The Einstein metrics with smooth scri

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Abstract
We consider solutions of the Einstein equations with cosmological constant \( \Lambda \neq 0 \) admitting conformal compactification with smooth scri. Metrics are written in the Bondi–Sachs coordinates and expanded into inverse powers of the affine distance \( r \). Unlike in the case \( \Lambda = 0 \) all free data are located on the scri. They are constrained by linear differential equations for the Bondi mass and angular momentum aspects. Given free data components of metric are defined in a recursive way.

Keywords The Einstein metrics · Conformal compactification · The Bondi–Sachs coordinates

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1 Introduction
In 1960 Bondi [1] proposed an approach to gravitational radiation based on a foliation of spacetime by null surfaces. The axially symmetric case was elaborated by Bondi, van der Burg and Metzner [2] and general case by Sachs [3]. This approach is very useful in studying asymptotic properties of spacetime in null directions (see [4] for a review of results). Their geometrical interpretation is provided by the Penrose conformal formalism [5]. It assumes that spacetime \( \tilde{M} \) with metric \( \tilde{g} \) admits a conformal compactification to manifold \( (M , g) \) with a boundary containing the future null
infinity (scri) $\mathcal{I}^+$. Originally metric $g$ was assumed to be smooth up to $\mathcal{I}^+$ what corresponds to the assumption in the Bondi–Sachs approach that metric components can be expanded with respect to the inverse of a radial distance $r$. Basic relations between above approaches were described by Tamburino and Winicour [6]. Much later Friedrich [7] and Winicour [8] realized that in more general setting one should admit so called polyhomogeneous expansions including powers of $1/r$ and $\ln r$ (see [9] for a more systematic approach).

A development of the AdS/CFT correspondence caused a great interest in asymptotically (anti) de Sitter metrics (see discussion of these metrics by Ashtekar, Bonga and Kesavan [10–12]). Friedrich [13, 14] proved the existence of solutions of this type for prescribed data. The Bondi–Sachs formalism was applied to them in the axisymmetric case by Poole, Skenderis and Taylor [15] and in general case by Compère, Fiorucci and Ruzziconi [16]. Several definitions of total energy and momentum for $\Lambda \neq 0$ were proposed (see a review by Szabados and Tod [17]).

In work [18] we reexamined the vacuum Einstein equations for the Bondi–Sachs metrics with smooth scri. Instead of traditional luminosity distance we used the affine parameter along geodesics as the radial coordinate. Assuming expansions in $1/r$ allowed to solve the equations and define coefficients of expansions in a recursive way. This scheme was more difficult for stationary metrics. Then a hierarchy of free constants (multipole moments) appeared in a natural way. In the present work we follow approach in [18] but now we admit a nonvanishing cosmological constant. Again we write metric in the Bondi–Sachs form using the affine distance $r$. Because of this choice, but not only, our approach differs from that in [16] or other references. We expand metric into inverse powers of $r$ and show that almost all Einstein equations can be solved in an algebraic way with respect to metric coefficients. Only the mass aspect $M$ and the angular momentum aspect $L_A$ undergo some linear differential constraints. In special coordinates they reduce to the Laplace-Beltrami equation related to unexpected metric on the scri. All free data are defined on the scri. For $\Lambda < 0$ this property is surprising in view of results of Friedrich [14], where data are defined on the scri and on a spacelike surface. If metric tends asymptotically to the (A)dS solution the constraints on $M$ and $L_A$ imply a preservation of the generalized Bondi energy and the oscillatory (for $\Lambda < 0$) or exponential (for $\Lambda > 0$) behaviour of the linear momentum. In Sect. 4 we consider stationary metrics when $\Lambda < 0$. In contrary to the case $\Lambda = 0$ solving the Einstein equations does not lead to a natural appearance of multipole moments. We show that arbitrariness of boundary metric may create problems with possible definitions of total mass.

2 Recursive solvability of the Einstein equations

Following the Penrose conformal approach we assume that

1. Spacetime $\tilde{M}$ with metric $\tilde{g}$ can be embedded in unphysical spacetime $M$ with metric $g = \Omega^2 \tilde{g}$.
2. The conformal factor $\Omega$ satisfies $\Omega > 0$ on $\tilde{M}$ and $\Omega = 0$, $d\Omega \neq 0$ on a 3-dimensional boundary (scri $\mathcal{I}$) of $\tilde{M}$ in $M$.
3. Metric $g$ is smooth in a neighborhood of $\mathcal{I}$.
4. Physical metric $\tilde{g}$ satisfies the Einstein equations with cosmological constant $\Lambda \neq 0$.

