ON A NONLOCAL DIFFERENTIAL EQUATION DESCRIBING ROOTS OF POLYNOMIALS UNDER DIFFERENTIATION

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Abstract. In this work we study the nonlocal transport equation derived recently by Steinerberger when studying how the distribution of roots of a polynomial behaves under iterated differentiation of the function. In particular, we study the well-posedness of the equation, establish some qualitative properties of the solution and give conditions ensuring the global existence of both weak and strong solutions. Finally, we present a link between the equation obtained by Steinerberger and a one-dimensional model of the surface quasi-geostrophic equation used by Chae, Córdoba, Córdoba & Fontelos.

1. Introduction and main results

In this paper, we consider the following one-dimensional nonlinear transport equation

\begin{equation}
\partial_t u + \partial_x \arctan \left( \frac{H u}{u} \right) = 0 \quad (x,t) \text{ on } \mathbb{S}^1 \times [0,T],
\end{equation}

where

\begin{equation*}
H u(x) = \frac{1}{2\pi \text{p.v.}} \int_{\mathbb{S}} \frac{u(y)}{\tan \left( \frac{y - x}{2} \right)} dy,
\end{equation*}

is the Hilbert transform. The previous equation needs to be supplemented with the initial data

\begin{equation}
\tag{2}
\label{InitialData}
u(x,0) = u_0(x).
\end{equation}

This equation has been derived by S. Steinerberger [31] when studying how the distribution of roots behaves under iterated differentiation. In particular, the purpose of this work is to study the properties of the transport equation (1). The study of such nonlocal and nonlinear one-dimensional equations is a wide research area with a large literature. For other similar equations and related results we refer to [4, 5, 8, 12, 13, 22, 14, 15, 23, 25, 28, 24].

In this paper we prove the following results:

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Theorem 1. Let $0 < u_0 \in H^2(S^1)$ be the initial data. Then there exists a time $0 < T \leq \infty$, $T = T(\|u_0\|_{H^2}, \min_x u_0(x))$ and a unique positive solution to (1)

\[ 0 < u \in C([0, T], H^2(S^1)). \]

Furthermore, this solution verifies the following properties:

- \( \|u(t)\|_{L^2}^2 + \int_0^t \mathcal{D}(s)ds = \|u_0\|_{L^2}^2 \),

where

\[ \mathcal{D} = \frac{1}{16\pi} \int_{\mathbb{R}} \text{p.v.} \int_{S^1} \frac{u(x) - u(y)}{\sin \left( \frac{x-y}{2} \right)} \log \left( \frac{u(x)^2 + (Hu(x))^2}{u(y)^2 + (Hu(y))^2} \right) dx dy, \]

- \( \|u(t)\|_{L^1} = \|u_0\|_{L^1} \),
- if $u_0(x)$ is even, $u(x, t)$ remains even,
- Maximum principle: $\max_x u(x, t) \leq \max_x u_0(x)$,
- Minimum principle: $\min_x u_0(x) \leq \min_x u(x, t)$.

Remark 1. We remark that, for an arbitrary $u(x, t)$, we are not able to give a sign to $\mathcal{D}$ (compare with [1, 17]). In other words, we are not able to show whether the $L^2$ norm decays.

Remark 2. The following scaling is invariant for the equation $u(x, t) = \lambda^\alpha u(\lambda x, \lambda^{1-\alpha} t) \forall \alpha \in \mathbb{R}$.

Under certain restrictions we can ensure that the solution is global:

Theorem 2. Let $0 < u_0 \in H^2(S^1)$ be the initial data and denote

\[ \langle u_0 \rangle = \frac{1}{2\pi} \int_{S^1} u_0(x)dx. \]

There exists $0 < \mathcal{C}$ such that if

\[ \frac{\|u_0 - \langle u_0 \rangle\|_{A^1}}{\langle u_0 \rangle} \leq \mathcal{C}, \]

then the solution (from Theorem 1) is global and satisfies

\[ \|u(t) - \langle u_0 \rangle\|_{A^1} \leq \|u_0 - \langle u_0 \rangle\|_{A^1} e^{-\delta t} \]

for certain $0 < \delta(\langle u_0 \rangle)$ small enough.

Remark 3. The explicit lower bound $0.17 < \mathcal{C}$ is obtained as a byproduct.

Finally, we study the existence of weak solutions i.e. solutions that satisfy the equation in the following sense:

\[ -\int_0^T \int_{S^1} u(x, s)\partial_t \phi(x, s) + \arctan \left( \frac{Hu(x, s)}{u(x, s)} \right) \partial_x \phi dx ds = \int_{S^1} u_0(x) \phi(x, 0) dx, \]

for all test functions $\phi(x, t) \in C^\infty(S^1 \times [0, T])$. 

\[ -\int_0^T \int_{S^1} u(x, s)\partial_t \phi(x, s) + \arctan \left( \frac{Hu(x, s)}{u(x, s)} \right) \partial_x \phi dx ds = \int_{S^1} u_0(x) \phi(x, 0) dx, \]
In that regards, we prove the global existence of weak solution for initial data satisfying certain size conditions in a scale invariant space (with respect to the scaling of the equation):

