D3-branes wrapped on a spindle

Pietro Ferrero, Jerome P. Gauntlett, Juan Manuel Pérez Ipiña, Dario Martelli and James Sparks

1 Mathematical Institute, University of Oxford, Woodstock Road, Oxford, OX2 6GG, U.K.
2 Blackett Laboratory, Imperial College, Prince Consort Road, London, SW7 2AZ, U.K.
3 Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
4 INFN, Sezione di Torino & Arnold-Regge Center, Via Pietro Giuria 1, 10125 Torino, Italy

We construct supersymmetric $AdS_3 \times \Sigma$ solutions of minimal gauged supergravity in $D = 5$, where $\Sigma$ is a two-dimensional orbifold known as a spindle. Remarkably, these uplift on $S^5$, or more generally on any regular Sasaki-Einstein manifold, to smooth solutions of type IIB supergravity. The solutions are dual to $d = 2$, $N = (0, 2)$ SCFTs and we show that the central charge for the gravity solution agrees with a field theory calculation associated with D3-branes wrapped on $\Sigma$. Unlike for smooth $\Sigma$ the superconformal R-symmetry mixes with the $U(1)$ isometry of the spindle.

**INTRODUCTION**

Important insights into strongly coupled supersymmetric conformal field theories (SCFTs) can be obtained by realising them as the renormalization group fixed points of compactifications of higher-dimensional field theories. Such SCFTs can be constructed by starting with the field theories arising on the worldvolumes of branes in string theory or M-theory, wrapping them on a compact manifold, which describes the ultraviolet (UV) dimensions that arises in the IR, where $\Sigma$ has dimension 2. Subsequently there have been many generalisations including wrapping manifolds of higher dimension, relaxing the constant curvature condition, allowing for punctures, and so on (e.g. [3–5]). In all of these developments, supersymmetry is preserved by demanding that the field theory arising on the brane worldvolume is a “topologically twisted theory” [6, 7]. This involves both a coupling to the curved metric on $\Sigma$, and also a specific coupling to external R-symmetry gauge fields. An important consequence of this twisting is that the Killing spinors preserved by the gravitational solution are independent of the coordinates of $\Sigma$.

In this paper we discuss a class of supersymmetric gravitational solutions which have fundamentally new features compared with all previous constructions. We present supergravity solutions of the form $AdS_3 \times \Sigma$ of $D = 5$ minimal gauged supergravity which we interpret as the near horizon limit of black brane solutions associated with D3-branes wrapped on $\Sigma$. The first new feature is that supersymmetry of this near horizon solution is *not* realised by a topological twist. The second new feature is that $\Sigma$ is not a compact manifold but an orbifold. Recall that while a manifold is a topological space that is locally modelled on open subsets of $\mathbb{R}^n$, an orbifold is locally modelled on open subsets of $\mathbb{R}^n/\Gamma$, where $\Gamma$ are finite groups. More specifically, we will consider the weighted projective space $\Sigma = \mathbb{WCP}^1_{[n_-, n_+]}$ also known as a spindle. This is topologically a two-sphere but with conical deficit angles $2\pi(1 - 1/n_\pm)$ at the poles, specified by two coprime positive integers $n_\pm$, with $n_- \neq n_+$. The poles are then locally modelled on $\mathbb{R}^2/\mathbb{Z}_{n_\pm}$ and $\mathbb{R}^2/\mathbb{Z}_{n_+}$. The spindle is a “bad” orbifold in the sense that it is not possible to move to a covering space that is a manifold. Moreover, it does not admit a metric of constant curvature, in contrast to smooth Riemann surfaces.

The $AdS_3 \times \Sigma$ solutions of $D = 5$ minimal gauged supergravity that we construct have orbifold singularities. However, quite remarkably, when the solutions are uplifted to $D = 10$ solutions of type IIB supergravity, in general on a regular Sasaki-Einstein five-manifold $SE_5$, they become completely smooth. Moreover, we find the resulting $AdS_3 \times M_7$ solutions are precisely those of [8].

