Weighted Spectral Gap and a Unique Continuation Result for the Magnetic Differential Elliptic Operator

Duni Y. F. Zongo¹, Bila A. Kyelem² & Arouna Ouédraogo³

¹ Ecole Normale Supérieure, Département de Mathématique, 01 BP.1757 Ouagadougou 01, Burkina Faso
² Université de Ouahigouya, 01 BP. 346 Ouahigouya 01, Burkina Faso
³ Université Norbert ZONGO, Unité de Formation et de Recherche en Sciences et Technologie, BP. 376 Koudougou, Burkina Faso

Correspondence: Duni Y. F. Zongo, Ecole Normale Supérieure, Département de Mathématique, 01 BP.1757 Ouagadougou 01, Burkina Faso

Received: November 9, 2021   Accepted: December 26, 2021   Online Published: January 19, 2022
doi:10.5539/jmr.v14n1p62   URL: https://doi.org/10.5539/jmr.v14n1p62

Abstract
We prove two different kind of results for the magnetic elliptic operator which are the spectral gap of the magnetic elliptic functional energy and the inverse Poincaré inequality which is used to prove that if \( u \) vanish on a set of positive measure \( E \) then, \( u \) has a zero of infinite order at almost every point of \( E \).  

Keywords: magnetic field, ellipticity condition, magnetic elliptic differential operator, spectral gap, Poincaré inverse inequality, zero of infinite order, UCP result

1. Introduction

In this paper we study the second order differential elliptic operator generalizing the magnetic Schrödinger operator namely the magnetic elliptic differential operator denoted \( D_\mathcal{A} = -\nabla_\mathcal{A}(M\nabla_\mathcal{A}) \), where \( \nabla_\mathcal{A} = \nabla + i\mathcal{A}, \mathcal{A} \) being a 1-form called the magnetic potential, \( \mathcal{B} = \text{Curl}\mathcal{A} \) being the magnetic field yielding from the potential \( \mathcal{A} \) and \( M = (g_{jk})_{1\leq j,k\leq N} \) a positive definite matrix on a domain \( \Omega \) subset of \( \mathbb{R}^N \). It is important to note that positivity of general differential operator of second order have been extensively studied over past decades, see for example (Agmon (1982), Pinchover, (1988), Pinsky (1995), Pinchover & Tintarev (2006)) and more recently the work of (Abbas & Ragusa (2021)), and when a magnetic field is added see (Melgaard(1996)).

The present paper generalize a results of (Ensted & Tintarev (2009)) which deal with the magnetic Schrödinger operator \( \Delta_\mathcal{A} \). We prove that if \( q_V \geq 0, \mathcal{A} \) satisfies a local condition of integrability and \( \mathcal{B} \neq 0 \) in the sense of distribution \( q_{\mathcal{A},V} \) has a weighted spectral gap that is there exists \( W > 0 \) such that \( q_{\mathcal{A},V}[u] \geq \int_\Omega W(x)\vert u \vert^2 \, dx, \quad u \in C_0^\infty(\Omega) \). The use of ellipticity condition lead us to set the problem in weighted spaces with weight function \( \gamma \). Let \( 0 \leq \gamma \in L^1_{\text{loc}}(\Omega) \), where \( \Omega \) is a sub-domain of \( \mathbb{R}^N \). Then, we will need some weighted embedding theorems see (Bourgain (2000), Kilpelinen (1997), Leonardi (1994)).

2. Preliminaries

Let us define a new measure denoted by \( d\mu = \gamma \, dx \). Then, we have the following weighted Laplace and Sobolev spaces
\[
L^p(\gamma, \Omega) = \{ u : \int_\Omega \vert u \vert^p \, d\mu < \infty \},
\]
\[
W^{m,p}(\gamma, \Omega) = \{ u \in L^p(\gamma, \Omega) : D^m u \in L^p(\gamma, \Omega) \} \forall m\text{ such that } \vert m \vert \leq m \}
\]
\[
H^1(\gamma, \Omega), \quad H^1_{\mathcal{A}}(\gamma, \Omega) \text{ respectively are the closure of } C_0^\infty(\Omega) \text{ with respect to the norms } \| u \|_p = (\int_\Omega \vert u \vert^2 + \vert \nabla u \vert^2 \, d\mu)^{\frac{1}{2}} \text{ and } \| u \|_{\mathcal{A}} = (\int_\Omega \vert u \vert^2 + \vert \nabla_\mathcal{A} u \vert^2 \, d\mu)^{\frac{1}{2}}. \]

We denote \( K \subset \subset \Omega, \) if \( K \) is relatively compact in \( \Omega \). To ensure the ellipticity condition of the operator \( D = \nabla(M\nabla) \), we suppose that for every \( K \subset \subset \Omega \) there exists \( \Lambda_K > 1 \) such that
\[
\Lambda_K^{-1} I_N \leq \Lambda_K I_N \quad \forall x \in K.
\]

Let \( V \in L^p(\Omega), \) where \( p > \frac{1}{2} \). Throughout the paper we assume that the bilinear forms
\[
q_V[u] = \int_\Omega M \nabla u \cdot \nabla u + V \vert u \vert^2 \, dx.
\]

62
\[
q_{\alpha, \nu}[u] = \int_{\Omega} M \nabla \alpha u \cdot \nabla \alpha u + V|u|^2 \, dx, \quad u \in C_0^\infty(\Omega)
\]
respectively associated with the Schrodinger operator \( D \) and \( D_{\alpha} \).

