Estimating the intrinsic dissipation using the second harmonic of the temperature signal in tension-compression fatigue: Part I. Theory

Giovanni Meneghetti | Mauro Ricotta

Department of Industrial Engineering, University of Padova, Padova, Italy

Correspondence
Giovanni Meneghetti, University of Padova, Department of Industrial Engineering, via Venezia, 1-35131, Padova, Italy.
Email: giovanni.meneghetti@unipd.it

Abstract
An analytical framework was developed to correlate the intrinsic dissipation to the second harmonic of the temperature signal evaluated by means of the Fourier transform. The theoretical model assumes, without loss of generality, total conversion of the input mechanical energy into heat. Elastic-perfectly plastic or elastic-plastic materials obeying a Ramberg–Osgood law are considered. For the sake of simplicity, analytical solution of the thermoelastic and plastic temperature evolutions are obtained in the case of uniform stress and temperature fields. It was found that the intrinsic dissipation is correlated to the second harmonic of the temperature signal by means of a parameter (the \( \beta \) parameter) that, in the case of an elastic–plastic material, depends on the cyclic hardening exponent \( n' \).

KEYWORDS
dissipated energy, energy approaches, Fourier transform, thermal methods

1 | INTRODUCTION

Fatigue is an irreversible process, accompanied by microstructural changes, localized plastic strains and energy dissipation. In the past, the input mechanical energy per cycle, \( W \) (i.e., the area of the hysteresis loop) has been proposed as a fatigue index to correlate the fatigue behavior of steels\(^1\)–\(^4\) and ductile cast irons.\(^5\)\(^,\)\(^6\)

The temperature increment of a metallic material undergoing a fatigue test is a manifestation of the thermal energy dissipation, and it has been experimentally observed that the higher the applied stress amplitude is, the more pronounced the temperature increase of the material is for a given set of mechanical and thermal boundary conditions (including load test frequency, stress ratio, room temperature, and specimen geometry).\(^7\) The material temperature has been adopted as an index for the rapid estimation of the material fatigue limit or the high cycle fatigue limit of components,\(^7\)–\(^21\) including the assessment of the fatigue curve scatter band on the basis of a probabilistic model,\(^22\)–\(^25\) for detecting fatigue damage and for monitoring crack propagation.\(^26\)–\(^31\) Moreover, temperature has recently been adopted as a fatigue damage indicator to analyze the fatigue life under constant amplitude,\(^32\)–\(^40\) multi-stage,\(^41\)–\(^43\) and multiaxial loading.\(^44\)\(^,\)\(^45\)

The dissipated heat energy density per cycle (the \( \bar{Q} \) parameter), on which the material temperature distribution depends, was proposed as a promising fatigue index\(^46\) because it is independent of mechanical and thermal boundary conditions\(^46\)\(^,\)\(^47\) for a given load cycle and stress state.\(^48\) Alternatively stated, \( \bar{Q} \) has been recognized...
as a material property in the same way that $\overline{W}$ is considered to be a material property. $Q$ can be easily evaluated from temperature measurements, as will be described later on. The $Q$ parameter was initially adopted to rationalize in a single scatter band approximately 160 experimental results generated from constant amplitude, completely reversed, stress- or strain-controlled fatigue tests on plain or notched hot rolled stainless steel specimens, with notch radii ranging from 0.5 to 8 mm, and from cold drawn un-notched bars of the same steel under fully reversed axial or torsional fatigue loading conditions. Finally, the $Q$ parameter was recently assumed to be a fatigue index that is capable of rapidly estimating the material fatigue limit by carrying out step-load fatigue tests, and it was successfully adopted to evaluate the fatigue limit of cold drawn AISI 304L bars.

Generally, during a fully reversed, elastic–plastic fatigue cycle, the material temperature oscillates due to the superposition of different phenomena: the thermoelastic effect, $\Delta T_{\text{the}}$, which is a reversible phenomenon, and the temperature change, $\Delta T_p$, which is related to the plastic energy dissipated as heat twice in one cycle, that is, during both the tensile and the compressive part of the fatigue cycle. An additional contribution to heat energy dissipation is due to anelastic phenomena, which are not damaging in a fatigue sense. In more detail, when the dependence of material properties, such as $E$, $\nu$, and $\alpha$, on temperature is negligible (as is generally the case for metallic materials fatigued at ambient temperature), $\Delta T_{\text{the}}$ is correlated with the applied stress range according to the following expression:

$$\Delta T_{\text{the}} = -T_0 \frac{\alpha}{\rho c} \Delta \sigma_{ii}$$

The above equation requires that the material is isotropic and undergoing adiabatic transformation, which is practically achieved by applying a cyclic loading above a suitable threshold frequency. If a cyclic sinusoidal loading is applied, the temperature change due to the thermoelastic effect is modulated at the same frequency of the applied load. Therefore, $\Delta T_{\text{the}}$ can be evaluated experimentally by considering the first harmonic of the discrete Fourier transform (DFT) of the temperature signal (see Figure 1A).

Regarding the temperature variation $\Delta T_p$ related to plastic deformations, it has been shown that in fully reversed tension-compression fatigue tests, $\Delta T_p$ is spread out into several harmonics of the temperature signal, starting from that modulated at the double frequency of the applied load, which however, retains the most significant contribution. On this basis, some researchers have investigated the DFT of the temperature signal during fatigue tests and in some cases evaluated the intrinsic plastic dissipation per unit volume of material per cycle, $Q_p$, starting from the range of the second harmonic $\Delta T_2$. Akai et al. proposed a rapid technique to estimate the fatigue limit of AISI 304 and AISI 316L stainless steel by carrying out step-load fatigue tests and by monitoring the dissipated energy, which is evaluated from $\Delta T_2$. They pointed out that satisfactory results were found in the case of AISI 316L, whereas in the case of AISI 304, the fatigue limit estimated by dissipated energy measurements provided a conservative value compared to that evaluated by means of conventional staircase fatigue tests.

Palumbo et al. presented an accurate experimental technique to estimate the heat dissipated at the crack tip in the cyclic plastic zone. The proposed approach investigates the heat source operating at twice the load.

---

**Figure 1** (A) Material temperature oscillations during a fully-reversed fatigue test and its first and second harmonic of Fourier series and (B) energy balance for a material undergoing fatigue loadings and evaluation of the cooling gradient during a fatigue test.
frequency by assuming a bilinear elastic–plastic material behavior, adiabatic conditions and the plastic strain energy (i.e., the area of the hysteresis cycle) converted to heat. The authors highlight that in real applications, care must be paid to evaluate the intrinsic dissipation from the second harmonic of the temperature signal because adiabatic conditions cannot be easily guaranteed due to the thermal diffusivity close to the crack tip. Similar conclusions were drawn by Bar et al. in fatigue tests carried out on flat specimens of oxygen free copper (OF-Cu) and aluminum alloy AA7475 T761. The fatigue tests were performed with a test frequency \( f_L = 1 \) Hz under pulsating tension \( (R = 0) \) and fully reversed axial loading conditions \( (R = -1) \).

Starting from the intuitive observation that, under completely reversed elastic–plastic fatigue, two temperature shots occur per cycle, then correlation between the intrinsic dissipation and the second harmonic of the temperature signal has been investigated by introducing the following general expression:

\[
\dot{W} = \dot{Q}_p \rightarrow = \rho \cdot c \cdot 2 \cdot \Delta T_2 \cdot \beta
\]  

where \( \beta \) is a coefficient to be determined, and the factor 2 takes into account that the intrinsic dissipation associated with plasticity is produced twice over one loading cycle. In particular, Part I of the paper presents an analytical framework for evaluating the intrinsic dissipation \( \dot{Q}_p \), starting from the second harmonic of the material temperature, here called “second harmonic approach,” evaluated by means of the Fourier Series. The theoretical model is based on the first principle of thermodynamics and the hypotheses that (i) the plastic strain energy is totally converted into heat, (ii) the stress state is uniform, and (iii) the temperature field is uniform; the model is developed for elastic-perfectly plastic and plastic materials obeying a Ramberg–Osgood law. In Part II of the paper, the theoretical approach is applied to experimental data obtained by carrying out step-load fatigue tests on AISI 304L cold drawn bars. The intrinsic dissipation has been evaluated starting from the DFT of the temperature signal and according to the procedure defined in Part I of this paper. Afterwards, the intrinsic dissipation has been successfully compared with the experimental intrinsic dissipation measured according to the experimental procedure proposed in a previous work, hereafter named “cooling gradient approach.”