Our construction suggests that the $d = 2$, $N = (0, 2)$ SCFTs dual to the $AdS_3 \times M_7$ solutions of [8] arise from compactifying $d = 4$, $N = 1$ SCFTs on a spindle, where the $d = 4$ theories are holographically dual to $AdS_5 \times SE_5$. This includes the case of $N = 4$ SYM, where $SE_5 = S^5$. We also make a precision test of this interpretation: we compute the central charge and superconformal R-symmetry of the $d = 2$ field theories using anomaly polynomials and c-extremization [9] and find exact agreement with the gravity result [8]. Another novel feature is that the $d = 4$ R-symmetry mixes with the isometry of $\Sigma$ in flowing to the $d = 2$ R-symmetry in the IR, which does not happen for two-sphere horizons [10].
\[ D = 5 \text{ SOLUTIONS} \]

The equations of motion for \( D = 5 \) minimal gauged supergravity [11] are given by

\[
\begin{align*}
R_{\mu\nu} &= -4g_{\mu\nu} + \frac{2}{3}F_{\mu\rho}F_{\nu}^{\rho} - \frac{1}{3}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, \\
\ast F &= -\frac{2}{3}F \wedge F,
\end{align*}
\]  

(1)

where \( F = dA \) is the field strength of the Abelian R-symmetry gauge field \( A \). This is Einstein-Maxwell theory with a negative cosmological constant \( \Lambda = -6 \) and a Chern-Simons-type interaction for the Maxwell field \( A \). A solution is supersymmetric if it admits a non-trivial solution to the Killing spinor equation

\[
(\nabla_{\mu} - \frac{1}{2}\Gamma_{\mu}^{\rho\sigma} - 4\delta_{\mu}^{\rho}(\Gamma^{\sigma})F_{\nu\rho} - \frac{1}{2}\Gamma_{\mu}^{\rho} - 1A_{\mu})\epsilon = 0.
\]  

(2)

Here \( \epsilon \) is a Dirac spinor field, and \( \Gamma_{\mu} \) generate the Clifford algebra \( \text{Cliff}(4,1) \), so \( \{\Gamma_{\mu}, \Gamma_{\nu}\} = 2g_{\mu\nu} \).

It is straightforward to verify that

\[
ds^{2}_{\text{AdS}} + ds^{2}_{\Sigma} = \frac{4y}{9}ds^{2}_{\text{AdS}_{3}} \wedge ds^{2}_{\Sigma} + A dz,
\]  

(3)

solves the equations of motion (1). Here \( ds^{2}_{\text{AdS}_{3}} \) is the metric on \( \text{AdS}_{3} \) with unit radius, while

\[
ds^{2}_{\Sigma} = \frac{y}{q(y)}dy^{2} + \frac{q(y)}{36y^{2}}dz^{2},
\]  

(4)

is the metric on the horizon that we refer to as \( \Sigma \), and

\[
q(y) = 4y^{3} - 9y^{2} + 6ay - a^{2},
\]  

(5)

where \( a \) is a constant. Moreover, we can explicitly construct a non-trivial solution \( \epsilon \) to (2), given below.

Assuming \( a \in (0,1) \) the three roots \( y_{i} \) of \( q(y) \) are all real and positive. Defining \( y_{1} < y_{2} < y_{3} \), we then take \( y \in [y_{1}, y_{2}] \) to obtain a positive definite metric (3) on \( \Sigma \). However, as \( y \) approaches \( y_{1} \) and \( y_{2} \) it is not possible to remove the conical deficit singularities at both roots by a single choice of period \( \Delta z \) for \( z \), to obtain a smooth two-sphere. Instead we find that

\[
a = \frac{(n_{-} - n_{+})^{2}(2n_{-} + n_{+})^{2}(n_{-} + 2n_{+})^{2}}{4(n_{-}^{2} + n_{-}n_{+} + n_{+}^{2})^{3}},
\]  

(6)

\[
\Delta z = \frac{2(n_{-}^{2} + n_{-}n_{+} + n_{+}^{2})}{3n_{-}n_{+}(n_{-} + n_{+})} 2\pi,
\]

then \( ds^{2}_{\Sigma} \) is a smooth metric on the orbifold \( \Sigma = \text{WCP}[n_{-}, n_{+}] \). Specifically, there are conical deficit angles \( 2\pi(1 - 1/n_{z}) \) at \( z = y_{1}, y_{2} \), respectively, where \( n_{\pm} \) are arbitrary coprime positive integers with \( n_{-} > n_{+} \).

