Difference-Cum-Ratio Estimators for Estimating Finite Population Coefficient of Variation in Simple Random Sampling

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Abstract

In this paper, three difference-cum-ratio estimators for estimating finite population coefficient of variation of the study variable using known population mean, population variance and population coefficient of variation of auxiliary variable were suggested. The biases and mean square errors (MSEs) of the proposed estimators were obtained. The relative performance of the proposed estimators with respect to that of some existing estimators were assessed using two populations’ information. The results showed that the proposed estimators were more efficient than the usual unbiased, ratio type, exponential ratio-type, difference-type and other existing estimators considered in the study.

Keywords: Auxiliary variable; MSE; coefficient of variation; study variable; simple random sampling.

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1 Introduction

Several studies in the theory of sampling survey have established the fact that the use of auxiliary information at the planning and estimation stages helps in enhancing the efficiency of estimators for estimating population parameters like population mean, population variance, standard deviation etc. as compared to the estimators which use not auxiliary information. Several authors like Singh HP et al. [1], Sahai A et al. [2], Srivastava SK et al. [3], Ahmed A et al. [4], Audu A et al. [5], Audu A et al. [6], Muili JO et al.[7] have worked extensively in that direction. Authors like Singh HP et al. [8], Sisodia BVS et al. [9], Khoshnevisan M et al. [10], Singh RVK et al. [11], Ahmed A et al.[12] and Audu A et al.[13] utilized coefficient of variation of auxiliary variable in the estimators formulation and obtained highly efficient estimators. Nevertheless the investigators did not emphasized on the problem of estimation of coefficient of variation. The coefficient of variation is one of the major parameters of population often used in comparing variability measured in different units. The coefficient of variation is expressed in percentages to indicate the extent of variability percent in the data. Some authors have worked on this, for instance [8,9] first proposed the estimator for coefficient of variation when samples were selected using SRSWOR. Das AK et al. [14], Das AK et al.[15], Patel PA et al. [16], Rajyaguru A et al. [17], Archana V et al. [19], Singh R et al. [20] also, worked on the problem of estimation of coefficient of variation (C.V) under simple random sampling and stratified random sampling. However, some of these existing estimators are ratio-based which are less efficient if the correlation between the study and auxiliary variables is weak. To address this flaw, concept of difference estimator which does not necessary required strong correlation was used to obtained new estimators.

In the present study, we have considered the problem of estimation of population coefficient of variation utilizing information on a single auxiliary variable in SRSWOR by proposing difference-cum-ratio estimators for estimating the population coefficient of variation of the study variable. These new estimators are expected to give a precise and efficient estimate of the population coefficient of variation of the study variable than existing estimators considered in the study.

2 Some Existing Estimators in Literature

The usual unbiased estimator using information on single auxiliary variable to estimate the population coefficient of variation is given by:

\[ t_0 = \hat{C}_y = \frac{s_y}{\bar{y}} \]  

where, \( \bar{y} = n^{-1} \sum_{i=1}^{n} y_i \) and \( s_y^2 = (n - 1)^{-1} \sum_{i=1}^{n} (y_i - \bar{y})^2 \) are the sample mean and variance respectively.

The mean square error (MSE) expression of the estimator \( t_0 \) is given by:

\[ \text{MSE}(t_0) = C_y^2 \gamma \left( C_y^2 + 0.25 (\lambda_{40} - 1) - C_y \lambda_{20} \right) \]  

where \( \gamma = n^{-1} - N^{-1}, \lambda_{rs} = \frac{\mu_{rs}}{\mu_{20}^{1/2} \mu_{02}^{1/2}}, \mu_{rs} = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y}) (x_i - \bar{x})^s, \lambda_{rs} = \frac{\mu_{rs}}{\mu_{20}^{1/2} \mu_{02}^{1/2}} \).

Archana V et al. [19] proposed ratio estimators of population coefficient of variation under simple random sampling without replacement (SRSWOR) as:

\[ t_{AR} = \hat{C}_y \left( \frac{\bar{X}}{\bar{x}} \right) \]
Where, \( \bar{X} = n^{-1} \sum_{i=1}^{n} x_i \) and \( X = N^{-1} \sum_{i=1}^{N} X_i \) are the sample and population mean of the auxiliary variable respectively.

The mean square error (MSE) expression of the estimator \( t_{AR} \) is given by:

\[
MSE(t_{AR}) = C_y^2 \gamma \left[ C_y^2 + 0.25 (\lambda_{40} - 1) - C_y \lambda_{30} - C_y \lambda_{21} + 0.25 (\lambda_{40} - 1) \right]
\]  

Singh R et al. [20] proposed ratio-type, exponential ratio-type and difference-type estimators for coefficient of variation of the study variable \( Y \) using mean of auxiliary variable and are given below with their MSEs as

\[
t_1 = \hat{C}_y \left( \frac{\bar{X}}{\bar{x}} \right)^a
\]  

\[
t_2 = \hat{C}_y \exp \left\{ \beta \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) \right\}
\]  

\[
t_3 = \hat{C}_y + d_i (\bar{X} - \bar{x})
\]  

\[
MSE(t_1) = C_y^2 \gamma \left[ C_y^2 + \frac{\lambda_{40} - 1}{4} + \alpha^2 C_y^2 - C_y \lambda_{30} + 2 \rho C_y C_x - \alpha C_y \lambda_{21} \right]
\]  

\[
MSE(t_2) = C_y^2 \gamma \left[ C_y^2 + \frac{\lambda_{40} - 1}{4} + \frac{\beta^2 C_y^2}{4} - C_y \lambda_{30} + \beta \rho C_y C_x - \frac{\beta}{2} C_y \lambda_{21} \right]
\]  

\[
MSE(t_3) = \gamma \left[ C_y^2 \left( C_y^2 - C_y \lambda_{30} + \frac{\lambda_{40} - 1}{4} \right) + d_i^2 \bar{X}^2 C_y^2 + 2 d_i \bar{X} \rho C_y^2 C_x - d_i \bar{X} C_y C_x \lambda_{21} \right]
\]  

Where \( \alpha = \frac{\lambda_{21} - 2 \rho C_y}{2 C_x}, \beta = \frac{\lambda_{21} - 2 \rho C_y}{C_x}, d_i = \frac{C_x \lambda_{21} - 2 \rho C_y^2}{2 \bar{X} C_y} \).

