AN ASYMPTOTIC SHARP SOBOLEV REGULARITY FOR PLANAR INFINITY HARMONIC FUNCTIONS

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Abstract. Given an arbitrary planar $\infty$-harmonic function $u$, for each $\alpha > 0$ we establish a quantitative $W^{1,2}_{\text{loc}}$-estimate of $|Du|^\alpha$, which is sharp as $\alpha \to 0$. We also show that the distributional determinant of $u$ is a Radon measure enjoying some quantitative lower and upper bounds. As a by-product, for each $p > 2$ we obtain some quantitative $W^{1,p}_{\text{loc}}$-estimates of $u$, and consequently, an $L^p$-Liouville property for $\infty$-harmonic functions in whole plane.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain (an open connected subset). A function $u \in C(\Omega)$ is $\infty$-harmonic in $\Omega$ if

$$-\Delta_{\infty} u := -u_{ii}u_{ij} = 0 \quad \text{in} \ \Omega$$

in viscosity sense; see [20]. In this paper, $v_i = \frac{\partial v}{\partial x_i}$ if $v \in C^1(\Omega)$, or $v_i$ denotes the distributional derivation in direction $i$ if $v \in L^2_{\text{loc}}(\Omega)$, and $v_{ij} = \frac{\partial^2 v}{\partial x_i \partial x_j}$ if $v \in C^2(\Omega)$. Write $Dv = (v_i)_{i=1}^n$, $D^2v = (v_{ij})_{i,j=1}^n$, and $D^2v Dv = (v_{ij}v_{ij})_{i=1}^n$. We always use the Einstein summation convention, that is, $f_i g_i = \sum_{i=1}^n f_i g_i$ for vectors $(f_i)_{i=1}^n$ and $(g_i)_{i=1}^n$.

The main purpose is to prove the following quantitative Sobolev regularity of $\infty$-harmonic functions in planar domains (that is, $n = 2$).

Theorem 1.1. Let $\Omega \subset \mathbb{R}^2$ be a domain and $u$ be an $\infty$-harmonic function in $\Omega$. For each $\alpha > 0$, we have $|Du|^\alpha \in W^{1,2}_{\text{loc}}(\Omega)$ with

$$\|D|Du|^\alpha\|_{L^2(V)} \leq C(\alpha) \frac{1}{\text{dist}(V, \partial U)} \|Du\|^\alpha_{L^2(U)} \quad \forall V \subset U \subset \Omega,$$

and

$$\left(|Du|^\alpha\right)_i u_i = 0 \quad \text{almost everywhere in} \ \Omega.$$  

The constant $C(\alpha)$ above depend only on $\alpha$.

As indicated by $\infty$-harmonic function

$$w(x_1, x_2) := x_1^{4/3} - x_2^{4/3} \quad \text{in} \ \mathbb{R}^2$$

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given by the Aronsson [7], we will see that $|Du|^\alpha \in W^{1,2}_{loc}(\Omega)$ in Theorem 1.1 is sharp when $\alpha \to 0$. Precisely, set

$$p_\alpha := \begin{cases} 3 & \text{if } \alpha \geq 1, \\
\frac{6}{3-\alpha} & \text{if } \alpha \in (0,1) \end{cases}$$

Note that $p_\alpha \to 2$ as $\alpha \to 0$. By directly calculation, we have the following result.

**Lemma 1.2.** For each $\alpha > 0$,

$$|Du|^\alpha \in W^{1,p}_{loc}(\mathbb{R}^2) \quad \forall p < p_\alpha \quad \text{but} \quad |Du|^\alpha \notin W^{1,p_\alpha}_{loc}(\mathbb{R}^2).$$

Moreover,

$$\log |Du| \in W^{1,p}_{loc}(\mathbb{R}^2) \cap BMO_{loc}(\mathbb{R}^2) \quad \forall p < 2 \quad \text{but} \quad \log |Du| \notin W^{1,2}_{loc}(\mathbb{R}^2).$$

Below, we show that the distributional determinant of any planar $\infty$-harmonic function is a Radon measure enjoying some lower and upper bounds. See Remark 2.2 for the definition of distributional determinant for functions in $W^{1,2}_{loc}(\Omega)$.

**Theorem 1.3.** Let $\Omega \subset \mathbb{R}^2$ be a domain and $u$ be an $\infty$-harmonic function in $\Omega$. Then the distributional determinant $-\det D^2udx$ is a Radon measure satisfying

$$-\det D^2u \geq |D|Du||^2$$

where $\Omega$ holds when $u \in C^2(\Omega)$, and

$$\| -\det D^2u \| (V) \leq C \frac{1}{\text{dist} (V, \partial U)^2} \|Du\|^2_{L^2(U)} \quad \forall V \subset U \subset \Omega,$$

where the constant $C$ above is absolute.

Above $\| -\det D^2u \| (V)$ denotes the total measure of the open set $V$ with respect to the Radon measure $-\det D^2u dx$. Here, we list some remarks about Theorem 1.1, Lemma 1.2 and Theorem 1.3.

**Remark 1.4.** (i) For planar $p$-harmonic function with $p > 2$, recall that $|Du|^{(p-2)/2}Du \in W^{1,2}_{loc}(\Omega)$ as proved in [10]. Theorem 1.1 can be viewed as some analogue for planar $\infty$-harmonic functions.

(ii) Note that the function $u(x) = |x| \in W^{1,\infty}_{loc}(\mathbb{R}^2)$ satisfies $|Du|^2 \in W^{1,2}_{loc}(\mathbb{R}^2)$ and (1.3), but is not $\infty$-harmonic.

(iii) We make the following conjecture.

**Conjecture.** Let $u$ be a planar $\infty$-harmonic function. Then the following hold:

(a) $|Du|^\alpha \in W^{1,p}_{loc}$ for $2 < p < p_\alpha$ and $\alpha > 0$, where $p_\alpha$ is given by (1.5);

(b) $-\det D^2u \in L^p_{loc}$ for $1 \leq p < 3/2$;

(c) $\log(|Du|^2 + \kappa) \in BMO_{loc} \cap W^{1,p}_{loc}$ for $p < 2$ uniformly in $\kappa > 0$.

The infinity Laplacian $\Delta_\infty$ is a highly degenerate nonlinear second elliptic partial differential operator. The equation (1.1) is derived by Aronsson 1960’s as the Euler-Lagrange’s equation when absolutely minimizing the $L^\infty$-functional

$$\mathcal{F}_\infty(u, \Omega) = \text{esssup}_{\Omega} |Du|^2;$$
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see [2, 3, 4, 5, 6]. A function \( u \in W^{1,\infty}_{\text{loc}}(\Omega) \) is an absolute minimizer in \( \Omega \) if

\[
\mathcal{F}_{\infty}(u, V) \leq \mathcal{F}_{\infty}(v, V)
\]

whenever \( v \in W^{1,\infty}(V) \) and \( v = u \) on \( \partial V \). Jensen in 1993 identified the viscosity solutions of the equation (1.1) (that is, \( \infty \)-harmonic functions) with absolute minimizers of such \( L^\infty \)-functional, see [20].

The \( \infty \)-harmonic functions (equivalently, absolute minimizers) are known to be differentiable almost everywhere by [20]; but not necessarily \( C^2 \) as shown by the Aronsson’s function in (1.4). The main issue in this direction to understand the possible regularity of \( \infty \)-harmonic functions. Crandall et al. [11] first obtained the linear approximation property. Later, for planar \( \infty \)-harmonic functions, the \( C^1 \)-regularity was proved by Savin [24], \( C^{1,\alpha} \)-regularity with \( 0 < \alpha < 1 \) by Evans-Savin [15] and boundary \( C^1 \)-regularity by [25]. The key idea is to establish a flatness estimate by the planar topology and comparison property with cones, as first observed by Savin [24]. When \( n \geq 3 \), the \( C^1 \) and \( C^{1,\alpha} \)-regularity of \( \infty \)-harmonic functions are still open. Recent progress is made by Evans-Smart [16, 17], who obtained the everywhere differentiability. Their approach is approximating the \( \infty \)-harmonic functions via exponential harmonic functions (originally given by Evans [13, 18]), and then establishing a weaker flatness estimate via a PDE argument.

Theorem 1.1 above gives the asymptotic sharp Sobolev \( W^{1,2}_{\text{loc}}(\Omega) \)-regularity of \( |Du|^\alpha \) for any \( \infty \)-harmonic function \( u \) in \( \Omega \subset \mathbb{R}^2 \) and \( \alpha > 0 \). To prove Theorem 1.1, we also approximate \( u \) via exponential harmonic functions. Precisely, given an arbitrary domain \( U \Subset \Omega \), for \( \varepsilon \in (0, 1] \) let \( u^\varepsilon \in C^\infty(U) \cap C(U) \) satisfy

\[
-\Delta_{\infty} u^\varepsilon - \varepsilon \Delta u^\varepsilon = 0 \text{ in } U, \quad u^\varepsilon = u \text{ on } \partial U.
\]

It is known that \( u^\varepsilon \to u \) uniformly in \( \overline{U} \), see [18, 17] (or Theorem 3.1 below). In this paper, we manage to show the following strong \( W^{1,p}_{\text{loc}}(\Omega) \)-convergence for \( 1 \leq p < \infty \), which may have its own interest.

**Theorem 1.5.** For each \( \alpha > 0 \), we have \( |Du^\varepsilon|^\alpha \to |Du|^\alpha \) in \( L^p_{\text{loc}}(U) \) for all \( p \in [1, \infty) \) and weakly in \( W^{1,2}_{\text{loc}}(U) \). In particular, \( u^\varepsilon \to u \) strongly in \( W^{1,p}_{\text{loc}}(U) \) for all \( p \in [1, \infty) \).

The proof of Theorem 1.5 relies on the uniform Sobolev regularity in Lemma 2.6 and the integral flatness estimate in Lemma 2.7 for approximating functions \( u^\varepsilon \). Observing the following identity

\[
-\det D^2 u^\varepsilon = |D|Du^\varepsilon|^2 + \varepsilon \frac{(\Delta u^\varepsilon)^2}{|Du^\varepsilon|^2} \text{ almost everywhere in } U
\]

in Lemma 2.3, and by integration against suitable test functions, we obtain Lemma 2.6 and Lemma 2.7, for the details see Section 5. The strong convergence in Theorem 1.5 permits us to conclude the Theorem 1.1 and Theorem 1.3 from the uniform Sobolev estimates in Lemma 2.6; see Section 3 for the proofs. The asymptotic sharpness of Theorem 1.1 (that is, Lemma 1.2) will be also given after Theorem 1.1 in Section 3.

Moreover, by integration (1.6) against some other test functions, we obtain some quantitative Sobolev estimate of \( u^\varepsilon \) in Lemma 2.8; for the details see also Section 5. The strong
convergence in Theorem 1.5 allows to conclude from them the following Theorem 1.6, see Section 3 for the proof.

**Theorem 1.6.** Let \( \Omega \subset \mathbb{R}^2 \) be a domain and \( p > 2 \). For any \( \infty \)-harmonic function \( u \) in \( \Omega \) we have

\[
\| Du \|_{L^p(V)} \leq C(p) \frac{1}{\text{dist} (V, \partial U)} \| u - a \|_{L^p(U)} \quad \forall V \subset U \subset \Omega, \quad \forall a \in \mathbb{R},
\]

(1.7)

where the constant \( C(p) \) depends only on \( p \).

As a consequence of Theorem 1.1 and Theorem 1.6, we obtain the following \( L^p \)-Liouville property with \( p > 2 \); see Section 3 for the proof. Below, for \( p \geq 1 \) a function \( u \in L^p_{\text{loc}}(\mathbb{R}^2) \) satisfies the \( L^p \)-vanishing condition if

\[
\liminf_{R \to \infty} \frac{1}{R^2} \left( \frac{1}{R^2} \int_{B(0,R)} |u(x)|^p \, dx \right)^{1/p} = 0.
\]

(1.8)

**Corollary 1.7.** If an \( \infty \)-harmonic function \( u \in C(\mathbb{R}^2) \) satisfies the \( L^p \)-vanishing condition for some \( p > 2 \), then \( u \) must be a constant function.

