DESCRIPTION OF UNITARY REPRESENTATIONS
OF THE GROUP OF INFINITE $p$-ADIC INTEGER MATRICES

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Abstract. We classify irreducible unitary representations of the group of all
infinite matrices over a $p$-adic field ($p \neq 2$) with integer elements equipped with
a natural topology. Any irreducible representation passes through a group GL
of infinite matrices over a residue ring modulo $p^k$. Irreducible representations
of the latter group are induced from finite-dimensional representations of cer-
tain open subgroups.

1. Introduction

1.1. Notations and definitions.

(a) Rings. Let $p$ be a prime,

$\quad p > 2$.

Let $\mathbb{Z}_{p^n} := \mathbb{Z}/p^n\mathbb{Z}$ be a residue ring, $\mathbb{F}_p := \mathbb{Z}_p$ be the field with $p$ elements. The
ring of $p$-adic integers $\mathcal{O}_p$ is the projective limit

$\quad \mathcal{O}_p = \lim_{\leftarrow n} \mathbb{Z}_{p^n}$

of the following chain (see, e.g., [32]):

$\quad \cdots \leftarrow \mathbb{Z}_{p^n-1} \leftarrow \mathbb{Z}_{p^n} \leftarrow \mathbb{Z}_{p^{n+1}} \leftarrow \cdots$,

we have $\mathbb{Z}_{p^n} = \mathcal{O}_p/p^n\mathcal{O}_p$. Denote by $\mathbb{Q}_p$ the field of $p$-adic numbers.

(b) The infinite symmetric group and oligomorphic groups. Let $\Omega$ be a
countable set. Denote by $S(\Omega)$ the group of all permutations of $\Omega$; denote $S_\infty := S(\mathbb{N})$. The topology on the infinite symmetric group $S(\Omega)$ is determined by the
condition: stabilizers of finite subsets are open subgroups and these subgroups
form a fundamental system of neighborhoods of the unit.$^1$ Equivalently, a sequence $g^\alpha$ converges to $g$ if for each $\omega \in \Omega$ we have $\omega g^\alpha = \omega g$ for sufficiently large $\alpha$.

A closed subgroup $G$ of $S(\Omega)$ is called oligomorphic if for each $k$ it has only a
finite number of orbits on the product $\Omega \times \cdots \times \Omega$ of $k$ copies of $\Omega$; see [5].

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$^1$Thus we get a structure of a Polish group. Moreover this topology is a unique separable
topology on the infinite symmetric group; see [13]. In particular, this means that a unitary
representation of $S_\infty$ in a separable Hilbert space is automatically continuous.

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(c) Modules \(l(\mathbb{Z}_p^n)\) and Groups \(\text{GL}(\infty, \mathbb{Z}_p^n)\). Define the module \(l(\mathbb{Z}_p^n)\) as the set of all sequences \(v = (v_1, v_2, \ldots)\), where \(v_j \in \mathbb{Z}_p^n\) and \(v_j = 0\) for sufficiently large \(j\). The set \(l(\mathbb{Z}_p^n)\) is countable; we equip it with a discrete topology. Denote by \(e_j\) the standard basis elements, i.e., \(e_j\) has a unit on \(j\)-th place, other elements are 0.

Define groups \(\text{GL}(\infty, \mathbb{Z}_p^n)\) as groups of infinite invertible matrices \(g\) over \(\mathbb{Z}_p^n\) such that:

- each row of \(g\) contains only a finite number of nonzero elements;
- each column contains only a finite number of nonzero elements;
- the inverse matrix \(g^{-1}\) satisfies the same conditions.

Notice that rows of a matrix \(g\) are precisely vectors \(e_j g\), and columns are \(e_j g^t\) (we denote by \(g^t\) a transposed matrix).

Actually, the topic of this paper is representations of \(\text{GL}(\infty, \mathbb{Z}_p^n)\).

This group is continual and we must define a topology on \(\text{GL}(\infty, \mathbb{Z}_p^n)\). A sequence \(g^{(\alpha)} \in \text{GL}(\infty, \mathbb{Z}_p^n)\) converges to \(g\) if all sequences \(e_j g^{(\alpha)}\) and \(e_i (g^{(\alpha)})^t\) are eventually constant and their limits are \(e_j g\) and \(e_j g^t\) respectively. Thus we get a structure of a totally disconnected topological group.

The group \(\text{GL}(\infty, \mathbb{Z}_p^n)\) acts on the countable set \(l(\mathbb{Z}_p^n) \oplus l(\mathbb{Z}_p^n)\) by transformations

\[
(v, w) \mapsto (v g, w(g^t)^{-1}).
\]

In particular, this defines an embedding of \(\text{GL}(\infty, \mathbb{Z}_p^n)\) to a symmetric group \(S(l(\mathbb{Z}_p^n) \oplus l(\mathbb{Z}_p^n))\). The image of the group \(\text{GL}(\infty, \mathbb{Z}_p^n)\) is a closed subgroup of \(S(l(\mathbb{Z}_p^n) \oplus l(\mathbb{Z}_p^n))\) and the induced topology coincides with the natural topology on \(\text{GL}(\infty, \mathbb{Z}_p^n)\). By \([27, \text{Lemma 3.7}]\), the group \(\text{GL}(\infty, \mathbb{Z}_p^n)\) is oligomorphic.

(d) Modules \(l(O_p)\) and Groups \(\text{GL}(\infty, O_p)\). Denote by \(l(O_p)\) the set of all sequences \(r = (r_1, r_2, \ldots)\), where \(r_j \in O_p\) and \(|r_j| \to 0\) as \(j \to \infty\). The space \(l(O_p)\) is a projective limit,

\[
l(O_p) = \lim_{\longrightarrow n} l(\mathbb{Z}_p^n),
\]

we equip it with the topology of the projective limit. In other words, a sequence \(r^{(j)} \in l(O_p)\) converges if for any \(p^n\) the reduction of \(r^{(j)}\) modulo \(p^n\) is eventually constant in \(\mathbb{Z}_{p^n}\).

We define \(\text{GL}(\infty, O_p)\) as the group of all infinite matrices \(g\) over \(O_p\) such that:

- each row of \(g\) is an element of \(l(O_p)\);
- each column of \(g\) is an element of \(l(O_p)\);
- the matrix \(g\) has an inverse and \(g^{-1}\) satisfies the same conditions.

We say that a sequence \(g^{(\alpha)} \in \text{GL}(\infty, O_p)\) converges to \(g\) if for any \(i\) the sequence \(e_i g^{(\alpha)}\) converges to \(e_i g\) and for any \(j\) the sequence \(e_i (g^{(\alpha)})^t\) converges to \(e_j g^t\). This determines a structure of a totally disconnected topological group on \(\text{GL}(\infty, O_p)\).

We have obvious homomorphisms \(\text{GL}(\infty, \mathbb{Z}_p^n) \to \text{GL}(\infty, \mathbb{Z}_{p^{n-1}})\), the group \(\text{GL}(\infty, O_p)\) is the projective limit

\[
\text{GL}(\infty, O_p) = \lim_{\longrightarrow n} \text{GL}(\infty, \mathbb{Z}_p^n)
\]

and its topology is the topology of projective limit.
1.2. Preliminary remarks. A priori we know the following statement:

**Theorem 1.1.**

(a) The group $GL(\infty, \mathbb{O}_p)$ is a type I group; it has a countable number of irreducible unitary representations. Any unitary representation of $GL(\infty, \mathbb{O}_p)$ is a sum of irreducible representations. Any irreducible unitary representation of $GL(\infty, \mathbb{O}_p)$ is in fact a representation of some group $GL(\infty, \mathbb{Z}_{p^n})$.

(b) Each irreducible representation of $GL(\infty, \mathbb{Z}_{p^n})$ is induced from a finite-dimensional representation of an open subgroup. More precisely, for any irreducible unitary representation of $GL(\infty, \mathbb{Z}_{p^n})$ there exists an open subgroup $\hat{Q} \subset GL(\infty, \mathbb{Z}_{p^n})$, a normal subgroup $Q \subset \hat{Q}$ of finite index and an irreducible representation $\nu$ of $\hat{Q}$, which is trivial on $Q$, such that $\rho$ is induced from $\nu$.

This is a special case of a theorem of Tsankov about unitary representations of oligomorphic groups and projective limits of holomorphic groups; see [34, Theorem 1.3].

It seems that [34], [2] are not sufficient to give a precise answer in our case.

Let us give a definition of an induced representation (see, e.g., [33, Sect. 7] and [15, Sect. 13]) which is appropriate in our case. Let $G$ be a totally disconnected separable group, $Q$ its open subgroup. Let $\nu$ be a unitary representation of $Q$ in a Hilbert space $V$. Consider the space $H$ of $V$-valued functions $f$ on a countable homogeneous space $Q \setminus G$ such that

$$\sum_{x \in Q \setminus G} \|f(x)\|^2 < \infty.$$ 

Equip $H$ with the inner product

$$\langle f_1, f_2 \rangle_H := \sum_{x \in Q \setminus G} \langle f_1(x), f_2(x) \rangle_V.$$ 

Let $U$ be a function on $G \times (Q \setminus G)$ taking values in the group of unitary operators in $V$ such that:

- Formula

$$\rho(g)f(x) = U(g, x)f(xg)$$

determines a representation of $G$ in $H$.

- Let $x_0$ be the initial point of $Q \setminus G$, i.e., $x_0Q = x_0$. Then for $q \in Q$ we have $U(q, x_0) = \nu(q)$.

The first condition implies that the function $U(g, x)$ satisfies the functional equation

$$U(x, g_1g_2) = U(x, g_1)U(xg_1, g_2).$$

It can be shown that $U(g, x)$ is uniquely defined up to a natural calibration

$$U(g, x) \sim A(gx)^{-1}U(g, x)A(x),$$

where $A$ is a function on $Q \setminus G$ taking values in the unitary group of $V$ (see, e.g., [15, Sect. 13.1]). For this reason, an induced representation $\rho(g) = \text{Ind}_Q^G(\nu)$ is canonically defined up to a unitary equivalence.

\[\text{A reduction of representations of } GL(\infty, \mathbb{O}_p) \text{ to representations of quotients } GL(\infty, \mathbb{Z}_{p^n}) \text{ easily follows from [20, Proposition VII.1.3]; see [27, Corollary 3.5]. In our proof of Theorem 1.3, Tsankov’s theorem is used in the proof of Proposition 2.1, which was done in [27].}\]
We also can choose $U(g,x)$ in the following way. For any $x \in Q \setminus G$ we choose an element $s(x) \in G$ such that $x_0 s(x) = x$. Then $U(g,x) = v(q)$, where $q$ is determined from the condition $s(x)g = q s(xg)$.

1.3. The statement. The result of the paper is Theorem 1.1.5 which claims that irreducible representations of $G$ are induced from finite dimensional representations of certain family of subgroups $G^0[L;M]$; these subgroups are described in Lemma 1.3.

Thus we fix a ring $\mathbb{Z}_p$ and examine the group

$$G := \text{GL}(\infty, \mathbb{Z}_p).$$

We consider two right actions of $G$ on $I(\mathbb{Z}_p)$, $g : v \mapsto vg$, $g : v \mapsto v(g^t)^{-1}$. Define a pairing

$$I(\mathbb{Z}_p) \times I(\mathbb{Z}_p) \to \mathbb{Z}_p$$

by

$$(1.1) \quad \{v, w\} := \sum v_jw_j = vw^t,$$

our action preserves this pairing, i.e.,

$$\{vg, v(g^t)^{-1}\} = \{v, w\}.$$

Let $L \subset I(\mathbb{Z}_p)$, $M \subset I(\mathbb{Z}_p)$ be finitely generated $\mathbb{Z}_p$-submodules. Denote by $\hat{G}[L;M]$ the subgroup of $G$ consisting of $g$ such that $Lg = L$ and $M(g^t)^{-1} = M$. By $G^0[L;M] \subset \hat{G}[L;M]$ we denote group of matrices fixing $L$ and $M$ pointwise. Obviously, the quotient group $\hat{G}[L;M]/G^0[L;M]$ is finite; it acts on the direct sum $L \oplus M$ preserving the pairing $\{f, g\}$. Any irreducible representation $\tau$ of $\hat{G}[L;M]/G^0[L;M]$ can be regarded as a representation $\hat{\tau}$ the group $\hat{G}[L;M]$, which is trivial on $G^0[L;M]$. For given $L$, $M$, $\tau$ we consider the representation

$$\text{Ind}_{\text{G}[L;M]}^G(\hat{\tau})$$

of $G$ induced from the representation $\hat{\tau}$ of the group $\hat{G}[L;M]$. Ol’shanski [30] obtained the following statement$^3$ for the group $\text{GL}(\infty, \mathbb{F}_p) = \text{GL}(\infty, \mathbb{Z}_p)$.

**Theorem 1.2.**

(a) Any irreducible unitary representation of the group $\text{GL}(\infty, \mathbb{F}_p)$ has this form. 

(b) Two irreducible representations can be equivalent only for a trivial reason, i.e.,

$$\text{Ind}_{G[L_1;M_1]}^G(\tau_1) \sim \text{Ind}_{G[L_2;M_2]}^G(\tau_2)$$

if and only if there exists $h \in G$ such that $L_1h = L_2h$, $M_1(h^t)^{-1} = M_2$ and $\tau_2(q) = \tau_1(hqh^{-1})$.

For groups $\text{GL}(\infty, \mathbb{Z}_p)$ with $\mu > 1$ the situation is more delicate. Let $L$, $M$ actually be contained in $(\mathbb{Z}_p)^m \subset I(\mathbb{Z}_p)$. Fix a matrix $b$ such that $\ker b = L$ and a matrix $c$ such that $\ker c^t = M$.

$^3$A proof in [30] is only sketched; other proofs were given by Dudko [8] and Tsankov [32].

$^4$We assume that each row of $b$ and each column of $c$ contain only a finite number of nonzero elements.
Lemma 1.3. The group $G^\circ[L;M]$ consists of all invertible matrices admitting the following representation as a block matrix of size $m+\infty$:

\[
g = \begin{pmatrix} a & bv \\ wc & z \end{pmatrix},
\]

where the block ‘a’ can be written in both forms

\[
a = 1 - bS, \quad a = 1 - Tc.
\]

Next, define a subgroup $G^\bullet[L;M] \subset G^\circ[L;M]$ consisting of matrices having the form

\[
g = \begin{pmatrix} 1 - bc & bv \\ wc & z \end{pmatrix}.
\]

Proposition 1.4. The group $G^\bullet[L;M]$ is the minimal subgroup of finite index in $\hat{G}[L;M]$, i.e., it is contained in any subgroup of finite index in $\hat{G}[L;M]$.

Theorem 1.5. (a) Any irreducible unitary representation of $G$ is induced from a representation $\tau$ of some group $\hat{G}[L;M]$ that is trivial on the subgroup $G^\bullet[L;M]$.

(b) Two irreducible representations of this kind can be equivalent only for the trivial reason as in Theorem 1.2.

Remark. Recall that $p \neq 2$. In several places of our proof we divide elements of residue rings $\mathbb{Z}_p\mu$ by 2. Usually, this division can be replaced by longer considerations. But in Lemma 6.8 this seems crucial.