We will suppose throughout the paper that the weight function \( \gamma \) satisfies the following two assumptions:

(A1) \( \gamma^{\frac{1}{2}} \in L^1_{\text{loc}}(\Omega) \)

(A2) There exists a real \( s \geq \frac{1}{p+1} \) such as \( \gamma^{-1} \in L^1_{\text{loc}}(\Omega) \).

The assumption (A1) guarantees that the weighted space \( W^{m,p}(\gamma, \Omega) \) is well defined and contains \( C_0^\infty(\Omega) \) as a subset and (A2) will allow us to pass from weighted Sobolev space to non-weighted Sobolev space. So we announce the following lemma see (Drabek & Als (1996)).

**Lemma 2.1.** Let \( s \) satisfying (A2). Then the space \( W^{m,p}(\gamma, \Omega) \) is continuously embedded into \( W^{m,p_1}(\Omega) \) where \( p_1 = \frac{sp}{s+1} \) for any \( |a| = m \).

**Picone type equality**

We give here a Picone type equality for the elliptic differential operator

\[
D = -\nabla(M \nabla)
\]

where \( M \) is a positive definite matrix with elements in \( C \).

**Lemma 2.2.** (Picone type equality) Let \( \nu \) a positive smooth function. And let \( u \in C_0^\infty(\Omega) \), then we have the following Picone type identity

\[
L(u, \nu) = R(u, \nu),
\]

where

\[
L(u, \nu) = M \nabla \left( \frac{u}{\nu} \right) \nabla \left( \frac{u}{\nu} \right) \quad \text{and} \quad R(u, \nu) = M \nabla u \nabla \nu - M \nabla \nu \nabla \left( \frac{|u|^2}{\nu} \right).
\]

**Proof:** We have

\[
L(u, \nu) = M \nabla \left( \frac{u}{\nu} \right) \nabla \left( \frac{u}{\nu} \right) \quad \text{and} \quad R(u, \nu) = M \nabla u \nabla \nu - M \nabla \nu \nabla \left( \frac{|u|^2}{\nu} \right).
\]

3. Existence of Spectral Gap

The main result of this section reads as follow.

**Theorem 3.1.** Suppose that \( \mathcal{A} \in L^\infty_{\text{loc}}(\Omega) \) and \( \text{Curl} \mathcal{A} \neq 0 \) in the sense of distributions on \( \Omega \) then the quadratic form \( q_{\mathcal{A}, \nu} \) admit a weighted gap in \( \Omega \).

Before proving this result, we need some auxiliary results.

**Lemma 3.2.** Assume that \( \mathcal{V} \in L^\infty_{\text{loc}}(\Omega) \). Let \( \nu \) be a positive solution of

\[
Dv + \mathcal{V}v = 0 \quad (3.1)
\]
i.e. for any $u \in C^\infty_0(\Omega)$

$$
\int_\Omega (M \nabla v \nabla u + V v \bar{u}) \, dx = 0
$$  \tag{3.2}

then,

$$
q_{\alpha, v}[u] = \int_\Omega M \nabla \left( \frac{\mu}{v} \right) \nabla \left( \frac{\mu}{v} \right) v^2 \, dx.
$$  \tag{3.3}

**Proof:** Let $v$ satisfies (3.2). Replacing $u$ by $\frac{|u|^2}{v}$ we get

$$
\int_\Omega (M \nabla v \nabla \left( \frac{|u|^2}{v} \right) + V v \left| \frac{u}{v} \right|^2 ) \, dx = 0
$$  \tag{3.4}

i.e

$$
- \int_\Omega M \nabla v \nabla \left( \frac{|u|^2}{v} \right) \, dx = \int_\Omega V |u|^2 \, dx.
$$  \tag{3.5}

And using the Picone type identity we obtain the desired result.

**Lemma 3.3.** Assume that $V \in L^\infty_{\text{loc}}(\Omega)$ and $A \in L^2_{\text{loc}}(\Omega)$. Let $v$ be a positive solution of Eq (3.1). Then we have

$$
q_{\alpha, v}[u] = \int_\Omega M \nabla _A \left( \mu \right) \nabla _A \left( \mu \right) v^2 \, dx.
$$

**Proof:** Let $u \in C^\infty_0(\Omega)$ and $v$ a positive function. We have

$$
q_{\alpha, v}[u] = \int_\Omega (M \nabla u \nabla \bar{u} + V |u|^2 ) \, dx
$$

$$
= \int_\Omega M (\nabla u + i A u) \cdot (\nabla \bar{u} - i A \bar{u}) + V |u|^2 \, dx
$$

$$
= \int_\Omega M \nabla u \cdot \nabla \bar{u} - i M \nabla u \cdot A \bar{u} + i M A u \cdot \nabla \bar{u} + M A u \cdot A \bar{u} + V |u|^2 \, dx
$$

$$
= \int_\Omega M \nabla \left( \frac{\mu}{v} \right) \cdot \nabla \left( \frac{\mu}{v} \right) v^2 \, dx + M A \left( \frac{\mu}{v} \right) \cdot A \left( \frac{\mu}{v} \right) v^2 + i (M A u \nabla \bar{u} - M \nabla u \cdot A \bar{u}) \, dx
$$

