THE MINIMUM PRINCIPLE FOR AFFINE FUNCTIONS WITH
THE POINT OF CONTINUITY PROPERTY AND
ISOMORPHISMS OF SPACES OF CONTINUOUS AFFINE
FUNCTIONS

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Abstract. Let $X$ be a compact convex set and let $\text{ext } X$ stand for the set of extreme points of $X$. We show that if $f : X \to \mathbb{R}$ is an affine function with the point of continuity property such that $f \leq 0$ on $\text{ext } X$, then $f \leq 0$ on $X$.

As a corollary of this minimum principle we obtain a generalization of a theorem by H.B. Cohen and C.H. Chu by proving the following result. Let $X, Y$ be compact convex sets such that every extreme point of $X$ and $Y$ is a weak peak point and let $T : \mathcal{A}^c(X) \to \mathcal{A}^c(Y)$ be an isomorphism such that $\|T\| \cdot \|T^{-1}\| < 2$. Then $\text{ext } X$ is homeomorphic to $\text{ext } Y$.

1. The minimum principle

We work within the framework of real vector spaces. If $X$ is a compact convex set in a Hausdorff locally convex space and $\text{ext } X$ is the set of all extreme points of $X$, the classical results assert that any semicontinuous affine function $f : X \to \mathbb{R}$ satisfying $f \leq 0$ on $\text{ext } X$ is actually smaller or less then 0 on $X$ (see e.g. [12, Corollary 4.8 and Section 3.9]). A generalization of this minimum principle can be found in [14] (see also [12, Section 10.8]). It is well known that any semicontinuous function $f : X \to \mathbb{R}$ has the point of continuity property, i.e., $f|_F$ has a point of continuity for each $F \subseteq X$ closed (see [8] or [12, Theorem A.121]). The first goal of our paper is a proof of the following result.

Theorem 1.1. Let $X$ be a compact convex set and $f : X \to \mathbb{R}$ be an affine function satisfying the point of continuity property. If $f \leq 0$ on $\text{ext } X$, then it follows that $f \leq 0$ on $X$.

This result prompts a question on validity of the minimum principle for strongly affine functions on a compact convex set $X$. Let us recall that any probability Radon measure $\mu \in \mathcal{M}^1(X)$ possesses its barycenter $r(\mu) \in X$, i.e., the point satisfying

$$h(r(\mu)) = \int_X h(x) \, d\mu(x), \quad h \in \mathcal{A}^c(X).$$

Here $\mathcal{A}^c(X)$ denotes the space of all real continuous affine functions on $X$. Then $f : X \to \mathbb{R}$ is said to be strongly affine, if for each $\mu \in \mathcal{M}^1(X)$, $f$ is $\mu$-integrable and (1.1) holds for $f$, i.e., $f(r(\mu)) = \int_X f \, d\mu$. Obviously, any strongly affine function

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is affine and moreover, it is bounded (see [3 Satz 2.1]). It is well known that any affine function with the point of continuity property is strongly affine (see e.g. [13 Chapter 14] or [12 Theorem 4.21]). As shown by M. Talagrand in [16] (see also [12 Theorem 12.65]), the minimum principle does not hold for strongly affine functions. Nevertheless, the following question seems to be open.

**Question 1.2.** Let \( f : X \to \mathbb{R} \) be a strongly affine Borel function on a compact convex set \( X \) such that \( f \leq 0 \) on \( \text{ext} \, X \). Does it follow that \( f \leq 0 \) on \( X \)?

Of course, the answer to this question is yes in case \( X \) is metrizable since then one can use that for any \( x \in X \) there exists a measure \( \mu \in \mathcal{M}^1(X) \) with \( r(\mu) = x \) such that \( \mu(\text{ext} \, X) = 1 \) (see [1 Corollary I.4.9], [13 Chapter 3] or [12 Section 3.8]). Also, if \( f \) is a Baire strongly affine function, the answer is also yes since for any \( x \in X \) there exists a measure \( \mu \in \mathcal{M}^1(X) \) with \( r(\mu) = x \) such that \( \mu(B) = 1 \) for each \( B \supseteq \text{ext} \, X \) Baire.

