Models of interacting dark energy

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Accepted 2008 August 18. Received 2008 August 11; in original form 2008 July 6

ABSTRACT

We investigate the interaction between dark energy and dark matter using a phenomenological model with an interaction proportional to the dark energy density and two physically motivated models, interacting holographic dark energy and mass varying neutrino dark energy. Constraints on the interaction parameters are obtained using observational data including the gold type Ia supernovae set, the cosmic microwave background (CMB) shift parameter $\Omega$ from the 3-yr Wilkinson Microwave Anisotropy Probe, the baryon acoustic oscillation parameter $A$ from the Sloan Digital Sky Survey and the time-dependent observables of the lookback time to high-redshift galactic clusters. We find that the available data can be satisfied reasonably well for a wide range of parameters in the models studied and that the CMB $\Omega$ parameter appears to have the most discriminating power between the models.

Key words: cosmology: theory – dark matter – large-scale structure of Universe.

1 INTRODUCTION

Recent cosmological observations such as the luminosity of high-redshift type Ia supernovae (SN Ia) (Riess et al. 1998; Perlmutter et al. 1999; Riess et al. 1999; Astier et al. 2006), the fluctuation spectrum of the cosmic microwave background (CMB) (Spergel et al. 2003, 2007b) and the baryon acoustic oscillations (BAO) (Eisenstein et al. 2005) in the galaxy power spectrum strongly indicate that the expansion of the Universe is accelerating. The most favoured explanation of this acceleration is that it is due to a dark energy component of the Universe which has caused the Universe to make a transition from a matter-dominated phase to a dark energy dominated phase. A good fit to observational data is obtained assuming dark energy is due to a cosmological constant $\Lambda$ (equivalent to a vacuum energy) with present densities of matter $\rho_0_\Lambda = 0.3\rho_c$ (comprised nearly entirely of cold dark matter (CDM)) and dark energy $\rho_0_\Lambda = 0.7\rho_c$ where $\rho_c = 3H^2/8\pi G$ is the critical density. However, this scenario (LCDM) gives a rapid transition to dark energy domination and the probability that we find ourselves in an era in which both matter and dark energy are present in comparable amounts is very low (the cosmic coincidence problem). One alternative is dynamical dark energy with negative pressure generated from a primordial scalar field $\phi$ (quintessence) with a self-interaction potential energy $V(\phi)$ that causes the dark energy density to track some component of the matter density.

With little known about the nature of dark matter and dark energy it seems implausible that these two sectors do not interact. An interesting solution of the cosmic coincidence problem is then to assume there is some mechanism that converts dark energy into matter such that the ratio $r = \rho_\Lambda/\rho_{DE}$ approaches a constant $r_0$ for large times (Zimdahl, Pavón & Chimento 2001; Chimento et al. 2003; Olivares, Atrio-Barandela & Pavón 2005). Coupling between dark energy and dark matter not only affects the expansion rate but also structure formation, distorting the virial theorem and influencing the dynamics of galaxy clusters. Abdalla et al. (2007) have analysed 33 galaxy clusters and find a weak preference for a small decay of dark energy into dark matter and Bertolami, Pedro & Le Delliou (2007) come to a similar conclusion for the Abell Cluster A586.

Interacting dark energy can be studied without specifying the physical origin of the interaction but there are models such as holographic and neutrino dark energy which are motivated by specific physical principles from quantum gravity and particle physics.

Holographic models of dark energy (Bekenstein 1973; Susskind 1995; Cohen, Kaplan & Nelson 1999) are based on the principle that the amount of matter and energy inside a given volume should not exceed that amount which would turn the volume into a black hole. Under this constraint, the dark energy density must satisfy

$$\rho_{DE} \leq \frac{3c^2M_p^2}{L^2},$$

where $M_p = 1/\sqrt{8\pi G}$ is the reduced Planck mass, $c$ is a constant of the order of 1 and $L$ is a cosmic length-scale. The most obvious choice for this scale is the Hubble scale $L = H^{-1}$ but Hsu (2004) has shown that this cannot lead to an accelerating universe. Li (2004) considered holographic dark energy models with the Hubble scale, the particle and the future event horizons as choices for $L$ and found that only the event horizon gave a viable model. However, Pavón & Zimdahl (2005) have pointed out that Li assumed independent evolution of $\rho_{DM}$ and $\rho_{DE}$ and showed that if the dark energy decayed into dark matter through an interaction of the form

$$\Gamma = \gamma\rho_{DE},$$

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then the choice $L = H^{-1}$ can produce both accelerated expansion and, because $r = r_0$, solve the coincidence problem. Dynamical evolution of the energy density ratio $r$ is possible (Wang, Gong & Abdalla 2005) in interacting holographic dark energy models if the Hubble scale is replaced by the future event horizon. Recently, Wu et al. (2007), Feng et al. (2007) and Sen & Pavón (2008) have reported a detailed study of the constraints on interacting holographic dark energy models from a wide range of observations. These models may even explain the observed suppression of the low-l multipoles in the CMB spectrum (Wang et al. 2007) if the length-scale $L$ is interpreted as the maximum possible wavelength.

