Superintegrable systems with spin and second-order integrals of motion

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Abstract
We investigate a quantum nonrelativistic system describing the interaction of two particles with spin-1/2 and spin 0, respectively. We assume that the Hamiltonian is rotationally invariant and parity conserving and identify all such systems which allow additional integrals of motion that are second-order matrix polynomials in the momenta. These integrals are assumed to be scalars, pseudoscalars, vectors or axial vectors. Among the superintegrable systems obtained, we mention a deformation of the Coulomb potential with the scalar potential \( V_0 = \frac{\alpha}{r} + \frac{3\hbar^2}{8r^2} \) and spin–orbital one \( V_1 = \frac{\hbar^2}{r^2} \).

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1. Introduction
This paper is part of a research program, the purpose of which is to study nonrelativistic integrable and superintegrable systems with spin in a real three-dimensional Euclidean space. A recent paper [1] was devoted to a system of two nonrelativistic particles with spin \( s = \frac{1}{2} \) and \( s = 0 \), respectively. Physically, this can be interpreted e.g. as a nucleon–pion interaction or an electron–\( \alpha \) particle one. An earlier paper [2] was devoted to the same problem in two dimensions.

We recall that a Hamiltonian system with \( n \) degrees of freedom is called integrable if it allows \( n \) independent commuting integrals of motion (including the Hamiltonian) and superintegrable if more then \( n \) independent integrals exist.

In this paper, we will consider the Hamiltonian,

\[
H = -\frac{\hbar^2}{2} \Delta + V_0(r) + V_1(r) (\vec{\sigma}, \vec{L}),
\]

in the real three-dimensional Euclidean space \( E_3 \). Here, \( H \) is a matrix operator acting on a two-component spinor and we decompose it in terms of the \( 2 \times 2 \) identity matrix \( I \) and Pauli
matrices (we drop the matrix $I$ whenever this does not cause confusion). We assume that the scalar potential $V_0(r)$ and the spin–orbital one $V_1(r)$ depend on the (scalar) distance $r$ only. The same system was already considered in [1] and the search for superintegrable systems was restricted to those that allow integrals that are first-order polynomials in the momenta.

Here we will concentrate on the case when the integrals of motion are allowed to be second-order matrix polynomials in the momenta. Our notations are the same as in [1], i.e.

$$p_k = -i\hbar \partial_{x_k}, \quad L_k = -i\hbar \epsilon_{klm} x_l \partial_{x_m},$$

(1.2)

are the linear and angular momenta, respectively, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(1.3)

are the Pauli matrices.

In the case of a purely scalar potential

$$H = \frac{1}{2} \vec{p}^2 + V_0(\vec{x}),$$

(1.4)

the same assumption, namely that the potential is spherically symmetric, $V_0 = V(r)$ would lead only to two superintegrable systems, namely the Kepler–Coulomb one $V_0 = \frac{\alpha}{r}$ and the harmonic oscillator, $V_0 = \omega r^2$. Indeed, Bertrand’s theorem [3, 4] tells us that the only two spherically symmetric potentials in which all bounded trajectories are closed are precisely these two. On the other hand, if a classical Hamiltonian (1.4) is maximally superintegrable $(2n - 1$ functionally independent integrals of motion in $E_n$ that are well-defined functions on phase space), then all bounded trajectories must be closed [5].

The superintegrability of the hydrogen atom in quantum mechanics is due to the existence of the Laplace–Runge–Lenz integral of motion. This was implicitly used by Pauli [6] and explicitly by Fock [7] and Bargmann [8] to calculate the energy levels and wavefunctions of the hydrogen atom. Similarly, the superintegrability of the isotropic harmonic oscillator is due to the existence of the quadrupole tensor [9] (also known as the Fradkin tensor [10]).

If the potential $V_0(\vec{x})$ in (1.4) is not spherically symmetric, then many new possibilities occur. The first one studied was the anisotropic harmonic oscillator with rational ratio of frequencies [9].

A systematic study of superintegrable systems in $E_n$ with $n = 2, 3$ and $n$ general was started more than 40 years ago [11–15]. Most of the earlier work was on second-order superintegrability, i.e. with integrals of motion that are second-order polynomials in the momenta. The existence of complete sets of commuting second-order integrals of motion is directly related to the separation of variables in the Hamilton–Jacobi or Schrödinger equation, respectively [11–17]. Superintegrability of the system (1.4) is thus related to multiseparability.

Second-order superintegrability has been studied in two- and three-dimensional spaces of constant and nonconstant curvature, respectively, [11–20], [21–28] and also in $n$ dimensions [29–31]. For third-order superintegrability, see [32–37].

Recently, infinite families of classical and quantum systems with integrals of arbitrary order have been discovered, shown to be superintegrable and solved [38–49].

Previous studies of superintegrable systems with spin have been of three types. One describes a particle with spin interacting with an external field, e.g. an electromagnetic one [50–56]. The second describes a spin-1/2 particle interacting with a dyon [57] or with a self-dual monopole [58]. The third one is our programme to study superintegrability in a system of two particles of which at least one has nonzero spin [1, 2].

In this paper, we look for superintegrable systems of the form (1.1). The system is integrable by construction. Since $V_0(r)$ and $V_1(r)$ depend on the distance $r$ alone and $(\vec{\sigma}, \vec{L})$
is a scalar, the Hamiltonian $H$ commutes with the total angular momentum $\vec{J}$. As a matter of fact, trivial integrals (present for arbitrary functions $V_0(r)$ and $V_1(r)$) are

$$H, \quad \vec{J} = \vec{L} + \frac{\hbar}{2}\vec{\sigma}, \quad (\vec{\sigma}, \vec{L}, \vec{E}^2).$$

For some potentials, $V_0(r)$ and $V_1(r)$ integrals of order 0 or 1 in the momenta exist [1]. They can be multiplied by one of the trivial integrals (1.5) (or by some function of them). This will provide further integrals which are nontrivial, but obvious. We will mention them whenever they occur and they may be useful for solving the corresponding superintegrable systems.

In view of the rotational and parity invariance of the Hamiltonian (1.1), we shall search for integrals of motion that have a well-defined behavior under these transformations. Thus, we will separately look for systems that allow integrals that are scalars, pseudoscalars, vectors or axial vectors. Tensor integrals are left for a future study.

To obtain nontrivial results, we impose from the beginning that the spin–orbital interaction be present ($V_1 \neq 0$). We also recall a result from [1], namely for

$$V_1(r) = \frac{\hbar}{r^2}, \quad V_0(r) \quad \text{arbitrary},$$

the Hamiltonian (1.1) allows 2 first-order axial vector integrals of motion. They are $\vec{J}$ and

$$\vec{S} = -\frac{\hbar}{2}\vec{\sigma} + \frac{\hbar}{r^2}(\vec{x} \times \vec{\sigma}).$$

For

$$V_1(r) = \frac{\hbar}{r^2}, \quad V_0(r) = \frac{\hbar^2}{r^2},$$

it allows 2 first-order axial vector integrals and 1 first-order vector integral. They are $\vec{J}, \vec{S}$ and

$$\vec{F} = \vec{p} - \frac{\hbar}{r^2}(\vec{x} \times \vec{\sigma}).$$

These two systems are first-order superintegrable and the term $V_1(r) = \frac{\hbar}{r^2}$ can be induced from a Hamiltonian with $V_1(r) = 0$ by a gauge transformation [1].

2. $O(3)$ multiplets of integrals of motion

Rotations and reflections in $E(3)$ leave the Hamiltonian (1.1) invariant, but can transform the integrals of motion into new invariants. Thus, instead of solving the whole set of determining equations, we simplify the problem by classifying the integrals of motion into irreducible $O(3)$ multiplets.

At our disposal are two vectors $\vec{x}$ and $\vec{p}$ and one pseudovector $\vec{\sigma}$. The integrals we are considering can involve at most second-order powers of $\vec{p}$ and first-order powers of $\vec{\sigma}$, but arbitrary powers of $\vec{x}$.

We shall construct scalars, pseudo-scalars, vectors and axial vectors in the space

$$\{[\vec{x}]^n \times \vec{p} \times \vec{\sigma}\}. \quad (2.1)$$

The quantities $\vec{x}$, $\vec{p}$ and $\vec{\sigma}$ allow us to define six independent ‘directions’ in the direct product of the Euclidean space and the spin one, namely

$$\{\vec{x}, \vec{p}, \vec{L} = \vec{x} \wedge \vec{p}, \vec{\sigma}, \vec{\sigma} \wedge \vec{x}, \vec{\sigma} \wedge \vec{p}\},$$

and any $O(3)$ tensor can be expressed in terms of these. The positive integer $n$ in (2.1) is arbitrary and any scalar in $\vec{x}$ space will be written as $f(r)$, where $f$ is an arbitrary function of $r = \sqrt{x^2 + y^2 + z^2}$. Since $\vec{p}$ figures at most quadratically and $\vec{\sigma}$ at most linearly, we can form
exactly seven linearly independent scalars and seven pseudoscalars out of the quantities (2.2) as follows.

Scalars:
\[
S_1 = 1, \quad S_2 = \vec{p}^2, \quad S_3 = (\vec{x}, \vec{p}), \quad S_4 = (\vec{\sigma}, \vec{L}),
\]
\[
S_5 = (\vec{x}, \vec{p}) (\vec{\sigma}, \vec{L}), \quad S_6 = \vec{L}^2, \quad S_7 = (\vec{x}, \vec{p})^2.
\]
(2.3)

Pseudoscalars:
\[
P_1 = (\vec{\sigma}, \vec{p}), \quad P_2 = (\vec{\sigma}, \vec{x}), \quad P_3 = \vec{p}^2 (\vec{x}, \vec{\sigma}), \quad P_4 = (\vec{x}, \vec{p}) (\vec{\sigma}, \vec{\sigma}),
\]
\[
P_5 = (\vec{x}, \vec{p})^2 (\vec{x}, \vec{\sigma}), \quad P_6 = (\vec{x}, \vec{p}) (\vec{\sigma}, \vec{\sigma}), \quad P_7 = (\vec{x}, \vec{\sigma}) \vec{L}^2.
\]
(2.4)

The independent vectors and axial vectors are as follows.

