Linear Sigma Models for the $R^8/\mathbb{Z}_k$ Orbifold

Kazumi Okuyama

Department of Physics, Shinshu University
Matsumoto 390-8621, Japan
kazumi@azusa.shinshu-u.ac.jp

We construct $\mathcal{N} = 4$ gauged linear sigma models in two dimensions whose Higgs branch has a $\mathbb{R}^8/\mathbb{Z}_k$ orbifold singularity or its generalization. Our linear sigma models have either ALF or ALE type hyperKähler 8-manifolds as their Higgs branch. For the ALE case, the matter content of our model is specified by a quiver diagram which is a union of two $A$-type extended Dynkin diagrams overlapping at one link.

July 2008
1. Introduction

It is well-known that some hyperKähler manifolds, such as Taub-NUT [1,2] and ALE spaces [3], can be realized as hyperKähler quotients of flat $\mathbb{R}^{4k}$ [4,5]. In this paper, following the general procedure in [6], we will consider the quotient construction of some 8-dimensional hyperKähler manifolds which have $\mathbb{R}^8/\mathbb{Z}_k$ orbifold singularity in a certain limit.

This is partly motivated by the recent excitement of the Bagger-Lambert-Gustavsson theory of multiple M2-branes [7,8,9,10] and the closely related model by Aharony, Bergman, Jafferis and Maldacena (ABJM) [11]. For the ABJM model, i.e. a three-dimensional $\mathcal{N} = 6 U(N) \times U(N)$ Chern-Simons-matter theory with level $(k, -k)$ [11], it is shown that its vacuum moduli space is $(\mathbb{R}^8/\mathbb{Z}_k)^N/S_N$, which suggests that this model is a theory on M2-branes in the orbifold $\mathbb{R}^8/\mathbb{Z}_k$ background. As argued in [11], this picture is also consistent with the brane construction of the model. Namely, the ABJM model is realized as a theory on the D3-branes wrapped around a circle in the presence of a NS5-brane and a $(k, 1)$ 5-brane transverse to the circle. The M-theory dual of this configuration is a collection of M2-branes in the background of intersecting KK monopoles. The corresponding 11-dimensional supergravity solution is given by an 8-dimensional toric hyperKähler manifold [8]. It is shown that the hyperKähler manifold appearing as the dual of NS5-(k, 1)5brane system has a $\mathbb{R}^8/\mathbb{Z}_k$ orbifold singularity [11]. Some generalizations of the ABJM model, which correspond to more general orbifold $\mathbb{R}^8/\Gamma$, were considered in [12,13].

In this paper, we will construct two-dimensional $\mathcal{N} = 4$ gauged linear sigma models (GLSMs) whose Higgs branch is a hyperKähler manifold which appears as the M-theory dual of a configuration of $n$ NS5-branes and $k$ $(1, 1)$ 5-branes, or $n$ NS5-branes and one $(k, 1)$ 5-brane. We should emphasize that our GLSM is not directly related to the theory on M2-branes in the orbifold background. We merely use GLSM as a tool to realize the hyperKähler quotient construction in the gauge theory language. Our GLSM is a natural generalization of the model for the Taub-NUT space studied in [14,15], which was shown to be dual to the GLSM for H-monopoles [16,17] applying the method of [17]. We consider both ALF and ALE type hyperKähler 8-manifolds, presented in section 2 and 3, respectively. For the ALE case, the matter content of our GLSM is described by a quiver diagram, which is a union of $\tilde{A}_{k-1}$ and $\tilde{A}_{n-1}$ Dynkin diagrams connected at one link (see Fig. 1).
2. ALF-type GLSM

We first construct an $\mathcal{N} = 4$ GLSM in two dimensions whose Higgs branch is an ALF-type hyperKähler 8-manifold, which appears as the M-theory dual of the type IIB 5-brane configurations. In the case of H-monopoles or its T-dual of KK-monopoles, the corresponding GLSMs were studied in [10,14,15]. Let us recall the matter content of the GLSM for the Taub-NUT space with KK-monopole charge $k$ [2,15]. The model has the gauge group \( \prod_{a=1}^{k} U(1)_a \) with $k$ hypermultiplets \((Q_a, \tilde{Q}_a)\) with charge \((+1, -1)\) under the gauge group \(U(1)_a\). Additionally, there is a linear-multiplet \((\Psi, P)\), where the shift symmetry of the imaginary part of $P$ is gauged under the diagonal part of \(\prod_a U(1)_a\).

