Classification of Fano manifolds containing a negative divisor isomorphic to projective space

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Abstract

We classify $n$-dimensional complex Fano manifolds $X$ ($n \geq 3$) containing a divisor $E$ isomorphic to $\mathbb{P}^{n-1}$ such that $\text{deg} N_{E/X}$ is strictly negative.

1 Introduction

A projective manifold $X$ is called Fano manifold if its anti-canonical bundle $-K_X$ is ample. In [BCW] the authors classified $n$-dimensional complex Fano manifolds $X$ ($n \geq 3$) containing a divisor $E$ isomorphic to $\mathbb{P}^{n-1}$ with normal bundle $N_{E/X} \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. The purpose of the present note is to generalize their result to the case $\text{deg} N_{E/X} < 0$. Note that such $X$ has automatically Picard number $\rho(X) \geq 2$.

Theorem 1. Let $X$ be a complex Fano manifold of dimension $n \geq 3$ and let $E$ be a divisor of $X$. We suppose that $E$ is isomorphic to $\mathbb{P}^{n-1}$ and let $E|_E \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(-d)$. If $d > 0$, the pair $(X, E)$ is (up to isomorphism) one of the following: 1

1. $X \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-d))$ ($0 < d < n$) and $E$ is the negative section such that $E|_E \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(-d);$  

2. (a) $X$ is the blow-up of $\mathbb{P}^n$ along $W$ smooth complete intersection of a hyperplane and a hypersurface of degree $r$ with $2 \leq r \leq n$, and $E$ is the strict transform of the hyperplane containing $W;$  

(b) $X$ is the blow-up of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(d'))$ ($d'$ is an integer satisfying $-n < d' < n$) along $W$ and $E$ is the strict transform of $E'$: here $E'$ is a section of the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(d'))$ with normal bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(d')$ and $W$ is a smooth divisor of degree $r$ ($d' < r < n + d'$) in $E' \simeq \mathbb{P}^{n-1}$.

As in [BCW] the proof of this theorem is based on the theory of extremal contractions, which was originally used in [MM] to classify Fano 3-folds with Picard number $\geq 2$. Recall that a fiber connected proper holomorphic map $\varphi : X \to Z$ to a normal projective variety is called extremal contraction if $-K_X$ is $\varphi$-ample. We say $\varphi$ is elemental if $\rho(X) - \rho(Z) = 1$. By the Mori theory, a Fano manifold has a finite number of elemental extremal contractions. So in our situation, we can take an extremal contraction $\varphi : X \to Z$ whose fiber meets the negative divisor $E$. The essential part of the classification is to show that the restriction map $\varphi|_E : E \to \varphi(E)$ is an isomorphism. The corresponding fact is shown in [BCW], but their proof relies heavily on the fact $\text{deg} N_{E/X} = -1$. In this note we will propose another approach which allows us to get our result.

1For a vector space $V$, we denote by $\mathbb{P}(V)$ the projective space of lines in $V$. 

1
2 Preliminary results

Let \( X \) be a complex Fano manifold of dimension \( n \geq 3 \) containing a divisor \( E \simeq \mathbb{P}^{n-1} \). We assume \( \deg N_{E/X} < 0 \) and let \( E|_E \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(-d) \) with \( d > 0 \). Note that \( d < n \). Indeed, since \(-K_X\) is ample (so is \(-K_X|_E\)), by the adjunction formula: \( K_E = (K_X + E)|_E \), we have

\[
0 < \deg(-K_X|_E) = \deg(-K_E) + \deg(E|_E) = n - d.
\]

A similar argument as \([BCW]\) (Lemme 1) shows that there exists an extremal ray \( \mathbb{R}^+[f] \) (where \( f \) is a minimal rational curve of the ray) such that \( E \cdot f > 0 \) and the fibers of the corresponding (elementary) extremal contraction \( \varphi : X \to Z \) are at most of dimension 1. By \([A]\), either:

1. \( \varphi \) is a conic bundle with smooth base \( Z \), or

2. \( \varphi \) is a smooth blow-up whose centre is smooth and of codimension 2.

We shall show that the restriction map \( \varphi|_E : E \to \varphi(E) \) is an isomorphism in each case.

