ALMOST EVERYWHERE CONVERGENCE OF FOURIER SERIES ON SU(2): THE CASE OF HÖLDER CONTINUOUS FUNCTIONS

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Abstract. We consider an aspect of the open problem: Does every square-integrable function on SU(2) have an almost everywhere convergent Fourier series? Let 0 < α < 1. We show that to each countable set E in SU(2) there corresponds an α-Hölder continuous function on SU(2) whose Fourier series diverges on E. We also show that the Fourier series of each α-Hölder continuous function on SU(2) converges almost everywhere.

1. Introduction

The Peter-Weyl theorem suggests the study of the formal Fourier series \( \sum d_\lambda (\chi_\lambda \ast f) \) of a function \( f \) on a compact, connected, semisimple Lie group \( G \). Here the sum is over the equivalence classes of continuous irreducible unitary representations of \( G \), \( d_\lambda \) is the degree of the representation, and \( \chi_\lambda \) is its character. The vast literature of Fourier analysis on \( G \) is primarily concerned with mean convergence or divergence of the Fourier series of \( f \in L^p(G) \) (e.g. [7,9,17,18]), uniform or absolute convergence of the partial sums if \( f \) is smooth (e.g. [3,9,11–13,15,16,20,21]), almost everywhere convergence or divergence of the partial sums if \( f \) is a central function in \( L^p(G) \) (e.g. [5,8]), and uniform, mean, or almost everywhere summability of the partial sums if \( f \) belongs to various subspaces of \( L^1(G) \) (e.g. [2,19,22]). The aim of this work is to advance the study of almost everywhere convergence or divergence of Fourier partial sums of nonsmooth, possibly noncentral functions in \( L^2(G) \).

Let \( SU(2) \) denote the two-dimensional special unitary group. We show that to each \( \alpha \) in \((0,1)\) and to each countable subset \( E \) of \( SU(2) \) there corresponds an \( \alpha \)-Hölder continuous function on \( SU(2) \) whose Fourier series diverges at each \( x \) in \( E \). Since it is possible to arrange that such a set \( E \) is dense in \( SU(2) \), the Fourier series of the corresponding function is divergent at infinitely many points in every nonempty open subset of \( SU(2) \). Nevertheless, the Fourier series of each \( \alpha \)-Hölder continuous function on \( SU(2) \) converges almost everywhere on \( SU(2) \).

In fact, relying on a general almost everywhere convergence result of Dai [6] for Fourier-Laplace series on spheres, it follows that if \( f \) in \( L^2(SU(2)) \) has an integral modulus of continuity \( \Omega(f,t) \) satisfying

\[
\int_0^1 \frac{\Omega^2(f,t)}{t} dt < \infty,
\]

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Date: 2020-03-24.

2010 Mathematics Subject Classification. 22E30, 43A50.

Key words and phrases. Fourier series, the two-dimensional special unitary group, Hölder continuous functions.
then the sequence of Fourier partial sums \( \{S_N f(x)\}_{N=1}^\infty \) converges to \( f(x) \) almost everywhere on \( SU(2) \). In particular, if \( f \) is an \( \alpha \)-Hölder continuous function on \( SU(2) \) for some \( \alpha \) in \( (0, 1) \), or more generally if \( \Omega(f, t) = O(t^\alpha) \) for some \( \alpha \) in \( (0, 1) \), then the Fourier partial sums of \( f \) converge to \( f(x) \) almost everywhere on \( SU(2) \).

It is still an open problem whether

\[
\lim_{N \to \infty} S_N f(x) = f(x)
\]

holds almost everywhere for every \( f \) in \( L^2(SU(2)) \). It follows from a result of Pollard \[14\] on Jacobi series that if \( f \) is a central function in \( L^p(SU(2)) \) for some \( p > 4/3 \), then (1.1) holds almost everywhere. On the other hand, a general theorem of Stanton and Tomas \[18\] for compact, connected, semisimple Lie groups shows that if \( p < 2 \) then there correspond an \( f \) in \( L^p(SU(2)) \) and a subset \( E \) of \( SU(2) \) of full measure such that (1.1) fails for all \( x \) in \( E \). Finally, the Peter-Weyl theorem implies that to each \( f \) in \( L^2(SU(2)) \) there corresponds an increasing sequence \( \{N_j\} \) of positive integers such that

\[
\lim_{j \to \infty} S_{N_j} f(x) = f(x)
\]

for almost every \( x \) in \( SU(2) \). This lends some hope for a positive answer to the problem (1.1), as do the results in this paper.

