High order PT calculations for heavy quarks near threshold

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Abstract

Results of analytical calculations for heavy quark systems in higher orders of perturbation theory are overviewed. I discuss baryons with one finite mass quark in the next-to-leading order and heavy quark pair production near the threshold within NRQCD.

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1 Introduction.

Recent years have witnessed some progress in theoretical description of heavy quark systems near the threshold which will be experimentally studied at future colliders [1]. It is mostly a technical progress within general approach of effective theories [2]. The computational difficulties of high order PT calculations have led to the necessity of introducing some approximations realized in the framework of the effective theories. Processes with a low momentum transfer for the particles containing one heavy quark are well approximated by HQET [3]. This approach was quite successful in describing transitions between heavy particles containing $b$- and $c$-quarks [4]. For the systems of two heavy particles near the threshold the effective theory is NRQCD [5]. Recent result obtained in its framework is the completion of NNLO analysis of heavy quark production e.g. [6]. In my talk I briefly present the results of computing perturbative corrections to physical quantities relevant for description of baryons with one heavy quark and of the heavy quark pair production near the threshold.

2 Baryons with one heavy quark.

Properties of the spectrum of heavy baryons can be obtained from the analysis of the correlator of two baryonic currents. Calculations with massive quarks are basically done in the leading order only. The NLO corrections in the massless case have been known since long ago [7]. The generic baryonic current is a local three-quark operator $j = \epsilon^{abc}(u^T_b C_d)\Gamma\Psi$, where $\Psi$ is a heavy quark with the mass $m$, $u,d$ are massless quarks, $C$ is the charge conjugation matrix, $\Gamma$ is a Dirac matrix. $\epsilon^{abc}$ is the totally antisymmetric tensor, $a,b,c$ are indices for the $SU(3)$ color group. Both functions $\Pi_{q,m}(q^2)$ of the correlator of two baryonic currents (for simplicity we take $\Gamma = 1$)

$$i \int \langle Tj(x)\bar{j}(0)\rangle e^{iqx}dx = \gamma_i q^\nu \Pi_q(q^2) + m \Pi_m(q^2)$$

are known up to NLO of PT (three-loop diagrams). I shall present results only for the function $\Pi_m(q^2)$ [8]. For the spectral density $\rho(s)$ in the dispersion relation

$$\Pi_m(q^2) = \frac{1}{128\pi^4} \int_{m^2}^{\infty} \frac{\rho(s)ds}{s-q^2}$$

we write

$$\rho(s) = s^2 \left( \rho_0(s)(1 + \frac{\alpha_s}{\pi} \ln \frac{\mu^2}{m^2}) + \frac{\alpha_s}{\pi} \rho_1(s) \right) \cdot$$

Here $\mu$ is the scale parameter, $\alpha_s = \alpha_s(\mu)$, $m$ is a pole mass of heavy quark [9]. The leading order term reads

$$\rho_0(s) = 1 + 9z - 9z^2 - z^3 + 6z(1+z) \ln z$$
with \( z = m^2/s \). In the \( \overline{\text{MS}} \)-scheme the NLO term is given by

\[
\rho_1(s) = 9 + \frac{665}{9} z - \frac{665}{9} z^2 - 9 z^3
- \left( \frac{58}{9} + 42 z - 42 z^2 - \frac{58}{9} z^3 \right) \ln(1 - z) + \left( 2 + \frac{154}{3} z - \frac{22}{3} z^2 - \frac{58}{9} z^3 \right) \ln z
+ 4 \left( \frac{1}{3} + 3z - 3z^2 - \frac{1}{3} z^3 \right) \ln(1 - z) \ln z + 12 z \left( 2 + 3z + \frac{1}{9} z^2 \right) \left( \frac{1}{2} \ln^2 z - \zeta(2) \right)
+ 4 \left( \frac{2}{3} + 12z + 3z^2 - \frac{1}{3} z^3 \right) \text{Li}_2(z) + 24z(1 + z) \left( \text{Li}_3(z) - \zeta(3) - \frac{1}{3} \text{Li}_2(z) \ln z \right)
\]

where \( \text{Li}_n(z) \) are polylogarithms and \( \zeta(n) \) is Riemann’s zeta function. The diagrams have generalized sunset topology which has been recently studied in some detail \[10\]. Such a topology of diagrams is rather frequent in phenomenological applications \[11\]. Since the anomalous dimension of the baryonic current is known to two-loop order \[12\] the result eq. (5) completes the NLO analysis.

