Generalization of The Results on Fixed Point For Couplings on Metric Spaces

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The purpose of this paper is to introduce the concept of self-cyclic maps, \( g \)-coupling, Banach type \( g \)-coupling which is the generalization of couplings introduced by Choudhury et al. \([5]\). In our main result we prove the existence theorem of coupled coincidence point for Banach type \( g \)-couplings which extends the results by choudhury et al. \([5]\). We give an example in support of our result.

Keywords : \( g \)-coupling, coupled coincidence point, strong coupled coincidence point, self-cyclic maps, Banach type \( g \)-coupling.

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1. Introduction and Preliminaries

The concept of coupled fixed point was introduced in the work of Guo et al. \([10]\). Then after the coupled contraction mapping theorem by Bhaskar and Lakshmikantham \([9]\), and the introduction of coupled coincidence point by Lakshmikantham.V and Ćirić.L \([13]\), the coupled fixed and coupled coincidence point results reported in papers \([2, 5, 6, 7, 11, 13, 14, 15, 16, 17, 18, 19, 20]\), Kirk et al. \([12]\) gave the concept of cyclic mapping. Recently Choudhury et al. \([5]\) introduced the concept of couplings which are actually coupled cyclic mappings with respect to two given subsets of a metric space. Choudhury et al. \([5]\) proved the existence of strong coupled fixed point for a coupling ( w.r.t. subsets of a complete metric space). H. Aydi et al. \([1]\) proved the theorem for existence and uniqueness of strong coupled fixed point in partial metric spaces. In this paper we extend the concept of Banach type coupling to Banach type \( g \)-coupling by extending the concept of coupling to \( g \)-coupling. We have introduced self-cyclic maps and generalize the results by Choudhury et al \([5]\). Now we recall some definitions.

**Definition 1.1 (Coupled fixed point) \([9]\).** An element \((x, y) \in X \times X\), where \(X\) is any non-empty set, is called a coupled fixed point of the mapping \(F : X \times X \to X\) if \(F(x, y) = x\) and \(F(y, x) = y\).

**Definition 1.2 (Strong coupled fixed point) \([5]\).** An element \((x, y) \in X \times X\), where \(X\) is any non-empty set, is called a strong coupled fixed point of the mapping \(F : X \times X \to X\) if \((x, y)\) is coupled fixed point and \(x = y\); that is if \(F(x, x) = x\).

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Definition 1.3 (Coupled Banach Contraction Mapping) [3]. Let \((X, d)\) be a metric space. A mapping \(F : X \times X \to X\) is called coupled Banach contraction if there exists \(k \in (0, 1)\) s.t \(\forall \ (x, y), (u, v) \in X \times X\), the following inequality is satisfied:

\[
d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)].
\]

Definition 1.4 (Cyclic mapping) [12]. Let \(A\) and \(B\) be two non-empty subsets of a given set \(X\). Any function \(f : X \to X\) is said to be cyclic (with respect to \(A\) and \(B\)) if

\[
f(A) \subseteq B \text{ and } f(B) \subseteq A.
\]

Definition 1.5 (Coupling) [8]. Let \((X, d)\) be a metric space and \(A\) and \(B\) be two non-empty subsets of \(X\). Then a function \(F : X \times X \to X\) is said to be a coupling with respect to \(A\) and \(B\) if

\[
F(x, y) \in B \text{ and } F(y, x) \in A \text{ whenever } x \in A \text{ and } y \in B.
\]

Definition 1.6 (Banach type coupling) [5]. Let \(A\) and \(B\) be two non-empty subsets of a complete metric space \((X, d)\). A coupling \(F : X \times X \to X\) is called a Banach type coupling with respect to \(A\) and \(B\) if it satisfies the following inequality:

\[
d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)].
\]

where \(x, v \in A, \ y, u \in B\) and \(k \in (0, 1)\).

Definition 1.7 (Coupled coincidence point of \(F\) and \(g\)) [13]. An element \((x, y) \in X \times X\) is called a coupled coincidence point of the mappings \(F : X \times X \to X\) and \(g : X \to X\) if \(F(x, y) = g(x)\) and \(F(y, x) = g(y)\).

Definition 1.8 (Commutative mappings) [13]. For any set \(X\), we say mappings \(F : X \times X \to X\) and \(g : X \to X\) are commutative if

\[
g(F(x, y)) = F(g(x), g(y)), \ \forall \ x, y \in X.
\]

2. Main Result

Here we give some definitions in support of our main result.

