Negative Magnetic Diffusivity $\beta$ Replacing the $\alpha$ Effect in the Helical Dynamo

Kiwan Park
Helmholtz-Zentrum Dresden Rossendorf, Dresden, Germany; park32@hzdr.de, pkwan@gmail.com
Received 2019 November 3; revised 2020 June 4; accepted 2020 June 9; published 2020 July 29

Abstract
In the Sun, the converting process of a poloidal magnetic field ($B_{\text{pol}}$) from a toroidal magnetic field ($B_{\text{tor}}$) is essential to sustaining the solar magnetic fields. However, the converting process, dominated by $\alpha$ and $\beta$ effects, is not yet clearly understood. Conventional theories expect that the $\alpha$ effect should be quenched as the magnetic field grows. Also, plasma kinetic energy is thought to diffuse magnetic energy (positive $\beta$ effect). Then, $B_{\text{pol}}$ is supposed to decay resulting in the dissipation of $B_{\text{tor}}$, followed by the diminishing dynamo process. But the solar magnetic field evolves periodically, as is observed. To solve this inconsistency between the theory and real nature, we first need to check if the $\alpha$ and $\beta$ effects indeed evolve as the conventional theories expect. However, these effects are theoretically or conceptually inferred quantities, and their exact expressions are not yet known. So, instead of their incomplete formulas, we used more practical representations composed of large-scale magnetic helicity $H_M (=\mathbf{A} \cdot \mathbf{B})$ and energy $E_M (=\mathbf{B}^2/2)$. We verified that the $\alpha$ effect quenches as the conventional theory expects. However, we also found that the $\beta$ effect can be negative. This negative $\beta$ apparently looks inconsistent with the conventional conclusion, but it can be a promising substitution for the decaying $\alpha$ effect. We discuss their physical bases and mechanisms using a field structure model supported by an analytic method. The model shows that the interaction between the poloidal velocity component ($U_{\text{pol}}$) and nonlocally transferred magnetic field ($\mathbf{B} \cdot \nabla U$) induces a current density $j_{\text{ind}}$ along with the magnetic field. Their combined structure yields magnetic helicity to the system, which is the $\alpha$ effect. However, $U_{\text{pol}}$ can also interact with the locally transferred magnetic field, i.e., $U_{\text{pol}} \times (-\mathbf{U} \cdot \nabla \mathbf{B})$ inducing a current density $j_{\text{diff}}$. This current density can produce additional magnetic helicity (negative $\beta$ effect) to the system. Simultaneously, the toroidal component $U_{\text{tor}}$ with $-\mathbf{U} \cdot \nabla \mathbf{B}$ leads to the usual positive $\beta$ effect, which diffuses the magnetic field. Finally, using the negative $\beta$ effect, we show how the plasma motion is suppressed in a helically forced dynamo system where Lorentz force ($\mathbf{J} \times \mathbf{B}$) apparently looks negligible.

Unified Astronomy Thesaurus concepts: Solar dynamo (2001); Magnetohydrodynamics (1964); Solar magnetic fields (1503); Plasma astrophysics (1261)

1. Introduction and Method
Magnetic field $B$ and plasma are ubiquitously observed phenomena. Interacting with the ionized particles, the $B$ field grows or decays to affect the evolution of a (celestial) plasma system significantly. Through electromotive force EMF ($\sim \mathbf{U} \times \mathbf{B}$), turbulent plasma energy can be converted into magnetic energy and cascades (is induced) toward the large-scale (large-scale dynamo, LSD) or small-scale regime (small-scale dynamo, SSD). As Maxwell equations imply, magnetic field propagates to infinity by nature. However, $B$ field in the plasma system is constrained with the massive charged particles so that the evolution and transport of magnetic field are limited and require some physical conditions.

According to dynamo theory, the converted magnetic energy ($E_M$) migrates toward a large scale through the effect of helicity, magnetorotational instability, or differential rotation (Moffatt 1978; Krause & Rädler 1980; Brandenburg & Subramanian 2005a; Park & Blackman 2012a, 2012b; Balbus & Hawley 1991). In contrast, $E_M$ in SSD propagates toward the small-scale regime forming its peak between the forcing scale and dissipation one. As the magnetic energy in the system grows, its constraint on the plasma system proportionally increases. The magnetic field controls the rate of collapse and formation of an accretion disk transporting angular momentum (magnetorotational instability; Balbus & Hawley 1991; Machida et al. 2005). Also, the balance between the thermal (kinetic) pressure and electromagnetic pressure decides the stability of plasma system (e.g., sausage, kink, or Kruskal–Schwarzschild instability; see Boyd & Sanderson 2003).

Furthermore, there may be many unknown effects; however, we do not discuss the general dynamo theory or the role of magnetic field in the astrosphere system. We study the mechanisms of the $\alpha$ and $\beta$ effects in the helical dynamo process. We investigate the physical basis of these temporally evolving pseudo tensors that linearize and parameterize the turbulent EMF with the large-scale magnetic field $\mathbf{B}$: ($\mathbf{a} \times \mathbf{b} \sim \alpha \mathbf{B} - \beta \nabla \times \mathbf{B}$). In particular, we focus on the property of negative $\beta$ replacing the quenched $\alpha$ effect depending on a magnetic Prandtl number $Pr_M$ and helicity ratio $f_\beta$. In addition, the negative magnetic diffusion shows the physical origin of the negative magnetic pressure instability (Jabbari et al. 2013).

In this paper, we first discuss the basic theory and introduce simulation results for the evolving $\alpha$ and $\beta$ in various cases. Then, we discuss the results using a theoretical model and analysis. Finally, using the negative $\beta$ effect, we explain why the plasma kinetic energy, constrained by magnetic energy through Lorentz force, is suppressed in the fully helical forcing system.

