Towards the two-loop $Lcc$ vertex in Landau gauge

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Abstract

We are interested in the structure of the $Lcc$ vertex in the Yang-Mills theory, where $c$ is the ghost field and $L$ the corresponding BRST auxiliary field. This vertex can give us information on other vertices, and the possible conformal structure of the theory should be reflected in the structure of this vertex. There are five two-loop contributions to the $Lcc$ vertex in the Yang-Mills theory. We present here calculation of the first of the five contributions. The calculation has been performed in the position space. One main feature of the result is that it does not depend on any scale, ultraviolet or infrared. The result is expressed in terms of logarithms and Davydychev integral $J(1, 1, 1)$ that are functions of the ratios of the intervals between points of effective fields in the position space. To perform the calculation we apply Gegenbauer polynomial technique and uniqueness method.

Keywords: Gegenbauer technique
1 Introduction

Recently it has been shown that the effective action of the $\mathcal{N} = 4$ SYM written in terms of the dressed mean fields does not depend on any scale, ultraviolet or infrared [1, 2]. The theory in terms of these variables is invariant conformally. Therefore, investigation of this action should be greatly simplified. It might be possible to fix all three-point correlation functions up to some coefficient. For example, the three-point function of dressed gluons in the Landau gauge could be found in such a way. Conformal symmetry does not help a lot in the case of the four-point function since an arbitrariness arises. However, in any case, it is important to check all these statements directly by the precise calculations of the vertices in terms of the dressed mean fields. $L_{cc}$ vertex is the most simple object for this calculation, especially in the Landau gauge. In that gauge it is simply totally finite. This fact has been indicated first in Refs. [1, 3].

The purpose of the study is to calculate the $L_{cc}$ vertex at two-loop level for $\mathcal{N} = 4$ super-Yang–Mills theory, which should confirm the statement of Refs. [1, 2] that were derived from the results of Refs. [4, 5, 6, 7, 8]. The present paper contains the calculation of the first needed diagram, and also all needed formulas. The evaluation of other diagrams is a subject of our future investigation. In our opinion, it is the first calculation of two-loop three-point diagrams in general kinematics for a real physical model. The calculation is very nontrivial and can be used by others in some different studies.

With a little modification, the investigation can be applied also for the finite (and of course, singular part) of the vertex at two-loop level for nonsupersymmetric theory. As to the singular part in higher orders for nonsupersymmetric case, in Ref. [9] the explicit three-loop computation of the anomalous dimension of the operator $cc$ has been carried out. The vertex $L_{cc}$ is convergent superficially in Landau gauge in any gauge theory, supersymmetric or nonsupersymmetric. This fact is a consequence of the possibility to integrate two derivatives by parts and put them outside the diagram on the external ghost legs due to transversality of the gauge propagator. If we make a change of the normalization $L$ to $Lg$, as it takes place in Refs. [9, 10], than the superficial divergence would mean that $Z_LZ_gZ_c = 1$. (Renormalization constant of $g$ is $Z_g$, and the renormalization constant of $c$ is $Z_c^{1/2})$. This is the condition of superficial convergence of this vertex. Formally, this result holds to all orders of the perturbation theory due to the so-called Landau ghost equation, stemming from the fact that in the Landau gauge the Yang-Mills action is left invariant by a constant shift of the Faddeev-Popov ghost $c$ (Ref. [11]). Note that it is not like the normalization used in Refs. [1, 2] where $Z_LZ_c = 1$. Moreover, the renormalization of ghost and antighost fields in our Refs. [1, 2] was chosen to be independent. This is different from Ref.[9] in which $Z_c = Z_c$. Thus, in terms of conventions of Ref. [1, 2] $Z_L = Z_c = 1$. It means the field $c$ does not get the renormalization in our convention. This coincides with the old results obtained from the antighost equation of Ref. [11]. Neither does the external field $L$ get the renormalization in this convention. Starting with the two loop order infinities reproduce the renormalization of the gauge coupling in nonsupersymmetric theory. If we change the convention of the renormalization, the relation between the renormalization constants of paper [9] for nonsupersymmetric case can be reproduced. Knowing the structure of the $L_{cc}$ vertex, one can obtain other vertices in terms of this one by using Slavnov-Taylor identity [12, 13, 14, 15, 16, 17] which is a consequence of the BRST symmetry [18, 19]. Moreover, the algorithm for obtaining these structures is expected to be sim-
ple, due to the simple structure of the $Lcc$ vertex, in particular due to its scale independence. Similar arguments can be applied to $\mathcal{N} = 8$ supergravity [20, 21], and to other theories which possess high level supersymmetry to guarantee good properties of the correlators. In some theories, for example Chern-Simons field theory, nonrenormalization of gauge coupling is protected by topological reasons and similar approach is valid near the fixed points [22] in the coupling space.

The article has the following structure. Section 2 contains the derivation of gluon and ghost propagators in the position space. A review of the one-loop results and all two-loop diagrams, contributing to the problem, are given in Section 3. Section 4 demonstrates a useful representation for the two-loop diagram (a) which is the subject of this study. In Section 5 we show basic formulas for calculation of the considered Feynman integrals. The calculation of diagram (a) is performed in Sections 6 and 7. Moreover, details of the calculations can be found in Appendices A and D. The most complicated Feynman integrals are evaluated in Appendices B and C. Section 8 contains conclusions and a summary of the results, and discussion about the future steps.

2 Landau and ghost propagators in the position space

In the momentum space the gluon propagator in Landau gauge is:

$$\left[ g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right] \frac{1}{(p^2)^a}, \quad (1)$$

where the case $a = 1$ corresponds to the free propagator in four dimensions. We will assume that the Wick rotation has been performed and will thus work in the Euclidean metric. The number of dimensions is $D = 4 - 2\epsilon$. We formulate the rules in the momentum space and then we go to the position space. The Fourier transform of the (1) to the position space can be done with the help of the following formulas [25]

$$\int d^Dp \frac{1}{(p^2)^a} e^{ipx} = 2^{D-2}\pi^{D/2}a(\alpha) \frac{1}{(x^2)^{D/2-\alpha}}, \quad a(\alpha) = \frac{\Gamma[D/2 - \alpha]}{\Gamma[\alpha]}.$$

Thus, the propagator is of the form

$$\frac{g_{\mu\nu}}{(x^2)^b} - c \frac{x_\mu x_\nu}{(x^2)^{b+1}},$$

with $b = D/2 - a$.

The transversality condition

$$\partial_\mu \left( \frac{g_{\mu\nu}}{(x^2)^b} - c \frac{x_\mu x_\nu}{(x^2)^{b+1}} \right) = 0.$$ 

allows to determine the coefficient $c$

$$\partial_\mu \left( \frac{g_{\mu\nu}}{(x^2)^b} - c \frac{x_\mu x_\nu}{(x^2)^{b+1}} \right) = -b \frac{2x_\mu}{(x^2)^{b+1}} - \frac{(D + 1)x_\mu - 2(b + 1)x_\nu}{(x^2)^{b+1}} =$$
\[ \begin{align*}
&= -b \frac{2x_\nu}{(x^2)^{b+1}} - c \frac{(D - 2b - 1)x_\nu}{(x^2)^{b+1}} = -\frac{(2b + c(D - 2b - 1))x_\nu}{(x^2)^{2-\epsilon}} \\
\Rightarrow c &= -\frac{2b}{D - 2b - 1}.
\end{align*} \]

Thus, the free propagator in the Landau gauge is:

\[ g_{\mu\nu} = \frac{g_{\mu\nu}}{(x^2)^{1-\epsilon}} + 2(1 - \epsilon) \frac{x_\mu x_\nu}{(x^2)^{2-\epsilon}} \]

The ghost propagator in the momentum space is

\[ \frac{p_\mu}{p^2}. \]

In the position space the Fourier transform is

\[ \partial_\mu \int d^Dp \frac{1}{p^2} e^{ipx} = 2^{D-2} \pi^{D/2} a(1) \partial_\mu \frac{1}{(x^2)^{1-\epsilon}} = 2^{D-2} \pi^{D/2} a(1)(\epsilon - 1) \frac{2x_\mu}{(x^2)^{2-\epsilon}}, \]

3 Diagram contributions

The one-loop contribution to the $L_{cc}$ correlator corresponds to the diagram of Fig. 1. This is

\[ \text{Figure 1: One-loop contribution to the } L_{cc} \text{ vertex. The wavy line represents the gluon propagator, the straight lines are for the ghosts.} \]

the only possible one-loop contribution, since the ghost field interacts only with the gauge field. This one-loop result is simple since it does not require the integration in the position space and is proportional to the following expression:

\[ f^{abc} \int d^4x_1 d^4x_2 d^4x_3 L^a(x_1)c^b(x_2)e^c(x_3) \frac{1}{(x_2 - x_3)^2} \times \]
\[ \times \left( g_{\mu\nu} + 2 \frac{(x_2 - x_3)_{\mu}(x_2 - x_3)_{\nu}}{(x_2 - x_3)^2} \right) \delta^{(2)}_{\mu} \partial^{(3)}_{\nu} \frac{1}{(x_1 - x_2)^2(x_1 - x_3)^2} = \]

\[ 2f^{abc} \int d^4x_1 d^4x_2 d^4x_3 \left[ (x_1 - x_2)^4(x_1 - x_3)^2(x_2 - x_3)^2 \right] \left[ \frac{1}{(x_1 - x_2)^2(x_1 - x_3)^4(x_2 - x_3)^2} - \frac{2}{(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^4} \right] + \frac{1}{(x_1 - x_2)^4(x_1 - x_3)^4(x_2 - x_3)^4} \]

As one can see, the one-loop contribution is simple, but it does not have a structure that is expected from conformal field theories \[1, 23, 24\]. This is because the external field \( L \) does not propagate, it is not in the measure of path integral. However, this vertex by Slavnov-Taylor identity can be related to the three-point function of dressed mean gluons and they are expected to have simple structure at least for the connected function in the Landau gauge in \( N = 4 \) super-Yang-Mills theory. For this reason, it is important to calculate the next order of \( L_{cc} \) vertex since poles disappear there. Poles do not disappear in the triple correlator of gluons since they must be absorbed into the dressing functions of the gluons.

Two-loop planar correction to \( L_{cc} \) vertex can be represented as combination of five diagrams depicted in Fig. 2.

