Reflection Symmetry of Cusps in Gravitational Lensing

Sun Hong Rhie (UND)

ABSTRACT

Criticality in gravitational microlensing is an everyday issue because that is what generates microlensing signals which may be of photon-challenged compact objects such as black holes or planetary systems ET calls home. The criticality of these quasi-analytic lenses is intrinsically quadratic, and the critical curve behaves as a mirror generating two mirror images along the image line (∥ ±E−) at the same distances from the critical curve in the opposite sides.

At the (pre)cusps where the caustic curve “reflects” and develops cusps, however, “would-be” two pairs of quadratic images “superpose” to produce three mirror images because of the degenerate criticality. The critical curve behaves as a parabolic mirror, and the image inside the parabola is indeed a superposed image having the sum of the magnifications of the other two that are outside the parabola. All three images lie on a parabolic image curve shaped by a property (∇J) of the (pre)cusp, and two distances of the system determine the position of the image curve and positions of the three images on the image curve. The triplet images satisfy \( \sum J^{-1} = 0 \), and the function \( \sum J^{-1} \) is discontinuous at a caustic crossing where a pair of quadratic images disappear into a critical point. The reflection symmetry of the image curve is a manifestation of the symmetry of the cusp which is also respected by a trio of parabolic curves that are tangent at the (pre)cusp and define the image domains. The symmetry is guaranteed when \( J_{+-} \) vanishes or can be ignored, and the cusps on the lens axis of the binary lenses are strongly symmetric having \( J_{+-} = 0 \) because of the global reflection symmetry of the binary lenses. “\( E_{\pm} \)-algebra” is laid out for users’ convenience.

Subject headings: gravitational lensing - binary stars, planets

0. “Forget About This Universe”

Q: Quazi-What lenses? Too special! What do you think is the probability that one of the baby universes will evolve into one just like this? Measure zero. A: Well, sir, one resolution is to preach a paradox. The other is to admit that you have not found the proper measure yet. – “Modo de Notre Dame”
1. Stationary Points of the Critical Curve and Cusps

The simplest and well-known cusp is the point caustic of a single lens. The critical curve of the single lens is a circle better known as Einstein image ring. A circle is locally parabolic in a small neighborhood of a point on the circle. If the circle is given by \( u^2 + v^2 = a^2 \), and \((u, v) = (a, 0)\) is a point on the circle, then the equation of the circle becomes \( 2at \approx v^2 \) where \( t = a - u \) and \( v \) is sufficiently small.

\[
2at = v^2 + t^2
\]  

(1)

The criticality of a single lens is highly degenerate as is shown in equation (13). In fact, the eigenvector \( E_+ \) is radial everywhere, and \( J_- \) and higher derivatives vanish. Aside from the single lens, quasi-analytic lenses do have non-vanishing second derivative \( J_{-} - \neq 0 \) at the cusps, and its relevance to the triple images near cusps can be seen in equation (32). One may refer to SEF for the cases where \( J_{-} = 0 \) with non-vanishing higher order derivatives. Here we are interested in quasi-analytic lenses that include \( n \)-point lenses and their variations such as constant shear (astro-ph/0103463).

We need two equations to examine the degenerate critical behavior of the lens equation. One is the differential of the lens equation, and the other is the differential of the Jacobian determinant of the lens equation. Since the binary lens equation is a typical quasi-analytic lens equation, we use it for a concrete book-keeping of the differential behavior.

\[
\omega = z - \frac{\epsilon_1}{\bar{z}_1} - \frac{\epsilon_2}{\bar{z}_2} \quad ; \quad z_j \equiv z - x_j \quad : \quad j = 1, 2
\]  

(2)

where \( \omega \) and \( z \) are the source and its image position, and \( x_1 \) and \( x_2 \) are the lens positions. Then, the Jacobian determinant is given by

\[
J = 1 - |\kappa|^2 \quad ; \quad \kappa \equiv \sum_j \frac{\epsilon_j}{z_j^2}
\]  

(3)

Now, we differentiate the two equations.

\[
d\omega = dz + \bar{\kappa}d\bar{z} + \frac{1}{2} \bar{\partial} \kappa \ dz^2 + \frac{1}{6} \bar{\partial}^2 \kappa \ d\bar{z}^3 + \mathcal{O}(dz^3) \equiv \delta \omega_1 + \delta \omega_2 + \delta \omega_3
\]  

(4)

\[
dJ = (dz \partial + d\bar{z} \bar{\partial})J + \frac{1}{2}(dz \partial + d\bar{z} \bar{\partial})^2 J + \mathcal{O}(dz^2) \equiv \delta J_1 + \delta J_2
\]  

