Spanning trees of a claw-free graph whose reducible stems have few leaves

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Abstract

Let $T$ be a tree, a vertex of degree one is a leaf of $T$ and a vertex of degree at least three is a branch vertex of $T$. For two distinct vertices $u, v$ of $T$, let $P_T[u, v]$ denote the unique path in $T$ connecting $u$ and $v$. For a leaf $x$ of $T$, let $y_x$ denote the nearest branch vertex to $x$. For every leaf $x$ of $T$, we remove the path $P_T[x, y_x]$ from $T$, where $P_T[x, y_x]$ denotes the path connecting $x$ to $y_x$ in $T$ but not containing $y_x$. The resulting subtree of $T$ is called the reducible stem of $T$. In this paper, we first use a new technique of Gould and Shull to state a new short proof for a result of Kano et al. on the spanning tree with a bounded number of leaves in a claw-free graph. After that, we use that proof to give a sharp sufficient condition for a claw-free graph having a spanning tree whose reducible stem has few leaves.

Keywords: spanning tree, leaf, claw-free graph, reducible stem

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1 Introduction

In this paper, we only consider finite graphs without loops or multiple edges. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v \in V(G)$, we use $N_G(v)$ and $\deg_G(v)$ (or $N(v)$ and $\deg(v)$ if there is no ambiguity) to denote the set of neighbors of $v$ and the degree of $v$ in $G$, respectively. For any $X \subseteq V(G)$, we denote by $|X|$ the cardinality of $X$. Sometime, we denote by $|G|$ instead of $|V(G)|$. 

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We define \( N_G(X) = \bigcup_{x \in X} N_G(x) \) and \( \deg_G(X) = \sum_{x \in X} \deg_G(x) \). The subgraph of \( G \) induced by \( X \) is denoted by \( G[X] \). We define \( G - uv \) to be the graph obtained from \( G \) by deleting the edge \( uv \in E(G) \), and \( G + uv \) to be the graph obtained from \( G \) by adding a new edge \( uv \) joining two non-adjacent vertices \( u \) and \( v \) of \( G \). For two vertices \( u \) and \( v \) of \( G \), the distance between \( u \) and \( v \) in \( G \) is denoted by \( d_G(u, v) \). We use \( K_n \) to denote the complete graph on \( n \) vertices. Win [20] obtained the following theorem, which confirms a conjecture of Las Vergnas [16]. Beside that, recently, the author [6] also gave an improvement of Win by giving an independence number condition, which confirms a conjecture of Las Vergnas [16].

For two distinct vertices \( u, v \) of \( T \), let \( P_T[u, v] \) denote the unique path in \( T \) connecting \( u \) and \( v \). For a leaf \( x \) of \( T \), let \( y_x \) denote the nearest branch vertex to \( x \). For every leaf \( x \) of \( T \), we remove the path \( P_T[x, y_x] \) from \( T \), where \( P_T[x, y_x] \) denotes the path connecting \( x \) to \( y_x \) in \( T \) but not containing \( y_x \). Moreover, the path \( P_T[x, y_x] \) is called the leaf-branch path of \( T \) incident to \( x \) and \( y_x \) and denoted by \( B_x \). The resulting subtree of \( T \) is called the reducible stem of \( T \) and denoted by \( R\text{Stem}(T) \).

There are several sufficient conditions (such as the independence number conditions and the degree sum conditions) for a graph \( G \) to have a spanning tree with a bounded number of leaves or branch vertices. Win [20] obtained the following theorem, which confirms a conjecture of Las Vergnas [16]. Beside that, recently, the author [6] also gave an improvement of Win by giving an independence number condition for a graph having a spanning tree which covers a certain subset of \( V(G) \) and has at most \( l \) leaves.

**Theorem 1.1** (Win [20]) Let \( m \geq 1 \) and \( l \geq 2 \) be integers and let \( G \) be a \( m \)-connected graph. If \( \alpha(G) \leq m + l - 1 \), then \( G \) has a spanning tree with at most \( l \) leaves.
Later, Broersma and Tuinstra gave the following degree sum condition for a graph to have a spanning tree with at most \( l \) leaves.

**Theorem 1.2** (Broersma and Tuinstra [1]) Let \( G \) be a connected graph and let \( l \geq 2 \) be an integer. If \( \sigma_2(G) \geq |G| - l + 1 \), then \( G \) has a spanning tree with at most \( l \) leaves.

Motivating by Theorem 1.1, a natural question is whether we can find sharp sufficient conditions of \( \sigma_{l+1}(G) \) for a connected graph \( G \) has a few leaves. This question is still open. But, in certain graph classes, the answers have been determined.

For a positive integer \( t \geq 3 \), a graph \( G \) is said to be \( K_{1,t} \)-free if it contains no \( K_{1,t} \) as an induced subgraph. If \( t = 3 \), the \( K_{1,3} \)-free graph is also called the claw-free graph. About this graph class, Kano, Kyaw, Matsuda, Ozeki, Saito and Yamashita proved the following theorem.

**Theorem 1.3** (Kano et al. [11]) Let \( G \) be a connected claw-free graph and let \( l \geq 2 \) be an integer. If \( \sigma_{l+1}(G) \geq |G| - l \), then \( G \) has a spanning tree with at most \( l \) leaves.

