Two-valued groups, Kummer varieties and integrable billiards

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Abstract

A natural and important question of study two-valued groups associated with hyperelliptic Jacobians and their relationship with integrable systems is motivated by seminal examples of relationship between algebraic two-valued groups related to elliptic curves and integrable systems such as elliptic billiards and celebrated Kowalevski top. The present paper is devoted to the case of genus 2, to the investigation of algebraic two-valued group structures on Kummer varieties.

One of our approaches is based on the theory of $\sigma$-functions. It enables us to study the dependence of parameters of the curves, including rational limits. Following this line, we are introducing a notion of $n$-groupoid as natural multivalued analogue of the notion of topological groupoid.

Our second approach is geometric. It is based on a geometric approach to addition laws on hyperelliptic Jacobians and on a recent notion of billiard algebra. Especially important is connection with integrable billiard systems within confocal quadrics.

The third approach is based on the realization of the Kummer variety in the framework of moduli of semi-stable bundles, after Narasimhan and Ramanan. This construction of the two-valued structure is remarkably similar to the historically first example of topological formal two-valued group from 1971, with a significant difference: the resulting bundles in the 1971 case were "virtual", while in the present case the resulting bundles are effectively realizable.

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1 Introduction

The study of structure of multivalued groups has been started in 1971 (see [5]) within the study of characteristic classes of vector bundles, at that time as the structure of formal (local) multivalued groups. In 1990, in [8] has been introduced algebraic two-valued group structure on $\mathbb{C}P^1$ by using addition theorems on elliptic curves. The structure of algebraic multivalued groups has been studied since then extensively (see [9] and references therein) in different contexts.

The deep connection between two-valued groups and integrable systems is well-known, see [7]. However, it has got a new impetus with a recent paper [18], where a connection between two-valued groups on $\mathbb{C}P^1$ and the celebrated Kowalevski top has been discovered. There, it was shown that the ”mysterious” Kowalevski change of variables (see, for example, [3] for the terminology) corresponds to the operation in two-valued group $W/\tau$, where $W$ is an elliptic curve, in the standard Weierstrass model of elliptic curve, with $\tau$ as the canonical involution. More detailed description of this two-valued group is presented in Section 2.1, see Example 3. This deep connection of Kowalevski integration procedure with a structure of elliptic curve, on the first glance, may be in surprising contrast to a well known fact that the integration of the Kowalevski top is performed on Jacobian of a genus two curve. Moreover, as it has been shown in [18], the Kowalevski integration procedure can be interpreted as a certain deformation of the two-valued group $p_2$ (see Example 2) toward the structure on $W/\tau$. The structure of $p_2$ is the rational limit of the one on $W/\tau$. In addition, in [18] it has been proven the equivalence between associativity of the two-valued group on $W/\tau$ with the Poncelet theorem of pencils of conics, in a case of triangles.

Higher genera Poncelet type problems, integrable billiards and so-called billiard algebra associated with pencils of quadrics in arbitrary dimensions have been studied in [20] and [19], where the relationship with hyperelliptic curves and their Jacobians has been considered (see also [21]).

From our previous experience and work with elliptic case, as a natural and important question arises the study two-valued groups associated with hyperelliptic Jacobians and their relationship with integrable systems. Especially important is connection with integrable billiards within pencils of quadrics.

The present paper is devoted to the case of genus 2. In this case one comes to the problem of investigation of two-valued group structures on Kummer varieties.

We describe the structure of algebraic two-valued groups on Kummer varieties, its rational limits and relationship with integrable systems. The Kummer variety is a classical and well-studied object of the algebraic geometry. It appears as a variety of orbits of a Jacobian of a curve of genus 2, factorized by the hyperelliptic involutive automorphism. Thus, there are various algebro-geometric and analytical structures on it, related to the moduli of curves of genus 2. We want to understand and present the structure of an algebraic two-valued group associated to different appearances of the Kummer varieties. Let us mention
that from the point of view of differential geometry and the theory of abelian functions of genus 2, Kummer varieties have been studied by Baker.

One side of our approach is based on the theory of $\sigma$-functions. The choice of $\sigma$ functions is motivated by our wish to study dependence of parameters of the curves. As we know from the theory of integrable systems, (see [22]), it is important to study not only a single curve and its Jacobian, but rather a whole class of curves and their Jacobians. Following this line, let $B$ denote a set parameterizing nonsingular curves of genus 2 and let $U$ be the analytic bundle over $B$ with a genus 2 curve as a fibre. We denote by $J(U)$ the associated bundle over $B$ with a Jacobian of a genus 2 curve as a fiber. Then, $J(U)$ has a natural structure of a topological groupoid, see [11].

In the next section, after recalling the definition of the $n$ valued group, we are introducing a notion of $n$-groupoid, which enables us to consider not only a single structure of a two-valued group, but also dependance of the parameters of the curve. Thus the notion of $n$-groupoid is a natural analogue of the notion of topological groupoid.

Another important reason of using $\sigma$-functions, lies in the fact that addition relations for them are well adjusted to two-valued structure, as one can easily see from the genus one case formula:

$$\sigma(u + v)\sigma(u - v) = \sigma(u)^2\sigma(v)^2(\wp(v) - \wp(u)),$$

The program of construction of $\sigma$-functions of higher genera had been proposed by Klein. In genus 2 case, the program got rather advanced development by Baker (see [4]), although in his last review of the program in 1923, Klein noted that program for genus $g > 2$ was still far from being completed. We base our first approach on recent results, see [11] and references therein.

Our second approach is geometric. Following [15], [24], [29] a geometric approach to addition laws on hyperelliptic Jacobians has been developed further in [20] (see also [21]). It has been based on the notion of billiard algebra and it has been connected to integrable billiard systems within confocal quadrics.

Our first approach is developed in Section 6 and the second one in Section 9. The Section 6 is preceded by the three sections of preparation. In Section 3 the derivation of the two-valued group law on $W/\tau$, where $W$ denotes an elliptic curve as above, is based on $\sigma$ - functions. The rational limit of the last structure is studied in the next Section 4. The study of the rational limit of a Kummer surface and related two-valued group law is performed in Section 5.

In Section 10 the geometric approach from the Section 9 develops further toward integrable systems, through the integrable billiards and the billiard algebra and is motivated by [20].

The final Section 11 gives yet another construction of the two-valued group law on the Kummer variety, based on the realization of the Kummer variety in the framework of moduli of semi-stable bundles. This framework has been developed by Narasimhan and Ramanan, see [28]. This last construction of the two-valued structure is remarkably similar to the historically first example of topological two-valued group from 1971 (see [5]), with a significant difference.
The resulting bundles in the 1971 case of \[5\] are "virtual", while in the present case the resulting bundles are effectively realizable.

2 From \(n\)-valued groups to \(n\)-groupoids.

2.1 Defining notions and basic examples of multivalued groups

Following \[9\], we give the definition of an \(n\)-valued group on \(X\) as a map:

\[ m : X \times X \rightarrow (X)^n \]
\[ m(x, y) = x \ast y = [z_1, \ldots, z_n], \]

where \((X)^n\) denotes the symmetric \(n\)-th power of \(X\) and \(z_i\) coordinates therein.

**Associativity** is the condition of equality of two \(n^2\)-sets

\[ [x \ast (y \ast z)_1, \ldots, x \ast (y \ast z)_n] \]
\[ [(x \ast y)_1 \ast z, \ldots, (x \ast y)_n \ast z] \]

for all triplets \((x, y, z) \in X^3\).

An element \(e \in X\) is a **unit** if

\[ e \ast x = x \ast e = [x, \ldots, x], \]

for all \(x \in X\).

A map \(\text{inv} : X \rightarrow X\) is an **inverse** if it satisfies

\[ e \in \text{inv}(x) \ast x, \quad e \in x \ast \text{inv}(x), \]

for all \(x \in X\).

Following \[9\], we say that \(m\) defines an **\(n\)-valued group structure** \((X, m, e, \text{inv})\) if it is associative, with a unit and an inverse.

An \(n\)-valued group \(X\) acts on the set \(Y\) if there is a mapping

\[ \phi : X \times Y \rightarrow (Y)^n \]
\[ \phi(x, y) = x \circ y, \]

such that the two \(n^2\)-multisubsets of \(Y\)

\[ x_1 \circ (x_2 \circ y) \quad (x_1 \ast x_2) \circ y \]

are equal for all \(x_1, x_2 \in X, y \in Y\). It is additionally required that

\[ e \circ y = [y, \ldots, y] \]

for all \(y \in Y\).
**Example 1** [A two-valued group structure on \( \mathbb{Z}_+ \), \([7]\)] Let us consider the set of nonnegative integers \( \mathbb{Z}_+ \) and define a mapping

\[
m : \mathbb{Z}_+ \times \mathbb{Z}_+ \to (\mathbb{Z}_+)^2,
m(x, y) = [x + y, |x - y|].
\]

This mapping provides a structure of a two-valued group on \( \mathbb{Z}_+ \) with the unit \( e = 0 \) and the inverse equal to the identity \( \text{inv}(x) = x \).

In \([7]\) sequence of two-valued mappings associated with the Poncelet porism was identified as the algebraic representation of this 2-valued group. Moreover, the algebraic action of this group on \( \mathbb{C}P^1 \) was studied and it was shown that in the irreducible case all such actions are generated by Euler-Chasles correspondences.

**Example 2** [2-valued group on \((\mathbb{C}, +)\)] Among the basic examples of multivalued groups, there are \( n \)-valued additive group structures on \( \mathbb{C} \). For \( n = 2 \), this is a two-valued group \( p_2 \) defined by the relation

\[
m_2 : \mathbb{C} \times \mathbb{C} \to (\mathbb{C})^2
\]

\[
x \ast_2 y = [(\sqrt{x} + \sqrt{y})^2, (\sqrt{x} - \sqrt{y})^2]
\]

The product \( x \ast_2 y \) corresponds to the roots in \( z \) of the polynomial equation

\[
p_2(z, x, y) = 0,
\]

where

\[
p_2(z, x, y) = (x + y + z)^2 - 4(xy + yz + zx).
\]

This two-valued group structure has been connected with degenerations of the Kowalevski top in \([18]\). Similar integrable systems were studied by Appel’rot, Delone, Mlodzevskii (see \([14]\), \([1]\), \([27]\), \([24]\).)

As it has been observed in \([18]\), the general Kowalevski case is connected with \( p_2 \) together with its deformation on \( \mathbb{C}P^1 \) as a factor of an elliptic curve, see the next example.

**Example 3** [2-valued group on \( S^2 = \hat{\mathbb{C}} \), associated with an elliptic curve] Suppose that a cubic \( W \) is given in the standard form

\[
W : t^2 = J(s) = 4s^3 - g_2s - g_3.
\]

Consider the mapping \( W \to S^2 = \hat{\mathbb{C}} : (s, t) \mapsto s \), where \( \hat{\mathbb{C}} \) represents a complex line extended by \( \infty \).

The curve \( W \) as a cubic curve has the group structure. Together with its canonical involutive automorphism \( \tau : (s, t) \mapsto (s, -t) \), it defines the standard
two-valued group structure of coset type (see [12], [9]) on $S^2$ with the unit at infinity in $S^2$. The product is defined by the formula:

$$[s_1] * [s_2] = \left[ -s_1 - s_2 + \left( \frac{t_1 - t_2}{2(s_1 - s_2)} \right)^2 \right], \quad \left[ -s_1 - s_2 + \left( \frac{t_1 + t_2}{2(s_1 - s_2)} \right)^2 \right],$$

where $t_i = J(s_i), i = 1, 2,$ and

$$[s_i] = \{ (s_i, t_i), (s_i, -t_i) \}, \quad s_i = \wp(u_i), t_i = \wp'(u_i),$$

by using addition theorem for the Weierstrass function $\wp(u)$:

$$\wp(u_1 + u_2) = -\wp(u_1) - \wp(u_2) + \left( \frac{\wp'(u_1) - \wp'(u_2)}{2(\wp(u_1) - \wp(u_2))} \right)^2.$$

The Kowalevski integration procedure was explained in [18] as certain deformation of $p_2$ to $(W, \tau)$.

The 2-group structure $(W, \tau)$ is also connected with the Poncelet and the Darboux theorem (see [13], [17], [20]).

In this example, $g_2, g_3$ are parameters of the curve, and they lead to a rational limit, in the standard limit procedure.

Related structure on $\mathbb{CP}^1$ as an algebraic mapping $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^2$ (see [12], [9]) is presented in Section 3.

**Example 4** Let $X$ be a topological space. Denote by $W(X)$ the set of all 2-dimensional complex vector bundles $\zeta = \eta + \bar{\eta}$ over $X$, where $\eta$ is a linear complex vector bundle over $X$. Then the formula

$$\zeta_1 \otimes \zeta_2 = (\eta_1 \otimes \eta_2 + \bar{\eta}_1 \otimes \bar{\eta}_2) + (\bar{\eta}_1 \otimes \eta_2 + \eta_1 \otimes \bar{\eta}_2)$$

gives the structure of 2-valued group on $W(X)$ defined by the 2-valued multiplication

$$\zeta_1 * \zeta_2 = [(\eta_1 \otimes \eta_2 + \bar{\eta}_1 \otimes \bar{\eta}_2), (\bar{\eta}_1 \otimes \eta_2 + \eta_1 \otimes \bar{\eta}_2)].$$

**2.2 Topological $n$-groupoids**

Following [11], let us fix a topological space $Y$. A space $X$ with a map $p_X : X \to Y$ is called a space over $Y$, and the map $p_X$ is called anchor. For $Y$, the anchor $p_Y$ is the identity map.

