A Logical Approach to Generating Test Plans

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Abstract—During the execution of a test plan, a test manager may decide to drop a test case if its result can be inferred from already executed test cases. We show that it is possible to automatically generate a test plan to exploit the potential to justifiably drop a test case and thus reduce the number of test cases. Our approach uses Boolean formulas to model the mutual dependencies between test results. The algorithm to generate a test plan comes with the formal guarantee of optimality with regards to the inference of the result of a test case from already executed test cases.

Index Terms—Test Planning; Test Prioritization; Requirement Dependencies; Test Dependencies; Automotive Testing

1. Introduction

In this paper, we present the theoretical investigation of a question which is potentially of practical relevance. The question is whether automatically generated test plans can help with the problem that, in practice, we always need to reduce the number of test cases (see, e.g., [2], [13], [29]). The problem is exacerbated in system testing in the automotive industry (see, e.g., [7]). A single test case for a high-level test platform such as HiL (Hardware-in-the-Loop), SiL (Software-in-the-Loop), or the vehicle itself, typically involves complex test setups with hours of human labor and hours of execution. Every single opportunity to justifiably drop a test case is valuable.

Recent work [3] shows that one can exploit logical dependencies between system requirements in this context. The idea here is that, during the execution of a test plan, a test manager may decide to drop a test case if its result can be inferred from the already executed test cases. The inference of this logical redundancy of a test case can be done automatically.

The redundancy of a test case depends on the ordering of the test cases; it may be inferred in one ordering, but not in another. To give a simple example, we consider the logical dependency between the requirement $\text{req}_0$: "rain sensor detects rain" and the requirement $\text{req}_1$: "sunroof closes automatically when it is raining." Given that the rain sensor provides the only way to detect rain, the second requirement can only be satisfied if the first one is, formally $\text{req}_1 \Rightarrow \text{req}_0$. We assume that the requirements are linked with the test cases $\text{test}_0$ and $\text{test}_1$, respectively. Assume that both test cases fail. If the test plan fixes the order $[\text{test}_0, \text{test}_1]$ then $\text{test}_1$ is inferred to be redundant after the execution of $\text{test}_0$. If, however, the test plan fixes the order $[\text{test}_1, \text{test}_0]$ then no test case becomes redundant.

The reader may have spotted an immediate issue here. In the case where the test cases $\text{test}_0$ and $\text{test}_1$ succeed, the test plan with the inverse ordering will be optimal (i.e., if the test plan fixes the order $[\text{test}_1, \text{test}_0]$ then $\text{test}_0$ becomes redundant after the execution of $\text{test}_1$, and if the test plan fixes the order $[\text{test}_0, \text{test}_1]$ then no test case becomes redundant). Apparently, in order to know what test plan provides an optimal ordering, we first have to execute the test cases.

The solution to the issue lies in the fact that a test manager, when trying to find an optimal test plan, has in mind her (more or less vague) expectation of what the results of the test cases will be. We will optimize the test plan with respect to the expectation of the test manager. In other words, the test manager will no longer need to find an optimal test plan by herself; instead, she specifies her expectation and the optimal test plan will be generated automatically from there.

To specify her expectation, the test manager can start with a default specification (see Section III-B). She can simply take one of the three specifications as is or she can take it as a basis and change it for individual test cases according to her personal insight and experience. She may also leave unspecified her expectation on a test case. An unspecified expectation (or one that turns out to be wrong) may possibly lead to missing a redundancy (in the case where the redundancy could have been inferred otherwise) but our formal criterion for the optimality of a test plan will still apply.

In the example, if the expectation is that both test cases fail, then the test plan with the optimal ordering is $[\text{test}_0, \text{test}_1]$. If the expectation is that both test cases succeed, then the test plan with inverse ordering will be optimal. (The expectation that one of the two fails and the other one succeeds would not be compatible with the logical dependencies between the requirements linked to the test cases; in that case, we would ignore the expectation for the two test cases for the purpose of generating a test plan.)

In our example, it is easy to generate an optimal test plan from the user’s expectation about the test results. In general, however, this cannot be done manually. The dependencies between requirements are complex (between more than two requirements), each test is linked to more than just one requirement, there are dependencies due to the different test platforms, and finally we have to take into account the expectation of the test manager. It is a priori not clear whether it is always possible, given an expectation on test results, to order the test cases such that the number of redundant test cases (and thus the number of opportunities to drop a
A test case (test) is optimal when the execution of the test cases yields the expected test results. Even if this is the case, it seems impossible to manually find a test plan with an optimal ordering, but it is a priori not clear that one can automatically generate such a test plan.

Our approach is to encode the expectation of the test manager in logical formulas in such a way that we can derive conditions on an optimal ordering by logical reasoning. The formulas are Boolean; the logical reasoning can be automated by calling a standard SAT solver. We add these logical formulas to the logical formulas that we use to formally model dependencies between results, links between test cases and requirements, and dependencies due to the different test platforms. We can prove a fundamental property of the resulting algorithm: its completeness. If there exists an ordering of test cases in which the result of a test case becomes redundant (in an execution of the test cases with the expected results), then the algorithm will infer such an ordering.

Generating a test plan is a difficult task which depends on many, often contradictory goals. Solving the task will always eventually rely on the human, her experience, insight, and intuition. Automatic support aims at helping the human to concentrate on the conceptual complexity of the task, by removing at least some of the burden of the combinatorial complexity. Our work is a first step in this direction. Using the logical approach to generate test plans allows the user to concentrate on specifying her expectation on test results, i.e., the one parameter that defines the specific optimality criterion, and frees her from the complexity of the above-mentioned dependencies.

II. Preliminaries

In Section III, we will explain how one can automatically generate test plans that are optimized for the possibility of identifying redundant test cases. In this section, we will explain how one can automatically identify when a test case is redundant (so that it can potentially be dropped without executing it). Thus, the purpose of this section is to make the paper self-contained. The technical content applies generally to every setting where the redundancy of a test case can be inferred from already executed test cases. The material in this section covers mostly work from [3]. This work shows how the documentation of requirements with links to tests on different test platforms can be exploited for identifying redundant test cases.