For other graph classes, we refer the readers to see [2], [3], [5], [14], [15] and [17] for examples.

The first main purpose of this paper is to give a new short proof for Theorem 1.3 base on the new technique of Gould and Shull in [5].

Moreover, many researchers studied spanning trees in connected graphs whose stems have a bounded number of leaves or branch vertices (see [7], [12], [13], [18] and [19] for more details). We introduce here some results on spanning trees whose stems have a few leaves or branch vertices.

**Theorem 1.4** (Tsugaki and Zhang [18]) Let \( G \) be a connected graph and let \( k \geq 2 \) be an integer. If \( \sigma_3(G) \geq |G| - 2k + 1 \), then \( G \) have a spanning tree whose stem has at most \( k \) leaves.

**Theorem 1.5** (Kano and Yan [12]) Let \( G \) be a connected graph and let \( k \geq 2 \) be an integer. If either \( \alpha_4(G) \leq k \) or \( \sigma_{k+1}(G) \geq |G| - k - 1 \), then \( G \) has a spanning tree whose stem has at most \( k \) leaves.

**Theorem 1.6** (Yan [19]) Let \( G \) be a connected graph and \( k \geq 0 \) be an integer. If one of the following conditions holds, then \( G \) have a spanning tree whose stem has at most \( k \) branch vertices.

(a) \( \alpha_4(G) \leq k + 2 \),

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Recently, Ha, Hanh and Loan gave a sufficient condition for a graph to have a spanning tree whose reducible stem has few leaves. In particular, they proved the following theorem.

**Theorem 1.7** (Ha et al. [8]) Let $G$ be a connected graph and let $k \geq 2$ be an integer. If one of the following conditions holds, then $G$ has a spanning tree whose reducible stem has at most $k$ leaves.

(i) $\alpha(G) \leq 2k + 2$,

(ii) $\sigma_{k+1}(G) \geq \left\lfloor \frac{|G|-k}{2} \right\rfloor$.

Here, the notation $\lfloor r \rfloor$ stands for the biggest integer not exceed the real number $r$.

After that, Ha, Hanh, Loan and Pham also gave a sufficient condition for a graph to have a spanning tree whose reducible stem has few branch vertices.

**Theorem 1.8** (Ha et al. [9]) Let $G$ be a connected graph and let $k \geq 2$ be an integer. If the following conditions holds, then $G$ has a spanning tree $T$ whose reducible stem has at most $k$ branch vertices.

$$\sigma_{k+3}(G) \geq \left\lfloor \frac{|G|-2k-2}{2} \right\rfloor.$$ 

Very recently, Hanh stated the following theorem.

**Theorem 1.9** (Hanh [10]) Let $G$ be a connected claw-free graph and let $k \geq 2$ be an integer. If one of the following conditions holds, then $G$ has a spanning tree whose reducible stem has at most $k$ leaves.

(i) $\alpha(G) \leq 3k + 2$,

(ii) $\sigma_{k+1}(G) \geq \left\lfloor \frac{|G|-4k-2}{2} \right\rfloor$.

The open question is whether we may give a sharp condition of $\sigma_{3k+3}(G)$ to show that $G$ has a spanning tree whose reducible stem has at most $k$ leaves.

For the last purpose of this paper, we will give an affirmative answer to this question. In particular, we prove the following theorem.

**Theorem 1.10** Let $G$ be a connected claw-free graph and let $k$ be an integer ($k \geq 2$). If $\sigma_{3k+3}(G) \geq |G| - k$, then $G$ has a spanning tree whose reducible stem has at most $k$ leaves.
To show that our result is sharp, we will give the following example. Let $k \geq 2$ and $m \geq 1$ be integers, and let $R_0, R_1, ..., R_k$ and $H_0, H_1, ..., H_k$ be $2k+2$ disjoint copies of the complete graph $K_m$ of order $m$. Let $D$ be a complete graph with $V(D) = \{w_i\}_{i=0}^k$.

Let $\{x_i, x_{iy}, x_{iz}\}_{i=0}^k$ be $3k+3$ vertices not contained in $\bigcup_{i=0}^k \left(V(R_i) \cup V(H_i) \cup \{w_i\}\right)$.

Join $x_{iy}$ to all the vertices of $V(R_i)$ and $x_{iz}$ to all the vertices of $V(H_i)$ for every $0 \leq i \leq k$. Adding $3k+3$ edges $x_i x_{iy}, x_i x_{iz}, x_{iy} x_{iz}$ and joining $x_i$ to $w_i$ for every $0 \leq i \leq k$. Let $G$ denote the resulting graph (see Figure 1). Then, we have $|G| = (k+1)(2m+4)$. Moreover, take a vertex $u_i \in V(R_i)$ and a vertex $v_i \in V(H_i)$ for for every $0 \leq i \leq k$. We obtain

$$\sigma_{3k+3}(G) = \sum_{i=0}^k \left( \deg_G(u_i) + \deg_G(v_i) + \deg_G(x_i) \right)$$

$$= (k+1)m + (k+1)m + 3(k+1) = (k+1)(2m+3)$$

$$= |G| - k - 1.$$ 

But $G$ has no spanning tree whose reducible stem has $k$ leaves. Hence the condition of Theorem 1.10 is sharp.
2 A new proof of Theorem 1.3

Before beginning to prove Theorem 1.3 we recall some definitions in [5].