**Lemma 1.** In the case (1), \( \varphi|_E \) is an isomorphism.

**Proof.** Since \( \deg N_{E/X} < 0 \), by Grauert’s criterion, there is a fiber connected holomorphic map \( \pi : X \to Y \) to an analytic variety such that \( \pi(E) \) is a point. We have

\[
K_X = \pi^*K_Y + \alpha E
\]

where \( \alpha := (n - d)/d \). Here \( \pi^*K_Y \) is to be considered as a \( \mathbb{Q} \)-divisor \((\pi^*(dK_Y))/d\) on \( X \) (note that \( dK_Y \) is Cartier divisor because \( \pi(E) \) is a cyclic quotient singular point of order \( d \)). This formal expression is useful in the following numerical calculations.

Since the smooth projective variety \( Z \) is dominated by \( E \simeq \mathbb{P}^{n-1} \) we have \( Z \simeq \mathbb{P}^{n-1} \) (see \([L]\)). Hence, if we define \( L := \varphi^*\mathcal{O}_Z(1) \), \( L^{n-1} \) is numerically equivalent to only one fiber of \( \varphi \). Therefore,

\[
\begin{aligned}
L^n &= 0 \\
(-K_X) \cdot L^{n-1} &= 2.
\end{aligned}
\]

Since the contraction map \( \varphi \) is supposed to be elemental, we have \( \rho(X) = \rho(Z) + 1 = 2 \). Hence there exist \( x, y \in \mathbb{Q} \) such that

\[
L \equiv x \pi^*(-K_Y) - y E.
\]

We have \( yd \in \mathbb{N} \), because \( 0 < L \cdot e = -y(E \cdot e) = yd \) where \( e \) is a line in \( E \simeq \mathbb{P}^{n-1} \). Remark that \( E^n = (-d)^{n-1} \) and \( \pi^*(-K_Y) \cdot E \equiv 0 \). We get

\[
L^n = (x \pi^*(-K_Y) - y E)^n = x^m - y^n d^{n-1},
\]

\[
(-K_X) \cdot L^{n-1} = (\pi^*(-K_Y) - \alpha E)(x \pi^*(-K_Y) - y E)^{n-1} = x^{n-1}m - \alpha y^{n-1} d^{n-1}
\]

where \( m := (\pi^*(-K_Y))^n \). Remark that \( m \in \mathbb{Q} \) and \( md^{n} \in \mathbb{N} \). We have

\[
\begin{aligned}
x^m m &= y^n d^{n-1} \\
x^{n-1} m &= 2 + \alpha y^{n-1} d^{n-1}
\end{aligned}
\]
We divide the first equality by the second:

\[ x = \frac{y^n d^{n-1}}{2 + \alpha y^{n-1} d^{n-1}}. \]  

(1)

By the first equation again,

\[ \left( \frac{y}{x} \right)^n = \frac{m}{d^{n-1}}. \]  

(2)

Since

\[ \frac{y}{x} = y \cdot \frac{2 + \alpha y^{n-1} d^{n-1}}{y^n d^{n-1}}, \]

we have

\[ \left( \frac{2 + \alpha y^{n-1} d^{n-1}}{y^{n-1} d^{n-1}} \right)^n = \frac{m}{d^{n-1}}. \]

We get

\[ \left( \frac{2d^2}{(yd)^{n-1}} + \alpha d^2 \right)^n = \left( \frac{2 + \alpha y^{n-1} d^{n-1}}{y^{n-1} d^{n-1}} \right)^n \cdot d^{2n} = md^n \cdot d \in \mathbb{N}, \]

because \( md^n \) and \( d \) are integers. It follows that

\[ \frac{2d^2}{(yd)^{n-1}} + (n - d)d \in \mathbb{N} \]

Hence \( l := \frac{2d^2}{(yd)^{n-1}} \) is an integer. We have

\[ 2d^2 = (yd)^{n-1} l. \]

On the other hand, \((l + (n - d)d)^n = d(md^n) \in d\mathbb{N}\). Hence \( l^n \in d\mathbb{N}\).