### 2 | THEORETICAL BACKGROUND: ENERGY BALANCE

According to the classical continuum mechanics reported in, the energy balance equation can be written in terms of power per unit volume by introducing the Helmholtz free energy as a thermodynamic potential (the dot symbol indicates the time derivative):

\[
\rho c \dot{T} - \lambda \nabla^2 T = \boldsymbol{\sigma} : \dot{\varepsilon}_p - \dot{E}_s - \alpha \cdot T \cdot \text{Tr}(\dot{\varepsilon}) + \dot{v}_e
\]  

where \( T \) is the material temperature, \( \dot{W} = \boldsymbol{\sigma} : \dot{\varepsilon}_p \) is the plastic strain power density, and \( \dot{E}_s \) the stored energy rate, including also the stored energy responsible for fatigue damage, \( \dot{Q}_{\text{the}} = -\alpha \cdot T \cdot \text{Tr}(\dot{\varepsilon}) \) is the heat rate per unit volume due to thermoelasticity and \( \dot{v}_e \) is the heat generation rate per unit volume supplied to or extracted from the material by external sources. The first two terms on the right-hand side of Equation 3 are the so-called intrinsic dissipation \( d_i \):

\[
d_i = \boldsymbol{\sigma} : \dot{\varepsilon}_p - \dot{E}_s
\]  

To derive Equation 3 starting from the general energy balance equation, it has been assumed that the dependence of the material state on temperature is negligibly small because temperature variations are too small to induce phase transformations, which would imply additional coupling heat sources further to thermoelasticity; as an example, Chrysochoos et al. tested dual phase steel specimens and noted that temperature variations were less than 15 K, therefore could not modify the internal state of the material. Second, it has been assumed that plasticity-induced phase transformations do not occur or can be considered negligible, as highlighted in previous work. These assumptions will be commented on the Part II of this paper, where experimental results are reported.

If the intrinsic dissipation and thermomechanical coupling in the form of thermoelasticity are null, Equation 3 simplifies to

\[
\rho c \dot{T} - \lambda \nabla^2 T = \dot{v}_e
\]  

which is the general heat conduction equation.
Letting $\lambda \nabla^2 T = \dot{Q}$ and $\dot{Q}_p = d_1$, Equation 3 can be written as follows:

$$\rho c T - \dot{Q} = \dot{Q}_p + \dot{Q}_{\text{THE}} + \dot{v}_e$$  \hfill (6)

Assuming that the material is homogeneous and that the temperature distribution is not affected by the nature of the internal heat source, $\dot{Q}_p$ and $\dot{Q}_{\text{THE}}$ can be thought of as equivalent plastic and thermoelastic power heat sources in a purely thermal problem, respectively. Concerning the regularity of the distribution of both dissipation and the thermoelastic heat sources, accurate analyses are reported in the works of Boulanger et al. and Morabito et al.

### 3 | THE COOLING GRADIENT APPROACH TO ESTIMATE THE INTRINSIC DISSIPATION

If the power quantities in Equation 6 are averaged over one loading cycle, then the contribution of the thermoelastic heat source vanishes because it consists of a reversible exchange between mechanical and thermal energy, which does not produce a net energy dissipation or absorption over one loading cycle; moreover, in absence of external heat sources $\dot{v}_e = 0$, Equation 6 can be written as follows:

$$\rho c \dot{T} - \dot{Q} = \dot{Q}_p$$  \hfill (7)

The energy balance Equation 7 is illustrated in Figure 1B, which shows the positive energy exchanges per cycle, that is, the mechanical energy $\dot{W}$ and the heat energy $\dot{Q}$.

When the temperature stabilizes during a fatigue test because equilibrium is achieved between heat production in the specimen and heat transfer to the surroundings, $\dot{T}$ becomes null, as shown in Figure 1B, and Equation 7 simplifies further to:

$$\dot{Q} = -\dot{Q}_p$$  \hfill (8)

If the fatigue test is stopped suddenly at $t = t^*$ (see Figure 1B), then just after $t^*$ (i.e., at $t = t^*+$), $\dot{Q}_p$ becomes zero (i.e., $\sigma = 0$ and $\dot{E}_s = 0$ in Equation 4). By re-writing the energy balance equation in Equation 7, one can obtain:

$$\dot{Q} = \rho c \lvert_{t^*+}^t \dot{T}$$  \hfill (9)

It is worth noting that the heat energy rate $\dot{Q}$ dissipated to the surroundings just before and just after $t^*$ is the same in Equation 8 and in Equation 9, respectively, because the temperature field does not change through $t^*$. Finally, the thermal energy released in a unit volume of material per cycle can be calculated by simply accounting for the load test frequency, $f_1$, as:

$$\dot{Q} = \frac{\rho c \lvert_{t^*+}^t \dot{T}}{f_1}$$  \hfill (10)

### 4 | THE SECOND HARMONIC APPROACH TO ESTIMATE THE INTRINSIC DISSIPATION

When equilibrium is achieved between heat production in the specimen and heat transfer to the surroundings, the stabilized material temperature can be assumed to be a periodic function and it can be expressed in terms of Fourier series, as follows:

$$T(t) = \bar{T} + \sum_{k=1}^{\infty} [A_k \cdot \cos(k \cdot \omega_L t) + B_k \cdot \sin(k \cdot \omega_L t)]$$  \hfill (11)

where

$$\bar{T} = \frac{1}{T} \int_0^T T(t) dt$$  \hfill (12a)

$$A_k = \frac{2}{T} \int_0^T T(t) \cdot \cos(k \cdot \omega_L t) dt$$  \hfill (12b)

$$B_k = \frac{2}{T} \int_0^T T(t) \cdot \sin(k \cdot \omega_L t) dt$$  \hfill (12c)

where $\omega_L = 2\pi f_1$ and $\bar{T} = 2\pi / \omega_L = \frac{1}{f_1}$. The second harmonic of temperature and its range are defined according to Equations 13a and 13b, respectively, and both are plotted in Figure 1A:

$$T_2(t) = A_2 \cdot \cos(2 \omega_L t) + B_2 \cdot \sin(2 \omega_L t)$$  \hfill (13a)

$$\Delta T_2 = 2\sqrt{A_2^2 + B_2^2}$$  \hfill (13b)

To evaluate the $\beta$ parameter in Equation 2 and for the sake of simplicity of the analytical developments, the following hypotheses are considered:

i. the plastic strain power density $\dot{W}$ is fully converted into specific heat energy rate $\dot{Q}_p$, (i.e., $\dot{E}_s = 0$ in
Equation 4, $\dot{W} = \dot{Q}_p$); this hypothesis, though idealized, is supported by different experimental outcomes\(^{49,74-77}\); however, it could also be assumed that $\dot{Q}_p$ is a fraction of $\dot{W}$, but the results in terms of $\beta$ coefficient obtained later on would not change because Equation 2 takes into account the thermal problem;

ii. a uniform stress state and a uniform temperature distribution exist in the material (i.e., internal adiabatic conditions exist, $\nabla^2 T = 0$, then $\dot{Q} = 0$ in Equation 6 and temperature $T$ becomes a function only of time);

iii. to achieve the stabilized thermal situation illustrated in Figure 1B, heat extraction from the material volume is simulated by a heat sink $\dot{v}_c$ such that $\dot{v}_c = -\dot{W} \cdot f_L$, that is, heat production per cycle owing to plasticity effects is exactly removed by the heat sink.

The instantaneous energy balance Equation 6 can be written as follows:

$$\rho \cdot c \cdot \dot{T}_p + \rho \cdot c \cdot \dot{T}_{\text{the}} + \rho \cdot c \cdot \dot{T}_{\text{sink}} = \dot{W} + \dot{Q}_{\text{the}} - \dot{W} \cdot f_L \quad (14)$$

In Equation 14, $T_p$ and $T_{\text{the}}$ are the plastic and thermoelastic temperatures, respectively, which originate from the plastic and thermoelastic heat sources, according to Equation 6; $T_{\text{sink}}$ is the temperature associated to the heat sink $\dot{v}_c$.

In what follows, the $Q_p$ parameter is evaluated theoretically by starting from the range of the second harmonic, $\Delta T_2$, of the temperature signal. The case of an elastic-perfectly plastic material is considered first, and then an elastic-plastic material obeying a Ramberg-Osgood law is analyzed.