Definition 2.1 ([5]) Let $T$ be a tree. For each $e \in E(T)$ and $u, v \in V(T)$, we denote $\{u,v\} = V(P_T[u,v]) \cap N_T(u)$ and $e_v$ as the vertex incident to $e$ which is the nearest vertex of $v$ in $T$.

Definition 2.2 ([5]) Let $T$ be a spanning tree of a graph $G$ and let $v \in V(G)$ and $e \in E(T)$. Denote $g(e,v)$ as the vertex incident to $e$ farthest away from $v$ in $T$. We say $v$ is an oblique neighbor of $e$ with respect to $T$ if $vg(e,v) \in E(G)$.

Definition 2.3 ([5]) Let $T$ be a spanning tree of a graph $G$. Two vertices are pseudoadjacent with respect to $T$ if there is some $e \in E(T)$ which has them both as oblique neighbors. Similarly, a vertex set is pseudoindependent with respect to $T$ if no two vertices in the set are pseudoadjacent with respect to $T$.

We note here that pseudoadjacency (with respect to any tree) is a weaker condition than adjacency, while pseudoindependence is a stronger condition than independence.

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Suppose that $G$ has no spanning tree with at most $l$ leaves. Choose some spanning tree $T$ of $G$ such that:

(T1) $|L(T)|$ is as small as possible.

By the assumption, $T$ must have at least $l + 1$ leaves.

We have the following claims.

Claim 2.4 $L(T)$ is pseudoindependent with respect to $T$. In particular, $L(T)$ is independent.

Proof. Suppose two leaves $s$ and $t$ are pseudoadjacent with respect to $T$. Then there is some edge $e \in E(T)$ such that $sg(e,s), tg(e,t) \in E(G)$. Let $b$ and $u$ be the nearest branch vertices of $s$ and $t$, respectively. Consider the following two cases.

Case 1: Suppose $g(e,s) \neq g(e,t)$. Then $e_s = g(e,t)$ and $e_t = g(e,s)$, so $se_t, te_s \in E(G)$. We consider the spanning tree

$$T' := \begin{cases} T - e + se_t, & \text{if } e = uu_t, \\ T - \{e, uu_t\} + \{se_t, te_s\}, & \text{if } e \neq uu_t. \end{cases}$$

Hence, $|L(T')| < |L(T)|$. This violates (T1). So case 1 does not happen.

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Case 2: Suppose \( g(e, s) = g(e, t) \). Define \( a := g(e, s) = g(e, t) \). Then \( e_s = e_t \notin \{s, t\} \) and denoted by vertex \( z \). Since \( G[a, z, s, t] \) is not \( K_{1,3} \)-free, so we have either \( st \in E(G) \) or \( zt \in E(G) \) or \( zs \in E(G) \). Consider the tree

\[
T' := \begin{cases} 
T - uu_t + st, & \text{if } st \in E(G), \\
T - \{e, uu_t\} + \{zt, sa\}, & \text{if } zt \in E(G), \\
T - \{e, bb_s\} + \{zs, ta\}, & \text{if } zs \in E(G).
\end{cases}
\]

Hence, \(|L(T')| < |L(T)|\). This violates the condition (T1). So case 2 does not happen.
Therefore, the claim 2.4 is proved.

**Claim 2.5** For each branch vertex \( b \in B(T) \), there are at least \( \deg_T(b) - 1 \) edges of \( T \) incident with \( b \) such that they have no oblique neighbor in \( L(T) \).

**Proof.** Set \( N_T(b) = \{s_1, s_2, ..., s_q\} \), \( q \geq 3 \).
Assume that there exist two vertices \( s_i, s_j \in N_T(b) \) such that \( s_is_i \notin E(G) \) for all \( t \in \{1, ..., q\} \setminus \{i\} \) and \( s_js_j \notin E(G) \) for all \( t \in \{1, ..., q\} \setminus \{j\} \). Then \( G[b, s_i, s_j, s_t] \) is \( K_{1,3} \)-free for every \( t \in \{1, ..., q\} \setminus \{i, j\} \). This is a contradiction. Therefore we conclude that there exists at most one vertex \( s \in N_T(b) \) such that \( ss \notin E(G) \) for all \( s_t \neq s \).

Let \( s_t \in N_T(b) \setminus \{s\} \). Then there exists some vertex \( s_i \in N_T(b) \setminus \{s_t, s\} \) such that \( s_ts_i \in E(G) \). Set \( e := bs_t \). To complete Claim 2.5 we will need only to prove that \( x \) is not an oblique neighbor of \( e \) with respect to \( T \) for every \( x \in L(T) \). Indeed, to the contrary, assume that there exists some vertex \( x \in L(T) \) such that \( x \) is an oblique neighbor of \( e \) with respect to \( T \). Consider the tree

\[
T' := \begin{cases} 
T - \{e, bs_t\} + \{bx, s_ts_i\}, & \text{if } g(e, x) = b, x \neq s_t, \\
T - bs_t + s_ts_i, & \text{if } g(e, x) = b, x = s_t, \\
T - e + xs_t, & \text{if } g(e, x) = s_t.
\end{cases}
\]

Hence, \(|L(T')| < |L(T)|\). This violates with (T1).
Claim 2.5 holds.