A map $f : X_1 \to X_2$ of two spaces over $Y$ is called a map over $Y$ if $p_{X_2} \circ f = p_{X_1}$. For two spaces $X_1, X_2$ over $Y$, their direct product over $Y$ is defined as

$$X_1 \times_X X_2 = \{ (x_1, x_2) \in X_1 \times X_2 \mid p_{X_1}(x_1) = p_{X_2}(x_2) \}$$
with the map $p_{X_1 \times_Y X_2}(x_1, x_2) = p_{X_1}(x_1)$. Along this line, one may define a product over $Y$ of $n$ spaces $X_1, \ldots, X_n$ over $Y$

\[ X_1 \times Y \cdots \times Y X_n = \{(x_1, \ldots, x_n) \in X_1 \times X_2 \times \cdots \times X_n \mid p_{X_1}(x_1) = p_{X_2}(x_2) = \cdots = p_{X_n}(x_n)\}. \]

In a special case $X_1 = \cdots = X_n = X$, we define the $n$-th power of $X$ over $Y$ and denote it as $X^\otimes_Y$.

For a space $X$ over $Y$ we define its $n$-th symmetric power over $Y$, denoted as $(X)^\nu_Y$

as the quotient of $X^n_Y$ by the action of the permutation group.

We define also the diagonal map over $Y$

\[ D : X \to (X)^\nu_Y, \quad x \mapsto (x, x, \ldots, x). \]

**Definition 1** A space $X$ with an anchor $p_X : X \to Y$ and structural maps over $Y$:

\[ \mu : X \times_Y X \to (X)^\nu_Y, \quad \text{inv} : X \to X, \quad e : Y \to X \]

is called an $n$-groupoid over $Y$ if the following conditions are satisfied:

1. For $x_1, x_2, x_3$ such that $p_X(x_1) = p_X(x_2) = p_X(x_3)$: if

   \[ \mu(x_1, x_2) = [z_1, \ldots, z_n], \quad \mu(x_2, x_3) = [w_1, \ldots, w_n], \]

   then

   \[ [\mu(x_1, w_1), \ldots, \mu(x_1, w_n)] = [\mu(z_1, x_3), \ldots, \mu(z_n, x_3)]. \]

2. For every $x \in X$ and $y = p_X(x) \in Y$:

   \[ \mu(e(y), x) = \mu(x, e(y)) = D(x). \]

3. For every $x \in X$ and $y = p_X(x) \in Y$:

   \[ e(y) \in \mu(x, \text{inv}(x)), \quad e(y) \in \mu(\text{inv}(x), x). \]

**Example 5** Let $X = \mathbb{C} \times \mathbb{C}$, $Y = \mathbb{C}$ and an anchor is defined as projection to the second component

\[ p_X := p_2 : X \to Y, \quad p_X(x, \lambda) := \lambda. \]

We define 2-groupoid over $Y$ starting with an operation over $Y$

\[ A((x_1, \lambda), (x_2, \lambda)) = (x_1 + x_2 - \lambda x_1 x_2, \lambda). \]

An involutive automorphism $I$ over $Y$ is defined by

\[ I : X \to X, \quad (\bar{u}, \lambda) = \left( \frac{u}{1 - \lambda u}, \lambda \right). \]
If we denote by $u\bar{u} = x$, $v\bar{v} = y$, then the 2-groupoid over $Y$ is defined by

$$\mu((u, \lambda), (v, \lambda)) = [(z_1, \lambda), (z_2, \lambda)]$$

where $z_i, i = 1, 2$ are solutions of the following quadratic equation

$$Z^2 - (2(x + y) - \lambda^2 xy) Z + (x - y)^2 = 0.$$  

Notice that we get $p_2$ for $\lambda = 0$. Thus the structure of the above 2-groupoid is certain deformation of $p_2$.

**Example 6** Similarly, we can consider a two-parameter deformation of elementary two-valued group $p_2$ and we get a structure of 2-groupoid over $Y_1 = C^2$. We are starting with an operation over $Y_1$ on $X_1 = C \times Y_1$, defined by

$$A_1((u, \lambda_1, \lambda_2), (v, \lambda_1, \lambda_2)) := \left(\frac{u + v - \lambda_1 uv}{1 - \lambda_2 uv}, \lambda_1, \lambda_2\right).$$

The involutive automorphism $I_1$, over $Y_1$, is defined by

$$I_1(u, \lambda_1, \lambda_2) = \left(-\frac{u}{1 - \lambda_1 u}, \lambda_1, \lambda_2\right).$$

One can easily deduce corresponding morphism $\mu_1$:

$$\mu_1((x, \lambda_1, \lambda_2), (y, \lambda_1, \lambda_2)) = [(z_1, \lambda_1, \lambda_2), (z_2, \lambda_1, \lambda_2)]$$

where $z_i, i = 1, 2$ are solutions of the following quadratic equation:

$$Z^2 - \frac{B(x, y, \lambda_1, \lambda_2)}{G(x, y, \lambda_1, \lambda_2)} Z + \frac{C(x, y, \lambda_1, \lambda_2)}{G(x, y, \lambda_1, \lambda_2)} = 0.$$  

Here we use the notation

$$x = uI_1(u), \quad y = vI_1(v)$$

and

$$B(x, y, \lambda_1, \lambda_2) = x^2 y^2 \lambda_1^2 \lambda_2^2 (-\lambda_1^2 + \lambda_2^2) + xy(x + y)\lambda_2^2 (2\lambda_2 - 3\lambda_1) + 2(x + y)$$
$$C(x, y, \lambda_1, \lambda_2) = (x - y)^2$$
$$G(x, y, \lambda_1, \lambda_2) = 1 + xy\lambda_2 (\lambda_1^2 - \lambda_2^2) + xy(x + y)\lambda_1^2 \lambda_2^2 + x^2 y^2 \lambda_2^3 (\lambda_2 - \lambda_1^2)$$
3 Structure of two-valued group on $\mathbb{CP}^1$ and sigma-functions

Let us consider the Weierstrass sigma-function $\sigma(u) = \sigma(u, g_2, g_3)$, associated with a curve

$$V = \{(t, s) \in \mathbb{C}^2 : t^2 = 4s^3 - g_2s - g_3\}.$$

On the Jacobian $J_1 = \mathbb{C}^1/\Gamma$ of the curve $V$, the $\wp$-function $\wp(u) = -\frac{d^2}{du^2} \ln \sigma(u)$ is defined. The function $\sigma(u)$ is odd, thus, the mapping

$$\pi : J_1 \to \mathbb{CP}^1 : \pi(u) = (x_1 : x_2),$$

with $x_1 = \sigma(u)^2$ and $x_2 = \sigma(u)^2 \wp(u)$, factorizes as a composition

$$\pi : J_1 \to J_1/\pm \xrightarrow{\tilde{\pi}} \mathbb{CP}^1,$$

where $\tilde{\pi}$ is a homeomorphism. Let us note that $x_1(u)$ and $x_2(u)$ are entire functions of $u$.

The canonical homeomorphism

$$\gamma : (\mathbb{CP}^1)^2 \to \mathbb{CP}^2 : ((x_1 : x_2), (y_1 : y_2)) \to (x_1y_1 : (x_1y_2 + x_2y_1) : x_2y_2),$$

corresponds to the mapping

$$[\gamma] : (x_1t_1 + x_2t_2, (y_1t_1 + y_2t_2)) \to x_1y_1t_1^2 + (x_1y_2 + x_2y_1)t_1t_2 + x_2y_2t_2^2.$$

**Theorem 1** Multiplication in the two-valued group

$$m : J_1/\pm \times J_1/\pm \to (J_1/\pm)^2$$

is defined by algebraic mapping

$$m : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^2 : (x_1 : x_2) \ast (y_1 : y_2) = (z_1 : z_2 : z_3),$$

where $z_1 = (x_1y_2 - x_2y_1)^2$, $z_2 = 2 \left[ (x_1y_2 + x_2y_1) (x_2y_2 - \frac{g_2}{4} x_1y_1) - \frac{g_3}{2} x_1^2 y_1^2 \right]$, $z_3 = (x_2y_2 + \frac{g_3}{4} x_1y_1)^2 + g_3x_1y_1(x_1y_2 + x_2y_1)$.

**Proof.** The composition of mappings

$$J_1/\pm \times J_1/\pm \xrightarrow{m} (J_1/\pm)^2 \xrightarrow{(\gamma)^2} (\mathbb{CP}^1)^2 \xrightarrow{\gamma} \mathbb{CP}^2$$

has the form

$$[u] \times [v] \to ([u + v], [u - v]) \to ((x_1 : x_2), (y_1 : y_2)) \to (z_1 : z_2 : z_3),$$

where $[u] = \{u, -u\}$, $x_1 = \sigma(u + v)^2$, $x_2 = x_1 \wp(u + v)$, $y_1 = \sigma(u - v)^2$, $y_2 = y_1 \wp(u - v)$.

Thus,

$$z_1 = \sigma(u + v)^2 \sigma(u - v)^2, \quad z_2 = z_1 (\wp(u + v) + \wp(u - v)), \quad z_3 = z_1 \wp(u + v) \wp(u - v).$$
Our goal is to express \( z_1, z_2, z_3 \) as functions of \( x_1, x_2, y_1, y_2 \). By using the classical addition theorem for sigma-functions
\[
\sigma(u + v)\sigma(u - v) = \sigma(u)^2\sigma(v)^2 (\wp(v) - \wp(u)),
\]
we get
\[
z_1 = (x_1 y_2 - x_2 y_1)^2.
\]

From
\[
\ln \sigma(u + v) + \ln \sigma(u - v) = 2\left(\ln \sigma(u) + \ln \sigma(v)\right) + \ln (\wp(v) - \wp(u)),
\]
applying the operator \( \partial = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \), we get
\[
2\zeta(u + v) = 2(\zeta(u) + \zeta(v)) + \frac{\wp'(v) - \wp'(u)}{\wp(v) - \wp(u)}
\]
where \( \zeta(u) = \frac{\partial}{\partial u} \ln \sigma(u) \). By differentiating previous identity in \( u \), one gets
\[
\wp(u + v) = w_1(u, v) - w_2(u, v),
\]
where
\[
w_1(u, v) = \wp(u) + \frac{1}{2} \frac{\wp''(u)(\wp(v) - \wp(u)) + \wp'(u)^2}{(\wp(v) - \wp(u))^2} \tag{3}
\]
\[
w_2(u, v) = \frac{1}{2} \frac{\wp'(u)\wp'(v)}{(\wp(v) - \wp(u))^2}. \tag{4}
\]

Observe, that \( w_1(u, -v) = w_1(u, v) \) and \( w_2(u, -v) = -w_2(u, v) \). Finally we get,
\[
\wp(u + v) + \wp(u - v) = 2w_1(u, v); \quad \wp(u + v)\wp(u - v) = w_1^2 - w_2^2.
\]

By using the Weierstrass uniformization of the elliptic curve, we get
\[
\wp'(u)^2 = 4\wp(u)^2 - g_2\wp(u) - g_3,
\]
and also,
\[
\wp''(u) = 6\wp(u)^2 - \frac{g_2}{2}.
\]

According to the formula \( \square \) we get
\[
2(\wp(v) - \wp(u))^2 w_1(u, v) =
\]
\[
= 2\wp(u)(\wp(v) - \wp(u))^2 + \left(6\wp(u)^2 - \frac{g_2}{2}\right) (\wp(v) - \wp(u)) + 4\wp(u)^3 - g_2\wp(u) - g_3 =
\]
\[
= 2\wp(u)\wp(v)(\wp(v) + \wp(u)) - \frac{g_2}{2} (\wp(v) + \wp(u)) - g_3.
\]

Thus,
\[
2z_1^2w_1(u, v) = 2\sigma^4(u)\sigma^4(v)(\wp(v) - \wp(u))^2 w_1(u, v) =
\]
\[
= \sigma^4(u)\sigma^4(v) \left[ (\wp(v) + \wp(u)) \left(2\wp(v)\wp(u) - \frac{g_2}{2}\right) - g_3 \right].
\]
As a consequence, we get

\[ z_2 = 2 \left[ (x_1 y_2 + x_2 y_1) \left( x_2 y_2 - \frac{g_2}{4} x_1 y_1 \right) - \frac{g_1}{2} x_1^2 y_1^2 \right]. \]

Further, we get

\[
(\varphi(v) - \varphi(u))^4 (w_1^2 - w_2^2) = \\
= \left[ (\varphi(v) + \varphi(u)) \left( 2\varphi(v)\varphi(u) - \frac{g_2}{2} \right) - \frac{g_3}{2} \right]^2 - \left[ 2\varphi(u)^3 - \frac{g_2}{2} \varphi(u) - \frac{g_3}{2} \right] \left[ 2\varphi(v) - \frac{g_2}{2} \varphi(v) - \frac{g_3}{2} \right].
\]

