Utility function estimation: the entropy approach

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Abstract

The maximum entropy principle can be used to assign utility values when only partial information is available about the decision maker’s preferences. In order to obtain such utility values it is necessary to establish an analogy between probability and utility through the notion of a utility density function. According to some authors [Soofi (1990), Abbas (2006a) (2006b), Sandow et al. (2006), Friedman and Sandow (2006), Darooneh (2006)] the maximum entropy utility solution embeds a large family of utility functions. In this paper we explore the maximum entropy principle to estimate the utility function of a risk averse decision maker.

Key words: Maximum entropy, utility functions, preferences, risk aversion.

Introduction

Utility function is one of the most useful concepts in decision analysis and may be computed empirically from analysis of a trading data of an agent which demonstrates its tolerance with respect to risk. This concept characterizes the excess demand in analogy to the potential energy in mechanical systems [Darooneh (2006)] and the maximization of utility shows the equilibrium condition of the respective market. The randomness in the market tends to increase with time, which is a consequence of the existent risks. Given this, the state of the market with maximum randomness or uncertainty is called equilibrium.

A possible approach to estimate utility functions and utility values using only partial information about the agent’s preferences is the Maximum Entropy (ME) principle. In this paper we refer to partial information when we only have inferred the utility values based on observed decisions.
The main assumption to derive the utility function of an agent, using the ME principle is the correspondence between the concept of equilibrium in physics (statistical) and economics (mechanical). According to some authors [namely Foley (1994), Candeal et al. (2001), Darooneh (2006)] the economic equilibrium can be viewed as an asymptotic approximation to physical equilibrium and some difficulties with mechanical picture (economic) of the equilibrium may be improved by considering the statistical (physical) description of it.

In this paper we explore the ME principle to estimate the utility values of a risk averse investor. The rest of the paper is organized as follows. In Section 1 we present a brief discussion of the background theory, namely the ME principle and its applications to economics and more specifically to decision analysis. Section 2 presents the analogy between utility and probability, and utility and entropy. Finally, Section 3 presents the main conclusions of this study.

1 Background theory

Suppose that we have a set of possible events whose probabilities of occurrence are \( p_1, p_2, \ldots, p_n \) and \( H \) is a measure of uncertainty. Shannon (1948) developed a measure of uncertainty associated with an outcome from a set of symbols that satisfy the following properties: (i) \( H \) should be continuous in \( p_i, i = 1, \ldots, n; \) (ii) if \( p_i = 1/n \), then \( H \) should be a monotonic increasing function of \( n; \) (iii) \( H \) is maximized in a uniform probability distribution context; (iv) \( H \) should be additive; (v) \( H \) should be the weighted sum of the individual values of \( H \).

According to Shannon (1948) a measure that satisfies all these properties is the entropy which is defined as \( H(X) = -\sum_i p_i \log p_i \). When the random variable has a continuous distribution, and \( p_X(x) \) is the density function of the random variable \( X \), the entropy (usually called differential entropy) is given by \( H(X) = -\int p_X(x) \log p_X(x) dx \).

The properties of the entropy of continuous (differential entropy) and discrete distributions are mainly alike. For continuous distributions, \( H(X) \) is not scale invariant \( (H(cX) = H(X) + \log |c|) \) but is translations invariant \( (H(c + X) = H(X)) \). The differential entropy may be negative and infinite [Shannon (1948), Soofi (1994)]. Entropy \( [H(X)] \) is a measure of the average amount of information provided by an outcome of a random variable and similarly, is a measure of uncertainty about a specific possible outcome before observing it [Golan (2002)].

Jaynes (1957) introduced the maximum entropy (ME) principle as a generalization of Laplace’s principle of insufficient reason. The ME principle appears
as the best way when we intend to make inference about an unknown distribution based only on few restrictions conditions, which represent some moments of the distribution. According to several authors [see for example Soofi (2000) and Golan (2002)] this principle uses only relevant information and eliminates all irrelevant details from the calculations by averaging over them.

The ME model is usually formulated to confirm the equality constraints on moments or cumulative probabilities of the distribution of the random variable \( X \), where \( h_j (X_i) \) is an indicator function over an interval for cumulative probability constraints and \( b_j \) are the moment \( j \) of the distribution.

\[
p^* = \arg \max \sum_i p_i \log p_i, \quad \text{s.t.} \\
\sum_i p_i = 1 \\
\sum_i h_j (X_i) p_i = b_j \\
p_i \geq 0 \quad j = 1, \ldots, m, \quad i = 1, \ldots, n.
\]

The density that respect all the conditions of the model (1) is defined by Entropy Density (ED). The Lagrangean of the problem is

\[
L = -\sum_i p_i \log p_i - \lambda_0 \left[ \sum_i p_i - 1 \right] - \sum_{j=1}^m \lambda_j \left[ \sum_i h_j (X_i) p_i - b_j \right],
\]

where \( \lambda_0 \) and \( \lambda_i \) are the Lagrange multipliers for each probability or moment constraint. The solution to this problem has the form

\[
p_i = \exp \left[ -\lambda_0 - 1 - \sum_{j=1}^m \lambda_j h_j (X_i) \right].
\]

For small values of \( m \) it is possible to obtain explicit solutions [Zellner (1996)]. If \( m = 0 \), meaning that no information is given, one obtains a uniform distribution. As one adds the first and the second moments, Golan, Judge and Miller (1996) recall that one obtains the exponential and the normal density, respectively. The knowledge of the third or higher moments does not yield to a density in a closed form and only numerical solutions may provide densities.