Vectors:
\[
\tilde{V}_1 = \vec{x}, \quad \tilde{V}_2 = \vec{p}, \quad \tilde{V}_3 = \vec{x} \wedge \vec{\sigma}, \quad \tilde{V}_4 = \vec{p} \wedge \vec{\sigma}, \quad \tilde{V}_5 = \vec{p}^2 \vec{x},
\]
\[
\tilde{V}_6 = \vec{p}^2 (\vec{x} \wedge \vec{\sigma}), \quad \tilde{V}_7 = (\vec{x}, \vec{p}) \vec{x}, \quad \tilde{V}_8 = (\vec{x}, \vec{p}) \vec{p}, \quad \tilde{V}_9 = (\vec{x}, \vec{p}) (\vec{\sigma} \wedge \vec{\sigma}),
\]
\[
\tilde{V}_{10} = (\vec{x}, \vec{p}) (\vec{p} \wedge \vec{\sigma}), \quad \tilde{V}_{11} = (\vec{\sigma}, \vec{L}) \vec{x}, \quad \tilde{V}_{12} = (\vec{\sigma}, \vec{L}) \vec{p}, \quad \tilde{V}_{13} = (\vec{x}, \vec{p})^2 \vec{x},
\]
\[
\tilde{V}_{14} = (\vec{x}, \vec{p})^2 (\vec{x} \wedge \vec{\sigma}), \quad \tilde{V}_{15} = (\vec{\sigma}, \vec{x}) (\vec{\sigma}, \vec{L}) \vec{x}, \quad \tilde{V}_{16} = \vec{L}^2 \vec{x}, \quad \tilde{V}_{17} = \vec{L}^2 (\vec{x} \wedge \vec{\sigma}),
\]
\[
\tilde{V}_{18} = (\vec{x} \wedge \vec{p}) \vec{x}, \quad \tilde{V}_{19} = (\vec{x}, \vec{p}) (\vec{\sigma}, \vec{\sigma}) \vec{L}.
\]
(2.5)

Axial vectors:
\[
\tilde{A}_1 = \vec{\sigma}, \quad \tilde{A}_2 = \vec{L}, \quad \tilde{A}_3 = \vec{p} \wedge \vec{\sigma}, \quad \tilde{A}_4 = (\vec{x}, \vec{p}) \vec{\sigma}, \quad \tilde{A}_5 = (\vec{x}, \vec{p}) \vec{L}, \quad \tilde{A}_6 = (\vec{\sigma}, \vec{L}) \vec{L},
\]
\[
\tilde{A}_7 = (\vec{x}, \vec{p})^2 \vec{\sigma}, \quad \tilde{A}_8 = (\vec{\sigma}, \vec{p}) \vec{x}, \quad \tilde{A}_9 = (\vec{\sigma}, \vec{p}) \vec{p}, \quad \tilde{A}_{10} = (\vec{\sigma}, \vec{x}) \vec{x}, \quad \tilde{A}_{11} = (\vec{\sigma}, \vec{p}) \vec{p},
\]
\[
\tilde{A}_{12} = \vec{p}^2 (\vec{x}, \vec{\sigma}) \vec{x}, \quad \tilde{A}_{13} = (\vec{x}, \vec{p}) (\vec{\sigma}, \vec{x}) \vec{x}, \quad \tilde{A}_{14} = (\vec{x}, \vec{p})^2 (\vec{x}, \vec{\sigma}) \vec{x}, \quad \tilde{A}_{15} = (\vec{x}, \vec{p}) (\vec{p}, \vec{\sigma}) \vec{x},
\]
\[
\tilde{A}_{16} = \vec{L}^2 \vec{\sigma}, \quad \tilde{A}_{17} = (\vec{x}, \vec{p}) (\vec{\sigma}, \vec{x}) \vec{p}, \quad \tilde{A}_{18} = (\vec{x}, \vec{\sigma}) \vec{L}^2 \vec{x}.
\]
(2.6)

An arbitrary function \(f(r)\) is also a scalar and each of the quantities in (2.3)–(2.6) can be multiplied by \(f(r)\) without changing its properties under rotations or reflections.

Even though all the above expressions are linearly independent, higher order polynomial relations between them exist. More importantly, we shall use linear relations with coefficients depending on the distance \(r\) that exist, namely
\[
S_1 = \vec{r} \vec{r} S_2 - S_6, \quad P_1 = \vec{r} \vec{r} P_2 - P_5, \quad V_{16} = \vec{r} \vec{r} V_5 - V_{13}, \quad V_{17} = \vec{r} \vec{r} V_6 - V_{14},
\]
\[
V_{18} = -\vec{r} \vec{r} V_4 + V_9 + V_{11}, \quad V_{19} = -\vec{r} \vec{r} V_{10} + V_{14} + V_{15}, \quad A_{16} = \vec{r} \vec{r} A_3 - A_7,
\]
\[
A_{17} = \vec{r} \vec{r} (A_9 - A_3) + A_6 + A_7 + A_{12} - A_{15}, \quad A_{18} = \vec{r} \vec{r} A_{12} - A_{14}.
\]
(2.7)

We mention that the vector \((\vec{\sigma}, \vec{p}) \vec{L}\) was eliminated from the list (2.5) using the nontrivial linear relation \((\vec{\sigma}, \vec{p}) \vec{L} = V_6 - V_{10} + V_{12}\). We use relations (2.7) to remove the left-hand sides of (2.7) from the analysis completely.

The relations determining \(S_3, P_3, V_{16}, V_{17}, A_{16}\) and \(A_{18}\) are all consequences of the simple vector relation \(\vec{L}^2 = \vec{p}^2 \vec{x}^2 - (\vec{x}, \vec{p})^2\). Those determining \(V_{18}\) and \(V_{19}\) follow from the identity \((\vec{\sigma}, \vec{x})(\vec{x} \wedge \vec{p}) = -\vec{x}^2 (\vec{p} \wedge \vec{\sigma}) + (\vec{x}, \vec{p})(\vec{x} \wedge \vec{\sigma}) + (\vec{\sigma}, \vec{x} \wedge \vec{p})\vec{x}\).

### 3. Symmetrization of the integrals of motion

In the rest of the paper, we separately take the linear combinations of all the scalars, pseudoscalars, vectors and axial vectors with coefficients \(f_1(r)\) that are real functions of \(r\). However, instead of having the bare linear combinations of these tensors, the full symmetric forms are needed for the analysis of the commutation relations. Thus, in this section, we briefly describe how this symmetrization process was carried out basically by working on the linear combination of the scalars chosen as a prototype.
In writing the most general scalar operator, one needs to symmetrize the linear combination of the scalars given in (2.3)

\[ X_S = \sum_{j=1}^{6} f_j(r)S_j, \quad f_j(r) \in \mathbb{R}, \]  

(3.1)
term by term. It is obvious that the terms, for example, \( x_i p_i \) (here and throughout the whole paper summation over the repeated indices through 1 to 3 is to be understood) and \( p_i x_i \) are in fact different scalars and their symmetric form is \( \frac{1}{2}(x_i p_i + p_i x_i) \). However, at this stage an immediate question arises: should we associate a single arbitrary function of \( r \) multiplying each term obtained from symmetrization or different weightings are necessary for each of them?

Let us consider the scalar \( S_3 \), for example. The most general possible form of it is achieved by giving an arbitrary function of \( r \) to each permutation:

\[ S_3 = f_a(r)x_i p_i + f_b(r)p_i x_i + x_i f_c(r)p_i + x_i p_i f_d(r) + p_i f_e(r)x_i + p_i x_i f_f(r). \]  

(3.2)

Requiring that the operator be Hermitian, that is, \( S_3 = S_3^\dagger \), we obtain

\[
\begin{align*}
fa(r)xip_i &+ fb(r)pixi + xifc(r)p_i + xipifd(r) + pfec(r)x_i + pixiff(r) \\
&= pixaf_a(r) + xipbf_b(r) + pfec(r)x_i + fdrpjpixi + xifc(rp_i) + ffrxpip_i.
\end{align*}
\]  

(3.3)

Hence, self-adjointness reduces the half of the arbitrary functions

\[
fa(r) = f_f(r), \quad fb(r) = fd(r), \quad fc(r) = f_e(r).
\]  

(3.4)
Thus, \( S_3 \) can now be rewritten as

\[
S_3 = (fa(r) + fb(r))xip_i + pxifd(r) + pfec(r) + fdrpjpixi + xipbf_b(r).\]  

(3.5)
Furthermore, it is always possible to move the derivative terms to the right:

\[
S_3 = 2(fa(r) + fb(r))xip_i - 3\hbar(fa(r) + fb(r) + fe(r)) \\
-ihrf_a^2(r) + f_f^2(r) + f_e^2(r)).
\]  

(3.6)
Defining

\[
f_3(r) = fa(r) + fb(r) + fe(r),\]

(3.7)
we write

\[
S_3 = 2f_3(r)xip_i - 3i\hbar f_3(r) - ihrf_3^2(r).
\]  

(3.8)
Thus, for the scalar \( S_3 \), it is explicitly shown that a single arbitrary function of \( r \) is enough to make the symmetric form as general as can be.

However, it is easy to see how difficult this process would become even for the scalars such as \( S_3 \), for which we would have to start by writing \( 5! = 120 \) arbitrary functions of \( r \). We have already shown that a single arbitrary function of \( r \) is enough to give the symmetric form of the operators that are first-order in the momentum \([1]\).

To investigate the operators that are second-order in the momentum, we wrote a computer program code working under Mathematica. The code organizes the permutations of a given operator into Hermitian couples, which does the equivalent steps given in the previous example (3.3)–(3.5). Hence, for an operator made of \( n \) terms, we are left with \( \frac{n}{2} \) Hermitian couples. Moving the derivative terms to the extreme right for each couple, we see that all of the Hermitian couples for a given operator have the same form (as in (3.6)) although some of them give a few extra terms. These extra terms can always be absorbed into already existing terms (of lower order in \( \hbar \)). Hence, a redefinition of the arbitrary functions associated with a given
operator (as done in (3.7)) assures that it is enough to have only one arbitrary function of \( r \) for each scalar.

For example, let us consider the symmetrization of the scalar \( S_2 \). There are two distinct Hermitian couples associated with \( S_2 \), namely:

\[
p_i f_2(r) p_i + p_i f_2(r) p_i = \frac{2}{r} \left( f_2(r) p_i^2 - i h f'_2(r) \frac{x_i}{r} p_i \right).
\]  

(3.9)

Moving the derivatives to the right in the first couple, we obtain

\[
p_i f_2(r) p_i + p_i f_2(r) p_i = 2 \left( f_2(r) p_i^2 - i h f'_2(r) \frac{x_i}{r} p_i \right).
\]  

(3.10)

The same process with the second couple gives

\[
p_i p_i f_2(r) + f_2(r) p_i p_i = 2 \left( f_2(r) p_i^2 - i h f'_2(r) \frac{x_i}{r} p_i \right) - \hbar^2 f''_2(r).
\]  

(3.11)

Here, we have an extra term in the second derivative of \( f_2(r) \). However, this is just another arbitrary function of \( r \) and it is already associated with the scalar \( S_1 \). Hence, we could eliminate it by redefining the function associated with \( S_1 \) as:

\[
f_1(r) = f_1(r) - \hbar^2 f''_2(r).
\]  

(3.12)

The requirement that \([H, X_S]\) = 0, gives us the determining equations for this case. The determining equations, obtained by equating the coefficients of the third-order terms to zero in the commutativity equation, become

\[
f_5 = 0, \quad f'_5 = 0, \quad f''_5 = 0.
\]  

(4.3)

The determining equations, obtained by equating the coefficients of the second-order and first-order terms to zero in the commutativity equation, read, respectively,

\[
f_3 = 0, \quad f'_3 = 4 f_2 V', \quad f''_3 = 0.
\]  

(4.4)

and

\[
f'_3 = 4 f_2 V'_0.
\]  

(4.5)
The rest of the determining equations are then satisfied identically. It is immediately seen that the only solutions for equations (4.3)–(4.5) are

\[ f_1 = 4c_1 V_0 + c_3, \quad f_2 = c_1, \quad f_4 = 4c_1 V_1 + c_4, \quad f_6 = c_2, \]  

where \( c_i, i = 1, \ldots, 4 \) are real constants.

Thus, the corresponding four integrals of motion are those given in (1.5), i.e. there are no nontrivial scalar integrals of motion.