From

\[ z_1 z_3 = \sigma^4(u) \sigma^4(v) (\varphi(v) - \varphi(u))^4 (w_1^2 - w_2^2) = \\
\left[ (x_1 y_2 + x_2 y_1) \left( x_2 y_2 - \frac{g_2}{4} x_1 y_1 \right) - \frac{g_3}{2} x_1^2 y_1^2 \right]^2 - \\
- x_1 y_1 \left[ 2x_2^2 - \frac{g_2}{2} x_2 x_1^2 - \frac{g_3}{2} x_1^2 \right] \left[ 2y_2^3 - \frac{g_2}{2} y_2 y_1^2 - \frac{g_3}{2} y_1^2 \right],
\]

we get

\[ z_1 z_3 = \left[ \left( x_2 y_2 + \frac{g_2}{4} x_1 y_1 \right)^2 + g_3 x_1 y_1 (x_1 y_2 + x_2 y_1) \right]. \]

\[ \square \]

**Corollary 1** The multiplication law of the two-valued group \( \mathbb{C} P^1 \) is defined in homogeneous coordinates by the formula

\[
(x_1 t_1 + x_2 t_2) \ast (y_1 t_1 + y_2 t_2) = \\
= (x_1 y_2 - x_2 y_1)^2 t_1^2 + 2 \left[ (x_1 y_2 + x_2 y_1) \left( x_2 y_2 - \frac{g_2}{4} x_1 y_1 \right) - \frac{g_3}{2} x_1^2 y_1^2 \right] t_1 t_2 + \\
+ \left[ \left( x_2 y_2 + \frac{g_2}{4} x_1 y_1 \right)^2 + g_3 x_1 y_1 (x_1 y_2 + x_2 y_1) \right] t_2^2.
\]

In the rational limit, defined with \( g_2 = g_3 = 0 \), we get

\[
(x_1 t_1 + x_2 t_2) \ast (y_1 t_1 + y_2 t_2) = \\
= (x_1 y_2 - x_2 y_1)^2 t_1^2 + 2(x_1 y_2 + x_2 y_1) x_2 y_2 t_1 t_2 + x_2^2 y_2^2 t_2^2 = \\
= [(x_1 y_2 + x_2 y_1) t_1 + x_2 y_2 t_2]^2 - 4x_1 x_2 y_1 y_2 t_1 t_2.
\]

**4 Elementary two-valued group on \((\mathbb{C}^2, +)\)**

Let us consider \( \mathbb{C}^2 \) together with its subgroup \( \mathbb{Z}^2 \). By the standard coset construction (see [9]),

\[ X = \mathbb{C}^2 / \pm \]
is equipped by two-valued group structure:

\[ \mu : X \times X \to (X)^2 \]
given in coordinates \( u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{C}^2 \) by the formula

\[ \mu([u, -u], [v, -v]) = ([u + v, -(u + v)], [(u - v), -(u - v)]). \]

Now, we pass to the embedding

\[ \pi : X \to \mathbb{C}^3, \quad \pi([(u_1, u_2), -(u_1, u_2)]) = (x_1, x_2, x_3) \]
given by the formula

\[ x_1 = u_1^2, \quad x_2 = u_1 u_2 \quad x_3 = u_2^2. \]

By checking, one sees that the inverse image of a point is a two element set:

\[ \pi^{-1}(x_1, x_2, x_3) = [(u_1, u_2), -(u_1, u_2)], \]

where

\[ u_1 = \pm \sqrt{x_1}, \quad u_2 = \pm \sqrt{x_3}. \]

The image of \( \pi \) is a quadric \( Q \) in \( \mathbb{C}^3 \) defined by the equation

\[ Q : x_1 x_3 = x_2^2. \]

On \( Q \), the multiplication \( \mu \) can be rewritten according to the formulae

\[ \mu([u, -u], [v, -v]) = ((X_2, X_4, X_6), (Y_2, Y_4, Y_6)), \]

where

\[ [u, -u] = \pi^{-1}(x_1, x_2, x_3), \quad [v, -v] = \pi^{-1}(y_1, y_2, y_3). \]

Indices 2, 4, 6 for variables \( X \) and \( Y \) are chosen to fit with a graduation in a sequel. Then

\[ X_2 = (\sqrt{x_1} + \sqrt{y_1})^2, \quad Y_2 = (\sqrt{x_1} - \sqrt{y_1})^2, \]

and \( X_2, Y_2 \) are the roots of the equation

\[ Z_1^2 - 2(x_1 + y_1)Z_1 + (x_1 - y_1)^2 = 0. \quad (5) \]

For \( X_4 \) and \( Y_4 \) we have

\[ X_4 = (\sqrt{x_1} + \sqrt{y_1})(\sqrt{x_3} + \sqrt{y_3}) \]

\[ Y_4 = (\sqrt{x_1} - \sqrt{y_1})(\sqrt{x_3} - \sqrt{y_3}). \]

Thus,

\[ X_4 + Y_4 = 2(x_2 + y_2) \]

\[ X_4 Y_4 = (x_1 - y_1)(x_3 - y_3), \]
and $X_4, Y_4$ are the solutions of the equation:

$$Z_2^2 - 2(x_2 + y_2)Z_2 + (x_1 - y_1)(x_3 - y_3) = 0.$$  \hfill (6)

The last equation can be written in the form

$$(Z_2 - (x_2 + y_2))^2 = y_1x_3 + x_1y_3 + 2x_2y_2.$$  

Finally, for $X_6, Y_6$ the situation is analogue to $X_2, Y_2$:

$$X_6 = (\sqrt{x_3} + \sqrt{y_3})^2, \quad Y_6 = (\sqrt{x_3} - \sqrt{y_3})^2.$$  

$X_6, Y_6$ are the roots of the equation

$$Z_3^2 - 2(x_3 + y_3)Z_3 + (x_3 - y_3)^2 = 0.$$  \hfill (7)

Three quadratic equations (5, 6, 7) determine four possible pairs of triplets of solutions. But, the constraints

$$X_2X_4 = X_6^2, \quad Y_2Y_4 = Y_6^2,$$

select a unique pair of triplets $[(X_2, X_4, X_6), (Y_2, Y_4, Y_6)]$, which is in accordance with the theory of two-valued groups.

## 5  Rational limit of a Kummer surface and two-valued addition law

We start with a genus two curve $V$ given by affine equation

$$V = \{(x,y) \in \mathbb{C}^2 : y^2 = x^5 + \lambda_4 x^3 + \lambda_6 x^2 + \lambda_8 x + \lambda_{10}\}.$$  

Corresponding Jacobian $\text{Jac}(V)$ as two-dimensional complex torus is a factor of $\mathbb{C}^2$ with a lattice $\Gamma$. The lattice $\Gamma$ is determined by the vector $(\lambda_4, \lambda_6, \lambda_8, \lambda_{10})$. Note that the indices $(4, 6, 8, 10)$ of $\lambda$ are chosen to fit with the graduation in sequel. The Kummer surface $K$ is the factor of the Jacobian by the group of automorphisms of order 2:

$$K = \text{Jac}(V)/\pm.$$  

Locally, in a vicinity of $0 = (0, 0)$ the Kummer surface $K$ is isomorphic to $\mathbb{C}^2/\pm$. Moreover, all constructions allow the rational limit: $\lambda \to 0$.

Starting from the vector $(\lambda_4, \lambda_6, \lambda_8, \lambda_{10})$ and a vector $u = (u_1, u_3)$, a function $\sigma(u, \lambda)$ is constructed as an entire function in $u$ and $\lambda$, such that all coefficients $c_{ij}(\lambda)$ in expansion in $u_1^iu_3^j$ are polynomials in $\lambda$.  

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5.1 Two-dimensional Weierstrass functions

Following [10], we introduce
\[ \zeta_k = \frac{\partial}{\partial u_k} \ln \sigma(u, \lambda), \quad k = 1, 3 \]
\[ \wp_{kl} = -\frac{\partial^2}{\partial u_k \partial u_l} \ln \sigma(u, \lambda), \quad k, l = 1, 3. \]

The last, Weierstrass functions are functions on the Jacobian \( \text{Jac}(V) \).

We are going to use the standard sigma function, the one defined by arf-invariants \((\ell, \ell')\) to be equal to \( \ell = (1, 1) \), \( \ell' = (2, 1) \).

The standard sigma function is odd in \( u \),
\[ \sigma(u, \lambda) = -\sigma(-u, \lambda), \]
while the \( \wp \)-functions are even:
\[ \wp_{k,l}(u, \lambda) = \wp_{k,l}(-u, \lambda). \]

Thus, the \( \wp \)-functions \( \wp_{k,l}(u, \lambda) \) generate well-defined functions on the Kummer surface \( K \).

There is an embedding with each fixed \( \lambda \):
\[ \pi_\lambda : K \rightarrow \mathbb{C}P^3 \]
\[ [u, -u] \mapsto [\sigma^2(u), \sigma^2(u)\wp_{11}(u), \sigma^2(u)\wp_{13}(u), \sigma^2(u)\wp_{33}(u)]. \]

Note that \( \sigma^2(u)\wp_{kl}(u) \) are entire functions. The embeddings \( \pi_\lambda \) serve to describe addition law on \( K \) in terms of coordinates on \( \mathbb{C}P^3 \).

One can easily compute the limit of the functions when \( \lambda \) tends to zero:
\[ \lim_{\lambda \to 0} \sigma(u, \lambda) = \sigma_0(u) = u_3 - \frac{1}{3} u_1^3 \]
\[ \zeta_1^{(0)} = \frac{\partial}{\partial u_1} \ln \sigma_0(u) = -\frac{u_1^2}{\sigma_0(u)} \]
\[ \zeta_3^{(0)} = \frac{\partial}{\partial u_3} \ln \sigma_0(u) = -\frac{1}{\sigma_0(u)} \]
\[ \wp_{11}^{(0)} = -\frac{\partial^2}{\partial u_1^2} \ln \sigma_0(u) = -\frac{2u_1 \sigma_0(u) + u_1^4}{\sigma_0^2(u)} \]
\[ \wp_{13}^{(0)} = -\frac{\partial}{\partial u_1} \zeta_3^{(0)}(u) = -\frac{u_1^2}{\sigma_0(u)^2} \]
\[ \wp_{33}^{(0)} = -\frac{\partial}{\partial u_3} \zeta_3^{(0)}(u) = \frac{1}{\sigma_0(u)}. \]

Thus,
\[ \sigma_0^2 \wp_{13}^{(0)} = -u_1^2, \quad \sigma_0^2 \wp_{33}^{(0)} = 1. \]
5.2 Rational limit embedding and two-valued group law

Now, we are going to construct a new two-valued group law on \( \mathbb{C}^2/\pm \) associated with an embedding \( \pi_K \) induced by the rational limit of a Kummer surface:

\[
\pi_K : \mathbb{C}^2/\pm \rightarrow \mathbb{C}^3 \\
\pi_K([u, -u]) = \left( u_3 - \frac{1}{3}u_1^3, 2u_1u_3 + \frac{1}{3}u_1^4, -u_1^2 \right),
\]

where \( u = (u_1, u_3) \).

One checks that an inverse image is a two element set if nonempty:

\[
x_1 = \left( u_3 - \frac{1}{3}u_1^3 \right)^2 \\
x_2 = 2u_1u_3 + \frac{1}{3}u_1^4 \\
x_3 = -u_1^2.
\]

The multiplication \( \mu \) given by the formula

\[
\mu([u, -u], [v, -v]) = [(u_1 + v_1), -(u_1 + v_1), (u_3 + v_3), -(u_3 + v_3), (u_1 - v_1), -(u_1 - v_1), (u_3 - v_3), -(u_3 - v_3)]
\]

after composition with the embedding \( (\pi_K)^2 \) leads to the formulæ

\[
\hat{X}_6 = \left( u_3 + v_3 - \frac{1}{3}(u_1 + v_1)^3 \right)^2 \\
\hat{X}_4 = 2(u_1 + v_1)(u_3 + v_3) + \frac{1}{3}(u_1 + v_1)^4 \\
\hat{X}_2 = -(u_1 + v_1)^2
\]

and

\[
\hat{Y}_6 = \left( u_3 - v_3 - \frac{1}{3}(u_1 - v_1)^3 \right)^2 \\
\hat{Y}_4 = 2(u_1 - v_1)(u_3 - v_3) + \frac{1}{3}(u_1 - v_1)^4 \\
\hat{Y}_2 = -(u_1 - v_1)^2.
\]

The last formulæ lead to the following change of variables:

\[
\hat{X}_2 = -X_2 \\
\hat{X}_4 = 2X_4 + \frac{1}{3}X_2^2 \\
\hat{X}_6 = X_6 - \frac{2}{3}X_2X_4 + \frac{1}{9}X_2^3.
\]
This is an algebraic change of variables and the inverse change is given by the formulae:

\[
X_2 = -\hat{X}_2 \\
X_4 = \frac{1}{2} \left( \hat{X}_4 - \frac{1}{3} \hat{X}_2^2 \right) \\
X_6 = \hat{X}_6 - \frac{1}{3} \hat{X}_2 \hat{X}_4 - \frac{2}{9} X_2^3
\]

By applying the last algebraic change of variables on the equation of the quadric \( Q : X_2 X_6 = X_4^2 \) we get the equation of the rational limit of the Kummer surface.