In many cases, precise values for moments and probabilities are unavailable. In face of this problem Abbas (2005) propose the use of the ME principle using upper and lower bonds in the moments constraints.

There are several research studies of ME in economics. Buchen and Kelly (1996) describe the application of ME principle to the estimation of the distribution of an underlying asset from a set of option prices. Samperi (1999)
selects a pricing measure that is consistent with observed market prices by
minimizing the relative entropy functional subject to linear constraints. This
author explores the relationship between entropy, utility theory and arbitrage
pricing theory. Gulko (1998) in its research work develops the *Entropy Pric-
ing Theory* as a main characteristic of an efficient market, where the entropy
is maximized in order to find the market equilibrium. Stutzer (1996, 2000)
derivates the Black-Scholes model by solution of a constrained minimization
of relative entropy and concludes that relative entropy minimization provides
a simple way to compute the martingale measure that yields the important
Black-Scholes option formula.

The ME principle has been more recently applied in decision analysis, specially
in the specification and estimation of utility values and utility functions. For
example, Fritelli (2000) derives the relative entropy minimizing martingale
measure under incomplete markets and demonstrates the connection between
it and the maximization of exponential utility. Herfert and La Mura (2004) use
a non-parametric approach based on the maximization of entropy to obtain a
model of consumer’s preferences using available evidence, namely surveys and
transaction data. In a different approach Abbas (2004) presents an optimal
question-algorithm to elicit von Neumann and Morgenstein utility values using
the ME principle. The same author [Abbas (2006a)] uses ME to assign utility
values when only partial information is available about the decision maker’s
preferences and [Abbas (2006b)] uses the discrete form of ME principle to
obtain a joint probability distribution using lower order assessments. Yang
and Qiu (2005) propose an expected utility-entropy measure of risk in portfolio
management, and the authors conclude that using this approach it is possible
to solve a class of decision problems which cannot be dealt with the expected
utility or mean-variance criterion.

Sandow et al. (2006) use the minimization of cross-entropy (or relative en-
tropy) to estimate the conditional probability distribution of the default rate
as a function of a weighted average bond rating, concluding that the mod-
eling approach is asymptotically optimal for an expected utility maximizing
investor. Friedman et al. (2007) explore an utility-based approach to some in-
formation measures, namely the Kullback-Leibler relative entropy and entropy
using the example of horse races. On the other way, Darooneh (2006) uses the
ME principle to find the utility function and the risk aversion of agents in a
exchange market.

According to Abbas (2006b), the ME principle presents several advantages
when we purpose to construct joint probability distributions and assign utility
values, namely: (i) it incorporates as much information as there is available
at the time of making the decision; (ii) it makes any assumptions about a
particular form or a joint distribution; (iii) it applies to both numeric and
nonnumeric variables; and (iv) it does not limits itself to the use of only mo-
ments and correlation coefficients, which may be difficult to obtain in decision analysis practice.

2 Utility and entropy

When a decision problem is deterministic, the order of the prospects is enough to define the optimal decision alternative. However, when uncertainty is present, it is necessary to assign the non Neumann and Morgenstein utility values. One of the basic assumptions of decision theory is that an agent’s observed behaviour can be rationalized in terms of the underlying preference ordering and if the observed behaviour is consistent with the ordering we can infer about the utility function using the available data. Sometimes the observations are not sufficient to identify clearly the orderings and one needs more general inference methods. La Mura (2003) presented a non-parametric method for preference estimation based on a set of axiomatic requirements: (i) no information; (ii) uniqueness; (iii) invariance; (iv) system independence, and (v) subset independence. The axioms characterize a unique inference rule, which amounts to the maximization of the entropy of the decision-maker’s preference ordering.

We extend an approach developed by Abbas (2006a) and also used before in a similar way by Herfert and La Mura (2004), the maximum entropy utility principle, where a utility function is normalized to range from zero to one and the utility density function is the first derivative of a normalized utility function. Based on such definition, the utility density function has two main properties: (i) is non-negative; and (ii) integrates to unity. The two properties allows the analogy between utility and probability, and consequently, with entropy [Abbas (2006a)].

For the discrete case, the utility vector has $K$ elements, defined as

$$U \triangleq (u_0, u_1, ..., u_{K-2}, u_{K-1}) = (0, u_1, ..., u_{K-2}, 1).$$

(4)

This vector of dimension $K$ can be represented as a point in a $(K - 2)$ dimensional space, which is defined by $0 \leq u_1 \leq ... \leq u_{K-2} \leq 1$. This region, called utility volume, has a volume equal to $1/(K-2)!$.

