QCD Spectra and Random Matrix Models

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Received 8 April 1997

Abstract. We summarize some recent results on the application of macroscopic spectral properties of random matrix models (RMM) to the QCD spectra. A comparison to existing lattice simulation is presented both for staggered and Wilson fermions for high but finite temperature. We consider two type of mixing between the four lowest Matsubara modes, corresponding to third and fifth order algebraic equation for the pertinent resolvent, respectively.

1. Introduction

Dedicated lattice simulations on the aspects of the QCD phase transition have triggered a number of theoretical investigations aimed at understanding some of the nonperturbative aspects of QCD whether in vacuum or matter \cite{1}. In particular, it was shown recently that the bulk character of the QCD Dirac spectra could be understood using simple random matrix models \cite{2,3}, and that the spectral-level correlations are consistent with the general lore of random matrix theory \cite{5,6}.

At zero temperature but for massive quarks, QCD simulations with Wilson quarks show that the Dirac spectrum exhibits a structural change with increasing quark mass, a feature that is reflected in the random matrix version in terms of heavy quarks becoming localized over their Compton wavelength \cite{3}. A similar behavior is also observed at finite temperature when the quark modes are restricted to the lowest Matsubara modes $\omega_0 = |\omega_{-1}| = \pi T$ \cite{2,7}.

At high temperature, the static screening lengths in QCD are dominated by the lowest Matsubara modes \cite{8}. This behavior is confirmed by lattice simulations for temperatures that are remarkably close to the critical temperature \cite{9}. At lower temperatures, the role of higher Matsubara modes become more relevant. One purpose of this paper, is to investigate the effects of few Matsubara modes (four) in random matrix models. In Section 2, we use the one-flavor version of the NJL
Lagrangian in Euclidean space to discuss the effects of mixing between various Matsubara modes in random matrix models \[10\]. In Section 3, we specialize to the case of Wilson fermions and derive Pastur’s equation \[11\] for the pertinent random matrix model for the two cases where the Matsubara modes are either mixed (model I), or decoupled (model II). In Section 4, the Dirac spectral distribution for model I is shown to follow from a fifth order algebraic equation (Quinto class), while the evolution of the end-points follow from a quartic equation. In Section 5, the Dirac spectrum associated to model II is discussed using a superposition of a solution to a cubic equation (Cardano class). Some further suggestions and conclusions are made in Section 6. In Section 7 we give a short overview of the applicability of such models to chiral fermions, and particularly we reproduce the valence quark condensate calculated on lattice \[12\]. Finally, in Section 8 we summarize.

2. Mapping of the NJL Model into RMM

The relevance of the Matsubara modes in random matrix models along with their mixing is best seen using one flavor NJL Lagrangian in Euclidean space, with anti-periodic boundary conditions. Specifically

\[
\mathcal{L}_4 = \psi^\dagger (i \gamma \cdot \nabla + im) \psi + \frac{g^2}{2} \left( (\psi^\dagger \psi)^2 + (\psi^\dagger i \gamma_5 \psi)^2 \right),
\]

(1)

where \( g \) is a fixed coupling constant. Using the standard bosonization prescription

\[
P \leftrightarrow -2i g^2 \psi^\dagger_L \psi_L, \quad P^\dagger \leftrightarrow -2i g^2 \psi^\dagger_R \psi_R,
\]

(2)

where \( P, P^\dagger \) stand for independent auxiliary fields, we may rewrite (1) in the form

\[
\mathcal{L}_4 = \psi_R^\dagger (i \gamma \cdot \nabla) \psi_R + \psi_L^\dagger (i \gamma \cdot \nabla) \psi_L + \psi_R^\dagger i(P + m) \psi_R + \psi_L^\dagger i(P^\dagger + m) \psi_L + \frac{1}{2g^2} PP^\dagger
\]

