ERRATUM TO: ‘YIELD CURVE SHAPES AND THE ASYMPTOTIC SHORT RATE DISTRIBUTION IN AFFINE ONE-FACTOR MODELS’

MARTIN KELLER-RESSEL

Abstract. I would like to thank Ralf Korn for alerting me to an error in the original paper [KRS08]. The error concerns the threshold at which the yield curve in an affine short-rate model changes from normal (strictly increasing) to humped (endowed with a single maximum). In particular, it is not true that this threshold is the same for the forward curve and for the yield curve, as claimed in [KRS08]. Below, the correct mathematical expression for the threshold is given, supplemented with a self-contained and corrected proof.

1. Setting

In [KRS08] affine short rate models for bond pricing were considered, i.e. models where the risk-neutral short rate process \( r_t \) is given by an affine process in the sense of [DFS03]. The process \( r_t \) takes values in a state space \( D \), which is either \( \mathbb{R} \geq 0 \) or \( \mathbb{R} \). In this setting, the price at time \( t \) of a zero-coupon bond with time-to-maturity \( x \), denoted by \( P(t, t + x) \), is of the form

\[
P(t, t + x) = \exp \left( A(x) + r_t B(x) \right),
\]

where \( A \) and \( B \) satisfy the generalized Riccati differential equations

\[
\begin{align*}
\partial_x A(x) &= F(B(x)), & A(0) &= 0 \\
\partial_x B(x) &= R(B(x)) - 1, & B(0) &= 0.
\end{align*}
\]

The functions \( F \) and \( R \) are of Lévy-Khintchine-form and their parameterization is in one-to-one correspondence with the infinitesimal generator of \( r \), cf. [KRS08, Sec. 2]. Derived from the bond price, are the yield curve

\[
Y(x, r_t) := \frac{-\log P(t, t + x)}{x} = \frac{-A(x)}{x} - r_t \frac{B(x)}{x}
\]

and the forward curve

\[
f(x, r_t) := -\partial_x \log P(t, t + x) = -A'(x) - r_t B'(x).
\]

The first objective of [KRS08] was to derive the long-term yield and long-term forward rate. It was shown that the equation \( R(c) = 1 \) has at most a single negative solution \( c \), and that under mild conditions

\[
b_{\text{asympt}} := \lim_{x \to \infty} Y(x, r_t) = \lim_{x \to \infty} f(x, r_t) = -F(c)
\]

if such a solution exists, cf. [KRS08, Thm. 3.7]. We remark that \( \lambda := -\frac{1}{c} > 0 \) was called quasi-mean-reversion of \( r \) in [KRS08], with the convention that \( \lambda = 0 \) if no negative solution \( c \) exists. The second objective of [KRS08] was to characterize...
all possible shapes of the yield and the forward curve. Recall that in common
terminology, the yield or the forward curve is called
- **normal** if it is a strictly increasing function of $x$,
- **inverse** if it is a strictly decreasing function of $x$,
- **humped** if it has exactly one local maximum and no local minimum in $(0, \infty)$.

Finally, we recall the technical condition [KRS08, Condition 3.1], in slightly rephrased form. The condition is necessary to guarantee finite bond prices, when negative val-
ues of the short-rate are allowed.

**Condition 1.1.** We assume that $r$ is regular and conservative. If $r$ has state space $D = \mathbb{R}$, which necessarily implies that $R$ is of the linear form $R(x) = \beta x$ (cf. [DFS03]), we require that

$$F(x) < \infty \quad \text{for all} \quad x \in \begin{cases} \frac{1}{\beta}, 0 \quad & \text{if } \beta < 0 \\ (-\infty, 0), & \text{else} \end{cases}.$$  

2. Corrections to results

**Theorem 2.1.** Let the risk-neutral short rate process be given by a one-dimensional affine process $(r_t)_{t \geq 0}$ satisfying Condition 1.1 and with quasi-mean-reversion $-1/c = \lambda > 0$. In addition suppose that $F \neq 0$ and that at least one of $F$ and $R$ is non-
linear. Then the following holds:

