ON THE STABILITY OF COMPACT PSEUDO-KÄHLER AND NEUTRAL CALABI-YAU MANIFOLDS

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Abstract. We study the stability of compact pseudo-Kähler manifolds, i.e. compact complex manifolds $X$ endowed with a symplectic form compatible with the complex structure of $X$. When the corresponding metric is positive-definite, $X$ is Kähler and any sufficiently small deformation of $X$ admits a Kähler metric by a well-known result of Kodaira and Spencer. We prove that compact pseudo-Kähler surfaces are also stable, but we show that stability fails in every complex dimension $n \geq 3$. Similar results are obtained for compact neutral Kähler and neutral Calabi-Yau manifolds. Finally, motivated by a question of Streets and Tian in the positive-definite case, we construct compact complex manifolds with pseudo-Hermitian-symplectic structures that do not admit any pseudo-Kähler metric.

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1. Introduction

Let $M$ be an even-dimensional manifold endowed with a complex structure $J$ and a symplectic form $F$. When $J$ and $F$ are compatible, i.e. the symplectic form $F$ is $J$-invariant, and the associated metric $g$ is Riemannian, then the manifold $(M, J, F)$ is Kähler. In the compact Kähler case, the positive-definiteness of $g$ imposes strong topological conditions on the manifold $M$; for instance, its Betti numbers $b_{2k+1}(M)$ are even and the manifold is formal [15]. Since there are compact complex manifolds with no Kähler metrics, many efforts have been done in understanding the properties of manifolds endowed with a pair $(J, F)$ satisfying weaker conditions than those of a Kähler structure. On the one hand, if we drop the closedness condition for the 2-form $F$, other special Hermitian structures arise, such as the strong Kähler with torsion (SKT) or balanced

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Hermitian structures. On the other hand, if we no longer require the positive definiteness of the metric $g$ but preserve the compatibility of the symplectic form $F$ with $J$, then a structure called pseudo-Kähler is obtained.

In this paper we focus on compact manifolds $M$ endowed with a pseudo-Kähler structure, i.e. a pair $(J, F)$ where $J$ is a complex structure and $F$ is a non-degenerate closed 2-form such that $F(JU, JV) = F(U, V)$, for any smooth vector fields $U, V$ on $M$. This is equivalent to $J$ being parallel, i.e. $\nabla J = 0$, where $\nabla$ is the Levi-Civita connection of the pseudo-Riemannian metric $g(U, V) = F(JU, V)$ (see [2]). If the real dimension of $M$ is $2n$, then the pseudo-Kähler metric has signature $(2k, 2n - 2k)$, where $k = n$ corresponds to the Kähler case. In other words, we have a compact complex manifold $X = (M, J)$ with a symplectic form $F$ of bidegree $(1, 1)$ with respect to the complex structure $J$. Notice that there are many compact pseudo-Kähler manifolds not admitting any Kähler metric, the simplest example being the compact complex surface known as the Kodaira-Thurston manifold [16].

Pseudo-Kähler structures appear in relation to other interesting structures on manifolds. For instance, compact complex homogeneous manifolds endowed with a pseudo-Kähler structure are classified in [16, 21]. Pseudo-Kähler Einstein metrics on compact complex surfaces are studied by Petean [39]. More recently, it is proved in [25] that there is a natural pseudo-Kähler structure on the universal intermediate $G_2$-Jacobi $\mathcal{J}$ of the moduli space of torsion-free $G_2$-structures on a fixed compact 7-manifold. Furthermore, any $4n$-dimensional hypersymplectic manifold in particular has a neutral Calabi-Yau structure [24] (see also [14]), which is a special type of pseudo-Kähler structure whose underlying moduli space is Ricci-flat and has signature $(2n, 2n)$.

Although several aspects in pseudo-Kähler geometry have been investigated, the stability of these structures under small holomorphic deformations of the complex manifold is, to our knowledge, only known in the positive-definite case. Indeed, if $X$ is a compact Kähler manifold, then any sufficiently small deformation of $X$ admits a Kähler metric due to a well-known result by Kodaira and Spencer [26]. In this paper we focus on the stability properties of compact pseudo-Kähler manifolds, as well as manifolds with related neutral (Kähler and Calabi-Yau) metrics.

We next explain in more detail the contents of the paper.

In Section 2 we firstly construct compact pseudo-Kähler manifolds of complex dimension $n \geq 3$ that are not stable under small holomorphic deformations of the complex structure. This result motivates the study of sufficient conditions for the stability of the pseudo-Kähler property. Our stability result involves the Bott-Chern cohomology of complex manifolds. We prove that if $X_0$ is pseudo-Kähler and the upper-semi-continuous function $t \mapsto h^{1,1}_{BC}(X_t)$ is constant, then the compact complex manifold $X_t$ admits a pseudo-Kähler metric for any small enough $t$ (see Proposition 2.9).

Combining this with a result of Teleman [45] on the complex invariant $\Delta^2$ introduced by Angella and Tomassini in [6], we prove in Theorem 2.13 that compact pseudo-Kähler surfaces are stable.

The results of Section 2 are illustrated with explicit constructions of pseudo-Kähler nilmanifolds and solvmanifolds, together with their small holomorphic deformations. In particular, we prove in Proposition 2.10 that the Iwasawa manifold and its small deformations do not admit any pseudo-Kähler metric. In contrast, the holomorphically parallelizable Nakamura manifold $X$ is pseudo-Kähler, as proved by Yamada in [50], and there exists a small deformation $X_t$ of $X$ admitting pseudo-Kähler metrics for every $t$ (see Proposition 2.9).

Section 3 is devoted to a special class of pseudo-Kähler manifolds, namely, neutral Calabi-Yau manifolds. They are neutral Kähler manifolds, i.e. manifolds with even complex dimension $2m$ and a metric $g$ of signature $(2m, 2m)$, that additionally have a nowhere vanishing form $\Phi$ of bidegree $(2m, 0)$ satisfying $\nabla \Phi = 0$, where $\nabla$ is the Levi-Civita connection of $g$. Observe that neutral Calabi-Yau manifolds are Ricci-flat. In Section 3.1 we prove the stability of neutral Kähler and neutral Calabi-Yau structures on compact complex surfaces. However, our Theorem 3.7 shows
that such structures are not stable on compact complex manifolds of any complex dimension \( n \geq 4 \). This result is in deep contrast to the case of Kähler Calabi-Yau manifolds, whose deformation space is unobstructed by the well-known Bogomolov-Tian-Todorov theorem [5, 47, 48].

For the proof of Theorem 3.7 we first construct a complex nilmanifold \( X \) of complex dimension 4 with (non-flat) neutral Calabi-Yau structures (see Proposition 3.4). An interesting class of neutral Calabi-Yau nilmanifolds, called Kodaira manifolds, was constructed by Fino, Pedersen, Poon and Sørensen in [20]. It is worth to note that the complex structure of the nilmanifold \( X \) is of a very different kind. Indeed, \( X \) has the special feature that it is 4-step and the center of its underlying Lie algebra has minimal dimension, which implies that \( X \) is far from being the total space of a torus bundle over a torus. Moreover, in Proposition 3.6 we show that \( X \) provides counterexamples to a conjecture in [13] about the type of invariant complex structures on nilmanifolds that admit compatible pseudo-Kähler metrics. By appropriately deforming the complex nilmanifold \( X \), we construct a holomorphic family of compact complex manifolds \( \{X_t\}_{t \in \Delta} \), with \( X_0 = X \), showing that the neutral Calabi-Yau and neutral Kähler properties are unstable.

In Section 4 we consider pseudo-Hermitian-symplectic structures, which are an indefinite version of Hermitian-symplectic structures. This geometry naturally arises from small deformations of pseudo-Kähler manifolds (see Proposition 4.5). Motivated by a question of Streets and Tian in the positive-definite case [44, Question 1.7], we prove that there are compact complex manifolds admitting pseudo-Hermitian-symplectic structures but no pseudo-Kähler metrics.

2. Pseudo-Kähler manifolds

This section starts showing that compact pseudo-Kähler manifolds of complex dimension \( n \geq 3 \) are not stable under small holomorphic deformations of the complex structure. This motivates the study of conditions under which a sufficiently small deformation \( X_t \) of a pseudo-Kähler manifold \( X \) is again pseudo-Kähler. Here, we present a stability result related to the Bott-Chern cohomology of complex manifolds. Several explicit constructions of pseudo-Kähler nilmanifolds and solvmanifolds are provided along the section, illustrating the behaviour of the pseudo-Kähler property under their small holomorphic deformations. In the final part of the section, we focus on the behaviour of small deformations of pseudo-Kähler compact complex surfaces.

Let us recall that a complex analytic family, or holomorphic family, of compact complex manifolds is a proper holomorphic submersion \( \pi: X \to \Delta \). This implies that the fibres \( X_t = \pi^{-1}(t) \) are compact complex manifolds of the same dimension. By a classical result of Ehresmann [18], any such family is locally \( C^\infty \) trivial (globally, if \( \Delta \) is contractible), so all the fibres \( X_t \) have the same underlying \( C^\infty \) manifold \( M \). Consequently, the holomorphic family can be viewed as a collection \( \{X_t\}_{t \in \Delta} \) of complex manifolds \( X_t = (M, J_t) \), where \( J_t \) is the complex structure of \( X_t \) for \( t \in \Delta \).

A classical result of Kodaira and Spencer [26] asserts that the property of being a compact Kähler manifold is stable under holomorphic deformations. In the following section we prove that such a stability result cannot be extended to compact pseudo-Kähler manifolds.

2.1. Instability of the pseudo-Kähler property. Here we prove that compact pseudo-Kähler manifolds of complex dimension \( n \geq 3 \) are in general not stable under small deformations of the complex structure. In the proof we will consider an appropriate holomorphic family consisting of complex nilmanifolds. We recall that a nilmanifold is a compact quotient \( N = \Gamma \backslash G \) of a connected and simply connected nilpotent Lie group \( G \) by a lattice \( \Gamma \) of maximal rank in \( G \). A complex nilmanifold \( X = (N, J) \) is a nilmanifold \( N = \Gamma \backslash G \) endowed with an invariant complex structure \( J \), i.e. \( J \) comes from a left-invariant complex structure on \( G \) by passing to the quotient \( \Gamma \backslash G \). (For results on complex nilmanifolds see for instance [3, 11, 41] and the references therein.)
Proposition 2.1. There is a holomorphic family of compact complex manifolds $\{X_t\}_{t \in \Delta}$ of complex dimension 3, where $\Delta = \{ t \in \mathbb{C} \mid |t| < 1 \}$, such that:

(i) $X_0$ is a pseudo-Kähler manifold;
(ii) $X_t$ does not admit pseudo-Kähler metrics for any $t \neq 0$.