(3)

in the chiral basis \( \psi = (\psi_R, \psi_L) \). Note that the Minkowski fields follow from the Euclidean fields through \((i\psi^\dagger, \psi) \rightarrow (\overline{\psi}, \psi)\). Equation (3) is defined on the strip \( \beta \times V_3 \) in Euclidean space, with \( P(\tau + \beta, \vec{x}) = P(\tau, \vec{x}) \), and \( \psi(\tau + \beta, \vec{x}) = -\psi(\tau, \vec{x}) \), and a 3-momentum cut-off \( \Lambda \). We simplify further the model by putting the left and right quark fields on a discrete grid spanned by the three-space points \( \vec{x} = 1, 2, \ldots, N \) and introducing the rescaled fields \( q_n^x = \sqrt{V_3} \psi_n^x \) with

\[
\psi(\tau, \vec{x}) = \sum_{n=-\infty}^{+\infty} e^{-i\omega_n \tau} q_n^x,
\]

(4)

where \( \omega_n = (2n+1)\pi T \) are the Matsubara frequencies \((T = 1/\beta)\). In what follows we choose \( P \tau \)-independent. This is equivalent to restricting the four-quark interaction

\[
\int_0^\beta d\tau \left( (\psi^\dagger \psi)^2 + (\psi^\dagger i \gamma_5 \psi)^2 \right) = 4 \sum_{n,m,k,l} \delta_{n+k,m+l} \psi_{Rn}^\dagger \psi_{Rm} \psi_{Lk}^\dagger \psi_{Ll}
\]

(5)
to only the terms with \( n = m \) and \( k = l \). Other choices are possible.

In the case where the auxiliary fields \( P \) are \( \tau \)-independent, we can bosonize the present model in a different but complementary way \[10\]. After integrating \( (3) \) over \( P \) and \( P^\dagger \), we bosonize the chirality-flipping pairs using

\[
R^{x,y}_{n,m} = q^x_{\text{Rm}} q^y_{\text{L}}
\]

which is a doubly banded, complex matrix \( R \) with dimensions \((N \times N) \otimes (\infty \times \infty)\). The upper indices refer to three-space \( x, y \) and the lower indices to frequency space \( n, m \). In terms of \( (6) \), the NJL Lagrangian \( (1) \) becomes

\[
L_0 = q^\dagger (\Omega \gamma_4 + im)q + N \Sigma \text{Tr}_{x,n}(RR^\dagger) + q^\dagger_R R q_L + q^\dagger_L R q_R,
\]

where the trace in \( (7) \) is over \( x \) and \( n \). The partition function associated to \( (7) \) is simply

\[
Z[T,\mu] = \int dR \ e^{-N \Sigma \text{Tr}_{x,n}(RR^\dagger)} \text{det}_{2,x,n} Q
\]

with the Dirac operator in a random background,

\[
Q_S = \begin{pmatrix}
im & \Omega \\
\Omega & -im
\end{pmatrix} + \begin{pmatrix}
0 & R \\
R^\dagger & 0
\end{pmatrix},
\]

where \( \Omega = \omega_n 1_n \otimes 1_x \). The determinant in \( (8) \) is over chirality (2), space (\( x \)), and frequency space (\( n \)). This is an example of a chiral random matrix model \[13\]. The structure of \( (9) \) generalizes to several flavors as discussed in \[3\]. The spectral distribution associated to \( (9) \) exhibits manifest chirality, and is suitable for describing Kogut-Susskind fermions on the lattice, as we will discuss below.

### 3. Massive Quark Spectra

The case of Wilson fermions is in a sense different. Indeed, on the lattice, Wilson fermions are not manifestly chirally symmetric due to the presence of the \( r \)-terms \[14\]. To compare with lattice Wilson spectra, we consider instead the hermitean operator \( Q = \gamma_5 (\not{\!D} + m) \), where \( Q \) is the Dirac Hamiltonian in five-dimensions and a single flavor. Here, the Dirac matrix \( \gamma_5 \) plays the role of the \( \beta \)-Dirac matrix in \( (3+1) \) dimensions. A number of simulations, using the analog of \( Q \) with Wilson fermions on the lattice have been carried out recently by Kalkreuter \[15\] for two-color and two-flavor QCD. His results are in qualitative agreement with random matrix theory \[3, 4, 16\]. In \[4\] the low energy QCD spectra was calculated taking into account the lowest Matsubara pair only. The lattice calculation \[15\] however shows the appearance of at least one more scale. One of the possible sources of another scale is the presence of the next Matsubara pair.