### 4.1 Elastic-perfectly plastic material

Let us consider a sinusoidal, fully reversed, strain-controlled fatigue test, according to

$$\varepsilon(t) = \varepsilon_a \cdot \sin(\omega_L \cdot t) \quad (15)$$

With the aim of simplifying the analytical frame, the coordinate reference system has been set at the lower tip of the hysteresis cycle (see Figure 2) by introducing the auxiliary co-ordinate axes $\Delta \sigma^*$ and $\Delta \varepsilon^*$. Therefore, the applied strain becomes

$$\Delta \varepsilon^*(t) = \varepsilon_a \cdot [1 - \cos(\omega_L \cdot t)] \quad (16)$$

The strain is fully elastic until the time $\tilde{t}$, when the stress reaches the value $2\sigma_y$, where $\sigma_y$ is the material yield strength, as shown in Figure 2. Considering that

$$\Delta \sigma^*(\tilde{t}) = E \cdot \Delta \varepsilon^* = E \cdot \varepsilon_a \cdot [1 - \cos(\omega_L \cdot \tilde{t})] \quad (17a)$$

for

$$0 \leq t \leq \tilde{t}$$

$\tilde{t}$ can be calculated as follows:

$$\Delta \sigma^*(\tilde{t}) = 2\sigma_y \Rightarrow \tilde{t} = \frac{1}{\omega_L} \arccos \left(1 - \frac{2 \cdot \sigma_y}{E \cdot \varepsilon_a}\right) \quad (17b)$$

![FIGURE 2 Hysteresis cycle for an elastic-perfectly plastic material](image-url)
For $t > \bar{\tau}$, the resulting stress remains equal to $2\sigma_y$, whereas the plastic strain increases from zero and reaches its maximum value when the applied strain is at its maximum, that is, at $t = \bar{T}/2$. Therefore, for $\bar{\tau} \leq t \leq \bar{T}$, the plastic strain is equal to

$$\Delta \epsilon_p^*(t) = \Delta \epsilon^*(t) - \frac{2\sigma_y}{E}$$  \hspace{1cm} (18)$$

At time $t = \bar{T}/2$, the plastic strain achieves its maximum value ($2\sigma_y - \frac{2\sigma_y}{E}$) and it remains constant as long as $\Delta \sigma^*$ returns again to zero, that is, for $t = \bar{T}/2 + \bar{\tau}$. For $\bar{T}/2 + \bar{\tau} \leq t \leq \bar{T}$, $\Delta \sigma^*$ and the elastic strain $\Delta \epsilon^*$ equal zero, whereas the applied strain $\Delta \epsilon^*(t)$ decreases to zero; accordingly, in this time window, $\Delta \epsilon_p^*(t) = \Delta \epsilon^*(t)$, as shown in Figure 3.

To manage the plastic strain evolution during a single fatigue cycle, let us consider the rectangular wave shown in Figure 3.

For $t > \bar{\tau}$, the resulting stress remains equal to $2\sigma_y$, whereas the plastic strain increases from zero and reaches its maximum value when the applied strain is at its maximum, that is, at $t = \bar{T}/2$. Therefore, for $\bar{\tau} \leq t \leq \bar{T}$, the plastic strain is equal to

$$\Delta \epsilon_p^*(t) = \Delta \epsilon^*(t) - \frac{2\sigma_y}{E}$$  \hspace{1cm} (18)$$

At time $t = \bar{T}/2$, the plastic strain achieves its maximum value ($2\sigma_y - \frac{2\sigma_y}{E}$) and it remains constant as long as $\Delta \sigma^*$ returns again to zero, that is, for $t = \bar{T}/2 + \bar{\tau}$. For $\bar{T}/2 + \bar{\tau} \leq t \leq \bar{T}$, $\Delta \sigma^*$ and the elastic strain $\Delta \epsilon^*$ equal zero, whereas the applied strain $\Delta \epsilon^*(t)$ decreases to zero; accordingly, in this time window, $\Delta \epsilon_p^*(t) = \Delta \epsilon^*(t)$, as shown in Figure 3.

To manage the plastic strain evolution during a single fatigue cycle, let us consider the rectangular wave reported in Figure 4A as having a period equal to $\bar{T}$, defined according to Equation 19:

$$R(t) = \delta + \frac{2}{\pi} \sum_{j=1}^{\infty} \sin \left( \frac{j \cdot \pi \cdot \delta}{\bar{T}} \right) \cdot \cos \left( j \cdot \omega_L \left[ t - \frac{1}{2} \left( \bar{T} + \bar{\tau} \right) \right] \right)$$  \hspace{1cm} (19)$$

where $\delta = \tau / \bar{T}$. Following Figure 4A, three rectangular wave functions are necessary, as reported in Figure 4B:

$$R_{\epsilon,L}(t) = \delta_e + \frac{2}{\pi} \sum_{j=1}^{\infty} \sin \left( \frac{j \cdot \pi \cdot \delta_e}{\bar{T}} \right) \cdot \cos \left( j \cdot \omega_L \left[ t - \frac{1}{2} \left( \bar{T} + \bar{\tau} \right) \right] \right)$$  \hspace{1cm} (20)$$

and the applied stress results $\Delta \sigma^*(t) = E \cdot \Delta \epsilon^*(t)$.

Let us evaluate the material temperature related to the thermoelastic effect in Equation 14. From the definition of $Q_{\text{the}}$, we have

$$\rho \cdot c \cdot \Delta T_{\text{the}} = -\alpha \cdot T_0 \cdot \Delta \sigma^*$$  \hspace{1cm} (25)$$

where $T_0$ is the material temperature when $\sigma = -\sigma_y$ (i.e., $\Delta \sigma^*(t) = 0$). Then, we have

$$\Delta T_{\text{the}}(t) = -\left( \frac{\alpha}{\rho \cdot c} \right) \cdot \Delta \sigma^*(t) \cdot T_0$$  \hspace{1cm} (26)$$

Figure 5A shows the trend of $\Delta T_{\text{the}}$ versus time according to Equation 26.

Let us consider now the material temperature related to plastic dissipation in Equation 14; we have

$$\rho \cdot c \cdot \Delta T_p = \dot{W}$$  \hspace{1cm} (27)$$

or alternatively

$$\rho \cdot c \cdot \Delta T_p = \sigma(t) \cdot \left| \Delta \epsilon_p^*(t) \right|$$  \hspace{1cm} (28)$$
From Equation 28 the temperature trend due to plasticity effects, \( \Delta T_p(t) \), can be calculated as follows:

\[
\Delta T_p(t) = \frac{1}{\rho \cdot C} \int_0^t \left( \Delta \sigma(t) - \sigma_y \right) \cdot \frac{d}{dt} \Delta \varepsilon_p(t) \cdot dt = \\
= \frac{1}{\rho \cdot c} \cdot \sigma_y \cdot \left\{ \varepsilon_a \cdot \left[ 1 - \cos(\omega t) - \frac{2\sigma_y}{E\varepsilon_a} \cdot R_{e,L}(t) + 2 \left( \varepsilon_a - \frac{\sigma_y}{E} \right) \right] \cdot R_{e,L}(t) + \left[ \varepsilon_a \cdot \cos(\omega t) + \left( 3\varepsilon_a - \frac{4\sigma_y}{E} \right) \right] \cdot R_{e,U}(t) \right\}
\]

Equation 29

Figure 5B shows the trend of \( \Delta T_p(t) \) vs. time according to Equation 29. From this figure, one can see that, in one loading cycle, the material temperature increases and achieves the maximum value when \( t = \hat{T} \), according to Equation 30:

\[
\Delta T_p(\hat{T}) = \frac{4\sigma_y}{\rho \cdot c} \left( \varepsilon_a - \frac{\sigma_y}{E} \right) 
\]

Under the hypothesis of a Masing material, the plastic strain energy per cycle, \( \dot{W} \), can be calculated according to Halford:

\[
\dot{W} = 4 \frac{1 - n'}{1 + n'} \sigma_a \cdot \varepsilon_{a,p} 
\]

where \( \varepsilon_{a,p} \) is the amplitude of the plastic strain, and \( n' \) is the cyclic strain hardening exponent. In the case of an elastic-perfectly plastic material, \( n' = 0 \), and the plastic strain amplitude is equal to \( \varepsilon_a - \frac{\sigma_y}{E} \); therefore,
\[ W = 4\sigma_y \left( \varepsilon_a - \frac{\sigma_y}{E} \right) \]  
\[ (32) \]

and Equation 30 can be expressed alternatively as

\[ \Delta T_p(\tilde{T}) = \frac{W}{\rho \cdot c} \]
\[ (33) \]

Regarding the heat sink temperature Equation 14, we have

\[ \rho \cdot c \cdot \Delta T_{sink} = -W \cdot f_L \]
\[ (34a) \]

\[ \Delta T_{sink}(t) = -\frac{W}{\rho \cdot c} \cdot t \]
\[ (34b) \]

Therefore, the solution of Equation 14 combines Equations 26, 29 and 34b:

\[ \Delta T(t) = \Delta T_{the}(t) + \Delta T_p(t) + \Delta T_{sink}(t) \]
\[ (35) \]

This idealized description of the heat extraction mechanism from the material is discussed in the Appendix A of this paper.

