The Curvature and Index of Completely Positive Maps

Paul S. Muhly*  
Department of Mathematics  
University of Iowa  
Iowa City, IA 52242  
muhly@math.uiowa.edu

Baruch Solel†  
Department of Mathematics  
Technion  
32000 Haifa  
Israel  
mabaruch@technion.ac.il

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Abstract

We study conjugacy invariants for completely positive maps that are inspired by the concept of curvature introduced for commuting \( d \)-tuples of contractions by Arveson.

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1 Introduction

In \[4\], Arveson defined a notion of curvature for a commutative \( d \)-tuple of operators \( T = (T_1, T_2, \ldots, T_d) \), where the \( T_i \) act on a Hilbert space \( H \) and where it is assumed that \( \sum_{i=1}^{d} T_i T_i^* \leq I_H \). He showed that his concept is fundamentally an artifact of the contractive, normal, completely positive map \( \Theta_T \) defined on

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$B(H)$ by the formula

$$\Theta_T(X) = \sum_{i=1}^{d} T_i X T_i^*, \quad (1)$$

$X \in B(H)$. Subsequently, Kribs [12] and Popescu [21] defined and studied a notion of curvature for arbitrary, not-necessarily-commuting, $d$-tuples of contractions $T = (T_1, T_2, \ldots T_d)$ such that $\sum_{i=1}^{d} T_i T_i^* \leq I_H$. Kribs’s definition [12, Definition 2.4] of this curvature is directly in terms of the map $\Theta_T$:

$$K(T) := (d-1) \lim_{k \to \infty} \frac{\text{tr}(I - \Theta_{kT}^k(I))}{d^k}. \quad (2)$$

(It of course needs to be proved that this limit always exists. It does, as both Kribs and Popescu show.) Popescu’s definition is different [21, Equation (0.1)]; it is based on his notion of Poisson transform (see [20]) and is closely aligned with Arveson’s definition. However, he shows in Theorem 2.3 of [21] that his definition is the same as Kribs’s. Kribs and Popescu show that the curvature $K(T)$ is a unitary invariant for $T$ that measures its “departure from being free”, whereby a “free $d$-tuple” we mean one, $T = (T_1, T_2, \ldots T_d)$, such that the $T_i$ are pure isometries, i.e. multiples of the unilateral shift, with orthogonal ranges. For such a $d$-tuple, $\Theta_T$ is an endomorphism of $B(H)$ and it is not difficult to see that $K(T)$ coincides with the rank of $I - \Theta_T(I)$. Further, owing to the Bunce-Frazho-Popescu dilation theorem (see [5, 7, 8, 19]), every $d$-tuple $T$ such that $\Theta_k^k(I) \to 0$ in the strong operator topology can be “dilated” to a “free $d$-tuple” $S$ and $K(T) \leq K(S)$, with equality holding if and only if $T$ is itself free. In this case, $T = S$.

These results and others in [12] and [21], together with Stinespring’s famous dilation theorem [23], suggest the intriguing possibility of defining the curvature of an arbitrary (contractive, normal) completely positive map $\Theta$ on $B(H)$ via a formula like (2). For after all, thanks to Stinespring’s analysis, every such map is a $\Theta_T$ for some $d$-tuple $T$. One might hope for a rich interplay between the new “geometric” invariants of $\Theta$, such as curvature, and conjugacy or dynamical invariants of $\Theta$ as a map on $B(H)$. The problem, however, is this: A contractive, normal completely positive map on $B(H)$ can be represented by many different $d$-tuples. Even the number $d$ is not uniquely determined. After all, the $d$-tuple $T = (T_1, T_2, \ldots T_d)$ and the $d+1$-tuple $\tilde{T} = (T_1, T_2, \ldots T_d, 0)$ determine the same completely positive map. However, if $K(T)$ is finite, then $K(\tilde{T})$ will be zero. So unless there is a canonical way to represent a completely positive map on $B(H)$ in terms of a $d$-tuple, there does not seem to be much hope in developing a notion of curvature for completely positive maps - at least not one that proceeds along the lines of the formula (2).

1We note in passing that $K(T)$ does not agree with Arveson’s definition of curvature when $T$ is a commutative $d$-tuple. However, Popescu shows that Arveson’s curvature can be gotten from his formula for defining $K(T)$ by “compressing to symmetric Fock space” [21, Corollary 2.4].
Fortunately, however, there is a canonical way to represent a normal, contractive, completely positive map in terms of a $d$-tuple. This was observed by Arveson in [1] and a generalization of his analysis was developed by us to study completely positive maps on general von Neumann algebras [17]. It is the starting point of the present paper. Our objective is to show how to define a concept of curvature that generalizes Kribs’s definition [2] for any completely positive map on any semifinite factor and which leads, more or less, to the same type of results that he and Popescu found. Here, roughly and incompletely, is a synopsis of what we do. (Full definitions and details will be given in the body of the paper.)

Let $N$ be a semifinite factor, with a faithful normal trace $tr$, acting on a Hilbert space $H$ and let $\Theta$ be a contractive, normal, completely positive map on $N$. Form the Stinespring dilation $\pi : N \to B(N \otimes_{\Theta} H)$ and let $E_{\Theta} := \{ X : H \to N \otimes_{\Theta} H \mid Xa = \pi(a)X, a \in N \}$. Then $E_{\Theta}$ has the structure of $W^*$-correspondence over the commutant of $N$, $N'$. We call $E_{\Theta}$ the Arveson-Stinespring correspondence associated with $N$. (See Proposition and Definition [4.1].) This means, roughly, that $E_{\Theta}$ is a bimodule over $N'$ and that there is $N'$-valued inner product on $E_{\Theta}$ making $E_{\Theta}$ a right Hilbert $C^*$-module over $N'$. The “identity representation” of $E$ on $H$ (Definition [4.2]) is a pair $(T, \sigma)$, where $\sigma$ is the identity representation of $N'$ on $H$ and where $T : E_{\Theta} \to B(H)$ is a completely contractive bimodule map that is constructed explicitly in terms of the ingredients of the Stinespring dilation. (The “bimodule” condition means that $T(ab) = aT(b)$, $a, b \in N'$ and $\xi \in E_{\Theta}$.) The map $T$, in turn, defines a Hilbert space contraction operator $\tilde{T} : E_{\Theta} \otimes_{\sigma} H \to H$ and we showed in [17, Corollary 2.23] that $\Theta$ is given by the formula

$$\Theta(a) = \tilde{T}((I_{E_{\Theta}} \otimes a)\tilde{T}^*)$$

$a \in N$ (see Proposition [4.3] below, also).

Now the correspondence $E_{\Theta}$ has a natural dimension $d$, which is a nonnegative real number or $+\infty$ (Definition [4.3]). This dimension is defined in terms of the trace $tr$ on $N$, but it is independent of how $N$ is represented, so long as the commutant of the representation is finite (Theorem [4.8]). Because of its invariance under representations of $N$, we call the dimension $d$ the index of $\Theta$ and write $d = d(\Theta)$ (Definition [4.9]). If $\Theta$ happens to be given by an $n$-tuple of operators $(t_1, t_2, \ldots, t_n)$ in $N$, i.e., if $\Theta(a) = \sum_{i=1}^n t_ia^*_it_i$, $a \in N$, then $d(\Theta)$ turns out to be the vector space dimension of the complex linear span of the $t_i$ in $N$ (Proposition [4.11]). In that case, too, the identity representation $(T, \sigma)$ of $E_{\Theta}$ gives a canonical $d$-tuple $(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_d)$ of operators in $N$ representing $\Theta$ through equation (3), i.e., such that $\Theta(a) = \sum_{i=1}^d \tilde{t}_ia^*_i\tilde{t}_i$, $a \in N$.

If $d = d(\Theta)$ is finite, then the limit

$$\lim_{k \to \infty} \frac{tr(I - \Theta^k(I))}{\sum_{j=0}^{k-1} d^j}$$

exists as a positive real number or $+\infty$. This limit is our definition of the curvature $K(\Theta, tr)$ of $\Theta$ (Definition [4.13]). Evidently, $K(\Theta, tr)$ depends on the trace.

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However, the quantity $K(\Theta, tr)/tr(I - \Theta(I))$ doesn’t and truly should be thought of as the curvature of $\Theta$; we call this quantity the normalized curvature of $\Theta$ (Definition 4.16) and show that it is a conjugacy invariant of $\Theta$ (Theorem 4.23). When the index of $\Theta, d$, is strictly larger than 1, then the limit defining $K(\Theta, tr)$ turns out to be the same as $(d-1) \lim_{k \to \infty} \frac{tr(I - \Theta^k(I))}{d^k}$, which is Kribs’s definition in the case when $N = B(H)$.

In [17], we showed how to dilate the completely positive map $\Theta$ on $N$ to an endomorphism $\alpha$ of a von Neumann algebra $R$ in which $N$ sits as a corner. This was based on our dilation theory for representations of correspondences, which, in turn, generalizes the Bunce-Frazho-Popescu dilation theorem (see [15, Theorem 3.3] and [17, Theorem and Definition 2.18]). We show in Theorem 4.25 that if $d(\Theta)$ is finite, then so is $d(\alpha)$ and the two are equal. Further, $K(\alpha, tr_R') \geq K(\Theta, tr_N')$. The curvature of the endomorphism $\alpha$ is $tr_R'(I - \alpha(I))$. If $\Theta$ is pure, in the sense that the sequence $(\Theta^k(I))_{k \geq 0}$ converges strongly to zero, then so is $\alpha$ and in this case $K(\alpha, tr_R') = K(\Theta, tr_N')$ if and only if $\Theta = \alpha$. Thus, given that endomorphisms may be expressed in terms of isometric representations of correspondences and that there is an analogue of “pure isometric representation” (see [14] and Definition 2.17 below), we find that our notion of curvature for a completely positive map may also be viewed a measure of how much the map deviates from a “free” representation. Thus, our notion of curvature for contractive, normal, completely positive maps on semifinite factors captures the salient features of the concept defined in equation (3).

The constructs we have defined are easy to calculate in some situations. We offer a number of examples in Section 3. Of particular note is Example 4.26 in which we show that if $N$ is a semifinite factor with normalized trace $tr$ and if $\Theta(a) = tat^*$ for a contraction $t \in N$ such that $tr(I - tt^*) < \infty$, then $d(\Theta) = 1$ and $K(\Theta, tr) = tr(I - tt^*) - tr((I - t^*t)^{1/2}(I - t^*t)(I - t^*t)^{1/2})$. This generalizes work of Parrott and Levy [13, 14].

The next section is devoted to assembling material which we will use, from various sources. We discuss the general theory of $W^*$-correspondences, their representation and dilation theory, and the notion of dimension for correspondences over semifinite factors. In the third section, we develop the notion of the curvature of a representation of a correspondence. We use this material in the last section to define the curvature of a completely positive map through the curvature of the identity representation of its Stinespring correspondence. We show that the constructs we define are independent of any representation and so are intrinsic features of completely positive maps. We also calculate a number of examples.

2 Preliminaries

In this section we collect or develop a variety of facts that will be used in the sequel. We organize them into several subsections which are somewhat disjoint as presented here, but which will be blended in the next two sections.
2.1 Dimensions of Representations of Semifinite Factors

Let $M$ be a semifinite factor with normal, faithful semifinite trace $\tau$. We do not preclude the possibility that $M$ is finite. However, when $M$ is finite, we do not necessarily assume that the trace $\tau$ is normalized so that $\tau(I) = 1$. We will make explicit our assumptions on the normalization of traces as they arise. We write $L^2(M)$, or $L^2(M, \tau)$, for the space obtained by the GNS construction applied to $M$ and $\tau$. Every element $a$ of $M$ then defines an operator of left multiplication on $L^2(M)$, denoted $\lambda(a)$, and an operator of right multiplication, denoted $\rho(a)$. The maps $\lambda$ and $\rho$ are $*$-representations of $M$ and the opposite algebra of $M$, $M^{\text{opp}}$, respectively. The symmetry between the left and right representations of $M$ is implemented by a conjugate linear isometry $\lambda_x$.

Lemma 2.2 Let $M$ be a semifinite factor represented (necessarily faithfully) on a Hilbert space $H$, with commutant $M'$. Let $e$ be a projection in $M'$ and let $x$ be a positive element of $eM'e$. Then

$$\text{tr}_{M'}(x) = \text{tr}_{(eM')',eH}(x)$$ (4)

where $p$ is as above.

Definition 2.1 Let $M$ be a semifinite factor with a prescribed normal semifinite trace $\tau$.

1. Given a representation $\sigma$ of $M$ on a Hilbert space $H$, the natural trace on $\sigma(M')$ is the trace $\text{tr}_{\sigma(M')}$ just defined.