Note that the \( L^p \)-vanishing condition in Corollary 1.7 is sharp in the sense that the planar \( \infty \)-harmonic function \( u(x) = x_1 \) satisfies

\[
\liminf_{R \to \infty} \frac{1}{R^2} \left( \frac{1}{R^2} \int_{B(0,R)} |u(x)|^p \, dx \right)^{1/p} > 0.
\]

Recall that if an \( \infty \)-harmonic function \( u \in C(\mathbb{R}^n) \) with \( n \geq 2 \) satisfying \( \lim_{x \to \infty} \frac{|u(x)|}{|x|} = 0 \) (that is, \( L^\infty \)-vanishing condition), then \( u \) must be a constant function as proved by Crandall et al [11]. Savin [24] further proved that if an \( \infty \)-harmonic function \( u \in C(\mathbb{R}^2) \) satisfying \( \sup_{x \in \mathbb{R}^2} \frac{|u(x)|}{1+|x|} < \infty \), then \( u \) must be a linear function; similar results in higher dimension are still unknown.

**Remark 1.8.** It would be interesting to show that Theorem 1.6, and hence Corollary 1.7, holds for \( p \in [1, 2] \).

Considering the duality relation between \( p \)-Laplacian and \( q \)-Laplacian when \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), an interesting question in this area is to understand the possible dual relation between 1-Laplacian and \( \infty \)-Laplacian. We show that \( |Du|^2 \) satisfies a certain type of 1-Laplacian equation for \( \infty \)-harmonic function \( u \) which is \( C^2 \) and does not have singular points; see Proposition 4.2. For Aronsson’s function \( w \), which is singular on the axes, there will be an extra term appearing in the 1-Laplacian equation for \( |Dw|^2 \); see Lemma 4.3.

Finally, we make some convention of notations. We often write the constants as positive real numbers \( C(\cdot) \) with the parenthesis including all the parameters on which the constant depends; we just simply write \( C \) if it is absolute and if there is no further explanation. The constant \( C(\cdot) \) may vary between appearances, even within a chain of inequalities. By \( V \subset U \) we mean that \( V \) is a bounded domain of \( U \) and \( \nabla \subset U \).
2. Determinants of approximating functions

Suppose that $U \subset \mathbb{R}^2$ is a bounded domain. We have the following general observation for the determinant of smooth functions.

**Lemma 2.1.** For any smooth function $v$ in $U$ we have

$$-\det D^2v = -\frac{1}{2} \text{div}(\Delta v Dv - D^2v Dv) = -\frac{1}{2}(v_i v_j)_{ij} + \frac{1}{2}(|Dv|^2)_{ii} \quad \text{in } U, \quad (2.1)$$

and

$$(-\det D^2v)|Dv|^2 = |D^2v Dv|^2 - \Delta v \Delta_\infty v \quad \text{in } U. \quad (2.2)$$

**Proof.** By direct calculation we have

$$-\det D^2v = (v_{12})^2 - v_{11} v_{22}$$

$$= \frac{1}{2} [v_{ij} v_{ij} - v_{ii} v_{jj}] = \frac{1}{2}(|D^2v|^2 - (\Delta v)^2)$$

$$= \frac{1}{2}(v_i v_j - v_{jj} v_i) = -\frac{1}{2} \text{div}(\Delta v Dv - D^2v Dv)$$

$$= \frac{1}{2}(v_j v_j)_{ii} - \frac{1}{2}(v_i v_j)_{ij} = -\frac{1}{2}(v_i v_j)_{ij} + \frac{1}{2}(|Dv|^2)_{ii},$$

which gives (2.1).

By direct calculation we have

$$|D^2v Dv|^2 = (v_{11} v_{12} + v_{21} v_{22})^2$$

$$= v_{11}((v_1)^2 v_{12} + 2 v_1 v_2 v_{12}) + v_{22}((v_2)^2 v_{22} + 2 v_1 v_2 v_{12}) + (v_{12})^2((v_1)^2 + (v_2)^2)$$

$$= (v_{11} + v_{22}) \Delta_\infty v - v_{11} v_{22}((v_2)^2 + (v_1)^2) + (v_{12})^2((v_1)^2 + (v_2)^2)$$

$$= \Delta v \Delta_\infty v + (-\det D^2v)|Dv|^2,$$

which gives (2.2). \qed

**Remark 2.2.** Given a function $v \in W^{1,2}_{\text{loc}}(U)$, the distributional determinant $\det D^2v$ of $v$ is well defined as given by

$$\int_U -\det D^2v \phi \, dx = \frac{1}{2} \int_U [-v_i v_j \phi_{ij} + |Dv|^2 \phi_{ii}] \, dx \quad \forall \phi \in C^2_c(U).$$

In the sequel of this section, for each $\epsilon > 0$ we let $u^\epsilon \in C^\infty(U)$ be a solution to

$$-\Delta_\infty u^\epsilon - \epsilon \Delta u^\epsilon = 0 \quad \text{in } U. \quad (2.3)$$

As a consequence of Lemma 2.1, we have the following results.

**Lemma 2.3.** For each $\epsilon > 0$ we have

$$(-\det D^2u^\epsilon)|Du^\epsilon|^2 = |D^2u^\epsilon Du^\epsilon|^2 + \epsilon(\Delta u^\epsilon)^2 \quad \text{in } U. \quad (2.4)$$

Moreover, $-\det D^2u^\epsilon \geq 0$ in $U$ and

$$-\det D^2u^\epsilon = |D|Du^\epsilon|^2 + \epsilon \frac{(\Delta u^\epsilon)^2}{|Du^\epsilon|^2} \quad \text{almost everywhere in } U. \quad (2.5)$$
Remark 2.4. If $Du^\epsilon(z) = 0$ at $z \in U$, we then have $\epsilon \Delta u^\epsilon(z) = -\Delta_\infty u^\epsilon(z) = 0$. For this reason, we define

$$\frac{\Delta u^\epsilon(z)}{|Du^\epsilon(z)|^2} = 0$$

for any $0 < \beta < 2$. In particular, the last term in (2.5) is well defined in $U$.

Proof. By (2.2) and $\Delta_\infty u^\epsilon = -\epsilon \Delta u^\epsilon$, we have

$$|D^2 u^\epsilon Du^\epsilon|^2 = -\epsilon (\Delta u^\epsilon)^2 + (-\det D^2 u^\epsilon)|Du^\epsilon|^2$$

which gives (2.4).

Now we use (2.4) to show $-\det D^2 u^\epsilon \geq 0$ in $U$. Let $\bar{x} \in U$ be an arbitrary fixed point. If $|Du^\epsilon|(\bar{x}) \neq 0$, then by (2.4) we have $-\det D^2 u^\epsilon(\bar{x}) \geq 0$. Assume that $Du^\epsilon(\bar{x}) = 0$. If there exist $\bar{x}^{(k)}$ such that $Du^\epsilon(\bar{x}^{(k)}) \neq 0$ and $\bar{x}^{(k)} \to \bar{x}$ as $k \to \infty$, then by continuity,

$$-\det D^2 u^\epsilon(\bar{x}) = \lim_{k \to \infty} -\det D^2 u^\epsilon(\bar{x}^{(k)}) \geq 0.$$

Otherwise, there exists some sufficiently small $r > 0$ such that $Du^\epsilon(x) = 0$ for all $x \in B(\bar{x}, r)$, and hence $u$ is a constant function in $B(\bar{x}, r)$, which implies that $-\det D^2 u^\epsilon(\bar{x}) = 0$.

Finally, (2.4) also implies (2.5). Indeed, notice that $|Du^\epsilon| \in \text{Lip}_{\text{loc}}(U)$, hence $|Du^\epsilon|$ is differentiable almost everywhere. Assume that $|Du^\epsilon|$ is differentiable at $z \in U$. If $Du^\epsilon(z) \neq 0$, then

$$|D|Du^\epsilon|(z)|^2 = \frac{|D^2 u^\epsilon Du^\epsilon(z)|}{|Du^\epsilon|^2} = -\det D^2 u^\epsilon - \frac{(\Delta u^\epsilon)^2}{|Du^\epsilon|^2}.$$

Assume now $Du^\epsilon(z) = 0$. Considering Remark 2.4, we are required to show

$$|D|Du^\epsilon|(z)|^2 = -\det D^2 u^\epsilon(z).$$

Since $|Du^\epsilon|$ and $Du^\epsilon$ are differentiable at $z$, applying Taylor’s expansion, we write

$$|Du^\epsilon(x)| = \langle D|Du^\epsilon|(z), x - z \rangle + o(|x - z|) \quad \forall x$$

and

$$Du^\epsilon(x) = D^2 u^\epsilon(z)(x - z) + o(|x - z|) \quad \forall x.$$

If $D|Du^\epsilon|(z) \neq 0$, plugging $x = z + tD|Du^\epsilon|(z)$ in both formula and letting $t \to 0$, we obtain

$$|D|Du^\epsilon|(z)|^2 = |D^2 u^\epsilon(z)D|Du^\epsilon|(z)|.$$

Assuming $D|Du^\epsilon|(z) = |D|Du^\epsilon|(z)|e_1$ without loss of generality, we have

$$|D|Du^\epsilon|(z)|^2 = |D^2 u^\epsilon e_1|^2 = (u^{11}_1)^2 + (u^{12}_2)^2.$$

Since $\Delta u^\epsilon(z) = -\frac{1}{\epsilon} \Delta_\infty u^\epsilon(z) = 0$, we obtain $u^{11}_1 = -u^{12}_2$, which yields that

$$|D|Du^\epsilon|(z)|^2 = -u^{11}_1u^{12}_2 + (u^{12}_2)^2 = -\det D^2 u^\epsilon(z).$$

If $D|Du^\epsilon|(z) = 0$, then

$$o(|x - z|) = |Du^\epsilon(x)| = |D^2 u^\epsilon(z)(x - z)| + o(|x - z|) \quad \forall x.$$

Hence $D^2 u^\epsilon(z) = 0$, and

$$|D|Du^\epsilon|(z)|^2 = 0 = -\det D^2 u^\epsilon(z).$$

This completes the proof of Lemma 2.3. □
Associated to such \( u^\epsilon \), we introduce a functional \( \mathbb{I}_\epsilon \) on \( C_c(U) \) defined by

\[
\mathbb{I}_\epsilon (\phi) = \int_U -\det D^2 u^\epsilon \phi \, dx \quad \forall \phi \in C_c(U) \quad (2.6)
\]

By (2.5) we write

\[
\mathbb{I}_\epsilon (\phi) = \int_U |D|Du^\epsilon|(z)|^2 \phi \, dx + \epsilon \int_U \frac{(\Delta u^\epsilon)^2}{|Du^\epsilon|^2} \phi \, dx \quad \forall \phi \in C_c(U). \quad (2.7)
\]

By (2.1) and integration by parts, we further write

\[
\mathbb{I}_\epsilon (\phi) = \frac{1}{2} \int_U |\Delta u^\epsilon u^\epsilon_\xi - u^\epsilon_\xi u^\epsilon_\eta \phi_i| \, dx \quad \forall \phi \in W^{1,2}_c(U) \quad (2.8)
\]

\[
= \frac{1}{2} \int_U \left[ -u^\epsilon_\xi u^\epsilon_\eta \phi_{ij} + |Du^\epsilon|^2 \phi_{ii} \right] \, dx \quad \forall \phi \in C^2_c(U). \quad (2.9)
\]

As a consequence of (2.5) and (2.9), we have the following apriori estimates, which is uniform in \( \epsilon > 0 \).

**Corollary 2.5.** We have

\[
\int_V |D|Du^\epsilon|(z)|^2 \, dx + \epsilon \int_V \frac{(\Delta u^\epsilon)^2}{|Du^\epsilon|^2} \, dx = \int_V -\det D^2 u^\epsilon \, dx \leq \frac{8}{[\text{dist} (V, \partial W)]^2} \int_W |Du^\epsilon|^2 \, dx \quad \forall V \in W \Subset U.
\]

Moreover, by testing \( \phi = (|Du^\epsilon|^2 + \kappa)^{\alpha-1} \xi^4 \) in (2.8) for some suitable cut-off functions \( \xi \in C^\infty_c(U) \) and \( \kappa > 0 \), applying (2.5) we have the following estimates of \( |Du^\epsilon|^\alpha \) for all \( \alpha > 0 \), which are uniform in \( \epsilon \). We postpone the details of the proof to Section 5.

