Robust Tensor Recovery using Low-Rank Tensor Ring

Huyan Huang, Yipeng Liu, Senior Member, IEEE, Ce Zhu, Fellow, IEEE

Abstract—Robust tensor completion recovers the low-rank and sparse parts from its partially observed entries. In this paper, we propose the robust tensor ring completion (RTRC) model and rigorously analyze its exact recovery guarantee via TR-unfolding scheme, and the result is consistent with that of matrix case. We propose the algorithms for tensor ring robust principle component analysis (TRRPCA) and RTCR using the alternating direction method of multipliers (ADMM). The numerical experiment demonstrates that the proposed method outperforms the state-of-the-art ones in terms of recovery accuracy.

Index Terms—Tensor ring robust principle component analysis, Robust tensor ring completion, alternating direction method of multipliers

I. INTRODUCTION

In practice, the acquired data are often slightly corrupted with small noises and hence present a rambling pattern, which results in a high-dimensional structure. As a widely used dimensionality reduction method, principle component analysis (PCA) extracts the most useful low-dimensional structure in observed data and reduces the unnecessary features. However, PCA suffers from the performance deterioration caused by grossly corruption. To address this issue, [1] considers the model as a superposition of low-rank matrix and the sparse noise and first gives the strong recovery guarantee for robust PCA (RPCA). Specifically, given an observed matrix $T$ and we wish to decompose it as $T = L_0 + S_0$, where $L$ is a low-rank matrix and $S$ is a sparse matrix. Then the exact $L_0$ and $S_0$ is obtained by solving the convex program

$$\min_{L, S} \|L\|_* + \lambda\|S\|_1, \text{ s. t. } L + S = T.$$  

The RPCA is applicable to a series of applications such as signal processing [2], machine learning [3], remote sensing [4], computer vision [5], etc.

A tensor is a multi-dimensional array that naturally represents the high-dimensional observation, and it is helpful for preserving more intrinsic structures and information than matrix when dealing with high-order data such as RGB images, light-field images and videos, etc [2], [6], [7]. It is ubiquitous that during data transmission parts of the data entries are not only missing but also (grossly) corrupted. RPCA is not able to perform well in this situation. Similar to RPCA, the most existing tensor methods are based on the assumption of low-rank property [8]. However, the extension for RPCA of matrix to tensor is hard because of its algebraic problem [9]. [10] proposes an algorithm for tensor RPCA (TRPCA) that utilizes the sum of Tucker nuclear norm (SNN) and hence is called TRPCA-SNN. The Tucker nuclear norm is a suboptimal convex surrogate [11], [12] of Tucker-rank which is defined as \(\text{rank}_{\text{Tucker}}(X) = \left[\text{rank}(X_{(1)}), \ldots, \text{rank}(X_{(d)})\right]^T\), where $X_{(i)}$ is the tensor unfolding along its $i$-th dimension [6]. [13] provides the exact recovery guarantee for TRPCA-SNN. Tensor singular value decomposition (t-SVD) factorizes a 3-way tensor into two orthogonal tensors and a f-diagonal tensor based on the tensor-tensor product [14], and the tubal rank is defined as the number of non-vanishing tubes in the f-diagonal tensor [15], [16] proves that the tensor nuclear norm (TNN) is the convex envelop of tubal rank and proposes a method called TRPCA-TNN for TRPCA with strong recovery guarantee. The corresponding model can be represented as

$$\min_{L, S} \|L\|_* + \lambda\|S\|_1, \text{ s. t. } L + S = T.$$  

The recently proposed tensor ring (TR) decomposition decomposes a high-order tensor as a sequence of cyclically contracted 3-order tensors [17], [18]. The cycle forces TR-rank to be small and TR-rank is consistently invariant under the cyclic permutation of the factors. [19], [20] show that the TR model has the advantage of powerful representation ability compared with other decompositions such as tensor train (TT) and Tucker. [21] proposes a balanced TR-unfolding scheme for tensor completion and shows performance improvement. [22] proposes an approach for tensor completion using TR nuclear norm minimization and provides the exact recovery guarantee. Motivated by [22], we extend their analysis to TRPCA. Our contributions are itemized as follows:

1) We rigorously analyze the sampling condition for TRPCA using the TR model, in which only one TR-unfolding suffices to yield the exact recovery. Inspired by this finding, we propose two algorithms that yield the exact recovery with high probability for robust PCA and robust completion using the TR decomposition. The two methods are called tensor ring robust PCA (TRRPCA) and robust TR completion (RTRC), respectively.

2) We use an alternating direction method of multipliers (ADMM) to solve two proposed models. The numerical experiments show the performance improvement in practical applications such as RGB image recovery and background modeling.

The organization for the remainder of this paper is arranged as follows. In section II, the basic notions of TR decomposition are introduced. In section III, we develop the analysis for robust TR completion. In section IV, we introduce
two algorithms for TRPCA and its robust version. Section V displays the experimental results. Finally we conclude our work in section VI.

