Entropy of semiclassical 2D dilaton black holes away from the 
Hawking temperature

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Recently we showed that in semiclassical 2D dilaton gravity the regular-
ity of a black hole horizon may be compatible with divergencies of Polyakov-
Liouville stresses on it, the temperature deviating from its Hawking value.
This makes the question about thermal properties of such solutions non-
trivial. We demonstrate that, adding to gravitation-dilaton part of the action
certain counterterms, which diverge on the horizon but are finite outside it,
one may achieve finiteness of the effective gravitation-dilaton couplings on the
horizon. This gives for the entropy $S$ the Bekenstein-Hawking value in the
nonextreme case and $S = 0$ in the extreme one similarly to what happens to
”standard ” black holes.
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I. INTRODUCTION

Two-dimensional (2D) dilaton theories of gravity [1] (for a recent review see [2], [3])
attract much attention for a number of reasons. In particular, they contain solutions of a
black hole type that gives a possibility to trace in details, exploiting simple models, rather
subtle effects of interaction between curvature and quantum fields, that in the more realistic
4D case is obscured by mathematical complexities. In the present Letter we touch upon
only one aspect of this subject, concerning connection between quantum backreaction and
black hole thermodynamics.
It is known that, inasmuch as backreaction is neglected, the Euclidean action and black hole entropy are finite at arbitrary temperature for both the non-extreme and extreme cases. Once backreaction is taken into account, this brings about severe restrictions into black hole thermodynamics. For non-extreme black holes this enforces the choice of the temperature $T = T_H$, where $T_H = \kappa/2\pi$ is the Hawking temperature, $\kappa$ is the surface gravity, i.e. a pure geometrical characteristics of a system. Otherwise, for $T \neq T_H$, quantum stresses blow up on the horizon that, typically, destroy a regular horizon completely. These circumstances were traced in detailed in [4], where the exact form of 2D stress-energy tensor was used explicitly. While the equality $T = T_H$ represents for non-extreme black holes well-established connection between geometry, quantum theory and thermodynamics, for the extreme case $T_H = 0$ the situation is not so obvious. It was suggested in [5] (see also [6] - [8]) to ascribe an arbitrary temperature $T$ to extreme black holes. Then one can calculate the Euclidean action and find the entropy $S = 0$. Such a prescription works well on a pure classical level but quantum corrections, caused by backreaction, however small they be, destroy this picture completely for the same reasons as in the non-extreme case. Therefore, one is forced to put $T = T_H = 0$, that makes thermodynamics questionable.

In previous works we showed that there exist exceptional situations, when infinite quantum backreaction on the horizon of dilaton black holes (in contrast to general relativity) may be compatible with regular geometry of a horizon both in the non-extreme [9] and extreme [10] cases. In doing so, infinite backreaction was in a sense compensated by infinite gravitation-dilaton coupling and infinite dilaton gradients. Meanwhile, for non-extreme black holes the entropy is proportional to the horizon value of this coupling, so it formally diverges as well as the Euclidean action as a whole. Thus, thermodynamic interpretation for such solutions fails. For extreme black holes, considered in [10], the entropy formally is zero, but the potential problem, caused by an infinite coupling, is connected with the energy, associated with the horizon since this energy is proportional to its gradient.

The aim of this Letter is to pay attention that there exists a possibility to combine the inequality $T \neq T_H$ (and, thus, infinite backreaction) not only with the regularity of a
horizon, but also with a well-defined entropy. This circumstance is especially important for the extreme case since it supplies us with examples, when the black hole entropy, whose very notion is the subject of intensive discussion in recent years [12] - [14], may be advocated on a *semiclassical* (not only pure classical) level.

II. NON-EXTREME BLACK HOLES AND EXACTLY SOLVABLE MODELS

Let us consider the system governed by the action

\[ I = I_0 + I_{PL}, \]  

where

\[ I_0 = \frac{1}{2\pi} \int_M d^2x \sqrt{-g} [F(\phi)R + V(\phi)(\nabla \phi)^2 + U(\phi)] + \frac{1}{\pi} \int_{\partial M} ds k F(\phi). \]  

Here the boundary term with the second fundamental form \( k \) makes the variational problem self-consistent, \( ds \) is the line element along the boundary \( \partial M \) of the manifold \( M \).

