Asymptotic representations for some functions and integrals connected with the Airy function

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Abstract

The asymptotic representations of the functions \( Ai_1(x), Gi(x), Gi'(x), Ai^2(x), Bi^2(x) \) are obtained. As a by-product, the factorial identity (21') is found. The derivation of asymptotic representations of the integral \( \int_0^\infty dx Ai(x)h(x,v) \) for \( v \to -\infty \) and integrals, differing from it by the change of \( Ai(x) \) by \( Ai'(x) \) or \( Ai_1(x) \), is presented. For the Airy function \( Ai(z) \), as an example, the Stokes' phenomenon is considered as a consequence of discontinuous behavior of steepest descent lines over the passes. When \( z \) crosses the Stokes ray, the steepest descent line over the higher pass abruptly changes the direction of its asymptotic approach to the steepest descent line over the lower pass to the direction of approach to the opposite end of this line. Therefore, when the integration contour, drawn along the steepest descent lines, goes over the higher pass, it begins or stops to go over the lower pass while \( z \) crosses the Stokes ray, and as a result the recessive series (contribution from the lower pass) discontinuously appears or disappears in the asymptotic representation of a function containing the dominant series.

1 Introduction

In the theory of quantum processes with particles in a constant or slowly varying field, the Airy function, related functions and integrals with these functions play an important role [1, 2]. In this paper [which is the Lebedev Phys. Inst. Preprint N 253 (1985)] we give the asymptotic expansions for some frequently occurring functions.

2 Asymptotic behavior of \( Hi(z), Gi(z), Ai_1(z) \) and derivatives \( Hi'(z), Gi'(z) \)

We define the Airy function and related functions as in [3, 4], but without the factor \( \pi^{-1} \). Starting from

\[
Hi(z) = \int_0^\infty dt e^{zt - \frac{t^3}{3}}
\]
and integrating by parts, we have

\[ H_i(z) = \sum_{k=1}^{n} q^{(k-1)}(0)(-z)^{-k} + \varepsilon_n(z), \quad \varepsilon_n(z) = (-z)^{-n} \int_0^\infty dt q^{(n)}(t)e^{zt}, \quad (2) \]

\[ q(t) = e^{-t^3/3}, \quad q'(t) = -t^2 q(t), \]

\[ q''(t) = (-2t + t^4)q(t), \quad q^{(3)}(t) = (-2 + 6t^3 - t^6)q(t), \ldots. \quad (3) \]

The relation (2) holds for any \( z \), but it is useful only for \( \text{Re} \, z \ll -1 \) (more exactly for \( |z| \to \infty, |\text{ph}(-z)| < 2\pi/3 \), see p. 432 in [4], when the remainder term \( \varepsilon_n(z) \) is small compared with the terms of the sum). This is not the case for \( z \gg 1 \) in (2). So the decrease of terms in the sum (2) does not provide any guarantee of smallness of the remainder term [4, 5]. Its magnitude depends on proximity of saddle points (in general, critical points) to the integration path. Assuming \( z \) fixed and \( \text{Re} \, z \ll -1 \) we shall increase \( n \) in (2). The terms of the sum first decrease in magnitude then increase, as the representation is asymptotic. It is clear that the sum of increasing terms, generally speaking, grows with \( n \) and this must be compensated by the remainder term; the representation with such a term is still exact. When \( n \to \infty \) and the remainder term is dropped, we get the asymptotic series and the problem of interpretation of diverging part of the series arises [5]. Dingle [5] suggested that the remainder term can be restored just from the general term in each considered series. Indeed, the remainder term can be restored using, when necessary, the additional information about the function. For this reason it is convenient to regard the divergent sum of late terms as a symbolic representation of the remainder term [5]. With this understanding the asymptotic series uniquely represents its function and is the complete asymptotic expansion (for a certain phase range of \( z \)). In general, the complete asymptotic expansion consist of several series (two for solutions to second-order homogeneous differential equations) each of which determines its own function [5]. In this case the complete asymptotic expansion expresses the linear dependence of the initial function upon other functions each of which is represented by its own series in the considered sector of \( z \). This linear dependence between the functions holds in all sectors of the complex variable \( z \), but outside the considered sector the representation of each function by one series may be unsatisfactory, that is the remainder term may be not small.

Asymptotic expansions with remainder terms exactly determine their functions. However, even in the case when the remainder terms are represented only symbolically by the increasing terms of infinite series, it is still possible to require, that complete asymptotic expansions satisfy the same relations as the functions themselves [5].

Returning now to eq.(2), for \( n \to \infty \) we have [4]

\[ H_i(z) \sim -\sum_{n=0}^{\infty} \frac{(3n)!}{3^n n! z^{3n+1}}, \quad |\text{ph}(-z)| < \frac{2\pi}{3}. \quad (4) \]

Another and more simple way of obtaining (4) consists of expanding \( \exp(-t^3/3) \) in power series and subsequent term by term integration. The integrable series is convergent everywhere, but nonuniformly, so the integrated series is asymptotic.

We consider now the real \( z = x \gg 1 \). The integrand in (1) first exponentially increases (up to the saddle point \( t_c = \sqrt{x} \)) then decreases. In accordance with this the asymptotic
expansion consists of a series of exponentially large terms (contribution from the saddle point) and, as we shall see below, of the series (4) (contribution from the lower limit of integration).

As mentioned above, complete asymptotic expansion should satisfy the same relations as the functions themselves. The relation

$$\text{Ai}_1(x) = \pi + \text{Hi}(x)\text{Ai}'(x) - \text{Hi}'(x)\text{Ai}(x),$$  \hspace{1cm} (5)

where

$$\text{Ai}_1(x) = \int_x^\infty dt \text{Ai}(t), \quad \text{Ai}(x) = \int_0^\infty dt \cos \left( xt + \frac{t^3}{3} \right),$$  \hspace{1cm} (6)

is important for us here. It is easily verified by differentiation and using the equations

$$\text{Hi}''(x) - x\text{Hi}(x) = 1, \quad \text{Ai}''(x) - x\text{Ai}(x) = 0.$$  \hspace{1cm} (7)

We shall obtain the asymptotic expansions for \(\text{Hi}(x)\), \(\text{Hi}'(x)\) and \(\text{Ai}_1(x)\) and check them up using (5) and the known asymptotic expansions for \(\text{Ai}(x)\) and \(\text{Ai}'(x)\) [3, 4].

The complete asymptotic expansion for \(\text{Hi}(x)\), \(x \to \infty\), can be obtained in the following manner. Substituting \(t = \tau + \sqrt{x}\), we have for the exponent in (1)

$$xt - \frac{t^3}{3} = \frac{2}{3} x^{3/2} - x^{1/2}\tau^2 - \frac{\tau^3}{3}, \quad \tau = t - \sqrt{x}.$$  \hspace{1cm} (8)

Next we expand \(\exp(-\tau^3/3)\) in power series and integrate term by term over \(\tau\) from \(-\sqrt{x}\) to \(\infty\). If we simply extend the region of integration to the whole real axis (using the condition \(\sqrt{x} \gg 1\)), we get only the exponentially large terms and the relation (5) will not be satisfied. So we write

$$\text{Hi}(x) = \int_{-\infty}^{-T} dt \, \exp\left( xt - \frac{t^3}{3} \right) - \int_{-T}^0 dt \, \exp\left( xt - \frac{t^3}{3} \right), \quad T \gg 1.$$  \hspace{1cm} (9)

In the first integral we use (8) and perform the indicated operations. In the second integral we expand \(\exp(-\tau^3/3)\) in power series and integrate term by term. Letting \(T \to \infty\), we get

$$\text{Hi}(x) \sim \pi^{1/2} x^{-1/4} e^{\zeta} \sum_{n=0}^\infty \frac{c_n}{\zeta^n} - \sum_{n=0}^\infty \frac{(3n)!}{3^n n! x^{3n+1}}, \quad x \to \infty,$$  \hspace{1cm} (10)

where

$$\zeta = \frac{2}{3} x^{3/2}, \quad c_n = \frac{\Gamma(n + \frac{1}{6})\Gamma(n + \frac{5}{6})}{2^{n+1} n! \pi} = \frac{(6n - 1)!!}{6^{3n} n!(2n - 1)!!}.$$  \hspace{1cm} (10')

The relation (10) agrees with the relation

$$\text{Hi}(x) = \text{Bi}(x) - \text{Gi}(x)$$  \hspace{1cm} (11)

for the functions

$$\text{Bi}(x) = \int_0^\infty dt \{ \exp(x t - \frac{t^3}{3}) + \sin(x t + \frac{t^3}{3}) \},$$  \hspace{1cm} (12)

$$\text{Gi}(x) = \int_0^\infty dt \sin(x t + \frac{t^3}{3}),$$  \hspace{1cm} (13)
for which the asymptotic expansions are known [4]

\[ \text{Bi}(x) \sim \pi^{1/2}x^{-1/4}e^{\zeta} \sum_{n=0}^{\infty} \frac{c_n}{\zeta^n}, \quad \zeta = \frac{2}{3}x^{3/2}, \quad (14) \]

\[ \text{Gi}(x) \sim \sum_{n=0}^{\infty} \frac{(3n)!}{3^n n! x^{3n+1}}, \quad x \to \infty. \quad (15) \]

(Assuming \( \text{Bi}(x) = \frac{1}{2}[\text{Bi}(x+ie) + \text{Bi}(x-ie)], \epsilon \to 0 \), it is easy to show that the asymptotic expansion for \( \text{Bi}(x), x \to +\infty \), consists of only one series (14), cf. Chap. 3 Sections 3.4 and 3.5 in [6] and p. 52 in [5].)