Definition 2.1 (Self-cyclic mapping). Let \(A\) and \(B\) be two non-empty subsets of any set \(X\). Then a mapping \(g : X \to X\) is said to be self-cyclic (with respect to \(A\) and \(B\)) if

\[
g(A) \subseteq A \text{ and } g(B) \subseteq B.
\]

Definition 2.2 (Strong coupled coincidence points of \(F\) and \(g\)). A coupled coincidence point \((x, y) \in X \times X\) of \(F : X \times X \to X\) and \(g : X \to X\) is said to be strong coupled coincidence point of \(F\) and \(g\), if \(x = y\) i.e. \(F(x, x) = g(x)\).
**Definition 2.3 (g-coupling).** Let \((X, d)\) be a metric space and \(A\) and \(B\) be two non-empty subsets of \(X\). Let functions \(F\) and \(g\) are such that \(F : X \times X \to X\) and \(g : X \to X\). Then \(F\) is said to be \(g\)-coupling (with respect to \(A\) and \(B\)) if
\[
F(x, y) \in g(A) \cap B \text{ and } F(y, x) \in g(B) \cap A, \text{ whenever } x \in A \text{ and } y \in B.
\]

**Remark 2.4:** It should be noted that every \(g\)-coupling is a coupling but converse is not true in general.

**Proof:** Let \(F : X \times X \to X\) be \(g\)-coupling, where \(g : X \to X\). Then by Definition (2.3), we have \(F(x, y) \in (g(A) \cap B) \subset B \text{ and } F(y, x) \in (g(B) \cap A) \subset A\).

This shows that \(F\) is a coupling (w.r.t. \(A\) and \(B\)). Clearly converse is not true in general. If \(F\) is a coupling (w.r.t \(A\) and \(B\)), it doesn’t imply that \(F\) is a \(g\)-coupling for any \(g : X \to X\) (w.r.t \(A\) and \(B\)) as \(F(x, y) \in B\) doesn’t imply that \(F(x, y) \in g(A)\), and similarly for \(F(y, x)\).

**Definition 2.5 (Banach type \(g\)-coupling).** Let \(A\) and \(B\) be two non-empty subsets of a complete metric space \((X, d)\). Then a \(g\)-coupling \(F : X \times X \to X\) is said to be Banach type \(g\)-coupling (with respect to \(A\) and \(B\)) if the following inequality holds:
\[
d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(gx, gu) + d(gy, gv)],
\]
whenever \(x, v \in A\), \(y, u \in B\) and \(k \in (0, 1)\). where \(g : X \to X\) is a self-cyclic mapping (with respect to \(A\) and \(B\)).

**Note:** If \(g = I\) (the identity mapping) which is also self-cyclic, then Banach type \(g\)-coupling becomes Banach type coupling.

**Theorem 2.6:** Let \(A\) and \(B\) be any two subsets of a complete metric space \((X, d)\). If there exists a Banach type \(g\)-coupling \(F : X \times X \to X\) (with respect to \(A\) and \(B\)), where \(g : X \to X\) is self-cyclic (with respect to \(A\) and \(B\)). If \(g(A)\) and \(g(B)\) are closed subsets of \((X, d)\). Then
(i) \(g(A) \cap g(B) \neq \emptyset\),
(ii) \(F\) and \(g\) have a coupled coincidence point in \(A \times B\), i.e. there exists \((a, b) \in A \times B\) s.t \(F(a, b) = g(a)\) and \(F(b, a) = g(b)\).

**Proof:** Here \(F : X \times X \to X\) is given to be Banach type \(g\)-coupling, i.e.
\[
d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(gx, gu) + d(gy, gv)], \tag{1}
\]
where \(x, v \in A\), \(y, u \in B\) and \(k \in (0, 1)\).

