Black hole entropy of new bigravity

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In \texttt{arXiv:1402.5737}, we proposed a new ghost-free massive spin two model. This model consists of a kinetic term and non-derivative interaction terms. Thus, the theory is expected to have different properties from de Rham-Gabadadze-Tolley (dRGT) massive gravity. In this paper, we couple this spin-2 theory with gravity and obtain a black hole solution in addition to (anti-) de Sitter space solution. Furthermore, by calculating the black hole entropy, we investigate the effect of this new spin-2 model to the Einstein gravity.

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I. INTRODUCTION

The consistent free massive spin-2 theory was first established by Fierz and Pauli \textsuperscript{1}. The mass term for spin-2 particles generally leads to a ghost mode, but they preserve the consistency of the theory by tuning coefficients of the mass term. Since the Fierz-Pauli theory does not have any gauge symmetry, it seems that arbitrary interactions can be added to the theory. Contrary to this naive expectation, Boulware and Deser \textsuperscript{2} showed that non-linear terms generally lead to another ghost called the Boulware-Deser ghost. There were another problem, that is, the appearance of the van Dam-Veltman-Zakharov (vDVZ) discontinuity \textsuperscript{3} in the massless limit, $m \to 0$ although the discontinuity can be screened by the Vainstein mechanism \textsuperscript{4} (see, for example, Ref. \textsuperscript{5}).

After these indications, the studies of massive spin 2 fields had not progressed until 2003. In 2003, Arkani-Hamed, Georgi, and Schwartz \textsuperscript{6}, however, regarded the theory as a low energy effective field theory, and revealed a cutoff scale. They considered a limit which focuses on the cutoff and they have shown that the special choice of the coefficients in the potential terms makes the cutoff scale larger. As the potential-tuned theory consists of infinite terms, it was unclear whether the theory contains the BD ghost or not. After that, de Rham, Gabadadze and Tolley (dRGT) \textsuperscript{7} succeeded in the resummation of the potential terms and Hassan and Rosen \textsuperscript{8} proved that the theory with the resummed potential terms does not contain any ghost. This theory is called the dRGT massive gravity. The most essential part in this theory is special forms of the fully non-linear potential terms eliminating the extra mode. Although the massive gravity models have non-dynamical background metric, the models have been extended to the models with dynamical metric \textsuperscript{9–11}, which are called as bigravity models.

Last year, Hinterbichler \textsuperscript{12} pointed out the possibility of new derivative interaction terms in the dRGT massive gravity. It was shown that new derivative interactions can be added to the Fierz-Pauli theory by the taking specific linear combination of interactions and conjectured fully non-linear counterparts of these interaction terms in dRGT massive gravity. In this context, it was also shown that the leading term of the dRGT potential term does not generate the ghost to the Fierz-Pauli theory. Thus, we constructed a new massive spin-2 model by adding the leading terms to Fierz-Pauli free theory in \textsuperscript{13}.

In this paper, we consider the model where the field of massive spin-2 particles couples with gravity. A reason why we consider this model is an application to the cosmology and black hole physics. We often consider the models of scalar fields to explain the expanding universe not to violate the isotropy while the condensation of the vector field violate the isotropy in general except the case that the model has a non-abelian gauge symmetry.\textsuperscript{1} See \textsuperscript{14–17} for the general review of the accelerating expansion of the universe by the modified gravities. The field of the massive spin two particle is given by rank 2 symmetric tensor. We should note that the condensation of the trace part of the rank 2 symmetric tensor (or $(t,t)$ component, or the trace of the spacial part) does not violate the isotropy and therefore we can use the rank 2 symmetric tensor in order to explain the expansion of the universe.

Such a cosmology has been studied in the massive gravity models \textsuperscript{18} by considering the decoupling limit where the models reduce to scalar-tensor theories. After that there follow several activities in the massive gravity models \textsuperscript{19,22} and in the bimetric gravity models \textsuperscript{23,30}.

\textsuperscript{1} Non-abelian gauge always contains $SU(2)$ or $SO(3)$ as a subgroup. The condensation of the vector field breaks both of the isotropy or rotational invariance and the gauge symmetry. Because the rotational symmetry is $SO(3)$, even if the vector field condensate, there remain the diagonal symmetry in the product of the rotational symmetry $SO(3)$ times the gauge symmetry $SO(3)$ and we can regard the diagonal symmetry as a new rotational symmetry.
In this paper, we mainly focus on the black hole physics, especially the black hole entropy. We have to note that this gravity coupled massive spin-2 model is essentially different from the Hassan-Rosen bigravity model \cite{9,11}. In the construction, we replace the Minkowski background metric with the dynamical metric and replace partial derivatives \( \partial_\mu \) with covariant derivatives \( \nabla_\mu \). Thus our model does contain one metric, not two. Since the effect of massive spin-2 particles to the black hole entropy has been already calculated in the Hassan-Rosen bigravity model \cite{31,32}, it is quite interesting to see how the result change depending on the model.