II. NOTATIONS AND PRELIMINARIES

A. Notations about tensor ring decomposition

This section introduces some basic notations of tensors. A scalar, a vector, a matrix and a tensor are denoted by normal letter, boldface lowercase letter, boldface uppercase letter and calligraphic letter, respectively. For instance, a d-order tensor is denoted as $\mathbf{X} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$, where $n_i$ is the size corresponding to dimension $i$, $i \in \{1, \ldots, d\}$. An entry of the tensor $\mathbf{X}$ is denoted as $x_{j_1, \ldots, j_d}$, where $j_i$ is the index with mode $i$, $1 \leq j_i \leq n_i$. A mode-i fiber of $\mathbf{X}$ can be denoted as $x_{j_1, \ldots, 1, \ldots, j_d}$ and $x_{j_1, \ldots, 1, \ldots, j_d}$ represents the slice along mode $i$.

$$\|\mathbf{X}\|_2 = \sigma_{\text{max}}(\mathbf{X})$$

denotes the spectral norm of the matrix $\mathbf{X}$, which is equal to its maximal singular value. We regard $\mathbf{E}$ as the identity matrix. The inner product of two tensors $\mathbf{X}$ and $\mathbf{Y}$ is defined as $\langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1}^{n_d} x_{j_1, \ldots, j_d} y_{j_1, \ldots, j_d}$. The Frobenius norm of a tensor $\mathbf{X}$ can be defined as $\|\mathbf{X}\|_F = \sqrt{\langle \mathbf{X}, \mathbf{X} \rangle}$. The Kronecker product is written as $\mathbf{X} \otimes \mathbf{Y}$. The Hadamard product $\circ$ is an element-wise product which is $a \circ b$ is an element-wise product which is $a_i b_i$.

Mode-i unfolding maps a tensor to a matrix $\mathbf{X}_{(i)} \in \mathbb{R}^{n_i \times d}$ by rearranging the fibers as the columns of the matrix, i.e., $x_{j_1, \ldots, j_d} = x_{j_1, \ldots, j_d}$, and $j = j_1 \cdots j_{i-1} j_{i+1} \cdots j_d = j_1 + \sum_{i \neq k} x_{j_1, \ldots, j_{i-1} j_i \ldots, j_d}$. Mode-i matricization unfolds a tensor along its first $i$ modes [19], i.e., $\mathbf{X}_{[i]} \in \mathbb{R}^{\prod_{k=1}^i n_k \times \prod_{k=i+1}^i n_k}$. In the context of TR decomposition, $\mathbf{X}_{(i)}$ denotes the $i$-shifting l-matricization of the tensor $\mathbf{X}$. It firstly permutes the tensor with order $[i, \ldots, d, 1, \ldots, i-1]$ and performs matricization along first $l$ modes. As a special case of l-shifting l-matricization, l-shifting balanced unfolding $\mathbf{X}_{(i)} \in \mathbb{R}^{n_i \times n_i + 1}$ for the tensor $\mathbf{X}$ permutates a tensor with order $[i, \ldots, d, 1, \ldots, i-1]$ and unfolds tensor along its first $[d/2]$ modes. The indices of $(\mathbf{X}_{[k]})_{pq}$ can be formulated as $p = 1 + \sum_{i=1}^{d-k+2} q = 1 + \sum_{i=1}^{d-k+2} (j_i - 1) \prod_{m=k+1}^{d} n_m$.

Let $\mathcal{G} = \{G^{(1)}, \ldots, G^{(d)}\}$ denote the cores of TR decomposition and the TR rank be $[r_1, \ldots, r_N]^T$, where $G^{(n)} \in \mathbb{R}^{r_n \times n_k \times n_{k+1}}$. Then the scalar form of TR decomposition can be written as $x_{j_1, \ldots, j_d} = \sum_{t_1=1}^{r_1} \cdots \sum_{t_d=1}^{r_d} g^{(1)}_{t_1} \cdots g^{(d)}_{t_d} \mathbf{g}^{(1)}_{t_1, \ldots, t_d} \mathbf{g}^{(d)}_{t_1, \ldots, t_d}$. Equivalently, it can be represented by a more compact form $x_{j_1, \ldots, j_d} = \text{tr} \left( \mathbf{G}^{(1)}_{j_1} \cdots \mathbf{G}^{(d)}_{j_d} \right)$, where $\mathbf{G}^{(k)}_{j_k}$ is the $j_k$-th mode-2 slice of core $\mathbf{G}^{(k)}$, and tr(·) is the trace function. The tensored representation is $\mathbf{X} = \sum_{t_1=1}^{r_1} \cdots \sum_{t_d=1}^{r_d} \mathbf{g}^{(1)}_{t_1} \cdots \mathbf{g}^{(d)}_{t_d} \mathbf{g}^{(1)}_{t_1, \ldots, t_d} \mathbf{g}^{(d)}_{t_1, \ldots, t_d}$, where $\mathbf{g}^{(k)}_{t_k t_{k+1}}$ is the $(t_k, t_{k+1})$-th mode-2 fiber of core $\mathbf{G}^{(k)}$, and $\otimes$ denotes the outer product.

We use $\otimes$ to denote the tensor connection product. It combines several TR-cores into a new core and the formula is $\bigotimes_{i=a}^{b} g^{(k)}_{i} = \sum_{i=a}^{b} \sum_{i=a}^{b} \cdots \sum_{i=a}^{b} \mathbf{g}^{(a)}_{t_a t_{a+1}} \otimes \cdots \otimes \mathbf{g}^{(b)}_{t_b t_{b+1}}$, where $\bigotimes_{i=a}^{b} g^{(k)}_{i} \in \mathbb{R}^{r_a \times \prod_{k=\max(a,b)}^{b} n_k}$.