Hence, the pseudo-Kähler property is not stable under deformations of the complex structure.

Proof. Let $X$ be the complex nilmanifold defined by the following complex structure equations

\[ d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12}. \]

Here $\omega^k$ has bidegree $(1,0)$ and $\omega^{jk}$ denotes the $(1,1)$-form $\omega^j \wedge \overline{\omega^k}$.

Observe that the compact complex manifold $X$ has pseudo-Kähler metrics. For instance, the 2-form

\[ F = i\omega^{11} + \omega^{23} - \omega^{32} \]

satisfies that $F = \bar{F}$, i.e. the 2-form $F$ is real, and $F \neq 0$, i.e. it is non-degenerate. The form $F$ is compatible with the complex structure $J$ of $X$ because it is of pure type $(1,1)$ on $X$. Moreover, from (1) we get that $dF = 0$, so the form $F$ is symplectic. Thus, $F$ defines a pseudo-Kähler metric on $X$.

Notice that the $(0,1)$-form $\omega^3$ is $\bar{\partial}$-closed on $X$. We will use the Dolbeault cohomology class $[\omega^3] \in H^{0,1}_\partial(X)$ to perform an appropriate holomorphic deformation of $X$. For each $t \in \mathbb{C}$ such that $|t| < 1$, let us consider the complex nilmanifold $X_t$ defined by the following complex basis of $(1,0)$-forms:

\[ \omega_1^t := \omega^1, \quad \omega_2^t := \omega^2, \quad \omega_3^t := \omega^3 + t \omega^3. \]

It follows from (1) and (2) that the complex structure equations for $X_t$ are

\[ d\omega_1^t = d\omega_2^t = 0, \quad d\omega_3^t = \omega_1^{12} - t \omega_2^{12}. \]

In order to prove that $X_t$ has no pseudo-Kähler metrics for any $t \neq 0$, we use Nomizu’s theorem [30] for the de Rham cohomology of nilmanifolds together with the symmetrization process introduced in [7]. More concretely, we take into account that, by [49, Remark 5], any closed $k$-form $\alpha$ on a nilmanifold is cohomologous to the invariant $k$-form $\bar{\alpha}$ obtained by the symmetrization process.

Fix any $t \in \Delta$. Since the complex structure on $X_t$ is invariant, if there exists a pseudo-Kähler structure $\Theta$ on $X_t$, then the form $\bar{\Theta}$, obtained by symmetrization of $\Theta$, would be an invariant closed 2-form of bidegree $(1,1)$ such that $[\bar{\Theta}] = [\Theta]$ in the second de Rham cohomology group $H^2_{\text{dR}}(X_t; \mathbb{R})$. In particular, this would imply $[\Theta^3] = [\Theta]^3 \neq 0$ in the de Rham cohomology group $H^6_{\text{dR}}(X_t; \mathbb{R})$, due to the non-degeneracy of $\Theta$ on the compact complex manifold $X_t$. However, for $t \neq 0$, it follows from (3) that any invariant closed $(1,1)$-form $\zeta$ on $X_t$ satisfies

\[ \zeta \in \mathbb{C}(\omega_1^{11}, \omega_1^{12}, \omega_2^{12}, \omega_2^{22}), \]

which implies that $\zeta^3 = \zeta \wedge \zeta \wedge \zeta = 0$, i.e. $\zeta$ is degenerate. In conclusion, there are no (invariant or not) pseudo-Kähler metrics on $X_t$ for $t \neq 0$. \hfill \Box

Remark 2.2. Let us consider $Y_t = X_t \times \mathbb{T}^m$, where $\{X_t\}_{t \in \Delta}$ is the holomorphic family given in Proposition 2.1 and $\mathbb{T}^m$ denotes the $m$-dimensional complex torus endowed with its standard Kähler metric. A similar argument as the one in the proof of the previous proposition shows that $Y_t$ does not admit any pseudo-Kähler metric for $t \neq 0$. Hence, the pseudo-Kähler property is not stable in any complex dimension $n \geq 3$ (see Theorem 3.7 for an irreducible example in complex dimension 4). In contrast, we will prove in Section 2.3 that compact pseudo-Kähler surfaces are stable.
The following example illustrates that, although the pseudo-Kähler property is in general not stable, one can find certain deformations where the existence of pseudo-Kähler metrics is preserved.

**Example 2.3.** Let us consider the differentiable family \( \{X_t\}_{t \in (-1,1)} \) of compact complex nilmanifolds defined by the complex structure equations

\[
d\omega^1_t = 0, \quad d\omega^2_t = \omega^1_{11}, \quad d\omega^3_t = \omega^1_{12} + t\omega^1_{12}.
\]

We note that this is a differentiable family of deformations of the compact complex nilmanifold \( X = X_0 \) determined by the equations

\[
d\omega^1 = 0, \quad d\omega^2 = \omega^{11}, \quad d\omega^3 = \omega^{12}.
\]

The manifolds \( X_t \) in this family are all pseudo-Kähler since, for instance, \( F_t = i(\omega^1_{13} + \omega^3_{31}) + i(1 + t)\omega^2_{22} \) defines a pseudo-Kähler metric on \( X_t \) for every \( t \in (-1,1) \).

Hence, one would like to study additional conditions under which the pseudo-Kähler property becomes stable. We next establish a condition in terms of the Bott-Chern cohomology.

### 2.2. Bott-Chern cohomology and stability of pseudo-Kähler manifolds.

Here, we show that the stability of the pseudo-Kähler property is closely related to the variation of the Bott-Chern cohomology. We recall that the Bott-Chern and the Aeppli cohomologies \([19]\) (see also \([3]\)) of a compact complex manifold \( X \) are defined, respectively, by

\[
H^{\bullet,\bullet}_{BC}(X) := \ker \partial \cap \ker \overline{\partial} \quad \text{and} \quad H^{\bullet,\bullet}_{A}(X) := \ker \partial + \ker \overline{\partial}.
\]

The dimensions of these cohomology groups will be denoted by \( h^{p,q}_{BC}(X) = \dim_C H^{p,q}_{BC}(X) \) and \( h^{p,q}_{A}(X) = \dim_C H^{p,q}_{A}(X) \).

Suppose that \( X \) admits a pseudo-Kähler metric defined by \( F \). Since the real form \( F^k \) is closed and has bidegree \((k,k)\), it defines a Bott-Chern class \( [F^k] \in H^{k,k}_{BC}(X) \) for any \( 1 \leq k \leq n \). Moreover:

**Lemma 2.4.** Let \( X \) be a compact complex manifold with \( \dim_C X = n \). If \( X \) admits a pseudo-Kähler metric, then \( h^{k,k}_{BC}(X) \geq 1 \) for any \( 1 \leq k \leq n \).

**Proof.** Let us consider a pseudo-Kähler metric on \( X \) defined by \( F \) and suppose that the class \( [F^k] \in H^{k,k}_{BC}(X) \) is zero for some \( 1 \leq k \leq n \). This fact implies that \( F^k = \partial \overline{\partial} \beta \) for some form \( \beta \in \Omega^{k-1,k-1}(X) \), so

\[
F^n = F^{n-k} \wedge F^k = F^{n-k} \wedge \partial \overline{\partial} \beta = \partial(F^{n-k} \wedge \overline{\partial} \beta) = d(F^{n-k} \wedge \overline{\partial} \beta),
\]

which contradicts the non-degeneracy of the form \( F \).

The following proposition is an extension of the Kodaira-Spencer stability result \([26]\) Theorem 15] for Kähler metrics. We recall that, for every \((p,q)\), the function \( t \mapsto h^{p,q}_{BC}(X_t) \) is upper-semi-continuous \([42]\).

**Proposition 2.5.** Let \( X \) be a compact pseudo-Kähler manifold, and let \( \{X_t\}_{t \in (-\varepsilon,\varepsilon)} \) be a differentiable family of deformations of \( X = X_0 \), where \( \varepsilon > 0 \). Suppose that the upper-semi-continuous function \( t \mapsto h^{1,1}_{BC}(X_t) \) is constant. Then, the compact complex manifold \( X_t \) admits a pseudo-Kähler metric for any \( t \) close enough to 0.

**Proof.** Let \( \{\omega_t\} \) be a family of Hermitian metrics on \( X_t \) for \( t \in (-\varepsilon,\varepsilon) \). For each \( t \), we consider the Bott-Chern Laplacian \( \Delta^B_t \) associated to the Hermitian metric \( \omega_t \) on \( X_t \) and the corresponding Green operator \( G_t \) \([42]\). Denote by \( H_t : \Omega^*_C(X_t) \to \ker \Delta^B_t \) the projection onto the space of harmonic forms with respect to \( \Delta^B_t \) (and with respect to the \( L^2_{\omega_t} \)-orthogonal decomposition
induced by $\omega_t$ \cite{[42]}, and by $\pi_t^{1,1}: \Omega^*_C(X) \to \Omega^{1,1}(X_t)$ the projection of the space of complex forms on X onto the space of (1,1)-forms on $X_t$.