2. The (left) dimension of $H$ (as an $M$-module) is

$$\dim_M(H) = \text{tr}_{\sigma(M')}(p)$$

where $p$ is as above.
Proof. As discussed above, we have an $M$-linear isometry $u$ from $H$ to $L^2(M) \otimes K$ whose range is a projection $p$ in $(\lambda(M) \otimes I_K)'$. The positive element $uxu^*$ also lies in $(\lambda(M) \otimes I_K)'$ and can be written as a matrix $(p(x_{ij}))$ for $x_{ij}$ in $M$. Note that the map $a \mapsto ae$ is a representation of $M$ on $eH$. We write $Q$ for the image of this representation, $Me$. If $\tau$ is the trace on $M$, then we transport it to one, $\tau_0$, on $Q$ via the formula $\tau_0(ane) = \tau(a)$. The map $V : L^2(M) \to L^2(Q)$ extending the map $a \mapsto ae$ is a Hilbert space isomorphism and $\lambda_Q(ane) = V\lambda_M(a)V^*$. Then the composition of $ae$ with the map $V \otimes I_K$ is a $Q$-module isometry from $eH$ to $L^2(Q) \otimes K$ and we denote it by $v$. A straightforward computation shows that $vxe^* = (V \otimes I)uxu^*(V^* \otimes I)$ has the matrix representation $(p(x_{ij}e))$ and, therefore, $tr_{(eM)'eH}(x) = \sum \tau_0(x_{ij}e) = \sum \tau(x_{ij}) = tr_M(x)$. \hfill $\blacksquare$

2.2 $W^*$-correspondences and their dimensions

We begin by recalling the notion of a $W^*$-correspondence. For the general theory of Hilbert $C^*$-modules which we use, we will follow [13]. In particular, a Hilbert $C^*$-module will be a right Hilbert $C^*$-module.

Definition 2.3 Let $M$ and $N$ be von Neumann algebras and let $\mathcal{E}$ be a (right) Hilbert $C^*$-module over $N$. Then $\mathcal{E}$ is called a Hilbert $W^*$-module over $N$ in case it is self dual (i.e. every continuous $N$-module map from $\mathcal{E}$ to $N$ is implemented by an element of $\mathcal{E}$). It is called a $W^*$-correspondence from $M$ to $N$ if it is also endowed with a structure of a left $M$-module via a normal $\ast$-homomorphism $\varphi : M \to \mathcal{L}(\mathcal{E})$. (Here $\mathcal{L}(\mathcal{E})$ is the algebra of all bounded, adjointable, module maps on $\mathcal{E}$. For a Hilbert $W^*$-module it is known to be a von Neumann algebra). A $W^*$-correspondence over $M$ is simply a $W^*$-correspondence from $M$ to $M$.

If $\mathcal{E}$ is a $W^*$-correspondence from $M$ to $N$ and if $\mathcal{F}$ is a $W^*$-correspondence from $N$ to $Q$, then the balanced tensor product, $\mathcal{E} \otimes_N \mathcal{F}$ is a $W^*$-correspondence from $M$ to $Q$. It is defined as the Hausdorff completion of the algebraic balanced tensor product with the internal inner product given by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \varphi((\xi_1, \xi_2)\xi)\eta_2 \rangle_{\mathcal{F}}$$

for all $\xi_1, \xi_2$ in $\mathcal{E}$ and $\eta_1, \eta_2$ in $\mathcal{F}$. The left and right actions are defined by

$$\varphi_{\mathcal{E} \otimes_N \mathcal{F}}(a)(\xi \otimes \eta)b = \varphi_{\mathcal{E}}(a)\xi \otimes \eta b$$

for $a$ in $M$, $b$ in $Q$, $\xi$ in $\mathcal{E}$ and $\eta$ in $\mathcal{F}$.

Given a $W^*$-correspondence $\mathcal{E}$ over $M$, the full Fock space over $\mathcal{E}$ will be denoted by $\mathcal{F}(\mathcal{E})$, so $\mathcal{F}(\mathcal{E}) = M \oplus \mathcal{E} \oplus \mathcal{E} \otimes 2 \oplus \cdots$. It is also a $W^*$-correspondence over $M$ with left action $\varphi_{\infty}$ (or $\varphi_{\infty}$) given by the formula

$$\varphi_{\infty}(a) = diag(a, \varphi(a), \varphi^{(2)}(a), \cdots),$$

where $\varphi^{(n)}(a)(\xi_1 \otimes \xi_2 \otimes \cdots \xi_n) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \xi_n$. For $\xi \in \mathcal{E}$ we write $T_\xi$ for the creation operator on $\mathcal{F}(\mathcal{E})$:

$$T_\xi \eta = \xi \otimes \eta, \eta \in \mathcal{F}(\mathcal{E}).$$
We also recall that any $W^*$-correspondence over $M$ carries a natural weak topology, called the $\sigma$-topology (see [13]). This is the topology defined by the functionals $f(\cdot) = \sum_{n=1}^{\infty} \omega_n((\eta_n, \cdot))$ where the $\eta_n$ lie in $E$, the $\omega_n$ lie in $M$, and $\sum \|\omega_n\|\|\eta_n\| < \infty$.

We shall need some of the concepts and results of Jones’ index theory and we will refer to [9] or [10] for the basic results.

**Definition 2.4** If $M$ is a semifinite factor, a Hilbert space $H$ is said to be an $M$-$M$ bimodule if $H$ is a left $M$-module (whose structure is given by a unital normal representation $\pi_1$ of $M$ on $H$), $H$ is a right $M$-module (whose structure is given by a unital normal representation $\pi_0^0$ of $M^{\text{opp}}$ or a unital normal antirepresentation $\pi_\sigma$ of $M$ on $H$) and the actions commute (i.e. $\pi_\sigma(M) \subseteq (\pi(M))'$). The dimensions of $H$ with respect to $\pi_1$ and $\pi_\sigma$ will be denoted $\dim_{M_1}(H)$ and $\dim_{-M}(H)$ respectively. We shall call the $M$-bimodule bifinite if both of these numbers are finite.

Observe that if $H$ is a left $M$-module, it can be viewed as a $W^*$-correspondence from $M$ to $C$. If $E$ is a $W^*$-correspondence over $M$ then the balanced tensor product $E \otimes_M H$ is also a left $M$-module. In particular, if $H$ is the $M-M$ bimodule $L^2(M)$ then the tensor product is an $M-M$ bimodule. The map $E \to E \otimes_M L^2(M)$ defines a bijection between $W^*$-correspondences over $M$ and $M-M$ bimodules. This map is explored in [12].

**Definition 2.5** If $E$ is a $W^*$-correspondence over a semifinite factor $M$, then the left dimension of $E$ is defined to be $\dim_{M_1}(E \otimes L^2(M))$ and will be written $\dim_l(E)$. Similarly, the right dimension of $E$ is $\dim_{-M}(E \otimes L^2(M))$ and will be written $\dim_r(E)$. A $W^*$-correspondence is said to be left-finite (respectively, right-finite) if $\dim_l(E)$ (respectively, $\dim_r(E)$) is finite. It is said to be bifinite if both these numbers are finite.

Let $E$ be a $W^*$-correspondence over the von Neumann algebra $M$ and let $H$ be a left $M$-module, with associated normal representation $\sigma$. Then there is an induced representation $\sigma^E : \mathcal{L}(E) \to B(E \otimes_M H)$ defined by the formula $\sigma^E(S) = S \otimes I$ [15, Lemma 3.4]. It is not hard to see that $\sigma^E$ is a normal representation. By [24, Theorem 6.23], the image of the induced representation is the commutant of the algebra of all operators of the form $I_E \otimes T$, where $T$ lies in the commutant of $\sigma(M)$. We have

$$\mathcal{L}(E) \otimes I_H = (I_E \otimes \sigma(M))' = (I_E \otimes \kappa(M))',$$

where $\kappa$ is the representation of $\sigma(M)'$ given by the formula $T \to I_E \otimes T$, $T \in \sigma(M)'$. In the special case when $H = L^2(M)$, so that $\sigma = \lambda$ and $\kappa = \rho$, we write $\pi_1$ and $\pi_\sigma$ for the representation and the antirepresentation that define the left and right actions of $M$ on $E \otimes L^2(M)$. We shall also, on occasion, write $\pi_1$ as $\varphi(\cdot) \otimes I$ and $\pi_\sigma$ as $I_E \otimes \rho(\cdot)$.

**Lemma 2.6** Let $E$ be a bifinite $W^*$-correspondence over the type $II_1$ factor $M$. Then
(1) The von Neumann algebra $\mathcal{L}(E)$ is a type $II_1$ factor and $\varphi(M)$ is a subfactor.

(2) If $\sigma$ is a representation of $M$ on $H$ then 
\[ [\mathcal{L}(E) \otimes I_H : \varphi(M) \otimes I_H] = \big[ \sigma^E([\mathcal{L}(E)] : \varphi(M)) \big] = [\mathcal{L}(E) : \varphi(M)]. \]

(3) $[\mathcal{L}(E) : \varphi(M)] = \dim_l(E) \dim_r(E) < \infty$

**Proof.** Suppose $\sigma$ is a normal representation of $M$ on $H$, as in part (2). Since $M$ is a factor, $\sigma$ is an isomorphism and it is easy to check that then $\sigma^E$ is an isomorphism, so that $L(E)$ is isomorphic to $L(E) \otimes I_H$. In particular, for $H = L^2(M)$, $L(E)$ is isomorphic to $L(E) \otimes I_{L^2(M)}$ and that algebra is the commutant of $I_E \otimes \rho(M)$. As $\rho(M)$ is a type $II_1$ factor and $I_E \otimes \rho(M)$ is isomorphic to it, $L(E)$ is a type $II$ factor. Since $I_E \otimes \rho(M)$ is a finite factor and $\dim_r(E)$ is finite it follows that $L(E) \otimes I_{L^2(M)}$ (and, hence also $L(E)$) is finite ($[R]$ Proposition 2.2.6(iii)). This proves (1). Part (2) follows from the fact that $\sigma^E$ is an isomorphism and part (3) is in $[R]$ Discussion following Corollary 2.3.6.

**Lemma 2.7** Let $M$ be a finite factor and let $\sigma$ be a normal representation of $M$ on $H$. If $E$ is a left-finite $W^*$-correspondence over $M$, then for every positive element $x$ in $\sigma(M)'$, we have 
\[ \text{tr}_{\sigma(M)'}(x) \cdot \dim_l(E) = \text{tr}_{\sigma(M) \otimes I_H'}(I_E \otimes x). \]

**Proof.** If $M = M_n(\mathbb{C})$, it is easy to check that there is a projection $e$ in $\sigma(M)'$ with $\text{tr}_{\sigma(M)'}(e) = 1/n$. If $M$ is of type $II_1$, the values of the trace $\text{tr}_{\sigma(M)'}$ on projections of $\sigma(M)'$ form an interval containing 0. In any case, one can always find a projection $e$ in $\sigma(M)'$ and a positive integer $m$ such that $\text{tr}_{\sigma(M)'}(e) = 1/m$. Fix such a projection. Then $\dim_{\sigma(M)} eH = 1/m$ and 
\[ \sum_{i=1}^{m} \oplus eH \cong L^2(M) \]
as left $M$-modules (where the action of $M$ on $eH$ is by $\sigma$ and on $L^2(M)$ by $\lambda$). Tensoring by $E$, we get 
\[ \sum_{i=1}^{m} \oplus (E \otimes eH) \cong E \otimes L^2(M). \]

It follows that 
\[ \text{tr}_{\sigma(M) \otimes I_H'}(I_E \otimes e) = \dim_{\sigma(M) \otimes I_H'} E \otimes eH = \frac{1}{m} \dim_l(E) = \dim_l(E) \cdot \text{tr}_{\sigma(M)'}(e). \]

Writing $\tau_1(x) = \text{tr}_{\sigma(M) \otimes I_H'}(I_E \otimes x)$ for $0 \leq x \in \sigma(M)'$, we get a faithful, normal, semifinite trace on $\sigma(M)'$. Thus it is a multiple of $\text{tr}_{\sigma(M)'}$. The computation above shows that the multiple is $\dim_l(E)$. 


Corollary 2.8 Let $M$ be a finite factor and let $\mathcal{E}$ and $\mathcal{F}$ be two left-finite $W^*$-correspondences over $M$. Then

$$\dim_l(\mathcal{E} \otimes_M \mathcal{F}) = \dim_l(\mathcal{E}) \dim_l(\mathcal{F}).$$

In particular, $\dim_l(\mathcal{E}^\otimes n) = (\dim_l(\mathcal{E}))^n$ for all positive integers $n$.

Proof. The corollary is proved by applying Lemma 2.7 several times. We write $H$ for the space $\mathcal{F} \otimes L^2(M)$ and $\sigma$ will be the representation $\varphi_\mathcal{E}(\cdot) \otimes I_{L^2(M)}$ of $M$ on $H$ (we write $\varphi_\mathcal{E}$, $\varphi_\mathcal{F}$ and $\varphi_{\mathcal{E} \otimes \mathcal{F}}$ for the left action maps on $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{E} \otimes \mathcal{F}$ respectively). Note that $\varphi_\mathcal{E}(M) \otimes I_H = \varphi_{\mathcal{E} \otimes \mathcal{F}}(M) \otimes I_{L^2(M)}$. Thus we have, for a positive element $x$ in $\lambda(M)' = \rho(M)$,

$$tr(\varphi_{\mathcal{E} \otimes \mathcal{F}}(M) \otimes I_{L^2(M)}) = tr(\varphi_\mathcal{E}(M) \otimes I_H \otimes x) = \dim_l(\mathcal{E}) \cdot tr_{\sigma(M)'}(I_F \otimes x) = \dim_l(\mathcal{E}) \dim_l(\mathcal{F}) \cdot tr_{\lambda(M)'}(x)$$

But, using Lemma 2.7 again, this will be equal to $tr_{\lambda(M)'}(x) \dim_l(\mathcal{E} \otimes \mathcal{F})$. Now set $x = I$ to complete the proof. ■

2.3 Representations of correspondences and completely positive maps

In this subsection discuss representations of $W^*$-correspondences and the completely positive maps associated with such representations. For more details, please refer to [15] and [17].

Definition 2.9 Let $\mathcal{E}$ be a $W^*$-correspondence over a von Neumann algebra $N$ and let $H$ be a Hilbert space.