What the reader should take away from this section is the following. We introduce a Boolean variable test for each test case; the Boolean value (true or false) for test stands for the result of executing the test case (the test case succeeds respectively fails). The result of a test case can become redundant, from the present test status S, i.e., the results of a set of other test cases (which have already been executed), and from the specific setting which is given by: the requirements specification R, the test suite T, and the test platforms P. The redundancy can be mechanically inferred (using a SAT solver), namely by logical inference of test = true resp. test = false from a logical formula R∧T∧P∧S. The logical formula is the conjunction of logical formulas which model the requirements specification, the test suite, the test platforms, and the present test status.

A. Logical Dependencies between Requirements

Our approach is generally applicable in every context where one can derive logical dependencies between requirements. For example, it is possible to use a formal specification of an ontology as in [30]. For concreteness, we will describe the setting of [3]. Here, the dependencies are derived (mechanically) from the documentation of (informal) requirements in a structured format (a format which is amenable to mechanical processing; see [13] and [33]).

The requirements document organizes the requirements in a hierarchy and classifies requirements on the same level in the hierarchy further by assigning a type. We have six types: Vehicle Function (VF), Sub Function (SF), End Condition (EC), Function Contribution (FC), Trigger (TR) and Pre Condition (PC). Formally, a structured requirements specification is a tuple (Reqs, parent, type) consisting of a set of requirements Reqs, a partial function parent which maps a requirement to its parent requirement if there is one, and a function type : Reqs → Types which assigns to each requirement a type in Types, where Types = {VF, SF, EC, FC, TR, PC}.

We will next explain how one derives the logical dependencies between requirements from the hierarchical relationship parent and the classification through types type. Intuitively, the logical dependencies between requirements reflect the hierarchical dependency and the temporal dependency (the function sequence) according to the function expressed by the type of the requirement (see Figure 1).

Fig. 1: Conjunctive vs. disjunctive logical dependencies between requirements according to their type.
Requirements Specification $\mathcal{R}$.
We introduce a Boolean variable $\text{req}$ for each requirement in $\text{Reqs}$. We will express the logical dependencies between requirements through a logical formula $\mathcal{R}$. Formally, $\mathcal{R}$ is the conjunction of all implications of the form $\text{req}_0 \Rightarrow \text{req}_1$ where:

- $\text{req}_0 = \text{parent}(\text{req}_1)$, $\text{type}(\text{req}_0) = \text{VF}$, and $\text{type}(\text{req}_1) = \text{SF}$
- or

\[
\text{parent}(\text{req}_0) = \text{parent}(\text{req}_1), \quad \text{type}(\text{req}_0) = \text{TR}, \quad \text{and} \quad \text{type}(\text{req}_1) = \text{PC}
\]

or

- $\text{parent}(\text{req}_0) = \text{parent}(\text{req}_1), \quad \text{type}(\text{req}_0) = \text{EC}, \quad \text{and} \quad \text{type}(\text{req}_1) = \text{FC}.$

and all implications of the form $\text{req}_1 \Rightarrow (\text{req}_1 \lor \ldots \lor \text{req}_n)$ where

- $\text{parent}(\text{req}_0) = \text{parent}(\text{req}_1) = \ldots = \text{parent}(\text{req}_n)$, $\text{type}(\text{req}_0) = \text{FC}$, and $\text{type}(\text{req}_1) = \ldots = \text{type}(\text{req}_n) = \text{TR}$
- or

$\text{parent}(\text{req}_0) = \text{parent}(\text{req}_1) = \ldots = \text{parent}(\text{req}_n), \quad \text{type}(\text{req}_0) = \text{SF}$, and $\text{type}(\text{req}_1) = \ldots = \text{type}(\text{req}_n) = \text{EC}.$

Thus, the formula $\mathcal{R}$ is the conjunction of implications of the form:

\[
\text{req}_0 \Rightarrow (\text{req}_1 \lor \ldots \lor \text{req}_n)
\]

where $n \geq 1$. We call an implication of the form above a conjunctive logical dependency if $n = 1$ and a disjunctive logical dependency if $n > 1$. The terminology ‘conjunctive’ stems from the fact that we express an implication of the form $\text{req}_0 = \text{req}_1 \land \ldots \land \text{req}_m$ by $\text{req}_0 \Rightarrow \text{req}_1, \ldots, \text{and} \text{req}_0 \Rightarrow \text{req}_m.$

B. Identifying Redundant Test Cases
The development process consists of a sequence of releases. Each release represents a specific development state and has its own testing phase with a given set of requirements and a corresponding test suite. We assume that test cases and requirements are linked with each other [6]. Each single requirement in the set of observed requirements for a specific release should be covered by a test case from the corresponding test suite at least once. Formally, we use a function that maps a test case to a set of requirements.

Test Suite $\mathcal{T}$.
A test suite $(\text{Tests}, \text{link})$ for a given set of requirements $\text{Reqs}$ consists of a set of test cases $\text{Tests}$ and a function

\[
\text{link} : \text{Tests} \rightarrow 2^{\text{Reqs}}
\]

which maps a test case to a set of requirements. Given a test suite $(\text{Tests}, \text{link})$ we use the test cases $\text{test} \in \text{Tests}$ as Boolean variables and we define the test suite $\mathcal{T}$ as a logical formula in the set of Boolean variables $\text{Tests}$, namely as a conjunction of equivalences of the form

\[
\text{test} \Leftrightarrow \text{req}_1 \land \ldots \land \text{req}_n
\]

for every $\text{test} \in \text{Tests}$ such that

\[
\text{link} (\text{test}) = \text{req}_1, \ldots, \text{req}_n
\]

holds.

Test Platform $\mathcal{P}$.
Test platforms are ordered according to their level. Intuitively, if a requirement is satisfied at a given level, then it is also satisfied at a lower level, but not necessarily on a higher level (the number of potential, non-modeled error causes rises with each higher level). Consequently, if a requirement is not satisfied at a given level, then it is also not satisfied at a higher level, but not necessarily on a lower level.

