LAPLACE TRANSFORMATIONS AND SPECTRAL THEORY OF TWO-DIMENSIONAL SEMI-DISCRETE AND DISCRETE HYPERBOLIC SCHRÖDINGER OPERATORS

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Abstract. We introduce Laplace transformations of 2D semi-discrete hyperbolic Schrödinger operators and show their relation to a semi-discrete 2D Toda lattice. We develop the algebro-geometric spectral theory of 2D semi-discrete hyperbolic Schrödinger operators and solve the direct spectral problem for 2D discrete ones (the inverse problem for discrete operators was already solved by Krichever). Using the spectral theory we investigate spectral properties of the Laplace transformations of these operators. This makes it possible to find solutions of the semi-discrete and discrete 2D Toda lattices in terms of theta-functions.

1. INTRODUCTION

The interest in the transformations

\[ L = \frac{1}{2} (\partial_x + A)(\partial_y + B) + W \mapsto \tilde{L} = \frac{1}{2} W(\partial_y + B)W^{-1}(\partial_x + A) + W, \]

\[ L = \frac{1}{2} (\partial_y + B)(\partial_x + A) + V \mapsto \hat{L} = \frac{1}{2} V(\partial_x + A)V^{-1}(\partial_y + B) + V \]

de the two-dimensional hyperbolic Schrödinger operator \( L = \frac{1}{2} \partial_x \partial_y + F \partial_x + G \partial_y + H \) goes back to Laplace. These transformations act also on the solutions of the equation \( L\psi = 0 \):

\[ L \mapsto \tilde{L}, \quad \psi \mapsto \tilde{\psi} = (\partial_y + B)\psi, \]

\[ L \mapsto \hat{L}, \quad \psi \mapsto \hat{\psi} = (\partial_x + A)\psi. \]

The Laplace transformations are useful in the theory of congruences of surfaces in \( \mathbb{R}^3 \) and they were studied by Darboux, Tzitzéica and others (references and a more extended exposition can be found in the paper [1]). It was remarked already then that the chain of Laplace

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transformations $\ldots, L_{-1}, L_0, L_1, \ldots$, where $L_{i+1} = \tilde{L}_i$, is equivalent to the non-linear equation

$$\frac{1}{2} \partial_x \partial_y g_n = e^{g_{n+1} - g_n} - e^{g_n - g_{n-1}}$$

now called the 2D Toda lattice. Its integrability from different points of view was discovered by Mikhailov \[2\], Fordy and Gibbons \[3\], Leznov and Savel’ev \[4\], Bulgadaev \[5\].

Various generalizations of the Laplace transformations were also studied (a review can be found in the paper \[1\]), among them the Laplace transformations of two-dimensional elliptic Schrödinger operators

$$L = \frac{1}{2} (\bar{\partial} + B)(\partial + A) + V, \quad \partial = \partial_x - i\partial_y,$$

and the Laplace transformations of discrete hyperbolic Schrödinger operators

$$(L\psi)_{n,m} = a_{n,m}\psi_{n,m} + b_{n,m}\psi_{n+1,m} + c_{n,m}\psi_{n,m+1} + d_{n,m}\psi_{n+1,m+1}.$$  

In both cases chains of Laplace transformations are related to the corresponding versions of the 2D Toda lattice. In the case of elliptic Schrödinger operators one of the principal results concerns the description of cyclic chains of Laplace transformations (such that $L_N = L_0$ for some $N$). It was proven by Novikov and Veselov \[6, 7\] that if we consider cyclic chains of Laplace transformations of periodic elliptic Schrödinger operators then the operators in such chains are topologically trivial algebro-geometric operators.

These results were the motivation for this paper. Our goal is to study Laplace transformations of two-dimensional semi-discrete

$$(L\psi)_n = a_n(y)\psi_n(y) + b_n(y)\psi_n'(y) + c_n(y)\psi_{n+1}(y) + d_n(y)\psi_{n+1}'(y)$$

and discrete \[2\] hyperbolic Schrödinger operators. To the best of the authors’ knowledge, the Laplace transformations of semi-discrete operators were not yet studied. We start by introducing Laplace transformations of semi-discrete Schrödinger operators and showing their relation to a semi-discrete 2D Toda lattice. Then we study spectral properties of the Laplace transformation of periodic operators.

Let us recall that the algebro-geometric spectral theory of 2D Schrödinger operators was introduced in 1976 by Dubrovin, Krichever and Novikov \[8\]. In this theory periodic 2D Schrödinger operators are considered. It turns out that the Floquet solution of the equation $L\psi = 0$ is a Baker-Akhiezer function on a spectral curve in the space of Floquet multipliers, and one can reconstruct the operator starting from its geometric spectral data including the spectral curve, the divisor of
poles of \( \psi \) etc. Later Novikov and Veselov studied the case of potential operators [9].

The inverse spectral problem for the discrete operator (2) was studied by Krichever [10]. As for the direct spectral problem for the operator (2), it was in fact implicitly studied by one of the authors in the paper [11]. Indeed, the direct spectral problem for 2D discrete elliptic operators with zero potential was in fact solved in [11] by reducing to the direct spectral problem for hyperbolic operators (2). We will show how to solve the direct spectral problem for operators (2) in this paper explicitly.

As far as we know, the algebro-geometric spectral theory of semi-discrete operators (3) was never studied. We develop the algebro-geometric spectral theory of 2D semi-discrete Schrödinger operators. The direct spectral problem is studied using Floquet theory of periodic first order linear ODEs, and this seems to be new. Despite the fact that an arbitrary periodic first order linear ODE cannot be solved explicitly, it turns out that using Floquet theory for linear ODEs we can obtain enough spectral information to understand the structure of the spectral data.

Using these algebro-geometric spectral theories we investigate the spectral properties of the Laplace transformations of semi-discrete (3) and discrete (2) operators. In both cases the Laplace transformations are described as shifts on the Jacobians of the spectral curves. This makes it possible to find solutions of the semi-discrete and discrete 2D Toda lattices in terms of theta-functions. We recall that solutions of the hyperbolic 2D Toda lattice (1) in term of theta-functions were found by Krichever [12].

It turns out that in the discrete case we can give a description of a cyclic chain of Laplace transformations in terms of the spectral data and, consequently, in terms of the linearizability of the dynamics.

2. LAPLACE TRANSFORMATIONS AND 2D TODA LATTICES

In this section we start by introducing the Laplace transformations of semi-discrete operators (3). We show their relation to a semi-discrete 2D Toda lattice. Then we recall briefly known results on the Laplace transformations of discrete operators (2).

2.1. Laplace transformations of two-dimensional semi-discrete hyperbolic Schrödinger operators. Let us consider an operator of the form (3) defined on the space of (in general complex-valued) functions \( \psi_n(y) = \psi(n, y) \) defined on \( \mathbb{Z} \times \mathbb{R} \). The coefficients \( a_n(y), \ldots, d_n(y) \) of the operator are also (in general complex-valued) functions on \( \mathbb{Z} \times \mathbb{R} \).
Let us define a shift operator $T\psi_n(y) = \psi_{n+1}(y)$.

**Lemma 1.** The operator [3] such that $b_n(y) \neq 0$ and $d_n(y) \neq 0$ can be uniquely presented in the form

$$L = f_n(y)((\partial_y + A_n(y))(1 + v_n(y)T) + w_n(y)).$$

or in the form

$$L = \hat{f}_n(y)((1 + \hat{v}_n(y)T)(\partial_y + \hat{A}_n(y)) + \hat{w}_n(y)).$$

**Proof.** Let us write for simplicity $\partial$ instead of $\partial_y$, $a_n$ instead of $a_n(y)$ etc. We will use $'$ as the derivation with respect to $y$. We can obtain by direct calculation that

$$a_n = f_nA_n + f_n w_n, \quad b_n = f_n, \quad c_n = f_n v'_n + f_n A_nv_n, \quad d_n = f_nv_n.$$

These equations can be easily solved if $b_n \neq 0$, $d_n \neq 0$:

$$f_n = b_n, \quad v_n = \frac{d_n}{b_n}, \quad A_n = \frac{c_n}{d_n} - \left(\log \frac{d_n}{b_n}\right)', \quad w_n = \frac{a_n}{b_n} \frac{c_n}{d_n} + \left(\log \frac{d_n}{b_n}\right)'.$$

In the same way we can see that if $b_n \neq 0$, $d_n \neq 0$ then

$$\hat{f}_n = b_n, \quad \hat{v}_n = \frac{d_n}{b_n}, \quad \hat{A}_n = \frac{c_{n-1}}{d_{n-1}}, \quad \hat{w}_n = \frac{a_n}{b_n} - \frac{c_{n-1}}{d_{n-1}}.$$

This completes the proof. □

In the following we will consider only operators satisfying the conditions $b_n \neq 0$ and $d_n \neq 0$.

Let us consider the equation $L\psi = 0$. Using Lemma 1 we can define the Laplace transformations.

**Definition 1.** Let us define a Laplace transformation of the first type as the transformation

$$L \mapsto \bar{L} = \bar{f}_n(w_n(1 + v_nT)\frac{1}{w_n}(\partial + A_n) + w_n), \quad \psi \mapsto \bar{\psi} = (1 + v_nT)\psi,$$

and a Laplace transformation of the second type as the transformation

$$L \mapsto \hat{L} = \hat{f}_n(\hat{w}_n(\partial + \hat{A}_n)\frac{1}{\hat{w}_n}(1 + \hat{v}_nT) + \hat{w}_n), \quad \psi \mapsto \hat{\psi} = (\partial + \hat{A}_n)\psi.$$

Here $\bar{f}_n(y)$ and $\hat{f}_n(y)$ are arbitrary functions.

Since $\bar{f}_n$ and $\hat{f}_n$ are arbitrary functions, these transformations define $\bar{L}$ and $\hat{L}$ up to a transformation $L \mapsto h_nL$. But the Laplace transformations are well-defined transformations of the equation $L\psi = 0$.

There are gauge transformations $L \mapsto \bar{L}$, $\psi \mapsto \bar{\psi}$ giving equivalent equations, they are defined by the formulas

$$\bar{\psi} = g_n^{-1}\psi, \quad \bar{a}_n = h_n a_n g_n + h_n b_n g_n', \quad \bar{b}_n = h_n b_n g_n,$$
\[ \bar{c}_n = h_n c_n g_{n+1} + h_n d_n g_{n+1}', \quad \bar{d}_n = h_n d_n g_{n+1}, \]

where \( h_n(y) \neq 0 \) and \( g_n(y) \neq 0 \) are arbitrary functions.

Let us remark that a gauge transformation \((L, \psi) \mapsto (L', \psi')\) such that \( L = L' \) is just a multiplication of \( \psi \) by a constant. For this reason we usually impose the normalization condition \( \psi_0(0) = 1 \).

As we will see in the next section, for a given periodic operator its normalized \( \psi \)-function with prescribed Floquet multipliers is unique. Hence a pair \((L, \psi)\) is uniquely defined by \( L \) in this case, and this case is the most interesting for us.

For this reason we will view both Laplace transformations and gauge transformations as transformations of operators \( L \) rather than transformations of pairs \((L, \psi)\) consisting of an operator \( L \) and its \( \psi \)-function.

**Lemma 2.** Laplace transformations of gauge equivalent operators are gauge equivalent.

**Lemma 3.** For each operator there exists a unique gauge equivalent operator such that

\[ b_n(y) \equiv 1, \quad d_n(y) \equiv 1. \]  

**Proofs** of both Lemmas can be obtained by a direct calculation. □

**Lemma 4.** The Laplace transformations of the first and the second type are inverse to each other (as transformations of gauge equivalence classes).

**Proof.** Let us take a gauge equivalence class and take the unique operator satisfying the condition \( \square \) in this class as its representative.

