SGD for Structured Nonconvex Functions:
Learning Rates, Minibatching and Interpolation

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Abstract

We provide several convergence theorems for SGD for two large classes of structured nonconvex functions: (i) the Quasar (Strongly) Convex functions and (ii) the functions satisfying the Polyak-Lojasiewicz condition. Our analysis relies on the Expected Residual condition which we show is a strictly weaker assumption as compared to previously used growth conditions, expected smoothness or bounded variance assumptions. We provide theoretical guarantees for the convergence of SGD for different step size selections including constant, decreasing and the recently proposed stochastic Polyak step size. In addition, all of our analysis holds for the arbitrary sampling paradigm, and as such, we are able to give insights into the complexity of minibatching and determine an optimal minibatch size. In particular we recover the best known convergence rates of full gradient descent and single element sampling SGD as a special case. Finally, we show that for models that interpolate the training data, we can dispense of our Expected Residual condition and give state-of-the-art results in this setting.

1 Introduction

We consider the unconstrained finite-sum optimization problem

$$\min_{x \in \mathbb{R}^d} \left[ f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x) \right].$$

(1)

We use $X^* \subset \mathbb{R}^d$ to denote the set of minimizers $x^*$ of (1) and assume that $X^*$ is not empty and that $f(x)$ is lower bounded. This problem is prevalent in machine learning tasks where $x$ corresponds to the model parameters, $f_i(x)$ represents the loss on the training point $i$ and the aim is to minimize the average loss $f(x)$ across training points.

When $n$ is large, stochastic gradient descent (SGD) and its variants are the preferred methods for solving (1) mainly because of their cheap per iteration cost. The standard convergence theory for SGD [47, 38, 39, 52, 37, 2, 14] in the smooth nonconvex setting shows slow sub-linear convergence to a stationary point. Yet in contrast, when applying SGD to many practical nonconvex problems of the form (1) such as matrix completion [49], deep learning [32], and phase retrieval [53] the iterates converge globally, and sometimes, even linearly. This is because these problems often have additional structure and properties, such as all local minimas are global minimas [49, 18], the model interpolates the data [32] or the function under study is unimodal on all lines through a minimizer [16]. By exploiting these structures and properties one can prove significantly tighter (and more useful) convergence bounds.

Here we present a general analysis of SGD for two large classes of structured nonconvex functions: (i) the Quasar (Strongly) Convex functions and (ii) the functions satisfying the Polyak-Lojasiewicz (PL) condition.
condition. In all of our results we provide convergence guarantees for SGD to the global minimum. We also develop several corollaries for functions that interpolate the data.

1.1 Background and Main Contributions

Classes of structured nonconvex functions. The last few years has seen an increased interest in exploiting additional structure prevalent in large classes of nonconvex functions. Such conditions include error bounds property [7], essential strong convexity [25], quasi strong convexity [34, 11], the restricted secant inequality [58], and the quadratic growth (QG) condition [1, 28]. We focus on two of the weakest conditions: the quasar (strongly) convex functions [16, 13, 12] and functions satisfying the PL condition [42, 31, 17]. The class of quasar convex functions include all convex functions as a special case, but it also includes several nonconvex functions. Recently there is also some evidences suggesting that the loss function of neural networks have a quasar-convexity structure [59, 21].

Contributions. We show that SGD converges at a $O(1/\sqrt{k})$ rate on the quasar convex functions and prove linear convergence to a neighborhood for PL functions without any bounded variance assumption or growth assumptions on the stochastic gradients. Instead, we rely on the recently introduced expected residual (ER) condition [10].

Assumptions on the gradient. The standard convergence analysis of SGD in the nonconvex setting relies on the bounded gradients assumption $E_i\|\nabla f_i(x^k)\|^2 < c [45, 15, 44]$ or a growth condition $E_i\|\nabla f_i(x^k)\|^2 \leq c_1 + c_2 K E \| \nabla f(x^k) \|^2$ [4, 5, 50]. There is now a line of recent works [41, 54, 11, 19, 24, 22] which aims at relaxing these assumptions.

Contributions. We use the recently introduced Expected Residual (ER) condition [10]. We give the first convergence proofs for SGD under the ER condition and we show that ER is a strictly weaker assumption than the SGC, Weak Growth (WGC) [54] or the Expected Smoothness (ES) [11] assumptions. Furthermore, we show that the ER condition holds for a large class of nonconvex functions including 1) smooth and interpolated functions 2) smooth and $x^+–$ convex functions.

PL condition. The PL condition [42, 31] was introduced as a sufficient condition for the linear convergence of Gradient Descent for nonconvex functions. Assuming bounded gradients, it was shown in [17] that SGD with a decreasing step size converges sublinearly at a rate of $O(1/\sqrt{k})$ for PL functions. In contrast, by using a step size which depends on the last iterate, the same convergence rate can be achieved without the need for the bounded gradient assumption [19]. Assuming in addition the interpolation condition and Strong Growth Condition (SGC) [54] showed that SGD converges linearly for PL functions, but the specialization of this last result to gradient descent results in a suboptimal dependence on the condition number of the function.

Contributions. We propose an analysis of minibatch SGD for PL functions which recovers the best known dependence on the condition number for Gradient Descent [17] while also matching the current state-of-the-art rate derived in [54, 24] for SGD for interpolated functions. All of which relies on the weaker ER condition. Moreover, we propose a switching step size scheme similar to [11] which does not require knowledge of the last iterate of the algorithm. Using this step size, we prove that SGD converges sublinearly at a rate of $O(1/k)$ for PL functions without any additional bounded gradient of bounded variance assumption or growth assumption.

Step-size selection for SGD. The most important parameter that one should select to guarantee the convergence of SGD is the step-size or learning rate. There are several choices that one can use including constant step-size [33, 55, 11, 36, 41], decreasing step-size [47, 8, 11, 37, 17] and adaptive step-size [6, 26, 20, 3, 55, 56].

Contributions. We provide convergence theorems for SGD under several step-size rules for minimizing Quasar (strongly) convex functions and functions satisfying the PL conditions. The proposed choices include constant and decreasing step-size as well as a recently proposed adaptive learning rate called the stochastic Polyak step [30].

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The $x^+$– convexity includes all convex functions and several nonconvex functions.

Theorem 4 in [54] specialized to GD gives a rate of $\mu^2/L^2$ where $L$ is the smoothness constant and $\mu$ the PL constant.
We work with two classes of nonconvex problems: the quasar-convex functions and the functions \( g \) with an unbiased estimate of the gradient \( v \). When the number of terms \( n \) is large calculating the full gradient becomes prohibitive. So instead, we assume we are given access to unbiased estimates \( g(x) \in \mathbb{R}^d \) of the gradient such that \( \mathbb{E}[g(x)] = \nabla f(x) \). For example, we can use a minibatch to form an estimate of the gradient such as \( g(x) = \frac{1}{b} \sum_{i \in B} \nabla f_i(x) \), where \( B \subset \{1, \ldots, n\} \) will be chosen uniformly at random and \( |B| = b \). To allow for any form of minibatching we use the arbitrary sampling notation
\[
g(x) = \nabla f_v(x) := \frac{1}{n} \sum_{i=1}^{n} v_i \nabla f_i(x),
\]
where \( v \in \mathbb{R}_+^n \) is a random sampling vector such that \( \mathbb{E}[v_i] = 1 \), for \( i = 1, \ldots, n \) and \( f_v(x) := \frac{1}{n} \sum_{i=1}^{n} v_i f_i(x) \). Note that it follows immediately from this definition of sampling vector that \( \mathbb{E}[g(x)] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[v_i] \nabla f_i(x) = \nabla f(x) \). In this work we mostly focus on the \( b \)-minibatch sampling, however we highlight that our analysis holds for any form of sampling vectors.

**Definition 1.1 (Minibatch sampling).** Let \( b \in [n] \). We say that \( v \in \mathbb{R}^n \) is a \( b \)-minibatch sampling if for every subset \( S \subset [n] \) with \( |S| = b \) we have that \( P[v = \frac{n}{b} \sum_{i \in S} e_i] = 1/(\binom{n}{b}) := b!(n-b)/n! \).

It is easy to verify by using a double counting argument that if \( v \) is a \( b \)-minibatch sampling, it is also a valid sampling vector (\( \mathbb{E}[v_i] = 1 \)) [11]. See [11] for other choices of sampling vectors \( v \).

With an unbiased estimate of the gradient \( g(x) \), we can now use Stochastic gradient descent (SGD) to solve (1) by sampling \( g(x^k) \) i.i.d and iterating
\[
x^{k+1} = x^k - \gamma^k g(x^k)
\]
We also make the following mild assumption on the gradient noise.

**Assumption 1.2.** The gradient noise \( \sigma^2 := \sup_{x^* \in \mathcal{X}^*} \mathbb{E} \left[ ||g(x^*)||^2 \right] \) is finite.

## 2 Classes of Structured Nonconvex Functions

We work with two classes of nonconvex problems: the quasar-convex functions and the functions that satisfy the Polyak-Lojasiewicz (PL) condition.

**Definition 2.1 (Quasar convex).** Let \( \zeta > 0 \) and \( x^* \in \mathcal{X}^* \). We that \( f \) is \( \zeta \)-quasar convex with respect to \( x^* \) if for all \( x \in \mathbb{R}^n \),
\[
f(x^*) \geq f(x) + \frac{1}{\zeta} \langle \nabla f(x), x^* - x \rangle.
\]

For shorthand we write \( f \in QC(\zeta) \) to mean (4). The class of quasar convex functions are parameterized by a positive constant \( \zeta > 0 \). In the case that \( \zeta = 1 \) then (4) is known as the star convexity [40] (generalization of convexity). One can think of \( \zeta \) as the value that controls the non-convexity of the function. In particular, as value of \( \zeta \) becomes smaller or larger than 1 the function becomes "more nonconvex" [16]. We highlight that the Quasar convex functions may have multiple solutions.

One of weakest possible assumptions that guarantee a global convergence of gradient descent to the global minimum is the PL condition [17]. This is largely due to the fact that all local minimas of a function satisfying the PL condition are also global minimas.
We say that $f$ as a direct consequence of Assumption 3.1 we have the following bound on the variance of $ER$.

These two assumptions are sufficient for the interpolation condition has drawn much attention recently because many overparametrized deep neural networks achieve a zero loss over all training data points [32] and thus satisfy (6).

### 3 Expected Residual (ER)

In all of our analysis of SGD we rely on the remarkably weak Expected Residual (ER) assumption. In this section we formally define ER, provide new sufficient conditions for it to hold and relate it to the existing assumptions on the gradient.

ER measures how far the gradient estimate $g(x)$ is from the true gradient in the following sense.

**Assumption 3.1 (Expected residual).** We say that the ER condition holds or $g \in \text{ER}(\rho)$ if

$$
\mathbb{E} \left[ \|g(x) - g(x^*) - (\nabla f(x) - \nabla f(x^*))\|^2 \right] \leq 2 \rho (f(x) - f(x^*)), \quad \forall x \in \mathbb{R}^d. \quad \text{(ER)}
$$

Note that ER depends on both how $g(x)$ is sampled and the properties of the $f(x)$ function.

As a direct consequence of Assumption 3.1 we have the following bound on the variance of $g(x)$.

**Lemma 3.2.** If $g \in \text{ER}(\rho)$ then

$$
\mathbb{E} \left[ \|g(x)\|^2 \right] \leq 4 \rho (f(x) - f^*) + \|\nabla f(x)\|^2 + 2 \sigma^2. \quad \text{(7)}
$$

It is this bound on the variance (7) that we use in our proofs and allows us to avoid the stronger bounded gradient or bounded variance assumptions.

**Connections to other Assumptions.** Let us provide some more familiar sufficient conditions which guarantee that the ER condition holds. In doing so, we will also provide simple and informative bounds on the expected residual constant $\rho$ when using minibatching.

We say that $f_i$ is $L_i$–smooth if

$$
f_i(z) - f_i(x) - \langle \nabla f_i(x), z - x \rangle \leq \frac{L_i}{2} \|z - x\|^2, \quad \forall x, z \in \mathbb{R}^d. \quad \text{(8)}
$$

Let $L_{\max} := \max_{i=1,\ldots,n} L_i$. For $x^* \in \mathcal{X}^*$, we say that $f_i$ is $x^*$–convex if

$$
f_i(x^*) - f_i(x) \leq \langle \nabla f_i(x^*), x^* - x \rangle, \quad \forall x \in \mathbb{R}^d. \quad \text{(9)}
$$

These two assumptions are sufficient for the $\text{ER}(\rho)$ condition to hold and give a useful bound on $\rho$, as we show in the following proposition.

**Proposition 3.3.** Let $v$ be a sampling vector. If $f_i$ is $L_i$–smooth and there exists $x^* \in \mathcal{X}^*$ such that $f_i$ is $x^*$–convex then $g \in \text{ER}(\rho)$. If in addition $v$ is the $b$–minibatch sampling then

$$
\rho(b) = L_{\max} \frac{n - b}{n - 1}, \quad \sigma^2(b) = \frac{1}{b} \frac{n - b}{n - 1} \sigma_1^2 \quad \text{where} \quad \sigma_1 := \sup_{x^* \in \mathcal{X}^*} \frac{1}{b} \sum_{i=1}^n \|\nabla f_i(x^*)\|. \quad \text{(10)}
$$

The bounds in Proposition 3.3 have been proven before but under the stronger assumption that each $f_i$ is convex[^4]. In this work by dropping the requirement that each $f_i$ is convex we are able to consider interesting classes of nonconvex functions. To this end, the following Theorem is of great importance. It establishes that only smoothness and the interpolation condition are sufficient for the ER to hold. Furthermore, we place the ER within a hierarchy of other popular assumptions used in the analysis of SGD for smooth nonconvex functions. We show that ER is the weakest condition.