**Theorem 3.** Let \( u_0 \in A^0(S^1) \) be the initial data and denote

\[
\langle u_0 \rangle = \frac{1}{2\pi} \int_{S^1} u_0(x)dx.
\]

There exists \( 0 < \tilde{C} \) such that if

\[
\frac{\|u_0 - \langle u_0 \rangle\|_{A^0}}{\langle u_0 \rangle} \leq \tilde{C},
\]

then there exists at least one global weak solution

\[
u \in L^\infty([0, T] \times \mathbb{S}^1) \cap L^2(0, T; H^{0.5}), \forall \ 0 < T < \infty
\]

and this solution satisfies

\[
\|u(t) - \langle u_0 \rangle\|_{L^\infty} \leq \|u_0 - \langle u_0 \rangle\|_{A^0} e^{-\delta t}
\]

for certain \( 0 < \delta(\langle u_0 \rangle) \) small enough.

**Remark 4.** The explicit lower bound \( 0.24 < \tilde{C} \) is obtained as a byproduct.

**Remark 5.** Similar results can be proved for the porous medium equation (see also \([30, 29]\))

\[
\partial_t u + \partial_x \left( \frac{H u}{u^m} \right) = 0, \quad m \in \mathbb{N}.
\]

The rest of the paper is devoted to the proofs of each results (sections 2-4) and the link between (1) and the equation

\[
\partial_t g + \Lambda g = \partial_x (g H g),
\]

(see section 5). We would like to remark that (4) was proposed as a one-dimensional model of the 2D surface quasi-geostrophic equation by Chae, Córdoa, Córdoba & Fontelos \([6]\) (see also the papers by Matsuno \([27]\) and Baker, Li & Morlet \([2]\)).

**Notation.** We denote

\[
\Lambda u = H \partial_x u(x) = \frac{1}{4\pi} \text{p.v.} \int_S \frac{u(x) - u(x - y)}{\sin^2(y/2)} dy.
\]

We define the \(L^2\) based Sobolev spaces

\[
H^s = \left\{ u(x) = \sum_{n \in \mathbb{Z}} \hat{u}(n)e^{inx} \right\} \text{ with } \sum_{n \in \mathbb{Z}} |n|^{2s} |\hat{u}(n)|^2 < \infty
\]

with norm \( \|u\|_{H^s} = \|\Lambda^s u\|_{L^2} \). Similarly, we recall the definition of the Wiener spaces

\[
A^s = \left\{ u(x) = \sum_{n \in \mathbb{Z}} \hat{u}(n)e^{inx} \right\} \text{ with } \sum_{n \in \mathbb{Z}} |n|^s |\hat{u}(n)| < \infty
\].
2. Proof of Theorem 1

Well-posedness. The existence will follow using the energy method [26] once the appropriate a priori estimates are obtained. We define the energy

\[ E(t) = \frac{1}{\min_x u(x, t)} + \|u(t)\|_{H^2}. \]

We have to prove an inequality of the type

\[ \frac{d}{dt} E(t) \leq C(1 + E(t))^p, \]

for certain \( C \) and \( p \).

To estimate the first term in the energy we use a pointwise argument (see [11, 9, 19, 3] for more details). The solution has at least a minimum:

\[ m(t) = \min_x u(x, t) = u(\mathcal{X}_t, t). \]

Because of the positivity of the initial data, we have that \( m(0) > 0 \). Following the argument in [11, 19, 3], we have that

\[ \frac{d}{dt} m(t) = \partial_t u(\mathcal{X}_t, t) = -\frac{m(t) \Lambda u(\mathcal{X}_t)}{m(t)^2 + (Hu(\mathcal{X}_t))^2} \quad \text{a.e.}. \]

Then,

\[ \frac{d}{dt} \frac{1}{\min_x u(x, t)} = -\frac{\partial_t u(\mathcal{X}_t, t)}{m(t)^2} \leq C \frac{\|u\|_{H^2}}{m(t)^3} \leq C (E(t))^4 \]

For the sake of brevity we only provide with the estimates for the higher order terms (being the rest of the terms straightforward). We take 2 derivatives of the equation and test against \( \partial_x^2 u \). We find that

\[
\frac{d}{dt} \|u(t)\|_{H^2}^2 = -\left( \frac{\partial_x^2 u \Lambda u + u \Lambda \partial_x^2 u}{u^2 + (Hu)^2} + \frac{u \Lambda u(2u \partial_x^2 u + 2Hu \partial_x \Lambda u) \partial_x^2 u}{(u^2 + (Hu))^2} \right)
\]

\[ + \left( \frac{\partial_x \Lambda u \partial_x^2 u + Hu \partial_x^2 u}{u^2 + (Hu)^2} - \frac{Hu \partial_x u(2u \partial_x^2 u + 2Hu \Lambda \partial_x u) \partial_x^2 u}{(u^2 + (Hu))^2} \right) + \text{l.o.t.} \]