We recall a particular class of Borel functions that satisfy the point of continuity property. For a topological space \( X \), let \( \text{Bos}(X) \) denote the algebra generated by closed (equivalently open) sets in \( X \). If \( Y \) is another topological space, then a mapping \( f : X \to Y \) is of the first Borel class if \( f^{-1}(U) \in \text{Bos}(X) \) for any open set \( U \subseteq Y \), i.e., \( f^{-1}(U) \) is a countable union of sets from \( \text{Bos}(X) \). Obviously, any semicontinuous function is of the first Borel class. It is proved in [8 Theorem 2.3] that any real-valued function of the first Borel class on a compact space has the point of continuity property.

We start the proof of Theorem 1.1 by the following result from [6].

**Lemma 1.3.** Let \( Y \) be a compact convex set and \( f : Y \to \mathbb{R} \) be an affine function such that the set \( C(f) \) of its points of continuity is dense in \( Y \). Then \( C(f) \cap \text{ext} \, Y \) is dense in \( \text{ext} \, Y \).

**Proof.** See [6 Lemma II.2]. \( \square \)

Below we use the following notation \( \overline{a, c} : = \text{co} \{a, c\} \setminus \{a, c\} \), where \( a, c \) are points in a vector space. The following lemma is inspired by the proof of [3 Proposition 3.1.1].

**Lemma 1.4.** Let \( E \) be a vector space, \( a, b, c, c_1, c_2 \) be distinct points in \( E \) and \( \Delta : = \text{co} \{a, c_1, c_2\} \). Let \( f : \Delta \to \mathbb{R} \) be an affine function, \( \eta \in \mathbb{R} \) and let

\[
(1.2) \quad b \in F : = \{x \in \Delta; f(x) \geq \eta\}, \quad a \notin F, \quad c, c_1, c_2 \in \{x \in \Delta; f(x) > \eta\}.
\]

Let the following assumptions be satisfied.

(A1) The vectors \( c_1 - a \) and \( c_1 - c_2 \) are linearly independent.

(A2) We have \( b \in \overline{a, c} \) and similarly \( c \in \overline{c_1, c_2} \).

Then \( b \notin \text{ext} \, F \).

**Proof.** First, we may assume that \( \eta = 0 \). Otherwise, it is enough to replace \( F \) by \( F + \eta \) in the following proof.

(i) Let \( f(b) > 0 \). As \( c \in F \) holds by assumption (1.2), it is enough to show that

\[
(1.3) \quad b \in \overline{c, c'} \quad \text{where} \quad c : = \frac{f(c) - f(a)}{f(c) - f(a)} c \in F.
\]

Since \( f(a) < 0 < f(c) \) holds by assumption (1.2), we get that \( c \) in (1.3) is well defined and that \( c \in \text{co} \{a, c\} \subseteq \Delta \) as \( c \in \text{co} \{c_1, c_2\} \) holds by assumption (A2).

Since \( f \) is assumed to be an affine function, we get that \( f(c) = 0 \), which ensures
that $e \in F$ and also that $e \neq c$ as $f(c) > 0$ holds by (1.2). By assumption (A2) there exists $\alpha \in (0, 1)$ such that we have the first equality in

$$b = \alpha a + (1 - \alpha) c = \varepsilon e + (1 - \varepsilon) c,$$

where $\varepsilon \overset{\text{def}}{=} \alpha [1 - f(a)/f(c)] > 0$.

The second equality can be derived from the definition of $e$ in (1.3), and to finish the first part of the proof, it is enough to show that $\varepsilon < 1$, but it follows from the following relations $0 < f(b) = \alpha f(a) + (1 - \alpha) f(c)$ based on affinity of $f$.

