Estimates for the Strong Approximation in Multidimensional Central Limit Theorem

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Abstract

In a recent paper the author obtained optimal bounds for the strong Gaussian approximation of sums of independent $\mathbb{R}^d$-valued random vectors with finite exponential moments. The results may be considered as generalizations of well-known results of Komlós–Major–Tusnády and Sakhanenko. The dependence of constants on the dimension $d$ and on distributions of summands is given explicitly. Some related problems are discussed.

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1. Introduction

Let $X_1, \ldots, X_n, \ldots$ be mean zero independent $\mathbb{R}^d$-valued random vectors and $D_n = \text{cov} S_n$ the covariance operator of the sum $S_n = \sum_{i=1}^{n} X_i$. By the Central Limit Theorem, under some simple moment conditions the distribution of normalized sums $D_n^{-1/2} S_n$ is close to the standard Gaussian distribution. The invariance principle states that, in a sense, the distribution of the whole sequence $D_n^{-1/2} S_1, \ldots, D_n^{-1/2} S_n, \ldots$ is close to the distribution of the sequence $D_n^{-1/2} T_1, \ldots, D_n^{-1/2} T_n, \ldots$, where $T_n = \sum_{i=1}^{n} Y_i$ and $Y_1, \ldots, Y_n, \ldots$ is a corresponding sequence of independent Gaussian random vectors (this means that $Y_i$ has the same mean and the same covariance operator as $X_i$, $i = 1, \ldots, n, \ldots$).

We consider here the problem of strong approximation which is more delicate than that of estimating the closeness of distributions. It is required to construct on a probability space a sequence of independent random vectors $X_1, \ldots, X_n$ (with
given distributions) and a corresponding sequence of independent Gaussian random vectors \( Y_1, \ldots, Y_n \) so that the quantity

\[
\Delta(X, Y) = \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} X_i - \sum_{i=1}^{k} Y_i \right\|
\]

would be so small as possible with large probability. Here \( \| \cdot \| \) is the Euclidean norm. It is clear that the vectors even with the same distributions can be very far one from another.

In some sense this problem is one of the most important in probability approximations because many well-known probability theorems can be considered as consequences of results about strong approximation of sequences of sums by corresponding Gaussian sequences. This is related to the law of iterated logarithm, to several theorems about large deviations, to the estimates for the rate of convergence of the Prokhorov distance in the invariance principles (Prokhorov [19], Skorokhod [26], Borovkov [4]), as well as to the Strassen-type approximations (Strassen [28], see, for example, Csörgő and Hall [8]).

The rate for strong approximation in the one-dimensional invariance principle was studied by many authors (see, e.g., Prokhorov [19], Skorokhod [26], Borovkov [4], Csörgő and Révész [6] and the bibliography in Csörgő and Révész [7], Csörgő and Hall [8], Shao [20]). Skorokhod [26] developed a method of construction of close sequences of sequential sums of independent random variables on the same probability space. For a long time the best rates of approximation were obtained by this method, known now as the Skorokhod embedding. However, Komlós, Major and Tusnády (KMT) [17] elaborated a new, more powerful method of dyadic approximation. With the help of this method they obtained optimal rates of Gaussian approximation for sequences of independent identically distributed random variables.

We restrict ourselves on the most important case, where the summands have finite exponential moments. Sakhanenko [24] generalized and essentially sharpened KMT results in the case of non-identically distributed random variables. He considered the following class of one-dimensional distributions:

\[
S_1(\tau) = \{ \mathcal{L}(\xi) : \mathbf{E} \xi = 0, \mathbf{E} |\xi|^3 \exp \left( \tau^{-1} |\xi| \right) \leq \tau \mathbf{E} |\xi|^2 \}
\]

(the distribution of a random vector \( \xi \) will be denoted by \( \mathcal{L}(\xi) \)). His main result is formulated as follows.