Note that there is a magnetic flux through \( \Sigma \) given by

\[
\frac{1}{2\pi} \int_{\Sigma} F = \frac{n_{-} - n_{+}}{2n_{-} - n_{+}}.
\]  

(7)

This may be contrasted with the Euler number:

\[
\chi(\Sigma) = \frac{1}{4\pi} \int_{\Sigma} R_{\Sigma} \text{vol}_{\Sigma} = \frac{n_{-} + n_{+}}{n_{-} - n_{+}},
\]  

(8)

where \( R_{\Sigma} \) is the Ricci scalar of \( \Sigma \), and \( \text{vol}_{\Sigma} \) is its volume form. In general the integral of the curvature for a \( U(1) \cong SO(2) \) gauge field on \( \Sigma = \text{WCP}[n_{-}, n_{+}] \) is necessarily \( 2\pi/(n_{-} + n_{+}) \) times an integer. This makes the gauge field \( A \), with flux given by (7), a spinor gauge field: \( n_{-} - n_{+} \) is even/odd precisely when \( n_{-} + n_{+} \) is even/odd, so that spinor fields with unit charge under \( A \) are always globally well-defined. We refer to [12] for more details.

The \( D = 5 \) Killing spinor \( \epsilon \) can be constructed as follows. We write the \( D = 5 \) gamma matrices as \( \Gamma^{\alpha} = \gamma^{a} \otimes \sigma^{\alpha} \), for \( a = 0, 1, 2 \) with \( \gamma^{0} = -i\sigma^{2}, \gamma^{1} = \sigma^{4}, \gamma^{2} = \sigma^{3} \) and \( \Gamma^{3} = 1 \otimes \sigma^{2}, \Gamma_{4} = 1 \otimes \sigma^{1} \), where \( \sigma^{i} \) are Pauli matrices. We then write \( \epsilon = \vartheta \otimes \chi \) with \( \vartheta \) a Killing spinor for \( \text{AdS}_{3} \) satisfying \( \nabla_{\vartheta} \vartheta = \frac{1}{2} \gamma_{\vartheta} \vartheta \). We find that the two-component spinor \( \chi \) on the spindle is given by

\[
\chi = \left( \sqrt{\frac{q_{1}(y)}{y}}, \frac{\sqrt{q_{2}(y)}}{\sqrt{y}} \right),
\]  

(9)

where we have defined

\[
q_{1}(y) = -a + 2y^{3/2} + 3y, \quad q_{2}(y) = a + 2y^{3/2} - 3y,
\]  

(10)

which satisfy \( q(y) = q_{1}(y)q_{2}(y) \). In contrast to the topological twist, this spinor depends on the coordinates of \( \Sigma \). Moreover, as shown in [12], the spinor is in fact a section of a non-trivial bundle over \( \Sigma \). We emphasise that the gauge choice used in (7) has been fixed by requiring the solution \( \epsilon \) to be independent of \( z \).

\[ \text{UPLIFT TO IIB STRING THEORY} \]

Any supersymmetric solution to minimal \( D = 5 \) gauged supergravity uplifts (locally) to a solution of type IIB supergravity via the following [13; 14]:

\[
ds^{2}_{10} = L^{2} \left[ ds^{2}_{5} + \left( \frac{1}{2}d\psi + \sigma + \frac{3}{2}A \right)^{2} + ds^{2}_{KE_{4}} \right]
\]

\[
g_{s}F_{5} = L^{4} \left[ 4\text{vol}_{5} - \frac{3}{8} \ast_{6} F \wedge J \\
+ (2J \wedge J - \frac{3}{4}F \wedge J) \wedge (\frac{1}{2}d\psi + \sigma + \frac{3}{2}A) \right].
\]  

(11)

Here \( F_{5} \) is the self-dual five-form, \( g_{s} \) is the string coupling constant, and \( L > 0 \) is an arbitrary length scale, that is fixed by flux quantization. \( KE_{4} \) is an arbitrary positively curved Kähler-Einstein four-manifold with Kähler form \( J \), with metric normalized so that the Ricci form is \( R = 6J \), and \( \sigma \) is a local one-form with \( d\sigma = 2J \).

Remarkably, substituting the form of the \( D = 5 \) solution (7) into (11) we find that the \( D = 10 \) metric may be written as

\[
ds^{2}_{10} = \frac{4}{9}L^{2}y \left[ ds^{2}_{\text{AdS}_{3}} + ds^{2}_{M_{7}} \right],
\]  

(12)
where the metric on the seven-manifold $\mathcal{M}_7$ is precisely that in equation (3) of [3] (with a different normalization of the $KE_4$ metric). The five-form flux also agrees (up to a typographical error in [3]).

