An iterative formula for the Kostka–Foulkes polynomials

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Abstract
An iterative formula for the Kostka–Foulkes polynomials is given using the vertex operator realization of the Hall–Littlewood polynomials. The operational formula can handle large Kostka–Foulkes polynomials, and a stability property for the Kostka–Foulkes polynomials is shown. We also use our algorithm to give a formula of \( K_{\lambda\mu}(t) \) for \( \mu \) being hook-shaped.

Keywords  Kostka–Foulkes polynomials · Jing operators · Hall–Littlewood polynomials

Mathematics Subject Classification Primary 05E05 · Secondary 17B69 · 05E10

1 Introduction

The Kostka–Foulkes polynomials \( K_{\lambda\mu}(t) \) associated with a pair of partitions \( \lambda, \mu \) have figured prominently in representation theory and algebraic combinatorics (cf. [1,14]). They have arisen in the context of character theory of the general linear group over the finite field \([ 6]\) in terms of the Hall–Littlewood polynomials [17]. Foulkes [4] conjectured that \( K_{\lambda\mu}(t) \) are positive integral polynomials, which was proved by Lascoux–Schützenberger [16] using plactic monoids. Lusztig later proved the Foulkes conjecture using geometric method [18] by identifying \( q^{n(\mu)}-n(\lambda)K_{\lambda\mu}(q^{-1}) \) as the affine Kazhdan–Lusztig polynomials of type A. It was also proved using crystal graphs [21]. More generally, it is a special case of the positivity of the \( (q, t) \)-Kostka–Foulkes polynomials [7]. The Kostka–Foulkes polynomials also calculate some fusion products for flag manifolds [12]. In combinatorics, \( K_{\lambda\mu}(1) \) counts the number of tableaux of...
shape \( \lambda \) with weight \( \mu \), which is the dimension of the irreducible representation of the symmetric group and also determines the characters of the irreducible representations of the special linear Lie algebras.

Theoretically, the Kostka–Foulkes polynomials can be expressed in several ways. The first combinatorial formula is Lascoux–Schützenberger’s charge statistic which shows that the Kostka–Foulkes polynomials are positive integral polynomials. In terms of Kostant’s partition function, Kostka–Foulkes polynomials are given inductively by symmetrizing over the symmetric group \([19, \text{Ch. 3}]\). Kirillov and Reshetikhin have derived a combinatorial formula \([15]\) via the Bethe Ansatz. Other combinatorial formulas \([2,5,8,13]\) have been derived since the introduction of the Macdonald symmetric functions. The Kostka–Foulkes polynomials can also be expressed in terms of crystal graphs \([21]\), which also shows the positivity. The Haglund–Haiman–Loehr statistic \([9]\) for the general Macdonald polynomial obviously implies the positivity in one-parameter case. Recently, a positive combinatorial formula is known for the symplectic Kostka–Foulkes polynomial \([3]\).

On the computational side, Morris has given an implicit iterative procedure to compute the Kostka–Foulkes polynomials \([20]\). The algorithm compiles previous Hall–Littlewood polynomials to compare with relevant Schur functions to find the next one (cf. \([1]\)), which is sometimes complicated to pin down the solution. This poses a question on how to find a direct and efficient formula for the Kostka–Foulkes polynomials.

The aim of this paper is to give an operational and iterative formula for the Kostka–Foulkes polynomials, using algebraic properties of the vertex operators of the Hall–Littlewood functions given by the second-named author \([10]\). Our explicit formula facilitates the computation and may help reveal intrinsic property of \( K_{\lambda\mu}(t) \), for instance, a stability property for the Kostka–Foulkes polynomial is derived (see Proposition 2.11).

Although the formula was established via the vertex operator techniques, the final computation can be formulated independently. One of the motivations is that our formula might shed some light on an iterative formula for the two-parameter Kostka–Foulkes polynomials.