Remark. Let $L, M \subset p \cdot l(\mathbb{Z}_p\mu)$. Then $G[L;M]$ contains a congruence subgroup $N$ consisting of elements of $G$ that are equal 1 modulo $p^{\mu-1}$. Since $N$ is a normal subgroup in $G$, it is normal in $\hat{G}[L;M]$. Let $\tau$ be trivial on $N$. Then the induced representation $\text{Ind}_{G[L;M]}(\hat{G}[L;M]$ is trivial on the congruence subgroup $N$ and actually we get representations of $\text{GL}(\infty, \mathbb{Z}_p\mu^{-1})$.

Remark. The statement (b) is a general fact for oligomorphic groups; see [34, Proposition 4.1(ii)]. So we omit a proof (in our case this can be easily established by examination of intertwining operators).

1.4. Remarks: Infinite-dimensional $p$-adic groups. Now there exists a well-developed representation theory of infinite symmetric groups and of infinite-dimensional real classical groups. Parallel development in the $p$-adic case meets some difficulties. However, infinite dimensional $p$-adic groups were a topic of sporadic attacks since late 1980s; see [19, 36, 13]. We indicate some works on $p$-adic groups and their parallels with nontrivial constructions for real and symmetric groups.

(a) An extension of the Weil representation of the infinite-dimensional symplectic group $\text{Sp}(2\infty, \mathbb{C})$ to the semigroup of lattices (Nazarov [19, 13]; see a partial exposition in [22, Sect. 11.1-11.2]).

(b) A construction of projective limits of $p$-adic Grassmannians and quasiinvariant actions of $p$-adic $\text{GL}(\infty)$ on these Grassmannians [24]. This is an analog of virtual permutations (or Chinese restaurant process, see, e.g., [11, 11.19]; they are a base of harmonic analysis related to infinite symmetric group, see [14]), and of projective limits of compact symmetric spaces (see [31, 21]); they are a standpoint for a harmonic analysis related to infinite-dimensional classical groups; see [3].
(c) An attempt to describe a multiplication of double cosets (see the next section) for \( p \)-adic classical groups in [25]. In any case this leads to a strange geometric construction, namely to simplicial maps of Bruhat–Tits buildings whose boundary values are rational maps of \( p \)-adic Grassmannians.

(d) The work [3] contains a \( p \)-adic construction in the spirit of exchangeability\(^5\) namely, descriptions of invariant ergodic measures on spaces of infinite \( p \)-adic matrices. By the Wigner–Mackey trick (see, e.g., [15, Sect. 13.3]), such kind of statements can be translated to a description of spherical functions on certain groups.

So during last years new elements of a nontrivial picture related to infinite-dimensional \( p \)-adic groups appeared. For this reason, understanding of representations \( \text{GL}(\infty, \mathcal{O}_p) \) becomes necessary.

1.5. Another completion of a group of infinite matrices over \( \mathbb{Z}_{p^n} \). Define a group \( \mathcal{G} \) consisting of infinite matrices \( g \) over \( \mathbb{Z}_{p^n} \) such that:

- \( g \) contains only a finite number of elements in each column;
- \( g^{-1} \) exists and satisfies the same property.

A sequence \( g^{(\alpha)} \) converges to \( g \) if for each \( j \) we have a convergence of \( e_j g^{(\alpha)} \).

Clearly, \( \mathcal{G} \supset \mathcal{G} \). Classification of irreducible unitary representations of \( \mathcal{G} \) is the following. For each finitely generated submodule in \( l(\mathbb{Z}_{p^n}) \) we consider the subgroup \( \widehat{\mathcal{G}}[L] \) consisting of transformations sending \( L \) to itself and the subgroup \( \mathcal{G}^0[L] \) fixing \( L \) pointwise.

**Proposition 1.6.** Any irreducible unitary representation of \( \mathcal{G} \) is induced from a representation of some group \( \widehat{\mathcal{G}}[L] \) trivial on \( \mathcal{G}^0[L] \).

This follows from Theorem 1.5; on the other hand this can be deduced in a straightforward way from Tsankov’s result [34].

2. Preliminaries: The category of double cosets

2.1. Multiplication of double cosets and the category \( \mathcal{K} \). Here we discuss a version of a general construction of multiplication of double cosets (see [29], [30], [20], [26], [27]).

Denote by \( \mathcal{G}_{\text{fin}} \subset \mathcal{G} \) the subgroup of \textit{finitary} matrices, i.e., matrices \( g \) such that \( g - 1 \) has only a finite number of nonzero elements. For \( \alpha = 0, 1, \ldots \) denote by \( \mathcal{G}(\alpha) \subset \mathcal{G} \) the subgroups consisting of matrices having the form \( \begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix} \), where \( 1_\alpha \) denotes the unit matrix of size \( \alpha \) and \( u \) is an arbitrary invertible matrix over \( \mathbb{Z}_{p^n} \). Obviously, \( \mathcal{G}(\alpha) \) is isomorphic to \( \mathcal{G} \). Consider double coset spaces \( \mathcal{G}(\alpha) \setminus \mathcal{G} / \mathcal{G}(\beta) \); their elements are matrices determined up to the equivalence

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1_\beta & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} a & bv \\ uc & udv \end{pmatrix},
\]

where a matrix \( g \) is represented as a block matrix of size \( (\alpha + \infty) \times (\beta + \infty) \). For a matrix \( g \) we write the corresponding double coset as

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{\alpha\beta},
\]

\(^5\)i.e., of higher analogs of the de Finetti theorem; see [1]
we will omit subscript $\alpha \beta$ if it is not necessary to indicate a size. We wish to define a natural multiplication

$$G(\alpha) \setminus G/G(\beta) \times G(\beta) \setminus G/G(\gamma) \to G(\alpha) \setminus G/G(\gamma).$$

Let $g_1 \in G(\alpha) \setminus G/G(\beta)$, $g_2 \in G(\beta) \setminus G/G(\gamma)$ be double cosets. By [27, Lemma 4.1], any double coset has a representative in $G_{\text{fin}}$. Choose such representatives $g_1$ and $g_2$ for $g_1$, $g_2$,

$$g_1 = \begin{bmatrix} a & b \\ c & d \\ 1_\infty \end{bmatrix}_{\alpha \beta}, \quad g_1 = \begin{bmatrix} p & q \\ r & t \\ 1_\infty \end{bmatrix}_{\beta \gamma}.$$

Let sizes of submatrices $(a \ b \ c \ d)$, $(p \ q \ r \ t)$, be $N \times N$. Denote by $\theta(\beta)(j)$ the following matrix

$$\theta(\beta)(j) = \begin{pmatrix} 1_\beta & 0 & 1_j \\ 0 & 1_j & 0 \\ 1_\infty \end{pmatrix} \in G(\beta).$$

Consider the sequence

$$G(\alpha) \cdot g_1 \theta(\beta)(j) g_2 \cdot G(\gamma) \in G(\alpha) \setminus G/G(\gamma).$$

It is more or less obvious that this sequence is eventually constant and its limit is

$$g_1 \circ g_2 = \begin{bmatrix} ap & aq & b \\ cp & cq & d \\ r \ t \ 0 & 1_\infty \end{bmatrix}_{\alpha \gamma},$$

where $L \geq N - \beta$. The final expression is

$$g_1 \circ g_2 = \begin{bmatrix} ap & aq & b \\ cp & cq & d \\ r \ t \ 0 & 1_\infty \end{bmatrix}_{\alpha \gamma} \sim \begin{bmatrix} ap & b & aq \\ cp & d & cq \\ r \ t \ 0 & 1_\infty \end{bmatrix}_{\alpha \gamma}.$$

In calculations below we use the last expression for $\circ$-product.

It is easy to verify that this multiplication is associative, i.e., for any

$$g_1 \in G(\alpha) \setminus G/G(\beta), \quad g_2 \in G(\beta) \setminus G/G(\gamma), \quad g_3 \in G(\gamma) \setminus G/G(\delta),$$

we have

$$(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3).$$

In other words, we get a category. Objects of this category are numbers $\alpha = 0, 1, 2, \ldots$. Sets of morphisms are

$$\text{Mor}(\beta, \alpha) := G(\alpha) \setminus G/G(\beta).$$

The multiplication is given by formula (2.4). Denote this category by $\mathcal{K}$. 

The group of automorphisms $\text{Aut}_K(\alpha)$ is $\text{GL}(\alpha, \mathbb{Z}_{p^\infty})$; it consists of double cosets of the form $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$.

Next, the map $g \mapsto g^{-1}$ induces maps

$$\mathbb{G}(\alpha) \setminus \mathbb{G}/\mathbb{G}(\beta) \to \mathbb{G}(\beta) \setminus \mathbb{G}/\mathbb{G}(\alpha),$$

de note these maps by $g \mapsto g^\ast$. It is easy to see that we get an involution in the category $\mathcal{K}$, i.e.,

$$(g_1 \circ g_2)^\ast = g_2^\ast \circ g_1^\ast.$$

The map $g \mapsto (g^t)^{-1}$ determines an automorphism of the category $\mathcal{K}$; denote it by $g \mapsto g^\ast$. It sends objects to themselves and

$$(g_1 \circ g_2)^\ast = g_2^\ast \circ g_1^\ast.$$

Remarks on notation.

(1) In formulas (2.2), (2.3), (2.4), the last columns, the last rows, and the blocks $1_{\infty}$ contain no information and only enlarge sizes of matrices. For this reason, below we will omit them. Precisely, for a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of finite size we denote

$$\begin{bmatrix} a & b \\ c & d \ast \end{bmatrix} := \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 0 & 1_{\infty} \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \ast \end{bmatrix} := \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 0 & 1_{\infty} \end{bmatrix}.$$

(2) We will denote a multiplication of $[g]$ by an automorphism $A$ as $A \cdot [g]$,

$$A \cdot \begin{bmatrix} a & b \\ c & d \ast \end{bmatrix} := \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \circ \begin{bmatrix} a & b \\ c & d \ast \end{bmatrix} = \begin{bmatrix} Aa & Ab \\ c & d \ast \end{bmatrix};$$

$$\begin{bmatrix} a & b \\ c & d \ast \end{bmatrix} \cdot A' := \begin{bmatrix} a & b \\ c & d \ast \end{bmatrix} \circ \begin{bmatrix} A' & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} aA' & b \\ cA' & d \ast \end{bmatrix}.$$

2.2. The multiplicativity theorem. Consider a unitary representation $\rho$ of the group $\mathbb{G}$ in a Hilbert space $H$. Denote by $H_{\alpha} \subset H$ the space of $\mathbb{G}(\alpha)$-fixed vectors. Denote by $P_{\alpha}$ the operator of orthogonal projection to $H_{\alpha}$.

**Proposition 2.1.**

(a) For any $\beta$ the sequence $\rho(\theta^\beta(j))$ converges to $P_{\beta}$ in the weak operator topology.

(b) The space $\cup H_{\alpha}$ is dense in $H$.

The first statement is Lemma 1.1 from [27]; the claim (b) is a special case of Proposition VII.1.3 from [20].

Let $g \in \mathbb{G}$, $\alpha, \beta \in \mathbb{Z}_+$. Consider the operator

$$\tilde{\rho}_{\alpha\beta}(g) : H_{\beta} \to H_{\alpha}$$

given by

$$\tilde{\rho}_{\alpha\beta}(g) := P_{\alpha} \rho(g) \bigg|_{H_{\beta}}.$$ 

It is easy to see that for $h_1 \in \mathbb{G}(\alpha)$, $h_2 \in \mathbb{G}(\beta)$ we have

$$\tilde{\rho}_{\alpha\beta}(g) = \tilde{\rho}_{\alpha\beta}(h_1 g h_2),$$

i.e., $\tilde{\rho}_{\alpha\beta}(g)$ actually depends on the double coset $g$ containing $g$. 


Theorem 2.2.
(a) The map \( g \mapsto \tilde{\rho}_{\alpha \beta}(g) \) is a representation of the category \( K \), i.e., for any \( \alpha, \beta, \gamma \) for any \( g_1 \in \text{Mor}(\beta, \alpha), g_2 \in \text{Mor}(\gamma, \beta) \) we have
\[
\tilde{\rho}_{\alpha \beta}(g_1) \tilde{\rho}_{\beta \gamma}(g_2) = \tilde{\rho}_{\alpha \gamma}(g_1 \circ g_2).
\]
(b) \( \tilde{\rho} \) is a \(*\)-representation, i.e.,
\[
\tilde{\rho}_{\alpha \beta}(g)^* = \tilde{\rho}_{\beta \alpha}(g^*).
\]

The statement (a) is an automatic corollary of Proposition 2.1; see [27, Theorem 2.1]. The statement (b) is obvious.

Remark. The considerations of Subsections 2.1, 2.2 are one-to-one repetitions of similar statements for real classical groups and symmetric groups; see [30], [28], [23], [26]. Further considerations drastically differ from these theories.

2.3. Structure of the paper. We derive the classification of unitary representations of \( G \) from the multiplicativity theorem and the following argumentation. The semigroups \( \Gamma(m) := \text{End}_K(m) \) are finite. It is known that a finite semigroup with an involution has a faithful \(*\)-representation in a Hilbert space if and only if it is an inverse semigroup (see discussion below, Subsection 3.3). More generally, if a category having finite sets of morphisms acts faithfully in Hilbert spaces, then it must be an inverse category; see [12]. However, semigroups \( \text{End}_K(\alpha) \) are not inverse\(^6\) and \(*\)-representations of \( K \) pass through a smaller category.

Section 3 contains preliminary remarks on inverse semigroup and construction of an inverse category \( L \), which is a quotient of \( K \). This provides us lower estimate of maximal inverse semigroup quotients of semigroups \( \Gamma(m) \).

In Section 4 we examine idempotents in maximal inverse semigroup quotients \( \text{inv}(\Gamma(m)) \) of \( \Gamma(m) \). In Section 5 we show that some of idempotents of \( \text{inv}(\Gamma(m)) \) act by the same operators in all representations of \( G \). Next, for any representation of \( G \) there is a minimal \( m \) such that \( H_m \neq 0 \). In Section 6 we examine the image of \( \Gamma(m) \) in such representation.

In Section 7 we discuss properties of the groups \( G^0[L; M] \) and \( G^*[L; M] \).

The final part of the proof is contained in Section 8.

3. THE REDUCED CATEGORY AND INVERSE SEMIGROUPS

3.1. Notation. Below we work only with the group \( G := \text{GL}(\infty, \mathbb{Z}_{p^\mu}) \). To simplify notation, we write
\[
\text{GL}(m) := \text{GL}(m, \mathbb{Z}_{p^\mu}), \quad \Gamma(m) := \text{End}_K(m), \quad \Gamma^m := (\mathbb{Z}_{p^\mu})^m.
\]

For a unitary representation \( \rho \) of \( G \) we define the height \( h(\rho) \) as the minimum of \( \alpha \) such that \( H_\alpha \neq 0 \).