**Proposition 1** The rational limit of the Kummer surface is given by the surface in \( \mathbb{C}^3 \) by the equation

\[
-9\hat{X}_4^2 - 36\hat{X}_2 \hat{X}_6 + 12\hat{X}_2^2 \hat{X}_4 + 7X_4^4 = 0. \tag{8}
\]

### 5.3 Rational Kummer two-valued group

In the previous section, we have constructed a two-valued group law in coordinates \((X_2, X_4, X_6)\) - the elementary two-valued group. Now, using the algebraic change of variables, we are going to construct a new two-valued group law, and we will call it the rational Kummer two-valued group.

First, we consider \((\hat{X}_2, \hat{Y}_2)\). We have

\[
\hat{X}_2 = -x_3 - 2u_1 v_1 - y_3 \\
\hat{Y}_2 = -x_3 + 2u_1 v_1 - y_3
\]

therefore

\[
\hat{X}_2 + \hat{Y}_2 = -2(x_3 + y_3) \\
\hat{X}_2 \hat{Y}_2 = (x_3 - y_3)^2.
\]

Thus we see that the pair \((\hat{X}_2, \hat{Y}_2)\) is the solution of the quadratic equation

\[
Z^2 + 2(x_3 + y_3)Z + (x_3 - y_3)^2 = 0. \tag{9}
\]

Now, we pass to the pair \((\hat{X}_4, \hat{Y}_4)\). They can be represented in the form

\[
\hat{X}_4 = \hat{X}_4^+ + \hat{X}_4^- \\
\hat{Y}_4 = \hat{X}_4^+ - \hat{X}_4^-
\]

where

\[
\hat{X}_4^+ = 2(u_1 u_3 + v_1 v_3) + \frac{1}{3}(u_1^4 + 6u_1^2 v_1^2 + v_1^4) \\
\hat{X}_4^- = 2(v_1 u_3 + u_1 v_3) + \frac{4}{3} u_1 v_1 (u_1^2 + v_1^2).
\]
Then, we have
\[ \hat{X}_4 + \hat{Y}_4 = 2\hat{X}_4^+ \]
\[ \hat{X}_4 \hat{Y}_4 = (\hat{X}_4^+)^2 - (\hat{X}_4^-)^2. \]
Thus, \((\hat{X}_4, \hat{Y}_4)\) are the roots of the quadratic equation
\[ Z^2 - 2\hat{X}_4^+ Z + (\hat{X}_4^+)^2 - (\hat{X}_4^-)^2 = 0. \] (10)
The last equation is equivalent to
\[ (Z - (\hat{X}_4^+))^2 = (\hat{X}_4^-)^2, \]
where \(\hat{X}_4^+, \hat{X}_4^-\) can be rewritten in the form
\[ \hat{X}_4^+ = 2(u_1 u_3 + v_1 v_3) + \frac{1}{3}(u_1^4 + 6u_1^2 v_1^2 + v_1^4) \]
\[ \hat{X}_4^- = 2(v_1 u_3 + u_1 v_3) + \frac{4}{3}u_1 v_1 (u_1^2 + v_1^2). \]
We pass to the last pair \(\hat{X}_6, \hat{Y}_6\):
\[ \hat{X}_6 = \left( (u_3 + v_3) - \frac{1}{3}(u_1 + v_1)^3 \right)^2 \]
\[ \hat{Y}_6 = \left( (u_3 - v_3) - \frac{1}{3}(u_1 - v_1)^3 \right)^2. \]
One can easily calculate
\[ \hat{X}_6 + \hat{Y}_6 = 2[u_3^2 + v_3^2 - \frac{2}{3}(u_3^4 u_3 + 3u_1 u_3 v_1 + 3u_1^2 v_1 v_3 + v_1^4) + \frac{1}{9}(u_1^6 + 15u_1^4 v_1^2 + 15u_1^2 v_1^4 + v_1^6)] \]
\[ \hat{X}_6 \hat{Y}_6 = \left( (u_3^2 - v_3^2) + \frac{1}{9}(u_1^2 - v_1^2)^3 - \frac{2}{3}(u_3 u_1^2 + 3u_1 v_1^2 u_3^2 - 3u_1^2 v_1 v_3 - v_1^3 v_3) \right)^2, \]
or, in the old coordinates
\[ \hat{X}_6 + \hat{Y}_6 = 2 \left( x_3 + y_3 - \frac{2}{3}(x_3 x_2 + 3x_2 y_1 + 3x_1 y_2 + y_1^2) + \frac{1}{9}(x_1^3 + 15x_1^2 y_1 + 15x_1 y_1^2 + y_1^3) \right) \]
\[ \hat{X}_6 \hat{Y}_6 = \left( (x_3 - y_3) + \frac{1}{9}(x_1 - y_1)^3 - \frac{2}{3}(x_2 x_1 + 3x_2 y_1 - 3x_1 y_2 - y_1 y_2) \right)^2. \]
In the new coordinates one may rewrite
\[ B_3 := \hat{X}_6 + \hat{Y}_6 \]
\[ C_3 := \hat{X}_6 \hat{Y}_6. \]
and to get finally

\[ B_3 = 2((\hat{x}_6 - \frac{1}{3}\hat{x}_4\hat{x}_2 - \frac{2}{9}\hat{x}_2^3) + (\hat{y}_6 - \frac{1}{3}\hat{y}_4\hat{y}_2 - \frac{2}{9}\hat{y}_2^3)) \]

\[ - \frac{1}{3}(-\hat{x}_2\hat{x}_4 + \frac{1}{3}\hat{x}_2^3 - 3\hat{x}_4\hat{y}_2 + \hat{x}_2^2\hat{y}_2 - 3\hat{x}_2\hat{y}_4 + \hat{x}_2\hat{y}_2^2 + 2\hat{y}_2^2) \]

\[ + \frac{1}{9}(-\hat{x}_2^3 + 15\hat{x}_2\hat{y}_2 + 15\hat{x}_2\hat{y}_2^2 - \hat{y}_2^3) \]

and

\[ C_3 = (\hat{x}_6 - \frac{1}{3}\hat{x}_4\hat{x}_2 - \frac{2}{9}\hat{x}_2^3 - \hat{y}_6 + \frac{1}{3}\hat{y}_4\hat{y}_2 + \frac{2}{9}\hat{y}_2^3 + \frac{1}{9}(\hat{y}_2 - \hat{x}_2)^3) \]

\[ - \frac{1}{9}(\frac{1}{3}\hat{x}_2^2 - \hat{x}_2\hat{x}_4 + \hat{x}_2^2\hat{y}_2 - 3\hat{x}_4\hat{y}_2 + 3\hat{y}_4\hat{x}_2 - \hat{y}_2^2\hat{x}_2 \]

\[ + (\hat{y}^2\hat{y}_4 - \frac{1}{3}\hat{y}_2^3)]^2. \]

Thus, we may conclude that the pair \((\hat{X}_6, \hat{Y}_6)\) is determined as the roots of the quadratic equation

\[ Z^2 - B_3Z + C_3 = 0, \]  

(11)

where \(B_3, C_3\) are functions of the coordinates \((\hat{x}_2, \hat{x}_4, \hat{x}_6, \hat{y}_2, \hat{y}_4, \hat{y}_6)\) given above.

6 Two-valued group structures on Kummer varieties and sigma-functions

We start with the sigma-function \(\sigma(u) = \sigma(u, \lambda)\), where \(u^\top = (u_1, u_3)\), \(\lambda^\top = (\lambda_4, \lambda_6, \lambda_8, \lambda_{10})\), associated with a curve

\[ V = \{(t, s) \in \mathbb{C}^2 : t^2 = s^5 + \lambda_4 s^3 + \lambda_6 s^2 + \lambda_8 s + \lambda_{10}\}. \]

We assume that \(\lambda \in \mathbb{C}^4\) is a non-discriminant point of the curve \(V\) and \(u \in \mathbb{C}^2\), where \(du_1 = \frac{dt}{t}\), \(du_3 = \frac{ds}{s}\). Indexation of the coordinates of the vector of the parameter \(\lambda\) is chosen according to the graduation \(\text{deg} s = -2\), \(\text{deg} t = -5\), \(\text{deg} \lambda_2 = 2i\). Moreover, \(\text{deg} u_1 = 1\), \(\text{deg} u_3 = 3\), and the sigma-function

\[ \sigma(u, \lambda) = u_3 - \frac{1}{3}u_1^3 + \frac{1}{6}\lambda_6 u_3^3 + (u^5) \]

is an entire and homogeneous of degree 3 in \(u\) and \(\lambda\). Recurrent description of the series for \(\sigma(u, \lambda)\) is given in [11].

We consider the Jacobian \(J_2 = \text{Jac}(V) = \mathbb{C}^2/\Gamma_2\) of the curve \(V\) and a vector-function is defined by the formulae

\[ \varphi(u) = (\varphi_{33}(u), \varphi_{13}(u), \varphi_{11}(u)), \]
where \( \wp_{kl}(u) = -\frac{\partial^2}{\partial u_k \partial u_l} \ln \sigma(u), \ k, l = 1, 3. \) Since the function \( \sigma(u) \) is odd, we get a mapping

\[
i : J_2 \rightarrow \mathbb{C}P^3 : i(u) = (x_0 : x_2 : x_4 : x_6),
\]

where \( x_0 = \sigma(u)^2 \wp_{33}(u), \ x_2 = \sigma(u)^2 \wp_{13}(u), \ x_4 = \sigma(u)^2 \wp_{11}(u), \ x_6 = \sigma(u)^2. \)

The last mapping factorizes through the Kummer variety \( K \) together with an embedding

\[
\hat{i} : K = (J_2/\pm) \rightarrow \mathbb{C}P^3.
\]

Let us note, that the embedding \( \hat{i} \) is defined with entire homogeneous functions \( x_{2k}(u), \ \deg x_{2k} = 2k, k = 0, \ldots, 3. \)

We are going to use a ramified covering \( \gamma : \mathbb{C}P^3 \times \mathbb{C}P^3 \rightarrow \mathbb{C}P^6, \)

defined by the relation

\[
[p(x, t), p(y, t)] \rightarrow p(x, t)p(y, t),
\]

where \( p(x, t) = x_0t_3 + x_2t_2t_1 + x_4t_3t_1^2 + x_6t_1^3. \) By putting \( \deg t_k = k, \) we get \( p(x, t) \) as a homogeneous polynomial of degree 9.

**Theorem 2** Multiplication \( m \) in the two-valued group on \( K \)

\[
m : K \times K \rightarrow (K)^2 : [u] * [u] = ([u + v], [u - v])
\]

is defined through algebraic mapping

\[
\mu : \mathbb{C}P^3 \times \mathbb{C}P^3 \rightarrow \mathbb{C}P^6 :
\]

and it is uniquely defined by the commuting condition of the following diagram

\[
\begin{array}{ccc}
K \times K & \xrightarrow{m} & (K)^2 \\
\downarrow \hat{i} \times \hat{i} & & \downarrow \gamma \\
\mathbb{C}P^3 \times \mathbb{C}P^3 & \xrightarrow{\mu} & \mathbb{C}P^6
\end{array}
\]

**Proof.** Set

\[
X(u) = (\wp_{33}(u), \wp_{31}(u), \wp_{11}(u), 1) \quad \mathcal{X}(u) = \sigma(u)^2 X(u).
\]

Consider the canonical projection

\[
\pi : \mathbb{C}^7 \setminus 0 \rightarrow \mathbb{C}P^6 : \pi(z) = (z_0 : z_2 : \ldots : z_{12}) = [z].
\]

According to the construction, we have

\[
\widehat{i} [u] = [\mathcal{X}(u)], \quad \gamma(\hat{i})^2 ([u] * [v]) = \gamma([\mathcal{X}(u + v), \mathcal{X}(u - v)]).
\]
Thus, we have to show that each coordinate \( z_{12}(u, v) \), \( k = 0, \ldots, 6 \), of the point 
\( \gamma([X(u + v), X(u - v)]) \) is a polynomial of the coordinates \( x_{2i}, y_{2i}, i = 0, \ldots, 3 \) 
of the points \( X(u), X(v) \).

Genus two sigma-function \( \sigma(u) \) satisfies the following addition theorem (see [10], [11]):

\[
\sigma(u + v)\sigma(u - v) = N(u) \mathcal{J} N(v).
\]

where \( \mathcal{J} = \begin{pmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{pmatrix} \) and \( \varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). We have,

\[
z_{12} = z_{12}(u, v) = \left( N(u) \mathcal{J} N(v) \right)^2.
\]

Thus, we get that in the mapping \( \mu(x, y) = [z] = (z_0 : z_2 : \ldots : z_{12}) \in \mathbb{C}P^6 \), 
the coordinate \( z_{12} \) is defined by the formula \( z_{12} = (x^\top \mathcal{J} y)^2 \), where \( (x, y) \in \mathbb{C}P^3 \times \mathbb{C}P^3 \).

Analogous result for the rest of the coordinates is based on deep facts about algebraic generators of the ring generated by logarithmic derivatives of order 2 and higher of the sigma-function \( \sigma(u) \) (see [10]).

