Asymptotic Expansions for the Lagrangian Trajectories from Solutions of the Navier–Stokes Equations

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Received: 9 April 2020 / Accepted: 25 July 2020
Published online: 16 September 2020 – © Springer-Verlag GmbH Germany, part of Springer Nature 2020

Dedicated to the memory of Ciprian Foias (1933–2020)

Abstract: Consider any Leray–Hopf weak solution of the three-dimensional Navier–Stokes equations for incompressible, viscous fluid flows. We prove that any Lagrangian trajectory associated with such a velocity field has an asymptotic expansion, as time tends to infinity, which describes its long-time behavior very precisely.

1. Introduction

We study the long-time dynamics of the incompressible, viscous fluid flows in the three-dimensional space. Theoretically speaking, there are two standard descriptions of fluid flows. One is the Lagrangian that is based on the trajectory \(x(t) \in \mathbb{R}^3\) of each initial fluid particle (or material point) \(x_0 = x(0)\), where \(t\) is the time variable. The other is the Eulerian which uses the velocity field \(u(x, t)\) and pressure \(p(x, t)\), where \(x \in \mathbb{R}^3\) is the independent spatial variable representing each fixed position in the fluid. The relation between the two descriptions is the following ordinary differential equations (ODE)

\[ x' = u(x, t). \]  

The solutions \(x(t)\) of (1.1) are called the Lagrangian trajectories.

The Eulerian description turns out to be simpler for deriving the set of equations that govern the fluid flows. They are called the Navier–Stokes equations (NSE),

\[
\begin{cases}
    u_t - \nu \Delta u + (u \cdot \nabla) u = -\nabla p, \\
    \text{div} \ u = 0.
\end{cases}
\]  

where \(\nu > 0\) is the kinematic viscosity, and the unknowns are the velocity \(u(x, t)\) and pressure \(p(x, t)\). For a solution \((u, p)\), we will conveniently say \(u(x, t)\) is a solution of (1.2).

The system (1.2) is subject to the initial condition \(u(x, 0) = u_0(x)\), where \(u_0\) is a given initial vector field.
From the mathematical point of view, the NSE is a system of nonlinear partial differential equations, and its understanding is still lacking. Even the basic question about its existence and uniqueness has not been answered completely. Because of this lack of information about the velocity $u(x, t)$ in (1.1), the analysis of the Lagrangian trajectories $x(t)$ is very limited. There have been results for the Lagrangian trajectories in small time intervals. See recent work [1,2,7–9,19,27] and references therein for short-time well-posedness, regularity, and analyticity, based on solutions of the Euler or Navier–Stokes related systems. See also [24] for studies of the topological structures of the flows in the two-dimensional case. Naturally, the long-term behavior of the Lagrangian trajectories is even less-known.

However, thanks to the remarkable result by Foias and Saut [17], the long-time behavior of a solution of the NSE, under the current consideration, can be described completely, bypassing its yet unknown global well-posedness. This is part of the Foias–Saut theory of asymptotic expansions and their associated normal form and nonlinear spectral manifolds for the NSE. See their work [15–18], which were developed further or extended in [4,5,10–12,20–22,25,26]. (The interested reader is referred to [13] for a brief survey on the subject.) The goal of this paper is to investigate (1.1) in this direction in order to gain knowledge of precise long-time dynamics of fluid flows in the Lagrangian description.

First, we recall the type of asymptotic expansions, as time tends to infinity, studied here as well as in previous work.

**Definition 1.1.** Let $(X, \| \cdot \|_X)$ be a normed space and $(\alpha_n)_{n=1}^\infty$ be a sequence of strictly increasing non-negative real numbers. A function $f : [T, \infty) \to X$, for some $T \geq 0$, is said to have an asymptotic expansion

$$f(t) \sim \sum_{n=1}^\infty f_n(t) e^{-\alpha_n t} \text{ in } X,$$

where each $f_n : \mathbb{R} \to X$ is a polynomial, if one has, for any $N \geq 1$, that

$$\| f(t) - \sum_{n=1}^N f_n(t) e^{-\alpha_n t} \|_X = O(e^{-(\alpha_N + \varepsilon_N) t})$$

as $t \to \infty$, for some $\varepsilon_N > 0$.

Clearly, if (1.4) holds for some $N \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and some polynomials $f_n$’s, for $n = 1, \ldots, N$, then those $f_n$’s are unique. Consequently, the polynomials $f_n$’s, for all $n \in \mathbb{N}$, in Definition 1.1 are unique. Moreover, in the case $X$ is finite dimensional, all of its norms are equivalent, hence, the expansion (1.3) is the same for any norm on $X$. Also, in many cases, the norm $\| \cdot \|_X$ is a standard and well-known one, hence will be implicitly understood.

Returning to the NSE, it was proved by Foias and Saut [17] that any solution $u(x, t)$ of the NSE processes an asymptotic expansion of type (1.3). Our goal is to establish the same result for solutions of (1.1), where $u(x, t)$ is a Leray–Hopf weak solution of the NSE (1.2). Indeed, we prove in Theorems 2.2 and 2.4 that when $u(x, t)$ satisfies the no-slip boundary condition, or is a spatially periodic solution with zero average, then system (1.1) has a solution $x(t)$, for sufficiently large $t$, which admits an asymptotic expansion in $\mathbb{R}^3$. The starting point is a simple realization in Proposition 2.1 that each trajectory $x(t)$ converges exponentially, as $t \to \infty$. The general case of spatially periodic solutions is treated in Sect. 3.
Our obtained results give very precise long-time dynamics for the Lagrangian trajectories for general weak solutions of the NSE. They contrast with the papers cited above which only yield short-time properties. Moreover, our approach draws strong conclusions with relatively simple proofs.

The rest of this section is focused on preliminaries. We consider the NSE (1.2) in one of the following two specified situations.

**Dirichlet boundary condition (DBC).** Let $\Omega$ be an bounded, open, connected set in $\mathbb{R}^3$ with $C^\infty$ boundary. We consider (1.2) in $\Omega \times (0, \infty)$ with the boundary condition $u = 0$ on $\partial\Omega \times (0, \infty)$.

