We are honoured and pleased to dedicate this paper to Professor Vince Moncrief on the occasion of his sixtieth birthday.

Abstract

There is derived, for a conformally flat three-space, a family of linear second-order partial differential operators which send vectors into tracefree, symmetric two-tensors. These maps, which are parametrized by conformal Killing vectors on the three-space, are such that the divergence of the resulting tensor field depends only on the divergence of the original vector field. In particular these maps send source-free electric fields into TT-tensors. Moreover, if the original vector field is the Coulomb field on $\mathbb{R}^3\setminus\{0\}$, the resulting tensor fields on $\mathbb{R}^3\setminus\{0\}$ are nothing but the family of TT-tensors originally written down by Bowen and York.

Keywords: TT-tensors, constraints of general relativity
1 Introduction

Bowen-York (BY-)tensors [4] are a particular class of TT-tensors. They have been a useful tool in constructing black-hole initial data satisfying the vacuum constraints of General Relativity. In the previous work [3] one of us (R.B.) described a scheme which gives BY-tensors a natural place in the context of the conformal geometry of conformally flat three-spaces.

Recall that the context in which BY-tensors appear is that of finding asymptotically flat solutions \((g'_{ab}, k'_{ab})\) to the vacuum constraints in the maximal, i.e. \(\text{tr}k' = 0\) - case. One starts with a “background” pair \((g_{ab}, k_{ab})\) and tries to solve a quasilinear elliptic PDE called the Lichnerowicz equation for a conformal factor \(\Phi\), so that \((g'_{ab} = \Phi^4 g_{ab}, k'_{ab} = \Phi^{-2} k_{ab})\) satisfies the Hamiltonian constraint. In addition, provided that \(k_{ab}\) is TT with respect to \(g_{ab}\), the tensor \(k'_{ab}\) will be TT with respect to the physical metric \(g'_{ab}\). In the process one assumes boundary conditions in order for the resulting initial data to be asymptotically flat.

To fix ideas let us for a start take the initial slice to be \(\mathbb{R}^3 \setminus \{0\}\) with flat background metric \(g_{ab}\). Consider the following 10-parameter set of symmetric, trace-free tensors:

\[
1\ k_{ab}(\vec{P}) = \frac{3}{2r^2}[P_a n_b + P_b n_a - (g_{ab} - n_a n_b)(P, n)]
\]

\[
2\ k_{ab}(\vec{S}) = \frac{3}{r^3}[(\epsilon_{cda} S^c n^d n_b + \epsilon_{cdb} S^c n^d n_a)]
\]

\[
3\ k_{ab}(C) = \frac{C}{r^3}[3n_a n_b - g_{ab}]
\]

\[
4\ k_{ab}(\vec{Q}) = \frac{3}{2r^4}[-Q_a n_b - Q_b n_a - (g_{ab} - 5n_a n_b)(Q, n)],
\]

where \((\vec{P}, \vec{S}, C, \vec{Q})\) are constants, \(r^2 = g_{ab} x^a x^b = (x, x)\) with \(x^a\) cartesian coordinates and \(n^a = x^a/r\). The tensors in (1.2) to (1.5) are all TT, i.e. they satisfy

\[
D^a k_{ab} = 0.
\]

