QUASILINEAR ELLIPTIC EQUATIONS WITH NATURAL GROWTH
AND QUASILINEAR ELLIPTIC EQUATIONS WITH SINGULAR
DRIFT

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Abstract. We prove an existence result for a quasilinear elliptic equation satisfying
natural growth conditions. As a consequence, we deduce an existence result for a quasi-
linear elliptic equation containing a singular drift. A key tool, in the proof, is the study
of an auxiliary variational inequality playing the role of “natural constraint”.

1. Introduction

Consider the quasilinear elliptic problem

\begin{equation}
\begin{aligned}
- \text{div} \left[ a(x, u, \nabla u) \right] + b(x, u, \nabla u) &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\end{equation}

where \( \Omega \) is a bounded and open subset of \( \mathbb{R}^n \) and

\( a : \Omega \times (\mathbb{R} \times \mathbb{R}^n) \to \mathbb{R}^n \), \quad b : \Omega \times (\mathbb{R} \times \mathbb{R}^n) \to \mathbb{R} \)

are two Carathéodory functions satisfying the natural growth conditions in the sense of [8].
More precisely, we assume that:

\[ \tag{N} \]
there exist \( 1 < p < \infty \) and, for every \( R > 0 \), \( a_R^{(0)} \in L^1(\Omega) \), \( a_R^{(1)} \in L^{p'}(\Omega) \), \( \beta_R > 0 \) and \( \nu_R > 0 \) such that

\[
|a(x, s, \xi)| \leq a_R^{(1)}(x) + \beta_R |\xi|^{p-1}, \quad |b(x, s, \xi)| \leq a_R^{(0)}(x) + \beta_R |\xi|^p,
\]

\[
a(x, s, \xi) \cdot \xi \geq \nu_R |\xi|^p - a_R^{(0)}(x),
\]

for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R}, \xi \in \mathbb{R}^n \) with \( |s| \leq R \); such a \( p \) is clearly unique;

\[ \tag{M} \]
we have

\[
\left[ a(x, s, \xi) - a(x, \hat{s}, \hat{\xi}) \right] \cdot (\xi - \hat{\xi}) > 0
\]

for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R}, \xi, \hat{\xi} \in \mathbb{R}^n \) with \( \xi \neq \hat{\xi} \).

Definition 1.1. We say that \( w \in W^{1,p}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \) is a supersolution (resp. subsolution) of the equation

\begin{equation}
- \text{div} \left[ a(x, u, \nabla u) \right] + b(x, u, \nabla u) = 0 \quad \text{in } \Omega,
\end{equation}

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if
\[
\int_{\Omega} [a(x, w, \nabla w) \cdot \nabla v + b(x, w, \nabla w)v] \, dx \geq 0 \quad \text{(resp.} \leq 0) \quad \text{for every } v \in C_c^\infty(\Omega) \text{ with } v \geq 0.
\]

Let us state our main result.

**Theorem 1.2.** Assume there exist \( u, \overline{u} \in W^{1,p}_\text{loc}(\Omega) \cap L^\infty(\Omega) \) and \( u_0 \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) such that \( u \) is a subsolution of (1.2), \( \overline{u} \) is a supersolution of (1.2) and \( u \leq u_0 \leq \overline{u} \) a.e. in \( \Omega \).

Then there exists \( u \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) satisfying \( u \leq u \leq \overline{u} \) a.e. in \( \Omega \) and (1.1) in a weak sense, namely
\[
\int_{\Omega} [a(x, u, \nabla u) \cdot \nabla v + b(x, u, \nabla u)v] \, dx = 0 \quad \text{for every } v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega).
\]

**Remark 1.3.** The previous result should be compared with [9, Theorem] and [4, Théorème 2.1]. The main feature is that in [9] a growth condition of the form
\[
|b(x, s, \xi)| \leq \alpha_R^{(0)}(x) + \beta_R |\xi|^{p-\varepsilon}
\]
with \( \alpha_R^{(0)} \in L^1(\Omega) \) and \( \varepsilon > 0 \) is required, so that the natural growth of order \( p \) in \( \xi \) is not allowed.

On the other hand, in [4] it is assumed that
\[
|b(x, s, \xi)| \leq \beta_R (1 + |\xi|^p)
\]
and the term \( \alpha_R^{(0)} \in L^1(\Omega) \) is not permitted (see also the remarks in [3]).