In \([23]\), Zabrocki has studied certain hook-shaped operator to compute the Hall–Littlewood functions. It would be interesting to study an analogous column operation similar to our formula for the Kostka–Foulkes polynomials.

The second-named author would like to thank Adriano Garsia for interesting discussions on using vertex operators to compute the Kostka–Foulkes polynomials back in 1989. Our current approach to the problem is based upon dual vertex operators developed in \([10,11]\).

2 Vertex operator realization of Hall–Littlewood polynomials

The original Hall–Littlewood vertex operators defined in \([10]\) used the infinite-dimensional Heisenberg algebra and its irreducible representation parameterized by \( t \). These vertex operators were subsequently utilized in \([11]\) to study more realizations of symmetric functions. To be consistent with combinatorial convention, we reformu-
late the vertex operators for the Hall–Littlewood symmetric functions as follows. For plethestic formulation, see [22]. Moreover, we will also recall a crucial vertex operator to realize the Schur functions in the same ring [10] as well as its dual vertex operator [11].

Let \( V \) be the space of symmetric functions over \( \mathbb{Q}(t) \). There are several well-known bases such as elementary symmetric functions, power-sum symmetric functions, monomial symmetric functions, and homogeneous symmetric functions. The power-sum symmetric functions are perhaps the most useful ones on the space, as they are group-like elements under the usual Hopf algebra structure.

A partition \( \lambda \) of \( n \), denoted \( \lambda \vdash n \), is a non-increasing sequence of nonnegative integers: \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0 \) such that \( \sum_i \lambda_i = n \), where \( l = l(\lambda) \) is the length of the partition \( \lambda \). The dominance order \( \geq \) is defined by \( \lambda \geq \mu \) if \( \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \) for all \( i \). For each partition \( \lambda \), we define

\[
n(\lambda) = \sum_{i=1}^{l(\lambda)} (i - 1)\lambda_i.
\] (2.1)

For partition \( \lambda = (\lambda_1, \ldots, \lambda_l) \), we define

\[
\lambda^{[i]} = (\lambda_{i+1}, \ldots, \lambda_l), \quad i = 0, 1, \ldots, l
\] (2.2)

\[
\lambda^{(i)} = (\lambda_1 + 1, \ldots, \lambda_{i-1} + 1, \lambda_{i+1}, \ldots, \lambda_l).
\] (2.3)

So \( \lambda^{[0]} = \lambda \) and \( \lambda^{[l]} = \emptyset \). It is readily seen that \( n(\lambda^{[i]}) - n(\lambda^{[i+1]}) = |\lambda^{[i+1]}| \).

The partition \( \lambda \) is usually visualized by its Young diagram of aligning \( l \) rows of boxes to the left where the \( i \)th row consists of \( \lambda_i \) boxes. The dual partition \( \lambda' = (\lambda'_1, \ldots, \lambda'_{l(\lambda)}) \) is the partition associated with the reflection of the Young diagram of \( \lambda \) along the diagonal.

Recall that the power-sum symmetric functions \( p_\lambda \) form a \( \mathbb{Q} \)-basis of the ring of the symmetric functions, where \( p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots \) with

\[
p_n = x_1^n + x_2^n + \cdots.
\] (2.4)

Therefore, the ring of symmetric functions \( V = \mathbb{Q}(t)[p_1, p_2, \ldots] \), the ring of polynomials in the \( p_n \). Using the degree gradation, \( V \) becomes a graded ring

\[
V = \bigoplus_{n=0}^{\infty} V_n.
\] (2.5)

A linear operator \( f \) is of degree \( n \) if \( f(V_m) \subset V_{m+n} \).

Let \( \langle \, , \, \rangle \) be the Hall–Littlewood bilinear form on \( V \) over \( \mathbb{Q}(t) \) defined by

\[
\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)! \frac{1}{1 - t^{\lambda_i}},
\] (2.6)
where \( m_i(\lambda) \) is the multiplicity of \( i \) in the partition \( \lambda \). The multiplication operator \( p_n : V \longrightarrow V \) is of degree \( n \). Clearly, the dual operator is the differential operator \( p_n^* = \frac{\partial}{\partial p_n} \) of degree \(-n\). Note that * is \( \mathbb{Q}(t) \)-linear and an anti-involution.