### 4.2. Pseudoscalars

As an integral of motion, we take a linear combination of the independent pseudoscalars given in (2.4):

\[ \tilde{X}_p = \sum_{j=1}^{6} f_j(r) P_j, \]  

which has the following fully symmetric form:

\[ X_p = (\vec{\sigma}, \vec{x}) \left( f_2 - i\hbar \left( \frac{1}{r} f_1^0 + 4 f_4 + r f_6^0 \right) \right) + \left( 2 f_1 - i\hbar (2 f_3 + 4 f_6 + r f_6^0) \right) (\vec{\sigma}, \vec{p}) \]

\[ + (\vec{\sigma}, \vec{x}) \left( 2 f_4 - i\hbar \left( \frac{2}{r} f_4^0 + 10 f_5 + 2 r f_5' + \frac{1}{r} f_6^0 \right) \right) (\vec{x}, \vec{p}) + 2 f_6 (\vec{x}, \vec{p}) (\vec{\sigma}, \vec{p}) \]

\[ + (\vec{\sigma}, \vec{x}) (2 f_3 (\vec{p}, \vec{p}) + 2 f_5 (\vec{x}, \vec{p}) (\vec{\sigma}, \vec{p})). \]  

(4.8)

Requiring that the commutator \([H, X_p] = 0\), we obtain the determining equations for this case. Those, obtained by equating the coefficients of the third-order terms to zero in the commutativity equation, read

\[ 2 r f_5 V_1 + \hbar f_5' = 0, \]

\[ \hbar f_4' + 2 r (\hbar f_5 + (f_3 + f_6) V_1) = 0, \]

\[ (\hbar r + 4 r^2 V_1) f_3 + \hbar f_6' - r (2 f_6 V_1 - 3 \hbar r f_5') = 0, \]

\[ (4 r^2 V_1 - \hbar) r f_5 - \hbar f_3' + \hbar f_6' - r (2 (f_3 + f_6) V_1 - 3 \hbar r f_5') = 0, \]

\[ h f_3 + \hbar f_6 + r (h f_3' + 2 r (h f_5 + f_3 V_1)) = 0, \]

\[ h f_5' + 2 h f_6' + r (2 (f_3 + f_6) V_1 + 2 f_5 (2 h + r^2 V_1) + 3 \hbar r f_5') = 0, \]

\[ h f_3' + \hbar f_6' + r (2 f_3 V_1 + f_5 (3 h + 2 r^2 V_1) + 2 h r f_5') = 0, \]

\[ f_3 + f_6 + r (3 r f_5 + f_3' + r^2 f_5' + f_6') = 0, \]

\[ h f_6 + f_3 (\hbar - 2 r^2 V_1) = 0, \]

\[ (2 r^2 V_1 - \hbar) (f_3 + r^2 f_5 + f_6) - r h f_6' = 0. \]

Equation (4.16) can immediately be integrated to give

\[ f_5 = \frac{c_1 - r (f_3 + f_6)}{r^2}, \]  

where \( c_1 \) is a real constant. If we introduce (4.19) into (4.18), then we obtain

\[ f_6' = c_1 \left( 2 V_1 - \frac{\hbar}{r^2} \right). \]  

(4.20)
Then writing $f_0$ from (4.17) and using the rest of the equations (4.9)–(4.15), as a compatibility condition we obtain a nonlinear second-order equation for $V_1$:

$$
3\hbar^4 r(h - 2V_1) V_1'' + 4\hbar^2 (3h^2 + r^2 V_1 - 5r^2 V_1 (2r^2 V_1 - 3h)) V_1 + 10\hbar^4 r^2 V_1^2 + 2r V_1^2 (45h^1 - 4r^2 V_1 (15h^1 + 4r^4 V_1^2 (r^2 V_1 - 3h))) = 0.
$$

(4.21)

Performing a standard symmetry analysis [59] for the equation (4.21), we find that the symmetry algebra is spanned by two vector fields:

$$
\vec{v}_1 = r \partial_r - 2V_1 \partial_{V_1}, \quad \vec{v}_2 = r^2 \partial_r + \left(\frac{\hbar}{2} - 3r^2 V_1\right) \partial_{V_1},
$$

(4.22)

with $[\vec{v}_1, \vec{v}_2] = 2\vec{v}_2$. It is now possible to lower the order of the equation (by two) using the standard method of symmetry reduction for ordinary differential equations [59]. In the case of equation (4.21), this leads to an implicit (general) solution which we do not find useful.

An alternative is to use the symmetry algebra to find particular solutions of (4.21), invariant under the subgroup generated by $\vec{v}_1$, or that generated by $\vec{v}_2$. The corresponding subgroup invariants are the potentials

$$
V_1 = \frac{C_1}{r^2} \quad \text{and} \quad V_1^* = \frac{C_2}{r^2} + \frac{\hbar}{2r^2},
$$

(4.23)

respectively. Substituting (4.23) into (4.21), we see that $V_1$ is a solution for

$$
C_1 = \begin{pmatrix}
-\frac{\hbar}{2} & 0 \\
\frac{\hbar}{2} & 0 \\
\frac{3\hbar}{2} & 0
\end{pmatrix}.
$$

(4.24)

The invariant $V_1^*$ is a solution only for $C_2 = 0$ and that solution is already included in (4.24) ($C_1 = \frac{\hbar}{2}$). The solutions $V_1$ for $C_1$ as in (4.24) (with $C_1 \neq \frac{\hbar}{2}$) can be extended to one-parameter classes of solutions by acting on them with the symmetry group generated by $\vec{v}_2$:

$$
\tilde{r} = \frac{r}{\sqrt{1 - 2\lambda r^2}}, \quad \tilde{V}_1 = \left(V_1 - \frac{\hbar}{2r^2}\right) (1 - 2\lambda r^2)^{\frac{1}{2}} + \frac{\hbar}{2r^2} (1 - 2\lambda r^2), \quad |\lambda| < \frac{1}{2r^2}.
$$

(4.25)

Substituting $V_1(r) = \frac{C_2}{r^2}$ into (4.25) and expressing $\tilde{V}_1$ in terms of $\tilde{r}$, we obtain

$$
\tilde{V}_1(\tilde{r}) = \frac{1}{2r^2} \left(h + \frac{2C_1 - h}{\sqrt{1 + 2\lambda r^2}}\right).
$$

(4.26)

The four values of $C_1 \neq \frac{\hbar}{2}$ in (4.24) lead to four new potentials:

$$
V_1(r) = \frac{h}{2r^2} \left(1 + \frac{2\epsilon}{\sqrt{1 + 2\lambda r^2}}\right), \quad V_1(r) = \frac{h}{2r^2} \left(1 + \frac{\epsilon}{\sqrt{1 + 2\lambda r^2}}\right),
$$

(4.27)

with $\epsilon^2 = 1$. Thus, we have seven potentials to consider: those in (4.27) and the original $V_1 = \frac{C_2}{r^2}$ with $C_1 = -\frac{\hbar}{2}, \frac{\hbar}{2}, \frac{3\hbar}{2}$ ($C_1 = 0$ is trivial and $C_1 = \hbar$ is gauge induced and was considered in [1]). Taking $\lambda = 0$ in (4.27), we recover the original cases with $C_1 = -\frac{\hbar}{2}, \frac{3\hbar}{2}$ (and $C_1 = 0, \hbar$), but we prefer to treat them separately.

Case I. $V_1 = \frac{h}{2r^2}$

For this type of potential, (4.17) and (4.19) immediately imply

$$
f_0 = 0, \quad \text{and} \quad f_5 = \frac{c_1}{r^3} - \frac{f_3}{r^2},
$$

(4.28)

where $c_1$ is an integration constant. Also the set of determining equations given in (4.9)–(4.18) together with (4.19) give us

$$
f_5 = -\frac{2c_2}{r},
$$

(4.29)
where \( c_j \) is an integration constant. Then the determining equations, obtained from lower order terms, provide us with

\[
f_1 = -r c_3, \quad f_2 = \frac{4c_1 V_0 + c_4}{r}, \quad f_4 = \frac{c_3}{r},
\]

(4.30)

where \( c_3 \) and \( c_4 \) are integration constants. For these values of \( f_j \) (for \( j = 1, \ldots, 6 \)), all the determining equations, obtained from the requirement that the commutator \([H, X_p]\) = 0, are satisfied for any \( V_0 = V_0(r) \). Since we have four arbitrary constants and none of them appear in the Hamiltonian, we have four different integrals of motion. Two of them are first-order operators and correspond to the ones that were found in [1], while the other two are second-order operators. They are given as

\[
X_p^1 = \frac{\langle \vec{\sigma}, \vec{x} \rangle}{r},
\]

(4.31)

\[
X_p^2 = -r <\vec{\sigma}, \vec{p}> + \frac{1}{r} <\vec{\sigma}, \vec{x}> (\vec{x}, \vec{p}) - \frac{i\hbar}{r} (\vec{\sigma}, \vec{x}),
\]

(4.32)

\[
X_p^3 = 4 \frac{\langle \vec{\sigma}, \vec{x} \rangle}{r} \left( \frac{1}{2} (\vec{p}, \vec{p}) + V_0 \right) + \frac{2i\hbar}{r^3} (\vec{\sigma}, \vec{x})(\vec{x}, \vec{p}) - \frac{2i\hbar}{r} (\vec{\sigma}, \vec{p}),
\]

(4.33)

\[
X_p^4 = \frac{\langle \vec{\sigma}, \vec{x} \rangle}{r} (6i\hbar (\vec{x}, \vec{p}) - 2 (\vec{x}, (\vec{x}, \vec{p}) \vec{p}) + 2r^2 (\vec{p}, \vec{p}) - 2i\hbar (\vec{\sigma}, \vec{p})).
\]

(4.34)

However, the only really independent pseudoscalar integral is \( X_p^1 \) since we have

\[
X_p^2 = -i X_p^1 ((\vec{\sigma}, \vec{L}) + h),
\]

(4.35)

\[
X_p^3 = 4 X_p^1 H,
\]

(4.36)

\[
X_p^4 = 2 X_p^1 ((X_p^1)^2 - h^2).
\]

(4.37)

**Case 2.** \( V_1 = -\frac{\hbar}{2\sigma} \)

For this type of potential, (4.20) implies

\[
f_6 = \frac{2c_1}{r} + c_2,
\]

(4.38)

where \( c_1 \) and \( c_2 \) are integration constants. Then the set of determining equations given in (4.9)–(4.18) gives

\[
f_1 = 0, \quad f_3 = -\frac{c_1}{r}, \quad f_4 = 0, \quad f_5 = 0, \quad c_2 = 0.
\]

(4.39)

Introducing (4.38) and (4.39) into the determining equations obtained from lower order terms, we find the following equations:

\[
2r (f_2 - r f_3^2) - \frac{12\hbar^2 c_1}{r^2} = 0, \quad 4c_1 r V_0' - \frac{\hbar^2 c_1 + 2c_3 r^2}{r^2} = 0.
\]

(4.40)

Their solutions are

\[
f_2 = \frac{3\hbar^2 c_1}{2r^3} + r c_3, \quad V_0 = -\frac{\hbar^2}{2r^2} + \alpha r^2, \quad \alpha = \frac{c_3}{4c_1},
\]

(4.41)

where \( c_3 \) is an integration constant. Hence, the integral of motion \( X_P \) depends on two constants \( c_1 \) and \( c_3 \). However, \( V_0 \) also depends on \( \alpha = \frac{c_3}{4c_1} \). Thus, we can choose \( c_3 = 4\alpha c_1 \) and set \( c_1 = 1 \). The integral of motion for this case can be written as

\[
X_P = \frac{\langle \vec{\sigma}, \vec{x} \rangle}{r} \left( \frac{3\hbar^2}{2r^2} + 4\alpha r^2 - 2(\vec{p}, \vec{p}) \right) + \frac{4}{r} ((\vec{x}, \vec{p}) - i\hbar (\vec{\sigma}, \vec{p})).
\]

(4.42)