1. The yield curve $Y(., r_t)$ can only be normal, inverse or humped.
2. Define

$$b_{y-norm} := \frac{1}{c} \int_c^0 \frac{F(u) - F(c)}{R(u) - 1} du,$$

$$b_{inv} := \begin{cases} -\frac{F'(0)}{R'(0)} & \text{if } R'(0) < 0 \\ +\infty & \text{if } R'(0) \geq 0. \end{cases}$$

The yield curve is normal if $r_t \leq b_{y-norm}$, humped if $b_{y-norm} < r_t < b_{inv}$ and inverse if $r_t \geq b_{inv}$.

**Remark 1.** The correction only concerns the expression for $b_{y-norm}$, which was called $b_{norm}$ in [KRS08] and erroneously given as $b_{norm} = -F'(c)/R'(0)$. All other parts of the theorem are the same as in [KRS08 Thm. 3.1].

**Corollary 3.11 in [KRS08] should be replaced by the following result:**

**Theorem 2.2.** Define $b_{inv}$ as in Thm. 2.1 and set

$$b_{fw-norm} := -\frac{F'(c)}{R'(c)}.$$  

Under the conditions of Theorem 2.1 the following holds:

1. The forward curve $f(., r_t)$ can only be normal, inverse or humped.
2. The forward curve is normal if $r_t \leq b_{fw-norm}$, humped if $b_{fw-norm} < r_t < b_{inv}$ and inverse if $r_t \geq b_{inv}$.
Remark 2. We have intentionally renamed the result from Corollary to Theorem, since the correction changes the logical structure of the proof. Note that the above result is equivalent to [KRS08, Cor. 3.11] up to the notational change from \( b_{\text{norm}} \) to \( b_{\text{fw-norm}} \). Note that \( b_{y-norm} \neq b_{\text{fw-norm}} \) in general, while in [KRS08] it was erroneously claimed that \( b_{y-norm} = b_{\text{fw-norm}} \).

Corollary 3.12 in [KRS08] should be replaced by the following result:

**Corollary 2.3.** Under the conditions of Theorem 2.1 it holds that

\[
(2.1) \quad b_{\text{fw-norm}} < b_{y-norm} < b_{\text{asymp}} < b_{\text{inv}}.
\]

In addition, the state space \( D \) of the short rate process satisfies

\[
D \cap (b_{y-norm}, b_{\text{inv}}) \neq \emptyset.
\]

The error also affects [KRS08, Figure 1], where the expression for \( b_{\text{norm}} \) should be replaced by the correct value of \( b_{y-norm} \). It also affects the application section [KRS08, Section 4], where the values of \( b_{\text{norm}} \) and \( b_{\text{inv}} \) are calculated in different models. The corrections to [KRS08, Section 4] are as follows:

In the **Vasicek model**, the short rate is given by

\[
(2.2) \quad dr_t = -\lambda(r_t - \theta) \, dt + \sigma dW_t, \quad r_0 \in \mathbb{R},
\]

with \( \lambda, \theta, \sigma > 0 \). This leads to the parameterization

\[
(2.3) \quad F(u) = \lambda \theta u + \frac{\sigma^2}{2} u^2,
\]

\[
(2.4) \quad R(u) = -\lambda u.
\]

By direct calculation we obtain

\[
(2.5) \quad b_{y-norm} = \theta - \frac{3\sigma^2}{4\lambda^2},
\]

\[
(2.6) \quad b_{\text{fw-norm}} = \theta - \frac{\sigma^2}{\lambda^2}.
\]

Note that the value of \( b_{y-norm} \) is now consistent with the results of [Vas77, p.186].

