On approximation methods generated by generalized Bochner-Riesz kernels

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Abstract

Some sharp results related to the convergence of means and families of operators generated by the generalized Bochner-Riesz kernels are obtained. The exact order of approximation of functions by these methods via $K$-functional (or its realization in the case of the space $L_p$, $0 < p < 1$) is derived.

1 Introduction

Let $\mathbb{T}^d = [0, 2\pi)^d$ be the $d$-dimensional torus. As usual, the space $L_p(\mathbb{T}^d)$, $0 < p < \infty$, consists of measurable real valued functions $f(x)$, $x \in \mathbb{R}^d$, which is $2\pi$-periodic in each variable and

$$\|f\|_p = \left( \int_{\mathbb{T}^d} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$ 

By $L_\infty(\mathbb{T}^d)$ denote the space of all real valued $2\pi$-periodic continuous functions on $\mathbb{T}^d$ which is equipped with the norm

$$\|f\|_\infty = \max_{x \in \mathbb{T}^d} |f(x)|.$$ 

Binomial coefficients of order $\beta > 0$ are given by

$$\binom{\beta}{k} = \frac{\beta(\beta-1)\ldots(\beta-k+1)}{k!}, \quad k \in \mathbb{N},$$

and $\binom{\beta}{0} = 1$. We will repeatedly use the following estimates (see, for example, [1, Ch.1, §1])

$$\left| \binom{\beta}{k} \right| \leq \frac{C(\beta)}{k^{\beta+1}}. \quad (1.1)$$
We denote by \( C \) and \( C_j, j = 1, 2, \ldots \), positive constants depending on the indicated parameters. The notation \( A(f,n) \lesssim B(f,n) \) means a two-sided inequality with positive constants independent of \( f \) and \( n \).

The generalized Bochner-Riesz kernel is defined as follows
\[
R_{n}^{\beta, \delta}(x) = \sum_{k \in \mathbb{Z}^d} \left(1 - \frac{|k|^{\beta}}{n^\beta}\right)^\delta e^{i(k,x)}, \tag{1.2}
\]
where \( (k,x) = k_1x_1 + \cdots + k_dx_d, |x| = (x,x)^{\frac{1}{2}}, \) and \( x_+ = \max\{x,0\} \). If \( \beta = 2 \), then in (1.2) we have the classical Bochner-Riesz kernel.

In the present paper we deal with the generalized Bochner-Riesz means given by
\[
S_{n}^{\beta, \delta}(f;x) = (2\pi)^{-d} \int_{T^d} f(x+y)R_{n}^{\beta, \delta}(y)dy, \quad n \in \mathbb{N}, \tag{1.3}
\]
and with the family of linear polynomials operators given by
\[
S_{n;\lambda}^{\beta, \delta}(f;x) = (2n+1)^{-d} \sum_{k=0}^{2n} f(t_{n}^{k}+\lambda)R_{n}^{\beta, \delta}(x-t_{n}^{k}-\lambda), \quad n \in \mathbb{N}, \tag{1.4}
\]
where
\[
t_{n}^{k} = \frac{2\pi k}{2n+1}, \quad k \in \mathbb{Z}^d; \quad \sum_{k=0}^{2n} = \sum_{k_1=0}^{2n} \cdots \sum_{k_d=0}^{2n}.
\]

The Bochner-Riesz means and the family of polynomial operators generated by the kernel \( R_{n}^{2, \delta} \) we denote by \( S_{n}^{\delta} \) and \( \{S_{n;\lambda}^{\delta}\} \), respectively.

The properties of the classical Bochner-Riesz means \( S_{n}^{\delta} \) are intensively studied by many authors (see, for example, [2, Ch.7], [3, Ch.3], [4], [5], [6], [7]). Some sharp results related to the approximation of functions by these means \( S_{n}^{\delta} \) and by the family \( \{S_{n;\lambda}^{\delta}\} \) were obtained in [8].

Which approximation properties do the means (1.3) have for different values of \( \beta \) and \( \delta \)? It is known that the regularity (convergence) of the Bochner-Riesz means depends on the parameter \( \delta \). In particular, it was shown in [7] and [6] that if \( \delta > (d - 1)/2 \), then the means (1.3) converge in \( L_p(T^d) \) for any \( p \in [1, \infty] \), for other values of \( \delta \) the convergence may not be achieved. The parameter \( \beta > 0 \) does not have any effect on the regularity of the means \( S_{n}^{\beta, \delta} \) in contrast to \( \delta \) (see, for example, [9, Ch.8] in the case of even \( \beta \)). However, the order of approximation of \( f \) by the means \( S_{n}^{\beta, \delta}(f) \) is better for large \( \beta \) and the order is the same while \( \delta \) is changing.

The special module of smoothness of order \( \beta > 0 \) is defined by
\[
\tilde{\omega}_\beta(f,h) = \left\| \int_{|u| \geq 1} \sum_{\nu=0}^{2r} (\nu r)^\nu f(\cdot + (\nu-r)uh) \frac{du}{|u|^{d+\beta}} \right\|_\infty,
\]
where \( r \in \mathbb{N} \) and \( r > d - 1 + \beta \). It was shown in [6] that the approximation error of \( f \in C(\mathbb{T}^d) \) by \( S_n^{\beta, \delta}(f) \) is equivalent to \( \tilde{\omega}_\beta(f, h)_\infty \) under certain restrictions on \( \beta \) and \( \delta \). This result is also valued in the space \( L_p(\mathbb{T}^d) \) (see, for example, [7, Theorem 7]). In particular, we have the following theorem.

**Theorem A.** Let \( f \in L_p(\mathbb{T}^d) \), \( 1 \leq p \leq \infty \), \( \beta > 0 \), and \( \delta > (d - 1)/2 \). Then

\[
\| f - S_n^{\beta, \delta}(f) \|_p \asymp \tilde{\omega}_\beta(f, 1/n)_p, \quad n \in \mathbb{N}.
\] (1.5)

Note that in the case of even \( \beta > 0 \) in (1.5) the module of smoothness \( \tilde{\omega}_\beta(f, h)_p \) can be replaced by the corresponding \( K \)-functional (see [9, Ch.8]) and other special moduli of smoothness (see [7]).

In [8] it was shown that for the classical Bochner-Riesz means \( S_n^{\delta} \) as well as for the families \( \{S_n^{\delta}\} \) there is an alternative: *either the means \( S_n^{\delta} \) (or the family \( \{S_n^{\delta}\} \)) diverge in \( L_p \) or its approximation error is equivalent to the \( K \)-functional (or its realization if \( 0 < p < 1 \)).*

In the present paper the results of [8] are extended to the case of the generalized Bochner-Riesz means \( S_n^{\beta, \delta} \) as well as to the family \( \{S_n^{\beta, \delta}\} \) with any \( \beta \in \mathbb{R} \), \( \beta > 0 \). In particular, it is proved that the above alternative holds for any positive \( \delta \) and \( \beta \). It turns out that in the case of \( 0 < p < 1 \) the regularity of the family \( \{S_n^{\beta, \delta}\} \) essentially depends on the parameter \( \beta \).