Since \( X_P \) is a pseudoscalar operator, it also commutes with the components of the total angular momentum \( J_i \) (\( i = 1, 2, 3 \)) \([H, X_P] = 0\), and \([J_i, X_P] = 0\), as do all the pseudoscalars obtained below.
Case 3, $V_1 = \frac{3\hbar}{r^2}$

For this type of potential, the set of determining equations given in (4.9)–(4.18) together with (4.20) give us

$$
\begin{align*}
f_1 &= 0, & f_3 &= -\frac{c_1}{r} + \frac{c_2}{2}, & f_4 &= 0, \\
f_5 &= \frac{4c_1}{r^3} - \frac{3c_2}{2r^2}, & f_6 &= -\frac{2c_1}{r} + c_2,
\end{align*}
$$

where $c_1$ and $c_2$ are integration constants. If we introduce these integrals of motion into the set of determining equations, then we obtain $c_2 = 0$. Using the determining equations, obtained from lower order terms, we obtain

$$
f_2 = \frac{5\hbar^2 c_1 + 2c_1 r^3 V_0'}{r^3},
$$

and upon introducing (4.44) back into the determining equations we obtain a second-order differential equation for $V_0$:

$$
r^3 (V_0' - r V_0'') + 15\hbar^2 = 0.
$$

Its solution is

$$
V_0 = \frac{15\hbar^2}{8r^2} + \alpha r^2,
$$

where $\alpha$ is an integration constant. Using (4.46), we obtain $f_2$ from (4.44):

$$
f_2 = c_1 \left( 4\alpha r - \frac{5\hbar^2}{2r^3} \right).
$$

Since there is only one arbitrary constant for this case ($\alpha$ appears in the Hamiltonian), we only have one second-order integral of motion, namely

$$
X_p = \frac{(\vec{\sigma} \times \vec{x})}{r} \left( -\frac{5\hbar^2}{2r^2} + 4\alpha r^2 - \frac{20\hbar}{r^2} (\vec{x} \cdot \vec{p}) - 2(\vec{p} \cdot \vec{p}) + \frac{8}{r^2} (\vec{x} \cdot (\vec{x} \cdot \vec{p}) \vec{p}) \right)
- \frac{4}{r} \left( (\vec{x} \cdot \vec{p}) - 2i\hbar (\vec{\sigma} \cdot \vec{p}) \right).
$$

Case 4, $V_1 = \frac{\hbar}{2r^2} \left( 1 + \frac{\beta}{\sqrt{1 + \beta}} \right)$, $\beta \equiv 2\lambda$

For this type of potential, the set of determining equations given in (4.9)–(4.18) together with (4.20) give us

$$
\begin{align*}
f_6 &= -\frac{\epsilon c_1 \sqrt{1 + \beta r^2}}{r} + c_2, & f_3 &= -\frac{c_1}{r} - c_1 r \beta + \epsilon c_2 \sqrt{1 + \beta r^2}, \\
f_5 &= c_1 (2 + r^2 \beta + \epsilon \sqrt{1 + \beta r^2}) - c_2 r (1 + \epsilon \sqrt{1 + \beta r^2})
- \frac{\sqrt{1 + \beta r^2}}{r^3},
\end{align*}
$$

where $c_1$ and $c_2$ are integration constants. Introducing (4.49) back into the determining equations we obtain $c_1 = 0$ and then rest of the determining equations give us

$$
\begin{align*}
f_1 &= \frac{c_3}{\beta} \sqrt{1 + \beta r^2}, & f_2 &= \frac{2\hbar^2 c_2 (1 + \epsilon \sqrt{1 + \beta r^2})}{r^2}, \\
f_4 &= -\frac{c_3}{-\epsilon + \sqrt{1 + \beta r^2}}, & V_0 &= \hbar V_1.
\end{align*}
$$

Finally, for this case, we have two arbitrary constants and hence two integrals of motion. One of them is a first-order operator and corresponds to the one already found in [1] and the other is a new second-order operator. They are given as

$$
X_p^1 = -\frac{1}{\beta} \sqrt{1 + \beta r^2} (\vec{\sigma} \cdot \vec{p}) + \frac{(\vec{\sigma} \cdot \vec{x})}{-\epsilon + \sqrt{1 + \beta r^2}} ((\vec{x} \cdot \vec{p}) - i\hbar),
$$
\[ X_p^2 = \frac{2}{r^2} (1 + \epsilon \sqrt{1 + \beta r^2}) (\sigma, \vec{x}) (\hbar^2 + 3i\hbar (\vec{x}, \vec{p}) - (\vec{x}, (\vec{x}, \vec{p}) \vec{p})) \]
\[ - 2i\hbar (2 + \epsilon \sqrt{1 + \beta r^2}) (\sigma, \vec{p}) + 2 (\vec{x}, \vec{p}) (\sigma, \vec{p}) + 2e \sqrt{1 + \beta r^2} (\sigma, \vec{x}) (\vec{p}, \vec{p}). \quad (4.52) \]
Also note that we obtain an obvious integral by multiplying \( X_p^1 \) by \((\sigma, \vec{L})\), however, not functionally independent of \( X_p^1 \) and \( X_p^2 \).

Case 5. \( V_1 = \frac{\hbar}{2\sigma^2} (1 + \frac{2\epsilon}{\sqrt{1 + \beta r^2}}) \), \( \beta = 2\lambda \).

For this type of potential, the set of determining equations given in (4.9)–(4.18) together with (4.20) give us
\[ f_6 = - \frac{2e c_1 \sqrt{1 + \beta r^2}}{r} + c_2, \quad f_3 = - \frac{c_1}{r} - c_1 r \beta + \frac{\epsilon}{2} c_2 \sqrt{1 + \beta r^2}, \]
\[ f_5 = \frac{2c_1 (2 + r^2 \beta + 2\epsilon \sqrt{1 + \beta r^2}) - c_2 (2 + e \sqrt{1 + \beta r^2})}{2r^3}, \quad (4.53) \]
where \( c_1 \) and \( c_2 \) are integration constants. Introducing (4.53) back into the determining equations we get \( c_2 = 0 \) and then rest of the determining equations imply
\[ f_1 = 0, \quad f_4 = 0, \]
\[ f_2 = \frac{-\hbar^2 c_1 (1 + 2r^2 \beta)}{2r^3 (1 + \beta r^2)^2}, \]
\[ V_0 = \frac{\hbar^2}{8\sigma^2 (1 + \beta r^2)^2} (7 + 10r^2 \beta + 8e (1 + \beta r^2)^{1/2}) - \alpha \frac{\frac{\alpha}{4\beta (1 + \beta r^2)}}{2r^3}. \quad (4.54) \]

Finally, for this case, we have only one arbitrary constant \( (c_3 = \alpha c_1) \) and hence only one second-order operator given as
\[ X_p = \frac{2r^4 \alpha (1 + \beta r^2)}{2r^3 (1 + \beta r^2)^2} - \frac{2i\hbar}{r^3} \left( 5 + 3\beta r^2 - \frac{\epsilon}{\sqrt{1 + \beta r^2}} + 6 \epsilon \sqrt{1 + \beta r^2} \right) (\sigma, \vec{x}) (\vec{x}, \vec{p}) \]
\[ + \frac{2i\hbar}{r} \left( 1 + \beta r^2 - \frac{\epsilon}{\sqrt{1 + \beta r^2}} + 4 \epsilon \sqrt{1 + \beta r^2} \right) (\sigma, \vec{p}) \]
\[ - \frac{2}{r} (1 + \beta r^2) (\sigma, \vec{x}) (\vec{p}, \vec{p}) + \frac{2}{r^3} (2 + \beta r^2 + 2 \epsilon \sqrt{1 + \beta r^2}) (\sigma, \vec{x}) (\vec{x}, (\vec{x}, \vec{p}) \vec{p}) \]
\[ - \frac{4\epsilon}{r} \sqrt{1 + \beta r^2} (\vec{x}, \vec{p}) (\vec{p}, \vec{p}). \quad (4.55) \]

5. Vector integrals of motion

Let us take a linear combination of the independent vectors given in (2.5):
\[ \tilde{X}_V = \sum_{j=1}^{15} f_j(r) \tilde{V}_j, \quad (5.1) \]
and fully symmetrize it as described in section 3. The symmetric form of (5.1) is written as
\[ \tilde{X}_V = \tilde{x} \left( 2f_1 - i\hbar \left( 4f_1 + \frac{f_1^2}{r} + rf_1 \right) \right) + \left( 2f_{11} - i\hbar \left( 4f_{15} + \frac{f_{15}^2}{r} + rf_{15} \right) \right) (\sigma, \vec{L}) \]
\[ + \left( 2f_1 - i\hbar \left( \frac{2f_1}{r} + \frac{f_1}{r} + 10f_{13} + 2rf_{13} \right) \right) (\vec{x}, \vec{p}) \]
where ϵ

\[ \epsilon \]

\[ 1 \]

\[ 12 \]

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\[ (\sigma, \tilde{L}) + f_{13}(\tilde{x}, (\tilde{x}, \tilde{p}) \tilde{p}) \]

\[ + 2 \left( \begin{array}{c} f_{13} (\tilde{x}, \tilde{p}) (\sigma, \tilde{L}) + f_{13}(\tilde{x}, (\tilde{x}, \tilde{p}) \tilde{p}) \end{array} \right) \]

\[ + \left( 2 \left( f_{2} - i h (f_{5} + 2 f_{8} + \frac{r f_{5}}{2}) + f_{12} (\sigma, \tilde{L}) + f_{8} (\tilde{x}, \tilde{p}) \right) \right) \tilde{p} \]

\[ + (\tilde{x} \wedge \tilde{\sigma}) \left( 2 f_{3} + i h \left( f_{11} - 4 f_{0} - \frac{f_{4}}{r} - r f_{4} \right) + 2 f_{6} \tilde{p}^{2} \right) \]

\[ + 2 f_{14} (\tilde{x}, (\tilde{x}, \tilde{p}) \tilde{p}) + \left( 2 f_{9} - i h \left( \frac{f_{10} + 2 f_{6}}{r} + 2 r f_{14} \right) \right) (\tilde{x}, \tilde{p}) \]

\[ - (2 f_{4} - i h (2 f_{6} + 4 f_{10} + r f_{10} + 2 r f_{10} (\tilde{x}, \tilde{p}) (\tilde{\sigma} \wedge \tilde{p})). \] \hfill (5.2)

The requirement \[ [H, \tilde{X}_{0}] = 0 \] gives us the determining equations for the vectors. The determining equations, obtained by equating the coefficients of the third-order terms to zero in the commutativity equation, are

\[ f_{5} + f_{8} = 0, \quad f_{5}^{'} = 0, \quad f_{13} = 0, \quad f_{14} - f_{15} = 0, \]

\[ f_{10} + r^{2} f_{14} + f_{6} = 0, \quad f_{6}^{'} - f_{15}^{'} = 0, \quad h f_{15} + 2 f_{6} V_{1} = 0. \] \hfill (5.3)

together with the following two differential equations:

\[ h f_{6} V_{1}^{'} + 2 r f_{6} V_{1}^{2} = 0, \]

\[ h f_{6}^{'} - 2 r^{2} (f_{6} V_{1})^{'} - 4 r V_{1} f_{6} = 0. \] \hfill (5.4)

It is obvious that \( f_{6} = 0 \) is a solution of the system given in (5.4). For \( f_{6} \neq 0 \), (5.4) implies a compatibility condition for \( V_{1} \) which reads as

\[ 6 h r V_{1}^{2} - 4 r^{3} V_{1}^{3} + h^{2} V_{1}^{'} = 0. \] \hfill (5.5)

The above differential equation for \( V_{1} \) had already been considered in [1]. Depending on the solutions of this equation we have several cases. The general solution is given by

\[ V_{1} = \frac{h}{2 r^{2}} \left( 1 + \frac{\epsilon}{\sqrt{1 + \beta r^{2}}} \right). \] \hfill (5.6)

where \( \epsilon = 1 \). Note that \( V_{1} = \frac{h}{r} \) and \( V_{1} = \frac{h}{2 r} \) are special solutions with \( (\epsilon, \beta) = (1, 0) \) and \( (1, \infty) \), respectively. The case \( V_{1} = \frac{h}{r} \) is induced by a gauge transformation had already been considered thoroughly in [1].