In terms of chiral random matrix models, the pertinent ensemble for investigating Wilson fermions for QCD is the Gaussian unitary ensemble for three and
more colors and is the Gaussian orthogonal ensemble \([16]\) for the case of two colors, provided that the quarks are in the fundamental representation of color. With this in mind, the pertinent random matrix model for spatially constant Wilson modes of one-flavor on a cylinder is

\[
Q_W = \left( \begin{pmatrix}
  m_f & -\partial_\tau \\
  -\partial_\tau & -m_f
\end{pmatrix} + R \right),
\]

(10)

where \(\partial_\tau\) is the \(\tau\)-derivative on a cylinder of length \(\beta = 1/T\), and \(R\) a random matrix. Equation (10) is defined with anti-periodic boundary conditions \([4]\).

The operator (10), rewritten in the basis \([4]\) is a the sum of a deterministic and a random piece,

\[
Q_W = \begin{pmatrix}
  m & i\Omega \\
  -i\Omega & -m
\end{pmatrix} + \begin{pmatrix} R \end{pmatrix}.
\]

(11)

Due to the presence of Wilson \(\tau\)-terms, the Dirac operator is only hermitean, the chiral structure is lost. In contrast to \(Q_S\) in \([9]\), \(Q_W\) in (11) is not block-off-diagonal.

The resolvent for the operator (11) is defined as

\[
G(z) = \frac{1}{2NN_*} \left\langle \text{Tr} \frac{1}{z - Q_W} \right\rangle,
\]

(12)

where \(N_*\) is the number of pairs of Matsubara frequencies retained. The averaging depends on the \(\tau\) dependence of the random matrix \(R\), as discussed above in the context of the NJL model. Two cases will be considered in this paper: (I) The case where \(R\) is \(\tau\)-dependent, in which case the various Matsubara modes can mix. (II) The case where the matrix \(R\) is \(\tau\)-independent, with a block diagonal structure

\[
R = \otimes R_n
\]

(13)
in which case, the various Matsubara frequencies are decoupled.

Throughout, and for simplicity, we choose Gaussian weights \(V(a) = \frac{1}{2}a^2\) for the random distributions

\[
\langle \ldots \rangle = \frac{1}{Z} \int \ldots \exp[-2NN_*^2 \text{Tr} V(R)] dR,
\]

(14)

although certainly higher (polynomial) weights are possible \([17]\). Note that since the dimension of the Gaussian matrix appears explicitly in the random weight, the widths of the Gaussians for the mixed and non-mixed case differ and are related by the factor \(\sqrt{N_*}\). In Sections 4 and 5 we use \(\Sigma_Q^2 = 1\) and \(\Sigma_C^2 = 1\) for the Quinto and Cardano classes, respectively. In Section 6, in comparing the two classes we set \(2\Sigma_Q^2 = \Sigma_C^2 = 1\).

The eigenvalue distribution of the Dirac operator (11) is related to the discontinuity of \(G(z)\) through the real axis

\[
\nu(\lambda) = -\frac{1}{\pi} \text{Im}G(z = \lambda + i0).
\]

(15)
Using either the law of addition of random matrices \[18\] or diagrammatic techniques \[19\], it follows that the resolvent $G(z)$ is given by the solution of the algebraic equation (Pastur equation) \[11\]. When the Matsubara frequencies are mixed (model I), the corresponding Pastur equation reads

$$G(z) = \frac{1}{2N^*} \sum_{i=1}^{2N^*} \left( z - G(z) - \epsilon_i \right), \quad (16)$$

where $\epsilon_i$ are the eigenvalues of the deterministic part of the Dirac operator \[9\]. The equation is a polynomial of $2N^* + 1$ order in $G(z)$. When the Matsubara frequencies are decoupled (model II), the corresponding Pastur equation is

$$G(z) = \frac{1}{N^*} \sum_{n=1}^{N^*} G_n(z), \quad (17)$$

where each of the $G_n(z)$ satisfies a cubic (Cardano class) equation

$$G_n(z) = \frac{1}{2} \left( \frac{1}{z - G_n(z) - M_n} + \frac{1}{z - G_n(z) + M_n} \right). \quad (18)$$

We note that $M_n(T) = \sqrt{m^2 + \omega_n^2}$, in overall agreement with dimensional reduction arguments from high temperature QCD \[8, 9\].

