Identifying the reversible thinking skill of students in solving function problems

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Abstract. This study aims to identify the reversible thinking skill of senior high school students in solving function problems. Thinking reversible is a mental activity to construct a reversible two-way correlation. One reason that makes students think reversible is that it is the primary requirement for them to solve mathematical problems; the core of school mathematics. Therefore, thinking reversible is necessary for every student to optimize their competence in solving mathematical problems. This study is qualitative using Test of Thinking Reversible (i.e., TTR). The subject is the 123 senior high school students, especially at the tenth grade. The test consists of two sections (i.e., TTR I and TTR II) with adversative contents. It is aimed to identify the students’ reversible thinking. TTR I contains a linear function, and the subject is asked to make the graphic. However, TTR II contains a graphic of a linear function in the form of the straight line with two identified points, and the subject is asked to define the function. The result shows that among 123 senior high school students, only five students are capable of constructing a reversible two-way correlation between a function and its graphic. The other 118 students are solely capable of drawing the graphic without being able to define the function of the identified graphics. This result shows that the problem related to reversible thinking on senior high school student still exist, especially for solving function problems.

1. Introduction

Thinking reversible, or commonly known as reversibility, is a part of the cognitive terms Piaget found when studying his three children [1]. Later, reversibility was developed or further identified by Krutetskii. As Flanders [2] argued, reversibility was introduced at first and studied by two prominent cognitive psychologists (Russian psychologists); Krutetskii and Jean Piaget.

This study is inspired by Lamon [3] and Fanders [2]. Lamon [3] argued that fewer kinds of literature and works were discussing reversible thinking. Hence, he asked other researchers, particularly ones in the education field, to pay attention, study, and examine students’ reversible thinking. However, Fanders [2] argued that reversibility began to have attention from the education field, since many education observers had already seen the importance of reversibility for students, especially in mathematics. Besides, this study is also motivated by some previous studies that described the importance of studying things related to cognitive psychology for developing and improving mathematics education [4-8].
Why is it important for students to have reversible thinking? First, thinking reversible is an individual’s competence that makes him/her think reasonably in a reversibly two-way manner [1, 2, 9, 10]. Through reversible thinking, an individual is capable of seeing things not only from one single perspective but also its reversal. When students have this competence, they are capable of solving complex problems and seeing from any position on the spectrum between two reversible lines. Second, thinking reversible is one primary requirement to solve mathematical problems [10], and problem-solving is the core of school mathematics [11]. Therefore, every student should have this competence to solve mathematical problems optimally. Third, thinking reversible may minimize errors in every decision. Through this kind of thinking, students are asked to think twice with reversal perspectives. Last, thinking reversible may minimize the error of answers, as students tend to recheck their answers by reversing the result to the base point [12].

Toward mathematical problems, the function is one basic material taught in high school level. It is about limits, continuity, derivative, integral, and other materials that involve function within. Students should completely understand all the materials. As Denbel [13] argued, the function was used at each branch of mathematics, as algebraic operations on numbers, transformations on points in the plane or space, intersection and union of pairs of sets, and so forth. The function is a unifying concept in all mathematics. Therefore, the function was selected for this study. Students’ comprehension will be complete when they involve their reversibility in learning function. Through reversible thinking, they are required to have a capability of two-way connection for two mathematical abilities. If the connection was constructed in two-way setting, understanding a concept will be easier for students. Therefore, this study focuses on the reversible thinking of high school students in solving function problems.

Thinking reversible is a mental capability to construct a reversible two-way correlation [1, 10, 14] from an initial condition to the target, and from the target to the base point, such as the correlation between derivative and integral. Thinking reversible is defined as an individual’s mental process to reverse to the base point after they got the result [10, 15]. Toward Polya’s procedures of problem-solving [16], being reversible has similar definition with looking back (i.e., the final stage of Polya’s problem-solving). For instance, a high school student had an equation $14 - \frac{15}{7-4} = 9$ and he was asked to define the $x$. Through the process of problem-solving, it found that $x = 4$. After finding the result, he reversed $x = 4$ to the base point in order to check his work. It found that $14 - \frac{15}{7-4} = 14 - \frac{15}{3} = 14 - 5 = 9$. Thus, $x = 4$ was correct.