Now we can show the equality \( yd = 1 \): Recall first \( yd \in \mathbb{N} \) and \( d < n \). If \( yd \geq 3 \), \( 2(n - 1)^2 \geq 2d^2 = (yd)^{n-1} l \geq 3^{n-1} l \). This is impossible because \( n \geq 3 \). We suppose now \( yd = 2 \). We have \( 2d^2 = 2^{n-1} l \), hence \( d^2 = 2^{n-2} l \). Since \( d \leq n - 1 \) and \( l^n \in d\mathbb{N} \), this equality is possible only when \((n, d, l) = (3, 2, 2) \) or \((5, 4, 2) \). These two cases can be ruled out by using the fact that \((-K_X)^n\) is a natural number (because \( X \) is smooth). Note first that

\[ (-K_X)^n = \left( \pi^*(-K_Y) - \frac{n - d}{d} E \right)^n = m - \frac{(n - d)^n}{d}. \]

If \((n, d, l) = (3, 2, 2)\) then \( y = yd/d = 1 \), \( x = 1 \) and \( m = 4 \) by \( \text{[1]} \) and \( \text{[2]} \). Therefore, \((-K_X)^3 = 7/2 \notin \mathbb{N}\), contradiction. Similarly, if \((n, d, l) = (5, 4, 2)\) we have \( y = yd/d = 1/2 \), \( x = 4/3 \) and \( m = 243/128 \), so that \((-K_X)^5 = 211/128 \notin \mathbb{N}\), contradiction. Finally, we conclude that \( yd = 1 \).

Now we can determine the intersection number \( E \cdot L^{n-1} \):

\[ E \cdot L^{n-1} = E \cdot (x\pi^*(-K_Y) - yE)^{n-1} = (yd)^{n-1} = 1. \]

Since \( L^{n-1} \) is a fiber of \( \varphi \), this implies that the restriction map \( \varphi|_E : E \to Z \) is an isomorphism.

\[ \blacksquare \]

Lemma 2. In the case (2) also, \( \varphi|_E \) is an isomorphism.
by the hypothesis of the proposition, we get 

If $C$ exists we have the example 1. 

In particular $E \subset B$ we deduce the exact sequence over $Z$ 

Theorem 1. 

The intersection $E \cap F$ is transversal. 

Proof: If not, we have an isomorphism of tangent bundles: $TE|_{\tilde{W}} \simeq TF|_{\tilde{W}}$ and so $N_{\tilde{W}}/E \simeq N_{\tilde{W}}/F$. But this is a contradiction. Indeed: $N_{\tilde{W}}/E$ is ample (because $\tilde{W} \subset E \simeq \mathbb{P}^{n-1}$) and $N_{\tilde{W}}/F$ is also ample (because $\tilde{W}$ is contracted by $\pi|_{F} : F \to \pi(F) \subset Y$). 

Hence $\tilde{W}$ is a section without multiplicity, namely $\varphi|_{E} : E \to E' := \varphi(E)$ is an isomorphism. In particular $E' \simeq \mathbb{P}^{n-1}$. 

Lemma 3. In the case (2), $Z$ is a Fano manifold. 

Proof. In fact, by the Proposition (see below), it is sufficient to show that for every curve $B \subset W$, we have $-K_{Z} \cdot B > 0$. Let $e' := \varphi_{*}e$. Since $\varphi|_{E}$ is an isomorphism, $e'$ is a line in $E' \simeq \mathbb{P}^{n-1}$. Since $d < n$, we get 

$$-K_{Z} \cdot e' = -K_{X} \cdot e + F \cdot e = n - d + r > 0$$

where $r$ is the degree of $W$ in $E' \simeq \mathbb{P}^{n-1}$. For each curve $B$ contained in $W$ (so in $E'$), there exists $b \in \mathbb{N}$ such that $B \equiv be'$. Hence $-K_{Z} \cdot B = b(-K_{Z} \cdot e') > 0$. 

Proposition 1. Let $X$ be a Fano manifold of dimension $n \geq 3$ and $\varphi : X \to Z$ a blow-up of center $W$ smooth subvariety of codimension $k$ ($2 \leq k \leq n$) in a projective manifold $Z$. If $-K_{Z} \cdot B > 0$ for any curve $B$ contained in $W$, then $Z$ is Fano. 