In [3] it was shown that these are smooth supersymmetric solutions of type IIB supergravity, with the compact seven-manifold $\mathcal{M}_7$ being the total space of a Lens space $S^3/\mathbb{Z}_q$ fibration over the $KE_4$, where the twisting is parametrized by another positive integer $p$. The Lens space fibre has coordinates $y, z, \psi$. In terms of our parameters $n_\pm$ we identify

$$p = kn_+, \quad q = \frac{k}{(n_- - n_+)},$$

where $p, q \in \mathbb{N}$ are coprime. Here $I$ is the largest positive integer (known as the Fano index) for which $\int\! c_1(I) \in \mathbb{Z}$, for all two-cycles $S$ in the $KE_4$, where we have introduced the cohomology class $c_1 = [\mathcal{R}/2\pi] \in H^2(KE_4, \mathbb{Z})$. We have also defined

$$k = \text{hcf}(I, p), \quad \text{(14)}$$

and identify $\psi$ with period $\Delta \psi$ given by [13]

$$\Delta \psi = \frac{2\pi I}{k}. \quad \text{(15)}$$

At a fixed point in the $M$-plane $\in \mathbb{R}$

$$\frac{(1 + \mathcal{R})}{2\pi}$$

and identify $\psi$ with period $\Delta \psi$ given by [13]

$$\Delta \psi = \frac{2\pi I}{k}. \quad \text{(15)}$$

At a fixed point in the $D = 5$ spacetime, the internal five-dimensional metric $ds^2_{5} = (\frac{1}{4}d\psi + \sigma)^2 + ds^2_{KE_4}$ in [11] is then a regular Sasaki-Einstein manifold, which is simply-connected when $k = 1$.

In order to obtain a string theory background one must also quantize the five-form flux $F_5$ through all five-cycles in $\mathcal{M}_7$. This was carried out in [3]. We define the integers

$$M = \int_{KE_4} c_1 \wedge c_1 = \frac{1}{4\pi^2} \int_{KE_4} \mathcal{R} \wedge \mathcal{R}, \quad \text{(16)}$$

and $h = \text{hcf}(M/I^2, q)$. Then if we choose the length scale $L$ to satisfy

$$\frac{L^4}{g_s\ell_s^4} = \frac{108\pi}{f^3h^3} k^2 n_+ n_- n_n, \quad \text{(17)}$$

where $\ell_s$ is the string length and $n \in \mathbb{N}$ is an arbitrary positive integer, then one finds that $1/(2\pi\ell_s)^4 \int_{\Sigma} F_5 \in \mathbb{Z}$, for all five-cycles $D$ in $\mathcal{M}_7$.

There is a finite set of choices for the positively curved $KE_4$. If $KE_4 = \mathbb{CP}^2$ then $I = 3, M = 9$. For this case, $k = 1$ gives $SE_5 = S^5$ as the internal space while $k = 3$ gives $S^3/\mathbb{Z}_3$. If $KE_4 = S^2 \times S^2$ we have $I = 2, M = 8$. Now $k = 1$ gives $SE_5 = T^{1,1}$, while $k = 2$ gives $SE_5 = T^{1,1}/\mathbb{Z}_2$. Finally, for $KE_4 = dP_3$, $3 \leq m \leq 8$, where $dP_m$ is a del Pezzo surface, we have $I = 1, M = 9 - m$.

A key observation in this paper is that the $AdS_3$ solutions of [3] may also be viewed as $SE_5$ fibrations over $\Sigma = \mathbb{WCPS}^{1}_{[n_- n_+]}$. In fact we may begin with any weighted projective space, with weights $n_- > n_+$, and then define $p$ and $q$ via [13], where we also define

$$k = \frac{I}{\text{hcf}(I, n_- - n_+)}. \quad \text{(18)}$$

With this definition, $p$ and $q$ are manifestly coprime integers, and one can check that [13] is equivalent to [18]. With this perspective, we may then calculate the flux of $F_5$ through the $SE_5$ fibre, which we denote as

$$N = \frac{1}{(2\pi\ell_s)^4} \int_{SE_5} F_5 = \frac{M}{f^3h} n_- n_+ n_n n_n \in \mathbb{N}. \quad \text{(19)}$$

Notice that for a given spindle, specified by $n_\pm$, and a given choice of $KE_4$ we only get a smooth type IIB solution for $k$ as in [13] and hence a specific $SE_5$. E.g. if $KE_4 = \mathbb{CP}^2$ and $n_+ = 2$, then for $n_- = 3, 7, \ldots$ and $n_- = 5, 9, \ldots$ we can uplift on $S^3/\mathbb{Z}_3$ and $S^5$, respectively.