In case of vacuum metrics Tamburino and Winicour [6] proved that existence of the conformal compactification allows to put physical metric into the Bondi–Sachs form. Below we first show what form of metric can be obtained under Assumptions 1–3. Then we impose the Einstein equations.

Let $\mathcal{I}$ has the form $\mathcal{I} = ]u_1, u_2[\times \Sigma$ (if not, we can take a subset of $\mathcal{I}$). The first factor is coordinated by $x^0 = u$ and the second by $x^A$, $A = 2, 3$. We assume that sections $u = \text{const}$ are spacelike and vector $\partial_u$ is orthogonal to them

$$g(\partial_u, \partial_A) = 0. \quad (1)$$

Now we transport coordinates $u, x^A$ from the scri to its neighborhood $U \subset M$ along null geodesics orthogonal to the leaves $u = \text{const}$. We complete these coordinates with $x^1 = \omega$ which is the affine parameter along the geodesics such that $\omega = 0$ on $\mathcal{I}$. For appropriately chosen $\omega$ and sign of $u$ the unphysical metric takes the form

$$g = du(g_{00}du - 2d\omega + 2g_{0A}dx^A) + g_{AB}dx^Adx^B. \quad (2)$$

It follows from assumption 2 that

$$\Omega = f\omega, \quad f \neq 0, \quad (3)$$

where $f$ is a function on $\tilde{M}$. On $U \cap \tilde{M}$ the physical metric is given by

$$\tilde{g} = \Omega^{-2}g = du(\tilde{g}_{00}du + 2dr + 2\tilde{g}_{0A}dx^A) + \tilde{g}_{AB}dx^Adx^B, \quad (4)$$

where $r$ is a coordinate defined by

$$r = -\int \frac{d\omega}{f^2\omega^2}. \quad (5)$$

From (5) one obtains an approximation

$$r \approx \frac{1}{f^2\omega} \quad (6)$$

for small values of $\omega$. Since metric $g$ is well defined at $\omega = 0$ and satisfies (1) components $\tilde{g}_{00}$ and $\tilde{g}_{AB}$ must be of the order $r^2$ and $\tilde{g}_{0A} = O(r)$. Now we can choose a new conformal factor

$$\Omega = \frac{1}{r}. \quad (7)$$
for which the unphysical metric reads
\[ g = d\rho (g_{00} d\rho - 2d\rho + 2g_{0A} dx^A) + g_{AB} dx^A dx^B. \] (8)

Thus, for this choice the conformal factor coincides with the affine parameter along geodesics. Condition (1) can be written as
\[ \hat{g}_{0A} = 0, \] (9)
where the hat denotes the pullback of a function or tensor on \( M \) to \( \mathcal{I} \).

We summarize this part in the following proposition.

**Proposition 2.1** Under Assumptions 1–3, in a neighborhood of the scri one can transform unphysical metric \( g \) to the form (8) with property (9).

We will use metric (8) as a basic object which defines the physical metric via
\[ \tilde{g} = \Omega^{-2} g. \] (10)

Both metrics are assumed to have signature \(+−−−\).

In accordance with assumption 4 we impose on \( \tilde{g} \) the Einstein equations with cosmological constant \( \Lambda \neq 0 \)
\[ \tilde{R}_{\mu\nu} + \Lambda \tilde{g}_{\mu\nu} = 0, \] (11)
where \( \tilde{R}_{\mu\nu} \) is the Ricci tensor of \( \tilde{g}_{\mu\nu} \). We will find consequences of (11) for coefficients of the Taylor series of metric \( g_{\mu\nu} \) with respect to \( \Omega \). In terms of the unphysical metric \( g \) the Einstein equations read
\[ R_{\mu\nu} - 2Y_{\mu\nu} - g_{\mu\nu} Y^\alpha_{\alpha} = 0, \] (12)
where
\[ Y_{\mu\nu} = -\frac{1}{\Omega} \Omega_{[\mu\nu]} + \frac{1}{2\Omega^2} \left( \Omega_{[\alpha\nu]} \Omega^{\alpha\mu} - \frac{\Lambda}{3} \right) g_{\mu\nu} \] (13)
and \( \Omega_{[\mu\nu]} \) denotes the covariant derivative defined by the Levi-Civita connection of \( g \).

