THE RELATIVE HERMITIAN DUALITY FUNCTOR

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Abstract. We extend to the category of relative regular holonomic modules on a manifold $X$, parametrized by a curve $S$, the Hermitian duality functor of Kashiwara. We prove that this functor is an equivalence with the similar category on the conjugate manifold $\overline{X}$, parametrized by the same curve.

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1. Introduction and statement of the main result

Let $X$ and $S$ be complex manifolds, $S$ having dimension at most one. We denote by $p_X : X \times S \to S$ (or simply by $p$) the projection. The ring of holomorphic differential operators on $X \times S$ relative to the projection $p$ is denoted by $\mathcal{D}_{X \times S/S}$. It is a subsheaf of the sheaf of holomorphic differential operators $\mathcal{D}_{X \times S}$.

We have introduced in [17, 5] the notion of regular holonomic $\mathcal{D}_{X \times S/S}$-module and the category $\mathcal{D}_{\text{hol}}^b(\mathcal{D}_{X \times S/S})$ of bounded complexes of $\mathcal{D}_{X \times S/S}$-modules having regular holonomic cohomology: a coherent $\mathcal{D}_{X \times S/S}$-module $M$ is said to be regular holonomic if it is holonomic, that is, its characteristic variety is contained in a product $\Lambda \times S$, $\Lambda$ being analytic lagrangian $\mathbb{C}^*$-homogeneous in $T^*X$, and the restriction (in the sense of $\mathcal{O}$-modules) to each fiber of $p$, denoted by $\mathcal{L}^*_s M$ for each $s \in S$, belongs to $\mathcal{D}_{\text{hol}}^b(\mathcal{D}_X)$. We have shown that the category $\mathcal{D}_{\text{hol}}^b(\mathcal{D}_{X \times S/S})$ is stable by the following functors.

(a) $\mathcal{D}_f^!$ for a projective morphism $f : Y \to X$, or for a proper morphism and restricting to $f$-good objects (17 Cor. 2.4), where the condition $f$-good or projective was mistakenly forgotten, cf. loc. cit., Th. 1.17).
(b) $d_j f^*$ for any holomorphic map $f : Y \to X$ (cf. [8, Th. 2]).
(c) Duality $D$ ([17, Prop. 2.3]).
(d) $R\Gamma_{[Z \times S]}$ for any $Z$ closed analytic subset of $X$ (this follows from [2, Prop. 2.6(b)] and the relative Riemann-Hilbert correspondence of loc. cit.).

The pair of quasi-inverse contravariant functors ($^p\text{Sol}, RH^S$), providing the relative Riemann-Hilbert correspondence of loc. cit. ($^p\text{Sol}$ denotes the derived solution functor shifted by the dimension of $X$), makes the above functors compatible with the corresponding functors on the category $D^b_{\mathcal{C},c}(p^{-1}\mathcal{O}_S)$ of $S$-$\mathcal{C}$-constructible complexes. This is the triangulated category of complexes $F$ of sheaves of $p^{-1}\mathcal{O}_S$-modules such that, for a suitable $\mathcal{C}$-analytical stratification $(X_\alpha)_{\alpha \in A}$ of $X$ and any $\alpha \in A$, each cohomology sheaf of $F|_{X_\alpha \times S}$ is, locally on $X_\alpha \times S$, isomorphic to $p^{-1}G$ for some coherent $\mathcal{O}_S$-module $G$.

It was proved in [16] and [17] that for all $s \in S$ the restriction functor $Li_s^*$ is conservative in both categories $D^b_{\mathcal{C},c}(p^{-1}\mathcal{O}_S)$ and $D^b_{\text{hol}}(D_{X \times S/S})$.

In this article, extending some definitions and results of [9] (cf. Notation 1 below for the notion of conjugate manifold), we define the contravariant relative Hermitian duality functor $C^S_{X \times X}$ and its covariant companion, the relative conjugation functor $C^S_{X \times X} := C^S_{X \times X} \circ D$, both from $D_{X \times S/S}$-modules to $D_{X \times S/S}$-modules.

For that purpose, we make use of the relative distributions $\mathcal{D}^b_{X \times S/S}$, consisting of those distributions on $X \times S$ which are holomorphic with respect to $S$ (i.e., killed by the $\overline{\partial}_S$ operator). A reminder on this sheaf is given in the appendix, for the sake of being complete.

**Notation 1.** For a complex manifold $X$, endowed with its sheaf $\mathcal{O}_X$ of holomorphic functions, we denote by $\overline{X}$ the same underlying $C^\infty$-manifold, endowed with the sheaf $\overline{\mathcal{O}}_X$ of anti-holomorphic functions, that we denote by $\overline{\mathcal{O}}$. We regard $\mathcal{O}_X$ as an $\mathcal{O}_X$-module by setting, for any holomorphic function $f$ on $X$, $f \cdot 1 := f$. Given an $\mathcal{O}_X$-module $M$, the naive conjugate module $\overline{M}$ is defined as $\overline{M} = \mathcal{O}_X \otimes_{\mathcal{O}_X} M$. In other words, $\overline{M}$ is equal to $M$ as a sheaf of $\mathcal{C}$-vector spaces, and the action of $\mathcal{O}_X$ is that induced by the action of $\mathcal{O}_X$. This naive conjugation functor can be regarded from $\text{Mod}(\mathcal{O}_X)$ to $\text{Mod}(\mathcal{O}_X)$ or vice-versa, and the composition of both is the identity.

In particular, the conjugate $Z$ of a closed analytic subspace $Z$ of $X$ with ideal sheaf $J_Z$ is the closed subset $|Z|$ endowed with the sheaf $\overline{J_Z} := \overline{J_Z} \subset \overline{\mathcal{O}_X}$, so that $Z$ and $\overline{Z}$ have the same support.

For a complex manifold $X$, the categories of sheaves of $\mathcal{C}$-vector spaces on $X$, resp. of $p^{-1}\mathcal{O}_S$-modules on $X \times S$, coincide with that on $\overline{X}$, resp. on $\overline{X} \times S$, since they only depend on the underlying topological space $|X|$. By the above remark, the categories of $\mathbb{R}$-constructible or $\mathcal{C}$-constructible such objects also coincide. As a consequence, we have a canonical identification $D^b_{\mathcal{C},c}(p_X^{-1}\mathcal{O}_S) = D^b_{\mathcal{C},c}(p_{\overline{X}}^{-1}\mathcal{O}_S)$.

The main result of this work is the following.
Theorem 2.
(a) The relative Hermitian duality functor
\[ CS_{X,X}(-) := R\text{Hom}_{D_{X \times S/S}}(-, Db_{X \times S/S}) \]
induces an equivalence
\[ CS_{X,X} : Db_{\text{rhol}}(D_{X \times S/S}) \sim \to Db_{\text{rhol}}(D_{X \times S/S})^{\text{op}} \]
such that
\[ CS_{X,X} \circ CS_{X,X} \simeq \text{Id} \]
Moreover, the relative conjugation functor \( c_{X,X} := CS_{X,X} \circ D \) induces an equivalence
\[ c_{X,X} : Db_{\text{rhol}}(D_{X \times S/S}) \sim \to Db_{\text{rhol}}(D_{X \times S/S})^{\text{op}}, \]
and there is an isomorphism of functors
\[ p_{\text{Sol}} \circ c_{X,X}^{S} \sim p_{\text{Sol}} : Db_{\text{rhol}}(D_{X \times S/S}) \to Db_{C,c}(p^{-1}O_{S}). \]
(b) If \( M \in \text{Mod}_{\text{rhol}}(D_{X \times S/S}) \) is strict, then so is \( H^0 CS_{X,X}(M) \) and \( H^i CS_{X,X}(M) = 0 \) for \( i \neq 0 \).
(c) If \( M \in \text{Mod}_{\text{rhol}}(D_{X \times S/S}) \) is torsion, then so is \( H^1 CS_{X,X}(M) \) and \( H^i CS_{X,X}(M) = 0 \) for \( i \neq 1 \).