**Definition 2.1** The vertex operators \( S(z) \) and \( H(z) \) and their dual operators \( S^*(z) \) and \( H^*(z) \) on the space \( V \) are defined as the following maps \( V \longrightarrow V[[z, z^{-1}]] \) given by

\[
S(z) = \exp \left( \sum_{n \geq 1} \frac{1}{n} p_n z^n \right) \exp \left( -\sum_{n \geq 1} \frac{\partial}{\partial p_n} z^{-n} \right) = \sum_{n \in \mathbb{Z}} S_n z^n,
\]

\[
S^*(z) = \exp \left( -\sum_{n \geq 1} \frac{1-t^n}{n} p_n z^n \right) \exp \left( \sum_{n \geq 1} \frac{z^{-n}}{1-t^n} \frac{\partial}{\partial p_n} \right) = \sum_{n \in \mathbb{Z}} S^*_n z^{-n}.
\]

\[
H(z) = \exp \left( \sum_{n \geq 1} \frac{1-t^n}{n} p_n z^n \right) \exp \left( -\sum_{n \geq 1} \frac{\partial}{\partial p_n} z^{-n} \right) = \sum_{n \in \mathbb{Z}} H_n z^n,
\]

\[
H^*(z) = \exp \left( -\sum_{n \geq 1} \frac{1-t^n}{n} p_n z^n \right) \exp \left( \sum_{n \geq 1} \frac{\partial}{\partial p_n} z^{-n} \right) = \sum_{n \in \mathbb{Z}} H^*_n z^{-n}.
\]

**Remark 2.2** The dual operators \( S^*(z) \) and \( H^*(z) \) \([10,11]\) will be the key for our later discussion. The indexing is different from that of \([10,11]\). Technically, \( S(z)^* = S^*(z^{-1}) \) in the current notation.

We collect some useful relations from \([10]\) for explicit computation.

**Lemma 2.3** We have the following relations:

\[
H_n H_{n+1} = t H_{n+1} H_n, \quad H_n^* H_{n-1}^* = t H_{n-1}^* H_n^*, \quad n \in \mathbb{Z};
\]

\[
H_{-n} 1 = \delta_{n,0}, \quad H_n^* 1 = \delta_{n,0}, \quad n \geq 0
\]

where \( \delta_{n,m} \) is the Kronecker delta and \( 1 \) is the vacuum vector in \( V \).

Observe that for \( n \geq 0 \), both \( H_{-n} \) and \( H_n^* \) are annihilation operators of degree \(-n\).

We introduce the symmetric function \( q_n = q_n(x; t) \) by the generating series

\[
\sum_{n \geq 0} q_n z^n = \exp \left( \sum_{n=1}^{\infty} \frac{1-t^n}{n} p_n z^n \right).
\]

(2.7)

Clearly, as a polynomial in the \( p_k \), the polynomial \( q_n \) is the Hall–Littlewood symmetric function \( Q_{(n)} \) associated with the one-row partition \((n)\). For any partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \), we define

\[
q_\lambda = q_{\lambda_1} q_{\lambda_2} \ldots
\]
then the set \{q_\lambda\} forms a basis of \(V\), which are usually called the generalized homogeneous polynomials.