Singh R et al. [20] proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on \( t_0 \) and \( t_1 \) estimators for estimating coefficient of variation of the study variable \( Y \) and are given below with their MSEs as

\[
t_{4AM} = \frac{\hat{C}_y}{2} \left[ 1 + \left( \frac{\bar{X}}{\bar{x}} \right)^a \right]
\]  

\[
t_{4GM} = \hat{C}_y \left( \frac{\bar{X}}{\bar{x}} \right)^{\frac{a}{2}}
\]
\[ t_{4}^{HM} = 2\hat{C}_{y} \left[ 1 + \left( \frac{\bar{X} - \bar{x}}{x} \right)^{a} \right]^{-1} \] \tag{13} \\

\[ \text{MSE}(t_{4}^{HM}) = C_{y}^{2} \gamma \left[ C_{y}^{2} + \frac{\lambda_{21}^{2} - 1}{4} + \frac{\alpha^{2} C_{y}^{2}}{4} - C_{y} \lambda_{30} + \alpha \rho C_{y} C_{x} - \frac{\alpha}{2} C_{x} \lambda_{21} \right] \] \tag{14} \\

Where \( \alpha = \frac{\lambda_{21} - 2 \rho C_{y} C_{x}}{C_{x}} \), \( j = AM, GM, HM \) .

Singh R et al. [20] proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on \( t_{1} \) and \( t_{2} \) estimators for estimating coefficient of variation of the study variable \( Y \) and are given below with their MSEs as

\[ t_{5}^{AM} = \frac{\hat{C}_{y}}{2} \left[ 1 + \exp \left\{ \beta \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) \right\} \right] \] \tag{15} \\

\[ t_{5}^{GM} = \hat{C}_{y} \exp \left\{ \frac{\beta}{2} \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) \right\} \] \tag{16} \\

\[ t_{5}^{HM} = 2\hat{C}_{y} \left[ 1 + \exp \left\{ -\beta \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) \right\} \right]^{-1} \] \tag{17} \\

\[ \text{MSE}(t_{5}^{HM}) = C_{y}^{2} \gamma \left[ C_{y}^{2} + \frac{\lambda_{21}^{2} - 1}{4} + \frac{\beta^{2} C_{y}^{2}}{16} - C_{y} \lambda_{30} + \frac{\beta}{2} \rho C_{y} C_{x} - \frac{\beta}{4} C_{x} \lambda_{21} \right] \] \tag{18} \\

where \( \beta = \frac{2 \left( \lambda_{21} - 2 \rho C_{y} \right)}{C_{x}} \).

Singh R et al. [20] proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on \( t_{1} \) and \( t_{2} \) estimators for estimating coefficient of variation of the study variable \( Y \) and are given below with their MSEs as

\[ t_{6}^{AM} = \frac{\hat{C}_{y}}{2} \left[ \left( \frac{\bar{X}}{\bar{x}} \right)^{a} + \exp \left\{ \beta \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) \right\} \right] \] \tag{19} \\

\[ t_{6}^{GM} = \hat{C}_{y} \left( \frac{\bar{X}}{\bar{x}} \right)^{\alpha} \exp \left\{ \frac{\beta}{2} \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) \right\} \] \tag{20}
\[
t_{6HM} = 2\hat{C}_y \left[ \left( \frac{\bar{x}}{\bar{x}} \right)^a + \exp \left\{ -\beta \left( \frac{\bar{x} - \bar{x}}{\bar{x} + \bar{x}} \right) \right\} \right]^{-1}
\]

\[
MSE(t_6) = C^2_y \gamma \left[ C_y^2 + \frac{\lambda_{40} - 1}{4} + \frac{1}{4} \left( \frac{C_y \lambda_{30}}{4} - \frac{1}{2} \left( \alpha + \frac{\beta}{2} \right) \right) C_y \lambda_{21} \right.
\]
\[
+ \left( \alpha + \frac{\beta}{2} \right) \rho C_y C_x \left[ \left( \frac{C_y \lambda_{30}}{4} - \frac{1}{2} \left( \alpha + \frac{\beta}{2} \right) \right) C_x \lambda_{21} \right] \]
\]

where \( \beta = 2 \left( \frac{\lambda_{21} - 2 \rho C_y}{C_x} - \alpha \right) \).

Singh R et al. [20] proposed ratio type, exponential ratio-type and difference-type estimators for estimating coefficient of variation of the study variable \( Y \) using variances of the auxiliary variables and are given below:

\[
t_7 = \hat{C}_y \left( \frac{S_x^2}{S_y^2} \right) \]

\[
t_8 = \hat{C}_y \exp \left\{ \beta \left( \frac{S_x^2 - S_y^2}{S_x^2 + S_y^2} \right) \right\}
\]

\[
t_9 = \hat{C}_y + d_2 \left( S_x^2 - S_y^2 \right)
\]

\[
MSE(t_7) = C^2_y \gamma \left[ C_y^2 + \frac{\lambda_{40} - 1}{4} + \alpha^2 \left( \lambda_{30} - \frac{1}{2} \right) - C_y \lambda_{30} + 2 \alpha C_y \lambda_{12} - \alpha \left( \lambda_{22} - 1 \right) \right]
\]

\[
MSE(t_8) = C^2_y \gamma \left[ C_y^2 + \frac{\lambda_{40} - 1}{4} + \frac{\beta^2 \left( \lambda_{30} - \frac{1}{2} \right)}{4} - C_y \lambda_{30} + \beta C_y \lambda_{12} - \frac{\beta}{2} \left( \lambda_{22} - 1 \right) \right]
\]

\[
MSE(t_9) = \gamma \left[ C_y^2 \left( \frac{C_y \lambda_{30} + \lambda_{40} - 1}{4} \right) + 2 C_y d_2 S_x^2 \lambda_{12} + d_2^2 S_x^4 \left( \lambda_{30} - \frac{1}{2} \right) \right]
\]

\[
- C_y d_2 S_x^4 \left( \lambda_{22} - 1 \right)
\]

