Renormalizable quantum field theory as a limit of a quantum field model on the loop space.

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**Abstract**

A nonlocal generalization of quantum field theory in which momentum space is the space of continuous maps of a circle into \( \mathbb{R}^4 \) is proposed. Functional integrals in this theory are proved to exist. Renormalized quantum field model is obtained as a local limit of the proposed theory.

Consider a quantum field model defined in the following way. Let the momentum space of the theory be the Banach space 

\[ \mathcal{P} = C(S^1, \mathbb{R}^4) \]

of all continuous maps of a circle of a unit length into \( \mathbb{R}^4 \) with the norm 

\[ \|p\|_P = \max_{\tau \in S^1} \|p(\tau)\|, \]

where \( \| \cdot \| \) is the canonical norm in \( \mathbb{R}^4 \).

An arbitrary element of this space can be represented in the form 

\[ p(\tau) = r + \frac{1}{\sqrt{\lambda}} \xi(\tau). \]  

(1)

Here \( \xi(\tau) \) satisfies the condition 

\[ \int_{S^1} \xi(\tau) d\tau = 0. \]  

(2)

We denote the space of maps \( \xi \) as \( C_0(S^1, \mathbb{R}^4) \).

Consider the group \( G \) of three times differentiable diffeomorphisms 

\[ G = Diff^3_+(S^1), \]

\[ g \in G \quad \{g: S^1 \rightarrow S^1, \quad g'(\tau) > 0\}. \]
Define the action of the group on the space $\mathcal{P}$ in the following way:

$$gp(\tau) = p \left( g^{-1}(\tau) \right) \frac{1}{\sqrt{(g^{-1})'(\tau)}}. \quad (3)$$

Consider the Wiener measure on the space $\mathcal{P}$ with the dispersion equal to $\sqrt{\lambda}$ and zero mathematical expectation

$$w_\lambda(dp) = \exp \left\{ -\frac{\lambda}{2} \int_{S^1} \|p'(\tau)\|^2 d\tau \right\} dp, \quad \lambda > 0. \quad (4)$$

For any continuous bounded functional $F(p)$, that satisfies the inequality $|F(p)| \leq \frac{C}{1+\|p\|}$, the following equation is valid

$$\int_\mathcal{P} F(p) \ w_\lambda(dp) = \int_{\mathbb{R}^4} \int_{C_0(S^1, \mathbb{R}^4)} F(r + \frac{1}{\sqrt{\lambda}} \xi) \ w_1(d\xi) \ dr. \quad (5)$$

From here, the local limit of the integral follows

$$\lim_{\lambda \to +\infty} \int_\mathcal{P} F(p) \ w_\lambda(dp) = \int_{\mathbb{R}^4} F(r) \ dr. \quad (5)$$

It is known that measures invariant with respect to the group $G$ do not exist. However, the measure $w_\lambda(dp)$ is quasi-invariant. \[1\]

It transforms as

$$w_\lambda(d(gp)) = \exp \left\{ \frac{\lambda}{4} \int_{S^1} S_g(\tau) \|p(\tau)\|^2 d\tau \right\} w_\lambda(dp), \quad (6)$$

Here $S_g$ denotes the Schwarz derivative

$$S_g(\tau) = \frac{g''''(\tau)}{g'('\tau)} - \frac{3}{2} \left( \frac{g''(\tau)}{g'(\tau)} \right)^2. \quad (7)$$

Consider the space $E$ of all square-integrable over the Wiener measure functions $\varphi : \mathcal{P} \to \mathbb{C}$, satisfying the equation

$$\varphi(p) = \bar{\varphi(-p)}$$

for all $p \in \mathcal{P}$.