\( I_{PL} \) is the Polyakov-Liouville action [15] incorporating effects of Hawking radiation and its backreaction on the black hole metric for a multiplet of \( N \) scalar fields. It is convenient to write it down in the form [16], [17]

\[ I_{PL} = -\frac{\kappa}{2\pi} \int_M d^2x \sqrt{-g} [\frac{(\nabla \psi)^2}{2} + \psi R] - \frac{\kappa}{\pi} \int_{\partial M} \psi k ds. \]  

The function \( \psi \) obeys the equation

\[ \Box \psi = R, \]  

where \( \Box = \nabla_\mu \nabla^\mu \), \( \kappa = N/24 \) is the quantum coupling parameter.

Varying the action with respect to metric gives us (\( T_{\mu\nu} = 2 \frac{\delta I}{\delta g_{\mu\nu}} \)):

\[ T_{\mu\nu}^{(tot)} = T_{\mu\nu}^{(0)} + T_{\mu\nu}^{(PL)} = 0, \]  

where
\[ T^{(0)}_{\mu\nu} = \frac{1}{2\pi} \{ 2(g_{\mu\nu} \square F - \nabla_\mu \nabla_\nu F) - U g_{\mu\nu} + 2V \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} V(\nabla \phi)^2 \}, \] (6)

\[ T^{(PL)}_{\mu\nu} = -\frac{\kappa}{2\pi} \{ \partial_\mu \psi \partial_\nu \psi - 2\nabla_\mu \nabla_\nu \psi + g_{\mu\nu} [2R - \frac{1}{2} (\nabla \psi)^2] \} \] (7)

Variation of the action with respect to \( \phi \) gives rise to the equation

\[ R\tilde{F}' + U' = 2V \square \phi + V'(\nabla \phi)^2, \] (8)

prime denotes derivative with respect to \( \phi \).

Inasmuch as the auxiliary function \( \psi \) can be expressed in terms of \( \phi \) only, the action \( I_0 \) and the Polyakov-Liouville action are combined in such a way that field equations (6) - (7) can be formally obtained from the action \( I_0 \) only but with the ”renormalized” coefficients which receive some shifts: \( F \to \tilde{F}, V \to \tilde{V} \) where

\[ \tilde{F} = F - \kappa \psi, \] (9)

\[ \tilde{V} = V - \frac{\kappa}{2} \psi'^2. \] (10)

The dilaton equation (8) can be rewritten, with account of (4), as

\[ R\tilde{F}' + U' = 2\tilde{V} \square \phi + \tilde{V}'(\nabla \phi)^2. \] (11)

Then there exists a class of exactly solvable models (see [18] and literature quoted there), for which

\[ V = \omega \left( u - \frac{\kappa \omega'}{2} \right) \] (12)

or, equivalently,

\[ \tilde{V} = \omega \tilde{F}', \omega \equiv \frac{U'}{U}. \] (13)

Then it turns out that in the Schwarzschild gauge the metric of an eternal black hole reads [18].
\[ ds^2 = -f dt^2 + f^{-1} dx^2, \]  

\[ f = \frac{4\lambda^2(\tilde{F} - \tilde{F}_h)}{U}, \quad x = \frac{\mu}{2\lambda}, \quad \mu' = \tilde{F}'e^{-\psi}, \quad U = 4\lambda^2 e^\psi, \quad \psi' = \omega, \]  

(14)

the curvature

\[ R = \frac{U}{F'}\left[\frac{U'(\tilde{F} - \tilde{F}_h)}{U F'}\right]. \]  

(16)

It was assumed in [18] that \( \tilde{F}' \neq 0 \) to ensure the regularity of a horizon. Meanwhile, the equality \( \tilde{F} = 0 \) may be compatible with the regularity of a horizon if, at the same time, \( U = 0 \). Consider the case, when near the horizon

\[ \tilde{F} - \tilde{F}_h = A(\phi - \phi_h)^2, \]  

(17)

\[ U = U_1(\phi - \phi_h), \]  

(18)

where \( A \) and \( U_1 \) are some constants.