Let \( \tilde{R}_{\mu\nu} + \Lambda \tilde{g}_{\mu\nu} \) be finite on \( \mathcal{I} \). Then \( Y_{\mu\nu} \) must be also finite. In order to avoid a second order pole at \( \Omega = 0 \) one has to assume that
\[ g_{00} = -\frac{\Lambda}{3} + a\Omega + b\Omega^2 - 2M\Omega^3 + O(\Omega^4). \] (14)

Thus, metric induced by \( g \) on the scri reads
\[ \hat{g} = -\frac{\Lambda}{3} d\rho^2 + \hat{g}_{AB} dx^A dx^B \] (15)
and $\mathcal{I}$ is spacelike if $\Lambda > 0$ and timelike if $\Lambda < 0$. For the (A)dS metric there is $\hat{g}_{AB} = -s_{AB}$.

In order to reduce a number of free functions in the boundary metric (15) we can change coordinates $u, x^A$ provided that $\hat{g}$ preserves form (15) modulo a conformal factor. It means that we can impose one condition on $\hat{g}_{AB}$, e.g. we can try to obtain

$$\det \hat{g}_{AB} = f, \quad f_{00} = 0$$

with a prescribed function $f$. For instance, if $\mathcal{I} = R \times S_2$, where $S_2$ is the 2-dimensional sphere, a natural condition would be

$$\det \hat{g}_{AB} = \det s_{AB}. \quad (17)$$

Regularity of $Y_{1A}$ yields

$$g_{0A} = q_A \Omega^2 + 2L_A \Omega^3 + O(\Omega^4). \quad (18)$$

Let us denote the first coefficients of the expansion of $g_{AB}$ in the following way

$$g_{AB} = \hat{g}_{AB} + n_{AB} \Omega + p_{AB} \Omega^2 + O(\Omega^3). \quad (19)$$

A lack of singularities in $Y_{AB}$ is equivalent to the relation

$$\frac{\Lambda}{3} n_{AB} = \hat{g}_{AB,0} - a \hat{g}_{AB}. \quad (20)$$

All other components of $Y_{\mu\nu}$ are nonsingular at $\Omega = 0$ under conditions (14), (18) and (20).

Unlike in the case $\Lambda = 0$, Eq. (20) defines $n_{AB}$ in terms of $\hat{g}_{AB}$ and $a$. This relation can be further simplified by means of a shift of $r$ leading to

$$a = 0. \quad (21)$$

We will not assume (21) at this stage since another condition

$$n = 0, \quad (22)$$

where $n = \hat{g}^{AB} n_{AB}$, may be more convenient. Note that (22) is equivalent to

$$a = \frac{1}{2} (\ln \det \hat{g}_{AB})_{0} \quad (23)$$

thanks to (20) and conditions (21) and (22) are equivalent if (16) is satisfied.

Following [18] we can reduce a number of the Einstein equations (11) by means of the Bianchi identity. To this end it is sufficient to replace $\bar{R}_{\mu\nu}$ by $\bar{R}_{\mu\nu} + \Lambda \bar{g}_{\mu\nu}$ and $\bar{G}_{\mu\nu}$
by $\tilde{G}_{\mu\nu} - \frac{1}{2} \Lambda \tilde{g}_{\mu\nu}$ in equation (24) in [18]. In this way one obtains a minimal system of equations given by

$$\tilde{R}^{(k)}_{11} = \tilde{R}^{(k)}_{1A} = 0, \quad (\tilde{R}_{AB} + \Lambda \tilde{g}_{AB})^{(k)} = 0$$

and

$$(\tilde{R}_{00} + \Lambda \tilde{g}_{00})^{(2)} = (\tilde{R}_{0A} + \Lambda \tilde{g}_{0A})^{(2)} = 0,$$

where the superscript $(k)$ denotes coefficient of order $k$ in the Taylor expansion with respect to $\Omega$, e. g.