2. Preliminaries

Let \( SU(2) \) denote the two-dimensional special unitary group. General matrices \( x, y \in SU(2) \) can be expressed as

\[
x = \begin{bmatrix} \alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\ -(\beta_1 - i\beta_2) & \alpha_1 - i\alpha_2 \end{bmatrix}, \quad y = \begin{bmatrix} \gamma_1 + i\gamma_2 & \delta_1 + i\delta_2 \\ -(\delta_1 - i\delta_2) & \gamma_1 - i\gamma_2 \end{bmatrix}
\]

where \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \) are real numbers satisfying

\[
\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1 = \gamma_1^2 + \gamma_2^2 + \delta_1^2 + \delta_2^2.
\]

Equip \( SU(2) \) with the left and right translation invariant metric \( d \) given by

\[
d(x, y) = \sqrt{\frac{1}{2} \text{tr}((x-y)(x-y)^*)} = \left( (\alpha_1 - \gamma_1)^2 + (\alpha_2 - \gamma_2)^2 + (\beta_1 - \delta_1)^2 + (\beta_2 - \delta_2)^2 \right)^{1/2}.
\]

Let \( 0 < \alpha < 1 \) and let \( f \) be a real function on \( SU(2) \). If there exists a real number \( M \geq 0 \) such that

\[
|f(x) - f(y)| \leq Md^\alpha(x, y)
\]

for all \( x, y \in SU(2) \), then we say that \( f \) is an \( \alpha \)-Hölder continuous function on \( SU(2) \) and write \( f \in \text{Lip}_\alpha(SU(2)) \).

Let \( \mu \) denote normalized Haar measure on \( SU(2) \). If \( f \) and \( g \) are Haar-integrable functions on \( SU(2) \), then their convolution product is defined for all \( x \in SU(2) \) by

\[
(f * g)(x) = \int_{SU(2)} f(xy^{-1})g(y)d\mu(y).
\]

If \( f \in L^1(SU(2)) \) and \( f * g = g * f \) for all \( g \in L^1(SU(2)) \), then we call \( f \) a central function on \( SU(2) \). This is equivalent to the property that for \( \mu \)-almost every
Then there exists a function

Proof. Recall that for \( \text{Theorem 3.1.} \)

Let \( \text{Theorem 3.1.} \)

where \( y \in SU(2) \) and \( e^{+i\theta} \) are the eigenvalues of \( x \), it follows that if \( f \) is central then for \( \mu \)-almost every \( x \in SU(2) \),

\[
f(x) = f(\omega(\theta))
\]

where \( \theta \in [0, \pi] \). Furthermore, we have

\[
\int_{SU(2)} f(x)d\mu(x) = \frac{2}{\pi} \int_0^\pi f(\omega(\theta)) \sin^2(\theta)d\theta
\]

when \( f \) is central; this is a special case of equation (4.1) below.

We denote the family of all (inequivalent) continuous, irreducible, unitary representations of \( SU(2) \) by \( \{\pi_n\}_{n=0}^\infty \) (cf. pp. 125-136 in vol. 2 of [10]). Observe that \( \pi_n \) has dimension \( n+1 \) and its character \( \chi_n \) is the continuous central function on \( SU(2) \) given by

\[
\chi_n(x) = \text{trace} (\pi_n(x)).
\]

If \( e^{+i\theta} \) are the eigenvalues of \( x \), then

\[
\chi_n(x) = \chi_n(\omega(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)};
\]

the rightmost member this identity is the \( n^{\text{th}} \) Chebyshev polynomial of the second kind on \([0, \pi]\).

The Dirichlet kernel \( \{D_N\}_{N=0}^\infty \) on \( SU(2) \) is the sequence of continuous central functions given by

\[
D_N(x) = \sum_{n=0}^N (n+1)\chi_n(x).
\]

The \( N^{\text{th}} \) Fourier partial sum of a function \( f \in L^1(SU(2)) \) is the continuous function on \( SU(2) \) given by

\[
S_Nf(x) = (f * D_N)(x).
\]

3. Divergence of Fourier partial sums on a countable subset

**Theorem 3.1.** Let \( \alpha \in (0, 1) \) and let \( \{x_i\}_{i=1}^\infty \) be any countable subset of \( SU(2) \). Then there exists a function \( f \in \text{Lip}_\alpha(SU(2)) \) such that

\[
\sup_{N \geq 1} |S_N f(x_i)| = \infty
\]

for all \( i = 1, 2, 3, \ldots \).