Here we take advantage of the framework of [1, 6] to allow the condition
\[
|b(x, s, \xi)| \leq \alpha_R^{(0)}(x) + \beta_R |\xi|^p,
\]
which seems to be the most general to guarantee that \( b(x, u, \nabla u) \in L^1(\Omega) \) whenever \( u \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \).

Just this level of generality will allow us to treat, as a particular case, a problem with singular drift, as we will see in the next result.

**Remark 1.4.** A more general equation of the form
\[
-\text{div} [a(x, u, \nabla u)] + b(x, u, \nabla u) = f_0 - \text{div} f_1,
\]
with \( f_0 \in L^1(\Omega) \) and \( f_1 \in L^{p'}(\Omega; \mathbb{R}^n) \), can be easily reduced to our case by setting
\[
\tilde{a}(x, s, \xi) = a(x, s, \xi) - f_1(x),
\]
\[
\tilde{b}(x, s, \xi) = b(x, s, \xi) - f_0(x).
\]
Of course, the key point is the existence of bounded super/subsolutions.

**Corollary 1.5.** Assume that \( a \) satisfies the further condition:
\[
a(x, s, 0) = 0 \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}.
\]
Let \( 0 < q \leq p, r > 0 \) and let \( b_1 \in L^{\frac{p}{q} - p}(\Omega; \mathbb{R}^n) \), and \( b_0, f \in L^1(\Omega) \) be such that:
\[
(1.3) \quad \text{there exists } Q \geq 0 \text{ satisfying } |f(x)| \leq Q b_0(x) \text{ for a.e. } x \in \Omega.
\]

\(^1\)We mean \( b_1 \in L^\infty(\Omega; \mathbb{R}^n) \) if \( q = p \).
Then there exists \( u \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) satisfying

\[-Q^{1/r} \leq u \leq Q^{1/r} \quad \text{a.e. in } \Omega\]

and

\[
\int_\Omega [a(x, u, \nabla u) \cdot \nabla v + b_1 \cdot (|\nabla u|^{p-1} \nabla u) v + b_0|u|^{r-1} uv] \, dx = \int_\Omega vf \, dx
\]

for every \( v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \).

**Remark 1.6.** Assumption (1.3) is in particular satisfied if \( f \in L^\infty(\Omega) \) and \( b_0(x) \geq b > 0 \). The previous corollary extends the results of [3], devoted to cases in which the principal part of the equation is linear (see also, when the equation is fully linear, the paper [10]). The technique of [3] is based on a duality approach which seems not to be easily adaptable when the principal part of the equation is not linear.

Concerning equations where condition (1.3) is assumed, Theorem 1.2 allows us also to prove the next corollary, which slightly generalizes some results of [2] obtained by a different technique.

**Corollary 1.7.** Assume that \( a \) and \( b \) satisfy the further condition:

\[ a(x, s, 0) = 0, \quad b(x, s, 0) = 0 \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}. \]

Let \( b_0, f \in L^1(\Omega) \) be such that (1.3) holds with \( Q > 0 \) and let \( g : \mathbb{R} \to \mathbb{R} \) be a continuous function such that

\[
\lim_{s \to -\infty} g(s) = -\infty, \quad \lim_{s \to +\infty} g(s) = +\infty.
\]

Then there exists \( u \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) satisfying

\[
\int_\Omega [a(x, u, \nabla u) \cdot \nabla v + b(x, u, \nabla u)v + b_0g(u)v] \, dx = \int_\Omega vf \, dx
\]

for every \( v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) and \( A \leq u \leq \overline{A} \) a.e. in \( \Omega \), provided that \( A \leq 0 \leq \overline{A} \) and \( g(A) \leq -Q, \ g(\overline{A}) \geq Q. \)

The proof of Theorem 1.2 is based on the study of an auxiliary variational inequality, which plays the role of “natural constraint”, in the sense that the solutions of the variational inequality are automatically solutions of the equation. This kind of device appears many times in the literature and goes back, to our knowledge, to [7]. A variant can be found in [11, Theorem 3.3] and [12, Theorem 2.3] (see also [6, 13]).

