ANGULAR INTRICACIES

IN

HOT GAUGE FIELD THEORIES

T. Grandou

Institut Non Linéaire de Nice UMR CNRS 6618; 1361, Route des Lucioles, 06560 Valbonne, France
e-mail:Thierry.Grandou@inln.cnrs.fr

ABSTRACT

It is argued that in hot gauge field theories, "Hard Thermal Loops" leading order calculations call for a definite sequence of angular averages and discontinuity (or Imaginary part prescription) operations, and run otherwise into incorrect results. The ten years old collinear singularity problem of hot QCD, provides a striking illustration of that fate.

PACS: 12.38.Cy, 11.10.Wx

Keywords: Hot QCD, Resummation Program, infrared, mass/collinear singularities...
1. Introduction

The intrinsic non perturbative nature of non zero temperature Quantum field Theories has long been recognized [1]. Naive thermal perturbation theory can nevertheless be devised, both in imaginary and real time formalisms [2], but then, it promptly appears that, under certain circumstances, the original perturbative series must be re-organized. Such an example of re-organization is provided by the so called Resummation Program [3]. This program, RP for short, is a resummation scheme of the leading order thermal fluctuations which, in the literature, are known under the spell of Hard Thermal Loops. Whenever one is calculating a physical process related to thermal Green’s functions whose external/internal legs are soft, it is mandatory to trade the naive thermal perturbation theory for the RP. The softness alluded to above, refers to momenta on the order of the soft scale $gT$, where $T$, the temperature, stands for the hard scale and $g$ for any relevant (bare/renormalized) and small enough coupling constant.

The RP which has been set up in order to remedy an obvious lack of completeness of the naive thermal perturbation theory, has produced interesting results, but has also met serious obstructions in the infrared regime of the theories [4],[5]. Within the Resummation Program itself, the solutions proposed so far [5,6,7], however interesting in their own respect, cannot be organized a systematic way and display a crucial dependence on the process under consideration.

In the past few years, another resummation scheme of the leading order thermal fluctuations has been proposed by the present author, and seems to avoid all of the infrared problems met by the standard RP, [8]. In loop calculations (where both resummation schemes are being used as effective perturbation theories, ruling the leading thermal fluctuations) the former only differs the standard RP, the sequence along which are performed an integration over a looping energy $p_0$, on the one hand, and a sum over the number $N$ of HTL-self energy insertions, on the other hand. This new resummation scheme, has accordingly been denoted PR for short, and is, by construction, a Perturbative Resummation scheme of the leading order thermal fluctuations.

Such is not the case of the RP. While HTL $n$-point vertices are purely perturbative objects, effective propagators are not, giving rise to pole residues and dispersion laws that cannot be obtained out of pure thermal perturbation theory. In the infrared limit, in particular, those effective propagators are known to give rise to series in the coupling constant that are no longer Taylor’s but Laurent’s series [8]. In any higher number of loops calculations, this
simple, still crucial difference, allows one to foresee easily, why, in the RP case and not in the PR one, some enhancement mechanisms may occur (and effectively will!) so as to make higher number of loops as important, and even bigger, than lower number of loops diagrams [9].

In this note, we take advantage of a comparison of the two RP and PR resummation schemes to point out some overlooked aspects concerning ”the historical derivations” of the famous collinear singularity problem of hot QCD, [5]. As many of our further calculations have displayed ever since, these aspects are generic and extend beyond the historical derivations to be recalled shortly.

We will be using the convention of upper case letters for quadrimomenta and lower case ones for their components, writing, for example $P = (p_0, \vec{p})$. Our conventions for labelling internal and external momenta can be read off Fig.1.

2. The collinear singularity problem of hot QCD

This almost ten years old issue is the following. The soft real photon emission rate out of a Quark-Gluon Plasma in thermal equilibrium involves, in particular, the calculation of the quantity

$$
\Pi_R(Q) = i \int \frac{d^4P}{(2\pi)^4} \left(1 - 2n_F(p_0)\right) \text{disc}_{\vec{p}} Tr \left\{ *S_R(P) *\Gamma_\mu(P_R, Q_R, -P'_A) \right. \\
\left. *S_R(P') *\Gamma_\mu(P_R, Q_R, -P'_A) \right\} 
$$