We define the test platforms $\mathcal{P}$ as a logical formula in the set of Boolean variables $\text{Reqs}$, namely as a conjunction of implications

\[
\text{req}_0 \Rightarrow \text{req}_1.
\]

We have the implication of the form above whenever the two requirements $\text{req}_0$ and $\text{req}_1$ express the same condition but refer to different test platforms, i.e., $\text{req}_0$ refers to a higher test platform than $\text{req}_1$.

Test Status $\mathcal{S}$.
A test status $(\text{Success, Fail, No result})$ for a given set of test cases $\text{Tests}$ consists of a triple of subsets (the subsets contain the test cases that have succeeded, failed, or that were not yet executed, thus have no result). Given a test status $(\text{Success, Fail, No result})$ we use the test cases $\text{test} \in \text{Tests}$ as Boolean variables and we define the test status $\mathcal{S}$ as a logical formula in the set of Boolean variables $\text{Tests}$, namely as equalities that bind the Boolean value of those test cases that have succeeded respectively failed so far. Formally,

\[
\text{test} \Leftrightarrow \text{true} \quad \text{if} \quad \text{test} \in \text{Success}
\]

\[
\text{test} \Leftrightarrow \text{false} \quad \text{if} \quad \text{test} \in \text{Fail}.
\]

From now on, when we use a test case test (i.e., an element of Tests) in a logical formula, it denotes the corresponding Boolean variable.

Redundancy of a Test Case.
Given the structured requirements specification $\mathcal{R}$, the test suite $\mathcal{T}$, the test platforms $\mathcal{P}$, and given the current test status $\mathcal{S}$, a test case test is redundant if its result can be inferred from $\mathcal{R}$, $\mathcal{T}$, $\mathcal{P}$ and $\mathcal{S}$, formally if either

\[
\mathcal{R} \land \mathcal{T} \land \mathcal{P} \land \mathcal{S} \models \text{test} = \text{true}
\]

or

\[
\mathcal{R} \land \mathcal{T} \land \mathcal{P} \land \mathcal{S} \models \text{test} = \text{false}.
\]

The condition means the value of (the Boolean variable corresponding to) the test case test is fixed in every model of $\mathcal{R} \land \mathcal{T} \land \mathcal{P} \land \mathcal{S}$. In other words, the value of test must
always be true or it must always be false in every valuation of (the Boolean variables corresponding to) the test cases in Tests and the requirements in Reqs which satisfies

- the logical dependencies derived from the structured requirements specification $\mathcal{R}$,
- the equivalences defined through the test suite $\mathcal{T}$,
- the logical dependencies defined through the test platforms $\mathcal{P}$, and
- the equalities defined through the current status $S$, i.e., the equalities binding the Boolean values of those test cases that have succeeded respectively failed so far.

Since the validity of entailment can be reduced to non-satisfiability, the condition amounts to the fact that either $\text{test} = \text{true}$ or $\text{test} = \text{false}$ is unsatisfiable in conjunction with the implications from $\mathcal{R}$, the equivalences from $\mathcal{T}$, the implications from $\mathcal{P}$, and the equalities from $S$. Thus, the redundancy of each test case can be inferred with an off-the-shelf SAT solver. The practical potential of the approach for reducing the number of test cases has been demonstrated in [3].

III. Generating Test Plans

In the previous section we have described how to identify the redundancy of a test case based on the present status of test results. In this section we give an algorithm to generate an optimal test plan (a sequence of test cases to be executed). The input of the algorithm is a) the dependencies between test results (dependencies entailed from $\mathcal{R}, \mathcal{T}$ and $\mathcal{P}$ e.g. “test$_0$ implies test$_1$”) and b) the user’s expectation on the outcome of the individual test cases (“I expect that test$_0$ succeeds”). Optimality here refers to the number of ordering constraints (“test$_0$ comes before test$_1$”) that are entailed from a) and b).

The algorithm consists of the following four steps (see Figure 2). The numbering A - D corresponds to the following subsection that explains the steps.

A. Compute the set of dependencies between test results that are entailed by the requirements specification $\mathcal{R}$, the test suite $\mathcal{T}$ and the test platforms $\mathcal{P}$. For example, this step could return the logical implication test$_0 \Rightarrow$ test$_1$.

B. Compute the set of dependencies between the expected test results that are entailed by the requirements specification $\mathcal{R}$, the test suite $\mathcal{T}$, the test platforms $\mathcal{P}$, the test status $S$ and the user’s expectation $E$. For example, this step could return the logical formula $\neg\text{xpctd}_0 \Rightarrow \neg\text{xpctd}_1$ where $\neg\text{xpctd}_0$ stands for the fact that the user expects a negative test results for test$_0$.

C. First, infer which test result becomes redundant from what set of expected test results. For example, this step could return that the test result of test$_1$ is redundant from the expected (negative) test result of test$_0$. Then, derive ordering constraints between test cases. For example this step could return that test$_0$ should be scheduled before test$_1$.

D. Generate a test plan. For example, the test plan could schedule test$_0$ before test$_1$.

Fig. 2: Algorithm for generating test plans. The inputs are the requirements specification $\mathcal{R}$, the test suite $\mathcal{T}$, the test platforms $\mathcal{P}$, and the expectation on test results $\mathcal{E}$.

For simplicity we took a simple example of a logical implication of the form test$_0 \Rightarrow$ test$_1$. As already indicated in the introduction, step A. can return more complex formulas than just a logical implication as in the example.

A. Dependencies between Test Results

Informal Discussion.

In this section we will explain how one can compute the dependencies between test results as discussed in the example in the introduction, in the scenario of the rain sensor. Formally, the dependencies are modelled by logical formulas that are entailed by the logical formulas that model the requirements, the test links, and the test platforms (see Section II).

Formal Description.

We now give the algorithm to compute all possible dependencies between test results. We will start by formally defining the notion of dependency between test results.

Definition 1 [Dependencies between Test Results]

We call a formula of the form

\[ \text{test}_1 \land \ldots \land \text{test}_n \Rightarrow \text{test}_{n+1} \lor \ldots \lor \text{test}_m \]

a dependency between test results (here, between the results of the test cases test$_1$,$\ldots$,$\text{test}_n$, test$_{n+1}$,$\ldots$,$\text{test}_m$).