In the **Cox-Ingersoll-Ross model**, the short rate is given by

\[
(2.7) \quad r_t = -a(r_t - \theta) \, dt + \sigma \sqrt{r_t} \, dW_t, \quad r_0 \in \mathbb{R}_{\geq 0},
\]

with \( a, \theta, \sigma > 0 \). This leads to the parameterization

\[
(2.8) \quad F(u) = a \theta u
\]

\[
(2.9) \quad R(u) = -\frac{\sigma^2}{2} u^2 - au.
\]

By direct calculation we obtain

\[
(2.10) \quad b_{y-norm} = \frac{2a \theta}{\gamma - a} \log \left( \frac{2\gamma}{a + \gamma} \right)
\]

\[
(2.11) \quad b_{\text{fw-norm}} = \frac{a \theta}{\gamma}
\]

where \( \gamma := \sqrt{2\sigma^2 + a^2} \).
In the gamma model, the short rate is given by an Ornstein-Uhlenbeck-type process, driven by a compound Poisson process with intensity $\lambda k$ and exponentially distributed jump heights of mean $1/\theta$, see [KRS08, Sec. 4.4] for details. In this model, we have

\begin{align}
F(u) &= \frac{\lambda \theta k u}{1 - \theta u} \\
R(u) &= -\lambda u.
\end{align}

and by direct calculation we obtain

\begin{align}
b_{y-norm} &= \frac{k\lambda}{1 + \theta/\lambda} \log (1 + \theta/\lambda) \\
b_{fw-norm} &= \frac{k\theta}{(1 + \theta/\lambda)^2}.
\end{align}

Since the resulting expressions are quite involved, we omit the calculations for the extended CIR model [KRS08, Eq. (4.7)].

3. Corrected proofs

To prepare for the corrected proofs, we collect the following properties from [KRS08 Sec. 2 and 3.1], which hold for the functions $F$, $R$, $B$ and for the state space $D$ under the assumptions of Theorem 2.1:

(P1) $F$ is either strictly convex or linear; the same holds for $R$. Both functions are continuously differentiable on the interior of their effective domain.

(P2) The function $B$ is strictly decreasing with limit $\lim_{x \to \infty} B(x) = c$.

(P3) $F(0) = R(0) = 0$ and $R'(c) < 0$. In addition, $F'(0) > 0$ if $D = \mathbb{R}_{\geq 0}$.

(P4) Either

(a) $D = \mathbb{R}_{\geq 0}$; or
(b) $D = \mathbb{R}$ and $R(u) = u/c$ with $c < 0$.

Note that Theorem 2.1 assumes that at least one of $F$ and $R$ is non-linear. Together with (P1) this implies the following:

(P1') At least one of $F$ and $R$ is strictly convex.

In addition, we introduce the following terminology: Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function. The zero set of $f$ is $Z := \{x \in (0, \infty) : f(x) = 0\}$. The sign sequence of $Z$ is the sequence of signs $\{+, -\}$ that $f$ takes on the complement of $Z$, ordered by the natural order on $\mathbb{R}$. For example, the function $x^2 - 1$ on $(0, \infty)$ has the finite sign sequence $(-+)$; the function $\sin(x)$ has the infinite sign sequence $(+ - + - \cdots)$. An obvious, but important property is the following: Let $g : (0, \infty) \to (0, \infty)$ be a positive continuous function. Then $fg$ has the same zero set and the same sign sequence as $f$.

Proof of Theorem 2.2 From the Riccati equations (1.1), we can write the derivative of the forward curve as

\begin{equation}
\partial_x f(x, r_t) = -B'(x) \cdot \underbrace{\{F'(B(x)) + r_t R'(B(x))\}}_{:= k(x)}.
\end{equation}

Note that by (P2) the factor $-B'(x)$ is strictly positive, and hence $\partial_x f$ has the same sign sequence as $k$. We distinguish cases (a) and (b) as in (P4):
(a) Assume that \( r_t \in D = \mathbb{R}_{\geq 0} \). By (P2) \( B(x) \) is strictly decreasing and by (P1') either \( F' \) or \( R' \) is strictly increasing. Thus, if \( r_t > 0 \), it follows that \( k(x) \) is a strictly decreasing function. If \( r_t = 0 \), then \( k \) is either strictly decreasing (if \( F' \) is strictly convex) or \( k \) is constant (if \( F \) is linear). By (P1) these are the only possibilities. In addition, the case \( F = 0 \) is ruled out by the assumptions.