II. NEW MODEL OF MASSIVE SPIN TWO PARTICLE

The Lagrangian of the Fierz-Pauli theory is given by \cite{1}

\[
\mathcal{L}_{FP} = -\frac{1}{2} \partial_\mu h_{\mu\nu} \partial^\lambda h^{\mu\nu} + \partial_\mu h_{\lambda\nu} \partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu} \partial_\nu h + \frac{1}{2} \partial_\lambda h \partial^\lambda h - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2). \tag{1}
\]

The relative sign of the mass term is tuned to eliminate a ghost. Hinterbichler pointed out that new interaction terms can be added to this model without introducing any ghost by taking the specific linear combination \cite{33}. In four dimensions, there are two kinds of non-derivative interactions:

\[
\mathcal{L}_3 \sim \eta^{\mu_1 \nu_1 \mu_2 \nu_2} h_{\mu_1 \nu_1} h_{\mu_2 \nu_2} h_{\mu_3 \nu_3},
\]

\[
\mathcal{L}_4 \sim \eta^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4} h_{\mu_1 \nu_1} h_{\mu_2 \nu_2} h_{\mu_3 \nu_3} h_{\mu_4 \nu_4}.
\]

Here \( \eta^{\mu_1 \nu_1 \cdots \mu_n \nu_n} \) is given by the product of \( n \) \( \eta_{\mu \nu} \) and anti-symmetrizing the indexes \( \nu_1, \nu_2, \cdots, \nu_n \), for examples,

\[
\eta^{\mu_1 \nu_1 \mu_2 \nu_2} \equiv \eta^{\mu_1 \nu_1} \eta^{\mu_2 \nu_2} - \eta^{\mu_1 \mu_2} \eta^{\nu_1 \nu_2},
\]

\[
\eta^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3} \equiv \eta^{\mu_1 \nu_1} \eta^{\mu_2 \nu_2} \eta^{\mu_3 \nu_3} - \eta^{\mu_1 \mu_2 \mu_3} \eta^{\nu_1 \nu_2 \nu_3} + \eta^{\mu_1 \nu_1 \nu_2} \eta^{\mu_2 \nu_2 \mu_3} - \eta^{\mu_1 \nu_1 \mu_2} \eta^{\nu_2 \nu_3 \mu_3} - \eta^{\mu_1 \nu_1 \nu_2} \eta^{\nu_3 \nu_3 \mu_2} - \eta^{\mu_1 \mu_2 \nu_2} \eta^{\nu_1 \nu_3 \mu_3}.
\]

In \cite{13}, we proposed the new model of massive spin-2 particles by adding the two terms \cite{2} and \cite{3} to the Fierz-Pauli Lagrangian \cite{1}.

\[
\mathcal{L}_{h\phi} = -\frac{1}{2} \eta^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4} \left( \partial_\mu h_{\nu_1 \nu_2} h_{\mu_2 \nu_2} h_{\mu_3 \nu_3} h_{\mu_4 \nu_4} \right) - \frac{\mu}{3!} \eta^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4} h_{\mu_1 \nu_1} h_{\mu_2 \nu_2} h_{\mu_3 \nu_3} h_{\mu_4 \nu_4} - \frac{1}{2} \left( h^{\mu \nu} \partial_\mu h^{\mu \nu} - h \partial_\mu h^{\mu \nu} \partial_\nu h^{\mu \nu} - h \partial_\mu \partial_\nu h^{\mu \nu} + 6 \partial_\mu h^{\lambda \nu} \partial_\nu h_{\lambda \mu} ight)
\]

\[
+ \frac{m^2}{2} \left( h^2 - \mu h^{\mu \nu} \right) \frac{\lambda}{4!} \left( h^3 - 3h h^{\mu \nu} h^{\mu \nu} + 2h \mu h^{\rho \sigma} h^{\rho \sigma} \right)
\]

\[
- \frac{\lambda}{4!} \left( h^4 - 6h^2 h^{\mu \nu} h^{\mu \nu} + 8h h^{\rho \sigma} h^{\rho \sigma} h^{\mu \nu} - 6h h^{\rho \sigma} h^{\rho \sigma} h^{\mu \nu} + 3 \left( h^{\mu \nu} h^{\mu \nu} \right)^2 \right). \tag{5}
\]

Here \( m \) and \( \mu \) are parameters with the dimension of mass and \( \lambda \) is a dimensionless parameter. We assume that \( \mu \) always takes positive values but cannot decide the sign of \( \lambda \), because it is non trivial to learn which sign for \( \lambda \) stabilizes this system as we will see later.