The intuitive meaning of these tensors is as follows: The tensor \(1 k_{ab}\) in Eq. (1.2) corresponds to a “source” at \(r = 0\) with linear momentum \(\vec{P}\) and zero angular momentum, these quantities being defined (“measured”) at \(r = \infty\). The tensor \(2 k_{ab}\) in Eq. (1.3) corresponds to a source at \(r = 0\) with zero linear momentum and angular momentum \(\vec{S}\). The tensor \(4 k_{ab}\) in Eq. (1.5) is
the result of acting on \( k_{ab} \) by spherical inversion at the sphere \( r = 1 \). To appreciate the meaning of \( 4k_{ab} \) one should first understand the role of the “puncture” at \( r = 0 \). Namely, if one solves the Lichnerowicz equation subject to the boundary condition that \( \Phi \) go to one at infinity and blow up near the origin like \( O(1/r) \), the resulting physical initial-data set is asymptotically flat near infinity AND \( r = 0 \). The interpretation, now, of \( 4k_{ab} \) is that it corresponds to a linear momentum \( \vec{Q} \) measured near the \( r = 0 \) - infinity. In fact it would conceptually have been much simpler to treat the two infinities on equal footing and start on the “conformally compactified” manifold \( S^3 \) with punctures at two antipodal points corresponding to \( r = 0 \) (“south pole”) and \( r = \infty \) (“north pole”) previously. With any TT-tensor on \( S^3 \) and any puncture one can associate a 10-tuple of numbers giving the values of certain “quantities” of the TT-tensor at these punctures. The physical meaning of these quantities is that of linear and angular momentum for the first six. The meaning of the remaining four - let us call them the “C-quantity” and the “\( \vec{Q} \)-quantity” - is less clear since these quantities are not preserved under time-evolution. The role of \( C \) can be seen e.g. in maximal slicings of the maximally extended Schwarzschild spacetime [2]. Anyway, using the above prescription in an appropriate sense, the tensor \( 1k_{ab} \) has linear momentum \( \vec{P} \) at infinity and \( \vec{Q} \)-quantity equal to \( \vec{P} \) at \( r = 0 \). The tensor \( 2k_{ab} \) has angular momentum \( \vec{S} \) at infinity and angular momentum equal to \( -\vec{S} \) at \( r = 0 \). The tensor \( 3k_{ab} \) has C-quantity equal to \( C \) both at infinity and at \( r = 0 \). Finally, the tensor \( 4k_{ab} \) has \( Q \)-quantity equal to \( \vec{Q} \) at infinity and linear momentum \( \vec{Q} \) at \( r = 0 \). All other quantities are zero. The above interpretation, which has already been given in [3], will be reviewed at the beginning of the next section which also contains our main new result. This consists of a formula, valid on arbitrary conformally flat three-manifolds, giving explicit BY-tensors in our generalized sense in terms of solutions of the Gauss-law constraint of electrodynamics. In the final section we mention some applications.

2 The main theorem

Suppose \((M, g_{ab})\) is a compact, locally conformally flat three-manifold. Conformal flatness is equivalent to the condition

\[
D_a L_{bc} = D_b L_{ac},
\]

(2.7)
where

$$L_{ab} = R_{ab} - \frac{1}{4} g_{ab} R,$$  \hspace{1cm} (2.8)$$

$R_{ab}$ and $R$ being the Ricci and scalar curvature respectively. A basic object for us is the quantity $X(\xi, \lambda)$ defined for arbitrary vector fields $(\xi, \lambda)$ by

$$X(\xi, \lambda) = 4 L_{ab} \xi^a \lambda^b + (D_{[a} \xi_{b]})(D^{[a} \lambda^{b]}) - \frac{2}{9} (D\xi)(D\lambda) + \frac{2}{3} [\xi^a D_b D\lambda + \lambda^a D_a D\xi],$$  \hspace{1cm} (2.9)$$

where the notation $D\xi$ denotes divergence of the vector field $\xi$. Eq.\(\text{(2.9)}\) defines a symmetric, bilinear functional on the space of all vector fields on $M$. It has the further property that it is invariant under conformal rescalings of $M$ in the sense that $\bar{X} = X$ when $\bar{g}_{ab} = \omega^2 g_{ab}$, $\bar{\xi}_a = \xi$, $\bar{\lambda}_a = \lambda$.

Furthermore, by virtue of Eq.\(\text{(2.7)}\), if $(\xi, \lambda)$ are both elements of the finite (at most 10-) dimensional vector space of conformal Killing vectors (CKV’s), $X$ is in fact constant on $M$. This constant, for a pair $(\xi, \lambda)$, is nothing but $1/3$ of the Killing metric evaluated on that pair $(\xi, \lambda)$, viewed as elements of the Lie algebra of CKVs on $M$. We will omit the verification of this fact. The constancy of $X$ on CKV’s will also follow from a more general statement which we will prove shortly.

