K3 surfaces with a symplectic automorphism of order 4

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Abstract
Given $X$, a K3 surface admitting a symplectic automorphism $\tau$ of order 4, we describe the isometry $\tau^*$ on $H^2(X, \mathbb{Z})$. Having called $\tilde{Z}$ and $\tilde{Y}$, respectively, the minimal resolutions of the quotient surfaces $Z = X / \tau^2$ and $Y = X / \tau$, we also describe the maps induced in cohomology by the rational quotient maps $X \rightarrow \tilde{Z}$, $X \rightarrow \tilde{Y}$ and $\tilde{Y} \rightarrow \tilde{Z}$: With this knowledge, we are able to give a lattice-theoretic characterization of $\tilde{Z}$, and find the relation between the Néron–Severi lattices of $X$, $Z$ and $Y$ in the projective case. We also produce three different projective models for $X$, $\tilde{Z}$ and $\tilde{Y}$, each associated to a different polarization of degree 4 on $X$.

Keywords
K3 surfaces, moduli spaces of projective K3 surfaces, symplectic automorphisms

INTRODUCTION

An automorphism $\alpha$ of a K3 surface $X$ is symplectic if it preserves its volume form: Therefore, the surface $\tilde{Y}$ that is the minimal resolution of the singularities of the quotient $Y = X / \alpha$ is again a K3 surface. Nikulin characterized K3 surfaces $X$ that admit a symplectic automorphism of order $n$ by the existence of a primitive embedding of a certain lattice $\Omega_n$ in $NS(X)$, and K3 surfaces $\tilde{Y}$ by the existence of a primitive embedding of another lattice $M_n$ in $NS(\tilde{Y})$ [13]. The first explicit description of the map that relates the lattices $H^2(X, \mathbb{Z})$ and $H^2(\tilde{Y}, \mathbb{Z})$ was given by Morrison [12] for a symplectic involution, and subsequent works by van Geemen, Garbagnati and Sarti [5, 7] produced a complete description of the correspondence between families of projective K3 surfaces that admit a symplectic involution, and those that arise as desingularization of their quotient. A similar lattice-theoretic approach was used by Garbagnati and Prieto [3] for symplectic automorphisms of order 3, and will be used in this work too. Indeed, the moduli space of K3 surfaces admitting a symplectic automorphism of order $n$ and a polarization of degree $2d$ splits in irreducible components, each determined by the Néron–Severi lattice of its general member $X$: This is an overlattice of finite index of $\Omega_n \oplus \langle 2d \rangle$, so for each $d$ there are a finite number of choices for $NS(X)$.

The aim of this paper is to study K3 surfaces $X$ with a symplectic automorphism of order 4 $\tau$. As one can expect, the order being nonprime presents new challenges, but also allows us to analyze surfaces not encountered before: Those that are minimal resolution of an intermediate quotient, as is in our case the K3 surface $\tilde{Z}$, the minimal resolution of $X / \tau^2$. The main results of this paper are the lattice-theoretic characterization of $\tilde{Z}$, and the comparison between its moduli space and those of $X$, $\tilde{Y}$:

Theorem 1 (see Theorem 4.5.1). A K3 surface $\tilde{Z}$ is the minimal resolution of $X / \tau^2$, for some K3 surface $X$ with a symplectic automorphism $\tau$ of order 4, if and only if $\tilde{Z}$ is $\Gamma$-polarized, where the lattice $\Gamma$ is defined in Definition 4.2.1.
**Theorem 2** (see Theorems 5.3.3, 5.4.3). *Let $X$ be a general projective K3 surface with a symplectic automorphism $\tau$ of order 4, and let $\tilde{Z}$ and $\tilde{Y}$ be, respectively, the minimal resolution of $X/\tau^2$ and $X/\tau$. Then, using the notation introduced for overlattices in Remark 5.6, we have the following correspondence between $NS(X)$, $NS(\tilde{Z})$ and $NS(\tilde{Y})$ depending on the value of $d$ modulo 4. For $d \equiv 2 \pmod{4}$, the two possible $NS(X)$ are not isomorphic, and the same holds for $NS(\tilde{Y})$.*

| $d \equiv 2 \pmod{4}$ | $NS(X)$ | $NS(\tilde{Z})$ | $NS(\tilde{Y})$ |
|------------------------|---------|----------------|----------------|
| $\Omega_4 \oplus (2d)^*$ | $(\Gamma \oplus (4d))^*$ | $(M \oplus (2d))^*$ |
| $\Omega_4 \oplus (2d)^{(1)}$ | $(\Gamma \oplus (4d))^{(1)}$ | $(M \oplus (2d))^{(1)}$ |
| $\Omega_4 \oplus (2d)^{(2)}$ | $(\Gamma \oplus (4d))^{(2)}$ | $(M \oplus (2d))^{(2)}$ |

Notice that there is a 1:1 correspondence between families of projective surfaces $X$ and $\tilde{Y}$, as it happens for automorphisms of order 2 and 3; however, when $d$ is even, two different families of $X$ (or $\tilde{Y}$) can correspond to the same family of $\tilde{Z}$. This only happens for projective surfaces, as in the nonprojective case the moduli spaces of $X$, $\tilde{Y}$, $\tilde{Z}$ are all irreducible.

The correspondence above is given by the maps induced in cohomology by the quotient maps from $X$ to $X/\tau^2$, $X/\tau$. To define these maps, one needs first to understand the isometry $\tau^*$ induced by $\tau$ on $H^2(X, \mathbb{Z})$. By [13, Theorem 4.7], this depends neither on $X$, nor on $\tau$, but only on its order: Therefore, we can use as a starting point a projective K3 surface on which $\tau^*$ is easy to describe. Our choice is the surface $X_4$ (see [20]), which has Picard rank 20 and a Jacobian fibration with Mordell–Weil group $\mathbb{Z}/4\mathbb{Z}$ [15]: This fibration provides us with a presentation of $\tau^*$ as permutation action on a sublattice $W \subset H^2(X, \mathbb{Z})$ of finite index, which can be then extended to the whole lattice.

Projective K3 surfaces appear again at the conclusion of our exposition. From our lattice-theoretic characterization, we know that the moduli space of K3 surfaces $X$ with a symplectic automorphism of order 4 and a polarization $L$ of degree 4 splits in three irreducible components: For each of them, we give equations for the general member and we then construct projective models of the quotient surfaces $Z$, $\tilde{Y}$; for these, the minimal degree of the polarization depends on the choice of $L$, accordingly to Theorem 2.

This paper is structured as follows: After an exposition of the necessary results of lattice theory in Section 1, in Section 2 we describe a Jacobian fibration $X_4 \to \mathbb{P}^1$ with a symplectic automorphism $\tau$ of order 4 acting on the fibers. In Section 3, we define the isometry $\tau^*$ on the lattice $H^2(X, \mathbb{Z})$: By [13, Theorem 4.7], this is a general result, valid for any K3 surface $X$ with a symplectic automorphism of order 4. Section 4 is then devoted to the description of the maps induced in cohomology by the rational maps, where $\tilde{Z}$ and $\tilde{Y}$ are the minimal resolutions of the quotient surfaces $Z = X/\tau^2$ and $\tilde{Y} = X/\tau$, respectively. In Section 5, we determine the families of projective K3 surfaces $X$ with an action of $\tau$; the correspondence between families of $X, \tilde{Z}$ and $\tilde{Y}$ is then given via the maps introduced in Section 4. In Section 6, we describe the action induced by $\tau$ on the projective space $\mathbb{P}(H^0(X, L)^*)$ for each of the families of $X$, concluding with projective models of $X$ and its quotients for $L^2 = 4$.

## 1 | LATTICES

In this section, we are going to recall some fundamental results on lattices and discriminant forms; most of them are due to Nikulin, and are exposed in [14, section 1].

**Definition 1.1.** An even lattice is a free $\mathbb{Z}$-module $S$ of finite rank, equipped with a nondegenerate quadratic form $q : S \to 2\mathbb{Z}$. Working in characteristic different than two, this is equivalent to giving an integral nondegenerate bilinear symmetric even form $b : S \times S \to \mathbb{Z}$; we will refer to $b$ as intersection form of $S$.

An isomorphism between lattices (or isometry) is an isomorphism of $\mathbb{Z}$-modules that preserves the intersection form. Denote $O(S)$ the group of isometries of $S$ into itself.

**Definition 1.2.** Define the K3 lattice

$$\Lambda_{K3} \cong E_8^{\oplus 2} \oplus U^{\oplus 3},$$

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where $E_8$ is the unique even negative definite unimodular lattice of rank 8, and $U$ is the unique even indefinite unimodular lattice of rank 2; the K3 lattice is isometric to the second integral cohomology group $H^2(X, \mathbb{Z})$ equipped with the cup product $H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to H^4(X, \mathbb{Z}) \cong \mathbb{Z}$, for any K3 surface $X$.

**Definition 1.3.** Let $S$ be an even lattice: Define the dual lattice $S^* = \{ x \in S \otimes \mathbb{Q} \mid \forall s \in S, b_\mathbb{Q}(x, s) \in \mathbb{Z} \}$ where $b_\mathbb{Q}$ denotes the $\mathbb{Q}$-linear extension of $b$.

Denote the discriminant group of $S$ by $A_S := S^*/S$, where $S \hookrightarrow S^*$ via $s \mapsto b(s, -)$; its cardinality is an invariant of $S$, called the discriminant of $S$ and denoted by $d(S)$. Another invariant of $S$ is the length $\ell(S)$, that is defined as the minimum number of generators of $A_S$; we are going to write

$$\lambda(A_S) = (n_1, n_2, ..., n_k)$$

if the first of the generators in a set that satisfies the minimum has order $n_1$, the second $n_2$, and so on, with $n_1 \leq n_2 \leq ... \leq n_k$.

Define the discriminant (quadratic) form $q_S : A_S \to \mathbb{Q}/2\mathbb{Z}$, induced on $A_S$ by the quadratic form $q$ of $S$. A subgroup $H \subset A_S$ is said to be isotropic if it is annihilated by the discriminant form $q_S$.

### 1.1 Overlattices and primitive embeddings

**Definition 1.1.1.** An embedding of (even) lattices $(S, q) \hookrightarrow (M, \bar{q})$ is an injective homomorphism of $\mathbb{Z}$-modules such that $\bar{q}(\bar{s}(s)) = q(s)$ for all $s \in S$. In this case, we say that $M$ is an overlattice of $S$. An embedding is primitive if $M/\ell(S)$ is free; an overlattice is of index $n$ if $M/\ell(S)$ is a (abelian) group of order $n$, and it is a cyclic overlattice if $M/\ell(S)$ is cyclic.

**Remark 1.1.2.** Let $M$ be an overlattice of index $n$ of $S$. The discriminant of $M$ is related to that of $S$ by

$$d(S)/d(M) = n^2.$$  

**Theorem 1.1.3** [14, Proposition 1.4.1.a]. Let $S$ be an even lattice, let $M$ be an overlattice of finite index of $S$, and define $H_M = M/S$. The correspondence $M \to H_M$ determines a bijection between overlattices of finite index of $S$ and isotropic subgroups of $A_S$.

**Definition 1.1.4.** Two embeddings $S \hookrightarrow M$, $S \hookrightarrow M'$ are isomorphic if there is an isometry between $M$ and $M'$ that restricted to $S$ is the identity.

Two overlattices of finite index of $S$, $Q$ and $Q'$ are isomorphic if there is an isometry $\alpha$ of $S$ into itself that extends $\mathbb{Q}$-linearly to an isometry $\overline{\alpha}$ between $Q$ and $Q'$.

**Theorem 1.1.5** [14, Proposition 1.6.1]. A primitive embedding of an even lattice $S$ into an even unimodular lattice $L$, in which the orthogonal complement of $S$ is isomorphic to $K$, is determined by an isomorphism $\gamma : A_S \sim A_K$ for which the discriminant quadratic forms satisfy $q_K \circ \gamma = -q_S$. Two such isomorphisms $\gamma$ and $\gamma'$ determine isomorphic primitive embeddings if and only if they are conjugate via an automorphism of $K$.

**Corollary 1.1.6.** Let $L$, $S$, and $K$ be as in Theorem 1.1.5. The isomorphism classes of overlattices $Q$ of $S \oplus K$ in $L$, such that $Q/(S \oplus K)$ is cyclic, are in bijection with the orbits for the action of $O(S)$ induced on $A_S$ (equivalently on $A_K$ via $\gamma$).

**Proof.** Using the notation of the previous theorem, fix the isomorphism $\gamma : A_S \simeq A_K$; let $s \in A_S$ such that $q_S(s) = d \in \mathbb{Q}/\mathbb{Z}$, let $k = \gamma(s)$; then $q_K(k) = -d$, and the cyclic subgroup generated by $s + k$ is isotropic in $A_S \oplus A_K$, so it determines an overlattice $Q$ of finite index of $S \oplus K$ that is by construction a sublattice of $L$. Consider an isometry $\alpha \in O(S)$, denote $\overline{\alpha}$ the induced isometry on $A_S$, and call $s' = \overline{\alpha}(s)$; Then $q_S(s') = d$, and $\beta := \gamma \circ \overline{\alpha} \gamma^{-1} \in O(A_K)$, hence the subgroup generated by $s' + \beta(k)$ determines an overlattice of $S \oplus K$ isomorphic to $Q$ thanks to the previous theorem.

On the other hand, consider $Q$ such that $S \oplus K \hookrightarrow Q \hookrightarrow L$ and $Q/(S \oplus K)$ is cyclic: Then $Q/(S \oplus K)$ is generated by an isotropic element in $A_S \oplus K = A_S \oplus A_K$, that is by construction of the form $s + k$, with $q_S(s) = d = -q_K(k)$. \(\square\)
Remark 1.1.7. With \( S, K, L \) as above, consider a primitive sublattice \( H \subset K \), with \( H = \langle h \rangle \), and suppose that \( h/n \in K^* \) for some integer \( n \): Then, \( L \) contains a cyclic overlattice \( Q \) of \( S \oplus K \), corresponding to the isotropic subgroup \( \langle \gamma^{-1}(h/n) + h/n \rangle \subset A_S \). Consider now an isometry \( \psi \in \text{O}(K) \) such that \( H = \psi(H) \) is isometric to \( H = \langle \tilde{h} \rangle \): Then \( \tilde{h}/n = \psi(h)/n \) belongs to the same isometry class of \( h/n \) in \( A_K \), and \( \langle \gamma^{-1}(\tilde{h}/n) + \tilde{h}/n \rangle \) is isotropic in \( A_K \), so it defines an overlattice \( \tilde{R} \) of \( S \oplus H \) as above: The relation between \( H \) and \( \tilde{R} \) consists in the fact that \( \tilde{R} \) is isomorphic to \( R \) as a sublattice of \( L \).

**Theorem 1.1.8** [14, Proposition 1.14.1]. For an even lattice \( S \) of signature \((s_+, s_-)\) and discriminant form \( q_S \), and an even unimodular indefinite lattice \( L \) of signature \((l_+, l_-)\), all primitive embeddings of \( S \) into \( L \) are isomorphic if and only if the lattice \( T \) with signature \((l_+ - s_+, l_- - s_-)\) and discriminant form \( q_T = -q_S \) is unique and the homomorphism \( \text{O}(T) \to \text{O}(q_T) \) is surjective.

Remark 1.1.9. Using the notation of the theorem, if \( \ell(S) > rk(L) - rk(S) \), no primitive embedding of \( S \) in \( L \) exists: Indeed, if it existed, then \( T \) would satisfy \( rk(T) < \ell(T) \), which is impossible.

Instead of proving directly that the conditions of Theorem 1.1.8 hold, we can rely on some sufficient conditions that are based only on the invariants of \( S, L, \) and \( T \):

**Proposition 1.1.10** [1, Corollary VIII.7.8]. Let \( T \) be an indefinite lattice such that \( rk(T) \geq 3 \). Write \( A_T = \mathbb{Z}/d_1\mathbb{Z} \oplus ... \oplus \mathbb{Z}/d_r\mathbb{Z} \) with \( d_i \geq 1 \) and \( d_i | d_{i+1} \). Suppose that one of the following holds:

1. \( d_1 = d_2 = 2 \);
2. \( d_1 = 2, d_2 = 4, \) and \( d_3 \equiv 4 \mod 8 \);
3. \( d_1 = d_2 = 2 \).

Then, \( T \) is uniquely determined by its signature and discriminant form, and the map \( \text{O}(T) \to \text{O}(q_T) \) is surjective.

**Theorem 1.1.11** (Nikulin, see [12, Theorem 2.8]). Let \( S \) be an even lattice of signature \((s_+, s_-)\) and discriminant form \( q_S \), and \( L \) an even unimodular lattice of signature \((l_+, l_-)\). Suppose that \( s_+ < l_+, s_- < l_- \), \( \ell(S) \leq rk(L) - rk(S) - 2 \). Then, there exists a unique primitive embedding of \( S \) into \( L \).

**Corollary 1.1.12** ([14, Remark 1.14.5]). If \( A_S \cong (\mathbb{Z}/2\mathbb{Z})^3 \oplus A' \), the conditions of the previous theorem are satisfied.

## 2 A SYMPLECTIC AUTOMORPHISM \( \tau \) OF ORDER 4 ON THE SURFACE \( X_4 \)

### 2.1 K3 surfaces and elliptic fibrations

**Definition 2.1.1.** A K3 surface \( X \) is a compact connected complex surface with trivial canonical bundle, such that \( H^{2,0}(X) \) is spanned by a nowhere degenerate (i.e., symplectic) form \( \omega_X \). An automorphism \( \alpha \in \text{Aut}(X) \) is called symplectic if \( \alpha^* \omega_X = \omega_X \).

Since by [13, Theorem 4.7] the isometry \( \alpha^* \in \text{O}(H^2(X, \mathbb{Z})) \) is determined up to isometries of the K3 lattice (see Definition 1.2) by the order of \( \alpha \), we can use a projective model of \( X \) of our choice to study \( \alpha^* \). Jacobian fibrations are useful to this end, because we can see the action of symplectic automorphisms of \( X \) on the singular fibers.

**Definition 2.1.2.** Define Jacobian fibration a fibration \( p : X \to \mathbb{P}^1 \) whose generic fiber is a genus 1 curve, and that admits a global section \( s : \mathbb{P}^1 \to X \) (it holds \( pos = id_X \)), denoted zero section. The fiber over a generic point \( F = p^{-1}(x) \) is an elliptic curve with the zero for the group law defined as \( s(x) \).

The Mordell–Weil group \( MW(p) \) of a Jacobian fibration is the group generated by all the sections, with the group law induced by that of the generic fiber.

**Remark 2.1.3.** The group \( MW(p) \) acts on \( X \) by translation on each fiber, therefore it acts as the identity on the symplectic form \( \omega_X \).
Given a Jacobian fibration, the Mordell–Weil group is linked to the Néron–Severi group of the surface by the following isomorphism [18, Theorem 6.3]:

\[ MW(p) \cong NS(X)/T(p), \tag{2.1.1} \]

where the trivial lattice \( T(p) \) is the sublattice of \( NS(X) \) generated by the generic fiber, the image of the zero section \( s = s(P^1) \), and the irreducible components of the reducible fibers which do not intersect the curve \( s \).