Although the model \cite{5} is power counting renormalizable, the model is not really renormalizable because the propagator behaves \( O \left( p^2 \right) \) for large momentum \( p \) instead of the naive expectation \( O \left( p^{-2} \right) \). In fact, the propagator has the following form:

\[
D_{\alpha \beta \rho \sigma}^m = -\frac{1}{2 \left( p^2 + m^2 \right)} \left( P_{\alpha \rho} P_{\beta \sigma}^m + P_{\alpha \beta} P_{\rho \sigma}^m - \frac{2}{3} P_{\alpha \beta} P_{\rho \sigma}^m \right), \tag{6}
\]

\[
P_{\mu \nu}^m \equiv \eta_{\mu \nu} + \frac{p_\mu p_\nu}{m^2}. \tag{7}
\]

Then when \( p^2 \) is large, the propagator behaves as \( D_{\alpha \beta \rho \sigma}^m \sim O \left( p^2 \right) \) due to the projection operator \( P_{\mu \nu}^m \), which makes the behavior for large \( p^2 \) worse and therefore the model should not be renormalizable.

Since this theory has no symmetry and is already non-renormalizable, it seems that there is no reason why we only consider the potential terms up to the quartic order. However, introducing higher order potential terms break the consistency as a quantum field theory in four dimensions. The potential terms described above does not generate any ghost due to the anti-symmetric property. Therefore, in four dimensions, we cannot construct similar ghost-free potential terms. Needless to say, we can add higher order terms in five or higher dimensions.
III. NEW BIGRAVITY

The bigravity model can be regarded as a model where the massive spin two field couples with gravity. Then we may consider the model where $h_{\mu\nu}$, whose lagrangian is given by (5), couples with gravity

\[ S = \int d^4 x \sqrt{-g} \left( -\frac{1}{2} \delta^{\mu\nu}_{\rho\sigma} \nabla_\mu \nabla_\nu h_{\rho\sigma} + \frac{1}{2} m^2 \gamma^{\mu\nu}_{\rho\sigma} h_{\rho\sigma} + \frac{\mu}{3!} \gamma^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} h_{\rho\sigma} h_{\alpha\beta} h_{\gamma\delta} \right) + S_{EH}, \]

(8)

which can be regarded as a new bigravity model because there appear two symmetric tensor fields $g_{\mu\nu}$ and $h_{\mu\nu}$. We should note that $h_{\mu\nu}$ is not the perturbation in $g_{\mu\nu}$ but $h_{\mu\nu}$ is a field independent of $g_{\mu\nu}$. We also note that $S_{EH}$ is the Einstein-Hilbert action without the cosmological constant.

\[ S_{EH} = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} R. \]

(9)

We consider the cosmological solution and the black hole solution for this theory. By the variations of $h_{\mu\nu}$, we obtain the equations of motion for $h_{\mu\nu}$,

\[ \frac{\delta S}{\delta h_{\mu\nu}} = \Box h_{\mu\nu} - \nabla_\lambda \nabla_\mu h^\lambda_{\nu} - \nabla_\lambda \nabla_\nu h^\lambda_{\mu} + g_{\mu\nu} \nabla_\lambda \nabla_\sigma h^{\lambda\sigma} + \nabla_\mu \nabla_\nu h - g_{\mu\nu} \Box h \]

\[ - m^2 (h_{\mu\nu} - g_{\mu\nu} h) - \frac{\mu}{3!} (3 g_{\mu\nu} h^2 - 3 g_{\mu\nu} h^\rho_{\rho\sigma} h^{\alpha\beta} + 6 h_{\mu\nu\rho} + 3 h_{\mu\nu} h^\rho_{\rho\nu} + 3 h_{\mu\nu} h^\rho_{\rho\mu}) \]

\[ - \frac{1}{4!} (4 g_{\mu\nu} h^3 - 12 g_{\mu\nu} h_{\rho\sigma} h^{\rho\sigma} - 12 h^2 h_{\mu\nu} + 8 g_{\mu\nu} h^\rho_{\rho\sigma} h_{\sigma\rho} h^\eta_{\rho\nu} + 12 h_{\mu\nu} h_{\nu\rho} + 12 h_{\mu\nu} h^\rho_{\rho\nu}) \]

\[ - 12 h_{\mu\nu} h^\rho_{\rho\sigma} h_{\sigma\nu} - 12 h_{\mu\nu} h^\rho_{\rho\sigma} h_{\sigma\mu} + 12 h_{\mu\nu} (h^\rho_{\rho\sigma} h_{\rho\sigma}) = 0. \]

