{1-1/s}B^{p/2-1})]. \]
Define $D^2 = B^{p/2-1}Z^{(s-1)/s}B^{p/2-1}$, so that $Z = (B^{1-p/2}D^2B^{1-p/2})s/(s-1)$. Then
\[ \Phi_{p,q,s}(A, B) = s \sup_{D \geq 0} \left\{ \text{Tr}[DBA^qB] + \left(\frac{1}{s} - 1\right) \text{Tr}[(B^{1-p/2}D^2B^{1-p/2})s/(s-1)] \right\} \]
\[ = s \sup_{D \geq 0} \left\{ \text{Tr}[DBA^qB] + \left(\frac{1}{s} - 1\right) \text{Tr}[(DB^{2-p}D)s/(s-1)] \right\}. \]
(4.4)

Since $-1 \leq q < 0$, $(A, B) \mapsto BA^qB$ is operator convex by Theorem 3.2, so $(A, B) \mapsto \text{Tr}[DBA^qB]$ is convex. By Hiai’s extension of Epstein’s Theorem [8, Thm. 4.1], $B \mapsto \text{Tr}[(DB^{2-p}D)s/(s-1)]$ is concave as long as $s/(s-1) \leq 1/(2 - p)$, which is the same as $s \geq 1/(p - 1)$. Thus, (4.4) represents \( \Phi_{p,q,s}(A, B) \) as a supremum of jointly convex functions and so $\Phi_{p,q,s}(A, B)$ is jointly convex for $s \geq 1/(p - 1)$. This completes the proof. \( \square \)

4.2 THEOREM. When $p = 2$, $\Phi_{p,q,s}(A, B)$ is jointly convex for all $-1 \leq q < 0$ and $s \geq 1/(2 + q)$.

This result yields the optimal range of convexity for $p = 2$. It had been conjectured in [2] for $s = 1/(2 + q)$.

Proof. The convexity for $s \geq 1$ follows from Theorem 4.1 and therefore we may assume that $1/(p + q) \leq s < 1$. Then, making use of 
\[ \Phi_{2,q,1}(A, B) = s \inf_{Z \geq 0} \left\{ \text{Tr}[BA^qBZ^{1-1/s}] + \left(\frac{1}{s} - 1\right) \text{Tr}[Z] \right\}. \]
(4.5)
The important distinction between this formula and formulas (4.3) and (4.4) is the infimum in place of the supremum. Joint convexity in $A, B$ no longer suffices. Instead we need joint convexity in $A, B, Z$, with which we can apply [4, Lemma 2.3].

Note that $1 - 1/s \leq 0$. By [11, Corollary 2.1], $(A, B, Z) \mapsto \text{Tr}[BA^qBZ^{1-1/s}]$ is jointly convex as long as $q + 1 - 1/s \geq -1$, which means $s \geq 1/(2 + q)$. For such $s$, the argument of the infimum in (4.5) is jointly convex in $A, B$ and $Z$. By [4, Lemma 2.3], the infimum itself is jointly convex in $A$ and $B$. This proves the assertion for $1/(2 + q) \leq s \leq 1$. \( \square \)
4.3 Remark. In the previous proof for the range \( s \geq 1 \) we referred to Theorem 4.1 which, in turn, was based on Hiai’s extension of Epstein’s theorem. For the case relevant for Theorem 4.2, however, there is a more direct proof. Indeed, let \( A_j, B_j \in P_n \), \( j = 1, 2 \), and \( \lambda \in (0, 1) \) and set \( A = \lambda A_1 + (1 - \lambda) A_2 \) and \( B = \lambda B_1 + (1 - \lambda) B_2 \). Then by Theorem 3.2 for \(-1 \leq q < 0\),
\[
BA_q^q B \leq \lambda B_1 A_q^q B_1 + (1 - \lambda) B_2 A_q^q B_2 .
\]
For all \( s \geq 0 \), \( X \mapsto \Tr[X^s] \) is monotone on \( P_n \). Hence, even for all \( s \geq 0 \),
\[
\Tr[(BA_q^q B)^s] \leq \Tr[(\lambda B_1 A_q^q B_1 + (1 - \lambda) B_2 A_q^q B_2)^s] .
\]
Finally, for \( s \geq 1 \), \( X \mapsto \Tr[X^s] \) is convex on \( P_n \). Therefore,
\[
\Tr[(\lambda B_1 A_q^q B_1 + (1 - \lambda) B_2 A_q^q B_2)^s] \leq \lambda \Tr[(B_1 A_q^q B_1)^s] + (1 - \lambda) \Tr[(B_2 A_q^q B_2)^s] .
\]
This proves the convexity for \( s \geq 1 \) and \(-1 \leq q < 0\).