(b) Assume that \( r_t \in D = \mathbb{R} \). In this case \( R(u) = u/c \) and hence \( R'(u) = 1/c \) is constant and \( F' \) is strictly increasing, by (P1'). We conclude that \( k \) is strictly decreasing.

In any case, \( k \) is either strictly decreasing or constant and non-zero. Thus the sign sequence of \( k \) can be completely characterized by its initial value \( k(0) \) and its asymptotic limit as \( x \) tends to infinity. Let us first show that

\[
(3.2) \quad k(0) \leq 0 \iff r_t \geq b_{\text{inv}} = \begin{cases} -\frac{F'(0)}{R'(0)} & \text{if } R'(0) < 0 \\ +\infty & \text{if } R'(0) \geq 0 \end{cases}.
\]

We have \( k(0) = F'(0) + r_t R'(0) \), such that the assertion follows immediately if \( R'(0) < 0 \). Consider the complementary case \( R'(0) \geq 0 \). This rules out case (b) in (P4) and hence we may assume that \( D = \mathbb{R}_{\geq 0} \). Since \( F'(0) > 0 \) by (P3), \( (3.2) \) follows. Next we show that

\[
(3.3) \quad \lim_{x \to \infty} k(x) \geq 0 \iff r_t \leq b_{\text{fw-norm}} = -\frac{F'(c)}{R'(c)}.
\]

This follows immediately from \( \lim_{x \to \infty} k(x) = F'(c) + r_t R'(c) \) and \( R'(c) < 0 \), by (P3). Combining \((3.2)\) with \((3.3)\), and using that \( k \) is either strictly decreasing or constant and non-zero we obtain

\[
(3.4) \quad r_t \geq b_{\text{inv}} \iff k \text{ has sign sequence } (-) \\
r_t \leq b_{\text{fw-norm}} \iff k \text{ has sign sequence } (+) \\
r_t \in (b_{\text{fw-norm}}, b_{\text{inv}}) \iff k \text{ has sign sequence } (+-).
\]

Since \( \partial_x f \) has the same sign sequence as \( k \), these statements can be directly translated into monotonicity properties of \( f \): In the first case the forward curve \( f \) is strictly decreasing, i.e. inverse; in the second case it is strictly increasing, i.e. normal. In the third case it is strictly increasing up to the unique zero of \( k \) and then strictly decreasing, i.e. humped. No other cases are possible. \( \square \)

**Proof of Theorem 2.1.** From the Riccati equations \((1.1)\), we can write the derivative of the yield curve as

\[
\partial_x Y(x, r_t) = \frac{1}{x^2} (A(x) + r_t B(x)) - \frac{1}{x} \{ F(B(x)) + r_t [R(B(x)) - 1] \}.
\]

Multiplying by the positive function \( x^2 \) we see that \( \partial_x Y(x, r_t) \) has the same zero set and the same sign sequence as

\[
M(x) := [A(x) - x F(B(x))] + r_t \{ B(x) - x [R(B(x)) - 1] \}.
\]

The derivative of \( M \) is given by

\[
M'(x) := -x B'(x) \cdot \{ F'(B(x)) + r_t R'(B(x)) \} = -x B'(x) \cdot k(x),
\]

with \( k \) as in \((3.1)\). Note that by (P2) the factor \(-x B'(x)\) is strictly positive, and hence \( M' \) has the same sign sequence as \( k \), which was already analyzed in \((3.4)\).
Since $M(0) = 0$, we can conclude that

$$r_t \geq b_{\text{inv}} \implies M \text{ has sign sequence } (-)$$

(3.5)  

$$r_t \leq b_{\text{fw-norm}} \implies M \text{ has sign sequence } (+)$$

$$r_t \in (b_{\text{fw-norm}}, b_{\text{inv}}) \implies M \text{ has sign sequence } (\text{+}) \text{ or } (+).$$

Essentially, the mistake in [KRS08] was to ignore the possible sign sequence (+) in the third case. Not repeating the same mistake, we take a closer look at the third case and note that the sign sequence of $M$ is $(+-)$ if and only if

$$\lim_{x \to \infty} M(x) < 0.$$ (3.6)