The space $E$ is a Hilbert space over the real field $\mathbb{R}$ with the scalar product

$$(\varphi, \phi)_E = \int_\mathcal{P} \varphi(p) \bar{\phi(p)} \ w_\lambda(dp).$$

Functions $\varphi$ and $\phi$ realize a regular unitary representation of the group $G$ in the Hilbert space $E$:

$$g \varphi(p) = \varphi(gp) \ \exp \left\{ \frac{\lambda}{8} \int_{S^1} S_g(\tau) \|p(\tau)\|^2 d\tau \right\}. \quad (8)$$

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Free action
\[ \mathcal{A}_0[\varphi] = \int_p |\varphi(p)|^2 \Omega^2(p) \ w_\lambda(dp) \] (9)
with
\[ \Omega^2(p) = \left( \int_{S^1} \frac{d\tau}{\|p(\tau)\|^2} \right)^{-1} + m^2 \] (10)
is invariant with respect to the group \( G \).

Note that
\[ \lim_{\lambda \to +\infty} \mathcal{A}_0[\varphi] = \int_{\mathbb{R}^4} |\varphi(r)|^2 \left( \|r\|^2 + m^2 \right) \ dr. \]

Thus, in the limit case we get the action of free scalar field.

Let us construct the interaction term. In the limit case \( \lambda \to +\infty \) it should give the usual interaction \( \varphi^4 \).

To get (quasi-) invariant expression we change \( p \) to \( gp \) and consider group averaging over \( G \) with the quasi-invariant measure \[ \mu_\beta(dg) = \exp \left\{ -\frac{\beta}{2} \int_{S^1} \left[ \frac{g''(\tau)}{g'(\tau)} \right]^2 d\tau \right\} \ dg, \] (11)
where \( \beta \) is an arbitrary positive parameter.

It is convenient to write the parameter \( \beta \) in the form \( \beta = \alpha \lambda \) with an arbitrary positive \( \alpha \).

Taking into account the transformation rules for the field and the measure we propose the following formula for interaction
\[ \mathcal{A}_1[\varphi] = \int_p \cdots \int_p \varphi(p_1)\varphi(p_2)\varphi(p_3)\varphi(p_4) \]
\[ \int_{S^1} \delta(p_1(\tau_1) - p_5(\tau_1)) \|p_1(\tau_1)\|^2 d\tau_1 \int_{S^1} \frac{d\tau}{\|p_1(\tau)\|^2} \]
\[ \int_{S^1} \delta(p_2(\tau_2) - p_6(\tau_2)) \|p_2(\tau_2)\|^2 d\tau_2 \int_{S^1} \frac{d\tau}{\|p_2(\tau)\|^2} \]
\[ \int_{S^1} \delta(p_3(\tau_3) - p_7(\tau_3)) \|p_3(\tau_3)\|^2 d\tau_3 \int_{S^1} \frac{d\tau}{\|p_3(\tau)\|^2} \]
\[ \int_{S^1} \delta(p_4(\tau_4) - p_8(\tau_4)) \|p_4(\tau_4)\|^2 d\tau_4 \int_{S^1} \frac{d\tau}{\|p_4(\tau)\|^2} \]
\[ \int_{S^1} \delta(p_5(\tau_5) + p_6(\tau_5) + p_7(\tau_5) + p_8(\tau_5)) \|p_5(\tau_5)\|^2 d\tau_5 \int_{S^1} \frac{d\tau}{\|p_5(\tau)\|^2} \]
\[ \int_G \exp \left\{ \frac{\lambda}{8} \int_{S^1} S_{g_1}(\tau) \|p_1(\tau)\|^2 d\tau \right\} \mu_\alpha(\lambda(dg_1) \ w_\lambda(dp_1) \]
We see that interaction in this model is nonlocal.