$$\tilde{R}_{11} = \Sigma_{0}^{\infty} \tilde{R}^{(k)}_{11} \Omega^{k}.$$ (26)

Equations $\tilde{R}_{11} = 0$ and $\tilde{R}_{1A} = 0$ are similar in character to those for $\Lambda = 0$. They yield

$$\hat{g}^{AB} g_{AB}^{(k+2)} = (g_{AB}^{(l)}, l \leq k + 1), \quad k \geq 0,$$ (27)

$$\hat{g}_{0A}^{(k+2)} = (g_{\mu\nu}^{(l)}, l \leq k + 1), \quad k \geq 0, \quad k \neq 1,$$ (28)

where brackets on the r. h. s. denote expressions depending on functions in the bracket. For $k = 0$ Eq. (28) reads

$$q_A = -\frac{1}{2} n_A^B + \frac{1}{2} n_A.$$ (29)

Equation $\tilde{R}^{(1)}_{1A} = 0$ does not define $s_{0A}^{(3)}$. It is equivalent to

$$\left( p_A^B + \frac{1}{4} n_B^A - \frac{1}{2} n_{BC} n_{AC} \right)_{|B} + \frac{1}{8} (n_{BC} n_{BC} - n^2)_{,A} = 0,$$ (30)

where operations on indices and covariant derivatives are defined by $\hat{g}_{AB}$. For $\Lambda = 0$ we used in [18] the topology of $\mathcal{G}^+ = R \times S_2$ to show that (30) yields

$$p_{AB} = \frac{1}{8} (n_{AB} n_{AB} - n^2) \hat{g}_{AB} + \frac{1}{4} n n_{AB}.$$ (31)

For $\Lambda \neq 0$ Eq. (31) follows algebraically from $(\tilde{R}_{AB} + \Lambda \tilde{g}_{AB})^{(0)} = 0$. The latter equation also implies

$$b = -\frac{1}{2} \hat{R} + \frac{1}{2} n_{,0} + \frac{1}{4} a n + \frac{1}{8} \left( n_{AB} n_{AB} - \frac{1}{3} n^2 \right).$$ (32)
where \( \tilde{R} \) is the Ricci scalar of \( \hat{g}_{AB} \). Note that from the point of view of Eqs. (31) and (32) the most convenient gauge condition is (22).

Let us define a trace of a tensor \( T_{AB} \) on \( \Sigma \) by

\[
T_A^A = \hat{g}^{AB} T_{AB}
\]

and its traceless part by

\[
\tilde{T}_{AB} = T_{AB} - \frac{1}{2} (T_C^C) \hat{g}_{AB}.
\]

In particular we can apply splitting (33)–(34) to equation

\[
(\tilde{R}_{AB} + \Lambda \tilde{g}_{AB})^{(k)} - \frac{\Lambda (k - 3)}{3(k + 1)} \hat{R}_{11}^{(k)} = 0, \quad k \geq 2.
\]

It allows to obtain \( \tilde{g}_{00}^{(k+2)} \) in terms of lower order coefficients

\[
\tilde{g}_{00}^{(k+2)} = (g_{\mu\nu}^{(l)}, l \leq k + 1), \quad k \geq 2.
\]

The traceless part of (35) yields

\[
\tilde{g}_{AB}^{(k+2)} = (g_{\mu\nu}^{(l)}, l \leq k + 1), \quad k \geq 2.
\]

Note that in the case \( \Lambda = 0 \) instead of (38) one obtains an expression for \( \tilde{g}_{AB,0}^{(k+1)} \) in terms of lower order coefficients [18]. Thus, (38) is the second equation, after (20), which makes a qualitative difference between \( \Lambda = 0 \) and \( \Lambda \neq 0 \). If \( \Lambda \neq 0 \) there is no need of initial data for \( g_{AB} \). Moreover, now tensor

\[
N_{AB} = \tilde{g}_{AB}^{(3)}
\]

is arbitrary.