The Mordell–Weil group is endowed with the height pairing [18, section 11.6 et seq.], a symmetric \( \mathbb{Q} \)-valued bilinear form induced by the projection of the intersection form of \( NS(X) \) onto \( NS(X)/T(p) \). In particular, for any \( t \in MW(p) \), one gets

\[ h(t) = 2\chi(X) - 2ts - K, \]

where \( K \) depends on the intersection of \( t \) with the reducible fibers of \( p \). The height of a torsion section is 0.

### 2.2 The surface \( X_4 \)

The surface \( X_4 \) is the unique K3 surface with transcendental lattice \( T(X_4) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \): It arises as desingularization of the quotient surface \( A/\langle \sigma \rangle \), where \( A \) is the abelian surface \( E_1 \times E_2 \), \( E_1 \) is the elliptic curve of lattice \( \langle 1, i \rangle \), and \( \sigma \) is the automorphism of \( A \) defined by \( \sigma(e_1, e_2) = (ie_1, -ie_2) \); this surface is well known, see, for instance, [19, 20].

A description of all the possible Jacobian fibrations on \( X_4 \) is given by Nishiyama [15, table 1.2]: In particular, there exists a fibration

\[ \pi : X_4 \to P^1 \quad \text{s.t.} \quad MW(\pi) \cong \mathbb{Z}/4\mathbb{Z}, \]

which provides a symplectic automorphism \( \tau \) of order 4 on \( X_4 \) by means of a section \( t_1 \) that generates \( MW(\pi) \). Moreover, by (2.1.1), the curves of the trivial lattice form a \( \mathbb{Q} \)-basis for \( NS(X_4) \).

The reducible fibers of \( \pi \) are one of type \( I_4 \) and one of type \( I_{16} \) [10, table IV.3.1]. Call \( B_0 \) (respectively, \( C_0 \)) the component of \( I_4 \) (resp. \( I_{16} \)) intersected by \( s \), and number the other components so that, for every \( i \in \mathbb{Z}/4\mathbb{Z}, \ j \in \mathbb{Z}/16\mathbb{Z}, \ B_i \) intersects only \( B_{i+1} \) and \( B_{i-1} \), and \( C_j \) intersects only \( C_{j+1} \) and \( C_{j-1} \). Thus, \( T(\pi) \) is generated by the class \( F \) of the generic fiber of \( \pi \), the curve \( s \), and the components \( B_i, C_j, \ i = 1, 2, 3, \ j = 1, \ldots, 15 \) of the reducible fibers: Except for \( F \), these curves are rational, so they have self-intersection \( -2 \); \( F \) satisfies \( F^2 = 0 \). The curves \( B_1, B_2, B_3 \) span the lattice \( A_3 \) [18, section 6.5], and \( C_1, \ldots, C_{15} \) span the lattice \( A_{15} \).

Using the height pairing, we can determine the components \( B_i, C_j \) of the reducible fibers that have nontrivial intersection with a nonzero section \( t \in MW(\pi) \). It holds \( \chi(X_4) = 2 \) because \( X_4 \) is a K3 surface, and since \( t \) is a torsion section, it holds \( ts = 0 \): therefore, \( i, j \) satisfy the equation

\[ 0 = h(t) = 4 - \left( \frac{i(4 - i)}{4} + \frac{j(16 - j)}{16} \right) \quad \text{(height formula).} \]

We will choose the following notation for the elements of \( MW(\pi) \): The zero section \( s \) intersects the components \( B_0 \) and \( C_0 \); the section \( t_1 \) intersects the components \( B_2 \) and \( C_4 \); the section \( t_2 \) intersects the components \( B_0 \) and \( C_8 \); the section \( t_3 \) intersects the components \( B_2 \) and \( C_{12} \). Notice that each of \( t_1 \) and \( t_3 \) generates \( MW(\pi) \), whereas \( t_2 \) has order 2.

We can write \( t_1, t_2, t_3 \) in function of the basis of the trivial lattice \( T(\pi) \) using the information about their intersections:

\[ t_1 = 2F + s - \frac{B_1 + 2B_2 + B_3}{2} - \frac{3C_1 + 6C_2 + 9C_3 + \sum_{j=1}^{12} jC_{16-j}}{4}, \]
\[ t_2 = 2F + s - \frac{\sum_{j=1}^{7} j(C_j + C_{16-j}) + 8C_8}{2}, \]
\[ t_3 = 2F + s - \frac{B_1 + 2B_2 + B_3}{2} - \frac{\sum_{j=1}^{12} jC_j + 9C_{13} + 6C_{14} + 3C_{15}}{4}. \]

Since the discriminant group of \( NS(X_4) \) is \( (\mathbb{Z}/2\mathbb{Z})^2 \) (it is indeed the orthogonal complement to \( T(X_4) \) in \( \Lambda_{K3} \)), from (2.1.1) and the equations above, it can be readily seen that \( NS(X_4) \) admits as a \( \mathbb{Z} \)-basis \( B = \{ F, s, t_1, B_1, B_2, B_3, C_1, \ldots, C_{14} \} \).
2.3 The action of $\tau^*$ on the second cohomology of $X_4$

The symplectic automorphism $\tau$ induces an isometry $\tau^*$ on $NS(X_4)$ such that

$$F \mapsto \tau^*F \quad s \mapsto \tau^*t_1 \mapsto \tau^*t_2 \mapsto \tau^*t_3 \mapsto \tau^*s$$

$$\tau^*(C_j) = C_{j+4} \quad \tau^*(B_i) = B_{i+2},$$

where $[a]_n$ is the class of $a$ modulo $n$. Therefore, we can easily identify two copies $\langle B_1 \rangle$ and $\langle B_3 \rangle$ of $A_1$, exchanged by the action of $\tau^*$, and a set of four copies of $D_4$ on which $\tau^*$ acts as a cycle of order 4:

$$\{\langle s, C_{15}, C_0, C_1 \rangle, \langle t_1, C_3, C_4, C_5 \rangle, \langle t_2, C_7, C_8, C_9 \rangle, \langle t_3, C_{11}, C_{12}, C_{13} \rangle\}.$$ All these lattices are pairwise orthogonal, and the orthogonal complement in $NS(X)$ of the direct sum $D_4^{\oplus 4} \oplus A_1^{\oplus 2}$ is generated over $\mathbb{Q}$ by the vectors

$$R_1 = -8F - 4s + 8t_1 + 4C_1 + 8C_2 + 13C_3 + 18C_4 + 15C_5 + 12C_6 + 10C_7 + 8C_8 + 6C_9 + 4C_{10} + 3C_{11} + 2C_{12} + C_{13} + 3B_1 + 6B_2 + 3B_3,$$

$$R_2 = -4F - 2s + 2t_1 + 2C_1 + 4C_2 + 5C_3 + 6C_4 + 5C_5 + 4C_6 + 4C_7 + 4C_8 + 4C_9 + 4C_{10} + 3C_{11} + 2C_{12} + C_{13} + B_1 + 2B_2 + B_3,$$

whose intersection form satisfies $R_1^2 = 4$, $R_2^2 = -4$, $R_1R_2 = 0$. It can be also verified that $\tau^*R_1 = R_1$, while $\tau^*R_2 = -R_2$. Therefore, we have the following description:

**Proposition 2.3.1.** Consider the sublattice $D_4^{\oplus 4} \oplus A_1^{\oplus 2} \oplus \langle 4 \rangle \oplus \langle -4 \rangle$ of $NS(X_4)$ generated as above. The isometry $\tau^*$ acts on this sublattice as the cyclic permutation of order 4 on $D_4 \oplus D_4 \oplus D_4 \oplus D_4$, as the cyclic permutation of order 2 on $A_1 \oplus A_1$, as $\text{id}$ (the identity) on $\langle 4 \rangle$ and as $-\text{id}$ on $\langle -4 \rangle$.

The lattice $D_4^{\oplus 4} \oplus A_1^{\oplus 2} \oplus \langle 4 \rangle \oplus \langle -4 \rangle$ has discriminant group

$$(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z},$$

so it has index $2^6$ in $NS(X_4)$; the latter can be obtained by adding the following generators to the generators of $D_4^{\oplus 4} \oplus A_1^{\oplus 2} \oplus \langle 4 \rangle \oplus \langle -4 \rangle$:

$$R := (R_1 + R_2)/2;$$

$$a := R_1/4 + R_2/4 - (s + C_{15})/2 - (t_1 + C_5)/2 = C_0 + C_1 + C_2 + C_3 + C_4;$$

$$b := R_1/4 - R_2/4 - (t_1 + C_3)/2 - (t_2 + C_0)/2 = C_4 + C_5 + C_6 + C_7 + C_8;$$

$$c := R_1/4 + R_2/4 - (t_2 + C_7)/2 - (t_3 + C_{13})/2 = C_8 + C_9 + C_{10} + C_{11} + C_{12};$$

$$d := R_1/4 - R_2/4 - (t_3 + C_{11})/2 - (s + C_1)/2 = C_{12} + C_{13} + C_{14} + C_{15} + C_0;$$

$$e := R_1/2 - (C_3 + C_5)/2 - (C_{11} + C_{13})/2 - B_1/2 - B_3/2 = t_1 + t_3 + C_4 + C_{12} + B_2.$$

Now, $H^2(X_4, \mathbb{Z})$ is an overlattice of index $2^2$ of the lattice $NS(X_4) \oplus T(X_4)$. Since $H^2(X_4, \mathbb{Z})$ is unimodular, following Theorem 1.1.3 we have to find a series of isotropic subgroups of $A_{NS(X_4) \oplus T(X_4)}$.

Denoting $\{\omega_1, \omega_2\}$ the $\mathbb{Z}$-basis of $T(X_4)$ for which the intersection matrix is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, the elements we have to add are:

$$f = (s + t_1 + C_1 + C_3 + B_1 + \omega_1)/2,$$

$$g = (s + t_1 + C_1 + C_3 + B_3 + \omega_2)/2.$$
THE ACTION OF $\tau^*$ AND $(\tau^2)^*$ ON THE K3 LATTICE

The main result of Nikulin’s paper [13] is that there are 14 finite abelian groups which can act symplectically on K3 surfaces, and the action of each one of these groups on the K3 lattice is unique up to isometry [13, Theorem 4.7]. This result enables us to deduce the action of any symplectic automorphism of order 4 (and of its square) on the second cohomology group of any K3 surface $X$ by looking at the surface $X_4$ with the action of the automorphism $\tau$ introduced in the previous section.

3.1 A convenient description of the K3 lattice

The isometry $\tau^*$ induced on $\Lambda_{K3}$ by an automorphism $\tau$ of order 4 acts on the sublattice of finite index of $\Lambda_{K3}$ $W := D_4^4 \oplus A_1^2 \oplus \langle -4 \rangle \oplus \langle 4 \rangle \oplus \langle 2 \rangle \oplus \langle 2 \rangle$ as the cycle $(1,2,3,4)$ on the four copies of $D_4$, as $(1,2)$ on the two copies of $A_1$, as $-id$ on $\langle -4 \rangle$, and as $id$ on the remaining orthogonal components. The following diagram describes the situation:

$$W := D_4 \oplus D_4 \oplus D_4 \oplus D_4 \oplus A_1 \oplus A_1 \oplus \langle -4 \rangle \oplus \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(3.1.1)

Denote $e_1, \ldots, e_4, f_1, \ldots, f_4, g_1, \ldots, g_4, h_1, \ldots, h_4$ the generators of the four copies of $D_4$, such that $e_3 e_1 = e_3 e_2 = e_3 e_4 = 1$ and similarly for the other copies of $D_4$, and $\tau^* : e_1 \mapsto f_1 \mapsto g_1 \mapsto h_1 \mapsto e_1$ for $i = 1, \ldots, 4$; $a_1$ and $a_2$ the generators of the two copies of $A_1$; $\sigma$ the generator of $\langle -4 \rangle$, $\rho$ the generator of $\langle 4 \rangle$, $\omega_1$ and $\omega_2$ the generators of $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Then, the K3 lattice is obtained by adding to $W$ the elements (cf. (2.3.1) and (2.3.2))

$$\chi = (\rho + \sigma)/2;$$
$$\alpha = (\rho + \sigma)/4 + (e_1 + e_2 + f_1 + f_4)/2;$$
$$\beta = (\rho - \sigma)/4 + (f_1 + f_2 + g_1 + g_4)/2;$$
$$\gamma = (\rho + \sigma)/4 + (g_1 + g_2 + h_1 + h_4)/2;$$
$$\delta = (\rho - \sigma)/4 + (h_1 + h_2 + e_1 + e_4)/2;$$
$$\varepsilon = (\rho + f_2 + f_4 + h_2 + h_4 + a_1 + a_2)/2;$$
$$\zeta = (e_1 + f_1 + e_4 + f_2 + a_1 + \omega_1)/2;$$
$$\eta = (e_1 + f_1 + e_4 + f_2 + a_2 + \omega_2)/2;$$

the action of $\tau^*$ and $(\tau^2)^*$ on these elements is deduced by the one on the sublattice $W$ described above by $\mathbb{Q}$-linear extension: Notice that $\tau^* : \alpha \mapsto \beta \mapsto \gamma \mapsto \delta \mapsto \alpha$, and that $(\tau^2)^*$ fixes $\varepsilon$ and $\chi$.

3.2 Invariant and coinvariant lattices for the action of $\tau$ and $\tau^2$

If $G$ is abelian and acts symplectically on a K3 surface, by uniqueness of the action of $G$ on $\Lambda_{K3}$, we can define the invariant lattice $\Lambda_{K3}^G$ and the coinvariant lattice $\Omega_G = (\Lambda_{K3}^G) \perp \Lambda_{K3}$. Moreover, the existence of a primitive embedding of $\Omega_G$ in the Néron–Severi lattice of a K3 surface is equivalent to the existence of a symplectic action of $G$ on that surface [13, Theorem 4.15]. The lattices $\Omega_G$ are known for all abelian groups $G$ acting symplectically on a K3 surface: See [12] and [7] for $G = \mathbb{Z}/2\mathbb{Z}$, [4] for $G = \mathbb{Z}/p\mathbb{Z}$, $p = 3, 5, 7, [6]$ for the remaining cases; a list of all the invariant and coinvariant lattices for $G$ finite is provided in [8].
From now on, denote $\Lambda^{(\tau)}_{K3}$ the invariant lattice, and $\Omega_4$ the coinvariant lattice for the action on $\Lambda_{K3}$ induced by the automorphism $\tau$ of order 4. The lattice $\Omega_4$ is a negative definite lattice of rank 14 and discriminant group $(\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2$ [13, section 10].

The invariant elements for the action of $\tau^\ast$ on $W$ (see (3.1.1)) are $\kappa_i = e_i + f_i + g_i + h_i$ for $i = 1, \ldots, 4$, $\kappa_5 = a_1 + a_2$, $\kappa_6 = \rho$, $\kappa_7 = \omega_1$, $\kappa_8 = \omega_2$. These elements span a sublattice of $\Lambda_{K3}$ of index 2. To obtain $\Lambda^{(\tau)}_{K3}$, add the generators $(\kappa_2 + \kappa_4)/2$ (i.e., $\alpha + \beta + \gamma + \delta - \kappa_1 - \kappa_6$), $(\kappa_5 + \kappa_7 + \kappa_8)/2$, and $(\kappa_1 + \kappa_4 + \kappa_5 + \kappa_7 + \kappa_8)/2$ (these are easily verified as independent integer elements in $\Lambda_{K3}$).

Its orthogonal complement $\Omega_4$ is a negative definite lattice of rank 14 and discriminant group $(\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2$ [13, section 10].

The discriminant groups of $\Lambda^{(\tau)}_{K3}$ and $\Omega_4$ satisfy $A_{\Lambda^{(\tau)}_{K3}} \cong (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2 \cong A_{\Omega_4}$.

The coinvariant lattice for a symplectic involution $\Omega_2 := \Omega_{\mathbb{Z}/2\mathbb{Z}}$ is isometric to the lattice $E_8(2)$ (see [7, section 1.3] and the proof of [12, Theorem 5.7]). Considering the involution $\tau^2$, $\Omega_2$ is obviously contained in the coinvariant lattice for $\tau$, $\Omega_4$: In the basis of $\Omega_4$ described above, $\Omega_2$ is generated by the elements $\alpha - \gamma$, $\beta - \delta$, $e_1 - g_1$, $e_3 - g_3$, $f_1 - h_1$, $f_2 - h_2$, $f_3 - h_3$, $f_4 - h_4$.

Denote $R$ the orthogonal complement to $\Omega_2$ in $\Omega_4$: Then $\Omega_4$ is an overlattice of $\Omega_2 \oplus R$ such that $\Omega_4/(\Omega_2 \oplus R) = (\mathbb{Z}/2\mathbb{Z})^4$.

4 | QUOTIENTS

For each of the abelian groups $G$ that act symplectically on a K3 surface $X$, Nikulin provides in [13, sections 5–7] a description of the singular locus of the quotient surface $X/G$, and of the exceptional lattice $M_G$: This is the minimal primitive sublattice of $\Lambda_{K3}$ containing all the exceptional curves of the minimal resolution $\tilde{X}/G$ of $X/G$. Denoting $q: X \to X/G$ the quotient map, $H^2(\tilde{X}/G, \mathbb{Z})$ is an overlattice of finite index of $q_! H^2(X, \mathbb{Z}) \oplus M_G$.

Remark 4.1. The lattices $\Omega_G$ and $M_G$ are closely related via the quotient map: This fact allowed Nikulin to compute the rank and the discriminant group of $\Omega_G$ starting from the (simpler) exceptional lattice [13, Lemma 10.2]. When $G$ is noncyclic, Nikulin’s results have been corrected by Garbagnati and Sarti in [6]; for a complete account of the relation between $\Omega_G$ and $M_G$, see Whitcher’s paper [21].

Consider a K3 surface $X$ that admits a symplectic automorphism $\tau$ of order 4, and the (singular) quotient surfaces $Y = X/\tau$, $Z = X/\tau^2$; resolve the singularities of $Y$ and $Z$ to obtain the K3 surfaces $\tilde{Y}$, $\tilde{Z}$: then $\tau$ induces an involution $\tilde{\tau}$ on $Z$ such that $Z/\tilde{\tau} \cong Y$, and this involution can be extended to $\tilde{Z}$, as we are going to show in the following sections. Denote the maps between these surfaces as in the following diagram:

\[
\begin{array}{c}
X \xrightarrow{\tilde{\tau}} \tilde{X} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\tilde{Y} \xrightarrow{\tilde{\tau}} \tilde{Z} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
Y \xrightarrow{\tau} Z
\end{array}
\]
Remark 4.2. The surfaces $\tilde{Y}$ and $\tilde{Z}/\tilde{\tau}$ are isomorphic, because they are birationally equivalent K3 surfaces.

We can describe the maps

$$\pi_{4*} : \Lambda_{K3} \cong H^2(X, \mathbb{Z}) \xrightarrow{q_{4*}} q_{4*}H^2(X, \mathbb{Z}) \hookrightarrow H^2(\tilde{Y}, \mathbb{Z}) \cong \Lambda_{K3}$$

$$\pi_{2*} : \Lambda_{K3} \cong H^2(X, \mathbb{Z}) \xrightarrow{q_{2*}} q_{2*}H^2(X, \mathbb{Z}) \hookrightarrow H^2(\tilde{Z}, \mathbb{Z}) \cong \Lambda_{K3}$$

by defining them on the sublattice $W$ (see (3.1.1)) in the first place, and subsequently on all of $\Lambda_{K3}$ by $\mathbb{Q}$-linear extension to the elements presented in (3.1.2).