(10)

We assume the solution of equations (10) is given by

\[ h_{\mu\nu} = C g_{\mu\nu}. \]

(11)

Here $C$ is a constant. Substituting (11) into the equations (10) gives

\[ (3 m^2 C - 3 \mu C^2 - \lambda C^3) g_{\mu\nu} = 0. \]

(12)

The solutions for (12) are given by

\[ C = 0, \quad -3 \mu \pm \sqrt{9 \mu^2 + 12 m^2 \lambda}. \]

(13)

Because the solution should be a real number, the parameters are constrained to be $\lambda \geq -3 \mu^2 / 4 m^2$.

By assuming (11), the action $S$ is reduced to

\[ S = - \int d^4 x \sqrt{-g} V(C) + S_{EH}, \quad V(C) \equiv -6 m^2 C^2 + 4 \mu C^3 + \lambda C^4, \]

(14)

because $\nabla_\mu g_{\mu\nu} = 0$. We may regard $V(C)$ as a potential for $C$. Then Eq. (12) is nothing but the condition $V'(C) = 0$. We should note that when $\mu = \lambda = 0$, which corresponds to the Fierz-Pauli model, the potential $V(C)$ is not unbounded below and $C = 0$ corresponds to the local maximum instead of the local minimum. As we know, however, that the massive spin two field is stable on the local maximum. On the other hand, on the local minimum of $C$, the fluctuation of the massive spin two field becomes tachyonic and unstable. Such a contradiction to the intuition occurs because $C$ does not correspond to the propagating mode and $C$ should be a constant. In fact, if we assume (11) and that $C$ could not be a constant, (10) tells

\[ 0 = g^{\mu\nu} (2 \Box C + 3 m^2 C - 3 \mu C^2 - \lambda C^3) - 2 \nabla^\mu \nabla^\nu C. \]

(15)

We may consider the local inertial frame of reference where all the connections vanish, $\Gamma_{\nu\rho} = 0$ and $g_{\mu\nu} = \eta_{\mu\nu}$. Then when $\mu \neq \nu$ in (13) gives

\[ \partial_\mu \partial_\nu C = 0, \]

(16)
which tells that $C$ is given by a sum of the functions of each of coordinates $C = \sum_\mu C^\mu(x^\mu)$. Eq. (15) also gives

$$\eta^{\mu\nu} \partial_\mu C = \eta^{\mu\nu} \partial_\nu C.$$  (17)

In Eq. (17), the indices $\mu$ in the l.h.s. and $\nu$ in the r.h.s. are not summed up. Eq. (17) tells that $C$ is linear to the coordinates, $C = \sum_\mu c_\mu x^\mu + C_0$. Here $c_\mu$'s and $C_0$ are constant. By substituting this expression into (15), we find $c_\mu = 0$ and therefore $C$ should be surely a constant. This tells that even if $C$ is on the local maximum of the potential (14), $C$ does not roll down.

As a next step in order to find the cosmological solution and the black hole solution, we derive the Einstein equation. The substitution of $h_{\mu\nu} = Cg_{\mu\nu}$ into (8) gives

$$S = -\int d^4x \sqrt{-g_{\text{eff}}} + \frac{1}{16\pi G} \int d^4x \sqrt{-g} R.$$  (18)

where

$$\Lambda_{\text{eff}} = V(C) = -6m^2C^2 + 4\mu C^3 + \lambda C^4.$$  (19)

Thus, if we define $\tilde{\Lambda}_{\text{eff}} = 8\pi G(-6m^2C^2 + 4\mu C^3 + \lambda C^4)$, we obtain

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - 2\tilde{\Lambda}_{\text{eff}} \right).$$  (20)

By taking the variation with respect to the metric, we find

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \tilde{\Lambda}_{\text{eff}} g_{\mu\nu} = 0.$$  (21)