(5)

Criticality refers to the fact that the linear terms of equation (4) can vanish, and two concepts need to be distinguished. One is the critical curve where the criticality occurs in the image plane (\( z \)-plane), and the other is the critical direction (\( \pm \bar{E}_- \)) at each (critical) point on the critical curve. The tangent of the critical curve at a critical point and critical direction at the
same critical point generally are not the same. The angle between the tangent and critical vector varies slowly along the critical curve, and the caustic develops a cusp when the two coincide. The cusps are the interest here. First, we run through the algebraic machinery of the criticality of quasi-analytic lenses.

1. The Jacobian matrix of the lens equation is an array of the linear coefficients of equation (4) and its complex conjugate, and $J$ is the determinant of the matrix.

$$J = \begin{pmatrix} 1 & \bar{\kappa} \\ \kappa & 1 \end{pmatrix} ; \quad J = \text{det}(J)$$

The analytic function $\kappa(z)$ completely determines the differential behavior, and we denote its phase angle by $2\varphi$.

$$\kappa = |\kappa|e^{i2\varphi}$$

2. The eigenvalues of the matrix (6) are given by

$$\lambda_{\pm} = 1 \pm |\kappa|.$$ When one ($\lambda_{-}$) of them vanishes, the Jacobian determinant $J$ vanishes, and the lens equation becomes quadratic (or degenerate) in that direction ($\pm E_{-}$). The critical condition ($J = 0 \iff |\kappa| = 1$) imposes one real constraint on the otherwise two degrees of freedom of the 2-d $z$-plane, and so the critical curve is a 1-d curve. We have emphasized that the critical direction is generally not the direction (or the tangent) of the critical curve. But, when they do coincide, the “motion” along the critical curve becomes stationary, and the curve in the $\omega$-space (caustic curve) develops a cusp. The critical direction ($\pm E_{-}$) is the eigendirection of $\lambda_{-}$, and we can use the orthogonal eigenvectors of the hermitian matrix (6) as an orthogonal basis vectors.

$$e_{\pm} = \begin{pmatrix} E_{\pm} \\ \bar{E}_{\pm} \end{pmatrix} ; \quad E_{+} = e^{-i\varphi} , \quad E_{-} = iE_{+}$$

The linear terms of the local lens equation in (2) reads as follows where $dz = dz_{+}E_{+} + dz_{-}E_{-}$.

$$\delta \omega_{1} = dz + \kappa d\bar{z} = (1 + |\kappa|)dz_{+}E_{+} + (1 - |\kappa|)dz_{-}E_{-}$$

At a critical point, $|\kappa| = 1$, and the equation is stationary in the critical direction ($E_{-}$). If $z$ is a critical point, $\omega(z, \bar{z})$ is called a caustic point, and the equation shows that the non-critical direction vector $E_{+}$ is tangent to the caustic curve: $\delta \omega_{1} \parallel \pm E_{+}$.

3. In order to discuss higher order terms, we need to do some book-keeping for orthogonal decompositions and differentiations in the bases of $E_{\pm}$. We note that $E_{\pm}(\varphi)$ are moving
frame bases, or curvilinear coordinate basis vectors, hence the corresponding tangent differential operators $\partial_\pm$ do not necessarily commute.

\[
(\partial_+, \partial_-) \equiv \partial_+ \partial_- - \partial_- \partial_+ \neq 0
\]  

(11)

The following is a list of notational definitions and useful relations.

\[
\partial = \partial/\partial z; \quad \bar{\partial} = \partial/\partial \bar{z};
\]

\[
dz = dz_+ E_+ + dz_- E_-; \quad \bar{d}z = dz_+ \bar{E}_+ + dz_- \bar{E}_-;
\]

\[
dz_\pm = \frac{1}{2} (\bar{E}_\pm dz + E_\pm d\bar{z});
\]

\[
\delta \omega_k = \delta \omega_k E_+ + \delta \omega_k E_-;
\]

\[
\partial_\pm = E_\pm \partial + \bar{E}_\pm \bar{\partial};
\]

\[
dz \partial = \partial J dz_\pm + \bar{\partial} J \bar{d}z_\pm = dz_+ \partial + dz_- \partial;\]

\[
dJ = \partial J dz + \bar{\partial} J \bar{d}z = J_+ dz_+ + J_- dz_-;
\]

\[
\partial J = J_+ E_+ + J_- E_-; \quad \bar{\partial} J = J_+ \bar{E}_+ + J_- \bar{E}_-;
\]