We assume from now on that $\xi$ is an arbitrary but fixed CKV. Then $X$ can be viewed as linear functional taking covectors on $M$ into $C^\infty$-functions on $M$. We write $X(\xi, \cdot) : \Lambda^{1,2} \rightarrow C^\infty$. The second superscript in these spaces refers to conformal weight, when the metric is given conformal weight 2. Taking, now, the natural $L^2$-adjoint we obtain a map $j(\xi; \cdot) : C^{\infty,-3} \rightarrow \Lambda^{1,-3}$, since the volume element on $M$ scales like $\omega^{-3}$. In other words there holds

$$\int_M X(\xi, \lambda) \rho \, dV = \int_M g^{ab} \lambda_a j_b(\xi; \rho) \, dV.$$  \hspace{1cm} (2.10)$$

Explicitly we have

$$j_b(\xi; \rho) = -D^a(D_{[a} \xi_{b]} \rho)) + \frac{2}{3} (D_b(D\xi)) \rho + \frac{2}{3} D_b D_a (\xi^a \rho) + \frac{2}{9} D_b((D\xi) \rho) + 4 L_{ba} \xi^a \rho.$$  \hspace{1cm} (2.11)$$
Next consider the equation

\[ D^a k_{ab} = j_b(\xi; \rho) \]  

(2.12)

on \( \mathbb{R}^3 \) with \( \rho = 4\pi \delta_0 \), the delta distribution concentrated at the origin. Then the ten TT-tensors given in Eq.'s (1.2)-(1.5) are solutions of Eq. (2.12) when the \( \xi \)'s run through CKV's on \( \mathbb{R}^3 \). Alternatively, after conformal compactification, we can view these tensors as solutions of Eq. (2.12) on \( S^3 \), with \( \rho \) given by \( \delta_{\text{northpole}} - \delta_{\text{southpole}} \). This choice of punctures is dictated by the following observation: If one contracts Eq. (2.12) with another CKV, say \( \eta \), and integrates over \( M \), one finds that

\[ 0 = X(\xi, \eta) \int_M \rho \, dV. \]  

(2.13)

Since the full conformal group acts on \( S^3 \) by conformal isometries and the Killing metric of the conformal group is non-degenerate, the integral of \( \rho \) over \( M \) has to vanish. Furthermore the right-hand side of (2.13), when integrated only over a region excluding one of the punctures, gives rise to the set of constants \( (\vec{P}, \vec{S}, C, \vec{Q}) \), mentioned in the Introduction. These things are more fully explained in [3].

We now come to the insight on which the main result of this paper is based: since \( X(\xi, \lambda) \) is constant when \( \lambda \) is a CKV, the gradient of \( X \), for general \( \lambda \), can only depend on \( l_{ab} = (CK)_{ab}(\lambda) \), where \( CK \) is the conformal Killing operator acting on \( \lambda \), i.e. the r.h. side of Eq. (A.5). Consequently there exists a linear operator \( i : \Sigma^{2,2} \to \Lambda^{1,0} \) - where \( \Sigma^{2,2} \) denotes covariant, valence-two, symmetric, trace-free tensors on \( M \) scaling like \( \omega^2 \) - so that

\[ D_a X(\xi, \lambda) = i_a(\xi; CK(\lambda)) \]  

(2.14)

In view of the definition of \( X \) and the formulae of the Appendix, the map \( i \) has to be a second-order partial differential operator in its second argument. Taking the \( L^2 \)-product of Eq. (2.14) with some 1-form \( E \) in \( \Lambda^{1,-1} \), using Eq. (2.10) on the left-hand side and integrating the right-hand side by parts twice, we see that there exists a second-order partial differential operator \( BY(\xi; \cdot) : \Lambda^{1,-1} \to \Sigma^{2,-1} \) so that

\[ \int_M (\text{div} \, E) \, X(\xi, \lambda) \, dV = \int_M g^{ab} g^{cd} (BY)_{ac}(\xi; E) (CK)_{bd}(\lambda) \, dV. \]  

(2.15)
Note that the operator $\text{div} : \Lambda^{1, -1} \rightarrow \mathcal{C}^{\infty, -3}$ is minus the $L^2$-adjoint of $\text{grad}$. Using Eq. (2.10) on the left-hand side of Eq. (2.15) and partially integrating the right-hand side of Eq. (2.15) once more, there finally follows that

$$
\int_M g^{ab} \lambda_a j_b(\xi; \text{div}E) \, dV = \int_M g^{ab} (\text{Div} \circ BY)_a(\xi; E) \lambda_b \, dV
$$

(2.16)