(ii) If $f(b) = 0$, it is enough show that

$$b \in \overline{b_1, b_2}, \quad \text{where} \quad b_i \overset{\text{def}}{=} \frac{f(c_i) - f(a)}{f(c_i) - f(a)} \in F.$$

Similarly as in (i) we would verify that $b_i$’s are well defined elements of $\Delta$ such that $f(b_i) = 0$, which ensures $b_i \in F, i = 1, 2$. Further, it follows from assumption (A2) that there are positive values $\gamma_1, \gamma_2$ and $\alpha \overset{\text{def}}{=} 1 - \gamma_1 - \gamma_2 > 0$ such that we have the first equality in

$$b = \alpha a + \gamma_1 c_1 + \gamma_2 c_2 = \beta_1 b_1 + \beta_2 b_2, \quad \text{where} \quad \beta_i \overset{\text{def}}{=} \gamma_i [1 - f(c_i)/f(a)] > 0.$$

The second equality can be verified just by computation using the definitions of $\alpha, b_i, \beta_i$’s and the following equality $0 = f(b) = \alpha f(a) + \gamma_1 f(c_1) + \gamma_2 f(c_2)$. This equality is the one that should be used together with the definition of $\alpha$ in order to verify that $\beta_1 + \beta_2 = 1$. Thus, in order to verify (1.4) it is enough to show that $b_1 \neq b_2$, which can be shown to be equivalent to the following inequality

$$[f(a) - f(c_1)](c_1 - c_2) \neq [f(c_1) - f(c_2)](a - c_1).$$

This is obviously satisfied by assumption (A1), since $f(a) < 0 < f(c_1)$ holds by assumption (1.2). □

**Proof of Theorem** (1.1) Let $f: X \to \mathbb{R}$ be an affine function with the point of continuity property and let $f \leq 0$ on ext $X$. Our aim is to show that also $f \leq 0$ on $X$. In other words, we assume that the following set is disjoint from ext $X$

$$F_{\eta} \overset{\text{def}}{=} \{x \in X; f(x) \geq \eta\}$$

whenever $\eta \in (0, \infty)$, and we are going to show that it is empty.

To this end, assume the contrary, i.e., that there exists $\eta > 0$ such that $F_\eta \neq \emptyset$. Then $Y \overset{\text{def}}{=} \overline{F_\eta}$ is a non-empty compact convex set, which ensures that ext $Y \neq \emptyset$. Since the set $C(f|_Y)$ of points of continuity of $f|_Y$ is of the second category in $Y$ (see [8, Theorem 2.3]), it is dense in $Y$, and we get from Lemma [1.3] that $C(f|_Y) \cap \text{ext } Y$ is a dense subset of ext $Y \neq \emptyset$. Then there has to exist a point

$$b \in C(f|_Y) \cap \text{ext } Y.$$

Since $f|_Y$ is continuous at $b$ and $b \in Y = \overline{F_\eta}$, we get that $f(b) \geq \eta$, i.e., $b \in F_\eta$. As $F_\eta$ is disjoint from ext $X$, we get that $b$ is not an extreme point of $X$. Then we can find $a, e \in X$ such that $b = \overline{a, e}$. Since $b \in F_\eta \cap \text{ext } Y \subseteq \text{ext } F_\eta$ and $f$ is affine, either $f(a) < \eta$ and $f(e) \geq \eta$ or vice versa. We assume that the former case holds. Then even

$$f(a) < \eta < f(e)$$

holds, since otherwise if $f(e) = \eta$, we would obtain that $f(b) < \eta$, which is impossible as $b \in F_\eta$. Put

