OPTIMAL DIVIDEND OF COMPOUND POISSON PROCESS UNDER A STOCHASTIC INTEREST RATE

LINLIN TIAN
School of Mathematical Sciences, Nankai University
Tianjin 300071, China

XIAOYI ZHANG*
School of Economics and Management, Hebei University of Technology
Tianjin 300401, China

YIZHOU BAI
School of Mathematical Sciences, Nankai University
Tianjin 300071, China

(Communicated by Kok Lay Teo)

ABSTRACT. In this paper we assume the insurance wealth process is driven by the compound Poisson process. The discounting factor is modelled as a geometric Brownian motion at first and then as an exponential function of an integrated Ornstein-Uhlenbeck process. The objective is to maximize the cumulated value of expected discounted dividends up to the time of ruin. We give an explicit expression of the value function and the optimal strategy in the case of interest rate following a geometric Brownian motion. For the case of the Vasicek model, we explore some properties of the value function. Since we can not find an explicit expression for the value function in the second case, we prove that the value function is the viscosity solution of the corresponding HJB equation.

1. Introduction. The optimal dividend problem has been discussed for a long time in the literature. In 1957 De Finetti [10] proposed that an insurance company should allow cash leakages and measure their performance during its life time instead of only focussing on ruin probability. These cash leakages can be interpreted as dividends. In the setting of constant interest rate, Asmussen and Taksar [3] solved the optimal dividend problem for the special case of Brownian motion. They found out that the optimal strategy is a constant barrier strategy in the case of unbounded dividend and a so-called threshold strategy in the case of restricted dividend rates. In the case of a surplus process following a compound Poisson process, Gerber and Shiu [14] showed that the optimal strategy is a threshold strategy when claim size are exponentially distributed for restricted dividend rates. For the more general case of claim size distribution, Azcue and Muler [4] studied the optimal reinsurance and dividend policy in the framework of Cramér-Lundberg model using viscosity.

2020 Mathematics Subject Classification. Primary: 93E20; Secondary: 91B30.
Key words and phrases. Hamilton-Jacobi-Bellman equation, Vasicek model, geometric Brownian motion, interest rate, viscosity solution, optimal dividends.
Research is supported by Chinese NSF Grants No.11471171 and No.11571189.
* Corresponding author: Xiaoyi Zhang.
solution. Later, Azcue and Muler, see [5], found the optimal dividend payment policy in the case of bounded dividend rates. In the setting of constant interest rate the optimal dividend problem has been studied quite well under various general reserve models, see e.g., [1, 15, 19]. We omit listing the existing literature and refer to a survey on the dividend problems by Albrecher and Thonhauser [2] and references therein.

The interest rate forms a key component of the financial market, influencing the firm’s cost and profit. There are a lot of factors influencing interest rate, such as inflation rate, monetary policy, exchange rate policy, international agreement, and international privity. The interest rate is also an important tool reflecting policy makers’ intentions and achieving economic objectives. As it changes over time, it is more reasonable to assume that the interest rate is a function of time instead of a deterministic constant. The changes of interest rate reflect the fluctuations of the monetary market. Eisenberg [11] solved optimal dividends problem in the setting of surplus following a drifted Brownian motion. The discounting factor is modelled as a stochastic process: at first as a geometric Brownian motion, then as an exponential function of an integrated Ornstein-Uhlenbeck process. They found an explicit expression for the value function of the optimal strategy for both restricted and unrestricted dividends in the case of geometric Brownian motion.

In our paper, we model the surplus process as a compound Poisson process. In section 3, we explore the dividend maximization problem under the Dothan model and find, similar to the case of deterministic interest rate, that the optimal strategy does not change (compared to the Gerber-Shiu case) in its form, but the parameters do. In Section 4, we consider the Vasicek model, for which the short rate is defined as an Ornstein-Uhlenbeck process. Here, the situation changed completely. It is not that easy to calculate the return function of the corresponding strategy. We explore the continuity of the value function but unfortunately we can not prove more regularity properties about the value function. It is natural to consider the problem in the framework of viscosity solutions.

2. Problem formulation. In our paper, the reserve \( X_t \) of an insurance company can be described by

\[
X_t = x + ct - \sum_{k=1}^{N(t)} Y_k,
\]

where \( x \geq 0 \) is the initial surplus, the constant \( c > 0 \) is the premium rate, \( N(t) \) is the Poisson process representing the frequency of the incoming claims, \( \{Y_i\}_{i=1}^{\infty} \) representing a sequence of independent, identically distributed (i.i.d.) random variables with distribution \( G : \mathbb{R}^+ \to \mathbb{R} \). Assume that the insurance company is allowed to pay out dividends, where the accumulated dividends until time \( t \) are given by \( L_t \). The surplus at time \( t \) is described as:

\[
X_t^L = x + ct - \sum_{k=1}^{N(t)} Y_k - L_t.
\]

Denote \( B_t \) a standard Brownian motion. All of the above defined quantities are defined on the same filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \), with \( (\mathcal{F}_t)_{t \geq 0} \) the filtration generated by \( \{B_t, X_t\} \). Here we only allow the restricted dividend, which means, the cumulative dividend up to time \( t \) is given by \( L_t = \int_0^t l_s ds \), with \( l_s \in \mathbb{R}^+ \).
[0, M] for some constant $M > 0$. We say that a strategy $L$ is admissible if it is predictable, nondecreasing, cadlag and it verifies $X_t^L \geq 0$ up to the ruin time. Denote $\mathcal{U}_{ad}$ the set of all admissible strategies. Our target is to find the optimal strategy maximizing the expectation of the cumulative discounted dividend under two different kinds of stochastic interest rates. First, we consider a geometric Brownian motion model and then we consider the Vasicek model.

3. Geometric Brownian motion as a discounting factor. In this section, we make the assumption that $M < c$ for mathematical convenience, which means the dividend rate can not exceeds the premium rate. We also specify $G(x) = 1 - e^{-\beta x}$, which means claims follow an exponential distribution with rate $\beta > 0$. As a risk measure, we consider that dividends are discounted by the geometric Brownian motion $\exp\{-r - mt - \delta B_t\}$.