Using the following notations

\[ \Delta(p_1, p_5) \equiv \int \frac{d\tau}{\| p_1(\tau) \|^2} \int \delta(p_1(\tau) - p_5(\tau)) \| p_1(\tau) \|^2 d\tau , \]

\[ \mathcal{U}_{\alpha\lambda}(p) \equiv \int G \exp \left\{ \frac{\lambda}{8} \int S_{g}(\tau) \| p(\tau) \|^2 d\tau \right\} \mu_{\alpha\lambda}(dg) , \]

we write it in a more compact form

\[ \mathcal{A}_1[\varphi] = \int \cdots \int \varphi(p_1) \mathcal{U}_{\alpha\lambda}(p_1) \Delta(p_1, p_5) \]

\[ \varphi(p_2) \mathcal{U}_{\alpha\lambda}(p_2) \Delta(p_2, p_6) \]

\[ \varphi(p_3) \mathcal{U}_{\alpha\lambda}(p_3) \Delta(p_3, p_7) \]

\[ \varphi(p_4) \mathcal{U}_{\alpha\lambda}(p_4) \Delta(p_4, p_8) \]

\[ \Delta(p_5, -p_6 - p_7 - p_8) \]

\[ \mathcal{U}_{\alpha\lambda}(p_5) \mathcal{U}_{\alpha\lambda}(p_6) \mathcal{U}_{\alpha\lambda}(p_7) \mathcal{U}_{\alpha\lambda}(p_8) \]

\[ w_{\lambda}(dp_1) \cdots w_{\lambda}(dp_8) . \]

Getting in mind the effects of renormalization we consider also the following addition to the action

\[ \mathcal{A}_2[\varphi] = \int \int \int \int \varphi(p_1) \varphi(p_2) \]

\[ \Delta(p_1, p_3) \Delta(p_2, p_4) \Delta(p_3, -p_4) \]

\[ \mathcal{U}_{\alpha\lambda}(p_1) w_{\lambda}(dp_1) \mathcal{U}_{\alpha\lambda}(p_2) w_{\lambda}(dp_2) \]

\[ \mathcal{U}_{\alpha\lambda}(p_3) w_{\lambda}(dp_3) \mathcal{U}_{\alpha\lambda}(p_4) w_{\lambda}(dp_4) . \]

Now the total action is given by the equation

\[ \mathcal{A}[\psi] = \mathcal{A}_0[\psi] + \kappa_1 \mathcal{A}_1[\psi] + \kappa_2 \mathcal{A}_2[\psi] . \]

The following theorem is valid

**Theorem 1.**

*For any positive \( \kappa_1 \) and any \( \psi \in \mathcal{E} \) there exists the integral*

\[ \int_{\mathcal{E}} e^{-\mathcal{A}[\varphi] - i(\varphi, \psi)_{\mathcal{E}}} d\varphi . \tag{12} \]
To prove this theorem note that for any \( \phi \in E \) the functional \( A_1[\phi] \) is nonnegative. Therefore, the functional \( e^{-\kappa_1 A_1[\phi]-\kappa_2 A_2[\phi]-i(\phi,\psi)_E} \) is bounded on \( E \). Hence, to prove that the integral (12) converges it is sufficient to verify the existence of the integrals

\[ \int_E A_1[\psi]e^{-A_0[\psi]}d\psi \]

and

\[ \int_E A_2[\psi]e^{-A_0[\psi]}d\psi \]

(3).

It can be done with the help of several bounds including those that are given by the lemmas.

**Lemma 1.** There exists \( c_1 > 0 \) such that \( \forall \ r \in \mathbb{R}^1 \) and \( \forall \ t_1, t_2 \in S^1 \) (\( t_1 \neq t_2 \)) the following inequality is valid

\[ \int_{\mathbb{R}} \delta(p(t_1) - r) \|p(t_1)\|^2 \|p(t_2)\|^2 \int_{S^1} \frac{d\tau}{\|p(\tau)\|^2} U_{\alpha \lambda}(p) \ w_\lambda(dp) \leq c_1 \left( 1 + r^4 \right) . \]

**Lemma 2.** There exists \( c_2 > 0 \) such that \( \forall \ t_1, t_2 \in S^1 \) (\( t_1 \neq t_2 \)) the following inequality is fulfilled

\[ \int_{\mathbb{R}} \left[ \Omega^2(p_1) \right]^{-1} \|p(t_1)\|^2 \left( \|p(t_2)\|^2 + 1 \right) \left( \int_{S^1} \frac{d\tau}{\|p(\tau)\|^2} \right)^2 (U_{\alpha \lambda}(p))^2 \ w_\lambda(dp) \leq c_2 . \]

We have already discussed that when \( \lambda \to +\infty \) the set of measures \( w_\lambda \) converges in a weak sense to Lebesgue measure on \( \mathbb{R}^4 \subset \mathcal{P} \).