1.1. Numerical Method
The evolving macroscopic magnetized plasma phenomena can be efficiently explained with the magnetohydrodynamic (MHD) model. The basic MHD equation set is composed of continuity, momentum, and magnetic induction equations as
\[ \frac{\partial \rho}{\partial t} = -\mathbf{U} \cdot \nabla \rho - \rho \nabla \cdot \mathbf{U}, \]  
(1)  
\[ \frac{\partial \mathbf{U}}{\partial t} = -\mathbf{U} \cdot \nabla \mathbf{U} - \nabla \ln \rho + \frac{1}{\rho} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \left( \nabla^2 \mathbf{U} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{U}) \right) + f_{\text{kin}}, \]  
(2)  
\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} + f_{\text{mag}}, \]  
(3)  
\[ \Rightarrow \frac{\partial \mathbf{A}}{\partial t} = (\mathbf{U} \times \mathbf{B}) - \eta \nabla \times \mathbf{B} + f'_{\text{mag}}, \]  
(4)

where \( \rho, \nu, \) and \( \eta \) indicate density, kinematic viscosity, and (molecular) magnetic diffusivity, respectively. The velocity field is in units of sound speed \( c_s \), and the magnetic field is normalized by \( (\mu_0 \mu_0)^{1/2} c_s \), where \( \mu_0 \) is magnetic permeability in vacuum. A forcing method is decided by \( f_{\text{kin}} \) or \( f_{\text{mag}} \).

To solve these coupled differential equations, we used Pencil-code (Brandenburg 2001). Pencil-code is a sixth-order finite-difference code for compressible fluid dynamics with magnetic field (MHD). The code solves the vector potential “A” equation (Equation (4)) instead of the magnetic induction equation (Equation (3)). One of the merits of solving \( A \) is that the constraint of divergence free magnetic field \( (\nabla \cdot \mathbf{B} = 0) \) is met in a natural way without any numerical manipulation. Moreover, in addition to magnetic energy \( \sim (\mathbf{B}^2) \), magnetic helicity \( (\mathbf{A} \cdot \mathbf{B}) \), a fundamental physical quantity, can be found without an integration requiring an unknown integral constant. For the detailed properties of the code, we suggest that readers refer to the manual (http://pencil-code.nordita.org/).

In our simulation, we forced the plasma system with the helical or nonhelical turbulent kinetic energy \( f_{\text{kin}} \) like

\[ f(k, t) = \frac{ik(t) \times (k(t) \times \dot{e}) - \lambda \epsilon(t) (k(t) \times \dot{e})}{k(t)^2 \sqrt{1 + \chi^2} - k(t) \cdot e^2 / k(t)^2}. \]

Here, “k” is a wavenumber, “\( \dot{e} \)” is an arbitrary unit vector, “\( \phi(t) \)” is a random phase \( |\phi(t)| \leq \pi \), and “\( \lambda \)” indicates a helicity ratio. If “\( \chi \)” is “\( \pm 1 \),” the forcing energy is fully helical \( i k \times f = \pm k f \). If “\( \chi \)” is “0,” “f” is fully nonhelical.

1.2. General Analytic Method

In principle, to explain the evolution of the plasma system, the MHD equation set is to be solved simultaneously. However, in theory, momentum Equation (2) and magnetic induction Equation (3) are mainly solved with appropriate closure assumptions based on the statistical equilibrium state (Pouquet et al. 1976; Yoshizawa 2011). Furthermore, some dynamo theories, where the evolving profile of magnetic field is a main research interest, solve only Equation (3) with the assumption of second-order moment \( \langle \mathbf{U}^2 \rangle \) and \( \langle \mathbf{B}^2 \rangle \) (Vainshtein 1970; Moffatt 1978; Krause & Rädler 1980). In particular, when the field is helical \( \langle \mathbf{F} \sim \nabla \times \mathbf{f} \rangle \), Equation (3) for the large-scale magnetic field \( \mathbf{B} \) can be more simplified with the pseudo tensor \( \alpha \) and \( \beta \) coefficients (Brandenburg & Subramanian 2005b; Park & Blackman 2012a, 2012b).

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{b}) + \eta \nabla^2 \mathbf{B}, \]  
(5)

\[ \sim \nabla \times (\alpha \mathbf{B}) + (\beta + \eta) \nabla^2 \mathbf{B}. \]  
(6)

With the \( \alpha \) and \( \beta \) coefficients, the nonlinear dynamo process can be handled in a linear way. So far, the exact \( \alpha \) and \( \beta \) that cover the whole MHD system are not yet found. Their first-order smoothing approximations on the basis of the mean field theory (MFT) are known as \( \alpha = 1/3 \int (\langle \mathbf{j} \cdot \mathbf{b} \rangle - (\mathbf{u} \cdot \mathbf{w})) \mathbf{d}r \), \( \beta = 1/3 \int \langle (\nabla^2) \mathbf{d}r \rangle \) (Moffatt 1978; Krause & Rädler 1980). However, since the derivation of these results assume small magnetic Reynolds number \( Re_M \) (=\( u_l / \eta \)) or Strouhal number \( St \) (=\( ur / \lambda \)), their validity in space has been under dispute. Also, as the \( \alpha \) coefficient implies, there is a possibility of \( \alpha \) quenching with the growing helical magnetic field \( \langle \mathbf{j} \cdot \mathbf{b} \rangle \sim \langle (\mathbf{u} \cdot \mathbf{w}) \rangle \). Moreover, the \( \beta \) coefficient is always positive so that it only diffuses the magnetic energy \( \beta \nabla^2 \mathbf{B} < 0 \). These theoretical inferences are inconsistent with the real dynamo phenomenon.