Also as \(g : X \to X\) is self-cyclic (w.r.t. \(A\) and \(B\)), so
\[
gx, gu \in g(A) \subseteq A \text{ and } gy, gu \in g(B) \subseteq B
\]
Let \(x_0 \in A\) and \(y_0 \in B\), then by definition of \(g\)-coupling, we have
\(F(x_0, y_0) \in g(A) \cap B\) and \(F(y_0, x_0) \in g(B) \cap A\)
in particular \(F(x_0, y_0) \in g(A)\) and \(F(y_0, x_0) \in g(B)\).
If
\[
F(x_0, y_0) = g(x_0) \text{ and } F(y_0, x_0) = g(y_0),
\]

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then \((x_0, y_0)\) is the coupled coincidence point of \(F\) and \(g\), so we are done in this case. Otherwise \(\exists x_1 \in A\) and \(y_1 \in B\), s.t.

\[
F(x_0, y_0) = g(x_1) \quad \text{and} \quad F(y_0, x_0) = g(y_1).
\]

Now if,

\[
F(x_1, y_1) = g(x_1) \quad \text{and} \quad F(y_1, x_1) = g(y_1),
\]

then \((x_1, y_1)\) is a coupled coincidence point of \(F\) and \(g\), and we are through otherwise \(\exists x_2 \in A\) and \(y_2 \in B\) s.t.

\[
F(x_1, y_1) = g(x_2) \quad \text{and} \quad F(y_1, x_1) = g(y_2).
\]

continuing in this way, we get sequences \(\{gx_n\}\) and \(\{gy_n\}\) in \(g(A)\) and \(g(B)\) respectively, such that

\[
g(x_{n+1}) = F(x_n, y_n) \quad \text{and} \quad g(y_{n+1}) = F(y_n, x_n). \tag{2}
\]

Now using (1) and (2), we get

\[
d(gx_1, gy_2) = d(F(x_0, y_0), F(y_1, x_1)) \leq \frac{k}{2}[d(gx_0, gy_1) + d(gy_0, gx_1)]
\]

and

\[
d(gy_1, gx_2) = d(F(y_0, x_0), F(x_1, y_1)) \leq \frac{k}{2}[d(gy_0, gx_1) + d(gx_0, gy_1)].
\]

from above two inequalities, we have

\[
d(gx_1, gy_2) + d(gy_1, gx_2) \leq \frac{k}{2}[d(gx_0, gy_1) + d(gy_0, gx_1) + d(gy_0, gx_1) + d(gx_0, gy_1)]
\]

or,

\[
\frac{d(gx_1, gy_2) + d(gy_1, gx_2)}{2} \leq \frac{k}{2}[d(gx_0, gy_1) + d(gy_0, gx_1)]. \tag{3}
\]

using (1), (2) and (3), we have

\[
d(gx_2, gy_3) = d(F(x_1, y_1), F(y_2, x_2)) \leq \frac{k}{2}[d(gx_1, gy_2) + d(gy_1, gx_2)] \leq \frac{k^2}{2}[d(gx_0, gy_1) + d(gy_0, gx_1)].
\]
and
\[ d(gy_2, gx_3) = d(F(y_1, x_1), F(x_2, y_2)) \]
\[ \leq \frac{k}{2} [d(gy_1, gx_2) + d(gx_1, gy_2)] \]
\[ \leq \frac{k^2}{2} [d(gx_0, gy_1) + d(gy_0, gx_1)]. \]

Let for some integer \( n \),
\[ d(gx_n, gy_{n+1}) \leq \frac{k^n}{2} [d(gx_0, gy_1) + d(gy_0, gx_1)]. \] \hspace{1cm} (4)
\[ d(gy_n, gx_{n+1}) \leq \frac{k^n}{2} [d(gx_0, gy_1) + d(gy_0, gx_1)]. \] \hspace{1cm} (5)

Now using (1), (4) and (5), we have
\[ d(gx_{n+1}, gy_{n+2}) = d(F(x_n, y_n), F(y_{n+1}, x_{n+1})) \]
\[ \leq \frac{k}{2} [d(gx_n, gy_{n+1}) + d(gy_n, gx_{n+1})] \]
\[ \leq \frac{k^n}{2} [d(gx_0, gy_1) + d(gy_0, gx_1)] \]
\[ = \frac{k^{n+1}}{2} [d(gx_0, gy_1) + d(gy_0, gx_1)]. \]

Similarly as above we can show that
\[ d(gy_{n+1}, gx_{n+2}) \leq \frac{k^{n+1}}{2} [d(gx_0, gy_1) + d(gy_0, gx_1)]. \]

Thus (4) and (5) remains also true for \( n + 1 \),

Hence by principle of mathematical induction, we have \( \forall \ n \geq 1 \),
\[ d(gx_n, gy_{n+1}) \leq \frac{k^n}{2} [d(gx_0, gy_1) + d(gy_0, gx_1)]. \] \hspace{1cm} (6)
\[ d(gy_n, gx_{n+1}) \leq \frac{k^n}{2} [d(gx_0, gy_1) + d(gy_0, gx_1)]. \] \hspace{1cm} (7)

