Gradient estimates for solutions of the Lamé system with partially infinite coefficients

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Abstract
We establish upper bounds on the blow up rate of the gradients of solutions of the Lamé system with partially infinite coefficients in dimension two as the distance between the surfaces of discontinuity of the coefficients of the system tends to zero.

1 Introduction
We consider the Lamé system in linear elasticity. Let \( \Omega \subset \mathbb{R}^d \), \( d \geq 2 \), be a bounded open set with \( C^2 \) boundary, and \( D_1 \) and \( D_2 \) be two disjoint strictly convex open sets in \( \Omega \) with \( C^{2,\gamma} \) boundaries, \( 0 < \gamma < 1 \), which are \( \epsilon \)-distance apart and far away from \( \partial \Omega \). More precisely,

\[
\overline{D_1}, \overline{D_2} \subset \Omega, \quad \text{the principle curvatures of } \partial D_1, \partial D_2 \geq \kappa_0 > 0, \\
\epsilon := \text{dist}(D_1, D_2) > 0, \quad \text{dist}(D_1 \cup D_2, \partial \Omega) > \kappa_1 > 0,
\]

(1.1)

where \( \kappa_0, \kappa_1 \) are constants independent of \( \epsilon \).

Denote

\[
\overline{\Omega} := \Omega \setminus D_1 \cup D_2.
\]

We assume that \( \overline{\Omega} \) and \( D_1 \cup D_2 \) are occupied by two different homogeneous and isotropic materials with different Lamé constants \( (\lambda, \mu) \) and \( (\lambda_1, \mu_1) \). Then the elasticity tensors for the inclusions and the background can be written, respectively, as \( C^1 \) and \( C^0 \), with

\[
C^1_{ijkl} = \lambda_1 \delta_{ij} \delta_{kl} + \mu_1 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),
\]

and

\[
C^0_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).
\]
where \( i, j, k, l = 1, 2, \cdots, d \) and \( \delta_{ij} \) is the Kronecker symbol: \( \delta_{ij} = 0 \) for \( i \neq j \), \( \delta_{ij} = 1 \) for \( i = j \).

Let \( u = (u^1, u^2, \cdots, u^d)^T : \Omega \to \mathbb{R}^d \) denote the displacement field. For a given vector-valued function \( \varphi \), we consider the following Dirichlet problem

\[
\begin{aligned}
&\nabla \cdot \left( (\chi_{\tilde{\Omega}} C^0 + \chi_{D_1 \cup D_2} C^1) e(u) \right) = 0, \quad \text{in } \Omega, \\
&u = \varphi, \quad \text{on } \partial \Omega, 
\end{aligned}
\]

(1.2)

where \( \chi_D \) is the characteristic function of \( D \),

\[
e(u) := \frac{1}{2} \left( \nabla u + (\nabla u)^T \right)
\]

is the strain tensor.

We assume that the standard ellipticity condition holds for (1.2), that is,

\[
\mu > 0, \quad d\lambda + 2\mu > 0; \quad \mu_1 > 0, \quad d\lambda_1 + 2\mu_1 > 0.
\]

For \( \varphi \in H^1(\Omega; \mathbb{R}^d) \), it is well known that there exists a unique solution \( u \in H^1(\Omega; \mathbb{R}^d) \) of the Dirichlet problem (1.2), which is also the minimizer of the energy functional

\[
J[u] = \frac{1}{2} \int_{\Omega} \left( (\chi_{\tilde{\Omega}} C^0 + \chi_{D_1 \cup D_2} C^1) e(u), e(u) \right) dx
\]

on

\[
H^1_{\varphi}(\Omega; \mathbb{R}^d) := \left\{ u \in H^1(\Omega; \mathbb{R}^d) \mid u - \varphi \in H^1_0(\Omega; \mathbb{R}^d) \right\}. 
\]

Babuška, Andersson, Smith, and Levin [10] computationally analyzed the damage and fracture in fiber composite materials where the Lamé system is used. They observed numerically that the size of the strain tensor \( e(u) \) remains bounded when the distance \( \epsilon \) tends to zero. Stimulated by this, there have been many works on the analogous question for the scalar equation

\[
\begin{aligned}
&\nabla \cdot (a_k(x) \nabla u_k) = 0 \quad \text{in } \Omega, \\
&u_k = \varphi \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.3)

where \( \varphi \) is given, and

\[
a_k(x) = \begin{cases} 
 k \in (0, \infty) & \text{in } D_1 \cup D_2, \\
 1 & \text{in } \tilde{\Omega}.
\end{cases}
\]

For touching disks \( D_1 \) and \( D_2 \) in dimension \( d = 2 \), Bonnetier and Vogelius [15] proved that \( |\nabla u_k| \) remains bounded. The bound depends on the value of \( k \). Li and Vogelius [28] extended the result to general divergence form second order elliptic equations with piecewise smooth coefficients in all dimensions, and they proved that \( |\nabla u| \) remains bounded as \( \epsilon \to 0 \). They also established stronger, \( \epsilon \)-independent, \( C^{1,\alpha} \) estimates for solutions in the closure of each of the regions \( D_1, D_2 \) and \( \tilde{\Omega} \). This extension covers domains \( D_1 \) and \( D_2 \) of arbitrary smooth shapes. Li and Nirenberg extended
in [27] the results in [28] to general divergence form second order elliptic systems including systems of elasticity. This in particular answered in the affirmative the question naturally led to by the above mentioned numerical indication in [10] for the boundedness of the strain tensor as $\epsilon$ tends to 0. For higher derivative estimates, we draw attention of readers to the open problem on page 894 of [27].

The estimates in [27] and [28] depend on the ellipticity of the coefficients. If ellipticity constants are allowed to deteriorate, the situation is very different. It was shown in various papers, see for example Budiansky and Carrier [17] and Markenscoff [31], that when $k = \infty$ in (1.3) the $L^\infty$-norm of $|\nabla u_\infty|$ generally becomes unbounded as $\epsilon$ tends to 0. The rate at which the $L^\infty$-norm of the gradient of a special solution blows up was shown in [17] to be $\epsilon^{-1/2}$ in dimension $d = 2$. Ammari, Kang and Lim [9] and Ammari, Kang, Lee, Lee and Lim [7] proved that when $D_1$ and $D_2$ are disks in $\mathbb{R}^2$, and when $k = \infty$ in (1.3), the blow up rate of $|\nabla u_\infty|$ is $\epsilon^{-1/2}$. This result was extended by Yun [36, 37] and Bao, Li and Yin [11] to strictly convex $D_1$ and $D_2$ in $\mathbb{R}^2$. In dimension $d = 3$ and $d \geq 4$, the blow up rate of $|\nabla u_\infty|$ turns out to be $(\epsilon \ln \epsilon)^{-1}$ and $\epsilon^{-1}$ respectively; see [11]. The results were extended to multi-inclusions in [12]. Further, more detailed, characterizations of the singular behavior of $\nabla u_\infty$ have been obtained by Ammari, Ciraolo, Kang, Lee and Yun [3], Ammari, Kang, Lee, Lim and Zribi [3], Bonnetier and Triki [13, 14], Kang, Lim and Yun [21, 22]. For related works, see [4, 5, 14, 16, 18, 19, 23, 24, 25, 26, 29, 30, 32, 33] and the references therein.

In this paper we obtain gradient estimates for the Lamé system with infinity coefficients in dimension $d = 2$. In a subsequent paper we treat higher dimensional cases $d \geq 3$.

The linear space of rigid displacements in $\mathbb{R}^2$ is

$$\Psi := \left\{ \psi \in C^1(\mathbb{R}^2; \mathbb{R}^2) \mid \nabla \psi + (\nabla \psi)^T = 0 \right\},$$

or equivalently [33],

$$\Psi = \text{span} \left\{ \psi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \psi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \psi^3 = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \right\}.$$

If $\xi \in H^1(D; \mathbb{R}^2)$, $e(\xi) = 0$ in $D$, and $D \subset \mathbb{R}^2$ is a connected open set, then $\xi$ is a linear combination of $\{\psi^\alpha\}$ in $D$. If an element $\xi$ in $\Psi$ vanishes at two distinct points of $\mathbb{R}^2$, then $\xi \equiv 0$.

For fixed $\lambda$ and $\mu$ satisfying $\mu > 0$ and $\lambda + \mu > 0$, denote $u_{\lambda,\mu}$ the solution of (1.2). Then, as proved in the Appendix,  

$$u_{\lambda,\mu} \to u \text{ in } H^1(\Omega; \mathbb{R}^2) \text{ as } \min\{\lambda_1, \lambda_1 + \mu_1\} \to \infty.$$  

(1.4)

where $u$ is a $H^1(\Omega; \mathbb{R}^2)$ solution of

$$\begin{cases}
\mathcal{L}_{\lambda,\mu} u := \nabla \cdot \left( \mathcal{C}^e(u) \right) = 0, & \text{in } \tilde{\Omega}, \\
u^+ = \nu^- = 0, & \text{on } \partial D_1 \cup \partial D_2, \\
\mathcal{E}(u) = 0, & \text{in } D_1 \cup D_2, \\
\int_{D_1} \frac{\partial u}{\partial n} \cdot \psi^\alpha = 0, & \alpha = 1, 2, 3, \ i = 1, 2, \\
u = \varphi, & \text{on } \partial \Omega,
\end{cases}$$

(1.5)
where
\[ \frac{\partial u}{\partial y_{0+}} := (\nabla e(u)) \vec{n} = \lambda (\nabla \cdot u) \vec{n} + \mu \left( \nabla u + (\nabla u)^T \right) \vec{n}, \]
and \( \vec{n} \) is the unit outer normal of \( D_i, i = 1, 2 \).

Here and throughout this paper the subscript \( \pm \) indicates the limit from outside and inside the domain, respectively. The existence, uniqueness and regularity of weak solutions to (1.5) are proved in the Appendix. In particular, the \( H^1 \) weak solution to (1.5) is in \( C^1(\Omega) \cap C^2(\overline{D_1 \cup D_2}) \).

The convergence (1.4) in the case \( \mu_1 \to \infty \) while \( \lambda_1 \) remains bounded was established in [6]. Our proof of (1.4) in the Appendix is different and is an extension to systems of that in [11].

The solution of (1.5) is also the unique function which has the least energy in appropriate functional spaces, characterized by
\[ I_\infty[u] = \min_{v \in \mathcal{A}} I_\infty[v], \]
where
\[ I_\infty[v] := \frac{1}{2} \int_{\overline{\Omega}} \left( \nabla e(v), e(v) \right) dx, \]
and
\[ \mathcal{A} := \{ u \in H^1_\varphi(\Omega; \mathbb{R}^2) \mid e(u) = 0 \text{ in } D_1 \cup D_2 \}. \]

A calculation gives
\[ (L_{\lambda, \mu} u)^i = \mu \Delta u^i + (\lambda + \mu) \left[ \partial_{x_i x_i} u^i + \partial_{x_j x_j} u^2 \right], \quad i = 1, 2. \tag{1.6} \]

Since \( D_1 \) and \( D_2 \) are two strictly convex subdomains of \( \Omega \), there exist two points \( P_1 \in \partial D_1 \) and \( P_2 \in \partial D_2 \) such that
\[ \text{dist}(P_1, P_2) = \text{dist}(\partial D_1, \partial D_2) = \epsilon. \tag{1.7} \]

We use \( P_1 P_2 \) to denote the line segment connecting \( P_1 \) and \( P_2 \). For readers’ convenience, we first assume that \( \partial D_1 \) near \( P_1 \) and \( \partial D_2 \) near \( P_2 \) are quadratic. For more general \( D_1 \) and \( D_2 \), we consider in Section 5.

Assume that for some \( \delta_0 > 0 \),
\[ \delta_0 \leq \mu, \lambda + \mu \leq \frac{1}{\delta_0}. \tag{1.8} \]

The main result in this paper is as follows.

\textbf{Theorem 1.1.} Assume that \( \Omega, D_1, D_2, \epsilon \) are defined in (1.1) with \( d = 2 \), \( \lambda \) and \( \mu \) satisfy (1.8), and \( \varphi \in C^{1,\gamma}(\partial \Omega; \mathbb{R}^2) \) for some \( 0 < \gamma < 1 \). Let \( u \in H^1(\Omega; \mathbb{R}^2) \cap C^1(\overline{\Omega}; \mathbb{R}^2) \) be a solution to (1.5). Then for \( 0 < \epsilon < 1 \), we have
\[ |\nabla u(x)| \leq \begin{cases} \frac{C}{\sqrt{\epsilon} + \text{dist}(x, P_1 P_2)} ||\varphi||_{C^{1,\gamma}(\partial \Omega; \mathbb{R}^2)}, & x \in \tilde{\Omega}, \\ C||\varphi||_{C^{1,\gamma}(\partial \Omega; \mathbb{R}^2)}, & x \in D_1 \cup D_2. \end{cases} \tag{1.9} \]

where \( C \) is a universal constant. In particular,
\[ ||\nabla u||_{L_\infty(\Omega)} \leq C \epsilon^{-1/2} ||\varphi||_{C^{1,\gamma}(\partial \Omega; \mathbb{R}^2)}. \tag{1.10} \]
Note that throughout the paper, unless otherwise stated, \( C \) denotes some constant, whose value may vary from line to line, depending only on \( \kappa_0, \kappa_1, \gamma, \delta_0, \|\partial D_1\|_{C^2}, \|\partial D_2\|_{C^2}, \|\partial \Omega\|_{C^2} \) and the Lebesgue measure of \( \Omega \), and is in particular independent of \( \epsilon \). Also, we call a constant having such dependence a universal constant.

Since the blow up rate of \( |\nabla u(\infty)| \) for solutions of (1.3) when \( k = \infty \) is known to reach the magnitude \( \epsilon^{-1/2} \), estimate (1.10) is expected to be optimal. This is also supported by the numerical indication in [20].

The paper is organized as follows. In Section 2, we first introduce the setup of the proof of Theorem 1.1. Then we state a proposition, Proposition 2.1, containing key estimates, and deduce Theorem 1.1 from the proposition. In Sections 3 and 4, we prove Proposition 2.1. In Section 5, we prove Theorem 5.1 which extends Theorem 1.1 in two aspects. One is that the strict convexity assumption on \( \partial D_1 \) and \( \partial D_2 \) can be replaced by a weaker relative strict convexity assumption. The other is an upper bound of the gradient when the flatness order near the closest points between \( \partial D_1 \) and \( \partial D_2 \) is \( m \geq 2 \) instead of \( m = 2 \) for the strictly convex \( \partial D_1 \) and \( \partial D_2 \). In the Appendix, we give a variational characterization of solutions of the Lamé system with infinity coefficients and prove the previously mentioned convergence result (1.4).

2 Outline of the proof of Theorem 1.1 and recall of Korn’s inequalities

The proof of Theorem 1.1 makes use of the following decomposition. By the third line of (1.5), \( u \) is a linear combination of \( \{\psi^\alpha\} \) in \( D_1 \) and \( D_2 \), respectively. Since \( \mathcal{L}_{\lambda,\mu} \xi = 0 \) in \( \bar{\Omega} \) and \( \xi = 0 \) on \( \partial \bar{\Omega} \) imply that \( \xi = 0 \) in \( \bar{\Omega} \), we decompose the solution of (1.5), in the spire of [11], as follows:

\[
\begin{align*}
\mathbf{u} &= \begin{cases}
\sum_{\alpha=1}^{3} C_1^\alpha \psi^\alpha, & \text{in } D_1, \\
\sum_{\alpha=1}^{3} C_2^\alpha \psi^\alpha, & \text{in } D_2, \\
\sum_{\alpha=1}^{3} C_1^\alpha \psi_1^\alpha + \sum_{\alpha=1}^{3} C_2^\alpha \psi_2^\alpha + \psi_3, & \text{in } \bar{\Omega},
\end{cases}
\end{align*}
\]

(2.1)

where \( \psi_i^\alpha \in C^1(\bar{\Omega}; \mathbb{R}^2) \cap C^2(\bar{\Omega}; \mathbb{R}^2), \alpha = 1, 2, 3, i = 1, 2, \) satisfy

\[
\begin{align*}
\mathcal{L}_{\lambda,\mu} \psi_i^\alpha &= 0, & \text{in } \bar{\Omega}, \\
\psi_i^\alpha &= \psi^\alpha, & \text{on } \partial D_i, \\
\psi_i^\alpha &= 0, & \text{on } \partial D_j \cup \partial \Omega, \quad j \neq i;
\end{align*}
\]

(2.2)

\( \psi_3 \in C^1(\bar{\Omega}; \mathbb{R}^2) \cap C^2(\bar{\Omega}; \mathbb{R}^2) \) satisfies

\[
\begin{align*}
\mathcal{L}_{\lambda,\mu} \psi_3 &= 0, & \text{in } \bar{\Omega}, \\
\psi_3 &= 0, & \text{on } \partial D_1 \cup \partial D_2, \\
\psi_3 &= \varphi, & \text{on } \partial \Omega,
\end{align*}
\]

(2.3)
We will use the following notations:

By the decomposition (2.1), we write

\[ \nabla u = \sum_{\alpha=1}^{2} (C_1^{\alpha} - C_2^{\alpha}) \nabla v_1^{\alpha} + \sum_{\alpha=1}^{2} C_2^{\alpha} (\nabla v_1^{\alpha} + \nabla v_2^{\alpha}) + \sum_{\alpha=1}^{2} C_3^{\alpha} \nabla v_3^{\alpha} \]  

in \( \bar{\Omega} \). \tag{2.4}

Theorem 1.1 can be deduced from the following proposition.