**Proof.** Recall that for \( \alpha \in (0, 1) \), \( \text{Lip}_\alpha(SU(2)) \) is a Banach space with norm

\[
\|f\|_{\text{Lip}_\alpha(SU(2))} = \sup_{x \in SU(2)} |f(x)| + \sup_{x,y \in SU(2)} \frac{|f(x) - f(y)|}{d^\alpha(x,y)}.
\]

Fix \( x \in SU(2) \) and \( n \in \mathbb{N} \), and set \( A_n^x(f) = S_n f(x) \). Each \( A_n^x \) is a bounded linear functional on \( \text{Lip}_\alpha(SU(2)) \) of norm

\[
\|A_n^x\| = \sup \{|S_n f(x)| : f \in \text{Lip}_\alpha(SU(2)), \|f\|_{\text{Lip}_\alpha(SU(2))} \leq 1\} \leq \|D_n\|_{L^1(SU(2))}.
\]
Specializing to the case when $x = e$, the identity matrix in $SU(2)$, we have

$$A^*_n(f) = (f * D_n)(e) = \int_{SU(2)} f(y) D_n(y) d\mu(y) = \frac{2}{\pi} \int_0^\pi f(\omega(\theta)) D_n(\omega(\theta)) \sin^2(\theta) d\theta$$

by (2.1). Note that

$$D_n(\omega(\theta)) = \frac{-1}{2\sin(\theta)} D'_{n+1}(\theta)$$

where

$$D_n(t) = \sum_{j=-n}^n e^{ijt} = 1 + 2 \sum_{j=1}^n \cos(jt) = \frac{\sin((2n+1)t/2)}{\sin(t/2)}$$

is the Dirichlet kernel on $[-\pi, \pi]$. It follows that

$$A^*_n(f) = \frac{1}{\pi} \int_0^\pi f(\omega(\theta)) \cos\left(\frac{\theta}{2}\right) D_{n+1}(\theta) d\theta$$

$$- \left(\frac{2n+3}{\pi}\right) \int_0^\pi f(\omega(\theta)) \cos\left(\left(n + \frac{3}{2}\right)\theta\right) \cos\left(\frac{\theta}{2}\right) d\theta.$$  

The absolute maxima and minima of the function $h_n(\theta) = \cos\left(\left(n + \frac{3}{2}\right)\theta\right)$ on $[0, \pi]$ occur at the endpoints of the intervals $I_k = \left[\frac{2k\pi}{2n+3}, \frac{2(k+1)\pi}{2n+3}\right]$ where $k \in \{0, 1, 2, \ldots, n\}$. Let $g_n$ be the sawtooth function on $[0, \pi]$ determined by $g_n\left(\frac{2k\pi}{2n+3}\right) = (-1)^k$ for $0 \leq k \leq n + 1$, $g_n(\pi/2) = 0$, and $g_n$ is piecewise linear between these points. Define a central function $f_n$ on $SU(2)$ by $f_n(\omega(\theta)) = g_n(\theta)$ for $\theta \in [0, \pi]$. It is easy to see that $f_n$ belongs to $\text{Lip}_n(SU(2))$; in fact,

$$|f_n(x) - f_n(y)| \leq \frac{(\pi)^{\alpha}}{2^{\alpha}(2n+3)} \leq \pi$$

for all distinct matrices $x$ and $y$ in $SU(2)$.

Since $g_n(\theta) \cos\left(\left(n + \frac{3}{2}\right)\theta\right) \geq g^2_n(\theta) \geq 0$ on each interval $I_k$ and on $\left[\frac{2(n+1)\pi}{2n+3}, \pi\right]$, and since the function $\theta \mapsto \cos(\theta/2)$ is positive and decreasing on $[0, \pi]$, it follows that

$$\int_{I_k} f_n(\omega(\theta)) \cos\left(\left(n + \frac{3}{2}\right)\theta\right) \cos(\theta/2) d\theta \geq \int_{I_k} g^2_n(\theta) \cos(\theta/2) d\theta$$

$$\geq \cos\left(\frac{(k+1)\pi}{2n+3}\right) \int_{I_k} g^2_n(\theta) d\theta$$

$$= \frac{2\pi}{3(2n+3)} \cos\left(\frac{(k+1)\pi}{2n+3}\right)$$

for all $k \in \{0, 1, 2, \ldots, n\}$ and

$$\int_{\frac{(2n+1)\pi}{2n+3}}^\pi f_n(\omega(\theta)) \cos\left(\left(n + \frac{3}{2}\right)\theta\right) \cos(\theta/2) d\theta \geq 0.$$  