For this operator \( f_n = v_n = 1, \ A_n = c_n \) and \( w_n = a_n - c_n \). Let us now find its Laplace transformation of the first type. If we take \( \tilde{f}_n = 1 \) then we obtain an operator with \( \tilde{a}_n = a_n, \ \tilde{b}_n = 1, \ \tilde{c}_n = c_{n+1} \frac{a_n - c_n}{a_{n+1} - c_{n+1}} \) and \( \tilde{d}_n = \frac{a_n - c_n}{a_{n+1} - c_{n+1}} \). For this operator \( \tilde{f}_n = 1, \ \tilde{v}_n = -\frac{a_n - c_n}{a_{n+1} - c_{n+1}}, \ \tilde{A}_n = c_n \) and \( \tilde{w}_n = a_n - c_n \). Let us now find its Laplace transformation of the second type. If we take \( \tilde{f}_n = 1 \) then we obtain an operator with \( \tilde{a}_n = a_n - (\log(a_n - c_n))', \ \tilde{b}_n = 1, \ \tilde{c}_n = \left( -\frac{a_n - c_n}{a_{n+1} - c_{n+1}} \right)' + c_{n+1} \frac{a_n - c_n}{a_{n+1} - c_{n+1}} - \frac{a_n - c_n}{a_{n+1} - c_{n+1}} (\log(a_n - c_n))', \ \text{and} \ \tilde{d}_n = \frac{a_n - c_n}{a_{n+1} - c_{n+1}} \). The gauge transformation with \( g_n = c_n - a_n \) and \( h_n = \frac{1}{c_n - a_n} \) transforms this operator into the initial one. □

It follows from Lemma \( \square \) that we can restrict ourselves to the Laplace transformations of the first type. Given a gauge equivalence class, let us choose \( A_n \) and \( w_n \) of the unique operator in this class satisfying the condition \( \square \) as gauge invariants. They give a complete set of
invariants, since if the operator satisfies the condition (4) then $f_n = 1$ and $v_n = 1$.

**Lemma 5.** In terms of the gauge invariants the Laplace transformation of the first type acts in the following way:

$$\tilde{A}_n(y) = A_{n+1}(y) + \frac{w'_{n+1}(y)}{w_{n+1}(y)},$$

$$\tilde{w}_n(y) = w_n(y) + A_n(y) + \frac{w'_n(y)}{w_n(y)} - A_{n+1}(y) - \frac{w'_{n+1}(y)}{w_{n+1}(y)}.$$

**Proof** can be obtained by direct calculation. □

Let us now consider a chain $\ldots, L_{-1}, L_0, L_1, \ldots$ of Laplace transformations of the first type, $L_{k+1} = \tilde{L}_k$. We obtain the system of equations

$$A_{n+1}^k = A_n^k + (\log w_{n+1}^k)' ,$$

$$w_{n+1}^k = w_n^k + A_n^k + (\log w_n^k)' - A_{n+1}^k - (\log w_{n+1}^k)' ,$$

where $A_n^k$ and $w_n^k$ are the gauge invariants of $L_k$. This system is equivalent to the system

$$(5) \quad (\log w_{n+1}^k(y))' = A_{n+1}^k(y) - A_n^k(y),$$

$$(6) \quad w_{n+1}^k(y) - w_n^k(y) = A_{n+1}^k(y) - A_n^k(y).$$

This system includes only differences of $A_n^k$, hence we should also fix some $A_0^k$ in order to find all $A_n^k$ from equations (5), (6). Let us now eliminate $A_n^k$. Let us remark that given $A_{n+1}^k$, we can find $A_n^k$ in two different ways: 1) Find $A_{n+1}^k$ using (5), then find $A_n^k$ using (6), or 2) Find $A_n^k$ using (6), then find $A_{n+1}^k$ using (5). Since the result should be the same, this gives us the compatibility condition

$$(7) \quad w_{n+1}^k - w_n^k - (w_{n+1}^k - w_{n+1}^{k-1}) = (\log w_n^k)' - (\log w_{n+1}^k)' .$$

Given a solution of the equation (7) and some fixed $A_{n_0}^k$, we can find all the $A_n^k$ using (5), (6). In this way solving our equations (5), (6) reduces to solving the equation (7).

**Theorem 1.** 1) Given a solution of the following semi-discrete 2D Toda lattice

$$(8) \quad (g_n^k - g_{n+1}^k)' = e^{g_{n+1}^k - g_n^k} - e^{g_{n}^k - g_{n+1}^k}$$

we can obtain a family of chains of Laplace transformations of the first type parameterized by one arbitrary function $A_0^k(y)$.

2) Given a chain of Laplace transformations of the first type we can obtain a family of solutions of the equation (8) parameterized by an arbitrary function $g_0^k(y)$ and a set of arbitrary constants $r^k, k \in \mathbb{Z}$. 


Proof. If we take the equation (3), subtract the same equation but with the shift $k \mapsto k - 1$, $n \mapsto n + 1$, and make the change of variables $w_n^k = e^{g_n^k - g_{n+1}^{k+1}}$, then we obtain the equation (7). As we explained before, given a solution of the equation (7) and an arbitrary function $A_0^n(y)$, we can construct a solution of the equations (5), (6) describing a chain of Laplace transformations. This proves the first statement of the theorem.

Given a solution $w_n^k(y)$ of the equation (7), we can construct $g_n^k$ starting from $g_0^0(y)$ using the equations

$$g_n^k(y) - g_{n+1}^{k+1}(y) = \log w_n^k(y),$$

$$g_n^k(y) - g_{n+1}^{k+1}(y) = \int_0^y (w_n^{k+1}(y') - w_n^k(y')) dy' + c_n^k,$$

where $c_n^k$ are constants. The solutions $g_n^k$ of the equations (9, 10) are clearly solutions of the equation (8). However, there is a compatibility condition

$$c_{n+1}^{k-1} = c_n^k + \int_0^y (w_n^{k+1}(y') - w_{n+1}^k(y') - w_n^k(y') + w_{n+1}^{k+1}(y')) dy' + \log \frac{w_{n+1}^k(y)}{w_n^k(y)}$$

for the equations (9, 10). This means that we can take arbitrary constants $c_0^k = r^k$; other constants $c_n^k$ are defined by this compatibility condition. This proves the second statement of the Theorem. □

Let us now describe possible modifications in the periodic case. Let us consider a periodic operator with some periods $N$ and $T$, i.e. such that $a_{n+N}(y) = a_n(y)$, $a_n(y + T) = a_n(y)$, ...

As was proven in Lemma 3, any operator is gauge equivalent to an operator satisfying the conditions (4). It turns out that this operator is, in general, non-periodic.

We could consider periodic gauge transformations, i.e. such that $h_{n+N}(y) = h_n(y)$, $h_n(y + T) = h_n(y)$, $g_{n+N}(y) = g_n(y)$, $g_n(y + T) = g_n(y)$, in order to preserve the periodicity of operators by gauge transformations. Let us now consider only periodic gauge transformations. Lemma 3 does not hold in this case, it is replaced by the following Lemma.

Lemma 6. The function $I(y) = \frac{b_1(y) \cdots b_N(y)}{d_1(y) \cdots d_N(y)}$ is a gauge invariant. Let $Z(y)$ be a function such that $Z^N(y) = I(y)$. We can transform a periodic operator by a periodic gauge transformation into a unique periodic operator such that for any $i$

$$b_i(y) = Z(y), \quad d_i(n) = 1.$$
Proof can be obtained by direct calculation. □

For an operator satisfying the conditions (11) we have $f_n = Z, v_n = \frac{1}{Z}$. We take $A_n$ and $w_n$ (as before) and $I$ as gauge invariants. The choice of $Z(y)$ is not unique, but $A_n$ and $w_n$ do not depend on this choice.

We can write now how the Laplace transformation acts in terms of $A_n, w_n$ and $Z$. We consider only the Laplace transformation of the first type.

**Lemma 7.** The function $I$ is preserved by the Laplace transformation.

Proof can be obtained by direct calculation. □

Lemma 2 holds in the periodic case. Hence we can always replace an operator by a gauge equivalent one.

Let us take a periodic operator. Let us choose $Z$ such that $Z^N = I$ and transform our operator to the operator satisfying the conditions (11). Let us apply to this operator the Laplace transformation. The invariant $I$ is preserved. We transform the resulting operator by a gauge transformation into the operator satisfying (11) with the same $Z$. This gives us a transformation of $A_n$ and $w_n$. We obtain the following Lemma

**Lemma 8.** The Laplace transformation acts in the following way:

$$
\tilde{A}_n(y) = A_{n+1}(y) + \frac{w'_{n+1}(y)}{w_{n+1}(y)} + (\log Z(y))',
$$

$$
\tilde{w}_n(y) = w_n(y) + A_n(y) + \frac{w'_{n}(y)}{w_{n}(y)} - A_{n+1}(y) - \frac{w'_{n+1}(y)}{w_{n+1}(y)} - (\log Z(y))'.
$$

The compatibility condition for these equations is given by the equation (7).

Proof can be obtained by a direct calculation. □

This means that in the periodic case we obtain the same equation (7) describing the compatibility condition, as in the general case. The only difference is that $w_n^k(y)$ should be periodic in $n$ and $y$.

2.2. Laplace transformations of two-dimensional discrete hyperbolic Schrödinger operators. We recall here briefly some already known results following the paper [1], where one can find proofs and a more extended exposition. Let us consider the shift operators

$$ T_1 \psi(n, m) = \psi(n + 1, m), \quad T_2 \psi(n, m) = \psi(n, m + 1) $$

acting on functions defined on $\mathbb{Z}^2$. We can rewrite the operator (2) as

$$ L\psi = (a_{n,m} + b_{n,m}T_1 + c_{n,m}T_2 + d_{n,m}T_1 T_2)\psi. $$
We are interested in the equation
\[ L\psi = (a_{n,m} + b_{n,m}T_1 + c_{n,m}T_2 + d_{n,m}T_1T_2)\psi = 0. \]

There are gauge transformations
\[ L \mapsto f_{n,m}Lg_{n,m}, \quad \psi_{n,m} = g^{-1}_{n,m}\psi_{n,m} \]
giving equivalent equations.

The operator (12) can be presented uniquely in the form
\[ L = f_{n,m}((1 + u_{n,m}T_1)(1 + v_{n,m}T_2) + w_{n,m}). \]

The operator (12) can also be presented uniquely in the form
\[ L = f'_{n,m}((1 + v'_{n,m}T_2)(1 + u'_{n,m}T_1) + w'_{n,m}). \]

We can define a Laplace transformation of the first type
\[ L \mapsto \tilde{L} = \tilde{f}_{n,m}(w_{n,m}(1 + v_{n,m}T_2)w_{n,m}^{-1}(1 + u_{n,m}T_1) + w_{n,m}), \]
\[ \psi \mapsto \tilde{\psi} = (1 + v_{n,m}T_2)\psi \]
and we can define a Laplace transformation of the second type
\[ L \mapsto L' = f'_{n,m}(w'_{n,m}(1 + u'_{n,m}T_1)(w'_{n,m})^{-1}(1 + v'_{n,m}T_2) + w'_{n,m}), \]
\[ \psi \mapsto \psi' = (1 + u'_{n,m}T_1)\psi. \]

These transformations transform gauge equivalent operators into gauge equivalent operators, i.e. they act on gauge equivalence classes. The transformations of the first and the second type are inverse to each other (as transformations of gauge equivalence classes). It follows that we can restrict ourselves to the Laplace transformations of the first type.

Let us introduce gauge invariants in the following manner. We can transform an operator by a gauge transformation to an operator such that\( f_{n,m} = 1 \). Then we can take \( w_{n,m} \) and \( H_{n,m} = \frac{w_{n,m}u_{n,m}+1}{u_{n,m}w_{n,m}+1} \) of this operator as gauge invariants.

In terms of the gauge invariants the Laplace transformation of the first type has the form
\[ 1 + \tilde{w}_{n+1,m} = (1 + w_{n+1,m})\frac{w_{n,m}w_{n+1,m+1}H_{n,m}^{-1}}{w_{n+1,m}w_{n,m+1}}, \]
\[ \tilde{H}_{n,m} = \frac{1 + w_{n,m+1}}{1 + \tilde{w}_{n+1,m}}. \]

Let us now consider a chain of Laplace transformations of the first type; we obtain \( H^{(k+1)}_{n,m} = \tilde{H}^{(k)}_{n,m}, \quad w^{(k+1)}_{n,m} = \tilde{w}^{(k)}_{n,m} \). After excluding \( H^{(k)}_{n,m} \) from
equations (17), (18) we obtain the so-called completely discretized 2D Toda lattice

\[
\frac{1 + w_{n+1,m}^{(k+2)}}{1 + w_{n+1,m}^{(k+1)}} \frac{1 + w_{n,m}^{(k+2)}}{1 + w_{n,m}^{(k+1)}} = \frac{w_{n+1,m}^{(k)} w_{n,m}^{(k+1)}}{w_{n,m}^{(k)} w_{n+1,m}^{(k+1)}}.
\]

We defined the Laplace transformation of the first type using the representation (15) where we used the shift \( T_1 \) and then the shift \( T_2 \). Let us denote this transformation by \( \Lambda_{12}^{++} \). The Laplace transformation of the second type was defined using the representation (16) where we used the shift \( T_2 \) and then the shift \( T_1 \). Let us denote this transformation by \( \Lambda_{21}^{++} \). We can however take any pair of orthogonal shifts \( T_i^\pm, T_j^\pm, i \neq j \). Hence we can introduce in an analogous way Laplace transformations corresponding to any pair of orthogonal shifts \( (T_1, T_2), (T_2, T_1) \rightarrow \Lambda_{12}^{++}, \Lambda_{21}^{++} \); \( (T_1, T_2^{-1}), (T_2, T_1^{-1}) \rightarrow \Lambda_{12}^{-+}, \Lambda_{21}^{-+} \); \( (T_1^{-1}, T_2), (T_2^{-1}, T_1) \rightarrow \Lambda_{12}^{+-}, \Lambda_{21}^{+-} \). It is easy to see that \( \Lambda_{st}^{12} \Lambda_{21}^{ts} = 1 \) for \( s, t = \pm \).