**Proposition 2.1.** Under the hypotheses of Theorem 1.1 and a normalization \( \|\varphi\|_{C^{1,\gamma}(\bar{\Omega})} = 1 \), we have, for \( 0 < \epsilon < 1 \),

\[ \|\nabla v_3\|_{L^{\infty}(\bar{\Omega})} \leq C; \tag{2.5} \]

\[ \|\nabla v_1^{\alpha} + \nabla v_2^{\alpha}\|_{L^{\infty}(\bar{\Omega})} \leq C, \quad \alpha = 1, 2, 3; \tag{2.6} \]

\[ |\nabla v_i^{\alpha}(x)| \leq \frac{C}{\epsilon + \text{dist}(x, \partial \Omega P_1 P_2)}, \quad i, \alpha = 1, 2, \quad x \in \bar{\Omega}; \tag{2.7} \]

\[ |\nabla v_i^{3}(x)| \leq \frac{C}{\epsilon + \text{dist}(x, \partial \Omega P_1 P_2)}, \quad i = 1, 2, \quad x \in \bar{\Omega}; \tag{2.8} \]

and

\[ |C_i^{\alpha}| \leq C, \quad i = 1, 2, \quad \alpha = 1, 2, 3; \tag{2.9} \]

\[ |C_1^{\alpha} - C_2^{\alpha}| \leq C \sqrt{\epsilon}, \quad \alpha = 1, 2. \tag{2.10} \]

**Proof of Theorem 1.1 by using Proposition 2.1.** Clearly, we only need to prove the theorem under the normalization \( \|\varphi\|_{C^{1,\gamma}(\bar{\Omega})} = 1 \).

Since

\[ \nabla u = C_i^{3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]  
in \( D_i \), \( i = 1, 2 \),

the second estimate in (1.9) follows easily from (2.9).

By (2.4) and Proposition 2.1 we have, for \( x \) in \( \Omega \),

\[ |\nabla u(x)| \leq \sum_{\alpha=1}^{2} |C_1^{\alpha} - C_2^{\alpha}| |\nabla v_1^{\alpha}(x)| + C \sum_{i=1}^{2} |\nabla v_i^{3}(x)| + C \leq \frac{C}{\sqrt{\epsilon + \text{dist}(x, \partial \Omega P_1 P_2)}}. \]

Theorem 1.1 follows. \( \square \)

To complete this section, we recall some properties of the tensor \( C \). For the isotropic elastic material, let

\[ C := (C_{ijkl}) = (\lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})), \quad \mu > 0, \quad d\lambda + 2\mu > 0. \]

The components \( C_{ijkl} \) satisfy the following symmetric condition:

\[ C_{ijkl} = C_{klij} = C_{klij}, \quad i, j, k, l = 1, 2, \cdots, d. \tag{2.11} \]

We will use the following notations:

\[ (CA)_{ij} = \sum_{k,l=1}^{d} C_{ijkl} A_{kl}, \quad \text{and} \quad (A, B) \equiv A : B = \sum_{i,j=1}^{d} A_{ij} B_{ij}, \]

\[ 6 \]
for every pair of $d \times d$ matrices $A = (A_{ij})$, $B = (B_{ij})$. Clearly

$$(CA, B) = (A, CB).$$

If $A$ is symmetric, then, by the symmetry condition (2.11), we have that

$$(CA, A) = C_{ijkl} A_{kl} A_{ij} = \lambda A^{kk} A_{ij} + 2 \mu A_{ij} A_{ij}.$$  

Thus $C$ satisfies the following ellipticity condition: For every $d \times d$ real symmetric matrix $A = (A_{ij})$,

$$\min \{2\mu, d\lambda + 2\mu\} |A|^2 \leq (CA, A) \leq \max \{2\mu, d\lambda + 2\mu\} |A|^2,$$  

(2.12)

where $|A|^2 = \sum_{ij} A_{ij}^2$.

For readers’ convenience, we recall some inequalities of Korn’s type, see, e.g. theorems 2.1, theorem 2.5, theorem 2.10 and theorem 2.14 in [33].

**Lemma A.** (First Korn inequality) Let $\Omega$ be a bounded open set of $\mathbb{R}^d$, $d \geq 2$. Then every $u \in H^1_0(\Omega, \mathbb{R}^d)$ satisfies the inequality

$$\|\nabla u\|_{L^2(\Omega)}^2 \leq 2\|e(u)\|_{L^2(\Omega)}^2.$$  

Next, a few versions of the Second Korn inequality

**Lemma B.** Suppose that $\Omega$ is a bounded open set of $\mathbb{R}^d$, $d \geq 2$, of diameter $R$, and it is star-shaped with respect to the ball $B_{R_1} = \{x : |x| < R_1\}$. Then for any $u \in H^1(\Omega, \mathbb{R}^d)$ we have the inequality

$$\|\nabla u\|_{L^2(\Omega)}^2 \leq C_1 \left( \frac{R}{R_1} \right)^{d+1} \|e(u)\|_{L^2(\Omega)}^2 + C_2 \left( \frac{R}{R_1} \right)^d \|\nabla u\|_{L^2(B_{R_1})}^2,$$

where $C_1, C_2$ are constants depending only on $d$.

We remark that the above inequality holds for a Lipschitz domain $\Omega$, with $C_1$ and $C_2$ depending on $\Omega$, since such a domain is a union of a finite number of star-shaped domains. The following lemma is an easy consequence of Lemma A and Lemma B.

**Lemma C.** Suppose that $\Omega$ satisfies the condition of Lemma B and $u \in H^1(\Omega, \mathbb{R}^d)$. Then

$$\|\nabla u\|_{L^2(\Omega)}^2 \leq C_1 \left( \frac{R}{R_1} \right)^{d+1} \|e(u)\|_{L^2(\Omega)}^2 + C_2 \left( \frac{R}{R_1} \right)^d \|u\|_{L^2(\Omega)}^2,$$

where $\gamma$ is the distance of $B_{R_1}$ from $\partial \Omega$, and $C_1, C_2$ depend only on $d$.

In applications it is often important to have the following version of the Second Korn inequality. We still use $\Psi$ to denote the linear space of rigid displacements in $\mathbb{R}^d$.

**Lemma D.** Let $\Omega$ be a bounded Lipschitz open set of $\mathbb{R}^d$, $d \geq 2$, and let $V$ be a closed subspace of $H^1(\Omega, \mathbb{R}^d)$, such that $V \cap \Psi = \{0\}$. Then every $v \in V$ satisfies

$$\|v\|_{H^1(\Omega)} \leq C \|e(v)\|_{L^2(\Omega)},$$

where $C$ depends only on $\Omega$ and $V$.  

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3 Estimates of $\nabla \nu_1^\alpha$, $\nabla \nu_2^\alpha$ and $\nabla \nu_3$

Before proceeding to prove Proposition 2.1, we first fix notations. By a translation and rotation of the coordinates if necessary, we may assume without loss of generality that the points $P_1$ and $P_2$ in (1.7) satisfy

$$P_1 = \left(0, \frac{\epsilon}{2}\right) \in \partial D_1, \quad \text{and} \quad P_2 = \left(0, -\frac{\epsilon}{2}\right) \in \partial D_2.$$  

Fix a small universal constant $R$, such that the portions of $\partial D_i$ near $P_i$ can be represented respectively by

$$x_2 = \frac{\epsilon}{2} + h_1(x_1), \quad \text{and} \quad x_2 = -\frac{\epsilon}{2} + h_2(x_1), \quad \text{for } |x_1| < 2R.$$  

Moreover, by the assumptions on $\partial D_i$, $h_i$ satisfies

$$\frac{\epsilon}{2} + h_1(x_1) > -\frac{\epsilon}{2} + h_2(x_1), \quad \text{for } |x_1| < 2R,$$

$$h_1(0) = h_2(0) = h_1'(0) = h_2'(0) = 0,$$

$$h_1''(0) \geq \kappa_0 > 0, \quad h_2''(0) \leq -\kappa_0 < 0,$$

and

$$\|h_1\|_{C^2(\{-2R,2R\})} + \|h_2\|_{C^2(\{-2R,2R\})} \leq C.$$  

For $0 < r \leq 2R$, denote

$$\Omega_r := \left\{ x \in \mathbb{R}^2 \mid -\frac{\epsilon}{2} + h_2(x_1) < x_2 < \frac{\epsilon}{2} + h_1(x_1), \ |x_1| < r \right\}.$$  

The top and bottom boundaries of $\Omega_r$ are

$$\Gamma^+_r = \left\{ x \in \mathbb{R}^2 \mid x_2 = \frac{\epsilon}{2} + h_1(x_1), \ |x_1| < r \right\},$$

and

$$\Gamma^-_r = \left\{ x \in \mathbb{R}^2 \mid x_2 = -\frac{\epsilon}{2} + h_2(x_1), \ |x_1| < r \right\}.$$  

Here $x = (x_1, x_2)$.

3.1 Estimates of $\nu_3$ and $\nu_1^\alpha + \nu_2^\alpha$, $\alpha = 1, 2, 3$

Lemma 3.1.  

$$\|\nu_3\|_{L^\infty(\tilde{\Omega})} + \|\nabla \nu_3\|_{L^\infty(\tilde{\Omega})} \leq C.$$  

$$\|\nu_1^\alpha + \nu_2^\alpha\|_{L^\infty(\tilde{\Omega})} + \|\nabla \nu_1^\alpha + \nabla \nu_2^\alpha\|_{L^\infty(\tilde{\Omega})} \leq C, \quad \alpha = 1, 2, 3.$$  

Proof. As mentioned before, we may assume without loss of generality that $\|\varphi\|_{C^{1,\gamma}(\overline{\Omega})} = 1$. Extending $\varphi$ to $\Phi \in C^{1,\gamma}(\overline{\Omega})$ satisfying $\Phi(x) = 0$ for all $\text{dist}(x, \partial \Omega) > \kappa_1/2$. In particular, $\Phi = 0$ near $D_1 \cup D_2$, and

$$\int_{\tilde{\Omega}} |\nabla \Phi|^2 \, dx \leq C\|\varphi\|_{C^{1,\gamma}(\Omega)} = C.$$  

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Then, in view of (2.3),

\[ I_\infty[v_3] := \frac{1}{2} \int_\Omega (C^0 e(v_3), e(v_3)) \, dx \leq I_\infty[\Phi] \leq C. \]

By the First Korn inequality (Lemma A) and (2.12),

\[
\| \nabla (v_3 - \Phi) \|^2_{L^2(\tilde{\Omega})} \leq 2 \| e(v_3 - \Phi) \|^2_{L^2(\tilde{\Omega})} \\
\leq C \left( \| e(v_3) \|^2_{L^2(\tilde{\Omega})} + \| e(\Phi) \|^2_{L^2(\tilde{\Omega})} \right) \\
\leq C (I_\infty[v_3] + I_\infty[\Phi]) \\
\leq C.
\]

It follows that

\[ \| \nabla v_3 \|_{L^2(\tilde{\Omega})} \leq C. \]

Consequently,

\[ \| v_3 \|_{L^2(\tilde{\Omega})} \leq C \| \nabla v_3 \|_{L^2(\tilde{\Omega})} \leq C. \]

Note that the constant \( C \) above is independent of \( \epsilon \). By the interior estimates and the boundary estimates for elliptic systems (see Agmon, Douglis and Nirenberg [1] and [2]), we have

\[ \| \nabla v_3 \|_{L^\infty(\tilde{\Omega}\setminus\Omega_R/2)} \leq C. \]

We apply theorem 1.1 in [26] to \( v_3 \) and obtain

\[ \| \nabla v_3 \|_{L^\infty(\Omega_R/2)} \leq C. \]

Since

\[
\begin{aligned}
& L_{t,\mu}(v_1^\alpha + v_2^\alpha - \psi^\alpha) = 0, \quad \text{in } \tilde{\Omega}, \\
& v_1^\alpha + v_2^\alpha - \psi^\alpha = 0, \quad \text{on } \partial D_1 \cup \partial D_2, \\
& v_1^\alpha + v_2^\alpha - \psi^\alpha = -\psi^\alpha, \quad \text{on } \partial \Omega,
\end{aligned}
\]

the above arguments yield, with \( \varphi = -\psi^\alpha \),

\[ \| \nabla v_1^\alpha + \nabla v_2^\alpha \|_{L^\infty(\tilde{\Omega})} \leq C, \quad \alpha = 1, 2, 3. \tag{3.4} \]

Lemma 3.1 follows from the above. \( \square \)

### 3.2 Estimates of \( v_i^\alpha, i, \alpha = 1, 2 \)

To estimate \( v_i^\alpha, i, \alpha = 1, 2 \), we introduce a scalar function \( \bar{u} \in C^2(\mathbb{R}^2) \), such that \( \bar{u} = 1 \) on \( \partial D_1 \), \( \bar{u} = 0 \) on \( \partial D_2 \cup \partial \Omega \),

\[
\bar{u}(x) = \frac{x_2 - h_2(x_1) + \frac{\xi}{\epsilon}}{\epsilon + h_1(x_1) - h_2(x_1)}, \quad \text{in } \Omega_R, \tag{3.5}
\]

and

\[ \| \bar{u} \|_{C^2(\mathbb{R}^2, \Omega_R)} \leq C. \tag{3.6} \]
A calculation gives
\[ |\partial_{x_1} \bar{u}(x)| \leq \frac{C|x_1|}{\epsilon + |x_1|^2}, \quad |\partial_{x_2} \bar{u}(x)| \leq \frac{C}{\epsilon + |x_1|^2}, \quad x \in \Omega_R, \]  
(3.7)
\[ |\partial_{x_1,x_1} \bar{u}(x)| \leq \frac{C}{\epsilon + |x_1|^2}, \quad |\partial_{x_1,x_2} \bar{u}(x)| \leq \frac{C|x_1|}{(\epsilon + |x_1|^2)^2}, \quad \partial_{x_2,x_2} \bar{u}(x) = 0, \quad x \in \Omega_R, \]  
(3.8)

Define
\[ \bar{u}_1 = (\bar{u}, 0)^T, \quad \bar{u}_2 = (0, \bar{u})^T, \quad \text{in } \Omega, \]  
(3.9)
then \( v_1^\alpha = \bar{u}_1^\alpha \) on \( \partial \Omega \). Similarly, we can define
\[ \bar{u}_2^1 = (u, 0)^T, \quad \bar{u}_2^2 = (0, u)^T, \quad \text{in } \Omega, \]  
(3.10)
where \( u \) is a scalar function in \( C^2(\mathbb{R}^2) \) satisfying \( u = 1 \) on \( \partial D_2 \), \( u = 0 \) on \( \partial D_1 \cup \partial \Omega \),
\[ u(x) = \frac{-x_2 + h_1(x_1) + \epsilon}{\epsilon + h_1(x_1) - h_2(x_1)}, \quad x \in \Omega_R, \]  
(3.11)
and
\[ \|u\|_{C^2(\mathbb{R}^2 \setminus \Omega_R)} \leq C. \]  
(3.12)

By (1.6), (3.7) and (3.8),
\[ |L_{\lambda,u} \bar{u}_i^\alpha(x)| \leq \frac{C}{\epsilon + |x_1|^2} + \frac{C|x_1|}{(\epsilon + |x_1|^2)^2}, \quad i, \alpha = 1, 2, \quad x \in \Omega_R. \]  
(3.13)

For \( |z_1| \leq R \), we always use \( \delta \) to denote
\[ \delta := \delta(z_1) = \frac{\epsilon + h_1(z_1) - h_2(z_1)}{2}. \]  
(3.14)

Clearly,
\[ \frac{1}{C}(\epsilon + |z_1|^2) \leq \delta(z_1) \leq C(\epsilon + |z_1|^2). \]  
(3.15)

For \( |z_1| \leq R/2, s < R/2 \), let
\[ \Omega_s(z_1) := \left\{ (x_1, x_2) \mid -\frac{\epsilon}{2} + h_2(x_1) < x_2 < \frac{\epsilon}{2} + h_1(x_1), \ |x_1 - z_1| < s \right\}. \]  
(3.16)

We denote
\[ w_i^\alpha := v_i^\alpha - \bar{u}_i^\alpha, \quad i, \alpha = 1, 2. \]  
(3.17)

In order to prove (2.7), it suffices to prove the following proposition.