The asymptotic expansions for \( \text{Ai}_1(x) \) can be obtained by integration of the asymptotic expansion for \( \text{Ai}(t) \), which for \( t \to +\infty \) has the form [3,4]

\[ \text{Ai}(t) \sim \frac{1}{2} \pi^{1/2} t^{-1/4} e^{-\zeta} \sum_{n=0}^{\infty} (-1)^n \frac{a_n}{\zeta^n}, \quad \zeta = \frac{2}{3}t^{3/2}, \quad (16) \]

cn are given in (10'). Using the definition (6) for \( \text{Ai}_1(x) \), let us replace the integration variable \( t \) by \( \zeta \). Then integrating by parts the leading term on the right-hand side of (16), we have

\[ \int_{v}^{\infty} dt e^{-\zeta} t^{-1/4} = \left( \frac{2}{3} \right)^{1/2} \int_{\xi}^{\infty} d\zeta e^{-\zeta} \zeta^{-1/2} = \]

\[ \left( \frac{2}{3} \right)^{1/2} e^{-\xi} \xi^{-3/2} \frac{3}{2} \int_{\xi}^{\infty} d\zeta e^{-\zeta} \zeta^{-3/2}, \quad \xi = \frac{2}{3}v^{3/2}. \]

We unite the last (integral) term with the integral of term with \( c_1 \) on the right-hand side of (16) and integrate by parts again. Repeating the process, we get

\[ \text{Ai}_1(v) \sim \frac{1}{2} \pi^{1/2} v^{-3/4} e^{-\zeta} \sum_{n=0}^{\infty} (-1)^n \frac{a_n}{\zeta^n}, \quad \zeta = \frac{2}{3}v^{3/2}, \quad v \to \infty. \quad (17) \]

Here \( a_n \) are connected with \( c_n \) by the recurrence relation

\[ a_n = c_n + \left( n - \frac{1}{2} \right) a_{n-1} = \sum_{k=0}^{n} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(k+\frac{1}{2})} c_k = \frac{\Gamma(n+\frac{1}{2})}{2^{3/2}} \sum_{k=0}^{n} 2^k B \left( k + \frac{1}{6}, k + \frac{5}{6} \right), \]

\[ a_0 = 1, \quad a_1 = \frac{41}{23 \cdot 3^2}, \quad a_2 = \frac{9241}{27 \cdot 3^4}, \quad a_3 = \frac{5^2 \cdot 203009}{210 \cdot 3^7}, \ldots \quad (18) \]

It can be shown now that the asymptotic series for \( \text{Ai}_1(x), \text{Hi}(x), \text{Ai}(x) \), see (17), (10), (16) and the series for derivatives \( \text{Hi}'(x), \text{Ai}'(x) \),

\[ \text{Hi}'(x) \sim \pi^{1/2} x^{1/4} e^{\zeta} \sum_{n=0}^{\infty} \frac{d_n}{\zeta^n} + \sum_{n=0}^{\infty} \frac{(3n+1)!}{3^n n! x^{3n+2}}, \quad (19) \]

\[ \text{Ai}'(x) \sim -\frac{1}{2} \pi^{1/2} x^{1/4} e^{-\zeta} \sum_{n=0}^{\infty} (-1)^n \frac{d_n}{\zeta^n}, \quad d_n = -\frac{6n+1}{6n-1} c_n, \quad (20) \]
satisfy the relation (5). (Each series of a complete asymptotic expansion corresponds to its own analytic function, so one can differentiate asymptotic series, see p. 21 in [4], and consequently the complete asymptotic expansion). To do this, we use the equality

\[ \sum_{n=0}^{2s} (-1)^n c_n d_{2s-n} = 0, \quad s \geq 1, \]  

equivalent to

\[ \sum_{k=0}^{2s} (-1)^k \frac{\Gamma(k + \frac{1}{6})\Gamma(k + \frac{5}{6})\Gamma(2s - k - \frac{1}{6})\Gamma(2s - k + \frac{7}{6})}{k!(2s-k)!} = 0, \]  

which can be regarded as a result of substitution of asymptotic series for Ai(x) and Bi(x) in the Wronskian

\[ Bi'(x)Ai(x) - Bi(x)Ai'(x) = \pi. \]  

Using (11) and (22), the relation (5) can be rewritten in the form

\[ Ai_1(x) = Gi'(x)Ai(x) - Gi(x)Ai'(x). \]  

This can easily be checked by differentiation and using the equations

\[ Gi''(x) - xGi(x) = -1, \quad Ai''(x) - xAi(x) = 0. \]  

With the help of (23) and asymptotic series (16), (20), (15) for Ai(x), Ai'(x), Gi(x) and the series

\[ Gi'(x) \sim -\sum_{n=0}^{\infty} \frac{(3n+1)!}{3^n n! x^{3n+2}}, \quad x \to \infty, \]  

obtained by differentiation, it is easy to recover the asymptotic expansions (17) for Ai_1(x). The agreement of the series for Ai_1(x), obtained with the help of (23) and by direct integration of series for Ai(x), is a check of correctness of series (15), (24) for Gi(x) and Gi'(x). Now we note that the leading term of the asymptotic series for Gi'(x) given in handbook [3],

\[ Gi'(x) \sim \frac{7}{96} x^{-2}, \quad x \to \infty, \]  

disagrees with (24) and its integration (integration of asymptotic series is always possible [4]) does not reproduce the leading term of series for Gi(x).

It is worth-while to get the asymptotic expansion for Ai_1(x) with the remainder term. Assuming \( v > 0 \) and using \( Ai''(x) = xAi(x) \), we have

\[ Ai_1(v) = \int_v^{\infty} x^{-1}dAi'(x) = x^{-1}Ai'(x)|_v^{\infty} + \int_v^{\infty} x^{-2}Ai'(x)dx = \]  

\[ -v^{-1}Ai'(v) + \int_v^{\infty} x^{-2}dAi(x) = -v^{-1}Ai'(v) - v^{-2}Ai(v) + 2 \int_v^{\infty} x^{-3}Ai(x)dx. \]
Continuing the integration by parts with the help of the relation
\[
\int_v^\infty x^{-n} \text{Ai}(x) \, dx = -v^{-n-1} \text{Ai}'(v) - (n+1)v^{-n-2} \text{Ai}(v) + (n+1)(n+2) \int_v^\infty x^{-n-3} \text{Ai}(x) \, dx,
\]
we easily find
\[
\text{Ai}_1(v) = -\sum_{k=0}^n \left[ \frac{(3k)!}{(3k)!!v^{3k+1}} \text{Ai}'(v) + \frac{(3k+1)!}{(3k)!!v^{3k+2}} \text{Ai}(v) \right] + \frac{(3n+2)!}{(3n)!!} \int_v^\infty x^{-3n-3} \text{Ai}(x) \, dx,
\]
(27)
For fixed \( n \) and \( v \to \infty \) the remainder term is negligible compared with the terms of the sum. For \( v \gg 1 \) and \( n \to \infty \) we obtain again the asymptotic expansion in the form (23) with the series for \( \text{Gi}(x) \) and \( \text{Gi}'(x) \) given in (15) and (24).

For \( v < 0 \) we proceed similarly
\[
\text{Ai}_1(v) = \int_{-\infty}^\infty dx \text{Ai}(x) - \int_{-\infty}^v dx \text{Ai}(x) = \pi - \int_{-\infty}^v x^{-1} d\text{Ai}'(x) =
\]
\[
\pi - v^{-1} \text{Ai}'(v) - v^{-2} \text{Ai}(v) - 2 \int_{-\infty}^v x^{-3} \text{Ai}(x) \, dx.
\]
(28)
Finally we get
\[
\text{Ai}_1(v) = \pi - \sum_{k=0}^n \left[ \frac{(3k)!}{(3k)!!v^{3k+1}} \text{Ai}'(v) + \frac{(3k+1)!}{(3k)!!v^{3k+2}} \text{Ai}(v) \right] + \frac{(3n+2)!}{(3n)!!} \int_{-\infty}^v x^{-3n-3} \text{Ai}(x) \, dx,
\]
(29)
Assuming \( v \ll -1 \) and letting \( n \to \infty \), we obtain the asymptotic expansion in the form of the right-hand side of (5) with the series (4) for \( \text{Hi}(v) \) and the series for \( \text{Hi}'(v) \), obtained from (4) by differentiation:
\[
\text{Hi}'(x) \sim \sum_{n=0}^\infty \frac{(3n+1)!}{3^n n! x^{3n+2}}, \quad x \to -\infty.
\]
(30)
We note that the leading term of the series for \( \text{Hi}'(x) \), given in [3],
\[
\text{Hi}'(x) \sim -\frac{3}{2} x^{-2}, \quad x \to -\infty,
\]
disagrees with (30) and its integration does not provide the leading term of (4) for \( \text{Hi}(x) \).
The analog of (17) for $v \to -\infty$ is

$$\text{Ai}_1(v) = \pi - \pi^{1/2} \left(-v\right)^{3/4} \sum_{k=0}^{\infty} (-1)^k \frac{a_{2k}}{\zeta^{2k}} \cos(\zeta + \frac{\pi}{4}) + \frac{a_{2k+1}}{\zeta^{2k+1}} \sin(\zeta + \frac{\pi}{4}),$$

$$\zeta = \frac{2}{3}(-v)^{3/2}, \quad v \to -\infty. \tag{31}$$

Here $a_n$ are the same as in (18).