[^4]: See Proposition 3.10 item (iii) in [11] and Lemma F.3 in [51]
Appendix C. In Appendix D, we present additional convergence results on quasar-strongly convex functions. To the best of our knowledge, the only prior result for the convergence of SGD for smooth quasar-convex functions was a finite horizon result similar to (13) but under the strong assumption of bounded gradient variance. Indeed, strictly stronger since due to Theorem 3.4 the ER condition holds when the gradient is bounded. Of particular importance is (14) which is the first $O\left(\frac{\log(k)}{\sqrt{k}}\right)$ any time convergence rate for quasar convex functions. Indeed, this rate has only been achieved before under the strictly stronger assumption that the $f_i$’s are smooth, convex and $g(x)$ has bounded variance [37]. Indeed, strictly stronger since due to Theorem 3.4 the ER condition holds when the $f_i$’s are smooth and convex without any bounded gradient assumption.

4 Convergence Analysis

In this section, we present the main convergence results. Proofs of all key results can be found in the Appendix C. In Appendix D, we present additional convergence results on quasar-strongly convex functions (Section D.1) and on convergence under expected smoothness (Section D.2).

4.1 Quasar Convex functions

4.1.1 Constant and Decreasing Stepsizes

Now we present our results for quasar-convex functions for SGD with a constant, finite horizon and decreasing step sizes.

Theorem 4.1. Assume $f(x)$ is $L$-smooth, $\zeta$-quasar-convex with respect to $x^*$ and $g \in ER(\rho)$. Let $0 < \gamma_k < \frac{\zeta}{2\rho+L}$ for all $k \in \mathbb{N}$ and let $r_0 = \|x^0 - x^*\|^2$. SGD (3) converges as

$$\min_{t=0,\ldots,k-1} \mathbb{E} \left[ f(x_t) - f(x^*) \right] \leq \frac{1}{\sum_{t=0}^{k-1} \gamma_t (\zeta - \gamma_t (2\rho + L))} \left[ \frac{r^0}{2} + \sigma^2 \sum_{t=0}^{k-1} \gamma_t^2 \right].$$

Moreover, for $\gamma < \frac{\zeta}{2\rho+L}$ we have that

1. If $\forall k \in \mathbb{N}$, $\gamma_k = \frac{\zeta}{2(2\rho+L)}$ then $\forall k \in \mathbb{N},$

$$\min_{t=0,\ldots,k-1} \mathbb{E} \left[ f(x_t) - f(x^*) \right] \leq 2r_0 \frac{2\rho+L}{\zeta^2} + \frac{\sigma^2}{2\rho+L}.$$

2. Suppose SGD (3) is run for $T$ iterations. If $\forall k = 0,\ldots,T-1$, $\gamma_k = \frac{\zeta}{\sqrt{T}}$ then

$$\min_{t=0,\ldots,T-1} \mathbb{E} \left[ f(x_t) - f(x^*) \right] \leq \frac{r_0 + 2\gamma^2\sigma^2}{\sqrt{T}}.$$

3. If $\forall k \in \mathbb{N}$, $\gamma_k = \frac{\zeta}{\sqrt{k+1}}$ then $\forall k \in \mathbb{N},$

$$\min_{t=0,\ldots,k-1} \mathbb{E} \left[ f(x_t) - f(x^*) \right] \leq \frac{1}{\sqrt{T}} \frac{r_0 + 2\gamma^2\sigma^2 (\log(k)+1)}{\zeta (k+1)} + \gamma (\rho+L/2) (\log(k)+1) \sim O\left(\frac{\log(k)}{\sqrt{k}}\right).$$

The importance of the assumptions for analyzing SGD in the nonconvex setting are the ones that are downstream from $L_i$ + Interpolated. This is because these exists a rich class of nonconvex functions that are smooth and satisfy the interpolation condition. In contrast, the WGC is only known to hold for smooth and convex functions that also satisfy the interpolation assumptions (Proposition 2 in [54]). A hierarchy of assumptions that also includes the SGC and a parametrized family of assumptions that include the ER condition as a special was also recently presented in [19].

To the best of our knowledge, the only prior result for the convergence of SGD for smooth quasar convex functions was a finite horizon result similar to (13) but under the strong assumption of bounded gradient variance. Of particular importance is (14) which is the first $O\left(\frac{\log(k)}{\sqrt{k}}\right)$ any time convergence rate for quasar convex functions. Indeed, this rate has only been achieved before under the strictly stronger assumption that the $f_i$’s are smooth, convex and $g(x)$ has bounded variance [37]. Indeed, strictly stronger since due to Theorem 3.4 the ER condition holds when the $f_i$’s are smooth and convex without any bounded gradient assumption.
When considering interpolated functions, we can completely drop the ER condition due to Theorem 3.4. In this next corollary we highlight this and show how the complexity of SGD is affected by increasing the minibatch size.

**Corollary 4.2.** Let \( f \) be \( \zeta \)-quasar convex with respect to \( x^* \), the interpolation assumption 2.3 hold and let each \( f_i \) be \( L_i \)-smooth. If \( v \) is a \( b \)-minibatch sampling and \( \gamma_k = \frac{1}{2TL_{\max}(n-b)+L(n-1)b} \) then

\[
\min_{t=0,\ldots,k-1} \mathbb{E} \left[ f(x^t) - f(x^*) \right] \leq \frac{2L_{\max}(n-b)+L(n-1)b}{\zeta(n-1)b} 2r^2 \eta_k. \tag{15}
\]

This shows that the total complexity to bring \( \min_{t=1,\ldots,k-1} \mathbb{E} \left[ f(x^t) - f^* \right] \leq \epsilon \), is given by

\[
\text{Total Complexity}(b) \geq \frac{2(n-b)L_{\max}+(n-1)bL}{\zeta(n-1)} 2r^2 \eta_0 \eta. \tag{16}
\]

Thus the optimal minibatch size \( b^* \) that minimizes this total complexity is given by

\[
b^* = \begin{cases} 
1 & \text{if } (n-1) \geq \frac{2L_{\max}}{\zeta} \\
\infty & \text{if } (n-1) < \frac{2L_{\max}}{\zeta}.
\end{cases} \tag{17}
\]

Specializing (15) to the full batch setting \( (n=b) \), we have that gradient descent (GD) with step size \( \gamma = \frac{\zeta}{2} \) converges as follows:

\[
f(x^t) - f(x^*) \leq \frac{2\|x-x^*\|^2}{\zeta \zeta} \frac{1}{k}. \tag{5}
\]

This is exactly the rate given recently for GD for quasar-convex functions in [12], with the exception that we have a squared dependency on \( \zeta \) the quasar convex parameter.

### 4.1.2 Stochastic Polyak Step-size (SPS) - Guarantee Convergence without tuning

The stochastic Polyak step size (SPS) is a recently proposed step size selection for SGD [30]. We generalize the SPS to the arbitrary sampling regime and provide a new convergence analysis of SGD with SPS for the class of smooth, quasar (strongly) convex functions.

Let \( v \) be a sampling vector and let \( f_v = \sum_{i=1}^n f_i(x)v_i \). Let \( f_v^* = \min_{x \in \mathcal{B}} f_v(x) \) which we assume exists. Just like the gradient, we have that \( f_v^* \) is an unbiased estimate of \( f \). Now given a sampling vector \( v \), we define the Stochastic Polyak Step size (SPS) as

\[
\text{SPS: } \gamma_k = \frac{f_v(x^k)-f_v^*}{\mathbb{E}[\|\nabla f_v(x^k)\|^2]}, \tag{18}
\]

where \( 0 < \gamma < \infty \). The SPS rule is particularly useful when the interpolation Assumption 2.3 holds and each \( f_i \) represents a loss function, since then we have that \( f_i^* = f_v^* = f(x^*) = 0 \) for every \( i \in [n] \) and realization of \( v \).

By assuming that every \( f_i \) is \( L_i \)-smooth, we have that \( f_v \) is \( L_v \)-smooth with \( L_v := \frac{1}{n} \sum_{i=1}^n v_i L_i \).

This smoothness combined with Lemma A.2 and Jensen’s inequality gives a lower bound on SPS (18):

\[
\mathbb{E} \left[ \gamma_k \right] \leq \mathbb{E} \left[ \frac{1}{E_{L_v}} \right] \leq \mathbb{E} \left[ \frac{1}{\mathbb{E}[L_v]} \right] \leq \mathbb{E} \left[ \frac{f_v(x^k)-f_v^*}{\mathbb{E}[\|\nabla f_v(x^k)\|^2]} \right]. \tag{19}
\]

This lower bound and the following new expected smoothness bound allows us to establish the forthcoming theorem for quasar convex functions.

**Lemma 4.3.** Assume interpolation 2.3 holds. Let \( f_i \) be \( L_i \)-smooth and let \( v \) be a sampling vector. It follows that there exists \( L_{\max} \geq 0 \) such that

\[
\mathbb{E} \left[ f(x^k) - f^* \right] \leq \mathbb{E} \left[ \frac{(f_v(x)-f_v^*)^2}{\|\nabla f_v(x)^\|^2} \right]. \tag{20}
\]

Furthermore, for \( B \subset \{1,\ldots,n\} \) let \( L_B \) be the smoothness constant of \( f_B := \frac{1}{n} \sum_{i \in B} p_i f_i \). If \( v \) is the \( b \)-minibatch sampling then \( L_{\max} = L_{\max}(b) = \max_{i=1,\ldots,n} \sum_{B \ni i \in B} \frac{(n-1)}{(n-1-b)} L_{B}^{-1} \).

With the above lemma we can now establish our main theorem.

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3Here we use that the smoothness of \( f \) guarantees that \( f(x^1), \ldots, f(x^t) \) for GD is a decreasing sequence.
Theorem 4.4. Let $v$ be a sampling vector. Assume interpolation 2.3 holds. Assume that each $f_i$ is $\zeta$-quasar convex with respect to $x^*$ and $L_i$-smooth. Then SGD with SPS (18) and $\epsilon > \frac{1}{\zeta^2}$ converges as follows:
\[
\min_{i=0,\ldots,K-1} \mathbb{E}[f(x^i) - f^*] \leq 2\epsilon \frac{\mathcal{L}_{\max}}{\zeta^2} L_{\max} \|x^0 - x^*\|^2,
\]
where $\mathcal{L}_{\max}$ is the expected smoothness constant defined in Lemma (4.3).

We now use $\mathcal{L}_{\max}(b)$ given in Lemma 4.3 to derive the importance sampling complexity. To the best of our knowledge, this is the first importance sampling result for SGD with SPS in any setting.

Corollary 4.5. Consider the setting of Theorem 4.4 with $c = 1/4\zeta$. Given $\epsilon > 0$ we have that
\[
k \geq \frac{\mathcal{L}_{\max} \|x^0 - x^*\|^2}{\epsilon} = O\left(\frac{\mathcal{L}_{\max}}{\zeta^2}\epsilon^{-1}\right) \Rightarrow \min_{i=0,\ldots,K-1} \mathbb{E}[f(x^i) - f^*] < \epsilon.
\]

1. (Full batch) If we use full batch sampling we have that $\mathcal{L}_{\max} = L$ and (21) becomes $O(L/\epsilon\zeta^2)$
2. (Importance sampling). If we use single element sampling with $p_i = L_i / \sum_j L_j$ we have that $\mathcal{L}_{\max} = \frac{1}{n} \sum_j L_j := L$ and (21) becomes $O(L/\epsilon\zeta^2)$.

4.2 PL condition

Here we present our convergence results for functions satisfying the PL condition.

Theorem 4.6. Let $f$ be $L$-smooth. Assume $f \in PL(\mu)$ and $g \in \mathbb{R}(\rho)$. Let $\gamma_k = \gamma \leq \frac{1}{1 + 2\rho/\mu} \frac{1}{L}$ for all $k$, then SGD given by (3) converges as follows:
\[
\mathbb{E}[f(x^k) - f^*] \leq (1 - \gamma \mu)^k \left[ f(x^0) - f^* \right] + \frac{L\gamma\sigma^2}{\mu}.
\]

Hence, given $\epsilon > 0$ and using the step size $\gamma = \frac{1}{L} \min \left\{ \frac{\mu\epsilon}{2\sigma^2}, \frac{1}{1 + 2\rho/\mu} \right\}$ gives
\[
k \geq \frac{L}{\mu} \max \left\{ \frac{2\sigma^2}{\mu\epsilon}, 1 + \frac{2\rho}{\mu} \right\} \log \left( \frac{2(f(x^0) - f^*)}{\epsilon} \right) \Rightarrow \mathbb{E}[f(x^k) - f^*] \leq \epsilon.
\]

In Appendix C.7 based on Theorem 4.6, we show how to obtain a $O(1/k)$ convergence for SGD using an insightful stepsize-switching rule. This stepsize-switching rule describes when one should switch from a constant to a decreasing stepsize regime. When the function is interpolated, a constant step size gives a linear rate of convergence, as we show in the next theorem.

