Regimes of photon generation in Dynamical Casimir Effect under various resonance conditions

Levchenko A.V., Trifanov A.I.
ITMO University, 49 Kronverksky Avenue, Saint Petersburg, 197101, Russia
E-mail: alextrifanov@gmail.com

Abstract. We investigate spectral and statistical properties of the field generated in dynamical Casimir effect under various resonance conditions appeared from periodical motion of cavity boundary. In particular we are interested in single and two-mode squeezing effects which take place for each resonant frequency under the cases of bounded and unbounded cavity spectrum.

1. Introduction
Dynamical Casimir effect (DCE) is the relativistic quantum parametric process by which the accelerated motion of the cavity boundaries excites vacuum fluctuations [1]. This process is accompanied by the generation of photons and intermode interaction. In spite of the fact that it requires relativistic velocities [2] at present there are experimental realization of DCE in superconducting schemes [3].

The aim of the presented work is investigation of spectral and statistical properties of DCE radiation under various resonant conditions in one dimensional cavity. These conditions may appear due to periodical law of boundary motion, which we define following [4] as

\[ q(t) = L \exp \left( \frac{q_0 \cos \Omega t}{L} \right), \] (1)

where \( L \) is the cavity length, \( \Omega \) is modulating frequency and \( q_0 \) is initial boundary coordinate. Under this conditions effective Hamiltonian of the system has the form

\[ H_{\text{eff}} = \sum_k \omega_k(t) a_k^+ a_k + i \sum_k \frac{\dot{q}(t)}{4q(t)} \left( a_k^+ a_k^2 - a_k^2 \right) + \frac{i}{2} \sum_{j \neq k} (-1)^{j+k} \frac{k j}{j^2 - k^2} \left( \frac{k}{j} \right)^{1/2} \frac{\dot{q}(t)}{q(t)} \left( a_k^+ a_j^+ + a_k^+ a_j - a_j a_k - a_j^+ a_k \right). \] (2)

Here \( \omega_k(t) \) is instantaneous frequency of k-th intracavity mode and \( a_k \) is its annihilation operator.

2. Resonance condition
In [4] it was only mentioned that the different regimes of photons generation and intermode interaction may occur and it depends on the choice of a particular resonant frequency of
boundary motion. Here we investigate thoroughly three regimes of DCE field generation, which takes place when "mechanical" frequency $\Omega$ takes the values $\omega_{k} = \frac{2\pi k}{T} (c = 1 \text{ is speed of light})$ of stationary cavity for $k = 1, 2$ and 4. Besides that, we derived the general form of the Hamiltonian depending on parity of $k$.

2.1. Phase modulator

Using resonance condition $\Omega = \omega_{1}$ and rotating wave approximation (RWA) one can easily obtain resonance part of Eq. (2), namely:

$$H_{PM} = \frac{q_{0}\pi}{4L^{2}} \sum_{k} \sqrt{k(k+1)}(a_{k+1}^{\dagger}a_{k} + a_{k}^{\dagger}a_{k+1}).$$

This Hamiltonian has the form of that which describe phase modulation process. Under approximations of bounded spectrum corresponding spectral problem has an analytical solution. It is worthwhile to mention that there is no photon creation in this process. The ultimate resonator quality factor will not grow infinitely as a function $\omega_{k}$, therefore we can assume approximately $f(k) \approx g(k) = \sqrt{(k-n_{min})(n_{max} - k)}$, where $n_{min}$ and $n_{max}$ are the minimum and maximum number of modes. In this case the problem can be solved analytically as in [5], where Hamiltonian of optical modulator was expressed in terms of SU(2) algebra generators:

$$A_{0} = \sum_{m=-S}^{S} m a_{m}^{\dagger}a_{m}, \quad A_{+} = \sum_{m=-S}^{S-1} f(m) a_{m}^{\dagger}a_{m+1}, \quad A_{-} = \sum_{m=-S}^{S} f(m) a_{m}^{\dagger}a_{m+1},$$

where $S = (n_{max} - n_{min} - 1)/2$. Then corresponding Hamiltonian has the form

$$H_{PM} = \frac{q_{0}\pi}{2L^{2}} (A_{+} + A_{-}),$$

and may be diagonalized by the unitary transformation $V$

$$V H_{PM} V^{\dagger} = \frac{q_{0}\pi}{2L^{2}} A_{0}, \quad V = \exp \left[ \frac{\pi}{4} (A_{+} - A_{-}) \right].$$

2.2. Single mode squeezing

Applying condition $\Omega = \omega_{2}$ together with RWA one can obtain the following Hamiltonian

$$H_{2} = H_{SQ} + H_{PM} = \frac{q_{0}\pi}{4L^{2}} \left( a_{1}^{\dagger 2} + a_{2}^{2} \right) + \frac{q_{0}\pi}{4L^{2}} \sum_{k} \sqrt{k(k+2)} \left( a_{k+2}^{\dagger}a_{k} + a_{k}^{\dagger}a_{k+2} \right),$$

which describes the process of photon creation in mode with $k = 1$ due to vacuum squeezing. We analyzed this process both theoretically and numerically. In particular the statistical properties of odd modes for single mode squeezing as a source of Casimir photons were investigated (since all even modes are in vacuum state).