**Claim 2.6** There are at least \( l \) distinct edges of \( T \) such that they have no oblique neighbor in \( L(T) \).

**Proof.** By Claim 2.4, we obtain that for each \( e \in E(T) \), \( e \) has at most an oblique neighbor in \( L(T) \). Moreover, if an edge \( e \) is incident with two branch vertices of \( T \)
then $e$ has to be an edge of the subgraph $T[\{b\}_{b \in B(T)}]$ of $T$. Then, there are at most $|B(T)| - 1$ edges which are adjacent with two branch vertices of $T$. Hence, combining with Claim 2.5 there exist at least
\[
\sum_{b \in B(T)} (\deg_T(b) - 1) - |B(T)| + 1 = \sum_{b \in B(T)} (\deg_T(b) - 2) + 1
\]
distinct edges in $E(T)$ which have no oblique neighbor in $L(T)$. On the other hand, we have
\[
|L(T)| = 2 + \sum_{b \in B(T)} (\deg_T(b) - 2)
\Rightarrow \sum_{b \in B(T)} (\deg_T(b) - 2) + 1 = |L(T)| - l \geq l.
\]
Therefore, the claim is proved. □

For any $v, x \in V(T)$, we now have $vx \in E(G)$ if and only if $v$ is an oblique neighbor of $xx_v$ with respect $T$. Therefore, the number of edges of $T$ with $v$ as an oblique neighbor equals the degree of $v$ in $G$. Combining with Claims 2.4 and 2.6 we obtain that
\[
\sigma_{l+1}(G) \leq |E(T)| - l = |V(T)| - 1 - l = |G| - 1 - l,
\]
which contradicts the assumption of Theorem 1.3. The proof of Theorem 1.3 is completed.

3 Proof of theorem 1.10

For two distinct vertices $u, v$ of $T$, let $P_T[u, v]$ denote the unique path in $T$ connecting $u$ and $v$. We define the orientation of $P_T[u, v]$ is from $u$ to $v$. For each vertex $x \in V(P_T[u, v])$, we denote by $x^+$ and $x^-$ the successor and predecessor of $x$ in $P_T[u, v]$, respectively, if they exist. For any $X \subseteq V(G)$, set $(N(X) \cap P_T[u, v])^- = \{x^- | x \in V(P_T[u, v]) \setminus \{u\} \text{ and } x \in N(X)\}$ and $(N(X) \cap P_T[u, v])^+ = \{x^+ | x \in V(P_T[u, v]) \setminus \{v\} \text{ and } x \in N(X)\}$.

Proof of Theorem 1.10. Suppose to the contrary that there does not exist a spanning tree $T$ of $G$ such that $|L(R_{\text{Stem}}(T))| \leq k$. Then every spanning tree $T$ of $G$ satisfies $|L(R_{\text{Stem}}(T))| \geq k + 1$.

Choose $T$ to be a spanning tree of $G$ such that
\[
(C0) \ |L(R_{\text{Stem}}(T))| \text{ is as small as possible,}
\]
Claim 3.1 For every $i \in \{1, 2, \ldots, l\}$, there exist at least two leaf-branch paths of $T$ which are incident to $x_i$.

Claim 3.2 For each $i \in \{1, 2, \ldots, l\}$, there exist $y_i, z_i \in L(T)$ such that $B_{y_i}, B_{z_i}$ are incident to $x_i$, and $N_G(y_i) \cap (V(R_{\text{Stem}}(T)) \setminus \{x_i\}) = \emptyset$ and $N_G(z_i) \cap (V(R_{\text{Stem}}(T)) \setminus \{x_i\}) = \emptyset$.

Proof. Let $\{a_{ij}\}_{j=1}^m$ be the subset of $L(T)$ such that $B_{a_{ij}}$ is adjacent to $x_i$. By Claim 3.1 we obtain $m \geq 2$. Suppose that there are more than $m - 2$ vertices $\{a_{ij}\}_{j=1}^m$ satisfying

$$N_G(a_{ij}) \cap (V(R_{\text{Stem}}(T)) \setminus \{x_i\}) \neq \emptyset.$$ 

Without loss of generality, we may assume that $N_G(a_{ij}) \cap (V(R_{\text{Stem}}(T)) \setminus \{x_i\}) \neq \emptyset$ for all $j = 2, \ldots, m$. Set $b_{ij} \in N_G(a_{ij}) \cap (V(R_{\text{Stem}}(T)) \setminus \{x_i\})$ and $v_{ij} \in N_T(x_i) \cap V(P_T[a_{ij}, x_i])$ for all $j \in \{2, \ldots, m\}$. Consider the spanning tree $T' := T + \{a_{ij}b_{ij}\}_{j=2}^m - \{x_iv_{ij}\}_{j=2}^m$.

Then $T'$ satisfies $|L(R_{\text{Stem}}(T'))| \leq |L(R_{\text{Stem}}(T))|$ and $|R_{\text{Stem}}(T')| < |R_{\text{Stem}}(T)|$, where $x_i$ is not in $V(R_{\text{Stem}}(T'))$. This contradicts either the condition (C0) or the condition (C1). Therefore, Claim 3.2 holds.