![Figure 2: The two-loop diagrams for the \( L_{cc} \) vertex. The wavy lines represent the gluons, the straight lines the ghosts. Black disk in diagram (d) stands for the total one-loop correction to the gluon propagator.](image)

### 4 Integral structure

As it was noted in the Introduction, in the present paper we analyse diagram (a) only. The derivatives on the ghost propagators at the points \( x_2 \) and \( x_3 \) can be integrated outside the diagram and be put on the external legs. Indeed, the result for the diagram (a) can be represented as the derivative

\[ \frac{1}{[23]} \left( g_{\mu\nu} + 2 \frac{[23]_{\mu}[23]_{\nu}}{[23]} \right) \partial^{(2)}_{\mu} \partial^{(3)}_{\nu} \]

(3)

to an integral \( T \) which contains numerators of the propagators independent of \( x_2 \) and \( x_3 \). We have introduced for the brevity the notation

\[ [yz] = (y - z)^2, \quad [y1] = (y - x_1)^2, \ldots \]

\[^{1}\text{I.K. thanks A. Jevicki for clarifying this point}\]
and so on.

These derivatives will simplify the Lorentz structure of the wavy gluon line and reduce the integrand to the scalar structure. The corresponding contribution is of the following form:

\[
\frac{(y_1)_{\mu}(z_1)_{\nu}}{[y_1]^{2-\epsilon}[z_1]^{2-\epsilon}} \left( \frac{g_{\mu\nu}}{[y_2]^{1-\epsilon}} + 2(1-\epsilon)(y_2)_{\mu}(z_2)_{\nu} \right) \frac{1}{[y_2]^{1-\epsilon}} \frac{1}{[z_3]^{1-\epsilon}} .
\]

Note that it is the one-loop contribution multiplied by the last two denominators.

The Lorentz structure can be simplified. Indeed, because the scalar product \(2(xy)_\mu(yz)_\mu = [xy] + [yz] - [xz]\), we have

\[
\frac{(y_1)_{\mu}(z_1)_{\nu}}{[y_1]^{2-\epsilon}[z_1]^{2-\epsilon}} \left( \frac{g_{\mu\nu}}{[y_2]^{1-\epsilon}} + 2(1-\epsilon)(y_2)_{\mu}(y_2)_{\nu} \right) = \frac{1}{2} \frac{[y_2][y_1] + [yz][z_1] - (2-\epsilon)[y_2]^2 + (1-\epsilon)([y_1] - [z_1])^2}{[y_2]^{2-\epsilon}[y_1]^{2-\epsilon}[z_1]^{2-\epsilon}},
\]

Thus, the integral we have to calculate is

\[
T = \int Dy \, Dz \, \frac{[y_2][y_1] + [yz][z_1] - (2-\epsilon)[y_2]^2 + (1-\epsilon)([y_1] + [z_1] - 2[y_1][z_1])}{[y_2]^{2-\epsilon}[y_1]^{2-\epsilon}[z_1]^{2-\epsilon}[y_2]^{1-\epsilon}[z_3]^{1-\epsilon}},
\]

where we use the notation \(Dy = \pi^{-D/2} d^Dy\).

We expect that the diagram \(T\) is finite in the limit \(\epsilon \to 0\). Infrared divergences in the position space can be analysed in the same manner as it has been done for the ultraviolet divergences in the momentum space in the Bogolubov-Parasiuk-Hepp-Zimmermann \(R\)-operation. From the expression above, for example, it can be seen that the infrared limit \(|x| \to \infty\) is safe in the position space in all the subgraphs and in the whole diagram. In the ultraviolet region of the position space each of the integrations is safe, too.

Since the diagram is finite, does not matter where precisely the “\(\epsilon\)” is. In a certain sense, it is possible to change the indices in the propagators by adding multiples of \(\epsilon\). In this way we can achieve the possibility to use the uniqueness relation [26] to calculate at least one of the two integrations by the bootstrap technique. Deviations in logarithms in the integrands, after changing the indices, cannot change the results in the limit \(\epsilon \to 0\) since they present finite construction times \(\epsilon\).

Thus, with the accuracy \(O(\epsilon^0)\), the integral above can be transformed as

\[
T = \int Dy \, Dz \, \frac{[y_2][y_1] + [yz][z_1] - (2-\epsilon)[y_2]^2 + (1-\epsilon)([y_1]^2 + [z_1]^2 - 2[y_1][z_1])}{[y_2]^{2-\epsilon}[y_1]^{2-\epsilon}[z_1]^{2-\epsilon}[y_2]^{1-\epsilon}[z_3]^{1-\epsilon}},
\]

where \(\tau = -2\epsilon/(1-2\epsilon)\), that corresponds to \(c = -2/(1-2\epsilon)\) and/or \(b = 1\). This change \(\epsilon \to \bar{\epsilon}\) is necessary, since we need to keep transversality in the position space after changing the index
of gluon propagator from $1 - \epsilon$ to 1, according to the formulas of Section 2. Transversality is a necessary requirement to avoid problems with ultraviolet divergence in the position space.

Moreover, the diagram is symmetric with replacement $\{y, 2\} \leftrightarrow \{z, 3\}$ and, thus, we replace Eq. (6) by

$$T = \left\{ \int Dz Dy \frac{[yz][z1] - (1 - \pi/2)[yz]^2 + (1 - \pi)([z1]^2 - [y1][z1])}{[y2][z3]} \right\} + \left\{ y, 2 \leftrightarrow z, 3 \right\}.$$  

(7)

Following to the Eq. (3), the final results for the first diagram $V$ in Fig. 1 can be represented as

$$V = \frac{1}{[23]} \left( g_{\mu\nu} + 2 \left( \frac{23}{[23]} \right)_{\mu} \partial^{(2)}_{\nu} \right) \partial^{(3)}_{\nu} T.$$  

(8)

5 Technique of calculation

To calculate expression (7), we need to use uniqueness method [26, 27, 25] and Gegenbauer polynomial technique (GPT) [28, 29]. Let us to give a short review of the uniqueness method. The GPT will be used only for the most complicated diagrams in Appendix A. All needed formulas for the GPT application can be found in [29].

The uniqueness method contains several rules to calculate massless chains and vertices algebraically, i.e. without a direct calculation of $D$-space integrals.

1. The results for chains $J(\alpha_1, \alpha_2)$ have the form

$$J(\alpha_1, \alpha_2) \equiv \int Dx \frac{1}{[x1]^{\alpha_1} [x2]^{\alpha_2}} = A(\alpha_1, \alpha_2, \alpha_3) \frac{1}{[12]^{\alpha_3}} ,$$  

(9)

where

$$A(\alpha_1, \alpha_2, \alpha_3) = a(\alpha_1) a(\alpha_2) a(\alpha_3), \quad Dx \equiv \pi^{-D/2} d^D x$$

and $\alpha_3 = D - \alpha_2 - \alpha_1$ and $\bar{\alpha}_i = D/2 - \alpha_i$. The point $x_1$ can be shifted to $x_1 = 0$. We have introduced new $D$-dimensional measure $Dx \equiv \pi^{-D/2} d^D x$. Note that chain can be considered as the vertex with one propagator having power 0, i.e. $J(\alpha_1, \alpha_2) = J(\alpha_1, \alpha_2, 0)$.

2. Uniqueness method [26, 27] (see also nice review [25]): if $\alpha_1 + \alpha_2 + \alpha_3 = D$, then

$$J(\alpha_1, \alpha_2, \alpha_3) \equiv \int Dx \frac{1}{[x1]^{\alpha_1} [x2]^{\alpha_2} [x3]^{\alpha_3}} = A(\alpha_1, \alpha_2, \alpha_3) \frac{1}{[12][13][23]^{\alpha_3} \bar{\alpha}_3} .$$  

(10)

3. Integration by parts procedure (IBP) [26, 27, 25].

*In the momentum space the equal relation for triangle is also very popular procedure (see [30]).
Including in the integrand of $J(\alpha_1, \alpha_2, \alpha_3)$ the function $\partial_\mu (x_1)_\mu$ and applying integration by parts, we obtain

$$DJ(\alpha_1, \alpha_2, \alpha_3) \equiv \int Dx \frac{1}{[x_1]^\alpha_1 [x_2]^\alpha_2 [x_3]^\alpha_3} \partial_\mu (x_1)_\mu$$

$$= \int Dx \left[ \partial_\mu \left\{ \frac{(x_1)_\mu}{[x_1]^\alpha_1 [x_2]^\alpha_2 [x_3]^\alpha_3} \right\} - (x_1)_\mu \partial_\mu \left\{ \frac{1}{[x_1]^\alpha_1 [x_2]^\alpha_2 [x_3]^\alpha_3} \right\} \right].$$

The first term on the r.h.s. is equal to zero. Performing the derivative in the second term, after little algebra we obtain IBP relation

$$\left(D - 2\alpha_1 - \alpha_2 - \alpha_3 \right) J(\alpha_1, \alpha_2, \alpha_3) = \alpha_2 \left( J(\alpha_1 - 1, \alpha_2 + 1, \alpha_3) - [12] J(\alpha_1, \alpha_2 + 1, \alpha_3) \right)$$

$$+ \alpha_3 \left( J(\alpha_1 - 1, \alpha_2, \alpha_3 + 1) - [13] J(\alpha_1, \alpha_2, \alpha_3 + 1) \right). \quad (11)$$

The relation has symmetry with respect to $\alpha_2 \leftrightarrow \alpha_3$. The index $\alpha_1$ has a special role, and the corresponding propagator having the power $\alpha_1$ will be called distinguish line.

### 6 Cancellation of poles

Now, using results of Appendices A and B, we obtain the final results for expression $T$. For this, it is convenient to consider the following combination:

$$T = \frac{(1 - 2\epsilon) [12]^{1-\epsilon} [13]^{1-\epsilon}}{\epsilon A(1, 1, 2 - 2\epsilon)} T = \sum_{k=1}^{4} T_k.$$  

The first term $T_1$ has the form

$$T_1 = -\frac{2(1 - \epsilon)}{\epsilon} A(1, 1, 2 - 2\epsilon),$$

where

$$A(1, 1, 2 - 2\epsilon) = \frac{\Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)}{\epsilon(1 - 2\epsilon) \Gamma(1 - 2\epsilon)} = \frac{\Gamma(1 + \epsilon) e^{-\zeta(2)\epsilon^2} + o(\epsilon)}{\epsilon(1 - 2\epsilon)},$$

The last identity holds because

$$\Gamma(1 + a\epsilon) = \exp \left[ -\gamma a\epsilon + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} (-a\epsilon)^k \right], \quad (12)$$

where $\gamma$ and $\zeta(k)$ are Euler constant and Euler numbers, respectively. Thus, we obtain

$$T_1 / \left\{ \Gamma(1 + \epsilon) e^{-\zeta(2)\epsilon^2} \right\} = -\frac{2(1 - \epsilon)}{\epsilon^2(1 - 2\epsilon)} + o(\epsilon).$$