$$c \overset{\text{def}}{=} e + t(e - a) \in X, \quad \text{where} \quad t \overset{\text{def}}{=} \max\{s \geq 0; e + s(e - a) \in X\}. $$
As $f$ is affine, we get from (1.6), (1.7) that $f(c) = f(e) + t[f(e) - f(a)] \geq f(e) > \eta$. Hence, $c \in F_\eta$. If $c \in \text{ext} X$, we have a contradiction with our assumption that $F_\eta \cap \text{ext} X = \emptyset$. So let us assume the contrary, i.e., that $c \in c_{1,2}$ for some $c_1, c_2 \in X$. We may assume that $c_1, c_2$ are chosen so that $f(c_i) > \eta$, $i = 1, 2$, since otherwise we would consider $\tilde{c}_i \equiv c + \varepsilon(c_i - c)$ for $\varepsilon > 0$ small enough instead. By the choice of $c$ in (1.7), the vectors $c - a$ and $c_1 - c_2$ are linearly independent, which obviously means that the same holds with $c$ replaced by $c_1$. Now we are at the situation of Lemma 1.4. Thus it follows that $b \in F \setminus \text{ext} F$, where $F \equiv F_\eta \cap \Delta$ and $\Delta \equiv \text{co} \{a, c_1, c_2\}$. Then $b$ cannot be an extreme point of any superset of $F$. In particular, $b \notin \text{ext} Y$, where $Y = F_\eta \supseteq F_\eta \supseteq F$, and we have a contradiction with (1.4), which finishes the proof. \hfill \Box

Using the Hahn-Banach theorem we can obtain the following corollary.

**Corollary 1.5.** Let $X$ be a compact convex set.

(a) If $F$ is a locally convex space and $f: X \to F$ is affine with the point of continuity property, then $f(X) \subseteq \overline{\text{co}} f(\text{ext} X)$.

(b) If $f: X \to \mathbb{C}$ is affine and has the point of continuity property, then $\sup_{x \in X} |f(x)| = \sup_{x \in \text{ext} X} |f(x)|$.

**Proof.** (a) We assume that there exists $x \in X$ such that $f(x) \notin \overline{\text{co}} f(\text{ext} X)$. By the Hahn-Banach theorem we can find $\tau \in F^*$ such that

$$\tau(f(x)) > \eta \equiv \sup \{\tau(z); z \in \overline{\text{co}} f(\text{ext} X)\} \in \mathbb{R}.$$ 

It straightforwardly follows from the definition that $\tau \circ f$ has the point of continuity property and the same holds for the function $\tau \circ f - \eta$ attaining values only in $(-\infty, 0]$ on $\text{ext} X$. Then we obtain by Theorem 1.1

$$\tau(f(x)) = (\tau \circ f)(x) \leq \eta < \tau(f(x)),$$

i.e., a contradiction.

(b) We identify $\mathbb{C}$ with $\mathbb{R}^2$. By (a) we have for each $x \in X$ that

$$f(x) \in \overline{\text{co}} f(\text{ext} X) \subseteq K_\eta \equiv \{\lambda \in \mathbb{C}; |\lambda| \leq \eta\}, \text{ where } \eta \equiv \sup_{x \in \text{ext} X} |f(x)|,$$

since $K_\eta$ is obviously a closed convex superset of $f(\text{ext} X)$. Thus $|f(x)| \leq \eta$, which finishes the proof. \hfill \Box

2. A generalization of the Cohen-Chu theorem

The aim of this part is a proof of a generalization of the result by C. H. Chu and H. B. Cohen in the spirit of the Banach-Stone theorem. They proved the following theorem (see [1]):

Let $X$ and $Y$ be compact convex sets and let $T: \mathcal{A}^c(X) \to \mathcal{A}^c(Y)$ be an isomorphism satisfying $\|T\| \cdot \|T^{-1}\| < 2$. If

- $X$ and $Y$ are metrizable and each point of $\text{ext} X$ and $\text{ext} Y$ is a weak peak point, or
- the sets $\text{ext} X$ and $\text{ext} Y$ are closed and each extreme point of $X$ and $Y$ is a split face,

then the sets $\text{ext} X$ and $\text{ext} Y$ are homeomorphic.

