Massive Gauge Field Theory Without Higgs Mechanism

I. Quantization

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According to the conventional concept of the gauge field theory, the local gauge invariance excludes the possibility of giving a mass to the gauge boson without resorting to the Higgs mechanism because the Lagrangian constructed by adding a mass term to the Yang-Mills Lagrangian is not only gauge-non-invariant, but also leads to an unrenormalizable theory. On the contrary, we argue that the principle of gauge invariance actually allows a mass term to enter the Lagrangian if the Lorentz constraint condition is taken into account at the same time. The Lorentz condition, which implies vanishing of the unphysical longitudinal field, defines a gauge-invariant physical space for the massive gauge field. The quantum massive gauge field theory without Higgs mechanism may well be established by using a BRST-invariant action which is constructed by incorporating the Lorentz condition and another condition constraining the gauge group into the original massive Yang-Mills action. The quantum theory established in this way shows good renormalizability and unitarity.

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1. INTRODUCTION

The gauge field theory which has been playing a leading role in developments of the contemporary theoretical physics had been beset with the gauge boson mass problem for a long time, because the requirement of local gauge invariance, according to the prevailing viewpoint, does not admit a mass term into the Yang-Mills Lagrangian. Historically, for building up meaningful strong and weak interaction theories in the gauge field framework, the gauge boson mass has to be necessarily introduced into the theory in spite of the gauge symmetry of the theory being destroyed. It was considered, however, to be unpleasant that such theories suffer from the difficulty of renormalization due to the inclusion of the boson mass term. A widely accepted solution to the mass problem was eventually found through introducing the spontaneous symmetry-breaking mechanism, i.e., the so-called Higgs mechanism, into the gauge field theory. In this way, Weinberg and Salam established the most successful unified model of weak and electromagnetic interactions which shows a great example followed subsequently by other extended unification theories and the charged meson field theory. In the standard model which is recognized to describe today’s physics, except for the intermediate bosons which acquire their masses through the Higgs mechanism, the gluons responsible for the strong interaction still remain massless owing to the assumption that the strong interaction respects an exact gauge symmetry.

The Higgs mechanism introduced into a gauge field theory, as we know, badly spoils the gauge symmetry of the theory due to the spontaneous symmetry-breaking of the vacuum. Equivalently, when we make the vacuum to be gauge-invariant by a translation of the scalar field, the Lagrangian will lose the original gauge symmetry. A question raised is that whether we have to live in the world with a broken gauge symmetry? In other words, whether a massive gauge field theory can be set up grounded on the gauge invariance principle without relying on any Higgs mechanism? This question has evoked continued efforts to study it. As will be commented in the last section, various attempts were made in the past. Among these, the Stueckelberg prescription was attracting most attention owing to the fact that the mass term in the Lagrangian is given a gauge-invariant form by introducing additional Stueckelberg scalar fields. However, all the formalisms presented previously were eventually abandoned as they are criticized to be either unrenormalizable or nonunitary. The failure of the previous efforts does not mean that there is no possibility of constructing a reasonable non-Abelian massive gauge field theory without Higgs mechanism. The possibility arises from the observation that in the functional space of the full vector potential of a gauge field, there is a physical subspace spanned by the Lorentz (four-dimensionally) transverse vector potential in which the mass term in the action is gauge-invariant. The transverse vector potential, as it completely describes the three degrees of freedom of the polarization of a massive gauge boson, is the genuine field variable of a massive gauge field, whereas the remaining component of the full vector potential, i.e., the Lorentz longitudinal vector potential, appears to be a redundant variable for describing the massive gauge field. The Yang-Mills Lagrangian usually is derived in whole space of the full vector potential by the requirement of gauge-invariance. This is a particularly necessary and important step of building up a gauge field theory in which the interaction terms are definitely given. For the massive gauge field, however, we only
need to write out a Lagrangian in the physical subspace of the transverse field. Such a Lagrangian may directly be obtained from the Yang-Mills Lagrangian by letting the longitudinal part of the vector potential in it vanish and then adding a mass term for the transverse field to it. It will be shown in the next section that the action given by this kind of Lagrangian is gauge-invariant. Therefore, the dynamics of massive gauge field described by the Lagrangian, which is expressed through the independent field variables, does not violate the gauge-invariance principle. Nevertheless, the Lagrangian given by including a mass term for the full vector potential in the Yang-Mills Lagrangian, as was done previously, can not make its action to be gauge-invariant owing to the inclusion of the longitudinal field variable in it. This kind of Lagrangian is actually not complete to describe the dynamics of the massive gauge field if the redundant field variable has not been limited by some constraint condition. From the physical viewpoint, we have no reasons to require the unphysical degree of freedom to possess a mass so that whether the Lagrangian expressed in terms of the full vector potential has some gauge symmetry or not is irrelevant to us. This point of view is easy to understand from the mechanics of a constrained system. Suppose a mechanical system is described by a Hamiltonian terms of the full vector potential has some gauge symmetry or not is irrelevant to us. This point of view is easy to understand from the mechanics of a constrained system. Suppose a mechanical system is described by a Hamiltonian $H(\mathbf{p}_i, \mathbf{q}_i)(i = 1, 2, \cdots, n)$ and constraint conditions $\varphi_\alpha(\mathbf{p}_i, \mathbf{q}_i) = 0(\alpha = 1, 2, \cdots, m < n)$. If the constrained variables can be solved out from the constraint conditions, we may write a Hamiltonian $H^*(p_i^\alpha, q_i^\beta)(j = 1, 2, \cdots, n - m)$ which is expressed via the independent variables. Obviously, it is not reasonable to demand the Hamiltonian $H^*(p_i^\alpha, q_i^\beta)$ to have the desired symmetry.

For convenience of theoretical treatments, it is necessary to transform the Lagrangian given by the transverse field variable into the Lagrangian represented in terms of the full vector potential. To do this, it is indispensable to introduce an appropriate constraint condition restricting the latter Lagrangian so as to ensure the unphysical degree of freedom being eliminated eventually from the theory. Such a constraint condition may be chosen to be the well-known Lorentz condition which naturally leads to vanishing of the longitudinal vector potential. This constraint condition may be incorporated into the Lagrangian by the Lagrange undetermined multiplier method. In order to make the resultant action given by this Lagrangian to be invariant under the gauge transformation of the vector potential, we have to impose an other constraint condition for the gauge group on the Lagrangian, just as was similarly done for the massless gauge field. This constraint may also be incorporated in the Lagrangian by the Lagrange undetermined multiplier approach. In this way, we obtain a Lagrangian which allows us to establish a correct quantum massive gauge field theory without any Higgs boson in it. When the Lagrangian thus obtained is used to construct the generating functional of Green’s functions, We derive an effective action which is invariant under a kind of BRST (Becchi-Rouet-Stora-Tyutin) transformations.23

This paper is devoted to elucidating the basic ideas which are important for setting up the massive gauge field theory and describing the procedure of quantization of the theory. The theory given in this paper is renormalizable and unitary. Detailed discussions of these problems will be presented in subsequent papers. The rest of this paper is arranged as follows. In Sect.2, we will discuss the gauge transformation and the classical dynamics of the massive gauge field. In addition, we will point out the reason why the previous effort of building the massive gauge field theory without the Higgs mechanism failed. In Sect.3, we will describe the quantization of the massive gauge field theory and the derivation of the BRST- transformation. In Sect.4, we will discuss equations of motion derived from the effective Lagrangian which is obtained from the quantum theory and give some results in the tree diagram approximation. The last section serves to make some comments and conclusions. In Appendix A, we will show the Fourier transformation of the transverse vector potential to help understanding of the meaning of the vector potential.

2. GAUGE TRANSFORMATION AND CLASSICAL DYNAMICS

The purpose of this section is to present an argument that the dynamics of a massive gauge field can, indeed, be constructed on the principle of local gauge symmetry. Before doing this, it is necessary to analyze the usual local gauge transformation. For more clearness, we would like to begin with the Abelian gauge field theory. The Lagrangian of the massless gauge field is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad (2. 1)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (2. 2)$$

is the field strength tensor and $A_\mu(x)$ is the vector potential of the field. As is well known, the Lagrangian above is invariant under the following gauge transformation
\[ A_\mu'(x) = A_\mu(x) + \partial_\mu \theta(x) \]  

(2.3)

where \( \theta(x) \) is the scalar gauge function of space and time variables. Eq.(2.3) clearly indicates that the gauge transformation only changes the longitudinal part of the vector potential because the second term in Eq.(2.3) is a longitudinal vector in the Minkowski space. Now let us split the vector potential \( A^\mu(x) \) into two Lorentz-covariant parts: the transverse vector potential \( A^\mu_T(x) \) and the longitudinal vector potential \( A^\mu_L(x) \)

\[ A^\mu(x) = A^\mu_T(x) + A^\mu_L(x) \]  

(2.4)

where

\[ A^\mu_T(x) = (g^{\mu\nu} - \frac{1}{\Box} \partial^\mu \partial^\nu) A_\nu(x) \]  

(2.5)

\[ A^\mu_L(x) = \frac{1}{\Box} \partial^\mu \partial^\nu A_\nu(x) \]  

(2.6)

here \( \Box = \partial^\mu \partial_\mu \) is the D’Alembertian operator. The vector potentials \( A^\mu_T(x) \) and \( A^\mu_L(x) \) satisfy the following transverse and longitudinal field conditions (identities):

\[ \partial_\mu A^\mu_T(x) = 0 \]  

(2.7)

\[ (g_{\mu\nu} - \frac{1}{\Box} \partial_\mu \partial_\nu) A^\nu_L(x) = 0 \]  

(2.8)

and the orthogonality relation

\[ \int d^4x A^\mu_T(x) A_{\mu\nu}(x) = 0 \]  

(2.9)

which characterizes the linear independence of the two field variables. Considering this independence and Eq.(2.4), Eq.(2.3) can be equivalently divided into two transformations:

\[ A^\mu_T(x) = A^\mu_T(x) \]  

(2.10)

\[ A^\mu_L(x) = A^\mu_L(x) + \partial^\mu \theta(x) \]  

(2.11)

Eqs.(2.10) and (2.11) clearly express the fact that only the longitudinal vector potential undergoes the gauge transformation, while the transverse vector potential is a gauge-invariant quantity.