As is well-known [10], the central charge $c$ is fixed by the $AdS_3$ radius $L$ and the Newton constant $G_N(3)$ of the effective three-dimensional theory obtained by compactifying type IIB string theory on $\mathcal{M}_7$ via $c = 3L/2G_N(3)$. We can rewrite the result derived in [3] as

$$c = \frac{4(n_- - n_+)^3}{3n_- n_+ (n_-^2 + n_- n_+ + n_+^2)} a_{4d}, \quad \text{(20)}$$

where we have introduced

$$a_{4d} = \frac{\pi^2 N^2}{4\text{vol}(SE_5)}. \quad \text{(21)}$$

In [3] the dual $d = 2, N = (0, 2)$ SCFTs were not identified, but our equivalent $D = 5$ construction of the solutions, together with the flux condition [13], leads to an immediate conjecture. Specifically, one should begin with the $d = 4$ SCFT dual to $AdS_5 \times SE_5$, describing $N$ D3-branes at the Calabi-Yau three-fold singularity with conical metric $dr^2 + r^2 ds^2_{5}$. The large $N$ central charge of this theory is precisely given by $a_{4d}$ in [21] [17]. One then compactifies that theory on $\Sigma = \mathbb{WCPS}^{1}_{[n_- n_+]}$, with a background $R$-symmetry gauge field with magnetic flux [7]. The supergravity solutions we have described suggest the theory flows to a strongly coupled $d = 2, N = (0, 2)$ SCFT in the IR, and we will give evidence for this in the next section by computing the central charge directly via a field theory calculation.

We conclude this section by noting that supersymmetric $AdS_3$ solutions with five-form flux were studied in [18]. The $U(1)_R$ symmetry of the dual $(0, 2)$ theory is realized geometrically by a canonical Killing vector field on the internal space $\mathcal{M}_7$, and for our solutions this vector field was found in [19] to be

$$R_{2d} = 2\partial_\psi + \frac{3n_- n_+ (n_- n_+)}{n_-^2 n_- n_+ n_+^2} \partial_\phi. \quad \text{(22)}$$
Here we have defined
\[ \varphi = \frac{2\pi z}{\Delta z}, \] (23)
so that \( \Delta \varphi = 2\pi \), and \( \partial_{\varphi} \) generates the \( U(1) \) isometry of the weighted projective space \( \Sigma \), which we shall refer to as \( U(1)_J \). We also note that the Killing spinor on the \( SE_3 \) has unit charge under \( R_{4d} = 2\partial_{\varphi} \), which may therefore be identified with the superconformal \( d = 4 \) SCFT before compactification on \( \Sigma \). In other words, equation (22) states that the disconnected \( d = 4 \) R-symmetry mixes with \( U(1)_J \) in flowing to the \( d = 2 \) R-symmetry in the IR. We shall also recover (22) from a field theory calculation in the next section.

\[ d = 4 \text{ SCFTs on} \ \Sigma \]

We begin with a general \( d = 4 \) SCFT with anomaly polynomial given by the 6-form
\[ A_{4d} = \frac{1}{6} \text{tr} R^3 c_1(R_{4d})^3 - \frac{1}{24} \text{tr} R c_1(R_{4d}) p_1(T Z_6). \] (24)

As is standard, in (even) dimension \( d \) the anomaly polynomial is a \( (d+2) \)-form on an abstract \( (d+2) \)-dimensional space, here called \( Z_6 \). \( c_1(R_{4d}) \) denotes the first Chern class of the \( d = 4 \) superconformal \( U(1)_R \) symmetry bundle over \( Z_6 \), and \( p_1 \) denotes the first Pontryagin class. The trace is over Weyl fermions when the theory has a Lagrangian description, and in any case we may always write
\[ \text{tr} R^3 = \frac{16}{9} (5a_{4d} - 3c_{4d}), \text{ tr} R = 16 (a_{4d} - c_{4d}), \] (25)
in terms of the central charges \( a_{4d} \), \( c_{4d} \). In the following we focus on the large \( N \) limit in which \( a_{4d} = c_{4d} \) to leading order, and hence we may write
\[ A_{4d} = \frac{16a_{4d}}{27} c_1(R_{4d})^3 \text{ (at large } N). \] (26)

We have computed the subleading correction, but we do not present the details as they are a little unwieldy.