Now, for any $t \in (-\varepsilon, \varepsilon)$, the operator $\Pi_t$ defined by

$$\Pi_t := \left( H_t + \partial_t \overline{\partial}_t (\partial_t \overline{\partial}_t)^* G_t \right) \circ \pi_t^{1,1}: \Omega^*_C(X) \to \ker \partial_t \cap \ker \overline{\partial}_t,$$

gives the projection of the space of complex forms on $X$ onto the space of $\partial_t$-closed and $\overline{\partial}_t$-closed (1,1)-forms on the compact complex manifold $X_t$. Here, $*_{\varepsilon}$ is the Hodge-operator with respect to the Hermitian metric $\omega_t$ on $X_t$. Since the function $t \mapsto h^{1,1}_{BC}(X_t)$ is constant, by elliptic theory \cite{[26]} Theorem 7 one has that the family $\{\Pi_t\}_t$ is smooth in $t$. Let $F_0$ be a pseudo-Kähler metric on $X_0 = X$. For $t \in (-\varepsilon, \varepsilon)$, we set

$$F_t := \frac{\Pi_t F_0 + \overline{\Pi_t F_0}}{2}.$$

Then, the family $\{F_t\}_t$ is smooth in $t$, and each $F_t$ is a real (1,1)-form on $X_t$ which is $d$-closed, because it is closed by $\partial_t$ and $\overline{\partial}_t$. Since $F_0^n$ is a nowhere vanishing $(n, n)$-form, we have that $F_t$ is non-degenerate for $t$ close enough to 0. Therefore, the form $F_t$ defines a pseudo-Kähler metric on the compact complex manifold $X_t$ for any $t$ close enough to 0.

**Example 2.6.** We here show that for the holomorphic family $\{X_t\}_{t \in \Delta}$ constructed in the proof of Proposition 2.4, the upper-semicontinuous function $t \mapsto h^{1,1}_{BC}(X_t)$ is not constant. According to Proposition 2.5 this fact gives a reason for the instability of the pseudo-Kähler property along the deformation of $X_0$. Indeed, for the compact complex manifold $X_0$ we have

$$H^{1,1}_{BC}(X_0) = \langle [i \omega^{11}], [i \omega^{22}], [\omega^{12} - \omega^{21}], [i (\omega^{12} + \omega^{21})], [\omega^{23} - \omega^{32}], [i (\omega^{23} + \omega^{32})] \rangle,$$

and, for any $t \in \Delta - \{0\}$, the Bott-Chern cohomology group of bidegree $(1,1)$ of $X_t$ is

$$H^{1,1}_{BC}(X_t) = \langle [i \omega_t^{11}], [i \omega_t^{22}], [\omega_t^{12} - \omega_t^{21}], [i (\omega_t^{12} + \omega_t^{21})] \rangle.$$

Therefore, $h^{1,1}_{BC}(X_0) = 6$, and $h^{1,1}_{BC}(X_t) = 4$, for any $t \neq 0$.

**Example 2.7.** Let us consider the differentiable family $\{X_t\}_{t \in (-1, 1)}$ of compact pseudo-Kähler manifolds given in Example 2.3. The Bott-Chern cohomology group of bidegree $(1,1)$ of $X_t$ is

$$H^{1,1}_{BC}(X_t) = \langle [i \omega_t^{11}], [\omega_t^{12} - \omega_t^{21}], [i (\omega_t^{12} + \omega_t^{21})], [i (\omega_t^{13} + \omega_t^{31}) + i (1 + t) \omega_t^{22}] \rangle.$$

Therefore, $h^{1,1}_{BC}(X_t) = 4$ for every $t$, i.e. the function $t \mapsto h^{1,1}_{BC}(X_t)$ is constant.

We next show that the result in Example 2.7 can indeed be extended to any family of deformations of $X = X_0$.

**Proposition 2.8.** Let $X$ be the compact pseudo-Kähler manifold in Example 2.3 defined by (1), and let $\{X_t\}_{t \in (-\varepsilon, \varepsilon)}$ be any differentiable family of deformations of $X = X_0$, where $\varepsilon > 0$. Then, $X_t$ admits a pseudo-Kähler metric for every $t$ close enough to 0.

**Proof.** Since $X$ is a nilmanifold endowed with an invariant complex structure, one has by \cite{[10]} Theorem 2.6 that the complex structure $J_t$ of any sufficiently small deformation $X_t$ of $X$ is also invariant. The dimension of the Bott-Chern cohomology groups of any invariant complex structure $J_t$ is given in \cite{[1]} Table 2 and \cite{[30]} Appendix 6. Note that the Lie algebra underlying the nilmanifold $X$ is precisely $\mathfrak{h}_{15}$, and $h^{1,1}_{BC}(\mathfrak{h}_{15}, J) \geq 4$ for any complex structure $J$ on $\mathfrak{h}_{15}$.

Since $h^{1,1}_{BC}(X_t) = h^{1,1}_{BC}(\mathfrak{h}_{15}, J_t) \geq 4$ varies upper-semi-continuously along differentiable families \cite{[42]}, for any $t$ close enough to 0 we have

$$4 = h^{1,1}_{BC}(X_0) \geq h^{1,1}_{BC}(X_t) \geq 4,$$
and thus $h_{BC}^{1,1}(X_t) = 4$. Hence, the function $t \mapsto h_{BC}^{1,1}(X_t)$ is constant and, by Proposition 2.5, $X_t$ admits a pseudo-Kähler metric for any $t$ close enough to 0.

We observe that the condition on $h_{BC}^{1,1}$ given in Proposition 2.5 is sufficient but not necessary. To illustrate this fact, we next study the existence of pseudo-Kähler metrics on the small deformations of the well-known Nakamura manifold. This manifold is a holomorphically parallelizable solvmanifold, i.e. a compact quotient $X = \Gamma \backslash G$ where $G$ is a simply connected complex solvable Lie group and $\Gamma$ a lattice in $G$.

Let $G$ be the simply-connected complex solvable Lie group given by

$$G = \left\{ \left( \begin{array}{ccc} e^{z_1} & 0 & z_2 \\ 0 & e^{-z_1} & z_3 \\ 0 & 0 & 1 \end{array} \right) \mid z_1, z_2, z_3 \in \mathbb{C} \right\},$$

i.e. $G$ is the semi-direct product $G = \mathbb{C} \ltimes \mathbb{C}^2$, where $(z_2, z_3)$ are the coordinates on $\mathbb{C}^2$ and

$$\varphi(z_1) = \left( \begin{array}{cc} e^{z_1} & 0 \\ 0 & e^{-z_1} \end{array} \right), \quad z_1 \in \mathbb{C}.$$

One can consider the symplectic form on the Lie group $G$ defined by

$$F = i\, dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_3 + d\bar{z}_2 \wedge dz_3,$$

which clearly has bidegree $(1,1)$ with respect to the complex structure of $G$. That is to say, $F$ is a pseudo-Kähler metric on $G$. Notice that $F$ is not left-invariant on $G$, indeed the forms

$$\omega^1 = dz_1, \quad \omega^2 = e^{-z_1}\, dz_2, \quad \omega^3 = e^{z_1}\, dz_3,$$

constitute a basis of left-invariant forms of bidegree $(1,0)$ on the complex Lie group $G$.

Yamada proved in [50] that there is a lattice $\Gamma$ of maximal rank in $G$ such that $F$ descends to a pseudo-Kähler metric on $X = \Gamma \backslash G$. Let $A \in \text{SL}(2,\mathbb{Z})$ be a unimodular matrix with distinct real eigenvalues $\lambda$ and $\lambda^{-1}$, and take $a = \log \lambda$. One can consider a lattice $\Gamma$ in $G$ of the form $\Gamma = \Gamma_1 \ltimes \varphi \Gamma_2$, with $\Gamma_1 = a\, \mathbb{Z} + 2\, \pi i\, \mathbb{Z}$ and $\Gamma_2$ a lattice in $\mathbb{C}^2$. The compact complex (solv)manifold $X = \Gamma \backslash G$ is known as the (holomorphically parallelizable) Nakamura manifold [35]. Yamada proved in [50] Theorem 2.1] that the symplectic form $F$ given in (5) descends to $X$, providing in this way the first example of a non-toral compact holomorphically parallelizable pseudo-Kähler solvmanifold. In fact, in terms of the $(1,0)$-basis $\{\omega^k\}_{k=1}^3$, the form $F$ expresses as

$$F = i\, \omega^1 \wedge \omega^1 + e^{2i\, \partial \bar{z}_1} \omega^2 \wedge \omega^3 + e^{-2i\, \partial \bar{z}_1} \omega^2 \wedge \omega^3,$$

where the functions $e^{2i\, \partial \bar{z}_1}$ and $e^{-2i\, \partial \bar{z}_1}$ are $\Gamma$-invariant. Therefore, $F$ induces a pseudo-Kähler metric on the Nakamura manifold $X$.

In [22] Hasagawa extended Yamada’s result to any compact holomorphically parallelizable solvmanifold of complex dimension 3, determining all the lattices of simply-connected complex solvable Lie groups. Hasagawa proves that such a complex solvmanifold admits a pseudo-Kähler metric if and only if the Hodge number $h_0^{1,1} = 3$ [22 Theorem 2]. Notice that for the Nakamura manifold $X$ one has $H_0^{1,1}(X) = \langle [\omega^1], [e^{-2i\, \partial \bar{z}_1} \omega^2], [e^{2i\, \partial \bar{z}_1} \omega^3] \rangle$.

In the following proposition we show that there is a small deformation $X_t$ of $X$ admitting pseudo-Kähler metrics. Note that by [5] the Nakamura manifold has $h_{BC}^{1,1}(X) = 7$, whereas our small deformation satisfies $h_{BC}^{1,1}(X_t) = 3$ for every $t \neq 0$.

**Proposition 2.9.** There exists a small deformation $X_t$ of the holomorphically parallelizable Nakamura manifold $X$ admitting pseudo-Kähler metrics for any $t$. 

Proof. In [5, Section 4] Angella and Kasuya studied some deformations of the Nakamura manifold $X = \Gamma \backslash G$. We here consider the deformation given by $t \frac{\partial}{\partial z_1} \otimes d\bar{z}_1 \in H^{0,1}(X; T^{1,0}X)$, which corresponds to the case (1) in their paper.