\[
\partial E_\pm = \frac{\partial J E_\pm}{4|\kappa|^2}; \quad \bar{\partial} E_\pm = -\frac{\bar{\partial} J E_\pm}{4|\kappa|^2};
\]

\[
\partial_+ E_\pm = \frac{\mp J_+ E_\pm}{4|\kappa|^2}; \quad \partial_- E_\pm = \frac{\mp J_+ E_\pm}{4|\kappa|^2};
\]

\[
\partial_+ dz_\pm = \frac{\mp J_+ dz_\pm}{4|\kappa|^2}; \quad \partial_- dz_\pm = \frac{\mp J_+ dz_\pm}{4|\kappa|^2};
\]

\[
(\partial_+, \partial_-) \equiv \partial_+ \partial_- - \partial_- \partial_+ = 2i (\partial J \bar{\partial} - \bar{\partial} J \partial);
\]

\[
(\partial_+, \partial_-) J = 0 \quad \Rightarrow \quad J_{++} = J_{--} = 0
\]  

(12)

And, also what the axial symmetry of the single lens does to the critical behavior. 1. Every point on the critical curve is a precusp. 2. The gradient $\nabla J$ is radial everywhere. 3. The criticality is highly degenerate.

\[
J_+ = \frac{4}{r^5}, \quad J_{++} = \partial_+ J = 0, \quad J_{++} = \partial_+^2 J = -\frac{20}{r^6}
\]  

(13)
5. **Quadratic Mirror Images:** We set $dz = dz_+ E_-$ to see the behavior of the lens equation in the critical direction.

\[
\delta \omega_2 = \frac{1}{2} \bar{\partial} \bar{\partial} dz^2 = \frac{1}{2} \bar{\partial} J dz^2 ; \\
J = dJ = \delta J_1 = J_+ dz_+ + J_- dz_- = J_- dz_- 
\]

It is quadratic, and the critical direction is at an angle with the critical curve where $J_- \neq 0$. Speaking in terms of images, two images on the image line along the critical direction comes from a same source. The image line is dissected by the critical point, and the source is located in the direction of $\nabla J$ from the caustic point. Since $\nabla J (=2 \bar{\partial} J)$ does not vanish on the critical curve unless the critical curve has bifurcation points, the quadratic criticality is defined almost everywhere except where $J_- = 0$.

At a (pre)cusp where $J_- = 0$, the lens equation appears to be stuck on the critical curve ($dJ = 0$), and that is an indication that there is a third image whose $J$-value vanishes (or its magnification diverges). Thus, the local lens equation (4) becomes a cubic equation in the neighborhood of the (pre)cusp. The quadratic behavior of the lens equation and its failure near (pre)cusps have been discussed for a quarter century and we also have detailed recently in astro-ph/0205067.

2. **Triple Mirror Images and Reflection Symmetry**

We calculate the second order derivatives of $J$ in order to examine the third order behavior of the lens equation (4). For visual simplicity, we use the following denotations.

\[
a \equiv J_+ , \quad b \equiv J_- ; \\
a_+ \equiv J_{++} , \quad a_- = b_+ \equiv J_{+-} , \quad b_- \equiv J_{--} ; \\
d\omega = (x + iy) E_+ , \quad dz = (u + iv) E_+ .
\]

\[
\partial \bar{\partial} J = \frac{1}{4} \left( (a_+ + b_-) + \frac{1}{4|\kappa|^2}(a^2 + b^2) \right) ; \\
\bar{\partial}^2 J = \frac{E_+^2}{4} \left( (a_+ - b_- + i2a_-) - \frac{1}{4|\kappa|^2}(a + ib)^2 \right) ; \\
\delta J = \delta J_1 + \delta J_2 = (au + bv) + \frac{1}{2}(a_+ u^2 + a_- 2uv + b_- v^2) + \frac{1}{8}(ub - va)^2 .
\]

For the quasi-analytic lenses we are interested in here, $J = 1 - |\kappa|^2$, and

\[
\partial \bar{\partial} J = -\frac{|\partial J|^2}{|\kappa|^2} \Rightarrow a_+ + b_- = -\frac{5(a^2 + b^2)}{4|\kappa|^2} < 0 .
\]
In the case of a single lens, $a_+ = -20$, $b_- = 0$, and $|\nabla J| = 4$ on the critical curve ($r = 1$). Away from a cusp, $\delta J \approx \delta J_1$. Near a cusp ($b = 0$), $\delta J$ dominantly depends linearly on $u = dz_+$ and quadratically on $v = dz_-$. 