Where \( s_x^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \), \( \alpha = \frac{\lambda_{22} - 1 - 2 C_y \lambda_{12}}{2 \left( \lambda_{30} - \frac{1}{2} \right)} \), \( \beta = \frac{\lambda_{22} - 1 - 2 C_y \lambda_{12}}{2 \left( \lambda_{30} - \frac{1}{2} \right)} \),

\[
d_2 = \frac{C_y \left( \lambda_{22} - 1 \right) - 2 C_y^2 \lambda_{12}}{2 S_x^2 \left( \lambda_{30} - \frac{1}{2} \right)}
\]

Singh R et al. [20] proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on \( t_9 \) and \( t_7 \) estimators for estimating coefficient of variation of the study variable \( Y \) and are given below with their
MSEs as

\[ t_{10}^{AM} = \frac{\hat{C}_y}{2} \left[ 1 + \left( \frac{S_y^2}{S_x^2} \right)^\alpha \right] \]  
(29)

\[ t_{10}^{GM} = \frac{\hat{C}_y}{2} \left( \frac{S_y^2}{s_x^2} \right)^\alpha \]  
(30)

\[ t_{10}^{HM} = 2\hat{C}_y \left[ 1 + \left( \frac{S_y^2}{S_x^2} \right)^\alpha \right]^{-1} \]  
(31)

\[ MSE(t_{10}) = C_y^2 \gamma \left[ C_y^2 + \frac{\lambda_{40} - 1}{4} + \alpha^2 \left( \frac{\lambda_{44} - 1}{4} \right) - C_y \lambda_{50} + \alpha C_y \lambda_{12} - \frac{\alpha}{2} (\lambda_{22} - 1) \right] \]  
(32)

where \( \alpha = \frac{\lambda_{22} - 1 - 2C_y \lambda_{12}}{(\lambda_{44} - 1)} \).

Rajyaguru A et al. [17] proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on \( t_0 \) and \( t_8 \) estimators for estimating coefficient of variation of the study variable Y and are given below with their MSEs as

\[ t_{11}^{AM} = \frac{\hat{C}_y}{2} \left[ 1 + \exp \left\{ \beta \left( \frac{S_y^2 - s_x^2}{S_x^2 + s_x^2} \right) \right\} \right] \]  
(33)

\[ t_{11}^{GM} = \frac{\hat{C}_y}{2} \exp \left\{ \beta \left( \frac{S_y^2 - s_x^2}{S_x^2 + s_x^2} \right) \right\} \]  
(34)

\[ t_{11}^{HM} = 2\hat{C}_y \left[ 1 + \exp \left\{ -\beta \left( \frac{S_y^2 - s_x^2}{S_x^2 + s_x^2} \right) \right\} \right]^{-1} \]  
(35)

\[ MSE(t_{11}) = C_y^2 \gamma \left[ C_y^2 + \frac{\lambda_{40} - 1}{4} + \frac{\beta^2 C_y^2}{16} - C_y \lambda_{50} + \frac{\beta}{2} C_y \lambda_{12} - \frac{\beta}{4} (\lambda_{22} - 1) \right] \]  
(36)

where \( \beta = \frac{2(\lambda_{22} - 1) - 4C_y \lambda_{12}}{(\lambda_{44} - 1)} \).

Singh R et al. [20] proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on \( t_7 \) and \( t_8 \) estimators for estimating coefficient of variation of the study variable Y and are given below with their MSEs as
\[ t_{12}^{AM} = \frac{\hat{C}_y}{2} \left[ \left( \frac{S_x^2}{s_x^2} \right)^a \exp \left\{ \beta \left( \frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right) \right\} \right] \] (37)

\[ t_{12}^{GM} = \hat{C}_y \left( \frac{S_x^2}{s_x^2} \right)^\frac{a}{2} \exp \left\{ \frac{\beta}{2} \left( \frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right) \right\} \] (38)

\[ t_{12}^{HM} = 2\hat{C}_y \left[ \left( \frac{S_x^2}{S_y^2} \right)^a + \exp \left\{ -\beta \left( \frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right) \right\} \right]^{-1} \] (39)

\[
\text{MSE}(t_{12}) = C_y^2 \gamma \left[ \frac{C_y^2 + \lambda_{40} - 1}{4} + \frac{1}{4} \left( \alpha + \frac{\beta}{2} \right)^2 \left( \lambda_{04} - 1 \right) - C_y \lambda_{30} - \frac{1}{2} \left( \alpha + \frac{\beta}{2} \right) \left( \lambda_{22} - 1 \right) \right] + \left( \alpha + \frac{\beta}{2} \right) C_y \lambda_{12} \] (40)

where \( \beta = 2 \left( \frac{\lambda_{22} - 1 - 2C_y\lambda_{42}}{\lambda_{04} - 1} \right) - \alpha \).

Singh R et al. [20] proposed ratio-type, exponential ratio-type and difference-type estimators for estimating coefficient of variation of the study variable \( Y \) using coefficients of variation of auxiliary variable, and are given below with their MSEs as

\[ t_{13} = \hat{C}_y \left( \frac{C_x}{c_x} \right)^a \] (41)

\[ t_{14} = \hat{C}_y \exp \left\{ \beta \left( \frac{C_x - c_x}{C_x + c_x} \right) \right\} \] (42)

\[ t_{15} = \hat{C}_y + d_3 \left( C_x - c_x \right) \] (43)

\[
\text{MSE}(t_{13}) = C_y^2 \gamma \left[ \frac{C_y^2 + \lambda_{40} - 1}{4} + \frac{\alpha^2 \left( \lambda_{04} - 1 \right)}{4} + \alpha^2 C_y^2 - C_y \lambda_{30} - 2\alpha \rho C_y C_x + \alpha C_y \lambda_{21} + \alpha C_y \lambda_{12} \right] \] (44)

\[
\text{MSE}(t_{14}) = C_y^2 \gamma \left[ \frac{C_y^2 + \lambda_{40} - 1}{4} + \frac{\beta^2 C_y^2}{4} - C_y \lambda_{30} - \beta \rho C_y C_x + \frac{\beta}{2} C_y \lambda_{21} + \frac{\beta}{2} C_y \lambda_{12} + \frac{\beta^3 \left( \lambda_{04} - 1 \right)}{16} \right] \] (45)
Singh R et al. [20] proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on $t_0$ and $t_{13}$ estimators for estimating coefficient of variation of the study variable $Y$ and are given below with their MSEs as

$$MSE(t_{16}) = \gamma \left[ C_y^2 \left( C_y^2 - C_z \lambda_{30} + \frac{\lambda_{30}-1}{4} \right) + \alpha d_1 C_z^2 \left( C_z^2 + \frac{\lambda_{30}-1}{4} - C_z \lambda_{30} \right) \right]$$