Note that this deformation defines a holomorphic family $\{X_t\}_{t \in \Delta}$, where $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$, such that $X_0 = X$. We have the $(1,0)$-forms on $X_t$ given by

$$\omega^1_t = dz_1 - t \cdot d\bar{z}_1, \quad \omega^2_t = e^{-z_3}dz_2, \quad \omega^3_t = e^{z_3}dz_3,$$

whose differentials satisfy

$$\begin{cases} 
   d\omega^1_t = 0, \\
   d\omega^2_t = -\frac{1}{1-|t|^2} \omega^1_t + \frac{t}{1-|t|^2} \omega^{21}_t, \\
   d\omega^3_t = \frac{1}{1-|t|^2} \omega^{13}_t - \frac{t}{1-|t|^2} \omega^{31}_t.
\end{cases}$$

A direct calculation shows that the real 2-form $F_t$ defined on $X_t$ by

$$F_t = i \omega^{11}_t + e^{2i\Im z_1} \omega^{23}_t + e^{-2i\Im z_1} \omega^{23}_t$$

is closed and non-degenerate. Since $F_t$ has bidegree $(1,1)$ with respect to the complex structure on $X_t$, we get a pseudo-Kähler metric on the compact complex manifold $X_t$, for any $t \in \Delta$. □

In complex dimension 3, there is another well-known complex holomorphically parallelizable solvmanifold, namely, the Iwasawa (nil)manifold. Although this manifold is symplectic, it is proved in [12, Theorem 3.2] (see also [50]) that a holomorphically parallelizable nilmanifold is pseudo-Kähler if and only if it is a complex torus. Hence, the Iwasawa manifold does not admit pseudo-Kähler metrics. Moreover:

Proposition 2.10. The Iwasawa manifold and its small deformations do not admit any pseudo-Kähler metric.

Proof. Recall that the Iwasawa manifold $X$ is the compact complex manifold obtained as a quotient of the 3-dimensional complex Heisenberg group. As we have previously stated, by [12, Theorem 3.2] the Iwasawa manifold is not pseudo-Kähler. Nakamura studied in [35] the small deformations of the Iwasawa manifold (see also [3] for more details). It turns out that the complex structure equations of any sufficiently small deformation $X_t$ of the Iwasawa manifold $X = X_0$ can be written as

$$\begin{cases} 
   d\omega^1_t = d\omega^2_t = 0, \\
   d\omega^3_t = \sigma_{12} \omega^{12}_t + \sigma_{11} \omega^{11}_t + \sigma_{12} \omega^{21}_t + \sigma_{21} \omega^{21}_t + \sigma_{22} \omega^{22}_t,
\end{cases}$$

where the coefficients $\sigma_{12}, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22} \in \mathbb{C}$ only depend on the parameter $t$ in the deformation space $\Delta = \{t = (t_{11}, t_{12}, t_{21}, t_{22}, t_{31}, t_{32}) \in \mathbb{C}^6 \mid |t| < \varepsilon\}$ for a sufficiently small $\varepsilon > 0$. Let $H^+(X_t) \subset H^2_{\text{dR}}(X_t; \mathbb{R})$ be the subspace determined by the second de Rham cohomology classes that can be represented by closed real forms of bidegree $(1,1)$ on the compact complex manifold $X_t$. As proved in [29, Proposition 3.4], this subspace is given by

$$H^+(X_t) = \langle [i \omega^{11}_t], [i \omega^{22}_t], [\omega^{12}_t - \omega^{21}_t], [i (\omega^{12}_t + \omega^{21}_t)] \rangle.$$

It is then clear that any de Rham cohomology class in $H^+(X_t)$ is degenerate, so $X_t$ does not admit any pseudo-Kähler metric. □
2.3. Compact pseudo-Kähler surfaces. We start this section applying our Proposition 2.5 to provide a stability result for the pseudo-Kähler property in terms of the complex invariant $\Delta^2(X)$ introduced in [6].

Let us recall that, for each $k \in \mathbb{N}$, Angella and Tomassini introduced the complex invariant

$$\Delta^k(X) := \sum_{p+q=k} (h^{p,q}_{BC}(X) + h^{p,q}_{A}(X)) - 2b_k,$$

which is a non-negative integer [6, Theorem A]. Furthermore, by [6, Theorem B], a compact complex manifold $X$ satisfies the $\partial\bar{\partial}$-Lemma if and only if $\Delta^k(X) = 0$ for any $k$. We are interested in the term $\Delta^2(X)$.

**Corollary 2.11.** Let $X$ be a compact pseudo-Kähler manifold, and let $\{X_t\}_{t \in (-\varepsilon, \varepsilon)}$ be a differentiable family of deformations of $X = X_0$, where $\varepsilon > 0$. If the upper-semi-continuous function $t \mapsto \Delta^2(X_t)$ is constant, then $X_t$ admits a pseudo-Kähler metric for any $t$ close enough to 0.

**Proof.** Suppose $\Delta^2(X_t) = c$ for any $t \in (-\varepsilon, \varepsilon)$, where $c$ is a non-negative integer. Then, expanding (6) for $k = 2$, we have

$$h^{2,0}_{BC}(X_0) + h^{1,1}_{BC}(X_0) + h^{0,2}_{BC}(X_0) + h^{2,0}_{A}(X_0) + h^{1,1}_{A}(X_0) + h^{0,2}_{A}(X_0)$$

$$= \Delta^2(X_0) + 2b_2 = c + 2b_2 = \Delta^2(X_t) + 2b_2$$

$$= h^{2,0}_{BC}(X_t) + h^{1,1}_{BC}(X_t) + h^{0,2}_{BC}(X_t) + h^{2,0}_{A}(X_t) + h^{1,1}_{A}(X_t) + h^{0,2}_{A}(X_t).$$

Since the functions $t \mapsto h^{p,q}_{BC}(X_t)$ and $t \mapsto h^{p,q}_{A}(X_t)$ are upper-semi-continuous for any $(p, q)$, they all must be constant. In particular, the function $t \mapsto \dim_{\mathbb{C}} H^{1,1}_{BC}(X_t)$ is constant, so Proposition 2.5 implies that the compact complex manifold $X_t$ admits a pseudo-Kähler metric for any $t$ close enough to 0.

The following result comes as a direct consequence.

**Corollary 2.12.** Any sufficiently small deformation of a compact pseudo-Kähler manifold $X$ satisfying $\Delta^2(X) = 0$ admits a pseudo-Kähler metric. In particular, any sufficiently small deformation of a compact pseudo-Kähler $\partial\bar{\partial}$-manifold is pseudo-Kähler.

Let us recall that a compact complex surface is Kähler if and only if its first Betti number $b_1$ is even (see Kodaira’s classification of surfaces, [31] and [43], or [10, 28] for a direct proof). To our knowledge, there is no classification of compact complex surfaces admitting a pseudo-Kähler metric. Petean found in [39] some obstructions to the existence of indefinite Kähler metrics on compact complex surfaces in terms of Seiberg-Witten invariants. Here “indefinite Kähler” means that the signature of the metric is $(2,2)$, i.e. the metric is pseudo-Kähler but non-Kähler (see [39, Theorem 4] for a list of the possible compact complex surfaces that might admit an indefinite Kähler metric).

Next, we make use of our previous results and a result by Teleman [45] to prove the stability of the pseudo-Kähler property on compact complex surfaces.

**Theorem 2.13.** Any sufficiently small deformation of a compact pseudo-Kähler surface admits a pseudo-Kähler metric.

**Proof.** Let $X$ be a compact pseudo-Kähler surface. By Teleman’s result [45, Lemma 2.3], the invariant $\Delta^2(X)$ is either 0 or 2. Moreover, Teleman also proves that $\Delta^2(X) = 0$ if and only if the first Betti number $b_1$ is even, which is equivalent to the existence of a Kähler metric on $X$. Hence, in the case $\Delta^2(X) = 0$, any sufficiently small deformation of $X$ is again Kähler by [26], so in particular pseudo-Kähler.
Let us now focus on the case $\Delta^2(X) = 2$. By [45] this is equivalent to $X$ having odd first Betti number, so the condition $\Delta^2(X) = 2$ is a topological property. Thus, $\Delta^2(X_t) = 2$ for any differentiable family $\{X_t\}_{t \in (-\varepsilon, \varepsilon)}$ of deformations of the compact pseudo-Kähler surface $X = X_0$. Since the upper-semi-continuous function $t \mapsto \Delta^2(X_t)$ is constant, by Corollary 2.11 the compact complex surface $X_t$ admits a pseudo-Kähler metric for any $t$ close enough to 0.

To finish this section we show that there is only one compact complex non-Kähler surface diffeomorphic to a solvmanifold that admits pseudo-Kähler metrics, namely the Kodaira-Thurston manifold.

Hasegawa classified in [23] the compact complex surfaces $X$ that are diffeomorphic to a 4-dimensional solvmanifold. Moreover, he proved that the complex structures on such surfaces are invariant (see [37] for a study of 4-dimensional Lie algebras admitting pseudo-Kähler metrics). In fact, in [23, Theorem 1] it is shown that $X$ must be one of the following surfaces: complex torus, hyperelliptic surface, Inoue surface of type $S_M$, primary Kodaira surface, secondary Kodaira surface, or Inoue surface of type $S^\pm$. Only the first two are Kähler, whereas the other ones have vanishing second Betti number, with the only exception of a primary Kodaira surface. It is well-known that the latter admits symplectic forms [36]. Consequently, a compact complex non-Kähler surface diffeomorphic to a solvmanifold admits a symplectic form if and only if it is a primary Kodaira surface.

We recall that a primary Kodaira surface, which we will denote by $KT$, admits pseudo-Kähler metrics. By [23], for any complex structure on $KT$ there is a global basis $\{\omega^1, \omega^2\}$ of $(1,0)$-forms satisfying

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{11}.$$  

A real $(1,1)$-form $F$ on $KT$ is closed if and only if

$$F = ir \omega^{11} + u \omega^{12} - \bar{u} \omega^{21},$$

for some $r \in \mathbb{R}$ and $u \in \mathbb{C}$. Since $F^2 = -2|u|^2 \omega^{12}\bar{12}$, we have that $F$ is non-degenerate if and only if $u \neq 0$. Thus, there are pseudo-Kähler metrics on $KT$.

As a consequence of our previous discussion, we have:

**Proposition 2.14.** Let $X$ be a compact complex non-Kähler surface diffeomorphic to a solvmanifold. If $X$ admits a pseudo-Kähler metric, then $X$ is a primary Kodaira surface $KT$.