Let consider the last term in the integral, we have

$$
M A u \cdot A \bar{u} = M A \left( \frac{\mu}{v} \right) \nabla \bar{u} \cdot \nabla \left( \frac{\mu}{v} \right)
$$

$$
= M A \left( \frac{\mu}{v} \right) [v^2 \nabla \left( \frac{\mu}{v} \right) + \bar{u} \nabla v]
$$

$$
= M A \left( \frac{\mu}{v} \right) [v^2 \nabla \left( \frac{\mu}{v} \right) + \nabla |v|^2]
$$

$$
= M A \left( \frac{\mu}{v} \right) [v^2 \nabla \left( \frac{\mu}{v} \right) + \nabla v \cdot A \left( \frac{\mu}{v} \right)
$$

$$
= M A \left( \frac{\mu}{v} \right) \cdot A \left( \frac{\mu}{v} \right) + M \nabla v \cdot A \left( \frac{\mu}{v} \right)
$$

$$
= M A \left( \frac{\mu}{v} \right) v^2 + M \nabla v \cdot A \left( \frac{\mu}{v} \right)
$$

$$
M \nabla u \cdot A \bar{u} = M (\nabla u) \cdot A \left( \frac{\mu}{v} \right)
$$

$$
= M [v^2 \nabla \left( \frac{\mu}{v} \right) + u \nabla v] \cdot A \left( \frac{\mu}{v} \right)
$$

$$
= M \nabla \left( \frac{\mu}{v} \right) \cdot A \left( \frac{\mu}{v} \right) v^2 + M \nabla v \cdot A \left( \frac{\mu}{v} \right) u
$$

$$
= M \nabla \left( \frac{\mu}{v} \right) \cdot A \left( \frac{\mu}{v} \right) v^2 + M \nabla v \cdot A \left( \frac{\mu}{v} \right)
$$

So,

$$
M A u \cdot A \bar{u} - M \nabla u \cdot A \bar{u} = M A \left( \frac{\mu}{v} \right) v^2 - M \nabla \left( \frac{\mu}{v} \right) \cdot A \left( \frac{\mu}{v} \right) v^2.
$$
And then

\[
q_{A,V}[u] = \int_{\Omega} \left[ M\nabla \left( \frac{u}{v} \right) \nabla \left( \frac{u}{v} \right)^2 + M\mathcal{A} \left( \frac{u}{v} \right) \mathcal{A} - M\nabla \left( \frac{u}{v} \right) \cdot \mathcal{A} \left( \frac{u}{v} \right)^2 \right] \ dx
\]

\[= \int_{\Omega} M\nabla \left( \frac{u}{v} \right) \nabla \left( \frac{u}{v} \right)^2 \ dx.\]

**Lemma 3.4.** Suppose \( \Omega \) be a smooth bounded domain of \( \mathbb{R}^N \) and let \( A \in L^N(\Omega) \), then \( H^1(\gamma, \Omega) \) is continuously embedded in \( H^1(\gamma, \Omega) \) i.e. there exists a constant \( C > 0 \) such that

\[
C\|u\|_{H^1(\gamma, \Omega)} \leq \|u\|_{H^1(\gamma, \Omega)}. \tag{3.6}
\]

**Proof:** This proof follows the same pattern as (Arioli & Szulkin (2003)). Let \( u \in H^1(\gamma, \Omega) \), we have

\[
\int_{\Omega} \gamma(|\nabla u|^2 + |u|^2) \ dx \geq \int_{\Omega} \gamma(||\nabla u||^2 - |A|u|^2 + |u|^2) \ dx
\]

\[
\geq \int_{\Omega} \gamma(|\nabla u|^2 - 2|A||\nabla u| + |A|^2|u|^2 + |u|^2) \ dx.
\]

It remains to show that

\[
\int_{\Omega} \gamma(|\nabla u|^2 - 2|A||\nabla u| + |A|^2|u|^2 + |u|^2) \ dx \geq \epsilon \int_{\Omega} \gamma(|\nabla u|^2 + |u|^2) \ dx
\]

for some \( \epsilon > 0 \).

Proceed by absurd, we suppose now

\[
\int_{\Omega} \gamma(|\nabla u|^2 - 2|A||\nabla u| + |A|^2|u|^2 + |u|^2) \ dx < \epsilon \int_{\Omega} \gamma(|\nabla u|^2 + |u|^2) \ dx
\]

for every \( \epsilon > 0 \), then we can find a sequence \( u_n \) in \( H^1(\gamma, \Omega) \) with

\[
\|u\|_{H^1(\gamma, \Omega)} = 1 \text{ such that }
\]

\[
\int_{\Omega} \gamma(|\nabla u_n|^2 - 2|A|u_n||\nabla u_n| + |A|^2|u_n|^2 + |u_n|^2) \ dx \leq \frac{1}{n}. \tag{3.7}
\]