B. Preliminaries

This subsection reviews the existing TRPCA methods. Following the RPCA, [10], [13] first propose the approach called TRPCA-SNN which is based on the Tucker-rank and its convex relaxation:

$$\min_{\lambda} \sum_{i=1}^{d} w_i \|x_{(i)}\|_1 + \|S\|_1, \text{ s. t. } \mathcal{P}_\Omega (\mathbf{X} + S) = \mathcal{P}_\Omega (T).$$

Then [16] proposes the TRPCA-TNN based on the tensor nuclear norm.

$$\min_{\lambda} \|\mathbf{X}\|_\Omega + \|S\|_1, \text{ s. t. } \mathcal{P}_\Omega (\mathbf{X} + S) = \mathcal{P}_\Omega (T).$$

III. SAMPLING FOR LOW-RANK TENSOR COMPLETION VIA TENSOR RING

This section introduces the main result of exact recovery guarantee for TRRTC. [22] rigorously proves the relationship between TR-rank and TR-unfolding’s rank, which is characterized in Fig. 1. Specifically, the rank of TR-unfolding $\mathbf{X}_{(i)}$ is rank $\mathbf{X}_{(i)} = r_i r_{i+1}, i \in \{1, \ldots, d\}$, $l \in \{1, \ldots, d-1\}$, provided that the TR-rank is $[r_1, \ldots, r_d]^T$. Their study also indicates that it suffices to consider a (sub)critical TR for the TRPCA, and in the next a TR is short for a (sub)critical TR wherever it appears.

**Theorem 1** (Robust TR completion, uniformly bounded model). Let a d-order tensor $\mathbf{X} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$ be sampled from a uniformly bounded model with TR-rank being $[r_1, \ldots, r_d]^T$. Define $\pi_{kl}$ and $\bar{\pi}_{kl}$ as the maximum and minimum values of $\prod_{i=k}^{d} n_i$. Then solving (5) with $\lambda_i = 1/\sqrt{\bar{\pi}_{kl}}$ gives the exact and unique solution $\mathbf{X}$ with probability at least $1 - C_1 \pi_{kl}^{-3}$, provided that

$$r_k r_{k+1} \leq C_2 \pi_{kl}^{-1} \left( \left[ d/2 \right] \ln \bar{\pi}_{kl} \right)^{-2}, \gamma \leq C_3,$$

where $C$, $C_r$ and $C_\gamma$ are positive numerical constants, and $p$ is probability of random sampling.
IV. ROBUST TENSOR RING COMPLETION METHOD

According to Theorem 1 and basic algebraic knowledge, a good strategy to improve the bound is to set \( l = \lfloor d/2 \rfloor \) within the \( \{i,l\} \) unfolding scheme. Then we derive model (4)

\[
\min_{X_{i}, S} \sum_{i=1}^{[d/2]} w_{i} \|X^{(i)}_{i}\|_{s} + \lambda_{i} \|S\|_{1}
\]

\[
\text{subject to } X^{(i)} + S = T,
\]

in which the subscript \( \{i,l\} \) is abbreviated to \( (i) \). This model is called tensor ring robust principle component analysis (TRRPCA).

Another model is an enhancement of model (4) since it incorporates the uniform and random sampling scheme. Consider the operator \( \varepsilon_{d/2} \) as the sampling process, the corresponding model can be modified as

\[
\min_{X_{i}, \varepsilon, S} \sum_{i=1}^{[d/2]} w_{i} \|X^{(i)}_{i}\|_{s} + \lambda_{i} \|S\|_{1}
\]

\[
\text{subject to } \varepsilon_{d/2} (L + S) = \varepsilon_{d/2} (T)
\]

\[
X^{(i)} = L \ (i = 1, \ldots, [d/2]).
\]

We term this model as robust tensor ring completion (RTRC).

A. Algorithm for TRRPCA

In order to solve (4) using ADMM, we consider its augmented Lagrangian (AL) function

\[
\min_{X_{i}, S} \lambda \|S\|_{1} + \sum_{i=1}^{[d/2]} w_{i} \|X^{(i)}_{i}\|_{s} + \langle Z^{(i)}, X^{(i)} + S - T \rangle + \frac{\mu}{2} \|X^{(i)} + S - T\|_{F}^{2}.
\]

The solution to problem (6) is given as follows.

1) Update of \( X^{(i)} \): Taking the first-order derivative of objective function in (6) we have

\[
\mathcal{O} = \partial \frac{\|X^{(i)}_{i}\|_{s}}{\mu} + X^{(i)} - \left( T - S - \frac{1}{\mu} Z^{(i)} \right),
\]

which is the optimality condition of

\[
\min_{X^{(i)}} \frac{w_{i}}{\mu} \|X^{(i)}_{i}\|_{s} + \frac{1}{2} \|X^{(i)} - \left( T - S - \frac{1}{\mu} Z^{(i)} \right)\|_{F}^{2},
\]

hence the optimal solution of \( X^{(i)} \) is

\[
X^{(i)*} = D_{\frac{\mu}{\|T - S - \frac{1}{\mu} Z^{(i)}\|_{F}}} ^{\lambda \varepsilon_{d/2}} \left( T - S - \frac{1}{\mu} Z^{(i)} \right).
\]

2) Update of \( S \): Follow the previous calculation there is

\[
\mathcal{O} = \partial \frac{\|S\|_{1}}{\mu} + \sum_{i=1}^{[d/2]} \left[ S - \left( T - X^{(i)} - \frac{1}{\mu} Z^{(i)} \right) \right]
\]

which is the optimality condition of

\[
\min_{S} \frac{\lambda}{\mu} \|S\|_{1} + \sum_{i=1}^{[d/2]} \frac{1}{2} \|S - \left( T - X^{(i)} - \frac{1}{\mu} Z^{(i)} \right)\|_{F}^{2},
\]