Let us also introduce a transformation \( S_1 \):

\[
L \mapsto \tilde{L} = \tilde{f}_{n,m}(a_{n-1,m} + b_{n-1,m} T_1 + c_{n-1,m} T_2 + d_{n-1,m} T_1 T_2),
\]

\[
\psi_{n,m} \mapsto \tilde{\psi}_{n,m} = \psi_{n-1,m}
\]

and a transformation \( S_2 \):

\[
L \mapsto \tilde{L} = \tilde{f}_{n,m}(a_{n,m-1} + b_{n,m-1} T_1 + c_{n,m-1} T_2 + d_{n,m-1} T_1 T_2),
\]

\[
\psi_{n,m} \mapsto \tilde{\psi}_{n,m} = \psi_{n,m-1}
\]

It is clear that \( S_k \) commutes with \( \Lambda_{ij}^{st} \). As usual, we consider all transformations as transformations of gauge equivalence classes. A direct calculation leads us to the following Lemma.

**Lemma 9.** The following identities hold:

\[
\Lambda_{12}^{++} = S_1 \Lambda_{12}^{++}, \quad \Lambda_{12}^{++} = S_2 \Lambda_{12}^{++}, \quad \Lambda_{12}^{++} = S_2 S_1 \Lambda_{12}^{--}
\]

This means that the group of transformations generated by \( \Lambda_{ij}^{st} \) has three generators.

### 3. Algebro-geometric spectral theory of two-dimensional semi-discrete and discrete hyperbolic Schrödinger operators

In this section we start by an investigation of the algebro-geometric spectral theory of semi-discrete operators (3). Then we consider an algebro-geometric theory of discrete operators (2). We recall known results on the inverse spectral problem for discrete operators (2) due to
Krichever [10]. We investigate the direct spectral problem for discrete operators. This was already done implicitly by one of the authors in the paper [11].

3.1. Algebro-geometric spectral theory of two-dimensional semi-discrete hyperbolic Schrödinger operators. Let us consider operators \( L \) of the form (3) and the corresponding equations \( L \psi = 0 \).

3.1.1. Direct spectral problem. Let us consider a periodic operator \( L \), i.e. such that the functions \( a_n(y), \ldots, d_n(y) \) satisfy the conditions \( a_{n+N}(y) = a_n(y), a_n(y+T) = a_n(y), \ldots \). We will also consider only periodic gauge transformations.

We will consider only periodic operators such that the gauge invariant \( I(y) \) is a constant. As any operator is gauge equivalent to the operator satisfying the conditions (11), we will consider only the operators satisfying these conditions. Operators with different \( I \) are not equivalent, but their spectral theory is the same. We will consider for simplicity the case of \( Z = -1 \), i.e. we consider periodic operators of the form

\[
(L \psi)_n = a_n(y) \psi_n(y) - \psi'_n(y) + c_n(y) \psi_{n+1}(y) + \psi'_{n+1}(y)
\]

and the corresponding equations \( L \psi = 0 \).

**Definition 2.** Let \( \rho \) and \( \mu \) be two complex numbers. A solution \( \psi_n(y) \) of the equation \( L \psi = 0 \) is said to be a Floquet solution with Floquet multipliers \( \rho \) with respect to \( n \) and \( \mu \) with respect to \( y \), if for any \( n \) and \( y \) we have

\[
\psi_{n+N}(y) = \rho \psi_n(y), \quad \psi_n(y+T) = \mu \psi_n(y).
\]

Since \( \psi_n(y) \) is defined up to multiplication by a constant, we will impose the normalization condition \( \psi_0(0) = 1 \).

Our first goal is to describe possible pairs \((\rho, \mu)\) of Floquet multipliers. Let us fix some Floquet multiplier \( \rho \). It follows from (21) that any \( \psi_n(y) \) can be expressed using \( \rho \) and \( \psi_0(y), \ldots, \psi_{N-1}(y) \). Thus the equation \( L \psi = 0 \) is equivalent to a finite number of linear ODEs on \( \psi_0(y), \ldots, \psi_{N-1}(y) \). It is easy to see that \( L \psi = 0 \) is equivalent to the equation \( B \Psi'(y) + C(y, \rho) \Psi(y) = 0 \), where

\[
\Psi(y) = \begin{pmatrix} \psi_0(y) \\ \vdots \\ \psi_{N-1}(y) \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ \rho & 0 & 0 & \cdots & 0 & -1 \end{pmatrix},
\]

and

\[
C(y, \rho) = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ \rho & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.
\]
\[ C(y, \rho) = \begin{pmatrix}
  a_0 & c_0 & 0 & \ldots & 0 & 0 \\
 0 & a_1 & c_1 & \ldots & 0 & 0 \\
 0 & 0 & 0 & \ldots & a_{N-2} & c_{N-2} \\
 \rho c_{N-1} & 0 & 0 & \ldots & 0 & a_{N-1}
\end{pmatrix}. \]

It follows that the equation \( L\psi = 0 \), where \( \psi \) has a Floquet multiplier \( \rho \), is equivalent to the linear ODE

\[
(22) \quad \Psi'(y) = A(y, \rho)\Psi(y),
\]

where \( A(y, \rho) = -B^{-1}(\rho)C(y) \). The Floquet multiplier \( \rho \) enters in this linear ODE as a parameter. It is easy to check that

\[
B^{-1} = \frac{1}{\rho - 1} \begin{pmatrix}
  1 & 1 & \ldots & 1 & 1 \\
  \rho & 1 & \ldots & 1 & 1 \\
  \ldots & \ldots & \ldots & \ldots & \ldots \\
  \rho & \rho & \ldots & \rho & 1
\end{pmatrix}.
\]

Thus for \( \rho \neq 1 \) the function \( A(y, \rho) \) is holomorphic with respect to \( \rho \). Let us remark that \( A(y, \rho) \) is periodic: \( A(y + T, \rho) = A(y, \rho) \). We will occasionally omit \( y \) to shorten the notation.

**Definition 3.** Let \( \mu \) be a complex number. A solution \( \Psi(y) \) of the equation \( \Psi'(y) = A\Psi(y) \) is said to be a Floquet solution with Floquet multiplier \( \mu \) if for any \( y \) we have

\[
(23) \quad \Psi(y + T) = \mu \Psi(y).
\]

We see that the question of describing possible Floquet multipliers \( \rho \neq 1, \mu \) for the equation \( L\psi = 0 \) can be restated in the following way: given \( \rho \neq 1 \), for which \( \mu \) does there exist a Floquet solution of the periodic equation \( \Psi' = A(\rho)\Psi \)? This permits us to use the Floquet theory of periodic linear differential equations.

Let us recall some standard facts from this theory, see e.g. [13]. Let us consider a homogeneous linear ODE

\[
(24) \quad \frac{dx(t)}{dt} = A(t)x(t),
\]

where \( x \in \mathbb{R}^n \) and \( A(t) \) is an \( n \times n \)-matrix. An \( n \times n \)-matrix \( \Phi(t, s) \) is called a resolvent of \( A(t) \) if \( \phi(t) = \Phi(t, t_0)x_0 \) is the solution of \( (24) \) satisfying the initial condition \( \phi(t_0) = x_0 \). The resolvent exists and is uniquely determined by the following properties:

\[
\forall t \quad \Phi(t, t) = I, \quad \forall s, t, u \quad \Phi(t, s) = \Phi(t, u)\Phi(u, s),
\]

\[
(25) \quad \frac{\partial}{\partial t}\Phi(t, s) = A(t)\Phi(t, s), \quad \frac{\partial}{\partial s}\Phi(t, s) = -\Phi(t, s)A(s).
\]
If \( \hat{\Phi}(t) \) is a fundamental matrix of the equation (24) (i.e. a matrix such that its columns form a basis in the space of solutions of this equation) then

\[
\Phi(t, s) = \hat{\Phi}(t)\hat{\Phi}^{-1}(s).
\]

Let us now consider an inhomogeneous linear ODE

\[
\frac{dx(t)}{dt} = A(t)x(t) + b(t).
\]

The solution \( \phi(t) \) of this equation satisfying the initial condition \( \phi(t_0) = x_0 \), can be found with the help of the resolvent of \( A(t) \):

\[
\phi(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s)b(s)\, ds.
\]

Let the matrix \( A(t) \) be periodic: \( A(t + T) = A(t) \). It follows that for any \( n \in \mathbb{N} \), \( t \) we have \( \Phi(t + nT, 0) = \Phi(t, 0)\Phi(nT, 0) \). This implies the following Lemma.

**Lemma 10.** [13] A solution \( g(t) \) of the periodic equation (24) is a Floquet solution with a Floquet multiplier \( \mu \) if and only if the initial condition \( g_0 = g(0) \) is an eigenvector of the matrix \( \Phi(T, 0) \) with the eigenvalue \( \mu \):

\[
\Phi(T, 0)g_0 = \mu g_0.
\]

Let \( L \) be a generic operator. Consider the equation (22). It follows from Lemma 10 that given \( \rho \neq 1 \), we have \( N \) Floquet multipliers \( \mu_1, \ldots, \mu_N \) (possibly coinciding) corresponding to independent Floquet solutions. Let us now recall that for \( \rho \neq 1 \) the matrix \( A(y, \rho) \) is holomorphic with respect to \( \rho \). It follows that solutions of the equation (22) are also holomorphic functions of \( \rho \). The formula (26) expressing \( \Phi(t, s) \) in terms of a fundamental matrix implies that the resolvent \( \Phi(t, s, \rho) \) of \( A(y, \rho) \) is also a holomorphic function of \( \rho \). We obtain the following Lemma.

**Lemma 11.** Let \( L \) be a generic periodic operator. The possible pairs \( (\rho, \mu) \) of Floquet multipliers of the equation \( L\psi = 0 \), such that \( \rho \neq 1 \), form an analytic curve \( \tilde{\Gamma} \), called a spectral curve. This curve is given by the equation \( \det(\Phi(T, 0, \rho) - \mu I) = 0 \). The natural projection \( \pi : (\rho, \mu) \mapsto \rho \) gives us an \( N \)-fold covering \( \pi : \tilde{\Gamma} \longrightarrow \mathbb{C} \setminus \{1\} \).
Let us now consider the eigenvectors $\Psi(0)$ of the matrix $\Phi(T,0,\rho)$. Since the matrix $\Phi(T,0,\rho)$ is holomorphic with respect to $\rho$, its eigenvectors $\Psi(0)$ (let us recall that we imposed the condition $(\Psi(0))_0 = \psi_0(0) = 1$) are meromorphic functions on the spectral curve $\tilde{\Gamma}$.

**Lemma 12.** The solution $\psi_n(y)$ of the equation $L\psi = 0$ is a meromorphic function on the spectral curve $\tilde{\Gamma}$. Its poles does not depend on $y$.

**Proof** follows from the fact that $\Psi(y) = \Phi(y,0,\rho)\Psi(0)$ and $\Phi(y,0,\rho)$ is a holomorphic function in $\rho$. □

We will consider $\psi_n(y)$ as a function defined on the spectral curve and depending on the parameters $n$ and $y$. Sometimes we will write explicitly $\psi_n(y,P)$, where $P$ is a point on the spectral curve.

Let us consider the simplest case when the coefficients $a_n(y), c_n(y)$ of the operator (22) do not depend on $y$. In this case the matrix $A(y,\rho)$ does not depend on $y$ and we can easily solve the equation (22): $\Psi(y) = e^{A(y,\rho)T}\Psi(0)$. It follows that the equation of the spectral curve $\tilde{\Gamma}$ is $\det(e^{A(y,\rho)T} - \mu I) = 0$. In this case it is better to use other coordinates, one can consider a curve $\det(A(y,\rho) - zI) = 0$ and consider $\mu = e^{zT}$ as a function on this curve.