Using that

\[ \left\| \frac{1}{u^2 + (Hu)^2} \right\|_{L^\infty} \leq \left\| \frac{1}{u^2} \right\|_{L^\infty} \leq \frac{1}{\min_x u^2} \leq \|E(t)\|_{H^2}, \]

we have that

\[ \text{l.o.t.} \leq C (E(t))^8 \|u(t)\|_{H^2}. \]

We recall the Córdoba-Córdoba inequality [10]

\[ \theta \Lambda \theta \geq \frac{1}{2} \Lambda (\theta^2), \]
to find that
\[
\int_{S^1} \frac{u \Lambda \partial_t^2 u \partial_x^2 u}{u^2 + (Hu)^2} \, dx \geq \frac{1}{2} \int_{S^1} H \partial_x \left( \frac{u}{u^2 + (Hu)^2} \right) \partial_x^2 u \, dx.
\]

\[
I_1 \leq \frac{\|u(t)\|_{H^1}^2 \|u\|_{A^1}}{m(t)^2} - \frac{1}{2} \int_{S^1} H \partial_x \left( \frac{u}{u^2 + (Hu)^2} \right) \partial_x^2 u \, dx
\]

\[
\leq \frac{\|u(t)\|_{H^2}^2 \|u\|_{A^1}}{m(t)^2} + \frac{\|u(t)\|_{H^1}^2 \|u\|_{A^1}}{m(t)^2}
\]

\[
\leq C(1 + \varepsilon(t))^8 \|u(t)\|_{H^2}.
\]

A similar use of Hölder inequality and Sobolev embedding leads to
\[
I_2 + I_4 \leq C(1 + \varepsilon(t))^8 \|u(t)\|_{H^2}.
\]

Finally, we observe that an integration by parts allow us to conclude
\[
I_3 \leq C(1 + \varepsilon(t))^8 \|u(t)\|_{H^2}.
\]

Then, we obtain that
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^2}^2 \leq C(1 + \varepsilon(t))^8 \|u(t)\|_{H^2}.
\]

Thus, collecting (6), (7) and (8), we conclude the desired inequality (5).

**An identity for the evolution of the L^2 norm.** We test (1) against u. We find that
\[
\frac{d}{dt} \|u(t)\|_{L^2}^2 = \int_{S^1} \left( - \frac{u \Lambda uu}{u^2 + (Hu)^2} + \frac{Hu \partial_x uu}{u^2 + (Hu)^2} \right) \, dx.
\]

We compute
\[
\mathcal{D} = -\frac{1}{2} \int_{S^1} Hu \partial_x \log \left( u^2 + (Hu)^2 \right) \, dx
\]

\[
= -\int_{S^1} Hu \frac{\partial_x uu + Hu \Lambda u}{u^2 + (Hu)^2} \, dx
\]

\[
= -\int_{S^1} \frac{\partial_x uu Hu}{u^2 + (Hu)^2} - \frac{u^2 \Lambda u}{u^2 + (Hu)^2} + \Lambda u \, dx
\]

\[
= -\int_{S^1} \frac{\partial_x uu Hu}{u^2 + (Hu)^2} - \frac{u^2 \Lambda u}{u^2 + (Hu)^2} \, dx.
\]

As a consequence,
\[
\frac{d}{dt} \|u(t)\|_{L^2}^2 = -\mathcal{D}.
\]

Furthermore,
\[
\mathcal{D} = \frac{1}{2} \int_{S^1} \Lambda u \log \left( u^2 + (Hu)^2 \right) \, dx
\]

\[
= \frac{1}{8\pi} \int_{S^1} \text{p.v.} \int_{S} \frac{u(x) - u(x - y)}{\sin(y/2)^2} \log \left( u(x)^2 + (Hu(x))^2 \right) \, dxdy.
\]
= \frac{1}{8\pi} \int_{S^1} \text{p.v.} \int_{S} \frac{u(x) - u(y)}{\sin((x - y)/2)} \log (u(x)^2 + (Hu(x))^2) \, dx \, dy

= \frac{1}{8\pi} \int_{S^1} \text{p.v.} \int_{S} \frac{u(y) - u(x)}{\sin((x - y)/2)} \log (u(y)^2 + (Hu(y))^2) \, dx \, dy.

Then, we have identity (3). This concludes with the existence part. The uniqueness follow from a standard contradiction argument.

**Propagation of the $L^1$ norm.** Once the solution remains positive, the $L^1$ norm is preserved due to the divergence form of the equation.

**Propagation of the even symmetry.** This is a straightforward consequence of the fact that the Hilbert transform $H$ maps even functions into odd functions.

