SELF-ADJOINT DIFFERENCE OPERATORS AND CLASSICAL SOLUTIONS
TO THE STIELTJES–WIGERT MOMENT PROBLEM

JACOB S. CHRISTIANSEN AND ERIK KOELINK

Abstract. The Stieltjes–Wigert polynomials, which correspond to an indeterminate moment prob-
lem on the positive half-line, are eigenfunctions of a second order $q$-difference operator. We con-
sider the orthogonality measures for which the difference operator is symmetric in the corresponding
weighted $L^2$-spaces. Under some additional assumptions these measures are exactly the solutions to
the $q$-Pearson equation. In the case of discrete and absolutely continuous measures the difference
operator is essentially self-adjoint, and the corresponding spectral decomposition is given explicitly.
In particular, we find an orthogonal set of $q$-Bessel functions complementing the Stieltjes–Wigert
polynomials to an orthogonal basis for $L^2(\mu)$ when $\mu$ is a discrete orthogonality measure solving the
$q$-Pearson equation. To obtain the spectral decomposition of the difference operator in case of an
absolutely continuous orthogonality measure we use the results from the discrete case combined with
direct integral techniques.

Key words and phrases: Difference operators, Stieltjes–Wigert polynomials, spectral an-
alysis, direct integrals of Hilbert spaces and self-adjoint operators.

AMS classification: Primary 47B36; Secondary 44A60.

1. Introduction

As part of the Askey-scheme [18] of basic hypergeometric orthogonal polynomials, the Stieltjes–
Wigert polynomials are eigenfunctions of a second-order $q$-difference operator. This operator is given
by

$$(Lf)(x) = f(xq) - \frac{1}{x} f(x) + \frac{1}{x} f(x/q)$$

or, in a more compact form,

$$L = T_q - x^{-1} (I - T_{q^{-1}}),$$

where $T_a$ denotes the operator defined by $(T_a f)(x) = f(ax)$ for fixed $a \neq 0$. We always take $q$ as a
fixed number in $(0, 1)$. Clearly, $L$ preserves the space of polynomials.

In this paper we consider $L$ as a (possibly) unbounded operator on $L^2(\mu)$, where $\mu$ is assumed to
be a solution to the Stieltjes–Wigert moment problem, i.e. a positive measure on $[0, \infty)$ such that

$$\int_0^\infty x^n d\mu(x) = q^{-\binom{n+1}{2}}, \quad n \geq 0. \quad (1.1)$$

Since the Stieltjes–Wigert moment problem is indeterminate, there are infinitely many positive mea-
sures to choose from. The operator $(L, P)$ with domain the space $P$ of polynomials is always sym-
metric on $L^2(\mu)$. However, the polynomials are only dense in $L^2(\mu)$ when $\mu$ is a so-called $N$-extremal
solution to the moment problem, see e.g. [11, Chapter 2]. So instead we consider $L$ with a larger
domain $L(D)$ which will be specified in (2.3). Under certain restrictions on $T_{q^{\pm 1}}$, this operator turns
out only to be symmetric for a special class of solutions to the moment problem, namely the solutions
that satisfy the $q$-Pearson equation or, in the setup of [10], the solutions that are fixed points of the
transformation $T$ defined in [10, Def. 2.4]. Such solutions are also called “classical” in [10]. We give
the precise condition that $\mu$ has to satisfy in Proposition 2.1.
The question now raises if $L$ can be extended to a self-adjoint operator on $L^2(\mu)$ when $\mu$ is a classical solution to the moment problem. We deal with the cases of discrete solutions, respectively absolutely continuous solutions, in Section 3 and Section 4.

In Section 3, where $\mu$ is supposed to be discrete, we show that $L$ is unitarily equivalent to a doubly infinite Jacobi operator acting on $\ell^2(\mathbb{Z})$. The theory of unbounded Jacobi operators then leads to the fact that $L$ is essentially self-adjoint. Starting from two explicit eigenfunctions of $L$ constructed in Section 2, the spectrum of $L$ is computed in Theorem 3.3. The spectrum is purely discrete (except for the point 0) and has an unbounded negative part and a bounded positive part. The positive part is simple and each point corresponds to a Stieltjes–Wigert polynomial of fixed degree. The negative part is also simple and each point corresponds now to a $q$-Bessel function of the second kind. This leads to orthogonality relations for the Stieltjes–Wigert polynomials and for Jackson’s second $q$-Bessel functions. None of the discrete measures under consideration are canonical solutions in the sense of [1, Def. 3.4.2, p. 115], and hence the space of polynomials has codimension $+\infty$ in the corresponding weighted $L^2$-spaces. Our analysis leads to an explicit set of orthogonal functions complementing the Stieltjes–Wigert polynomials to a basis for $L^2(\mu)$.

In the case where $\mu$ is absolutely continuous, the operator $L$ is again essentially self-adjoint. We show this in Section 4 using direct integrals of Hilbert spaces and the results of Section 3. The spectrum of $L$ has a purely discrete positive part, where each point is of infinite multiplicity and corresponds to a Stieltjes–Wigert polynomials of fixed degree times an arbitrary $q$-periodic function, i.e. a function $f$ satisfying $f(xq) = f(x)$ for all $x > 0$. In case $\text{supp}(\mu) = [0, \infty)$, the continuous spectrum of $L$ is $(-\infty, 0]$ and each point here is simple. We also give an explicit formula for the spectral measure. The approach in Section 4 should be compared with related ideas of Berg [5].

The indeterminate cases within the Askey-scheme have been classified in [11] and one may ask if a similar construction is possible for other cases as well. For the $q$-Laguerre polynomials the analysis is already done in [12], where the motivation comes from quantum groups and limit transitions of the big $q$-Jacobi polynomials. Formal limit results of [12] lead to the results of Section 3, and we note that the methods of Section 4 can be used for the $q$-Laguerre case as well. See also [9] for the transformation corresponding to the $q$-Pearson equation. For other cases in the indeterminate part of the Askey-scheme several problems arise, and it is not clear if symmetry of the difference operator for the corresponding orthogonal polynomials has a clear-cut meaning for solutions to the moment problem.

Acknowledgement. We thank the referee for useful suggestions, and Barry Simon for a remark that led to an improvement of Section 2.

2. Difference Operator

2.1. Difference operator. Consider the second order $q$-difference operator

$$
(Lf)(x) = f(xq) - \frac{1}{x} f(x) + \frac{1}{x} f(x/q).
$$

The motivation for studying $L$ is the fact that the Stieltjes–Wigert polynomials

$$
S_n(x; q) = \frac{1}{(q; q)_n} \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{k^2} x^k, \quad n = 0, 1, \ldots
$$

are eigenfunctions of $L$ corresponding to the eigenvalues $q^n$, see Proposition 2.6 below. Here we use the notation

$$
(q; q)_0 = 1, \quad (q; q)_n = \prod_{k=1}^{n} (1 - q^k), \quad n = 1, 2, \ldots
$$
and

\[
\begin{bmatrix}
  n \\
  k
\end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \quad 0 \leq k \leq n.
\]

Throughout the paper we assume that $0 < q < 1$ and follow the notation of Gasper and Rahman [15] for basic hypergeometric series.

Recall that the image measure $\tau(\mu)$ of a finite positive measure $\mu$ under a measurable map $\tau$ is defined by

\[
\tau(\mu)(A) = \mu(\tau^{-1}(A))
\]

for any measurable set $A$. Recall also that integration with respect to $\tau(\mu)$ is carried out via the rule

\[
\int f \, d\tau(\mu) = \int (f \circ \tau) \, d\mu.
\]

In what follows we denote by $\tau_a : (0, \infty) \to (0, \infty)$ the map given by $x \mapsto ax$ for fixed $a > 0$.

Writing $M$ for the operator of multiplication by $1/q$, we see that $L$ can be written as

\[
L = T_q - M + M \circ T_q^{-1}.
\]

Our first task is therefore to define and discuss the operators $M$ and $T_{q^\pm 1}$ as possibly unbounded operators on $L^2(\mu)$, where $\mu$ for the time being is supposed to be any finite positive (Borel) measure on $(0, \infty)$. We define the operator $M$ on the maximal domain

\[
D(M) = \left\{ f \in L^2(\mu) \mid \int_0^\infty \frac{1}{x^2} |f(x)|^2 \, d\mu(x) < \infty \right\}.
\]

As regards the operators $T_{q^\pm 1}$, it may happen that one (or both) of them is identically zero on $L^2(\mu)$. This happens if $xq$ (or $x/q$) never belongs to supp($\mu$) when $x \in$ supp($\mu$) (and hence for example if $\mu$ is discrete and supported on $\{tq^{2n} \mid n \in \mathbb{Z}\}$ for some $t > 0$). To avoid this situation we require that $T_{q^\pm 1}$, defined on the maximal domains

\[
D(T_{q^\pm 1}) = \left\{ f \in L^2(\mu) \mid T_{q^\pm 1} f \in L^2(\mu) \right\},
\]

have trivial kernels, i.e. Ker($T_{q^\pm 1}$) = $\{0\}$. For any Borel set $A \subset (0, \infty)$, the indicator function $\chi_A$ belongs to $D(T_{q^\pm 1})$ since

\[
\int_0^\infty |(T_{q^\pm 1} \chi_A)(x)|^2 \, d\mu(x) = \mu(q^{\pm 1} A) < \infty.
\]

When $\mu(A) > 0$, we have $\chi_A \neq 0$ in $L^2(\mu)$ and the requirement on the kernels therefore implies that $\mu(q^{\pm 1} A) = \tau_{q^\pm 1}(\mu)(A) > 0$. In other words, $\mu$ is absolutely continuous with respect to $\tau_{q^\pm 1}(\mu)$, that is, $\tau_{q^\pm 1}$ preserve the support of $\mu$. Note that the domains $D(T_{q^\pm 1})$ are dense in $L^2(\mu)$ since the set of finite linear combinations of indicator functions is dense in $L^2(\mu)$.