**Theorem 1** (Sakhanenko [24]). Suppose that \( \tau > 0 \), and \( \xi_1, \ldots, \xi_n \) are independent random variables with \( \mathcal{L}(\xi_j) \in S_1(\tau) \), \( j = 1, \ldots, n \). Then one can construct on a probability space a sequence of independent random variables \( X_1, \ldots, X_n \) and a sequence of independent Gaussian random variables \( Y_1, \ldots, Y_n \) so that \( \mathcal{L}(X_j) = \mathcal{L}(\xi_j) \), \( \mathbf{E} Y_j = 0 \), \( \mathbf{E} Y_j^2 = \mathbf{E} X_j^2 \), \( j = 1, \ldots, n \), and

\[
\mathbf{E} \exp \left( c \Delta(X, Y) / \tau \right) \leq 1 + B / \tau,
\]

(1.1)

where \( c \) is an absolute constant and \( B^2 = \mathbf{E} \xi_1^2 + \cdots + \mathbf{E} \xi_n^2 \).
KMT [17] supposed that $\xi, \xi_1, \ldots, \xi_n$ are identically distributed and $\mathbf{E} e^{\langle h, \xi \rangle} < \infty$, for $h \in V$, where $V \subset \mathbb{R}^d$ is some neighborhood of zero. The KMT (1975–76) result follows from Theorem 1. It is easy to see that there exists $\tau(F)$ such that $F = \mathcal{L}(\xi) \in \mathcal{S}_1(\tau(F))$. Applying the Chebyshev inequality, we observe that (1.1) imply that

$$
P (c_1 \Delta(X, Y) / \tau(F) \geq x) \leq \exp \left( \log \left( 1 + \sqrt{n \mathbf{E} \xi^2 / \tau(F)} \right) - x \right), \quad x > 0. \quad (1.2)$$

Inequality (1.2) provides more information than the original KMT formulation which contains unspecified constants depending on $F$. In (1.2) the dependence of constants on the distribution $F$ is written out in an explicit form. The quantity $\tau(F)$ can be easily calculated or estimated for any concrete distribution $F$.

The first attempts to extend the KMT and Sakhanenko approximations to the multidimensional case (see Berkes and Philipp [3], Philipp [18], Berger [2], Einmahl [10, 11]) had a partial success only. Comparatively recently U. Einmahl [12] obtained multidimensional analogs of KMT results which are close to optimal. Zaitsev [33, 34] removed an unnecessary logarithmic factor from the result of Einmahl [12] and obtained multidimensional analogs of KMT results (see Theorem 2 below). In Theorem 2 the random vectors are, generally speaking, non-identically distributed. However, they have the same identity covariance operator $\Lambda$. Therefore, the problem of obtaining an adequate multidimensional generalization of the main result of Sakhanenko [24] remained open. This generalization is given in Theorem 3 below.

## 2. Main results

For formulations of results we need some notations. Let $\mathcal{A}_d(\tau)$, $\tau \geq 0$, $d \in \mathbb{N}$, denote classes of $d$-dimensional distributions, introduced in Zaitsev [29], see as well Zaitsev [33–35]. The class $\mathcal{A}_d(\tau)$ (with a fixed $\tau \geq 0$) consists of $d$-dimensional distributions $F$ for which the function $\varphi(z) = \varphi(F, z) = \log \int_{\mathbb{R}^d} e^{\langle z, x \rangle} F(dx) \left( \varphi(0) = 0 \right)$ is defined and analytic for $\|z\|_\tau < 1$, $z \in \mathbb{C}^d$, and $\left| d_n d^2 \varphi(z) \right| \leq \|u\|_\tau \langle D u, v \rangle$ for all $u, v \in \mathbb{R}^d$ and $\|z\|_\tau < 1$, where $D = \text{cov} F$, the covariance operator corresponding to $F$, and $d_n \varphi$ is the derivative of the function $\varphi$ in direction $u$.

**Theorem 2** (Zaitsev, [33, 34]). Suppose that $\tau \geq 1$, $\alpha > 0$ and $\xi_1, \ldots, \xi_n$ are random vectors with distributions $\mathcal{L}(\xi_k) \in \mathcal{A}_d(\tau)$, $\mathbf{E} \xi_k = 0$, $\text{cov} \xi_k = \mathbb{I}$, $k = 1, \ldots, n$. Then one can construct on a probability space a sequence of independent random vectors $X_1, \ldots, X_n$ and a sequence of independent Gaussian random vectors $Y_1, \ldots, Y_n$ so that

$$
\mathcal{L}(X_k) = \mathcal{L}(\xi_k), \quad \mathbf{E} Y_k = 0, \quad \text{cov} \mathcal{L}(Y_k) = \mathbb{I}, \quad k = 1, \ldots, n,
$$

and

$$
\mathbf{E} \exp \left( \frac{c_1(\alpha) \Delta(X, Y)}{\tau d^{\alpha+1}} \right) \leq \exp \left( c_2(\alpha) d^{\alpha+1} \log^* (n/\tau^2) \right),
$$

where $c_1(\alpha), c_2(\alpha)$ are positive quantities depending on $\alpha$ only and $\log^* b = \max\{1, \log b\}$, for $b > 0$. 

