THE COLORFUL HELLY THEOREM AND COLORFUL RESOLUTIONS OF IDEALS

GUNNAR FLOYSTAD

Abstract. We demonstrate that the topological Helly theorem and the algebraic Auslander-Buchsbaum may be viewed as different versions of the same phenomenon. Using this correspondence we show how the colorful Helly theorem of I.Barany and its generalizations by G.Kalai and R.Meshulam translates to the algebraic side. Our main results are algebraic generalizations of these translations, which in particular give a syzygetic version of Helly’s theorem.

2000 MSC : Primary 13D02. Secondary 13F55, 05E99.

1. Introduction

The classical Helly theorem is one of the basic results of convex geometry. It was subsequently generalized by Helly to a topological setting. In the last decade topological techniques has been infused into the study of resolutions of a monomial ideal \( I \) in a polynomial ring \( S \) by the technique of cellular resolutions [11]. Such resolutions are constructed from a cell complex with monomial labellings on the cells satisfying certain topological conditions. We show that these topological conditions are the same as the hypothesis of Helly’s theorem, and that the conclusion of Helly’s theorem corresponds to the weaker version of the Auslander-Buchsbaum theorem one gets if one does not involve the concept of depth, but only that of dimension of the quotient ring \( S/I \).

The Helly theorem was generalized by I.Bárány [2] to a colorful version. By considering nerve complexes this has again recently been abstracted and generalized by G.Kalai and R.Meshulam [10]. Their result involves a pair \( M \subseteq X \) where \( M \) is a matroidal complex and \( X \) is a simplicial complex on a vertex set \( V \). In the case when \( M \) is a transversal matroid on a partition \( V = V_1 \cup \cdots \cup V_{d+1} \) it corresponds to the colorful version of Barany. This latter result may be translated to the algebraic side in two ways. Either by the notion of cellular resolutions, or by the Stanley-Reisner ring of the nerve complex. In fact these two translations are related by Alexander duality.

The translation by the Stanley-Reisner ring of the nerve complex admits algebraic generalizations to multigraded ideals. We consider a polynomial ring \( S \) whose set of variables \( X \) is partitioned into a disjoint union \( X = X_1 \cup \cdots \cup X_r \) and the variables in \( X_i \) have multidegree \( e_i \), the \( i \)’th coordinate vector. (One may think of the variables in \( X_i \) as having color \( i \).) We
investigate $\mathbb{N}^r$-graded ideals $I$ of $S$ and their resolutions. The following is our first main result. (The notion of $(d+1)$-regular ideal is explained in the beginning of Section 4.)

**Theorem 4.1.** Let $S$ be a polynomial ring where the variables have $r$ colors. Let $I$ be a $(d+1)$-regular ideal in $S$, homogeneous for the $\mathbb{N}^r$-grading. Suppose $I$ has elements of pure color $i$ for each $i = 1, \ldots, r$. Then for each color vector $(a_1, \ldots, a_r)$ where $\sum a_i = d+1$, there exists an element of $I$ with this color vector.

This provides a generalization of the result of Kalai and Meshulam [10] in the case of transversal matroids, since in our situation it applies to multi-graded ideals and not just to squarefree monomial ideals. Also the number of colors $r$ and the regularity $d+1$ may be arbitrary, and not the same as in [10]. (But in the monomial case this generalization would have been a relatively easy consequence of [10].) The theorem above is however a special case of our second main result which is the following syzygetic version.

**Theorem 4.3.** Let $S$ be a polynomial ring where the variables have $r$ colors. Let $I$ be an ideal in $S$ which is homogeneous for the $\mathbb{N}^r$-grading, is generated in degree $d+1$ and is $d+1$-regular, so it has linear resolution. Suppose $I$ has elements of pure color $i$ for each $i = 1, \ldots, r$, and let $\Omega^l$ be the $l$’th syzygy module in an $\mathbb{N}^r$-graded resolution of $S/I$ (so $\Omega^1 = I$). For each $l = 1, \ldots, r$, and each color vector $a = (a_1, \ldots, a_r)$ where $\sum a_i = d+l$ and $s$ the number of nonzero coordinates of $a$, the vector space dimension of $(\Omega^l)_a$ is greater or equal to $\left(\binom{s}{l} - 1\right)$.

It is noteworthy that we do not prove Theorem 4.1 directly and do not know how to do it. Rather we show Theorem 4.3 by descending induction on the $\Omega^l$, starting from $\Omega^r$, and then Theorem 4.1 is easily deduced from the special case of $\Omega^1$.

Letting $T = k[y_1, \ldots, y_r]$ be the polynomial ring in $r$ variables, our technique involves comparing the ideal $I \subseteq S$ and its natural image in $T$ by mapping the variables in $X_i$ to sufficiently general constant multiples of $y_i$. Along this vein we also prove the following.

**Theorem 4.7.** Let $I \subseteq S$ be a multigraded ideal with elements of pure color $i$ for each $i$, and let $J \subseteq T$ be an ideal containing the image of $I$ by the map $S \to T$ (described above). Then the regularity of $I$ is greater than or equal to the regularity of $J$.