2.1. M-theory Dual of $n$ NS5-branes and $k$ (1,1) 5-branes

We first consider the GLSM for the 8-manifold which is dual to a configuration of $n$ NS5-branes and $k$ (1,1) 5-branes. Since this brane configuration of 5-branes is U-dual to the intersecting KK-monopoles, we expect that the GLSM for this background is obtained by a simple generalization of the Taub-NUT case. We will show that this is indeed the case following the general recipe for the quotient construction of toric hyperKähler 8-manifolds [3]. The matter content of our GLSM is the same as the two sets of GLSMs for Taub-NUT spaces with charge $k$ and $n$, which we call A-part and B-part, respectively:

\[
\begin{align*}
\text{A-part} & \quad \begin{cases} 
\text{vector : } (\Sigma_a, \Phi_a) \\
\text{hyper : } (Q_a, \tilde{Q}_a) \\
\text{linear : } (\Psi_A, P_A)
\end{cases} \\
\text{B-part} & \quad \begin{cases} 
\text{vector : } (\Sigma_i, \Phi_i) \\
\text{hyper : } (H_i, \tilde{H}_i) \\
\text{linear : } (\Psi_B, P_B)
\end{cases}
\end{align*}
\]

(2.1)

Here and in what follows, we use the $\mathcal{N} = 2$ language as in [18]. For instance, $\Sigma_a$ and $\Phi_a$ are the twisted chiral multiplets in the $\mathcal{N} = 2$ language. All other fields such as $Q_a$ and $\Phi_a$ are \(\mathcal{N} = 2\) chiral multiplets. The gauge groups of the A-part and the B-part are \(\prod_{a=1}^{k} U(1)_{A,a}\) and \(\prod_{i=1}^{n} U(1)_{B,i}\). The only difference from the naive direct sum of two Taub-NUT models is that the linear-multiplet in the B-part is shifted by the diagonal part of the total gauge group \(\prod_{a=1}^{k} U(1)_{A,a} \times \prod_{i=1}^{n} U(1)_{B,i}\), while the linear-multiplet of the A-part is shifted only by the diagonal of \(\prod_{a=1}^{k} U(1)_{A,a}\) as in the original Taub-NUT model.
The Lagrangian of our model (2.1) is given by $\mathcal{L} = \mathcal{L}_D + \mathcal{L}_F + \mathcal{L}_\tilde{F}$, where the D-term $\mathcal{L}_D$ is

$$\mathcal{L}_D = \int d^4 \theta \left\{ \frac{1}{g_A^2} \Phi_A \Phi_A + \frac{g_A^2}{2} \left( P_A + \frac{1}{k} \sum_{a=1}^k V_a \right) \right\}^2 + \frac{1}{g_B^2} \Psi_B \Psi_B + \frac{g_B^2}{2} \left( P_B + \frac{1}{k} \sum_{a=1}^k V_a + \sum_{i=1}^n V_i \right)^2 \right\} \right\} + \sum_{a=1}^k \left\{ \frac{1}{e_A^2} \left( -\Sigma^+_a \Sigma_a + \Phi^+_a \Phi_a \right) + Q_a e^{V_a} Q_a + \tilde{Q}_a e^{-V_a} \tilde{Q}_a \right\} + \sum_{i=1}^n \left\{ \frac{1}{e_i^2} \left( -\Sigma^+_i \Sigma_i + \Phi^+_i \Phi_i \right) + H^+_i e^{V_i} H_i + \tilde{H}^+_i e^{-V_i} \tilde{H}_i \right\} ,$$

and the F-term $\mathcal{L}_F$ and the twisted F-term $\mathcal{L}_\tilde{F}$ are

$$\mathcal{L}_F = \int d\theta^+ d\theta^- \sum_{a=1}^k \left\{ \tilde{Q}_a \Phi_a Q_a + (s_a - \Psi_A) \Phi_a \right\} + \sum_{i=1}^n \left\{ \tilde{H}_i \Phi_i H_i + (s_i - \Psi_B) \Phi_i \right\} + c.c.$$

$$\mathcal{L}_\tilde{F} = \int d\theta^+ d\tilde{\theta}^- \sum_{a=1}^k t_a \Sigma_a + \sum_{i=1}^n t_i \Sigma_i + c.c. .$$

In the above equations, $e_A^2$ and $e_i^2$ denote the gauge couplings, and $g_A^2$ and $g_B^2$ are some parameters. The parameters $(s_a, t_a)$ and $(s_i, t_i)$ appearing in (2.3) are the $N = 4$ FI-parameters. They are naturally decomposed into the triplets $(\tilde{r}_a, \tilde{r}_i)$ and the singlets $(\theta_a, \theta_i)$ under the $SU(2)_R$ R-symmetry:

$$s_a = r_a^1 + i r_a^2, \quad t_a = r_a^3 + i \theta_a, \quad s_i = r_i^1 + i r_i^2, \quad t_i = r_i^3 + i \theta_i. \quad (2.4)$$