(46)

Where

$$\alpha = \frac{2 \rho C_x C_y + 0.5 \left( \lambda_{22} - 1 \right) - C_y \lambda_{32} - C_x \lambda_{21}}{2 C_x^2 + 0.5 \left( \lambda_{04} - 1 \right) - 2 C_x \lambda_{03}}, \quad \beta = \frac{8 \rho C_x C_y + 2 \left( \lambda_{22} - 1 \right) - 4 C_y \lambda_{12} - 4 C_x \lambda_{21}}{4 C_x^2 + \left( \lambda_{04} - 1 \right) - 4 C_x \lambda_{03}}.$$

$$d_1 = -\frac{C_y \left( \rho C_x C_y - 0.5 \left( C_y \lambda_{12} + C_x \lambda_{21} \right) - 0.25 \left( \lambda_{22} - 1 \right) \right)}{C_x \left( C_x^2 + 0.25 \left( \lambda_{04} - 1 \right) - C_x \lambda_{03} \right)}.$$

The MSEs of the estimators $t_{16}^{AM}$, $t_{16}^{GM}$ and $t_{16}^{HM}$ are given by

$$MSE(t_{16}^{AM}) = C_y^2 \gamma \left[ C_y^2 + \frac{\lambda_{30}-1}{4} - C_y \lambda_{30} - \alpha \rho C_x C_y + \frac{\alpha}{2} C_x \lambda_{21} + \frac{\alpha}{2} C_x \lambda_{12} + \frac{\alpha^2 \left( \lambda_{30}-1 \right)}{16} \right]$$

(50)

$$MSE(t_{16}^{GM}) = C_y^2 \gamma \left[ C_y^2 + \frac{\lambda_{30}-1}{4} - C_y \lambda_{30} - \frac{\alpha^2 C_x^2}{4} - C_x \lambda_{30} - \alpha \rho C_x C_y + \frac{\alpha}{2} C_x \lambda_{21} + \frac{\alpha}{2} C_x \lambda_{12} + \frac{\alpha^2 \left( \lambda_{30}-1 \right)}{16} \right]$$

(51)

where

$$\alpha = \frac{8 \rho C_x C_y + 2 \left( \lambda_{22} - 1 \right) - 4 C_y \lambda_{12} - 4 C_x \lambda_{21}}{4 C_x^2 + \left( \lambda_{04} - 1 \right) - 4 C_x \lambda_{03}}.$$

Singh R et al. [20] proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on $t_0$ and $t_{14}$ estimators for estimating coefficient of variation of the study variable $Y$ and are given below with their MSEs as

$$t_{16}^{AM} = \frac{\hat{C}_y}{2} \left[ 1 + \left( \frac{C_x}{C_y} \right)^{\alpha} \right]$$

(47)

$$t_{16}^{GM} = \hat{C}_y \left( \frac{C_x}{C_y} \right)^{\frac{\alpha}{2}}$$

(48)

$$t_{16}^{HM} = 2 \hat{C}_y \left[ 1 + \left( \frac{C_x}{C_y} \right)^{\alpha} \right]^{-1}$$

(49)

$$MSE(t_{16}^{AM}) = C_y^2 \gamma \left[ C_y^2 + \frac{\lambda_{30}-1}{4} - C_y \lambda_{30} - \alpha \rho C_x C_y + \frac{\alpha}{2} C_x \lambda_{21} + \frac{\alpha}{2} C_x \lambda_{12} + \frac{\alpha^2 \left( \lambda_{30}-1 \right)}{16} \right]$$

(50)

$$MSE(t_{16}^{GM}) = C_y^2 \gamma \left[ C_y^2 + \frac{\lambda_{30}-1}{4} - C_y \lambda_{30} - \frac{\alpha^2 C_x^2}{4} - C_x \lambda_{30} - \alpha \rho C_x C_y + \frac{\alpha}{2} C_x \lambda_{21} + \frac{\alpha}{2} C_x \lambda_{12} + \frac{\alpha^2 \left( \lambda_{30}-1 \right)}{16} \right]$$

(51)

where

$$\alpha = \frac{8 \rho C_x C_y + 2 \left( \lambda_{22} - 1 \right) - 4 C_y \lambda_{12} - 4 C_x \lambda_{21}}{4 C_x^2 + \left( \lambda_{04} - 1 \right) - 4 C_x \lambda_{03}}.$$
\[ t^{GM}_{17} = \hat{C}_y \exp \left\{ \frac{\beta}{2} \left( \frac{C_x - c_x}{C_x + c_x} \right) \right\} \]  
\[ \text{(52)} \]

\[ t^{HM}_{17} = 2\hat{C}_y \left[ 1 + \exp \left\{ -\beta \left( \frac{C_x - c_x}{C_x + c_x} \right) \right\} \right]^{-1} \]  
\[ \text{(53)} \]

\[
\text{MSE}(t_{17}^{i}) = C^2_y \left[ \frac{C_x^2 + \frac{\lambda_{12} - 1}{4} + \beta^2 \left( \frac{\lambda_{12} - 1}{4} \right)^2}{16} \right. \\
- \frac{\beta \rho C_x}{2} \left[ \frac{\beta (\lambda_{12} - 1)}{8} + \frac{BC_x \lambda_{12}}{4} + \frac{BC_x \lambda_{21}}{4} \right] \]  
\[ \text{(54)} \]

where \( \beta = \frac{16 \rho C_x}{4 C_x^2 + (\lambda_{02} - 1) - 4C_x \lambda_{03}}. \)