With the above assumptions in mind we define $L$ as the possibly unbounded operator on $L^2(\mu)$ with domain

\[
D(L) = \left\{ f \in L^2(\mu) \mid f \in D(T_q) \cap D(M) \cap D(T_{q^{-1}}), T_{q^{-1}} f \in D(M) \right\}.
\]

(2.3)

**Proposition 2.1.** Let $\mu$ be a positive measure on $(0, \infty)$ such that

\[
m_n := \int_0^\infty x^n \, d\mu(x) < \infty \quad \text{for } n \geq -2.
\]
Assume that $T_{q^{+1}} : D(T_{q^{+1}}) \rightarrow L^2(\mu)$ have trivial kernels. Then the domain $D(L)$ defined in (2.3) is dense in $L^2(\mu)$ and the operator $(L,D(L))$ is symmetric on $L^2(\mu)$ if and only if the measure $\tau_q(\mu)$ is absolutely continuous with respect to $\mu$ and the Radon–Nikodym derivative is given by

$$\frac{d\tau_q(\mu)}{d\mu} = \frac{1}{x} \text{ a.e. with respect to } \mu. \quad (2.4)$$

**Remark 2.2.** When $\mu$ is a finite positive measure on $(0,\infty)$ satisfying (2.3), it follows by induction that $\tau_{q^n}(\mu)$ is absolutely continuous with respect to $\mu$ for all $n \in \mathbb{Z}$ and

$$\frac{d\tau_{q^n}(\mu)}{d\mu} = \frac{q(\frac{3}{2})}{x^n} \text{ a.e. with respect to } \mu.$$

This in particular means that $\mu$ has moments of all orders and if $\mu$ is a probability measure, then

$$\int_0^\infty x^n d\mu(x) = q^{-\left(\frac{n+1}{2}\right)} \quad \text{for all } n \in \mathbb{Z}.$$

So the requirement in Proposition 2.1 on the existence of the first two negative moments is actually implied by (2.3). Moreover, we see that $\mu$ is uniquely determined by its restriction $\mu|_{(q,1]}$ to the interval $(q,1]$ (or any other interval of the form $(tq^{k+1}, tq^k]$ for $t > 0$ and $k \in \mathbb{Z}$). See [10, Section 2] for more details.

**Proof.** Since by assumption $m_{-2} < \infty$, we see that $\chi_A \in D(M)$ for any Borel set $A \subset (0,\infty)$. We have already observed that $\chi_A \in D(T_{q^{+1}})$ and that $T_{q^{-1}} \chi_A = \chi_{qA} \in D(M)$. Hence, all indicator functions are contained in $D(L)$, and finite linear combinations of these functions are dense in $L^2(\mu)$.

Suppose that $f,g \in D(L)$, then

$$\langle Lf, g \rangle = \int_0^\infty (Lf)(x) \overline{g(x)} \, d\mu(x)$$

$$= \int_0^\infty \left( f(qx) - \frac{1}{x} f(x) + \frac{1}{x} f(x/q) \right) \overline{g(x)} \, d\mu(x)$$

$$= \int_0^\infty f(x) \frac{g(x/q)}{x} \, d\tau_q(\mu)(x) - \int_0^\infty f(x) \frac{g(x)}{x} \, d\mu(x) + \int_0^\infty f(x) \frac{g(xq)}{qx} \, d\tau_{q^{-1}}(\mu)(x),$$

using the fact that each term is integrable. The right-hand side can be written as $\langle f, Lg \rangle$ if and only if

$$\int_0^\infty f(x) g(qx) \, d\mu(x) + \int_0^\infty f(x) \frac{g(x/q)}{x} \, d\mu(x) = \int_0^\infty f(x) \frac{g(x/q)}{x} \, d\tau_q(\mu)(x) + \int_0^\infty f(x) \frac{g(xq)}{qx} \, d\tau_{q^{-1}}(\mu)(x). \quad (2.5)$$

Now, if $\tau_q(\mu)$ and $\tau_{q^{-1}}(\mu)$ are both absolutely continuous with respect to $\mu$ and the conditions

$$\frac{d\tau_q(\mu)}{d\mu} = \frac{1}{x} \quad \text{and} \quad \frac{d\tau_{q^{-1}}(\mu)}{d\mu} = xq \text{ a.e. with respect to } \mu$$

are met, then (2.5) is satisfied. Since $\tau_{q^{-1}} = \tau_q^{-1}$, these conditions are equivalent and the “if” part of the proposition follows.

Conversely, if $(L,D(L))$ is symmetric, then (2.5) holds for all $f,g \in D(L)$. Take $f = \chi_A$, $g = \chi_B$, then

$$\int_{A \cap q^{-1} B} d\mu(x) + \int_{A \cap q B} \frac{1}{x} d\mu(x) = \int_{A \cap q B} d\tau_q(\mu)(x) + \int_{A \cap q^{-1} B} \frac{1}{xq} d\tau_{q^{-1}}(\mu)(x)$. 

Now take $A \subset (q^{k+1},q^k]$ for some $k \in \mathbb{Z}$, and set $B = q^{-1}A$ or $A = qB$. This gives $A \cap q^{-1}B = \emptyset$ and therefore

$$\int_A \frac{1}{x} \, d\mu(x) = \tau_q(\mu)(A).$$

Since any Borel set $A \subset (0,\infty)$ can be written as a disjoint union $A = \bigcup_{k \in \mathbb{Z}} A_k$, where $A_k = A \cap (q^{k+1},q^k]$, we find that

$$\tau_q(\mu)(A) = \sum_{k \in \mathbb{Z}} \tau_q(\mu)(A_k) = \sum_{k \in \mathbb{Z}} \int_{A_k} \frac{1}{x} \, d\mu(x) = \int_A \frac{1}{x} \, d\mu(x),$$

recalling that $1/x$ is integrable with respect to $\mu$. In particular, $\tau_q(\mu)$ is absolutely continuous with respect to $\mu$ and (2.4) is satisfied.

**Remark 2.3.** When $\mu$ is an $N$-extremal (or $m$-canonical) solution to the Stieltjes–Wigert moment problem, then $\tau_{q^{\pm 1}}$ do not preserve the support of $\mu$. See [10, Section 3] for details. So the assumptions on $T_{q^{\pm 1}}$ in Proposition 2.1 exclude canonical solutions of all orders.

In this paper we shall mainly focus on discrete and absolutely continuous measures and state therefore the following consequence of Proposition 2.1. As for notation, we denote by $\delta_x$ the unit mass at the point $x$.

**Corollary 2.4.** (i) Suppose that $t > 0$ and let $\mu_t$ be a positive discrete measure of the form

$$\mu_t = \sum_{k=-\infty}^{\infty} m_t(k) \delta_{t^{q^k}},$$

where $m_t(k) > 0$ for all $k \in \mathbb{Z}$ and $\sum_{k=-\infty}^{\infty} m_t(k) < \infty$. The operator $L$ is symmetric on $L^2(\mu_t)$ if and only if

$$m_t(k+1) = t^{q^k+1} m_t(k) \quad \text{for all } k \in \mathbb{Z}. \quad (2.6)$$

(ii) Let $\mu$ be an absolutely continuous measure on $(0,\infty)$ given by a positive density function $w$ satisfying $\int_0^\infty w(x) \, dx < \infty$. Assume that $\mu$ and $\tau_{q^{\pm 1}}(\mu)$ have the same support. The operator $L$ is symmetric on $L^2(\mu)$ if and only if

$$w(xq) = xw(x) \quad \text{for all } x \in (0,\infty). \quad (2.7)$$

**Remark 2.5.** (i) The condition (2.6) is equivalent to $m_t(k) = t^{q^{k+1}} m_t(0)$ for $k \in \mathbb{Z}$. If we set $1/m_t(0) = (-tq,-1/t,q,q)_\infty$, it follows by the triple product identity [15, (1.6.1)] that $\mu_t$ becomes a probability measure.

(ii) The condition (2.7) is the $q$-Pearson equation for the Stieltjes–Wigert polynomials, see e.g. [21] and [2]. This equation is for example satisfied by the log-normal density

$$w(x) = \frac{1}{\sqrt{x}} e^{\frac{1}{2} \frac{(\log x)^2}{\log q}}, \quad x > 0$$

and (for fixed $c > 0$) by the infinite products

$$w_c(x) = \frac{x^{c-1}}{(-q^{1-c},-q^c/x;q)_\infty}, \quad x > 0.$$

Note also that (2.7) is invariant under multiplication with $q$-periodic functions, that is, functions which satisfy $f(xq) = f(x)$ for $x > 0$. 

\[\square\]
In the setting of Proposition 2.1 we find
\[ \int_0^\infty |f(x)|^2 \, dx = \int_0^\infty \frac{1}{z} |f(x)|^2 \, dx = q \int_0^\infty \frac{1}{z^2} |f(x/q)|^2 \, dx, \]
showing that \( L \) is well-defined on any continuous function \( f \) satisfying \( f(x) = O(x^N) \) as \( x \to \infty \) and \( f(x) = O(x^{-M}) \) as \( x \to 0 \) for some \( N, M \geq 0 \), cf. Remark 2.2.

2.2. Eigenfunctions. The \( 1 \varphi_1 \)-series with lower parameter equal to zero, say \( 1 \varphi_1 \left( \frac{a}{0} ; q, y \right) \), satisfies the second order \( q \)-difference equation
\[ -a y f(yq) + (y - q) f(y) + q f(y/q) = 0. \]  
This result can be obtained from the second order \( q \)-difference equation for the \( 2 \varphi_1 \)-series \([15\text{ Exerc. 1.13}]\) by taking a limit.

By looking for solutions of the form \( \sum_{k=0}^\infty c_k y^{\lambda+k} \), respectively \( \sum_{k=0}^\infty c_k y^{\lambda-k} \), with \( c_0 = 1 \), we see that
\[ 1 \varphi_1 \left( \frac{a}{0} ; q, y \right) \quad \text{and} \quad y^\alpha \, 1 \varphi_1 \left( \frac{a}{0} ; q, \frac{q^2}{y} \right), \quad q^\alpha a = 1 \]
both satisfy (2.8).

**Proposition 2.6.** The functions defined by
\[ \phi_z(x) = 1 \varphi_1 \left( \frac{1}{z} ; q, -xzq \right), \quad \Phi_z(x) = x^{\ln z/ \ln q} \, 1 \varphi_1 \left( \frac{1}{z} \right) \]
are solutions to the eigenvalue equation \( Lf = zf \). Here \( \phi_z(x) \) is defined for \( x, z \in \mathbb{C} \), where the case \( z = 0 \) has to be interpreted as the limit
\[ \phi_0(x) = 0 \varphi_1 \left( - \frac{1}{0} ; q, -xz \right), \]
and \( \Phi_z(x) \) is defined for \( x \in (0, \infty) \) and \( z \in \mathbb{C} \setminus (-\infty, 0] \).

In particular, the Stieltjes–Wigert polynomials are solutions to the eigenvalue equations
\[ L S_n(\cdot ; q) = q^n S_n(\cdot ; q), \quad n = 0, 1, \ldots. \]

**Remark 2.7.** The function
\[ \phi_0(x) = \sum_{n=0}^\infty \frac{(-1)^n q^n x^n}{(q;q)_n}, \quad x \in \mathbb{C} \]
is also known as the entire Rogers–Ramanujan function, since its values at \(-1\) and \(-q\) appear in the celebrated identities \([15\text{ (2.7.3/4)}]\]
\[ \sum_{n=0}^\infty q^n (q; q)_n = \frac{1}{(q, q^3, q^5)_\infty} \quad \text{and} \quad \sum_{n=0}^\infty q^{n(n+1)} (q; q)_n = \frac{1}{(q^2, q^3, q^5)_\infty}. \]
The reader is referred to \([3]\) and \([16]\) for interesting results about the zeros of \( \phi_0 \), which are all positive and simple.