**Proposition 3.2.** Assume the above, let \( v_i^\alpha \in C^2(\Omega; \mathbb{R}^2) \cap C^1(\Omega; \mathbb{R}^2) \) be the weak solution of (2.2). Then, for \( i, \alpha = 1, 2 \),
\[ \int_{\Omega_s} |\nabla w_i^\alpha|^2 \leq C, \]  
(3.18)
\[ \int_{\Omega_s(z_1)} |\nabla w_i^\alpha|^2 \leq \begin{cases} C(\epsilon^2 + |z_1|^2), & |z_1| \leq \sqrt{\epsilon}, \\ C|z_1|^2, & \sqrt{\epsilon} < |z_1| \leq R, \end{cases} \]  
(3.19)
and
\[ |\nabla w_i^\alpha(x)| \leq \begin{cases} \frac{C(\epsilon + |z_1|)}{\epsilon}, & |x_1| \leq \sqrt{\epsilon}, \\ \frac{C}{|z_1|}, & \sqrt{\epsilon} < |x_1| \leq R. \end{cases} \]  
(3.20)
Corollary 3.3. For $i, \alpha = 1, 2$,

$$
\left| \nabla v_i^\alpha(x) \right| \leq \frac{C}{\epsilon + \text{dist}^2(x, P_1P_2)}, \quad x \in \overline{\Omega}. \quad (3.21)
$$

Proof of Corollary 3.3. A consequence of (3.18) is

$$
\int_{\overline{\Omega}\setminus\Omega_{R/2}} |\nabla v_i^\alpha|^2 \, dx \leq 2 \int_{\overline{\Omega}\setminus\Omega_{R/2}} \left( |\nabla \tilde{u}_i^\alpha|^2 + |\nabla w_i^\alpha|^2 \right) \, dx \leq C,
$$

With this we can apply classical elliptic estimates to obtain

$$
\left\| \nabla v_i^\alpha \right\|_{L^\infty(\overline{\Omega}\setminus\Omega_R)} \leq C, \quad i, \alpha = 1, 2. \quad (3.22)
$$

Under assumption (1.1),

$$
\frac{1}{C}(\epsilon + |x_1|^2) \leq \text{dist}(x, P_1P_2) \leq C(\epsilon + |x_1|^2).
$$

Estimate (3.21) in $\Omega_R$ follows from (3.20) and the fact that

$$
\left| \nabla \tilde{u}_i^\alpha(x) \right| \leq \frac{C}{\epsilon + |x_1|^2}, \quad \text{in } \Omega_R.
$$

\[ \square \]

Proof of Proposition 3.2. The iteration scheme we use in the proof is similar in spirit to that used in [26]. We only prove it for $i = \alpha = 1$, since the same proof applies to the other cases. For simplicity, denote $w := w_1^1$. We divide into three steps.

**STEP 1.** Proof of (3.18).

By (3.17),

$$
\begin{cases}
\mathcal{L}_{\lambda, \mu} w = - \mathcal{L}_{\lambda, \mu} \tilde{u}_1^1, & \text{in } \overline{\Omega}, \\
w = 0, & \text{on } \partial\overline{\Omega}.
\end{cases} \quad (3.23)
$$

Multiplying the equation in (3.23) by $w$ and integrating by parts, we have

$$
\int_{\overline{\Omega}} \left( \frac{\partial}{\partial x} e(w), e(w) \right) \, dx = \int_{\overline{\Omega}} w \mathcal{L}_{\lambda, \mu} \tilde{u}_1^1 \, dx. \quad (3.24)
$$

By the mean value theorem, there exists $r_0 \in (R/2, 2R/3)$ such that

$$
\int_{|x_1| = r_0} |w| \, dx_2 \leq \frac{6}{R} \int_{-\epsilon/2 + h_2(x_1) < x_2 < \epsilon/2 + h_1(x_1)} |w| \, dx \leq C \int_{\Omega_{2R/3} \setminus \Omega_{R/2}} |\nabla w| \, dx \leq C \left( \int_{\overline{\Omega}} |\nabla w|^2 \, dx \right)^{1/2}. \quad (3.25)
$$
It follows from (2.12), (3.24) and the First Korn inequality that
\[
\int_{\Omega} |\nabla w|^2 \, dx \\
\leq 2 \int_{\Omega} |e(w)|^2 \, dx \\
\leq C \left( \int_{\Omega_0} w \left( \mathcal{L}_{\lambda,\mu} \bar{u}_1 \right) \, dx \right) + C \left( \int_{\Omega_1 \setminus \Omega_0} w \left( \mathcal{L}_{\lambda,\mu} \bar{u}_1 \right) \, dx \right) \\
\leq C \left( \int_{\Omega_0} \left| \partial_{x_1} w \right| \, dx \right) + C \int_{\Omega_1 \setminus \Omega_0} |w| \, dx \\
\leq C \left( \left( \int_{\Omega_0} |\partial_{x_1} \bar{u}|^2 \, dx \right)^{1/2} \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^{1/2} \right) + C \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^{1/2}. \tag{3.26}
\]

First,
\[
\int_{\Omega_0} w^{(1)} \partial_{x_1} \bar{u} \, dx = - \int_{\Omega_0} \partial_{x_1} w^{(1)} \partial_{x_1} \bar{u} \, dx + \int_{|x_1| = r_0, -\epsilon/2+h_2(x_1) \leq x_2 \leq \epsilon/2+h_1(x_1)} (\partial_{x_1} \bar{u}) w^{(1)} \, dx_2 \\
:= I + II.
\]

Then, by (3.7),
\[
|I| \leq C \left( \int_{\Omega_0} |\partial_{x_1} \bar{u}|^2 \, dx \right)^{1/2} \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^{1/2} \leq C \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^{1/2}.
\]

By (3.25), we have
\[
|II| \leq C \int_{|x_1| = r_0, -\epsilon/2+h_2(x_1) \leq x_2 \leq \epsilon/2+h_1(x_1)} |w| \, dx_2 \leq C \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^{1/2}.
\]

Hence
\[
\left| \int_{\Omega_0} w^{(1)} \partial_{x_1} \bar{u} \, dx \right| \leq C \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^{1/2}. \tag{3.27}
\]

Similarly, using \(w = 0\) on \(\partial D_1 \cup \partial D_2\),
\[
\left| \int_{\Omega_0} w^{(2)} \partial_{x_1} \bar{u} \, dx \right| = \left| \int_{\Omega_0} \partial_{x_2} w^{(2)} \partial_{x_1} \bar{u} \, dx \right| \\
\leq C \left( \int_{\Omega_0} |\partial_{x_1} \bar{u}|^2 \, dx \right)^{1/2} \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^{1/2} \\
\leq C \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^{1/2}.
\]

Therefore, combining this estimate with (3.27) and (3.26),
\[
\int_{\Omega} |\nabla w|^2 \, dx \leq C \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^{1/2},
\]

which implies (3.18).

**STEP 2. Proof of (3.19).**

For $0 < t < s < R$, let $\eta$ be a smooth function satisfying $\eta(x_1) = 1$ if $|x_1 - z_1| < t$, $\eta(x_1) = 0$ if $|x_1 - z_1| > s$, $0 \leq \eta(x_1) \leq 1$ if $t \leq |x_1 - z_1| \leq s$, and $|\eta'(x_1)| \leq \frac{2}{s-t}$. Multiplying the equation in (3.23) by $w\eta^2$ and integrating by parts lead to

$$\int_{\tilde{\Omega}_s(z_1)} (C^0 e(w), e(w\eta^2))dx = -\int_{\tilde{\Omega}_s(z_1)} (w\eta^2) \mathcal{L}_{\lambda,\mu} \bar{u}_1 dx.$$ \hspace{1cm} (3.28)

Using the First Korn inequality and some standard arguments, we have

$$\int_{\tilde{\Omega}_s(z_1)} (C^0 e(w), e(w\eta^2))dx \geq \frac{1}{C} \int_{\tilde{\Omega}_s(z_1)} |\nabla (w\eta)|^2 dx - C \int_{\tilde{\Omega}_s(z_1)} |w|^2 |\nabla \eta|^2 dx,$$ \hspace{1cm} (3.29)

and

$$\left| \int_{\tilde{\Omega}_s(z_1)} (w\eta^2) \mathcal{L}_{\lambda,\mu} \bar{u}_1 dx \right| \leq \frac{C}{(s-t)^2} \int_{\tilde{\Omega}_s(z_1)} |w|^2 dx + (s-t)^2 \int_{\tilde{\Omega}_s(z_1)} |\mathcal{L}_{\lambda,\mu} \bar{u}_1|^2 dx.$$ \hspace{1cm} (3.30)

**Case 1. For $\sqrt{\epsilon} \leq |z_1| \leq R$.**

Note that for $0 < s < \frac{2|z_1|}{3}$, we have

$$\int_{\tilde{\Omega}_s(z_1)} |w|^2 dx = \int_{|x_1 - z_1| \leq s} \int_{\frac{s + h_1(x_1)}{2} + h_2(x_1)} |w(x_1, x_2)|^2 dx_2 dx_1$$

$$\leq \int_{|x_1 - z_1| \leq s} (\epsilon + h_1(x_1) - h_2(x_1))^2 \int_{\frac{s + h_1(x_1)}{2}} \left| \partial_{x_2} w(x_1, x_2) \right|^2 dx_2 dx_1$$

$$\leq C |z_1|^4 \int_{\tilde{\Omega}_s(z_1)} |\nabla w|^2 dx,$$ \hspace{1cm} (3.31)

By (3.13), we have

$$\int_{\tilde{\Omega}_s(z_1)} |\mathcal{L}_{\lambda,\mu} \bar{u}_1|^2 dx \leq \frac{C \epsilon}{|z_1|^4}, \quad 0 < s < \frac{2|z_1|}{3}.$$ \hspace{1cm} (3.32)

Denote

$$\bar{F}(t) := \int_{\tilde{\Omega}_s(z_1)} |\nabla w|^2 dx.$$ \hspace{1cm} (3.33)

It follows from the above that

$$\bar{F}(t) \leq \left( \frac{C_0 |z_1|^2}{s-t} \right)^2 \bar{F}(s) + C(s-t)^2 \frac{s}{|z_1|^4}, \quad \forall \ 0 < t < s < \frac{2|z_1|}{3},$$ \hspace{1cm} (3.33)

where $C_0$ is also a universal constant.
Let \( t_i = 2C_0|i|z_1|^2 \), \( i = 1, 2, \cdots \). Then
\[
\frac{C_0|i|z_1|^2}{t_{i+1} - t_i} = \frac{1}{2}.
\]
Let \( k = \left[ \frac{1}{4C_0|i|z_1|^2} \right] \). Then by (3.33) with \( s = t_{i+1} \) and \( t = t_i \), we have
\[
\tilde{F}(t_i) \leq \frac{1}{4} \tilde{F}(t_{i+1}) + \frac{C(t_{i+1} - t_i)^2t_{i+1}}{|i|z_1|^4} \leq \frac{1}{4} \tilde{F}(t_{i+1}) + C(i+1)|i|z_1|^2,
\]
After \( k \) iterations, we have, using (3.18),
\[
\tilde{F}(t_i) \leq \left( \frac{1}{4} \right)^k \tilde{F}(t_{k+1}) + C|z_1|^2 \sum_{l=1}^{k} \left( \frac{1}{4} \right)^l (l+1) \leq C|z_1|^2. \quad (3.34)
\]
This implies that
\[
\int_{\tilde{\Omega}_{\epsilon(t)}} |\nabla w|^2 \, dx \leq C|z_1|^2. \quad (3.34)
\]
**Case 2.** For \( |z_1| \leq \sqrt{\epsilon} \).
For \( 0 < t < s < \sqrt{\epsilon} \), we still have (3.30). Estimate (3.31) becomes
\[
\int_{\tilde{\Omega}_{\epsilon(t)}} |w|^2 \, dx \leq C \epsilon^2 \int_{\tilde{\Omega}_{\epsilon(t)}} |\nabla w|^2 \, dx, \quad 0 < s < \sqrt{\epsilon}. \quad (3.34)
\]
Estimate (3.32) becomes
\[
\int_{\tilde{\Omega}_{\epsilon(t)}} |L_{i,\mu \nu} \tilde{u}^1|^2 \, dx \leq \frac{Cs}{\epsilon} + \frac{C|z_1|^2s}{\epsilon^3} + \frac{Cs^3}{\epsilon^3}, \quad 0 < s < \sqrt{\epsilon}. \quad (3.35)
\]
Estimate (3.33) becomes, in view of (3.30),
\[
\tilde{F}(t) \leq \left( \frac{C_0\epsilon}{s-t} \right)^2 \tilde{F}(s) + C(s-t)^2 s \left( \frac{1}{\epsilon} + \frac{|z_1|^2}{\epsilon^3} + \frac{s^2}{\epsilon^3} \right), \quad \forall \ 0 < t < s < \sqrt{\epsilon}. \quad (3.36)
\]
Let \( t_i = 2C_0it\epsilon \), \( i = 1, 2, \cdots \). Then
\[
\frac{C_0\epsilon}{t_{i+1} - t_i} = \frac{1}{2}.
\]
Let \( k = \left[ \frac{1}{4C_0|z_1|^2} \right] \). Then by (3.36) with \( s = t_{i+1} \) and \( t = t_i \), we have
\[
\tilde{F}(t_i) \leq \frac{1}{4} \tilde{F}(t_{i+1}) + Ci^2(\epsilon^2 + |z_1|^2). \quad (3.36)
\]
After \( k \) iterations, we have, using (3.18),
\[
\tilde{F}(t_i) \leq \left( \frac{1}{4} \right)^k \tilde{F}(t_{k+1}) + C \sum_{l=1}^{k} \left( \frac{1}{4} \right)^l \tilde{F}(\epsilon^2 + |z_1|^2) \leq C \left( \frac{1}{4} \right)^{\frac{1}{2}} + C(\epsilon^2 + |z_1|^2) \leq C(\epsilon^2 + |z_1|^2). \quad (3.36)
\]
This implies that
\[ \int_{\Omega_{d}(z_1)} |\nabla w|^2 dx \leq C(e^2 + |z_1|^2). \]

**STEP 3. Proof of (3.20).**

Making a change of variables
\[ \begin{cases} x_1 - z_1 = \delta y_1, \\ x_2 = \delta y_2, \end{cases} \tag{3.37} \]
then \( \tilde{Q}_d(z_1) \) becomes \( Q'_1 \), where
\[ Q'_1 = \left\{ y \in \mathbb{R}^2 \mid -\frac{e}{2^\delta} + \frac{1}{\delta} h_2(\delta y_1 + z_1) < y_2 < \frac{e}{2^\delta} + \frac{1}{\delta} h_1(\delta y_1 + z_1), |y_1| < r \right\}, \]
for \( r \leq 1 \), and the boundaries \( \Gamma_1^+ \) become
\[ y_2 = \hat{h}_1(y_1) := \frac{1}{\delta} \left( \frac{e}{2} + h_1(\delta y_1 + z_1) \right), \quad |y_1| < 1, \]
and
\[ y_2 = \hat{h}_2(y_1) := \frac{1}{\delta} \left( -\frac{e}{2} + h_2(\delta y_1 + z_1) \right), \quad |y_1| < 1. \]

Then
\[ \hat{h}_1(0) - \hat{h}_2(0) := \frac{1}{\delta} (e + h_1(z_1) - h_2(z_1)) = 2, \]
and by (3.1) and (3.2),
\[ |\hat{h}'_1(0)| + |\hat{h}'_2(0)| \leq C|z_1|, \quad |\hat{h}''_1(0)| + |\hat{h}''_2(0)| \leq C\delta. \]

Since \( R \) is small, \( \|\hat{h}_1\|_{C^{1,1}((-1,1))} \) and \( \|\hat{h}_2\|_{C^{1,1}((-1,1))} \) are small and \( \frac{1}{2} Q'_1 \) is essentially a unit square as far as applications of Sobolev embedding theorems and classical \( L^p \) estimates for elliptic systems are concerned. Let
\[ U^1_1(y_1, y_2) := \bar{u}_1^1(x_1, x_2), \quad W^1_1(y_1, y_2) := w^1_1(x_1, x_2), \quad y \in Q'_1, \tag{3.38} \]
then by (3.23),
\[ \mathcal{L}_{\lambda, \mu} W^1_1 = -\mathcal{L}_{\lambda, \mu} U^1_1, \quad y \in Q'_1, \tag{3.39} \]
where
\[ |\mathcal{L}_{\lambda, \mu} U^1_1| = \delta^2 |\mathcal{L}_{\lambda, \mu} \bar{u}_1| . \]

Since \( W^1_1 = 0 \) on the top and bottom boundaries of \( Q'_1 \), we have, using Poincaré inequality, that
\[ \|W^1_1\|_{H^1(Q'_1)} \leq C \|\nabla W^1_1\|_{L^2(Q'_1)} . \]

By \( W^{2,p} \) estimates for elliptic systems (see [2]) and Sobolev embedding theorems, we have, with \( p = 3 \),
\[ \|\nabla W^1_1\|_{L^\infty(Q'_{1/2})} \leq C \|W^1_1\|_{W^{2,p}(Q'_{1/2})} \leq C \left( \|\nabla W^1_1\|_{L^2(Q'_1)} + \|\mathcal{L}_{\lambda, \mu} U^1_1\|_{L^\infty(Q'_1)} \right) . \]
It follows that
\[ \|\nabla w_i^1\|_{L^\infty(\tilde{\Omega}_{\delta}(z_1))} \leq \frac{C}{\delta} \left( \|\nabla w_i^1\|_{L^2(\tilde{\Omega}_{\delta}(z_1))} + \delta^2 \|\mathcal{L}_{1,\mu} \bar{u}_i^1\|_{L^\infty(\tilde{\Omega}_{\delta}(z_1))} \right). \] (3.40)

**Case 1.** For \( \sqrt{\epsilon} \leq |z_1| \leq R. 

By (3.19),
\[ \int_{\tilde{\Omega}_{\delta}(z_1)} |\nabla w_i^1|^2 \, dx \leq C|z_1|^2. \]

By (3.13),
\[ \delta^2 |\mathcal{L}_{1,\mu} \bar{u}_i^1| \leq \delta^2 \left( \frac{C}{|z_1|^2} + \frac{C}{|z_1|^T} \right) \leq C|z_1|, \quad \text{in } \tilde{\Omega}_{\delta}(z_1). \]