While we study colored homogeneous ideals, until now they have mostly been studied in the monomial setting. One of the first is perhaps Stanley [15] studying balanced simplicial complexes. Then followed Björner, Frankl, Stanley [1] which classified flag $f$-vectors of $a$-balanced Cohen-Macaulay simplicial complexes where $a \in \mathbb{N}^r$. A more recent paper is Babson, Novik [1] where they develop shifting theory for colored homogeneous ideals, but rather focus on monomial ideals to give another approach to [1]. Nagel and
Reiner [12] give a very nice construction of the cellular resolution of colored monomial ideals associated to strongly stable ideals generated in a single degree.

To give some more perspective on our approach, in studying graded ideals in a polynomial ring much attention has been given to homogeneous (i.e. \( \mathbb{N} \)-graded) ideals and to monomial (i.e. \( \mathbb{N}^n \)-graded) ideals. One way to approach intermediate cases is toric ideals where the variables are attached degrees in some \( \mathbb{N}^r \). A toric ideal is uniquely determined by this association of degrees in \( \mathbb{N}^r \), but their class does not encompass the classes of monomial or homogeneous ideals. Our approach is probably the simplest way of building a bridge encompassing these two extremal cases: let the variables only have the unit vectors in \( \mathbb{N}^r \) as degrees.

The organization of the paper is as follows. In Section 2 we show that the topological Helly theorem and the algebraic Auslander-Buchsbaum theorem are basically different versions of the same phenomenon. In Section 3 we consider the colorful Helly theorem and how it translates to the algebraic side, either via cellular resolutions or via the nerve complex. In Section 4 we state and prove our main results, Theorems 4.1, 4.3, and 4.7.

Acknowledgements. We thank B. Sturmfels for comments on the results and in particular for providing the idea to Theorem 4.7.

2. The topological Helly theorem and the Auslander-Buchsbaum theorem

2.1. The Helly theorem. The classical Helly theorem is one of the founding theorems in convex geometry, along with Radon’s theorem and Carathéodory’s theorem, and is the one which has been generalized in most directions.

**Theorem 2.1** (Helly 1908). Let \( \{K_i\}_{i \in B} \) be a family of convex subsets of \( \mathbb{R}^d \). If \( \bigcap_{i \in B} K_i \) is empty, then there is \( A \subseteq B \) of cardinality \( d + 1 \) such that \( \bigcap_{i \in A} K_i \) is empty.

Helly generalized this result in 1930 to a topological version. We shall be interested in it in the context of polyhedral complexes. Let \( X \) be a bounded polyhedral complex. So \( X \) is a finite family of convex polytopes with the following properties.

- If \( P \) is in \( X \), then all the faces of \( P \) are in \( X \).
- If \( P \) and \( Q \) are in \( X \), then \( P \cap Q \) is a face in both \( P \) and \( Q \).

Let \( V \) be the vertices of \( X \), i.e. the 0-dimensional faces. Let \( \{V_i\}_{i \in B} \) be a collection of subsets of \( V \), and let \( C_i \) be the subcomplex of \( X \) induced from the vertex set \( V_i \), consisting of the faces of \( X \) whose vertices are all in \( V_i \). Now if \( X \) can be embedded in \( \mathbb{R}^d \), and for every \( B' \subseteq B \) the intersection \( \bigcap_{i \in B'} C_i \) is either empty or acyclic, then the topological realizations \( |C_i| \) fulfill the conditions of the topological Helly theorem.
Theorem 2.2 (Helly 1930). Let \( \{K_i\}_{i \in B} \) be a family of closed subsets in \( \mathbb{R}^d \) with empty intersection, such that for each \( B' \subseteq B \) the intersection \( \cap_{i \in B'} K_i \) is either empty or acyclic over the field \( \mathbb{R} \). Then there is \( A \subseteq B \) of cardinality \( \leq d + 1 \) such that \( \cap_{i \in A} K_i \) is empty.

Corollary 2.3. Every minimal subset \( A \) of \( B \) such that \( \cap_{i \in A} K_i \) is empty, has cardinality \( \leq d + 1 \).

Remark 2.4. Note that in Theorem 2.2 we only require the existence of an \( A \) with \( \cap_{i \in A} K_i \) empty while in Corollary 2.3 we say that every minimal \( A \) with \( \cap_{i \in A} K_i \) is empty has cardinality \( \leq d + 1 \). Given the hypothesis these two properties are really equivalent.

2.2. Cellular resolutions. We now make a monomial labelling of the vertices in \( X \) by letting the vertex \( v \) be labeled by

\[
x^{a_v} = \prod_{v \in C_i} x_i.
\]

In other words the variable \( x_i \) is distributed to all vertices outside \( C_i \).

Before proceeding we recall some basic theory of monomial labellings of cell complexes and their associated cellular complexes of free modules over a polynomial ring, following [11, Ch.4].

Let \( k \) be a field and let \( \tilde{C} \cdot (X; k) \) be the reduced chain complex of \( X \) over \( k \). The term \( \tilde{C}_i (X; k) \) is the vector space \( \oplus_{\text{dim } F = i} k F \) with basis consisting of the \( i \)-dimensional polytopes of \( X \), and differential

\[
\partial_i (F) = \sum_{\text{facets } G \subseteq F} \text{sign}(G, F) \cdot G,
\]

where \( \text{sign}(G, F) \) is either 1 or \(-1\) and can be chosen consistently making \( \partial \) a differential.