In the general case we cannot solve the equation (22) and find the spectral curve explicitly. However, we can obtain enough information about the spectral curve and the solution $\psi_n(y)$ in order to understand what kind of spectral data we should consider in the inverse spectral problem.

For negative $n$ we define a pole of order $n$ as a zero of order $-n$ and a zero of order $n$ as a pole of order $-n$.

Let us make the following observation: if $\rho = 0$ then the matrix $A(y,\rho)$ is an upper-triangular matrix. This permits us to prove the following Lemma

**Lemma 13.** The fiber $\pi^{-1}(0)$ of the projection $\pi : \tilde{\Gamma} \to \mathbb{C} \setminus \{1\}$ consists of points $P_i^+ = (0, e^{\int_0^\tau a_i-1(\xi)d\xi})$, $i = 1, \ldots, N$. For a generic operator $L$ the function $\psi_n(y)$ has a zero of order $\left[\frac{n-1}{N}\right] + 1$ at the point $P_i^+$.

If there are coinciding points $P_i^+$ we should treat these points with multiplicities, i.e. we should add the corresponding orders of zero.

**Proof.** We will consider for simplicity the case $N = 2$. The proof is analogous for $N > 2$.

We have

$$A(y,0) = \begin{pmatrix} a_0(y) & a_1(y) + c_0(y) \\ 0 & a_1(y) \end{pmatrix}.$$
Let us put \( \alpha = a_0, \beta = a_1 + c_0, \gamma = a_1 \) in order to shorten the notation. Then our equation (22) for \( \rho = 0 \) becomes the equation

\[
(29) \quad \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}' = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix},
\]

The second equation \( \psi'_1(y) = \gamma(y)\psi_1(y) \) is easy to solve: \( \psi_1(y) = e^{\int_0^y \gamma(\xi)\,d\xi}\psi_1(y_0) \). This means that \( \hat{\Phi}(t, s) = e^{\int_t^s \gamma(\xi)\,d\xi} \) is a resolvent of \( \gamma(y) \). Let us now consider the first equation in (29):

\[
(30) \quad \psi'_0(y) = \alpha(y)\psi_0(y) + \beta(y)\psi_1(y).
\]

The resolvent of \( \alpha(y) \) is \( \hat{\Phi}(t, s) = e^{\int_t^s \alpha(\xi)\,d\xi} \). Using \( \hat{\Phi}(t, s) \) and the formula (28) we can solve the equation (30):

\[
\psi_0(y) = e^{\int_0^y \alpha(\xi)\,d\xi}\psi_0(y_0) + \int_{y_0}^y e^{\int_\eta^y \alpha(\xi)\,d\xi} \beta(\eta)\psi_1(\eta)\,d\eta = e^{\int_0^y \alpha(\xi)\,d\xi}\psi_0(y_0) + \int_{y_0}^y e^{\int_\eta^y \alpha(\xi)\,d\xi + \int_0^\eta \gamma(\xi)\,d\xi} \beta(\eta)\,d\eta \psi_1(y_0).
\]

These explicit formulas for \( \psi_i(y) \) at \( \rho = 0 \) give immediately an explicit formula for \( \Phi(T, 0, 0) \):

\[
(31) \quad \Phi(T, 0, 0) = \begin{pmatrix} e^{\int_0^T \alpha(\xi)\,d\xi} & \int_0^T e^{\int_\eta^T \alpha(\xi)\,d\xi + \int_0^\eta \gamma(\xi)\,d\xi} \beta(\eta)\,d\eta \\ 0 & e^{\int_0^T \gamma(\xi)\,d\xi} \end{pmatrix}.
\]

This formula gives us explicitly that the fiber \( \pi^{-1}(0) \) of the projection \( \pi : \tilde{\Gamma} \mapsto \mathbb{C} \setminus \{1\} \) consists of points \( P_1^+ = (0, e^{\int_0^T \alpha(\xi)\,d\xi}) \), and \( P_2^+ = (0, e^{\int_0^T \gamma(\xi)\,d\xi}) \).

To shorten the notation, let us denote the matrix elements of \( \Phi(T, 0, 0) \) by \( \mu_1, \mu_2 \) and \( \delta \), i.e.

\[
\Phi(T, 0, 0) = \begin{pmatrix} \mu_1 & \delta \\ 0 & \mu_2 \end{pmatrix}.
\]

Let us consider the generic case where \( \mu_1 \neq \mu_2 \) and \( \delta \neq 0 \). The matrix \( \Phi(T, 0, 0) \) has two eigenvectors (normalized as usual by the condition \( \psi_0(0) = 1 \))

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ \frac{\mu_2 - \mu_1}{\delta} \end{pmatrix}
\]

with the eigenvalues \( \mu_1 \) and \( \mu_2 \) respectively.

Let us now consider a neighborhood of \( \rho = 0 \). Since \( \Phi(T, 0, \rho) \) is holomorphic and hence can be expanded in a power series in \( \rho \), the
normalized eigenvectors can also be expanded in a series in powers of $\rho$:

\[
\begin{pmatrix} 1 \\ C_1 \rho + \ldots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{\mu_2 - \mu_1}{\delta} + C_2 \rho + \ldots \end{pmatrix},
\]

where $C_1$ and $C_2$ are constants.

In the generic case $\mu_1 \neq \mu_2$, i.e. $\rho = 0$ is not a branching point. It follows that $\rho$ is a local parameter in a neighborhood of $P_1^+ = (0, \mu_1)$ and $P_2^+ = (0, \mu_2)$.

We obtain that in a neighborhood of $P_1^+$

\[
\psi_0(0) = 1, \quad \psi_1(0) = C_1 \rho + \ldots,
\]

\[
\psi_2(0) = \rho \psi_1(0) = \rho, \quad \psi_3(0) = \rho^2 + \ldots, \quad \ldots
\]

i.e. $\psi_0(0)$ has a zero of order 0 in $P_1^+$, $\psi_1(0)$ has a zero of order 1 in $P_1^+$, $\psi_2(0)$ has a zero of order 1 in $P_1^+$, $\psi_3(0)$ has a zero of order 2 in $P_1^+$, ...

Let us now recall that $\Phi(y, 0, 0)$ is an upper-triangular matrix. It follows that

\[
\Phi(y, 0, \rho) = \begin{pmatrix} C_1 + \ldots & C_2 + \ldots \\ \rho C_3 + \ldots & C_4 + \ldots \end{pmatrix},
\]

where $C_1, \ldots, C_4$ do not depend on $\rho$. Then the function $\psi_0(y) = C_1 \psi_0(0) + C_2 \psi_1(0) = C_1 + \ldots$ has a zero of the same order in $P_1^+$, as the function $\psi_0(0)$. The same argument works for any $\psi_i(y)$. We see that the orders of zeroes of $\psi_i(y)$ in $P_1^+$ are as described in the statement of this Lemma. The proof for $P_2^+$ is analogous. □

In general the number of branching points of the covering $\pi : \tilde{\Gamma} \to \mathbb{C} \setminus \{1\}$ is infinite. We will say that an operator is algebro-geometric if this covering has only a finite number of branching points. Let us now consider only this case. We will show that in this case we can compactify $\tilde{\Gamma}$ and obtain a compact Riemann surface $\Gamma$ and a covering $\pi : \Gamma \to \mathbb{CP}^1$.

**Lemma 14.** Let $L$ be a generic algebro-geometric operator. Then we can add to $\tilde{\Gamma}$ points $P_i^+ = (\infty, e^{-f_i^0 c_{N-i}(\xi)} d\xi)$, $i = 1, \ldots, N$, in such a way that we obtain an analytic curve $\tilde{\Gamma}$ with a projection $\pi : \tilde{\Gamma} \to \mathbb{CP}^1 \setminus \{1\}$. The function $\psi_n(y)$ can be continued to all of $\tilde{\Gamma}$. The function $\psi_n(y)$ has a pole of order $\left\lfloor \frac{n-1}{N} \right\rfloor + 1$ at the point $P_i^+$.

**Proof.** Let us compactify the $\rho$-plane $\mathbb{C}$ by adding the point $\infty$ at infinity with local parameter $t = \frac{1}{\rho}$. It is easy to check that the matrix $A(y, \rho)$ has a limit when $\rho$ tends to infinity. We can analytically continue $A(y, \rho)$ to a holomorphic function on $\mathbb{CP}^1 \setminus \{1\}$. Moreover, the matrix $A(y, \infty)$ is a lower-triangular matrix. We can prove in
the same way as in Lemma 13 that the Floquet multipliers at $\infty$ are $e^{-\int_{\mathcal{C}N-1(\xi)}^{T} \xi^T d\xi}$, $i = 1, \ldots, N$. Since our operator is algebro-geometric, there is a neighborhood of $\infty$ without branching points. Using this fact we can add points $P_i^-$ in order to obtain an analytic curve $\hat{\Gamma}$. If all Floquet multipliers are different, we add these points with the local parameter $t$. If there are coinciding $P_i^-$, we add them with the local parameter equal to the root of $t$ of degree corresponding to the multiplicity of these Floquet multipliers. The calculation of the order of poles of $\psi_n(y)$ in these points can be done in the same way as in Lemma 13. □

Lemma 15. Let $L$ be a generic algebro-geometric operator. Then we can add the point $(1,1)$ and a point $Q$ to $\hat{\Gamma}$ in such a way that we obtain an analytic compact Riemann surface $\Gamma$ with a projection $\pi : \Gamma \to \mathbb{C}P^1$. The function $\psi_n(y)$ can be continued to all of $\Gamma$. The function $\psi_n(y)$ is a Baker-Akhiezer function on $\Gamma \setminus Q$ and has an exponential singularity at the point $Q$. The function $\psi_n(y)$ can be presented in a neighborhood of $Q$ as $e^{Kt} h_n(y,t)$, where $K$ is a constant, $t$ is a local parameter at $Q$ and $h_n(y,t)$ is a function holomorphic with respect to $t$, such that $h(y,0) \neq 0$. The Floquet multiplier $\mu$ in a neighborhood of $Q$ can be presented as $e^{\frac{M}{t} g(t)}$, where $M$ is a constant and $g(t)$ is a holomorphic function, such that $g(0) \neq 0$.

Proof. We will consider for simplicity the case $N = 3$. The proof is analogous for $N \neq 3$.

The matrix $A(y,\rho)$ has a pole at $\rho = 1$. Let $t = \rho - 1$, then

$$A(y,t) = \frac{1}{t} A_{-1}(y) + A_0(y),$$

where

$$A_{-1}(y) = - \begin{pmatrix} a_0(y) + c_2(y) & a_1(y) + c_0(y) & a_2(y) + c_1(y) \\ a_0(y) + c_2(y) & a_1(y) + c_0(y) & a_2(y) + c_1(y) \\ a_0(y) + c_2(y) & a_1(y) + c_0(y) & a_2(y) + c_1(y) \end{pmatrix}.$$ 

The matrix $A(y,t)$ tends to $\frac{1}{t} A_{-1}(y)$ as $t$ tends to zero, thus the solution of the equation (22) tends to the solution of the equation

$$\Psi'(y) = \frac{1}{t} A_{-1}(y) \Psi.$$ 

The key observation is that the matrix $A_{-1}(y)$ has the same rows. It follows that $\psi'_0(y) - \psi'_1(y) = 0$, $\psi'_0(y) - \psi'_2(y) = 0$. This implies that
\[ \psi_0'(y) - \psi_1'(y) \text{ and } \psi_0'(y) - \psi_2'(y) \text{ are constants that can be expressed in terms of the initial conditions. It follows that} \]
\[ \psi_1(y) = \psi_0(y) - \psi_0(0) + \psi_1(0), \quad \psi_2(y) = \psi_0(y) - \psi_0(0) + \psi_2(0). \]

We substitute these formulas in the equation for \( \psi_0'(y) \). Let us put
\[ E(y) = -a_0(y) - c_2(y), \quad F(y) = -a_1(y) - c_0(y), \quad G(y) = -a_2(y) - c_1(y) \]
in order to shorten the notation. We obtain the equation
\[ \psi_0'(y) = \frac{E(y) + F(y) + G(y)}{t} \psi_0(y) - \frac{F(y) + G(y)}{t} \psi_0(0) + \frac{F(y)}{t} \psi_1(0) + \frac{G(y)}{t} \psi_2(0). \]
This is a first-order linear ODE which can be explicitly solved. Using its solution we can find \( \psi_1(y) \) and find explicitly the resolvent \( \Phi(y, 0) \).