It would be emphasized that the Lagrangian expressed in terms of the full vector potential, actually, is only related to the transverse field variable \( A^\mu_T \). In fact, when Eq.(2.4) is substituted in Eq.(2.2) and noticing Eq.(2.6), or the general expression \( A^\mu_L(x) = \partial^\mu \varphi(x) \) where \( \varphi(x) \) is an arbitrary scalar function, it is easy to find that the longitudinal vector potential \( A^\mu_L \) is cancelled in the strength tensor. Therefore, we have

\[ F^{\mu\nu} = \partial^\mu A^\nu_T - \partial^\nu A^\mu_T = F_T^{\mu\nu} \]  

(2.12)

This tells us that all physical observables of the massless Abelian gauge field are only related to the transverse vector potential. Eq.(2.12) enables us to write the Lagrangian given in Eq.(2.1) in the form

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{\mu\nu} \]  

(2.13)

According to Eq.(2.10), the above Lagrangian manifestly shows its gauge invariance property. Since the transverse vector potential \( A^\mu_T(x) \), as it contains three independent components, may entirely describe the three degrees of freedom of polarization of a massive gauge field, it appears to be the genuine field variable of the massive gauge field (see Appendix A). Particularly, the Lorentz-covariance of the potential \( A^\mu_T \) implies that the massive gauge field only exists in the subspace spanned by the potential \( A^\mu_T(x) \). It is obvious that the gauge invariance of the \( A^\mu_T(x) \) allows us to write, in a gauge-invariant manner, a massive gauge field Lagrangian given by adding a mass term to the Lagrangian shown in Eq.(2.13)

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A^\mu_T A_{\mu T} \]  

(2.14)

It is no doubt that the above Lagrangian expressed by the independent field variable completely represents the classical dynamics of massive \( U(1) \) gauge field. If we want to express the Lagrangian through the full vector potential, it is
necessary to introduce a constraint condition imposed on the Lagrangian so as to guarantee the redundant degree of freedom to be eliminated from the Lagrangian. A suitable constraint condition is the covariant Lorentz condition

$$\partial^\mu A_\mu = 0$$  \hspace{1cm} (2.15)

This condition, as we see from the definition given in Eq.(2.6), directly leads to vanishing of the unphysical longitudinal vector potential. With the above gauge condition constraining the Lagrangian, we may rewrite the Lagrangian in terms of the full vector potential

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$$  \hspace{1cm} (2.16)

where the longitudinal field is also given a mass term formally, but it is eventually cancelled out by the constraint condition. We emphasize here that the Lagrangian in Eq.(2.16) together with the constraint in Eq.(2.15) is, in essence, gauge-invariant because it is equivalent to the Lagrangian denoted in Eq. (2.14). It is noted that even for the massless U(1) gauge field, the introduction of the Lorentz constraint is also necessary as we know from the quantum theory. At classical level, due to the relation in Eq.(2.12), it happens to be no problem when the Lagrangian is expressed by the full vector potential and the Lorentz condition is not considered. In this case, the Lagrangian in Eq. (2.1) appears to be complete for formulating the dynamics. However, such a formulation cannot be generalized to the massive case and the non-Abelian case because in these cases the longitudinal field variable can not automatically disappear if the Lorentz condition is not introduced.

Now let us turn to the non-Abelian gauge field, It will be found that the situation is much similar to the Abelian case. Firstly, we write down the Yang-Mills Lagrangian for a massless non-Abelian gauge field

$$\mathcal{L} = -\frac{1}{4} F^{a\mu\nu} F_{a\mu\nu}$$  \hspace{1cm} (2.17)

where

$$F_{a\mu\nu} = \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + gf^{abc} A_{b\mu} A_{c\nu}$$  \hspace{1cm} (2.18)

in which \(g\) is the coupling constant, \(f^{abc}\) are the structure constants of a simple compact non-Abelian group \(G\) and \(A_{a\mu}(a = 1, 2, \cdots, n)\) are the vector potentials of the non-Abelian gauge field. The above Lagrangian is invariant under the following infinitesimal gauge transformation which corresponds to the finite gauge transformation as given by \(e^{ig\theta^a T^a}\)

$$A^a_\mu(x) = A^a_\mu(x) + B^a_\mu(x) + \partial_\mu \theta^a(x)$$  \hspace{1cm} (2.19)

where \(\theta^a(x)(a = 1, 2, \cdots, n)\) are the parametric functions of the local gauge group and \(B^a_\mu(x)\) is defined as

$$B^a_\mu(x) = -gf^{abc} \theta^b(x) A^c_{\mu}(x)$$  \hspace{1cm} (2.20)

which characterizes the non-Abelian property of the gauge transformation and is, in general, neither transverse nor longitudinal. When we decompose the vector \(B^{a\mu}(x)\) into a transverse part \(B^T_{\mu}\) and a longitudinal part \(B^L_{\mu}\), the transformation in Eq. (2.19) may be rewritten as a sum of the following two transformations

$$\delta A^a_{T\mu} = B^a_{T\mu}$$  \hspace{1cm} (2.21)

$$\delta A^a_{L\mu} = D^{ab}_{\mu} \theta^b - B^a_{T\mu}$$  \hspace{1cm} (2.22)

where

$$D^{ab}_{\mu} = \delta^{ab} \partial_\mu - gf^{abc} A^c_{\mu}$$  \hspace{1cm} (2.23)

is the covariant derivative. As exhibited above, the non-Abelian gauge transformation not only alters the longitudinal vector potential, but also the transverse vector potential. This is an essential feature of the non-Abelian gauge field which is different from the Abelian gauge field. However, in the physical subspace of the transverse vector potential, we find, the action of the mass term for a non-Abelian gauge field is of gauge-invariance. For convincing of this point, let us compute the variation of the mass term in the action under the gauge transformation given in Eq.(2.19). Noticing the orthogonality relation shown in Eq.(2.9) and the following identity

$$f^{abc} A^{a\mu} A^{b}_{\mu} \theta^b = 0$$  \hspace{1cm} (2.24)
it is easy to derive
\[ \delta S_m = m^2 \int d^4x A^{a\mu}(x) \delta A^a_\mu(x) = m^2 \int d^4x A^{a\mu}_L(x) \partial_\mu \theta^a(x) \] (2. 25)

From the above variation, it is clearly seen that in the whole space of vector potential, the mass term in the action is, indeed, not gauge-invariant. The origin of the gauge-non-invariance is merely due to the presence of the redundant variable \( A^{a\mu}_L(x) \) (the situation for Abelian gauge field is the same). It is, however, evident that in the physical space restricted by the Lorentz gauge condition:

\[ \partial^\mu A^a_\mu(x) = 0, a = 1, 2, ..., n \] (2. 26)

which holds before and after the gauge transformation and implies the longitudinal vector potential to be zero, the above variation of the action vanishes. Therefore, we may write out a gauge-invariant action given by the following Lagrangian which is expressed via the independent field variable \( A^a_\mu(x) \) for the massive non-Abelian gauge field

\[ \mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a}_{\mu\nu} + \frac{1}{2} m^2 A^a_\mu A^a_\mu \] (2. 27)

where \( F^a_{\mu\nu} \) are defined as Eq.(2.18) with substitution of \( A^a T^\mu_\mu \) for the \( A^a_\mu \). In the physical space, the gauge transformation of the transverse vector potential should be

\[ A^a T^\mu_\mu = A^a T^\mu_\mu - g f^{abc} \theta^b A^c T^\mu_\mu + \partial_\mu \theta^a \] (2. 28)

which is written out from Eq.(2.19) by letting the longitudinal part of the vector potential vanish. It is easy to verify that the action given by the Lagrangian in Eq.(2.27) is invariant under the above gauge transformation

\[ \delta S = m^2 \int d^4x A^{a\mu}_T(x) \partial_\mu \theta^a(x) = 0 \] (2. 29)

where the orthogonality between the transverse field \( A^{a\mu}_T(x) \) and the longitudinal field \( \partial^\mu \theta^a(x) \) has been considered. The action given by the Lagrangian in Eq.(2.27), as it is Lorentz-invariant and gauge-invariant, forms a suitable basis of formulating the massive gauge field dynamics. Just as we have done for the Abelian gauge field, the Lagrangian of the massive non-Abelian gauge field may be expressed in terms of the full vector potentials if the Lorentz gauge condition is treated as a constraint imposed on the Lagrangian

\[ \mathcal{L} = Tr\{ -\frac{1}{2} F^{\mu\nu} F_{\mu\nu} + m^2 A^a A^a \} \] (2. 30)

where \( F^{\mu\nu} = F^{a}_{\mu\nu} T^a, A^a = A^a T^a, T^a \) are the generators of the gauge group and \( "Tr" \) is the symbol of trace. Obviously, the combination of the above Lagrangian with the Lorentz constraint condition is equivalent to the Lagrangian written in Eq.(2.27) and describes the gauge-invariant dynamics of the massive non-Abelian gauge field. As mentioned before, the gauge-invariance of the dynamics can be directly seen, in this formulation, from the variation in Eq.(2.25) and the Lorentz condition in Eq.(2.26) if the differentiation in Eq.(2.25) is performed by part.