For convenience, we also denote \(s_n = q_n(0)\), the homogeneous symmetric function or the Schur function associated with the row partition \((n)\). The generating series of the \(s_n\) is

\[
s(z) = \sum_{n \geq 0} s_n z^n = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} p_n z^n \right). \tag{2.8}\]

In this paper, we adopt the combinatorial \(t\)-integer \([n]\), \(n \in \mathbb{N}\) defined by

\[
[n] = \frac{1 - t^n}{1 - t} = 1 + t + \cdots + t^{n-1}. \tag{2.9}\]

Therefore, \([n]! = [n][n-1] \cdots [1]\). For simplicity, we define \([0] = 1\). We also use the Gauss \(t\)-binomial symbol

\[
\left[ \begin{array}{c} n \\ k \end{array} \right] = \frac{[n]!}{[k]![n-k]!} \tag{2.10}\]

which can also be defined inductively by:

\[
\left[ \begin{array}{c} n + 1 \\ k \end{array} \right] = t^k \left[ \begin{array}{c} n \\ k \end{array} \right] + \left[ \begin{array}{c} n \\ k - 1 \end{array} \right]. \tag{2.11}\]

**Proposition 2.4** [10] (1) Given a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \), the product \(H_{\lambda_1}H_{\lambda_2}\cdots H_{\lambda_l}.1\) can be expressed as

\[
H_{\lambda_1}H_{\lambda_2}\cdots H_{\lambda_l}.1 = \prod_{i < j} \frac{1 - R_{ij}}{1 - t R_{ij}} q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda_l}
\]

where \(R_{ij}\) is the raising operator given by

\[
R_{ij} q(\mu_1, \mu_2, \ldots, \mu_l) = q(\mu_1, \mu_2, \ldots, \mu_i+1, \ldots, \mu_j-1, \ldots, \mu_l)
\]

which ensures that the expression has finitely many terms. Moreover, the products \(H_{\lambda}.1 = H_{\lambda_1}H_{\lambda_2}\cdots H_{\lambda_l}.1\) are orthogonal such that

\[
\langle H_{\lambda}.1, H_{\mu}.1 \rangle = \delta_{\lambda\mu} b_{\lambda}(t), \tag{2.12}\]

where \(b_{\lambda}(t) = (1 - t)^{l(\lambda)} \prod_{i \geq 1} [m_i(\lambda)]!\).

(2) Similarly for composition \( \mu = (\mu_1, \ldots, \mu_k) \), the elements \(S_{\mu_1} S_{\mu_2} \cdots S_{\mu_k}.1\) can be expressed as

\[
S_{\mu_1} S_{\mu_2} \cdots S_{\mu_k}.1 = \prod_{i < j} (1 - R_{ij}) s_{\mu_1} s_{\mu_2} \cdots s_{\mu_k} = s_{\mu}.
\]
which is the Schur function associated with the composition \( \mu \). In general, \( s_\mu = 0 \) or \( \pm s_\lambda \) for a partition \( \lambda \) such that \( \lambda \in \mathcal{S}_l(\mu + \delta) - \delta \). Here, \( \delta = (l - 1, l - 2, \ldots, 1, 0) \), where \( l = l(\lambda) \).

The vertex operator \( H_\lambda \cdot 1 \) is the Hall–Littlewood polynomial \( Q_\lambda(t) \) defined in [19]. Similarly \( S_\lambda \cdot 1 \) is the Schur polynomial \( s_\lambda \), therefore we will sometimes write \( S_\lambda = S_{\lambda_1} \cdots S_{\lambda_l} \cdot 1 \) (similarly for \( H_\lambda = H_{\lambda} \cdot 1 \)) if there is no confusion.

**Definition 2.5** The Kostka–Foulkes polynomials \( K_{\lambda \mu}(t) \) are defined for all partitions \( \mu, \lambda \) by

\[
s_\lambda = \sum_\mu \frac{1}{b_\mu(t)} K_{\lambda \mu}(t) Q_\mu(t)
\]

where \( b_\lambda(t) = (1 - t)^{l(\lambda)} \prod_{i \geq 1} [m_i(\lambda)]! \) was defined in (2.12).