We deduce from (3.40) that
\[ |\nabla w_i^1(z_1, x_2)| = \frac{C|z_1|}{\delta} \leq \frac{C}{|z_1|}, \quad \forall - \frac{\epsilon}{2} + h_2(z_1) < x_2 < \frac{\epsilon}{2} + h_1(z_1). \]

**Case 2.** For \( |z_1| \leq \sqrt{\epsilon}. 

By (3.19),
\[ \int_{\tilde{\Omega}_{\delta}(z_1)} |\nabla w_i^1|^2 \, dx \leq C(\epsilon^2 + |z_1|^2). \]

By (3.13),
\[ \delta^2 |\mathcal{L}_{1,\mu} \bar{u}_i^1| \leq C\delta^2 \left( \frac{1}{\epsilon} + \frac{|z_1|}{\epsilon^2} \right) \leq C(\epsilon + |z_1|), \quad \text{in } \tilde{\Omega}_{\delta}(z_1). \]

We deduce from (3.40) that
\[ |\nabla w_i^1(z_1, x_2)| = \frac{C(\epsilon + |z_1|)}{\delta} \leq \frac{C\epsilon + |z_1|}{\epsilon}, \quad \forall - \frac{\epsilon}{2} + h_2(z_1) < x_2 < \frac{\epsilon}{2} + h_1(z_1). \]

Proposition 3.2 is established. \( \square \)

### 3.3 Estimates of \( v_i^3, i = 1, 2 \)

Define
\[ \bar{u}_i^3 = (x_2 \bar{u}, -x_1 \bar{u})^T, \quad \text{and} \quad \bar{u}_2^3 = (x_2 u, -x_1 u)^T \] (3.41)
then \( v_i^3 = \bar{u}_i^3 \) on \( \partial \Omega, i = 1, 2. \) Using (3.7), (3.1) and (3.3), we obtain
\[ |\nabla \bar{u}_i^3(x)| \leq \frac{C(\epsilon + |x_1|)}{\epsilon + |x_1|^2}, \quad i = 1, 2, \quad x \in \Omega_R, \] (3.42)
and
\[ |\nabla \bar{u}_i^3(x)| \leq C, \quad i = 1, 2, \quad x \in \tilde{\Omega} \setminus \Omega_R. \] (3.43)

It follows from (3.41), (1.6), (3.7) and (3.8) that
\[ |\mathcal{L}_{1,\mu} \bar{u}_i^3| \leq \frac{C}{\epsilon + |x_1|^2}, \quad i = 1, 2, \quad x \in \Omega_R. \] (3.44)

We estimate the energy of \( v_i^3, i = 1, 2. \)
Lemma 3.4.  
\[
\int_{\hat{\Omega}} |v_i^3|^2 \, dx + \int_{\hat{\Omega}} |\nabla v_i^3|^2 \, dx \leq C, \quad i = 1, 2,  \tag{3.45}
\]
and
\[
\|\nabla v_i^3\|_{L^\infty(\hat{\Omega}; \Omega_t)} \leq C, \quad i = 1, 2.  \tag{3.46}
\]

Proof. By (3.42) and (3.43), we have
\[
I_{\infty}[v_i^3] \leq I_{\infty}[\tilde{u}_i^3] \leq C \|\nabla \tilde{u}_i^3\|_{L^2(\hat{\Omega})} \leq C,
\]
and, by (1.8) and (2.12) and the First Korn inequality,
\[
\|\nabla v_i^3\|_{L^2(\hat{\Omega})} \leq \|\nabla(v_i^3 - \tilde{u}_i^3)\|_{L^2(\hat{\Omega})} + \|\nabla \tilde{u}_i^3\|_{L^2(\hat{\Omega})} \leq \sqrt{2} \|e(v_i^3 - \tilde{u}_i^3)\|_{L^2(\hat{\Omega})} + C
\leq C \|e(v_i^3)\|_{L^2(\hat{\Omega})} + C \leq C I_{\infty}[v_i^3] + C \leq C.
\]

We know from the Poincaré inequality that
\[
\int_{\hat{\Omega}} |v_i^3|^2 \, dx \leq C \int_{\hat{\Omega}} |\nabla v_i^3|^2 \, dx \leq C.
\]

Note that the above constant $C$ is independent of $\epsilon$.

With (3.45), we can apply classical elliptic estimates, see [1] and [2], to obtain (3.46). \hfill \Box

Denote
\[
w_i^3 := v_i^3 - \tilde{u}_i^3, \quad i = 1, 2.
\]

It is easy to see from (3.42), (3.43) and (3.45) that
\[
\int_{\hat{\Omega}} |\nabla w_i^3|^2 \leq C. \tag{3.47}
\]

Lemma 3.5. With $\delta = \delta(z_1)$ in (3.14), we have, for $i = 1, 2,$
\[
\int_{\hat{\Omega}_\delta(z_1)} |\nabla w_i^3|^2 \, dx \leq \begin{cases} 
Ce^2, & |z_1| < \sqrt{\epsilon}, \\
C|z_1|^4, & \sqrt{\epsilon} \leq |z_1| < R.
\end{cases} \tag{3.48}
\]

Proof. The proof is similar to that of (3.19). We will only prove it for $i = 1,$ since the proof for $i = 2$ is the same. For simplicity, denote $w := w_1^3,$ then
\[
\begin{cases} 
\mathcal{L}_{1,\mu} w = -\mathcal{L}_{1,\mu} \tilde{u}_1^3, & \text{in } \hat{\Omega}, \\
w = 0, & \text{on } \partial \hat{\Omega}.
\end{cases} \tag{3.49}
\]

As in the proof of (3.19), we have, instead of (3.30),
\[
\int_{\hat{\Omega}_\delta(z_1)} |\nabla w|^2 \, dx \leq \frac{C}{(s-t)^2} \int_{\hat{\Omega}_\delta(z_1)} |w|^2 \, dx + (s-t)^2 \int_{\hat{\Omega}_\delta(z_1)} |\mathcal{L}_{1,\mu} \tilde{u}_1|^2 \, dx. \tag{3.50}
\]

Case 1. $\sqrt{\epsilon} < |z_1| < R.$
We still have (3.31) for $0 < s < \frac{2|z_1|}{3}$. Instead of (3.32), we have, using (3.44),

$$
\int_{\Omega_{(z_1)}} |L_{A_1} \tilde{u}_1|^2 \, dx \leq \frac{Cs}{|z_1|^2}.
$$

(3.51)

Instead of (3.33), we have

$$
\tilde{F}(t) \leq \left( \frac{C_0|z_1|^2}{s-t} \right)^2 \tilde{F}(s) + \frac{Cs}{|z_1|^2} s, \quad \forall \ 0 < t < s < \frac{2|z_1|}{3}.
$$

(3.52)

We define $\{t_i\}$, $k$ and iterate as in the proof of (3.19), right below formula (3.33), to obtain, using (3.47),

$$
\tilde{F}(t_1) \leq \left( \frac{1}{4} \right)^k \tilde{F}(t_k+1) + C|z_1|^4 \sum_{l=1}^{k} \left( \frac{1}{4} \right)^{l-1} l \leq C|z_1|^4.
$$

This implies that

$$
\int_{\Omega_{(z_1)}} |\nabla w|^2 \, dx \leq C|z_1|^4.
$$

Case 2. $|z_1| < \sqrt{\epsilon}$.

Estimate (3.34) remains the same. Estimate (3.35) becomes

$$
\int_{\Omega_{(z_1)}} |L_{A_1} \tilde{u}_1|^2 \, dx \leq \frac{Cs}{\epsilon}, \quad 0 < s < \sqrt{\epsilon}.
$$

(3.53)

Estimate (3.36) becomes

$$
\tilde{F}(t) \leq \left( \frac{C_0 \epsilon}{s-t} \right)^2 \tilde{F}(s) + \frac{C(s-t)^2 s}{\epsilon}, \quad \forall \ 0 < t < s < \sqrt{\epsilon}.
$$

(3.54)

Define $\{t_i\}$, $k$ and iterate as in the proof of (3.19), right below formula (3.36), to obtain

$$
\tilde{F}(t_1) \leq \left( \frac{1}{4} \right)^k \tilde{F}(t_k+1) + C \sum_{l=1}^{k} \left( \frac{1}{4} \right)^{l-1} l \epsilon^2 \leq C\epsilon^2.
$$

This implies that

$$
\int_{\Omega_{(z_1)}} |\nabla w|^2 \, dx \leq C\epsilon^2.
$$

□

Lemma 3.6.

$$
\|\nabla w_i^3\|_{L^\infty(\Omega)} \leq C, \quad i = 1, 2.
$$

(3.55)

Consequently,

$$
|\nabla v_i^3(x)| \leq \frac{C(\epsilon + |x_1|)}{\epsilon + |x_1|^2}, \quad i = 1, 2, \quad x \in \Omega_R.
$$

(3.56)
Proof. The proof is the same as that of (3.20). In Case 1, $\sqrt{\epsilon} \leq |z_1| \leq R$, we use estimates
\[
\int_{\hat{\Omega}(z_1)} |\nabla w_3|^2 \, dx \leq C|z_1|^4,
\]
and
\[
\delta^2 |L_{\lambda,\mu} \bar{u}_1| \leq C|z_1|^2.
\]
In Case 2, $|z_1| \leq \sqrt{\epsilon}$, we use
\[
\int_{\hat{\Omega}(z_1)} |\nabla w_3|^2 \, dx \leq C\epsilon^2,
\]
and
\[
\delta^2 |L_{\lambda,\mu} \bar{u}_1| \leq C\epsilon.
\]
\[\square\]

4 Estimates of $C_1^\alpha$ and $C_2^\alpha$

In this section, we first prove that $C_1^\alpha$ and $C_2^\alpha$ are uniformly bounded with respect to $\epsilon$, and then estimate the difference $C_1^\alpha - C_2^\alpha$.

4.1 Boundedness of $C_i^\alpha$, $i = 1, 2$, $\alpha = 1, 2, 3$

Lemma 4.1. Let $C_i^\alpha$ be defined in (2.1). Then
\[
|C_i^\alpha| \leq C, \quad i = 1, 2; \quad \alpha = 1, 2, 3.
\]

Proof. We only need to prove it for $i = 1$, since the proof for $i = 2$ is the same. Let $u_\epsilon$ be the solution of (1.5). By Theorem 6.5 and Theorem 6.6 in the Appendix,
\[
I_\infty[u_\epsilon] := \frac{1}{2} \int_\Omega \left( C(0)e(u_\epsilon), e(u_\epsilon) \right) \leq I_\infty[\Phi] \leq C
\]
where $\Phi$ is the one in the proof of Lemma 3.1.

It follows that
\[
\|u_\epsilon\|_{H^1(\hat{\Omega})} \leq C\|e(u_\epsilon)\|_{L^2(\hat{\Omega})} \leq CI_\infty[u_\epsilon] \leq C.
\]
By the trace embedding theorem,
\[
\|u_\epsilon\|_{L^2(\partial D_1 \setminus B_R)} \leq C.
\]
On $\partial D_1$,
\[
u \epsilon = \sum_{\alpha=1}^3 C_i^\alpha \psi^\alpha.
\]
If $C_1 := (C_1^1, C_1^2, C_1^3) = 0$, there is nothing to prove. Otherwise
\[
C \geq |C_1| \left\| \sum_{\alpha=1}^3 \tilde{C}_j^\alpha \psi^\alpha \right\|_{L^2(\partial D_1 \setminus B_R)} \quad (4.1),
\]

where $\tilde{C}_1^\alpha = \frac{C_1^\alpha}{|C_1|}$ and $|\tilde{C}_1| = 1$. It is easy to see that
\[
\left\| \sum_{\alpha=1}^{3} \tilde{C}_1^\alpha \psi_\alpha \right\|_{L^2(\partial D_1 \backslash B_\epsilon)} \geq \frac{1}{C}.
\] (4.2)

Indeed, if not, along a subsequence $\epsilon \to 0$, $\tilde{C}_1^\alpha \to \tilde{C}_1^\alpha$, and
\[
\left\| \sum_{\alpha=1}^{3} \tilde{C}_1^\alpha \psi_\alpha \right\|_{L^2(\partial D_1 \backslash B_\epsilon)} = 0,
\]
where $\partial D_1^\epsilon$ is the limit of $\partial D_1$ as $\epsilon \to 0$ and $|\tilde{C}_1| = 1$. This implies $\sum_{\alpha=1}^{3} \tilde{C}_1^\alpha \psi_\alpha = 0$ on $\partial D_1 \backslash B_R$. But $\{\psi_\alpha_{|\partial D_1 \backslash B_\epsilon}\}$ is easily seen to be linear independent, we must have $\tilde{C}_1 = 0$. This is a contradiction. Lemma 4.1 for $i = 1$ follows from (4.1) and (4.2). \hfill \Box

### 4.2 Estimates of $|C_1^\alpha - C_2^\alpha|$, $\alpha = 1, 2$

In the rest of this section, we prove

**Proposition 4.2.** Let $C_i^\alpha$ be defined in (2.1). Then
\[
|C_1^\alpha - C_2^\alpha| \leq C \sqrt{\epsilon}, \quad \alpha = 1, 2.
\]

By the fourth line of (1.5),
\[
\sum_{\alpha=1}^{3} C_1^\alpha \int_{\partial D_1} \frac{\partial \nu^\alpha}{\partial \nu_0} \cdot \psi_\beta + \sum_{\alpha=1}^{3} C_2^\alpha \int_{\partial D_1} \frac{\partial \nu^\alpha}{\partial \nu_0} \cdot \psi_\beta + \int_{\partial D_1} \frac{\partial \nu_3}{\partial \nu_0} \cdot \psi_\beta = 0,
\]
whenever $j = 1, 2; \beta = 1, 2, 3$. (4.3)

Denote
\[
a_{ij}^{\alpha\beta} = -\int_{\partial D_1} \frac{\partial \nu^\alpha}{\partial \nu_0} \cdot \psi_\beta, \quad b_j^\beta = \int_{\partial D_1} \frac{\partial \nu_3}{\partial \nu_0} \cdot \psi_\beta, \quad i, j = 1, 2; \alpha, \beta = 1, 2, 3.
\]

Integrating by parts over $\tilde{\Omega}$ and using (2.2), we have
\[
a_{ij}^{\alpha\beta} = \int_{\tilde{\Omega}} (C^\alpha e(v_i^\beta), e(v_j^\beta)) dx, \quad b_j^\beta = -\int_{\tilde{\Omega}} (C^\alpha e(v_3^\beta), e(v_j^\beta)) dx.
\]

Then (4.3) can be written as
\[
\begin{cases}
\sum_{\alpha=1}^{3} C_1^\alpha a_{11}^{\alpha\beta} + \sum_{\alpha=1}^{3} C_2^\alpha a_{21}^{\alpha\beta} - b_1^\beta = 0, \\
\sum_{\alpha=1}^{3} C_1^\alpha a_{12}^{\alpha\beta} + \sum_{\alpha=1}^{3} C_2^\alpha a_{22}^{\alpha\beta} - b_2^\beta = 0, \\ 
\end{cases}
\] (4.4)

\[\beta = 1, 2, 3.\]
For simplicity, we use $a_{ij}$ to denote the $3 \times 3$ matrix $(a_{ij}^{\alpha \beta})$. To estimate $|C_1^\alpha - C_2^\alpha|$, $\alpha = 1, 2$, we only need to use the first three equations in (4.4):

$$a_{11}C_1 + a_{21}C_2 = b_1,$$

where

$$C_1 = (C_1^1, C_1^2, C_1^3)^T, \quad C_2 = (C_2^1, C_2^2, C_2^3)^T, \quad b_1 = (b_1^1, b_1^2, b_1^3)^T.$$

We write the equation as

$$a_{11}(C_1 - C_2) = p := b_1 - (a_{11} + a_{21})C_2. \quad \text{(4.5)}$$

Namely,

$$a_{11}(C_1 - C_2) = \begin{pmatrix} a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \end{pmatrix} \begin{pmatrix} C_1^1 - C_2^1 \\ C_1^2 - C_2^2 \\ C_1^3 - C_2^3 \end{pmatrix} = \begin{pmatrix} p_1^1 \\ p_2^1 \\ p_3^1 \end{pmatrix}. \quad \text{(4.6)}$$

We will show that $a_{11}$ is positive definite, which we assume for the time being. By Cramer’s rule, we see from (4.6),

$$C_1^1 - C_2^1 = \frac{1}{\det a_{11}} \begin{vmatrix} p_1 & a_{11}^{12} & a_{11}^{13} \\ p_2 & a_{11}^{12} & a_{11}^{13} \\ p_3 & a_{11}^{12} & a_{11}^{13} \end{vmatrix}, \quad C_1^2 - C_2^2 = \frac{1}{\det a_{11}} \begin{vmatrix} a_{11}^{11} & p_1 & a_{11}^{13} \\ a_{11}^{11} & p_2 & a_{11}^{13} \\ a_{11}^{11} & p_3 & a_{11}^{13} \end{vmatrix}.$$

Therefore

$$C_1^1 - C_2^1 = \frac{1}{\det a_{11}} \begin{vmatrix} p_1 & a_{11}^{12} & a_{11}^{13} \\ p_2 & a_{11}^{12} & a_{11}^{13} \\ p_3 & a_{11}^{12} & a_{11}^{13} \end{vmatrix} - p^2 \begin{vmatrix} a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \end{vmatrix} + p^3 \begin{vmatrix} a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \end{vmatrix}, \quad \text{(4.7)}$$

and

$$C_1^2 - C_2^2 = \frac{1}{\det a_{11}} \begin{vmatrix} a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \end{vmatrix} - p^2 \begin{vmatrix} a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \end{vmatrix} + p^3 \begin{vmatrix} a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \end{vmatrix}. \quad \text{(4.8)}$$

In order to prove Proposition 4.2, we first study the right hand side of (4.6) and have the following estimates.