**Spatial periodicity condition (SPC).** Fix a vector $L = (L_1, L_2, L_3) \in (0, \infty)^3$. We consider (1.2) in $\mathbb{R}^3 \times (0, \infty)$ with $u(\cdot, t)$ and $p(\cdot, t)$ being $L$-periodic for $t > 0$.

Here, a function $g$ defined on $\mathbb{R}^3$ is called $L$-periodic if

$$g(x + L_ie_i) = g(x) \text{ for } i = 1, 2, 3 \text{ and all } x \in \mathbb{R}^3,$$

where $\{e_1, e_2, e_3\}$ is the standard canonical basis of $\mathbb{R}^3$.

Define domain $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$ in this case. A function $g$ is said to have zero average over $\Omega$ if

$$\int_\Omega g(x) dx = 0.$$

We recall some needed basic elements from the theory of the NSE. For details, the reader is referred to the books [6,14,28,29]. Below, $H^m = W^{m,2}$, for $m \in \mathbb{N}$, denote the standard Sobolev spaces.

In the (DBC) case, let $\mathcal{V}$ be the set of divergence-free vector fields in $C^\infty_c(\Omega^3)$. Define $\mathcal{X}$ to be the set of functions in $\bigcap_{m=1}^\infty H^m(\Omega^3)$ that are divergence-free and vanish on the boundary $\partial\Omega$, and denote $\Omega^* = \Omega$. Let $L^2(\Omega) = L^2(\Omega^3)$ and $H^m(\Omega) = H^m(\Omega^3)$.

In the (SPC) case, let $\mathcal{V}$ be the set of $L$-periodic trigonometric polynomial vector fields on $\mathbb{R}^3$ which are divergence-free and have zero average over $\Omega$. Define $\mathcal{X} = \mathcal{V}$, and denote $\Omega^* = \mathbb{R}^3$. Let $L^2(\Omega)$ (respectively, $H^m(\Omega)$) be the space of $L$-periodic vector fields on $\mathbb{R}^3$ that belong to $L^2_{\text{loc}}(\mathbb{R}^3)$ (respectively, $H^m_{\text{loc}}(\mathbb{R}^3)$), and is endowed with the inner product and norm of $L^2(\Omega)$ (respectively, $H^m(\Omega)$).

In both cases, define space $H$ (respectively, $V$) to be the closure of $\mathcal{V}$ in $L^2(\Omega)$ (respectively, $\mathcal{V}$). The Leray projection $\mathbb{P}$ is the orthogonal projection from $L^2(\Omega)$ to $H$. The Stokes operator is $(-\mathbb{P} \Delta)$ defined on $V \cap H^1(\Omega)$.

Denote the spectrum of Stokes operator by $\Lambda = \{\Lambda_k : k \in \mathbb{N}\}$, where $\Lambda_k$’s are positive, strictly increasing to infinity. Let $\mathcal{S}$ be the additive semigroup generated by $\nu \Lambda_k$’s, that is,

$$\mathcal{S} = \left\{ \nu \sum_{j=1}^N \Lambda_k : N, k_1, \ldots, k_N \in \mathbb{N} \right\}.$$

We arrange the set $\mathcal{S}$ as a sequence $(\mu_n)_{n=1}^\infty$ of positive, strictly increasing numbers. Clearly,

$$\lim_{n \to \infty} \mu_n = \infty, \quad \mu_n + \mu_k \in \mathcal{S} \quad \forall n, k \in \mathbb{N}.$$

For convenience, we will write $f(t) = g(t) + O(h(t))$ to indicate

$$|f(t) - g(t)| = O(h(t)) \text{ as } t \to \infty.$$
2. Main Results

For any \( u_0 \in H \), there exists a Leray–Hopf weak solution \( u(x, t) \) of (1.2) on \([0, \infty)\) with initial condition \( u(x, 0) = u_0(x) \). By its eventual regularity, there is \( T \geq 0 \) such that \( u \in C^\infty(\Omega^* \times [T, \infty)) \) and satisfies the corresponding (DBC) or (SPC).

(A) Throughout this section, let us fix such a Leray–Hopf weak solution \( u(x, t) \) and a Lagrangian trajectory \( x(t) \in C^1([T, \infty), \Omega) \) in the (DBC) case, or \( x(t) \in C^1([T, \infty), \mathbb{R}^3) \) in the (SPC) case.

A discussion about assumption (A) is given in Remark 2.5 below.

It is proved in [17] that the solution \( u(x, t) \) has an asymptotic expansion, in the sense of Definition 1.1,

\[
    u(\cdot, t) \sim \sum_{n=1}^{\infty} q_n(\cdot, t)e^{-\mu_n t} \text{ in } L^m(\Omega), \tag{2.1}
\]

for any \( m \in \mathbb{N} \), where \( q_j(\cdot, t)'s \) are polynomials in \( t \) with values in \( \mathcal{X} \subset C^\infty(\Omega^*)^3 \).

One can write each polynomial \( q_n(x, t) \), for \( n \geq 1 \), explicitly as

\[
    q_n(x, t) = \sum_{k=0}^{d_n} t^k q_{n,k}(x), \quad \text{where } d_n \geq 0, \text{ and } q_{n,k} \in \mathcal{X}. \tag{2.2}
\]

In fact, \( q_1(x, t) \) is independent of \( t \), hence we write

\[
    q_1(x, t) = q_1(x) \in \mathcal{X}. \tag{2.3}
\]

According to the expansion (2.1) with \( m = 2 \) and Definition 1.1, we have

\[
    \left\| u(\cdot, t) - \sum_{n=1}^{N} q_n(\cdot, t)e^{-\mu_n t} \right\|_{H^2(\Omega)^3} = \mathcal{O}(e^{-(\mu_N+\delta_N)t}),
\]

for any \( N \in \mathbb{N} \), and some \( \delta_N > 0 \).