By \( x(\text{mod } p) \) we denote a reduction of an object (a scalar, a vector, a matrix) defined over \( \mathbb{Z}_{p^\mu} \), modulo \( p \), i.e. to the field \( \mathbb{F}_p \). Notice that a square matrix \( A \) of finite size over \( \mathbb{Z}_{p^\mu} \) is invertible if and only if \( A(\text{mod } p) \) is invertible. A matrix \( B \) is nilpotent (i.e., \( B^N = 0 \) for sufficiently large \( N \)) if and only if \( B(\text{mod } p) \) is nilpotent. Indeed, if \( B^k = 0(\text{mod } p) \), then \( B^k \) has the form \( pC \) for some matrix \( C \). Hence, \((B^k)^{\mu+1} = p^{\mu+1}C^{\mu+1} = 0\).

\(^6\)This was observed by Ol’shanski [30] for \( \text{GL}(\infty, \mathbb{F}_p) \).
We use several symbols for equivalences in $\text{Mor}_K(\beta, \alpha)$; the $\sim$ was defined by (2.1); the symbols

$\equiv, \approx, \approx_m$

are defined in the next two subsections.

### 3.2. The reduced category $\text{red}(K)$

Let $g_1, g_2 \in \text{Mor}(\beta, \alpha)$. We say that they are $\approx$-equivalent if for any unitary representation of $G$ we have $\bar{\rho}_{\alpha\beta}(g_1) = \bar{\rho}_{\alpha\beta}(g_2)$.

The reduced category $\text{red}(K)$ is the category whose objects are nonnegative integers and morphisms $\beta \to \alpha$ are $\approx$-equivalence classes of $\text{Mor}(\beta, \alpha)$. Denote by $\text{red}(\Gamma(m))$ the semigroups of endomorphisms of $\text{red}(K)$.

Also we define a weaker equivalence, $g_1 \approx_m g_2$ if $\bar{\rho}_{\alpha\beta}(g_1) = \bar{\rho}_{\alpha\beta}(g_1)$ for all $\rho$ of height $\geq m$. Denote by $\text{red}_m(K)$ the corresponding $m$-reduced category.

Our proof of Theorem 1.5 is based on an examination of the categories $\text{red}(K)$ and $\text{red}_m(K)$. We obtain an information sufficient for a classification of representations of $G$. However, the author does not know an answer to Question 3.1.

**Question 3.1.** Find a transparent description of the category $\text{red}(K)$.

### 3.3. Inverse semigroups

Let $\mathcal{P}$ be a finite semigroup with an involution $x \mapsto x^*$. Then the following conditions are equivalent.

(A) $\mathcal{P}$ admits a faithful representation in a Hilbert space.

(B) $\mathcal{P}$ admits an embedding to a semigroup of partial bijections of a finite set compatible with the involutions in $\mathcal{P}$ and in partial bijections.

(C) $\mathcal{P}$ is an inverse semigroup (see [6], [17], [16]), i.e., for any $x$ we have

$$xx^*x = x, \quad x^*xx^* = x^*$$

and any two idempotents in $\mathcal{P}$ commute.

Discuss briefly some properties of inverse semigroups. Any idempotent in $\mathcal{P}$ is self-adjoint, and for any $x$, the element $x^*x$ is an idempotent. Since idempotents commute, a product of idempotents is an idempotent. The semigroup of idempotents has a natural partial order,

$$x \preceq y \quad \text{if} \quad xy = x.$$

We have $xy \preceq x$. If $x \preceq y$ and $u \preceq v$, then $xu \preceq yv$. Since our semigroup is finite, the product of all idempotents is a minimal idempotent $0$; we have $0x = x0 = 0$ for any $x$.

Let $\mathcal{R}$ be a finite semigroup with involution. Then there exists an inverse semigroup $\text{inv}(\mathcal{R})$ and epimorphism $\pi : \mathcal{R} \to \text{inv}(\mathcal{R})$ such that any homomorphism $\psi$ from $\mathcal{R}$ to an inverse semigroup $\mathcal{Q}$ has the form $\psi = \zeta \pi$ for some homomorphism $\zeta : \text{inv}(\mathcal{R}) \to \mathcal{Q}$. We say that $\text{inv}(\mathcal{R})$ is the maximal inverse semigroup quotient of $\mathcal{R}$.

**Lemma 3.1.** The semigroups $\Gamma(m)$ are finite.

This is a corollary of the following statement; see [27, Lemma 4.1.a].

**Lemma 3.2.** Any double coset in $G(m) \\setminus G/G(m)$ has a representative in $GL(3m)$.

---

7Recall that a *partial bijection* $\sigma$ from a set $A$ to a set $B$ is a bijection from a subset $S$ of $A$ to a subset $T$ of $B$; see e.g., [17], or [20, Sect. VIII.1]. The adjoint partial bijection $\sigma^* : B \to A$ is the inverse bijection $T$ to $S$. 
We consider the following quotients of $\Gamma(m)$:
(1) $\text{inv}(\Gamma(m))$ is the maximal inverse semigroup quotient of $\Gamma(m)$;
(2) $\text{red}(\Gamma(m)) := \text{End}_{\text{red}}(\Gamma(m));$
(3) $\text{red}_m(\Gamma(m)) := \text{End}_{\text{red}_m}(\Gamma(m)).$

We have the following sequence of epimorphisms:
$$\Gamma(m) \to \text{inv}(\Gamma(m)) \to \text{red}(\Gamma(m)) \to \text{red}_m(\Gamma(m)).$$

For $g \in G_{\text{fin}}$ we denote by $[g]_m$ the corresponding element of $\Gamma(m)$ and by $[[g]]_m$ the corresponding element of $\text{inv}(\Gamma(m))$. The equality in $\Gamma(m)$ we denote by $\sim$, in $\text{inv}(\Gamma(m))$ by $\equiv$, in $\text{red}(\Gamma(m))$ by $\approx$, in $\text{red}_m(\Gamma(m))$ by $\approx_m$. Denote by $[[g_1]] \circ [[g_2]]$ the product in $\text{inv}(\Gamma(m))$.

Our next purpose is to present some (non-maximal) inverse semigroup quotients of $\Gamma(m)$.

3.4. The category $\mathcal{L}$ of partial isomorphisms. Let $V, W$ be modules over $\mathbb{Z}_{p^\nu}$. A partial isomorphism $p : V \to W$ is an isomorphism of a submodule $A \subset V$ to a submodule $B \subset W$. We denote $\text{dom} p := A, \text{im} p := B$. By $p^*$ we denote the inverse map $B \to A$. Let $p : V \to W, q : W \to Y$ be partial isomorphisms. Then the product $pq$ is defined in the following way:
$$\text{dom} pq := p^*(\text{dom} q) \cap \text{dom} p,$$
for $v \in \text{dom} pq$ we define $v(pq) = (vp)q$.

A partial isomorphism $p$ is an idempotent if $\text{dom} p = \text{im} p$ and $p$ is an identical map.

Objects of the category $\mathcal{L}$ are modules
$$\mathfrak{l}_+^\alpha \oplus \mathfrak{l}_-^\alpha := (\mathbb{Z}_{p^\nu})^\alpha \oplus (\mathbb{Z}_{p^\nu})^\alpha$$
equipped with the following pairing
$$\{v_+; v_-\} := \sum_j v_+^j v_-^j = v_+(v_-)^t,$$
where $v_+ \in \mathfrak{l}_+^\alpha$. We say that two partial isomorphisms
$$\xi_+ : \mathfrak{l}_+^\alpha \to \mathfrak{l}_+^\beta, \quad \xi_- : \mathfrak{l}_-^\alpha \to \mathfrak{l}_-^\beta$$
are compatible if for any $y_+ \in \text{dom} \xi_+$ and $y_- \in \text{dom} \xi_-$, we have
$$\{\xi_+(y_+), \xi_-(y_-)\} = \{y_+, y_-\}.$$

Next, we define a category $\mathcal{L}$. Its objects are spaces $\mathfrak{l}_+^\alpha \oplus \mathfrak{l}_-^\alpha$ and morphisms are pairs of compatible partial isomorphisms $\xi_+ : \mathfrak{l}_+^\alpha \to \mathfrak{l}_+^\beta, \xi_- : \mathfrak{l}_-^\alpha \to \mathfrak{l}_-^\beta$.

The category $\mathcal{L}$ is equipped with an involution
$$(\xi_+, \xi_-)^* = (\xi_+^*, \xi_-^*)$$
and an automorphism
$$(\xi_+, \xi_-)^\star = (\xi_-, \xi_+).$$

Lemma 3.3. The semigroups $\text{End}_{\mathcal{L}}(m)$ are inverse.

Indeed, $\text{End}_{\mathcal{L}}(m)$ is a semigroup of partial bijections of a finite set $\mathfrak{l}_+^m \oplus \mathfrak{l}_-^m$. The whole category $\mathcal{L}$ is inverse for the same reason.

---

8 All these semigroups are different.
3.5. The functor $\Pi : \mathcal{K} \to \mathcal{L}$. Consider $g \in \mathbb{G}_{\text{fin}}$. Let actually $g$ be contained in $\text{GL}(N)$. Represent $g$ as a block $(\beta + (N - \beta)) \times (\alpha + (N - \alpha))$ matrix and $g^{-1}$ as an $(\alpha + (N - \alpha)) \times (\beta + (N - \beta))$-matrix,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$ 

Define maps $\xi_{\pm} : l^\alpha \to l^\beta$ by:

- $\text{dom} \xi_+ := \ker b$ and $\xi_+$ is the restriction of $a$ to $\ker b$;
- $\text{dom} \xi_- := \ker C^t$ and $\xi_-$ is the restriction of $A^t$ to $\ker C^t$.

Proposition 3.4.

(a) The pair $\xi_+, \xi_-$ depends only on the double coset containing $g$.

(b) Partial isomorphisms $\xi_+, \xi_-\,$ are compatible.

(c) The map $g \mapsto (\xi_+, \xi_-)$ determines a functor from the category $\mathcal{K}$ to the category $\mathcal{L}$.

Denote this functor by $\Pi$. By $\Pi(g)$ we denote the morphism of $\mathcal{L}$ corresponding to $g$. We have

$$\Pi(g^*) = (\Pi(g))^*, \quad \Pi(g^\bullet) = (\Pi(g))^\bullet.$$ 

Proof. For any invertible matrix $v$ we have, $\ker b = \ker bv$. Therefore $\xi_+$ depends only on a double coset. For $\xi_-$ we apply (3.2).

(b) Let $v \in \ker b, w \in \ker C^t$. Then

$$\{v, w\} = vw^t = v(aA + bC)w^t = va \cdot (wA^t)^t + vb \cdot (wC^t)^t = \{va, wA^t\} + 0.$$

(c) We look to formula (2.4) for a product in $\mathcal{K}$. The new $\xi_+$ is a restriction of $ap$ to $\ker b \cap \ker aq$. This is the product of two $\xi$-es. \hfill $\square$

Remark. According Ol’shanskiĭ [30], for the case $\text{GL}(\infty, \mathbb{F}_p)$ the functor $\Pi : \mathcal{K} \to \mathcal{L}$ determines an isomorphism of categories $\text{red}(\mathcal{K}) \to \mathcal{L}$. However, for $\mu > 1$ the maps $\Pi : \text{red}(\Gamma(m)) \to \text{Mor}_\mathcal{L}(m)$ are neither surjective nor injective. However we will observe that $\Pi$ induces isomorphisms of semigroups of idempotents; this provides us an important argument for the proof of Proposition 6.1.

4. Idempotents in $\text{inv}(\Gamma(m))$

Here we examine idempotents in the semigroup $\text{inv}(\Gamma(m))$. The main statement of the section is Proposition 4.10.

4.1. Projectors\footnote{This subsection contains generalities; $\mathcal{K}$ is an ordered category in the sense of [20, Sect. III.4]; this implies all statements of the subsection.} $P_\alpha$. Consider an irreducible representation $\rho$ of $\mathbb{G}$; let subspaces $H_m \subset H$ and orthogonal projectors $P_m : H \to H_m$ be as above.

Lemma 4.1.

(a) The projector

$$P_\alpha|_{H_m} : H_m \to H_\alpha$$
is given by the operator \( \bar{\rho}_{mm}(\Theta_{[m]}^\alpha) \), where

\[
\Theta_{[m]}^\alpha := \begin{bmatrix}
1_\alpha & 0 & 0 & 0 \\
0 & 0 & 1_{m-\alpha} & 0 \\
0 & 1_{m-\alpha} & 0 & 0 \\
0 & 0 & 0 & 1_\infty
\end{bmatrix}_{mm} \in \Gamma(m).
\]

(b) The tautological embedding \( H_\alpha \to H_m \) is defined by the operator \( \bar{\rho}_{ma}(\Lambda_{[m]}^\alpha) \), where

\[
\Lambda_{[m]}^\alpha := \begin{bmatrix}
1_\alpha & 0 & 0 & 0 \\
0 & 1_{ma} & 0 & 0 \\
0 & 0 & 0 & 1_\infty
\end{bmatrix}_{am} \in \text{Mor}_K(\alpha, m).
\]

(c) The orthogonal projector \( H_m \to H_\alpha \) is given by \( \bar{\rho}_{am}(\Lambda_{[m]}^\alpha)^* \)

\[
(\Lambda_{[m]}^\alpha)^* := \begin{bmatrix}
1_\alpha & 0 & 0 & 0 \\
0 & 1_{m-\alpha} & 0 & 0 \\
0 & 0 & 0 & 1_\infty
\end{bmatrix}_{ma} \in \text{Mor}_K(m, \alpha).
\]

Proof. (a) We apply Proposition 2.1(a). For \( j > m - \alpha \) we have \( [\theta^\alpha(j)]_{mm} = \Theta_{[m]}^\alpha \).

The same argument proves (b) and (c).

Lemma 4.2.

(a) The map \( \iota_m^\alpha : \begin{bmatrix} a & b \\ c & d_* \end{bmatrix} \to \begin{bmatrix} a & 0 & b & 0 \\ 0 & 0 & 0 & 1_{m-\alpha} \\ c & 0 & d & 0 \\ 0 & 1_{m-\alpha} & 0 & 0_* \end{bmatrix}_{mm} \) is a homomorphism \( \Gamma(\alpha) \to \Gamma(m) \).

(b) We have

\[
\iota_m^\alpha(g) \sim \Lambda_n^\alpha \circ g \circ (\Lambda_n^\alpha)^*.
\]

This follows from a straightforward calculation.

Corollary 4.3. The map \( \iota_m^\alpha \) is compatible with representations \( \bar{\rho} \) of \( \Gamma(\alpha) \) and \( \Gamma(m) \). Namely, operators \( \bar{\rho}_{mm}(\iota_m^\alpha(g)) \) have the following block structure with respect to the decomposition \( H_m = H_\alpha \oplus (H_m \ominus H_\alpha) \):

\[
\bar{\rho}_{mm}(\iota_m^\alpha(g)) = \begin{pmatrix} \bar{\rho}_{\alpha\alpha}(g) & 0 \\ 0 & 0 \end{pmatrix}.
\]

4.2. Idempotents in \( \text{inv}(\Gamma(m)) \). Here we formulate several lemmas (their proofs occupy Subsections 4.3, 4.7); as a corollary we get Proposition 4.10.