**Proof.** The result follows from (2.8) and (2.9) if we replace \( a \) by \( 1/z \) and \( y \) by \(-xzq\). Since
\[ \sum_{k=0}^n \binom{n}{k/q} (-1)^k q^{k^2} x^k = 1 \varphi_1 \left( \frac{q^n}{0} ; q, -q^{n+1} x \right), \]
the last assertion follows immediately from (2.2). \( \Box \)
To get hold of the behavior of \( \Phi_z(x) \) as \( x \downarrow 0 \), we need the following result.

**Lemma 2.8.** As \( x \downarrow 0 \), we have

\[
0\varphi_1 \left( \frac{-\frac{z^2 q}{x}}{-qz/x} : q, -\frac{z^2 q}{x} \right) \longrightarrow 0\varphi_0 \left( \frac{-q}{qz} : q, z \right) = (z;q)_\infty,
\]

and the convergence is uniform for \( z \) in compact subsets of \( \mathbb{C} \setminus (-\infty,0) \).

**Proof.** Notice that

\[
0\varphi_1 \left( \frac{-\frac{z^2 q}{x}}{-qz/x} : q, -\frac{z^2 q}{x} \right) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2} z^{2n}}{(q;q)_n (x+qz) \cdots (x+qz^n)}
\]

for \( z \in \mathbb{C} \setminus (-\infty,0) \) and \( x > 0 \). The termwise convergence is thus obvious. Let \( K \) be a compact subset of \( \mathbb{C} \setminus (-\infty,0) \) and take \( \delta > 0 \) such that \( |z-t| \geq \delta \) for all \( z \in K \) and \( t < 0 \). Clearly,

\[
|(x+qz) \cdots (x+qz^n)| \geq \delta^n q^{\binom{n+1}{2}}
\]

and since the right-hand side is independent of \( z \in K \) and \( x > 0 \), we have dominated convergence. \( \square \)

A limit case of Heine’s transformation formula for the \( 2\varphi_1 \)-series \( \text{[IS] (0.6.8/9)} \) tells us that

\[
1\varphi_1 \left( \frac{1}{q} : q, -\frac{q}{x} \right) = (-q/xz; q)_\infty 0\varphi_1 \left( \frac{\frac{-q}{x}}{-q/x} : q, -\frac{q}{xz} \right)
\]

and according to Lemma 2.8, the \( 0\varphi_1 \)-series on the right-hand side converges to \((1/z; q)_\infty\) as \( x \downarrow 0 \). We follow the convention that in a fraction the part to the right of \( / \) is the denominator. So in (2.8), for example, we write \((-q/xz; q)_\infty\) instead of \((-\frac{q}{xz}; q)_\infty\). The infinite product \((-q/xz; q)_\infty\) does not have a limit as \( x \to 0 \), but for \( x = tq^n \) we have

\[
(-q/xz; q)_\infty = (-q^{1-n}/tz; q)_\infty = \frac{(-tz; q)_n (-q/tz; q)_\infty}{(tz)^n q(\frac{t}{z})}.
\]

### 3. Spectral Analysis for the Discrete Case

In this section we consider \( L \) as an unbounded symmetric operator on the Hilbert space \( L^2(\mu_t) \), where \( \mu_t \) is the discrete measure from Corollary 2.4 (i). Throughout the section the parameter \( t > 0 \) will be fixed.

#### 3.1. \( \ell^2(\mathbb{Z}) \) Setup.

Since \( L^2(\mu_t) \) essentially is a weighted \( \ell^2 \)-space over the integers, we start by defining a unitary operator \( U : L^2(\mu_t) \to \ell^2(\mathbb{Z}) \) by

\[
Uf = \sum_{k=-\infty}^{\infty} f(tq^k) \sqrt{m_t(k)} e_k,
\]

where \( \{e_k\}_{k \in \mathbb{Z}} \) denotes the standard orthonormal basis for \( \ell^2(\mathbb{Z}) \). The adjoint of \( U \) is given by

\[
(U^* e_k)(tq^r) = \frac{1}{\sqrt{m_t(k)}} \delta_{k,r}
\]

and the operator \( J = ULU^* \) becomes a doubly infinite Jacobi operator on \( \ell^2(\mathbb{Z}) \). More precisely, \( J \) has the form

\[
Je_k = a_k e_{k+1} + b_k e_k + a_{k-1} e_{k-1}, \quad k \in \mathbb{Z}
\]

with

\[
a_k = \frac{1}{\sqrt{tq^{k+1}}} \quad \text{and} \quad b_k = -\frac{1}{tq^k}.
\]
In what follows, we denote by $\mathcal{D}$ the subspace of $\ell^2(\mathbb{Z})$ consisting of finite linear combinations of the basis elements. Clearly, $(J, \mathcal{D})$ is a densely defined symmetric operator on $\ell^2(\mathbb{Z})$. But more importantly, we have the following result.

**Theorem 3.1.** The operator $(J, \mathcal{D})$ is essentially self-adjoint.

By the unitary intertwiner $U$, the operator $(J, \mathcal{D})$ corresponds to $(L, U^*DU)$ which is a restriction of the operator $(L, D(L))$ considered in Proposition 2.1. The domain $U^*DU$ consists of the compactly supported functions in $L^2(\mu)$, and it is straightforward to check that this is a core for the closure of $(L, D(L))$. So by the above theorem, $(L, D(L))$ is essentially self-adjoint in the case $\mu = \mu_t$.

**Proof.** We employ a theorem of Masson and Repka [22], see also [19, Thm. 4.2.2]. For this we define the operators

$$J^\pm := P^\pm J|_{\mathcal{D}^\pm},$$

where $P^+$ and $P^-$ are the orthogonal projections onto $\text{span}\{e_k \mid k \geq 0\}$, respectively $\text{span}\{e_k \mid k < 0\}$, and

$$\mathcal{D}^+ = \mathcal{D} \cap \text{span}\{e_k \mid k \geq 0\}, \quad \mathcal{D}^- = \mathcal{D} \cap \text{span}\{e_k \mid k < 0\}.$$

Notice that $J^\pm$ are Jacobi operators on $\ell^2(\mathbb{N})$ with finite linear combinations of the basis vectors as domain. The theorem of Masson and Repka states that the deficiency indices of $J$ can be obtained by adding the deficiency indices of $J^+$ and $J^-$, see e.g. Akhiezer [11, Ch. 4] or Berezanski˘ı [4, Ch. 7] for more information. The deficiency indices of $J^+$ and $J^-$ are $(0, 0)$ since the coefficients $a_k$ and $b_k$ are bounded as $k \to -\infty$. For the deficiency indices of $J^+$ we observe that $a_k + b_k + a_{k-1}$ is bounded from above for $k \geq 0$, and by [11, Addenda and problems to Chap. 1] or [41, Thm. 1.4, p. 505] this implies that $J^+$ is essentially self-adjoint. Hence, the deficiency indices of $J^+$ are $(0, 0)$ and we conclude that the deficiency indices of $J$ are also $(0, 0)$. The statement follows. \hfill \square

The closure of $(J, \mathcal{D})$ thus coincides with the adjoint operator $(J^*, \mathcal{D}^*)$, which is defined on the maximal domain

$$\mathcal{D}^* = \left\{ v \in \ell^2(\mathbb{Z}) : \sum_{k=-\infty}^{\infty} |a_kv_{k+1} + b_kv_k + a_{k-1}v_{k-1}|^2 < \infty \right\}.$$

### 3.2. Wronskian and Green function.

We now aim at finding the spectrum of the self-adjoint operator $(J^*, \mathcal{D}^*)$. In this connection the functions from Proposition 2.6 become very useful. We set

$$\psi_k(z) = t^{k/2} q^{(k+1)/4} \phi_z(tq^k),$$

respectively

$$\Psi_k(z) = t^{k/2} q^{(k+1)/4} \Phi_z(tq^k)/t^{\ln z/\ln q},$$

and consider the two sequences $\psi(z) = \{\psi_k(z)\}_{k \in \mathbb{Z}}$ and $\Psi(z) = \{\Psi_k(z)\}_{k \in \mathbb{Z}}$. Notice that $\psi(z)$ belongs to $\ell^2$ as $k \to \infty$ for all $z \in \mathbb{C}$, whereas $\Psi(z)$ belongs to $\ell^2$ as $k \to -\infty$ for $z \in \mathbb{C} \setminus \{0\}$. However, except for special values to be determined later on, neither $\psi(z)$ nor $\Psi(z)$ is an element of $\ell^2(\mathbb{Z})$. Since we divide by $t^{\ln z/\ln q}$ in the definition of $\Psi(z)$, the sequence $\Psi(z)$ is well-defined for all $z \in \mathbb{C} \setminus \{0\}$.

It follows from Proposition 2.6 that $\psi(z)$ and $\Psi(z)$ are solutions to the eigenvalue equation $Jv = zv$. Their Wronskian, i.e. the sequence defined by

$$[\psi(z), \Psi(z)]_k = a_k(\psi_{k+1}(z)\Psi_k(z) - \psi_k(z)\Psi_{k+1}(z)), \quad k \in \mathbb{Z}, \tag{3.1}$$

is therefore independent of $k$.