The description of

$$\tilde{\pi}_{2*} : H^2(Z, \mathbb{Z}) \rightarrow H^2(\tilde{Y}, \mathbb{Z})$$

will require some more effort: In fact, $H^2(Z, \mathbb{Z})$ is an overlattice of finite index of $q_{2*}H^2(X, \mathbb{Z}) \oplus M_{\mathbb{Z}/2\mathbb{Z}}$, while $H^2(\tilde{Y}, \mathbb{Z})$ is an overlattice of finite index of $q_{4*}H^2(X, \mathbb{Z}) \oplus M_{\mathbb{Z}/4\mathbb{Z}}$; for now, notice that

$$\tilde{\pi}_{2*} |_{M_{\mathbb{Z}/2\mathbb{Z}}} : M_{\mathbb{Z}/2\mathbb{Z}} \rightarrow M_{\mathbb{Z}/4\mathbb{Z}},$$

$$\tilde{\pi}_{2*} |_{q_{2*}H^2(X, \mathbb{Z})} : q_{2*}H^2(X, \mathbb{Z}) \rightarrow q_{4*}H^2(X, \mathbb{Z}).$$

(4.2)

4.1 The image of $H^2(X, \mathbb{Z})$ via the maps $\pi_{4*}$ and $\pi_{2*}$

Proposition 4.1.1. The maps $\pi_{2*}$, $\tilde{\pi}_{2*}$, and $\pi_{4*} = \tilde{\pi}_{2*} \circ \pi_{2*}$ act in the following way on $W$ and its image in $\pi_{2*}H^2(X, \mathbb{Z})$:

Proof. Since $\pi_4$ is a finite morphism of degree 4, we can compute the intersection form of $\pi_{4*}W$ via the push–pull formula:

$$\pi_{4*}x \cdot \pi_{4*}y = \frac{1}{4} \sum_{\kappa=0}^{3} (\tau^\kappa)^* (x),$$

where $\pi_{4*}x = \sum_{\kappa=0}^{3} (\tau^\kappa)^* (x)$. The behavior of $\pi_{2*}$ and $\tilde{\pi}_{2*}$ can be similarly determined. □

Corollary 4.1.2. The embedding $q_{4*}H^2(X, \mathbb{Z}) \hookrightarrow H^2(\tilde{Y}, \mathbb{Z})$ is unique up to isometries of the latter, and the same holds for $q_{2*}H^2(X, \mathbb{Z}) \hookrightarrow H^2(\tilde{Z}, \mathbb{Z})$. 
Proof. Computing \(q_4 \ast H^2(X, \mathbb{Z})\) and \(q_2 \ast H^2(X, \mathbb{Z})\) by \(\mathbb{Q}\)-linear extension of the maps in the proposition above to the elements in \((3.1.2)\), it can be seen that the lattice \(q_4 \ast H^2(X, \mathbb{Z})\) is even, indefinite, and it has rank 8 and length 4, so it satisfies the conditions of \([14, \text{Theorem 1.14.2}]\); similarly, \(q_2 \ast H^2(X, \mathbb{Z})\) is even, indefinite, it has \(rk = 14\) and \(\ell = 6\), and the same result can be applied. \(\square\)

4.2 The resolution of \(Z = X/\tau^2\), and the lattice \(\Gamma\)

For any symplectic involution \(\iota\) on a K3 surface \(X\), the quotient surface \(X/\iota\) has eight isolated singularities, that are ordinary double points \([13, \text{section 5}]\): To resolve it, it is sufficient to blow up these points once. Therefore, the exceptional lattice \(N := M_{Z/\mathbb{Z}}\) for the quotient \(X/\iota\) (usually called Nikulin lattice) satisfies \(N \otimes \mathbb{Q} = A_1^{\oplus 8} \otimes \mathbb{Q}\): More precisely, if \(\{n_1, \ldots, n_8\}\) is a \(\mathbb{Z}\)-basis of \(A_1^{\oplus 8}\), then a set of generators over \(\mathbb{Z}\) for \(N\) can be obtained by adding to this list the element \(v = (n_1 + \cdots + n_8)/2\) \([13, \text{Definition 6.2, case 1a}]\).

The second integral cohomology of the K3 surface \(\tilde{Z}\), the minimal resolution of the quotient \(Z = X/\tau^2\), can be described as an overlattice of index \(2^6\) of \(\pi_2 \ast H^2(X, \mathbb{Z}) \oplus N\): This is done via an isomorphism of the discriminant groups of \(\pi_2 \ast H^2(X, \mathbb{Z})\) and \(N\), as described in Theorem 1.1.5. The generators that we need to add are the following:

\[
\begin{align*}
z_1 &= (\hat{\beta} + \hat{f}_1 + \hat{f}_2 + \hat{a}_2 + \hat{\eta})/2 + (n_2 + n_8)/2, \\
z_2 &= \hat{\epsilon}/2 + (n_3 + n_8)/2, \\
z_3 &= \hat{a}_2/2 + (n_4 + n_8)/2, \\
z_4 &= (\hat{\epsilon} + \hat{a}_2 + \hat{\rho})/2 + (n_5 + n_8)/2, \\
z_5 &= (\hat{\beta} + \hat{f}_1 + \hat{f}_2 + \hat{\rho} + \hat{\chi} + \hat{\eta})/2 + (n_6 + n_8)/2, \\
z_6 &= (\hat{\beta} + \hat{f}_1 + \hat{f}_2 + \hat{\epsilon} + \hat{a}_2 + \hat{\rho} + \hat{\zeta})/2 + (n_7 + n_8)/2,
\end{align*}
\]

where \(\hat{\beta} = \pi_2 \ast \beta\) (\(= \pi_2 \ast \delta\)), and similarly the cap over the other elements of \(\Lambda_{K3}\) denotes their image via \(\pi_2\). These generators were already known in the general case of a symplectic involution \([7, \text{Lemma 1.10}]\).

Definition 4.2.1. Notice that, while \(\Omega_2 \subset \Omega_4\) is annihilated by \(\pi_2\), the same is not true for its orthogonal complement in \(\Omega_4\), the lattice \(R\) (see Section 3.2). Define \(\hat{R} := \pi_2 \ast \Omega_4\): It is an overlattice of \(\pi_2 \ast R\), and it is spanned by \(\hat{\epsilon}_1 - \hat{f}_1, \hat{\alpha} - \hat{\beta}, \hat{\epsilon}_3 - \hat{f}_3, \hat{\epsilon}_4 - \hat{f}_4, \hat{\sigma}, (\hat{\alpha}_1 - \hat{\alpha}_2 + \hat{\delta})/2\). It has rank 6 and discriminant group \((\mathbb{Z}/2\mathbb{Z})^4 \times (\mathbb{Z}/4\mathbb{Z})^2\).

Define \(\Gamma\) the negative definite lattice of rank 14 and discriminant group \((\mathbb{Z}/2\mathbb{Z})^6 \times (\mathbb{Z}/4\mathbb{Z})^2\), obtained as an overlattice of \(\hat{R} \oplus N\) by adding to the list of generators the elements

\[
\begin{align*}
x_1 &= n_3 + n_4 + n_5 + n_8 + \hat{\sigma}/2, \\
x_2 &= n_3 + n_4 + (\hat{\epsilon}_1 - \hat{f}_1) + (\hat{\epsilon}_4 - \hat{f}_4) + (\hat{\alpha}_1 - \hat{\alpha}_2 + \hat{\sigma})/4.
\end{align*}
\]

(4.2.2)

The lattice \(\Gamma\) can be primitively embedded in \(H^2(Z, \mathbb{Z})\) with the lattice \(S = \langle \hat{\epsilon}_1 + \hat{f}_1, \hat{\alpha} + \hat{\beta}, \hat{\epsilon}_3 + \hat{f}_3, \hat{\epsilon}_2 + \hat{f}_2, \hat{\rho}, (\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\rho})/2, (\hat{\rho} + \hat{\omega}_1 + \hat{\omega}_2)/2, \hat{\omega}_2 \rangle\) as orthogonal complement; indeed, we can obtain \(H^2(Z, \mathbb{Z})\) as overlattice of finite index of \(S \oplus \Gamma\) by adding the generators:

\[
\begin{align*}
z'_1 &= (\hat{\alpha} + \hat{\rho})/2 + (\hat{\alpha} - \hat{\rho})/2, \\
z'_2 &= (\hat{\epsilon}_3 + \hat{f}_3)/2 + (\hat{\epsilon}_3 - \hat{f}_3)/2, \\
z'_3 &= (\hat{\epsilon}_2 + \hat{f}_2)/2 + (\hat{\epsilon}_4 - \hat{f}_4)/2 + (n_3 + n_4 + n_5 + n_8)/2, \\
z'_4 &= (\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\rho})/4 + (n_2 + n_3 + n_4 + n_6)/2 + (\hat{\epsilon}_1 - \hat{f}_1 + \hat{\epsilon}_4 - \hat{f}_4)/2, \\
z'_5 &= (\hat{\rho} + \hat{\omega}_1 + \hat{\omega}_2)/4 + (n_2 + n_7)/2 + (\hat{\epsilon}_1 - \hat{f}_1 + \hat{\epsilon}_4 - \hat{f}_4)/2, \\
z'_6 &= \hat{\omega}_2/2 + (n_2 + n_6)/2,
\end{align*}
\]

(4.2.3)
\[ z'_7 = (\hat{e}_1 + \hat{f}_1 + \hat{e}_2 + \hat{f}_2 + (\hat{a}_1 + \hat{a}_2 + \hat{\rho})/2 + \hat{\omega}_2)/4 + (\hat{\alpha} - \hat{\beta})/2 + (3n_5 + 2n_6 + n_8)/4 + (x_1 + x_2)/2, \]

\[ z'_8 = \hat{\rho}/4 + (2n_2 + n_3 + 3n_4 + n_5 + 2n_6 + 3n_8)/4 + x_1/2. \]

Remark 4.2.2. The primitive embedding \( \Gamma \hookrightarrow \Lambda_{K3} \) is unique up to isometries of \( \Lambda_{K3} \), because Theorem 1.1.8 holds: In fact the orthogonal complement \( S \) of \( \Gamma \) satisfies the first condition of Proposition 1.1.10.

4.3 The map \( \hat{\pi}_2^* \) and the resolution of \( Y = X/\tau \)

The action of a symplectic automorphism of order 4 \( \tau \) on a K3 surface \( X \) has always exactly eight isolated points on \( X \) with nontrivial stabilizer: Four of them are fixed by \( \tau \), and four more exchanged by \( \tau \) (so they are fixed by \( \tau^2 \)) [13, section 5, case 2]; therefore, the singular locus of the quotient \( X/\tau \) consists of six isolated points, two of which are resolved by blowing up once (thus introducing two rational curves in the quotient surface), and each of the other four by three curves in \( A_3 \) configuration.

The exceptional lattice \( M := M_{\mathbb{Z}/4\mathbb{Z}} \) therefore satisfies \( M \otimes \mathbb{Q} = (A_3^{\oplus 1} \oplus A_1^{\oplus 2}) \otimes \mathbb{Q} \); calling \( \hat{m}_1, \hat{m}_2 \) the generators of the two copies of \( A_1 \), and \( m_1^1, m_1^3, m_2^1 \) the generators of the \( i \)-th copy of \( A_3 \), such that \( m_2^1 \) is the curve that intersects both \( m_1^1 \) and \( m_1^3 \), then a set of \( \mathbb{Z} \)-generators for \( M \) consists of all these elements, and (see [13, Definition 6.2, case 1b]) the class

\[ \mu = \frac{\sum_{i=1}^{4}(m_1^i + 2m_2^i + 3m_3^i)}{4} + \hat{m}_1 + \hat{m}_2/2. \] (4.3.1)

We can now describe the second integral cohomology of the K3 surface \( \tilde{Y} \), the minimal resolution of the quotient \( Y = X/\tau \):

The discriminant group of each of the orthogonal summands \( M \) and \( \pi_4^*H^2(X, \mathbb{Z}) \) is isomorphic to \((\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2 \), and the elements that define \( \Lambda_{K3} \) as an overlattice of \( M \oplus \pi_4^*H^2(X, \mathbb{Z}) \) are:

\[ y_1 = (m_1^1 + 2m_2^1 + m_3^1 + m_4^1)/4 + \hat{m}_2/2 + (\tilde{e}_1 + \tilde{e}_4 + \tilde{e}_7 + \tilde{\eta})/2 + (\tilde{a} + \tilde{r})/4, \]

\[ y_2 = (m_1^2 + 2m_2^2 + 3m_3^2 + 3m_4^2)/4 + m_2^1/2 + (\tilde{e}_1 + \tilde{e}_4 + \tilde{e}_7 + \tilde{\eta})/2 + (\tilde{a} + \tilde{r} + 3\tilde{\eta})/4, \]

\[ y_3 = (m_1^3 + m_2^3 + m_3^3)/2 + (\tilde{e}_1 + \tilde{e}_4 + \tilde{e}_7 + \tilde{r} + \tilde{\eta})/2, \]

\[ y_4 = (\hat{m}_1^1 + \hat{m}_2^3)/2 + \tilde{a}/2, \] (4.3.2)

where \( \tilde{e}_i = \pi_{4e}e_i = \pi_{4e}f_i = \pi_{4e}g_i = \pi_{4e}h_1, \tilde{a} = \pi_{4e}a_1 = \pi_{4e}a_2, \) and similarly \( \tilde{\eta} = \pi_{4e}(\eta) \).

The exceptional lattice \( M \) can be also computed from the image of \( N \) via \( \hat{\pi}_2^* \) (see Remark 4.2) with the resolution of the singularities that arise from the quotient: In fact, the involution \( \hat{\tau} \) (that is induced on \( \tilde{Z} \) by the action of \( \tau \) on \( X \)) acts by fixing two points on each of the four exceptional curves of \( Z \) corresponding to the four points of \( X \) fixed by \( \tau \), and by exchanging pairwise the remaining four exceptional curves (these correspond to the four points fixed only by \( \tau^2 \)). Therefore, the invariant lattice for the action of \( \hat{\tau}^* \) on \( N \) is the sublattice spanned by the four invariant curves, and the sum of the pairs of exchanged curves.

In the lattice \( \Gamma \) (see Definition 4.2.1), the orthogonal complement of the invariant lattice for the action of \( \hat{\tau}^* \) is a copy of \( \Omega_2 \): This is indicative of the fact that the surface \( \tilde{Z} \) admits a symplectic involution, which is indeed \( \hat{\tau} \).

Remark 4.3.1. The curves of \( N \) were numbered in such a way that the gluing between \( N \) and \( \pi_2^*H^2(X, \mathbb{Z}) \) is described by the elements in (4.2.1): Since the action of \( \hat{\tau}^* \) on \( \pi_2^*H^2(X, \mathbb{Z}) \) determines the action of \( \hat{\tau}^* \) on \( N \) via this gluing, we find accordingly that \( \hat{\tau}^* \) fixes \( n_1, n_2, n_6, n_7 \) and exchanges \( n_3 \) with \( n_4, n_5 \) with \( n_8 \).

Proposition 4.3.2. It holds \( \hat{\pi}_2^*(\Gamma) \subset M \): More precisely, \( \hat{\pi}_2^* \) annihilates \( \hat{R} \), because \( \hat{\pi}_2^*\hat{R} = \hat{\pi}_2^*\pi_2^*\Omega_4 = \pi_4^*\Omega_4 = 0 \), and is defined on the \( \mathbb{Q} \)-generators of \( N \) as follows:

\[ \bar{n}_1 := \hat{\pi}_2^*n_1 = m_1^1 + 2m_2^1 + m_3^1, \quad \bar{n}_2 := \hat{\pi}_2^*n_2 = m_1^3 + 2m_2^3 + m_3^3. \]
so that  \( \hat{\pi}_2^* A^{\oplus 8} = A_1(2)^{\oplus 4} \oplus A_1^{\oplus 2} \).

**Proof.** Let  \( k = 1, 2, 6, 7, j = 1, \ldots, 8 \). The surface  \( \mathcal{Z}/\tau \) is singular in eight points, two on each of the curves  \( \hat{q}_2^* n_k \); consider the blow-up  \( \beta : \mathcal{Y} \to \mathcal{Z}/\tau \) of the singular points: then, the curve  \( \overline{n}_j = \hat{\pi}_2^* n_j \) is the strict transform of  \( \hat{q}_2^* n_j \). By push–pull, we have  \( \overline{n}_k = -4, \overline{n}_3 = -2 = \overline{n}_5 \). Consider the following diagram:

\[
\begin{array}{cccccccc}
1 & n_2 & n_6 & n_7 & n_3 & n_4 & n_5 & n_8 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\mathcal{Y} \simeq \mathcal{Z}/\tau & \beta & \mathcal{Z}/\tau
\end{array}
\]

First, notice that either  \( (\overline{n}_3, \overline{n}_5) = (\overline{m}^1, \overline{m}^2) \) or  \( (\overline{n}_3, \overline{n}_5) = (\overline{m}^3, \overline{m}^1) \).

The eight exceptional curves introduced with the blow-up  \( \beta \) span a new copy of the Nikulin lattice

\[
N_Y = \langle m_1^i, m_3^i \rangle_{i=1} \subset M;
\]

the pullback  \( \beta^* \hat{q}_2^* n_k \) of each singular curve is an  \( A_3 \) lattice, and the class  \( \overline{n}_k \) (being the strict transform) is by definition orthogonal to the exceptional curves: thus we find that  \( \overline{n}_k = m_1^i + 2m_2^i + m_3^i \) for some  \( i \). To determine which copy of  \( A_3, A_1 \) in  \( M \) each  \( \overline{n}_j \) corresponds to, we still have to require that the image of the elements  \( z_i \) defined in (4.2.1) be integer elements in  \( H^2(\mathcal{Y}, \mathbb{Z}) \); this forces the definition of  \( \hat{\pi}_2^* \) as stated.

**Corollary 4.3.3.** The lattice  \( M \) is an overlattice of index  \( 2^5 \) of the lattice  \( \hat{\pi}_2^* N \oplus N_Y \), obtained by adding as generators the elements  \( m_2^1 = (\overline{n}_1 - m_1^1 - m_3^1)/2, m_2^2 = (\overline{n}_6 - m_1^1 - m_3^1)/2, m_2^3 = (\overline{n}_2 - m_1^1 - m_3^1)/2, m_2^4 = (\overline{n}_7 - m_1^1 - m_3^1)/2, \) and  \( \mu \) (see (4.3.1)).