Set

\[
M(u, v) = X(u)^\top \mathcal{J} X(v).
\]

We have

\[
\ln \sigma(u + v) + \ln \sigma(u - v) = 2(\ln \sigma u + \ln \sigma v) + \ln M(u, v).
\]

Apply the operators \( \partial_k = \frac{\partial}{\sigma u_k} + \frac{\partial}{\sigma v_k} \), for \( k = 1 \) and 3. We get

\[
2\zeta_k(u + v) = 2(\zeta_k(u) + \zeta_k(v)) + \frac{1}{M(u, v)} \left( X_k(u)^\top \mathcal{J} X(v) + X(u)^\top \mathcal{J} X_k(v) \right),
\]

where \( \zeta_k(u) = \frac{\partial}{\sigma u_k} \ln \sigma(u) \) \( X_k(u) = \frac{\partial}{\sigma u_k} X(u) \). Now, we apply the operator \( \frac{\partial}{\sigma u_l} \),

and we get

\[
\varphi_{kl}(u + v) = -\frac{\partial}{\sigma u_l} \zeta_k(u + v) = \varphi_{kl}(u, v) - \psi_{kl}(u, v),
\]

where

\[
\varphi_{kl}(u, v) = \varphi_{kl}(u) - \frac{1}{2M(u, v)^2} \left\{ (X_{kl}(u)^\top \mathcal{J} X(v)) M(u, v) - X_k(u)^\top B(v) X_l(u) \right\},
\]

\[
\psi_{kl}(u, v) = \frac{1}{2M(u, v)^2} \left( X_l(u)^\top C(u, v) X_k(v) \right)
\]

and \( B(v) = \mathcal{J} X(v) X(v)^\top \mathcal{J} \), \( C(u, v) = (M(u, v) - \mathcal{J} X(v) X(u)^\top) \mathcal{J} \).

Note \( B(-v) = B(v) \) \( C(u, v) = C(u, -v) = C(u, v) \). Thus, \( \varphi_{kl}(u, -v) = \varphi_{kl}(u, v) \) \( \psi_{kl}(u, v) = -\psi_{kl}(u, v) \). We get

\[
N(u + v) = \sigma(u + v)^2(X_1 - X_2) \quad N(u - v) = \sigma(u - v)^2(X_1 + X_2),
\]

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where \( X_1 = (\varphi_{33}, \varphi_{13}, \varphi_{11}, 1) \) \( X_2 = (\psi_{33}, \psi_{13}, \psi_{11}, 0) \). We obtain

\[
p(X(u + v), t)p(X(u - v), t) = z_1 \cdot (p(X_1, t)^2 - p(X_2, t)^2)
\]

where

\[
p(X_1, t) = \varphi_{33} t_3^3 + \varphi_{13} t_3^2 t_1 + \varphi_{11} t_3 t_1^2 + t_1^3,
\]

\[
p(X_2, t) = t_3(\psi_{33} t_3^3 + \psi_{13} t_3 t_1 + \psi_{11} t_1^2).
\]

From the above formulae for \( \varphi_{kl}(u, v) \) \( \psi_{kl}(u, v) \psi_{pq}(u, v) \), it immediately follows that these functions are polynomials of \( \varphi_{ij}(u, v) \), \( \varphi_{ijk}(u) \psi_{ij}(u) \), \( \varphi_{ijk}(u) \psi_{ij}(v) \), \( \varphi_{ij}(v) \) \( \psi_{ij}(v) \psi_{ij}(v) \). The functions \( \varphi_{ijk}(u) \psi_{ij}(v) \) \( \varphi_{ijkl}(u) \psi_{ij}(v) \), where \( i, j, k, l \) take values 1 or 3 independently, are polynomials of \( \varphi_{ij}(u) \) (see [16]). Consequently, using \( (x, y)_{1 \times 7} = (\mathcal{X}(u), \mathcal{X}(v)) \), we get coordinates \( z_{2k} \) of the vector \( \mu(x, y) \) as polynomials of the coordinates of the vectors \( x \) and \( y \).

\[ \square \]

### 7 Homomorphism of rings of functions, induced by Abel mapping in genus 2

Let \( V = \{(s, \mu) \in \mathbb{C}^2 : \mu^2 = s^5 + \lambda_4 s^3 + \lambda_6 s^2 + \lambda_8 s + \lambda_{10} \} \) denotes a hyperelliptic curve.

**Theorem 3** *The Abel mapping*

\[ \mathcal{A} : (V)^2 \longrightarrow \text{Jac} V \]

*induces a homomorphism of rings of functions*

\[ \mathcal{A}^* : \mathcal{F}(\text{Jac} V) \longrightarrow \mathcal{F}((V)^2) \]

*such that*

\[ \varphi_{11}(u) = s_1 + s_2, \quad \varphi_{13}(u) = -s_1 s_2, \quad \varphi_{33}(u) = \frac{F(s_1, s_2) - 2\mu_1 \mu_2}{(s_1 - s_2)^2}, \]

*where*

\[ F(s_1, s_2) = 2\lambda_{10} + \lambda_8 (s_1 + s_2) + s_1 s_2 (2\lambda_6 + \lambda_4 (s_1 + s_2)), \]

\[ \varphi_{111}(u) = 2\frac{\mu_1 - \mu_2}{s_1 - s_2}, \quad \varphi_{113}(u) = 2\frac{s_1 \mu_2 - s_2 \mu_1}{s_1 - s_2}, \]

\[ \varphi_{331}(u) = -2\frac{s_1 \mu_2 - s_2 \mu_1}{s_1 - s_2}, \quad \varphi_{333}(u) = 2\frac{\psi(s_1, s_2) \mu_2 - \psi(s_2, s_1) \mu_1}{(s_1 - s_2)^3}, \]

*and*

\[ \psi(s_1, s_2) = 4\lambda_{10} + \lambda_8 (3s_1 + s_2) + 2\lambda_6 s_1 (s_1 + s_2) + \lambda_4 s_1^3 (s_1 + s_2) + s_1^3 s_2 (3s_1 + s_2). \]
The proof of the Theorem can be found in [10]. Let us note that we use the
indexation here different from [10], in correspondence with the graduation:

$$\deg s = -2; \deg \mu = -5; \deg \lambda_{2k} = -2k, \ k = 2, 3, 4, 5; \ \deg u_i = i, \ i = 1, 3.$$ 

This provides additional opportunity to check the formulae. Observe that
$$\deg \varphi_{kl}(u) = -(k + l), \ \deg \varphi_{klp}(u) = -(k + l + p).$$

Any Abelian function on Jac$V$ represents a linear function of $\varphi_{111}(u)$ with coefficients which are rational functions of $\varphi_{11}(u)$ and $\varphi_{13}(u)$. On the other hand, from the theory of polysymmetric functions (see [23] and [12]), it is known
that the field of rational functions on $(\mathbb{C}^2)^2$ in the coordinates $[(s_1, \mu_1), (s_2, \mu_2)]$ is generated by polysymmetric functions

$$e_{10} = s_1 + s_2, \ e_{20} = s_1s_2, \ e_{01} = \mu_1 + \mu_2, \ e_{02} = \mu_1\mu_2, \ e_{11} = s_1\mu_2 + s_2\mu_1,$$

which are related by unique (for $(\mathbb{C}^2)^2$) relation

$$(e_{10}^2 - 4e_{20})(e_{01}^2 - 4e_{02}) = (e_{10}e_{01} - 2e_{11}).$$

The mapping $\mathcal{A}$ is a birational equivalence, thus $\mathcal{A}^*$ induces isomorphism of the field of Abelian functions $\mathcal{F}(\text{Jac}V)$ on Jac$V$ with the field of rational functions $\mathcal{F}((V)^2)$ on $(V)^2$. We have:

$$\mathcal{A}^*(\varphi_{11}(u)) = e_{10}, \ \mathcal{A}^*(\varphi_{13}(u)) = -e_{20}, \ \mathcal{A}^*(\varphi_{111}(u)) = \frac{e_{10}e_{01} - 2e_{11}}{e_{10}^2 - 4e_{20}}.$$ 

In this way, the Theorem [3] completely describes the isomorphism $\mathcal{A}^*$ and gives an opportunity to express explicitly, for example, the even functions $\varphi_{klp}(u)\varphi_{k'\ell'}(u)$ as polynomials of $\varphi_{kl}(u)$. The explicit formulae for those polynomials are given in the book [10].

The hyperelliptic involution acts on $(V)^2$ according to the formula

$$[(s_1, \mu_1), (s_2, \mu_2)] \rightarrow [(s_1, -\mu_1), (s_2, -\mu_2)].$$

The Abel mapping is invariant with respect to this involution on $(V)^2$ and the involution $u \rightarrow -u$ on Jac$V$.

From the above formulae one can see that the images of the functions $\varphi_{kl}(u)$ and $\varphi_{klp}(u)\varphi_{k'\ell'}(u)$ are even, while the images of the functions $\varphi_{klp}(u)$ are odd. Thus, the mapping

$$\hat{A}: (V)^2/_{\pm} \rightarrow K = (\text{Jac}V)/_{\pm},$$

is defined and it induces a homomorphism between rings of functions.

There is an addition law on $(V)^2$, with the Abel mapping $\mathcal{A}$ as a homomorphism. Explicit form of this operation in the coordinates $[(s_1, \mu_1), (s_2, \mu_2)]$ has been described in [11].

Thus, on $(V)^2/_{\pm}$ there is corresponding two-valued addition, such that $\hat{A}$ is a homomorphism with respect to the two-valued group structure on the Kummer variety $K$, defined above.
8 Solutions of the system of equations of S. V. Kowalevski in genus two \( \wp \)-functions

We follow chapter IV of the Golubev book [24]. Kowalevski introduced variables \( s_1, s_2 \), which satisfy the system of equations (see equations (17) and (18) from [24]):

\[
\frac{ds_1}{\sqrt{\Phi(s_1)}} + \frac{ds_2}{\sqrt{\Phi(s_2)}} = 0, \quad (12)
\]

\[
\frac{s_1 ds_1}{\sqrt{\Phi(s_1)}} + \frac{s_2 ds_2}{\sqrt{\Phi(s_2)}} = \frac{i}{2} dt, \quad (13)
\]

where \( \Phi(s) \) is a polynomial of fifth degree. The Abel mapping

\[ A : (V)^2 \rightarrow \text{Jac}V \]

where \( V = \{ (s, \mu) \in \mathbb{C}^2 : \mu^2 = \Phi(s) \} \), \( \Phi(s) = s^5 + \lambda_2 s^3 + \lambda_4 s^2 + \lambda_6 s + \lambda_10 \), is defined by the system of equations

\[
\frac{ds_1}{\mu_1} + \frac{ds_2}{\mu_2} = du_3, \quad (14)
\]

\[
\frac{s_1 ds_1}{\mu_1} + \frac{s_2 ds_2}{\mu_2} = du_1. \quad (15)
\]

We are going to describe solutions of the system of equations [12], [13], following the book [10].

Let us consider the sigma-function \( \sigma(u) = \sigma(u, \lambda) \), associated with the curve \( V \), and corresponding Abelian functions

\[ \wp_{k,l}(u) = -\frac{\partial}{\partial u_k \partial u_l} \ln \sigma(u), \ k = 1, 3. \]

The general solution of the system of equations [14], [15], which is the solution of the Jacobi inversion problem for the Abel mapping, is represented by the point \([ (s_1, \mu_1), (s_2, \mu_2) ] \), where \((s_1, s_2)\) are the solutions of the equation

\[
 s^2 - \wp_{11}(u)s - \wp_{13}(u) = 0, \quad (16)
\]

and for given \( s_l \) we have

\[
 2\mu_l = \wp_{11l}(u)s_l + \wp_{13l}(u), \ l = 1, 2. \quad (17)
\]

Thus, we get the solution \([ (s_1(t), \mu_1(t)), (s_2(t), \mu_2(t)) ] \in (V)^2 \) of the Kowalevski system of equations [12], [13] in terms of genus 2 \( \wp \)-functions, where \((s_l(t), \mu_l(t)) = (s_l(t, c_1, c_3), \mu_l(t, c_1, c_3)), \ l = 1, 2, \) is the solution of the Jacobi problem for
\[ u = (u_1, u_3) \in \text{Jac}V, \text{ where } u_1 = u_1(t) = c_1 + it, \ u_3 = c_3 \text{ and } c_1, \ c_3 \text{ are constants.} \]

The solution of the system (12), (13) based on the sigma - functions, unlike the classical solutions in terms of the theta-functions, is stable under the limit procedure when \( \lambda \to 0. \)

Without changing the notation, let us consider corresponding functions, associated with the curve \( V_0 = \{(s, \mu) \in \mathbb{C}^2 : \mu^2 = s^5\}. \) We have

\[ \sigma(u) = u_3 - \frac{1}{3} u_1^3, \]
\[ \sigma(u)^2 \varphi_{11}(u) = 2u_1 \left(u_3 + \frac{1}{6} u_1^3\right), \]
\[ \sigma(u)^2 \varphi_{13}(u) = -u_1^2. \]

Thus, for \( \sigma(u) \neq 0, \) the equation (16) with \( \lambda \to 0 \) is equivalent to

\[ \hat{s}^2 - 2u_1 \left(u_3 + \frac{1}{6} u_1^3\right) \hat{s} + u_1^2 \sigma(u)^2 = 0, \tag{18} \]

where \( \hat{s} = \sigma(u)^2 s. \) Further,

\[ \sigma(u)^3 \varphi_{111}(u) = \sigma(u)^3 \frac{\partial}{\partial u_1} \varphi_{111}(u) = 2 \left[u_3^2 + \frac{7}{3} u_1^3 u_3 + \frac{1}{9} u_1^6\right], \]
\[ \sigma(u)^3 \varphi_{113}(u) = \sigma(u)^3 \frac{\partial}{\partial u_1} \varphi_{113}(u) = -2u_1 \left(u_3 + \frac{2}{3} u_1^3\right). \]

In this way, for \( \sigma(u) \neq 0, \) the formula (17) with \( \lambda \to 0 \) is equivalent to

\[ \hat{\mu}_1 = \left[u_3^2 + \frac{7}{3} u_1^3 u_3 + \frac{1}{9} u_1^6\right] \hat{s} - u_1 \left(u_3 + \frac{2}{3} u_1^3\right) \sigma(u)^2, \tag{19} \]

where \( \hat{\mu}_1 = \sigma(u)^5 \mu. \) Observe, that \( \hat{\mu}_1^2 = \hat{s}^5. \)

Thus, the general solution of the inversion problem \( [(s_1, \mu_1), (s_2, \mu_2)] \in (V_0)^2, \) where \( V_0 = \{(s, \mu) \in \mathbb{C}^2 : \mu^2 = s^5\}, \) has the form

\[ \sigma(u)^2 s_{1,2} = u_1 \left[u_3^3 + \frac{1}{6} u_1^3\right] \pm \sqrt{u_1^3 \left[u_3^3 - \frac{1}{12} u_1^3\right]}, \tag{20} \]

and \( \hat{\mu}_1 = \sigma(u)^5 \mu_1 \) are given by the formula (19).