The next result concerns the concavity of \( \Phi_{p,q,s}(A, B) \).

4.4 THEOREM. The trace function \( \Phi_{p,q,s}(A, B) \) is jointly concave if and only if \( 0 \leq p, q \leq 1 \) and \( 0 \leq s \leq 1/(p + q) \).

Proof. The necessity of the condition is proved in [8, Prop. 5.1] and the sufficiency for \( 1/2 \leq s \leq 1/(p + q) \) is proved in [8, Thm. 2.1]. Our task is to prove sufficiency in the case \( 0 < s < 1/2 \). We write, using (4.2),
\[
\Phi_{p,q,s}(A, B) = s \inf_{X > 0} \Tr \left\{ A^{q/2} B^p A^{q/2} X^{1-1/s} + \left( \frac{1}{s} - 1 \right) X \right\}
\]
\[
= s \inf_{Y > 0} \Tr \left\{ B^p Y + \left( \frac{1}{s} - 1 \right) A^{q/2} Y^{-1} A^{q/2} \right\}^{s/(1-s)}
\]
\[
= s \inf_{Y > 0} \Tr \left\{ B^p Y + \left( \frac{1}{s} - 1 \right) Y^{-1/2} A^{q/2} Y^{-1/2} \right\}^{s/(1-s)} .
\]
Since \( 0 \leq p \leq 1 \), \( B \mapsto B^p \) is operator concave and so \( B \mapsto \Tr B^p Y \) is concave. By the extension of Epstein’s Theorem proved in [8, Theorem 4.1], \( A \mapsto \Tr(Y^{-1/2} A^{q/2} Y^{-1/2} s/(1-s)) \) is concave if \( s/(1-s) \leq 1/q \). This condition is satisfied since \( s \leq 1/2 \leq 1/(1 + q) \). We conclude that \( \Phi_{p,q,s}(A, B) \) as an infimum of concave functions is concave.

We conclude with a corollary of Theorem 4.2. For \( \rho, \sigma \in P_n \) and \( \alpha, z > 0 \), we introduce the so-called \( \alpha - z \)-relative Rényi entropies
\[
D_{\alpha,z}(\rho||\sigma) = \frac{1}{\alpha - 1} \ln \frac{\Tr (\sigma^{(1-\alpha)/(2z)} \rho^{\alpha/z} \sigma^{(1-\alpha)/(2z)})^z}{\Tr \rho} .
\]
(For \( \alpha = 1 \), a limit has to be taken.) These functionals appeared in [9, Sec. 3.3] and were further studied in [2], where the question was raised whether the \( \alpha - z \)-relative Rényi entropies are monotone under completely positive, trace preserving maps. Currently this is known for \( 0 < \alpha \leq 1 \) and \( z \geq \max \{ \alpha, 1 - \alpha \} \), and for \( 1 \leq \alpha \leq 2 \) and \( z = 1 \), and for \( 1 \leq \alpha < \infty \) and \( z = \alpha \). See [2] for these cases. In this paper Audenaert and Datta conjecture that monotonicity holds for \( 1 \leq \alpha \leq 2 \) and \( \alpha/2 \leq z < \alpha \), and for \( 2 \leq \alpha < \infty \) and \( \alpha - 1 \leq z < \alpha \). Our contribution here is to prove their conjecture for \( 1 < \alpha = 2z \leq 2 \).
4.5 COROLLARY. Let α = 2z ∈ (1, 2] and let ρ, σ ∈ P_n. Then for any completely positive, trace preserving map E on P_n,

\[ D_{\alpha,\alpha/2}(\rho||\sigma) \geq D_{\alpha,\alpha/2}(E(\rho)||E(\sigma)). \]

Proof. By a classical argument due to Lindblad and Uhlmann, see, e.g., [5, 7], the monotonicity follows once it is shown that

\( (\rho, \sigma) \mapsto \text{Tr} \left( \sigma^{(1-\alpha)/\alpha} \rho^{2(1-\alpha)/\alpha} \right)^{\alpha/2} = \Phi_{2,2(1-\alpha)/\alpha,\alpha/2}(\sigma, \rho) \)

is jointly convex. For \( \alpha \in (1, 2] \) this convexity follows from Theorem 4.2. □

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