By Morrey’s embedding theorem, it follows that

\[
    \sup_{x \in \Omega^*} \left| u(x, t) - \sum_{n=1}^{N} q_n(x, t)e^{-\mu_n t} \right| = \mathcal{O}(e^{-(\mu_N+\delta_N)t}). \tag{2.4}
\]

In particular, letting \( N = 1 \) in (2.4) and using the fact (2.3), we infer

\[
    \sup_{x \in \Omega^*} |u(x, t)| \leq \sup_{x \in \Omega^*} |q_1(x)|e^{-\mu_1 t} + \mathcal{O}(e^{-(\mu_1+\delta_1)t}) = \mathcal{O}(e^{-\mu_1 t}).
\]

Therefore, there is \( C_0 > 0 \) such that

\[
    \sup_{x \in \Omega^*} |u(x, t)| \leq C_0 e^{-\mu_1 t} \text{ for all } t \geq T. \tag{2.5}
\]

Taking \( x = x(t) \) in (2.4) and (2.5), one has

\[
    \left| u(x(t), t) - \sum_{n=1}^{N} q_n(x(t), t)e^{-\mu_n t} \right| = \mathcal{O}(e^{-(\mu_N+\delta_N)t}), \tag{2.6}
\]

\[
    |u(x(t), t)| \leq C_0 e^{-\mu_1 t} \text{ for all } t \geq T. \tag{2.7}
\]
Proposition 2.1. The limit \( x_* \stackrel{\text{def}}{=} \lim_{t \to \infty} x(t) \) exists and belongs to \( \Omega^* \), and
\[
|x(t) - x_*| = \mathcal{O}(e^{-\mu_1 t}). \tag{2.8}
\]

Proof. For \( t \geq T \), we have
\[
x(t) = x(T) + \int_T^t u(x(\tau), \tau) \, d\tau.
\tag{2.9}
\]

It follows (2.9) and estimate (2.7) that
\[
x_* = \lim_{t \to \infty} x(t) = x(T) + \int_T^\infty u(x(\tau), \tau) \, d\tau \quad \text{which exists in } \mathbb{R}^3. \tag{2.10}
\]

Obviously, \( x_* \in \Omega^* \). By (2.9), (2.10), and (2.7) again, we obtain, for \( t \geq T \),
\[
|x(t) - x_*| = \left| \int_T^\infty u(x(\tau), \tau) \, d\tau \right| \leq \int_T^\infty C_0 e^{-\mu_1 t} \, d\tau = C_0 \mu_1^{-1} e^{-\mu_1 t},
\]
which proves (2.8). \( \square \)

Notation. For \( x \in \mathbb{R}^3 \), denote \( x^{(0)} = 1 \), and by \( x^{(k)} \) the \( k \)-tuple \((x, \ldots, x)\) for \( k \geq 1 \).

If \( m \in \mathbb{N} \) and \( L \) is an \( m \)-linear mapping from \((\mathbb{R}^3)^m\) to \( \mathbb{R}^3 \), the norm of \( L \) is defined by
\[
\|L\| = \max\{|L(y_1, y_2, \ldots, y_m)| : y_j \in \mathbb{R}^3, |y_j| = 1, \text{ for } 1 \leq j \leq m\}.
\]

It is known that the norm \( \|L\| \) belongs to \([0, \infty) \), and one has
\[
|L(y_1, y_2, \ldots, y_m)| \leq \|L\| \cdot |y_1| \cdot |y_2| \cdots |y_m| \quad \forall y_1, y_2, \ldots, y_m \in \mathbb{R}^n. \tag{2.11}
\]

Below are consequences of expansion (2.1) and Proposition 2.1. Let \( x_* \) be as in Proposition 2.1.

Consideration I: The (SPC) case, or \( x_* \in \Omega \) in the (DBC) case.

Consideration II: The (DBC) case with \( x_* \in \partial \Omega \).

We focus on Consideration I first. By using the Taylor expansion for each \( q_{n,k}(x) \), see e.g. [23, Chapter XVI, Sect. 6], we obtain, for any \( s \geq 0 \),
\[
q_{n,k}(x) = \sum_{m=0}^{s} \frac{1}{m!} D_x^m q_{n,k}(x_*)(x - x_*)^{(m)} + g_{n,k,s}(x), \tag{2.12}
\]

where \( D_x^m q_{n,k} \) denotes the \( m \)-th order derivative of \( q_{n,k} \), and \( g_{n,k,s} \in C(\Omega^*)^3 \) satisfying
\[
g_{n,k,s}(x) = \mathcal{O}(|x - x_*|^{s+1}) \text{ as } x \to x_* \tag{2.13}
\]

Here, \( D_x^m q_{n,k} \) is \( q_{n,k} \) for \( m = 0 \), and is an \( m \)-linear mapping from \((\mathbb{R}^3)^m\) to \( \mathbb{R}^3 \), for \( m \geq 1 \).

Substituting (2.12) into (2.2) gives
\[
q_n(x, t) = \sum_{k=0}^{d_n} t_k \left[ \sum_{m=0}^{s} \frac{1}{m!} D_x^m q_{n,k}(x_*)(x - x_*)^{(m)} + g_{n,k,s}(x) \right].
\]
which can be rewritten as

\[ q_n(x, t) = \sum_{m=0}^{s} Q_{n,m}(x_*, t)(x - x_*)^{(m)} + \sum_{k=0}^{d_n} t^k g_{n,k,s}(x), \]  \(2.14\)

where

\[ Q_{n,m}(x_*, t) = \sum_{k=0}^{d_n} \frac{t^k}{m!} D_x^m q_{n,k}(x_*) = \frac{1}{m!} D_x^m q_n(x_*, t). \]  \(2.15\)

In particular,

\[ Q_{n,0}(x_*, t) = q_n(x_*, t), \quad Q_{n,1}(x_*, t) = D_x q_n(x_*, t), \quad Q_{n,2}(x_*, t) = \frac{1}{2} D_x^2 q_n(x_*, t). \]  \(2.16\)