Here we denote $r_t = r + mt + \delta B_t$ with initial value $r$ and assume that $m > \frac{\delta^2}{2} > 0$. Denoting by $\tau^L$ the ruin time of the surplus process under some admissible strategy $L = \{l_s\}$, we define the return function corresponding to $L$ to be

$$J^L(r, x) = E \left[ \int_0^{\tau^L} e^{-r - ms - \delta B_s} l_s \, ds \right].$$

The objective is to find an optimal dividend policy $L_t$ to maximize the expectation of cumulative discounted dividends. We denote $V(r, x)$ the optimal value function

$$V(r, x) = \sup_{L \in \mathcal{U}_{ad}} J^L(r, x),$$

here $\mathcal{U}_{ad}$ denotes the set of all admissible strategies. We note that for any strategy $L$,

$$J^L(r, x) = E \left[ \int_0^{\tau^L} e^{-r - ms - \delta B_s} l_s \, ds \right] \leq E \left[ \int_0^{\tau^L} e^{-r - ms - \delta B_s} M \, ds \right] = \frac{Me^{-r}}{m - \frac{\delta^2}{2}}.$$

This means $V(r, x)$ is bounded. The HJB equation corresponding to the problem is

$$mV_r + \frac{\delta^2}{2}V_{rr} + cV_x - \lambda V + \lambda \int_0^x V(r, x - y)\beta e^{-\beta y}dy + \max_{l \in [0, M]} (e^{-r} - V_x(r, x))l = 0.$$  

(3.4)

3.1. Solving HJB equation. Now we focus on solving the HJB equation. Denote $C^1(\mathbb{R}^+)$ the set of all continuously differentiable function on $\mathbb{R}^+$. We conjecture that $V(r, x) = e^{-r} F(x)$. We only need to find a function $F(x) \in C^1(\mathbb{R}^+)$ such that $F(x)$ satisfies

$$\left(-m + \frac{\delta^2}{2} - \lambda\right) F(x) + cF'(x) + \lambda \int_0^x F(x - y)\beta e^{-\beta y}dy + \max_{l \in [0, M]} (1 - F'(x))l = 0.$$  

(3.5)

We suppose that there exists a concave function $F(x)$ satisfying equation (3.5). Because of the linearity in the control $l$, we get a critical point $b^*$ with $F'(x) > 1$ for $x < b^*$, $F'(b^*) = 1$ and $F'(x) < 1$ for $x > b^*$. It is possible that $b^* = 0$. Under these assumptions the HJB equation (3.5) becomes
We can see that $e^{1}$, then $F(x)$, where $\beta e^{-\beta y}dy = 0,$ $0 < x < b^*$; (3.6)

$$\left(-m + \frac{\delta^2}{2} - \lambda\right)F(x) + cF'(x) + \lambda \int_{0}^{x} F(x-y)\beta e^{-\beta y}dy = 0,$$ $0 < x < b^*$

Equation (3.6) can be written as

$$cF''(x) + \left[\beta c - \lambda - \left(m - \frac{\delta^2}{2}\right)\right]F'(x) - \beta \left(m - \frac{\delta^2}{2}\right)F(x) = 0,$$ $0 < x < b^*$

(3.8)

with a general solution of the form

$$F(x) \triangleq F_1(x) = C_1e^{R_{1}x} + C_2e^{R_{2}x},$$ (3.9)

where $R_1 > 0$ and $R_2 < 0$ are the roots of the characteristic equation

$$c\xi^2 + \left[\beta c - \lambda - \left(m - \frac{\delta^2}{2}\right)\right] \xi - \beta \left(m - \frac{\delta^2}{2}\right) = 0.$$ Similarly, for all $x > b^*$, equation (3.7) can be written as

$$(c - M)F''(x) + \left[\beta(c - M) - \lambda - \left(m - \frac{\delta^2}{2}\right)\right]F'(x) - \beta \left(m - \frac{\delta^2}{2}\right)F(x) + \beta M = 0.$$ (3.10)

Combining (3.10) with the fact that $0 \leq F(x) \leq \frac{M}{m^2}$, we know that (3.10) has a solution of the form

$$\forall x \geq b^*, \quad F(x) \triangleq F_2(x) = \frac{M}{m - \frac{\delta^2}{2}} + De^{S_{2}x},$$ (3.11)

where $D \leq 0$ is a constant, and $S_{2}$ denotes the negative root of the following equation

$$(c - M)\xi^2 + \left[\beta(c - M) - \lambda - (m - \frac{\delta^2}{2})\right] \xi - \beta \left(m - \frac{\delta^2}{2}\right) = 0.$$ (3.12)

It is possible that $b^* = 0$. If $b^* = 0$, then (3.11) satisfies (3.7) for all initial capital $x \geq 0$. Putting (3.11) into (3.7) implies that (3.7) has a solution of the form

$$F_2(x) = \frac{M}{m - \frac{\delta^2}{2}} \left[1 - e^{S_{2}x} \left(1 + \frac{S_{2}}{\beta}\right)\right].$$

This function is increasing and concave because $\beta + S_2 > 0$. If $(-S_2) \frac{M}{m - \frac{\delta^2}{2}}(1 + \frac{S_{2}}{\beta}) \leq 1$, then $F_2(0) \leq 1$, in this case $F_2(x) \leq 1$ for all $x \geq 0$ and $F_2(x)$ is the solution of (3.5). We can see that $e^{-\gamma}F_2(x)$ is the solution of HJB equation (3.4).

From now on we consider the opposite case $(-S_2) \frac{M}{m - \frac{\delta^2}{2}}(1 + \frac{S_{2}}{\beta}) > 1$. We need to find a differentiable solution of (3.6) and (3.7). Substituting (3.9) into (3.6) and setting the coefficient of $e^{-\beta y}$ with 0, we obtain that there exists a constant $\gamma > 0$ ($\gamma$ is independent of $x$) such that

$$F_1(x) = \gamma([R_1 + \beta)e^{R_1x} - (R_2 + \beta)e^{R_2x}], \quad 0 \leq x \leq b^*.$$ (3.13)
Now we should notice that for all $x > b^*$, the term $\int_0^x F(x - y)e^{-\gamma y}dy$ in (3.7) satisfies that

$$\int_0^x F(x - y)e^{-\gamma y}dy = \int_0^b F(y)e^{-\gamma (x-y)}dy = \int_0^{b^*} F(y)e^{-\gamma (x-y)}dy + \int_{b^*}^x F(y)e^{-\gamma (x-y)}dy$$

and $S_2$ is the negative root of the equation (3.12). Thus, substituting (3.11) and (3.13) into (3.7), we can get that

$$e^{-\beta x} \left[ \gamma(e^{R_1b^*} - e^{R_2b^*}) - \frac{M}{\beta(m - \frac{\delta}{\gamma})} - \frac{De^{S_2b^*}}{\beta + S_2} \right] = 0.$$  

Since $x > b^*$ is arbitrary, we can get another condition for $b^*$, which is

$$\gamma(e^{R_1b^*} - e^{R_2b^*}) - \frac{M}{\beta(m - \frac{\delta}{\gamma})} - \frac{De^{S_2b^*}}{\beta + S_2} = 0. \quad (3.14)$$

From the continuity of $F(x)$ at $b^*$, which means $F_1(b^*) = F_2(b^*)$, we obtain

$$\gamma[(R_1 + \beta)e^{R_1b^*} - (R_2 + \beta)e^{R_2b^*}] = \frac{M}{m - \frac{\delta}{\gamma}} + De^{S_2b^*}. \quad (3.15)$$

To determine $\gamma, D, b^*$, we can use the condition

$$F_1'(b^*) = F_2'(b^+) = 1. \quad (3.16)$$

Combining (3.15), (3.14) and (3.16), we can obtain closed-form expressions for $\gamma, D,$ and $b^*$. It is not hard to see that $F_1'(x) > 1$ on $(0, b^*)$ and $F_2'(x) \leq 1$ on $[b^*, +\infty)$, we omit the details here. As a summary, we give out the following theorem.