The paper is organized as follows. In Section 2 we formulate the main results. In Subsection 2.1 the convergence theorems are formulated; in Subsection 2.2 the theorems on equivalence of error of approximation of functions by corresponding method are formulated. The auxiliary results are formulated and proved in Section 3. In Section 4 we prove the main results of the paper.

## 2 Main Results

### 2.1 Convergence Theorems

Often we will deal with functions in \( L_p(\mathbb{T}^{2d}) \) which depend additionally on parameter \( \lambda \in \mathbb{T}^d \). We denote by \( \| \cdot \|_p \) the \( p \)-(quasi-)norm with respect to both the main variable \( x \in \mathbb{T}^d \) and the parameter \( \lambda \in \mathbb{T}^d \), i.e.

\[
\| \cdot \|_p = \| \|_{p, x} \|_{p, \lambda},
\]
where $\| \cdot \|_{p,x}$ and $\| \cdot \|_{p,\lambda}$ are the $p$-norms (quasi-norms, if $0 < p < 1$) with respect to $x$ and $\lambda$, respectively.

Let $T_n$ be the set of all real valued trigonometric polynomials of order $n$:

$$T_n = \left\{ T(x) = \sum_{k \in \mathbb{Z}^d, |k| \leq n} c_k e^{i(k,x)} : c_{-k} = \overline{c_k} \right\}.$$ 

A sequence of linear operators $\{L_n\}_{n \in \mathbb{N}}$, mapping $L_p$, $1 \leq p \leq \infty$, into the space $T_n$ is said to be convergent (or converges) in $L_p$, if

$$\lim_{n \to \infty} \| f - L_n(f) \|_p = 0$$

for each $f \in L_p(\mathbb{T}^d)$. By analogy, a family of linear operators $\{L_n; \lambda\}_{n \in \mathbb{N}, \lambda \in \mathbb{T}^d}$, mapping $L_p$, $0 < p \leq \infty$, into $T_n$, converges in $L_p$, if

$$\lim_{n \to \infty} \| f - L_n; \lambda(f) \|_{\frac{p}{p-1}} = 0$$

for each $f \in L_p(\mathbb{T}^d)$.

In order to formulate the main results, we split the domain $\mathbb{R}^2_+$ of pairs $(1/p, \beta)$ into three parts:

$$\Sigma(d) = \left\{ \left( \frac{1}{p}, \delta \right) \in \mathbb{R}^2_+ : \delta > \max \left\{ \frac{d-1}{2}, d \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{1}{2} \right\} \right\},$$

$$\Gamma(d) = \left\{ \left( \frac{1}{p}, \delta \right) \in \mathbb{R}^2_+ : 0 \leq \delta \leq d \left| \frac{1}{p} - \frac{1}{2} \right| - \frac{1}{2} \right\},$$

$$\Omega(d) = \left\{ \left( \frac{1}{p}, \delta \right) \in \mathbb{R}^2_+ : 0 \leq \delta \leq \frac{d-1}{2}, \delta > d \left| \frac{1}{p} - \frac{1}{2} \right| - \frac{1}{2} \right\}.$$

**Theorem 2.1.** Let $1 \leq p \leq \infty$ and $\beta, \delta > 0$. The means $S_n^{\beta, \delta}$ converge in $L_p$ if and only if the means $S_n^{\delta}$ converge in the same space $L_p$. In particular, the means $S_n^{\beta, \delta}$ converge in $L_p$ if $(1/p, \delta) \in \Sigma(d)$ and they diverge in $L_p$ if $(1/p, \delta) \in \Gamma(d)$.

Note that the problem of convergence in the domain $\Omega(d)$ is not completely studied even for the classical Bochner-Riesz means $S_n^{\delta}$ (see, for example, [10]).

From Theorem 2.1 it follows that the parameter $\beta > 0$ does not affect on the convergence of the means $S_n^{\beta, \delta}$ in $L_p$, $1 \leq p \leq \infty$. In contrast to the case $p \geq 1$, the dependence on the parameter $\beta$ is essential for the family (1.4) in $L_p$, $0 < p < 1$. To state the next theorem we introduce the following set

$$B(d) = \left\{ \left( \frac{1}{p}, \beta \right) \in \mathbb{R}^2_+ : \beta \in 2\mathbb{N}, \beta > d \left( \frac{1}{p} - 1 \right)_+ \right\}.$$
Theorem 2.2. 1) Let $1 \leq p \leq \infty$ and $\beta, \delta > 0$. Then the family $\{S_{n:\lambda}^{\beta,\delta}\}$ converges in $L_p$ if and only if the family $\{S_{n:\lambda}^{\beta}\}$ converges in the same space $L_p$. In particular, the family $\{S_{n:\lambda}^{\beta,\delta}\}$ converges in $L_p$ if $(1/p,\delta) \in \Sigma(d)$ and it diverges in $L_p$ if $(1/p,\delta) \in \Gamma(d)$.

2) Let $0 < p < 1$ and $\beta, \delta > 0$. Then the family $\{S_{n:\lambda}^{\beta,\delta}\}$ converges in $L_p$ if and only if $(1/p,\delta) \in \Sigma(d)$ and $(1/p,\beta) \in B(d)$.

2.2 Two-sided estimates of approximation

For our purpose we will use a $K$-functional related to the power of the Laplacian $\Delta^{\beta/2}$, which we define by

$$\Delta^{\beta/2} f(x) \sim \sum_k |k|^\beta c_k(f)e^{-i(k,x)}, \quad \beta \in \mathbb{R}, \quad \beta > 0,$$

where

$$c_k(f) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(x)e^{-i(k,x)}dx, \quad k \in \mathbb{Z}^d,$$

are the Fourier coefficients of the function $f$. The corresponding $K$-functional is given by

$$K_\beta(f,t)_p = \inf_g \{\|f - g\|_p + t^\beta \|\Delta^{\beta/2} g\|_p\}. \quad (2.1)$$

It should be noticed that in the case $0 < p < 1$ the $K$-functional given in (2.1) is identically zero (see [11]). However, in accordance with the concept of “Realization” (see [12]), the $K$-functional can be replaced by the quantity

$$\tilde{K}_\beta(f,t)_p = \inf_{T \in \mathbb{T}_{1/t}} \{\|f - T\|_p + t^\beta \|\Delta^{\beta/2} T\|_p\} \quad (2.2)$$

(see also [11] for properties of $\tilde{K}_\beta$ in the case $0 < p < 1$).