Lemma 4.4. Let for

\[
[g] = \begin{bmatrix} a & b \\ c & d_* \end{bmatrix}_{mm} \in \Gamma(m)
\]

one of the blocks \( a, d \) be degenerate. Then \( [g]_{mm} \in \text{inv}(\Gamma(m)) \) has a representative \( [g'] \), for which both blocks \( a, d \) are degenerate.

Denote by \( \Gamma^\circ(m) \) the subsemigroup in \( \Gamma(m) \) consisting of all \( [g] \), for which both blocks \( a, d \) are nondegenerate.
Lemma 4.5. Any idempotent in \(\text{inv}(\Gamma(m))\) has a representative of the form \(q \cdot [[R]] \cdot q^{-1}\) with \(q\) ranging \(\text{GL}(m)\) and \(R\) having the form

\[
[R] := \begin{bmatrix}
1_\alpha & 0 & \varphi & 0 \\
0 & 0 & 0 & 1_{m-\alpha} \\
\psi & 0 & \chi & 0 \\
0 & 1_{m-\alpha} & 0 & 0_\ast
\end{bmatrix}_{mm} \in \Gamma(m),
\]

where

\[
\begin{bmatrix}
1 \\
\psi \\
\chi \\
\end{bmatrix} \in \Gamma(\alpha)
\]

represents an idempotent in \(\text{inv}(\Gamma^0(\alpha))\). The parameter \(\alpha\) ranges in the set \(0, 1, 2, \ldots, m\).

Remark. Denote

\[
R^\Box := \begin{bmatrix}
1_\alpha & 0 & \varphi \\
0 & 1_{m-\alpha} & 0 \\
\psi & 0 & \chi_\ast
\end{bmatrix}_{mm}.
\]

Then the following elements of \(\Gamma(m)\) coincide:

\[
R = R^\Box \Theta_m^\alpha = \Theta_m^\alpha R^\Box = \Theta_m^\alpha \Theta_m^\alpha.
\]

Denote

\[
X(b, c) := \begin{pmatrix}
1_m & b \\
0 & 1 \\
c & 0
\end{pmatrix} \in \mathbb{G}_{\text{fin}}.
\]

Lemma 4.6. Elements of the form \([X(b, c)]\) are idempotents in \(\Gamma^0(m)\). They depend only on \(\ker b\) and \(\ker c^\ast \subset \mathbb{I}^m\).

Let \(L := \ker b\) and \(M := \ker c^\ast\). Denote

\[
X[L, M] := [X(b, c)].
\]

Lemma 4.7. We have

\[
X[L_1, M_1] \cdot X[L_2, M_2] = X[L_1 \cap L_2, M_1 \cap M_2].
\]

Lemma 4.8. Any idempotent in \(\text{inv}(\Gamma^0(m))\) has the form \(X[L, M]\).

Corollary 4.9. Idempotents \(X[L, M]\) are pairwise distinct in \(\text{inv}(\Gamma^0(m))\).

Proof. Indeed, \(\text{End}_L(m)\) is an inverse semigroup; therefore we have a chain of maps

\[
\Gamma^0(m) \rightarrow \text{inv}(\Gamma^0(m)) \rightarrow \text{inv}(\Gamma(m)) \rightarrow \text{Mor}_L(m).
\]

The image of \(X(b, c)\) in \(\text{Mor}_L(m)\) is precisely the pair of identical partial isomorphisms \(M \rightarrow M, L \rightarrow L\). Therefore for nonequivalent \(X(b, c)\) we have different images. \(\Box\)

Proposition 4.10. Any idempotent in \(\text{inv}(\Gamma(m))\) has a representative of the form

\[
q \cdot \begin{bmatrix}
1_\alpha & 0 & b & 0 & 0 \\
0 & 0 & 0 & 1_{m-\alpha} & 0 \\
0 & 0 & 1 & 0 & 0 \\
c & 0 & 0 & 1 & 0 \\
0 & 1_{m-\alpha} & 0 & 0 & 0_\ast
\end{bmatrix}_{mm} \cdot q^{-1},
\]

where \(q \in \text{GL}(m) = \text{Aut}_X(m)\).
Proof. Lemma 4.2 defines a canonical embedding \( i_{\alpha}^{m} : \Gamma(\beta) \to \Gamma(m) \) for \( \alpha < m \). By Lemma 4.5 any idempotent in \( \text{inv}(\Gamma(m)) \) is equivalent to an idempotent lying in some \( i_{\alpha}^{m}(\Gamma^{\circ}(\alpha)) \). Lemma 4.8 gives us a canonical form of this idempotent. \( \square \)

Now we start proofs of Lemmas 4.4–4.8.

4.3. Proof of Lemma 4.4. Clearly \( \Gamma(m) \setminus \Gamma^{\circ}(m) \) is a two-sided ideal in \( \Gamma(m) \). Since \([[[g \circ (g^{-1} \circ g)]]]_{m} = [[[g]]]_{m} \), it is sufficient to prove the statement for idempotents.

Let

\[(4.5) \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad g^{-1} =: \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]

Then

\[
[[[g]]] \circ [[[g]]]^{*} = [[[g \circ g^{-1}]]] = \begin{bmatrix} aA & * \\ * & * \end{bmatrix}.
\]

If \( a \) is degenerate, then \( aA \) is degenerate. Now let \( a \) be non-degenerate, \( d \) degenerate. Since the matrices \((4.5)\) are inverse one to another, we have

\[
aA = 1 - bC, \quad Dd = 1 - Cb.
\]

We see that \((1 - Cb)(\text{mod } p)\) is degenerate, \((1 - bC)(\text{mod } p)\) also is degenerate, and therefore \( aA \) is degenerate.

4.4. Proof of Lemma 4.5.

Step 1.

**Lemma 4.11.** Let \( x \) be an idempotent in \( \text{inv}(\Gamma(m)) \). Then it can be represented as \([[[u]]]\), where \( u = u^{-1} \).

**Proof.** Let \( x = [[[g]]] \). Then

\[
x = [[[g]]] \circ [[[g]]]^{*} = [[[g \circ g^{-1}]]] = [[[g \theta^{m}(j)g^{-1}]]]
\]

for sufficiently large \( j \). We set \( u := g^{m}(j)g^{-1} \). \( \square \)

**Lemma 4.12.** Let \( g = g^{-1} \in G_{\text{fin}} \). For any \( N > 0 \) there exists a representative \( r \in G_{\text{fin}} \) of \([g]^{2N}\) such that \( r = r^{-1} \).

**Proof.** Let actually \( g \in \text{GL}(m + l) \). Then we choose the following representative of \([g]^{2N}\):

\[
r = g \theta^{m}(l) g \theta^{m}(2l) g \theta^{m}(4l) g \theta^{m}(8l) g \theta^{m}(4l) g \theta^{m}(2l) g \theta^{m}(l) g.
\]

Step 2.

**Lemma 4.13.** Let \( g = g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \). Then there exists a matrix

\[
Z = \begin{pmatrix} \zeta & 0 \\ 0 & 1_{*} \end{pmatrix} \in \text{Aut}_{\mathbb{K}}(m) \text{,} \quad \text{where } \zeta \in \text{GL}(m),
\]

and \( N \) such that

\[
[[[Z \cdot g \cdot Z^{-1}]]]^{\circ N} = [[\left( \begin{pmatrix} \zeta & 0 \\ 0 & 1_{*} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left( \begin{pmatrix} \zeta & 0 \\ 0 & 1_{*} \end{pmatrix} \right)^{-1} \right)]^{\circ N}]
\]
has a form
\[
r = \begin{bmatrix}
0 & 0 & * \\
0 & 1_k & * \\
* & * & *
\end{bmatrix},
\]
where \( k \) is the rank of the reduced matrix \( a^m \pmod{p} \).

Clearly our lemma is a corollary of the following statement:

**Lemma 4.14.** For any \( m \times m \) matrix \( a \) over \( \mathbb{Z}_p^\mu \) there exists \( \zeta \in \text{GL}(m) \) and \( N \) such that
\[
(\zeta a \zeta^{-1})^N = \begin{pmatrix} 0 & 0 \\ 0 & 1_k \end{pmatrix}.
\]

**Proof.** We split the operator \( a \pmod{p} \) over the field \( \mathbb{F}_p \) as a direct sum of a nilpotent part \( S \) and an invertible part \( T \). For sufficiently large \( M \) the matrix \( \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}^M \) has the form \( \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix} \) with a nondegenerate \( P \). Since the group \( \text{GL}(k, \mathbb{F}_p) \) is finite, \( P^L = 1_k \) for some \( L \).

Thus without a loss of generality, we can assume that \( a \) has a form
\[
a = \begin{pmatrix} p\alpha & p\beta \\ p\gamma & 1 + p\delta \end{pmatrix},
\]

where \( \alpha, \beta, \gamma, \delta \) are matrices over \( \mathbb{Z}_p^\mu \). We conjugate it as follows
\[
\begin{pmatrix} 1 & pu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p\alpha & p\beta \\ p\gamma & 1 + p\delta \end{pmatrix} \begin{pmatrix} 1 & -pu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & -p^2(\alpha u + u\gamma u) + p(\beta + u(1 + p\delta)) \\ * & * \end{pmatrix}.
\]

We wish to choose \( u \) to make zero in the boxed block. It is sufficient to find a matrix \( u \) satisfying the following equation:
\[
(4.6) \quad u = (-\beta + p(\alpha u + u\gamma u))(1 + p\delta)^{-1} = -\beta + p(-\delta + \alpha u + u\gamma u)(1 + p\delta)^{-1}.
\]

We look for a solution in the form
\[
u = \sum_{k=0}^{\mu} p^k S_k.
\]

First, we consider \( S_k \) as formal noncommutative variables. Then we get a system of equations of the form
\[
S_0 = -\beta, \quad S_k = F_k(\alpha, \beta, \gamma, \delta; S_0, S_1, \ldots, S_{k-1}),
\]

where \( F_k \) are polynomial expressions with integer coefficients. These equations can be regarded as recurrence formulas for \( S_k \). In this way we get a solution \( u \).

Thus without a loss of generality we can assume that \( a \) has the form
\[
a = \begin{pmatrix} p\alpha' & 0 \\ p\gamma' & 1 + p\delta' \end{pmatrix}.
\]

Raising it to \( \mu \)-th power, we come to a matrix of the form
\[
a = \begin{pmatrix} 0 & 0 \\ p\gamma'' & 1 + p\delta'' \end{pmatrix}.
\]
We conjugate it as
\[
\begin{pmatrix}
1 & 0 \\
-pv & 1
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
-p'0 & 1 + p\delta''
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-pv & 1
\end{pmatrix} =
\begin{pmatrix}
0 & 0 \\
p(\gamma'' - (1 + p\delta'')v) & (1 + \delta''v)
\end{pmatrix}.
\]
Taking \(v = (1 + p\delta'')^{-1}\gamma''\) we kill the left lower block and come to a matrix of the form \(\begin{pmatrix} 0 & 0 \\ 0 & 1 + p\delta'' \end{pmatrix}\). Raising it in \(p^\mu\)-th power we come to \(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\).

**Step 3.** Thus the element \([ [g] ]^{02N}\) from Lemma 4.13 has a representative of the following block \((m - k) + k + (m - k) + \infty\) form:
\[
r = r^{-1} = \begin{pmatrix}
0 & 0 & \beta_{11} & \beta_{12} \\
0 & 1_k & \beta_{21} & \beta_{22} \\
\gamma_{11} & \gamma_{12} & \delta_{11} & \delta_{12} \\
\gamma_{21} & \gamma_{22} & \delta_{21} & \delta_{22}
\end{pmatrix}.
\]

**Lemma 4.15.** There is a matrix \(U = \begin{pmatrix} 1_m & 0 \\ 0 & u_* \end{pmatrix}\) such that \(UrU^{-1}\) has the form
\[
\tilde{r} = \begin{pmatrix}
0 & 0 & 1_{m-k} & 0 \\
0 & 1_k & 0 & \varphi \\
1_{m-k} & 0 & 0 & 0 \\
0 & \psi & 0 & \kappa_*
\end{pmatrix}.
\]

Recall that \([r] \sim [UrU^{-1}]\).

**Proof.** Since the matrix \((\beta_{11} \beta_{12})\) is nondegenerate (otherwise \(r\) is degenerate), we can choose a conjugation of \(r\) by matrices \(U = \begin{pmatrix} 1_m & 0 \\ 0 & 0 \end{pmatrix}\) reducing this block to the form \(\begin{pmatrix} 1 & 0 \end{pmatrix}\). We have \(r^2 = 1\); evaluating \(r^2\) we get \(\gamma_{11} = 1\). Thus we come to new \(r\),
\[
r^\sim = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1_k & \beta_{21} & \beta_{22} \\
1 & \gamma_{12} & \delta_{11} & \delta_{12} \\
\gamma_{21} & \gamma_{22} & \delta_{21} & \delta_{22}
\end{pmatrix}
\]
with new \(\beta, \gamma, \delta\). Next, we conjugate this matrix by
\[
\begin{pmatrix}
1_m & 0 & 0 \\
0 & 1_m-k & 0 \\
0 & -\gamma_{21} & 1_*
\end{pmatrix}
\]
and kill \(\gamma_{21}\). Thus we come to new \(r\),
\[
r^{\sim\sim} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1_k & \beta_{21} & \beta_{22} \\
1 & \gamma_{12} & \delta_{11} & \delta_{12} \\
0 & \gamma_{22} & \delta_{21} & \delta_{22}
\end{pmatrix}.
\]
But \((r^{\sim\sim})^2 = 1\). Looking to third row and third column of \((r^{\sim\sim})^2\) we observe that \(\beta_{21}, \delta_{11}, \delta_{21}, \gamma_{12}, \delta_{12}\) are zero.

Thus, \(r^{\sim\sim}\) has the desired form. \(\square\)
4.5. **Proof of Lemma 4.6.** Denote
\[ [X_+(A)] := \begin{bmatrix} 1 & A \\ 0 & 1 \end{bmatrix}. \]
We can conjugate this matrix by \( \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix} \). Therefore a matrix \( A \) is defined up to multiplications \( A \sim Au \), where \( u \) is an invertible matrix. The invariant of this action is \( \ker A \) (this is more or less clear; formally we can refer to Lemma 7.3 proved below).

Next,
\[ [X_+(A)] \circ [X_+(A)] = \begin{bmatrix} 1 & A & A \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]
We have \( \ker (A \quad A) = \ker A \) and therefore \( [X_+(A)] \) is an idempotent. In the same way, \( [X_-(B)] := \begin{bmatrix} 1 & 0 \\ B & 1 \end{bmatrix} \) is an idempotent. It remains to notice that
\[ [X(A, B)] = [X_+(A)] \circ [X_-(B)]. \]

Thus \( [X(A, B)] \) is an idempotent.

