Tensor distributions and their derivatives are described without assuming the presence of a metric. This provides a natural framework for discussing tensor distributions on manifolds with degenerate metrics, including in particular metrics which change signature.

I. INTRODUCTION

The ordinary (scalar) distributions on a manifold are the continuous linear functionals on test functions on the manifold, and so require for their definition only the (topological) space of test functions. Similarly, tensor distributions are continuous linear functionals on test tensors. The additional structure of a volume element allows the association of a tensor distributions to each locally integrable tensor field. On a manifold with (nondegenerate) metric, it is natural to use the metric volume element for this purpose. Furthermore, the Levi-Civita connection can be extended to act on tensor distributions.

In the absence of a metric, one can still associate tensor distributions to tensor fields by introducing a (nondegenerate) volume element. Of particular interest is the case of a metric which is degenerate at a hypersurface. While much of the standard theory of tensor distributions carries over directly using this volume element, the lack of a (global) metric-compatible connection appears to preclude the existence of a preferred notion of covariant differentiation of tensor distributions. We investigate here the extent to which the covariant derivative corresponding to the (local) metric-compatible connection can be extended to tensor distributions. Our main result is that this can indeed be done for a large class of degenerate metrics.

We consider here two types of degenerate metrics, those which are discontinuous at a hypersurface, and those which are continuous but have vanishing determinant at a hypersurface. In both cases, we assume that the metric is piecewise smooth, and that the pullback of the metric to the hypersurface is the same from both sides and nondegenerate. These two cases are typical of signature-changing metrics, where the orthogonal direction to the hypersurface changes character from timelike to spacelike as one crosses the hypersurface. However, the formalism we construct does not involve signature change explicitly, and is thus also applicable to degenerate metrics which do not change signature provided they satisfy the above conditions.

The bulk of the paper is devoted to deriving the properties of differential operations on tensor distributions in the case where the association of tensor distributions to tensor fields is made via an arbitrary volume element. Our presentation parallels that of [1], which presents similar results using elegant mathematics. We opt here instead for a more pedagogical, self-contained approach. Thus, in Section II we first review the definition of tensor distributions and the role of the volume element, then introduce the Heaviside and Dirac distributions and show that they have the expected properties. In Section III we define the partial, exterior and covariant derivatives of tensor distributions. In Section IV we briefly review currents, and discuss their relation to the usual formulation [3, 4] of tensor distributions in the presence of a metric.

Having established our formalism, in Section V we apply it to tensor distributions corresponding to regularly discontinuous tensors, and, more importantly, we give a possible generalization to the case of a regularly discontinuous connection. In Section VI we consider the Levi-Civita connection and possible choices of a volume element in spacetimes having a degenerate metric. Finally, a detailed example involving a signature-changing metric is given in Section VII, and we discuss our results in Section VIII.

*Permanent address
II. TENSOR DISTRIBUTIONS

Let $M$ be a smooth ($C^\infty$) $n$-dimensional manifold without boundary. Defining the test functions on $M$ to be the space $\mathcal{F}$ of smooth (real or complex) functions with compact support on $M$, then the (ordinary) distributions on $M$ are the continuous linear maps from $\mathcal{F}$ to $\mathbb{R}$. This naturally leads to the definition of test tensor fields as being the smooth tensor fields with compact support on $M$. Equivalently, the components of test tensors in any admissible coordinate chart are test functions. Tensor distributions $T$ are then the continuous linear maps from test tensors $U$ to $\mathbb{R}$, with the result being denoted $T[U].$

The product of an ordinary distribution $D$ and a (smooth) tensor field $S$ may be defined by

$$(SD)[U] = D[(S,U)]$$

where $(S,U)$ denotes the total contraction of $S$ with the test tensor $U$. Furthermore, tensor distributions can be multiplied by (smooth) functions $f$ via

$$(fT)[U] = T[fU]$$

to produce a tensor distribution of the same type as the original. These products extend naturally to the tensor product of tensor distributions with (smooth) tensors via

$$(S \otimes T)[U \otimes W] = T[(S,U)W] = (T \otimes S)[W \otimes U]$$

This tensor product can be further generalized to the wedge product of antisymmetric tensor distributions with antisymmetric tensors.

For example, given a local frame $\{e^a\}$ and its dual coframe $\{\omega^a\}$, consider a distribution $\alpha$ which maps test vector fields $V$ to real numbers $\alpha[V]$. The components $\alpha_a$ of $\alpha$ in this frame are ordinary distributions defined by

$$\alpha_a[f] = \alpha[fe^a]$$

for any test function $f$. Using linearity, this yields

$$\alpha[V] = \alpha[V^a e_a] = \alpha_a[V^a] = \alpha_a[\omega^a(V)]$$

where $\omega^a(V) = \langle \omega^a, V \rangle$ is the contraction of $\omega^a$ with $V$, so that

$$\alpha = \omega^a \alpha_a = \alpha_a \omega^a$$

Thus $\alpha$ is a covariant tensor of degree 1, a 1-form (with distributional components). The contraction of a (suitably smooth) vector field $W$ with a 1-form distribution is analogous to its contraction with a 1-form field,

$$\alpha(W) = \omega^a(W) \alpha_a = W^a \alpha_a$$

or equivalently

$$\alpha(W)[f] = \alpha[fW]$$

---

1See e.g. \[3\] or \[4\] for definitions of (equivalent) appropriate topologies on the space of test functions.

2The topology on the space of test tensors can be invariently defined using an auxiliary (Riemannian) metric \[3\] or via the topology on test functions \[4\].

3One can also define the product of two tensor distributions as

$$(S \times T)[U,W] = S[U] T[W]$$

but this does not define a tensor distribution since it acts on pairs of tensors rather than (the tensor product of) tensors. For example,

$$S[fU] T[W] \neq S[U] T[fW]$$

for arbitrary functions $f$. 

2
The extension to general tensor distributions is straightforward. In this manner, any tensor distribution can be written as a sum of terms, each of which is the product of an ordinary distribution and a tensor, and is thus a “distribution-valued tensor”. The tensorial type of a tensor distribution is “dual” to that of the test tensors it acts on, i.e. contravariant indices of the test tensors correspond to covariant indices of the tensor distribution and vice versa. We could in fact have used this property to define tensor distributions in terms of their components in a given basis. In terms of components, the operations defined in the previous paragraph are identical to the usual formulae for tensors. One also sees that the tensor product of two tensor distributions cannot be defined, because it would involve products of the component distributions.

A volume element on $M$ is a nowhere vanishing $n$-form $\omega$ on $M$. Given a volume element $\omega$, we can associate a tensor distribution $\hat{T}$ to each locally integrable tensor field $T$ via

$$\hat{T}[U] = \int_M \langle T, U \rangle \omega$$

The tensorial type of $\hat{T}$ is the same as that of $T$.

### A. Heaviside and Dirac Distributions

We now define the Heaviside and Dirac distributions. We will assume that a volume element $\omega$ is given on $M$, and that $N$ is a given ($n$-dimensional) submanifold of $M$. The Heaviside 0-form distribution $\Theta_0^N$ with support on $N$ is defined by

$$\Theta_0^N[f] = \int_N f \omega$$

for test functions $f$, and the Dirac 1-form distribution $\delta_1^N$ with support on $\partial N$ is defined by

$$\delta_1^N[V] = \int_{\partial N} i_V \omega$$

where $V$ is a test vector field and $i_V$ denotes the interior product (e.g. $i_V(df) = df(V) = V(f)$ for any function $f$).