### 4. Coupled Wilson Fermions

For Wilson fermions, we consider \[11\] in the case of two lowest Matsubara frequencies. All our considerations will be made for a single flavor. For the coupled case (model I), the Dirac operator reads explicitly

$$Q = \begin{pmatrix} m & 0 & i\omega_0 & 0 \\ 0 & m & 0 & i\omega_1 \\ -i\omega_0 & 0 & -m & 0 \\ 0 & -i\omega_1 & 0 & -m \end{pmatrix} + \begin{pmatrix} R \end{pmatrix}. \quad (19)$$

The resolvent of the deterministic part for the two lowest pairs of frequencies ($\omega_{-n} = -\omega_{n-1}$) reads

$$G_D = \alpha \frac{z}{z^2 - M_0^2} + (1 - \alpha) \frac{z}{z^2 - M_1^2} \quad (20)$$

with

$$M_n^2 = m^2 + ((2n + 1)\pi T)^2 \quad (21)$$

Here $n = 0, 1$ and $\alpha$ (hereafter equal to $1/2$) refers to the relative weight of the modes. Adding the corresponding Blue’s functions of the deterministic and the random part one arrives at a fifth order algebraic equation \[20\].
Fig. 1. Spectral function for Quinto class for the lowest two Matsubara frequencies with small mass $m = 0.1$ in units of $\Sigma_Q = 1$

$$G^5 + a_4 G^4 + a_3 G^3 + a_2 G^2 + a_1 G + a_0 = 0$$  \hspace{1cm} (22)

with the following coefficients

$$a_4 = -4z$$
$$a_3 = 1 - (M_0^2 + M_1^2) + 6z^2$$
$$a_2 = -4z^3 - 3z + 2z(M_0^2 + M_1^2)$$
$$a_1 = \alpha(M_0^2 - M_1^2) - M_0^2(1 - M_1^2) + z^2 \left[ 3 - (M_0^2 + M_1^2) \right] + z^4$$
$$a_0 = z \left[ M_0^2 - \alpha(M_0^2 - M_1^2) - z^2 \right].$$  \hspace{1cm} (23)

The spectral function generated by the solutions to (22) through (15) allow for a rich phase structure (Quinto class). In general, four possible phases $P_1, P_2, P_3, P_4$ defined by the number of allowed disconnected arcs or supports of the eigenvalue distribution, are possible. In Fig. 1 we show the distribution of eigenvalues as a function of temperature and for a single light quark flavor of mass $m = 0.1$ in units of $1/\Sigma_Q$. At zero temperature, the spectral function is peaked around zero virtuality, with a nonzero condensate. The system is in the $P_1$ phase, and the distribution is Wigner’s semicircle. As the temperature increases, the condensate decreases, followed by the decoupling of the Matsubara modes. The heaviest, the first. The structural changes are: $P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow P_4$ or $P_1 \rightarrow P_3 \rightarrow P_4$ or $P_1 \rightarrow P_4$, depending on whether the restoration of chiral symmetry precedes, follows or parallels the splitting of the frequencies, respectively. Figure 2 shows the behavior of the spectral distribution for a heavier quark flavor. In this case, the quarks are always localized with no condensate whatever the temperature. An increase in the temperature
causes only a decoupling of the two Matsubara modes, that is a structural phase change from $P_2 \rightarrow P_4$.