2. Parametric quasilinear elliptic variational inequalities with natural growth conditions

Throughout this section, we still consider two Carathéodory functions \( a, b \) satisfying (N) and (M) and, moreover, a \( p \)-quasi upper semicontinuous function \( \underline{a} : \Omega \to \overline{\mathbb{R}} \) and a \( p \)-quasi lower semicontinuous function \( \overline{a} : \Omega \to \overline{\mathbb{R}}. \) It is well known that every \( u \in W^{1,p}_0(\Omega) \) admits a Borel and \( p \)-quasi continuous representative \( \hat{u} \), defined up to a set of null \( p \)-capacity, which we still denote by \( u \) (see e.g. [5]).
For every $t \in [0, 1]$, we set
\[
\begin{align*}
\underline{u}_t &= u - t, \\
\overline{u}_t &= \overline{u} + t,
\end{align*}
\]
and
\[
K_t = \{ u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) : \underline{u} \leq u \leq \overline{u}_t \text{ p.q.e. in } \Omega \}.
\]
We aim to consider the solutions $u$ of the parametric variational inequality
\[
(VI_t)
\]
\[
\begin{cases}
u \in K_t, \\
\int_{\Omega} [a(x,u,\nabla u) \cdot \nabla (v - u) + b(x,u,\nabla u) (v - u)] \, dx \geq 0
\end{cases}
\]
for every $v \in K_t$.

**Theorem 2.1.** Assume that $\underline{u}$, $\overline{u}$ are bounded and that there exists $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\underline{u} \leq u_0 \leq \overline{u}$ p.q.e. in $\Omega$.

Then the following facts hold:

(a) for every $t \in [0, 1]$, there exists a solution $u$ of $(VI_t)$;

(b) the set
\[
\{ (u,t) \in (W_0^{1,p}(\Omega) \cap L^\infty(\Omega)) \times [0,1] : u \text{ is a solution of } (VI_t) \}
\]
is strongly compact in $W_0^{1,p}(\Omega) \times [0,1]$.

**Proof.** We aim to apply the results of [6]. Let us denote by $Z_t^{\text{tot}}$ the set of solutions of $(VI_t)$. Since $K_t \neq \emptyset$, from [6, Theorem 5.10] we infer that $\text{ind}(Z_t^{\text{tot}}) = 1$. Then [6, Theorem 5.6] implies that $Z_t^{\text{tot}} \neq \emptyset$ and assertion (a) follows.

To prove assertion (b), we want to apply (a) of [6, Theorem 5.8]. The only assumption we need to check is the continuity of $t \mapsto K_t$ with respect to the Mosco convergence $(MC)$.

Let $t_k \to t$ in $[0,1]$ and $(v_k)$ be a sequence weakly convergent to $v$ in $W_0^{1,p}(\Omega)$ with $v_k \in K_{t_k}$. Up to a subsequence, we may assume that $(t_k)$ is monotone.

If $(t_k)$ is increasing, we have $v_{t_k} \in K_t$ for every $k \in \mathbb{N}$. Since $K_t$ is weakly closed, it follows that $v \in K_t$. If $(t_k)$ is decreasing, for every fixed $h \in \mathbb{N}$ it is $v_k \in K_{t_h}$ for every $k \geq h$, whence $v \in K_{t_h}$, namely
\[
\underline{u} - t_h \leq v \leq \overline{u} + t_h \quad \text{p.q.e. in } \Omega.
\]
Since $t_h$ is converging to $t$, it follows that
\[
\underline{u} - t \leq v \leq \overline{u} + t \quad \text{p.q.e. in } \Omega,
\]
namely $v \in K_t$.

Let now $t_k \to t$ in $[0,1]$ and $v \in K_t$. As before, up to a subsequence we may assume that $(t_k)$ is monotone.

If $(t_k)$ is decreasing, we set $v_k = v$ for every $k \in \mathbb{N}$ and of course $(v_k)$ is strongly convergent to $v$ in $W_0^{1,p}(\Omega)$ with $v_k \in K_{t_k}$. If $(t_k)$ is increasing, we set
\[
v_k = u_0 + (v - u_0 - t + t_k)^+ - (v - u_0 + t - t_k)^-.
\]
Since it is easily seen that $(v_k)$ converges to $v$ strongly in $W_0^{1,p}(\Omega)$, we have only to check that $v_k \in K_{t_k}$ eventually as $k \to \infty$. If
\[
v(x) - u_0(x) - t + t_k > 0,
\]
we have
\[ v(x) - u_0(x) + t - t_k \geq v(x) - u_0(x) - t + t_k > 0, \]
whence
\[ v_k(x) = v(x) - t + t_k \geq u_0(x). \]
It follows
\[ u(x) - t_k \leq u_0(x) \leq v_k(x) = v(x) - t + t_k \leq \overline{u}(x) + t_k. \]
If
\[ v(x) - u_0(x) + t - t_k < 0, \]
we infer in a similar way that
\[ u(x) - t_k \leq v_k(x) \leq u_0(x) \leq v_k(x) \leq \overline{u}(x) + t_k. \]
Otherwise \( v_k(x) = u_0(x) \), which yields the same conclusion. Therefore \( v_k \in K_{t_k} \) and the proof is complete.