To plot the temperature function \( T(t) \) according to the strain function \( \varepsilon(t) \) of Equation 15, and not \( \Delta \varepsilon^p(t) \) of Equation 16, we have to shift \( \Delta T(t) \) (Equation 35) by a factor \( \frac{T}{4} \) and refer the thermoelastic temperature to the unstressed state \( \sigma = 0 \):

\[ T(t) = \Delta T \left( t + \frac{T}{4} \right) + T_0 \]
\[ (36) \]

Given \( T(t) \), then \( \tilde{T}_a \), \( A_1 \), \( B_1 \), \( A_2 \) and \( B_2 \) were analytically evaluated according to Equation (12a-c), as follows:

\[ T_a = -\alpha \cdot T_0 \cdot \frac{\sigma_y}{\rho \cdot c} + \frac{\sigma_y}{\rho \cdot c} \cdot \left[ 2\varepsilon_a \cdot \left( 1 - \frac{i}{2\pi} - \frac{1}{2\pi} \sqrt{1 - \left( 1 - \frac{2\varepsilon_y}{E \cdot \varepsilon_a} \right) \cdot \frac{\sigma_y}{E}} \right) + \frac{\sigma_y}{E} \cdot \left( 4 \cdot \frac{i}{T} - 3 \right) \right] - \frac{1}{2} \cdot \frac{\tilde{T}}{\rho \cdot c} + T_0 \]
\[ (37a) \]

\[ A_1 = \frac{\alpha \cdot T_0 \cdot E}{\rho \cdot c} \cdot \left[ \varepsilon_a + \sqrt{1 - \left( 1 - \frac{2\varepsilon_y}{E \cdot \varepsilon_a} \right) \cdot \frac{2\varepsilon_y}{E \cdot \varepsilon_a} + \varepsilon_a \cdot \left( \frac{\varepsilon_a}{\pi \cdot E} - \frac{1}{\pi} \right) } \right] \]
\[ (37b) \]

\[ B_1 = -\frac{\alpha \cdot T_0}{\rho \cdot c} \cdot \frac{4 \cdot \sigma_y \cdot \left( E \cdot \varepsilon_a - \sigma_y \right)}{\pi \cdot E \cdot \varepsilon_a} \]
\[ (37c) \]

\[ A_2 = \frac{\sigma_y}{\pi \cdot \rho \cdot c} \cdot \left\{ \left( 1 - \frac{2\varepsilon_y}{E \cdot \varepsilon_a} \right) \cdot \left( \frac{\sigma_y}{E} - 2 \cdot \varepsilon_a \right) + \frac{1}{2} \cdot \frac{\sigma_y}{E} \cdot \left( 1 - \frac{2\varepsilon_y}{E \cdot \varepsilon_a} \right) \right\}^{3/2} \]
\[ (37d) \]

\[ B_2 = \frac{\sigma_y}{\pi \cdot \rho \cdot c} \cdot \left\{ \left( 1 - \frac{2\varepsilon_y}{E \cdot \varepsilon_a} \right)^2 - 1 \right\} \cdot \left\{ 2 \cdot \varepsilon_a - \frac{4\sigma_y}{E} \right\} \]
\[ (37e) \]

It is worth noting that thermo-elasticity involves odd harmonics \((k = 1, 3, 5...\)), whereas plasticity involves even harmonics \((k = 2, 4, 6, etc.)\). Moreover, the presence of the idealized heat sink (Equation 34a) influences only the even \( B_k \) coefficients.

Figure 6 shows the \( T(t) \) versus time graph evaluated according to Equation 36, the first, second, third, and fourth harmonic waves of the temperature signal \( T(t) \) and the plastic temperature, which was evaluated by subtracting from \( T(t) \) the thermoelastic harmonics, that is, the odd ones, according to Equation 38:

\[ T_p(t) = T(t) - \sum_{k=1}^{\infty} \left[ A_{2k-1} \cdot \cos((2k-1)\omega_L t) \right] + B_{2k-1} \cdot \sin((2k-1)\omega_L t) \]
\[ (38) \]

Even though the heat extraction from the material has been idealized by means of a heat sink, Figure 6 highlights that the range of the second harmonic wave \( \Delta T_2 \) is not equal to the range of plastic temperature \( \Delta T_p \) and the temperature signal obtained after cleaning the thermoelastic signal, Equation 38, is not a sine wave, differently from the applied strain, as pointed out by previous works.\(^{57,58,61}\)

To evaluate the intrinsic dissipation per cycle, \( \dot{Q}_p \), using \( \Delta T_2 \), the \( \beta \) parameter was evaluated according to Equation 2, and it was found to depend on the ratio \( \frac{\varepsilon_a}{\left( \frac{\sigma_y}{E} \right) \cdot \left( \frac{\varepsilon_a}{\pi} - \frac{1}{\pi} \right) } \), that is, the ratio between the applied strain amplitude
and the yield strain of the elastic-perfectly plastic material. Conversely, $\beta$ is independent of $\rho$, $c$, and $f_{L}$. $\beta$ is plotted against the strain amplitude ratio in Figure 7 and it is seen that it ranges from $\pi/2$ and $3\pi/2$ when $\frac{\epsilon_a}{\sigma_y/E} > 1$.

The open and filled symbols reported in this figure will be discussed in Appendix A.

It is worth remembering that the parameter $\beta$ reported in Figure 7 has been evaluated starting from the idealized temperature signal in Equations 35 and 36, where thermal equilibrium with the surroundings has been imposed by means of a heat sink, which linearly removes the heat generated by the intrinsic dissipation during one loading cycle (see Equation 34b). In Appendix A, more realistic hypotheses regarding the heat transfer from the specimen to the surroundings have been considered. Even though Appendix A illustrates that strictly speaking $\beta$ depends on the thermal boundary conditions of the experimental test, the difference of the $\beta$ parameter with respect to the values reported in Figure 7 is limited to a few percentage points, at least by considering usual laboratory test conditions. Therefore, the results reported in Figure 7 are substantially robust while varying the thermal boundary conditions of the tests.

### 4.2 | Elastic–plastic material

Let us consider an elastic–plastic material obeying a Ramberg-Osgood law in a fully reversed, force-controlled fatigue test with a sinusoidal waveform. By evaluating the macroscopic stress as $\sigma(t) = \sigma_a \sin(\omega t)$, with a stress amplitude $\sigma_a$, the elastic and plastic deformations can be written according to Equations 39a and 39b, respectively:

$$\epsilon_e(t) = \frac{\sigma(t)}{E} = \frac{\sigma_a \sin(\omega t)}{E}$$

$$\epsilon_p(t) = \left(\frac{\sigma(t)}{K'}\right)^{1/n'} = \left[\frac{\sigma_a \sin(\omega t)}{K'}\right]^{1/n'}$$

where $K'$ and $n'$ are the cyclic strength coefficient and the cyclic strain hardening exponent of the material, respectively. Equation 39b can be defined only when the sinusoidal function is positive. To analytically evaluate the plastic strain power density over one load cycle, the coordinate reference system has been set once again at the lower tip of the hysteresis cycle (see Figure 8). As a consequence, in the time-domain, the stress-range can be written as

$$\Delta\sigma^*(t) = \sigma_a[1 - \cos(\omega t)]$$
Accordingly, the time evolution of the elastic and plastic strain ranges is

\[
\Delta \varepsilon^e(t) = \frac{\Delta \sigma^e(t)}{E} = \sigma_a[1 - \cos(\omega t)]
\]

\[
\Delta \varepsilon^p(t) = 2 \left[ \frac{\sigma(t)}{2 \cdot K'} \right]^{\frac{1}{n'}} = 2 \left[ \frac{\sigma_a(1 - \cos(\omega t))}{2 \cdot K'} \right]^{\frac{1}{n'}} \text{ for } 0 \leq t \leq \frac{T}{2}
\]

\[
\Delta \varepsilon^p(t) = \Delta \varepsilon^p \left( \frac{T}{2} \right) - 2 \left[ \frac{\sigma_a(1 - \cos(\omega t))}{2 \cdot K'} \right]^{\frac{1}{n'}} \text{ for } \frac{T}{2} \leq t \leq \frac{T}{3}
\]

The rates of the plastic strain energy for \(0 \leq t \leq \frac{T}{4}\), \(W_C(t)\), and \(\frac{T}{3} \leq t \leq \frac{T}{4}, W_C \left( t + \frac{T}{2} \right)\), are equal to (see Figure 9):

\[
\dot{W}_C(t) = \left[ \sigma_a - \Delta \sigma^e(t) \right] \cdot \frac{d}{dt} \Delta \varepsilon^p(t)
\]

\[
= \frac{\omega_t}{n'} \cdot \frac{\sigma_a^{\frac{1}{n'}}}{2^\frac{1}{n'} \cdot K'} \cdot \sin(\omega t) \cdot \cos(\omega t) \cdot \left[ (1 - \cos(\omega t)) \right]^{\frac{1}{n'} - 1}
\]

\[
\dot{W}_C \left( t + \frac{T}{2} \right) = \frac{\omega_t}{n'} \cdot \frac{\sigma_a^{\frac{1}{n'}}}{2^\frac{1}{n'} \cdot K'} \cdot \sin(\omega t + \pi) \cdot \cos(\omega t + \pi) \cdot \left[ (1 - \cos(\omega t + \pi)) \right]^{\frac{1}{n'} - 1}
\]

To manage a single equation defining the evolution of the plastic strain power density during a single fatigue cycle, proper rectangular waves were defined once again. According to Equation 19, a first rectangular wave was defined as equal to one in the \(0 \leq t \leq \frac{T}{4}\) time window and equal to zero in the remaining part of the fatigue cycle. In this case, we have \(\tau = T/4\) and therefore \(\delta_C = \frac{T}{T/4}\).