Adding these inequalities we obtain

$$\int_0^\pi f_n(\omega(\theta)) \cos\left(\left(n + \frac{3}{2}\right)\theta\right) \cos\left(\frac{\theta}{2}\right) d\theta \geq \frac{2}{3} \left(\frac{\pi}{2n+3}\right) \sum_{k=1}^{n+1} \cos\left(\frac{k\pi}{2n+3}\right)$$

$$= \frac{2}{3} \left(\frac{\pi}{2n+3}\right) \left\{ D_{n+1}\left(\frac{\pi}{2n+3}\right) - 1\right\}.$$
and hence
\[
\left| \frac{(2n+3)}{\pi} \int_0^\pi f_n(\omega(\theta)) \cos \left( \left( n + \frac{3}{2} \right) \theta \right) \cos \left( \frac{\theta}{2} \right) d\theta \right| \geq \frac{2}{3} \left\{ D_{n+1} \left( \frac{\pi}{2n + 3} \right) - 1 \right\}.
\]

Since the function \( \theta \mapsto f_n(\omega(\theta)) \cos^2(\theta/2) \) is uniformly bounded by 1 on \([0, \pi]\),
\[
\frac{1}{\pi} \int_0^\pi f_n(\omega(\theta)) \cos^2(\theta/2) D_{n+1}(\theta) d\theta \leq \frac{1}{\pi} \int_0^\pi |D_{n+1}(\theta)| d\theta = \frac{4}{\pi^2} \log(n + 1) + o(1)
\]
as \( n \to \infty \). Consequently
\[
\frac{|A_n^\alpha(f_n)|}{\|f_n\|_{\text{Lip}_\alpha(SU(2))}} \geq \frac{2}{3} \left\{ D_{n+1} \left( \frac{\pi}{2n + 3} \right) - 1 \right\} - \left( \frac{4}{\pi^2} \log(n + 1) + o(1) \right) \frac{1}{1 + \pi}
\]
But \( D_{n+1} \left( \frac{\pi}{2n + 3} \right) = \left( \sin \left( \frac{\pi}{2(2n + 3)} \right) \right)^{-1} \geq \frac{2(2n + 3)}{\pi} \) and hence
\[
|A_n^\alpha| = \sup \left\{ \frac{|A_n^\alpha(f)|}{\|f\|_{\text{Lip}_\alpha(SU(2))}} : f \in \text{Lip}_\alpha(SU(2)), f \neq 0 \right\}
\]
is asymptotically bounded below by
\[
\frac{2(2n + 3) - \frac{4}{\pi^2} \log(n + 1)}{1 + \pi}
\]
as \( n \to \infty \). Consequently, the sequence of bounded linear functionals
\[
A_n^\alpha(f) = S_n f(e)
\]
is not uniformly bounded on the Banach space \( \text{Lip}_\alpha(SU(2)) \) as \( n \to \infty \). By the uniform boundedness principle
\[
\sup_{n \geq 1} |S_n f(e)| = \infty
\]
for all \( f \) belonging to some dense \( G_3 \) set in \( \text{Lip}_\alpha(SU(2)) \).

If \( z \in SU(2) \), define the left translation operator \( L_z \) on \( \text{Lip}_\alpha(SU(2)) \) by \( L_z f(y) = f(zy) \) for all \( y \in SU(2) \). For each element of the countable subset \( \{ x_i \}_{i=1}^\infty \) of \( SU(2) \), observe that
\[
\left| A_n^\alpha \left( L_{x_i}^{-1} f_n \right) \right| = \left| A_n^\alpha(f_n) \right| \frac{\|f_n\|_{\text{Lip}_\alpha(SU(2))}}{\|f_n\|_{\text{Lip}_\alpha(SU(2))}}
\]
so there corresponds a dense \( G_3 \) subset \( E_{x_i} \) of \( \text{Lip}_\alpha(SU(2)) \) such that
\[
\sup_{n \geq 1} |S_n f(x_i)| = \infty
\]
for all \( f \in E_{x_i} \). By the Baire category theorem \( E = \bigcap_{n=1}^\infty E_{x_i} \) is dense in \( \text{Lip}_\alpha(SU(2)) \). In particular, \( E \) is nonempty and any \( f \in E \) gives the desired conclusion. \( \square \)