Remark 3. According to [4], \( p_{\text{Sol}} \) is \( t \)-exact with respect to the perverse \( t \)-structure on \( Db_{C,c}(p^{-1}O_{S}) \) and the dual \( t \)-structure of the natural one in \( Db_{\text{rhol}}(D_{X \times S/S}) \). This entails that the functor \( c_{X,X}^{S} \) is \( t \)-exact with respect to the natural \( t \)-structures in both origin and target and the functor \( CS_{X,X} \) is \( t \)-exact with respect to the natural one in \( Db_{\text{rhol}}(D_{X \times S/S}) \) and the dual \( t \)-structure on \( Db_{\text{rhol}}(D_{X \times S/S})^{\text{op}} \), as seen on [5] and [6].

Recall (cf. [10]) that a \( D_{X \times S/S} \)-module is said to be strict if it has non nonzero \( p^{-1}O_{S} \)-torsion. It is said to be torsion if it is a torsion \( p^{-1}O_{S} \)-module.

We can express the main theorem by emphasizing the functor \( RH^{S} \) instead of \( p_{\text{Sol}} \). For example, by the relative Riemann-Hilbert correspondence of [5, Th. 1] and [2], [3] can be expressed as an isomorphism of functors
\[ RH^{S}_{X} \simeq c_{X,X}^{S} \circ RH^{S}_{X} : Db_{C,c}(p^{-1}O_{S}) \to Db_{\text{rhol}}(D_{X \times S/S}). \]

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2. Review on Kashiwara’s conjugation functor

The case where \( S \) is a point (that we call the “absolute case”) was solved by Kashiwara in [7], where \( CS_{X,X} \) is simply denoted by \( C_{X} \). We recall here the main results in [3] (cf. also [2] for further study of this functor).
Properties 4.

(a) Denoting by $\mathcal{D}b_X$ the left $(\mathcal{D}X, \mathcal{D}Y)$-bimodule of distributions on $X$, the Hermitian duality functor $C_X$ is defined by

$$C_X(M) := \mathcal{H}om_{\mathcal{D}X}(M, \mathcal{D}b_X),$$

endowed with the $\mathcal{D}X$-module structure induced by that of $\mathcal{D}b_X$. It is an equivalence $\text{Mod}_{\text{hol}}(X) \simeq \text{Mod}_{\text{hol}}(X)^{\text{op}}$. We recall that regularity is in fact not needed, and that $C_X$ extends as an equivalence $\text{Mod}_{\text{hol}}(X) \simeq \text{Mod}_{\text{hol}}(X)^{\text{op}}$, cf. [18, §II.3], [13, §4.4], [20, §12.6], but we will not develop this direction in this article. Let us also recall that the nondegenerate associated Hermitian pairing $M \otimes_{\mathbb{C}} C_X(M) \to \mathcal{D}b_X$ plays an instrumental role in the theory of mixed twistor $\mathcal{D}$-modules (cf. [19, 14, 15]).

(b) In order to keep the same underlying manifold $X$ in the target category, we compose $C_X$ with naive conjugation and obtain the equivalence $C_X(\cdot) : \text{Mod}_{\text{hol}}(X) \simeq \text{Mod}_{\text{hol}}(X)^{\text{op}}$.

(c) Denoting by $D$ the duality functor for holonomic $\mathcal{D}X$-modules, the conjugation functor $c_X(M)$ is defined as $C_X(DM)$. For $M \in \text{Mod}_{\text{hol}}(X)$ (and also in $\text{Mod}_\text{hol}(X)$), $c_X(M) = \mathcal{H}om_{\mathcal{D}X}(DM, \mathcal{D}b_X)$. It is thus a covariant functor, inducing an equivalence $\text{Mod}_{\text{hol}}(X) \simeq \text{Mod}_{\text{hol}}(X)$ (and similarly for holonomic modules). This functor satisfies

$$p\text{Sol}(c_X(M)) \simeq p\text{Sol}(M) \quad \text{and} \quad p\text{DR}(c_X(M)) \simeq p\text{DR}(M),$$

where, on the right-hand sides, we consider the naive conjugation of complexes of $\mathbb{C}$-vector spaces.

(d) For $M$ regular holonomic (holonomic is enough, according to (a)) the local cyclicity of $C_X(M)$ (because it is holonomic) is equivalent to the property that $M$ can be locally embedded as a $\mathcal{D}X$-submodule of $\mathcal{D}b_X$.

(e) If $u \in \Gamma(X, \mathcal{D}b_X)$ is a regular holonomic distribution on $X$, i.e., is such that the submodule $\mathcal{D}Xu \subset \mathcal{D}b_X$ is regular holonomic, then the $\mathcal{D}X$-module $\mathcal{D}Xu \simeq C_X(\mathcal{D}Xu) \subset \mathcal{D}b_X$ is also regular holonomic (cf. [9, §3], [3, Prop. 7.4.3]). In other words, the conjugate distribution $\overline{u}$ also generates a regular holonomic $\mathcal{D}X$-submodule of $\mathcal{D}b_X$.

(The same property holds for holonomic distributions.)

3. Applications: relative regular holonomic distributions

In this section we extend to the relative case some applications given in [9] and [3]. From [3] we adapt two functorial applications but a number of other applications in loc. cit. should be generalizable to the relative setting. We will say for short that a morphism of $\mathcal{D}X_{\times S}/S$-modules is an isomorphism up to $S$-torsion if its kernel and cokernel are $S$-torsion $\mathcal{D}X_{\times S}/S$-modules.

Concerning the analogue of [10], the following is obtained as in [9] Prop. 5].
Proposition 5. Let $\mathcal{M}$ be a strict regular holonomic $\mathcal{D}_{X \times S/S}$-module and let $\phi$ be a local section of $C^S_{X,X}(\mathcal{M})$, hence a $\mathcal{D}_{X \times S/S}$-linear morphism from $\mathcal{M}$ to $\mathcal{D}b_{X \times S/S}$. The following properties are equivalent:

(a) $\phi$ is injective,

(b) the inclusion $\mathcal{D}_{X \times S/S}\phi \subset C^S_{X,X}(\mathcal{M})$ is an equality up to $S$-torsion.

Sketch of proof. Set $N = \mathcal{D}_{X \times S/S}\phi$ and let $\alpha : N \hookrightarrow C^S_{X,X}(\mathcal{M})$ denote the inclusion. Since $\mathcal{M}$ is regular holonomic and strict, so is $C^S_{X,X}(\mathcal{M})$ by Theorem 2 hence so is $N$ (cf. [6, Prop. 3.1(iii)]). We consider the exact sequence of regular holonomic $\mathcal{D}_{X \times S/S}$-modules:

$$0 \longrightarrow N \xrightarrow{\alpha} C^S_{X,X}(\mathcal{M}) \longrightarrow N' \longrightarrow 0.$$ 

Applying $C^S_{X,X}$ to it and using Theorem 2, we find an exact sequence

$$0 \longrightarrow \mathfrak{H}^d C^S_{X,X}(N') \longrightarrow M \xrightarrow{\beta} C^S_{X,X}(N) \longrightarrow \mathfrak{H}^1 C^S_{X,X}(N') \longrightarrow 0,$$

with $\beta = C^S_{X,X}(\alpha)$. The argument given in the proof of [6, Prop. 5] shows that $\ker \beta = \ker \varphi$.

If $\phi$ is injective, so is $\beta$, and [6] and [9] in Theorem 2 imply that $\mathfrak{H}^1 C^S_{X,X}(N')$ is of $S$-torsion, and therefore so is $N'$.

Conversely, if $N'$ is of $S$-torsion, we deduce similarly that $\ker \beta = 0$, hence $\phi$ is injective. 

q.e.d.

Let $u$ be a nonzero relative distribution, that is, a local section of $\mathcal{D}b_{X \times S/S}$. It generates a $\mathcal{D}_{X \times S/S}$-submodule

$$\mathcal{D}_{X \times S/S} \cdot u \subset \mathcal{D}b_{X \times S/S}$$

and a $\mathcal{D}b_{X \times S/S}$-submodule $\mathcal{D}b_{X \times S/S} \cdot u \subset \mathcal{D}b_{X \times S/S}$. Let us already notice that both are strict, i.e., do not have nonzero $p^{-1}\mathcal{O}_S$-torsion elements, as follows from (2) in the appendix.

We denote by $\mathcal{C}_{M\times T/T}$ the sheaf of relative currents of maximal degree, that is, the sheaf of relative forms on $X \times S$ of maximal degree with coefficients in $\mathcal{D}b_{X \times S/S}$ (cf. Section A.3 of the appendix).