By introducing the vector operator \( 1 \) the model becomes

\[
\min_{S} \frac{\lambda}{\mu} \|S\|_{1} + \frac{1}{2} \|S - \left( T - \frac{1}{\mu} Z \right)\|_{F}^{2},
\]

the corresponding optimality condition is

\[
\mathcal{O} = \partial \frac{\|S\|_{1}}{\mu} + \left( T - X^{(i)} - \frac{1}{\mu} Z^{(i)} \right).
\]

Rewrite the above representation as

\[
\mathcal{O} = \partial \frac{\|S\|_{1}}{\mu} + \frac{1}{2} \|S - \left( T - X^{(i)} - \frac{1}{\mu} Z^{(i)} \right)\|_{F}^{2},
\]

thereby the update is

\[
S^{*} = S - \frac{1}{\mu} d_{\frac{\lambda}{\|T - X^{(i)} - \frac{1}{\mu} Z^{(i)}\|_{F}}} \left( \frac{1}{[d/2]} \sum_{i=1}^{[d/2]} \left( T - X^{(i)} - \frac{1}{\mu} Z^{(i)} \right) \right).
\]

3) Update of \( Z^{(i)} \): Consider the scheme of dual ascent, the update of \( Z^{(i)} \) is given by

\[
Z^{(i)} = Z^{(i)} + \mu \left( X^{(i)} + S - T \right).
\]

The pseudocode of TR robust PCA (TRRPCA) is outlined in Algorithm 1. A simplified version is taking one balanced unfolding within the model (4).

Algorithm 1 Tensor Ring Robust PCA (TRRPCA)

**Input:** Full observed tensor \( T \), penalty coefficient \( \mu \), number of maximal iterations \( K \).

**Output:** Recovered low-rank tensor \( \bar{T} \) and sparse tensor \( \bar{S} \).

1. Initialization \( L_{0} = T, S_{0} = O, \mathcal{X} = L_{0}, \mathcal{Z} = O \).
2. for \( k = 1 \) to \( K \) do
3. \hspace{1em} for \( i = 1 \) to \( [d/2] \) do
4. \hspace{2em} Update \( X^{(i)} \) according to (7)
5. \hspace{1em} end for
6. Update \( S \) according to (8)
7. \hspace{1em} for \( i = 1 \) to \( [d/2] \) do
8. \hspace{2em} Update \( Z^{(i)} \) according to (9)
9. \hspace{1em} end for
10. end for
11. \( \bar{L} = \frac{1}{[d/2]} \sum_{i=1}^{[d/2]} X^{(i)} \)
12. return \( \bar{L} \) and \( \bar{S} \)

B. Algorithm for RTRC

The AL function of model (5) is

\[
\min_{X_{i}, \varepsilon, S} \lambda \|S\|_{1} + \sum_{i=1}^{[d/2]} w_{i} \|X^{(i)}_{i}\|_{s} + \langle Z^{(i)}, X^{(i)} + S - \varepsilon \rangle + \frac{\mu}{2} \|X^{(i)} - \varepsilon\|_{F}^{2} + \langle w, \varepsilon_{d/2} (L + S - T) \rangle + \frac{\mu}{2} \|\varepsilon_{d/2} (L + S) - \varepsilon_{d/2} (T)\|_{F}^{2}.
\]
1) Update of $\mathcal{X}^{(i)}$: Similar to the solution given in (7), the optimality condition

$$ \mathcal{O} \in \partial \frac{w_i}{\mu} \| \mathcal{X}^{(i)} \|_* + \mathcal{X}^{(i)} - \left( \mathcal{L} - \frac{1}{\mu} \mathcal{Z}^{(i)} \right) $$

of (5) is also that of

$$ \min_{\mathcal{X}^{(i)}} \frac{w_i}{\mu} \| \mathcal{X}^{(i)} \|_* + \frac{1}{2} \| \mathcal{X}^{(i)} - \left( \mathcal{L} - \frac{1}{\mu} \mathcal{Z}^{(i)} \right) \|_F^2, $$

and the solution is

$$ \mathcal{X}^{(i)} = \mathbb{D} \mathbb{W}_i \left( \mathcal{L} - \frac{1}{\mu} \mathcal{Z}^{(i)} \right), \quad (11) $$

2) Update of $\mathcal{L}$: Denote by $\mathcal{A}_{\Omega}^*$ the adjoint of $\mathcal{A}_{\Omega}$, the optimality condition is

$$ \mathcal{O} \in \sum_{i=1}^{[d/2]} \mathcal{Z}^{(i)} + \mathcal{A}_{\Omega}^* (\mathcal{W}) + \sum_{i=1}^{[d/2]} \mu \left( \mathcal{L} - \mathcal{X}^{(i)} \right) + \mu \mathcal{A}_{\Omega} \mathcal{A}_{\Omega}^* \left( \mathcal{L} + \mathcal{S} - \mathcal{T} \right). $$