**Lemma 4.3.**

$$|a_{11}^{\alpha \beta} + a_{21}^{\alpha \beta}| \leq C, \quad \alpha, \beta = 1, 2, 3;$$

$$|b_1^\beta| \leq C, \quad \beta = 1, 2, 3.$$

Consequently

$$|p| \leq C. \quad \text{(4.9)}$$
Proof. For $\beta = 1, 2, 3$, using (3.21) and (3.45),
\[
\int_{\Omega} |\nabla v_1^\beta| \, dx \leq \int_{\Omega_{W/2}} |\nabla v_1^\beta| \, dx + \int_{\Omega \setminus \Omega_{W/2}} |\nabla v_1^\beta| \, dx \leq C.
\]
(4.10)

For $\alpha, \beta = 1, 2, 3$, by Lemma 3.1 and (4.10), we have
\[
|a_{11}^{\alpha\beta} + a_{21}^{\alpha\beta}| = \left| \int_{\Omega} (C^0 e(v_1^\alpha + v_2^\beta), e(v_1^\beta)) \, dx \right|
\leq C \|\nabla (v_1^\alpha + v_2^\beta)\|_{L^2(\tilde{\Omega})} \int_{\Omega} |\nabla v_1^\beta| \, dx
\leq C.
\]
Similarly, it follows from Lemma 3.1 and (4.10) that
\[
|b_{11}^\beta| = \left| \int_{\Omega} (C^0 e(v_1^\beta), e(v_3)) \, dx \right| \leq C \|\nabla v_3\|_{L^2(\tilde{\Omega})} \int_{\Omega} |\nabla v_1^\beta| \, dx \leq C, \quad \beta = 1, 2, 3.
\]
Lemma 4.3 follows immediately, in view of Lemma 4.1. \qed

Lemma 4.4. $a_{11}$ is positive definite, and
\[
\frac{1}{C \sqrt{\epsilon}} \leq a_{11}^{\alpha\alpha} \leq \frac{C}{\sqrt{\epsilon}}, \quad \alpha = 1, 2,
\]
(4.11)
\[
\frac{1}{C} \leq a_{11}^{33} \leq C, \quad \alpha = 1, 2;
\]
(4.12)
\[
|a_{11}^{12}| = |a_{11}^{21}| \leq \frac{C}{\epsilon^{1/4}},
\]
(4.13)
\[
|a_{11}^{33}| = |a_{11}^{33}| \leq C, \quad \alpha = 1, 2;
\]
(4.14)
and
\[
\frac{1}{C \epsilon} \leq \det a_{11} \leq \frac{C}{\epsilon}.
\]
(4.15)

Proof. **STEP 1.** Proof of (4.11) and (4.12).

For any $\xi = (\xi^{(1)}, \xi^{(2)}, \xi^{(3)})^T \neq 0$,
\[
\xi^T a_{11} \xi = \int_{\Omega} (C^0 e(\xi^{(\alpha)} v_1^\alpha), e(\xi^{(\beta)} v_1^\beta)) \, dx \geq \frac{1}{C} \int_{\Omega} e(\xi^{(\alpha)} v_1^\alpha) \, dx > 0.
\]

In the last inequality we have used the fact that $e(\xi^{(\alpha)} v_1^\alpha)$ is not identically zero. Indeed if $e(\xi^{(\alpha)} v_1^\alpha) = 0$, then $\xi^{(\alpha)} v_1^\alpha = a \psi^1 + b \psi^2 + c \psi^3$ in $\tilde{\Omega}$ for some constants $a, b$ and $c$. On the other hand, $\xi^{(\alpha)} v_1^\alpha = 0$ on $\partial D_2$, and $\psi^1|_{\partial D_2}$, $\psi^2|_{\partial D_2}$ and $\psi^3|_{\partial D_2}$ are clearly independent. This implies that $a = b = c = 0$. Thus on $\partial D_1$, $\xi^{(\alpha)} v_1^\alpha = 0$, violating the linear independence of $\psi^1|_{\partial D_1}$, $\psi^2|_{\partial D_1}$ and $\psi^3|_{\partial D_1}$. We have proved that $a_{11}$ is positive definite.

By (1.8), (2.12) and (2.7),
\[
a_{11}^{\alpha\alpha} = \int_{\Omega} (C^0 e(v_1^\alpha), e(v_1^\alpha)) \, dx \leq C \int_{\Omega} |\nabla v_1^\alpha|^2 \, dx \leq \frac{C}{\sqrt{\epsilon}}, \quad \alpha = 1, 2.
\]
With (3.17), we have, by (3.18),
\[
a^{11}_{11} = \int_\Omega (C^0 e(v^1_1), e(v^1_1)) dx \geq \frac{1}{C} \int_\Omega \left| e(v^1_1) \right|^2 dx \\
\geq \frac{1}{2C} \int_\Omega \left| e(\bar{u}_1) \right|^2 dx - C \int_\Omega \left| e(w_1^1) \right|^2 dx \\
\geq \frac{1}{2C} \int_\Omega \left| e(\bar{u}_1) \right|^2 dx - C.
\]
Since
\[
\left| e(\bar{u}_1) \right|^2 \geq \frac{1}{4} \left| \partial_x \bar{u} \right|^2,
\]
we have
\[
\int_\Omega \left| e(\bar{u}_1) \right|^2 dx \geq \frac{1}{4} \int_\Omega \left| \partial_x \bar{u} \right|^2 dx \geq \frac{1}{4} \int_{\Omega_\epsilon} \frac{dx}{\epsilon + h_1(x_1) - h_2(x_1)^2} \geq \frac{1}{C \sqrt{\epsilon}}.
\]
Thus
\[
a^{11}_{11} \geq \frac{1}{C \sqrt{\epsilon}}.
\]
Similarly, we have
\[
a^{22}_{11} \geq \frac{1}{C \sqrt{\epsilon}}.
\]
Estimate (4.11) is proved.

By Lemma 3.4
\[
a^{33}_{11} = \int_\Omega (C^0 e(v^3_1), e(v^3_1)) dx \leq C. \tag{4.17}
\]
Claim: There exists \( C \) which is independent of \( \epsilon \) such that for any \( v \in H^1(\Omega_R \setminus \Omega_{R/2}) \) satisfying \( v = 0 \) on \( \Gamma^1_R \setminus \Gamma^-_{R/2} \), it holds
\[
\|\nabla v\|_{L^2(\Omega_{\epsilon} \setminus \Omega_{R/2})} \leq C \|e(v)\|_{L^2(\Omega_{\epsilon} \setminus \Omega_{R/2})}. \tag{4.18}
\]
Proof of the claim. Suppose the contrary, along a sequence of \( \epsilon_j \to 0^+ \), there exist \( \{v_j\}_{j=1}^\infty \subset H^1(\Omega_R \setminus \Omega_{R/2}) \) (we still omit the superscript \( \epsilon_j \)) satisfying \( v_j = 0 \) on \( \Gamma^1_R \setminus \Gamma^-_{R/2} \), and
\[
\|\nabla v_j\|_{L^2(\Omega_{\epsilon} \setminus \Omega_{R/2})} \geq j \|e(v_j)\|_{L^2(\Omega_{\epsilon} \setminus \Omega_{R/2})}. \tag{4.19}
\]
By Lemma 3.4 we have
\[
\|\nabla v_j\|_{L^2(\Omega_{\epsilon} \setminus \Omega_{R/2})} \leq C \left( \|e(v_j)\|_{L^2(\Omega_{\epsilon} \setminus \Omega_{R/2})} + \|v_j\|_{L^2(\Omega_{\epsilon} \setminus \Omega_{R/2})} \right), \tag{4.20}
\]
where \( C \) is independent of \( j \). Replacing \( v_j \) by \( \frac{v_j}{\|v_j\|_{L^2(\Omega_{\epsilon} \setminus \Omega_{R/2})}} \), we may assume without loss of generality that
\[
\|v_j\|_{L^2(\Omega_{\epsilon} \setminus \Omega_{R/2})} = 1. \tag{4.21}
\]
It follows from \((4.19), (4.20)\) and \((4.21)\) that
\[
\lim_{j \to \infty} \|v_j\|_{H^{1}(\Omega^e \setminus \Omega_{R/2})} = 0, \\
\|v_j\|_{H^{1}(\Omega^e \setminus \Omega_{R/2})} \leq C. 
\]

Let
\[
\Omega^e_r := \{ x \in \mathbb{R}^2 \mid h_2(x_1) < x_2 < h_1(x_1), |x_1| < r \}
\]
and
\[
(\Gamma^-)^e_r := \{ x \in \mathbb{R}^2 \mid x_2 = h_2(x_1), |x_1| < r \}
\]
denote the limits of \(\Omega_e\) and \(\Gamma^-\) as \(\epsilon \to 0\).

We can easily construct a \(C^1\) diffeomorphism \(\phi_{e} : \Omega_r \setminus \Omega_{R/2} \to \Omega^e_r \setminus \Omega^*_{R/2}\) satisfying
\[
\phi_{e}(\Gamma^-_R \setminus \Gamma^-_{R/2}) = (\Gamma^e)^{-}_R \setminus (\Gamma^e)^{-}_{R/2}
\]
and
\[
\| \nabla \phi_{e} - I\|_{C^{0}(\Omega_r \setminus \Omega_{R/2})}, \| \nabla (\phi_{e})^{-1} - I\|_{C^{0}(\Omega^e_r \setminus \Omega^*_{R/2})} \to 0, \quad \text{as } \epsilon \to 0^+, 
\]
where \(I\) denotes the identity matrix. Let \(\tilde{v}_j := v_j \circ (\phi_{e})^{-1}\). We deduce from \((4.22)\) and \((4.23)\) that, along a subsequence, \(\tilde{v}_j \rightharpoonup v^*\) weakly in \(H^1(\Omega^e_{R/2} \setminus \Omega_{R/2})\), where \(v^*\) satisfies
\[
e(v^*) = 0, \quad \text{in } \Omega_{R} \setminus \Omega_{R/2}^e, \tag{4.25}
\]
and
\[
v^* = 0, \quad \text{on } (\Gamma^-)^e_R \setminus (\Gamma^e)^{-}_{R/2}. \tag{4.26}
\]

By \((4.25)\), \(v^* \in \Psi\) in \(\Omega^e_{R/2} \setminus \Omega^*_{R/2}\). Thus, in view of \((4.26)\), \(v^* \equiv 0\) in \(\Omega^*_{R} \setminus \Omega^*_{R/2}\). By the compact embedding theorem of \(H^1(\Omega^e_{R/2} \setminus \Omega_{R/2})\) to \(L^2(\Omega^e_{R} \setminus \Omega^*_{R/2})\),
\[
\| \tilde{v}_j \|_{L^2(\Omega^e_{R} \setminus \Omega^*_{R/2})} \to \| v^* \|_{L^2(\Omega^e_{R} \setminus \Omega^*_{R/2})} = 0,
\]
By \((4.24)\) and \((4.22)\),
\[
\| \tilde{v}_j \|_{L^2(\Omega^e_{R} \setminus \Omega^*_{R/2})} \to 1.
\]

These lead to a contradiction. The claim has been proved.

With the claim \((4.18)\), we obtain from \((1.8)\) that
\[
a_{11}^{33} = \int_{\Omega} (c^0 e(v_1^3), e(v_1^3)) dx \geq \frac{1}{C} \int_{\Omega_{R} \setminus \Omega_{R/2}} |e(v_1^3)|^2 dx \geq \frac{1}{C} \int_{\Omega_{R} \setminus \Omega_{R/2}} |\nabla v_1^3|^2 dx \geq \frac{1}{C}.
\]
Combining with \((4.17)\), estimate \((4.12)\) is proved.

**STEP 2.** Proof of \((4.13)\).

Notice that
\[
a_{11}^{12} = a_{11}^{21} = \int_{\Omega} (c^0 e(v_1^1), e(v_1^2)) dx = \int_{\Omega} (c^0 \nabla v_1^1, \nabla v_1^2) dx.
\]
With \((3.17)\), we have
\[
\int_{\Omega_{R} \setminus \Omega_{R/2}} (c^0 \nabla v_1^1, \nabla v_1^2) dx = \int_{\Omega_{R} \setminus \Omega_{R/2}} (c^0 \nabla (u_1^1 + w_1^2), \nabla (u_1^2 + w_1^2)) dx
\]
\[
= \int_{\Omega_{R} \setminus \Omega_{R/2}} (c^0 \nabla u_1^1, \nabla u_1^2) dx + \int_{\Omega_{R} \setminus \Omega_{R/2}} (c^0 \nabla u_1^2, \nabla w_1^2) dx
\]
\[
+ \int_{\Omega_{R} \setminus \Omega_{R/2}} (c^0 \nabla u_1^2, \nabla w_1^1) dx + \int_{\Omega_{R} \setminus \Omega_{R/2}} (c^0 \nabla w_1^1, \nabla w_1^1) dx. \tag{4.27}
\]
By the definition $\tilde{u}_1 = (\tilde{u}, 0)^T$ and $\tilde{u}_1^2 = (0, \tilde{u})^T$, we have

$$
\nabla \tilde{u}_1 = \begin{pmatrix} \partial_{x_1} \tilde{u} & \partial_{x_2} \tilde{u} \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \nabla \tilde{u}_1^2 = \begin{pmatrix} 0 & \partial_{x_1} \tilde{u} \\ \partial_{x_1} \tilde{u} & \partial_{x_2} \tilde{u} \end{pmatrix}.
$$

(4.28)

By (3.18),

$$
\left| \int_{\Omega_{R/2}} (C^0 \nabla w_1^1, \nabla w_1^2) \, dx \right| \leq C \left( \int_{\Omega_{R/2}} |\nabla w_1^1|^2 \, dx \right)^{1/2} \left( \int_{\Omega_{R/2}} |\nabla w_1^2|^2 \, dx \right)^{1/2} \leq C,
$$

and

$$
\left| \int_{\Omega_{R/2}} (C^0 \nabla \tilde{u}_1, \nabla w_1^2) \, dx \right| \leq C \left( \int_{\Omega_{R/2}} |\nabla \tilde{u}_1|^2 \, dx \right)^{1/2} \left( \int_{\Omega_{R/2}} |\nabla w_1^2|^2 \, dx \right)^{1/2} \leq \frac{C}{\epsilon^{1/4}},
$$

(4.29)

Similarly,

$$
\left| \int_{\Omega_{R/2}} (C^0 \nabla \tilde{u}_1^2, \nabla w_1^1) \, dx \right| \leq \frac{C}{\epsilon^{1/4}}.
$$

(4.30)

On the other hand,

$$
(C^0 \nabla \tilde{u}_1, \nabla \tilde{u}_1^2) = \begin{pmatrix} (\lambda + 2\mu)\partial_{x_1} \tilde{u} & \mu \partial_{x_2} \tilde{u} \\ \mu \partial_{x_2} \tilde{u} & \lambda \partial_{x_1} \tilde{u} \end{pmatrix} : \begin{pmatrix} 0 & \partial_{x_1} \tilde{u} \\ \partial_{x_1} \tilde{u} & \partial_{x_2} \tilde{u} \end{pmatrix} = (\lambda + \mu)\partial_{x_1} \tilde{u} \partial_{x_2} \tilde{u}.
$$

Thus,

$$
\left| \int_{\Omega_{R/2}} (C^0 \nabla \tilde{u}_1, \nabla \tilde{u}_1^2) \, dx \right| \leq C \int_{\Omega_{R/2}} |\partial_{x_1} \tilde{u}| |\partial_{x_2} \tilde{u}| \, dx \leq C \int_{\Omega_{R/2}} \frac{|x_1| \, dx}{(\epsilon + |x_1|^2)^2} \leq C |\ln \epsilon|.
$$

Substituting these estimates above into (4.27), and using (3.21), we have

$$
|a_{11}^{12}| = |a_{11}^{21}| = \left| \int_{\Omega} (C^0 \nabla v_1^1, \nabla v_1^2) \, dx \right| \leq \left| \int_{\Omega_{R/2}} (C^0 \nabla v_1^1, \nabla v_1^2) \, dx \right| + C \leq \frac{C}{\epsilon^{1/4}}.
$$

The proof of (4.13) is finished.