The discontinuity is to be taken in the energy variable $p_0$, by forming the difference of $R$ and $A$-indiced $P$-dependent quantities, and within standard notations, fermionic HTL self energies, effective propagators and vertices are respectively given by

$$
*S_\alpha(P) = \frac{i}{P - \Sigma_\alpha(P) + i\epsilon_\alpha p_0} , \quad \alpha = R, A , \quad \epsilon_R = -\epsilon_A = \epsilon 
$$

$$
\Sigma_\alpha(P) = m^2 \int \frac{d\hat{K}}{4\pi} \frac{\hat{K}}{\hat{K} \cdot P + i\epsilon_\alpha} , \quad m^2 = C_F g^2 T^2 8
$$

$$
*\Gamma_\mu(P_\alpha, Q_\beta, P'_\delta) = -ie \left( \gamma_\mu + \Gamma_\mu^{HTL}(P_\alpha, Q_\beta, P'_\delta) \right) 
$$

$$
\Gamma_\mu^{HTL}(P_\alpha, Q_\beta, P'_\delta) = m^2 \int \frac{d\hat{K}}{4\pi} \frac{\hat{K}_\mu \hat{K}}{(\hat{K} \cdot P + i\epsilon_\alpha)(\hat{K} \cdot P' + i\epsilon_\delta)} 
$$
where $\hat{K}$ is the lightlike four vector $(1, \hat{k})$. In view of (2.4), four terms come about, three of them proportional to a collinear singularity. These singular terms are the two terms with one bare vertex $\gamma_\mu$, the other $\Gamma_\mu^{HTL}$, plus the term including two $HTL$ vertices, $\Gamma_\mu^{HTL}$. Thanks to an abelian Ward identity peculiar to the high temperature limit, a partial cancellation of these collinear singularities occurs, but out of the term including two $\Gamma_\mu^{HTL}$ vertices, a collinear singularity remains,

$$\begin{align*}
-2i \frac{e^2 m^2}{q^2} \left( \int \frac{d\hat{K}}{4\pi} \frac{1}{\hat{Q} \cdot \hat{K} + i\epsilon} \right) \int \frac{d^4 P}{(2\pi)^3} \delta(P \cdot \hat{Q}) (1 - 2n_F(p_0)) \times [Tr \left( \ast S_A(P \hat{Q}) \right) - Tr \left( \ast S_R(P' \hat{Q}) \right)]
\end{align*}$$

where, the soft photon being real, $Q$ is the lightlike 4-vector $Q = q \hat{Q} = q(1, \hat{q})$. In the literature, this result is written in the form

$$C_{st} \varepsilon \int \frac{d^4 P}{(2\pi)^4} \delta(\hat{Q} \cdot P) (1 - 2n_F(p_0)) \sum_{s=\pm 1, V=P, P'} \pi(1 - s\frac{v_0}{v})\beta_s(V)$$

where the overall $1/\varepsilon$ comes from a dimensionally regularized evaluation of the factored out angular integration appearing in (2.6), and where $\beta_s(V)$ is related to the effective fermionic propagator usual parametrization [2],

$$\ast S_\alpha(P) = \frac{i}{2} \sum_{s=\pm 1} \hat{P}_s \Delta_s(p_0 + i\epsilon_\alpha, p)$$

with $\hat{P}_s = (1, s\hat{p})$, the label $s$ referring to the two dressed fermion propagating modes. One has then

$$\Delta_s(p_0 + i\epsilon_\alpha, p) \equiv \Delta_\alpha_s(p_0, p) = \alpha_s(p_0, p) - i\pi\epsilon(\epsilon_\alpha)\beta_s(p_0, p)$$

where $\epsilon(x)$ is the distribution ”sign of $x$”.

3. On an improper derivation

The third term, with the two $HTL$ vertices, $\Gamma_\mu^{HTL}$, involves two angular integrals $W_i$ that are, with $i \in \{1, 2\},$

$$W_i(P, P') = \int \frac{d\hat{K}}{4\pi} \int \frac{d\hat{K}'}{4\pi} \frac{(\hat{K} \cdot \hat{K}')^i}{(\hat{K} \cdot P + i\epsilon)(\hat{K} \cdot P' + i\epsilon)(\hat{K}' \cdot P + i\epsilon)(\hat{K}' \cdot P' + i\epsilon)}$$
While $W_2$ only is met in the standard $RP$ calculation, $W_1$ and $W_2$ come into play within the $PR$ scheme. As shown in [10], these integrals are given by very cumbersome expressions, so that, following [5], we will illustrate our point on the two diagrams including one bare vertex $\gamma_\mu$, the other $\Gamma^{HTL}_\mu$. The $R/A$ real time formalism conveys us to the expression