**Theorem 3.1.** The solution of HJB equation (3.4) is organised as follows.

If $(-S_2)\frac{M}{m - \frac{\delta}{\gamma}}(1 + \frac{S_2}{\beta}) \leq 1$,

$$V(r, x) = e^{-r} \frac{M}{m - \frac{\delta}{\gamma}} \left[ 1 - e^{S_2x} \left( 1 + \frac{S_2}{\beta} \right) \right],$$

and the optimal dividend strategy $L^* = \{l^*_s\}$ is

$$l^*_s = M1_{\{X^*_s \geq 0\}}.$$  

If $(-S_2)\frac{M}{m - \frac{\delta}{\gamma}}(1 + \frac{S_2}{\beta}) > 1$,

$$V(r, x) = \begin{cases} 
-e^{-r} \frac{S_2}{m - \frac{\delta}{\gamma}} \frac{(\beta + R_1)e^{R_1x} - (\beta + R_2)e^{R_2x}}{(R_1 - S_2)e^{R_1x} - (R_2 - S_2)e^{R_2x}}, & x < b^*; \\
-e^{-r} \left[ \frac{M}{m - \frac{\delta}{\gamma}} + \frac{1}{S_2}e^{S_2(x-b^*)} \right], & x \geq b^*;
\end{cases} \quad (3.17)$$

where $b^* = \frac{1}{R_1 - R_2} \log \frac{R_2^2 - S_2R_2}{R_1^2 - S_2R_1}$. And the optimal dividend strategy $L^* = \{l^*_s\}_{s \geq 0}$ is

$$l^*_s = M1_{\{X^*_s \geq b^*\}},$$

Here $1_{\{X^*_s > b^*\}}$ is the indicator function, which means that the optimal strategy is such that dividends are paid at the maximum rate $M$ whenever $X^*_s \geq b^*$. 
Proof. First, we show that $V(r, x)$ is a continuously differentiable solution of (3.4). If $(-S_2) \frac{M}{m - \frac{\beta}{2}} (1 + \frac{S_2}{\beta}) \leq 1$, denote $V(r, x) = e^{-r} F(x)$, where

$$F(x) = e^{-r} \frac{M}{m - \frac{\beta}{2}} \left[ 1 - e^{S_2 x} (1 + \frac{S_2}{\beta}) \right].$$

Since $F(x)$ is a continuously differentiable solution of equation (3.5), it is easy to obtain that $e^{-r} F(x)$ is a solution of (3.4). Similarly, if $(-S_2) \frac{M}{m - \frac{\beta}{2}} (1 + \frac{S_2}{\beta}) > 1$, denote $V(r, x) = e^{-r} F(x)$, where

$$F(x) = \begin{cases} \frac{S_2}{\beta} \frac{M}{m - \frac{\beta}{2}} \frac{(\beta + R_1)e^{R_1 x} - (\beta + R_2)e^{R_2 x}}{(R_1 - S_2)e^{R_1 x} - (R_2 - S_2)e^{R_2 x}}, & x < b^*; \\ \frac{M}{m - \frac{\beta}{2}} + \frac{1}{S_2} e^{S_2 (x-b^*)}, & x \geq b^*. \end{cases}$$

(3.18)

From the fact that $F(x)$ is a continuously differentiable solution of equation (3.5), we obtain that $V(r, x) = e^{-r} F(x)$ is a solution of (3.4).

From now on, we prove the optimality of strategy $L^*$. Let $L$ be an admissible strategy with dividend rate $\{l_s\}_{s \geq 0}$. Let $\tau^L$ denotes the ruin time of the surplus process. From the Itô formula we obtain

$$E[V(r_{t \wedge \tau^L}, X_{t \wedge \tau^L})] = V(r, x) + E \left[ \int_0^{t \wedge \tau^L} \left( -m V_r + \frac{\delta^2}{2} V_{rr} + c V_x - l_s V_x \right) (r_s, X_s) ds \right]$$

(3.19)

Thus, we obtain

$$V(r, x) = -E \left[ \int_0^{t \wedge \tau^L} \left( -m V_r + \frac{\delta^2}{2} V_{rr} + c V_x - l_s V_x \right) (r_s, X_s) ds \right]$$

$$- E \left[ \sum_{0 \leq s \leq t \wedge \tau^L} (V(r_s, X_s) - V(r_s, X_{s -})) \right]$$

$$= -E \left[ \int_0^{t \wedge \tau^L} \left( -m V_r + \frac{\delta^2}{2} V_{rr} + c V_x - l_s V_x \right) (r_s, X_s) ds \right]$$

$$- \lambda \int_0^{X_{s -}} V(r_s, X_{s -} - y) \beta e^{-\beta y} dy + \lambda V(r_s, X_{s -}) ds \right]$$

$$\geq E \left[ \int_0^{t \wedge \tau^L} e^{-r_s l_s} ds \right].$$

(3.20)

We let $t \to \infty$ and use the dominated convergence theorem to get

$$V(r, x) \geq E \left[ \int_0^{t \wedge \tau^L} e^{-r_s l_s} ds \right].$$

(3.21)

If we use the strategy $\{l_s^*\}_{s \geq 0}$, we get the equality in (3.20) which leads to $V(r, x) = J^L^*(r, x)$. This completes the proof. \qed
Example 3.1. Let $\lambda = 2, \delta = 1, \beta = 1, m = 1, c = 2, M = 1$. In this case, $(-S_2) \frac{M}{m-\delta^2}(1 + \frac{S_2}{\beta}) = 0.403882 < 1$ and the corresponding value function is illustrated in Figure 1 (a).

Example 3.2. Let $\lambda = 8, \delta = 4, \beta = 3, m = 12.5, c = 100, M = 80$. In this case, $(-S_2) \frac{M}{m-\delta^2}(1 + \frac{S_2}{\beta}) = 5.746322 > 1$. The value function is illustrated in Figure 1 (b). In this case the optimal threshold is $b^* = 0.6479102$.

Example 3.3. Let $\lambda = 2, \delta = 1, m = 1, c = 2, M = 1, x = 2, r = 1$. We show the sensitivity analysis of $V$ about the parameter $\beta$, see Figure 2 (a). We can see that the value function is a non-decreasing function of $\beta$. Actually, since claims follow an exponential distribution with rate $\beta$, as $\beta$ increases, the expectation of each claim decreases and the company can get more dividend.

Example 3.4. Let $\delta = 1, m = 1, c = 2, M = 1, x = 2, r = 1, \beta = 1$. We show the sensitivity of $V$ about the Poisson intensity $\lambda$, see Figure 2 (b). The value function decreases as the intensity $\lambda$ increases. Because when the intensity $\lambda$ increases, the surplus of the insurance company decreases and eventually dividend payment decreases.