Theorem 2.3. Let $f \in L_p(\mathbb{T}^d)$, $1 \leq p \leq \infty$, $\beta > 0$, and $\delta > (d - 1)/2$. Then

$$\|f - S_{n:\lambda}^{\beta,\delta}(f)\|_p \asymp \|f - S_{n:\lambda}^{\beta,\delta}(f)\|_p \asymp K_\beta(f,1/n)_p \asymp \tilde{K}_\beta(f,1/n)_p \asymp \tilde{\omega}_\beta(f,1/n)_p. \quad (2.3)$$

The next theorem holds in the case $0 < p < 1$.

Theorem 2.4. Let $f \in L_p(\mathbb{T}^d)$, $0 < p < 1$, $(1/p,\delta) \in \Sigma(d)$, and $(1/p,\beta) \in B(d)$. Then

$$\|f - S_{n:\lambda}^{\beta,\delta}(f)\|_p \asymp \tilde{K}_\beta(f,1/n)_p, \quad n \in \mathbb{N}. \quad (2.4)$$
The following theorems contain approximation properties of the methods (1.3) and (1.4) for \((1/p, \delta) \in \Omega(d)\). We will denote the subdomains of \(\Omega(d)\), where the Bochner-Riesz means \(S^\delta_n\) and the corresponding families \(\{S^\delta_{n; \lambda}\}\) have the convergence property with \(\Omega'(d)\) and \(\Omega''(d)\), respectively.

**Theorem 2.5.** Let \(f \in L_p(\mathbb{T}^d), 1 \leq p \leq \infty, \beta, \delta > 0, \text{ and } (1/p, \delta) \in \Omega(d)\). Then
\[
\|f - S^\beta_n, \delta(f)\|_p \leq (2\pi)^{-\frac{d}{p}} \|f - S^\beta_{n; \lambda}(f)\|_\mathcal{P}, \quad n \in \mathbb{N},
\]
(2.5) in particular, \(\Omega''(d) \subset \Omega'(d)\).

**Theorem 2.6.** Let \(f \in L_p(\mathbb{T}^d), 1 \leq p \leq \infty, d > 1, \beta, \delta > 0, \text{ and } (1/p, \delta) \in \Omega'(d)\). Then
\[
\|f - S^\beta_n, \delta(f)\|_p \asymp K_\beta(f, 1/n)_p \asymp \tilde{K}_\beta(f, 1/n)_p \asymp \tilde{\omega}_\beta(f, 1/n)_p.
\]
(2.6)

Thus, from Theorems 2.1 and 2.6 it follows that the approximation error of the means \(S^\beta_n, \delta\) is equivalent to the corresponding \(K\)-functional if and only if the means \(S^\beta_n, \delta\) converge in \(L_p\).

**Theorem 2.7.** Let \(f \in L_p(\mathbb{T}^d), 1 \leq p \leq \infty, d > 1, \beta, \delta > 0, \text{ and } (1/p, \delta) \in \Omega''(d)\). Then
\[
\|f - S^\beta_{n; \lambda}(f)\|_\mathcal{P} \asymp \|f - S^\beta_n, \delta(f)\|_p \asymp K_\beta(f, 1/n)_p \asymp \tilde{K}_\beta(f, 1/n)_p \asymp \tilde{\omega}_\beta(f, 1/n)_p.
\]
(2.7)

Similarly to the case considered above, from Theorems 2.2 and 2.7 it follows that the approximation error of the family \(\{S^\beta_n; \lambda\}\) is equivalent to the corresponding \(K\)-functional (or its realization \(\tilde{K}\) if \(0 < p < 1\)) if and only if the family \(\{S^\beta_{n; \lambda}\}\) converges in \(L_p\).

### 3 Auxiliary assertions

Let us present some facts related to multipliers for trigonometric polynomials. Let \(g\) be a real or complex valued function defined on \(\mathbb{R}^d\). It generates operators \(\{A_n(g)\}_{n \geq 1}\) given by
\[
A_n(g)T(x) = \sum_{k \in \mathbb{Z}^d} g\left(\frac{k}{n}\right) c_k e^{i(k,x)}, \quad T(x) = \sum_{k \in \mathbb{Z}^d} c_k e^{i(k,x)} \in \mathcal{T},
\]
where $\mathcal{T}$ is the set of all trigonometric polynomials.

Consider the inequality
\[
\|A_n(g)T\|_p \leq C\|T\|_p, \quad T \in \mathcal{T}_n, \ n \geq 1.
\] (3.1)
We say that (3.1) is valid for the function $g$ (this we denote by $g \in M_p(\mathcal{T})$), if it is valid in the $L_p$-norm for all $T \in \mathcal{T}_n$ and $n \geq 1$ with some positive constant $C$ independent of $T$ and $n$.

The following two lemmas are evident.

**Lemma 3.1.** Let $g, h \in M_p(\mathcal{T})$. Then the functions $g + h$ and $g \cdot h$ belong to $M_p(\mathcal{T})$.

We will also use the inequalities of type
\[
\|A_n(g)T\|_p \leq C\|A_n(h)T\|_p, \quad T \in \mathcal{T}_n, \ n \geq 1.
\] (3.2)
In the next we suppose that $h(\xi) \neq 0$ for $\xi \neq 0$. Put
\[
\mathcal{X}(\xi) = \frac{g(\xi)}{h(\xi)}, \quad \xi \in \mathbb{R}^d \setminus \{0\}.
\]
We assume that $\mathcal{X}$ is somehow defined at the point $\xi = 0$.

**Lemma 3.2.** Let $g(0) = h(0) = 0$ and let $\mathcal{X} \in M_p(\mathcal{T})$. Then the inequality in (3.2) is valid in $L_p$ independently of the value $\mathcal{X}(0)$.

As usual, the Fourier transform of a function $f \in L_1(\mathbb{R}^d)$ is given by
\[
\hat{f}(x) = \int_{\mathbb{R}^d} f(y)e^{-i(x,y)}dy.
\]

The next lemma (see [7], [13, p. 150-151]) gives sufficient conditions for the validity of (3.1) in the space $L_p(\mathbb{T}^d)$.

**Lemma 3.3.** Let $0 < p \leq \infty$ and let $g$ be a continuous function with compact support. If $\hat{g} \in L_{p^*}(\mathbb{R}^d)$ ($p^* = \min(1, p)$), then $g \in M_p(\mathcal{T})$.