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Corollary 1. In the conditions of Theorem 2 for all \( x \geq 0 \) the following inequality is valid
\[
P \left\{ \Delta(X, Y) \geq \frac{c_2(\alpha) \tau d^{3/4 + \alpha} \log^* d \log^* (n/\tau^2)}{c_1(\alpha)} + x \right\} \leq \exp \left( -\frac{c_1(\alpha) x}{\tau d^{7/2} \log^* d} \right).
\]

It is easy to see that if \( V \subset \mathbb{R}^d \) is some neighborhood of zero and \( \mathbb{E} e^{\langle h, \xi \rangle} < \infty \), for \( h \in V \), then \( F = L(\xi) \in \mathcal{A}_d(c(F)) \). Below we list some simple and useful properties of classes \( \mathcal{A}_d(\tau) \) which are essential in the proof of Theorem 2. Theorem 2 implies in one-dimensional case Sakhanenko’s Theorem 1 for identically distributed random variables with finite exponential moments as well as the result of KMT [17].

Corollary 2. Suppose that a random vector \( \xi \) has finite exponential moments \( \mathbb{E} e^{\langle h, \xi \rangle} \), for \( h \in V \), where \( V \subset \mathbb{R}^d \) is some neighborhood of zero. Then one can construct on a probability space a sequence of independent random vectors \( X_1, X_2, \ldots \) and a sequence of independent Gaussian random vectors \( Y_1, Y_2, \ldots \) so that
\[
L(X_k) = L(\xi), \quad \mathbb{E} Y_k = 0, \quad \text{cov}(Y_k) = \text{cov}(\xi), \quad k = 1, 2, \ldots,
\]
and
\[
\sum_{k=1}^n X_k - \sum_{k=1}^n Y_k = O(\log n) \quad \text{a.s.}
\]

As it is noted in KMT [17], from the results of Bártfai [1] that the rate of approximation in Corollary 2 is the best possible for non-Gaussian vectors \( \xi \). An analog of Corollary 2 was obtained by Einmahl [12] under additional smoothness-type restrictions on the distribution \( L(\xi) \). The following statement is a sharpening of Corollary 2.

Corollary 3 (Zaitsev [36]). Suppose that a random vector \( \xi \) has the distribution such that \( L(D^{-1/2} \xi) \in \mathcal{A}_d(\tau) \), where \( D = \text{cov} L(\xi) \) is a reversible operator. Let \( \sigma^2, \sigma > 0, \) be the maximal eigenvalue of \( D \). Then for any \( \alpha > 0 \) there exists a construction from Corollary 2 such that
\[
P \left\{ \limsup_{n \to \infty} \frac{1}{\log n} \left\| \sum_{k=1}^n X_k - \sum_{k=1}^n Y_k \right\| \leq c_3(\alpha) \sigma \tau d^{3/4 + \alpha} \log^* d \right\} = 1 \quad (2.1)
\]
with \( c_3(\alpha) \) depending on \( \alpha \) only.

In Theorems 2 and Corollary 3 we consider the case \( \tau \geq 1 \). The case of small \( \tau \) was investigated by Götze and Zaitsev [16]. It is shown that under additional smoothness-type restrictions on the distribution \( L(\xi) \) the expression in the right-hand side of the inequality in (2.1) can be arbitrarily small if the parameter \( \tau \) is small enough. It is clear that the statements of Theorem 2 and Corollary 3 becomes stronger for small \( \tau \). In Götze and Zaitsev [16] one can find simple examples in
which the sufficiently complicated smoothness condition is satisfied. The approximation is better in the case when summands have smooth distributions which are close to Gaussian ones (see inequalities (3.1) and (3.2) below).