Following Inhelder & Piaget, Ramful, and Hackenberg [1, 17, 18] reversibility can be applied in two ways; inverse and reciprocity. Reversibility with inverse is thinking reversible that involves inverse within. Reversibility with reciprocity is thinking reversible that involves a reciprocal correlation or compensation or equilibrium. Consistently, Maf’ulah, Juniati, and Siswono [12] identified the aspects of thinking reversible, as following in Table 1.

| Aspect of thinking reversible | Explanation |
|-----------------------------|-------------|
| Negation                    | It was when a subject used inversion toward the related operation |
| Reciprocity                 | It was when a subject used compensation or any other relationships equivalent with a given equation |
| Capability to reverse the ending back into the base point after obtaining the result | It was when a subject could reverse the equation he made into its base point using correct procedures |

Krutetskii [10] suggested two definitions of thinking reversible, as follow.
1. Two-way process; capability to construct a reversible two-way correlation
2. Mental process in reasoning; capability of two-way connection for two mathematical abilities, or a capability derived from a mental process by an individual to go back to the base point after finding the result.

To reveal students’ reversible thinking on function, this study points to Krutetskii’s definition on reversible thinking. He argued that it was a two-way process; a capability to construct a reversible two-way correlation between a function and its graphic. Therefore, the criteria of students’ reversible thinking in solving function problems are as follow.

1. Student’s capability to make the graphic of a function presented.
2. Student’s capability to define the function of the graphic presented.

2. Method
This study is qualitative. The subject is 123 senior high school students, particularly at the tenth grade, with high, moderate, and low mathematical capabilities. The instrument applied here is the test of thinking reversible (i.e., TTR) which consists of two sections; TTR I and TTR II. Those two tests have reversal content. It aims to identify students’ reversible thinking. The subject is asked to complete the TTR. Their work is then analyzed based on the criteria of students’ reversible thinking as described in the theoretical framework.

3. Result and Discussion
The result of this study derived from providing Tests of Thinking Reversible (i.e., TTR) to 123 senior high school students, particularly at the tenth grade. This test was provided in two sections with reversal content. TTR I contained a linear function, and the students were asked to make the graphic of that function. However, TTR II contained the graphic of a linear function in the form of a straight line with two identified points, and the students should define the function of that graphic. These two tests aimed to see and identify their reversible thinking in solving function problems. With these tests, students’ capability to construct a reversible two-way correlation between function and its graphic is revealed. TTR I was designed to identify students’ capability to draw the graphic of a function presented, while TTR II was designed to identify their capability to define the function of a graphic presented. Table 2 presented the tests.

| TTR I | TTR II |
|-------|--------|
| Complete the following problem along with the process of its solution! | Complete the following problem along with the process of its solution! |
| Identified that function \( f(x) = x - 2 \), with \( x \) a set of real numbers. Make the graphic of that function! | Given the graphic of a function as follow. Define the function of that graphic! |

Providing TTR I and TTR II to 123 senior high school students, some results are found, as presented in Table 3 below.
Table 3. The result of TTR

| The Result of TTR I | The Result of TTR II |
|---------------------|----------------------|
| The number of students with correct answer | The number of students with wrong answer |
| 101                 | 22                   |
| The number of students with correct answer | The number of students with wrong answer |
| 5                   | 118                  |

Table 3 showed a gap between TTR I and TTR II. It found that the student drew the graphic easily and fast when the function was identified in prior. However, they felt difficult to define the function of the identified graphic, although the function presented was a linear function relatively easier rather than the other functions.

According to the students’ works with the correct answer in TTR I, in general, there were two categories of the process of solution they did in making the graphic of a function presented, as follow.

1. The student determined two coordinate points to make the graphic of function \( f(x) = x - 2 \); when \( x = 0 \) the point found was \((0, -2)\) and when \( f(x) = 0 \) the point found was \((2,0)\). Then, the students made the graphic in the form of straight line through those two points. Figure 1 is one of students’ works (i.e., FPG, the initials) who completed TTR I through this process.

   Figure 1. The example of student’s work drawing the graphic of function through two points

2. The student defined the number of points (at least 5 coordinate points) to make the graphic of function \( f(x) = x - 2 \). When \( x = 0 \) it found \((0, -2)\), when \( x = 1 \) the point was \((1, -1)\), when \( x = 2 \) it found \((2,0)\), when \( x = 3 \) it found \((3,1)\), when \( x = 4 \) it found \((4,2)\). Then, the student drew a straight line through those points. Figure 2 is one of students’ works (i.e., YOK, the initials) who completed TTR I through such process.

