On nonexistence of non-constant volatility in the Black-Scholes formula

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Abstract. We prove that if the Black-Scholes formula holds with the spot volatility for call options with all strikes, then the volatility parameter is constant. The proof relies some result on semimartingales (Theorem 2) of independent interest.

Key words: Black-Scholes formula, stochastic volatility, stochastic implied volatility, local volatility models

Mathematics Subject Classification (1991): 60G44, 60H30, 90A09

1 Introduction

One of the most important applications of stochastic dynamics has occurred in finance as a model for evolution of prices of stocks and their options. This model in its simple form is described by a randomly perturbed exponential growth. If $S_t$ denotes the price of stock at time $t$, then its evolution is given by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

with $B_t$ denoting the Brownian motion process. The strength of the random perturbation is determined by the parameter $\sigma$, which is known as the volatility of the stock in finance. The above model was used by Merton, Black and Scholes to find the price of an option on stock, such as an agreement to buy the stock at some future time $T$ for the specified at time $t < T$ price $K$. Their formula states that the price of such an option at time $t$ is given by

$$C_t = S_t \Phi \left( \frac{\log S_t K + \left( r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma \sqrt{T-t}} \right) - Ke^{-r(T-t)} \Phi \left( \frac{\log S_t K + \left( r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma \sqrt{T-t}} \right),$$

1This work was supported by the Australian Research Council.
Date: September 2004
where \( \Phi \) denotes the standard normal distribution function. Remarkably, the parameter \( \mu \) does not enter the formula, but \( \sigma \) does, as well as \( r \), the riskless rate available in a savings account. The Black-Scholes formula is widely used in financial markets and risk management.

The volatility of the stock is seen to be the parameter in the quadratic variation of the return on the stock process, \( \sigma^2 = d[R, R]/dt \), where \( dR_t = dS_t/S_t \), see [1]. It is widely believed and experimentally verified that stocks do not have a constant volatility, rather this parameter varies with time, see e.g. [4], [6], [11]. In spite of this fact, the Black-Scholes options pricing formula (2) is still used with some adjustments to the volatility. Thus the question of existence of a model with non-constant volatility in which the Black-Scholes formula remains valid is of interest in financial mathematics as well as in practical applications. A non-constant volatility model has the form

\[
dS_t = \mu S_t dt + \theta_t S_t dB_t,
\]

where \( \theta_t \) is a function of time that can be random, for example, it can be a function of stock \( S_t \), as well as include other independent source of randomness. There is a large literature on non-constant volatility models, both deterministic and stochastic (see e.g. Fouque et.al. (2001)). In the next paragraph we need to recall the basic facts on options pricing (see e.g. Shiryaev (1999)). We can assume in without loss of generality that the riskless interest rate \( r = 0 \), otherwise work with discounted prices \( S_t e^{-rt} \).

The First Fundamental theorem of asset pricing states that a model does not admit arbitrage if and only if there exists an equivalent probability measure \( Q \) such that \( S_t \) is a \( Q \)-martingale. The price at time \( t \) of a call option that pays \( (S_T - K)^+ \) at time \( T \) is given by

\[
C_t = E_Q((S_T - K)^+ | F^S_t),
\]

where \( E_Q \) is the expectation under \( Q \) and \( F^S_t \) is the \( \sigma \)-field generated by the process \( S_u, u \leq t \).

Next we comment on recent work to reconcile the non-constant volatility with the use of the Black-Scholes formula by constructing the so-called stochastic implied volatility (SIV) market models. A market model is a model that returns Black-Scholes option prices. Such models are important in practical applications for calculations of non-standard options. The idea in SIV models is to use equation (3) together with the volatility surface \( \sigma(t, T, K) \). These \( \sigma(t, T, K) \) are implied by the Black-Scholes option prices by equating the theoretical price in (4) \( C_t(T, K) = E_Q((S_T - K)^+ | F^S_t) \) with the Black-Scholes price with \( \sigma = \sigma(t, T, K) \). The spot volatility \( \theta_t \) is obtained from the volatility surface \( \sigma(t, T, K) \) by \( \theta_t = \sigma(t, t, S_t) \). This approach results in a nonlinear system of stochastic differential equations with delay, see Brace et.al. (2001), also Schonbucher (1998), Carr (2000), Brace et.al. (2002). While existence of SIV is still an open problem, the purpose of this note is to show that, unfortunately, non-constant volatility models are not compatible with the Black-Scholes formula.