Since \( u_n \) is bounded in \( H^1(\gamma, \Omega) \) we can extract a subsequence still denoted \( u_n \) converging weakly to \( u \) in \( H^1(\gamma, \Omega) \), and since \( A \in L^N(\Omega) \) then

\[
\int_{\Omega} \gamma|A|u_n||\nabla u_n| \ dx \rightarrow \int_{\Omega} \gamma|A||\nabla u| \ dx.
\]

Hence passing to the limit in (1.7) we get

\[
\int_{\Omega} \gamma(|\nabla u| - |A||u|^2 + |u|^2) \ dx = \int_{\Omega} \gamma(|\nabla u|^2 - 2|A||\nabla u| + |A|^2|u|^2 + |u|^2) \ dx \leq 0
\]

If \( u \neq 0 \) this is a contradiction, and if \( u = 0 \) again a contradiction since (1.6) can be write

\[
1 - \int_{\Omega} \gamma(2|A|u_n||\nabla u_n| - |A|^2|u_n|^2) \ dx \leq \frac{1}{n}, \tag{3.8}
\]

and passing to the limit we get \( 1 \leq 0 \).

**Lemma 3.5.** Let \( B \subset \Omega \) and \( \overline{B} \) is compact in \( \Omega \), \( V \in L^\infty_{\text{loc}}(\Omega) \) and \( A \in L^N_{\text{loc}}(\Omega) \). If \( \text{Curl}A \neq 0 \) as a distribution on \( B \), then

\[
C_B = \inf \{ q_{A,V}[u] : u \in C^0_0(\Omega), \int_B |u|^2 \ dx = 1 \} > 0.
\]

**Proof:** Assume that \( C_B = 0 \). Then, there exists a sequence \( (u_k) \subset C^0_0(\Omega) \) with

\[
\int_{\Omega} |u_k|^2 \ dx = 1 \tag{3.9}
\]

such that

\[
q_{A,V}[u_k] \rightarrow 0. \tag{3.10}
\]
By the uniform ellipticity condition we get
\[ \int_{\Omega} \gamma(x)|\nabla (\frac{u_k}{v})|^2 v^2 \, dx \to 0. \] (3.11)

So, \((u_k)_{k \in \mathbb{N}}\) is a bounded sequence in the separable weighted Hilbert space \(H^1_{\gamma}(\gamma, B)\). Therefore by the Banach-Alaoglu theorem we can extract from \((u_k)_{k \in \mathbb{N}}\) a subsequence still denoted by \(u_k\) converging weakly to \(w\) in \(H^1_{\gamma}(\gamma, B)\). Taking into account (1.11) and the weakly lower semi-continuity of the form \(\int_{\Omega} \gamma(x)|\nabla (\frac{u_k}{v})|^2 v^2 \, dx\) we have
\[ \int_{\Omega} \gamma(x)|\nabla (\frac{w}{v})|^2 v^2 \, dx \leq \liminf_{k} \int_{\Omega} \gamma(x)|\nabla (\frac{u_k}{v})|^2 v^2 \, dx \leq \liminf_{k} q_{\gamma, v}[u_k] = 0, \]
thus
\[ \int_{\Omega} \gamma(x)|\nabla (\frac{w}{v})|^2 v^2 \, dx = 0. \] (3.12)

Now using the diamagnetic inequality we get
\[ \int_{\Omega} \gamma(x)|\nabla (\frac{|w|}{v})|^2 v^2 \, dx = 0 \]
that implies \(\frac{|w|}{v}\) is constant on \(\Omega\), let \(\frac{|w|}{v} = C\). Since \(u_k \rightharpoonup w\) in \(H^1_{\gamma}(\gamma, B)\) and the embedding \(H^1_{\gamma}(\gamma, B) \subset L^2(\gamma, B)\) is compact then \(u_k \rightharpoonup w\) in \(L^2(\gamma, B)\). We may therefore conclude from (1.9) that \(C > 0\). Moreover applying successively Lemma 1.4 and Lemma 1.1 we conclude that \(w\) belongs to a non-weighted Sobolev space say \(W^{1,p}(B)\). By the lifting theorem by (Bethuel & Zheng (1988)) there exists a function \(\phi \in W^{1,p}(B)\) such that \(\frac{w}{v} = C \exp[i\phi]\). Hence applying (1.12) we get
\[ 0 = \int_{B} |\nabla C \exp[i\phi]|^2 v^2 \, d\mu = C \int_{B} |\nabla \phi + \mathcal{A}|^2 v^2 \, d\mu. \]

And thus \(\nabla \phi + \mathcal{A} = 0\) as an element of \(L^2(\gamma, B)\). Since \(\text{curl} \nabla \phi = 0\) (also in the sense of weak derivatives) we conclude that \(\text{curl} \mathcal{A} = 0\) in the sense of distributions on \(B\), which is a contradiction. We can now prove Theorem 1.1.

**Proof of theorem 1.1:** Let \((B_k)\) be a partition of the domain \(\Omega\) such that any \(B_k\) is i.e. a relatively compact subset of \(\Omega\) with Lipschitz boundary and \(B_k \subset B_{k+1}\).