**STEP 3. Proof of (4.14).**

$$
a_{11}^{3\alpha} = a_{11}^{3\alpha} = \int_{\Omega} (C^0 e(v_1^\alpha), e(v_1^\alpha)) \, dx = \int_{\Omega} (C^0 \nabla v_1^\alpha, \nabla v_1^\alpha) \, dx, \quad \alpha = 1, 2.
$$

Similarly to the above, using (3.18) and (3.47), we have, for $\alpha = 1$,

$$
a_{11}^{13} = \int_{\Omega_{R/2}} (C^0 \nabla v_1^1, \nabla v_1^3) \, dx + O(1)
$$

$$
= \int_{\Omega_{R/2}} (C^0 \nabla \tilde{u}_1, \nabla \tilde{u}_1^3) \, dx + \int_{\Omega_{R/2}} (C^0 \nabla \tilde{u}_1, \nabla w_1^3) \, dx
$$

$$
+ \int_{\Omega_{R/2}} (C^0 \nabla \tilde{u}_1^3, \nabla w_1^3) \, dx + \int_{\Omega_{R/2}} (C^0 \nabla w_1^3, \nabla v_1^3) \, dx + O(1)
$$

$$
= \int_{\Omega_{R/2}} (C^0 \nabla \tilde{u}_1, \nabla \tilde{u}_1^3) \, dx + \int_{\Omega_{R/2}} (C^0 \nabla \tilde{u}_1^3, \nabla \tilde{u}_1^3) \, dx + \int_{\Omega_{R/2}} (C^0 \nabla \tilde{u}_1^3, \nabla \tilde{u}_1^3) \, dx + O(1)
$$

$$
= : I + II + III + O(1).
$$
By the definition of $\tilde{u}_1^3 = (x_2 \tilde{u}, -x_1 \tilde{u})^T$, we have

$$
\nabla \tilde{u}_1^3 = \begin{pmatrix} x_2 \partial_{x_1} \tilde{u} & \tilde{u} + x_2 \partial_{x_2} \tilde{u} \\ -\tilde{u} - x_1 \partial_{x_1} \tilde{u} & -x_1 \partial_{x_2} \tilde{u} \end{pmatrix}.
$$

Then

$$
(C^0 \nabla \tilde{u}_1^1, \nabla \tilde{u}_1^3) = \begin{pmatrix} (\lambda + 2\mu) \partial_{x_1} \tilde{u} & \mu \partial_{x_2} \tilde{u} \\ \mu \partial_{x_1} \tilde{u} & (\lambda + \mu) \partial_{x_2} \tilde{u} \end{pmatrix} : \begin{pmatrix} x_2 \partial_{x_1} \tilde{u} & \tilde{u} + x_2 \partial_{x_2} \tilde{u} \\ -\tilde{u} - x_1 \partial_{x_1} \tilde{u} & -x_1 \partial_{x_2} \tilde{u} \end{pmatrix} = (\lambda + 2\mu)x_2 (\partial_{x_1} \tilde{u})^2 + \mu x_2 (\partial_{x_2} \tilde{u})^2 - (\lambda + \mu)x_1 \partial_{x_1} \tilde{u} \partial_{x_2} \tilde{u}.
$$

Hence, by (3.7),

$$
|II| = \left| \int_{\Omega_{R/2}} (C^0 \nabla \tilde{u}_1^1, \nabla \tilde{u}_1^3) \right| dx \\
\leq C \left( \int_{\Omega_{R/2}} |x_2||x_1|^2 dx + \int_{\Omega_{R/2}} \frac{|x_2|}{(\epsilon + |x_1|^2)^2} dx + \int_{\Omega_{R/2}} \frac{|x_1|^2}{(\epsilon + |x_1|^2)^2} dx \right) \\
\leq C.
$$

By (3.18) and (3.42),

$$
|II| = \left| \int_{\Omega_{R/2}} (C^0 \nabla \tilde{u}_1^1, \nabla \tilde{w}_1^1) \right| dx \leq C \left( \int_{\Omega_{R/2}} |\nabla \tilde{u}_1^1|^2 dx \right)^{1/2} \left( \int_{\Omega_{R/2}} |\nabla \tilde{w}_1^1|^2 dx \right)^{1/2} \leq C.
$$

While, by (3.55),

$$
|III| = \left| \int_{\Omega_{R/2}} (C^0 \nabla \tilde{u}_1^1, \nabla \tilde{w}_1^3) \right| dx \leq C \int_{\Omega_{R/2}} |\nabla \tilde{u}_1^1| dx \leq C.
$$

Therefore

$$
|a_{11}^{13}| \leq C.
$$

Similarly, using (3.18) and (3.47),

$$
a_{11}^{23} = \int_{\Omega_{R/2}} (C^0 \nabla \tilde{v}_1^2, \nabla \tilde{v}_1^3) dx + O(1) \\
= \int_{\Omega_{R/2}} (C^0 \nabla \tilde{u}_1^1, \nabla \tilde{u}_1^3) dx + \int_{\Omega_{R/2}} (C^0 \nabla \tilde{u}_1^2, \nabla \tilde{w}_1^1) dx + \int_{\Omega_{R/2}} (C^0 \nabla \tilde{w}_1^2, \nabla \tilde{w}_1^3) dx + O(1) \\
\leq \int_{\Omega_{R/2}} (C^0 \nabla \tilde{u}_1^1, \nabla \tilde{u}_1^3) dx + \int_{\Omega_{R/2}} (C^0 \nabla \tilde{u}_1^2, \nabla \tilde{w}_1^3) dx + O(1).
$$

By the definition $\tilde{u}_1^2$ and $\tilde{u}_1^3$, we have

$$
(C^0 \nabla \tilde{u}_1^2, \nabla \tilde{u}_1^3) = \begin{pmatrix} \lambda \partial_{x_1} \tilde{u} & \mu \partial_{x_2} \tilde{u} \\ \mu \partial_{x_1} \tilde{u} & (\lambda + 2\mu) \partial_{x_2} \tilde{u} \end{pmatrix} : \begin{pmatrix} x_2 \partial_{x_1} \tilde{u} & \tilde{u} + x_2 \partial_{x_2} \tilde{u} \\ -\tilde{u} - x_1 \partial_{x_1} \tilde{u} & -x_1 \partial_{x_2} \tilde{u} \end{pmatrix} = (\lambda + \mu)x_2 \partial_{x_1} \tilde{u} \partial_{x_2} \tilde{u} - \mu x_1 (\partial_{x_1} \tilde{u})^2 - (\lambda + 2\mu)x_1 (\partial_{x_2} \tilde{u})^2.
$$
Hence, using (3.7), we have
\[
\int_{\Omega_{R/2}} \left( C_0 \nabla \bar{u}^2, \nabla \bar{u}^3 \right) dx
= -(\lambda + 2\mu) \int_{\Omega_{R/2}} x_1 (\partial_{x_2} \bar{u})^2 dx + O(1)
= -(\lambda + 2\mu) \int_{|x_1| < R/2} x_1 \left( \frac{1}{\epsilon + h_1(x_1) - h_2(x_1)} - \frac{1}{\epsilon + \frac{1}{2}(h_1''(0) - h_2''(0))x_1^2} \right) dx_1 + O(1)
= O(1).
\]
Therefore
\[
|a_{11}^{23}| \leq C.
\]
Lemma 4.4 is proved. □

**Proof of Proposition 4.2.** By (4.7), Lemma 4.3 and Lemma 4.4,
\[
C_1^1 - C_2^1 = \frac{1}{\det a_{11}} \left( \left( p^2 a_{11}^{11} a_{11}^{33} - p^3 a_{11}^{22} a_{11}^{13} \right) + O\left( \frac{1}{\epsilon^{1/4}} \right) \right).
\]
Therefore
\[
|C_1^1 - C_2^1| \leq C \sqrt{\epsilon}.
\]
Similarly, using (4.8),
\[
C_1^2 - C_2^2 = \frac{1}{\det a_{11}} \left( \left( p^2 a_{11}^{11} a_{11}^{33} - p^3 a_{11}^{22} a_{11}^{13} \right) + O\left( \frac{1}{\epsilon^{1/4}} \right) \right).
\]
Therefore
\[
|C_1^2 - C_2^2| \leq C \sqrt{\epsilon}.
\]
The proof is completed. □

**Proof of Proposition 2.1.** Estimates (2.5) and (2.6) have been proved in Lemma 3.1; estimate (2.7) has been proved in Corollary 3.3; estimate (2.8) has been proved in Lemma 3.4 and Lemma 3.6; estimate (2.9) has been proved in Lemma 4.1; and estimate (2.10) has been proved in Proposition 4.2. The proof of Proposition 2.1 is completed. □

5 More general $D_1$ and $D_2$

As mentioned in the introduction, the strict convexity assumption on $\partial D_1$ and $\partial D_2$ can be weakened. In fact, our proof of Theorem 1.1 applies, with minor modification, to more general situations.

In $\mathbb{R}^2$, under the same assumptions in the beginning of Section 3 except for the strict convexity condition, $\partial D_i$ near $P_i$ can be represented by the graphs of $x_2 = \frac{x_1}{\lambda} + h_1(x_1)$, and $x_2 = -\frac{x_1}{\lambda} + h_2(x_1)$, for $|x_1| < 2R$. We assume that $h_1, h_2 \in C^2([-2R, 2R])$ and (3.1) still holds. Instead of the convexity assumption, we assume that
\[
\Lambda_0 |x_1|^m \leq h_1(x_1) - h_2(x_1) \leq \Lambda_1 |x_1|^m, \quad \text{for } |x_1| < 2R, \quad (5.1)
\]
and

\[ |h'_i(x_1)| \leq C|x_1|^{m-1}, \quad |h''_i(x_1)| \leq C|x_1|^{m-2}, \quad i = 1, 2, \quad \text{for } |x_1| < 2R, \tag{5.2} \]

for some \( \epsilon \)-independent constants \( 0 < \Lambda_0 < \Lambda_1 \), and \( m \geq 2 \). Define \( \delta := \delta(z_1) \) as \( (3.14) \). Clearly,

\[ \frac{1}{C} (\epsilon + |z_1|^m) \leq \delta(z_1) \leq C(\epsilon + |z_1|^m). \tag{5.3} \]

Then

**Theorem 5.1.** Under the above assumptions with \( m \geq 2 \), let \( u \in H^1(\Omega; \mathbb{R}^2) \cap C^1(\Omega; \mathbb{R}^2) \) be a solution to \((1.5)\). Then for \( 0 < \epsilon < 1 \), we have

\[ |\nabla u(x)| \leq \begin{cases} C \epsilon^{1-\frac{1}{\alpha}} + \text{dist}(x, P_1 P_2) \|\varphi\|_{C^{1, \gamma}(\partial \Omega; \mathbb{R}^2)}, & x \in \overline{\Omega}, \\ C \|\varphi\|_{C^{1, \gamma}(\partial \Omega; \mathbb{R}^2)}, & x \in D_1 \cup D_2. \end{cases} \tag{5.4} \]

where \( C \) is a universal constant. In particular,

\[ \|\nabla u\|_{L^\infty(\Omega)} \leq C \epsilon^{\frac{1}{\alpha} - 1} \|\varphi\|_{C^{1, \gamma}(\partial \Omega; \mathbb{R}^2)}. \tag{5.5} \]

In the following, we only list the main differences. We define \( \tilde{u} \) by \( (3.5) \) as before. A calculation gives

\[ |\partial_{x_i} \tilde{u}(x)| \leq \frac{C|x_i|^{m-1}}{\epsilon + |x_i|^m}, \quad |\partial_{x_i} \tilde{u}(x)| \leq \frac{C}{\epsilon + |x_i|^m}, \quad x \in \Omega_R, \tag{5.6} \]

by \((3.5)\), we have

\[ |\partial_{x_1 x_1} \tilde{u}(x)| \leq \frac{C|x_1|^{m-2}}{\epsilon + |x_i|^m}, \quad |\partial_{x_1 x_2} \tilde{u}(x)| \leq \frac{C|x_1|^{m-1}}{(\epsilon + |x_i|^m)^2}, \quad \partial_{x_2 x_2} \tilde{u}(x) = 0, \quad x \in \Omega_R. \tag{5.7} \]

Define \( \tilde{u}^\alpha_i, i, \alpha = 1, 2 \) as in \((3.9)\) and \((3.10)\). By \((1.6)\), \((5.6)\) and \((5.7)\), we have

\[ |L_{1, \alpha} \tilde{u}^\alpha_i(x)| \leq \frac{C|x_i|^{m-2}}{\epsilon + |x_i|^m} + \frac{C|x_i|^{m-1}}{(\epsilon + |x_i|^m)^2}, \quad i, \alpha = 1, 2, \quad x \in \Omega_R. \tag{5.8} \]

Instead of Proposition \(2.1\) we have

**Proposition 5.2.** Under the hypotheses of Theorem 5.1 and a normalization \( \|\varphi\|_{C^{1, \gamma}(\partial \Omega)} = 1 \), we have, for \( 0 < \epsilon < 1 \),

\[ \|\nabla V_3\|_{L^\infty(\Omega)} \leq C; \tag{5.9} \]

\[ \|\nabla V^\alpha_i + \nabla V^\alpha_2\|_{L^\infty(\Omega)} \leq C, \quad \alpha = 1, 2, 3; \tag{5.10} \]

\[ |\nabla V^\alpha_i(x)| \leq \frac{C}{\epsilon + \text{dist}(x, P_1 P_2)}, \quad i, \alpha = 1, 2, \quad x \in \overline{\Omega}; \tag{5.11} \]

\[ |\nabla V^\beta_i(x)| \leq \frac{C}{\epsilon + \text{dist}(x, P_1 P_2)} \tag{5.12} \]

and

\[ |C^\alpha_i| \leq C, \quad i = 1, 2, \quad \alpha = 1, 2, 3; \tag{5.13} \]

\[ |C^\alpha_i - C^\alpha_2| \leq C \epsilon^{-\frac{1}{\alpha}}, \quad \alpha = 1, 2. \tag{5.14} \]
Denote 
\[ w_i^\alpha := v_i^\alpha - \bar{u}_i^\alpha, \quad i = 1, 2, \quad \alpha = 1, 2, 3. \]
Then, instead of Proposition 3.2, we have

**Proposition 5.3.** Assume the above, let \( v_i^\alpha \in C^2(\Omega; \mathbb{R}^2) \cap C^1(\overline{\Omega}; \mathbb{R}^2) \) be the weak solution of (2.2). Then, for \( i, \alpha = 1, 2 \),

\[ \int_\Omega |\nabla w_i^\alpha|^2 \, dx \leq C, \quad (5.15) \]

\[ \int_{\hat{\Omega}_0(z_i)} |\nabla w_i^\alpha|^2 \, dx \leq \begin{cases} C \left( \epsilon^{2m-2} + |z_1|^{2m-2}, \right), & |z_1| \leq \sqrt{\epsilon}, \\ C|z_1|^{2m-2}, & \sqrt{\epsilon} < |z_1| \leq R, \end{cases} \quad (5.16) \]

and

\[ |\nabla w_i^\alpha(x)| \leq \begin{cases} C \frac{\epsilon^{m-1}|z_1|^{m-1}}{|x|}, & |x| \leq \sqrt{\epsilon}, \\ C \frac{1}{|x|}, & \sqrt{\epsilon} < |x| \leq R. \end{cases} \quad (5.17) \]

**Proof.** The proof of (5.15) is the same as that of (5.18). We only list the main differences from STEP 2 and STEP 3 in the proof of Proposition 3.2.

**STEP 2.** Proof of (5.16).

**Case 1.** For \( \sqrt{\epsilon} \leq |z_1| \leq R/2. \)

Note that for \( 0 < s < \frac{2|z_1|}{3} \), we have

\[ \int_{\hat{\Omega}_0(z_i)} |w|^2 \, dx \leq \int_{|x_1| \leq s} \left( \epsilon + h_1(x_1) - h_2(x_1) \right)^2 \int_{\frac{\alpha + h_1(x_1)}{\epsilon}}^{\frac{\alpha + h_2(x_1)}{\epsilon}} |\partial_{x_2} w(x_1, x_2)|^2 \, dx_2 \, dx_1 \]

\[ \leq C|z_1|^{2m} \int_{\hat{\Omega}_0(z_i)} |\nabla w|^2 \, dx, \quad (5.18) \]

By (5.8), we have

\[ \int_{\hat{\Omega}_0(z_i)} |\mathcal{L}_{\alpha} w|^2 \, dx \leq \int_{\hat{\Omega}_0(z_i)} \left( \frac{C|z_1|^{m-2}}{\epsilon + |z_1|^m} + \frac{C|z_1|^{m-1}}{\epsilon + |x|^m} \right)^2 \, dx \]

\[ \leq C \frac{|z_1|^m s}{|z_1|^{2|m| + 2}}, \quad 0 < s < \frac{2|z_1|}{3}. \quad (5.19) \]

As before, it follows from the above and (3.30) that

\[ \tilde{F}(t) \leq \left( \frac{C_0|z_1|^{m}}{s - t} \right)^2 \tilde{F}(s) + C(s - t)^2 \frac{s}{|z_1|^{m+2}}, \quad \forall \ 0 < t < \frac{2|z_1|}{3}, \quad (5.20) \]

where \( C_0 \) is also a universal constant.