Let us denote by $W^m_1(\mathbb{R}^d)$ the Sobolev space of all integrable functions whose derivatives up to the order $m$ belong to $L_1(\mathbb{R}^d)$. By $\overset{\circ}{W}^m_1(\mathbb{R}^d)$ denote the set of functions in $W^m_1(\mathbb{R}^d)$ having compact support (see details, for example, in [14, Ch.1]).

The following lemma is proved in [8] (more general statements are proved in [9, Ch.6]).

**Lemma 3.4.** Let $0 < p \leq \infty$, $m = \lfloor d/p^* \rfloor + 1$ ($p^* = \min(1, p)$, $[a]$ is integral part of $a$). If $g \in \overset{\circ}{W}^m_1(\mathbb{R}^d)$, then $\hat{g} \in L_{p^*}(\mathbb{R}^d)$ and, therefore, $g \in M_p(\mathcal{T})$. 7
In the next by $\mathcal{C}^d$ we denote the class of real or complex valued $C^\infty$-functions with a compact support contained in the set \( \{ x \in \mathbb{R}^d : |x| \leq 1 \} \). We use the symbol $\mathcal{R}^d$ to denote the class of real valued radial $C$-functions $\psi$ with a compact support and $\psi(0) = 1$.

The following two lemmas are proved in [15].

**Lemma 3.5.** Suppose $f \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is a homogeneous function of order $\beta > 0$, it is not a polynomial, and $h \in \mathcal{R}^d$. Then $\hat{fh} \in L_p(\mathbb{R}^d)$ if and only if \( \frac{d}{d+\beta} < p \leq \infty \).

**Lemma 3.6.** Let $0 < p \leq \infty$, $\beta \in B(d)$, $n \in \mathbb{N}$, and $T \in \mathcal{T}_n$. Then
\[
\| \Delta^{\beta/2}T \|_p \leq Cn^\beta \| T \|_p,
\]
where $C$ is a constant independent of $T$ and $n$.

The following lemma is easily obtained by using Theorem 6.1.1 in [9].

**Lemma 3.7.** Let $0 < p \leq 1$, let $f$ and $g$ be bounded functions with compact supports contained in $\{ x \in \mathbb{R}^d : |x| < 3 \}$, and let $\hat{f}, \hat{g} \in L_p(\mathbb{R}^d)$. Then $\hat{fg} \in L_p(\mathbb{R}^d)$.

The item 1) of the next lemma is well known (see, for example, [16], [10, Ch.9]); the item 2) is proved in [8].

**Lemma 3.8.** 1) Let $1 \leq p \leq \infty$. Then the means $S_{n}^{\delta}$ converge in $L_p$ if $(1/p, \delta) \in \Sigma(d)$ and they diverge in $L_p$ if $(1/p, \delta) \in \Gamma(d)$.

2) Let $0 < p \leq \infty$. Then the family $\{S_{n;\lambda}^{\delta}\}$ converges in $L_p$ if $(1/p, \delta) \in \Sigma(d)$ and it diverges in $L_p$ if $(1/p, \delta) \in \Gamma(d)$.

Throughout what follows we will use the following notation:

\[
\varphi_{\beta,\delta}(x) = (1 - |x|^\beta)^\delta,
\]
\[
h_0(x) = \begin{cases} 
1, & |x| \leq 4/3; \\
0, & |x| > 2,
\end{cases}
\]
\[
h_1(x) = \begin{cases} 
1, & |x| \leq 1/2; \\
0, & |x| \geq 3/4,
\end{cases}
\]
\[
h_2(x) = h_0(x) - h_1(x).
\]

In addition, suppose that $h_0$ and $h_1$ belong to $C^\infty(\mathbb{R}^d) \cap \mathcal{R}^d$.

**Lemma 3.9.** For $\delta > 0$ we have
\[
\int_{\mathbb{R}^d} h_2(x) \varphi_{2,\delta}(x)e^{-i(x,y)}dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\cos(2\pi|y| - \pi \frac{\delta}{2} - \pi \frac{\delta}{4})}{|y|^{\frac{3}{2}+\delta}} + O(|y|^{-\frac{d+3}{2}-\delta}), \quad |y| \geq 1.
\]
Proof. Using the well-known equality (see, for example, [2, Ch. IV])
\[
\int_{\mathbb{R}^d} (1 - |x|^2) e^{-i(x,y)} dx = \pi^{-\delta} \Gamma(\delta + 1) \frac{J_{\frac{d}{2} + \delta}(2\pi|y|)}{|y|^\frac{d}{2} + \delta},
\]
and \((h_2(x) - 1)(1 - |x|^2) e^{-i(x,y)} \in C^\infty(\mathbb{R}^d)\), we get
\[
\int_{\mathbb{R}^d} h_2(x) \varphi_{2,\delta}(x) e^{-i(x,y)} dx = \pi^{-\delta} \Gamma(\delta + 1) \frac{J_{\frac{d}{2} + \delta}(2\pi|y|)}{|y|^\frac{d}{2} + \delta} + O(|y|^{-r}),
\]
where \(r\) is large enough. It remains to use the asymptotics of the Bessel function (see, for example, [2, Ch. IV])
\[
J_\nu(u) = \sqrt{\frac{2}{\pi u}} \cos \left( u - \frac{\nu\pi}{2} - \frac{\pi}{4} \right) + O(u^{-\frac{3}{2}}), \quad u \to \infty.
\]  
\[(3.7)\]

The statement of the next lemma for \(p = 1\) see in [17] or [18]; for \(\beta = 2\) this result follows immediately from the asymptotics of Bessel functions \((3.7)\).

Lemma 3.10. Let \(0 < p \leq 1, \beta > 0, \) and \(\delta > 0\). Then \(\hat{\varphi}_{\beta,\delta} \in L_p(\mathbb{R}^d)\) if and only if \((1/p, \beta) \in B(d)\) and \(\delta > d(1/p - 1/2) - 1/2\).

Proof. Let us prove the sufficiency. We put
\[
\Phi_j = \varphi_{\beta,\delta} \cdot h_j, \quad j = 1, 2,
\]
where \(\varphi_{\beta,\delta}\) and \(h_j\) are defined by \((3.3)\) and \((3.4)\), respectively.

We first show that \(\hat{\Phi}_1 \in L_p(\mathbb{R}^d)\). To see this we use the representation:
\[
\Phi_1 = \Phi_{1,1} + \Phi_{1,2},
\]
where
\[
\Phi_{1,1}(x) = \sum_{\nu=1}^{\sigma-1} \binom{\delta}{\nu} (-1)^\nu |x|^\beta \nu h_1(x),
\]
\[
\Phi_{1,2}(x) = h_1(x) \left( 1 + \sum_{\nu=\sigma}^{\infty} \binom{\delta}{\nu} (-1)^\nu |x|^\beta \nu \right),
\]
\[(3.10)\]
and \( \sigma > 2(d/p + 1)/\beta + 2 \).