Nonetheless, Equation (6) itself leads to a formally valid statistical representation (see Equation (10) and refer to dyadic). If we take the scalar product of \( \mathbf{B} \) to \( \partial \mathbf{B} / \partial t \), we get

\[ \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \sim \alpha (\mathbf{j} \cdot \mathbf{B}) - (\beta + \eta) (\mathbf{B}^2), \]  
(7)

for the helical large-scale field \( (k = 1) \). This relation implies that the growth rate of magnetic energy is decided by the diagonal terms \( \sim (\mathbf{B}, \mathbf{B}) \), energy density and off diagonal terms \( \sim (\mathbf{B}, \mathbf{B}), i \neq j \), helicity. Similarly, the growth rate of helicity can be represented with the diagonal terms and off diagonal ones. In our previous work (Park 2014), we gave the initially same magnetic energy \( \sim (\mathbf{B}, \mathbf{B}) \) to the system. And the growth rate of large-scale magnetic energy proportionally depends on the helicity ratio \( \sim (\mathbf{B}, \mathbf{B}) \) in the initial energy.

In solar or planetary physics, Equation (6) is divided into the poloidal component and the toroidal one to explain the observed magnetic fields as follows (Charbonneau 2014):

\[ \frac{\partial \mathbf{A}}{\partial t} = (\eta + \beta) \left( \nabla^2 - \frac{1}{\omega^2} \right) \mathbf{A} - \frac{u_p}{\omega} \cdot \nabla (\dot{\mathbf{v}} \cdot \mathbf{A}) \]
\[ + \alpha \mathbf{B}_{\text{tor}} \]  
(8)

\[ \frac{\partial \mathbf{B}_{\text{tor}}}{\partial t} \]
\[ = (\eta + \beta) \left( \nabla^2 - \frac{1}{\omega^2} \right) \mathbf{B}_{\text{tor}} + \frac{1}{\omega} \frac{\partial (\dot{\mathbf{v}} \cdot \mathbf{B}_{\text{tor}})}{\partial r} \frac{\partial (\eta + \beta)}{\partial r} \]
\[ - \omega u_p \cdot \nabla \mathbf{B}_{\text{tor}} \]
\[ - \omega \left( \nabla \times (\mathbf{A} \dot{\mathbf{v}}) \right) \cdot \nabla \Omega + \nabla \times \left( \alpha \nabla \times (\mathbf{A} \dot{\mathbf{v}}) \right), \]  
(9)

where \( \mathbf{B}_{\text{pol}} = \nabla \times \mathbf{A}, \omega \) is r sin \( \theta \), and \( \Omega \) is the angular velocity.

As these equations show, the \( \alpha \) effect or some corresponding energy source \( S \) is a prerequisite for the sustainable poloidal magnetic field \( \mathbf{B}_{\text{pol}} \). The \( \alpha \) effect has been considered as a main energy source to generate \( \mathbf{B}_{\text{pol}} \) in Parker’s solar dynamo model (Parker 1955). This model is based on the direct mechanical actions of buoyancy and Coriolis force to \( \mathbf{B}_{\text{tor}} \), and \( \alpha \) is assumed to be its proportional tensor coefficient. But, the \( \alpha \) effect that we discuss here is of the electromagnetic interaction between \( \mathbf{U} \) and \( \mathbf{B} \), i.e., \( \sim \mathbf{J} \) (Park 2017a). On the other hand,
Babcock–Leighton’s model (Babcock 1961; Leighton 1969) considers the sunspot effect \( \sim S(r, \theta, \vec{B}_{\text{ext}}) \) including the effects of buoyancy and internal convective flow as a primary source of \( B_{\text{ext}} \). Numerically, this effect is included with the assumed sunspot effect near the surface and magnetic field at the tachocline \( (r_c = 0.7 R_{\odot}, S = C_S B(r_c, \theta)) \) (Jouve et al. 2008). The basic form of the equation apparently looks similar to Parker’s model. But, the exact \( C_S \) is not yet known. Recently, observational data of the Sun’s polar magnetic field were directly applied to Equation (9) to reproduce the periodic solar cycle. It expects a more complete pattern of solar magnetic field in a short-term period, not in the long-term period (Choudhuri et al. 2007).

In direct numerical simulation (DNS), the external energy is applied to the system through the forcing source \( f \) in Equations (2) and (3). And according to the forcing method, dynamo can be divided into kinetically forced dynamo (KFD) or magnetically forced dynamo (MFD). The effect is implicitly reflected in \( \alpha \) and \( \beta \), which provide some parameterized information of the internal plasma motion. To derive \( \alpha \) and \( \beta \), a few rigorous analytical methods such as eddy damped quasi-normalized Markovianized approximation (EDQNM; Pouquet et al. 1976) and direct interactive approximation (DIA; Yoshizawa 2011) were suggested and applied to Equations (2) and (3), and EMF \((\mathbf{u} \times \mathbf{B})\). They yielded qualitatively the same \( \alpha \) and \( \beta \) coefficient as those of MFT in the level of the first-order approximation. It is a reasonable result because the second-order moments \( \langle \mathbf{U} \mathbf{U} \rangle \) or \( \langle \mathbf{B}^2 \rangle \) are replaced by the same statistical relation during calculation (Frisch et al. 1975; Yoshizawa 2011; Park 2014):

\[
\langle X_i(k)X_m(-k) \rangle = P_{lm}(k)E(k) + \frac{i k_i k_m}{2 k^2} \epsilon_{lmn}H(k),
\]

Here, \( E(k) \) indicates the trace of the moment, i.e., energy density \( \langle X^2 \rangle / 2 \) in Fourier space, and \( P_{lm}(k) \) is a projection operator \( \delta_{lm} - k k_m / k^2 \). The physical meaning of \( H(k) \) becomes clear if we apply it to helicity.