Corollary 4.7. Consider the setting of Theorem 4.6 and let assume interpolation 2.3 holds. Then SGD with $\gamma_k = \gamma \leq \frac{1}{1 + 2\rho/\mu} \frac{1}{L}$, converges linearly at a rate of $(1 - \gamma \mu)$. Consequently for every $\epsilon > 0$, the iteration complexity of SGD to achieve $\mathbb{E}[f(x^k) - f^*] \leq \epsilon$ is
\[
k \geq \frac{L}{\mu} \left( 1 + \frac{2\rho}{\mu} \right) \log \left( \frac{f(x^0) - f^*}{\epsilon} \right).
\]

If $v$ is a $b$-minibatch sampling then the total complexity is given by
\[
\text{Total Complexity}(b) \geq \frac{L}{\mu} \left( b + 2\frac{\mathcal{L}_{\max}}{\mu} \frac{n - b}{n - 1} \right) \log \left( \frac{f(x^0) - f^*}{\epsilon} \right).
\]

Finally, let $\kappa_{\max} := L_{\max}/\mu$. The minibatch size $b^*$ that optimizes the total complexity is given by
\[
b^* = \begin{cases} 1 & \text{if } n - 1 \geq 2\kappa_{\max} \\ n & \text{if } n - 1 < 2\kappa_{\max}. \end{cases}
\]

Note that Corollary 4.7 recovers the linear convergence rate of the gradient descent algorithm under the PL condition [17] as a special case. Indeed for gradient descent we have that $\sigma = 0 = \rho$. Thus by choosing $\gamma = \frac{1}{L}$ the resulting iteration complexity is $\frac{L}{\mu} \log(\epsilon^{-1})$ which was already proven in [17] and is the tightest known convergence result for gradient descent under the PL condition. On the other extreme, we see that for $b = 1$, that is SGD without minibatching, we obtain the convergence rate $1 - \mu^2/3L_{\max}$ which matches the current state-of-the-art rate [54, Thm. 4], [19, Thm. 2] and [24, Thm 4] known under the exact same assumptions. Thus we recover the best known rate on either
We establish a hierarchy between the expected residual (ER) condition and a host of other assumptions as an immediate extension of Theorem 4.4, if we were to consider the constrained optimization problem \( \min_{x \in C} f(x) \) where \( C \subset \mathbb{R}^d \) and the corresponding projected SGD method, then assuming that each \( f_i(x) \) is smooth reduces to assuming that \( f_i \) is twice continuously differentiable over \( C \) \(^7\) thus greatly expanding the number of applications we can consider.

\(^6\) In [17] the authors claim that \( x^2 + 3 \sin^2(x) \) is PL. We then used computer aided analysis to show that \( x^2 + 3b \sin^2(x) \) satisfies the PL condition for \( 0 < b < 4 \). \(^7\) Since \( f_i \) would be \( \max_{x \in C} \| \nabla^2 f(x) \| \)-smooth.
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The Supplementary Material is organized as follows: In Section A, we give some lemmas and consequences of smoothness. In Section B we present the proofs of the proposition, lemma and theorem related to the Expected Residual condition as presented in Section 3 of the main paper. In Section C we present the proofs of the main theorems. In Section D we provide additional convergence results under the strongly quasar convex assumption (Section D.1), the Expected Smoothness assumption (Section D.2) and a minibatch analysis that does not rely on the interpolation condition (Section D.3).

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A Technical Lemmas on Smoothness

Here we give some lemmas and consequences of smoothness. For all of our analysis we do not need that the $f_i$ functions be smooth in all directions. Rather, we just need them to be smooth along the $x^*$–direction, as we define next.

**Definition A.1.** We say that $f : \mathbb{R}^d \mapsto \mathbb{R}$ is $L$–smooth function along the $x^*$–direction if there exists $x^*$ such that
\[
    f(z) - f(x) \leq \langle \nabla f(x), z - x \rangle + \frac{L}{2} \|z - x\|^2, \quad \forall x \in \mathbb{R}^d, \tag{28}
\]
where
\[
    z = x - \frac{1}{L} (\nabla f(x) - \nabla f(x^*)). \tag{29}
\]

By inserting $z$ into (28) we can equivalently write (28) as
\[
    f(x - (1/L)(\nabla f(x) - \nabla f(x^*))) \leq f(x) - \frac{1}{2L} \|\nabla f(x)\|^2 + \frac{1}{2L} \|\nabla f(x^*)\|^2. \tag{29}
\]

**Lemma A.2.** Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be differentiable and suppose $f$ has a minimizer $x^* \in \mathbb{R}^d$. Furthermore, let $f$ be $L$–smooth function along the $x^*$–direction according to Definition A.1. It follows that
\[
    \|\nabla f(x)\|^2 \leq 2L(f(x) - f(x^*)). \tag{30}
\]

**Proof.** Since $x^*$ is a minimizer of $f$ we have that $\nabla f(x^*) = 0$. Furthermore, since $f$ is $L$–smooth function along the $x^*$–direction we have by re-arranging (29) that
\[
    f(x^*) - f(x) \leq f(x - (1/L)\nabla f(x)) - f(x) \leq -\frac{1}{2L} \|\nabla f(x)\|^2.
\]
Re-arranging the above gives (30). \qed

Now we provide a lemma that will then be used to establish the simplest and most minimalistic assumptions that imply the expected residual (ER) condition (Assumption 3.1).

**Lemma A.3.** Suppose there exists $x^* \in \mathbb{R}^d$ where
\[
    x^* \in \arg \min \left\{ f(x) := \frac{1}{n} \sum_{i=1}^n f_i(x) \right\},
\]
such that each $f_i$ is convex around $x^*$, that is
\[
    f_i(x^*) - f_i(x) \leq \langle \nabla f_i(x^*), x^* - x \rangle, \quad \forall x \in \mathbb{R}^d, \tag{31}
\]
and each $f_i$ is $L_i$–smooth along the $x^*$–direction according Definition A.1. It follows for every $i \in \{1, \ldots, n\}$ that
\[
    \|\nabla f_i(x) - \nabla f_i(x^*)\|^2 \leq 2L_i(f_i(x) - f_i(x^*) - \langle \nabla f_i(x^*), x^* - x \rangle), \quad \forall x \in \mathbb{R}^d. \tag{32}
\]

**Proof.** Fix $i \in \{1, \ldots, n\}$. To prove (32), it follows that
\[
    f_i(x^*) - f_i(x) \leq f_i(x^*) - f_i(z) + f_i(z) - f_i(x) \leq \frac{1}{L_i} \langle \nabla f_i(x^*), x^* - z \rangle + \langle \nabla f_i(x), z - x \rangle + \frac{L_i}{2} \|z - x\|^2, \tag{33}
\]
where
\[
    z = x - \frac{1}{L_i} (\nabla f_i(x) - \nabla f_i(x^*)). \tag{34}
\]
Substituting this in $z$ into (33) gives
\[
    f_i(x^*) - f_i(x) = \langle \nabla f_i(x^*), x^* - x + \frac{1}{L_i} (\nabla f_i(x) - \nabla f_i(x^*)) \rangle - \frac{1}{L_i} \langle \nabla f_i(x), \nabla f_i(x) - \nabla f_i(x^*) \rangle + \frac{1}{2L_i} \|\nabla f_i(x) - \nabla f_i(x^*)\|^2
\]
\[
    = \langle \nabla f_i(x^*), x^* - x \rangle - \frac{1}{L_i} \|\nabla f_i(x) - \nabla f_i(x^*)\|^2 + \frac{1}{2L_i} \|\nabla f_i(x) - \nabla f_i(x^*)\|^2
\]
\[
    = \langle \nabla f_i(x^*), x^* - x \rangle - \frac{1}{2L_i} \|\nabla f_i(x) - \nabla f_i(x^*)\|^2.
\]
Now we present a corollary of the previous lemma for over-parametrized functions. We now develop an immediate consequence of each $f_i$ being convex around $x^*$ and smooth along the $x^*$-direction.

**Corollary A.4.** Suppose these exists $x^* \in \mathbb{R}^d$ where

$$x^* \in \arg\min \left\{ f(x) := \frac{1}{n} \sum_{i=1}^n f_i(x) \right\}.$$ 

Suppose the interpolated Assumption 2.3 holds. Furthermore, suppose that for each $f_i$ there exists $L_i$ such that

$$f_i \left( x - \frac{1}{L_i} \nabla f_i(x) \right) \leq f_i(x) - \frac{1}{2L_i} \| \nabla f_i(x) \|^2 .$$ \hspace{1cm} (35)

It follows for every $i \in \{1, \ldots, n\}$ that

$$\| \nabla f_i(x) - \nabla f_i(x^*) \|^2 \leq 2L_i(f_i(x) - f_i(x^*)) . \hspace{1cm} \forall x \in \mathbb{R}^d .$$ \hspace{1cm} (36)

**Proof.** Note that for interpolated functions we have that each $f_i$ is convex around $x^*$. Furthermore, since each $\nabla f_i(x^*) = 0$ we have that (29) holds, and thus $f_i$ is smooth in the $x^*$-direction according to Definition A.1. Finally all the conditions of Lemma A.3 holds, and thus so does (36) holds.

---

**B Proofs of results on Expected Residual**

**B.1 Proof of Lemma 3.2**

**Proof.** Using

$$\| g(x) - \nabla f(x) \|^2 \leq 2 \| g(x) - g(x^*) - \nabla f(x) \|^2 + 2 \| g(x^*) \|^2 ,$$

and taking expectation together with (ER) and $\nabla f(x^*) = 0$ gives

$$\mathbb{E} \left[ \| g(x) - \nabla f(x) \|^2 \right] \leq 4\rho(f(x) - f(x^*)) + 2\mathbb{E}_D \left[ \| g(x^*) \|^2 \right] .$$

Taking the supremum over $x^* \in X^*$ and using Assumption 1.2 and that $\mathbb{E} \left[ \| X - \mathbb{E}[X] \|^2 \right] = \mathbb{E} \left[ \| X \|^2 \right] - \| \mathbb{E}[X] \|^2$ with $X = g(x)$ gives (7).

**B.2 Proof of Proposition 3.3 and its expansion to all samplings.**

In this section we give an expanded version of Proposition 3.3 that also gives bounds for the Expected Smoothness assumption (ES), a closely related assumption to the Expected Residual condition.

**Assumption B.1 (Expected smoothness).** We say that the stochastic gradient $g$ satisfy the expected smoothness assumption if for all $x \in \mathbb{R}^d$, there exists $\mathcal{L} = \mathcal{L}(g) > 0$ such that

$$\mathbb{E}_D \left[ \| g(x) - g(x^*) \|^2 \right] \leq 2\mathcal{L} \left( f(x) - f(x^*) \right) .$$ \hspace{1cm} (ES)

We use $g \in \text{ES}(\mathcal{L})$ as shorthand for expected smoothness.

Here we show that a sufficient condition for the expected smoothness and the expected residual conditions B.1 and 3.1 to hold if that each $f_i$ is convex around $x^*$ and smooth. Furthermore, we give tight bounds on the expected smoothness $\mathcal{L}$ and the expected residual constant $\rho$ for when $\nu$ is an independent sampling and, in particular, a $b$-minibatch sampling.

In the main text our minibatch results are stated only for $b$-minibatching. But they actually hold for a large family of sampling that we refer to as the independent samplings.
we show that only convexity around $x$. The following Proposition is based on the proof of Proposition 3.8 in [11] with the exception that now

\[ \mathbb{P}[i \in S] = \frac{b}{n} = \mathbb{P}[j \in S] \quad \text{and} \quad \mathbb{P}[i, j \in S] = \frac{b}{n} \left[ \frac{b-1}{n} - \frac{1}{n-1} \right]. \]  

(37)

In [11] it was proven that an independent sampling vector is indeed a valid sampling vector. For completeness we also give the proof in Lemma C.2. Furthermore, all the samplings presented in [11] are examples of an independent sampling vector. In particular the minibatch sampling in Definition 1.1 is also an independent sampling. Finally, note that (37) does not imply that $i \in S$ and $j \in S$ are independent events unless $c_2 = 1$. Indeed, for $b$-minibatch sampling we have that

\[ \mathbb{P}[i \in S] = \frac{b}{n} = \mathbb{P}[j \in S] \quad \text{and} \quad \mathbb{P}[i, j \in S] = \frac{b}{n} \left[ \frac{b-1}{n} - \frac{1}{n-1} \right] \]  

and thus they are not independent events yet satisfy (37) with $c_2 = \frac{n-b}{n}$. 

The following Proposition is based on the proof of Proposition 3.8 in [11] with the exception that now we show that only convexity around $x^*$ is required for the proof to follow, as opposed to assuming convexity everywhere.

**Proposition B.3.** Let $f$ be a finite sum problem $f = \frac{1}{n} \sum_{i=1}^{n} f_i$. Let $f_i$ be $L_i$-smooth and convex around $x^*$ according to (A.1) and (31), respectively. It follows that

1. If $v$ is a sampling vector then the expected smoothness and expected residual conditions hold $g \in \text{ES}(\mathcal{L})$ and $g \in \text{ER}(\rho)$ with $\mathcal{L} = \max_i L_i$ and $\rho = \frac{1}{n} \sum_{i=1}^{n} \rho_i$, respectively.