Eq. (21) has the (anti-) de Sitter solutions depending on the sign of $\tilde{\Lambda}_{\text{eff}}$. The de Sitter solution describes the accelerating expansion of the universe and it may be related with the inflation in the early universe and the dark energy problem in the present universe. Therefore, we investigate when the solution becomes the de Sitter or anti-de Sitter space-time depending on $C$. The condition for the de Sitter solution is given by

$$\Lambda_{\text{eff}} = -6m^2 C^2 + 4\mu C^3 + \lambda C^4 > 0.$$  (22)

On the other hand, the condition for the anti-de Sitter solution is

$$\Lambda_{\text{eff}} = -6m^2 C^2 + 4\mu C^3 + \lambda C^4 < 0.$$  (23)

As the sign of $\lambda$ is undetermined while $\mu$ takes positive values, we consider the two cases where $\lambda > 0$ and $\lambda < 0$.

(i) $\lambda > 0$

Neglecting the trivial solution $C = 0$, the remaining solutions for $C$ are given by

$$C_1 = \frac{-3\mu + \sqrt{9\mu^2 + 12m^2\lambda}}{2\lambda} > 0, \quad C_2 = \frac{-3\mu - \sqrt{9\mu^2 + 12m^2\lambda}}{2\lambda} < 0.$$  (24)

Let us consider which solution corresponds to the (anti-) de Sitter solution under the assumption $\mu > 0$ and $\lambda > 0$. For this purpose, we have to solve inequalities (22) and (23). The solutions are given by

$$C < C_- \quad \text{or} \quad C_+ < C \quad \text{for} \quad \Lambda_{\text{eff}} > 0,$$

$$C_+ < C < C_- \quad \text{for} \quad \Lambda_{\text{eff}} < 0.$$  (25)

Here $C_+$ and $C_-$ are defined by

$$C_+ = \frac{-2\mu + \sqrt{4\mu^2 + 6\lambda m^2}}{\lambda} > 0, \quad C_- = \frac{-2\mu - \sqrt{4\mu^2 + 6\lambda m^2}}{\lambda} < 0.$$  (27)

For $C_\pm$ to be real numbers, $\lambda$ should be larger than $-2\mu^2/3m^2$, but we assume the positivity of $\lambda$ here. $C_1$ and $C_+$ are both positive and $C_2$ and $C_-$ are both negative. Thus, what we should do is to compare $C_1$ to $C_+$ and $C_2$ to $C_-$ respectively.
1. $C_+$ and $C_1$
Let us consider the following quantity.

\[
\frac{C_+}{C_1} - 1 = \frac{\mu^2}{m^2\lambda} \left( -1 + \sqrt{1 + \frac{3m^2\lambda}{2\mu^2}} \right) \left( 1 + \sqrt{1 + \frac{4m^2\lambda}{3\mu^2}} \right) - 1 \\
= \frac{1}{A} \left( -1 + \sqrt{1 + \frac{3A}{2}} \right) \left( 1 + \sqrt{1 + \frac{4A}{3}} \right) - 1,
\]

(28)

where $A := m^2\lambda/\mu^2 > 0$ because of $\lambda > 0$. For $A > 0$, the quantity (28) is always positive, which means $C_1 < C_+$.

2. $C_2$ and $C_-$
In this case, we define the quantity

\[
\left| \frac{C_-}{C_2} \right| - 1 = \frac{\mu^2}{m^2\lambda} \left( 1 + \sqrt{1 + \frac{3m^2\lambda}{2\mu^2}} \right) \left( -1 + \sqrt{1 + \frac{4m^2\lambda}{3\mu^2}} \right) - 1 \\
= \frac{1}{A} \left( 1 + \sqrt{1 + \frac{3A}{2}} \right) \left( -1 + \sqrt{1 + \frac{4A}{3}} \right) - 1.
\]

(29)

For $A > 0$, (29) is always positive. Since $C_-$ and $C_2$ take negative values, $C_- < C_2$ is held.

Therefore, we see that both solutions satisfy the condition for the anti-de Sitter solution.

