Schwarzschild solution in extended teleparallel gravity

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Abstract – A tetrad field with two unknown functions of the radial coordinate and an angle Φ (the polar angle φ times a function of the radial coordinate), is applied to the field equation of the modified theory of gravity. An exact vacuum solution is derived; its scalar torsion, \( T = S_{\alpha \mu \nu} \), is constant. When the angle Φ coincides with the polar angle φ, the derived solution will be a solution only for the linear form of the \( f(T) \) gravitational theory.

Introduction. – The common belief in the scientific community is that the gravitational field is governed by Einstein’s general relativity (GR) theory — although this can only be true on super-Planckian scales, where quantum effects can be neglected. Furthermore, as an ultimate explanation of physical phenomena, GR also faces a curious problem related to the late accelerated expansion stage of the Universe. Due to these problems and other apparent inadequacies (e.g., the hitherto elusive dark-matter issue), GR has been the subject of many modifications. These modifications have attempted to supply more satisfying descriptions of the gravitational field in the aforementioned extreme regimes. One of the recent modified gravitational theories is \( f(T) \) gravity. This sort of theory is constructed in a space-time having absolute parallelism [1–6], with the curvature contributions vanishing identically and the only contribution is due to the anti-symmetric part of the non-symmetric affine connection. This procedure involves the so-called Weitzenböck’s, Teleparallel Equivalent of General Relativity (TEGR), space-time.

Recently, the specifics of \( f(T) \) gravity have begun to be elaborated (e.g., [7–11]). It is found that the \( f(T) \) gravity theory is not dynamically equivalent to the TEGR Lagrangian through conformal transformation [12]. Birkhoff’s theorem in \( f(T) \) gravity has been studied [13]. Stationary solutions having spherical symmetry have been derived for \( f(T) \) theories [14]. Relativistic stars and the cosmic expansion have been studied [15,16]. A Schwarzschild solution is derived in the \( f(T) \) theory using a tetrad field having four unknown functions [17]. This solution is characterized by the vanishing of the scalar torsion tensor. Many observational constraints have also been examined [18–21]. Cosmological perturbations in \( f(T) \) gravity have been examined [22–25] and the cosmological large-scale structure has been analyzed [26].

\( f(T) \) gravitational theories have been the subject of many studies, as it has been indicated that the Lagrangian and the equations of motion of those theories are not invariant under local Lorentz transformations [27]. Indeed it has been shown that setting back local Lorentz symmetry in \( f(T) \) theories cannot upgrade to credible dynamics, even if one relinquishes teleparallelism [28]. The equations of motion of \( f(T) \) theories differ from those of \( f(R) \) theories [29–44], because they are of second order rather than of fourth order. Therefore, dealing with the field equations of \( f(T) \) is believed to be much easier than with those of \( f(R) \). Because of the non-locality of these theories, \( f(T) \), it seems that these theories contain more degrees of freedom.

Here, in this letter, we will derive the Schwarzschild solution using a simple tetrad field having two unknown functions in addition to angle Φ which is a function in the azimuthal angle φ. The advantage of the tetrad used in this letter is that it has no inertial matrix like the one used in [17]. The aim of the present study is to find an analytic vacuum spherically symmetric solution, in the framework of the \( f(T) \) gravitational theory.

In the next section, a brief review of the \( f(T) \) gravitational theory is provided. Then, a non-diagonal, spherically symmetric tetrad field with two unknown functions of radial coordinate in addition to the angle Φ is introduced and the application of such a tetrad to the field equation of \( f(T) \) is presented. Finally, an analytic vacuum spherically
symmetric solution with one constant of integration is derived. In the third section, the physical properties of the derived solution—i.e., the decomposition of the derived solution—is achieved, and the energy is calculated, in order to understand the physical meaning of the constant of integration. The final section is devoted to discussing the results.