For the second term $T_2$ we have

$$T_2 = \frac{2}{(1 - 2\epsilon)} A(\epsilon, 2, 2 - 3\epsilon) \frac{[12]^{\epsilon} [13]^{\epsilon}}{[23]^{2\epsilon}},$$
where

\[ A(\epsilon, 2, 2 - 3\epsilon) = -\frac{1 - 2\epsilon}{2\epsilon(1 - 3\epsilon)} \frac{\Gamma(1 - \epsilon)\Gamma(1 - 2\epsilon)\Gamma(1 + 2\epsilon)}{\Gamma(1 + \epsilon)\Gamma(1 - 3\epsilon)} = -\frac{\Gamma(1 + \epsilon)}{2\epsilon} \frac{1 - 2\epsilon}{1 - 3\epsilon} e^{-\zeta(2)\epsilon^2} + o(\epsilon). \]

Thus, we obtain

\[ T_2/\{\Gamma(1 + \epsilon) e^{-\zeta(2)\epsilon^2}\} = -\frac{1}{\epsilon(1 - 3\epsilon)} \frac{[12][13][23]}{[23]^2} + o(\epsilon) = -\frac{1}{\epsilon(1 - 3\epsilon)} \left(1 + \epsilon L\right) + o(\epsilon), \]

where

\[ L = \ln \frac{[12][13]}{[23]^2}. \]

The third term is very simple

\[ T_3 = ([12] + [13]) J(1, 1, 1), \]

where for integral \( J(1, 1, 1) \) we can use Davydychev formula (see [31]):

\[ J(1, 1, 1) = \frac{2}{B} \left[ \zeta(2) - \text{Li}_2 \left( \frac{[23] + [12] - [13] - B}{2[23]} \right) - \text{Li}_2 \left( \frac{[23] + [13] - [12] - B}{2[23]} \right) \right] + \ln \left( \frac{[23] + [12] - [13] - B}{2[23]} \right) \ln \left( \frac{[23] + [13] - [12] - B}{2[23]} \right) - \frac{1}{2} \ln \left( \frac{[12]}{[23]} \right) \ln \left( \frac{[13]}{[23]} \right), \tag{13} \]

where

\[ B^2 = ([12] - [13])^2 - 2([12] + [13])([23] + [23]^2). \]

Note that the results (13) have a clear symmetry \( \{2 \leftrightarrow 3\} \).

The last term \( T_4 \) has the form

\[ T_4 = \frac{2 - 3\epsilon}{\epsilon} \left[ [13]^{1 - \epsilon} J(2 - 3\epsilon, \epsilon, 1) + [12]^{1 - \epsilon} J(2 - 3\epsilon, 1, \epsilon) \right], \]

where \( J(2 - 3\epsilon, 1, \epsilon) = \left\{ J(2 - 3\epsilon, \epsilon, 1), 1 \leftrightarrow 2 \right\}, \)

\[ J(2 - 3\epsilon, \epsilon, 1) = \frac{A(1 + \epsilon, 1 - 2 - 3\epsilon)}{A(1 + \epsilon, 1 - \epsilon, 2 - 2\epsilon)} \frac{1}{\Gamma(1 - \epsilon)(2\epsilon - 1)} \frac{[12]^{-\epsilon}}{[13]^{1 - 2\epsilon}} \tilde{J}(2 - 3\epsilon, \epsilon, 1), \]

and (see Appendix A)

\[ \tilde{J}(2 - 3\epsilon, \epsilon, 1) = -\frac{1}{\epsilon} + \epsilon \left[ \ln \frac{[12]}{[23]} \ln \frac{[12]}{[23]} + \left([13] + [23] - [12]\right) J(1, 1, 1) \right]. \tag{14} \]

Because

\[ \frac{A(1 + \epsilon, 1 - 2 - 3\epsilon)}{A(1 + \epsilon, 1 - \epsilon, 2 - 2\epsilon)} = \frac{1 - 2\epsilon}{2(1 - 3\epsilon)} \frac{\Gamma^2(1 - \epsilon)\Gamma(1 - 2\epsilon)\Gamma(1 + 3\epsilon)}{\Gamma(1 + \epsilon)\Gamma(1 - 3\epsilon)} = \frac{\Gamma(1 + \epsilon)}{2} \frac{1 - 2\epsilon}{1 - 3\epsilon} e^{-\zeta(2)\epsilon^2} + o(\epsilon), \]

8
we have
\[
\mathcal{T}_4/\{\Gamma(1 + \epsilon)e^{-\zeta(2)\epsilon^2}\} = -\frac{2 - 3\epsilon}{2\epsilon(1 - 3\epsilon)} \left[\frac{[13]^\epsilon}{[12]^\epsilon} \tilde{J}(2 - 3\epsilon, \epsilon, 1) + \frac{[12]^\epsilon}{[13]^\epsilon} \tilde{J}(2 - 3\epsilon, 1, \epsilon)\right] + o(\epsilon).
\]

Consider terms in the brackets. All \(O(1)\) terms are cancelled:
\[-\ln \frac{[12]}{[13]} - \ln \frac{[13]}{[12]} = 0 .\]

At \(O(\epsilon)\) level, all logarithms are also canceled. Indeed
\[
\left\{-\frac{1}{2} \ln^2 \frac{[12]}{[13]} + \ln \frac{[12]}{[23]} \ln \frac{[12]}{[13]} \right\} + \left\{1 \leftrightarrow 2\right\} = \ln \frac{[12]}{[13]} \left[-\ln \frac{[12]}{[13]} + \ln \frac{[12]}{[23]} - \ln \frac{[13]}{[23]}\right] = 0 .
\]

So, the terms in brackets are
\[-\frac{2}{\epsilon} + 2\epsilon[23]J(1, 1, 1) = -2 \left[\frac{1}{\epsilon} - \epsilon[23]J(1, 1, 1)\right] .
\]

Thus, we have
\[
\mathcal{T}_4/\{\Gamma(1 + \epsilon)e^{-\zeta(2)\epsilon^2}\} = \frac{2 - 3\epsilon}{(1 - 3\epsilon)} \left[\frac{1}{\epsilon^2} - [23]J(1, 1, 1)\right] + O(\epsilon) ,
\]
and, for the sum
\[
\left(\mathcal{T}_1 + \mathcal{T}_4\right)/\{\Gamma(1 + \epsilon)e^{-\zeta(2)\epsilon^2}\} = \frac{1}{\epsilon(1 - 2\epsilon)(1 - 3\epsilon)} - 2[23]J(1, 1, 1) + O(\epsilon) = \left(\mathcal{T}_1 + \mathcal{T}_4\right)/\Gamma(1 + \epsilon) .
\]

Combination of three terms
\[
\left(\mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_4\right)/\Gamma(1 + \epsilon) = 2 - L - 2[23]J(1, 1, 1) + O(\epsilon) = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_4
\]
is finite and, thus,
\[
\mathcal{T} = 2 - L + \left([12] + [13] - 2[23]\right)J(1, 1, 1) .
\]

Because \(\epsilon A(1, 1, 2 - 2\epsilon) = 1 + o(\epsilon)\), we obtain
\[
T = \frac{1}{[12][13]} \left[2 - L + \left([12] + [13] - 2[23]\right)J(1, 1, 1)\right] .
\]

(15)
7 Final result

To obtain the results (8) for the diagram (a) of Fig. 1, we apply the differentiation procedure (3) to the r.h.s. of Eq. (15).

To write it in the more convenient for the differentiating form it is convenient to represent the expression in terms of the function $J(1, 1, 1)$.

It is better to represent the integral $T$ as three terms:

$$T = 2T^{(1)} - T^{(2)} + T^{(3)}, \quad T^{(1)} = \frac{1}{[12][13]}.$$

To obtain the final results for the expression $V$ representing the first diagram (a), we apply the projector

$$P_{\mu\nu}\partial^{(2)}_{\mu}\partial^{(3)}_{\nu} \equiv \frac{1}{[23]} \left( g_{\mu\nu} + 2\frac{(23)_{\mu}(23)_{\nu}}{23} \right) \partial^{(2)}_{\mu}\partial^{(3)}_{\nu},$$

to the integral $T$, i.e.

$$V = 2V^{(1)} - V^{(2)} + V^{(3)}, \quad V^{(i)} \equiv P_{\mu\nu}\partial^{(2)}_{\mu}\partial^{(3)}_{\nu}T^{(i)}.$$

1. The result for $V^{(1)}$ has the form

$$V^{(1)} = P_{\mu\nu}\partial^{(2)}_{\mu}\partial^{(3)}_{\nu} \frac{1}{[12][13]} = \frac{2B_1}{[12][13][23]^2},$$

where

$$B_1 = 2(12)_{\mu}(13)_{\mu} + 4(12)_{\mu}(23)_{\mu}(13)_{\nu}(23)_{\nu} = ([12] - [13])^2 + ([12] + [13])[23] - 2[23]^2.$$

2. To obtain $V^{(2)}$, we represent it as the sum of three terms:

$$V^{(2)} = \sum_{k=1}^{3} V^{(2)}_i.$$

The first term $V^{(2)}_1$ is proportional to $V^{(1)}$:

$$V^{(2)}_1 = \left( P_{\mu\nu}\partial^{(2)}_{\mu}\partial^{(3)}_{\nu} \frac{1}{[12][13]} \right) L = V^{(1)} L.$$

The third term is zero. Indeed

$$\partial^{(2)}_{\mu}\partial^{(3)}_{\nu}L = \partial^{(2)}_{\mu} \left[ -\frac{2(13)_{\nu}}{13} + 4\frac{(23)_{\nu}}{23} \right] = 4\partial^{(2)}_{\mu} \left[ \frac{(23)_{\nu}}{23} - 4 \right] = 4 \left[ g_{\mu\nu} - 2\frac{(23)_{\mu}(23)_{\nu}}{23} \right]$$

$$\Rightarrow V^{(2)}_3 = \frac{1}{[12][13]} P_{\mu\nu}\partial^{(2)}_{\mu}\partial^{(3)}_{\nu}L = \frac{4}{[12][13][23]^2} (D + 2 - 2 - 4) = 0.$$
The most complicated second part of \( V^{(2)} \) has the form

\[
V_{2}^{(2)} = P_{\mu\nu} \left\{ \frac{1}{[12][13]} \partial_{\mu}^{(2)} L \right\} + \left\{ 2 \leftrightarrow 3 \right\} = \tilde{V}_{2}^{(2)} + \left\{ \tilde{V}_{2}^{(2)}, 2 \leftrightarrow 3 \right\}.
\]

The result in brackets is

\[
\frac{2(12)_\mu}{[12]^2[13]} \left[ -2(13)_\nu + 4(23)_\nu \right]
\]

and, so,

\[
\tilde{V}_{2}^{(2)} = \frac{4}{[12]^2[13]^2[23]^2} \left[ -2(12)_\mu(13)_\mu + 6(12)_\mu(23)_\mu - 2(12)_\mu(23)_\mu(13)_\nu(23)_\nu \right]
\]

\[
= \frac{4}{[12]^2[13]^2[23]^2} \left[ 2[23]^2 - (7[13] + [12])[23] + ([13] - [12])(5[13] + [12]) \right].
\]

Thus, we have

\[
V_{2}^{(2)} = \frac{8B_2}{[12]^2[13]^2[23]^2}, \quad V^{(2)} = \frac{2(B_1L + 4B_2)}{[12]^2[13]^2[23]^2},
\]

where

\[
B_2 = ([12] - [13])^2 - 2([12] + [13])[23] + [23]^2 \equiv B^2.
\]

3. The third term \( V^{(3)} \) can be also represented as the sum of three terms:

\[
V^{(3)} = V^{(3)}_1 + V^{(3)}_2 + V^{(3)}_3.
\]

where the terms \( V^{(3)}_1, V^{(3)}_2 \) and \( V^{(3)}_3 \) can be found in Appendix D.