Note that $\mathcal{A}_{\Omega}^* (\mathcal{W}) = \mathcal{P}_{\Omega} (\mathcal{W})$ and $\mathcal{A}_{\Omega} \mathcal{A}_{\Omega}^* = \mathcal{P}_{\Omega}$, we have

$$ \mathcal{O} \in \sum_{i=1}^{[d/2]} \mathcal{Z}^{(i)} + \mathcal{P}_{\Omega} (\mathcal{W}) + \sum_{i=1}^{[d/2]} \mu \left( \mathcal{L} - \mathcal{X}^{(i)} \right) + \mu \mathcal{P}_{\Omega} \left( \mathcal{T} - \mathcal{S} - \mathcal{L} \right). $$

Reformulate the formula as

$$ \left( \left[ \frac{d}{2} \right] \mathcal{Z} + \mathcal{P}_{\Omega} (\mathcal{W}) \right) \left( \mathcal{L} \right) = \sum_{i=1}^{[d/2]} \left( \mathcal{X}^{(i)} + \frac{1}{\mu} \mathcal{Z}^{(i)} \right) + \mathcal{P}_{\Omega} \left( \mathcal{T} - \mathcal{S} - \mathcal{W} \right), $$

and this leads to the rule of update

$$ \mathcal{L}^* = \left\{ \sum_{i=1}^{[d/2]} \left( \mathcal{X}^{(i)} + \frac{1}{\mu} \mathcal{Z}^{(i)} \right) + \mathcal{P} \left( \mathcal{T} - \mathcal{S} - \mathcal{W} \right) \right\} \otimes \left( \left[ \frac{d}{2} \right] \mathcal{I} + \mathcal{P} \right), \quad (12) $$

where $\otimes$ is the identity operator, $\mathcal{I}$ is a tensor that pulls with ones and $\otimes$ represents the element-wise division.

3) Update of $\mathcal{S}$: Similar to the update of $\mathcal{L}$, the optimality condition is

$$ \mathcal{O} \in \partial \frac{\lambda}{\mu} \| \mathcal{S} \|_1 + \mathcal{P}_{\Omega} \left( \mathcal{S} \right) - \mathcal{P}_{\Omega} \left( \mathcal{T} - \mathcal{L} - \frac{1}{\mu} \mathcal{W} \right), $$

which induces the equivalent model

$$ \min_{\mathcal{S}} \frac{1}{2} \| \mathcal{A}_{\Omega} \left( \mathcal{S} \right) - \mathcal{A}_{\Omega} \left( \mathcal{T} - \mathcal{L} - \frac{1}{\mu} \mathcal{W} \right) \|_F^2 + \frac{\lambda}{\mu} \| \mathcal{S} \|_1. $$

According to Lemma 1, the optimal solution is

$$ \mathcal{S}^* = \mathcal{S} \left( \mathcal{P} \left( \mathcal{T} - \mathcal{L} - \frac{1}{\mu} \mathcal{W} \right) \right), \quad (13) $$

Lemma 1. The solution to optimization

$$ \min_{\mathcal{X}} \frac{1}{2} \| \mathcal{A}_{\Omega} \left( \mathcal{X} \right) - \mathcal{A}_{\Omega} \left( \mathcal{B} \right) \|_F^2 + \tau \| \mathcal{X} \|_1 $$

is $\mathcal{X}^* = \mathcal{S} \left( \mathcal{P} \right)$, where $\mathcal{P}$ is the binary sampling tensor.

4) Update of $\mathcal{Z}^{(i)}$: It is easy to calculate the update of dual variable $\mathcal{Z}^{(i)}$

$$ \mathcal{Z}^{(i)} = \mathcal{Z}^{(i)} + \mu \left( \mathcal{X}^{(i)} - \mathcal{L} \right), \quad (14) $$

5) Update of $\mathcal{W}$: According to the rule of ADMM, the vector form of update is

$$ \mathcal{W} = \mathcal{W} + \mu \mathcal{P} \left( \mathcal{L} - \mathcal{S} - \mathcal{T} \right), \quad (15) $$

Rewrite the formulation in a tensor form we derive

$$ \mathcal{W} = \mathcal{W} + \mu \mathcal{P} \otimes \left( \mathcal{L} - \mathcal{S} - \mathcal{T} \right). $$

Algorithm 2 Robust Tensor Ring Completion (RTRC)

**Input:** Zero-filled observed tensor $\mathcal{T}$, observation set $\Omega$, penalty coefficient $\mu$, number of maximal iterations $K$.

**Output:** Recovered low-rank tensor $\mathcal{L}$ and sparse tensor $\mathcal{S}$.

1: Initialization $\mathcal{P}, \mathcal{L}_0 = \mathcal{P} \otimes \mathcal{T}, \mathcal{S}_0 = \mathcal{O}, \{ \mathcal{X} \} = \mathcal{L}_0, \{ \mathcal{Z} \} = \mathcal{W} = \mathcal{O}$.
2: for $k = 1$ to $K$ do
3: \hspace{0.5cm} for $i = 1$ to $\left[ \frac{d}{2} \right]$ do
4: \hspace{1cm} Update $\mathcal{X}^{(i)}$ according to (11)
5: \hspace{0.5cm} end for
6: Update $\mathcal{L}$ according to (12)
7: Update $\mathcal{S}$ according to (13)
8: for $i = 1$ to $\left[ \frac{d}{2} \right]$ do
9: \hspace{1cm} Update $\mathcal{Z}^{(i)}$ according to (14)
10: \hspace{0.5cm} end for
11: Update $\mathcal{W}$ according to (15)
12: end for
13: return $\mathcal{L}$ and $\mathcal{S}$

C. Algorithmic complexity

We adopt the Lanczos algorithm introduced in [23] and [24] for the fast computation of SVD, since they mentioned it has a linear complexity $O(p + q)$ for a $p$-by-$q$ matrix. For a $d$-order hypercubic tensor $\mathcal{X} \in \mathbb{R}^{n \times \cdots \times n}$ with TR-rank being $[r, \ldots, r]^T$, the complexities of TRRPCA and RTRC algorithms mainly depend on the updates of $\mathcal{X}$ which involve $d/2$ soft thresholds and hence own $O(dn^{d/2})$ complexity. The storage complexities of two algorithms are $dnl^{d/2}r^2$, since $d/2$ outcomes of SVDs are stored.