**Lemma 3.2.** The Wronskian of $\psi(z)$ and $\Psi(z)$ is given by

$$[\psi(z), \Psi(z)] = -z(-tzq, -1/tz, 1/z; q)_\infty.$$
Proof. Inserting the expressions for $a_k$, $\psi_k(z)$ and $\Psi_k(z)$ in \eqref{integral}, we get after a few computations

\[
[\psi(z), \Psi(z)] = z^k q^k q^{(k+1)} \left\{ \begin{array}{l}
1\varphi_1 \left( \frac{1}{z} ; q, -t z q^{k+2} \right) 1\varphi_1 \left( \frac{1}{z} ; q, -q^{1-k} \right) \\
- z 1\varphi_1 \left( \frac{1}{z} ; q, -t z q^{k+1} \right) 1\varphi_1 \left( \frac{1}{z} ; q, -q^{-k} \right) \end{array} \right. 
\]

Since the Wronskian is independent of $k$, we evaluate the expression by taking the limit $k \to \infty$. Clearly, the $1\varphi_1$-series with argument $-t z q^{k+2}$ (or $-t z q^{k+1}$) converges to 1 as $k \to \infty$. Combining \eqref{2.10} with Lemma 2.8 and \eqref{2.11}, we find that

\[
1\varphi_1 \left( \frac{1}{z} ; q, -q^{1-k} \right) \sim \frac{(-t z, -q/t z, 1/z; q)_\infty}{(t z)^k q^{(k+1)_2}} \text{ as } k \to \infty,
\]

respectively

\[
1\varphi_1 \left( \frac{1}{z} ; q, -q^{-k} \right) \sim \frac{(-t z, -q/t z, 1/z; q)_\infty}{(t z)^{k+1} q^{(k+1+1)_2}} \text{ as } k \to \infty,
\]

where $\sim$ means that the ratio of the right-hand side and the left-hand side converges to 1 as $k \to \infty$. Therefore,

\[
[\psi(z), \Psi(z)] = \lim_{k \to \infty} \left( q^k - 1/t \right) (-t z, -q/t z, 1/z; q)_\infty = -z (-t z q, -1/t z, 1/z; q)_\infty
\]

and the desired result is established. \hfill \Box

With the Wronskian of $\psi(z)$ and $\Psi(z)$ at hand, we define the Green function by

\[
G_z(j,l) = \frac{1}{[\psi(z), \Psi(z)]} \begin{cases} 
\psi_j(z) \Psi_l(z), & l \leq j, \\
\psi_l(z) \Psi_j(z), & l > j.
\end{cases}
\]

The resolvent of $(J^*, D^*)$ is closely related to the Green function, see e.g. \cite{19} Section 4.3. For any sequence $v \in \ell^2(\mathbb{Z})$, we have

\[
((J^* - z)^{-1} v)_j = \sum_{l=-\infty}^{\infty} G_z(j,l) v_l, \quad z \in \mathbb{C} \setminus \mathbb{R}.\tag{3.2}
\]

### 3.3. Spectral decomposition

We denote by $E$ the resolution of the identity corresponding to the self-adjoint operator $(J^*, D^*)$. From general theory (see e.g. \cite{14} Thm. XII.2.10) we know that

\[
\langle E((a,b)v, w) \rangle = \lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{a+i\delta}^{b-i\delta} \langle (J^* - s - i\epsilon)^{-1} v, w \rangle - \langle (J^* - s + i\epsilon)^{-1} v, w \rangle \, ds\tag{3.3}
\]

for $v, w \in \ell^2(\mathbb{Z})$ and because of \eqref{3.2}, the inner products in the integral can be written as

\[
\langle (J^* - (s + i\epsilon))^{-1} v, w \rangle = \sum_{l \leq j} \frac{\psi_j(s + i\epsilon) \Psi_l(s + i\epsilon)}{|\psi(s + i\epsilon)|^2} (v_l \overline{w}_j + v_j \overline{w}_l) (1 - \frac{1}{2} \delta_{j,l}).\tag{3.4}
\]

Since $\psi_k(z)$ is entire and $\Psi_k(z)$ is analytic in $\mathbb{C} \setminus \{0\}$, it therefore follows that the spectral measure is discrete and supported on the zeros of the Wronskian $[\psi(z), \Psi(z)]$. We can read off these zeros from Lemma 3.2 and get $0, -q^r/t$ for $r \in \mathbb{Z}$ and $q^n$ for $n \in \mathbb{Z}_+$. \hfill \Box

**Theorem 3.3.** The spectrum of $J^*$ is given by $\sigma(J^*) = -q^2/t \cup \{0\} \cup q^{2n}$. The accumulation point 0 does not belong to the point spectrum $\sigma_p(J^*)$. 


Proof. It is only left to prove that 0 does not belong to the point spectrum of $J^*$. We show that no non-trivial solution to the equation $Jv = 0$ belongs to $\ell^2(\mathbb{Z})$. In the end of the proof we use the implication $\phi_0(t) = 0 \Rightarrow \phi_0(tq) \neq 0$, which follows from the fact that the zeros of $\phi_0$ are very well separated, see e.g. [10, Section 3].

The space of solutions to the equation $a_k v_{k+1} + b_k v_k + a_{k-1} v_{k-1} = 0$ or, more explicitly,

$$v_{k+1} = \frac{1}{\sqrt{tq^{k-1}}} v_k - \sqrt{q} v_{k-1}, \quad k \in \mathbb{Z}$$

(3.5)

is two-dimensional. We already know one solution, namely $\psi(0)$, which is given by $\psi_k(0) = t^{k/2}q^{k(k+1)/4}\phi_0(tq^k)$, $k \in \mathbb{Z}$. Clearly $\psi(0)$ belongs to $\ell^2$ as $k \rightarrow \infty$ but recalling that $\phi_0(tq^{-2n}) \sim (-1)^n t^n q^{-n^2}K(t)$ as $n \rightarrow \infty$ for some constant $K(t) > 0$, see e.g. [14], it follows that $\psi_{-2n}(0) \sim (-1)^n q^{-n/2}K(t)$ as $n \rightarrow \infty$.

Therefore, $\psi(0)$ does not belong to $\ell^2(\mathbb{Z})$.

The sequence $\Psi(z)$ is not defined for $z = 0$ so we need to look for other solutions to (3.5). Note that if $v_k$ has the form

$$v_{k+1} = \frac{F_{k+1}}{t^{k/2}q^{k(k-1)/4}},$$

then (3.5) is equivalent to

$$F_{k+1} = F_k - tq^{k-1} F_{k-1}, \quad k \in \mathbb{Z}.$$

With $F_0 = 0$ and $F_1 = 1$ (or, equivalently, $v_0 = 0$ and $v_1 = 1$) we see that $F_k$, $k = 0, 1, \ldots$, essentially are $q$-Fibonacci polynomials in $t$, see e.g. [12]. In particular,

$$F_{k+1} = \sum_{n=0}^{k-1} \left[ \begin{array}{c} k-n \\ n \end{array} \right] q^n t^{k-n} \quad \text{and} \quad F_k \rightarrow \phi_0(t) \text{ as } k \rightarrow \infty.$$

There are two cases to be considered. 1) When $\phi_0(t) \neq 0$, the solution to (3.5) with $v_0 = 0$ and $v_1 = 1$ does not belong to $\ell^2$ as $k \rightarrow \infty$. Moreover, since this solution is linearly independent of $\psi(0)$, there are no solutions to (3.5) in $\ell^2(\mathbb{Z})$. 2) In the case $\phi_0(t) = 0$, the solution to (3.5) with $v_0 = 0$ and $v_2 = 1$ is linearly independent of $\psi(0)$. This solution behaves like $\phi_0(tq)/t^{k/2}q^{k(k-1)/4}$ as $k \rightarrow \infty$ and as before we see that no solution to (3.5) belongs to $\ell^2(\mathbb{Z})$. \hfill \Box

3.4. Orthogonality relations. In this section we determine the spectral measure $E(\{\xi\})$ for $\xi$ in the point spectrum of $J^*$. Our considerations will lead to explicit orthogonality relations for the Stieltjes–Wigert polynomials and the second $q$-Bessel functions of Jackson.

Along the way we will need the following auxiliary result.

Lemma 3.4. For $c \in \mathbb{C}$ and $k, m \in \mathbb{Z}$, we have

$$(-c)^{m+k} \varphi_1 \left( -cq^{-m} : q, q^{1+m+k} \right) = q^{m(m+k)} \varphi_1 \left( -cq^{-m} : q, q^{1-m-k} \right).$$

(3.6)

Proof. Because of symmetry it suffices to establish the identity for $m+k \geq 0$. Applying the transformation [13 (0.6.8/9)], we see that the right-hand side of (3.6) can be written as

$$q^{m(m+k)} \sum_{n=m+k}^{\infty} \frac{(q^{1-m-k+n}; q)_\infty (-c)^n q^{n(n-2m-k)}}{(q; q)_n} = (-c)^{m+k} \sum_{n=0}^{\infty} \frac{(q^{1+m+k+n}; q)_\infty (-c)^n q^{n(n+k)}}{(q; q)_n},$$

$$q^{m(m+k)} \sum_{n=m+k}^{\infty} \frac{(q^{1-m-k+n}; q)_\infty (-c)^n q^{n(n-2m-k)}}{(q; q)_n} = (-c)^{m+k} \sum_{n=0}^{\infty} \frac{(q^{1+m+k+n}; q)_\infty (-c)^n q^{n(n+k)}}{(q; q)_n},$$
which is exactly the left-hand side of (3.6). The special case \( c = -1 \) can also be obtained by reversing the order of summation.

From (3.3) and (3.4) it follows that
\[
\langle E(\{q^n\})v, w \rangle = -\frac{1}{2\pi i} \int_{\{q^n\}} \langle (J^* - s)^{-1}v, w \rangle \, ds
= -\frac{1}{2\pi i} \sum_{t \leq j} (w \bar{w}_j + v_j \bar{v}_l) (1 - \frac{1}{2} \delta_{j,l}) \int_{\{q^n\}} \frac{\psi_j(s)\Psi_l(s)}{[\psi(s), \Psi(s)]} \, ds.
\]
The integral on the right-hand side is given by
\[
-\frac{1}{2\pi i} \int_{\{q^n\}} \frac{\psi_j(s)\Psi_l(s)}{[\psi(s), \Psi(s)]} \, ds = \psi_j(q^n)\Psi_l(q^n) \text{ Res}_{z=q^n} \frac{1}{[\psi(z), \Psi(z)]}
\]
and by Lemma 3.4 (with \( c = -1 \)), we have \( \psi_k(q^n) = (-1)^n q^n q^{2} \Psi_k(q^n) \). Combining this with the fact that
\[
\text{Res}_{z=q^n} \frac{1}{[\psi(z), \Psi(z)]} = \frac{(-1)^{n+1} \nu n q^{n(n+1)}}{(q; q)_n} \frac{1}{(-tq, -1/t, q; q)_\infty},
\]
we end up with
\[
\langle E(\{q^n\})v, w \rangle = q^n \frac{\langle v, \psi(q^n) \rangle \langle \psi(q^n), w \rangle}{(q; q)_n (-tq, -1/t, q; q)_\infty}.
\]
In particular, it follows that
\[
\|\psi(q^n)\|^2 = \frac{(q; q)_n}{q^n} (-tq, -1/t, q; q)_\infty \quad \text{and} \quad \langle \psi(q^n), \psi(q^n) \rangle = \|\psi(q^n)\|^2 \delta_{m,n}
\] (3.7)
if we set \( v = w = \psi(q^n) \), respectively \( v = w = \psi(q^m) \).