$$\delta J = au + \frac{1}{2} \left( b_- + \frac{a^2}{4} \right) v^2$$  \hspace{1cm} (23)

Therefore, the critical curve ($\delta J = 0$; $J_o = 0$) is parabolic near a cusp. 

$$u = -\frac{1}{2a} \left( b_- + \frac{a^2}{4} \right) v^2$$  \hspace{1cm} (24)

Now we choose the cusp under consideration as the origin of the lens plane, and the eigendirection $E_+$ as the real axis so that $E_+ = 1 \ (\Leftrightarrow \varphi = 0)$. Then, $\bar{\kappa}_o = 1$, and the lens equation (4) reads as follows.

$$\omega = z + \bar{z} + \alpha_2 \bar{z}^2 + \alpha_3 \bar{z}^3;$$  \hspace{1cm} (25)

$$\alpha_2 = -\frac{a}{4}, \quad \alpha_3 = \frac{1}{12} \left( b_- + \frac{3a^2}{4} \right) - \frac{i}{12} a_-.$$  \hspace{1cm} (26)

The coefficients are almost real. In fact, we can ignore the imaginary component of $\alpha_3$, $\Im(\alpha_3) = -a_-/12$, when we consider only the lowest order terms.

When there is a reflection symmetry in the system, $a_- = b_+ = 0$ at the cusps on the symmetry axis. That is the case for the cusps on the lens axis of the binary lenses, as we can directly calculate from the binary equation (2). Without loss of generality, the real axis has been chosen to be the lens axis, and $c.c$ stands for complex conjugate as usual. Then, 

$$b_+ = \partial_+ \left( -E_- \bar{\kappa} \partial \kappa + c.c \right) = -E_- \partial_+ (\bar{\kappa} \partial \kappa) + c.c = -i\bar{\kappa} \partial^2 \kappa + c.c = 0.$$  \hspace{1cm} (27)

The last equality holds because $\bar{\kappa} \partial^2 \kappa = \partial^2 \kappa$ is real on the real axis.

$$\partial^2 \kappa = \sum_j \frac{e_j}{z_j^2}; \quad \bar{\kappa} = 1$$  \hspace{1cm} (28)

Thus, the cusps on the lens axis of the binary lenses have strong reflection symmetries.

### 2.1. A Source on the Symmetry Axis:  $\omega = \bar{\omega}$

The lens equation is factorized for a source on the symmetry axis, $\omega = \bar{\omega}$.

$$0 = \omega - \bar{\omega} = (z - \bar{z}) \left( \alpha_2 (\bar{z} + z) + \alpha_3 (\bar{z}_+^2 + \bar{z}z + z^2) \right).$$  \hspace{1cm} (29)
Therefore, there is one image on the symmetry axis \((z = \bar{z})\), and there can be two images off the symmetry axis depending on the position of the source.

\[
0 = \alpha_2 2u - \alpha_3 v^2
\]  

(30)

1. **One image on the symmetry axis:**

\[
\begin{align*}
    u_1 &= \frac{\omega}{2} = \frac{x}{2}, \quad v_1 = 0; \quad J_1 = au = \frac{a\omega}{2}.
\end{align*}
\]  

(31)

2. **Two images exist off the symmetry axis when** \((ab_-)\omega < 0\):

\[
\begin{align*}
    u_{2,3} &= \frac{\omega}{2} \left( 1 + \frac{3a^2}{4b_-} \right), \quad v_{2,3} = \pm \left( -\frac{3a\omega}{b_-} \right)^{1/2}; \quad J_{2,3} = -a\omega
\end{align*}
\]  

(32)

The two images are twin images with the same magnification and parity because of the reflection symmetry retained by the source on the symmetry axis. The symmetry axis is mapped to a parabolic curve (see figure 1).

\[
\begin{align*}
    u_{2,3} &= -\frac{a}{6} v_{2,3}^2 \left( \frac{3}{4} + \frac{b_-}{a^2} \right)
\end{align*}
\]  

(33)

3. The three images satisfy \(\sum J^{-1} = 0\). The image on the symmetry axis has the same magnification as the sum of the magnifications of the images off the symmetry axis: \(|J_1|^{-1} = |J_2|^{-1} + |J_3|^{-1}\). The off axis images have the same parity \((J_2 J_3 > 0)\) which is the opposite to the parity of the image on the symmetry axis \((J_1 J_2 J_3 < 0\) in figure 1).