**Maximum principle.** We define

$$M(t) = \max_x u(x, t) = u(\overline{x}_t, t)$$

Then (see [16, 19, 3] for more details) we have that

$$\frac{d}{dt} M(t) = \partial_t u(\overline{x}_t, t) \, \text{ a.e.},$$

Then

$$\frac{d}{dt} M(t) + \frac{M(t)\Lambda u(\overline{x}_t)}{M(t)^2 + (Hu(\overline{x}_t))^2} = 0,$$

We observe that

$$\Lambda u(\overline{x}_t) \geq 0.$$ 

Thus, using $0 \leq M(t)$, we obtain that

$$M(t) \leq M(0).$$

**Minimum principle.** With the previous definition for $m(t)$, we have that

$$\frac{d}{dt} m(t) + \frac{m(t)\Lambda u(x_t)}{m(t)^2 + (Hu(x_t))^2} = 0,$$

Thus, using $\Lambda u(x_t) \leq 0$, we find that

$$0 \leq m(0) \leq m(t).$$

**3. Proof of Theorem 2**

The proof of this Theorem follows the approach in [18]. We define the new variable

$$v(x, t) = u(x, t) - \langle u_0 \rangle.$$ 

This variable quantifies the difference between the steady state $u_x = \langle u_0 \rangle$ and $u$. The idea of the theorem is first to linearize around the steady state $\langle u_0 \rangle$. Secondly, we obtain an inequality of the form

$$\frac{d}{dt} \|v(t)\|_{A^1} + \frac{\|v(t)\|_{A^2}}{\langle u_0 \rangle} \leq F \left( \frac{\|v(t)\|_{A^1}}{\langle u_0 \rangle} \right) \|v(t)\|_{A^2},$$
with $\mathcal{F}(0) = 0$ and $\mathcal{F}$ continuous. We observe that this inequality guarantees $v(t) \to 0$ in $A^1$ for small enough $\|v_0\|_{A^1}/\langle u_0 \rangle$.

In what follows we assume that

$$r = \frac{\|v\|_{A^1}}{\langle u_0 \rangle} < \frac{1}{2},$$

so that

$$\frac{\|v\|_{A^1}}{\langle u_0 \rangle - \|v\|_{A^1}} = \frac{r}{1-r} < 1.$$ 

Since we have the following Poincaré type inequality

$$\|v\|_{A^s} \leq \|v\|_{A^r}, \forall 0 \leq s < r,$$

we observe that

$$\frac{\|H u\|}{u} = \frac{\|H v\|}{u} \leq \frac{\|v\|_{A^0}}{\langle u_0 \rangle - \|v\|_{A^0}} \leq \frac{\|v\|_{A^1}}{\langle u_0 \rangle - \|v\|_{A^1}} < 1.$$ 

As a consequence, we can expand the nonlinearity as a power series

$$\arctan \left( \frac{H u}{u} \right) = \sum_{n \in \mathbb{Z}^+ \cup \{0\}} (-1)^n \left( \frac{H u}{u} \right)^{2n} \left( \frac{\Lambda v}{v + \langle u_0 \rangle} - \frac{H v \hat{c}_x v}{(v + \langle u_0 \rangle)^2} \right),$$

so

$$\hat{c}_t u = - \sum_{n \in \mathbb{Z}^+ \cup \{0\}} (-1)^n \left( \frac{H u}{u} \right)^{2n} \left( \frac{\Lambda v}{v + \langle u_0 \rangle} - \frac{H u \hat{c}_x u}{u^2} \right).$$

In the new variable, this latter equation reads

$$\hat{c}_t v = - \sum_{n \in \mathbb{Z}^+} (-1)^n \left( \frac{H v}{v + \langle u_0 \rangle} \right)^{2n} \left( \frac{\Lambda v}{v + \langle u_0 \rangle} - \frac{H v \hat{c}_x v}{(v + \langle u_0 \rangle)^2} \right)$$

$$- \left( \frac{\Lambda v}{v + \langle u_0 \rangle} - \frac{H v \hat{c}_x v}{(v + \langle u_0 \rangle)^2} \right).$$

We recall the following Taylor series

$$\frac{1}{\langle u_0 \rangle + v} = \frac{1}{\langle u_0 \rangle} + \frac{1}{\langle u_0 \rangle} \sum_{n \in \mathbb{Z}^+} (-1)^n \left( \frac{v}{\langle u_0 \rangle} \right)^n,$$

$$\frac{1}{(\langle u_0 \rangle + v)^2} = \frac{1}{\langle u_0 \rangle^2} + \frac{1}{\langle u_0 \rangle^2} \sum_{n \in \mathbb{Z}^+} (-1)^n (1 + n) \left( \frac{v}{\langle u_0 \rangle} \right)^n.$$ 

We define

$$\mathfrak{S}_1 = \frac{H v}{\langle u_0 \rangle} + \frac{H v}{\langle u_0 \rangle} \sum_{m \in \mathbb{Z}^+} (-1)^m \left( \frac{v}{\langle u_0 \rangle} \right)^m,$$

$$\mathfrak{S}_2 = \frac{H v \hat{c}_x v}{\langle u_0 \rangle^2} + \frac{H v \hat{c}_x v}{\langle u_0 \rangle^2} \sum_{m \in \mathbb{Z}^+} (-1)^m (1 + m) \left( \frac{v}{\langle u_0 \rangle} \right)^m,$$

$$\mathfrak{S}_3 = \frac{\Lambda v}{\langle u_0 \rangle} \sum_{m \in \mathbb{Z}^+} (-1)^m \left( \frac{v}{\langle u_0 \rangle} \right)^m.$$
Using the previous Taylor series together with the previous definitions, we find that
\[
\partial_t v + \frac{\Lambda v}{\langle u_0 \rangle} = - \sum_{n \in \mathbb{Z}^+} (-1)^n (\mathcal{G}_1)^{2n} \left( \frac{\Lambda v}{\langle u_0 \rangle} + \mathcal{G}_3 \right) \\
+ \sum_{n \in \mathbb{Z}^+} (-1)^n (\mathcal{G}_1)^{2n} \mathcal{S}_2 - \mathcal{S}_3 + \mathcal{S}_2.
\]
(9)