4. Ornstein-Uhlenbeck process as a interest rate. In this section, we consider the Vasicek model as the interest rate model. This model is based on the idea of mean-reversion, it tends to revert to a constant in the long run. This characteristic can also be justified by economic arguments. We refer the interested readers to the article of Vasicek [21] for more details about the Vasicek model. The Vasicek model assumes the current short interest follows an Ornstein-Uhlenbeck process. Denote $\{r_s\}$ an Ornstein-Uhlenbeck process, we can write it as a stochastic differential equation of a standard Brownian motion.

$$dr_s = a(\hat{b} - r_s)ds + \hat{\delta}dB_s,$$  \hspace{1cm} (4.22)

$a, \hat{\delta}, \hat{b} > 0$ are constants. Here, $\hat{b}$ is the long-term mean of the process $\{r_s\}$, i.e. the interest rate process $\{r_s\}$ will evolve around $\hat{b}$ in the long run. The solution of the
stochastic differential equation (4.22) can be found by applying Itô lemma to $e^{rt}r_t$, which leads to

$$r_s = r e^{-as} + \hat{b} (1 - e^{-as}) + \hat{\delta} e^{-as} \int_0^s e^{au}dB_u,$$

with initial condition $r_0 = r$. Let $L = \{l_s\}_{s \geq 0}$ be an admissible strategy and $\tau^L$ denotes the ruin time of surplus process $X^L_s$ with initial wealth $X_0 = x$. The return function corresponding to $L$ is

$$V^L(r, x) = \mathbb{E} \left[ \int_0^{\tau^L} e^{-\int_0^s r_u du} l_s ds \right], \quad (r, x) \in \mathbb{R} \times \{\mathbb{R}^+ \cup 0\}. \quad (4.23)$$

It means the dividend rate $l_s$ at time $s$ is discounted by the factor $e^{-\int_0^s r_u du}$. In the following, we write $U^L_s$ as $U^L_s = \int_0^s r_u du$ with initial value $r_0 = r$. In Figure 3 one sees 2 realizations of the process $\exp\{-U^L_s\}$, i.e. $r_0 = 1, a = 1, \delta = 1, \hat{b} = 2$(the left picture) and $\hat{b} = -2$(the right picture). Our target is to maximize the expected discounted dividends given the preference rate $\{r_t\}$. We define the value function as

$$V(r, x) = \sup_{L \in \mathcal{U}_ad} V^L(r, x), \quad (r, x) \in \mathbb{R} \times \{\mathbb{R}^+ \cup 0\}. \quad (4.24)$$

The corresponding Hamilton-Jacobi-Bellman equation is

$$\left[ -(r + \lambda)V + a(\hat{b} - r)V_r + \frac{\hat{\delta}^2}{2} V_{rr} + cV_x \right] (r, x)$$

$$+ \lambda \int_0^x V(r, x - y) dG(y) + \max_{0 \leq l \leq M} l(1 - V_x(r, x)) = 0. \quad (4.25)$$

Given a continuously differentiable function $\varphi(r, x) : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$, we define the operator

$$\mathcal{L} [\varphi] = \left[ -(r + \lambda)\varphi + a(\hat{b} - r)\varphi_r + \frac{\hat{\delta}^2}{2} \varphi_{rr} + c\varphi_x \right] (r, x)$$
\[ + \lambda \int_0^x \varphi(r, x - y) dG(y) + \max_{0 \leq t \leq M} l (1 - \varphi_x(r, x)). \tag{4.26} \]

This definition will make it easier for us to state the definition of viscosity solution.

\textbf{Figure 3.} the realization of \( \exp\{-U_r^s\} \), \( r = 1, a = 1, \hat{\delta} = 1 \) for \( \hat{b} = 2 \) and \( \hat{b} = -2 \).

\section*{4.1. Properties of the value function.} In this subsection we prove the boundedness and continuity of the value function \( V \) which is defined in (4.24). The continuity makes it easier for us to define viscosity solution.

\textbf{Lemma 4.1.} The value function \( V \) is bounded.

\textbf{Proof.} Via Fubini’s theorem, the value function satisfies

\[ V(r, x) = \sup_{L \in \mathcal{U}_{ad}} V^L(r, x) \leq E \left[ \int_0^\infty e^{-U_r^s} M ds \right] = E[\int_0^\infty E[e^{-U_r^s}] M ds]. \tag{4.27} \]

Thanks to Borodin and Salminen (1998, p.525)\cite{7}, we can use the fact that \( E[e^{-U_r^s}] = e^{f(r, s)} \), where

\[ f(r, s) := -bs + \frac{\hat{\delta}^2}{2a^2} s - \frac{r - \hat{b}}{a} (1 - e^{-as}) + \frac{\hat{\delta}^2}{4a^3} (1 - (2 - e^{-as})^2). \tag{4.28} \]

Let \( b = \hat{b} - \frac{\hat{\delta}^2}{2a^2} \) and \( \hat{\delta} = \frac{\delta}{\sqrt{2a}} \). We can rewrite (4.28) as

\[ f(r, s) = -bs - \frac{r - \hat{b}}{a} (1 - e^{-as}) - \frac{\hat{\delta}^2}{2a^2} (1 - e^{-as})^2. \tag{4.29} \]

Note that we can estimate the function \( f \) as follows

\[ f(r, s) \geq -bs - \frac{\hat{\delta}^2}{2a^2} - \max\{ \frac{r - \hat{b}}{a}, 0 \}. \]

\[ f(r, s) \leq -bs - \min\{ \frac{r - \hat{b}}{a}, 0 \}. \tag{4.30} \]

From (4.27), (4.30) and the assumption \( b > 0 \), we obtain

\[ V(r, x) \leq ME \left[ \int_0^\infty e^{f(r, s)} ds \right] \leq M e^{-\min\{ \frac{r - \hat{b}}{a}, 0 \}}. \tag{4.31} \]
This shows that $V$ is bounded. \hfill \square

**Remark 1.** We assume $b > 0$ because it helps us to obtain the boundedness of the value function.

**Lemma 4.2.** The value function is locally Lipschitz continuous in $r$ and it is continuous in $x$.

**Proof.** The value function $V$ is obviously strictly increasing in $x$ and decreasing in $r$. Let $h \in \mathbb{R}^+$, $r \in \mathbb{R}$ and $L$ be an $\varepsilon$-optimal strategy for the initial point $(r, x)$. Then $L = \{l_t\}_{t \geq 0}$ is also an admissible strategy for $(r + h, x)$. In particular, $X^L$ denotes the wealth process with control strategy $L$, and $\tau^L$ denotes the time of ruin of the surplus process $X^L$. Therefore, one has

$$0 \geq V(r + h, x) - V(r, x)$$

$$\geq \mathbb{E} \left[ \int_0^{\tau^L} e^{-r_{s+h}} l_s ds - \int_0^{\tau^L} e^{-r_s} l_s ds \right] - \varepsilon$$

$$= \mathbb{E} \left[ \int_0^{\tau^L} e^{-r_s} l_s \left( e^{-h} \left( 1 - e^{-as} \right) - 1 \right) ds \right] - \varepsilon.$$ 

Using the fact that for all $s$, $e^{-h} \left( 1 - e^{-as} \right) - 1 \geq \frac{h}{a} (e^{-as} - 1)$ holds, we can see

$$0 \geq V(r + h, x) - V(r, x) \geq \frac{h}{a} \mathbb{E} \left[ \int_0^{\tau^L} e^{-r_s} l_s (e^{-as} - 1) ds \right] - \varepsilon.$$

From $e^{-as} - 1 \geq -1$, we can see

$$0 \geq V(r + h, x) - V(r, x)$$

$$\geq -\frac{h}{a} \mathbb{E} \left[ \int_0^{\tau^L} e^{-r_s} l_s ds \right] - \varepsilon$$

$$\geq \frac{h}{a} V(r, x) - \varepsilon$$

$$\geq -hM \frac{e^{-\min\left\{ \frac{r+h}{a}, 0 \right\}}}{ab} - \varepsilon.$$ 

Here, in the last step, we used the fact that inequality (4.31) holds. This shows that $V$ is locally Lipschitz in $r$.