(ii) $-3m^2/4\mu^2 < \lambda < 0$

We can continue the same analysis in this case. However, the sign of $C_{1,2}$ and $C_\pm$ are modified.

\[
C_{1,2} > 0 \quad \text{and} \quad C_\pm > 0 \quad (C_2 > C_1, \quad C_- > C_+).
\]

(30)

Since $\lambda$ is the coefficient of $C^4$, the solutions for the inequality are also changed.

\[
C_+ < C < C_- \quad \text{for} \quad \Lambda_{\text{eff}} > 0,
\]

(31)

\[
C < C_+ \quad \text{or} \quad C_- < C \quad \text{for} \quad \Lambda_{\text{eff}} < 0.
\]

(32)

The condition for $\lambda$ to be real is given by $-2\mu^2/3m^2$ which is larger than the condition for the reality of $C_{1,2}$. As we assume the reality of $C$ in this analysis, $C_\pm$ does not exist for the case $-3\mu^2/4m^2 < \lambda < -2\mu^2/3m^2$. Therefore, we have to divide the parameter region $-3m^2/4\mu^2 < \lambda < 0$ into $-3\mu^2/4m^2 < \lambda < -2\mu^2/3m^2$ and $-2\mu^2/3m^2 < \lambda < 0$.

(ii-1) $-2\mu^2/3m^2 < \lambda < 0$

Let us compare $C_{1,2}$ with $C_\pm$.

1. $C_+$ and $C_1$
As in the previous case, we consider the quantity

\[
\frac{C_+}{C_1} - 1 = \frac{\mu^2}{m^2|\lambda|} \left( 1 - \sqrt{1 - \frac{3m^2|\lambda|}{2\mu^2}} \right) \left( 1 + \sqrt{1 + \frac{4m^2|\lambda|}{3\mu^2}} \right) - 1 \\
= \frac{1}{A} \left( 1 - \sqrt{1 - \frac{3A}{2}} \right) \left( 1 + \sqrt{1 + \frac{4A}{3}} \right) - 1.
\]

(33)

Here $A := m^2|\lambda|/\mu^2 > 0$. (33) is positive under the $A > 0$, which leads to $C_1 < C_+ < C_-$. 

2. $C_-$ and $C_2$

\[
\frac{C_-}{C_2} - 1 = \frac{\mu^2}{m^2|\lambda|} \left( 1 + \sqrt{1 - \frac{3m^2|\lambda|}{2\mu^2}} \right) \left( 1 - \sqrt{1 + \frac{4m^2|\lambda|}{3\mu^2}} \right) - 1
\]
To summarize, in the case of stability of the solution, which will be discussed in the next section. We show the relation among considering the negative cosmological constant, we find the anti-de Sitter solution

\[
\tilde{\Lambda} = \frac{\mu^2}{m^2|\lambda|} \left( 1 - \sqrt{1 - \frac{3m^2|\lambda|}{2\mu^2}} \right) \left( 1 - \sqrt{1 - \frac{4m^2|\lambda|}{3\mu^2}} \right) - 1.
\]

\[\text{(34)}\]

is positive when \( A > 0 \). Thus, \( C_2 < C_- \).

3. \( C_+ \) and \( C_2 \)

\[
\frac{C_+}{C_2} - 1 = \frac{\mu^2}{m^2|\lambda|} \left( 1 - \sqrt{1 - \frac{3m^2|\lambda|}{2\mu^2}} \right) \left( 1 - \sqrt{1 - \frac{4m^2|\lambda|}{3\mu^2}} \right) - 1
\]

\[
= \frac{1}{A} \left( 1 - \sqrt{1 - \frac{3A}{2}} \right) \left( 1 - \sqrt{1 - \frac{4A}{3}} \right) - 1. \tag{35}
\]

Eq. \( \text{(35)} \) takes negative values when \( A > 0 \). This means \( C_+ < C_2 \).

According to these analysis, \( C_2 \) and \( C_1 \) correspond to the de Sitter solution and the anti-de Sitter solution respectively.

\( \text{(ii-2)} \) \(-3\mu^2 / 4m^2 < \lambda < -2\mu^2 / 3m^2\)

As mentioned above, \( C_{\pm} \) is no longer real in this case. Thus, \( \Lambda_{\text{eff}} \) takes negative values only, which means that \( C_1 \) and \( C_2 \) produce the anti-de Sitter solution.

We show the relation among \( \lambda, C_{1,2} \) and the space-time in the table I. The table I also includes the results on the stability of the solution, which will be discussed in the next section.

