Wave-Function renormalization and the Hopf algebra of Connes and Kreimer

Florian Girelli (1), Thomas Krajewski (2), Pierre Martinetti (1)

(1) Centre de Physique Théorique, CNRS - Luminy, Case 907, 13288 Marseille Cedex, France
(2) Scuola Internazionale Superiore di Studi Avanzati, Via Beirut 2-4, I-34014, Trieste, Italy

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Abstract

In this talk, we show how the Connes-Kreimer Hopf algebra morphism can be extended when taking into account the wave-function renormalization. This leads us to a semi-direct product of invertible power series by formal diffeomorphisms.

1 The Connes-Kreimer formalism

As has already been mentioned by A. Connes in his lecture [1], the standard BPHZ recursion formula for the substraction of ultraviolet divergent graphs can be interpreted as an algebraic analogue of the Birkhoff decomposition of loops with values in a infinite dimensional group associated to Feynman diagrams [2]. Although the proof of the existence of the Birkhoff decomposition for an infinite dimensional group seems to be a rather difficult task, it is fairly easy to obtain, in the case under consideration, if one works at the level of the associated algebra of polynomial functions. Besides, it turns out to hold for the group of characters of general graded Hopf algebras [3].

\textbf{Theorem 1.1} Let $\mathcal{H}$ be a graded Hopf algebra, $\mathcal{A}$ an arbitrary commutative algebra which is written as a sum (in the sense of vector spaces) of two commutative sub-algebras $\mathcal{A}_-$ and $\mathcal{A}_+$. Then any character $\gamma$ of $\mathcal{H}$ with values in $\mathcal{A}$ can be decomposed as

$$\gamma = \gamma_-^{-1} \ast \gamma_+,$$

where $\gamma_\pm$ is a character of $\mathcal{H}$ with values in $\mathcal{A}_\pm$. Moreover, the decomposition is unique up to multiplication on the left by a character with values in $\mathcal{A}_- \cap \mathcal{A}_+$.

This is easily applied to the Hopf algebra $\mathcal{H}$ of Feynman diagrams of a given field theory (say $\phi^3$ in six dimensions). In this case we can choose $\mathcal{A}$ to be the algebra of formal Laurent series of finite negative order, $\mathcal{A}_+$ to be the subalgebra corresponding to positive powers and $\mathcal{A}_-$ the subalgebra of power series with exponent of strictly negative degree. Evaluation of the diagrams in dimensional regularization yields a character with values in $\mathcal{A}$ and its Birkhoff decomposition is nothing but a way to rewrite the splitting between...
counterterms and renormalized amplitudes in the minimal subtraktion scheme. Note that
the convolution product encodes all the intricacies of the subtraction of subdivergences.
In this example, the intersection between the two algebras is empty, so that the de-
composition is unique. However, we could as well include the constants in $A_-$, so that the
ambiguities of the decomposition are measured by characters of $\mathcal{H}$. These characters just
associate a number to each Feynman diagram and their inverses are nothing but arbitrary
finite counterterms whose effects we want to compensate by suitable finite changes of the
parameters of the theory.

Of course, in physically interesting situations we impose normalization conditions to
the Green’s functions of the theory and it is a change of the normalization conditions
that yields to a finite character. Alternatively, we can use the minimal subtraction
scheme in dimensional regularization so that we have a unique solution for the Birkhoff
decomposition. However, this theory has a hidden parameter, the unit of mass $\mu$, and
characters for different values of $\mu$ still differ by a finite counterterm.

A first step involves the associated change of the coupling constant in the massless
$\phi^3$ theory and has been obtained in [4]. Before we give this result, let us recall that the
formal power series in $g$ of the type

$$g + \sum_{n=2}^{\infty} a_n g^n$$

form a group $G_{diff}$ for the usual composition of power series. We shall denote by $G_{diff}^0$
the opposite group and by $\mathcal{H}_{diff}$ the Hopf algebra of polynomial functions on $G_{diff}^0$. The
latter is generated as an algebra by the functions $\alpha_n$ with $n > 1$, defined on $G_{diff}$ by

$$\alpha_n \left( \sum_{k=2}^{\infty} a_k g^k \right) = a_n,$$

and its Hopf algebra structure is derived from the group law of $G_{diff}$.