These special distributions can be multiplied not just by smooth tensors, but by suitable locally integrable tensors. Specifically, for any locally integrable tensor $S$ on $N$, we can define

$$S \Theta_0^N[U] = \int_N \langle S, U \rangle \omega$$

which is essentially (10), while for any locally integrable tensor $\overline{S}$ on $\partial N$ we can define

$$\overline{S} \otimes \delta_1^N[W \otimes V] = \int_{\partial N} \langle \overline{S}, W \rangle i_V \omega$$

Suppose further that a hypersurface $\Sigma$ is given by $\{\lambda = 0\}$ with $d\lambda \neq 0$. This determines a volume element $\sigma$ on $\Sigma$ via the pullback of the Leray form $\sigma$ satisfying

$$\omega = d\lambda \wedge \sigma$$

so that $\sigma$ depends on the choice of $\lambda$. We can replace $\lambda$ by $-\lambda$ if necessary to ensure that $d\lambda(Y) > 0$ for all vector fields $Y$ which point away from $N$; this fixes the orientation of $\sigma$. The Dirac 0-form distribution $\delta_0^\Sigma$ with support on $\Sigma$ is defined via

$$\delta_0^\Sigma[f] = \int_\Sigma f \sigma$$

4More generally, let $\mu \neq 0$ be a 1-form which is normal to $\Sigma$ in the sense that $\mu(X) = 0$ for all $X$ tangent to $\Sigma$. A simple and common example is $\mu = d\lambda$. This determines a volume element $\sigma$ on $\Sigma$ via $\omega = \mu \wedge \sigma$, which of course depends on the choice of $\mu$. 

3
But we now have
\[ i_V \omega = d\lambda(V) \sigma - d\lambda \wedge i_V \sigma \] (16)
and the last term pulls back to zero on \( \Sigma \), so that if \( \Sigma = \partial N \) we have
\[ \delta^N_1 [V] = \delta^N_0 [d\lambda(V)] \] (17)
or equivalently
\[ \delta_1 = \delta_0 \, d\lambda \] (18)
where we have dropped the indices \( N \) and \( \partial N \).

**III. DIFFERENTIATION OF DISTRIBUTIONS**

Before discussing in turn partial, exterior, and covariant differentiation, we first recall the definition of the divergence of a (smooth) vector field \( X \) with respect to the (smooth) volume element \( \omega \), namely
\[ \text{div}(X) \, \omega = \mathcal{L}_X \omega \] (19)
where
\[ \mathcal{L}_X = i_X d + di_X \] (20)
denotes Lie differentiation. For example, in a coordinate basis with
\[ \omega = k \, dx^1 \wedge ... \wedge dx^n \] (21)
we have
\[ \text{div} \left( X^a \frac{\partial}{\partial x^a} \right) = \frac{1}{k} \frac{\partial}{\partial x^a} (kX^a) \] (22)
In Cartesian coordinates on \( \mathbb{R}^n \), for which \( k = 1 \), we recover the usual Euclidean expression for the divergence of a vector. More generally, the covariant divergence \( \nabla \cdot X \) of \( X \) with respect to any torsion-free connection leaving \( \omega \) invariant coincides with \( \text{div}(X) \), \( \nabla \cdot X = \text{div}(X) \); an example is the Levi-Civita connection determined by a metric \( g \), for which \( k = \sqrt{|\det g|} \).

We now derive some useful properties of the divergence which we will need later. From the identity
\[ \mathcal{L}_{fX} \omega = di_{fX} \omega = di_X(f \omega) = \mathcal{L}_X(f \omega) = X(f) \, \omega + f \text{div}(X) \, \omega \] (23)
we recover the usual formula
\[ \text{div}(fX) = f \text{div}(X) + df(X) \] (24)
Similarly,
\[ \mathcal{L}_X \mathcal{L}_Y \omega = \mathcal{L}_X (\text{div}(Y) \, \omega) = \text{div}(\text{div}(Y)X) \, \omega \] (25)
and
\[ \mathcal{L}_{[X,Y]} \omega = [\mathcal{L}_X, \mathcal{L}_Y] \omega = \mathcal{L}_X \mathcal{L}_Y \omega - \mathcal{L}_Y \mathcal{L}_X \omega \] (26)
so that
\[ \text{div}([X,Y]) = \text{div}(\text{div}(Y)X) - \text{div}(\text{div}(X)Y) \] (27)
A. Partial Differentiation

The distribution corresponding to \( X(k) \), where \( X \) is a vector field and \( k \) is a (locally integrable and differentiable) function on \( M \) is

\[
\hat{X}(k)[f] = \int_M X(k) f \omega = \int_M \mathcal{L}_X(k) \omega \\
= \int_M \left( \mathcal{L}_X(k\omega) - k \mathcal{L}_X \omega \right) \\
= \int_M \left( d_{i,k} X(k\omega) - k \div(fX) \omega \right) = \int_{\partial M} k i_{X} \omega - \hat{k}[\div(fX)]
\]

(28)

where \( d\omega = 0 \) was used and the surface term vanishes because \( f \) has compact support. This motivates the definition \( \hat{k} \) (which is equivalent to that of \( \hat{k} \))

\[
X(D)[f] = -D[\div(fX)]
\]

(29)

for the action of \( X \) on an ordinary distribution \( D \). Using the identities (24) and (27) it is easy to check that the definition (29) is linear in \( X \) and \( D \), respects the action of vector fields on functions

\[
X(\hat{k}) = \hat{X}(k)
\]

(30)

satisfies the product rule

\[
X(kD) = X(k)D + kX(D)
\]

(31)

and preserves the commutator algebra of vector fields

\[
X(Y(D)) - Y(X(D)) = [X,Y](D)
\]

(32)

In Cartesian coordinates on \( \mathbb{R}^n \) we recover the usual expression

\[
\left( \frac{\partial}{\partial x^a} D \right)[f] = -D \left[ \frac{\partial}{\partial x^a} f \right]
\]

(33)

or equivalently

\[
X(D)[f] = -D \left[ \frac{\partial}{\partial x^a} (fX^a) \right]
\]

(34)

For example, with \( \Sigma = \partial N \) as above and \( \partial \lambda \) defined by \( \partial \lambda(\lambda) = 1 \) and e.g. \( i_{\partial \lambda} \sigma = 0 \), we have

\[
\div(f\partial \lambda) \omega = di_{\partial \lambda} \sigma = d(f\sigma)
\]

(35)

so that

\[
\partial \lambda(\Theta_0^N)[f] = -\Theta_0^N[\div(f\partial \lambda)] \\
= -\int_N d(f\sigma) \\
= -\int_{\partial N} f\sigma = -\delta_0^{\partial N}[f]
\]

(36)

or in other words

Both \( X \) and \( \omega \) must be at least \( C^{k+1} \) if \( D \) is a distribution of order \( k \). Unless explicitly stated, all tensor fields, including \( \omega \), are assumed to be smooth \( (C^\infty) \).
\[ \partial_\lambda (\Theta^N_0) = -\delta^\partial_0 \]

More generally,
\[ X(\Theta^N_0) = -\delta^\partial_0 d\lambda(X) = -i_X \delta^\partial_1 \]

where
\[ i_X \alpha[U] = p \alpha[X \otimes U] \]
defines the interior product acting on \( p \)-form distributions. We can continue in this manner to define further partial derivatives of these distributions. For instance, defining
\[ \delta'_0 = \partial_\lambda (\delta_0) \]
leads to
\[ \delta'_0[f] = -\delta_0[\text{div}(f \partial_\lambda)] \]

where we have again dropped the labels \( N \) and \( \partial N \). In the special case where \( d\sigma = 0 \), i.e. \( \sigma \) is independent of \( \lambda \), then \( \text{div}(f \partial_\lambda) = \partial_\lambda (f) \) and we recover the familiar result
\[ \delta'_0[f] = -\int_S \partial_\lambda (f) \sigma = -\delta_0[f'] \]
where \( f' = \partial_\lambda (f) \). Note that distributions are not necessarily infinitely differentiable; their differentiability is limited by that of \( \omega \).