Since the general solutions of a fifth order algebraic equation do not exist in analytical form, the results shown in Figs. [12] were obtained numerically. Out of the five possible solutions, four are ruled out by enforcing normalizability and positivity of the spectral distribution. To check that the presented solution is indeed correct and unique, we have constructed analytically the equations for the evolution of the endpoints. These equations are best amenable in the form of Blu’s functions, i.e. the functional inverses of the resolvents: $G[B(z)] = z$. Due to the cuts of the Green’s functions, the evolution equations stem from the condition $B'(z)_{e.p.} = 0$, where prime denotes differentiation with respect to $z$ and the derivative is taken at the endpoint (e.p.). Pastur equations rewritten in terms of the Blu’s functions are always one degree lower than those for the Green’s functions [18], therefore the evolution equations for the endpoints in our case are solution to a forth order (Ferrari) equation.

In the case considered here, the endpoints $\pm A_i$ ($A_1 > A_2 > A_3 > A_4$) of the four arcs $[-A_1, -A_2]$, $[-A_3, -A_4]$, $[A_4, A_3]$, $[A_2, A_1]$ are explicitly given by

$$A_i = \sqrt{X_i \frac{X_i^2 - (M_0^2 + M_i^2 - 1)X_i + F(\alpha, M_0^2, M_i^2)}{(X_i - M_0^2)(X_i - M_i^2)}},$$

$$F(\alpha, M_0^2, M_i^2) = \alpha(M_0^2 - M_i^2) - M_i^2 + M_0^2M_i^2$$

where $X_i$ are real positive roots of the Ferrari equation

$$X^4 - b_3X^3 - b_2X^2 - b_1X - b_0 = 0$$

with coefficients

$$b_3 = 1 + 2(M_0^2 + M_i^2)$$
Fig. 3. Evolution of the endpoints of the spectral function following numerically from (22) (dashed curve), and analytically from (24) (solid lines)

\[ b_2 = 3\alpha (M_0^2 - M_1^2) - (M_0^2 + M_1^2)^2 - 2M_0^2(1 + M_1^2) \]
\[ b_1 = -\alpha (M_0^4 - M_1^4) + 2M_0^2 M_1^2 (M_0^2 + M_1^2 + 1) + M_0^4 \]
\[ b_0 = -\alpha M_0^2 M_1^2 (M_0^2 - M_1^2) + M_0^4 M_1^2 (1 - M_1^2) . \]  

The critical points are determined from the condition that the real positive points \( A_4, A_3, A_2 \) melt or vanish. In Fig. 3 we show the evolution of the endpoints for the numerically generated solutions above (dashed curve) versus the analytically generated solutions (solid curve) from (24). Numerically, the endpoint curve was generated using the limit of the norm \( \lim |\text{Im}G(z) - \epsilon| \to 0 \), with \( \epsilon = 0.01 \) and \( G(z) \) the normalizable solution to (22). The agreement of the two curves, confirms the uniqueness of the spectral distribution discussed above.

5. Decoupled Wilson Fermions

The case of Wilson fermions with two lowest but decoupled Matsubara modes (model II), follows from (17) through

\[ G_n(z) = \frac{z - G_n(z)}{(z - G_n(z))^2 - M_n^2} . \]  

The evolution equations for the endpoints in this case come from the overlap of individual supports for each frequency. Since for each frequency the equation for
the resolvent is cubic, the corresponding equation for the endpoints is quadratic, with the following positive solutions

$$A_{n}^{1,2} = \frac{1}{\sqrt{2}} \cdot \frac{3 \pm \sqrt{1 + 8M_{n}^{2}}}{1 \pm \sqrt{1 + 8M_{n}^{2}}} \cdot \sqrt{1 + 2M_{n}^{2} \pm \sqrt{1 + 8M_{n}^{2}}}$$

(29)

for $\alpha = 1/2$. Figures 4 and 5 show the spectral functions versus temperature, for a light and heavy quark flavor in units of $1/\Sigma_C$.

6. Comparison of the Models

Despite the overall structural similarities between the spectral functions discussed for model I and II, there are a number of quantitative differences upon closer look.
These differences may be helpful for distinguishing these two scenarios in a lattice simulation for instance.