\[ \square \]

3. Solutions of equations versus solutions of variational inequalities

Throughout this section, \( \hat{\Omega} \) will denote an open subset of \( \mathbb{R}^n \) and
\[ \hat{a} : \hat{\Omega} \times (\mathbb{R} \times \mathbb{R}^n) \to \mathbb{R}^n, \quad \hat{b} : \hat{\Omega} \times (\mathbb{R} \times \mathbb{R}^n) \to \mathbb{R} \]
two Carathéodory functions such that:

(i) there exist \( 1 < p < \infty \) and, for every compact subset \( C \) of \( \hat{\Omega} \) and every \( R > 0 \),
\[ \alpha_{C,R}^{(0)} \in L^1(C), \alpha_{C,R}^{(1)} \in L^p(C) \text{ and } \beta_{C,R} \geq 0 \text{ such that} \]
\[ |\hat{a}(x, s, \xi)| \leq \alpha_{C,R}^{(1)}(x) + \beta_{C,R} |\xi|^{-1}, \]
\[ |\hat{b}(x, s, \xi)| \leq \alpha_{C,R}^{(0)}(x) + \beta_{C,R} |\xi|^p; \]
for a.e. \( x \in C \) and every \( s \in \mathbb{R}, \xi \in \mathbb{R}^n \) with \( |s| \leq R; \)
(ii) we have
\[ \left[ \hat{a}(x, s, \xi) - \hat{a}(x, s, \hat{\xi}) \right] \cdot (\xi - \hat{\xi}) \geq 0 \]
for a.e. \( x \in \hat{\Omega} \) and every \( s \in \mathbb{R}, \xi, \hat{\xi} \in \mathbb{R}^n. \)

We also denote by \( L_c^\infty(\hat{\Omega}) \) the set of \( v \)'s in \( L^\infty(\hat{\Omega}) \) vanishing a.e. outside some compact subset of \( \hat{\Omega}. \)

Definition 3.1. We say that \( w \in W^{1,p}_{\text{loc}}(\hat{\Omega}) \cap L^\infty_{\text{loc}}(\hat{\Omega}) \) is a supersolution (resp. subsolution) of the equation
\[ (3.1) \quad - \text{div } [\hat{a}(x, u, \nabla u)] + \hat{b}(x, u, \nabla u) = 0 \quad \text{in } \hat{\Omega}, \]
if
\[ \int_{\hat{\Omega}} \left[ \hat{a}(x, w, \nabla w) \cdot \nabla v + \hat{b}(x, w, \nabla w) v \right] dx \geq 0 \quad \text{(resp. } \leq 0) \]
for every \( v \in C_c^\infty(\hat{\Omega}) \) with \( v \geq 0. \)
**Theorem 3.2.** Let \( \underline{u}, u, \overline{u} \in W^{1,p}_0(\hat{\Omega}) \cap L^\infty_c(\hat{\Omega}) \) be such that \( \underline{u} \) is a subsolution of (3.1), \( \overline{u} \) is a supersolution of (3.1) and
\[
\int_{\hat{\Omega}} \left[ \hat{a}(x, u, \nabla u) \cdot \nabla (v - u) + \hat{b}(x, u, \nabla u) (v - u) \right] \, dx \geq 0
\]
for every \( v \in \hat{K} \) with \( (v - u) \in L^\infty_c(\hat{\Omega}) \),
where
\[
\hat{K} = \left\{ v \in W^{1,p}_0(\hat{\Omega}) \cap L^\infty_c(\hat{\Omega}) : \underline{u} \leq v \leq \overline{u} \text{ a.e. in } \hat{\Omega} \right\}.
\]

Suppose also that:

(iii) for every compact subset \( C \) of \( \hat{\Omega} \), there exist \( r_C > 0 \) and \( \gamma_C \in L^p(C) \) such that
\[
|\hat{a}(x, s, \nabla u(x)) - \hat{a}(x, u(x), \nabla u(x))| \leq \gamma_C(x) (s - u(x)),
\]
\[
|\hat{a}(x, u(x), \nabla \overline{u}(x)) - \hat{a}(x, \sigma, \nabla \overline{u}(x))| \leq \gamma_C(x) (\overline{u}(x) - \sigma),
\]
for \( \text{a.e. } x \in C \) and every \( s, \sigma \in \mathbb{R} \) with \( \underline{u}(x) \leq s \leq \overline{u}(x) + r_C \) and \( \overline{u}(x) - r_C \leq \sigma \leq \overline{u}(x) \).

Then we have
\[
\int_{\hat{\Omega}} \left[ \hat{a}(x, u, \nabla u) \cdot \nabla v + \hat{b}(x, u, \nabla u) v \right] \, dx = 0 \quad \text{for every } v \in W^{1,p}_0(\hat{\Omega}) \cap L^\infty_c(\hat{\Omega}).
\]

**Proof.** Let \( v \in C^\infty_c(\hat{\Omega}) \) with \( v \geq 0 \), let \( t > 0 \) and let
\[
u_t = \min \{ u + tv, \overline{u} \}.
\]
Clearly \( u_t \in \hat{K} \) and \( (u_t - u) \in L_c^\infty(\hat{\Omega}) \), whence
\[
\int_{\hat{\Omega}} \left[ \hat{a}(x, u, \nabla u_t) \cdot \nabla (u_t - u) + \hat{b}(x, u, \nabla u) (u_t - u) \right] \, dx \geq 0.
\]
Taking into account assumption (iii), we get
\[
\int_{\hat{\Omega}} \left[ \hat{a}(x, u, \nabla u_t) \cdot \nabla (u_t - u) + \hat{b}(x, u, \nabla u) (u_t - u) \right] \, dx \geq 0,
\]

namely
\[
\int_{\hat{\Omega}} \left[ \hat{a}(x, u_t, \nabla u_t) \cdot \nabla (u_t - u - tv) + \hat{b}(x, u_t, \nabla u_t) (u_t - u - tv) \right] \, dx
\]
\[
+ t \int_{\hat{\Omega}} \left[ \hat{a}(x, u_t, \nabla u_t) \cdot \nabla v + \hat{b}(x, u_t, \nabla u_t) v \right] \, dx
\]
\[
\geq \int_{\hat{\Omega}} \left[ \hat{a}(x, u_t, \nabla u_t) - \hat{a}(x, u, \nabla u_t) \right] \cdot \nabla (u_t - u - tv) \, dx
\]
\[
+ t \int_{\hat{\Omega}} \left[ \hat{a}(x, u_t, \nabla u_t) - \hat{a}(x, u, \nabla u_t) \right] \cdot \nabla v \, dx
\]
\[
+ \int_{\hat{\Omega}} \left[ \hat{b}(x, u_t, \nabla u_t) - \hat{b}(x, u, \nabla u) \right] (u_t - u) \, dx.
\]
Since $u_t = \overline{u}$ where $u_t - u - tv \neq 0$, we have

$$
\int_{\hat{\Omega}} \left[ \hat{a}(x, u_t, \nabla u_t) \cdot \nabla (u_t - u - tv) + \hat{b}(x, u_t, \nabla u_t) (u_t - u - tv) \right] dx
$$

$$
= \int_{\hat{\Omega}} \left[ \hat{a}(x, \overline{u}, \nabla \overline{u}) \cdot \nabla (u_t - u - tv) + \hat{b}(x, \overline{u}, \nabla \overline{u}) (u_t - u - tv) \right] dx \leq 0,
$$
as $\overline{u}$ is a supersolution of (3.1) and $u_t - u - tv \leq 0$. That leads to the final inequality

$$
\int_{\hat{\Omega}} \left[ \hat{a}(x, u_t, \nabla u_t) \cdot \nabla v + \hat{b}(x, u_t, \nabla u_t)v \right] dx
$$

$$
\geq \int_{\hat{\Omega}} \frac{\hat{a}(x, \overline{u}, \nabla \overline{u}) - \hat{a}(x, u, \nabla u)}{t} \cdot \nabla (u_t - u - tv) dx
$$

$$
+ \int_{\hat{\Omega}} [\hat{a}(x, u_t, \nabla u_t) - \hat{a}(x, u, \nabla u)] \cdot \nabla v dx
$$

$$
+ \int_{\hat{\Omega}} [\hat{b}(x, u_t, \nabla u_t) - \hat{b}(x, u, \nabla u)] \frac{u_t - u}{t} dx.
$$