2. If $v$ is an independent sampling vector according to Definition B.2 then we have that

\[ \mathcal{L} = c_2 L + \max_{i=1, \ldots, n} \frac{L_i}{n} \left( 1 - p_i c_2 \right). \]  

(38)

\[ \rho = \frac{\lambda_{\max}(\mathbb{E}[v-1])}{n} \mathcal{L}_{\max}, \]  

(39)

3. If $v$ is the $b$-minibatch sampling with replacement then

\[ \sigma^2 = \frac{1}{n} \left( \frac{n-b}{n-1} \right) \sigma_i^2 \]  

(40)

\[ \rho = \frac{1}{n} \left( \frac{n-b}{n-1} \right) \mathcal{L}_{\max}, \]  

(41)

\[ \mathcal{L} = \frac{n-b}{n-1} L + \frac{1}{n} \left( \frac{n-b}{n-1} \right) \mathcal{L}_{\max}. \]  

(42)

**Proof.**

1. Assume that $v$ is any sampling vector. Since $f_i$ is $L_i$-smooth and convex around $x^*$ we have that by multiplying each side of

\[ f_i(z) - f_i(x) \leq \langle \nabla f_i(x), z - x \rangle + \frac{L_i}{2} \|z - x\|^2 \]  

and

\[ f_i(x^*) - f_i(x) \leq \langle \nabla f_i(x^*), x^* - x \rangle, \]  

by $v_i/n$ and summing up over $i = 1, \ldots, n$ bearing in mind that $v_i \geq 0$ we have that

\[ f_v(z) - f_v(x) \leq \langle \nabla f_v(x), z - x \rangle + \frac{1}{n} \sum_{i=1}^{n} v_i L_i \|z - x\|^2 \]  

and

\[ f_v(x^*) - f_v(x) \leq \langle \nabla f_v(x^*), x^* - x \rangle. \]  

Consequently $f_v$ is convex and $x^*$ is also $L_v$-smooth where $L_v := \frac{1}{n} \sum_{i=1}^{n} v_i L_i$. Applying Lemma A.3 we thus have that

\[ \|\nabla f_v(x) - \nabla f_v(x^*)\|^2 \leq L_v (f_v(x) - f_v(x^*) - \langle \nabla f_v(x^*), x - x^* \rangle), \quad \forall x \in \mathbb{R}^d. \]  

(43)

Taking expectation gives

\[ \mathbb{E} [\|\nabla f_v(x) - \nabla f_v(x^*)\|^2] \leq \mathbb{E} [L_v (f_v(x) - f_v(x^*) - \langle \nabla f_v(x^*), x - x^* \rangle)] \]  

\[ \leq \max_v \mathbb{E} [(f_v(x) - f_v(x^*) - \langle \nabla f_v(x^*), x - x^* \rangle)] \]  

\[ = \max_v L_v (f(x) - f(x^*)). \]  

(44)

This proves that the expected smoothness assumption holds with $\mathcal{L} = \max_v L_v$. Consequently by Theorem 3.4 we have that the expected residual condition holds with $\rho = \mathcal{L}$. 

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2. Assume that \(v_i\) is an independent sampling. First we prove (38).

Since \(f_i\) is \(L_i\)-smooth and convex around \(x^*\) we have that \(f\) is \(L\)-smooth and convex around \(x^*\) and by Lemma A.3

\[
\|\nabla f_i(x) - \nabla f_i(x^*)\|^2 \leq 2L_i(f_i(x) - f_i(x^*) - \langle \nabla f_i(x^*), x - x^* \rangle) \tag{44}
\]

\[
\|\nabla f(x) - \nabla f(x^*)\|^2 \leq 2L(f(x) - f(x^*) - \langle \nabla f(x^*), x - x^* \rangle). \tag{45}
\]

Noticing that

\[
\|\nabla f_i(x) - \nabla f_i(x^*)\|^2 = \frac{1}{n^2} \left\| \sum_{i \in S} \frac{1}{p_i} (\nabla f_i(x) - \nabla f_i(x^*)) \right\|^2 = \sum_{i,j \in S} \left( \frac{1}{np_i} (\nabla f_i(x) - \nabla f_i(x^*)) \right),
\]

we have

\[
\mathbb{E}[\|\nabla f_i(x) - \nabla f_i(x^*)\|^2] = \sum_{C} p_C \sum_{i,j \in C} \left( \frac{1}{np_i} (\nabla f_i(x) - \nabla f_i(x^*)) \right) = \sum_{i,j=1, C : i,j \in C} \left( \frac{1}{np_i} (\nabla f_i(x) - \nabla f_i(x^*)) \right) = \sum_{i,j=1, p_i,p_j} \left[ \frac{1}{n} (\nabla f_i(x) - \nabla f_i(x^*)) \right],
\]

where we used a double counting argument in the 2nd equality. Now since \(\mathbb{P}[i,j \in S] / (p_i p_j) = c_2\) for \(i \neq j\). Recalling that \(\mathbb{P}[i,i \in S] = p_i\) we have from the above that

\[
\mathbb{E}[\|\nabla f_i(x) - \nabla f_i(x^*)\|^2] = \sum_{i \neq j} c_2 \left( \frac{1}{n} (\nabla f_i(x) - \nabla f_i(x^*)) \right) + \sum_{i=1}^n \frac{1}{n^2} \| \nabla f_i(x) - \nabla f_i(x^*) \|^2
\]

\[
= \sum_{i,j=1}^n c_2 \left( \frac{1}{n} (\nabla f_i(x) - \nabla f_i(x^*)) \right) + \sum_{i=1}^n \frac{1}{n^2} \| \nabla f_i(x) - \nabla f_i(x^*) \|^2 \leq c_2 \| \nabla f(x) - \nabla f(x^*) \|^2
\]

\[
+ 2 \sum_{i=1}^n \frac{1}{np_i} (1 - p_i c_2) \| \nabla f_i(x) - \nabla f_i(x^*) \|^2 \leq 2 \left( c_2 L + \max_{i=1, \ldots, n} \frac{1}{np_i} (1 - p_i c_2) \right) (f(x) - f(x^*) - \langle \nabla f(x^*), x - x^* \rangle). \tag{46}
\]

Comparing the above to the definition of expected smoothness (ES) we have that

\[
\mathcal{L} \leq c_2 L + \max_{i=1, \ldots, n} \frac{1}{np_i} (1 - p_i c_2). \tag{46}
\]

Now we will prove that

\[
\mathbb{E} \left[ \| \nabla f_i(w) - \nabla f_i(x^*) - (\nabla f(w) - \nabla f(x^*)) \|^2 \right] \leq 2\rho (f(w) - f(x^*)), \tag{47}
\]

holds with the constant given in (39). First we expand the squared norm on the left hand side of (47). Define \(DF(w) = [\nabla f_1(w), \ldots, \nabla f_n(w)] \in \mathbb{R}^{d \times n}\) as the Jacobian of
\( F(w) \overset{def}{=} [f_1(w), \ldots, f_n(w)] \). We denote \( \mathbf{R} := (DF(w) - DF(x^*)) \). It follows that
\[
C := \|\nabla f_i(w) - \nabla f_i(x^*) - (\nabla f(w) - \nabla f(x^*))\|^2
= \frac{1}{n^2} \|DF(w) - DF(x^*) - (v - 1)\|^2
= \frac{1}{n^2} \|\mathbf{R}(v - 1)\|^2_{\mathbb{R}^d}
= \frac{1}{n^2} \text{Trace} \left( (v - 1)\mathbf{R}^\top \mathbf{R}(v - 1) \right)
= \frac{1}{n^2} \text{Trace} \left( \mathbf{R}^\top \mathbf{R}(v - 1)(v - 1)^\top \right).
\]

Let \( \text{Var} [v] := \mathbb{E} \left[ (v - 1)(v - 1)^\top \right] \). Taking expectation,
\[
\mathbb{E} [C] = \frac{1}{n^2} \text{Trace} \left( \mathbf{R}^\top \mathbf{R} \text{Var} [v] \right)
\leq \frac{1}{n^2} \text{Trace} \left( \mathbf{R}^\top \mathbf{R} \right) \lambda_{\max} (\text{Var} [v]).
\]

Moreover, since the \( f_i \)'s are convex around \( x^* \) and \( L_i \)-smooth, it follows from (32) that
\[
\text{Trace} \left( \mathbf{R}^\top \mathbf{R} \right) = \sum_{i=1}^n \|\nabla f_i(w) - \nabla f_i(x^*)\|^2
\leq 2 \sum_{i=1}^n L_i(f_i(w) - f_i(x^*) - \langle \nabla f_i(x^*), w - x^* \rangle)
\leq 2nL_{\max} (f(w) - f(x^*)).
\]
Therefore,
\[
\mathbb{E} [C] \overset{(48)+(49)}{\leq} \frac{2\lambda_{\max} (\text{Var} [v])}{n} L_{\max} (f(w) - f(x^*)).
\]
Which means
\[
\rho = \frac{\lambda_{\max} (\text{Var} [v])}{n} L_{\max}.
\]

3. Finally, if \( v \) is a \( b \)-minibatch sampling, the specialized expressions for \( \mathcal{L} \) in (42) follows by observing that \( \mathbb{P} \{i \in S\} = p_i = \frac{b}{n}, \mathbb{P} \{i, j \in S\} = \frac{b(b-1)}{n(n-1)} \) and consequently \( c_2 = \frac{b(n-1)}{n-1} \).
The specialized expressions for \( \sigma \) and \( \rho \) in (40) and (41) follow from Proposition 3.8 [11] and Lemma F.3 in [51], respectively.

\[
\square
\]

B.3 Proof of Theorem 3.4

First we include the formal definition of each of these assumptions named in Theorem 3.4. Let \( g(x) = \nabla f_i(x) \) denote the stochastic gradient. The results in this section carry over verbatim by using \( g(x) = \nabla f_i(x) \) and \( f_i = f_i \) instead, where \( v \) is a sampling vector. But since the sampling only affects the constants in each of the forthcoming assumptions, and here we are only interested in a hierarchy between assumptions, we omit the proof for a general sampling vector.

First we repeat the definitions of \( ES, WGC \) and \( SGC \) from Assumption 2 [19], Assumption 2.1 in [11], Eq (7) and Eq (2) in [54], respectively.

**SGC: Strong Growth Condition.** We say that \( SGC \) holds with \( \rho_{\text{SGC}} > 0 \) if
\[
\mathbb{E} \left[ \|g(x)\|^2 \right] \leq \rho_{\text{SGC}} \|\nabla f(x)\|^2.
\]

**WGC: Weak Growth Condition.** We say that \( WGC \) holds with \( \rho_{\text{WGC}} > 0 \) if
\[
\mathbb{E} \left[ \|g(x)\|^2 \right] \leq 2\rho_{\text{WGC}} (f(x) - f(x^*)).
\]

**ES: Expected Smoothness.** We say that \( ES \) holds with \( \mathcal{L} > 0 \) if
\[
\mathbb{E} \left[ \|g(x) - g(x^*)\|^2 \right] \leq 2\mathcal{L} (f(x) - f(x^*)).
\]
ER: Expected Residual. We say that ER holds with $\rho > 0$ if
\[
E \left[ \| g(x) - g(x^*) - (\nabla f(x) - \nabla f(x^*)) \|^2 \right] \leq 2\rho (f(x) - f(x^*)).
\] (55)
In addition we will use

$x^*$–convex. We say that $x^*$–convex holds if
\[
f_i(x^*) - f_i(x) \leq \langle \nabla f_i(x^*), x^* - x \rangle,
\]
for $i = 1, \ldots, n.$
(56)

$L_i$–smoothness. We say that $L_i$–smoothness holds for $L_i > 0$ if
\[
f_i(z) - f_i(x) \leq \langle \nabla f_i(x), z - x \rangle + \frac{L_i}{2} \| z - x \|^2,
\]
for $x,z \in \mathbb{R}^d$, $i = 1, \ldots, n.$
(57)

Interpolated. We say that the interpolation condition holds at $x^*$ if
\[
f_i(x^*) \leq f_i(x),
\]
for $i = 1, \ldots, n$, and for every $x \in \mathbb{R}^d$.
(58)

Now we repeat the statement of Theorem 3.4 for convenience.

Theorem B.4. The following hierarchy holds

\[
\begin{array}{|c|c|c|c|}
\hline
SGC + L–smooth & \Rightarrow & WGC & \Rightarrow \quad ES & \Rightarrow \quad ER \\
\hline
\end{array}
\]

In addition we have that $ES(L) + PL(\mu) \Rightarrow ER(L - \mu)$ and $ER \not\Rightarrow ES.$

Proof. We first prove the top row of implications.

1. **SGC + L–smooth $\Rightarrow$ WGC.** Using Lemma A.2 and (52) we have that

\[
E \left[ \| g(x) \|^2 \right] \leq \rho_{SGC} \| \nabla f(x) \|^2
\]
\[
\leq 2L\rho_{SGC} (f(x) - f(x^*)).
\]

Thus (53) holds with $\rho_{WGC} = 2L\rho_{SGC}$.

2. **WGC $\Rightarrow$ ES.**

Plugging in $x = x^*$ in WGC (53) gives $g(x^*) = 0$ almost surely. Since $g(x^*) = 0$ we have that (53) gives (54).

3. **ES $\Rightarrow$ ER.** Expanding the squares of the left hand side of (55) gives

\[
\| g(x) - g(x^*) - (\nabla f(x) - \nabla f(x^*)) \|^2 = \| g(x) - g(x^*) \|^2 + \| \nabla f(x) - \nabla f(x^*) \|^2
\]
\[
- 2 \langle g(x) - g(x^*), \nabla f(x) - \nabla f(x^*) \rangle.
\]

Now assuming that $ES$ (54) holds, taking expectation and using that $E [g(x)] = \nabla f(x)$ we have that

\[
E \left[ \| g(x) - g(x^*) - (\nabla f(x) - \nabla f(x^*)) \|^2 \right] = E \left[ \| g(x) - g(x^*) \|^2 \right] - \| \nabla f(x) - \nabla f(x^*) \|^2
\]
\[
\leq E \left[ \| g(x) - g(x^*) \|^2 \right]
\]
\[
\leq 2\mathcal{L}(f(x) - f^*).
\]

In addition, if the PL condition holds, then we can upper bound $-\| \nabla f(x) - \nabla f(x^*) \|^2 \leq -2\mu(f(x) - f^*)$ which combined with the above gives

\[
E \left[ \| g(x) - g(x^*) - (\nabla f(x) - \nabla f(x^*)) \|^2 \right] \leq 2(\mathcal{L} - \mu)(f(x) - f^*).
\]

Thus ER holds with $\rho = \mathcal{L} - \mu$.