The problem of pointwise convergence for the Fourier series of central functions in \( \text{Lip}_\alpha(SU(2)) \) for some \( \alpha \in (0, 1) \) is related to an analogous problem for Fourier-Jacobi series of functions \( f : [-1, 1] \to \mathbb{R} \). Recall that the Chebyshev polynomials of the second kind \( U_n(n = 0, 1, 2, \ldots) \) are a special case of the Jacobi polynomials \( P_n^{(\alpha, \beta)} \) with \( \alpha = \beta = 1/2 \); i.e.
\[
U_n(\cos(\theta)) = P_n^{(1/2, 1/2)}(\cos(\theta)) = \frac{\sin((n + 1)\theta)}{\sin(\theta)} \quad (n = 0, 1, 2, \ldots; \theta \in [0, \pi]).
\]
Clearly \( \{U_n\}_{n=0}^{\infty} \) is an orthogonal set of functions with respect to the inner product
\[
\langle F, G \rangle = \int_{-1}^{1} F(t)G(t)\sqrt{1-t^2}dt.
\]

If \( F \in L^2([-1, 1], \sqrt{1-t^2}dt) \), let \( s_N(F; t) \) denote the \( N \)th partial sum of the Fourier series of \( F \) with respect to \( \{U_n\}_{n=0}^{\infty} \). If \( F \) has a modulus of continuity \( \omega(F,h) \) satisfying the Dini-Lipschitz condition
\[
\lim_{h \to 0^+} \left( \omega(F,h) \log(h^{-1}) \right) = 0,
\]
then a classical theorem [11] assures \( s_N(F; t) \to F(t) \) uniformly on any interval \([-1+\delta, 1-\delta]\) with \( \delta \in (0,1) \). Furthermore, a general theorem of Belen’kii [11] guarantees that if \( F \) satisfies a Dini-Lipschitz condition and \( \{s_N^{(\alpha,\beta)}(F; \pm 1)\}_{N=0}^{\infty} \) converge for some \( \alpha > -1 \) and \( \beta > -1 \), then the Fourier-Jacobi series of \( F \) converges uniformly to \( F \) on \([-1,1]\). Observe that a central function \( f \in L^2(SU(2)) \) corresponds to a function \( F \in L^2([-1, 1], \sqrt{1-t^2}dt) \) via \( F(\cos(\theta)) = f(x) \) where \( e^{i\theta}, e^{-i\theta} \) are the eigenvalues of \( x \in SU(2) \). It is easy to check that the Fourier partial sums of \( F \) and \( f \) satisfy the identity
\[
s_N(F; \cos(\theta)) = (S_N f)(x) \quad (N = 0, 1, 2, \ldots),
\]
and if \( f \in \text{Lip}_\alpha(SU(2)) \) for some \( \alpha \in (0, 1) \) then \( F \) satisfies a Dini-Lipschitz condition. This leads to the following result.

**Theorem 3.2.** Let \( f \) be a central function in \( \text{Lip}_\alpha(SU(2)) \) for some \( \alpha \in (0, 1) \). Then:
(a) \( S_N f(x) \to f(x) \) uniformly outside any open set containing \( \{e, -e\} \);
(b) \( S_N f(x) \to f(x) \) uniformly on \( SU(2) \) if \( \{S_N f(\pm x)\}_{N=0}^{\infty} \) converge.

The authors of this paper originally wondered if they could delete the word “central” from the hypothesis of Theorem 3.2 and still obtain conclusions (a) and (b). Theorem 3.1 shows that there is no possibility of such an analogue of Theorem 3.2 for general functions in \( \text{Lip}_\alpha(SU(2)) \) for \( 0 < \alpha < 1 \). This is so because the points of divergence for the Fourier partial sums of such a noncentral function need no longer be at the “poles” \( \pm e \). According to Theorem 3.1, points of divergence for \( \text{Lip}_\alpha(SU(2)) \) functions can be dense in \( SU(2) \).