It is known that \( K_{\lambda \mu}(t) = 0 \) unless \( \lambda \geq \mu \). Using Proposition 2.4, one immediately gets the following:

\[
K_{\lambda \mu}(t) = \langle S_\lambda \cdot 1, H_\mu \cdot 1 \rangle = \langle H^{*}_{\mu_1} S_\lambda \cdot 1, H_{\mu_1} \cdot 1 \rangle.
\]

Therefore, if \( H^{*}_{\mu_1} S_\lambda \cdot 1 \) is straightened out as a linear combination of \( S_\tau \cdot 1 \) with \( |\tau| = |\lambda| - \mu_1 \), then one can compute \( K_{\lambda \mu}(t) \) iteratively.

**Proposition 2.6** The commutation relations between the vertex operators realizing Hall–Littlewood and Schur symmetric functions are:

\[
S_m s_n = s_n S_m - s_{n-1} S_{m+1},
\]

\[
H^*_m S_n = t^{-1} S_n H^*_m + t^{-1} H^*_n S_{n-1} + (t^{n-m} - t^{n-m-1}) s_{n-m}.
\]

**Proof** The first relation follows from the relation

\[
S(z)s(w) = \left(1 - \frac{w}{z}\right) s(w)S(z)
\]

where \( s(z) \) is the generating series of the Schur function (2.8).

The second relation follows from the commutation relation

\[
H^*(z)S(w)\frac{w - tz}{z} + S(w)H^*(z) = (1 - t)s(tz)\delta\left(\frac{w}{z}\right)
\]

which can be checked by the same method in [10] and \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \) is the delta function.

We remark that in applying Proposition 2.6 one often utilizes the simple fact that

\[
H^*_n S_{\lambda_1} \cdots S_{\lambda_l} \cdot 1 = 0
\]
whenever $n > |\lambda|$, which helps terminate the iteration.

We are now ready to give our main result, an explicit formula for the Kostka–Foulkes polynomials utilizing the vertex operator for the Hall–Littlewood polynomials. In the theorem and subsequent proof, we will make repeated use of partitions $\lambda^{(i)}$ defined in (2.2).

**Theorem 2.7** For partition $\lambda \vdash n$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and natural number $k$,

$$H_k^* S_\lambda = \sum_{i=1}^{l} (-1)^{i-1} t^{\lambda_1 - k - i + 1} s_{\lambda_i - k - i + 1} S_{\lambda^{(i)}},$$  

(2.17)

where $s_n$ is the multiplication operator by the Schur polynomial $s_n$.

**Proof** We argue by induction on $|\lambda| + k$, where $\lambda$ is a partition and $k$ is the degree of the dual vertex operator for the Hall–Littlewood symmetric function. First of all, the initial step is clear. Thus, the inductive hypothesis implies that

$$H_k^* S_{\lambda_1 - 1} S_{\lambda_2} \cdots S_{\lambda_l}$$

$$= t^{\lambda_1 - k} s_{\lambda_1 - k} S_{\lambda_2} \cdots S_{\lambda_l} - t^{\lambda_2 - k} s_{\lambda_2 - k} S_{\lambda_3} \cdots S_{\lambda_l} - \cdots$$

$$H_k^* S_{\lambda_2} \cdots S_{\lambda_l} = t^{\lambda_2 - k} s_{\lambda_2 - k} S_{\lambda_3} \cdots S_{\lambda_l} - t^{\lambda_3 - k - 1} s_{\lambda_3 - k - 1} S_{\lambda_4} \cdots S_{\lambda_l} - \cdots.$$

Now we have

$$H_k^* S_{\lambda} = t^{-1} S_{\lambda_1} H_k^* S_{\lambda_2} \cdots S_{\lambda_l} + t^{-1} H_{k-1}^* S_{\lambda_1} - 1 S_{\lambda_2} \cdots S_{\lambda_l}$$

$$+ (t^{\lambda_1 - k} - t^{\lambda_1 - k - 1}) S_{\lambda_1 - k} S_{\lambda_2} \cdots S_{\lambda_l}$$