Two limiting cases of interest are the near-threshold and high-energy asymptotics. With eq. (5) both limits can be taken explicitly. Effective theories can be viewed as special devices for calculations in such limiting cases.

In the high energy (small mass) limit \( z \to 0 \) the correction reads

\[
\rho_1(s) = 9 + 83 z - 4 \pi^2 z + 2 \ln(z) + 50 z \ln(z) + 12 z \ln(z)^2 - 24 z \zeta(3) + O(z^2).
\]

This leads to the small mass expansion of the spectral density in the form

\[
m \rho(s) = m_{\overline{\text{MS}}} (\mu) \rho_{\text{massless}} (s) = m_{\overline{\text{MS}}} (\mu) s^2 \left( 1 + \frac{\alpha_s}{\pi} \left( 2 \ln \frac{\mu^2}{s} + \frac{31}{3} \right) \right)
\]

where \( \rho_{\text{massless}} (s) \) is the result of calculating the correlator in the effective theory of massless quarks. The relation between the pole mass \( m \) and the \( \overline{\text{MS}} \) mass \( m_{\overline{\text{MS}}} (\mu) \) reads

\[
m = m_{\overline{\text{MS}}} (\mu) \left( 1 + \frac{\alpha_s}{\pi} \left( \ln \frac{\mu^2}{m^2} + \frac{4}{3} \right) \right).
\]

The massless effective theory cannot reproduce the mass singularities – terms like \( z \ln(z) \) in eq. (6). Within the massless effective theory such contributions can be parametrized with condensates of local operators. The first \( m^2 \) correction in eq. (6) can be found if the perturbative value of the heavy quark condensate \( \langle \overline{\Psi} \Psi \rangle \) taken from the full theory is added. The vacuum expectation value \( \langle \overline{\Psi} \Psi \rangle \) of the composite operator \( \langle \overline{\Psi} \Psi \rangle \) should be understood within a mass independent renormalization scheme such as the \( \overline{\text{MS}} \)-scheme. This value (perturbative, \( \langle \overline{\Psi} \Psi \rangle \sim m^3 \ln(\mu^2/m^2) \)) cannot be computed within the effective theory of massless quarks. It represents the proper matching between the perturbative expressions for the correlators of the full (massive) and effective (massless) theories. This matching procedure allows one to restore
higher order terms of the mass expansion in the full theory from the effective massless theory with the mass term treated as a perturbation.

In the near-threshold limit \( E \to 0 \) with \( s = (m + E)^2 \) one explicitly obtains

\[
\rho_{th}(m, E) = \frac{16E^5}{5m} \left( 1 + \frac{\alpha_s}{\pi} \ln \frac{\mu^2}{m^2} + \frac{\alpha_s}{\pi} \left( \frac{54}{5} + \frac{4\pi^2}{9} + 4 \ln \frac{m}{2E} \right) \right) + O(E^6). \tag{9}
\]

In this region the appropriate device to compute the limit of the correlator is HQET. Writing

\[
m\rho_{th}(m, E) = C\left(\frac{m}{\mu}, \alpha_s\right) \rho_{HQET}(E, \mu) \tag{10}
\]

we obtain the known result for \( \rho_{HQET}(E, \mu) \) \[13\]

\[
\rho_{HQET}(E, \mu) = \frac{16E^5}{5} \left( 1 + \frac{\alpha_s}{\pi} \left( \frac{182}{15} + \frac{4\pi^2}{9} + 4 \ln \frac{\mu}{2E} \right) \right) + O(E^6) \tag{11}
\]

with matching coefficient \( C(m/\mu, \alpha_s) \) \[14\]

\[
C(m/\mu, \alpha_s) = 1 + \frac{\alpha_s}{\pi} \left( \frac{1}{2} \ln \frac{m^2}{\mu^2} - \frac{2}{3} \right). \tag{12}
\]