1. A completely contractive covariant representation of $\mathcal{E}$ (or, simply, a representation of $\mathcal{E}$) in $B(H)$ is a pair $(T, \sigma)$, where

(a) $\sigma$ is a normal representation of $N$ in $B(H)$.

(b) $T$ is a linear, completely contractive map from $\mathcal{E}$ to $B(H)$ that is continuous with respect to the $\sigma$-topology of $\mathcal{B}(\mathcal{E})$ on $\mathcal{E}$ and the $\sigma$-weak topology on $B(H)$.

(c) $T$ is a bimodule map in the sense that $T(\varphi(a)\xi b) = \sigma(a)T(\xi)\sigma(b)$, $\xi \in \mathcal{E}$, and $a, b \in N$.

2. A completely contractive covariant representation $(T, \sigma)$ of $\mathcal{E}$ in $B(H)$ is called isometric in case

$$T(\xi)^*T(\eta) = \sigma(\langle \xi, \eta \rangle),$$

for all $\xi, \eta$ in $\mathcal{E}$. 

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The theory developed in [15] applies here to prove that if a representation \((T, \sigma)\) of \(\mathcal{E}\) is given, then it determines a contraction \(\tilde{T} : \mathcal{E} \otimes_\sigma H \to H\) defined by the formula

\[
\tilde{T}(\xi \otimes h) = T(\xi)h.
\]

Moreover, for every \(a\) in \(N\) we have

\[
\tilde{T}(\varphi(a) \otimes I(= \tilde{T}\sigma^E(\varphi(a)))) = \sigma(a)\tilde{T},
\]

i.e., \(\tilde{T}\) intertwines \(\sigma\) and \(\sigma^E \circ \varphi\). In fact, it is shown in [15] that there is a bijection between representations \((T, \sigma)\) of \(\mathcal{E}\) and intertwining operators \(\tilde{T}\) of \(\sigma\) and \(\sigma^E \circ \varphi\).

It is also shown in [15] that \((T, \sigma)\) is isometric if and only if \(\tilde{T}\) is an isometry.

**Definition 2.10** A representation \((T, \sigma)\) of a \(\mathcal{W}\)-correspondence \(E\) is called fully coisometric if \(\tilde{T}\) is a coisometry (i.e. if \(\tilde{T}\tilde{T}^* = I_H\)).

Associated with every representation \((T, \sigma)\) we can define a completely positive, contractive, map \(\Theta = \Theta_T\) on the commutant of \(\sigma(N)\). It is given in the following proposition. The proof and more details can be found in [17, Proposition 2.21].

**Proposition and Definition 2.11** Let \(N\) be a von Neumann algebra, let \(E\) be a \(\mathcal{W}\)-correspondence over \(N\) and let \((T, \sigma)\) be a completely contractive covariant representation of \(E\) on a Hilbert space \(H\). For \(S \in \sigma(N)'\), set

\[
\Theta(S) = \Theta_T(S) = \tilde{T}(I_E \otimes S)\tilde{T}^*.
\]

Then \(\Theta\) is a contractive, normal completely positive map from \(\sigma(N)\) into itself. It is unital if and only if \((T, \sigma)\) is fully coisometric and it is multiplicative (i.e. a *-endomorphism of \(\sigma(N)\)) if \((T, \sigma)\) is isometric.

In addition to \(\tilde{T}\) we also define the maps \(\tilde{T}_n : \mathcal{E}^\otimes_n \otimes H \to H\) by \(\tilde{T}_n(\xi_1 \otimes \ldots \otimes \xi_n \otimes h) = T(\xi_1) \cdots T(\xi_n)h\) and then we have \(\tilde{T}_{n+m} = \tilde{T}_n(I_n \otimes \tilde{T}_m) = \tilde{T}_m(I_m \otimes \tilde{T}_n)\), where \(I_n\) is the identity map on \(\mathcal{E}^\otimes_n\) [16]. It follows that

\[
\Theta^n(S) = \tilde{T}_n(I_n \otimes S)\tilde{T}_n^*
\]

for \(S\) in \(\sigma(M)'\).

**Proposition 2.12** Let \(E\) be a left-finite \(\mathcal{W}\)-correspondence over a finite factor \(M\) and let \((T, \sigma)\) be a representation of \(E\) in \(B(H)\). Then for every positive \(x\) in \(\sigma(M)\)' we have

\[
\text{tr} \sigma(M)'(\Theta_T(x)) \leq ||\tilde{T}||^2 \dim(\mathcal{E}) \text{tr} \sigma(M)'(x).
\]
Proof. Let \( \tilde{\sigma} \) be the representation of \( M \) on \( H \oplus (E \otimes H) \) defined by the formula \( \tilde{\sigma}(a) = \begin{pmatrix} \sigma(a) & 0 \\ 0 & \varphi(a) \otimes I_H \end{pmatrix} \) and let \( B \) be the image of \( \tilde{\sigma} \):

\[
B = \tilde{\sigma}(M) = \left\{ \begin{pmatrix} \sigma(a) & 0 \\ 0 & \varphi(a) \otimes I_H \end{pmatrix} \in B(H \oplus (E \otimes H)) : a \in M \right\}.
\]

Write \( tr_B' \) for the natural trace on \( B' \) (Definition 2.1).

For \( y \) in \( \sigma(M)' \) and \( z \) in \( (\varphi(M) \otimes I)' \) it is easy to check that \( \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix} \) lie in \( B' \) and that the equations,

\[
tr_B' \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = tr_{\sigma(M)'}(y)
\]

and

\[
tr_B' \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix} = tr_{(\varphi(M) \otimes I)'}(z),
\]

are valid.

Note also that equation (5) implies that \( \begin{pmatrix} 0 & \tilde{T} \\ 0 & 0 \end{pmatrix} \) lies in \( B \). We therefore have, for a positive element \( x = c^*c \) in \( \sigma(M)' \), that

\[
tr_{\sigma(M)'}(\Theta_T(x)) = tr_{\sigma(M)'}(\tilde{T}(I \otimes x)\tilde{T}^*) = tr_{B'} \begin{pmatrix} \tilde{T}(I \otimes x)\tilde{T}^* & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
= tr_{B'} \begin{pmatrix} 0 & \tilde{T}(I \otimes c^*) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ (I \otimes c)\tilde{T}^* & 0 \end{pmatrix}
\]

\[
= tr_{B'} \begin{pmatrix} 0 & 0 \\ (I \otimes c)\tilde{T}^* & 0 \end{pmatrix} \begin{pmatrix} 0 & \tilde{T}(I \otimes c^*) \\ 0 & 0 \end{pmatrix}
\]

\[
= tr_{B'} \begin{pmatrix} 0 & 0 \\ 0 & (I \otimes c)\tilde{T}^*(I \otimes c^*) \end{pmatrix} = tr_{(\varphi(M) \otimes I)'}((I \otimes c)\tilde{T}^*(I \otimes c^*))
\]

\[
\leq ||\tilde{T}||^2 tr_{(\varphi(M) \otimes I)'}(I \otimes x) = ||\tilde{T}||^2 \dim_l(E) tr_{\sigma(M)'}(x),
\]

where the last equality follows from Lemma 2.7.

Corollary 2.13 Let \( E \) be a left-finite \( W^* \)-correspondence over a finite factor \( M \) and let \( (T, \sigma) \) be a representation of \( E \), as in Proposition 2.12. Write \( \Delta = (I - \tilde{T}^*\tilde{T})^{1/2} \). Then, for every positive element \( x \) in \( \sigma(M)' \),

\[
\dim_l(E) \cdot tr_{\sigma(M)'}(x) = tr_{\sigma(M)'}(\Theta_T(x)) + tr_{(\varphi(M) \otimes I)'}(\Delta(I \otimes x)\Delta).
\]
In particular, if \((T, \sigma)\) is an isometric representation of \(E\) then
\[
tr_{\sigma(M)'}(\Theta_T(x)) = \diml(E) \cdot tr_{\sigma(M)'}(x)
\]
for every positive \(x\) in \(\sigma(M)'\).

**Proof.** It follows from the computation in Proposition 2.12 that we have,
\[
tr_{\sigma(M)'}(\Theta_T(x)) + tr_{(\varphi(M) \otimes \mathbb{I})'}((I \otimes c)\Delta^2(I \otimes c^*)) = \diml(E) \cdot tr_{\sigma(M)'}(x)
\]
where \(c^*c = x\). Using the trace property of \(tr_{\varphi(M) \otimes \mathbb{I}}\) we see that the second summand on the left hand side is equal to \(tr_{\varphi(M) \otimes \mathbb{I}}'\left(\Delta(I \otimes x)\Delta\right)\) and this completes the proof of the main assertion. When the representation is isometric, we have \(\tilde{T}^*\tilde{T} = I\) and \(\Delta = 0\); thus the conclusion follows. \(\blacksquare\)

**Remark 2.14** Using the notation of Proposition 2.12 the result of the corollary can also be written as
\[
\diml(E) = tr_{\mathcal{B}'} \left( \Theta_T(x) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) + tr_{\mathcal{B}'} \left( \begin{pmatrix} 0 & 0 \\ \Delta(I \otimes x)\Delta & 0 \end{pmatrix} \right)
\]
for any positive element \(x\) in \(\sigma(M)'\) whose trace is finite. (We can, in fact, let \(tr_{\mathcal{B}'}\) be any (faithful, normal, semifinite) trace on \(\mathcal{B}'\)).

### 2.4 Dilations of representations of correspondences

We conclude our preliminary discussion by recalling and developing some results concerning the minimal isometric dilation \((V, \rho)\) of a representation \((T, \sigma)\) of a correspondence \(E\) over a von Neumann algebra \(M\). For details beyond those presented here see [15] and [17].

Assume that \((T, \sigma)\) is a representation of \(E\) on \(H\) and let \(\tilde{T}\) be the contraction defined above. Write \(\Delta = (I - \tilde{T}^*\tilde{T})^{1/2}\) (in \(B(E \otimes H)\)) and let \(\mathcal{D}\) be the closure of the range of \(\Delta\). Owing to formula (3), \(\Delta\) commutes with \(\sigma \circ \varphi\) and so \(\mathcal{D}\) reduces \(\sigma \circ \varphi\). We let \(\sigma_1 := \sigma \circ \varphi|\mathcal{D}\). Also let \(L_\xi : H \rightarrow E \otimes \mathcal{D}\) be the map defined by \(L_\xi h = \xi \otimes h\) and write \(D(\xi) = \Delta \circ L_\xi\). Note that \(T(\xi) = \tilde{T} \circ L_\xi\). The representation space \(K\) of \((V, \rho)\) is
\[
K = H \oplus \mathcal{D} \oplus (E \otimes \mathcal{D}) \oplus (E \otimes^2 \mathcal{D}) \oplus \ldots
\]
Let \(\sigma_{n+1} = \sigma_1^{\otimes n} \circ \varphi_n, n \geq 1\), so that \(\sigma_{n+1}\) is a representation of \(M\) on \(E \otimes^2 \mathcal{D}\). The representation \(\rho\) of \(M\) on \(K\) that we want is \(\rho = \sigma \oplus \sum_{n \geq 1} \sigma_n\). The map \(V : E \rightarrow B(K)\) is defined by
\[
V(\xi) = \begin{pmatrix}
T(\xi) & 0 & 0 & \ldots \\
0 & D(\xi) & 0 & \ldots \\
0 & 0 & L_\xi & \ldots \\
\end{pmatrix}
\]
where $L_ξ$ here is the obvious map from $E^⊗m ⊗_σ D$ to $E^⊗(m+1) ⊗_σ D$. Letting $\tilde{V} : E ⊗_ρ K \to K$ be the map sending $ξ ⊗ k$ to $V(ξ)k$, we can write

$$
\tilde{V} = \begin{pmatrix}
\dd 0 0 & \ldots \\
\Delta 0 0 & \ldots \\
0 I 0 & \\
0 0 I & \\
\ldots & \\
\end{pmatrix}
$$

(7)

where the identity operators in this matrix should be interpreted as the operators that identify the spaces $E ⊗_σ (E^⊗n ⊗_σ D)$ with $E^⊗(n+1) ⊗_σ D$.

We shall refer to this $(V, ρ)$ as the minimal isometric dilation of $(T, σ)$ as it satisfies the following properties.

1. $(V, ρ)$ is an isometric covariant representation of $E$ on $K$,
2. $H$ reduces $ρ$, and $ρ(a)|H = P_H ρ(a)|H = σ(a)$ for all $a$ in $M$,
3. $H^⊥ = K ⊕ H$ is invariant under each $V(ξ)$, $ξ ∈ E$; i.e., $P_H V(ξ)|H^⊥ = 0$,
4. $P_H V(ξ)|H = T(ξ)$, for all $ξ ∈ E$, and
5. the smallest subspace of $K$ containing $H$ and invariant under any $V(ξ)$ is all of $K$.

It is shown in [13, Proposition 3.2] that a minimal isometric dilation is unique up to unitary equivalence. Hence we refer to the representation defined above as the minimal isometric representation of $(T, σ)$.

From the isometric property of $(V, ρ)$ it follows that $\tilde{V}$ is an isometry and, thus, the map $Θ_ν$, which, recall, is defined on $ρ(M)^\prime$, is in fact a *-endomorphism of $ρ(M)^\prime$. By Proposition and Definition 2.11, $Θ_ν$ is unital precisely when $V$ is fully coisometric and this happens if and only if $V$ is a coisometry. However, a simple calculation shows that $\tilde{V}$ is a coisometry if and only if $T$ is a coisometry, i.e., by Proposition and Definition 2.11 again, if and only if $Θ_T$ is unital.