**Brief review of the \( f(T) \) gravitational theory and spherically symmetric solution.** – The equations of motion of the \( f(T) \) gravitational theory have the form \[7\]

\[ S_{\mu
u} T_{\mu} f(T)_{TT} + \left[ e^{-1} e_{\alpha} \partial_\rho \left( e_{\alpha} S_{\mu}^{\rho \nu} \right) + T^{\alpha \mu \nu} S_{\alpha}^{\nu \lambda} \right] f(T)_{T} + \frac{1}{4} \delta_{\mu}^{\nu} f(T) = 4 \pi T^{\nu}_{\mu}, \]

where \( T^{\alpha \mu \nu} \) is the torsion tensor, \( \Gamma^{\alpha \mu \nu} \) is the non-symmetric affine connection, \( e_{\alpha} \) is the tetrad field which is the main block in the modified theories of teleparallel of gravity and \( e = \det(e_{\alpha}) = \sqrt{g} \), is the determinant of the tetrad, the tensor \( S_{\alpha}^{\nu \gamma} \) is defined as follows:

\[ S_{\alpha}^{\nu \gamma} \overset{\text{def}}{=} \frac{1}{2} \left( K^{\nu \gamma} + \delta_{\nu}^{\gamma} (T_{\delta \gamma}^{\alpha} - \delta_{\delta}^{\gamma} T_{\nu \gamma}^{\alpha}) \right) \text{ where } K^{\nu \gamma} \overset{\text{def}}{=} -\frac{1}{2} \left( T^{\nu \gamma} - T^{\nu \nu} T^{\gamma \gamma} - T^{\gamma \gamma} T^{\nu \nu} \right) \text{ is the contortion and } T \text{ is the scalar torsion which is defined as } T^{\alpha \mu \nu} S_{\alpha}^{\nu \mu} \text{ where } T_{\mu} = \frac{\partial \Omega}{\partial r} f(T)_{T} = \frac{\partial f(T)}{\partial r} f(T)_{TT} = \frac{\partial f(T)}{\partial r} f(T)_{TT} \text{ and } T^{\nu}_{\mu} \text{ is the energy-momentum tensor.}

The total energy-momentum of \( f(T) \) gravitational theory contained in a three-dimensional volume \( V \) has the form \[45\]

\[ P^{\mu} = \int_{V} d^{3} x e^{\alpha} e_{\alpha}^{\mu} T^{\nu}_{\mu} = \frac{1}{4 \pi} \int_{V} d^{3} x \partial_{\rho} \left[ e^{\alpha} S^{\mu \nu \alpha} f(T)_{T} \right]. \]

In this study we are interested in studying the vacuum case of the \( f(T) \) theory, i.e., \( T^{\nu}_{\mu} = 0 \).

Let us assume that the space-time possessing a stationary and spherical symmetry has the form

\[ (e_{\mu}) = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & B(r) \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ 0 & B(r) \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ 0 & B(r) \cos \theta & -r \sin \theta & 0 \end{pmatrix}, \]

where \( A(r) \) and \( B(r) \) are two unknown functions of the radial coordinate, \( r \) and \( \Phi = \phi L(r) \).

Using eq. (1) and \( (e_{\mu}) \) given by eq. (3), one can obtain \( e = \det(e_{\alpha}) = r^{2} A B \sin \theta \) (see footnote\(^1\)) and the torsion and its derivatives in the form

\[ T(r) = \frac{-2 \left( AB^{2} - AB - r B A' - A B L_{\phi} + A + 2 r A' - r B A' L_{\phi} \right)}{r^{2} A B^{2}}, \]

where

\[ \begin{align*}
L_{\phi} &= \frac{\partial L(\phi)}{\partial \phi} , \quad A' = \frac{\partial A(r)}{\partial r} , \\
B' &= \frac{\partial B(r)}{\partial r} , \\
T' &= \frac{\partial T}{\partial r} = - \frac{1}{r^{2} A B^{2}} \left( 2 r^{2} B [ A B' - A^{2} + B L_{\phi} ] - 2 r A A' \left[ B L_{\phi} [ B + r B' ] + r B' [ B - 4 ] - 2 B - B^{2} \right] - 2 A^{2} \left[ B L_{\phi} [ 2 B - 2 B^{2} + r B' ] + r B' [ B - 2 ] + 2 B^{2} - 2 B \right] \right), \\
T_{\phi} &= \frac{\partial T}{\partial \phi} = 2 \frac{L_{\phi} [ r A' - A (B - 1) ]}{r^{2} A B^{2}} .
\end{align*} \]