Hamiltonian $H_{2}$ may be written in interaction picture of operator $H_{PM}$ which in fact is Hamiltonian of phase modulation process for odd modes. Using technique described in eq’s (4)-(6) one can obtain time dependent operator $H_{2}^{\text{int}}$ in the from

$$H_{s} = \frac{q_{0}\pi}{L^{2}} \left( \left( \sum_{\Delta \rho = -S}^{S} a_{\Delta \rho} R_{\Delta m(1)\Delta \rho}(t) \right)^{2} + \left( \sum_{\Delta \rho = -S}^{S} a_{\Delta \rho}^{\dagger} R_{\Delta m(1)\Delta \rho}(t) \right)^{2} \right),$$

were function $R_{\Delta m(1)\Delta \rho}$ is expressed in terms of Wigner D-matrices.
Figure 1. Photon generation in DCE: variance of quadrature for mode with number $m = 1$ (left) and mean photon number per mode under the process described by Hamiltonian from Eq.(7) (right) at time $T = 2mcs$

2.3. Two mode squeezing
Resonance condition $\Omega = \omega_4$ with applied RWA gives the following form of effective Hamiltonian

$$H_4 = \frac{q_0\pi}{2L^2} \left( a_2^\dagger a_2^2 + a_2^2 \right) + \frac{\sqrt{3}q_0\pi}{4L^2} \left( a_1^\dagger a_3^\dagger + a_1 a_3 \right) + \frac{q_0\pi}{L^2} \sum_k \sqrt{k(k+4)} \left( a_{k+4}^\dagger a_k + a_k a_{k+4}^\dagger \right).$$

There are two processes in this case which insert into photon generation: single mode squeezing (the first term) and two-mode squeezing (the second term). Under assumption of weak interaction between modes with numbers $k$ and $(k+4)$ spectral problem become solvable analytically since each term may be treated separately. Here the process of two-mode squeezing is of the most interest. Its Hamiltonian may be written in terms of SU(1, 1) algebra generators

$$K_+ = a_1^\dagger a_3^\dagger, \quad K_- = a_1 a_3, \quad K_0 = \frac{1}{2} \left( a_1^\dagger a_1 + a_3^\dagger a_3 + 1 \right)$$

and diagonalized using coherent representations of this algebra.

2.4. General case
Resonance condition $\Omega = \frac{m\pi}{L} = \frac{2l\pi}{L}$ for $m$ even gives the following Hamiltonian, which describes processes discussed above.

$$H_m' = \frac{q_0l\pi}{4L^2} \left( a_l^\dagger a_l^2 + a_l^2 \right) + \frac{q_0m\pi}{4L^2} \sum_{k,j,k+j=m} \frac{k_j}{2} \frac{1}{k_j - k} \left( k_j \frac{k_j}{2} \right)^{1/2} \left( a_k^\dagger a_k^\dagger a_k a_k + a_j a_j a_j a_j \right) - \frac{q_0m\pi}{4L^2} \sum_k \sqrt{k(k+m)} \left( a_k^\dagger a_{k+m} + a_k a_{k+m}^\dagger \right).$$

It is worthwhile to notice that the first $m$ modes undergo squeezing due to the DCE, while other modes become excited through the interaction mechanism similar to phase modulation. The mode with number $l = m/2$ undergoes single mode squeezing while others are being squeezed by pairs located symmetrically relative to “central” mode indexed by $l$. Similar situation takes place for resonance condition when $m$ is odd. But in this case there is no process of single mode
**Figure 2.** Photon generation in DCE; two mode variance of quadrature for modes with numbers $m = 1$ and $3$ (left) and mean photon number per mode under the process described by Hamiltonian from Eq.(9) (left) at time $T = 5mcs$

squeezing

$$H''_m = -\frac{q_0m\pi}{4L^2} \sum_{k,j,k\neq j,k+m} \frac{k_j}{j^2-k^2} \left(\frac{k}{j}\right)^{1/2} \left(a_j^\dagger a_k^\dagger + a_j a_k\right) + \frac{q_0m\pi}{4L^2} \sum_k \sqrt{k(k+m)} \left(a_k a_{k+m} + a_k^\dagger a_{k+m}^\dagger\right).$$

**(12)**

2.5. Conclusion

Here we investigated the evolution of intracavity electromagnetic field in Dynamical Casimir Effect. Using effective Hamiltonian we show that there are three processes which takes place in the cavity: intermode interaction which is similar to phase modulation, single mode squeezing for the case of resonance with the mode indexed by even number and two mode squeezing which takes place for each resonance.

**References**

[1] Moore G 1970 *J. Math. Phys.* 11 2679-2691.
[2] Dodonov V V 2010 *Physica Scripta* 82 038105
[3] Wilson C M et all 2011 *Nature* 479 376-379
[4] Law C K 1994 *Phys. Rev. A* 49 433-537
[5] Miroshnichenko G P 2015 arXiv:1504.01632