Exceptional points as a route to magnetic nano-oscillators

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One of the most fascinating and puzzling aspects of non-Hermitian systems is their spectral degeneracies, i.e., exceptional points (EPs), at which both eigenvalues and eigenvectors coalesce to form a defective state space. While coupled magnetic systems are natural hosts of EPs, the relation between the linear and nonlinear spin dynamics in the proximity of EPs remains relatively unexplored. Here we theoretically investigate the spin dynamics of easy-plane magnetic bilayers in proximity of exceptional points. We show that the interplay between the intrinsically dissipative spin dynamics and external drives can yield a rich dynamical phase diagram. In particular, we find that, in antiferromagnetically coupled bilayers, a periodic oscillating dynamical phase emerges in the region enclosed by EPs. Our findings have practical utility for engineering magnetic nano-oscillator with very large amplitude oscillations and have the potential to shed light on a plethora of non-Hermitian systems displaying EPs and nonlinearities.

Introduction. The degeneracies of Hermitian Hamiltonians are diabolic points, i.e., points at which two (or more) real eigenenergies coalesce under a system parameter variation, while the eigenstates still span the full Hilbert space. Non-Hermitian degeneracies, i.e., exceptional points (EPs), display properties that are radically different from their Hermitian counterpart. At an EP, two (or more) complex eigenvalues and the corresponding eigenvectors simultaneously coalesce, resulting into a defective Hamiltonian that cannot span the entire Hilbert space [1,3]. The incompleteness of the eigenbases at second-order EPs leads to a square root dependence on external perturbations, resulting in a giant sensitivity-factor enhancement [4,7].

As non-Hermitian systems are recently under comprehensive research [8–12], intense efforts have been put forward to explore the properties of EPs. Particular emphasis has been placed on PT-symmetric systems [6,13–15], where EPs signal a PT-symmetry-breaking transition at which a system’s eigenvalues turn from real to complex conjugate pairs. The emergence of EPs does not, however, require a fine-tuned balance of gain and loss. EPs have been reported in a plethora of open systems, ranging from optics and photonics [1,3,13–15] to superconducting quantum circuits [16,17], semimetals [18,19] and magnetic systems [20–31].

Magnetic systems are intrinsically open due to the ubiquitous dissipation of magnetization dynamics [31–33]. The gain can be tuned via experimentally established techniques such as, e.g., spin current injection via an adjacent metal [30,31,34–37]. Exceptional points naturally emerge in the description of coupled magnetization dynamics and have been recently observed in magnonic PT-symmetric devices [31]. Second-order and higher-order EPs displaying higher-order roots singularities [38–44], which can yield further ultra-sensitivity, have been reported in magnetic multilayers [20]. While the potential of EPs in magnetic sensing has been under intense scrutiny, the role that EPs play in dynamical magnetic phase transitions is yet relatively unexplored. Coupled magnetization dynamics can be described, in the long-wavelength limit, via coupled Landau-Lifshitz-Gilbert (LLG) equations [45]. By linearizing the LLG equations of motion, one can derive an effective non-Hermitian Hamiltonian quadratic in second-quantized magnon operators. The EPs appear as singularities of the quadratic Hamiltonian, signaling a dynamical phase transition of the linearized dynamics due to a width bifurcation [46–49]. If signatures of such transition survive in the nonlinear LLG-like classical dynamics, the analysis of the corresponding quadratic magnon Hamiltonian can unveil unforeseen dynamical regimes as function of experimentally tunable parameters.

In this Letter, we explore the connection between linear and nonlinear spin dynamics in proximity of EPs by taking as an example an easy-plane magnetic bilayer. The ratio between gain and loss is modulated by spin injection in the bottom layer and the loss of magnetization dynamics is taken to be larger than the overall gain. As function of the interlayer coupling, we find that the linearized spectrum displays two regions circled by exceptional points, emerging around, respectively, vanishing and strong antiferromagnetic (AFM) interlayer coupling. The non-linear dynamics in proximity of the region with vanishing interlayer coupling displays a ferromagnetic (FM)-to-AFM dynamical phase transition. Such transition has been reported in a magnonic PT-symmetric system [25]; our results show that fine-tuned balance of gain and loss is not necessary for the transition to take place.