**Proof.** The Nikulin lattice  \( N \) is an overlattice of index  \( 2 \) of  \( A_1^{\oplus 8} = \langle a_1, \ldots, a_8 \rangle \) obtained by adding as generator the element  \( v = (a_1^+, \ldots, +a_8)/2 \): In our case

\[
v_Y = \frac{\Sigma_{i=1}^4 (m_1^i + m_3^i)}{2};
\]

the element  \( \mu \) defined in (4.3.1) can then be written as

\[
\mu = \frac{v_Y + m_2^1 + m_2^2 + m_2^3 + m_2^4 + m_3^1 + m_3^2 + m_3^3 + m_3^4 + m_1^1 + m_1^2}{2},
\]

so it generates an overlattice (of index  \( 2 \)) of the overlattice (of index  \( 2^4 \)) of  \( \mathcal{Z}/\tau \) generated by the  \( m_2^i, i = 1, \ldots, 4 \). □

Notice that we have the following equalities:

\[
H^2(\mathcal{Y}, \mathbb{Q}) = (\pi_4^* H^2(\mathcal{X}, \mathbb{Z}) \oplus M) \otimes \mathbb{Q} = (\pi_4^* H^2(\mathcal{X}, \mathbb{Z}) \oplus \hat{\pi}_2^* N \oplus N_Y) \otimes \mathbb{Q} = (\hat{\pi}_2^* H^2(\mathcal{Z}, \mathbb{Z}) \oplus N_Y) \otimes \mathbb{Q}.
\]
Working on \( \mathbb{Z} \), we recover the first equality using the \( y_i \)s in (4.3.2), and the second one as in Proposition 4.3.3; the next one is trivial, and for the last one, we use the \( z_i \)s in (4.2.1). Thus, the lattice \( H^2(\tilde{Y}, \mathbb{Z}) \) can be also described directly as an overlattice of finite index of \( \tilde{\pi}_2^*H^2(\mathbb{Z}, \mathbb{Z}) \oplus N_Y \), allowing for an easier computation of the map \( \tilde{\pi}_2^* \) in Section 4.4: To do this, use for \( \tilde{\pi}_2^*H^2(\mathbb{Z}, \mathbb{Z}) \) the \( \mathbb{Z} \)-basis \([e_1, \alpha, e_3, \tilde{\alpha}, \gamma, \tilde{\gamma}, \tilde{\pi}_2^*n_6, \tilde{\pi}_2^*z_1, \tilde{\pi}_2^*z_2, \tilde{\pi}_2^*z_5, \tilde{\pi}_2^*z_6]\), with the \( z_i \)s defined in (4.2.1), and for \( N_Y \) the \( \mathbb{Z} \)-basis \([\nu_Y, m_1^1, m_2^2, m_3^3, m_4^4] \); then, the gluing elements are

\[
\begin{align*}
y'_1 &= (a + \gamma + \tilde{\pi}_2^*\nu + m_1^1 + m_2^1 + m_3^1 + m_4^1)/2, \\
y'_2 &= (\alpha + \zeta + \tilde{\pi}_2^*z_1 + \tilde{\pi}_2^*z_5 + m_2^1 + m_3^1)/2, \\
y'_3 &= (\gamma + \tilde{\alpha} + \tilde{\gamma} + \tilde{\pi}_2^*n_6 + m_1^1 + m_4^1)/2, \\
y'_4 &= (e_4 + a + \gamma + \tilde{\pi}_2^*z_1 + \tilde{\pi}_2^*z_6 + m_3^1 + m_4^1)/2, \\
y'_5 &= (\alpha + a + \gamma + \tilde{\pi}_2^*z_1 + \tilde{\pi}_2^*z_5 + m_1^1 + m_3^1 + m_4^1 + m_5^1)/2, \\
y'_6 &= (\tilde{\pi}_2^*n_6 + m_1^2 + m_2^2)/2.
\end{align*}
\]

(4.3.3)

### 4.4 The dual maps

We are now going to define the dual maps

\[
\begin{align*}
\pi_4^* &: H^2(\tilde{Y}, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \\
\tilde{\pi}_2^* &: H^2(\mathbb{Z}, \mathbb{Z}) \to H^2(Z, \mathbb{Z}) \\
\pi_2^* &: H^2(\tilde{Z}, \mathbb{Z}) \to H^2(X, \mathbb{Z})
\end{align*}
\]

using the descriptions of \( H^2(\tilde{Y}, \mathbb{Z}) \) as an overlattice, respectively, of \( \pi_4^*\Lambda_{K3} \oplus M \) and \( \tilde{\pi}_2^*\Lambda_{K3} \oplus N_Y \), and of \( H^2(Z, \mathbb{Z}) \) as an overlattice of \( \pi_2^*\Lambda_{K3} \oplus N \). These maps are used in Section 6 to find the dimension of the eigenspaces for the action induced by the automorphism \( \tau \) on \( H^0(X, L) \) for any possible choice of polarization \( L \) on \( X \), and they are given here explicitly for completeness of our exposition.

**Proposition 4.4.1.**

1. The map \( \pi_4^* \) annihilates \( M \), and acts on \( \pi_4^*W \subset \pi_4^*\Lambda_{K3} \) as

\[
\begin{align*}
\pi_4^*: D_4 \oplus A_1(2) \oplus \left[ \begin{array}{ccc}
16 & 0 & 0 \\
0 & 8 & 0 \\
0 & 0 & 8
\end{array} \right] &\to D_4^{\oplus 4} \oplus A_1^{\oplus 2} \oplus \left[ \begin{array}{ccc}
4 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{array} \right] \\
\left[ \begin{array}{c}
e_1 \\
e_2 \\
e_3 \\
e_4
\end{array} \right] &\to \\
\left[ \begin{array}{c}
e_1 + f_1 + g_1 + h_1 \\
e_2 + f_2 + g_2 + h_2 \\
e_3 + f_3 + g_3 + h_3 \\
e_4 + f_4 + g_4 + h_4
\end{array} \right]
\end{align*}
\]

Its action can be extended to \( \pi_4^*\Lambda_{K3} \) adding these elements (and their respective images to the image lattice): \( \alpha = \rho/4 + \bar{e}_1 + (\bar{e}_2 + \bar{e}_4)/2, \bar{\alpha} = \bar{\rho}/2, \zeta = \bar{e}_1 + (\bar{e}_2 + \bar{e}_4 + \bar{\alpha} + \bar{\omega}_1)/2, \bar{\zeta} = \bar{e}_1 + (\bar{e}_2 + \bar{e}_4 + \bar{\alpha} + \bar{\omega}_2)/2; \) to extend the action to \( H^2(\tilde{Y}, \mathbb{Z}) \), add also \( y_1, \ldots, y_4 \) (see (4.3.2)).
2. The map $\pi_2^*$ annihilates $N$, and acts on $\pi_2^* W \subset \pi_2^* \Lambda_{K3}$ as

$$
\pi_2^* : D^G_4 \otimes A_1(2) \otimes \langle -8 \rangle \otimes \left[ \begin{array}{ccc} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{array} \right] \longrightarrow D^G_4 \otimes A_1 \langle -4 \rangle \otimes \left[ \begin{array}{ccc} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{array} \right]
$$

Its action can be extended to $\pi_2^* \Lambda_{K3}$ adding the following elements (and their respective image to the image lattice):

$\hat{\alpha} = (\hat{\rho} + \hat{\sigma})/4 + (\hat{e}_1 + \hat{e}_2 + \hat{f}_1 + \hat{f}_3)/2$, $\hat{\beta} = (\hat{\rho} - \hat{\sigma})/4 + (\hat{e}_1 + \hat{e}_4 + \hat{f}_1 + \hat{f}_2)/2$, $\hat{\varepsilon} = \hat{\rho}/2 + \hat{f}_2 + \hat{f}_4 + (\hat{a}_1 + \hat{a}_2)/2$, $\hat{\chi} = (\hat{\rho} + \hat{\sigma})/2$, $\hat{\xi} = (\hat{e}_1 + \hat{f}_1 + \hat{e}_4 + \hat{f}_2 + \hat{a}_1 + \hat{a}_2)/2$, $\hat{\eta} = (\hat{e}_1 + \hat{f}_1 + \hat{e}_4 + \hat{f}_2 + \hat{a}_1 + \hat{a}_2 + \hat{\omega}_1)/(2)$; to extend the action to $H^2(Z, \mathbb{Z})$, add also $z_1, \ldots, z_6$ (see (4.2.1)).

3. Recall from Corollary 4.3.3 that $H^2(\tilde{Y}, \mathbb{Z})$ as an overlattice of finite index of $\widehat{\pi}_2^* H^2(\tilde{Z}, \mathbb{Z}) \oplus N_Y$. The lattice $N_Y \subset H^2(\tilde{Y}, \mathbb{Z})$ is annihilated by $\widehat{\pi}_2^*$, and for the generators of $\widehat{\pi}_2^* H^2(\tilde{Z}, \mathbb{Z})$ the map $\widehat{\pi}_2^*$ is defined as follows:

$$
\widehat{\pi}_2^* e_i = \hat{e}_i + \hat{f}_i \ (\text{for } i = 1, \ldots, 4, \text{ but } e_2 \text{ is not needed as generator}),
$$

$$
\widehat{\pi}_2^* \hat{\rho} = \hat{e}_1 + \hat{f}_1 + (\hat{e}_2 + \hat{f}_2 + \hat{e}_4 + \hat{f}_4 + \hat{a}_1 + \hat{a}_2 + 2\hat{\omega}_1)/2,
$$

$$
\widehat{\pi}_2^* \hat{\sigma} = \hat{e}_1 + \hat{f}_1 + (\hat{e}_2 + \hat{f}_2 + \hat{e}_4 + \hat{f}_4 + \hat{a}_1 + \hat{a}_2 + 2\hat{\omega}_2)/2,
$$

$$
\widehat{\pi}_2^* \hat{\alpha} = \hat{\alpha} + \hat{\beta}, \quad \widehat{\pi}_2^* \hat{\omega}_1 = \hat{\alpha} + \hat{\omega}_1,
$$

$$
\widehat{\pi}_2^* \hat{\omega}_2 = \hat{\alpha} + \hat{\omega}_2,
$$

$$
\widehat{\pi}_2^* n_6 = 2n_6, \quad \widehat{\pi}_2^* \hat{v} = 2\hat{v},
$$

$$
\widehat{\pi}_2^* \hat{z}_1 = (\hat{\alpha} + \hat{\beta} + \hat{e}_1 + \hat{f}_1 + \hat{e}_2 + \hat{f}_2 + \hat{a}_1 + \hat{a}_2 + 2\hat{\omega}_1 + 2n_2 + n_5 + n_8)/2,
$$

$$
\widehat{\pi}_2^* \hat{z}_2 = (\hat{\rho} + \hat{e}_2 + \hat{f}_2 + \hat{e}_4 + \hat{f}_4 + \hat{a}_1 + \hat{a}_2 + n_3 + n_4 + n_5 + n_8)/2,
$$

$$
\widehat{\pi}_2^* \hat{z}_3 = (\hat{\alpha} + \hat{\beta} + \hat{e}_1 + \hat{f}_1 + \hat{e}_2 + \hat{f}_2 + \hat{p} + 2\hat{\omega}_1 + 2n_6 + n_5 + n_8)/2 + \hat{\rho},
$$

$$
\widehat{\pi}_2^* \hat{z}_4 = (\hat{\alpha} + \hat{\beta} + \hat{e}_1 + \hat{f}_1 + \hat{e}_4 + \hat{f}_4 + \hat{p} + 2\hat{\omega}_1 + 2n_7 + n_5 + n_8)/2 + \hat{\rho} + \hat{\alpha} + \hat{\omega}_2 + \hat{f}_2.
$$

Notice that $\widehat{\pi}_2^* \hat{z}_i = z_i + \tau^* z_i$, and that to obtain the whole image of $\widehat{\pi}_2^*$ the images of the elements $y'_i$ of (4.3.3) are also to be considered.

Proof. We are going to prove only that the map $\pi_2^*$ acts on $\pi_4^* W$ as stated above; the other cases are similar.

Since $\pi_2^*$ and $\pi_4^*$ are dual maps, $\pi_2^* a = b$ iff $(b \cdot c)_X = (a \cdot \pi_4^* c)_Y$ for every $a \in \pi_4^* W$, $c \in W$. Hence $\pi_2^* a \cdot c = a \cdot \pi_4^* c$. Take $e_j = (e_j, 0, 0, 0) \in D^G_4$: then $\pi_4^* \hat{e}_i \cdot e_j = \hat{e}_i \cdot \pi_4^* e_j = \hat{e}_i \cdot \hat{e}_j$, but it holds also $\hat{e}_i \cdot \hat{e}_j = \hat{e}_i \cdot \pi_4^* f_j = \hat{e}_i \cdot \pi_4^* g_j = \hat{e}_i \cdot \pi_4^* h_j$; therefore, $\pi_4^* \hat{e}_i = e_i + f_i + g_i + h_i$.

Take $a_1 = (a_1, 0, 0) \in A_1 \langle 8 \rangle$: then $\pi_4^* \hat{a}_1 = \hat{a}_1 \cdot \pi_4^* a_1 = 2(\hat{a}_1 \cdot \hat{a}_1)$ because $\pi_4^*$ doubles the intersection form on $\Lambda_{A_1} \oplus A_1$; moreover, $\pi_4^* a_1 = \hat{a}_1 = 2a_1 = 2a_2$. Similarly, since $\pi_4^*$ multiplies by 4 the intersection form of the sublattice of $W$ invariant for the action of $\tau$, we can conclude that $\pi_4^* \hat{\rho} = 4\hat{\rho}$, $\pi_4^* \hat{\omega}_1 = 4\omega_1$, $\pi_4^* \hat{\omega}_2 = 4\omega_2$.

Corollary 4.4.2. The image of $H^2(Y, \mathbb{Z})$ via the map $\pi_2^*$ coincides with the invariant lattice $\Lambda_{K3}^{(c)}$ described in Section 3.2. In other words, it holds

$$
\pi_2^* H^2(Y, \mathbb{Z}) = \Omega_{K3}^{1/4, K3}.
$$
Similarly, we obtain:

\[ \pi_2^* H^2(\tilde{Z}, \mathbb{Z}) = \Omega^1_{-H_2(X, \mathbb{Z})} = \Omega^1_{-\Lambda_{K3}} , \]

\[ \tilde{\pi}_2^* H^2(\tilde{Y}, \mathbb{Z}) = \Omega^1_{-H_2(\tilde{Z}, \mathbb{Z})} = \Omega^1_{-\Lambda_{K3}} . \]

**Proof.** It holds \( \pi_1^* \tilde{e}_1 = e_i + f_i + g_i + h_i \) for \( i = 1, \ldots, 4 \), \( \pi_4^* \tilde{a} = 2(a_1 + a_2) \), \( \pi_4^* \tilde{\rho} = 4\rho \), \( \pi_4^* \tilde{\omega}_1 = 4\omega_1 \), \( \pi_4^* \tilde{\omega}_2 = 4\omega_2 \); however, these elements generate only a sublattice of finite index of \( \pi_4^* H_2(\tilde{Y}, \mathbb{Z}) \). Adding as generators the images via \( \pi_4^* \) of the elements \( \alpha, \chi, \eta \) and \( y_1, \ldots, y_4 \), we obtain the whole invariant lattice for the action of \( \tau \) on \( \Lambda_{K3} \). \( \square \)

### 4.5 Characterization of the surface \( \tilde{Z} \): The nonprojective case

Nikulin’s seminal work [13] provides a lattice-theoretic characterization of K3 surfaces \( X \) that admit a symplectic action of a cyclic group \( G = \mathbb{Z}/n\mathbb{Z} \) (for \( n = 2, \ldots, 8 \)), and of surfaces \( \tilde{Y} \) that are the resolution of the quotient \( X/G \), by providing a relation between some (even, negative definite) lattices \( \Omega_G \) and \( M_G \) that have to be primitively embedded in their respective Néron–Severi lattices; we want to show that similarly, in the case \( G = \mathbb{Z}/4\mathbb{Z} \) (generated by an automorphism \( \tau \) of \( X \)), the lattice \( \Gamma \) characterizes the surface \( \tilde{Z} \) that is the resolution of \( Z := X/\tau^2 \). For simplicity, we are going to state our result for the most general K3 surface \( X \), which is not projective: In this case, both \( \text{NS}(X) = \Omega_4 \) and \( \text{NS}(\tilde{Y}) = M \) have rank 14.

**Theorem 4.5.1.** Let \( Z \) be a K3 surface such that \( \text{rk}(\text{NS}(Z)) = 14 \). There exists a pair \((X, \tau)\) where \( X \) is a K3 and \( \tau \) is a symplectic automorphism of order 4 such that \( Z \) is birationally equivalent to the quotient \( X/\tau^2 \) if and only if \( \text{NS}(Z) = \Gamma \) (see Definition 4.2.1).

**Proof.** The “only if” is true by construction (see Section 4.2). Conversely, suppose \( \text{NS}(Z) = \Gamma \). The embedding \( \Omega_2 \subset \Gamma \) described in Remark 4.3.1 defines a symplectic involution \( \tilde{\tau} \) on \( \tilde{Z} \), and the Néron–Severi lattice of the resolution \( \tilde{Z}/\tilde{\tau} \) is naturally a copy of \( M \), as proved in Corollary 4.3.3; therefore, by the results of Nikulin, the surface \( \tilde{Z}/\tilde{\tau} \) is the resolution of the quotient of a K3 surface \( X \) for a symplectic automorphism \( \tau \) of order 4, and it holds \( \text{NS}(X) = \Omega_4 \). The action of \( \tau \) on \( \Omega_4 \) naturally defines a copy of \( \Omega_2 \subset \Omega_4 \) by \( \Omega_2 = (\Omega_4^2)^{1+4} \), as described in Section 3.2; taking the quotient map \( \pi_{\tilde{2}} : X \to X/\tau^2 \) and the resolution \( X/\tau^2 \), it holds \( \text{NS}(X/\tau^2) \approx \text{NS}(Z) \). \( \square \)

### 5 PROJECTIVE K3 SURFACES WITH A SYMPLECTIC AUTOMORPHISM OF ORDER 4 AND THEIR QUOTIENTS

It was already known by Nikulin that the correspondence between surfaces \( X \) that admit a symplectic action of an abelian group \( G \), and surfaces \( \tilde{Y} \) that are the resolution of the quotient \( X/G \), is actually a moduli spaces correspondence [13, Proposition 2.9]; the same idea was later generalized to the nonabelian case by Whitcher [21, section 3].

We can therefore refine the characterization of \( X, Z, \) and \( \tilde{Y} \) by their Néron–Severi to the projective case. The approach we follow mimics the one used in [5, 7] for symplectic involutions, and in [3] for symplectic automorphisms of order 3.

**Definition 5.1** [2, §1]. Let \( \Lambda \) be an even lattice. A K3 surface \( X \) is \( \Lambda \)-polarized if there is a primitive embedding \( \Lambda \hookrightarrow \text{NS}(X) \).

**Remark 5.2.** The moduli space of \( \Lambda \)-polarized K3 surfaces has dimension \( 20 - \text{rk}(\Lambda) \).

**Proposition 5.3** (see [13, Proposition 2.9] and [7, Proposition 2.2]). Projective K3 surfaces \( X \) that admit a symplectic action of an abelian group \( G \) are polarized with a lattice of rank \( 1 + \text{rk}(\Omega_G) \) that contains primitively both the lattice \( \Omega_G \) and an ample class \( L \) of square \( 2d \); projective K3 surfaces that are the resolution of \( X/G \) are polarized with a lattice of rank \( 1 + \text{rk}(M_G) \) that contains primitively both the lattice \( M_G \) and an ample class \( K \) of square \( 2f \).