$$
\Pi_R^{(\ast,\ast;1)}(Q) = -ie^2 m^2 \int \frac{d^4 P}{(2\pi)^4} (1 - 2n_F(p_0))
$$

$$
\text{disc}_P \int \frac{d\hat{K}}{4\pi} \frac{Tr \left( *S_R(P)\hat{K}^* S_R(P')\hat{K} \right)}{(\hat{K} \cdot P + i\epsilon)(\hat{K} \cdot P' + i\epsilon)}
$$

(3.2)

where the superscript $(\ast,\ast;1)$ in the left hand side refers to a self energy diagram involving two effective propagators and one vertex $HTL$ correction. The steps leading to the collinear singularity of (3.2) are as follows (the second reference of [5], using the $R/A$ real time formalism, is followed here). The discontinuity in $p_0$ is taken in consistency with the $R/A$ formalism, by writing

$$
\text{disc}_P \frac{Tr \left( *S_R(P)\hat{K}^* S_R(P')\hat{K} \right)}{(\hat{K} \cdot P + i\epsilon)(\hat{K} \cdot P' + i\epsilon)} = \frac{Tr \left( *S_R(P)\hat{K}^* S_R(P')\hat{K} \right)}{(\hat{K} \cdot P + i\epsilon)(\hat{K} \cdot P' + i\epsilon)} - \frac{Tr \left( *S_A(P)\hat{K}^* S_R(P')\hat{K} \right)}{(\hat{K} \cdot P - i\epsilon)(\hat{K} \cdot P' + i\epsilon)}
$$

(3.3)

Closing the $p_0$ integration contour in the upper half complex $p_0$ plane, one selects a pole term coming from the vertex $\Gamma^{HTL}_\mu$ which reads

$$
\Pi_R^{(\ast,\ast;1)}(Q) = -e^2 m^2 \int \frac{d^4 P}{(2\pi)^3} (1 - 2n_F(p_0))
$$

$$
\int \frac{d\hat{K}}{4\pi} \frac{\delta(\hat{K} \cdot P)}{\hat{K} \cdot P + i\epsilon} Tr \left( *S_A(P)\hat{K}^* S_R(P')\hat{K} \right)
$$

(3.4)

Since $P' = P + Q$, we have indeed, as a building block of (3.4), the expression

$$
\int \frac{d\hat{K}}{4\pi} \frac{\delta(\hat{K} \cdot P)}{\hat{K} \cdot Q + i\epsilon} Tr \left( *S_A(P)\hat{Q}^* S_R(P')\hat{Q} \right)
$$

(3.5)

The angular average therefore develops a collinear singularity in a neighbourhood of $\hat{K} = \hat{Q}$, not compensated for by the numerator, and whose "residue" reads

$$
\frac{C^{(st)}}{\epsilon} \frac{\delta(\hat{Q} \cdot P)}{q} Tr \left( *S_A(P)\hat{Q}^* S_R(P')\hat{Q} \right)
$$

(3.6)
The first factor, singular at \( \varepsilon = 0 \), is obtained by using a dimensional regularization of the angular integral. Up to regular contributions that we do not consider here, this singular piece of (3.5) translates, for \( \Pi_R^{(\ast,\ast;1)}(Q) \), into the singular result

\[
\Pi_R^{(\ast,\ast;1)}(Q) = -\frac{C^{(st)} e^2 m^2}{\varepsilon} \int \frac{d^4 P}{(2\pi)^3} (1 - 2n_F(p_0)) \times \delta(\hat{Q} \cdot P) \text{Tr} \left( \ast S_A(P) \hat{Q} \ast S_R(P' \hat{Q}) \right)
\]

(3.7)

The contribution of (3.7) to the soft photon emission rate being proportional to its imaginary part, the emission rate is thus plagued with a collinear singularity. In its original (published) version [5], this result is obtained within the imaginary time formalism, continued to real energies, where the \( RP \) has been first devised [3].

