CMC HIERARCHY I: COMMUTING SYMMETRIES AND LOOP ALGEBRA

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Abstract. We propose an extension of the structure equation for constant mean curvature (CMC) surfaces in a three dimensional Riemannian space form to the associated CMC hierarchy of evolution equations by the higher-order commuting symmetries. Via the canonical formal Killing field, considered as an infinitely prolonged and loop algebra valued Gauß map, the CMC hierarchy is obtained by the assembly of a pair of Adler-Kostant-Symes bi-Hamiltonian hierarchies to the original CMC system. The infinite sequence of higher-order conservation laws of the CMC system admits the corresponding extension, and we find a formula for the generating series of the representative 1-forms. We also introduce a class of generalized (complexified) CMC surfaces as the phase space of the CMC hierarchy.

Contents

1. Introduction 2
1.1. Symmetry of a differential equation 2
1.2. Symmetry extension 3
1.3. CMC hierarchy 4
1.4. Results 4
1.5. Contents 5
1.6. Remarks 6
2. Summary of previous results 6
2.1. Differential system 6
2.2. Structure equations 7
2.3. Formal Killing field 8
2.4. Jacobi fields and conservation laws 9
2.5. Formal moduli spaces 10
3. Complexified CMC surfaces 10
3.1. Curvature of a (1, 1)-form on a Riemann surface 10
3.2. Complexified CMC surfaces 11
3.3. Local normal form 12
3.4. Real involution 12
3.5. Remarks 13
4. Infinitely prolonged Gauß map 14
4.1. Twisted loop algebra 14

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1. INTRODUCTION

1.1. Symmetry of a differential equation.

1.1.1. Classical symmetry. The classical notion of symmetry of a differential equation is a change of variable; a change of dependent and independent variables which maps solutions to solutions. From the inception of Lie groups and Lie algebras, Lie himself viewed a Lie group as a transformation group of symmetries of a differential equation, and more generally of a differential geometric object, [14, 15] translated in [11, 12].

On the other hand, a classical symmetry satisfies the uniform jet-order constraints and it is generated by the (infinitesimal) transformations defined on a finite jet space.

1.1.2. Generalized symmetry. A conceptual working definition of the symmetry of a differential equation would be:

\[
\text{a (local) Lie group or a Lie algebra which acts on the (formal) moduli space of solutions.}
\]

In this generalized sense, many new forms of symmetries, which are rooted in the deeper structural properties of a differential equation, become available. For example, one of the initial discoveries regarding the integrable hierarchies was that a differential equation may admit another compatible (commuting) evolution equation as a symmetry, [16][3][4] and the references therein.
1.1.3. **Infinite prolongation and integrable extension.** There exist many differential equations of interest which admit the various kinds of generalized, higher-order symmetries. To accommodate these, it is necessary to consider the infinite jets of a differential equation as a whole, and the infinite prolongation space provides the adequate background for analysis.

Furthermore, there are occasions when it is relevant to introduce the auxiliary non-local variables by integrable extension, [13]. Simply stated, this amounts to supplementing the given differential equation with an additional system of compatible ODE’s.

| Differential equation | Algebraic equation |
|-----------------------|--------------------|
| integrable extension  | field extension     |
| infinite prolongation | completion          |

**Table 1. Analogy with algebraic equation**

In this extended setting, the space of symmetries corresponds to the kernel of the linearization of the infinitely prolonged differential equation. The foundational works of Tsjishita [22], Vinogradov [24, 25], and Bryant and Griffiths [2] provide the general methods of commutative algebraic analysis to compute the symmetries and other cohomological invariants of a differential equation.

Regarding the integrable extension, we mention for an example that the log of tau function of KP hierarchy is defined as the potential for a non-local closed 1-form obtained by dressing, [4].

1.2. **Symmetry extension.** One of the characteristic defining properties of an integrable equation is that it admits an infinite sequence of higher-order commuting flows (evolution equations) as symmetries. This in turn leads to the extension of the given differential equation to the associated infinite hierarchy of equations. Compared to infinite prolongation and integrable extension, which are vertical extensions in a sense, the symmetry extension of a differential equation, which increases the number of independent variables, can be considered as a horizontal extension.

An important consequence of the symmetry extension is that it may lead to the additional symmetries of a differential equation. By iterating the two processes of finding generalized symmetries and symmetry extension, one ultimately hopes to gain an insight into solving the given differential equation. Note that the stationary solutions to the additional symmetries of an integrable equation provide a new class of solutions different from the finite type, algebro-geometric solutions, [23][17][20].
1.3. **CMC hierarchy.** The elliptic Monge-Ampère system for constant mean curvature (CMC) surfaces in a three dimensional Riemannian space form is the typical example of an integrable elliptic equation; in particular, it possesses an infinite sequence of higher-order symmetries and conservation laws, [7]. We wish to apply the idea of symmetry extension described above to find the additional symmetries, and subsequently to understand their stationary solutions.

In this first part of the series on CMC hierarchy, we will discuss the infinite sequence of commuting symmetries of the CMC system which are based on a twisted loop algebra \( g \subset \text{sl}(2, \mathbb{C})((\lambda)) \), (20).

1.3.1. **Purpose.** The purpose of this paper is to show that the CMC system admits a symmetry extension by the higher-order commuting symmetries to the compatible CMC hierarchy. Although in a slightly different context, the general model for our investigation is the Frenkel’s work [8] on Drinfeld-Sokolov hierarchies.

1.3.2. **CMC hierarchy.** We will find that, as a system of PDE’s, the proposed CMC hierarchy is locally equivalent to,

\[-\text{mKdV hierarchy} \oplus \text{elliptic sinh-Gordon} \oplus \text{mKdV hierarchy}.

For a related work on sine Gordon \( \oplus \) mKdV hierarchy, we refer to [10].

From this, it is expected that a substantial part of the existing theory of integrable systems can be introduced to the study of CMC surfaces.

1.4. **Results.** In the previous work [7], we gave a differential algebraic inductive formula for the \( g_{\geq 1} \)-valued\(^1\) canonical formal Killing field, denoted by \( Y \). The Jacobi fields and conservation laws of the CMC system were embedded in the coefficients of \( Y \), and accordingly the infinite sequence of higher-order Jacobi fields and conservation laws were completely determined.

The algebraic basis of this results lies in the compatibility of the prolongation structure of the CMC system with the recursive structure equation of the loop algebra \( g_{\geq 1} \). We claim that the consequences of this compatibility go beyond the effective calculation of the Jacobi fields and conservation laws.

1.4.1. **Complexified CMC surface.** We introduce a class of complexified CMC surfaces as a generalization of the ordinary CMC surfaces, Defn 3.1. They serve as the phase space of the CMC hierarchy.

1.4.2. **Infinitely prolonged and loop algebra valued Gauß map.** The canonical formal Killing field \( Y \) induces a map,

\[\hat{F}^{(\infty)}_{\infty} \xrightarrow{(-Y, Y)} -\frac{\partial}{\partial \lambda} \times g_{\geq 1},\]

which can be considered as an infinitely prolonged version of Gauß map, Fig 3, Defn 4.1. Here \( \hat{F}^{(\infty)}_{\infty} \) is roughly the infinite jet space of the CMC system, and \( \hat{F}^{(\infty)}_{\infty} \subset \hat{F}^{(\infty)} \) is a certain Zariski open set, §4.2.

\(^{1}\)Here \( g_{\geq 1} = g \cap \text{sl}(2, \mathbb{C})[[\lambda]] \lambda \) is the subalgebra of formal power series of \( \lambda \)-degree \( \geq 1 \), (21), (23).
1.4.3. **CMC hierarchy.** The twisted loop algebra \( \mathfrak{g} \) supports the Adler-Kostant-Symes (AKS) bi-Hamiltonian hierarchy, which is induced from the vector space decomposition, \((21)\),

\[
\mathfrak{g} = \mathfrak{g}_{\leq -1} + \mathfrak{g}_{\geq 0},
\]

and the associated R-matrix, Defn.5.1. The fundamental observation for the construction of the CMC hierarchy is that the image of the map \((−\overrightarrow{Y}, \overrightarrow{Y})\) is tangent to (or agrees with) the first two flows of the pair of AKS hierarchies on \((−\overrightarrow{g}_{\geq 1}, g_{\geq 1})\) respectively, Lem.6.1. From the formal symmetry of the Maurer-Cartan form, (33), the CMC hierarchy is obtained by attaching the pair of AKS hierarchies via \((−\overrightarrow{Y}, \overrightarrow{Y})\) to a combined system of equations on \(-\overrightarrow{g}_{\geq 1} \times g_{\geq 1}\), Thm.8.1.

\[\text{Figure 2. Anatomy of CMC hierarchy} \simeq \overrightarrow{\text{AKS hierarchy}} \oplus \text{AKS hierarchy}\]

In this schematic picture of \(\{t_m, t_{m}\}_{m \geq 0}\) are the time variables for the AKS hierarchies. Note that the original CMC system is embedded as the \(t_0, t_{\theta}\)-flows.

1.4.4. **Extension of conservation laws.** The infinite sequence of higher-order conservation laws of the CMC system admits the corresponding extension to the CMC hierarchy. We find an explicit formula for the generating series of the representative 1-forms, Thm.9.1.

1.4.5. **Linear finite type surfaces.** For an application, we show that the linear finite type CMC surfaces are characterized by the property that the canonical formal Killing field \(\overrightarrow{Y}\) is stationary with respect to a higher-order symmetry, Cor.10.1.

1.5. **Contents.** After a summary of the results from [7] in §2, we introduce in §3 a class of generalized CMC surfaces as a complexification of the CMC surfaces. In §4, we examine the algebraic properties of the \(\mathfrak{g}_{\geq 1}\)-valued canonical formal Killing field, considered as an infinitely prolonged version of Gauß map. In §5, the AKS construction of bi-Hamiltonian hierarchy is adapted to the twisted loop algebra \(\mathfrak{g}\). Based on this, we propose in §6 an ansatz for the CMC hierarchy in terms of an \(\mathfrak{sl}(2, \mathbb{C})[[\lambda^{-1}, \lambda]]\)-valued extended Maurer-Cartan form. In §7, the CMC hierarchy is translated into the \(\mathfrak{so}(4, \mathbb{C})\)-setting. In §8, we give a proof by direct computation that the proposed structure equation for the CMC hierarchy is compatible. In §9, we show that there exists the corresponding extension of the infinite sequence of higher-order conservation laws. In §10, we give a geometric characterization of the linear finite type CMC surfaces.

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2. Strictly speaking, the CMC hierarchy is an integrable extension of \(−\text{AKS hierarchy} \oplus \text{AKS hierarchy}, \) Fig.4.