The matching procedure allows one to restore the near-threshold limit of the full QCD correlator \[9\] starting from the result obtained in a simpler HQET valid near the threshold only. Higher order terms of \( E/m \) expansion can be easily obtained from the explicit result eq. \[5\]. The NLO correction in low energy expansion reads

\[
\Delta \rho_{th}(m, E) = \frac{-88E^6}{5m^2} \left\{ 1 + \frac{\alpha_s}{\pi} \left( \ln \frac{\mu^2}{m^2} + \frac{376}{33} + \frac{4\pi^2}{9} + \frac{140}{33} \ln \left( \frac{m}{2E} \right) \right) \right\}. \tag{13}
\]

To obtain this result starting from HQET is a more difficult task requiring the analysis of contributions of higher dimension operators.

Note that the simple interpolation between the massless and HQET limits gives a rather good approximation of the ratio \( \rho_1(s)/\rho_0(s) \) for all \( m^2 < s \). For the moments of the spectral density

\[
\mathcal{M}_n = \int_{m^2}^{\infty} \frac{\rho(s) ds}{s^n} = m^{6-2n} M_n, \quad M_n = M_n^{(0)} \left( 1 + \frac{\alpha_s}{\pi} \ln \frac{\mu^2}{m^2} + \delta_n \right) \tag{14}
\]

one finds

\[
M_n^{(0)} = \frac{12}{n(n-1)^2(n-2)^2(n-3)}, \quad \delta_n = A_n + \frac{4}{3} \zeta(2) = A_n + \frac{2\pi^2}{9}. \tag{15}
\]

The coefficients \( A_n \) are rational numbers, \( A_4 = 3, A_5 = 13/2, A_6 = 17/2, A_7 = 535/54 \) \[8\]. The expression for \( \delta_n \) with arbitrary \( n \) is long.
3  Two heavy quarks near threshold.

With two heavy quarks involved in a physical process it is becoming more difficult to perform PT calculations. Still many results in finite order PT for vector current correlators are known both analytically and numerically [15]. For applications near the threshold where \( v = \sqrt{1 - 4m_t^2/s} \ll 1 \) (\( m_t \) is a generic notation for the heavy quark mass) the ordinary perturbation theory (with free quarks as the lowest order approximation) breaks down. The ratio \( \alpha_s/v \) is not small and all terms of the order \( (\alpha_s/v)^n \) should be summed. The motion of the heavy quark-antiquark pair near the production threshold is nonrelativistic to high accuracy that justifies the use of nonrelativistic quantum mechanics for describing such a system [5]. Being much simpler than the comprehensive relativistic treatment of the bound state problem with Bethe-Salpeter amplitude [16], this approach allows one to take into account Coulomb interaction exactly [17, 18]. However, for this approximation to work the full QCD should first be mapped onto NRQCD which is done in PT. For the analysis of heavy quark production by the vector current the basic quantity is the correlator

\[
\Pi(E) = i \int \langle T j_{em}(x) j_{em}(0) \rangle e^{iqx} dx, \quad q^2 = (2m_t + E)^2. \tag{16}
\]

Near the threshold (for small energy \( E \)) NRQCD is used. Up to the NNLO accuracy in NRQCD one has (cf. eq. (10))

\[
\Pi(E) = \frac{2\pi}{m_t^2} C_h(\alpha_s) C_O(E/m_t) G(E; 0, 0). \tag{17}
\]

Here \( C_h(\alpha_s) \) is the high energy coefficient (analog of eq. (12)) which has been known in the NLO since long ago [19]. \( G(E; 0, 0) \) is the nonrelativistic Green’s function (an object of the effective theory as \( \rho_{HQET}(E, \mu) \) in eq. (14)), \( E = \sqrt{s} - 2m_t \). Up to NNLO of NRQCD the contribution of higher dimension operators can be written as a total factor \( C_O(E/m_t) = 1 - 4E/3m_t \) [20].