4. The three images are on the parabolic curve (image curve) given by the following equation.

\[
\begin{align*}
    u &= u_1 - \frac{a}{8} v^2
\end{align*}
\]  

(34)

The shape of the image curve is determined by \(\nabla J = aE_+\) at the cusp, and the (vertex) position of the curve is determined by the source position \(\omega\): \(u_1 = \omega/2\). The image curve intersects with the critical curve at \((u, v) = (u_c, v_c)\) when \((ab_-)\omega < 0\), or equivalently when there are two images off the symmetry axis.

\[
\begin{align*}
    v_c &= \pm \left( -\frac{a\omega}{b_-} \right)^{1/2}; \quad u_c = \frac{\omega}{2} \left( 1 + \frac{a^2}{4b_-} \right)
\end{align*}
\]  

(35)

The corresponding caustic points \((x, y_c)\) can be found from equation (37).

\[
\begin{align*}
    y_c &= \pm \left( -\frac{aw}{b_-} \right)^{1/2} \frac{a\omega}{6}.
\end{align*}
\]  

(36)

One trivial nonetheless useful observation is that the source position satisfies \(0 = |y| \leq |y_c|\).
2.2. An Arbitrary Source Near a Cusp

The images of an arbitrary source near a cusp can be discussed just as easily. In the lowest order approximation, where \( u \sim v^2 \), the lens equation (26) near a cusp reads as follows.

\[
x = 2u - \alpha_2 v^2 ; \quad y = -v(\alpha_2 2u - \alpha_3 v^2)
\]

(37)

1. **Image Curve:** The first equation is nothing but the parabolic image curve in equation (34) with the vertex position \( \beta = x/2 \) determined by the \( E_+ \)-component \( x \) of the source position.

\[
u = \frac{x}{2} - \frac{a}{8} v^2
\]

(38)

The images are on the parabolic image curve parameterized by \( x \) and shaped by the properties of the cusp – reflection symmetry and the value of \( \nabla J \). Since the image curve is independent of \( y \), an arbitrary source on the line \((x, y)\) where \( x = 2\beta \) produces images on the same image curve determined by \( x \). The image curve intersects with the critical curve when \((ab-)x < 0\).

\[
v_c = \pm \left( -\frac{ax}{b_-} \right)^{1/2} ; \quad u_c = \frac{x}{2} \left( 1 + \frac{a^2}{4b_-} \right) ; \quad y_c = \frac{ax}{6} = \pm \left( -\frac{ax}{b_-} \right)^{1/2} \frac{ax}{6}
\]

(39)

These are virtually identical equations with (35) and (36). The source on the symmetry axis belongs to a family of sources that produce images on the image curve determined by the \( E_+ \)-position of the source \( x \).

2. **The Number of Images:** We eliminate \( u \) from the two equations in (37) and obtain a standard third order real polynomial equation satisfied by \( v \).

\[
0 = v^3 + 3ax b_- v - \frac{12y}{b_-} = v^3 + pv - q
\]

(40)

So, there can be one, two, or three images on the image curve depending on the value of \( y \). The number of real solutions of the cubic equation (40) is related to the sign of the discriminant \( D \): \# = 2 - sgn(D).

\[
D = \left( \frac{ax}{b_-} \right)^3 + \left( \frac{6y}{b_-} \right)^2 = \frac{36}{b^2} \left( y^2 - y_c^2 \right)
\]

(41)

where

\[
y_c^2 \equiv -\frac{a^3 x^3}{36b_-} ; \quad y_c^2 (axb-) < 0
\]

(42)

If \((axb-) < 0\), then \( y_c^2 > 0 \) and the number of images is three, two, or one depending on \(|y| < |y_c|, |y| = |y_c|, \) or \(|y| > |y_c|\). If \((axb-) > 0\), then \( D > 0 \), and there is one
image. We recall that \((ab_\perp)x < 0\) is the condition that the image curve intersect with the critical curve. Since a source on the symmetry axis \((y = 0)\) have three images, we can expect from continuity that sources with \(|y| < |y_c|\) are inside the caustic and produce three images. It is consistent with the count of the solutions determined by \(D\).