Decomposing $M(x) = L_1(x) + r_t L_2(x)$ it remains to study the asymptotic properties of $L_1$ and $L_2$. We have

$$L_1(x) = A(x) - x F(B(x)) = \int_0^x (F(B(s)) - F(B(x))) \, ds =$$

$$= \int_0^c \frac{F(u) - F(B(x))}{R(u) - 1} du \xrightarrow{x \to \infty} \int_0^c \frac{F(u) - F(c)}{R(u) - 1} du.$$ (3.7)

In addition

$$L_2(x) = B(x) - x [R(B(x)) - 1] = \int_0^x (R(B(s)) - R(B(x))) \, ds =$$

$$= \int_0^c \frac{R(u) - R(B(x))}{R(u) - 1} du \xrightarrow{x \to \infty} \int_0^c \frac{R(u) - 1}{R(u) - 1} du = c.$$ (3.8)

Since $c < 0$, we conclude that

$$\lim_{x \to \infty} M(x) < 0 \iff r_t > b_{\text{y-norm}} = \frac{1}{c} \int_0^c \frac{F(u) - F(c)}{R(u) - 1} du.$$ (3.9)

By convexity of $F$ and $R$ and using that $c < 0$ we observe that

$$b_{\text{y-norm}} = \frac{1}{c} \int_0^c \frac{F(u) - F(c)}{R(u) - 1} du \geq \frac{1}{c} \int_0^c \frac{F'(c)}{R'(c)} du = -\frac{F'(c)}{R'(c)} = b_{\text{fw-norm}}.$$ (3.10)

Together with (3.5) this completes the proof. \qed

Proof of Corollary 2.3. Recall that $R(c) = 1$ and $c < 0$. By convexity of $F$ and $R$ we have

$$F'(c) \leq \frac{F(u) - F(c)}{u - c} \leq \frac{F(c)}{c} \leq F'(0)$$ (3.11)

$$R'(c) \leq \frac{R(u) - 1}{u - c} \leq \frac{1}{c} \leq R'(0).$$ (3.12)

for all $u \in (c, 0)$. Note that by (P1') either $F$ or $R$ is strictly convex, such that strict inequalities must hold in either the first or the second line. If $R'(0) < 0$, then applying the strictly increasing transformation $x \mapsto -\frac{1}{2}$ to the second line in (3.11) and multiplying term-by-term with the first, we obtain

$$-\frac{F'(c)}{R'(c)} < -\frac{F(u) - F(c)}{R(u) - 1} < -\frac{F(c)}{c} < -\frac{F'(0)}{R'(0)}.$$ (3.13)

Applying the integral $\frac{1}{2} \int_0^c du$ to all terms, (2.1) follows. If $R'(0) \geq 0$, this approach is still valid for the first two inequalities in each line of (3.11), but not for the last one. However, in the case $R'(0) \geq 0$ we have set $b_{\text{inv}} = +\infty$ in (3.2), and the
last inequality in (2.1) holds trivially. It remains to show that $D \cap (b_{y-norm}, b_{inv})$ is non-empty. $F$ is a convex function and by Condition 1.1 finite at least on the interval $(c, 0)$. It follows that $F'(0) > -\infty$ and thus that $b_{inv} > -\infty$ in general. If $D = \mathbb{R}_{\geq 0}$ then $F'(0) > 0$ by (P3) and hence $b_{inv} > 0$. Moreover $b_{y-norm} < b_{asymp} = - F(c) < \infty$, completing the proof. □

References

[DFS03] D. Duffie, D. Filipovic, and W. Schachermayer. Affine processes and applications in finance. The Annals of Applied Probability, 13(3):984–1053, 2003.

[KRS08] M. Keller-Ressel and T. Steiner. Yield curve shapes and the asymptotic short rate distribution in affine one-factor models. Finance and Stochastics, 12(2):149 – 172, 2008.

[Vas77] O. Vasicek. An equilibrium characterization of the term structure. Journal of Financial Economics, 5:177–188, 1977.