It is remarked here that the gauge-invariance, usually, is required to the Lagrangian. From the dynamical viewpoint, the action is of more essential significance than the Lagrangian. Another point we would like to mention is that in examining the gauge invariance of the mass term in the action, we confine ourself to consider the infinitesimal gauge transformation. The reason for this arises from the fact that the Lorentz condition defining the physical space, generally, does not fix the gauge uniquely and limits the gauge transformation only to take place in the vicinity of the unity of the gauge group. This fact was clarified in the previous studies of the quantum massless gauge field theory and becomes a basis of establishing that theory. 24-28 The non-uniqueness (usually called Gribov ambiguity) was firstly revealed by Gribov and investigated further by other authors. 26. That is to say, in the physical space, the non-Abelian gauge field still undergoes nontrivial gauge transformations, or say, has residual gauge degrees of freedom, as denoted in Eq.(2.21). Certainly, this fact is closely related to the property of coupled nonlinear partially differential equations satisfied by the parametric functions of the gauge group. Such equations may be derived from the gauge-invariance of the Lorentz condition and the finite gauge transformation. 25 What are the solutions of these equations look like? This is a difficult problem in Mathematics. Nevertheless, there indeed exist regular solutions to the equations which are linearized in the neighborhood of unit of the group. 25 As will be exhibited in the next section, to achieve a correct formulation of the quantum theory, the infinitesimal gauge transformations are only needed to be considered, That
is to say, in the physical space defined by the Lorentz condition, only the infinitesimal gauge transformations are admissible (see the statement given in Ref. (24)).

It should be mentioned that the Lagrangian in Eq. (2.30) itself is, ordinarily, considered to form a complete description of the classical massive gauge field. From this Lagrangian, one may derive the following equation of motion

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = j^\nu$$ (2.31)

where

$$j^\nu = ig[A_\mu, F^{\mu\nu}]$$ (2.32)

which is the current caused by the gauge field itself. This current is conserved. In fact, by applying Eq. (2.31) and the Jacobian identity:

$$[A_\mu, [A_\nu, F^{\mu\nu}]] + [A_\nu, [F^{\mu\nu}, A_\mu]] + [F^{\mu\nu}, [A_\mu, A_\nu]] = 0$$ (2.33)

it is not difficult to prove

$$\partial_\nu j^\nu = 0$$ (2.34)

Furthermore, let us examine the conservation of the canonical energy-momentum tensor for the massive gauge field. The symmetric expression of the energy-momentum tensor may be written out by the usual procedure. The result is

$$T^{\mu\nu} = -2Tr\{F^{\mu\lambda}F_\lambda^\nu - \frac{1}{4}g^{\mu\nu}F^{\lambda\tau}F_{\lambda\tau} + m^2(\frac{1}{2}g^{\mu\nu}A_\lambda - A^\mu A^\nu)\}$$ (2.35)

In the light of the equation of motion presented in Eq. (2.31) and the Jacobian identity:

$$[D^{\mu}, F^{\nu\lambda}] + [D^{\nu}, F^{\lambda\mu}] + [D^{\lambda}, F^{\mu\nu}] = 0$$ (2.36)

where $D^{\mu}$ is the familiar covariant derivative

$$D^{\mu} = \partial^{\mu} - igA^{\mu}$$ (2.37)

it is easy to derive the following result from Eq. (2.35)

$$\partial_\mu T^{\mu\nu} = 2m^2 Tr(\partial^{\mu} A_\mu A^\nu)$$ (2.38)

We see, only under the Lorentz gauge condition, the conservation of the energy-momentum tensor holds

$$\partial_\mu T^{\mu\nu} = 0$$ (2.39)

The Lorentz gauge condition usually is considered as a consequence or a part of the equations of motion in the previous literature. In fact, when we take divergence of the both sides of Eq. (2.31), in the case of $m \neq 0$, Eq. (2.34) immediately gives rise to Eq. (2.26). This result will still be preserved when the gauge field is coupled to a matter field because the total current, in this case, is conserved.

It is noted that since the Lorentz condition implies vanishing of the longitudinal vector potential, it essentially plays the role of a constraint. The above result seems to indicate that this constraint has already been included in the Lagrangian by some Lagrange undetermined multiplier method. However, we cannot see what is the Lagrange multiplier in the Lagrangian which should appear as an independent variable. We emphasize that the Lagrangian in Eq. (2.30) itself, actually, is not suitable to describe the dynamics of the massive gauge field. This is because the Lagrangian is not gauge-invariant owing to the inclusion of the longitudinal vector potential, As indicated in Eq. (2.12), this potential loses its kinetic energy term in the Lagrangian and hence has no any dynamical meaning. Such a vector potential can only be a constrained variable. As we learn from Mechanics, if this variable is not initially excluded from the Lagrangian in Eq. (2.30), the Lagrangian can not serve as a complete description of the massive gauge field dynamics. If it can, otherwise, one may, as did in the early time, start from it to perform quantization of the theory, for instance, to use it to construct the generating functional of Green’s functions. As one knows, from this kind of generating functional, we will derive a wrong propagator for the massive vector boson like this.

$$D_{\mu\nu}^{ab}(k) = \frac{-i\delta^{ab}}{k^2 - m^2 + i\epsilon}(g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2})$$ (2.40)
As pointed out previously by Feynman\textsuperscript{4}, there occurs a severe contradiction that in the zero-mass limit, the Lagrangian in Eq.(2.30) is converted to the massless one, but the propagator is not and of a singular behavior which makes the theory to be nonrenormalizable. This contradiction reveals nothing but the incompleteness of the Lagrangian in Eq.(2.30). This just is the reason why the original attempt of establishing the massive gauge field theory from the Lagrangian in Eq.(2.30) had failed.

In accordance with the general principle established well in Mechanics for constrained systems, the correct procedure of setting up the massive gauge field theory is to start with the Lagrangian expressed via the independent dynamical variables, as written in Eq.(2.27). When we work in the whole space of the full vector potential, i.e. the Lagrangian in Eq.(2.30) is used, the Lorentz condition must be treated as a necessary constitutive ingredient of the dynamics and introduced at the first onset so as to restrict the unphysical degrees of freedom in the Lagrangian. This constraint may be incorporated into the Lagrangian by the Lagrange undetermined multiplier method, as will be stated in the next section. Since the Lagrangian in Eq.(2.27) gives a complete description of the dynamics, certainly, we are allowed to derive from this Lagrangian all the equations as shown in Eq.(2.31)–(2.38) with replacement of the full vector potential by the transverse one. In this case, the right hand side (RHS) of Eq.(2.38) automatically vanishes owing to the transversality condition written in Eq.(2.7). When taking the divergence of the equation of motion, due to the transversality condition and the current conservation, we are left with a trivial identity.

### 3. Quantization and BRST Transformation

There are several approaches to the quantization of gauge field theories\textsuperscript{24–32}. They may be used to quantize the massive gauge field stated in the preceding section. The quantization given in the Hamiltonian path-integral formalism has been described in a separate paper of ours\textsuperscript{34}. In this section, we plan to discuss the quantization given in the Lagrangian formalism. In the massless gauge field theory, as one knows, an elegant method devised in the Lagrangian path-integral formalism was firstly proposed by Faddeev and Popov\textsuperscript{24}. Analyzing the Faddeev-Popov approach, it is easy to find that this approach just gives a way of introducing a constraint condition on the gauge field and a constraint condition on the gauge group into the generating functional of Green’s functions and then incorporating the constraint conditions in the effective action. In the case of massless gauge field theory, this approach is equivalent to another procedure of quantization\textsuperscript{32}. The basic idea of the latter quantization is to seek a BRST-invariant action. The action is given by a generalized Lagrangian which may be obtained by incorporating all the constraint conditions into the original Lagrangian by means of the Lagrange undetermined multiplier method and can be directly used to construct the generating functional. It will be shown that this method of quantization is efficient for quantizing the massive gauge field.