Let us consider important particular cases:

1. Let \( u_1 = 0, \) then \( \sigma(u) = u_3 \) and

\[ s_1 = s_2 = 0, \ \mu_1 = \mu_2 = 0. \]

2. Let \( u_3 = 0, \) then \( \sigma(u) = -\frac{1}{3} u_1^3 \) and

\[ \hat{s}_{1,2} = \frac{1}{6} \left(1 \pm i \sqrt{3}\right) u_1^4, \ \hat{\mu}_{1,2} = \frac{1}{18} \left(-1 \pm \frac{i}{\sqrt{3}}\right) u_1^{10}. \]
Check: the identity holds
\[
\frac{1}{18^2} \left( -1 + \frac{i}{\sqrt{3}} \right)^2 = \frac{1}{6^4} \left( 1 + i \sqrt{3} \right)^5 .
\]

By taking \( u_1 = c_1 + \frac{i}{2} t \) and \( u_3 = 0 \), we get particular solution of the Kowalevski system in the rational limit.

3. Let \( u_3 + \frac{2}{3} u_1^3 = 0 \), then \( \sigma(u) = u_1^3 \)
\[
\hat{s}_{1,2} = \frac{1}{2} \left( -1 \pm i \sqrt{3} \right) u_1^4, \quad \hat{\mu}_i = -\hat{s}_i u_1^6 .
\]
Check: the identity holds \((-1 \pm i \sqrt{3})^3 = 8.

4. Let \( u_3 = \frac{1}{12} u_1^3 \), then \( \sigma(u) = -\frac{1}{3} u_1^3 \)
\[
\hat{s}_{1,2} = \frac{1}{2^2} u_1^4, \quad \hat{\mu}_{1,2} = \frac{1}{2^5} u_1^{10} .
\]

9 Geometric two-valued group laws and Kummer varieties

9.1 A quadric in \( \mathbb{C}P^5 \) and a line complex in \( \mathbb{C}P^3 \)

Following classics, let us consider a three-dimensional projective space \( \mathbb{C}P^3 = P(\mathbb{C}^4) \) and corresponding Grassmannian \( Gr(2, 4) \) of all lines in \( \mathbb{C}P^3 \). By Plücker embedding, the Grassmannian \( Gr(2, 4) \) can be realized as a quadric \( G \) in \( \mathbb{C}P^5 = P(\wedge^2 V) \), where \( V = \mathbb{C}^4 \):
\[
Gr(2, 4) \hookrightarrow \mathbb{C}P^5
\]
\[
\ell = \ell < v_1, v_2 > \mapsto v_1 \wedge v_2, \quad v_1, v_2 \in V^4.
\]
The quadric \( G \) parameterizes all decomposable elements \( w = v_1 \wedge v_2 \) of \( P(\wedge^2 V) \) and the quadric is described by the Plücker quadratic relation:
\[
G : w \wedge w = 0.
\]
For a given element \( x \in G \), denote by \( \ell_x \) the line in \( \mathbb{C}P^3 \) which maps to \( x \) by the above embedding.
We consider so-called Schubert cycles:

\[ \sigma_1(\ell) = \{ x \in G \mid \ell_x \cap \ell \neq \emptyset \} \]
\[ \sigma_2(p) = \{ x \in G \mid \ell_x \ni p \} \]
\[ \sigma_{1,1}(h) = \{ x \in G \mid \ell_x \subset h \} \]
\[ \sigma_{2,1}(p, h) = \{ x \in G \mid p \in \ell_x \subset h \} \]

with the intersection table

\[
\begin{align*}
\sigma_1 \cdot \sigma_1 &= \sigma_2 + \sigma_{1,1} \\
\sigma_1 \cdot \sigma_2 &= \sigma_1 \cdot \sigma_{1,1} = \sigma_{2,1} \\
\sigma_2 \cdot \sigma_2 &= \sigma_{1,1} \cdot \sigma_{1,1} = \sigma_1 \cdot \sigma_{2,1} = 1 \\
\sigma_2 \cdot \sigma_{1,1} &= 0.
\end{align*}
\]

One can easily see that every cycle \( \sigma_1(\ell) \) is a hyperplane section of the quadric \( G \). If the line \( \ell \in \mathbb{C}P^3 \) is determined by vectors \( v_1, v_2 \in V^4 \) then the hyperplane of intersection is of the form \( H_{v_1 \wedge v_2} = \{ w \mid w \wedge v_1 \wedge v_2 = 0 \} \).

Every cycle \( \sigma_{2,1}(p, h) \) is a line in \( \mathbb{C}P^5 \). Every line \( L \subset G \) is of the form \( L = \sigma_{2,1}(p, h) \).

A line \( L \subset G \) is a pencil of lines in \( \mathbb{C}P^3 \). This is a confocal pencil with the common point, the focus \( p \in \mathbb{C}P^3 \). At the same time this pencil is coplanar with the common plane \( h \in \mathbb{C}P^3^* \).

Every cycle of the form \( \sigma_2(p) \) or of the form \( \sigma_{1,1}(h) \) is a two-plane in \( \mathbb{C}P^5 \). Conversely, every two-plane in \( G \) is of the form \( \sigma_2(p) \) or of the form \( \sigma_{1,1}(h) \).

Let us recall some general properties of a quadric \( Q \) in \( \mathbb{C}P^m \). The rank of the quadric is equal to the rank of any of its symmetric \( (m+1) \times (m+1) \) matrices. The quadric is smooth if its rank is maximal, i.e. if rank of \( Q \) is \( m+1 \). If the rank of \( Q \) is \( r \) then it is a cone over a smooth quadric in \( \mathbb{C}P^{r-1} \) with a vertex \( \mathbb{C}P^{m-r} \). Quadrics of rank \( m+1 \) and \( m \) are called general.

**Lemma 1** Let \( Q \in \mathbb{C}P^{2m+1} \) be a general quadric.

(a) The dimension of a maximal linear subspace of \( Q \) is equal to \( m \).

(b) The collection of maximal linear subspaces \( C(Q) \) forms an algebraic variety of dimension \( m(m+1)/2 \).

(c) If the rank of \( Q \) is even, then \( C(Q) \) has two components. Otherwise, the component is unique. A component is unirational variety.

(d) Let \( L \subset Q \) be a linear subspace of dimension \( m-1 \) which does not contain the vertex of \( Q \). For an irreducible family of maximal subspaces \( A \in C(Q) \), there is a unique maximal subspace \( M = M(L, A) \) of dimension \( m \) which belongs to \( A \) and contains \( L \).
(e) Suppose that \( Q \) is smooth and let \( M_1, M_2 \) be two of its maximal linear subspaces. Then
\[
\dim M_1 \cap M_2 \equiv m \pmod{2}
\]
is equivalent to the fact that \( M_1, M_2 \) belong to the same irreducible component of \( C(Q) \).

Now, we specialize previous statement for the case of a smooth quadric in \( \mathbb{C}P^5 \).

**Proposition 2** Let \( G \) be a smooth quadric in \( \mathbb{C}P^5 \). Then:

(a) There exists a four-dimensional vector space \( V^4 \) such that \( G \) is the Plucker embedding of Grassmannian of lines in \( P(V^4) \).

(b) Maximal linear subspaces of \( G \) are two-dimensional and they are of the form \( \sigma_2(p) \) or of the form \( \sigma_{1,1}(h) \), where \( p \in \mathbb{C}P^3 = P(V^4) \) and \( h \in \mathbb{C}P^{3*} \).

(c) Variety \( C(Q) \) of all two-dimensional subspaces of \( G \) is three-dimensional and it has two irreducible components, \( A \) and \( B \):
\[
A = \{ \sigma_2(p) \mid p \in \mathbb{C}P^3 = P(V^4) \}
\]
\[
B = \{ \sigma_{1,1}(h) \mid h \in \mathbb{C}P^{3*} \}.
\]

### 9.2 Smooth intersection of two quadrics and Abelian varieties

Now we consider the intersection \( X \) of two quadrics \( G \) and \( F \) in \( \mathbb{C}P^5 \). Such a set is classically called quadratic complex of lines, if \( G \) is understood as a Grassmannian of lines in some \( \mathbb{C}P^3 \). Together with two quadrics \( G \) and \( F \), one may consider the whole pencil of quadrics:
\[
F_\lambda := F + \lambda G,
\]
and \( X \) is the base set for the pencil, the common intersection of quadrics from the pencil.

A pencil of quadrics is *generic* if associated pencil of \( 6 \times 6 \) symmetric matrices contains six different singular matrices. For a generic pencil \( F_\lambda \) denote by \( \lambda_1, \ldots, \lambda_6 \) corresponding values of the pencil parameter associated with singular matrices.

The condition that \( X \) is smooth is equivalent to the condition that the pencil \( F_\lambda \) is generic. Smoothness of \( X \) is also equivalent to the fact that all quadrics \( F_\lambda \) are general and that exactly six of them, \( F_{\lambda_1}, \ldots, F_{\lambda_6} \), are singular.

**Proposition 3** Suppose \( X \) is smooth intersection of two quadrics \( X = G \cap F \) in \( \mathbb{C}P^5 \). Then:
(a) Maximal linear subspaces of $X$ are one-dimensional.
(b) There is a maximal linear subspace through each point of $X$.
(c) There are four maximal linear subspaces passing through a generic point of $X$.

Given a smooth intersection of quadrics $X = G \cap F$ in $\mathbb{CP}^5$, following Narasimhan, Ramanan, Reid and Donagi, let us consider the set of one-dimensional linear subspaces

$$A(X) = \{ L \mid L \in X \cap \text{Gr}(2,6) \}.$$  

Suppose $G$ is realized as a Grassmannian of lines in some $\mathbb{CP}^3$, and denote as before the two components of $C(G)$ of two-dimensional linear subspaces of $G$ as

$$A = \{ \sigma_2(p) \mid p \in \mathbb{CP}^3 = P(V^4) \}$$
$$B = \{ \sigma_{1,1}(h) \mid h \in \mathbb{P}^3 \}.$$  

**Lemma 2** Given $L \in A(X)$. Then:

(a) There is a unique two-dimensional linear subspace of $G$, $\sigma_2(p) \in A$ such that $L \subset \sigma_2(p)$. There is a unique one-dimensional linear subspace of $X$

$$L_1 \in A(X),$$

such that

$$\sigma_2(p) \cap F = L \cup L_1.$$  

(b) There is a unique two-dimensional linear subspace of $G$, $\sigma_{1,1}(h) \in A$ such that $L \subset \sigma_{1,1}(h)$. There is a unique one-dimensional linear subspace of $X$

$$L_2 \in A(X),$$

such that

$$\sigma_{1,1}(h) \cap F = L \cup L_2.$$  

The last Lemma introduces two involutions

$$i_1 : A(X) \to A(X)$$
$$i_1 : L \mapsto L_1$$
$$i_2 : A(X) \to A(X)$$
$$i_2 : L \mapsto L_2.$$  

As pencils of lines in $\mathbb{CP}^3$ the subspaces $L$ and $L_1$ are confocal, they have the same focus $p$. In the same manner, the subspaces $L$ and $L_2$ are coplanar, they have the same plane $h$.  

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Moreover, there are two mappings

\[ k_1 : \mathcal{A}(X) \to \mathbb{CP}^3 \]

\[ k_1 : L \mapsto p \]

\[ k_2 : \mathcal{A}(X) \to \mathbb{CP}^{3*} \]

\[ k_2 : L \mapsto h. \]

The mapping \( k_1 \) maps a pencil \( L \) to its focus in \( \mathbb{CP}^3 \) while \( k_2 \) maps a pencil to its plane in \( \mathbb{CP}^{3*} \).

Denote by \( K \subset \mathbb{CP}^3 \) the image of \( \mathcal{A}(X) \) by \( k_1 \). We see that \( k_1 \) is a double covering of \( \mathcal{A}(X) \) over \( K \) and that the involution \( i_1 \) interchanges the leaves of the covering. We are going to call \( K \) the Kummer variety of \( \mathcal{A}(X) \). It is associated to the choice of a quadric \( G \) from the pencil and to the choice of a connected component \( A \) of \( C(G) \).