Now given a polynomial ring \( S = k[x_i]_{i \in B} \) we may label each vertex \( v \) of \( X \) by a monomial \( x^{a_v} \). Each face \( F \) is then labeled by \( x^{a_F} \) which is the least common multiple \( \text{lcm}_{v \in F} \{x^{a_v}\} \). Now we construct the cellular complex \( F(X; k) \) consisting of free \( S \)-modules. The term \( F_i (X; k) \) is the free \( S \)-module \( \oplus_{\text{dim } F = i} k F \) with basis consisting of the \( i \)-dimensional polytopes of \( X \). The basis element \( F \) is given degree \( a_F \). The differential is given by

\[
\partial_i (F) = \sum_{\text{facets } F \subseteq G} \text{sign}(G, F) \frac{x^{a_F}}{x^{a_G}} \cdot G,
\]

which makes it homogeneous of degree 0.

We are interested in the case that \( F(X; k) \) gives a free resolution of \( \text{coker } \partial_0 = S/I \) where \( I \) is the monomial ideal generated by the monomials \( x^{a_v} \). For each \( b \in \mathbb{N}^r \), let \( X_{\leq b} \) be the subcomplex of \( X \) induced on the vertices \( v \) such that \( x^{a_v} \) divides \( x^b \). This is the subcomplex consisting of all faces \( F \) such that \( x^{a_F} \) divides \( x^b \). According to [11 Ch.4], \( F(X; k) \) is a free cellular resolution iff \( X_{\leq b} \) is acyclic over \( k \) or empty for every \( b \in \mathbb{N}^r \). Let \( e_i \) be the \( i \)'th unit coordinate vector in \( \mathbb{N}^r \) and let \( 1 = \sum_{i=1}^r e_i \).
The following provides an unexpected connection between Helly’s theorem and resolutions of square free monomial ideals.

**Theorem 2.5.** Let $X$ be an acyclic polyhedral complex. There is a one-to-one correspondence between the following.

- Finite families $\{C_i\}_{i \in B}$ of induced subcomplexes of $X$ such that for each $B' \subseteq B$, the intersection $\cap_{i \in B'} C_i$ is either empty or acyclic.
- Monomial labellings of $X$ with square free monomials in $k[x_i]_{i \in B}$ such that $F(X; k)$ gives a cellular resolution of the ideal generated by these monomials.

The correspondence is given as follows. Given a monomial labeling define $C_i, i \in B$, by letting $C_i = X_{\leq 1 - e_i}$, and given a family $\{C_i\}$ define the monomials $x^{a_v}$, by $x^{a_v} = \prod_{v \notin C_i} x_i$.

**Proof.** Assume part a. To show part b. we must show that $X_{\leq b}$ is either acyclic or empty for $b \in \mathbb{N}$. It is enough to show this for $b \in \{0, 1\}$ since we have a square free monomial labelling. If $b = 1$, then $X_{\leq b} = X$ which is acyclic. If $b < 1$ we have an intersection $X_{\leq b} = \cap_{\{i : b_i = 0\}} X_{\leq 1 - e_i}$ and $C_i = X_{\leq 1 - e_i}$ by (1).

Now given part b., then $\cap_{i \in B} C_i$ is equal to $\cap_{i \in B} X_{\leq 1 - e_i}$ which is $X_{\leq 1 - \sum_{e_i} e_i}$ and therefore is empty or acyclic. □

**2.3. The Auslander-Buchsbaum theorem.** We shall now translate the statements of the Helly theorem to statements concerning the monomial ideal. First we have the following.

**Lemma 2.6.** Let $A \subseteq B$. Then $\cap_{i \in A} C_i$ is empty if and only if the ideal $I$ is contained in $\langle x_i, i \in A \rangle$.

**Proof.** Since $C_i = X_{\leq 1 - e_i}$, that the intersection is empty means that for each monomial $x^{a_v}$ there exists $i \in A$ such that $x_i$ divides $x^{a_v}$. But this means that $x^{a_v}$ is an element of $\langle x_i, i \in A \rangle$ and so $I$ is included in this. □

Condition a. in Theorem 2.5 is the same as the condition in the topological Helly theorem. Taking Corollary 2.3 and the lemma above into consideration, the topological Helly theorem, Theorem 2.2, translates to the following.

**Theorem 2.7.** Suppose the monomial labelling (1) gives a cellular resolution of the ideal $I$ generated by these monomials. Then every minimal prime ideal of $I$ has codimension $\leq d + 1$.

On the other hand we have the following classical result.

**Theorem 2.8 (Auslander-Buchsbaum 1955).** Let $I$ be an ideal in the polynomial ring $S$. Then

$$\text{projdim}(S/I) + \text{depth}(S/I) = |B|.$$
Without the concept of depth but using only the concepts of dimension and projective dimension this takes the following form.

**Corollary 2.9.**

\[
\text{codim } I \leq \text{projective dimension}(S/I).
\]

We see that in the situation we consider where \(X\) gives a cellular resolution of \(I\), the projective dimension of \(I\) is less or equal to \(d+1\), with equality if the resolution is minimal. We thus see that the Auslander-Buchsbaum theorem and Helly’s theorem are simply different versions of the same phenomenon. Note however that not every monomial ideal has a cellular resolution giving a minimal free resolution of the ideal, see [16].