3. Here \(\mathfrak{sl}(2, \mathbb{C})[[\lambda^{-1}, \lambda]]\) is the Lie algebra of \(\mathfrak{sl}(2, \mathbb{C})\)-valued formal power series in \(\lambda^{-1}, \lambda\).
1.6. Remarks.

1.6.1. Extension sequence of the underlying Lie algebras. In terms of the underlying Lie algebras only, the extension process of the CMC system up to the CMC hierarchy can be summarized as follows:

\[ \text{sl}(2, \mathbb{C}) \rightarrow \text{sl}(2, \mathbb{C})[[\lambda^{-1}, \lambda]] \rightarrow \text{sl}(2, \mathbb{C})[[\lambda^{-1}, \lambda]]. \]

The next step of the extension involves the Virasoro type of non-commuting symmetries (called spectral symmetries) and the associated generalized affine Kac-Moody algebras. This will be reported in Part II of the series.

1.6.2. Application. The construction of compact, high genus CMC surfaces so far has relied on the analytic existence results of PDE’s; either to find the fundamental domains for reflection, or to perturb an approximate CMC surface obtained by gluing to an actual CMC surface, \cite{I} for a survey of the related works. One of the initial objectives of \cite{I} was to find a class of generalized (nonlinear) finite type CMC surfaces. We hope that the stationary constraints from the additional symmetries may lead to a new class of CMC surfaces which can be analyzed by the methods of ODE’s.

2. Summary of previous results

We recall the relevant notations and results from \cite{I}. We only give a brief description and refer the reader to \cite{I} for the details.

2.1. Differential system.

[Grassmann bundle of oriented 2-planes]

\[ M : \text{three dimensional Riemannian space form of constant curvature } \epsilon, \]

\[ \mathcal{F} := \text{Iso}(M) : \text{group of isometries of } M, \]

\[ X := \text{Gr}^+(2, TM) : \text{Grassmann bundle of oriented 2-planes}. \]

They fit into the commutative diagram:

\[
\begin{array}{ccc}
\text{SO}(2) & \to & \mathcal{F} \\
\downarrow & & \downarrow \\
X & \to & \text{SO}(3) \\
\downarrow & & \downarrow \\
\text{SO}(3)/\text{SO}(2) = S^2 & \to & M
\end{array}
\]

[Structure constant \( \gamma \)]

We shall consider the immersed oriented surfaces in \( M \) of constant mean curvature \( \delta \).

\[ \gamma^2 := \epsilon + \delta^2 : \text{structure constant}, \]

assumption : \( \gamma^2 > 0 \) and \( \gamma \) is real.

The case \( \gamma^2 < 0 \) appears to be incompatible with the certain aspects of the theory of integrable systems applied in this work.
[CMC system]

$(X, I) : \text{original CMC system on } X,$

$(X^{(\infty)}, I^{(\infty)}) : \text{infinite prolongation of } (X, I),$

$\mathcal{F}^{(\infty)} \rightarrow \mathcal{F} : \text{pulled back bundle},$

$\hat{\mathcal{F}}^{(\infty)} \rightarrow \mathcal{F}^{(\infty)}, \hat{X}^{(\infty)} \rightarrow X^{(\infty)} : \text{double covers}.$

They fit into the commutative diagrams:

\[
\begin{array}{ccc}
\mathcal{F}^{(\infty)} & \rightarrow & X^{(\infty)} \\
\downarrow & & \downarrow \\
\mathcal{F} & \stackrel{\text{SO(2)}}{\rightarrow} & X,
\end{array}
\quad
\begin{array}{ccc}
\hat{\mathcal{F}}^{(\infty)} & \rightarrow & \hat{X}^{(\infty)} \\
\downarrow & & \downarrow \\
\hat{\mathcal{F}} & \stackrel{\text{SO(2)}}{\rightarrow} & \hat{X}^{(\infty)}.
\end{array}
\]

The corresponding differential ideals on $\mathcal{F}^{(\infty)}, \hat{X}^{(\infty)}, \hat{\mathcal{F}}^{(\infty)}$ are denoted by $I^{(\infty)}$, and $\hat{I}^{(\infty)}$ respectively.

2.2. Structure equations. The structure equations recorded below are written modulo the appropriate differential ideals, see §6.5.4 for a related remark. The meaning is generally clear from the context, and we omit the specific descriptions.

[Basic structures]

$\xi$ : tautological unitary (1,0)-form,

$\rho$ : connection 1-form,

$d\xi = i \rho \wedge \xi,$

$d\rho = R^1_2 \xi \wedge \xi,$

$I I = h^2_2 \xi^2 : \text{Hopf differential},$

$R = \gamma^2 - h_2 \bar{h}_2 : \text{Gauß curvature}.$

[Infinite prolongation]

\[
dh_j + ij h_{j} \rho = h_{j+1} \xi + T_j \bar{\xi}, \quad j \geq 2,
\]

$T_2 = 0,$

\[
T_{j+1} = \sum_{s=0}^{j-2} a_{js} h_{j-s} \partial^s_\xi R, \quad \text{for } j \geq 2,
\]

\[
a_{js} = \frac{(j + s + 2)}{2} \frac{(j - 1)!}{(j - s - 2)! (s + 2)!} = \frac{(j + s + 2)}{2} \binom{j}{s+2},
\]

$\partial^s_\xi R = \delta_{0s} \gamma^2 - h_2 \bar{h}_2.$

Here $i = \sqrt{-1}$ denotes the unit imaginary number.
[√II, and balanced coordinates]

\[ \omega := \sqrt{II} = h_2^\frac{1}{2} \xi, \]
\[ d\omega = 0, \]
\[ z_j := h_2^{-j} h_j, \quad j \geq 3, \]
\[ \mathcal{R} := \mathbb{C}[z_3, z_4, ...], \quad \overline{\mathcal{R}} := \mathbb{C}[\overline{z}_3, \overline{z}_4, ...]. \]

Assign the spectral weights by,

| \omega \quad | \text{weight} \quad | \overline{\omega} \quad | \text{weight} \\
|---|---|---|---|
| \omega \quad | -1 \quad | \overline{\omega} \quad | +1 \\
| z_j \quad | j - 2 \quad | \overline{z}_j \quad | -(j - 2) \\
| h_2 \overline{h}_2 \quad | 0 \quad | \quad | \\

2.3. Formal Killing field.

[\text{sl}(2, \mathbb{C})[\lambda^{-1}, \lambda]-valued Maurer-Cartan form]

\( (2) \)

\[ \phi_+ = \left( \frac{i}{2} \overline{h}_2, -\frac{1}{2} \gamma \right) \overline{\xi}, \quad \phi_0 = \left( \frac{i}{2} \rho , -\frac{1}{2} \rho \right), \quad \phi_- = \left( \frac{i}{2} \gamma, -\frac{1}{2} h_2 \right) \xi, \]
\[ \phi_{\lambda} := \lambda \phi_+ + \phi_0 + \lambda^{-1} \phi_- = \left( \frac{i}{2} \gamma h_2 \overline{\xi} + \lambda^{-1} \frac{1}{2} \gamma \xi, -\frac{1}{2} h_2 \overline{\xi} - \lambda^{-1} \frac{1}{2} \gamma \xi \right). \]
\[ d\phi_{\lambda} + [\phi_{\lambda}, \phi_{\lambda}] = 0. \]

[\text{sl}(2, \mathbb{C})[[\lambda]] \lambda-valued formal Killing field]

\( (3) \)

\[ Y := \begin{pmatrix} -ia & 2c \\ 2b & ia \end{pmatrix}, \]
\[ a = \sum_{n=0}^{\infty} \lambda^{2n} a^{2n+1}, \quad b = \sum_{n=0}^{\infty} \lambda^{2n+1} b^{2n+2}, \quad c = \sum_{n=0}^{\infty} \lambda^{2n+1} c^{2n+2}. \]
\[ (4) \]

\[ dY + [\phi_{\lambda}, Y] = 0. \]

\[ (5) \]

\[ a^1 = 0, \quad b^2 = -i \gamma h_2^{-\frac{3}{2}}, \quad c^2 = i h_2^{-\frac{1}{2}}, \quad \{a^{2n+1}, h_2^2 b^{2n+2}, h_2^{-2} c^{2n+2}\}_{n \geq 0} \subset \mathcal{R}. \]

\[ (6) \]

\[ \det(Y) = -4b^2 c^2 \lambda^2 = -4 \gamma \lambda^2. \]
[\mathfrak{so}(4, C)[\lambda^{-1}, \lambda]-valued Maurer-Cartan form]

\begin{equation}
(7) \quad \psi_+ = \frac{1}{2} \begin{pmatrix}
\cdot & -\gamma & -i\gamma & \cdot \\
\gamma & \cdot & \cdot & -\bar{h}_2 \\
i\gamma & \cdot & \cdot & i\bar{h}_2 \\
\cdot & \bar{h}_2 & -i\bar{h}_2 & \cdot
\end{pmatrix} \xi,
\psi_0 = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & -\rho & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot
\end{pmatrix},
\psi_- = \frac{1}{2} \begin{pmatrix}
\cdot & -\gamma & i\gamma & \cdot \\
\gamma & \cdot & \cdot & -h_2 \\
i\gamma & \cdot & \cdot & ih_2 \\
\cdot & h_2 & ih_2 & \cdot
\end{pmatrix} \xi,
\end{equation}

\begin{equation}
\psi_\lambda := \lambda \psi_+ + \psi_0 + \lambda^{-1} \psi_-.
\end{equation}

\begin{equation}
\mathcal{D} \psi_\lambda + \psi_\lambda \wedge \psi_\lambda = 0.
\end{equation}