Claim 3.3 For every $i, j \in \{1, 2, \ldots, k + 1\}$, $i \neq j$, if $u \in V(P_T[y_i, x_i])$ and $v \in V(P_T[y_j, x_j])$, $w \in V(P_T[z_i, x_i])$, then $uv \notin E(G)$ and $uw \notin E(G)$. In particular, we have $N_G(y_i) \cap V(B_{y_j}) = \emptyset$ and $N_G(y_i) \cap V(B_{z_j}) = \emptyset$.

Proof. By the same role of $y_j$ and $z_j$, we only need to prove $uv \notin E(G)$. Suppose the assertion of the claim is false. Set $T' := T + uv$. Then $T'$ is a subgraph of $G$ including a unique cycle $C$, which contains both $x_i$ and $x_j$. Since $k \geq 2$, then $|L(R_{\text{Stem}}(T))| \geq 3$. Hence, we obtain $|B(R_{\text{Stem}}(T))| \geq 1$. Then there exists a branch vertex of $R_{\text{Stem}}(T)$ contained in $C$. Let $e$ be an edge incident to such a vertex in $C$ and $R_{\text{Stem}}(T)$. By removing the edge $e$ from $T'$ we obtain a spanning
tree $T''$ (see Figure 2). Hence $T''$ satisfies $|L(R_{\text{Stem}}(T''))| < |L(R_{\text{Stem}}(T))|$, the reason is that either $R_{\text{Stem}}(T'')$ has only one new leaf and $x_i, x_j$ are not leaves of $R_{\text{Stem}}(T'')$ or $x_i$ (or $x_j$) is still a leaf of $R_{\text{Stem}}(T'')$ but $R_{\text{Stem}}(T'')$ has no new leaf and $x_j$ (or $x_i$ respectively) is not a leaf of $R_{\text{Stem}}(T'')$. This is a contradiction with the condition (C0). So Claim 3.3 is proved.

We obtain the following claim as a corollary of Claim 3.3.

**Claim 3.4** $L(R_{\text{Stem}}(T))$ is an independent set in $G$.

Set $U_1 = \{y_i, z_i\}_{i=1}^l$. For each $i \in \{1, ..., l\}$ we also set $x_{iy} \in N_T(x_i) \cap V(B_{y_i})$ and $x_{iz} \in N_T(x_i) \cap V(B_{z_i})$.

**Claim 3.5** $U_1$ is an independent set in $G$.

**Proof.** Suppose that there exist two vertices $u, v \in U_1$ such that $uv \in E(G)$. Without lost of generality, we may assume that $v = y_i$ for some $i \in \{1, 2, ..., l\}$. Consider the spanning tree $T' := T + uy_i - x_{iy}x_i$. Then $|L(R_{\text{Stem}}(T'))| \leq |L(R_{\text{Stem}}(T))|$. If $\deg_T(x_i) = 3$ then $x_i$ is not a branch vertex of $T'$. Hence $|R_{\text{Stem}}(T')| < |R_{\text{Stem}}(T)|$, this contradicts either the conditions (C0) or (C1). Otherwise, we have $|L(R_{\text{Stem}}(T'))| = |L(R_{\text{Stem}}(T))|$, $|R_{\text{Stem}}(T')| = |R_{\text{Stem}}(T)|$ and $|L(T')| < |L(T)|$, where either $T'$ has only one new leaf and $y_i, u$ are not leaves of $T'$ or $y_i$ is still a leaf of $T'$ but $T'$ has no any new leaf and $u$ is not a leaf of $T'$. This contradicts the condition (C2). The proof of Claim 3.5 is completed.

Now, we choose $T$ to be a spanning tree of $G$ satisfying

(C3) $\sum_{i=1}^l \deg_T(x_i)$ is as small as possible, subject to (C0)-(C2),
(C4) \[ \sum_{i=1}^{l} (|B_{y_i}| + |B_{z_i}|) \] is as large as possible, subject to (C0)-(C3).

Set \( U = U_1 \cup L(R_{\text{Stem}}(T)) \).

\textbf{Claim 3.6} \( U \) is an independent set in \( G \).

\textbf{Proof.} Suppose that there exist two vertices \( u, v \in U \) such that \( uv \in E(G) \). By Claims 3.4 and 3.5 without lost of generality, we may assume that \( u \in L(R_{\text{Stem}}(T)) \) and \( v = y_i \in U_1 \) for some \( i \in \{1, 2, ..., l\} \). Moreover, by Claim 3.3, we now only need to consider the case \( u = x_i \).

Set \( t \in N_T(x_i) \cap V(R_{\text{Stem}}(T)) \).