D. Algorithmic convergence

The ADMM algorithm has a linear rate of convergence when one of the objective terms is strongly convex [25]. Reference [26] provides a rather simple but efficient strategy to improve convergence, in which the penalty coefficient $\mu$ increases geometrically with iterations, i.e., $\mu_{k+1} = \beta \mu_k$, where $\beta$ is a numerical constant.

V. Numerical Experiments

In this section, three groups of datasets are used for tensor completion experiments, i.e., synthetic data, real-world images and videos. Two algorithms are used to test the performance on real-world data, consisting of tensor robust PCA via tensor
nuclear norm minimization (TRPCA-TNN) [27] and the proposed one. All the experiments are conducted in MATLAB 9.3.0 on a computer with a 2.8GHz CPU of Intel Core i7 and a 16GB RAM.

There are several evaluations for the quality of visual data. Relative error (RE), short for the root of relative squared error, is a common indicator for recovery accuracy, which is defined as

$$\text{RE} = \frac{\| \hat{\mathbf{x}} - \mathbf{x} \|_F}{\| \mathbf{x} \|_F}$$

where \( \mathbf{x} \) is the ground truth and \( \hat{\mathbf{x}} \) is the recovered tensor.

The second quality metric is peak signal-to-noise ratio, often abbreviated PSNR, is the ratio between the maximum possible power of a signal and the power of corrupting noise [28]. Given the ground truth \( \mathbf{x} \) and the estimation \( \hat{\mathbf{x}} \), the mean squared error (MSE) is defined as

$$\text{MSE} = \frac{\| \hat{\mathbf{x}} - \mathbf{x} \|_F^2}{\text{card}(\mathbf{x})}$$

then the PSNR (in dB) is defined as

$$\text{PSNR} = 20 \log \left( \frac{M}{\sqrt{\text{MSE}}} \right)$$

where \( M \) is the maximal pixel value which is 255 for the RGB images and videos, and \( \text{card}(\cdot) \) represents the number of elements in a set. A higher PSNR usually indicates a higher quality of the reconstruction.

The third assessment is called structural similarity index (SSIM) which is used for measuring the similarity between the recovered image and original image [29]. It is calculated on various windows of an image. The measure between two windows \( X \) and \( Y \) of common size \( N \times N \) is

$$\text{SSIM}_{XY} = \frac{(2\mu_X \mu_Y + c_1)(2\sigma_{XY} + c_2)}{(\mu_X^2 + \mu_Y^2 + c_1)(\sigma_X^2 + \sigma_Y^2 + c_2)}$$

where \( \mu_X \) and \( \mu_Y \) are the averages of \( X \) and \( Y \), \( \sigma_X \) and \( \sigma_Y \) are the variances of \( X \) and \( Y \), \( c_1 = (k_1 L)^2 \) and \( c_2 = (k_2 L)^2 \) are two variables to stabilize the denominator (default values for \( k_1 \) and \( k_2 \) are 0.01 and 0.03), \( L = 2^w \times \text{bits per pixel} - 1 \) is the dynamic range of pixel-values.

The last one quantifying the algorithmic complexity is the computational CPU time (in seconds).

The sampling ratio (SR) is defined as the ratio of the number of sampled entries to the number of the elements in tensor \( \mathbf{x} \), noted as \( \text{SR} = \text{card}(\mathbf{O}) / \text{card}(\mathbf{x}) \).

For fair comparisons, the parameters in each algorithm are tuned to give optimal performance, and all trials are repeated adequate times for avoiding fortuitous results. In our algorithm, \( \mu_0 \) is set to be 1 \( \times 10^{-3} \). The convergence is determined by the relative change (RC) \( \text{RC} = \| X_k - X_{k-1} \|_F / \| X_{k-1} \|_F \), where the tolerance is set to be 1 \( \times 10^{-5} \). The number of maximal iterations is 100.

In the next section, the experiments on real-world data including images and videos are used to test the performance of the proposed algorithms and others.

A. Color images

A RGB image lena is tested in this section’s experiment. In image recovery, we set \( \beta = 1.1 \) and \( \mu_0 = 1 \times 10^{-2.4} \).

The visual data tensorization (VDT) method introduced in [19] and [30] can improve the performance, as a higher-order tensor makes it more efficient to exploit the local structures in original tensor and, if a tensor is slightly correlated, the tensorized one is more likely to have a low rank [19]. VDT first transforms an image into a real ket of a Hilbert space by casting the image to a higher-order tensor with an appropriate block structured addressing, i.e., tensorizing an image of size \( M \times N \times 3 \) to a tensor of size \( m_1 \times \cdots \times m_K \times n_1 \cdots n_K \times 3 \). Then VDT permutes and reshapes the resulting tensor into another one with size \( m_1 n_1 \cdots m_K n_K \times 3 \). For this image, we use \([2 \times \text{ones}(1, 16), 3]^{T}\) to reshape it and finally get a \([4 \times \text{ones}(1, 8), 3]^{T}\) sized tensor. Note that the tensorizations are manually operated, and different operations will cause other results.