Assume without loss of generality that \(\text{curl} \mathcal{A} \neq 0\) as a distribution on \(B_1\), then \(\text{curl} \mathcal{A} \neq 0\) as a distribution on each \(B_k\). Let
\[ c_k = \inf [q_{\gamma, v}[u], u \in C^0_{\infty}(\Omega), \int_{B_k} |u|^2 = 1]. \]

We can always find a function \(u \in C^0_{\infty}(\Omega)\) with \(\int_{B_k} |u|^2 = 1\) such that \(q_{\gamma, v}[u] \leq 1\) this fact with Lemma 1.1 imply \(0 < c_k \leq 1\). Now, choosing \(0 \leq \psi_k \leq 1\) in \(C^0_{\infty}(\Omega)\), with \(\psi_k = 1\) on \(B_k\) and \(\text{supp} \psi_k \subset B_{k+1}\) we get
\[ q_{\gamma, v}[u] \geq \int_{\Omega} W(x)|u|^2, \]
where \(W(x) = \sum_{k=1}^{\infty} \frac{c_k}{2^k} \psi_k(x)\). Indeed,
\[ \int_{\Omega} W(x)|u|^2 = \sum_{k=1}^{\infty} \frac{c_k}{2^k} \int_{\Omega} \psi_k(x)|u|^2 \]
\[ \leq \sum_{k=1}^{\infty} \frac{1}{2^k} c_k \int_{\Omega} |u|^2 \]
\[ \leq q_{\gamma, v}[u] \sum_{k=1}^{\infty} \frac{1}{2^k} \]
\[ \leq q_{\gamma, v}[u]. \]
3. An UCP Result

In this part, we prove a weak ucp property of the magnetic differential elliptic operator $D_A$. First of all, let us give the definitions of different notion of UCP. For more details see (Regbaoui (2001)).

**Definition 3.6.** A function $u \in L^2(\Omega)$ has a zero of infinite order at $x_0 \in \Omega$ if for each $n \in \mathbb{N}$, there exists a constant $C > 0$ such that

$$
\int_{B(x_0,R)} |u|^2 \leq cR^n.
$$

**Definition 3.7.** A family of functions enjoys the unique continuation property for short U.C.P. , if no function besides possibly the zero function vanishes in a set of positive measure of $\Omega$.

**Definition 3.8.** A family of functions has the strong unique continuation property for short S.U.C.P. , if no function besides possibly the zero function has a zero of infinite order.

**Definition 3.9.** A family of functions enjoys the weak unique continuation property for short W.U.C.P. , if no function besides possibly the zero function vanishes in an open subset of $\Omega$.

Now, we state the main result of this part.

**Theorem 3.10.** Suppose that $u \in H^1_\gamma(\gamma, \Omega)$ be a solution of

$$
D_A u + Vu = 0 \quad (3.13)
$$

If $u = 0$ on a set of positive measure $E$, then $u$ has a zero of infinite order.

**Lemma 3.11.** Let the magnetic potential $A$ be such that $A \in L^N_{\text{loc}}(\Omega)$. Then for any $u \in H^1(\Omega)$ we have

$$
\int |A|^2 |u|^2 \leq C||u||^2 + \epsilon_2 ||\nabla u||^2.
$$

**Proof:** Let $A \in L^N_{\text{loc}}(\Omega)$ and $u \in H^1(\Omega)$. For any $M > 0$, let decompose $A = A_1 + A_2$ where

$$
A_1 = A. 1_{E |A| < M} \quad \text{and} \quad A_2 = A. 1_{E |A| > M},
$$

here $1_E$ denote the characteristic function of the set $E$.

Clearly, $A_1 \in L^\infty$. As for $A_2$, by the dominated convergence theorem it converges to zero in $L^N$. Indeed, $|A_2| \leq |A|$ and $A_2$ converges to 0 when $M \to +\infty$. Therefore, choosing $M$ sufficiently large we can make the $L^N$-norm of $A_2$ arbitrarily small.

Now by Hölder and Sobolev inequalities

$$
\int |A|^2 |u|^2 \leq \frac{||A_1||_{L^1}||u||^2 + ||A_2||_{L^1}||u||^2}{M^2} \\
\leq \frac{||A_1||_{L^1}||u||^2 + C||A_2||_{L^1}||\nabla u||^2}{M^2} \\
\leq C||u||^2 + \epsilon_2 ||\nabla u||^2.
$$

Now, let give the following inverse Poincaré’s Inequality.

**Lemma 3.12.** Let $B_r$ and $B_{2r}$ be two concentric balls contains in $\Omega$. If $u$ is a solution of 3.1 then, we have

$$
\int_{B_r} |\nabla u|^2 \leq \frac{C}{r^2} \int_{B_{2r}} |u|^2. \quad (3.15)
$$

**Proof:** First of all, we rewrite the operator $D_A$ as following

$$
D_A = -(\nabla + iA)(M\nabla + iA)) \\
= -\nabla(M\nabla) - i\nabla(MA) - iA(M\nabla) + A(MA)
$$