Note from \(2.15\) that \(Q_{n,m}(x_*, t)\) is a polynomial in \(t\) valued in the space of \(m\)-linear mappings from \((\mathbb{R}^3)^m\) to \(\mathbb{R}^3\). Therefore, one has, for any \(k \geq 1\) and \(m \geq 0\),

\[ \|Q_{k,m}(x_*, t)\| = \mathcal{O}(e^{\delta t}) \quad \forall \delta > 0. \]  \(2.17\)

Denote \(z(t) = x(t) - x_*\). Then \(2.8\) reads as

\[ |z(t)| = \mathcal{O}(e^{-\mu_1 t}). \]  \(2.18\)

Combining \(2.14\) for \(x = x(t)\) with \(2.13\) and \(2.18\) yields

\[ q_n(x(t), t) = \sum_{m=0}^{s} Q_{n,m}(x_*, t)z(t)^{(m)} + \sum_{k=0}^{d_n} t^k \mathcal{O}(e^{-\mu_1(s+1)t}), \]

thus

\[ q_n(x(t), t) = \sum_{m=0}^{s} Q_{n,m}(x_*, t)z(t)^{(m)} + \mathcal{O}(e^{-\mu_1(s+1)t - \delta t}) \quad \forall \delta > 0. \]  \(2.19\)

Our main result is the following.

**Theorem 2.2.** Under Consideration I, there exist polynomials \(\zeta_n : \mathbb{R} \to \mathbb{R}^3\), for \(n \geq 0\), such that solution \(x(t)\) has an asymptotic expansion, in the sense of Definition 1.1,

\[ x(t) \sim x_* + \sum_{n=1}^{\infty} \zeta_n(t) e^{-\mu_n t} \text{ in } \mathbb{R}^3, \]  \(2.20\)

where each \(\zeta_n\), for \(n \geq 1\), is the unique polynomial solution of the following differential equation

\[ \zeta_n'(t) - \mu_n \zeta_n(t) = \sum_{\mu_k + \mu_j_1 + \mu_j_2 + \cdots + \mu_j_m = \mu_n} Q_{k,m}(x_*, t)(\zeta_j_1(t), \ldots, \zeta_j_m(t)). \]  \(2.21\)

for all \(t \in \mathbb{R}\). More explicitly, \(\zeta_n(t)\) can be calculated recursively by formula \(2.29\) below.
Before proving Theorem 2.2, we explain the formulas appearing there.

(a) Formula (2.21) is the concise form of the following

\[
\zeta_n(t) = q_n(x_s, t) + \sum_{m, k, j_1, \ldots, j_m \geq 1, \mu_k + \mu_{j_1} + \cdots + \mu_{j_m} = \mu_n} Q_{k, m}(x_s, t)(\zeta_{j_1}(t), \ldots, \zeta_{j_m}(t))
\]

(2.22)

Indeed, when \( m = 0 \), the indices \( j_1, \ldots, j_m \), numbers \( \mu_{j_1}, \ldots, \mu_{j_m} \), and functions \( \zeta_{j_1}(t), \ldots, \zeta_{j_m}(t) \) are not needed in (2.21), then \( \mu_k = \mu_n \), which implies \( k = n \), and \( Q_{k, m}(x_s, t)(\zeta_{j_1}(t), \ldots, \zeta_{j_m}(t)) \) is just \( q_n(x_s, t) \). When \( m \geq 1 \), the indices \( j_1, \ldots, j_m \) are present in (2.21) and are in \( \mathbb{N} \).

(b) The sum on the right-hand side of (2.21) is a finite one. Indeed, when \( m = 0 \), one has, again, \( k = n \). Consider \( m \geq 1 \). Since \( \mu_k, \mu_{j_1}, \ldots, \mu_{j_m} \geq \mu_1 \), we have

\[
(1 + m)\mu_1 \leq \mu_k + \mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_m} = \mu_n,
\]

\[
m_k, \mu_{j_1}, \ldots, \mu_{j_m} < \mu_n.
\]

Thus, \( m \leq \mu_n / \mu_1 - 1 \) and \( j_1, \ldots, j_m \leq n - 1 \). (Observe that \( k \leq n \) for both cases of \( m \).) Therefore, we can explicitly rewrite (2.21), via (2.22), as

\[
\zeta_n(t) = q_n(x_s, t) + \sum_{m=1}^{s_n} \sum_{k, j_1, \ldots, j_m=1}^{n-1} Q_{k, m}(x_s, t)(\zeta_{j_1}(t), \ldots, \zeta_{j_m}(t)),
\]

(2.23)

where \( s_n = \min\{s \in \mathbb{N} : s \geq \mu_n / \mu_1 - 1\} \).

(c) To give examples, we write equation (2.21) explicitly, by using the identities in (2.16), for \( n = 1, 2, 3 \) as

\[
\zeta_1(t) - \mu_1 \zeta_1(t) = q_1(x_s),
\]

\[
\zeta_2(t) - \mu_2 \zeta_2(t) = D_x q_1(x_s) \zeta_1(t) + q_2(x_s, t),
\]

\[
\zeta_3(t) - \mu_3 \zeta_3(t) = D_x^2 q_1(x_s) (\zeta_1(t), \zeta_1(t)) + D_x q_2(x_s, t) \zeta_1(t) + q_3(x_s, t).
\]

(2.24)

(d) Equation (2.21) comes from the following approximation lemma. It is the particular Case (iii) of [21, Lemma 4.2], which essentially originates from Foias–Saut’s work [17].