In terms of the component fields, the bosonic part our Lagrangian is written as a sum of the kinetic term $\mathcal{L}_{\text{kin}}$, the potential term $\mathcal{L}_{\text{pot}}$ and the topological term $\mathcal{L}_{\text{top}}$:

$$\mathcal{L}_{\text{kin}} = \frac{1}{2g_A^2} (\partial \vec{A})^2 + \frac{g_A^2}{2} \left( \partial \gamma_A + \sum_{a=1}^k A_a \right)^2 + \frac{1}{2g_B^2} (\partial \vec{B})^2 + \frac{g_B^2}{2} \left( \partial \gamma_B + \sum_{a=1}^k A_a + \sum_{i=1}^n B_i \right)^2$$

$$+ \sum_{a=1}^k \left\{ \frac{1}{e_A^2} \left( (F^a_{01})^2 + |\partial \phi_a|^2 + |\partial \sigma_a|^2 \right) + |\mathcal{D} q_a|^2 + |\mathcal{D} \tilde{q}_a|^2 \right\}$$

$$+ \sum_{i=1}^n \left\{ \frac{1}{e_i^2} \left( (F^i_{01})^2 + |\partial \phi_i|^2 + |\partial \sigma_i|^2 \right) + |\mathcal{D} h_i|^2 + |\mathcal{D} \tilde{h}_i|^2 \right\} .$$

(2.5)
\[ \mathcal{L}_{\text{pot}} = - \sum_{a=1}^{k} \left\{ \frac{e_{a}^{2}}{2} \left( |q_{a}|^{2} - |\tilde{q}_{a}|^{2} - x_{a}^{3} - x_{a}^{3} + r_{a}^{3} \right)^{2} \right\} \]
\[ + \frac{e_{a}^{2}}{2} \left| 2q_{a}\tilde{q}_{a} - (x_{A}^{1} + x_{B}^{1} + ix_{A}^{2} + ix_{B}^{2}) + r_{a}^{1} + ir_{a}^{2} \right|^{2} \]
\[ + \left( |\phi_{a}|^{2} + |\sigma_{a}|^{2} \right) \left( |q_{a}|^{2} + |\tilde{q}_{a}|^{2} + g_{A}^{2} \right) \}
\[ - n \sum_{i=1}^{n} \left\{ \frac{\epsilon_{i}^{2}}{2} \left( |h_{i}|^{2} - |\tilde{h}_{i}|^{2} - x_{B}^{3} + r_{i}^{3} \right)^{2} \right\} \]
\[ + \frac{\epsilon_{i}^{2}}{2} \left| 2h_{i}\tilde{h}_{i} - (x_{B}^{1} + ix_{B}^{2}) + r_{i}^{1} + ir_{i}^{2} \right|^{2} \]
\[ + \left( |\phi_{i}|^{2} + |\sigma_{i}|^{2} \right) \left( |h_{i}|^{2} + |\tilde{h}_{i}|^{2} + g_{B}^{2} \right) \} \quad (2.6) \]
\[ \mathcal{L}_{\text{top}} = - \sum_{a=1}^{k} \theta_{a} F_{01}^{a} - \sum_{i=1}^{n} \theta_{i} F_{01}^{i}. \quad (2.7) \]

Here we used the lower case letters to denote the scalar components of the corresponding (twisted) chiral superfields, except for the linear-multiplets. For the linear-multiplets, the scalar components are denoted as

\[ \Psi_{A} = x_{A}^{1} + ix_{A}^{2}, \quad \Psi_{B} = x_{B}^{1} + ix_{B}^{2}, \quad P_{A} = \frac{1}{g_{A}^{2}}x_{A}^{3} + i\gamma_{A}, \quad P_{B} = \frac{1}{g_{B}^{2}}x_{B}^{3} + i\gamma_{B}. \quad (2.8) \]
\[ A_{a} = A_{a,\mu}dx^{\mu} \text{ and } B_{i} = B_{i,\mu}dx^{\mu} \text{ in } (2.3) \text{ are the gauge fields for the gauge groups } U(1)_{A,a} \text{ and } U(1)_{B,i}, \text{ respectively. } \tilde{x}_{A} \text{ and } \tilde{x}_{B} \text{ appearing in } (2.3) \text{ denote the } SU(2)_{R} \text{ triplet parts of the scalar components of the linear-multiplets } (2.8) \]
\[ \tilde{x}_{A} = (x_{A}^{1}, x_{A}^{2}, x_{A}^{3}), \quad \tilde{x}_{B} = (x_{B}^{1}, x_{B}^{2}, x_{B}^{3}). \quad (2.9) \]

The kinetic term in (2.3) such as \((\partial\tilde{x}_{A})^{2}\) means \( \sum_{\mu=0,1} \partial_{\mu}\tilde{x}_{A} \cdot \partial^{\mu}\tilde{x}_{A} \). \( \gamma_{A} \) and \( \gamma_{B} \) are normalized to have the period \( 2\pi \)
\[ \gamma_{A} \sim \gamma_{A} + 2\pi, \quad \gamma_{B} \sim \gamma_{B} + 2\pi. \quad (2.10) \]