The following remark expresses the relevance of the notion defined in Definition 1.
Remark 1 [Dependencies between Test Results]
If a dependency between the \( n + m \) test cases \( test_1, \ldots, test_m \) is entailed by the requirements specification \( R \), the test suite \( T \), and the test platforms \( P \), then each of the \( n + m \) test cases can be made redundant by the other ones.

By “test can be made redundant” we mean that there exists a test status \( S \) such that \( R, T, P \) and \( S \) entail the result of \( test \), and thus \( test \) is redundant if all the other test cases \((n + m - 1 \text{ many})\) are scheduled before. If the other test cases have been executed, then the test status \( S \) assigns a Boolean value modelling their test results.

Often, each disjunction or its negation is called a clause and each disjunction (a Boolean variable \( test \) or its negation \( \neg test \)) is called a literal. The conjunction is often written as a set; i.e., the resulting formula is a set of clauses.

Algorithm to Compute Dependencies between Test Results.
We next explain how we compute the set of all dependencies that are entailed by \( R, T \) and \( P \).

As explained in Section II, we use the requirements specification \( R \) as a conjunction of implications of the form

\[
\text{req} \Rightarrow (\text{req}_1 \lor \ldots \lor \text{req}_n)
\]

for every Boolean variable \( \text{req} \in \text{Reqs} \). The test suite \( T \) is a conjunction of equivalences of the form

\[
test \Leftrightarrow \text{req}_1 \land \ldots \land \text{req}_n
\]

for every Boolean variable \( test \in \text{Tests} \). The test platforms \( P \) are implications of the form

\[
\text{req}_1 \Rightarrow \text{req}_2
\]

Thus the conjunction

\[
R \land T \land P
\]

is a Boolean formula in variables from the set \( \text{Reqs} \cup \text{Tests} \), i.e., in variables that stand for requirements and in variables that stand for test results.

The formula

\[
\exists \text{Reqs}(R \land T \land P)
\]

stands for the formula with an existential quantification \( \exists \text{req} \) for every variable \( \text{req} \in \text{Reqs} \).

In the first step of the algorithm, we apply a quantifier elimination procedure and obtain an equivalent formula (without existential quantifiers) in variables only from the set \( \text{Tests} \), i.e., in variables that stand for test results.

In the second step for the algorithm, we transform the resulting formula into conjunctive normal form, i.e., into a conjunction of disjunctions, where each disjunct is of the form \( test \) or of the form \( \neg test \), for \( test \in \text{Tests} \).

In the third and last step of the algorithm, we saturate the set of clauses; i.e., we add clauses that are derivable under the resolution rule until no more new clauses can be added. For example, given the clauses \( test_0 \lor test_1 \) and \( test_0 \lor \neg test_1 \), we will derive and add the clause \( test_0 \).

The resulting set is then exactly the set of dependencies (dependencies between test results) that are entailed by \( R, T \) and \( P \). That is, a dependency between test results is entailed by \( R, T \) and \( P \) if and only if it lies in the set.

B. Dependencies between Expected Test Results

Informal Discussion.
We come back to the example discussed in the introduction. The example indicates that already a test plan with an ordering of only two test cases cannot be optimal independently of the results of the test cases. We have the test case \( test_0 \) for the requirement \( \text{sunroof closes automatically when it is raining} \) and the test cases \( test_1 \) for the requirement \( \text{rain sensor detects rain} \), and we have the dependency between the two test results \( test_0 \lor \neg test_1 \), formally

\[
R \land T \land P \models test_0 \lor \neg test_1.
\]

If the user expects that both test cases are going to succeed (both boolean variables are \( true \)) then the test plan should start with the execution of \( test_1 \).

If the user expects that both test cases are going to fail (both boolean variables are \( false \)) then the test plan should start with the execution of \( test_0 \).

We now come back to Remark 1. The remark says that each test case \( test \) that appears among the \( n + m \) test cases involved in a dependency of the form

\[
\text{test}_1 \land \ldots \land \text{test}_m \Rightarrow \text{test}_{m+1} \lor \ldots \lor \text{test}_n
\]

can be made redundant if all the other test cases \((n + m - 1 \text{ many})\) are scheduled before and their results are corresponding, i.e., if the test status \( S \) models their results, then \( R, T, P \) and \( S \) entail the result of \( test \).

Obviously, the user is not able to make a prophecy of the test results, i.e., she does not know \( S \) before executing the test cases. However, the user has an expectation on the test results and she wants to know what test case \( test \) will be redundant if the other test cases are scheduled before and the test results are according to her expectation.

We next introduce the concept of a default specification. As mentioned in the introduction, the test manager can use a default specification to specify her expectation (and can take it as is or take it as a basis, changing individual test cases according to her personal insight and experience).

(1) Each test case will fail. This pessimistic default expectation seems applicable at an early level of maturity, towards the beginning of a development process\(^1\).

(2) Each test case will succeed. This optimistic default expectation seems applicable at an advanced level of maturity, towards the end of a development process.

\(^1\)It is interesting here to contrast the initial expectation in software testing (the software will be buggy) with, say, the initial expectation in a medical check-up (the patient will be healthy).
(3) Each test case will have the same result as the previous time. This is the default expectation that seems applicable at an intermediate level of maturity.

As mentioned in the introduction, the test manager may also leave unspecified her expectation on a test case. An unspecified expectation (or one that turns out to be wrong) may possibly lead to missing a redundancy (in the case where the redundancy could have been inferred otherwise) but our formal criterion for the optimality of a test plan will still apply.

**Formal Description.**

We assume that the user has formalized her expectation on the test results as a function \(X_{\text{xpctd}}\) that maps each test case \(\text{test}\) to \(\text{success}\) or to \(\text{fail}\). For each test case \(\text{test} \in \text{Tests}\) we introduce a Boolean variable \(x_{\text{xpctd}}\) which we will use in the encoding of the user expectation as a logical formula. Notation: Given \(\text{test}\), we will write \(x_{\text{xpctd}}\) instead of \(x_{\text{xpctd}}(\text{test})\).