If \( y_i x_{iz} \in E(G) \). Consider the spanning tree \( T' := T + y_i x_{iz} - x_{iz}x_i \). Then \( |L(R_{\text{Stem}}(T'))| \leq |L(R_{\text{Stem}}(T))| \). If \( \deg_T(x_i) = 3 \) then \( x_i \) is not a branch vertex of \( T' \). Hence \( |R_{\text{Stem}}(T')| < |R_{\text{Stem}}(T)| \), this contradicts either the conditions (C0) or (C1). Otherwise, we have \( |L(R_{\text{Stem}}(T'))| = |L(R_{\text{Stem}}(T))| \), \( |R_{\text{Stem}}(T')| = |R_{\text{Stem}}(T)| \) and \( |L(T')| < |L(T)| \), this contradicts the condition (C2). Now, since \( G[x_i, t, x_{iz}, y_i] \) is not \( K_{1,3}\)-free, we obtain that \( tx_{iz} \in E(G) \) or \( ty_i \in E(G) \). We consider the spanning tree

\[ T' := \begin{cases} T + tx_{iz} - x_{iz}x_i, & \text{if } tx_{iz} \in E(G), \\ T + ty_i - x_{iz}x_i, & \text{if } ty_i \in E(G). \end{cases} \]

If \( \deg_T(x_i) = 3 \) then we obtain \( |L(R_{\text{Stem}}(T'))| \leq |L(R_{\text{Stem}}(T))| \) and \( |R_{\text{Stem}}(T')| < |R_{\text{Stem}}(T)| \), a contradiction with (C0) or (C1). Otherwise, we have \( L(R_{\text{Stem}}(T')) = L(R_{\text{Stem}}(T)) = \{x_i\}_{i=1}^{k+1}, |R_{\text{Stem}}(T')| = |R_{\text{Stem}}(T)|, |L(T')| = |L(T)| \) and \( \sum_{i=1}^{l} \deg_T(x_i) < \sum_{i=1}^{l} \deg_T(x_i) \). This also violates the condition (C3).

Therefore, the proof of Claim 3.6 is completed. \( \square \)

By Claim 3.6 we conclude that \( \alpha(G) \geq 3l \geq 3k + 3 \).

\textbf{Claim 3.7} For every \( p \in L(T) \setminus U_1 \), then \( \sum_{u \in U} |N_G(u) \cap V(B_p)| \leq |B_p| \).

\textbf{Proof.} Set \( v_p \in B(T) \) such that \( (V(P_T[p, v_p]) \setminus \{v_p\}) \cap B(T) = \emptyset \). Let \( V(B_p) \cap N_T(v_p) = \{v_p^{-}\} \). Then we consider \( B_p = P_T[p, v_p^-] \).

Assume that there exists a vertex \( x \in V(B_p) \) such that \( xu \in E(G) \) for some \( u \in U_1 \). Consider the spanning tree

\[ T'' := \begin{cases} T + xu - v_p^- v_p, & \text{if } x \in \{v_p^-, p\}, \\ T + xu - xx+, & \text{if } x \notin \{v_p^-, p\}. \end{cases} \]
This contradicts either the condition (C2) if \( x \in \{v_p^-, p\} \) or the condition (C4) for otherwise. Therefore, we conclude that \( \sum_{u \in U_1} |N_G(u) \cap V(B_p)| = 0 \).

Assume that there exist \( x_i, x_j \in L(R_{\text{Stem}}(T)) \) for some \( i \neq j \) and \( x \in V(B_p) \) such that \( xx_i, xx_j \in E(G) \). Set

\[
G' := \begin{cases} 
T + xx_j, & \text{if } x_i = v_p, \\
T + \{xx_i, xx_j\} - \{v_p^-\}, & \text{if } x_i \neq v_p.
\end{cases}
\]

Then \( G' \) is a subgraph of \( G \) including a unique cycle \( C \), which contains both \( x_i \) and \( x_j \). Since \( k \geq 2 \), then \( |L(R_{\text{Stem}}(T))| \geq 3 \). Hence, we obtain \( |B(R_{\text{Stem}}(T))| \geq 1 \). Then there exists a branch vertex of \( R_{\text{Stem}}(T) \) contained in \( C \). Let \( e \) be an edge incident to such a vertex in \( C \). By removing the edge \( e \) from \( G' \) we obtain a spanning tree \( T' \) of \( G \) satisfying \( |L(R_{\text{Stem}}(T'))| < |L(R_{\text{Stem}}(T))| \), the reason is that either \( R_{\text{Stem}}(T') \) has only one new leaf and \( x_i, x_j \) are not leaves of \( R_{\text{Stem}}(T') \) or \( x_i \) (or \( x_j \)) is still a leaf of \( R_{\text{Stem}}(T') \) but \( R_{\text{Stem}}(T') \) has no new leaf and \( x_j \) (or \( x_i \), respectively) is not a leaf of \( R_{\text{Stem}}(T') \) (see Figure 3 for an example). This is a contradiction with the condition (C0). Therefore, we concludes that \( \sum_{u \in L(R_{\text{Stem}}(T))} |N_G(u) \cap \{x\}| \leq 1 \) for every \( x \in V(B_p) \).

Now we obtain the following

\[
\sum_{u \in U} |N_G(u) \cap V(B_p)| = \sum_{u \in U_1} |N_G(u) \cap V(B_p)| + \sum_{u \in L(R_{\text{Stem}}(T))} |N_G(u) \cap V(B_p)| \leq |B_p|.
\]

Claim 3.7 is proved.

Claim 3.8 For every \( 1 \leq i \leq k+1 \), then \( \sum_{u \in U} |N_G(u) \cap V(B_{y_i})| \leq |B_{y_i}| \) and \( \sum_{u \in U} |N_G(u) \cap V(B_{z_i})| \leq |B_{z_i}| \).
Proof. By the same role of $y_i$ and $z_i$, we only need to prove
\[ \sum_{u \in U} |N_G(u) \cap V(B_{y_i})| \leq |B_{y_i}|. \]

We consider $B_{y_i} = P_T[y_i, x_{iy}]$.