Let us begin with the Lagrangian given in Eq.(2.30) and the Lorentz constraint condition written in Eq.(2.26). In order to incorporate the Lorentz condition into the Lagrangian and finally into the effective action in the generating functional of Green’s functions, it is convenient, as usual, to introduce additional variables $\lambda^a(x)$ to enlarge the Lorentz gauge condition in such a manner\textsuperscript{24,25}

$$\partial^\mu A^a_\mu + a\lambda^a(x) = 0 , a = 1, 2, \cdots, n$$

where $a$ is a gauge parameter. According to the Lagrange undetermined multiplier method, the above constraint condition may be inserted into the Lagrangian shown in Eq.(2.30) to obtain a generalized Lagrangian in which both the field variables $A^a_\mu(x)$ and the variables $\lambda^a(x)$ can all be handled as free ones. The form of the generalized Lagrangian is determined by the requirement that when varying all the variables in the Lagrangian, we can exactly recover the constraint condition and derive proper field equations of motion from the stationary condition of the action given by the Lagrangian. In this way, we may write

$$\mathcal{L} = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + \frac{1}{2} m^2 A^a_\mu A^a_{\mu} + \lambda^a \partial^\mu A^a_\mu + \frac{1}{2} a(\lambda^a)^2$$

(Note: speaking more specifically, the above Lagrangian is obtained by incorporating the constraint in Eq.(3.1) into an extended Lagrangian: $\mathcal{L} = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + \frac{1}{2} m^2 A^a_\mu A^a_{\mu} - \frac{1}{2} a(\lambda^a)^2$ by the usual Lagrange multiplier method). It is easy to check that minimizing the action built by the above Lagrangian, the constraint condition in Eq.(3.1) will, indeed, be given, while, owing to the enlarged form of the constraint in Eq.(3.1), the equation of motion derived will be of the form

$$\partial^\mu F^a_{\mu\nu} + m^2 A^a_\nu - \partial^\nu \lambda^a = j^a_\nu$$

where the current $j^a_\nu$ was defined in Eq.(2.32). Taking the divergence of the above equation and using Eqs.(3.3) and (2.33), it may be found
\[ \Box \lambda^a - m^2 \partial^\alpha A^a_{\alpha} + g f^{abc} A^b_{\mu} \partial^\mu \lambda^c = 0 \]

As we see, the Lorentz condition no longer appears to be a consequence or a part of the equation of motion at present. When the constraint condition in Eq.(3.1) is applied to Eqs.(3.3) and (3.4), we will obtain two coupled sets of equations

\[
\partial^\mu F^a_{\mu\nu} + m^2 A^a_{\nu} + \frac{1}{\alpha} \partial_\nu \partial_\mu A^a_{\mu} = j^a_\nu
\] (3.5)

\[
[(\Box + \alpha m^2) \delta^{ab} - g f^{abc} A^c_{\mu} \partial^\mu] \lambda^b = 0
\] (3.6)

We see, only in the Abelian case, the two equations decouple.

It would be noted here that the action given by the Lagrangian in Eq.(3.2), generally, is not gauge-invariant under the condition in Eq.(3.1). However, for establishing a correct gauge field theory, a key point is to require the action to be invariant under the gauge transformation. To fulfill this requirement, it is necessary to introduce an additional constraint on the gauge group which is indispensable to eliminate the residual gauge degrees of freedom contained in the space defined by the condition in Eq.(3.1). The correctness of this procedure has been verified by all the present gauge theories, as will be elucidated in the last part of this section. Now let us examine the invariance condition of the action given by the Lagrangian in Eq.(3.2) under the usual gauge transformation

\[
\delta A^a_\mu = D^{ab}_\mu \theta^b
\] (3.7)

which is identical to Eqs.(2.21) and (2.22). That is to say, in the stationary condition of the action, the variations of the vector potentials \( A^a_\mu(x) \) are definitely given, while the variations of the functions \( \lambda^a(x) \) are still arbitrary. In this case, noticing the gauge-invariance of the first term in Eq.(3.2), we have

\[
\delta S = \int d^4 x \{ \lambda^a(x) \partial^\mu (D^{ab}_\mu(x) \theta^b(x)) - m^2 \theta^a(x) \partial^\mu A^a_\mu(x)
+ \delta \lambda^a(x)(\partial^\mu A^a_\mu(x) + \alpha \lambda^a(x)) \} = 0
\] (3.8)

where the second term in the above is derived by employing the identity given in Eq.(2.24). This term is identical to that as shown in Eq.(2.25) and, as we see, it vanishes in the Landau gauge \( (\alpha = 0) \) in which the condition in Eq.(3.1) reduces to the ordinary Lorentz condition. Considering the arbitrariness of the variation \( \delta \lambda^a \), we still get the constraint written in Eq.(3.1) from Eq.(3.8). On applying this constraint to the first term of Eq.(3.8), we obtain

\[
\delta S = -\frac{1}{\alpha} \int d^4 x \Box \lambda^a(x) \partial^\mu (D^{ab}_\mu(x) \theta^b(x)) = 0
\] (3.9)

where

\[
D^{ab}_\mu(x) = \delta^{ab} \frac{\mu^2}{\Box_x} \partial^\mu + D^{ab}_\mu(x)
\] (3.10)

\[
\mu^2 = \alpha m^2
\] (3.11)

the \( D^{ab}_\mu(x) \) was defined in Eq.(2.23). Because \( \frac{1}{\alpha} \partial^\mu A^a_\mu = -\lambda^a \neq 0 \), in order to make the action invariant, the following constraint condition on the gauge group is necessary to be required

\[
\partial^\mu (D^{ab}_\mu(x) \theta^b(x)) = 0, \quad a, b = 1, 2, \cdots, n
\] (3.12)

These are the coupled equations satisfied by the parametric functions \( \theta^a(x) \) of the gauge group. Since the Jacobian matrix is not singular

\[
det M \neq 0
\] (3.13)

where

\[
M^{ab}(x, y) = \frac{\delta (\Box_x D^{ac}_\mu(x) \theta^c(x))}{\delta \theta^b(y)}
= \delta^{ab} (\Box_x + \mu^2) \delta^4(x - y) - g f^{abc} \partial^\mu (A^c_\mu(x) \delta^4(x - y))
\] (3.14)
the above equations are solvable and would give a set of solutions which express the functions \( \theta^a(x) \) as functionals of the vector potentials \( A^a_\mu(x) \). In the Abelian case, Eq.(3.12) will reduce to a Klein-Gordon equation such that

\[
(\Box_x + \mu^2)\theta(x) = 0 \tag{3.15}
\]

In the Landau gauge \((\alpha = 0)\), Eq.(3.12) becomes

\[
\partial^\mu(D^{ab}_\mu\theta^b) = 0 \tag{3.16}
\]

This also is the constraint condition imposed on the gauge group for the massless gauge field theory.

The constraint conditions in Eq.(3.12) may also be inserted into the Lagrangian by the Lagrange undetermined multiplier approach. In doing this, it is convenient, as usually done, to introduce the ghost field variables in such a fashion

\[
\theta^a(x) = \xi C^a(x), \quad a = 1, 2, \ldots, n \tag{3.17}
\]

where \( \xi \) is an infinitesimal Grassmann's parameter independent of the space-time and \( C^a(x) \) are the ghost field variables which are the elements of a Grassmann algebra. In accordance with Eq.(3.17), the gauge transformations in Eq.(3.7) will be rewritten as

\[
\delta A^a_\mu = \xi D^{ab}_\mu C^b \tag{3.18}
\]

and the constraint in Eq.(3.12) becomes

\[
\partial^\mu(D^{ab}_\mu C^b) = 0 \tag{3.19}
\]

where the parameter \( \xi \) has been dropped.

When the constraint condition in Eq.(3.19) is incorporated in the Lagrangian shown in Eq.(3.2) by the Lagrange multiplier procedure, we obtain a more generalized Lagrangian as follows

\[
L = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \frac{1}{2}m^2 A^a_\mu A^a_\mu + \lambda^a \partial^\mu A^a_\mu + \frac{1}{2}\alpha(\lambda^a)^2 + \bar{C}^a \partial^\mu(D^{ab}_\mu C^b) \tag{3.20}
\]

where \( \bar{C}^a(x) \), acting as Lagrange undetermined multipliers, are the new variables which are complex conjugate to the ghost variables \( C^a(x) \). (Note: we may directly incorporate the constraint in Eq.(3.12) into the Lagrangian, obtaining a term \( \chi^a \partial^\mu(D^{ab}_\mu \theta^b) \) where \( \chi^a \) are the Lagrange multipliers. If define the ghost field variables as in Eq.(3.17) and \( \bar{C}^a = \chi^a \xi \), this term will become the last term in Eq.(3.20)). It is easy to verify that the stationary condition of the action given by the above Lagrangian and the arbitrariness of the variables \( \lambda^a(x) \) and \( \bar{C}^a(x) \) will surely yield the constraint conditions written in Eq.(3.1) and (3.19) and, furthermore, these constraints as well as the transformations presented Eq.(3.18) really ensure the action stationary. At this stage, we have a particular interest in other possibility to ensure the action mentioned above to be invariant. In fact, in order to make the action invariant, except for Eq.(3.1), Eq.(3.19) may not be necessary to use, instead, we may make the ghost field functions \( \bar{C}^a(x) \) and \( C^a(x) \) undergo a certain transformations. Let us evaluate the variation of the action again. When the transformations in Eq.(3.18) and the constraint conditions in Eq.(3.1) are employed, the variation of the action will be of the form

\[
\delta S = \int d^4x \{ [\delta \bar{C}^a - \frac{\xi}{\alpha} \partial^\nu A^a_\nu] \partial^\mu(D^{ab}_\mu C^b) - \partial^\mu \bar{C}^a \delta(D^{ab}_\mu C^b) \} \tag{3.21}
\]

This expression suggests that if we set

\[
\delta \bar{C}^a = \frac{\xi}{\alpha} \partial^\nu A^a_\nu \tag{3.22}
\]

and

\[
\delta(D^{ab}_\mu C^b) = 0 \tag{3.23}
\]