Proof. Let $F$ be the exceptional divisor of $\varphi$ and let $R$ be the extremal ray defining $\varphi$. Let $C$ be a curve of $X$ such that $[C] \notin R$. If $C \notin F$, $F \cdot C \geq 0$ so that 

$$\varphi^{*}(-K_{Z}) \cdot C = -K_{X} \cdot C + (k - 1)F \cdot C > 0.$$ 

If $C \subset F$, $\varphi_{*}C$ is an effective 1-cycle (because $[C] \notin R$) whose support is contained in $W$. So, by the hypothesis of the proposition, we get $-K_{Z} \cdot \varphi_{*}C > 0$. By Lemma (3.1), this means that $-K_{Z}$ is ample. 

3 Proof of Theorem 

In this section, we prove two propositions which imply our main Theorem. 

Proposition 2. In the case (1) ($\varphi$ is a conic bundle), we have the example 1 in the list of Theorem 

Proof. Since $\varphi|_{E} : E \to Z$ is an isomorphism, $\varphi$ is necessarily a $\mathbb{P}^{1}$-bundle and $E \cdot f = 1$ where $f \simeq \mathbb{P}^{1}$ is any fiber. From the exact sequence 

$$0 \to \mathcal{O}_{X} \to \mathcal{O}_{X}(E) \to \mathcal{O}_{E}(E) \to 0,$$

we deduce the exact sequence over $Z \simeq \mathbb{P}^{n-1}$: 

$$0 \to \mathcal{O}_{\mathbb{P}^{n-1}} \to \varphi_{*}\mathcal{O}_{X}(E) \to \mathcal{O}_{\mathbb{P}^{n-1}}(-d) \to 0,$$

so that $\varphi_{*}\mathcal{O}_{X}(E)$ is identified to $\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-d)$ and $X$ to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-d))$. Hence we have the example 1. 

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Proposition 3. In the case (2) (\(\varphi\) is a blow-up along a centre of codimension 2), we have the examples 2-(a) or 2-(b).

Proof. Since \(Z\) is Fano, there exists an elementary extremal contraction \(\mu := \text{cont}_{\mathbb{R}^+[m]} : Z \to Z'\) such that \(E' \cdot m > 0\) where \(m\) is a minimal rational curve of the ray.

The case where there exists a fiber \(M = \mu^{-1}(z')\) of dimension \(\geq 2\). In this case there exists a curve \(B \subset E' \cap M\). So \([B] \in \mathbb{R}^+[e']\) and \([B] \in \mathbb{R}^+[m]\) (\(e'\) is a line in \(E' \simeq \mathbb{P}^{n-1}\)). Hence \(\mathbb{R}^+[e'] = \mathbb{R}^+[m]\). It follows that (the numerical class of) any curve in \(E'\) is on the ray \(\mathbb{R}^+[m]\), namely, \(\mu(E')\) is a point in \(Z'\). On the other hand, \(E' \cdot e' > 0\) because \(E' \cdot m > 0\). By Proposition [4] (see below) we have \(\rho(Z) = 1\) and the effective divisor \(E' \simeq \mathbb{P}^{n-1}\) is then ample. Finally, by [BCW] Lemme 4, we conclude that \((Z, E') \simeq (\mathbb{P}^n, O_{\mathbb{P}^n}(1))\). In particular, \(E' \cdot e' = 1\).

Now we estimate the number \(r := \text{deg of } W\) as a divisor in \(E' \simeq \mathbb{P}^{n-1}\). Since \(\varphi^*E' = E + F\), we have \(1 = E' \cdot e' = E \cdot e + F \cdot e = -d + r\). But since \(0 < d < n\), we obtain \(2 \leq r \leq n\). It follows that \(W = E' \cap L\) where \(E' \in |O_{\mathbb{P}^n}(1)|\) and \(L \in |O_{\mathbb{P}^n}(r)|\) (\(2 \leq r \leq n\)). So we get the example (2)-(a).

The case where every fiber of \(\mu\) is at most of dimension 1 (the following argument is essentially due to [BCW]). By [A], the elementary extremal contraction \(\mu : Z \to Z'\) is one of the following:

1. a \(\mathbb{P}^1\)-bundle,

2. a conic bundle (with singular fibers),

3. a smooth blow-up whose centre is smooth and of codimension 2.

