The Anderson Model as a Matrix Model

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Abstract

In this paper we describe a strategy to study the Anderson model of an electron in a random potential at weak coupling by a renormalization group analysis. There is an interesting technical analogy between this problem and the theory of random matrices. In $d = 2$ the random matrices which appear are approximately of the free type well known to physicists and mathematicians, and their asymptotic eigenvalue distribution is therefore simply Wigner's law. However in $d = 3$ the natural random matrices that appear have non-trivial constraints of a geometrical origin. It would be interesting to develop a general theory of these constrained random matrices, which presumably play an interesting role for many non-integrable problems related to diffusion. We present a first step in this direction, namely a rigorous bound on the tail of the eigenvalue distribution of such objects based on large deviation and graphical estimates. This bound allows to prove regularity and decay properties of the averaged Green’s functions and the density of states for a three dimensional model with a thin conducting band and an energy close to the border of the band, for sufficiently small coupling constant.

This paper is dedicated to Claude Itzykson.

1 Introduction and Warning!

The reader might be slightly surprised (perhaps even baffled?) by this contribution. Indeed it has nothing to do directly with the main topic of these Proceedings, namely strings, membranes and duality. It is however a tradition (and a good one!) to include in this cycle of conferences some applications of
quantum field theory to condensed matter physics. But in the case of our contribution, not only its subject, the mathematical analysis of disordered systems, is quite marginal from the point of view of the main theme of the conference, but also its form. It is not a “pedagogical introductory review” on the subject (for such a review we refer e.g. to [A][S]). We chose indeed to give here the main technical elements of the proof of a non-perturbative regularity theorem on the Anderson model of an electron scattered by a random potential. We hope that glancing at this paper might nevertheless be amusing or even useful to the reader for two reasons. First the arid details of the mathematical bounds below will remind us how much effort is necessary to transform into proof even what certainly looks as the most trivial of all theoretical issues. Second we hope that the reader might be interested by the tool that was recently introduced in this area, namely a new category of random matrices which have constraints of a geometrical origin. The free random matrices with independent entries considered by Wigner, Dyson and others (and to the theory of which Claude Itzykson contributed so successfully) held the key to many unexpected theoretical developments which range from nuclear physics and confinement to quantum gravity and non-perturbative strings. There is now a general mathematical frame for such matrices, namely the free non commutative probability theory of Voiculescu [V]. In this frame, Wigner’s law appears very general: it is simply the non commutative analog of the Gaussian law of large numbers for independent ordinary (commutative) random processes. But the theory of constrained non commutative processes remains to be developed. There are many three dimensional problems (the BCS theory, the Anderson model, scattering of particles in 3 or 4 dimensional Minkovski space) that can be discretized as problems of random matrices with geometrical constraints of the type considered below in section 4. These matrices seem definitely to have quite different scaling properties from the usual ones with independent entries. Perhaps they might also become useful tools in many other domains, when integrability and two dimensional conservation laws are no longer available. Therefore we think that the scaling properties of such random matrices of non-free type deserve to be further investigated, even numerically.

2 The Anderson Model

The Anderson model of a single electron in a random potential was introduced to study the concept of localization.

Rigorous results in dimensions not restricted to 1 up to now concern the localization regime. The first result of this kind was the multiscale analysis of Fröhlich and Spencer, who proved absence of diffusion for large disorder or energy out of the conduction band [FS]. It was later proved that in this regime localization holds, namely the expectation value of the modulus of the Green’s function decays exponentially and the spectrum of the Hamiltonian is almost
surely pure point \[DLS\], \[FMSS\]. The proofs have been simplified more recently by an argument due to Aizenman and Molchanov \[AM\], which is based on considering the expectation value of the modulus of the Green's function to a power strictly smaller than one.

However almost nothing rigorous is known about the “extended states” regime where the energy lies in the middle of the spectrum of the free Hamiltonian and the disorder is weak. This is the regime where diffusive conduction should take place, at least in 3 dimensions. There are only results at \(d = \infty\) by Klein \[K\].

We think that the program of renormalization group around singular manifolds should be useful to study this regime. This program was originally developed for the study of many Fermions and in particular for a rigorous understanding of the BCS theory of superconductivity \[FT\], \[FMRT1\]. In that context, the singular manifold was the Fermi surface. In the case of the Anderson model, the singular manifold is simply defined by the equation \(p^2 = E\), namely it is a sphere for a free rotation invariant Hamiltonian; in the case of a lattice model breaking rotation invariance, one would have a slightly more complicated manifold such as e.g. \(\sum_{i=1}^{d} (2 - 2 \cos p_i) = E\).

This renormalization group analysis consists in slicing the free Green’s function around the singular locus in momentum space, and performing a phase space analysis (by means of cluster expansions) which allows to compute the long range behavior governed by the effective theory near the singularity.

In order to control this renormalization group analysis it is useful to understand the behavior of the theory in a single momentum slice, hence slightly away from the singularity, and to prove that perturbation theory in this case is meaningful (i.e. asymptotic), using the fact that the coupling is weak to control the cluster expansion.

In a previous paper \[P\] this single slice analysis was performed in two dimensions. In that situation the main problem consists in perturbing the identity matrix by a random matrix with independent coefficients, and this problem can be controlled using e.g. the asymptotics to the Wigner’s law for the density of eigenvalues of this kind of matrices.

In three dimensions the analogous problem is more difficult since the matrix no longer has independent random entries. It belongs to this new class of random matrices alluded above, for which large size asymptotics are not known. In this paper we initiate the study of such matrices by large deviation techniques which ultimately rely on Feynman graphs estimates. We prove a theorem which states that the probability that such random matrices develop a large eigenvalue is sufficiently small, so that a single slice analysis again can be controlled.

This single slice analysis can be considered either as a step in the program to analyze by the RG method the situation where the energy lies in the middle of the conduction band, or as a full result in its own right, but for an energy out of the conduction band and close to its border. In this last case our probabilistic bounds imply via a cluster expansion that the averaged Green’s function has
fast decay (for a $C_0^\infty$ band cutoff, this decay is polynomial of arbitrarily high order), and that it is asymptotic to its perturbative expansion.