To get the asymptotic expansion for $\text{Gi}(-x), x \to \infty$, we use (11) and the asymptotic expansion for $\text{Bi}(x)$, see [3] or (35) below. Then we obtain

$$\text{Gi}(-x) \sim \pi^{1/2} \left(-x\right)^{1/4} \sum_{n=0}^{\infty} (-1)^n \left[c_{2n} \cos(\zeta + \frac{\pi}{4}) + c_{2n+1} \sin(\zeta + \frac{\pi}{4})\right] + \sum_{n=0}^{\infty} \frac{(3n)!}{3^n n! (-x)^{3n+1}},$$

$$\zeta = \frac{2}{3}x^{3/2}, \quad x \to \infty. \tag{32}$$

Differentiating it, we have

$$\text{Gi}'(-x) \sim \pi^{1/2} x^{1/4} \sum_{n=0}^{\infty} (-1)^n \left[d_{2n} \zeta^{-2n} \sin(\zeta + \frac{\pi}{4}) - d_{2n+1} \zeta^{-2n-1} \cos(\zeta + \frac{\pi}{4})\right] -$$

$$- \sum_{n=0}^{\infty} \frac{(3n + 1)!}{3^n n! (-x)^{3n+2}}. \tag{33}$$

c_n and $d_n$ are the same as in (10′) and (20).

### 3 Asymptotic expansions for $\text{Ai}^2(x)$ and $\text{Bi}^2(x)$ for $x \to -\infty$

In this case [3]

$$(-x)^{1/4} \text{Ai}(x) \sim S \sin(\zeta + \frac{\pi}{4}) - C \cos(\zeta + \frac{\pi}{4}), \tag{34}$$

$$(-x)^{1/4} \text{Bi}(x) \sim S \cos(\zeta + \frac{\pi}{4}) + C \sin(\zeta + \frac{\pi}{4}), \quad \zeta = \frac{2}{3}(-x)^{3/2}, \quad x \to -\infty, \tag{35}$$

where

$$S = \sqrt{\pi} \sum_{k=0}^{\infty} (-1)^k c_{2k} \zeta^{-2k}, \quad C = \sqrt{\pi} \sum_{k=0}^{\infty} (-1)^k c_{2k+1} \zeta^{-2k-1}. \tag{36}$$

So the combinations

$$\frac{1}{2} [\text{Ai}^2(x) \pm \text{Bi}^2(x)] \equiv w_{1,2}(x) \tag{37}$$

are respectively the nonoscillatory and oscillatory parts of $\text{Ai}^2(x)$. Writing

$$w_1(x) = \frac{1}{2} [\text{Ai}^2(x) + \text{Bi}^2(x)] \sim \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{e_n}{(-x)^{3n+1/2}} \tag{38}$$
and substituting it in the equation

\[ w'''(x) - 4xw'(x) - 2w(x) = 0, \]  

satisfied by \( \text{Ai}^2(x) \) and \( \text{Bi}^2(x) \) [3], we get

\[ e_n = -\frac{(6n-1)(6n-3)(6n-5)}{2^5 \cdot 3n} e_{n-1} = (-1)^n \frac{(6n-1)!!}{2^{5n} \cdot 3^n \cdot n!}. \]  

According to (34)-(36) \( e_0 = 1 \). Consequently

\[ w_1(x) = \frac{1}{2}[\text{Ai}^2(x) + \text{Bi}^2(x)] \sim \frac{\pi}{2} \sum_{n=0}^\infty \frac{(-1)^n (6n-1)!!}{2^{5n} \cdot 3^n \cdot n! (-x)^{3n+1/2}}. \]  

Similarly for the oscillatory part of \( \text{Ai}^2(x) \) we write

\[ w_2(x) = \frac{1}{2}[\text{Ai}^2(x) - \text{Bi}^2(x)] \sim \frac{\pi}{2} \sum_{n=0}^\infty \frac{g_{2n+1}}{(-x)^{3n+2}} \sin 2\zeta + \frac{g_{2n+1}}{(-x)^{3n+2}} \cos 2\zeta, \quad \zeta = \frac{2}{3} (-x)^{3/2}. \]  

Substitution in (39) gives the recurrence relation

\[ g_n = \frac{(3n-5)(3n-3)(3n-1)}{2^5 \cdot 3 \cdot n} g_{n-2} + (-1)^n \frac{27n^2 - 27n + 5}{2^3 \cdot 3 \cdot n} g_{n-1}. \]  

The initial \( g_0 = 1 \) and \( g_1 = -\frac{5}{2^2 \cdot 3} \) are easily obtainable from (34)-(36). The rest are determined from (43):

\[ g_2 = -\frac{5 \cdot 41}{2^7 \cdot 3^2}, \quad g_3 = \frac{5 \cdot 7 \cdot 11 \cdot 59}{2^{10} \cdot 3^4}, \quad g_4 = \frac{5 \cdot 7 \cdot 11 \cdot 12769}{2^{15} \cdot 3^5}, \quad \ldots. \]  

4 Stokes’ phenomenon and the choice of saddle points

In this Section using as an example the function \( w(z) \), defined by the contour integral (45), we consider the Stokes’ phenomenon, that is an abrupt appearance or disappearance of component series of a complete asymptotic expansion at certain phases of \( z \). The complete asymptotic expansion consists of several series; the number of these component series is different in different sectors of the complex variable \( z \). Each series corresponds to a contribution from a single saddle point and determines its own function. So in different sectors of the complex plane \( z \) the asymptotic expansion is determined by different number of saddle points. The problem of choosing of the saddle points, determining an asymptotic expansion, is connected with the topology of steepest decent lines and the disposition of ends of the integration path in the integral representation for the function. For this reasons we draw the integration path along the steepest descent lines and will watch over its deformation with the change of \( z \) in the complex plane. For definiteness we consider the solution of the Airy equation \( w'' = zw \) given by the integral

\[ w(z) = i \int_C dt e^{-i(zt^3/3)}, \]  

\[ 8 \]
where the contour $C$ goes from infinity in the sector $\pi/3 < \text{ph } t < 2\pi/3$ to infinity in the sector $-\pi/3 < \text{ph } t < 0$ (sectors 1 and 3 in fig.1-6).

We note that $w(z)$ can be represented by the sum of two integrals over imaginary and real positive half-axes:

$$w(z) = g(z) + f(z), \quad g(z) = i \int_{\infty}^{0} dt \ e^{-i(zt + \frac{t^3}{3})}, \quad f(z) = i \int_{0}^{\infty} dt \ e^{-i(zt + \frac{t^3}{3})}. \quad (46)$$

By substitution $t = i\tau$ the integral $g(z)$ is reduced to $\text{Hi}(z)$. The integral $f(z) = \text{Gi}(z) + i\text{Ai}(z)$. Then from (11) it follows

$$w(z) = \text{Hi}(z) + \text{Gi}(z) + i\text{Ai}(z) = \text{Bi}(z) + i\text{Ai}(z). \quad (47)$$

Introducing the modulus and phase of the parameter $z = |z|e^{i\phi}$ and the integration variable $t = |t|e^{i\theta}$, we write the exponent in (45) in the form

$$-i(zt + \frac{t^3}{3}) = |t||z| \sin(\phi + \theta) + \frac{1}{3}|t|^2 \sin 3\theta - i|t||z| \cos(\phi + \theta) + \frac{1}{3}|t|^2 \cos 3\theta. \quad (48)$$

The steepest decent lines are determined by the constancy of imaginary part of (48), that is by the condition

$$-|t||z| \cos(\phi + \theta) + \frac{1}{3}|t|^2 \cos 3\theta = \text{Const}. \quad (49)$$

The saddle points $t_{1,2} = |z|^{1/2} \exp[i(\phi + \pi)/2]$. For the steepest decent lines, passing over the saddle points $t_1, t_2$, the values of the constants on the right-hand side of (49) differ in sign:

$$\text{Const} = \pm \frac{2}{3}|z|^{3/2} \sin \frac{3\phi}{2}. \quad (50)$$

Therefore the steepest decent lines from the passes $t_1, t_2$ meet and coincide below the lower pass only in exceptional cases, when the constant (50) becomes zero, i.e. when phase of the parameter $z$ is equal to

$$\phi = 0, \pm \frac{2\pi}{3}, \pm \frac{4\pi}{3}, \ldots. \quad (51)$$

The lines of these exceptional values in the complex plane of the parameter $z$ are called the Stokes rays. Where go the ends of steepest descent lines for $|t| \to \infty$? From (49) it follows

$$|z| \cos(\phi + \theta) + \frac{1}{3}|t|^2 \cos 3\theta \to 0, \quad |t| \to \infty, \quad (52)$$

i.e.

$$\theta \to \theta_{\infty} = \pm \frac{\pi}{6}(2k + 1), \quad k = 0, 1, 2, \ldots.$$ Of these directions those ones, for which the real part of (48) goes to $-\infty$ for $|t| \to \infty$ (i.e. $\sin \theta < 0$), just correspond to descent and not to ascent. But

$$\sin[\pm \frac{\pi}{2}(2k + 1)] = \pm (-1)^k$$
is negative at odd \( k \) for the upper sign and at even \( k \) for the lower sign. Hence

\[
\theta_\infty = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \ldots.
\]  

(53)

In fig. 1-6 in the complex \( t \)-plane the locations of saddle points \( t_1, t_2 \) and the behavior of steepest descent lines from the passes are qualitatively indicated for the specific values of phase of the parameter \( z \), which changes in its own complex plane, see fig. 7. The arrows mark off the integration path in (45) going from the sector \( \pi/3 < \phi t < 2\pi/3 \) into the sector \( -\pi/3 < \phi t < 0 \) along the steepest descent lines. When the parameter \( z \) happens to be on a Stokes ray, the steepest descent lines, passing over two saddle points, meet one another at the lower saddle point. The integration path going over this saddle point either retains its direction or abruptly changes it by \( \pi/2 \).