The rates of the plastic strain energy for \(\frac{T}{4} \leq t \leq \frac{T}{2}\), \(W_L(t)\), and for \(\frac{3}{4} \cdot T/4 \leq t \leq T\), \(W_U = W_L \left( t + \frac{T}{2} \right)\), are equal to

\[
\dot{W}_L(t) = \left[ \Delta \sigma^e(t) - \sigma_a \right] \cdot \frac{d}{dt} \Delta \varepsilon^p(t)
\]

\[
= - \frac{\omega_t}{n'} \cdot \frac{\sigma_a^{\frac{1}{n'}}}{2^\frac{1}{n'} \cdot K'} \cdot \sin(\omega t) \cdot \cos(\omega t) \cdot \left[ (1 - \cos(\omega t)) \right]^{\frac{1}{n'} - 1}
\]

\[
\dot{W}_U = \left[ \Delta \sigma^e \left( t + \frac{T}{2} \right) - \sigma_a \right] \cdot \frac{d}{dt} \Delta \varepsilon^p \left( t + \frac{T}{2} \right)
\]

\[
= - \frac{\omega_t}{n'} \cdot \frac{\sigma_a^{\frac{1}{n'}}}{2^\frac{1}{n'} \cdot K'} \cdot \sin(\omega t + \pi) \cdot \cos(\omega t + \pi) \cdot \left[ (1 - \cos(\omega t + \pi)) \right]^{\frac{1}{n'} - 1}
\]

\[
R_C(t) = \delta_C \sum_{j=1}^{\infty} \frac{\sin(j \cdot \pi \cdot \delta_C)}{j} \cdot \cos \left[ j \cdot \omega_t \left( t - \frac{T}{2} \right) \right]
\]
The second rectangular wave was defined to be equal to one when \( \dot{T}/4 \leq t \leq \dot{T}/2 \) and equal to zero in the remaining part of the fatigue cycle. In this case, \( \tau = \frac{T}{4} \) and \( \delta_{L} = \frac{T}{4} \):

\[
R_{L}(t) = \delta_{L} + \frac{2}{\pi} \sum_{j=1}^{\infty} \sin(j \cdot \pi \cdot \delta_{L}) \cdot \cos\left[j \cdot \omega_{L} \left( t - \frac{1}{2} \left( \frac{T}{2} + \frac{T}{4} \right) \right) \right]
\]

Then, the rate of plastic strain energy can be defined as follows:

\[
\dot{W}(t) = \dot{W}_{C}(t) \cdot R_{C}(t) + \dot{W}_{L}(t) \cdot R_{L}(t) + \dot{W}_{U}(t) \cdot R_{L}(t + \frac{T}{2})
\]

\[
\cdot R_{C} \left( t + \frac{T}{2} \right) + \dot{W}_{U}(t) \cdot R_{L} \left( t + \frac{T}{2} \right)
\]

(46)

where \( R_{C} \left( t + \frac{T}{2} \right) \) and \( R_{L} \left( t + \frac{T}{2} \right) \) are the rectangular functions of Equations 44 and 45, respectively, which are translated of \( \frac{T}{4} \), as shown in Figure 10. It is worth noting that \( \dot{W}_{C} = 0 \), when the material experiences a purely elastic behavior during the unloading phase. In this case, the plastic strain energy density averaged over one loading cycle, \( \dot{W} \), is equal to the area of the hysteresis cycle (\( \dot{W} = W_{hyst} \), \( \dot{W}_{hyst} = \dot{W}_{L} + \dot{W}_{U} - 2 \cdot \dot{W}_{C} \) in Figure 9).

Recalling Equation 14, the thermoelastic temperature can be evaluated again by using previous Equation 26, where now \( \Delta \sigma(t) \) is given by Equation 40. As a result, Figure 11A shows the elastic strain and the thermoelastic temperature evolution over one loading cycle.

Regarding the temperature due to plastic dissipation, \( \Delta T_{P}(t) \), it can be calculated according to Equation 27, by using \( \dot{W} \) Equation 46, as follows:

\[
\Delta T_{P}(t) = \frac{1}{\rho \cdot c} \int_{0}^{T} \dot{W}(t) \cdot dt = \frac{1}{\rho \cdot c} \left[ 2^{1-\frac{\dot{T}}{2}} \cdot \sigma_{a}^{\beta+1} \right] \cdot \left[ n' + \cos(\omega_{L} \cdot t) \right] \cdot \left[ 1 - \cos(\omega_{L} \cdot t) \right]^{\beta} \cdot \dot{W}_{C}(t) + \dot{W}_{L}(t) \cdot \dot{W}_{L}(t + \frac{T}{2}) + \dot{W}_{U}(t) \cdot \dot{W}_{L}(t + \frac{T}{2}) \right]
\]

(47)

Figure 11B plots \( \Delta T_{P} \) versus time according to Equation 47. From this figure, one can see that, in one loading cycle, the plastic strain energy increases and reaches its maximum value when \( t = \dot{T} \), as in Equation 48:

\[
\Delta T_{P}(\dot{T}) = \frac{1}{\rho \cdot c} \left[ 2^{1-\frac{\dot{T}}{2}} \cdot \sigma_{a}^{\beta+1} \right] \cdot \left[ 4n' - 2 \cdot 2^{\beta}(n' - 1) \right]
\]

(48)

Following the track adopted for the elastic-perfectly plastic material, the material temperature \( \Delta T(t) \) was calculated according to Equation 35 and eventually \( T(t) \) was evaluated according to Equation 36 and reported in Figure 12. Once \( T(t) \) was defined by means of Equation 36, the parameters of the Fourier Transform \( \hat{T}_{s} \), \( A_{1}, B_{1}, A_{2} \), and \( B_{2} \) were calculated according to Equation (12). Differently from the elastic-perfectly plastic material, a closed form solution for the \( A_{k} \) and \( B_{k} \) coefficients was not found, therefore they were calculated numerically. Taking advantage of a sensitivity analysis, it was found that convergence is guaranteed for \( j \geq 500 \) (see Equations 44 and 45). The first, second, and fourth harmonics of \( T(t) \) are reported in Figure 12; in this case the odd FT terms, starting from the third one, are null because the thermoelastic temperature is a sine function. The plastic temperature was evaluated once again according to Equation 38, that is, by subtracting the thermoelastic harmonics from \( T(t) \). It is seen that also in the present case of an elastic-plastic material, the temperature evolution related to plasticity is not a sine wave,
as reported by previous works. Furthermore, Figure 12 highlights that the range of the second harmonic wave \( \Delta T_2 \) is not equal to the plastic temperature range \( \Delta T_p \).

Finally, the \( \beta \) parameter was evaluated according to Equation 2, and it was found that it depends only on the strain hardening coefficient, \( n' \). Figure 13 reports \( \beta \) versus \( n' \), where it should be kept in mind that usually, in the case of metals, \( n' \) ranges from 0.05 to 0.6.4

![Figure 11](image1.png) (A) Elastic strain and thermoelastic temperature and (B) plastic strain and plastic temperature during a fully-reversed, stress-controlled fatigue cycle. Frame of reference (\( \Delta \varepsilon^*, \Delta \sigma^* \)) according to Figure 8

![Figure 12](image2.png) Material temperature evolution due to the superposition of thermoelastic and plasticity effects over one loading cycle in the case of elastic–plastic material; figure is according to the \( \sigma-\varepsilon \) co-ordinate system and it compares the first, second and fourth harmonic of FT (Equation 38, second and fourth harmonics are magnified 10 times, \( n' = 0.2 \)) [Colour figure can be viewed at wileyonlinelibrary.com]

![Figure 13](image3.png) \( \beta \) parameter as a function of the cyclic strain hardening exponent \( n' \) for an elastic–plastic material

### 5 | CONCLUSIONS

The evaluation of the intrinsic dissipation per cycle (the \( Q_\rho \) parameter) by means of the Fourier Transform of the material temperature has been addressed. In view of this, an analytical model was developed based on the first principle of thermodynamics under the hypotheses that the rate of plastic strain energy is totally converted into the rate of specific heat energy, and uniform stress as well as temperature fields exist.