3. The Third Image on the Precaustic Curve: The equation has the standard form of Vieta where the coefficient of the second order term vanishes. For our purpose, it means that \(\sum_j v_j = 0\) where \(v_j : j = 1, 2, 3\) are the \(E_-\) positions of the images two of which can be complex. For example, the images of a source at a caustic point \((x, y_c)\) can be found trivially: A degenerate image at a critical point \((u_c, v_c)\) and the third image at \((u_3, v_3)\) where \(v_3 = -2v_c\) \((0 = \sum_j v_j = v_c + v + v_3)\).

\[
 u_3 = \frac{x}{2} \left( 1 + \frac{a^2}{b_-} \right) ; \quad v_3 = -2v_c .
\] (43)

The degenerate image is on the critical curve and so has \(J = 0\). For the third image, \(J = \frac{1}{2} (ax + b_-v_3^2) = -\frac{3ax}{2} .\) (44)

4. The Precaustic Curve Tangent to the Critical Curve at the PreCusp: We recall that the critical curve is not the only curve that is mapped onto the caustic curve under the lens equation (see astroph.0103463.fig.10). The curves in the image plane that are mapped onto the caustic curve are called precaustic curves and the precaustic curves constitute the boundaries of the image domains. The critical curve is one of the precaustic curves, and that smooth one. The non-critical precaustic curves are cuspy at the cusps of the caustic curve, except at the precusps where they are tangent to the critical curve. At a precusp, the critical curve and one non-critical precaustic curve are tangent. In other words, the third image of the source \((x, y_c)\) on the caustic curve is on the non-critical precaustic curve that is tangent to the critical curve at the precusp under consideration (at the origin here). By eliminating the parameter \(x\) from \(u_3\) and \(v_3\), we find that the precaustic curve is a parabolic curve with the vertex at the origin.

\[
 u_3 = -\frac{a}{8} v_3^2 \left( 1 + \frac{b_-}{a^2} \right) .
\] (45)

If \(|y| > |y_c|\), then \(|v| > 2|v_c|\), and the third image is outside the parabola of the precaustic curve. See figure 1. The precaustic curve in equation (45) defines the boundary of the image domains that are mapped to inside the caustic and outside – or the image domains of triplets and of singlets.
5. **Sum Rule** $\sum J^{-1} = 0$ for **Triple Images**: We intuitively expect that the triple critical images near a precusp should satisfy $\sum J^{-1} = 0$ because they should be a “superposition” of the “would-be” two pairs of quadratic images. A pair of quadratic images connected by the imageline bisected by the critical curve have the same magnification and opposite parities, i.e., $\sum_1^2 J^{-1} = 0$. In fact, the triplet images of a source on the symmetry axis satisfy $\sum J^{-1} = 0$. On the other hand, the images of a source on the caustic curve fail to satisfy the relation $\sum J^{-1} = 0$. Thus, we expect that the function $\sum J^{-1}$ is not smooth but discontinuous at a caustic crossing. From equation (40), we obtain

$$J = \frac{1}{2} (ax + b_\pm v^2) = -ax \left(1 - \frac{t}{v}\right); \quad t \equiv \frac{6y}{ax}$$

$$\Rightarrow \quad \sum J^{-1} = \frac{-2Pt + 3Q}{\Pi_j(t - v_j)}.$$ (47)

The numerator vanishes. Therefore, the sum rule holds, and the total magnification of the positive images ($A_+$) is the same as the total magnification of the negative images ($A_-$).

$$\sum J^{-1} = 0 \quad \Rightarrow \quad A_+ = A_-; \quad \Pi_j(t - v_j) \neq 0 \quad (48)$$

Since the “middle image” has the opposite parity to the others, the magnification of the “middle image” is the same as the total magnification of the other two images. In figure 1, the “middle image” is the negative image, and that is always the case for the cusps on the symmetry axis of the binary lenses.

(a) **Breakdown of the Sum Rule**: We note that $t = v_c$ when $y = y_c$. Thus, the denominator $\Pi_j(t - v_j)$ vanishes for a source on the caustic, and $\sum J^{-1} = 0/0$ is indeterminate. In other words, the sum rule for the triplet breaks down at a caustic crossing.

6. **Decoupling of the Triplet to Doublet $\oplus$ Singlet**: If we consider that the two images disappearing into the critical curve are a pair of quadratic images, it should be clear that the sum rule $\sum_{\text{doublet}} J^{-1} = 0$ holds between the two quadratic images, and it is impossible to satisfy the sum rule for the triplet. For practical purpose, the transition from a triplet to doublet $\oplus$ singlet occurs where the magnification of the “middle image” becomes much larger than that of the third image $|J^{-1}(u_3, v_3)| = |2/3ax|$. We can consider the inverse transition to be where the behavior of a pair of images near the critical curve fails to be of a quadratic mirror image pair. Since the magnification of the third image is inversely proportional to $|x/2|$, the (vertex) position of the image curve, the validity range of the quadratic images converges to zero as the source nears the cusp. In figure 1, as the two images $z_-$ and $z_1^+$ move toward the critical curve, $z_2^+$
moves toward the non-critical caustic curve, and the magnifications of the $z_-$ and $z_{1+}$ become almost equal.