Collecting all these terms together, we obtain, after some algebra

\[
V^{(3)} = \frac{2}{[12]^2[13]^2[23]^2} \left[ 2B_4 + 2B_5 \ln \frac{[12]}{23} + 2B_6 \ln \frac{[13]}{23} + B_7J(1,1,1) \right],
\]

where

\[
B_4 = ([13] + [12])^2 - 3([13] + [12])[23] + 2[23]^2,
B_5 = [12]^2 - [13]^2 + [12][13] + 3[13][23] - 2[23]^2,
B_6 = \left\{ B_4, 2 \leftrightarrow 3 \right\},
B_7 = 4[23]^3 - 6([13] + [12])[23]^2 + \left( ([13] + [12])^2 - 2[12][13] \right)[23] + \left( ([13] + [12])^2 - 6[12][13] \right)\left( [13] + [12] \right).
\]

4. Now the result for the expression \( V \) representing diagram (a) has the form

\[
V = 2V^{(1)}_1 - V^{(2)} + V^{(3)} = \frac{2}{[12]^2[13]^2[23]^2} \left[ A_1 + A_2 \ln \frac{[12]}{23} + A_3 \ln \frac{[13]}{23} + A_4J(1,1,1) \right]. \quad (16)
\]
where

\[ A_1 = 2B_1 - 4B_2 + 2B_4 = 8[12][13] + 4([13] + [12])[23] - 4[23]^2, \]

\[ A_2 = -B_1 + 2B_5 = [12]^2 - 3[13]^2 + 4[12][13] + (5[13] - [12])[23] - 2[23]^2, \]

\[ A_3 = \{A_2, 2 \leftrightarrow 3\}, \]

\[ A_4 = B_7 = 4[23]^3 - 6([13] + [12])[23]^2 + \left(\left([13] + [12]\right)^2 - 2[12][13]\right)[23]
\quad + \left(\left([13] + [12]\right)^2 - 6[12][13]\right)([13] + [12]). \]

8 Conclusions

In this paper we have shown, among other things, that the first of the five two-loop contributions in the correlator $L_{cc}$ does not depend on any scale. The calculation has been performed in the position space and in the Euclidean metric. For this particular contribution it is a direct consequence of the transversality of the gluon propagator in the Landau gauge. The same is true for the two other vertex-type contributions. The $\mathcal{N} = 4$ supersymmetry does not play any role in the scale-independence of the first three contributions. It is important only for the cancellation of poles between the other two contributions of propagator-type corrections. The present study was necessary to investigate the precise structure of the two-loop contributions. By the ST identity the $L_{cc}$ correlator can be transformed to the correlator of three dressed gluons. It is natural to expect that the conformal symmetry of the theory fixes the correlator of this triple gluon vertex completely up to some coefficient (that depends on the gauge coupling and number of colours). Since it is expected that the structure of the correlators of three dressed gluons is simple in the position space, the precise structure of the $L_{cc}$ vertex in the position space is also simple. There are also other motivations in favour of the expected simple structure of the $L_{cc}$ vertex.

Indeed, the results for $V$ and $T$ contain the terms with different values of the transcendentality level. It was shown in [32] that some values of $N = 4$ SYM variables have only terms with the same value of the transcendentality level at any order of perturbation theory (see [32, 33, 34]). It is possible that the same property is applicable to the $L_{cc}$ vertex. If this is the case, then the final two-loop result for the $L_{cc}$ vertex should contain [in the numerator of the r.h.s. of Eqs. (15) and (16)] only $J(1, 1, 1)$ vertex and/or $\zeta(2)$ Euler number.

Finally, we would like to mention another obvious consequence of the considerations made here and in Refs. [1, 2]. All this can be applied to any gauge theory with only one coupling (gauge coupling) whose beta function is vanishing at every order. It means that the correlators of the dressed gauge bosons in that theory do not depend on any scale in the transversal gauge.

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9 Appendix A

The Appendix A is devoted to evaluate the expression for $T$.

The integrand in brackets on the r.h.s. of (7) is a sum of several parts. The result, coming from term $\sim [yz]^2$ in numerator, is

$$I_2 = \int DyDz \frac{1}{[y1]^{2-2\epsilon}[z1]^{2-2\epsilon}[y2][z3]} = A^2(1,2-2\epsilon,1) \frac{1}{[12]^{1-\epsilon}[13]^{1-\epsilon}}.$$ 

This integral is very simple and requires only the use of Eq. (9). The term $\sim [y1][z1]$ leads to

$$I_4 = \int DyDz \frac{1}{[yz][y1]^{1-2\epsilon}[z1]^{1-2\epsilon}[y2][z3]} = A(1,2,1-2\epsilon) \frac{1}{[12]^{1-\epsilon}} \frac{1}{[12]^{1-2\epsilon}[13]^{1-2\epsilon}[23]^{2\epsilon}}J(2-3\epsilon,1+\epsilon,1) = A(1,2,1-2\epsilon)A(2-3\epsilon,1+\epsilon,1) \frac{1}{[12]^{1-2\epsilon}[13]^{1-2\epsilon}[23]^{2\epsilon}},$$

where Eqs. (9) and (10) have been used simultaneously. Because of the relations

$$A(1,2,1-2\epsilon) = -(1-2\epsilon)A(1,1,2-2\epsilon), \quad A(2-3\epsilon,1+\epsilon,1) = -\frac{1}{1-2\epsilon}A(2-3\epsilon,\epsilon,2),$$

$$A(1-3\epsilon,1+\epsilon,2) = \frac{2(1-3\epsilon)}{1-2\epsilon}A(2-3\epsilon,\epsilon,2) \quad (A1)$$

we obtain

$$I_4 = A(1,1,2-2\epsilon)A(2-3\epsilon,\epsilon,2) \frac{1}{[12]^{1-2\epsilon}[13]^{1-2\epsilon}[23]^{2\epsilon}}.$$

Using the uniqueness relation (10), we have for the term $\sim [yz][z1]$

$$I_1 = \int DyDz \frac{1}{[yz][y1]^{2-2\epsilon}[z1]^{1-2\epsilon}[y2][z3]} = A(1,1,2-2\epsilon) \frac{1}{[12]^{1-\epsilon}} \frac{1}{[12]^{1-2\epsilon}}J(2-3\epsilon,\epsilon,1).$$

The last integral

$$I_3 = \int DyDz \frac{1}{[yz][y1]^{2-2\epsilon}[z1]^{1-2\epsilon}[y2][z3]}$$

can be reduced by integration by parts to the basic integrals. It is better to consider first the integral $I_1$ and to apply IBP to the vertex with the center in point $z$ and the distinguished line with the power $1-2\epsilon$. The result has the form

$$2\epsilon I_1 = I_3 - I_4 + \int DyDz \frac{[z1] - [31]}{[yz][y1]^{2-2\epsilon}[z1]^{1-2\epsilon}[y2][z3]^2}. $$
Evaluating the integrals on the r.h.s., we obtain

\[ I_3 = A(1, 1, 2 - 2\epsilon) \frac{1}{[12]^{1 - \epsilon}} \left[ 2\epsilon J(2 - 3\epsilon, \epsilon, 1) - J(1 - 3\epsilon, \epsilon, 2) \right] + \left[ A(1, 2, 1 - 2\epsilon) A(1 + \epsilon, 1, 2 - 3\epsilon) + A(1, 1, 2 - 2\epsilon) A(\epsilon, 2, 2 - 3\epsilon) \right] \frac{1}{[12]^{1 - 2\epsilon} [13]^{1 - 2\epsilon} [23]^{2\epsilon}}. \]

Because of the relations (A1), the integral \( I_3 \) has the form

\[ \frac{I_3}{A(1, 1, 2 - 2\epsilon)} = \frac{1}{[12]^{1 - \epsilon}} \left[ 2\epsilon J(2 - 3\epsilon, \epsilon, 1) - J(1 - 3\epsilon, \epsilon, 2) \right] + 2A(\epsilon, 2, 2 - 3\epsilon) \frac{1}{[12]^{1 - 2\epsilon} [13]^{1 - 2\epsilon} [23]^{2\epsilon}}. \] (A2)

Both r.h.s. integrals \( J(2 - 3\epsilon, \epsilon, 1) \) and \( J(1 - 3\epsilon, \epsilon, 2) \) have singularities at \( \epsilon \to 0 \) and it is convenient to express \( J(1 - 3\epsilon, \epsilon, 2) \) through \( J(2 - 3\epsilon, \epsilon, 1) \) and \( J(1 - 3\epsilon, 1 + \epsilon, 1) \). The last integral is finite at \( \epsilon \to 0 \).