Now we prove the remaining implications.
4. $L_i +$ Interpolated $\implies L_i + x^*$-convex. A direct consequence of the interpolation assumption (2.3) is that $\nabla f_i(x^*) = 0$ and $f_i(x^*) \leq f_i(x)$. Consequently $f_i(x^*) \leq f_i(x) + \langle \nabla f_i(x^*), x - x^* \rangle$.

5. $L_i + x^*$-convex $\implies$ ES. Follows from Proposition B.3.

Finally

6. ER $\not\Rightarrow$ ES. Since when $v$ encodes the full batch sampling where $g(x) = \nabla f(x)$, the expected residual condition always holds for any $\rho > 0$ since the left hand side of (ER) is zero and $0 \leq \rho(f(x) - f^*)$. On the other hand, in the full batch case the expected smoothness assumption is equivalent to claiming that $f$ is $L$-smooth, and clearly there exist differentiable functions that have gradients that are not Lipschitz. For instance $f(x) = x^4$. 

An important assumption created recently [19] is the following ABC- assumption

**ABC.** We say that ABC holds with $A, B, C > 0$ if

$$
\mathbb{E} \left[ \|g(x)\|^2 \right] \leq 2A(f(x) - f(x^*) + B \|\nabla f(x)\|^2 + C. \tag{59}
$$

The ABC condition (59) includes all previous assumptions SGC, WGC, ES and ER as a special case by choosing the three parameters $A, B$ and $C$ appropriately. In this sense, it is rather a family of assumptions. See [19] for more details on this assumption and how it linked to all the other assumptions.

C Proofs of Main Convergence Analysis Results

C.1 Proof of Theorem 4.1

First we need the following lemma.

**Lemma C.1.** Assume $g \in ER(\rho)$. Then for all $x \in \mathbb{R}^d$,

$$
\mathbb{E}_D \left[ \|g(x)\|^2 \right] \leq 2(2\rho + L)(f(x) - f(x^*)) + 2\sigma^2. \tag{60}
$$

**Proof.** Since $f$ is $L$-smooth, we have $\|\nabla f(x)\|^2 \leq 2L(f(x) - f(x^*))$. Using this inequality together with (7) gives (60).

Proof. We have:

$$
\|x^{k+1} - x^*\|^2 = \|x^k - x^*\|^2 - 2\gamma_k \langle g(x^k), x^k - x^* \rangle + \gamma_k^2 \|g(x^k)\|^2.
$$

Hence, taking expectation conditioned on $x_k$, we have:

$$
\mathbb{E}_D \left[ \|x^{k+1} - x^*\|^2 \right] = \|x^k - x^*\|^2 - 2\gamma_k \langle \nabla f(x^k), x^k - x^* \rangle + \gamma_k^2 \mathbb{E}_D \left[ \|\nabla f_{x_k}(x_k)\|^2 \right]
$$

$$
\leq \|x^k - x^*\|^2 - 2\gamma_k (\zeta - \gamma_k(2\rho + L))(f(x^k) - f^*) + 2\gamma_k^2 \sigma^2.
$$

Rearranging and taking expectation, we have

$$
2\gamma_k (\zeta - \gamma_k(2\rho + L)) \mathbb{E} \left[ f(x_k) - f^* \right] \leq \mathbb{E} \left[ \|x^k - x^*\|^2 \right] - \mathbb{E} \left[ \|x^{k+1} - x^*\|^2 \right] + 2\gamma_k^2 \sigma^2.
$$

Summing over $k = 0, \ldots, t - 1$ and using telescopic cancellation gives

$$
2 \sum_{k=0}^{t-1} \gamma_k (\zeta - \gamma_k(2\rho + L)) \mathbb{E} \left[ f(x_k) - f^* \right] \leq \mathbb{E} \left[ \|x^0 - x^*\|^2 \right] - \mathbb{E} \left[ \|x^{t+1} - x^*\|^2 \right] + 2\sigma^2 \sum_{k=0}^{t-1} \gamma_k^2.
$$

Since $\mathbb{E} \left[ \|x^k - x^*\|^2 \right] \geq 0$ and $(\zeta - \gamma_k(2\rho + L)) \geq 0$, dividing both sides by $2 \sum_{i=0}^{t-1} \gamma_i (\zeta - \gamma_i(2\rho + L))$ gives:

$$
\sum_{k=0}^{t-1} \mathbb{E} \left[ \frac{\gamma_k (\zeta - \gamma_k(2\rho + L))}{\sum_{i=0}^{t-1} \gamma_i (\zeta - \gamma_i(2\rho + L))} \left( f(x_k) - f^* \right) \right] \leq \frac{\mathbb{E} \left[ \|x^0 - x^*\|^2 \right]}{2 \sum_{i=0}^{t-1} \gamma_i (\zeta - \gamma_i(2\rho + L))} + \frac{\sigma^2 \sum_{k=0}^{t-1} \gamma_k^2}{\sum_{i=0}^{t-1} \gamma_i (\zeta - \gamma_i(2\rho + L))}.
$$
Thus,
\[
\min_{k=0, \ldots, t-1} \mathbb{E} \left[ f(x^k) - f(x^*) \right] \leq \frac{\|x^0 - x^*\|^2}{2 \sum_{i=0}^{k-1} \gamma_i (\zeta - \gamma_i (2\rho + L))} + \frac{\sigma^2 \sum_{i=0}^{k-1} \gamma_i^2}{\sum_{i=0}^{k-1} \gamma_i (\zeta - \gamma_i (2\rho + L))}.
\]

For the different choices of step sizes:

1. If \(\forall k \in \mathbb{N}, \gamma_k = \frac{\zeta}{2(2\rho + L)}\), then it suffices to replace \(\gamma_k = \gamma\) in (11).

2. Suppose algorithm (3) is run for \(T\) iterations. Let \(\forall k = 0, \ldots, T - 1, \gamma_k = \zeta \sqrt{T}\) with \(\gamma \leq \frac{\zeta}{2(2\rho + L)}\). Notice that since \(\gamma \leq \frac{\zeta}{2(2\rho + L)}\), we have \(\zeta - \gamma (2\rho + L) \leq \frac{1}{2}\). Then it suffices to replace \(\gamma_k = \frac{\zeta}{\sqrt{T}}\) in (11).

3. Let \(\forall k \in \mathbb{N}, \gamma_k = \frac{\zeta}{\sqrt{k+1}}\) with \(\gamma \leq \frac{\zeta}{2\rho + L}\). Note that that since \(\gamma_t = \frac{\zeta}{\sqrt{t+1}}\) and using the integral bound, we have that
\[
\sum_{t=0}^{k-1} \gamma_t^2 = \gamma^2 \sum_{t=0}^{k-1} \frac{1}{t+1} \leq \gamma^2 \left( \log(k) + 1 \right). \tag{61}
\]
Furthermore using the integral bound again we have that
\[
\sum_{t=0}^{k-1} \gamma_t \geq 2\gamma \left( \sqrt{k} - 1 \right). \tag{62}
\]
Now using (61) and (62) we have that
\[
\sum_{i=0}^{k-1} \gamma_i (\zeta - \gamma_i (2\rho + L)) = \zeta \sum_{i=0}^{k-1} \gamma_i - (2\rho + L) \sum_{i=0}^{k-1} \gamma_i^2 \geq 2\gamma \left( \zeta (\sqrt{k} - 1) - \gamma \left( \rho + \frac{b}{2} \right) \left( \log(k) + 1 \right) \right).
\]

It remains to replace bound the sums in (11) by the values we have computed.

\[\square\]

C.2 Proof of Corollary 4.2

Proof. The interpolated assumption 2.3 implies that \(\nabla f_i(x^*) = g(x^*) = 0\) and thus \(\sigma = 0\). Furthermore from (10) we have that the ER condition holds with \(\rho = \frac{n-b}{n(1-b)}\). Combining these two observations with (12) gives (15). The total complexity (16) follows from computing the iteration complexity via (15) and multiplying it by \(b\).

Finally for the optimal minibatch size, since (16) is a linear function in \(b\), the minimum depends on the sign of its slope. Taking the derivative in \(b\) we have the slope is given by \(2 \frac{L-2L_{max}}{n-1} \). If the slope is negative, we want \(b\) to be a large as possible, that is \(b = n\). Otherwise if the slope is positive \(b = 1\) is optimal.

\[\square\]

C.3 Proof of Lemma 4.3

Before presenting our proof for Lemma C.3, we need to present a large family of sampling vectors called the arbitrary samplings.

**Lemma C.2** (Lemma 3.3 [11]). Let \(S \subset \{1, \ldots, n\}\) be a random set. Let \(P[i \in S] = p_i\). It follows that \(v = \sum_{i \in S} \frac{1}{P} e_i\) is a sampling vector. We call \(v\) the arbitrary sampling vector.

An arbitrary sampling is sufficiently flexible as to model almost all samplings and minibatching schemes of interest, see Section 3.2 iii [11]. For example the \(b\)-minibatch sampling is a special case where \(p_i = \frac{b}{n}\) and \(P[S = B] = 1/\binom{n}{b}\) for every \(B \in \{1, \ldots, n\}\) that has \(b\) elements.

Now we prove Lemma 4.3 and some additional results.

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Lemma C.3. Assume interpolation 2.3 holds. Let \( f_i \) be \( L_i \)-smooth and let \( v \) be a sampling vector as defined in Lemma C.2. It follows that there exists \( \mathcal{L}_{\max} > 0 \) such that

\[
\frac{1}{2\mathcal{L}_{\max}} (f(x) - f^*) \leq \mathbb{E} \left[ \frac{(f_v(x) - f^*_v)^2}{\|\nabla f_v(x)\|^2} \right].
\] (63)

For \( B \subset \{1, \ldots, n\} \) let \( L_B \) be the smoothness constant of \( f_B := \frac{1}{n} \sum_{i \in B} p_i f_i \). It follows that

1. If \( v \) is an arbitrary sampling vector (Lemma C.2) then \( \mathcal{L}_{\max} = \max_{i=1, \ldots, n} \sum_{B:i \in B} \frac{p_B}{L_B} \).

2. If \( v \) is the \( b \)-minibatch sampling then \( \mathcal{L}_{\max} = \mathcal{L}_{\max}(b) = \max_{i=1, \ldots, n} \sum_{B:i \in B} \frac{1}{b-1} \frac{1}{L_B} \).

Proof. Since \( f_i \) is \( L_i \)-smooth, we have that \( f_v \) is \( L_v \)-smooth with \( L_v := \frac{1}{n} \sum_{i=1}^n v_i L_i \). Thus according to Lemma A.3 we have that

\[
\|\nabla f_v(x)\|^2 \leq 2L_v (f_v(x) - f^*_v).
\]

Consequently we have that

\[
\frac{1}{\|\nabla f_v(x)\|^2} \geq \frac{1}{2L_v (f_v(x) - f^*_v)}.
\] (64)

Using this we have the following bound

\[
\mathbb{E} \left[ \frac{(f_v(x) - f^*_v)^2}{\|\nabla f_v(x)\|^2} \right] \geq \mathbb{E} \left[ \frac{f_v(x) - f^*_v}{2L_v} \right].
\] (65)

Let \( S \) be the random set associated to the arbitrary sampling vector \( v \). We use \( B \subset \{1, \ldots, n\} \) to denote a realization of \( S \) and \( p_B := \mathbb{P}[B = S] \). Thus with this notation we have that

\[
\mathbb{E} \left[ \frac{(f_v(x) - f^*_v)^2}{\|\nabla f_v(x)\|^2} \right] \geq \sum_{B \subset \{1, \ldots, n\}} p_B \frac{f_B(x) - f^*_B}{2L_B}.
\] (66)

Now let \( p_i := \mathbb{P}[i \in S] \). Due to the interpolation condition we have that and the definition of \( f_B \) we have that

\[
f^*_B = f_B(x^*) = \frac{1}{n} \sum_{i \in B} p_i f_i(x^*) = \frac{1}{n} \sum_{i \in B} p_i f_i^*.
\]

Consequently

\[
\mathbb{E} \left[ \frac{(f_v(x) - f^*_v)^2}{\|\nabla f_v(x)\|^2} \right] \geq \sum_{B \subset \{1, \ldots, n\}} p_B \sum_{i \in B} \frac{f_i(x) - f_i^*}{2L_B p_i}.
\]

\[
\geq \min_{i=1, \ldots, n} \left\{ \sum_{B:i \in B} \frac{p_B}{p_i L_B} \right\} \frac{1}{2n} \sum_{i=1, \ldots, n} (f_i(x) - f_i^*)
\]

\[
= \min_{i=1, \ldots, n} \left\{ \sum_{B:i \in B} \frac{p_B}{p_i L_B} \right\} \left( \frac{1}{2n} \sum_{i=1, \ldots, n} (f(x) - f^*) \right),
\] (67)

where in the first equality we used a double counting argument to switch the order of the sum over subsets \( B \) and elements \( i \in B \). The main result (63) now follows by observing that

\[
\frac{1}{\min_{i=1, \ldots, n} \left\{ \sum_{B:i \in B} \frac{p_B}{p_i L_B} \right\}} = \max_{i=1, \ldots, n} \left\{ \frac{p_i}{\sum_{B:i \in B} \frac{p_B}{L_B}} \right\} = \mathcal{L}_{\max}.
\]

Finally, for a \( b \)-minibatch sampling we have that

\[
p_i = \frac{b}{n}, \quad p_B = 1/\binom{n}{b} \quad \text{and} \quad L_B \leq \frac{1}{b} \sum_{j \in B} L_j,
\]

which in turn gives

\[
\frac{1}{\mathcal{L}_{\max}} = \min_{i=1, \ldots, n} \sum_{B:i \in B} \frac{1}{b} \frac{1}{L_B} = \min_{i=1, \ldots, n} \sum_{B:i \in B} \frac{1}{(b-1)} \frac{1}{L_B}.
\]
C.4 Proof of Theorem 4.4