The last equations to consider are given by (25). They are equivalent to the equations

\[
(\tilde{R}_{00} + \Lambda \tilde{g}_{00})^{(2)} - \frac{\Lambda}{6} \tilde{g}^{AB} (\tilde{R}_{AB} + \Lambda \tilde{g}_{AB})^{(2)} - \frac{\Lambda^2}{18} \tilde{R}_{11}^{(2)} - \frac{\Lambda}{9} a \tilde{R}_{11}^{(1)} = 0,
\]

\[
(\tilde{R}_{0A} + \Lambda \tilde{g}_{0A})^{(2)} - \frac{\Lambda}{3} \tilde{R}_{1A}^{(2)} + \frac{\Lambda}{6} \tilde{R}_{11,A}^{(1)} = 0
\]

which have the following structure

\[
M_{0} + \frac{3}{4} (\ln |\hat{g}|)_{0} M + \frac{\Lambda}{2} L_{|A}^A = \frac{\Lambda^2}{24} n^{AB} N_{AB} + \langle a, \hat{g}_{AB} \rangle.
\]
\[ L_{A,0} + \frac{1}{2} \left( \ln |\tilde{g}| \right)_0 L_A + \frac{1}{3} M_{A} = \frac{\Lambda}{6} N_{AB}^B + \langle a, \hat{g}_{AB} \rangle. \] (43)

Given \( a, \hat{g}_{AB} \) and \( N_{AB} \), Eqs. (42) and (43) constitute a conjugate system of linear equations for \( M \) and \( L_A \). Condition (16) allows to simplify them to

\[ M_{,0} + \frac{\Lambda}{2} L_A^A = f, \] (44)

\[ L_{A,0} + \frac{1}{3} M_A = h_A. \] (45)

where \( f \) and \( h_A \) are defined by \( a, \hat{g}_{AB} \) and \( N_{AB} \). Let us introduce a function \( M' \) such that

\[ M = M'_{,0}. \] (46)

Then (45) yields

\[ L_A = -\frac{1}{3} M'_{,A} + \int h_A du \] (47)

and (44) becomes

\[ -\frac{6}{\Lambda} M'_{,00} + \hat{\Delta} M' = -\frac{6}{\Lambda} f + 3 \nabla^A \int h_A du, \] (48)

where \( \hat{\Delta} \) is the covariant Laplace operator related to \( \hat{g}_{AB} \). Thus, Eqs. (42) and (43) reduce to hyperbolic (\( \Lambda < 0 \)) or elliptic (\( \Lambda > 0 \)) Eq. (48) for \( M' \). Note that the operator acting on \( M' \) is not the Laplace-Beltrami operator of the induced metric (15) on \( \mathcal{J} \) because of the “wrong” coefficient \( \frac{6}{\Lambda} \) instead of \( \frac{3}{\Lambda} \).

In the case \( \Lambda = 0 \) Eq. (44) is responsible for diminishing of the total mass defined as the integral over \( S_2 \) from \( M \) plus corrections. If \( \Lambda \neq 0 \) there are no local (in \( u \)) corrections assuring the mass loss property (see [19] for non-local corrections). This is the main problem with a definition of total energy for (A)dS-like metrics.

Let us summarize consequences of the Einstein equations expanded into powers of \( 1/r \).

**Proposition 2.2** The Einstein metric with \( \Lambda \neq 0 \) and smooth scri \( \mathcal{J} \) is given by

\[ \tilde{g} = du(\tilde{g}_{00} du + 2dr + 2\tilde{g}_{0A} dx^A) + \tilde{g}_{AB} dx^A dx^B, \] (49)

\[ \tilde{g}_{00} = -\frac{\Lambda}{3} r^2 + ar + b - \frac{2M}{r} + \sum_2 \tilde{g}_{00}^{(k+2)} r^{-k}, \] (50)

\[ \tilde{g}_{0A} = q_A + \frac{2L_A}{r} + \sum_2 \tilde{g}_{0A}^{(k+2)} r^{-k}, \] (51)

\[ \tilde{g}_{AB} = r^2 \tilde{g}_{AB} + r n_{AB} + p_{AB} + \frac{1}{r} (N_{AB} + N_{AB}) + \sum_2 \tilde{g}_{AB}^{(k+2)} r^{-k}. \] (52)
where coefficients are defined in the following way:

- \( \hat{g}_{AB} \)-dependent metric and a traceless (with respect to \( \hat{g}_{AB} \)) tensor \( N_{AB} \) can be arbitrarily chosen up to a gauge condition, e.g. (16).
- Function \( a \) is arbitrary but it can be gauged away, e.g. by means of (23).
- Components \( n_{AB}, p_{AB}, N, q_A \) and \( b \) are defined by \( a \) and \( \hat{g}_{AB} \) according to (20), (31), (27) with \( k = 0 \), (29) and (32).
- Coefficients \( M \) and \( L_A \) are defined by their initial values at \( u = u_0 \) via Eqs. (42) and (43). Under condition (16) these equations reduce to Eq. (48) for \( M' \) and relations (46) and (47).
- All other components are defined in a recursive way by Eqs. (27), (28), (37) and (38).