In the utility increment vector ($\Delta U$) the elements are equal to the difference between consecutive elements in the utility vector, it has $K - 1$ elements and is defined by

$$\Delta U \triangleq (u_1 - 0, u_2 - u_1, ..., 1 - u_{K-2}) = (\Delta u_1, \Delta u_2, ..., \Delta u_{K-1}).$$

The coordinates of $\Delta U$ are all non-negative and sum to one.
According to Abbas (2006a) the knowledge of the preference order alone do not inform at all about the location of the utility increment vector. In this conditions is reasonable to assume that the respectively location is uniformly distributed over the domain. The assumptions gives equal likelihood to all utility values and satisfy agent’s preference order, adding no further information than the knowledge of the order of the prospects.

For the continuous case, the concepts are similar, but the number of prospects $K$ can be infinite. Is this case the utility vector is a utility curve $[U(x)]$, and has the same mathematical properties as a cumulative probability distribution. The utility increment vector (or in this case, utility density function) is now a derivative of the utility curve

$$u(x) \triangleq \frac{\partial U(x)}{\partial x}$$ (5)

which is non-negative and integrates to unity.

Given the analogy between utility and probability, the concept of entropy can be used as a measure of spread for the coordinates of the utility increment vector

$$H(\Delta u_1, \Delta u_2, ..., \Delta u_{K-1}) = -\sum_{i=1}^{K-1} \Delta u_i \log \Delta u_i.$$ (6)

The utility increment vector that maximizes this measure is the uniform distribution. There are other measures that can be used to spread the utility increment vector, although, the entropy satisfies the following 3 axioms: (1) the measure of spread of the utility increment vector is a monotonically increasing function of the number of prospects $K$, when the utility increments are all equal; (2) the measure of spread of a utility increment vector should be a continuous functions of the increments; (3) the order in which we calculate the measure of spread should not influence the results.

The differential entropy can also be applied to a utility density function

$$H(u(x)) = -\int_a^b u(x) \log u(x) dx,$$

and this function is maximized when $u(x) = 1/(b-a)$. The uniform density integrates to a linear (risk neutral) utility function.
The maximum entropy utility problem is described by

\[ u_{\text{max} \text{ent}} (x) = - \int_a^b u(x) \log u(x) \, dx, \ s.t. \]
\[ \int_a^b u(x) \, dx = 1 \]
\[ \int_a^b h_i (x) u(x) \, dx = b_i \]
\[ u(x) \geq 0, \ i = 1, ..., n. \] (7)

Abbas (2006a) used a CARA utility density to show that the differential entropy has a unique maximum, that occurs exactly when the agent is risk neutral.

This approach is also defended by Darooneh (2006), that considers that the equilibrium condition may be expressed by the maximum entropy utility, since the risk of the market induce the randomness. The solution for this problem is given by the following expression

\[ u_{\text{max} \text{ent}} (x) = \exp \left[ -\lambda_0 - 1 - \lambda_1 h_1 (x) - \lambda_2 h_2 (x) - ... - \lambda_n h_n (x) \right], \] (8)

where \([a, b]\) are the domain of the prospects, \(h_i (x)\) is a given preference constraint, \(b_i\)'s are a given sequence of utility values or moments of the utility function and \(\lambda_i\) is the Lagrangean multiplier for each utility value. The uniform utility density is a special case of equation (8) where the constraints \(h_i (x)\) do not exist. When \(h_1 (x) = x\) and the remaining constraints are zero, the maximum entropy utility is a CARA utility on the positive domain. When \(h_1 (x) = x\) and \(h_2 (x) = x^2\) the maximum entropy utility is a Gaussian utility density, which integrates to a S-shaped prospect theory utility function on the real domain.

The risk aversion parameter \((\gamma)\), using the Arrow-Pratt definition, of the agent is given by

\[ \gamma_{\text{max} \text{ent}} (x) = - \frac{\partial \ln [u_{\text{max} \text{ent}} (x)]}{\partial x} = \lambda_1 h_1 \prime (x) + \lambda_2 h_2 \prime (x) + ... + \lambda_n h_n \prime (x), \] (9)

where \(h_i \prime (x) = \partial h_i (x) / \partial x\). The equation (9) shows the linear effect contributed by the derivative of each preference constraint on the overall risk aversion function.

Abbas (2006a) presents several examples of application of maximum entropy utility principle, namely for cases when we know some utility values, cases when we need to infer utility values by observing decisions and for the case of multiattribute utility. For all the cases explored, Abbas (2006a) concludes that the maximum entropy utility principle presents advantages and satisfies
the important assumption of utility and probability independence that stems from the foundations of normative utility theory.

3 Conclusions

This paper presents an efficient alternative way to estimate the utility function of any agent when there is only partial available information about the decision maker’s preferences. The maximum entropy approach here presented provides a unique utility function that makes no assumptions about the structure, unless there is preference information to support it.

Based on the recent literature on this research area, we show that the analogy probability - utility can be explored in order to use the information theory measures, and obtain a more robust estimation of the utility function.

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