3 Notations and Results

3.1 Definitions

We consider an electron in a weak random potential in three dimensions. The mass of the electron is set to $m = 1/2$ so that in these units the ordinary kinetic energy in Fourier space is simply $p^2$. To model a conduction band, we consider a free Hamiltonian $H_0$ which is simply a multiplication operator in Fourier space:

$$\hat{H}_0(p) = p^2 / \eta(p)$$  \hspace{1cm} (1)

where $\eta$ is some rotation invariant $C_0^\infty$ function which is 1 on a certain interval $[a, b]$ and zero on an interval $[a', b']$, with $a' > 0$. Therefore in this model $H_0$, the energy, is both infinite for large momenta $|p| > b'$ and for small momenta ($|p| < a'$), so there is both an ultraviolet cutoff, as should be the case for any non-relativistic problem, and an infrared cutoff on small momenta. Lattice cutoffs could also be treated by our methods, replacing for instance the function $p^2$ by $\sum_i (2 - 2 \cos p_i)$ (hence they are not rotation invariant).

The interacting Hamiltonian is obtained by adding to $H_0$ a random potential $V$:

$$H = H_0 + \lambda V$$  \hspace{1cm} (2)

We take $V$ as a Gaussian ultra-local field in $x$-space, i.e. each $V(x)$ is an independent Gaussian variable. More precisely, it means that $V$ is distributed according to a normalized Gaussian measure $d\mu_\kappa$ of covariance $\kappa$. We also include a volume cutoff: this means that we choose a large box $\Lambda$ of volume $|\Lambda|$ and side size $L = |\Lambda|^{1/3}$, and restrict the operator $V$ to that box. We can also use e.g. periodic boundary conditions for the operator $H_0$ on the boundary $\partial\Lambda$ (although this last condition is not essential). In that case the Fourier-transform of a $\Lambda$-periodic function $f(x)$ is

$$f(p) = \int_\Lambda e^{ip\cdot x} f(x) \, dx$$  \hspace{1cm} (3)

and verifies

$$f(x) = \frac{1}{|\Lambda|} \sum p \in (2\pi \mathbb{Z})^3 e^{-ip\cdot x} f(p) \text{ with } p \in (2\pi \mathbb{Z})^3$$  \hspace{1cm} (4)

We are interested in computing the averaged Green’s function at energy $E$ (with a Feynman prescription) as a kernel in $x$-space, or the density of states, which is the imaginary part of this kernel at coinciding points. Therefore we want to analyze

$$\langle G_\pm \rangle = \lim_{\Lambda \to \infty} \lim_{\varepsilon \to \pm 0} \langle G_{\Lambda, \varepsilon} \rangle$$  \hspace{1cm} (5)
\[
\langle G_{\lambda,\varepsilon} \rangle = \int \frac{1}{H_0 + \lambda V - (E + i\varepsilon)} \, d\mu_\kappa(V)
\] (6)

In the following we will note \( < . > \) for integration with respect to \( V \), and \text{Prob} for the probability of an event, measured with respect to \( d\mu_\kappa \).

It is important to remark that we can put some kind of ultraviolet cutoff on the random potential \( V \) as well, because we are treating a non-relativistic problem. Usually this is done by considering the potential on a lattice, but we can also choose to stay in the continuum and to take the covariance \( \kappa \) of \( V \) not to be a \( \delta \) distribution but a smooth positive type kernel with compact support or fast decay at the unit scale (the scale of our ultra-violet cutoff).

### 3.2 Expected Results

In the standard Anderson model at weak coupling in dimension 3 the first significant open mathematical problem is to prove that for a conduction band of finite width and an energy \( E \) inside the band (that is \( E \in [a, b] \)) the thermodynamic limit of Green’s functions defined above exist and has fast decay (exponential decay in the case of the lattice cutoff) for real small enough coupling \( \lambda \). One want also to prove that the decay rate behaves, as a function of the coupling, as:

\[
\Delta = \pi^2 \lambda^2 + O(\lambda^4)
\] (7)

All these conjectures stem from the well-known perturbative analysis of the model. It is based on expanding (6) as a power series in \( \lambda \) (resolvent expansion), then integrating on \( V \). This perturbative analysis is of course plagued by the usual divergence of perturbation theory.

To obtain the leading behavior \( \Delta \simeq \pi^2 \lambda^2 \), it is sufficient to compute a self-consistent renormalization of the divergent tadpole graph, which leads to:

\[
\Delta \simeq \lambda^2 \Im \int d^d p \frac{\eta(p)}{p^2 - (E + i\Delta)}
\] (8)

so that

\[
1 \simeq \lambda^2 \int d^d p \frac{\eta(p)}{(p^2 - E)^2 + \Delta^2}.
\] (9)

This equation gives the behavior (7) since \( E \) belongs to \( [a, b] \), where the cutoff \( \eta \) is 1. This self-consistent computation is the so-called “rainbow graphs” approximation.

Thus the first main effect of \( V \) is to add a mass term in \( O(\lambda^2) \) which screens the singularity of the free propagator \( \frac{1}{p^2 - E} \). Physically this means that the electron wave function loses memory of its phase by the incoherent reflections on \( V \).

To prove this fast decay of \( \langle G_{\pm} \rangle \) or equivalently to analyze the behavior at small \( \lambda \) of the density of states is our first natural goal in the small coupling
analysis of the Anderson model. Remark however that this result would still fail to distinguish between localization or the existence of extended states, which can be distinguished only by the decay properties of \( <G_+G_-> \). Let us recall that in the localization regime (which in \( d = 3 \) means for \( \lambda \) large) the function \( <G_+(x,y)G_-(x,y)> \) decays exponentially, whether in the diffusive or extended states regime (which in \( d = 3 \) should exist for \( \lambda \) small), it is conjectured that the function \( <G_+(x,y)G_-(x,y)> \) decays only as \( |x-y|^{-1} \). To distinguish rigorously between these two regimes at \( d = 3 \), hence to establish the existence of the Mott-Anderson transition is the long term goal for our program.