\section*{B. Exterior Differentiation}

Antisymmetric tensor distributions of degree \((0, p)\) will be called \( p \)-form distributions. It is customary to write the action of \( p \)-forms \( \alpha \) on \( p \) vector fields \( \{X^i, i = 1...p\} \) as \( \alpha(X^1, ..., X^p) \), where of course
\[ \alpha(X^1, ..., X^p) = \langle \alpha, X^1 \otimes ... \otimes X^p \rangle \]
The volume element \( \omega \) provides us through (11) with an action of \( p \)-forms \( \alpha \) on \( p \) vector fields \( \{X^i, i = 1...p\} \) as
\[ \hat{\alpha}[X^1 \otimes ... \otimes X^p] = \int_M \alpha(X^1, ..., X^p) \omega \]
which makes \( \hat{\alpha} \) a \( p \)-form distribution.

For the special case where \( \alpha \) is a 1-form \( df \), we note the identity
\[ \hat{df}[X] = \int_M df(X) \omega = \int_M i_X(df) \omega = \int_M \mathcal{L}_X(f) \omega = \int_M \left( \mathcal{L}_X(f \omega) - f \mathcal{L}_X \omega \right) = \int_M \left( di_X(f \omega) - f \text{div}(X) \omega \right) = \int_{\partial M} f i_X \omega - \hat{f}[\text{div}(X)] \]
and the surface term vanishes since the test vector field \( X \) is supported away from \( \partial M \). This motivates the definition
\[ dF[V] = -F[\text{div}(V)] \]
for (ordinary) distributions \( F \), which satisfies
\[ d\hat{f} = \hat{df} \]
Equivalently, we can define exterior differentiation of (ordinary) distributions \( F \) using (29) via

\[
dF(X) = X(F)
\]

(48)

Thus, introducing a local frame \( \{e_a\} \) and its dual coframe \( \{\omega^a\} \) as before

\[
dF[V] = dF[V^ae_a] = dF(e_a)[V^a] = e_a(dF)[V^a] = -F[\text{div}(V^ae_a)] = -F[\text{div}(V)]
\]

(49)

which agrees with (13). The exterior calculus of \( p \)-form distributions can therefore be built up in complete analogy with the exterior calculus of \( p \)-forms by regarding tensor distributions as distribution-valued tensors. For instance, in a coordinate basis, any \( p \)-form distribution can be written as a sum of terms of the form \( F dx^1 \wedge \ldots \wedge dx^p \), whose exterior derivative can then be defined to be

\[
d(F dx^1 \wedge \ldots \wedge dx^p) = dF \wedge dx^1 \wedge \ldots \wedge dx^p
\]

(50)

In any case, applying (48) to the Heaviside distribution, we have

\[
d\Theta^N_0[V] = -\Theta^N_0[\text{div}(V)] = - \int_N \text{div}(V) \omega
\]

\[
= - \int_N di_v \omega
\]

\[
= - \int_{\partial N} i_v \omega = -\delta^N_1[V]
\]

(51)

or in other words

\[
d\Theta^N_0 = -\delta^N_1
\]

(52)

The formula (46) generalizes recursively to higher degree forms as follows. If \( \alpha \) is a \( p \)-form, the analog of (43) is

\[
(p + 1) \widehat{d\alpha}[X \otimes U] = (p + 1) \int_M \langle d\alpha, X \otimes U \rangle \omega = \int_M \langle i_X d\alpha, U \rangle \omega
\]

\[
= \int_M \left( \langle \mathcal{L}_X \alpha, U \rangle - \langle di_X \alpha, U \rangle \right) \omega
\]

\[
= \int_M \left( \mathcal{L}_X (\langle \alpha, U \rangle \omega) - \langle \alpha, \mathcal{L}_X U \rangle \omega - \langle \alpha, U \rangle \mathcal{L}_X \omega - \langle di_X \alpha, U \rangle \omega \right)
\]

\[
= \int_M d(\langle \alpha, U \rangle i_X \omega) - \widehat{\alpha} \left[ \text{div}(X)U + \mathcal{L}_X U \right] - di_X \alpha[U]
\]

(53)

where \( X \) is a test vector field and \( U \) is a test tensor field of type \( (p, 0) \), and where the surface term vanishes since \( U \) is a test tensor. This motivates the following recursive definition

\[
(p + 1) d\alpha[X \otimes U] = -\alpha \left[ \text{div}(X)U + \mathcal{L}_X U \right] - di_X \alpha[U]
\]

(54)

for the \( p \)-form distribution \( \alpha \). This equation can alternatively be derived from the more elegant approach starting from (48) and (50). In any case, (54) again satisfies the compatibility condition

\[
d\widehat{\alpha} = \widehat{d\alpha}
\]

(55)

In the special case when \( \alpha \) is a 1-form distribution, we have

\[
2d\alpha[X \otimes Y] = -\alpha \left[ \text{div}(X)Y - \text{div}(Y)X + [X, Y] \right]
\]

(56)

The usual properties of exterior differentiation, such as \( d^2 = 0 \) and the product rule, follow directly from (48) and (50). They can also be verified directly using the identities (24) and (27). For instance, for any (ordinary) distribution \( D \) we have

\[
2d^2 D[X \otimes Y] = -D \left[ \text{div}(X)Y - \text{div}(Y)X + [X, Y] \right]
\]

\[
= D \left[ \text{div}(\text{div}(X)Y) - \text{div}(\text{div}(Y)X) + \text{div}([X, Y]) \right] = 0
\]

(57)
and
\[ d(fD)[X] = -fD[\text{div}(X)] = -D[f\text{div}(X)] = -D[\text{div}(fX) - df(X)] = dD[fX + D[df(X)]] \] (58)

so that
\[ d(fD) = fdD + dfD \] (59)
as expected. The product rule
\[ d(D\alpha) = dD \wedge \alpha + D d\alpha \] (60)
can then be established by induction. Since as previously noted any tensor distribution can be written as a sum of tensor products of tensors with ordinary distributions, we have in fact established the full product rule, namely
\[ d(\alpha \wedge \Gamma) = d\alpha \wedge \Gamma + (-1)^p \alpha \wedge d\Gamma \] (61)
for smooth p-forms \( \alpha \) and q-form distributions \( \Gamma \).

C. Covariant Differentiation

Now suppose that not only a volume element \( \omega \) is given on \( M \), but also a connection \( \nabla \) (i.e. a connection whose covariant derivative operator is \( \nabla \)). The standard way to extend \( \nabla \) to act on tensor distributions is simply to work with tensor components, Christoffel symbols, and partial differentiation [5]. Alternatively, one can construct the “adjoint” of \( \nabla \) with respect to contraction, as follows.