[\mathfrak{so}(4, C)[[\lambda]]\lambda-valued formal Killing field]

\begin{equation}
(8) \quad X := \begin{pmatrix}
i(c_2 + b_4) & -(c_2 - b_4) & -a_3 \\
-ia_1 & -i(b_2 + c_4) & (c_2 - b_4) \\
a_3 & i(b_2 + c_4) & (b_2 - c_4)
\end{pmatrix},
\end{equation}

\begin{align*}
a_1 &= \sum_{n=0}^{\infty} (-1)^n \lambda^{4n+1} a_{4n+1}, \\
b_2 &= \sum_{n=0}^{\infty} (-1)^n \lambda^{4n+1} b_{4n+2}, \\
c_2 &= \sum_{n=0}^{\infty} (-1)^n \lambda^{4n+1} c_{4n+2}, \\
a_3 &= \sum_{n=0}^{\infty} (-1)^n \lambda^{4n+2} a_{4n+3}, \\
b_4 &= \sum_{n=0}^{\infty} (-1)^n \lambda^{4n+3} b_{4n+4}, \\
c_4 &= \sum_{n=0}^{\infty} (-1)^n \lambda^{4n+3} c_{4n+4}.
\end{align*}

\begin{equation}
(9) \quad \mathcal{D} X + [\psi_\lambda, X] = 0.
\end{equation}

[Recursive structure equation for the coefficients of Y]

\begin{equation}
(10) \quad \mathcal{D} a^{2n+1} = (i\gamma c^{2n+2} + i\bar{h}_2 b^{2n+2}) \xi + (i\gamma b^{2n} + i\bar{h}_2 c^{2n}) \bar{\xi},
\end{equation}

\begin{equation}
\mathcal{D} b^{2n+2} - ib^{2n+2} \rho = \frac{i\gamma}{2} a^{2n+3} \xi + \frac{i}{2} \bar{h}_2 a^{2n+1} \bar{\xi},
\end{equation}

\begin{equation}
\mathcal{D} c^{2n+2} + ic^{2n+2} \rho = \frac{i}{2} h_2 a^{2n+3} \xi + \frac{i\gamma}{2} a^{2n+1} \bar{\xi}.
\end{equation}

2.4. Jacobi fields and conservation laws.

[Jacobi fields]

Each coefficient $a^{2n+1}$ is a Jacobi field which lies in the kernel of the Jacobi operator,

\begin{equation}
(11) \quad \mathcal{E} := \partial_\xi \partial_{\bar{\xi}} + \frac{1}{2}(\bar{\rho}^2 + h_2 \bar{h}_2).
\end{equation}

Here $\partial_\xi, \partial_{\bar{\xi}}$ are the covariant derivative operators in the $\xi, \bar{\xi}$ directions (mod $\hat{\lambda}^{(\infty)}$) respectively. Jacobi fields are the generating functions of the generalized symmetries of $(\hat{\lambda}^{(\infty)}, \hat{\lambda}^{(\infty)})$.

The set \{ $a^{2n+1}, a^{2n+1}$ \}_{n \geq 1} spans the space of higher-order Jacobi fields. The corresponding higher-order symmetries commute with each other.

[Conservation laws]
Set
\[(12) \quad \varphi^n := c^{2n+2} \xi + b^{2n} \xi, \quad n \geq 0.\]
Then
\[d\varphi^n = 0,\]
and each \(\varphi^n\) represents a nontrivial conservation law.
The set \([\varphi^n, [\varphi^n]]\) \(n \geq 0\) spans the space of higher-order conservation laws.

2.5. Formal moduli spaces. The differential ideal on each of the infinite prolongation spaces \(X^{(\infty)}, \mathcal{F}^{(\infty)}, \hat{X}^{(\infty)}, \hat{\mathcal{F}}^{(\infty)}\) is formally Frobenius. Denote the formal moduli spaces of the integral foliation respectively by,
\[
\hat{M}_F := \hat{\mathcal{F}}^{(\infty)}/(\hat{I}^{(\infty)})^\perp, \quad \hat{M} := \hat{X}^{(\infty)}/(\hat{I}^{(\infty)})^\perp, \quad M_F := \mathcal{F}^{(\infty)}/(I^{(\infty)})^\perp, \quad M := X^{(\infty)}/(I^{(\infty)})^\perp.
\]
They fit into the commutative diagram:
\[
\begin{array}{ccc}
\hat{M}_F & \longrightarrow & \hat{M} \\
\downarrow & & \downarrow \\
M_F & \longrightarrow & M.
\end{array}
\]
Any one of \(\hat{M}_F, M_F, \hat{M}, M\) will be used as the formal moduli space of CMC surfaces as convenient.

3. Complexified CMC surfaces

The purpose of this paper is to extend the CMC system to the CMC hierarchy of evolution equations by the higher-order commuting symmetries generated by \(Y\). It turns out that the deformations induced by the CMC hierarchy do not preserve exactly the class of CMC surfaces. The structure equation shows that it is necessary to generalize and consider a class of complexified CMC surfaces for the phase space of the CMC hierarchy.

To this end, we give in this section a precise definition of the complexified CMC surfaces. It will be on the formal moduli space of such generalized CMC surfaces that the CMC hierarchy will be realized as the hierarchy of commuting flows.

3.1. Curvature of a (1, 1)-form on a Riemann surface. We first record a preliminary analysis on the curvature associated with a nowhere zero (1, 1)-form on a Riemann surface.

Let \(\Sigma\) be a Riemann surface. Let \(\Omega^{p,q} \rightarrow \Sigma\) be the bundle of \((p, q)\)-forms. Let \(K = \Omega^{1,0}\) denote the canonical line bundle.

Suppose \(\Upsilon \in H^0(\Sigma, \Omega^{1,1})\) be a nowhere zero (1, 1)-form. At each point of \(\Sigma\), there exists a pair of (1, 0)-form \(\xi\) and (0, 1)-form \(\bar{\xi}\) such that
\[\Upsilon = \frac{i}{2} \xi \wedge \bar{\xi}.\]
(note that $\bar{\xi}$ is a notation for a $(0,1)$-form and does not necessarily mean the complex conjugate $(\bar{\xi})$). Such a pair of 1-forms $(\xi, \bar{\xi})$ is defined up to scaling by
\[(\xi, \bar{\xi}) \rightarrow (s\xi, s^{-1}\bar{\xi}), \quad s \in \mathbb{C}^*.\]

Let $\pi : F_\Upsilon \to \Sigma$ be the associated principal $\mathbb{C}^*$-bundle. From the general theory of $G$-structures, let $(\xi, \bar{\xi})$ be the tautological pair of 1-forms on $F_\Upsilon \to \Sigma$ (which we denote by the same notations) such that
\[\pi^*\Upsilon = \frac{i}{2} \xi \wedge \bar{\xi}.\]

A standard equivalence method argument shows that there exists a unique torsion-free (complex) connection 1-form $\rho$ on $F_\Upsilon$ such that
\[d\xi = i\rho \wedge \xi, \quad d\bar{\xi} = -i\rho \wedge \bar{\xi}.\]

The (scalar) curvature $R_\Upsilon$ of the $(1,1)$-form $\Upsilon$ is then defined by the equation
\[d\rho = R_\Upsilon \Upsilon.\]

### 3.2. Complexified CMC surfaces

With this preparation, we give a definition of the complexified CMC surfaces.

**Definition 3.1.** Let $\gamma^2 \in \mathbb{R}$ be a given real structural constant. A **complexified** CMC surface consists of the triple of data $(\Sigma, \Upsilon, \Phi)$, where $\Sigma$ is a Riemann surface, $\Upsilon \in H^0(\Sigma, \Omega^{1,1})$ is a nowhere zero $(1,1)$-form, and $\Phi \in H^0(\Sigma, K^2)$ is a holomorphic quadratic differential. They must satisfy the following compatibility condition; suppose we write (locally)
\[\Upsilon = \frac{i}{2} \xi \wedge \bar{\xi},\]
for a $(1,0)$-form $\xi$ and a $(0,1)$-form $\bar{\xi}$. Let
\[\Phi = h_2 \xi^2, \quad \bar{\Phi} = \bar{h}_2 \bar{\xi}^2, \quad \text{(complex conjugate of } \Phi).\]

Here $h_2, \bar{h}_2$ are the scalar coefficients (note again that $\bar{h}_2$ does not necessarily mean the complex conjugate $(\bar{h}_2)$). Then,
\[R_\Upsilon = \gamma^2 - h_2 \bar{h}_2.\]

Here $R_\Upsilon$ is the curvature of the $(1,1)$-form $\Upsilon$.

Let $\mathcal{M}_C$ denote the **formal moduli space** of the complexified CMC surfaces.

A version of the classical Bonnet theorem holds and a complexified CMC surface admits a local embedding into a homogeneous space of $\text{SO}(4, \mathbb{C})$. We do not pursue to give the precise description of this space, nor the related extrinsic geometry of a complexified CMC surface.

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4Here $\mathbb{C}^*$ is considered as a multiplicative group.
3.3. **Local normal form.** The compatibility equation (15) can be written in the familiar local normal form of (complex) sinh-Gordon equation.

Away from the zero divisor (called “umbilics”) of $\Phi$, choose a local holomorphic coordinate $z$ on $\Sigma$ such that

$$\Phi = (dz)^2.$$  

Without loss of generality, let

$$(16) \quad \xi = e^{u} dz, \quad \bar{\xi} = e^{u} d\bar{z}$$

for a (complex) scalar function $u = u(z, \bar{z})$ such that

$$\Upsilon = e^{2u} \frac{1}{2} dz \wedge d\bar{z}.$$  

By definition, we have

$$h_2 = \bar{h}_2 = e^{-2u}.$$  

Differentiate (16), and the corresponding section of the torsion-free connection 1-form $\rho$ is given by

$$\rho = i(u_\bar{z} dz - u_\bar{\xi} d\bar{z}).$$

Here $u_\bar{z}, u_\bar{\xi}$ denote the partial derivatives, etc. Differentiate the given $\rho$ again, and the curvature $R_\Upsilon$ is given by

$$R_\Upsilon = -4e^{-2u} u_{\bar{\xi}}.$$  

Eq. (15) is now reduced to the sinh-Gordon equation,

$$u_{\bar{\xi}} + \frac{1}{4} (\gamma^2 e^{2u} - e^{-2u}) = 0.$$  

3.4. **Real involution.** Let $\mathcal{M}$ be the formal moduli space of ordinary CMC surfaces. In the analysis above, note that $u$ is real whenever $\bar{\xi} = (\xi)$ (complex conjugate). We elaborate on this observation and give a geometric description of how $\mathcal{M}$ sits inside $\mathcal{M}^C$.

Let $(\Sigma, \Upsilon, \Phi)$ be a complexified CMC surface. Consider the associated triple

$$(\Sigma, \Upsilon, \Phi).$$

From the definition, it is easily checked that the compatibility equation for this triple is given by (following the notations above)

$$(17) \quad R_\Upsilon = \gamma^2 - (\bar{h}_2)(h_2), \quad \text{(complex conjugation).}$$

On the other hand, by definition of the curvature of a $(1, 1)$-form,

$$R_\Upsilon = \bar{R}_\Upsilon.$$  

Since $\gamma^2$ is real, this implies that (17) holds and $(\Sigma, \Upsilon, \Phi)$ is also a complexified CMC surface.

As a result, the map

$$(18) \quad (\Sigma, \Upsilon, \Phi) \mapsto (\Sigma, \bar{\Upsilon}, \Phi)$$
defines an involution on $M^\mathbb{C}$;

$$i : M^\mathbb{C} \to M^\mathbb{C}, \quad i^2 = 1_{M^\mathbb{C}}.$$ 

The fixed point loci of the involution, i.e., the complexified CMC surfaces with the real $(1,1)$-form $\Upsilon = \overline{\Upsilon}$, then exactly correspond to the ordinary CMC surfaces.