In the latter case the integration path passes over both saddle points; the lower one (situated at the break) appears at the integration path when the parameter \( z \) crosses the Stokes ray. This is a manifestation of the Stokes’phenomenon: when \( z \) crosses the Stokes ray the integration path begins or stops to pass over the second (the lower) saddle point.

At the absence of break on the integration contour it contains only one (lower) saddle point. In this case both before and after \( z \) crosses the Stokes ray, the integration contour goes over only one (the lower) saddle point and Stokes phenomenon does not occur.

In other words, the steepest descent lines, going over different saddle points, generally do not meet, but in one of the sectors their ends asymptotically approach each other. When the parameter \( z \) approaches the Stokes ray, these ends come closer and closer at a greater length. For \( z \) at the Stokes ray, they merge up to the lower saddle point; in this case the steepest descent line from the higher pass (active line) approaches the steepest descent line from the lower pass (passive line) at the angle of \( \pi/2 \). When \( z \) goes beyond the Stokes ray, the active line breaks away from the half of the passive line and goes to infinity along the other half of the line. So, when \( z \) crosses the Stokes ray, the passive line changes limply, but the active one drastically reverses the direction of its approach to the end of the passive line in order to approach the other end of that line. If the integration path for \( z \) near the Stokes ray contains the active line, then at \( z \) crossing the Stokes ray the passive line is included in (see fig. 6, 1, 2) or excluded from (see fig. 2, 3, 4) the integration path, bringing in it or out of it the lower saddle point. This is the Stokes’phenomenon.

If the integration path for \( z \) near the Stokes ray contains only the passive line of descent (see fig. 4, 5, 6), then at \( z \) crossing the Stokes ray, the active line (with its higher pass) is not contained in the integration path, which, as before, goes only over the lower pass.

So for \( z \) outside the Stokes ray, the integration path contains one or two lines of steepest descent with the ends going to infinity. Asymptotic expansion for the integral over each such line is obtained in usual manner by the saddle point method. For this reason, in order to get the asymptotic expansion for the considered integral, it is convenient to start from one of the expressions

\[
w(z) = w_{13}(z),
\]

(54)

\[
w(z) = w_{12}(z) + w_{23}(z),
\]

(55)

where \( w_{ij} \) are defined by the same representation (45) as \( w(z) \) but with integration path \( C_{ij} \) beginning in the \( i- \) and ending in the \( j- \)sector at infinity. These expressions hold for any \( z \),
but only one of them becomes suitable for the asymptotic expansion of the initial function in that sector of the complex $z$–plane for which each of the contours $C_{ij}$ can be drawn along the steepest descent line over the single saddle point.

As seen from fig. 1-6 for the sector $0 < \phi h < 2\pi/3$ the contours $C_{12}$, $C_{23}$ can be drawn along the steepest descent lines over the passes $t_2$, $t_1$. So in this sector the formula (55) is suitable. For the sector $2\pi/3 < \phi h < 2\pi$ or $-4\pi/3 < \phi h < 0$ this expression is unsuitable as neither $C_{13}$, nor $C_{23}$ can be drawn along the steepest descent line over a single saddle point. However, for $z$ in this sector the contour $C_{13}$ can be drawn along the steepest descent line over the saddle point $t_1$. So here the expression (54) is suitable.

If $z$ is on the Stokes ray and the integration contour, drawn along the steepest descent lines, does not suffer a break at the lower pass, then such a contour goes over the one, lower saddle point only, see fig. 5, and for the asymptotic representation of integral $w(z)$ the formula (54) is suitable.

If $z$ is on the Stokes ray and the integration contour, drawn along the steepest descent lines, suffers a break at the lower saddle point (and therefore passes the higher saddle point, see fig. 1 or 3), then it is difficult to obtain the asymptotic expansion for the integral over this path. Yet this integral can be regarded as a limit of the half-sum of integrals with parameters $z_1$, $z_2$ lying on different sides of the Stokes ray and tending to $z$. Hence, for the asymptotic expansion we can use the formula

$$w(z) = \frac{1}{2} [w_{12} + w_{23} + w_{13}]. \quad (56)$$

For $z$ in the sector $-4\pi/3 < \phi h < 0$ and $|z| \gg 1$, using (54) and the saddle point method, we represent $w(z)$ by the asymptotic series

$$w(z) = w_{13}(z) = W_n^{(13)}(z) + R_n^{(13)}(z) \sim W_n^{(13)}(z), \quad (57)$$

where $W_n^{(13)}(z)$ is the sum of the first $n$ terms of the asymptotic series $W_n^{(13)}(z)$, and $R_n^{(13)}(z)$ is the remainder term, small in comparison with $W_n^{(13)}(z) \equiv 2S_n^{(2)}(z)$. The explicit terms of $S_n^{(2)}(z)$ and $S_n^{(1)}(z)$, connected by the relation $S_n^{(2)}(z) = \pm i S_n^{(1)}(ze^{\pm 2\pi i})$, are given in equations (68), (66).

Similarly for $z$ in the sector $0 < \phi h < 2\pi/3$ and $|z| \gg 1$ the function $w(z)$ can be represented by two asymptotic series

$$w(z) = w_{12}(z) + w_{23}(z) = \quad \frac{1}{2} [w_{12} + w_{23} + w_{13}]. \quad (58)$$

forming together with (57) the complete asymptotic expansion.

Near the Stokes ray $\phi h = 0$ the series $W_n^{(12)}(z)$, representing the contribution from the higher pass, is dominant and the series $W_n^{(23)}(z)$, representing the contribution from the lower pass, is recessive. On the ray $\phi h = \pi/3$ their role in the representation for $w(z)$ is equal and near the Stokes ray $\phi h = 2\pi/3$ the series $W_n^{(23)}(z)$ becomes dominant and the series $W_n^{(12)}(z)$ recessive.

Near the Stokes ray $\phi h = 0$ the series $W_n^{(13)}(z)$ in the representation (57) and the dominant series $W_n^{(12)}(z)$ in (58) represent the contribution from the higher pass.
they differ only in their remainder terms, i.e. near the Stokes ray their $n$–th partial sums are equal:

$$W_n^{(13)}(z) = W_n^{(12)}(z) \equiv 2S_n^{(2)}(z),$$  

but the remainder terms suffer a jump

$$R_n^{(13)}(z) - R_n^{(12)}(z) = w_{23}(z), \quad \text{ph} \ z = 0,$$

equal to the analytic function $w_{23}(z)$. As this one is recessive, its asymptotic representation and the remainder term does not suffer any jump at this Stokes ray, cf. with the asymptotic behavior of $w_{13}(z)$ at the Stokes ray $\text{ph} \ z = -2\pi/3$, eq. (57).

So on the Stokes ray $\text{ph} \ z = 0$ the asymptotic representation for $w(z)$ is given by the $n$–th partial sum (59) of the dominant series and the remainder term

$$R_n^{(13)}(z) = R_n^{(12)}(z) + w_{23}(z) = \frac{1}{2}[R_n^{(12)}(z) + R_n^{(13)}(z) + w_{23}(z)].$$

All three representations (61), which are the limits of representations (57) and (58) from each side of the Stokes ray and their half-sum, are equivalent.

Near the Stokes ray $\text{ph} \ z = 2\pi/3$, due to dominance of the higher pass, the $n$–th partial sums $W_n^{(23)}(z)$ and $W_n^{(13)}(z)e^{-2\pi i}$ are equal (the branching of the sums at $z = 0$ is taken into account):

$$W_n^{(13)}(z)e^{-2\pi i} = W_n^{(23)}(z) = 2S_n^{(2)}(z)e^{-2\pi i} = 2iS_n^{(1)}(z),$$

and the remainder terms suffer a jump at this ray

$$R_n^{(13)}(z)e^{-2\pi i} - R_n^{(23)}(z) = w_{12}(z), \quad \text{ph} \ z = \frac{2\pi}{3}.$$

On the Stokes ray $\text{ph} \ z = 2\pi/3$ the asymptotic expansion for $w(z)$ is given by the $n$-th partial sum (62) of the dominant series and the remainder

$$R_n^{(13)}(z)e^{-2\pi i} = R_n^{(23)}(z) + w_{12}(z) = \frac{1}{2}[R_n^{(13)}(z)e^{-2\pi i} + R_n^{(23)}(z) + w_{12}(z)].$$

Note that the functions $w_{ij}(z)$ can be turned one into another by rotation of the integration path over the angle of $\pm 2\pi/3$:

$$w_{13}(z) = -e^{2\pi i/3}w_{23}(ze^{2\pi i/3}), \quad w_{12}(z) = e^{-2\pi i/3}w_{23}(ze^{-2\pi i/3}).$$