\[
\langle \mathbf{u} \cdot \nabla \times \mathbf{u} \rangle = \int (-ik_i)\xi_{ij} \{ u_i(k)u_j(-k) \} \, dk
\]

\[
= \int \mathbf{H}(k) \, dk. \tag{11}
\]

As discussed, the quenching \( \alpha \) effect and positive \( \beta \) effect cannot explain the sustainable evolution of large-scale magnetic field. There were some trials to derive the negative diffusion effect using the \( \alpha \) coefficient. Moffatt derived the \( \alpha \) and \( \beta \) in Lagrangian formation (Moffatt 1974), and Kraichnan derived the magnetic induction equation in the strongly helical system (Kraichnan 1976):

\[
\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B} + \tau_2 \nabla \times (\alpha \nabla \times \alpha \mathbf{B})
\]

\[
= (\eta - \tau_2 A) \nabla^2 \mathbf{B}. \tag{12}
\]

(\( \eta = \int u^2_0 \, dt \sim u_0^2, \alpha(x, t) = (-1/3)(\mathbf{u} \cdot \omega) \gamma, \alpha(x, t) \alpha(x', t') = A(x - x')D_2(t - t'), \tau_2 = \int u^2 \, dt \). This result implicitly assumes the long-lasting stability of the helical field and memory effect in the large eddy. However, as the result implies, \( \langle \mathbf{B}^2 \rangle \) and \( \langle \mathbf{A} \cdot \mathbf{B} \rangle \) become mutually independent, which is not correct.

### 1.3. Semianalytic Method

To calculate \( \alpha \) and \( \beta \) numerically, the test field method (TFM) has been studied (Schrinner et al. 2005). Repeated simulations with the embedded arbitrary large-scale magnetic field \( \mathbf{B}^T \) can produce the data for \( \mathbf{u} \) and \( \mathbf{b} \). Then, using the basic relation \( \xi_i = \langle \mathbf{u} \times \mathbf{b} \rangle_i = \alpha_i \mathbf{B}^T + \beta_i \partial \mathbf{B}^T / \partial x_k, \alpha_i \) and \( \beta_i \) can be found. This method indeed provides detailed information on \( \alpha_i \) and \( \beta_i \), which will be very useful to analytical theory. However, there are a couple of issues to be made clear. First, it should be checked how \( \alpha \) and \( \beta \) are affected by \( \mathbf{B}^T \). Basically, \( \alpha \) and \( \beta \) are small-scale quantities that can be easily constrained by \( \mathbf{B}^T \). If \( \mathbf{B}^T \) is applied, the fluid motion parallel to \( \mathbf{B}^T \) is not influenced. However, the motion perpendicular to \( \mathbf{B}^T \) is affected. Larmor radii of the flowing charged particles decrease in proportion to the strength of \( \mathbf{B}^T \), which also diminishes the Coulomb effect in the system. All of these effects may change the binding energy between electrons and charged particles so that the atoms or particles can become of a needle shape (Zeldovich 1983). It is desirable to study how that change affects the velocity of MHD flow \( \mathbf{v} = \vec{v}(\mathbf{R}, \mathbf{V}, t) \, d\mathbf{R} \, d\mathbf{V} \), \( \langle \mathbf{r}(\mathbf{R}, \mathbf{V}, t) \rangle \) (distribution function).

Second, some methods to apply TFM to astrophysical data need to be found. Being different from the lab experiment, there are practically few things that we can do to the astrophysical data aside from observation and measurement.

Instead of applying the artificial \( \mathbf{B}^T \) field to the system, we can calculate \( \alpha \) and \( \beta \) using the data of large-scale magnetic helicity \( \mathbf{H}_M(= \langle \mathbf{A} \cdot \mathbf{B} \rangle) \) and energy \( \mathbf{E}_M(= \langle \mathbf{B}^2 \rangle / 2) \). Applying dot product of \( \mathbf{A} \) and \( \mathbf{B} \) to Equation (6), we get

\[
\frac{d}{dt} \mathbf{H}_M = 4\alpha \mathbf{E}_M - 2(\beta + \eta) \mathbf{H}_M, \tag{13}
\]

\[
\frac{d}{dt} \mathbf{E}_M = \alpha \mathbf{H}_M - 2(\beta + \eta) \mathbf{E}_M. \tag{14}
\]

Diagonalizing the coupled equations, these equations can be calculated exactly (Park 2019)

\[
2\mathbf{H}_M(t) = (2\mathbf{E}_M(0) + \mathbf{H}_M(0)) e^{2\int_0^t (\alpha - \beta - \eta) \, dr}
- \int_0^t (2\mathbf{E}_M(0) - \mathbf{H}_M(0)) e^{2\int_0^r (\alpha + \beta + \eta) \, dr} \, dr, \tag{15}
\]

\[
4\mathbf{E}_M(t) = (2\mathbf{E}_M(0) + \mathbf{H}_M(0)) e^{2\int_0^t (\alpha - \beta - \eta) \, dr}
+ \int_0^t (2\mathbf{E}_M(0) - \mathbf{H}_M(0)) e^{2\int_0^r (\alpha + \beta + \eta) \, dr} \, dr. \tag{16}
\]

(Here, we used the Fourier transformed \( \langle \mathbf{A} \cdot \mathbf{B} \rangle = \langle \mathbf{J} \cdot \mathbf{B} \rangle \) in the large scale \( \langle \nabla \rightarrow i k, k = 1 \rangle \). The nonhelical component drops when the average is taken to this second-order moment. \( \mathbf{H}_M \) is always smaller than \( 2\mathbf{E}_M \), but \( \mathbf{H}_M \rightarrow 2\mathbf{E}_M \) with time in the helically forced system.)