Let us introduce the notation
\[ P(y) = \int_0^y e^{\int_{E(t) + F(t) + G(t)}} d\xi + \int_0^y e^{\int_{E(t) + F(t) + G(t)}} d\xi - \frac{F(\eta) - G(\eta)}{t} d\eta, \]
\[ Q(y) = \int_0^y e^{\int_{E(t) + F(t) + G(t)}} d\xi \frac{F(\eta)}{t} d\eta, \]
\[ R(y) = \int_0^y e^{\int_{E(t) + F(t) + G(t)}} d\xi \frac{G(\eta)}{t} d\eta. \]

We obtain
\[ \Phi(T, 0) = \begin{pmatrix} P(T) & Q(T) & R(T) \\ P(T) - 1 & Q(T) + 1 & R(T) \\ P(T) - 1 & Q(T) & R(T) + 1 \end{pmatrix}. \]

This matrix has eigenvalues \( \mu_1 = 1, \mu_2 = 1 \) and \( \mu_3 = P(T) + Q(T) + R(T) \). We see that \( \mu_3 \) tends to the infinity as \( t \) tends to zero. We can add to the curve \( \Gamma \) the point \((1, 1)\) with a local parameter \( \sqrt{t} \) on the branches where \( \mu \) tends to 1 and a point \( Q \) with a local parameter \( t \) on the branch where \( \mu \) tends to the infinity. We obtain an analytic compact Riemann surface \( \Gamma \) and a holomorphic projection \( \pi : \Gamma \to \mathbb{C}P^1 \). One can easily verify that \( \psi_\mu(y) \) can be continued as a meromorphic function to the point \((1, 1)\).

Let us now consider the point \( Q \). The eigenvector corresponding to \( \mu_3 \) is \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \). Using formulas for \( P, Q \) and \( R \) we see that for the solution of the equation (32) we have the formula
\[ \begin{pmatrix} \psi_0(y) \\ \psi_1(y) \\ \psi_2(y) \end{pmatrix} = \begin{pmatrix} P(y) & Q(y) & R(y) \\ P(y) + 1 & Q(y) + 1 & R(y) \\ P(y) - 1 & Q(y) & R(y) + 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \]
\[ = e^{\int_0^y E(t) + F(t) + G(t)} d\xi \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \]
We see that the solution $\psi_n(y)$ of the equation (22) in a neighborhood of $Q$ has the same behavior as $e^{\int_0^y {K(y)+G+G(y)}} a_\xi$ as $t$ tends to zero. It is easy to see that $E + F + G = -(c_0 + c_1 + c_2 - a_0 + a_1 + a_2)$. Let $K = -(c_0(0) + c_1(0) + c_2(0) - a_0(0) + a_1(0) + a_2(0))$. We see that the solution $\psi_n(y)$ of the equation $L\psi = 0$ has the same behavior as $e^{K(y)} h(y,t)$, where $h$ is a holomorphic with respect to $t$ function such that $h(y,0) \neq 0$ for the generic operator $L$. It is easy to see that the Floquet multiplier $\mu$ in a neighborhood of $Q$ has the same behavior as $\mu_3 = e^{\int_0^T E(y)+F(y)+G(y)} d\xi$. This implies that $\mu$ can be presented as $e^{\int_0^T Q(y,t)}$, where $M$ is a constant and $g(t)$ is a holomorphic function such that $g(t) \neq 0$. □

Further we will consider only generic algebro-geometric operators.

Let us now study the divisor $\mathcal{D}(n,y)$ of the poles of $\psi_n(y)$ on $\Gamma \setminus \{Q, P_{i}^\pm\}$. As it was proven in Lemma 12, $\mathcal{D}(n,y)$ does not depend on $y$, thus we will write $\mathcal{D}(n)$.

We need to consider the formal adjoint operator $L^+$ of $L$. It is easy to see that

$$(L^+\psi^+) = a_n\psi^+_n + (\psi^+_n)' + c_{n-1}\psi^+_{n-1} - (\psi^+_{n-1})'.$$

Let us also consider the adjoint equation $L^+\psi^+ = 0$. We can as before consider Floquet solutions of this equation normalized by the condition $\psi^+_0(0) = 1$ and see that $\psi^+_n(y)$ is a Baker-Akhiezer function on the corresponding spectral curve.

**Lemma 16.** The spectral curves of the equations $L\psi = 0$ and $L^+\psi^+ = 0$ are isomorphic. We can identify them and consider $\psi^+$ as a function on $\Gamma$, in the sense that a point $P = (\rho, \mu)$ corresponds to $\psi$ with Floquet multipliers $\rho$ and $\mu$, and to $\psi^+$ with Floquet multipliers $\frac{1}{\rho}$ and $\frac{1}{\mu}$. In a neighborhood of the point $Q$ the function $\psi^+_n(y)$ can be presented as $e^{-K(y)} h^+(y,t)$, where $K$ is the same constant and $t$ is the same local parameter at $Q$ as in Lemma 13, and $h^+(y,t)$ is a holomorphic with respect to $t$ function such that $h^+(y,0) \neq 0$.

**Proof.** As was explained above, if $\psi$ is a Floquet solution then the equations $L\psi = 0$ can be rewritten as $A(y,\rho)\psi(y)$, where $A(y,\rho) = -B^{-1}(\rho)C(y,\rho)$. In the same way if $\psi^+$ is a Floquet solution with the Floquet multiplier $\frac{1}{\rho}$ then the adjoint equation $L^+\psi^+ = 0$ can be written as $(\Psi^+(y))' = A^+(y,\rho)\Psi^+(y)$, where $A^+(y,\rho) = (B^{-1}(\rho))^T(C(y,\rho))^T$. 

\[\text{LAPLACE TRANSFORMATIONS AND SPECTRAL THEORY 19}\]
Let $\Phi(y, 0)$ and $\Phi^+(y, 0)$ be the resolvents of $A$ and $A^+$. We know that
\begin{equation}
\frac{d}{dy} \Phi(y, 0) = A(y)\Phi(y, 0), \quad \frac{d}{dy} \Phi^+(y, 0) = A^+(y)\Phi^+(y, 0).
\end{equation}

Let $Q(y) = B^{-1}(\Phi^+(y, 0))^T B \Phi(y, 0)$. It is easy to see that $Q(0) = I$. It follows from (33) that
\begin{equation}
\frac{d}{dy} Q(y) = B^{-1}(\Phi^+(y, 0))^T C B^{-1} B \Phi(y, 0) - B^{-1}(\Phi^+(y, 0))^T B B^{-1} C \Phi(y, 0) = 0.
\end{equation}

This implies that $Q(y) \equiv I$. Hence $B^{-1}(\Phi^+(y, 0))^T B$ is the inverse of $\Phi(T, 0)$. This means that the eigenvalues of $\Phi^+(T, 0)$ and $\Phi^+(y, 0)$ are inverse to each other. This implies that the spectral curves of the equations $L \psi = 0$ and $L^+ \psi^+ = 0$ are isomorphic. We can think that $\psi^+$ is a function on $\Gamma$ in the sense that a point $P = (\rho, \mu)$ corresponds to $\psi$ with Floquet multipliers $\rho$ and $\mu$, and to $\psi^+$ with Floquet multipliers $\frac{1}{\rho}$ and $\frac{1}{\mu}$.

The statement about the behavior of $\psi^+$ in a neighborhood of $Q$ can be proven in the same way as in Lemma 15. □

**Lemma 17.** For any $k$ and $y$ the following identity holds
\begin{equation}
\frac{d\rho}{\rho} \sum_{n=k+1}^{k+N} (\psi_n(y)\psi_n^+(y) - \psi_n(y)\psi_n^-(y)) = \frac{d\mu}{\mu} \int_y^{y+T} [c_k(y')\psi_{k+1}(y')\psi_k^+(y') + \psi_{k+1}'(y')\psi_k^+(y')] dy'.
\end{equation}

It is easy to see that this differential does not depend on $y$ since on the right hand side we integrate an expression periodic in $y'$. For an analogous reason this differential does not depend on $k$. We will denote this differential by $\Omega$. Let us define
\begin{equation}
R_\rho = \sum_{n=k+1}^{k+N} (\psi_n(y)\psi_n^+(y) - \psi_n(y)\psi_n^-(y)),
\end{equation}
and
\begin{equation}
R_\mu = \int_y^{y+T} [c_k(y')\psi_{k+1}(y')\psi_k^+(y') + \psi_{k+1}'(y')\psi_k^+(y')] dy'.
\end{equation}

Thus $\Omega = \frac{d\rho}{\rho R_\rho} = \frac{d\mu}{\mu R_\mu}$.

**Proof of the Lemma.** Let us multiply our equation
\begin{equation}
a_n \psi_n(y, P) - \psi_n'(y, P) + c_n \psi_{n+1}(y, P) + \psi_n'(y, P) = 0,
\end{equation}

by \( \psi_n^+(y, \tilde{P}) \), and subtract the adjoint equation
\[
a_n \psi_n^+(y, \tilde{P}) + (\psi_n^+(y, \tilde{P}))' + c_{n-1} \psi_{n-1}^+(y, \tilde{P}) - (\psi_{n-1}^+(y, \tilde{P}))' = 0,
\]
multiplied by \( \psi_n(y, P) \). In the following formulas we will presume that \( \psi_n \) is taken at the point \( P \), and \( \psi_n^+ \) is taken at the point \( \tilde{P} \).

Let us sum the resulting formula
\[
-(\psi_n \psi_n^+) + c_n \psi_{n+1}^+ - c_{n-1} \psi_n \psi_{n-1}^+ + \psi_n^' + \psi_{n+1}^+ + \psi_{n-1}^+ \psi_n^+ = 0
\]
from \( k + 1 \) to \( k + N \). We can rewrite the resulting formula using the fact that \( \psi \) and \( \psi^+ \) are Floquet solutions. We obtain
\[
- \sum_{n=k+1}^{k+N} (\psi_n \psi_n^+) + \left( \frac{\rho(P)}{\rho(\tilde{P})} - 1 \right) c_k \psi_{k+1} \psi_k^+ + \\
+ \sum_{n=k+1}^{k+N} (\psi_n \psi_{n-1}^+) + \left( \frac{\rho(P)}{\rho(\tilde{P})} - 1 \right) \psi_{k+1}^+ \psi_k^+ = 0.
\]
We integrate this formula and use again the fact that \( \psi \) and \( \psi^+ \) are Floquet solutions. We obtain
\[
- \left( \frac{\mu(P)}{\mu(\tilde{P})} - 1 \right) \sum_{n=k+1}^{k+N} \psi_n \psi_n^+ + \left( \frac{\rho(P)}{\rho(\tilde{P})} - 1 \right) \int_{y}^{y+T} c_k \psi_{k+1} \psi_k^+ dy' + \\
+ \left( \frac{\mu(P)}{\mu(\tilde{P})} - 1 \right) \sum_{n=k+1}^{k+N} \psi_n \psi_{n-1}^+ + \left( \frac{\rho(P)}{\rho(\tilde{P})} - 1 \right) \int_{y}^{y+T} \psi_{k+1}^+ \psi_k^+ dy' = 0.
\]
Taking a limit when \( \tilde{P} \) tends to \( P \) we obtain the formula in the statement. \( \square \)

**Lemma 18.** The functions \( R_\rho \) and \( R_\mu \) have no common zeroes in \( \Gamma \setminus \{Q, P_\rho^\pm\} \).

**Proof** is similar to the proof of the previous Lemma. Let us consider the curves \( a_n(y, t), c_n(y, t), \rho(t), \mu(t), \psi_n(y, t) \) such that \( \psi_n(y, t) \) is a solution of the equation \( L(t) \psi = 0 \) with Floquet multipliers \( \rho(t) \) and \( \mu(t) \) and \( a_n(y, 0) = a_n(y) \) etc.