$$= t^{-1} S_{\lambda_1} (t^{\lambda_2 - k} s_{\lambda_2 - k} S_{\lambda_3} \cdots S_{\lambda_l} - t^{\lambda_3 - k - 1} s_{\lambda_3 - k - 1} S_{\lambda_4} \cdots S_{\lambda_l} + t^{\lambda_2 - k - 2} s_{\lambda_4 - k - 2} S_{\lambda_5} \cdots S_{\lambda_l} - \cdots)$$

$$+ t^{-1} (t^{\lambda_1 - k} s_{\lambda_1 - k} S_{\lambda_2} \cdots S_{\lambda_l} - t^{\lambda_2 - k} s_{\lambda_2 - k} S_{\lambda_3} \cdots S_{\lambda_l} + t^{\lambda_3 - k - 1} s_{\lambda_3 - k - 1} S_{\lambda_4} \cdots S_{\lambda_l} - \cdots)$$

Using the commutation relation (2.14) in Proposition 2.6, the first parenthesis can be put into

$$(t^{\lambda_2 - k - 1} s_{\lambda_2 - k} S_{\lambda_3} \cdots S_{\lambda_l} - t^{\lambda_3 - k - 2} s_{\lambda_3 - k - 1} S_{\lambda_4} \cdots S_{\lambda_l} + t^{\lambda_4 - k - 3} s_{\lambda_4 - k - 3} S_{\lambda_5} \cdots S_{\lambda_l} - \cdots)$$

$$- (t^{\lambda_2 - k - 1} s_{\lambda_2 - k} S_{\lambda_3} \cdots S_{\lambda_l} - t^{\lambda_3 - k - 2} s_{\lambda_3 - k - 2} S_{\lambda_4} \cdots S_{\lambda_l} + t^{\lambda_4 - k - 3} s_{\lambda_4 - k - 3} S_{\lambda_5} \cdots S_{\lambda_l} - \cdots).$$
Combining with the other terms, they are exactly the following sum
\[
\sum_{i=1}^{l}(-1)^{i-1}t^{\lambda_i - k-i + 1}s_{\lambda_i - k-i + 1}S_{\lambda(i)}.
\]

Using the Pieri rule, we get the following direct and iterative formula for the Kostka–Foulkes polynomials.

**Corollary 2.8** For partition \( \lambda, \mu \vdash n \) with \( \lambda \geq \mu \), one has that
\[
K_{\lambda \mu}(t) = \sum_{i=1}^{l(\lambda)}(-1)^{i-1}t^{\lambda_i - \mu_1 - i + 1} \sum_{\tau^i} K_{\tau^i \mu[1]}(t),
\]
where \( \mu[1] = (\mu_2, \mu_3, \ldots) \) and \( \tau^i \) runs through the partitions such that \( \tau^i / \lambda^{(i)} \) are horizontal \((\lambda_i - \mu_1 - i + 1)\)-strips.

**Remark 2.9** Morris [20] has given an implicit iterative algorithm to compute \( K_{\lambda \mu}(t) \) by listing all Schur functions corresponding to previous partitions in dominance order and then extract the relevant ones (see [1, Sect. 4] for a clear exposition), while ours is a direct iterative one based on the upper partition \( \lambda \).

In the following remark, we highlight several interesting features of the Kostka–Foulkes polynomial formula in Theorem 2.7.

**Remark 2.10** For partitions \( \lambda, \mu \vdash n \) with \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \),
- The formula has at most \( l \) nonzero summands, although in many cases there are fewer by \( s_{-m} = 0 \) for a positive integer \( m \).
- The formula depends only upon \( \mu_1 \) from the content partition.
- The exponent \( \lambda_i - k - i + 1 \) of \( t \) is evocative of the exponents of the determinant definition of the Schur symmetric functions.
- Despite the alternating signs in the formula, positivity is assured by [16]. It would be interesting how to show the positivity directly from Corollary 2.8.