4.6. **Proof of Lemma 4.6.** In notation of the previous subsection
\[ X_-(A_1) \circ X_-(A_2) \sim X(A_1, A_2), \]
i.e.,
\[ X[\ker A_1, 0] \circ X[\ker A_2, 0] \equiv X[\ker A_1 \cap \ker A_2, 0], \]
or
\[ X[L_1, 0] \circ X[L_2, 0] \equiv X[L_1 \cap L_2, 0]. \]
On the other hand, we have
\[ X[L, 0] \circ X[0, M] \equiv X[L, M], \]
and now the statement becomes obvious.

**Proof of Lemma 4.7.** Indeed,
\[ [X(b_1, c_1)] \circ [X(b_1, c_1)] \sim \begin{bmatrix} 1 & A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ c_1 \end{bmatrix}. \]
and \( \ker (b_1 \quad b_2) = \ker b_1 \cap \ker b_2. \)

4.7. **Proof of Lemma 4.8.**

*Step 1.* Any idempotent \( [[g]] \in \text{inv}(\Gamma (m)) \) has a representative of the form \( \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} \), where \( ab = 0 \).

Let \( [[g]] = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \) be an idempotent; let \( \alpha, \delta \) be nondegenerate. By Lemma 4.11 without loss of generality we can assume \( g = g^{-1} \). Taking an appropriate power \( r = [g]^{2N} \), we can achieve \( \alpha = 1 \). By Lemma 4.12 we can assume \( r = r^{-1} \).

Set \( r = \begin{bmatrix} 1 & -a \\ b & c \end{bmatrix} \). Evaluating \( r^2 = 1 \) we get the following collection of conditions
\[ ab = 0, \quad ac = -a, \quad cb = -b, \quad c^2 - ba = 1. \]
We replace \( r \) by an equivalent matrix
\[
[1 \ -a \\
\ b \ c_\star] (1 \ 0 \ c_*^{-1}) = \begin{pmatrix} 1 & -ac^{-1} \\ b & 1_* \end{pmatrix},
\]
here we used the identity \(-ac^{-1} = a\).

**Step 2.** We evaluate \([r]^{\circ 2}\),
\[
[r]^{\circ 2} = \left[ \begin{pmatrix} 1 & a & 0 \\ \ b & 1 & 0 \\ 0 & 0 & 1_\star \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ b & 0 & 1_\star \end{pmatrix} \right] = \begin{pmatrix} 1 & a & a \\ b & 1 & ba \\ b & 0 & 1_\star \end{pmatrix},
\]
\[
\sim \left[ \begin{pmatrix} 1 & a & a \\ b & 1 & ba \\ b & 0 & 1_\star \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -ba \\ 0 & 0 & 1_\star \end{pmatrix} \right] = \begin{pmatrix} 1 & a & a - aba \\ b & 1 & 0 \\ b & 0 & 1_\star \end{pmatrix}.
\]
But \( ab = 0 \) and therefore \( aba = 0 \). Repeating the same reasoning, we get
\[
[[r]]^{\circ N} \equiv [[q]] = \begin{pmatrix} 1_m & a \ldots a \\ \ b & 1 \ldots 0 \\ \vdots & \vdots & \ddots \\ \ b & 0 \ldots 1_\star \end{pmatrix}.
\]

**Step 3.** Next, we set \( N = p^\mu \) in formula (4.8). Consider the following block matrix \( u \) of size \( p^\mu \),
\[
u := \begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \\ \vdots & \vdots & \ddots \\ 0 & 0 & \ldots 1 \end{pmatrix}, \quad u^{-1} := \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{pmatrix}.
\]
We conjugate the matrix \( q \) defined by (4.8) as
\[
\begin{pmatrix} 1 & 0 \\ 0 & u_* \end{pmatrix} q \begin{pmatrix} 1 & 0 \\ 0 & u_*^{-1} \end{pmatrix}.
\]
We have
\[
u \begin{pmatrix} b \\ b \\ \vdots \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ \vdots \end{pmatrix}, \quad (a \ a \ a \ \ldots) u^{-1} = (0 \ -a \ -2a \ -3a \ \ldots),
\]
and we get a matrix of the form \( X(A, B) \).

5. Idempotents in \( \text{red}(\Gamma(m)) \)

Here the main statement is Proposition 5.1, which shows that all idempotents in \( \text{red}(\Gamma(m)) \) have representatives in \( \text{red}(\Gamma^\circ(m)) \); therefore they have the form \( X[L, M] \), where \( L \subset t^m, M \subset t^m \). The second fact (Proposition 5.2), which is important for the proof below, is a coherence of elements \( X[L, M] \) in different semigroups \( \text{red}(\Gamma(n)) \).
5.1. Coincidence of idempotents.

**Proposition 5.1.** The following idempotents in \( \text{inv}(\Gamma(m)) \) coincide as elements of \( \text{red}(\Gamma(m)) \):

\[
[[X_\alpha^\circ (b, c)]] := \left[ X (\begin{pmatrix} b & 0 \\ 0 & 1_{m-\alpha} \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & 1_{m-\alpha} \end{pmatrix}) \right]
\]

\[
= \begin{bmatrix}
1_\alpha & 0 & b & 0 & 0 & 0 \\
0 & 1_{m-\alpha} & 0 & 1_{m-\alpha} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
c & 0 & 0 & 0 & 1 & 0 \\
0 & 1_{m-\alpha} & 0 & 0 & 0 & 1_*
\end{bmatrix}
\]

and

\[
[[X_\alpha^\square (b, c)]] := \begin{bmatrix}
1_\alpha & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & 1_{m-\alpha} \\
0 & 0 & 1 & 0 & 0 \\
c & 0 & 0 & 1 & 0 \\
0 & 1_{m-\alpha} & 0 & 0 & 0_*
\end{bmatrix}
\]

**Corollary 5.2.** Any idempotent in \( \text{red}(\Gamma(m)) \) has the form \( X[L, M] \).

**Proof of corollary.** The semigroup \( \text{red} \Gamma(m) \) is a quotient of \( \text{inv}(\Gamma(m)) \); the semigroup of idempotents also is a quotient of the semigroup of idempotents. By Proposition 4.10 all idempotents in \( \text{inv}(\Gamma(m)) \) have \( [[X_\alpha^\circ [b, c]]] \). By Proposition 5.1 they also can be written as \( [[X_\alpha^\square [b, c]]] \). \( \square \)

Proposition will be proved in Subsection 5.3.

**Remarks.**

(a) The idempotents \( [[X_\alpha^\circ (b, c)]] \) and \( [[X_\alpha^\square (b, c)]] \) are different in \( \text{inv}(\Gamma(m)) \). Indeed, we have the following homomorphism from \( \Gamma(m) \) to the inverse semigroup \( \text{End}_L(m) \). On \( \Gamma^\circ(m) \) we define it as the map \( \Pi \) described in Subsection 3.5. On the other hand, we send \( \Gamma(m) \backslash \Gamma^\circ(m) \) to 0, i.e., to a pair of partial bijections with empty domains of definiteness. This map separates our idempotents.

(b) Idempotents \( X[L, M] \) are pairwise different in \( \text{red}(\Gamma(m)) \). To verify this, consider the representation of \( G \) in \( \ell^2(G/G[L; M]) \). It is easy to show that \( X(L, M) \) is the minimal idempotent of \( \text{red}(\Gamma(m)) \) acting in this representation nontrivially.

5.2. Coherence. Let \( L, M \subset \ell^m \) be submodules. Formula (4.4) defines the idempotent \( X[L, M] = X(b, c) \) as an element of \( \Gamma(m) \); recall that \( L = \ker b, M = \ker c \). However, for \( n > m \) we can regard \( L, M \subset \ell^n \) as submodules \( L \) in \( \ell^n \supset \ell^m \). In the larger space we have

\[
L = \ker \begin{pmatrix} b & 0 \\ 0 & 1_{n-m} \end{pmatrix}, \quad M = \ker \begin{pmatrix} c & 0 \\ 0 & 1_{n-m} \end{pmatrix}.
\]

Consider a unitary representation \( \rho \) of \( G \) in a Hilbert space \( H \). For any \( n \geq m \) we have an operator

\[
\bar{\rho}_{nn} \left( X \left( \begin{pmatrix} b & 0 \\ 0 & 1_{n-m} \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & 1_{n-m} \end{pmatrix} \right) \right) : H_n \to H_n.
\]

We claim that these operators as operators \( H \to H \) depend only on \( L, M \) and not on \( n \). Precisely, we have the following statement.
Proposition 5.3.
(a) Let \( n \geq m \). Then a block matrix structure of the operator (5.1) with respect to the orthogonal decomposition \( H_n = H_m \oplus (H_n \ominus H_m) \) is

\[
\tilde{\rho}_{nm}(X \left( \begin{pmatrix} 1 & 0 \\ 0 & 1_{n-m} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1_{m-n} \end{pmatrix} \right)) = \left( \begin{pmatrix} \tilde{\rho}_{mm}(X(b,c)) & 0 \\ 0 & 0 \end{pmatrix} \right).
\]

(b) For any \( L, M \subset \mathfrak{m} \) we have a well-defined operator \( \tilde{\rho}(X[L,M]) \) in \( \mathcal{H} \), which sends \( H_m \) to \( H_m \) as \( \tilde{\rho}_{mm}(X[L,M]) \) and is zero on the orthocomplement \( H \ominus H_m \).

Proof. According Corollary 4.3, the right hand side of (5.2) is \( \tilde{\rho}_{nn}(X_n(b,c)) \). By Proposition 5.1, this operator coincides with \( \tilde{\rho}_{nn}(X^n(b,c)) \). \( \square \)

5.3. Proof of Proposition 5.1

Lemma 5.4. Let \( g \in \text{red}(\Gamma(m)) \) be an idempotent. Let \( g \) be a representative of \( g \) in \( \mathcal{G}(m) \). Then for any unitary representation \( \rho \) of \( \mathcal{G} \) in a Hilbert space \( \mathcal{H} \) the image of the orthogonal projector \( \tilde{\rho}_{mm}(g) \) coincides with the space of fixed points of the subgroup in \( \mathcal{G} \) generated by \( \mathcal{G}(m) \) and \( g \).

Proof. Let \( v \in \text{im} \tilde{\rho}_{mm}(g) \), i.e.,

\[ P_m \rho(g) P_m v = v. \]

This happens if and only if \( P_m v = v, \rho(g)v = v \). The condition \( P_m v = v \) means that \( \rho(h)v = v \) for all \( h \in \mathcal{G}(m) \).

Therefore, it is sufficient to show that the group generated by \( \mathcal{G}(m) \) and \( X \cap (b,c) \) coincides with the group generated by \( \mathcal{G}(m) \) and \( X \cap (b,c) \).

Lemma 5.5. The group generated by the subgroup \( \mathcal{G}(\beta) \) and the matrix

\[
X(1,1) = \begin{pmatrix}
1_{\beta} & 1_{\beta} & 0 \\
0 & 1_{\beta} & 0 \\
1_{\beta} & 0 & 1_{\beta^*}
\end{pmatrix}
\]

coincides with \( \mathcal{G} \).

Proof. Denote by \( G \) the group generated by \( X(1,1) \) and \( \mathcal{G}(\beta) \). Conjugating \( X(1,1) \) by block diagonal matrices we can get any matrix of the form \( X(A,B) \) with non-degenerate \( A, B \). Multiplying such matrices we observe that elements of the form \( X(A_1 + A_2, B_1 + B_2) \) are contained in \( G \). In particular, \( X(0,2) \in G \). Since \( p \neq 2 \), conjugating \( X(0,2) \) by a block scalar matrix we come to \( X(0,1) \in G \). In the same way \( X(1,0) \in G \). Now the statement became more-or-less obvious. \( \square \)

Lemma 5.6. The group generated by \( \mathcal{G}(\beta) \) and the matrix

\[
\begin{pmatrix}
0 \\
1_{\beta} \\
1_{\beta^*}
\end{pmatrix}
\]

coincides with \( \mathcal{G} \).

Proof. Denote this group by \( G \). Denote \( S_\infty(\beta) := S_\infty \cap \mathcal{G}(\beta) \). Multiplying the matrix (5.3) from the left and right by elements of \( S_\infty(\beta) \) we can get an arbitrary matrix of the form \( \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ 0_{\beta^*} \end{pmatrix} \) with \( \sigma_1, \sigma_2 \in S_\beta \). Multiplying two matrices of this type we can get any matrix \( \begin{pmatrix} \sigma \\ 1_{\beta} \\ 1_{\beta^*} \end{pmatrix} \), where \( \sigma \in S_\beta \). Therefore our group
contains the subgroup $S_\beta \times S_{\infty}(\beta)$, which is maximal in $S_{\infty}$. Therefore $G \supset S_{\infty}$. But $S_{\infty}$ and $\mathbb{G}(\beta)$ generate $G$; see [27, Lemma 3.6].

Proof of Proposition 5.1. Denote by

- $G^\circ$ the group generated by $\mathbb{G}(m)$ and $X_\beta(b, c)$;
- $G^\square$ the group generated by $\mathbb{G}(m)$ and $X_\alpha(b, c)$;
- $G$ the group generated by $\mathbb{G}(\alpha)$ and the matrix $X_\phi(b, c)$ defined by

$$X_\phi(b, c) := \begin{pmatrix} 1_\alpha & 0 & b & 0 \\ 0 & 1_{m-\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c & 0 & 0 & 1_\ast \end{pmatrix}. $$

Obviously, $G \supset G^\circ$, $G \supset G^\square$. Let us verify the opposite inclusions.

The inclusion $G^\circ \supset G$. Clearly $X_\alpha(-b, -c) \in G^\circ$. Therefore $G^\circ$ contains

$$X_\alpha(b, c)X_\alpha(-b, -c) = X \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \sim X \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =: Y. $$

By Lemma 5.5 the group generated by $Y$ and $\mathbb{G}(m)$ is $\mathbb{G}(\alpha)$. On the other hand, $Y^{-1}X^\circ(b, c) \sim X_\phi(b, c)$.

5.3.1. The inclusion $G^\square \supset G$. We have

$$X_\square(b, c)^2 \sim X_\phi(2b, 2c) \sim X_\phi(b, c).$$

Next, $X_\phi(b, c)^{-1}X_\square(b, c) \sim X_\alpha(0, 0)$ and we refer to Lemma 5.6. Thus, $G^\circ = G^\square$. By Lemma 5.4 for any unitary representation $\rho$ of $\mathbb{G}$ we have

$$\tilde{\rho}_{mm}(X^\circ(b, c)) = \tilde{\rho}_{mm}(X^\square(b, c))$$

and this completes the proof of Proposition 5.1.

6. The semigroup $\text{red}_m(\Gamma(m))$

6.1. Structure of the semigroup $\text{red}_m(\Gamma(m))$. Denote by $0$ the minimal idempotent of the semigroup $\text{red}_m(\Gamma(m))$.

Proposition 6.1. Any element $\neq 0$ in $\text{red}_m(\Gamma(m))$ has a representative of a form $aX(b, c)$, where $a \in \text{GL}(m)$.

The proof occupies the rest of the section. As a byproduct of Lemma 6.3 we will get the following statement.