Furthermore, we unveil a distinct dynamical phase transition occurring in the AF-coupled region circled by the EPs. Simulations of the nonlinear dynamics show, that upon crossing the EP in parameter space, the damped magnetization dynamics enters a regime of steady self-oscillations with large amplitude that can be described by a supercritical Hopf-Bifurcation [50,51]. According to our estimates, this dynamical phase transition can be observed in bilayer CrCl₃ [52], opening up a new route to engineer large-amplitude magnetic nano-
oscillators \([34, 53–59]\). Our findings have also the potential to shed light on the interplay between linear and nonlinear dynamics in a plethora of non-Hermitian systems with nonlinearities.

**Model.** We consider the magnetic bilayer shown in Fig. 1(a) whose spin Hamiltonian can be written, in the long-wavelength limit, as

\[
\mathcal{H} = \sum_{i=A,B} (K S_i^z + \gamma B_0 \cdot \mathbf{S}_i) + J \mathbf{S}_A \cdot \mathbf{S}_B,
\]

where \(\mathbf{S}_{A(B)}\), with \(|\mathbf{S}_{A,B}| = S\), is the dimensionless (macro) spin operator of the top (bottom) layer, \(B_0\) the applied magnetic field, \(\gamma > 0\) the gyromagnetic ratio, \(J\) the interlayer coupling, and \(K \geq 0\) parametrizes the easy-plane anisotropy. To introduce loss and gain, we recast the magnetization dynamics in the form of coupled Landau–Lifshitz–Gilbert (LLG) equations, i.e.,

\[
\begin{align*}
\frac{d\mathbf{S}_A}{dt} &= -\gamma \mathbf{S}_A \times \mathbf{B}_A^\text{eff} - \frac{\alpha_A}{S} \mathbf{S}_A \times \frac{d\mathbf{S}_A}{dt}, \\
\frac{d\mathbf{S}_B}{dt} &= -\gamma \mathbf{S}_B \times \mathbf{B}_B^\text{eff} - \frac{\alpha_B}{S} \mathbf{S}_B \times \frac{d\mathbf{S}_B}{dt},
\end{align*}
\]

where we have introduced the effective field \(\gamma \mathbf{B}_i^\text{eff} = -\partial \mathcal{H}/\partial \mathbf{S}_i\), with \(i = A, B\). Here \(\alpha_A > 0\) represents the effective damping parameter of the top layer. The effective gain \(\alpha_B < 0\) can be introduced by injecting spin current into the bottom layer.

To investigate the non-Hermitian spin-wave spectrum as function of the exchange coupling \(J\) and magnetic field \(B_0\), we orient the spin-space Cartesian coordinate system such that the \(\hat{z}\) axis locally lies along the classical orientation of the macrospin \(\mathbf{S}_i\). The latter can be related to the spin operator \(\mathbf{S}_i\) in the global frame of reference via the transformation

\[
\mathbf{S}_i = \mathcal{R}_z(\phi_i) \mathcal{R}_y(\theta_i) \hat{\mathbf{S}}_i,
\]

where the matrix \(\mathcal{R}_z(\eta)\mathcal{R}_y(\varphi)\) describes a right-handed rotation by an angle \(\varphi\) about the \(\hat{z}(\hat{y})\) axis, and \(\theta_i(\phi_i)\) is the polar (azimuthal) angle of the classical orientation of the spin \(\mathbf{S}_i\). We then solve self-consistently Eqs. (2) and (3) in the linear approximation, i.e., we consider \(\hat{\mathbf{S}}_i = \hat{\mathbf{S}}_i^+ + i \hat{\mathbf{S}}_i^-\) and invoke the Holstein-Primakoff transformation \([60, 61]\) \(\hat{\mathbf{S}}_i^+ \approx \sqrt{2} \mathbf{a}(\mathbf{b})\), where the second-quantized operator \(\mathbf{a}(\mathbf{b})\) annihilates a magnon in the top (bottom) layer and obeys bosonic commutation relations. By invoking the Heisenberg equation for \(\mathbf{a}(\mathbf{b})\), we obtain the non-Hermitian Hamiltonian \(\mathcal{H}_{nh}\). The resulting Hamiltonian is not block-diagonal and a Bogoliubov transformation is required to obtained the spin-wave spectrum.