Definition 6. Let $u$ be a section of $\mathcal{D}b_{X \times S/S}$ on an open subset $\Omega$ of $X \times S$. We say that $u$ is a regular holonomic relative distribution on $\Omega$ if $\mathcal{D}_{X \times S/S}u$ is a (strict) regular holonomic $\mathcal{D}_{X \times S/S}\mathcal{O}_S$-module. Similarly, a section $u$ of $\mathcal{C}_{M\times T/T}$ on an open subset $\Omega \subset X \times S$ is a regular holonomic relative current of maximal degree on $\Omega$ if $u\mathcal{D}_{X \times S/S}$ is a regular holonomic right $\mathcal{D}_{X \times S/S}\mathcal{O}_S$-module.

We obtain the analogue of Property (c) above in the relative setting.

Corollary 7. Let $u$ be a section of $\mathcal{D}b_{X \times S/S}$ on an open subset $\Omega$ of $X \times S$. Then the following conditions are equivalent:

(a) $\mathcal{D}_{X \times S/S}u$ is regular holonomic on $\Omega$,

(b) $\mathcal{D}b_{X \times S/S}u$ is regular holonomic on $\Omega$.

Furthermore, in such a case, $\mathcal{D}_{X \times S/S}u \simeq C^S_{X,X}(\mathcal{D}_{X \times S/S}u)$ up to $S$-torsion.
Proof. Assume that (a) holds for \( u \) and let \( \mathcal{J} \subset \mathcal{D}_{X \times S/S} \) be the left ideal consisting of operators \( P \) such that \( Pu = 0 \) in \( \mathcal{D}_{X \times S/S} \). Let us set \( \mathcal{M} = \mathcal{D}_{X \times S/S}/\mathcal{J} \). By definition, the \( \mathcal{D}_{X \times S/S} \)-linear morphism

\[
\mathcal{D}_{X \times S/S} \longrightarrow \mathcal{D}_{X \times S/S} u, \quad 1 \mapsto u,
\]

induces an isomorphism \( \mathcal{M} \xrightarrow{\sim} \mathcal{D}_{X \times S/S} u \), that we regard as an injective \( \mathcal{D}_{X \times S/S} \)-linear morphism \( \phi : \mathcal{M} \hookrightarrow \mathcal{D}_{X \times S/S} \), in particular a local section of \( \text{Hom}_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathcal{D}_{X \times S/S}) \). Thus the coherent \( \mathcal{D}_{X \times S/S} \)-submodule \( \mathcal{D}_{X \times S/S} \phi \) of \( \mathcal{C}^S_{X,X}(\mathcal{M}) \) is regular holonomic. Since \( \phi \) is injective, Proposition 5 implies that \( \mathcal{C}^S_{X,X}(\mathcal{M})/\mathcal{D}_{X \times S/S} \phi \), which is also regular holonomic, is of \( S \)-torsion. By construction, \( \mathcal{D}_{X \times S/S} \phi \) is nothing but \( \mathcal{D}_{X \times S/S} u \). This proves (b) and the supplementary assertion of the proposition. Changing \( u \) to \( \overline{u} \) yields the converse.

q.e.d.

Remark 8. Embedding locally a regular holonomic \( \mathcal{D}_{X \times S/S} \)-module \( \mathcal{M} \) in \( \mathcal{D}_{\mathcal{D}_{X \times S/S}} \) by a morphism \( \phi \) can only occur if \( \mathcal{M} \) is strict, and even then, as we saw in Proposition 5, this is not sufficient to conclude that \( \mathcal{C}^S_{X,X} \mathcal{M} \) is locally cyclic as a \( \mathcal{D}_{X \times S/S} \)-module (an analogue of Property (b)); this holds only up to \( S \)-torsion, that is, away from the support of the cokernel of \( \mathcal{D}_{X \times S/S} \phi \). On the other hand, again in contrast with the absolute case, we do not know whether the local cyclicity property holds for any strict regular holonomic \( \mathcal{D}_{X \times S/S} \) or \( \mathcal{D}_{X \times S/S} \)-module. We consider examples in Section 5.

Nevertheless, Corollary 8 implies that, for \( \mathcal{M} \) regular holonomic and strict, the conjunction of \( \mathcal{M} \) and \( \mathcal{C}^S_{X,X} \mathcal{M} \) being locally cyclic implies that both are, up to \( S \)-torsion, locally generated (over \( \mathcal{D}_{X \times S/S} \) resp. \( \mathcal{D}_{X \times S/S} \)) by the same distribution \( u \).

By definition, a strict holonomic \( \mathcal{D}_{X \times S/S} \)-module \( \mathcal{M} \) is regular if and only if \( i^*_u \mathcal{M} \) is a regular holonomic \( \mathcal{D}_X \)-module. This translates as follows for relative distributions.

Proposition 9. Let \( \Omega' \) be an open subset in \( \mathbb{C}^n \), \( V \) an open disc centered at the origin in \( \mathbb{C} \) and \( \Omega = \Omega' \times V \). Let \( u \in \Gamma(\Omega; \mathcal{D}_{X \times S/S}) \) and let

\[
u = \sum_{m \geq 0} u_m s^m
\]

be the expansion of \( u \) provided by Corollary 3. If \( u \) is regular holonomic, then each \( u_m \) is regular holonomic. Conversely, any finite sum \( \sum_{m=0}^N u_m s^m \), with \( u_m \in \Gamma(\Omega'; \mathcal{D}_{X}) \) regular holonomic for each \( m \), is regular holonomic.

Proof. We first notice that, for \( v \in \Gamma(\Omega; \mathcal{D}_{X \times S/S}) \) and \( f \in \mathcal{O}(V) \) nonzero, if \( f(s)v \) is regular holonomic, then so is \( v \), since for \( P \in \Gamma(\Omega; \mathcal{D}_{X \times S/S}) \), the relation \( f(s)Pv = 0 \) in \( \Gamma(\Omega; \mathcal{D}_{X \times S/S}) \) is equivalent to \( Pv = 0 \). We can therefore assume in the proof that \( u_0 \neq 0 \). If \( J_u \subset \Gamma(\Omega; \mathcal{D}_{X \times S/S}) \) denotes the ideal of operators satisfying \( Pu = 0 \) and if we write for such an operator \( P = P_0 + sP' \) with \( P_0 \in \Gamma(\Omega', \mathcal{D}_X) \), we have \( i^*_u J_u = \{ P_0 \mid P \in J_u \} \). The relation \( Pu = 0 \) implies \( P_0 u_0 = 0 \), so \( u_0 = i^*_0 J_u \), hence is
regular holonomic. It follows that \( u - u_0 \) is also relatively regular holonomic, and processing that way we obtain the first assertion.

For the second assertion, we note that \( \mathcal{D}_{X \times S/S}u \) is a coherent submodule of \( \mathcal{D}_{X \times S/S} \mathcal{D}_X \left( \bigoplus_m \mathcal{D}_X u_m \right) \) and the assertion follows since \( \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \) is stable by sub-quotients in \( \text{Mod}_{\text{coh}}(\mathcal{D}_{X \times S/S}) \) (cf. [6, Prop. 3.1(iii)]). q.e.d.

We also prove, by mimicking the proof given by Björk [6 §VII.4 & VII.5], that regular holonomicity of relative distributions (and currents) is preserved under pushforward with proper support and non-characteristic pullback.

We say that an embedding \( i_Y : Y \times S \to X \times S \) of a closed submanifold \( Y \) of \( X \) is non-characteristic for a coherent \( \mathcal{D}_{X \times S/S} \)-module \( M \) if

\[
\text{Char}(M) \cap T^*_Y X \times S \subset T^*_X X \times S.
\]

In such a case, \( \nabla^n i^*_Y M = \mathcal{H}^0 \nabla^n i^*_Y M \) is \( \mathcal{D}_{Y \times S/S} \)-coherent. Moreover, according to [5, Th. 2] it follows that, if \( M \) is regular holonomic, then so is \( \mathcal{H}^0 \nabla^n i^*_Y M \).