Moreover, from Theorem 4.5.1, we deduce that in the projective case \( \tilde{Z} \) is polarized with a lattice of rank 15 that contains primitively both the lattice \( \Gamma \) and an ample class \( K \) of square \( 2f \).
Remark 5.4. Let $S$ be either $\Omega_G$ or $M_G$: The only lattices that satisfy the proposition above are $S \oplus \langle 2d \rangle$ and its cyclic overlattices of finite index \cite[Proposition 6.1]{6}. Each nonisomorphic primitive embedding of any of these lattices in $\Lambda_{K3}$ gives a different irreducible component of a moduli space of projective K3 surfaces: either of surfaces $X$ that admit a symplectic action of $G$ (if $S = \Omega_G$), or of surfaces $\tilde{Y}$ that are the minimal resolution of $X/G$ (if $S = M_G$).

For $G = \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$, it is already known that there exists a bijection between the irreducible components of the moduli spaces of $X$ and $\tilde{Y}$ \cite{3, 7}. We are going to show that this also holds for $\mathbb{Z}/4\mathbb{Z}$, but not when considering irreducible components of the moduli spaces of $X$ (or $\tilde{Y}$) and of intermediate quotient surfaces $\tilde{Z}$.

Remark 5.5. If $G = \mathbb{Z}/4\mathbb{Z}$, we have to study $S = \Omega_{4, \Gamma}$ \cite[Sections 3.2, 4.3, 4.2]{4}. The moduli spaces of projective K3 surfaces $X$ that admit a symplectic automorphism $\tau$ of order 4, and projective K3 surfaces are the resolution of $X/\tau^2$ or $X/\tau$ all have dimension 5.

Remark 5.6. Notation. Consider the lattice $S \oplus \langle k \rangle$, where $S$ is one between $\Omega_{4, \Gamma}$ and $M$, and $\langle k \rangle$ is an even positive definite lattice of rank 1 and intersection matrix $[k]$.

Denote $(S \oplus \langle k \rangle)'$ and $(S \oplus \langle k \rangle)\ast$ any cyclic overlattices of $S \oplus \langle k \rangle$ such that the following holds:

\[
\frac{(S \oplus \langle k \rangle)'}{S \oplus \langle k \rangle} \simeq \mathbb{Z}/2\mathbb{Z}, \quad \frac{(S \oplus \langle k \rangle)\ast}{S \oplus \langle k \rangle} \simeq \mathbb{Z}/4\mathbb{Z}.
\]

5.1 Families of projective K3 surfaces with a symplectic automorphism $\tau$ of order 4

We are going to find all the nonisomorphic overlattices of $\Omega_4 \oplus \langle 2d \rangle$ as follows.

In Proposition 5.1.2, we look at the orbits for the induced action of $O(\Omega_4)$ on $A^{\Omega_4}$, and we fix a representative $s \in A^{\Omega_4}$ for each of them in Corollary 5.1.3, using the embedding of $\Omega_4$ in $\Lambda_{K3}$ defined in Section 3.2. In Theorems 5.1.4 and 5.1.5, we find all the overlattices of $\Omega_4 \oplus \langle 2d \rangle$, and prove that each one admits a unique primitive embedding in $\Lambda_{K3}$. In Example 5.1.6, we then give for each $s$ defined in Corollary 5.1.3 a primitive class $L \in \Lambda_{\tau}^{K3}$ of square $2d$, $d \in \mathbb{Z} > 0$ such that $L/m + s$ is an integral class in $\Lambda_{K3}$: The maximum $m$ for which this happens is the index of the overlattice of $\Omega_4 \oplus \langle 2d \rangle$ that we obtain choosing $L$ as generator of $\langle 2d \rangle$. According to Corollary 1.1.6 and Remark 1.1.7, elements $\tilde{s}$ in the same orbit of $s$ for the action of $O(S)$, and the corresponding $\tilde{L}$, will give the same irreducible component of the moduli space.

Definition 5.1.1. Consider an even lattice $S$, its group of isometries $O(S)$, and its discriminant group $A_S$ with discriminant form $q_S$. We define two equivalence relations on $A_S$:

1. **By order and square**: two elements $r, s \in A_S$ are in relation ($r \sim_S s$) if they have the same order and square, that is, $\langle r \rangle \simeq \langle s \rangle \simeq \mathbb{Z}/k\mathbb{Z} \subset A_S$ and $q_S(r) = q_S(s) = g \in \mathbb{Q}/2\mathbb{Z}$; we will denote each equivalence class for this relation with the pair $(k, g)$.
2. **By (induced) isometry**: two elements $r, s \in A_S$ are in relation ($r \approx_S s$) if there exists an isometry $\varphi \in O(S)$ induced by an isometry $\varphi \in O(A_S)$ such that $\varphi(r) = s$; we will denote each equivalence class for this relation with the triple $(k, g, n)$, where $k, g$ are as above, and $n$ is the cardinality of the class (in our case, this is sufficient to uniquely identify each of them).

Proposition 5.1.2. The relation $\sim_{\Omega_4}$ divides $A^{\Omega_4}$ in seven nontrivial equivalence classes (plus the trivial one $\{0\}$), whose cardinality is displayed below. Each of them corresponds to an equivalence class for $\approx_{\Omega_4}$, except for $(2, 1)$, which is the union of two classes: $(2,1,6)$ and $(2,1,10)$.

|     | 0     | 1/2   | 1     | 3/2   |
|-----|-------|-------|-------|-------|
| 2   | 15    | 32    | 16    | 0     |
| 4   | 240   | 240   | 240   | 240   |

Proof. The equivalence classes for $\sim_{\Omega_4}$ can be computed from a basis of $A^{\Omega_4}$ and its discriminant form. The generators of $O(\Omega_4)$ can be computed using the Integral Lattices package in SAGE \cite{16}: Then, choosing for each of the classes $(k, g)$
of $\sim_{\Omega_4}$ a representative element $x_{(k,g)}$, their orbit for the induced action of $O(\Omega_4)$ on $A_{\Omega_4}$ is computed recursively [9, Algorithm I.4]. □

**Corollary 5.1.3.** We give a representative element $x_{(k,g,n)}$ for each nontrivial equivalence class $(k, g, n)$ for the relation $\approx_{\Omega_4}$, in terms of the generators of $\Omega_4$ introduced in Section 3.2.

| Class $(k, g, n)$ | Representative $x_{(k,g,n)}$ |
|------------------|-------------------------------|
| $(2,0,15)$       | $\frac{5e_3-f_3-g_3-h_3}{2}$ |
| $(2,1/2,32)$    | $\frac{\sigma+5e_3-f_3-g_3-h_3}{2}$ |
| $(2,1,10)$      | $\frac{5(3e_3-f_3-g_3-h_3)+2(2e_1-f_1-g_1+e_4-f_4+\alpha-\gamma)+e_2-g_2+e_4-g_4+a_1-a_2+\sigma}{4}$ |
| $(4,0,240)$     | $\frac{5(3e_1-f_1-g_1-h_1)+2(2e_1-f_1-g_1+e_4-f_4+\alpha-\gamma)+e_2-g_2+e_4-g_4+a_1-a_2+\sigma}{4}$ |
| $(4,1/2,240)$   | $\frac{5(3e_1-f_1-g_1-h_1)}{2}$ |
| $(4,1,240)$     | $\frac{5(3e_1-f_1-g_1-h_1)}{4}$ |
| $(4,3/2,240)$   | $\frac{5(3e_1-f_1-g_1-h_1)}{2}$ |

**Theorem 5.1.4.** Let $X$ be a projective $K3$ surface that admits a symplectic automorphism of order 4, such that $rk(NS(X)) = 15$. Then, using the notation introduced in Remark 5.6, $NS(X)$ is one of the following lattices:

1. For every $d \in \mathbb{N}$, $NS(X) = \Omega_4 \oplus \langle 2d \rangle$.
2. For $d \equiv 1 \pmod{4}$, $NS(X) = (\Omega_4 \oplus \langle 2d \rangle)^*$; for $d \equiv 2 \pmod{4}$, there are two nonisometric possibilities.
3. For $d \equiv 0 \pmod{4}$, $NS(X) = (\Omega_4 \oplus \langle 2d \rangle)^{**}$.

**Proof.** By Corollary 1.1.6, overlattices of index 2 of $\Omega_4 \oplus \langle 2d \rangle$ correspond to isotropic elements in $A_{\Omega_4} \oplus \langle 2d \rangle$ of the form $(L + v)/2$, where $L$ generates $\langle 2d \rangle$ and $v \in \Omega_4$ is chosen up to the action of $O(\Omega_4)$ on $A_{\Omega_4}$. Requiring

$$\left(\frac{L + v}{2}\right)^2 = \frac{d}{2} + \left(\frac{v}{2}\right)^2 = 0 \quad \text{in} \quad \mathbb{Q}/2\mathbb{Z},$$

we see that for each value of $d$ modulo 4, $v/2$ belongs to one of the classes of $\approx_{\Omega_4}$ described in Proposition 5.1.2 containing elements of order 2. Therefore, for $d = 4h + 1$, no overlattice of index 2 of $\Omega_4 \oplus \langle 2d \rangle$ exists; for $d = 4h$ or $d = 4h + 3$, there exists one overlattice of index 2 of $\Omega_4 \oplus \langle 2d \rangle$; for $d = 4h + 2$, there are two equivalence classes for $\approx_{\Omega_4}$, and the corresponding overlattices of $\Omega_4 \oplus \langle 2d \rangle$ are not in the same genus. This can be proved using [14, Proposition 1.15.1]: The overlattices corresponding to $(2,1,10)$ and $(2,1,6)$ have discriminant group $(\mathbb{Z}/4\mathbb{Z})^4 \times \mathbb{Z}/4\mathbb{Z}$ and discriminant form, respectively,

$$\begin{pmatrix} 0 & 1/4 & 1/4 & 1/2 & 1/4 & 1/4 & 1/4 & 1/2 & 1/4 & 1/2 \\ 1/4 & 0 & 1/4 & 1/4 & 1/2 & 1/4 & 1/4 & 1/2 & 1/4 & 1/2 \\ \end{pmatrix} \oplus \begin{pmatrix} d+1 \\ 2d \\ \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/4 & 1/4 & 1/2 & 1/4 & 1/4 & 1/2 & 1/4 & 1/2 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 1/2 & 1/4 & 1/4 & 1/2 & 1/4 & 1/2 \\ \end{pmatrix} \oplus \begin{pmatrix} d+1 \\ 2d \\ \end{pmatrix}.$$ 

By a similar argument, overlattices of type $(\Omega_4 \oplus \langle 2d \rangle)^*$ exist only for $d \equiv 0 \pmod{4}$; to each value of $d$ modulo 16 corresponds one equivalence class of $\approx_{\Omega_4}$ of those containing elements of order 4. □

**Theorem 5.1.5.** Each of the lattices presented in Theorem 5.1.4 admits a unique primitive embedding in $\Lambda_{K3}$ up to isometries of $\Lambda_{K3}$.

**Proof.** Let $NS(X) = \Omega_4 \oplus \langle 2d \rangle$, let $T(X) = NS(X)_{\Lambda_{K3}}$; it holds $\lambda(A_{T(X)}) = \lambda(A_{NS(X)}) = (2, 2, 4, 4, 4, 4, 2d)$ $(2,2,4,4,4,4,4d)$ for $d = 1$; if $d$ is odd, then $Z/2dZ = Z/2Z \times Z/dZ$, so we have $A_{NS(X)} = (Z/2Z)^3 \times A'$ and $NS(X)$ satisfies Corollary 1.1.12; if $d$ is even, then $\lambda(A_{T(X)}) = (2, 2, 4, 4, 4, 4, 4d')$, so $T(X)$ satisfies Proposition 1.1.10, and therefore $NS(X)$ satisfies Theorem 1.1.8.
Let \( NS(X) = (\Omega_4 \oplus \langle 2d \rangle)' \), let \( T(X) = NS(X)^{4\times K_3} \): For \( d \equiv 2, 3 \pmod{4} \) \( A_{NS(X)} \) has length 5, so there exists a unique primitive embedding of \( NS(X) \) in \( \Lambda_{K_3} \), thanks to Theorem 1.1.11; for \( d = 4d' \), \( T(X) \) satisfies Proposition 1.1.10, for \( \lambda(A_{T(X)}) = (2, 2, 2, 4, 4, 8d') \); therefore \( NS(X) \) satisfies Theorem 1.1.8.

Let \( NS(X) = (\Omega_4 \oplus \langle 2d \rangle)^* \): Then \( A_{NS(X)} \) has length 5, so it satisfies Theorem 1.1.11. \( \square \)

**Example 5.1.6.** We now use Remark 1.1.7 (and the notation of Section 3.1) to provide for each \( x_{(k,g,n)} \) in Corollary 5.1.3 a primitive class \( L \in \Omega_4^{H^2(X,\mathbb{Z})} \) such that \( L^2 = 2d \) and \( L/k + x_{(k,g,n)} \) is an integral class in \( H^2(X, \mathbb{Z}) \). By Theorems 5.1.4 and 5.1.5, the general member of each irreducible component of the moduli space of projective K3 surfaces with a symplectic automorphism of order 4 is obtained as one of these examples.

1. For every \( d \in \mathbb{N} \setminus \{0\} \), the class
   \[
   L_0 = L_0(d) = \frac{a_1 + a_2 + \omega_1 + \omega_2}{2} + \frac{d}{2}(\frac{-a_1 - a_2 + \omega_1 - \omega_2}{2})
   \]
   generates the lattice \( \langle 2d \rangle \) such that \( \Omega_4 \oplus \langle 2d \rangle \) is primitively embedded in \( H^2(X, \mathbb{Z}) \).

2. For \( d = 4(h - 1) \), \( h \in \mathbb{N} \setminus \{0, 1\} \) the class
   \[
   L_{2,0}(h) = 2L_0(h) + e_3 + f_3 + g_3 + h_3
   \]
   generates the lattice \( \langle 2d \rangle \) such that \( (\Omega_4 \oplus \langle 2d \rangle)' \) is primitively embedded in \( H^2(X, \mathbb{Z}) \); \( L_{2,0}/2 + x_{(2,0,15)} \) is in fact an integral class in \( H^2(X, \mathbb{Z}) \).

3. For \( d = 4h + 2 \), \( h \in \mathbb{N} \), the class
   \[
   L_{2,2}^{(1)}(h) = 2L_0(h) + \rho
   \]
   generates the lattice \( \langle 2d \rangle \) such that \( (\Omega_4 \oplus \langle 2d \rangle)' \) is primitively embedded in \( H^2(X, \mathbb{Z}) \); \( L_{2,2}^{(1)}/2 + x_{(2,1,6)} \) is in fact an integral class in \( H^2(X, \mathbb{Z}) \).

4. For \( d = 4(h - 1) + 2 \), \( h \in \mathbb{N} \setminus \{0\} \), the class
   \[
   L_{2,2}^{(2)}(h) = 2L_0(h) + \rho + e_3 + f_3 + g_3 + h_3
   \]
   generates the lattice \( \langle 2d \rangle \) such that \( (\Omega_4 \oplus \langle 2d \rangle)' \) is primitively embedded in \( H^2(X, \mathbb{Z}) \); \( L_{2,2}^{(2)}/2 + x_{(2,1,10)} \) is in fact an integral class in \( H^2(X, \mathbb{Z}) \).

5. For \( d = 4h + 3 \), \( h \in \mathbb{N} \), the class
   \[
   L_{2,3}(h) = 2L_0(h) + \omega_2 + \frac{e_1 + f_1 + g_1 + h_1 + e_4 + f_4 + g_4 + h_4 + a_1 + a_2 + 3\rho}{2}
   \]
   generates the lattice \( \langle 2d \rangle \) such that \( (\Omega_4 \oplus \langle 2d \rangle)' \) is primitively embedded in \( H^2(X, \mathbb{Z}) \); \( L_{2,3}/2 + x_{(2,1/2,32)} \) is in fact an integral class in \( H^2(X, \mathbb{Z}) \).

6. For \( d = 16(h - 1) \), \( h \in \mathbb{N} \setminus \{0, 1\} \), the class
   \[
   L_{4,0}(h) = 4L_0(h) + e_1 + f_1 + g_1 + h_1 + 3(e_3 + f_3 + g_3 + h_3) + e_4 + f_4 + g_4 + h_4 + a_1 + a_2 + \rho + 2\omega_2
   \]
   generates the lattice \( \langle 2d \rangle \) such that \( (\Omega_4 \oplus \langle 2d \rangle)^* \) is primitively embedded in \( H^2(X, \mathbb{Z}) \); \( L_{4,0}/4 + x_{(4,0,240)} \) is in fact an integral class in \( H^2(X, \mathbb{Z}) \).

7. For \( d = 16h + 4 \), \( h \in \mathbb{N} \), the class
   \[
   L_{4,4}(h) = 4L_0(h) + a_1 + a_2 + \rho + 2\omega_2
   \]
generates the lattice \langle 2d \rangle such that \((\Omega_4 \oplus \langle 2d \rangle)^* \) is primitively embedded in \(H^2(X, \mathbb{Z})\); \(L_{4,4}/4 + x_{4(3/2,2,240)}\) is in fact an integral class in \(H^2(X, \mathbb{Z})\).

8. For \(d = 16(h - 4) + 8, \ h \in \mathbb{N} \setminus \{0, 1, 2, 3\}\), the class

\[
L_{4,8}(h) = 4L_0(h) + \rho + \frac{3(e_2 + f_2 + g_2 + h_2) + 7(e_4 + f_4 + g_4 + h_4)}{2}
\]

generates the lattice \langle 2d \rangle such that \((\Omega_4 \oplus \langle 2d \rangle)^* \) is primitively embedded in \(H^2(X, \mathbb{Z})\); \(L_{4,8}/4 + x_{4(1,2,240)}\) is in fact an integral class in \(H^2(X, \mathbb{Z})\).

9. For \(d = 16h + 12, \ h \in \mathbb{N}\), the class

\[
L_{4,12}(h) = 4L_0(h) + e_1 + f_1 + g_1 + h_1 + 2(e_4 + f_4 + g_4 + h_4 + \rho + \omega_1 + \omega_2)
\]

generates the lattice \langle 2d \rangle such that \((\Omega_4 \oplus \langle 2d \rangle)^* \) is primitively embedded in \(H^2(X, \mathbb{Z})\); \(L_{4,12}/4 + x_{4(1/2,2,240)}\) is in fact an integral class in \(H^2(X, \mathbb{Z})\).

### 5.2 Relations between the families associated to the action of \(\tau\), and those associated to the action of \(\tau^2\)

Let \(X\) be a projective K3 surface which admits a symplectic automorphism \(\tau\) of order 4, and suppose \(NS(X)\) is one of the lattices in Theorem 5.1.4. Since \(\tau^2\) is a symplectic involution, \(NS(X)\) has to primitively contain \(\Omega_2 \oplus \langle 2d \rangle\) or some overlattice of it: Indeed, \(X\) is a special member of one of the families of projective K3 surfaces that admit a symplectic involution, whose general element has Picard number 9.

**Theorem 5.2.1** [7, Proposition 2.2]. Let \(X\) be a projective K3 surface with a symplectic involution \(\iota\), such that \(rk(NS(X)) = 9\). Then we have the following possible cases for \(NS(X)\):

(a) for every \(d\), \(NS(X) = \Omega_4 \oplus \langle 2d \rangle\);
(b) for \(d\) even, \(NS(X) = (\Omega_2 \oplus \langle 2d \rangle)'\).