From Lemma 3.5 for \( \beta \not\in 2\mathbb{N} \) and Lemma 3.4 for \( \beta \in 2\mathbb{N} \) it follows immediately that \( \hat{\Phi}_{1,1} \in L_p(\mathbb{R}^d) \). Using Lemma 3.4 and (1.1) it is easy to verify that for any positive \( \beta \) and \( \delta \)

\[
\hat{\Phi}_{1,2} \in L_p(\mathbb{R}^d). \tag{3.12}
\]

Thus, taking into account (3.9), we obtain that

\[
\hat{\Phi}_1 \in L_p(\mathbb{R}^d). \tag{3.13}
\]

Now we check that \( \hat{\Phi}_2 \in L_p(\mathbb{R}^d) \). Observe that for any positive \( \beta \) and \( \delta \) the following expansion holds:

\[
(1 - |x|^\beta)^\delta = \sum_{\nu=0}^{\infty} a_\nu (1 - |x|^2)^{\delta + \nu}, \tag{3.14}
\]

where \( a_\nu \in \mathbb{R} \) and \( a_0 = (\beta/2)^\delta \). Consequently, the function \( \Phi_2 \) can be represented as follows:

\[
\Phi_2 = \Phi_{2,1} + \Phi_{2,2},
\]

where

\[
\Phi_{2,1}(x) = h_2(x) \sum_{\nu=0}^{\lambda} a_\nu (1 - |x|^2)^{\delta + \nu},
\]

\[
\Phi_{2,2}(x) = h_2(x) \sum_{\nu=\lambda+1}^{\infty} a_\nu (1 - |x|^2)^{\delta + \nu},
\]

and \( \lambda > d/p - \delta \). Using equality (3.6) and asymptotic formula (3.7), it is easy to see that \( \hat{\Phi}_{2,1} \in L_p(\mathbb{R}^d) \). Calculating the partial derivatives of the function \( \Phi_{2,2} \), we obtain that \( \Phi_{2,2} \in W_1^m(\mathbb{R}^d) \), where \( m = [d/p] + 1 \). Thus, by Lemma 3.4 we have that \( \hat{\Phi}_{2,2} \in L_p(\mathbb{R}^d) \). Therefore,

\[
\hat{\Phi}_2 \in L_p(\mathbb{R}^d). \tag{3.15}
\]

Combining (3.13) and (3.15), we obtain \( \hat{\varphi}_{\beta,\delta} \in L_p(\mathbb{R}^d) \).

Now, let us prove the necessity. From Lemma 3.7 it follows that

\[
\hat{\Phi}_j \in L_p(\mathbb{R}^d), \quad j = 1, 2. \tag{3.16}
\]

We claim that under condition (3.16), the pair \( (1/p, \beta) \) belongs to \( B(d) \). Indeed, from (3.12), (3.16) and (3.9) it follows immediately that \( \hat{\Phi}_{1,1} \in L_p(\mathbb{R}^d) \). Taking into account that

\[
\Phi_{1,1}(x) = |x|^\beta \phi(x),
\]
where
\[
\phi(x) = \sum_{\nu=1}^{\sigma-1} \left( \frac{\delta}{\nu} \right) (-1)^{\nu} |x|^{\beta(\nu-1)} h_1(x)
\]
and applying Lemma 3.5 we conclude that \((1/p, \beta) \in \mathcal{B}(d)\).

Now we show that \(\delta > d(1/p - 1/2) - 1/2\). Similarly to the previous arguments, we have \(\hat{\Phi}_{2,1} \in L_p(\mathbb{R}^d)\). Using Lemma 3.9, we get
\[
\int_1^N |\hat{\Phi}_{2,1}(x)|^p dx \geq C \int_1^N r^{d-1} \left| \frac{\cos(2\pi r - \frac{\pi \delta}{2} - \frac{\pi}{4})}{r^{\frac{d+1}{2} + \delta}} \right|^p dr - O\left( \int_1^N r^{d-1-p\left(\frac{d+3}{2} + \delta\right)} dr \right).
\]

The last inequality implies that \(\delta > d(1/p - 1/2) - 1/2\). Otherwise, we would have that \(\hat{\Phi}_2 \not\in L_p(\mathbb{R}^d)\). ■

Let us consider general approximation methods generated by the kernel
\[
\mathcal{K}_n^\varphi(x) = \sum_k \varphi\left( \frac{k}{n} \right) e^{i(k,x)},
\]
where \(\varphi \in C(\mathbb{R}^d)\) is a real valued centrally symmetric function with a compact support in \(\{ x \in \mathbb{R}^d : |x| \leq 1 \}\) and \(\varphi(0) = 1\). By analogy with the definition of the methods (1.3) and (1.4), we put:
\[
\mathcal{L}_{n}^\varphi(f; x) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(x + y) \mathcal{K}_n^\varphi(y) dy, \quad n \in \mathbb{N},
\]
\[
\mathcal{L}_{n; \lambda}^\varphi(f; x) = (2n + 1)^{-d} \sum_{k=0}^{2n} f(t_k^n + \lambda) \mathcal{K}_n^\varphi(x - t_k^n - \lambda), \quad n \in \mathbb{N}.
\]

As usual, the norm of a linear operator \(\mathcal{L}_n^\varphi\) is given by
\[
\|\mathcal{L}_n^\varphi\| = \sup_{\|f\|_p \leq 1} \|\mathcal{L}_n^\varphi(f)\|_p.
\]

By analogy, we define the (quasi-)norm of a family \(\{\mathcal{L}_{n; \lambda}^\varphi\}\) by
\[
\|\{\mathcal{L}_{n; \lambda}^\varphi\}\| = (2\pi)^{-d/p} \sup_{\|f\|_p \leq 1} \|\mathcal{L}_{n; \lambda}^\varphi(f; x)\|_p.
\]

The proof of the lemma below is standard (see, for example, [8]).

**Lemma 3.11.** 1) The means \(\mathcal{L}_n^\varphi\) converge in \(L_p, 1 \leq p \leq \infty\), if and only if the sequence of their norms \(\{\|\mathcal{L}_n^\varphi\|_p\}_{n \in \mathbb{N}}\) is bounded.
2) The family \( \{L_{n;\lambda}^{\varphi}\} \) converges in \( L_p \), \( 0 < p \leq \infty \), if and only if the sequence \( \{\|L_{n;\lambda}^{\varphi}\|_p\}_n \) is bounded.

The general conditions of convergence for the methods (3.17) and (3.18) are formulated in the next lemma (see [19] and [20]).