Similarly, denote by \( K^* \subset \mathbb{CP}^{3*} \) the image of \( \mathcal{A}(X) \) by \( k_2 \). The mapping \( k_2 \) is a double covering of \( \mathcal{A}(X) \) over \( K \) and the involution \( i_2 \) interchanges the leaves of the covering. We are going to call \( K^* \) the dual Kummer variety of \( \mathcal{A}(X) \). It is associated to the choice of a quadric \( G \) from the pencil and to the choice of a connected component \( B \) of \( C(G) \).

It can be shown that \( K^* \) is dual to \( K \), which means that every plane from \( K^* \) is tangent to \( K \). From the previous considerations it can also be seen that the degree of \( K \) is equal to 4.

For a general point \( p \in \mathbb{CP}^4 \) the plane \( \sigma_2(p) \subset G \) intersects \( F \) along a smooth conic. The Kummer variety can be described as a set of the points in \( \mathbb{CP}^3 \) for which this intersection is a conic which is not smooth. For general point of \( K \) this intersection is a degenerate conic which is a union of two different lines \( L \) and \( i_1(L) \). But, there is a subset \( R \subset K \) of sixteen points, for which this intersection is a degenerate conic of double line. These sixteen points from \( R \) correspond to the fixed points of the involution \( i_1 \) and to the ramification points of the double covering. We also denote \( R^* \subset K^* \) the set of points that correspond to the fixed points of the involution \( i_2 \).

### 9.3 Pencils of quadrics, hyperelliptic curves and geometric group laws

With a generic pencil of quadrics \( F_\lambda \) in \( \mathbb{CP}^5 \) with six singular quadrics \( F_{\lambda_1}, \ldots, F_{\lambda_6} \) one may associate a genus two curve

\[ \Gamma : y^2 = \prod_{i=1}^{6}(x - \lambda_i). \]

As it was shown by Reid and Donagi, this correspondence between a generic pencil of quadrics and a hyperelliptic curve is not just formal. After Donagi,
denote by $E$ the family of all connected components of $C(F\lambda)$ of all quadrics from the pencil together with a projection

$$p : E \to \mathbb{CP}^1$$

which maps a given irreducible family of two-subspaces of a quadric $F\mu$ to the value $\mu$ of the pencil parameter. The projection $p$ is obviously double covering. It is ramified over the points $\lambda_i, i = 1, \ldots, 6$ since the singular quadrics are the only one with unique component of maximal linear subspaces. Thus, Donagi showed the isomorphism between $E$ and $\Gamma$.

But, there is yet another natural realization of the hyperelliptic curve $\Gamma$ in the context of the pencil $F\lambda$, as it was demonstrated by Reid.

For $L \in \mathcal{A}(X)$ denote by $\mathcal{A}_L(X)$ the closure of the set $\{L' \in \mathcal{A}(X) \mid L \cap L' \neq \emptyset\}$. There is a natural projection

$$q : \mathcal{A}_L(X) \setminus \{L\} \to \mathbb{CP}^1$$

which maps $L'$ to the parameter $\mu$ of a quadric $F\mu$ if the space $<L, L'>$ belongs to $C(F\mu)$. The mapping $q$ is double covering ramified over the six points $\lambda_i, i = 1, \ldots, 6$ and $\mathcal{A}_L(X)$ is isomorphic to the hyperelliptic curve $\Gamma$. The natural involution on $\mathcal{A}_L(X)$ which interchanges the folds of $q$ will be denoted by $\tau_L$.

Moreover, as it was shown by Reid and Donagi, $\mathcal{A}(X)$ is an Abelian variety, isomorphic to the Jacobian of the curve $\Gamma$.

We will refer to the curves $\mathcal{A}_L(X)$ and $E$ as Donagi-Reid-Knörrer curves (DRK) associated with intersection of quadrics $X$ of a generic pencil.

It can easily be shown that for any hyperelliptic curve $\Gamma$, there exists a pencil of quadrics with the base set $X$ such that $\mathcal{A}(X)$ is isomorphic to the Jacobian of $\Gamma$.

The addition laws on the Abelian varieties $\mathcal{A}(X)$ are such that

$$L_1 + L_2 = M_1 + M_2$$

if there exists $\mu$ such that

$$<L_1, L_2>, <M_1, M_2> \in C(F\mu)$$

and if the two two-dimensional spaces $<L_1, L_2>, <M_1, M_2>$ belong to the same component of $C(F\mu)$.

**Proposition 4** Every point $e \in E$ determines its Kummer variety involution $i_e$ and its Kummer variety $K_e$. The hyperelliptic involution on $E$, $\tau$ interchanges a Kummer variety and its dual:

$$K_{\tau(e)} = K_e^*.$$
We will fix a point $e_0 \in E$ and a line $L_0 \in \mathcal{A}(X)$ as the origin of a group structure on $\mathcal{A}(X)$ such that
\[ L_0 + \hat{L}_0 = 0, \]
whenever $< L_0, \hat{L}_0 > \in e_0$.

As above, denote by $\tau_{L_0}$ the natural involution on $\mathcal{A}_{L_0}(X)$. Then we have

**Lemma 3** The involutions $\tau_{L_0}$ and $i_{e_0}$ are related according to the formula
\[ i_{e_0}|_{\mathcal{A}_{L_0}(X)} = \tau_{L_0}. \]

Now, we are ready to define a two-valued group structure on the Kummer variety $K_{e_0}$. We will define a mapping
\[ \star : K_{e_0} \times K_{e_0} \longrightarrow (K_{e_0})^2 \]
by the following procedure.

Take a pair
\[ (p_1, p_2) \in K_{e_0} \times K_{e_0}. \]
Denote by
\[ [L_1] = \{ L_1, \hat{L}_1 \}, \quad L_1, \hat{L}_1 \in \mathcal{A}(X) \]
the class of lines in $\mathcal{A}(X)$ which represents the two confocal pencils of lines in $\mathbb{C}P^3$ with the focal point $p_1$. Similarly, denote by
\[ [L_2] = \{ L_2, \hat{L}_2 \}, \quad L_2, \hat{L}_2 \in \mathcal{A}(X) \]
the class of lines in $\mathcal{A}(X)$ which represents the two confocal pencils of lines in $\mathbb{C}P^3$ with the focal point $p_2$.

Assume that $L_1, L_2$ don’t intersect $L_0$. Denote by $N_1, N_2$ the two lines in $\mathcal{A}_{L_0}(X)$ of intersection of the space $< L_0, L_1 >$ with $X$ and by $e_1, e_2 \in E$ denote the classes determined by
\[ < L_0, N_1 > \in e_1, \quad < L_0, N_2 > \in e_2. \]
Denote also by
\[ \mu_1 = p(e_1), \quad \mu_2 = p(e_2), \]
and by
\[ N'_1, N''_1, N'_2, N''_2 \in \mathcal{A}_{L_2}(X) \]
the lines which intersect $L_2$ and which are uniquely defined by the conditions
\[ < N'_1, L_2 > \in e_1, \quad < N''_1, L_2 > \in \tau(e_1), \quad < N'_2, L_2 > \in e_1, \quad < N''_2, L_2 > \in \tau(e_2). \]
In other words, \( N'_1, N''_1 \) belong to the two two-dimensional spaces of the different classes of \( C(F_{\mu_1}) \) which contain \( L_2 \); \( N'_2, N''_2 \) belong to the two two-dimensional spaces of the different classes of \( C(F_{\mu_2}) \) which contain \( L_2 \).

Let \( M_1 \) be the fourth intersection line in the intersection of \( X \) with the space generated with \( L_2, N'_1, N''_1 \) and let \( M_2 \) be the fourth intersection line in the intersection of \( X \) with the space generated with \( L_2, N''_1, N''_2 \). The line \( M_1 \) represents a pencil of lines in \( CP^3 \) with the focal point \( w_1 \) and \( M_2 \) represents a pencil of lines in \( CP^3 \) with the focal point \( w_2 \).

If we repeat the above procedure with \( \hat{L}_1, \hat{L}_2 \) instead of \( L_1, L_2 \) we come to the lines \( \hat{M}_1 \) and \( \hat{M}_2 \) which are confocal with \( M_1 \) and \( M_2 \).

One can easily adjust previous construction to the case where \( L_1, L_2 \) intersect \( L_0 \); denote by \( M_1 \) the fourth line of the intersection of \( X \) with the space generated with \( \hat{L}_1, \hat{L}_2, L_0 \); denote by \( M_2 \) the fourth line of the intersection of \( X \) with the space generated with \( L_1, \hat{L}_2, L_0 \).

Thus we get

**Theorem 4** The mapping defined by the formulae

\[
\star : K_{e_0} \times K_{e_0} \longrightarrow (K_{e_0})^2
\]

\[p_1 \star p_2 = (w_1, w_2)\]

defines a structure of two-valued group on the Kummer variety \( K_{e_0} \).

**Definition 2** The mapping \( \star \) defined by previous construction defines the geometric two-valued group law on the Kummer variety \( K_{e_0} \).

### 10 Integrable billiards and two-valued laws

#### 10.1 Pencils of quadrics and billiards, an overview

We begin this Section by repeating basic definitions related to billiard systems of confocal quadrics from [19], [20].

Let \( Q_1 \) and \( Q_2 \) be two quadrics. Denote by \( u \) the tangent plane to \( Q_1 \) at point \( x \) and by \( z \) the pole of \( u \) with respect to \( Q_2 \). Suppose lines \( \ell_1 \) and \( \ell_2 \) intersect at \( x \), and the plane containing these two lines meet \( u \) along \( \ell \).

**Definition 3** If lines \( \ell_1, \ell_2, xz, \ell \) are coplanar and harmonically conjugated, we say that rays \( \ell_1 \) and \( \ell_2 \) obey the reflection law at the point \( x \) of the quadric \( Q_1 \) with respect to the confocal system which contains \( Q_1 \) and \( Q_2 \).

If we introduce a coordinate system in which quadrics \( Q_1 \) and \( Q_2 \) are confocal in the usual sense, reflection defined in this way is same as the standard one.
Theorem 5 (One Reflection Theorem) Suppose rays $\ell_1$ and $\ell_2$ obey the reflection law at $x$ of $Q_1$ with respect to the confocal system determined by quadrics $Q_1$ and $Q_2$. Let $\ell_1$ intersects $Q_2$ at $y'_1$ and $y_1$, $u$ is a tangent plane to $Q_1$ at $x$, and $z$ its pole with respect to $Q_2$. Then lines $y'_1z$ and $y_1z$ respectively contain intersecting points $y'_2$ and $y_2$ of ray $\ell_2$ with $Q_2$. Converse is also true.

Corollary 2 Let rays $\ell_1$ and $\ell_2$ obey the reflection law of $Q_1$ with respect to the confocal system determined by quadrics $Q_1$ and $Q_2$. Then $\ell_1$ is tangent to $Q_2$ if and only if is tangent $\ell_2$ to $Q_2$; $\ell_1$ intersects $Q_2$ at two points if and only if $\ell_2$ intersects $Q_2$ at two points.

Next assertion is crucial for applications to the billiard dynamics.

Theorem 6 (Double Reflection Theorem) Suppose that $Q_1$, $Q_2$ are given quadrics and $\ell_1$ line intersecting $Q_1$ at the point $x_1$ and $Q_2$ at $y_1$. Let $u_1, v_1$ be tangent planes to $Q_1, Q_2$ at points $x_1, y_1$ respectively, and $z_1, w_1$ their poles with respect to $Q_2$ and $Q_1$. Denote by $x_2$ second intersecting point of the line $w_1x_1$ with $Q_1$, by $y_2$ intersection of $y_1z_1$ with $Q_2$ and by $\ell_2, \ell'_1, \ell'_2$ lines $x_1y_2, y_1x_2, x_2y_2$. Then pairs $\ell_1, \ell_2; \ell_1, \ell'_1; \ell_2, \ell'_2; \ell'_1, \ell'_2$ obey the reflection law at points $x_1$ (of $Q_1$), $y_1$ (of $Q_2$), $y_2$ (of $Q_2$), $x_2$ (of $Q_1$) respectively.

Corollary 3 If the line $\ell_1$ is tangent to a quadric $Q$ confocal with $Q_1$ and $Q_2$, then rays $\ell_2, \ell'_1, \ell'_2$ also touch $Q$.

For the conclusion, we recall the notion of generalized Cayley’s curve from [19], [20].

Definition 4 The generalized Cayley curve $C_\ell$ is the variety of hyperplanes tangent to quadrics of a given confocal family in $\mathbb{C}P^d$ at the points of a given line $\ell$.

This curve is naturally embedded in the dual space $\mathbb{C}P^d^*$.  

Proposition 5 The generalized Cayley curve in $\mathbb{C}P^d$, for $d \geq 3$ is a hyperelliptic curve of genus $g = d - 1$. Its natural realization in $\mathbb{C}P^d^*$ is of degree $2d - 1$.

The natural involution $\tau_\ell$ on the generalized Cayley’s curve $C_\ell$ maps to each other the tangent planes at the points of intersection of $\ell$ with any quadric of the confocal family.

It was observed in [19] that this curve is isomorphic to the Veselov-Moser isospectral curve.

Now we are going to mention a connection obtained in [20], between generalized Cayley’s curve defined above and the curves (see the previous Section) studied by Knörrer, Donagi, Reid. This connection traces out the relationship
between billiard constructions and the algebraic structure of the corresponding Abelian varieties.