The analytical model was then applied in the case of an elastic-perfectly plastic material and an elastic–plastic...
material. In both circumstances, it was found that the range of the second harmonic is correlated to the $Q_p$ parameter by means of a multiplicative parameter, $\beta$, which was found to be dependent on the assumed material behavior and the thermal boundary conditions. Nevertheless, assuming usual laboratory test conditions, the dependence of $\beta$ on the thermal boundary conditions appeared very weak. Conversely, a strong dependence was found on the material stress–strain constitutive law. Specifically, in the case of the elastic–perfectly plastic material, $\beta$ depends on the ratio between the applied strain amplitude and the yield strain, whereas in case of an elastic–plastic material obeying a Ramberg-Osgood type behavior, $\beta$ depends on the cyclic strain hardening exponent.

**DATA AVAILABILITY STATEMENT**
The data that support the findings of this study are available from the corresponding author upon reasonable request.

**NOMENCLATURE**
- $A_k, B_k$: coefficients of the Fourier transform of the temperature signal
- $c$: material specific heat [J/(kg·K)]
- $d_1$: intrinsic dissipation [W/m$^3$]
- $E$: material elastic modulus [MPa]
- $\dot{E}_s$: rate of accumulation of stored energy [W/m$^3$]
- $f_l$: load test frequency [Hz]
- $\dot{Q}$: heat energy rate exchanged by a unit volume of material, by conduction, convention and radiation [W/m$^3$]
- $\bar{Q}$: heat energy exchanged by a unit volume of material per cycle, by conduction, convention and radiation (Q averaged over one loading cycle) [J/(m$^3$·cycle)]
- $\dot{\bar{Q}}$: heat energy rate exchanged by a unit volume of material per cycle, by conduction, convention and radiation (Q averaged over one loading cycle) [W/m$^3$]
- $\dot{Q}_{\text{conv}}$: heat power extracted by convection [W/m$^3$]
- $Q_p$: intrinsic dissipation per unit volume [J/m$^3$]
- $\ddot{Q}_p$: intrinsic dissipation per unit volume of material per cycle ($Q_p$, averaged over one loading cycle) [J/(m$^3$·cycle)]
- $Q_{\text{th}}$: thermoelastic strain energy per unit volume [J/m$^3$]
- $\dot{v}_c$: heat generation rate per unit volume supplied to or extracted from the material by external sources [W/m$^3$]
- $R$: nominal stress ratio (ratio between the minimum and the maximum applied nominal stresses)
- $t^*$: time at which the fatigue test is suddenly stopped [s]
- $i$: time at which the yield strength $\sigma_y$ is achieved [s]
- $T$: material temperature [K]
- $T_0$: material temperature when $\Delta \sigma(t) = 0$ [K]
- $\bar{T}$: material temperature averaged over one loading cycle [K]
- $\bar{T}^*$: period of the loading cycle ($\bar{T}^* = 1/f_l$) [s]
- $T_s$: average material temperature under stabilized thermal conditions [K]
- $T_{\text{amb}}$: room temperature [K]
- $T_{\text{sink}}$: temperature associated to the heat sink $\dot{v}_c$ [K]
- $\dot{W}$: plastic strain power density [W/m$^3$]
- $\dot{W}$: plastic strain energy density per cycle [J/(m$^3$·cycle)]
- $\alpha$: isotropic coefficient of thermal expansion [K$^{-1}$]
- $\alpha_{\text{conv}}$: heat transfer coefficient by convection [W/(m$^2$·K)]
- $\beta$: parameter correlating the intrinsic dissipation and the second harmonic of the temperature signal [/]
- $\varepsilon(t), \varepsilon_a$: applied strain versus time, its amplitude [m/m]
- $\varepsilon_e(t)$: elastic part of strain function [m/m]
- $\varepsilon_p(t)$: plastic part of strain function [m/m]
- $\lambda$: material thermal conductivity [W/(m·K)]
- $\nu$: material Poisson’s coefficient
- $\rho$: material density [kg/m$^3$]
- $\sigma(t), \sigma_a$: applied stress versus time, its amplitude [MPa]
- $\sigma_{\text{ii}}$: first invariant of the stress tensor (sum of normal stresses) [MPa]
material yield strength [MPa]

range of the temperature variation of the material due to the thermoelastic effect [K]

range of material temperature versus time due to plasticity effects, evaluated starting from the lower tip of the hysteresis cycle [K]

range of the second harmonic of the Fourier Transform of the temperature signal [K]

applied strain versus time, evaluated starting from the lower tip of the hysteresis cycle, its elastic and plastic component [m/m]

applied stress versus time, evaluated starting from the lower tip of the hysteresis cycle [MPa]

angular frequency of the applied stress or strain [rad/s]

REFERENCES

1. Feltner CE, Morrow JD. Microplastic strain hysteresis energy as a criterion for fatigue fracture. Trans. ASME, Series D. J Basic Engineering. 1961;83(1):15-22.

2. Morrow JD. Cyclic plastic strain energy and fatigue of metals. In: International Friction, Damping and Cyclic Plasticity. Philadelphia, PA: ASTM STP 378, American Society for Testing and Materials; 1965:45-84.

3. Halford GR. The energy required for fatigue. J Mater. 1966;19:3-18.

4. Ellyin F. Fatigue Damage, Crack Growth and Life Prediction. London: Chapman & Hall; 1997.

5. Atzori B, Meneghetti G, Ricotta M. Unified material parameters based on full compatibility for low-cycle fatigue characterization of as-cast and austempered ductile irons. Int J Fatigue. 2014;68:111-122.

6. Ricotta M. Simple expressions to estimate the Manson–Coffin curves of ductile cast irons. Int J Fatigue. 2015;78:38-45.

7. Curti G, Geraci A, Risitano A. A new method for rapid determination of the fatigue limit. Ingegneria Automotoristica. 1989;42:634-636. in Italian

8. Dengel D, Harig H. Estimation of the fatigue limit by progressively-increasing load tests. Fatigue Fract Eng Mater Struct. 1980;3(2):113-128.

9. Luong MP. Infrared thermographic scanning of fatigue in metals. Nuc Eng des. 1995;158(2-3):365-376.

10. Luong MP. Fatigue limit evaluation of metals using an infrared thermographic technique. Mech Mater. 1998;28(1-4):155-163.

11. La Rosa GA, Risitano A. Thermographic methodology for rapid determination of the fatigue limit of materials and mechanical components. Int J Fatigue. 2000;22(1):65-73.

12. Krapez JC, Pacou D, Gardette G. Lock-in thermography and fatigue limit of metals. In: Proceedings of Q.I.R.T, July 18–21. Reims (France); 2000.

13. Boulander T, Chrysochoos A, Mabru C, Galtier A. Calorimetric analysis of dissipative and thermoelastic effects associated with the fatigue behavior of steels. Int J Fatigue. 2004;26(3):221-229.

14. Curà F, Curti G, Sesana R. A new iteration method for the thermographic determination of fatigue limit in steels. Int J Fatigue. 2005;27(4):453-459.

15. Fernández-Canteli A, Castillo E, Argüelles A, Fernández P, Canales M. Checking the fatigue limit from thermographic techniques by means of a probabilistic model of the epsilon-N field. Int J Fatigue. 2012;39:109-115.

16. Li XD, Zhang H, Wu DL, Liu X, Liu JY. Adopting lock-in infrared thermography technique for rapid determination of fatigue limit of aluminum alloy riveted component and affection to determined result caused by initial stress. Int J Fatigue. 2012;36(1):18-23.

17. Wang XG, Crupi V, Guo XL, Guglielmino E. A thermography-based approach for structural analysis and fatigue evaluation. P I Mech Eng C-J Mec. 2012;226(5):1173-1185.

18. Risitano A, Risitano G. Determining fatigue limits with thermal analysis of static traction tests. Fatigue Fract Engng Mater. Struct. 2013;36(7):631-639.

19. Fan J, Guo X, Wu C, Zhao Y, Guo Q. Stress assessment and fatigue behavior evaluation of components with defects based on the finite element method and lock-in thermography. P I Mech Eng C-J Mec. 2015;7:1194-1205.

20. De Finis R, Palumbo D, Ancona F, Galietti U. Fatigue limit evaluation of various martensitic stainless steels with new robust thermographic data analysis. Int J Fatigue. 2015;74:88-96.

21. Palumbo D, Galietti U. Thermoelastic Phase Analysis (TPA): a new method for fatigue behaviour analysis of steel. Fatigue Fract Eng Mater Struct. 2017;40(4):523-534.

22. Doudard C, Calloch S, Hild F, Cugy P, Galtier A. Identification of the scatter in high cycle fatigue from temperature measurements. Comptes Rendus Mecanique. 2004;332:785-801.

23. Doudard C, Calloch S, Hild F, Cugy P, Galtier A. A probabilistic two-scale model for high cycle fatigue life predictions. Fatigue Fract Eng Mater Struct. 2005;3:279-288.