Let \( M \) be a real analytic manifold, let \( u \) be a section of \( \mathcal{D}M_{X \times S/S} \) on an open subset \( \Omega \subset M \times S \). We say that \( u \) is regular holonomic on \( \Omega \) if, for each \( (m, s) \in \Omega \), there exists a neighborhood \( V \times W \subset \Omega \) of \( (m, s) \), a complexification \( X \) of \( V \), a coherent left ideal \( J \) of \( \mathcal{D}_{X \times S/S} \) defined in a neighborhood \( M' := V' \times W \subset V \times W \) of \( (m, s) \) in \( X \times S \) such that \( J|_{\Omega' \cap M \times S} \cdot u|_{\Omega' \cap M \times S} = 0 \) and \( \mathcal{D}_{X \times S/S} \cdot J \) is regular holonomic. In that situation, we denote \( \mathcal{D}_{X \times S/S}u \) instead of \( \mathcal{D}_{X \times S/S}/J \).

We finally recall that the analytical wave front set of a distribution \( u \) on a real manifold \( Z \), noted by \( \mathcal{W}F_A(u) \), is an \( \mathbb{R}_+ \)-conic closed subset of \( T^*Z \) which can be defined as the support of the Sato’s microfunction defined by \( u \).

**Proposition 10.**

(a) Let \( u \) be a relative regular holonomic current of maximal degree on \( X \times S \) such that \((f \times \text{Id})|_{\text{Supp} u} \) is proper. Then \( \int_f u \) is a regular holonomic relative current on \( Y \times S \).

(b) Let \( N \subset M \) be real analytic manifolds and let \( i_Y : Y \subset X \) be their respective germs of complexifications (in particular, \( Y \cap M = N \)). Let \( \Omega \) be an open subset of \( M \times S \) and let \( u \in \Gamma(\Omega, \mathcal{D}M_{X \times S/S}) \) be such that \( M := \mathcal{D}_{X \times S/S} \) is regular holonomic. Assume that \( Y \times S \) is non-characteristic for \( M \), so that \( \nabla^n i^*_Y M = \mathcal{H}^0 \nabla^n i^*_Y M \) is regular holonomic. Then the restriction \( u|_{N \times S} \) is a relative distribution on \( (\Omega \cap N) \times S \) such that \( \mathcal{D}_{Y \times S/S} \cdot u|_{N \times S} \simeq \mathcal{H}^0 \nabla^n i^*_Y M|_{N \times S} \).

**Proof.**

(a) This part is completely similar to that of [3 Th.7.4.11]. One shows that the integral of \( u \) along \( f \times \text{Id} \), as defined in Section 11.3, is a section of the 0th pushforward of the \( \mathcal{D}_{X \times S/S} \)-module \( \mathcal{D}_{X \times S/S} \cdot u \). This uses Proposition 11.6. Note that, since \( \mathcal{D}_{X \times S/S} \cdot u \) is globally generated, it is \( f \)-good. It follows then from the stability of regular holonomic \( \mathcal{D}_{X \times S/S} \)-modules by \( f \)-good pushforward (cf. [17, Cor. 2.4]) that the latter is a regular holonomic \( \mathcal{D}_{Y \times S/S} \)-module. We conclude
by using the property that the category $\text{Mod}_{\text{coh}}(\mathcal{D}_{X \times S/S})$ is stable by sub-quotients in $\text{Mod}_{\text{coh}}(\mathcal{D}_{X \times S/S})$, as already mentioned in the proof of Proposition \cite{9}.

(b) Let us prove \cite{10,11}. We first remark that $T^s_N M = T^s_M X \cap T^s_Y X$. Hence, by the assumption, according to a well-known result by M. Sato, the analytical wave front set of $u$ seen as a distribution on $M \times S$, $WF_A(u)$, satisfies

$$WF_A(u) \cap (T^s_N M \times S) \subset T^s_M M \times S$$

The proof then proceeds like in \cite{3, 7.5.4} in view of the fact that $\phi \in \mathcal{D}_S$ is regular holonomic in the absolute sense, that is, as a section of $D_{S/S, \text{reg}}$. Therefore, by construction, \( u_{|N \times S} \) is a relative distribution on $(N \cap \Omega) \times S$ such that $\mathcal{D}_{Y \times S/S} \cdot u_{|N \times S} \simeq \mathcal{H}^0_{\mathbb{D}^*M}|_{N \times S}$.

In the absolute case, Andronikof in \cite{T} Th. 1.2] and Björk in \cite{3} Th. 8.11.8] proved that, if $u$ is a regular holonomic distribution, then the characteristic variety of the regular $\mathcal{D}_X$-module $\mathcal{D}_{X} u$ satisfies

$$WF_A(u) = \text{Char}(\mathcal{D}_X u).$$

For relative regular holonomic distributions, the inclusion $\subset$ is obvious:

**Proposition 11.** Given a regular holonomic relative distribution $u$, we have $WF_A(u) \subset \text{Char}(\mathcal{D}_{X \times S/S} u)$ (where $S$ is identified to $T^s_S S$).

**Proof.** Regarding $u$ as a distribution solution of the coherent $\mathcal{D}_{X \times S}$-module $\mathcal{D}_{X \times S} \otimes_{\mathcal{D}_{X \times S/S}} \mathcal{D}_{X \times S/S} u$, we have $WF_A(u) \subset \Lambda \times T^s S$. On the other hand, since $u$ satisfies $\partial_S u = 0$, it follows that $WF_A(u) \subset \Lambda \times S$. q.e.d.

In order to prove the converse inclusion in Proposition \cite{11}, one should develop a relative variant of the microlocal techniques used in \cite{T} which is out of the scope of this work.

**Remark 12.** In the conditions of Corollary \cite{A2} if $\phi = \sum_{i=0}^L \phi_i s^i$ then $\phi$ is regular holonomic in the absolute sense, that is, as a section of $\mathcal{D}_S$. Indeed each term $\phi_i s^i$ is regular holonomic in the absolute sense since it satisfies the ideal generated in $\mathcal{D}_S$ by the annihilator ideal of $\phi_i$ in $\mathcal{D}_S$ and by $\partial^{i+1}_s$.

**Example 13.** Let us assume $X = \mathbb{C} = S$, and let us consider the relative regular holonomic distribution $\phi = x + \delta(0)s$. Let $\mathcal{D}_{X \times S/S} \delta = \mathcal{D}_{X \times S/S} \delta$. If $P \in \mathcal{J}$ that is, $P x = -P \delta(0)s$, then clearly $P x = P \delta(0) = 0$ and conversely. Hence $\text{Char}(\mathcal{D}_{X \times S/S} \delta) = (T^s_{(0)} X \cup T^s_{X^*} X) \times S = WF_A(\delta)$. 

**Example 14.** For $X = S = \mathbb{C}$, $\phi(z, z, s) = e^{iz}$ and $|z|^s$ are examples of relative regular holonomic distributions which are not regular holonomic in the absolute sense (they are not even holonomic).
4. Proof of the main theorem

We first reduce the proof of Theorem 2 to the following three statements.

(i) \( C_{X,X}^S \) sends \( D_{\text{hol}}(\mathcal{D}x X/S)^{\text{op}} \) to \( D_{\text{hol}}(\mathcal{D}x X/S)^{\text{op}} \).

(ii) For any \( s_o \in S \), \( Li_{s_o} \circ C_{X,X}^S \simeq C_X \).

(iii) (iii) (iii) holds.

Proof of Theorem 2 assuming (i), (ii) and (iii). Since the family of functors \( (Li_{s_o}^*)_{s_o \in S} \) is conservative on \( D_{\text{hol}}(\mathcal{D}x X/S) \), we can apply \( [16, (5) \& \text{Cor. 3.9}] \) to obtain

\[
P(Drhol(\mathcal{D}x X/S) (M, Drhol(\mathcal{D}x X/S)) [d_X]
\]

\[
\simeq R\text{Hom}_{Dx X/S} (M, Drhol(\mathcal{D}x X/S) (\mathcal{O}_{Dx X/S}, Db X/X/S)) [d_X]
\]

\[
\simeq p\text{Sol}_{Dx} (M),
\]

where \((*)\) follows from a standard argument (cf. [10, p. 241]) and \((**)*\) follows from the relative Dolbeault-Grothendieck lemma \( DR_{x X} (Db X/X/S) \simeq \mathcal{O}_{X/S} \) (cf. [21, §3.1]). Since \( M \) and \( C_{X,X}^S(M) \) have holonomic cohomologies (according to \( [4] \) for the latter), we can apply \( [16, (5) \& \text{Cor. 3.9}] \) to obtain

\[
p\text{Sol}_{x X} (C_{X,X}^S(M)) \simeq D p\text{DR}_{x X} (C_{X,X}^S(M))
\]

\[
\simeq D p\text{Sol}_{x} (DM) \simeq p\text{Sol}_{x} (M).
\]

The proof of (iii) will follow the same strategy as for that of [5 Th. 3]. We consider the statement \( P_X \) defined, for any complex manifold \( X \) and any \( M \in D_{\text{hol}}^b(\mathcal{D}x X/S) \), by

\[
P_X(M) := C_{X,X}^S(M) \in D_{\text{hol}}^b(\mathcal{D}x X/S).
\]

in other words, (iii) holds for \( M \). Properties (a)–(d) of [5 Lem. 3.6] clearly hold for \( P_X \). We need to check Property (f) of loc. cit., that is, the truth of \( P_X(M) \) when \( M \) is torsion.