The following Theorem 3 is a generalization of Theorem 2 to the case of multivariate random variables. In one-dimensional situation, Theorem 3 implies Theorem 1.

**Theorem 3** (Zaitsev [35]). Suppose that $\alpha > 0$, $\tau \geq 1$, and $\xi_1, \ldots, \xi_n$ are independent random vectors with $E \xi_j = 0$, $j = 1, \ldots, n$. Assume that there exists a strictly increasing sequence of non-negative integers $m_0 = 0, m_1, \ldots, m_s = n$ satisfying the following conditions. Write

$$
\xi_k = \xi_{m_{k-1} + 1} + \cdots + \xi_{m_k}, \quad k = 1, \ldots, s,
$$

and suppose that (for all $k = 1, \ldots, s$) $L(\xi_k) \in A_d(\tau)$, $\text{cov} \xi_k = B_k$ and, for all $u \in \mathbb{R}^s$,

$$
c_4 \|u\|^2 \leq \langle B_k u, u \rangle \leq c_5 \|u\|^2
$$

with some constants $c_4$ and $c_5$. Then one can construct on a probability space a sequence of independent random vectors $X_1, \ldots, X_n$ and a corresponding sequence of independent Gaussian random vectors $Y_1, \ldots, Y_n$ so that $L(X_j) = L(\xi_j)$, $E Y_j = 0$, $\text{cov} L(Y_j) = \text{cov} L(X_j)$, $j = 1, \ldots, n$, and

$$
E \exp \left( \frac{a_1 \Delta(X, Y)}{\tau d^{3/2} \log^* d} \right) \leq \exp \left( a_2 d^{3+\alpha} \log^* (s/\tau^2) \right),
$$

where $a_1, a_2$ are positive quantities depending only on $\alpha, c_4, c_5$.

### 3. Properties of classes $A_d(\tau)$

Let us consider elementary properties of classes $A_d(\tau)$ which are essentially used in the proof of Theorems 2 and 3, see Zaitsev [29, 31, 33–35]. It is easy to see that $\tau_1 < \tau_2$ implies $A_d(\tau_1) \subset A_d(\tau_2)$. Moreover, the class $A_d(\tau)$ is closed with respect to convolution: if $F_1, F_2 \in A_d(\tau)$, then $F_1 F_2 = F_1 * F_2 \in A_d(\tau)$. Products of measures are understood in the convolution sense. Note that the condition $L(\xi_k) \in A_d(\tau)$ in Theorem 3 is satisfied if $L(\xi_j) \in A_d(\tau)$, for $j = 1, \ldots, n$.

Let $\tau \geq 0$, $F = L(\xi) \in A_d(\tau)$, $y \in \mathbb{R}^m$, and $\Lambda : \mathbb{R}^d \to \mathbb{R}^m$ is a linear operator. Then

$$
L(\Lambda \xi + y) \in A_m(\|\Lambda\| \tau), \quad \text{where} \quad \|\Lambda\| = \sup_{x \in \mathbb{R}^d, \|x\| \leq 1} \|\Lambda x\|.
$$

Suppose that $\tau \geq 0$, $F_k = L(\xi^{(k)}) \in A_{d_k}(\tau)$, and the vectors $\xi^{(k)}$, $k = 1, 2$, are independent. Let $\xi \in \mathbb{R}^{d_1 + d_2}$ be the vector with the first $d_1$ coordinates coinciding with those of $\xi^{(1)}$ and with the last $d_2$ coordinates coinciding with those of $\xi^{(2)}$. Then $F = L(\xi) \in A_{d_1 + d_2}(\tau)$.

The classes $A_d(\tau)$ are closely connected with other naturally defined classes of multidimensional distributions. From the definition of $A_d(\tau)$ it follows that if
\( \mathcal{L}(\xi) \in \mathcal{A}_d(\tau) \) then the vector \( \xi \) has finite exponential moments \( \mathbb{E} e^{(h, \xi)} < \infty \), for \( h \in \mathbb{R}^d, \| h \| \tau < 1 \). This leads to exponential estimates for the tails of distributions.