3.3 A Result

Here we do not even prove yet this fast decay of \( <G_+> \) when \( E \) is inside the band, but establish a weaker result. We consider an energy \( E \) not inside the band but near its edge. Also the coupling constant \( \lambda \) is taken real and small enough so that both the width of the band \( b' - a' \) and the distance between the energy and the border of the band (say, \( a' - E \) for an energy close to the lower edge of the band) are large compared to \( \lambda^2 \). Under these conditions we shall prove that the averaged Green’s function has fast decay on a spatial scale of order \( r^{-1} \), where \( r = \inf\{b' - a', a' - E\} \). More precisely we can state the:

**Theorem I** Let \( r = \inf\{b' - a', a' - E\} \). There exists some small constant \( \epsilon \) such that if \( \lambda^2 \leq \epsilon r \), then \( <G_\pm> \) exist (the convergence of its kernel as \( \Lambda \to \infty \) and \( \epsilon \to 0 \) being uniform on all compact sets) and for all integer \( m \) there exists a constant \( K_m \) such that:

\[
|<G_\pm>(x,y)| \leq K_m \frac{1}{(1 + r|x-y|)^m}.
\]

Furthermore this averaged Green’s function is asymptotic to all orders its perturbative expansion in \( \lambda \). This expansion is obtained by expanding the resolvent (see (19) below) and performing the Gaussian integration on \( V \) in \( <G_\pm>(x,y) \).

This result in \( d = 2 \) is essentially the subject of [P]. The purpose of these notes is to extend it to \( d = 3 \), and explain the difference. The proof combines two aspects, a large/small field probabilistic analysis which is given here in detail, and a cluster expansion which is a rather standard method ([R][P]), so not repeated here.

3.4 Next steps...

The same result can be proved to hold also for a band width no longer small but finite, of order 1, the distance of the energy to the border of the band being still small (hence \( \lambda^2 \leq \epsilon r = a' - E \)); since the proof is technically slightly more difficult, we do not include it here.
Finally the much more difficult problem of an energy within the band, is somewhat analogous to spontaneous mass generation in field theory or in the BCS theory. Our result which concerns the energy outside the band is nevertheless a relevant step towards this more difficult result, since one can expect to analyze the case of the energy inside the band by cutting slices closer and closer to it, as in a renormalization group analysis [P]-[FT]. We must however add that the situation is quite different in dimensions 2 and 3. In two dimensions we have a precise scenario in progress to solve this problem of an energy within the band, using the specific two dimensional rules of conservation of momentum [MPR]. In three dimensions we have not yet such a precise scenario on how to attack the problem of an energy within the band.

3.5 Complex translation and limit $\varepsilon \to 0$

We can make an \textit{a priori} partial renormalization and generate a non vanishing imaginary part in the denominator thanks to a complex translation of the integration contour of $V$, using the formulas

$$
\int d\mu C(X) F(X) = e^{(aCa)/2} \\
\int d\mu C(X) F(X + iCa) e^{-iaX} \cdot (11)
$$

These formulas are easily proved for $F$ polynomial and then extended with a density argument to meromorphic functions provided we meet no pole.

Let us translate $V$ by $-i\delta$, and get:

$$
G_{\Lambda, \varepsilon} = e^{|A|\delta^2/2}\left\langle \frac{e^{i\delta V/\Lambda}}{H_0 - (E + i\delta + i\varepsilon) + \lambda V} \right\rangle (12)
$$

(The exact value of $\delta$ will be fixed later on). Now we can take the limit $\varepsilon \to 0$ which yields

$$
G_{\Lambda} = e^{|A|\delta^2/2\lambda^2}\left\langle \frac{e^{i\delta V/\Lambda}}{H_0 - (E + i\delta) + \lambda V} \right\rangle . (13)
$$

On the lattice this formula is exact but in the continuum it has to be slightly corrected to take into account the ultraviolet cutoff on $V$, but this is inessential [P].

The cost of this operation is the bad factor $e^{|A|\delta^2/2\lambda^2}$ that we may have to compensate if some absolute value in the bounds cancels the oscillating term $e^{i\delta V/\Lambda}$. However let us explain the strategy: space will be divided into so-called large/small field regions. In the small field regions, perturbation theory works. The eigenvalues of $\lambda V$ are too small to cancel $H_0 - E$. Therefore the imaginary
term $i\delta$ is not really needed and the complex translation can be reversed, so that the bad factor has not to be compensated. In the large field region, this is no longer true, but the bad factor is more than compensated by the probabilistic factor associated to the measure $d\mu$ which is very small.

3.6 The resolvent expansion

From now on the number $r$ is chosen of the form $r = M^{-j}$, where $M$ is some fixed number and $j$ a large positive integer. These are the usual notations which come from the renormalization group point of view [FT], [FMRT], [P]. We also set $E = 1$ for simplicity. Let us also define $\eta(p) = |\eta_j(p)|^3$ (the subscript $j$ simply recalls that the width of the support of the function $\eta$ is $M^{-j}$). We introduce the free Green’s function

$$\Gamma = (H_0 - (1 + i\delta))^{-1} = \eta_j C \eta_j ; \quad (14)$$

where

$$C = \frac{\eta_j}{p^2 - \eta_j^3 (1 + i\delta)} ; \quad (15)$$

Then we can factorize $C$ and rewrite $G$ as

$$G_\Lambda = e^{[\Lambda i\delta^2/2\lambda^2} \left( e^{i\delta_1 V/\lambda} \eta_j \frac{1}{1 + \lambda C \eta_j V \eta_j} C \eta_j \right)$$

$$\equiv A \left\langle e^{i\delta_1 V/\lambda} R(V) \right\rangle \quad (16)$$

where we used the operator product conventions

$$(AB)(x, y) = \int A(x, z)B(z, y) \, dz$$

$$\Leftrightarrow (AB)(p, q) = \frac{1}{|\Lambda|} \sum A(p, k)B(-k, q) \quad (17)$$

where

$$A(p, q) = \int e^{ip \cdot x - iq \cdot y} A(x, y) \, dx \, dy . \quad (18)$$

Because of its smooth covariance $V$ is almost surely bounded. Thus for a given $V$, $R(V)$ is analytic in $\lambda$ in a small domain around the real axis. This implies that $R$ is the unique analytic continuation of the operator series

$$S = \sum_{n=0}^{\infty} (-\lambda)^n \eta_j \langle CHC \ldots HC \rangle \eta_j \quad (19)$$

with $H = \eta_j V \eta_j$ . \quad (20)
4 Phase space Analysis

We want to study $G$ by means of a phase space analysis. In order to do so, we will cut the propagator $C$ according to cells (called sectors) in the space of angular coordinates of the momentum $p/p$ (see subsection below). These sectors are also the main tool in the constructive renormalization group analysis of [FMRT1-4]. Space will also be cut according to a lattice of cubes of size $M^j = R^{-j}$, which is adapted to the decay of the propagator, or of the cutoff $\eta_j$. In momentum space $V$, seen as an operator, is $V(p, q) \equiv V(p - q)$.