If we introduce the solutions (5.6) back into the system (5.4), then we obtain differential equations for \( f_{6} \) which can be solved. Hence, bearing in mind that \( f_{6} = 0 \) is also a solution of the system (5.4), we have several cases to be considered separately.

5.1. \( V_{1} \) is a solution of (5.5)

Subcase 1. \( V_{1} = \frac{h}{2 r} \)

For this type of potential (5.4) implies \( f_{6} = c_{1} r \). Thus, introducing this together with the relations given in (3.3) into the determining equations obtained by equating the coefficients of the second-order terms to zero in the commutativity equation, we obtain

\[ f_{2} = c_{3} - r^{2} f_{7}, \quad f_{4} = c_{4} - r^{2} f_{9}, \quad f_{7} = - \frac{c_{1}}{2 r} + \frac{c_{5}}{2 r^{2}}, \quad f_{8} = - c_{6}, \]

\[ f_{9} = \frac{c_{7}}{r}, \quad f_{11} = \frac{c_{6}}{2 r^{2}} + f_{9}, \quad f_{12} = \frac{c_{1} r}{2} + r^{3} f_{7}, \quad c_{6} = 2 c_{4}. \] \hfill (5.7)

where \( c_{i}, (i = 3, \ldots, 7) \) are integration constants.
The determining equations obtained by equating the coefficients of the first- and zeroth-order terms to zero in the commutativity equation give us

\[ c_4(3h^2 - 4r^3(2V_0' + rV_0'')) = 0. \]  

(5.8)

Hence we have two possibilities. Either \( c_4 = 0 \), or \( 3h^2 - 4r^3(2V_0' + rV_0'') = 0 \), which has the following solution:

\[ V_0 = \frac{3h^2}{8r^2} - \frac{\alpha}{r}. \]  

(5.9)

where \( \alpha \) is an integration constant and we have set an irrelevant additive constant equal to zero.

\( I_1 \): \( V_0 \) is given as in (5.9)

Upon introducing \( V_0 \) back into the determining equations coming from first- and zeroth-order terms, we have

\[ f_1 = \frac{c_4(2h^2 - 4\alpha r) + c_7r}{2r^2}, \quad f_3 = -\frac{h^2c_3 - 2h^2c_1r - 4c_3\alpha r}{4r^2}, \quad c_5 = -c_3. \]  

(5.10)

Then all the determining equations are satisfied. We have four arbitrary constants \((c_1, c_3, c_4, c_7)\). Thus, we have four integrals of motion which read

\[ \hat{X}^0 = -(\hat{\sigma},\hat{L})\hat{\rho} + \frac{3h}{2}\hat{\rho} - \frac{h\overline{\hat{x}}}{2r^2}(\overline{\hat{x}},\overline{\hat{\rho}}) + \frac{i\hbar}{2}(\hat{\sigma} \wedge \hat{\rho}) + i\hbar^2\frac{\overline{\hat{x}}^2}{2r^2} + \frac{4\alpha r - h^2}{4r^2}(\overline{\hat{x}} \wedge \hat{\sigma}), \]  

(5.11)

\[ \hat{X}^2 = 2\overline{\hat{x}}\hat{\rho}^2 - 2(\overline{\hat{x}},\overline{\hat{\rho}})\hat{\rho} - \frac{h\overline{\hat{x}}}{r^2}(\overline{\hat{x}},\overline{\hat{L}}) + 2i\hbar\overline{\hat{\rho}} - h(\hat{\sigma} \wedge \hat{\rho}) + i\hbar^2\frac{\overline{\hat{x}} \wedge \hat{\sigma}}{2r^2} + \frac{\overline{\hat{x}}}{r^2}(h^2 - 2\alpha r), \]  

(5.12)

\[ \hat{X}^3 = \frac{(\overline{\hat{x}},\hat{\sigma})}{r}\hat{L} + \frac{\hbar}{2r}((\overline{\hat{x}} - i(\overline{\hat{x}} \wedge \hat{\sigma})), \]  

(5.13)

\[ \hat{X}^4 = r\hat{L}(\hat{\sigma},\hat{\rho}) + \frac{hr}{2}\hat{\rho} - \frac{ihr}{2}(\hat{\sigma} \wedge \hat{\rho}) + \hat{X}^3_0(i\hbar - (\overline{\hat{x}},\overline{\hat{\rho}})). \]  

(5.14)

However, the vector integrals of motion (5.13) and (5.14) can be written as anticommutators of the pseudoscalar integrals of motion given in (4.31) and (4.32), respectively, i.e.

\[ \{\hat{J},\hat{X}^3_0\} = 2\hat{X}^3_0, \quad \text{and} \quad \{\hat{J},\hat{X}^3_0\} = -2\hat{X}^4_0. \]  

(5.15)

\( I_2 \): \( V_0 \) unspecified, \( c_4 = 0 \)

From the determining equations obtained by equating the coefficients of the first- and zeroth-order terms to zero in the commutativity equation, we obtain

\[ c_5 = -c_3, \quad f_1 = \frac{c_7}{2r}, \quad c_3(3h^2 - 4r^3(2V_0' + rV_0'')) = 0. \]  

(5.16)

Thus, we must have \( c_3 = 0 \) because the other possibility gives us the previous potential for \( V_0 \). Upon introducing back all information into the determining equations, we obtain

\[ f_3 = \frac{c_1}{2r}, \]  

(5.17)

and then all the determining equations are satisfied for arbitrary values of \( V_0(r) \). We have two integrals of motion for this case which are given as in (5.13) and (5.14). It is worth noting that the commutativity condition is satisfied for arbitrary scalar potentials \( V_0(r) \).
Subcase II. $V_1 = \frac{\hbar}{2\pi^2} \left( 1 + \frac{\epsilon}{\sqrt{1 + \beta r^2}} \right)$

For this type of potential (5.4) implies $f_6 = c_2 \sqrt{1 + \beta r^2}$. Thus, introducing this together with the relations given in (5.3) into the determining equations obtained by equating the coefficients of the second-order terms to zero in the commutativity equation, we obtain

\[
\begin{align*}
    f_2 &= c_3 - r^2 f_1, \\
    f_4 &= c_4 - r^2 f_1, \\
    f_8 &= 0, \\
    f_9 &= f_{11}, \\
    f_{11} &= \frac{c_4 + \epsilon c_4 \sqrt{1 + \beta r^2}}{r^2}, \\
    f_{12} &= \frac{\epsilon r^2 (1 + \beta r^2) (c_2 \beta - 2 \epsilon (1 + \beta r^2) f_7)}{1 + \epsilon \sqrt{1 + \beta r^2} + \beta r^2 (2 + \beta r^2 + 2 \epsilon \sqrt{1 + \beta r^2})}.
\end{align*}
\]  

(5.18)

where $c_3$ and $c_4$ are integration constants.

Upon introducing the relations given in (5.18) into the remaining determining equations, we have

\[
f_1 = 0,
\]

(5.19)

together with $c_2 = 0$ and $c_4 = 0$. However, $c_2 = 0$ reduces $f_6$ to zero which will be analyzed below.

5.2. $V_1$ unspecified, $f_6 = 0$

For this case, let us continue to analyze the determining equations obtained by equating the coefficients of the second-order terms to zero in the commutativity equation. Introducing $f_6 = 0$ together with the relations given in (5.3) into the determining equations, we obtain

\[
\begin{align*}
    f_2 &= c_3 - r^2 f_1, \\
    f_4 &= c_4 - r^2 f_1, \\
    f_7 &= f_{12} (V_1 + rV'_1), \\
    f_9 &= \frac{2\hbar c_4 + h f_8}{2r^2 V_1 - h}, \\
    f_{11} &= \frac{f_1 + f_8 (V_1 + rV'_1)}{r V_1 - h}.
\end{align*}
\]  

(5.20)

together with the relations

\[
\begin{align*}
    2r(2c_4 + f_8) (2r^2 V_1 - 3h) V_1^2 + 2f_8 (h^2 + 6r^2 V_1 (r^2 V_1 - h)) - h^2 c_4) V'_1 = 0, \\
    2r(2c_4 + f_8) (2r^2 V_1 - 3h) V_1^2 - 2f_8 (2h^2 + 6r^2 V_1 (r^2 V_1 - h)) + h^2 c_4) V'_1 = 0, \\
    -rf_8 (2r^2 V_1 - h V_1^2 V'_1 = 0, \\
    (2c_4 + f_8) (-6hr V_1^2 + 4r^3 V_1^3 - h^2 V'_1) = 0, \\
    f_{12} (3V'_1 + rV''_1) + f'_{12} (V_1 + rV'_1) = 0, \\
    f_8 (3V'_1 + V''_1) = 0.
\end{align*}
\]  

(5.21)

Here, $c_3$ and $c_4$ are arbitrary constants of integration. From the system (5.21), we have four subcases.

Subcase I. $f_8 = -2c_4$, $f_{12} = c_7$, $3V'_1 + rV''_1 = 0$, which implies

\[
V_1 = \frac{\alpha}{2r^2} + \beta.
\]  

(5.22)

This is a new spin–orbital potential.

When we introduce all the information we have found up to now into the remaining determining equations, we obtain

\[
\begin{align*}
    c_4 (\alpha^2 - 4 \beta r^2 (\beta r^2 - h)) + r^2 (2h^2 f_1 + 4c_4 r V'_0) = 0, \\
    c_7 = c_3, \\
    c_4 (4r V''_0 + 8V_0' - 12 \beta^2 r - \frac{\alpha (\alpha + 2h)}{r^3}) = 0.
\end{align*}
\]  

(5.23)

(5.24)

From equation (5.24) it is immediately seen that

\[
4r V''_0 + 8V_0' - 12 \beta^2 r - \frac{\alpha (\alpha + 2h)}{r^3} = 0.
\]  

(5.25)
because the other possibility $c_4 = 0$, either gives us the known potentials ($V_1 = \frac{h}{\beta r}$ and $V_1 = \frac{h}{2\beta r}$) or reduces the number of the integral of motions. The solution of (5.25) is given as

$$V_0 = \frac{\alpha (\alpha + 2h) + 4\beta^2 r^4 - 8r\gamma}{8 r^2},$$

(5.26)

where $\gamma$ is an integration constant. Then, introducing back the solution given in (5.26) into the remaining determining equations, we obtain

$$f_1 = \frac{2c_4 (\alpha - 2r (\beta + h\beta r))}{2r^2}, \quad f_3 = \frac{c_3 (4r (\gamma + \beta^2 r^3) - \alpha^2)}{4 r^2 (\alpha + 2\beta r^2)}, \quad \gamma = 0.$$

(5.27)

For these values of $f_j$, ($j = 1, \ldots, 15$), all the determining equations are satisfied for the potentials (5.22) and (5.26). We have two arbitrary constants, and hence we have two different integrals of motion. They are given as

$$\vec{X}_1 = \vec{x} \left( \frac{2\beta r^2 - \alpha}{r^2} \right) \left( (\vec{\sigma}, \vec{L}) + h \right) + 2i(h - (\vec{x}, \vec{p})) \vec{p} - h(\vec{\sigma} \land \vec{p})$$

(5.28)

$$+ i\hbar \left( \frac{2\beta r^2 - \alpha}{r^2} \right) (\vec{\sigma} \land \vec{x}),$$

$$\vec{X}_2 = \left( \frac{1}{2} (2h + \alpha - 2\beta r^2) + (\vec{\sigma}, \vec{L}) \right) \vec{p} + \vec{x} \left( \frac{2\beta r^2 - \alpha}{r^2} \right) \left( (\vec{x}, \vec{p}) - ih \right) + \frac{i\hbar}{2} (\vec{\sigma} \land \vec{p})$$

(5.29)

$$- \frac{\hbar}{4} \left( \frac{2\beta r^2 - \alpha}{r^2} \right) (\vec{\sigma} \land \vec{x}).$$

Subcase 2. $f_8 = 0, \quad f_{12} = 0, \quad 6\hbar V_1^2 - 4rV_1^3 + h^2 V_0^2 = 0.$

In this subcase, depending on the solutions of (5.5) we have two possibilities.