We shall write $P_n, n ≥ 0$, for the projection $Θ_ν^0 (I) (= V_n V_n^*)$. Then the sequence $\{P_n\}_{n≥0}$ is a decreasing sequence of projections and we write $P_∞ := ∧ P_n$. Moreover, if $Q_n := P_n - P_{n+1}, n ≥ 0$, then the $Q_n$ are mutually orthogonal (they are “wandering” projections) and $\sum_{k=0}^∞ Q_k = I - P_∞$.

The following terminology comes from [13].

**Definition 2.15** An isometric representation $(V, ρ)$ of $E$ is said to be induced if there is a (normal) representation $π_0$ on a Hilbert space $H_0$ such that $(V, ρ)$ is unitarily equivalent to the representation $(R, σ)$ defined on $F(E) ⊗_π H_0$ by the formulae $σ = π_0^F(E) ◦ φ_∞$ and $R(ξ) = π_0^F(E)(T_ξ), ξ ∈ E$; i.e.,

$$
σ(a) = φ_∞(a) ⊗ I_{H_0}, a ∈ M,
$$

13
and
\[ R(\xi) = T_\xi \otimes I_{H_0}, \quad \xi \in \mathcal{E}, \]
where, recall, \( T_\xi \) is the creation operator on \( \mathcal{F}(\mathcal{E}) \) determined by \( \xi \) and \( \varphi_\infty \) is the diagonal representation of \( M \) on \( \mathcal{F}(\mathcal{E}) \) defined in the discussion following Definition 2.3.

We state for reference a result from [16] that will prove useful in our analysis; it is a generalization of the Wold decomposition of an isometry.

**Proposition 2.16** [16, Theorem 2.9] Every isometric representation \((V, \rho)\) of \( \mathcal{E} \) on \( K \) decomposes as the direct sum
\[ (V, \rho) = (V_{\text{ind}}, \rho_{\text{ind}}) \oplus (V_\infty, \rho_\infty) \] (8)
where \((V_\infty, \rho_\infty)\) is the restriction of \((V, \rho)\) to \( P_\infty(K) \) and is fully coisometric (and isometric), while \((V_{\text{ind}}, \rho_{\text{ind}})\) is the restriction of \((V, \rho)\) to \( K \ominus P_\infty(K) \) and is an induced representation. In fact, the representation \( \pi_0 \) of \( M \) appearing in the definition of induced representation is, in this case, the restriction of \( \rho \) to the range of \( Q_0 \). (Here \( P_\infty \) and \( Q_0 \) are as defined above).

**Definition 2.17** The direct sum decomposition of an isometric representation of \( \mathcal{E} \), \((V, \rho)\), given in equation (8) is called the Wold decomposition of \((V, \rho)\): \((V_{\text{ind}}, \rho_{\text{ind}})\) is called the pure part or induced part of \((V, \rho)\) and \((V_\infty, \rho_\infty)\) is called the fully coisometric part or residual part of \((V, \rho)\). If \((V_\infty, \rho_\infty)\) reduces to zero, we say that \((V, \rho)\) is pure or induced.

If \((V, \rho)\) is the minimal isometric dilation of a representation \((T, \sigma)\) of \( \mathcal{E} \), we will say that \((T, \sigma)\) is pure if \((V, \rho)\) is pure.

The “purity” of \( T \) is reflected in \( \Theta_T \) and \( \Theta_V \) as the next proposition indicates.

**Proposition 2.18** Let \((T, \sigma)\) be a representation of \( \mathcal{E} \) on a Hilbert space \( H \) and let \((V, \rho)\) be its minimal isometric dilation acting on the Hilbert space \( K \) containing \( H \). Then the following conditions are equivalent.

1. \((T, \sigma)\) is pure.
2. \( \Theta_V^k(I) \to 0 \) in the strong operator topology on \( K \).
3. \( \Theta_T^k(I) \to 0 \) in the strong operator topology on \( H \).

**Proof.** The equivalence of conditions (1) and (2) is proved in Corollary 2.10 of [14]. Condition (2) implies condition (3) simply because \( \tilde{T}_h\tilde{T}_h^* = P_H\tilde{V}_h\tilde{V}_h^*P_H \).

For the other direction note that, since \( \|\tilde{T}_h^*h\| = \|\tilde{V}_h^*h\| \) for all \( h \in H \), condition (3) implies that \( H \) is orthogonal to the range of \( P_\infty = \wedge\tilde{V}_h\tilde{V}_h^* \). Hence it follows from the minimality of the dilation that \( P_\infty = 0 \). \( \blacksquare \)
3 The Curvature of a Representation

In this section we define and study the curvature invariant for representations of a given $W^*$-correspondence $\mathcal{E}$ over a semifinite factor $M$.

**Definition 3.1** Let $M$ be a semifinite factor and let $\mathcal{E}$ be a $W^*$-correspondence over $M$ with finite left dimension $d = \dim_l(\mathcal{E})$. For every representation $(T, \sigma)$ of $\mathcal{E}$ we define the curvature of $(T, \sigma)$ to be

$$K(T, \sigma, \mathcal{E}) = \lim_{k \to \infty} \frac{\text{tr}_{\sigma(M)'}(I - \Theta^k_T(I))}{\sum_{j=0}^{k-1} d_j}$$

where and $\Theta_T$ is the contractive, normal, completely positive map defined in Proposition 2.11.

When $\sigma$ and $\mathcal{E}$ are fixed, we sometimes abbreviate $K(T, \sigma, \mathcal{E})$ as $K(T)$.

In order to prove the existence of the limit in the definition of the curvature we require the following “summability result”, which stated by Popescu in [21, p. 280]. It is reminiscent of the fact that if a sequence is convergent, then the sequence of its arithmetic means converges to its limit.

**Lemma 3.2** Let $\{a_j\}_{j=0}^\infty$ and $\{b_j\}_{j=0}^\infty$ be two sequences of real numbers such that $b_j > 0$ for all $j \geq 0$. Write $A_k$ and $B_k$ for the partial sums, $A_k = \sum_{j=0}^{k-1} a_j$ and $B_k = \sum_{j=0}^{k-1} b_j$, and assume that $B_k \to \infty$ as $k \to \infty$. If the limit $L = \lim_{j \to \infty} a_j/b_j$ exists (and finite), then

$$L = \lim_{k \to \infty} \frac{A_k}{B_k}.$$

**Theorem 3.3** Let $M$ be a semifinite factor, let $\mathcal{E}$ be a $W^*$-correspondence over $M$, with finite left dimension $d = \dim_l(\mathcal{E})$, and let $(T, \sigma)$ be a representation of $\mathcal{E}$ on a Hilbert space. Then:

1. The limit defining $K(T, \sigma, \mathcal{E})$ (in Definition 3.1) exists, either as a positive number or $+\infty$.

2. $K(T, \sigma, \mathcal{E}) = \infty$ if and only if $\text{tr}_{\sigma(M)'}(I - \Theta_T(I)) = \infty$.

3. If $\text{tr}_{\sigma(M)'}(I - \Theta_T(I)) \neq \infty$, so $K(T, \sigma, \mathcal{E}) < \infty$, then the following formulae hold.

   (3a) If $d$ is larger than or equal to 1, then

   $$K(T, \sigma, \mathcal{E}) = \lim_{k \to \infty} \frac{\text{tr}_{\sigma(M)'}(I - \Theta^k_T(I))}{d^k}.$$

   and, if $d$ is strictly larger than 1, then this limit is also given by the equation $K(T, \sigma, \mathcal{E}) = (d - 1)\lim_{k \to \infty} \frac{\text{tr}_{\sigma(M)'}(I - \Theta^k_T(I))}{d^k}$.
(3b) If $d$ is strictly less than 1, then $A := \lim_{k \to \infty} tr_{\sigma(M)^t}(I - \Theta_k^t(I))$ exists (and is finite) and

$$K(T, \sigma, E) = (1 - d)A.$$ 

\textbf{Proof.} Write $tr$ for $tr_{\sigma(M)^t}$, $\Theta$ for $\Theta_T$ and $a_j$ for $tr(\Theta_j^t(I) - \Theta_{j+1}^t(I))$. Then it follows from Proposition 2.12 that

$$a_{j+1} \leq da_j$$

for $j \geq 0$. This shows that $a_0 = \infty$ if and only if $a_j = \infty$ for all $j \geq 0$. Now assume $a_0 < \infty$ and consider the sequence $\{a_j/d^j\}_{j=0}^{\infty}$. It follows from equation (4.7) that this is a non-increasing sequence of nonnegative numbers. We write $L$ for its limit. Then $0 \leq L \leq a_0$.

Note that $tr(I - \Theta^k(I)) = \sum_{j=0}^{k-1} a_j$.

Suppose $d \geq 1$. Then we can use Lemma 3.3 (with $b_j = d^j$) to conclude that in this case, $K(T, \sigma, E)$ exists and equals $L$. This proves one equality in part (3). For the other one note that

$$\sum_{j=0}^{k-1} d^j = \frac{d^k - 1}{d - 1}$$

and $\lim_{k \to \infty} d^k/(d^k - 1) = 1$ when $d > 1$. This proves the assertions in (3a).

If $d < 1$, the denominator in the definition of the curvature has a finite limit $1/(1 - d)$ as $k \to \infty$. Since it follows from equation (4.7) that $a_j \leq d^j a_0$ for all $j \geq 0$, the numerator in the definition of the curvature tends to a finite limit (denoted $A$ in (3b)). This completes the proof of assertion (3b) and also the proofs of assertions (1) and (2) because we just showed that, whenever $a_0 \neq \infty$, the limit defining the curvature exists and is finite. \hfill $\blacksquare$

As we noted in the Introduction, this result “captures” Kribs’s definition of the curvature of a $d$-tuple [12, Definition 2.4] and Popescu’s Corollary 2.7 in [21].

Lemma 3.4 Let $M$ be a finite factor, let $E$ be a $W^*$-correspondence over $M$ and let $(T, \sigma)$ be a representation of $E$ on a Hilbert space. Then:

(1) The curvature of $T$, $K(T, \sigma, E)$, satisfies the inequality

$$K(T, \sigma, E) \leq tr_{\sigma(M)^t}(I - \tilde{T}\tilde{T}^*)$$

and if the representation is isometric, equality holds.

(2) Suppose $tr_{\sigma(M)^t}(I - \tilde{T}\tilde{T}^*) < \infty$. Then $K(T, \sigma, E) = tr_{\sigma(M)^t}(I - \tilde{T}\tilde{T}^*)$ if and only if, for every $j \geq 1$,

$$tr_{\sigma(M)^t}(\Theta_j^t(I) - \Theta_{j+1}^t(I)) = \dim(E) \cdot tr_{\sigma(M)^t}(\Theta_{j-1}^t(I) - \Theta_j^t(I)).$$

(11)
Proof. Write \( a_j = tr(\Theta^j(I) - \Theta^{j+1}(I)) \) as in the proof of Theorem 3.3. We have \( a_{j+1} \leq da_j \) where \( d = dim_1(\mathcal{E}) \) and (using Corollary 2.13) for an isometric representation we have equality. Hence

\[
tr(I - \Theta^k(I)) = \sum_{j=0}^{k-1} a_j \leq a_0 \sum_{j=0}^{k-1} d^j
\]

and equality holds for an isometric representation. This proves assertion (1).

In fact, it shows that equality holds whenever \( a_{j+1} = da_j \) for all \( j \geq 0 \) and this proves one implication in assertion (2). For the other implication, assume that \( K(T, \sigma, \mathcal{E}) = a_0 \). If \( d \geq 1 \) it follows that \( a_0 \) is the limit of the decreasing sequence \( \{a_j/d^j\} \) whose first term is \( a_0 \). Hence the sequence is constant. If \( d < 1 \) then it follows that \( a_0 = \sum a_j - \sum da_j = \sum (a_{j+1} - da_j) + a_0 \) and, again, we get \( a_{j+1} = da_j \) for all \( j \).

The notion of “pure rank” which we define next is motivated by the concept first presented by Davidson, Kribs and Shpigel in [6]. Their concept coincides with ours in the case when the factor \( M \) reduces to the scalars \( \mathbb{C} \).

**Definition 3.5** Let \( M \) be a semifinite factor, let \( \mathcal{E} \) be a \( W^* \)-correspondence over \( M \), and let \( (T, \sigma) \) be a representation of \( \mathcal{E} \) on a Hilbert space.

1. For an element \( a \) in \( \sigma(M)' \), we write \( rank_{\sigma(M)'}(a) \) (or, simply, \( rank(a) \)) for \( tr_{\sigma(M)'}(r(a)) \) where \( r(a) \) is the range projection of \( a \).

2. We define the pure rank of \( (T, \sigma) \) to be

\[
pure\ rank(T, \sigma) = rank_{\sigma(M)'}(I - T\tilde{T}^*).
\]

The inequalities displayed in the next proposition generalize Theorem 2.7 of [12].