7. **Magnifications of the Three Images:** The sum rule $\sum J^{-1}$ is an analytic topological quantity, and we only had to manipulate the cubic equation without solving it. The magnification of the individual images requires the solutions, and the solutions of the cubic equation (40) can be found in mathematics encyclopedia such as by E. Weisstein (1999). The trick is simple. Start with the cubic equation $0 = v^3 + pv - q$. Substitute $v$ by $\sigma - p/3\sigma$ (Vieta) and solve the quadratic equation for $\sigma^3$. For three images, $D < 0$.

$$v = \sigma - \frac{p}{3\sigma} = \sigma + \bar{\sigma} = 2|\sigma| \cos \theta; \quad |\sigma| = \left(-\frac{p}{3}\right)^{1/3} = \left(-\frac{ax}{b_-}\right)^{1/3},$$

and $\theta$ can be (numerically easily) found from the following.

$$\theta = \frac{1}{3} \tan^{-1}\left(\frac{2|D|^{1/2}}{q}\right) \quad \Rightarrow \quad |\sigma|^3 \cos 3\theta = \frac{q}{2}; \quad |\sigma|^3 \sin 3\theta = |D|^{1/2} \quad (50)$$

Then, the three solutions are $\theta$, and $\theta \pm 2\pi/3$. The Jacobian determinant follows easily.

$$J = \frac{1}{2} \left(ax + b_- v^2\right) = -ax + \frac{6y}{2|\sigma| \cos \theta} \quad (v \neq 0) \quad (51)$$

If $y = 0$, then $\theta = \pi/6$, and we recover the solutions of a source on the symmetry axis: $v = 0, \pm \sqrt{3}|\sigma|; \quad J = ax/2, -ax$ where the latter two exist when $(ab_-)x < 0$. For $y \neq 0$, the algebraic expressions may not be so illuminating than figures. Figures 2 and 3 show the general behavior of the positive and negative cusps of a binary lens caustic. The cusp at $\omega = 0.757$ shows a typical orderly distribution of the iso-$J$ curves and how the multiplicity of images occurs. The cusp $\omega = 0.757$ is a positive cusp and each positive iso-$J$ curve self-intersects inside the cusp. Thus, there is only one positive image outside and two positive images inside. Figure 3 shows that there is one negative image and none outside. Being inside the binary lens caustic, the source should produce five images (two positive and three negative). The other two images are negative fainter images near the two lens positions and are not represented in the figures because the $J$-value is cut at $J \sim -1.0$. The images at the lens positions have $J = -\infty$. The three bright images are the ones that are obtained as three solutions of the cubic equation when the source is near the cusp. The iso-$J$ contours can be found from equation (51). Also, see figure 4 in astro-ph/0206162 (Gaudi and Petters).

8. **Singlet Images Outside the Non-critical Precaustic Parabola:** There is one image when $D > 0$, and the image and its magnification can be found using the same procedure.

$$v = \sigma - \frac{p}{3\sigma} = \sigma_+ + \sigma_-; \quad \sigma_{\pm} = \left(\frac{q}{2} \pm \sqrt{D}\right)^{1/3} \quad (52)$$
\[ J = \frac{1}{2} \left( ax + b_- (\sigma_+ + \sigma_-)^2 \right) \] (53)

If \( x = 0 \), then \( p = 0 \) and \( v = q^{1/3} \). Therefore, \( J \propto y^{2/3} \).

\[ J = (18b_- y^2)^{1/3} \] (54)

(a) The parity of the singlet image outside the non-critical precaustic parabola is determined by the sign of \( b_- \). A cusp is referred to as a positive cusp if \( b_- > 0 \) and a negative cusp if \( b_- < 0 \). The “superposed image” of the triplet near a positive (negative) cusp is a negative (positive) image.

(b) Figures 2, 3, and 5 show the six-cusped connected caustic with lens parameters \( \epsilon_2 = 0.1 \) and separation \( \ell = 1.3 \). The cusps on the lens axis are positive cusps and very strong. The cusps off the lens axis are negative cusps. The negative cusps nearer to the smaller are weaker. The cusp near the larger mass is very strong and mimicks the behavior of a single lens producing iso-\( J \) curves (or “balloons”) that are almost circular (2).