Comment that the “Black-Scholes” equation with non-constant volatility \( dS_t = \theta_t S_t dB_t \) holds for a wide class of positive martingales. If \( S_t \) is a positive
martingale with \( P(S_t > 0) = 1 \) for any \( t \leq T \) and the predictable representation property holds then there exists a process \( \theta_t \), such that \( dS_t = \theta_t S_t dB_t \). We prove the statement even for wider class of processes and do not assume any dynamics on \( S_t \), namely if for a positive semimartingale \( S_t \) and for all values of \( K \) and three different maturities the Black-Scholes formula holds with some adapted \( \theta_t \), then \( \theta_t \) must be constant. This \( \theta_t \) can be taken as any adapted functional of the spot volatility, including the spot volatility itself.

2 Results

Let \((\Omega, \mathcal{F}, \mathbb{F}, Q)\) be the filtered probability space, with the general conditions, supporting a Brownian motion \( B_t \). Let for a constant \( \sigma > 0 \),

\[
dZ_t = \sigma Z_t dB_t, \quad Z_0 = z_0
\]

and

\[
C(T, t, K, \sigma, z) = \mathbb{E}\left[(Z_T - K)^+ | Z_t = z \right].
\] (5)

**Theorem 1** Let \( S_t \) and \( \theta_t \) be two adapted processes such that \( \theta_0 = \sigma \) and \( S_0 = z_0 \). Assume that \( S_t \) is strictly positive, that \( \mathcal{F}_0 \) is trivial and that there exist three equidistant terminal times, \( T_1 < T_2 < T_3 \) such that, for all \( K \) and all \( t \leq T_i \),

\[
\mathbb{E}[(S_{T_i} - K)^+ | \mathcal{F}_t] = C(T_i, t, K, \theta_t, S_t).
\]

Then \( \theta_t^2 = \sigma^2 \) for all \( t \leq T_1 \).

The proof relies on the following result.

**Theorem 2** Let \( M_t \) and \( X_t \) be two semimartingales, \( M_t \) is strictly positive. Assume that \( M_t \), \( M_t X_t \) and \( M_t X_t^2 \) are local martingales. Then \( X_t \equiv X_0 \).

Note that Theorem 1 can be formulated for three terminal non-equidistant times \( T_i \), \( i = 1, 2, 3 \), and in this case Theorem 2 can be generalized to the case when \( M_t \), \( M_t X_t \) and \( M_t X_t^\alpha \), \( \alpha > 1 \) are local martingales. For clarity of presentation we use equidistant times.

**Proof** of Theorem 2. Denote by \( X^c \) the continuous martingale component of \( X \), by \( M^c \) that of \( M \), and by \( V \) any optional compensator of \( X \). By the Itô formula we find that

\[
M_t X_t = M_0 X_0 + \int_0^t M_s - dX_s + \int_0^t X_s - dM_s + (X^c, M^c)_t + \sum_{s \leq t} \triangle X_s \triangle M_s.
\]

\[
(MX)_t X_t = (MX)_0 X_0 + \int_0^t X_s - d(XM)_s + \int_0^t M_s - X_s - dX_s
\]

\[
\quad + \int_0^t M_s - d(X^c, X^c)_s + \int_0^t X_s - d(M^c, X^c)_s
\]

\[
\quad + \sum_{s \leq t} \triangle (MX)_s \triangle X_s.
\]
Extracting martingales and using the assumptions of the theorem it follows that the random processes

\[ L_t^{(1)} = \int_0^t M_s - dV_s + \langle X^c, M^c \rangle_t + \sum_{s \leq t} \triangle X_s \triangle M_s, \]

\[ L_t^{(2)} = \int_0^t M_s - X_s - dV_s + \int_0^t M_s - d\langle X^c, M^c \rangle_s \]

are local martingales. Therefore the process

\[ L_t^{(2)} - \int_0^t X_s - dL_s^{(1)} = \int_0^t M_s - d\langle X^c, X^c \rangle_s \]

\[ + \sum_{s \leq t} [\triangle (MX)_s \triangle X_s - X_s - \triangle M_s \triangle X_s] \]

\[ = \int_0^t M_s - d\langle X^c, X^c \rangle_s + \sum_{s \leq t} M_s (\triangle X_s)^2 \] (6)

is a local martingale. Since \( M_t > 0 \), it is an increasing process. Therefore it is zero. It follows that \( \triangle X_t = 0 \) and \( \int_0^t M_s - d\langle X^c, X^c \rangle_s = 0 \).