In a recent article [10], the analogous \( PR \) calculation has been proven to be mass/collinear singularity free. In particular, such is the case of any of the contributions to the soft real photon emission rate due to \( \Pi_R^{(N,N';1)}(Q) \) quantities, the sum of which, on \( N \) and \( N' \) corresponding to \( \Pi_R^{(\ast,\ast;1)}(Q) \), as depicted on Fig.1. Thus, at face value, a collinear singularity shows up in a resummation scheme, not in the other. Let us now examine how can this be so.

In a \( PR \) scheme, one has indeed for the potentially dangerous most part of any \( \Pi_R^{(N,N';1)}(Q) \) quantity, the expression [10],

\[
-i e^2 m^2 \left( \ldots \right) \sum_{n,n'} (m^2)^{N+N'-2n-2n'} \int \frac{d^4 P}{(2\pi)^3} (1 - 2n_F(p_0)) \frac{1}{p'^p} \left( -P'^2 \Sigma_R(P') \right)^{n'} \frac{1}{(P'^2 + i\epsilon p_0 N')} \text{disc}_p \left( \frac{-P^2 \Sigma_R(P)}{(P^2 + i\epsilon p_0)^N} \right) \int \frac{d\hat{K}}{4\pi} \frac{1}{\hat{K} \cdot P + i\epsilon} \frac{1}{\hat{K} \cdot P' + i\epsilon} (3.8)
\]

where the discrete sums on \( n \) and \( n' \) extend over a finite set of integers and need not be further specified here. Performing the angular integration, one finds

\[
\int \frac{d\hat{K}}{4\pi} \frac{1}{\hat{K} \cdot P + i\epsilon} \frac{1}{\hat{K} \cdot P' + i\epsilon} = \frac{1}{2Q \cdot P + i\epsilon q} \ln \frac{P^2 + 2Q \cdot P + i\epsilon p_0'}{P^2 + i\epsilon p_0} (3.9)
\]

The collinear domain is included in the phase space region where \( \hat{Q} \cdot P \simeq 0 \), where one has

\[
\frac{1}{2Q \cdot P + i\epsilon q} \ln \frac{P^2 + 2Q \cdot P + i\epsilon p_0'}{P^2 + i\epsilon p_0} = \frac{1}{P^2 + i\epsilon p_0} \frac{1}{2} \frac{2Q \cdot P + i\epsilon q}{(P^2 + i\epsilon p_0)^2} + \ldots (3.10)
\]

Obviously, this \( HTL \)-vertex induced behaviour is potentially mass singular by the light cone region \( P^2 \simeq 0 \). This means that in a neighbourhood of \( \hat{Q} \cdot P \simeq 0 \), (3.9) mixes up with partial
propagators, $S_R^{(N^{(N')})}(P(P'))$, own potentially mass singular behaviours. Recalling that we have

$$S_R^{(N)}(P) = \frac{iP \Sigma_R(P)P}{(P^2 + i\epsilon p_0)^{N+1}}$$  \hspace{1cm} (3.11)

the whole integrand structure of (3.8), in the collinear domain, is readily seen to "boil down" to a simple shift of inverse power

$$\frac{1}{(P^2 + i\epsilon p_0)^M} \mapsto \frac{1}{(P^2 + i\epsilon p_0)^{M+1}}, \quad M = N + N'$$  \hspace{1cm} (3.12)

As demonstrated in [10], similar shifts, as well as many more transformations, are proven to leave totally unaffected the robust mass singularity cancellation patterns which guarantee the regular character of the PR calculation.

That is, in contradistinction with the RP scheme, a picture emerges out of the PR scheme, where effective vertices potentially mass singular behaviours, melt with partial effective propagators own potentially mass singular behaviours into structural patterns which rule the overall compensation of actual mass/collinear singularities, \[8,10\].

However, it is important to remark that, apart from their constitutive difference, not exactly the same steps have been followed in either RP and PR schemes. Both resummation schemes are here being developed within the $R/A$ real time formalism, so that, starting from the PR expression (3.8), exactly the same procedure as followed from (3.3) to (3.7) may be applied, with, as a straightforward result

$$-\frac{C^{(st)}}{\varepsilon} \frac{e^2 m^2}{q} (\ldots) \sum_{n,n'} (m^2)^{(N'+N-2n'-2n)} \int \frac{d^4p}{(2\pi)^3} (1 - 2n_F(p_0)) \frac{1}{p'p}$$

$$\times \delta(\hat{Q} \cdot P) \frac{[-P'^2 \Sigma_R(P')]^{n'}}{P'^2 + i\epsilon p_0)^{N'}} \frac{[-P^2 \Sigma_A(P)]^n}{P^2 - i\epsilon p_0)^N} + \ldots$$  \hspace{1cm} (3.13)

In words, (3.8) is now discovered to display the same collinear singularity as the one plaguing the RP result (3.7), in sharp contradiction with our claim that any $\Pi_R^{(N,N',1)}(Q)$ contributes mass/collinear singularity free quantities to the soft photon emission rate.