To summarize, in the case of \(-2\mu^2 / 3m^2 < \lambda < 0\), \( C_2 \) corresponds to the de Sitter solution

\[
ds^2 = -\left( 1 - \frac{\tilde{\Lambda}_{\text{eff}}}{3} r^2 \right) dt^2 + \left( 1 - \frac{\tilde{\Lambda}_{\text{eff}}}{3} r^2 \right)^{-1} dr^2 + r^2 d\Omega_2^2.
\]

while \( C_1 \) generates the anti-de Sitter solution.

\[
ds^2 = -\left( 1 + \frac{\tilde{\Lambda}_{\text{eff}}}{3} r^2 \right) dt^2 + \left( 1 + \frac{\tilde{\Lambda}_{\text{eff}}}{3} r^2 \right)^{-1} dr^2 + r^2 d\Omega_2^2. \tag{37}
\]

In the other parameter region for \( \lambda \), both solutions \( C_{1,2} \) give the anti-de Sitter solution \( \text{(37)} \). The Einstein equations with the effective cosmological constant \( \Lambda_{\text{eff}} \) also admit the Schwartzchild-(anti) de Sitter black hole solution. The Schwartzchild-de Sitter solution is given by

\[
ds^2 = -\left( 1 - \frac{2M}{r} - \frac{\tilde{\Lambda}_{\text{eff}}}{3} r^2 \right) dt^2 + \left( 1 - \frac{2M}{r} - \frac{\tilde{\Lambda}_{\text{eff}}}{3} r^2 \right)^{-1} dr^2 + r^2 d\Omega_2^2. \tag{38}
\]

Considering the negative cosmological constant, we find the anti-de Sitter solution

\[
ds^2 = -\left( 1 - \frac{2M}{r} + \frac{\tilde{\Lambda}_{\text{eff}}}{3} r^2 \right) dt^2 + \left( 1 - \frac{2M}{r} + \frac{\tilde{\Lambda}_{\text{eff}}}{3} r^2 \right)^{-1} dr^2 + r^2 d\Omega_2^2. \tag{39}
\]

The Schwartzchild-de Sitter solution \( \text{(35)} \) has the BH horizon \( r_{\text{BH}} \) and the cosmological horizon \( r_c \).

\[
r_{\text{BH}} = \frac{2}{\sqrt{\tilde{\Lambda}_{\text{eff}}} \cos \frac{\pi + \psi}{3}}, \quad r_c = \frac{2}{\sqrt{\tilde{\Lambda}_{\text{eff}}} \cos \frac{\pi - \psi}{3}}, \tag{40}
\]

\[\text{TABLE I: The relation between } C_{1,2} \text{ and the corresponding space-time and the stability of the solutions}\]

| space-time | parameters | 0 < \lambda | \(-2\mu^2 / 3m^2 < \lambda < 0\) | \(-3\mu^2 / 4m^2 < \lambda < -2\mu^2 / 3m^2\) |
|------------|------------|-------------|----------------|-----------------|
| de Sitter  | no solution| \( C_2 \) (stable) | no solution |
| Anti-de Sitter | \( C_1 \) (unstable) and \( C_2 \) (unstable) | \( C_1 \) (unstable) | \( C_2 \) (stable) |
where $\psi := \cos^{-1}\left(3M \sqrt{\Lambda_{\text{eff}}}ight)$. On the other hand, the Schwartzchild-anti-de Sitter solution has one event horizon only:

$$r_{\text{BH}} = \frac{2}{\sqrt{|\Lambda_{\text{eff}}|}} \sinh \left[\frac{1}{3} \sinh^{-1}\left(3M \sqrt{|\Lambda_{\text{eff}}|}\right)\right].$$

(41)

The area of the event horizon for (40) is given by

$$A_{\text{SdS}} = \frac{16\pi}{|\Lambda_{\text{eff}}|} \cos^2 \frac{\pi + \psi}{3}.\quad \text{(42)}$$

For the Schwartzchild-anti de Sitter black hole, the area of the event horizon is

$$A_{\text{SAdS}} = \frac{16\pi}{|\Lambda_{\text{eff}}|} \sinh^2 \left[\frac{1}{3} \sinh^{-1}\left(3M \sqrt{|\Lambda_{\text{eff}}|}\right)\right].$$

(43)

Thus, from the Bekenstein-Hawking formula, we find the black hole entropy for (38)

$$S_{\text{SdS}} = \frac{A_{\text{SdS}}}{4G} = \frac{4\pi}{G|\Lambda_{\text{eff}}|} \cos^2 \frac{\pi + \psi}{3}.\quad \text{(44)}$$

Similarly, the entropy for (39) is given by

$$S_{\text{SAdS}} = \frac{A_{\text{SAdS}}}{4G} = \frac{4\pi}{G|\Lambda_{\text{eff}}|} \sinh^2 \left[\frac{1}{3} \sinh^{-1}\left(3M \sqrt{|\Lambda_{\text{eff}}|}\right)\right].$$

(45)

Then the expressions in (44) and (45) are not changed from those in the Einstein gravity. These results can be compared with those [31, 32] in the Hassan-Rosen bigravity model [9–11], where the entropy is given by the sum of the contributions from two metric sectors.