Let \( T, U \) be tensors of “dual” degree, so that the contraction \( T(U) = \langle T, U \rangle \) is defined. We have
\[ \nabla_X T(U) = X(\langle T, U \rangle) - T(\nabla_X U) = \mathcal{L}_X(\langle T, U \rangle) - T(\nabla_X U) \] (62)

Thus, the tensor \( \nabla_X T \), reinterpreted as a tensor distribution, acts on the test tensor \( U \) as
\[ \hat{\nabla}_X T[U] = \int_M \nabla_X T(U) \omega \]
\[ = \int_M \mathcal{L}_X(\langle T, U \rangle) \omega - \hat{T}[\nabla_X U] \]
\[ = \int_M \mathcal{L}_X(\langle T, U \rangle) \omega - \int_M \langle T, U \rangle \mathcal{L}_X \omega - \hat{T}[\nabla_X U] \]
\[ = \int_M d(\langle T, U \rangle i_X \omega) - \int_M \langle T, U \rangle \text{div}(X) \omega - \hat{T}[\nabla_X U] \]
\[ = \int_{\partial M} \langle T, U \rangle i_X \omega - \hat{T}[\text{div}(X) U + \nabla_X U] \] (63)

where the surface term vanishes since \( U \) is a test tensor. This motivates the definition of the covariant derivative of any tensor distribution \( T \) as
\[ \nabla_X T[U] = -T[\text{div}(X) U + \nabla_X U] \] (64)

This expression agrees with the usual component definition [30] of \( \nabla_X T \) using partial differentiation as given by (29), which makes it clear that \( \nabla_X \) is indeed a tensor derivation.

It is important to note that \( \text{div}(X) \) in (64) refers to the divergence defined in (19), and not to the divergence \( \nabla \cdot X \) defined by the connection \( \nabla \). As already noted, if \( \nabla \) leaves \( \omega \) invariant, these two notions of divergence agree; (64) generalizes the results of (30) to the case where they differ. (This generalization is obtained in the component definition of the covariant derivative by interpreting the ordinary derivatives in the sense of equation (29) above, so that they depend on the choice of \( \omega \).)
IV. MANIFOLDS WITH OTHER STRUCTURE

A. Manifolds without Volume Elements

Even without a volume element, there is an action of $k$-forms on $(n-k)$-forms, given by

$$\tilde{\alpha}[\beta] = \int_M \alpha \wedge \beta$$  \hfill (65)

which defines the current $\tilde{\alpha}$ of degree $k$. Currents of degree $k$ can be uniquely associated with totally antisymmetric contravariant tensor distributions of degree $n-k$. We can thus define the Heaviside current $\Theta^N$ with support on a submanifold $N$ by

$$\Theta^N[\beta] = \int_N \beta$$  \hfill (66)

for test $n$-forms $\beta$. Similarly, we define the Dirac current $\delta^\Sigma$ with support on a hypersurface $\Sigma$ via

$$\delta^\Sigma[\gamma] = \int_\Sigma \gamma$$  \hfill (67)

for test $(n-1)$-forms $\gamma$.

Since test tensors vanish on the (topological) boundary of $M$, integration by parts yields

$$\tilde{d}\alpha[\gamma] = \int_M d\alpha \wedge \gamma = \int_M d(\alpha \wedge \gamma) - (-1)^k \alpha \wedge d\gamma$$

$$= 0 - (-1)^k \tilde{\alpha}[d\gamma]$$  \hfill (68)

for $k$-forms $\alpha$ and $(n-k-1)$-forms $\gamma$. We use this formula to define exterior differentiation of currents. For the Heaviside current, we obtain

$$d\Theta^N[\gamma] = -\Theta^N[d\gamma] = -\int_N d\gamma$$

$$= -\int_{\partial N} \gamma = -\delta^{\partial N}[\gamma]$$  \hfill (69)

and we thus again have a relation of the form

$$d\Theta^N = -\delta^{\partial N}$$  \hfill (70)

B. Manifolds with Metrics

If we now assume that a nondegenerate $C^0$ metric tensor $g$ is specified on $M$, then there is a natural volume element $\omega = *1$ on $M$, where $*$ denotes the Hodge dual determined by $g$. This in turn determines the Heaviside 0-form distribution $\Theta_0$ and the Dirac 1-form distribution $\delta_1$, which can now be expressed in terms of the Heaviside current $\Theta$ and the Dirac current $\delta$ as

$$\Theta_0[f] = \Theta[*f]$$  \hfill (71)

$$\delta_1[V] = \delta[*V^\flat]$$  \hfill (72)

where $V^\flat$ denotes the 1-form which is the metric dual of $V$ and we have used the identity

$$*V^\flat = i_V *1 = i_V \omega$$  \hfill (73)

So long as $\Sigma$ is not null, there is also a natural induced volume element $\sigma$ on $\Sigma$, obtained as above as the pullback of the Leray form $\sigma$ satisfying $\omega = m \wedge \sigma$. Here $m$ is the unique 1-form normal to $\Sigma$ having unit norm and satisfying $m(Y) > 0$ for appropriately oriented vector fields $Y$. Normal coordinates can be used to ensure that $m$ takes the form
\[ m = d\lambda. \] Equivalently, \( \sigma = \pi_1 \), where \( \pi \) is the Hodge dual determined by the pullback \( h \) of \( g \) to \( \Sigma \); \( h \) is nondegenerate since \( \Sigma \) is not null. This enables us to determine the Dirac 0-form distribution \( \delta_0 \), which can now be written

\[ \delta_0[f] = \delta[\pi f] \] (74)

Thus, in this case there are preferred forms for the basic distributions, arising from the preferred volume elements.

Even if \( \Sigma \) is null, the volume element \( \omega \) is nondegenerate, and we can still define the distributions \( \delta_0, \delta'_0, \) and \( \delta'_1 \) provided we are willing to choose a function \( \lambda \) such that \( \Sigma = \{ \lambda = 0 \} \) as before, or equivalently provided we are willing to specify the “induced” volume element on \( \Sigma \). The scale freedom in this choice reflects the lack of unit normal vector to a null surface, or equivalently the nonexistence of normal coordinates.

As these formulas hint, the entire theory of \( p \)-form distributions can in fact be elegantly rewritten in terms of yet another action of \( p \)-forms, this time on \( p \)-forms, namely

\[ \hat{\alpha} [\beta] = \int_M \alpha \wedge * \beta \] (75)

Using the identity

\[ i_X \alpha = * (\ast^{-1} \alpha \wedge X^\flat) = (-1)^{p+1} \ast^{-1} (X^\flat \wedge \ast \alpha) \] (76)

it is straightforward to show that

\[ (p!) \ast (\alpha(X_1, ..., X_p)) \equiv \alpha \wedge * \left( X_1^\flat \wedge ... \wedge X_p^\flat \right) \] (77)

Taking \( \ast^{-1} \) of both sides of this formula results in an expression for contractions in terms of \( \ast \), which is remarkable since it shows that the resulting right-hand-side is thus independent of the choice of metric and corresponding Hodge dual operator appearing in it. This formula immediately yields for any \( p \)-form \( \alpha \)

\[ (p!) \hat{\alpha}[U] = \hat{\alpha}[U^\flat] \] (78)

where as usual

\[ (X_1 \otimes ... \otimes X_p)^\flat = X_1^\flat \otimes ... \otimes X_p^\flat \] (79)

and where the action (75) has been extended by antisymmetrization of the argument to an action on covariant tensors of type \((0, p)\). Explicitly,

\[ \hat{\alpha} [X_1^\flat \otimes ... \otimes X_p^\flat] = \hat{\alpha} [X_1^\flat \wedge ... \wedge X_p^\flat] \] (80)

so that (75) is equivalent to the original action (8) defined by contraction. This equivalence allows us to interpret \( p \)-form distributions \( \alpha \) as \((n-p)\)-currents, also denoted \( \alpha \), via

\[ \alpha[\beta] := (p!) \alpha[\beta^\flat] \] (81)

where \( \flat \) is the inverse of (79).