Using the above calculations, the field equations (1) take the form

\[ \begin{align*}
4 \pi T^{0}_{0} &= - \frac{f_{TT} T'^{0} [ B - 2 - B L_{\phi} ]}{r^{2} A B^{2}}, \\
+ \frac{f_{TT} B^{2} r^{2}}{2 r^{2} A B^{2}} \left( B L_{\phi} ( r A' - A (B - 1) ) + r B ( A B' - 2 B + B^{3} + B^{2} ) \right) + \frac{f}{4}, \quad \text{see eq. (6) above}
\end{align*} \]

\[ \begin{align*}
4 \pi T^{2}_{2} &= - \frac{f_{TT} T'^{2} | r A' - A B L_{\phi} + A |}{r^{2} A B^{2}}, \\
4 \pi T^{1}_{3} &= \frac{f_{TT} L_{\phi} ( A B - A - r A'^{2} )}{r^{2} A B^{2}}, \quad \text{see eq. (7) above}
\end{align*} \]

\[ \begin{align*}
4 \pi T^{2}_{2} &= - \frac{f_{TT} T'^{2} [ r A' - A B L_{\phi} + A ]}{r^{2} A B^{2}}, \\
+ \frac{f_{TT} B^{2} r^{2}}{2 r^{2} A B^{2}} \left( r B A' - r A' [ B^{2} - 3 B + r B' + B^{2} L_{\phi} ] - A [ B^{2} (1 - B) L_{\phi} + r B' + B^{2} - B ] \right) + \frac{f}{4}, \quad \text{see eq. (8) above}
\end{align*} \]
From eqs. (5)–(10), it is clear that $A \neq 0$ and $B \neq 0$. To solve the differential equations (5)–(10) we put the following constraints:

$$
T = \text{const} = T_0 \Rightarrow T^0 = 0, \quad L_\phi = 0 \Rightarrow \phi = 0,
$$

$$
B^2 L_\phi (r A' - A (B - 1)) + r B A' (B - 2)
+ A (2 r B' - 2 B + B^3 + B^2) = T_0,
$$

$$
2 A B^2 + 2 r B A' L_\phi - 8 r A' - 4 A + 2 A B
- 2 A B^2 L_\phi + 2 B r A' + 2 A B L_\phi = T_0,
$$

$$
r^2 B A' - r A' (B^2 - 3 B + r B' + B^2 L_\phi)
- A [B^2 (1 - B) L_\phi + r B' + B^2 - B] = T_0.
$$

The first constraint of eq. (11) ensures the vanishing of the right-hand side of $T^0_0$ and $T^j_0$ and also the disappearance of the $f_{TT}$ term in eqs. (5), (8) and (10). The rest of the constraints of eq. (11) comes from the coefficient of the $f_{TT}^j$ term of the differential equations (5)–(10). The constraints of eq. (11) constitute three non-linear differential equations in three unknown functions, $A(r)$, $B(r)$ and $L(\phi)$. Therefore, the constraints of (11) ensure the disappearance of the two terms $f_{TT}^j$ and $f_T$, which means that we are dealing with the equations of motion of TEG. The solution of these differential equations has the form

$$
A = \frac{1}{B} = \sqrt{\frac{1 - c_1}{r}},
$$

$$
L(\phi) = \left\{ \frac{2 \sqrt{r} - 2 c_1 + \sqrt{7 (2 + 2 r^2 T_0)} \phi}{2 \sqrt{r} - 2 c_1 + 2 \sqrt{r}} \right\} \phi,
$$

where $c_1$ is a constant of integration. Using eq. (12) in eq. (4) we get a constant value of the scalar torsion which gives a vanishing quantity of the second of eqs. (4). Therefore, eq. (12) is an exact vacuum solution to eqs. (5)–(10) provided that

$$
f(T_0) = -T_0, \quad f_T(T_0) = 1, \quad f_{TT} \neq 0.
$$

To understand the nature of the constant appearing in eq. (12), we are going to discuss the physics related to this solution and calculate the energy associated with the tetrad field (3) after using eq. (12).

**Physical properties of the derived solution.** – To understand the construction of the derived solution, let us rewrite tetrad (3), after using solution (12), in the following form:

$$
\phi (e^i_\mu) = (A^i_j) (e^j_\mu)_d,
$$

where

$$
(A^i_j) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \sin \theta \cos \Phi & \cos \theta \cos \Phi & -\sin \Phi \\
0 & \sin \theta \sin \Phi & \cos \theta \sin \Phi & \cos \Phi \\
0 & \cos \theta & -\sin \theta & 0
\end{pmatrix},
$$

$$
(e^i_\mu)_d = \begin{pmatrix}
\sqrt{1 - \frac{\nu}{r}} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \sqrt{1 - \frac{\nu}{r}} & 0 \\
0 & 0 & 0 & r \sin \theta
\end{pmatrix}.
$$

Equation (14) shows that the tetrad (3) with solution (12) consists of a diagonal tetrad in addition to "so(3)". The diagonal tetrad alone is not a solution to the field equations of $f(T)$. Therefore, "so(3)" plays an important role in $f(T)$, along with the angle $\Phi$ which is a function of the radial coordinate $r$. It is of interest to note that when $\Phi = \phi L(r) = \phi$, the derived solution will be a solution only to the linearized form of $f(T)$, i.e., $f(T) = T$.