**Proposition 3.6** Let \( M \) be a semifinite factor and let \( \mathcal{E} \) be a \( W^* \)-correspondence over \( M \). If \( (T, \sigma) \) is a representation of \( \mathcal{E} \) on a Hilbert space and if \( (V, \rho) \) is its minimal isometric dilation, then

\[
K(V, \rho, \mathcal{E}) = tr_{\rho(M)'}(I - \tilde{V}\tilde{V}^*) = pure\ rank(V, \rho) = pure\ rank(T, \sigma)
\]

\[
\geq tr_{\sigma(M)'}(I - \tilde{T}\tilde{T}^*) \geq K(T, \sigma, \mathcal{E}).
\]

**Proof.** Most of the asserted inequalities have already been proven. What needs to be proved here is the equality of the pure ranks of \( (T, \sigma) \) and \( (V, \rho) \). For this we shall analyze the construction of \( (V, \rho) \). Let the Hilbert space of \( (T, \sigma) \) be \( H \) and let the Hilbert space of \( (V, \rho) \) be \( K \) constructed in Section 3. Write the matrix of \( \tilde{V} \) as in equation (7), write \( P \) for the projection of \( K \) onto \( H \), and \( Q \) for the projection \( I - \tilde{V}\tilde{V}^* \). Note that \( Q \in \rho(M)' \). Further, denote the span \( span\{V(\xi)k : \xi \in \mathcal{E}, \ k \in K\} \) by \( L(K) \). Then, for \( \xi \) in \( \mathcal{E} \) and \( k \) in \( K \), we have...
\[ QV(\xi)k = (I - \tilde{V}\tilde{V}^*)\tilde{V}(\xi \otimes k) = 0, \]
since \( \tilde{V} \) is an isometry. Hence \( Q \) vanishes on \( L(K) \). Since \( (V, \rho) \) is minimal, we have \( K = H \vee L(K) \) and, thus, \( Q(H) \) is dense in the range of \( Q \). Setting \( S = QP \) we conclude that the range projection of \( S \) is \( Q \). We now turn to showing that the range projection of \( S^* = PQ \) is equal to the range projection of \( I - \tilde{T}\tilde{T}^* \). For this we use equation (7) to compute

\[
Q = I - \tilde{V}\tilde{V}^* = \begin{pmatrix}
I_H - \tilde{T}\tilde{T}^* & -\tilde{T}\Delta & 0 & \ldots \\
-\Delta\tilde{T}^* & I_D - \Delta^2 & 0 & \\
0 & 0 & \ddots & \\
\vdots & \ddots & \ddots & 
\end{pmatrix}.
\]

(12)

It follows from this that the range of \( PQ \) is the space

\[
\{(I_H - \tilde{T}\tilde{T}^*)h + \tilde{T}\Delta^2(\xi \otimes f) : f, h \in H, \xi \in E\}.
\]

But \( \tilde{T}\Delta^2 = \tilde{T}(I - \tilde{T}^*\tilde{T}) = (I - \tilde{T}\tilde{T}^*)\tilde{T} \). Hence the range of \( PQ \) is contained in the range of \( I - \tilde{T}\tilde{T}^* \). Since the other containment is obvious, the range projection of \( PQ = S^* \) is the range projection of \( I - \tilde{T}\tilde{T}^* \). As the range projections of \( S \) and \( S^* \) are equivalent in \( \rho(M)' \), they have the same trace. More precisely, this argument shows that they have the same trace with respect to \( tr_{\rho(M)'} \). But then Lemma \( \ref{lem:trace} \) shows that the traces in \( \rho(M)' \) and in \( \sigma(M)' \) coincide on the range projection of \( I - \tilde{T}\tilde{T}^* \). □

We conclude this section with a generalization of [12, Theorem 3.4] and [21, Theorem 3.4].

**Theorem 3.7** Let \( M \) be a semifinite factor, let \( E \) be a correspondence over \( M \) with finite left dimension and let \( (T, \sigma) \) be a pure representation of \( E \) with finite pure rank. Then \( (T, \sigma) \) is isometric (hence, necessarily, an induced representation) if and only if \( K(T, \sigma, E) = \text{pure rank}(T, \sigma) \).

**Proof.** If the representation is isometric (and pure) it follows from Proposition \( \ref{prop:isometric} \) that it is induced. It also follows from Proposition \( \ref{prop:purerank} \) that the equality holds.

So we now assume that the equality holds. It follows from Proposition \( \ref{prop:purerank} \) that

\[
K(T, \sigma, E) = tr_{\rho(M)'}(I - \tilde{T}\tilde{T}^*).
\]

The argument in the proof of Theorem \( \ref{thm:isometric} \) shows that this holds only if, for every \( j \geq 1 \),

\[
tr(\Theta^j(I - \Theta(I))) = tr(\Theta^j(I) - \Theta^{j+1}(I)) = d^j \cdot tr(I - \Theta(I))
\]

where \( tr \) stands for \( tr_{\rho(M)'} \), \( \Theta = \Theta_T \) and \( d \) is \( \text{dim}_\text{u}(E) \). Note that the assumption we made that the pure rank of the representation is finite means that the trace of \( I - \Theta(I) \) and, thus, that the trace of \( I - \Theta^j(I) \) is finite for all \( j > 0 \). The computation in the proof of Proposition \( \ref{prop:purerank} \) applied to \( x_j = I - \Theta^j(I) \), now
shows that, letting $c_j$ be $(I - \Theta^j(I))^{1/2}$, the traces of $I \otimes c_j$ and of $(I \otimes c_j)\tilde{T}^*\tilde{T}(I \otimes c_j)$ (both elements of $(\varphi(M) \otimes I)'$) are equal. Since $\tilde{T}$ is a contraction and the two elements have the same finite trace, it follows that the elements are equal. Thus

$$(I \otimes c_j)(I - \tilde{T}^*\tilde{T})(I \otimes c_j) = 0, \quad j > 0.$$ 

Since $\Theta^j(I) \to 0$ in the strong operator topology (as $(T, \sigma)$ was assumed to be pure), we conclude that $c_j \to 0$ in the strong operator topology. Hence $\tilde{T}^*\tilde{T} = I$; i.e., $(T, \sigma)$ is isometric.

4 The Index and Curvature for Completely Positive Maps

We shall now apply the curvature invariant for representations of correspondences to the study of contractive, completely positive maps on semifinite factors. We start by describing how one can associate, to a contractive completely positive map $\Theta$ on a von Neumann algebra $N$, a $W^*$-correspondence $E$ and a representation $(T, \sigma)$ of $E$ such that $\Theta = \Theta_T$, where $\Theta_T$ is the completely positive map associated to the representation $(T, \sigma)$ as in Proposition and Definition 2.11.

The construction we describe was presented in [17] and details can be found there. We call attention to the fact that in [17] we assumed that the completely positive map $\Theta$ under discussion is unital. However, the arguments and conclusions, with minor modifications, only require that $\Theta$ be contractive (as well as normal and completely positive).

So, let $N$ be a von Neumann algebra acting on a Hilbert space $H$ and let $\Theta : N \to N$ be a contractive, normal and completely positive map. View $\Theta$ also as a completely positive map from $N$ into $B(H)$ and write $N \otimes \Theta H$ for the space of its Stinespring dilation (see [23]). Recall that $N \otimes \Theta H$ is obtained from the algebraic tensor product $N \otimes H$ via the process of completion using the sesquilinear form $\langle \cdot, \cdot \rangle$ defined by the formula

$$\langle a_1 \otimes h_1, a_2 \otimes h_2 \rangle = \langle h_1, \Theta(a_1^*a_2)h_2 \rangle,$$

where $a_i \otimes h_i \in N \otimes H$. The Hausdorff completion of $N \otimes H$ is a Hilbert space which will be denoted $N \otimes_{\Theta} H$. The Stinespring representation of $N$ on this space is given by the formula

$$\pi(a)(b \otimes h) = ab \otimes h,$$

for $a \in N$ and $b \otimes h \in N \otimes_{\Theta} H$. Also the formula

$$W_\Theta(h) = I \otimes h, \quad h \in H,$$

defines a bounded operator $W_\Theta$ mapping $H$ into $N \otimes_{\Theta} H$ satisfying the equations $\|W_\Theta\| = \|\Theta(I)\|$ and

$$\Theta(a) = W_\Theta^*\pi(a)W_\Theta, \quad a \in N.$$
One also finds, in particular, that \( W_\Theta^*(a \otimes h) = \Theta(a)h \). The \( W^* \)-correspondence of importance to us is the following one.

**Proposition and Definition 4.1** Let \( N \) be a von Neumann algebra acting on the Hilbert space \( H \) and let \( \Theta \) be a contractive, normal, completely positive map on \( N \). We define \( \mathcal{E}_\Theta \) to be the space of all operators from \( H \) to \( N \otimes \Theta H \) that intertwine the identity representation of \( N \) on \( H \) and the Stinespring representation \( \pi \) just described; i.e.,

\[
\mathcal{E}_\Theta = \{ X : H \to N \otimes \Theta H : Xa = \pi(a)X, \ a \in N \}.
\]

Then \( \mathcal{E}_\Theta \) is a \( W^* \)-correspondence over \( N' \) (the commutant of \( N \) on \( H \)), where the right and the left actions are defined by the formulae

\[
X \cdot b = X \circ b, \ \varphi(b)X = (I \otimes b) \circ X, \ b \in N',
\]

and the \( N' \)-valued inner product is given by the formula

\[
\langle X, Y \rangle = X^* Y, \ X, Y \in \mathcal{E}_\Theta.
\]

We call \( \mathcal{E}_\Theta \) the Arveson-Stinespring correspondence associated to \( \Theta \).

**Proof.** This is proved in [17, Proposition 2.3] under the assumption that \( \Theta \) is unital. However, a moment’s reflection reveals that this assumption plays no material role in the construction of \( \mathcal{E}_\Theta \).

**Definition 4.2** Let \( \Theta \) be a contractive, normal, completely positive map on the von Neumann algebra \( N \) acting on the Hilbert space \( H \) and let \( \mathcal{E}_\Theta \) be the Arveson-Stinespring correspondence associated to \( \Theta \). We define a representation \((T, \sigma)\) of \( \mathcal{E}_\Theta \) on \( H \) by letting \( \sigma \) be the identity representation of \( N' \) on \( H \) and by setting

\[
T(X)h = W_\Theta^* Xh, \ X \in \mathcal{E}_\Theta, \ h \in H.
\]

It is shown in [17, Discussion following Theorem and Definition 2.18] that this is indeed a representation of \( \mathcal{E}_\Theta \), which will be called the identity representation of \( \mathcal{E}_\Theta \).

For future reference, we record the following fact that justifies the terminology. It is essentially [17, Corollary 2.23]. The proof presented there works here as well, and so will be omitted.

**Proposition 4.3** Let \( N \) be a von Neumann algebra acting on a Hilbert space \( H \) and let \( \Theta \) be a contractive, normal, completely positive map acting on \( N \). If \((T, \sigma)\) is the identity representation of the Arveson-Stinespring correspondence \( \mathcal{E}_\Theta \), then \( \Theta = \Theta_T \).

Our next major objective, Theorem 4.8, is to show that the left dimension, \( \dim_l(\mathcal{E}_\Theta) \), of the Arveson-Stinespring correspondence associated with a contractive, normal, completely positive map \( \Theta \) on a semifinite factor \( N \) is independent
of the Hilbert space on which $N$ is represented. Along the way, we shall make some useful auxiliary observations.

First observe that if $N \otimes H$ is the Hilbert space of the Stinespring dilation of a contractive, normal, completely positive map $\Theta$ acting on a von Neumann algebra $N \subseteq B(H)$, and if $a$ is an operator in $N'$, then the operator $I \otimes a$, acting on $N \otimes H$ according to the formula $(I \otimes a)(b \otimes h) = b \otimes ah$ is well defined and bounded, with $\|I \otimes a\| \leq \|a\|$. In fact, the map $a \mapsto I \otimes a$ is a normal representation of $N'$ and we have the following result that identifies this representation with the induced representation $\sigma^{\otimes N} \circ \varphi_{\otimes \Theta}$.

**Lemma 4.4** If $E_\Theta$ is the Arveson-Stinespring correspondence associated with a contractive, normal, completely positive map $\Theta$ on a von Neumann algebra $N$, then the operator $u : E_\Theta \otimes H \rightarrow N \otimes H$ defined by

$$u(X \otimes h) = X(h), \ X \in E_\Theta, \ h \in H$$

is a Hilbert space isomorphism that intertwines the actions of $N'$ on the two spaces; i.e.,

$$u(\sigma^{\otimes \Theta} \circ \varphi_{\otimes \Theta}(a))u^* = u(\varphi_{\otimes \Theta}(a) \otimes I)u^* = I \otimes a, \ a \in N'.$$

**Proof.** We compute:

$$\langle X \otimes h, X \otimes h \rangle = \langle h, X^*Xh \rangle = \langle X(h), X(h) \rangle.$$

Hence $u$ is an isometry. The fact that $u$ is surjective follows from [[17], Lemma 2.10]. The intertwining property is a simple computation:

$$u(\varphi(a) \otimes I)(X \otimes h) = u(\varphi(a)X \otimes h) = (\varphi(a)X)(h) = X \otimes ah$$

$$= (I \otimes a)(X \otimes h) = (I \otimes a)u(X \otimes h)$$

for $X \in E_\Theta$, $h \in H$ and $a \in N'$, where we have written $\varphi$ for $\varphi_{\otimes \Theta}$. ■

Let $C = \{ \begin{pmatrix} a & 0 \\ 0 & I \otimes \Theta a \end{pmatrix} \in B(H \oplus (N \otimes H)) : a \in N' \}$, a subalgebra of $B(H \oplus (N \otimes H))$ that is isomorphic to $N'$. Let $(T, \sigma)$ be the identity representation of $E_\Theta$ (see [(13)]) and let $B$ be the algebra defined in formula [(1)], associated with this representation (with $N'$ playing the role of $M$ there). Then the operator $U = I \oplus u$ implements an isomorphism between $C$ and $B$. Explicitly, we have

$$U \begin{pmatrix} a & 0 \\ 0 & \varphi(a) \otimes I_H \end{pmatrix} U^* = \begin{pmatrix} a & 0 \\ 0 & I \otimes \Theta a \end{pmatrix}$$

for $a$ in $N'$. (We suppressed the $\sigma$ in this formula.) Recall that $\tilde{T}$ is the map from $E_\Theta \otimes H$ to $H$ defined by $\tilde{T}(X \otimes h) = \tilde{T}(X)h = W^*_\Theta X(h)$. We shall write $S$ for the map $S = \tilde{T}u^*$ and then we have,

$$Su(X \otimes h) = \tilde{T}(X \otimes h) = W^*_\Theta X(h)$$

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and, if \( X(h) \) may be expressed as \( X(h) = b \otimes h \), then,

\[
S(b \otimes h) = W_\Theta^*(b \otimes h) = \Theta(b)h.
\]

Also,

\[
U \begin{pmatrix} 0 & \hat{T} \\ 0 & 0 \end{pmatrix} U^* = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}.
\] (15)

In particular, the matrix in the right hand side of this equation lies in \( C' \).