### 3. Neutral Calabi-Yau manifolds

In this section we focus our attention on a special kind of pseudo-Kähler manifolds, namely, neutral Calabi-Yau manifolds. Moreover, the intermediate class constituted by neutral Kähler manifolds is also studied. We first prove that compact neutral, Kähler or Calabi-Yau, surfaces are stable by small deformations of the complex structure. In higher dimensions, we construct an 8-dimensional nilmanifold endowed with a neutral Calabi-Yau metric that allows us to prove the instability of the neutral Kähler and neutral Calabi-Yau properties in any even complex dimension $n \geq 4$. It is worth to note that such nilmanifold also provides a counterexample to a conjecture in [13].

We first recall some definitions. Let $X = (M, J)$ be a complex manifold of complex dimension $n = 2m$. Following [20], a neutral Kähler structure on $X$ is a neutral metric $g$, i.e. of signature $(2m, 2m)$, such that

- $g$ is compatible with $J$, i.e. $g(JU, JV) = g(U, V)$ for any vector fields $U, V$ on $M$; and
- $J$ is parallel with respect to the Levi-Civita connection $\nabla$ of $g$, i.e. $\nabla J = 0$. 


These conditions imply that the 2-form \( F(U, V) = g(U, JV) \) is closed, i.e. a neutral Kähler structure is in particular pseudo-Kähler.

A neutral Kähler structure is said to be neutral Calabi-Yau if there exists a nowhere vanishing form \( \Phi \) of bidegree \((2m,0)\) with respect to \( J \) satisfying \( \nabla \Phi = 0 \). Neutral Calabi-Yau manifolds are Ricci-flat.

### 3.1. Stability on compact complex surfaces.

In this section we study the stability of the neutral Kähler and neutral Calabi-Yau properties in complex dimension 2.

**Proposition 3.1.** Let \( X \) be a compact complex non-Kähler surface. Suppose that \( X \) admits a neutral Kähler metric. Then, any sufficiently small deformation of \( X \) also admits neutral Kähler metrics.

**Proof.** By Theorem 2.13 any sufficiently small deformation \( X_t \) of \( X \) admits pseudo-Kähler metrics. Since \( X \) is non-Kähler by hypothesis, the first Betti number of \( X_t \) is odd, so a pseudo-Kähler metric on \( X_t \) cannot have signature \((4,0)\) or \((0,4)\). Hence, the pseudo-Kähler metrics are necessarily neutral. \( \square \)

Let us now observe the following. On the one hand, Petean proves in [39, Proposition 5] that if a compact complex surface \( X \) admits a Ricci-flat neutral Kähler metric, then its Kodaira dimension and its first Chern class are both zero. Moreover, \( X \) must be a complex torus, a hyperelliptic surface, or a primary Kodaira surface [39, Corollary 2]. On the other hand, any compact complex surface with holomorphically trivial canonical bundle is isomorphic to a K3 surface, a torus, or a primary Kodaira surface. Hence, as a consequence of these results, one can ensure that the only compact complex surfaces that can admit neutral Calabi-Yau structures are a complex torus and a primary Kodaira surface. We know by [23] that these two are diffeomorphic to a 4-dimensional nilmanifold with an invariant complex structure. It is easy to check that they are both neutral Calabi-Yau. In fact, the result is clear for the torus, whereas for a primary Kodaira surface it suffices to remark that, according to the complex equations (7), the \((2,0)\)-form \( \Phi = \omega^{12} \) is nowhere vanishing and parallel with respect to the Levi-Civita connection of any neutral Kähler metric [8]. Concerning deformations, it is well-known that the small deformations of the invariant complex structures on the complex torus or on the primary Kodaira surface are again invariant, so they admit neutral Calabi-Yau structures.

We sum up our previous discussion in the following result.

**Proposition 3.2.** Let \( X \) be a compact neutral Calabi-Yau surface. Then, \( X \) is a complex torus or a primary Kodaira surface. Moreover, any sufficiently small deformation of \( X \) admits neutral Calabi-Yau structures.

Our next goal is to prove the instability of the neutral Calabi-Yau and neutral Kähler properties in complex dimension \( n \geq 4 \). We begin constructing an 8-dimensional nilmanifold endowed with neutral Calabi-Yau metrics. To our knowledge, this neutral Calabi-Yau nilmanifold is new and, in addition, it provides counterexamples to a conjecture on pseudo-Kähler nilpotent Lie algebras, as we will shortly see.

### 3.2. A neutral Calabi-Yau nilmanifold in eight dimensions.

Let \( M \) be a nilmanifold endowed with an invariant complex structure \( J \). Let \( n \) be the complex dimension of \( X = (M, J) \), and suppose that \( F \) is an invariant pseudo-Kähler metric on \( X \). By [11], there always exists a closed (non-zero) invariant form \( \Phi \) of bidegree \((n,0)\) with respect to \( J \), so \( \nabla \Phi = 0 \) for the Levi-Civita connection \( \nabla \) of the invariant metric. Therefore, any invariant pseudo-Kähler metric \( F \) on a complex nilmanifold \( X \) is Ricci-flat (see [19]).
Proposition 3.3. Let $X = (M, J)$ be the 8-dimensional nilmanifold $M$ endowed with the invariant complex structure $J$ defined by a basis of $(1,0)$-forms $\{\omega^k\}_{k=1}^4$ satisfying the structure equations

$$
\begin{align*}
\begin{cases}
    d\omega^1 & = 0, \\
    d\omega^2 & = -i \omega^{14} + i \omega^{11}, \\
    d\omega^3 & = \omega^{12} + \omega^{12} - \omega^{21}, \\
    d\omega^4 & = -\omega^{13} + \omega^{31}.
\end{cases}
\end{align*}
$$

Then, $X$ admits pseudo-Kähler metrics. Moreover, any invariant pseudo-Kähler metric $F$ on $X$ is given by

$$
F = i (r \omega^{11} + s \omega^{41}) + u (\omega^{12} - \omega^{21}) + v (\omega^{13} - \omega^{31}) - s (\omega^{23} - \omega^{32}),
$$

where $r, s, u, v \in \mathbb{R}$ and $rs \neq 0$.

Proof. It is easy to check that the equations (9) satisfy the Jacobi identity, i.e. $d^2 \omega^k = 0$ for $1 \leq k \leq 4$. Thus, they define a simply-connected, connected, nilpotent Lie group $G$ of real dimension 8. Moreover, since the structure constants belong to $\mathbb{Q}[\bar{i}]$, the well-known Malcev’s theorem ensures the existence of a lattice $\Gamma$ of maximal rank in $G$. We consider the nilmanifold $M = \Gamma \backslash G$, which is endowed by construction with the complex structure $J$ defined by the $(1,0)$-forms $\{\omega^k\}_{k=1}^4$.

Any invariant real 2-form $F$ of bidegree $(1,1)$ on $X = (M, J)$ can be written as

$$
F = \sum_{k=1}^4 i x_{kk} \omega^{kk} + \sum_{1 \leq k < l \leq 4} (x_{ki} \omega^{kl} - \bar{x}_{ki} \omega^{lk}),
$$

where $x_{kk} \in \mathbb{R}$ and $x_{kl} \in \mathbb{C}$, for $1 \leq k, l \leq 4$. Since we are looking for pseudo-Kähler metrics, we will study the condition $dF = 0$ and the non-degeneracy condition $F^4 \neq 0$.

By a direct computation using the complex structure equations (9) we get

$$
dF = \Theta + \overline{\Theta},
$$

where $\Theta = \partial F$ is the complex 3-form of bidegree $(2,1)$ given by

$$
\Theta = 2i \text{Im} x_{13} \omega^{121} + 2i \text{Im} x_{23} \omega^{122} - (x_{24} - i x_{33}) \omega^{123} + x_{34} \omega^{124} - x_{14} \omega^{131} + i x_{33} \omega^{132}
$$

$$
- x_{34} \omega^{133} + 2 \text{Im} x_{13} \omega^{141} + (x_{23} - \bar{x}_{34}) \omega^{142} - i (x_{23} + \bar{x}_{44}) \omega^{143} - i x_{24} \omega^{144}
$$

$$
- (x_{24} + i x_{33}) \omega^{231} + (x_{22} + \bar{x}_{34}) \omega^{241} + i (x_{44} + \bar{x}_{23}) \omega^{341}.
$$

Now, the closedness condition $dF = 0$ is equivalent to $\Theta = 0$. It is straightforward to check that the latter is satisfied if and only if

$$
x_{22} = x_{33} = x_{14} = x_{24} = x_{34} = \text{Im} x_{12} = \text{Im} x_{13} = \text{Im} x_{23} = 0, \quad \text{and} \quad x_{23} = -x_{44}.
$$

Replacing these values in (11) and denoting $x_{11} = r$, $x_{44} = s$, $x_{12} = u$, $x_{13} = v$, with $r, s, u, v \in \mathbb{R}$, one directly gets the expression (10).

We finally need to ensure the non-degeneracy condition for $F$. Using (10), it is easy to see that

$$
F^4 = -24 r s^2 \omega^{12341234}.
$$

Therefore, $F^4 \neq 0$ if and only if $rs \neq 0$, as stated in the proposition. □

In the following result we prove that the family of pseudo-Kähler metrics (10) provides neutral Calabi-Yau metrics in eight dimensions that are not flat (although they all are Ricci flat).
Proposition 3.4. The complex nilmanifold \( X = (M, J) \) constructed in Proposition 3.3 has non-flat neutral Calabi-Yau structures.