High energy coefficient \( C_h(\alpha_s) \) is given by the expression

\[
C_h(\alpha) = 1 - 4\frac{\alpha_s}{\pi} + C_F \left( \frac{\alpha_s}{\pi} \right)^2 \left( -\frac{35}{9} \pi^2 \ln \frac{\mu_f}{m_t} + c_2 \right). \tag{18}
\]

The most difficult part of eq. (18) to obtain is the coefficient \( c_2 \) [21]

\[
c_2 = \frac{103}{36} + \frac{1003}{216} \pi^2 - \frac{56}{9} \pi^2 \ln 2 - \frac{125}{6} \zeta(3) + \frac{11}{18} n_f,
\]

\( n_f \) is the number of light quarks. An explicit dependence of high and low energy quantities on the factorization scale \( \mu_f \) is a general feature of effective theories which are valid only for a given region of energy. A physical quantity, which is given by a proper combination of results obtained in different energy regions within respective approximations, is factorization scale independent e.g. [22]. The basic dynamical quantity in the analysis of the near-threshold effects is the nonrelativistic Green’s function \( G(E) = (H - E)^{-1} \) where

\[
H = \frac{p^2}{m_t} + V(r) \tag{19}
\]
is the nonrelativistic Hamiltonian. A part of Hamiltonian (19) which is difficult to find is the heavy quark static potential \( V_{\text{pot}}(r) \) entering into the potential \( V(r) \). The static potential \( V_{\text{pot}}(r) \) is computed in perturbation theory

\[
V_{\text{pot}}(r) = -C_F \frac{\alpha_s}{r} \left(1 + \frac{\alpha_s}{4\pi} \left(C_1^1 \ln(\mu r) + C_0^1\right) + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(C_2^1 \ln^2(\mu r) + C_1^2 \ln(\mu r) + C_0^2\right)\right). \tag{20}
\]

Nontrivial coefficients of this expansion are \( C_0^1 \) and \( C_0^2 \). The numerical value for \( C_0^1 \) was obtained long ago [23], while \( C_0^2 \) has been recently computed [24]. In order to exactly take into account the Coulomb effects near the threshold the Hamiltonian (19) is represented in the form [25]

\[
H = H_C + \Delta H, \quad H_C = \frac{p^2}{m_t} - C_F \frac{\alpha_s}{r} \tag{21}
\]

with

\[
\Delta H = \Delta V_{\text{pot}} - \frac{H_C^2}{4m_t^2} - \frac{3C_F\alpha_s}{4m_t} \left[H_C, \frac{1}{r}\right]_+ - \frac{4\pi\alpha_s}{m_t^2} \left(C_F + \frac{C_A}{2}\right) \delta(\vec{r}). \tag{22}
\]

For QCD one has \( C_A = 3, C_F = 4/3 \). Constructing the Green’s function is straightforward and can be done analytically within perturbation theory in \( \Delta H \) near the Coulomb Green’s function \( G_C(E) \) [26, 27, 28] or numerically by solving the Schrödinger equation with Hamiltonian (19) (for complex values of \( E \) that can be used to describe the production of particles with nonzero width) [29, 30, 31, 32, 33, 34]. I’ll discuss the analytical solution only. Some terms in eq. (22) can be handled efficiently. The shift of the parameters of the Coulomb Green’s function \( E \to E + E^2/4m_t \) and \( \alpha_s \to \alpha_s(1 + 3E/2m_t) \) accounts for the relativistic \( H_C^2 \) and anticommutator corrections in eq. (22). In this respect the modified Coulomb approximation can be used as the leading order approximation. The Hamiltonian can be written in the form