2.4. The nerve complex. Recall that a simplicial complex \(\Delta\) on a subset \(B\) is a family of subsets of \(B\) such that if \(F \in \Delta\) and \(G \subseteq F\) then \(G \in \Delta\). Simplicial complexes on \(B\) are in one-to-one correspondence with square free monomial ideals in \(k[x_i]_{i \in B}\), where \(\Delta\) corresponds to the ideal \(I_{\Delta}\) generated by \(x_\tau\) where \(\tau\) ranges over the nonfaces of \(\Delta\). If \(R \subseteq B\) the restriction \(\Delta_R\) consist of all faces \(F\) in \(\Delta\) which are contained in \(R\). The simplicial complex is called \(d\)-Leray if the reduced homology groups \(\tilde{H}_i(\Delta_R, k)\) vanish whenever \(R \subseteq B\) and \(i \geq d\).

The nerve complex of the family \(\{C_i\}_{i \in B}\) is the simplicial complex consisting of the subsets \(A \subseteq B\) such that \(\cap_{i \in A} C_i\) is nonempty. The following are standard facts.

**Proposition 2.10.** Let \(\{C_i\}_{i \in B}\) be a family of induced subcomplexes of a polyhedral complex \(X\), with nerve complex \(N\). Suppose that any intersection of elements in this family is empty or acyclic.

a. The union \(\bigcup_{i \in B} C_i\) is homotopy equivalent to \(N\).

b. If \(X\) has dimension \(d\) and \(\tilde{H}_d(X, k) = 0\), the nerve complex \(N\) is \(d\)-Leray.

**Proof.** Part a. follows for instance from [3, Theorem 10.7]. Part b. follows because for \(R \subseteq B\), the restriction \(N_R\) is the nerve complex of \(\{C_i\}_{i \in R}\), and so \(N_R\) is homotopy equivalent to \(\bigcup_{i \in R} C_i\) which is a subcomplex of \(X\). Hence \(\tilde{H}_i(\bigcup_{i \in R} C_i)\) must vanish for \(i \geq d\). \(\square\)

The Alexander dual simplicial complex of \(\Delta\) consist of those subsets \(F\) of \(B\) such that their complements, \(F^c\), in \(B\) are not elements of \(\Delta\).

**Proposition 2.11.** Given any finite family \(\{C_i\}_{i \in B}\) of induced subcomplexes of \(X\), let \(I_{\Delta}\) be generated by the monomials in \(I\). The nerve complex \(N\) of the family \(\{C_i\}_{i \in B}\) is the Alexander dual of the simplicial complex \(\Delta\).

**Proof.** We must show that a square free monomial \(x^A\) is in \(I_{\Delta}\) iff \(A^c\) is in \(N\). Given a generator \(x^{A_v}\), we have \(v \in C_i\) for every \(i \notin \text{supp } a_v\) (the support is the set of positions of nonzero coordinates). Hence \(v \in \cap_{i \in (\text{supp } a_v)^c} C_i\) so this intersection is not empty and hence \((\text{supp } a_v)^c\) is in the nerve complex.

Conversely, if \(v\) is in \(\cap_{i \in A^c} C_i\), then \(a_v \leq A\) so \(x^A\) is in \(I_{\Delta}\). \(\square\)
Corollary 2.12. If $I_\Delta$ is a Cohen-Macaulay monomial ideal of codimension $d+1$, then $I_N$ has $(d+1)$-linear resolution.

Proof. This follows from [9], since $N$ is the Alexander dual of $\Delta$. \qed

3. THE COLORFUL HELLY THEOREM

Helly’s theorem was generalized by I.Barany [2] to the so called colorful Helly theorem. This was again generalized by G.Kalai and R. Meshulam [10]. The following version of the colorful Helly theorem is a specialization of their result, which we recall in Theorem 4.4. It follows from Corollary 4.5 by letting $Y$ there be the nerve complex of the $C_i$ and letting the $V_p$ be $B_p$.

Theorem 3.1. Let $X$ be a polyhedral complex of dimension $d$ with $\tilde{H}_d(X; k) = 0$. Let $\{C_i\}_{i \in B_p}$ for $p = 1, \ldots , d+1$ be $d+1$ finite families of induced subcomplexes of $X$, such that for every $A \subseteq \cup_p B_p$ the intersection $\cap_{i \in A} C_i$ is empty or acyclic. If every $\cap_{i \in B_p} C_i$ is empty, there exists $i_p \in B_p$ for each $p = 1, \ldots , d+1$ such that $\cap_{p=1}^{d+1} C_{i_p}$ is empty.

Note that we think of the index sets as disjoint although $C_i$ may be equal to $C_j$ for $i$ and $j$ in distinct index sets, or even in the same index set.

Remark 3.2. The topological Helly theorem follows from the colorful theorem above by letting all the families $\{C_i\}_{i \in B_p}$ be equal.

Let $B$ be the disjoint union of the $B_p$ and let $S = k[x_{i}]_{i \in B}$. (One may think of the variables $x_i$ for $i \in B_p$ as having a given color $p$.) Let $I_\Delta \subseteq S$ be the associated Stanley-Reisner ideal of the family $\{C_i\}_{i \in B}$, given by the correspondence in Theorem 2.5. The following is then equivalent to the theorem above.