After applying the VDT manipulation, we compare the proposed method with the state-of-the-art algorithms. The recovery results (REs, PSNRs, SSIMs and CPU times) of 2 algorithms, based on an average of 5 repetitions, are exhibited in Fig. 2.

B. Real-world videos

In this group of experiments, a video are used to test the algorithms and each video recovery is repeated 5 times. The first is a color video called pendulum can be found in MATLAB with the size of 288 \( \times 352 \times 3 \times 50 \). The video is further reshaped into a 9D tensor with the size of \( 4 \times 4 \times 4 \times 4 \times 9 \times 11 \times 3 \times 2 \times 5 \times 5 \) by VDT method. We set \( \beta = 1.1 \) and \( \mu_0 = 1 \times 10^{-3} \).

Fig. 3 gives the completion results of seven methods.

VI. Conclusion

Based on existing analysis of TR completion, we extend the analysis of exact recovery condition for robust tensor ring completion. The result shows a similar lower bound to that of matrix case. Equipped with this foundation and based on a balanced TR-unfolding scheme, two algorithms named TRRPCA and RTRC are proposed to deal with the robust tensor completion, which show better performance against other state-of-the-art methods in application of visual data recovery and background modeling.

Appendix A

Proof of Theorem 1

**Proof.** The dual certificate for RTRC is that there is an unfolding \( Y_{\{k,l\}} \) such that the following conditions are satisfied:

$$\left\| \frac{1}{2(1-2\gamma)} \mathcal{P}_{T_{kl}} \mathcal{P}_{\Psi_{k,l}} \mathcal{P}_{T_{kl}} - \mathcal{P}_{T_{kl}} \right\|_2 < \frac{1}{2}$$

$$\left\| \mathcal{P}_{T_{kl}} (Y_{\{k,l\}}) - \mathcal{P}_{T_{kl}} (o_k N_{\{k,l\}} - R_{\{k,l\}}) \right\|_F \leq \sum_{i=1}^{d} n_i^{-1}$$

$$\left\| \mathcal{P}_{T_{kl}} (Y_{\{k,l\}}) - \mathcal{P}_{T_{kl}}^+ (o_k N_{\{k,l\}}) \right\|_2 \leq \frac{w_k}{2}$$

$$\left\| \mathcal{P}_{\Psi_{k,l}} (Y_{\{k,l\}}) \right\|_2 \leq \frac{\lambda}{2}$$

(21)
Define \( R = \sum_{i=1}^{[d/2]} \lambda_i \sum_{i=1}^{[d/2]} w_i r_i^{-1} r_i^{-1} \) and \( \lambda = \sum_{i=1}^{[d/2]} \lambda_i \) and note that \( \partial ||S_0||_1 = E + N \), we now show that a small perturbation will increase the value of \( f(L_0, S_0) \).

\[
\begin{align*}
  f(L_0 + \Delta, S_0 - R_{\Omega}(\Delta)) - f(L_0, S_0) &
  \geq \sum_{i=1}^{[d/2]} w_i \varphi_{i} \left( \partial ||L_{(i)}^2||_1, \Delta \right) - \langle \lambda \partial ||S_0||_1, R_{\Omega}(\Delta) \rangle \\
  &= \langle R + \sum_{i=1}^{[d/2]} w_i \varphi_{i} \left( W_{(i)}, \Delta \right), \Delta \rangle - \langle \lambda \partial ||S_0||_1, \Delta \rangle \\
  &= w_k \| P_{T_{k,l}}^+(\Delta_{(k,l)}) \|_* + \lambda \| P_{\Psi}(\Delta) \|_1 + \langle Y + R - \lambda N, \Delta \rangle - \langle Y, \Delta \rangle \\
  &\geq w_k \| P_{T_{k,l}}^+(\Delta_{(k,l)}) \|_* + \lambda \| P_{\Psi}(\Delta) \|_1 + \langle P_{T_{k,l}}^+(Y_{(k,l)} + R_{(k,l)} - \lambda N_{(k,l)}), P_{T_{k,l}}^+(\Delta_{(k,l)}) \rangle + \langle P_{T_{k,l}}^+(Y_{(k,l)}), P_{T_{k,l}}^+(\Delta_{(k,l)}) \rangle - \langle Y, P_{\Psi}(\Delta) \rangle \\
  &\geq \frac{w_k}{2} \| P_{T_{k,l}}^+(\Delta_{(k,l)}) \|_* + \frac{\lambda}{2} \| P_{\Psi}(\Delta) \|_1 - \\
  &\prod_{i=1}^{d} n_i^{-1} \| P_{T_{k,l}}(\Delta_{(k,l)}) \|_F
\end{align*}
\]

The first inequality is attributed to the convexity of nuclear norm and \( \ell_1 \) norm using Taylor expansion. The second equality is due to \( R_{\Omega^\perp}(\partial ||S||) = 0 \). Since \( P_{T_{k,l}}(W_{(k,l)}) = 0 \) and \( P_{\Psi^\perp}(E) = 0 \), let \( W_{(i,l)} = 0 \), \( i \neq k \) and picking up \( W_{(k,l)} \) and \( E \) such that \( \langle T_{k,l}(W_{(k,l)}), \Delta \rangle = \| P_{\Psi^\perp}(\Delta_{(k,l)}) \|_* \), and \( \langle -E, \Delta \rangle = \| P_{\Psi}(\Delta) \|_1 \) the third equality holds. Expanding the inner product and note that \( P_{T_{k,l}^+}(R_{(k,l)}) = 0 \) the forth equality holds. The last inequality is because of (21) and

Fig. 2. Image lena recovered by TRRPCA and LRTC-TNN. The image recovered by TRRPCA has 0.0530 RE, 30.6257 PSNR and 0.9834 SSIM, and the image recovered by TRPCA-TNN has 0.0535 RE, 30.5527 PSNR and 0.9841 SSIM.