For a set $F \subseteq X$, the complementary set $F^{cs}$ is defined as the union of all faces of $X$ disjoint from $F$. A face $F$ of $X$ is said to be a split face if its complementary set $F^{cs}$ is convex (and hence a face, see [1] p. 132]) and every point in $X \setminus (F \cup F^{cs})$
can be uniquely represented as a convex combination of a point in \( F \) and a point in \( F^{cs} \).

We call \( x \in \text{ext} X \) a weak peak point if for each \( \varepsilon \in (0, 1) \) and an open neighborhood \( U \) of \( x \) there exists \( h \in \mathcal{A}^c(X) \) such that
\[
\|h\| \leq 1, \ h(x) > 1 - \varepsilon \quad \text{and} \quad |h| < \varepsilon \quad \text{on} \quad \text{ext} X \setminus U.
\]

Let us also recall that if \( x \) is a weak peak point of a compact convex set \( X \), then \( \{x\} \) is a split face and the converse holds if ext \( X \) is closed; see [12, Proposition 1].

We refer the reader to [11 pp. 72, 73, 75] for notions of the theory of compact convex sets (see also [12, Section 4.3]). We just mention that \( X \) can be embedded to \((\mathcal{A}^c(X))^*\) via the evaluation mapping \( \phi: X \to (\mathcal{A}^c(X))^* \) defined as \( \phi(x)(f) = f(x) \), \( f \in \mathcal{A}^c(X), \ x \in X \). The dual unit ball \( B_{(\mathcal{A}^c(X))^*} \) equals the convex hull \( \text{co}(X \cup -X) \) and \((\mathcal{A}^c(X))^*\) coincides with span \( X \), the linear span of \( X \). Further, any affine bounded function \( f \) on \( X \) has the unique extension to span \( X \), and this provides an identification of \((\mathcal{A}^c(X))^*)^{**}\) with the space \( \mathcal{A}^b(X) \) of all bounded affine functions on \( X \).

We use Theorem 1.1 to show the following theorem.

**Theorem 2.1.** Let \( X, Y \) be compact convex sets such that every extreme point of \( X \) and \( Y \) is a weak peak point and let \( T: \mathcal{A}^c(Y) \to \mathcal{A}^c(X) \) be an isomorphism with \( \|T\| \cdot \|T^{-1}\| < 2 \). Then \( X \) is homeomorphic to ext \( Y \).

Example 1 on [11, p. 83] shows that Theorem 2.1 need not hold even for compact convex sets in finite dimensional spaces if we omit the assumption that extreme points are weak peak points. An example due to H. U. Hess (see [7]) shows that for every \( \varepsilon > 0 \) there exist metrizable simplices \( X, Y \) and an isomorphism \( T: \mathcal{A}^c(X) \to \mathcal{A}^c(Y) \) such that \( \|T\| \cdot \|T^{-1}\| < 1 + \varepsilon \) and ext \( X \) is not homeomorphic to ext \( Y \).

Also, the bound \( 2 \) is optimal by a result of H. B. Cohen (see [5]). We also mention paper [10] where Theorem 2.1 is proved under condition that the sets of extreme points are Lindelöf.

**Proof of Theorem 2.1.** We follow the proof of [10, Theorem 1.1]. The main difference is that we use Theorem 1.1 instead of [10, Lemma 2.1] and thus we need to verify its assumptions in Claim 2. Let \( T: \mathcal{A}^c(X) \to \mathcal{A}^c(Y) \) be an isomorphism satisfying \( \|T\| \cdot \|T^{-1}\| < 2 \). We may assume that there exists \( 1 < c' < 2 \) such that
\[
(2.1) \quad \|T\| < 2 \quad \text{and} \quad \|Tf\| \geq c'\|f\| \quad \text{for all} \quad f \in \mathcal{A}^c(X).
\]
Otherwise, we would find \( 1 < c' < 2 \) such that \( \|T\| \cdot \|T^{-1}\| < \frac{c'}{2} < 2 \) and consider \( T' \triangleq \frac{c'}{2}\|T^{-1}\|T \) instead; see [4, p. 76]. Further, we fix \( 1 < c < c' \).