Proof. Let us take the basis of real 1-forms \( \{e^1, \ldots, e^8\} \) on \( X \) defined by \( e^{2k-1} + ie^{2k} = \omega^k \), for \( 1 \leq k \leq 4 \), where \( \{\omega^1, \ldots, \omega^4\} \) is the basis of \((1,0)\)-forms in Proposition 3.3 satisfying the complex structure equations (9). In terms of this real basis, the complex structure \( J \) and the pseudo-Kähler metrics \( F \) given in (10) express as

\[
J e^1 = -e^2, \quad J e^3 = -e^4, \quad J e^5 = -e^6, \quad J e^7 = -e^8,
\]

\[
F = 2r e^{12} + 2u e^{13} + 2v e^{15} + 2u e^{24} + 2v e^{26} - 2s e^{35} - 2s e^{46} + 2s e^{78},
\]

where \( r, s, u, v \in \mathbb{R} \) with \( rs \neq 0 \). The pseudo-Riemannian metric \( g(x, y) = F(Jx, y) \) is then given in terms of this real basis by the matrix

\[(g_{ij})_{i,j} = \begin{pmatrix}
2r & 0 & 0 & -2u & 0 & -2v & 0 & 0 \\
0 & 2r & 2u & 0 & 2v & 0 & 0 & 0 \\
0 & 2u & 0 & 0 & 0 & 2s & 0 & 0 \\
-2u & 0 & 0 & 0 & -2s & 0 & 0 & 0 \\
0 & 2v & 0 & -2s & 0 & 0 & 0 & 0 \\
-2v & 0 & 2s & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2s & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2s 
\end{pmatrix}.
\]

It is easy to see that there are metrics in the family (12) with neutral signature. In fact, taking for instance \( u = v = 0 \), one has that \( rs < 0 \) is equivalent to the signature being \((4,4)\). Since the \((4,0)\)-form \( \Phi = \omega^{1234} \) is parallel, we conclude that \( X \) has neutral Calabi-Yau structures.

Let us now prove that any pseudo-Kähler metric (10) on \( X \) is non-flat. Since the pseudo-Kähler structures are invariant, the (complexified) Koszul formula for the Levi-Civita connection \( \nabla \) of the metric \( g \) reduces to

\[
2g(\nabla_U V, W) = g([U, V], W) - g([V, W], U) + g([W, U], V),
\]

for (invariant) complex vector fields \( U, V, W \) on the complex nilmanifold \( X \). Let \( \{Z_j\}_{j=1}^4 \) denote the basis of complex vector fields of bidegree \((1,0)\) dual to the basis \( \{\omega^j\}_{j=1}^4 \). Notice that, by complex conjugation, it suffices to compute \( \nabla_{Z_i} Z_j \) and \( \nabla_{\overline{Z}_i} Z_j \) for \( 1 \leq j \leq 4 \). Furthermore, since \( \nabla J = 0 \), one has that \( \nabla_U V \) is of bidegree \((1,0)\) whenever \( V \) is. In particular, \( \nabla_U Z_j \) has type \((1,0)\) for every \( 1 \leq j \leq 4 \).

Let \( R \) be the curvature tensor of the pseudo-Kähler metric, i.e. \( R \) is given by

\[
R(U, V, W, T) = g(\nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U, V]} W, T),
\]

for \( U, V, W, T \) complex vector fields on \( X \). Taking into account the observation in the previous paragraph, complex conjugation and the symmetries of the curvature tensor, one concludes that the metric \( g \) is non-flat if and only if \( R(Z_i, \overline{Z}_j, Z_k, \overline{Z}_l) \neq 0 \) for some \( i, j, k, l \). In our case, we will prove that \( R(Z_2, \overline{Z}_3, Z_2, \overline{Z}_3) \neq 0 \).

A direct calculation shows:

\[
\nabla_{Z_2} Z_1 = Z_3, \quad \nabla_{\overline{Z}_2} Z_2 = 0, \quad \nabla_{\overline{Z}_2} Z_3 = 0, \quad \text{and} \quad \nabla_{Z_2} Z_2 = -\frac{is}{r} Z_1 - \frac{iv}{r} Z_2 + \frac{iu}{r} Z_3.
\]
From the complex equations (3) we have $[Z_2, \bar{Z}_2] = 0$. Hence,
\[
R(Z_2, \bar{Z}_2, Z_2, \bar{Z}_2) = g(\nabla_{Z_2} \nabla_{\bar{Z}_2} Z_2 - \nabla_{\bar{Z}_2} \nabla_{Z_2} Z_2 - \nabla_{[Z_2, \bar{Z}_2]} Z_2, \bar{Z}_2)
\]
\[
= -g(\nabla_{Z_2} \nabla_{\bar{Z}_2} Z_2, \bar{Z}_2)
\]
\[
= g(\frac{i}{r} Z_3, \bar{Z}_2)
\]
\[
= -\frac{2}{r} \neq 0.
\]

Note that in [20] neutral Calabi-Yau structures on a specific class of nilmanifolds are constructed. This class is given by the so-called Kodaira manifolds, which are $4m$-dimensional 2-step nilmanifolds whose underlying Lie algebras have center of dimension $2m$. Moreover, their complex structure is invariant and preserves the center. Kodaira manifolds are a generalization of the Kodaira-Thurston manifold $KT$, and they have the structure of a principal torus bundle over a torus, with fiber the central torus.

It is worth to remark that, for the neutral Calabi-Yau nilmanifold $X = (M, J)$ constructed in Proposition 3.3, the center of the Lie algebra $g$ underlying $M$ is 1-dimensional. Hence, it is not invariant under the complex structure $J$ (see Section 3.2.1 for more details). Furthermore, $g$ has nilpotency step equal to 4. By [35], this implies that the neutral Calabi-Yau nilmanifold $X = (M, J)$ is far from being the total space of a principal torus bundle over a torus.

3.2.1. Counterexamples to a conjecture on pseudo-Kähler nilmanifolds. Here we show that the new pseudo-Kähler nilmanifold constructed in Proposition 3.3 provides counterexamples to a conjecture in [13]. The conjecture states that an invariant complex structure $J$ on a nilmanifold $M = \Gamma\backslash G$ must satisfy a certain property so that $(M, J)$ admits pseudo-Kähler metrics. Let us first recall some results on complex structures on nilpotent Lie algebras (which can be found in [31] and the references therein) and then formulate the conjecture in precise terms.

Let $g$ be a nilpotent Lie algebra. Complex structures on $g$ can be classified into different types attending to the behaviour of the ascending $J$-compatible series of $g$, which is defined inductively as

$$a_0(J) = \{0\}, \quad a_k(J) = \{X \in g \mid [X, g] \subseteq a_{k-1}(J) \text{ and } [JX, g] \subseteq a_{k-1}(J)\}, \quad \text{for } k \geq 1.$$

Note that $a_k(J)$ is an even-dimensional $J$-invariant ideal of $g$. In particular, $a_1(J)$ is the largest subspace of the center of $g$ which is $J$-invariant.

Unlike the usual ascending central series $\{g_k\}_k$ of $g$, the series $\{a_k(J)\}_k$ is adapted to the complex structure $J$, and it allows to introduce the following partition of the space of complex structures:

**Definition 3.5.** [31] A complex structure $J$ on a nilpotent Lie algebra $g$ is said to be

(i) nilpotent, if there exists an integer $t > 0$ such that $a_t(J) = g$;

(ii) non-nilpotent, if $a_t(J) \neq g$ for every $t \geq 0$; moreover, $J$ is called

(ii.1) strongly non-nilpotent, if $a_1(J) = \{0\}$ (which implies $a_t(J) = \{0\}$ for every $t$);

(ii.2) weakly non-nilpotent, if there is an integer $t > 0$ satisfying $\{0\} \neq a_t(J) = a_t(J) \neq g$, for every $t \geq t$.

Notice that $a_1(J) \neq \{0\}$ for any nilpotent or weakly non-nilpotent complex structure $J$. This allows to construct such structures from other complex structures defined on lower dimensional nilpotent Lie algebras (see [31] for details). This fact leaves strongly non-nilpotent complex structures as the essentially new complex structures that arise in each even real dimension. In [31] Section 3.1] it is proved that if $g$ admits a strongly non-nilpotent complex structure $J$, then
the nilpotency step of \( g \) is at least 3 (see [31] for other general properties on Lie algebras with strongly non-nilpotent complex structures and structure results up to real dimension 8).

We can now formulate the following conjecture proposed in [13]:

**Conjecture** [13, page 123]: a complex structure on a \( 2n \)-dimensional nilpotent Lie algebra must be of nilpotent type in the presence of a compatible symplectic form.

It is proved in [13] that the conjecture holds for \( n \leq 3 \). However, the complex structure \( J \) given in Proposition 3.3 provides a counterexample for \( n = 4 \). In fact, from the equations (9) it is straightforward to prove that \( a_1(J) = \{0\} \), that is to say, \( J \) is strongly non-nilpotent according to Definition 3.5, and it admits the compatible symplectic forms given in (10). Furthermore, in every complex dimension \( n \geq 4 \) we have the following result.

**Proposition 3.6.** For each \( n \geq 4 \), there exists a \( 2n \)-dimensional nilmanifold endowed with a non-nilpotent complex structure that admits pseudo-Kähler metrics.

**Proof.** Let us consider \( Y = X \times T^k \), where \( X = (M, J) \) is the 8-dimensional pseudo-Kähler nilmanifold given in Proposition 3.3 and \( T^k \) denotes the \( k \)-dimensional complex torus endowed with any invariant pseudo-Kähler metric. Then, \( Y \) is a pseudo-Kähler nilmanifold of real dimension \( 2n = 8 + 2k \) with invariant complex structure \( J_Y = J \times J_{T^k} \) satisfying \( \{0\} \neq b = a_t(J_Y) \neq g \times b \), for every \( t > 0 \), where \( b \) denotes the abelian Lie algebra underlying the torus \( T^k \), and \( g \) the Lie algebra of \( M \). In particular, \( J_Y \) is (weakly) non-nilpotent. \( \square \)

### 3.3. Instability in complex dimension \( n \geq 4 \)

In contrast to the stability results for compact complex surfaces proved in Section 3.1, we will next show that the neutral Kähler and neutral Calabi-Yau properties are both unstable in every complex dimension \( n \geq 4 \). This constitutes a deep difference with the Kähler Calabi-Yau case, for which the deformation space is unobstructed by the well-known Bogomolov-Tian-Todorov theorem [8, 47, 48]. We begin with the following result in complex dimension 4.

**Theorem 3.7.** There exists a holomorphic family of compact complex manifolds \( \{X_t\}_{t \in \Delta} \) of complex dimension 4, where \( \Delta = \{t \in \mathbb{C} | ||t|| < 1\} \), satisfying the following properties:

(i) \( X_0 \) is a neutral Calabi-Yau manifold;
(ii) \( X_t \) does not admit any pseudo-Kähler structure for \( t \in \Delta \setminus \mathcal{C} \), where \( \mathcal{C} \) is the real curve through 0 given by \( \mathcal{C} = \{t \in \Delta | \Re t = 0\} \).