**Lemma 2.6.** For \( \alpha > 0, \epsilon \in (0,1] \) and \( V \Subset W \Subset U \), we have

\[
\int_V |D|Du^\epsilon|^\alpha|^2 \, dx + \epsilon \int_V |Du^\epsilon|^{2\alpha-4} (\Delta u^\epsilon)^2 \, dx \leq C(\alpha) \frac{1}{[\text{dist} (V, \partial W)]^2} \int_W |Du^\epsilon|^{2\alpha} \, dx.
\]

By testing \( \phi = (u^\epsilon - P)^2 \xi^4 \) in (2.8) for some suitable cut-off functions \( \xi \in C^\infty_c(U) \) and any linear function \( P \), applying \(-\det D^2 u^\epsilon \geq 0 \) in \( U \) given by Lemma 2.3 and Lemma 2.6 we have the following uniform integral flatness. The detailed proof is postpone to Section 5.

**Lemma 2.7.** For any \( \bar{x} \in U \), \( 0 < r < \text{dist} (\bar{x}, \partial U)/4 \) and linear function \( P \), we have

\[
\int_{B(\bar{x},r)} \left( |Du^\epsilon|^2 - \langle DP, Du^\epsilon \rangle \right)^2 \, dx 
\]

\[
\leq C \left[ \int_{B(\bar{x},2r)} |Du^\epsilon|^4 \, dx \right]^{1/2} \left[ \int_{B(\bar{x},2r)} \left( \frac{|u^\epsilon - P|^2}{r^2} (|DP| + |Du^\epsilon|^2) + \frac{|u^\epsilon - P|^4}{r^4} \right) \, dx \right]^{1/2}
\]

Finally, by testing \( \phi = (|Du^\epsilon|^2 + \kappa)^{\alpha-1} |u^\epsilon|^2 \xi^{2(\alpha+1)} \) in (2.8) for some suitable cut-off functions \( \xi \in C^\infty_c(U) \) and \( \kappa > 0 \), applying (2.5), we obtain the following estimates. The detailed proof is postpone to Section 5.
Lemma 2.8. For any $\alpha > 0$ and $\kappa > 0$, we have
\[
\int_V (|Du|^{2} + \kappa)^{\alpha+1} \, dx \\
\leq C(\alpha) \frac{1}{[\text{dist}(V, \partial W)]^{2(\alpha+1)}} \int_W |u|^{2\alpha+2} \, dx + (8\kappa + \tilde{C}(\alpha)\epsilon) \int_W (|Du|^{2} + \kappa)^{\alpha} \, dx.
\]
for all $V \subseteq W \subseteq U$.

3. Proofs of main results

Let $u \in C(\Omega)$ be an $\infty$-harmonic function in $\Omega \subset \mathbb{R}^2$. It is known that $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ and $u$ is differentiable almost everywhere, that is, $Du$ exists almost everywhere. Note that $Du$ also coincides with the weak derivative of $u$, and we abuse of the notation here for convenience.

By Evans [13] (see also [18, 17]), we know that, on subdomains of $\Omega$, $u$ is approximated by exponential harmonic functions. To be precise, fix an arbitrary domain $U \subseteq \Omega$. For $\epsilon \in (0, 1]$, consider the following Dirichlet problem:
\[
\begin{cases}
-\Delta_{\infty} u^\epsilon - \epsilon \Delta u^\epsilon = 0 & \text{in } U \\
 u^\epsilon = u & \text{on } \partial U.
\end{cases}
\] (3.1)

See [17, Theorem 2.1] for the following properties.

Theorem 3.1. For each $\epsilon \in (0, 1]$, there exists a unique solution $u^\epsilon \in C^\infty(U) \cap C(\overline{U})$ to (3.1). Moreover for all $\epsilon \in (0, 1]$, we have
\[
\max_{\overline{U}} |u^\epsilon| \leq \max_{\partial U} u
\]
and for every open set $V \subseteq U$
\[
\max_{V} |Du^\epsilon| \leq C(\max_{\partial U} |u|, \text{dist}(V, \partial U)),
\]
where $C$ is independent of $\epsilon$. Furthermore, $u^\epsilon \rightarrow u$ uniformly on $\overline{U}$.

As a consequence of Lemma 2.7 and Theorem 3.1, we have the following flatness.

Corollary 3.2. Given $\epsilon \in (0, 1]$, $\bar{x} \in U$ and $r < \text{dist}(\bar{x}, \partial U)/4$, if
\[
\sup_{B(\bar{x}, 2r)} \frac{|u^\epsilon(x) - P(x)|}{r} \leq \lambda
\]
for some linear function $P$ and $0 < \lambda < 1$, then
\[
\int_{B(\bar{x}, r)} (|Du^\epsilon|^2 - \langle DP, Du^\epsilon \rangle)^2 \, dx \leq C(\text{dist}(\bar{x}, \partial U)) \lambda
\]

With the help of Theorem 3.1, Lemma 2.6 and Corollary 3.2 (or Lemma 2.7), we are able to show the strong $L^p_{\text{loc}}(\Omega)$-convergence of $Du^\epsilon$ for $1 \leq p < \infty$, that is, Theorem 1.5.
Proof of Theorem 1.5. Fix $\alpha > 0$. By Lemma 2.6 and Theorem 3.1, we know that $D|Du^\epsilon|^\alpha \in L_{\text{loc}}^2(U)$ locally uniformly. From the weak compactness of $W^{1,2}(U)$, it follows that $|Du^\epsilon|^\alpha$ converges, up to some subsequence, to some function $f^{(\alpha)}$ in $L_{\text{loc}}^p(U)$ and weakly in $W^{1,2}_{\text{loc}}(U)$. On the other hand, from $u^\epsilon \to u$ in $C(\bar{U})$ it follows that $Du^\epsilon$ converges to $Du$ weakly in $L^p(U)$.

We claim that $f^{(2)} = |Du|^2$ almost everywhere. Assume that the claim holds for the moment. Then for all $\alpha > 0$, we have

$$|Du^\epsilon|^\alpha = (|Du^\epsilon|^2)^{\alpha/2} \to (|Du|^2)^{\alpha/2} = |Du|^\alpha$$

almost everywhere as $\epsilon \to 0$. By Theorem 3.1 and Lebesgue’s dominated convergence theorem, for any $p \in [1, \infty)$ and $\epsilon \in U$,

$$\lim_{\epsilon \to 0} \int_V |Du^\epsilon|^\alpha - |Du|^\alpha \, dx = 0,$$

that is, $|Du^\epsilon|^\alpha \to |Du|^\alpha$ in $L_{\text{loc}}^p(U)$. When $\alpha = 1$, this together with the weak convergence $Du^\epsilon \to Du$ in $L^p(U)$ shows that $Du^\epsilon \to Du$ strongly in $L_{\text{loc}}^p(U)$, that is, $u^\epsilon \to u$ strongly in $W^{1,p}_{\text{loc}}(U)$ for all $p \in [1, \infty)$.

We prove the above claim below, i.e. $f^{(2)} = |Du|^2$ almost everywhere. Assume that $u$ is differentiable at $\bar{x}$, and also assume that $\bar{x}$ is Lebesgue point of $f^{(2)}$ and $Du$; the set of such $\bar{x}$ has full measure in $U$. Then for any $\lambda \in (0, 1)$, there exists $r_{\lambda, \bar{x}} \in (0, \text{dist}(\bar{x}, \partial U)/8)$ such that for any $r \in (0, r_{\lambda, \bar{x}})$, we have

$$\sup_{B(\bar{x}, 2r)} \frac{|u(x) - u(\bar{x}) - \langle Du(\bar{x}), (x - \bar{x})\rangle|}{r} \leq \lambda.$$

By Theorem 3.1, for arbitrary $r \in (0, r_{\lambda, \bar{x}})$, there exists $\epsilon_{\lambda, \bar{x}, r} \in (0, 1]$ such that for all $\epsilon \in (0, \epsilon_{\lambda, \bar{x}, r})$, we have

$$\sup_{B(\bar{x}, 2r)} \frac{|u^\epsilon(x) - u^\epsilon(\bar{x}) - \langle Du(\bar{x}), (x - \bar{x})\rangle|}{r} \leq 2\lambda.$$

Letting $P(x) = u^\epsilon(\bar{x}) - \langle Du(\bar{x}), (x - \bar{x})\rangle$ in Corollary 3.2, we arrive at

$$\int_{B(\bar{x}, r)} (|Du^\epsilon|^2 - \langle Du(\bar{x}), Du^\epsilon\rangle)^2 \, dx \leq C(u, \text{dist}(\bar{x}, \partial U))\lambda \quad \forall r \in (0, r_{\lambda, \bar{x}}), \, \epsilon \in (0, \epsilon_{\lambda, \bar{x}, r}).$$

On the other hand, since $|Du^\epsilon|^2 \to f^{(2)}$ in $L_{\text{loc}}^2(U)$ and $Du^\epsilon \to Du$ weakly in $L_{\text{loc}}^2(U)$, for any $r \in (0, \text{dist}(\bar{x}, \partial U)/4)$ we have

$$\int_{B(\bar{x}, r)} (f^{(2)} - \langle Du(\bar{x}), Du\rangle)^2 \, dx \leq \liminf_{\epsilon \to 0} \int_{B(\bar{x}, r)} (|Du^\epsilon|^2 - \langle Du(\bar{x}), Du^\epsilon\rangle)^2 \, dx.$$

Therefore,

$$\int_{B(\bar{x}, r)} (f^{(2)} - \langle Du(\bar{x}), Du\rangle)^2 \, dx \leq C(u, \text{dist}(\bar{x}, \partial U))\lambda \quad \forall r \in (0, r_{\lambda, \bar{x}}).$$

Since $\bar{x}$ is a Lebesgue point of $f^{(2)}$ and $Du$, via Hölder’s inequality, we obtain

$$|f^{(2)}(\bar{x}) - |Du|^2(\bar{x})| = \lim_{r \to 0} \int_{B(0, r)} |f^{(2)} - \langle Du(\bar{x}), Du\rangle| \, dx \leq C(u, \text{dist}(\bar{x}, \partial U))\lambda^{1/2}.$$
By letting $\lambda \to 0$, we have $f^{(2)}(\bar{x}) = |Du|^2(\bar{x})$, and conclude the claim. \qed

Now we are ready to prove Theorem 1.1 and Theorem 1.3 using Theorem 1.5 and Lemma 2.6.

**Proof of Theorem 1.1.** Assume that $u \in C(\Omega)$ is $\infty$-harmonic in $\Omega \subset \mathbb{R}^2$. Fix an arbitrary domain $U \subset \Omega$ and let $u^\varepsilon$ be the solution to the Dirichlet problem (3.1) in $U$.

Let us show the first part of the theorem. For $\alpha > 0$, by Theorem 1.5, we know that $|Du^\varepsilon|^\alpha$ weakly converges to $|Du|^\alpha$ in $W^{1,2}_{\text{loc}}(U)$, and hence, together with Lemma 2.6 and Theorem 3.1, we obtain

$$
\|D|Du|^\alpha\|_{L^2(V)} \leq \liminf_{\varepsilon \to 0} \|D|Du^\varepsilon|^\alpha\|_{L^2(V)} \\
\leq C(\alpha) \liminf_{\varepsilon \to 0} \|Du^\varepsilon\|_{L^2(U)} \\
\leq C(\alpha) \lim_{\varepsilon \to 0} \|Du\|_{L^2(U)}.
$$

Given $\alpha > 0$, by the local strong convergence $Du^\varepsilon \to Du$ and the local weak convergence $D|Du^\varepsilon|^\alpha \rightharpoonup D|Du|^\alpha$ in $L^2_{\text{loc}}(U)$ shown in Theorem 1.5, we have

$$
\int_U \langle D|Du|^\alpha, Du \rangle \phi \, dx = \lim_{\varepsilon \to 0} \int_U \langle D|Du^\varepsilon|^\alpha, Du^\varepsilon \rangle \phi \, dx \quad \forall \phi \in C_c^\infty(U)
$$

Note that $D(|Du^\varepsilon|^2 + \kappa)^{\alpha/2}$ converges to $D|Du|^\alpha$ weakly in $L^2_{\text{loc}}(U)$ as $\kappa \to 0$, we have

$$
\int_U \langle D|Du|^\alpha, Du \rangle \phi \, dx = \lim_{\varepsilon \to 0} \int \left( \frac{\alpha}{2} (|Du^\varepsilon|^2 + \kappa)^{\alpha/2 - 1} D|Du^\varepsilon|^2, Du^\varepsilon \right) \phi \, dx \\
= \lim_{\varepsilon \to 0} \int \alpha (|Du^\varepsilon|^2 + \kappa)^{\alpha/2 - 1} \Delta u^\varepsilon \phi \, dx \\
= \lim_{\varepsilon \to 0} \int \alpha |Du^\varepsilon|^2 \Delta u^\varepsilon \phi \, dx \quad \forall \phi \in C_c^\infty(U)
$$