**Lemma 2.3.** Let \((X, \| \cdot \|_X)\) be a Banach space. Let \( p : \mathbb{R} \rightarrow X \) be a polynomial, and \( g \in C([t_*, \infty), X) \), for some \( t_* \geq 0 \), satisfy

\[
\|g(t)\|_X \leq Me^{-\delta t} \quad \forall t \geq t_*, \text{ for some } M, \delta > 0.
\]

Let \( \gamma \) be a positive real number. Suppose that \( y \in C([t_*, \infty), X) \cap C^1((t_*, \infty), X) \) solves the equation

\[
y'(t) - \gamma y(t) = p(t) + g(t) \quad \text{for } t > t_*,
\]

and satisfies

\[
\lim_{t \to \infty} (e^{-\gamma t} \|y(t)\|_X) = 0.
\]

(2.25)
Then there exists a unique polynomial \( q : \mathbb{R} \to X \) such that
\[
\| y(t) - q(t) \|_X \leq \frac{M}{\gamma + \delta} e^{-\delta t} \quad \text{for all } t \geq t_* .
\] (2.26)

More precisely, \( q(t) \) is the unique polynomial solution of
\[
q'(t) - \gamma q(t) = p(t) \quad \text{for } t \in \mathbb{R},
\] (2.27)
and can be explicitly defined by
\[
q(t) = -\int_t^\infty e^{\gamma (t-\tau)} p(\tau) d\tau .
\] (2.28)

In fact, the statements in \[21, \text{Lemma 4.2}\] are for \( t_* = 0 \), but they can be easily generalized and proved for any \( t_* \geq 0 \), see e.g. \[3, \text{Lemma 2.2}\].

(a) For each \( n \in \mathbb{N} \), the right-hand side of (2.21) is an \( \mathbb{R}^3 \)-valued polynomial. Then polynomial solution \( \zeta_n \) exists and is unique, see (2.27) and (2.28) in Lemma 2.3. Explicitly, for \( n \geq 1 \) and \( t \in \mathbb{R} \),
\[
\zeta_n(t) = -\int_t^\infty \mu_n(t-\tau) \left\{ q_n(x_*, \tau) + \sum_{m=1}^{s_n} \sum_{\substack{k, j_1, \ldots, j_m = 1, \mu_k + \mu_j = \mu_n}} Q_{k,m}(x_*, \tau)(\zeta_{j_1}(\tau), \ldots, \zeta_{j_m}(\tau)) \right\} d\tau .
\] (2.29)

In particular, when \( n = 1 \), one has
\[
\zeta_1(t) = -q_1(x_*)/\mu_1 \quad \text{for } t \in \mathbb{R} .
\] (2.30)

(b) We explicate (b) above. One has, for each \( n \geq 1 \), and integers \( M \geq \mu_n/\mu_1 - 1, K \geq n, J \geq n - 1 \), that
\[
\sum_{\mu_k + \mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_m} = \mu_n} Q_{k,m}(x_*, t)(\zeta_{j_1}(t), \ldots, \zeta_{j_m}(t)) = \sum_{m=0}^M \sum_{k=1}^K \sum_{\mu_k + \mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_m} = \mu_n} Q_{k,m}(x_*, t)(\zeta_{j_1}(t), \ldots, \zeta_{j_m}(t)) ,
\] (2.31)
with the right-hand side being interpreted for \( m = 0 \) in the same way as in (a) above.

Indeed, the right-hand side of (2.31) is part of the left-hand side. Reversely, the arguments in (b) show that the left-hand side of (2.31) equals the right-hand side of (2.23), which, in turn, is part of the right-hand side of (2.31). Hence, both sides of (2.31) must be the same.
Proof of Theorem 2.2. The proof follows the general scheme of Foias–Saut [17] with simplified presentation as in [21]. By the virtue of (2.8), it suffices to prove that

\[ z(t) = x(t) - x_* \sim \sum_{n=1}^{\infty} \xi_n(t)e^{-\mu_n t}. \]

To this end, we will prove, by induction, that given any \( N \in \mathbb{N} \), there exists \( \varepsilon_N > 0 \) such that

\[ \left| z(t) - \sum_{n=1}^{N} \xi_n(t)e^{-\mu_n t} \right| = O(e^{-\left(\mu_N + \varepsilon_N\right)t}). \]  

(2.32)

Consider \( N = 1 \). We have from (2.6) that

\[ z'(t) = x'(t) = u(x(t), t) = q_1(x(t))e^{-\mu_1 t} + O(e^{-\left(\mu_1 + \delta_1\right)t}). \]

Writing \( q_1(x(t)) \) by (2.19) for \( n = 1 \), \( s = 0 \) and \( \delta = \mu_1/2 \), we obtain

\[ z'(t) = [q_1(x_*) + O(e^{-\mu_1 t/2})]e^{-\mu_1 t} + O(e^{-\left(\mu_1 + \delta_1\right)t}) = q_1(x_*)e^{-\mu_1 t} + O(e^{-\left(\mu_1 + \varepsilon_1\right)t}), \]

where \( \varepsilon_1 = \min\{\mu_1/2, \delta_1\} \). Let \( w_0(t) = e^{\mu_1 t}z(t) \). Then

\[ w_0'(t) = \mu_1 e^{\mu_1 t}z(t) + e^{\mu_1 t}z'(t) = \mu_1 w_0(t) + q_1(x_*) + O(e^{-\varepsilon_1 t}). \]

Thus,

\[ w_0'(t) - \mu_1 w_0(t) = q_1(x_*) + O(e^{-\varepsilon_1 t}). \]  

(2.33)

We apply Lemma 2.3 to equation (2.33), that is, \( t_* = T, y(t) = w_0(t), \gamma = \mu_1, \rho(t) = q_1(x_*) \), \( \delta = \varepsilon_1 \), and note that

\[ \lim_{t \to -\infty} (e^{-\gamma t}|y(t)|) = \lim_{t \to -\infty} |z(t)| = 0, \]

hence condition (2.25) is met. It follows (2.26) that

\[ |w_0(t) - \xi_1(t)| = O(e^{-\varepsilon_1 t}), \]  

(2.34)

where \( \xi_1(t) \) is the unique polynomial solution of (2.24). Multiplying (2.34) by \( e^{-\mu_1 t} \), we obtain (2.32) for \( N = 1 \).

Now, given \( N \in \mathbb{N} \), assume (2.32) holds with \( \xi_n \) being the unique polynomial solutions of (2.21) for \( n = 1, \ldots, N \). Let \( z_N(t) = \sum_{n=1}^{N} \xi_n(t)e^{-\mu_n t} \) and \( \tilde{z}_N(t) = z(t) - z_N(t) \).