**Definition 2 [Expectation on Test Results \(E\)]**

The expectation on test results is a logical formula \(E\) over the set of Boolean variables \(\{x_{\text{xpctd}} \mid \text{test} \in \text{Tests}\}\), namely a conjunction of equivalences. The equivalence \(x_{\text{xpctd}} \leftrightarrow \text{test}\) encodes that user expects a \text{positive}\ test result for \(\text{test}\). The equivalence \(x_{\text{xpctd}} \leftrightarrow \neg \text{test}\) encodes that user expects a \text{negative}\ test result for \(\text{test}\).

Using the convention that a set denotes the conjunction of its elements, we can write:

\[
E = \{x_{\text{xpctd}} \leftrightarrow \text{test} \mid \text{test} \in \text{Tests}, X_{\text{xpctd}}(\text{test}) = \text{success}\} \\
\cup \{x_{\text{xpctd}} \leftrightarrow \neg \text{test} \mid \text{test} \in \text{Tests}, X_{\text{xpctd}}(\text{test}) = \text{fail}\}.
\]

**Remark 2 [Expectation on Test Results \(E\)]**

The variable \(x_{\text{xpctd}}\) has the value \text{true} if and only if the value of \(\text{test}\) corresponds to the expected result. Formally, if \(b\) stands for the Boolean constant \text{true} if \(X_{\text{xpctd}}(\text{test}) = \text{success}\), and for the Boolean constant \text{false} if \(X_{\text{xpctd}}(\text{test}) = \text{fail}\), then we have:

\[
E \models x_{\text{xpctd}} \leftrightarrow (\text{test} \leftrightarrow b)
\]

Introducing the concept of the \textit{expectation on test results} is the key idea in our paper. Adding the formula \(E\) to the formulas \(R, T, P\), and \(E\) will allow us to compute what test case \(\text{test}\) can be inferred as redundant (and potentially be dropped) and what test cases have to be scheduled before \(\text{test}\) (because the result of \(\text{test}\) can be inferred from the results of those test cases), always under the assumption that the results of those test cases are as expected.

**Algorithm to Compute Dependencies between Expected Test Results.**

We can now give the algorithm to compute all dependencies between expected test results that are entailed by \(R, T, P,\) and \(E\). We first apply the algorithm described in Section III-A and compute the set of all dependencies between test results. We transform each dependency between test results (an implication) into a disjunction, i.e., a clause. For each (positive or negative) occurrence of \(\text{test}\) in a clause we replace \(\text{test}\) by \(x_{\text{xpctd}}\) if \(X_{\text{xpctd}}(\text{test})\) is equal to \text{success} and by \(\neg x_{\text{xpctd}}\) if \(X_{\text{xpctd}}(\text{test})\) is equal to \text{fail}. We then obtain a set of clauses over variables \(x_{\text{xpctd}}\) where \(\text{test} \in \text{Tests}\). We now transform each clause into an equivalent implication.

The resulting set is then exactly the set of dependencies (dependencies between expected test results) that are entailed by \(R, T, P,\) and \(E\). That is, a dependency between expected test results is entailed by \(R, T, P,\) and \(E,\) if and only if it lies in the set.

**C. Ordering Constraints**

**Informal Discussion.**

Using the formula for computing dependencies between test results and the user’s expectation on test results we are able to automatically identify implied test results.

An example, assume that \(R, T, P\) entails the dependency between test results

\[
\text{test}_2 \land \text{test}_1 \Rightarrow \text{test}_0 \lor \text{test}_1
\]

which is equivalent to the clause

\[
\text{test}_0 \lor \text{test}_1 \lor \neg \text{test}_2 \lor \neg \text{test}_3.
\]

Assume that the user’s expectation is as follows.

\[
X_{\text{xpctd}}(\text{test}_0) = \text{fail} \\
X_{\text{xpctd}}(\text{test}_1) = \text{fail} \\
X_{\text{xpctd}}(\text{test}_2) = \text{success} \\
X_{\text{xpctd}}(\text{test}_3) = \text{fail}
\]

As described in Section III-B we encode the user’s expectation in the logical formula \(E\). We then have that \(R, T, P\) and \(E\) entail the dependency between expected test results

\[
x_{\text{xpctd}}(\text{test}_0) \land x_{\text{xpctd}}(\text{test}_1) \land x_{\text{xpctd}}(\text{test}_2) \Rightarrow x_{\text{xpctd}}(\text{test}_3)
\]

which is equivalent to the clause

\[
\neg x_{\text{xpctd}}(\text{test}_0) \lor \neg x_{\text{xpctd}}(\text{test}_1) \lor \neg x_{\text{xpctd}}(\text{test}_2) \lor x_{\text{xpctd}}(\text{test}_3).
\]

We use the dependency between expected test results in order to infer the ordering constraints \(\text{test}_0 < \text{test}_1, \text{test}_1 < \text{test}_3\) and \(\text{test}_2 < \text{test}_1\). The ordering constraints tell us that we should schedule \(\text{test}_0, \text{test}_1\) and \(\text{test}_2\) before \(\text{test}_3\). Indeed, if the results of \(\text{test}_0, \text{test}_1\) and \(\text{test}_2\) are as expected then \(\text{test}_3\) is redundant (its result can be inferred from \(\text{test}_0, \text{test}_1\) and \(\text{test}_2\)).

The above example shows that we can infer ordering constraints from a dependency between expected test results if the dependency is expressed by an implication which has only one disjunct on the right hand side (this is equivalent to the fact that the corresponding clause has exactly one positive disjunct). The next example shows that we cannot infer ordering constraints when there is more than one disjunct in the right hand side of the implication (if the clause has more than one positive disjunct).
Assume now that the users expectation is as follows.

\[ \text{xpctd}(\text{test}_0) = \text{success} \]
\[ \text{xpctd}(\text{test}_1) = \text{success} \]
\[ \text{xpctd}(\text{test}_2) = \text{success} \]
\[ \text{xpctd}(\text{test}_3) = \text{success} \]

We then have that \( R, T, P \) and \( E \) entail the dependency between expected test results

\[ \text{xpctd}_2 \land \text{xpctd}_3 \Rightarrow \text{xpctd}_0 \lor \text{xpctd}_1 \]

which is equivalent to the clause

\[ \text{xpctd}_0 \lor \text{xpctd}_1 \lor \neg \text{xpctd}_2 \lor \neg \text{xpctd}_3. \]

Now, the dependency between expected test results does not allow us to infer any ordering constraints. In fact, there exists no execution order between test_0, test_1, test_2, and test_3 where one of the four test results could be inferred by the three other ones (if the three other ones are executed before and their results are as expected).