Lemma 6.2. Any idempotent $[X(b, c)]$ by a conjugation by $a \in \text{GL}(m)$ can be reduced to a form

$$[X \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \gamma & 0 \end{pmatrix}]$$

where $\gamma \beta = 0 (\text{mod } p)$, $\beta \gamma = 0 (\text{mod } p)$. 
6.2. Proof of Proposition 6.1

Step 1.

**Lemma 6.3.**

(a) Let $B$ be an $m \times N$ matrix over $\mathbb{Z}_{p^\mu}$, $C$ an $N \times m$ matrix. Then transformations

$$B \mapsto u^{-1}Bv, \quad C \mapsto v^{-1}Cu$$

allow to reduce them to the form

$$\tilde{B} = \begin{pmatrix} 0 & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} 0 & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

where $b_{12}, c_{21}$ are square nondegenerate matrices of the same size, products $b_{21}c_{12}, c_{12}b_{21}$ are nilpotent and $b_{22} = 0 \pmod{p}, c_{22} = 0 \pmod{p}$.

(b) The transformations

$$B \mapsto u^{-1}Bv, \quad C \mapsto w^{-1}Cu,$$

where $u, v, w$ are invertible, allow to reduce a pair $(B, C)$ to the form

$$\tilde{B} = \begin{pmatrix} 0 & b_{12} \\ 1 & 0 \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} 0 & 1 \\ c_{21} & 0 \end{pmatrix},$$

where $c_{21}b_{12} = 0 \pmod{p}, b_{12}c_{21} = 0 \pmod{p}$.

**Proof.** (a) Reduce our matrices modulo $p$. A canonical form of a pair of counter operators $P : \mathbb{F}_p^m \to \mathbb{F}_p^N$ and $Q : \mathbb{F}_p^N \to \mathbb{F}_p^m$ is a standard problem of linear algebra; see, e.g., [7], [11]. In particular, such operators in some bases admit block decompositions

$$P = \begin{pmatrix} P_r & 0 \\ 0 & P_n \end{pmatrix}, \quad Q = \begin{pmatrix} Q_r & 0 \\ 0 & Q_n \end{pmatrix},$$

where $P_Q, Q_P, P_QP_R$ are nondegenerate and $P_nQ_n, Q_nP_n$ are nilpotent.

Thus the matrices $B, C$ can be reduced to the form

$$B' = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad C' = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

where

1. $b_{21}, c_{12}$ are invertible matrices of the same size;
2. products $b_{21}c_{21}, c_{21}b_{12}$ are nilpotent;
3. the matrices $b_{11}, b_{22}, c_{11}, c_{22}$ reduced (mod $p$) are zero.

Set

$$u_1 := \begin{pmatrix} 1 & b_{11}b_{12}^{-1} \\ 0 & 1 \end{pmatrix},$$

notice that $u_1 \pmod{p}$ is 1. We pass to new matrices

$$B'' = u_1^{-1}B', \quad C'' = C'u_1.$$

For new $B$ the block $b_{11} = 0$; other properties (1)–(3) of matrices $B, C$ are preserved. Next, we take a unique matrix of the form $u_2 = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ such that $C''u_2$ has zero block $c_{12}$. On the other hand the block $b_{11}$ of $u_2^{-1}B''$ is zero. We come to a desired form.

(b) We apply statement (a) and reduce $(B, C)$ to the form (6.1). Next, we multiply $\tilde{B}$ from right by $\left(\begin{smallmatrix} b_{21} \\ 1 \end{smallmatrix}\right)^{-1}$ and get 1 on the place of $b_{21}$. After this, we
multiply new \(B\) from right by \(\begin{pmatrix} 1 & -b_{22} \\ 0 & 1 \end{pmatrix}\) and kill \(b_{22}\). Finally, we repeat the same transformations with \(\tilde{C}\).

Now the problem is reduced to the same question for a pair \(b_{12}, c_{21}\). If \(c_{21}b_{12} \neq 0 \pmod{p}\), then we choose an invertible matrix \(U\) such that \(b_{12}Uc_{21}\) is not nilpotent and again repeat (a). Etc.

\(\square\)

**Step 2.**

**Lemma 6.4.** Let \([g] \in \Gamma(m)\) have the form

\[
[g] = \begin{pmatrix} 1 & b \\ c & 1_\ast \end{pmatrix}_{mm}
\]

and \([[g]] \not\equiv_m 0\). Then \(bc\) and \(cb\) are nilpotent.

**Proof.** We apply the previous lemma and represent \([g]\) as

\[
[g] = \begin{pmatrix} 1_\alpha & 0 & 0 & b_{12} \\ 0 & 1_{m-\alpha} & b_{21} & b_{22} \\ 0 & c_{12} & 1_{m-\alpha} & 0 \\ c_{21} & c_{22} & 0 & 1_\ast \end{pmatrix}_{mm}.
\]

Set

\[
[h^\alpha_m] := \begin{pmatrix} 1_\alpha & 0 & 0 & 0 \\ 0 & 1_{m-\alpha} & 1_{m-\alpha} & 0 \\ 0 & 0 & 0 & 1_{m-\alpha} \\ 0 & 1_{m-\alpha} & 0 & 1_{m-\alpha} \end{pmatrix}.
\]

Let us show that

\[(6.2) \quad [g] \circ [h^\alpha_m] \sim [g].\]

Indeed,

\[(6.3) \quad [g] \circ [h^\alpha_m] = \begin{pmatrix} 1_\alpha & 0 & 0 & b_{12} & 0 & 0 \\ 0 & 1_{m-\alpha} & b_{21} & b_{22} & 1 & 0 \\ 0 & c_{12} & 1_{m-\alpha} & 0 & c_{12} & 0 \\ c_{21} & c_{22} & 0 & 1 & c_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1_\ast \end{pmatrix}_{mm}.
\]

\[
\sim \begin{pmatrix} 1_\alpha & 0 & 0 & b_{12} & 0 & 0 \\ 0 & 1_{m-\alpha} & b_{21} & b_{22} & 1_{m-\alpha} & 0 \\ 0 & c_{12} & 1_{m-\alpha} & 0 & 0 & 0 \\ c_{21} & c_{22} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1_{m-\alpha} & 0 & 0 & 0 & 1_\ast \end{pmatrix}_{mm} =: r,
\]

to establish the equivalence we multiply \([g] \circ [h^\alpha_m]\) from the left by

\[
\begin{pmatrix} 1_m \\ 1_0 & -c_{12} \\ 0 & 1 & -c_{22} \\ 0 & 0 & 1 \\ & & & 1_\ast \end{pmatrix}.
\]
Next, denote
\[ v_1 := \begin{pmatrix} 1_m & 1 & 1 \\ 0 & 0 & 1_m \end{pmatrix}, \quad v_2 := \begin{pmatrix} 1_m & 1 & 1 \\ 0 & 0 & 1_m \end{pmatrix}. \]

We have
\[ [r] \sim [v_2v_1^{-1}rv_1v_2^{-1}], \]
the latter matrix is obtained from \( r \), see (6.3), by removing two boxed blocks \( 1_m - \alpha \); all other blocks are the same. Thus \([r] \sim [g]\), i.e., we established (6.2).

Suppose that \( \alpha \neq m \). Then by Proposition 5.1
\[ [g] \sim [g] \circ [h^\alpha_m] \approx [g] \circ \Theta^\alpha_m. \]
But \( \Theta^\alpha_m \approx m \); therefore \([g] \not\approx m \). \( \square \)

**Step 3.** Thus it is sufficient to prove Proposition 6.1 for \([g] \circ \Gamma^\gamma(m)\). We must verify the following statement:

**Lemma 6.6.** Under our conditions,
\[ [g^{-1}] \circ [g] \equiv X(b, c). \]

Next,
\[ [g^{-1}] \circ [g] = \begin{pmatrix} (1 - bc)^{-1} & b & (1 - bc)^{-1}b \\ c & 0 & 1_m \end{pmatrix} \approx \begin{pmatrix} (1 - bc)^{-1} & b & c \\ c(1 - bc)^{-1} & 1 & 0 \end{pmatrix}. \]

This matrix defines an idempotent in \( \text{inv} \Gamma^\gamma(m) \). We must verify the following statement:

**Lemma 6.6.** Under our conditions,
\[ [g^{-1}] \circ [g] \equiv X(b, c). \]

---

This is equivalent to invertibility of \((1 - bc)^{-1}\) or invertibility of \((1 - cb)^{-1}\). Here we do not need a nilpotency of \(bc\).
Proof. By Corollary 4.39 we can identify an idempotent in \(\text{inv}(\Gamma^\circ)\) evaluating its image in \(\text{Mor}_L(m)\). So we get
\[
[g] \circ [g^{-1}] \equiv [[X(B, C)]],
\]
where
\[
B := (b \ b), \quad C := \left(\frac{c}{(1 - cb)^{-1}c}\right).
\]
We have \(\ker B = \ker b, \ker C^t = \ker c^t\); therefore by Lemma 4.6 we have \([[X(B, C)]] \equiv [[X(b, c)]].\)

Corollary 6.7. Let
\[
[g] = \left[\begin{array}{c|c}
1 & b \\
\hline c & 1_*
\end{array}\right]_{mm}, \quad [g'] = \left[\begin{array}{c|c}
1 & bu \\
\hline c & 1_*
\end{array}\right]_{mm}
\]
be invertible and \(u\) also be invertible. Then
\[
[g^{-1}] \circ [g] \equiv ((g')^{-1}) \circ [g'].
\]

Proof. Indeed, \(\ker bu = \ker b\). So both sides are \([[X(b, c)]].\)

Step 4.

Lemma 6.8. Let \([g] = \left[\begin{array}{c|c}
1 & b \\
\hline c & 1_*
\end{array}\right]_{mm}\); let \(bc\) and \(cb\) be nilpotent. Then there exists \(u\) having the form
\[
(6.5) \quad u = -\frac{1}{2} + \sum_{j>0} \sigma_j \frac{\sigma_j (cb)^j}{2n_j}, \quad \text{where } \sigma_j \in \mathbb{Z}, \ n_j \in \mathbb{Z}_+,
\]
such that
\[
(6.6) \quad \left(\begin{array}{c|c}
1 & bu \\
\hline c & 1_*
\end{array}\right)^{-1} \circ \left(\begin{array}{c|c}
1 & bu \\
\hline c & 1_*
\end{array}\right) \circ [g^{-1}]
\equiv \left[\begin{array}{c|c}
(1 - buc)^{-1}(1 - bc)^{-1} & b & 0 \\
\hline 0 & 1 & 0 \\
0 & 0 & 1_*
\end{array}\right]_{mm}.
\]

Proof. The product is
\[
\left[\begin{array}{c|c|c|c}
(1 - buc)^{-1}(1 - bc)^{-1} & bu & bu & (1 - buc)^{-1}b \\
\hline c(1 - bc)^{-1} & 1 & 0 & cb \\
\hline c(1 - buc)^{-1} & 0 & 1 & c(1 - buc)^{-1} \\
\hline c(1 - bc)^{-1} & 0 & 0 & 1_*
\end{array}\right]_{mm}
\sim \left[\begin{array}{c|c|c|c}
(1 - buc)^{-1}(1 - bc)^{-1} & bu & bu & (1 - buc)^{-1}b \\
\hline c & 1 & 0 & 0 \\
\hline c(1 - buc)^{-1} & 0 & 1 & 0 \\
\hline c(1 - bc)^{-1} & 0 & 0 & 1_*
\end{array}\right]_{mm} =: \left[\begin{array}{c|c|c}
A & br \\
\hline qc & 1_*
\end{array}\right]_{mm},
\]
here
\[
r := \left(\begin{array}{c|c}
u & (1 - cbu)^{-1} \\
\hline 1 & (1 - cb)^{-1}
\end{array}\right), \quad q := \left(\begin{array}{c|c}
1 & (1 - cbu)^{-1} \\
\hline (1 - cb)^{-1}
\end{array}\right).
\]
We claim that there exists a unique \(u\) such that \(rq = 0\). A straightforward calcula-
tion shows that
\[ rq = 2u - ucbu + (1 - cb)^{-1}. \]
Since \( cb \) is nilpotent, we can write the equation \( rq = 0 \) as
\[ 2u + 1 = ucbu - \sum_{j>0} (cb)^j, \]
the sum actually is finite. Clearly we can find a solution in the form \( u = -1/2 + \sum_{j>0} s_j (cb)^j \), where \( s_j \) are dyadic rationals; for coefficients \( s_j \) we have a system of recurrent equations. This \( u \) is invertible (since we can write a finite series for \( u^{-1} \)).

Next, we must show that the matrix \( \begin{pmatrix} 1 & b \nu \\ c & 1_* \end{pmatrix} \) is invertible. Indeed, this is equivalent to existence of \((1 - cbu)^{-1}\) and this is clear since by (6.5) \( cbu \) is nilpotent.

Next we wish to simplify the matrix \( \begin{pmatrix} A & br \\ qc & 1_* \end{pmatrix} \) by conjugations by matrices of the form \( \begin{pmatrix} 1 & 0 \\ 0 & D_* \end{pmatrix} \). In fact, we have transformations
\[ r \mapsto r' = pD^{-1}, \quad q \mapsto q' = Dq. \]
For such transformations we have \( r'q' = rq \). Set
\[ D = \begin{pmatrix} 1 & 1 & u^{-1}(1 - cbu)^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1_* \end{pmatrix}. \]
Then \( r' = (u \ 0 \ 0) \). But \( u \) is invertible and \( r'q' = 0 \). Therefore \( q' \) has the form \( \begin{pmatrix} 0 \\ * \\ * \end{pmatrix} \); on the other hand multiplication \( q \mapsto Dq \) does not change the second and third elements of the column \( q \). Thus we came to the matrix
\[ R :=\begin{pmatrix} (1 - buc)^{-1}(1 - bc)^{-1} & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ (1 - cbu)^{-1} & c & 0 & 1 \\ (1 - cb)^{-1} & c & 0 & 0 & 1_* \end{pmatrix} \]
Consider the following matrices:
\[ S := \begin{pmatrix} 1 \\ u \\ 1 - cbu \\ (1 - cb)_* \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ -1 & 1_* \end{pmatrix}. \]
The conjugation \( R \mapsto TRT^{-1} \) kills boxed elements of \( R \). The conjugation \( R \mapsto STRT^{-1}S^{-1} \) reduces the matrix to the desired form. \( \square \)

**Proof.** Proof of Proposition 6.1 Thus we have
\[ \left( \begin{pmatrix} 1 & b \\ c & 1_* \end{pmatrix} \right)^{-1} \equiv (1 - buc)^{-1}(1 - bc)^{-1} \cdot \left[ X((1 - bc)(1 - buc)b, c) \right]. \] \( \square \)
The second factor is
\[ [X(b(1 - cb)(1 - cbu), c)] \equiv [X(b, c)]. \]
Passing to adjoint elements we get
\[
\left[ \frac{1}{c} \begin{array}{c} a \\ b \\ 1 \end{array} \right]_{mn} \equiv [[X(b, c)] \cdot (1 - bc)(1 - bu)
\equiv (1 - bc)(1 - bu) \cdot [X((1 - bu)^{-1}(1 - bc)^{-1}b, c(1 - bc)(1 - bu))]
\equiv (1 - bc)(1 - bu) \cdot [X(b, c)].
\]
It remains to notice that
\[
\left[ \frac{a}{c} \begin{array}{c} b \\ 1 \end{array} \right] = a \cdot \left[ \frac{1}{c} \begin{array}{c} a^{-1}b \\ 1 \end{array} \right].
\]

6.3. **Proof of Lemma 6.2** We refer to Lemma 6.3

7. **The groups \( G^*[L; M] \)**

In this section we examine subgroups \( G^*[L; M] \), \( G^*[L; M] \subset G \) defined in Subsection 1.3. We prove that \( G^*[L; M] \) is well-defined. Lemma 7.5 shows that it is generated by \( G(m) \) and the element \( X(b, c) \). Also we prove that it is a minimal subgroup of finite index in \( G[L; M] \) (equivalently, \( G^*[L; M] \) has no subgroups of finite index, Proposition 7.11).