![Figure 1](image)

The key difference between the constructive or phase space analysis in 2 and 3 dimensions is that in two dimensions when $p$ and $q$ have the same fixed norm (here close to $E = 1$, since $|p| - 1 \leq 2r = 2M^{-j}$), the sum $p+q$ defines in a unique way the pair $\{p, q\}$, whether in three dimensions it defines it only up to a rotation by an angle (see Figure 1). The operator $V$ can be considered as a matrix whose entries are labeled by this discrete set of sector indices. In this point of view, the geometrical behavior depicted in Figure 1 translates essentially in the following property: the matrix obtained in $d = 2$ is approximately an orthogonal ensemble with Wigner’s semi-circle law as asymptotic behavior of the eigenvalues density $[P]$, whether in three dimensions the matrix is of a constrained type,
and the asymptotic distribution for the eigenvalues can be probed only more indirectly, for instance, as will be shown below, through large order deviations which amount to the computation of particular Feynman graphs. We go on to make now this statement more precise.

4.1 The discretized “single cube problem” as a Random Constrained Matrix

The usual point of view in localization theory is to treat space as fundamental and to expand the off diagonal part of $C$, which couples different points. But for diffusion theory and extended states regimes, it is better to keep momentum space as fundamental. Since $C$ is a diagonal operator in momentum space, the important part of the problem is to understand the behavior of the random operator $H$ in momentum space. We show now how to discretize this operator as a matrix to understand better its qualitative behavior.

From now on we restrict ourselves to dimension $d = 3$. Suppose that we cut the momentum shell support of the cutoff $\eta_j$ into cells $\sigma$ (isotropic sectors) which have size $M^{-j}$ in all directions: $\eta_j = \sum_\sigma \eta_j,\sigma$. There will be roughly $N = M^2$ such sectors (neglecting factors such as $4\pi$). Consider the covariance $\kappa$ of $V$. Since $V$ is almost ultralocal, this covariance is almost 1 on a ball of large radius. But by momentum conservation, the operator $H_{\sigma,\sigma'} = \eta_j,\sigma V \eta_j,\sigma'$ is zero unless the momentum of $V$, which should be approximately $\sigma - \sigma'$, lies inside the ball of radius $2 + 4r$, so approximately the ball of radius 2. In this ball $\kappa(p) = 1$ with a very good approximation. Cut the interior of this three dimensional ball of radius 2 into about $M^{3j} \approx N^{3/2}$ cells $c$ of side size $M^{-j}$: $\kappa = \sum_c \kappa_c$. There will be an associated decomposition of $V$ as a sum of random variables $V_c$, each independent and distributed with covariance $\kappa_c$.

When this decomposition of momentum space both for the electron cutoff $\eta_j$ and for $V$ has been made with smooth $C_0^\infty$ functions, there is fast dual spatial decay of the propagators $\eta_j,\sigma$ and $\kappa_c$ on length scales $M^{j}$. Therefore the heart of the theory is given by the problem restricted to a single spatial cube of side size $M^j$. To understand this single cube theory we can neglect now the space dependence of $\eta_j,\sigma$ and $\kappa_c$, keeping only the constraint of momentum conservation.

In this approximation the different random variables $V_c$ (which are no longer fields but real random variables) become identically distributed. The matrix $H$ has now $N^2$ entries, namely all pairs $(\sigma, \sigma')$ where $\sigma$ and $\sigma'$ are isotropic sectors. We understand that since this $N$ by $N$ matrix depends on $N^{3/2}$ independent variables only, it cannot be of the free type usually considered in theoretical physics (orthogonal, hermitian or symplectic ensemble). It is of a new constrained type, and the constraints stem from the geometry of momentum addition on the singular manifold (here the sphere). These constraints are related to the absence, in dimension 3 or more, of conservation rules for diffusion and of completely integrable models. Because of these constraints, the matrices
obtained are not invariant under a group such as the orthogonal, unitary and
symplectic group; the statistics do no longer factorize and reduce to eigenvalue
statistics (namely there are “preferred eigenvectors” as well), orthogonal polyno-
mial methods do not apply, etc..., and nothing seems to be known up to now
about such matrices.

These matrices can nevertheless be described and studied numerically on
a computer very easily with the following program (here we simplified again
slightly the initial problem, which involved complex matrices, replacing it by an
analogous real problem with sums of momenta instead of differences):

Take a small number \( r = M^{-j} \). Divide the unit sphere into \( N \approx M^{2j} \)
sectors of approximately similar area \( 4\pi r^2 \). Divide the ball of radius two into
\( N^{3/2} \approx M^{3j/2} \) cubic cells of volume approximately \( (4/3)8\pi r^3 \). Generate with a
random number generator \( N^{3/2} \) independent random variables \( V_c \), one for each
cell of the ball. Associate to any pair of sectors \((\sigma, \sigma')\) the unique cell
\( c = c(\sigma, \sigma') \) such that the sum of the two momenta at the center of \( \sigma \) and \( \sigma' \) fall into the
cell \( c \), and plug the corresponding variable \( V_c(\sigma, \sigma') \) at the corresponding entry
of the matrix, that is consider the (real symmetric) \( N \) by \( N \) matrix
\( H(\sigma, \sigma') = V_c(\sigma, \sigma') \). Diagonalize it, plot the eigenvalues, and repeat the operation a lot of
times to get some statistics.

We have implemented this program and run it on a computer for \( N \) up
to about 600 [G. Poirot, unpublished]. We cannot yet predict clearly scaling
laws or tail estimates from this numerical study, but the results indicate that
the eigenvalue distribution is differently peaked and extends more towards large
eigenvalues that in the free case. This is not surprising, since constraints typi-
ically increase both the degeneracy hence the small eigenvalues of a matrix, and
allow for larger exceptional eigenvalues. For instance the completely constrained
matrix with the same random value at every entry has \( N - 1 \) eigenvalues 0, and
a single large eigenvalue of size \( N \).

We close this parenthesis on this simple discretization of the random matrix
model, and return now to the true problem which is complicated by spatial de-
pendence. For this true problem one gets better bounds by considering angular
sectors, called anisotropic, which are wider in the tangential than in the radial
directions [FMRT1]. We introduce now these sectors and the precise tools to
prove Theorem I below.