Again by (1) and (2), we have
\[ d(gx_1, gy_1) = d(F(x_0, y_0), F(y_0, x_0)) \]
\[ \leq \frac{k}{2} [d(gx_0, gy_0) + d(gy_0, gx_0)] \]
\[ = kd(gx_0, gy_0). \]

that is,
\[ d(gx_1, gy_1) \leq kd(gx_0, gy_0). \] \hspace{1cm} (8)
Then from (1), (2) and (8), we have

\[ d(gx_2, gy_2) = d(F(x_1, y_1), F(y_1, x_1)) \]
\[ \leq \frac{k}{2} [d(gx_1, gy_1) + d(gy_1, gx_1)] \]
\[ = kd(gx_1, gy_1) \]
\[ \leq k^2 d(gx_0, gy_0). \]

Let for some integer \( n \), we have

\[ d(gx_n, gy_n) \leq k^n d(gx_0, gy_0). \] (9)

Then from (1), (2) and (9), we get

\[ d(gx_{n+1}, gy_{n+1}) = d(F(x_n, y_n), F(y_n, x_n)) \]
\[ \leq \frac{k}{2} [d(gx_n, gy_n) + d(gy_n, gx_n)] \]
\[ = kd(gx_n, gy_n) \]
\[ \leq k^{n+1} d(gx_0, gy_0). \]

This shows that (9) remains also true for \( n + 1 \), thus by principle of mathematical induction, we say that

\[ d(gx_n, gy_n) \leq k^n d(gx_0, gy_0), \forall n \geq 1. \] (10)

Now by (6), (7) and (10) and by triangular inequality, we have \( \forall n \geq 1 \)

\[ d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) \leq d(gx_n, gy_n) + d(gy_n, gx_n) + d(gx_n, gx_{n+1}) \]
\[ = 2d(gx_n, gy_n) + [d(gy_n, gx_{n+1}) + d(gx_n, gy_{n+1})] \]
\[ \leq 2k^n d(gx_0, gy_0) + k^n [d(gx_0, gy_1) + d(gy_0, gx_1)]. \]

since \( k \in (0, 1) \), it follows that \( \sum d(gx_n, gx_{n+1}) + \sum d(gy_n, gy_{n+1}) < \infty \).

Thus sequences \( \{gx_n\} \) and \( \{gy_n\} \) are Cauchy sequences in \( g(A) \) and \( g(B) \).

As \( g(A) \) and \( g(B) \) are closed subsets of complete metric space \( (X, d) \), so \( \{g(x_n)\} \) and \( \{g(y_n)\} \) are convergent in \( g(A) \) and \( g(B) \) respectively.

Therefore \( \exists u \in g(A) \) and \( v \in g(B) \), s.t.

\[ gx_n \to u \text{ and } gy_n \to v \text{ as } n \to \infty. \] (11)

using (10), as \( k \in (0, 1) \), we have

\[ d(gx_n, gy_n) \to 0 \text{ as } n \to \infty. \]

therefore from (11), we have

\[ u = v. \] (12)

As \( u \in g(A) \) and \( v \in g(B) \) \( \Rightarrow u \in g(A) \cap g(B) \)

This proves part (i) that \( g(A) \cap g(B) \neq \emptyset. \)
Now, since \( u \in g(A) \) and \( v \in g(B) \), therefore \( \exists \ a \in A \) and \( b \in B \), s.t \( u = g(a) \) and \( v = g(b) \).
then from (11) and (12), we have
\[
gx_n \to g(a) \text{ and } gy_n \to g(b). 
\] (13)
and
\[
g(a) = g(b). 
\] (14)
Now by (1), (2), (13), (14) and triangular inequality, we have
\[
d(g(a), F(a, b)) \leq d(g(a), gy_{n+1}) + d(gy_{n+1}, F(a, b))
\]
\[
= d(g(a), gy_{n+1}) + d(F(x_n, y_n), F(a, b))
\]
\[
\leq d(g(a), gy_{n+1}) + \frac{k}{2}[d(gy_n, g(a)) + d(gx_n, g(b))]
\]
\[
\to 0 \text{ as } n \to \infty.
\]
thus, we have
\[
F(a, b) = g(a). 
\] (15)
Again from (1), (2), (13), (14) and triangular inequality, we have
\[
d(g(b), F(b, a)) \leq d(g(b), gx_{n+1}) + d(gx_{n+1}, F(b, a))
\]
\[
= d(g(b), gx_{n+1}) + d(F(x_n, y_n), F(b, a))
\]
\[
\leq d(g(b), gx_{n+1}) + \frac{k}{2}[d(gx_n, g(b)) + d(gy_n, g(a))]
\]
\[
\to 0 \text{ as } n \to \infty.
\]
thus, we get
\[
F(b, a) = g(b). 
\] (16)
Hence from (15) and (16) we get, \( F(a, b) = g(a) \) and \( F(b, a) = g(b) \),
where \( a \in A \) and \( b \in B \).
Thus \( (a, b) \in A \times B \) is the coupled coincidence point of \( F \) and \( g \).