\[
H = H_{LO} + \Delta V_{\text{pot}} + \alpha_s V_0 \delta(\vec{r}), \quad V_0 = -\frac{4\pi}{m_t^2} \left(C_F + \frac{C_A}{2}\right) = -\frac{2\pi}{m_t^2} \frac{35}{9} \tag{23}
\]

where \( H_{LO} \) is the (modified) Coulomb approximation, \( \Delta V_{\text{pot}} \) is the perturbation theory correction to \( V_{\text{pot}}(r) \). The \( \delta(\vec{r}) \)-part is a separable potential and can be taken into account exactly [35]. The solution reads

\[
G(E; 0, 0) = \frac{G_{tr}(E; 0, 0)}{1 + \alpha_s V_0 G_{tr}(E; 0, 0)}, \quad G_{tr}(E) = (H_{LO} + \Delta V_{PT} - E)^{-1}. \tag{24}
\]

Therefore one is left only with the necessity to construct corrections due to \( \Delta V_{\text{pot}} \)-part which modify \( G_{tr}(E) \). An efficient calculational framework for perturbation theory in \( \Delta V_{\text{pot}} \) near the Coulomb Green’s function \( G_C(E) \) has been developed [26]. The key point is the use of a partial wave decomposition for the Green’s function

\[
G(E; x, y) = \sum_{l=0}^{\infty} (2l + 1)(xy)^l P_l(xy/xy)G_l(E; x, y) \tag{25}
\]
where \( P_l(z) \) is a Legendre polynomial. The Coulomb partial waves \( G^C_l(x, y, k) \) are known
\[
G^C_l(E; x, y) = \frac{m_t k}{2\pi} (2k)^2 e^{-k(x+y)} \sum_{m=0}^{\infty} \frac{L_m^{l+1}(2kx)L_m^{l+1}(2ky)m!}{(m + l + 1 - \nu)(m + 2l + 1)!}
\]  

(26)

where \( k^2 = -m_t E, \nu = \lambda/k, \lambda = \alpha_s C_F m_t/2, L_m^\nu(z) \) is a Laguerre polynomial. The result of the evaluation of the NLO correction in S-wave is
\[
\Delta_1 G_0(E) = \frac{\alpha_s \beta_0 \lambda m_t k^2}{2\pi} \left( \sum_{m=0}^{\infty} F(m)^2 (m + 1) (L_1(k) + \Psi_1(m + 2)) - 2 \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} F(m) \right)
\]
\[
\times F(n) \left( \frac{n + 1}{m - n} + 2 \sum_{m=0}^{\infty} F(m) (L_1(k) - 2\gamma_E - \Psi_1(m + 1)) - \gamma_E L_1(k) + \frac{1}{2} L_1(k)^2 \right)
\]

where \( \Psi_n(x) = d^n \ln \Gamma(x)/dx^n, \beta_0 \) is the first coefficient of the \( \beta \)-function. Here
\[
L_1(k) = \ln \left( \frac{\mu_s e^{C^0_1/C^1_1}}{2k} \right), \quad F(m) = \frac{\nu}{(m + 1)(m + 1 - \nu)}.
\]

The NLO correction to the \( l = 1 \) partial wave was found in [30]
\[
\Delta_1 G_1(E) = \frac{\alpha_s \beta_0 \lambda m_t k^2}{18\pi} \left( \sum_{m=0}^{\infty} \tilde{F}(m)^2 (m + 1)(m + 2)(m + 3) (L_1(k) + \Psi_1(m + 4)) \right)
\]
\[
- 2 \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \tilde{F}(m) \tilde{F}(n) \frac{(n + 1)(n + 2)(n + 3)}{m - n} + 2 \sum_{m=0}^{\infty} \tilde{F}(m) \left( 2\tilde{J}_0(m) + (m + 1)(m + 2)L_1(k) \right)
\]
\[
+ (1 + \nu)(\tilde{J}_1(m) + (m + 1)L_1(k)) + \frac{\nu(\nu + 1)}{2}(\tilde{J}_2(m) + 2L_1(k)) \right) \right)
\]