Neutrino models of dark energy are motivated by two main factors: (i) the desire to relate the small energy scale $E_0 \sim 2 \times 10^{-3}$ eV associated with the dark energy density $\rho_{DE} = \rho_{vac} = E_0^3$ to particle physics and (ii) to resolve the coincidence problem by assuming $\rho_{DE}$ tracks a component of $\rho_M$. The scenario proposed by Fardon, Nelson & Weiner (2004, hereafter FNW) is that $\rho_{DE}$ tracks the density of neutrinos $\rho_\nu$, thereby relating $E_0$ to the scale of neutrino masses $m_\nu$, coupling the dark energy and neutrinos such that $\rho_{DE}$ depends on $m_\nu$ which in turn are variable and a function of $\rho_\nu$.

In this paper, we study the viability of models with a general interaction of the form (equation 2) and the two physically motivated models of interacting holographic dark energy and neutrino dark energy using a combination of recent observations. These include the data sets of 182 gold SN Ia (Riess et al. 2007), the ESSENCE SN Ia data (Riess et al. 2007; Davis et al. 2007; Wood-Vasey et al. 2007), the shift parameter from the 3-yr Wilkinson Microwave Anisotropy Probe (WMAP3) (Spergel et al. 2007a) and the BAO A parameter determined (Eisenstein et al 2005) from the Sloan Digital Sky Survey (SDSS) sample of 46 748 luminous red galaxies. In addition, we incorporate a test based on time-dependent observables which involve the lookback time to high-redshift clusters of galaxies and the age of the Universe (Dalal et al. 2001; Capozziello et al. 2004; Feng et al. 2007).

2 DARK ENERGY MODELS

2.1 Background

The field equations of general relativity for a Robertson–Walker metric give the Friedmann equation for the evolution of the cosmic scalefactor $a(t)$:

$$H^2 = \frac{8\pi G}{3} \rho_{tot} - \frac{k}{a^2},$$

where $H = \dot{a}/a$ is the Hubble parameter, $k$ is the curvature scalar, $\rho_{tot} = \rho_M + \rho_{rad} + \rho_{DE}$ is the total energy density and $\rho_{rad}$ is the radiation density. In addition, conservation of energy–momentum gives

$$\dot{\rho} = -3H(\rho + p),$$

where $p$ is the pressure. The matter component is assumed non-relativistic and pressureless and the dark energy to have an equation of state $w = p_{DE}/\rho_{DE}$ where, for an accelerating universe, $w < -1/3$. For the cosmological constant model $w_{\Lambda} = -1$. The recent analysis of the 5-yr WMAP data gives (Komatsu et al. 2008) $w = -0.984^{+0.018}_{-0.016}$ for a constant $w$.

It is convenient to introduce the dimensionless ratios $\Omega_i = \rho_i/\rho_c$, and rewrite equation (3) as

$$\Omega_M + \Omega_{rad} + \Omega_{DE} = 1 + \frac{k}{a^2H^2}.$$  

Current observations indicate that the Universe has zero spatial curvature ($k = 0$), $\Omega_{rad}$ is negligible and that $\Omega_M \approx \Omega_{DM}$ so that we can take

$$\Omega_{DE} + \Omega_{DM} = 1. \hspace{1cm} (6)$$

2.2 Interacting dark energy and dark matter

2.2.1 General equations

For generic models of interacting dark energy and dark matter, the continuity equations for the evolution of the component densities are

$$\dot{\rho}_{DE} + 3H(1 + w)\rho_{DE} = -\Gamma,$$

$$\dot{\rho}_{DM} + 3H\rho_{DM} = \Gamma,$$

where $\Gamma$ specifies the interaction between the dark energy and dark matter sectors. The interaction typically takes the form (equation 2) which leads to the equations

$$\dot{\rho}_{DE} + 3H(1 + w + \frac{\gamma}{3H})\rho_{DE} = 0,$$

$$\dot{\rho}_{DM} + 3H\frac{1 - \gamma}{r}\rho_{DM} = 0,$$

where $r = \rho_{DM}/\rho_{DE}$. These equations are of the form (equation 4) with the effective parameters

$$w_{DE}^{eff} = w + \frac{\gamma}{3H}, \hspace{1cm} w_{DM}^{eff} = -\frac{\gamma}{3rH}.$$  

From equation (8), the evolution of $r$ is

$$\dot{r} = 3Hr \left[ w + \frac{\gamma}{3H} \left( 1 + \frac{1}{r} \right) \right]$$

or, equivalently, in terms of the redshift $z \equiv a_0/a - 1$,

$$\frac{dr}{dz} = -3r \left[ w + \frac{\gamma}{3H} \left( 1 + \frac{1}{r} \right) \right].$$