In a similar way as above, one can show that
\[
\langle E(-q^r/t) v, w \rangle = \frac{q^r}{(-q/t; q)_r} \frac{\langle v, \psi(-q^r/t) \rangle \langle \psi(-q^r/t), w \rangle}{(-t, q, q; q)_\infty}.
\]
For by Lemma 3.4 we have \( \psi_k(-q^r/t) = (-1)^{r} q^{r^2 t^{-r}} \Psi_k(-q^r/t) \) and
\[
\text{Res}_{z=-q^r/t} \frac{1}{[\psi(z), \Psi(z)]} = \frac{(-1)^{r+1} q^{r(r+1)}}{t^r (-q/t; q)_r} \frac{1}{(-t, q, q; q)_\infty}.
\]
It thus follows that
\[
\langle \psi(-q^r/t), \psi(-q^r/t) \rangle = \frac{(-q/t; q)_r}{q^r} (-t, q, q; q)_\infty \delta_{r,s}.
\] (3.8)
Moreover, we clearly have
\[
\langle \psi(q^n), \psi(-q^r/t) \rangle = 0.
\] (3.9)

Recall now that the Stieltjes–Wigert polynomials are given by
\[
S_n(x; q) = \frac{1}{(q; q)_n} \phi_{q^n}(x)
\]
and consider also the functions \( M_{r}^{(t)}(x; q) \) defined by
\[
M_{r}^{(t)}(x; q) = \frac{1}{(q; q)_\infty} \phi_{-q^r/t}(x), \quad r \in \mathbb{Z}.
\]
These functions are closely related to the second $q$-Bessel function [15, Exerc. 1.24] defined by

$$J^{(2)}_{\nu}(z; q) = \frac{(z/2)^{\nu}}{(q; q)_{\infty}} \, \varphi_{1} \left( -\frac{z^2}{4}; q, q^{-1}; q, \frac{z^2}{4} \right).$$

Indeed, we have

$$t^{k+r} M^{(t)}_{r}(tq^{k}; q) = q^{r(r+k)/2} J^{(2)}_{k+r}(2\sqrt{tq}^{-r}; q).$$

It follows immediately from Proposition 2.6 that

$$LS_{n}(\cdot; q) = q^n S_{n}(\cdot; q) \quad \text{for } n \in \mathbb{Z}_{+}$$

and

$$LM_{r}(\cdot; q) = -\frac{q^{r}}{t} M^{(t)}_{r}(\cdot; q) \quad \text{for } r \in \mathbb{Z}.$$

Furthermore, since the spectral decomposition is unique, these eigenfunctions form an orthogonal basis for $L^{2}(\mu_{t})$. We put together the results from (3.7), (3.8) and (3.9) in the following theorem which is a formal limit transition of [12, Thm. 4.1].

**Theorem 3.5.** The Stieltjes–Wigert polynomials $S_{n}(x; q)$, respectively the $q$-Bessel functions $M^{(t)}_{r}(x; q)$, are orthogonal in $L^{2}(\mu_{t})$. The orthogonality relations are given by

$$\frac{1}{(tq, -1/t, q; q)_{\infty}} \sum_{k=-\infty}^{\infty} t^{k} q^{(k+1)/2} S_{n}(tq^{k}; q) S_{m}(tq^{k}; q) = \frac{\delta_{m,n}}{q^{n}(q; q)_{n}}$$

and

$$\frac{1}{(t, q)_{\infty}} \sum_{k=-\infty}^{\infty} t^{k} q^{(k+1)/2} M^{(t)}_{r}(tq^{k}; q) M^{(t)}_{s}(tq^{k}; q) = \frac{(-q/t; q)_{\infty}}{q^{r}} \delta_{r,s},$$

Moreover, $S_{n}(x; q)$ and $M^{(t)}_{r}(x; q)$ are mutually orthogonal in $L^{2}(\mu_{t})$, that is,

$$\sum_{k=-\infty}^{\infty} t^{k} q^{(k+1)/2} S_{n}(tq^{k}; q) M^{(t)}_{r}(tq^{k}; q) = 0 \quad \text{for all } n, r$$

and \{ $S_{n}(x; q)$ $\} \cup \{ M^{(t)}_{r}(x; q)$ $\}_{r \in \mathbb{Z}}$ form an orthogonal basis for $L^{2}(\mu_{t})$.

**Remark 3.6.** The orthogonality relation (3.10) is due to Chihara [8], whereas (3.11) is the Hansen–Lommel orthogonality relation for the second $q$-Bessel function, see [20, Thm. 3.1]. The above theorem contradicts [20, Thm. 3.3], and the flaw in the proof of [20, Thm. 3.3] is contained in [20, Lemma 3.4], where the unbounded operator $S$ as constructed there is not symmetric as claimed.

The statement in (3.12) can also be proved directly in the following way. Use [18, (0.6.8/9)] to write $M^{(t)}_{r}(x; q)$ as

$$M^{(t)}_{r}(x; q) = \frac{(xq^{r+1}/t; q)_{\infty}}{(q; q)_{\infty}} \varphi_{1} \left( -\frac{xq^{r+1}/t}{q, -xq} \right),$$

so that

$$\sum_{k=-\infty}^{\infty} t^{k} q^{(k+1)/2} S_{n}(tq^{k}; q) M^{(t)}_{r}(tq^{k}; q) = \sum_{k=-\infty}^{\infty} t^{k} q^{(k+1)/2} \frac{1}{(q; q)_{n}} \varphi_{1} \left( q^{n}; q, -tq^{k+n+1} \right) \frac{(q^{k+r+1}; q)_{\infty}}{(q; q)_{\infty}} \varphi_{1} \left( -\frac{q^{k+r+1}/q}{q, -tq^{k+1}} \right).$$
Because of absolute convergence we can interchange the order of summation to get
\[
\frac{1}{(q; q)_n} \sum_{m=0}^{n} \frac{(q^{-n}; q)_m (q^{m(n+1)} \sum_{l=0}^{\infty} \frac{(-1)^l q^{2l+1}}{(q; q)_l} \sum_{k=-\infty}^{\infty} (q^{k+r+l+1}; q)_\infty t^k q^{(k)_2} + k(m+l+1)).
\]

The inner sum (over \( k \)) reduces to
\[
\sum_{k=-r-l}^{\infty} (q^{k+r+l+1}; q)_\infty t^k q^{(k)_2} + k(m+l+1) = \frac{(q; q)_\infty q^{(r+1)_2}}{t^{r+l} q^{(r+l)(m+l)}} \sum_{k=0}^{\infty} t^k q^{(k)_2} + k(m+1-r)
\]
and the sum over \( l \) then becomes
\[
\sum_{l=0}^{\infty} (-1)^l q^{(l)_2} t^m = (q^{-m}; q)_\infty.
\]

Since \((q^{-m}; q)_\infty = 0\) for \( m \geq 0\), the relation (3.12) is established.

**Remark 3.7.** Using the explicit expression for \( M_r(t)(x; q) \) and Lemma 3.1, we see that \(|M_r(t)(tq^k; q)|\) is bounded by some constant, say \( M(r, t) \), for all \( k \in \mathbb{Z} \) provided \( t < q^r \). By the construction of Berg [6] it thus follows from Theorem 3.3 that the measure
\[
\nu_{s,t} = \frac{1}{(-tq, -1/t; q)_\infty} \sum_{k=0}^{\infty} t^k q^{(k+1)_2} \left( 1 + \frac{s}{M(r, t)} M_r(t)(tq^k; q) \right) \delta_{tq^k}
\]
is a solution to the Stieltjes–Wigert moment problem for all \(|s| \leq 1\) and \( t < q^r \).

### 4. Spectral analysis for the continuous case

We now work on the Hilbert space \( L^2(\mu) \), where \( \mu \) is the absolutely continuous measure from Corollary 2.3 (ii). The density of \( \mu \), which will be denoted \( w \), thus satisfies the functional equation
\[
w(xq) = xw(x), \quad x > 0.
\]
We remind the reader that a function \( g \) is called \( q \)-periodic if \( g(xq) = g(x) \) for all \( x > 0 \).

#### 4.1. Direct integral decomposition

Consider the Hilbert space \( l^2(\mathbb{Z}) \) equipped with its standard orthonormal basis \( \{e_k\}_{k \in \mathbb{Z}} \). For a compactly supported measurable function \( f \) on \((0, \infty)\) we define
\[
(q, 1] \ni t \mapsto (If)(t) = \sum_{k=0}^{\infty} f(tq^k) q^{k/2} \sqrt{w(tq^k)} e_k
\]
\[
= \sqrt{w(t)} \sum_{k=-\infty}^{\infty} f(tq^k) t^{k/2} q^{(k+1)/4} e_k \in l^2(\mathbb{Z}).
\]
Clearly, \((Igf)(t) = g(t)(If)(t)\) whenever \( g \) is a \( q \)-periodic function.

**Proposition 4.1.** The operator \( I \) defined in (4.2) extends to a unitary isomorphism
\[
I: L^2(\mu) \rightarrow \int_{\Omega} l^2(\mathbb{Z}) dt
\]
with \( \Omega = (q, 1] \cap \text{supp}(\mu) \).
Remark 4.2. The direct integral Hilbert space $\int_\Omega \ell^2(\mathbb{Z}) \, dt$ consists of all measurable functions $f : \Omega \to \ell^2(\mathbb{Z})$ with $\int_\Omega \|f(t)\|_{\ell^2(\mathbb{Z})} \, dt < \infty$. The term measurable means that $t \mapsto \langle f(t), e_k \rangle_{\ell^2(\mathbb{Z})}$ is measurable for all $k \in \mathbb{Z}$. In particular, the constant vector fields $t \mapsto e_j$ are measurable. The inner product on $\int_\Omega \ell^2(\mathbb{Z}) \, dt$ is given by

$$\langle f, g \rangle_{\int_\Omega \ell^2(\mathbb{Z}) \, dt} = \int_\Omega \langle f(t), g(t) \rangle_{\ell^2(\mathbb{Z})} \, dt$$

and we have $\int_\Omega \ell^2(\mathbb{Z}) \, dt \cong L^2(\Omega) \otimes \ell^2(\mathbb{Z})$ as Hilbert spaces. The space of all $t \mapsto g(t)e_j$, $g$ bounded measurable function on $\Omega$, is therefore dense in $\int_\Omega \ell^2(\mathbb{Z}) \, dt$. Notice that $t \mapsto (If)(t)$ as defined in (4.2) is measurable. See e.g. [13, Part II, Ch. 1] for more information.

Proof. For $f, g$ compactly supported functions in $L^2(\mu)$, we have

$$\langle If, Ig \rangle_{\int_\Omega \ell^2(\mathbb{Z}) \, dt} = \int_\Omega \langle (If)(t), (Ig)(t) \rangle_{\ell^2(\mathbb{Z})} \, dt = \int_\Omega \sum_{k=-\infty}^{\infty} f(tq^k)\overline{g(tq^k)}q^kw(tq^k) \, dt$$

where interchanging summation and integration is allowed since $f, g$ being compactly supported implies that the sum is finite. Moreover, we can switch from $\int_\Omega$ to $\int_q$ since $w$ satisfies the functional equation (4.1).

Recalling that the compactly supported measurable functions are dense in $L^2(\mu)$, the operator $I$ from (4.2) extends to an isometry $I : L^2(\mu) \to \int_\Omega \ell^2(\mathbb{Z}) \, dt$. Since the image of $I$ contains any element of the form $t \mapsto h(t)e_k$, $h$ bounded measurable function on $\Omega$, and these elements are dense in $\int_\Omega \ell^2(\mathbb{Z}) \, dt$, we conclude that $I : L^2(\mu) \to \int_\Omega \ell^2(\mathbb{Z}) \, dt$ is surjective and thus unitary.