**Proof.**

\[
\|x^{k+1} - x^*\|^2 = \|x^k - \gamma_k \nabla f_o(x^k) - x^*\|^2 \\
= \|x^k - x^*\|^2 - 2\gamma_k \langle x^k - x^*, \nabla f_o(x^k) \rangle + \gamma_k^2 \|\nabla f_o(x^k)\|^2 \\
\leq (4) \|x^k - x^*\|^2 - 2\zeta\gamma_k [f_o(x^k) - f_o(x^*)] + \gamma_k^2 \|\nabla f_o(x^k)\|^2 \\
\leq (18) \|x^k - x^*\|^2 - 2\zeta\gamma_k [f_o(x^k) - f_o(x^*)] + \frac{2k}{\gamma} [f_o(x^k) - f_o(x^*)] \\
= \|x^k - x^*\|^2 - \gamma_k (2\zeta - \frac{1}{\epsilon}) [f_o(x^k) - f_o(x^*)].
\] (68)

By rearranging we have that

\[
\gamma_k (2\zeta - \frac{1}{\epsilon}) [f_o(x^k) - f_o(x^*)] \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2.
\] (69)

Taking expectation, and since \(2\zeta - \frac{1}{\epsilon} > 0\) we have by Lemma 4.3 we have that

\[
\frac{2\zeta - 1}{\epsilon}\frac{1}{\lambda_{\max}} \mathbb{E} [f(x^k) - f(x^*)] \leq (2\zeta - \frac{1}{\epsilon}) \mathbb{E} \left[ \frac{(f_o(x) - f_o(x^*))^2}{\|\nabla f_o(x)\|^2} \right] \\
\leq (18) \mathbb{E} [\gamma_k (f_o(x) - f_o(x^*))] \\
\leq (69) \mathbb{E} [\|x^k - x^*\|^2] - \mathbb{E} [\|x^{k+1} - x^*\|^2].
\]

Summing from \(k = 0, \ldots, K - 1\) and using telescopic cancellation gives

\[
\frac{2\zeta - 1}{\epsilon}\frac{1}{\lambda_{\max}} \sum_{k=0}^{K-1} \mathbb{E} [f(x^k) - f(x^*)] \leq \|x^0 - x^*\|^2 - \mathbb{E} [\|x^K - x^*\|^2].
\]

Multiplying through by \(\lambda_{\max} \frac{\epsilon}{2\zeta - 1} \frac{1}{K}\) gives

\[
\min_{i=0,\ldots,K-1} \mathbb{E} [f(x^k) - f(x^*)] \leq \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E} [f(x^k) - f(x^*)] \leq \frac{2\epsilon}{2\zeta - 1} \lambda_{\max} \|x^0 - x^*\|^2.
\]

\[\square\]

C.5 Proof of Theorem 4.6

In the following proof, for ease of reference, we repeat the step-size choice here:

\[
\gamma \leq \frac{1}{1 + 2\mu/\mu L} \frac{1}{L}.
\] (71)

**Proof.** By combining the smoothness of function \(f\) with the update rule of SGD we obtain:

\[
f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{\mu}{2} \|x^{k+1} - x^k\|^2 \\
= f(x^k) - \gamma (\nabla f(x^k), \nabla f_{\epsilon^k}(x^k)) + \frac{\mu^2}{2} \|\nabla f_{\epsilon^k}(x^k)\|^2.
\] (72)

By taking expectation conditioned on \(x^k\) we obtain:

\[
\mathbb{E} [f(x^{k+1}) \mid x^k] \leq f(x^k) - \gamma \|\nabla f(x^k)\|^2 + \frac{\mu^2}{2} \mathbb{E} \|\nabla f_{\epsilon^k}(x^k)\|^2 \\
\leq (7) f(x^k) - \gamma \|\nabla f(x^k)\|^2 + 2L\gamma^2 \rho(f(x^k) - f^*) \\
+ \frac{\mu^2}{2} \|\nabla f(x^k)\|^2 + L\gamma^2 \sigma^2 \\
= f(x^k) - \gamma (1 - \frac{\mu^2}{2L}) \|\nabla f(x^k)\|^2 \\
+ 2L\gamma^2 \rho(f(x^k) - f^*) + L\gamma^2 \sigma^2 \\
\leq (5) f(x^k) - 2\mu \gamma (1 - \frac{\mu^2}{2L}) ([f(x^k) - f^*]) \\
+ 2L\gamma^2 \rho(f(x^k) - f^*) + L\gamma^2 \sigma^2,
\] (73)
where the last inequality holds because $1 - \frac{L \gamma}{2} > 0$ since $\gamma \leq \frac{1}{1 + 2\rho/\mu} < \frac{1}{L}$.

Taking expectations again and subtracting $f^*$ from both sides yields:

$$
\mathbb{E}[f(x^{k+1}) - f^*] \leq \left(1 - 2\gamma \left(\mu(1 - \frac{L^2}{2}) - L \gamma \rho\right)\right)\mathbb{E}[f(x^k) - f^*] + L \gamma^2 \sigma^2.
$$

(71)

Recursively applying the above and summing up the resulting geometric series gives:

$$
\mathbb{E}\left[f(x_i) - f^*\right] \leq (1 - \mu \gamma)^i \mathbb{E}\left[f(x_0) - f^*\right] + L \gamma^2 \sigma^2.
$$

(74)

Using $\sum_{i=0}^{k-1} (1 - \mu \gamma)^i = \frac{1 - (1 - \mu \gamma)^k}{1 - (1 - \mu \gamma)} \leq \frac{1}{\mu \gamma}$, in the above gives (22).

On Iteration Complexity: For ease of reference, we repeat the step-size choice for the iteration complexity result

$$
\gamma = \frac{1}{\mu} \min \left\{ \frac{\mu \epsilon}{\sigma^2}, \frac{1}{1 + 2\rho/\mu} \right\}
$$

(76)

To analyze the iteration complexity, let $\epsilon > 0$ and let us divide the right hand side of (22) into two parts and bound each of them separately by $\frac{\epsilon}{2}$. For the right most part we have that

$$
\frac{L \gamma^2 \sigma^2}{\mu} \leq \frac{\epsilon}{2} \Rightarrow \gamma \leq \frac{1}{\mu} \frac{\mu \epsilon}{\sigma^2}.
$$

(77)

The derivation in (77) gives us the restriction (76) on the step size.

For the other remaining part we have that

$$
(1 - \mu \gamma)^k (f(x_0) - f^*) < \frac{\epsilon}{2}.
$$

Taking logarithms and re-arranging the above gives

$$
\log \left(\frac{2(f(x_0) - f^*)}{\epsilon}\right) \leq k \log \left(\frac{1}{1 - \gamma \mu}\right).
$$

(78)

Now using that $\log \left(\frac{1}{\rho}\right) \geq 1 - \rho$, for $0 < \rho \leq 1$ gives

$$
k \geq \frac{1}{\mu \gamma} \log \left(\frac{2(f(x_0) - f^*)}{\epsilon}\right).
$$

Thus restricting the step size according to (76) and inserting $\gamma$ into the above gives the result (23).

C.6 Proof of Theorem 4.7

Proof. By Theorem 3.4 we have that the ER condition holds. Thus Theorem 4.6 holds. Furthermore, since $f$ is interpolated we have that $\sigma = 0$, which when combined with Theorem 4.6 and (23) gives (24).

The total complexity (25) follows by using Lemma 3.3 and the expression for $\rho$ in (10) and plugging into (24). Since (25) is a linear function in $b$, the minimum depends on the sign of its slope. Taking the derivative in $b$ we have the sign slope is given by $\left(1 - \frac{2s_{\text{max}}}{n \mu}\right)$. If the slope is negative, we want $b$ to be a large as possible, that is $b = n$. Otherwise if the slope is positive $b = 1$ is optimal.

C.7 Theorem of PL with switching stepsize

Theorem C.4 (Decreasing step sizes/switching strategy). Let $f$ be an $L$-smooth. Assume $f \in PL(\mu)$ and $g \in ER(\rho)$. Let $k^* := 2 \frac{L}{\mu} \left(1 + 2 \frac{L}{\rho}\right)$ and

$$
\gamma^k = \begin{cases} 
\frac{\mu}{\mu(\mu + 2\rho)} & \text{for } k \leq \lfloor k^* \rfloor \\
\frac{2k+1}{(k+1)^2 \mu} & \text{for } k > \lfloor k^* \rfloor 
\end{cases}
$$

(79)
If \( k \geq \lceil k^* \rceil \), then SGD given by (3) satisfies:

\[
\mathbb{E}[f(x^k) - f^*] \leq \frac{4L}{{\mu}^2} \frac{1}{k} + \frac{(k^*)^2}{k+\rho} [f(x^0) - f^*].
\] (80)

**Proof.** Let \( \gamma_k := \frac{2k+1}{(k+1)^2} \) and let \( k^* \) be an integer that satisfies

\[
\gamma_k^* \leq \frac{\mu}{L(\mu + 2\rho)}.
\] (81)

Note that \( \gamma_k \) is decreasing in \( k \) and consequently \( \gamma_k \leq \frac{\mu}{L(\mu + 2\rho)} \) for all \( k \geq k^* \). This in turn guarantees that (74) holds for all \( k \geq k^* \) with \( \gamma_k \) in place of \( \gamma \), that is

\[
\mathbb{E}[f(x^{k+1}) - f^*] \leq \frac{k^2}{(k+1)^2} \mathbb{E}[f(x^k) - f^*] + \frac{L\sigma^2}{\mu^2} \frac{(2k+1)^2}{(k+1)^2}.
\] (82)

Multiplying both sides by \((k+1)^2\) we obtain

\[
(k+1)^2 \mathbb{E}[f(x^{k+1}) - f^*] \leq k^2 \mathbb{E}[f(x^k) - f^*] + \frac{L\sigma^2}{\mu^2} (2k+1)^2 \]

\[
\leq k^2 \mathbb{E}[f(x^k) - f^*] + 4L\sigma^2, \]

where the second inequality holds because \( \frac{2k+1}{k+1} < 2 \). Rearranging and summing from \( t = k^* \ldots k \) we obtain:

\[
\sum_{t=k^*}^{k} (t+1)^2 \mathbb{E}[f(x^{t+1}) - f^*] - t^2 \mathbb{E}[f(x^t) - f^*] \leq \sum_{t=k^*}^{k} 4L\sigma^2. \] (83)

Using telescopic cancellation gives

\[
(k+1)^2 \mathbb{E}[f(x^{k+1}) - f^*] \leq (k^*)^2 \mathbb{E}[f(x^{k^*}) - f^*] + 4L\sigma^2(k-k^*)
\]

Dividing the above by \((k+1)^2\) gives

\[
\mathbb{E}[f(x^{k+1}) - f^*] \leq \frac{(k^*)^2}{(k+1)^2} \mathbb{E}[f(x^{k^*}) - f^*] + 4L\sigma^2(k-k^*)/\mu^2(k+1)^2. \] (84)

For \( k \leq k^* \) we have that (22) holds, which combined with (84), gives

\[
\mathbb{E}[f(x^{k+1}) - f^*] \leq \frac{(k^*)^2}{(k+1)^2} \left(1 - \frac{\mu^2}{1+2\rho} \right)^k [f(x^0) - f^*] + \frac{L\sigma^2}{\mu^2(k+1)^2} \left(4(k-k^*) + \frac{(k^*)^2\mu^2}{1+2\rho} \right), \] (85)

It now remains to choose \( k^* \). Choosing \( k^* \) that minimizes the second line of (85) gives \( k^* = 2L \left(1 + \frac{2L}{\mu} \right) \). With this choice of \( k^* \) it is easy to show that (81) holds. Furthermore, by using that \( \frac{2}{k^*} = \frac{\mu^2}{\mu^2 + 2\rho} \frac{1}{L} \) in (85) gives

\[
\mathbb{E}[f(x^{k+1}) - f^*] \leq \frac{(k^*)^2}{(k+1)^2} \left(1 - \frac{\mu^2}{2L} \right)^k [f(x^0) - f^*] + \frac{L\sigma^2}{\mu^2(k+1)^2} (4(k-k^*) + 2k^*)
\]

\[
\leq \frac{(k^*)^2}{(k+1)^2} [f(x^0) - f^*] + \frac{2L\sigma^2}{\mu^2(k+1)^2} (2k-k^*)
\]

\[
\leq \frac{(k^*)^2}{(k+1)^2} [f(x^0) - f^*] + \frac{4L\sigma^2}{\mu^2(k+1)^2}. \] (86)

where in the second inequality we have used that \( \left(1 - \frac{1}{2^x} \right)^{2^x} \leq \frac{1}{4^x} \) for all \( x \geq 1 \), and in the third inequality we used that \( \frac{2k-k^*}{k+1} \leq \frac{2k}{k+1} \leq 2 \).

**C.8 Proofs of Section 5**

**Separable, smooth and PL.** Let \( f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x_i) \), which are smooth and interpolated. If in addition each \( f_i(x_i) \) satisfies the PL condition with constant \( \mu_i \), then there exists \( x^* \in \mathcal{X}^* \) such that
f(x) satisfies the PL condition with \( \mu = \min_{i=1, \ldots, n} \frac{\mu_i}{n} \). Indeed since

\[
\| \nabla f(x) \|^2 = \sum_{i=1}^{n} \frac{1}{n^2} \| \nabla f_i(x_i) \|^2 \geq \sum_{i=1}^{n} \frac{\mu_i}{n^2} (f_i(x_i) - f_i(x^*)) \geq \min_{i=1, \ldots, n} \frac{\mu_i}{n} (f(x) - f(x^*)). 
\]

### C.8.1 Proof of Lemma 5.1

Consider the problem

\[
\min_{x \in \mathbb{R}^d} f(x) := \frac{1}{2n} \| F(x) - y \|^2 = \frac{1}{2n} \sum_{i=1}^{n} (F_i(x) - y_i)^2 \tag{87}
\]

where \( y \in \mathbb{R}^n \).