The evolution of the endpoints of the spectra as obtained from model I and II are different. Figure 6 shows the behavior of the endpoints versus temperature for I (solid line) and II (dashed line) for the light quark flavor case \(m = 0.1\) in units where \(2\Sigma_Q = \Sigma_C = 1\). The solid curves follow from (29), while the dashed curves follow from (24). At high temperature \((T \geq 3T_c)\), the evolution equation becomes degenerate. This is expected following the thermal localization. Around the critical temperature, however, the evolution of the endpoints are qualitatively different.

The critical masses and temperatures, are of course different for the two cases considered. For model I and at zero virtuality \((\lambda = 0)\) the Quinto equation reduces to a second order equation, with the solution

\[
G^2 = \frac{\Sigma_Q^2(M_0^2 + M_1^2) - 1}{2} \pm \frac{1}{2} \sqrt{1 + \Sigma_Q^2(M_0^2 - M_1^2)^2}
\]

for \(\alpha = 1/2\). The support for the spectral density vanishes when \(G^2\) becomes real and positive, that is for

\[
M_0^2 + M_1^2 - 2\Sigma_Q^2 M_0^2 M_1^2 = 0.
\]

This yields the critical temperature

\[
4\pi^2 \Sigma_Q^2 T_c^2 = \frac{10}{9}(1 - 2\Sigma_Q^2 m^2) + 
\sqrt{\frac{100}{9}(1 - 2\Sigma_Q^2 m^2)^2 - \frac{16}{9}\Sigma_Q^2 m^2(\Sigma_Q^2 m^2 - 1)}.
\]

For zero quark mass (32) gives \(\pi^2 T_c^2 = 5/9\Sigma_Q^2\). The critical mass, at which the condensate vanishes at zero temperature is \(m_c = 1/\Sigma_Q\). Figure 7 shows the behavior of the critical temperature versus \(m\) for model I or Quinto class (solid curve). The dashed curve is the expected result from model II, with

\[
\pi^2 T_c^2 = 1 - m_c^2.
\]

For the massless case the critical temperature is in this case \(1\), while at zero temperature the critical mass is 1. The comparison between the two models, was carried for \(2\Sigma_Q = \Sigma_C = 1\). Note that the ratio of the critical parameters depend on the mixing scenario

\[
\frac{\pi T_c}{m_c} = \left\{ \begin{array}{ll}
1 & \text{model I} \\
\sqrt{\frac{5}{9}} & \text{model II}.
\end{array} \right.
\]

The spectral distributions around the critical temperature are also qualitatively different for the two models considered, as shown in Fig. 8. The spectral function in model I following from the Quinto equation (solid curve) is smooth throughout,
while the one for model II following from the superposition of two Cardano solutions (dashed curve) has sharp edges (discontinuous derivatives). The wiggly behavior seen in both spectra is reminiscent of a similar feature seen in Kalkreuter lattice spectra with Wilson fermions [15]. This is suggestive of some additional (finite size) scale in the latter. This point is worth investigating.

Finally, in both models the critical exponents are mean-field [7, 10, 21]. This is expected in large $N$ and for Gaussian randomness for all polynomial classes (Cardano, Ferrari, Quinto and higher).

### 7. Staggered Quark Spectra

For chiral fermions (9) with decoupled Matsubara frequencies, consider the case with the dimensional reduction argument [8]

$$Q_{S,n}^I = \otimes_{\pm,n} Q_{\tilde{S},n}^I = \otimes_{\pm,n} \left( \begin{array}{cc} 0 & \pm M_n(T) \\ \pm M_n(T) & 0 \end{array} \right) + \left( \begin{array}{cc} 0 & R \\ R^\dagger & 0 \end{array} \right)$$

(35)

with $M_n(T) = \sqrt{m_s^2 + \omega_n^2}$. The spectral distribution associated to (33) follows from the combination of discontinuities of the pertinent solution to a cubic (Cardano) equation, Eqs. (17,18). The case considered here at high enough temperature is equivalent to the one of Wilson fermions with decoupled Matsubara modes.