Therefore, neither the neutral Calabi-Yau property nor the neutral Kähler property are stable under deformations of the complex structure.

**Proof.** The proof is based on an appropriate deformation of the neutral Calabi-Yau nilmanifold \( X = (M, J) \) found in Proposition 3.3.

Let \( \{\omega^k\}_{k=1}^4 \) be the basis of \((1,0)\)-forms on \( X \) satisfying (9). Observe that the \((0,1)\)-form \( \omega^\dagger \) defines a Dolbeault cohomology class on \( X \). We consider the class \( [\omega^\dagger] \in H^{0,1}_{\phi}(X) \) to perform an appropriate holomorphic deformation of \( X \). For each \( t \in \mathbb{C} \) such that \( ||t|| < 1 \), we define the complex structure \( J_t \) on \( M \) given by the following basis \( \{\eta^k_t\}_{k=1}^4 \) of \((1,0)\)-forms:

\[
\eta^1_t := \omega^1, \quad \eta^2_t := \omega^2 - t \omega^\dagger, \quad \eta^3 := \omega^3, \quad \eta^4_t := \omega^4.
\]
The complex structure equations for $X_t = (M, J_t)$ are

$$
\begin{align*}
    d\eta_t^1 &= 0, \\
    d\eta_t^2 &= -i\eta_t^{14} + i\eta_t^{13}, \\
    d\eta_t^3 &= \eta_t^{12} + t\eta_t^{11} + \eta_t^{13} - \eta_t^{21}, \\
    d\eta_t^4 &= -\eta_t^{13} + \eta_t^{31}.
\end{align*}
$$

Observe that the initial structure $J$ is recovered for $t = 0$. Therefore, $X_0 = X$ and one immediately gets part (i) of the statement.

To prove (ii), since $X_t$ is a complex nilmanifold, a similar argument as in the proof of Proposition 2 allows us to reduce the study of existence of pseudo-Kähler metrics on $X_t$ to the study of invariant ones. We first analyze the existence of invariant closed 2-forms $\Omega$ on $X_t$, not necessarily real. Any such $\Omega$ belongs to $\bigwedge_{\eta_t}^{1,1}(g^*)$, where $g$ denotes the Lie algebra underlying the nilmanifold $M$, and it is given by

$$
\Omega = \sum_{k=1}^{4} (a_k \eta_t^1 + b_k \eta_t^2 + c_k \eta_t^3 + f_k \eta_t^4) \wedge \eta_t^k,
$$

where $a_k, b_k, c_k, f_k \in \mathbb{C}$, for $1 \leq k \leq 4$. Making use of the complex structure equations (14), one has

$$
d\Omega = \partial_t \Omega + \bar{\partial}_t \Omega,
$$

where

$$
\partial_t \Omega = (a_3 - \bar{t}b_3 + c_1) \eta_t^{121} + (b_3 + c_2) \eta_t^{122} + (c_3 - b_4) \eta_t^{123} + c_4 \eta_t^{124} - (a_4 + \bar{t}c_3) \eta_t^{131}
$$

$$
+ c_3 \eta_t^{132} - c_4 \eta_t^{133} - i(a_2 + b_1 - i\bar{t}f_3) \eta_t^{141} + (f_3 - i b_2) \eta_t^{142} - (f_4 + i b_3) \eta_t^{143}
$$

$$
- i b_4 \eta_t^{144} - (b_1 + c_3) \eta_t^{231} - (f_3 + i b_2) \eta_t^{241} + (f_4 - i c_2) \eta_t^{341}.
$$

and

$$
\bar{\partial}_t \Omega = -(a_3 + c_1 - t c_2) \eta_t^{112} + (f_1 + t c_3) \eta_t^{113} - i(a_2 + b_1 + i t c_4) \eta_t^{114} + (c_3 + f_2) \eta_t^{123}
$$

$$
+ (c_4 - i b_2) \eta_t^{124} - (f_4 + i b_3) \eta_t^{131} - (b_3 + c_2) \eta_t^{212} - c_3 \eta_t^{213} - (c_4 + i b_2) \eta_t^{214}
$$

$$
+ (f_2 - c_3) \eta_t^{212} + f_3 \eta_t^{213} + (f_4 - i c_2) \eta_t^{314} - f_3 \eta_t^{315} - i f_2 \eta_t^{314}.
$$

From these expressions we see that $\Omega$ is closed, i.e. $\partial_t \Omega = 0 = \bar{\partial}_t \Omega$, if and only if

$$
a_4 = b_2 = b_4 = c_3 = c_4 = f_1 = f_2 = f_3 = 0,
$$

and

$$
 b_1 = -a_2, \quad c_2 = -b_3, \quad f_4 = -i b_3, \quad -a_3 + \bar{t}b_3 = c_1 = -a_3 - t b_3.
$$

Now, the latter two equalities imply the equation

$$
b_3 \Re t = 0,
$$

which gives rise to the following two cases.

If $\Re t = 0$, then pseudo-Kähler metrics exist on $X_t$; for instance

$$
F = i \eta_t^{11} - i \eta_t^{44} - \frac{t}{2} \eta_t^{13} + \eta_t^{23} + \frac{\bar{t}}{2} \eta_t^{31} - \eta_t^{32}
$$

is a real closed $(1,1)$-form which is non-degenerate.

However, if we assume $\Re t \neq 0$, then one has $b_3 = 0$ and consequently $c_2 = f_4 = 0$. Thus, every closed 2-form $\Omega$ of bidegree $(1,1)$ with respect to the complex structure $J_t$ is given by

$$
\Omega = a_1 \eta_t^{11} + a_2 (\eta_t^{12} - \eta_t^{21}) + a_3 (\eta_t^{13} - \eta_t^{31}),
$$
with $a_1, a_2, a_3 \in \mathbb{C}$.

Hence, when $\mathfrak{Re} t \neq 0$, the space of real closed 2-forms on $g$ compatible with $J_t$, namely $Z^+_t(g) = \{ \alpha \in \wedge^2 g^* \mid d\alpha = 0 \text{ and } J_t \alpha = \alpha \} = \{ \alpha \in \wedge^1_{J_t} g^*_C \mid d\alpha = 0 \text{ and } \bar{\alpha} = \alpha \}$, is generated by

$$Z^+_t(g) = \{ i \eta^1_1, \eta^1_2 - \eta^2_1, \eta^3_1 - \eta^3_2 \}.$$ 

Since every element in $Z^+_t(g)$ is degenerate, no pseudo-Kähler metrics exist on $X_t = (M, J_t)$ when $\mathfrak{Re} t \neq 0$. This clearly implies that $X_t$ cannot admit any neutral Kähler or neutral Calabi-Yau structure for $t \in \Delta \setminus C$, where $C = \{ t \in \Delta \mid \mathfrak{Re} t = 0 \}$. Hence, both properties are unstable by small deformations. \(\square\)

In the following theorem we sum up the main results about instability found along the preceding sections.

**Theorem 3.8.** On compact complex manifolds of complex dimension $\geq 3$, the properties of “being pseudo-Kähler”, “being neutral Kähler” and “being neutral Calabi-Yau” are not stable under small deformations of the complex structure.

**Proof.** For the pseudo-Kähler property, the result follows from Proposition 2.1 and Remark 2.2. For the neutral Kähler and neutral Calabi-Yau properties in complex dimension 4, the result is given in Theorem 3.7. In higher dimensions, it suffices to consider the product $Y_t = X_t \times \mathbb{T}^{2m}$, where $X_t = (M, J_t)$ is the holomorphic family given in Theorem 3.7 and $\mathbb{T}^{2m}$ the $2m$-dimensional complex torus endowed with an invariant neutral Calabi-Yau metric. If $g$ and $b$ denote the Lie algebras underlying $M$ and $\mathbb{T}^{2m}$, respectively, then the space of invariant closed real 2-forms on $Y_t$ compatible with the product complex structure $J_{Y_t} = J_t \times J_{\mathbb{T}^{2m}}$ is given by

$$Z^+_t(g \times b) = Z^+_t(g) \oplus \left\{ \eta^1_1 \wedge \bar{\alpha} - \alpha \wedge \eta^3_1 \mid \alpha \in \Lambda^{1,0}_{J_t} b^* \right\} \oplus Z^+_{J_{\mathbb{T}^{2m}}}(b),$$

where the space $Z^+_t(g)$ is described in (15) for any $t \in \Delta$ such that $\mathfrak{Re} t \neq 0$. It is easy to see that any element in $Z^+_t(g \times b)$ is degenerate, so $Y_t$ does not admit pseudo-Kähler metrics for any $t \in \Delta \setminus C$. Since $Y_0 = X_0 \times \mathbb{T}^{2m}$ is neutral Calabi-Yau, the result follows immediately. \(\square\)

### 4. Pseudo-Hermitian-symplectic structures

In this section we consider an indefinite version of the Hermitian-symplectic geometry. The motivation comes from the fact that the small deformations of any pseudo-Kähler manifold always posses what we will call a pseudo-Hermitian-symplectic structure. We will show that there are compact complex manifolds with pseudo-Hermitian-symplectic structure but not admitting any pseudo-Kähler metric.

Recall that a complex structure $J$ on a symplectic manifold $(M, \Omega)$ is said to be *tamed* by the symplectic form $\Omega$ if the condition $\Omega(x, Jx) > 0$ is satisfied for all non-zero tangent vectors $x$. Following the terminology of [11], we will refer to the pair $(\Omega, J)$ as a *Hermitian-symplectic structure*. Note that the tamed condition is equivalent to require that the $(1,1)$-component $\Omega^{1,1}$ of the symplectic form $\Omega$ is positive, i.e. $\Omega^{1,1}$ is a Hermitian metric on the complex manifold $X = (M, J)$. No example of a non-Kähler compact complex manifold admitting a Hermitian-symplectic structure is known (see [32] page 678 and [11] Question 1.7).

By analogy, in the pseudo-Hermitian setting, we introduce the following notion:

**Definition 4.1.** A complex manifold $X = (M, J)$ is called pseudo-Hermitian-symplectic if there exists a symplectic form $\Omega$ on $M$ such that its component of bidegree $(1, 1)$ with respect to $J$ is non-degenerate. In such case we will say that the pair $(\Omega, J)$ is a pseudo-Hermitian-symplectic structure.
From the definition, it is clear that any pseudo-K"ahler manifold is pseudo-Hermitian-symplectic.