In the rest of this section, we will analyze the Higgs branch of our model. From the expression of the potential energy in (2.6), the vacuum moduli space\footnote{Strictly speaking, there is no moduli space of vacua in two dimensions because of the Coleman theorem\cite{19}. We analyze the low energy theory in the spirit of Born-Oppenheimer approximation.} is characterized by
\[ F_{01}^{a} = F_{01}^{i} = \sigma_{a} = \phi_{a} = \sigma_{i} = \phi_{i} = 0 \]
\[ |q_{a}|^{2} - |\tilde{q}_{a}|^{2} = x_{a}^{3} + x_{a}^{3} - r_{a}^{3} \]
\[ 2q_{a}\tilde{q}_{a} = x_{A}^{1} + x_{B}^{1} + i(x_{A}^{2} + x_{B}^{2}) - r_{a}^{1} - ir_{a}^{2} \]
\[ |h_{i}|^{2} - |\tilde{h}_{i}|^{2} = x_{B}^{3} - r_{i}^{3}, \quad 2h_{i}\tilde{h}_{i} = x_{B}^{1} + ix_{B}^{2} - r_{i}^{1} - ir_{i}^{2}. \quad (2.11) \]
In the IR limit \( e^2_a, e_i^2 \to \infty \), the vector multiplets and the charged hypermultiplets become massive and they can be integrated out. To find the low energy action, the crucial step is to rewrite the kinetic term of hypermultiplet restricted on the vacuum locus (2.11)

\[
\begin{align*}
|Dq^a|^2 + |\tilde{D}q^a|^2 &= \frac{(\partial \vec{x}^A + \partial \vec{x}^B)^2}{4|\vec{x}^A + \vec{x}^B - \vec{r}^a|} + \frac{|\vec{x}^A + \vec{x}^B - \vec{r}^a|}{4}\left\{2A_a + 2\partial \varphi_a + \vec{\omega}_a \cdot (\partial \vec{x}^A + \partial \vec{x}^B)\right\}^2, \\
|Dh^i|^2 + |\tilde{D}\tilde{h}^i|^2 &= \frac{(\partial \vec{x}^B)^2}{4|\vec{x}^B - \vec{r}^i|} + \frac{|\vec{x}^B - \vec{r}^i|}{4}\left(2B_i + 2\partial \varphi_i + \vec{\tau}_i \cdot \partial \vec{x}^B\right)^2,
\end{align*}
\]

(2.12)

where \( \varphi_a = -\arg(iq^a) \) and \( \varphi_i = -\arg(ih^i) \). \( \vec{\omega}_a \) and \( \vec{\tau}_i \) in the above equations are given by

\[
\vec{\nabla} \times \vec{\omega}_a = \vec{\nabla} \frac{1}{|\vec{x}^A + \vec{x}^B - \vec{r}^a|}, \quad \vec{\nabla} \times \vec{\tau}_i = \vec{\nabla} \frac{1}{|\vec{x}^B - \vec{r}^i|}.
\]

(2.13)

Due to the gauge symmetry, the low energy theory depends only on the gauge invariant combinations

\[
\theta^A = \gamma^A - \sum_{a=1}^{k} \varphi_a, \quad \theta^B = \gamma^B - \sum_{a=1}^{k} \varphi_a - \sum_{i=1}^{n} \varphi_i.
\]

(2.14)

In the IR limit the gauge kinetic term can be ignored, hence the gauge fields \( A_a \) and \( B_i \) become auxiliary fields. After integrating out the gauge fields, we arrive at the effective Lagrangian on the Higgs branch

\[
\mathcal{L}_\text{eff} = \frac{1}{2} \sum_{i,j=A,B} (U_{ij} \partial \vec{x}^i \cdot \partial \vec{x}^j + (U^{-1})_{ij} \beta^i \beta^j)
\]

(2.15)

where \( \beta^A \) and \( \beta^B \) are given by

\[
\beta^A = \partial \theta^A - \frac{1}{2} \sum_{a=1}^{k} \vec{\omega}_a \cdot (\partial \vec{x}^A + \partial \vec{x}^B), \quad \beta^B = \partial \theta^B - \frac{1}{2} \sum_{a=1}^{k} \vec{\omega}_a \cdot (\partial \vec{x}^A + \partial \vec{x}^B) - \frac{1}{2} \sum_{i=1}^{n} \vec{\tau} \cdot \partial \vec{x}^B,
\]

(2.16)

and the matrix \( U \) in (2.13) is

\[
U = \begin{pmatrix} U_{AA} & U_{AB} \\ U_{BA} & U_{BB} \end{pmatrix} = \begin{pmatrix} \frac{1}{g^2_a} + H & H \\ H & \frac{1}{g^2_i} + K + H \end{pmatrix},
\]

\[
H = \frac{1}{2} \sum_{a=1}^{k} \frac{1}{|\vec{x}^A + \vec{x}^B - \vec{r}^a|}, \quad K = \frac{1}{2} \sum_{i=1}^{n} \frac{1}{|\vec{x}^B - \vec{r}^i|}.
\]

(2.17)