Since \( \xi_1(t) \) is a constant vector, see (2.30), we have from the definition of \( z_N(t) \) that

\[ |z_N(t)| = O(e^{-\mu_1 t}). \]  

(2.35)

Also, estimate (2.32) reads as

\[ |\tilde{z}_N(t)| = O(e^{-\left(\mu_N + \varepsilon_N\right)t}). \]  

(2.36)

Define \( w_N(t) = e^{\mu_1 t + \varepsilon_N} \tilde{z}_N(t) \). Taking derivative of \( w_N(t) \) gives

\[ w'_N = \mu_{N+1}w_N + e^{\mu_{N+1} t} \left( z' - \sum_{n=1}^{N} e^{-\mu_n t} (\xi_n' - \mu_n \xi_n) \right). \]  

(2.37)
We will find an appropriate expansion for \( z'(t) \) in (2.37). By (2.6), we have

\[
z'(t) = x'(t) = u(x(t), t) = \sum_{k=1}^{N+1} q_k(x(t), t) e^{-\mu_k t} + \mathcal{O}(e^{-\left(|\mu_{N+1}| + \delta_{N+1}\right)t}).
\]

We make use of the approximation (2.19) for each \( q_k(x(t), t) \), for \( k = 1, 2, \ldots, N+1 \), with \( s = s_{N+1} \) defined in (2.23), and \( \delta = \mu_1/2 \). Noticing that \( \mu_1(s_{N+1} + 1) \geq \mu_{N+1} \), we obtain

\[
z'(t) = \sum_{k=1}^{N+1} \sum_{m=0}^{s_{N+1}} Q_{k,m}(x_*, t) z(t)^{(m)} e^{-\mu_k t} + \sum_{k=1}^{N+1} \mathcal{O}(e^{-\left(|\mu_{N+1} - \mu_1/2\right)t}) e^{-\mu_k t} + \mathcal{O}(e^{-\left(|\mu_{N+1} + \delta_{N+1}\right)t}).
\]

For the middle sum on the right-hand side, we estimate \( \mu_k - \mu_1/2 \geq \mu_1 - \mu_1/2 = \mu_1/2 \). Then it follows that

\[
z'(t) = \sum_{k=1}^{N+1} \sum_{m=0}^{s_{N+1}} Q_{k,m}(x_*, t) z(t)^{(m)} e^{-\mu_k t} + \mathcal{O}(e^{-\left(|\mu_{N+1} + \delta_{N+1}\right)t}),
\]

where \( \hat{\delta}_{N+1} = \min\{\mu_1/2, \delta_{N+1}\} \).

Denote \( Q = Q_{k,m}(x_*, t) \) in the calculations below. For \( m, k \geq 1 \), we write

\[
Q z(t)^{(m)} = Q z_N(t) + Q z_N(t) + \mathcal{O}(Q y_1, \ldots, y_m),
\]

where the last sum is a finite sum, and the vectors \( y_1, \ldots, y_m \) belong to \( \{z_N(t), \tilde{z}_N(t)\} \) with at least one of them being \( \tilde{z}_N(t) \). We estimate each \( Q(y_1, \ldots, y_m) \) by (2.11), with \( \|Q\| \) being bounded by (2.17) for \( \delta = \varepsilon_{N+1}/2 \), and use estimates (2.36), (2.37) for \( |z_N(t)|, |\tilde{z}_N(t)| \), respectively. We obtain

\[
Q z(t)^{(m)} = Q z_N(t)^{(m)} + \mathcal{O}(e^{(\varepsilon_{N+1})t} e^{-\left(|\mu_N + \varepsilon_N\right)t})
\]

\[
= Q \left( \sum_{j_1=1}^{N} \xi_{j_1} e^{-\mu_j t}, \sum_{j_2=1}^{N} \xi_{j_2} e^{-\mu_j t}, \ldots, \sum_{j_m=1}^{N} \xi_{j_m} e^{-\mu_j t} \right) + \mathcal{O}(e^{-\left(|\mu_{N+1} + \varepsilon_{N+1}\right)t})
\]

\[
= \sum_{j_1, \ldots, j_m=1}^{N} Q(\xi_{j_1}, \xi_{j_2}, \ldots, \xi_{j_m}) e^{-\left(\mu_{j_1} + \cdots + \mu_{j_m}\right)t} + \mathcal{O}(e^{-\left(|\mu_{N+1} + \varepsilon_{N+1}\right)t}).
\]

Combining this with (2.38), we have

\[
z'(t) = \sum_{k=1}^{N+1} \sum_{m=0}^{s_{N+1}} \sum_{j_1, \ldots, j_m=1}^{N} Q_{k,m}(x_*, t) (\xi_{j_1}, \ldots, \xi_{j_m}) e^{-\left(\mu_k + \mu_{j_1} + \cdots + \mu_{j_m}\right)t}
\]

\[
+ \sum_{k=1}^{N+1} \sum_{m=0}^{s_{N+1}} \left( e^{-\left(|\mu_{N+1} + \varepsilon_{N+1}\right)t} \right) e^{-\mu_k t} + \mathcal{O}(e^{-\left(|\mu_{N+1} + \hat{\delta}_{N+1}\right)t}).
\]
In the middle terms on the right-hand side, the number $\mu_N + \mu_k$ is in $\mathcal{S}$, which is due to (1.6), greater than $\mu_N$, and hence, greater or equal to $\mu_{N+1}$. Therefore,

$$z'(t) = \sum_{k=1}^{N+1} \sum_{m=0}^{N} \sum_{j_1, \ldots, j_m=1}^{N} Q_{k,m}(x_*, t) (\xi_{j_1}, \ldots, \xi_{j_m}) e^{-(\mu_k + \mu_{j_1} + \cdots + \mu_{j_m}) t} + \mathcal{O}(e^{-(\mu_{N+1} + \epsilon'_{N+1}) t}),$$

(2.39)

where $\epsilon'_{N+1} = \min\{\epsilon_N/2, \hat{\delta}_{N+1}\}$.