**Proposition 3.1.** Let $M$ be the formal moduli space of CMC surfaces, and let $M^\mathbb{C}$ be the formal moduli space of complexified CMC surfaces. There exists an involution $i : M^\mathbb{C} \to M^\mathbb{C}$ defined by (18) such that $M = (M^\mathbb{C})^i$ is the fixed point loci of $i$. In this sense, $M^\mathbb{C}$ is the complexification of $M$.

### 3.5. Remarks.

Let us make a few relevant remarks.

a) Most of the results of [7] summarized in §2, including infinite prolongation, structure equations, formal Killing field, Jacobi fields, and conservation laws, have their obvious analogues for the complexified CMC surfaces. One only needs to re-consider the complex conjugation notation (overline) as the formal complex conjugation, §6.1.1. We leave the rest of details of the necessary changes for the transition from the CMC surfaces to the complexified CMC surfaces.

For simplicity, we use the same notations for the corresponding objects for the complexified CMC surfaces.

b) For a complexified CMC surface, we generally have (following the notations above)

$$\xi \neq (\xi), \quad \rho \neq (\rho), \quad \overline{\hbar}_2 \neq \overline{(\hbar_2)}, \quad \text{(complex conjugate)}$$

and hence

$$\overline{\hbar}_j \neq \overline{(\hbar_j)}, \quad j \geq 2.$$ 

It follows that, on the infinite prolongation space $\hat{F}^{(\infty)}$ for the complexified CMC surfaces, the sequences of functions $[\hbar_j]$ and $[\overline{\hbar}_j]$ are independent.

We call $[h_j, z_k]$ and $[\overline{h}_j, \overline{z}_k]$ the functions of type $(1,0)$ and $(0,1)$ respectively.

c) For example, there exist two canonical formal Killing fields for the generalized CMC surfaces, $\Upsilon$ of type $(1,0)$, and $-\overline{\Upsilon'}$ of type $(0,1)$. They satisfy the structure equations,

$$d\Upsilon + [\phi_\lambda, \Upsilon] = 0,$$

$$d(-\overline{\Upsilon'}) + [\phi_\lambda, (-\overline{\Upsilon'})] = 0.$$ 

Here $\overline{\Upsilon}$ should be understood as the formal complex conjugation of $\Upsilon$, §6.1.1. For the second equation, note the formal symmetry, $\phi_\lambda = -\overline{\phi_\lambda}$.

With these being understood, a CMC surface would mean a complexified CMC surface from now on.
4. Infinitely prolonged Gauß map

Consider the Lie algebra decomposition,

\[ \mathfrak{so}(4, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}). \]

One of the simplifying factors in the study of the CMC system is that, due to this decomposition, the entire analysis can be based on the simpler Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \). We utilize this and formulate the CMC hierarchy in terms of a twisted loop algebra \( \mathfrak{g} \subset \mathfrak{sl}(2, \mathbb{C})(\langle \lambda \rangle) \), (20).

In this section, we give a description of the \( \mathfrak{g} \)-valued formal Killing field \( Y \) as a part of an infinitely prolonged version of Gauß map. This interpretation will play a role in connecting the CMC system to the AKS bi-Hamiltonian hierarchies on \((-\tilde{g}, \mathfrak{g})\).

4.1. Twisted loop algebra. Let \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \). Let

\[ \mathfrak{g}((\lambda)) := \{ \text{\( \mathfrak{g} \)-valued formal Laurent series in } \lambda \}. \]

Here \( \lambda \in \mathbb{C}^* \) is the spectral parameter. Define the twisted loop algebra,

\[ (20) \quad \mathfrak{g} := \left\{ h(\lambda) \in \mathfrak{g}((\lambda)) \mid h_1^1(\lambda) = -h_2^2(\lambda) \text{ is even in } \lambda; h_1^2(\lambda), h_2^1(\lambda) \text{ are odd in } \lambda \right\} \subset \mathfrak{g}((\lambda)). \]

Here \( h_i^j(\lambda) \)'s denote the components of \( h(\lambda) \).

The formal Killing field \( Y \) for the CMC system, and the extended Maurer-Cartan form for the CMC hierarchy, etc, will either take values in \( \mathfrak{g} \), or at least satisfy the twistedness condition given in (20).

4.1.1. Vector space decomposition. Consider the vector space decomposition of \( \mathfrak{g} \) into the subalgebras,

\[ (21) \quad \mathfrak{g} = \mathfrak{g}_{\leq -1} \oplus^\text{ps} \mathfrak{g}_{\geq 0} \subset \mathfrak{g}[\langle \lambda \rangle]. \]

Here "\( +^\text{ps} \)" means the direct sum as a vector space and not as a Lie algebra. The notation \( \mathfrak{g}_{\leq -1} \) means the subalgebra of polynomial loops of \( \lambda \)-degree \( \leq -1 \), and \( \mathfrak{g}_{\geq 0} \) similarly means the subalgebra of formal power series loops of \( \lambda \)-degree \( \geq 0 \).

4.1.2. Dual decomposition. We adopt the standard invariant inner product on \( \mathfrak{g} \) defined by

\[ (22) \quad \langle Y_1, Y_2 \rangle := \text{Res}_{\lambda=0}(\text{tr}(Y_1 Y_2)), \quad Y_1, Y_2 \in \mathfrak{g}. \]

Here \( \text{Res}_{\lambda=0} \) is the residue operator which takes the terms of \( \lambda \)-degree 0. The corresponding decomposition of \( \mathfrak{g} = \mathfrak{g}^* \) dual to (21) is given by

\[ (23) \quad \mathfrak{g} = \mathfrak{g}_{\geq 1} \oplus^\text{ps} \mathfrak{g}_{\leq 0} \subset \mathfrak{g}[\langle \lambda \rangle], \mathfrak{g}[\langle \lambda \rangle]. \]

Here \( \mathfrak{g}_{\geq 1}, \mathfrak{g}_{\leq 0} \) denote the subalgebras which are defined similarly as above according to their \( \lambda \)-degrees.
4.1.3. Determinantal subvarieties in $\mathfrak{g}_{\geq 1}$. Recall $C((\lambda))$ is the space of formal Laurent series in $\lambda$. By definition, the product map $C((\lambda)) \times C((\lambda)) \to C((\lambda))$ is well defined. This implies that the determinant function

$$\text{det} : \mathfrak{g} \mapsto C((\lambda^2))$$

is also well defined.

Recall the identity, (6),

$$\det(Y) = -4\gamma\lambda^2 \in C[[\lambda^2]].$$

Set $P_{4\gamma\lambda^2} : \mathfrak{g} \mapsto C((\lambda^2))$ be the defining function for $Y$,

$$P_{4\gamma\lambda^2}(Y) := \det(Y) + 4\gamma\lambda^2, \quad \text{for } Y \in \mathfrak{g}.$$

Let $Y_{4\gamma\lambda^2} \subset \mathfrak{g}$ be the corresponding subvariety,

$$Y_{4\gamma\lambda^2} := \{ Y \in \mathfrak{g} | P_{4\gamma\lambda^2}(Y) = 0 \}.$$

We record the following elementary property of the relevant subset $Y_{4\gamma\lambda^2} \cap \mathfrak{g}_{\geq 1}$ without proof.

**Lemma 4.1.** Let $G_{\geq 0}$ be the formal loop group with Lie algebra $\mathfrak{g}_{\geq 0}$. Then, under the adjoint action, $G_{\geq 0}$ acts transitively on $Y_{4\gamma\lambda^2} \cap \mathfrak{g}_{\geq 1}$.

4.1.4. Infinite sequence of quadratic constraints. From Lem.4.1, consider the restriction of the function $P_{4\gamma\lambda^2}$ to $\mathfrak{g}_{\geq 1}$. When expanded as a formal power series in $\lambda^2$, it gives rise to an infinite sequence of quadratic constraints for the subvariety $Y_{4\gamma\lambda^2} \cap \mathfrak{g}_{\geq 1}$. This sequence of quadratic functions will serve as the commuting Hamiltonians for the CMC hierarchy, §5.

4.2. Zariski open sets. Recall the commutative diagram from (1),

$\begin{array}{ccc}
\hat{\mathcal{F}}^{(\infty)} & \rightarrow & \hat{X}^{(\infty)} \\
\downarrow & & \downarrow \\
\mathcal{F} & \rightarrow & X.
\end{array}$

Define the Zariski open sets,

$\hat{\mathcal{F}}_{\infty}^{(\infty)} := \hat{\mathcal{F}}^{(\infty)} \setminus \bigcup \{ h_2 = 0, \infty \} \cup \{ h_2 = 0, \infty \}$,

$\hat{\mathcal{F}}_{\infty}^{(\infty)} := \hat{\mathcal{F}}_{\infty}^{(\infty)} \setminus \bigcup \{ h_j = \infty \} \cup \{ h_j = \infty \}$.

The corresponding open subsets of $\hat{X}^{(\infty)}$ are denoted by

$$\hat{X}_{\infty}^{(\infty)}, \hat{X}_{\infty}^{(\infty)} \subset \hat{X}^{(\infty)}.$$

Recall

$$\{ a^{2n+1}, b^{2n+2}, c^{2n+2} \}_{n \geq 0} \subset R = C[z_3, z_4, \ldots].$$
It follows that the formal Killing fields $-\overline{Y}, \overline{Y}$ are well defined and smooth on $\hat{F}_{\infty}^{(\infty)}$. They can be considered as a map

$$(-\overline{Y}, \overline{Y}): \hat{F}_{\infty}^{(\infty)} \mapsto \overline{g}_{\geq 1}^t \times g_{\geq 1}.$$

4.3. **Infinitely prolonged Gauß map.** With this preparation, we give a definition of the infinitely prolonged version of Gauß map.