Applying IBP to the integral \( J(1 - 3\epsilon, \epsilon, 2) \) with the distinguished line having the power 2, we obtain

\[ -J(1 - 3\epsilon, \epsilon, 2) = \epsilon \left[ J(1 - 3\epsilon, 1 + \epsilon, 1) - [23]J(1 - 3\epsilon, 1 + \epsilon, 2) \right] + (1 - 3\epsilon) J(2 - 3\epsilon, \epsilon, 1) \]

\[ -[13]J(2 - 3\epsilon, \epsilon, 2) = \epsilon J(1 - 3\epsilon, 1 + \epsilon, 1) + (1 - 3\epsilon) J(2 - 3\epsilon, \epsilon, 1) \]

\[ -\left[ \epsilon A(1 + \epsilon, 2, 1 - 3\epsilon) + (1 - 3\epsilon) A(\epsilon, 2, 2 - 3\epsilon) \right] \frac{1}{[12]^{1 - 2\epsilon} [13]^{1 - 2\epsilon} [23]^{2\epsilon}}. \]

Because of relations (A1), integral \( J(1 - 3\epsilon, \epsilon, 2) \) obtains the form

\[ -J(1 - 3\epsilon, \epsilon, 2) = \epsilon J(1 - 3\epsilon, 1 + \epsilon, 1) + (1 - 3\epsilon) J(2 - 3\epsilon, \epsilon, 1) \]

\[ -\frac{1 - 3\epsilon}{1 - 2\epsilon} A(\epsilon, 2, 2 - 3\epsilon) \frac{1}{[12]^{-\epsilon} [13]^{1 - 2\epsilon} [23]^{2\epsilon}}. \]

Then, for the integral \( I_3 \) we obtain

\[ \frac{I_3}{A(1, 1, 2 - 2\epsilon)} = \frac{1}{[12]^{1 - \epsilon}} \left[ (1 - \epsilon) J(2 - 3\epsilon, \epsilon, 1) + J(1 - 3\epsilon, 1 + \epsilon, 1) \right] \]

\[ + \frac{1 - \epsilon}{1 - 2\epsilon} A(\epsilon, 2, 2 - 3\epsilon) \frac{1}{[12]^{1 - 2\epsilon} [13]^{1 - 2\epsilon} [23]^{2\epsilon}}. \]

Combining all the results, we obtain for expression \( T \):

\[ T = I + I(2 \leftrightarrow 3), \] (A3)

where \( I = I_1 - (1 - \epsilon/2)I_2 + (1 - \epsilon)(I_3 - I_4) \) has the following form:

\[ \frac{1 - 2\epsilon}{A(1, 1, 2 - 2\epsilon)} I = \frac{1}{[12]^{1 - \epsilon}} \left[ (2 - 3\epsilon) J(2 - 3\epsilon, \epsilon, 1) + J(1 - 3\epsilon, 1 + \epsilon, 1) \right] \]

\[ + \left[ \frac{\epsilon}{1 - 2\epsilon} A(\epsilon, 2, 2 - 3\epsilon) \right] [12]^{\epsilon} [13]^{\epsilon} [23]^{2\epsilon} - (1 - \epsilon) A(1, 1, 2 - 2\epsilon) \frac{1}{[12]^{1 - 2\epsilon} [13]^{1 - \epsilon}}. \] (A4)
This formula presents the result for the expression $T$. It contains two integrals $J(2 - 3\epsilon, 1\epsilon)$ and $J(1 - 3\epsilon, 1, 1 + \epsilon)$ which cannot be calculated by rules from the previous section and will be calculated in Appendix B by using GPT.

10 Appendix B

In Appendix B we consider the integral $J(2 - 3\epsilon, 1\epsilon)$.

1. There is a relation between two integrals $J(2 - 3\epsilon, 1\epsilon)$ and $J(\lambda, \lambda, 2\lambda)$ where $\lambda = 1 - \epsilon$. The last integral is more convenient to calculate by GPT.

First, we transform using the uniqueness relation

\[
J(2 - 3\epsilon, 1\epsilon) = \frac{1}{[23]^{1-2\epsilon}} J(2 - 3\epsilon, 1\epsilon)[23]^{1-2\epsilon} = \int Dx \frac{1}{[23]^{1-2\epsilon}[x]^{2-3\epsilon}[x^2][x^3][23]^{1-2\epsilon}} = \int Dy \frac{1}{[y]^{1+\epsilon}[y^2]^{1-\epsilon}[y^3]^{2-2\epsilon} A(1 + \epsilon, 1 - \epsilon, 2 - 2\epsilon)} = \int Dy \frac{1}{[y]^{1-\epsilon}[y^2]^{1-\epsilon}[y^3]^{2-2\epsilon} A(2 - 3\epsilon, 1 + \epsilon, 1)}.
\]

2. Thus, we calculate the integral appearing above, by applying GPT (one can put $x_3 = 0$ by shifting the arguments), following [29]

\[
J(\lambda, \lambda, 2\lambda) = \int Dx \sum_{n=0}^\infty M_n(\lambda) x^{\lambda^1\mu_2\ldots\mu_n} x_1^{\lambda^1\mu_2\ldots\mu_n} \left[ \frac{\theta(x_1)}{[x]^{\lambda+n}} + \frac{\theta(x_2)}{[x^2]\lambda^2} \right] \frac{1}{[x]^{2\lambda}} = \frac{1}{[1]^{3\lambda+n-1}(n+\lambda)} \left( \frac{\theta(12)}{3\lambda + n - 1} - \frac{\theta(12)}{2\lambda - 1} \right) + \frac{1}{[2]^{3\lambda+n-1} \lambda(n+\lambda)} \left[ \frac{\theta(12)}{2\lambda - 1} - \frac{\theta(21)}{n - \lambda + 1} \right].
\]

We introduce notation

\[
M_n(\lambda) \equiv \frac{2^n \Gamma(n + 1)}{n! \Gamma(\lambda)}, \quad \theta(12) \equiv \theta(x_1^2 - x_2^2).
\]

Here, $M_n(\lambda)$ is the coefficient at the traceless products appearing in the Gegenbauer polynomials $\mathcal{C}_{n}^{\lambda}(\hat{x}_1 \hat{x}_2)$

\[
\hat{C}_{n}^{\lambda}(\hat{x}_1 \hat{x}_2) = M_n(\lambda) \frac{x_1^\lambda x_2^\lambda}{(x_1^n)(x_2^n)}
\]

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Transforming the previous expression one obtains:

\[
J(\lambda, \lambda, 2\lambda) = \frac{1}{\Gamma(\lambda)} \frac{1}{(1 - 2\lambda)} \sum_{n=0}^{\infty} M_n(\lambda) x_1^{\mu_1 \mu_2 \cdots \mu_n} x_2^{\mu_2 \cdots \mu_n} \left[ \theta(21) \left( \frac{1}{[2]^{3\lambda + n - 1}(3\lambda + n - 1)} - \frac{1}{[1]^{2\lambda - 1}[2]^{n + \lambda}} \right) + \theta(12) \left( \frac{1}{[1]^{3\lambda + n - 1}(3\lambda + n - 1)} - \frac{1}{[2]^{2\lambda - 1}[1]^{\lambda + n}} \right) \right]
\]

\[
\equiv \frac{1}{\Gamma(\lambda)} \frac{1}{(1 - 2\lambda)} \left[ \theta(21) j_{21}(\lambda, \lambda, 2\lambda) + \theta(12) j_{12}(\lambda, \lambda, 2\lambda) \right],
\]

where \( j_{12}(\lambda, \lambda, 2\lambda) = j_{21}(\lambda, \lambda, 2\lambda) [1] \leftrightarrow [2] \).

Thus, we can consider below only the case \( \theta(21) \). It is convenient to use Gegenbauer polynomial itself to reconstruct the results for \( J(\lambda, \lambda, 2\lambda) \).

Note that we can re-present the expression for \( j_{12}(\lambda, \lambda, 2\lambda) \) as

\[
J(\lambda, \lambda, 2\lambda) = \sum_{n=0}^{\infty} C_n(\hat{x}_1 \hat{x}_2) \left( \frac{1}{[2]^{2 - 3\epsilon}} \frac{1}{2 + n - 3\epsilon} - \frac{1}{[1]^{1 - 2\epsilon}} \frac{1}{[2]^{1 - \epsilon} n + \epsilon} \right).
\]

where \( \xi \equiv \sqrt{[1]/[2]} \). We would like to note that GPT has been used before only for the propagator-type diagrams (or for the vertex ones with very specific kinematics – see the last entry of Refs. [28]), where the results could be represented as some numbers, i.e., for example, with \( \xi = 1 \) and \( C_n(1) \) in above formula. Here, GPT is applied for the first time for the diagrams having two independent arguments. So, we need a technique for reconstruction of the final results from the expansion in Gegenbauer polynomials. To obtain it, we represent the above series in terms of integrals

\[
\sum_{n=0}^{\infty} C_n(\hat{x}_1 \hat{x}_2) \left( \frac{1}{[2]^{2 - 3\epsilon}} \frac{1}{2 + n - 3\epsilon} - \frac{1}{[1]^{1 - 2\epsilon}} \frac{1}{[2]^{1 - \epsilon} n + \epsilon} \right) = \int_0^\xi d\omega \omega^{n+1-3\epsilon} \frac{1}{[1]^{1 - \frac{3\epsilon}{2}} [2]^{1 - \frac{1}{2}\epsilon}} \sum_{n=0}^{\infty} C_n(\hat{x}_1 \hat{x}_2) \left( \omega^{n+1-3\epsilon} - \omega^{n+\epsilon-1} \right)
\]

\[
\frac{1}{[1]^{1 - \frac{3\epsilon}{2}} [2]^{1 - \frac{1}{2}\epsilon}} \int_0^\xi d\omega \omega^{n+1-3\epsilon} \frac{1}{[1]^{1 - \frac{3\epsilon}{2}} [2]^{1 - \frac{1}{2}\epsilon}} \int_0^\xi d\omega \omega^{n+\epsilon-1}
\]

New notation is introduced

\[
(\hat{x}_1 \hat{x}_2) \equiv \cos \theta.
\]

The second integral over \( \omega \) in the above expression can be re-written as

\[
\int_0^\xi d\omega \omega^{\epsilon-1} = \int_0^\xi d\omega \omega^{\epsilon} \frac{1}{1 - 2 \cos \theta \omega + \omega^2} \left[ \frac{1}{\omega} + \frac{2 \cos \theta - \omega}{1 - 2 \cos \theta \omega + \omega^2} \right] = \int_0^\xi d\omega \frac{1}{\epsilon} (1 - 2 \cos \theta \omega + \omega^2)^\epsilon + \int_0^\xi d\omega \omega^{\epsilon} \left[ \frac{2 \cos \theta - \omega}{(1 - 2 \cos \theta \omega + \omega^2)^{1-\epsilon}} \right]
\]
\[
\xi^\varepsilon \frac{1}{\varepsilon} (1 - 2 \cos \theta \xi + \xi^2)^\varepsilon + \int_0^\xi d\omega \frac{4 \cos \theta \omega^\varepsilon - 3 \omega^{1+\varepsilon}}{(1 - 2 \cos \theta \omega + \omega^2)^{1-\varepsilon}} = \\
\frac{1}{\varepsilon} \left( \frac{1}{1^{\varepsilon/2}} \right)^{1/2} \left( \frac{1}{2} \right)^{1/2} \int_0^\xi d\omega \frac{4 \cos \theta \omega^\varepsilon - 3 \omega^{1+\varepsilon}}{(1 - 2 \cos \theta \omega + \omega^2)^{1-\varepsilon}}.
\]

In such a way we have extracted the pole in \(\varepsilon\). The integral in the last line is not singular. Now the total expression for \(j_{21}\) is

\[
q_{j21}(\lambda, \lambda, 2\lambda) = \frac{[12]^{-\varepsilon}}{[1]^{1-2\varepsilon} [2]^{1-2\varepsilon}} \left( -\frac{1}{\varepsilon} \left( \frac{[12]}{2} \right) \right)^{2\varepsilon} + \frac{[12]^\varepsilon}{[1]^{1/2} [2]^{1/2}} \int_0^\xi d\omega \frac{-4 \cos \theta \omega^\varepsilon + 3 \omega^{1+\varepsilon} + \omega^{1-3\varepsilon}}{(1 - 2 \cos \theta \omega + \omega^2)^{1-\varepsilon}}.
\]