4. **Almost everywhere convergence of Fourier partial sums**

Note that \( SU(2) \) is isometrically homeomorphic to the unit sphere \( S^3 \) in \( \mathbb{R}^4 \) via the isometry \( \eta \) from \( SU(2) \) onto \( S^3 \) given by
\[
\begin{bmatrix}
\alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\
-\beta_1 + i\beta_2 & \alpha_1 - i\alpha_2
\end{bmatrix} \mapsto (\alpha_1, \alpha_2, \beta_1, \beta_2).
\]

Furthermore, the spherical coordinate system on \( S^3 \):
\[
\begin{align*}
\alpha_1 &= \cos(\theta), \quad \alpha_2 = \sin(\theta) \cos(\phi), \\
\beta_1 &= \sin(\theta) \sin(\phi) \cos(\psi), \\
\beta_2 &= \sin(\theta) \sin(\phi) \sin(\psi),
\end{align*}
\]
where \( \phi \in [0, \pi], \theta \in [0, \pi], \) and \( \psi \in [0, 2\pi] \), can be transferred to \( SU(2) \) and forms a convenient parametrization thereof:
\[
x(\phi, \theta, \psi) = \begin{bmatrix}
\cos(\theta) + i\sin(\theta) \cos(\phi) & \sin(\theta) \sin(\phi) e^{i\psi} \\
-\sin(\theta) \sin(\phi) e^{-i\psi} & \cos(\theta) - i\sin(\theta) \cos(\phi)
\end{bmatrix}.
\]
We denote by Φ the mapping from \([0, \pi] \times [0, \pi] \times [0, 2\pi]\) onto \(SU(2)\) given by 
\((\phi, \theta, \psi) \mapsto x(\phi, \theta, \psi)\).

In spherical coordinates, normalized Haar measure \(\mu\) on \(SU(2)\) satisfies
\[
(4.1) \quad \int_{SU(2)} f(x) d\mu(x) = \frac{1}{4\pi^2} \int_0^\pi \int_0^{2\pi} (f \circ \Phi)(\phi, \theta, \psi) \sin^2(\theta) \sin(\phi) d\psi d\phi d\theta
\]
for every \(f \in L^1(SU(2), \mu) = L^1(SU(2))\) (cf. pp. 133-134 in vol. 2 of [10]). Using spherical coordinates, the \(N\)th Fourier partial sum of a function \(f \in L^1(SU(2))\) can be written as
\[
(S_N f)(x(\phi_0, \theta_0, \psi_0)) = \int_{SU(2)} D_N(y) f(x(\phi_0, \theta_0, \psi_0) y^{-1}(\phi, \theta, \psi)) d\mu(y)
\]
\[
= \frac{-1}{4\pi^2} \int_0^\pi \int_0^{2\pi} D_N^{+1}(\theta) f(x(\phi_0, \theta_0, \psi_0) y^{-1}(\phi, \theta, \psi)) \sin(\theta) \sin(\phi) d\psi d\phi d\theta
\]
\[
= \frac{-1}{4\pi} \int_0^\pi D_N^{+1}(\theta) \sin(\theta) [Q_x f](\theta) d\theta,
\]
where
\[
[Q_x f](\theta) = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} f(x(\phi_0, \theta_0, \psi_0) y^{-1}(\phi, \theta, \psi)) \sin(\phi) d\psi d\phi.
\]
The above identities for the Fourier partial sums of \(f \in L^p(SU(2))\) reduce convergence problems at a point \(x\) in the three-dimensional manifold \(SU(2)\) to the behavior of the real function \([Q_x f]\) on the interval \([0, \pi]\). See [12] and [13] for applications of this principle.

Because \(SU(2)\) is isometrically homeomorphic to the unit sphere \(S^3\), the theory of Fourier series of functions on \(SU(2)\) is closely connected with that of Fourier-Laplace series on \(S^3\). Specifically, if \(\sigma_3\) denotes normalized surface measure on \(S^3\), then the \(N\)th partial sum of the Fourier-Laplace series of \(f \in L^2(S^3, \sigma_3)\) is equal to the \(N\)th partial sum of the Fourier series of \(f = \Phi \circ \eta \in L^2(SU(2))\) [12 pp. 93-94]. Moreover, it follows from [11] that \(\sigma_3(E) = \mu(\eta^{-1}(E))\) for all spherical boxes
\[
E = \{ \varphi(\phi, \theta, \psi) \in S^3 : (\phi, \theta, \psi) \in [\alpha, \beta] \times [\gamma, \delta] \times [\epsilon, \nu]\}
\]
in \(S^3\), in particular, sets of zero Haar measure in \(SU(2)\) correspond to sets of zero surface measure in \(S^3\).