The main difference between present situation and that for \( \Lambda_1 = 0 \) (see Proposition 2.1 in [18]) is that now all free data are located on \( \mathcal{I} \). For \( \Lambda > 0 \) the scri \( \mathcal{I} \) is spacelike (either \( \mathcal{I}^+ \) or \( \mathcal{I}^- \) in de Sitter metric), so the free data play a role of initial data. For \( \Lambda < 0 \) the scri is timelike and our data are boundary data. In this case only half of them agree with data introduced by Friedrich [14]. The other half should be equivalent to Friedrich’s data on a spacelike surface.

### 3 Solutions with the (A)dS boundary metric

Metric \( \tilde{g} \) tends asymptotically to the (anti) de Sitter solution if \( \Sigma = S_2 \) and

\[
\tilde{g}_{AB} = -s_{AB}.
\]  

(53)

Then we can embed \( S_2 \) in \( R^3 \) and introduce spherical coordinates \( x^A = \theta, \varphi \) corresponding to an orthonormal frame in \( R^3 \).

For such metrics it follows from Eqs. (14)–(20) and (27)–(32) that in the gauge \( a = 0 \) there is

\[
\tilde{g}_{00} = -\frac{\Lambda}{3} r^2 + 1 - \frac{2M}{r} + O \left( \frac{1}{r^2} \right),
\]

(54)

\[
\tilde{g}_{0A} = \frac{2L_A}{r} + O \left( \frac{1}{r^2} \right),
\]

(55)

\[
\tilde{g}_{AB} = -r^2 s_{AB} + \frac{1}{r} N_{AB} + O \left( \frac{1}{r^2} \right).
\]

(56)

The r.h.s. of Eqs. (44) and (45) is given by

\[
h_A = \frac{\Lambda}{6} N_{|B}^A, \quad f = 0.
\]

(57)

Let us introduce a tensor \( N'_{AB} \) such that

\[
N_{AB} = N'_{AB,0}, \quad s^{AB} N'_{AB} = 0.
\]

(58)
Equation (47) yields
\[ L_A = -\frac{1}{3} M_{,A}^\prime + \frac{\Lambda}{6} N_{AB}^\prime |_{AB} + \tilde{L}_A, \quad \tilde{L}_{A,0} = 0. \] (59)

Using decomposition
\[ \tilde{L}_A = f_{,A} + \eta^C_A h_{,C} \] (60)
we can incorporate \( f \) into \( M' \) and \( h \) into \( N'_{AB} \) except the dipole component of \( h \) (see Lemma 3.1 in [18]). Thus, without loss of generality we can assume that
\[ \tilde{L}_A = \eta^C_A h_{,C}, \quad h = \alpha^m Y_{1m}, \] (61)
where \( \alpha^m = \text{const} \) and \( Y_{lm} \) are spherical harmonics. We can pass to coordinates \( \theta, \varphi \) related to frame in \( R^3 \) such that \( \alpha^m = \alpha \delta^m_0 \). Then one obtains
\[ \tilde{L}_A dx^A = \alpha \sin^2 \theta d\varphi, \] (62)
which is the angular momentum term known from the Kerr solution. In view of (59) and (61) Eq. (44) transforms into
\[ \frac{6}{\Lambda} M_{,00}^\prime + \Delta M' = -\frac{\Lambda}{2} N_{AB}^\prime |_{AB}, \] (63)
where \( \Delta \) is the standard Laplace operator on the sphere.