The condition \( \mathcal{L}(\xi) \in \mathcal{A}_1(\tau) \) is equivalent to Statulevičius’ [27] conditions on the rate of increasing of cumulants \( \gamma_m \) of the random variable \( \xi \):

\[
|\gamma_m| \leq \frac{1}{2} m! \tau^{m-2}\gamma_2, \quad m = 3, 4, \ldots.
\]

This equivalence means that if one of these conditions is satisfied with parameter \( \tau \), then the second is valid with parameter \( c\tau \), where \( c \) denotes an absolute constant. However, the condition \( \mathcal{L}(\xi) \in \mathcal{A}_d(\tau) \) differs essentially from other multidimensional analogs of Statulevičius’ conditions, considered by Rudzkiš [23] and Saulis [25].

Zaitsev [30] considered classes of distributions

\[
B_d(\tau) = \left\{ F = \mathcal{L}(\xi) : \mathbb{E} \xi = 0, \left\| \mathbb{E} (\xi, v)^2 (\xi, u)^{m-2} \right\| \leq \frac{1}{2} m! \tau^{m-2} \right\}
\]

satisfying multidimensional analogs of the Bernstein inequality condition. Sakhanenko’s condition \( \mathcal{L}(\xi) \in \mathcal{B}_1(\tau) \) is equivalent to the condition \( \mathcal{L}(\xi) \in \mathcal{B}_d(\tau) \). Note that if \( F \{ \{ x \in \mathbb{R}^d : \| x \| \leq \tau \} \} = 1 \) then \( F \in \mathcal{B}_d(\tau) \).

Let us formulate a relation between classes \( \mathcal{A}_d(\tau) \) and \( \mathcal{B}_d(\tau) \). Denote by \( \sigma^2(F) \) the maximal eigenvalue of the covariance operator of a distribution \( F \). Then

a) If \( F = \mathcal{L}(\xi) \in \mathcal{B}_d(\tau) \), then \( \sigma^2(F) \leq 12 \tau^2 \), \( \mathbb{E} \xi = 0 \) and \( F \in \mathcal{A}_d(c\tau) \).

b) If \( F = \mathcal{L}(\xi) \in \mathcal{A}_d(\tau) \), \( \sigma^2(F) \leq \tau^2 \) and \( \mathbb{E} \xi = 0 \), then \( F \in \mathcal{B}_d(c\tau) \).

If \( F \) is an infinitely divisible distributions with spectral measure concentrated on the ball \( \{ x \in \mathbb{R}^d : \| x \| \leq \tau \} \) then \( F \in \mathcal{A}_d(c\tau) \), where \( c \) is an absolute constant. It is obvious that the class \( \mathcal{A}_d(0) \) coincides with the class of all \( d \)-dimensional Gaussian distributions. The following inequality was proved in Zaitsev [29] and can be considered as an estimate of stability of this characterization:

\[
\text{if } F \in \mathcal{A}_d(\tau), \text{ then } \pi (F, \Phi(F)) \leq cd^2 \tau \log^*(\tau^{-1}); \quad (3.1)
\]

where \( \pi(\cdot, \cdot) \) is the Prokhorov distance and \( \Phi(F) \) denotes the Gaussian distribution whose mean and covariance operator coincide with those of \( F \). The Prokhorov distance between distributions \( F, G \) may be defined by means of the formula

\[
\pi(F, G) = \inf \{ \lambda : \pi(F, G, \lambda) \leq \lambda \},
\]

where

\[
\pi(F, G, \lambda) = \sup_X \max \{ F\{X\} - G\{X^\lambda\}, G\{X\} - F\{X^\lambda\} \}, \quad \lambda > 0,
\]

and \( X^\lambda = \{ y \in \mathbb{R}^d : \inf_{x \in X} \| x - y \| < \lambda \} \) is the \( \lambda \)-neighborhood of the Borel set \( X \).

Moreover, in Zaitsev [29] it was established that

\[
\pi(F, \Phi(F), \lambda) \leq cd^2 \exp \left( - \frac{\lambda}{cd^2 \tau} \right). \quad (3.2)
\]
It is very essential (and important) that the inequality (3.2) is proved for all \( \tau > 0 \) and for arbitrary \( \text{cov} F \), in contrast to Theorems 2 and 3, where \( \tau \geq 1 \) and covariance operators satisfy condition (2.2). The question about the necessity of condition (2.2) in Theorems 2 and 3 remains open. In Zaitsev [30] inequalities (3.1) and (3.2) were proved for convolutions of distributions from \( B_d(\tau) \).