\text{II\textsubscript{1}: $V_1 = \frac{h}{2\beta r}$}

Since for this type of potential $f_8$ in (5.20) becomes undefined, we need to analyze this case from the beginning. Analysis of the determining equations obtained by equating the coefficients of the second-order terms to zero in the commutativity equation gives

$$f_2 = c_3, \quad f_4 = -r^2 f_9, \quad f_7 = 0, \quad f_{11} = f_9, \quad f_9 = \frac{c_4}{r},$$

(5.30)

where $c_3$ and $c_4$ are integration constants. For this case, we already have $f_8 = 0$ and $f_{12} = 0$.

From the other determining equations, we have

$$f_1 = \frac{c_4}{2r}, \quad f_3 = 0, \quad c_3 = 0.$$

(5.31)

All the determining equations are satisfied for arbitrary potentials $V_0(r)$ and since we have only one arbitrary constant, namely $c_4$, there is only one integral of motion for this case. It is given by equation (5.13) and is first-order in the momenta.

\text{II\textsubscript{2}: $V_1 = \frac{h}{2\beta r} (1 + \frac{1}{\sqrt{1 + \beta r^2}})$}

Upon introducing all the information we have gathered so far into the determining equations obtained from equating the first- and zeroth-order terms to zero in the commutativity equation, we find

$$f_1 = \frac{c_4}{2r^2} \left( 2 + \epsilon \frac{2 + \beta r^2 (4 + \beta r^2)}{(1 + \beta r^2)^2} \right), \quad f_3 = -\frac{c_4 (1 + \epsilon \sqrt{1 + \beta r^2})}{2r^2}.$$

(5.32)

If we introduce the above values of $f_1$ and $f_3$ back into the determining equations, then we see that we must have $c_3 = 0$ and $c_4 = 0$. Then all the determining equations are satisfied. Hence, we do not have any integral of motion for this case except the obvious one which is the anticommutator of the pseudoscalar integral of motion (4.51) with $\mathcal{J}$, namely

$$\vec{X}_V = \{\vec{X}_V^\dagger, \mathcal{J}\}.$$

(5.33)
Subcase III.
In this subcase, we have \( f_8 = 0, \ c_4 = 0 \) and \( f_{12} = 0 \). Upon introducing this information into the determining equations obtained from the first- and zeroth-order terms in the commutativity equation, we have
\[
f_1 = 0, \quad f_3 = \frac{c_3 V_i^1}{2 r V_i^1}, \quad V_1 = \frac{\hbar + \epsilon h \sqrt{1 + 4 C r^2}}{2 r^2},
\]
where \( C \) is an integration constant. Now if we introduce these back into the determining equations we see that we either have \( c_3 = 0 \) or \( C = 0 \). If \( c_3 = 0 \), we have no integral of motion since \( c_4 \) is already zero. If \( C = 0 \), then we have \( V_1 = \frac{\hbar}{2 r^2} \), which has been investigated thoroughly. Hence, we conclude that no new information is obtained from this case.

Subcase IV.
\[
f_{12} (3 V_i^i + r V_i^i) + f_{12}^0 (V_1 + r V_i^i) = 0 \tag{5.35}
\]
If we solve the above equation for \( f_{12} \) and introduce back into the determining equations obtained from first- and zeroth-order terms in the commutativity equation, then we have
\[
V_1 (V_1 + r V_i^i) (3 V_i^i + r V_i^i) f_{12}'' = 0. \tag{5.36}
\]
Thus, we either have \( V_1 + r V_i^i = 0 \) or \( f_{12}'' = 0 \). The other choice \( 3 V_i^i + r V_i^i = 0 \) has already been investigated.
\[
V_1 + r V_i^i = 0 \implies V_1 = \frac{C}{r}, \text{ where } C \text{ is an integration constant. Thus } (5.35) \text{ becomes }
\]
\[
C \frac{f_{12}}{r^2} = 0. \tag{5.37}
\]
Hence, we either have \( C = 0 \), but then \( V_1 = 0 \) or \( f_{12} = 0 \) which has been investigated in the previous case.

The condition \( f_{12}'' = 0 \) implies
\[
f_{12} = c_{10} r + c_{11}, \tag{5.38}
\]
where \( c_{10} \) and \( c_{11} \) are integration constants. Introducing this back into the determining equations, we see that we have \( c_{10} = 0 \) and then (5.35) becomes
\[
c_{11} (3 V_i^i + r V_i^i) = 0. \tag{5.39}
\]
The two cases \( c_{11} = 0 \) and \( 3 V_i^i + r V_i^i = 0 \) have been previously investigated.

Hence, we conclude that no new information is obtained from this case.

6. Axial vector integrals of motion
Let us take a linear combination of the axial vectors given in (2.6):
\[
\tilde{X}_A = \sum_{j=1}^{15} f_j(r) \tilde{A}_j, \tag{6.1}
\]
and fully symmetrize it as described in section 3. The symmetric form of (6.1) is
\[
\tilde{X}_A = \left( (\tilde{\sigma}, \tilde{x}) \left( 2 f_{11} - i \hbar \left( 2 f_{12} + f_6 + \frac{1}{r} f_9 \right) \right) + 2 f_9 (\tilde{\sigma}, \tilde{p}) \tilde{p} \right.
\]
\[
+ (2 f_2 - i \hbar (5 f_5 + r f_2) + 2 f_5 (\tilde{x}, \tilde{p}) + 2 f_6 (\tilde{L}, \tilde{\sigma}) \tilde{L} + \tilde{\sigma}) \left( 2 f_{11} - i \hbar (f_{11} + 3 f_4 + f_8 + r f_3) 
\]
\[
+ (2 f_4 - i \hbar (f_{15} - 2 f_6 + 8 f_5 + \frac{2}{r} f_3 + 2 r f_7) \right) \left( \tilde{x}, \tilde{p} \right) + 2 f_7 (\tilde{x}, (\tilde{x}, \tilde{p}) \tilde{p} + 2 f_3 \tilde{p}^2 \right) \right)
\]
The determining equations obtained by equating the coefficients of the third-order terms to zero in the commutativity equation are

\begin{align}
  f_5 &= 0, \\
  2rf_1 + f_3 + r^2f_7 &= 0, \\
  2rf_1 + f_3 + r^2 f_6 &= 0, \\
  hf_{12} + 2(f_0 + f_3)V_1 &= 0, \\
  2f_3V_1 + 2r^2f_7V_1 - (f_1 + f_{12} + r^2 f_{14})(\hbar - 2r^2 V_1) - hr(f_{15} + f_6) &= 0, \\
  2rf_{14}V_1 + hf_{14} &= 0, \\
  hf_{12} + r(3hf_{14} + 2(f_{12} + f_{15} + f_7)V_1) &= 0, \\
  hf_6 + r(hf_{12} + hf_{15} + 2f_3V_1 - hr f_6) &= 0, \\
  hf_{12} + 2(f_0 + f_3)V_1 + r(hr f_{14} + 2rf_1V_1 + hf_6) &= 0, \\
  2rf_{15} + r^2f_{14} + 2f_7 V_1 - hf_{15} &= 0.
\end{align}

Equation (6.4) can be integrated to give

\begin{equation}
  f_5 = -r^2 f_1 - c_1,
\end{equation}

where $c_1$ is a real constant. Introducing (6.13) into equation (6.5) and integrating, we obtain

\begin{equation}
  f_7 = f_6 + c_2,
\end{equation}

where $c_2$ is an integration constant. We also solve (6.6) for $f_{12}

\begin{equation}
  hf_{12} = 2(c_1 - f_0 + c_1 r^2 + f_0 r^2)V_1.
\end{equation}

With the obtained knowledge in equations (6.13)–(6.15), we solve algebraically for derivatives of the functions. Equations (6.4)–(6.12) are then satisfied for arbitrary values of $V_1(r)$.

Satisfying the determining equations, obtained by equating the coefficients of the third-order terms to zero in the commutativity equation, for arbitrary values of $V_1(r)$ we continue with the determining equations, obtained by equating the coefficients of the second-order terms to zero in the commutativity equation. We note that four of the second-order determining equations involve only the functions $f_4$, $f_8$, $f_{11}$ and $f_{13}:

\begin{align}
  f_4 &= 0, \\
  (f_{11} - f_5)V_1 &= 0, \\
  h(f_{11} + f_4 + f_8) - 2r^2 V_1 f_{11} &= 0, \\
  hf_{11}' + hr f_{13} + 2rV_1 (f_{11} + f_4) &= 0.
\end{align}

They immediately imply that

\begin{equation}
  f_4 = 0, \\
  f_8 = 0, \\
  f_{11} = 0, \\
  f_{13} = 0,
\end{equation}

which greatly simplifies the rest of the analysis.
The complete set of determining equations is too long to present here. Instead, we present a nonlinear equation for $V_1$ which is obtained as a compatibility condition for $f_0$ in the analysis. Different solutions of this condition will form the different cases to be analyzed separately in which the full set of determining equations are fully investigated. The compatibility condition reads as

$$c_1 \frac{\Gamma_1 \Gamma_2}{\Gamma_3} = 0,$$

where

$$\Gamma_1 = -20\hbar^2 r^2 V_1^3 - 60\hbar^2 r^2 V_1^4 + 16r^2 V_1^6 - 5\hbar^4 r V_1^2 - 3h^4 V_1 (V'_1 - rV''_1),$$

$$\Gamma_2 = 3\hbar^3 V'_1 + r \left( -3\hbar^3 V'_1 (h - r^2 V_1) + h^2 r (V_1 V'_1 (-h^2 63 + 20r^2 V_1 (3h - r^2 V_1)) - 5\hbar^4 V_1^2 \right. + 4r V_1^3 (-30h^3 + 4r^2 V_1 (6h - r^2 V_1))),$$

$$\Gamma_3 = -3h^4 r^2 V''_1 (2V_1 (3h^2 - 6hr^2 V_1 + 4r^2 V_1^2) + h^2 r V'_1) + 6r \left( h^2 V_1 V'_1 (3h^4 + 4r^2 V_1 (-9h^3 + 2r^2 V_1 (3h^2 + r^2 V_1 (5h^2 - 2r^2 V_1)))) + 2r V_1^4 (-15h^4 + 8r^2 V_1^2 (9h^2 - 2r^2 V_1)) \right. - 2r V_1^2 (4h^2 - r^2 V_1))) + h^2 r V'_1^2 (3h^2 - 20r^2 V_1 (h - r^2 V_1))).$$

The compatibility condition (6.18) is satisfied if $c_1 = 0$ or $\Gamma_1 = 0$ or $\Gamma_2 = 0$. For $\Gamma_1 = 0$ or $\Gamma_2 = 0$, a standard symmetry analysis is performed in a similar fashion as for equation (4.21) and we find invariant solutions for $V_1$.