The limit of \( A_0[\phi_\varphi] \) gives the action of free scalar field.

Similarly, when \( \alpha \to +\infty \ \lambda \to +\infty \), the measure \( U_{\alpha \lambda}(p) \ w_\lambda(dp) \) converges in a weak sense to a Lebesgue measure. And for the limits of \( A_1[\phi_\varphi] \) and \( A_2[\phi_\varphi] \) we have the action of \( \varphi^4 \) model.

\[ \lim_{\lambda \to +\infty, \alpha \to +\infty} A_1[\phi_\varphi] = \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \delta(r_1 + r_2 + r_3 + r_4) \ \varphi(r_1)\varphi(r_3)\varphi(r_3)\varphi(r_4) \ dr_1dr_2dr_3dr_4 . \]

\[ \lim_{\lambda \to +\infty, \alpha \to +\infty} A_2[\phi] = \int_{\mathbb{R}^4} |\varphi(r)|^2 \ dr . \]

It can be proven that the local limit of our model is the renormalized \( \varphi^4 \) theory. That is, the following theorem is valid.

**Theorem 2.**

There are functions \( \kappa_1(\theta, \alpha) \) and \( \kappa_2(\theta, \alpha) \) such that for all integer \( n \geq 0 \) the following limits exist

\[ \lim_{\lambda \to +\infty} \lim_{\alpha \to +\infty} \frac{\partial^n}{\partial \theta^n} \left[ \int_E e^{-A_0[\psi]-\kappa_1(\theta, \alpha) A_1[\psi]-\kappa_2(\theta, \alpha) A_2[\psi]-i(\psi,\phi)_E} \ d\psi \right]_{\theta=0} . \]
To prove this theorem we make a substitution

\[ f(\tau) = \sqrt{\lambda} \left( \frac{g''(\tau)}{g'(\tau)} \right). \] (13)

The measure \( \mu \) looks like

\[ \mu_{\alpha \lambda}(dg) = \exp \left\{ -\frac{\alpha}{2} \int \frac{(f'(\tau))^2}{\lambda} d\tau \right\} \, df = w_\alpha(df). \] (14)

Schwarz derivative \( S \) takes the form

\[ S_g(\tau) = \frac{1}{\sqrt{\lambda}} f'(\tau) - \frac{1}{2\lambda} f^2(\tau). \] (15)

Also, we have

\[
\lim_{\lambda \to +\infty} \exp \left\{ \frac{\lambda}{8} \int_{S^1} S_g(\tau) \|p(\tau)\|^2 \, d\tau \right\} = \exp \left\{ -\frac{1}{16} \int_{S^1} f^2(\tau) \, d\tau \|r\|^2 + \frac{1}{4} \int_{S^1} f'(\tau) (r, \xi(\tau)) \, d\tau \right\}.
\]

Now, for the limit \( \lim_{\lambda \to +\infty, \alpha = const} A_1 \) we get

\[
\lim_{\lambda \to +\infty, \alpha = const} A_1[\varphi] = \int_{R^4} \cdots \int_{R^4} \int_{F} \cdots \int_{F} \varphi(r_1)\varphi(r_2)\varphi(r_3)\varphi(r_4) \delta(r_1 - r_5) \delta(r_2 - r_6) \delta(r_3 - r_7) \delta(r_4 - r_8) \delta(r_5 + r_6 + r_7 + r_8) \]