Next we repeatedly use the following simple argument. If for a positive function \( f \) and an increasing function \( g \), \( \int_0^t f(s)dg(s) \equiv 0 \) then \( g(t) - g(0) = \int_0^t f(s)^{-1}dh(s) \equiv 0 \) where \( h(t) = \int_0^t f(s)dg(s) \). In other words \( g \) is constant.

Applying this argument to \( \int_0^t M_s - d\langle X^c, X^c \rangle_s = \int_0^t M_s - d\langle X^c, X^c \rangle_s \), we see that \( \langle X^c, X^c \rangle_t \equiv 0 \), that is \( X \) is a continuous process of finite variation. Since

\[ M_t X_t = M_0 X_0 + \int_0^t X_s dM_s + \int_0^t M_s - dX_s, \]

it follows that \( \int_0^t M_s dX_s = \int_0^t M_s - dX_s \) is a continuous local martingale, and since it is of finite variation \( \int_0^t M_s dX_s \equiv 0 \). Applying the above argument to \( \int_0^t M_s dY_s \), where \( Y \) is the variation process of \( X \), we complete the proof. \( \square \)

The proof of Theorem 1 is broken into a number of propositions. All of them assume the conditions and notations of Theorem 1.

**Proposition 3** \( S_t \) is a martingale and \( E[S_t^2] < +\infty \) for all \( t \).
Proof Take any $i$. By condition (5), $S_{T_i}$ is integrable and
\[
E[S_{T_i} | \mathcal{F}_i] = E[(S_{T_i} - 0)^+ | \mathcal{F}_i] = C(T_i, t, 0, \theta_t, S_t)
\]
which proves that $S_t$ is a martingale.

Also $E[(S_{T_i} - K)^+ | \mathcal{F}_0] = C(T_i, 0, K, \theta_0, S_0) = C(T_i, 0, K, \sigma, z_0)$ using Lemma 3 we deduce that $S_{T_i}$ and $Z_{T_i}$ have the same distribution. The integrability of $Z_{T_i}^2$ induces that $S_{T_i}^2$ is integrable.

Proposition 4 \( S_{T_i}^2 e^{\theta_i^2(T_i-t)} \) is a martingale, for each $i = 1, 2, 3$.

Proof The proof is based on the representation of $E[X^2] \sigma$ in terms of integrals $\int_0^{+\infty} E[(X - K)^+ | \sigma] dK$ and $\int_0^{+\infty} E[(X + K)^- | \sigma] dK$ given in Lemma 4. Using this lemma we obtain
\[
E[S_{T_i}^2 | \mathcal{F}_i]
= 2 \int_0^{+\infty} E[(S_{T_i} - K)^+ | \mathcal{F}_i] dK
+ 2 \int_{-\infty}^0 \left( E[(S_{T_i} - K)^+ | \mathcal{F}_i] - E[S_{T_i} | \mathcal{F}_i] + K \right) dK
= 2 \left[ \int_0^{+\infty} C(T_i, t, K, \sigma, z) dK + \int_{-\infty}^0 \left( C(T_i, t, K, \sigma, z) - z + K \right) dK \right]_{\sigma=\theta_t, z=S_t}
= 2 \left[ \int_0^{+\infty} E[(Z_{T_i} - K)^+ | Z_t = z] dK
+ \int_{-\infty}^0 \left( E[(Z_{T_i} - K)^+ | Z_t = z] - E[Z_{T_i} | Z_t = z] + K \right) dK \right]_{\sigma=\theta_t, z=S_t}
= [E[Z_{T_i}^2 | Z_t = z] \sigma=\theta_t, z=S_t
= [z^2 e^{\theta_i^2(T_i-t)}] \sigma=\theta_t, z=S_t
= S_{T_i}^2 e^{\theta_i^2(T_i-t)}
\]
Thus for each $i$, \( S_{T_i}^2 e^{\theta_i^2(T_i-t)} \) is a martingale of the Doob's form.

Proposition 5 \( \theta_i^2 \) is a semimartingale.

Proof Since $S > 0$, $e^{\theta_i^2(T_i-T)} = \frac{S_{T_i}^2 e^{\theta_i^2(T_i-t)}}{S_{T_i}^2 e^{\theta_i^2(T_i-t)}}$ is a semimartingale in $t \in [0, T_i]$.