Note that each $Q_{k,m}(x_*, t) (\xi_{j_1}, \ldots, \xi_{j_m})$ is a $\mathbb{R}^3$-valued polynomial, hence

$$|Q_{k,m}(x_*, t) (\xi_{j_1}, \ldots, \xi_{j_m})| = \mathcal{O}(e^{\hat{\delta} t}) \quad \forall \hat{\delta} > 0,$$

(2.40)

For the first sums on the right-hand side of (2.39), one has, thanks to (1.6), $\mu_k + \mu_{j_1} + \cdots + \mu_{j_m} \in \mathcal{S}$, hence there is a unique $n \in \mathbb{N}$ such that

$$\mu_n = \mu_k + \mu_{j_1} + \cdots + \mu_{j_m}.$$

We split the sums into two parts: $I_1(t)$ corresponds to $n \leq N+1$, and $I_2(t)$ corresponds to $n \geq N + 2$. Clearly, by using the estimate (2.40), we have $I_2(t) = \mathcal{O}(e^{-(\mu_{N+2} - \epsilon_{N+1}) t})$, for any $\delta > 0$. Taking $\delta = \hat{\epsilon}_{N+1} \overset{\text{def}}{=} (\mu_{N+2} - \mu_{N+1})/2$ gives

$$I_2(t) = \mathcal{O}(e^{-(\mu_{N+1} + \hat{\epsilon}_{N+1}) t}) = \mathcal{O}(e^{-(\mu_{N+1} + \epsilon_{N+1}) t}),$$

where $\epsilon_{N+1} = \min\{\epsilon'_{N+1}, \hat{\epsilon}_{N+1}\}$. We also rewrite $I_1(t) = \sum_{n=1}^{N+1} J_n(t) e^{-\mu_n t}$, where

$$J_n(t) = \sum_{k=1}^{N+1} \sum_{m=0}^{N} \sum_{j_1, \ldots, j_m=1, \mu_k + \mu_{j_1} + \cdots + \mu_{j_m} = \mu_n}^{N} Q_{k,m}(x_*, t) (\xi_{j_1}, \ldots, \xi_{j_m}), \text{ for } n = 1, \ldots, N+1.$$

We obtain

$$z'(t) = \sum_{n=1}^{N+1} J_n(t) e^{-\mu_n t} + \mathcal{O}(e^{-(\mu_{N+1} + \epsilon_{N+1}) t}).$$

(2.41)

Combining (2.37) with (2.41) gives

$$w'_N = \mu_{N+1} w_N + e^{\mu_{N+1} t} \sum_{n=1}^{N} e^{-\mu_n t} \left[ J_n - (\xi'_n - \mu_n \xi_n) \right] + J_{N+1} + \mathcal{O}(e^{-\epsilon_{N+1} t}).$$

(2.42)

For $n = 1, \ldots, N+1$, applying formula (2.31) to $M = s_{N+1} \geq \mu_{N+1}/\mu_1 - 1 \geq \mu_n/\mu_1 - 1$, $K = N+1 \geq n$, and $J = N \geq n - 1$, we have

$$J_n(t) = \sum_{\mu_k + \mu_{j_1} + \cdots + \mu_{j_m} = \mu_n}^{N} Q_{k,m}(x_*, t) (\xi_{j_1}, \ldots, \xi_{j_m})).$$

(2.43)

For $n = 1, \ldots, N$, it follows relation (2.43) and equation (2.21) that $J_n - (\xi'_n - \mu_n \xi_n) = 0$. Thus, equation (2.42) yields

$$w'_N - \mu_{N+1} w_N = J_{N+1} + \mathcal{O}(e^{-\epsilon_{N+1} t}).$$

(2.44)
We apply Lemma 2.3 to equation (2.44), i.e.,
\[ t^* = T, \ y(t) = w_N(t), \ \gamma = \mu_{N+1}, \]
\[ p(t) = J_{N+1}(t) = \sum_{\mu_k + \mu_j_1 + \mu_j_2 + \cdots + \mu_j_m = \mu_{N+1}} Q_{k,m}(x_*, t)(\xi_j_1(t), \ldots, \xi_j_m(t)), \]
and \( \delta = \varepsilon_{N+1} \). Note that
\[ \lim_{t \to \infty} (e^{-\gamma t} |y(t)|) = \lim_{t \to \infty} |\tilde{z}_N(t)| = 0, \]
which verifies condition (2.25). Then one has
\[ \left| w_N(t) - \zeta_{N+1}(t) \right| = O(e^{-\varepsilon_{N+1}t}), \quad (2.45) \]
where \( \zeta_{N+1} : \mathbb{R} \to \mathbb{R}^3 \) is the unique polynomial solution of equation (2.21) for \( n = N+1 \).

Multiplying (2.45) by \( e^{-\mu_{N+1}t} \) gives
\[ \left| \tilde{z}_N(t) - \zeta_{N+1}(t)e^{-\mu_{N+1}t} \right| = O(e^{-(\mu_{N+1}+\varepsilon_{N+1})t}), \]
which proves (2.32) for \( N := N + 1 \).

By the induction principle, (2.32) holds for all integers \( N \in \mathbb{N} \). The proof is complete. \( \square \)

Next is the result corresponding to Consideration II.

**Theorem 2.4.** Under Consideration II, one has
\[ |x(t) - x_*| = O(e^{-\mu t}) \text{ for all } \mu > 0. \quad (2.46) \]

**Proof.** Let \( N_* > 1 \) be any integer, denote \( s_* = s_{N_*} \in \mathbb{N} \) which is defined in (2.23). For \( 1 \leq n \leq N_* \) and \( 0 \leq k \leq d_n \), there exists an extension \( Q_{n,k} \in C_{s_*+1}^0(\mathbb{R}^3) \) such that
\[ q_{n,k} = Q_{n,k} |\Omega_. \]
Define \( Q_n(x, t) = \sum_{k=0}^{d_n} t^k Q_{n,k}(x) \). Then (2.12) and (2.15) hold true with \( D^m_s q_{n,k} \) being replaced with \( D^m_s Q_{n,k} \), and \( D^m_s q_n(x, t) \) with \( D^m_s Q_n(x, t) \). Repeat the above proof of Theorem 2.2 with finite induction, we assert that (2.32) holds for \( N = N_* \), i.e.,
\[ \left| z(t) - \sum_{n=1}^{N_*} \xi_n(t)e^{-\mu_n t} \right| = O(e^{-(\mu_{N_*}+\varepsilon_{N_*})t}). \quad (2.47) \]