The following result also proves (iii) of Theorem 2.

Lemma 15. Assume that \( M \in \text{Mod}_{\text{hol}}(\mathcal{D}x X/S) \) is torsion. Then \( P_X(M) \) is true. Moreover, \( C_{X,X}^S(M)[1] \simeq \mathcal{H}^1 C_{X,X}^S(M) \).

Proof. The property is local with respect to \( X \) and \( S \), and we can assume that \( S = \mathbb{C} \) with coordinate \( s \) and that \( s^N M = 0 \) for some \( N \geq 1 \). By a
standard extension argument, and due to Property (c) of [5] Lem. 3.6], we can assume that \( sM = 0 \).

Then, by division by \( s \), we can write \( M \simeq q_X^* M' \otimes_{p_X^{-1}O_S} p_X^{-1}(O_S/sO_S) \) where \( q_X \) denotes the projection \( X \times S \rightarrow X \), for some regular holonomic \( D_X \)-module \( M' \). Hence

\[
C^S_X(X)(M) \simeq R^X\text{hom}_{D_X \times S/S}(q_X^* M' \otimes_{p_X^{-1}O_S} p_X^{-1}(O_S/sO_S), D\text{b}_{X \times S/S})
\]

\[
\simeq R^X\text{hom}_{D_X \times S/S}(q_X^* M', R^X\text{hom}_{p_X^{-1}O_S}(p_X^{-1}(O_S/sO_S), D\text{b}_{X \times S/S}))
\]

\[
\simeq R^X\text{hom}_{D_X \times S/S}(q_X^* M', D\text{b}_{X \times S/S} / S_{/S} \text{b}_{X \times S/S})[-1]
\]

\[
\simeq R^X\text{hom}_{D_X \times S/S}(q_X^* M'/s q_X^* M', D\text{b}_{X \times S/S} / S_{/S} \text{b}_{X \times S/S})[-1]
\]

\[
\simeq R^X\text{hom}_{D_X}(M', D\text{b}_X)[-1],
\]

which has regular holonomic cohomology over \( D_X \) concentrated in degree 1 according to [9] Th. 2., q.e.d.

**Proof of (i).** We consider the analogue of the statement \( P \) for right \( D_X \times S/S \)-modules:

\[
P_X(N) : \quad N \otimes_{D_{X \times S/S}} D\text{b}_{X \times S/S} \in \text{D}_{\text{rhol}}(D_{X \times S/S}).
\]

We first remark that, by biduality, (i) is equivalent to the following condition, working with right \( D_X \times S/S \)-modules \( N \):

(i') For all \( N \in \text{D}_{\text{rhol}}(D_{X \times S/S}) \), \( P_X(N) \) holds true.

In what follows we will prove (i'). We argue in a way similar to that of [3] End of the proof of Th. 3], and we only emphasize the modifications of the argument.

**Step 1. Proof of (i') in the torsion case.** This follows from Lemma [13].

**Step 2.** In a way identical to that used in [3] End of the proof of Th. 3], we reduce to proving \( P_X(N) \) for those regular holonomic \( D_X \times S/S \)-modules \( N \) which are strict and satisfy

(a) the support \( Z \subset X \) of \( N \) is pure dimensional of dimension \( k \),

(b) there exists a closed hypersurface \( Y \subset X \) such that

- \( Y \cap Z \) contains the singular locus of \( Z \),
- \( N \simeq N(* (Y \times S)) \).

**Step 3. Reduction to the case of D-type.** Let us recall the definition of D-type (cf. [17] Def. 2.10]).

**Definition 16.** Let \( Y \) be a normal crossing hypersurface in \( X \). We recall that a regular holonomic \( D_X \times S/S \)-module \( N \) is said to be of D-type along \( Y \) if it is strict, if its relative characteristic variety is contained in \((T^*X|_Y \cup T^*_XX) \times S \) and if \( N = N(* (Y \times S)) \). Equivalently (cf. [17] Prop. 2.11]), there exists a locally free \( p_X^{-1}O_S \)-module \( F \) such that the left \( D_X \times S/S \)-module associated with \( N \) is equal to the extension by moderate growth of \( F \otimes_{p_X^{-1}O_S} \mathcal{O}(X \times Y) \times S \).
In the situation of Step 2, there exists a commutative diagram

\[
\begin{array}{ccc}
X' \setminus Y' & \xrightarrow{j'} & X' \\
\pi Z^* & \downarrow \pi & \\
Z^* \setminus Y & \xrightarrow{j} & X
\end{array}
\]

where \( \pi \) is a projective morphism, \( X' \) is smooth, \( Y' \) is a normal crossing hypersurface, \( Z^* := Z \setminus Y \) and \( \pi_Z^* \) is biholomorphic. Let

\[
\pi^* N = N \otimes_{D_X/Y} D(X \leftarrow X')/S
\]

be the pullback of \( N \). Then \( N' := \pi^*N[d_{Y'} - d_X] \) is concentrated in degree zero and is of D-type along \( Y' \) and we have \( N \simeq \pi^* N' \). We assume that \( P_X(N') \) holds and we prove that \( P_X(N) \) holds.

According to [17 Cor. 2.4], the pushforward \( \pi^* (N \otimes_{D_{X'/S}} Db_{X'/S}) \) is an object of \( Db_{hol}(D_{X/S}) \). Then we can argue as in [9, p. 206] to conclude that \( P_X(N) \) holds.

**Step 4. Proof of [5] in the case of D-type.** Let \( Y \) be a normal crossing divisor in \( X \) and assume that \( N \) is a right \( D_{X/S} \)-module of D-type along \( Y \). In particular, \( N \) is strict and \( N \simeq N(\{Y \times S\}) \). We are thus reduced to proving that

\[
N \otimes_{D_{X/S}} Db_{X/S}(\{Y \times S\})
\]

has regular holonomic cohomology as a \( D_{X \times S/S} \)-module. The main point is to prove coherence and holonomicity.

Since the proof is local, we may assume that, in a neighbourhood of \((x_0, s_0) \in X \times S, S = \mathbb{C}^d \) with a coordinate \( s \) vanishing at \( s_0 \), \( X = \mathbb{C}^d \) with coordinates \((x_1, \ldots, x_d)\) vanishing at \( x_0 \), \( Y = \{f(x) = 0\} \) with \( f = x_1 \cdots x_l \) and, by strictness (cf. [17, Prop. 2.11 & proof of Cor. 2.8]), after a convenient ramification \( \rho: S' \to S \) at \( s_0 \) of finite order, the pullback \( \rho_N \)

(a) has a finite filtration whose successive quotients take the form

\[
D_{X \times S'/S'}/(P_1, \ldots, P_d),
\]

with

\[
P_1 = x_1 \partial_1 - \alpha_1(s), \ldots, P_l = x_l \partial_l - \alpha_l(s), P_{l+1} = \partial_{l+1}, \ldots, P_d = \partial_d,
\]

for some holomorphic functions on \( S' \) such that \( \alpha_j(s_0) \notin \mathbb{Z}_- \) (since \( N \) is localized along \( Y \times S \) as a right \( D_{X \times S/S} \)-module).