67
Then, if \( u \) is a solution of 3.1 we have for any \( v \in C^0_0(\Omega) \)
\[
\langle (D\phi + V) u, v \rangle = 0
\]
i.e.
\[
\int (M\nabla u) \cdot \nabla v + i \int (\mathcal{A}) \cdot \nabla \nabla u - i \int \nabla (\mathcal{A}) \cdot (M\nabla u) v + \int (\mathcal{A} \cdot (M\mathcal{A}) u v + \int Vu v = 0
\]
(3.16)
Let \( \phi \in C^0_0(\Omega) \), with \( \text{supp} \phi \subseteq B_{2r}, \phi \equiv 1 \) for \( x \in B_r \), \( \phi \leq 1 \) on \( B_2 \), and \( |\nabla \phi| \leq \frac{2}{C_0} \).
Taking \( v = \phi^2 u \) as test function in 3.16, we have
\[
\int (M\nabla u) \cdot \nabla (\phi^2 u) + i \int (\mathcal{A}) \cdot \nabla (\phi^2 u) u - i \int \nabla (\mathcal{A}) \cdot (M\nabla u) \phi^2 u + \int (\mathcal{A} \cdot (M\mathcal{A}) |u|^2 \phi^2 + \int V|u|^2 \phi^2 = 0.
\]
After derivation, we get:
\[
\int (M\nabla u \cdot \nabla u) \phi^2 + 2 \int (M\nabla u \cdot \nabla \phi) \phi u + 2i \int (\mathcal{A}) \cdot (\nabla \phi) |u|^2 \phi
\]
\[
+ i \int (\mathcal{A}) \cdot (\phi^2 u) u - i \int \nabla (\mathcal{A} \cdot (M\nabla u) \phi^2 u + \int (\mathcal{A} \cdot (M\mathcal{A}) |u|^2 \phi^2 + \int V|u|^2 \phi^2 = 0.
\]
Using the symmetry of \( M \), we finally get
\[
\int (M\nabla u \cdot \nabla u) \phi^2 = -2 \int (M\nabla u \cdot \nabla \phi) \phi u - 2i \int (\mathcal{A}) \cdot (\nabla \phi) |u|^2 \phi
\]
\[
+ 2i \int (\mathcal{A}) \cdot (\phi^2 u) u - i \int (\mathcal{A} \cdot (M\nabla u) \phi^2 u + \int (\mathcal{A} \cdot (M\mathcal{A}) |u|^2 \phi^2 + \int V|u|^2 \phi^2 = 0.
\]
Let
\[
I_1 := -2 \int (M\nabla u \cdot \nabla \phi) \phi u, \quad I_2 := -2i \int (\mathcal{A}) \cdot (\nabla \phi) |u|^2 \phi
\]
\[
I_3 := 2i \int (\mathcal{A}) \cdot (\phi^2 u) u, \quad I_4 := - \int (\mathcal{A} \cdot (M\nabla u) |u|^2 \phi^2 \quad \text{and} \quad I_5 := - \int V|u|^2 \phi^2.
\]
We estimate now each integral \( I_i \). By the symmetry of \( M \) and the Cauchy inequality, we have
\[
|I_1| \leq 2 \int \sqrt{(M\nabla u \cdot \nabla u)} \sqrt{(M\nabla \phi \cdot \nabla \phi) \phi u
\]
\[
\leq 2 \epsilon \int (M\nabla u \cdot \nabla u) \phi^2 + 2C(\epsilon) \int (M\nabla \phi \cdot \nabla \phi) |u|^2
\]
\[
\leq 2 \epsilon_i \lambda \int |\nabla u|^2 \phi^2 + 2C(\epsilon_i) \lambda \int |\phi|^2 |u|^2 \quad \text{(where we used the ellipticity condition)}
\]
\[
\leq 2 \epsilon_i \lambda \int B_{2r} |\nabla u|^2 + 4 \frac{C(\epsilon_i)}{\epsilon^2} \lambda \int B_{2r} |u|^2
\]
(3.18)
As for \( I_2 \),
\[
|I_2| \leq 2 \int (\mathcal{A}) \cdot (\nabla \phi) |u|^2 \phi
\]
\[
\leq 2 \int \sqrt{(\mathcal{A}) \cdot \mathcal{A} \sqrt{M\nabla u \cdot \nabla u} \phi |u|^2 \phi
\]
\[
\leq 2 \epsilon_i \lambda \int |\mathcal{A}| \phi^2 |u|^2 + 2 \lambda C(\epsilon_i) \int |\nabla \phi|^2 |u|^2
\]
\[
\leq 2 \epsilon_i \lambda C \int B_{2r} |u|^2 + 2 \epsilon_i \epsilon_2 \lambda \int B_{2r} |\nabla u|^2 + 4 \lambda \frac{C(\epsilon_i)}{\epsilon^2} \int B_{2r} |u|^2
\]
\[
\leq 2 \lambda (\epsilon_i C \lambda + \frac{2C(\epsilon_i)}{\epsilon^2}) \int B_{2r} |u|^2 + 2 \lambda \epsilon_i \epsilon_2 \int B_{2r} |\nabla u|^2
\]
(3.19)
Concerning $I_3$, we have

$$I_3 \leq 2 \int (M \mathbf{A} \cdot (\nabla u)) \varphi^2 u$$

$$\leq 2 \int \sqrt{(M \mathbf{A} : \mathbf{A}) \nabla u : \nabla u} \varphi^2 u$$

$$\leq 2 \lambda C(\epsilon_1) \int |\mathbf{A}|^2 |u|^2 + 2 \lambda \epsilon_1 \int |\nabla u|^2 \varphi^4$$

$$\leq 2 \lambda C(\epsilon_1) C_\mathbf{A} \int_{B_r} |u|^2 + 2 \lambda \epsilon_2 C(\epsilon_1) \int_{B_r} |\nabla u|^2 + 2 \lambda \epsilon_1 \int_{B_r} |\nabla u|^2$$

$$\leq 2 \lambda C(\epsilon_1) C_\mathbf{A} \int_{B_r} |u|^2 + 2 \lambda \epsilon_2 (C(\epsilon_1) + 1) \int_{B_r} |\nabla u|^2.$$  \hspace{1cm} (3.20)