**Theorem 1.2** Let $Z_1(g)$ and $Z_3(g)$ be the formal of power series in the coupling constant
$g$ that correspond to sum of the 1PI diagrams of the 2 and 3 point functions. Then the
expansion

$$gZ_1(g)Z_3^{-3/2}(g) = g + \sum_{n=2}^{\infty} z_n g^n$$

generates a Hopf algebra homomorphism $\Psi(\alpha_n) = z_n$ from $\mathcal{H}_{diff}$ to $\mathcal{H}$.

Using the coproduct of $\mathcal{H}$ one can define a right action of $G$ on $\mathcal{H}$ by

$$f^\gamma = (\gamma \otimes id) a \Delta$$

for any $\gamma \in G$ and $f \in \mathcal{H}$. This is an action by algebra homomorphisms and it extends to
the algebra of formal power series with coefficients in $\mathcal{H}$.

On the other side, the Hopf algebra morphism $\Psi$ induces at the level of characters a
group morphism from $G$ to $G_{diff}$. Since any element of $G_{diff}$ acts on formal power series
by composition, we have another action of $G$ on formal power series with coefficients in $\mathcal{H}$.
The combination of the two actions leave the coupling constant invariant in the following
sense.

**Corollary 1.1** Let $\gamma \in G$ and denote by $\Psi_\gamma$ the associated formal diffeomorphism. Then

$$\left( \psi_\gamma(g)^{-1} Z_g(\psi_\gamma(g)^{-1}) \right)^\gamma = gZ_g(g).$$
When evaluated on the character $\gamma_+$, this is just the statement that the renormalized theory is invariant provided the effect of the finite counterterm $\gamma$ has been compensated by trading the old coupling constant $g$ for the new one $\psi_\gamma(g)$.

To complete the picture of reparametrization invariance, we have to study the effect of the combined change of counterterm and coupling constant on the Green’s functions. This will be achieved by extending the previous morphism to take into account finite wave function renormalization.

## 2 Wave-function renormalization

Suppose that we consider two renormalization schemes $\mathcal{R}$ and $\mathcal{R}'$ and denote by primed indices the renormalization factors associated to the scheme $\mathcal{R}'$. By this we simply mean that we have two ways to compute the renormalized Feynman diagrams that differ only by finite values. Then the new coupling constant $g'$ is determined as a function of the old one by requiring that the bare coupling constants are the same for both schemes

$$g'_0(g'(g)) = g_0(g)$$

which yields directly to the definition of the previous morphism.

As is well known, the Green’s functions are not invariant under this transformation since it involve also a change of the normalization of the field. Accordingly, the $N$-point functions in the two renormalization schemes are related by

$$\Gamma'_N(p, g') = \zeta(g) \Gamma_N(p, g)$$

with

$$\zeta(g, \mathcal{R}', \mathcal{R}) = \frac{Z^\mathcal{R}'}{Z^\mathcal{R}}.$$

Note that physics of the system (i.e. the $S$-matrix) is invariant by such a transformation.

We need now to extend $\mathcal{H}_{\text{Diff}}$ to take account of the effect of wave-function renormalization, in the spirit of what has been done by Connes and Kreimer.

First let $G_{\text{pow}}$ be the group (for usual multiplication) of power series of the type $1 + \sum_{n \geq 1} c_n g^n$ with $c_n \in \mathbb{C}$.

Define $\tilde{G} = G_{\text{diff}} \ltimes G_{\text{pow}}$ with the group law

$$(S, T)(S', T') = (S o S', T o S'T') \quad \forall S, S' \in G_{\text{diff}}, T, T' \in G_{\text{pow}}.$$  

Let us denote by $\gamma$ the character of $\mathcal{H}$ associated to the change from $\mathcal{R}$ to $\mathcal{R}'$. Motivated by the previous considerations on wave function renormalization, we define the following function on $G$:

$$\zeta_\gamma(x) = \frac{Z^\gamma_3[\psi^{-1}_\gamma(x)]}{Z_3(x)}.$$  

It describes wave function renormalization and leads to the following result.