For the \( n = k = 1 \) case, one can easily see that the effective metric on the Higgs branch is nothing but the metric studied in [11], which was shown to be the M-theory dual of a
NS5-brane and a \((1,1)\) 5-brane. For the general case, the metric becomes singular when \(K \to \infty\) or \(H \to \infty\). This implies that when we set \(\vec{r}_a = \vec{r}_i = 0\) there is a singularity at the origin \(\vec{x}_A = \vec{x}_B = 0\). Near the origin the metric behaves as

\[
\mathcal{L}_{\text{eff}} \sim \frac{1}{2} \left\{ H (\partial \vec{x}_A + \partial \vec{x}_B)^2 + K (\partial \vec{x}_B)^2 + H^{-1} (\beta_A)^2 + K^{-1} (\beta_A - \beta_B)^2 \right\}.
\] (2.18)

From this expression, one can see that the moduli space has a \(\mathbb{R}^4/\mathbb{Z}_k \times \mathbb{R}^4/\mathbb{Z}_n\) orbifold singularity. From the constant part \(U_\infty\) of the matrix \(U\)

\[
U_\infty = \begin{pmatrix}
\frac{1}{g_A} & 0 \\
0 & \frac{1}{g_B}
\end{pmatrix},
\] (2.19)

we can read off the moduli of the torus (or type IIB axio-dilaton) as \[1\]

\[
\tau = \chi + \frac{i}{g_s} \frac{g_B}{g_A}.
\] (2.20)

The singularity at the origin is factorized \(\mathbb{R}^4/\mathbb{Z}_k \times \mathbb{R}^4/\mathbb{Z}_n\) since the configuration with \(n\) NS5-brane and \(k\) \((1,1)\) 5-branes becomes equivalent to the configuration of \(n\) NS5-brane and \(k\) D5-branes by the shift \(\tau \to \tau + 1\). The latter configuration is dual to the orthogonal KK-monopoles, hence the singularity is factorized.

As discussed in \[14,15\], we can perform T-duality along one of the \(S^1\) direction, say \(\theta_B\), by using the method of \[17\]. In this duality, the linear-multiplet \((\Psi_B, P_B)\) is replaced by the twisted hypermultiplet \((\Psi_B, \Theta)\) where \(\Theta\) is a twisted chiral multiplet in the \(\mathcal{N} = 2\) language. The resulting model describes the configuration of \(n\) NS5-branes intersecting with KK-monopoles. As argued in \[16,14,15\], the low energy effective action receives instanton corrections, which leads to the localization of brane positions along the \(S^1\) direction. It would be interesting to study such instanton corrections in our model.

2.2. \textit{M-theory Dual of }\(n\) NS5-branes and one \((k,1)\) 5-brane

Next we consider the GLSM for the configuration of \(n\) NS5-branes and one \((k,1)\) 5-brane. This is obtained by replacing the \(A\)-part in the previous subsection with the following model of single \(U(1)_A\) gauge symmetry: one hypermultiplet with charge 1 under the gauge group \(U(1)_A\), and the linear-multiplet with shift charge \(k\) under \(U(1)_A\). The
linear multiplet in the $B$-part is charged under the diagonal of $U(1)_A \times \prod_{i=1}^n U(1)_{B,i}$. The D-term for the linear multiplet reads

$$L_D^{\text{linear}} = \int d^4 \theta \frac{1}{g_A^2} \Psi_A^\dagger \Psi_A + \frac{g_A^2}{2} \left( P_A + P_A^\dagger + k V_A \right)^2$$

$$+ \frac{1}{g_B^2} \Psi_B^\dagger \Psi_B + \frac{g_B^2}{2} \left( P_B + P_B^\dagger + V_A + \sum_{i=1}^n V_i \right)^2$$

(2.21)

where $V_A$ is the vector superfield for the gauge group $U(1)_A$. After a similar analysis as in the previous subsection, we find that the effective metric on the Higgs branch has the same form as (2.18) with

$$U = \begin{pmatrix} \frac{1}{g_A^2} + k^2 H & k H \\ k H & \frac{1}{g_B^2} + K + H \end{pmatrix}, \quad H = \frac{1}{2|k \vec{x}_A + \vec{x}_B|}, \quad K = \frac{1}{2} \sum_{i=1}^n \frac{1}{|\vec{x}_B - \vec{r}_i|}$$

$$\beta_A = \partial \theta_A - \frac{k}{2} \vec{\omega} \cdot (k \partial \vec{x}_A + \partial \vec{x}_B), \quad \beta_B = \partial \theta_B - \frac{1}{2} \vec{\omega} \cdot (k \partial \vec{x}_A + \partial \vec{x}_B) - \frac{1}{2} \sum_{i=1}^n \vec{r}_i \cdot \partial \vec{x}_B.$$  

(2.22)

By the similar analysis as in [11], we find that the metric has the orbifold singularity $\mathbb{C}^4/\Gamma$, where $\Gamma$ is generated by $g_1$ and $g_2$

$$g_1 : (z_1, z_2, z_3, z_4) \sim (e^{\frac{2\pi i}{k}} z_1, e^{-\frac{2\pi i}{k}} z_2, e^{\frac{2\pi i}{k}} z_3, e^{-\frac{2\pi i}{k}} z_4)$$

$$g_2 : (z_1, z_2, z_3, z_4) \sim (e^{\frac{2\pi i}{n}} z_1, e^{\frac{2\pi i}{n}} z_3, e^{-\frac{2\pi i}{n}} z_4).$$  

(2.23)

In particular, the singularity for the $n = 1$ case is $\mathbb{R}^8/\mathbb{Z}_k$ [11]. For the general case, (2.23) is in agreement with [13].