By the Strassen–Dudley theorem (see Dudley [9]) coupled with inequality (3.2), one can construct on a probability space the random vectors \( \xi \) and \( \eta \) with \( L(\xi) = F \) and \( L(\eta) = \Phi(F) \) so that

\[
P \{ \| \xi - \eta \| > \lambda \} \leq c d^2 \exp \left( - \frac{\lambda}{c d^2 \tau} \right).
\]

For convolutions of bounded measures, this fact was used by Rio [21], Einmahl and Mason [13], Bovier and Mason [5], Gentz and L"owe [15], Einmahl and Kuelbs [14].

The scheme of the proof of Theorems 2 and 3 is very close to that of the main results of Sakhanenko [24] and Einmahl [12]. We suppose that the Gaussian vectors \( Y_1, \ldots, Y_n, \) \( n = 2^N \), are already constructed and construct the independent vectors which are bounded with probability one, have sufficiently smooth distributions and the same moments of the first, second and third orders as the needed independent random vectors \( X_1, \ldots, X_n \). For the construction we use the dyadic scheme proposed by KMT [17]. Firstly we construct the sum of \( 2^N \) summands using the Rosenblatt [22] quantile transform for conditional distributions (see Einmahl [12]). Then we construct blocks of \( 2^{N-1}, 2^{N-2}, \ldots, 1 \) summands. The rate of approximation is estimated using the fact that, for smooth summands distributions, the corresponding conditional distribution are smooth and close to Gaussian ones. Then we construct the vectors \( X_1, \ldots, X_n \) in several steps. After each step the number of \( X_k \) which are not constructed becomes smaller in \( 2^p \) times, where \( p \) is a suitably chosen positive integer. In each step we begin with already constructed vectors which are bounded with probability one and have sufficiently smooth distributions and the needed moments up to the third order. Then we construct the vectors such that, in each block of \( 2^p \) summands, only the first vector has the initial bounded smooth distribution. The rest \( 2^p - 1 \) vectors have the needed distributions \( L(\xi_k) \). These \( 2^p - 1 \) vectors from each block will be chosen as \( X_k \) and will be not involved in the next steps of the procedure. The coincidence of third moments will allow us to use more precise estimates of the closeness of quantiles of conditional distributions contained in Zaitsev [32]. In the estimation of closeness of random vectors in the steps of the procedure described above, we use essentially properties of classes \( A_d(\tau) \).

4. Infinitely divisible approximation

Let us finally mention a result about strong approximation of sums of independent random vectors by infinitely divisible distributions. Theorem 4 below follows from the main result of Zaitsev [32] coupled with the Strassen–Dudley theorem. Inequality (4.1) can be considered as a generalization of inequality (3.3) to convolution of distribution with unbounded supports.
Theorem 4. Let $d$-dimensional probability distributions $F_i$, $i = 1, \ldots, n$, be represented as mixtures of $d$-dimensional probability distributions $U_i$ and $V_i$:

$$F_i = (1 - p_i)U_i + p_iV_i,$$

where

$$0 \leq p_i \leq 1, \quad \int x U_i\{dx\} = 0, \quad U_i \{x \in \mathbb{R}^d : \|x\| \leq \tau\} = 1,$$

and $V_i$ are arbitrary distributions. Then for any fixed $\lambda > 0$ one can construct on the same probability space the random vectors $\xi$ and $\eta$ so that

$$P\{\|\xi - \eta\| > \lambda\} \leq c(d) \left( \max_{1 \leq i \leq n} p_i + \exp\left(-\frac{\lambda}{c(d)\tau}\right) \right) + \sum_{i=1}^n p_i^2 \quad (4.1)$$

and

$$L(\xi) = \prod_{i=1}^n F_i, \quad L(\xi) = \prod_{i=1}^n e(F_i),$$

where $c(d)$ depends on only and $e(F_i)$ denotes the compound Poisson infinitely divisible distribution with characteristic function $\exp(\hat{F}_i(t) - 1)$, where $\hat{F}_i(t) = \int e^{itx} F_i\{dx\}$. If the distributions $V_i$ are identical, the term $\sum_{i=1}^n p_i^2$ in (4.1) can be omitted.

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