It follows from (16) that \( R \) is finite. Near the horizon \( \tilde{F} \) is finite, \( \tilde{V} \sim 2A \) is finite, \( T^{\mu(\text{tot})}_{\nu} \) are finite. It is instructive to note that for our exactly solvable models

\[ \frac{\partial f}{\partial x} = 2\lambda\left[1 - \frac{(\tilde{F} - \tilde{F}_h)\psi'}{F'}\right], \]  

(19)

\[ T_H = \frac{1}{4\pi} \lim_{x \to x_h} \frac{\partial f}{\partial x} = \frac{\lambda}{\pi}. \]  

(20)

As a result, our Hawking temperature is as twice as little as compared to the standard case [18], when \( T_H = \frac{\lambda}{2\pi} \). We have also

\[ \frac{\partial \tilde{F}}{\partial x} = (2\lambda)^{-1}U, \quad \tilde{V} = \tilde{F}'\psi'. \]  

(21)

Consider, for example, the spatial component. Then direct calculations give us that

\[ 2\pi T_1^{(\text{tot})} = 0, \]  

as it should be. The gravitation-dilaton part of the field equations

\[ 2\pi T_1^{(0)} = \frac{\partial f}{\partial x} \frac{\partial F}{\partial x} - U + \frac{Vf}{\left(\frac{\partial \phi}{\partial x}\right)^2}, \]  

(22)
near the horizon behaves like

$$\frac{\pi T_1^{(0)}}{2\lambda^2} = \frac{3U_1}{8A} \kappa (\phi - \phi_h)^{-1} + ...$$  \hspace{1cm} (23)

In a similar way,

$$\frac{\pi T_1^{(PL)}}{2\lambda^2} = -\kappa \left( \frac{f}{2} \frac{\partial \psi}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial \psi}{\partial x} \right).$$  \hspace{1cm} (24)

Near the horizon

$$\frac{\pi T_1^{(PL)}}{2\lambda^2} = -\kappa \frac{3U_1}{8A} (\phi - \phi_h)^{-1} + ...$$  \hspace{1cm} (25)

We see that divergent parts of gravitation-dilaton and Polyakov-Liouville parts mutually compensate each other, as it should be. Thus, the entropy (up to the constant) $S = 2\tilde{F}_h$ [11] is finite; $\tilde{F}$, $\tilde{V}$ are finite on the horizon, $S$ is well defined. The "bare" quantities $F$, $V$ diverge: according to (3), (13), (18), near the horizon

$$F = \tilde{F} + \kappa \ln(\phi - \phi_h) + \text{const},$$ \hspace{1cm} (26)

$$V = \tilde{V} + \frac{\kappa}{2} (\phi - \phi_h)^{-1}.$$ \hspace{1cm} (27)

As $T_\mu^{(PL)}$ diverges, it means in turn that $T \neq T_H$. Indeed, this component may be represented as (see, e.g. [4] for details) $T_1^{(PL)} = -\pi N \left[ T^2 - \left( \frac{1}{4\pi} \frac{\partial f}{\partial x} \right)^2 \right]$. If spacetime is flat at infinity, $\frac{\partial f}{\partial x} \to 0$, and the parameter $T$ has the meaning of temperature. Near the horizon, where $\frac{1}{4\pi} \frac{\partial f}{\partial x} \to T_H$, only the choice $T = T_H$ makes $T_1^{(PL)}$ regular there. If $T_1^{(PL)}$ diverges on the horizon, this is inconsistent with the equality $T = T_H$.

III. EXTREME CASE

Exactly solvable models, exploited for the analysis of the non-extreme case, are now unsuitable since they give $T_H \neq 0$ according to (20). Fortunately, the essence of matter becomes clear even without resorting to exactly solvable models. Basic equations for 00 and 11 components read:
\[ 2f \frac{\partial^2 \tilde{F}}{\partial x^2} + \frac{\partial f}{\partial x} \frac{\partial \tilde{F}}{\partial x} - U - \tilde{V} f \left( \frac{\partial \phi}{\partial x} \right)^2 = 0, \]  
(28)

\[ \frac{\partial f}{\partial x} \frac{\partial \tilde{F}}{\partial x} - U + \tilde{V} f \left( \frac{\partial \phi}{\partial x} \right)^2 = 0. \]  
(29)

It is also convenient to take the difference of (28), (29) to get

\[ \frac{\partial^2 \tilde{F}}{\partial x^2} = \tilde{V} \left( \frac{\partial \phi}{\partial x} \right)^2. \]  
(30)