Since $|u_t - u| \leq v$, from assumption (i) it follows that

$$
\lim_{t \to 0^+} \int_{\hat{\Omega}} \left[ \hat{a}(x, u_t, \nabla u_t) \cdot \nabla v + \hat{b}(x, u_t, \nabla u_t)v \right] dx
$$

$$
= \int_{\hat{\Omega}} \left[ \hat{a}(x, u, \nabla u) \cdot \nabla v + \hat{b}(x, u, \nabla u)v \right] dx,
$$

$$
\lim_{t \to 0^+} \int_{\hat{\Omega}} [\hat{a}(x, u_t, \nabla u_t) - \hat{a}(x, u, \nabla u)] \cdot \nabla v dx = 0,
$$

$$
\lim_{t \to 0^+} \int_{\hat{\Omega}} [\hat{b}(x, u_t, \nabla u_t) - \hat{b}(x, u, \nabla u)] \frac{u_t - u}{t} dx = 0.
$$

Let now $C$ be a compact subset of $\hat{\Omega}$ such that $v = 0$ outside $C$ and let $r_C > 0$ and $\gamma_C \in L^p(C)$ be as in assumption (iii). Without loss of generality, we may assume that $tv \leq r_C$ on $C$. Then, since $0 \leq \overline{u} - u < tv \leq r_C$ where $u_t - u - tv \neq 0$, we get

$$
\left| \frac{\hat{a}(x, \overline{u}, \nabla \overline{u}) - \hat{a}(x, u, \nabla u)}{t} \cdot \nabla (u_t - u - tv) \right| \leq \gamma_C v \left| \nabla (\overline{u} - u - tv) \right|.
$$

Again from assumption (i) we infer that

$$
\lim_{t \to 0^+} \int_{\hat{\Omega}} \frac{\hat{a}(x, \overline{u}, \nabla \overline{u}) - \hat{a}(x, u, \nabla u)}{t} \cdot \nabla (u_t - u - tv) dx = 0.
$$

Therefore we have

$$
\int_{\hat{\Omega}} \left[ \hat{a}(x, u, \nabla u) \cdot \nabla v + \hat{b}(x, u, \nabla u)v \right] dx \geq 0
$$

for every $v \in C_{c}^\infty(\hat{\Omega})$ with $v \geq 0$.

If $v \in C_{c}^\infty(\hat{\Omega})$ with $v \leq 0$, we consider $t > 0$ and

$$
u_t = \max \{ u + tv, \overline{u} \}.$$
Arguing in a similar way, we get
\[ \int_\hat{\Omega} \left[ \hat{a}(x, u, \nabla u) \cdot \nabla v + \hat{b}(x, u, \nabla u) v \right] \, dx \geq 0 \]
for every \( v \in C^\infty_c(\hat{\Omega}) \) with \( v \leq 0 \).

It follows
\[ \int_\hat{\Omega} \left[ \hat{a}(x, u, \nabla u) \cdot \nabla v + \hat{b}(x, u, \nabla u) v \right] \, dx \geq 0 \]
for every \( v \in C^\infty_c(\hat{\Omega}) \), whence the equality, as we can swap \( v \) with \(-v\).

The case \( v \in W^{1,p}_0(\hat{\Omega}) \cap L^\infty_c(\hat{\Omega}) \) can be treated by a standard approximation argument.

\( \square \)

**Remark 3.3.** Assumption (iii) is obviously satisfied in the following cases:

(a) the function \( \hat{a}(x, s, \xi) \) is independent of \( s \);
(b) we have \( \hat{a}(x, s, 0) = 0 \) and \( \underline{u}, \overline{u} \) are constant.

On the other hand, we do not know whether Theorem 3.2 holds true without assumption (iii).