Again, \( \alpha \) and \( \beta \) can be found

\[
\alpha(t) = \frac{1}{4} \frac{d}{dt} \log \left. \frac{2\mathbf{E}_M(t) + \mathbf{H}_M(t)}{2\mathbf{E}_M(t) - \mathbf{H}_M(t)} \right|, \tag{17}
\]

\[
\beta(t) = -\frac{1}{4} \frac{d}{dt} \log \left| (2\mathbf{E}_M(t) - \mathbf{H}_M(t))(2\mathbf{E}_M(t)
+ \mathbf{H}_M(t)) \right| - \eta. \tag{18}
\]

The results may look rather complicated, but what actually we need are \( \mathbf{E}_M(t), \mathbf{H}_M(t), \Delta \mathbf{E}_M / \Delta t, \) and \( \Delta \mathbf{H}_M / \Delta t \): temporally changing three-dimensional data \( \mathbf{J}(t) \) and \( \mathbf{B}(t) \) (i = 1, 2, 3).
In Equation (6) there is a tricky sign relation between $\mathbf{A} \cdot \mathbf{B}$ and $\alpha$. However, the sign issue becomes clear when Equations (13) and (14) are derived from Equation (6). For example, if the system is driven with positive kinetic helicity, $\alpha (\sim (\mathbf{j} \cdot \mathbf{b}) - (\mathbf{u} \cdot \mathbf{\omega}))$ becomes negative so that the second terms in the right-hand side of Equations (15) and (16) become dominant. Since $2E_M$ is always larger than $H_M$, the signs of $H_M$ and $E_M$ become consistent with the simulation result. Moreover, because of the magnetic helicity conservation, the sign of the large-scale magnetic helicity and the small-scale one become opposite. In the system forced with helical kinetic energy, the sign relation can be used to separate the large-scale field from the small-scale one without ambiguity. In contrast, when the system is forced with helical magnetic energy, the sign of $\alpha$ and magnetic helicity are the same, which is also reflected in the equations.

2. Result and Analysis

2.1. Numerical Result

Figures 1(a)–(c) show the evolving $\alpha$ (dotted line), $\beta$ (dashed line), and large-scale magnetic energy $E_M$ (solid line) in the systems forced with the fully right-handed helicoidal kinetic energy ($f_h = 1$). Their (molecular) magnetic diffusivities $\eta$ are the same, but kinematic viscosities $\nu$ vary to make $PrM(=\nu/\eta)$ 10, 1, and 0.33 in each system. In each simulation, “$\nu$” and “$\eta$” are chosen to get reliable results with the limited computational resources (80 CPUs, Resolution=400$^3$).

The evolution of the $\alpha$ effect is consistent with the theoretical expectation. However, its quenching position is too fast for the slowly growing $E_M$. To the contrary, $\beta$ keeps some negative value until it becomes quenched with the arising $E_M$. The negative $\beta$ is contradictory to the conventional dynamo theory. However, with the negative Laplacian $\nabla^2 B$ in Fourier space ($-k^2$), it makes sense that the negative $\beta$ effect plays an actual role of energy source for the large-scale magnetic field (see Equation (6)).

Figure 1(d) shows the effect of negative $f_h$ on $\alpha$, $\beta$, and $E_M$. The system itself is the same as that of Figure 1(b) except the forcing helicity ratio $f_h = -1$. The comparison of these two plots shows that the evolving patterns of $E_M$ and $\beta$ are independent of the sign of $f_h$. The $\alpha$ coefficients evolve with the opposite sign, but their magnitudes eventually converge to zero. Their temporal profiles are consistent with Equations (15)–(18).

Figure 1(e) shows the effect of changing $f_h$. The system starts with the fully helical kinetic energy $f_h = 1$. Then, the helicity ratio is dropped to be zero after $\alpha$ is quenched to separate their effects. This sudden change makes $\beta$ elevate up to a positive value. Considering that $\beta$ is independent of the sign of $f_h (+1$ or $-1$) and there is no change of kinetic energy in the system, the elevation of $\beta$ with decreasing $f_h$ is unusual. This implies that the $\beta$ effect is also related to the conservation of helicity or some other quantities. The positive $\beta$ has the effect of diffusing the magnetic energy, as the conventional dynamo theory expects $(\beta \sim u^2 \tau, \beta \nabla^2 \mathbf{B} \rightarrow - \beta k^2 \mathbf{B}, k = 1)$. We will explain its physical mechanism with the analysis of field structure of $U$ and $B$.

Figure 1(f) shows the evolution of typical SSD forced with the fully nonhelical kinetic energy ($f_h = 0$). Most of the magnetic energy is cascaded toward the small-scale regime, and only partial energy is inversely cascaded to the large scale $E_M(\lesssim 10^{-5})$. The initial flip-flop $\alpha$ and negative $\beta$ effect in this fully nonhelical forcing system may be responsible for this negligible growth. The nontrivial $\alpha$ effect seems to be caused by the naturally generated magnetic helicity (Woltjer 1958) or some helical component existing in the nonhelical forcing energy in the code. However, the reason is not clear at the moment.

2.2. Analysis with Field Structure

Equations (15)–(18) and simulation data explain the temporal evolution of $\alpha$, $\beta$, and $E_M(t)$ consistently. From now on, we will discuss the origins of the $\alpha$ and $\beta$ effects and their physical meaning in terms of the amplification or decay of the $B$ field. In addition to the semi-analytic approach, we will use the field structure model for the intuitive understanding of magnetic amplification (Park 2017b, 2019). The model is based on the conceptual but exact definitions of electromagnetism. Conventional dynamo theory does not strictly discriminate mechanical force and electromagnetic force. However, these two forces are different. The mechanical force cannot change the magnetic energy directly. In the level of a single fluid model (MHD), magnetic energy changes only through the curl of electromotive force (EMF) $\nabla \times (U \times B)$, the average force that accelerates the charged particles. This electromagnetic force leads to the actual motion of the charged particles, i.e., current density, which induces magnetic field and radiates the energy in the system.