Theorem 3.3. Suppose $I_\Delta$ is contained in each ideal $\langle x_i; i \in B_p \rangle$ generated by variables of the same color $p$, for $p = 1, \ldots , d+1$. Then there are $i_p \in B_p$ for each $p$ such that $I_\Delta$ is contained in the ideal $\langle x_{i_1}, \ldots , x_{d+1} \rangle$ generated by variables of each color.

Proof. Immediate from the theorem above when we take into consideration Lemma 2.6 and Theorem 2.5. \qed

If $A \subseteq B$ let $a_p$ be the cardinality $|A \cap B_p|$. We say that the ideal $\langle x_i; i \in A \rangle$ has color vector $(a_1, \ldots , a_{d+1})$.

Corollary 3.4. Suppose $I_\Delta$ is a Cohen-Macaulay ideal of codimension $d+1$. If $I_\Delta$ has associated prime ideals of pure color $(d+1) \cdot e_i$ for each $i = 1, \ldots , d+1$, then it has an associated prime ideal with color vector $\sum_{i=1}^{d+1} e_i$.

Proof. In this case all associated prime ideals are generated by $d+1$ variables. \qed

The following is an equivalent formulation of Theorem 3.1 and is the form which will inspire the results in the next Section 4. Say that a square free
monomial has color vector \((a_1, \ldots, a_d)\) if it contains \(a_i\) variables of color \(i\).

**Theorem 3.5.** Let \(N\) be the nerve complex of the family \(\{C_i\}_{i \in B}\) of Theorem 3.1. If \(I_N\) has monomials of pure color \(p\) for \(p = 1, \ldots, d+1\), then \(I_N\) has a monomial of color \(\sum_{i=1}^{d+1} e_i\).

**Proof.** The intersection \(\cap_{i \in A} C_i\) is empty iff \(\Pi_{i \in A} x_i\) is a monomial in \(I_N\). □

4. Colorful resolutions of ideals

4.1. **Main results.** Let \(X = X_1 \cup X_2 \cup \cdots \cup X_r\) be a partition of the variables in a polynomial ring \(S\). Letting the variables in \(X_i\) have multidegree \(e_i \in \mathbb{N}^r\), the \(i\)'th coordinate vector, we get an \(\mathbb{N}^r\)-grading of the polynomial ring. A homogeneous polynomial for this grading with multidegree \(a = (a_1, \ldots, a_r)\) will be said to have color vector \((a_1, \ldots, a_r)\). A polynomial with color vector \(r_i e_i\) where \(r_i\) is a positive integer is said to have pure color \(i\).

If \(I\) is an ideal in \(S\), homogeneous for this grading and containing polynomials of pure color \(i\) for \(i = 1, \ldots, r\), then it may be a complete intersection of these, in which case \(I\) does not have more generators. However if we put conditions on the regularity of \(I\) this changes. We recall this notion. Let \(F^*\) be a minimal free resolutions of \(I\) with terms \(F_p^* = \oplus_{i \in \mathbb{Z}} S(-i)^{\beta_p,i}\). We say \(I\) is \(m\)-regular if \(i \leq m + p\) for every nonzero \(\beta_p,i\). This may be shown to be equivalent to the truncated ideal \(\oplus_{p \geq m} I_p\) having linear resolution. For a simplicial complex \(\Delta\) it follows by the description of the Betti numbers of \(I_\Delta\) by reduced homology groups, see [11], Corollary 5.12, that \(I_\Delta\) is \((d+1)\)-regular if and only if \(\Delta\) is \(d\)-Leray. The latter hypothesis is fulfilled in Theorem 3.5 for the nerve complex \(N\), and the conclusion is that \(I_N\) contains an element with color vector \(\sum_{i=1}^{d+1} e_i\). The following generalizes this.

**Theorem 4.1.** Let \(S\) be a polynomial ring where the variables have \(r\) colors. Let \(I\) be a \((d+1)\)-regular ideal in \(S\), homogeneous for the \(\mathbb{N}^r\)-grading. Suppose \(I\) has elements of pure color \(i\) for each \(i = 1, \ldots, r\). Then for each color vector \((a_1, \ldots, a_r)\) where \(\sum a_i = d+1\), there exists an element of \(I\) with this color vector.

**Example 4.2.** Let there be \(r = 3\) colors and suppose \(I\) contains polynomials of multidegree \((3, 0, 0)\), \((0, 4, 0)\), and \((0, 0, 5)\). If \(I\) is generated by them, it is a complete intersection and the regularity is \(3+4+5-2=10\). Suppose then that the regularity is \(9, 8, 7\), and so on. What can be said of the generators of \(I\)?

If the regularity is 9, by the above theorem it must contain a generator of multidegree \(\leq (2, 3, 4)\) (for the partial order where \(a \leq b\) if in each coordinate \(a_i \leq b_i\)).

If the regularity is 8, it must by the above contain a generator of multidegree \(\leq (1, 3, 4)\), a generator of multidegree \(\leq (2, 2, 4)\), and a generator of multidegree \(\leq (2, 3, 3)\). In the same way we may continue and get requirements on the generators of \(I\) if its regularity is 7, 6, and 5 also.
The most obvious example of an ideal as in the theorem above is of course the power $m^{d+1} = (x_1, \ldots, x_r)^{d+1}$ in $k[x_1, \ldots, x_r]$. Let $a = (a_1, \ldots, a_r)$ be a multidegree with $\sum a_i = d + j$ and support $s = \text{supp}(a)$ defined as the number of coordinates $a_i$ which are nonzero. Then the multigraded Betti number $\beta_{j,a}(S/m^{d+1})$ is equal to $(s-j-1)$.