Consider the diagram in Fig.3

\[
\begin{array}{ccc}
\hat{F}_{\infty}^{(\infty)} & \xrightarrow{(-\overline{Y}, \overline{Y})} & \left(-Y_{4y;\lambda;2} \times Y_{4y;\lambda;2}\right) \cap \left(-\overline{g}_{\geq 1}^t \times g_{\geq 1}\right) \\
\pi \downarrow & & \\
F & & 
\end{array}
\]

**Figure 3.** Infinitely prolonged Gauß map

**Definition 4.1.** Let $\hat{F}_{\infty}^{(\infty)} \subset \hat{F}^{(\infty)}$ be the Zariski open set defined in (26). The **infinitely prolonged Gauß map** is defined by

$$(-\overline{Y}, \overline{Y}): \hat{F}_{\infty}^{(\infty)} \mapsto \left(-Y_{4y;\lambda;2} \times Y_{4y;\lambda;2}\right) \cap \left(-\overline{g}_{\geq 1}^t \times g_{\geq 1}\right).$$

Each component $-\overline{Y}, \overline{Y}$ satisfies the Killing field equation with respect to $\phi_\lambda$,

$$dY + [\phi_\lambda, Y] = 0,$$

$$d(-\overline{Y}) + [\phi_\lambda, (-\overline{Y})] = 0.$$

A relevant observation is that, from the construction, the horizontal map $(-\overline{Y}, \overline{Y})$ in Fig.3 is an isomorphism when restricted to a fiber of the projection $\pi$, which can be considered as the higher-order jet space of the CMC system. Then Lem. [4.1] implies that $\hat{F}_{\infty}^{(\infty)}$ is homogeneous under the combined action of the (formal) Lie groups

$$(\text{Iso}(M)^{(\infty)}, \overline{G}_{\geq 0}, G_{\geq 0}).$$

Here $\text{Iso}(M)^{(\infty)}$ denotes the infinitely prolonged representation of $\text{Iso}(M)$ in $\text{Diffeo}(\hat{F}_{\infty}^{(\infty)})$.

**Corollary 4.2.** The infinite jet space $\hat{F}^{(\infty)}$ of the CMC system is a quasi-homogeneous variety under the combined action of the Lie groups $(\text{Iso}(M)^{(\infty)}, \overline{G}_{\geq 0}, G_{\geq 0})$.

The homogeneity of the infinite jet space of Drinfeld-Sokolov hierarchies is originally due to Frenkel, [6, 8].
5. Adler-Kostant-Symes bi-Hamiltonian hierarchy

In this section, we digress from the CMC system and give a description of the Adler-Kostant-Symes (AKS) bi-Hamiltonian hierarchy on \( g \) via R-matrix approach. For the general reference on R-matrices and AKS hierarchies, we refer to [21].

5.1. R-matrix. Recall the vector space decomposition (21),

\[
\mathfrak{g} = \mathfrak{g}_{\leq -1} + \mathfrak{g}_{\geq 0}.
\]

Let \( \pi_{\leq -1}, \pi_{\geq 0} \) denote the respective projection maps. Let

\[
R := -\pi_{\leq -1} + \pi_{\geq 0} : \mathfrak{g} \rightarrow \mathfrak{g}
\]

be the corresponding R-matrix. An R-matrix defines a new Lie bracket \([\ , \, ]_R\) on \( \mathfrak{g} \) given by

\[
[Y_1, Y_2]_R := \frac{1}{2} ( [R(Y_1), Y_2] + [Y_1, R(Y_2)] ), \quad Y_1, Y_2 \in \mathfrak{g}.
\]

In the present case, the new Lie bracket \([\ , \, ]_R\) splits into the direct sum and one gets

\[
\mathfrak{g}_R = \mathfrak{g}_{\leq -1} + \mathfrak{g}_{\geq 0} \quad \text{(as a Lie algebra)}.
\]

Here \( \mathfrak{g}_R \) denotes the vector space \( \mathfrak{g} \) equipped with the new Lie bracket \([\ , \, ]_R\). The “\( \oplus \)" sign indicates that the Lie bracket is given by

\[
[ (Y_-, Y_+), (Y'_-, Y'_+) ]_R = (-[Y_-, Y'_-], +[Y_+, Y'_+]),
\]

for \( (Y_-, Y_+), (Y'_-, Y'_+) \in \mathfrak{g}_{\leq -1} + \mathfrak{g}_{\geq 0} \).

5.2. Bi-Poisson structures. Let

\[
\sigma_k : \mathfrak{g} \rightarrow \mathfrak{g}, \quad k \in 2\mathbb{Z},
\]

\[
\sigma_k(Y) := \lambda^k Y, \quad Y \in \mathfrak{g},
\]

be the sequence of intertwining operators\(^5\) Let

\[
R_k = R \circ \sigma_k
\]

be the corresponding sequence of R-matrices.

It is clear that for any non-trivial finite linear combination \( \sigma = \sum_{k=1}^{i_2} c_k \sigma_k \), we have that Ker(\( \sigma \)) is trivial. Hence, the sequence of R-matrices \{R_k\} define a family of Lie algebra structures on \( \mathfrak{g} \). As a consequence, they induce an infinite dimensional linear family of compatible Lie-Poisson structures on \( \mathfrak{g}^* = \mathfrak{g} \).

For our purpose, the relevant bi-Poisson structures are given by the pairs of R-matrices

\[
(R, R_{\pm 2}).
\]

\(^5\)An intertwining operator is an endomorphism of a Lie algebra which commutes with the adjoint action.
5.3. AKS bi-Hamiltonian hierarchy. Recall the dual decomposition (23).

\[ \mathfrak{g}' = \mathfrak{g} = \mathfrak{g}_{\geq 1} + \mathfrak{r} \mathfrak{s} \mathfrak{g}_{\leq 0}. \]

Here \( \mathfrak{g}_{\geq 1} = (\mathfrak{g}_{\leq -1})', \) and the co-adjoint action of \( X_- \in \mathfrak{g}_{\leq -1} \) on \( Y_+ \in \mathfrak{g}_{\geq 1} \) is given by

\[ [X_-, Y_+]_{\geq 1}. \]

The subscript \( \geq 1 \) denotes the \( \mathfrak{g}_{\geq 1} \)-component of the Lie bracket, etc.

With this preparation, consider the following set of coadjoint invariant Hamiltonian functions on \( \mathfrak{g}' \),

\[ \frac{1}{n} \text{Res}_{\lambda=0} (\lambda^{-m} \text{tr}(Y^n)), \quad n, m \in \mathbb{Z}. \]

Since \( \mathfrak{g}' \subset \text{sl}(2, \mathbb{C})(\lambda) \), choose the set of nontrivial and functionally independent ones,

\[ H_m := -\left( \frac{1}{2i} \right) \frac{1}{2} \text{Res}_{\lambda=0} (\lambda^{-2m-2} \text{tr}(Y^2)), \quad m \geq 0 \]

(the scaling constants are ornamental). Its differential is given by

\[ dH_m = -\left( \frac{1}{2i} \right) \lambda^{-2m-2} Y \in \mathfrak{g} = (\mathfrak{g}')'. \]

With respect to the Lie-Poisson structure on \( \mathfrak{g}_R' = \mathfrak{g}_R \), the Hamiltonian equation for \( H_m \) is given by \((21)\) Theorem 2.5]

\[ \frac{dY}{dt_m} = -\text{ad}^*_\mathfrak{g} U_m(Y), \quad m \geq 0, \]

where \( t_m \) is the time variable, and the formula for the element \( U_m = \frac{1}{2} R(dH_m) \in \mathfrak{g} \) is

\[ U_m = \frac{1}{2} \left( \frac{-1}{2i} \right) \left( (-\lambda^{-2m-2} Y)_{\leq -1} + (\lambda^{-2m-2} Y)_{\geq 0} \right). \]

Under the identification \( \mathfrak{g} = \mathfrak{g}' \), we get

\[ \frac{dY}{dt_m} = -[U_m, Y] = -[U_m + \left( \frac{1}{4i} \right) \lambda^{-2m-2} Y, Y]. \]

Hence the Hamiltonian equation becomes

\[ \frac{dY}{dt_m} = -\left( \frac{1}{2i} \lambda^{-2m-2} Y \right)_{\leq -1}, \quad m \geq 0. \]

It is clear that the hierarchy of \( t_m \)-flows defined by Eq.(31) is bi-Hamiltonian with respect to the bi-Poisson structures \((R, R_{\pm 2})\). As a consequence, we obtain a commuting bi-Hamiltonian hierarchy of evolution equations on \( \mathfrak{g} \).

**Definition 5.1.** Let \( \mathfrak{g} \subset \text{sl}(2, \mathbb{C})(\lambda) \) be the twisted loop algebra \((20)\). The AKS hierarchy on \( \mathfrak{g} \) is the sequence of commuting bi-Hamiltonian system of equations \((31)\).

Consequently, the resulting AKS hierarchy will involve the "time" variables \( \{t_m\}_{m \geq 0} \).
Since $Y$ takes values in $\mathfrak{g}_{\geq 1}$, it suffices for the construction of the CMC hierarchy to consider the restriction of the AKS hierarchy to the strictly positive part $\mathfrak{g}_{\geq 1} = (\mathfrak{g}_{\leq -1})'$. 

5.4. Liouville tori.

There exists an obvious family of Liouville tori for the AKS hierarchy.

Let $P_q : \mathfrak{g} \mapsto \mathbb{C}((\lambda^2))$ be a function defined by,

$$P_q(Y) := \det(Y) + q, \quad q \in \mathbb{C}((\lambda^2)).$$

Consider the corresponding determinantal variety defined by

$$Y_q = \{ P_q = 0 \} \subset \mathfrak{g}.$$ 

Note by definition that all the Hamiltonians $H_m$ are constant on this subvariety, and $Y_q$ is clearly invariant under the flow \(Y_t\). As the constant element $q \in \mathbb{C}((\lambda^2))$ varies, the set of subvarieties $\{Y_q\}$ forms an analogue of the Liouville foliation by invariant tori on $\mathfrak{g}$ for the AKS hierarchy.

Since $\det(Y) = -4\gamma\lambda^2$, the relevant subset for our analysis is

$$Y_{4\gamma\lambda^2} \cap \mathfrak{g}_{\geq 1}.$$ 

As noted earlier, $Y_{4\gamma\lambda^2} \cap \mathfrak{g}_{\geq 1}$ is an adjoint orbit of the formal loop group $G_{\geq 0}$. On the other hand, by construction of the AKS hierarchy, the trajectories of the CMC hierarchy lie in the co-adjoint orbits of $G_{\leq -1}$ at the same time. Here $G_{\leq -1}$ denotes the loop group corresponding to the polynomial loop algebra $\mathfrak{g}_{\leq -1}$.

6. CMC hierarchy

In this section, the preceding analyses are combined to yield the structure equations for the CMC hierarchy.