2.2.1. Field Structure of $u$ and $b$

In Figure 2(a), the field structure for the nonhelical dynamo is analyzed. The structures of $u$ and $b$ are based on the mathematical interpretation of $\nabla \times (u \times b) = -u \cdot \nabla b + b \cdot \nabla u$ from Equation (3). These two terms indicate the growth rate of magnetic field due to the local (advective) and nonlocal dynamo processes, respectively. The magnetic energy at “$b$” is locally transferred (induced) to “$\mathbf{B}$” through $\int (-u \cdot \nabla b) \; d\tau = \int \mathbf{b} \; d\tau$, and the magnetic energy at “$u$” is nonlocally transferred to “$U$” through $\int \mathbf{u} \; d\tau$, which is the strongest near the intersection of “$u$” and “$\mathbf{B}$.” This nonuniform distribution of $j_{\text{diff,1}}$ generates $\mathbf{b}_{\text{diff,1}}$ along $U$. This consequent process forms a net magnetic field $\mathbf{b}_{\text{net}} = \mathbf{b}_{\text{loc,1}} + (\mathbf{b}_{\text{rad}} + \mathbf{b}_{\text{diff,1}}) \zeta$, which is used as a new seed field for the next dynamo process. The outgoing magnetic field along the velocity field $(\sim \zeta)$ results in $\mathbf{b}_{\text{net}}$ closer to the velocity field, which has the effect of decreasing EMF and dynamo efficiency. Physically, the conventional $\beta$ effect originates from this secondary interaction as follows:

$$\int u \; (u \times (\mathbf{u} \cdot \nabla \mathbf{B})) \; d\tau = -\int \varepsilon_{ijk} \langle u_i (\mathbf{u}) u_j (\mathbf{u}) \rangle \partial_i \mathbf{B}_k \; d\tau \rightarrow -\frac{1}{3} \int \langle u^2 \rangle \; d\tau \nabla \times \mathbf{B}.$$  

The $\beta$ coefficient is always positive and has an effect of diffusing magnetic energy to make the system homogeneous.
We forced six plasma systems with the same kinetic energy but different helicity ratios ($\nu = 6 \times 10^{-2}$, $f_h = 1$, HKFD) and magnetic Prandtl number $Pr_M$ (0.33, 1, 10). Resolution is 400$^3$.

Figure 1. We forced six plasma systems with the same kinetic energy but different helicity ratios $f_h = (u \cdot \omega) / k_T (u^2) (-1, 0, 1)$ and magnetic Prandtl number $Pr_M$ (0.33, 1, 10). Resolution is 400$^3$. 
Figures 2(b) and (c) show the evolution of a system forced with the left-handed kinetic helicity. Compared to the nonhelical dynamo process in Figure 2(a), an additional poloidal velocity field \( u_{\text{pol}} \) interacts with \( \mathbf{B}_{\text{pol}} \) and induces the current density \( J_{\text{ind,1}} \) which is parallel to \( \mathbf{B}_{\text{pol}} \). Then, \( J_{\text{ind,1}} \) generates the toroidal magnetic field \( \mathbf{B}_{\text{tor}} \) around \( \mathbf{B}_{\text{pol}} \) forming a right-handed magnetic helicity. \( \mathbf{B}_{\text{tor}} \) and other magnetic fields interact mutually to induce another current density \( J_{\text{ind,2}} \) parallel to \( \mathbf{B}_{\text{pol}} \). \( J_{\text{ind,2}} \) (not included in Figure 2) consequently amplifies \( \mathbf{B}_{\text{pol}} \). These toroidal and poloidal magnetic fields amplify each other through the magnetic helicity with the left-handed kinetic helicity. The growth of magnetic structure grows to surpass those of other “B” field eddies (see Table 1).

When the strength of \( \mathbf{B} \) is larger than those of other magnetic eddies \( b \), the direction of \( \mathbf{B}_{\text{loc}} \) \((-\mathbf{u} \cdot \nabla \mathbf{B})\) gets reversed from \( \hat{\mathbf{z}} \) to \(-\hat{\mathbf{z}}\). The magnetic energy in \( \mathbf{B} \) begins to cascade toward \( \mathbf{b} \) through the local transfer term.

### 2.2.2. Helical \( \beta \) Effect

The growth of \( \mathbf{B}_{\text{pol}} \) does not directly affect the essential property of \( u_{\text{pol}} \times \mathbf{B}_{\text{pol}} \sim J_{\text{ind,1},1} \) but amplifies the local \( J_{\text{diff}} \) is influenced by the relative strength of \( \mathbf{B}_{\text{pol}} \). Referring to Figures 2(b), (c), we can expand \( J_{\text{diff}} \sim (\mathbf{u}(t) \times \nabla \mathbf{B}) \) with the assumption of the slowly evolving \( \mathbf{B} \).

\[
J_{\text{diff}} \sim -\zeta_{ijk} \left\langle u_{ij}(r, t) u_{lm}(r, t) \right\rangle \frac{\partial \mathbf{B}_{k}}{\partial r} \delta_{im} \right) \frac{\partial \mathbf{B}_{k}}{\partial r} \delta_{im},
\]

\[
= \frac{1}{3} \left\langle u_{ij}(t) u_{lm}(t) \right\rangle \frac{\partial \mathbf{B}_{k}}{\partial r} \delta_{jm} = \frac{1}{6} \mathbf{H}(r) \zeta_{ijk} \frac{\partial \mathbf{B}_{k}}{\partial r} \delta_{im}, \tag{21}
\]

\[
\Rightarrow -\int \frac{1}{3} \left\langle u^2 \right\rangle \frac{\partial \mathbf{B}}{\partial r} \delta_{ij} + \frac{1}{6} \int \frac{\mathbf{H}(r)}{\partial \mathbf{B}} \frac{\partial \mathbf{B}}{\partial r} \delta_{ij}.
\]

In the second term, we used the identity \( \left\langle u_{ij} \partial_j u_{lm} \right\rangle = \left\langle \zeta_{ijn} \mathbf{H}(r) \right\rangle /6 \), which is a correlation length scale. The left-handed kinetic helicity (\( f_0 = (\mathbf{u} \cdot \nabla \times \mathbf{u})/k (\mathbf{u}^2) = -1, k = 5 \)) is used for the visual simplicity in the plot. There is no practical difference from the right-handed helical dynamo.