We take a derivative of (9) to obtain that
\[
\partial_t \partial_x v + \frac{\Lambda \partial_x v}{\langle u_0 \rangle} = \sum_{n \in \mathbb{Z}^+} (-1)^{n+1} 2n (\mathcal{G}_1)^{2n-1} \partial_x \mathcal{G}_1 \left( \frac{\Lambda v}{\langle u_0 \rangle} + \mathcal{G}_3 \right) \\
- \sum_{n \in \mathbb{Z}^+} (-1)^n (\mathcal{G}_1)^{2n} \left( \frac{\Lambda \partial_x v}{\langle u_0 \rangle} + \partial_x \mathcal{G}_3 \right) \\
+ \sum_{n \in \mathbb{Z}^+} (-1)^n 2n (\mathcal{G}_1)^{2n-1} \partial_x \mathcal{G}_1 \mathcal{S}_2 + \sum_{n \in \mathbb{Z}^+} (-1)^n (\mathcal{G}_1)^{2n} \partial_x \mathcal{S}_2 \\
- \partial_x \mathcal{S}_3 + \partial_x \mathcal{S}_2.
\]
(10)

We want to estimate \( \| v \|_{A^0} \leq \| \partial_x v \|_{A^0} \).

To do that we first observe that \( A^0 \) is an algebra, thus,
\[
\| \mathcal{G}_1^{2n} \|_{A^0} \leq \| \mathcal{G}_1 \|_{A^0}^{2n} \leq \left( \frac{\| v \|_{A^0}}{\langle u_0 \rangle} + \frac{\| v \|_{A^0}}{\langle u_0 \rangle} \sum_{m \in \mathbb{Z}^+} \left( \frac{\| v \|_{A^0}}{\langle u_0 \rangle} \right)^m \right)^{2n}.
\]

Summing up the series, we find the estimate
\[
\| \mathcal{G}_1^{2n} \|_{A^0} \leq \left( \frac{\| v \|_{A^0}}{\langle u_0 \rangle} - \| v \|_{A^0} \right)^{2n}.
\]