We wish to compactify the \( d = 4 \) theory on \( \Sigma = \mathbb{CP}^{[n,...,n]} \), with a magnetic flux \( 7 \) for the \( d = 4 \) R-symmetry gauge field. The resulting \( d = 2 \) anomaly polynomial will then capture the right-moving central charge \( c_r \), but crucially we need to include the \( U(1)_J \) global symmetry in \( d = 2 \) that comes from the isometry of \( \Sigma \). Geometrically, this involves taking \( Z_6 \) to be the total space of a \( \Sigma \) fibration over a four-manifold \( Z_4 \) \( 14 \). More precisely, we let \( J \) be a \( U(1) \) bundle over \( Z_6 \), with connection corresponding to a background gauge field \( A_J \) for the \( d = 2 \) \( U(1)_J \) global symmetry, and then fibre \( \Sigma \) over \( Z_4 \) using the \( U(1)_J \) action and connection \( A_J \). In practice this amounts to the replacement \( d\varphi \to d\varphi + A_J \).

Incorporating the magnetic flux \( 7 \) into this construction amounts to “gauging” the \( U(1) \) gauge field \( A \) in \( 8 \), as just described. Thus, we define the following one-form on \( Z_6 \):
\[ \mathcal{A} = \frac{1}{4} \left( 1 - \frac{a}{g} \right) \frac{\Delta z}{2\pi} (d\varphi + A_J) \equiv \rho(y)(d\varphi + A_J), \] (27)
where recall that the gauge choice we made is such that the Killing spinors are uncharged under the \( U(1)_J \) symmetry generated by \( \partial_{\varphi} \). This is necessary for the twisting to make sense. \( \mathcal{A} \) defined by (27) is a gauge field on \( Z_6 \), which restricts to the supergravity gauge field \( A \) \( 3 \) on each \( \Sigma \) fibre. We compute the curvature
\[ \mathcal{F} = d\mathcal{A} = \rho(y)dy \wedge (d\varphi + A_J) + \rho(y)F_J, \] (28)
where \( F_J = dA_J \). The one-form \( d\varphi + A_J \) is precisely the global angular form for the \( U(1) \) bundle, and so is globally defined on \( Z_6 \) away from the poles of \( \Sigma = \mathbb{CP}^{[n,...,n]} \) at \( y = y_1, y_2 \). Moreover, one can verify that \( \rho(y)dy \wedge d\varphi \) vanishes smoothly at the poles, where the angular coordinate \( \varphi \) is not defined, implying that \( \mathcal{F} \) is a globally defined closed two-form on \( Z_6 \). By construction the integral of \( \mathcal{F} \) over a fibre \( \Sigma \) of \( Z_6 \) satisfies (7). More generally, the integrals of wedge products over the fibres are given, for \( s \in \mathbb{N} \), by
\[ \int_{\Sigma} \left( \frac{\mathcal{F}}{2\pi} \right)^s = \frac{1}{2\pi} \left( \frac{1}{n_+} - \frac{1}{n_-} \right) \left( -\frac{F_J}{2\pi} \right)^{s-1}. \] (29)

The curvature form \( \mathcal{F} \) defines a \( U(1) \) bundle \( \mathcal{L} \) over \( Z_6 \) by taking \( c_1(\mathcal{L}) = \lvert \lvert \mathcal{F} / 2\pi \rvert \rvert \in H^2(Z_6, \mathbb{R}) \). This is different from \( 10 \), where the \( U(1) \cong SO(2) \) bundle was taken to be the tangent bundle to the fibres \( T_{fibres} Z_6 \), which gives the Euler class \( 8 \), rather than (7). We note that at the poles we have \( c_1(\mathcal{L}) \bigg|_{y=y_1,y_2} = -\frac{1}{2\pi} c_1(J) \), where we have defined \( c_1(J) = \lvert F_J / 2\pi \rvert \in H^2(Z_4, \mathbb{Z}) \). In the anomaly polynomial we then write
\[ c_1(R_{4d}) = c_1(R_{2d}) + c_1(\mathcal{L}), \] (30)
where \( R_{2d} \) is the pull-back of a \( U(1) \) bundle over \( Z_4 \). Notice that the twisting \( 30 \) will make sense globally only if the \( d = 4 \) R-charges of fields satisfy appropriate quantization conditions, and for gauge-invariant operators this is equivalent to the global regularity and flux quantization conditions imposed on the supergravity solutions, cf. the discussion below equation (10).