The adjoint of the unitary operator $I$ is given explicitly by

$$I^*(t \mapsto \sum_{k=-\infty}^{\infty} h_k(t)e_k)(x) = \sum_{k=-\infty}^{\infty} \chi_{(q^{k+1},q^k)}(x) \frac{h_k(xq^{-k})}{q^{k/2}\sqrt{w(x)}}, \quad (4.3)$$

where $\chi_A$ denotes the indicator function of the set $A$. The right-hand side of (4.3) only makes sense when $w(x) > 0$, but there is no need to specify the value of a function in $L^2(\mu)$ at points where $w(x) = 0$. Formally calculating $I\phi_z$, with $\phi_z$ the eigenfunction of $L$ from Proposition 2.6 gives

$$\langle I\phi_z(t), \psi(z; t) \rangle = \sqrt{w(t)} \sum_{k=-\infty}^{\infty} \phi_z(tq^k)q^{-k/2}q^{k(k+1)/4}e_k = \sqrt{w(t)} \psi(z; t),$$

with $\psi(z; t)$ the formal, i.e. in general not contained in $\ell^2(\mathbb{Z})$, eigenvectors of $J_t$ as in Section 5.2. Conversely, by (4.3) we have for any function $f$ on $\Omega$ that

$$I^*(t \mapsto f(t) \sum_{k=-\infty}^{\infty} \phi_z(tq^k)q^{-k/2}q^{k(k+1)/4}e_k) = \text{Per}(f/\sqrt{w}) \phi_z,$$
where Per maps a function on \( \Omega \) to a \( q \)-periodic function on \( \text{supp}(\mu) \) such that they are equal on \( \Omega \), explicitly
\[
\text{Per}(f)(x) = \sum_{k=-\infty}^{\infty} \chi_{(q^{k+1},q^k]}(x) f(xq^{-k}). \quad (4.4)
\]

Recall from Section 3.1 the unbounded symmetric operator \((J_t,\mathcal{D})\) on \( \ell^2(\mathbb{Z}) \) defined by
\[
J_t e_k = a_k(t)e_{k+1} + b_k(t)e_k + a_{k-1}(t)e_{k-1}, \quad k \in \mathbb{Z}
\]
with
\[
a_k(t) = \frac{1}{\sqrt{tq^{k+1}}} \quad \text{and} \quad b_k(t) = -\frac{1}{tq^k}.
\]
Note that \( a_k \) and \( b_k \) are bounded continuous functions of \( t \in (q,1] \) for fixed \( k \in \mathbb{Z} \). It follows from Theorem 3.1 that \((J_t,\mathcal{D})\) is essentially self-adjoint, and we denote by \((J_t^*,\text{dom}(J_t^*))\) its unique self-adjoint extension.

Let \( L^2(\Omega) \otimes \mathcal{D} \) be the (algebraic) tensor product of the space \( L^2(\Omega) \) and the space \( \mathcal{D} \) of finite linear combinations of the basis vectors. By Remark 4.2 this tensor product is dense in \( \int_{\Omega} \ell^2(\mathbb{Z}) dt \) since it contains \( B(\Omega) \otimes \mathcal{D}, \) with \( B(\Omega) \) the space of bounded measurable functions on \( \Omega \). Observe that for \( h \otimes v \in L^2(\Omega) \otimes \mathcal{D} \), the field \( t \mapsto h(t)J_t v \) is measurable because the inner product
\[
t \mapsto \langle h(t)J_t v, e_k \rangle = h(t)\langle v, J_t e_k \rangle = h(t)(a_k(t)\langle v, e_{k+1} \rangle + b_k(t)\langle v, e_k \rangle + a_{k-1}(t)\langle v, e_{k-1} \rangle)
\]
is measurable for any \( k \in \mathbb{Z} \). Moreover, this inner product is only non-zero for finitely many values of \( k \), so the vector field \( t \mapsto h(t)J_t v \) is an element of \( \int_{\Omega} \ell^2(\mathbb{Z}) dt \). We now define \( \int_{\Omega} J_t dt \) as the operator with domain \( L^2(\Omega) \otimes \mathcal{D} \) mapping the element \( h \otimes v \) considered as the vector field \( t \mapsto h(t)v \) to \( t \mapsto h(t)J_t v \). Note that \( h \otimes v \) is identified with \( f \otimes v \) whenever \( f = h \text{ a.e. in } \Omega \).

**Proposition 4.3.** Consider \( L \) as an unbounded operator with domain the compactly supported functions in \( L^2(\mu) \). Then \( I \) intertwines \( L \) with \( J = \int_{\Omega} J_t dt \).

**Proof.** For \( f \) compactly supported, take \( N, M \in \mathbb{Z} \) such that \( \text{supp}(f) \subset (q^{N+1}, q^M] \) and identify
\[
(I_f)(t) = \sum_{k=N}^{M} f(tq^k)q^{k/2} \sqrt{w(tq^k)} e_k
\]
with \( \sum_{k=N}^{M} h_k \otimes e_k \in L^2(\Omega) \otimes \mathcal{D} \), where \( h_k(t) = f(tq^k)q^{k/2} \sqrt{w(tq^k)} \). Since
\[
\int_{\Omega} |h_k(t)|^2 dt = \int_{q^k}^{q^{k+1}} |f(x)|^2 w(x) dx < \infty,
\]
we have indeed \( h_k \in L^2(\Omega) \). So \( I \) maps the domain of \( L \) into \( L^2(\Omega) \otimes \mathcal{D} \). Conversely, \( I^* \) of an element \( h \otimes e_k \in L^2(\Omega) \otimes \mathcal{D} \) gives by (4.3) a compactly supported function on \((0,\infty)\) and
\[
\int_{0}^{\infty} |I^* (h \otimes e_k)(x)|^2 w(x) dx = \int_{\Omega} |h(t)|^2 dt < \infty.
\]
The intertwining property is a straightforward calculation. For \( f \in \text{dom}(L) \) and fixed \( t \in \Omega \), we have
\[
I(Lf)(t) = \sqrt{w(t)} \sum_{k=-\infty}^{\infty} \left( f(tq^{k+1}) - \frac{1}{tq^k} f(tq^k) + \frac{1}{tq^k} f(tq^{-k-1}) \right) t^{k/2} q^{k(k+1)/4} e_k
\]
\[
= \sqrt{w(t)} \sum_{k=-\infty}^{\infty} f(tq^k) \left( \frac{1}{\sqrt{tq^k}} e_{k-1} - \frac{1}{\sqrt{tq^{k+1}}} e_k + \frac{1}{\sqrt{tq^{k+1}}} e_{k+1} \right) t^{k/2} q^{k(k+1)/4} = J_t(I_f)(t).
\]
Note that the infinite sums only contain a finite number of non-zero terms, so that all rearrangements are valid. □

Since the operator \( L \) from Proposition 4.3 is symmetric and commutes with complex conjugation, it has a self-adjoint extension. We aim at finding its adjoint for which we want to give a direct integral representation. Because of Proposition 4.3 and the fact that each \((J^*_t, \text{dom}(J^*_t))\) is self-adjoint we consider the operator \( J^* = \int_{\Omega} J^*_t dt \). The next paragraph justifies this notation.

According to [23, Def. p. 283] we need to check that the field of operators \( t \mapsto (J^*_t + i)^{-1} \) is measurable, i.e. that \( t \mapsto \langle (J^*_t + i)^{-1} e_k, e_l \rangle_{\ell^2(\mathbb{Z})} \) is measurable for all \( k, l \in \mathbb{Z} \). By the functional calculus for \( J^*_t \) established in Section 3 we have

\[
\langle (J^*_t + i)^{-1} e_k, e_l \rangle_{\ell^2(\mathbb{Z})} = \int_{\mathbb{R}} \frac{1}{\lambda + i} dE_{e_k, e_l}(\lambda),
\]

where the right-hand side can be written as

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \frac{\langle e_k, \psi(q^n t) \rangle \langle \psi(q^n t), e_l \rangle}{\| \psi(q^n t) \|^2} + \sum_{r=-\infty}^{\infty} \frac{1}{i - q^r t} \frac{\langle e_k, \psi(-q^r t) \rangle \langle \psi(-q^r t), e_l \rangle}{\| \psi(-q^r t) \|^2}.
\]

The desired measurability hence follows. Now define

\[
dom(J^*) = \left\{ t \mapsto u(t) \in \int_{\Omega} \ell^2(\mathbb{Z}) dt \left| u(t) \in \text{dom}(J^*_t) \text{ a.e.}, \int_{\Omega} \| J^*_t u(t) \|^2 dt < \infty \right\},
\]

\[
J^* = \int_{\Omega} J^*_t dt : \text{dom}(J^*) \ni (t \mapsto u(t)) \mapsto (t \mapsto J^*_t u(t)).
\]

By [23, Thm. XIII.85, p. 284] the operator \( J^* = \int_{\Omega} J^*_t dt \) is the adjoint of \( J \) and \( J^* \) is self-adjoint. Moreover, the functional calculus is given by

\[
f(J^*) = f \left( \int_{\Omega} J^*_t dt \right) = \int_{\Omega} f(J^*_t)dt \tag{4.5}
\]

for any bounded measurable function \( f \) on \( \mathbb{R} \).

**Proposition 4.4.** The adjoint operator \((L^*, \text{dom}(L^*))\) is intertwined with \((J^*, \text{dom}(J^*))\) by the unitary isomorphism \( I \).

As an immediate consequence, we have

**Corollary 4.5.** \((L^*, \text{dom}(L^*))\) is the unique self-adjoint extension of \((L, \text{dom}(L))\), and for any bounded Borel function \( f \) on \( \mathbb{R} \) the functional calculus is given by

\[
f(L^*) = I^* \int_{\Omega} f(J^*_t)dt I.
\]

**Proof of Proposition 4.4.** The domain of \( L^* \) consists of all functions \( g \in L^2(\mu) \) such that

\[
f \mapsto \langle Lf, g \rangle_{L^2(\mu)} = \int_{\Omega} \langle I(Lf)(t), (Ig)(t) \rangle_{\ell^2(\mathbb{Z})} dt = \int_{\Omega} \langle J_t(If)(t), (Ig)(t) \rangle_{\ell^2(\mathbb{Z})} dt
\]

defines a continuous linear functional on \( \text{dom}(L) \). We have used Proposition 4.3 to replace \( I(Lf) \) with \( J_t(If) \) in the inner product on the right-hand side. So for \( g \in \text{dom}(L^*) \) there exists a constant \( C = C(g) > 0 \) such that

\[
|\langle Lf, g \rangle_{L^2(\mu)}| = \left| \int_{\Omega} \langle J_t(If)(t), (Ig)(t) \rangle_{\ell^2(\mathbb{Z})} dt \right| \leq C \| f \|_{L^2(\mu)} = C \left( \int_{\Omega} \| If(t) \|^2_{\ell^2(\mathbb{Z})} dt \right)^{1/2} \tag{4.6}
\]
Theorem 4.6. The spectrum of the self-adjoint operator $(L^*, \text{dom}(L^*))$ consists of point spectrum $q^{Z^+}$, each point having infinite multiplicity, and continuous spectrum $\bigcup_{l \in \mathbb{Z}} \mathcal{G}_l$, where $\mathcal{G}_l = \{-q/l \mid t \in \mathbb{Q}\}$. In particular, we have $\sigma(L^*) = (-\infty, 0] \cup q^{Z^+}$ when $\Omega = (q, 1]$.