Note first that only the first case is possible. In fact in the other cases, \(K_Z \cdot m = -1\) and there exists a minimal rational curve \(m\) such that \(m \cap W\) is not empty. If \(\tilde{m}\) is the strict transform of \(m\), we have

\[
K_X \cdot \tilde{m} = K_Z \cdot m + F \cdot \tilde{m} = -1 + F \cdot \tilde{m} \geq 0,
\]

which is a contradiction because \(X\) is Fano.

So, it is sufficient to study the case where \(\mu\) is a \(\mathbb{P}^1\)-bundle. In this case each fiber (= \(m\)) meets \(W\) transversally at a single point: in fact, if not, there exists \(m\) such that \(F \cdot \tilde{m} \geq 2\). Then, as above, \(K_X \cdot \tilde{m} = K_Z \cdot m + F \cdot \tilde{m} = -2 + F \cdot \tilde{m} \geq 0\), contradiction. So \(\mu_W : W \to \mu(W) \subset Z'\) is an isomorphism (by [L], \(Z'\) is isomorphic to \(\mathbb{P}^{n-1}\) because \(Z'\) is dominated by \(E' \simeq \mathbb{P}^{n-1}\).

We have a standard commutative diagram of Picard groups:

\[
\begin{array}{ccc}
\text{Pic}(Z') & \longrightarrow & \text{Pic}(E') \\
\downarrow & & \downarrow \\
\text{Pic}(\mu(W)) & \cong & \text{Pic}(W)
\end{array}
\]

By a theorem of Lefchetz, the two vertical maps (induced by inclusions: \(\mu(W) \subset Z'\) and \(W \subset E'\)) are isomorphisms. Therefore \(\mu^* : \text{Pic}(Z') \to \text{Pic}(E')\) is an isomorphism, so is \(\mu_{|E'} : E' \to Z'\).

Let \(E'_{|E'} \simeq O_{\mathbb{P}^{n-1}}(d')\) (\(d' = -d + r\) where \(r\) is the degree of \(W \subset E' \simeq \mathbb{P}^{n-1}\), so \(d'\) might be positive). Finally \(Z \simeq \mathbb{P}(O_{\mathbb{P}^{n-1}} \oplus O_{\mathbb{P}^{n-1}}(d'))\) and \(E'\) is a section with normal bundle \(O_{\mathbb{P}^{n-1}}(d')\).

Now we estimate the possibilities of the integers \(d'\) and \(r\). Since \(Z\) is Fano, we get immediately \(-n < d' < n\). We recall now that \(0 < d < n\). Since \(r = d' - d\), we have finally \(d' < r < n + d'\). So we get the example (2)-(b). ■

Proposition 4. Let \(X\) be a smooth projective variety and let \(\varphi : X \to Z\) be an elementary extremal contraction of ray \(R\). We assume that there exists a prime divisor \(E\) such that \(\varphi(E)\) is a point. If there exists a curve \(C \subset E\) such that \(E \cdot C > 0\) then \(\rho(X) = 1\).
Proof. By assumption, $E \cdot R > 0$ (ie: for all $[\Gamma] \in R$, we have $E \cdot \Gamma > 0$). If $\varphi$ is birational, $E = \text{Exc}(\varphi)$ and $E \cdot R < 0$ \footnote{Let $H$ be a hyperplane section of $Z$ passing through the point $\varphi(E)$. We can write: $\varphi^*H = \tilde{H} + kE$ ($k > 0$) where $\tilde{H}$ is the strict transform of $H$. Note that there exists a curve $A \subset E$ such that $\tilde{H} \cdot A > 0$. We get $0 = (\varphi^*H) \cdot A = \tilde{H} \cdot A + k(E \cdot A)$. Therefore $E \cdot A < 0$. It follows that $E \cdot R < 0$ because $\varphi$ is an elementary contraction.}, a contradiction. Hence $\varphi$ is of fiber type. If $\dim Z > 0$, we take a point $z \neq \varphi(E)$. Then for any curve $B$ contained in $\varphi^{-1}(z)$, we get $E \cdot B = 0$ although $[B] \in R$. This contradicts our assumption. Therefore $Z$ is a point, namely $\rho(X) = 1$. ■

4 Comments on the bound of Picard number

By the classification result of Theorem 1, the Picard number of a Fano manifold $X$ containing a divisor $E \simeq \mathbb{P}^{n-1}$ with $\deg N_{E/X} < 0$ is less than or equal to 3. This is in fact true in more general situation.