**Proof.** The Jacobian of \( F \) is given by \( DF(x)^\top = [\nabla F_1(x), \ldots, \nabla F_n(x)] \in \mathbb{R}^{d \times n} \). Note that

\[
\begin{align*}
\nabla f(x) &= \frac{1}{n} DF(x)^\top (F(x) - y), \\
\nabla f_i(x) &= \nabla F_i(x)(F_i(x) - y_i). \tag{88}
\end{align*}
\]

Consequently \( \nabla f(x^*) = \nabla f_i(x^*) = 0 \). Finally, we suppose that the \( F_i(x) \) functions are Lipschitz and the Jacobian \( DF(x) \) has full row rank, that is,

\[
\begin{align*}
\| \nabla F_i(x) \| &\leq u, \quad \forall i \in \{1, \ldots, n\}, \forall k. \tag{90} \\
\| DF(x)^\top v \| &\geq \ell \| v \|, \quad \forall v. \tag{91}
\end{align*}
\]

Under these assumptions, our objective (87) satisfies the PL condition and the expected smoothness condition. Indeed (ES) holds using

\[
E_i \left[ || \nabla f_i(x) - \nabla f_i(x^*) ||^2 \right] = E_i \left[ || \nabla f_i(x) ||^2 \right] = \frac{1}{n} \sum_{i=1}^{n} || \nabla F_i(x) ||^2 (F_i(x) - y_i)^2 \\
\leq \frac{u}{n} \sum_{i=1}^{n} (F_i(x) - y_i)^2 = \frac{u}{n} \| F(x) - y \|^2 \\
= 2u(f(x) - f(x^*)), \tag{92}
\]

where we used (90) in the inequality. This shows that the expected smoothness condition hold with \( u = \mathcal{L} \).

By using the lower bound (91) we have that

\[
\| \nabla f(x) \|^2 = \frac{1}{n^2} \left\| DF(x)^\top (F(x) - y) \right\| \\
\geq \frac{1}{n^2} \| F(x) - y \|^2 \min_{v} \frac{\| DF(x)^\top v \|^2}{\| v \|^2} \\
\geq \ell f(x) = \ell (f(x) - f(x^*)),
\]

which shows that the PL condition holds with \( \mu = \ell \). \( \Box \)

The condition (91) is hard to verify, and somewhat unlikely to hold for all \( x \in \mathbb{R}^d \). Though if we had consider a closed and bounded constraint \( X \subset \mathbb{R}^d \), and applied the projected SGD method, then (90) is more likely to hold. For instance, assuming that (91) holds in neighborhood of the solution is the typical assumption used to prove the asymptotic convergence of the Gauss-Newton method (see Theorem 10.1 in [57]).
D Additional Convergence Analysis Results

D.1 Convergence of SGD for Quasar Strongly Convex functions

In this section we develop specialized theorems for quasar strongly convex functions.

**Definition D.1** (Quasar strongly convex). Let $\zeta > 0$ and $\lambda \geq 0$. Let $x^* \in \mathcal{X}$. We that $f$ is $(\zeta, \mu)$-quasar strongly convex with respect to $x^*$ if for all $x \in \mathbb{R}^n$,

$$f(x^*) \geq f(x) + \frac{\zeta}{2} \langle \nabla f(x), x^* - x \rangle + \frac{\mu}{2} \|x^* - x\|^2. \quad (93)$$

For shorthand we write $f \in \text{QSC}(\zeta, \lambda)$.

Note that if (93) holds with $\lambda = 0$ we say that the function $f$ is $\zeta$-quasar convex and we write $f \in \text{QC}(\zeta)$ (same to (4) from the main paper).

D.1.1 Constant Step-size

When $\lambda = 1$ we say that $f$ is star-strong convexity, but it is also known in the literature as quasi-strong convexity [34, 11] or weak strong convexity [17]. Star-strong convexity is also used in [34] for the analysis of gradient and accelerated gradient descent and in [11] for the analysis of SGD. The following theorem is a generalization of Theorem 1 in [11] to quasar strongly convex functions and under the assumption of expected residual.

**Theorem D.2.** Let $\zeta > 0$. Assume $f$ is $(\zeta, \lambda)$-quasar strongly convex with respect to $x^*$ and $g \in ER(\rho)$. Choose $\gamma^k = \gamma \in (0, \frac{\zeta}{2(\rho + L)})$ for all $k$. Then iterates of SGD given by (3) satisfy:

$$\mathbb{E}\|x^k - x^*\|^2 \leq (1 - \gamma^k \lambda) \|x^0 - x^*\|^2 + \frac{2\gamma^2 \sigma^2}{\zeta \lambda}. \quad (94)$$

**Proof.** Let $r^k = x^k - x^*$. From (3), we have

$$\|r^{k+1}\|^2 \overset{(3)}{=} \|r^k - x^* - \gamma \nabla f_{\mu^k}(x^k)\|^2$$

Taking expectation conditioned on $x^k$ we obtain:

$$\mathbb{E}_D\|r^{k+1}\|^2 = \|r^k\|^2 - 2\gamma \langle r^k, \nabla f(x^k) \rangle + \gamma^2 \mathbb{E}_D\|\nabla f_{\mu^k}(x^k)\|^2 \overset{(93)}{\leq} (1 - \gamma \lambda)\|r^k\|^2 - 2\zeta \gamma [f(x^k) - f(x^*)] + \gamma^2 \mathbb{E}_D\|\nabla f_{\mu^k}(x^k)\|^2.$$

Taking expectations again and using (60)

$$\mathbb{E}\|r^{k+1}\|^2 \overset{(60)}{\leq} (1 - \gamma \lambda)\|r^k\|^2 - 2\zeta \gamma [f(x^k) - f(x^*)] + \gamma^2 (2\rho + L)(f(x) - f(x^*)) + 2\gamma^2 \sigma^2$$

$$\leq (1 - \gamma \lambda)\|r^k\|^2 + 2\gamma (2\rho + L) - \zeta \mathbb{E}[f(x^k) - f(x^*)] + 2\gamma^2 \sigma^2$$

$$\leq (1 - \gamma \lambda)\|r^k\|^2 + 2\gamma^2 \sigma^2,$$

where we used in the last inequality that $\gamma (2\rho + L) \leq \zeta$ since $\gamma \leq \frac{\zeta}{2(\rho + L)}$. Recursively applying the above and summing up the resulting geometric series gives

$$\mathbb{E}\|r^k\|^2 \leq (1 - \gamma \lambda)^k \|r^0\|^2 + 2 \sum_{j=0}^{k-1} (1 - \gamma \lambda)^j \gamma^2 \sigma^2$$

$$\leq (1 - \gamma \lambda)^k \|r^0\|^2 + \frac{2\gamma^2 \sigma^2}{\zeta \lambda}. \quad (95)$$

D.1.2 Stochastic Polyak Step-size (SPS)

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Taking expectations again and using the tower property:

\[ \mathbb{E}[|x^k - x^*|^2] \leq \left(1 - \frac{\lambda}{2\sigma(x^* - x)}\right)^k \|x^0 - x^*\|^2. \tag{96} \]

\[ \text{This implies that function } f(x) = \sum_{i=1}^n f_i(x) \text{ is also } (\zeta, \lambda) \text{- strongly quasar convex function with respect to } x^* \in X \text{ (see [16]).} \]

**Proof.**

\[
\|x^{k+1} - x^*\|^2 = \|x^k - \gamma_k \nabla f_v(x^k) - x^*\|^2 = \|x^k - x^*\|^2 - 2\gamma_k \langle x^k - x^*, \nabla f_v(x^k) \rangle + \gamma_k^2 \|\nabla f_v(x^k)\|^2 \tag{93} \\
\leq (1 - \lambda\gamma_k)\|x^k - x^*\|^2 - 2\gamma_k \langle f_v(x^k) - f_v^*, \nabla f_v(x^k) \rangle + \gamma_k^2 \|\nabla f_v(x^k)\|^2 \tag{18} \\
= (1 - \lambda\gamma_k)\|x^k - x^*\|^2 + \gamma_k \left(\frac{1}{\lambda} - 2\zeta\right) \|f_v(x^k) - f_v^*\| \tag{19} \\
= (1 - \lambda\gamma_k)\|x^k - x^*\|^2.
\]

Since \( f_i \) is \( L_i \)-smooth and \( v \in \mathbb{R}^n_+ \) we have that \( f_v \) is \( L_v \)-smooth where \( L_v := \frac{1}{n} \sum_{i=1}^n v_i L_i \). Consequently, taking expectation condition on \( x^k \)

\[
\mathbb{E}_D[\|x^{k+1} - x^*\|^2] \leq (1 - \lambda\mathbb{E}_D[\gamma_k]) \|x^k - x^*\|^2 \leq (1 - \frac{\lambda}{2\sigma(x^* - x)}) \|x^k - x^*\|^2. \tag{98}
\]

Taking expectations again and using the tower property:

\[
\mathbb{E}[\|x^{k+1} - x^*\|^2] \leq \left(1 - \frac{\lambda}{2\sigma(x^* - x)}\right) \mathbb{E}[\|x^k - x^*\|^2]. \tag{99}
\]

\[ \square \]

**Corollary D.4.** If \( v \) is the \( b \)-minibatch sampling, we have that \( \mathbb{E}[L_v] \leq L \frac{n(b-1)}{(n-1)b} + L_{\max} \frac{n-b}{(n-1)b} \). Consequently, by selecting \( c = \frac{1}{2\zeta} \) and given \( \epsilon > 0 \), if we take

\[ k \geq \left( L \frac{n(b-1)}{(n-1)b} + L_{\max} \frac{n-b}{(n-1)b} \right) \log \left( \frac{\|x^0 - x^*\|^2}{\epsilon} \right), \]

steps of SGD with the SPS step size then \( \mathbb{E}[\|x^k - x^*\|^2] \leq \epsilon \).

\[ \square \]

**Proof.** Since the interpolation condition implies that \( f_i \) is convex around \( x^* \), we have by Proposition B.3 that the expected smoothness condition ES holds with \( \mathcal{L}(b) \) given in (42). Furthermore, we have that from Lemma E.1 in [11] that \( \mathbb{E}[L_v] \leq \mathcal{L}(b) \). Finally, from (96) we have that the iteration complexity is given by

\[ k \geq \frac{2\mathbb{E}[L_v]}{\lambda} \log \left( \frac{\|x^0 - x^*\|^2}{\epsilon} \right). \]

Plugging in \( c = \frac{1}{2\zeta} \) and the upperbound (42) for \( \mathcal{L}(b) \) gives the result.

\[ \square \]

### D.2 Convergence Analysis Results under Expected Smoothness

#### D.2.1 Quasar Convex and Expected Smoothness

For this next theorem we first need the following lemma.

**Lemma D.5.** Suppose \( f \) satisfies the expected smoothness Assumption 54. It follows that

\[ \mathbb{E}_D[\|g(x)\|^2] \leq 4\mathcal{L}(f) f^* + 2\sigma^2, \tag{100} \]

For this next theorem we first need the following lemma.

**Theorem D.3.** Assume interpolation 2.3 holds. Let all \( f_i \) be \( L_i \)-smooth and \( (\zeta, \lambda) \)-quasar strongly convex functions (4) with respect to \( x^* \in X^* \). SGD with SPS with \( c = 1/2\zeta \) converges as:

\[ \mathbb{E}[|x^k - x^*|^2] \leq \left(1 - \frac{\lambda}{2\sigma(x^* - x)}\right)^k \|x^0 - x^*\|^2. \]

### D.2 Convergence Analysis Results under Expected Smoothness

#### D.2.1 Quasar Convex and Expected Smoothness

For this next theorem we first need the following lemma.