Applying $\Delta u^\varepsilon = -\varepsilon \Delta u^\varepsilon$, we have

$$
\int_U \langle D|Du|^\alpha, Du \rangle \phi \, dx = \lim_{\varepsilon \to 0} \alpha \int_U (|Du^\varepsilon|^2 + \kappa)^{\alpha/2 - 1} \Delta u^\varepsilon \phi \, dx \\
= \lim_{\varepsilon \to 0} -\alpha \varepsilon \int_U |Du^\varepsilon|^{2 \alpha - 4} \Delta u^\varepsilon \phi \, dx \quad \forall \phi \in C_c^\infty(U)
$$

Let $V = \text{supp} \phi \subset W \subset U$. Notice that by Lemma 2.6 and Theorem 3.1, we obtain

$$
\varepsilon \int_U |Du^\varepsilon|^{2 \alpha - 4} (\Delta u^\varepsilon)^2 (\phi)^2 \, dx \leq C(\phi) \varepsilon \int_V |Du^\varepsilon|^{2 \alpha - 4} (\Delta u^\varepsilon)^2 \, dx \\
\leq C(\alpha, \phi, \text{dist}(V, \partial W)) \int_W |Du^\varepsilon|^{2 \alpha} \, dx \\
\leq C(\alpha, \phi, \text{dist}(V, \partial W), \text{dist}(W, \partial U)).
$$
Thus via Young’s inequality, we further have
\[
-\alpha \epsilon \int_{U} |Dv^{\epsilon}|^{\alpha-2} \Delta v^{\epsilon} \phi \, dx \leq C \epsilon^{1/2} |V| + C \epsilon^{3/2} \int_{U} |Dv^{\epsilon}|^{2\alpha-4} (\Delta v^{\epsilon})^{2} (\phi)^{2} \, dx \to 0
\]
as \epsilon \to 0. Therefore we conclude that
\[
\int_{U} \langle D|Du|^{\alpha}, Du \rangle \phi \, dx = 0 \quad \forall \phi \in C_{c}(U)
\]
as desired. \qed

**Proof of Lemma 1.2.** A direct calculus gives that
\[
|Du| = \frac{4}{3}(x_{1}^{2/3} + x_{2}^{2/3})^{1/2} \quad \text{and} \quad |Dw|^{2} = \frac{4^{2}}{3^{3}}(x_{1}^{-2/3} + x_{2}^{-2/3})^{1/2}.
\]
For any domain \( U \subset \mathbb{R}^{2} \setminus \{0\} \), since \(|Du|\) and \(|Dw|^{2}\) have upper and lower bounds on \( U \), we know that \(|Dw|^{\alpha} = \alpha 2^{-1}|Dw|^{\alpha-2}\) \(|Dw|^{2}\) \in \( L_{p}(U) \) for all \( p \geq 1 \).

Now assume that \( 0 \in U \subset \mathbb{R}^{2} \), and without loss of generality let \( U = (-1,1)^{2} \). We have
\[
\int_{1}^{1} \int_{-1}^{1} \frac{|DW|^{2p}|Dw|^{2p}}{p(2-\alpha)} \, dx \, dt = 8 \cdot 4^{2p} \frac{3^{3p}}{3^{3p}} \int_{x}^{1} \int_{0}^{x_{1}} \frac{(x_{1}^{-2/3} + x_{2}^{-2/3})^{p/2}}{(x_{1}^{2/3} + x_{2}^{2/3})^{p(2-\alpha)/2}} \, dx_{2} \, dx_{1}
\]
\[
= 8 \cdot 4^{2p} \frac{3^{3p}}{3^{3p}} \int_{0}^{1} \int_{0}^{(1+t^{-2/3})^{p/2}} \frac{(1+t^{-2/3})^{p/2}}{(1+t^{2/3})^{p(2-\alpha)/2}} \, dt \, dx_{1}.
\]
The first integral is finite if and only if \( p < 3 \), the second integral is finite if and only if \( p < \frac{6}{7-\alpha} \) when \( \alpha < 3 \) and \( p < \infty \) when \( \alpha \geq 3 \).

Therefore, we conclude that
\[
|Dw|^{\alpha} = \alpha 2^{-1}|Dw|^{\alpha-2}|Dw|^{2} \in \( L_{p}^{loc}(\mathbb{R}^{2}) \) \quad \forall p < p_{\alpha} \quad \text{but} \quad \notin \( L_{p}^{\alpha}(\mathbb{R}^{2}) \)
\]
and
\[
|D \log |Du|| = \frac{1}{2}|Dw|^{-2}|Dw|^{2} \in \( L_{p}^{loc}(\mathbb{R}^{2}) \) \quad \forall p < 2 \quad \text{but} \quad \notin \( L_{p}^{2}(\mathbb{R}^{2}) \).
\]
Moreover,
\[
\log |Du| = \log \frac{4}{3} + \frac{1}{2} \log (x_{1}^{2/3} + x_{2}^{2/3}).
\]
Since
\[
|x|^{2/3} \leq x_{1}^{2/3} + x_{2}^{2/3} \leq C|x|^{2/3},
\]
by \( \log |x| \in BMO(\mathbb{R}^{2}) \) we also have \( |D \log |Du|| \in BMO_{loc}(\mathbb{R}^{2}) \) as desired. \qed

**Proof of Theorem 1.3.** By Remark 2.2, \( u \in W^{1,\infty}_{loc}(\Omega) \) allows to defined the distributional determinant \( D^{2}v \), that is,
\[
\int_{\Omega} -\det D^{2}u \phi \, dx = \frac{1}{2} \int_{\Omega} [\nabla \cdot (u_{i}u_{j}\phi_{ij}) + |Du^{2}|^{2} \phi_{ii}] \, dx \quad \forall \phi \in C_{c}^{2}(\Omega).
\]
Hence by Theorem 1.5,
\[
\int_{\Omega} -\det D^{2}u \phi \, dx = \lim_{\epsilon \to 0} \frac{1}{2} \int_{\Omega} [\nabla \cdot (u_{i}^{\epsilon}u_{j}^{\epsilon}\phi_{ij}) + |Du_{i}^{\epsilon}|^{2} \phi_{ii}] \, dx = \lim_{\epsilon \to 0} \int_{\Omega} -\det D^{2}u^{\epsilon} \phi \, dx \quad \forall \phi \in C_{c}^{2}(\Omega).
\]
Fix an arbitrary domain \( U \Subset \Omega \) and let \( u^\epsilon \) be the solution to the Dirichlet problem (3.1) in \( U \). A density argument shows that for all \( \phi \in C_c(\Omega) \), we can define
\[
\int_U - \det D^2 u^\epsilon \phi \, dx = \lim_{\epsilon \to 0} \int_U - \det D^2 u^\epsilon \phi \, dx.
\]
Recalling \(- \det D^2 u^\epsilon \geq 0 \) in \( U \) given by Lemma 2.3, we know that \(- \det D^2 u \) is indeed a nonnegative Radon measure. Moreover the uniform upper estimates of \(- \det D^2 u^\epsilon \) yields that
\[
\| - \det D^2 u \|(V) \leq \liminf_{\epsilon \to 0} \int_V - \det D^2 u^\epsilon \, dx \leq \frac{C}{\text{dist} (V, \partial W)^2} \int_W |Du^\epsilon|^2 \, dx \quad \forall V \Subset W \Subset U.
\]
Since \(- \det D^2 u^\epsilon \geq |D|Du^\epsilon|^2 \) as given in Lemma 2.3, for \( \phi \in C_c(\Omega) \) with \( \phi \geq 0 \) by Theorem 1.5 we have
\[
\int_U - \det D^2 u \phi \, dx = \lim_{\epsilon \to 0} \int_U - \det D^2 u^\epsilon \phi \, dx \geq \limsup_{\epsilon \to 0} \int_U |D|Du^\epsilon|^2 \phi \, dx \geq \int_U |D|Du|^2 \phi \, dx,
\]
which yields that \(- \det D^2 u \, dx \geq |D|Du|^2 \, dx \) in \( U \). By the arbitrariness of \( U \Subset \Omega \), we know that \(- \det D^2 u \, dx \) is a Radon measure enjoys the desired upper bounds and lower bounds.

Finally, assume that \( u \in C^2(\Omega) \). If \( Du(z) = 0 \) for some \( z \in U \), then by [5] (see also [26]), \( u \) is a constant function in \( \Omega \), and hence, we have \( |D|Du|^2 = 0 = - \det D^2 u \) in \( \Omega \). Now, we assume that \( |Du| > 0 \) in \( \Omega \). Up to approximating \( u \) in \( C^2_{\text{loc}}(\Omega) \) by smooth functions, applying Lemma 2.1, we have
\[
(- \det D^2 u) |Du|^2 = |D^2 u Du|^2 \quad \text{in } \Omega.
\]

Since \( |Du| > 0 \) in \( \Omega \), we have
\[
- \det D^2 u = \frac{|D^2 u Du|^2}{|Du|^2} = |D|Du|^2 \quad \text{in } \Omega.
\]
This completes the proof of Theorem 1.3.

**Proof of Theorem 1.6.** By Theorem 3.1, we know that the following term
\[
\int_W (|Du^\epsilon|^2 + \kappa)^\alpha \, dx + \frac{1}{\text{dist} (V, \partial W)^2} \int_W (|Du^\epsilon|^2 + \kappa)^{\alpha-1} |u^\epsilon|^2 \, dx
\]
appeared in Lemma 2.8 is bounded uniformly in \( \epsilon \) for each fixed \( \kappa > 0 \) when \( \alpha \in (0, 1) \) and for all \( \kappa \in (0, 1) \) when \( \alpha \geq 1 \). Letting \( \epsilon \to 0 \) in Lemma 2.8, by Theorem 1.5 and Theorem 3.1 we have
\[
\int_V (|Du|^2 + \kappa)^{\alpha+1} \, dx \leq C(\alpha) \frac{1}{\text{dist} (V, \partial W)^{2(\alpha+1)}} \int_W |u^{2\alpha+2} \, dx + 8\kappa \int_W (|Du|^2 + \kappa)^\alpha \, dx.
\]
Letting \( \kappa \to 0 \), we further obtain
\[
\int_V |Du|^{2\alpha+2} \, dx \leq C(\alpha) \frac{1}{\text{dist} (V, \partial W)^{2(\alpha+1)}} \int_W |u|^{2\alpha+2} \, dx.
\]
Observing \( u - a \) is also \( \infty \)-harmonic, we know that the above also holds by replacing \( u \) with \( u - a \) for any \( a \in \mathbb{R} \). Thus Theorem 1.6 holds with \( p = 2\alpha + 2 \). \( \square \)

Finally, we prove Corollary 1.7 using Theorem 1.1 and Theorem 1.6.

**Proof of Corollary 1.7.** Assume that \( u \in C(\mathbb{R}^2) \) is \( \infty \)-harmonic satisfying \( L^p \)-vanishing condition. By Theorem 1.1 with \( \alpha = p/2 \) and Theorem 1.6, we know that

\[
\|D|Du|^\alpha\|_{L^2(\mathbb{R}^2)} = \liminf_{R \to \infty} \frac{1}{R} \|D|Du|^\alpha\|_{L^2(B(0,R))} \leq C \liminf_{R \to \infty} \frac{1}{R} \|u\|_{L^p(B(0,2R))} \leq C \liminf_{R \to \infty} \frac{1}{R} \|u\|_{L^p(B(0,2R))}.
\]

By \( L^p \)-vanishing condition, we have \( \|D|Du|^\alpha\|_{L^2(\mathbb{R}^2)} = 0 \) and hence \( D|Du|^\alpha = 0 \) almost everywhere. Thus \( |Du| = c \) almost everywhere. By Theorem 1.6 again, we have

\[
c^\alpha \leq C \liminf_{R \to \infty} \frac{1}{R} \|u\|_{L^p(B(0,2R))} = C \liminf_{R \to \infty} \frac{1}{R} \|u\|_{L^p(B(0,2R))}.
\]

By \( L^p \)-vanishing condition again, we have \( c = 0 \), that is, \( u \) must be a constant function. \( \square \)

4. **The duality between the 1-Laplacian and the \( \infty \)-Laplacian in the plane**

Let \( U \subset \mathbb{R}^2 \) be a bounded domain. For a given pair of continua \( E, F \subset \overline{U} \) and \( 1 \leq p \leq \infty \), one defines the \( p \)-capacity between \( E \) and \( F \) in \( U \) as

\[
\text{Cap}_p(E, F; U) = \inf\{\|\nabla u\|_{L^p(\Omega)}^p : u \in \Delta(E, F; U)\},
\]

where \( \Delta(E, F; U) \) denotes the class of all \( u \in W^{1,p}(\Omega) \) that are continuous in \( \Omega \cup E \cup F \) and satisfy \( u = 1 \) on \( E \), and \( u = 0 \) on \( F \). The following duality of capacities in the plane was established in [23, pp.888-891], which originally follows from [27].