The \( d = 2 \) anomaly polynomial is obtained by integrating \( A_{4d} \) in (29) over \( \Sigma \). Using (29) and (30) we compute
\[ A_{2d} = \frac{2a_{4d}}{27} \left[ 12 \left( \frac{1}{n_+} - \frac{1}{n_-} \right) c_1(R_{2d})^2 
- \left( \frac{1}{n_+} - \frac{1}{n_-} \right) c_1(R_{2d}) c_1(\mathcal{L}) 
+ \left( \frac{1}{n_+} - \frac{1}{n_-} \right) c_1(J)^2 \right]. \] (31)
The coefficient of $\frac{1}{2}c_1(L_i)c_1(L_j)$ in $\mathcal{A}_{2d}$ is $\text{tr} \gamma^3 Q_i Q_j$, where the global symmetry $Q_i$ is associated to the $U(1)$ bundle $L_i$ over $Z_4$, and $\gamma^3$ is the $d=2$ chirality operator. On the other hand, $c$-extremization implies that the $d=2$ superconformal $U(1)_R$ extremizes

$$c_{\text{trial}} = 3 \text{tr} \gamma^3 R_{\text{trial}}^2,$$

over the space of possible R-symmetries. We set

$$R_{\text{trial}} = R_{2d} + \epsilon J,$$

and extremize the quadratic function of $\epsilon$ one obtains from (31) and (32). The extremal value is

$$\epsilon_* = \frac{3n_- n_+ (n_- + n_+)}{n_-^2 + n_- n_+ + n_+^2}. \quad (34)$$

The right-moving central charge is given by (33) evaluated on the superconformal R-symmetry. Substituting (31) into (32), (33) we find

$$c_r = \frac{(n_- - n_+)^3}{n_- n_+ (n_-^2 + n_- n_+ + n_+^2)} \frac{4d_{4d}}{3}. \quad (35)$$

At leading order in the large $N$ limit, $c_r = c_0 \equiv c$ is the central charge of the SCFT, and we see that the field theory result (35) precisely matches the gravity result (20). Moreover, the R-symmetry (33), with $\epsilon = \epsilon_*$, precisely matches the supergravity R-symmetry (22).

**DISCUSSION**

We have constructed a novel class of gravity solutions that are associated with D3-branes wrapped on a spindle $\Sigma = \text{WCP}^1_{[n_- n_+]}$. We have argued that these theories flow to $d=2$, $\mathcal{N} = (0,2)$ SCFTs in the IR, and provided exact quantitative tests of this conjecture.

The new solutions exhibit a number of new features, raising several directions for future research. Firstly, despite having orbifold singularities in $D=5$, when uplifted to $D=10$ the solutions are completely regular. What type of singularities are permitted in lower-dimensional supergravity theories that have this property? Secondly, it is often claimed in the literature that supersymmetry requires a topological twist when branes wrap a compact manifold, so that the “twisted spinors” are constant on the manifold. The near horizon solutions we have presented are a counterexample. It would be interesting to understand this more systematically, both for uplifts that are not on regular Sasaki-Einstein manifolds, but also more broadly in other theories, and in different dimensions. For example, are there instances for which the compactification space is not an orbifold? Thirdly, we have provided the first holographic examples in which there is a mixing of the higher-dimensional and lower-dimensional R-symmetries (with no rotation). We note that [10] raised the question of whether there are supergravity solutions where the isometry of the internal space mixes with the R-symmetry of the $d=2$ CFT, and how this is realized, which we have answered in this paper.

Fourthly, the results of this paper strongly suggest there should exist black string solutions which approach $AdS_5$ in the UV and $AdS_5 \times \Sigma$ in the IR. In fact we expect many such solutions, differing in the detailed approach to the UV, as in [3]. It is possible that the simplest such solution can be constructed in minimal $D=5$ gauged supergravity, but it may be necessary to construct them within type IIB supergravity. The conformal boundary of such solutions will illuminate the precise deformations of the $d=4$, $\mathcal{N} = 1$ SCFTs that can then flow to the $d=2$, $\mathcal{N} = (0,2)$ SCFTs dual to the $AdS_3 \times M_7$ solutions of [8]. In particular, it will clarify the precise way in which the D3-brane is wrapping the spindle while preserving supersymmetry.