Following Chandrasekharan and Christ [12], let us define the valence quark condensate,

$$<\bar{\zeta}\zeta> = 2m_\zeta \int_0^\infty d\lambda \frac{\nu(\lambda)}{\lambda^2 + m_\zeta^2},$$

(36)

1 Quadratic equations do not allow for a phase transition. The system is always in the $P_1$ phase (Wigner semicircle).
Fig. 7. Critical temperature as a function of the current mass \( m \) for model I (solid curve) and model II (dashed curve). The units are fixed by \( \Sigma^2_C = 2\Sigma^2_Q = 1 \).

Fig. 8. Comparison of the spectral functions for model I (solid line) and model II (dashed line) with \( m = 0 \) at \( \pi T = 0.9 \).

where \( m_\zeta \) is the valence quark mass, and \( \nu(\lambda) \) the spectral density associated to (35) for massless sea quarks, that is \( m_s = 0 \). Although we have discussed a single quark flavor, the analysis carries through unchanged in the massless case.

In Fig. 9 we show the behavior of the valence quark condensate as obtained by Chandrasekharan and Christ [12], using two-flavor QCD with staggered fermions and \( m_s a = 0.01 \) (or \( m_s = 6 \) MeV in physical units). In Fig. 10 we display the results following from (36) with \( m_s = 0 \), with the two lowest Matsubara pairs (right figure) and for comparison the same result but for one Matsubara mode (left figure) as discussed in [2]. The results are in overall agreement with each other, showing mild sensitivity to \( m_s \) and the addition of further Matsubara modes. The largest sensitivity appears around large values of \( m_\zeta \) in units of \( 1/\Sigma \).

Model I for the mixing of the two lowest Matsubara modes is not trivial in the case of chiral fermions. Since for the decoupled case model II was equivalent to the one of Wilson fermions, we used as model I for chiral fermions the spectral density obtained from (19) with the proper scaling \( 2\Sigma^2_Q = \Sigma^2_C \). Comparing the large mass part of the spectra we may conclude that the model with the coupling between the Matsubara modes agrees better with the lattice calculation (Fig. 9).

8. Conclusion

We have investigated the role of few Matsubara modes on random matrix models as inspired from QCD spin and flavor symmetry. After showing how the mixing be-
Fig. 9. Semi-quenched condensate versus the valence quark mass $m_\zeta$ for two flavor QCD with a sea mass $m_s a = 0.01$.

Fig. 10. Semi-quenched condensate versus the valence quark mass $m_\zeta$ obtained from model II (36) for the lowest Matsubara modes (left) and the one corresponding to the coupling (model I) (right) with zero sea mass.
tween the various Matsubara modes may affect, the generic structure of the random matrix formulation at finite temperature, we have considered two complementary models. One with mixing (I) and the other without (II). The spectral function associated to the former was shown to follow from a quintic equation (Quinto class), while the one for the latter was shown to follow from the superposition of two solutions to quartic equations (Cardano class).

For a light quark mass, the spectral distribution at low temperature has a support at zero virtuality. Randomness prevails over temperature. As the temperature is increased, the thermal effects cause the spectral distribution to vanish. Thermal localizations are found to follow the thermal wavelengths. For a heavy quark mass, the spectral distribution shows no support at zero virtuality even at zero temperature owing to the localization over the Compton wavelength. With increasing temperature, the thermal localization takes over, with a segregation along the thermal frequencies.

While schematic, the present analysis shows that around the critical temperature the effects of the higher Matsubara modes cause quantitative changes not only in the critical parameters and their ratios, but also in the endpoints evolution and the spectral shapes. These observations are of relevance for lattice simulations of Dirac spectra for both Kogut-Susskind and Wilson fermions.

Acknowledgments

I thank Maciej A. Nowak and Ismail Zahed for the valuable discussions. This work was supported by the Hungarian Research Foundation OTKA grants T022931 and F019689.

Notes

Dedicated to G. Marx and K. Nagy on the occasion of their 70th birthdays

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