We prove the statement for \( N \) assuming it for \( \rho^* N \). Let us note that

\[
\rho^*(N \otimes_{D_{X \times S}} Db_{X \times S/S}) \simeq \rho^* N \otimes_{D_{X \times S'}} Db_{X \times S'/S'}
\]

functorially on \( N \). In fact, this holds for any \( D_{X \times S} \)-coherent module \( N \); to see that, we can locally reduce to the case were \( N = D_{X \times S} \), that is, to prove that \( \rho^* Db_{X \times S} = Db_{X \times S'/S'} \), which is true by Corollary 1.5. Since \( N \simeq N(\{Y \times S\}) \), we also have

\[
(5) \quad \rho^* N \otimes_{D_{X \times S'}} Db_{X \times S'/S'} \simeq \rho^* N \otimes_{D_{X \times S'}} Db_{X \times S'/S'}(\{Y \times S'\}).
\]
Since $N = N((Y \times S))$, the localized pushforward (in the sense of $\mathcal{D}$-modules) $(\rho_{s}', \rho^* N)((Y \times S))$ is equal to the sheaf pushforward $\rho_s^* N$, and by Corollary [A.5] and the projection formula, we have functorial isomorphisms of $\mathcal{D}_{X \times S/S}((Y \times S'))$-modules

$$\mathcal{D}_{X \times S/S} \rho_{s}^* (\rho^* N \otimes_{\mathcal{D}_{X \times S/S'}} L_{\mathcal{D}_{X \times S/S'}}^{L} \mathcal{D}_{X \times S'/S'}((Y \times S')))$$

$$\approx \rho_{s}^* (\rho^* N \otimes_{\mathcal{D}_{X \times S/S'}} L_{\mathcal{D}_{X \times S/S'}}^{L} \mathcal{D}_{X \times S'/S'}((Y \times S')))$$

$$\approx \rho_{s}^* (\rho^* N \otimes_{\mathcal{D}_{X \times S/S'}} L_{\mathcal{D}_{X \times S/S'}}^{L} \mathcal{D}_{X \times S/S'}((Y \times S')).$$

Hence $N \otimes_{\mathcal{D}_{X \times S/S'}} L_{\mathcal{D}_{X \times S/S'}}^{L} \mathcal{D}_{X \times S/S'}((Y \times S'))$, being a direct summand of the holonomic $\mathcal{D}_{X \times S/S'}$-module $\rho_{s}^* N \otimes_{\mathcal{D}_{X \times S/S'}} L_{\mathcal{D}_{X \times S/S'}}^{L} \mathcal{D}_{X \times S'/S'}((Y \times S'))$, is holonomic.

To check the regularity of $N \otimes_{\mathcal{D}_{X \times S/S'}} L_{\mathcal{D}_{X \times S/S'}}^{L} \mathcal{D}_{X \times S/S'}((Y \times S'))$, we apply $Li_*^s$ for each $s_i \in S$ and recall ([10 Prop. 2.1]) that the result is functorially isomorphic to

$$Li_*^s N \otimes_{\mathcal{D}_{X}} Li_*^s \mathcal{D}_{X \times S/S}$$

hence, according to Proposition [A.3] and to [4 Th. 1], $N \otimes_{\mathcal{D}_{X \times S/S'}} L_{\mathcal{D}_{X \times S/S'}}^{L} \mathcal{D}_{X \times S/S'}$ is regular holonomic.

It remains therefore to prove the statement for $N$ assuming that it satisfies [4]. It is then enough to prove the statement for each successive quotient of the corresponding filtrations, so we assume that $N$ is one such quotient. In such a case, $N$ has a resolution by free $\mathcal{D}_{X \times S'/S'}((Y \times S'))$-modules obtained by taking the simple complex associated with the $d$-cube complex having vertices equal to $\mathcal{D}_{X \times S'/S'}((Y \times S'))$ and arrows in the $i$th direction equal to $P_i$. It follows that the later complex is isomorphic to the simple complex associated with the $d$-cube complex having vertices equal to $\mathcal{D}_{X \times S'/S'}((Y \times S'))$ and arrows in the $i$th direction equal to the left action of $P_i$ on $\mathcal{D}_{X \times S'/S'}((Y \times S'))$. Note also that $\mathcal{D}_{X \times S'/S'}((Y \times S')) = \mathcal{D}_{X \times S'/S'}((\mathcal{T} \times S'))$.

The function $(x, s) \mapsto |x|^{2\alpha(s)} := |x_1|^{2\alpha_1(s)}|x_2|^{2\alpha_2(s)} \cdots |x_\ell|^{2\alpha_\ell(s)}$ is real analytic on $(X \setminus Y) \times S'$, holomorphic in $s$, and defines, by multiplication, an $(\mathcal{O}_{X \times S'}, \mathcal{O}_{X \times S'})$-linear isomorphism from $\mathcal{D}_{X \times S'/S'}((Y \times S'))$ to itself. Regarding $P_i$ as $|x|^{2\alpha_i(s)} \circ (x_i \partial_i) \circ |x|^{-2\alpha_i(s)}$ for $i = 1, \ldots, \ell$, [3] is isomorphic to the simple complex associated with the $d$-cube complex having vertices equal to $\mathcal{D}_{X \times S'/S'}((Y \times S'))$ and arrows in the $i$th direction equal to the left action of $Q_i$ on $\mathcal{D}_{X \times S'/S'}((Y \times S'))$, with $Q_i = x_i \partial_i, Q_{i+1} = \partial_{i+1}, \ldots, Q_d = \partial_d$.

By the relative Dolbeault-Grothendieck lemma, the latter complex has cohomology in degree zero only. It follows that so does [3], and its zeroth cohomology is isomorphic to $\mathcal{O}_{\mathcal{T} \times S'}((\mathcal{T} \times S'))$ endowed with its usual connection twisted by $|x|^{-2\alpha(s)}$. In other words, it is of $D$-type on $(\mathcal{T} \times S') \times S$ and, is thus holonomic.
5. Some examples

**Proposition 17.** Let us assume that $\delta X > 0$. Any regular holonomic $\mathcal{D}_{X \times S/S}$-module $\mathcal{M}$ of D-type along a normal crossing divisor $Y \subset X$ (Definition 16) locally admits an injective $\mathcal{D}_{X \times S/S}$-linear morphism taking values in $\mathfrak{D}b_{X \times S/S}$. Furthermore, $C_{X,X}(\mathcal{M})$ and $\mathcal{D}\mathcal{M}$ are locally cyclic up to $S$-torsion.

**Proof.** In the proof of Lemma 4.2 of [17] it is shown that a regular holonomic $\mathcal{D}_{X \times S/S}$-module $\mathcal{M}$ is of D-type along $Y$ if and only if $p\text{Sol} \mathcal{M}$ takes the form $j!F[d_X]$ for some $S$-locally constant sheaf of locally free $p^{-1}\mathcal{O}_S$-modules $F$ on $X \setminus Y$, where $j : X \setminus Y \hookrightarrow X$ denotes the inclusion, and then $\mathcal{M} \simeq \text{RH}^S(j_!F[d_X])$, according to the Riemann-Hilbert correspondence of loc. cit. Then, by Theorem 2.6 in loc. cit., we can assume that $\partial_S \mathcal{M}$ is of D-type along $X \setminus S$. Therefore, if the first statement of the proposition is proved, we conclude from Proposition 5 that $C_{X,X}(\mathcal{M})$ as well as $C_{X,X}(\partial_S \mathcal{M}) \simeq \mathcal{D}\mathcal{M}$ are locally cyclic up to $S$-torsion, according to Theorem 2.2.

Let us prove the first statement. Since it is local on $X$ and $S$, we can choose local coordinates $x_1, \ldots, x_d$ on $X$ such that $Y = \{x_1 \cdots x_{\ell} = 0\}$ and, according to [17] Prop.2.11 and the first part of the proof of Theorem 2.6 in loc. cit., we can assume that $\mathcal{M}$ is the free $\mathcal{O}_{X \times S}(\ast(Y \times S))$-module $\mathcal{O}_{X \times S}(\ast(Y \times S))^k$ endowed with the relative connection $\nabla$ with matrix $\sum_{i=1}^\ell A_i(s)dx_i$ in the canonical $\mathcal{O}_{X \times S}(\ast(Y \times S))$-basis $(e_1, \ldots, e_k)$ of $\mathcal{O}_{X \times S}(\ast(Y \times S))^k$ (where $A_i(s)$ are matrices depending holomorphically on $s$).

