Optimal Lower Bound for Itemset Frequency Indicator Sketches

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Abstract

Given a database, a common problem is to find the pairs or $k$-tuples of items that frequently co-occur. One specific problem is to create a small space “sketch” of the data that records which $k$-tuples appear in more than an $\epsilon$ fraction of rows of the database.

We improve the lower bound of Liberty, Mitzenmacher, and Thaler [LMT14], showing that $\Omega(\frac{1}{\epsilon d \log(\epsilon d)})$ bits are necessary even in the case of $k = 2$. This matches the sampling upper bound for all $\epsilon \geq 1/d^{\alpha_0}$, and (in the case of $k = 2$) another trivial upper bound for $\epsilon = 1/d$.

1 Introduction

[Check out [LMT14] for a more complete introduction.]

We are concerned with sketches for itemset frequencies in databases. The “itemset frequency” is the fraction of rows in a database where a set of items co-occur:

**Definition 1.1** (Itemset Frequency). For a database $\mathcal{D} \in \{(0,1)^d\}^n$ and an itemset $T \subseteq [d]$, the frequency of $T$ in $\mathcal{D}$ is

$$f_T(\mathcal{D}) = \frac{1}{n} \left| \{i : \forall j \in T, (\mathcal{D}_i)_j = 1\} \right|$$

An itemset frequency indicator sketch is a smaller space representation of $\mathcal{D}$ that lets us identify the itemsets with large frequency:

**Definition 1.2** (Itemset-Frequency-Indicator sketches). An Itemset-Frequency-Indicator sketching scheme is a pair of algorithms: one receives $k, \epsilon$ and a database $\mathcal{D} \in \{(0,1)^d\}^n$ and outputs a sketch $S \in \{0,1\}^m$, and another takes $S$, $\epsilon$, and a set $T \subseteq [d]$ with $|T| = k$, and returns an estimate of whether $f_T(\mathcal{D}) > \epsilon$. In particular, it must output YES if

$$f_T(\mathcal{D}) \geq \epsilon$$

and NO if

$$f_T(\mathcal{D}) \geq \epsilon/2.$$  

For this problem, we require that the first algorithm “succeed” with $3/4$ probability, and if it does then the second algorithm should always output the correct answer for every query $T$.

The question is: how large must $m$ to solve this problem? If we allowed the queries to fail with a small constant probability, then per [LMT14] the space complexity is $\Theta(d/\epsilon)$. The goal of this paper is to get an extra $\log d$ factor from needing to union bound over $d^k$ queries.

There are two trivial upper bounds, for constant $k$: 
Sampling takes $O(\frac{1}{\epsilon}d \log d)$ bits of space.

Storing all the answers takes $O(d^k)$ bits of space.

We show that $\Omega(\frac{1}{\epsilon}d \log(\epsilon d))$ bits are necessary even in the case of $k = 2$. This means that sampling is optimal for all $\epsilon \geq 1/d^{1-\alpha}$ for any constant $\alpha > 0$, while storing all answers is optimal for $\epsilon \leq 1/d$ and $k = 2$.

**Theorem 3.2.** Any sketch for the Itemset-Frequency-Indicator problem must take $\Omega(\frac{1}{\epsilon}d \log(\epsilon d))$ space for all $1/d \leq \epsilon \leq 1/8$, even in the case of $k = 2$.

For $k = 2$, in the relatively minor intermediate regime of $\epsilon = 1/d^{1-\alpha(1)}$, it seems likely that neither trivial upper bound is quite tight. For $k > 2$, one can probably extend the result to show that sampling is optimal for $\epsilon > 1/d^{k-1-\alpha}$; we leave these questions to future work.

A more interesting open question is for itemset frequency estimation. If we want to estimate $f_T(D)$ to $\pm \epsilon$, then sampling requires $O(\frac{1}{\epsilon^2}d \log d)$ space but we don’t know any better lower bound than the above $\Omega(\frac{1}{\epsilon^2}d \log d)$ bound. ([LMT14] first showed this for $1/d^{1-\alpha} \ll \epsilon \ll 1/\log d$, and our Theorem 3.2 removes the upper limit on $\epsilon$).

To the best of our knowledge, [LMT14] contains the only previous space lower bound for this type of problem. A number of other aspects of the problem have been studied, however; see [LMT14] for an overview of related work. Our theorem is a strict improvement over their Theorem 18, which gets $\Omega(\frac{1}{\epsilon^{1-1/k}}d \log d)$ for a restricted range of $\epsilon$.

### 2 Notation

We use $[n]$ to denote $\{1, 2, \ldots, n\}$. For two vectors $v \in \mathbb{R}^d$ and $w \in \mathbb{R}^{d'}$, we use $v \parallel w$ to denote the $d + d'$ dimensional vector that is the concatenation of $v$ and $v'$.

### 3 Proof

For simplicity of exposition, we begin with the $\epsilon = \Theta(1)$ case, which was not previously known ([LMT14] required $\epsilon \ll 1$). The general $\epsilon$ case follows a very similar outline.