Let \( t_i = 2C_0|z_1|^m, \ i = 1, 2, \ldots \). Then

\[ \frac{C_0|z_1|^m}{t_{i+1} - t_i} = \frac{1}{2}. \]

Let \( k = \left\lfloor \frac{1}{sC_0|z_1|^{m}} \right\rfloor \). Then by (5.20) with \( s = t_{i+1} \) and \( t = t_i \), we have

\[ \tilde{F}(t_i) \leq \frac{1}{4} \tilde{F}(t_{i+1}) + \frac{C(t_{i+1} - t_i)^2 t_{i+1}}{|z_1|^{m+2}} \leq \frac{1}{4} \tilde{F}(t_{i+1}) + C(i + 1)|z_1|^{2m-2}, \]

29
After \( k \) iterations, we have, using (5.15),
\[
\tilde{F}(t_l) \leq \left( \frac{1}{4} \right)^k \tilde{F}(t_{k+1}) + C|z_1|^{2m-2} \sum_{l=1}^{k} \left( \frac{1}{4} \right)^{l-1} (l + 1)
\]
\[
\leq C|z_1|^{2m-2}.
\]
This implies that
\[
\int_{\hat{\Omega}(z_1)} |\nabla w|^2 dx \leq C|z_1|^{2m-2}.
\]

**Case 2.** For \(|z_1| \leq \sqrt{\epsilon} \).
For \(0 < t < \sqrt{\epsilon} \), estimate (5.18) becomes
\[
\int_{\hat{\Omega}(z_1)} |w|^2 dx \leq C \epsilon^2 \int_{\hat{\Omega}(z_1)} |\nabla w|^2 dx, \quad 0 < s < \sqrt{\epsilon}; \quad (5.21)
\]
Estimate (5.19) becomes
\[
\int_{\hat{\Omega}(z_1)} |L_{\epsilon, u}\hat{u}_1|^2 dx \leq \int_{\hat{\Omega}(z_1)} \left( \frac{|x_1|^{m-2}}{\epsilon + |x_1|^m} + \frac{|x_1|^{m-1}}{(\epsilon + |x_1|^m)^2} \right)^2 dx
\]
\[
\leq \frac{Cs}{\epsilon} + \frac{C(|z_1|^{2m-2} + s^{2m-2})s}{\epsilon^3}, \quad \text{for } 0 < s < \sqrt{\epsilon}; \quad (5.22)
\]
Estimate (5.20) becomes, in view of (5.30),
\[
\tilde{F}(t) \leq \left( \frac{C_0 \epsilon}{s - t} \right)^2 \tilde{F}(s) + C(s - t)^2 \left( \frac{1}{\epsilon} + \frac{|z_1|^{2m-2}}{\epsilon^3} + \frac{s^{2m-2}}{\epsilon^3} \right), \quad \forall \ 0 < t < \sqrt{\epsilon}. \quad (5.23)
\]
Let \( t_i = 2C_0 i \epsilon, \ i = 1, 2, \ldots \). Then
\[
\frac{C_0 \epsilon}{t_{i+1} - t_i} = \frac{1}{2}.
\]
Let \( k = \left\lfloor \frac{1}{4C_0 \epsilon^{1-\frac{1}{m}}} \right\rfloor \). Then by (3.36) with \( s = t_{i+1} \) and \( t = t_i \), we have
\[
\tilde{F}(t_i) \leq \frac{1}{4} \tilde{F}(t_{i+1}) + C_i^3 \left( \epsilon^{2m-2} + |z_1|^{2m-2} \right).
\]
After \( k \) iterations, we have, using (5.15),
\[
\tilde{F}(t) \leq \left( \frac{1}{4} \right)^k \tilde{F}(t_{k+1}) + C \sum_{l=1}^{k} \left( \frac{1}{4} \right)^{l-1} l^3 \left( \epsilon^{2m-2} + |z_1|^{2m-2} \right)
\]
\[
\leq C \left( \frac{1}{4} \right)^{\frac{1}{4}} + C \left( \epsilon^{2m-2} + |z_1|^{2m-2} \right) \leq C \left( \epsilon^{2m-2} + |z_1|^{2m-2} \right).
\]
This implies that
\[
\int_{\hat{\Omega}(z_1)} |\nabla w|^2 dx \leq \epsilon^{2m-2} + |z_1|^{2m-2}.
\]

**STEP 3.** Proof of (5.17).
Using a change of variables $(3.37)$, define $Q', \tilde{h}_1$, and $\tilde{h}_2$ as in the proof of Proposition $3.2$. Then by $(5.2)$,

\[ |\tilde{h}'_1(0)| + |\tilde{h}'_2(0)| \leq C|\Omega_1|^{m-1}, \quad |\tilde{h}''_1(0)| + |\tilde{h}''_2(0)| \leq C\delta|\Omega_1|^{m-2}. \]

Since $R$ is small, $\|\tilde{h}_1\|_{C^{1,1}((-1,1))}$ and $\|\tilde{h}_2\|_{C^{1,1}((-1,1))}$ are small and $\frac{1}{2}Q'_{\delta}$ is essentially a unit square as far as applications of Sobolev embedding theorems and classical $L^p$ estimates for elliptic systems are concerned. By the same argument as in the proof of Proposition $3.2$ $(3.40)$ still holds. We divide into two cases to proceed.

**Case 1.** For $\sqrt{\varepsilon} \leq |z_1| \leq R/2$.

By $(5.16)$,

\[ \int_{\Omega_{\delta}(z_1)} |\nabla w_1|^2 \, dx \leq C|\Omega_1|^{2m-2}. \]

By $(5.8)$,

\[ \delta^2 |L_{\lambda,\mu} u_1|^2 \leq \delta^2 \left( \frac{C}{|\Omega_1|^2} + \frac{C}{|\Omega_1|^{m+1}} \right) \leq C|\Omega_1|^{m-1}, \quad \text{in } \Omega_{\delta}(z_1). \]

We deduce from $(3.40)$ that

\[ |\nabla w_1(z_1, x_2)| = \frac{C|\Omega_1|^{m-1}}{\delta} \leq \frac{C}{|\Omega_1|}, \quad \forall \ x_2 < \frac{\varepsilon}{2} + h_2(z_1) \leq \frac{\varepsilon}{2} + h_1(z_1). \]

**Case 2.** For $|z_1| \leq 2 \sqrt{\varepsilon}$.

By $(5.16)$,

\[ \int_{\Omega_{\delta}(z_1)} |\nabla w_1|^2 \, dx \leq C(e^{2m-2} + |\Omega_1|^{2m-2}). \]

By $(5.8)$,

\[ \delta^2 |L_{\lambda,\mu} u_1|^2 \leq C\delta^2 \left( \frac{(\varepsilon + |\Omega_1|)^{m-2}}{\varepsilon} + \frac{(\varepsilon + |\Omega_1|)^{m-1}}{\varepsilon^2} \right) \leq C (\varepsilon + |\Omega_1|)^{m-1}, \quad \text{in } \Omega_{\delta}(z_1). \]

We deduce from $(3.40)$ that

\[ |\nabla w_1(z_1, x_2)| = \frac{C}{\delta} \left( e^{m-1} + |\Omega_1|^{m-1} \right) \leq \frac{C(e^{m-1} + |\Omega_1|^{m-1})}{\varepsilon}, \quad \forall \ x_2 < \frac{\varepsilon}{2} + h_2(z_1) \leq \frac{\varepsilon}{2} + h_1(z_1). \]

Proposition $5.3$ is established. $\square$

Define $\tilde{u}_3$, $i = 1, 2$ by $(3.41)$. Using $(5.1)$, $(5.2)$ and $(5.6)$, we have

\[ |\nabla \tilde{u}_3^i(x)| \leq \frac{C(\varepsilon + |x_1|)}{\varepsilon + |x_1|^m}, \quad i = 1, 2, \quad x \in \Omega_R. \]

and

\[ |\nabla \tilde{u}_3^i(x)| \leq C, \quad i = 1, 2, \quad x \in \Omega \setminus \Omega_R. \]

It follows from $(1.6)$, $(5.6)$ and $(5.7)$ that

\[ |L_{\lambda,\mu} \tilde{u}_3^i| \leq \frac{C}{\varepsilon + |x_1|^m}, \quad i = 1, 2, \quad x \in \Omega_R. \]

Then Lemma $5.4$ still holds, while Lemma $5.5$ and Lemma $5.6$ become
Lemma 5.4. With \( \delta = \delta(z_1) \) in (3.14), we have, for \( i = 1, 2 \),

\[
\int_{\widehat{\Omega}_{\delta}(z_1)} |\nabla w_3|^2 \, dx \leq \begin{cases} 
Ce^2, & |z_1| < \sqrt[6]{\epsilon}, \\
C|z_1|^{2n}, & \sqrt[6]{\epsilon} \leq |z_1| < R/2. 
\end{cases}
\] (5.28)

Proof. The proof is very similar to that of Lemma 3.5. By the same argument, we still have (3.50) holds.

Case 1. \( \sqrt[6]{\epsilon} < |z_1| < R/2. \)

We still have (5.18) for \( 0 < s < \frac{2|z_1|}{3} \). Instead of (5.19), we have, using (5.27),

\[
\int_{\Omega_{\delta}(z_1)} s(z_1) \left| L_{\lambda, \mu} \tilde{u}_3 \right|^2 \, dx \leq Cs |z_1|^m. 
\] (5.29)

Instead of (5.20), we have

\[
\mathcal{F}(t) \leq \left( \frac{C_0 |z_1|^m}{s-t} \right)^2 \mathcal{F}(s) + C(s-t)^2 s |z_1|^{2m}, \quad \forall \ 0 < t < s < \frac{2|z_1|}{3}. 
\] (5.30)

We define \( \{t_i\}, k \) and iterate as in the proof of (3.16), right below formula (3.36), to obtain, using (3.47),

\[
\mathcal{F}(t_1) \leq \left( \frac{1}{4} \right)^k \mathcal{F}(t_{k+1}) + C |z_1|^{2m} \sum_{l=1}^{k} \left( \frac{1}{4} \right)^l \leq C |z_1|^{2m}. 
\]

This implies that

\[
\int_{\widehat{\Omega}_{\delta}(z_1)} |\nabla w|^2 \, dx \leq C |z_1|^{2m}. 
\]

Case 2. \( |z_1| < \sqrt[6]{\epsilon}. \)

Estimate (5.21) remains the same. Estimate (5.22) becomes

\[
\int_{\Omega_{\delta}(z_1)} |L_{\lambda, \mu} \tilde{u}_3|^2 \, dx \leq \frac{Cs}{\epsilon}, \quad 0 < s < \sqrt[6]{\epsilon}. 
\] (5.31)

Estimate (5.23) becomes

\[
\mathcal{F}(t) \leq \left( \frac{C_0 \epsilon}{s-t} \right)^2 \mathcal{F}(s) + \frac{C(s-t)^2 s}{\epsilon}, \quad \forall \ 0 < t < \sqrt[6]{\epsilon}. 
\] (5.32)

Define \( \{t_i\}, k \) and iterate as in the proof of (3.19), right below formula (3.36), to obtain

\[
\mathcal{F}(t_1) \leq \left( \frac{1}{4} \right)^k \mathcal{F}(t_{k+1}) + C \sqrt{k} \left( \frac{1}{4} \right)^l \epsilon^2 \leq Ce^2. 
\]

This implies that

\[
\int_{\Omega_{\delta}(z_1)} |\nabla w|^2 \, dx \leq Ce^2. 
\]

It is not difficult to obtain
Lemma 5.5.
\[ \| \nabla w_i^3 \|_{L^\infty(\tilde{\Omega})} \leq C, \quad i = 1, 2. \]  
(5.33)

Consequently,
\[ |\nabla v_i^3(x)| \leq \frac{C(\epsilon + |x_1|)}{\epsilon + |x_1|^m}, \quad i = 1, 2, \quad x \in \Omega_R. \]  
(5.34)

The last main difference is the computation of \( a^{\alpha \alpha}_{11}, \alpha = 1, 2 \). In fact, By (1.8), (2.12), (2.7) and (5.15),
\[
a^{\alpha \alpha}_{11} = \int_{\tilde{\Omega}} \left( C^0 e(v_1^\alpha), e(v_1^\alpha) \right) dx \leq C \int_{\Omega} |\nabla v_1^\alpha|^2 dx \leq C \int_{\Omega} |\nabla u_1^\alpha|^2 dx + C \int_{\tilde{\Omega}} |\nabla w_1^\alpha|^2 dx \\
\leq C \int_{-R}^{R} \frac{1}{\epsilon + h_1(x_1) - h_2(x_1)} dx_1 + C \\
\leq C \int_{0}^{R} \frac{1}{\epsilon + |x_1|^m} dx_1 + C \\
\leq C \frac{\epsilon^{\frac{1}{m} - 1}}{\epsilon + |x_1|^m}, \quad \alpha = 1, 2.
\]

Using (5.15) again, we have
\[
a^{11}_{11} = \int_{\tilde{\Omega}} \left( C^0 e(v_1^1), e(v_1^1) \right) dx \geq \frac{1}{C} \int_{\tilde{\Omega}} |e(v_1^1)|^2 dx \\
\geq \frac{1}{2C} \int_{\tilde{\Omega}} |e(\tilde{u}_1^1)|^2 dx - C \int_{\tilde{\Omega}} |e(w_1^1)|^2 dx \\
\geq \frac{1}{2C} \int_{\tilde{\Omega}} |e(\tilde{u}_1^1)|^2 dx - C.
\]

In view of (4.16), we have
\[
\int_{\tilde{\Omega}} |e(\tilde{u}_1^1)|^2 dx \geq \frac{1}{4} \int_{\Omega} |\partial_{x_2} \tilde{u}_1|^2 dx \geq \frac{1}{4} \int_{\Omega_R} \frac{dx}{(\epsilon + h_1(x_1) - h_2(x_1))^2} \\
\geq \frac{1}{C} \int_{0}^{R} \frac{1}{\epsilon + |x_1|^m} dx_1 + C \\
\geq \frac{\epsilon^{\frac{1}{m} - 1}}{C}.
\]

Thus
\[ a^{11}_{11} \geq \frac{\epsilon^{\frac{1}{m} - 1}}{C}. \]

Similarly, we have
\[ a^{22}_{11} \geq \frac{\epsilon^{\frac{1}{m} - 1}}{C}. \]

By the argument as in the proof of Lemma 4.4, we have
\[ \frac{\epsilon^{\frac{1}{m} - 2}}{C} \leq \det a_{11} \leq C \epsilon^{\frac{1}{m} - 2}. \]

Then, we have
\[ |C_1^\alpha - C_2^\alpha| \leq C \epsilon^{\frac{1}{m}}, \quad \alpha = 1, 2. \]

The proof of Theorem 5.1 is finished.
6 Appendix: Some results on the Lamé system with infinity coefficients

Assume that in \( \mathbb{R}^d \), \( \Omega \) and \( \omega \) are bounded open sets with smooth boundaries satisfying
\[
\overline{\omega} = \bigcup_{s=1}^m \overline{\omega}_s \subset \Omega,
\]
where \( \{\omega_s\} \) are connected components of \( \omega \). Clearly, \( m < \infty \) and \( \omega_s \) is open for all \( 1 \leq s \leq m \). Given \( \varphi \in C^{1,\gamma}(\partial \Omega; \mathbb{R}^d) \), \( 0 < \gamma < 1, \mu > 0, d\lambda + 2\mu > 0 \), and
\[
\mu^{(s)}_{n} \to \infty, \quad d\lambda^{(s)}_{n} + 2\mu^{(s)}_{n} \to \infty, \quad \text{as} \quad n \to \infty.
\]
We denote
\[
C_{n}^{(s)} := \lambda^{(s)}_{n} \delta_{ij} \delta_{kl} + \mu^{(s)}_{n} \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right), \quad 1 \leq s \leq m,
\]
\[
C^{(0)} := \lambda \delta_{ij} \delta_{kl} + \mu \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right),
\]
and
\[
C_n(x) = \begin{cases} 
C_n^{(s)}, & \text{in } \omega_s, \quad 1 \leq s \leq m, \\
C^{(0)}, & \text{in } \Omega \setminus \overline{\omega}.
\end{cases}
\]
Consider for every \( n \)
\[
\begin{cases}
\nabla \cdot (C_n e(u_n)) = 0, & \text{in } \Omega, \\
u = \varphi, & \text{on } \partial \Omega.
\end{cases} \tag{6.1}
\]

Let \( \Psi \) be the linear space of rigid displacements of \( \mathbb{R}^d \), i.e. the set of all vector-valued functions \( \eta = (\eta^1, \ldots, \eta^d)^T \) such that \( \eta = a + Ax \), where \( a = (a_1, \ldots, a_d)^T \) is a vector with constant real components, \( A \) is a skew-symmetric \((d \times d)\)-matrix with real constant elements. It is easy to see that \( \Psi \) is a linear space of dimension \( d(d + 1)/2 \). Denote
\[
\Psi = \text{span} \left\{ \psi^\alpha \mid 1 \leq \alpha \leq \frac{d(d + 1)}{2} \right\}.
\]

Equation (6.1) can be rewritten in the following form to emphasize the transmission condition on \( \partial \omega \):
\[
\begin{cases}
\nabla \cdot \left( C_n^{(s)} e(u_n) \right) = 0, & \text{in } \omega_s, \quad 1 \leq s \leq m, \\
\nabla \cdot \left( C^{(0)} e(u_n) \right) = 0, & \text{in } \Omega \setminus \overline{\omega}, \\
\frac{\partial u_n}{\partial v_0} \bigg|_+ \cdot \psi^\alpha = \frac{\partial u_n}{\partial v_0} \bigg|_- \cdot \psi^\alpha, & \text{on } \partial \omega_s, \quad 1 \leq s \leq m; \quad 1 \leq \alpha \leq \frac{d(d + 1)}{2},
\end{cases} \tag{6.2}
\]
where
\[
\frac{\partial u_n}{\partial v_0} \bigg|_+ := \left( C^{(0)} e(u) \right) \tilde{n} = \lambda (\nabla \cdot u_n) \tilde{n} + \mu \left( \nabla u_n + (\nabla u_n)^T \right) \tilde{n}, \quad \text{on } \partial \omega_s,
\]
\[
\frac{\partial u_n}{\partial v_0} \bigg|_- := \left( C_n^{(s)} e(u) \right) \tilde{n} = \lambda^{(s)}_{n} (\nabla \cdot u_n) \tilde{n} + \mu^{(s)}_{n} \left( \nabla u_n + (\nabla u_n)^T \right) \tilde{n}, \quad \text{on } \partial \omega_s,
\]
and the subscript \( \pm \) indicates the limit from outside and inside \( \omega_s \), respectively.
Theorem 6.1. If $u_n \in H^1(\Omega; \mathbb{R}^d)$ is a solution of equation (6.1), then $u_n \in C^1(\Omega \setminus \omega; \mathbb{R}^d) \cap C^1(\overline{\omega}; \mathbb{R}^d)$ and satisfies equation (6.2).