The key observation is that the $t_0, t_0$-flows of the $(\overline{-\text{AKS}})$-hierarchies on $(\overline{-\mathfrak{g}_{\geq 1}}, \mathfrak{g}_{\geq 1})$ are tangent to the infinitely prolonged Gauss map $(\overline{-Y}, Y)$ respectively, Lem.\[6.1]\]. The formal symmetry of the Maurer-Cartan form $\phi_\lambda$,

$$\phi_\lambda = -\overline{\phi_\lambda},$$

then dictates that the proposed CMC hierarchy should be obtained by attaching the pair of AKS hierarchies via $(\overline{-Y}, Y)$ to a combined system of equations on $\overline{\mathfrak{g}_{\geq 1}} \times \mathfrak{g}_{\geq 1}$. The original CMC system, which corresponds to the $t_0, t_0$-flows, serves as the connecting neck for the operation. The $\mathfrak{g}[[\lambda^{-1}, \lambda]]$-valued extended Maurer-Cartan form for the CMC hierarchy is given by the formulas \(35), (40).\)

6.1. Construction plan. Consider the diagram in Fig.\[3\]. Based on this, the construction plan for the CMC hierarchy can be summarized by the following diagram, Fig.\[4\].

Here the appearance of the mKdV hierarchies is explained by fact that the AKS hierarchy on $\mathfrak{g}_{\geq 1}$ generated by the $t_m$-flows for $m \geq 0$ is a matrix representation of the mKdV hierarchy, \[26].
CMC hierarchy = \(-mKdV \text{ hierarchy } \oplus \text{ CMC system } \oplus mKdV \text{ hierarchy}\)

\[ \begin{array}{c}
(-\nabla, Y) \\
\rightarrow \\
-\text{AKS hierarchy } \oplus \text{ AKS hierarchy.}
\end{array} \]

**Figure 4.** CMC hierarchy

6.1.1. *Formal complex conjugation.* A technical remark is in order. Define the operation of formal complex conjugation \(\bar{(\quad)}\) by,

\[
\begin{pmatrix}
\lambda^\pm 1 \\
h_j \\
t_m \\
\xi \\
\rho
\end{pmatrix}
\quad \xrightarrow{\text{conjugation}}
\begin{pmatrix}
\lambda^\mp 1 \\
\bar{h}_j \\
\bar{t}_m \\
\bar{\xi} \\
\bar{\rho}
\end{pmatrix}
\]

(34)

For example, the notation \(-\bar{Y}t\) appeared above means the negative transpose of the formal complex conjugate of \(Y\).

6.2. **Connecting neck.**

6.2.1. *Decomposition of \(Y\).* For \(m \geq 0\), set

\[
\frac{1}{2i} \lambda^{-2m-2} Y =: U_m + U_{(m+1)}
\]

\[
\subset g_{\leq -1} +^{rs} g_{\geq 0}
\]

be the decomposition of the scaled canonical formal Killing field into the respective parts. The \(g_{\leq -1}\)-part \(U_m\) is given explicitly by

\[
U_m = \begin{pmatrix}
-i U^a_m & 2 U^c_m \\
2 U^b_m & i U^b_m
\end{pmatrix},
\]

where

\[
U^a_m = \frac{1}{2i} \sum_{j=0}^{m} \lambda^{(2j+1)-(2m+2)} a^{2j+1},
\]

\[
U^c_m = \frac{1}{2i} \sum_{j=0}^{m} \lambda^{(2j+1)-(2m+2)} c^{2j+2},
\]

\[
U^b_m = \frac{1}{2i} \sum_{j=0}^{m} \lambda^{(2j+1)-(2m+2)} b^{2j+2}.
\]

6.2.2. **Key lemma.** Recall from (2),

\[
\phi_\lambda = \lambda \phi_+ + \phi_0 + \lambda^{-1} \phi_-.
\]

**Lemma 6.1.** Suppose,

\[
d\bar{t}_0 = -\frac{1}{2} \bar{h}_2^\perp \xi, \quad dt_0 = -\frac{1}{2} h_2^\perp \xi.
\]

(37)
Under this relation,

\[ \lambda \phi = -U_t dt_0, \quad \lambda^{-1} \phi = U_0 dt_0. \]  

(38)

Note that the consistency of Eqs. (37) imposes the following constraints on the proposed CMC hierarchy,

\[ d(\bar{h}^2 z) = 0, \quad d(h^2 z) = 0. \]  

(39)

**Corollary 6.2.** The infinitely prolonged Gauß map \((-\bar{Y}, Y)\) is tangent to the \(t_0, t_0\)-flows of the \((-\text{AKS}, \text{AKS})\) hierarchies on \((-\bar{g}_{\geq 1}, g_{\geq 1})\) respectively.

6.3. **Extended Maurer-Cartan form.** Motivated by this and the relation (33), set the \(\mathfrak{g}[[\lambda^{-1}, \lambda]]\)-valued extended Maurer Cartan form \(\phi\) by

\[ \phi := -\sum_{m=1}^{\infty} U_m dt_m + \phi_1 + \sum_{m=1}^{\infty} U_m dt_m, \]

\[ = -\sum_{m=0}^{\infty} U_m dt_m + \phi_0 + \sum_{m=0}^{\infty} U_m dt_m. \]  

(40)

Note that \(\phi\) satisfies the twistedness condition given in (20). Note also the formal identity,

\[ \phi = -\phi^t. \]  

(41)

6.3.1. **Extended structure equation.** The resulting structure equations for the CMC hierarchy are\(^6\):

\[ dY + [\phi, Y] = 0, \quad d(-\bar{Y}) + [\phi, (-\bar{Y})] = 0, \]

\[ d\phi + \phi \wedge \phi = 0. \]  

(42)

We claim that this system of equations is compatible.

6.4. **Assembly.** In order to check the consistency of the resulting set of structure equations, we wish to extract a subset of generating equations for (42). In particular, we are interested in the extension (deformation) of the structure equations for the objects

\{\xi, \xi, \rho, h_2, \bar{h}_2\}.

In a sense, these structure equations are the connection for the assembly. The rest of the equations shall be accounted for by the extended Killing field equations for \(-\bar{Y}, Y\).

In view of the analysis above, and by imposing the condition that the deformations induced by the CMC hierarchy are conformal and preserve Hopf differential\(^7\), we propose

---

\(^6\)The set of equations (42) is sometimes referred to as the central system, [5].

\(^7\)6.5.1.
the following ansatz for the extension of the structure equations for \{\xi, \xi, \rho, h_2, \bar{h}_2\}:

\[
\begin{align*}
\text{Eq.}(\xi) & \quad \begin{cases} 
   d\xi - i\rho \wedge \xi = \sum_{m=1}^{\infty} a^{2m+3} dt_m \wedge \xi, \\
   d\bar{\xi} + i\rho \wedge \bar{\xi} = \sum_{m=1}^{\infty} a^{2m+3} d\bar{t}_m \wedge \bar{\xi},
\end{cases} \\
   d\rho & \equiv R^\chi \xi \wedge \xi \mod dt, d\bar{t}, \\
   dh_2 + 2ih_2 \rho & = h_3 \xi - 2 \sum_{m=1}^{\infty} h_2 a^{2m+3} dt_m, \\
   d\bar{h}_2 - 2i\bar{h}_2 \rho & = \bar{h}_3 \bar{\xi} - 2 \sum_{m=1}^{\infty} \bar{h}_2 \bar{a}^{2m+3} d\bar{t}_m.
\end{align*}
\]

Let us rewrite the extended Killing field equations,

\[
\text{Eq.}(Y) \quad dY + [\phi, Y] = 0, \quad d(-\bar{Y}) + [\phi, (-\bar{Y})] = 0.
\]

\[
\text{Eq.}(\phi) \quad d\phi + \phi \wedge \phi = 0.
\]

The claim is that,

a) Eq.\((\xi)\) and Eq.\((Y)\) imply Eq.\((\phi)\),

b) Eq.\((\xi)\) and Eq.\((Y)\) are compatible, i.e., the identity \(d^2 = 0\) is a formal consequence of these equations.

The proof is postponed to §8.

6.5. Remarks.

6.5.1. Conformal, and preserving Hopf differential. Note Eq.\((\xi)\) implies \(d(h_2^1 \xi) = 0, d(\bar{h}_2^1 \bar{\xi}) = 0\). Hence, the deformations induced by the CMC hierarchy are conformal and preserving Hopf differential.

6.5.2. Well definedness of \([\phi, Y], \phi \wedge \phi\). Although the extended Maurer-Cartan form \(\phi\) takes values in \(g[[\lambda^{-1}, \lambda]]\), note that each coefficient of the 1-forms \(\rho, dt_m, d\bar{t}_m, m \geq 0\), in \(\phi\) is \(g[\lambda^{-1}, \lambda]\)-valued. Since the multiplication map

\[
C[\lambda^{-1}, \lambda] \times C((\lambda)) \longrightarrow C((\lambda))
\]

is well defined, the structure equations Eq.\((\xi)\) Eq.\((Y)\) make sense.

6.5.3. Commuting symmetries. Note that the extended structure equations Eq.\((\xi)\) Eq.\((Y)\) reduce to the original CMC system if we set

\[
d\bar{t}, dt \equiv 0.
\]

Hence, the compatibility implies that the CMC hierarchy induces a pair of commuting hierarchies of formal symmetry vector fields \(\{\partial_{r_m} \}_{m=0}^{\infty}, \{\partial_{t_m} \}_{m=0}^{\infty}\) (formally dual to \(\{d\bar{t}_m \}_{m=0}^{\infty}, \{dt_m \}_{m=0}^{\infty}\)) on the moduli space \(\hat{\mathcal{M}}_F\), (13).

---

8Here “\(d\bar{t}, dt = 0\)” means “modulo \(d\bar{t}_m, dt_m, \forall m \geq 1\)."
6.5.4. Differential system. Consider the product space
\[ \hat{F}^{(\infty)}_+ := \hat{F}^{(\infty)} \times \{ t_n \}_{n \geq 0} \times \{ t_m \}_{m \geq 0}. \]

Let \( \tilde{I}^{(\infty)}_+ \) be the differential ideal on \( \hat{F}^{(\infty)}_+ \) which cut out the equations Eq.(ξ), Eq.(Y).

Strictly speaking, the equality signs in Eq.(ξ), Eq.(Y) should be replaced with “≡ mod \( \tilde{I}^{(\infty)}_+ \)”.

For simplicity, we omit this. The meaning will be clear from the context.

6.5.5. Affine Toda equation. The preceding analysis shows that the CMC system (or sinh-Gordon equation) arises as the compatibility equation to join a pair of AKS hierarchies.

From the Frenkel’s work [8], it is evident that such a characterization exists for the general affine Toda field equations.

Before we proceed to the proof of compatibility, we translate the structure equations for the CMC hierarchy into the original \( so(4, \mathbb{C}) \)-setting.

7. Translation into \( so(4, \mathbb{C}) \)-setting

Recall the \( so(4, \mathbb{C})[\lambda^{-1}, \lambda] \)-valued Maurer-Cartan form \( \psi_\lambda \) (7), and the corresponding \( so(4, \mathbb{C})[[\lambda]] \lambda \)-valued formal Killing field \( X_\lambda \) (8). In order to define the extension of \( \psi_\lambda \), we introduce the deformation coefficients \( V_m \) analogous to \( U_m \) for \( \phi \).