By Claim 3.3, we obtain the following.

**Subclaim 3.8.1.** $N_G(U_i) \cap V(B_{y_i}) = N_G\{y_i, z_i\} \cap V(B_{y_i})$.

**Subclaim 3.8.2.** We have $x_{iy} x_{iz} \in E(G)$.

Indeed, if $x_{iy} x_{iz} \notin E(G)$ we set $t \in N_T(x_i) \cap V(R_{\text{Stem}}(T))$. Then since $G[x_i, t, x_{iy}, x_{iz}]$ is not $K_{1, 3}$-free we obtain either $x_{iy} t \in E(G)$ or $x_{iz} t \in E(G)$. Without loss of generality, we may assume that $x_{iy} t \in E(G)$. Consider the spanning tree $T' = T - x_i x_{iy} + x_{iy} t$. If $\deg_T(x_i) = 3$ then we obtain $|L(R_{\text{Stem}}(T'))| \leq |L(R_{\text{Stem}}(T))|$ and $|R_{\text{Stem}}(T')| < |R_{\text{Stem}}(T)|$, a contradiction with (C0) or (C1). Otherwise, we have $L(R_{\text{Stem}}(T')) = L(R_{\text{Stem}}(T))$, $|R_{\text{Stem}}(T')| = |R_{\text{Stem}}(T)|$, $|L(T')| = |L(T)|$ and $\sum_{i=1}^{l} \deg_{T}(x_i) < \sum_{i=1}^{t} \deg_{T}(x_i)$. This violates the conditions (C3). Subclaim 3.8.2 is proved.

**Subclaim 3.8.3.** If $x \in N_G(y_i) \cap V(B_{y_i})$ then $x^- \notin N_G(z_i) \cap V(B_{y_i})$.

Suppose that there exists $x \in N_G(y_i) \cap B_{y_i}$ such that $x^- \in N_G(z_i) \cap B_{y_i}$. Consider the spanning tree $T' := T + \{x_{iy}, z_i x^-\} - \{x x^-, x_{iy} x_i\}$. Then $|L(R_{\text{Stem}}(T'))| \leq |L(R_{\text{Stem}}(T))|$. If $\deg_T(x_i) = 3$ then $x_i$ is not a branch vertex of $T'$. Hence $|R_{\text{Stem}}(T')| < |R_{\text{Stem}}(T)|$, this contradicts the condition (C1). Otherwise, we have $|L(R_{\text{Stem}}(T'))| = |L(R_{\text{Stem}}(T))|$, $|R_{\text{Stem}}(T')| = |R_{\text{Stem}}(T)|$ and $|L(T')| < |L(T)|$. This is a contradiction with the condition (C2). Therefore, Subclaim 3.8.3 holds.

**Subclaim 3.8.4.** If $x \in N_G(y_i) \cap V(B_{y_i})$ then $x^- \notin N_G(x_i) \cap V(B_{y_i})$.

Suppose that there exists $x \in N_G(y_i) \cap B_{y_i}$ such that $x^- \in N_G(x_i) \cap V(B_{y_i})$ for some $w \in L(R_{\text{Stem}}(T))$. By Subclaim 3.8.2, consider the spanning tree $T' := T + \{x_{iy}, x x^-, x_{iy} x_i\} - \{x x^-, x_{iy} x_i, x_{iz}\}$. Then $|L(R_{\text{Stem}}(T'))| \leq |L(R_{\text{Stem}}(T))|$ and $|R_{\text{Stem}}(T')| \leq |R_{\text{Stem}}(T)|$, this contradicts the conditions either (C0) or (C1). Otherwise, we have $|L(R_{\text{Stem}}(T'))| = |L(R_{\text{Stem}}(T))|$, $|R_{\text{Stem}}(T')| = |R_{\text{Stem}}(T)|$ and $|L(T')| < |L(T)|$. This contradicts with the condition (C2). Therefore, Subclaim 3.8.4 holds.

**Subclaim 3.8.5.** We have $x_{iy} \notin N_G(z_i)$.

Indeed, suppose to the contrary that $x_{iy} z_i \in E(G)$. We consider the spanning tree $T' := T + x_{iy} z_i - x_i x_{iy}$. Hence, $T'$ is a spanning tree of $G$ satisfying $|L(R_{\text{Stem}}(T'))| \leq |L(R_{\text{Stem}}(T))|$, $|R_{\text{Stem}}(T')| \leq |R_{\text{Stem}}(T)|$ and $|L(T')| < |L(T)|$, where $z_i$ is not a leaf of $T'$. This contradicts the conditions (C0), (C1) or (C2). Subclaim 3.8.5 is
Then we have $\sum_{3.8.1, 3.8.6-3.8.7}$, we obtain this completes the proof of Claim 3.8.

Indeed, assume that $x_i x \in E(G)$. By Subclaim 3.8.5 and Claim 3.6 we obtain $x_i x \notin E(G)$ and there exists $x^+$. Combining with $G[x, x_i, x^+, z_i]$ is not $K_{1,3}$-free we get $x^+ x_i \in E(G)$ or $x^+ z_i \in E(G)$.