Similarly,
\[
\| \mathcal{G}_1^{2n-1} \|_{A^0} \leq \left( \frac{\| v \|_{A^0}}{\langle u_0 \rangle} - \| v \|_{A^0} \right)^{2n-1},
\]
\[
\| \mathcal{G}_2 \|_{A^0} \leq \frac{\| v \|_{A^0}}{\langle u_0 \rangle} + \frac{\| v \|_{A^0}}{\langle u_0 \rangle^2} \sum_{m \in \mathbb{Z}^+} (1 + m) \left( \frac{\| v \|_{A^0}}{\langle u_0 \rangle} \right)^m \leq \frac{\| v \|_{A^0} \| v \|_{A^1}}{\langle u_0 \rangle - \| v \|_{A^0}^2},
\]
\[
\| \mathcal{G}_3 \|_{A^0} \leq \frac{\| v \|_{A^1}}{\langle u_0 \rangle} \sum_{m \in \mathbb{Z}^+} \left( \frac{\| v \|_{A^0}}{\langle u_0 \rangle} \right)^m \leq \frac{\| v \|_{A^1} \| v \|_{A^0}}{\langle u_0 \rangle - \| v \|_{A^0}},
\]
\[
\| \partial_x \mathcal{G}_1 \|_{A^0} \leq \frac{\| v \|_{A^1}}{\langle u_0 \rangle - \| v \|_{A^0}} + \frac{\| v \|_{A^0}}{\langle u_0 \rangle - \| v \|_{A^0}^2},
\]
\[
\| \partial_x \mathcal{G}_2 \|_{A^0} \leq \frac{\| v \|_{A^2}^2 + \| v \|_{A^0} \| v \|_{A^2}^2}{\langle u_0 \rangle - \| v \|_{A^0}^2} + \frac{\| v \|_{A^0} \| v \|_{A^1}^2}{\langle u_0 \rangle - \| v \|_{A^0}^3},
\]
\[
\| \partial_x \mathcal{G}_3 \|_{A^0} \leq \frac{\| v \|_{A^2}^2 \| v \|_{A^0} + \| v \|_{A^1}^2}{\langle u_0 \rangle - \| v \|_{A^0}} + \frac{\| v \|_{A^1}^2}{\langle u_0 \rangle - \| v \|_{A^0}^2}.\]
We obtain that
\[
\frac{d}{dt} \|v\|_{A^1} + \|v\|_{A^2}^2 \leq \frac{\|v\|_{A^2}^2}{\langle u_0 \rangle} \left\{ \sum_{n \in \mathbb{Z}^+} 2n \left( \frac{r}{1-r} \right)^{2n-1} \left[ \frac{r}{1-r} + \frac{r^2}{(1-r)^2} \right] \frac{1}{1-r} + \sum_{n \in \mathbb{Z}^+} \left( \frac{r}{1-r} \right)^{2n} \left( 1 + \frac{r}{1-r} + \frac{r^2}{(1-r)^2} \right) \right. \\
+ \sum_{n \in \mathbb{Z}^+} 2n \left( \frac{r}{1-r} \right)^{2n-1} \left[ \frac{r}{1-r} + \frac{r^2}{(1-r)^2} \right] \frac{r}{(1-r)^2} \\
+ 2 \sum_{n \in \mathbb{Z}^+} \left( \frac{r}{1-r} \right)^{2n} \left( \frac{r}{(1-r)^2} + \frac{r^2}{(1-r)^3} \right) \\
\left. + \frac{r}{1-r} + \frac{r}{(1-r)^2} \right\}. \\
\tag{11}
\]
Using
\[
\frac{z^2}{1-z^2} = \sum_{n \in \mathbb{Z}^+} z^{2n},
\]
we find
\[
\frac{d}{dt} \|v\|_{A^1} + \|v\|_{A^2}^2 \leq \frac{\|v\|_{A^2}^2}{\langle u_0 \rangle} \left\{ \frac{2 \frac{r}{1-r}}{1 - \left( \frac{r}{1-r} \right)^2} \left[ \frac{r}{1-r} + \frac{r^2}{(1-r)^2} \right] \frac{1}{1-r} \\
+ \frac{\left( \frac{r}{1-r} \right)^2}{1 - \left( \frac{r}{1-r} \right)^2} \left( 1 + \frac{r}{1-r} + \frac{r^2}{(1-r)^2} \right) \right. \\
+ \frac{2 \frac{r}{1-r}}{1 - \left( \frac{r}{1-r} \right)^2} \left[ \frac{r}{1-r} + \frac{r^2}{(1-r)^2} \right] \frac{r}{(1-r)^2} \\
+ 2 \frac{\left( \frac{r}{1-r} \right)^2}{1 - \left( \frac{r}{1-r} \right)^2} \left( \frac{r}{(1-r)^2} + \frac{r^2}{(1-r)^3} \right) \\
\left. + \frac{r}{1-r} + \frac{r}{(1-r)^2} \right\}. \\
\tag{12}
\]
Finally, we can simplify the previous expression and find that
\[
\mathcal{F}(r) = \frac{2 \frac{r}{1-r}}{1 - \left( \frac{r}{1-r} \right)^2} \left( \frac{r}{(1-r)^3} + \frac{r}{(1-r)^4} \right) + \frac{\left( \frac{r}{1-r} \right)^2}{1-2r}
\]
We observe that $F$ is a continuous function in a neighborhood of $r = 0$ and satisfies $F(0) = 0$. Thus, it exists $0 < C$ such that $F(C) < 1$. We finally observe that if $\|v_0\|_{A^1}/\langle u_0 \rangle < C$ this condition propagates in time and ensures the following bound

$$\|v(t)\|_{A^1} \leq \|v_0\|_{A^1} e^{-\delta t},$$

for small enough $0 < \delta < 1$. This last inequality together with a close inspection of the energy estimates in Theorem 1 lead to the following inequality

$$\frac{d}{dt}\|u\|_{H^2}^2 \leq C(u_0)\|u\|_{H^2}^2,$$

and then we conclude the global bound for the $H^2$ norm using Gronwall’s inequality.

4. Proof of Theorem 3

In this section we prove the existence of global weak solutions for certain initial data satisfying appropriate size restriction in the space $A^0$. We emphasize that this space is scale invariant with respect to the scaling of the equation. First we obtain a priori estimates, then we consider a vanishing viscosity approximation and prove the convergence of the approximate solutions.

A priori estimates. Following the previous ideas, the first nonlinear term contributes with

$$\|NL_1\|_{A^0} = \left\| \sum_{n \in \mathbb{Z}^+} (-1)^n (\mathcal{S}_1)^{2n} \left( \frac{\Lambda v}{\langle u_0 \rangle} + \frac{\Lambda v}{\langle u_0 \rangle} \right) \sum_{m \in \mathbb{Z}^+} (-1)^m \left( \frac{v}{\langle u_0 \rangle} \right)^m \right\|_{A^0}$$

$$\leq \sum_{n \in \mathbb{Z}^+} \left( \frac{\|v\|_{A^0}}{\langle u_0 \rangle} - \|v\|_{A^0} \right)^{2n} \left( \frac{\|v\|_{A^1}}{\langle u_0 \rangle} - \|v\|_{A^1} \right)$$

$$\leq \left( \frac{\|v\|_{A^1}}{\langle u_0 \rangle} - \|v\|_{A^1} \right) \left( \frac{1}{1 - \left( \frac{|v|_{A^0}}{\langle u_0 \rangle - |v|_{A^0}} \right)^2} - 1 \right)$$

$$\leq \left( \frac{\|v\|_{A^1}}{\langle u_0 \rangle} - \|v\|_{A^1} \right) \left( \frac{|v|_{A^0}^2}{\langle u_0 \rangle - |v|_{A^0}} \right) \left( \frac{|v|_{A^0}^2}{\langle u_0 \rangle - |v|_{A^0}} \right).$$