For a universe with zero spatial curvature, the Friedmann equation (6) gives

$$\frac{r}{\Omega_{DE}} = 1 - \Omega_{DE}.$$  

Hence, $dr/dz = -\Omega_{DE}^2 d\Omega_{DE}/dz$ and the dimensionless dark energy density satisfies

$$\frac{d\Omega_{DE}}{dz} = 3\Omega_{DE} \left[ w(1 - \Omega_{DE}) + \frac{\gamma}{3H} \right].$$

Alternatively, from the definition of $\Omega_{DE}$, we find

$$\frac{d\Omega_{DE}}{dz} = \frac{d}{dz} \left( \frac{\rho_{DE}}{\rho_c} \right) = \Omega_{DE} \left[ 3(w + 1) + \frac{\gamma}{1 + z} \right] - \frac{2}{H} \frac{dH}{dz}.$$

A comparison with equation (13) gives

$$\frac{2}{H} \frac{dH}{dz} = \frac{3}{1 + z}(1 + w\Omega_{DE}),$$

which can be integrated to obtain

$$E(z) = \frac{H(z)}{H_0} = \exp \left[ \frac{3}{2} \int_0^z \frac{1 + w\Omega_{DE}(z')}{1 + z'} dz' \right].$$  

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When an interaction of the form (Kim, Lee & Myung 2006)

\[
\frac{\gamma}{3H} = \frac{b^2}{\Omega_{DE}}
\]

is used in equation (13), where \(b^2\) measures the strength of the interaction, the dimensionless dark energy density evolves according to

\[
\frac{d\Omega_{DE}}{dz} = \frac{3}{1+z} \left[ w\Omega_{DE}(1 - \Omega_{DE}) + b^2 \right].
\]

Note that this interaction reduces to the widely used form \(\gamma = 3Hb^2(1 + r)\), i.e. \(\Gamma = 9b^2M_p^2H^4\).

The dark energy evolution with redshift can be calculated when values for the parameters \((w, b^2)\) are specified in the above.

Recently, Guo, Ohta & Tsujikawa (2007) have considered interactions of the form \(\gamma = H\delta\) for a constant \(\delta\) and a redshift-dependent \(\delta(z) = -\langle \xi + 3w \rangle \Omega_{DE}\).

Using the SN Ia data, the CMB shift parameter and the BAO A parameter, they find the constraints \(-0.08 < \delta < 0.03\) and \(-0.4 < \delta(z = 0) < 0.1\). Similarly, Feng et al. (2008) have constrained models with interactions of the form \(\gamma = H\delta\rho_{DE}, \Gamma = H\delta\rho_{DM}\) and \(\Gamma = H\delta(\rho_{DE} + \rho_{DM})\) using these data sets together with the lookback time data and find a tendency for small positive \(\delta\).

2.2.2 Analytic solution for interacting models

In the case of constant \(w\) and \(b^2\), equation (18) can be solved analytically (see Appendix):

\[
\Omega_{DE} = \frac{1}{2}(1 - s) + s \left[ 1 - \frac{\Omega_{DE}^0 - \frac{s}{2}(1 + s) - \frac{3}{2}(1 + z)^{-4w}}{\Omega_{DE}^0 - \frac{1}{2}(1 + s)} \right]^{-1},
\]

where \(s = \sqrt{1 + 4b^2/w}\). It follows that the dimensionless density of dark energy is constrained within the bound

\[
\frac{1}{2}(1 - s) < \Omega_{DE} < \frac{1}{2}(1 + s).
\]

Since \(w < 0\) and \(\Omega_{DE} < 1\), this places limits on the interaction parameter, given by \(0 \leq b^2 < -w/4\).

2.3 Holographic dark energy

2.3.1 General features

The holographic principle relates \(\rho_{DE}\) to a cosmic length-scale \(L\) for which possible choices are (i) the Hubble horizon \(L_{BH} = H^{-1}\), (ii) the particle horizon \(L_{PH} = a(t)\int_0^t dt'\), or (iii) the future event horizon \(L_{FH} = a(t)\int_t^\infty dt'\).

By taking the upper bound of equation (1) for \(\rho_{DE}\), we have

\[
\Omega_{DE} = \frac{c^2}{H^2L^2}.
\]

and

\[
\rho_{DE} = -\frac{6c^2M_p^2}{L^3}L = -2\rho_{DE} \frac{L}{L},
\]

Equation (8) then gives the relation

\[
3H(1 + w) + \gamma - \frac{2L}{L} = 0
\]

connecting the three quantities \((w, \gamma, L)\). For \(L = H^{-1}, \dot{L}/L = -H/H\) whereas for \(L\) taken to be the particle horizon \(R_{PH}\) or the future horizon \(R_{FH}\), we have \(L = H\pm 1\) which, using equation (24), gives

\[
\frac{1}{H} \frac{\dot{L}}{L} = 1 \pm \sqrt{\Omega_{DE}}.
\]

where the upper (lower) sign refers to the choice \(L = R_{PH}(R_{FH})\).