**Lemma 3.1.** Any sketch for the Itemset-Frequency-Indicator problem with $\epsilon = 1/8$ must take $\Omega(d \log d)$ space.

**Proof.** Let $m = d/2$. We will encode an arbitrary permutation $\Pi$ of $[m]$ into the results of Itemset-Frequency-Indicator. This forces Itemset-Frequency-Indicator to store at least $\log(m!) = \Theta(m \log m) = \Theta(d \log d)$ bits.

For each $i$, define $e_i \in \{0, 1\}^m$ to be the elementary unit vector with a 1 in position $i$. Given a subset $S$ of $[m]$, we associate a vector $v_S := \left(\sum_{i \in S} e_i\right) \parallel \left(\sum_{i \in \overline{S}} e_{\Pi(i)}\right)$

where $\parallel$ denotes concatenation and $\overline{S} = [m] \setminus S$.

Our database simply consists of $n = \Theta(\log d)$ vectors $v_S$ for independent, randomly chosen $S$. In particular, each $S$ contains each element of $[m]$ with probability $1/2$.

Now, for each row $v_S$ and any $i, j \in [m]$ consider the distribution on the co-occurrence of the itemset $\{i, m + j\}$. If $j = \Pi(i)$, this conjunction never appears. If $j \neq \Pi(i)$, on the other hand, then the conjunction appears with $1/4$ probability.
After looking at $n = \Theta(\log d)$ such vectors, with high probability all itemsets $\{i, m + j\}$ with $j \neq \Pi(i)$ will have more than $n/8$ appearances. Then $f_{\{i, m+j\}}(D)$ will be 0 if $j = \Pi(i)$ and at least 1/8 if $j \neq \Pi(i)$. Therefore an $\epsilon = 1/8$ Itemset-Frequency-Indicator algorithm will return NO for $\{i, m + j\}$ precisely when $j = \Pi(i)$, so we can recover $\Pi$ from the sketch. Hence the sketch must have $\Omega(d \log d)$ bits.

We now extend this approach to general $\epsilon$ with $1/d \leq \epsilon \leq 1$.

**Theorem 3.2.** Any sketch for the Itemset-Frequency-Indicator problem must take $\Omega(\frac{1}{\epsilon} d \log(\epsilon d))$ space for all $1/d \leq \epsilon \leq 1/8$, even in the case of $k = 2$.

**Proof of Theorem 3.2.** Let $m = ed/2$, which we can assume is an integer by rescaling constants. We will encode $1/\epsilon^2$ permutations $\Pi_{k,l}$ of $[m]$, for $k,l \in [1/\epsilon]$. This requires $(1/\epsilon^2) \log((ed/2)!)) = \Theta(\frac{1}{\epsilon} d \log(d))$ bits, giving the result.

Let $e_i \in \{0,1\}^m$ denote the elementary unit vector with a 1 in position $i$. For any $S \subset [m]$ and $k \in [1/\epsilon]$, we first define $u_{k,S} \in \{0,1\}^d/2$ by

$$u_{k,S}^i = 1 \text{ if and only if } i = (k-1)m + j \text{ for some } j \in S$$


to represent the set $S$ in “block” $k$. We then define the associated vector $v_{k,S} \in \{0,1\}^d$ by

$$v_{k,S} := u_{k,S} \parallel (\sum_{i \in S} e_{\Pi_{k,1}(i)}) \parallel (\sum_{i \in S} e_{\Pi_{k,2}(i)}) \parallel \cdots \parallel (\sum_{i \in S} e_{\Pi_{k,1}(i)})$$

We then choose $n = \Theta(\frac{1}{\epsilon} \log d)$ vectors for the database by, for each $k \in [1/\epsilon]$, choosing $\Theta(\log d)$ $v_{k,S}$ for uniformly random $S \subseteq [m]$.

Given the database, to figure out $\Pi_{k,l}(i)$ we query the itemset $T_{k,l}(i,j) = \{(k-1)m + i, d/2 + (l-1)m + j\}$ for all $j \in [m]$. We have that $T_{k,l}(i,j)$ appears in $v_{k',S}$ exactly when $k' = k$ with $i \in S$ and $\Pi_{k,l}^{-1}(j) \not\in S$. Thus it never appears if $j = \Pi_{k,l}(i)$, but otherwise it appears in each sampled $v_{k,S}$ with probability 1/4. Thus with high probability, it will appear in at least $\epsilon n/8$ of the rows. By a union bound, with high probability $f_{T_{k,l}(i,j)}(D) \geq \epsilon/8$ for all $i,j,k,l$ with $j \neq \Pi_{k,l}(i)$, while it is zero when $j = \Pi_{k,l}(i)$. Hence an $\epsilon/8$-approximate solution to Itemset-Frequency-Indicator would let us recover all the $\Pi_{k,l}$ with high probability, retrieving $\Theta(\frac{d}{\epsilon} \log(\epsilon d))$ bits of information. Therefore the sketch must store this many bits.

**References**

[LMT14] Edo Liberty, Michael Mitzenmacher, and Justin Thaler. Space lower bounds for itemset frequency sketches. arXiv preprint arXiv:1407.3740, 2014.