where
\[
\tilde{F}(m) = \frac{\nu(\nu^2 - 1)}{(m + 2 - \nu)(m + 1)(m + 2)(m + 3)},
\]
\[
\tilde{J}_0(m) = -2\Psi_1(m + 1) - 4\gamma_E + 3,
\]
\[
\tilde{J}_1(m) = (m + 1)(-\Psi_1(m + 2) - 2\gamma_E + 2),
\]
\[
\tilde{J}_2(m) = \frac{(m + 1)(m + 2)}{2} \left( -\Psi_1(m + 3) - 2\gamma_E + \frac{3}{2} \right),
\]
\[
\tilde{I}(k) = -\frac{(\gamma_E - 1)^2}{2} - \frac{\pi^2}{12} - (4 - 3\gamma_E)\nu + \frac{1}{4} \left( -9\gamma_E + 6\gamma_E^2 + \frac{\pi^2}{2} \nu^2 + \frac{1 - 3\gamma_E \nu^3}{2} + \frac{1 - 3\gamma_E}{4} \nu^4 \right)
\]
\[
+ \left( \gamma_E - 1 - 3\nu + \frac{9 - 12\gamma_E}{4} \nu^2 + \frac{3}{2} \nu^3 + \frac{1}{4} \nu^4 \right) L_1(k) + \left( -\frac{1}{2} + \frac{3}{2} \nu^2 \right) L_1(k)^2.
\]

The Green’s function at the origin can be written in the form with single poles only
\[
G(E; 0, 0) = \sum_{m=0}^{\infty} \frac{|\psi_{0m}(0)|^2}{E_0m - E} + \frac{1}{\pi} \int_0^{\infty} \frac{|\psi_{0E'}(0)|^2}{E' - E} dE'
\]  

(27)
where $\psi_{0m, E}(0)$ is the wave function at the origin. Up to NNLO one writes

$$
E_{0m} = E_{0m}^C (1 + \Delta_1 E_{0m} + \Delta_2 E_{0m}),
$$

$$
|\psi_{0m}(0)|^2 = |\psi_{0m}^C(0)|^2 \left(1 + \Delta_1 \psi_{0m}^2 + \Delta_2 \psi_{0m}^2\right)
$$

where

$$
E_{0m}^C = -\frac{\lambda^2}{m_t(m+1)^2}, \quad |\psi_{0m}^C(0)|^2 = \frac{\lambda^3}{\pi(m+1)^3}.
$$

The analytical expression for the NLO corrections to the bound state parameters has the form

$$
\Delta_1 E_{0m} = \frac{\alpha_s\beta_0}{\pi} \left(\bar{L}_1(m) + \Psi_1(m + 2)\right),
$$

$$
\Delta_1 \psi_{0m}^2 = \frac{\alpha_s\beta_0}{2\pi} \left(3\bar{L}_1(m) + \Psi_1(m + 2) - 2(m + 1)\Psi_2(m + 1) - 1 - 2\gamma_E + \frac{2}{m + 1}\right)
$$

where $\bar{L}_1(m) = L_1(\lambda/(m + 1))$. The expressions of the NNLO corrections to the energy levels and wave functions at the origin are rather long [27, 37, 38]. As an example I give the correction $\Delta_2^{(1)} \psi_{0m}^2$ due to the second iteration of the $O(\alpha_s)$ correction to the static potential [32]