Here $\text{Div}$ is the divergence-operator mapping $\Sigma^{2,1}$ into $\Lambda^{1, -3}$. Since Eq. (2.16) holds for arbitrary fields $\lambda$, we have obtained the result that the operator $BY$ satisfies the identity

$$
\text{Div} \circ BY(\xi; E) = j(\xi; \text{div}E).
$$

(2.17)

(Eq. (2.17) is of course an entirely local statement: to derive it the integrations by parts have been used for convenience.) In other words we have obtained the following result

**Theorem:** Suppose that $E$ satisfies $\text{div}E = \rho$. Then the symmetric, trace-free tensor $k_{ab}$ defined by

$$
k_{ab} = (BY)_{ab}(\xi; E)
$$

(2.18)

satisfies the Bowen-York equation (2.12), namely

$$
D^a k_{ab} = j_b(\xi; \rho).
$$

(2.19)

In order for this result to be useful it remains to give the explicit expression for $BY$. We first have to find $i_a$ by taking the gradient of $X$. Using Eq.’s (A.5), (A.6) and (A.7) together with the remark following it, we find after a lengthy but straightforward calculation

$$
i_a(\xi; l) = \xi^b[(-2)\Delta l_{ab} + 4D_{(a}D^c l_{b)c} - g_{ab}D^cD^d l_{cd} -
- 3g_{ab} l^{cd} L_{cd} + 6L_{a} c^l_b c + 10L_{b} c^l_a +
+ 2(D^b \xi^c)(D_{[b}c_{d]a}) + \frac{2}{3}(D^b D\xi) l_{ab}
$$

(2.20)

Recall that $BY$ is the $L^2$-adjoint of $i$. Using that CK($\xi$) is zero we obtain by another lengthy computation that
\[(BY)_{ab}(\xi; E) = -\xi^c D_{(a} D_{b)} E_c - 2\xi_{(a} \Delta E_{b)} + g_{ab} \xi^c \Delta E_c + 2\xi_c D_{(a} D^c E_{b)} +
+ 2\xi_{(a} D_{b)} (DE) - \frac{4}{3} g_{ab} \xi^c D_c (DE) + \frac{4}{3} (D\xi) D_{(a} E_{b)} - 8F_{c(a} D^c E_{b)} + 4F^c_{(a} D_{b)} E_c -
- \frac{4}{9} g_{ab} (D\xi)(DE) + 4g_{ab} F^{cd} D_c E_d + 4E_{(a} D_{b)} D\xi - \frac{4}{3} g_{ab} E^c D_c (D\xi) + 9\xi^c L_{c(a} E_{b)} +
+ \xi_{(a} L_{b)c} E^c + 4L \xi_{(a} E_{b)} - 2g_{ab} L \xi^c E_c, \tag{2.21}\]

where \(L = g^{ab} L_{ab}\) and \(F_{ab} = D_{[a} E_{b]}\).

The space of TT-tensors is infinite-dimensional. This is due to the fact that the operator \(\text{Div}\) is underdetermined-elliptic. Thus the equation (2.21) has many solutions. If \(M\) is asymptotically flat and appropriate boundary conditions are assumed or when \(M\) is compact, a unique solution can be found by means of the York-decomposition. This consists of the observation that \(\Sigma^2\) splits into an \(L^2\)-orthogonal sum of tensors which are TT and “longitudinal ones”, i.e. ones of the form \(k = CK(W)\) for some covector \(W\). Note that this splitting breaks conformal invariance (hence the second superscript in \(\Sigma\) was omitted). Since \(\text{Div} \circ CK\) is elliptic with null space solely the CKV’s, Eq. (2.12) has a unique longitudinal solution. Similarly the equation \(\text{div} E = \rho\) has a unique longitudinal solution given by \(E = \text{grad} G\) for some scalar function \(G\). One is thus led to the question whether \(BY\) maps longitudinal solutions of the respective spaces into one another. There is an affirmative answer to this question in the case where \(g_{ab}\) is a space form, i.e. has constant curvature rather than merely being conformally flat. Namely, in that case, there exists an operator \(h(\xi; \cdot) : C^\infty \to \Lambda^1\) such that \(CK \circ h(\xi, G) = BY(\xi, \text{grad} G)\). The operator \(h\) appears already in the work [3]. The restriction to constant curvature for the compactified manifold leaves one with the case of \(T^3, S^3\) and quotients thereof. The present work was motivated by the desire to do similar things on \(S^2 \times S^1\), and this finally led us to the operator \(BY\).