It follows from (63) that \( M,0 \) is a divergence of a vector on \( S_2 \). Integrating this relation over the sphere yields
\[ \oint_{S_2} M d\sigma = \text{const}. \] (64)
The l.h.s. of (64) is, modulo \( 4\pi \), rather unique candidate for the total energy in this case. One can interpret (64) as a lack of gravitational radiation (see [17] and references therein for more information). If we integrate (63) with \( Y_{1k} \) we obtain equation
\[ \frac{6}{\Lambda} M_{,00}^\prime - 2 M_k^\prime = 0 \] (65)
for dipole moments
\[ M_k^\prime = \oint_{S_2} M' Y_{1k} d\sigma. \] (66)
The Bondi linear momentum \( P_k \) is proportional to \( M_k^\prime,0 \), so it satisfies
\[ P_{k,00} = \frac{\Lambda}{3} P_k. \] (67)
Hence, it either oscillates when $u$ changes (for $\Lambda < 0$) or behaves in an exponential way (for $\Lambda > 0$). In the second case its square must exceed the total energy either for increasing or decreasing $u$, so a reasonable physical assumption would be $M_k = 0$. Higher moments of $M'$ satisfy nonhomogeneous equations coming from (63). Their solutions are defined in quadratures up to oscillatory or exponential functions depending on sign of $\Lambda$.

An alternative approach to Eq. (63) is to treat it as an equation for tensor $N'_{AB}$. We can represent this tensor by two scalar functions $f$ and $h$ [18]

$$N'_{AB} = f_{|AB} - \frac{1}{2} \Delta f s_{AB} + \eta^C_{(A} h_{|B)C}.$$  

(68)

Substituting (68) into (63) yields

$$\frac{6}{\Lambda} M'_{,00} + \Delta M' = -\Lambda \frac{\Lambda}{4} (\Delta + 2) \Delta f.$$  

(69)

Function $f$ exists since conditions (64) and (65) assure that the l.h.s. of (72) does not contain $Y_{00}$ nor $Y_{1m}$. We can easily find its expansion into spherical harmonics if such expansion is given for $M'$. In this approach functions $M'$ and $h$ are arbitrary modulo conditions (64) and (65). We summarize above results in the following proposition.

**Proposition 3.1** The Einstein metric which has a smooth scri $I$ and tends asymptotically to the (A)dS metric is given by (49) and (54)–(56). Components $M, N_{AB}$ and $L_A$ are defined by solution $(M', N'_{AB})$ of Eq. (63) via relations (46), (58), (59) and (62). Higher order components are defined in a recursive way by Eqs. (27), (28), (37) and (38). The Bondi energy is constant and the Bondi linear momentum satisfies (67).

### 4 Stationary metrics

If physical metric admits a timelike Killing vector $K$ then we can transform it to the form (49) with $u$-independent coefficients but we cannot assume (9) (see Sect. 3 in [18]) The Killing vector is given by $K = \partial_0$. Let us denote low order coefficients of expansions of $g_{\mu\nu}$ in agreement with (19) and Let us assume (19) and expansions

$$g_{00} = \hat{g}_{00} + a\Omega + b\Omega^2 - 2M\Omega^3 + O(\Omega^4),$$  

(70)

$$g_{0A} = \hat{g}_{0A} + v_A \Omega + q_A \Omega^2 + 2L_A \Omega^3 + O(\Omega^4).$$  

(71)

All coefficients now depend on coordinates $x^A$. Tensor (13) is finite on the boundary iff

$$\hat{g}_{00} = -\frac{\Lambda}{3} + \hat{g}_{0A} \hat{g}_{0A},$$  

(72)

$$v_A = n^B_A \hat{g}_{0B}$$  

(73)
and
\[
\frac{\Lambda}{3} n_{AB} = -2 \hat{g}_0 (A|B) + (n_{CD} \hat{g}_0^C \hat{g}_0^D - a) \hat{g}_{AB},
\] (74)

where, as in Sect. 2, indices $A$, $B$ are raised by means of $\hat{g}^{AB}$. Since $\hat{g}_0 A \hat{g}_0^A \leq 0$ and we exclude $\Lambda = 0$ it follows from (72)–(74) and timelike property of $K = \partial_0$ that
\[
\Lambda < 0, \quad |\hat{g}_0 A \hat{g}_0^A| \leq |\Lambda| \frac{3}{3}.
\] (75)
Thus, metrics must be of the AdS type.