Let us multiply
\[
a_n(y, t) \psi_n(y, t) - \psi_n'(y, t) + c_n(y, t) \psi_{n+1}(y, t) + \psi_{n+1}'(y, t) = 0,
\]
by \( \psi_n^+(y) \), and subtract the adjoint equation
\[
a_n(y) \psi_n^+(y) + (\psi_n^+(y))' + c_{n-1}(y) \psi_{n-1}^+(y) - (\psi_{n-1}^+(y))' = 0,
\]
multiplied by $\psi_n(y, t)$. Let us sum and integrate as before. Taking the derivative with respect to $t$ at $t = 0$ we obtain

$$
\sum_{n=k+1}^{k+N} \int_y^{y+T} \left[ \frac{\partial a_n}{\partial t}(y', 0)\psi_n(y')\psi_n^+(y') + \frac{\partial c_n}{\partial t}(y', 0)\psi_{n+1}(y')\psi_{n+1}^+(y') \right] dy' - \frac{\partial \mu}{\partial t}(0) \int_y^{y+T} [\psi_n(y)\psi_n^+(y) - \psi_n(y)\psi_{n-1}^+(y)] \, dy' = 0.
$$

If $R_\rho$ and $R_\mu$ have a common zero then in this point

$$
\sum_{n=k+1}^{k+N} \int_y^{y+T} \left[ \frac{\partial a_n}{\partial t}(y', 0)\psi_n(y') + \frac{\partial c_n}{\partial t}(y', 0)\psi_{n+1}(y') \right] \psi_n^+(y') \, dy' = 0
$$

for any $\frac{\partial a_n}{\partial t}(y, 0) \frac{\partial c_n}{\partial t}(y, 0)$. It follows that for any $n$ and $y$ we have $\psi_n^+(y) = 0$, but this contradicts the normalization condition $\psi_0^+(0) = 1$. □

It follows from Lemma 15 that $\frac{d\mu}{\mu}$ has a pole of order 2 at $Q$. Let us consider $\Omega = \frac{d\mu}{\mu R_\mu}$, where $R_\mu$ is written for example for $k = 0$ and $y = 0$:

$$
R_\mu = \int_0^T [c_0(y')\psi_1(y') + \psi_1'(y')]\psi_0^+(y') \, dy'.
$$

It follows from the behavior of $\psi$ and $\psi^+$ at the point $Q$ described in Lemmas 13 and 14 that $\Omega$ has a pole of first order at $Q$.

From Lemmas 13 and 14 we know the structure of poles and zeroes of $\psi_n$ at the points $P_i^\pm$. We can prove in the same way that $\psi_n^+(y)$ has no zeroes or poles at $P_i^\pm$. It follows that in these points $\Omega$ has only a first order pole at $P_i^+$.

It is easy to see that $\frac{d\rho}{\rho}$ and $\frac{d\mu}{\mu}$ have no poles in $\Gamma \setminus \{Q, P_i^\pm\}$. It follows from Lemma 18 that $\Omega$ is holomorphic in $\Gamma \setminus \{Q, P_i^\pm\}$.

For a generic operator the functions $a_n\psi_n - \psi_n'$ and $c_{n-1}\psi_n + \psi_n'$ have in $\Gamma \setminus \{Q, P_i^\pm\}$ the same poles as $\psi_n$. It follows from the equation $L\psi = 0$ that $\mathcal{D}(n)$ does not depend on $n$ and we will write $\mathcal{D}$. We can give the same argument with the divisor $\mathcal{D}^+$ corresponding to $\psi^+$. Since for a generic operator $d\rho$ and $d\mu$ have no common zeroes, we obtain the following Lemma.

**Lemma 19.** $(\Omega) = -Q - P_i^+ + \mathcal{D} + \mathcal{D}^+$. 
Let \( g \) be the genus of \( \Gamma \). Since the canonical class has degree \( 2g - 2 \), we see that \( D \) is an effective divisor of degree \( g \).

We can now summarize the results of this Section in the following Theorem.

**Theorem 2.** Let \( L \) be a generic algebro-geometric operator. Then its spectral curve \( \Gamma \) is a compact Riemann surface. Let \( g \) be the genus of \( \Gamma \). The Floquet solution \( \psi_n(y) \) is a Baker-Akhiezer function on \( \Gamma \). There are points \( P^\pm_i, i = 1, \ldots, N \), such that \( \psi_n(y) \) has a zero of order \( \left\lceil \frac{n-i}{N} \right\rceil + 1 \) at \( P^+_i \) and a pole of the same order at \( P^-_i \). There is a point \( Q \) such that the function \( \psi_n(y) \) is meromorphic in \( \Gamma \setminus \{Q\} \). The function \( \psi_n(y) \) can be presented in a neighborhood of \( Q \) as \( e^{-\frac{y}{K}} h_n(y, t) \), where \( K \) is a constant, \( t \) is a local parameter at \( Q \) and \( h_n(y, t) \) is a function that is holomorphic with respect to \( t \), such that \( h(y, 0) \neq 0 \). The divisor \( D \) of poles of \( \psi_n(y) \) in \( \Gamma \setminus \{Q, P^\pm_i\} \) is an effective divisor of degree \( g \) and does not depend on \( n \) or \( y \).

### 3.1.2. Inverse spectral problem.

Let us consider a non-singular curve \( \Gamma \) of genus \( g \), labelled points \( P^\pm_i, i = 1, \ldots, N \), and \( Q \) on the curve \( \Gamma \), and also a divisor \( D = R_1 + \cdots + R_g \). Let us also fix a 1-jet \( [\lambda]_1 \) of a local parameter at the point \( Q \), i.e. a local parameter at the point \( Q \) up to a transformation \( \tilde{\lambda} = \lambda + O(\lambda^2) \). The set \((\Gamma, P^\pm_i, Q, [\lambda]_1, D = R_1 + \cdots + R_g)\) is called the spectral data.

As we have seen in Theorem 2, we can build spectral data starting from a generic periodic algebro-geometric operator \( L \) (we should take \( \lambda = \frac{t}{K} \)). Our goal is to prove the inverse theorem.

**Theorem 3.** Let \((\Gamma, P^\pm_i, Q, [\lambda]_1, D = R_1 + \cdots + R_g)\) be spectral data with a generic \( D \). There exists a Baker-Akhiezer function \( \psi_n(y) \) defined on \( \Gamma \) and depending on two parameters \( n \in \mathbb{N} \) and \( y \in \mathbb{R} \) such that

1. \( \psi_0(0) = 1 \).
2. The function \( \psi_n(y) \) has a zero of order \( \left\lceil \frac{n-i}{N} \right\rceil + 1 \) at \( P^+_i \) and a pole of the same order at \( P^-_i \).
3. The function \( \psi_n(y) \) is meromorphic in \( \Gamma \setminus \{Q\} \).
4. The poles of the function \( \psi_n(y) \) in \( \Gamma \setminus \{P^\pm_i, Q\} \) can be only first-order poles at points \( R_i \).
5. The product \( \psi_n(y)e^{-\frac{y}{K}} \) is a holomorphic function in a neighborhood of \( Q \) and it is not equal to zero at \( Q \).
6. There exists an operator \( L \) of the form \( [3] \) such that \( L \psi = 0 \).
7. The function \( \psi_n(y) \) and \( L \) are defined by spectral data uniquely up to a gauge transformation.

**Proof.** We will use the same notation and conventions as in the paper [14]. Let us choose a basis in cycles. Let \( \Omega_{P^+_i P^-_i} \) be the Abelian
differential of the third kind with poles at points $P^+_i$ and $P^-_i$ and residues $+1$ and $-1$ respectively. Let $\Omega$ be the Abelian differential of the second kind with a pole of second order at $Q$ and the Laurent series expansion $\Omega = (\frac{-1}{\lambda^2} + \ldots) d\lambda$. We recall that $\alpha$-periods of $\Omega$ and $\Omega_{P^+_iP^-_i}$ are equal to zero. It is easy to see that $\Omega$ depends only on the 1-jet $[\lambda]_1$.

Let us define $P^\pm_i$ for all $i$ by periodicity: $P^\pm_i + N = P^\pm_i$. Let $U$ and $U_{P^+_iP^-_i}$ denote the vectors of $b$-periods of $\Omega$ and $\Omega_{P^+_iP^-_i}$ respectively.

Let us define $\sum'$ in the following way:

$$\sum_{i=1}^n a_i = \begin{cases} \sum_{i=1}^n a_i & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\sum_{i=n}^{-1} a_i & \text{if } n < 0. \end{cases}$$

It is easy to check that the function

$$(34) \quad \exp\left(\int_{P_0}^P y\Omega + \sum_{i=1}^n \Omega_{P^+_iP^-_i} \right) \frac{\Theta(A(P) + yU + \sum_{i=1}^n U_{P^+_iP^-_i} - A(D) - K)}{\Theta(A(P) - A(D) - K)},$$

depending on two parameters $n$ and $y$, satisfies the conditions (1)–(5) from the statement of this theorem. Hence the existence of $\psi_n(y)$ with properties (1)–(5) is proven.

Let us now prove that any function $\psi_n(y)$ satisfying the conditions (1)–(5) has the form

$$(35) \quad \psi_n(y, P) = r_n(y) \exp\left(\int_{P_0}^P y\Omega + \sum_{i=1}^n \Omega_{P^+_iP^-_i} \right) \frac{\Theta(A(P) + yU + \sum_{i=1}^n U_{P^+_iP^-_i} - A(D) - K)}{\Theta(A(P) - A(D) - K)},$$

where $r_n(y)$ are constants (i.e. do not depend on a point on $\Gamma$) such that $r_0(0) = 1$.

Indeed, the ratio of a function $\psi_n(y)$ and the function (34) is a meromorphic function on $\Gamma$ with a pole divisor $D - yU - nU_{P^+_iP^-_i}$, which is generic. It follows that the space of such meromorphic functions has dimension 1, i.e. this ratio is a constant function on $\Gamma$, depending on the parameters $n$ and $y$. Let us denote this constant as $r_n(y)$. It follows from the condition (1) that $r_0(0) = 1$. 
Let us now construct the operator \( L \). We already have the function \( \psi_n(y) \) given by the formula (35). Let us consider the equation \( L\psi = 0 \) as an equation for unknown \( a_n(y), \ldots, d_n(y) \):

\[
(36) \quad a_n(y)\psi_n + b_n(y)\psi'_n + c_n(y)\psi_{n+1} + d_n(y)\psi'_{n+1} = 0.
\]

Let us take the Laurent series expansion of (36) at the point \( P_{n+1}^- \). The first term of this expansion gives us the identity

\[
c_n(y) r_{n+1}(y) \exp \frac{\int_{P_0}^{P_{n+1}^-} y \Theta(A(P^-_{n+1}) + yU + \sum_{i=1}^{n+1} U_{p_i^+} - A(D) - K)}{\Theta(A(P^-_{n+1}) - A(D) - K)} + \]

\[
d_n(y) r'_{n+1}(y) \exp \frac{\int_{P_0}^{P_{n+1}^-} y \Theta(A(P^-_{n+1}) + yU + \sum_{i=1}^{n+1} U_{p_i^+} - A(D) - K)}{\Theta(A(P^-_{n+1}) - A(D) - K)} + \]

\[
r_{n+1}(y) \int_{P_0}^{P_{n+1}^-} y \exp \frac{\int_{P_0}^{P_{n+1}^-} y \Theta(A(P^-_{n+1}) + yU + \sum_{i=1}^{n+1} U_{p_i^+} - A(D) - K)}{\Theta(A(P^-_{n+1}) - A(D) - K)} + \]

\[
r'_{n+1}(y) \exp \frac{\int_{P_0}^{P_{n+1}^-} y \Theta(A(P^-_{n+1}) + yU + \sum_{i=1}^{n+1} U_{p_i^+} - A(D) - K)}{\Theta(A(P^-_{n+1}) - A(D) - K)} = 0.
\]

This implies that

\[
(37) \quad \frac{c_n(y)}{d_n(y)} = \frac{r'_{n+1}(y)}{r_{n+1}(y)} - \int_{P_0}^{P_{n+1}^-} y \Theta(A(P^-_{n+1}) + yU + \sum_{i=1}^{n+1} U_{p_i^+} - A(D) - K).
\]

In the same way the term of order \( \lambda^{-1} \) in the Laurent series expansion of the equation \( (L\psi)_n e^{-\frac{y}{\lambda}} = 0 \) at \( Q \) gives us the formula

\[
(38) \quad \frac{d_n(y)}{b_n(y)} = - \frac{r_n(y)}{r'_{n+1}(y)} \exp(- \int_{P_0}^{Q} y \Omega_{P_{n+1}^-} - P_{n+1}^-) \times \frac{\Theta(A(Q) + yU + \sum_{i=1}^{n} U_{p_i^+} - A(D) - K)}{\Theta(A(Q) + yU + \sum_{i=1}^{n+1} U_{p_i^+} - A(D) - K)}.
\]
Considering the Laurent series expansion of (36) at the point $P_{n+1}^+$ we obtain the formula

\begin{equation}
\frac{a_n(y)}{b_n(y)} = \frac{r'_n(y)}{r_n(y)} - \int_{P_0}^{P_{n+1}^+} \frac{\Omega}{\partial y} \Theta(A(P_{n+1}^+) + yU + \sum_{i=1}^n U_{P_i^+P_i^-} - A(D) - K).
\end{equation}

If $a_n(y), \ldots, d_n(y)$ satisfy identities (37), (38) and (39) then the equation (36) holds at any point on $\Gamma$. To prove this, let us consider the function

$$
\varphi_n(y) = -\frac{b_n(y)}{a_n(y)} \psi'_n(y) - \frac{c_n(y)}{a_n(y)} \psi_{n+1}(y) - \frac{d_n(y)}{a_n(y)} \psi'_{n+1}(y).
$$

It follows from the formulas (37) and (38) that $\varphi_n(y)$ satisfies the same conditions (1)--(5) as $\psi_n(y)$. As explained before, it follows that these two functions are proportional. The formula (39) means that the coefficient of proportionality is equal to 1, i.e. $\varphi_n(y) \equiv \psi_n(y)$. This implies that the equation (36) holds at any point on $\Gamma$.