If $x^+ x_i \in E(G)$. Combining with Subclaim 3.8.2, we consider the spanning tree

$$T' := \begin{cases} \{T + \{x_i x_{iz}, z_i x\} - \{xx^+, x_i x_{iz}\} \} & \text{if } x^+ = x_i y, \\ T + \{x^+ x_i, x_i x_{iz}, z_i x\} - \{xx^+, x_i x_{iy}, x_i x_{iz}\} & \text{if } x^+ \neq x_i y. \end{cases}$$

Hence $|L(R_{Stem}(T'))| \leq |L(R_{Stem}(T'))|$. If $deg_T(x_i) = 3$ then $x_i$ is not a branch vertex of $T'$. Hence $|R_{Stem}(T')| < |R_{Stem}(T)|$, this contradicts the condition (C1). Otherwise, we have $|L(R_{Stem}(T'))| = |L(R_{Stem}(T))|, |R_{Stem}(T')| = |R_{Stem}(T)|$ and $|L(T')| < |L(T)|$, this contradicts the condition (C2).

Otherwise, we have $x^+ z_i \in E(G)$. We set $t \in N_T(x_i) \cap V(R_{Stem}(T))$. Since $G[x_i, t, x_{iz}, x]$ is not $K_{1,3}$-free we obtain either $xt \in E(G)$ or $x_{iz} t \in E(G)$ or $xx_{iz} \in E(G)$. Consider the spanning tree

$$T' := \begin{cases} T + \{xt, x^+ z_i\} - \{xx^+, x_i x_{iz}\}, & \text{if } xt \in E(G), \\ T + x_{iz} t - x_i z_i, & \text{if } x_{iz} t \in E(G), \\ T + \{xx_{iz}, x^+ z_i\} - \{xx^+, x_i x_{iz}\}, & \text{if } xx_{iz} \in E(G). \end{cases}$$

Then we have $|L(R_{Stem}(T'))| \leq |L(R_{Stem}(T))|, |R_{Stem}(T')| \leq |R_{Stem}(T)|, |L(T')| \leq |L(T)|$ and $\sum_{i=1}^l \deg_T(x_i) < \sum_{i=1}^l \deg_T(x_i)$. This violates the conditions (C0), (C1), (C2) or (C3).

Subclaim 3.8.6 holds.

On the other hand, it follows from Claim 3.3 that.

Subclaim 3.8.7. We have $N_G(w) \cap V(B_{y_i}) = \emptyset$ for all $w \in L(R_{Stem}(T)) \setminus \{x_i\}$.

By Subclauses 3.8.3-3.8.4 we conclude that $\{y_i\}, N_G(y_i) \cap V(P_T[y_i, x_i])$ and $(N_G(\{z_i, x_i\}) \cap (P_T[y_i, x_i])^+), \{y_i\}$ are pairwise disjoint subsets in $P_T[y_i, x_i]$. Combining with Claim 3.6 and Subclaims 3.8.1, 3.8.6-3.8.7, we obtain

$$\sum_{u \in U} |N_G(u) \cap V(B_{y_i})| = |N_G(y_i) \cap V(B_{y_i})| + |N_G(z_i) \cap V(B_{y_i})| + |N_G(x_i) \cap V(B_{y_i})|$$

$$= |N_G(y_i) \cap V(B_{y_i})| + |N_G(\{z_i, x_i\}) \cap V(B_{y_i})|$$

$$= |N_G(y_i) \cap V(P_T[y_i, x_i])| + |(N_G(\{z_i, x_i\}) \cap P[y_i, x_i])^+|$$

$$\leq |P_T[y_i, x_i]| - 1 = |B_{y_i}||.$$

This completes the proof of Claim 3.8. □
Now, repeating the proof of Theorem 1.3 for the subtree $R_{Stem}(T)$ we obtain the following claim.

**Claim 3.9** $|N_G(L(R_{Stem}(T))) \cap V(R_{Stem}(T))| \leq |R_{Stem}(T)| - |L(R_{Stem}(T))|$. 

By Claim 3.2 and Claims 3.7-3.9 we obtain that

$$\deg_G(U) = \sum_{i=1}^{l} \left( \sum_{u \in U} |N_G(u) \cap V(B_{y_i})| + \sum_{u \in U} |N_G(u) \cap V(B_{z_i})| \right) +$$

$$+ \sum_{p \in L(T) \setminus U_1} \sum_{u \in U} |N_G(u) \cap V(B_p)| + |N_G(L(R_{Stem}(T))) \cap V(R_{Stem}(T))|$$

$$\leq \sum_{i=1}^{l} |B_{y_i}| + \sum_{i=1}^{l} |B_{z_i}| + \sum_{p \in L(T) \setminus U_1} |B_p| + |R_{Stem}(T)| - |L(R_{Stem}(T))|$$

$$= |G| - |L(R_{Stem}(T))|$$

$$= |G| - l.$$ 

Hence

$$\sigma_{3k+3}(G) \leq \sigma_3(G) \leq \deg_G(U)$$

$$\leq |G| - l \leq |G| - k - 1.$$ 

This contradicts the assumption of Theorem 1.10. Therefore, the proof of Theorem 1.10 is completed.

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