More generally, let \(S^n\) denote the unit sphere in \(\mathbb{R}^{n+1}\). For \(f \in L^2(S^n, \sigma_n)\) and \(\theta \in (0, \pi)\), define the spherical translation operator \(T_\theta\) by
\[
(T_\theta F)(\varphi) = \frac{1}{|S^{n-1}|} \int_{\varphi \in S^n : \varphi^T \varphi = 0} F(\varphi \cos(\theta) + \psi \sin(\theta)) d\nu(\psi)
\]
where \(\nu\) denotes surface measure on the \(n-1\) dimensional sphere \(\{ \varphi \in S^n : \varphi^T \varphi = 0\}\). Let
\[
(\Delta_\theta F)(\varphi) = F(\varphi) - (T_\theta F)(\varphi)
\]
denote the difference operator acting on \(F \in L^2(S^n)\), and for \(t > 0\) let
\[
\omega(F, t) = \sup\{ ||\Delta_\theta F||_{L^2(S^n)} : 0 < \theta \leq t\}
\]
denote the integral modulus of continuity of \(F \in L^2(S^n)\). The following result is the \(r = 2\) case of Theorem 2 obtained by Feng Dai in [6].
Theorem 4.1. Let $F \in L^2(S^n)$. If
\[ \int_0^1 \frac{\omega^2(F,t)}{t} \, dt < \infty \]
then the Fourier-Laplace partial sums of $F$ at $\vec{x}$ converge to $F(\vec{x})$ $\sigma_n$-almost everywhere.

Let us specialize to the case $n = 3$ of the previous theorem. If $\vec{x} = \vec{\eta}(x(\phi,\theta,\psi))$ then
\[ (T_\theta F)(\vec{x}) = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} F(\cos(\theta)\vec{c}_1 + \sin(\theta)\vec{v}) \sin(\phi) \, d\psi \, d\phi \]
\[ = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} F(\cos(\theta)\vec{c}_1 + \sin(\theta)\vec{1}) \sin(\phi) \, d\psi \, d\phi \]
where $\vec{c}_1 = [1,0,0,0]^T$, $\vec{v} = [0,\cos(\phi),\sin(\phi)\cos(\psi),\sin(\phi)\sin(\psi)]^T$, and
\[ O = \cos(\theta_0) \, \text{diag}(1,-1,-1,1) + \sin(\theta_0)S \]
with
\[ S = \begin{bmatrix}
0 & \cos(\phi_0) & \sin(\phi_0) \cos(\psi_0) & \sin(\phi_0) \sin(\psi_0) \\
\cos(\phi_0) & 0 & \sin(\phi_0) \cos(\psi_0) & -\sin(\phi_0) \sin(\psi_0) \\
\sin(\phi_0) \cos(\psi_0) & -\sin(\phi_0) \sin(\psi_0) & 0 & \cos(\phi_0) \\
\sin(\phi_0) \sin(\psi_0) & \sin(\phi_0) \cos(\psi_0) & -\cos(\phi_0) & 0
\end{bmatrix}. \]

If $F = f \circ \eta^{-1}$ then a routine computation yields
\[ (T_\theta F)(\vec{x}) = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} f(x(\phi,\theta,\psi)g^{-1}(\phi,\theta,\psi)) \sin(\phi) \, d\psi \, d\phi \]
and hence it follows that
\[ (T_\theta F)(\vec{x}) = [Q_xf](\theta) \]
for all $0 < \theta < \pi$.

Let $\theta \in [0,\pi]$. Define the difference operator $\delta_\theta$ on $L^2(SU(2))$ by
\[ \delta_\theta(f)(x) = f(x) - [Q_xf](\theta) \]
and let
\[ \Omega(f,t) = \sup\{\|\delta_\theta f\|_{L^2(SU(2))} : 0 < \theta \leq t\} \]
denote the integral modulus of continuity of $f \in L^2(SU(2))$. Using Theorem 4.1 and identity 4.2 yields the following almost everywhere convergence result on $SU(2)$.

Theorem 4.2. Let $f \in L^2(SU(2))$. If
\[ \int_0^1 \frac{\Omega^2(f,t)}{t} \, dt < \infty \]
then the Fourier partial sums of $f$ at $x$ converge to $f(x)$ $\mu$-almost everywhere on $SU(2)$.

The next three corollaries are immediate consequences.