Fig. 3. The recovery result of background separation for pendulum video yielded by two methods.
Since (20) indicates \( \| \frac{1}{\sqrt{(1-2 \gamma) p}} P_{T_{k_i}} P_{\Psi_{(k,1)}} \|_F < \sqrt{3/2} \), thus

\[
\| P_{T_{k_i}} (\Delta_{(k,l)}) \|_F \leq 2 \| \frac{1}{(1-2 \gamma) p} P_{T_{k_i}} P_{\Psi_{k_i}} P_{T_{k_i}} (\Delta_{(k,l)}) \|_F
\]

\[
\leq 2 \| \frac{1}{(1-2 \gamma) p} P_{T_{k_i}} P_{\Psi_{k_i}} P_{T_{k_i}} (\Delta_{(k,l)}) \|_F + 2 \| \frac{1}{(1-2 \gamma) p} P_{T_{k_i}} P_{\Psi_{k_i}} (\Delta_{(k,l)}) \|_F + \sqrt{\frac{6}{(1-2 \gamma) p} \| P_{T_{k_i}} (\Delta_{(k,l)}) \|_F^2}
\]

\[
\leq \sqrt{\frac{6}{(1-2 \gamma) p} \| P_{\Psi_{k_i}} (\Delta_{(k,l)}) \|_F^2}
\]

Without loss of generality, assume \( \prod_{i=1}^{d} n_i \) is sufficient large and we have

\[
f((L_0 + \Delta, S_0 - P_{\Psi_{k_i}} (\Delta)) - f(L_0, S_0) \geq \left( \frac{w_k}{2} - \sqrt{(\frac{6}{(1-2 \gamma) p} \prod_{i=1}^{d} n_i^{-1}} \| P_{T_{k_i}} (\Delta_{(k,l)}) \|_F^2 + \chi \| P_{\Psi} (\Delta) \|_F \right) \geq 0.
\]

Since the \( P_{\Psi_{k_i}} P_{T_{k_i}} \) satisfies injective property when \( p \) and \( \gamma \) are small such that \( \| P_{\Psi_{k_i}} P_{T_{k_i}} \|_2 \leq \sqrt{3(1-2 \gamma)} p/2 < 1 \), the equality holds if and only if \( \Delta = \mathcal{O} \).

The next step is to construct the dual certificate via the Golf scheme introduced in [11].

\section*{Appendix B

Proof of Lemma 1

\textbf{Proof.} Denote by \( A \in \mathbb{R}^{n \times n \times n} \) the matrix form of linear mapping \( \mathcal{A}_{\Omega} \), where \( \Omega = \{ j_1, \ldots, j_m \} \) and \( A = [e_{j_1}, \ldots, e_{j_m}]^T \), \( 1 \leq j_i \leq m, \forall i \in [m] \). The aforementioned optimization becomes

\[
\min_{A} \frac{1}{2} \| A \text{ Vec} (X) - A \text{ Vec} (B) \|_F^2 + \tau \| \text{Vec} (X) \|_1,
\]

thus its first-order optimality condition is

\[
0 \in A^T A \text{ Vec} (X) - A \text{ Vec} (B) + \tau \text{Vec} (X),
\]

which can be reformulated as

\[
0 \in A^T A \left[ \text{ Vec} (X) - \text{ Vec} (B) \right] + \tau \text{Vec} (X) + \tau \partial \| \text{Vec} (X) \|_1
\]

Define the projection operator \( P_{\Omega} = \mathcal{A}_{\Omega}^* \mathcal{A}_{\Omega} \) and its matrix expression \( P = A^T A \), where \( X^* \) represents the adjoint of \( A \). The formula of optimality condition is

\[
0 \in P_{\Omega} (\text{ Vec} (X) - \text{ Vec} (B)) + \tau \partial \| P_{\Omega} (\text{Vec} (X)) \|_1 + \tau \partial \| P_{\Omega^+} (\text{ Vec} (X)) \|_1.
\]

In order to minimize the \( l_1 \) norm, the value of component under projection \( \Omega^+ \) should be 0. Note that the projection satisfies

\[
A^T A = \sum_{i=1}^{m} e_{j_i} e_{j_i}^T = \text{diag} \left( \ldots, j_1, 0, \ldots, j_m, 0, \ldots \right) = \text{diag} \left( \text{Vec} (P) \right).
\]

Rewrite the condition as

\[
0 \in P \odot X - P \odot B + \tau \partial \| P \odot X \|_1,
\]

which is also the optimality condition of

\[
\min_{A} \frac{1}{2} \| P \odot X - P \odot B \|_F^2 + \tau \| P \odot X \|_1.
\]

Since \( \text{sgn}(0) = 0 \), the optimal solution is \( X^* = S_{\tau} (P \odot B) \). \qed

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