**Claim 1.** For any \( f \in \mathcal{A}^b(X) \) and \( g \in \mathcal{A}^b(Y) \) non-zero, \( \|T^{**}f\| > c\|f\| \) and \( \|((T^{-1})^{**}g)\| > \frac{1}{2}\|g\| \).

**Proof of Claim 1.** See [10, Proof of Claim 1].

If \( x \in \text{ext} X \), we recall that \((\mathcal{A}^c(X))^* = \text{span}\{x\} \oplus \text{span}\{x\}^{cs}\) because \( \{x\} \) is a split face; see [4, p. 72]. Hence, given \( y \in Y \), following [4, p. 76] we can write
\[
(2.2) \quad T^*y = \lambda x + \mu \quad \text{for some} \ \lambda \in \mathbb{R} \ \text{and} \ \mu \in \text{span}\{x\}^{cs}.
\]
Similarly as in [4, p. 77], for \( y \in Y \) satisfying (2.2), we have that
\[
(2.3) \quad |\lambda| > c \ \Rightarrow \ \|\mu\| = \|T^*y\| - |\lambda| < 2 - c.
\]
Given $x \in \text{ext } X$, we denote by $\chi_{\{x\}}$ the characteristic function of the set $\{x\}$. Then the upper envelope function $h_x \triangleq \tilde{\chi}_{\{x\}}$, defined as

$$\tilde{\chi}_{\{x\}}(z) \triangleq \inf \{ h(z) : h \in \mathfrak{F}(X), h > \chi_{\{x\}} \} \quad \text{for } z \in X,$$

is upper semicontinuous and affine (see [4, p. 73]), and thus strongly affine (see [2, Theorem 1.6.1(ix)]). Further, we note (see [4, p. 77]) that (2.7)

$$\{x\} = h_x^{-1}(1) \quad \text{and} \quad \{x\}^c = h_x^{-1}(0).$$

Claim 2. For any $x \in \text{ext } X$, $T^{**} h_x$ is a strongly affine function of the first Borel class and thus it has the point of continuity property.

Proof of Claim 2. Since $T : \mathfrak{F}(X) \to \mathfrak{F}(Y)$, we have $T^* : \text{span } Y \to \text{span } X$. If $f \in \mathfrak{F}(X)$ and $\tilde{f}$ is the linear extension of $f$ to span $X$, then $T^{**} f = \tilde{f} \circ T^*$. Since $||T|| < 2$,

$$T^* Y \subseteq 2B(\mathfrak{F}(X))^* = \text{co}(2X \cup -2X).$$

The function $f \triangleq h_x$, being upper semicontinuous and affine, is strongly affine on $X$. The sets $2X$ and $-2X$ are affinely homeomorphic to $X$, and hence $\tilde{f}$ is strongly affine on both of them. By [13, Lemma 2.4(b)], $\tilde{f}$ is strongly affine on $2B(\mathfrak{F}(X))^* = \text{co}(2X \cup -2X)$.

Since $Y$ is affinely homeomorphic to $T^* Y$ and $T^{**} f = \tilde{f} \circ T^*$, we obtain that $T^{**} f$ is strongly affine on $Y$.

Further, $h_x$ is upper semicontinuous on $X$ and thus it is of the first Borel class on $X$. Since $2X$ and $-2X$ are affinely homeomorphic to $X$, $\tilde{f}$ is of the first Borel class on $2X \cup -2X$. Now we can use [11, Theorem 3.5(b)] to conclude that $\tilde{f}$ is of the first Borel class on $2B(\mathfrak{F}(X))^* = \text{co}(2X \cup -2X)$. As above we obtain that $T^{**} h_x$ is of the first Borel class on $Y$. The final statement now follows from [8, Theorem 2.3].