The holographic model of dark energy is generally specified (Berger & Shojaei 2006a; Wu et al. 2007) by the choice of parameters \((\gamma, L)\), resulting in a time-dependent \(w\). The choice \(L = H^{-1}\) gives \(\Omega_{DE} = c^2\) and therefore \(\Omega_{DM} = 1 - c^2\), independent of any interaction \(\gamma\). As both the dark energy and the dark matter density ratios are constant, \(\rho_{DE}\) and \(\rho_{DM}\) must evolve identically, implying \(w_{DE}^{\text{eff}} = w_{DM}^{\text{eff}}\), where

\[
\frac{w_{DE}^{\text{eff}}}{3H} = w + \frac{\gamma}{3H} = -\frac{1}{3} - \frac{2H}{3H^2}.
\]

This scenario is effectively a one-component model with \(w_{DE}^{\text{eff}} = \gamma/(3\Omega_0)\). Berger & Shojaei (2006b) have shown that this model can give a good fit to the gold SN Ia data for an interaction \(\gamma = \kappa H_0\) with \(\Omega_0 \kappa \approx 0.5\). However, Wu et al. (2007) dismiss the case \(L = H^{-1}\), claiming it cannot explain the transition from a decelerating to accelerating universe. For the other horizon options, equation (26) can be rewritten as

\[
w + \frac{\gamma}{3H} = -\frac{1}{3} \pm \frac{2\sqrt{\Omega_{DE}}}{3}.
\]

so that the particle horizon option gives \(w_{DE}^{\text{eff}} > -1/3\), in conflict with the observation that the Universe is accelerating. When an interaction of the form (equation 17) is chosen, the dimensionless dark energy density for the particle and future horizon options evolves according to

\[
\frac{d\Omega_{DE}}{dz} = -\frac{\Omega_{DE}(1 - \Omega_{DE})}{1+z} \left[ 1 + \frac{2\sqrt{\Omega_{DE}}}{c} - \frac{3b^2}{(1 - \Omega_{DE})} \right].
\]

Berger & Shojaei (2006a), Wu et al. (2007) and Feng et al. (2007) have studied this equation for the future horizon case. Berger & Shojaei (2006a) find an acceptable value of \(\Omega_{DE}\) for \(c = 1, b^2 = 0.2\) whereas more extensive studies using a range of data sets to bound the \((c, b^2)\) parameter space give the best-fitting values \(c = 0.88_{-0.07}^{+0.08}, b^2 = 0.18_{-0.01}^{+0.02}\) (Wu et al. 2007) and \(c = 0.83_{-0.25}^{+0.43}, b^2 = 0.033_{-0.012}^{+0.012}\) (Feng et al. 2007). It is of interest to note that a similar study of non-interacting \((b = 0)\) holographic dark energy based upon the future horizon (Zhang & Wu 2007) gives \(c = 0.91_{-0.18}^{+0.26}\).

2.3.2 Specifying \((w, L)\)

Since the form of the possible interaction between dark energy and dark matter is entirely unknown, alternative parameters may be chosen to study the dark energy evolution. The state parameter is not tightly constrained by current data, so it is therefore possible that \(w\) takes a constant value. To eliminate \(\gamma\) from our equations, we rearrange equation (26),

\[
\frac{\gamma}{3H} = \frac{2}{3H} \frac{\dot{L}}{L} - (1 + w).
\]
and use equation (27). The evolution of the dimensionless dark energy parameter then satisfies

\[
\frac{d\Omega_{DE}}{dz} = -\frac{\Omega_{DE}}{1+z} \left( 1 \mp \frac{2\sqrt{\Omega_{DE}}}{c} + 3w\Omega_{DE} \right). \tag{32}
\]

We expect the dark energy component to become more dominant in the future and this requires

\[
\left. \frac{d\Omega_{DE}}{dz} \right|_{z=0} < 0 \tag{33}
\]
or

\[
\Omega_{DE}^0 \left( 1 \mp \frac{2\sqrt{\Omega_{DE}}}{c} + 3w\Omega_{DE}^0 \right) > 0. \tag{34}
\]

For the future horizon case, this condition places a bound on the parameter \( c \):

\[
c < \left( \frac{3}{2} w \sqrt{\Omega_{DE}^0} = \frac{1}{2\sqrt{\Omega_{DE}^0}} \right)^{-1}. \tag{35}
\]

### 2.3.3 \( w = -1 \) cosmology

A ‘cosmological constant’ holographic dark energy may be parametrized by \((w, L) = (-1, L)\). For this case,

\[
\frac{\gamma}{3H} = \frac{2L}{3HL}. \tag{36}
\]

The equation for \( d\Omega_{DE}/dz \) then assumes the form

\[
\frac{d\Omega_{DE}}{dz} = -\frac{\Omega_{DE}}{1+z} \left( 1 \mp \frac{2\sqrt{\Omega_{DE}}}{c} - 3\Omega_{DE} \right). \tag{37}
\]

Maintaining that the dimensionless dark energy density increases in the future yields the constraint on \( c \):

\[
c < \left( \frac{3}{2} \sqrt{\Omega_{DE}^0} = \frac{1}{2\sqrt{\Omega_{DE}^0}} \right)^{-1}. \tag{38}
\]

Taking \( \Omega_{DE}^0 = 0.7 \) which is supported by current observations, we obtain \( c < 1.52 \).