4.2 Anisotropic Sector Decomposition

We noticed that the cutoff \( \eta_j \) implies the condition:

\[
||p| - 1| \leq 2M^{-j}.
\]  

(21)

We construct a partition of the unit sphere into anisotropic sectors [FMRT2]
\( S_\alpha \), by projecting e.g. a standard division of the faces of the cube \([-1,1]^3\) into
\( 6 \times 4M^{(2j)/2} = 24M^j \) square plaquettes of side \( M^{-j/2} \), hence area \( M^{-j} \), onto
the sphere. This gives a sharp partition of the sphere into \( 24M^j \) *spherical
plaquettes” or sectors $\alpha$ of variable area (between $M^{-j}$ and $M^{-j-3}$, since the maximal distance of the cube to the origin is $\sqrt{3}$). We call $k_\alpha$ the center of mass of $\alpha$. From this sharp partition one can build an associated smooth $C^\infty$ scaled partition of unity $\bar{\chi}_\alpha$ on the sphere with Urison’s lemma. Finally we transport this partition into a partition of the support of the cutoff $\eta_j$ by writing:

$$\eta_j = \sum_{\alpha} \eta_\alpha$$

(22)

where $\eta_\alpha(p) = \eta_j(p) \cdot \bar{\chi}_\alpha(p/|p|)$.

The support $S_\alpha$ of a sector $\alpha$, or of an $\eta_\alpha$ function, is therefore included in a parallelepipedic slab, of center $k_\alpha$, which has small thickness $l = M^{-j}$ in the direction parallel to $k_\alpha$ and which is longer in the two other perpendicular directions where it has length $L = 4M^{-j/2}(1 + \frac{2}{M}) = O(1)M^{-j/2}$.

The cube and our sector partition is invariant under parity: $p \to -p$. Therefore for any sector $\alpha$ with center of mass $k_\alpha$ we can define the antipodal sector $\bar{\alpha} = -\alpha$. All other sectors have support at distance at least say $(1/4)M^{-j/2}$ of the antipodal center $k_\bar{\alpha} = -k_\alpha$, if the partition of unity $\chi_\alpha$ does not spread too much onto the nearest neighbors, a fact we can assume from now on.

Then if $\beta \neq \alpha$, we define

$$k_{\alpha-\beta} = k_\alpha - k_\beta = 2\gamma_{\alpha-\beta} \cos x_{\alpha,\beta}$$

(23)

where $x_{\alpha,\beta} = \theta_{\alpha,\beta}/2$ is half the angle $\theta_{\alpha,\beta}$ (modulo $\pi$) between $k_\alpha$ and $-k_\beta$, so $x_{\alpha,\beta} \in [0, \pi/2]$, and $\gamma_{\alpha-\beta}$ is a well defined unit vector.

### 4.3 Triplet of cubes operator

It is natural to consider a smooth partition of the space by a lattice $D$ of cubes $\Delta$ of side $M^j$, with $C^\infty$ “characteristic functions” $\chi_\Delta$. There is a corresponding orthogonal decomposition of the random variable $V = \sum_\Delta V_\Delta$, by writing its covariance $\kappa(x, y) = \sum_\Delta \kappa^{1/2}(x, z)\chi_\Delta(z)\chi_\Delta(z)\kappa^{1/2}(z, y)$.

We define $H_\Delta(x, y) = \eta_j V_\Delta \eta_j$ and for each triplet $\tilde{\Delta} = (\Delta, \Delta', \Delta'') \in D^3$ or each pair $(\Delta, \Delta') \in D^3$

$$H_{\tilde{\Delta}}(x, y) \equiv H_{\Delta''} = \chi_\Delta(x)H_{\Delta''}(x, y)\chi_{\Delta'}(y) + \chi_{\Delta'}(x)H_{\Delta''}(x, y)\chi_\Delta(y)$$

(24)

Similarly we define

$$C(\Delta, \Delta')(x, y) \equiv \chi_\Delta(x)C(x, y)\chi_{\Delta'}(y) + \chi_{\Delta'}(x)C(x, y)\chi_\Delta(y)$$

(25)

to keep the operators symmetric.
5 Size of the operator H

Let us repeat that because we are in dimension $d = 3$, the operator $H$ localized in a box, considered as a matrix whose entries are the sectors of ingoing and outgoing momenta, does not, even approximately, look like a classical ensemble such as the orthogonal or Hermitian ensembles of random matrices which appeared in dimension 2 [P]. So Wigner’s semi-circle law and the usual asymptotics do not apply to this problem. We must use a different, less accurate, method to bound the tail of the distribution, namely the probability that some eigenvalue becomes large.

Let $N_j = M^{2j}$, and

$$D(\Delta) = (1 + M^{-j}d(\Delta, \Delta') + M^{-j}d(\Delta', \Delta''))$$

(26)

Our main probabilistic bound is

**Theorem II**

Let $p \geq 1$ be some integer. There exists a constant $C_p > 1$ such that for $A > C_p$

$$\text{Prob}\{\|H_\Delta\| \geq A. D(\Delta)^{-p} N^{-1/4}\}$$

$$\leq (AD(\Delta)^{2p})^{-N^{3/16}}.$$  

(27)

The proof is quite lengthy, and is the subject of the rest of the paper.

Recall that for a compact symmetric trace class operator such as $H$ (it is easily proved trace class because of the cutoffs), the ordinary norm is bounded by $(trH^{2n})^{1/2n}$ for any positive integer $n$. Therefore

$$\text{Prob}\{\|H_\Delta\| \geq A. D(\Delta)^{-p} N^{-1/4}\}$$

$$\leq (A.D(\Delta)^{-p} N^{-1/4})^{-2n} \int d\mu(V^{\Delta''}) tr H^{2n}.$$  

(28)

Performing the Gaussian integration we obtain a sum over all Feynman graphs obtained by contracting $2n$ insertions of $V^{\Delta''}$ on a loop of $2n$ propagators which are $\eta_j(\chi_\Delta + \chi_\Delta')^2 \eta_j$. $n$ is some integer that will be adjusted later.

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More precisely

\[
\int d\mu(V^{\Delta''}) \text{tr} H^{2n} = \sum_{G \in \mathcal{G}_m} I_G ,
\]

where the amplitude for the graph \( G \) is noted \( I_G \).