The function $w_{23}$ up to a factor coincides with the Airy function: $w_{23}(z) = 2i\text{Ai}(z)$. As seen from fig. 5, 6, 1, 2, 3 for $z$ in the sector $-2\pi/3 < \text{ph} \ z < 2\pi/3$, the integration path $C_{23}$, drawn along the steepest descent line, goes through one saddle point. So for $|z| \gg 1$ the Airy function is represented by one asymptotic series, the recessed one near the Stokes ray $\text{ph} \ z = 0$, cf. (57). By the saddle point method we get for this series

$$\text{Ai}(z) = S_n^{(1)}(z) + R_n(z), \quad |\text{ph} \ z| \leq \frac{2\pi}{3},$$

$$S_n^{(1)}(z) = \frac{\pi^{1/2}}{2z^{1/4}}e^{-\zeta} \sum_{k=0}^{n-1} c_k (-\zeta)^{-k}, \quad \zeta = \frac{2}{3}z^{3/2}.$$
For \(|\text{ph} \ z| \leq 2\pi/3\) the remainder term \(R_n(z)\) is small in comparison with at least first few terms of \(S_n^{(1)}(z)\). Outside this sector the value of the representation (66) diminishes as \(z\) approaches the negative half-axis because \(R_n(z)\) increases. In order to get better representation for the complementary sector \(2\pi/3 < \text{ph} \ z \leq 4\pi/3\), we use the relation

\[
\text{Ai}(z) = -e^{-2\pi i/3} \text{Ai}(ze^{-2\pi i/3}) - e^{2\pi i/3} \text{Ai}(ze^{-4\pi i/3}),
\]

(67)

following from (55) and (65). For variable \(z\) in the complementary sector the arguments of the Airy functions in the right-hand side of (67) does not leave the main sector, where (66) holds. It is easy to verify that

\[
-e^{-2\pi i/3} \text{Ai}(ze^{-2\pi i/3}) = iS_n^{(2)}(z) - e^{-2\pi i/3} R_n(ze^{-2\pi i/3}),
\]

(68)

and

\[
-e^{2\pi i/3} \text{Ai}(ze^{-4\pi i/3}) = S_n^{(1)}(z) - e^{2\pi i/3} R_n(ze^{-4\pi i/3}).
\]

(69)

Hence

\[
\text{Ai}(z) = S_n^{(1)}(z) + S_n^{(1)}(ze^{-2\pi i}) - e^{-2\pi i/3} R_n(ze^{-2\pi i/3}) - e^{2\pi i/3} R_n(ze^{-4\pi i/3}),
\]

(70)

As seen from here, the remainder, represented by the last two terms in the right-hand side of (70), does not exceed the double maximum value of \(R_n(z)\) in the main sector.

Similarly, we find

\[
w(z) = -2ie^{2\pi i/3} \text{Ai}(ze^{2\pi i/3}) = 2S_n^{(2)}(z) + 2e^{i\pi/6} R_n(ze^{2\pi i/3}), \quad -4\pi/3 \leq \text{ph} \ z < 0,
\]

(71)

\[
w(z) = 2S_n^{(2)}(z) + 2S_n^{(2)}(ze^{-2\pi i}) + 2iR_n(z) + 2e^{-i\pi/6} R_n(ze^{-2\pi i/3}), \quad 0 \leq \text{ph} \ z < 2\pi/3.
\]

(72)

Asymptotic representations for \(\text{Bi}(z)\) follow from the given relations with account of equality \(\text{Bi}(z) = w(z) - i\text{Ai}(z)\).

As seen from (70) and (72), the asymptotic representation in the complementary sector is the sum of two analytical continuations of the asymptotic series from the main sector, continuations corresponding to the right and the left detours of the branch point \(z = 0\). Thereby the asymptotic representation does not depend on the position of branch cut.

5 Asymptotic behavior of \(\int_v^\infty dx \text{Ai}(x) h(x, v)\) for \(v \to -\infty\)

The probabilities of many processes with elementary particles in a constant external field are reduced to the integral \([1, 2, 7]\)

\[
\int_v^\infty dx \text{Ai}(x) h(x, v).
\]

(73)
For the processes taking place also in the absence of a field, the parameter \( v \) is negative and \( v \to -\infty \) for the field tending to zero. Let us consider the asymptotic behavior of the integral (73) for \( v \to -\infty \). It is assumed that the function \( h(x, v) \) has no singularity in the integration interval and can be expanded at \( x = v \) in the series

\[
h(x, v) = \sum_{k=-1}^\infty h_k (x - v)^{k/2} = \sum_{k=-1}^\infty f_k \left( \frac{x - v}{-v} \right)^{k/2},
\]

(74)
in which the coefficients \( f_k \) weakly depend on \( v \),

\[
f_k \equiv h_k \cdot (-v)^{k/2} \sim f_0.
\]

(75)

In the following we omit for brevity the second argument of the function \( h(x, v) \). Then we can write

\[
\int_v^\infty dx \, \text{Ai}(x)h(x) = h(0)\text{Ai}_1(v) + \int_v^\infty dx \, x\text{Ai}(x)\tilde{h}(x),
\]

\[
\tilde{h}(x) = \frac{h(x) - h(0)}{x} = \sum_{k=-1}^\infty \tilde{h}_k (x - v)^{k/2} = \sum_{k=-1}^\infty \tilde{f}_k \left( \frac{x - v}{-v} \right)^{k/2}.
\]

(76)
The function \( \tilde{h}(x) \) has the property of \( h(x) \): it is finite at \( x = 0 \) and behaves as \( \tilde{f}_{-1} \left( \frac{x-v}{-v} \right)^{-1/2} \) near \( x = v \).

Introducing the function

\[
H(x) = \tilde{h}(x) - 2 \sum_{k=-1}^2 \tilde{f}_k \left( \frac{x - v}{-v} \right)^{k/2} = \sum_{k=3}^\infty \tilde{f}_k \left( \frac{x - v}{-v} \right)^{k/2},
\]

(77)
vanishing at \( x = v \) not weaker than \( (x - v)^{3/2} \), and using equation \( x\text{Ai}(x) = \text{Ai}''(x) \), let us transform the second term in (76) by integration by parts:

\[
\int_v^\infty dx \, x\text{Ai}(x)\tilde{h}(x) = \tilde{f}_{-1}\sqrt{-v} \frac{d^2}{dv^2} [2^{2/3}\text{Ai}^2(t)] \bigg|_{t=2-2/3v} - \tilde{f}_0 \text{Ai}'(v) - \frac{\tilde{f}_1}{2\sqrt{-v}} \frac{d}{dv} [2^{2/3}\text{Ai}^2(t)] - \frac{\tilde{f}_2}{v} \text{Ai}(v) + \int_v^\infty dx \, \text{Ai}(x)H''(x), \quad t = 2^{-2/3v}.
\]

(78)
Here we have used the equation [1]

\[
\int_v^\infty \frac{dx \, \text{Ai}(x)}{\sqrt{x - v}} = \int_v^\infty \frac{dx \, \text{Ai}(x + v)}{\sqrt{x}} = 2^{2/3}\text{Ai}^2(t), \quad t = 2^{-2/3v}.
\]

(79)
It can be shown that the coefficients \( \tilde{f}_k \) are connected with \( f_k \) in (74) by formulae

\[
\tilde{f}_{2k-1} = \frac{1}{v} \sum_{s=0}^k f_{2s-1}, \quad \tilde{f}_{2k} = \frac{1}{v} \sum_{s=0}^k f_{2s} - \frac{1}{v} h(0).
\]

(80)
Using (80), we obtain from (76) and (78)

\[
\int_v^\infty dx \, \text{Ai}(x)h(x) = h(0)\pi - 2 \int_{-\infty}^v \frac{dx \, \text{Ai}(x)}{x^3} + f_{-1} \left[ \frac{\text{Ai}(t)\text{Ai}'(t)}{(-v)^{3/2}} - \frac{t\text{Ai}^2(t) + \text{Ai}'^2(t)}{\sqrt{-t}} \right] - \frac{\tilde{f}_1}{2\sqrt{-v}} \frac{d}{dv} [2^{2/3}\text{Ai}^2(t)] - \frac{\tilde{f}_2}{v} \text{Ai}(v) + \int_v^\infty dx \, \text{Ai}(x)H''(x), \quad t = 2^{-2/3v}.
\]
\[
f_0\left[\frac{\text{Ai}(v)}{v^2} + \frac{\text{Ai}'(v)}{v}\right] + f_1 \frac{\text{Ai}(t)\text{Ai}'(t)}{(-v)^{3/2}} - f_2 \frac{\text{Ai}(v)}{v^2} + \int_v^\infty dx \text{Ai}(x)h_1(x), \quad t = 2^{-2/3}v. \tag{81}
\]

Here \(h_1(x) \equiv H''(x)\) and in the first term the equation (28) was used. The function \(h_1(x)\) near \(x = v\) and \(x = 0\) has the properties of \(h(x)\), but is \((-v)^3\) times less than the last one:

\[
h_1(x) \sim (-v)^{-3}h(x). \tag{82}
\]

Applying to the last integral on the right-hand side of (81) the same procedure as to the function:

\[
h(x) = \sum_{k=0}^\infty \{h_k(0)[\pi - 2 \int_{-\infty}^v \frac{dx}{x^3}\text{Ai}(x)] + \int_{-\infty}^\infty \frac{dx}{x^3}\text{Ai}(x)\} - \int_v^\infty dx \text{Ai}(x)h_1(x) - f_k, 1 \frac{\text{Ai}(t)\text{Ai}'(t)}{(-v)^{3/2}} - f_{k, 2} \text{Ai}(v), \quad t = 2^{-2/3}v. \tag{83}
\]

Here, similarly to (74) we have used the notation

\[
h_k(x, v) \equiv h_k(x) = \sum_{m=-1}^\infty h_k(m)(x - v)^m = \sum_{m=-1}^\infty f_{k, m} \left(\frac{x - v}{-v}\right)^{m/2}; \tag{83'}
\]

\(h_k(x)\) is obtained from \(h_{k-1}(x)\) in the same manner as \(h_1(x)\) from \(h_0(x) \equiv h(x) \equiv h(x, v)\).