To check the self-consistency of the result, we have to check that the theta-functions disappear and that we left with a Lorentz-invariant structure. For the poles this is obvious. Let us check this for the zeroth order in \(\varepsilon\). The integral can be decomposed as

\[
\int_0^\xi d\omega \frac{-4 \cos \theta \omega^\varepsilon + 3 \omega^{1+\varepsilon} + \omega^{1-3\varepsilon}}{(1 - 2 \cos \theta \omega + \omega^2)^{1-\varepsilon}} = \int_0^\xi d\omega \frac{-4 \cos \theta + 4\omega}{1 - 2 \cos \theta \omega + \omega^2} + \varepsilon \left( -4 \cos \theta \int_0^\xi d\omega \frac{\ln \omega}{1 - 2 \cos \theta \omega + \omega^2} + \int_0^\xi d\omega \frac{(-4 \cos \theta + 4\omega) \ln(1 - 2 \cos \theta \omega + \omega^2)}{1 - 2 \cos \theta \omega + \omega^2} \right) + o(\varepsilon).
\]

Since we have to calculate the finite part of the initial diagram, the higher orders in \(\varepsilon\) do not contribute in the limit \(\varepsilon \to 0\). The first and the third integral can be easily calculated

\[
\int_0^\xi d\omega \frac{-4 \cos \theta + 4\omega}{1 - 2 \cos \theta \omega + \omega^2} = 2 \int_0^\xi d\ln(1 - 2 \cos \theta \omega + \omega^2) = 2 \ln(1 - 2 \cos \theta \xi + \xi^2) = 2 \ln \left[ \frac{[12]}{2} \right],
\]

\[
\int_0^\xi d\omega \frac{(-4 \cos \theta + 4\omega) \ln(1 - 2 \cos \theta \omega + \omega^2)}{1 - 2 \cos \theta \omega + \omega^2} = \int_0^\xi d\ln^2(1 - 2 \cos \theta \omega + \omega^2) = \ln^2 \left[ \frac{[12]}{2} \right].
\]

Moreover, by the conformal substitution \(\omega \to 1/\omega\) it can be immediately verified that

\[
\int_0^\xi d\omega \frac{\ln \omega}{1 - 2 \cos \theta \omega + \omega^2} = \int_0^{1/\xi} d\omega \frac{\ln \omega}{1 - 2 \cos \theta \omega + \omega^2},
\]

i.e. the integral is symmetric under the exchange \([1] \leftrightarrow [2]\).

Expanding the r.h.s. of (B2) in powers of \(\varepsilon\), we obtain that the terms of zero order in \(\varepsilon\) are absent

\[
-2 \ln \left[ \frac{[12]}{2} \right] + 2 \ln \left[ \frac{[12]}{2} \right] = 0,
\]

and that the logarithms of the first order in \(\varepsilon\) are

\[
-\ln^2 \left[ \frac{[12]}{2} \right] + \ln \left[ \frac{[12]}{2} \right] \ln \left[ \frac{[12]^2}{1[2]} \right] = \ln \left[ \frac{[12]}{2} \right] \ln \left[ \frac{[12]}{1} \right].
\]
Thus, we present expression (B2) in the form

$$j_{21}(\lambda, \lambda, 2\lambda) = \frac{[12]^{-\epsilon}}{[1]^{1/2}[2]^{1/2}} \left[ -\frac{1}{\epsilon} + \epsilon \left( \ln \frac{[12]}{[2]} \ln \frac{[12]}{[1]} - 4 \cos \theta \int_0^\xi d\omega \frac{\ln \omega}{1 - 2 \cos \theta \omega + \omega^2} \right) \right]. \quad \text{(B3)}$$

The result (B3) has completely symmetric form under the exchange $[1] \leftrightarrow [2]$

$$j_{21}(\lambda, \lambda, 2\lambda) = j_{12}(\lambda, \lambda, 2\lambda)$$

and therefore

$$J(\lambda, \lambda, 2\lambda) = \frac{1}{\Gamma(1-\epsilon)} \frac{1}{2\epsilon - 1} \frac{[12]^{-\epsilon}}{[1][1^{1/2}[2]^{1/2}]} \left[ -\frac{1}{\epsilon} + \epsilon \left( \ln \frac{[12]}{[2]} \ln \frac{[12]}{[1]} - 4 \cos \theta \int_0^\xi d\omega \frac{\ln \omega}{1 - 2 \cos \theta \omega + \omega^2} \right) \right]. \quad \text{(B4)}$$

All the theta functions disappeared and the result is Lorentz invariant. Now we restore $x_3$ component by shifting back the space-time coordinates $x_1$ and $x_2$. Taking this into account, we obtain

$$\cos \theta = \frac{(x_1 x_2)}{[1^{1/2}[2]^{1/2}]^2} = \frac{1}{2} \frac{[1][2] - [12]}{[1][1^{1/2}[2]^{1/2}]} ,$$

and introducing the notation

$$\int_0^\xi d\omega \frac{\ln \omega}{1 - 2 \cos \theta \omega + \omega^2} \equiv I(x_1, x_2, x_3) \equiv I_{123}$$

we can write Eq. (B4) in the form

$$J(\lambda, \lambda, 2\lambda) = \frac{1}{\Gamma(1-\epsilon)} \frac{1}{2\epsilon - 1} \frac{[12]^{-\epsilon}}{[1][1^{1/2}[2]^{1/2}]} \left[ -\frac{1}{\epsilon} + \epsilon \left( \ln \frac{[12]}{[2]} \ln \frac{[12]}{[1]} - 2 \frac{[13] + [23] - [12]}{[1][1^{1/2}[2]^{1/2}]} I_{123} \right) \right]. \quad \text{(B5)}$$

3. Now we want to express the integral $I_{123}$ through $J(1,1,1)$. It is convenient to start with the integral $J(\lambda, \lambda, 2\lambda - 1)$, with the purpose to use the results of the previous subsection. Applying GPT we calculate the integral (one can put $x_3 = 0$ by shifting the arguments)

$$J(\lambda, \lambda, 2\lambda - 1) = \int Dx \sum_{n=0}^\infty M_n(\lambda)x_1^{\mu_1 \mu_2 \cdots \mu_n} x_2^{\mu_1 \mu_2 \cdots \mu_n} \left[ \frac{\theta(x_1)}{[x]^{\lambda+n}} + \frac{\theta(1x)}{[1]^{\lambda+n}} \right] \frac{1}{[x2]^{\lambda}} \frac{1}{[2]^{2\lambda-1}} =$$

$$\frac{1}{\Gamma(\lambda)} \sum_{n=0}^\infty M_n(\lambda) x_1^{\mu_1 \mu_2 \cdots \mu_n} x_2^{\mu_1 \mu_2 \cdots \mu_n} \left[ \frac{\theta(21)}{[2]^{3\lambda+n-2}} \frac{1}{[2]^{2\lambda-1}} \right] +$$

$$\frac{1}{[1]^{3\lambda+n-2}(n+\lambda)} \left( \frac{\theta(12)}{3\lambda+n-2} \frac{\theta(21)}{2-2\lambda} \left( \frac{[1]}{[2]} \right)^{n+\lambda} \right) +$$

$$\frac{1}{[2]^{2\lambda-2}} \frac{\theta(12)}{[1]^{\lambda+n}} \frac{1}{(2\lambda-2)(n-\lambda+2)} - \frac{1}{[1]^{3\lambda+n-2}(n+\lambda)} \left( \frac{\theta(12)}{2\lambda-2} \frac{\theta(21)}{n-\lambda+2} \left( \frac{[1]}{[2]} \right)^{n+\lambda} \right).$$
Transforming the previous expression one obtains

\[ J(\lambda, \lambda, 2\lambda - 1) = \]

\[
\frac{1}{\Gamma(\lambda) (2 - 2\lambda)} \sum_{n=0}^\infty M_n(\lambda) x_1^{\mu_1 + \mu_2 - \mu_n} x_2^{\mu_1 + \mu_2 - \mu_n} \left[ \theta(21) \left( \frac{1}{[2]^{3\lambda+n-2} (3\lambda+n-2)} - \frac{1}{[2]^{2\lambda-2} [1]^{3\lambda+n-2} 3\lambda+n-2} \right) \right]
\]

\[
- \frac{1}{[1]^{2\lambda-2} [2]^{n+\lambda} n - \lambda + 2} + \theta(12) \left( \frac{1}{[1]^{3\lambda+n-2} 3\lambda+n-2} - \frac{1}{[2]^{2\lambda-2} [1]^{\lambda+n} n - \lambda + 2} \right)
\]

\[
\equiv \frac{1}{\Gamma(\lambda) (2 - 2\lambda)} \left[ \theta(21) j_{21}(\lambda, \lambda, 2\lambda - 1) + \theta(12) j_{12}(\lambda, \lambda, 2\lambda - 1) \right],
\]

where \( j_{12}(\lambda, \lambda, 2\lambda - 1) = j_{21}(\lambda, \lambda, 2\lambda - 1) \). Thus, we can consider below only the case \( \theta(21) \). As in the previous subsection, it is convenient to use Gegenbauer polynomial itself to reconstruct the results for \( J(\lambda, \lambda, 2\lambda) \) obtained above. Applying the GPT technique one can represent the above expression as

\[
j_{21}(\lambda, \lambda, 2\lambda - 1) = \sum_{n=0}^\infty C_n^\lambda(\hat{x}_1 \hat{x}_2) \left( \frac{1}{[2]^{1-3\epsilon}} \frac{\xi^n}{1 + n - 3\epsilon} - \frac{1}{[1]^{1-2\epsilon} [2]^{1-\epsilon}} \frac{\xi^n}{n + 1 + \epsilon} \right).
\]