\[
\int_{S^1} \exp \left\{ -\frac{1}{16} \int_{S^1} f_1^2(\tau) \, d\tau \|r\|^2 + \frac{1}{4} \int_{S^1} f_1'(\tau) (r_1, \xi_1(\tau)) \, d\tau \right\} \, w_\alpha(df_1) \, w_1(d\xi_1) \, dr_1 \]

\[
\cdots \]

\[
\int_{S^1} \exp \left\{ -\frac{1}{16} \int_{S^1} f_8^2(\tau) \, d\tau \|r\|^2 + \frac{1}{4} \int_{S^1} f_8'(\tau) (r_8, \xi_8(\tau)) \, d\tau \right\} \, w_\alpha(df_8) \, w_1(d\xi_8) \, dr_8.
\]

Integrations over \( r_5, ..., r_8 \) and \( \xi_1, ..., \xi_8 \) result in

\[
\lim_{\lambda \to +\infty, \alpha = const} A_1[\varphi] = \int_{R^4} \cdots \int_{R^4} \varphi(r_1)\varphi(r_2)\varphi(r_3)\varphi(r_4) \int_{F} \cdots \int_{F} \exp \left\{ -\frac{1}{32} \int_{S^1} \left[ f_1^2(\tau) + f_5^2(\tau) \right] \, d\tau \|r_1\|^2 \right\} \exp \left\{ -\frac{1}{32} \int_{S^1} \left[ f_2^2(\tau) + f_6^2(\tau) \right] \, d\tau \|r_2\|^2 \right\} \]

\[
\exp \left\{ -\frac{1}{32} \int_{S^1} \left[ f_3^2(\tau) + f_7^2(\tau) \right] \, d\tau \|r_3\|^2 \right\} \exp \left\{ -\frac{1}{32} \int_{S^1} \left[ f_4^2(\tau) + f_8^2(\tau) \right] \, d\tau \|r_4\|^2 \right\}.
\]
Finally, evaluating functional integrals over $f_i$ we obtain

$$\lim_{\lambda \to +\infty, \alpha = \text{const}} A_1[\varphi] =$$

$$\int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \varphi(r_1) I(\alpha, \frac{1}{4} \|r_1\|) \cdots \varphi(r_4) I(\alpha, \frac{1}{4} \|r_4\|)$$

$$\delta(r_1 + r_2 + r_3 + r_4) dr_1 \cdots dr_4.$$  

Here

$$I(\alpha, a) = \frac{a}{\sqrt{\alpha}} \frac{\exp\left(-\frac{a}{\sqrt{\alpha}}\right)}{1 - \exp\left(-\frac{a}{\sqrt{\alpha}}\right)}.$$  

Similarly, for $\lim_{\lambda \to +\infty, \alpha = \text{const}} A_2$ we obtain

$$\lim_{\lambda \to +\infty, \alpha = \text{const}} A_2[\varphi] =$$

$$\int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \varphi(r_1) I(\alpha, \frac{1}{4} \|r_1\|) \varphi(r_2) I(\alpha, \frac{1}{4} \|r_2\|)$$

$$\delta(r_1 + r_2) dr_1 dr_2.$$  

Note that the factors $I$ decrease very fast at large $\|r\|$. Therefore, we can consider the above formulae as a regularization of $\varphi^4$ model with the regularization parameter $\alpha$. It belongs to a set of regularizations that can be used for the proof of renormalizability of $\varphi^4$ model [4].

In a similar manner, it is possible to prove the existence of the functional integral in $\varphi^4$ model in 3-dimensional space-time

$$\lim_{\lambda \to +\infty} \lim_{\alpha \to +\infty}$$

$$\int_{E} e^{-A_0[\varphi] - \kappa_1 A_1[\varphi] - \kappa_2 (\theta, \alpha) A_2[\varphi] - i(\varphi, \chi) E} d\varphi$$

$$\int_{E} e^{-A_0[\varphi] - \kappa_1 A_1[\varphi] - \kappa_2 (\theta, \alpha) A_2[\varphi]} d\varphi.$$  

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