Singh R et al. [20] proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on \( t_{13} \) and \( t_{15} \) estimators for estimating coefficient of variation of the study variable \( Y \) and are given below with their MSEs as

\[ t^{AM}_{18} = \frac{\hat{C}_y}{2} \left\{ \frac{C_x}{c_x} \right\}^{\alpha} + \exp \left\{ \beta \left( \frac{C_x - c_x}{C_x + c_x} \right) \right\} \]  
\[ \text{(55)} \]

\[ t^{GM}_{18} = \hat{C}_y \left( \frac{C_x}{c_x} \right)^{\frac{\alpha}{2}} \exp \left\{ \frac{\beta}{2} \left( \frac{C_x - c_x}{C_x + c_x} \right) \right\} \]  
\[ \text{(56)} \]

\[ t^{HM}_{18} = 2\hat{C}_y \left[ \left( \frac{C_x}{C_y} \right)^{\frac{\alpha}{2}} + \exp \left\{ -\beta \left( \frac{C_x - c_x}{C_x + c_x} \right) \right\} \right]^{-1} \]  
\[ \text{(57)} \]

\[
\text{MSE}(t_{18}^{i}) = C^2_y \left[ \frac{C_x^2 + \frac{\lambda_{12} - 1}{4} + \frac{\beta (\lambda_{12} - 1)}{2}}{4} \right. \\
- \frac{\beta \rho C_x}{2} \left[ \frac{\beta (\lambda_{12} - 1)}{8} + \frac{BC_x \lambda_{12}}{4} + \frac{BC_x \lambda_{21}}{4} \right] \]  
\[ \text{(58)} \]

where \( \beta = 2 \left[ \frac{2 \rho C_x C_y - C_x \lambda_{12} - C_x \lambda_{21} + 0.5 (\lambda_{12} - 1)}{C_x^2 + 0.25 (\lambda_{04} - 1) - C_x \lambda_{03}} - \alpha. \right] \)
3 Methodology

3.1 Proposed estimators

Having studied the estimators in section 2, we proposed three difference-cum-ratio estimators for coefficient of variation based on information on a single auxiliary variable in three sections.

\[
T_{M1} = \left[ \frac{\hat{C}_y}{2} \left( \frac{\bar{X}}{\bar{X}} + \frac{X}{\bar{X}} \right) + w_1 \left( \bar{X} - \bar{X} \right) + w_2 \hat{C}_y \right] \left[ \frac{\bar{X}}{\bar{X}} \right]
\]

\[
T_{M2} = \left[ \frac{\hat{C}_y}{2} \left( \frac{s^2_x}{s^2_x} + \frac{S^2_x}{s^2_x} \right) + w_1 \left( \frac{s^2_x}{s^2_x} - \frac{S^2_x}{s^2_x} \right) + w_2 \hat{C}_y \right] \left[ \frac{S^2_x}{s^2_x} \right]
\]

\[
T_{M3} = \left[ \frac{\hat{C}_y}{2} \left( \frac{C_x}{c_x} + \frac{C_x}{c_x} \right) + w_1 \left( C_x - c_x \right) + w_2 \hat{C}_y \right] \left[ \frac{C_x}{c_x} \right]
\]

where \( w_k, k = 1, 2, ..., 6 \) are unknown functions to be estimated by minimizing the MSEs of \( T_{Mj}, j = 1, 2, 3 \)

3.2 Properties (Bias and MSE) of \( T_{Mj}, j = 1, 2, 3 \)

Now let us define:

\[
e_0 = \bar{Y} - \bar{Y}, \quad e_1 = \bar{X} - \bar{X}, \quad e_2 = \frac{s^2_x - S^2_x}{s^2_x}, \quad e_3 = \frac{s^2_x - S^2_x}{S^2_x}\text{ such that } \lim_{n \to \infty} |e_k| \approx 0, \quad h = 0, 1, 2, 3
\]

\[
E(e_h) = 0, h = 0, 1, 2, 3, E(e_0^2) = \gamma C_x^2, E(e_1^2) = \gamma C_x^2, E(e_2^2) = \gamma (\lambda_{40} - 1),
\]

\[
E(e_3^2) = \gamma (\lambda_{10} - 1), E(e_0 e_1) = \gamma p c_x c_x, E(e_0 e_2) = \gamma c_x C_x, E(e_0 e_3) = \gamma c_x \lambda_{40}, E(e_0 e_3) = \gamma c_x \lambda_{42},
\]

\[
E(e_1 e_2) = \gamma C_x \lambda_{21}, E(e_1 e_3) = \gamma C_x \lambda_{43}, E(e_2 e_3) = \gamma (\lambda_{22} - 1)
\]

Expressing (59), (60) and (61) in terms of error terms, we have:

\[
T_{M1} = \left[ \frac{S_x (1+e_2)^{1/2}}{2Y (1+e_0)^{1/2}} \left( \frac{X}{X} + \frac{\bar{X}}{X} \right) + \frac{\bar{X} (1+e_1)}{\bar{X}} \right] \left[ \frac{\bar{X}}{\bar{X} (1+e_1)} \right]
\]

\[
+ w_1 \left( \bar{X} - \bar{X} (1+e_1) \right) + w_2 \frac{S_x (1+e_2)^{1/2}}{Y (1+e_0)} \left[ \frac{\bar{X}}{\bar{X} (1+e_1)} \right]
\]