**Remark 5.2.2.** The action of \(O(\Omega_2)\) on \(A_{\Omega_2}\) has two nontrivial orbits, that consist of elements of order 2 and square, respectively, 0 or 1. Depending on the value of \(d\) (modulo 4, the overlattice \((\Omega_2 \oplus \langle 2d \rangle)'\) is obtained using one or the other (compare to the proof of Theorem 5.1.4).

To determine which of these families our Néron–Severi groups belong to, we fix the embedding of \(\Omega_2\) in \(\Omega_4\) as in Section 3.2, so that the symplectic involution we are considering is indeed \(\tau^2\); as class \(L\) of square \(2d\), we take again those defined in Example 5.1.6.

**Theorem 5.2.3.**

1. For every \(d \in \mathbb{N}\), \(NS(X) = \Omega_4 \oplus \langle 2d \rangle\) corresponds to case (a) of Theorem 5.2.1.
2. For \(d \equiv 2, 3 \pmod{4}\), \(NS(X) = (\Omega_4 \oplus \langle 2d \rangle)'\) corresponds to case (a) of Theorem 5.2.1.
3. For \(d \equiv 0 \pmod{4}\), \(NS(X) = (\Omega_4 \oplus \langle 2d \rangle)'\) corresponds to case (b) of Theorem 5.2.1.
4. For \(d \equiv 0 \pmod{4}\), \(NS(X) = (\Omega_4 \oplus \langle 2d \rangle)^*\) corresponds to case (b) of Theorem 5.2.1

The following table describes the situation: Saying that \(L\) glues to \(\Omega_2\) (and similarly for \(R\)), we mean there exists an element in \(NS(X)\) of the form \((L + v)/2, v \in \Omega_2\).
Proof. The class $L_0$ does not glue to $\Omega_4$, so it cannot glue either to $\Omega_1$ or to $R$. Since $L_2 = 2d$ with $d \equiv 3 \pmod{4}$, this case corresponds necessarily to case (a) of Theorem 5.2.1; $\Omega_2 \oplus R \oplus \langle 2d \rangle$ has index $2^5$ in $\text{NS}(X)$, because there exists $r \in R$ such that $(L_2, r) \in \text{NS}(X)$. Since $L_1 \equiv 2, 3 \pmod{4}$, this case corresponds necessarily to case (a) of Theorem 5.2.1; $\Omega_2 \oplus R \oplus \langle 2d \rangle$ has index $2^5$ in $\text{NS}(X)$, because there exists $r \in R$ such that $(L_1, r) \in \text{NS}(X)$.

The gluings for the cases in which $d = 4k$ are described as follows: $(L_2, 0) \in \text{NS}(X)$ and $(L_2, 0) \in \text{NS}(X)$ for $\tilde{v} = e_1 - g_1 + f_1 - h_1 \in \Omega_2$, $r = e_1 - f_1 + g_1 - h_1 \in R$; since $(\tilde{v} + r) \in \text{NS}(X)$, this case corresponds necessarily to case (a) of Theorem 5.2.1; $\text{NS}(X)$ is obtained as over lattice of $\Omega_2 \oplus R \oplus \langle 2d \rangle$ by gluing first $\Omega_2$ with $R$ to get $\Omega_4$, and then $L_{4, k}$ with $\Omega_4$ as in Example 5.1.6.

5.3 | Families of projective K3 surfaces which arise as desingularization of $X/\tau$

Projective surfaces $\tilde{Y}$ that are the resolution of $X/\tau$ have to primitively contain in their Néron–Severi group both the exceptional lattice $M$ described in Section 4.3 [13, sections 5–7] and a positive class $H$ of square $2e$ that generates $M \bot \text{NS}(\tilde{Y})$; therefore, $\tilde{Y}$ is polarized with the lattice $M \oplus \langle 2e \rangle$ or one of its cyclic overlattices.

**Theorem 5.3.1.** The relation $\sim_M$ (see Definition 5.1.1) divides $A_M$ in seven nontrivial equivalence classes (plus the trivial one $\{0\}$):

| 0     | 1/2 | 1     | 3/2 |
|-------|-----|-------|-----|
| 2     | 3   | 8     | 4   | 0   |
| 4     | 12  | 12    | 12  | 12  |

Each of them corresponds to an equivalence class for $\approx_M$, except for (2,1), which is the union of two classes: (2,1,1) and (2,1,3). We give a representative element $x_{(k,g,n)}$ for each nontrivial equivalence class $(k,g,n)$ in terms of the generators of $M$ introduced in Section 4.3.

| class $(k, g, n)$ | representative $x_{(k,g,n)}$ |
|-------------------|-----------------------------|
| (2,0,3)           | $m_1^2 + m_2^2 + m_3^2 + m_4^2$ |
| (2,1,2,8)         | $m_1^4 + m_2^4 + m_3^4 + m_4^4$ |
| (2,1,1)           | $m_1^2 + m_2^2 + m_3^2 + m_4^2$ |
| (2,1,3)           | $m_1^2 + m_2^2 + m_3^2 + m_4^2$ |
| (4,0,12)          | $m_1^4 + m_2^4 + m_3^4 + m_4^4$ |
| (4,1,2,12)        | $m_1^2 + m_2^2 + m_3^2 + m_4^2$ |
| (4,1,12)          | $m_1^4 + m_2^4 + m_3^4 + m_4^4$ |
| (4,3/2,12)        | $m_1^2 + m_2^2 + m_3^2 + m_4^2$ |
Theorem 5.3.2. Let $\tilde{Y}$ be a projective $K3$ surface such that $rk(\text{NS}(\tilde{Y})) = 15$ and $\text{NS}(\tilde{Y})$ contains primitively $M$ and $\langle 2e \rangle$, $e \in \mathbb{N} \setminus \{0\}$. Then, using the notation introduced in Remark 5.6, $\text{NS}(\tilde{Y})$ is one of the following:

1. For every $e$, $\text{NS}(\tilde{Y}) = M \oplus \langle 2e \rangle$.
2. For $e \equiv 1 \pmod{4}$, $\text{NS}(\tilde{Y}) = (M \oplus \langle 2e \rangle)'$; for $e \equiv 2 \pmod{4}$, there are two nonisometric possibilities.
3. For $e \equiv 0 \pmod{4}$, $\text{NS}(\tilde{Y}) = (M \oplus \langle 2e \rangle)^*$.

Each of these lattices admits a unique primitive embedding in $\Lambda_{K3}$.

Proof. The overlattices of $M \oplus \langle 2e \rangle$ are in bijection with the equivalence classes for $\approx M$. Fix the primitive embedding $M \hookrightarrow \Lambda_{K3}$ as in Section 4.3: Since the orthogonal complement of $M$ is the lattice $\pi_4^*H^2(X, \mathbb{Z})$, we can use as generators of the lattice $\langle 2e \rangle$ the primitive classes $L \in \text{NS}(\tilde{Y})$ obtained from $\pi_4^*L$ (with $L$ as in Example 5.1.6) as follows. Refer to Section 4.1 for the computation of $\pi_4^*L$, and see also the corresponding classes $D_1$ in Section 6.1 for the explicit gluing element $(L + m) \div k$, $m \in \mathbb{A}_M$, $k = 2, 4$:

| $\text{NS}(\tilde{Y})$ | $e$ | $\overline{L}$ |
|-------------------------|-----|---------------|
| $M \oplus \langle 2e \rangle$ | $4(h - 1)$ | $\pi_4^*L_{4,0}(h)/4$ |
| | $4h - 4 + 2$ | $\pi_4^*L_{4,2}(h)/4$ |
| | $4h + 3$ | $\pi_4^*L_{4,3}(h)/4$ |
| $(M \oplus \langle 2e \rangle)'$ | $4(h - 1)$ | $\pi_4^*L_{2,1}(h)/2$ |
| | $4h + 2$ | $\pi_4^*L_{2,1}(h)/2$ |
| | $4h - 1 + 2$ | $\pi_4^*L_{2,3}(h)/2$ |
| | $4h + 3$ | $\pi_4^*L_{2,3}(h)/2$ |
| $(M \oplus \langle 2e \rangle)^*$ | $4h$ | $\pi_4^*L_0(h)$ |

Notice that for every choice of $e \equiv 2 \pmod{4}$, there exist two nonisometric lattices of the form $(M \oplus \langle 2e \rangle)'$, realized using alternatively as generator of $\langle 2e \rangle$ the classes $\pi_4^*L(i)$, $i = 1, 2$; indeed, for $h$ odd, both of them glue to the class $(2,1,1)$, while for $h$ even, they both glue to $(2,1,3)$. The resulting lattices belong to different genera.

Each of the possible lattices $\text{NS}(\tilde{Y})$ admits a unique primitive embedding in $H^2(X, \mathbb{Z})$, because $\ell(\text{NS}(\tilde{Y})) \leq 5$, so Theorem 1.1.11 holds.

Theorem 5.3.3. There is a 1:1 correspondence between families of $K3$ surfaces $X$ with $\text{NS}(X)$ as in Theorem 5.1.4, and families of $K3$ surfaces $\tilde{Y}$ with $\text{NS}(\tilde{Y})$ as in Theorem 5.3.2. The primitive classes $L \in \text{NS}(\tilde{Y})$ that generate the sublattices $\langle nd \rangle$ as stated are indicated in curly brackets. For $d \equiv 2 \pmod{4}$, the lattices $S^{(1)}$, $S^{(2)}$ are not isometric.

| $\text{NS}(X)$ | $\text{NS}(\tilde{Y})$ |
|----------------|------------------|
| $\forall d$ | $(\Omega_4 \oplus (2d))$ |
| $d \equiv 2$ | $(\Omega_4 \oplus (2d))^{(1)}$ |
| | $(\Omega_2 \oplus (2d))^{(2)}$ |
| $d \equiv 3$ | $(\Omega_4 \oplus (2d))'$ |
| | $(\Omega_4 \oplus (2d))'$ |
| $d \equiv 0$ | $(\Omega_4 \oplus (2d))^*$ |
| | $(M \oplus (d/2))$ |

Proof. The map $\pi_4$ kills $\Omega_4$, so the possible Néron–Severi groups for the general smooth quotient surface $\tilde{Y} = \tilde{X}/\tau$ are determined by how the image of the ample class $L$ glues to the exceptional lattice $M$. For each of the $L$ in Example 5.1.6, we compute $\pi_4^*L$ using the description of the image lattice $\pi_4^*H^2(X, \mathbb{Z})$ given in Section 4.1; we find the unique integral and primitive $\overline{L} = \pi_4^*L/k$, where $k$ can be 1, 2, or 4 depending on the case, and we then compare $\overline{L}$ to $L$.  

5.4 Families of projective K3 surfaces which arise as desingularization of $X/\tau^2$

The process used in the previous section can be also applied to describe the K3 surfaces $Z$ that are resolution of $X/\tau^2$, and the relations between $\text{NS}(X)$ and $\text{NS}(Z)$; for the general symplectic involution, this was already done by Garbagnati and Sarti:
**Theorem 5.4.1** [5, Corollary 2.2]. Let $X$ be an algebraic K3 surface with $rk(NS(X)) = 9$ admitting a Nikulin involution $\iota$, and let $\tilde{Z}$ be the desingularization of the quotient $X/\iota$. Then:

(a) $NS(X) = \Omega_2 \oplus (2d)$ if and only if $NS(\tilde{Z}) = (\Omega_2 \oplus (4d))'$;
(b) $NS(X) = (\Omega_2 \oplus (2d))'$ if and only if $NS(\tilde{Z}) = N \oplus (d)$.

The analog to this theorem when $\iota = \tau^2$ is as follows.

**Theorem 5.4.2.** Let $Z$ be a K3 surface such that $\text{rk}(NS(Z)) = 15$; suppose $NS(Z)$ admits a primitive embedding of both $\Gamma$ (see Definition 4.2.1) and of a class of positive square $2d$ that generates $\Gamma^\perp \cap NS(Z)$. Then $d = 2N$, and $NS(Z)$ is one of the following:

1. for every $x$, $NS(Z) = (\Gamma \oplus (4x))^*$, uniquely determined;
2. for $x \equiv 2, 3$ (mod 4), $NS(Z) = (\Gamma \oplus (4x))^*$, uniquely determined.

Moreover, there exists a unique primitive embedding of these lattices in $H^2(Z, \mathbb{Z})$ up to isometries of the latter.

**Proof.** The lattice $\Gamma \oplus (2d)$ cannot be the Néron–Severi of a K3 surface, since $\lambda(A_{\Gamma \oplus (2d)}) = (2, 2, 2, 2, 2, 2, 4, 4, 2d)$, so its length is $9 > 22 - \text{rk}(\Gamma \oplus (2d))$ (see Remark 1.1.9).

Consider the table of nontrivial equivalence classes for $\sim_{\Gamma}$:

| class $(k, g, n)$ | representative $x_{(k,g,n)}$ | glues to: |
|-------------------|--------------------------------|-----------|
| $(2,0,1)$         | $(n_1 + n_2 + n_3 + n_4)/2$   | $L_{2,1}^{(1)}/2$ for $i = 1, 2$ |
| $(2,0,6)$         | $(n_1 + n_2 + n_3 + n_4)/2$   | $\hat{L}_{0}(h)/2, h = z_0$ |
| $(2,0,30)$        | $(n_1 + n_2 + n_3 + n_4)/2$   | $\hat{L}_{4,0}/2$ |
| $(2,0,90)$        | $(n_1 + n_2 + n_3 + n_4)/2$   | $\hat{L}_{4,8}/2$ |
| $(2,1,2)$         | $(n_1 + n_2 + n_3 + n_4)/2$   | $\hat{L}_{2,0}(h)/2, h = z_1$ |
| $(2,1,6)$         | $(n_1 + n_2 + n_3 + n_4)/2$   | $\hat{L}_{2,2}(h)/2, h = z_1$ |
| $(2,1,30)$        | $(n_1 + n_2 + n_3 + n_4)/2$   | $\hat{L}_{4,4}$ |
| $(2,1,90)$        | $(n_1 + n_2 + n_3 + n_4)/2$   | $\hat{L}_{4,12}$ |
| $(4,1/4,256)$     | $(n_1 + n_2 + n_3 + n_4)/2$   | $\hat{L}_{2,3}(h)/4, h = z_2$ |
| $(4,1/2,24)$      | $(n_1 + n_2 + n_3 + n_4)/2$   | $\hat{L}_{2,3}(h)/4, h = z_1$ |
| $(4,1/2,120)$     | $(n_1 + n_2 + n_3 + n_4)/2$   | $\hat{L}_{2,3}(h)/4, h = z_0$ |
| $(4,5/4,256)$     | $(n_1 + n_2 + n_3 + n_4)/2$   | $\hat{L}_{2,3}(h)/4, h = z_0$ |
| $(4,3/2,40)$      | $(n_1 + n_2 + n_3 + n_4)/2$   | $\hat{L}_{2,3}(h)/4, h = z_1$ |
| $(4,3/2,72)$      | $(n_1 + n_2 + n_3 + n_4)/2$   | $\hat{L}_{2,3}(h)/4, h = z_0$ |
Now, the classes (2,0,1) and (2,1,2) produce overlattices of \( \Gamma \oplus \langle 4x \rangle \) that are not admissible as Néron–Severi of a K3 surface, because they have \( \ell = 9 \) (see Remark 1.1.9). For the remaining classes \((k,g,n)\), those contained in the same equivalence class \((k,g)\) for the relation \(\sim_G\) give rise to isomorphic lattices: Indeed, having fixed \(x\), it can be proved that all the lattices of the form \((\Gamma \oplus \langle 4x \rangle)^*\) are in the same genus, and the same holds for all the lattices of the form \((\Gamma \oplus \langle 4x \rangle)^*\); however, since \(\lambda(A_{\Gamma \oplus \langle 4x \rangle}^*) = (2,2,2,4,4,4,4x)\), and \(\lambda(A_{\Gamma \oplus \langle 4x \rangle}^*) = (2,2,2,2,2,4x)\), actually \((\Gamma \oplus \langle 4x \rangle)^*\) and \((\Gamma \oplus \langle 4x \rangle)^*\) are unique in their genus by Proposition 1.1.10. Furthermore, they admit a unique primitive embedding in \(\Lambda_{K3}\), as we can apply Proposition 1.1.10 to the corresponding transcendental lattices, both of signature \((2,5)\) and length 7, \(T' = ((\Gamma \oplus \langle 4x \rangle)^*)^\perp\), \(T^* = ((\Gamma \oplus \langle 4x \rangle)^*)^\perp\): Indeed \(\lambda(A_{T'}^*) = \lambda(A_{\Gamma \oplus \langle 4x \rangle}^*)\), and \(\lambda(A_{T^*}^*) = \lambda(A_{\Gamma \oplus \langle 4x \rangle}^*)\).

**Theorem 5.4.3.** Let \(\tau\) be a symplectic automorphism of order 4 on a projective K3 surface \(X\) such that \(rk(\text{NS}(X)) = 15\), and let \(\tilde{Z}\) be the resolution of the quotient \(X/\tau^2\): The following table describes the correspondence between \(\text{NS}(X)\) and \(\text{NS}(\tilde{Z})\). The primitive classes \(\tilde{L}\) in \(\text{NS}(\tilde{Z})\) that generate the sublattices \((md)\) as stated are indicated in curly brackets. For \(d \equiv 2 \pmod{4}\), the lattices \(S^{(1)}, S^{(2)}\) are not isometric. Saying \(\tilde{L}\) glues to \(N\) means there exists an element in \(\text{NS}(\tilde{Z})\) of the form \((\tilde{L}+n)/2, n \in N\).

| \(\text{NS}(X)\) | \(\text{NS}(\tilde{Z})\) | \(\tilde{L}\) glues to \(N\) |
|-----------------|-----------------|-----------------|
| \(\forall d\)   | \(\Omega_4 \oplus \langle 2d \rangle\) | \((\langle 4d \rangle \oplus \Gamma)^*\) | \([L = \pi_2, L]\) Yes |
| \(d \equiv 2\)  | \(\langle \Omega_4 \oplus \langle 2d \rangle \rangle^{(1)}\) | \([\langle 4d \rangle \oplus \Gamma\}]^*\) | \([L = \pi_2, L]\) Yes |
|                 | \(\langle \Omega_4 \oplus \langle 2d \rangle \rangle^{(2)}\) | \([\langle 4d \rangle \oplus \Gamma\}]^*\) | \([L = \pi_2, L]\) Yes |
| \(d \equiv 0\)  | \(\langle \Omega_4 \oplus \langle 2d \rangle \rangle^*\) | \([\langle d \rangle \oplus \Gamma\}]^*\) | \([L = \pi_2, L]\) No |
| \(d \equiv 0\)  | \(\langle \Omega_4 \oplus \langle 2d \rangle \rangle^*\) | \([\langle d \rangle \oplus \Gamma\}]^*\) | \([L = \pi_2, L]\) No |

**Proof.** Recall from Theorem 5.2.3 the possible Néron–Severi groups of \(X\). We use the map \(\pi_{2*}\) (see Section 4.1) to compute \(\tilde{L}\) for each of the \(L\) in Example 5.1.6, and we check their eventual gluing to \(N\) following Section 4.2.