$$
\Delta_2^{(1)} \psi_{0m}^2 = \left(\frac{\alpha_s}{4\pi}\right)^2 \left(3(C_0^1 + (\bar{L}(m) + \Psi_1(m + 2))C_1^1) + 2C_1^1 \sum_{n=0}^{m-1} \frac{(n+1)(m+1)}{(n-m)^3} \left(C_0^1 + \left((\bar{L}(m) + \Psi_1(n + 2) + \frac{1}{2(n-m)(m+1)}\left(C_1^1 + \left((\bar{L}(m) + \Psi_1(n + 2) - \frac{1}{2(n-m)}\right)C_1^1\right)\right)\right) + 2C_1^1 \left(C_0^1 + (\bar{L}(m) + \Psi_1(m + 2))C_1^1\right) \left(-\frac{5}{2} + \sum_{n=0}^{m-1} \frac{n+1}{(n-m)^2} U(m, n) - \sum_{n=m+1}^{\infty} \frac{m+1}{(n-m)^2} U(m, n) + \sum_{n=0}^{m-1} \frac{n+1}{(n-m)^2} \sum_{n=m+1}^{\infty} \frac{m+1}{(n-m)^2} U(m, n) + \sum_{n=m+1}^{m-1} \sum_{l=0}^{n-1} \frac{(l+1)(m+1)}{(n-m)^2 (l-m)^2} - \frac{(l+1)(m+1)}{(n-m)^2 (l-m) (n-l)} + \frac{(l+1)(m+1)}{(n-m)^2 (n-m) (n-l)} + \frac{(l+1)(m+1)}{(l-m)^2 (n-m) (n-l)} + \sum_{n=2}^{m-1} \sum_{l=m+1}^{n-1} \frac{(m+1)^2}{(n-m)^2 (l-m)^2}\right)
$$
\[
\left( \frac{(l+1)(m+1)^2}{(n-m)^2(l-m)(n-l)(n+1)} + \frac{(m+1)^2}{(l-m)^2(n-m)(n-l)} \right) \right)
\]

where \( \bar{L}(m) = L(\lambda/(m+1)) \) and

\[
U(m,n) = 3 + \frac{n+1}{m+n+2} - 2 \left( \frac{n+1}{n-m}(n+m+2) \right).
\]

4 Physical results.

There are basically two application areas of the near-threshold calculations within NRQCD: the analysis of \( \Upsilon \)-resonances in the \( b\bar{b} \) system and the \( t\bar{t} \) production cross section near the threshold.

4.1 \( \Upsilon \)-resonances.

For the \( b\bar{b} \) system moments of the spectral density of different kinds can be studied (cf. eq. (14)). For instance,

\[
\int_{4m_b^2}^{\infty} \frac{\rho(s)ds}{s^n}
\]

are meaningful in the near-threshold Coulomb PT calculations \[39\]. In the NNLO we obtain the numerical value for the \( b \)-quark pole mass \[27\]

\[
m_b = 4.80 \pm 0.06 \text{ GeV}.
\]

With this result we obtain the value of the matrix element \(|V_{cb}|\) from the analysis of the \( B \)-meson semileptonic width \[27\]

\[
|V_{cb}| = 0.0423 \left( \frac{\text{BR}(B \to X_c l\nu_l)}{0.105} \right)^{\frac{1}{2}} \left( \frac{1.55 \text{ps}}{\tau_B} \right)^{\frac{1}{2}} \left( 1 - 0.01 \frac{\alpha_s(M_Z) - 0.118}{0.006} \right) (1 \pm \Delta_{npt})
\]

where \( \Delta_{npt} \sim 0.02 \) is the uncertainty of nonperturbative contributions. We use the analysis in the pole mass scheme. Other approaches based on redefinition of the mass near threshold have been suggested. The problem of infrared instability of the pole mass and its relation to \( \overline{\text{MS}} \)-mass has been discussed \[37, 40\].

4.2 \( t\bar{t} \) cross section near threshold.