7.1. **Several remarks on submodules in \( t^k \)**

**Lemma 7.1.** Let \( L \subset t^k \) be a submodule. Then there exists a basis \( e_j \in t^k \) such that \( M := \oplus p^{s_j}Z_p e_j \). The collection \( s_1, s_2, \ldots \) is a unique \( \text{GL}(m) \)-invariant of a submodule \( L \).

This is equivalent to a classification of sublattices in \( (O_p)^k \) under the action of \( \text{GL}(k, O_p) \) or equivalently to a classification of pairs of lattices in \( Q_p^k \) under \( \text{GL}(k, Q_p) \); the latter question is standard; see, e.g., [35, Theorem I.2.2].

**Corollary 7.2.** Any submodule \( L \subset t^k \) is a kernel of some endomorphism \( t^k \to t^k \).

Indeed, we pass to a canonical basis \( e_j \) as in the lemma and consider the map sending \( e_j \) to \( p^{s_j-s_j} e_j \).

**Lemma 7.3.**

(a) Let \( L \) be a submodule in \( t^m \). Let \( b, b' : t^m \to t^N \) be morphisms of modules such that \( L = \ker b = \ker b' \). Then there is a transformation \( u \in \text{GL}(N) \) such that \( b' = bu \).

(b) Let \( \ker b = L' \supset L \). Then there is an endomorphism \( u : t^N \to t^N \) such that \( b' = bu \).

**Proof.** (a) The modules \( \text{im} b \simeq \text{im} b' \simeq t^m/L \) are isomorphic. By the previous lemma there is an automorphism of \( t^N \) identifying these submodules.

(b) \( L \) is a submodule of \( L' \); therefore \( \text{im} b' \) is a quotient module of \( \text{im} b \). Therefore there is a projection map \( \pi : \text{im} b \to \text{im} b' \); orders of elements do not increase under this map. By Lemma 7.1 we have a basis \( e_j \in t^N \) such that \( p^{s_j}e_j \), where \( j = 1, \ldots, m \), is the system of generators of \( \text{im} b \). Choose arbitrary vectors \( v_j \) such that \( p^{s_j}v_j = \pi(p^{s_j}e_j) \) and consider the map sending \( e_j \) to \( v_j \). \( \square \)
7.2. **The group** $\mathbb{G}^\bullet$. Here we show that $\mathbb{G}^\bullet[L; M]$ is a group, and its definition does not depend on the choice of matrices $b, c$.

**Lemma 7.4.**

(a) Fix a matrix $B$ of size $l \times N$. Then the set of invertible matrices $g$ of the form $1 - BS$, where $S$ ranges in the set of $N \times l$ matrices, is a group.

(b) Fix matrices $B, C$ of sizes $l \times N$ and $N \times l$ respectively. Then the set of invertible matrices $g$ of the form $g = 1 - BuC$ is a group.

**Proof.** Clearly, both sets are closed with respect to multiplication. We must show that $g^{-1}$ satisfies the same property. In the first case,

$$1 - g^{-1} = 1 - (1 - BS)^{-1} = -BS(1 - BS)^{-1}.$$  

In the second case,

$$1 - g^{-1} = 1 - (1 - BuC)^{-1} = -BuC(1 - BuC)^{-1} = -Bu(1 - CBu)^{-1}C.$$  

\[\square\]

**Lemma 7.5.** Fix matrices $b, c$ of sizes $m \times N$ and $N \times m$ respectively.

(a) The set of invertible matrices $g = \begin{pmatrix} a & bv \\ wc & z \end{pmatrix}$ such that the block ‘$a$’ admits representations $a = 1 - bS, a = 1 - Tc$ is a group.

(b) The set $\mathbb{G}^\bullet [L; M]$, i.e., the set of all invertible matrices of the form $g = \begin{pmatrix} 1 - buc & bv \\ wc & z \end{pmatrix}$, is a group.

**Proof.** In the first case we write

$$g = \begin{pmatrix} 1 - bS & bv \\ wc & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} bS & -bv \\ -wc & 1 - z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S & -v \\ -wc & 1 - z \end{pmatrix},$$

and reduce the statement to the previous lemma.

In the second case we write

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & -v \\ -w & 1 - z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and again we apply the previous lemma.  

\[\square\]

7.3. **The group** $\mathbb{G}^\circ [L; M]$.

**Proof of Lemma 7.3** Let $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{G}^\circ [L, M]$, i.e., $g$ fix pointwise $L \subset \ell^m$ and $g^t$ fix pointwise of $M \subset \ell^n$. Then $L \subset \ker \beta$ and by Lemma 7.3(b) we have $\beta = bv$ for some matrix $v$. Also $L \subset \ker (1 - \alpha)$ and therefore $\alpha = 1 - bS$ for some $S$.  

\[\square\]

7.4. **Changes of coordinates.**

**Lemma 7.6.** Let $L, M \subset \ell^m$. Let $a \in \text{GL}(m)$. Then

$$a \mathbb{G}^\circ [L; M]a^{-1} = \mathbb{G}^\circ [aL, (a^t)^{-1} M], \quad a \mathbb{G}^\bullet [L; M]a^{-1} = \mathbb{G}^\bullet [aL, (a^t)^{-1} M].$$

The first statement is an immediate consequence of the definition; the second is straightforward.
7.5. **Generators of** $\mathbb{G}^*[L; M]$. Let $m$, $b$, $c$ be as in Subsection 1.3, i.e., $L = \ker b$, $M = \ker c < \mathfrak{m}$.

**Proposition 7.7.** The group $\mathbb{G}^*[L; M]$ is generated by $\mathbb{G}(m)$ and the matrix $X(b, c)$.

**Proof.** Consider the group $G$ generated by $\mathbb{G}(m)$ and $X(b, c)$. Clearly, $\mathbb{G}^*[L, M] \supset G$. Let us prove the converse.

1) Conjugating $X(b, c)$ by block diagonal matrices $\in \mathbb{G}(m)$ we get arbitrary matrices of the form $X(bv, wc)$, where $v$, $w$ are invertible matrices. Consider products

\[
X(bv, wc) X(b', v, wc') = X((b + b')v, w(c + c')).
\]

We set $b = -b'$; for any matrix $\sigma$ we can find invertible matrices $c$, $c'$ such that $c + c' = \sigma$. Thus $G$ contains all matrices of the form

\[
\begin{pmatrix}
1_m & bv \\
wc & 1_{m*}
\end{pmatrix}, \quad \begin{pmatrix}
1_m & \beta \\
0 & 1_{m*}
\end{pmatrix}, \quad \begin{pmatrix}
1_m & 0 \\
w & 1_{m*}
\end{pmatrix},
\]

where $v$, $w$ are arbitrary matrices.

2) In virtue of Lemma 6.2, conjugating the matrices (7.2) by elements of $\GL(m)$ and multiplying from the left and the right by elements of $\mathbb{G}(m)$ we can reduce the matrices (7.2) to the forms

\[
Y[\beta] := \begin{pmatrix}
1_{m-\alpha} & 0 & 0 & \beta \\
0 & 1_\alpha & 1_\alpha & 0 \\
0 & 0 & 1_\alpha & 0 \\
0 & 0 & 0 & 1_{m-\alpha*}
\end{pmatrix},
\]

and we can represent any matrix $r$ as a sum of 3 invertible matrices.

3) Multiplying the matrices (7.2), we get

\[
\begin{pmatrix}
1 - bwv & bv \\
w & 1_{1*}
\end{pmatrix} \in G \quad \text{for any } v, w.
\]

\[\text{It is sufficient to verify this statement for matrices over } \mathbb{F}_p. \text{ Without loss of generality we can assume that } \sigma \text{ is diagonal. For } p \neq 2 \text{ any element of } \mathbb{F}_p \text{ is a sum of two nonzero elements, where } \sigma \text{ can be represented as a sum of two diagonal matrices.}\]
We represent our matrix as

\[
\begin{pmatrix}
1 & 0 \\
wc(1-\text{bvwc})^{-1} & 1_+
\end{pmatrix}
\begin{pmatrix}
1-\text{bvwc} & 0 \\
0 & 1_+
\end{pmatrix}
\times
\begin{pmatrix}
1 & 0 \\
0 & 1_+
\end{pmatrix}
\begin{pmatrix}
1-\text{bvwc}^{-1}\text{bv} \\
0 & 1_+
\end{pmatrix}.
\]

Since the whole product and three factors are contained in \(G\), the fourth factor also is contained in \(G\),

\[
\begin{pmatrix}
1-\text{bvwc} & 0 \\
0 & 1_+
\end{pmatrix} \in G
\]

for any \(v, w\).

(4) Now consider an arbitrary element of \(\mathbb{G}^*[L; M]\),

\[
\begin{pmatrix}
1-\text{buc} & \text{bv} \\
\text{wc} & z_+
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
wc(1-\text{buc})^{-1} & 1_+
\end{pmatrix}
\begin{pmatrix}
1-\text{buc} & 0 \\
0 & 1_+
\end{pmatrix}
\times
\begin{pmatrix}
1 & 0 \\
0 & 1_+
\end{pmatrix}
\begin{pmatrix}
1-\text{buc}^{-1}\text{bv} \\
0 & 1_+
\end{pmatrix}.
\]

All factors of the right hand side are contained in \(G\), and therefore \(\mathbb{G}^*[L; M]\) is contained in \(G\).

**Corollary 7.8.** The group \(\mathbb{G}^*[L; M]\) does not depend on a choice of \(m\).

**Proof.** Let \(L, M \subseteq \mathfrak{l}^m\); let \(L = \ker b, M = \ker c^i\). Let us regard \(L, M\) as submodules \(L', M'\) of \(\mathfrak{l}^m \oplus \mathfrak{k}^e\). Then

\[L' = \ker b', M' = \ker(c')t, \text{where } b' = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}, c' = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}.\]

Clearly the subgroup \(G_m\) generated by \(\mathbb{G}(m)\) and \(X(b, c)\) and the subgroup \(G_{m+k}\) generated by \(\mathbb{G}(m + k)\) and \(X(b', c')\) coincide. Formally, we must repeat the first two steps of the previous proof. \(\square\)

**7.6. The quotient \(\mathbb{G}/\mathbb{G}^*\).**

**Lemma 7.9.** A group \(\mathbb{G}^*[L; M]\) has finite index in \(\mathbb{G}[L; M]\).

**Proof.** Without loss of generality we can assume that \(cb = 0(\mod p)\), \(bc = 0(\mod p)\). Denote by \(A^o \subseteq \text{GL}(m)\) the subgroup consisting of matrices \(a\) admitting representations \(a = 1-bS, a = 1-Tc\). Notice that \(1-a\) is a nilpotent, since \(TcbS = 0(\mod p)\). Therefore \(a\) is invertible. Denote by \(A^*\) the subgroup consisting of elements of the form \(1-buc\).

The subgroup \(A^*\) is normal in \(A^o\). Indeed, let \(a \in A^o, a = 1-bS, a^{-1} = 1-Tc\). Then

\[a(1-buc)a^{-1} = 1-abuca^{-1} = 1-(1-bS)buc(1-Tc) = 1-b(1-Sb)u(1-cT)c.\]

Let \(g = \begin{pmatrix} a & \text{bv} \\ wc & z \end{pmatrix} \in \mathbb{G}^*[L; M]\). Let us show that the map \(g \mapsto a\) induces a homomorphism from \(\mathbb{G}^*[L; M] \rightarrow A^o/A^*\). Indeed,

\[g_1g_2 = \begin{pmatrix} a_1 & \text{bv}_1 \\ w_1 & z_1 \end{pmatrix} \begin{pmatrix} a_2 & \text{bv}_2 \\ w_2c & z_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + \text{bv}_1w_2c & * \\ * & * \end{pmatrix}.\]
In the left upper block we have
\[ a_1 a_2 (1 + a_2^{-1} a_1^{-1} b v_1 w_2 c). \]
We represent \( a_1^{-1} = 1 - b S_1 \), \( a_2^{-1} = 1 - b S_2 \) and get
\[ a_1 a_2 (1 + (1 - b S_2)(1 - b S_1) b v_1 w_2 c) = a_1 a_2 \{ 1 + b(1 - S_2 b)(1 - S_1 b) v_1 w_2 c \}. \]
The expression in the curly brackets is contained in \( A^\bullet \).

Clearly, the kernel of the homomorphism is \( G^\bullet \{ L, M \} \). Thus we have an isomorphism of quotient groups,
\[ G^\circ \{ L; M \} / G^\bullet \{ L; M \} \cong A^\circ \{ L; M \} / A^\bullet \{ L; M \}. \]
The group on the right-hand side is finite. \( \square \)

7.7. Absence of subgroups of finite index.

**Lemma 7.10.** The group \( G \) has not proper open subgroups of finite index.

**Proof.** Let \( P \) be a proper open subgroup. Then it contains some group \( G(\nu) \). On the other hand \( G \) contains a complete infinite symmetric group \( S_\infty \), and \( S_\infty \) has no subgroups of finite index. Therefore \( P \) contains \( S_\infty \). But the subgroup in \( G \) generated by \( G(\nu) \) and \( S_\infty \) is the whole group \( G \); see [27, Lemma 3.6]. \( \square \)

**Proposition 7.11.** The subgroup \( G^\bullet \{ L; M \} \) has no proper open subgroups of finite index.

**Proof.** Let \( Q \) be such subgroup. By the previous lemma, \( G(m) \) has not open subgroups of finite index; we have \( Q \supset G(m) \). Hence \( Q \) contains a minimal normal subgroup \( R \) containing \( G(m) \). The quotient \( Q/R \) is generated by the image \( \xi \) of \( X(b, c) \); therefore \( Q/R \) is a cyclic group. But
\[ X(b, c)^2 = X(2b, 2c) = \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1/2 \end{pmatrix}. \]
Since \( p \neq 2 \) the elements \( X(b, c)^2 \) and \( X(b, c) \) have the same images in \( Q/R \). Therefore the image of \( X[b, c] \) is 1. \( \square \)

**Corollary 7.12.** Any subgroup of finite index in \( G[L, M] \) contains \( G^\bullet \{ L; M \} \).

8. END OF THE PROOF

This section contains the end of the proof of Theorem 1.5. We know that all idempotents in semigroups \( \text{red}(\Gamma(n)) \) have the form \( \chi[L, M] \), see Corollary 5.2, for different \( n \) they can be identified in a natural way; see Proposition 5.3. We also know that any non-zero element of \( \text{red}_m(\Gamma(m)) \) is a product of an invertible element and an idempotent \( \chi[L, M] \); see Proposition 6.1. This implies that all irreducible representations of \( G \) are induced from representations \( \tau \) of groups \( G[L, M] \). Proposition 7.11 implies that such \( \tau \) must be trivial on \( G^\bullet \{ L; M \} \).
8.1. A preliminary remark.