Since for \( p \)-forms \( \alpha \) and \((p+1)\)-forms \( \beta \)

\[ d(\alpha \wedge * \beta) = d\alpha \wedge * \beta + (-1)^{p+1} \alpha \wedge d* \beta \]
\[ = d\alpha \wedge * \beta - (-1)^{p} \alpha \wedge * (\ast^{-1}d* \beta) \] (82)

we see that the adjoint of \( d \) with respect to the action (7) is given by

\[ \hat{\omega} [\beta] = \hat{\alpha} [\delta \beta] \] (83)

where the operator \( \delta \) (no relation to the tensor distribution \( \delta \)) is defined on \( p \)-forms by

\[ \delta = (-1)^{p} \ast^{-1} d* \] (84)

The action (83) leads to an alternative notion of exterior differentiation of \( p \)-form distributions, namely
It now follows immediately that

$$\overline{d}^2 \alpha[\beta] = \overline{d} \alpha[\delta \beta] = \alpha[\delta^2 \beta] = 0$$  \hspace{1cm} (86)$$

since \( \delta^2 = 0 \) is a consequence of \( d^2 = 0 \) and \( ** = \pm 1 \). We next note the identity

$$\alpha[i \beta] = (X^\flat \wedge \alpha)[\beta]$$  \hspace{1cm} (87)$$

which follows from (76).

We further compute

$$\overline{d}(f \alpha)[\beta] = f \alpha[\delta \beta]$$  \hspace{1cm} = \hspace{1cm} \alpha[f \delta \beta]$$  \hspace{1cm} = \hspace{1cm} \alpha[\delta(f \beta) + id f \beta]$$  \hspace{1cm} = \hspace{1cm} \overline{d} \alpha[f \beta] + (df \wedge \alpha)[\beta]$$  \hspace{1cm} (88)$$

where \( f \) is the inverse of \( \beta \) and where we have used the identity

$$\delta(f \beta) = f \delta \beta - id f \beta$$  \hspace{1cm} (89)$$

which again follows from (76). In other words

$$\overline{d}(f \alpha) = df \wedge \alpha + f \overline{d} \alpha$$  \hspace{1cm} (90)$$

It only remains to relate the two notions of exterior derivative \( d \) and \( \overline{d} \). To avoid confusion, we think of \( \overline{d} \) as acting on \( p \)-form distributions via the equivalence (81), i.e.

$$\overline{d} \alpha[U] = \alpha[(\delta U^\flat)^2]$$  \hspace{1cm} (91)$$

We claim that

$$\overline{d} \equiv d$$  \hspace{1cm} (92)$$

This is obvious for ordinary distributions \( F \), for which

$$\overline{d} F[X] = F[(\delta X^\flat)^2] = F[\delta X^\flat] = -F[\ast^{-1}d\ast X^\flat] = -F[\text{div}(X)] = dF[X]$$  \hspace{1cm} (93)$$

But since \( d \) and \( \overline{d} \) agree on ordinary distributions and since both have the usual properties of exterior differentiation, they must be identical, as claimed. \[ \]

**V. REGULARLY DISCONTINUOUS TENSORS**

In many applications, tensors which are smooth on either side of a given hypersurface play an important role. This notion is made precise by the notion of *regularly discontinuous* tensors.

---

6 Repeated use of this equation can in fact be used to *define* the exterior product of differential forms with \( p \)-forms.

7 This makes sense, since they agree on tensor distributions arising from locally integrable tensors, and in particular from test tensors, by virtue of (81), and since such tensor distributions are dense in the space of all tensor distributions (81). This argument could perhaps be used to *prove* \( d^2 = 0 \) and the product rule from (86) and (90). If so, then even in the absence of a metric one could introduce a local “auxiliary metric” whose metric volume element agrees with the given volume element. For instance, choose any 1-forms \( \{ \omega^i \} \) such that \( \omega = \omega^1 \wedge ... \wedge \omega^n \) and define \( \omega^i \) to be orthonormal, so that

$$g = \omega^1 \otimes \omega^1 + ... + \omega^n \otimes \omega^n$$

The above proofs would then hold locally, and would be independent of the metric chosen.
We assume that we are given a hypersurface $\Sigma$ which partitions $M$ into two disjoint open regions $M^\pm$. A tensor field $T$ is said to be regularly discontinuous across $\Sigma$ if $T$ is continuous on $M^\pm$ and $T$ converges uniformly to tensors $T^\pm_\Sigma$ in the limit to $\Sigma$ from $M^\pm$. Note that $T$ need only be defined on $M^\pm$. A tensor $T$ is thus regularly discontinuous if and only if its components in any coordinate chart are regularly discontinuous functions. The discontinuity $[T]$ of a regularly discontinuous tensor $T$ is the (ordinary) continuous tensor on $\Sigma$ given by

$$[T] = T^+_\Sigma - T^-_\Sigma$$

(94)

More generally, we shall call $T$ piecewise $C^k$ if the components of $T$ in any coordinate chart as well as their first $k$ derivatives are all regularly discontinuous functions. This agrees with the notion of regularly $C^k$ discontinuous tensors in $[5]$.

### A. Heaviside and Dirac Distributions

We now further assume that a volume element $\omega$ is given on $M$. Define the Heaviside 0-form distributions $\Theta^\pm_0$ by

$$\Theta^\pm_0[f] = \Theta^M_0[f] = \int_{M^\pm} f \omega$$

(95)

and the Dirac 1-form distribution $\delta_1$ by

$$\delta_1[V] = \delta^\Sigma_1[V] = \int_\Sigma i_V \omega$$

(96)

where we have given $\Sigma$ the orientation of $\partial M^-$. Thus,

$$d\Theta^\pm_0 = d\Theta^M_0 = -\delta_1 \Theta^\mp_0 = \pm \delta_X^\Sigma = \pm \delta_1$$

(97)

Since any regularly discontinuous tensor $T$ is locally integrable, the tensor distribution $\hat{T}$ is defined by (9), which is equivalent to

$$\hat{T} = T^- \Theta^-_0 + T^+ \Theta^+_0$$

(98)

where $T^\pm = T|_{M^\pm}$. Furthermore, since $[T]$ is locally integrable on $\Sigma$, products such as $[T] \otimes \delta_1$ are defined.

### B. Differentiation of Regularly Discontinuous Tensors and the Associated Distributions

Using (98) it is straightforward to compute the partial and exterior derivatives of regularly discontinuous tensors. For instance, if $f$ is a regularly discontinuous function, then

$$\hat{f} = f^- \Theta^-_0 + f^+ \Theta^+_0$$

(99)

and if $f$ is piecewise $C^1$ we have, using (100),

$$d\hat{f} = df^- \Theta^-_0 + df^+ \Theta^+_0 + [f] \delta_1$$

(100)

Extending the action of the exterior derivative to piecewise $C^1$ differential forms $\alpha$ by defining $d\alpha$ to be the piecewise $C^0$ form satisfying

$$(d\alpha)^\pm = (d\alpha)|_{M^\pm} = d(\alpha|_{M^\pm}) = d(\alpha^\pm)$$

(101)

allows this to be rewritten as

$$d\hat{f} = \hat{df} + [f] \delta_1$$

(102)

We turn now to covariant differentiation. Assuming that there is a smooth connection on $M$, we can again use (98) to determine the covariant derivatives of piecewise smooth tensors. In analogy with (101), we extend the covariant derivative to piecewise $C^1$ tensors $T$ by defining the piecewise $C^0$ tensor $\nabla_x T$ via
\[(\nabla_x T)^\pm = (\nabla_x T)|_{M^\pm} = \nabla_x (T|_{M^\pm}) =: \nabla_x T^\pm \quad \text{(103)}\]