2.2.3. \( \beta \) in the Kinematic Regime

For \( 0 < t < 10 \) in Figures 1 and 2(a), (b), \( \partial \mathbf{B}/\partial \mathbf{z} \) is negative. The first term in Equation (21) can be written as \( J_{\text{diff,1}} \sim 1/3 \left\langle u^2 \right\rangle \mathbf{H} \zeta_{ijk} \mathbf{B}_{k} /\partial \mathbf{z} \) which is in the strong regime increase of \( \mathbf{B}_{\text{pol}} \). However, with the helical component, the effective \( \beta \) coefficient becomes \( \int \left\langle u^2 \right\rangle /l(6) \mathbf{H} \zeta_{ijk} \mathbf{B}_{k} /\partial \mathbf{z} \). This result indicates that any sign of kinetic helicity can amplify the large-scale field, as is shown in Figures 1(b), (d) (also refer to Equations (15), (16)). We check how these \( \beta \) effects evolve in the kinematic and nonlinear regimes separately.
approaches comes from the fact that $u_{pol}$ included in $H_V$ is calculated separately from the large-scale magnetic field.

2.2.4. $\beta$ in the Nonlinear Regime

As the strength of $B_{pol}$ outgrows that of other magnetic eddies, $\partial B / \partial t$ turns into a positive value (Figure 2(c)). Then, the direction of local transfer term reverses from “$\hat{x}$” to “$-\hat{x}$,” and the energy in $\mathbf{B}$ cascades toward $\mathbf{b}$. The first term in Equation (21) becomes $\beta_{diff,1} \sim -1/3(u^2) |\partial B / \partial t| \hat{y}$ decreasing $B_{nl} (\beta_1 > 0)$. The second term becomes $\sim 1/6 \beta |\nabla B / \partial \xi| \hat{y}$ increasing $B_{nl} (\beta_2 < 0)$. And, the overall $\beta_{net}$ is negative, which constrains “$l > \lambda_f / 2$” ($\lambda_f$, forcing eddy scale, see Figure 2). The field analysis also shows that the mutual interaction of $u \times b_{loc}$ yields $\beta_{diff,1}$ heading for “$-\hat{y}$.” On the contrary, $u_{pol} \times b_{loc}$ induces $\beta_{diff,2}$ parallel to $\mathbf{B}_{nl}$ generating the right-handed magnetic helicity.

On the other hand, the extraordinary change of $\beta > 0$ in Figure 1(e) where $f_{hl} = 1 \rightarrow 0$ can be explained with a virtual poloidal velocity field. The nonhelical forcing $f_{hl} = 0$ can be realized as applying the left-handed kinetic helicity to the right-handed helical system. We can assume a new poloidal velocity field $-u_{pol} \hat{y}$ with the same $\beta_{net}$. This new poloidal field can interact with $-u \cdot \nabla b$ to generate $-\beta_{diff,2} (\hat{\xi})$ producing the left-handed magnetic helicity, i.e., $\beta > 0$.

2.3. $\beta$ Effect that Constrains the Helical Plasma Motion

As Figure 3 shows, the growing magnetic energy suppresses the plasma motion regardless of forcing method, kinetic, magnetic, helical (LSD with $\alpha$ effect), or nonhelical (SSD). For the SSD with nonhelical energy, Lorentz force in momentum Equation (2) is divided into magnetic tension $\mathbf{B} \cdot \nabla \mathbf{B}$ and pressure $-\nabla B^2 / 2$ that can quench the plasma motion with thermal pressure $-\nabla p$. However, if the system is forced with helical energy, Lorentz force becomes negligible $\mathbf{J} \times \mathbf{B} \rightarrow -\mathbf{J} \cdot \mathbf{B} \times \mathbf{B} = \mathbf{J} \cdot (\mathbf{J} \times \mathbf{B})$, which is negligible. In the same way, the third term can also be dropped. The fourth term can be ignored using $j = (\sum_{f} f_{jl} v_{jl}) = \rho u$. The second term can be rewritten with $\alpha$ and $\beta$.

$$
\mathbf{J} \cdot \mathbf{B} \rightarrow -\nabla \mathbf{B} \cdot (\mathbf{U} \times \mathbf{B}) \\
\mathbf{J} \cdot \mathbf{B} \rightarrow \mathbf{J} \cdot (\mathbf{J} \times \mathbf{B}), \text{ which is negligible.}
$$

This result shows that the negative $\beta$ can dynamically suppress the mean plasma motion $-\beta \mathbf{B} \cdot \nabla^2 \mathbf{B} \rightarrow \beta \mathbf{B} \cdot \nabla^2 \mathbf{B}$. Also, the sign relation between $\alpha$ and $\mathbf{B} \cdot \nabla \times \mathbf{B}$, $\alpha$ term suppresses the plasma motion until it is quenched. But, its effect is not significant compared to the $\beta$ effect. The replacement of Lorentz force in the momentum equation is useful to deriving a new analytical MHD formula.
3. Summary