**Theorem 2.1** The application $\gamma \to (\psi^{-1}_\gamma(x), \zeta_\gamma(x))$ is a group morphism from $G$ to $G_{\text{diff}} \ltimes G_{\text{pow}}$.

**Proof:** We prove this in two steps. First we show that the application $\zeta_\gamma(x)$ is well defined because it must be a formal power series with scalar coefficients whereas its definition involves $\mathcal{H}$ valued coefficients. Then we prove that the group law is satisfied.
By construction we have

$$\psi^{-1}_\gamma(x)Z^\gamma_1(\psi^{-1}_\gamma)(Z^{3/2}_3)^\gamma(\psi^{-1}_\gamma) = xZ_1(x)Z^{3/2}_3(x).$$  \hfill (12)

So by reordering, we get

$$\frac{(Z^\gamma_3)^\gamma(\psi^{-1}_\gamma)}{Z_3(x)} = \left(\frac{\psi^{-1}_\gamma Z_1(\psi^{-1}_\gamma)}{xZ_1(x)}\right)^{2/3}. \hfill (13)$$

This just asserts that a diagram with 2 external lines has to be equal to a diagram with 3 external lines, the only possibility is that it is a scalar. So $\zeta_\gamma(x)$ is well defined.

Moreover, it is invariant under the action of $G$, so that

$$\zeta_{\gamma\gamma'}(x) = \frac{Z^\gamma_3(\psi^{-1}_\gamma(x))}{Z_3(x)} \left[\frac{Z^\gamma_3(\psi^{-1}_\gamma(x))}{Z^\gamma_3(\psi^{-1}_\gamma'(x))}\right]^{\gamma'} \frac{Z^\gamma_3(\psi^{-1}_\gamma'(x))}{Z_3(x)} \hfill (14)$$

$$= \zeta_\gamma(\psi^{-1}_\gamma(x))\zeta_\gamma'(x) \hfill (15)$$

which is just the product law.

Accordingly, after evaluation of $Z_1(x)$ and $Z_3(x)$ on a character, we get the transformation law of the 2- and 3-points functions by the following rules:

$$Z^\gamma_1(\psi^{-1}_\gamma(x)) = c^{3/2}_\gamma(x)Z_1(x) \hfill (17)$$
$$Z^\gamma_3(\psi^{-1}_\gamma(x)) = \zeta_\gamma(x)Z_3(x). \hfill (18)$$

Note that $\zeta_\gamma(x)$ is independent of any regulation procedure: it is a purely diagrammatical object.

### 3 Concluding remarks

We have shown that the renormalization of the wave-function yields also to a morphism from $G$ to $\tilde{G} = G_{\text{diff}} \ltimes G_{\text{pow}}$ which is purely diagrammatical.

We should also point out that this is a very particular case of a general statement about the reparametrization invariance of the functional integral that is currently under investigation using Hopf algebraic techniques. For instance, the previous morphism can be obtained through a two variable power series as follows: let us denote by $Z_{1,2i}$ (resp. $Z_{3,2i}$) the sum of all 1PI diagrams with $i$ loops, weighted with their symmetry factor, that contribute to the 3-point function (resp. the 2-point function). Then

$$X(x,y) = x(1 + Z_{1,2i}x^2y^3 + Z_{1,4i}x^4y^6 + \ldots + Z_{1,2i}x^2y^{3i} + \ldots) \hfill (19)$$
$$Y(x,y) = y(1 - Z_{3,2i}x^2y^3 - Z_{3,4i}x^4y^6 + \ldots - Z_{3,2i}x^2y^{3i} + \ldots)^{-1} \hfill (20)$$

defines a morphism of two variables that preserves the foliation of the first quadrant by the curves $x^2y^3$ from which we recover the coupling constant renormalization (change of the curve) and wave function (modification of the coordinate on the curve).
References

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