It is now straightforward to compute
\[
\nabla_x (\Theta^\pm T^\pm)[U] = -(\Theta^\pm T^\pm) \left[ \text{div}(X)U + \nabla_x^\pm U \right] \\
= -\int_{M^\pm} \langle T^\pm, \text{div}(X)U + \nabla_x^\pm U \rangle \omega \quad \text{(104)}
\]

and
\[
(\Theta^\pm \nabla_x T^\pm)[U] = \int_{M^\pm} \langle \nabla_x^\pm T^\pm, U \rangle \omega \\
= -\int_{M^\pm} \langle T^\pm, \text{div}(X)U + \nabla_x^\pm U \rangle \omega \mp \int_{\Sigma} \langle T^\pm, U \rangle i_x \omega \\
= \nabla_x (\Theta^\pm T^\pm)[U] \mp \delta_1 \langle \langle T^\pm, U \rangle X \rangle \quad \text{(105)}
\]

which verifies the product rule
\[
\nabla_x (\Theta^\pm T^\pm) = \Theta^\pm \nabla_x T^\pm + \nabla_x \Theta^\pm \otimes T^\pm \quad \text{(106)}
\]

For piecewise $C^1$ tensors $T$, $\nabla_x T$ is regularly discontinuous, and we can rewrite this as
\[
\overline{\nabla_x T}[U] = \nabla_x \hat{T}[U] - i_x \delta_1 [\langle \langle T, U \rangle \rangle] \quad \text{(107)}
\]

which finally yields the expected formula for differentiating tensor distributions associated with regularly discontinuous tensors, namely
\[
\nabla_x \hat{T} = \overline{\nabla_x T} + i_x \delta_1 \otimes \langle T \rangle \quad \text{(108)}
\]

C. Regularly Discontinuous Connections

A regularly discontinuous connection is a connection whose components are regularly discontinuous (piecewise $C^0$). The definition of the covariant derivative on tensor distributions can be generalized to such connections, provided its action is restricted to a subclass of the tensor distributions. We investigate here the consequences of taking this approach.

We would like to define the action on tensor distributions of the covariant derivative corresponding to a regularly discontinuous connection. There are at least two ways to do this: 1) Use the component definition. This succeeds for all distributions such that the products of the connection symbols with the component distributions are distributions. While this does not work for all distributions, the class for which it does work includes all distributions associated with locally integrable tensors. 2) Use (64). This amounts to repeating the derivation in the previous subsection for this case, and is outlined below. In either approach, one recovers (108). Alternatively, (108) itself could have been used to define covariant differentiation of tensor distributions associated with piecewise $C^1$ tensor fields.

Given (continuous) connections $\nabla^\pm$ on $M^\pm$, we can construct the (piecewise continuous) connection $\nabla$ on $M$ whose restrictions to $M^\pm$ are just $\nabla^\pm$. Given a piecewise $C^1$ tensor $T$ on $M$, (103) again defines a piecewise $C^0$ tensor $\nabla_x T$, which can be expressed directly in terms of $\nabla^\pm$ as
\[
(\nabla_x T)|_{M^\pm} = \nabla_x^\pm T^\pm \quad \text{(109)}
\]

Using this definition of $\nabla_x U$, we can use (64) to define the covariant derivative $\nabla_x T$ of some tensor distributions $T$. Specifically, we can do so for $T$ such that the map $U \to T|_{\nabla_x U}$ is a distribution. Again, this includes all distributions $T$ associated with locally integrable tensors. If the connection is regularly discontinuous, $\nabla_x^\pm T$ and $\nabla_x^\pm U$ are locally integrable, and the derivation of (108) in the preceding subsection can be used without modification.
VI. TENSOR DISTRIBUTIONS WITH DEGENERATE METRICS

We consider both metrics $g$ which are discontinuous at a hypersurface $\Sigma = \{ \lambda = 0 \}$, and those which are continuous but have vanishing determinant at $\Sigma$. We assume in both cases that $g$ is piecewise smooth, and that the pullback $h$ of $g$ to $\Sigma$ is the same from both sides and nondegenerate. In particular, we assume that $\Sigma$ is not null. In both cases, $g$ takes the form

$$g = N \, d\lambda \otimes d\lambda + h_\lambda$$

(110)

where $h_0 = h$ and with $N$ being discontinuous (and nonzero) at $\Sigma$ in one case and zero there in the other.

One way of treating the Levi-Civita connection determined by such a metric would be to think of it as being singular on $\Sigma$, and perhaps to give it the status of a distribution. Instead, we wish to investigate the consequences of thinking of it as being a regularly discontinuous function (in the case of discontinuous metrics) or as the inverse of a smooth function having a zero (in the continuous metric case). This allows the covariant derivative on tensor distributions previously defined to be used.

A. Discontinuous Metrics

For discontinuous metrics of the form (110), one still has the induced metric $h$ on $\Sigma$. It thus seems natural to define the induced volume element on $\Sigma$ to be $\sigma = \overline{\epsilon}$ as before, which leads to $\delta_0$ being well-defined. Furthermore, the metric volume element on $M$

$$\omega = \sqrt{|N|} \, d\lambda \wedge \overline{\epsilon}$$

(111)

which is discontinuous except when $N$ changes sign but not magnitude. It is remarkable that this special case of a discontinuous signature-changing metric leads to a natural, continuous choice of volume element everywhere.

For discontinuous metrics which do not change signature, there are two different 1-sided volume elements $\omega^\pm$ and 1-sided unit normal 1-forms $\mu^\pm$, satisfying

$$\omega^\pm = \mu^\pm \wedge \sigma$$

(112)

One must now either choose a continuous, and hence non-metric, volume element $\omega$ (e.g. by extending the metric volume element on one side), in which case the volume elements $\sigma^\pm$ induced on $\Sigma$ by $\omega = \mu^\pm \wedge \sigma^\pm$ will differ, or accept the existence of separate metric volume elements $\omega^\pm$ on $M^\pm$, which leads to two different distributions $\delta^\pm_1 = \pm d\Theta^\pm_0$.

Since the discontinuous metric $g$ admits nondegenerate 1-sided limits to $\Sigma$, one can define the Levi-Civita connections $\nabla^\pm$ on the manifolds-with-boundary $M^\pm \cup \Sigma$. In particular, $\nabla^\pm$ are well-defined at $\Sigma$, although they will not in general agree there. Thus, the connection defined by (109) is regularly discontinuous. As discussed above, provided a smooth volume element has been chosen, partial and exterior differentiation of tensor distributions can be defined. Furthermore, covariant differentiation of distributions associated with regularly discontinuous tensors can be defined by the usual component formula or invariantly by (64). Should the connection in fact be smooth, of course, covariant differentiation of all tensor distributions can be defined. We emphasize that there is a natural choice of smooth volume element for discontinuous, signature-changing metrics, namely (111).

B. Continuous Metrics

If however $N|_{\Sigma} = 0$, then the metric volume element (111) is continuous, but zero on $\Sigma$. Furthermore, there is no notion of unit normal vector to $\Sigma$. One way around this problem is again to work with a non-metric volume element, the most obvious one being

$$\omega = d\lambda \wedge \sigma$$

(113)

$^8$If we further assume that $dN|_{\Sigma} \neq 0$, then we have a typical continuous signature-changing metric. In this case, Kossowski and Kriele have shown that $N$ can always be chosen to be $\lambda$ near $\Sigma$.
There is no difficulty defining tensor distributions with respect to this volume element, using $\sigma$ as the natural volume element on $\Sigma$.