The \( t\bar{t} \)-pair near the production threshold is just a system that satisfies the requirement of being nonrelativistic. Therefore the description of \( t\bar{t} \)-system near the production threshold \( \sqrt{s} \approx 2m_t \) (\( \sqrt{s} \) is a total energy of the pair, \( m_t \) is the top quark mass) is quite precise within NRQCD. The top quark is very heavy \( m_t = 175 \text{ GeV} \) \[41\] and there is an energy region of about \( 8 - 10 \text{ GeV} \) near the threshold where the nonrelativistic approximation for the kinematics is very precise.
Relativistic effects are small and can be taken into account perturbatively. The strong coupling constant at the high energy scale is small $\alpha_s(m_t) \approx 0.1$ that makes the mapping of QCD onto the low energy effective theory (NRQCD), which is perturbative in $\alpha_s(m_t)$, numerically precise.

The top quark decay width is large, $\Gamma_t = 1.43 \text{ GeV}$; the infrared (small momenta) region is suppressed and PT calculations for the cross section near the threshold are reliable even point-wise in energy \[12\]. One can study the $t\bar{t}$ system near the threshold in the processes $e^+e^- \rightarrow t\bar{t}$ \[18\] and $\gamma\gamma \rightarrow t\bar{t}$ \[43\]:

- $e^+e^- \rightarrow t\bar{t}$: the production vertex is local, the basic observable is a production cross section which is saturated by S-wave (for the vector current), NNLO analysis is available.

- $\gamma\gamma \rightarrow t\bar{t}$: the production vertex is nonlocal (T-product of two electromagnetic currents), both S- and P-waves can be studied for different helicity photons, the number of observables is larger (cross sections $\sigma_S$, $\sigma_P$, S-P interference). Because of nonlocality of the production vertex the high energy coefficient (necessary for mapping the QCD quantities to NRQCD ones) is more difficult to obtain. Some calculations were done in NLO \[14, 13\] and the full analysis of the cross section is available in NLO of NRQCD only \[36\]. The low energy part of the process can be studied in NNLO without a strict normalization to full QCD (see \[16\] for relativistic corrections).

The top quark width $\Gamma_t$ plays a crucial role in the calculation of the $t\bar{t}$ production cross section near the threshold. It is taken into account by the shift $E \rightarrow E + i\Gamma_t$ \[12\]. One has

$$\sigma(E) \sim \text{Im } \Pi(E + i\Gamma_t) = \text{Im } \int \frac{\rho(E')dE'}{E' - E - i\Gamma_t} = \Gamma_t \int \frac{\rho(E')dE'}{(E' - E)^2 + \Gamma_t^2}.$$  

(31)

Because the point $E + i\Gamma_t$ lies sufficiently far from the positive semiaxis in the complex energy plane the cross section eq. (31) is calculable point-wise in energy. The hadronic cross section $\sigma(E)$ was obtained by many authors (as a review see, \[3\]). The normalized cross sections $R^\nu(E)$ are plotted in Fig.1 \[32\]. The curves have characteristic points which are usually considered as basic observables. They are: $E_p$ – the position of the peak in the cross section and $H_p$ – the height of the peak. The convergence for $E_p$ and $H_p$ in consecutive orders of PT near the Coulomb solution is not fast in the $\overline{\text{MS}}$-scheme \[8, 38, 47\]. For typical values $m_t = 175 \text{ GeV}$, $\Gamma_t = 1.43 \text{ GeV}$, $\alpha_s(M_Z) = 0.118$ one finds \[32\]

$$E_p = E_0(1 + 0.58 + 0.38 + \ldots)$$

$$H_p = H_0(1 - 0.15 + 0.12 + \ldots).$$

(32)

The way of dealing with the PT expansion of the static potential is an important issue in getting stable results within NRQCD \[48\]. The convergence in the $\overline{\text{MS}}$-scheme is not fast which reflects the physical situation that the observables $E_p$ and $H_p$ are sensitive to different scales.

The finite-order perturbation theory expansion of the static potential given in eq. (20) cannot handle several distinct scales with the same accuracy.
5 Conclusion.

Some calculations of PT corrections to systems with heavy particles are presented. The use of effective theories for computing high order corrections is crucial in obtaining precise results near the threshold. The calculation are rather tedious but still can be completed analytically in some cases.

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