**Antiferromagnetic to ferromagnetic transition.** As a first instructive example, we turn off the easy-plane anisotropy, i.e., \(K = 0\), and we take a damping coefficient of the same order of magnitude of the ones reported for chromium trihalide crystals \([62]\), i.e., \(\alpha_A = 0.06\), while we set \(\alpha_B = -0.04\) \([63]\). We set \(B_0 = 0.1\) T and take \(B_0 \parallel \hat{x}\). It is worth noting that our results do not depend on the field direction since the Hamiltonian \([1]\) is \(SO(3)\)-symmetric for \(K = 0\). The real and imaginary energy spectra of \(\mathcal{H}_{nh}\) as a function of \(J\) are shown, respectively, in Fig. 1(b) and 1(c). Near \(J = 0\), region I is enclosed by EPs. On the left side of the red dashed line, the ground state of the Hermitian Hamiltonian (i.e., Eq. (1) for \(\alpha_A = \alpha_B = 0\)) is collinear and oriented along the magnetic field. On the right side of the dashed line, the interplay between the magnetic field and the antiferromagnetic coupling \(J\) leads to a noncollinear ground state, while increasing \(J\) further yields an AFM ground state.
magnon energy are shown in Fig. 2(a) and 2(b) respectively. We find two regions enclosed by EPs: region I near $J = 0$ and region II near $J = 12.2 \mu eV$, i.e., the exchange interaction of CrCl$_3$. Region I corresponds to region I shown in Figs. 1(b) and 1(c). Region II emerges instead in correspondence with a noncollinear ground state and, as we will show in details, its nonlinear magnetization dynamics display very different features from the ones observed in region I.

Figures 2(c)–2(e) show the time evolution of the relative alignment of the macrospins $S_{AB}(t)$ for, respectively, $J = 9, 12.2$, and $16 \mu eV$. Similarly to region I, passing through the EPs yields a dynamical phase transition. However, around region II, the exchange interaction is too strong for a FM-to-AFM switching to take place. Instead, while for $J = 9.0 \mu eV$ and $J = 16.0 \mu eV$ we observe damped dynamical phases, see Figs. 2(c) and 2(e) inside region II (i.e., $J = 12.2 \mu eV$) a periodic dynamical phase emerges, as shown in Fig. 2(d). Within the periodic dynamical phase, the value of $S_{AB}$ ranges from $0.7$ to $-0.7$, signaling unusual large-amplitude oscillations. Our results show that, although the overall loss is larger than the effective gain, i.e., $\alpha_A > |\alpha_B|$, the system can still survive in a steady periodic state in a EP-enclosed region. The dynamical phase transition can be understood as a supercritical Hopf-Bifurcation $[50, 51]$. When crossing the EPs and entering in region II, the fixed point of the dynamical system, which corresponds to the damped magnetization dynamics, bifurcates into a stable orbit. **Tunability.** We proceed to investigate the dependence of the periodic stable magnetization dynamics on the system’s parameters. Not surprisingly, the stability of the periodic solution strongly depends on the ratio between the effective gain and loss. Setting $J = 12.2 \mu eV$ and $\alpha_A = 0.06$, in Fig. 3(a-d) we show the time evolution of $S_A$ (upper panel) and $S_B$ (lower panel) on the Bloch sphere decreasing the effective gain $|\alpha_B|$ from $0.055$ to $0.01$. The colors in Fig. 3(a-d) are in direct correspondence with the time intervals of the time-evolution of $S_{AB}$ shown in Fig. 2(c-e). For larger values of gain, e.g., $\alpha_B = -0.055$, the dynamics of both macrospins $S_A$ and $S_B$ flow to a fixed point, as shown by Fig. 3(a). We have verified that the same scenario is realized at the $PT$-symmetric point. For lower values of the gain, the spin dynamics evolve into a steady-state oscillations, see Figs. 3(b-d). Since the macrospin $S_B$ is directly subjected to gain while $S_A$ experiences it indirectly via the coupling to $S_A$, the amplitude of oscillations of the macrospin $S_A$ is smaller than the one of $S_B$. For decreasing $\alpha_B$, the amplitude of both limit cycles shrink.