**Formal Description.**

We will next define the notion of an ordering constraint and we will present the algorithm to infer ordering constraints.

**Definition 3 [Ordering constraint]**

An ordering constraint is a conjunction of inequalities of the form

\[ \text{test}_0 < \text{test} \land \ldots \land \text{test}_n < \text{test} \]

between the test cases test_0, ..., test_n on the left-hand side and the test case test on the right-hand side.

A test plan satisfies the ordering constraint of the form above if the test plan schedules test_0, ..., test_n before test (i.e., in the sequential ordering specified by the test plan, test_0, ..., test_n occurs before test).

**Algorithm to Compute Ordering Constraints.**

We can now give the algorithm to compute ordering constraints from a given requirements specification, a given test suite, the test platforms, and the user’s expectation, modelled by \( R, T, P, \) and \( E, \) respectively.

In the first step of the algorithm, we apply the algorithm described in Section [III-A] and compute all dependencies between expected test results.

In the second step, we take the set of all dependencies between expected test results that can be expressed by a Horn clause, i.e., an implication with exactly one disjunct on the right hand side (equivalently a clause with exactly one positive disjunct). We compute its subset of minimal dependencies by eliminating each dependency that is subsumed by another one in the set. A first dependency test_1 \&\& ... \&\& test_n \Rightarrow test is subsumed by a second dependency test_1 \&\& ... \&\& test_m \Rightarrow test if the set of conjuncts of the first is contained by the second, i.e., if \( \{\text{test}_1, \ldots, \text{test}_n\} \subseteq \{\text{test}_1, \ldots, \text{test}_m\}. \)

In the third and final step of the algorithm, we form the ordering constraint

\[ \text{test}_1 < \text{test} \land \ldots \land \text{test}_n < \text{test} \]

for each minimal dependency

\[ \text{xpctd}_1 \land \ldots \land \text{xpctd}_n \Rightarrow \text{xpctd}_{\text{test}} \]

in the subset obtained after the second step (which eliminates each dependency that is not minimal).

**Theorem [Completeness of the Algorithm]**

For every test case test, if there exists an ordering of test cases in which the result of test becomes redundant, then the algorithm will infer such an ordering. More precisely, the algorithm will infer an ordering constraint such that the test case test becomes redundant in every sequence of test cases (with the expected results) that satisfies the ordering constraint.

**Proof [Completeness of the Algorithm]**

We assume that a test result test is redundant in the sequential ordering test_1, ..., test_n followed by test. Moreover, we assume that \{test_1, ..., test_n\} is a minimal set of test cases; i.e., if we dropped one test case from the set, then test would no longer be redundant. We need to show that our algorithm will infer the ordering constraint test_1 < test \land \ldots \land test_n < test which are satisfied by the minimal sequential ordering test_1, ..., test_n, test.

The redundancy means that the result of the test case test (say, false) can be inferred from \( R, T, P, \) and from the present test status S after executing the sequence of test cases test_1, ..., test_n and right before executing the test case test. That is, we have

\[ R \land T \land P \land S \models \text{test} = \text{false}. \]

The test status S fixes the Boolean value b_i for each of the test cases test_1, ..., test_n. That is, S is equivalent to the conjunction of the equivalences \( \text{test}_i \Leftrightarrow b_i, \) formally

\[ S \equiv \text{test}_1 \Leftrightarrow b_1 \land \ldots \land \text{test}_n \Leftrightarrow b_n. \]

Thus, we have

\[ R \land T \land P \models (\text{test}_1 \Leftrightarrow b_1) \land \ldots \land (\text{test}_n \Leftrightarrow b_n) \Rightarrow \text{test} = \text{false}. \]

Since we assume that the execution of the test cases test_1, ..., test_n produces the expected result, and since the equivalence \( \text{xpctd}_i \Leftrightarrow \text{test}_i \) lies in \( E \) whenever \( b_i \) is equal to true, and the equivalence \( \text{xpctd}_i \Leftrightarrow \neg \text{test}_i \) lies in \( E \) whenever \( b_i \) is equal to false, we have that (see also Remark 2)

\[ E \models \text{xpctd}_i \Leftrightarrow (\text{test}_i \Leftrightarrow b_i). \]

Thus, we have that

\[ R \land T \land P \land E \models \text{xpctd}_1 \land \ldots \land \text{xpctd}_n \Rightarrow \text{test} = \text{false}. \]

Since we assume that the execution of all test cases (i.e., including the redundant test case test) produces the result corresponding to the expectation and the result of test can already be inferred to be false from the results of the test cases preceding test, we also have that \( E \models \text{xpctd}_{\text{test}} \Leftrightarrow \neg \text{test} \).
and hence $\mathcal{E} \models \text{xpctd}_{\text{test}} \Leftrightarrow (\text{test} \Leftrightarrow \text{false})$. Thus, we have that
\[ R \land T \land P \land \mathcal{E} \models \text{xpctd}_1 \land \ldots \land \text{xpctd}_n \Rightarrow \text{xpctd}_{\text{test}}. \]

By our assumption on the minimality of the set of test cases \{test_1, \ldots, test_n\} (i.e., if we dropped one test case from the set, then test would no longer be redundant), we have that the Horn clause above is minimal (i.e., if we dropped one of the conjuncts from the body, the implication would no longer be entailed by $R \land T \land P \land \mathcal{E}$).

As a consequence, our algorithm infers the Horn clause above as a dependency between expected test results in Step B, and the algorithm will infer the ordering constraint
\[ \text{test}_1 < \text{test} \land \ldots \land \text{test}_n < \text{test} \]
which are satisfied by the minimal sequential ordering test_1, \ldots, test_n, test. This terminates the proof.