### IV. STABILITY

As we mentioned in the section III, the Fierz-Pauli theory is stable on the local maximum. Therefore, it is plausible to assume that the theory is stable on the local maximum even though the parameters $\mu$ and $\lambda$ take non-zero values. Under this assumption, we check the stability of the solution $C_{1,2}$ in (24). For this purpose, we have to obtain the expression of the second derivative of the potential:

$$V''(C) = -12m^2 + 24\mu C + 12\lambda C^4.$$

(46)

We find the stability by substituting the solutions into (46) for each parameter region.

(i) $0 < \lambda$

In this case, both solutions corresponds to the anti-de Sitter solutions. Plugging these solution yields

1. $C = C_1$

$$V''(C = C_1) = \frac{3}{\lambda}(6\mu^2 - 2\mu \sqrt{9\mu^2 + 12m^2\lambda}) + 24m^2 > 0.$$

(47)

2. $C = C_2$

$$V''(C = C_2) = \frac{3}{\lambda}(6\mu^2 + 2\mu \sqrt{9\mu^2 + 12m^2\lambda}) + 24m^2 > 0.$$

(48)

Because $V''(C)$ in (46) is positive, these solutions are unstable.

(ii) $-2\mu^2/3m^2 < \lambda < 0$

$C_1$ and $C_2$ are linked with the de Sitter and the anti-de Sitter solutions respectively. As the above case, we find (46) for each solution.
1. \( C = C_1 \)

\[
V''(C = C_1) = \frac{3}{|\lambda|} (-6\mu^2 + 2\mu\sqrt{9\mu^2 - 12m^2|\lambda|}) + 24m^2 > 0.
\] (49)

2. \( C = C_2 \)

\[
V''(C = C_2) = \frac{3}{|\lambda|} (-6\mu^2 - 2\mu\sqrt{9\mu^2 - 12m^2|\lambda|}) + 24m^2 < 0.
\] (50)

This result means that \( C_1 \) corresponding to the anti-de Sitter solution is unstable while \( C_2 \) corresponding to the de Sitter solution is stable.

(iii) \(-3\mu^2/4m^2 < \lambda < -2\mu^2/3m^2\)

The anti-de Sitter space-time is realized for the both solutions \( C_1 \) and \( C_2 \).

1. \( C = C_1 \)

\[
V''(C = C_1) = \frac{3}{|\lambda|} (-6\mu^2 + 2\mu\sqrt{9\mu^2 - 12m^2|\lambda|}) + 24m^2 > 0.
\] (51)

2. \( C = C_2 \)

\[
V''(C = C_2) = \frac{3}{|\lambda|} (-6\mu^2 - 2\mu\sqrt{9\mu^2 - 12m^2|\lambda|}) + 24m^2 < 0.
\] (52)

Although both solutions lead to the anti-de Sitter solution, \( C_1 \) is unstable and the other solution is stable.

From the discussion, we see that solutions \( C_{1,2} \) generate the same space-time (the anti-de Sitter) in the both cases (i) and (iii), but the stability is different. Both solutions are unstable in the case (i) while there exists the stable solution in the case (ii). The case (iii) have one stable de Sitter solution and one unstable anti-de Sitter solution. These results are also summarized in the table I.

V. SUMMARY

In this paper, we have investigate the classical solutions in a new ghost-free field theory of symmetric tensor field describing massive spin two particles, which was proposed in [13], by coupling the model with gravity. Then we obtained solutions describing the (anti)-de Sitter space-time. The obtained de Sitter space-time might correspond to the inflation in the early universe or the accelerating expansion in the present universe. These solutions correspond to the extrema of the potential for the trace of the symmetric tensor field. In conflict with the intuition, massive spin two particle becomes tachyon on the local minimum of the potential and the particle is stable on the local maximum, that is, the local minimum induces the instability although the local maximum corresponding to the stability. In addition to the solutions describing the (anti)-de Sitter space-time, we find the solutions describing the black hole, which could be the (anti)-de Sitter-Schwarzschild or the (anti)-de Sitter-Kerr space-time. By calculating the black hole entropy, furthermore, we find the expressions of the black hole entropy in (38) and (39) are identical with in the Einstein gravity. In case of the Hassan-Rosen bigravity model [9–11], the entropy is given by the sum of the contributions from two metric sectors, which is different from the results in the model proposed in [13] and in this paper.

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