The symmetry algebra for $\Gamma_1 = 0$ is spanned by two vector fields:

$$\vec{v}_1 = r \partial_r - 2V_1 \partial_1, \quad \vec{v}_2 = r^3 \partial_r - 3r^2 V_1 \partial_1,$$

with $[\vec{v}_1, \vec{v}_2] = 2\vec{v}_2$. The invariant spin–orbit potentials are, respectively,

$$V_1 = \frac{C_1}{r^2}, \quad \text{and} \quad V_1 = \frac{C_2}{r^2},$$

where $C_1$ and $C_2$ are integration constants and only for the following values:

$$C_1 = \left\{ -h, -\frac{h}{2}, 0, \frac{h}{2}, h \right\}, \quad \text{and} \quad C_2 = \left\{ 0 \right\}.$$

(6.22)

These two invariants are also solutions of $\Gamma_1 = 0$. Thus, these values of the constants $C_1$ and $C_2$ give us the special solutions. The group transformations generated by $\vec{v}_2$ are

$$\vec{r} = \frac{r}{\sqrt{1 - 2\lambda r^2}}, \quad \vec{V}_1 = V_1 (1 - 2\lambda r^2)^{\frac{1}{2}}, \quad \left| \lambda \right| < \frac{1}{2r^2}.$$

(6.23)

Upon introducing the value of $V_1$ in the equation (6.23), $\vec{V}_1$ becomes

$$\vec{V}_1 = \frac{C_1}{r^2} \left( \frac{1}{1 + 2\lambda r^2} \right)^{\frac{1}{2}}.$$

(6.24)

New special solutions of $\Gamma_1 = 0$ can be obtained from its known solutions, say $V_1 = \frac{C_1}{r^2}$ with the constant $C_1$ taking values from the set given in (6.22), since $\vec{V}_1 (\vec{r})$ is also a solution if $V_1 (r)$ is. Hence, we have the following solutions:

$$C_1 = -h \quad \Rightarrow \quad \vec{V}_1 = \frac{h}{r^2} \left( \frac{1}{1 + 2\lambda r^2} \right)^{\frac{1}{2}},$$

(6.25)

$$C_1 = -\frac{h}{2} \quad \Rightarrow \quad \vec{V}_1 = -\frac{h}{2r^2} \left( \frac{1}{1 + 2\lambda r^2} \right)^{\frac{1}{2}},$$

(6.26)

$$C_1 = 0 \quad \Rightarrow \quad \vec{V}_1 = 0.$$

(6.27)
\[ C_1 = \frac{\hbar}{2} \quad \Rightarrow \quad \tilde{V}_1 = \frac{\hbar}{2\sqrt{2}} \left( 1 + \frac{1}{1 + 2\lambda r^2} \right)^{\frac{1}{2}}, \quad (6.28) \]

\[ C_1 = \hbar \quad \Rightarrow \quad \tilde{V}_1 = \frac{\hbar}{r} \left( 1 + \frac{1}{1 + 2\lambda r^2} \right)^{\frac{1}{2}}. \quad (6.29) \]

Together with the four solutions obtained from \( V_1 = \frac{\tilde{C}}{r^2} \) with the constant \( C_1 \) taking values from the set given in (6.22), we have eight different type of potentials \( V_1 \) for \( \Gamma_1 = 0 \).

The symmetry algebra for \( \Gamma_2 = 0 \) is spanned by two vector fields:
\[ \vec{V}_1 = r \partial_r - 2V_1 \partial_{\lambda}, \quad \vec{V}_2 = r^2 \partial_r + (\hbar - 3\lambda^2 V_1) \partial_{\lambda}, \quad (6.30) \]

with \([\vec{V}_1, \vec{V}_2] = 2\vec{v}_2\). The invariants are
\[ V_1 = \frac{C_1}{r^2} \quad \text{and} \quad V_1 = \frac{C_2}{r^2} + \frac{\hbar}{r}, \quad (6.31) \]

where \( C_1 \) and \( C_2 \) are integration constants and only for the following values:
\[ C_1 = \left\{ 0, \frac{\hbar}{2}, \frac{3\hbar}{2}, 2\hbar \right\}, \quad \text{and} \quad C_2 = \{ 0 \}. \quad (6.32) \]

these two invariants are also solutions of \( \Gamma_2 = 0 \). Thus, these values of the constants \( C_1 \) and \( C_2 \) give us the special solutions. The group transformations induced by \( \vec{v}_2 \) are
\[ \tilde{r} = \frac{r}{\sqrt{1 - 2\lambda r^2}}, \quad \tilde{V}_1 = \frac{1 - 2\lambda r^2}{r^2} - (\hbar + (\hbar - r^2)\sqrt{1 - 2\lambda r^2}), \quad |\lambda| < \frac{1}{2r^2}. \quad (6.33) \]

Upon introducing the value of \( V_1 \) in the equation (6.33), \( \tilde{V}_1 \) becomes
\[ \tilde{V}_1 = \frac{1}{r^2} \left( \hbar + (\hbar - C_1)\sqrt{1 - 2\lambda r^2} \right). \quad (6.34) \]

New special solutions of \( \Gamma_2 = 0 \) can be obtained from its known solutions, say \( V_1 = \frac{\tilde{C}}{r^2} \) with the constant \( C_1 \) taking values from the set given in (6.32), since \( \tilde{V}_1(\tilde{r}) \) is also a solution if \( V_1(r) \) is. Hence, we have the following solutions:
\[ C_1 = 0 \quad \Rightarrow \quad \tilde{V}_1 = \frac{\hbar}{r^2} \left( 1 + \frac{1}{\sqrt{1 + 2\lambda r^2}} \right), \quad (6.35) \]
\[ C_1 = \frac{\hbar}{2} \quad \Rightarrow \quad \tilde{V}_1 = \frac{\hbar}{2r^2} \left( 2 + \frac{1}{\sqrt{1 + 2\lambda r^2}} \right), \quad (6.36) \]
\[ C_1 = \hbar \quad \Rightarrow \quad \tilde{V}_1 = \frac{\hbar}{r^2}, \quad (6.37) \]
\[ C_1 = \frac{3\hbar}{2} \quad \Rightarrow \quad \tilde{V}_1 = \frac{\hbar}{2r^2} \left( 2 - \frac{1}{\sqrt{1 + 2\lambda r^2}} \right), \quad (6.38) \]
\[ C_1 = 2\hbar \quad \Rightarrow \quad \tilde{V}_1 = \frac{\hbar}{r^2} \left( 1 - \frac{1}{\sqrt{1 + 2\lambda r^2}} \right). \quad (6.39) \]

Together with the two new solutions of the form \( V_1 = \frac{\tilde{C}}{r^2} \), we have six different type of potentials \( V_1 \) for this case.

Finally, if we consider the simultaneous solutions of \( \Gamma_1 = 0 \) and \( \Gamma_2 = 0 \) by substituting \( V_1'' \) from \( \Gamma_1 = 0 \) in \( \Gamma_2 = 0 \), we obtain
\[ 6\hbar r V_1'' - 4r^3 V_1' + \hbar^2 V_1 = 0. \quad (6.40) \]

Solving this equation provides us with two new potentials
\[ V_1 = \frac{\hbar}{2r^2} \left( 1 + \frac{\epsilon}{\sqrt{1 + \beta r^2}} \right), \quad \epsilon^2 = 1, \quad (6.41) \]

where \( \beta \) is a constant of integration.
Although we have 16 different type of potentials which satisfy the compatibility condition (6.18), we note that for the following potentials:

\[ V_1 = \frac{\epsilon h}{2r^2}, \quad V_1 = \frac{3\epsilon h}{2r^2}, \quad V_1 = \frac{\epsilon h}{2r^2\sqrt{1 + 2\lambda r^2}}, \]

\[ V_1 = \frac{h}{2r^2\left(2 + \frac{\epsilon}{\sqrt{1 + 2\lambda r^2}}\right)}, \quad V_1 = \frac{h}{2r^2\left(1 + \frac{\epsilon}{\sqrt{1 + \beta r^2}}\right)}. \] (6.42)

\( \Gamma_3 \) becomes identically zero. Hence, we do not consider these potentials in the rest of the analysis. Indeed, if we consider them and start the investigation of the determining equations from the very beginning the analysis gives \( c_1 = 0 \). This possibility will be analyzed separately.

Thus, excluding the potentials given in (6.42) as well as the potential \( V_1 = \frac{h}{2} \), which is a gauge induced one and was completely investigated in [1], we now investigate the following five cases separately.

**Case I.** \( c_1 = 0 \)

For \( c_1 = 0 \), the third-order determining equations (6.4)–(6.12) are satisfied by

\[ f_7 = 0, \quad f_8 = 0, \quad f_{12} = 0, \quad f_{14} = 0, \quad f_{15} = 0, \] (6.43)

together with equations (6.13) and (6.14). Then the determining equations, obtained from lower order terms, provide us with

\[ f_1 = c_3, \quad f_2 = c_4, \quad f_{10} = 0, \quad c_4 = 2c_3 - \frac{c_2^2}{2}, \] (6.44)

where \( c_3 \) and \( c_4 \) are integration constants. For these values of \( f_j \), all the determining equations, obtained from the requirement that the commutator \([H, \vec{X}_A] = 0\), are satisfied for any \( V_0 = V_0(r) \) and \( V_1 = V_1(r) \). Since we have two arbitrary constants and none of them are appeared in the Hamiltonian, we have the following integrals of motion:

\[ \vec{X}_A^1 = i\hbar \left(\vec{\sigma}, \vec{x}\right) \vec{p} + \vec{x}(\vec{\sigma}, \vec{p}) - \hbar \vec{L} - 2(\vec{\sigma}, \vec{L})\vec{L} - 2i\hbar \vec{\sigma}(\vec{x}, \vec{p}) = \{\vec{\sigma}, \vec{L}\}, \vec{J}, \] (6.45)

\[ \vec{X}_A^2 = 4\vec{L} + 2\hbar \vec{\sigma} = 4\vec{J}. \] (6.46)

They are however both trivial since \( \vec{J} \) and \( \{\vec{\sigma}, \vec{L}\} \) are in the set (1.5). From here on, we shall list only nontrivial integrals.