The contradiction is of course only apparent and entirely rooted in the fact that passing from (3.1) to (3.3), the prescription of discontinuity in $p_0$ has been commuted with the angular integration on $\hat{K}$. As we have just come to see, in both RP and PR schemes, the effect of that improper commutation (the discontinuity of the integral is prescribed by the formalisms, and not the integral of the discontinuity!) is to provide the collinear singularity with a somewhat
absolute status. In either schemes in effect, the collinear singularity just factors out, in total independence of the remaining integrations to be performed.

The lack of commutativity of the discontinuity and angular integration can be the matter of a direct observation. From (3.9), one has

$$\text{disc}_{\mathbf{P}} \int \frac{d\mathbf{K}}{4\pi} \frac{1}{\mathbf{K} \cdot \mathbf{P} + i\epsilon} \frac{1}{\mathbf{K} \cdot \mathbf{P}' + i\epsilon} = \frac{i\pi\epsilon(p_0)\Theta(-P^2)}{Q \cdot \mathbf{P} + i\epsilon q} - \frac{i\pi}{q} \delta(\hat{Q} \cdot \mathbf{P}) \ln \frac{P^2 + 2Q \cdot \mathbf{P} + i\epsilon p_0'}{P^2 + i\epsilon p_0}$$

$$= \frac{i\pi\epsilon(p_0)\Theta(-P^2)}{Q \cdot \mathbf{P} + i\epsilon} \tag{3.14}$$

In the right hand side, the term proportional to the distribution $\delta(\hat{Q} \cdot \mathbf{P})$, has coefficient zero, and not the collinear singularity $C^{(st)}/\epsilon$, as historically derived within the reverse, improper sequence. Though the illustration above, (3.5)-(3.6), is given within the R/A real time formalism only, it must be noticed that, mutatis mutandis, both the historical derivations of [5] hinge on a reverse sequence calculation,

$$\int \frac{d\mathbf{K}}{4\pi} \left( \text{disc}_{\mathbf{P}}, \text{or } I_{\text{Im}} \right) \frac{1}{\mathbf{K} \cdot \mathbf{P} + i\epsilon} \frac{1}{\mathbf{K} \cdot \mathbf{P}' + i\epsilon} = \int d\mathbf{K} \frac{\delta(\hat{K} \cdot \mathbf{P})}{\hat{K} \cdot \mathbf{Q} + i\epsilon} \tag{3.15}$$

along which a collinear singularity is identified at $\hat{K} = \hat{Q}$. Then, as advertised after (2.7), one relies on a dimensionally regularized angular integration to find out the singular counterpart

$$-i \left\{ \frac{1}{2}, \text{or } \frac{1}{4} \right\} \int d\hat{K} \frac{\delta(\hat{K} \cdot \mathbf{P})}{\hat{K} \cdot \mathbf{Q} + i\epsilon} = \frac{-iC^{(st)}}{\epsilon} \delta(\hat{Q} \cdot \mathbf{P}) \tag{3.16}$$

where $\epsilon$ is the regularizing parameter of an angular integration performed at $D = 3 + 2\epsilon$ spatial dimensions. This derivation, however, is inaccurate. This can be verified by a direct calculation of (3.15)’s right hand side, which gives instead

$$\int \frac{d\mathbf{K}}{4\pi} \left( \text{disc}_{\mathbf{P}}, \text{or } I_{\text{Im}} \right) \frac{1}{\mathbf{K} \cdot \mathbf{P} + i\epsilon} \frac{1}{\mathbf{K} \cdot \mathbf{P}' + i\epsilon} = \frac{i\pi\Theta(-P^2)\delta(\hat{p} - \hat{q})}{Q \cdot \mathbf{P} - i\epsilon} \tag{3.17}$$

A first important remark is that under subsequent angular (\hat{p}) and energy (p_0) integrations, the above reverse sequence result (3.17) will lead to more singular expressions than the proper sequence one, (3.14). Likewise, the correct prescription of $+i\epsilon$ is not preserved by the sequence (3.17). Now, the point is that nothing like the "absolute", overall factoring out collinear singularity of (3.16), i.e. the term $C^{(st)}/\epsilon$, ever appears in the final result, and this underlines the illicit character of the manipulations leading to it.