Proof. The theorem follows from [23, Thm. XIII.85] and Proposition 4.3. We only need to consider the point 0 which is in the closure of $q^{Z^+}$ and in the closure of $\bigcup_{l \in \mathbb{Z}} \mathcal{G}_l$. Since $(L^*, \text{dom}(L^*))$ is self-adjoint, 0 is either in the point spectrum or in the continuous spectrum. In case 0 is in the point spectrum, it is also contained in the point spectrum of $(J^*, \text{dom}(J^*))$, so there exists a non-trivial function $t \mapsto v(t)$ such that $J^* v(t) = 0$ a.e. on $\Omega$. By Theorem 3.3, however, the point 0 is not contained in the point spectrum of $(J^*, \text{dom}(J^*))$ for any $t \in \mathcal{G}_l$, so $v(t) = 0$ a.e. and 0 belongs to the continuous spectrum.

4.2. Spectral decomposition for $L^*$. We start this section by presenting the spectrum of $L^*$.
In order to make Theorem 4.6 more explicit we establish the corresponding spectral decomposition. Following the ideas of the proof of [23, Thm. XIII.86] we define
\[ \mathcal{H}^+_n = \left\{ v \in \int_{\Omega} \ell^2(\mathbb{Z})dt \mid v(t) = f(t) \frac{\psi(q^n t)}{N_{q^n}(t)} \text{ for some } f \in L^2(\Omega) \right\}, \quad n \in \mathbb{Z}_+ \tag{4.8} \]
and
\[ \mathcal{H}^-_r = \left\{ v \in \int_{\Omega} \ell^2(\mathbb{Z})dt \mid v(t) = f(t) \frac{\psi(-q^r t)}{N_{-q^r}(t)} \text{ for some } f \in L^2(\Omega) \right\}, \quad r \in \mathbb{Z}, \tag{4.9} \]
using the notation \( N_\xi(t) = \|\psi(\xi t)\|_{\ell^2(\mathbb{Z})} \) for \( \xi \) in the point spectrum of \( J^*_t \). Then \( \mathcal{H}^+_n, \mathcal{H}^-_r \) are mutually orthogonal closed subspaces of \( \int_{\Omega} \ell^2(\mathbb{Z})dt \) and, moreover,
\[ \int_{\Omega} \ell^2(\mathbb{Z})dt = \mathcal{H}^+ \oplus \mathcal{H}^-, \quad \text{with } \mathcal{H}^+ = \bigoplus_{n=0}^{\infty} \mathcal{H}^+_n \text{ and } \mathcal{H}^- = \bigoplus_{r=-\infty}^{\infty} \mathcal{H}^-_r. \]
Note that the subspaces \( \mathcal{H}^\pm \) are contained in \( \text{dom}(J^*) \) and \( J^* \) preserves each of them. By \( U^\pm_t : \mathcal{H}^\pm \to L^2(\Omega) \) we denote the unitary operator defined by \( U^\pm_t v = f \) for \( v \in \mathcal{H}^\pm \) of the form as in (4.8) or (4.9). It follows that \( U^\pm_t \) intertwines \( J^* \) with multiplication by \( \lambda^\pm_t \) on \( L^2(\Omega) \), where \( \lambda^+_t(t) = q^t \) and \( \lambda^-_t(t) = -q^t/t \). We put \( J^\pm_t = U^\pm_t J^* (U^\pm_t)^* \) so that \( J^\pm_t f = \lambda^\pm_t f \) for all \( f \in L^2(\Omega) \). In particular, it follows that \( \ker(J^* - q^t) = \mathcal{H}^+_n \) so that \( q^\mathcal{H}^+_n \) is contained in the point spectrum of \( J^* \), and each point of this form has infinite multiplicity.

For the case of negative eigenvalues we define \( \tilde{\Omega}_t = \{-q^t/t \mid t \in \Omega\} \subseteq (-q^{-1}, -q^t] \) for \( t \in \mathbb{Z} \). Then \( V_t : L^2(\Omega) \to L^2(\tilde{\Omega}_t) \) given by
\[ (V_t f)(\lambda) = \frac{q^{t/2}}{|\lambda|} f(-q^t/\lambda), \quad \lambda \in \tilde{\Omega}_t \]
is a unitary operator and its adjoint \( V^*_t \) is almost given by the same formula,
\[ (V^*_t g)(t) = \frac{q^{t/2}}{t} g(-q^t/t), \quad t \in \Omega. \]
By a straightforward calculation we see that
\[ (V^*_t J^*_t V_t^* g)(\lambda) = \lambda g(\lambda), \quad \lambda \in \tilde{\Omega}_t \tag{4.10} \]
for any \( g \in L^2(\tilde{\Omega}_t) \). It thus follows that \( \tilde{\Omega} = \cup_{t \in \mathbb{Z}} \tilde{\Omega}_t \subseteq (-\infty, 0] \) is contained in the continuous spectrum of \( J^* \), and this part of the spectrum is simple. Using the notation \( E(T|A) \) for the spectral projection corresponding to the Borel set \( A \subset \mathbb{R} \) for a (possibly unbounded) self-adjoint operator \( T \), we see that \( E(V_t J^-_t V_t^*|A) \) is just multiplication by the characteristic function \( \chi_{A \cap \tilde{\Omega}_t} \). Tracing the steps back it follows that
\[ E(J^*_t|\mathcal{H}^-) v(t) = \chi_{A \cap \tilde{\Omega}_t}(-q^t/t) v(t), \]
with the notation as in (4.8) and (4.9). By considering \( J^* \) restricted to \( \mathcal{H}^- \), we see that \( \sigma(J^*|\mathcal{H}^-) = \cup_{t \in \mathbb{Z}} \tilde{\Omega}_t \).

To obtain the spectral decomposition \( E \) of \( (L^*, \text{dom}(L^*)) \) we use Proposition 4.4 and Theorem 4.6. The idea is to get the results from the spectral decomposition for \( J^* \) using the unitary isomorphism \( T \). First we consider the spectral decomposition corresponding to the point spectrum \( \sigma_p(L^*) \). It follows that \( L^* \) preserves \( I^* \mathcal{H}^+_n \) and
\[ \text{ran}(E\{q^n\}) = I^* \mathcal{H}^+_n = \{ \text{Per}(f/\sqrt{w}) \cdot s_n \mid f \in L^2(\Omega) \}, \]
where \( s_n \) is the orthonormal Stieltjes–Wigert polynomial of degree \( n \). Note that by the functional equation (4.1), we have

\[
\text{Per}\left(f/\sqrt{w}\right)(x) = \frac{(Pf)(x)}{\sqrt{w(x)}}, \quad (Pf)(x) = \sum_{k=-\infty}^{\infty} \chi_{(q^{k+1},q^k]}(x) x^{k/2}q^{-k(k+1)/4}f(xq^{-k})
\]

and \((Pf)(xq) = \sqrt{x}(Pf)(x)\). In particular, by taking any orthonormal basis \( \{f_j\}_{j \in \mathbb{N}} \) of \( L^2(\Omega) \) we obtain from the orthonormality of \( t \mapsto f_j(t)\psi(q^j; t)/N_{q^j}(t) \) in \( \mathcal{H}^+ \) and the unitarity of \( I \) the orthogonality relations

\[
\int_{\supp(\mu)} (Pf_i)(x)(Pf_j)(x) s_n(x)s_m(x)dx = \delta_{n,m}\delta_{i,j}.
\]

(4.11)

The special case \( i = j \) tells us that the Stieltjes–Wigert polynomials are orthogonal with respect to any absolutely continuous measure whose density satisfies the functional equation (4.1). This result is also obtained in [10, Prop. 2.1].

To sum up, we denote by \( P\text{Pol} \subset L^2(\mu) \) the closure of the space of functions of the form \( \sum f_np_n \in L^2(\mu) \), with \( f_n \) a \( q \)-periodic function and \( p_n \) a polynomial. It follows that \( P\text{Pol} = I^*\mathcal{H}^+ \subset \text{dom}(L^*) \) and \( L^*|_{P\text{Pol}} \) is a bounded linear operator on \( P\text{Pol} \) with spectrum \( q^{2\mathbb{Z}} + \{0\} \).

We now take a closer look at the spectral decomposition corresponding to the continuous spectrum of \( L^* \). For any Borel set \( A \subset (-q^{-1}, -q) \) we have \( E(A)I^*\mathcal{H}^+_r = \{0\} \) unless \( r = l \). Since \( E(A)F = I^*(J^*|A)IF \) for \( F \in L^2(\mu) \) with compact support, it thus follows that

\[
E(J^*|A)(IF)(t) = \chi_{A \cap \Phi_l}(-q^l/t) \frac{\langle (IF)(t), \psi(-q^l/t; t) \rangle_{L^2(\mathbb{Z})}}{N_{-q^l/t}(t)} \frac{\psi(-q^l/t; t)}{N_{-q^l/t}(t)}.
\]