The proof of Theorem 4.8, which shows that the left dimension of the Arveson-Stinespring correspondence associated with a completely positive map on a semifinite factor is independent of the Hilbert space on which the factor is represented, will involve a few steps. We start by fixing a (faithful, normal) representation \( \pi \) of the semifinite factor \( N \) (on which the normal, contractive, completely positive map \( \Theta \) is defined) and we write \( E \) for \( E_\Theta \). We also write \( H_\infty \) for the direct sum of infinitely many copies of \( H \) and \( \pi_\infty \) for infinite ampliation of \( \pi \) acting on \( H_\infty \). (When it is clear in the discussion to follow which representation we have in mind, we will suppress \( \pi \) or \( \pi_\infty \).) We shall write \( \pi_\infty(N) \)' is isomorphic to \( M_\infty(\pi(N)) \)' (i.e., \( \pi(N) \otimes B(l^2) \)) and \( \sigma_\infty \) is its identity representation on \( H_\infty \). Our aim is to show that \( \dim_l(E) = \dim_l(E_\infty) \).

Let \( v \) be the map from \( N \otimes_\Theta H_\infty \) to \( (N \otimes_\Theta H_\infty)_\infty \) defined by the formula

\[
v(b \otimes (h_i)) = (b \otimes h_i),
\]

\( b \otimes (h_i) \in N \otimes_\Theta H_\infty \). Then \( v \) evidently is a Hilbert space isomorphism. We write \( V \) for the Hilbert space isomorphism \( V = I \oplus v \) mapping \( H_\infty \oplus (N \otimes_\Theta H_\infty) \) to \( H_\infty \oplus (N \otimes_\Theta H_\infty)_\infty \). Given an operator \( R \) from one Hilbert space to another, we write \( R(\infty) \) for its infinite ampliation.

The following lemma is easy to check and so we omit the proof.

**Lemma 4.5** With \( V \) as above we have,

\[
VC_\infty V^* = M_\infty(C),
\]

and

\[
VC_\infty' V^* = M_\infty(C)' \cong C' \otimes I_\infty \cong C'.
\]
We shall identify $M_\infty(C)'$ with $C' \otimes I_\infty$, write $\Psi$ for the isomorphism of $C'$ onto $C' \otimes I_\infty$ and write $\Phi$ for the isomorphism of $C'$ onto $C'_\infty$ defined by

$$\Phi(R) = V^* \Psi(R)V, \quad R \in C'.$$

**Lemma 4.6** For $b \in N$, $S$ defined above and for $D = (I - S^*S)^{1/2}$ and $D_\infty = (I - S^*_\infty S_\infty)^{1/2}$, we have the following equations:

1. $\Phi\left( \begin{pmatrix} \pi(b) & 0 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} \pi(\infty)(b) & 0 \\ 0 & 0 \end{pmatrix}$, $b \in N$,
2. $\Phi\left( \begin{pmatrix} 0 & 0 \\ 0 & b \otimes I_H \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & b \otimes I_{H_\infty} \end{pmatrix}$, $b \in N$,
3. $\Phi\left( \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & S_\infty \\ 0 & 0 \end{pmatrix}$, and
4. $\Phi\left( \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & D_\infty \end{pmatrix}$.

**Proof.** The proof of equations (1) and (2) is straightforward. For equation (3), write $S(\infty)$ for the diagonal map from $(N \otimes_H H_\infty)$ to $H_\infty$ induced by $S$ and compute, for $h = (h_i)$ in $H_\infty$ and $b \in N$,

$$S(\infty)v(b \otimes h) = S(\infty)((b \otimes h_i)) = (S(b \otimes h_i)) = (\pi(\Theta(b))h_i) = \pi(\Theta(b))h = S_\infty(b \otimes h).$$

This proves equation (3). Equation (4) follows immediately from equation (3).

**Lemma 4.7** Let $N$ and $\Theta$ be as above. Fix a (faithful, normal) representation $\pi$ of $N$ on $H$ with $M = \pi(N)'$ finite. Let $E = (E_H)$, $C$, $S$ and $D = (I - S^*S)^{1/2}$ be as above and let $tr$ be any (normal, faithful, semifinite) trace on $C'$. Then, for every positive $b$ in $N$ with finite trace (with respect to a semifinite trace on $N$),

$$\dim(E) = \frac{tr\left( \begin{pmatrix} \Theta(b) & 0 \\ 0 & 0 \end{pmatrix} \right) + tr\left( \begin{pmatrix} 0 & 0 \\ 0 & D(b \otimes I_H)D \end{pmatrix} \right)}{tr\left( \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \right)}.$$

**Proof.** The lemma follows from Remark 2.14 using equations (14), (15) and (16) (to translate from $B$ to $C$).
Theorem 4.8 Let $\Theta$ be a normal, contractive, completely positive map on the semifinite factor $N$. Let $\pi_1$ and $\pi_2$ be two faithful, normal $*$-representations of $N$ on Hilbert spaces $H_1$ and $H_2$ respectively such that both $\pi_1(N)'$ and $\pi_2(N)'$ are finite. Write $E_1$ and $E_2$ for the $W^*$-correspondences associated to $\Theta$ and the representations $\pi_1$ and $\pi_2$ respectively. Then

$$\dim_l(E_1) = \dim_l(E_2).$$

Proof. Using Lemma 4.7, it suffices to show that the quotient on the right hand side of equation (17) is the same for $\pi_1$ and for $\pi_2$. We shall show this for every $\pi_1$, $\pi_2$ (regardless of whether the commutants of the images of these representations are finite or not). So we now drop the assumption on the commutants.

If the two representations are equivalent then the conclusion of the theorem certainly holds. This is the case if $\pi_1(N)'$ and $\pi_2(N)'$ are both infinite ([11, Exercises 9.6.30 and 9.6.31]).

Thus, it suffices to consider only the case where $\pi_2$ is $\pi_1,\infty$. So we now write $\pi$ for $\pi_1$ and $\pi_\infty$ for $\pi_2$. But then the analysis above (Lemma 4.5 and Lemma 4.6) is applicable and the equality of the expressions on the right hand side of equation (17) follows from Lemma 4.6.

Definition 4.9 Let $N$ be a semifinite factor and let $\Theta$ be contractive, normal, completely positive map on $N$. Then we define the index of $\Theta$ to be the left dimension of the Arveson-Stinespring correspondence associated with any (necessarily faithful) normal representation of $N$ having finite commutant. We denote the index of $\Theta$ by $d(\Theta)$.

Theorem 4.8 shows that our notion of “index” is well defined. In [11] Arveson defined a notion of index for continuous semigroups of normal, contractive completely positive maps on $B(H)$. Our definition here seems to be the analogous one for a single map on an arbitrary semifinite factor.

The following lemma gives a convenient computation of the index.

Lemma 4.10 Let $\Theta$ be a contractive, normal completely positive map on the semifinite factor $N$. Fix a semifinite, normal, faithful trace $tr$ on $N$ and a projection $e$ in $N$ with $tr(e) = 1$ (if $N$ is finite we can assume that $tr$ is normalized and $e = 1$). Write $L^2(N)$ for $L^2(N, tr)$ and denote by $\lambda$ and $\rho$ the left and right actions of $N$ on $L^2(N)$. Let $H$ be $\rho(e)L^2(N)$ (to be written $L^2(N,e)$) and let $M$ be the algebra $\rho(eN,e)$. Consider the Hilbert space $eN \otimes_H H$ (a subspace of $N \otimes_H H$) on which $M$ acts according to the equation

$$\rho(eae) \cdot (eb \otimes h) = eb \otimes \rho(eae)h, \quad a, b \in N, \quad h \in H. \quad (18)$$

Then

$$d(\Theta) = \dim_M(eN \otimes H).$$
Proof. Let $\pi$ be the restriction of the representation $\lambda$ of $N$ on $L^2(N)$ to $H$. Then $M = \pi(N)'$ and it is a finite factor. The operator $u$ of Lemma 4.4 maps $E_\Theta \otimes H$ onto $N \otimes_\Theta H$. Since $L^2(M)$ can be identified with $\lambda(e)L^2(Ne)$, the space $E_\Theta \otimes L^2(M)$ is $(I_E \otimes \lambda(e))(E_\Theta \otimes H)$. As $u(I_E \otimes \lambda(e))u^* = e \otimes I$ (see equation (14)), $u$ is a unitary operator mapping $E_\Theta \otimes L^2(M)$ onto $eN \otimes_\Theta H$. It follows from Lemma 4.4 that it is a module isomorphisms (when the latter space is viewed as a module over $M$ as in equation (18)). Thus they have the same dimension over $M$. \qed

Proposition 4.11 Let $N$ be a semifinite factor, let $t_1, t_2, \ldots t_n$ be elements of $N$ such that $\sum t_i t_i^* \leq I$, and $\Theta$ be the contractive, normal, completely positive map on $N$ defined by the formula

$$\Theta(a) = \sum_{i=1}^n t_i a t_i^*. $$

Then

$$d(\Theta) = \dim_{\mathbb{C}} \text{span}\{t_i : 1 \leq i \leq n\} \in \{0, 1, \ldots n\}$$

where $\dim_{\mathbb{C}}$ is the dimension as a complex vector space.

Proof. By Lemma 4.10 we have (once we fix a projection $e$ with trace equal to 1),

$$d(\Theta) = \dim_{\mathbb{C}} \rho(eN) \otimes L^2(Ne).$$

Define the Hilbert space $K$ by the formula

$$K = \text{span}\{\sum_i \oplus eat_i^* : a, b \in N \} \subseteq \sum_i \oplus L^2(eNe).$$

In fact, $\sum \oplus L^2(eNe)$ is a module over $M = \rho(eNe)$ and $K$ is a submodule. Define a linear operator $w : eN \otimes_\Theta L^2(Ne) \to K$ by the formula $w(ea \otimes be) = \sum \oplus eat_i^* be$. This defines $w$ on a dense subspace and we compute, for $a, b, c$ and $f$ in $N$,

$$\langle \sum \oplus eat_i^* be, \sum \oplus ect_i^* fe \rangle = \sum \langle eat_i^* be, e^*ct_i^* fe \rangle = \sum \langle be, t_i a^* e^*ct_i^* fe \rangle$$

$$= \langle be, \Theta(a^* ec)fe \rangle = \langle ea \otimes_\Theta be, ec \otimes_\Theta fe \rangle.$$

Hence $w$ can be extended to a unitary operator (also denoted $w$). It is also a module isomorphism and, thus, we have $d(\Theta) = \dim_M(K)$. Now write $Q$ for the projection, in $B(\sum \oplus L^2(eNe))$, whose range is $K$. Identifying $B(\sum \oplus L^2(eN))$ with $B(L^2(eNe)) \otimes M_n(\mathbb{C})$, we see that $Q$ lies in the commutant of both $\lambda(eNe) \otimes I$ and $\rho(eNe) \otimes I$. Hence $Q = I \otimes q$ for some projection $q$ in $M_n(\mathbb{C})$. Clearly, $d(\Theta) = \dim(q)$ and, in particular, $d(\Theta)$ is an integer between 0 and $n$. Write

$\ldots$
Thus \( D \otimes N \) of \( S \).

For every choice of \( l \) and \( p \), the vector \( \sum_l \otimes w_l t^*_l w^*_p \) lies in \( K \). We shall write this vector as a column vector \((w_l t^*_l w^*_p)\) and then we have \((I - q)(w_l t^*_l w^*_p) = 0\). Since \( t^*_l = \sum_{i,p} w^*_i (w_l t^*_l w^*_p) w_p \), we find that the (column) vector \((t^*_l)\) satisfies \((I - q)(t^*_l) = 0\). As \( \text{rank}(I - q) = n - k \), we conclude that \( m (= \dim \text{span}\{t^*_l\}) \leq k \). For the other inequality note that since \( \dim \text{span}\{t^*_l\} = m \), we can find an \((n - m) \times n\) matrix \( A \) (over \( \mathbb{C} \)) with \( \text{rank}A = n - m \) such that \( A(t^*_l) = 0 \). But then \( K \) lies in the kernel of \( A \otimes I \) and, consequently, \( Aq = 0 \). We conclude that \( m = n - (n - m) = \dim \text{Ker}A \geq \dim q = k \), and, therefore, that \( m = k \).

In the next proposition, \( D \) is the operator on \( N \otimes \Theta_H \) described in Lemma 4.7.