We now use the GTP formulas to present series in terms of integrals

\[
\sum_{n=0}^\infty C_n^\lambda(\hat{x}_1 \hat{x}_2) \left( \frac{1}{[2]^{1-3\epsilon}} \frac{1}{[1]^{1-\epsilon} [2]^{1-\epsilon}} \int_0^\xi d\omega \omega^{n-3\epsilon} - \frac{1}{[1]^{1-2\epsilon} [2]^{1-\epsilon}} \int_0^\xi d\omega \omega^{n+\epsilon} \right)
\]

\[
= \frac{1}{[1]^{1-\frac{3\epsilon}{2}} [2]^{1-\frac{\epsilon}{2}}} \int_0^\xi d\omega \sum_{n=0}^\infty C_n^\lambda(\hat{x}_1 \hat{x}_2) \left( \omega^{n-3\epsilon} - \omega^{n+\epsilon} \right)
\]

\[
= \frac{1}{[1]^{1-\frac{3\epsilon}{2}} [2]^{1-\frac{\epsilon}{2}}} \int_0^\xi d\omega \left[ \frac{(\omega^{-3\epsilon} - \omega^\epsilon)}{(1 - 2 \cos \theta \omega + \omega^2)\lambda} \right]
\]

\[
= \frac{1}{[1]^{1-\frac{3\epsilon}{2}} [2]^{1-\frac{\epsilon}{2}}} \left[ -4\epsilon \int_0^\xi d\omega \ln \omega \frac{1 - 2 \cos \theta \omega + \omega^2 + o(\epsilon)}{1 - 2 \cos \theta \omega + \omega^2 + o(\epsilon)} \right]
\]

\[
= \frac{1}{[1]^{1-\frac{3\epsilon}{2}} [2]^{1-\frac{\epsilon}{2}}} \left[ -4\epsilon I_{123} + o(\epsilon) \right].
\]

The result has completely symmetric form under \([1] \leftrightarrow [2]\):

\[
j_{21}(\lambda, \lambda, 2\lambda - 1) = j_{12}(\lambda, \lambda, 2\lambda - 1)
\]

and

\[
J(\lambda, \lambda, 2\lambda - 1) = \frac{1}{[1]^{1/2} [2]^{1/2}} \left[ -2I_{123} + o(1) \right].
\]

Now we restore the \( x_3 \)-dependence and obtain the relation

\[
J(1, 1, 1) = -\frac{2}{[13]^{1/2}[23]^{1/2}} I_{123}.
\]
The following consequences can be derived from here:

\[
\frac{1}{[13]^{1/2} [23]^{1/2}} I_{123} = \frac{1}{[12]^{1/2} [23]^{1/2}} I_{132} \quad \Rightarrow \quad I_{123} = \frac{[13]^{1/2}}{[12]^{1/2}} I_{132}.
\]

Thus, we can write Eq. (B5) in the form

\[
J(\lambda, \lambda, 2\lambda) = \frac{1}{\Gamma(1 - \epsilon)} \frac{1}{2\epsilon - 1} \frac{[12]^{-\epsilon}}{[13]^{1 - 2\epsilon} [23]^{-2\epsilon}} \left[ -\frac{1}{\epsilon} + \epsilon \left( \ln \frac{[12]}{[23]} \ln \frac{[12]}{[13]} + (13 + [23] - [12]) J(1, 1, 1) \right) \right].
\]

11 Appendix C

In Appendix C we consider the integrals \(J(1 - 2\epsilon, 2, 2)\) and \(J(2\epsilon, 1, 2)\).

1. The integral \(J(1 - 2\epsilon, 2, 2)\) can be represented as

\[
J(1 - 2\epsilon, 2, 2) = \frac{1}{4\epsilon} \partial^{(3)}_\mu \partial^{(3)}_\mu J(1 - 2\epsilon, 2, 1) = \frac{1}{4\epsilon} A(1 - 2\epsilon, 2, 1) \partial^{(3)}_\mu \partial^{(3)}_\mu \frac{[13]^{\epsilon}}{[12]^{1 - \epsilon} [23]^{1 + \epsilon}}.
\]

Because

\[
8\epsilon(1 + \epsilon) \left[ \frac{[13]^{\epsilon}}{[12]^{1 - \epsilon} [23]^{2 + \epsilon}} - \frac{(23)_\mu (13)_\mu}{[12]^{1 - \epsilon} [13]^{1 - \epsilon} [23]^{2 + \epsilon}} \right] + 4\epsilon \frac{1}{[12]^{1 - \epsilon} [13]^{1 - \epsilon} [23]^{2 + \epsilon}} = 4\epsilon(1 + \epsilon) \left[ \frac{[13]^{\epsilon}}{[12]^{1 - \epsilon} [23]^{2 + \epsilon}} + \frac{[12]^{\epsilon}}{[13]^{1 - \epsilon} [23]^{2 + \epsilon}} \right] - 4\epsilon^2 \frac{1}{[12]^{1 - \epsilon} [13]^{1 - \epsilon} [23]^{2 + \epsilon}},
\]

we have

\[
J(1 - 2\epsilon, 2, 2) / A(1 - 2\epsilon, 2, 1) = (1 + \epsilon) \left[ \frac{[13]^{\epsilon}}{[12]^{1 - \epsilon} [23]^{2 + \epsilon}} + \frac{[12]^{\epsilon}}{[13]^{1 - \epsilon} [23]^{2 + \epsilon}} \right] - \epsilon \frac{1}{[12]^{1 - \epsilon} [13]^{1 - \epsilon} [23]^{2 + \epsilon}}.
\]

(C1)

2. The integral \(J(-2\epsilon, 1, 2)\) can be rewritten, using uniqueness, as

\[
J(-2\epsilon, 1, 2) = [12]^{1 + \epsilon} \int Dx \frac{1}{[12]^{1 + \epsilon} [x1]^{-2\epsilon} [x2] [x3]^{2}} = \frac{1}{A(2 + \epsilon, 1 - \epsilon, 1 - 2\epsilon)} \int Dx \frac{1}{[x3]^{2}} \int Dy \frac{1}{[y1]^{1 - \epsilon} [y2]^{2 + \epsilon} [yx]^{1 - 2\epsilon}} = [12]^{1 + \epsilon} \frac{A(2, 1 - 2\epsilon, 1)}{A(2 + \epsilon, 1 - \epsilon, 1 - 2\epsilon)} \int Dy \frac{1}{[y1]^{1 - \epsilon} [y2]^{2 + \epsilon} [y3]^{1 - \epsilon}} = [12]^{1 + \epsilon} \frac{A(2, 1 - 2\epsilon, 1)}{A(2 + \epsilon, 1 - \epsilon, 1 - 2\epsilon)} J(1 - \epsilon, 2 + \epsilon, 1 - \epsilon).
\]
The integral \( J(1 - \epsilon, 2 + \epsilon, 1 - \epsilon) \) can be treated in a similar manner

\[
J(2 - 3\epsilon, \epsilon, 1) = [13]^e \int Dx \frac{1}{x^2|2\epsilon|} \int Dy \frac{1}{y^2|2\epsilon|} = [13]^e \frac{A(2 + \epsilon, 2 - 2\epsilon, -\epsilon)}{A(1, 1, 2 - 2\epsilon)} J(1, 2, 1). 
\]

So, for the initial diagram \( J(-2\epsilon, 1, 2) \) we have

\[
J(-2\epsilon, 1, 2) = [12]^{1+\epsilon}[13]^e J(1, 2, 1),
\]

because

\[
\frac{A(2, 1 - 2\epsilon, 1)}{A(2 + \epsilon, 1 - \epsilon, 1 - 2\epsilon)} = 1.
\]

Evaluation of the integral \( J(1, 1, 1) \) is very simple. Following Ref. [35], we apply IBP to \( J(1, 1, 1) \) with different distinguished lines \(^4\):

\[
(D - 4)J(1, 1, 1) = 2J(0, 2, 1) - [12]J(1, 2, 1) - [13]J(1, 1, 2), \quad (C2)
\]

\[
(D - 4)J(1, 1, 1) = 2J(1, 0, 2) - [23]J(1, 1, 2) - [12]J(2, 1, 1), \quad (C3)
\]

\[
(D - 4)J(1, 1, 1) = 2J(1, 2, 0) - [23]J(1, 2, 1) - [13]J(2, 1, 1). \quad (C4)
\]

Considering the combination

\[
[23] < (B2) > - [13] < (B3) > + [12] < (B4) > ,
\]

where symbols \( < (B2) >, < (B3) > \) and \( < (B4) > \) represent the results of Eqs. (C2), (C3) and (C4), respectively, we have

\[
(D - 4)
\left( [23] - [13] + [12] \right)
J(1, 1, 1) = 2
\left( [23]J(0, 2, 1) - [13]J(1, 0, 2) + [12]J(1, 2, 0) \right)
\]

\[
- 2[23][12]J(1, 2, 1). 
\]

The expression on the l.h.s. is negligible when \( D \to 4 \). Thus

\[
J(1, 2, 1) = \frac{1}{[12][23]}
\left( [23]J(0, 2, 1) - [13]J(1, 0, 2) + [12]J(1, 2, 0) \right)
\]

and

\[
J(1, 2, 1)/A(1, 2, 1 - 2\epsilon) = \frac{[12]^e[13]^e}{[23]}
\left[ [12]^{-\epsilon} - [13]^{-\epsilon} + [23]^{-\epsilon} \right]. \quad (C5)
\]

\(^4\)In momentum space, similar analysis has been done in [36, 31].
In Appendix C we calculate the most complicated term \( V^{(3)} \) contributed to the first diagram in Fig. 1.

1. The first term \( V_1^{(3)} \) has the form:

\[
V_1^{(3)} / J(1, 1, 1) = P_{\mu\nu} \partial^{(2)}_\mu \partial^{(3)}_\nu \frac{1}{[12][13]} \left( [12] + [13] - 2[23] \right).
\]

The derivatives generate

\[
\partial^{(2)}_\mu \partial^{(3)}_\nu \frac{1}{[12][13]} \left( [12] + [13] - 2[23] \right) = \frac{4}{[12]^2[13]^2} \left( [12][13]g_{\mu\nu} - 2[12](23)_{\mu}(13)_{\nu} + 2[13](23)_{\mu}(12)_{\nu} - 2[23](12)_{\mu}(13)_{\nu} \right).
\]

After simple algebra, we have

\[
V_1^{(3)} = \frac{8B_3}{[12]^2[13]^2[23]} J(1, 1, 1),
\]

where

\[
B_3 = ([12] + [13])^2 - [12][13] - 2([12] + [13])[23] + [23]^2.
\]

2. The third term \( V_3^{(3)} \) is

\[
V_3^{(3)} \frac{[12][13][23]^2}{[12] + [13] - 2[23]} = \left( [23]^2 P_{\mu\nu} \partial^{(2)}_\mu \partial^{(3)}_\nu J(1, 1, 1) = \right.
\]

\[
2 \left( \frac{[23]}{2} \partial^{(2)}_\mu \partial^{(3)}_\nu J(1, 1, 1) + (23)_{\mu}(23)_{\nu} \partial^{(2)}_\mu \partial^{(3)}_\nu J(1, 1, 1) \right).
\]

The first term in brackets generates the additional factor

\[
\frac{2(2x)_{\mu}(3x)_{\mu}}{[2x][3x]} = \frac{2x + [3x] - [23]}{[2x][3x]}
\]

in the subintegral expression of \( J(1, 1, 1) \), where \( x \) is the variable of integration. The term has the form\(^5\)

\[
\frac{1}{2} \partial^{(2)}_\mu \partial^{(3)}_\mu J(1 - 2\epsilon, 1, 1) = J(1 - 2\epsilon, 1, 2) + J(1 - 2\epsilon, 2, 1) - [23]J(1 - 2\epsilon, 2, 2).
\]

The second term can be calculated similarly: \( (23)_{\mu}(23)_{\nu} \partial^{(2)}_\mu \partial^{(3)}_\nu J(1, 1, 1) \) generates

\[
\frac{4(23)_{\mu}(2x)_{\mu}(3x)_{\nu}}{[2x][3x]}
\]

---

\(^5\)Hereafter we change \( J(1, 1, 1) \rightarrow J(1 - 2\epsilon, 1, 1) \) to have regularization and uniqueness of the “star”. [25]
in the subintegral expression of $J(1, 1, 1)$.