The second nonlinear term can be estimated as

$$\|NL_2\|_{A^0} = \left\| \sum_{n \in \mathbb{Z}^+} (-1)^n (\mathcal{S}_1)^{2n} \left( \frac{H v \partial_x v}{\langle u_0 \rangle^2} + \frac{H v \partial_x v}{\langle u_0 \rangle^2} \right) \sum_{m \in \mathbb{Z}^+} (-1)^m (1 + m) \left( \frac{v}{\langle u_0 \rangle} \right)^m \right\|_{A^0}$$
\[ \sum_{n \in \mathbb{Z}^+} \left( \frac{\|v\|_{A^0}}{\langle u_0 \rangle} \right)^{2n} \left( \frac{(\|v\|_{A^0} - \|v\|_{A^1})^2}{(\langle u_0 \rangle - \|v\|_{A^0})^2} \right) \]

\[ \left( \frac{\|v\|_{A^0} \|v\|_{A^1}}{(\langle u_0 \rangle - \|v\|_{A^0})^2} \right) \left( \frac{(\|v\|_{A^0})^2}{1 - (\|v\|_{A^0})^2} \right). \]

Finally, we find that

\[ \|NL_3\|_{A^0} = \left\| A v \frac{1}{\langle u_0 \rangle} \sum_{n \in \mathbb{Z}^+} (-1)^n \left( \frac{v}{\langle u_0 \rangle} \right)^n \right\|_{A^0} \]

\[ \leq \left\| \frac{v}{\langle u_0 \rangle} \right\|_{A^1} \sum_{n \in \mathbb{Z}^+} \left( \frac{\|v\|_{A^0}}{\langle u_0 \rangle} \right)^n \]

\[ \leq \left\| \frac{v}{\langle u_0 \rangle} \right\|_{A^1} \left( \frac{1}{\langle u_0 \rangle - \|v\|_{A^0}} - \frac{1}{\langle u_0 \rangle} \right). \]

\[ \|NL_4\|_{A^0} = \left\| \left( \frac{H v \hat{\psi}_0 v}{\langle u_0 \rangle^2} + \frac{H v \hat{\psi}_0 v}{\langle u_0 \rangle^2} \sum_{n \in \mathbb{Z}^+} (-1)^n (1 + n) \left( \frac{v}{\langle u_0 \rangle} \right)^n \right) \right\|_{A^0} \]

\[ \leq \left\| \frac{v}{\langle u_0 \rangle} \right\|_{A^1} \left( \frac{1}{\langle u_0 \rangle^2} + \frac{v}{\langle u_0 \rangle^2} \right) \sum_{n \in \mathbb{Z}^+} (1 + n) \left( \frac{\|v\|_{A^0}}{\langle u_0 \rangle} \right)^n \]

\[ \leq \frac{\left\| \frac{v}{\langle u_0 \rangle} \right\|_{A^0} \left\| \frac{v}{\langle u_0 \rangle} \right\|_{A^1}}{(\langle u_0 \rangle - \|v\|_{A^0})^2}. \]

We define

\[ s = \frac{\|v\|_{A^0}}{\langle u_0 \rangle}. \]

Collecting the previous estimates, we find that

\[ \frac{d}{dt} \left( \frac{\|v\|_{A^0}}{\langle u_0 \rangle} \right) + \frac{\|v\|_{A^1}}{\langle u_0 \rangle} \leq \left[ \frac{s}{(1 - s)^2} + \frac{s}{1 - s} + \left( \frac{s}{1 - s} \right)^2 \right] \left( \frac{\left( \frac{s}{1 - s} \right)^2}{1 - \left( \frac{s}{1 - s} \right)^2} \right) \]

\[ + \left( \frac{1}{1 - s} \right) \left( \frac{\left( \frac{s}{1 - s} \right)^2}{1 - \left( \frac{s}{1 - s} \right)^2} \right) \]

Using the hypotheses on \( \hat{\psi} \), we conclude that

\[ s \leq \hat{\psi} \]

implies

\[ \frac{d}{dt} \|v\|_{A^0} + \delta \|v\|_{A^1} \leq 0, \]

and that, thank to a Poincaré-type inequality, leads to

\[ \|v(t)\|_{L^\infty} \leq \|v_0\|_{A^0} e^{-\delta t}. \]
Furthermore, the solution also enjoys the following parabolic gain of regularity
\[ \int_0^t \|v(s)\|^2_{H^{0.5}} \, ds \leq \int_0^t \|v(s)\|_{A^1} \, ds \sup_s \|v(s)\|_{L^1} \leq \int_0^t \|v(s)\|_{A^1} \, ds \|v_0\|_{A^0} 2 \pi. \]

**Approximated solutions.** To construct the approximate solutions, we consider the following vanishing viscosity approximated problem
\[ \bar{\partial}_t u^\varepsilon + \bar{\partial}_x \arctan \left( \frac{H u^\varepsilon}{u^\varepsilon} \right) = \varepsilon \bar{\partial}_x^2 u^\varepsilon \quad (x, t) \text{ on } S^1 \times [0, T], \]
with a mollified initial data
\[ u^\varepsilon(x, 0) = \mathcal{M}_\varepsilon * u_0(x). \]

The corresponding approximate solution exists globally and remains smooth.