Corollary 4.3. Let $f \in L^2(SU(2))$. If $\Omega(f,t) = O\left(\frac{1}{\log^{\beta}(1/t)}\right)$ for some $\beta > 1/2$ then $S_Nf(x) \to f(x)$ for almost every $x \in SU(2)$. 

Corollary 4.4. Let \( f \in L^2(SU(2)) \). If \( \Omega(f,t) = O(t^\alpha) \) for some \( \alpha \in (0,1) \) then \( S_N f(x) \to f(x) \) for almost every \( x \in SU(2) \).

Corollary 4.5. Let \( f \in \text{Lip}_\alpha(SU(2)) \) for some \( \alpha \in (0,1) \). Then \( S_N f(x) \to f(x) \) for almost every \( x \in SU(2) \).

REFERENCES

[1] Belen’kii, A. M., Uniform convergence of Fourier-Jacobi series on the orthogonality segment, Mathematical notes of the Academy of Sciences of the USSR, 46 (1989), 901-906.
[2] Chen, X. and D. Fan, On almost everywhere divergence of Bochner-Riesz means on compact Lie groups, Math. Z. 289 (2018), 961-981.
[3] Clerc, J.L., Sommes de Riesz et multiplicateurs sur un groupe de Lie, Ann. Inst. Fourier (Grenoble) 24 (1974), 149-172.
[4] Colzani, L., S. Giulini, and G. Travaglini, Sharp results for the mean summability of Fourier series on compact Lie groups, Math. Ann. 285 (1989), 75-84.
[5] Colzani, L., S. Giulini, G. Travaglini, and M. Vignati, Pointwise convergence of Fourier series on compact Lie groups, Coll. Math. 61 (1990), 379-386.
[6] Dai, F., A note on a.e. convergence of Fourier-Laplace series in \( L^2 \), J. Beijing Normal Univ. (Natur. Sci.), 35 (1999), 6-9.
[7] Giulini, S., P.M. Soardi, and G. Travaglini, Norms of characters and Fourier series on compact groups, J. Funct. Anal. 46 (1982), 88-101.
[8] Giulini, S. and G. Travaglini, Central Fourier analysis for Lorentz spaces on compact Lie groups, Monatsh. Math. 107 (1989), 207-215.
[9] Gong, S., S.X. Li, and X.A. Zheng, Harmonic analysis on classical groups, Proceedings of the Analysis Conference, Singapore 1986, S.T.L. Choy et al (ed.), Elsevier, 1988.
[10] Hewitt, E. and K.A. Ross, Abstract Harmonic Analysis (two volumes), Springer, 1963 and 1970.
[11] Mayer, R.A., Fourier series of differentiable functions on SU(2), Duke Math. J. 3 (1967), 549-554.
[12] Myers, D., Pointwise and Uniform Convergence of Fourier Series (PhD dissertation), Missouri University of Science and Technology, 2016.
[13] Myers, D. and D. Grow, Lipschitz functions on SU(2) have uniformly convergent Fourier series, J. Math. Anal. Appl., 458 (2018), 730-741.
[14] Pollard, H., The convergence almost everywhere of Legendre series, Proc. Amer. Math. Soc. 35 (1972), 442-444.
[15] Raguzin, D.L., Polynomial approximation on compact manifolds and homogeneous spaces, Trans. Amer. Math. Soc. 150 (1970), 41-53.
[16] Raguzin, D.L., Approximation theory, absolute convergence, and smoothness of random Fourier series on compact Lie groups, Math. Ann. 219 (1976), 1-11.
[17] Stanton, R.J., Mean convergence of Fourier series on compact Lie groups, Trans. Amer. Math. Soc., 218 (1976), 61-87.
[18] Stanton, R.J. and P.A. Tomas, Convergence of Fourier series on compact Lie groups, Bull. Amer. Math. Soc. 82 (1976), 61-62.
[19] Suetin, P. K., Classical Orthogonal Polynomials [in Russian], pp. 146, 300. Nauka, Moscow.
[20] Sugiuira, M., Fourier series of smooth functions on compact Lie groups, Osaka J. Math. 8 (1971), 33-47.
[21] Taylor, M.E., Fourier series on compact Lie groups, Proc. Amer. Math. Soc. 19 (1968), 1103-1105.
[22] Založnik, A., Function spaces generated by blocks associated with spheres, Lie groups, and spaces of homogeneous type, Trans. Amer. Math. Soc., 309 (1988), 139-164.

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