\( (59) \)
estimators as
Simplify (63), (64) and (65) up to second degree approximation, we obtained
\[ T_{M2} = \frac{C_y}{2} \left[ \frac{1 - e_0}{1 + e_0} \right] \left[ \frac{\left(1 + e_0\right)^{1/2}}{\left(1 + e_0\right)^{1/2}} \right]^2 - w_e C_y \left[ \left(1 - e_0\right) - e_0 e_0 \right] \left( C_y \right) \left( e_3 - e_3 \right) \]
\[ + w_e \left( 1 - e_0 - e_0 + e_0 e_0 \right) \]
\[ T_{M1} - C_y = -C_y \left[ \left( e_3 - e_3 \right) \right] \]
\[ T_{M2} - C_y = -C_y \left[ \left( -e_0 - e_0 + e_0 e_0 \right) \right] \]
\[ T_{M3} - C_y = C_y \left[ \left( e_3 - e_3 \right) \right] \]
Taking expectation of (66), (67) and (68) and apply the results of (62) to obtain the bias of the suggested estimators as
Squaring and taking expectation of (67), (68) and (69) and apply the results of (62) to obtain the MSEs of the suggested estimators as

\[ \text{MSE}(T_{m1}) = C_y \left[ A + w_1^2 B + w_1^2 C + 2w_1 D - 2w_1 E - 2w_1 F \right] \]

\[ \text{MSE}(T_{m2}) = C_y \left[ A + w_1^2 B_1 + w_1^2 C_1 + 2w_1 D_1 - 2w_1 E_1 - 2w_1 F_1 \right] \]

\[ \text{MSE}(T_{m3}) = C_y \left[ A + w_1^2 B_2 + w_1^2 C_2 - 2w_1 D_2 + 2w_1 E_2 - 2w_1 F_2 \right] \]

where

\[ A = \gamma \left( C_y^2 + C_y^2 + 2\rho C_y C_x - C_y \lambda_{21} - C_y \lambda_{30} + \frac{(\lambda_{40} - 1)}{4} \right), \quad B = \gamma \delta^2 \left( \lambda_{40} - 1 \right), \quad \delta = \frac{X}{C_y} \]

\[ C = 1 + \gamma \left( C_y^2 + 4C_y C_x - 2C_y \lambda_{21} - 2C_y \lambda_{30} \right), \quad D = \gamma \delta^2 \left( C_y^2 + \rho C_y C_x - \frac{C_y \lambda_{21}}{2} \right) \]

\[ E = \gamma \left( \frac{3C_y \lambda_{21}}{2} - 3\rho C_y C_x - \frac{5C_y^2}{2} - 2C_y^2 + \frac{3C_y \lambda_{30}}{2} - \frac{(\lambda_{40} - 1)}{8} \right), \quad F = \gamma \delta^2 \left( \frac{C_y \lambda_{21}}{2} - \rho C_y C_x - 2C_y^2 \right) \]

\[ A_1 = \gamma \left( C_y + 2C_y \lambda_{12} - (\lambda_{22} - 1) - C_y \lambda_{30} + \frac{(\lambda_{40} - 1)}{4} \right), \quad B_1 = \gamma \delta^2 \left( \lambda_{22} - 1 \right), \quad \delta_1 = \frac{S^2}{C_y} \]

\[ C_1 = 1 + \gamma \left( 3(\lambda_{40} - 1) + 3C_y^2 + 4C_y \lambda_{12} - 2(\lambda_{22} - 1) - 2C_y \lambda_{30} \right) \]
Differentiating (72) partially with respect to \( w_1 \) and \( w_2 \), equate to zero and solve for \( w_1 \) and \( w_2 \) simultaneously, we obtained:

\[
D_1 = \gamma \delta_1 \left[ (\lambda_{04} - 1) + C_y \lambda_{12} - \frac{(\lambda_{22} - 1)}{2} \right]
\]

\[
E_1 = \gamma \left[ \frac{3(\lambda_{22} - 1)}{2} - 3C_y \lambda_{12} - \frac{5}{2} \frac{5(\lambda_{04} - 1)}{2} - 2C^2 + \frac{3C_y \lambda_{30}}{2} - \frac{(\lambda_{40} - 1)}{8} \right]
\]

\[
F_1 = \gamma \delta_1 \left[ \frac{(\lambda_{22} - 1)}{2} - C_y \lambda_{12} - 2(\lambda_{04} - 1) \right]
\]

\[
A_2 = \gamma \left[ C_y^2 - C_y \lambda_{03} - 2 \rho C_y C_x + \frac{3(\lambda_{04} - 1)}{4} + C_y \lambda_{21} + \frac{(\lambda_{22} - 1)}{2} + C_y - C_y \lambda_{30} + \frac{(\lambda_{40} - 1)}{4} \right]
\]

\[
B_2 = \gamma \left[ C_y^2 - C_y \lambda_{03} + \frac{3(\lambda_{04} - 1)}{4} \right] \quad \delta_2 = \frac{C_y^2}{C_x^2}
\]

\[
C_2 = 1 + \gamma \left[ C_y^2 - 2C_y \lambda_{03} - 4 \rho C_y C_x + 2C_y \lambda_{21} + \frac{(\lambda_{04} - 1)}{4} + 2C_y \lambda_{12} - (\lambda_{22} - 1) + 3C_y^2 - 2C_y \lambda_{30} \right]
\]

\[
D_2 = \gamma \left[ C_y^2 - C_y \lambda_{03} - \rho C_y C_x + \frac{3(\lambda_{04} - 1)}{4} + C_y \lambda_{21} + \frac{3C_y \lambda_{22}}{2} - \frac{(\lambda_{22} - 1)}{2} \right]
\]

\[
E_2 = \gamma \left[ \frac{3C_y^2}{2} - 2C_y \lambda_{03} - 3 \rho C_y C_x - \frac{3C_y \lambda_{21}}{4} + \frac{3(\lambda_{04} - 1)}{2} - \frac{3C_y \lambda_{22}}{2} - \frac{3(\lambda_{22} - 1)}{4} - \frac{2C_y \lambda_{30}}{2} + \frac{(\lambda_{40} - 1)}{8} \right]
\]

\[
F_2 = \gamma \left[ C_y^2 - \frac{3C_y \lambda_{03}}{2} - \rho C_y C_x + \frac{C_y \lambda_{21}}{2} + \frac{5(\lambda_{04} - 1)}{8} + C_y \lambda_{12} + \frac{C_y \lambda_{12}}{2} - \frac{(\lambda_{22} - 1)}{4} \right]
\]

Differentiating (72) partially with respect to \( W_1 \) and \( W_2 \), equate to zero and solve for \( W_1 \) and \( W_2 \) simultaneously, we obtained:

\[
W_1 = \frac{CD - EF}{F^2 - BC}
\]

and

\[
W_2 = \frac{DF - BE}{F^2 - BC}
\]