Calculating \( I^* \) on \( \mathcal{H}^+_r \) gives

\[
I^* \left( t \mapsto f(t) \frac{\psi(-q^l/t; t)}{N_{-q^l/t}(t)} \right)(x) = I^* \left( t \mapsto \frac{f(t)}{N_{-q^l/t}(t)} \sum_{k=-\infty}^{\infty} t^{k/2}q^{k(k+1)/4}\phi_{-q^j/t}(tq^k)e_k \right)(x)
\]

\[
= \sum_{k=-\infty}^{\infty} \chi_{(q^{k+1},q^k]}(x) \frac{f(xq^{-k})x^{k/2}}{N_{q^j+k/x}(xq^{-k})} q^{-k(k+1)/4} \phi_{q^{-k}/x}(x),
\]

so when \( f \) has the form

\[
f(t) = \chi_{A \cap \Phi_l}(-q^l/t) \frac{\langle (IF)(t), \psi(-q^l/t; t) \rangle_{L^2(\mathbb{Z})}}{N_{-q^l/t}(t)},
\]

we obtain for \( G \in L^2(\mu) \) with compact support that

\[
\langle E(A)F, G \rangle_{L^2(\mu)} = \int_0^\infty (I^*E(J^*|A)IF)(x) G(x)w(x)dx
\]

\[
= \int_0^\infty \sum_{k=-\infty}^{\infty} \chi_{(q^{k+1},q^k]}(x) \frac{\chi_A(-q^{l+k}/x)}{N_{q^j+k/x}(xq^{-k})} x^{k/2}q^{-k(k+1)/4} \phi_{q^{-k}/x}(x)
\]

\[
\times \langle (IF)(xq^{-k}), \psi(-q^{l+k}/x; xq^{-k}) \rangle_{L^2(\mathbb{Z})} G(x)\sqrt{w(x)}dx.
\]
Expanding the inner product in the integrand, the integral can be written as

\[
\sum_{j,k=-\infty}^{\infty} \int_{-\infty}^{\infty} q^k \chi_A(-q^{l+k}/x) \phi_{-q^{l+k}/x}(x) \phi_{-q^{l+k}/x}(xq^{-k}) N_{-q^{l+k}/x}(xq^{-k})^2 x^2 q^{l+j+1-k} j \int \phi_{\lambda}(-q^{l+j}/\lambda) \phi_{\lambda}(-q^{l+k}/\lambda) N_{\lambda}(-q^{l}/\lambda)^2 \lambda^{j+k} F(q^{l-j}/\lambda)G(q^{l+k}/\lambda) w(q^{l}/\lambda) q^{l} \lambda \ d\lambda,
\]

using the functional equation (4.11), switched to \( \lambda = -q^{l+k}/x \). Note that

\[
\sum_{j=-\infty}^{\infty} F(-q^{l+j}/\lambda)(-q^{l}/\lambda)^{j} q^{l+j+1}/\lambda) \phi_{\lambda}(-q^{l+j}/\lambda) = \frac{(-\lambda)^j}{q^{l+j+1}} \sum_{j=-\infty}^{\infty} F(-q^{l}/\lambda)(-\lambda)^{-j} q^{l+j+1}/\lambda) \phi_{\lambda}(-q^{l}/\lambda).
\]

and define

\[
(FF)(\lambda) = \sum_{j=-\infty}^{\infty} F(-q^{l}/\lambda)(-\lambda)^{-j} q^{l+j+1}/\lambda) \phi_{\lambda}(-q^{l}/\lambda).
\]

By means of (1.13) we can write (4.12) as

\[
\langle E(A)F, G \rangle_{L^2(\mu)} = \int_A (FF)(\lambda)(FG)(\lambda) \lambda^{2l} q^{-l(l+1)} \frac{q^l w(-q^l/\lambda)}{\lambda^2 N_{\lambda}(-q^l/\lambda)^2} d\lambda
\]

using the functional equation (4.11) once more. Now define

\[
\nu(\lambda) = \sum_{l=-\infty}^{\infty} \chi_{(-q^{l+1},-q^l]}(\lambda) |\lambda|^l q^{-l(l+1)/2} N_{\lambda}(-q^l/\lambda)^2.
\]

and use (4.14) to obtain

\[
\langle E(A)F, G \rangle_{L^2(\mu)} = \int_A (FF)(\lambda)(FG)(\lambda) \nu(\lambda) w(-1/\lambda) d\lambda
\]

for an arbitrary Borel set \( A \subset (-\infty, 0) \). It follows that the complex measure \( \langle E(A)F, G \rangle_{L^2(\mu)} \) is absolutely continuous with respect to the Lebesgue measure on \((0, \infty)\), and for any \( F, G \in \mathcal{H}^{-} \) we have

\[
\langle F, G \rangle_{L^2(\mu)} = \int_{-\infty}^{0} (FF)(\lambda)(FG)(\lambda) \nu(\lambda) w(-1/\lambda) d\lambda.
\]

Taking into account the discrete spectrum of \( L^* \) on the space \( \text{PPol} \) as well, we obtain the following Plancherel type theorem.
Theorem 4.7. Consider an absolutely continuous positive measure $\mu$ on $(0, \infty)$ with density $w$ satisfying the functional equation \((4.1)\). Let $\Omega = (q, 1] \cap \text{supp}(\mu)$ and suppose that $\{f_i\}_{i=0}^{\infty}$ is an arbitrary fixed orthonormal basis of $L^2(\Omega)$. For all $F, G \in L^2(\mu)$, we have the Plancherel equality
\[
\int_0^\infty F(x)G(x)w(x)dx = \sum_{i,n=0}^{\infty} F_{in} \overline{G_{in}} + \int_{-\infty}^0 (FF)(\lambda)(FG)(\lambda) \nu(\lambda)w(-1/\lambda) \frac{d\lambda}{\lambda^2},
\]
where
\[
F_{in} = \int_0^\infty F(x) \text{Per}(f_i/\sqrt{w})(x)s_n(x)w(x)dx
\]
and $F$, respectively $\nu$, are defined in \((4.13)\) and \((4.15)\).

We can rewrite the above result in terms of a corresponding transform. Consider the Hilbert space
\[
\mathcal{K} = \ell^2(\mathbb{Z}_+ \times \mathbb{Z}_+) \oplus L^2((-\infty, 0), \nu(\lambda)w(-1/\lambda) \frac{d\lambda}{\lambda^2})
\]
and define
\[
(F^*g)(x) = \sum_{i,n=0}^{\infty} g_{in} \text{Per}(f_i/\sqrt{w})(x)s_n(x) + \sum_{j=-\infty}^{\infty} g(-q^j/x) \phi_{-q^j/x}(x)\nu(-q^j/x), \quad x > 0
\]
for compactly supported functions $g \in \mathcal{K}$. If we consider $F$ as defined in \((4.13)\) as $F : I^*H^- \to L^2((-\infty, 0), \nu(\lambda)w(-1/\lambda) \frac{d\lambda}{\lambda^2})$ and extend it to an operator $F : L^2(\mu) \to \mathcal{K}$ by defining $F : I^*H^+ \to \ell^2(\mathbb{Z}_+ \times \mathbb{Z}_+)$ by $FF = \{F_{in}\}_{i,n \in \mathbb{Z}_+}$ with $F_{in}$ as in Theorem 4.7 then we have the following result.

Corollary 4.8. $F : L^2(\mu) \to \mathcal{K}$ is a unitary isomorphism with adjoint given by \((4.18)\).

References

[1] Naum Ilyich Akhiezer, *The classical moment problem and some related questions in analysis*, Translated by N. Kemmer, Hafner Publishing Co., New York, 1965.

[2] Renato Álvarez-Nodarse and Juan Carlos Medem, *q-classical polynomials and the q-Askey and Nikiforov-Uvarov tableaus*, J. Comput. Appl. Math. 135 (2001), no. 2, 197–223.

[3] George E. Andrews, *Ramanujan’s “lost” notebook. VIII: The entire Rogers-Ramanujan function*, Adv. Math. 191 (2005), no. 2, 393–407.

[4] Yurij M. Berezans’kiï, *Expansions in eigenfunctions of selfadjoint operators*, Translated from the Russian by R. Bolstein, J. M. Danskin, J. Rosnyak and L. Shulman. Translations of Mathematical Monographs, Vol. 17, American Mathematical Society, Providence, R.I., 1968.

[5] Christian Berg, *From discrete to absolutely continuous solutions of indeterminate moment problems*, Arab J. Math. Sci. 4 (1998), no. 2, 1–18.

[6] ____, *On some indeterminate moment problems for measures on a geometric progression*, J. Comput. Appl. Math. 99 (1998), no. 1, 67–75.

[7] Leonard Carlitz, *Fibonacci notes. IV. q-Fibonacci polynomials*, Fibonacci Quart. 13 (1975), 97–102.

[8] Theodore Seio Chihara, *A characterization and a class of distribution functions for the Stieltjes-Wigert polynomials*, Canad. Math. Bull. 13 (1970), 529–532.

[9] Jacob Stordal Christiansen, *The moment problem associated with the q-Laguerre polynomials*, Constr. Approx. 19 (2003), no. 1, 1–22.

[10] ____, *The moment problem associated with the Stieltjes-Wigert polynomials*, J. Math. Anal. Appl. 277 (2003), no. 1, 218–245.

[11] ____, *Indeterminate moment problems within the Askey-scheme*, Ph.D. thesis, University of Copenhagen (2004), http://www.math.ku.dk/~stordal/thesis.pdf

[12] Nicola Ciccoli, Erik Koelink, and Tom H. Koornwinder, *q-Laguerre polynomials and big q-Bessel functions and their orthogonality relations*, Methods Appl. Anal. 6 (1999), no. 1, 109–127.

[13] Jacques Dixmier, *von Neumann algebras*, North-Holland Mathematical Library, vol. 27, North-Holland Publishing Co., Amsterdam, 1981, Translated from the second French edition by F. Jellett.
[14] Nelson Dunford and Jacob T. Schwartz, *Linear operators. Part II*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1988, Spectral theory. Selfadjoint operators in Hilbert space, Reprint of the 1963 original, A Wiley-Interscience Publication.

[15] George Gasper and Mizan Rahman, *Basic hypergeometric series*, 2nd ed., Encyclopedia of Mathematics and its Applications, vol. 96, Cambridge University Press, Cambridge, 2004.

[16] Walter K. Hayman, *On the zeros of a q-Bessel function*, to appear in Contemp. Math.

[17] Mourad E. H. Ismail, *On Jackson's third q-Bessel function and q-exponentials*, Preprint (2000).

[18] Roelof Koekoek and René F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Tech. Report no. 98-17, TU–Delft (1998).

[19] Erik Koelink, *Spectral theory and special functions*, Laredo Lectures on Orthogonal Polynomials and Special Functions, Adv. Theory Spec. Funct. Orthogonal Polynomials, Nova Sci. Publ., Hauppauge, NY, 2004, pp. 45–84.

[20] H. T. Koelink, *Hansen-Lommel orthogonality relations for Jackson’s q-Bessel functions*, J. Math. Anal. Appl. 175 (1993), no. 2, 425–437.

[21] Francisco Marcellán and Juan Carlos Medem, *q-classical orthogonal polynomials: a very classical approach*, Electron. Trans. Numer. Anal. 9 (1999), 112–127 (electronic), Orthogonal polynomials: numerical and symbolic algorithms (Leganés, 1998).

[22] David R. Masson and Joe Repka, *Spectral theory of Jacobi matrices in $l^2(\mathbb{Z})$ and the su(1,1) Lie algebra*, SIAM J. Math. Anal. 22 (1991), no. 4, 1131–1146.

[23] Michael Reed and Barry Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.

Katholieke Universiteit Leuven, Departement Wiskunde, Celestijnenlaan 200B, B-3001 Leuven, Belgium

E-mail address: stordal@wis.kuleuven.ac.be

Technische Universiteit Delft, DIAM, PO Box 5031, 2600 GA Delft, the Netherlands

E-mail address: h.t.koelink@ewi.tudelft.nl