Coming to $I_4$, we have

$$I_4 \leq \int (\mathbf{A} : M \mathbf{A}) |u|^2 \varphi^2$$

$$\leq \lambda \int |\mathbf{A}|^2 |u|^2 \varphi^2$$

$$\leq \lambda C_\mathbf{A} \int_{B_r} |u|^2 + \epsilon_2 \int_{B_r} |\nabla u|^2.$$ \hspace{1cm} (3.21)

By a result in [?] we can estimate $I_5$ as follow:

$$I_5 \leq \epsilon_1 \int |\nabla (\varphi u)|^2 + C(\epsilon_1) \int |\varphi u|^2$$

$$\leq \epsilon_1 \int |\varphi \nabla u + u \nabla \varphi|^2 + C(\epsilon_1) \int |\varphi u|^2$$

$$\leq 2 \epsilon_1 \int |\varphi \nabla u|^2 + |u \nabla \varphi|^2 + C(\epsilon_1) \int |\varphi u|^2$$

$$\leq 2 \epsilon_1 \int |\nabla u|^2 + \left(\frac{4 \epsilon_1}{r^2} + C(\epsilon_1)\right) \int_{B_r} |u|^2.$$ \hspace{1cm} (3.22)

From (3.17), (3.18), (3.19), (3.20), (3.21), (3.22) and the ellipticity condition we have

$$\frac{1}{A} \int_{B_r} |\nabla u|^2 \leq C_1(\epsilon_1, \epsilon_2) \int_{B_r} |\nabla u|^2 + \frac{r^2 C_{21}(\epsilon_1) + C_{22}(\epsilon_1)}{r^2} \int_{B_r} |u|^2,$$

where

$$C_1(\epsilon_1) = \lambda [3 \epsilon_1 + 2 \epsilon_1 \epsilon_2 + \epsilon_2 (3 + 2 C(\epsilon_1))]$$

$$C_{21}(\epsilon_1) = C(\epsilon_1) + \lambda C_\mathbf{A} (1 + 2 \epsilon_1 + C(\epsilon_1)) \text{ and } C_{22}(\epsilon_1) = 8 \lambda C(\epsilon_1) + 4 \epsilon_1.$$

Now, we choose appropriately $\epsilon_1$ and $\epsilon_2$ such that $C_1(\epsilon_1, \epsilon_2) \leq \frac{1}{\lambda}$. ($\epsilon_1, \epsilon_2$ can be take small enough). We thus get (3.15).  

**Proof of theorem:** Suppose $u \in H^1_{\text{loc}}(\Omega)$ vanishes on a set $E$ of positive measure. Almost every point of $E$ is a density point i.e. if $x_0$ is such a point then

$$\lim_{r \to 0} \frac{|E \cap B_r(x_0)|}{|B_r(x_0)|} = 1$$

where $B_r(x_0)$ denotes the ball of radius $r$ centered at $x_0$ and $|S|$ the Lebesgue’s measure of a set $E$. In other words, if $x_0$ is a density point of $E$, given some $\epsilon > 0$, there is an $r_0 = r_0(\epsilon)$ such that for any $r \leq r_0$ we have

$$\frac{|E^C \cap B_r(x_0)|}{|B_r(x_0)|} < \epsilon,$$ \hspace{1cm} (3.23)

where $E^C$ is the complement set of $E$.  

\[69]
Taking $r_0$ smaller if necessary we can assume $B_{r_0}(x_0) \subset \Omega$. Since $u = 0$ on $E$, the Hölder inequality and inequality (3.23) yield

$$\int_{B_r} |u|^2 = \int_{B_r \cap E^c} |u|^2 \leq \left( \int_{B_r \cap E^c} |u|^{\frac{2^*}{2^*-2}} \right)^{\frac{2^*}{2^*-2}} |E^c \cap B_r(x_0)|^{\frac{2}{2^*}} \leq \epsilon^{\frac{2}{2^*}} |B_r|^{\frac{2}{2^*}} \left( \int_{B_r \cap E^c} |u|^{\frac{2^*}{2^*-2}} \right)^{\frac{2^*}{2^*-2}}$$

(3.24)

From (3.24), the Sobolev and the inverse Poincaré inequalities yield

$$\int_{B_r} |u|^2 \leq C \epsilon^{\frac{2}{2^*}} (r_N)^{\frac{2}{2^*-2}} \left( \int_{B_r} |u|^2 + \int_{B_r} |\nabla u|^2 \right) \leq \epsilon^{\frac{2}{2^*}} r_0^2 \int_{B_r} |u|^2 + C \epsilon^{\frac{2}{2^*}} r^2 \int_{B_r} |\nabla u|^2 \leq C' \epsilon^{\frac{2}{2^*}} r^2 \int_{B_r} |\nabla u|^2$$

where $C' > 0$ is independent of $\epsilon$ and of $r$. Now, by the inverse-Poincaré inequality (3.15) we have for any $r \leq r_0$

$$\int_{B_r} |u|^2 \leq C'' \epsilon^{\frac{2}{2^*}} \int_{B_{2r}} |u|^2.$$  (3.25)