Now let $L$ be an $\varepsilon$-optimal strategy for the initial point $(r, x + h)$, with a slight abuse of notation $\tau^L$, the ruin time of surplus process $X^L$ with initial value $x + h$ is denoted by $\tau^L$. $T_1$ denotes the first claim time of compound Poisson process. Define $\tau = \inf\{t \geq 0 | \tilde{X}_t \not\in [0, x + h), \tilde{X}_0 = x\}$, where $\tilde{X}_t$ denotes the surplus process driven by $\tilde{L}$ with initial value $x$. Define $\tilde{L} = \{\tilde{l}_t\}_{t \geq 0}$ to be

$$\tilde{l}_t = \begin{cases} 0, & t \leq \tau, \\ l_{t-\tau}, & t > \tau \text{ and } \tilde{X}_\tau = x + h. \end{cases} \quad (4.32)$$

Strategy $\tilde{L}$ means that $\tilde{X}_t$ will not pay dividend until $\tilde{X}_t$ attains $x + h$. From now on, denote $h = \frac{b}{a}$ for simplicity. Then

$$0 \leq V(r, x + h) - V(r, x)$$
\[ V^L(r, x + h) + \varepsilon - \mathbb{E}\left[ e^{-U_r^L} 1_{\tilde{X}_t = x+h} \int_0^{T_r} \exp\left\{-\frac{1}{a}(r_s - r)(1 - e^{-as})\right\} e^{-U_r^L} l_s ds \right] \]

\[ = V^L(r, x + h) + \varepsilon - \mathbb{E}\left[ e^{-U_r^L} 1_{\tilde{X}_t = x+h} \int_0^{T_r} \exp\left\{-\frac{1}{a}(r_s - r)(1 - e^{-as})\right\} e^{-U_r^L} l_s ds \right] \]

Since on \( \{T_1 \geq \frac{h}{\varepsilon}\} \), there is no claims between time 0 and time \( h \), thus surplus process \( \tilde{X}_t \) attains \( x + h \) at time \( h \), i.e. \( r = h \) on \( \{T_1 \geq \frac{h}{\varepsilon}\} \). We can obtain

\[ 0 \leq V(r, x + h) - V(r, x) \]

\[ \leq V^L(r, x + h) + \varepsilon - \mathbb{E}\left[ e^{-U_r^L} 1_{T_1 \geq h} \int_0^{T_r} \exp\left\{-\frac{1}{a}(r_s - r)(1 - e^{-as})\right\} e^{-U_r^L} l_s ds \right] \]

\[ = V^L(r, x + h) - \mathbb{E}\left[ e^{-U_r^L} 1_{T_1 \geq h} \int_0^{T_r} \exp\left\{-\frac{1}{a}(r_s - r)(1 - e^{-as})\right\} e^{-U_r^L} l_s ds \right] \]

\[ - \mathbb{E}\left[ e^{-U_r^L} 1_{T_1 \geq h} 1_{\{r_s < r\}} \int_0^{T_r} \exp\left\{-\frac{1}{a}(r_s - r)(1 - e^{-as})\right\} e^{-U_r^L} l_s ds \right] + \varepsilon. \]

From the fact that \( T_1 \) is independent of \( \{r_t\} \), we can deduce that

\[ 0 \leq V(r, x + h) - V(r, x) \]

\[ \leq V^L(r, x + h) - \mathbb{E}\left[ e^{-U_r^L} 1_{\{r_s \geq r\}} \int_0^{T_r} (1 + \frac{1}{a}(r - r_s)(1 - e^{-as})) e^{-U_r^L} l_s ds \right] e^{-\lambda h} \]

\[ - \mathbb{E}\left[ e^{-U_r^L} 1_{\{r_s < r\}} \int_0^{T_r} e^{-U_r^L} l_s ds \right] e^{-\lambda h} + \varepsilon \]

\[ \leq V^L(r, x + h) \left[ 1 - \mathbb{E}[e^{-U_r^L}] e^{-\lambda h} \right] + V^L(r, x + h) \mathbb{E}\left[ e^{-U_r^L} 1_{\{r_s > r\}} (r_s - r) \right] \frac{1}{a} e^{-\lambda h} + \varepsilon. \]  

(4.33)

In Borodin and Salminen (1998, p525) [7], we can find the distribution of \( \exp\{-U_r^L\} \) and \( r_s \exp\{-U_r^L\} \). Calculating the expectation in the square brackets directly, we find that there exists a constant \( Q_1 \) such that, for \( h \) small enough, we have

\[ \mathbb{E}\left[ e^{-U_r^L} 1_{\{r_s > r\}} (r_s - r) \right] \frac{1}{a} e^{-\lambda h} \leq Q_1 \sqrt{h}. \]  

(4.34)

And there also exists a constants \( Q_2 \) such that

\[ 1 - \mathbb{E}\left[ e^{-U_r^L} \right] e^{-\lambda h} \leq Q_2 h. \]  

(4.35)

Substituting (4.34) and (4.35) into (4.33), we obtain that there exists a constant \( Q \) such that

\[ 0 \leq V(r, x + h) - V(r, x) \leq V(r, x + h) Q \sqrt{h} + \varepsilon \leq \frac{Me^{-\min\{\frac{r_s}{a}, 0\}}}{b} Q \sqrt{h} + \varepsilon. \]

This proves the continuity of the value function. \( \Box \)

We do want to explore more regularity properties about the value function, but unfortunately, in many applications the value function \( V(r, x) \) is not necessarily smooth, or it can be very difficult to prove its differentiability. Therefore we need to introduce the notion of weak solutions, namely viscosity solutions.
We recall that the notion of viscosity solutions was introduced by Crandall and Lions [9] for the first order equations and Lions [16, 17] for the second order equations. The notion of viscosity solution of integro-differential equations was pursued by Soner [20]. The viscosity solution concept of fully nonlinear partial differential equations has been proving to be extremely useful for control theory due to the fact that it does not need the differentiability of the value function. It merely requires continuity of the value function to define the viscosity solution. We refer to the user’s guide of Crandall, Ishii and Lions [8] for an overview of the theory of viscosity solutions and their applications. Using the notion of viscosity solution we prove that the value function is the (viscosity) solution of the corresponding equation (4.25). The viscosity solution approach is becoming a well established approach to study stochastic control problem, see, e.g. the books [13, 23].