24. Doudard C, Calloch S. Influence of hardening type of self-heating of metallic materials under cyclic loadings at low amplitude. Eur J Mech-A/ Solids. 2009;28(2):233-240.

25. Ezanno A, Doudard C, Calloch S, Heuzé JL. A new approach to characterizing and modelling the high cycle fatigue properties of cast materials based on self-heating measurements under cyclic loadings. Int J Fatigue. 2013;47:232-243.
26. Plekhov O, Palin-Luc T, Saintier N, Uvarov S, Naimark O. Fatigue crack initiation and growth in a 35CrMo4 steel investigated by infrared thermography. *Fatigue Fract Eng Mater Struct*. 2005;28(1-2):169-178.

27. Morabito AE, Chrysochoos A, Dattoma V, Galietti U. Analysis of heat sources accompanying the fatigue of 2024 T3 aluminium alloys. *Int J Fatigue*. 2007;29(5):977-984.

28. Ummenhofer T, Medgenberg J. On the use of infrared thermography for the analysis of fatigue damage process in welded joints. *Int J Fatigue*. 2009;31(1):130-137.

29. Jones R, Krishnapillai M, Cairns K, Matthews N. Application of infrared thermography to study crack growth and fatigue life extension procedures. *Fatigue Fract Eng Mater Struct*. 2010;33(12):871-884.

30. Wagner D, Ranc N, Bathias C, Paris PC. Fatigue crack initiation detection by an infrared thermography method. *Fatigue Fract Eng Mater Struct*. 2010;33:12-21.

31. Wang C, Blanche A, Wagner D, Chrysochoos A, Bathias C. Dissipative and microstructural effects associated with fatigue crack initiation on an Armco iron. *Int J Fatigue*. 2014;58:152-157.

32. Fargione G, Geraci A, La Rosa G, Risitano A. Rapid determination of the fatigue curve by the thermographic method. *Int J Fatigue*. 2002;24(1):11-19.

33. Yang B, Liaw PK, Morrison M, et al. Temperature evolution during fatigue damage. *Intermetallics*. 2005;13(3-4):419-428.

34. Starke P, Walther F, Euller D. Fatigue assessment and fatigue life calculation of quenched and tempered SAE 4140 steel based on stress–strain hysteresis, temperature and electrical resistance measurements. *Fatigue Fract Eng Mater Struct*. 2007;30(11):1044-1051.

35. Charkaluk E, Constantinescu A. Dissipative aspects in high cycle fatigue. *Mech Mater*. 2009;5:483-494.

36. Amiri M, Khonsari MM. Rapid determination of fatigue failure based on temperature evolution: fully reversed bending load. *Int J Fatigue*. 2010;32(2):382-389.

37. Crupi V, Chiofalo G, Guglielmino E. Infrared investigations for the analysis of low cycle fatigue processes in carbon steels. *PI Mech Eng C-J Mech*. 2011;225:833-842.

38. Jegou L, Marco Y, Le Saux V, Calloch S. Fast prediction of the Wöhler curve from heat build-up measurements on Short Fiber Reinforced Plastics. *Int J Fatigue*. 2013;47:259-267.

39. Zhang L, Liu XS, Wu SH, Ma ZQ, Fang HY. Rapid determination of fatigue life based on temperature evolution. *Int J Fatigue*. 2013;54:1-6.

40. Munier R, Doudard C, Calloch S, Weber B. Determination of high cycle fatigue properties of a wide range of steel sheet grades from self-heating measurements. *Int J Fatigue*. 2014;63:46-61.

41. Risitano A, Risitano G. Cumulative damage evaluation of steel using infrared thermography. *Theor Appl Fract Mec*. 2010;54(2):82-90.

42. Fan J, Guo X, Wu C. A new application of the infrared thermography for fatigue evaluation and damage assessment. *Int J Fatigue*. 2012;44:1-7.

43. Risitano A, Risitano G. Cumulative damage evaluation in multiple cycle fatigue tests taking into account energy parameters. *Int J Fatigue*. 2013;48:214-222.

44. Poncelet M, Doudard C, Calloch S, Hild F, Weber B, Hild F. Prediction of self-heating measurements under proportional and non-proportional multiaxial cyclic loadings. *Comptes Rendus Mecanique*. 2007;2:81-86.

45. Poncelet M, Doudard C, Calloch S, Hild F, Hild F, Weber B. Probabilistic multiscale models and measurements of self-heating under multiaxial high cycle fatigue. *J Mech Phys Solids*. 2010;4:578-593.

46. Meneghetti G. Analysis of the fatigue strength of a stainless steel based on the energy dissipation. *Int J Fatigue*. 2007;29(1):81-94.

47. Meneghetti G, Ricotta M, Atzori B. A synthesis of the push-pull fatigue behaviour of plain and notched stainless steel specimens by using the specific heat. *Fatigue Fract Engng Mater Struct*. 2013;36(12):1306-1322.

48. Meneghetti G, Ricotta M, Atzori B. A two-parameter, heat energy-based approach to analyse the mean stress influence on axial fatigue behaviour of plain steel specimens. *Int J Fatigue*. 2016;82:60-70.

49. Meneghetti G, Ricotta M. The use of the specific heat loss to analyse the low- and high cycle fatigue behaviour of plain and notched specimens made of a stainless steel. *Eng Fract Mech*. 2012;81:2-16.

50. Rigon D, Ricotta M, Meneghetti G. An analysis of the specific heat loss at the tip of severely notched stainless steel specimens to correlate the fatigue strength. *Theor Appl Fract Mech*. 2017;92:240-251.

51. Meneghetti G, Ricotta M, Negrisolo L, Atzori B. A synthesis of the fatigue behaviour of stainless steel bars under fully reversed axial or torsion loading by using the specific heat loss. *Key Eng Mater*. 2013;577-578:453-456.

52. Ricotta M, Meneghetti G, Atzori B, Risitano G, Risitano A. Comparison of experimental thermal methods for fatigue limit evaluation of a stainless steel. *Metals*. 2019;9(6):677-700.

53. Feltner CE, Morrow JD. Microplastic strain hysteresis energy as a criterion for fatigue fracture. *Trans. ASME, Series D, J. Basic Eng*. 1961;83(1):15-22.

54. Biotny R, Kaleta J. A method for determining the heat energy of the fatigue process in metals under uniaxial stress. Part 1. *Int. J. Fatigue*. 1986;8(1):29-33.

55. Wong AK, Sparrow JG, Dunn SA. On the revised theory of the thermoelastic effect. *J Phys Chem Solid*. 1988;49(4):395-400.

56. Pitarresi G, Patterson EA. A review of the general theory of thermoelastic stress analysis. *J Strain Anal Eng des*. 2003;38(5):405-417.

57. Bar J, Seilnacth L, Urbanek R. Determination of dissipated energies during fatigue tests on copper and AA7475 with infrared thermography. *Procedia Structural Integrity*. 2019;17:308-315.

58. Bar J, Urbanek R. Determination of dissipated energy in fatigue crack propagation experiments with lock-in thermography. *Fracture and Structural Integrity*. 2019;48:563-570.

59. Brémond P. New developments in Thermo Elastic Stress Analysis by Infrared Thermography. IV Pan-American Conference for Non-Destructive Testing, 2007.

60. Akai A, Shiozawa D, Sakagami T. Fatigue limit evaluation for austenitic stainless steel. *J of Soc Mat Sci, Japan*. 2012;61(12):953-959.

61. Palumbo D, De Finis R, Demelo GP, Galietti U. Damage monitoring in fracture mechanics by evaluation of the heat dissipated in the cyclic plastic zone ahead of the crack tip with thermal measurements. *Eng Fract Mech*. 2017;181:65-76.
APPENDIX A

In Equation 35, an idealized thermal boundary condition has been used, to simulate the achievement of thermal stabilization over time, which consists in a space- and time-independent heat sink (Equation 34a). Let us analyze the dependence of the β parameter on the thermal boundary conditions; two scenarios were considered referring to the elastic-perfectly plastic material:

a. a constant heat power equal to \( \dot{W}/2t \) is extracted from the material by the heat sink only during the two linear elastic paths of the stress–strain loop reported in Figure 2; instead, during the perfectly plastic path adiabatic conditions are supposed to exist;

b. to approach little more realistic thermal boundary conditions, the heat power is extracted steadily by convection during cyclic loading, by keeping the hypothesis of uniform temperature distribution valid; in this case the heat sink does not work any longer (\( \dot{v}_c = 0 \)).