Lemma 8.1. Consider an irreducible \( \ast \)-representation \( \sigma \) of the category \( \mathcal{K} \) in a sequence of Hilbert spaces \( H_j \). Let \( \xi \in H_m \) be a nonzero vector. Then the matrix element
\[
c(\mathbf{g}) = \langle \sigma(\mathbf{g})\xi, \xi \rangle_{H_m}, \quad \text{where \( \mathbf{g} \) ranges in \( \text{End}_\mathcal{K}(m) \)},
\]
determines \( \sigma \) up to equivalence.

This is a general statement on \( \ast \)-representations of categories (and a copy of a similar statement for unitary representations of groups); we give a proof for completeness.

Proof. For each \( \mathbf{g} \in \text{Mor}_\mathcal{K}(m, \alpha) \) we define a vector
\[
\omega_\mathbf{g}^\alpha = \sigma(\mathbf{g})\xi \in H_\alpha.
\]
Since \( \sigma \) is irreducible, vectors \( \omega_\mathbf{g}^\alpha \), where \( \mathbf{g} \) ranges in \( \text{Mor}_\mathcal{K}(m, \alpha) \), generate the space \( H_\alpha \). Their inner products are determined by the function \( c \):
\[
\langle \omega_\mathbf{g}_1^\alpha, \omega_\mathbf{g}_2^\alpha \rangle_{H_\alpha} = \langle \sigma(\mathbf{g}_1)\xi, \sigma(\mathbf{g}_2)\xi \rangle_{H_\alpha} = \langle \sigma(\mathbf{g}_2 \circ \mathbf{g}_1)\xi, \xi \rangle_{H_m} = c(\mathbf{g}_2 \circ \mathbf{g}_1).
\]
Next, let \( \mathbf{h} \in \text{Mor}_\mathcal{K}(\alpha, \beta) \). Let \( \mathbf{g}, \mathbf{f} \) range respectively in \( \text{Mor}_\mathcal{K}(m, \alpha), \text{Mor}_\mathcal{K}(m, \beta) \). Then
\[
\langle \sigma(\mathbf{h})\omega_\mathbf{g}, \omega_\mathbf{f} \rangle_{H_\beta} = \langle \sigma(\mathbf{h})\sigma(\mathbf{g})\xi, \sigma(\mathbf{f})\xi \rangle_{H_\beta} = \langle \sigma(\mathbf{f} \circ \mathbf{h} \circ \mathbf{g})\xi, \xi \rangle_{H_m} = c(\mathbf{f} \circ \mathbf{h} \circ \mathbf{g}).
\]
Clearly an operator \( \sigma(\mathbf{h}) \) is uniquely determined by such inner products. \( \square \)

8.2. Representations of the semigroup \( \text{red}_m(\Gamma(m)) \). Consider an irreducible representation of \( \mathcal{K} \) of height \( m \) and the corresponding representation \( \lambda \) of the semigroup \( \text{End}_\mathcal{K}(m) \) in \( H_m \). Recall that \( \tau \) passes through semigroup \( \text{red}_m(\Gamma(m)) \). By Proposition 6.1 any nonzero element of the latter semigroup can be represented as \( a \cdot \mathcal{X}[L, M] \), where \( a \in \text{GL}(m) \). Denote
\[
\hat{\mathcal{G}}_n[L, M] = \text{GL}(n) \cap \hat{\mathcal{G}}[L, M], \quad \hat{\mathcal{G}}_{\text{fin}}[L, M] = \mathcal{G}_{\text{fin}} \cap \hat{\mathcal{G}}_m[L, M].
\]

Lemma 8.2 is a special case of general description of representations of finite inverse semigroups; see, e.g., \cite{10}. However, due to Proposition 6.1 our case is simpler than general inverse semigroups. We show that the representation of \( \text{GL}(m) \) in \( H_m \) is induced from an irreducible representation of some subgroup \( \hat{\mathcal{G}}_m[L, M] \) and idempotents \( \mathcal{X}[N, K] \) act in the induced representation as multiplications by indicator functions of certain sets. Precisely,

Lemma 8.2. Let \( \mathcal{X}[L, M] \) be the minimal idempotent in \( \text{red}_m(\Gamma(m)) \) such that \( \lambda(\mathcal{X}[L, M]) \neq 0 \). Then there is an irreducible representation \( \tau_m \) of \( \hat{\mathcal{G}}_m[L, M] \) in a space \( V \) such that \( H_m \) can be identified with the space \( \ell_2 \) of \( V \)-valued functions on the homogeneous space \( \hat{\mathcal{G}}_m[L, M] \setminus \text{GL}(m) \) and

1. The group \( \text{GL}(m) \) acts by transformations of the form
\[
\lambda(g)f(x) = R(g, x)f(xg),
\]
and for \( q \in \hat{\mathcal{G}}_m[L, M] \) we have \( R(p, x_0) = \tau_m(q) \) (where \( x_0 \) denotes the initial point of \( \hat{\mathcal{G}}_m[L, M] \setminus \text{GL}(m) \)).
(2) The semigroup of idempotents acts by multiplications by indicator functions. Namely $\mathcal{X}[K, N]$ acts by multiplication by the function

$$I_{K, N}(x_0a) = \begin{cases} 1, & \text{if } K \supset aL, 
N \supset (a^t)^{-1}M; \\
0, & \text{otherwise.} \end{cases}$$

**Proof.** Consider the image $V$ of the projector $\lambda(\mathcal{X}[L, M])$. The idempotent $\mathcal{X}[L, M]$ commutes with $\hat{\mathcal{G}}_m[L, M]$. Indeed, for $q \in \hat{\mathcal{G}}_m[L, M]$ we have

$$q \cdot \mathcal{X}[L, M] \cdot q^{-1} = \mathcal{X}[Lq, M(q^t)^{-1}] = \mathcal{X}[L, M].$$

Therefore the subspace $V$ is $\hat{\mathcal{G}}_m[L, M]$-invariant. Denote by $\tau_m$ the representation of the group $\hat{\mathcal{G}}_m[L, M]$ in $V$. We need Lemma 8.3.

**Lemma 8.3.** For any $g \in \text{red}_m(\Gamma(m))$ we have $\lambda(g)V = V \lor \lambda(g)V \perp V$.

**Proof of Lemma 8.3.** Let us apply an arbitrary element of $\text{red}_m(\Gamma(m))$ to $v \in V$,

$$\lambda(a \cdot \mathcal{X}[K, N]) v = \lambda(a) \cdot \lambda(\mathcal{X}[K, N] \mathcal{X}[L, M]) v = \lambda(a) \cdot \lambda(\mathcal{X}[K \cap L, N \cap M]) v.\]$$

We have the following cases:

1. If $K \nsubseteq L$ or $N \nsubseteq M$, then by our choice of $\mathcal{X}[L, M]$, we have

$$\lambda(\mathcal{X}[K \cap L, N \cap M]) = 0.$$

2. Otherwise we come to $\lambda(a) \lambda(\mathcal{X}[L, M]) v = \lambda(a)v$.

   2.1 If $a \in \hat{\mathcal{G}}_m[L, M]$, we get $\lambda(a)v \in V$.

   2.2 Let $a \notin \hat{\mathcal{G}}_m[L, M]$. Then

$$\lambda(\mathcal{X}[L, M]) \lambda(a) \lambda(\mathcal{X}[L, M]) v = \lambda(a) \{\lambda(a^{-1}) \lambda(\mathcal{X}[L, M]) \lambda(a)\} \lambda(\mathcal{X}[L, M]) v$$

$$= \lambda(a) \lambda(\mathcal{X}[La, M(a^t)^{-1}]) \lambda(\mathcal{X}[L; M]) v$$

$$= \lambda(a) \lambda(\mathcal{X}[La \cap L, M(a^t)^{-1} \cap M]) v = 0.$$}

Since an idempotent $\mathcal{X}[a^{-1}L \cap L, a^tM \cap M]$ is strictly smaller than $\mathcal{X}[L, M]$, the $\lambda(\mathcal{X}[\ldots]) = 0$. □

**End of proof of Lemma 8.2** Thus $H_m$ is an orthogonal direct sum of spaces $V_x$, where $x$ ranges in the homogeneous space $\hat{\mathcal{G}}_m[L, M] \setminus \text{GL}(m)$, and $\lambda(a)$ sends each $V_x$ to $V_{xa}$. This means that the representation $\lambda$ of GL$(m)$ is induced from the representation of $\hat{\mathcal{G}}_m[L, M]$ in $V$; see, e.g., [33 Sect.7.1].

Operators

$$\lambda(\mathcal{X}[La^{-1}, a^tM]) = \lambda(a)\lambda(\mathcal{X}[L, M])\lambda(a^{-1})$$

act as orthogonal projectors to $V_{xa}$. A projector $\lambda(\mathcal{X}[K, N])$ is identical on $V_{xa}$ if and only if $\mathcal{X}[K, N] \mathcal{X}[La^{-1}, Ma^t] = \mathcal{X}[La^{-1}, Ma^t]$ and this gives us the action of the semigroup of idempotents.

It remains to show the representation of $\hat{\mathcal{G}}_m[L, M]$ in $V$ is irreducible. Assume that it contains a $\hat{\mathcal{G}}_m[L, M]$-invariant subspace $W$; then each $V_x$ contains a copy $W_x$ of $W$ and $\oplus_x W_x$ is a $\text{GL}(m)$-invariant subspace in the whole $H_m$. □

**Corollary 8.4.** Let $\lambda(g)$ be a nonzero operator leaving $V$ invariant. Then there is $b \in \hat{\mathcal{G}}_m[L, M]$ such that

$$\lambda(g)|_V = \rho(b)|_V.$$
Proof. This operator can be represented as $\lambda(a)\lambda(X[N,K])$. An operator $\lambda(X[N,K])$ restricted to $V$ is 0 or 1. Let this operator be 1. Then $\lambda(a)$ preserves $V$ only if $a \in \hat{G}_m[L,M]$. In this case we set $b = a$. □

Keeping in mind Lemma 8.1 we get the following statement:

**Corollary 8.5.** An irreducible $*$-representation of the category $\mathcal{K}$ is determined by its height $m$, a minimal idempotent $X[L,M]$ acting nontrivially in $H_m$ and an irreducible representation $\tau$ of the group $\hat{G}_m[L,M]$.

We do not claim an existence of representation corresponding to given data of this kind.

8.3. **End of proof.** Let $\rho$ be an irreducible unitary representation of $G$ of height $m$ in a Hilbert space $H$. Then we have a chain of subspaces in $H$:

$$H_m \rightarrow H_{m+1} \rightarrow H_{m+2} \rightarrow \ldots$$

Lemma 4.2 defines a chain of semigroups

$$\Gamma(m) \rightarrow \Gamma(m+1) \rightarrow \Gamma(m+2) \rightarrow \ldots$$

Each semigroup $\Gamma(n)$ acts in $H$ as follows: in $H_n$ it acts by operators $\tilde{\rho}_{nn}(\cdot)$; on $H_n^\perp$ these operators are zero (see Lemma 4.1).

On the other hand, we have a chain of groups

$$\text{GL}(m) \rightarrow \text{GL}(m+1) \rightarrow \text{GL}(m+2) \rightarrow \ldots$$

acting by unitary operators; their inductive limit is the group $G_{\text{fin}}$. Each group $\text{GL}(n)$ preserves the subspace $H_n$; on this subspace the action of $\text{GL}(n)$ coincides with the action of the group $\text{Aut}_\mathcal{K}(n) = \text{GL}(n)$.

Consider the data listed in Corollary 8.5. We regard the subspace $V = \text{im} \tilde{\rho}_{nm}(X[L,M]) \subset H_m$ as a subspace in $H$. Denote the $\text{GL}(n)$-cyclic of $V$ by $W_n$; it is a subspace in $H_n$.

**Lemma 8.6.** Let $g \in \text{GL}(n)$. If $g \in \tilde{G}_n[L;M]$, then $\rho(g)V = V$. Otherwise, $\rho(g)V \perp V$.

Proof. In the first case, we have

$$\tilde{\rho}_{nn}(X[L,M])(\tilde{\rho}_{nn}(g))^{-1}\tilde{\rho}_{nn}(X[Lg,M(g^t)^{-1}]) = \tilde{\rho}_{nn}(X[L,M])$$

and therefore the image $V$ of $\tilde{\rho}_{nn}(X[L,M])$ is invariant with respect to $\rho(g)$.

In the second case we repeat the line 8.1. □

Thus the representation of $\text{GL}(n)$ in $W_n$ is induced from the subgroup $\tilde{G}_n[L,M]$. If $k > n$, then we have embeddings

$$\text{GL}(n) \rightarrow \text{GL}(k), \quad \tilde{G}_n[L,M] \rightarrow \tilde{G}_k[L,M]$$

and therefore the map of homogeneous spaces

$$\Xi_{n,k} : \tilde{G}_n[L,M] \setminus \text{GL}(n) \rightarrow \tilde{G}_k[L,M] \setminus \text{GL}(k).$$

On the other hand, we have an embedding $W_n \rightarrow W_k$ regarding the orthogonal decompositions of these spaces into copies of $V$; therefore the map $\Xi_{n,k}$ is an embedding.
Finally, we get a representation of $\hat{G}_{\text{fin}}$ induced from the subgroup $\hat{G}_{\text{fin}}[L,M]$. By continuity, $G$ acts regarding the same orthogonal decomposition $\oplus V_{xa}$. Hence a representation of $G$ is induced from closure of $\hat{G}_{\text{fin}}[L,M]$, i.e., $\hat{G}[L,M]$.

**Lemma 8.7.** The image of $\hat{G}_{\text{fin}}[L,M]$ in the group of operators in $V$ coincides with the image of $\hat{G}_{m}[L,M]$.

**Proof.** Let $u \in \hat{G}_{n}[L,M]$. Then $\rho(u)$ preserves $V \subset H_{m}$. Therefore
\[ \rho(u)|_{V} = P_{m}\rho(u)P_{m} = \bar{\rho}(\chi_{m} u) |_{V}. \]

By Corollary 8.4 this operator has the form $\rho(u')|_{V}$, where $u' \in \hat{G}_{m}[L,M]$. □

Thus the representation $\tau$ of $\hat{G}_{\text{fin}}[L,M]$ in $V$ has a finite image. Its continuous extension to $G[L,M]$ has the same image. The kernel of the representation $\tau$ is a closed subgroup. Since it has a finite index, it is open. By Proposition 7.11 $\tau$ is trivial on the subgroup $G^{\cdot}[L,M]$.

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