It is convenient to shift coordinate $r$ in order to obtain
\[
n^A_A = 0.
\] (76)
Equation (74) is then equivalent to the following relations defining $n_{AB}$ and $a$
\[
\frac{\Lambda}{3} n_{AB} = -2 \hat{g}_0 (A|B) + \hat{g}_0^C \hat{g}_{AB},
\] (77)
\[
a = n_{CD} \hat{g}_0^C \hat{g}_0^D - \hat{g}_0 A.\] (78)
Thus, $n_{AB}$ is proportional to the traceless part of the exterior curvature of sections $u = \text{const}$ of $\mathcal{I}$.

A transformation of coordinates $x^A$ allows us to write
\[
\hat{g}_{AB} = -\gamma^2 s_{AB},
\] (79)
where $\gamma$ is a positive function of $x^A$. Still there is a freedom of supertranslation
\[
u \rightarrow u + f(x^A)
\] (80)
which can be used to impose a condition on $\hat{g}_0 A$, $\gamma$ or higher order coefficients in metric $g$. All the Einstein equations except (25) can be solved algebraically as in the general case in Sect. 2. Equations (25) are now a complicated system of constraints on $M$, $L_A$, $\gamma$, $\hat{g}_0 A$ and $N_{AB}$, which is linear in $M$, $L_A$ and $N_{AB}$.

A space of stationary solutions is much bigger than in the case $\Lambda = 0$ [18]. In general, solutions do not tend asymptotically to the AdS metric. They do if, upon application of the Einstein equations, the boundary metric
\[
\hat{g} = -\frac{\Lambda}{3} du^2 - \gamma^2 s_{AB} (dx^A + \hat{g}_0 A du)(dx^B + \hat{g}_0 B du)
\] (81)
is conformally equivalent to
\[
\hat{g}' = -\frac{\Lambda}{3} du'^2 - s_{AB} dx'^A dx'^B.
\] (82)
Metric (82) is conformally flat (note that it coincides with the conformal compactification of the 3-dimensional Minkowski metric). Thus, metric (81) must be also conformally flat. Since \( K = \partial_u \) is the Killing field of (81) it is a conformal Killing vector of the flat metric. Such vectors compose the 10-dimensional algebra \( so(2, 3) \). Given one of them in terms of the Cartesian coordinates of flat space one can find (at least in principle) coordinates \( u, x^A \) such that \( K = \partial_u \). Writing flat metric in these coordinates allows to find all u-independent metrics which are conformally equivalent to (82). For all of them the integral of the Bondi mass aspect related to coordinates present in (82) will be constant despite the fact that a transformation leading to these coordinates depends on time.

Existence of solutions of the Einstein equations with boundary metric which is not conformally equivalent to (82) may be considered as an obstacle to a reasonable definition of total energy. A crucial ingredient in every such definition should be the mass aspect \( M \) related to coordinates such that Eqs. (15) and (17) (or some other gauge condition) are satisfied. Unfortunately, in these coordinates metric coefficients usually depend on time. It seems unavoidable that for generic solutions the integral of \( M \) and the total energy are non-constant. This property is rather unacceptable for stationary spacetimes.

5 Summary

In this paper we considered the Einstein metrics admitting a conformal compactification with smooth scri. For non-vanishing cosmological constant the scri is either space-like (\( \Lambda > 0 \)) or time-like (\( \Lambda < 0 \)). Metrics can be transformed to the Bondi–Sachs form with components which can be expanded into powers of a radial distance (Proposition 2.1). The Einstein equations imply recursive formulas for the expansion coefficients and constraints for the analog of the Bondi mass and angular momentum aspect (see Proposition 2.2). For a special foliation of the scri the constraints can be reduced to one second order equation (48) for one function. All free data are located on the scri.

An asymptotic form of metric simplifies considerably if metric is assumed to tend to the (anti) de Sitter solution (see Proposition 3.1). In this case the total energy calculated according to the Bondi prescription is constant and the total momentum oscillates or behaves exponentially.

Finally we considered solutions with \( \Lambda < 0 \) admitting the time-like Killing field. In this case, in contrary to \( \Lambda = 0 \), there is no big difference in solving the Einstein equations with respect to the general case. Existence of solutions with nontrivial boundary metric leads to conceptual problems with defining total energy.

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Data availability All data generated or analyzed during this study are included in this published article.

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