The famous Chasles theorem (see [2]) states that any line in the space $\mathbb{R}^d$ is tangent to exactly $d - 1$ quadrics from a given confocal family:

$$Q_\lambda : Q_\lambda(x) = 1. \quad (21)$$

where we denote:

$$Q_\lambda(x) = \frac{x_1^2}{a_1 - \lambda} + \cdots + \frac{x_d^2}{a_d - \lambda}.$$

We assume that the family (21) is generic, i.e. the constants $a_1, \ldots, a_d$ are all distinct.

Suppose a line $\ell$ is tangent to quadrics $Q_{\alpha_1}, \ldots, Q_{\alpha_{d-1}}$ from the given confocal family. Denote by $A_\ell$ the family of all lines which are tangent to the same $d - 1$ quadrics. Note that according to the corollary of the One Reflection Theorem, the set $A_\ell$ is invariant to the billiard reflection on any of the confocal quadrics.

For what follows, the next simple observation is important.

**Lemma 4** Let the lines $\ell$ and $\ell'$ obey the reflection law at the point $z$ of a quadric $Q$ and suppose they are tangent to a confocal quadric $Q_0$ at the points $z_1$ and $z_2$. Then the intersection of the tangent spaces $T_{z_1}Q \cap T_{z_2}Q$ is contained in the tangent space $T_zQ$.

Following [25], together with $d - 1$ affine confocal quadrics $Q_{\alpha_1}, \ldots, Q_{\alpha_{d-1}}$, one can consider their projective closures $Q^p_{\alpha_1}, \ldots, Q^p_{\alpha_{d-1}}$ and the intersection $X$ of two quadrics in $\mathbb{C}P^{2d-1}$:

$$x_1^2 + \cdots + x_d^2 - y_1^2 - \cdots - y_{d-1}^2 = 0, \quad (22)$$

$$a_1 x_1^2 + \cdots + a_d x_d^2 - \alpha_1 y_1^2 - \cdots - \alpha_{d-1} y_{d-1}^2 = x_0^2. \quad (23)$$

Denote by $\mathcal{A}(X)$ the set of all $(d - 2)$-dimensional linear subspaces of $X$. For a given $L \in \mathcal{A}$, denote by $\mathcal{A}_L(X)$ the closure in $\mathcal{A}(X)$ of the set $\{L' \in F | \dim L \cap L' = d - 3\}$. It was shown in [29] that $F_L$ is a nonsingular hyperelliptic curve of genus $d - 1$.

The projection

$$\pi' : \mathbb{C}P^{2d-1} \setminus \{(x, y) | x = 0\} \to \mathbb{C}P^d, \quad \pi'(x, y) = x,$$

maps $L \in \mathcal{A}(X)$ to a subspace $\pi'(L) \subset \mathbb{C}P^d$ of the codimension 2. $\pi'(L)$ is tangent to the quadrics $Q_{\alpha_1}^p, \ldots, Q_{\alpha_{d-1}}^p$ that are dual to $Q_{\alpha_1}^*, \ldots, Q_{\alpha_{d-1}}^*$.

Thus, the space dual to $\pi'(L)$, denoted by $\pi^*(L)$, is a line tangent to the quadrics $Q_{\alpha_1}^*, \ldots, Q_{\alpha_{d-1}}^*$.

We can reinterpret the generalized Cayley’s curve $C_\ell$, which is a family of tangent hyperplanes, as a set of lines from $\mathcal{A}_\ell$ which intersect $\ell$. Namely, for almost every tangent hyperplane there is a unique line $\ell'$, obtained from $\ell$ by the billiard reflection. Having this identification in mind, it is easy to prove the following
Corollary 4 There is a birational morphism between the generalized Cayley’s curve $C_\ell$ and Reid-Donagi-Knörrer’s curve $F_L$, with $L = \pi^{*-1}(\ell)$, defined by

$$ j : \ell' \mapsto L', \quad L' = \pi^{*-1}(\ell'), $$

where $\ell'$ is a line obtained from $\ell$ by the billiard reflection on a confocal quadric.

Lemma 4, giving a link between the dynamics of ellipsoidal billiards and algebraic structure of certain Abelian varieties, provides a two way interaction: to apply algebraic methods in the study of the billiard motion, but also vice versa, to use billiard constructions in order to get more effective, more constructive and more observable understanding of the algebraic structure.

10.2 Two-valued billiard structure

Thus, we are going to apply this relationship to construct a billiard analogue of the geometric two-valued group structure defined on the Kummer variety $K_{e_0}$ from the previous Section, see Theorem 7 below.

In order to fit together notations from the previous Section and from the last Subsection, we set $d = 3$ and

$$ \{\lambda_1, \ldots, \lambda_6\} = \{a_1, a_2, a_3, a_1, a_2, \infty\}. $$

We consider the confocal pencil of quadrics $Q_\lambda$ in $\mathbb{C}P^3$ and the set $\mathcal{A}_\ell$ of lines in $\mathbb{C}P^3$ which are tangent to both quadrics $Q_{a_1}, Q_{a_2}$.

We select a quadric $Q_{p(e_0)}$ together with a family of reflections which corresponds to $e_0$ and a line $\ell_0 \in \mathcal{A}_\ell$.

We introduce an involution $I_{e_0}$ on $\mathcal{A}_\ell$ induced by $Q_{p(e_0)}$ with $e_0$, which maps a line $m \in \mathcal{A}_\ell$ to a line $m' \in \mathcal{A}_\ell$ uniquely defined by the condition that $m$ and $m'$ are obtained by billiard reflection of each other from $Q_{p(e_0)}$ from the given family $e_0$.

We suppose that the restriction of this involution $I_{e_0}$ restricted on $C_{e_0}$ coincides with the natural involution on the generalized Cayley curve $C_{e_0}$.

Denote by

$$ [\ell_1] = \{\ell_1, \hat{\ell}_1\}, \quad \ell_1, \hat{\ell}_1 \in \mathcal{A}_\ell $$

and by

$$ [\ell_2] = \{\ell_2, \hat{\ell}_2\}, \quad \ell_2, \hat{\ell}_2 \in \mathcal{A}_\ell $$

two classes of lines in $\mathcal{A}_\ell/I_{e_0}$.

Assume that $\ell_1, \ell_2$ don’t intersect $\ell_0$. Denote by $n_1, n_2$ the two lines in $C_{e_0}(X)$ which form a double reflection configuration with $\ell_0, \ell_1$. Denote by $Q_{\mu_1}$ the quadric of billiard reflection of $\ell_0, n_1$ and of billiard reflection of $\ell_1, n_2$. Similarly, denote by $Q_{\mu_2}$ the quadric of billiard reflections of pairs $\ell_0, n_2$ and $\ell_1, n_1$. 
The quadrics $Q_{\mu_1}, Q_{\mu_2}$ intersect line $\ell_2$. Denote by

$$n_1', n_1'', n_2', n_2'' \in C_{\ell_2}$$

the lines of billiard reflection of the line $\ell_2$ from the quadrics $Q_{\mu_1}, Q_{\mu_2}$.

The lines $n_1', n_1''$ are obtained by reflection at $Q_{\mu_1}$ and reflection of $n_1'$ belongs to the same family as the reflection of $\ell_0, n_1$; the lines $n_2', n_2''$ are obtained by reflection from $Q_{\mu_2}$ and reflection of $n_2'$ belongs to the same family as the reflection $\ell_0, n_2$.

Let $m_1$ be the fourth line of the double reflection configuration determined with $\ell_2, n_1', n_2'$ and let $m_2$ be the fourth line of the double reflection configuration generated by $\ell_2, n_1'', n_2''$.

If we repeat the above procedure with $\hat{\ell}_1, \hat{\ell}_2$ instead of $\ell_1, \ell_2$ we come to the lines $\hat{m}_1$ and $\hat{m}_2$ which are obtained from $m_1$ and $m_2$ by the involution $I_{e_0}$.

One can easily adjust previous construction to the case where $\ell_1, \ell_2$ intersect $\ell_0$: denote by $m_1$ the fourth line of the double reflection configuration determined with $\hat{\ell}_1, \hat{\ell}_2, \ell_0$; denote by $m_2$ the fourth line of the double reflection configuration determined with $\ell_1, \hat{\ell}_2$ and $\ell_0$.

Thus we get

**Theorem 7** The mapping defined by the formulae

$$\ast_b : \mathcal{A}_{\ell}/I_{e_0} \times \mathcal{A}_{\ell}/I_{e_0} \longrightarrow (\mathcal{A}_{\ell}/I_{e_0})^2$$

$$[\ell_1] \ast_b [\ell_2] = ([m_1], [m_2])$$

defines a structure of two-valued group on the variety $\mathcal{A}_{\ell}/I_{e_0}$.

**Definition 5** The mapping $\ast_b$ defined by previous construction defines the billiard two-valued group law on the variety $\mathcal{A}_{\ell}/I_{e_0}$.

11 Moduli of semi-stable bundles and two-valued group structure on Kummer varieties

As we have mentioned before, historically first examples of 2-valued groups appeared in topological context in the study of the characteristic classes of vector bundles, see [3]. There one-dimensional symplectic bundles over $\mathbb{H}P^n$, and together with the canonical projection $\mathbb{C}P^{2n+1} \rightarrow \mathbb{H}P^n$, the associated two-dimensional complex vector bundles over $\mathbb{C}P^{2n+1}$ with $c_1 = 0$, were considered.

For a pair of such two-dimensional bundles,

$$\xi_1, \xi_2, \quad c_1(\xi_i) = 0, \quad i = 1, 2$$
its tensor product
\[ \xi_1 \otimes_c \xi_2 \]
is a four-dimensional bundle, with the first two Pontryagin classes \( p_1(\xi_1 \otimes_c \xi_2) \) and \( p_2(\xi_1 \otimes_c \xi_2) \). In [3], to the initial pair of bundles, a pair of virtual two-bundles, were associated according to the formula
\[ Z^2 - p_1(\xi_1 \otimes_c \xi_2)Z + p_2(\xi_1 \otimes_c \xi_2) = 0. \]
The solutions \( Z_{1,2} \) of the last quadratic equation play a role of the first Pontryagin classes of two virtual two-bundles. The last quadratic equation defines a two-valued group structure in \( x = p_1(\xi_1) \) and \( y = p_1(\xi_2) \), since \( p_1(\xi_1 \otimes_c \xi_2) = \Theta_1(x, y) \) and \( p_2(\xi_1 \otimes_c \xi_2) = \Theta_2(x, y) \), where \( \Theta_i \) are certain series.

In this Section, we are going to construct the two-valued group structure on the Kummer variety, in the context of two-dimensional varieties, semi-stable in the sense of Mumford and Seshadri. The significance of the present situation, lies in the fact that obtained resulting two-bundles are not virtual - they are realized as a pair of two-bundles, semi-stable but not stable.

To get this, yet another interpretation of Kummer varieties, we are going back to [28]. Let \( X \) be a genus two curve. Following Mumford and Seshadri, the notions of stable and semi-stable vector bundles of rank \( n \) and degree \( d \) have been introduced respectively. For a holomorphic nonzero vector bundle \( W \) on \( X \) one introduces a rational-valued function \( \mu(W) = \deg W/\text{rank} W \). A vector bundle \( W \) is stable if for every proper subbundle \( V \) the condition
\[ \mu(V) \leq \mu(W) \]
is satisfied. Similarly, a bundle is semi-stable if in the last inequality the sign \( < \) is replaced by \( \leq \). Any semi-stable bundle \( W \) has a strictly decreasing filtration
\[ W = W_0 \supset W_1 \supset \cdots \supset W_n = \{0\} \]
such that \( W_{i-1}/W_i \) are stable and \( \mu(W_{i-1}/W_i) = \mu(W) \). Denote by \( \text{Gr} W = \oplus W_{i-1}/W_i \). As Seshadri defined, two semi-stable bundles \( W_1 \) and \( W_2 \) are S-equivalent if \( \text{Gr} W_1 \approx \text{Gr} W_2 \); a normal, projective \( (n^2 + 1) \)-dimensional variety of S-equivalence classes of semi-stable bundles of degree \( d \) and rank \( n \) denote \( U(n, d) \).

Following [28], for \( U(2, 0) \), denote by \( S \) its three-dimensional sub-variety of bundles with trivial determinant. The non-stable bundles in \( S \) are of the form
\[ j \oplus j^{-1}, \]
with \( j \) a line bundle of degree 0. The Kummer surface \( K \) associated to the Jacobian of \( X \) is isomorphic to the set of all non-stable bundles in \( S \).

Now, we define a structure of two-valued group on \( K \). It is an important development of the Example [3] from Section [2]. Denote \( a, b \in K \), where
\[ a = j \oplus j^{-1}, \quad b = l \oplus l^{-1}, \]
with \( j \) a line bundle of degree 0.
where \( j, l \) are line bundles on \( X \) of degree 0. Then:

\[
\alpha *_{s} \beta := (j \otimes l \oplus j^{-1} \otimes l^{-1}, j \otimes l^{-1} \oplus j^{-1} \otimes l).
\]

(24)

**Proposition 6** The operation

\[
*_{s} : K \times K \longrightarrow (K)^{2}
\]

determined by the relation (24) defines a two-valued group structure on \( K \).

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