Similarly as in [4, p. 77] we consider mappings $\rho : \text{ext } Y \to \text{ext } X$, $\tau : \text{ext } X \to \text{ext } Y$ defined as follows

$$(2.5) \quad \rho \triangleq \{(y, x) \in \text{ext } Y \times \text{ext } X : |T^{**} h_x(y)| > c\},$$

$$(2.6) \quad \tilde{\tau} \triangleq \{(x, y) \in \text{ext } X \times \text{ext } Y : |(T^{-1})^{**} h_y(x)| > \frac{1}{2}\}.$$

By [4, p. 77], $\rho$ is a mapping and we denote its domain as $\tilde{Y} \triangleq \text{dom}(\rho)$. Analogously, we would get that also $\tilde{\tau}$ is a mapping and we put $\tilde{X} \triangleq \text{dom}(\tilde{\tau})$.

Note that if $x \in \text{ext } X, y \in \text{ext } Y$ and $\lambda$ are as in (2.2), then for the linear extension $\tilde{h}_x$ of $h_x$ on span $X = (\mathfrak{F}(X))^*$ we have

$$(2.7) \quad T^{**} h_x(y) = \tilde{h}_x(T^* y) = \tilde{h}_x(\lambda x + \mu) = \lambda h_x(x) + h_x(\mu) = \lambda

\text{as } \tilde{h}_x \text{ is linear and } h_x(x) = 1 \text{ and } h_x(\mu) = 0 \text{ hold by (2.3)}.$$

Claim 3. The mappings $\tilde{\rho} : \tilde{Y} \to \text{ext } X$ and $\tilde{\tau} : \tilde{X} \to \text{ext } Y$ are surjective.

Proof of Claim 3. Let $x \in \text{ext } X$ be given and assume that $|T^{**} h_x(y)| \leq c$ for all $y \in \text{ext } Y$. By Theorem [14] and Claim 2, $|T^{**} h_x| \leq c$ on $Y$. Then

$$c \geq ||T^{**} h_x|| > c ||h_x|| = c$$

gives a contradiction. Hence $\tilde{\rho}$ is surjective. Analogously, using the second part of Claim 1 we would obtain that $\tilde{\tau}$ is surjective.
The following claim is essentially Lemma 6 of [4] and Claim 4 in [10]. However, we recall its proof since it uses Theorem 1.1.

Claim 4. We have $\hat{X} = \text{ext } X$ and $\hat{Y} = \text{ext } Y$ and, for any $x \in \text{ext } X$ and $y \in \text{ext } Y$, $\hat{\rho}(\hat{x}) = x$ and $\hat{\tau}(\hat{y}) = y$.

Proof of Claim 4. We will show that $(\hat{\rho}(\hat{y}), \hat{\tau}(\hat{x})) \in \hat{\tau}$ holds for any $\hat{y} \in \hat{Y}$, i.e.,

$$|(T^{-1})^{**}h_{\hat{y}}(\hat{\rho}(\hat{y}))| > \frac{1}{2}. \leqno{(2.8)}$$

By Claim 1, $\| (T^{-1})^{**}h_{\hat{y}} \| > \frac{1}{2} \| h_{\hat{y}} \| = \frac{1}{2}$. Then Claim 2 and Theorem 1.1 yield

$$d \triangleq \sup_{\hat{x} \in \text{ext } X} |(T^{-1})^{**}h_{\hat{y}}(\hat{x})| = \sup_{\hat{x} \in \text{ext } X} |(T^{-1})^{**}h_{\hat{y}}(\hat{x})| = |(T^{-1})^{**}h_{\hat{y}}| > \frac{1}{2}. \leqno{(2.9)}$$