3. ALE-type GLSM (or Quiver Gauge Theory)

In this section, we will consider the ALE analogue of the model. The ALE-type GLSM can be obtained from the ALF-type cousin studied in section 2.1. Let us first consider the $A$-part. We replace the hypermultiplet $(Q_a, \tilde{Q}_a)$ charged under $U(1)_{A,a}$ by the “bi-fundamental” hypermultiplet charged under $U(1)_{A,a} \times U(1)_{A,a+1}$. In order to have the $A_{k-1}$ model, we have to reduce the number of hypermultiplets by one, i.e. $a$ runs from 2 to $k$. We should also promote the linear-multiplet to a “bi-fundamental” hypermultiplet.
charged under $U(1)_{A,1} \times U(1)_{A,2}$. Then the gauge field appearing the Lagrangian (2.3) is replaced as

$$A_a \rightarrow A_a - A_{a+1}$$

$$\sum_{a=1}^{k} A_a \rightarrow \sum_{a=2}^{k} (A_a - A_{a+1}) = A_2 - A_1$$

(3.1)

where we identified $k + 1 \equiv 1$. Then the resulting theory is described by the $\widehat{A}_{k-1}$ Dynkin diagram. Note that the link between the node 1 and node 2 represents the hypermultiplet coming from the linear-multiplet in the ALF-type model in the previous section.

We can do the same replacement in the $B$-part. Then we get a matter content specified by the $\widehat{A}_{n-1}$ Dynkin diagram. However, there is an important difference for the link between the node 1 and node 2 from the rest of the links. Since the linear-multiplet for the $B$-part is charged under the gauge field $\sum_a A_a + \sum_i B_i$ for the ALF case, this becomes a hypermultiplet in the ALE model charged under the gauge field

$$\sum_{a=1}^{k} A_a + \sum_{i=1}^{n} B_i \rightarrow \sum_{a=2}^{k} (A_a - A_{a+1}) + \sum_{i=2}^{n} (B_i - B_{i+1}) = A_2 - A_1 + B_2 - B_1.$$ 

(3.2)

Therefore, the hypermultiplet on the link between the node 1 and node 2 in the $B$-part is charged under $U(1)_{A,1} \times U(1)_{A,2} \times U(1)_{B,1} \times U(1)_{B,2}$.

---

**Fig. 1:** The quiver diagram for the ALE-type GLSM is a union of the $\widehat{A}_{k-1}$ Dynkin diagram (labeled $A$, black) and the $\widehat{A}_{n-1}$ Dynkin diagram (labeled $B$, blue). The link between the node 1 and 2 in the diagram $B$ (the dashed line between the node 1 and 2) is charged under $U(1)_{A,1} \times U(1)_{A,2} \times U(1)_{B,1} \times U(1)_{B,2}$, while the link between the node 1 and 2 in the diagram $A$ (the solid line between the node 1 and 2), is charged only under $U(1)_{A,1} \times U(1)_{A,2}$. 

8
The resulting matter content of the ALE-type GLSM is summarized by the quiver diagram in Fig. 1. Namely, the quiver diagram of our theory is a union of two $\hat{A}$ Dynkin diagrams overlapping at the link between the node 1 and node 2. The only difference from the usual ALE quiver is that the link between the node 1 and 2 in the diagram $B$ is charged under both $U(1)_{B,1} \times U(1)_{B,2}$ and $U(1)_{A,1} \times U(1)_{A,2}$. Other links in the diagram $A$ (resp. diagram $B$) are charged only under the gauge group $U(1)_{A,a} \times U(1)_{A,a+1}$ (resp. $U(1)_{B,i} \times U(1)_{B,i+1}$).

3.1. Singularity of the Higgs branch

Now we consider the singularity of the moduli space. It is straightforward to study the low energy effective metric on the Higgs branch as in the previous section. The resulting metric is not the one obtained from the ALF case [2.15] by setting the constant part $U_\infty$ of the matrix $U$ to zero. Instead of analyzing the metric, let us consider the singularity from the complex viewpoint by looking at the F-term constraints for the Higgs branch:

\begin{align}
q_{a,a+1} \bar{q}_{a+1,a} - q_{a-1,a} \bar{q}_{a,a-1} &= \mu_a \quad (a = 3, \ldots, k) \\
q_{1,2} \bar{q}_{1,2} - q_{k,1} \bar{q}_{k,1} + h_{1,2} \bar{h}_{1,2} &= \mu_1 \\
q_{2,3} \bar{q}_{2,3} - q_{1,2} \bar{q}_{1,2} - h_{1,2} \bar{h}_{1,2} &= \mu_2 \\
h_{i,i+1} \bar{h}_{i+1,i} - h_{i-1,i} \bar{h}_{i,i-1} &= \zeta_i \quad (i = 1, \ldots n),
\end{align}