The identities (37), (38) and (39) determine $a_n(y), \ldots, d_n(y)$ up to a multiplication by a constant depending on $n$ and $y$. It is easy to check that this fact together with the fact that $\psi_n(y)$ are determined up to constants $r_n(y)$ means exactly that $L$ and $\psi$ are defined up to a gauge transformation. □

We should remark that operators obtained by this Theorem from spectral data are not necessarily gauge equivalent to periodic ones.

### 3.2. Algebro-geometric spectral theory of two-dimensional discrete hyperbolic Schrödinger operators.

Let us consider a discrete operator $L$ of the form (2) and the corresponding equation $L\psi = 0$ (13) with the following matrix of periods

\begin{equation}
\Xi = \begin{pmatrix} P & R \\ S & T \end{pmatrix},
\end{equation}

i.e.

$$
a_{n+P,m+R} = a_{n+m}, b_{n+P,m+R} = b_{n+m},
$$

$$
c_{n+P,m+R} = c_{n+m}, d_{n+P,m+R} = d_{n+m}.
$$

Let $\Delta = \det(\Xi) > 0$. The matrix $\Xi$ defines a sub-lattice of $\mathbb{Z} \times \mathbb{Z}$. We call it a period sub-lattice. Let us fix two specific choices of basis of the period sub-lattice.
Lemma 20. For the period sub-lattice given by the matrix $\mathfrak{T}$ there exists a unique basis such that the corresponding matrix

$$\mathfrak{T}_1 = \begin{pmatrix} \tilde{\delta} & 0 \\ -\tilde{\zeta} & \delta \end{pmatrix}$$

satisfies conditions $\delta = (R, T)$, $\tilde{\delta} = \Delta/\delta$, $0 \leq \zeta < \tilde{\delta}$.

There also exists a unique basis such that the corresponding matrix

$$\mathfrak{T}_2 = \begin{pmatrix} \varepsilon & -\xi \\ 0 & \bar{\varepsilon} \end{pmatrix}$$

satisfies the conditions $\varepsilon = (P, S)$, $\bar{\varepsilon} = \Delta/\varepsilon$, $\Delta = \det(\mathfrak{T})$, $0 \leq \xi < \bar{\varepsilon}$.

Proof is a direct calculation. □

We consider only the case when $\delta < \Delta$, $\varepsilon < \Delta$. It is a reasonable assumption. Indeed, if, for example, $\delta = \Delta$ then $\tilde{\delta} = 1$. It means that one of the periods is equal to 1. This degenerate case is of little interest.

The main object of our interest is the Floquet solution $\psi$ of the equation $L\psi = 0$. Since we have two choices of basis of the period sub-lattice, we can define two pairs of corresponding Floquet multipliers of the function $\psi_{n,m}$:

$$\psi_{n+\tilde{\delta},m} = \nu_1 \psi_{n,m}, \quad \psi_{n-\zeta,m+\delta} = \mu_1 \psi_{n,m},$$
$$\psi_{n,m+\bar{\varepsilon}} = \mu_2 \psi_{n,m}, \quad \psi_{n+\varepsilon,m-\xi} = \nu_2 \psi_{n,m}.$$  \hspace{1cm} (41)  \hspace{1cm} (42)

Since $\mathfrak{T}_1 \mathfrak{T}_2^{-1} \in SL_2(\mathbb{Z})$, it follows that $1 - \bar{\zeta} \bar{\xi} = \varkappa \Delta$, where $\xi = \bar{\xi} \varepsilon$, $\zeta = \bar{\zeta} \delta$, $\Delta = \bar{\Delta} \varepsilon \delta$, $\varkappa \in \mathbb{Z}_+$. Thus we can express $\nu_1, \mu_1$ in terms of $\nu_2, \mu_2$ and vice versa by the formulas

$$\nu_1 = \nu_2 \tilde{\delta} \mu_2 \bar{\zeta}, \quad \mu_1 = \nu_2 \bar{\xi} \mu_2 \varkappa,$$  \hspace{1cm} (43)  \hspace{1cm} (44)
$$\nu_2 = \nu_1 \varkappa \mu_1 \tilde{\zeta}, \quad \mu_2 = \nu_1 \bar{\xi} \mu_1.$$

3.2.1. The inverse spectral problem. The inverse spectral problem for the discrete operators was solved by Krichever [10]. Let us recall (in our notation) the main theorem of the paper [10].

**Theorem 4.** Let the matrix $\mathfrak{S} \in SL_2(\mathbb{Z})$ define a period sub-lattice (and hence $\mathfrak{T}_1, \mathfrak{T}_2$). Let $C$ be a curve of genus $g = \Delta - \delta - \varepsilon + 1$ and $P_n^\pm, Q_n^\pm$ be points on $C$ such that

$$P_n^\pm = P_{n'}^\pm, \quad Q_k^\pm = Q_{k'}^\pm \quad \text{for} \quad n-n' = 0 \mod \delta, \quad k-k' = 0 \mod \varepsilon.$$  \hspace{1cm} (45)

Let $D$ be a generic effective divisor of degree $g$. 

Let one of the following equivalent conditions be satisfied

\begin{align}
\sum_{i=1}^{\delta}(Q_i^+ - Q_i^-) &\sim (\nu_1), \quad \sum_{j=1}^{\delta}(P_j^+ - P_j^-) - \sum_{i=1}^{\zeta}(Q_i^+ - Q_i^-) \sim (\mu_1), \\
\sum_{i=1}^{\varepsilon}(P_i^+ - P_i^-) &\sim (\mu_2), \quad \sum_{j=1}^{\epsilon}(Q_j^+ - Q_j^-) - \sum_{i=1}^{\xi}(P_i^+ - P_i^-) \sim (\nu_2),
\end{align}

where \(\nu_i, \mu_i\) are meromorphic functions on the curve.

Then the following statements hold.

(1) The space

\[ \mathcal{L}_{m,n}(\mathcal{D}) = \mathcal{L}(\sum_{i=1}^{m}(Q_i^- - Q_i^+) + \sum_{i=1}^{n}(P_i^- - P_i^+)) + \mathcal{D}. \]

is one-dimensional.

(2) For any choice of nonzero functions \(\psi_{m,n} \in \mathcal{L}_{m,n}(\mathcal{D})\) satisfying the Floquet conditions (41), (42) there exists an operator \(L\) with period matrix \(\Sigma\) such that \(L\psi_{n,m} = 0\). The operator \(L\) and the function \(\psi_{n,m}\) are unique up to a gauge transformation (14).

The explicit formulas for \(L\) and \(\psi_{n,m}\) in terms of theta-functions can be found in [10].

This means that given the spectral data \((\Gamma, P_n^\pm, Q_n^\pm, \mathcal{D})\) satisfying the conditions (45), (46) and (47), one can reconstruct \(L\) and \(\psi_{nm}\) uniquely up to a gauge transformation in terms of theta-functions.

Counting of parameters performed in the paper [10] shows that varying the curve \(C\) and the divisor \(\mathcal{D}\) we can get a dense subset of the set of periodic discrete operators. In this paper this fact is proved effectively by solving direct spectral problem.

3.2.2. The direct spectral problem. For discrete operators the direct spectral problem can be solved explicitly. Let us consider a generic discrete operator \(L\) with a period matrix \(\Sigma\) (40) and the equation \(L\psi = 0\). As in the semi-discrete case, we consider the Floquet solution.

**Theorem 5.** [10] The possible pairs of Floquet multipliers form a curve which can be compactified. The compactified curve \(\Gamma\) (called a spectral curve) has genus \(g = \Delta - \delta - \varepsilon + 1\). There exist an effective divisor \(\mathcal{D}\) of degree \(g\) and points \(P_k^\pm, Q_l^\pm\) satisfying the conditions (45), (46) and (47), such that the function \(\psi_{n,m}\) normalized by the condition \(\psi_{0,0} = 1\) is meromorphic function belonging to the space \(\mathcal{L}_{n,m}(\mathcal{D})\).
Proof. Let us introduce some notations:

\[ A_{i,j} = \prod_{i=1}^{\delta} a_{i,j}, \quad B_{i,j} = \prod_{i=1}^{\delta} b_{i,j}, \quad C_{i,j} = \prod_{i=1}^{\delta} c_{i,j}, \quad D_{i,j} = \prod_{i=1}^{\delta} d_{i,j}, \]

\[ A_{i,\varepsilon} = \prod_{j=1}^{\varepsilon} a_{i,j}, \quad B_{i,\varepsilon} = \prod_{j=1}^{\varepsilon} b_{i,j}, \quad C_{i,\varepsilon} = \prod_{j=1}^{\varepsilon} c_{i,j}, \quad D_{i,\varepsilon} = \prod_{j=1}^{\varepsilon} d_{i,j}. \]

Let the matrix \( M \) be the matrix of the linear system \( L\psi = 0 \) written in some basis. In more details, let us put the coefficients of the equation \( (L\psi)_{i,j} = 0 \) in the \( i + j\delta + 1 \)-th row \((0 \leq i < \delta, 0 \leq j < \delta)\) of \( M \), where \( \psi_{k,l} \) are written in the form \( \psi_{k,l} = \nu_1^k \mu_1^l \psi_{k',l'}, 0 \leq k' < \delta, 0 \leq l' < \delta \). The \( 1 + i + j\delta \)-th column \((0 \leq i < \delta, 0 \leq j < \delta)\) of \( M \) corresponds to \( \psi_{i,j} \). The constructed matrix \( M \) has a block triangular structure and the equation \( R(\mu_1, \nu_1) := \det M = 0 \) is the equation of an affine part of the spectral curve.

We also need a matrix \( \hat{M} \) constructed in an analogous way. Let us put the coefficients of the equation \( (L\psi)_{i,j} = 0 \) in the \( i\varepsilon + j + 1 \)-th row \((0 \leq i < \varepsilon, 0 \leq j < \varepsilon)\) of the matrix \( \hat{M} \), where \( \psi_{k,l} \) are presented in the form \( \psi_{k,l} = \nu_2^k \mu_2^l \psi_{k',l'}, 0 \leq k' < \varepsilon, 0 \leq l' < \varepsilon \). The \( 1 + i\varepsilon + j \)-th column \((0 \leq i < \varepsilon, 0 \leq j < \varepsilon)\) corresponds to \( \psi_{i,j} \). Let \( \hat{R}(\nu_2, \mu_2) = \det \hat{M} \).

Reformulating Lemma 1 from the paper [11] we get the following Lemma.