**Proposition 4.12** Let \( \Theta \) be a completely positive map with finite index acting on a semifinite factor \( N \) and let \( x \) be a positive element of \( N \) with finite trace. Then the following assertions are equivalent.

1. \( \text{tr}(\Theta(x)) = d(\Theta) \cdot \text{tr}(x) \).
2. \( D(x \otimes I_H)D = 0 \).
3. The element \( c = x^{1/2} \) satisfies the equation,

\[
\Theta(ac)\Theta(cb) = \Theta(ac^2b),
\]

for all \( a \) and \( b \) in \( N \).

Moreover, if any of these conditions holds for \( x \) then they hold for all positive \( y \) in \( xNx \) (in particular, they hold for all \( y \in N \) such that \( 0 \leq y \leq x \)).

**Proof.** The equivalence of parts (1) and (2) follows from Lemma 4.7. Suppose (2) holds and write \( c = x^{1/2} \). Then \((c \otimes I)D = 0\) and, thus, \((c \otimes I)D^2(c \otimes I) = 0\). Since \( D^2 = I - S^*S \), we have \((c \otimes I)S^*S(c \otimes I) = c^2 \otimes I \) and

\[
S(ac \otimes I)S^*S(\overline{c}b \otimes I)S^* = S(ac^2b \otimes I)S^*
\]

for all \( a \) and \( b \) in \( N \). Recall that \( S \) is the operator from \( N \otimes \Theta H \) to \( H \) (for a fixed representation of \( N \) on \( H \)) defined by \( S(b \otimes h) = \Theta(b)h \), \( b \in N \), \( h \in H \).

It is easy to check that \( S^*h = I \otimes h \) for \( h \) in \( H \). Thus, for \( y \) in \( N \),

\[
S(y \otimes I)S^*h = S(y \otimes I)(I \otimes h) = \Theta(y)h , \quad h \in H.
\]

Thus, \( S(y \otimes I)S^* = \Theta(y) \) and this completes the proof of (3). For the other direction, condition (3) implies that \( S(a^*c \otimes I)D^2(c \otimes I)S^* = 0 \) for all \( a \in N \). Thus \( D(c \otimes I)(a \otimes I)S^*S = 0 \) for all \( a \in N \). Note that the range of \( S^*S \) is \( I \otimes [\Theta(N)(H)] \) and, in order to prove (2), it suffices to show that the subspace of \( N \otimes \Theta_H \) spanned by elements of the form \((a \otimes I)(I \otimes \Theta(b)k)\) for \( a \) and \( b \) in
\[ \text{and} \quad k \text{ in } H \text{ is dense. Suppose } \sum n_i \otimes h_i \text{ in } N \otimes H \text{ is orthogonal to this subspace. Then} \]
\[ 0 = \sum \langle n_i \otimes h_i, a \otimes \Theta(b)k \rangle = \sum \langle \Theta(a^*n_i)h_i, \Theta(b)k \rangle. \]
Since this holds for all \( b \in N \) and \( k \in H \),
\[ \sum \Theta(a^*n_i)h_i = 0. \]
\[ \text{From this we conclude that } \sum n_i \otimes h_i \text{ is orthogonal to } a \otimes h \text{ for all } a \in N \text{ and } h \in H; \text{ i.e., it is equal to } 0. \text{ This shows that the subspace above is indeed dense and completes the proof of } (2). \]

The last statement of the proposition is clear by considering condition (2).

We now can define our principle object of study, the curvature of a completely positive map.

**Definition 4.13** Let \( N \) be a semifinite factor with faithful, normal semifinite trace, \( tr \), and let \( \Theta \) be a contractive, normal completely positive map on \( N \) with finite index \( d = d(\Theta) \). We define the curvature of \((\Theta, tr)\), \( K(\Theta, tr) \), by the formula
\[ K(\Theta, tr) = \lim_{k \to \infty} \frac{tr(I - \Theta^k(I))}{\sum_{j=0}^{k-1} d^j}. \]

Theorem 4.8 and Theorem 3.3 guaranty that the curvature is well defined. Of course, the statements of Theorem 3.3 have obvious analogues for \( K(\Theta, tr) \).

Observe that if \( \Theta \) is a contractive, normal, completely positive map on a von Neumann algebra, then the sequence \( \{\Theta^k(I)\}_{k \geq 0} \) is a decreasing sequence of positive operators and so converges to a positive operator in the strong operator topology. The following terminology is consistent with Definition 2.17 and will be useful in the sequel.

**Definition 4.14** Let \( \Theta \) be a contractive, normal, completely positive map on a von Neumann algebra. We write \( \Theta^\infty(I) \) for the limit of \( \{\Theta^k(I)\}_{k \geq 0} \) in the strong operator topology. Then \( \Theta \) will be called full if \( \Theta^\infty(I) = I \) and pure if \( \Theta^\infty(I) = 0 \).

**Theorem 4.15** Let \( \Theta \) be a normal, contractive, completely positive map with finite index acting on a semifinite factor \( N \). Then
\[ \text{(1) } K(\Theta, tr) \leq tr(I - \Theta(I)). \]
\[ \text{(2) } K(\Theta, tr) = \infty \text{ if and only if } tr(I - \Theta(I)) = \infty. \]
Suppose $\text{tr}(I - \Theta(I)) < \infty$. Then $K(\Theta, \text{tr}) = \text{tr}(I - \Theta(I))$ if and only if, for all $a$ and $b$ in $N$,

$$\Theta(ac_\infty)\Theta(c_\infty b) = \Theta(ac^2_\infty b)$$

where $c_\infty = (I - \Theta^\infty(I))^{1/2}$. In particular, if $\Theta$ is pure, then the equality holds in the first assertion, (1), if and only if $\Theta$ is a $*$-endomorphism.

Proof. Assertion (1) follows from Lemma 3.4 and the second assertion follows from assertion (2) of Theorem 3.3. Now assume that $\text{tr}(I - \Theta(I))$ is finite. For every $k$ write $x_k = \Theta^{k-1}(I) - \Theta^k(I)$. Then $c^2_\infty = \sum_{k=1}^{\infty} x_k$ (where the convergence is in the strong operator topology). The condition

$$\Theta(ac_\infty)\Theta(c_\infty b) = \Theta(ac^2_\infty b)$$

for all $a$ and $b$ in $N$, is equivalent to condition (2) of Proposition 4.12 for $x = c^2_\infty$. But this holds if and only if it holds for all $x_k$. Using that proposition, this is equivalent to the condition

$$\text{tr}(\Theta(x_k)) = d(\Theta) \cdot \text{tr}(x_k), \quad k \geq 1.$$

Applying part (2) of Lemma 3.4 we find that the latter condition is equivalent to $K(\Theta, \text{tr}) = \text{tr}(I - \Theta(I))$.

Theorem 4.15 suggests that, for non unital maps, it is better to “normalize” $K$ and study the normalized curvature of a completely positive map.

Definition 4.16 Let $\Theta$ be a normal, contractive completely positive non unital map acting on a semifinite factor $N$ and assume that $\Theta$ has finite index and finite curvature. Then the normalized curvature of $\Theta$, denoted $K_1(\Theta)$, is to be

$$K_1(\Theta) = K(\Theta, \text{tr})/\text{tr}(I - \Theta(I))$$

where $\text{tr}$ is any semifinite, normal, faithful trace on $N$.

With the definition of $K_1$ in hand, the following corollary of Theorem 4.15 is immediate.

Corollary 4.17 Let $\Theta$ be a normal $*$-endomorphism of $N$ with finite index. Then

1. For all positive $x$ in $N$, $\text{tr}(\Theta(x)) = d(\Theta) \cdot \text{tr}(x)$.
2. $K(\Theta, \text{tr}) = \text{tr}(I - \Theta(I))$; hence either $\Theta(I) = I$ or $K_1(\Theta) = 1$.
3. If $\Theta_1$ is another normal $*$-endomorphism of $N$ with finite index, then $d(\Theta \circ \Theta_1) = d(\Theta)d(\Theta_1)$.

Specializing still further, we have the following corollary, in which $\text{mod}(\alpha)$ denotes the modulus of an automorphism $\alpha$ of a type $II_1$ factor (See [11].)
Corollary 4.18 Let $N$ be a factor of type $II_1$ with normalized trace $\tau$, and let $\alpha$ be a normal $*$-automorphism of $N$. Then $d(\alpha) = \text{mod}(\alpha)$ and $K(\alpha, \tau) = 0$.

Example 4.19 Let $N$ be a type $II_1$ factor and $p$ be a projection such that $pNp$ is isomorphic to $N$. Write $\Theta : N \to pNp$ for this isomorphism and view $\Theta$ as an endomorphism on $N$. Then $d(\Theta) = \tau(p)$ and $K(\Theta, \tau) = 1 - \tau(p)$, where $\tau$ is the normalized trace on $N$.

This follows easily from Corollary 4.17 since $\Theta(I) = p$.

Example 4.20 Let $N$ be a type $II_1$ factor with normalized trace $\tau$ and let $N_0 \subseteq N$ be a subfactor of finite index, $[N : N_0]$. If $E$ is the (trace preserving) conditional expectation from $N$ onto $N_0$, then $d(E) = [N : N_0]$ and $K(E, \tau) = 0$.

Proof. The statement about $K$ is obvious since the map is unital. As for its index, recall that, using Lemma 4.10, we have

$$d(E) = \dim_{\rho(N)}(N \otimes_E L^2(N)).$$

Let $e_0 = e_{N_0}$ be the projection of $L^2(N, \tau)$ onto $L^2(N_0, \tau)$ (extending the map $E$). Let $N_1$ be the von Neumann algebra generated by $N$ and $e_0$. It is the next algebra in the basic construction of Jones. Write $\tau_1$ for the normalized trace of $N_1$. Then $\tau_1(xe_0) = [N : N_0]^{-1}\tau(x)$ for all $x \in N$ ([Pi, Proposition 3.1.2]). We define a linear map $S : N \otimes_E L^2(N, \tau) \to L^2(N_1, \tau_1)$ by $S(a \otimes_E b) = ae_0b$ for $a, b$ in $N$, and compute: For $a, b, c$ and $d$ in $N$,

$$\langle a \otimes_E b, c \otimes_E d \rangle = \langle b, E(a^*c)d \rangle = \tau(b^* E(a^*c)d) = \tau(db^* E(a^*c))$$

$$= \tau_1(db^* E(a^*c)e_0)[N : N_0] = \tau_1(db^*e_0a^*ce_0)[N : N_0]$$

$$= \tau_1(b^*e_0a^*ce_0d^*)[N : N_0] = [N : N_0](ae_0b, ce_0d)_{L^2(N_1, \tau_1)}.$$ 

Hence the map $V = [N : N_0]^{-1/2}S$ extends to a Hilbert space isomorphism from $N \otimes_E L^2(N, \tau)$ onto $L^2(N_1, \tau_1)$. (The surjectivity of $V$ uses [Pi, 2.6(d)]). In fact, $V$ intertwines the actions of $\rho(N)$:

$$V(\rho(c)(a \otimes_E b)) = V(ax \otimes_E b) = [N : N_0]^{-1/2}ae_0bc = \rho(c)V(a \otimes_E b)$$

for $a, b$ and $c$ in $N$. Thus, if we write $N_2$ for the next algebra in Jones’s basic construction (containing $N_1$), we have

$$d(E) = \dim_{\rho(N)}L^2(N_1, \tau_1) = \dim_{N_2}L^2(N_1, \tau_1) = [N_1'] [N_2'] \dim_{N_1'} L^2(N_1, \tau_1)$$

$$= [N_1' : N_2'] [N_2 : N_1] = [N : N_0].$$
In the computation above we used the fact that, on $L^2(N_1, \tau_1)$, $N'_2 = \rho(N)$ (see Proposition 3.1.2 (ii) of [10]) and the equality $\dim N'_2 H = [N'_1 : N'_2] \dim N'_1 H$, which holds for every $N'_1$-module $H$. ■

In order to show that $d(\Theta)$ is an outer conjugacy invariant we need first to recall some results from [17]. Let $\Theta_1$ and $\Theta_2$ be two normal, contractive, completely positive maps on $N$ and let $H$ be a left $N$-module. Recall the definition of $E_{\Theta_i}$: $E_{\Theta_i} = \{ X : H \to N \otimes_{\Theta_i} H : Xa = (a \otimes I)X, \ a \in N \}$

for $i = 1, 2$. We can also define the space $N \otimes_{\Theta_1} N \otimes_{\Theta_2} \otimes H$ as the Hausdorff completion of the space we get when we define the inner product

$(a \otimes_{\Theta_1} b \otimes_{\Theta_2} h, c \otimes_{\Theta_1} d \otimes_{\Theta_2} k) = (h, \Theta_2(b^* \Theta_1(a^*c)d)k)$

on the algebraic tensor product. We now set

$\mathcal{E} = \{ X : H \to N \otimes_{\Theta_1} N \otimes_{\Theta_2} H : Xa = (a \otimes I \otimes I)X, \ a \in N \}$.

Define the map $\Psi : E_{\Theta_2} \otimes E_{\Theta_1} \to \mathcal{E}$ by

$\Psi(X \otimes Y) = (I \otimes X)Y$

where $I \otimes X$ is the map from $N \otimes_{\Theta_2} H$ to $N \otimes_{\Theta_1} N \otimes_{\Theta_2} H$ given by the equation $(I \otimes X)(a \otimes h) = a \otimes Xh$. We also define a map $V_0 : N \otimes_{\Theta_2 \Theta_1} H \to N \otimes_{\Theta_1} N \otimes_{\Theta_2} H$ via the equation $V_0(a \otimes h) = a \otimes I \otimes h$ and we let $V : E_{\Theta_2 \Theta_1} \to \mathcal{E}$ be the map defined by $V(X) = V_0X$.