4. PROOF OF THE RESULTS STATED IN THE INTRODUCTION

**Proof of Theorem 1.2.**

It is easily seen that
\[ \hat{a}(x, s, \xi) = a(x, \min\{s, \max\{u(x), \underline{u}(x)\}\}, \xi) , \]
\[ \hat{b}(x, s, \xi) = b(x, \min\{s, \max\{u(x), \underline{u}(x)\}\}, \xi) \]
are still two Carathéodory functions satisfying assumptions (N) and (M) and \( \underline{u}, \overline{u} \) are respectively a subsolution and a supersolution of
\[ -\text{div} [\hat{a}(x, u, \nabla u)] + \hat{b}(x, u, \nabla u) = 0 \quad \text{in} \ \Omega . \]

On the other hand, if \( u \) satisfies the assertion with respect to \( \hat{a} \) and \( \hat{b} \), then it does the same with respect to \( a \) and \( b \), as \( \underline{u} \leq u \leq \overline{u} \) a.e. in \( \Omega \). Therefore, without loss of generality, we may assume that \( a(x, s, \xi) \) and \( b(x, s, \xi) \) are independent of \( s \) for \( s \leq \underline{u}(x) \) and for \( s \geq \overline{u}(x) \).

As in Section 2, for every \( t \in [0, 1] \) we set
\[ \underline{u}_t = u - t , \quad \overline{u}_t = \overline{u} + t . \]

Then, assumptions (i) and (ii) of Section 3 are obviously satisfied and \( \underline{u}_t, \overline{u}_t \) are respectively a subsolution and a supersolution of (1.2) for any \( 0 \leq t \leq 1 \). If \( t > 0 \), also the hypothesis (iii) of Theorem 3.2 holds true, as \( a(x, s, \xi) \) is independent of \( s \) for \( s \leq \underline{u}(x) \) and for \( s \geq \overline{u}(x) \).

From Theorem 2.1 we infer that for every \( t \in [0, 1] \) there exists a solution \( u \) of \((VI_t)\) and that the set
\[ \{ (u, t) \in (W^{1,p}_0(\Omega) \cap L^\infty(\Omega)) \times [0, 1] : u \text{ is a solution of } (VI_t) \} \]
is strongly compact in \( W^{1,p}_0(\Omega) \times [0, 1] \).

Let now, for every \( m \geq 1 \), \( u_m \) be a solution of \((VI_t)\) with \( t = 1/m \). Then \( (u_m) \) is bounded in \( L^\infty(\Omega) \) and, up to a subsequence, strongly convergent in \( W^{1,p}_0(\Omega) \) to some \( u \) satisfying \((VI_t)\) with \( t = 0 \). In particular, we have \( \underline{u} \leq u \leq \overline{u} \) a.e. in \( \Omega \).
From Theorem 3.2 we infer that each $u_m$ actually satisfies
\[ \int_\Omega [a(x, u_m, \nabla u_m) \cdot \nabla v + b(x, u_m, \nabla u_m)v] \, dx = 0 \quad \text{for every } v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega). \]

Going to the limit as $m \to \infty$, it easily follows that
\[ \int_\Omega [a(x, u, \nabla u) \cdot \nabla v + b(x, u, \nabla u)v] \, dx = 0 \quad \text{for every } v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega). \]

By a standard density argument, the equation holds for any $v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$.

\[ \square \]

Proof of Corollary 1.5.
If $Q = 0$, we have $f = 0$ and the assertion is satisfied by $u = 0$. Assume that $Q > 0$, so that $b_0(x) \geq 0$ a.e. in $\Omega$, and set
\[ b(x, s, \xi) = b_1(x) \cdot (|\xi|^{q-1} \xi) + b_0(x)|s|^{r-1}s - f(x). \]

In the case $0 < q < p$ we have
\[ |b(x, s, \xi)| \leq \frac{p-q}{p} |b_1(x)|^{\frac{p}{p-q}} + \frac{q}{p} |\xi|^p + R^{r} b_0(x) + |f(x)|, \]
whenever $|s| \leq R$. Therefore assumptions (N) and (M) are satisfied. In the case $q = p$ we have $b_1 \in L^\infty(\Omega; \mathbb{R}^n)$ and the argument is even simpler.

On the other hand, if we set $u(x) = -Q^{1/r}$, $\overline{u}(x) = Q^{1/r}$,

it follows that $a(x, u, \nabla u) = 0$ and
\[ b(x, u, \nabla u) = -Q b_0 - f \leq -Q b_0 + Q b_0 = 0, \]

so that $u$ is a subsolution of (1.2). The proof that $\overline{u}$ is a supersolution of (1.2) is similar.