It is easy to obtain a connection of the coefficients \(f_{k, m}\) with the coefficients of the previous function:

\[
f_{k, 2m-1} = \frac{(m + \frac{1}{2})(m + \frac{3}{2})}{v^3} \sum_{s=0}^{m+2} f_{k-1, 2s-1},
\]

\[
f_{k, 2m} = \frac{(m + 1)(m + 2)}{v^3} \left[\sum_{s=0}^{m+2} f_{k-1, 2s} - h_{k-1}(0)\right]. \tag{84}
\]

With the help of these equations the odd coefficients \(f_{k, 2m-1}\) can be expressed via odd coefficients \(f_{2s-1}\) of the initial function, and the even coefficients \(f_{k, 2m}\) - via even coefficients \(f_{2s}\) of function \(h(x)\) and via values of functions \(h_n(x), n < k, at x = 0\).

As the connection of \(f_{k, 2m}\) with \(f_{2s}\) and \(h_n(0)\) is linear, let us first find the dependence of \(f_{k, 2m}\) on \(h_n(0)\) putting at first \(f_{2s} = 0\). So, using the second equation in (84) first for \(k = 1\), then for \(k = 2\), we express \(f_{2, 2m}\) through \(h(0)\) and \(h_1(0)\). Continuing this process, we find

\[
f_{k, 2m} = -\frac{(m + 2)!}{m!v^3} \left\{h_{k-1}(0) + \sum_{n=0}^{m+2} \frac{(n + 2)!h_{k-2}(0)}{n!v^3} + \sum_{n=0}^{m+2} \frac{(n + 2)!h_{k-2}(0)}{n!v^3} + \sum_{l=0}^{n+2} \frac{(l + 2)!h_{k-3}(0)}{l!v^6} + \ldots \right\}
\]

\[
+ \sum_{n_k-1=0}^{m+2} \frac{(n_k-1) + 2)!}{n_k-1!} \ldots \sum_{n_k-2=0}^{m+2} \frac{(n_k-2) + 2)!}{n_k-2!} \sum_{n_1=0}^{n_k-2} \frac{(n_1 + 2)!h_0(0)}{n_1!v^3-3} \right\} = -\frac{1}{v^3} \sum_{s=0}^{k-1} Q_{m, s} h_{k-1-s}(0), \tag{85}
\]

\[
Q_{m, s} = \frac{(m + 2)!}{m!} \sum_{n_s=0}^{m+2} \frac{(n_s + 2)!}{n_s!} \ldots \sum_{n_2=0}^{m+2} \frac{(n_2 + 2)!}{n_2!} \sum_{n_1=0}^{n_2+2} \frac{(n_1 + 2)!}{n_1!} = \frac{(m + 2 + 3s)!}{(3s)!!m!}. \tag{86}
\]
The final formula for $Q_{m,s}$ is easily proved by the use of known formula

$$\sum_{n=0}^{m} \frac{(n+p)!}{n!} = \frac{(m+p+1)!}{m!(p+1)}$$

(87)

and the induction method. By definition, $(3n)!!! = 3 \cdot 6 \cdot 9 \cdot \ldots \cdot (3n)$, $0!!! = 1$. Thus

$$\sum_{k=0}^{\infty} f_{k,2m} = -\frac{1}{v^3 \cdot m!} \sum_{s=0}^{\infty} \frac{(3s+m+2)!}{(3s)!!!v^{3s}} \sum_{n=0}^{\infty} h_n(0).$$

(88)

Then the combination of terms with even coefficients $f_{k,2m}$ in formula (83) transforms into the following expression

$$-\sum_{k=0}^{\infty} f_{k,0} \left[ \frac{\text{Ai}(v)}{v^2} + \frac{\text{Ai}'(v)}{v} \right] - \sum_{k=0}^{\infty} f_{k,2} \frac{\text{Ai}(v)}{v^2} =$$

$$= \sum_{n=0}^{\infty} h_n(0) \left\{ \frac{\text{Ai}'(v)}{v^4} \sum_{s=0}^{\infty} \frac{(3s+2)!}{(3s)!!!v^{3s}} + \frac{\text{Ai}(v)}{v^5} \sum_{s=0}^{\infty} \frac{(3s+4)!}{(3s+3)!!!v^{3s}} \right\}. \tag{89}$$

Using the relation $\text{Ai}(x) = x\text{Ai}''(x)$ and twice integrating by parts, we find

$$\int_{-\infty}^{v} \frac{dx}{x^n} \text{Ai}(x) = \frac{\text{Ai}(x)}{v^n+1} + (n+1) \frac{\text{Ai}(x)}{v^{n+1}} + (n+1)(n+2) \int_{-\infty}^{v} \frac{dx}{x^{n+3}} \text{Ai}(x).$$

Repeatedly using this equation, we obtain

$$2 \int_{-\infty}^{v} \frac{dx}{x^3} \text{Ai}(x) = \frac{\text{Ai}'(v)}{v^4} \sum_{s=0}^{n} \frac{(3s+2)!}{(3s)!!!v^{3s}} + \frac{\text{Ai}(v)}{v^5} \sum_{s=0}^{n} \frac{(3s+4)!}{(3s+3)!!!v^{3s}} + \frac{(3n+5)!}{(3n+3)!!!} \int_{-\infty}^{v} \frac{dx}{x^{3n+6}} \text{Ai}(x). \tag{90}$$

Thus the expression in braces in (89) is the asymptotic representation of integral (90). So for the case $f_{2s} = 0$, i.e. when only odd $k$ are present in (74), the expression (83) becomes simpler ($t = 2^{-2/3}v$):

$$\int_{v}^{\infty} \frac{dx}{x^n} \text{Ai}(x) \text{h}(x) = \sum_{k=0}^{\infty} \left\{ \pi h_k(0) - \frac{t \text{Ai}'(t)}{\sqrt{-t}} \frac{\text{Ai}^2(t)}{(v^2)^{3/2}} (f_{k,-1} + f_{k,1}) \right\}. \tag{91}$$

From the expression

$$h_{k+1}(x) = \left[ \frac{h_k(x) - h_k(0)}{x} \right]^n - \frac{3}{4v^3} f_{k,-1} \left( \frac{x-v}{-v} \right)^{-5/2} + \frac{1}{4v^3} (f_{k,-1} + f_{k,1}) \left( \frac{x-v}{-v} \right)^{-3/2}, \tag{92}$$

connecting the function $h_{k+1}(x)$ with the previous one, it follows

$$h_{k+1}(0) = \frac{1}{3} h_k^{(3)}(0) - \frac{3}{4v^3} f_{k,-1} + \frac{1}{4v^3} (f_{k,-1} + f_{k,1}). \tag{93}$$
Substituting $k + 1$ by $k$ in (92) and differentiating three times the obtained $h_k(x)$, we find for $x = 0$:

$$h_k^{(3)}(0) = \frac{1}{6} h_k^{(6)}(0) - \frac{3}{4v^3} f_{k-1} \cdot \left( \frac{5}{2}, \frac{7}{2}, \frac{9}{2} \right) + \frac{f_{k-1} + f_{k-1,1}}{4v^6} \frac{3}{2}, \frac{5}{2}, \frac{7}{2}. \tag{94}$$

Continuing to use formula (92) in such a way, it can be shown that

$$h_{k+1}(0) = \frac{1}{(3k + 3)!!!} h_{(3k+3)}(0) + \sum_{n=0}^{k} \frac{1}{(3n)!!!} \left[ -\frac{3}{4v^{3n+3}} f_{k-n-1} \cdot \left( \frac{5}{2}, \frac{7}{2}, \frac{9}{2} \right) + \frac{f_{k-n-1} + f_{k-n,1}}{4v^{3n+3}} \left( \frac{3}{2} \right)_{3n} \right], \tag{95}$$

where the products of half-integer numbers are written in terms of Pochhammer’s symbol

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{96}$$

Then

$$\sum_{k=0}^{\infty} h_k(0) = \sum_{k=0}^{\infty} \left( \frac{h_{(3k)}(0)}{(3k)!!!} - \frac{3}{4v^3} f_{k-1} \cdot \phi_{5/2}(v) + \frac{f_{k-1} + f_{k,1}}{4v^3} \phi_{3/2}(v) \right), \tag{97}$$

Thus the asymptotic representation of the initial integral takes the form

$$\frac{1}{\pi} \int_{v}^{\infty} dx \, \text{Ai}(x) h(x) = \sum_{k=0}^{\infty} \left( \frac{h_{(3k)}(0)}{(3k)!!!} - \frac{t \text{Ai}^2(t) + \text{Ai}'^2(t)}{\pi \sqrt{-t}} + \frac{3}{4v^3} \phi_{5/2}(v) \right) f_{k-1} + \left[ \frac{\text{Ai}(t) \text{Ai}'(t)}{\pi (-v)^{3/2}} + \frac{\phi_{3/2}(v)}{4v^3} \right] (f_{k-1} + f_{k,1}), \quad t = 2^{-2/3} v, \quad (f_{2s} = 0). \tag{98}$$

It only remains to express the coefficients $f_{k,\pm 1}$ via the odd coefficients $f_{2s-1}$ of function $h(x)$. It can be done by repeated employment of the first formula in (84):

$$f_{k,2m-1} = \frac{(m+1/2)(m+3/2)}{v^{3k}} \sum_{n_{k-1}=0}^{m+2} (n_{k-1} + 1/2)(n_{k-1} + 3/2) \cdots \sum_{n_2=0}^{n_{2}+2} (n_2 + 1/2)(n_2 + 3/2) \sum_{n_1=0}^{n_{1}+2} (n_1 + 1/2)(n_1 + 3/2) \sum_{s=0}^{n_{1}+2} f_{2s-1}. \tag{99}$$