**Lemma 4.1.** Let \( U \subset \mathbb{R}^2 \) be a Jordan domain enclosed by four arcs \( \gamma_1, \gamma_2, \gamma_3 \) and \( \gamma_4 \) counterclockwise. Then we have

\[
\left[\text{Cap}_p(\gamma_1, \gamma_3; U)\right]^\frac{1}{p} \left[\text{Cap}_q(\gamma_2, \gamma_4; U)\right]^\frac{1}{q} = 1
\]

for \( 1 \leq p \leq \infty \) and \( q = \frac{p}{p-1} \).

When \( 1 < p < \infty \) and \( q = \frac{p}{p-1} \), this duality between capacities is related to the following equation system

\[
\begin{cases}
  v_x = |Du|^{p-2}u_y \\
  v_y = -|Du|^{p-2}u_x
\end{cases}
\]

in a domain \( U \subset \mathbb{R}^2 \); see [23]. The function \( u \) is \( p \)-harmonic and \( v \) is \( q \)-harmonic, and their gradients are orthogonal to each other. This is a generalization of classical Cauchy-Riemann equations and was applied in e.g. [21]. Also it is related to the hodograph transformation, which, for example, was applied to show the sharp Hölder regularity of solutions to certain equations involving the \( p \)-Laplacian; see e.g. [19] and [1]. We also refer to [8, Chapter 16] for more applications of hodograph transformation. Especially, our Lemma 2.3 is partially motivated by the lower estimate on the determinant of the Jacobian of the hodograph transformation (see e.g. [9, Lemma 2.1]).
However, when $p = 1$ or $\infty$ we no longer have such a nice equation system, even though the duality of capacities in Lemma 4.1 still holds. The reason is that a 1-harmonic function may not be continuous; it can even be a summation of several characteristic functions of sets. Then it is not very meaningful to talk about the orthogonality of gradient between 1-harmonic functions and $\infty$-harmonic functions.

Nevertheless, when $u$ is a smooth infinity harmonic function, notice that $|Du|^2$ is constant along the gradient trajectory of $u$ and $D|Du|^2$ is orthogonal to it. Then $|Du|^2$ behaves similar to a dual function of $u$ in the above sense. Indeed, motivated by [12, 14] we have the following observation.

**Proposition 4.2.** Let $U \subset \mathbb{R}^2$ be a domain. If $u \in C^2(U)$ is an $\infty$-harmonic function so that $Du \neq 0$ and $\det D^2u \neq 0$, then $v = \frac{1}{2}|Du|^2$ satisfies the following equation

$$-\text{div} \left( \frac{Du}{|Du|} \right) = \frac{|Du|^2}{2v}. \tag{4.1}$$

The geometric meaning of the equation is that, the mean curvature of the level set of $v$, equivalently that of the gradient trajectory of $u$, is $|D^2uDu|/|Du|^2$.

**Proof.** First of all, by [5, Lemma 2] we know that $u$ is smooth. Then a direct calculation via the equation of $u$ shows that

$$-\langle D^2vDu, Du \rangle = -(u_{ijk}u_i + u_{ij}u_{ik})u_ju_k = |D^2uDu|^2 = |Dv|^2.$$

Since we have assumed that $\det D^2u \neq 0$, then $Dv \neq 0$. By the orthogonality between $Du$ and $Dv$, we then have

$$-2v\langle D^2v\frac{Du}{|Du|^2}, \frac{Du}{|Du|^2} \rangle = |Dv|^2.$$

In the plane we further deduce

$$-|Dv|\text{div} \left( \frac{Du}{|Du|} \right) = \frac{|Dv|^2}{2v}.$$

As $Dv \neq 0$, consequently we conclude the proposition. \hfill $\Box$

However in general (4.1) is not true; one can check that for $w = x_1^{4/3} - x_2^{4/3}$ in any neighborhood of the set where $D|D^2w| = \infty$, i.e. the $x_1$-axis and $x_2$-axis. Indeed, there is another singular term on the right-hand side of (4.1); see below.

**Lemma 4.3.** The function $v = \frac{1}{2}|Dw|^2$ is a weak solution of the equation

$$-\text{div} \left( \frac{Dv}{|Dv|} \right) = \frac{|Dv|}{2v} - 2(\mathcal{H}^1\{x_1=0\} + \mathcal{H}^1\{x_2=0\}) \text{ in } \mathbb{R}^2.$$

**Proof.** For $\phi \in C_c(\mathbb{R}^2)$, write

$$F(\phi) = \int_{\mathbb{R}^2} \langle \frac{Dv}{|Dv|}, D\phi \rangle dx - \frac{|Dv|}{2v} \phi dx.$$

For $\eta \in \mathbb{R}$, write

$$S_\eta = \{(x_1, \eta) : x_1 \in \mathbb{R}^1\}.$$
If \( \{0, x_2| x_2 \in \mathbb{R}\} \) \( \cap \) \( \text{spt} \phi = \emptyset \), by the Green identity and Proposition 4.1, we have
\[
F(\phi) = -2 \lim_{\eta \to 0+} \int_{x_1 \in \mathbb{R}} \frac{v_2}{|D\nu|} \phi \, dx_1.
\]
Note that
\[
v = \frac{4^2}{2.3^2} (x_1^{2/3} + x_2^{2/3})
\]
and
\[
v_2 = \frac{4^2}{3^3} x_2^{-1/3}, |Dv| = \frac{4^2}{3^3} (x_1^{-2/3} + x_2^{-2/3})^{1/2}.
\]
It follows that
\[
F(\phi) = -2 \lim_{\eta \to 0+} \int_{x_1 \in \mathbb{R}} \frac{x_1^{-1/3}}{(x_1^{-2/3} + \eta^{-2/3})^{1/2}} \phi(x_1, \eta) \, dx_1 = -2 \int_{x_1 \in \mathbb{R}} \phi(x_1, 0) \, dx_1
\]
as desired.

If \( \{(x_1, 0)| x_1 \in \mathbb{R}\} \) \( \cap \) \( \text{spt} \phi = \emptyset \), we have similar result. If \( (0, 0) \in \text{spt} \phi = \emptyset \), similarly, we have
\[
F(\phi) = -2 \int_{x_1 \in \mathbb{R}} \phi(x_1, 0) \, dx_1 - 2 \int_{x_2 \in \mathbb{R}} \phi(0, x_2) \, dx_2
\]
as desired. \( \square \)

5. PROOFS OF THE LEMMAS 2.6 TO 2.8

Suppose that \( U \subset \mathbb{R}^2 \) is a bounded domain, and for \( \epsilon \in (0, 1) \), let \( u^\epsilon \in C^\infty(U) \) be a solution to the equation (2.3).

**Lemma 5.1.** For any \( \alpha > 0 \) and \( \xi \in C_\infty(U) \), we have
\[
\int_U |D(|Du^\epsilon|^{2\alpha})|^2 \xi^2 \, dx + \epsilon \int_U |Du^\epsilon|^{2\alpha-4} |\Delta u^\epsilon|^2 \xi^2 \, dx \leq C(\alpha) \int_U |Du^\epsilon|^{2\alpha}(|D\xi|^2 + |D^2 \xi| |\xi|) \, dx,
\]
where the constant \( C \) is absolute.

We obtain Lemma 2.6 as immediate consequence by choosing \( \xi \in C_c(U) \) so that \( \xi = 1 \) on \( U \) with
\[
|D\xi| \leq \frac{2}{\text{dist}(V, \partial U)} \quad \text{and} \quad |D^2 \xi| \leq \frac{C}{[\text{dist}(V, \partial U)]^2}.
\]

**Proof of Lemma 5.1.** The case \( \alpha = 1 \) is already proved in Corollary 2.5 via taking \( \phi = \xi \).

Now we assume that \( \alpha \in (0, 1) \cup (1, \infty) \).

Let \( \phi = (|Du^\epsilon|^2 + \kappa)^{\alpha-1} \xi^2 \) for \( \kappa > 0 \) and \( \xi \in C_\infty(U) \). Then \( \phi \in W^{1,2}_c(U) \). By (2.7), we write
\[
I_\epsilon(\phi) = \int_U |D|Du^\epsilon|(z)|^2 \phi \, dx + \epsilon \int_U \frac{(\Delta u^\epsilon)^2}{|Du^\epsilon|^2} \phi \, dx
\]
\[
= \int_U |D|Du^\epsilon|(z)|^2 (|Du^\epsilon|^2 + \kappa)^{\alpha-1} \xi^2 \, dx + \epsilon \int_U \frac{(\Delta u^\epsilon)^2}{|Du^\epsilon|^2} (|Du^\epsilon|^2 + \kappa)^{\alpha-1} \xi^2 \, dx.
\]
Since
\[ |D|Du|^{(z)}|^2 \geq |D^2u|D(u^{(z)})|^2|D(u)|^2 + \kappa \] \quad and \quad \frac{(\Delta u^\varepsilon)^2}{|D^2u^\varepsilon|^2} \geq (\Delta u^\varepsilon)^2(|D^2u^\varepsilon|^2 + \kappa)^{-1}
almost everywhere we obtain
\[ \mathbb{I}_\varepsilon(\phi) \geq \int_U |D^2u^\varepsilon Du^\varepsilon|^2(|Du^\varepsilon|^2 + \kappa)^{\alpha-2}\xi^2 dx + \varepsilon \int_U (\Delta u^\varepsilon)^2(|Du^\varepsilon|^2 + \kappa)^{\alpha-2}\xi^2 dx. \tag{5.1} \]

On the other hand, note that
\[ \phi_i = 2(\alpha - 1)(|Du^\varepsilon|^2 + \kappa)\alpha^{-2}u^\varepsilon_{ik}u^\varepsilon_{ki} + 2\xi \xi_i(|Du^\varepsilon|^2 + \kappa)^{-1}. \]

Plugging again \( \phi \) in (2.8), we obtain a second expression for \( I_\varepsilon \),
\[ \mathbb{I}_\varepsilon(\phi) = \frac{1}{2} \int_U [\Delta u^\varepsilon u^\varepsilon_i\phi_i - u^\varepsilon_{ij}u^\varepsilon_{ji}\phi_i] dx \]
\[ = - (\alpha - 1) \int_U (|Du^\varepsilon|^2 + \kappa)^{\alpha-2}|D^2u^\varepsilon Du^\varepsilon|^2\xi^2 dx \]
\[ - \int_U (|Du^\varepsilon|^2 + \kappa)^{-1}u^\varepsilon_{ij}u^\varepsilon_{ji}\xi \]
\[ + (\alpha - 1) \int_U (|Du^\varepsilon|^2 + \kappa)^{\alpha-2} \Delta u^\varepsilon u^\varepsilon_{ik}u^\varepsilon_{ki}\xi^2 dx \]
\[ + \int_U (|Du^\varepsilon|^2 + \kappa)^{\alpha-1} \Delta u^\varepsilon u^\varepsilon_{i}\xi \xi dx. \]

Replacing \( \Delta_\infty u^\varepsilon = u^\varepsilon_{ik}u^\varepsilon_{ki} \) by \( -\varepsilon \Delta u^\varepsilon \) in the third term - which we may since \( u^\varepsilon \) satisfies (2.3), we further have
\[ (\alpha - 1) \int_U (|Du^\varepsilon|^2 + \kappa)^{\alpha-2} \Delta u^\varepsilon u^\varepsilon_{ik}u^\varepsilon_{ki}\xi^2 dx = - (\alpha - 1) \varepsilon \int_U (|Du^\varepsilon|^2 + \kappa)^{\alpha-2}(\Delta u^\varepsilon)^2\xi^2 dx. \]