**Remark 2.7.** It is worth noting that the above theorem also gives the condition for the existence of symmetric point of \( F \) in \( A \times B \), i.e. \( \exists (a, b) \in A \times B \) s.t \( F(a, b) = F(b, a) \).
As from (14) \( g(a) = g(b) \) so, \( F(a, b) = F(b, a) \).

**Existence and Uniqueness of Strong coupled coincidence point of \( F \) and \( g \).**

**Theorem 2.8:** If in addition to above condition in Theorem 2.6, \( g \) is one-one, then
(i) \( A \cap B \neq \emptyset \), and
(ii) \( F \) and \( g \) have unique strong coupled coincidence point in \( A \cap B \).

**Proof:** since \( g \) is given to be one-one
then from (14) of Theorem 2.6, we get
\[
a = b.
\]
since, \( a \in A \) and \( b \in B \)  
\[ \Rightarrow a = b \in A \cap B. \]  
Hence \( A \cap B \neq \emptyset \), this proves our part (i).  
Also from (15) of Theorem 2.6, we have  
\[ F(a, a) = g(a). \]  
which shows that \( F \) and \( g \) have strong coupled coincidence point in \( A \cap B \).

**Uniqueness:** Let us suppose if possible there exists two strong coupled coincidence points \( l, m \in A \cap B \) of \( F \) and \( g \)  
then by definition we have,  
\[ F(l, l) = g(l) \quad \text{and} \quad F(m, m) = g(m). \]  
then from (1) of Theorem 2.6, we have  
\[ d(g(l), g(m)) = d(F(l, l), F(m, m)) \leq \frac{k}{2}[d(g(l), g(m)) + d(g(l), g(m))] = k[d(g(l), g(m))]. \]  
Which is a contradiction as \( k \in (0, 1) \) and is only possible if \( d(g(l), g(m)) = 0 \), i.e. \( g(l) = g(m) \) or \( l = m \) because \( g \) is one-one.

Hence \( F \) and \( g \) have a unique strong coupled coincidence point in \( A \cap B \).

**Corollary 2.9:** If \( F : X \times X \to X \) and \( g : X \to X \) are commutative and \( g \) is one-one.  
if \((gx, gy)\) is a coupled coincidence point of \( F \) and \( g \), where \( x, y \in X \), then \((x, y)\) is also a coupled coincidence point of \( F \) and \( g \).

**Proof:** As \((gx, gy)\) is a coupled coincidence point of \( F \) and \( g \), we have  
\begin{align*}  
F(gx, gy) &= gx, \quad (18) 
F(gy, gx) &= gy. \quad (19) 
\end{align*}  
Now by (18) and commutativity of \( F \) and \( g \), we have  
\[ g(F(x, y)) = F(gx, gy) = gx. \]  
as \( g \) is one-one, we have from above  
\[ F(x, y) = x. \]  
Now again by (19) and commutativity of \( F \) and \( g \), we have  
\[ g(F(y, x)) = F(gy, gx) = gy. \]  
as \( g \) is one-one, we have from above  
\[ F(y, x) = y. \]
Thus from (20) and (21), we get

\[ F(x, y) = x \quad \text{and} \quad F(y, x) = y. \]

i.e. \((x, y)\) is the coupled coincidence point of \(F\) and \(g\).

The following example illustrates our results.

**Example 2.10.** Let \(X = \mathbb{R}\) with the metric defined as \(d(x, y) = |x - y|\), where \(x, y \in X\).

Let \(A = [0, 2]\) and \(B = [0, 3]\).

Let \(F\) be defined as \(F(x, y) = \frac{x + y}{10}\), where \(x, y \in X\).

and let \(g : X \to X\) is defined by \(g(x) = \frac{x}{2}\).