From equation (15), the Hubble parameter is given by

\[
\frac{dH}{dz} = \frac{3}{2} \left( 1 + z \right) \left( 1 - \Omega_{DE} \right) \frac{H}{1 + z} \Omega_{DM}. \tag{39}
\]

The quantity \( E(z) \) now satisfies

\[
E(z) = H(z)/H_0 = \exp \left[ \frac{3}{2} \int_0^z \frac{\Omega_{DM}(z')}{1 + z'} dz' \right]. \tag{40}
\]

### 2.4 Neutrino dark energy

#### 2.4.1 Basic model

The FNW model (FNW) assumes that the dark energy sector consists of two components, one due to a form of dark energy such as quintessence with density \( \rho_{DE} \) and the other due to neutrinos. The total dark energy density is therefore

\[
\rho_{DE} = \rho_{DE} + \rho_\nu. \tag{41}
\]

For one generation of neutrinos and antineutrinos generalized (Peccei 2005) to neutrinos with arbitrary velocities, the energy density at temperature \( T \) is given by

\[
\rho_\nu = T^4 F(\xi), \tag{42}
\]

where \( \xi = m_\nu(T)/T \) and

\[
F(\xi) = \frac{1}{\pi^2} \int_0^\infty dy y^2 \sqrt{y^2 + \xi^2} \frac{e^y + 1}{e^y - 1}. \tag{43}
\]

Imposing the constraint that \( \rho_{DE} \) be stationary with respect to the changes in \( m_\nu \) and using the conservation equation

\[
\dot{\rho}_{DE} + 3H(1 + w)\rho_{DE} = 0 \tag{44}
\]
yields the equation of state:

\[
1 + w = \frac{\rho_\nu [4 - h(\xi)]}{3\rho_{DE}}. \tag{45}
\]

where

\[
h(\xi) = \frac{\xi}{F(\xi)} \frac{\partial F}{\partial \xi}. \tag{46}
\]

Expressing \( \rho_{DE} \) and \( \rho_{DE} = w\rho_{DE} \) in terms of the two components of the dark energy gives (Peccei 2005)

\[
\dot{\rho}_{DE} + \rho_\nu = 0 \tag{47}
\]

which is only compatible with a scenario where the component dark energy is described by a pure potential energy term:

\[
\dot{\rho}_{DE}(m_\nu) = V(m_\nu). \tag{48}
\]

Thus, the FNW scenario is equivalent to having a cosmological constant which varies with neutrino mass.

#### 2.4.2 Specifying the potential

Peccei (2005) considers a power-law dependence of the form

\[
V = -m_\nu^0 n_\nu^0 \left( \frac{w_0}{1 + w_0} \right) \left( \frac{m_\nu}{m_\nu^0} \right)^{1+w_0/w_0}. \tag{49}
\]

for which the equation of state becomes

\[
1 + w = \frac{4 - h(\xi)}{3 \left[ 1 - \frac{w_0}{1 + w_0} h(\xi) \right]}. \tag{50}
\]

Defining

\[
g(\xi) = \int_0^\infty \frac{d\nu \nu^2}{\sqrt{\nu^2 + 2(\xi + 1)}} \int_0^\infty \frac{d\nu \nu^2}{\sqrt{\nu^2 + 2(\xi + 1)}} = \int_0^\infty \frac{d\nu \nu^2}{\sqrt{\nu^2 + 2(\xi + 1)}}. \tag{51}
\]

we obtain

\[
m_\nu = m_\nu^0 \left[ \left( \frac{T}{T_0} \right) \frac{3}{g(\xi)} \right]^{w_0}. \tag{52}
\]

Since \( \xi = m_\nu(T)/T \), equation (52) can be solved numerically to give the variation of the neutrino mass with \( T \) throughout the cosmic history. The present value \( m_\nu^0 \) of the neutrino mass can be expressed in terms of \( w_0 \) and \( \rho_{DE}^0 \) using the non-relativistic limit \( \xi \gg 1 \) of equation (45):

\[
\rho_\nu^0 = m_\nu^0 m_\nu^0 = (1 + w_0) \rho_{DE}^0. \tag{53}
\]

Given \( m_\nu(T), w(T) \) can be obtained from equation (50) and the evolution with redshift of the dark energy density from equation (44) using \( z = T/T_0 - 1 \).
2.5 Predictions for cosmic evolution

Each of the interacting dark energy models considered above is capable of producing the cosmic evolution of $\Omega_{\text{DE}}$. In Fig. 1, the difference in the evolution of the dimensionless dark energy density $\Omega_{\text{DE}}$ for the various models and that for the $\Lambda$CDM model is shown as a function of redshift. In the upper panel, the interaction parameter $b$ is set equal to zero whereas $b^2 = 0.05$ in the lower panel. We see that the neutrino model exhibits the least difference while the holographic model with constant $w$ shows the greatest difference. For no interaction, the dimensionless densities converge in the past and the future. Interacting models exhibit larger differences and no longer approach the same value as the $\Lambda$CDM model in the future, generally having densities less than that of the cosmological constant model.