Motivated by this we have a syzygetic version of the colorful Helly theorem.

**Theorem 4.3.** Let $S$ be a polynomial ring where the variables have $r$ colors. Let $I$ be an ideal in $S$ which is homogeneous for the $\mathbb{N}^r$-grading, is generated in degree $d + 1$ and is $(d+1)$-regular, so it has linear resolution. Suppose $I$ has elements of pure color $i$ for each $i = 1, \ldots, r$, and let $\Omega_l$ be the $l$'th syzygy module in an $\mathbb{N}^r$-graded resolution of $S/I$ (so $\Omega_1 = I$). For each $l = 1, \ldots, r$ and each color vector $a = (a_1, \ldots, a_r)$ where $\sum a_i = d + l$ and $s = \text{supp}(a)$, the vector space dimension of $(\Omega_l)_a$ is greater than or equal to $(s-l-1)$.

Theorem 4.1 may be deduced from this. Simply apply Theorem 4.3 to the truncated ideal $I_{\geq d+1} = \oplus_{p \geq d+1} I_p$. Then Theorem 4.1 is the special case $l = 1$. Our goal is therefore now to prove Theorem 4.3.

### 4.2. The result of Kalai and Meshulam.

Let $M$ be a matroid on a finite set $V$ (see Oxley [13] or, relating it more directly to simplicial complexes, Stanley [17, III.3]). This gives rise to a simplicial complex consisting of the independent sets of the matroid. If $\rho$ is the rank function of the matroid, this simplicial complex consist of all $S \subseteq V$ such that $\rho(S) = |S|$. Kalai and Meshulam [10, Thm. 1.6] show the following.

**Theorem 4.4** (Kalai, Meshulam 2004). Let $Y$ be a $d$-Leray complex on $V$ and $M$ a matroid complex on $V$ such that $M \subseteq Y$. Then there is a simplex $\tau \in Y$ such that $\rho(V - \tau) \leq d$.

In case we have a partition $V = V_1 \cup \cdots \cup V_{d+1}$ we get the following.

**Corollary 4.5.** Let $M$ be the transversal matroid on the sets $V_1, \ldots, V_{d+1}$, i.e. the bases consist of all $S \subseteq V$ such that the cardinality of each $S \cap V_i$ is one, and let $Y$ be a $d$-Leray complex containing $M$. Then there is a simplex $\tau$ such that $(V - \tau) \cap V_i$ is empty for some $i$, or in other words $\tau \supseteq V_i$.

In the monomial case it is not difficult, by polarizing, to prove Theorem 4.1 as a consequence of Corollary 4.5. However our result is a simultaneous generalization of this fact to arbitrary multigraded ideals, to two parameters, $r$ the number of colors, and $d + 1$ the regularity, and to higher syzygies of the ideal.

It is particularly worth noting that we do not prove Theorem 4.1 in any direct way, and we do not know how to do it. Rather, it comes out as a special case of Theorem 4.3, which is proved by showing that it holds for the $\Omega^l$ by descending induction on $l$, starting from $\Omega^r$. 
4.3. **Comparing the resolution of $I$ to resolutions of monomial ideals.** Let $T = k[y_1, \ldots, y_r]$. To start with we will look at monomial ideals $J$ in $T$ such that $T/J$ is artinian, i.e. $J$ contains an ideal $K = (y_1^{a_1}, \ldots, y_r^{a_r})$ generated by powers of variables. There is then a surjection $T/K \xrightarrow{\bar{p}} T/J$ which lifts to a map of minimal resolutions $B \xrightarrow{\tilde{p}} A$.

**Lemma 4.6.** Let $T$ be the last term in the minimal resolution $B$ of $T/K$ (so $e$ has multidegree $(a_1, \ldots, a_r)$), and let $A_r = \oplus_1^n T e_i$ be the last term in the minimal free resolution $A$ of $T/J$. Suppose in the lifting $B \xrightarrow{\tilde{p}} A$ of $T/K \xrightarrow{p} T/J$ that $e \mapsto \sum m_i e_i$. Then each $m_i \neq 0$.

**Proof.** Let $\omega_T \cong S(-1)$ be the canonical module of $T$. Dualizing the map $\tilde{p}$ we get a map
\[
\text{Hom}_T(A, \omega_T) \xrightarrow{\text{Hom}_T(\tilde{p}, \omega_T)} \text{Hom}_T(B, \omega_T).
\]
Since $T/K$ and $T/J$ are artinian, this map will be a lifting of the map
\[
\text{Ext}_T^r(T/J, \omega_T) \to \text{Ext}_T^r(T/K, \omega_T)
\]
to their minimal free resolutions. But this map is simply
\[
(2) \quad (T/J)^* \to (T/K)^*,
\]
where $(\cdot)^*$ denotes the vector space dual $\text{Hom}_k(\cdot, k)$, and so this map is injective.