**Definition 4.3.** We say that a continuous function $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is a viscosity subsolution of (4.25) at $(r, x) \in \mathbb{R} \times \mathbb{R}^+$ if any continuously differentiable function $\varphi : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ with $\varphi(r, x) = u(r, x)$ such that $u - \varphi$ reaches the maximum at $(r, x)$ satisfies

$$\mathcal{L}[\varphi](r, x) \leq 0.$$  

We say that a continuous function $\bar{u} : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is a viscosity supersolution of (4.25) at $(r, x) \in \mathbb{R} \times \mathbb{R}^+$ if any continuously differentiable function $\varphi : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ with $\varphi(r, x) = \bar{u}(r, x)$ such that $\bar{u} - \varphi$ reaches the minimum at $(r, x)$ satisfies

$$\mathcal{L}[\varphi](r, x) \geq 0.$$ 

Finally, we call a continuous function $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is a viscosity solution of (4.25) if it is both a viscosity subsolution and a viscosity supersolution at any $(r, x) \in \mathbb{R} \times \mathbb{R}^+$.

**Theorem 4.4.** The value function $V$ defined in (4.24) is a viscosity solution of (4.25) on $(0, +\infty)$.

**Proof.** First, we show that the value function is a viscosity supersolution of (4.25). Here we claim that the dynamic programming principle holds: i.e., for any $(r, x) \in \mathbb{R} \times [0, +\infty)$ and any stopping time $\tau$, we have

$$V(r, x) = \sup_{L \in \mathcal{L}_{ad}} E \left[ \int_0^{\tau \land \hat{\tau}} e^{-\int_0^s r_u du} l_\sigma ds + e^{-\int_0^{\tau \land \hat{\tau}} r_u du} V(r_{\tau \land \hat{\tau}}, X_{\tau \land \hat{\tau}}) \right]. \quad (4.36)$$

This principle can be proving by similar methods from Azcue and Muler [4]. We consider the following strategy: The company always pays dividends at rate $l_0$ until time of ruin, where $l_0 \in [0, M]$ is a positive constant. Let $X^0_\tau$ denotes the surplus process controlled by strategy $l_0$. Denote $\tau_1$ the first claim time of the surplus process. Let $\phi$ be a continuously differentiable function on $\mathbb{R} \times [0, +\infty)$ such that $V - \phi$ attains its minimum 0 at $(r, x)$. By the dynamic programming principle, we get

$$0 \geq l_0 E \left[ \int_0^{\tau_1 \land h} e^{-U_r} ds \right] + E \left[ e^{-U_{\tau_1 \land h}} V(r_{\tau_1 \land h}, X^0_{\tau_1 \land h}) \right] - V(r, x)$$

$$\geq l_0 E \left[ \int_0^{\tau_1 \land h} e^{f(r, s)} ds \right] + E \left[ e^{-U_{\tau_1 \land h}} \phi(r_{\tau_1 \land h}, X^0_{\tau_1 \land h}) \right] - \phi(r, x).$$
\[ l_0 E \int_0^{\tau_1 \wedge h} e^{f(r,s) ds} + E \left[ e^{-U_{\tau_1 \wedge h} \phi(r_{\tau_1 \wedge h}, X_{\tau_1 \wedge h})} \alpha(r_{\tau_1 \wedge h}, X_{\tau_1 \wedge h}) \right] + E \left[ e^{-U_{\tau_1 \wedge h} \phi(r_{\tau_1 \wedge h}, X_{\tau_1 \wedge h})} \right] = I_1 + I_2 + I_3, \]

where \( I_i, i = 1, 2, 3 \) are the three terms on the right hand side above. Clearly, we have

\[ I_1 = l_0 E \int_0^{\tau_1 \wedge h} e^{f(r,t) dt} = l_0 E \int_0^h 1_{\{\tau_1 \geq t\}} e^{f(r,t) dt} = l_0 \int_0^h e^{-\lambda t} e^{f(r,t) dt}, \]

\[ I_2 = E \left[ \int_0^h \lambda e^{-\lambda t} \int_0^{X_t} e^{-f(r,t) dr} \{\phi(r_t, X_t) - \phi(r_0, X_0)\} dG(y) dt \right], \]

\[ I_3 = E \left[ \int_0^h 1_{\{\tau_1 \geq t\}} e^{-f(r,t) dt} \left\{ -r_t \phi(r_t, X_t) + \alpha(b-r_t) \phi_r(r_t, X_t) + \frac{\delta^2}{2} \phi_{rr}(r_t, X_t) + c \phi_x(r_t, X_t) - l_0 \phi_x(r_t, X_t) \right\} dt \right]. \]

Let us sum those three together and divide by \( h \). Letting \( h \to 0 \) and using the fact that \( l_0 \) is arbitrary, we obtain

\[ \mathcal{L}(\phi)(r, x) \leq 0. \]

This proves that the value function is a viscosity supersolution of equation (4.25).

Now we prove that the value function is a viscosity subsolution of the corresponding HJB equation. Assume the contrary, i.e. there exists a point \((r_0, x_0) \in \mathbb{R} \times \mathbb{R}^+ \) such that \( V \) is not a viscosity subsolution. By the definition of viscosity solution, there exists \( \eta > 0 \) and a continuously differentiable function \( \varphi^0 \) such that \( V(r_0, x_0) = \varphi^0(r_0, x_0) \), \( \varphi^0(r, x) \geq V(r, x) \) on \( \mathbb{R} \times \mathbb{R}^+ \) and

\[ \mathcal{L}(\varphi^0)(r_0, x_0) = -\eta < 0. \]

First, we assume that \( r_0 \geq 0 \) (\( r_0 < 0 \) can be proved similarly). Consider the function

\[ \hat{\varphi}(r, x) = \varphi^0(r, x) + \frac{\eta}{x_0 \lambda} (x - x_0)^2 + \frac{\eta}{\lambda} (r - r_0)^4, \tag{4.37} \]

then we can notice that \( \hat{\varphi}(r_0, x_0) = \varphi^0(r_0, x_0) \), \( \hat{\varphi}_x(r_0, x_0) = \varphi^0_x(r_0, x_0) \), \( \hat{\varphi}_r(r_0, x_0) = \varphi^0_r(r_0, x_0) \), \( \hat{\varphi}_{rr}(r_0, x_0) = \varphi^0_{rr}(r_0, x_0) \), and

\[ \lambda \int_0^{x_0} \hat{\varphi}(r_0, x_0 - y) dG(y) \leq \lambda \int_0^{x_0} \left[ \varphi^0(r_0, x_0 - y) + \frac{\eta}{x_0 \lambda} y^2 \right] dG(y) \]

\[ \leq \lambda \int_0^{x_0} \varphi^0(r_0, x_0 - y) dG(y) + \eta. \]

We can get

\[ \mathcal{L}(\hat{\varphi})(r_0, x_0) \leq -\eta < 0. \]