In this expression we reverse the order of summation first over $n_1$ and $s$, then over $n_2$ and $s$, and so on. Finally, we obtain

$$f_{k,2m-1} = \frac{(m+1/2)(m+3/2)}{v^{3k}} \sum_{s=0}^{m+2k} R_{2s-1}(k,m)f_{2s-1}, \tag{100}$$

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We note now that all terms of asymptotic representation (104), except the expression (85) for even coefficients $f_k$, the quadratic in $\phi h$ functions of $v$, and nonoscillatory parts, but the last are canceled with nonoscillatory functions $f, f_k, m = 1$. The formula (98) is correct if the expansion of $h(x) \equiv h(x, v)$ at $x = v$ contains only the odd coefficients $f_k$, see (74). Just such functions were considered in paper [2], where the representation (98) was obtained. For $f_{2s} \neq 0$ it follows from the lower formula (84) that to the expression (85) for even coefficients $f_k, m$, the sum

$$f_{k, 2m}^+ = \frac{(m + 1)(m + 2)}{\nu^{3k}} \sum_{s=0}^{m+2k} T_{2s}(k, m) f_{2s},$$

analogous to the expression (100) for odd coefficients $f_{k, 2m-1}$, is added. In this sum

$$T_{2s}(k, m) = \sum_{n_{k-1}=n_{k-1}(s)}^{m+2} (n_{k-1} + 1)(n_{k-1} + 2) \cdot \sum_{n_{k-1}=n_{k-1}(s)}^{m+2} (n_2 + 1)(n_2 + 2) \cdot \sum_{n_1=n_1(s)}^{m+2} (n_1 + 1)(n_1 + 2),$$

and the $n_i(s)$ are the same as in (101). As a result in asymptotic representation of integral (83), the terms linear in $\Ai(v), \Ai'(v)$ remain:

$$\int_{v}^{\infty} dx \Ai(x)h(x) = \sum_{k=0}^{\infty} h^{(3k)}(0) \cdot \frac{t\Ai^2(t) + \Ai'^2(t)}{\pi\sqrt{t}} + \frac{3}{4\nu^3} \cdot \text{oscillation terms},$$

$$\left[ \frac{\Ai(t)\Ai'(t)}{\pi(-v)^{3/2}} + \frac{\phi_{3/2}(v)}{4\nu^3} \right] (f_{k-1} - f_{k,1}) - \frac{\Ai'(v)}{\nu v} f_{k,0}^+ - \frac{\Ai(v)}{\nu v^2} (f_{k,0}^+ + f_{k,2}^+), \quad t = 2^{-2/3} v. \quad (104)$$

We remind that $\phi_a(v), a = 3/2, 5/2$, is given in (97) and also that $h(x) \equiv h(x, v)$ and $h^{(3k)}(0) = \left[ 3k h(x, v) / (dx)^{3k} \right]_{x=0}$ is a function of $v$.

6 Comments to the formula (104)

We note now that all terms of asymptotic representation (104), except $h^{(3k)}(0)$, are oscillatory functions of $v$. Indeed, the linear in $\Ai(v), \Ai'(v)$ terms for $v \ll -1$ are pure oscillatory according to (34). The quadratic in $\Ai(v), \Ai'(v)$ terms for $v \ll -1$ contain both oscillatory and nonoscillatory parts, but the last are canceled with nonoscillatory functions $\phi_a(v)$.

Really, as it was shown in Section 3, $\Ai^2(t)$ is the sum

$$\Ai^2(t) = w_1(t) + w_2(t)$$

(105)
of nonoscillatory \( w_1 \) and oscillatory \( w_2 \) functions, the asymptotic series of which are given by formulas (41),(42). Differentiating (105), we obtain

\[
\text{Ai}(t)\text{Ai}'(t) = \frac{1}{2}w'_1(t) + \frac{1}{2}w'_2(t),
\]

(106)

\[
t\text{Ai}^2(t) + \text{Ai}'^2(t) = \frac{1}{2}w''_1(t) + \frac{1}{2}w''_2(t),
\]

(107)

It follows from (41) and (97) that

\[
\frac{w'_1(t)}{2\pi(-v)^{3/2}} = -\frac{\phi_{3/2}(v)}{4v^3}, \quad \frac{w''_1(t)}{2\pi(-v)^{3/2}} = -\frac{3\phi_{5/2}(v)}{4v^3},
\]

(108)

so that combinations in square brackets in (104) are pure oscillatory functions:

\[
\frac{\text{Ai}(t)\text{Ai}'(t)}{\pi(-v)^{3/2}} + \frac{\phi_{3/2}(v)}{4v^3} = \frac{w'_1(t)}{2\pi(-v)^{3/2}} = \left[ -\frac{1}{2^{8} \cdot 3 \cdot v^3} - \frac{5 \cdot 7 \cdot 89}{2^{8} \cdot 3 \cdot v^3} + \cdots \right] \sin 2\zeta +
\]

(109)

\[
\frac{t\text{Ai}^2(t) + \text{Ai}'^2(t)}{\pi\sqrt{-t}} + \frac{3}{4v^3}\phi_{5/2}(v) = \frac{w''_1(t)}{2\pi\sqrt{-t}} =
\]

\[
\left[ -1 + \frac{1}{2^{5} \cdot 3 \cdot v^3} - \frac{5 \cdot 7 \cdot 19}{2^{11} \cdot 3 \cdot v^3} + \cdots \right] \sin 2\zeta +
\]

(110)

\[
\left[ -\frac{1}{2^{3} \cdot 3 \cdot (-v)^{3/2}} - \frac{5 \cdot 7 \cdot 19}{2^{7} \cdot 3 \cdot (-v)^{3/2}} + \cdots \right] \cos 2\zeta, \quad \zeta = \frac{2}{3}(-t)^{3/2} = \frac{1}{3}(-v)^{3/2}.
\]

Let us further simplify the coefficients \( R_{2s-1}(k, m) \) and \( T_{2s}(k, m) \) in formulas for \( f_{k,2s-1}^* \)
and \( f_{k,2s}^* \) at least for the lower values of \( k \), see (100-103). We begin with \( T_{2s}(k, m) \). Using (87), we obtain for the first sum in (103)

\[
\sum_{n_1=n_1(s)}^{n_2+2} (n_1 + 1)(n_1 + 2) = \sum_{n_1=0}^{n_2+2} \frac{(n_1 + 2)!}{n_1!} - \theta \left( s - \frac{5}{2} \right) \sum_{n_1=0}^{s-3} \frac{(n_1 + 2)!}{n_1!} =
\]

(111)

\[
= \frac{(n_2 + 5)!}{(n_2 + 2)!3} - \theta \left( s - \frac{5}{2} \right) \frac{s!}{(s-3)!3}.
\]

As \( s \) takes only integer positive values, the step function \( \theta \) may be omitted because \( (s-3)! \) turns into infinity at integer negative \( s - 3 \). Putting \( n_2 = m \), we have from (111) and (103)

\[
T_{2s}(2, m) = \frac{(m + 5)!}{(m + 2)!3} - \frac{s!}{(s-3)!3}.
\]

(112)

For the second sum in (103) we obtain on account of (111)

\[
\sum_{n_2=n_2(s)}^{n_3+2} (n_2 + 1)(n_2 + 2) \sum_{n_1=n_1(s)}^{n_2+2} (n_1 + 1)(n_1 + 2) =
\]

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Hence, from (103) and (113) it follows for \( k = 3 \)
\[
T_{2s}(3, m) = \frac{(m + 8)!}{3 \cdot 6 \cdot (m + 2)!} - \frac{s!(m + 5)!}{3^2 \cdot (s - 3)!(m + 2)!} + \frac{s!}{3 \cdot 6 \cdot (s - 6)!},
\]
(114)

Similarly,
\[
T_{2s}(4, m) = \frac{(m + 11)!}{3 \cdot 6 \cdot 9 \cdot (m + 2)!} - \frac{s!(m + 8)!}{3^2 \cdot 6 \cdot (s - 3)!(m + 2)!} + \frac{s!(m + 5)!}{3^2 \cdot 6 \cdot (s - 6)!(m + 2)!} - \frac{s!}{3 \cdot 6 \cdot 9 \cdot (s - 9)!},
\]
(115)

and so on.