**Lemma 3.12.** 1) The means \( L_{n}^{\varphi} \) and the family \( \{L_{n;\lambda}^{\varphi}\} \) converge in \( L_p \) for all \( 1 \leq p \leq \infty \) if and only if \( \hat{\varphi} \in L_1(\mathbb{R}^d) \).

2) Let \( 0 < p \leq 1 \). The family \( \{L_{n;\lambda}^{\varphi}\} \) converges in \( L_p \) if and only if \( \hat{\varphi} \in L_p(\mathbb{R}^d) \).

The following two lemmas are the main tools for proving the theorems from Section 2.2.

**Lemma 3.13.** Let \( \delta > 0 \), \((1/p, \beta) \in B(d)\), and \((1/p, \delta) \in \Sigma(d) \cup \Omega'(d)\). Then

\[
\|T - S_{n}^{\beta,\delta}(T)\|_p \asymp n^{-\beta}\|\Delta_{\beta/2}^{\beta}T\|_p, \quad T \in \mathcal{T}_n, \quad n \in \mathbb{N}. \tag{3.19}
\]

In (3.19) the operator \( S_{n}^{\beta,\delta} \) can be replaced by \( S_{n;\lambda}^{\beta,\delta} \) for any fixed \( \lambda \in \mathbb{R}^d \) without affecting the constants.

**Proof.** Let us prove the upper estimate. It is easy to check that for each polynomial \( T \in \mathcal{T}_n, \ n \in \mathbb{N} \) and \( \lambda \in \mathbb{R}^d \)

\[
S_{n}^{\beta,\delta}(T; x) = S_{n;\lambda}^{\beta,\delta}(T; x), \quad x \in \mathbb{T}^d. \tag{3.20}
\]

From (3.20) it follows that the operator \( S_{n}^{\beta,\delta} \) in (3.19) can be replaced by \( S_{n;\lambda}^{\beta,\delta} \).

According to Lemma 3.11 for \( p \geq 1 \), we have

\[
\|S_{n}^{\beta,\delta}(T)\|_p \leq C\|T\|_p, \quad T \in \mathcal{T}_n, \quad n \in \mathbb{N}. \tag{3.21}
\]

We claim that the inequality (3.21) holds also for \( 0 < p < 1 \). Indeed, from the equality (3.20) and the conditions of Lemma 3.13 we obtain that for every polynomial \( T \in \mathcal{T}_n \) and \( n \in \mathbb{N} \)

\[
\|S_{n}^{\beta,\delta}(T)\|_p = (2\pi)^{-d/p}\|S_{n;\lambda}^{\beta,\delta}(T)\|_p \leq \|S_{n;\lambda}^{\beta,\delta}(T)\|_p \leq C\|T\|_p.
\]

Thus, we have for \( 0 < p \leq \infty \)

\[
\varphi_{\beta,\delta} \in M_p(\mathcal{T}). \tag{3.22}
\]

Put

\[
\xi(x) = \begin{cases} 
(1 - \varphi_{\beta,\delta}(x))|x|^{-\beta}, & x \neq 0; \\
\delta, & x = 0,
\end{cases} \quad \xi_j(x) = h_j(x)\xi(x), \quad j = 1, 2, \tag{3.23}
\]

\[
\xi(x) = \begin{cases} 
(1 - \varphi_{\beta,\delta}(x))|x|^{-\beta}, & x \neq 0; \\
\delta, & x = 0,
\end{cases} \quad \xi_j(x) = h_j(x)\xi(x), \quad j = 1, 2,
\]

\[
\xi_j(x) = h_j(x)\xi(x), \quad j = 1, 2,
\]
where \( h_1 \) and \( h_2 \) are defined by (3.4). Note that

\[
\xi_1(x) = h_1(x) \sum_{\nu=1}^{\infty} \left( \frac{\delta}{\nu} \right) (-1)^\nu |x|^\beta (\nu-1).
\]

By analogy with the proof of sufficiency in Lemma 3.10 we have that \( \hat{\xi}_1 \in L_{p^*}(\mathbb{R}^d) \) \( (p^* = \min(p, 1)) \) and hence, by Lemma 3.3, we get

\[
\xi_1 \in M_p(T). \tag{3.24}
\]

Now we consider the function \( \xi_2 \). It is obvious that

\[
(h_2(x)|x|^{-\beta})_\wedge \in L_{p^*}(\mathbb{R}^d). \tag{3.25}
\]

Applying Lemma 3.3, (3.25), Lemma 3.1, and (3.22), we get

\[
\xi_2 \in M_p(T). \tag{3.26}
\]

Thus, by (3.24), (3.26) and Lemma 3.1, we obtain that \( \xi \in M_p(T) \) and hence the application of Lemma 3.2 yields

\[
\|T - S_n^{\beta, \delta}(T)\|_p \leq C_3 n^{-\beta} \|\Delta^{\beta/2}T\|_p, \quad T \in T_n, \quad n \in \mathbb{N}. \tag{3.27}
\]

In order to prove the lower estimate in (3.19) we put

\[
\eta_j(x) = h_j(x)(\xi(x))^{-1}, \quad j = 1, 2.
\]

We claim that \( \eta_j \in M_p(T), j = 1, 2 \). To see that \( \eta_2 \in M_p(T) \), we represent \( \eta_2 \) in the following form:

\[
\eta_2 = \eta_{2,1} + \eta_{2,2},
\]

where

\[
\eta_{2,1}(x) = |x|^{\beta} h_2(x) \sum_{\nu=0}^{\sigma} \varphi^{\nu}(x), \quad \eta_{2,2}(x) = |x|^{\beta} h_2(x) \sum_{\nu=\sigma+1}^{\infty} \varphi^{\nu}(x),
\]

and \( \sigma > (d/p + 1)/\delta \). Using Lemmas 3.4 and 3.3, it is easy to check that

\[
\eta_{2,2} \in M_p(T) \tag{3.28}
\]

Consider the function \( \eta_{2,1} \). It is obvious that

\[
(h_2(x)|x|^{\beta})_\wedge \in L_{p^*}(\mathbb{R}^d). \tag{3.29}
\]

Applying Lemma 3.3, (3.29), Lemma 3.1, and (3.22), we get

\[
\eta_{2,1} \in M_p(T). \tag{3.30}
\]
Thus, by (3.28), (3.31) and Lemma 3.1, we obtain

$$\eta_2 \in M_p(\mathcal{T}).$$  \hspace{1cm} (3.31)