As we have shown, given any smooth volume element on $M$, we can define the basic distributions, specifically the Heaviside and Dirac distributions, as well as partial and exterior differentiation of distributions. Consider now covariant differentiation. The Levi-Civita connection $\nabla^\pm$ on $M^\pm$ for the metric (110) is easily computed. Note, however, that since

$$d\lambda(\nabla^\pm_\partial\lambda) = \frac{1}{2N} \frac{\partial N}{\partial \lambda}$$

which is singular on $\Sigma$, $\nabla^\pm$ do not lead to a regularly discontinuous connection on $M$. But this means that it is not possible to define covariant differentiation of distributions using (64).

As we have already pointed out, one can alternatively define covariant differentiation of tensor distributions in terms of Christoffel symbols and components. For smooth connections, of course, both approaches agree. For continuous degenerate metrics, this approach requires giving meaning as a distribution to the product of the Christoffel symbols with an arbitrary distribution. This in turn is essentially the same as defining the distribution $\frac{1}{\lambda}D$ for any distribution $D$. This can be done in many ways, as there are many distributions $E$ satisfying $\lambda E = D$: choosing one is equivalent to choosing a preferred test function [2]. It is far from obvious to what extent the resulting theory of tensor distributions, and in particular of covariant differentiation thereof, depends on this choice.

VII. EXAMPLE WITH SIGNATURE CHANGE

A typical discontinuous signature-changing metric on $\mathbb{R}^2$ is

$$g_1 = \text{sgn}(\tau) d\tau \otimes d\tau + a(\tau)^2 dx \otimes dx$$

while a typical continuous signature-changing metric on $\mathbb{R}^2$ is

$$g_2 = t dt \otimes dt + a(t)^2 dx \otimes dx$$

where $a > 0$. These metrics can be identified away from $\Sigma = \{ t = 0 = \tau \}$ via the transformation

$$\tau = \int_0^t \sqrt{|t|} dt$$

but this fails to be a coordinate transformation at $t = 0$, so that $t$ and $\tau$ correspond to different differentiable structures on $\mathbb{R}^2$. It is instructive to note that in the “$t$ differentiable structure”, in which $t$ is an admissible coordinate, $d\tau = \sqrt{|t|} dt$ is zero at $\Sigma$, whereas in the $\tau$ differentiable structure it is a basis 1-form and nowhere zero.

A. Heaviside and Dirac Distributions

The metric volume elements determined by these metrics away from $\Sigma$ are

$$\omega_1 = a d\tau \wedge dx$$
$$\omega_2 = a \sqrt{|t|} dt \wedge dx$$

which agree up to a coordinate transformation away from $\Sigma$. However, $\omega_2|_\Sigma = 0$, so that it is a degenerate volume element on $\mathbb{R}^2$. The pullback metric on $\Sigma$ is in both cases

$$h = a^2 dx \otimes dx$$

with volume element

$$\sigma = a dx$$

Even though the metric $g_1$ is discontinuous, not only the volume element $\omega_1$ but also the unit normal 1-form $d\tau$ is continuous, and there is no difficulty satisfying the relationship
\[ \omega_1 = d\tau \wedge \sigma \] everywhere. This leads to the distributions

\[ \Theta_{\pm}^\pm[f] = \int_{\pm\tau > 0} f \, d\tau \wedge dx \]  \hspace{1cm} \text{(123)}

\[ \delta_1[V] = \int_{\tau = 0} d\tau(V) \, a \, dx \]  \hspace{1cm} \text{(124)}

\[ \delta_0[f] = \int_{\tau = 0} f \, a \, dx \]  \hspace{1cm} \text{(125)}

which satisfy

\[ d\Theta_{\pm}^\pm = \delta_1 = \delta_0 \, d\tau \]  \hspace{1cm} \text{(126)}

For the continuous metric \(g_2\), things are not so simple. One possibility is to choose the non-metric but nowhere degenerate volume element

\[ \omega_2 = a \, dt \wedge dx \]  \hspace{1cm} \text{(127)}

which leads to the distributions

\[ \Theta_{\pm}^\pm[f] = \int_{\pm t > 0} f \, dt \wedge dx \]  \hspace{1cm} \text{(128)}

\[ \delta_1[V] = \int_{t = 0} dt(V) \, a \, dx \]  \hspace{1cm} \text{(129)}

\[ \delta_0[f] = \delta_0[f] = \int_{t = 0} f \, a \, dx \]  \hspace{1cm} \text{(130)}

These distributions satisfy

\[ d\Theta_{\pm}^\pm = \delta_1 = \delta_0 \, dt \]  \hspace{1cm} \text{(131)}

\section*{B. Differentiation of Distributions}

Once an appropriate volume element is chosen, the definition of partial and exterior differentiation of tensor distributions for signature-changing metrics follows immediately from the general results in Section III. Turning to covariant differentiation, the Christoffel symbols \(\Gamma_{\pm}^i\) for the Levi-Civita connections \(\nabla_{\pm}^i\) associated with the metrics \(g_i\) restricted to \(M_{\pm}\) take the form

\[ (\Gamma_{\pm}^2)^{tt} = \frac{1}{2t} \quad (\Gamma_{\pm}^1)^{\tau \tau} = 0 \]  \hspace{1cm} \text{(132)}

\[ (\Gamma_{\pm}^2)^{xt} = \frac{a'}{a} \quad (\Gamma_{\pm}^1)^{x \tau} = \frac{\dot{a}}{a} \]  \hspace{1cm} \text{(133)}

\[ (\Gamma_{\pm}^2)^{xx} = -\frac{aa'}{t} \quad (\Gamma_{\pm}^1)^{xx} = -\ddot{a} \]  \hspace{1cm} \text{(134)}

in a coordinate basis, where derivatives with respect to \(t\) (\(\tau\)) have been denoted with a prime (dot).

As already noted, connections which are well-behaved on \(\Sigma\) can be extended via (14) to a connection on (a restricted set of) tensor distributions. The connections \(\nabla_{\pm}^i\) associated with the discontinuous signature-changing metric (115) each admit a well-defined limit to \(\tau = 0\). Thus, for discontinuous signature-changing metrics, the Levi-Civita connection extends naturally to a definition of covariant differentiation of distributions associated with regularly discontinuous tensor fields. Furthermore, this implies that the distributional curvature (thought of as an operator on smooth tensors) can always be defined.

\[ \text{If a satisfies the additional restriction } \dot{a}(0) = 0, \text{ the connections on } M_{\pm} \text{ can be extended to a smooth connection on all of } M, \]  

and thus to all tensor distributions, in a natural way, and the distributional curvature corresponds to a (smooth) tensor. This is not, however, a necessary condition for the distributional part of the curvature tensor to correspond to a locally integrable tensor. The implications of this for surface layers in signature-changing spacetimes has been discussed elsewhere [8].
The connections $\nabla_T^\pm$ associated with the continuous signature-changing metric (116) are singular at $t = 0$, which means that (64) cannot be used to extend them to tensor distributions. As discussed above, however, they can be so extended using the component approach, provided one chooses a definition of $\nabla D$. Having done that, the Levi-Civita connection can be extended to all distributions. To what extent this definition depends on the choice made, and whether the resulting notion of distributional curvature does so, are open questions.