(c) Given a value of \( J \), \( y \propto |b_-|^{1/2} \). Thus, the larger is \( |b_-| \), the tighter are the contours of iso-\( J \). The iso-\( J \) curves tend to be oblong in the direction of symmetry axis because \( y \sim J^{3/2} \) and \( x \sim J \).

9. The Position of the Cusp: Let’s calculate the position of the cusp (and precusp) on the lens axis near the smaller mass in figure 2. The position of the precusp in the image plane can be found by solving the following quartic equation where \( \epsilon_2 = 0.1 \) and \( |x_2 - x_1| = \ell \).

\[
1 = \frac{\epsilon_1}{(z - x_1)^2} + \frac{\epsilon_2}{(z - x_2)^2}; \quad z = \bar{z} \tag{55}
\]

When caustic curve is connected, there are two real solutions, and the one near the smaller mass is at \( z = 1.553 \) and the corresponding caustic position \( \omega = 0.759 \) (in the center of mass coordinate system).

10. Figure 5 shows an example of “perfectly good” line caustics that are by definition clear of cusp “balloons”. Dotted curves are \( J = 0.88 \) and 0.9 and tracing their variations requires good quality photometry available only from space microlensing. For ground-based alert networks, such line caustic crossing will be without warning unless it is the exit crossing. It is necessary to pin down the lens parameters in order to utilize the exit line caustic crossing for limb darkening measurements, and that requires sufficient information of the entry caustic crossing, which in turn requires a reasonable time resolution survey (\( \gtrsim 2/\text{day} \)) and immediate follow-up capabilities. In addition, better than 1% photometry is required to be able to discern the linear limb darkening parameter by 10% (rb99).
REFERENCES

Rhie, S. H. 2002, astro-ph/0205067

Gaudi, S., and Petters, A. 2002, astro-ph/0206162

Rhie, S. H. 2001, astro-ph/0103463

Rhie, S. H., and Bennett, D. P. 1999 (rb99), astro-ph/9912050

Schneider, P., Elhers, J., and Falco, E. 1992 (SEF), “Gravitational Lenses”, Springer-Verlag
Fig. 1.— Three curves are tangent at the precusp: critical curve, a non-critical precaustic curve, and the image of the symmetry axis. Precaustic curves in the image plane define image domains, and the critical curve is one of them. The critical curve is tangent to a non-critical precaustic curve at a cusp and the neighborhood of the precusp is divided into four image domains. The degeneracy is two fold and defines the reflection symmetry axis. The parabolic image curve at $\omega/2$ is the image of the source line at $\omega$ and the intersection points with the non-critical precaustic curve define the range of the image curve. Three source positions and their images are shown to guide the eyes. The $1:3$ correspondence is shown for an area inside the caustic curve. Images $z_-$ and $z_{1+}$ resemble a pair of quadratic images, but they are part of the triplet with $z_{2+}$, and the magnifications of $z_-$ and $z_{1+}$ are not the same. $A(z_-) = A(z_{1+}) + A(z_{2+})$. 
Fig. 2.— Contours of $J = \text{constant}$ for $J \geq 0$. The two cusps on the lens axis are positive cusps. The cusp at $\omega = 1.553$ shows the typical distribution of the $J$-curve distribution near it. The cusp near the larger mass shows the same behavior but for much smaller $J$-values not shown here. Near the cusps, the iso-$J$ curves can be obtained by solving the cubic equation.
Fig. 3.— Contours of $J = \text{constant}$ for $J \leq 0$. The four cusps off the lens axis are negative cusps. The distribution of the negative iso-$J$ contours is compact in comparison to the positive contours. Negative values $J = (-\infty, 0)$ are confined to the finite area inside the critical curve while the positive values are distributed over the entire image plane outside the critical curve, and the steep gradient of the distribution of negative iso-$J$ contours should be expected. The curves neatly lining the caustic curve from inside represent the “widths” of the line caustics.
Fig. 4.— Blow-up of the negative cusps nearer to the smaller mass.
Fig. 5.— Arrows point at examples of “perfectly good” line caustics that are by definition clear of cusp “balloons”. Dotted curves are $J = 0.88$ and 0.9. The area clear of iso-$J$ curves centered around $\sim (0.45, 0)$ is the location of the lowest magnifications inside the caustic and is responsible for glacier-carved $U$-shape valley in the light curves. That is also typical of diamond shape four-cusped caustics. Five images of a source inside the caustic can have the absolute minimum magnification 3 when the separation $\ell = \sqrt{2}$. In that case, two of the five images are at the so-called finite limit points marked by *’s.