Now to check

$$\eta_1 \in M_p(\mathcal{T}),$$  \hspace{1cm} (3.32)

we introduce the function

$$\phi(x) = \frac{|x|^\beta}{1 - (1 - |x|^\beta)^\delta} - \sum_{\nu=0}^{\sigma} a_\nu |x|^\nu \delta,$$  \hspace{1cm} (3.33)

where $\sigma > \frac{2}{p}(d/p + 1) + 2$. We put also $\gamma(x) = \phi(x)h_1(x)$. From Lemmas 3.1 and 3.5 it follows that $\gamma - \eta_1 \in M_p(\mathcal{T})$. Thus, to check (3.32), we need to check

$$\gamma \in M_p(\mathcal{T}).$$  \hspace{1cm} (3.34)

Note that, for $u \in (0, 1)$

$$\frac{u^\beta}{1 - (1 - u^\beta)^\delta} = \left( \sum_{\nu=0}^{\infty} c_\nu u^{\beta \nu} \right)^{-1},$$

where $c_0 = \delta \neq 0$. Therefore, the numbers $\{a_\nu\}$ in (3.33) can be chosen such that

$$\phi(x) = \left( \sum_{\mu=0}^{\infty} c_\mu |x|^\mu \right)^{-1} \sum_{\nu=\sigma+1}^{\infty} b_\nu |x|^\nu.$$  \hspace{1cm} (3.35)

Calculating the partial derivatives of $\phi(x)$ and taking into account (3.35), we get $h_1 \phi \in \tilde{W}^m_1(\mathbb{R}^d)$, where $m = [d/p] + 1$. Thus, by Lemma 3.4 we have (3.34) and therefore (3.32).

Combining (3.31) and (3.32), using Lemmas 3.1 and 3.2, we have the following inequality:

$$n^{-\beta} \|\Delta^{\beta/2} T\|_p \leq C_4 \|T - S_n^{\beta, \delta}(T)\|_p, \quad T \in \mathcal{T}_n,\quad n \in \mathbb{N}.$$  \hspace{1cm} (3.36)

Thus, from (3.27) and (3.36) we have the two-sided inequality (3.19). \hfill \Box

**Lemma 3.14.** Let $f \in L_p(\mathbb{T}^d)$, $1 \leq p \leq \infty$, $\beta, \delta > 0$, and $(1/p, \delta) \in \Sigma(d) \cup \Omega'(d)$. Then

$$\|f - S_n^{\beta, \delta}(f)\|_p \leq C n^{-\beta} \|\Delta^{\beta/2} f\|_p, \quad n \in \mathbb{N},$$

where $C$ is a constant independent of $f$ and $n$.

**Proof.** The proof of Lemma 3.14 is similar to the proof of Lemma 3.13 (the upper inequality in (3.19)). The main difference is that it is necessary to use multipliers of
Fourier series instead of multipliers of trigonometric polynomials. We give a brief proof of Lemma 3.14 by using the corresponding theorems in [7] (see also Chapters 7 and 8 in [9]).

Denote by $M_p$ the algebra of locally Riemann-integrable functions with the norm

$$
\|\varphi\|_{M_p} = \sup_n \left\| \left\{ \varphi \left( \frac{k}{n} \right) \right\}_{L^p \to L^p} < \infty,
$$

where $\|\{\lambda_k\}_{L^p \to L^p}$ is the norm of Fourier multiplier $\Lambda = \{\lambda_k\}$, acting from $L_p$ to $L_p$ (for the precise definition see [7]). By the comparison principle for Fourier multipliers (see Theorem 6 in [7]) it suffices to show that the function $\xi$ defined by (3.23) belongs to $M_p$.

Put $\xi_1(x) = h_1(x)\xi(x)$ and $\xi_2(x) = (1 - h_1(x))\xi(x)$. In the proof of Lemma 3.13 it was shown that $\hat{\xi}_1 \in L_1(\mathbb{R}^d)$, hence using Theorem 1 in [7] and the inequality

$$
\|\{\lambda_k\}_{L^p \to L^p} \leq \|\{\lambda_k\}_{L^\infty \to L^\infty},
$$

which holds for any $p \in [1, \infty)$ (see, for example, [21, p. 284]), we get $\xi_1 \in M_p$.

To conclude the proof, it remains to show that $\xi_2 \in M_p$. In accordance with the conditions of Lemma 3.14, we obtain $\varphi_{\beta, \delta} \in M_p$. Note also that the function $\psi(x) = (1 - h_1(x)) |x|^{-\beta}$ can be represented as an absolutely convergent Fourier integral (see Theorem 4 in [7]), hence $\psi \in M_p$ (see Theorem 1 in [7]). Thus, by using elementary properties of multipliers, we obtain $\xi_2 \in M_p$. ■

4 Proofs of the Main Results

Proof of Theorem 2.1. Let $\alpha$ and $\delta$ be some positive numbers. Let us show that the convergence of the means $S_n^{\alpha, \delta}$ in $L_p$ implies the convergence of $S_n^{\beta, \delta}$ in the same space $L_p$ for any $\beta > 0$.

First we show that the convergence of $S_n^{\alpha, \delta}$ implies the convergence of $S_n^{\alpha, \delta+1}$. Indeed, from the equality

$$
S_n^{\alpha, \delta+1} = S_n^{\alpha, \delta} - n^{-\alpha} \Delta^{\alpha/2} \circ S_n^{\alpha, \delta}
$$

and from Lemma 3.6 we get that for each $f \in L_p$:

$$
\|S_n^{\alpha, \delta+1}(f)\|_p \leq \|S_n^{\alpha, \delta}(f)\|_p + n^{-\alpha} \|\Delta^{\alpha/2} S_n^{\alpha, \delta}(f)\|_p \leq C\|S_n^{\alpha, \delta}(f)\|_p.
$$

(4.1)
From the last inequality and Lemma 3.11 it follows that the convergence of $S_n^{\alpha, \delta}$ implies the convergence of $S_n^{\alpha, \delta+1}$.

Next, using the expansion
\[(1 - |x|^{\beta})_+^\delta = \sum_{\nu=0}^{\infty} b_\nu (1 - |x|^{\alpha})_+^{\delta+\nu},\]
we can represent the means $S_n^{\beta, \delta}$ in the following form
\[S_n^{\beta, \delta} = \sum_{\nu=0}^{\lambda} b_\nu S_n^{\alpha, \delta+\nu} + \mathcal{P}_n, \tag{4.2}\]
where
\[\mathcal{P}_n(f; x) = \sum_k \psi \left( \frac{k}{n} \right) c_k(f) e^{i(k,x)}\]
and
\[\psi(x) = (1 - |x|^{\beta})_+^\delta - \sum_{\nu=0}^{\lambda} b_\nu (1 - |x|^{\alpha})_+^{\delta+\nu}. \tag{4.3}\]
We choose the parameter $\lambda$ in (4.3) such that $\psi(0) \neq 0$ and $\lambda > d$. Repeating the proof of Lemma 3.10 it is easy to check that $\hat{\psi} \in L_1(\mathbb{R}^d)$. Whence, by Lemma 3.12 we have that $\mathcal{P}_n$ with an appropriate normalization converge in $L_q$ for any $q \in [1, \infty]$. Thus, by using the equality (4.2), the inequality (4.1), and Lemma 3.11 we get that the means $S_n^{\beta, \delta}$ converge in $L_p$ for any $\beta > 0$. To complete the proof of the theorem, it remains only to use Lemma 3.8.