Taking into account (5.1) we conclude that
\[ \alpha \int_U (|Du^\varepsilon|^2 + \kappa)^{\alpha-2}|D^2u^\varepsilon Du^\varepsilon|^2\xi^2 dx + \alpha \int_U \frac{(\Delta u^\varepsilon)^2}{|D^2u^\varepsilon|^2}(|Du^\varepsilon|^2 + \kappa)^{\alpha-1}\xi^2 dx \]
\[ \leq - \int_U u^\varepsilon_{ij}u^\varepsilon_{ji}(|Du^\varepsilon|^2 + \kappa)^{\alpha-1}\xi^2 dx + \int_U \Delta u^\varepsilon u^\varepsilon_{i}\xi (|Du^\varepsilon|^2 + \kappa)^{\alpha-1}\xi dx. \tag{5.2} \]

For the second term of the right hand side of (5.2), via integration by parts we have
\[ \int_U \Delta u^\varepsilon u^\varepsilon_{i}\xi (|Du^\varepsilon|^2 + \kappa)^{\alpha-1}\xi dx = - \int_U u^\varepsilon_{ik}(u^\varepsilon_{i}\xi (|Du^\varepsilon|^2 + \kappa)^{\alpha-1}\xi)_k dx \]
\[ = - \int_U u^\varepsilon_{ik}u^\varepsilon_{ki}(|Du^\varepsilon|^2 + \kappa)^{\alpha-1}\xi dx - \int_U (u^\varepsilon_{i}\xi)^2(\Delta u^\varepsilon + \kappa)^{\alpha-1} dx \]
\[ - \int_U \xi_{ik}u^\varepsilon_{ik}(|Du^\varepsilon|^2 + \kappa)^{\alpha-1}\xi dx - (\alpha - 1) \int_U u^\varepsilon_{i}\xi u^\varepsilon_{jk}u^\varepsilon_{j}(|Du^\varepsilon|^2 + \kappa)^{\alpha-2}\xi dx. \tag{5.3} \]
For the sum of the two first term of the right hand side of (5.2) and (5.3), via integration by parts we have
\[
-2 \int_{U} u_{ij}^\epsilon u_{si}^\epsilon (|Du^\epsilon|^2 + \kappa)^{\alpha-1} \xi \, dx = -\frac{2}{\alpha} \int_{U} [(|Du^\epsilon|^2 + \kappa)^{\alpha}]_{i} \xi \, dx \\
= \frac{2}{\alpha} \int_{U} (|Du^\epsilon|^2 + \kappa)^{\alpha} (|D\xi| + \Delta \xi \cdot \xi) \, dx \\
\leq \frac{2}{\alpha} \int_{U} (|Du^\epsilon|^2 + \kappa)^{\alpha} (|D\xi|^2 + |D^2 \xi| |\xi|) \, dx. \tag{5.4}
\]
Observing the fact that $-\xi_{ik} u_{i}^\epsilon u_{k}^\epsilon \leq |Du^\epsilon|^2 |D\xi|$, for the third term in the right hand side of (5.3) we have
\[
- \int_{U} \xi_{ik} u_{i}^\epsilon u_{k}^\epsilon (|Du^\epsilon|^2 + \kappa)^{\alpha-1} \xi \, dx \leq C \int_{U} (|Du^\epsilon|^2 + \kappa)^{\alpha} |D^2 \xi| |\xi| \, dx.
\]
Noting
\[
u_{ij}^{\epsilon} u_{i}^\epsilon u_{j}^\epsilon u_{k}^\epsilon \leq |D^2 u^\epsilon D u^\epsilon^\epsilon| |D\xi|
\]
and applying Young’s inequality, for the forth term in the right hand side of (5.3) we obtain
\[
-(\alpha - 1) \int_{U} u_{i}^\epsilon \xi_{i} u_{j}^\epsilon u_{k}^\epsilon u_{l}^\epsilon (|Du^\epsilon|^2 + \kappa)^{\alpha-2} \xi^3 \, dx \\
\leq \eta \int_{U} |D^2 u^\epsilon D u^\epsilon^\epsilon^\epsilon|^2 (|Du^\epsilon|^2 + \kappa)^{\alpha-2} \xi^2 \, dx + \frac{\alpha - 1}{4\eta} \int_{U} (|Du^\epsilon|^2 + \kappa)^{\alpha} |D\xi|^2 \, dx.
\]
for $\eta > 0$. We collect all the estimates starting from (5.3)
\[
\int_{U} \Delta u_{i}^\epsilon u_{i}^\epsilon \xi_{i} (|Du^\epsilon|^2 + \kappa)^{\alpha-1} \xi \, dx \\
\leq \eta \int_{U} |D^2 u^\epsilon D u^\epsilon^\epsilon|^2 (|Du^\epsilon|^2 + \kappa)^{\alpha-2} \xi^2 \, dx \\
+ C(\eta, \alpha) \int_{U} (|Du^\epsilon|^2 + \kappa)^{\alpha} (|D\xi|^2 + |D^2 \xi| |\xi|) \, dx
\]
We use the estimate with $\eta = \alpha/2$ and arrive at
\[
\int_{U} (|Du^\epsilon|^2 + \kappa)^{\alpha-2} |D^2 u^\epsilon D u^\epsilon^\epsilon|^2 |\xi|^4 \, dx + \epsilon \int_{U} (|Du^\epsilon|^2 + \kappa)^{\alpha-2} (\Delta u^\epsilon)^2 |\xi|^4 \, dx \\
\leq C(\alpha) \int_{U} (|Du^\epsilon|^2 + \kappa)^{\alpha} (|D\xi|^2 + |D^2 \xi| |\xi|) \xi^2 \, dx
\]
If $\alpha \geq 2$ we conclude, by letting $\kappa \to 0$, that
\[
\int_{U} |D^2 u^\epsilon D u^\epsilon|^2 |Du^\epsilon|^2 |\xi|^4 \, dx + \epsilon \int_{U} (\Delta u^\epsilon)^2 |Du^\epsilon|^2 |\xi|^4 \, dx \\
\leq C(\alpha) \int_{U} |Du^\epsilon|^2 |\xi|^2 (|D\xi|^2 + |D^2 \xi| |\xi|) \, dx.
\]
Since
\[
|D|Du^\epsilon|^{\alpha} = \alpha |Du^\epsilon|^{\alpha-2} |D^2 u^\epsilon|,
\]
we obtain the desired result.

If \( \alpha < 2 \) then \((\Delta u^\epsilon(z))^2|Du^\epsilon(z)|^{2\alpha-4}\) is well-defined by Remark 2.4. Since \( Du^\epsilon(z) = 0 \) implies \( \Delta u^\epsilon(z) = 0 \), we have

\[
(\Delta u^\epsilon)^2 |Du^\epsilon|^{2\alpha-4} = \lim_{\kappa \to 0} (|Du^\epsilon|^2 + \kappa)^{\alpha-2}(\Delta u^\epsilon)^2
\]
almost everywhere in \( U \). Moreover, note that

\[
(|Du^\epsilon|^2 + \kappa)^{\alpha-2} |D^2 u^\epsilon| D u^\epsilon = \frac{1}{\alpha^2} |D(|Du^\epsilon|^2 + \kappa)^{\alpha/2})|^2.
\]
Choosing suitable functions \( \xi \) we deduce that \((|Du^\epsilon|^2 + \kappa)^{\alpha/2} \in W^{1,2}_{\text{loc}}(U)\) with a uniform bound for \( \kappa \in (0,1) \) on compact sets. Since \((|Du^\epsilon|^2 + \kappa)^{\alpha} \to |Du^\epsilon|^\alpha \) almost everywhere as \( \kappa \to 0 \), we deduce that \(|Du^\epsilon|^\alpha \in W^{1,2}_{\text{loc}}(U)\) and this convergence is indeed weakly in \( W^{1,2}_{\text{loc}}(U)\).

Therefore we conclude

\[
\int_U |D(Du^\epsilon)|^2\xi^2 \, dx + \epsilon \int_U |Du^\epsilon|^{2\alpha-4}|D^2 u^\epsilon|^2 \xi^2 \, dx
\]

\[
\leq \lim_{\kappa \to 0} \left\{ \int_U |D(|Du^\epsilon|^2 + \kappa)^{\alpha/2})^2\xi^2 \, dx + \epsilon \int_U (|Du^\epsilon|^2 + \kappa)^{\alpha-2}|D\xi|^2 \xi^2 \, dx \right\}
\]

\[
\leq C(\alpha) \int_U |Du^\epsilon|^{2\alpha}(|D\xi|^2 + |D^2 \xi| |\xi|) \, dx
\]
as desired. \( \square \)

We then show the following uniform flatness estimate.

**Lemma 5.2.** For any \( \xi \in C^\infty_c(U) \) and linear function \( P \), we have

\[
\int_U (|Du^\epsilon|^2 - \langle DP, Du^\epsilon \rangle)^2\xi^2 \, dx
\]

\[
\leq \left[ \int_U |Du^\epsilon|^4(|D\xi|^2 + |D^2 \xi| |\xi|) \, dx \right]^{1/2} \left[ \int_U |u^\epsilon - P|^2(|DP|^2 + |Du^\epsilon|^2)\xi^2 \, dx + \int_U |u^\epsilon - P|^4(|D\xi|^2 + |D^2 \xi| |\xi|) \, dx \right]^{1/2},
\]

where the constant \( C \) is absolute.

Lemma 2.7 follows from Lemma 5.2 via suitable choice of \( \xi \).

**Proof of Lemma 5.2.** Without loss of generality, we may assume that \( P(x) = cx_2 \). Then \( |c| = |DP|, \) \( DP = c\partial_2 \) and \( \langle Du^\epsilon, DP \rangle = cu_2^\epsilon \). Let \( \phi = (u^\epsilon - cx_2)^2\xi^2 \in W^{1,2}_{\text{loc}}(U) \). Since \( -\det D^2 u^\epsilon \geq 0 \) by Lemma 2.3, we have

\[
\mathbb{I}_u(\phi) = \int_U (-\det D^2 u^\epsilon)\phi \, dx \geq 0.
\]

Now

\[
\phi_i = 2(u_i^\epsilon - c\partial_2)\xi^2 + 2(u - cx_2)^2\xi_i.
\]
By (2.8), we obtain
\[
\mathbb{I}_\varepsilon(\phi) = \frac{1}{2} \int_U [\Delta u^\varepsilon u_i^\varepsilon \phi_i - u_{ij}^\varepsilon \phi_i u_j^\varepsilon] \, dx
\]
(5.5)
\[
= \int_U \Delta u^\varepsilon u_i^\varepsilon (u_i^\varepsilon - c\delta_{2i}) \xi^2 (u^\varepsilon - cx_2) \, dx + \int_U \Delta u^\varepsilon u_i^\varepsilon \xi (u^\varepsilon - cx_2)^2 \, dx
\]
\[
- \int_U u_{ij}^\varepsilon (u_i^\varepsilon - c\delta_{2i}) \xi^2 (u^\varepsilon - cx_2) \, dx - \int_U u_{ij}^\varepsilon \xi (u^\varepsilon - cx_2)^2 \, dx.
\]
(5.6)

We apply the Cauchy-Schwartz inequality to the third and forth terms in the right hand side of (5.6)
\[
- \int_U u_{ij}^\varepsilon (u_i^\varepsilon - c\delta_{2i}) \xi^2 (u^\varepsilon - cx_2) \, dx - \int_U u_{ij}^\varepsilon \xi (u^\varepsilon - cx_2)^2 \, dx
\]
\[
\leq \left[ \int_U |D^2 u^\varepsilon D u^\varepsilon|^2 |\xi^2 \, dx \right]^{1/2} \left[ \int_U (u^\varepsilon - cx_2)^2 (|c| + |Du|^2)^2 \, dx + \int_U (u^\varepsilon - cx_2)^4 |D\xi|^2 \, dx \right]^{1/2}
\]
(5.7)

Keep in mind below that \( u_i^\varepsilon (u_i^\varepsilon - c\delta_{2i}) = (|Du|^2 - cu_2^\varepsilon) \). Then via integration by parts, we write the first term in the right hand side of (5.6) as
\[
\int_U \Delta u^\varepsilon u_i^\varepsilon (u_i^\varepsilon - c\delta_{2i}) \xi^2 (u^\varepsilon - cx_2) \, dx = -\int_U u_i^\varepsilon (\Delta u^\varepsilon)^2 (|Du|^2 - cu_2^\varepsilon) \xi^2 (u^\varepsilon - cx_2) \, dx
\]
\[
= -\int_U (|Du|^2 - cu_2^\varepsilon)^2 \xi^2 \, dx
\]
\[
- \int_U u_i^\varepsilon (|Du|^2 - cu_2^\varepsilon) \xi^2 (u^\varepsilon - cx_2) \, dx
\]
\[
- 2 \int_U u_i^\varepsilon (|Du|^2 - cu_2^\varepsilon) \xi \xi (u^\varepsilon - cx_2) \, dx.
\]