Now $\text{Hom}_T(A_r, \omega_T)$ is $\oplus_1^n T u_i$ where the $u_i$ are a dual basis of the $e_i$, and correspondingly let $u$ be the dual basis element of $e$. Then we will have
\[
\text{u}_i \xrightarrow{\text{Hom}_T(\tilde{p}, \omega_T)} m_i u.
\]
Each $u_i$ maps to a minimal generator of $(T/J)^*$. If $m_i$ was 0, this generator would map to 0 in $(T/K)^*$, but since (2) is injective this does not happen. \hfill $\square$

Let $\lambda : X \to k$ be a function associating to each variable in $X$ a constant in $k$. There is then a map
\[
(3) \quad p_\lambda : S = k[X] \to k[y_1, \ldots, y_r] = T
\]
sending each element $x$ in $X$ to $\lambda(x)y_i$. Given a multigraded ideal $I$ in $S$ with elements of pure color $i$ for each $i$, we can compare its regularity to the regularity of ideals in $T$.

**Theorem 4.7.** Let $I \subseteq S$ be a multigraded ideal with elements of pure color $i$ for each $i$. The image $p_\lambda(I)$ by the map (3) is then a monomial ideal. Let $\lambda$ be sufficiently general so that for each $i$ some element of pure color $i$ in $I$ has nonzero image. If $J \subseteq T$ is an ideal containing the image $p_\lambda(I)$, its regularity is less than or equal to the regularity of $I$. 

Proof. Let $P_i$ be elements of pure color $i$ in $I$ for $i = 1, \ldots, r$, such that $P_i$ maps to a nonzero multiple of $g_i^{\nu_i}$. Let $H$ be the ideal $(P_1, \ldots, P_r)$. We get a commutative diagram

$$
\begin{array}{c}
S/H \longrightarrow T/K \\
\downarrow \quad \downarrow \\
S/I \longrightarrow T/J.
\end{array}
$$

In the minimal free resolution $A_r$ of $T/J$ let $A_r = \oplus T e_i$. Since $T/J$ is artinian, its regularity is $\max \{\deg(e_i) - r\}$. Let $F_r$ be the minimal free resolution of $S/I$ and let $F_r = \oplus S f_j$. The maps in the diagram above lift to maps of free resolutions. In homological degree $r$ of the resolution of $S/H$ we have a free $S$-module of rank one. If we consider its image in $T/J$ and use Lemma 4.6 together with the commutativity of the diagram, we see that for every $e_i$ exists an $f_j$ such that the composition

$$
Sf_j \rightarrow F_r \rightarrow A_r \rightarrow T e_i
$$

is nonzero. But then

$$
\max \{\deg(f_j) - r\} \geq \max \{\deg(e_i) - r\}
$$

so we get our statement. □

Corollary 4.8. If $I$ has linear resolution, then $p_\lambda(I)$ has linear resolution (with the generality assumption on $\lambda$).

Question 4.9. Is the corollary above true without the hypothesis on $I$ that it contains elements of pure color $i$ for each $i$?

Now we shall proceed to prove Theorem 4.3 and thereby also the special case Theorem 4.1.

Proof of Theorem 4.3. Let $J = m^{d+1}$ be the power of the maximal ideal in $T$. There is then a completely explicit form of the minimal free resolution $A_r$ of $m^{d+1}$, the Eliahou-Kervaire resolution, [8] or [14], which we now describe in this particular case. For a monomial $m$ let $\max(m) = \max \{q | y_q \text{ divides } m\}$. Let $A_p$ be the free $T$-module with basis elements $(m; j_1, \ldots, j_{p-1})$ where $m$ is a monomial of degree $d + 1$ and $1 \leq j_1 < \cdots < j_{p-1} < \max(m)$. These basis elements are considered to have degree $d + p$. The differential of $A_r$ is given by sending a basis element $(my_k; j_1, \ldots, j_{p-1})$ with $\max(my_k) = k$ to

$$
\sum_q (-1)^q y_{j_0} (my_k; j_1, \ldots, \hat{j}_q, \ldots, j_{p-1})
$$

$$
- \sum_q (-1)^q y_{j_0} (my_{j_0}; j_1, \ldots, j_q, \ldots, j_{p-1}).
$$

(If a term in the second sum has $\max(my_{j_0}) \geq j_{p-1}$, the term is considered to be zero.) In homological degree $r$ we know by the proof of Theorem 4.7
that for each basis element $e = (my_r; 1, 2, \ldots, r - 1)$ of $A_r$ there is a basis element $f$ of $F_r$ such that the composition
\[ Sf \rightarrow F_r \rightarrow A_r \rightarrow e \]
maps $f$ to $ne$ where $n$ is a nonzero monomial. Since the $f$ and $e$ have the same total degree (the resolution is linear), we may assume that the monomial $n$ is 1. Since the map $F_r \rightarrow A_r$ is multihomogeneous and every two of the basis elements of $A_r$ have different multidegrees, we will in fact have $f \mapsto e$. This shows that the theorem holds when $l = r$.

We will now show Theorem 4.3 by descending induction on $l$. Let $(my_k; J')$ be a given basis element of $A_r$ where $|J'| = l - 1$ and $\max(my_k) = k > \max(J')$.

1. Suppose $k > l$. Then there is an $r < k$ not in $J'$, and let $J = J' \cup \{r\}$. Consider now the image of $(my_k; J)$ given by (4). No other basis element involved in this image has the same multidegree as $(my_k; J')$. By induction we assume there is an element $f$ in $F_{l+1}$, that forms part of a basis of $F_{l+1}$, such that $f \mapsto (my_k; J)$. The differential in $F$ maps $f$ to $\sum_{i=1}^{\lvert X \rvert} x_i f_i$ where each $f_i$ may be considered a basis element of $F_l$ if nonzero. Since the map $\tilde{p}_l$ is homogeneous, a scalar multiple of some $f_i$ must map to $(my_k; J')$.