For each \( m \geq 1 \), define the \( so(4, \mathbb{C}) \)-valued function \( V_m \) depending on the parity of \( m \) as follows. Here we set \( a^{-1} = 0 \).

[case \( m \) is even]

Define
\[ \epsilon(m) = \begin{cases} +1 & \text{if } m \equiv 0 \\ -1 & \text{if } m \equiv 2 \end{cases} \pmod{4}. \]

Let
\begin{align*}
V_{m}^{a_1} &= \epsilon(m) \sum_{j=0}^{m} (-1)^{j} \lambda^{(4j-2)-(2m+2)} d^{4j-1}, \\
V_{m}^{a_3} &= \epsilon(m) \sum_{j=0}^{m} (-1)^{j} \lambda^{(4j+0)-(2m+2)} d^{4j+1}, \\
V_{m}^{b_2} &= \epsilon(m) \sum_{j=0}^{m} (-1)^{j} \lambda^{(4j-1)-(2m+2)} b^{4j+0}, \\
V_{m}^{b_4} &= \epsilon(m) \sum_{j=0}^{m} (-1)^{j} \lambda^{(4j+1)-(2m+2)} b^{4j+2}, \\
V_{m}^{c_2} &= \epsilon(m) \sum_{j=0}^{m} (-1)^{j} \lambda^{(4j-1)-(2m+2)} c^{4j+0}, \\
V_{m}^{c_4} &= \epsilon(m) \sum_{j=0}^{m} (-1)^{j} \lambda^{(4j+1)-(2m+2)} c^{4j+2}.
\end{align*}

[case \( m \) is odd]

Define
\[ \epsilon(m) = \begin{cases} -1 & \text{if } m \equiv 1 \\ +1 & \text{if } m \equiv 3 \end{cases} \pmod{4}. \]
Let

\[ V_{m}^{a_{1}} = e(m) \sum_{j=1}^{m+1} (-1)^{j} \lambda^{(4j-4)-(2m+2)} a^{4j-3}, \quad V_{m}^{a_{3}} = e(m) \sum_{j=1}^{m+1} (-1)^{j} \lambda^{(4j-2)-(2m+2)} a^{4j-1}, \]

\[ V_{m}^{b_{2}} = e(m) \sum_{j=1}^{m+1} (-1)^{j} \lambda^{(4j-3)-(2m+2)} b^{4j-2}, \quad V_{m}^{b_{4}} = e(m) \sum_{j=1}^{m+1} (-1)^{j} \lambda^{(4j-1)-(2m+2)} b^{4j+0}, \]

\[ V_{m}^{c_{2}} = e(m) \sum_{j=1}^{m+1} (-1)^{j} \lambda^{(4j-3)-(2m+2)} c^{4j-2}, \quad V_{m}^{c_{4}} = e(m) \sum_{j=1}^{m+1} (-1)^{j} \lambda^{(4j-1)-(2m+2)} c^{4j+0}. \]

Now set

\[ V_{m} = \begin{pmatrix}
    i(V_{m}^{c_{2}} + V_{m}^{b_{1}}) & -(V_{m}^{c_{2}} - V_{m}^{b_{1}}) & -V_{m}^{a_{3}} \\
    -i(V_{m}^{c_{2}} + V_{m}^{b_{1}}) & iV_{m}^{a_{1}} & -i(V_{m}^{b_{2}} + V_{m}^{c_{1}}) \\
    V_{m}^{a_{3}} & i(V_{m}^{b_{2}} + V_{m}^{c_{1}}) & (V_{m}^{b_{2}} - V_{m}^{c_{1}})
\end{pmatrix}. \]

Define the so(4, C)[[\lambda^{-1}, \lambda]]-valued extended Maurer-Cartan form \( \psi \) by

\[ \psi := - \sum_{m=1}^{\infty} V_{m} dt^{m} + \psi_{\lambda} + \sum_{m=1}^{\infty} V_{m} dt_{m}. \]

Note the formal identity

\[ \psi = - \bar{\psi}. \]

The corresponding extended structure equations are:

\[ dX + [\psi, X] = 0, \quad d(-X) + [\psi, (-X)] = 0, \]

\[ d\psi + \psi \wedge \psi = 0. \]

It can be checked that these equations are equivalent to (42).

**8. Proof of Compatibility**

Let us first rewrite the compatibility equation [Eq.(\phi)] in such a way that is suitable for the computation in this section.

Consider the decomposition

\[ \phi = - \sum_{n=0}^{\infty} U_{n} dt_{n} + \phi_{0} + \sum_{m=0}^{\infty} U_{m} dt_{m} \]

\[ := \phi_{+} + \phi_{0} + \phi_{-}. \]

The \( \phi_{+} \)-terms have \( \lambda \)-degree \( \geq 1 \), and the \( \phi_{-} \)-terms have \( \lambda \)-degree \( \leq -1 \). The \( \phi_{0} \)-term, (2), has \( \lambda \)-degree 0.
In terms of this decomposition, Eq.\((\xi)\), Eq.\((Y)\), Eq.\((\phi)\) can be organized as follows.

\[
\begin{align*}
\frac{d\xi}{dt} - i\rho \wedge \xi &= \sum_{m=1}^{\infty} a_{2m+3}^m d\bar{t}_m \wedge \xi, \\
\frac{d\xi}{dt} + i\rho \wedge \xi &= \sum_{m=1}^{\infty} a_{2m+3}^m d\bar{t}_m \wedge \xi, \\
\frac{d\rho}{dt} &= R\frac{1}{2}\xi \wedge \xi \quad \text{mod} \ dt, d\bar{t}, \\
dh_2 + 2ih_2 \rho &= h_3 \xi - 2 \sum_{m=1}^{\infty} a_{2m+3}^m dt_m, \\
dh_2 - 2i\bar{h}_2 \rho &= \bar{h}_3 \xi - 2 \sum_{m=1}^{\infty} \bar{a}_{2m+3}^m d\bar{t}_m.
\end{align*}
\]

Here the equation \(d\phi + \phi \wedge \phi = 0\) is decomposed into the three parts \((B_\oplus),(B_0),(B_\ominus)\) according to their \(\lambda\)-degrees. The subscripts “\(\oplus, 0, \ominus\)” denote the terms of \(\lambda\)-degree \(\geq 1, = 0, \leq -1\) respectively.

We now state the main theorem of this paper.

**Theorem 8.1.** The system of equations Eq.\((A)\), Eq.\((C)\) for the CMC hierarchy is compatible, i.e., \(d^2 = 0\) is a formal consequence of the structure equations.

Note that the compatibility equation of Eq.\((C)\) is Eqs.\((B_\oplus),(B_0),(B_\ominus)\). For a proof of the theorem, we first show that Eqs.\((B_\oplus),(B_0),(B_\ominus)\) vanish modulo Eq.\((A)\), Eq.\((C)\). Then, we check that Eq.\((A)\) is compatible with Eqs.\((B_\oplus),(B_0),(B_\ominus)\), and Eq.\((C)\).

8.1. Eqs.\((B_\oplus),(B_\ominus)\). The claim is that Eqs.\((B_\oplus),(B_\ominus)\) \(\equiv 0\) mod Eq.\((A)\), Eq.\((C)\).

This follows from the commuting property of the AKS bi-Hamiltonian hierarchy. We record a proof for completeness.

8.1.1. \(dt_m \wedge dt_\ell, d\bar{t}_m \wedge d\bar{t}_\ell\)-terms. It is clear that this part of the claim is equivalent to the following lemma and its formal complex conjugate.

**Lemma 8.2.** For all \(m, \ell \geq 0\),

\[
\partial_{\ell m} U_\ell - \partial_{m \ell} U_m + [U_m, U_\ell] = 0.
\]

**Proof.** We give a proof by \(\lambda\)-degree counting.

**Step 1.** Recall the decomposition

\[
Y = 2i\lambda^{2m+2}(U_m + U_{(m+1)}).
\]

From Eq.\((C)\), we have

\[
\partial_{\ell} Y = -[U_\ell, Y].
\]
Here $\partial_{t_1} = \frac{\partial}{\partial t_1}$ denotes the partial derivative operator.

Step 2. Substitute (50) to (51), and one gets

$$(51) \quad \partial_{t_1} U_m + \partial_{t_1} U_{(m+1)} = -[U_{t_1}, U_m + U_{(m+1)}]$$

Interchange $t, m$ and take the difference, and one gets

$$(52) \quad (\partial_{t_1} U_t - \partial_{t_1} U_m) + (\partial_{t_1} U_{(t+1)} - \partial_{t_1} U_{(m+1)})$$

$$+ (2[U_m, U_t]) + ([U_m, U_{(t+1)}] - [U_t, U_{(m+1)}]) = 0.$$  

**Lemma 8.3.** For all $m, t \geq 0$,

$$(53) \quad [U_m, U_t] + ([U_m, U_{(t+1)}] - [U_t, U_{(m+1)}]) + [U_{(m+1)}, U_{(t+1)}] = 0.$$  

**Proof.** This follows from the trivial identity,

$$[Y, Y] = 0 = [U_m + U_{(m+1)}, U_t + U_{(t+1)}].$$

□

Step 3. Substitute (53) to (52), and one gets

$$(54) \quad \left(\partial_{t_1} U_t - \partial_{t_1} U_m\right) + [U_m, U_t]$$

$$+ \left(\partial_{t_1} U_{(t+1)} - \partial_{t_1} U_{(m+1)}\right) - [U_{(m+1)}, U_{(t+1)}] = 0.$$  

In this equation, the $\lambda$-degree of the first line is $\leq -1$, whereas the $\lambda$-degree of the second line is $\geq 0$. It follows that the equation above holds separately, i.e.,

$$(55) \quad (\partial_{t_1} U_t - \partial_{t_1} U_m) + [U_m, U_t] = 0,$$

$$(\partial_{t_1} U_{(t+1)} - \partial_{t_1} U_{(m+1)}) - [U_{(m+1)}, U_{(t+1)}] = 0.$$  

This completes the proof of Lem. 8.2. □

8.1.2. $dt_{m,n} \wedge dU_{m,n}$-terms. Similarly as above, the claim is equivalent to the following lemma.