Since \( \hat{\varphi} \) is nonnegative and continuously differentiable, we can find \( h \in (0, \frac{\eta}{2}) \) such that

\[ \mathcal{L}(\hat{\varphi})(r, x) \leq -\frac{\eta}{2} < 0 \tag{4.38} \]

on \((r, x) \in [r_0 - 2h, r_0 + 2h] \times [x_0 - 2h, x_0 + 2h] \). Let \( \psi \) be an even and nonnegative continuously differentiable function with support included in \((-1, 1) \times (-1, 1) \).
such that \( \int_{-1}^{1} \int_{-1}^{1} \psi(r,y) dr dy = 1 \). We define \( \nu_n : (-\infty, \infty) \times [0, \infty) \rightarrow \mathbb{R} \) as the convolution
\[
\nu_n(r,y) = \frac{1}{n^2} \int \int_{\sqrt{|y-x|^2 + |r-s|^2} < \frac{1}{n}} \psi(n(r-s), n(y-x)) \left( V(s,x) + \frac{\eta h^2}{2\lambda x_0^2} + \frac{\eta h^4}{2\lambda} \right) ds dx.
\] (4.39)

Since \( V \) is not defined on \( \mathbb{R} \times \mathbb{R}^- \) in this integral, we can extend \( V \) as \( V(r,0) + y \) for \( (r,y) \in \mathbb{R} \times \mathbb{R}^- \). By standard techniques (e.g., see Wheeden and Zygmund [22]), we have that \( \nu_n \) is a smooth function and \( \nu_n \) converges to \( V + \frac{\eta h^2}{2\lambda x_0^2} + \frac{\eta h^4}{2\lambda} \) uniformly on \( [r_0 - 2h, r_0 + 2h] \times [0, x + h] \). Then, we can find \( n_0 \) large enough such that
\[
V(r,y) + \frac{\eta h^2}{\lambda x_0^2} + \frac{\eta h^4}{\lambda} \geq \nu_{n_0}(r,y) \geq V(r,y) + \frac{\eta h^2}{4\lambda x_0^2} + \frac{\eta h^4}{4\lambda}.
\] (4.40)

Let \( \chi \) be a continuously differentiable function satisfying the following conditions
\begin{enumerate}
\item \( 0 \leq \chi \leq 1 \),
\item \( \chi(r,y) = 1 \) for \( (r,y) \in [r_0 - h, r_0 + h] \times [x_0 - h, x_0 + h] \),
\item \( \chi(r,y) = 0 \) for \( (r,y) \notin [r_0 - 2h, r_0 + 2h] \times [x_0 - 2h, x_0 + 2h] \).
\end{enumerate}

Define the function
\[
\varphi(r,y) = \chi(r,y) \hat{\varphi}(r,y) + (1 - \chi(r,y)) \nu_{n_0}(r,y).
\] (4.41)

Take \( \varepsilon = \min \left\{ \frac{n}{2(r_0 + h)}, \frac{\eta h^4}{\lambda x_0^2}, \frac{\eta h^4}{4\lambda x_0^2} \right\} \), from (4.37), (4.40), (4.41) we can see that function \( \varphi(r,y) \) satisfies
\[
[V - \varphi](r,y) \leq -\varepsilon
\] (4.42)
on \( [r_0 - h] \times [x_0 - h, x_0 + h] \cup \{r_0 + h\} \times [x_0 - h, x_0 + h] \cup [r_0 - h, r_0 + h] \times [0, x_0 - h] \cup [r_0 - h, r_0 + h] \times [x_0 + h] \). From (4.38), we obtain
\[
L[\varphi](r,y) \leq -r \varepsilon
\] (4.43)on \( [r_0 - h, r_0 + h] \times [x_0 - h, x_0 + h] \). For any strategy \( L = \{l_t\}_{t \geq 0} \), denote
\[
\tau = \inf \left\{ t > 0 : X^L_t \geq x_0 + h \text{ or } r_t \notin [r_0 - h, r_0 + h] \right\},
\]
\[
\bar{\tau} = \inf \left\{ t > 0 : X^L_t \leq x_0 - h \text{ or } r_t \notin [r_0 - h, r_0 + h] \right\}.
\]

Take \( \tau = \bar{\tau} \wedge \tau \). Since \( \varphi \) is continuously differentiable, we can see that
\[
E \left[ \varphi(X_{\tau}, r_{\tau}) e^{-\int_{0}^{\tau} r_{u} ds} \right] - \varphi(r_0, x_0)
\]
\[
= E \left\{ \int_{0}^{\tau} e^{-\int_{0}^{u} r_{s} ds} \left[ a(\hat{b} - r_{u}) \varphi_{r} - ru_{u} \varphi_{x} + c\varphi_{x} + \frac{1}{2} \delta^2 \varphi_{rr} \right] (r_{u}, X_{u-}) du \right. 
\]
\[
+ \int_{0}^{\tau} e^{-\int_{0}^{u} r_{s} ds} \left[ \lambda \int_{0}^{r_{u}} \varphi(r_{u}, X_{u-} - y) dG(y) - \lambda_{2} \varphi(r_{u}, X_{u-}) \right] du \right\} 
\]
\[
\leq E \left[ \int_{0}^{\tau} e^{-\int_{0}^{u} r_{s} ds} L[\varphi](r_{u}, X_{u-}) du - \int_{0}^{\tau} e^{-\int_{0}^{u} r_{s} ds} l_{u} du \right] 
\]
\[
\leq -\varepsilon E \left[ \int_{0}^{\tau} e^{-\int_{0}^{u} r_{s} ds} r_{u} du \right] - E \left[ \int_{0}^{\tau} e^{-\int_{0}^{u} r_{s} ds} l_{u} du \right].
\]
The last inequality holds because of (4.43). Combining with (4.42), we can see
\[
E\left[ e^{-\int_0^t r_s ds} V(r_t, X_t) \right] 
\leq E\left[ e^{-\int_0^t r_s ds} (\varphi(r_t, x_t) - \varepsilon) \right] 
= E\left[ e^{-\int_0^t r_s ds} \varphi(r_t, x_t) - e^{-\int_0^t r_s ds} \varphi(r_0, x_0) \right] + E\left[ \varphi(r_0, x_0) - e^{-\int_0^t r_s ds} \varepsilon \right] 
\leq -\varepsilon E\left[ \int_0^T e^{-\int_0^u r_s ds} l_u du \right] - E\left[ \int_0^T e^{-\int_0^u r_s ds} l_u du \right] + E\left[ \varphi(r_0, x_0) - e^{-\int_0^t r_s ds} \varepsilon \right].
\]
Since
\[
E\left[ \int_0^T e^{-\int_0^u r_s ds} l_u du \right] = 1 - E\left[ e^{-\int_0^t r_s ds} \right],
\]
we obtain
\[
E\left[ e^{-\int_0^t r_s ds} V(r_t, X_t) \right] 
\leq \varphi(r_0, x_0) - \varepsilon - E\left[ \int_0^T e^{-\int_0^u r_s ds} l_u du \right] 
= V(r_0, x_0) - \varepsilon - E\left[ \int_0^T e^{-\int_0^u r_s ds} l_u du \right].
\]
Since strategy \( L \) is arbitrary, using the Dynamic Programming Principle (4.36), we can see that
\[
V(r_0, x_0) = \sup_{L \in \mathcal{U}_{ad}} E\left[ \int_0^T e^{-\int_0^u r_s ds} l_u du + e^{-\int_0^t r_s ds} V(r_t, X_t) \right] \leq V(r_0, x_0) - \varepsilon.
\]
This is a contradiction. This shows that the value function is also a viscosity subsolution of (4.25).