The action will be invariant. Eq.(3.22) gives the transformation law of the ghost field variable \( \bar{C}^a(x) \) which is the same as the one in the massless gauge field theory. From Eq.(3.23), we may derive a transformation law of the ghost variables \( C^a(x) \). Noticing
\[ \delta(D_{\mu}^{ab}(x)C^{b}(x)) = \frac{\mu^{2}}{\Box_{x}}\partial^{\mu}_{x}\delta C^{a}(x) + \delta(D_{\mu}^{ab}(x)C^{b}(x)) \] (3.24)

and the following result
\[ \delta(D_{\mu}^{ab}(x)C^{b}(x)) = D_{\mu}^{ab}(x)[\delta C^{b}(x) + \frac{\xi}{2}gf^{bcd}C^{c}(x)C^{d}(x)] \] (3.25)

which we are familiar with in the massless theory, we can get from Eq.(3.23)
\[ D_{\mu}^{ab}(x)\delta C^{b}(x) = D_{\mu}^{ab}(x)[-\frac{\xi}{2}gf^{bcd}C^{c}(x)C^{d}(x)] \] (3.26)

Differentiating the above equation with respect to the coordinate \( x \), we have
\[ M^{ab}(x)\delta C^{b}(x) = M_{0}^{ab}(x)\delta C_{0}^{b}(x) \] (3.27)

where we have defined
\[ M^{ab}(x) = \partial_{\mu}D_{\mu}^{ab}(x) \]
\[ = \delta^{ab}(\Box_{x} + \mu^{2}) - gf^{abc}A_{\mu}(x)\partial_{x}^{\mu} \] (3.28)
\[ M_{0}^{ab}(x) = \partial_{\mu}D_{\mu}^{ab}(x) = M^{ab}(x) - \mu^{2}\delta^{ab} \] (3.29)

and
\[ \delta C_{0}^{a}(x) = -\frac{\xi}{2}gf^{abc}C^{b}(x)C^{c}(x) \] (3.30)

It is indicated that Eq.(3.19) is, precisely, the equation of motion for the ghost field \( C^{a}(x) \) (see Eq.(4.3)). Corresponding to this equation of motion, we may write an equation satisfied by the Green’s function \( \Delta^{ab}(x - y) \)
\[ M^{ac}(x)\Delta^{cb}(x - y) = \delta^{ab}\delta^{4}(x - y) \] (3.31)

The function \( \Delta^{ab}(x - y) \) is nothing but the exact propagator of the massive ghost field which is the inverse of the operator \( M^{ab}(x) \) (see the next paper).

In the light of Eq.(3.31) and noticing Eq.(3.29) we may solve out the \( \delta C^{a}(x) \) from Eq.(3.27)
\[ \delta C^{a}(x) = (M^{-1}M_{0}\delta C_{0})^{a}(x) = \delta C_{0}^{a}(x) - \mu^{2}\int dy\Delta^{ab}(x - y)\delta C_{0}^{b}(y) \] (3.32)

This just is the transformation law for the ghost variables \( C^{a}(x) \). When the mass tends to zero, Eq.(3.32) immediately goes over to the corresponding transformation given in the massless gauge field theory. It is interesting that in the Landau gauge \( (\alpha = 0) \), due to \( \mu = 0 \), the above transformation also reduces to the form as given in the massless theory. This result is natural since in the Landau gauge the gauge field mass term is gauge-invariant. However, in general gauges, the mass term is no longer gauge-invariant. In this case, to maintain the action to be gauge-invariant, it is necessary to give the ghost field a mass \( \mu \) so as to counteract the gauge-non-invariance of the gauge field mass term. As a result, in the transformation in Eq.(3.32), appears a term proportional to \( \mu^{2} \).

At present, we are ready to formulate the quantization of the massive gauge field theory. As we learn from the Lagrange indetermined multiplier method, all the variables, namely, the dynamical variables, the constrained variables, and the Lagrange multipliers in the Lagrangian shown in Eq.(3.20) can be viewed as free ones, varying arbitrarily. Therefore, we are allowed to use this kind of Lagrangian to construct the generating functional of Green’s functions as follows
\[ Z[J^{\mu}, \bar{K}^{a}, K^{a}] = \frac{1}{N} \int D(A^{\mu}_{\mu}, \bar{C}^{a}, C^{a}, \lambda^{a})\exp\{i\int d^{4}x[L(x) + \lambda^{a}(x)A^{\mu}_{\mu}(x) + \bar{K}^{a}(x)C^{a}(x) + C^{a}(x)K^{a}(x)]\} \] (3.33)

where \( D(A^{\mu}_{\mu}, \cdots, \lambda^{a}) \) denotes the functional integration measure, \( L(x) \) was given in Eq.(3.20), \( J^{\mu}_{\mu} \), \( \bar{K}^{a} \) and \( K^{a} \) are the external sources coupled to the gauge and ghost fields and \( N \) is a normalization constant. Looking at the expression of the Lagrangian in Eq.(3.20), we see, the integral over the \( \lambda^{a}(x) \) is of Gaussian type. Upon completing the calculation of this integral, we arrive at.
\[
Z[J^{a\mu}, \bar{K}^a, K^a] = \frac{1}{N} \int D(A^a_{\mu}, \bar{C}^a, C^a) \exp \{i \int d^4x [\mathcal{L}_{\text{eff}}(x) + J^{a\mu}(x)A^a_{\mu}(x) + \bar{K}^a(x)C^a(x) + \bar{C}^a(x)K^a(x)] \}
\]

(3. 34)

where

\[
\mathcal{L}_{\text{eff}} = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + \frac{1}{2} m^2 A^{a\mu} A^a_{\mu} - \frac{1}{2\alpha} (\partial^{\mu} A^a_{\mu})^2 - \partial^{\mu} \bar{C}^a \partial^\mu C^b
\]

(3. 35)
in which the tensor \( F^{a}_{\mu\nu} \) was defined in Eq.(2.18), the third and forth terms are usually referred to respectively as gauge-fixing term and ghost term which arise from the constraints written in Eqs.(3.1) and (3.12) respectively and play the role of quenching the unphysical degrees of freedom contained in the remaining terms in Eq.(3.35). Eq.(3.35) just express the effective Lagrangian in the quantum non-Abelian massive gauge field theory. When the mass tends to zero, Eq.(3.35) straightforwardly reaches the Lagrangian encountered in the massless gauge field theory\(^{24,25}\).

An important property of the effective action built by the Lagrangian shown in Eq.(3.35) is that this action is also invariant under the transformations written in Eqs.(3.18), (3.22) and (3.32). In fact, when we evaluate the variation of this action by using the transformations given in Eq.(3.18), we still obtain the expression as represented in Eq.(3.21). Obviously, once the transformations shown in Eqs.(3.22) and (3.32) are applied, the variation vanishes. The transformations written in Eqs.(3.18), (3.22) and (3.32) commonly are called BRST-transformations\(^{23}\).

In the last part of this section, we would like to describe the quantization by the Faddeev-Popov’s approach\(^{24,25}\). For the massless gauge field theory, as mentioned before, the constraint condition on the gauge group is represented by Eq.(3.16). This condition together with the Lorentz condition in Eq.(3.1) would ensure the action, which is given by the Lagrangian in Eq.(3.2) without the mass term, to be invariant under the gauge transformation given in Eq.(3.7). We shall see that the constraint condition in Eq. (3.16) is also a consequence of the gauge-invariance of the Lorentz condition. The gauge-invariance of the Lorentz condition arises from the requirement that this condition holds for arbitrary vector potentials including the ones before and after gauge transformations. In the general gauge, the gauge-invariance mentioned above is expressed as

\[
\partial^{\mu}(A^{\theta})_{\mu}^{a} + \alpha(\lambda^{\theta})^{a} = \partial^{\mu} A^{a}_{\mu} + \alpha \lambda^{a} = 0
\]

(3. 36)

where \((A^{\theta})^{a}_{\mu}\) and \((\lambda^{\theta})^{a}\) represent the gauge-transformed ones. The \((\lambda^{\theta})^{a} = \lambda^{a} + \delta \lambda^{a}\) may be determined by the requirement that the following action

\[
S = \int d^4x \left\{ -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} - \frac{1}{2} \alpha (\lambda^{a})^2 \right\}
\]

(3. 37)
is to be gauge-invariant with respect to the transformations of \(A^{a}_{\mu}\) and \(\lambda^{a}\). In this way, it is easy to find \((\lambda^{\theta}) = \lambda^{a}\). With this relation, when the gauge-transformation in Eq.(3.7) is inserted into Eq.(3.36), we immediately obtain the condition in Eq.(3.16). Therefore, the condition in Eq.(3.16) can equally be replaced by the gauge-invariance condition of the Lorentz constraint. This fact allows us, according to the Faddeev-Popov’s approach, to insert the following identity\(^{24}\)

\[
det M[A] \int D(\theta^{a}) \delta[\partial^{\mu} A^{a}_{\mu} + \alpha \lambda^{a}] = 1
\]

(3. 38)

into the vacuum-to-vacuum transition amplitude, giving

\[
Z[0] = \langle 0^+ | 0^- \rangle = \frac{1}{N} \int D(A^a_{\mu}, \lambda^a, \theta^a) \det M[A] \delta[\partial^{\mu} A^{a}_{\mu} + \alpha \lambda^{a}] \exp(iS)
\]

(3. 39)

where \(S\) was given in Eq.(3.37) and the matrix \(M[A]\) which is completely determined by the condition in Eq.(3.16), was defined in Eq.(3.14) with the mass being zero. The delta-functional in Eq.(3.39) just means the identity in Eq.(3.36). When the gauge transformation: \(A^a_{\mu} \rightarrow (A^{-\theta})^a_{\mu}\) is made to the integral in Eq.(3.39), considering the gauge-invariance of the integration measure, the determinant and the action, the integral over \(\theta^a\), as a constant, can be factored out and put in the normalization constant \(N\). Thus, we have

\[
Z[0] = \frac{1}{N} \int D(A^a_{\mu}, \lambda^a) \det M[A] \delta[\partial^{\mu} A^{a}_{\mu} + \alpha \lambda] \exp(iS)
\]

(3. 40)

In the above, the delta-functional and the determinant, as one knows, would contribute a gauge-fixing term and a ghost term to the effective Lagrangian appearing in the generating functional, playing the same role as the constraint
conditions in Eqs.(3.1) and (3.16) do. From the above statement, as one can see, in the case of massless gauge field theory, the gauge-invariance required for the Lorentz constraint condition coincides with the requirement of the action being gauge-invariant. Therefore, the Faddeev-Popov’s quantization is equivalent to the quantization of employing the Lagrange undetermined multiplier method as described in this section.