**Lemma 8.4.** For all $m, n \geq 0$,

$$\partial_{t_{m,n}} U_{m,n} + \partial_{t_{m,n}} U_{m} + [U_{m}, U_{m,n}]_{\ominus} + [U_{m}, U_{m,n}]_{\ominus} = 0.$$  

**Proof.** By Eq.(C), we have

$$\partial_{t_{m,n}} (U_m + U_{m+1}) = [U_{m}, U_{m} + U_{m+1}].$$  

Since the terms in $U_{m+1}$, $U_{m,n}$ are of $\lambda$-degree $\geq 0$, take the $\ominus$-terms (of $\lambda$-degree $\leq -1$) only and one gets

$$(56) \quad \partial_{t_{m,n}} U_{m} = [U_{m,n}, U_{m}]_{\ominus}.$$  

Take the conjugate transpose of this equation and interchange $m, n$, and one gets

$$\partial_{t_{m,n}} U_{m,n} = [U_{m,n}, U_{m}]_{\ominus}.$$ □
8.2. Eq. (B0). This gives the formula for $d\rho$ in Eq. (A).

In order to show the compatibility, one needs to verify that $d^2\rho = 0$ is an identity. This is equivalent to,

$$d(B_0) \equiv 0 \mod \text{Eqs.}(B_0),(B_0),(B_0), \text{Eq.}(C).$$

8.3. $d^2\rho$, or $d(B_0)$. From Eq. (B0), we have

$$d\phi_0 + (\phi_+ \land \phi_- + \phi_- \land \phi_+) = 0.\quad (57)$$

Differentiate this equation using the given formulas for $d\phi_\pm$. After collecting terms, one gets

$$\left\{ [\phi_- \land \phi_+ + [\phi_+, \phi_- \land \phi_-] + [\phi_-, [\phi_+, \phi_-]_\Theta] + [\phi_+, [\phi_+, \phi_-]]_\Theta \right\}_0 = 0.$$

Considering the $\lambda$-degrees, this is equivalent to

$$\left\{ [\phi_- \land \phi_+ + [\phi_+, \phi_- \land \phi_-] + [\phi_-, [\phi_+, \phi_-]_\Theta] + [\phi_+, [\phi_+, \phi_-]] \right\}_0 = 0.$$

The expression inside the parenthesis vanishes by cancellation.

For the remainder of proof, we first derive Eq. (A) from Eqs. (B_0),(B_0),(B_0), Eq. (C). Then, we show that Eq. (A) is compatible.

8.4. Eq. (A). The analysis thus far shows the compatibility of the ($-\text{AKS}$, AKS)-hierarchy on $-g_{\geq 1} \times g_{\geq 1}$, under the constraints that

$$d\bar{t}_0 = -\frac{1}{2} h_{2}^\frac{1}{2} \xi, \quad dt_0 = -\frac{1}{2} h_{2}^\frac{1}{2} \xi.$$

From (5), the formula for $dh_2$ is included in Eq. (C), and hence it is compatible. Note that Eq. (A) implies,

$$d(h_{2}^\frac{1}{2} \xi) = 0, \quad dh_2^\frac{1}{2} \xi = 0.$$

The formulas for $d\xi, d\bar{\xi}$ will follow from these equations.

8.4.1. Formula for $dh_2$. We first derive the formula for $dh_2$.

Recall

$$U_0 = \left( \begin{array}{ccc} h_{2}^\frac{1}{2} & \cdots \\ \cdots & \cdots \\ -\gamma h_{2}^{-\frac{1}{2}} & \cdots \end{array} \right) \lambda^{-1}.$$

Apply the formula (56) for the case $m = 0$,

$$\partial_{\tilde{t}_n} U_0 = [\tilde{U}_{n}, U_0]_\Theta.$$ 

Since the terms in $\tilde{U}_{n}$ have $\lambda$-degree $\geq 1$, whereas $U_0$ has $\lambda$-degree $-1$, we have $[\tilde{U}_{n}, U_0]_\Theta = 0$. Hence,

$$\partial_{\tilde{t}_n} U_0 = 0, \quad \forall n \geq 0.$$
On the other hand, collecting the terms of \( \lambda \)-degree \(-1\) from (51) for the case \( m = 0 \), one gets
\[
\partial_t U_0 = -[U_\ell, U_{(1)}]_{\ominus 1}.
\]
Here the subscript “\( \ominus 1 \)” means the terms of \( \lambda \)-degree \(-1\). Consider the identity
\[
[U_\ell + U_{(\ell+1)}, U_0 + U_{(1)}] = 0,
\]
(this is, up to constant scale, the trivial equation \([Y, Y] = 0\)). Collecting the terms of \( \lambda \)-degree \(-1\), one gets
\[
-[U_\ell, U_{(1)}]_{\ominus 1} = [U_{(\ell+1)}, U_0]_{\ominus 1} = [(U_{(\ell+1)})_0, U_0].
\]
Here "\( (U_{(\ell+1)})_0 \)" denotes the terms of \( \lambda \)-degree 0 in \( U_{(\ell+1)} \). This gives the desired formula for \( dh_2 \).

8.4.2. Formula for \( d\xi \). Given the formula for \( dh_2 \), the formula for \( d\xi \) is determined from the equations,
\[
\xi = -2h_2^{-1} dt_0,
\]
\[
d\xi = h_2^{-\frac{1}{2} - 1} dh_2 \wedge dt_0 = -\frac{1}{2} h_2^{-1} dh_2 \wedge \xi.
\]
The compatibility equation \( d^2 \xi = 0 \) follows from this and the compatibility of \( dh_2 \). This completes the proof for the compatibility of Eq.(A).

9. Extension of conservation laws
Recall the sequence of higher-order conservation laws \( \varphi^n \), (12). We show that they admit an extension to the conservation laws of the CMC hierarchy.

Let us introduce a relevant notation. Let
\[
D := L_{\lambda, \frac{\partial}{\partial \lambda}}
\]
be the Euler operator with respect to the spectral parameter \( \lambda \). For a scalar function, or a differential form \( A \), the notation \( \dot{A} \) (upper-dot) would mean the application of the Euler operator,
\[
\dot{A} = D(A).
\]
Set
\[
(58) \quad \varphi_Y := \text{tr}(Y\dot{\phi}).
\]

**Theorem 9.1.** Consider the \( \mathbb{C}[[\lambda^{-2}, \lambda^2]] \)-valued 1-form \( \varphi_Y \), (58).

a) The 1-form \( \varphi_Y \) is closed,
\[
d\varphi_Y = 0.
\]
When expanded as a formal series in \( \lambda^{-2}, \lambda^2 \), each coefficient represents a conservation law of the CMC hierarchy.
b) $\varphi_Y$ represents an extension of the sequence of conservation laws $\varphi^n$ in the following sense;

$$\varphi_Y + \text{id}a \equiv -2\gamma \sum_{n=0}^{\infty} \lambda^{2n} \varphi^n \mod \text{d}\bar{t}, \text{dt}. \tag{59}$$

Proof. a) Differentiate $\varphi_Y$, and one gets

$$d\varphi_Y = \text{tr}(dY \wedge \dot{\phi} + Yd\dot{\phi})$$
$$= \text{tr}((-\phi Y + Y\phi) \wedge \dot{\phi} - Y(\phi \wedge \dot{\phi} + \dot{\phi} \wedge \phi))$$
$$= 0.$$

b) Modulo $\text{d}\bar{t}, \text{dt},$

$$\varphi_Y = \text{tr}(Y\dot{\phi}) \equiv \text{tr}[(-ia \ 2c)(\lambda \frac{1}{2}h_2 \xi - \lambda^{-1} \frac{1}{2}h_2 \xi)] \mod \text{d}\bar{t}, \text{dt},$$
$$= c(-\lambda^{-1} \gamma \xi + \lambda h_2 \xi) + b(-\lambda \gamma \xi + \lambda^{-1} h_2 \xi)$$
$$= -\gamma(\lambda^{-1} c \xi + \lambda b \xi) + (\lambda^{-1} h_2 b \xi + \lambda h_2 c \xi).$$

On the other hand, we have

$$da \equiv \lambda^{-1}(i\gamma c + i\dot{h}_2 b)\xi + \lambda(i\dot{c} \xi + i\ddot{h}_2 c \xi) \mod \text{d}\bar{t}, \text{dt},$$

$$\sum_{n=0}^{\infty} \lambda^{2n} \varphi^n = \sum_{n=0}^{\infty} \lambda^{2n}(c^{2n+2} \xi + b^{2n} \xi), \quad \text{(here we set } b^0 = c^0 = 0) \tag{62}$$
$$= \lambda^{-1} c \xi + \lambda b \xi.$$

Eq. (59) follows from (60), (61), (62). \hfill \Box

10. Linear finite type surfaces

The class of linear finite type (ordinary) CMC surfaces are characterized by the property that a higher-order Jacobi field vanishes, [19],

$$a^{2N_0+3} = 0, \quad N_0 \geq 0. \tag{63}$$

This implies that, up to scaling by an element in $\mathbb{C}[[\lambda^2]]$, the formal Killing field $Y$ factors into a polynomial Killing field.

In this section, we give a geometric interpretation of this characterization in terms of the invariance property of $Y$ under the higher-order symmetry.

Let $\{\partial_{tm}\}_{m=0}^{\infty}$ be the frame formally dual to $\{\text{d}t_m\}_{m=0}^{\infty}$. The CMC hierarchy defines a representation of $\{\partial_{tm}\}_{m=0}^{\infty}$ as a sequence of commuting symmetry vector fields on $\mathcal{M}_F$, §6.5.3.

For a finite set of constants $c_i, 0 \leq i \leq N$, let

$$\mathcal{V} = \sum_{i=0}^{N} c_i \partial_{t_i}. $$
The canonical formal Killing field $Y$ of a CMC surface is stationary with respect to $\mathcal{V}$ whenever

\begin{equation}
\mathcal{V}(Y) = 0.
\end{equation}

From the initial data (5) for the coefficients $b^2, c^2$ of $Y$, and the structure equation for $h_2$ in Eq.(A), Eq.(64) implies $\mathcal{V}(h_2) = 0$ and hence

$$
\sum_{i=0}^{N} c_i a^{2i+3} = 0.
$$

It is known that this is equivalent to the linear finite type condition, [19].

Conversely, since the deformations induced by the CMC hierarchy are conformal and preserve Hopf differential, it is easily checked that, for a vector field $\mathcal{V}$ as above,

$$
\mathcal{V}(h_2) = 0 \quad \rightarrow \quad \mathcal{V}(Y) = 0.
$$

It follows that the formal Killing field $Y$ of a linear finite type CMC surface defined by the equation(63) is invariant under the higher-order symmetry $\mathcal{V} = \partial_{N_0}$.

**Corollary 10.1.** The linear finite type (ordinary) CMC surfaces are characterized by the property that the canonical formal Killing field $Y$ is stationary with respect to a higher-order symmetry.

This shows that, in a sense, the linear finite type CMC surfaces generalize such surfaces as Delaunay surfaces and the twizzlers, which are invariant under a one parameter group of motions of the ambient space form, [18].

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