Fix \(d = 4h + 2 = 4(k - 1) + 2\), and consider the ample classes \(L_{1,1}'(h), L_{2,2}'(k)\) of \(X\) that generate the two nonisomorphic overlattices of index 2 of \(\Omega_4 \oplus \langle 2d \rangle\): Denote these lattices \(\text{NS}(X)^{(1)}\) and \(\text{NS}(X)^{(2)}\). Now take \(\tilde{Z}\) the resolution of \(X/\tau^2\): from the previous theorem, we have \(\text{NS}(\tilde{Z})^{(1)} = \langle \Gamma, \tilde{L}_{1,1}'(h) \rangle \simeq \text{NS}(\tilde{Z})^{(2)} = \langle \Gamma, \tilde{L}_{2,2}'(k) \rangle\). Therefore, for \(d \equiv 2 \pmod{4}\), there is a 2:1 correspondence between \((\Omega_4 \oplus \langle 2d \rangle)^*\)-polarized families of \(X\) and \((\Gamma \oplus \langle 4d \rangle)^*\)-polarized families of \(\tilde{Z}\). Similar considerations apply to \(d \equiv 0 \pmod{4}\).  

## 6 Projective Models

Given a nef and big divisor \(L\) on \(X\), there is a natural map \(\phi_{[L]} : X \rightarrow \mathbb{P}(H^0(X,L)^* ) \simeq \mathbb{P}^n\), with \(n = L^2/2 + 1\). Any automorphism \(\sigma\) of \(X\) that preserves \(L\) induces an action on \(H^0(X,L)^\ast\): In particular, if \(\sigma\) is finite of order \(m\), we can split \(H^0(X,L)^\ast\) in eigenspaces corresponding to the \(m\)-roots of unity.

**Remark 6.1.** Notice that the action of \(\sigma\) on \(H^0(X,L)^\ast\) could have order \(km\) for some integer \(k > 1\): Indeed, if

\[
\sigma^m : (x_0, \ldots, x_n) \mapsto \xi_k(x_0, \ldots, x_n)
\]

for \(\xi_k\) a root of unity, then on \(\mathbb{P}(H^0(X,L)^\ast)\), it will hold \(\sigma^m = id\); but this does not matter when we study the action of the cyclic group \(\langle \sigma \rangle\) on the projective surface \(X\), as we can just take the action of \(\sigma^k\) on \(H^0(X,L)^\ast\) without loss of generality.

Thus, considering a symplectic automorphism \(\tau\) of order 4 on a K3 surface \(X\), we have

\[
H^0(X,L) = V_1 \oplus V_1 \oplus V_{-1} \oplus V_{-1} = W_+ \oplus W_-, 
\]

where \(V_\ast\) are the eigenspaces relative to the action of \(\tau^\ast\), and \(W_\ast\) are relative to \((\tau^2)^\ast\), so that \(W_+ = V_1 \oplus V_{-1}\), and \(W_- = V_1 \oplus V_{-1}\).
6.1 | Eigenspaces of $\tau^*$ and classes in $\text{NS}(\tilde{Y})$

The purpose of this section is to prove the following proposition:

**Proposition 6.1.1** see [7, Proposition 2.7] and [3, Theorem 5.6]. There exist divisors $D_1, \ldots, D_4 \in \text{NS}(\tilde{Y})$ such that

$$H^0(X, L) = \pi_4^*H^0(\tilde{Y}, D_1) \oplus \pi_4^*H^0(\tilde{Y}, D_2) \oplus \pi_4^*H^0(\tilde{Y}, D_3) \oplus \pi_4^*H^0(\tilde{Y}, D_4)$$

and every $\pi_4^*H^0(\tilde{Y}, D_i)$ corresponds to one of the eigenspaces for the action of $\tau^*$ on $H^0(X, L)$.

We start by defining some divisors $D_1, \ldots, D_4$ associated to each $L$ of Example 5.1.6. The proof of the proposition can be found below, and amounts to show that these divisors are indeed the ones in the statement.

Consider the following elements in $M^*$, for $i = 1, \ldots, 4$, $j = 1, 2$ (also see Section 4.3):

$$\alpha_i = \frac{3m_1^i + 2m_2^i + m_3^i}{4}, \quad \beta_i = \frac{m_1^i + 2m_2^i + m_3^i}{2}, \quad \gamma_i = \frac{m_1^i + 2m_2^i + 3m_3^i}{4}, \quad \delta_j = \frac{\tilde{m}_j^2}{2};$$

notice that $(\alpha_i)^2 = (\gamma_i)^2 = -3/4$, $(\beta_i)^2 = -1$, $(\delta_j)^2 = -1/2$ with respect to the intersection form of $M$ extended $\mathbb{Q}$-linearly to $M^*$.

Consider $L_0(d)$; depending on the value of $d$ modulo 4, we define $D_1, \ldots, D_4$ as follows:

| $L_0(d)$ | $d = 0$ | $d = 1$ |
|----------|---------|---------|
| $D_1$    | $\pi_4^*L_0/4 - \gamma^2 - \gamma^4 - \delta^2$ | $\pi_4^*L_0/4 - \gamma^2 - \alpha^2 - \delta^1 - \delta^2$ |
| $D_2$    | $\pi_4^*L_0/4 - \alpha^1 - \alpha^3 - \delta^1$ | $\pi_4^*L_0/4 - \alpha^1 - \beta^1 - \alpha^4$ |
| $D_3$    | $\pi_4^*L_0/4 - \beta^1 - \alpha^2 - \beta^3 - \alpha^4 - \delta^2$ | $\pi_4^*L_0/4 - \beta^1 - \alpha^2 - \gamma^3 - \beta^4 - \delta^1 - \delta^2$ |
| $D_4$    | $\pi_4^*L_0/4 - \gamma^1 - \beta^2 - \gamma^3 - \beta^4 - \delta^1$ | $\pi_4^*L_0/4 - \gamma^1 - \beta^2 - \gamma^4$ |

Consider $L_0(d)$; depending on the value of $d$ modulo 4, we define $D_1, \ldots, D_4$ as follows:

| $L_0(d)$ | $d = 2$ | $d = 3$ |
|----------|---------|---------|
| $D_1$    | $\pi_4^*L_0/4 - \gamma^2 - \beta^3 - \alpha^4 - \delta^2$ | $\pi_4^*L_0/4 - \gamma^2 - \beta^3 - \alpha^4 - \delta^2$ |
| $D_2$    | $\pi_4^*L_0/4 - \alpha^1 - \gamma^3 - \beta^4 - \delta^1$ | $\pi_4^*L_0/4 - \alpha^1 - \gamma^4$ |
| $D_3$    | $\pi_4^*L_0/4 - \beta^1 - \alpha^2 - \gamma^4 - \delta^2$ | $\pi_4^*L_0/4 - \beta^1 - \alpha^2 - \gamma^4 - \delta^2$ |
| $D_4$    | $\pi_4^*L_0/4 - \gamma^1 - \beta^2 - \gamma^3 - \alpha^4$ | $\pi_4^*L_0/4 - \gamma^1 - \beta^2 - \gamma^4$ |

Consider $L_2,h(\sigma)$, whose square is $2d = 8(h - 1)$; depending on the value of $h$ modulo 2, we define $D_1, \ldots, D_4$ as follows:

Consider $L_{2,j}(h)$, $j = 1, 2$; recall that $L_{2,1}(h)^2 = 8h + 4$, while $L_{2,2}(h)^2 = 8h - 4$: thus, any value of $d \equiv 2 (\mod 4)$ can be realized both with $h$ even and $h$ odd, using one between $L_{2,1}^{(1)}$, $L_{2,2}^{(2)}$ alternatively, giving two nonisomorphic cases.

| $L_{2,j}(h)$ | $h = 0$ | $h = 1$ |
|--------------|---------|---------|
| $D_1$        | $\pi_4^*L_{2,j}^{(1)}/4 - \beta^1 - \beta^3 - \delta^1 - \delta^2$ | $\pi_4^*L_{2,j}^{(1)}/4 - \beta^1 - \beta^3 - \delta^1 - \delta^2$ |
| $D_2$        | $\pi_4^*L_{2,j}^{(1)}/4 - \alpha^1 - \alpha^3 - \gamma^3 - \gamma^4$ | $\pi_4^*L_{2,j}^{(1)}/4 - \alpha^1 - \alpha^3 - \alpha^4$ |
| $D_3$        | $\pi_4^*L_{2,j}^{(1)}/4 - \beta^1 - \beta^2 - \delta^1 - \delta^2$ | $\pi_4^*L_{2,j}^{(1)}/4 - \beta^1 - \beta^2 - \beta^3 - \delta^1 - \delta^2$ |
| $D_4$        | $\pi_4^*L_{2,j}^{(1)}/4 - \gamma^1 - \gamma^2 - \alpha^3 - \alpha^4$ | $\pi_4^*L_{2,j}^{(1)}/4 - \gamma^1 - \gamma^2 - \gamma^3 - \gamma^4$ |

Consider $L_{2,3}(h)$, whose square is $2d = 2(4h + 3)$; depending on the value of $h$ modulo 2, we define $D_1, \ldots, D_4$ as follows:
TABLE 1  Euler characteristics.

| No. | L  | $\chi(D_1)$ | $\chi(D_2)$ | $\chi(D_3)$ | $\chi(D_4)$ |
|-----|----|-------------|-------------|-------------|-------------|
| $d = 4$ | 1 | $L_0$ | $(d + 3)/4$ | $(d + 3)/4$ | $(d - 1)/4$ | $(d + 3)/4$ |
| 2 | $L_0$ | $(d + 2)/4$ | $(d + 2)/4$ | $(d + 2)/4$ | $(d + 2)/4$ |
| 3 | $L_1$ | $(d + 2)/4$ | $(d + 2)/4$ | $(d + 2)/4$ | $(d + 2)/4$ |
| 4 | $L_2$ | $(d + 2)/4$ | $(d + 2)/4$ | $(d + 2)/4$ | $(d + 2)/4$ |
| $d = 4$ | 5 | $L_0$ | $(d + 5)/4$ | $(d + 1)/4$ | $(d + 1)/4$ | $(d + 1)/4$ |
| 6 | $L_2$ | $(d + 5)/4$ | $(d + 1)/4$ | $(d + 1)/4$ | $(d + 1)/4$ |
| $d = 4$ | 7 | $L_0$ | $d/4 + 1$ | $d/4 + 1$ | $d/4$ | $d/4$ |
| 8 | $L_2$ | $d/4 + 1$ | $d/4$ | $d/4 + 1$ | $d/4$ |
| 9 | $L_4$ | $d/4 + 2$ | $d/4$ | $d/4$ | $d/4$ |

\[ \begin{align*}
L_{2,3}(h) & = \begin{cases} 
\pi_4 L_2/4 - \beta^1 - \delta^2 
\end{cases} \\
D_1 & = \begin{cases} 
\pi_4 L_2/4 - \alpha^1 - \alpha^2 - \alpha^3 - \gamma^4 - \delta^4
\end{cases} \\
D_2 & = \begin{cases} 
\pi_4 L_2/4 - \alpha^1 - \alpha^2 - \gamma^4 - \delta^4
\end{cases} \\
D_3 & = \begin{cases} 
\pi_4 L_2/4 - \alpha^1 - \alpha^2 - \gamma^4 - \delta^4
\end{cases}
\end{align*} \]

Consider $L_{4,j}$ for $j = 0, 4, 8, 12$; in this case, $\pi_4 L_{4,j}/4$ is primitive in $\text{NS}(Y)$, and we can define $D_1, ..., D_4$ simultaneously for any $j$ and any value of $h$, as follows:

\[ \begin{align*}
L_{4,j}(h) & = \begin{cases} 
\pi_4 L_4/4 - \gamma^1 - \gamma^3 - \gamma^4 - \delta^4
\end{cases} \\
D_1 & = \begin{cases} 
\pi_4 L_4/4 - \beta^1 - \gamma^3 - \gamma^4 - \delta^4
\end{cases} \\
D_2 & = \begin{cases} 
\pi_4 L_4/4 - \beta^1 - \gamma^3 - \gamma^4 - \delta^4
\end{cases} \\
D_3 & = \begin{cases} 
\pi_4 L_4/4 - \beta^1 - \gamma^3 - \gamma^4 - \delta^4
\end{cases} \\
D_4 & = \begin{cases} 
\pi_4 L_4/4 - \gamma^1 - \gamma^3 - \gamma^4 - \delta^4
\end{cases}
\end{align*} \]

Proof of Proposition 6.1.1. Consider $L$ as any of the ample divisors of $X$ presented in Example 5.1.6, and the corresponding $D_1, ..., D_4$ as in the tables above. Notice that for every $i$, the relation $\pi_4^* (D_i) = L$ is satisfied (since $\pi_4^* M = 0$); Therefore, we always have $\pi_4^* H^0(Y, D_i) \subset H^0(X, L)$. Moreover, $\pi_4^* H^0(Y, D_i(L))$ is entirely contained in one of the eigenspaces $V_i$, (the sections of $D_i$ are in fact well defined on the quotient surface $Y$), and for $i \neq j$, $\pi_4^* H^0(Y, D_i(L))$ and $\pi_4^* H^0(Y, D_j(L))$ are in different eigenspaces, since $D_i(L)$ and $D_j(L)$ intersect differently the exceptional lattice for $i \neq j$.

It remains to show that $H^0(X, L) = \bigoplus_{i=1}^4 \pi_4^* H^0(Y, D_i)$; To do this, it is enough to compute the Euler characteristics $\chi(D_i) = D_i^2/2 + 2$, and check that

\[ d + 2 = \chi(L) = \sum_i \chi(D_i). \]

The results are displayed in Table 1.
such that \( L^2 = 2d \), that generates \( \Omega_{2,4}^{L,NS(X)} \). Then, \( H^0(X, L) \simeq \pi^*H^0(Z, E_1) \oplus \pi^*H^0(Z, E_2) \), with \( E_1, E_2 \) described as follows, for suitable numbering \( n_1, \ldots, n_8 \) of the exceptional curves of \( Z \):

1. if \( NS(X) = \Omega_2 \oplus \langle L \rangle \), and \( d \equiv 0 \pmod{2} \), then \( E_1 = \pi_*L/2 - (n_1 + n_2 + n_3 + n_4)/2 \), \( E_2 = \pi_*L/2 - (n_5 + n_6 + n_7 + n_8)/2 \);
2. if \( NS(X) = \Omega_2 \oplus \langle L \rangle \), and \( d \equiv 1 \pmod{2} \), then \( E_1 = \pi_*L/2 - (n_1 + n_2)/2 \), \( E_2 = \pi_*L/2 - (n_3 + n_4 + n_5 + n_6 + n_7 + n_8)/2 \);
3. if \( NS(X) = (\Omega_2 \oplus \langle L \rangle)' \) (this case occurs only if \( d \equiv 0 \pmod{2} \)), then \( E_1 = \pi_*L/2 \), \( E_2 = \pi_*L/2 - \sum_{i=1}^{8} n_i/2 \).

If \( X \) admits an automorphism \( \tau \) of order 4 and \( \tilde{Z} \) is the minimal resolution of \( X/\tau^2 \), taking \( L \) ample that generates \( \Omega_{2,4}^{NS(X)} \), the Nef divisors \( E_1, E_2 \in NS(Z) \) that satisfy this equality for the examples of ample classes introduced in Example 5.1.6 are defined in the following tables, with the exceptional curves numbered as in Sections 4.2 and 4.3:

| \( L_0(d) \) | \( d = 0 \) | \( d = 1 \) |
|---|---|---|
| \( E_1 \) | \( \pi_*L/2 - (n_1 + n_2 + n_3 + n_4)/2 \) | \( \pi_*L/2 - (n_3 + n_4 + n_5 + n_6 + n_8)/2 \) |
| \( E_2 \) | \( \pi_*L/2 - (n_1 + n_2 + n_3 + n_4)/2 \) | \( \pi_*L/2 - (n_1 + n_5)/2 \) |

| \( L_{2,0}(h) \) | any \( h \) | \( L_{2,2}^{(j)}(h) \) | any \( h, j = 1, 2 \) |
|---|---|---|---|
| \( E_1 \) | \( \pi_*L_{2,0}/2 \) | \( E_1 \) | \( \pi_*L_{2,2}/2 - (n_3 + n_4 + n_6 + n_8)/2 \) |
| \( E_2 \) | \( \pi_*L_{2,0}/2 - \sum_{i=1}^{8} n_i/2 \) | \( E_2 \) | \( \pi_*L_{2,2}/2 - (n_1 + n_2 + n_3 + n_5 + n_7)/2 \) |

| \( L_{2,3}(h) \) | any \( h \) | \( L_{4,4}(h) \) | any \( h, j = 0, 4, 8, 12 \) |
|---|---|---|---|
| \( E_1 \) | \( \pi_*L_{2,3}/2 - (n_3 + n_6)/2 \) | \( E_1 \) | \( \pi_*L_{4,4}/2 \) |
| \( E_2 \) | \( \pi_*L_{2,3}/2 - (n_1 + n_2 + n_3 + n_4 + n_6 + n_7)/2 \) | \( E_2 \) | \( \pi_*L_{4,4}/2 - \sum_{i=1}^{8} n_i/2 \) |

**Proposition 6.2.2.** It holds \( \pi_*^*H^0(Z, E_1) = \pi_*^*H^0(\tilde{Y}, D_1) \oplus \pi_*^*H^0(\tilde{Y}, D_3) \), while \( \pi_*^*H^0(Z, E_2) = \pi_*^*H^0(\tilde{Y}, D_2) \oplus \pi_*^*H^0(\tilde{Y}, D_4) \).

**Proof.** It is easy to see that the dimensions agree. Moreover, notice that if \( E_1 \) intersects positively \( n_i \in \{ n_1, n_2, n_6, n_7 \} \) (classes fixed by \( \hat{\tau}^* \)), then \( \phi|_{E_1}(n_i) \) is a curve \( C \) in \( \phi|_{E_1}(\tilde{Z}) \); consider now the induced automorphism \( \hat{\tau} \) on \( \phi|_{E_1}(\tilde{Z}) \): It fixes two points \( p_1, p_2 \) on \( C \), each belonging to one (or the other) of the eigenspaces for the action of \( \hat{\tau}^* \) on \( H^0(\tilde{Z}, E_1) \), that are

\[
H^0(\tilde{Z}, E_1) = \pi_*^*H^0(\tilde{Z}/\hat{\tau}, F_1) \oplus \pi_*^*H^0(\tilde{Z}/\hat{\tau}, F_2)
\]

for some divisors \( F_1, F_2 \) of \( \tilde{Z}/\hat{\tau} \). Therefore, \( F_1 + F_2 \) intersects positively the two curves \( C_1, C_2 \), that resolve the singular points image of \( p_1, p_2 \) in \( \tilde{Z}/\hat{\tau} \). If \( E_1 \) intersects trivially \( n_i \), then \( \phi|_{E_1}(n_i) \) is a point \( p \) in \( \phi|_{E_1}\tilde{Z} \), which is fixed by \( \hat{\tau} \) and thus belongs to an eigenspace: Its image in \( \tilde{Z}/\hat{\tau} \) is resolved by a curve \( C_p \), that is intersected positively by either \( F_1 \) or \( F_2 \), and so by their sum. A similar argument can be applied for \( n_i \in \{ n_3, n_4, n_5, n_8 \} \), thus proving that how \( F_1^{(i)} + F_2^{(j)} \) intersects the exceptional lattice of the resolution of \( \tilde{Z}/\hat{\tau} \) depends on how \( E_i \) intersects that of \( \tilde{Z} \).
Since the surfaces $\tilde{Z}/\tau$ and $Y$ are isomorphic (see Remark 4.2), we have

$$H^0(X, L) = \bigoplus_{i=1,2} \pi^*_4 H^0(\tilde{Z}/\tau, F_i^{(1)}) \oplus \pi^*_4 H^0(\tilde{Z}/\tau, F_i^{(2)})$$

and a correspondence between each $H^0(Y, D_j)$ and one of the $H^0(\tilde{Z}/\tau, F_i^{(k)})$.