Let us consider extreme black holes for which \( \tilde{F} \) and \( \tilde{V} \) are finite on the horizon. Let also on the horizon \( x = x_h \) the dilaton \( \phi = \phi_h \) and \( \frac{\partial \phi}{\partial x} \) be finite. Then it follows from (29) that \( U(\phi_h) = 0. \) For a static metric eq. (4) gives us

\[ \frac{\partial \psi}{\partial x} = \frac{a - \frac{\partial f}{\partial x}}{f}. \]  
(31)

Then it follows from (24) that

\[ 2\pi T_1^{(PL)} = - \frac{\kappa (a + \frac{\partial f}{\partial x})(a - \frac{\partial f}{\partial x})}{2 f}. \]  
(32)

If \( a = 0 \), the quantity \( T_1^{(PL)} \) is finite on the extreme horizon since the denominator and numerator have the same order \( (x - x_h)^2 \). Let, however, \( a \neq 0 \). Then near the horizon \( \frac{\partial \phi}{\partial x} = \) \( \frac{A_2}{(x - x_h)^2} \) and

\[ \psi = - \frac{A_1}{x - x_h} = - \frac{A_2}{\phi - \phi_h}, \]  
(33)

\( A_2 = A_1 \lim_{x \to x_h} \left( \frac{\partial \phi}{\partial x} \right). \) As, by assumption, \( \tilde{F} \) is finite, the quantity \( F = \tilde{F} + \kappa \psi \) behaves like \( (x - x_h)^{-1} \sim (\phi - \phi_h)^{-1} \), \( V = \tilde{V} + \frac{\kappa}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 \) behaves like \( (\phi - \phi_h)^{-4} \). (Note that quantities \( F \) and \( V \) should diverge in any case, if we want to have finite \( \tilde{F} \) and \( \tilde{V} \) but for \( a = 0 \) divergencies are milder: \( \psi \sim \ln(\phi - \phi_h), F \sim \ln(\phi - \phi_h), V \sim (\phi - \phi_h)^2 \).) On one hand, eqs. (28) and (30) have the solutions which can be obtained by Taylor series in \( (\phi - \phi_h) \) or \( (x - x_h) \), with finite \( \tilde{F}_h \) and \( \tilde{V}_h \). On the other hand, for \( a \neq 0 \), \( T_1^{(PL)} \) diverges on the horizon according to (32). Direct calculations show that \( T_1^{(0)} \) have the leading divergent term that compensates exactly that in (32), as it should be. Thus, we get the regular solutions different parts of
which diverge. The total Euclidean action of the quantum-corrected system is well defined since $\tilde{F}$ is finite on the horizon (in this situation the energy, as usual, is due to the term on the physical boundary and does not contain the contribution from the horizon). Then all the arguments of [5] - [8] apply, with the only change that the classical coupling $F$ should be replaced now by the quantum-corrected one $\tilde{F}$. Thus, the value $S = 0$ is advocated in that case. In other words, we should introduce appropriate counterterms into the gravitation-dilaton part of the action in such away, that they diverge on the horizon. Then, if they behave like we described it above, they compensate divergencies in Polyakov-Liouville part and give the quite definite well-defined answer for the entropy. The divergencies in $I_0$ is an inevitable price for the finiteness of the total action. There is no need to adjust very special logarithmic dependence for $\tilde{F}$ to obtain semiclassical regular extreme black holes, as was done in [10] - now $\tilde{F}$ is finite on the horizon.

IV. SUMMARY

A natural way to understand better fundamentals of black hole thermodynamics is to try to violate some “obvious” assumptions, which are usually tacitly assumed and remain unspoken, and to look at the consequences to which this violation can lead. As a result, we can either realize, why these assumptions were necessary or, otherwise, extend some basic notions beyond their original region of validity. In this Letter we abandoned the assumption of the finiteness of the action coefficients on the horizon in the bare gravitation-dilaton part. We demonstrated, how divergencies in the corresponding couplings are connected with the violation of the condition $T = T_H$ and showed that this can be compatible with the well-defined black hole entropy. In particular, for corresponding solutions the prescription for the entropy value $S = 0$ of extremal black holes [5] can be advocated.