Now we are going to calculate the energy associated with the derived solution using formula (2). The necessary non-vanishing components of the tensor $S^\mu_\nu$ are

$$
S^{001} = -\frac{r + 2 \sqrt{r^2 - 2 r c_1}}{\sqrt{r^4 - 2 r^3 c_1}},
$$

where

$$
L_\phi = \frac{\partial L(\phi)}{\partial \phi},
$$

$$
E = P^0 = \frac{1}{4 \pi} \int_V \sqrt{g} \left( \partial_\nu [e e^\mu_\nu S^{\mu \nu} f_T] \right),
$$

$$
fr \approx 1 + T + T^2 + \ldots,
$$

$$
E \approx \left( M - r - \frac{r M^2 T_0}{2} - 2 T_0 r^3 \right) \left[ 1 + T_0 + T_0^2 + \ldots \right] \approx \left( M - r - \frac{r M^2 T_0}{2} - 2 T_0 r^3 \right),
$$

which is the divergence. To remove such divergence we use the following expression:

$$
P_\phi^{\text{Regularized}} \approx \frac{1}{4 \pi} \int_V \sqrt{g} \left( \partial_\nu [e S^{\mu \nu} f(T)] \right) \left( \{\partial_\nu [e S^{\mu \nu} f(T)]\}_{\text{vanishing physical quantity}} \right),
$$

where the physical quantity here is $c_1$. Using eq. (15) we get

$$
E = P_0^{\text{Regularized}} \approx c_1,
$$

which is the energy of Schwarzschild provided that $c_1 = M$, where $M$ is the gravitational mass [46].
Main results and discussion. – In this study we have considered the modified teleparallel theory of gravitation, \( f(T) \), in the vacuum case. The field equations have been applied to a non-diagonal tetrad field having two unknown functions —in the radial coordinate, and an angle \( \Phi \), which is a function of the azimuthal angle. Six non-linear differential equations have been derived. These differential equations are not easily solved; therefore, some constraints have been imposed. These are: 1) The torsion scalar is constant. These constraints constitute three non-linear differential equations in three unknown functions. The solution contains one constant of integration. Therefore, an exact vacuum spherically symmetric solution to the field equations of the \( f(T) \) gravitational theory has been derived. This solution has a constant scalar torsion, i.e. \( T = T_0 \), and satisfies the field equations of \( f(T) \) if eq. (13) is satisfied. To understand what the nature of the constant of integration is, we calculated the energy associated with the derived solution. We have shown that such constant is related to the mass of gravitation.

We have also shown that the tetrad of the derived solution can be rewritten as two matrices. The first matrix is \( \text{so}(3) \), which is a special case of Euler’s angle [47]. On the other hand, the second matrix is a diagonal matrix of the Schwarzschild metric space-time. It is demonstrated that, when \( L(\phi) = \phi \), the derived solution will be a solution to the first order of \( f(T) \).

Repeating the same procedure done in the derivation of the solution, we can derive for tetrad (3) another solution in the framework of \( f(T) \). This solution has the following form:

\[
A = \frac{1}{B} = \sqrt{1 - \frac{c_2}{r}}, \quad L(\phi) = -\phi, \quad (18)
\]

where \( c_2 \) is another constant related to the gravitational mass. Solution (18) gave a vanishing value of the scalar torsion, \( T = 0 \), and is a solution to the field equations of \( f(T) \) provided that

\[
f(0) = 0, \quad f_T(0) = 1, \quad f_{TT} \neq 0. \quad (19)
\]

Solution (18) justifies that “any GR solution remains valid in \( f(T) \) theories having \( f_T(0) = 1 \) whenever the geometry admits a tetrad field with vanishing scalar torsion, \( T^r \) [42].

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