To evaluate the amplitude of a given graph \( I_G \), we split the spatial integrals into cubes of side size \( M^{1/2} \), hence volume \( N^{3/4} \). In the language of the \( \phi^4 \) field theory, the “vertices” correspond to the contractions of the initial \( V \) random field. They are propagators \( W(x, y) = \kappa^{1/2}(x, z) \chi_{\Delta}(z) \kappa^{1/2}(z, y) \). They are represented by the dotted lines in Figure 2. There are also the “ordinary lines” which correspond to the electron propagators \( \eta_j \chi_{\Delta \cup \Delta'} \eta_j \). They are represented by solid lines in Figure 2.

**Figure 2:** A graph \( G \) and its associated \( \phi^4 \) graph obtained by contracting the dotted lines
We split these electron propagators as \( \Gamma_{\Delta, \Delta'} = \sum_{\alpha, \alpha'} \Gamma_{\Delta, \Delta'} \chi_{\delta} \chi_{\Delta'} \eta_{\alpha} \).

Since the functions \( \chi_{\Delta} \) are scaled and smooth, we have momentum conservation, which means that \( \alpha \) must be close to \( \alpha' \), and then we have spatial decay of \( \Gamma \) dual to the length of the cell. Therefore:

**Lemma 1**

For any integer \( q \), there exists a constant \( C_q \) such that

\[
|\Gamma_{\Delta, \Delta'}(x, y)| \leq \frac{C_q M^{-2j}}{[1 + M^{-2j} d^2(x, \Delta \cup \Delta')]^q [1 + M^j d^2(\alpha, \alpha')]^q}
\]

where \( d^2(\alpha, \alpha') \) is the distance on the sphere between sectors \( \alpha \) and \( \alpha' \), and \( d_{/\parallel} \) and \( d_{/\perp} \) are the distances in the axis parallel to the center of \( \alpha \) and in the plane perpendicular to it.

**Proof** This is an easy exercise in integration by parts, similar to Lemma 4 in [FMRT2] (pg 689).

Applying also standard rules of integration by parts (see e.g. [FMRT2]), we can extract a very small factor if sectors \( \alpha_1, \alpha_2, \alpha_3 \) and \( \alpha_4 \) meet at a given vertex, and do not contain points which add up to 0. This mean that we can bound every vertex, extracting such a momentum conservation factor, plus some distance factor, by

\[
\frac{C_q}{[1 + M^j |\bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3 + \bar{\alpha}_4|]^q}
\]

(31)

where \( \bar{\alpha}_i \) is the center of the sector \( \alpha_i \).

Substituting these bounds we obtain our general bound for a graph of order \( n = 2m \), taking into account the fact that each half vertex joins successively \( \Delta \), \( \Delta'' \) and \( \Delta' \) or \( \Delta', \Delta'' \) and \( \Delta \):

\[
I_G \leq C_q^{3n} D(\Delta)^{-qn}
\]

\[
\sum_{(\alpha_i, i \in G)} \int_{\Delta \cup \Delta'} dx_1 \int_{R^3} dx_2 \ldots dx_n
\]

\[
\prod_{v} \frac{1}{[1 + M^j |\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4|^q]}
\]
\[ \prod_l \frac{N^{-1}}{[1 + M j d^2(\alpha_l, \alpha'_l)]^q [1 + M^{-2j} d^2_{\alpha_l/\perp} (x_l, y_l)]^q} \]

\[ \frac{1}{[1 + M^{-j} d^2_{\alpha_l/\perp} (x_l, y_l)]^q}. \]

(32)

where the sum runs over attributions of sectors to the lines, and spatial integrations over the vertices have been performed, except at one point per vertex.

Now by a combinatoric factor 2 per vertex we can decide if there exists at each vertex a sum or a difference between two incident sectors which is small in the sense that \(|\alpha_v \pm \alpha'_v| \leq M^{-(1/2) - a}\). \(a\) is some small number that we will fix later to \(a = 1/32\). In this way we cut the amplitude \(I_G\) in \(2^n\) elements as \(I_G = \sum_t I_{G,t}\), where the sum over \(t\) runs over the choices at each vertex whether there is or not such a small sum or difference.

Vertices such that there is no such small sum or difference are called “twisted” or non planar. The vertices with such a small difference are called “untwisted” since their four momenta typically will lie approximately in a common plane, or otherwise the vertex factor in (31) will be very small. The number of twisted vertices is called \(n_t\), and the number of untwisted vertices is \(n_u\), with \(n = n_t + n_u\).

Now for each “untwisted vertex”, we choose the particular pair of momenta realizing the smallest sum or difference (or an arbitrary one if several pair realize this minimum) and we split the vertex into two half vertices and an intermediate wavy line according to this splitting. The graph has then \(n_t\) quartic vertices and \(n_u\) pairs of trilinear vertices. Then we cut all the corresponding wavy lines. The graph splits then into \(c\) “ordinary” connected components which contain at least one twisted vertex, plus possibly other twisted vertices and chains of untwisted half vertices, and into \(c'\) pure “cycles” of untwisted half vertices, with \(c + c' \geq 1\). The result of this splitting process is pictured in Figure 3, for a graph with 15 vertices, 8 of which are untwisted and seven twisted. It splits into \(c = 5\) connected components if the wavy lines (untwisted vertices) are removed. We have \(c' = 3\) of these 5 components which are cycles which do not contain twisted vertices.

We select a tree of wavy lines relating together all these components, and will perform the spatial integration and attribution of momentum sectors according to what we call “painting rules” for each connected components. The tree of wavy lines is ordered by choosing a particular root, which correspond to a special connected component \(G_0\) (either ordinary or cycle).
The standard weight for integration of a vertex with the help of a single propagator hooked to it is the volume of a tube of size $M^{j/2}$ in the two perpendicular directions to a sector, and $M^j$ in the parallel direction, hence it is $N$. However if the vertex is twisted and integrated by taking into account the decay of two propagators hooked to it, remark that the corresponding weight is at most $N^{1-\alpha/2}$, since we gain the fact that the integration is located at the intersection of two tubes who make an angle at least $M^{-j((1/2)-\alpha)}$. Finally there is one single vertex in the root component whose integration costs the full volume of $\Delta \cup \Delta \cup \Delta''$, hence $M^{3j} = N^{3/2}$.