### D. Optimal Test Plan

#### Informal Discussion.

We give an example that illustrates a difficulty in the generation of a test plan that satisfies all ordering constraints. Assume that we have the following two dependencies between test results
\[
\begin{align*}
\text{test}_0 \land \text{test}_1 & \Rightarrow \text{test}_2 \\
\text{test}_3 & \Rightarrow \text{test}_0 \lor \text{test}_1
\end{align*}
\]
and assume the user’s expectation
\[
\begin{align*}
\text{xpctd(test}_0) &= \text{success} \\
\text{xpctd(test}_1) &= \text{fail} \\
\text{xpctd(test}_2) &= \text{fail} \\
\text{xpctd(test}_3) &= \text{success}.
\end{align*}
\]

Our algorithm computes the following ordering constraints.
\[
\begin{align*}
\text{test}_0 < \text{test}_1 \\
\text{test}_1 < \text{test}_0 \\
\text{test}_2 < \text{test}_1 \\
\text{test}_3 < \text{test}_0
\end{align*}
\]

There does not exist a test plan that satisfies all these ordering constraints (because there is no sequential ordering that satisfies the first two ordering constraints, i.e., test_0 < test_1 and test_1 < test_0).

#### Formal Description.

Since in general it is not possible to generate a test plan that satisfies all ordering constraints computed by the algorithm, we can only ask for a test plan which maximizes the number of ordering constraints that are satisfied by the test plan.

#### Algorithm to Compute an Optimal Test Plan.

In the first step, given the set of ordering constraints computed by the algorithm of Section III-C, we compute a subset with the maximal number of ordering constraints that can be satisfied simultaneously. The problem of computing the maximal number of sets of ordering constraints that can be satisfied simultaneously, can be reduced to a variant of a well known NP-complete problem, namely the problem to compute a minimum feedback arc set in a directed graph. As usual, each inequality in an ordering constraint is translated into an edge of the directed graph, but now we have additional hyperedges which go from a set of source nodes to a (single) target node. Namely, each ordering constraint of the form
\[ \text{test}_1 < \text{test} \land \ldots \land \text{test}_n < \text{test} \]
is translated to a hyperedge that goes from the set of nodes \{test_1, \ldots, test_n\} to the node test.

For efficient implementations of algorithms to solve this problem, see e.g., [5].

In the second step, we compute an optimal test plan, i.e., a sequential ordering of test cases that satisfies all ordering constraints in the set that we have computed in the first step, a set with the maximal number of ordering constraints that can be satisfied simultaneously. Here, we can apply classical algorithms for topological sorting (see, e.g., [1]).

#### Online Update of a Test Plan.

If during the execution of a test plan the execution of a test case leads to a result that is different from the expectation on the result then the user may consider to adapt the test plan. In this case, all she has to do is to update the function Xpctd (which leads to an update of the logical formula for the expectation \(\mathcal{E}\)) and re-execute the algorithm to generate new ordering constraints and a new optimal test plan.

### IV. Related Work

Generating an optimal test plan seems related to the prioritization of test cases, which is a topic of very active research which we will discuss in greater detail below.

The goal of prioritization is generally phrased as obtaining a maximum amount of information about the maturity of the system under development as early as possible. During the execution of the test cases, at the moment when resources have run out, the test manager will have to drop test cases; i.e., the decision depends on external factors.

We next discuss work on prioritization in the setting of regression testing. The formal setup makes our approach \textit{a priori} independent of a particular practical setting. However, the setup of regression testing may seem particularly suitable for our approach because it can facilitate the task of the user to revise her expectation of test results.

The work in [11], [12], [17], [22], [26] uses the history of test results for the prioritization. In contrast, the work of [31], [16] uses the explicit knowledge about what fault can be revealed by what test case. Alternatively, the prioritization can be based on coverage criteria; see, e.g., [10], [21], [23], [24], [52]. Another line of research investigate prioritization under the header of increasing the failure detection potential (FDP) by concentrating on the mutations the program [8], [25], [28].
The work in \cite{14,18,19,20} goes into the same direction, using system models.

The work in \cite{27} is related to ours in that it also starts with the requirements. In fact, it first prioritizes the requirements and then derives the prioritization of the test cases from the prioritization of the requirements.

The work in \cite{9,29} bases the prioritization on the cost of the execution of the test cases.

The work in \cite{4} uses dataflow analysis in order to eliminate redundant test cases. The dataflow analysis is done offline; i.e., the results of the test cases are not taken into account.

V. Conclusion

As the discussion of related work shows, the work presented in this paper is quite different from existing work, by its topic, and also in its style.

We present the formal foundation for what could become a new line of work, namely the automatic generation of test plans from a specification (here, the specification of the expectation on test results) and according to a criterion for optimality.

We introduce a novel algorithmic problem. We give an algorithm approach to solve the problem. The logical setting allows us to formulate the approach in a concise manner.

We formally characterize the contribution of the approach. This is quite different from work that experimentally validates the contribution of a new approach.

As for future work, our approach cannot be viewed stand-alone. The automatic generation of an optimal test plan can only be the first step. The conception of a test plan will take into account many different objectives, the reduction of the set of test cases being only one of these. The conception of a test plan will also take into account constraints on the execution of the test plan, the cost of preparing a test case (so that, e.g., we may have sets of test cases that must be grouped together), the availability of a testing resource, etc. Furthermore, the redundancy of a test case seems orthogonal to the criteria used in the methods for prioritization discussed above. The automatic generation of test cases can in principle be integrated with each of these methods. All this indicates interesting directions for future research.