For the massive gauge field, the quantization, certainly, may also be carried out by the Faddeev-Popov’s method. In this case, the transition amplitude in Eq.(3.39) formally remains unchanged except that the action contains a mass term such that

\[
S = \int d^4x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - \frac{1}{2} \alpha(\lambda^a)^2 \right]
\]

(3.41)

The matrix \( M[A] \) may still be determined from the equation shown in Eq.(3.36) where the transformation \((\lambda^\theta)^a = \lambda^a + \delta\lambda^a \) should be now derived from the gauge-invariance condition of the action in Eq.(3.41). In the Landau gauge \((\alpha = 0)\), as we have known, due to the restriction of the delta-functional \( \delta[\theta^\mu A_\mu] \) in Eq.(3.39), the mass term in the action is gauge-invariant so that we still have \( \delta\lambda^a = 0 \). As a result, the \( M[A] \) is the same as given in the massless gauge theory. However, in the general gauges \((\alpha \neq 0)\), the mass term in the action is no longer gauge-invariant under the gauge transformation in Eq.(3.7) and the constraint condition in Eq.(3.1). In this case, in order to make the action gauge-invariant, it is necessary to give the function \( \lambda^a(x) \) a gauge transformation, as easily seen from the gauge-invariance condition of the action:

\[
\delta S = \int d^4x \phi^\mu A_\mu (\delta\lambda^a - m^2\theta^a) = 0
\]

(3.42)

where the condition in Eq.(3.1) has been considered. From the above condition, noticing \( \delta^\mu A_\mu \neq 0 \) in the general gauges, we see, it must be \( \delta\lambda^a = m^2\theta^a \). When this gauge transformation and that written in Eq.(3.7) is inserted into Eq.(3.36), we may obtain a constraint condition which is identical to the one denoted in Eq.(3.12). and hence the matrix \( M[A] \) in Eq.(3.39) now is the same as given in Eq.(3.14). It is easy to verify that the determinant of the \( M[A] \), the integration measure and the action in Eq.(3.39) are invariant with respect to the gauge transformations \( A_\mu^a \rightarrow (A^\theta)^a_\mu \) and \( \lambda^a \rightarrow (\lambda^\theta)^a \) to the functional integral in Eq.(3.39), the integral over \( \theta^\mu(x) \) may also be factored out and put in the normalization constant \( N \), giving formally the same expression as written in Eq.(3.40) with the \( M[A] \) and \( S \) being represented in Eqs.(3.14) and (3.41) respectively. On completing the integration over \( \lambda^a \) in Eq.(3.40), we get

\[
Z[0] = \frac{1}{N} \int D(A_\mu^a) det M[A] \exp \left\{ i S - \frac{i}{2\alpha} \int d^4x (\partial^\mu A_\mu)^2 \right\}
\]

(3.43)

As one knows, the determinant may be represented by an integral over the ghost field functions

\[
det M[A] = \frac{1}{N} \int D(C^\alpha, C^a) \exp \left\{ i \int d^4x d^4y C^a(x) M^{ab}(x,y) C^b(y) \right\}
\]

(3.44)

Upon substituting the above expression into Eq.(3.43) and introducing the external sources in Eq.(3.43), we exactly recover the generating functional as shown in Eq.(3.34) and (3.35). In comparison with the Faddeev-Popov’s quantization stated above, the quantization by the Lagrange multiplier method as described in the former part of this section looks more simple and direct, and its physical meaning is much clear.

In the end, we note that the quantized result shown in Eqs.(3.34) and (3.35) was derived by utilizing the infinitesimal gauge transformations denoted in Eq.(3.7). It has been shown that this result is identical to that obtained by the quantization in the Hamiltonian path-integral formalism. In the latter quantization, we only need to calculate the classical Poisson brackets, without concerning any usage of the gauge transformation. This fact reveals that to get the correct quantum result by the method formulated in this section, the infinitesimal gauge transformations are only necessary to be taken into account and thereby confirms that in the physical subspace restricted by the Lorentz condition, only the infinitesimal gauge transformations are possible to exist.

IV. EQUATIONS OF MOTION AND FEYNMAN RULES

In this section, we first show the role played by the gauge-fixing term and the ghost term in the effective Lagrangian shown in Eq.(3.35) from the viewpoint of equations of motion. These equations of motion may be derived from the stationary condition of the action built by the Lagrangian in Eq.(3.35) and are given in the following
\[\partial_\nu F^{\nu \mu} + \frac{1}{\alpha} \partial^\mu \partial^\nu A_\nu + m^2 A^\mu = j^\mu \]  
\[(\Box + \mu^2) \bar{C} = ig[A_\mu, \partial^\mu \bar{C}] \]  
\[(\Box + \mu^2) C = ig\partial^\mu [A_\mu, C] \]

where

\[j^\mu = ig[A_\nu, F^{\nu \mu}] + ig[\partial^\mu \bar{C}, C] \]  

and all the field quantities above are represented as vectors in the space spanned by the generators of the gauge group. Taking the divergence of Eq.(4.1), we have

\[\frac{1}{\alpha} \Box + m^2 \partial_\mu A^\mu = \partial_\mu j^\mu \]  

The expression of the current divergence \(\partial_\mu j^\mu\) is not difficult to be derived by making use of Eqs.(4.1)–(4.4) and the Jacobian identities. From the expression thus derived and noticing \(\partial_\mu j^\mu = \partial_\mu j^\mu_L\) where \(j^\mu_L\) being a longitudinal quantity, we can get

\[j^\mu_L = \frac{g}{\alpha}[A^\mu, \partial^\nu A_\nu] + ig[\partial^\mu \bar{C}, (\partial^\nu C - ig[A^\mu, C])] \]

in which the first term and the second term arise respectively from the gauge-fixing term and ghost term in Eq.(3.35). Let us rewrite Eq.(4.5) in the form

\[\partial_\mu (A^\mu_L + \frac{1}{\alpha m^2} \partial^\mu \partial^\nu A_\nu - \frac{1}{m^2} j^\mu_L) = 0 \]

This equation implies that the sum of the quantities contained in the parenthesis should be a transverse quantity. However, all the terms in the parenthesis are longitudinal. Therefore, the only possibility is

\[A^\mu_L + \frac{1}{\alpha m^2} \partial^\mu \partial^\nu A_\nu - \frac{1}{m^2} j^\mu_L = 0 \]

This result clearly states that there is a counteraction among the longitudinal field variable, the gauge-fixing term and the ghost term. We are now interested in the effect of the counteraction on the effective Lagrangian. When Eqs.(2.4), (2.9) and (4.8) are noticed, the part of the action given by the mass term and the gauge-fixing term in Eq.(3.35) will become

\[\int d^4x Tr \{m^2 A^\mu A_\mu - \frac{1}{\alpha} (\partial^\mu A_\mu)^2\} = \int d^4x Tr \{m^2 A^\mu_T A_T^\mu + A^\mu_L j^\mu_L\} \]

This expression indicates that only the transverse field has a mass term in the Lagrangian. This point can also be seen from the equation of motion. Let us rewrite Eq.(4.1) separately for the transverse and longitudinal vector potentials as displayed below

\[(\Box + \mu^2) A_T^\mu = J^\mu_T \]
\[(\Box + \mu^2) A_L^\mu + (\frac{1}{\alpha} - 1) \partial^\mu \partial^\nu A_\nu = j^\mu_L \]

In Eq.(4.10), the current is defined as

\[J^\mu_T = j^\mu_T + j^\mu \]

where \(j^\mu_T\) is the transverse part of the current given in Eq.(4.4) and

\[j^\mu_T = ig\partial^\mu [A_\mu, A_\nu] \]  

which obviously is a transverse quantity. Particularly, when Eq.(4.8) is applied to Eq.(4.11), we obtain a trivial equation, namely, the longitudinal field condition given in Eq.(2.8) which the function \(A^\mu_L\) must satisfy. Thus, we are left with only one equation of motion denoted in Eq.(4.10) for the transverse field.