C. Massless Klein-Gordon Equation

Dray et al. [9] postulate the massless Klein-Gordon equation for (continuous) signature-changing metrics in the form

$$dF = 0 \quad (135)$$

where

$$F = *d\Phi^- \Theta^-_0 + *d\Phi^+ \Theta^+_0 \quad (136)$$

and where a regularity condition is assumed on $*d\Phi^\pm$, namely that their 1-sided limits to $\Sigma$ exist. In other words, “$*d\Phi$” is regularly discontinuous, where care must be taken to distinguish the different Hodge dual operators on $M^\pm$, both denoted by $*$. Using (60), the field equation (135) implies

$$\Theta^\pm_0 d*\Phi^\pm = 0 = \delta_1 \wedge [*[d\Phi]] \quad (137)$$

which implies

$$d*\Phi^\pm = 0 \quad (138)$$

$$\Sigma^*[d\Phi] = 0 \quad (139)$$

where $\Sigma^*$ denotes the pullback to $\Sigma$. While the Heaviside distributions $\Theta^\pm_0$, and thus the distributional field $F$, certainly depend on the choice of (nondegenerate) volume element used to define tensor distributions, the resulting field equations (138) and (139) do not. Furthermore, since only limits of fields to $\Sigma$ occur in the crucial equation (139), and since the continuous and discontinuous approaches are smoothly related away from $\Sigma$, one obtains the same equations starting from a discontinuous metric.

VIII. DISCUSSION

We began by constructing the Heaviside and Dirac tensor distributions corresponding to an arbitrary volume element, and then showed in detail how to differentiate tensor distributions. We then applied our formalism to manifolds with degenerate metrics, where the usual, metric-based definitions fail. In particular, we have shown that partial and exterior differentiation can always be extended to tensor distributions; our extension depends only on the volume element, and does not require a metric.

Our results concerning covariant differentiation are more subtle. There are situations in which the requirement that the connection be continuous is too strong. In particular, this would lead to the curvature tensor being at worst discontinuous, thus eliminating the possibility of distributional curvature. We have therefore investigated the circumstances under which this restriction can be weakened.

We can summarize our results in more informal language as follows. Since the component expansion of any discontinuous tensor on a smooth manifold consists of discontinuous components multiplying smooth basis tensors, the usual rules for a covariant derivative operator of course imply that

$$\nabla_X(fT) = X(f)T + f\nabla_XT \quad (140)$$

Thus, for any distribution associated with a regularly discontinuous tensor field of the form

$$T = \Theta^- T^- + \Theta^+ T^+ \quad (141)$$

and using the action of vector fields implicit in (100) (i.e. obtained by contracting (100) with the vector field), we must have
\n
\[
\n\nabla X T = \Theta^{-} \nabla X T^{-} + \Theta^{+} \nabla X T^{+} + [T] \delta(X) \tag{142}
\]

With the convention that $\Theta^2 = \Theta$ and $\Theta^+ \Theta^- = 0$, we can formally rewrite this as

\[
\nabla = \Theta^{-} \nabla^{-} + \Theta^{+} \nabla^{+} \tag{143}
\]

and this formal prescription accurately reproduces our results provided we use (140) to differentiate the Heaviside functions. Note that it is only necessary here that $\nabla^\pm$ admit 1-sided limits to $\Sigma$. If the connection should be smooth, then $\nabla$ may be applied to any tensor distribution, whereas if it is not then undefined products such as $\delta \Theta$ would occur. As a special case, our results can be applied to the case of a continuous (nondegenerate) metric which is not differentiable, yielding the usual formalism for the distributional curvature in that case.

One unusual feature of the connection (143) is that even if $\nabla^\pm$ are metric compatible, $\nabla$ will not be. To see this, simply compute

\[
\nabla(\Theta^{-} g^{-} + \Theta^{+} g^{+}) = [g] \otimes \delta \tag{144}
\]

where the other two terms vanish due to the assumed metric-compatibility of $\nabla^\pm$. This result is unchanged even if $\nabla$ is assumed to be smooth. To avoid this, one might consider other extensions of $\nabla^\pm$, which would differ from (143) by the presence of a term proportional to $\delta$. However, such a connection could only be applied to smooth tensors and their associated tensor distributions; covariant differentiation of discontinuous tensors using such a connection could not be defined, as it would lead to unacceptable $\Theta \delta$ terms. Not only does this mean that the distributional curvature could not be defined, but the metric itself could not be differentiated, and there would be no way to establish that such a connection is metric compatible in the first place! If one instead directly modifies (142), e.g. by removing the last term, the argument leading up to (142) shows that such a connection would not reduce to the correct action on discontinuous functions. While we find the introduction of a connection which is not metric compatible disconcerting, we are forced to the conclusion that it is a necessary feature of covariant differentiation for discontinuous metrics. It is after all the unique extension of $\nabla^\pm$ for which the question of metric-compatibility can even be asked!

For continuous signature-changing metrics, the Levi-Civita connection can be extended in several ways to all tensor distributions, depending on the implicit choice of test function made when “dividing a distribution by zero”. It is not clear what role this choice plays, and in particular it is not clear what structure is independent of this choice. For discontinuous signature-changing metrics, it is remarkable that the Levi-Civita connection can be naturally extended to tensor distributions, albeit only to a restricted class and with the novel feature of metric incompatibility. While this distinction does not affect field equations such as the Klein-Gordon equation, which can be expressed without using a connection, other field equations on signature-changing manifolds, for instance Einstein’s equations, may turn out to be more naturally discussed using one of these approaches than the other. This issue is further discussed in [5], where a variational treatment of Einstein’s equations using the discontinuous approach is given.
ACKNOWLEDGMENTS

It is a pleasure to thank George Ellis, David Hartley, Charles Hellaby, Corinne Manogue, Jörg Schray, Robin Tucker, and Philip Tuckey for extensive helpful discussions. Further thanks are due the School of Physics & Chemistry at Lancaster University and the Department of Physics and Mathematical Physics at the University of Adelaide for kind hospitality. This work was partially supported by NSF Grant PHY-9208494, as well as a Fulbright Grant under the auspices of the Australian-American Educational Foundation.

[1] David Hartley, Robin W. Tucker, Philip Tuckey, and Tevian Dray, *Tensor Distributions on Signature-Changing Space-Times*, gr-qc/9701046, J. Math. Phys. (submitted).

[2] Yvonne Choquet-Bruhat, Cecile DeWitt-Morette, with Margaret Dillard-Bleick, *Analysis, Manifolds, and Physics* (revised edition), North-Holland, Amsterdam and New York, 1982, Section VI B

[3] A. Lichnerowicz, *Propagateurs, Commutateurs et Anticommutateurs en Relativité Générale*, Publ. math. IHES No. 10, Presses Universitaires de France, Paris, 1961.

[4] F. G. Friedlander, *The wave equation on a curved space-time*, Cambridge University Press, Cambridge, 1975.

[5] Yvonne Choquet-Bruhat, *Applications of Generalized Functions to Shocks and Discrete Models*, Proceedings of the International Symposium on *Generalized Functions and their Applications* (Varanasi, 1991), ed. R. S. Pathak, Plenum, New York, 1993.

[6] Georges de Rham, *Differentiable Manifolds*, Springer Verlag, Berlin and Heidelberg, 1984. (Originally published in French as: *Variétés Différentiables*, Actualités Scientifiques et Industrielles 1222, Hermann, Paris, 1955.)

[7] M. Kossowski and M. Kriele, *Smooth and Discontinuous Signature Type Change in General Relativity*, Class. Quant. Grav. 10, 2363 (1993).

[8] Tevian Dray, *Einstein’s Equations in the Presence of Signature Change*, J. Math. Phys. 37, 5627-5636 (1996).

[9] Tevian Dray, Corinne A. Manogue, and Robin W. Tucker, *The Scalar Field Equation in the Presence of Signature Change*, Phys. Rev. D48, 2587 (1993).