The Kostka number \( K_{\lambda \mu} \) satisfies the stability property [19, Ex.1.6.4], which implies that if \( \mu_1 \geq \lambda_2 \) then \( K_{\lambda+(r),\mu+(r)} = K_{\lambda,\mu} \) for all \( r \geq 1 \). Here, \( \lambda + (r) = (\lambda_1 + r, \lambda_2, \ldots) \). We have the following \( t \)-analog:

**Proposition 2.11** Suppose \( \mu_1 \geq \lambda_2 \), then for all \( r \geq 1 \) we have the stability \( K_{\lambda+(r),\mu+(r)}(t) = K_{\lambda,\mu}(t) \).

**Proof** This follows immediately from Theorem 2.7:
\[
K_{\lambda+(r),\mu+(r)}(t) = t^{\lambda_1 - \mu_1} s_{\lambda_i - \mu_1} S_{\lambda^{(1)}} H_{\mu[1]} = K_{\lambda,\mu}(t).
\]
The following simple facts are easy consequences of Theorem 2.7. The first two are well known (see [19]).

**Proposition 2.12** For $\mu \vdash n$, we have that

\[
K_{\mu, (n)} = \delta_{\mu, (n)},
\]

(2.19)

\[
K_{(n), \mu} = t^{n(\mu)},
\]

(2.20)

\[
K_{(r, s), \mu} = t^{r - \mu_1^1} (K_{(r - \mu_1^1 + s), \tilde{\mu}} + K_{(r - \mu_1^1 + s - 1, 1), \tilde{\mu}} + \cdots + K_{(r - \mu_1^1, s), \tilde{\mu}})
- t^{s - \mu_1^1 - 1} (K_{(r - \mu_1^1 + s), \tilde{\mu}} + K_{(r - \mu_1^1 + s - 1, 1), \tilde{\mu}} + \cdots + K_{(r + 1, s - \mu_1^1 - 1), \tilde{\mu}})
\]

(2.21)

where $\tilde{\mu} = \mu^{[1]}$.

Morris has given a table of $K_{\lambda, \mu}$ for $|\lambda| \leq 4$ and derived a formula for $K_{(21^{n-2}), \mu}$ in $\mathbb{Z}[t]$ [20]. A general compact formula is known for $\lambda$ being hook-shaped [14, Lemma 7.12]:

\[
K_{(n-k, 1^k), \mu} = t^{n(\mu) + \frac{k(k+1-2l)}{2}} \left[ \frac{l!}{k} \right]
\]

(2.22)

where $n = |\mu|$, $l = l(\mu)$ and $(n-k, 1^k) \geq \mu$. The formula can also be easily proved by induction on $|\mu|$ using Corollary 2.8.

We can give another formula for the Kostka–Foulkes polynomials of hook-shaped partitions (lower partition $\mu$). Recall the formula [19, p. 243]:

\[
K_{\lambda, (1^n)} = t^{n(\lambda')} \frac{[n]!}{\prod_{x \in \lambda} [h(x)]}
\]

(2.23)

where $h(x) = \lambda_i + \lambda_j' - i - j + 1$ is the hook length of $x = (i, j)$.

**Proposition 2.13** For all $\lambda \geq (n-k, 1^k)$

\[
K_{\lambda, (n-k, 1^k)} = \sum_{i=1}^{l(\lambda)} (-1)^i t^{\lambda_i - n + k - i + 1} \sum_{\tau^j} t^{n((\tau^j'))} \frac{[k]!}{\prod_{x \in \tau^j} [h(x)]},
\]

(2.24)

where $\tau^j$ runs through each partition such that $\tau^j / \lambda^{(i)}$ is a horizontal $(\lambda_i - n + k - i + 1)$-strip.

**Proof** It follows from Corollary 2.8 that $K_{\lambda, (n-k, 1^k)}$ is written as an alternative sum in terms of $K_{\tau^j, 1^k}$, where $\tau^j$ runs through each partition such that $\tau^j / \lambda^{(i)}$ is a horizontal $(\lambda_i - n + k - i + 1)$-strip, the formula is then proved by using (2.23). \qed

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