Now we complete the proof that the value function is a viscosity solution of HJB equation (4.25), but we don’t show that it is the unique viscosity solution of HJB equation. Since in the HJB equation (4.25), the coefficient of the term \( V \) is \(-(r + \lambda)\) and \(-(r + \lambda) \in \mathbb{R}\), we can’t prove the uniqueness using the standard method. The standard way of proving uniqueness can be seen in Azcue and Muler [4] and Bai, Ma and Xing [6]. Both papers used the fact that the discount factor is positive. In our paper, the discount rate can be negative or positive. The most recent work about the uniqueness of viscosity solutions for integro-differential equations is in Mou and Święch [18]. They showed that the uniqueness of viscosity solutions for nonlocal integro-differential equations can be proved when the coefficient of the term \( V \) is non-negative. When the coefficient of the term \( V \) is negative, the uniqueness of viscosity solutions is still an opening problem. The uniqueness of viscosity solutions for our HJB equation is still an tricky problem for us and we will keep working on it in our future work.

**Remark 2.** We compared our model with Julia Eisenberg [12]. In [12], there is no claim, which means that there is no need to worry about bankruptcy. Besides, there is no restriction on the dividend rate. Intuitively, it is clear that when starting with a negative initial discount rate, one should stop paying dividend, because the discounting factor \( \exp\{-U_t^+\} \) will increase at least until \( r_t \) becomes positive. On the other hand, if \( r_0 > r^* \), where \( r^* \) is a positive constant, then \( -U_t^+ \) will remain negative and will keep decreasing in time. In this case, it is reasonable to pay all the current wealth as dividend at once. This strategy’s corresponding cost function
can be well analyzed by stochastic calculus and thus the author can show that it is the solution of the corresponding HJB equation. In our model, there is a compound Poisson claim process and a restriction constant $M > 0$ on the dividend rate, which means we can not pay dividend to 0 without thinking about the risk of ruin. It is hard to follow Julia Eisenberg [12] to study the optimal dividend strategy. We can only conjecture that the optimal dividend threshold must rely on the current value of $\{r_t\}$.

5. Concluding remarks. In this paper we investigate the optimal dividend of insurance company under the assumption of stochastic interest rate and give out the explicit expression of the optimal strategy when the interest rate follows a geometric Brownian motion and the claim sizes follow the exponential distribution. For the case of the Vasicek model, we did not give out the solution of the value function but we explored its properties and we used the notion of viscosity solution to create the connection between the value function and the HJB equation, which is important for the future study about the optimal strategy.

When the discounting factor is given by a geometric Brownian motion, we can see that the optimal strategy is still a threshold strategy, except some changes in the parameters compared with the case of deterministic interest rate. This partly used the fact that the surplus process is independent of the discounting factor, which provides a convenient condition for us to prove the optimality. Only exponential claims are considered in section 3, but we already started to explore more general cases of claim distributions. We conjecture that in the setting of geometric Brownian motion, the optimal dividend is a band strategy if the claim follows a more general continuous distribution function $G(y)$.

In section 4, we consider the dividend maximization problem when stochastic interest rate follows an Ornstein-Uhlenbeck Process. But we do not show more regularity properties of the value function. It is quite hard to find an explicit expression of the dividend strategy. We will focus on comparison principle and optimal strategy in future research.

Acknowledgments. This work is supported by the NSF of China (No. 11471171 and No. 11571189). Here we want to express our thanks to Lihua Bai and Junyi Guo for their valuable insights and suggestions. Thanks to Jacques Rioux for his dedication to the improvement of this paper.

REFERENCES

[1] H. Albrecher and S. Thonhauser, Optimal dividend strategies for a risk process under force of interest, Insurance Math. Econom., 43 (2008), 134–149.
[2] H. Albrecher and S. Thonhauser, Optimality results for dividend problems in insurance, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, 103 (2009), 295–320.
[3] S. Asmussen and M. Taksar, Controlled diffusion models for optimal dividend pay-out, Insurance Math. Econom., 20 (1997), 1–15.
[4] P. Azcue and N. Muler, Optimal reinsurance and dividend distribution policies in the Cramér-Lundberg model, Math. Finance, 15 (2005), 261–308.
[5] P. Azcue and N. Muler, Optimal dividend policies for compound Poisson processes: The case of bounded dividend rates, Insurance Math. Econom., 51 (2012), 26–42.
[6] L. Bai, J. Ma and X. Xing, Optimal dividend and investment problems under Sparre Andersen model, Ann. Appl. Probab., 27 (2017), 3588–3632.
[7] A. N. Borodin and P. Salminen, Handbook of Brownian motion-facts and formulae, Birkhäuser Verlag, Basel, 2002.
[8] M. G. Crandall and H. Ishii, User’s guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)*, 27 (1992), 1–67.

[9] M. G. Crandall and P. L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.*, 277 (1983), 1–42.

[10] F. De. Finetti, Su un’impostazione alternativa della teoria collettiva del rischio, *Transactions of the XVth International Congress of Actuaries*, II (1957), 33–443.

[11] J. Eisenberg, Optimal dividends under a stochastic interest rate, *Insurance Math. Econom.*, 65 (2015), 259–266.

[12] J. Eisenberg, Unrestricted consumption under a deterministic wealth and an Ornstein-Uhlenbeck process as a discount rate, *Stoch. Models*, 34 (2018), 139–153.

[13] W. H. Fleming and H. M. Soner *Controlled Markov processes and Viscosity Solutions*, 2nd edition, Springer, New York, 2006.

[14] H. U. Gerber and E. S. W. Shiu, On optimal dividend strategies in the compound Poisson model, *N. Am. Actuar. J.*, 10 (2006), 76–93.

[15] R. Loeffen, On optimality of the barrier strategy in de Finetti’s dividend problem for spectrally negative Lévy processes, *Ann. Appl. Probab.*, 18 (2008), 1669–1680.

[16] P.-L. Lions, Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. I. The dynamic programming principle and applications, *Comm. Partial Diff. Eqs.*, 8 (1983), 1101–1174.

[17] P.-L. Lions, Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. II. Viscosity solutions and uniqueness, *Comm. Partial Diff. Eqs.*, 8 (1983), 1229–1276.

[18] C. Mou and A. Świąch, Uniqueness of viscosity solutions for a class of integro-differential equations, *NoDea-Nonlinear Differ. Equ. Appl.*, 22 (2015), 1851–1882.

[19] J. Smoller, *Stochastic Control in Insurance*, Springer, New York, 2008.

[20] H. M. Soner, Optimal control with state-space constraint. II, *SIAM J. Control Optim.*, 24 (1986), 1110–1122.

[21] O. A. Vasicek, An equilibrium characterization of the term structure, *Finance, Economics and Mathematics*, 5 (1977), 177–188.

[22] R. L. Wheeden and A. Zygmund, *Measure and Integral*, Marcel Dekker, Inc., New York, 1977.

[23] J. Yong and X. Y. Zhou, *Stochastic Controls. Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York, 1999.

Received January 2018; revised November 2018.