The variation of $w$ with respect to $z$ for the holographic and neutrino models is shown in Fig. 2. These models with time-dependent $w$ show quite different trends. For the holographic models based on the future horizon, $w$ increases with decreasing $z$ until $z \approx 3$ and then decreases to an approximately constant value in the future. In the neutrino model, $w$ evolves from $w = 1/3$ for large $z$ to $w = w_0 = -0.9$ for $z \lesssim 5$.

3 CONFRONTING MODELS WITH DATA

3.1 Data sets

We seek to constrain our models by using the observations of the redshift dependence of the luminosity distance of distant SN Ia, the shift parameter from measurements on the CMB, the BAO $A$ parameter derived from the SDSS of large-scale distribution of galaxies and the lookback time to high-redshift galactic clusters.

![Figure 1. The difference in evolution of the dimensionless dark energy density $\Omega_{\text{DE}}$ for various dark energy models and $\Omega_\Lambda$ for the $\Lambda$CDM model. The cases shown are the general interaction model (equation 18) (solid line), holographic model with specified interaction (equation 30) (dotted line), and the neutrino model with $w_0 = -0.8$ (dot-dashed line). Panel a corresponds to the no-interaction case $b^2 = 0$ and panel b to $b^2 = 0.05$. Consequently, the $\Lambda$CDM and interacting model are equivalent in panel a. For both the panels, $\Omega_{\text{DE}}^0 = 0.7$, $c = 1$ and $w = -1$.](https://academic.oup.com/mnras/article-abstract/390/4/1719/980775)

![Figure 2. Evolution of equation of state parameter, $w$, for the future horizon holographic model with $c = 1$ and $b^2 = 0$ (dashed line), $b^2 = 0.01$ (dotted line) and $b^2 = 0.05$ (solid line). Also shown is the neutrino model with $w_0 = -0.9$ (dot-dashed line).](https://academic.oup.com/mnras/article-abstract/390/4/1719/980775)

Fitting of the models to the SN Ia data points requires the distance modulus, which is given by

$$\mu(z) = 25 + 5 \log \left[ H_0^{-1}(1+z) \int_0^z \frac{dz'}{E(z')} \right].$$  (54)

The data used for comparison are the 182 gold measurements from Riess et al. (2007), spanning $0.024 < z < 1.76$. The CMB shift parameter is calculated using

$$\mathcal{R} = \sqrt{\Omega_{\text{DM}}^0} \int_0^{z_{\text{BAO}}} \frac{dz}{E(z)},$$  (55)

where $z_{\text{BAO}} = 1089 \pm 1$ is the redshift of the surface of last scattering. The WMAP3 analysis (Spergel et al. 2007a), which assumes $\Lambda$CDM cosmology, yields $\mathcal{R} = 1.70 \pm 0.03$. However, Wang & Mukherjee (2006) have shown that this value holds independent of the dark energy model.

Eisenstein et al. (2005) have used the SDSS of 46748 luminous red galaxies to determine the BAO parameter

$$A = \sqrt{\Omega_{\text{DM}}^0 E(z_{\text{BAO}})}^{-1/3} \left[ \frac{1}{z_{\text{BAO}}} \int_0^{z_{\text{BAO}}} \frac{dz}{E(z)} \right]^{2/3}.$$  (56)

They find $A = 0.469 (n_s/0.98)^{-0.35} \pm 0.017$, independent of the dark energy model, where $n_s = 0.95$ for the typical redshift $z_{\text{BAO}} = 0.35$ of their sample.

The lookback time of an object is defined as the difference between the present age of the Universe and its age at redshift $z$

$$t_l(z) = H_0^{-1} \int_0^{z_l} \frac{dz'}{(1+z')E(z')}.$$  (57)

The lookback time is not sensitive to the present age $t_0$ of the Universe so, to ensure both $t_l$ and $t_0$ are used to test a given model, the quantity (Capozziello et al. 2004; Feng et al. 2007)

$$t_l(z) = H_0^{-1} \left[ \int_z^{\infty} \frac{dz'}{(1+z')E(z')} - \int_{z_l}^{\infty} \frac{dz'}{(1+z')E(z')} \right]$$

is used where $z_l$ is the redshift when the object was formed. The lookback time for an object with redshift $z_i$ is then

$$t_l^{\text{obs}}(z_i) = t_l(z_i) - t_i = t_l^{\text{obs}} - t_i(z) - df,$$  (59)
where $\epsilon_0^{obs}$ is the estimated age of the Universe and
\[
d_j = t_0^{obs} - t_L(z_j)
\]
is a delay factor. This factor accounts for our ignorance of $z_0$ and is treated as an unknown parameter in the fitting. We use the data set of Capozziello et al. (2004) derived from a sample of 160 galactic clusters which consists of cluster ages at six redshifts in the range $0.10 \leq z \leq 1.27$.