Discarding thus (3.16), one may still compare the right hand sides of (3.14) and (3.17). Then, apart from the fact that a non vanishing result is obtained for strictly collinear $\hat{p}$
and \( \vec{q} \) momenta only, another essential difference comes about immediately, which is the sign distribution \( \epsilon(p_0) \). In real time formalisms, sign distributions have long been noticed to be crucial so as to preserve integrable and non-integrable mass singularity compensations [11], and as observed here again, they are a natural outcome of the calculational operations proper sequence (3.14).

Indeed, this is the very place where intricacies are taking place, between angular averages peculiar to \( n \)-points HTL-vertices, and discontinuity or Imaginary part prescriptions. Staring at (3.9), for example, one can observe how the angular average is able to correctly reproduce all of the relevant internal legs \( R/A \) specifications, the \( i\epsilon q \), \( i\epsilon p_0 \) and \( i\epsilon p'_0 \), which do not appear at all in the left hand side. As can be read off [10], sections 4 and 5, this is not an isolated situation and rather reveals to be general a mechanism.

That is, the proper sequence of angular average and discontinuity operations complies with the required, exact sign distributions which otherwise, are clearly endangered.

By the way, this elucidates also the intriguing point made in the third reference of [8], after equation (3.32), which was then left as an issue. There, the discontinuity in \( p_0 \) of the angular integral \( W_2 \) was taken to be

\[
\text{disc } W_2(P, P') = -4i\pi \epsilon(p_0) \int \frac{d\hat{K}}{4\pi} \frac{\delta(\hat{K} \cdot P)}{\hat{K} \cdot P' + i\epsilon} \int \frac{d\hat{K}'}{4\pi} \frac{(\hat{K} \cdot \hat{K}')^2}{(\hat{K}' \cdot P + i\epsilon)(\hat{K}' \cdot P' + i\epsilon)}
\]

whereas it is manifest that in conformity with the common use, the reverse improper sequence had been followed. In order to preserve mass singularity cancellations though, the correct sign distribution, \( \epsilon(p_0) \), had to be restored by hand (this restoration was motivated a heuristic way, by recalling that the vertex corrections \( \Gamma^{HTL}_\mu \), were after all nothing but leading order approximations of full order \( g^2 \) expressions, endowed with explicit \( i\epsilon p_0 \) specifications).

Since further, higher number of loops calculations [12] enforce this peculiarity too, it seems reasonable to conclude that in any \( RP \) or \( PR \) calculations, and any real or imaginary time formalism, angular averages should definitely be performed first.

4. The collinear problem and the proper sequence

A full, extensive treatment of the soft real photon emission rate \( RP \) calculation, as the proper sequence is followed, falls far beyond the scope of the short present note, and could be
postponed to a future publication. We will here content ourselves with an illustration of how the collinear singularity problem gets translated when the angular integration is performed first.

From (3.2) and (2.8), one obtains

\[
i \Im \Pi_R^{(s, s')}(Q) = -ie^2 m^2 \int \frac{d^4 P}{(2\pi)^2} (1 - 2n_F(p_0)) \frac{1}{pp'} \sum_{s, s' = \pm 1} ss' \beta_s(p_0, p) \beta_{s'}(p'_0, p') \\
+ \frac{e^2 m^2}{2} \int \frac{d^4 P}{(2\pi)^3} (1 - 2n_F(p_0)) \frac{1}{pp'} \sum_{s, s' = \pm 1} \left( s' \beta_{s'}(p'_0, p') \frac{1}{1 - sp_0/p} \right) \ln \frac{p_0 + p}{p_0 - p} + (P \leftrightarrow P')
\]

\[
\times (1 - s p_0/p) \text{disc}_P \Delta_R^s(P) \frac{1}{2Q \cdot P} \ln \frac{P^2}{P^2}
\]

where the distribution \( \beta_s(p_0, p) \) of (2.9), though textbook material, may be worth recalling here in view of the forthcoming comments

\[
\beta_s(p_0, p) = Z_s(p) \delta (p_0 - \omega_s(p)) + Z_{-s}(p) \delta (p_0 + \omega_{-s}(p)) \\
+ \frac{m^2}{2p} \left( p(1 - s p_0/p) - \frac{m^2}{2p} \left( (1 - s p_0/p) \ln \frac{p_0 + p}{p_0 - p} + 2s \right) \right)^2 + \frac{\pi^2 m^4}{4p^2} (1 - s p_0/p)^2
\]