**Proof of Theorem 2.2.** The proof of item 1) of Theorem 2.2 is similar to the proof of Theorem 2.1. The proof of item 2) follows from Lemmas 3.10 and 3.12.

**Proof of Theorem 2.3.** By using Lemma 3.14 for any $g$ satisfying $\Delta^{\beta/2} g \in L_p(\mathbb{T}^d)$, we get
\[\|f - S_n^{\beta, \delta}(f)\|_p \leq \|(f - g) - S_n^{\beta, \delta}(f - g)\|_p + \|g - S_n^{\beta, \delta}(g)\|_p \leq (1 + \|S_n^{\beta, \delta}\|_p)\|f - g\|_p + C_1 n^{-\beta} \|\Delta^{\beta/2} g\|_p. \tag{4.4}\]
Passing to the infimum on $g$ in (4.4), we obtain
\[\|f - S_n^{\beta, \delta}(f)\|_p \leq C_2 K_\beta(f, 1/n)_p \leq C_2 \tilde{K}_\beta(f, 1/n)_p. \tag{4.5}\]
We now prove the lower estimate. Using the lower estimate in (3.19), we have
\[
K_\beta(f, 1/n)_p \leq \tilde{K}_\beta(f, 1/n)_p \leq \| f - S_n^{\beta, \delta}(f) \|_p + n^{-\beta} \| \Delta^{\beta/2} S_n^{\beta, \delta}(f) \|_p \leq \| f - S_n^{\beta, \delta}(f) \|_p + C_3 \| S_n^{\beta, \delta}(f - S_n^{\beta, \delta}(f)) \|_p \leq C_4 \| f - S_n^{\beta, \delta}(f) \|_p.
\]
(4.6)

The equivalence
\[
\| f - S_n^{\beta, \delta}(f) \|_p \asymp \| f - S_n^{\beta, \delta}(f) \|_p
\]
(4.7)
follows directly from Lemma 2.2 in [20] (see also the proof of Theorem 2 in [8]).

Thus, the equivalences in (2.3) follow from (4.5), (4.6), (4.7) and Theorem A. ■

**Proof of Theorem 2.4** The proof of Theorem 2.4 follows from Lemma 3.13 and Theorem 2.2. It coincides with the proof of Theorem 3 in [8]. ■

**Proof of Theorem 2.5** The proof of inequality (2.5) is similar to the proof of Theorem 6 in [8]. ■

**Proof of Theorem 2.6 and Theorem 2.7** The proof is similar to the proof of Theorem 2.3. ■

**References**

[1] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Integrals and derivatives of fractional order and some of their applications*, Nauka i Tekhnika, Minsk, 1987 (Russian).

[2] E. M. Stein, G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton, 1971.

[3] L. Grafakos, *Classical Fourier Analysis*, Springer-Verlag, Graduate Texts in Mathematics. 249, Second Edition, 2008.

[4] B. I. Golubov, *Multiple series and Fourier integrals*, Itogi Nauki i Tekhn., Ser. Mat. Anal. VINITI, Moscow, 19 (1982), 3-54 (Russian).

[5] E. S. Belinsky, *Approximation by the Bochner-Riesz means and spherical modulus of continuity*, Dopov. Akad. Nauk. Ukr. RSR, Ser. A, 7 (1975), 579-581 (Russian).

[6] O. I. Kuznetsova, R. M. Trigub, *Two-sided estimates of the approximation of functions by Riesz and Marcinkiewicz means*, Doklady Akad. Nauk SSSR, 251 (1980), 34-36 (Russian).
[7] R. M. Trigub, "Absolute convergence of Fourier integrals, summability of Fourier series, and polynomial approximation of functions on the torus," Izv. Akad. Nauk SSSR, Ser. Mat. (6), 44 (1980), 1378-1409 (Russian).

[8] K. Runovski, H.-J. Schmeisser, "On Approximation Methods Generated by Bochner-Riesz Kernels," J. Fourier Anal. Appl., 14 (2008), 16-38.

[9] R. M. Trigub, E. S. Belinsky, "Fourier Analysis and Approximation of Functions," Kluwer, 2004.

[10] E. M. Stein, "Harmonic Analysis: Real-variable Methods, Orthogonality and Oscillatory Integrals," Princeton University Press, 1993.

[11] Z. Ditzian, V. Hristov, K. Ivanov, "Moduli of smoothness and K-functional in $L_p$, $0 < p < 1$," Constr. Approx., 11 (1995), 67-83.

[12] V. Hristov, K. Ivanov, "Realization of K-functionals on subsets and constrained approximation," Math. Balkanica. (New Series), 4 (1990), 236-257.

[13] H.-J. Schmeisser, H. Triebel, "Topics in Fourier Analysis and Function Spaces," Wiley, Chichester, 1987.

[14] H. Triebel, "Higher Analysis," J.A. Barth, Leipzig, 1992.

[15] K. Runovski, H.-J. Schmeisser, "On some extensions of Berenstein’s inequality for trigonometric polynomials," Functiones et Approximatio, XXIX (2001), 125-142.

[16] C. Herz, "On the mean inversion of Fourier and Hankel transforms," Proc. Nat. Acad. Sci., 40 (1954), 996-999.

[17] J. Löfström, "Some theorems on interpolation spaces with application to approximation on $L_p$," Math. Anal., 172 (1967), 176-196.

[18] R. M. Trigub, "Linear methods of summation of simple and multiple Fourier series and their approximative properties," in: Theory of the Approximation of Functions, (Proc. Intern. Conf. Kaluga.), Nauka, Moscow, (1977), 383-390 (Russian).

[19] K. Runovski, H.-J. Schmeisser, "On the convergence of Fourier means and interpolation means," J. Comp. Anal. and Appl. (3), 6 (2004), 211-220.

[20] V. I. Rukasov, K. V. Runovski, H.-J. Schmeisser, "On convergence of families of linear polynomial operators," Funct. Approx. Comment. Math. (1), 41 (2009), 41-54.

[21] A. Zigmund, "Trigonometric series," Vol.1, Mir, Moscow, 1965 (Russian).