3.2 Fitting procedure

As the models considered in our study are quite generic and lack detailed structure, our interest is in ascertaining whether or not these classes of models are tenable in light of the observational data, not in determining the model which best fits the data. Thus, it is sufficient to use simple $\chi^2$ statistics rather than more sophisticated techniques such as Bayesian or Akaike information criteria (Davis et al. 2007). We use $\chi^2$ statistics to find the best-fitting parameters for each of the models considered. As the data sets are effectively independent (Davis et al. 2007), we minimize the function (Zhang & Wu 2005)
\[
\chi^2 = \sum_j \chi^2_j,
\]
where $j$ runs over the different data sets. For Gaussian errors, this is related to the maximum likelihood $L$ by $\chi^2 = -2 \ln L$. Confidence limits on the fitted parameters are obtained using Monte Carlo simulation to create synthetic data sets (Press et al. 1992) and the Levenberg–Marquardt method to calculate the best-fitting parameters for each synthetic data set. The value of $\chi^2$ for the actual data set is obtained using these parameters. In this way, it is straightforward to identify the desired percentage confidence region as the area containing that percentage of points with the smallest $\chi^2$ values.

To study the effects of an interaction between dark energy and dark matter, we chose to fix the interaction parameter $b^2$ at sensible values and, for ease of display, allowed two parameters to vary while other parameters (if any) were held fixed. The nuisance parameter $H_0$ has been marginalized over.

Wherever possible we assumed no priors on the parameters, however at times the choice of model and data sets tended to iterate towards unrealistic best-fitting parameters, e.g. the future horizon confronted with the ESSENCE data tends to give the dimensionless dark energy density at present greater than unity. In such cases, a prior is imposed in the form of a $\chi^2$ penalty on parameter $a$:
\[
\chi^2_a = \left( \frac{a - a_{mod}}{\Delta a} \right)^2,
\]
where $a = a_0 \pm \Delta a$ is the prior and $a_{mod}$ is given by the model.

3.3 Results

Confidence regions for the interacting holographic dark energy model are shown in Figs 3 and 4 for the ($\Omega^0_{DE}, w$) plane with $b^2 = 0.005$ and the ($\Omega^0_{DE}, c$) plane with $b^2 = 0.01$. We see that the SN data set alone does not constrain the parameter space very tightly. However, the combined analyses with the BAO A and/or the CMB R parameter considerably decrease the range of preferred parameter values. At this level of interaction, the 95 per cent confidence regions for SN + BAO A and SN + CMB R overlap. We do not show fits to the ESSENCE SN data as they differ insignificantly from those for the gold SN data.

Table 1 displays the results for $\chi^2_{dat} = \chi^2/n$, where $n$ is the number of degrees of freedom, for different models of dark energy constrained by the various data sets. The analysis of the constant $w$ holographic model includes the weak prior $\Omega^0_{DE} = 0.7 \pm 0.15$. The various models fit the gold SN data almost equally well, and the same can be said when the models are confronted with the combined SN data + BAO A parameter. The CMB R parameter appears to have the most discriminating power between the models as it appears to rule out both the interacting holographic model with $b^2 = 0.05$ and the interacting model with $b^2 = 0.1$ as viable candidates.

Table 2 is similar to Table 1, however the analysis now includes the lookback time data set. The introduction of extra degrees of freedom produces a decrease in most of the $\chi^2_{dat}$ values compared...
Models of dark interacting energy

Table 1. Values for $\chi^2_{\text{dof}}$ for dark energy models confronted with gold supernova, BAO parameter and CMB $R$ data.

| Model                      | Fixed values | SN  | SN+BAO | SN+$R$ | SN+BAO+$R$ |
|----------------------------|--------------|-----|--------|--------|------------|
| Interacting holographic    | $b^2 = -0.004$ | 0.88| 0.88   | 0.88   | 0.88       |
|                            | $b^2 = 0.01$  | 0.88| 0.88   | 0.89   | 0.90       |
|                            | $b^2 = 0.05$  | 0.88| 0.89   | 1.8    | 1.8        |
| Constant $w$ holographic   | $w = -0.8$   | 0.90| 0.90   | 0.90   | 0.90       |
|                            | $w = -0.9$   | 0.89| 0.89   | 0.89   | 0.89       |
|                            | $w = -0.95$  | 0.89| 0.89   | 0.89   | 0.89       |
| Interacting model          | $b^2 = 0.001$| 0.87| 0.89   | 0.89   | 0.89       |
|                            | $b^2 = 0.005$| 0.87| 0.89   | 0.89   | 0.89       |
|                            | $b^2 = 0.1$  | 0.88| 0.91   | 5.7    | 6.0        |

Table 2. Values for $\chi^2_{\text{dof}}$ for dark energy models confronted with gold supernova, BAO parameter, CMB $R$ data and the lookback time data set.