Since $c > 1$, we have $d > \max\{\frac{d}{c}, \frac{1}{2}\}$. Hence, there exists $x \in \text{ext } X$ such that

$$|(T^{-1})^{**}h_{\hat{y}}(x)| > \max\{\frac{d}{c}, \frac{1}{2}\} \geq \frac{1}{2}, \text{ i.e. } (x, \hat{y}) \in \hat{\tau}. \leqno{(2.10)}$$

Let us assume that (2.8) does not hold. Then $\hat{\rho}(\hat{y}) \neq x$, and by Claim 3 we can find $y \in \hat{Y}$ with $\hat{\rho}(y) = x$. Then $y \in \{\hat{y}\}^{c}$, and thus $h_{\hat{y}}(y) = 0$. Since $x \in \text{ext } X$ and $y \in Y$, we can use decomposition (2.2) in order to get that

$$0 = h_{\hat{y}}(y) = (T^{-1})^{**}h_{\hat{y}}(Ty) = (T^{-1})^{**}h_{\hat{y}}(\lambda x) + (T^{-1})^{**}h_{\hat{y}}(\mu). \leqno{(2.11)}$$

Since $\lambda = T^{**}h_{\hat{y}}(y)$ holds by (2.7), and as $(y, x) \in \hat{\rho}$, we get by (2.5) that $|\lambda| > c$. Then we get from (2.8), (2.9), (2.10) and (2.11) that

$$d < |\lambda| \frac{d}{c} < |\lambda| \cdot |(T^{-1})^{**}h_{\hat{y}}(x)| = |(T^{-1})^{**}h_{\hat{y}}(\lambda x)| = |(T^{-1})^{**}h_{\hat{y}}(\mu)| \leq \| \mu \| < d(2 - c) < d. \leqno{(2.12)}$$

This is a contradiction with assumption that (2.8) does not hold, hence (2.8) holds, and we have that

$$\hat{\tau}(\hat{\rho}(y)) = y, \text{ } y \in \hat{Y}. \leqno{(2.13)}$$

Now, let $x \in \text{ext } X$ be given. By Claim 3 there exists $\hat{y} \in \hat{Y}$ with $\hat{\rho}(\hat{y}) = x$. Then we get from (2.13) that $\hat{\tau}(\hat{x}) = \hat{y}$, which ensures that $x \in \hat{X} \subseteq \text{ext } X$ and finally that also $\hat{X} = \text{ext } X$.

Let $y \in \text{ext } Y$ be given. From Claim 3 we obtain first that there exists $x \in \hat{X} = \text{ext } X$ such that $\hat{\tau}(\hat{x}) = y$ and then that there exists $\hat{y} \in \hat{Y}$ such that $\hat{\rho}(\hat{y}) = x$. Then we get from (2.12) that $y = \hat{\tau}(\hat{x}) = \hat{\tau}(\hat{\rho}(\hat{y})) = \hat{y} \in \hat{Y} \subseteq \text{ext } Y$, and finally that $\hat{Y} = \text{ext } Y$.

If $x \in \text{ext } X = \hat{X}$, it is enough to use Claim 3 once again and property (2.12) in order to get that $\hat{\rho}(\hat{\tau}(\hat{x})) = \hat{\rho}(\hat{\tau}(\hat{\rho}(\hat{y}))) = \hat{\rho}(y) = x$ holds for some $y \in \text{ext } Y$.

By the proof of Theorem 7 on p. 78 in [4], the mappings $\hat{\rho}$ and $\hat{\tau}$ are continuous ($\hat{\rho}$ and $\hat{\tau}$ are denoted as $\rho$ and $\tau$ in [4]). This finishes the proof of Theorem 2.1. \qed

As in [4] Corollaries 13 and 14 we obtain the following corollary.

**Corollary 2.2.** Let $A$ and $B$ be function algebras, and let $T: \text{Re } A \to \text{Re } B$ be an isomorphism satisfying $\| T \| \cdot \| T^{-1} \| < 2$. Then the Choquet boundaries of $A$ and $B$ are homeomorphic.
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