By the Cauchy-Schwartz inequality,
\[
- \int_U u_i^\varepsilon (|Du|^2 - cu_2^\varepsilon) \xi \xi (u^\varepsilon - cx_2) \, dx
\]
\[
= -\int_U 2u_i^\varepsilon u_{ij}^\varepsilon \xi^2 (u^\varepsilon - cx_2) \, dx + c \int_U u_{ij}^\varepsilon u_i^\varepsilon \xi^2 (u^\varepsilon - cx_2) \, dx
\]
\[
\leq \left[ \int_U |D^2 u^\varepsilon D u^\varepsilon|^2 |\xi^2 \, dx \right]^{1/2} \left[ \int_U (u^\varepsilon - cx_2)^2 (|c| + |Du|)^2 \xi^2 \, dx \right]^{1/2}
\]

and
\[
- 2 \int_U u_i^\varepsilon (|Du|^2 - cu_2^\varepsilon) \xi \xi (u^\varepsilon - cx_2) \, dx
\]
\[
\leq 2 \left[ \int_U (u_i^\varepsilon \xi)^2 |Du|^2 \, dx \right]^{1/2} \left[ \int_U (u^\varepsilon - cx_2)^2 (|c| + |Du|^2)^2 \xi^2 \, dx \right]^{1/2}.
\]
Therefore, using Lemma 5.1 to estimate \( \int_U |D^2 u |^2 \) we arrive at
\[
\int_U \Delta u \cdot u_i (u_i - c \delta_{2i}) \xi (u - cx) \xi (u - cx) \leq - \int_U (|D u|^2 - c u_{i j}^2) \xi^2 dx
\]
\[+ C \left[ \int_U |D u|^4 (|D \xi|^2 + |D^2 \xi||\xi|) dx \right]^{1/2} \left[ \int_U (u - cx)^2 (|\xi| + |D u|^2) \xi^2 dx \right]^{1/2}. \tag{5.8} \]

Finally, again via integration by parts, for the second term in the right hand side of (5.6) we have
\[
\int_U \Delta u \cdot u_i \xi_i (u - cx) \xi (u - cx) dx = - \int_U u_j \xi_i (u - cx) \xi (u - cx)^2 j dx
\]
\[= - \int_U (u_i \xi_i)^2 (u - cx)^2 dx - \int_U u_j u_{ij} \xi_i \xi (u - cx)^2 dx
\]
\[- \int_U u_j u_i \xi_i \xi (u - cx)^2 dx - 2 \int_U (|D u|^2 - c u_{i j}^2) u_i \xi_i \xi (u - cx) dx. \tag{5.9} \]

By the Cauchy-Schwartz inequality, we have
\[
- \int_U u_j u_{ij} \xi_i \xi (u - cx)^2 dx \leq \left[ \int_U |D^2 u \cdot D u|^2 \xi^2 dx \right]^{1/2} \left[ \int_U (u - cx)^4 |D \xi|^2 dx \right]^{1/2},
\]
\[- \int_U u_j u_i \xi_i \xi (u - cx)^2 dx \leq \left[ \int_U |D u|^4 |D \xi||\xi| dx \right]^{1/2} \left[ \int_U (u - cx)^4 |D \xi||\xi| dx \right]^{1/2},
\]
and
\[
- \int_U (|D u|^2 - c u_{i j}^2) u_i \xi_i \xi (u - cx) dx \leq \frac{1}{4} \int_U (|D u|^2 - c u_{i j}^2)^2 \xi^2 dx + 4 \int_U |D u|^2 |D \xi|^2 (u - cx)^2 dx
\]
\[\leq \frac{1}{4} \int_U (|D u|^2 - c u_{i j}^2)^2 \xi^2 dx + 4 \left[ \int_U |D u|^4 |D \xi|^2 dx \right]^{1/2} \left[ \int_U (u - cx)^4 |D \xi|^2 dx \right]^{1/2}. \]

Thus (since the first term on the right hand side of (5.9) can easily be estimated)
\[
\int_U \Delta u \cdot u_i \xi_i \xi (u - cx)^2 dx \leq \frac{1}{4} \int_U (|D u|^2 - c u_{i j}^2)^2 \xi^2 dx
\]
\[+ C \left[ \int_U |D u|^4 (|D \xi|^2 + |D^2 \xi||\xi|) dx \right]^{1/2} \left[ \int_U (u - cx)^4 (|D \xi|^2 \xi^2 + |D^2 \xi||\xi|) dx \right]^{1/2}. \tag{5.10} \]

Combining (5.6) together with (5.7), (5.8) and (5.10), we complete the proof of Lemma 5.2. \( \square \)

**Lemma 5.3.** Let \( \alpha > 0 \). For any \( \kappa > 0 \) and \( \xi \in C^\infty_c(U) \), we have
\[
\int_U \left[ (|D u|^2 + \kappa)|\xi|^2 \right]^{\alpha + 1} dx + \epsilon \alpha \int_U (|D u|^2 + \kappa)^{\alpha - 2} (\Delta u)^2 |u|^2 \xi^2 dx
\]
\[ \leq C(\alpha) \int_U |u'|^{2\alpha+2}(|D\xi|^2 + |D^2\xi||\xi|)^{\alpha+1} \, dx \]
\[ + \tilde{C}(\alpha) \epsilon \int_U (|Du'|^2 + \kappa)^{\alpha-1}|u'|^2|D\xi|^2\xi^{2\alpha} \, dx \]
\[ + (8\kappa + \tilde{C}(\alpha)\epsilon) \int_U (|Du'|^2 + \kappa)^{\alpha}|\xi|^{2(\alpha+1)} \, dx. \]

Lemma 2.8 follows from Lemma 5.3 by choosing a suitable cut-off functions \( \xi \).

**Proof of Lemma 5.3.** We write the desired inequality as
\[ K_1 + \varepsilon \alpha K_2 \leq C(\alpha)J + C(\alpha)\epsilon E_1 + (8\kappa + \tilde{C}(\alpha)\epsilon)E_2. \] (5.11)
Let \( \phi = (|Du'|^2 + \kappa)^{\alpha-1}|u'|^2|\xi|^{2(\alpha+1)} \). Then \( \phi \in W^{1,2}_c(U) \). By (2.7), we write
\[ I_\epsilon(\phi) = \int_U |D|Du'|(\xi)|^2(|Du'|^2 + \kappa)^{\alpha-1}|u'|^2\xi^{2(\alpha+1)} \, dx \]
\[ + \epsilon \int_U (\Delta u')^2(|Du'|^2 + \kappa)^{\alpha-1}|u'|^2\xi^{2(\alpha+1)} \, dx =: J_1 + J_2. \]

We compute the derivative of \( \phi \),
\[ \phi_i = 2(\alpha - 1)(|Du'|^2 + \kappa)^{\alpha-2}u^j_i u^j_k |u'|^2 \xi^{2(\alpha+1)} \
\[ + 2(\alpha + 1)|\xi|^{2\alpha} \xi_i(|Du'|^2 + \kappa)^{\alpha-1}|u'|^2 + 2(|Du'|^2 + \kappa)^{\alpha-1}u^j_i u^j_k \xi^{2(\alpha+1)}. \]

As above we have by (2.8),
\[ I_\epsilon(\phi) = -(\alpha - 1) \int_U (|Du'|^2 + \kappa)^{\alpha-2}|D^2 u' u' Du'|^2 |u'|^2|\xi|^{2(\alpha+1)} \, dx \]
\[ - (\alpha + 1) \int_U (|Du'|^2 + \kappa)^{\alpha-1}u^j_i u^j_k |\xi|^{2\alpha} \xi_i |u'|^2 \, dx \]
\[ - \int_U (|Du'|^2 + \kappa)^{\alpha-1}u^j_i u^j_k |u'|^2 \xi^{2(\alpha+1)} \, dx \]
\[ + (\alpha - 1) \int_U (|Du'|^2 + \kappa)^{\alpha-2} \Delta u' u^j_i u^j_k |u'|^2 \xi^{2(\alpha+1)} \, dx \]
\[ + (\alpha + 1) \int_U (|Du'|^2 + \kappa)^{\alpha-1} \Delta u' u^j_i \xi_i |\xi|^{2\alpha} |u'|^2 \, dx \]
\[ + \int_U (|Du'|^2 + \kappa)^{\alpha-1} \Delta u' u^j_i u^j_k |\xi|^{2(\alpha+1)} \, dx \]
\[ = I_1 + \cdots + I_6. \]

Notice that
\[ J_1 - I_1 \geq \alpha \int_U (|Du'|^2 + \kappa)^{\alpha-2}|D^2 u' Du'|^2 |u'|^2|\xi|^{2(\alpha+1)} \, dx \geq 0. \]
Since \( u^j_i u^j_k = \Delta_{\infty} u' = -\varepsilon \Delta u' \), we write \( I_4 \) as
\[ I_4 = -\epsilon(\alpha - 1) \int_U (|Du'|^2 + \kappa)^{\alpha-2}(\Delta u')^2 |u'|^{2\alpha} |\xi|^{2(\alpha+1)} \, dx. \]
Hence,

\[ J_2 - I_4 \geq \epsilon \alpha K_2. \]

To complete the proof it suffices to show that

\[ I_2 \leq \frac{1}{8} K_1 + C(\alpha)J \]  \hspace{1cm} (5.12)

\[ I_3 \leq \frac{1}{8} \epsilon \alpha K_2 + C(\alpha)\epsilon E_1 \]  \hspace{1cm} (5.13)

\[ I_5 \leq \frac{1}{4} K_1 + \frac{1}{4} \epsilon \alpha K_2 + C(\alpha)J + C(\alpha)\epsilon E_1 \]  \hspace{1cm} (5.14)

and

\[ I_6 \leq -\frac{7}{8} K_1 + \frac{1}{8} \epsilon \alpha K_2 + C(\alpha)J + (C(\alpha)\epsilon + 4\kappa)E_2. \]  \hspace{1cm} (5.15)

Below we prove (5.12) to (5.15) in order. Recall that by Hölder’s and Young’s inequality we have for \( \frac{1}{p} + \frac{1}{q} = 1 \), \( 1 < p, q < \infty \) and \( \eta > 0 \)

\[ \int fg dx \leq \|f\|_{L^p} \|g\|_{L^q} \leq \eta \|f\|_{L^p}^{\frac{p}{q}} + C(\eta, p) \|g\|_{L^q}^{\frac{q}{p}}, \]  \hspace{1cm} (5.16)

which will be used later for different choice of \( p \) (and hence \( q \)) and \( \eta \).