In an experimental setup, the effective gain $\alpha_B$ can be controlled via spin injection. The ratio $\alpha_A/\alpha_B$ is determined by the spin current transport efficiency through the magnetic layers which, to our knowledge, has not been yet thoroughly investigated in van der Waals magnets. Our results show that, however, the periodic oscilla-
with values of the magnetic field’s strength and cant angle Θ, does not require fine-tuning: there is a broad range of namics. As shown by Fig. 3(f), accessing the region II II, i.e., it displays periodic oscillatory coupled spin dy-

P exceptional point, the two eigenstates coalesce, i.e., \( \psi_1 \) \( \to \psi_2 \). The above (below) panels shows the time evolution of \( \mathbf{S}_A \) (\( \mathbf{S}_B \)). The color on curves are in direct correspondence with the time intervals of the time-evolution of \( \mathbf{S}_{AB} \) in Figs.2(c)-(d), i.e., they label the earliest to the latest time by ordering purple, blue, gray, green, yellow, orange, and red. (a) For \( \alpha_B = -0.055 \), the dynamics of \( \mathbf{S}_A \) and \( \mathbf{S}_B \) flow into fixed points. (b-d): When \( |\alpha_B| \leq 0.05 \), the system drops on steady orbitals through the supercritical Hopf-Bifurcation. (e) Frequency \( f \) of the coupled oscillations \( \mathbf{S}_{AB} \) as a function of the effective gain \( \alpha_B \) for different values of \( J \). For \( J = 4.2 \mu \text{eV} \) \( (J = 8.2 \mu \text{eV}) \), steady periodic dynamical phases exist only for \( |\alpha_B| \leq 0.035 \) \( (|\alpha_B| \leq 0.045) \). (f) The dependence of the square of the overlap of the two right eigenvectors, i.e., \( \mathcal{P}_{EP} \equiv \left| \langle \psi_1^R | \psi_2^R \rangle \right|^2 \), on the magnetic field strength \( B_0 \) and cant angle \( \Theta \).

Summary and outlook. In this Letter, we investigate the interplay between the linear and nonlinear spin dynamics in proximity of exceptional points. We show that the emergence of exceptional points in the linearized magnon Hamiltonian underlies a dynamical phase transition of the nonlinear spin dynamics. As an example, we consider on an easy-plane bilayer in which, while one layer experiences effective gain, the overall spin dynamics is lossy. An analysis of the linearized long-wavelength magnetization dynamics of the bilayer shows that two regions encircled by EPs can appear as function of the interlayer coupling. One region, characterized by small values of the interlayer coupling, displays a FM-to-AFM dynamical phase transition. The second region, appearing for larger values of the AFM interlayer coupling, displays large-amplitude self-sustained oscillations. We show that this oscillatory dynamical regime might be accessed via spin injection in CrCl\(_3\), opening a concrete route for engineering large-amplitude magnetic nano-oscillators.

Future works should address the interplay between linear and non-linear spin dynamics beyond the long-wavelength limit and explore dynamical phase transitions signaled by EPs in other non-Hermitian systems characterized by nonlinearities.

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[63] These values can be chosen with a certain flexibility as long as the loss is larger than the gain, as we shall discuss in detail later.