Substituting the results in (72), we obtained the minimum mean square error of \( T_{M1} \) denoted by \( MSE(T_{M1})_{\text{min}} \):

\[
MSE(T_{M1})_{\text{min}} = C_y^2 \left[ A + \frac{(CD^2 + BE^2 - 2DEF)}{(F^2 - BC)} \right] \quad (75)
\]

Differentiating (73) partially with respect to \( W_3 \) and \( W_4 \), equate to zero and solve for \( W_3 \) and \( W_4 \) simultaneously, we obtained:

\[
W_3 = \frac{C_1D_1 - E_1F_1}{F_1^2 - B_1C_1}
\]

and

\[
W_4 = \frac{D_1F_1 - B_1E_1}{F_1^2 - B_1C_1}
\]

Substitute the results in (73), we obtained the minimum mean square error of \( T_{M2} \) denoted by \( MSE(T_{M2})_{\text{min}} \):

\[
MSE(T_{M2})_{\text{min}} = C_y^2 \left[ A_1 + \frac{\left( C_1D_1^2 + B_1E_1^2 - 2D_1E_1F_1^1 \right)}{(F_1^2 - B_1C_1)} \right] \quad (76)
\]

Differentiating (74) partially with respect to \( W_3 \) and \( W_4 \), equate to zero and solve for \( W_3 \) and \( W_4 \) simultaneously, we obtained:

\[
W_3 = \frac{C_1D_1 - E_1F_1}{F_1^2 - B_1C_1}
\]

and

\[
W_4 = \frac{D_1F_1 - B_1E_1}{F_1^2 - B_1C_1}
\]

Substitute the results in (74), we obtained the minimum mean square error of \( T_{M3} \) denoted by \( MSE(T_{M3})_{\text{min}} \):

\[
MSE(T_{M3})_{\text{min}} = C_y^2 \left[ A_2 + \frac{\left( C_1D_1^2 + B_1E_1^2 - 2D_1E_1F_1^1 \right)}{(F_1^2 - B_1C_1)} \right] \quad (77)
\]
Similarly, by differentiating (74) partially with respect to \( w_5 \) and \( w_6 \), equate to zero and solve for \( w_5 \) and \( w_6 \) simultaneously, we obtained

\[
\begin{align*}
  w_5 &= \frac{E_2 F_2 - C_2 D_2}{F_2^2 - B_2 C_2} \\
  w_6 &= \frac{B_2 E_2 - D_2 F_2}{F_2^2 - B_2 C_2}
\end{align*}
\]

Substituting the results in (74), we obtained the minimum mean square error of \( T_{M3} \) denoted by \( \text{MSE}(T_{M3})_{\text{min}} \)

\[
\text{MSE}(T_{M3})_{\text{min}} = C_y^2 \left[ A_2 + \frac{(C_2 D_2^2 + B_2 E_2^2 - 2D_2 E_2 F_2)}{(F_2^2 - B_2 C_2)} \right] 
\]

(77)

4 Empirical Study

In this section, we carry out an empirical study to elucidate the performance of our proposed estimators with respect to some existing related estimators using two (2) data sets below.

Population 1: [Source: [21], p.399]

X: Area under wheat in 1963, Y: Area under wheat in 1964

\[
N = 34, n = 15, \bar{X} = 208.88, \bar{Y} = 199.44, C_x = 0.72, C_y = 0.75, \rho = 0.98, \lambda_{21} = 1.0045, \lambda_{12} = 0.9406, \\
\lambda_{40} = 3.6161, \lambda_{44} = 2.8266, \lambda_{30} = 1.1128, \lambda_{50} = 0.9206, \lambda_{22} = 3.0133
\]

Population 2: [Source: [22], p.1116]

X: Number of fish caught in year 1993, Y: Number of fish caught in year 1995

\[
N = 69, n = 40, \bar{X} = 4591.07, \bar{Y} = 4514.89, C_x = 1.38, C_y = 1.38, \rho = 0.96, \lambda_{21} = 2.19, \lambda_{12} = 2.30, \\
\lambda_{40} = 7.66, \lambda_{44} = 9.84, \lambda_{30} = 1.11, \lambda_{50} = 2.52, \lambda_{22} = 8.19
\]

Table 1. MSEs and PREs of proposed and other estimators in the study

| Estimators | Population 1 | Population 2 |
|------------|--------------|--------------|
|            | MSE | PRE | MSE | PRE |
| **Auxiliary Information:** \( \bar{X}, \bar{Y} \) |     |     |     |     |
| \( t_0 \)  | 0.008003575 | 100.00   | 0.03808827 | 100.00 |
| \( t_{\theta R} \) | 0.02715658 | 29.47 | 0.07645918 | 49.82 |
| \( t_1 \)  | 0.006868341 | 116.53 | 0.03731461 | 102.07 |
| \( t_2 \)  | 0.006868341 | 116.53 | 0.03731461 | 102.07 |
| \( t_3 \)  | 0.006868341 | 116.53 | 0.03731461 | 102.07 |
| \( t_4 \)  | 0.006868341 | 116.53 | 0.03731461 | 102.07 |
| \( t_5 \)  | 0.006868341 | 116.53 | 0.03731461 | 102.07 |
| \( t_6 \)  | 0.006868341 | 116.53 | 0.03731461 | 102.07 |
| Estimators | Population 1 | Population 2 |
|------------|--------------|--------------|
|            | MSE | PRE | MSE | PRE |
| $T_{M1}$   | 0.006737495 | 118.79 | 0.03404568 | 111.87 |
| Auxiliary Information: $S^2_X, S^2_y$ |  |  |  |  |
| $t^j_1$    | 0.006962763 | 114.95 | 0.037568156 | 101.38 |
| $t^j_8$    | 0.006962763 | 114.95 | 0.037568156 | 101.38 |
| $t^j_9$    | 0.006962763 | 114.95 | 0.037568156 | 101.38 |
| $t^j_{10}$ | 0.006962763 | 114.95 | 0.037568156 | 101.38 |
| $t^j_{11}$ | 0.006962763 | 114.95 | 0.037568156 | 101.38 |
| $t^j_{12}$ | 0.006962763 | 114.95 | 0.037568156 | 101.38 |
| $T_{M2}$   | 0.006013652 | 133.09 | 0.02810758 | 135.51 |
| Auxiliary Information: $C_X = S_X / \bar{X}, c_X = s_X / \bar{x}$ |  |  |  |  |
| $t^j_{13}$ | 0.001208508 | 662.27 | 0.02988236 | 127.46 |
| $t^j_{14}$ | 0.001208508 | 662.27 | 0.02988236 | 127.46 |
| $t^j_{15}$ | 0.001208508 | 662.27 | 0.02988236 | 127.46 |
| $t^j_{16}$ | 0.001208508 | 662.27 | 0.02988236 | 127.46 |
| $t^j_{17}$ | 0.001208508 | 662.27 | 0.02988236 | 127.46 |
| $t^j_{18}$ | 0.001208508 | 662.27 | 0.02988236 | 127.46 |
| $T_{M3}$   | 0.000787631 | 1016.16 | 0.02984082 | 127.64 |