From the other hand, the procedure under discussion cannot be considered only as a methodical exercise for elucidating, what happens if some ”obvious” assumption about the bare Lagrangian are violated. The point is that for a quantum-corrected system it is just
quantities like $\tilde{F}$ and $\tilde{V}$ have direct meaning - meanwhile, they are finite in our case. (To some extent, the relationship between $F$, $V$ and $\tilde{F}$, $\tilde{V}$ can be considered in our case as some analogue of the renormalization procedure, but with the wording that this ”renormalization” should remove divergencies only on the horizon - outside the horizon all quantities are finite separately.) Only if one wishes to trace back the behavior of different parts in field equation and splits them explicitly to gravitation-dilaton part and quantum backreaction, the thermal divergencies on the horizon come into play (otherwise they remain hidden, since geometry near the horizon is perfectly smooth). This means that if spacetime is asymptotically flat, at infinity $T^{\nu}_{\mu}^{(PL)} \rightarrow \frac{\pi T^2 N}{6} diag(1, -1)$ has the form, inherent to thermal radiation, but with temperature $T \neq \kappa/2\pi$.

Combining the results of the previous works [9], [10] with the present ones, we see that dilaton gravity gives us a more rich set of possibilities than general relativity in what concerns thermal properties of black holes and relationship between the character of quantum stresses on the horizon and the regularity of the geometry.

For convenience, different cases are brought together in a table, where ” + ” means ”exists”, while ” - ” means ”does not exists”, the row ”0” represents the classical case, all other rows refer to the semiclassical situation. Here case 1 can be called typical and in fact discussion in [6] applies to it, case 1’ showing, what happens if deviation of the temperature from the Hawking value occurs. Case 2 refers to models of [9], [10], case 3 is the subject of the present article. Case 1 for extreme black hole means that $T = T_H$, so a regular horizon is possible but, as $T_H = 0$, there is no sensible thermodynamics. That for extreme semiclassical black holes there two possibilities in case 2, is explained by the fact that thermodynamics fails if $\tilde{F}$ and, correspondingly, the energy associated with the horizon grow too rapidly near the horizon (cf. [13]).
Non-extreme black holes

| $T$, $T_H$ | $F_h, V_h$ | $\tilde{F}_h, \tilde{V}_h$ | $T_{\mu}^{(PL)}$ on horizon | Regular horizon | Thermodynamics |
|------------|-------------|-----------------|-------------------|-----------------|----------------|
| 0 = or $\neq$ finite | $\tilde{F} = F$, $\tilde{V} = V$ | 0 | + | +( $S = 2F_h$) |
| 1 | finite | finite | finite | + | $(S = 2\tilde{F}_h)$ |
| 1' $\neq$ finite | infinite | infinite | infinite | $- - -$ |
| 2 $\neq$ infinite | infinite | infinite | infinite | + $-$ |
| 3 $\neq$ infinite | finite | infinite | infinite | + $(S = 2\tilde{F}_h)$ |

Extreme black holes

| $T$, $T_H = 0$ | $F_h, V_h$ | $\tilde{F}_h, \tilde{V}_h$ | $T_{\mu}^{(PL)}$ on horizon | Regular horizon | Thermodynamics |
|----------------|-------------|-----------------|-------------------|-----------------|----------------|
| 0 $\neq$ finite | $\tilde{F} = F$, $\tilde{V} = V$ | 0 | + | $(S = 0)$ |
| 1 | finite | finite | finite | + | $-$ |
| 1' $\neq$ finite | infinite | infinite | infinite | $- - -$ |
| 2 $\neq$ infinite | infinite | infinite | infinite | + $(S = 0)$ or $-$ |
| 3 $\neq$ infinite | finite | infinite | infinite | + $(S = 0)$ |

It is seen from the table that in 2D dilaton gravity black hole thermodynamics becomes, on one hand, more restricted in that it fails to exist at all in certain dilaton theories. However, from the other hand, it extends to a more vast region than usually since it may withstand the violation of the property $T = T_H$ (that manifests that an intimate link between geometry and thermodynamics is now broken) and the appearance of infinite stresses on the horizon. In the given context it is worth mentioning that black hole thermodynamics fails to be mandatory also in some 4D theories with a scalar field (dilaton), even in the absence of quantum effects, because of an infinite area of an event horizon or divergencies in the coupling $F$; if $F$ grows too rapidly, the Euclidean action turns out to be infinite [19]. Thus, divergencies in the gravitation-dilaton coupling do not deprive the corresponding models of physical sense but, rather, give rise to unusual and non-trivial relationship between the existence of thermal properties, quantum backreaction and/or concrete nature of the aforementioned divergencies.
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