In the companion paper [12] we present analogous supergravity solutions in $D=4$, associated with M2-branes wrapped on a “spinning spindle”. In that case the full black hole solution is known, not just the near horizon geometry. The $D=4$ black holes approach $AdS_4$ in the UV and $AdS_4 \times \Sigma$ in the IR, with the horizon $\Sigma$ of the black hole a spindle. The $D=4$ black holes are not only charged and rotating, but also accelerating, and this leads to the conical deficit singularities on $\Sigma$. Once lifted to $D=11$ these orbifold singularities are removed and, in particular, the $AdS_4 \times \Sigma$ solutions become completely regular. Supersymmetry is again not realised by the topological twist for the $AdS_4 \times \Sigma$ solution, similar to our $AdS_3 \times \Sigma$ solutions. In the UV, the black holes at finite temperature have a conformal boundary consisting of a spindle. In the supersymmetric and extremal limit, however, the spindle degenerates into “two halves”, each of which is associated with a topological twist, but with a different constant spinor on each component! It is another fascinating open question to determine how typical such a novel realisation of supersymmetry is for branes wrapping spindles, as well as spaces of higher dimension.

**Acknowledgments**

This work was supported in part by STFC grants ST/P000762/1, ST/T000791/1 and ST/T000864/1. JGP is supported as a KIAS Scholar and as a Visiting Fellow at the Perimeter Institute.

[1] In general $M$ will be fibred over $\Sigma$, and the metric will not be a direct product, but a warped product.

[2] J. M. Maldacena and C. Nunez, Int.J.Mod.Phys. A16, 822 (2001) [arXiv:hep-th/0007018 [hep-th]].

[3] J. P. Gauntlett, N. Kim, and D. Waldram, Phys. Rev. D63, 126001 (2001) [arXiv:hep-th/0012195].
[4] D. Gaiotto and J. Maldacena, (2009), arXiv:0904.4466 [hep-th].

[5] M. T. Anderson, C. Beem, N. Bobev, and L. Rastelli, Commun. Math. Phys. 318, 429 (2013), arXiv:1109.3724 [hep-th].

[6] E. Witten, Commun. Math. Phys. 117, 353 (1988).

[7] M. Bershadsky, C. Vafa, and V. Sadov, Nucl. Phys. B 463, 420 (1996), arXiv:hep-th/9511222.

[8] J. P. Gauntlett, O. A. P. Mac Conamhna, T. Mateos, and D. Waldram, Phys. Rev. Lett. 97, 171601 (2006), arXiv:hep-th/0606221 [hep-th].

[9] F. Benini and N. Bobev, Phys. Rev. Lett. 110, 061601 (2013), arXiv:1211.4030 [hep-th].

[10] S. M. Hosseini, K. Hristov, Y. Tachikawa, and A. Zaffaroni, JHEP 09, 167 (2020) arXiv:2006.08629 [hep-th].

[11] M. Gunaydin, G. Sierra, and P. Townsend, Nucl. Phys. B 242, 244 (1984).

[12] P. Ferrero, J. P. Gauntlett, J. M. Perez Ipiña, D. Martelli, and J. Sparks, To appear.

[13] A. Buchel and J. T. Liu, Nucl. Phys. B771, 93 (2007), arXiv:hep-th/0608002.

[14] J. P. Gauntlett and O. Varela, Phys. Rev. D76, 126007 (2007), arXiv:0707.2315 [hep-th].

[15] Notice that $\Delta \psi \Delta z$ agrees with the value in [8], but because we have identified the coordinates $z, \psi$ in the opposite order they cover the corresponding torus differently.

[16] J. Brown and M. Henneaux, Commun. Math. Phys. 104, 207 (1986).

[17] S. S. Gubser, Phys. Rev. D 59, 025006 (1999), arXiv:hep-th/9807164.

[18] N. Kim, JHEP 01, 094 (2006), arXiv:hep-th/0511029 [hep-th].

[19] J. P. Gauntlett, N. Kim, and D. Waldram, JHEP 04, 005 (2007) arXiv:hep-th/0612253 [hep-th].