Let

$$f(r) = \int_{B_r} |u|^2$$

and fix $n \in \mathbb{N}$, choose $\epsilon > 0$ such that $C'' \epsilon^{\frac{2}{2^*}} = 2^{-n}$. Since $r_0$ depends on $\epsilon$ then $r_0$ will also depends on $n$. (3.25) can be written as

$$f(r) \leq 2^{-n} f(2r), \quad \text{for } r \leq r_0. \quad (3.26)$$

By iterations of (3.26) we get

$$f(r') \leq 2^{-kn} f(2^k r'), \quad \text{for } r' \leq 2^{1-k} r_0. \quad (3.27)$$

Now fix $r$ and choose $k$ such that $2^{-k} r_0 \leq r \leq 2^{1-k} r_0$. Then, it follows from (3.27) that

$$f(r) \leq 2^{-kn} f(2^k r).$$

(3.28)

Moreover, since the function $f$ is an increasing one we have

$$f(2^k r) \leq f(2r).$$

(3.29)

Putting together (3.28) and (3.29) we obtain

$$f(r) \leq 2^{-kn} f(2r_0).$$

(3.30)

Since, $2^{-k} \leq \frac{r}{r_0}$ we have

$$f(r) \leq \left( \frac{r}{r_0} \right)^n f(2r_0)$$

(3.31)

which shows that $x_0$ is zero of infinite order of $u$. Then, the unique continuation property for the magnetic elliptic operation will follow from the Carleman estimates see (Tataru (2004), Garrigue (2018)).

**Acknowledgements**

The first author thanks Michael Melgaard who introduce him to the UCP result and for valuable discussions.

**References**

Abbas, M., & Ragusa, M. A. (2021). On the hybrid fractional differential equations with fractional proportional derivatives of a function with respect to certain function. *Symmetry, 13*(2), 264.
Agmon, S. (1982). On positivity and decay of solutions of second order elliptic equations in riemannian manifolds, *In Method of Functional Analysis and Theory of Elliptic Equations*, MR 819005 — Zbl 0595.58044. http://doi.org/10.1515/9781400853076

Arioli, G., & Szulkin, A. (2003). A semilinear Schrödinger equation in the presence of magnetic field, *Arch. Rational. Mech. Anal.*, 170(4), 277-295. https://doi.org/10.1007/s00205-003-0274-5

Bethuel, F., & Zheng, X. M. (1988). Density of smooth functions between two manifolds in Sobolev spaces. *J. Funct. Anal.*, 80(1), 60-75. https://doi.org/10.1016/0022-1236(88)90065-1

Bourgain, J., Brezis, H., & Minorescu, H. (2000). Lifting in Sobolev spaces. *J. Anal. Math.*, 80, 37-86. http://doi.org/10.1007/BF02791533

Drabek, P., Kufner, A., & Nicolosi, F. (1996). Weak solutions of degenerated quasilinear elliptic equations of higher order. *Internat. J. Math. & Math. Sci.*, 19(4), 689-706. https://doi.org/10.1155/S0161171296000956.

Enstedt, M., & Tintarev, K. (2009). Weighted spectral gap for magnetic Schrodinger operators with a potential term. *Potential Anal.*, 3, 215-226.

de Figueiredo, D. G., & Gossez, J. P. (1992). Strict monotonicity of eigenvalues and unique continuation. *In Djairo G. de Figueiredo-Selected Papers* (pp. 361-368). Springer, Cham.

Garrigue, L. (2018). Unique continuation for many-body Schrodinger operators and the Hohenberg-Kohn theorem. *hal-01770288v2*.

Kilpelinen, T. (1997). Smooth approximation in weighted Sobolev spaces, *Comment. Math. Univ. Carolina.*, 38(1), 29-35.

Leonardi, S. (1994). On embedding theorems and Nemytskii’s operators in weighted Sobolev spaces. *Ricerche di Matematica*.

Melgaard, M., & Rozenblum, V. G. (1996). Spectral estimates for magnetic operators in: *Math. Scand.*, 79, 237-254.

Pinchover, Y. (1988). On the positivity of solutions of second order elliptic equations, stability results and classification. *Duke Math. J.*, 57.

Pinchover, Y., & Tintarev, K. (2006). A ground state alternative for singular Schrodinger operators, *J. Funct. Anal.*, 230, 65-77.

Pinsky, R. G. (1995). *Positive harmonic functions and diffusion*, Cambridge Studies in Advanced Mathematics, vol. 45, Cambridge University Press, Cambridge 1995.

Regbaoui, R. (2001). Unique Continuation from Sets of Positive Measure, in: Colombini F., Zuiy C. (Eds.), *Carleman Estimates and Applications to Uniqueness and Control Theory*. (Birkhäuser), 179-190.

Tataru, D. (2004). Unique continuation problems for partial differential equations. *Geometric methods in inverse problems and PDE control*, 137, 239-255.

Copyrights

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).