**Lemma D.5.** Suppose \( f \) satisfies the expected smoothness Assumption 54. It follows that

\[ \mathbb{E}_D[\|g(x)\|^2] \leq 4\mathcal{L}(f) f^* + 2\sigma^2, \tag{100} \]
Proof. Using
\[ \|g(x)\|^2 \leq 2\|g(x) - g(x^*)\|^2 + 2\|g(x^*)\|^2, \]
and taking the supremum over \(x^* \in X^*\) and expectation together with (ES) gives the result. \(\square\)

**Theorem D.6.** Assume \(f(x)\) is \(\zeta\)-quasar-convex (4) and \(g \in ES(\mathcal{L})\). Let \(0 < \gamma_k < \frac{\zeta}{2\mathcal{L}}\) for all \(k \in \mathbb{N}\). Then, for every \(x^* \in X^*\)
\[
\min_{t=0,\ldots,k-1} \mathbb{E} \left[ f(x^t) - f(x^*) \right] \leq \frac{\|x_0 - x^*\|^2}{2\sum_{i=0}^{k-1} \gamma_i (\zeta - 2\gamma_i \mathcal{L})} + \sigma^2 \sum_{i=0}^{k-1} \gamma_i (\zeta - 2\gamma_i \mathcal{L}),
\]
(101)
Moreover:
1. if \(\forall k \in \mathbb{N}, \gamma_k = \gamma \leq \frac{\zeta}{2\mathcal{L}}\), then \(\forall k \in \mathbb{N}\),
\[
\min_{t=0,\ldots,k-1} \mathbb{E} \left[ f(x^t) - f(x^*) \right] \leq \frac{\|x_0 - x^*\|^2}{2\gamma (\zeta - 2\gamma \mathcal{L})} + \frac{\gamma^2 \sigma^2}{\zeta - 2\gamma \mathcal{L}},
\]
(102)
2. suppose algorithm (3) is run for \(T\) iterations. If \(\forall k = 0,\ldots,T-1, \gamma_k = \frac{\zeta}{\sqrt{T}}\) with \(\gamma \leq \frac{\zeta}{2\mathcal{L}}\),
\[
\min_{t=0,\ldots,T-1} \mathbb{E} \left[ f(x^t) - f(x^*) \right] \leq \frac{\|x_0 - x^*\|^2 + 2\gamma^2 \sigma^2}{\gamma \sqrt{T}},
\]
(103)
3. \(\forall k \in \mathbb{N}, \gamma_k = \frac{\zeta}{\sqrt{k+1}}\) with \(\gamma \leq \frac{\zeta}{2\mathcal{L}}\), then \(\forall k \in \mathbb{N}\),
\[
\min_{t=0,\ldots,k-1} \mathbb{E} \left[ f(x^t) - f(x^*) \right] \leq \frac{\|x_0 - x^*\|^2 + 2\gamma^2 \sigma^2 (\log(k+1)+1)}{4\gamma (\zeta - 2\gamma \mathcal{L}) (\log(k+1))} \sim O \left( \frac{\log(k)}{\sqrt{k}} \right).
\]
(104)

**Proof.** We have:
\[ \|x^{k+1} - x^*\|^2 = \|x^k - x^*\|^2 - 2\gamma_k (g(x^k), x^k - x^*) + \gamma_k^2 \|g(x^k)\|^2 \]
Hence, taking expectation conditioned on \(x_k\), we have:
\[
\mathbb{E}_D \left[ \|x^{k+1} - x^*\|^2 \right] = \|x^k - x^*\|^2 - 2\gamma_k \langle \nabla f(x^k), x^k - x^* \rangle + \gamma_k^2 \mathbb{E}_D \left[ \|\nabla f_{\nu_k}(x^k)\|^2 \right] \leq \|x^k - x^*\|^2 - 2\gamma_k (\zeta - 2\gamma \mathcal{L}) (f(x^k) - f^*) + 2\gamma_k^2 \sigma^2.
\]
Rearranging and taking expectation, we have
\[
2\gamma_k (\zeta - 2\gamma \mathcal{L}) \mathbb{E} \left[ f(x^k) - f^* \right] \leq \mathbb{E} \left[ \|x^k - x^*\|^2 \right] - \mathbb{E} \left[ \|x^{k+1} - x^*\|^2 \right] + 2\gamma_k^2 \sigma^2.
\]
Summing over \(k = 0,\ldots,t - 1\) and using telescopic cancellation gives
\[
2 \sum_{k=0}^{t-1} \gamma_k (\zeta - 2\gamma \mathcal{L}) \mathbb{E} \left[ f(x^k) - f^* \right] \leq \|x_0 - x^*\|^2 - \mathbb{E} \left[ \|x^k - x^*\|^2 \right] + 2\gamma_k^2 \sigma^2.
\]
Since \(\mathbb{E} \left[ \|x^k - x^*\|^2 \right] \geq 0\), dividing both sides by \(2 \sum_{k=0}^{t-1} \gamma_k (\zeta - 2\gamma \mathcal{L})\) gives:
\[
\sum_{k=0}^{t-1} \mathbb{E} \left[ \frac{\gamma_k (\zeta - 2\gamma \mathcal{L})}{\sum_{i=0}^{k} \gamma_i (\zeta - 2\gamma_i \mathcal{L})} (f(x^k) - f^*) \right] \leq \frac{\|x_0 - x^*\|^2}{2 \sum_{i=0}^{k} \gamma_i (\zeta - 2\gamma_i \mathcal{L})} + \frac{\sigma^2 \sum_{k=0}^{t-1} \gamma_k^2}{\sum_{i=0}^{k} \gamma_i (\zeta - 2\gamma_i \mathcal{L})}.
\]
Thus,
\[
\min_{k=0,\ldots,t-1} \mathbb{E} \left[ f(x^k) - f(x^*) \right] \leq \frac{\|x_0 - x^*\|^2}{2 \sum_{i=0}^{k} \gamma_i (\zeta - 2\gamma_i \mathcal{L})} + \frac{\sigma^2 \sum_{k=0}^{t-1} \gamma_k^2}{\sum_{i=0}^{k} \gamma_i (\zeta - 2\gamma_i \mathcal{L})}.
\]
For the different choices of step sizes:
1. If \(\forall k \in \mathbb{N}, \gamma_k = \gamma \leq \frac{\zeta}{2\mathcal{L}}\), then it suffices to replace \(\gamma_k = \gamma\) in (101).
2. Suppose algorithm (3) is run for $T$ iterations. Let $\forall k = 0, \ldots, T - 1$, $\gamma_k = \frac{\zeta}{\sqrt{T}}$ with $\gamma \leq \frac{\zeta}{4L}$. Notice that since $\gamma \leq \frac{\zeta}{4L}$, we have $\zeta - 2\gamma L \leq \frac{1}{2}$. Then it suffices to replace $\gamma_k = \frac{\gamma}{\sqrt{T}}$ in (101).

3. Let $\forall k \in \mathbb{N}$, $\gamma_k = \frac{\gamma}{\sqrt{k+1}}$ with $\gamma \leq \frac{\zeta}{2L}$. Note that that since $\gamma_t = \frac{\gamma}{\sqrt{t+1}}$ and using the integral bound, we have that

$$k - 1 \sum_{t=0}^{k-1} \gamma_t^2 = \gamma^2 \sum_{t=0}^{k-1} \frac{1}{t+1} \leq \gamma^2 \left( \log(k) + 1 \right). \quad (105)$$

Furthermore using the integral bound again we have that

$$\sum_{t=0}^{k-1} \gamma_t \geq 2\gamma \left( \sqrt{k} - 1 \right). \quad (106)$$

Now using (105) and (106) we have that

$$k - 1 \sum_{t=0}^{k-1} \gamma_t (\zeta - 2\gamma tL) = \zeta \sum_{t=0}^{k-1} \gamma_t - 2L \sum_{t=0}^{k-1} \gamma_t^2 \geq 2\gamma \left( \zeta (\sqrt{k} - 1) - \gamma L (\log(k) + 1) \right).$$

It remains to replace bound the sums in (101) by the values we have computed.

\[\square\]

**Specialized results for Interpolated Functions (with expected smoothness)** Analogously to Corollary 4.2, when the interpolated Assumption 2.3 holds, we can drop the expected smoothness assumption in Theorem D.6 in lieu for standard smoothness assumptions.

**Theorem D.7.** Let $f_i(x)$ be $L_i$-smooth for $i = 1, \ldots, n$, $f(x)$ be $\zeta$-quasar-convex (4) and assume that the interpolated Assumption 2.3 holds. Consequently $g \in \text{ES}(L)$. If we use the step size

$$\gamma_k \equiv \gamma \leq \frac{\zeta}{2L}, \quad (107)$$

for all $k$, then SGD given by (3) converges

$$\min_{i=1,\ldots,k} \mathbb{E} \left[ f(x^i) - f^* \right] \leq \frac{1}{k} \frac{\|x^0 - x^*\|^2}{2\zeta (\zeta - 2\gamma L)}.$$

Hence, if $\gamma = \frac{\zeta}{4L}$ and given $\epsilon > 0$ we have that

$$k \geq \frac{4L}{\epsilon \zeta^2} \left\| x^0 - x^* \right\|^2,$$  

implies $\min_{i=1,\ldots,k} \mathbb{E} \left[ f(x^i) - f^* \right] \leq \epsilon.$  

**Proof.** By Theorem 3.4 we have that the ES condition holds. Thus Theorem D.6 holds. Furthermore, since $f$ is interpolated we have that $\sigma = 0$, which when combined with (102) from Theorem D.6 gives the result. \[\square\]

Specializing Theorem D.7 to the full batch setting, we have that gradient descent (GD) with step size $\gamma = \frac{\zeta}{4L}$ converges at a rate of

$$f(x^i) - f(x^*) \leq \frac{4L \left\| x^0 - x^* \right\|^2}{\zeta \gamma},$$

where we have used that $L = L$ in the full batch setting and the smoothness of $f$ guarantees that the sequences $f(x^1), \ldots, f(x^t)$ for GD is a decreasing sequence. Similar to the result of the main paper on GD, we note that this is exactly the rate given recently for GD for quasar-convex functions in [12], with the exception that we have a squared dependency on $\xi$ the quasar convex function.
**Quasar-strongly convex functions and Expected smoothness**  
Similar to Theorem D.2 we present below the convergence of SGD with constant step-size for $(\zeta, \lambda)$-quasar-strongly convex functions under the expected smoothness.

**Theorem D.8.** Let $\zeta > 0$. Assume $f$ is $(\zeta, \lambda)$-quasar-strongly convex and that $(f, D) \sim ES(\mathcal{L})$. Choose $\gamma_k = \gamma \in (0, \zeta \mathcal{L}/2\zeta]$ for all $k$. Then iterates of SGD given by (3) satisfy:

$$E\|x_k - x^*\|^2 \leq (1 - \gamma \zeta \lambda)^k \|x^0 - x^*\|^2 + \frac{2\sigma^2}{\zeta \lambda}. \quad (110)$$

**Proof.** Similar to the proof of Theorem D.2 but using ES instead of ER.

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**D.2.2 PL and Expected Smoothness.**

In this section we present four main theorems for the convergence of SGD with constant and decreasing step size. Through our results we highlight the limitations of using expected smoothness in the PL setting and explain why one needs to have the expected residual to prove efficient convergence.

**Theorem D.9.** Let $f(x)$ be $L$-smooth and PL function and that $g \in ES(\mathcal{L})$. Choose $\gamma_k = \gamma \leq \frac{\mu}{2L}$ for all $k$. Then SGD given by (3) converges as follows:

$$E[f(x^k) - f^*] \leq (1 - \gamma \mu)^k [f(x^0) - f^*] + \frac{L \gamma \sigma^2}{\mu} \quad (111)$$

**Proof.** By Proposition B.4 we have that the expected smoothness condition holds with $\rho = L - \mu$. Thus by Theorem 4.6 we have that with a step size

$$\gamma_k = \gamma \leq \frac{1}{1 + 2L/\mu} \frac{1}{L} = \frac{1}{1 + 2(L-\mu)/\mu} \frac{1}{L} = \frac{\mu}{2L}$$

the iterates converge according to (111).

**Limitation of Theorem D.9.** Let us consider the case where $|S| = n$ with probability one. That is, each iteration (3) uses the full batch gradient. Thus $\sigma = 0$ and the expected smoothness parameter becomes $\mathcal{L} = L$. Consequently, from Theorem D.9 we obtain:

$$E[f(x^k) - f^*] \leq (1 - \gamma \mu)^k [f(x^0) - f^*]. \quad (112)$$

For $\gamma = \frac{\mu}{2L}$ (larger possible value) the rate of the gradient descent is $\rho = 1 - \frac{\mu^2}{2L^2}$. Thus the resulting iteration complexity (number of iterations to achieve given accuracy) for gradient descent becomes $k \geq 2L^2/\mu^2$. However it is known that for minimizing PL functions, the iteration complexity of gradient descent method is $k \geq 2L/\mu$. Thus the result of Theorem D.9 give as a suboptimal convergence for gradient descent and the gap between the predicted behavior and the known results could potentially be very large.

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**D.3 Minibatch Corollaries without Interpolation**

In this section we state the corollaries for the main theorems when $v$ is a $b$ minibatch sampling. Differently than what we did in the main paper, we will not assume interpolation. Instead, we will use the weaker assumptions that each $f_i$ is $x^*$-convex.

**D.3.1 Quasar Convex**

**Corollary D.10.** Assume $f$ is $\zeta$-quasar-strongly convex and that each $f_i$ is $L_i$-smooth and $x^*$-convex. If $v$ is a $b$-minibatch sampling and $\gamma_k \equiv \frac{1}{2} \frac{\zeta}{2\gamma + L}$ then,

$$\min_{t = 0, \ldots, k - 1} E[f(x^t) - f(x^*)] \leq 2 \|x^0 - x^*\|^2 \frac{2 L_{\max} \frac{n - b}{(n - 1)b} + L}{\zeta^2 k} + \frac{1}{b} \frac{n - b}{n - 1} \frac{2L_{\max}}{(n - 1)b + L}. \quad (113)$$

**Proof.** By Theorem 3.4 we have that the ER condition holds. Thus, the main Theorem 4.1 holds. Replacing the constants $\rho$ and $\sigma$ by their corresponding minibatch constants in (10) gives the result.
D.3.2 PL Function

**Corollary D.11.** Let $b \in \{1, \ldots, n\}$ and let $v$ be a $b$-minibatch sampling with replacement. Furthermore let each $f_i$ be $L_i$-smooth and convex around $x^*$. If $f$ satisfies the PL condition (5), then by Theorem 4.6 if

$$\gamma = \frac{\mu(n-1)b}{\mu(n-1)b + 2L_{\text{max}}(n-b) \frac{1}{L}};$$

then

$$\mathbb{E} \left[ f(x^k) - f^* \right] \leq \left( 1 - \frac{\rho^2(n-1)b}{\mu(n-1)b + 2L_{\text{max}}(n-b) \frac{1}{L}} \right)^k (f(x^0) - f^*) + \frac{n-b}{\mu(n-1)b + 2L_{\text{max}}(n-b)} \sigma^2.$$  \hspace{1cm} (115)

**Proof.** The proof follows by plugging in the values of $\rho$ and $\sigma$ given in Proposition 3.3 into (76) and (22).