**Case II.** \( V_1 = -\frac{h^2}{2} \)

For this type of potential, the set of determining equations given in (6.4)–(6.12), is satisfied by

\[ f_6 = -c_2, \quad f_8 = 2c_1, \quad f_{14} = 0, \quad f_{15} = -\frac{4c_1}{r^2}, \] (6.47)

together with the equations given in (6.13)–(6.15). Here, \( c_1 \) and \( c_2 \) are integration constants. If we introduce these integrals of motion into the set of all determining equations, then we obtain

\[ f_2 = c_3, \quad f_{10} = 0, \quad f_1 = \frac{c_2}{4} + \frac{c_3}{2} + c_4 r^2, \] (6.48)

where \( c_3 \) and \( c_4 \) are integration constants. Upon introducing all the information obtained so far back into the determining equations, we find

\[ V_0 = \alpha r^2, \] (6.49)

with \( \alpha = -\frac{c_2}{4} \). Hence, the integral of motion \( \vec{X}_A \) depends on three constants \( c_1, c_2 \) and \( c_3 \). The nontrivial integral of motion for this case can be written:

\[ \vec{X}_A = - \left(2\alpha r^2 + \vec{p}^2\right)\vec{\sigma} + 2(\vec{\sigma}, \vec{p})\vec{p} + \frac{2}{r^2}(\vec{x}(\vec{\sigma}, \vec{x})\vec{p}^2 + 2i\hbar(\vec{\sigma}, \vec{p}) - 2(\vec{x}, \vec{p})(\vec{\sigma}, \vec{p}) + i\hbar(\vec{\sigma}, \vec{p}) - -(\vec{\sigma}, \vec{x})\vec{p}). \] (6.50)
Case III: $V_1 = \frac{2\hbar}{r^2}$

For this type of potential, the set of determining equations given in (6.4)–(6.12), is satisfied by

$$
f_6 = -4c_1 \frac{c_2}{r^2} - c_2, \quad f_9 = -2c_1, \quad f_{14} = \frac{8c_1}{r^2}, \quad f_{15} = 0, \quad (6.51)
$$

together with the equations given in (6.13)–(6.15). Here, $c_1$ and $c_2$ are integration constants.

If we introduce these integrals of motion into the set of determining equations, obtained from lower order terms, then we obtain

$$
f_2 = -4c_1 \frac{c_2}{r^2} + c_3, \quad f_{10} = -6c_1 \frac{c_3}{r^2} + c_4, \quad f_1 = \frac{c_2}{4} + \frac{c_3}{2} - \frac{c_3}{2} r^2, \quad (6.52)
$$

where $c_3$ and $c_4$ are integration constants. Upon introducing all the information obtained so far back into the determining equations, we find

$$
V_0 = \frac{3\hbar^2}{r^2} + \alpha r^2, \quad (6.53)
$$

with $\alpha = \frac{c_2}{4}$. Hence, again the integral of motion $X_A$ depends on three constants $c_1$, $c_2$ and $c_3$.

Two of the integrals of motion are trivial and the third one is

$$
X_A = \left(3\vec{p}^2 - 2\alpha r^2 + \frac{4}{r^2} (i\hbar(\vec{x}, \vec{p}) - (\vec{x}, (\vec{x}, \vec{p})\vec{p}))\right) \vec{q}
$$

$$
= \frac{2}{r^2} (h + (\vec{L}, \vec{q})) \vec{L} - 2 \left((\vec{q}, \vec{p}) - \frac{3i\hbar(\vec{q}, \vec{x})}{r^2}\right) \vec{p}
$$

$$
+ \frac{2c}{r^2} (3i\hbar r^3 (\vec{q}, \vec{p}) - (\vec{q}, \vec{x}) (3\hbar^2 - 2r^2 + 2r \vec{p}^2 + 12i\hbar(\vec{x}, \vec{p}) - 4(\vec{x}, (\vec{x}, \vec{p})\vec{p})). \quad (6.54)
$$

Case IV: $V_1 = \frac{\hbar}{r^2 \sqrt{1 + \beta^2}}$, $\beta \equiv 2\lambda, \epsilon^2 = 1$

For this type of potential, the set of determining equations given in (6.4)–(6.12), is satisfied by

$$
f_6 = -c_2 - \frac{2c_1}{r^2} \left(1 + \frac{\beta}{2} r^2 + \epsilon \sqrt{1 + \beta r^2}\right), \quad f_9 = -2\epsilon c_1 \sqrt{1 + \beta r^2}, \quad f_{14} = 0, \quad f_{15} = \frac{4\epsilon c_1 \sqrt{1 + \beta r^2}}{r^2}, \quad \quad (6.55)
$$

together with the equations given in (6.13)–(6.15). Here, $c_1$ and $c_2$ are integration constants.

If we introduce these integrals of motion into the set of determining equations, obtained from lower order terms, then we obtain

$$
f_1 = \frac{c_1}{4} + \frac{c_3}{2} + \frac{c_1}{4} r^2 \left(1 + \frac{3}{1 + \beta r^2} - \frac{8 (1 + \alpha r^2)}{1 + \alpha r^2} - \frac{4\epsilon}{\sqrt{1 + \beta r^2}}\right),
$$

$$
f_2 = c_3 - \frac{2c_1}{r^2} \left(1 + 2\alpha r^2 + \frac{\epsilon}{\sqrt{1 + \beta r^2}}\right), \quad f_{10} = 0, \quad (6.56)
$$

where $c_3$ and $\alpha$ are integration constants. Upon introducing all the information obtained so far back into the determining equations, we find

$$
V_0 = \frac{\hbar^2}{8r^2 \left(1 + \beta r^2\right)^2} \left(4 + 6\beta r^2 - r^4 \beta^2 + 4\epsilon \left(1 + \beta r^2\right)^2\right) + \frac{\alpha}{1 + \beta r^2}. \quad (6.57)
$$

Hence, again the integral of motion $X_A$ depends on three constants $c_1$, $c_2$ and $c_3$. The only nontrivial integral of motion is
by

\[ F_0 = \frac{3h^2 (4 + 5\beta r^2 + 4\alpha (1 + \beta r^2)^2)}{8r^2 (1 + \beta r^2)^2} - \frac{\alpha}{2\beta (1 + \beta r^2)}. \]

Hence, again the integral of motion \( \tilde{X}_A \) depends on three constants \( c_1, c_2 \) and \( c_3 \). The only nontrivial integral of motion is

\[
\tilde{X}_A = \left( \frac{2\hbar^2 \langle \vec{\sigma}, \vec{\xi} \rangle}{r^2} Q_+ - 4e \sqrt{1 + \beta r^2} \langle \vec{\sigma}, \vec{p} \rangle \right) \vec{p} - \frac{4}{r^2} \left( \hbar + \frac{\epsilon \hbar}{\sqrt{1 + \beta r^2}} + q(\vec{\sigma}, \vec{L}) \right) \vec{L} \\
+ \frac{2\vec{\sigma}}{r^2} \left( 2\hbar q(\vec{x}, \vec{p}) \vec{Y} - 2q(\vec{x}, (\vec{x}, \vec{p}) \vec{p} + r^2 (2q - 1) \vec{p}^2 \right) + \frac{2\vec{\xi}}{r^2} ((\vec{\sigma}, \vec{x})(4q(\vec{x}, (\vec{x}, \vec{p}) \vec{p})) - 2r^2 (2q - 1 - \epsilon \sqrt{1 + \beta r^2}) \vec{p}^2 - Z - 4 \hbar W(\vec{x}, \vec{p}) + i \hbar \tilde{Q}(\vec{\sigma}, \vec{p})). \]

(6.64)
where \( q \) is given in (6.59) and \( \tilde{Q}, \tilde{Y}, Z \) and \( W \) are given by the following relations:

\[
\tilde{Q} = 3 + \frac{5\beta}{2} r^2 + \epsilon \frac{3 + 4\beta r^2}{\sqrt{1 + \beta r^2}}, \quad W = 3 + 2\beta r^2 + \epsilon \frac{6 + 7\beta r^2}{2\sqrt{1 + \beta r^2}},
\]

\[
\tilde{Y} = -\frac{h^2 r^4(4\alpha - 6\beta^2) + h^2 \epsilon (3\beta^3 + 4\beta \alpha) - 6h^2 \beta r^2}{4(1 + \beta r^2)^2} + \frac{2\epsilon h^2 \beta r^2}{\sqrt{1 + \beta r^2}},
\]

\[
Z = -\frac{4r^4\alpha(1 + \beta r^2) + 3h^2 (2 + \beta r^2(7 + 6\beta r^2))}{2(1 + \beta r^2)^2} + \epsilon \frac{3h^2 (1 + 2\beta r^2)}{\sqrt{1 + \beta r^2}}.
\]

### 7. Conclusions

The main results of this paper are summed up in table 1. We list all potentials \( V_0(r) \) and \( V_1(r) \) that have at least one integral of motion in addition to the ‘trivial’ ones, i.e. those that exist for all \( V_0(r) \) and \( V_1(r) \) (see equation (1.5)). We have included the obvious integrals obtained by multiplying a lower order integral by the trivial ones.

The results are ordered according to the value of \( V_1(r) \) (in column 2). In column 3, \( V_0(r) \) denotes an arbitrary function. Throughout, \( \alpha \) and \( \beta \) are real arbitrary constants, and we have \( \epsilon = \pm 1 \). The nontrivial pseudoscalar, vector and axial vector integrals are listed in columns 4, 5 and 6. We have included integrals of order 0 and 1 (in the momenta) already found in \([1]\). Indeed, the potentials in row 1 are first order superintegrable.

We mention that the nonlinear ordinary differential equations (4.21) and \( \Gamma_1 = 0, \Gamma_2 = 0 \) with \( \Gamma_1 \) and \( \Gamma_2 \) defined in (6.19), are compatibility conditions for the existence of pseudoscalar and axial vector integrals, respectively. We have obtained the general solutions of these equations in implicit form. Since a potential must be explicit in order to be useful, we only used the particular explicit solutions obtained by requiring that they be invariant under a subgroup of the symmetry groups of these equations. Whenever possible, we enlarged the

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**Table 1. Superintegrable potentials and their integrals of motion.**

| No | \( \frac{\hbar}{\sqrt{7}} \) | \( V_0(r) \) | Pseudoscalars | Vectors | Axial Vectors |
|----|-------------------------------|----------------|---------------|---------|--------------|
| 1  | \( \frac{\hbar}{\sqrt{7}} \) | \( V_0(r) \) | – | – | (1.7) |
| 2  | \( \frac{\hbar}{\sqrt{7}} \) | \( V_0(r) \) | (4.31), (4.32) | (5.13), (5.14) | – |
| 3  | \( \frac{\hbar}{\sqrt{7}} \) | \( \beta \alpha - \frac{\hbar}{\sqrt{7}} \) | (4.31), (4.32) | (5.11)–(5.14) | – |
| 4  | \( \frac{\hbar}{\sqrt{7}} \) | \( \alpha r^2 \) | (4.42) | – | – |
| 5  | \( \frac{\hbar}{\sqrt{7}} \) | \( \alpha r^2 \) | – | – | (6.50) |
| 6  | \( \frac{\hbar}{\sqrt{7}} \) | \( \alpha r^2 \) | – | – | (6.54) |
| 7  | \( \frac{\hbar}{\sqrt{7}} \) | \( \sqrt{1 + \beta r^2} \) | (4.51), (4.52) | (5.33) | – |
| 8  | \( \frac{\hbar}{\sqrt{7}} \) | \( \sqrt{1 + \beta r^2} \) | (4.54) | (4.55) | – |
| 9  | \( \frac{\hbar}{\sqrt{7}} \) | \( \sqrt{1 + \beta r^2} \) | (6.57) | – | – |
| 10 | \( \frac{\hbar}{\sqrt{7}} \) | \( \sqrt{1 + \beta r^2} \) | (6.63) | – | – | (6.64) |
class of solutions by acting on the subgroup invariant solution with the entire symmetry group of the auxiliary equations.

We consider the most interesting cases to be those with vector or axial vector integrals, specially those that can be viewed as a Coulomb atom or harmonic oscillator with a spin–orbital interaction.

A study of the algebras of integrals of motion and of the corresponding solutions of the Pauli–Schrödinger equation is in progress, as is a systematic search for integrals that are two index tensors or pseudotensors.

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