Now let us perform momentum attributions and vertex integrations. We perform these attributions in an order induced from the tree chosen and from fixing rules associated to each connected components. Let $G_k$ be pure cycles of $p_k$ untwisted half vertices. Let $r_k$ be the number of such half-vertices whose other half vertex is not yet fixed in the process (i.e. is lower in the tree). We have $r_k \geq 1$, except possibly for the last cycle $G_0$ if it is the root of the tree.
Then the first sector attribution to a line of $G_k$ is made on a half vertex of the type $r_k$ (except possibly if $k = 0$). It costs $N^{1/2}$ (total number of sectors). Each next attribution costs either $N^a$ (number of sectors in the cone of opening angle $M^{-j((1/2) - a))}$), or $O(1)$, depending whether the sectors of the other half vertex to which this half vertex is associated has been already attributed or not. So the total factor is $N^{(1/2)+a(r_k-1)} = N^{(1/2) - a + ar_k}$, except possibly for $k = 0$, where it can be $N^{1/2}$ if $r_0 = 0$, so there is an extra $N^a$ in that case, with respect to the formula $N^{(1/2) - a + ar_k}$.

Finally it remains to attribute the momenta in the ordinary components. Let us contract the chains of untwisted vertices to single lines (but with an index $p$ for the number of insertions of half untwisted vertices, and an index $r$ for the number of such insertions whose other half vertex is not yet fixed). Fixing all the sectors of such a line is like fixing one of them at the end, plus a factor $N^a r_k$.

To complete the bound, it remains therefore to compute the weight of fixing sectors in a vacuum connected $\phi^4$ graph solely made of twisted vertices. Let $\bar{G}$ be such a graph, and $s$ its number of vertices.

A fixing rule $P$ on $\bar{G}$ is an ordering of the vertices of $\bar{G}$ as $v_1, \ldots, v_s$ such that for every $k > 1$ there exists a line hooked at one end to $v_k$ and at the other end to one vertex among $\{v_1, \ldots, v_{k-1}\}$. There always obviously exist such fixing rules. For instance to any tree $T$ of $G$ and any way of “turning around the tree” (i.e. choice of a root and relative ordering of the non trivial forks going up in the tree) is associated a fixing rule corresponding to the ordering in which vertices of $G$ are met by “climbing around the tree” in that way.

The $k$-th vertex $v_k$ for the fixing rule $P$ is called of type 4, 3, 2, 1 or 0 with respect to $P$ if the number of half lines hooked at one end to $v_k$ and at the other end to a vertex not among $\{v_1, \ldots, v_{k-1}\}$ is 4, 3, 2, 1, or 0. The number of vertices of type $r$ is called $n_r$. Obviously $n_4 + n_3 + n_2 + n_1 + n_0 = s$, and $n_4 = 1$ since the first vertex of the fixing rule is the only one of type 4.

To perform the sum over the values of sectors in the graph we fix first the indices of vertex $v_1$, which is the only one of type 4. This costs $N^{1/2} N^{1/2} N^{1/4} = N^{5/4}$. Then vertices of type 3 cost $N^{1/2} N^{1/4} = N^{3/4}$. Vertices of type 2 cost $N^{1/4}$, because they are twisted. Vertices of type 1 or zero cost nothing. Finally we should take into account the fact that because of the twisting, vertices of type 2, 1 or 0, when integrated, are integrated with the decay of two propagators whose sectors are known and have a minimum angle. This allows a gain of $N^{-a}$, as explained above, for the spatial integration of these vertices.

Collecting all factors, the key point is to remark that the factors $N^a r_k$ and $N^a r$ combine into $N^a n_u$, since one of the two half vertices of any untwisted vertex is always fixed before the other. Adding the extra $N^a$ eventual factor for $G_0$, we arrive at the following estimate, where we recall that $n_u$ is the total number of untwisted vertices:

$$I_{G,t} \leq C_q^{3n} D(\Delta)^{-qn} N^{-2n} N^{n+1/2} N^a(n_u+1)$$
\[ N^{(1/2-a)c' + (3/4)c + (3/4)n_3 + ((1/4) - a)n_2 - a(n_1 + n_0)} \]  

where we have \( n = n_u + c + n_3 + n_2 + n_1 + n_0 \), and \( 2n = 2n_u + 4c + 3n_3 + 2n_2 + n_1 \), counting respectively the vertices and the lines. Taking into account the second relation we find

\[ I_{G,t} \leq C_q^{3n} D(\tilde{\Delta})^{-qn} N^{1/2 - n/2} N^{c'/2 - n_u/2} \]

\[ N^{c/4 - n_2/4 - n_1/4} N^{a(n_u+1-(n_0+n_1+n_2+c'))} \]  

(34)

Taking \( q = 4p \), the factor \( N^{-n/2} D(\tilde{\Delta})^{-qn/2} \) kills the \( N^{n/2} D(\tilde{\Delta})^{2pn} \) in formula (27). It remains:

\[ N^{n/2} D(\tilde{\Delta})^{2pn} |I_{G,t}| = J_{g,t} \]

\[ \leq K_q^n D(\tilde{\Delta})^{-2pn} N^{a+1/2} N^{c'/2} \]

\[ N^{-n_u/2 + c/4 - n_2/4 - n_1/4} N^{a(n_u+1-(n_0+n_1+n_2+c'))} \]  

(35)

From the fact that there is a tree of untwisted “wavy lines” between the connected components of type \( c \) and \( c' \), we know that \( c + c' \leq 1 + n_u \). Therefore

\[ J_{g,t} \leq K_q^n D(\tilde{\Delta})^{-2pn} N^{a+3/4} \]

\[ N^{-((1/4-a)(n_u-c')-n_2/4-n_1/4) N-a(n_0+n_1+n_2)} \]  

(36)

Choosing \( a=1/16 \), we have:

\[ J_{g,t} \leq K_q^n D(\tilde{\Delta})^{-2pn} N^{a+3/4} \]

\[ N^{-(3/16)(n_u-c')-(1/16)(n_0+n_1+n_2)} \]  

(37)