   Figure 2. The example of student’s work drawing the graphic of function through many points
The students’ work as in Figure 1 and Figure 2 show that they easily drew the graphic based on the identified function on TTR 1, as they just substituted variable $x$ and $y$ with any number to reach the coordinates. Furthermore, the problem on TTR 1 is often presented by their teacher as the example to define the graphic of a function. Toward the kind of mathematics problem and based on the range of difficulty within, such problem is classified into “very easy problem-exercise” category. As Hudgson and Sullivan argue in Silver [19] argue, there are three kinds of problems based on the range of difficulty, including very easy problem-exercise, problem with a clear context, and problem without a clear context.

According to the subject’s work with correct answer in TTR II, it found two categories of process of problem-solving they had done in defining the function of the graphic presented, as follow.

1. Trial and error. Students were having trial and error by defining any functions before making the graphic. If the graphic is not similar to the graphic presented in TTR II, they would define another function similar to the given graphic. Among 5 high school students with correct answers for TTR II, 2 students had trial and error. Figure 3 showed the examples student’s work showing the student’s work with trial and error strategy (i.e., EDAP, the initials).

![Figure 3. The example of student’s work in defining function through trial and error process](image)

2. Through a formula: $y - y_1 = m(x - x_1)$, with $m$ is gradient that $m = \frac{y_2 - y_1}{x_2 - x_1}$. the first thing they did was defining the gradient $m$. Then, defining the line equation through $y - y_1 = m(x - x_1)$, and changing $y$ into $f(x)$ to result in function $f(x)$. Figure 4 showed the example of student’s work (i.e., RMS, the initial) on TTR II through the process of formulation.

![Figure 4. The example of student’s work in defining function through the process of formulation](image)

TTR II is considered more difficult rather than TTR I. It is apparent from the students’ work as seen in Table 3. Some factors seem to be the reasons. First, the students’ comprehension is less related to the concept of function. Second, the students do not usually encounter reversible problems. Third, the problem on TTR II is not familiar for the students. Thus, only 5 among 123 students may complete TTR II correctly, that three of them use problem-solving heuristics and two others use trial-and-error method. In relation to trial-and-error, NCTM [11] explains that it is one of students’ initial attempt in problem-solving.
solving. Moreover, through this method, the students have chances to explore their insight by trying out all cases and thus making them able to understand the most appropriate and systematic procedures to solve problems.

To identify the reversible thinking of high school students in solving function problems, 2 of 5 students completing TTR I and TTR II correctly were selected. They were EDAP and RMS. The reversible thinking skill of those two high schools students were presented in Table 4, as follow.

**Table 4. Identifying the Reversible Thinking Skill of Senior High Schools Students in Solving Function Problems**

| The Reversible Thinking Skill of Senior High Schools Students in Solving Function Problems |  |
| --- | --- |
| **Student’s Capability to Make the Graphic of the Given Function** | To construct the graphic of function \( f(x) = x - 2 \), those two students defined some coordinate points (at least 5 points) at first before constructing a straight line based on those points as the graphic of function \( f(x) = x - 2 \) |
| **Student’s Capability to Define the Function of the Given Graphic** | To define the function of the given graphic. Two different categories of the process were applied, as follow. |
|  | 1. The EDAP used trial-and-error strategy by defining any random functions to find the graphic similar to the given one. |
|  | 2. The RMS used the strategy of the formulation with straight line equation. |

Figure 5 presented the illustration, as follow.

**Figure 5. Two-Way Process of Reversibility**

Most of the previous studies on the reversible problems, such as Maf’ulah, Juniati, Siswono [9], Heckenberg [18], Ramful and Olive [20], Fuson [21], and Wong [22], stress on arithmetic. They focus on how important reversible thinking on arithmetic school. However, students’ reversible thinking on this present study focuses on function material, as the author wants to convey that reversible thinking is likely to be developed through any materials.

The result of this study is consistent with Haciomeroglu Aspinwall’s and Presmeg’s [23] study that aims to see how students may construct a reversible relationship between derivative and anti-derivative. Their finding shows that students just understand the relationship on one-way only; from derivative to anti-derivative. However, they have less understanding to describe the concept of “anti-derivative to derivative.” That is, their reversible thinking should be drilled more to make them able to construct two-way relationship between concepts.
4. Conclusion
The issue of reversible thinking skill in the part of senior high school students still existed, particularly in the context of the function. Among 123 senior high school students, only five students were capable of constructing a reversible two-way correlation between a function and its graphic. The other 118 students were not able to construct the correlation. In fact, they were capable of making the graphic but failed to define the function of the given graphic.

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