We can now clarify the connection between our map \(BY\) and the Bowen-York expressions in (1.2)-(1.5). Let us take \(M\) to be \(\mathbb{R}^3\). The CKV’s are explicitly given by
\[
1\xi^a(\vec{\pi}) = \pi^a \tag{2.22}
\]
\[
2\xi^a(\vec{\sigma}) = \sigma^a_{\ bc}x^c \tag{2.23}
\]
\[
3\xi^a(\zeta) = \zeta x \tag{2.24}
\]
\[
4\xi^a(\vec{\gamma}) = (x, x)\gamma^a - 2(x, \gamma)x^a, \tag{2.25}
\]

where \(\pi^a, \sigma^a, \zeta, \gamma^a\) are constants. Let \(E\) be the Coulomb solution of \(\text{div}E = 4\pi\delta_0\), i.e.

\[
E = \frac{n}{r^2}. \tag{2.26}
\]

Then

\[
1k(\vec{P}) = -\frac{1}{2}BY(4\xi(\vec{P}), E) \tag{2.27}
\]
\[
2k(\vec{S}) = -BY(2\xi(\vec{S}), E) \tag{2.28}
\]
\[
3k(C) = -BY(3\xi(C), E) \tag{2.29}
\]
\[
4k(\vec{Q}) = \frac{1}{2}BY(1\xi(\vec{Q}), E) \tag{2.30}
\]

The main result of this paper is part of a more elaborate scheme. To describe it recall that the operators \(\text{grad}\) and \(\text{div}\) and are respectively the left and right end of the de Rham complex on \(M\), which has the operator \(\text{rot}\), given by \((\text{rot } \mu)_a = \epsilon_{a\ bc}\partial_b\mu_c\), in the middle. There exists a similar elliptic complex, introduced in \([1]\), with \(\textsc{CK}\) and \(\text{Div}\) at the left and right end, respectively. The operator in the middle is \((1/2\text{ of})\) a third-order partial differential operator called \(\text{H}\) in \([1]\) which is the linearization, at the conformally flat metric afforded by \(g_{ab}\), of the Cotton-York tensor applied to trace-free symmetric tensors. These two elliptic complexes are related by the following
The two lower-left and upper-right vertical maps are algebraic: The one on the far left is simply multiplication of $\mathcal{C}^\infty$-functions by $\xi$. The next one is the tensor product with $\xi$ combined with taking the symmetric, trace-free part. The right upper arrows denote contraction with $\xi$. Note that there is, in this diagram, a symmetry: If one simultaneously permutes top and bottom and left and right, one obtains the adjoint map of the original one.

After completion of this work we learnt that the above structure should be a special case of a general construction known as the “Bernstein-Gelfand-Gelfand” resolution [5]. For a more pedagogical exposition of this material we refer to the works [7] and [8].

### 3 Applications

We are interested in finding BY-tensors on manifolds with nontrivial topology. Take for example the background conformal structure to be the standard one on $\mathbb{S}^2 \times \mathbb{S}^1(a)$, i.e. the unit-two sphere times the circle with radius $a$. Solving the Lichnerowicz equation with one puncture and vanishing extrinsic curvature, one obtains the Misner wormhole (see [10]), with the radius $a$ determining the “distance” between the wormholes in units of the total mass. Taking, on that manifold, two punctures which lie in the same fibre of the $\mathbb{S}^2$-factor and antipodally on the circle, one obtains Einstein-Rosen data with two bridges. (For a nice review of these initial data see [9].) The aim is to construct non-time symmetric generalizations of such initial data.
(For known such data and their use in Numerical Relativity see the review in \cite{6}.) We do this by taking for the background extrinsic curvature BY-data as described in the present work. One is then left with the task of solving the Gauss law $div E = \rho$ with appropriate delta function sources and applying to the result the BY-map found in this paper. In this way one is able to give new expressions for existing BY-type data and also to construct new ones. Details will be given in the forthcoming PhD-thesis by one of us (W.K.)