**Compactness.** Fix \( 0 < T < \infty \). We have that \( u^\varepsilon \) is uniformly bounded in
\[ L^\infty(0, T; A^0) \cap L^2(0, T; H^{0.5}). \]
This implies weak-* convergence
\[ u^\varepsilon \rightharpoonup u, \]
in
\[ L^\infty([0, T] \times S^1), \]
and weak convergence
\[ u^\varepsilon \to u, \]
in
\[ L^2(0, T; H^{0.5}(S^1)). \]
Furthermore, \( \bar{\partial}_t u^\varepsilon \) is uniformly bounded in
\[ L^2(0, T; H^{-1.5}). \]
A standard application of Aubin-Lions Theorem [32] ensures the strong convergence (after maybe taking a subsequence)
\[ u^\varepsilon \to u, \quad H u^\varepsilon \to H u \]
in
\[ L^2(0, T; L^2). \]
Taking another subsequence if necessary, we obtain that
\[ u^\varepsilon(x, t) \to u(x, t) \text{ a.e in } S^1 \times [0, T] \]
In particular, we conclude the lower bound
\[ \min_x u_0(x) \leq u(x, t) \text{ a.e in } S^1 \times [0, T]. \]
Passing to the limit. Being the other terms linear, we only have to take into consideration the convergence of
\[ J = \int_0^T \int_{S^1} \left( \arctan \left( \frac{H u'}{u^2} \right) - \arctan \left( \frac{H u}{u} \right) \right) \partial_x \phi dx ds. \]
We have that
\[ J \leq \int_0^T \int_{S^1} \left| \frac{H u'}{u^2} - \frac{H u}{u} \right| |\partial_x \phi| dx ds. \]
Using the lower bounds for \( u \) and \( u' \) together with Hölder inequality, we conclude that
\[ J \to 0. \]
This concludes the proof of the existence of a global weak solution \( u \).

5. Link between (1) and (4)

We now look for a solution of (1) having the following form
\[ u(x, t) = \langle u_0 \rangle + \varepsilon \sum_{j=0}^{\infty} \varepsilon^j f^{(j)}(x, t), \]
(here \( \varepsilon \) can be thought as the displacement from the homogeneous state \( \langle u_0 \rangle \)).

The idea is to truncate the series up to certain order, two say,
\[ f(x, t) = \varepsilon f^{(0)}(x, t) + \varepsilon^2 f^{(1)}(x, t), \]
and see what \( f \) solves. In this way we will obtain that (up to \( O(\varepsilon^3) \)), \( f \) solves (4). A similar approach has been used in the study of free boundary problems for incompressible fluids (see [7, 20, 21] and the references therein).

First, we observe that (1) can be equivalently written as
\[ \partial_t u + \frac{u \Lambda u - H u \partial_x u}{u^2 + (H u)^2} = 0 \quad (x, t) \text{ on } S^1 \times [0, T]. \]
Thus,
\[ \partial_t u \left( \langle u_0 \rangle^2 + 2(u - \langle u_0 \rangle)\langle u_0 \rangle + (u - \langle u_0 \rangle)^2 + (H u)^2 \right) + u \Lambda u - H u \partial_x u = 0. \]

Forcing the previous ansatz and matching the powers of \( \varepsilon \), we find that \( f^{(0)} \) solves
\[ \partial_t f^{(0)} + \frac{\Lambda f^{(0)}}{\langle u_0 \rangle} = 0. \]

Similarly, \( f^{(1)} \) solves
\[ \partial_t f^{(1)} \langle u_0 \rangle^2 + 2 \partial_t f^{(0)} f^{(0)} \langle u_0 \rangle + \langle u_0 \rangle \Lambda f^{(1)} + f^{(0)} \Lambda f^{(0)} - H f^{(0)} \partial_x f^{(0)} = 0. \]
Thus, substituting \( \langle u_0 \rangle \partial_t f^{(0)} \) by \(- \Lambda f^{(0)} \), we find that \( f \) solves
\[ \partial_t f + \frac{1}{\langle u_0 \rangle} \Lambda f - \frac{1}{\langle u_0 \rangle^2} \partial_x (H f f) = O(\varepsilon^3). \]
Thus, neglecting the $O(\varepsilon^3)$ terms we find that
\[ g(x, t) = \frac{f(x, t\langle u_0 \rangle)}{\langle u_0 \rangle} \]
solves (4).

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