5 Results and Discussion

Table 1 shows the MSEs and PREs of the proposed and other estimators considered in the study using information of the two populations 1 and 2. Results obtained from each category revealed that proposed estimators under each category has minimum MSEs and higher PREs compared to other competing existing estimators. These imply that the suggested estimators are more efficient than their counterparts and have higher chances to produce estimates closer to the true values of means for any population of interest.

6 Conclusions

In our study, we have suggested three difference-cum-ratio estimators for estimating the coefficient of variation of the study variable $Y$, and these estimators utilized information on population mean, population mean square error and population coefficient of variation of the auxiliary variable $X$. From the empirical study, the results showed that the proposed estimators were more efficient than the existing estimators considered in the study. Hence we recommend that the proposed estimators should be used in both theoretical and real life applications.

Competing Interests

Authors have declared that no competing interests exist.
References

[1] Singh HP, Solanki RS. An efficient class of estimators for the population mean using auxiliary information in systematic sampling. Journal of Statistical Theory and Practice. 2012;6(2):274-285.

[2] Sahai A, Ray SK. And efficient estimator using auxiliary information, Metrika. 1980;27(4):271-275.

[3] Srivastava SK, Jhajj HA. A class of estimators of the population mean in survey sampling using auxiliary information Biometrika. 1981;68(1):341-343.

[4] Ahmed A, Adewara AA, Singh RVK. Class of ratio estimators with known functions of auxiliary variable for estimating finite population variance. Asian Journal of Mathematics and Computer Research. 2016;12(1):63-70.

[5] Audu A, Adewara AA. Modified factor-type estimators under two-phase sampling. Punjab Journal of Mathematics. 2017;49(2):59-73. ISSN: 1016-2526.

[6] Audu A, Singh R, Khare S, Dauran NS. Almost unbiased estimators for population mean in the presence of non-response and measurement error. Journal of Statistics & Management Systems; 2020. DOI: 10.1080/09720510.2020.1759209

[7] Muili JO, Agwamba EN, Erinola YA, Yunusa MA, Audu A, Hamzat MA. Modified ratio-cum-product estimators of finite population variance. International Journal of Advances in Engineering and Management. 2020;2(4):309-319. DOI: 10.35629/5252-0204309319.

[8] Singh HP, Tailor R. Estimation of finite population mean with known coefficient of variation of an auxiliary character. Statistica. 2005;65(3):301-313.

[9] Sisodia BVS, Dwivedi VK. Modified ratio estimator using coefficient of variation of auxiliary variable. Journal-Indian Society of Agricultural Statistics. 1981;33:13-18.

[10] Khoshnevisan M, Singh R, Chauhan P, Sawan N, Smarandache F. A general family of estimators for estimating population means using known value of some population parameter(s). Far East Journal of Theoretical Statistics. 2007;22(2):181-191.

[11] Singh RVK, Audu A. Efficiency of ratio estimators in stratified random sampling using information on auxiliary attribute. International Journal of Engineering Science and Innovative Tecnology. 2013;2(1):166-172.

[12] Ahmed A, Singh RVK, Adewara AA. Ratio and product type exponential estimators of population variance under transformed sample information of study and supplementary variables. Asian Journal of Mathematics and Computer Researh. 2016;11(3):175-183.

[13] Audu A, Singh RVK. Exponential-type regression compromised imputation class of estimators. Journal of Statistics and Management Systems. 2020;1-15. DOI: 10.1080/09720510.2020.1814501

[14] Das AK, Tripathi TP. A class of estimators for co-efficient of variation using knowledge on co-efficient of variation of an auxiliary character. In annual conference of Ind. Soc. Agricultural Sdtatistics. Held at New Delhi, India; 1981.

[15] Das AK, Tripathi TP. Use of auxiliary information in estimating the coefficient of variation. Alig. J. of Statist. 1992;12:51-58.
[16] Patel PA, Rina S. A Monte Carlo comparison of some suggested estimators of coefficient of variation in finite population. Journal of Statistics Science. 2009;1(2):137-147.

[17] Rajyaguru A, Gupta P. On the estimation of the coefficient of variation from finite population-I, Model Assisted Statistics and Application. 2002;36(2):145-156.

[18] Rajyaguru A, Gupta P. On the estimation of the coefficient of variation from finite population –II, Model Assisted Statistics and Application. 2006;1(1):57-66.

[19] Archana V, Rao A. Same improved estimators of co-efficient of variation from bivariate normal distribution. a monte carlo comparison. Pakistan Journal of Statistics and Operation Research. 2014;10(1).

[20] Singh R, Mishra M, Singh BP, Singh P, Adichwal NK. Improved estimators for population coefficient of variation using auxiliary variables. Journal of Statistics & Management Systems. 2018;21(7):1335-1335.

[21] Murthy MN. Sampling theory and methods. Sampling Theory and Methods; 1967.

[22] Singh S. Advanced sampling theory with applications. How Michael “Selected” Amy. Springer Science and Media. 2003;2.

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