Then \(g(A) = [0, 1]\) and \(g(B) = [0, \frac{3}{2}]\), so \(g(A)\) and \(g(B)\) are closed subsets of \(X\).

Also \(g(A) \subseteq A\) and \(g(B) \subseteq B\), so \(g\) is self-cyclic.

Now we show that \(F\) is \(g\)-coupling.

As \(g(A) \cap B = [0, 1]\) and \(g(B) \cap A = [0, 1]\), so \(\forall x \in A\) and \(y \in B\), we have \(0 \leq F(x, y) \leq \frac{1}{2}\) and \(0 \leq F(y, x) \leq \frac{1}{2}\).

i.e. \(F(x, y) \in g(A) \cap B\) and \(F(y, x) \in g(B) \cap A\), which shows that \(F\) is \(g\)-coupling with respect to \(A\) and \(B\).

Again for \(x, v \in A\) and \(y, u \in B\), we have

\[
d(F(x, y), F(u, v)) = |\frac{x + y}{10} - \frac{u + v}{10}| = \frac{1}{5} |\frac{x - u}{2} + \frac{y - v}{2}| \\
\leq \frac{1}{5} \left| \frac{x - u}{2} \right| + \frac{1}{5} \left| \frac{y - v}{2} \right| = \frac{1}{5} \left| \frac{x - u}{2} \right| + \frac{1}{5} \left| \frac{y - v}{2} \right| \\
= k \frac{d(gx, gu) + d(gy, gv)}{2}, \text{where } k = \frac{2}{5} \in (0, 1),
\]

which shows that \(F\) is a Banach type \(g\)-coupling (w.r.t. \(A\) and \(B\)).

Thus all the conditions of Theorem (2.6) are satisfied, therefore \(\exists (a, b) \in A \times B\) s.t.

\(F(a, b) = g(a)\) and \(F(b, a) = g(b)\).

i.e.

\[
\frac{a + b}{10} = \frac{a}{2} \quad \text{and} \quad \frac{b + a}{10} = \frac{b}{2}
\]

or

\[
\frac{a + b}{5} = a = b.
\]

which is only possible with \(a = 0\) and \(b = 0\).

thus \((0, 0)\) is the unique strong \(g\)-coupled coincidence point of \(F\) and \(g\).

**Note:** The uniqueness of the strong \(g\)-coupled coincidence point in the above example is because of \(g\) is one-one.
**Corollary 2.11:** Let \( g : X \to X \) is a Banach contraction, Then every Banach type \( g \)-coupling (w.r.t. \( A \) and \( B \)) is a Banach type coupling (w.r.t. \( A \) and \( B \)), where \( A \) and \( B \) are subsets of \( X \).

**Proof:** since \( g \) is a contraction, therefore \( \exists \alpha \in (0, 1), \) s.t

\[
d(gx, gy) \leq \alpha d(x, y) \quad \forall \ x, y \in X.
\]

(23)

Let \( F : X \times X \to X \) be a Banach type \( g \)-coupling (w.r.t. \( A \) and \( B \)), i.e.

\[
d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(gx, gu) + d(gy, gu)], \text{ where } x, v \in A, \ y, u \in B \text{ and } k \in (0, 1).
\]

(24)

Also \( F \) is a \( g \)-coupling (w.r.t. \( A \) and \( B \)) and hence by remark (2.4), \( F \) is a coupling (w.r.t. \( A \) and \( B \)).

Now using (23) in (24), we get

\[
d(F(x, y), F(u, v)) \leq \frac{k}{2} [\alpha d(x, u) + \alpha d(y, v)] \\
\leq \frac{\alpha k}{2} [d(x, u) + d(y, v)]
\]

as \( k \in (0, 1) \) and \( \alpha \in (0, 1), \) therefore \( \alpha k = k_1 \in (0, 1), \) so

\[
d(F(x, y), F(u, v)) \leq \frac{k_1}{2} [d(x, u) + d(y, v)].
\]

where \( x, v \in A, \ y, u \in B \) and \( k_1 \in (0, 1). \)

Hence \( F \) is a Banach type coupling (w.r.t. \( A \) and \( B \)).

3. Conclusion

In this paper we introduce some new results that extend the concept of coupling and Banach type coupling introduced by Choudhury et al. [5] to \( g \)-coupling and Banach type \( g \)-coupling resp. and proved the existence and uniqueness theorem for coupled coincidence point and strong coupled coincidence point.

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