Write

\[ I_2 = -\left( \alpha + 1 \right) \int_U (|Du|^{2} + \kappa)^{\alpha-1} u_{ij} u_j^i |\xi|^{2\alpha} |\xi| |u^i|^2 dx \]

\[ = -\left( \frac{\alpha + 1}{2\alpha} \right) \int_U ((|Du|^{2} + \kappa)^{\alpha} |\xi|^2 |\xi|^{2\alpha-1} |u^i|^2 dx \]

\[ = \frac{\alpha + 1}{2\alpha} \int_U (|Du|^{2} + \kappa)^{\alpha} |\xi|^2 |\xi|^{2\alpha-1} |u^i|^2 dx \]

\[ \leq \frac{\alpha + 1}{2\alpha} \int_U [||\xi|^2 (|Du|^{2} + \kappa)|^{\alpha} [\Delta \xi + (2\alpha + 1)|D\xi|^2]^{1/2} |u^i|^2 dx \]

\[ + \left( \frac{\alpha + 1}{\alpha} \right) \int_U [(|\xi|^2 (|Du|^{2} + \kappa))^{\alpha+1/2} |D\xi||u^i| dx. \]  \hspace{1cm} (5.17)

Applying (5.16) with \( \eta = \frac{\alpha}{16} \) and \( p = \frac{\alpha+1}{\alpha} \) in the first term of (5.17) or \( p = \frac{\alpha+2}{\alpha+1} \) in the second term of (5.17), we obtain (5.12), that is,

\[ I_2 \leq \frac{1}{8} K_1 + C(\alpha)J. \]

Replacing \( u_{ij}^i u_j^i u^i \) by \(-\epsilon \Delta u^i\) in \( I_3 \) and using (5.16) with \( p = 2 \) and \( \eta = \frac{\alpha}{8} \), we have

\[ I_3 = \epsilon \int_U (|Du|^{2} + \kappa)^{\alpha-1} \Delta u^i u^i \xi^{2(\alpha+1)} dx \]

\[ \leq \frac{1}{8} \epsilon \alpha K_2 + C(\alpha) \epsilon \int_U (|Du|^{2} + \kappa)^{\alpha} \xi^{2(\alpha+1)} dx \]

\[ = \frac{1}{8} \epsilon \alpha K_2 + C(\alpha)\epsilon E_1, \]

which gives (5.13).
By integration by parts, write

\[ I_5 = -(\alpha + 1) \int_U u_j^\epsilon [(|Du^\epsilon|^2 + \kappa)^{\alpha-1} u_i^\epsilon \xi_j \xi^{2\alpha+1} |u^\epsilon|^2]_j \, dx \]

\[ = -2(\alpha + 1)(\alpha - 1) \int_U (|Du^\epsilon|^2 + \kappa)^{\alpha-1} a_j^\epsilon u_i^\epsilon u_j^\epsilon \xi_i \xi_j \xi^{2\alpha+1} |u^\epsilon|^2 \, dx \]

\[ - (\alpha + 1) \int_U u_j^\epsilon u_j^\epsilon \xi_i (|Du^\epsilon|^2 + \kappa)^{\alpha-1} \xi^{2\alpha+1} |u^\epsilon|^2 \, dx \]

\[ - (\alpha + 1) \int_U u_j^\epsilon u_i^\epsilon \xi_j (|Du^\epsilon|^2 + \kappa)^{\alpha-1} \xi^{2\alpha+1} |u^\epsilon|^2 \, dx \]

Replacing \( u_j^\epsilon u_j^\epsilon u_i^\epsilon = \Delta_{\infty} u^\epsilon \) by \( -\epsilon \Delta u^\epsilon \) in \( I_{5,1} \) and using using (5.16) with \( p = 2 \) and \( \eta = \frac{\alpha}{8} \), we have

\[ I_{5,1} = 2(\alpha + 1)(\alpha - 1) \epsilon \int_U (|Du^\epsilon|^2 + \kappa)^{\alpha-2} \Delta u^\epsilon u_i^\epsilon \xi_j \xi^{2\alpha+1} |u^\epsilon|^2 \, dx \]

\[ \leq \frac{1}{8} \epsilon \alpha K_2 + C(\alpha) \epsilon \int_U (|Du^\epsilon|^2 + \kappa)^{\alpha-1} D\xi^2 \xi^{2\alpha} |u^\epsilon|^2 \, dx \]

\[ = \frac{1}{8} \epsilon \alpha K_2 + C(\alpha) \epsilon E_1. \]

Note that \( I_{5,2} = I_2 \) and hence have the same estimate as \( I_2 \) above. Applying (5.16) with \( p = \frac{\alpha+1}{\alpha} \) and \( \eta = \frac{\gamma(2\gamma-1)}{16} \) we have

\[ I_{5,3} = -(\alpha + 1) \int_U u_j^\epsilon u_i^\epsilon \xi_j \xi_i (|Du^\epsilon|^2 + \kappa)^{\alpha-1} \xi^{2\alpha+1} |u^\epsilon|^2 \, dx \leq \frac{1}{16} K_1 + C(\alpha) J. \]

Moreover,

\[ I_{5,4} = -(\alpha + 1)(2\alpha + 1) \int_U \langle Du^\epsilon, D\xi \rangle^2 (|Du^\epsilon|^2 + \kappa)^{\alpha-1} \xi^{2\alpha+1} |u^\epsilon|^2 \, dx \leq 0. \]

By (5.16) with \( p = \frac{2\alpha+2}{2\alpha+1} \) and \( \eta = \frac{1}{8} \) we also have

\[ I_{5,5} \leq 2(\alpha + 1) \int_U (|Du^\epsilon|^2 + \kappa)^{\alpha+1/2} D\xi |\xi|^{2\alpha+1} (|u^\epsilon|^2 + \tau)^{\gamma-1/2} \, dx \leq \frac{1}{16} K_1 + C(\alpha) J. \]

Combining the estimates for \( I_{5,1} \) to \( I_{5,5} \), we conclude (5.14).

By integration by parts, we have

\[ I_6 = - \int_U u_j^\epsilon [(|Du^\epsilon|^2 + \kappa)^{\alpha-1} u_i^\epsilon u_j^\epsilon \xi^{2(\alpha+1)}]_j \, dx \]

\[ = -2 \int_U u_j^\epsilon u_j^\epsilon u_i^\epsilon (|Du^\epsilon|^2 + \kappa)^{\alpha-2} u_i^\epsilon u_j^\epsilon \xi^{2(\alpha+1)} \, dx \]
\[ -2 \int_U u_j^\varepsilon |Du^\varepsilon|^2 + \kappa \alpha^2 u_j^\varepsilon u_i^\varepsilon \xi^{2(\alpha+1)} \, dx \\
- \int_U u_j^\varepsilon |Du^\varepsilon|^2 + \kappa \alpha^2 u_j^\varepsilon u_i^\varepsilon \xi^{2(\alpha+1)} \, dx \\
- 2(\alpha + 1) \int_U u_j^\varepsilon |Du^\varepsilon|^2 + \kappa \alpha \alpha^2 u_i^\varepsilon u_i^\varepsilon \xi^{2(\alpha+1)} \, dx \\
= I_{6,1} + \cdots + I_{6,4}. \]

Replacing \( u_j^\varepsilon u_j^\varepsilon u_i^\varepsilon = \Delta_\infty u^\varepsilon \) by \(-\epsilon \Delta u^\varepsilon\) in \( I_{6,1}, I_{6,2} \) and using Young’s inequality (5.16) with \( p = 2 \) and \( \eta = \alpha/16 \), we have

\[ I_{6,1} + I_{6,2} = 2 \epsilon \int_U \Delta u^\varepsilon (|Du^\varepsilon|^2 + \kappa)^{\alpha-2} |Du^\varepsilon|^2 u^\varepsilon \xi^{2(\alpha+1)} \, dx \]
\[ + 2 \epsilon \int_U \Delta u^\varepsilon (|Du^\varepsilon|^2 + \kappa)^{\alpha-1} u^\varepsilon \xi^{2(\alpha+1)} \, dx \]
\[ \leq \frac{1}{8} \epsilon \alpha K_2 + C(\alpha) \epsilon \int_U (|Du^\varepsilon|^2 + \kappa)^{\alpha} \xi^{2(\alpha+1)} \, dx \]
\[ = \frac{1}{8} \epsilon \alpha K_2 + C(\alpha) \epsilon E_2. \]

Write

\[ I_{6,3} = - \int_U (|Du^\varepsilon|^2 + \kappa)^{\alpha-1} |Du^\varepsilon|^4 \xi^{2(\alpha+1)} \, dx \]
\[ \leq -K_1 + 2 \kappa \int_U (|Du^\varepsilon|^2 + \kappa)^{\alpha} \xi^{2(\alpha+1)} \, dx \]
\[ = -K_1 + 4 \kappa E_2. \]

By (5.16) with \( p = \frac{2\alpha + 2}{2\alpha + 1} \) and \( \eta = \frac{1}{8} \) we have

\[ I_{6,4} \leq 2(\alpha + 1) \int_U (|Du^\varepsilon|^2 + \kappa)^{\alpha+1/2} |D\xi| u^\varepsilon \xi^{2\alpha+1} \, dx \leq \frac{1}{8} K_1 + C(\alpha) J. \]

Combining the estimates for \( I_{6,1} \) to \( I_{6,4} \), we conclude (5.15). This complete the proof of Lemma 5.3. \( \square \)

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References

[1] D. J. Araújo, E. V. Teixeira, J. M. Urbano, A proof of the \( C^{1,1}_p \)-regularity conjecture in the plane. Adv. Math. 316 (2017), 541–553.
[2] G. Aronsson, Minimization problems for the functional sup\( x \) \( F(x, f(x), f'(x)) \). Ark. Mat. 6 (1965), 33–53.
[3] G. Aronsson, Minimization problems for the functional sup\( x \) \( F(x, f(x), f'(x)) \). II. Ark. Mat. 6 (1966), 409–431.
[4] G. Aronsson, Extension of functions satisfying Lipschits conditions. Ark. Mat. 6 (1967), 551–561.
G. Aronsson, On the partial differential equation $ux^2u_{xx} + 2ux_yu_{xy} + u_y^2u_{yy} = 0$. Ark. Mat. 7 1968 395–425 (1968).

G. Aronsson, Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$. III. Ark. Mat. 7 (1969), 599–512.

G. Aronsson, On certain singular solutions of the partial differential equation $u^2xu_{xx} + 2u_xu_yu_{xy} + u_y^2u_{yy} = 0$. Manuscripta Math. 47 (1984), 133-151.

K. Astala, T. Iwaniec, G. Martin, Elliptic partial differential equations and quasiconformal mappings in the plane. Princeton Mathematical Series, 48. Princeton University Press, Princeton, NJ, 2009.

A. H. Baernstein, L. V. Koval’ev, On Hölder regularity for elliptic equations of non-divergence type in the plane. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 4 (2005), no. 2, 295–317.

B. Bojarski and T. Iwaniec, $p$-harmonic equation and quasiregular mappings. Partial Differential Equations (Warsaw 1984), pp. 25-38, Banach Center Publications 19 (1987).

M. G. Crandall, L. C. Evans and R. F. Gariepy, Optimal Lipschitz extensions and the infinity Laplacian. Calc. Var. Partial Differential Equations 13 (2001), 123–139.

L. C. Evans, Estimates for smooth absolutely minimizing Lipschitz extensions. Electron. J. Differential Equations 1993, No. 03, approx. 9 pp.

L. C. Evans, L. C. Three singular variational problems. Viscosity Solutions of Differential Equations and Related Topics. RIMS Kokyuroku 1323. Research Institute for the Mathematical Sciences, 2003.

L. C. Evans, The 1-Laplacian, the $\infty$-Laplacian and differential games. Perspectives in nonlinear partial differential equations, 245–254, Contemp. Math., 446, Amer. Math. Soc., Providence, RI, 2007.

L. C. Evans and O. Savin, $C^{1,\alpha}$ regularity for infinity harmonic functions in two dimensions. Calc. Var. Partial Differential Equations 32 (2008), 325–347.

L. C. Evans and C. K. Smart, Everywhere differentiability of infinity harmonic functions. Calc. Var. Partial Differential Equations 42 (2011), 289–299.

L. C. Evans and C. K. Smart, Adjoint methods for the infinity Laplacian partial differential equation. Arch. Ration. Mech. Anal. 201 (2011), 87–113.

L. C. Evans and Y. Yu, Various properties of solutions of the infinity-Laplace equation. Comm. Partial Differential Equations 30 (2005), 1401–1428.

T. Iwaniec, J. J. Manfredi, Regularity of $p$-harmonic functions on the plane. Rev. Mat. Iberoamericana 5 (1989), 1–19.

R. Jensen, Uniformity of Lipschitz extensions: minimizing the sup norm of the gradient. Arch. Ration. Mech. Anal. 123 (1993), 51–74.

John L. Lewis, Approximation of Sobolev functions in Jordan domains. Ark. Mat. 25 (1987), no. 2, 255–264.

P. Lindqvist and J. J. Manfredi, The Harnack inequality for $\infty$-harmonic functions. Electron. J. Differential Equations 04 (1995), 1–5.

A. S. Romanov, Capacity relations in a planar quadrilateral. [Russian] Sibirsk. Mat. Zh. 49 (2008), no. 4, 886–897; translation in Sib. Math. J. 49 (2008), no. 4, 709–717.

O. Savin, $C^1$ regularity for infinity harmonic functions in two dimensions. Arch. Ration. Mech. Anal. 176 (2005), 351–361.

C. Y. Wang and Y. F. Yu, $C^1$-boundary regularity of planar infinity harmonic functions. Math. Res. Lett. 19 (2012), 823–835.

Y. F. Yu, A remark on $C^2$ infinity-harmonic functions. Electron. J. Differential Equations 122 (2006) 35–60.

W. P. Ziemer, Extremal length and conformal capacity. Trans. Amer. Math. Soc. 126 1967 460–473.

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