References

\begin{thebibliography}{10}
\footnotesize
\bibitem{1} D. Ajwani, A. Cosgaya-Lozano, and N. Zeh. A topological sorting algorithm for large graphs. \textit{Journal of Experimental Algorithms (JEA)}, 2012.
\bibitem{2} C. Andrés and A. Cavalli. How to reduce the cost of passive testing. In \textit{High-Assurance Systems Engineering (HASE)}, 2012.
\bibitem{3} S. Arlt, T. Morciniec, A. Podelski, and S. Wagner. If A fails, can B still succeed? Inferring dependencies between test results in automotive system testing. In \textit{International Conference of Software Testing (ICST)}, 2015.
\bibitem{4} S. Arlt, A. Podelski, C. Bertolini, M. Schäf, I. Banerjee, and A. M. Memon. Lightweight static analysis for GUI testing. In \textit{23rd IEEE International Symposium on Software Reliability Engineering, ISSRE 2012, Dallas, TX, USA, November 27-30, 2012, 2012.}
\bibitem{5} A. Baharev, H. Schichl, and A. Neumaier. An exact method for the minimum feedback arc set problem. \textit{Ibid}, 2015.
\bibitem{6} E. Bauer and J. M. Küster. Combining specification-based and code-based coverage for model transformation chains. In \textit{International Conference on Model Transformation (ICMT)}, 2011.
\bibitem{7} M. Broy, A. Pretschner, and C. Salzmann. Engineering automotive software. \textit{IEEE Proceedings}, 2007.
\bibitem{8} T. A. Buel. Mutation analysis of program test data. Ph.D. Thesis, 1980.
\bibitem{9} S. Elbaum, A. Malishedevsky, and G. Rothermel. Incorporating varying test costs and fault severities into test case prioritization. In \textit{International Conference on Software Engineering (ICSE)}, 2001.
\bibitem{10} S. Elbaum, A. G. Malishedevsky, and G. Rothermel. \textit{Prioritizing Test Cases for Regression Testing.} Association for Computing Machinery (ACM), 2000.
\bibitem{11} E. Engstrom, P. Runeson, and A. Ljung. Improving regression testing transparency and efficiency with history-based prioritization - an industrial case study. In \textit{International Conference on Software Testing, Verification and Validation (ICST)}, 2011.
\bibitem{12} Y. Fazlalizadeh, A. Khalilian, M. A. Azgomi, and S. Parsa. Prioritizing test cases for resource constraint environments using historical test case performance data. In \textit{International Conference on Computer Science and Information Technology (ICCSIT)}, 2009.
\bibitem{13} P. Filipović, M. Nyberg, and G. Rodríguez-Navas. Reassessing the pattern-based approach for formalizing requirements in the automotive domain. In \textit{International Requirements Engineering Conference (RE)}, 2014.
\bibitem{14} G. Fraser and F. Wotawa. Test-case prioritization with model-checkers. In \textit{International Association of Science and Technology for Development (IASTED)}, 2007.
\bibitem{15} A. Gotlieb and D. Marijan. FLOWER: optimal test suite reduction as a network maximum flow. In \textit{International Symposium on Software Testing and Analysis (ISSTA)}, 2014.
\bibitem{16} M. J. Harrold. Testing evolving software. \textit{Journal of Systems and Software}, 1999.
\bibitem{17} J.-M. Kim and A. Porter. A history-based test prioritization technique for regression testing in resource constrained environments. In \textit{International Conference on Software Engineering (ICSE)}, 2002.
\bibitem{18} B. Korel, G. Koutsogiannakis, and L. H. Tahat. Model-based test prioritization heuristic methods and their evaluation. In \textit{International workshop on Advances in model-based testing (A-MOST)}, 2007.
\bibitem{19} B. Korel, G. Koutsogiannakis, and L. H. Tahat. Application of system models in regression test suite prioritization. In \textit{International Conference on Software Maintenance (ICSM)}, 2008.
\bibitem{20} B. Korel, L. H. Tahat, and M. Harman. Test prioritization using system models. In \textit{International Conference on Software Maintenance (ICSM)}, 2005.
\bibitem{21} Z. Li, M. Harman, and R. M. Hierons. Search algorithms for regression test case prioritization. \textit{Transactions on Software Engineering (TSE)}, 2007.
\bibitem{22} B. Qu, C. Nie, B. Xu, and X. Zhang. Test case prioritization for black box testing. In \textit{International Computer Software and Applications Conference (COMPSAC)}, 2007.
\bibitem{23} G. Rothermel, R. H. Untch, C. Chu, and M. J. Harrold. Test case prioritization: An empirical study. In \textit{International Conference on Software Maintenance (ICSM)}, 1999.
\bibitem{24} G. Rothermel, R. H. Untch, C. Chu, and M. J. Harrold. Prioritizing test cases for regression testing. \textit{Transactions on Software Engineering (TSE)}, 2001.
\bibitem{25} R. K. Saha, L. Zhang, S. Khurshid, and D. E. Perry. An information retrieval approach for regression test prioritization based on program changes. In \textit{International Conference on Software Engineering (ICSE)}, 2015.
\bibitem{26} M. Sherriff, M. Lake, and L. Williams. Prioritization of regression tests using singular value decomposition with empirical change records. In \textit{International Symposium on Software Reliability (ISSRE)}, 2007.
\bibitem{27} H. Srikanth, L. Williams, and J. Osborne. System test case prioritization of new and regression test cases. In \textit{International Symposium on Empirical Software Engineering and Measurement (ESEM)}, 2005.
\bibitem{28} A. Srivastava and J. Thiagarajan. Effectively prioritizing tests in development environment. In \textit{SIGSOFT Software Engineering Notes}, 2002.
\bibitem{29} K. R. Wallace, M. L. Soffa, G. M. Kapfhammer, and R. S. Roos. Timeaware test suite prioritization. In \textit{International Symposium on Software Testing and Analysis (ISSTA)}, 2006.
\end{thebibliography}
[30] Y. Wang, X. Bai, J. Li, and R. Huang. Ontology-based test case generation for testing web services. In International Symposium on Autonomous Decentralized Systems (ISADS), 2007.

[31] W. E. Wong, J. R. Horgan, S. London, and A. P. Mathur. Effect of test set minimization on fault detection effectiveness. In International Conference on Software Engineering (ICSE), 1995.

[32] Q. Yang, J. J. Li, and D. M. Weiss. A survey of coverage-based testing tools. The Computer Journal, 2009.

[33] S. Yeganefard and M. Butler. Structuring functional requirements of control systems to facilitate refinement-based formalisation. Electronic Communications of the EASST, 2011.