It is easy to check that the first line of (4.1), with the two functions \( \beta_s \), is safe. For example, one may rely on the integration contour technique used for energy sum rules, [2], chapter 6, to show that \( p_0 \) and \( \hat{\beta} \) integrations are well defined. Likewise, inspection shows that no problem is inherited from the second and third lines of (4.1). At \( p_0 = \pm p \), in effect, the potentially dangerous term of \( \ln |p_0 + p/p_0 - p| \) is, according to the value of \( s = \pm 1 \), either depleted by its own squared expression, or cancelled by a factor of \( (1 - sp_0/p)^2 \).

For the last term of (4.1), a potentially singular most contribution to \( \Im \Pi_R^{(s, s')}(Q) \) is generated by taking the discontinuity in \( p_0 \) of the first factor, \( \Delta_R^s(P) \). One gets

\[
-e^2 m^2 \int \frac{d^4 P}{(2\pi)^2} (1 - 2n_F(p_0)) \sum_{s, s' = \pm 1} (1 - s' p_0/p') \beta_{s'}(p'_0, p') \\
\times (1 - s p_0/p) \beta_s(p_0, p) \frac{1}{2Q \cdot P} \ln \frac{P^2}{P^2}
\]
The Dirac pieces of both spectral densities \( \beta_s \) and \( \beta_{s'} \), pose no problem, whereas the other parts, proportional to the Heaviside distributions \( \Theta(-P^2) \) and \( \Theta(-P'^2) \), and hereafter denoted by \( \hat{\beta}_{s,s'} \), yield

\[
-e^2 m^2 \int \frac{d^3 p}{(2\pi)^2} \int_{-p}^{+p} dp_0 \left( 1 - 2n_F(p_0) \right) \sum_{s' = \pm 1} \left( 1 - s' \frac{p'_0}{p'} \right) \hat{\beta}_{s'}(p'_0, p') \\
\times \sum_{s = \pm 1} \left( 1 - s \frac{p_0}{p} \right) \hat{\beta}_s(p_0, p) \frac{1}{2Q \cdot P} \ln \frac{P'^2}{P^2} \tag{4.4}
\]

In the phase space domain where \( 2Q \cdot P \) is vanishing, the integrand of (4.4) behaves like

\[
(1 - 2n_F(pz)) \sum_{s' = \pm 1} \left( 1 - s' \frac{q + pz}{p'} \right) \hat{\beta}_{s'}(q + pz, p') \times -\frac{1}{p'^2} \sum_{s = \pm 1} \hat{\beta}_s(pz, p) \frac{1}{1 + sz} \tag{4.5}
\]

where we have introduced the scaling variable \( z = p_0/p \), and where \( p'^2(z) = p^2 + 2qpz + q^2 \).

Writing

\[
1 - 2n_F(pz) = 2[n_F(p) - n_F(pz)] + (1 - 2n_F(p)) \tag{4.6}
\]

the last term only, in the right hand side, can possibly induce a singularity by the light cone region \( (z = \pm 1) \), reached from below, and this would be singular contribution can be expressed as

\[
\frac{e^2 m^2}{2\pi} \Theta(-P^2) \int p \, dp \left( 1 - 2n_F(p) \right) \int_0^1 dz \frac{\hat{\beta}_-(pz, p)}{1 - z} \sum_{\eta = \pm 1} \left( 1 + \frac{q + \eta pz}{p'(\eta z)} \right) \eta \hat{\beta}_-(q + \eta pz, p'(\eta z)) \tag{4.7}
\]

where use has been made of the relation \( \beta_s(-p_0, p) = \beta_{-s}(p_0, p) \). Now, against all odds, the light cone \( (1 - z \simeq 0) \) behaviours of \( \hat{\beta}_-(pz, p) \), \( \left( 1 + \frac{q + \eta pz}{p'} \right) \), and \( \hat{\beta}_-(q + \eta pz, p'(\eta z)) \), make it straightforward to check that the above potentially dangerous most contribution is a regular one indeed.