With the help of these expressions it is easy to find several first coefficients \( f_{k,2m}^+ \) in (102),
\[
f_{1,0}^+ = 2v^{-3}(f_0 + f_2 + f_4),
\]
\[
f_{1,2}^+ = 6v^{-3}(f_0 + f_2 + f_4 + f_6),
\]
\[
f_{2,0}^+ = 4v^{-6}[10(f_0 + f_2 + f_4) + 9f_6 + 6f_8],
\]
\[
f_{2,2}^+ = 12v^{-6}[20(f_0 + f_2 + f_4) + 19f_6 + 16f_8 + 10f_{10}],
\]
\[
f_{3,0}^+ = 2^4 \cdot 5v^{-9}[2^2 \cdot 7(f_0 + f_2 + f_4) + 3^3f_6 + 2^3 \cdot 3f_8 + 2 \cdot 3^2f_{10} + 3^2f_{12}],
\]
\[
f_{3,2}^+ = 2^4 \cdot 3^3v^{-9}[2^2 \cdot 3 \cdot 7(f_0 + f_2 + f_4) + 2 \cdot 41f_6 + 2^2 \cdot 19f_8 + 2^6f_{10} + 3^2 \cdot 5f_{12} + 3 \cdot 7f_{14}].
\]
(116)

To calculate \( R_{2s-1}(k, m) \), we proceed similarly. Using the relation [8]
\[
\sum_{k=0}^{n} \frac{\Gamma(k + a)}{\Gamma(k + b)} = \frac{\Gamma(n + a + 1)}{(a - b + 1)\Gamma(n + b)} - \frac{\Gamma(a)}{(a - b + 1)\Gamma(b - 1)},
\]
(117)

we obtain for the first sum in (101)
\[
\sum_{n_1=n_1(s)}^{n_2+2} \left( n_1 + \frac{1}{2} \right) \left( n_1 + \frac{3}{2} \right) = \frac{\Gamma(n_2 + \frac{11}{2})}{3\Gamma(n_2 + \frac{5}{2})} + \frac{1}{8} - \theta \left( s - \frac{5}{2} \right) \left[ \frac{\Gamma(s + \frac{1}{2})}{3\Gamma(s - \frac{5}{2})} + \frac{1}{8} \right],
\]
(118)

Putting here \( n_2 = m \), we obtain \( R_{2s-1}(2, m) \). To calculate \( i \)-th sum, the relation
\[
\sum_{n_i=n_i(s)}^{n_i+1+2} \frac{\Gamma(n_i + \frac{5}{2} + 3l)}{\Gamma(n_i + \frac{1}{2})} = \frac{1}{3(l + 1)} \left[ \frac{\Gamma(n_i+1 + \frac{5}{2} + 3l + 3)}{\Gamma(n_i+1 + \frac{5}{2})} + \frac{(6l + 3)!!}{2^{3l+1}} \right] - \theta \left( s - \frac{1}{2} - 2i \right) \left[ \frac{\Gamma(s - 2i + \frac{5}{2} + 3l)}{\Gamma(s - 2i - \frac{1}{2})} + \frac{(6l + 3)!!}{2^{3l+1}} \right]
\]
(119)
is useful.
For the second sum in (101), we find with the help of (118), (119)

\[ \sum_{n_2=n_2(s)}^{n_3+2} \frac{\Gamma(n_2 + \frac{5}{2})}{\Gamma(n_2 + \frac{1}{2})} \sum_{n_1=n_1(s)}^{n_2+2} \frac{\Gamma(n_1 + \frac{5}{2})}{\Gamma(n_1 + \frac{1}{2})} = \frac{\Gamma(n_3 + \frac{11}{2})}{3 \cdot 6 \Gamma(n_3 + \frac{5}{2})} + \frac{\Gamma(n_3 + \frac{11}{2})}{3 \Gamma(n_3 + \frac{9}{2})} \cdot \frac{1}{8} \]

\[ \theta \left( s - \frac{5}{2} \right) \left[ \frac{\Gamma(s + \frac{1}{2})}{3 \Gamma(s - \frac{5}{2})} + \frac{9!!}{3 \cdot 6 \cdot 2^6} \right] + \frac{1}{2^6} \theta \left( s - \frac{5}{2} \right) \left[ \frac{\Gamma(s + \frac{1}{2})}{3 \cdot 8 \Gamma(s - \frac{5}{2})} + \frac{1}{2^6} \right] + \theta \left( s - \frac{9}{2} \right) \left[ \frac{\Gamma(s + \frac{1}{2})}{3 \cdot 6 \Gamma(s - \frac{11}{2})} + \frac{\Gamma(s + \frac{1}{2})}{3 \cdot 8 \Gamma(s - \frac{9}{2})} - \frac{9!!}{3 \cdot 6 \cdot 2^6} \right]. \quad (120) \]

At \( n_3 = m \) the right-hand side of (120) equals to \( R_{2s-1}(3, m) \). The first few coefficients \( f_{k,2s-1} \) in (100) are

\[ f_{0,-1} = f_{-1}, \quad f_{0,1} = f_{1}, \quad f_{1,-1} = \frac{3}{4v^3}(f_{-1} + f_{1} + f_{3}), \quad f_{1,1} = \frac{15}{4v^3}(f_{-1} + f_{1} + f_{3} + f_{5}), \]

\[ f_{2,-1} = \frac{3}{16v^6}[53(f_{-1} + f_{1} + f_{3}) + 50f_{5} + 35f_{7}], \]

\[ f_{2,1} = \frac{15}{16v^6}[116(f_{-1} + f_{1} + f_{3}) + 113f_{5} + 98f_{7} + 63f_{9}]. \quad (121) \]

7 \hspace{1em} \textbf{Asymptotic behavior of integrals related to the integral (73)}

In applications the asymptotic behavior for \( v \ll -1 \) of integrals obtained from (73) by the replacement of Airy function \( \text{Ai}(x) \) by the function \( \text{Ai}_1(x) \) or \( \text{Ai}'(x) \) are needed. In the first case, integrating by parts, we obtain

\[ \int_v^\infty dx \text{Ai}_1(x)h(x) = \int_v^\infty dx \text{Ai}(x)g(x), \quad h(x) \equiv h(x, v), \quad g(x) \equiv g(x, v), \quad (122) \]

\[ g(x) = \int_v^x dx h(x) = 2h_{-1}(x-v)^{1/2} + h_0(x-v) + \frac{2}{3}h_1(x-v)^{3/2} + \frac{1}{2}h_2(x-v)^2 + \cdots. \quad (123) \]

It is seen that \( g(x) \) has the same structure as \( h(x) \) in (74). Hence the above consideration is applied to the right-hand side of (122).

In the second case we have

\[ \int_v^\infty dx \text{Ai}'(x)[h_{-1}(x-v)^{-1/2} + \varphi(x)] = \]

\[ = 2h_{-1}\text{Ai}(t)\text{Ai}'(t) - h_0\text{Ai}(v) - \int_v^\infty dx \text{Ai}(x)\varphi'(x), \quad t = 2^{-2/3}v, \quad (124) \]

\[ h(x) = h_{-1}(x-v)^{-1/2} + h_0 + h_1(x-v)^{1/2} + h_2(x-v) + \cdots = h_{-1}(x-v)^{-1/2} + \varphi(x). \quad (125) \]
The first term in the right-hand side of (124), arisen from the first term on the right-hand side of (125), is found by differentiation with respect to \( v \) of the expression

\[
\int_v^\infty \text{d}x \, \text{Ai}(x)(x - v)^{-1/2} = \int_0^\infty t^{-1/2} \text{Ai}(t + v) = 2^{2/3} \text{Ai}^2(2^{-2/3}v).
\]  

The function \( \varphi'(x) \) according to (125) again has the structure of \( h(x) \) in (74) and the preceding consideration is applicable to (124).

Note now that

\[
- \int_v^\infty \text{d}x \, \text{Ai}(x) \varphi'(x) = -\pi \sum_{n=0}^{\infty} \frac{\varphi^{3n+1}(0)}{(3n)!!} + O.T.,
\]  

where \( O.T. \) are oscillatory terms. As

\[
\frac{d^n}{dx^n} (x - v)^{-1/2} = \frac{(-1)^n(2n - 1)!!}{2^n(x - v)^{n + 1/2}},
\]

then

\[
\varphi^{(3n+1)}(0) = h^{(3n+1)}(0) + (-1)^n \frac{(6n + 1)!! h_{-1}}{2^{3n+1}(-v)^{3n+3/2}}.
\]  

Hence,

\[
-\pi \sum_{n=0}^{\infty} \frac{\varphi^{(3n+1)}(0)}{(3n)!!} = -\pi \sum_{n=0}^{\infty} \frac{h^{(3n+1)}(0)}{(3n)!!} - \pi \sum_{n=0}^{\infty} \frac{(-1)^n(6n + 1)!! h_{-1}}{(3n)!!2^{3n+1}(-v)^{3n+3/2}}.
\]

On the other hand, for the nonoscillatory part of the term \( 2h_{-1}\text{Ai}(t)\text{Ai}'(t) \) in the right-hand side of (124) we have according to (41) and (106):

\[
2h_{-1} \frac{1}{2} w_1(t) \equiv \pi \sum_{n=0}^{\infty} \frac{(-1)^n(6n + 1)!! h_{-1}}{(3n)!!2^{3n+1}(-v)^{3n+3/2}}, \quad t = 2^{-2/3}v.
\]

So, the all nonoscillatory part (130) is cancelled by the second sum on the right-hand side of (129). Consequently, the nonoscillatory part of (124) is given by the first sum in the right-hand side of (129), and the oscillatory part can be found according to the previous consideration.

Note added to the electronic version of this paper.

The consideration of the recessive series switching on and off, carried by us in the Lebedev Phys. Inst. Preprint N 253 (1985), and connected with the abrupt behavior of the steepest decent line over the lower pass, led us later to the notion of the natural Stokes’ line width [9-11].

The natural width of the Stokes line is defined by the dimension of the saddle of the lower pass, i.e. by such change \( \Delta \alpha \) of the parameter \( \alpha \) for which the steepest decent line from the higher pass essentially changes its behavior near the lower pass, so that the difference of the phases

\[
\text{Im}(f_1 - f_2) = \Delta \omega \cdot \alpha + \cdots, \quad \Delta \omega = \omega_2(0) - \omega_1(0), \quad \omega_{1,2}(\alpha) = -\partial f_{1,2}(\alpha)/\partial \alpha,
\]

of the contributions \( e^{f_1(\alpha)} \) and \( e^{f_2(\alpha)} \) of the lower and higher passes becomes perceptible quantity of the order of 1 and satisfies the uncertainty relation:

\[
\Delta \omega \cdot \Delta \alpha \gtrsim 1.
\]
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