The following proposition can be found in [17, Propositions 2.12 and 2.14]. It was presented there under the assumption that the maps $\Theta_1$ and $\Theta_2$ are unital, but the proof given holds without this assumption.

**Proposition 4.21** In the notation just established, we have the following.

1. The map $\Psi$ is an isomorphism of correspondences.
2. The map $V$ is an isometric bimodule map.
3. The map $m = V^* \Psi$ is a coisometry and $m^* : E_{\Theta_2 \Theta_1} \to E_{\Theta_2} \otimes E_{\Theta_1}$ is an isometric bimodule map.
4. If either $\Theta_2$ is an endomorphism (of $N$) or $\Theta_1$ is an automorphism, then the map $m$ is an isomorphism of correspondences. Hence, in either case,

$$E_{\Theta_2 \Theta_1} \simeq E_{\Theta_2} \otimes E_{\Theta_1}.$$
Proposition 4.22 For two normal, contractive, completely positive maps \( \Theta_1 \) and \( \Theta_2 \) on a semifinite factor \( N \), we have

\[
d(\Theta_1 \Theta_2) \leq d(\Theta_1)d(\Theta_2).
\]

We also obtain

Theorem 4.23 Let \( \Theta_1 \) and \( \Theta_2 \) be two normal, contractive, completely positive maps on a semifinite factor \( N \).

1. If the maps are outer conjugate (i.e., if there is an automorphism \( \alpha \) of \( N \) and a unitary operator \( u \) in \( N \) such that \( \text{ad}(u) \circ \Theta_1 = \alpha^{-1} \circ \Theta_2 \circ \alpha \)), then \( d(\Theta_1) = d(\Theta_2) \). In particular, one has a finite index if and only if the other one has.

2. If they both have a finite index and they are conjugate (i.e., if \( \Theta_1 = \alpha^{-1} \circ \Theta_2 \circ \alpha \) for some automorphism \( \alpha \) on \( N \)), where \( \text{tr} \) is any faithful normal trace on \( N \) and where, recall, \( d(\alpha) \) is the index of \( \alpha \). Further, if the curvature is finite and the maps are non unital, then \( K_1(\Theta_1) = K_1(\Theta_2) \).

Proof. If the maps are outer conjugate and we write \( \beta \) for \( \text{ad}(u) \), it follows from Proposition 4.21 that

\[
\mathcal{E}_\alpha \otimes \mathcal{E}_\beta \otimes \mathcal{E}_{\Theta_1} \simeq \mathcal{E}_{\Theta_2} \otimes \mathcal{E}_\alpha.
\]

Using Corollary 2.8 we find that

\[
d(\alpha)d(\beta)d(\Theta_1) = d(\Theta_2)d(\alpha)
\]

and, since \( d(\beta) = 1 \) (by Example 4.11), the assertions in part (1) follow. The assertions in part (2) now result from the following computation.

\[
\text{tr}(I - \Theta_2^k(I)) = \text{tr}(I - \alpha \circ \Theta_2^k(\alpha^{-1}(I))) = \text{tr}(\alpha(I - \Theta_1^k(I))) = d(\alpha)\text{tr}(I - \Theta_1^k(I)).
\]

Given a completely positive map \( \Theta \) on \( N \), we showed in [17] that we can construct a “dilation” of \( \Theta \) to a \(*\)-endomorphism \( \alpha \) on another von Neumann algebra \( R \) in which \( N \) “sits” as a corner \( R \). We turn to describing this construction and relating the invariants of \( \Theta \) to those of \( \alpha \). So, let \( \Theta \) be a contractive, normal, completely positive map on the semifinite factor \( N \) and let \( N \) act on the Hilbert space \( H \). Form the Arveson-Stinespring correspondence \( E_{\Theta} \) and its identity representation \( (T, \sigma) \) and recall that \( \Theta = \Theta_T \) by Proposition 2.13. Let \((V, \rho)\) be the minimal isometric dilation of \((T, \sigma)\) and write \( K \) for its representation space (see the discussion preceding Definition 2.15). We let \( R \) be \( \rho(N')' \) and we let \( \alpha \) be the endomorphism of \( R \), \( \Theta_V \). If we write \( W \) for the isometric embedding of \( H \) onto \( K \), then \( N \) is \( W^*RW \) and \( \alpha \) is a “dilation” of \( \Theta \) in the sense portrayed in the following proposition which can be found in [17, Theorem 2.24].
Proposition 4.24  With the notation just established, we may assert that:

(1) \( W^*RW = N \) and \( R' \) is a normal homomorphic image of \( N' \).

(2) \( \alpha \) is a normal \( * \)-endomorphism of \( R \).

(3) For every non-negative integer \( n \),

\[
\Theta^n(a) = W^*\alpha^n(WaW^*)W
\]

and

\[
\Theta^n(W^*bW) = W^*\alpha(b)W
\]

for all \( a \in N \) and \( b \in R \).

It should be noted that this proposition was proved in [17] under the assumption that \( \Theta \) is unital and the proof presented there seems to use this fact by virtue of [17, Theorem 2.20], which requires that \( \Theta \) be unital. However, one can use instead [17, Theorem and Definition 2.18] directly, which does not require that \( \Theta \) be unital. The dilation endomorphism \( \alpha \) that is produced, when \( \Theta \) is not unital, is not unital either.

Theorem 4.25  Let \( \Theta \) be a contractive, normal, completely positive map on the semifinite factor \( N \) and let \( \alpha \) be the dilation of \( \Theta \) constructed above from the identity representation \((T, \sigma)\) of the Arveson-Stinespring correspondence \( \mathcal{E}_\Theta \) associated with \( \Theta \).

(1) If \( \Theta \) has finite index, then so does \( \alpha \) and \( d(\Theta) = d(\alpha) \).

(2) If \( d(\Theta) < \infty \), then \( K(\alpha, \text{tr}_{\rho(N')}) \geq K(\Theta, \text{tr}_{\sigma(N')}) \).

(3) If \( d(\Theta) < \infty \) and \( \Theta \) is pure, then \( K(\alpha, \text{tr}_{\rho(N')}) = \text{tr}_{\rho(N')}(I - \alpha(I)) \).

(4) If \( d(\Theta) < \infty \), if \( \Theta \) is pure and if \( K(\alpha, \text{tr}_{\rho(N')}) = K(\Theta, \text{tr}_{\sigma(N')}) \), then \( \Theta = \alpha \).

Proof. Write \( \mathcal{E} \) for \( \mathcal{E}_\Theta \) and \( \mathcal{E}_V \) for \( \mathcal{E}_{\Theta_V} = \mathcal{E}_\alpha \) (where \((V, \rho)\) is the isometric representation of \( \mathcal{E} \) dilating \((T, \sigma)\)). In order to prove that \( d(\alpha) = d(\Theta) \) note that \( \mathcal{E} \) is a \( W^* \)-correspondence over \( N' \) while \( \mathcal{E}_V \) is a \( W^* \)-correspondence over \( \rho(N') \). We shall construct a map \( U : \mathcal{E} \to \mathcal{E}_V \) satisfying

(i) \( U \) is a linear bijection,

(ii) it is a bimodule map in the sense that, for every \( X \) in \( \mathcal{E} \) and \( a, b \) in \( N' \),

\[
U(aXb) = \rho(a)U(X)\rho(b)
\]

and,
(iii) $\langle U(X_1), U(X_2) \rangle = \rho(\langle X_1, X_2 \rangle)$.

Once we construct such a map, assertion (1) will follow. To define $U$ we let $u : R \otimes_{\alpha} K \to K$ be the linear map defined by the equation $u(a \otimes k) = \alpha(a)k$. It is easy to check that $u$ is an isometry onto $[\alpha(R)K]$. Also note that, for every $b \otimes g$ in $R \otimes_{\alpha} K$ and every $k$ in $K$, we have $\langle u^*k, b \otimes g \rangle = \langle k, \alpha(b)g \rangle = \langle I \otimes k, b \otimes g \rangle$; hence $u^*k = I \otimes k$. It also follows from this that $u^*\rho(a) = (I \otimes \rho(a))u^*$ for $a$ in $N'$. Now set

$$U(X) = u^*V(X), \quad X \in \mathcal{E}.$$  

For $X$ in $\mathcal{E}$, $U(X)$ is a map from $K$ to $R \otimes_{\alpha} K$. For $a$ in $R$ and $k$ in $K$, we have

$$U(X)ak = u^*V(X)ak = u^*\tilde{V}(X \otimes ak) = u^*\tilde{V}(I \otimes a)\tilde{V}^*(X \otimes k)$$

$$= u^*\alpha(a)V(X)k = a \otimes V(X)k = (a \otimes I)(I \otimes V(X)k)$$

$$= (a \otimes I)u^*V(X)k = (a \otimes I)U(X)k.$$  

Hence $U$ maps $\mathcal{E}$ into $\mathcal{E}_V$.

For $a, b$ in $N'$ and $X$ in $\mathcal{E}$ we have

$$U(axb) = u^*V(axb) = u^*\rho(a)V(X)\rho(b)$$

$$= (I \otimes \rho(a))u^*V(X)\rho(b) = \rho(a) \cdot U(X)\rho(b).$$  

Thus (ii) holds. To prove (iii) we compute (using the fact that $uu^* = \alpha(I) = \tilde{V}V^*$)

$$\langle U(X_1), U(X_2) \rangle = U(X_1)^*U(X_2) = V(X_1)^*uu^*V(X_2)$$

$$= V(X_1)^*V(X_2) = \rho(\langle X_1, X_2 \rangle).$$  

This proves (iii). It is now clear that $U$ is injective and it is left to prove surjectivity. For this assume that $Y$ in $\mathcal{E}_V$ is orthogonal to the range of $U$. Then $Y^*u^*V(X)k = 0$ for all $k$ in $K$ and $X$ in $\mathcal{E}$. But the vectors of the form $V(X)k = \tilde{V}(X \otimes k)$ span a dense subspace of the range of $\tilde{V}$ which is equal to the range of $u$. Since $u$ is an isometry, vectors of the form $u^*V(X)k$ span a dense subspace of $K$. Thus $Y = 0$ and, since $\mathcal{E}_V$ is self dual, this implies that $U$ is surjective completing the proof of (i), (ii) and (iii). Thus $d(\Theta) = dim(\mathcal{E}) = dim(\mathcal{E}_V) = d(\alpha)$. Assertion (2) now follows from Proposition 3.6. Assertions (3) and (4) are immediate from Theorem 3.7. Alternatively, use Theorem 4.1.

In Parrott studied the curvature of a single contraction $T$. His work was supplemented by Levy in [14]. In our terminology the curvature of a single contraction is the curvature of the contractive, normal, completely positive map $\Theta(S) = TST^*$, for $S$ in $B(H)$. Our analysis allows us to replace $B(H)$ by a general semifinite factor and to extend Parrott’s results.
Proposition 4.26 Let $N$ be a semifinite factor with a normal semifinite trace $\text{tr}$. Let $t \in N$ be a non zero contraction with $\text{tr}(I- tt^*) < \infty$ and set $\Theta(a) = tat^*$ for $a$ in $N$. Then

1. $d(\Theta) = 1$,
2. $K(\Theta, \text{tr}) = \lim_{k \to \infty} \frac{\text{tr}(I - t^k t^{k+*})}{k} = \lim_{k \to \infty} \text{tr}(t^k t^{k*} - t^{k+1} t^{(k+1)*})$
3. $\Theta$ is pure (i.e., if $t_{\infty} = 0$) then $K(\Theta, \text{tr}) = \text{tr}(I - tt^*) - \text{tr}(I - t^* t)$.

Proof. The fact that $d(\Theta) = 1$ follows from Example 4.11. Let $H$ be an $N$-module as in Lemma 4.7. Then the map $v : N \otimes_H \rightarrow H$ defined by $v(a \otimes \Theta h) = at^*h$ is a unitary operator. The map $S$ of Lemma 4.6 is defined by the equation $S(a \otimes h) = \Theta(a) h = tat^* h$. Hence $Sv^* = t$ and, consequently, $vDv^* = (I - t^* t)^{1/2}$. Also, for $b$ in $N$, $v(b \otimes I_H)v^* = b$. It now follows from Lemma 4.7 that, for a positive $b$ in $N$ with finite trace,

$\text{tr}(b) = d(\Theta) \text{tr}(b) = \text{tr}(\Theta(b)) + \text{tr}((I - t^* t)^{1/2}b(I - t^* t)^{1/2})$.

Thus, for $x = I - \Theta(I)$ and $j \geq 0$, we have

$\text{tr}(\Theta^j(x)) - \text{tr}(\Theta^{j+1}(x)) = \text{tr}((I - t^* t)^{1/2} \Theta^j(x)(I - t^* t)^{1/2})$.

Summing this up for $j$ from 0 to $k$ we find that

$\text{tr}(x) = \text{tr}(\Theta^{k+1}(x)) + \text{tr}((I - t^* t)^{1/2}(x + \ldots + \Theta^k(x))(I - t^* t)^{1/2})$.

But $x + \ldots + \Theta^k(x) = I - \Theta^{k+1}(I)$. Hence, when we take the limit, as $k \to \infty$, and use the fact that $\text{tr}(\Theta^{k+1}(x)) \to K(\Theta, \text{tr})$ (Theorem 3.3(3)), we get part (2). Part (3) follows from (2). □

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