By Theorem 1.2 the assertion follows.

\[ \square \]

Proof of Corollary 1.5.
If we set $\tilde{b}(x, s, \xi) = b(x, s, \xi) + b_0(x)g(s) - f(x)$,

it is easily seen that $a$ and $\tilde{b}$ still satisfy assumptions (N) and (M).

Let $A \leq 0 \leq \overline{A}$ be such that
\[ g(A) \leq -Q, \quad g(\overline{A}) \geq Q. \]

Then, if we set
\[ u(x) = A, \quad \overline{u}(x) = \overline{A}, \]

it follows that $a(x, u, \nabla u) = 0$, $b(x, u, \nabla u) = 0$ and
\[ \tilde{b}(x, u, \nabla u) = b_0 g(A) - f \leq -Q b_0 + Q b_0 = 0, \]

so that $u$ is a subsolution of (1.2) with $b$ replaced by $\tilde{b}$. The proof that $\overline{u}$ is a supersolution is similar.

By Theorem 1.2 the assertion follows.

\[ \square \]
References

[1] S. Almi and M. Degiovanni, On degree theory for quasilinear elliptic equations with natural growth conditions, in Recent Trends in Nonlinear Partial Differential Equations II: Stationary Problems (Perugia, 2012), J.B. Serrin, E.L. Mitidieri and V.D. Rădulescu eds., 1–20, Contemporary Mathematics, 595, Amer. Math. Soc., Providence, R.I., 2013.

[2] D. Arcoya and L. Boccardo, Regularizing effect of the interplay between coefficients in some elliptic equations, J. Funct. Anal. 268 (2015), no. 5, 1153–1166.

[3] L. Boccardo, Finite energy weak solutions to some Dirichlet problems with very singular drift, 2018.

[4] L. Boccardo, F. Murat, and J.-P. Puel, Résultats d’existence pour certains problèmes elliptiques quasilinéaires, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 11 (1984), no. 2, 213–235.

[5] G. Dal Maso, On the integral representation of certain local functionals, Ricerche Mat. 32 (1983), no. 1, 85–113.

[6] M. Degiovanni and A. Pluda, Nontrivial solutions of quasilinear elliptic equations with natural growth term, in Solvability, Regularity, and Optimal Control of Boundary Value Problems for PDEs, P. Colli, A. Favini, E. Rocca, G. Schimperna and J. Sprekels eds., 183–215, Springer INdAM Series, 22, Springer, Cham, 2017.

[7] J. Deuel and P. Hess, A criterion for the existence of solutions of non-linear elliptic boundary value problems, Proc. Roy. Soc. Edinburgh Sect. A 74 (1976), 49–54.

[8] M. Giaquinta, “Multiple integrals in the calculus of variations and nonlinear elliptic systems”, Annals of Mathematics Studies, 105, Princeton University Press, Princeton, NJ, 1983.

[9] P. Hess, On a second-order nonlinear elliptic boundary value problem, in Nonlinear analysis (collection of papers in honor of Erich H. Rothe), L. Cesari, R. Kannan and H. F. Weinberger eds., 99–107, Academic Press, New York, 1978.

[10] H. Kim and Y.-H. Kim, On weak solutions of elliptic equations with singular drifts, SIAM J. Math. Anal. 47 (2015), no. 2, 1271–1290.

[11] A. Marino, The calculus of variations and some semilinear variational inequalities of elliptic and parabolic type, in Partial differential equations and the calculus of variations, Vol. II, F. Colombini, A. Marino, L. Modica and S. Spagnolo eds., 787–822, Progr. Nonlinear Differential Equations Appl., 2, Birkhäuser Boston, Boston, MA, 1989.

[12] D. Passaseo, Molteplicità di soluzioni per certe disequazioni variazionali di tipo ellittico, Boll. Un. Mat. Ital. B (7) 3 (1989), no. 3, 639–667.

[13] C. Saccon, Multiple positive solutions for a nonsymmetric elliptic problem with concave convex nonlinearity, in Analysis and topology in nonlinear differential equations (João Pessoa, 2012), D.G de Figueiredo, J.M. do Ó and C. Tomei eds., 387–403, Progr. Nonlinear Differential Equations Appl., 85, Birkhäuser/Springer, Cham, 2014.