**Example 4.2.** For a primary Kodaira surface, with equations given by (7), the pseudo-Hermitian-symplectic (indeed pseudo-K"ahler) structures are defined in (8). Observe that the form \( \Omega = \omega^{12} + \omega^{1\bar{2}} \) is symplectic, but the pair \((\Omega, J)\) is not pseudo-Hermitian-symplectic because the \((1,1)\)-component of \(\Omega\) is identically zero.

**Remark 4.3.** In the positive-definite case, [17, Proposition 2.1] provides an important characterization of the Hermitian-symplectic condition. More precisely, the existence of such a structure on a complex manifold \(X\) is shown to be equivalent to the existence of a Hermitian metric \(F\) satisfying \(\partial F = \partial \alpha\), for some \(\partial\)-closed \((2,0)\)-form \(\alpha\) on \(X\). In the following example we illustrate that a similar result does not hold in the pseudo-Hermitian-symplectic setting.

**Example 4.4.** Let us consider a compact complex nilmanifold defined by the complex equations
\[
d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho \omega^{1\bar{2}} + \omega^{1\bar{1}} + \rho \omega^{1\bar{2}},
\]
for some \(D \in \mathbb{C}\) and \(\rho \in \{0, 1\}\).

Let \(F = \frac{1}{r} \omega^{1\bar{1}} - \omega^{2\bar{3}} + \omega^{3\bar{2}}\), where \(r \in \mathbb{R}^*\). Then, \(F\) is a real form of bidegree \((1,1)\) and \(F^3 = 3i r \omega^{1\bar{2}3\bar{3}} \neq 0\). Moreover, the \((2,0)\)-form \(\alpha = -\omega^{2\bar{3}}\) is \(\partial\)-closed and satisfies
\[
\partial F = \omega^{1\bar{2}1} + \rho \omega^{1\bar{2}2} = -\bar{\partial} \alpha.
\]
Take now the real 2-form \(\Omega = \alpha + F + \bar{\alpha}\). By the previous condition we have \(\partial \alpha = \bar{\partial} \alpha + \partial F = 0\), which implies \(d\Omega = 0\).

However, the closed 2-form \(\Omega\) is not symplectic. In fact,
\[
\Omega^3 = (\alpha + F + \bar{\alpha})^3 = F^3 + 6 \alpha \wedge \bar{\alpha} \wedge F = 0.
\]

Even more, if \(\rho = 0\) and \(D \in \mathbb{R}^*\), then the corresponding nilmanifold does not admit any symplectic structure (indeed, the underlying Lie algebra is \(h_3\) in the notation of [39]).

For the other values of \(\rho\) and \(D\) we have by [11, Proposition 2.4] that the nilmanifold has underlying Lie algebra \(h_2, h_4, h_6\) or \(h_8\). These nilmanifolds are symplectic.

We next prove that, similarly to the Hermitian case [51, Proposition 2.4], the pseudo-Hermitian-symplectic property is open under holomorphic deformations.

**Proposition 4.5.** For compact complex manifolds, the pseudo-Hermitian-symplectic property is stable. Therefore, any sufficiently small deformation of a pseudo-K"ahler manifold is pseudo-Hermitian-symplectic.

**Proof.** Let us denote by \(M\) the real manifold underlying a complex manifold \(X\) and by \(J\) the complex structure on \(M\) such that \(X = (M, J)\). Let \(\Omega\) be a symplectic form on \(M\) whose \((1,1)\)-component with respect to \(J\) is non-degenerate, i.e. \((\Omega, J)\) is a pseudo-Hermitian-symplectic structure. We consider a holomorphic family of compact complex manifolds \(\{X_t = (M, J_t)\}_{t \in \Delta}\), with \(\Delta\) containing 0, such that \(X_0 = X\).

The symplectic form \(\Omega\) decomposes on the compact complex manifold \(X_t\) as
\[
\Omega = \alpha_t + F_t + \beta_t,
\]
where \(\alpha_t\) has bidegree \((2,0)\), \(F_t\) is the \((1,1)\) component of \(\Omega\), and \(\beta_t = \bar{\alpha}_t\). By hypothesis, for \(t = 0\) the form \(F_0\) is non-degenerate. Hence, one concludes that \(F_t\) is also non-degenerate for any \(t \in \Delta\) sufficiently close to 0 \((0 \in \Delta\), so \(X_t\) satisfies the pseudo-Hermitian-symplectic property.

The second assertion in the proposition is clear since any pseudo-K"ahler manifold is pseudo-Hermitian-symplectic. \(\square\)
As we recalled above, Streets and Tian pose in [44, Question 1.7] the following question, which is still an open problem: Does there exist a compact complex manifold, of complex dimension $\geq 3$, admitting a Hermitian-symplectic structure but no Kähler metrics?

In the following result we prove that the indefinite counterpart of this problem has an affirmative answer.

**Proposition 4.6.** There exist compact complex manifolds with pseudo-Hermitian-symplectic structure but not admitting any pseudo-Kähler metric.

**Proof.** Let us consider the family of compact complex manifolds \( \{X_t\}_{t \in \Delta} \), of complex dimension 3, constructed in Proposition 2.1. We have:

(i) \( X_0 \) is a pseudo-Kähler manifold,

(ii) \( X_t \) does not admit any pseudo-Kähler metric for \( t \neq 0 \).

Since \( X_0 \) is a pseudo-Kähler manifold, by Proposition 4.5 and (ii) we conclude that, for sufficiently small values of \( t \neq 0 \), the compact complex manifold \( X_t \) is a pseudo-Hermitian-symplectic manifold with no pseudo-Kähler metrics.

In the next examples we consider pseudo-Kähler structures on a compact complex manifold \( X_0 \) and illustrate their behaviour along a small holomorphic deformation \( X_t \) of \( X_0 \).

**Example 4.7.** Let us consider the family of compact complex manifolds \( \{X_t\}_{t \in \Delta=\{t \in \mathbb{C} | |t|<1\}} \) of complex dimension 3 constructed in Proposition 2.1. Using the complex equations (1) of the □

There exist compact complex manifolds with pseudo-Hermitian-symplectic structure but not admitting any pseudo-Kähler metric.

Proof. Let us consider the family of compact complex manifolds \( \{X_t\}_{t \in \Delta=\{t \in \mathbb{C} | |t|<1\}} \) of complex dimension 3, constructed in Proposition 2.1. We have:

\[
\begin{align*}
F &= i (r \omega^{11} + s \omega^{22}) + u \omega^{12} - \bar{u} \omega^{21} + v \omega^{23} - \bar{v} \omega^{32},
\end{align*}
\]

for some \( r, s \in \mathbb{R} \) and \( u, v \in \mathbb{C} \) satisfying \( rv \neq 0 \).

A direct calculation using (2) shows that the 2-form \( F \) decomposes along the deformation \( X_t \) as

\[
F = \alpha_t + F_t + \beta_t = -\frac{v \bar{t}}{1-|t|^2} \omega_t^{23} + i (r \omega_t^{11} + s \omega_t^{22}) + u \omega_t^{12} - \bar{u} \omega_t^{21} + \frac{1}{1-|t|^2} (v \omega_t^{23} - \bar{v} \omega_t^{32}) - \frac{\bar{v} \bar{t}}{1-|t|^2} \omega_t^{23},
\]

where \( \alpha_t = -\frac{-v \bar{t}}{1-|t|^2} \omega_t^{23} \) is the \((2,0)\)-component of \( F \), and \( \beta_t = \bar{\alpha}_t \). The real \((1,1)\)-form \( F_t \) is non-degenerate on the manifold \( X_t \), i.e. \( F \) defines a pseudo-Hermitian-symplectic structure on \( X_t \), which is in accord to Proposition 4.5. Notice that \( dF_t = \frac{1}{1-|t|^2} (v \bar{t} \omega_t^{22} + \bar{v} t \omega_t^{12}) \neq 0 \) for any \( t \in \Delta \setminus \{0\} \), i.e. the \((1,1)\)-form \( F_t \) is not closed. Furthermore, by Proposition 2.1(ii), \( X_t \) does not admit any pseudo-Kähler structure for \( t \in \Delta \setminus \{0\} \).

**Example 4.8.** Let \( \{X_t\}_{t \in \Delta=\{t \in \mathbb{C} | |t|<1\}} \) be the family of compact complex manifolds of complex dimension 4 constructed in Theorem 3.7. The manifold \( X_0 \) is neutral Calabi-Yau, hence pseudo-Kähler, and by (10) any invariant pseudo-Kähler metric \( F \) on \( X_0 \) is given by

\[
F = i (r \omega^{11} + s \omega^{44}) + u (\omega^{12} - \omega^{21}) + v (\omega^{13} - \omega^{31}) - s (\omega^{23} - \omega^{32}),
\]

for some \( r, s, u, v \in \mathbb{R} \) with \( rs \neq 0 \). A direct calculation using (13) shows that the 2-form \( F \) decomposes along the deformation \( X_t \):

\[
F = \alpha_t + F_t + \beta_t = -s \bar{t} \eta_t^{13} + i (r \eta_t^{11} + s \eta_t^{44}) + u (\eta_t^{12} - \eta_t^{21}) + v (\eta_t^{13} - \eta_t^{31}) - s (\eta_t^{23} - \eta_t^{32}) - s \bar{t} \eta_t^{13},
\]

where \( \alpha_t = -s \bar{t} \eta_t^{13} \) is the \((2,0)\)-component of the form \( F \), and \( \beta_t = \bar{\alpha}_t \). The real \((1,1)\)-form \( F_t \) defines a pseudo-Hermitian-symplectic structure on \( X_t \) as it is non-degenerate, accordingly to Proposition 4.5. Note that \( F_t \) is not closed for any \( t \in \Delta \setminus \{0\} \), since \( dF_t = s (t \eta_t^{121} + \bar{t} \eta_t^{112}) \). Moreover, by Theorem 3.7(ii), the compact complex manifold \( X_t \) does not admit any pseudo-Kähler structure for every \( t \in \Delta \setminus C \), where \( C = \{t \in \Delta | \Re t = 0\} \).
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