We have discussed the physical meaning of the $\alpha$ and $\beta$ effects and how to find the coefficients using $E_M(t)$ and $H_M(t)$. We showed the evolving profiles of these effects with the large-scale magnetic energy in the helical and nonhelical forcing dynamo. Figure 1 implies that the negative $\beta$ effect is a de facto dynamo generator after the $\alpha$ effect is quenched. To explain the results, we used the field structure model and analytic method. The $\beta$ effect in the helical system is composed of the toroidal component $\beta_1 \sim \mathbf{u} \times (-\mathbf{u} \cdot \nabla \mathbf{B})$ and the poloidal component $\beta_2 \sim \mathbf{u}_{\text{pol}} \times (-\mathbf{u} \cdot \nabla \mathbf{B})$. We showed that these $\beta$ effects are not fixed but are influenced by the spatially inhomogeneous large-scale magnetic field along with plasma motion: $-\mathbf{u} \cdot \nabla \mathbf{B}$. We have checked that the $\beta_1$ effect is dominant in the kinematic regime, and the $\beta_2$ effect is dominant in the nonlinear regime. For $-\mathbf{u} \cdot \nabla \mathbf{B} > 0$ in the kinematic regime, $\mathbf{J}_{\text{diff,1}}$ from $\beta_1$ amplifies magnetic field additionally, and $\mathbf{J}_{\text{diff,2}}$ from $\beta_2$ suppresses the growth of magnetic helicity inducing the oppositely handed magnetic helicity. In contrast, for $-\mathbf{u} \cdot \nabla \mathbf{B} < 0$ (nonlinear regime) $\mathbf{J}_{\text{diff,1}}$ reduces the dynamo efficiency, but $\mathbf{J}_{\text{diff,2}}$ elevates the helical dynamo efficiency producing the same sign of magnetic helicity. Briefly, the molecular magnetic diffusivity “$\eta$” in Equations (3) and (6) always diffuses magnetic energy. However, the turbulent magnetic diffusivity “$\beta$” can diffuse magnetic energy in a positive or negative way depending on the helical ratio and relative energy level. We also showed that the role of the negative $\beta$ effect can suppress the plasma motion.

Now, we may be tempted to ignore the $\alpha$ effect in dynamo. However, for the negative $\beta$ to be a helical dynamo generator, the nontrivial $\alpha$ effect that can amplify the large-scale magnetic field beyond the kinematic regime is a prerequisite. Moreover, without the $\alpha$ effect the poloidal magnetic field “$\mathbf{B}_{\text{pol}} = \nabla \times \mathbf{A}$” can play just a passive role in the generation of “$\mathbf{B}_{\text{tor}}$” (see Equations (8), (9)). This is inconsistent with the observed relation between “$\mathbf{B}_{\text{pol}}$” and “$\mathbf{B}_{\text{tor}}$.” The physical role and meaning of the $\alpha$ effect in solar or stellar dynamo models will be discussed in our next work.

This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation program (grant agreement No. 787544). The author is thankful for support from bwForCluster for numerical simulation. K.P. also appreciates Mr. Jungbae Park and Ms. Hyesoon Im for their support.

References

Babcock, H. W. 1961, ApJ, 133, 572
Balbus, S. A., & Hawley, J. F. 1991, ApJ, 376, 214
Boyd, T. J. M., & Sunderson, J. J. 2003, The Physics of Plasmas (Cambridge: Cambridge Univ. Press)
Brandenburg, A. 2001, ApJ, 550, 824
Brandenburg, A., & Subramanian, K. 2005a, PhR, 417, 1
Brandenburg, A., & Subramanian, K. 2005b, A&A, 349, 835
Charbonneau, P. 2014, ARA&A, 52, 251
Choudhuri, A. R., Chatterjee, P., & Jiang, J. 2007, PhRvL, 98, 131103
Frisch, U., Pouquet, A., Leorat, J., & Mazure, A. 1975, JFM, 68, 769
Jabbari, S., Brandenburg, A., Kleedorin, N., Mitra, D., & Rogachevskii, I. 2013, A&A, 556, A106
Jouve, L., Brun, A. S., Arlt, R., et al. 2008, A&A, 483, 949
Kraichnan, R. H. 1976, JFM, 75, 657
Krause, F., & Rädler, K. 1980, Mean-field Magnetohydrodynamics and Dynamo Theory (Oxford: Pergamon Press)
Leighton, R. B. 1969, ApJ, 156, 1
Machida, M. N., Matsumoto, T., Tomisaka, K., & Hanawa, T. 2005, MNRAS, 362, 369
Moffatt, H. K. 1974, JFM, 65, 1
Moffatt, H. K. 1978, Magnetic Field Generation in Electrically Conducting Fluids (Cambridge: Cambridge Univ. Press)
Park, K. 2014, MNRAS, 444, 3837
Park, K. 2017a, PhRvD, 96, 083505
Park, K. 2017b, MNRAS, 472, 1628
Park, K. 2019, ApJ, 872, 132
Park, K., & Blackman, E. G. 2012a, MNRAS, 419, 913
Park, K., & Blackman, E. G. 2012b, MNRAS, 423, 2120
Parker, E. N. 1955, ApJ, 122, 293
Pouquet, A., Frisch, U., & Leorat, J. 1976, JFM, 77, 321
Schrinner, M., Rädler, K.-H., Schmitt, D., Rheinhardt, M., & Christensen, U. 2005, AN, 326, 245
Valmstein, S. I. 1970, JETP, 31, 87
Woltjer, L. 1958, PNAS, 44, 489
Yoshizawa, A. 2011, Hydrodynamic and Magnetohydrodynamic Turbulent Flows (Dordrecht: Springer)
Zeldovich, Y. B. 1983, Magnetic Fields in Astrophysics (New York: Gordon and Breach)