Proposition 5. Let $X$ be an $n$-dimensional Fano manifold ($n \geq 3$). We assume that $X$ contains a prime divisor $E$ with $\rho(E) = 1$. Then $\rho(X) \leq 3$.

Proof. Since $X$ is Fano, we can take an extremal ray $\mathbb{R}^+[f]$ such that $E \cdot f > 0$. Let $\varphi := \text{cont}_{\mathbb{R}^+[f]} : X \to Z$ be the associated elementary extremal contraction.

If there exists $z \in Z$ such that $\dim \varphi^{-1}(z) \geq 2$, then there exists a curve $B \subset E \cap \varphi^{-1}(z)$. We have $[B] \in \mathbb{R}^+[f]$, because $B \subset \varphi^{-1}(z)$. This implies that $E \cdot B > 0$. Since $B \subset E$, we have $\rho(X) = 1$ by Proposition 4.

If $\dim \varphi^{-1}(z) \leq 1$ for all $z \in Z$, by $\varphi$ is either a conic bundle or a smooth blow-up of a smooth center of codimension 2 (here, $\mathbb{P}^1$-bundle is considered as a special case of conic bundles).

- If $\varphi$ is a conic bundle (or a $\mathbb{P}^1$-bundle), the restriction map $\varphi|_E : E \to Z$ is surjective (and moreover finite). Since $\rho(E) = 1$, $\rho(Z) = 1$. It follows that $\rho(X) = \rho(Z) + 1 = 2$ (because $\varphi$ is elemental).

- We treat now the case where $\varphi$ is a blow-up along a smooth center $W$ of codimension 2. Let $E' = \varphi(E)$. Since there exists a surjective map $\varphi|_E : E \to E'$, we have $\rho(E') = 1$.

Claim. The smooth variety $Z$ is Fano.

Proof: By Proposition 4 it is sufficient to show that for any curve $B \subset W$ we have $-K_Z \cdot B > 0$. Let $A$ be a curve in $E'$ not contained in $W$. Since $\rho(E') = 1$, there exists a positive real number $a$ such that $B \equiv aA$ in $E'$. Since $E' \subset Z$, this numerical equivalence holds also in $Z$. Therefore $-K_Z \cdot B > 0$ if and only if $-K_Z \cdot A > 0$. This is in fact the case, because

$$-K_Z \cdot A = -K_X \cdot \tilde{A} + F \cdot \tilde{A} > 0$$

where $\tilde{A}$ is the strict transform of $A$ by $\varphi$.

So we can take an extremal ray $\mathbb{R}^+[m]$ such that $E' \cdot m > 0$. Let $\mu := \text{cont}_{\mathbb{R}^+[m]} : Z \to V$ be the associated elementary extremal contraction.

- If there exists $v \in V$ such that $\dim \varphi^{-1}(v) \geq 2$, by the same argument as above, we get $\rho(Z) = 1$. Hence $\rho(X) = \rho(Z) + 1 = 2$. 


If \( \dim \varphi^{-1}(v) \leq 1 \) for all \( v \in V \), as mentioned above, \( \mu \) is either a conic bundle or a blow-up of codimension 2. In the first case, we have a surjective map \( \mu|_{E'} : E' \to V \), so \( \rho(V) = 1 \) and \( \rho(Z) = \rho(V) + 1 = 2 \). It follows that \( \rho(X) = \rho(Z) + 1 = 3 \).

Now we rule out the birational case: let \( M \) be the exceptional divisor of \( \mu \). Since \( \rho(E') = 1 \), \( M \cap W \neq \emptyset \). It follows that there is a fiber \( m \) of the exceptional divisor \( M \) such that \( m \cap W \neq \emptyset \). If \( \tilde{m} \) is the strict transform of \( m \) by \( \varphi \), we get

\[
K_X \cdot \tilde{m} = K_Z \cdot m + F \cdot \tilde{m} \geq -1 + 1 = 0,
\]

a contradiction because \( X \) is Fano.

We conclude that in every case, we have \( \rho(X) \leq 3 \). ■

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