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### A Appendix

In this appendix we collect a few formulae which are important in the text. Let $(M, g_{ab})$ be a three-manifold. Define the Schouten tensor by

$$ L_{ab} = R_{ab} - \frac{1}{4} g_{ab} R. \quad (A.1) $$

In three dimensions the Riemann tensor can be expressed in terms of the Ricci tensor by means of

$$ R_{abcd} = 2L_{c[a}g_{b]d} - 2L_{d[a}g_{b]d}. \quad (A.2) $$

Suppose $\xi$ is a conformal Killing vector. There is then a well-known way by means of which one can write down a sequence of integrability conditions, sometimes called “conformal Killing transport equations” which imply that $\xi$ is given uniquely by its “conformal Killing data” at some arbitrary point in $M$. These Killing data are furnished by

$$ (\xi_a, D_{[a}\xi_{b]}, D\xi, D_a D\xi), \quad (A.3) $$

where $D\xi$ and $D_a D\xi$ are respectively the divergence and the gradient of the divergence of $\xi$. What we do here is to perform the same procedure on the equation

$$ (CK)_{ab}(\lambda) = D_{(a}\lambda_{b)} - \frac{1}{3} g_{ab} D\lambda = l_{ab}, \quad (A.4) $$
which is simply the definition of the conformal Killing form for an arbitrary
vector field $\lambda$. Equivalently we can write

$$D_a \lambda_b = l_{ab} + D_{[a} \lambda_{b]} + \frac{1}{3} g_{ab} D \lambda$$  \hspace{1cm} (A.5)$$

Using the Ricci identity and (A.2) we obtain

$$D_a D_{[b} \lambda_{c]} = -2 L_{a[b} D_{c]} (D \lambda) - 2 g_{a[b} L_{c]d} \lambda^d - \frac{2}{3} g_{a[b} D_{c]} D \lambda + 2 D_{[b} \lambda_{c]a} \hspace{1cm} (A.6)$$

and

$$D_a D_b D \lambda = -3 L \lambda L_{ab} + 18 L^c (a \lambda_b)_{c} - \frac{9}{2} g_{ab} l^{cd} L_{cd} - 3 \Delta l_{ab} + 6 D_{(a} D^{c} \lambda_{b)_{c}} - \frac{3}{2} g_{ab} D^{c} D^{d} l_{cd}$$  \hspace{1cm} (A.7)$$

Note that the first term on the r.h. side of Eq. (A.7) can be expressed in terms
of $l_{ab}$ and the would-be conformal Killing data of $\lambda$ if $\lambda$ is a CKV.

References

[1] Beig R 1997 TT-Tensors and Conformally Flat Structures on 3-
Manifolds in: Mathematics of Gravitation, Part 1, Lorentzian Geometry
and Einstein Equations P T Chrusciel (Ed.) Banach Center Publications

[2] Beig R and ´O Murchadha N 1998 Late time behaviour of the maximal
slicing of the Schwarzschild black hole Phys. Rev. D 57 4728-4737

[3] Beig R 2000 Generalized Bowen-York Initial Data, in: Mathematical
and Quantum Aspects of Relativity and Cosmology S Cotsakis and G W
Gibbons (Eds.) LNP 537 Berlin: Springer

[4] Bowen J M and York Jr J W 1980 Time-asymmetric initial data for
black holes and black-hole collisions Phys. Rev. D 21 2047-2056
[5] Cap A, Slovak J and Soucek V 2001 Bernstein-Gelfand sequences  *Ann. Math (2)*** **154** 97-113

[6] Cook G 2000 Initial Data for Numerical Relativity  *Living Rev. Relativity*** 3, 5. [Online article]:cited on 15 Aug 2001 [http://www.livingreviews.org/Articles/Volume3/2000-5cook/](http://www.livingreviews.org/Articles/Volume3/2000-5cook/)

[7] Eastwood M 1999 Variations on the de Rham Complex  *Notices AMS*** **46** 1368-1376

[8] Eastwood M 2000 A complex from linear elasticity  *Suppl. di Rend. del Circolo Mat. di Palermo ser. II Suppl.* **63** 23-29

[9] Giulini D 1998 On the Construction of Time-Symmetric Black Hole Initial Data in:  *Black Holes: Theory and Observation* F Hehl C Kiefer and R Metzler (Eds.) LNP 514 Berlin: Springer

[10] Misner C W 1960 Wormhole Initial Conditions  *Phys. Rev.* **118** 1110-1111