**Step 1.** We first embed $\mathcal{O}_{X \times S}(\ast(Y \times S))^k$ with its standard $\mathcal{D}_{X \times S/S}$-module structure in $\mathfrak{D}b_{X \times S/S}$. Let us set $f(x) = x_1 \cdots x_\ell$ and $u_j = \frac{\partial_f}{\partial_f} (j = 1, \ldots, k)$. Then $u_j$ is a section of $\mathfrak{D}b_{X \times S/S}$ and $1/f \mapsto u_j$ induces an isomorphism $\mathcal{O}_{X \times S}(\ast(Y \times S)) \simeq \mathcal{D}_{X \times S/S} \cdot u_j$. The surjective $\mathcal{D}_{X \times S/S}$-linear morphism

$$\mathcal{O}_{X \times S}(\ast(Y \times S))^k \longrightarrow \sum_{j=1}^k \mathcal{D}_{X \times S/S}u_j \subset \mathfrak{D}b_{X \times S/S}$$

$$e_j \longmapsto u_j$$

is an isomorphism: indeed, it suffices to show that its restriction to $(X \setminus Y) \times S$ is injective. The target of the latter is equal to $\sum_{j} \mathcal{O}_{(X \setminus Y) \times S}u_j$; if $\sum_{j} g_j u_j = 0$ is a relation with $g_j$ holomorphic, then applying successively $\partial_{x_1}, \partial_{x_1}^{-1}, \ldots$ to this relation gives successively $g_k = 0, g_{k-1} = 0, \ldots$

**Step 2.** Locally on $X \times S$ there exists $N_i > 0$ such that each entry of the matrix $|x_i|^{2A_i(s)}$ becomes locally bounded after being multiplied by $\mathcal{P}_{i}^{-N_i}$. We then twist the $\mathcal{O}_{X \times S}(\ast(Y \times S))$-basis $(u_1, \ldots, u_k)$ by setting

$$(v_1, \ldots, v_k) = (u_1, \ldots, u_k) \cdot \prod_{i=1}^\ell \mathcal{P}_{i}^{-N_i} |x_i|^{2A_i}.$$
The \( \mathcal{O}_{X \times S}\)-linear morphism
\[
\mathcal{O}_{X \times S}(s(Y \times S))^k \longrightarrow \mathcal{D}b_{X \times S/S}
\]
\[
e_j \longrightarrow v_j
\]
is thus \( \mathcal{D}_{X \times S/S}\)-linear when using the \( \mathcal{D}_{X \times S/S}\)-structure induced by \( \nabla \) on the left-hand side. Since it is obtained by applying to (1) a matrix which is invertible on \( (X \setminus Y) \times S \), it is also injective. q.e.d.

**Example 18.** We give an example, in general not of D-type, for which both \( \mathcal{M} \) and \( C^S_{X,X}(\mathcal{M}) \) are strict, regular and locally cyclic up to \( S\)-torsion.

Let \( X \) be a relatively compact open subset of \( \mathbb{C}^d \) and let \( u \) be a regular holonomic distribution in the sense of [8] on some open neighbourhood of the closure of \( X \). Let \( f \) be a holomorphic function on \( X \). Let \( M = \mathcal{D}_X[s] \cdot "f^su" \) be the \( \mathcal{D}_X[s]\)-module introduced in [8, 2.2]. Recall that \( M \) is the quotient of \( \mathcal{D}_X[s] \) by the left ideal \( \mathcal{J} \) consisting of the operators \( P(s) \) such that the operator \( f^{ord(P)} \cdot s P(s) f^s \in \mathcal{D}_X[s] \) satisfies \( (f^{ord(P)} - s) P(s) f^s u = 0 \). Let \( \mathcal{M} := \mathcal{D}_X \cdot "f^su" \) be the analytification of \( M \), i.e., \( \mathcal{M} = \mathcal{O}_{X \times \mathbb{C}} \otimes_{\mathcal{O}_X[s]} M \). Then \( \mathcal{M} \) is holonomic (cf. [11, Prop. 13]), strict and regular since \( \mathcal{D}_X \cdot u \) is regular.

Let \( p \) denote the finite order of \( u \) on the closure of \( X \). Then \( |f|^2 u \) is a relative distribution on \( \mathcal{O}_X \times \{ s \in \mathbb{C} \mid \text{Re} s > p \} \). By using a Bernstein relation for \( "f^su" \), one finds by a standard procedure a finite set \( A \subset \mathbb{C} \) (the roots of a Bernstein equation for \( "f^su" \) and \( "f^su" \)) such that \( |f|^2 u \) extends as a relative distribution \( v \) on \( X \times S \), with \( S := \mathbb{C} \setminus (A - \mathbb{N}) \).

Let \( P(s) \) be a differential operator in \( \mathcal{J} \) of order \( ord(P) \). Assume that \( \text{Re} s > \max\{ p, ord(P) \} \). Then, since \( P(s) \) commutes with \( f^s \), we have
\[
P(s)v = P(s)|f|^2 u = f^s \cdot f^{ord(P)} (f^{ord(P)} - s) P(s) f^s u = 0.
\]
This identity extends to \( s \in \mathbb{S} \), according to [9] in the appendix. There is thus a well-defined \( \mathcal{D}_{X \times S/S}\)-linear injective morphism \( \phi : \mathcal{M} \vert_{X \times S} \hookrightarrow \mathcal{D}b_{X \times S/S} \):
\[
\phi : \mathcal{M} \vert_{X \times S} \hookrightarrow \mathcal{D}_{X \times S/S} \cdot v \subset \mathcal{D}b_{X \times S/S}
\]
\[
"f^su" \longmapsto v.
\]
Hence \( \phi \) is a generator of \( C^S_{X,X}(\mathcal{M}) \vert_{X \times S} \) up to \( S\)-torsion. From Corollary [6] we also deduce an isomorphism up to \( S\)-torsion:
\[
C^S_{X,X}(\mathcal{M}) \vert_{X \times S} \cong \mathcal{D}_{X \times S/S} \cdot v \cong \mathcal{D}_{X \times S/S} \cdot f^s u.
\]
If \( \alpha : S' \rightarrow \mathbb{C} \) is any holomorphic function on a Riemann surface \( S' \), then the pullback \( \alpha^* \mathcal{M} = \mathcal{D}_{X \times S'/S'} \cdot "f^{\alpha(s')}u" \) with respect to \( S' \) is regular holonomic and \( C^S_{X,X}(\mathcal{M}) = \alpha^* C^S_{X,X}(\mathcal{M}) \). As a consequence, still denoting \( S' := \alpha^{-1}(S) \), we find an isomorphism up to \( S\)-torsion
\[
C^S_{X,X}(\mathcal{D}_{X \times S'/S'} \cdot "f^{\alpha(s')}u") \cong \mathcal{D}_{X \times S'/S'} \cdot "f^{\alpha(s')}u".
\]
For example, consider the case where \( X \subset \mathbb{C} \) with coordinate \( z \), \( f(z) = z \) and \( u = \overline{z}^m \) for some \( m \geq 0 \), so that \( \text{ord}(u) = 0 \). Then we set \( S = \mathbb{C} \setminus \mathbb{N}^* \), \( S' = \alpha^{-1}(S) \) and \( v \) is the extension to \( S' \) of \( |z|^{2\alpha(s')} \overline{z}^m \) defined for \( \text{Re}\alpha(s') + m/2 > -1 \). Then \( v \) is a global section of the sheaf \( \mathcal{D} \mathcal{B}_{X \times S'/S'} \) satisfying \( (z\partial z - \alpha(s'))v = 0 \), so that \( \mathcal{D} \mathcal{X}_{X \times S'/S'}/(z\partial z - \alpha(s')) \cong \mathcal{D} \mathcal{X}_{X \times S'/S'} \cdot v \), and we have

\[
C^S_{X,\mathcal{X}}[\mathcal{D} \mathcal{X}_{X \times S'/S'}/(z\partial z - \alpha(s'))] \cong \mathcal{D} \mathcal{X}_{X \times S'/S'}/(\overline{z}\partial \overline{z} - \alpha(s') - m).
\]

### Appendix. The sheaf of relative distributions

In this appendix, we give details on the sheaf of partially holomorphic distributions \( \mathcal{D} \mathcal{B}_{X \times S'/S'} \). This sheaf was already considered by Schapira and Schneider in [21] (cf. also [19, 12]). The notion of relative current of maximal degree leads to the sheaf \( \mathcal{C}_{M \times T/T} \) as mentioned in the introduction.