Thus \( \theta_i^2 \) is also a semimartingale.
Proof of Theorem \[ \text{1} \]

Let \( M_t = S_t^2 e^{(T_2 - T_1)\theta_t^2}, X_t = e^{(T_2 - T_1)\theta_t^2} = e^{(T_3 - T_2)\theta_t^2} \). It is easy to see using the previous propositions that \( M_t \) and \( X_t \) satisfy the conditions of Theorem \[ \text{2} \]. By its conclusion \( X_t \equiv X_0 \) and \( \theta_t^2 = \sigma^2 \) for all \( t \leq T_1 \), which completes the proof. \( \square \)

Remarks

1. Note that although in Theorem \[ \text{1} \] we do not assume any dynamics on \( S_t \), the assumptions of theorem imply that \( S_t \) is a strictly positive martingale, see Proposition \[ \text{3} \]. If in addition, the predictable representation property with respect to \( B_t \) holds, then \( S_t \) can be represented as a stochastic volatility model \( dS_t = h_t S_t dB_t \), see e.g. \[ \text{7} \], p.286.

2. Let \( dS_t = h_t S_t dB_t \). While we may primarily think of \( \theta_t \) in Theorem \[ \text{1} \] as the volatility \( \theta_t = h_t \), Theorem \[ \text{1} \] applies for any adapted functional of \( h_u \) and \( S_u \), \( u \leq t \), for example, the average volatility on \([0, t]\) is given by \( \theta_t = \sqrt{ \frac{1}{t} \int_0^t h_s^2 ds } \).

3. Let \( dS_t = h_t S_t dB_t \), where \( h_t \) is a deterministic function. Then \( S_t = S_0 \exp(\int_0^t h_s dB_s - \frac{1}{2} \int_0^t h_s^2 ds) \) has a lognormal distribution and the price of the option is given by the Black-Scholes formula \( E[(S_T - K)^+ | \mathcal{F}_t] = C(T, t, K, \theta(t, T), S_t) \) with \( \theta(t, T) = \frac{1}{T-t} \int_t^T h_s^2 ds \), see e.g. \[ \text{9} \]. This example does not contradict Theorem \[ \text{1} \] since in this case \( \theta_t \) depends also on \( T \).

3 Auxiliary Results

Lemma 6 If \( E[|X|] < \infty \), then \( G(K) = E[(X - K)^+ | \mathcal{G}] \) is a convex function \((G'_-, \text{ and } G'_+ \text{ are increasing, respectively left and right-continuous and } \{K : G'_-(K) \neq G'_+(K)\} \text{ is countable})\) with

\[
G'_-(K) = -P[X \geq K|\mathcal{G}] \quad \text{and} \quad G'_+(K) = -P[X > K|\mathcal{G}].
\]

Proof The proof easily follows from the fact that, for \( \varepsilon > 0 \),

\[
(x - (K + \varepsilon))^+ - (x - K)^+ = \begin{cases} 0 & x \leq K \\ -(x - K) & K < x < K + \varepsilon \\ -\varepsilon & x \geq K + \varepsilon \end{cases}
\]

and

\[
(x - (K - \varepsilon))^+ - (x - K)^+ = \begin{cases} 0 & x \leq K - \varepsilon \\ x - K + \varepsilon & K - \varepsilon < x < K \\ \varepsilon & x \geq K \end{cases}
\]

Note that \( 1_{K < x < K + \varepsilon} = 1_{K - \varepsilon < x < K} = 0 \) for \( \varepsilon \) small enough. \( \square \)
Lemma 7 If $E[X^2] < +\infty$, then

$$\frac{1}{2} E[X^2 | \mathcal{G}] = \int_{0}^{+\infty} E[(X-K)^+ | \mathcal{G}] dK + \int_{0}^{+\infty} E[(X+K)^- | \mathcal{G}] dK$$

$$= \int_{0}^{+\infty} E[(X-K)^+ | \mathcal{G}] dK + \int_{0}^{+\infty} (E[(X-K)^+ | \mathcal{G}] - E[X | \mathcal{G}] + K) dK$$

In particular, if $X$ is non-negative,

$$E[X^2 | \mathcal{G}] = 2 \int_{0}^{+\infty} E[(X-K)^+ | \mathcal{G}] dK.$$

Proof It is easily checked that for any $x$,

$$x^2 = 2 \int_{0}^{+\infty} (x-K)^+ dK + 2 \int_{0}^{+\infty} (x+K)^- dK.$$

The result immediately follows.

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