2. Suppose $k = l$. Then $J' = \{1, 2, \ldots, k - 1\}$ and let $J = J' \cup \{k\}$. The image of $(my_{k+1}; J)$ by the differential in $A$ will be
\begin{equation}
(5) \quad \sum_{q=1}^{k} (-1)^q y_q (my_{k+1}; J \setminus \{q\}) - \sum_{q=1}^{k} (-1)^q y_{k+1} (my_q; J \setminus \{q\}) .
\end{equation}
Since $\max(my_q) \leq k$ and $\max J \setminus \{q\} = k$ when $q \neq k$, all the terms in the second sum become zero save
\[ (-1)^k y_{k+1} (my_k; J') \]
and this term is the only one in (5) involving a basis element with the multidegree of $(my_k; J')$. By the same argument as in 1. above, there is a basis element $f$ in $F_l$ that maps to $(my_k; J')$. This concludes the proof of the theorem.

The following is a consequence of Theorem 4.1.

**Corollary 4.10.** Let $J$ in $k[y_1, \ldots, y_r]$ be a monomial ideal with linear resolution generated in degree $d + 1$. If $J$ contains the pure powers $y_i^{d+1}$ for each $i$, then $J$ is $(y_1, \ldots, y_r)^{d+1}$.

This also follows from the fact that the vector space dimensions of the graded pieces of the socle of $S/J$ is determined by the last term of the resolution of $S/J$. The resolution being linear means that the socle of $S/J$ is concentrated in degree one lower than the generators of $J$.

Let $\Delta(d+1)$ be the geometric simplex in $\mathbb{R}^r$ defined as the convex hull of all $r$-tuples $(a_1, \ldots, a_r)$ of non-negative integers with $\sum a_i = d + 1$. The
monomials of degree $d + 1$ of a monomial ideal in $k[y_1, \ldots, y_r]$ may be identified with a subset of $\Delta(d+1)$. Under the hypothesis of $d + 1$-linearity the corollary above says that if the ideal contains the extreme points of $\Delta(d+1)$ then it contains all of its lattice points. This suggests the following more general problem.

**Problem 4.11.** Let $J$ in $k[y_1, \ldots, y_r]$ be a monomial ideal with linear resolution generated in degree $d + 1$. What can be said of the “topology” of the generating monomials of $J$ considered as elements of $\Delta(d+1)$?

**References**

[1] E.Babson, I.Novik *Face numbers of nongeneric initial ideals*, Electronic Journal of Combinatorics, Research paper 25,11, no.2, (2004-2006), 1-23.

[2] I.Bárány, *A generalization of Caratheodory's theorem*, Discrete Math. 40 (1982), 141-152.

[3] A.Björner, *Topological methods*, in: R.Graham, M.Grötschel, L.Lovász(Eds.), Handbook of Combinatorics, North-Holland, Amsterdam, 1995, 1819-1872.

[4] A.Björner, P.Frankl, R.Stanley, *The number of faces of balanced Cohen-Macaulay complexes and a generalized Macaulay theorem*, Combinatorica 7 (1987), 23-34.

[5] W.Bruns and J.Herzog, *Cohen-Macaulay rings*, Cambridge studies in advanced mathematics 39, Cambridge University Press 1993.

[6] D.Eisenbud, *Commutative algebra with a view towards algebraic geometry*, GTM 150, Springer-Verlag, 1995.

[7] J. Eckhoff, *Helly, Radon and Carathéodory type theorems*, in: P.M. Gruber, J.M. Wills (Eds.), Handbook of Convex Geometry, North-Holland, Amsterdam, 1993.

[8] S.Eliashou, M.Kervaire, *Minimal resolutions of some monomial ideals*, Journal of Algebra 129 (1990), 1-25.

[9] J.Eagon, V.Reiner, *Resolutions of Stanley-Reisner rings and Alexander duality*, Journal of Pure and Applied Algebra 130 (1998), 265-275.

[10] G.Kalai, R.Meshulam, *A topological colorful Helly theorem*, Advances in Mathematics, 191 (2005), 305-311.

[11] E.Miller, B.Sturmfels, *Combinatorial commutative algebra*, GTM 2005, Springer-Verlag, 2005.

[12] U.Nagell, V.Reiner *Betti numbers of monomial ideals and shifted skew shapes*, Research paper 3, 16, no.2, 59 pp.

[13] J.G.Oxley *Matroid theory*, Oxford Graduate Texts in Mathematics 1992.

[14] I.Peeva, M.Stillman *The minimal free resolution of a Borel ideal*, Expositiones Mathematicae, 26, no.3 (2008), 237-247.

[15] R.Stanley *Balanced Cohen-Macaulay complexes*, Trans. Amer. Math. Soc. 249, no.1, (1979), 139-157.

[16] M.Velasco *Minimal free resolutions that are not supported on a CW-complex*, Journal of Algebra, 319, no.1 (2008), 102-114.

[17] R.Stanley, *Combinatorics and Commutative Algebra*, Second Edition, Birkhäuser 1996.