If $u_n \in C^1(\Omega \setminus \omega; \mathbb{R}^d) \cap C^1(\overline{\omega}; \mathbb{R}^d)$ is a solution of equation (6.2), then $u_n \in H^1(\Omega; \mathbb{R}^d)$ and satisfies equation (6.1).

Proof. The first part of the theorem follows from Proposition 1.4 of [27]. The proof of the rest is standard.

Theorem 6.2. There exists at most one solution $u_n \in H^1(\Omega; \mathbb{R}^d)$ to equation (6.1).

Proof. We only need to prove that if $\varphi = 0$ then a solution $u_n$ of (6.1) is zero. Indeed it follows from (6.1) that

$$
\int_{\Omega} (C_n e(u_n), e(\psi)) \, dx = 0, \quad \forall \, \psi \in C^\infty_c(\Omega; \mathbb{R}^d).
$$

This implies by density of $C^\infty_c(\Omega; \mathbb{R}^d)$ in $H^1_0(\Omega; \mathbb{R}^d)$ that $\int_{\Omega} (C_n e(u_n), e(u_n)) \, dx = 0$. By the property of $C_n$ and the First Korn inequality, we have $\nabla u_n = 0$, and therefore $u_n = 0$.

Define the functional

$$
I_n[v] := \frac{1}{2} \int_{\Omega} (C_n(x)e(v), e(v)) \, dx,
$$

where $v$ belongs to the set

$$
H^1_\varphi(\Omega; \mathbb{R}^d) := \left\{ v \in H^1(\Omega; \mathbb{R}^d) \mid v = \varphi, \text{ on } \partial \Omega \right\},
$$

where $\varphi \in C^{1,\gamma}(\partial \Omega; \mathbb{R}^d)$, $0 < \gamma < 1$.

Theorem 6.3. For every $n$, there exists a minimizer $u_n \in H^1_\varphi(\Omega; \mathbb{R}^d)$ satisfying

$$
I_n[u_n] := \min_{v \in H^1_\varphi(\Omega; \mathbb{R}^d)} I_n[v].
$$

Moreover, $u_n \in H^1(\Omega; \mathbb{R}^d)$ is a solution of equation (6.1).

The proof of Theorem 6.3 is standard. The existence of a minimizer $u_n$ follows from the lower semi-continuity property of the functional with respect to the weak convergence in $H^1(\Omega; \mathbb{R}^d)$ and the First Korn inequality.

Comparing equation (6.1), the Lamé system with infinity coefficients is

$$
\begin{align*}
\nabla \cdot \left( C^{(0)} e(u) \right) &= 0, & \text{in } \Omega \setminus \overline{\omega}, \\
|u|_+ &= u_- \quad & \text{on } \partial \omega, \\
e(u) &= 0 & \text{in } \omega, \\
\int_{\partial \omega_0} \frac{\partial u}{\partial \nu_0} \cdot \psi^\alpha &= 0, & 1 \leq s \leq m; \quad 1 \leq \alpha \leq \frac{d(d+1)}{2}, \\
u &= \varphi & \text{on } \partial \Omega.
\end{align*}
$$

(6.4)

We have similar results:
Theorem 6.4. If \( u \in H^1(\Omega; \mathbb{R}^d) \) satisfies (6.4) except for the fourth line, then \( u \in C^1(\Omega \setminus \omega; \mathbb{R}^d) \cap C^1(\overline{\omega}; \mathbb{R}^d). \)

Proof. By the third line of equation (6.4), \( u \) is a linear combination of \( \{ \psi^\alpha \} \), and therefore \( u \in C^\infty(\partial \omega) \). Since \( \nabla \cdot (C(0)e(u)) = 0 \) on \( \Omega \setminus \overline{\omega} \), the regularity of \( u \) in \( \Omega \setminus \omega \) follows from [2]. \( \square \)

Theorem 6.5. There exists at most one solution \( u \in H^1(\Omega; \mathbb{R}^d) \cap C^1(\Omega \setminus \omega; \mathbb{R}^d) \cap C^1(\omega; \mathbb{R}^d) \) of (6.4).

Proof. It is equivalent to showing that if \( \varphi = 0 \), equation (6.4) only has the solution \( u = 0 \). We know from the third and the second lines of equation (6.4) that \( u|_{\partial \omega} \) is a linear combination of \( \{ \psi^\alpha \} \). Multiplying the first line of equation (6.4) by \( u \) and integrating by parts leads to, using a version of the Second Korn inequality (Lemma D),

\[
0 = \int_{\Omega \setminus \overline{\omega}} \left( C(0)e(u), e(u) \right) dx \geq \frac{1}{C} \int_{\Omega \setminus \overline{\omega}} |e(u)|^2 dx \geq \frac{1}{C} \int_{\Omega \setminus \overline{\omega}} |\nabla u|^2 dx.
\]

It follows that \( u = 0 \). \( \square \)

The existence of a solution can be obtained by using the variational method. Define the energy functional

\[
I_\infty[v] := \frac{1}{2} \int_{\Omega \setminus \overline{\omega}} \left( C(0)e(u), e(u) \right) dx,
\]

where \( v \) belongs to the set

\[
\mathcal{A} := \left\{ u \in H^1_{\psi}(\Omega; \mathbb{R}^d) \mid e(u) = 0 \text{ in } \omega \right\}.
\]

Theorem 6.6. There exists a minimizer \( u \in \mathcal{A} \) satisfying

\[
I_\infty[u] = \min_{v \in \mathcal{A}} I_\infty[v],
\]

Moreover, \( u \in H^1(\Omega; \mathbb{R}^d) \cap C^1(\Omega \setminus \omega; \mathbb{R}^d) \cap C^1(\omega; \mathbb{R}^d) \) is a solution of equation (6.4).

Proof. By the lower semi-continuity of \( I_\infty \) and the weakly closed property of \( \mathcal{A} \), it is not difficult to see that a minimizer \( u \in \mathcal{A} \) exists and satisfies \( \nabla \cdot (C(0)e(u)) = 0 \) in \( \Omega \setminus \overline{\omega} \). The only thing needs to shown is the fourth line of (6.4), i.e.

\[
\int_{\partial \omega} \frac{\partial u}{\partial \nu_0} \cdot \psi^\alpha = 0, \quad 1 \leq s \leq m.
\]

Indeed, since \( u \) is a minimizer, for any \( 1 \leq s \leq m, 1 \leq \alpha \leq d(d + 1)/2 \), and any \( \phi \in C_c^\infty(\Omega; \mathbb{R}^d) \) satisfying \( \phi \equiv \psi^\alpha \) on \( \overline{\omega} \), and \( \phi = 0 \) on \( \overline{\omega} \), \( t \neq s \), let

\[
i(t) := I_\infty[u + t\phi], \quad t \in \mathbb{R},
\]

we have

\[
0 = i'(0) := \frac{di}{dt}_{t=0} = \int_{\Omega \setminus \overline{\omega}} \left( C(0)e(u), e(\phi) \right) dx.
\]
Therefore
\[ 0 = -\int_{\Omega^c} \nabla \cdot \left( C(0)e(u) \right) \cdot \phi \, dx = \int_{\Omega^c} \left( C(0)e(u), e(\phi) \right) \, dx + \int_{\partial \Omega} \frac{\partial u}{\partial \nu_0} \cdot \phi \, dy. \]

Finally, we give the relationship between \( u_n \) and \( u \).

**Theorem 6.7.** Let \( u_n \) and \( u \) in \( H^1(\Omega; \mathbb{R}^d) \) be the solutions of equations (6.2) and (6.4), respectively. Then
\[ u_n \rightarrow u \quad \text{in} \quad H^1(\Omega; \mathbb{R}^d), \quad \text{as} \quad n \rightarrow \infty, \quad (6.6) \]
and
\[ \lim_{n \rightarrow \infty} I_n[u_n] = I_\infty[u], \quad (6.7) \]
where \( I_n \) and \( I_\infty \) are defined by (6.3) and (6.5).

**Proof.** **Step 1.** Prove that \( \{u_n\} \) weakly converges in \( H^1(\Omega; \mathbb{R}^d) \) to a solution \( u \) of (6.4).

Due to the uniqueness of the solution to (6.4), we only need to show that after passing to a subsequence, \( \{u_n\} \) weakly converges in \( H^1(\Omega; \mathbb{R}^d) \) to a solution \( u \) of (6.4).

Let \( \eta \in H^1_0(\Omega; \mathbb{R}^d) \) be fixed and satisfy \( \eta \equiv 0 \) on \( \overline{\omega} \). Since \( u_n \) is the minimizer of \( I_n \) in \( H^1_0(\omega; \mathbb{R}^d) \), we have, for some constant \( C \) independent of \( n \),
\[ \frac{1}{C} \lVert e(u_n) \rVert_{L^2(\Omega)}^2 \leq I_n[u_n] \leq I_n[\eta] = \frac{1}{2} \int_{\Omega^c} \left( C(0)e(\eta), e(\eta) \right) \, dx \leq C \lVert \eta \rVert_{H^1(\Omega)}^2. \]

Using the Second Korn inequality and the fact that \( u_n = \varphi \) on \( \partial \Omega \), we obtain
\[ \lVert u_n \rVert_{H^1(\Omega)} \leq C, \]
and therefore, along a subsequence,
\[ u_n \rightarrow u \quad \text{in} \quad H^1_\varphi(\Omega; \mathbb{R}^d), \quad \text{as} \quad n \rightarrow \infty. \]

Next we show that \( u \) is a solution of equation (6.4). In fact, we only need to prove the following three conditions:

\[ \nabla \cdot \left( C(0)e(u) \right) = 0, \quad \text{in} \quad \Omega \setminus \overline{\omega}, \quad (6.8) \]
\[ e(u) = 0, \quad \text{in} \quad \omega, \quad (6.9) \]
\[ \int_{\partial \omega} \frac{\partial u}{\partial \nu_0} \cdot \psi^s = 0, \quad 1 \leq s \leq m, \quad 1 \leq \alpha \leq d(d+1)/2. \quad (6.10) \]

(i) Since \( u_n \in H^1(\Omega; \mathbb{R}^d) \) is a solution of equation (6.1) and \( u_n \rightarrow u \) in \( H^1_\varphi(\Omega; \mathbb{R}^d) \), we have, for any \( \phi \in C^\infty_c(\Omega \setminus \overline{\omega}; \mathbb{R}^d) \), that
\[ 0 = \int_{\Omega^c} \left( C(0)e(u_n), e(\phi) \right) \, dx \rightarrow \int_{\Omega^c} \left( C(0)e(u), e(\phi) \right) \, dx. \]
Therefore
\[ \int_{\Omega} (C^{(0)}e(u), e(\phi)) \, dx = 0, \quad \forall \phi \in C_0^\infty (\Omega \setminus \overline{\omega}), \]
that is (6.8).

(ii) Let \( \eta \in H^1_\Omega (\Omega; \mathbb{R}^d) \) be fixed and satisfy \( \eta \equiv 0 \) on \( \overline{\omega} \), then since \( u_n \) is a minimizer of \( I_n \) in \( H^1_\phi (\Omega; \mathbb{R}^d) \), we have
\[ I_n[u_n] \leq I_n[\eta] \leq \frac{1}{2} \int_{\Omega} (C^{(0)}e(\eta), e(\eta)) \, dx \leq C. \]

On the other hand,
\[ I_n[u_n] \geq \sum_{s=1}^m \min(2\mu_n^{(s)}, d\lambda_n^{(s)} + 2\mu_n^{(s)}) \int_{\omega_s} |e(u_n)|^2 \, dx. \]

Since \( \mu_n^{(s)} \to \infty \) and \( d\lambda_n^{(s)} + 2\mu_n^{(s)} \to \infty \) as \( n \to \infty \), we have
\[ \|e(u_n)\|_{L^2(\omega)} \to 0, \quad \text{as} \quad n \to \infty. \]

By (1), \( u_n \to u \) in \( H^1(\Omega; \mathbb{R}^d) \). Therefore
\[ \|e(u)\|_{L^2(\omega)} = 0, \]
i.e. \( e(u) = 0 \) in \( \omega \), which is (6.9).

(iii) By (i) and (ii), \( u \) satisfies (6.9) and is a linear combination of \( \{\psi^s\} \) on each \( \partial \omega_s \), and is equal to \( \varphi \) on \( \partial \Omega \). By the elliptic regularity theorems, \( u \in C^1(\Omega \setminus \omega; \mathbb{R}^d) \cap C^2(\Omega \setminus \overline{\omega}; \mathbb{R}^d) \). For each \( s = 1, 2, \ldots, m, \ 1 \leq \alpha \leq d(d + 1)/2 \), we construct a function \( \rho \in C^2(\Omega \setminus \omega; \mathbb{R}^d) \) such that \( \rho = \psi^s \) on \( \partial \omega_s \), \( \rho = 0 \) on \( \partial \omega_t \) for \( t \neq s \), and \( \rho = 0 \) on \( \partial \Omega \). By Green’s identity, we have the following:
\[ 0 = -\int_{\Omega} \nabla \cdot (C^{(0)}e(u_n)) \cdot \rho \, dx \]
\[ = \int_{\Omega} (C^{(0)}e(u_n), e(\rho)) \, dx + \int_{\partial \omega_s} \frac{\partial u_n}{\partial \nu_0} \cdot \psi^s \]
\[ = \int_{\Omega} (C^{(0)}e(u_n), e(\rho)) \, dx + \int_{\partial \omega_s} \frac{\partial u_n}{\partial \nu_0} \cdot \psi^s \]
\[ = \int_{\Omega} (C^{(0)}e(u_n), e(\rho)) \, dx. \]

Similarly,
\[ 0 = -\int_{\Omega} \nabla \cdot (C^{(0)}e(u)) \cdot \rho \, dx = \int_{\Omega} (C^{(0)}e(u), e(\rho)) \, dx + \int_{\partial \omega_s} \frac{\partial u}{\partial \nu_0} \cdot \psi^s. \]

Since \( u_n \to u \) in \( H^1(\Omega) \), it follows that
\[ 0 = \int_{\Omega} (C^{(0)}e(u_n), e(\rho)) \, dx \to \int_{\Omega} (C^{(0)}e(u), e(\rho)) \, dx. \]
Thus
\[
\int_{\partial\omega_s} \frac{\partial u}{\partial v_0} \cdot \psi^n = 0, \quad 1 \leq s \leq m, \quad 1 \leq \alpha \leq d(d+1)/2.
\]

Step 1 is completed.

Step 2. Prove (6.6) and (6.7).

Since \( u_n \) is a minimizer of \( I_n \) and \( e(u) = 0 \) in \( \omega \), we have
\[
I_n[u_n] \leq I_n[u] = I_\infty[u].
\]

Thus
\[
\limsup_{n \to \infty} I_n[u_n] \leq I_\infty[u].
\]

On the other hand, since \( e(u) = 0 \) and \( u_n \rightharpoonup u \) in \( H^1(\Omega; \mathbb{R}^d) \),
\[
I_\infty[u] = \frac{1}{2} \int_{\Omega^{(1)}} (C^{(0)} e(u), e(u)) \, dx
\]
\[
\leq \liminf_{n \to \infty} \frac{1}{2} \int_{\Omega^{(1)}} (C^{(0)} e(u_n), e(u_n)) \, dx
\]
\[
\leq \liminf_{n \to \infty} \frac{1}{2} \int_{\Omega^{(1)}} (C^{(0)} e(u_n), e(u_n)) \, dx + \limsup_{n \to \infty} \frac{1}{2} \sum_s \int_{\omega_s} (C^{(s)} e(u_n), e(u_n)) \, dx
\]
\[
\leq \limsup_{n \to \infty} I_n[u_n].
\]

With the help of the first Korn’s inequality, we easily deduce (6.7) and (6.6) from the above. The proof of Theorem 6.7 is completed.

Acknowledgments. The first author was partially supported by NNSF (11071020) (11371060), SRFDPHE (20100003110003), and Beijing Municipal Commission of Education for the Supervisor of Excellent Doctoral Dissertation (20131002701). The second author was partially supported by SRFDPHE (20100003120005) and NNSF (11201029). The research of the third author is partially supported by NSF grant DMS-1065971, DMS-1203961. All authors were partially supported by the Fundamental Research Funds for the Central Universities. We thank an anonymous referee for helpful suggestions which improve the exposition.

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