Since \( x_* \in \partial \Omega \) and \( q_{n,k} \in \mathcal{X} \), one has \( q_{n,k}(x_*) = 0 \), hence \( q_n(x_*, t) = 0 \) for \( 1 \leq n \leq N_* \). By (2.30), \( \xi_1 = 0 \). One can verify by the recursive formula (2.29) that \( \xi_n = 0 \) for \( n = 1, \ldots, N_* \). Combining this with (2.47) yields
\[ |z(t)| = O(e^{-\mu_{N_*} t}) \text{ for any integer } N_* > 1. \]

By letting \( N_* \to \infty \) and using property (1.5), we obtain (2.46). \( \square \)
Remark 2.5. This is a discussion about assumption (A).

In the (SPC) case, by its regularity and spatial periodicity, \( u \) is bounded on \( \mathbb{R}^3 \times [T, T'] \) for any \( T' > T \). Then, given any \( x_0 \in \mathbb{R}^3 \), there is a unique solution \( x(t) \in C^1([T, T'], \mathbb{R}^3) \) of (1.1) with \( x(T) = x_0 \).

In the (DBC) case, for any \( x_0 \in \Omega \), there exists a unique solution \( x(t) \in C^1([T, T_{\text{max}}], \Omega) \) of (1.1) with \( x(T) = x_0 \), where \([T, T_{\text{max}}]\) is the maximal interval of existence. In case \( T_{\text{max}} < \infty \), we have, by the boundedness of \( u \) on \( \Omega \times [T, T_{\text{max}}] \),
\[
\lim_{t \to T_{\text{max}}} x(t) \overset{\text{def}}{=} x(T) + \int_T^{T_{\text{max}}} u(x(\tau), \tau) d\tau,
\]
which exists and must belong to \( \partial \Omega \). Then \( u(\bar{x}, t) = 0 \), and by defining \( x(t) = \bar{x} \) for \( t \geq T_{\text{max}} \) we obtain a solution \( x(t) \in \Omega \) of (1.1) for all \( t \geq T \). However, its longtime behavior is not interesting. Therefore, in assumption (A) above, we only consider \( T_{\text{max}} = \infty \), that is, \( x(t) \in \Omega \) for all \( t \geq T \).

3. General Spatial Periodicity Case

We consider the general case of (SPC), when the velocity field is not required to have zero average over \( \Omega \).

Let \( u(x, t) \in C_x^{2,1}([0, \infty)) \cap C([0, \infty)) \) and \( p(x, t) \in C_x^1([0, \infty)) \) be \( L \)-periodic functions that form a solution \( (u, p) \) of the NSE (1.2).

Let \( x(t) \in \mathbb{R}^3 \) be a solution of (1.1) on \( (0, \infty) \). The next theorem shows that \( x(t) \) has a similar asymptotic expansion to (2.20) in Theorem 2.2.

**Theorem 3.1.** There exist \( x_* \in \mathbb{R}^3 \) and polynomials \( X_n : \mathbb{R} \to \mathbb{R}^3 \), for \( n \in \mathbb{N} \), such that
\[
x(t) \sim (x_* + U_0 t) + \sum_{n=1}^{\infty} X_n(t) e^{-\mu_n t} \quad \text{in} \ \mathbb{R}^3,
\]
where \( U_0 = (L_1 L_2 L_3)^{-1} \int_{\Omega} u(x, 0) dx \).

**Proof.** We use the standard Galilean transformation. Set
\[
v(X, t) = u(X + U_0 t, t) - U_0 \quad \text{and} \quad P(X, t) = p(X + U_0 t, t), \quad X \in \mathbb{R}^3, t \geq 0.
\]

One can verify that \((v, P)\) is a classical solution of the NSE (1.2) on \(\mathbb{R}^3 \times (0, \infty)\). Moreover, \((v, P)\) is \( L \)-periodic, and \( v(\cdot, t) \) has zero average for each \( t \geq 0 \).

Let \( X(t) = x(t) - U_0 t \). We have
\[
X'(t) = x'(t) - U_0 = u(x(t), t) - U_0 = v(x(t) - U_0 t, t) + U_0 - U_0 = v(X(t), t).
\]

Applying Theorem 2.2 to \(v(X, t)\) and \(X(t)\) yields
\[
X(t) \sim x_* + \sum_{n=1}^{\infty} X_n(t) e^{-\mu_n t},
\]
where \( x_* \) is a constant vector in \( \mathbb{R}^3 \), and \( X_n \)'s are \( \mathbb{R}^3 \)-valued polynomials on \( \mathbb{R} \). Consequently, we obtain
\[
x(t) = X(t) + U_0 t \sim x_* + U_0 t + \sum_{n=1}^{\infty} X_n(t) e^{-\mu_n t},
\]
which proves (3.1).
\[\square\]
Remark 3.2. We end the paper with the following comments.

(a) The asymptotic expansion is uniquely determined for each given Leray–Hopf weak solution, but does not imply the uniqueness of the Leray–Hopf weak solutions starting from the same initial condition.

(b) The expansion of $x(t)$ in Theorem 2.2 is not driven by a dissipative ODE, but rather by the exponential decay in the time-dependent expansion (2.1) of $u(x, t)$. This is different from the previous results obtained for the NSE [4,5,17,21,22] or general nonlinear differential equations considered in [3,25,26].

(c) The above proof of Theorem 2.2 can be adapted to study the Navier–Stokes equations in different contexts when $u(x, t)$ processes an asymptotic expansion similar to (2.1) such as those in [5,21,22].

(d) Although the above presentation is focused on the three-dimensional space, the obtained results are equally true for the two-dimensional case. Moreover, Theorem 2.2 is not restricted to just the velocity field and space $\mathbb{R}^3$, but in fact, applies to general differential equations in Banach spaces.

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Communicated by A. Ionescu