Let us turn to discuss Feynman rules in perturbative calculations of the generating functional given in Eq.(3.34). In the tree diagram approximation, as one knows, the effective action appearing in Eq.(3.34) acts as the proper vertex.
generating functional of zeroth order. The Feynman rules for the massive gauge field theory are easily derived from such a generating functional by the conventional procedure. It is obvious that the lowest order vertices including the three and four-line gauge boson vertices and the three-line ghost vertex are completely the same as those given in the massless gauge field theory. In the momentum space, corresponding to Figs.(1)-(3), these vertices are respectively represented in the following.

\[\Gamma^{(0)abc}_{\mu\nu\lambda}(k_1, k_2, k_3) = -(2\pi)^4\delta^4\left(\sum_{i=1}^{3} k_i\right)g f^{abc}[g_{\mu\nu}(k_1 - k_2)\lambda + g_{\nu\lambda}(k_2 - k_3)\mu + g_{\lambda\mu}(k_3 - k_1)\nu]\]

(4.14)

\[\Gamma^{(0)abc}_{\mu\nu\lambda}(k_1, k_2, k_3, k_4) = -i(2\pi)^4\delta^4\left(\sum_{i=1}^{4} k_i\right)g^2[f^{abc}f^{e\sigma\delta}(g_{\mu\nu}g_{\rho\tau} - g_{\mu\tau}g_{\nu\rho}) + \text{face} f^{edh}(g_{\mu\tau}g_{\nu\lambda} - g_{\mu\nu}g_{\lambda\tau}) + f^{edh}f^{f\sigma\epsilon\lambda}(g_{\mu\nu}g_{\lambda\tau} - g_{\mu\lambda}g_{\nu\tau})]\]

(4.15)

\[\Gamma^{(0)abc}_{\mu\nu\lambda}(k_1, k_2, k_3) = (2\pi)^4\delta^4\left(\sum_{i=1}^{3} k_i\right)g f^{abc}k_1\lambda\]

(4.16)

However, due to the massive property, the gauge boson propagator and the ghost particle one are different from those in the massless theory. The gauge boson propagator derived, in the momentum space, is of the following form

\[iD^{ab}_{\mu\nu}(k) = -i\delta^{ab}\left(\frac{g_{\mu\nu} - k_\mu k_\nu/k^2}{k^2 - m^2 + i\varepsilon}\right.\]

\[\left.\frac{\alpha k_\mu k_\nu/k^2}{k^2 - \mu^2 + i\varepsilon}\right)\]

(4.17)

where \(\mu\) was defined in Eq.(3.11). The ghost particle propagator is

\[i\Delta^{ab}(q) = \frac{-i\delta^{ab}}{q^2 - \mu^2 + i\varepsilon}\]

(4.18)

As we see from Eqs.(4.17) and (4.18), these propagators formally are similar to those appearing in the unified electroweak interaction theory. Particularly, the longitudinal part of the gauge boson propagator and the ghost particle propagator have the same pole determined by the mass \(\mu\). They are actually related to each other by a Ward-Takahashi identity which will be derived in the next paper. It is clear that when the mass tends to zero, Eqs.(4.17) and (4.18) immediately go over to the propagators for the massless bosons just as the Lagrangian in Eq.(3.35) does. So, there is not the contradiction mentioned in Sect. 2. In particular, In the ultraviolet limit, as long as the gauge parameter \(\alpha\) is not chosen at infinity, the massive gauge boson and ghost particle propagators have the same behavior as the massless ones do since the mass term occurring in the denominators of the propagators can be ignored in this limit.

While, in the infrared limit, the behavior of the massive particle propagators is obviously better than the massless ones because the mass term may avoid the infrared divergence. These asymptotic properties tell us that the massive particle propagators can not cause more divergences than the massless ones in perturbative calculations and hence the renormalizability of the quantum theory of the massive gauge field is at least the same as the massless gauge field theory. If we notice the fact that in a given Feynman diagram, the gauge boson and ghost particle internal lines have the same pole determined by the mass \(\mu\), the propagators for the massless bosons just as the Lagrangian in Eq.(3.35) does. So, there is not the contradiction in Sect. 2. In particular, In the ultraviolet limit, as long as the gauge parameter \(\alpha\) is not chosen at infinity, the massive gauge boson and ghost particle propagators have the same behavior as the massless ones do since the mass term occurring in the denominators of the propagators can be ignored in this limit.

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5 Comments and Conclusions

In this paper, it has been shown that the massive non-Abelian gauge field theory may really be set up on the gauge invariance principle with the help of the Higgs mechanism. In achieving this conclusion, we have emphasized the essential points: (1) the massive gauge field only exists in the physical space described completely by the transverse vector potential. In the physical space, the dynamics of massive gauge field is, indeed, of gauge-invariance. Therefore, to build up a massive gauge field theory, we should initially write a Lagrangian in the physical space and then extend the Lagrangian to the whole space of the full vector potential by introducing appropriate constraint conditions imposed on the Lagrangian. That is to say, the massive gauge field must be treated as a constrained system in the space of full vector potential. This is the key point stressed in this paper; (2) The gauge-invariance of gauge field dynamics should be more generally required to the action other than the Lagrangian because the action is of more fundamental dynamical meaning; (3) In the physical space restricted by the Lorentz condition, only the infinitesimal
gauge transformations are possible to exist and necessary to be taken into account; (4) To construct the quantum theory, the residual gauge degrees of freedom existing in the physical space must be eliminated by the constraint condition on the gauge group. This constraint condition may be determined by requiring the action to be gauge-invariant. Thus, the theory would be set up from beginning to end on the gauge-invariance principle. These points are important to establish a correct quantum massive gauge field theory. But, they were not realized clearly in the previous literature. In some earlier investigations\(^3\)–\(^6\) on the massive gauge field theory, authors all started with the Lagrangian shown in Eq.(2.30) without imposing any constraint on it. As was pointed out in Sect.2, this Lagrangian alone can not give a complete formulation of the massive gauge field dynamics because it contains an unphysical longitudinal variable and hence is not gauge-invariant. Therefore, the conclusion of unrenormalizability drawn from such investigations is not true for the massive gauge field.

In Sect.3, we have shown that the quantum massive gauge field theory can be well established in the Lagrangian formalism. The procedure of the quantization contains the following steps: (1) incorporation of the constraint condition on the gauge field into the original Lagrangian by the Lagrange undetermined multiplier method; (2) to find the constraint condition on the gauge group from the requirement of the action given in step(1) being gauge-invariant; (3) the above constraint is also incorporated in the Lagrangian by the Lagrange multiplier method; (4) use of the Lagrangian thus obtained to construct the generating functional. It has been shown that this method of quantization is equivalent to the Faddeev-Popov’s approach.

We would like now to comment on the previous works concerning the massive gauge field theory without the Higgs bosons. It is firstly mentioned that in Ref.(6), authors started from the Lagrangian written in Eq.(2.30) to perform the quantization and introduced a gauge condition to the generating functional by the Faddeev-Popov operation. However, the introduction of the gauge condition was thought to be not necessary. They did it only for the purpose of improving the behavior of the gauge boson propagator. Besides, in contradiction with the procedure that the ghost term in the effective Lagrangian is introduced from the gauge condition by the infinitesimal transformation, they made a finite transformation to the mass term irrespective of what the gauge condition means. Just due to this transformation, there occur an infinite number of terms in the effective Lagrangian which lead to the bad nonrenormalizability. The above viewpoint and treatment are not correct. The reason is obvious. We firstly note that according to the general procedure of constructing the generating functional, we should initially start from the Lagrangian expressed through the independent dynamical field variables as given in Eq.(2.27) other than the Lagrangian in Eq.(2.30) which contains the unphysical variables \(A^a_\mu\). If we want to represent the generating functional in the whole space of the vector potential, as argued in Sect.2, a definite constraint, such as the Lorentz condition, is necessary to be introduced. How to introduce the Lorentz condition into the generating functional by the Faddeev-Popov operation? The correct procedure was stated in the last part of Sect.3. However, if doing this in the fashion as given in Ref.(6), one can not get a BRST-invariant effective Lagrangian which includes a ghost field mass term in the general gauge. Lack of this mass term would not make the theory to be self-consistent in the general gauges. This point will be demonstrated in the subsequent paper. Another point we stress is that the Lorentz condition defining the physical space only permits the gauge transformation to be infinitesimal as was pointed out in Sect.2. Any consideration of finite transformations actually is an inconsistent procedure that ought to be excluded. Noticing this point, the unrenormalizable terms are impossible to appear in the effective Lagrangian.