(3.3)

where $\mu_a$ and $\zeta_i$ are the complex FI-parameters. For the consistency of these relations, the FI-parameters should satisfy

\begin{equation}
\sum_{a=1}^{k} \mu_a = \sum_{i=1}^{n} \zeta_i = 0. \quad (3.4)
\end{equation}

Then the equations (3.3) can be solved as

\begin{align*}
q_{1,2} \bar{q}_{1,2} &= u - v, \\
h_{1,2} \bar{h}_{1,2} &= v, \\
q_{a,a+1} \bar{q}_{a+1,a} &= u + c_a \quad (a = 2, \ldots k), \\
c_a &= \sum_{b=2}^{a} \mu_b, \\
h_{i,i+1} \bar{h}_{i+1,i} &= v + d_i \quad (i = 2, \ldots n), \\
d_i &= \sum_{j=2}^{i} \zeta_j.
\end{align*}

(3.5)
By introducing the baryonic operators

\[ x = q_1,2q_2,3 \cdots q_{k,1} \]
\[ y = \tilde{q}_2,1\tilde{q}_3,2 \cdots \tilde{q}_{1,k} \]
\[ z = h_{1,2}h_{2,3} \cdots h_{n,1} \]
\[ w = \tilde{h}_{2,1}\tilde{h}_{3,2} \cdots \tilde{h}_{1,n} \]

the vacuum moduli space is written as

\[
\begin{cases}
  xy = (u - v) \prod_{a=2}^{k} (u + c_a), \\
  zw = v \prod_{i=2}^{n} (v + d_i).
\end{cases}
\]

This moduli space becomes singular when we set some of the FI-parameters to zero. The most singular case occurs when all FI parameters are zero. In this case, the moduli space becomes

\[
\begin{cases}
  xy = (u - v)u^{k-1}, \\
  zw = v^n.
\end{cases}
\]

To see the nature of the singularity of (3.8), let us recall the case of 4-dimensional $A_{k-1}$ singularity described by the equation

\[ xy = u^k. \]

This equation can be parametrized by the two complex numbers $z_1, z_2 \in \mathbb{C}$

\[ x = z_1^k, \quad y = z_2^k, \quad u = z_1z_2. \]

This parametrization of the variety (3.9) by $(z_1, z_2) \in \mathbb{C}^2$ is $k$ to 1, hence we have to mod out by the $\mathbb{Z}_k$ identification

\[ (z_1, z_2) \sim (e^{2\pi i/k}z_1, e^{-2\pi i/k}z_2). \]

Therefore, (3.9) has $\mathbb{C}^2/\mathbb{Z}_k$ singularity at the origin.

Now we go back to the analysis of the singularity of (3.8). Let us first consider the case $n = 1$. Strictly speaking, the $n = 1$ case does not follow from the quiver gauge theory, since we need two distinguished nodes in order to connect two $\tilde{A}$ Dynkin diagrams, which
implies \( k, n \geq 2 \). However, we can formally set \( n = 1 \) in the equation (3.8) without asking where it comes from. When \( n = 1 \), the moduli space (3.8) becomes

\[ xy = (u - zw)u^{k-1}. \]  

(3.12)

When \( zw \neq 0 \), there is a \( \mathbb{C}^2/\mathbb{Z}_{k-1} \) singularity at \( x = y = u = 0 \). When \( z \) or \( w \) vanishes, the singularity at \( x = y = u = 0 \) is enhanced to \( \mathbb{C}^2/\mathbb{Z}_k \). Let us consider the singularity at the origin \( x = y = z = w = 0 \). In analogy with the \( A_{k-1} \) ALE space reviewed in the previous paragraph, we parametrize (3.12) as

\[ x = z_1^k, \quad y = z_2^k, \quad z = z_1 z_4, \quad w = z_2 z_3, \quad u = z_1 z_2 t. \]  

(3.13)

Then the equation (3.12) becomes

\[ 1 = (t - z_3 z_4)t^{k-1}. \]  

(3.14)

Since this space is regular, (3.14) does not introduce any constraint on the variables \( z_3 \) and \( z_4 \). Therefore, the space (3.12) is parametrized by \( (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \) with the identification

\[ (z_1, z_2, z_3, z_4) \sim \left(e^{\frac{2\pi i}{k}} z_1, e^{-\frac{2\pi i}{k}} z_2, e^{\frac{2\pi i}{k}} z_3, e^{-\frac{2\pi i}{k}} z_4\right). \]  

(3.15)

Namely, the space (3.12) has the orbifold singularity \( \mathbb{C}^4/\mathbb{Z}_k \) at the origin.