| Model                      | Fixed values | SN  | SN+BAO | SN+$R$ | SN+BAO+$R$ |
|----------------------------|--------------|-----|--------|--------|------------|
| Interacting holographic    | $b^2 = -0.004$ | 0.88| 0.87   | 0.87   | 0.87       |
|                            | $b^2 = 0.01$  | 0.89| 0.90   | 0.88   | 0.89       |
|                            | $b^2 = 0.05$  | 1.7 | 1.7    | –      | –          |
| Constant $w$ holographic   | $w = -0.8$   | 0.89| 0.90   | 0.88   | 0.89       |
|                            | $w = -0.9$   | 0.89| 0.88   | 0.88   | 0.88       |
|                            | $w = -0.95$  | 0.88| 0.88   | 0.88   | 0.87       |
| Interacting model          | $b^2 = 0.001$| 0.87| 0.88   | 0.88   | 0.88       |
|                            | $b^2 = 0.005$| 0.87| 0.88   | 0.88   | 0.89       |
|                            | $b^2 = 0.1$  | 0.87| 0.89   | 5.5    | 5.8        |

to Table 1. However, the interacting holographic model with $b^2 = 0.05$ now fails to adequately match any of the data sets. The remaining values are all very similar to the values already obtained without the use of the lookback time.

Our conclusion that $b^2 \sim 0.005$ for the interacting model is consistent with the finding of Feng et al. (2008) that $\delta = 3b^2/\Omega_{\text{DE}}$ is small, positive and of the order of a few per cent. Similarly, for the interacting holographic model, our result that $b^2$ could lie between $-0.004$ and $0.01$ is consistent with the results $b^2 = 0.0\pm 0.01$ (Wu et al. 2007) and $b^2 = -0.003^{+0.006}_{-0.01}$ (Feng et al. 2007). However, our results for the interacting holographic model disagree with the range $0.05 < b^2 < 0.2$ obtained by Wang et al. (2007).

Results for the neutrino model of dark energy are shown in Table 3. For each of the $w_0$ values chosen the model matches the data quite well with the values of $\chi^2_{\text{dof}}$ quite similar to those in Table 1 for the other models.

4 CONCLUDING REMARKS

It is apparent that the available data can be met reasonably well by a wide range of parameters in various models. To explain why this is the case, it is informative to examine the quantity $1 + w\Omega_{\text{DE}}$, which appears in the numerator of the integrand in equation (16). Each of the tests employed in the previous section constrains the cosmological models by using this quantity.

The $z$-dependence of $1 + w\Omega_{\text{DE}}$ for all the models examined so far is shown in Fig. 5. Although Figs 1 and 2 show $\Omega_{\text{DE}}$ and $w$ evolve quite differently for the various models, it is evident here that the combination involved in the dimensionless Hubble parameter tends to suppress the differences between these models. In particular, we see that the neutrino model closely follows the $\Lambda$CDM model throughout cosmic history.

We conclude with some comments on the use of the particle horizon in holographic models. Li (2004) has shown that accelerated expansion cannot occur when the particle horizon is used within the holographic regime because the equation of state parameter is not sufficiently negative. However, it is possible to obtain $\chi^2 \approx 200$ when the particle horizon model is confronted with the gold SN Ia data set. This is larger than other models considered in this paper but this value is equivalent to $\chi^2_{\Lambda} \approx 1.1$, which is not large enough to suggest that this model is completely inadequate at describing the current cosmology.
The key difference between this model and others is shown in Fig. 6. For the particle horizon model, the integrand in equation (16) becomes negative for large $z$, which means that any data requiring the dimensionless Hubble parameter $\mathcal{E}(z)$ at high redshift will not be accurately matched by this cosmology.

APPENDIX A: ANALYTIC SOLUTION FOR INTERACTING DARK ENERGY

Consider equation (18) for the dimensionless dark energy density interacting with dark matter:

$$\frac{d\Omega}{dz} = \frac{3}{1+z} \left[ w\Omega(1-\Omega) + b^2 \right], \quad (A1)$$

where the DE subscript has been omitted. It follows that

$$\frac{3dz}{1+z} = -\frac{1}{w} \frac{d\Omega}{\Omega^2 - \Omega - b^2/w}, \quad (A2)$$

or, after expansion by partial fractions,

$$\frac{3dz}{1+z} = -\frac{1}{ws} \left[ \frac{1}{\Omega - \frac{1}{2}(1+s)} - \frac{1}{\Omega - \frac{1}{2}(1-s)} \right] d\Omega, \quad (A3)$$

where $s = \sqrt{1+4b^2/w}$. If we now assume that both $w$ and $b^2$ are constants, we can integrate to obtain

$$3ws \ln(1+z) = -\ln \left\{ \frac{\Omega - \frac{1}{2}(1+s)}{\Omega - \frac{1}{2}(1-s)} \right\}, \quad (A4)$$

which yields equation (20) in the text.

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