By analogy with above case we obtain, after some algebra

$$
(23)_\mu(23)_\nu \partial_\mu^{(2)} \partial_\nu^{(3)} J(1 - 2\epsilon, 1, 1) = J(1 - 2\epsilon, 0, 2) + J(1 - 2\epsilon, 2, 0) - 2J(1 - 2\epsilon, 1, 1) - [23]^2 J(1 - 2\epsilon, 2, 2) .
$$

Then

$$
V_3^{(3)} \frac{[12][13][23]^2}{[12] + [13] - 2[23]} = 2 \left\{ [23] J(1 - 2\epsilon, 2, 1) + J(1 - 2\epsilon, 2, 0) - J(1 - \epsilon, 1, 1) - [23]^2 J(1 - 2\epsilon, 2, 2) \right\}
$$

+ \left\{ 2 \leftrightarrow 3 \right\} = 2 \left. V_3^{(3)} + \left\{ V_3^{(3)}, 2 \leftrightarrow 3 \right\} \right].
$$

Taking the result for $J(1 - 2\epsilon, 2, 2)$ from Appendix B and using

$$
J(1 - 2\epsilon, 2, 2) = A(1 - 2\epsilon, 2, 1) \frac{1}{[12]^{1-\epsilon}} ,
$$

$$
J(1 - 2\epsilon, 2, 1) = A(1 - 2\epsilon, 2, 1) \frac{[13]^\epsilon}{[12]^{1-\epsilon} [23]^{1+\epsilon}} ,
$$

we have

$$
\left( V_3^{(3)} + J(1, 1, 1) \right)/A(1 - 2\epsilon, 2, 1) = \frac{[13]^\epsilon}{[12]^{1-\epsilon} [23]^{1+\epsilon}} + \frac{1}{[12]^{1-\epsilon}} - (1 + \epsilon) \left[ \frac{[13]^\epsilon}{[12]^{1-\epsilon} [23]^{1+\epsilon}} + \frac{[12]^\epsilon}{[13]^{1-\epsilon} [23]^{1+\epsilon}} + \epsilon \frac{[23]^{1-\epsilon}}{[12]^{1-\epsilon} [13]^{1-\epsilon}} \right] = 1 - \epsilon \frac{1}{[12]^{1-\epsilon}} - \frac{1}{[13]^{1-\epsilon}} \left( 1 + \epsilon \left[ 1 + \ln \frac{[12]}{[23]} \right] \right) + \frac{\epsilon [23]}{[12][13]} .
$$

Then

$$
\left( V_3^{(3)} + \left\{ V_3^{(3)}, 2 \leftrightarrow 3 \right\} + 2J(1, 1, 1) \right)/A(1 - 2\epsilon, 2, 1) = \epsilon \left( 2 \frac{[23]}{[12][13]} - \frac{1}{[12]} \left( 2 + \epsilon \ln \frac{[13]}{[23]} \right) - \frac{1}{[13]} \left( 2 + \epsilon \ln \frac{[12]}{[23]} \right) \right) .
$$

Because $\epsilon A(1 - 2\epsilon, 2, 1) = -1 + o(\epsilon)$, we have

$$
V_3^{(3)} + \left\{ V_3^{(3)}, 2 \leftrightarrow 3 \right\} + 2J(1, 1, 1) = \frac{1}{[12][13]} \left( 2 ([12] + [13] - [23]) + [12] \ln \frac{[12]}{[23]} + [13] \ln \frac{[13]}{[23]} \right) .
$$

Thus,

$$
V_3^{(3)} = \frac{2([12] + [13] - 2[23])}{[12]^2 [13]^2 [23]^2} \left( 2 ([12] + [13] - [23]) + [12] \ln \frac{[12]}{[23]} + [13] \ln \frac{[13]}{[23]} \right) - \frac{2[12][13] J(1, 1, 1)}{[12][13]} .
$$

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3. The most complicated second part of $V^{(3)}$ has the form

$$V_2^{(3)} = P_{\mu\nu} \left[ \partial_\mu^{(2)} \left( \frac{1}{[12][13]} ([12] + [13] - 2[23]) \right) \partial_\nu^{(3)} J(1, 1, 1) \right] + \left\{ 2 \leftrightarrow 3 \right\}.$$ 

The derivative $\partial_\mu^{(2)}$ generates

$$\partial_\mu^{(2)} \left( \frac{1}{[12][13]} ([12] + [13] - 2[23]) \right) = \frac{2}{[12]^2[13]} \left( ([13] - 2[23])(12)_\mu - 2[12](23)_\mu \right)$$

and after little algebra we have

$$P_{\mu\nu} \partial_\mu^{(2)} \left( \frac{1}{[12][13]} ([12] + [13] - 2[23]) \right) = \frac{2}{[12]^2[13][23]^2} \left[ \Phi_1(12)_\nu + \Phi_2(23)_\nu \right],$$

where

$$\Phi_1 = [23]([13] - 2[23]), \quad \Phi_2 = [13]([13] - [12]) - [23](3[13] + 4[12]) + 2[23]^2.$$ 

It is convenient to represent $(12)_\nu$ as $(12)_\nu = (13)_\nu - (23)_\nu$. Then

$$P_{\mu\nu} \partial_\mu^{(2)} \left( \frac{1}{[12][13]} ([12] + [13] - 2[23]) \right) = \frac{2}{[12]^2[13][23]^2} \left[ \Phi_1(13)_\nu + \Phi_2(23)_\nu \right],$$

where

$$\Phi_2 = [13]([13] - [12]) - 4([13] + [12])[23] + 4[23]^2.$$ 

Thus, the considered term $V_2^{(3)}$ has the form

$$V_2^{(3)} = 2 \left[ [13] \left[ \Phi_1(13)_\nu + \Phi_2(23)_\nu \right] \partial_\nu^{(3)} J(1, 1, 1) \right] + \left\{ 2 \leftrightarrow 3 \right\}$$

$$= 2 \left[ \tilde{V}_2^{(3)} + \left\{ \tilde{V}_2^{(3)}, 2 \leftrightarrow 3 \right\} \right],$$

where

$$\tilde{V}_2^{(3)} = [13] \left[ \Phi_1 W_1^{(3)} + \Phi_2 W_2^{(3)} \right].$$

Consider first $W_2^{(3)} = (23)_\nu \partial_\nu^{(3)} J(1, 1, 1)$. The derivative generates the term

$$\frac{2(23)_\nu (3x)_\nu}{[3x]} = -\frac{[23] + [3x] - [2x]}{[3x]}$$

in the subintegral expression of $J(1, 1, 1)$.

Thus, $W_2^{(3)}$ is

$$W_2^{(3)} = J(1, 1, 1) = -J(1 - 2\epsilon, 0, 2) + [23]J(1 - 2\epsilon, 1, 2).$$

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Using Eq.(D1) we obtain
\[
\left( W_2^{(3)} - J(1, 1, 1) \right) / A(1 - 2\epsilon, 2, 1) = \frac{[12][13]^{1-\epsilon}[23]^\\epsilon}{[23]} - \frac{1}{[13]^{1-\epsilon}} = \frac{\epsilon}{[13]} \ln \frac{[12]}{[23]} .
\]
Because \( \epsilon A(1 - 2\epsilon, 2, 1) = -1 + o(\epsilon) \), we have
\[
W_2^{(3)} = -\frac{1}{[13]} \ln \frac{[12]}{[23]} + J(1, 1, 1) .
\]

The operation \( W_1^{(3)} = (13)_x \partial_x J(1, 1, 1) \) generates the term
\[
\frac{2(13)_x(3x)_x}{[3x]} = -\frac{[13] + [3x] - [1x]}{[3x]}
\]
in the subintegral expression of \( J(1, 1, 1) \).

Thus, \( W_1^{(3)} \) is
\[
W_1^{(3)} = J(1, 1, 1) = -J(2\epsilon, 1, 2) + [13]J(1 - 2\epsilon, 1, 2) .
\]

Taking the integral \( J(2\epsilon, 1, 2) \) from Appendix B and using Eq.(D1), we obtain
\[
\left( W_1^{(3)} - J(1, 1, 1) \right) / A(1 - 2\epsilon, 2, 1) = \frac{[12][13]^{1-\epsilon}[23]^\\epsilon}{[23]} - \frac{1}{[12]^{1-\epsilon}} - \frac{1}{[13]^{1-\epsilon}} + \frac{1}{[23]^{1-\epsilon}} = \frac{\epsilon}{[23]} \ln \frac{[12]}{[13]} .
\]

Thus,
\[
W_1^{(3)} = -\frac{1}{[23]} \ln \frac{[12]}{[13]} + J(1, 1, 1) .
\]

Then
\[
\tilde{V}_2^{(3)} = -\frac{[13]}{[23]} \Phi_1 \ln \frac{[12]}{[13]} - \Phi_2 \ln \frac{[12]}{[23]} + [13] (\Phi_1 + \Phi_2) J(1, 1, 1)
\]
and
\[
V_2^{(3)} = \frac{2}{[12]^2[13]^2[23]^2} \left[ \phi_1 \ln \frac{[12]}{[23]} + \phi_2 \ln \frac{[13]}{[23]} + \phi_3 J(1, 1, 1) \right] ,
\]
where
\[
\phi_1 = -\Phi_2 + \left( [12] - [13] \right) \left( [13] + [12] - 2[23] \right) = \left( [12] - [13] \right) \left( 2[13] + [12] \right) + 2 \left( [12] + [13] \right) [23] - 4[23]^2 ,
\]
\[
\phi_2 = \left\{ \phi_1, 2 \leftrightarrow 3 \right\} ,
\]
\[
\phi_3 = [13] \left( \Phi_1 + \Phi_2 \right) + [12] \left\{ \left( \Phi_1 + \Phi_2 \right), 2 \leftrightarrow 3 \right\} = \left( [13] + [12] \right) \left( \left( [13] - [12] \right)^2 + 2[23]^2 \right) - \left( 3 \left( [13] + [12] \right)^2 + 2[12][13] \right) [23] .
\]
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