Let us turn to the widely discussed Lagrangian in which the mass term is given a gauge-invariant form through introduction of Stueckelberg-type functions \(\omega^{a\mu}\)\(^{12}\)–\(^{18}\)

\[
L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 (A^{a\mu} - \omega^{a\mu})(A^{a\mu} - \omega^{a\mu})
\]

(5.1)

where the \(\omega^{a\mu}\) are defined as

\[
\omega^{a\mu} = \frac{i}{g} (\partial^{\mu} U^{-1} U)^a
\]

(5.2)

with \(U\) being the representation of the gauge group

\[
U = e^{ig\phi^a T^a}
\]

(5.3)

When another group element \(S = e^{ig\phi^a T^a}\) is used to make the gauge transformation, \(U' = US\), one may find that the functions \(\omega^{a\mu}\) comply with the same transformation as the vector potential. From Eqs.(5.2) and (5.3), it is not difficult to derive the following expression\(^15\)

\[
\omega^{a\mu} = \left(\frac{e^{i\omega} - 1}{i\omega}\right)^{ab} \partial^{\mu} \phi^b
\]

(5.4)
where $\omega$ is a matrix whose elements are
\[ \omega^{ab} = ig f^{abc} \delta^c \]  
(5.5)

Eq.(5.4) contains an infinite number of terms when we expand the exponential function $e^{i\omega}$ as a series. As was indicated in Ref.(15) that the quantum theory built by the Lagrangian in Eq.(5.1) is nonrenormalizable owing to the nonpolynomial nature of the function $\omega^{a}_\mu$. In order to make the theory to be renormalizable, it is necessary to introduce a number of subsidiary conditions into the theory so as to strike off the effects arising from the function $\omega^{a}_\mu$ and the others.\(^ {16-18}\). Let us comment on the Lagrangian in Eq.(5.1) from other aspects. As one knows, the functions $\omega^{a}_\mu$ represent a pure gauge field which has a vanishing strength tensor. In the Lagrangian written in Eq.(5.1), this field is also given a mass. Therefore, if it is considered to be physical, having three polarized states, we should also impose on it a constraint, for example, the Lorentz condition
\[ \partial^{\mu} \omega^{a}_\mu = 0 \]  
(5.6)

Substituting Eq.(5.4) in Eq.(5.6), one may find a necessary and sufficient condition to content Eq.(5.6) such that
\[ \partial_{\mu} \phi^{a} = 0 \]  
(5.7)

From the expression shown in Eq.(5.4), it is seen that Eq.(5.7) directly leads to
\[ \omega^{a}_\mu = 0 \]  
(5.8)

Thus, in the physical space defined by the Lorentz condition, the Lagrangian in Eq.(5.1) will reduce to the one denoted in Eq.(2.30). On the other hand, if the function $\omega^{a}_\mu$ are treated as free variables, not receiving any constraint, there should be an integral over them in the generating functional. The integral is of Gaussian type and hence easily calculated. As a result of the integration, the mass term of the vector potential $A^{a}_\mu$ will be cancelled out from the Lagrangian. Thus, we are left with only the massless gauge field theory. To avoid the above trivial situations, the authors in Ref.(18) introduced a special constraint imposed on the functions $\omega^{a}_\mu$:
\[ \partial^{\mu} A^{a}_\mu = D^{ab}_\mu (A) \omega^{b\mu} \]  
(5.9)

so as to eliminate the Stueckelberg field in favor of the gauge potential $A^{a}_\mu$. The solution of Eq.(5.9) was thought of a complicated nonpolynomial for which one is no better of. So, the authors limited themselves to treat the model given in the Landau gauge and claimed that in this gauge, the Stueckelberg field vanishes, just as the case denoted in Eqs.(5.6)-(5.8). Thus, we are back to the original formalism without the Stueckelberg field. In general, the condition in Eq.(5.9) cannot make the theory to be self-consistent unless we set $\omega^{a}_\mu = 0$. In fact, the condition (5.9) may follow from the gauge-invariance of the mass term in Eq.(5.1) if the requirement of $\delta \omega^{a}_\mu = 0$ is enforced. Obviously, this requirement conflicts with the original assumption that to make the mass term gauge-invariant, the function $\omega^{a}_\mu$ must transforms according to the same law as the vector potential. In view of these reasons, we can say, although the Landau gauge result given in Ref.(18) is correct, the theoretical logic is not reasonable. Frankly speaking, the Stueckelberg field is unnecessary to be introduced in building the massive gauge field theory as the Stueckelberg field has no physical meaning. This point may also be seen from the fact that when the Lagrangian in Eq.(5.1) is written in the physical space where the vector potentials are replaced by the transverse ones and compared with the corresponding Lagrangian shown in Eq.(2.27), we see, the occurrence of the functions $\omega^{a}_\mu$ in the Lagrangian is completely redundant.

It should be mentioned that in a previous literature, the authors proposed a formalism which is constructed by the Lagrangian written in Eq.(3.2) plus a scalar field Lagrangian. The latter Lagrangian is devised in such a way that it directly yields the equation of motion shown in Eq.(3.6). The Feynman rule derived from this formalism requires that the closed ghost loop possesses an extra factor $\frac{i}{2}$, similar to the theory presented earlier in Ref. (13). In the latter theory, the factor $\frac{i}{2}$ was attached to the ghost vertex. This kind of formalism is not correct. The correct result was given in Ref.(19) with the aid of the solution of Eq.(3.6) even though the quantization method used looks unusual. It is pointed out that the Feynman rule concerning the ghost vertex is uniquely determined by the property of the gauge transformation taking place in the physical space of transverse vector potential. The ghost vertex does not exist in the Abelian theory because in this case, the physical space does not undergo any gauge transformation. While, for the non-Abelian theory, the physical space undergoes nontrivial gauge transformations. In this case, the role played by the ghost vertex is just to counteract the residual gauge degrees of freedom contained in the gauge boson vertex. Since the gauge transformation is the same for the both of massive and massless theories, the Feynman rule related to the ghost vertex should be identical in the both theories. All the formalisms mentioned above were not accepted in the past and even negated by the authors themselves owing to the criticism that the theories violate the unitarity condition (the unitarity problem will be discussed in one of subsequent papers). Apart from this, the theoretical basis, we think, was not well understood and clearly explicated in the previous works.
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APPENDIX A: Fourier transformation of the transverse vector potential

This Appendix is pedagogical, written for the purpose of understanding why it is said that the transverse vector potential is the independent field variable for the massive gauge field. With the Fourier transformation of the full vector potential

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega(k)}} [A_\mu(k)e^{-ikx} + A_\mu^*(k)e^{ikx}] \quad (A. 1)$$

where \(\omega(k) = k_o = (\vec{k}^2 + m^2)^{1/2}\), the transverse vector potential defined in Eq.(2.5) may be written as

$$A_T^\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega(k)}} [A_T^\mu(k)e^{-ikx} + A_T^{\mu*}(k)e^{ikx}] \quad (A. 2)$$

where

$$A_T^\mu(k) = (g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2})A_\nu(k) \quad (A. 3)$$

is the spectral representation of the transverse vector potential given in the momentum space, and \(A_T^{\mu*}(k)\) is the complex conjugate of the \(A_T^\mu(k)\). Let us introduce the four-dimensional polarization vectors defined in the following:

$$e^\mu_i(k) = \begin{cases} \vec{\varepsilon}_i(\vec{k}), & if \quad \mu = 1, 2, 3; \\ 0, & if \quad \mu = 0 \end{cases} \quad i = 1, 2 \quad (A. 4)$$

$$e^\mu_3(k) = \begin{cases} \frac{k^\mu \vec{\varepsilon}_i(\vec{k})}{k^2|\vec{k}|}, & if \quad \mu = 1, 2, 3; \\ \frac{i\kappa}{k^2}, & if \quad \mu = 0 \end{cases} \quad (A. 5)$$

$$e^\mu_0(k) = \frac{k^\mu}{\sqrt{k^2}} \quad (A. 6)$$

where \(\vec{\varepsilon}_i(\vec{k})(i = 1, 2)\) denote the two three-dimensional vectors which satisfy the transversality and orthonormality conditions

$$\vec{k} \cdot \vec{\varepsilon}_i(\vec{k}) = 0 \quad (A. 7)$$

$$\vec{\varepsilon}_i(\vec{k}) \cdot \vec{\varepsilon}_j(\vec{k}) = \delta_{ij} \quad (A. 8)$$

We note that the definitions given in Eqs.(A.5) and (A.6) allow us to discuss the off-shell polarization states. It is easy to verify that the polarization vectors defined in Eq.(A.4)-(A.6) are of the following orthonormality, completeness and transversality

$$\sum_{\lambda=0}^3 e^\mu_\lambda(k)e^{\lambda\nu}(k) = g^{\mu\nu} \quad (A. 9)$$

$$\sum_{\lambda=0}^3 e^\mu_\lambda(k)e_{\lambda\mu}(k) = g_{\lambda\lambda} \quad (A. 10)$$

$$k^\mu e^\mu_\lambda(k) = 0, \lambda = 1, 2, 3. \quad (A. 11)$$

Particularly, from Eqs.(A.9) and (A.6), we get
\[
\sum_{\lambda=1}^{3} e_\lambda^\mu(k) e^{\lambda\nu}(k) = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \tag{A.12}
\]

Upon inserting Eq.(A.12) into Eq.(A.3), it is seen that

\[
A_\mu^T(k) = \sum_{\lambda=1}^{3} e_\lambda^\mu(k) A^\lambda(k) \tag{A.13}
\]

where

\[
A^\lambda(k) = e^{\lambda\nu}(k) A_\nu(k) \tag{A.14}
\]

Eq.(A.13) is the expansion in the three polarization vectors given in Eqs.(A.4) and (A.5) for the transverse field. The unit vector in Eq.(A.4) describes the two three-dimensionally transverse polarization states for a massive vector boson, while the vector in Eq.(A.5) represents the three-dimensional longitudinal polarization of the boson since the time-component of the vector \( e_3^\mu(k) \) may be expressed in terms of the spatial component through Eq.(A.11) with \( \lambda = 3 \). This clearly indicates that the transverse vector potential \( A_\mu^T(k) \), as it completely describes the three polarization states of the vector boson, precisely, is the independent field variable for the massive gauge field.

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**Figure Caption**

Fig.1: The three-line gauge boson vertex. Each wavy line directed outward from the vertex represents a gluon line of momentum $k_1$, $k_2$ or $k_3$, color index $a$, $b$ or $c$ and Lorentz index $\mu$, $\nu$ or $\lambda$.

Fig.2: The four-line gauge boson vertex.

Fig.3: The three-line ghost vertex in which the dashed line represents a ghost particle line of momentum $k_2$ or $k_3$ and color index $b$ or $c$.
Figure 1  

Figure 2  

Figure 3