In both cases, two different virtual tests were considered, as follows:

1. \( \varepsilon_a = 2\% \), \( \sigma_y = 300 \text{ MPa} \), \( E = 194,700 \text{ MPa} \), \( f_l = 0.1 \text{ Hz} \), \( \rho = 7,940 \text{ kg/m}^3 \); \( c = 507 \text{ J/(kg K)} \); \( \alpha = 16 \times 10^{-6} \text{ K}^{-1} \); \( T_{amb} = 293.15 \text{ K} \)

2. \( \varepsilon_a = 0.15\% \), \( \sigma_y = 250 \text{ MPa} \), \( E = 194,700 \text{ MPa} \), \( f_l = 5 \text{ Hz} \), \( \rho = 7,940 \text{ kg/m}^3 \); \( c = 507 \text{ J/(kg K)} \); \( \alpha = 16 \times 10^{-6} \text{ K}^{-1} \); \( T_{amb} = 293.15 \text{ K} \)

Case (a)

In this case, the material temperature evolution governed by Equation 35 can be re-written as

\[
\Delta T(t) = \Delta T_{the}(t) + \Delta T_p(t) + \Delta T_{sink}(t) \rightarrow = \Delta T_{the}(t)
\]

\[
+ \frac{1}{\rho \cdot c} \cdot \sigma_y \cdot \left\{ \varepsilon_a \cdot \left[ 1 - \cos(a_0 \cdot \dot{t}) \right] - \frac{2\gamma}{E\varepsilon_a} \cdot \frac{\dot{W}}{2\sigma_y} \right\} \cdot R_{c,1}(t) +
\]

\[
+ 2 \left[ \varepsilon_a - \frac{\sigma_y}{E} \right] \cdot \frac{\dot{W}}{2\sigma_y} \cdot R_c(t) + \left[ \varepsilon_a \cdot \cos(a_0 \cdot \dot{t}) + \left( 3\varepsilon_a - \frac{4\sigma_y}{E} \right) \cdot \frac{\dot{W}}{2\sigma_y} \right] \cdot R_{c,2}(t) +
\]

\[
- \frac{\dot{W}}{\rho \cdot c} \cdot \frac{t^2}{2t} \cdot R_1(t) - \frac{\dot{W}}{\rho \cdot c} \cdot \frac{t^2}{2t} \cdot R_1 \left( \frac{t + \dot{t}}{2} \right)
\]

(A.1)

where \( R_c(t) \) is a rectangular function, with \( \tau = \dot{t} \) and \( \delta = \dot{t}/\dot{T} \) (see Equation 19) and \( \Delta T_{the}(t) \) is Equation 26.

1. From Equation 17b, \( \dot{t} = 8.95 \times 10^{-1} \text{s} \), and from Equation 32, \( \dot{T} = 22.15 \text{MJ/(m}^3\text{•cycle)} \). Figure A.1A reports the temperature T(t) derived from Equation A.1 taking into account Equation 36 and the plastic

---

How to cite this article: Meneghetti G, Ricotta M. Estimating the intrinsic dissipation using the second harmonic of the temperature signal in tension-compression fatigue: Part I. Theory. Fatigue Fract Eng Mater Struct. 2021;44: 2168–2185. https://doi.org/10.1111/ffe.13485
temperature evaluated in light of Equation 38 as $T_p(t) = T(t) - T_{pe}(t)$. It is seen that the temperature shot occurring twice in a strain cycle due to plasticity has a range equal to $\frac{W}{2\pi c} = 2.75\,\text{K}$, according to the hypothesis of adiabatic conditions when the material undergoes plastic deformation. However, the range of the 2nd harmonic $\Delta T_2 = 2.35\,\text{K}$ is lower than $2.75\,\text{K}$ because the temperature due to plasticity is not a sine function, as explained in the paper. Finally, according to Equation 2, $\beta = 1.17$, as shown in Figure 7.

2. From Equation 17b, $\bar{t} = 7.52 \cdot 10^{-2}\,\text{s}$, and from Equation 32, $\bar{W} = 0.216\,\text{MJ/}[\text{m}^3\cdot\text{cycle}]$. Figure A.1B shows the results, similarly to previous Figure A.1A and again it is seen that $\Delta T_2 = 2.084 \cdot 10^{-2}\,\text{K}$ is lower than $\bar{W} \approx 2.683 \cdot 10^{-2}\,\text{K}$. According to Equation 2, $\beta = 1.29$ in this case, as shown in Figure 7.

Case (b)
To account for the heat power extracted by convection, Equation 6 is written as follows:

$$\rho c T - \dot{Q}_{\text{conv}} = \dot{Q}_p + \dot{Q}_{\text{the}}$$  \hspace{1cm} (A.2)

where $\dot{Q}_{\text{conv}}$ is the heat power extracted by convection\textsuperscript{46}:

$$\dot{Q}_{\text{conv}} = -\alpha_{\text{conv}} \cdot (T(t) - T_{\text{amb}}) \cdot \frac{p}{A}$$  \hspace{1cm} (A.3)

where $\alpha_{\text{conv}}$, T(t), $T_{\text{amb}}$, p and A are the heat transfer coefficient by convection, the material temperature, the ambient temperature, the perimeter and the cross-sectional area of the plain specimen shown in Figure A.2, respectively.

Therefore, by integrating Equation A.2, the material temperature is

$$\Delta T(t) = \int_0^t \dot{Q}_p(t) / \rho c \, dt + \int_0^t \dot{Q}_{\text{the}}(t) / \rho c \, dt + \int_0^t \dot{Q}_{\text{conv}}(t) / \rho c \, dt$$  \hspace{1cm} (A.4)

Equation A.4 was numerically solved assuming a typical specimen geometry, with $t_k = 6\,\text{mm}$ and $w = 12\,\text{mm}$ (see Figure A.2). Assuming a realistic heat transfer coefficient of natural convection on the order of $\alpha_{\text{conv}} = 10\,\text{W/} (\text{m}^2\cdot\text{K})$, it was found $\beta = 3.732$ (to be compared to $\beta = 3.707$ evaluated from Figure 7 by using Equation 35) and $\beta = 1.631$ (to be compared to $\beta = 1.682$ evaluated from Figure 7 by using Equation 35) for cases (1) and (2), respectively, as shown in Figure 7. To conclude, although

![Figure A.1](http://wileyonlinelibrary.com)  \hspace{1cm} FIGURE A.1  \hspace{1cm} Material temperature evolution due to the superposition of thermoelasticity and plasticity effects over one strain cycle in the case of an elastic-perfectly plastic material, evaluated according to case (a) hypothesis; (A) $\varepsilon_a = 2\%$, $\sigma_y = 300\,\text{MPa}$, $E = 194,700\,\text{MPa}$, $f_t = 0.1\,\text{Hz}$ and (B) $\varepsilon_a = 0.15\%$, $\sigma_y = 250\,\text{MPa}$, $E = 194,700\,\text{MPa}$, $f_t = 5\,\text{Hz}$ (in Figure A.1B, $T_p(t)$ and second and harmonic are magnified 10 times) [Colour figure can be viewed at wileyonlinelibrary.com]

![Figure A.2](http://wileyonlinelibrary.com)  \hspace{1cm} FIGURE A.2  \hspace{1cm} Plain specimen with a rectangular cross section having area $A$ and perimeter $p$
the $\beta$ parameter depends on the thermal boundary conditions of the test, this dependence is very limited in the case of usual laboratory test conditions, which have been simulated here in the case (b). To investigate further the stability of $\beta$ parameter to the thermal boundary conditions, Table A1 reports some scenarios where the load test frequency $f_L$ and the convection coefficient $\alpha_{\text{conv}}$ were varied very significantly from the usual laboratory test conditions. Simulations were performed by using the heat transfer mode case (b). As an example, Table A1 demonstrates that by increasing the convection coefficient by two orders of magnitude (from the realistic value 10 W/(m$^2$ K) to the unrealistic value 1,000 W/(m$^2$ K) for usual laboratory testing conditions) $\beta$ increases from 3.732 to 3.891, respectively, that is, only 4.3%.

| $\varepsilon_a$ | $\sigma_y$ [MPa] | $W$ [MJ/(m$^3$ cycle)] | $\beta$ Equation 35 | $f_L$ [Hz] | $\alpha_{\text{conv}}$ [W/(m$^2$ K)] | $\beta$ |
|-----------------|-----------------|------------------------|---------------------|-------------|---------------------------------|--------|
| 2%              | 300             | 22.15                  | 3.707               | 0.1         | /                               | 1.17$^a$|
|                 |                 |                        |                     | 0.1         | 10                              | 3.732$^b$|
|                 |                 |                        |                     | 0.001       | 10                              | 3.891$^b$|
|                 |                 |                        |                     | 0.1         | 1,000                           | 3.891$^b$|
| 0.15%           | 250             | 0.216                  | 1.682               | 5           | /                               | 1.29$^a$|
|                 |                 |                        |                     | 5           | 10                              | 1.631$^b$|
|                 |                 |                        |                     | 0.001       | 10                              | 1.753$^b$|
|                 |                 |                        |                     | 0.1         | 1,000                           | 1.755$^b$|

$^a$Case (a).
$^b$Case (b).