#### A.a. Partially holomorphic distributions

Let \( M \) be a \( C^\infty \) manifold of real dimension \( m \) and let \( T \) be a complex manifold (we also denote by \( T \) the underlying \( C^\infty \) manifold). Let \( p : M \times T \to T \) denote the projection. Let \( W \) be an open set of \( M \times T \). We say that a distribution \( u \in \mathfrak{D} \mathfrak{b}(W) \) is partially holomorphic with respect to \( T \), or \( T \)-holomorphic, if it satisfies the partial Cauchy-Riemann equation \( \overline{\partial}_T u = 0 \). The subsheaf of \( \mathfrak{D} \mathfrak{b}_{M \times T} \) consisting of \( T \)-holomorphic distributions is denoted by \( \mathfrak{D} \mathfrak{b}_{M \times T/T} \). For example, if \( u \in \mathfrak{D} \mathfrak{b}(M) \), then \( u \) defines a section of \( \mathfrak{D} \mathfrak{b}_{M \times T/T}(M \times T) \). The following result is obtained by adapting the proof of [4 Th. 4.4.7].

**Proposition A.1.** Let \( U \) be an open subset of \( \mathbb{R}^m \), \( K \) a compact subset of \( T \) and let \( u \) be a distribution defined on a neighbourhood of \( U \times K \) in \( U \times T \). If \( \overline{\partial}_T u = 0 \), then there exists a neighbourhood \( V \) of \( K \) and finite family of functions \( f_\alpha \in C^0(U \times V) \) which are \( V \)-holomorphic and a decomposition

\[
u = \sum_{\alpha} \partial_{\overline{z}}^\alpha f_\alpha.
\]

Let us mention some obvious consequences.

(a) A partially holomorphic distribution \( u \in \mathfrak{D} \mathfrak{b}_{M \times T/T}(M \times T) \) extends as a continuous linear map from the space of \( C^\infty \) functions on \( M \times T \) which are \( T \)-holomorphic and have \( p \)-proper support to the space of holomorphic functions \( \mathcal{O}(T) \), endowed with the usual family of semi-norms (sup of absolute value on compact subsets of \( T \)).

(b) As a consequence, a \( T \)-holomorphic distribution on \( M \times T \) can be restricted to each \( t_0 \in T \), giving rise to a distribution \( u|_{t_0} \in \mathfrak{D} \mathfrak{b}(M) \). More generally, for any holomorphic function \( g : T' \to T \), the pullback \( g^*u \) is defined as a \( T' \)-holomorphic distribution on \( M \times T' \).

(c) If \( T \) is connected and if two \( T \)-holomorphic distributions on \( M \times T \) coincide on \( M \times V \) for some nonempty open subset \( V \) of \( T \), they coincide: this follows from the first point above.

(d) Let \( f \in \mathcal{O}(T) \) not vanishing on a dense open subset of \( T \). For \( u \in \mathfrak{D} \mathfrak{b}_{M \times T/T}(M \times T) \), if \( fu = 0 \), then \( u = 0 \).
Corollary A.2. Assume that $T$ is an open polydisc centered at the origin with coordinates $t_1, \ldots, t_n$. Let $u \in \mathcal{Db}_{M \times T/T}(M \times T)$. Then $u$ admits a termwise $T$-differentiable convergent expansion

$$u = \sum_{m \in \mathbb{N}^n} u_m t^m$$

where $u_m$ in $\mathcal{Db}(M)$ (regarded in $\mathcal{Db}_{M \times T/T}(M \times T)$) are uniquely determined from $u$, and convergence meaning convergence in $\mathcal{Db}(M \times T)$.

**Proof.** We apply Proposition [A.1] on each relatively compact open polydisc $T'$ in $T$. Each $f_a(x, t)$ is $T'$-holomorphic on $M \times T'$ and can be expanded as $\sum_m f_{a,m}(x)t^m$ with $f_{a,m}(x)$ continuous. The existence of the expansion follows, as well as its $T'$-differentiability. Uniqueness is then clear: for example, $u_0 = u_{t=0}$. By uniqueness, applying the result to an increasing sequence of such subpolydiscs converging to $T$, we obtain the assertion. \[\text{q.e.d.}\]

Corollary A.3. For any $t_o \in T$, we have a natural identification

$$Li_{t_o}^* \mathcal{Db}_{M \times T/T} \simeq \mathcal{Db}_M.$$

**Proof.** Let us start by proving that the complex $Li_{t_o}^* \mathcal{Db}_{M \times T/T}$ is concentrated in degree zero. This a local statement and by induction on $\dim T$, it is enough to check that, for any local coordinate $t$ on $T$, $t : \mathcal{Db}_{M \times T/T} \to \mathcal{Db}_{M \times T/T}$ is injective. Let $u$ be a section of $\mathcal{Db}_{M \times T/T}$ in the neighbourhood of $(x_0, 0)$ such that $tu = 0$. According to [G] above we get $u = 0$ hence the vanishing of $\mathcal{F}^{-1}Li_{t_o}^* \mathcal{Db}_{M \times T/T}$.

That $i_{t_o}^* \mathcal{Db}_{M \times T/T} = \mathcal{Db}_M$ follows from [D] above. \[\text{q.e.d.}\]

Let $\rho : T' \to T$ be a finite ramification around the coordinate axes, that we write $\rho(t'_1, \ldots, t'_n) = (t'^{k_1}_1, \ldots, t'^{k_n}_n)$. If $u$ is $T'$-holomorphic, it follows from [E] above that $\rho^* u$ is $T'$-holomorphic. Moreover, the expansion of $\rho^* u$ is obtained from that of $u$ as

$$(A.1) \quad \rho^* u = \sum m u_m t^{k_1 a_1} \ldots t^{k_n a_n}.$$

Corollary A.4. Under these assumption, the natural morphism

$$\mathcal{O}(T') \otimes_{\mathcal{O}(T)} \mathcal{Db}_{M \times T/T}(M \times T) \to \mathcal{Db}_{M \times T'/T'}(M \times T')$$

$h \otimes u \mapsto h \cdot \rho^* u$

is an isomorphism.

**Proof.** Surjectivity is obvious, by using the expansion of Corollary [A.2] on $M \times T'$ and [A.1]. Since $\mathcal{O}(T')$ is free over $\mathcal{O}(T)$ with basis $t'^r$, $0 \leq r_i < k_i$ ($i = 1, \ldots, n$), injectivity follows from uniqueness of the decomposition in Corollary [A.2] on $M \times T'$. \[\text{q.e.d.}\]

By sheafifying Corollary [A.4] we obtain:

**Corollary A.5.** Let $T, T'$ be polydiscs and let $\rho : T' \to T$ be a ramification along the coordinate axes. Then

$$\rho^* \mathcal{Db}_{M \times T/T} := \mathcal{O}_{T'} \otimes_{\mathcal{O}_T} \mathcal{Db}_{M \times T/T} \simeq \mathcal{Db}_{M \times T'/T'}.$$
A.b. **Integration and pushforward of relative currents by a proper map.** The sheaf $\mathcal{C}_{M \times T/T}$ of $T$-holomorphic currents of maximal degree is the subsheaf of the sheaf of degree $m$ currents which are killed by $\partial_T$.

Let $f : M \to N$ be a $C^\infty$ map between $C^\infty$ manifolds. Let $W' \subset N \times B$ be an open set and let $W = f^{-1}(W')$. Let $u \in \mathcal{C}_{M \times T/T}(W)$ and assume that $f$ is proper on $\text{Supp } u$. Then the integral $\int_f u$, which is a current of degree $\text{dim } N$ on $N \times T$, is also killed by $\partial_T$, hence belongs to $\mathcal{C}_{N \times T/T}(W')$.

Let us now consider the sheaf-theoretic pushforward of $\mathcal{C}_{M \times B/B}$ by $f \times \text{Id}$.

**Proposition A.6.** The sheaf $\mathcal{C}_{M \times T/T}$ is $(f \times \text{Id})_!$-acyclic.

**Proof.** For any $t_o \in T$, the sheaf-theoretic restriction $i_t^{-1}\mathcal{C}_{M \times T/T}$ is c-soft, being a $\mathcal{C}_M^c$-module. It follows that for any $(y_o, t_o) \in N \times T$, denoting by $i : f^{-1}(y_o) \times \{t_o\} \to M \times T$ the inclusion,

$$i_{(y_o, t_o)}^{-1}(f \times \text{Id}!)\mathcal{C}_{M \times T/T} = H^j_c(f^{-1}(y_o) \times \{t_o\}, i^{-1}\mathcal{C}_{M \times T/T}) = 0$$

for any $j > 0$. \hspace{1cm} q.e.d.

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