The fact that this correspondence is exactly as stated comes from a comparison between how the $E_i$ and the $D_j$ intersect the exceptional lattices of $Z$ and $Y$, respectively.

### 6.3 Examples with $L^2 = 4$

There are three families of K3 surfaces $X$ polarized with an ample class $L$ such that $L^2 = 4$, corresponding to no. 2, no. 3, no. 4 of Table 1: Since it holds $\chi(D_i) = h^0(D_i)$, as we proved in Proposition 6.1.1, we can read from Table 1 the dimension of the eigenspaces of the action induced by $\tau$. The automorphisms thus constructed are indeed symplectic, as we find that the quotient surfaces $X/\tau$ are birational to K3 surfaces.

Recall from Remark 5.2 that the projective dimension of each of the families of $X$ is $5 = 20 - (rk(\Omega_4) + 1)$.

**Remark 6.3.1.** A symplectic automorphism $\tau$ of order 4 on a K3 surface $X$ always fixes four points, but an automorphism $\alpha$ of order 4 that fixes four points is not necessarily symplectic, as it can also hold $\alpha^* \omega_X = -\omega_X$ [1, Proposition 2].

The projective models of $X$ and its quotients are summarized in the following table:

| No. | $X$                 | $X/\tau^2$                                      | $X/\tau$                                      |
|-----|---------------------|------------------------------------------------|------------------------------------------------|
| 2   | quartic in $\mathbb{P}^3$ | complete intersection of 3 quadrics in $\mathbb{P}^5$ | $(2,2) \cap (1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ |
| 3   | double cover of a quadric | complete intersection of 3 quadrics in $\mathbb{P}^5$ | quartic in $\mathbb{P}^3$ |
| 4   | quartic in $\mathbb{P}^3$ | complete intersection of 3 quadrics in $\mathbb{P}^5$ | double cover of a quadric |

**No. 2:** The divisor $L_0(2)$ has square 4, and in $NS(X) = \Omega_4 \oplus \mathbb{Z}L$, there exists no class $E$ such that $E^2 = 0$ and $EL_0(2) = 2$: therefore by [17, Theorem 5.2] the map $\phi_{L_0(2)} : X \hookrightarrow \mathbb{P}^3$ is an embedding of $X$ in $\mathbb{P}^3$ as a quartic surface. Consider the automorphism of $\mathbb{P}^3$:

$$\psi_2 : (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : ix_1 : -x_2 : -ix_3);$$

quartic surfaces of the form

$$Q_2 : x_0^3x_2 + x_0^2(x_1^2 + x_3^2) + x_0(x_2^3 + \delta x_1x_2x_3) + x_2^3(\epsilon x_1^2 + \zeta x_3^2) + \eta x_1^3x_3 + \theta x_1x_3^3 = 0$$

are invariant under the action of $\psi_2$, and they depend on five projective parameters up to projectivities of the form $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : ax_1 : bx_2 : cx_3)$, which commute with $\psi_2$. Moreover, $Q_2$ contains exactly eight points fixed by $\psi_2$, of which four are fixed also by $\psi_2$; we have therefore

$$\phi_{L_0(2)} : X \xrightarrow{\cong} Q_2 \subset \mathbb{P}^3.$$  

To find models for the quotient surfaces, as in [7, section 3.4], we consider the map given by the degree 2 invariants under the action of $\psi_2^2$

$$(x_0 : x_1 : x_2 : x_3) \mapsto (x_0^2 : x_1^2 : x_2^2 : x_0x_2 : x_1x_3) = (z_0 : z_1 : z_2 : z_3 : z_4 : z_5);$$
then the surface $Q_2$ maps to the complete intersection of quadrics in $\mathbb{P}^5$

\[
R_2 : \begin{cases}
z_0z_2 - z_4^2 = 0 \\
z_3^2 - z_1z_3 = 0 \\
z_0z_4 + z_0(\alpha z_1 + \beta z_3) + z_4(\gamma z_2 + \delta z_3) + z_2(\varepsilon z_1 + \zeta z_3) + z_5(\eta z_1 + \theta z_3) = 0.
\end{cases}
\]

which is a projective model for $Q_2/\psi_2^2$. Since $L_0(2) = \pi_2^*L_0(2)$ has self-intersection 8, it holds

\[
\phi|_{L_0(2)} : Z \overset{\sim}{\longrightarrow} R_2 \subset \mathbb{P}^5.
\]

The automorphism induced by $\psi_2$ on $\mathbb{P}^5$ is

\[
\tilde{\psi}_2 : (z_0 : z_1 : z_2 : z_3 : z_4 : z_5) \mapsto (z_0 : -z_1 : z_2 : -z_3 : -z_4 : z_5) : \n\]

Since the surface $R_2$ has the same form as in [7, section 3.6], then the quotient of $R_2$ under the action of $\tilde{\psi}_2$ is described by a complete intersection in $\mathbb{P}_2^2(z_0 : z_2 : z_3) \times \mathbb{P}_2^2(z_1 : z_3 : z_4)$ of two polynomials of bidegree, respectively, (2,2), (1,1), that is,

\[
S_2 : \begin{cases}
z_0z_2z_1z_3 - z_4^2z_5^2 = 0 \\
z_0z_4 + z_0(\alpha z_1 + \beta z_3) + z_4(\gamma z_2 + \delta z_3) + z_2(\varepsilon z_1 + \zeta z_3) + z_5(\eta z_1 + \theta z_3) = 0.
\end{cases}
\]

**No. 3:** The divisor $L_{1,2,2}^{(1)}(0)$ is ample but not very ample: indeed by [17, Theorem 5.2], we have $L_{1,2,2}^{(1)}(0) = H_1 + H_2$ with

\[
H_1 = \frac{L_0(0) + \rho + \sigma}{2}, \quad H_2 = \frac{L_0(0) + \rho - \sigma}{2}; \quad \langle H_1, H_2 \rangle = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}; \quad \tau^*(H_1) = H_2.
\]

Hence,

\[
\phi|_{L_{1,2,2}^{(1)}(0)} = \phi|_{H_1+H_2} : X \overset{2:1}{\longrightarrow} \mathbb{P}^1 \times \mathbb{P}^1
\]

is a double cover ramified along a curve $C$ of bidegree (4,4) invariant for the automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$

\[
\overline{\psi}_3 : (x_0 : x_1)(y_0 : y_1) \mapsto (y_0 : iy_1)(x_0 : ix_1),
\]

which switches the two copies of $\mathbb{P}^1$ (as prescribed by $\tau^*H_1 = H_2$). The curve $C$ depends on five projective parameters when taking into account the action of the group of projectivities of the form $(x_0 : x_1)(y_0 : y_1) \mapsto (x_0 : ax_1)(y_0 : ay_1)$, which are the only ones that commute with $\overline{\psi}_3$. We embed $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ via the Segre map

\[
(x_0 : x_1)(y_0 : y_1) \mapsto (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1) = (z_0 : z_1 : z_2 : z_3);
\]

Now $X$ is a double cover of the quadric surface $z_0z_3 = z_1z_2$ ramified along a curve of degree 4 invariant for the automorphism $\psi_3$ of $\mathbb{P}^3$ induced by $\overline{\psi}_3$ via the Segre map,

\[
\psi_3 : (z_0 : z_1 : z_2 : z_3) \mapsto (z_0 : iz_2 : iz_1 : -z_3);
\]

notice that $\psi_3$ has eigenspaces of the same dimension, accordingly to Table 1. The surface $X$ is therefore described in $\mathbb{P}(2, 1, 1, 1, 1)$ by

\[
Q_3 : \begin{cases}
z_0z_3 = z_1z_2 \\
w^2 = \alpha z_0^4 + \beta z_0^2z_2^2 + \gamma z_3^4 + z_0z_3(\delta z_1^2 + \varepsilon z_1z_2 + \delta z_2^2) + \zeta z_1^4 + \eta z_1^2z_2^2 + \zeta z_2^4.
\end{cases}
\]
The fixed locus of $\psi_3$ on $\mathbb{P}^3$ is $\{(1:0:0:0),(0:0:0:1),(0:1:1:0)\}$: Only the first two of these points belong to the branch curve, so to have four points fixed by $\psi_3$ on $Q_3$, the action induced by $\psi_3$ on $\mathbb{P}(2,1,1,1)$ has to be

$$(w;z_0 : z_1 : z_2 : z_3) \mapsto (w;iz_2 : iz_1 : -z_3).$$

To find a projective model of the quotient surface $Z$, we consider the degree 2 invariants for the action of $\psi_3^2$, that form a projective space of dimension 6:

$$(w; z_0 : z_1 : z_2 : z_3) \mapsto (w : z_0^2 : z_1^2 : z_2^2 : z_3 : z_0z_3 : z_1z_2) = (w : a_0 : a_1 : a_2 : a_3 : a_4 : a_5);$$

the surface $Q_3$ maps to

$$R_3 : \begin{cases}
a_0a_3 = a_4^2 \\
a_1a_2 = a_5^2 \\
a_4 = a_5 \\\nw^2 = \alpha a_3^2 + \beta a_0a_3 + \gamma a_2^2 + z_0z_3(\delta a_1 + \varepsilon a_5 + \delta a_2) + \zeta a_1^2 + \eta a_1a_2 + \xi a_2^2,
\end{cases}$$

the complete intersection of three quadrics in the hyperplane defined by $a_4 = a_5$ in $\mathbb{P}^6$.

Eliminate $a_5$, and change coordinates to

$$b_0 = a_0, \quad b_1 = a_1 + a_2, \quad b_2 = a_1 - a_2, \quad b_3 = a_3, \quad b_4 = a_4;$$

then the automorphism induced by $\psi_3$ on $\mathbb{P}^5$ is

$${\hat{\psi}}_3 : (w : b_0 : b_1 : b_2 : b_3 : b_4) \mapsto (w : b_0 : -b_1 : b_2 : b_3 : -b_4),$$

and in the new coordinates, we can write $R_3$ as

$$R_3 : \begin{cases}
b_1b_4 = b_2^2 + 4b_0b_3 =: \rho_2(w, b_0, b_2, b_3) \\
b_1^2 =: \rho_1(w, b_0, b_2, b_3) \\
b_4^2 =: \rho_3(w, b_0, b_2, b_3),
\end{cases}$$

where $\rho_1, \rho_2, \rho_3$ are quadrics, similarly to [7, section 3.7]: The quotient surface $S_3 = R_3/{\hat{\psi}}_3$ is therefore the quartic surface $\rho_1\rho_3 - \rho_2^2 = 0$ in $\mathbb{P}^3$.

**No. 4:** The eigenspaces associated to the action of $\tau$ on $H^0(X, L_{2,2}(1))$ have dimensions 2,1,0,1: Consider the automorphism of $\mathbb{P}^3$:

$$\psi_4 : (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 : ix_2 : -ix_3);$$

quartic surfaces of the form

$$Q_4 : f_4(x_0, x_1) + x_2x_3f_2(x_0, x_1) + \alpha x_2^4 + \beta x_2^2x_3^2 + \gamma x_3^4 = 0,$$

where $f_4, f_2$ are, respectively, homogeneous quartic and quadric polynomials, are invariant under the action of $\psi_4$, and they form a family of projective dimension 5 when taking into account the action of the group of projectivities of the form $(x_0 : x_1 : x_2 : x_3) \mapsto (ax_0 + bx_1 : cx_0 + dx_1 : ex_2 : x_3)$, that commute with $\psi_4$. Moreover, $Q_4$ contains exactly four points fixed by $\psi_4$, and four more fixed only by $\psi_4^2$. Therefore, $L_{2,2}(1)$ is very ample, and

$$\phi_{|L_{2,2}(1)}^\infty : X \xrightarrow{\sim} Q_4 \subset \mathbb{P}^3.$$
Proceeding as in no. 2, we consider the map given by the degree 2 invariants under the action of $\psi_4^3$

$$(x_0 : x_1 : x_2 : x_3) \mapsto (x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_2x_3) = (z_0 : z_1 : z_2 : z_3 : z_4 : z_5);$$

then the quotient $Q_4/\psi_4^3|_{Q_4}$ is the complete intersection of quadrics in $\mathbb{P}^5$

$$R_4 : \begin{cases} 
\rho_1 : z_0z_1 - z_4^2 = 0 \\
\rho_2 : z_3^2 - z_2z_3 = 0 \\
\rho_3 : \bar{f}_4(z_0, z_1, z_4) + z_5\bar{f}_2(z_0, z_1, z_4) + \alpha z_2^2 + \beta z_5^2 + \gamma z_3^2 = 0,
\end{cases}$$

where $\bar{f}_4, \bar{f}_2$ are, respectively, homogeneous quadric and linear polynomials such that $\bar{f}_4(x_0^2, x_1^2, x_0x_1) = f_4(x_0, x_1)$ (similarly $\bar{f}_2$ and $f_2$). The automorphism induced by $\psi_4$ on $\mathbb{P}^5$ is

$$\psi_4 : (z_0 : z_1 : z_2 : z_3 : z_4 : z_5) \mapsto (z_0 : z_1 : -z_2 : -z_3 : z_4 : z_5).$$

The surface $R_4$ is singular in eight points, four obtained as $R_4 \cap \{z_0 = z_1 = z_4 = 0\}$, which are not fixed by $\psi_4$, and four obtained as $R_4 \cap \{z_2 = z_3 = z_5 = 0\}$, which are fixed by $\psi_4$.

To find the quotient $R_4/\psi_4|_{R_4}$, we can consider the projection $\mathbb{P}^5 \to \mathbb{P}^3$

$$\pi : (z_0 : z_1 : z_2 : z_3 : z_4 : z_5) \mapsto (z_0 : z_1 : -z_2 : -z_3 : z_4 : z_5)$$

from the line $\ell = (0 : 0 : s : t : 0 : 0)$. Notice that $\pi(\rho_1) = \rho_1$, and that for every $(\bar{z}_0 : \bar{z}_1 : \bar{z}_4 : \bar{z}_5) \in \mathbb{P}^3$, we can compute its preimage as

$$\begin{cases} 
\bar{z}_5^2 = st \\
\bar{f}_4(\bar{z}_0, \bar{z}_1, \bar{z}_4) + \bar{z}_5\bar{f}_2(\bar{z}_0, \bar{z}_1, \bar{z}_4) + \alpha s^2 + \beta st + \gamma t^2 = 0;
\end{cases}$$

setting $B = \beta \bar{z}_5^2 + \bar{z}_5\bar{f}_2 + \bar{f}_4$, this gives

$$\begin{cases} 
s = \bar{z}_5^2/t \\
t^2 = -B\pm\sqrt{B^2 - 4\alpha\gamma\bar{z}_5^4}/2\gamma.
\end{cases}$$

There are generally four solutions, pairwise identified by the action of $\psi_4$: We can therefore define a surface $S_4$ that completes the diagram

$$\begin{array}{c}
R_4 \\
\downarrow \pi \\
4:1 \\
\downarrow /\psi_4 \\
\rho_1 \\
\downarrow 2:1 \\
S_4
\end{array}$$

that is, the quotient $S_4 = R_4/\psi_4|_{R_4}$ is a double cover of the quadric $\rho_1 \subset \mathbb{P}^3$ ramified over the curve defined by $B^2 - 4\alpha\gamma\bar{z}_5^4 = 0$, and thus is a K3 surface.

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REFERENCES
[1] M. Artebani and A. Sarti, Symmetries of order four on K3 surfaces, J. Math. Soc. Japan 67 (2015), no. 2, 503–533.
[2] I. V. Dolgachev, Mirror symmetry for lattice polarized K3 surfaces, J. Math. Sci. 81 (1996), 2599–2630.
[3] A. Garbagnati and Y. Prieto, Order 3 symplectic automorphisms on K3 surfaces, Math. Z. 301 (2022), 225–253.
[4] A. Garbagnati and A. Sarti, Symplectic automorphisms of prime order on K3 surfaces, J. Algebra 318 (2007), no. 1, 323–350.
[5] A. Garbagnati and A. Sarti, Projective models of K3 surfaces with an even set, Adv. Geom. 8 (2008), no. 3, 413–440.
[6] A. Garbagnati and A. Sarti, Elliptic fibrations and symplectic automorphisms on K3 surfaces, Comm. Algebra 37 (2009), 3601–3631.
[7] B. van Geemen and A. Sarti, Nikulin involutions on K3 surfaces, Math. Z. 255 (2007), 731–753.
[8] K. Hashimoto, Finite symplectic actions on the K3 lattice, Nagoya Math. J. 206 (2012), 99–153.
[9] A. Hulpke, Notes on computational group theory, 2010. https://www.math.colostate.edu/~hulpke/CGT/cgtnotes.pdf
[10] R. Miranda, The basic theory of elliptic surfaces, Dip. di matematica Univ. Pisa, Edizioni ETS, Pisa. https://www.math.colostate.edu/~miranda/BTES-Miranda.pdf, 1989.
[11] R. Miranda and D. R. Morrison, Embeddings of integral quadratic forms. 2009 https://web.math.ucsb.edu/~drm/manuscripts/eiqf.pdf.
[12] D. R. Morrison, On K3 surfaces with large Picard number, Invent. Math. 75 (1984), 105–121.
[13] V. V. Nikulin, Finite groups of automorphisms of Kählerian K3 surfaces, Russian: Trudy Moskov. Mat. Obshch. 38 (1979), 75–137. English translation: Trans. Moscow Math. Soc. 38 (1980), 71–135.
[14] V. V. Nikulin, Integral symmetric bilinear forms and some of their applications, Russian: Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 1, 111–177. English translation: Math. USSR Izv. 14 (1980), no. 1, 103–167.
[15] K. Nishiyama, The Jacobian fibrations on some K3 surfaces and their Mordell-Weil groups, Japan. J. Math. 22 (1996), no. 2, 293–347.
[16] SageMath, The Sage mathematics software system (Version 9.2), The Sage Developers. 2020 https://www.sagemath.org.
[17] B. Saint-Donat, Projective models of K-3 surfaces, Am. J. Math. 96 (1974), 602–639.
[18] M. Schütt and T. Shioda, Elliptic surfaces, Adv. Stud. Pure Math. 60 (2010), 51–160.
[19] T. Shioda and H. Inose, On singular K3 surfaces, In W. Baily & T. Shioda (eds.), Complex Analysis and Algebraic Geometry: A Collection of Papers Dedicated to K. Kodaira, Cambridge University Press, Cambridge, 1977, pp. 119–136.
[20] È. B. Vinberg, The two most algebraic K3 surfaces, Math. Ann. 265 (1983), 1–21.
[21] U. Whitcher, Symplectic Automorphisms and the Picard group of a K3 surface, Comm. Algebra 39 (2011), 1427–1440.

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