Similarly, we can analyze the singularity of (3.8) for the case \( n \geq 2 \) by rewriting \( (x, y, z, w, u, v) \) as

\[ x = z_1^k, \quad y = z_2^k, \quad z = (z_1 z_4)^n, \quad w = (z_2 z_3)^n, \quad u = z_1 z_2 t, \quad v = z_1 z_2 z_3 z_4. \]  

(3.16)

Again, the equation for the moduli space (3.8) reduces to the regular equation (3.14). Therefore, the moduli space (3.8) is parametrized by \( (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \) with the identification

\[ \mathbb{Z}_k : \quad (z_1, z_2, z_3, z_4) \sim \left(e^{\frac{2\pi i}{k}} z_1, e^{-\frac{2\pi i}{k}} z_2, e^{\frac{2\pi i}{k}} z_3, e^{-\frac{2\pi i}{k}} z_4\right), \]

\[ \mathbb{Z}_n : \quad (z_1, z_2, z_3, z_4) \sim \left(z_1, z_2, e^{\frac{2\pi i}{n}} z_3, e^{-\frac{2\pi i}{n}} z_4\right). \]  

(3.17)

Namely, the moduli space (3.8) has the orbifold singularity \( \mathbb{C}^4/(\mathbb{Z}_k \times \mathbb{Z}_n) \) at the origin. The moduli space (3.7) with generic FI parameters \( c_a, d_i \neq 0 \) can be thought of as a hyperKähler resolution of the orbifold \( \mathbb{C}^4/(\mathbb{Z}_k \times \mathbb{Z}_n) \).

**Acknowledgment**

This work is supported in part by MEXT Grant-in-Aid for Scientific Research #19740135.
References

[1] S. W. Hawking, “Gravitational Instantons,” Phys. Lett. A 60, 81 (1977).
[2] G. W. Gibbons and S. W. Hawking, “Classification Of Gravitational Instanton Symmetries,” Commun. Math. Phys. 66, 291 (1979).
[3] T. Eguchi and A. J. Hanson, “Asymptotically Flat Selfdual Solutions To Euclidean Gravity,” Phys. Lett. B 74, 249 (1978).
[4] P. Kronheimer, “The construction of ALE spaces as hyper-kahler quotients,” J. Diff. Geom. 28, 665 (1989); “A Torelli-Type Theorem for Gravitational Instantons,” J. Diff. Geom. 29, 685 (1989).
[5] G. W. Gibbons and P. Rychenkova, “HyperKaehler quotient construction of BPS monopole moduli spaces,” Commun. Math. Phys. 186, 585 (1997) [arXiv:hep-th/9608083].
[6] J. P. Gauntlett, G. W. Gibbons, G. Papadopoulos and P. K. Townsend, “Hyper-Kaehler manifolds and multiply intersecting branes,” Nucl. Phys. B 500, 133 (1997) [arXiv:hep-th/9702202].
[7] J. Bagger and N. Lambert, “Comments On Multiple M2-branes,” JHEP 0802, 105 (2008) [arXiv:0712.3738 [hep-th]].
[8] J. Bagger and N. Lambert, “Gauge Symmetry and Supersymmetry of Multiple M2-Branes,” Phys. Rev. D 77, 065008 (2008) [arXiv:0711.0955 [hep-th]].
[9] J. Bagger and N. Lambert, “Modeling multiple M2’s,” Phys. Rev. D 75, 045020 (2007) [arXiv:hep-th/0611108].
[10] A. Gustavsson, “Algebraic structures on parallel M2-branes,” [arXiv:0709.1260 [hep-th]].
[11] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, “N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” arXiv:0806.1218 [hep-th].
[12] M. Benna, I. Klebanov, T. Klose and M. Smedback, “Superconformal Chern-Simons Theories and AdS4/CFT3 Correspondence,” arXiv:0806.1519 [hep-th].
[13] Y. Imamura and K. Kimura, “Coulomb branch of generalized ABJM models,” arXiv:0806.3727 [hep-th].
[14] J. A. Harvey and S. Jensen, “Worldsheet instanton corrections to the Kaluza-Klein monopole,” JHEP 0510, 028 (2005) [arXiv:hep-th/0507204].
[15] K. Okuyama, “Linear sigma models of H and KK monopoles,” JHEP 0508, 089 (2005) [arXiv:hep-th/0508097].
[16] D. Tong, “NS5-branes, T-duality and worldsheet instantons,” JHEP 0207, 013 (2002) [arXiv:hep-th/0204180].
[17] M. Rocek and E. P. Verlinde, “Duality, quotients, and currents,” Nucl. Phys. B 373, 630 (1992) [arXiv:hep-th/9110053].
[18] E. Witten, “Phases of \( N = 2 \) theories in two dimensions,” Nucl. Phys. B 403, 159 (1993) [arXiv:hep-th/9301042].

[19] S. R. Coleman, “There are no Goldstone bosons in two-dimensions,” Commun. Math. Phys. 31, 259 (1973).