We have \( n_t = c + n_3 + n_2 + n_1 + n_0 \), and \( 2n_t = 4c + 3n_3 + 2n_2 + n_1 \), therefore \( 2c + n_3 = n_1 + 2n_0 \). Therefore \( c + n_3 \leq n_1 + 2n_0 \), so \( n_t \leq n_2 + 2n_1 + 3n_0 \leq 3(n_2 + n_1 + n_0) \). Hence

\[ J_{g,t} \leq K_q^n D(\tilde{\Delta})^{-2pn} N^{a+3/4} \]

\[ N^{-(3/16)[(n_u-c')-(3/16)n_t]} \]

\[ \leq K_q^n D(\tilde{\Delta})^{-2pn} N^{a+3/4} N^{-(3/16)(n-c')} \]  

(38)

Now we consider the following reduction process. We contract a certain number \( x \leq c' \) of pure cycles in the following inductive way: first we contract every tadpole (pure cycle with \( p = 1 \)), i.e. suppress it together with one of the two external lines hooked to the tadpole. Then we repeat this operation inductively (since the contraction at first stage can create new tadpoles), until
complete exhaustion. In the case of the graph of Figure 3, there is a single tadpole so the operation stops after one step, but in general it can be more complicated (see Figure 4).

![Figure 4: The contraction of tadpoles](image)

Let $x_1$ be the number of cycles contracted in this way. This contraction process continues until either we arrive at the empty graph because at some stage we obtained a graph with $c' = 2$ made of two tadpoles, which we then delete or until we arrive at a graph with $c' - x_1$ cycles without any tadpole. In this last case either the graph is a pure loop of $x_2$ bubbles that we destroy (see Figure 5), or it is not. In this second case, the graph may still contain pure bubbles, i.e. loops without twisted vertices of length 2. For instance in the case of the graph of Figure 3, after contraction of the tadpole, there is one such bubble which appears.
We perform then a second operation of contraction, which contracts every maximal chain of such bubbles (pure cycles of length 2 without twisted vertices) to a single new type of wavy line. This second operation has not to be repeated inductively in contrast with the first, since it cannot regenerate any bubble (see Figure 6). (Wavy lines have been omitted in Figure 4-5-6 for simpler pictures).

Let $x_2$ be the total number of bubbles contracted in this way. We arrive at a graph with $c' - x_1 - x_2$ cycles, each of them having at least three lines.

Let $x = x_1 + x_2$.

**Lemma 2** We have $x = x_1 + x_2 \leq f_2(G) + 2$, where $f_2(G)$ is the cardinal of the largest forests of connected 4-point subgraphs of $G$, defined in [CR].

Proof: Obviously each contracted cycle can be associated to a new 4 point subgraph in a forest, except if $x = c'$, in which case when one destroys the two tadpoles there is no new 4-point subgraph associated, or when one destroys the loop of $k$ bubbles, there is only $k - 1$ corresponding 4-point subgraphs in a forest. This two cases are exclusive, which explains the 2 in the lemma.

We have obviously $x \leq c'$ hence $n - c' \geq n - x \geq n - f_2(G) - 2$. We use now:

**Lemma 3 (particular case of [CR], theorem II)** The number of Wick contractions of half-vertices giving rise to a vacuum connected graph with $n$ vertices and $f_2(G) + 2 = f$, is bounded by $K^n n! / f! \leq K^n n^{n-f}$, for some constant $K$.

Proof See ([CR], Theorem II).

Combining this, with the previous bound we obtain (since $f \leq n + 2$ ([CR])):

$$
\sum_G J_G \leq \sum_{f \leq n+2} (KK_q)^{n-n-f} D(\tilde{\Delta})^{-2pn} N^{a+3/4} N^{-(3/16)(n-f)}
$$
\[ C^n D(\vec{\Delta})^{-2pN^{a+3/4}(n/N^{3/16})^{-1}} \]  
\[ \text{(39)} \]

where \( C = 2K.K_q \) is a constant. We choose now \( n = N^{3/16} \) and obtain:

\[
\begin{align*}
\text{Prob} \{ ||H_{\vec{\Delta}}|| \geq A.D(\vec{\Delta})^{-pN^{-1/4}} \} \\
\leq A^{-2N^{3/16}} D(\vec{\Delta})^{-2pN^{3/16}} N^{a+3/4} C N^{3/16}
\end{align*}
\[ \text{(40)} \]

If \( A^{1/2} \) is larger than \( C \), we obtain easily the bound (32) of the theorem (since \( A^{-N^{3/16}/2N^{a+3/4}} \leq 1 \), \( A \) being large and \( N \geq 1 \), and we can also choose \( A \) such that \( CA^{-1/2} \leq 1 \). We have therefore in that case:

\[
\begin{align*}
\text{Prob} \{ ||H_{\vec{\Delta}}|| \geq A.D(\vec{\Delta})^{-pN^{-1/4}} \} \\
\leq A^{-N^{3/16}} D(\vec{\Delta})^{-2pN^{3/16}}
\end{align*}
\[ \text{(41)} \]

which completes the proof of Theorem II.

In order to obtain a theorem no longer on \( H \) but on the full operator \( \lambda CH \) that occurs in our resolvent expansions, we need to add to the estimate of Theorem I the power counting coming from the coupling constant and the norm of the propagator. This adds a factor \( \epsilon N^{1/4} \) to the norm of \( H \). The factor \( \epsilon \) comes from the distance to the last slice, since recall that in section 3 we assumed that \( \lambda^2 < \epsilon M^{-1} \). Finally we obtain, taking \( \epsilon = A^{-2} \) small, the following estimate:

**Theorem III**

*If the coupling is kept small enough (i.e. \( \epsilon \leq A^{-2} \), where \( A \) is defined by Theorem II) we have:*

\[
\text{Prob} \{ ||\lambda CH_{\vec{\Delta}}|| \geq A^{-1} D(\vec{\Delta})^{-p} \} \\
\leq A^{-N^{3/16}} D(\vec{\Delta})^{-2pN^{3/16}}
\]  
\[ \text{(42)} \]

To complete the proof of Theorem I, one combines the estimate of Theorem III with a large versus small field cluster expansion. This expansion is exactly similar to the one of [P], to which we refer the reader. The decay factor \( D(\vec{\Delta})^{-p} \) is roughly speaking needed to sum the various triplets \( \vec{\Delta} \) with respect to one of their elements, and the factor \( A^{-N^{3/16}} \) is sufficient to compensate the bad factors due to the imaginary translation (see (13)) which come from the estimation of the resolvent in the large field regions.

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