Since, out of the last term of (4.1), no other sign of a possible singularity shows up, it appears that the two \( \Pi_{R}^{(+, +; 1)}(Q) \) contributions to the soft real photon emission rate are regular, in contradistinction with the usual, singular results.
5. Conclusion

In this note, we have stressed that whatever the formalism (real or imaginary time) and the resummation scheme (standard \((RP)\) or perturbative \((PR)\)), a definite sequence of angular averages and discontinuity (or Imaginary part) operations, must definitely be preserved. Moreover, this proper sequence is nothing exotic, as it simply corresponds to the sequence naturally prescribed by any real or imaginary time formalism one may use.

Ignoring that proper sequence may lead (and has led) to incorrect derivations and/or results. In particular, infrared singularity cancellation patterns have long revealed to be particularly sensitive to the angular averages and discontinuity operations proper sequence. This is also a point which, we think, should be kept in mind within the context of numerical simulations.

For the hot \(QCD\) collinear problem met in the \(RP\) scheme, this point certainly matters a lot. Our rapid analysis of the one effective vertex topologies, \(\Pi^{(\star\star;1)}\), has not allowed us to detect any relic of a mass/collinear singularity in the soft photon emission rate, the proper sequence being followed.

Now, as advertised in section 3, this indicates only that the proper sequence is less singular, under subsequent integrations, than the reverse improper one, and let totally open the issue concerning the two effective vertex topology, \(\Pi^{(\star\star;2)}\), contribution to the emission rate.

In other words, the whole correct settings of the hot \(QCD\) collinear singularity \(RP\)-problem.. if any! .. has to be worked out again.
References

[1] N.P. Landsman, “Quark Matter 90”, *Nucl. Phys. A* **525**, (1991) 397.

[2] M. Le Bellac, “Thermal Field Theory” (Cambridge University Press, 1996).

[3] E. Braaten and R. Pisarski, *Phys. Rev. Lett.* **64**, (1990) 1338;

*Nucl. Phys. B** **337**, (1990) 569.

J. Frenkel and J.C Taylor, *Nucl. Phys. B** **334**, (1990) 199.

[4] R.D. Pisarski, *Phys. Rev. Lett.* **63**, (1989) 1129.

[5] R. Baier, S. Peigné and D. Schiff, *Z. Phys. C* **62**, (1994) 337;

Aurenche, T. Becherrawy and E. Petitgirard, [hep-ph 9403320](https://arxiv.org/abs/hep-ph/9403320) (unpublished).

[6] A. Niegawa, *Mod. Phys. Lett. A* **10**, (1995) 379;

F. Flechsig and A. Rebhan, *Nucl. Phys. B* **464**, (1996) 279.

[7] P. Arnold, G.D. Moore and L.G. Yaffee, JHEP **0206**, (2002) 030.

[8] B. Candelpergher and T. Grandou, *Ann. Phys. (NY)* **283**, (2000) 232;

[arXiv:hep-ph/0009349](https://arxiv.org/abs/hep-ph/0009349); *Nucl. Phys. A* **699**, (2002) 887.

T. Grandou, *Acta Physica Polonica B* **32**, (2001) 1185.

[9] P. Aurenche, F. Gelis, R. Kobes and E. Petitgirard, *Z. Phys. C* **75**, (1997) 315;

P. Aurenche, F. Gelis, R. Kobes and H. Zaraket,

*Phys. Rev. D* **58**, (1998) 085003, and references therein.

F. Gelis, Thèse présentée à l’Université de Savoie, le 10 Décembre 1998.

[10] T. Grandou, J. Math. Phys. **44**, (2003) 611.

[11] T. Grandou, M. Le Bellac and D. Poizat, *Nucl. Phys. B* **358**, (1991) 408.

[12] T. Grandou, work in progress.
**Figure caption**

**Fig.1:** A graph denoted by $(N, N'; 1)$, with $N(N')$ insertions of HTL self energy along the $P(P')$-line, one bare vertex $-ie\gamma_\mu$, and one HTL vertex correction (2.5). It is a PR scheme object. In the standard RP scheme, the two internal $P(P')$-lines are to be replaced with the full effective propagators $\ast S_\alpha(P(P'))$ of Equation (2.2), and this corresponds to the two RP graphs denoted by $(\ast, \ast; 1)$ in the text.