Lemma 21. For any operator \( L \) the following identities hold.

\[ R(\mu_1, \nu_1) = \hat{R}(\mu_2, \nu_2), \]

\[ R(\nu_1, \mu_1) = \sum_{0 \leq i < \delta, 0 \leq j < \delta} r_{i,j} \nu_1^i \mu_1^j, \quad \hat{R}(\nu_2, \mu_2) = \sum_{0 \leq i < \varepsilon, 0 \leq j < \varepsilon} \hat{r}_{i,j} \nu_2^i \mu_2^j, \]

\[ \sum_{i=0}^{\delta+\zeta} r_{i,0} \nu_1^i = \prod_{j=0}^{\delta-1} \left( B_{j,0} \nu_1 - (-1)^\delta A_{j,0} \right), \]

\[ \sum_{i=0}^{\delta+\zeta} r_{i,\delta} \nu_1^i = \nu_1^\delta \prod_{j=0}^{\delta-1} \left( D_{j,\delta} \nu_1 - (-1)^\delta C_{j,\delta} \right), \]

\[ \sum_{j=0}^{\epsilon+\zeta} \hat{r}_{0,j} \mu_2^j = \prod_{i=0}^{\epsilon-1} \left( C_{i,0} \mu_2 - (-1)^\epsilon A_{i,0} \right), \]

\[ \sum_{j=0}^{\epsilon+\zeta} \hat{r}_{\epsilon,j} \mu_2^j = \mu_2^\epsilon \prod_{i=0}^{\epsilon-1} \left( D_{i,\epsilon} \mu_2 - (-1)^\epsilon B_{i,\epsilon} \right). \]
Proof is by a direct calculation. □

From this Lemma we see that for a generic operator \( L \) the natural compactification of the affine part of the spectral curve 
\[ \{(\nu_1, \mu_1)|R(\nu_1, \mu_1) = 0, \nu_1 \mu_1 \neq 0\} \] is a curve of genus \( \Delta - \delta - \epsilon + 1 \). Compacting this curve we add four groups of points \( P_i^\pm, Q_i^\pm \). In the coordinates \( \nu_1, \mu_1 \) we have
\[ P_i^+ = ((-1)^i \frac{A_{-i,j}}{B_{-i,j}}, 0), \quad P_i^- = ((-1)^i \frac{C_{-i,j}}{D_{-i,j}}, \infty); \]
and in the coordinates \( \nu_2, \mu_2 \) we have
\[ Q_i^+ = (0, (-1)^i \frac{A_{i,-}}{C_{i,-}}), \quad Q_i^- = (\infty, (-1)^i \frac{B_{i,-}}{D_{i,-}}). \]

It is easy to see that
\[ \frac{\psi_{n+1,m}}{\psi_{n,m}} = \frac{\partial R}{\partial a_{n,m}}, \quad \frac{\psi_{n,m+1}}{\psi_{n,m}} = \frac{\partial R}{\partial c_{n,m}}. \]

Using this formula together with Lemma 21 we can prove that there exists an effective divisor \( D \) of degree \( g \) such that \( \psi \in \mathcal{L}_{n,m}(D) \). For the proof see Lemma 2 of the paper [11].

To get more information about the divisor \( D \) we need the formal adjoint operator
\[ (L^+ \psi^+)(n,m) = a_{n,m} \psi_{n,m}^+ + b_{n-1,m} \psi_{n,m-1}^+ + c_{n,m-1} \psi_{n,m-1}^+ + d_{n-1,m-1} \psi_{n-1,m-1}^+. \]

We are interested in a Floquet solution of the equation \( L^+ \psi^+ = 0 \) such that
\begin{align*}
\psi_{n+\delta,m}^+ &= (\nu_1)^{-1} \psi_{n,m}^+, \quad \psi_{n-\delta,m+\delta}^+ = (\mu_1)^{-1} \psi_{n,m}^+, \\
\psi_{n,m+\varepsilon}^+ &= (\mu_2)^{-1} \psi_{n,m}^+, \quad \psi_{n+\varepsilon,m-\varepsilon}^+ = (\nu_2)^{-1} \psi_{n,m}^+. \end{align*}

It is easy to see that the \( 1 + i + j \tilde{\delta} \)-th row of the matrix \( M^T \) corresponds to the equation \( (L^+ \psi^+),i,j = 0 \) (\( 0 \leq i < \tilde{\delta}, 0 \leq j < \delta \)), where \( \psi^+ \) is written in the form \( \psi_{n,m}^+ = \nu_i^{\alpha} \mu_2^{\beta} \psi_{n,m'}^{+}, \) with \( 0 \leq n' < \tilde{\delta}, 0 \leq m' < \delta \).

The \( 1 + i + j \tilde{\delta} \)-th column of \( M^T \) corresponds to \( \psi_{i,j}^+ \). The operator \( L^+ \) is of a form similar to the form of \( L \). Moreover, the spectral curves of these operators are isomorphic. The function \( \psi^+ \) belong to the space \( \mathcal{L}_{n,m}(D^+) \) for some effective divisor \( D^+ \). Using this observation we get (see Lemma 5 of the paper [11]) the following

Lemma 22. Let \( L \) be a generic operator. Then the differential
\[ \Omega = \frac{\tilde{\Delta} d\nu_1}{\mu_1 \nu_1 R}\mu_1, \]
where \( \tilde{\Delta} \) is the determinant of the algebraic complement of the element \( M_{11} \) of the matrix \( M \), satisfies the identity

\[
(\Omega) = -P_0^+ - Q_0^+ + \mathcal{D} + \mathcal{D}^+.
\]

It follows that \( |\mathcal{D}| = |\mathcal{D}^+| = \Delta - \delta - \epsilon + 1 = g \).

This lemma completes the proof of the Theorem. \( \square \)

4. Spectral properties of the Laplace transformations

4.1. Spectral properties of the Laplace transformations of algebro-geometric two-dimensional semi-discrete hyperbolic Schrödinger operators. Since the Laplace transformations act on gauge equivalence classes of operators, and operators can be obtained from spectral data, we can ask how to describe the Laplace transformations in terms of spectral data. It turns out that the Laplace transformations are shifts on the Jacobian of a spectral curve.

**Theorem 6.** The Laplace transformations act on spectral data in the following way: \( \Gamma, P_i^+, Q, \) and \( [\lambda]_1 \) are not changing. The points \( P_i^- \) and the divisor \( \mathcal{D} \) are changing according to the rule

\[
\tilde{\mathcal{D}} = \mathcal{D} + P_i^- - Q, \quad \tilde{P}_i^- = P_{i+1}^- 
\]

for the Laplace transformation of the first type and according to the rule

\[
\tilde{\mathcal{D}} = \mathcal{D} - P_{i^-} - Q, \quad \tilde{P}_i^- = P_{i^-} - 1
\]

for the Laplace transformation of the second type.

When we write a formula like \( \tilde{\mathcal{D}} = \mathcal{D} + P_i^- - Q \) we mean that \( \tilde{\mathcal{D}} \) is an effective divisor equivalent to \( \mathcal{D} + P_i^- - Q \).

**Proof.** Let us recall that after the Laplace transformation of the first type the new \( \psi \)-function is

\[
\tilde{\psi}_n = (1 + v_n T) \psi_n = \psi_n + v_n \psi_{n+1}.
\]

But \( \tilde{\psi} \) does not satisfy the normalization condition \( \psi_0(0) = 1 \). Dividing by \( \tilde{\psi}_0(0) \) we obtain

\[
\tilde{\psi}_n(y) = \frac{\psi_n(y) + v_n(y) \psi_{n+1}(y)}{1 + v_0(0) \psi_1(0)}.
\]

All terms in this formula can be expressed in terms of theta-functions since \( v_n = \frac{d_n}{b_n} \). It follows from the consideration of poles and zeroes that up to multiplication by a constant \( 1 + v_0(0) \psi_1(0) \) is equal to

\[
\exp\left( \int_{P_0}^{P_i} \Theta(A(P) + A(Q) - A(P_i^-) - A(\mathcal{D}) - \mathcal{K}) \right) \Theta(A(P) - A(\mathcal{D}) - \mathcal{K}).
\]
In the same way up to multiplication by a constant $\psi_n(y) + v_n(y)\psi_{n+1}(y)$ is equal to

$$\exp\left(\int_{P_0}^{P} y\Omega + \sum_{i=1}^{n} \Omega_{P^+_i P^-_i} + \Omega_{QP-n+1} \right) \times$$

$$\Theta(A(P) + yU + \sum_{i=1}^{n} U_{P^+_i P^-_i} + A(Q) - A(P_{n+1}^-) - A(D) - K)$$

$$\times \frac{\Theta(A(P) - A(D) - K)}{\Theta(A(P) + A(Q) - A(P^-_1) - A(D) - K)},$$

where $\hat{r}_n(y)$ are constants.

Since $\sum_{i=1}^{n} \Omega_{P^+_i P^-_i} + \Omega_{QP-n+1} - \Omega_{QP_1} = \sum_{i=1}^{n} \Omega_{P^+_i P^-_{i+1}}$, we obtain

$$\hat{\psi}_n(y) = \hat{r}_n(y) \exp\left(\int_{P_0}^{P} y\Omega + \sum_{i=1}^{n} \Omega_{P^+_i P^-_{i+1}} \right) \times$$

$$\Theta(A(P) + yU + \sum_{i=1}^{n} U_{P^+_i P^-_{i+1}} - A(\tilde{D}) - K)$$

$$\times \frac{\Theta(A(P) - A(\tilde{D}) - K)}{\Theta(A(P) - A(D) - K)},$$

where $\tilde{D} = D + P^-_1 - Q$. This implies the statement of this Theorem for the Laplace transformation of the first type.

The formula for the Laplace transformations of the second type can be easily obtained from the formula for the Laplace transformation of the first type since they are inverse to each other. □

This Theorem makes it possible to easily construct chains of Laplace transformations and, therefore, solutions of the equations (7) and (8) in terms of theta-functions.

4.2. Spectral properties of the Laplace transformations of algebro-geometric two-dimensional discrete hyperbolic Schrödinger operators. As was already explained in Section 2.2, the group generated by the Laplace transformations has three generators. Let us now describe how these generators act on the spectral data.
Theorem 7. The following identities hold:

\[ \lambda_{12}^{++}(P^-_i) = P^-_{i+1}, \quad \lambda_{12}^{++}(P^+_i) = P^+_i, \quad \lambda_{12}^{++}(Q^-_i) = P^-_{i-1}, \quad \lambda_{12}^{++}(P^+_i) = P^+_i, \]
\[ S_1(Q^-_i) = Q^-_{i-1}, \quad S_1(Q^+_i) = Q^+_{i-1}, \quad S_1(P^-_i) = P^-_i, \quad S_1(P^+_i) = P^+_i, \]
\[ S_2(Q^-_i) = Q^-_i, \quad S_2(Q^+_i) = Q^+_i, \quad S_2(P^-_i) = P^-_{i-1}, \quad S_1(P^+_i) = P^+_i, \]
\[ \lambda_{12}^{++}(D) = D + P^-_0 - Q^-_1, \quad S_1(D) = D + Q^+_1 - Q^-_1, \]
\[ S_2(D) = D + P^+_1 - P^-_1. \]

Proof is analogous to the Proof of Theorem 6 \( \square \)

As it was mentioned in the Introduction, cyclic chains of Laplace transformations were studied in the paper [7] (we should also mention the paper [15] where cyclic chains of Darboux transformations were studied).

In our case we define a cyclic chain of Laplace transformations as a chain such that \((\lambda_{12}^{++})^\alpha(L) = S_1^\beta S_2^\gamma(L)\).

Let us say that an operator \( L \) is an integrable operator if \( P^\pm_i = P^\pm \), \( Q^\pm_i = Q^\pm \). This condition means that the dynamics of the iterated Laplace transformation is linearizable on the Jacobian of the spectral curve. Indeed, it follows from the Theorem 7 that in this case the points \( P^\pm, Q^\pm \) are invariant and the divisor \( D \) is shifted by a fixed vector \( P^- - Q^- \).

The condition of integrability is absolutely explicit in terms of the coefficients of a periodic operator \( L \). Indeed, it follows from the explicit formulas for the points \( P^\pm_n \) and \( Q^\pm_n \) that an operator \( L \) is integrable if and only if \( \frac{A_{ij}}{B_{ij}}, \frac{C_{ij}}{D_{ij}}, \frac{A_{ik}}{C_{ik}}, \text{ and } \frac{B_{ik}}{D_{ik}} \) are constants.

We have the following Theorem similar to the one from [7].

Theorem 8. If \((\lambda_{12}^{++})^\alpha(L) = S_1^\beta S_2^\gamma(L) \) and \((\beta, \delta) = 1\), \((\gamma, \epsilon) = 1\), \((\alpha + \gamma, \epsilon) = 1\), \((\alpha - \beta, \delta) = 1\), then operator \( L \) is an integrable operator.

Proof can be done by direct calculation. \( \square \)

As in the semi-discrete case, we can construct solutions of the completely discretized 2D Toda lattice \([19]\). In this case we can do it for arbitrary generic periodic initial data since any periodic operator is algebro-geometric. This gives us the following Lemma.

Lemma 23. The family of solutions of the completely discretized 2D Toda lattice \([19]\) with a generic \( \Xi \)-periodic initial data \( w^{(0)}_{n,m} \) can be written explicitly in terms of \( \theta \)-functions of the corresponding spectral curve. This family is parameterized by a set of arbitrary \( \Xi \)-periodic constants \( H^{(0)}_{n,m} \).
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