GL(2,\mathbb{R}) Structures, G_2 Geometry and Twistor Theory

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Abstract. A GL(2,\mathbb{R}) structure on an (n+1)-dimensional manifold is a smooth point-wise identification of tangent vectors with polynomials in two variables homogeneous of degree n. This, for even n = 2k, defines a conformal structure of signature (k, k+1) by specifying the null vectors to be the polynomials with vanishing quadratic invariant. We focus on the case n = 6 and show that the resulting conformal structure in seven dimensions is compatible with a conformal G_2 structure or its non-compact analogue. If a GL(2,\mathbb{R}) structure arises on a moduli space of rational curves on a surface with self-intersection number 6, then certain components of the intrinsic torsion of the G_2 structure vanish. We give examples of simple 7th order ODEs whose solution curves are rational and find the corresponding G_2 structures. In particular we show that Bryant’s weak G_2 holonomy metric on the homology seven-sphere SO(5)/SO(3) is the unique weak G_2 metric arising from a rational curve.

1. Introduction

Consider the three-dimensional space M of holomorphic parabolas in \mathbb{C}^2. Each parabola is of the form

\[ y = ax^2 + 2bx + c \]

and (a, b, c) serve as local holomorphic coordinates on M. Two parabolas generically intersect at two points, and we can define a holomorphic conformal structure on M by declaring two points p and \tilde{p} to be null separated iff the corresponding parabolas are tangent. The tangency condition is equivalent to a polynomial equation

\[ v^3x^2 + 2v^2x + v^1 = 0 \]

having a double root. Here (v^1, v^2, v^3) = (\tilde{c} - c, \tilde{b} - b, \tilde{a} - a) is the vector connecting p and \tilde{p}. Calculating the discriminant shows that this vector is null if (v^3)^2 - v^1v^3 = 0. This quadratic condition defines a flat conformal structure on M = \mathbb{C}^3.

An immediate question is whether this approach can be generalised to curved conformal structures. One answer goes back to Wünschmann [29] who worked in the real category. The parabolas are integral curves of a third order ODE \( y'' = 0 \). Wünschmann has found the necessary and sufficient condition for a general third order ODE so that the conformal structure induced on the solution space by the tangency condition is well defined. The question has also been considered in the context of twistor theory [14].
where one is not concerned with differential equations but rather with the algebro-geometric properties of rational curves in a complex two-fold.

How about higher dimensions? It turns out that one can define conformal structures on certain odd-dimensional moduli spaces of rational curves, but the discriminant (which is not quadratic for higher degree curves) needs to be replaced by another invariant. In this paper we shall consider the seven-dimensional case and answer the following questions:

- Given a seven-dimensional family of rational curves, can one define a conformal complexified $G_2$ structure on the moduli space $M$ on these curves? Does this structure admit a real form of Riemannian signature?
- Can one characterise the curves and the corresponding $G_2$ structures in terms of a 7th order ODE $y^{(7)} = F(x, y, y', \ldots, y^{(6)})$ with $M$ as its solution space?

The methods employed in the paper form a mixture of ‘old’ and ‘new’. To define the conformal structure on $M$ we use the 19th century classical invariant theory (formula (3.7) Section 3), but the characterisation of curves builds on the Penrosean holomorphic twistor methods. The allowed rational curves must (after complexification) have self-intersection number 6 in some complex two-fold or a normal bundle $O(5) \oplus O(5)$ in a complex contact three-fold. This allows a point-wise identification of tangent vectors in $M$ with sextic homogeneous polynomials in two variables. Now the invariant theory can be applied to construct a conformal structure, and the associated $G_2$ three-form $\phi$ (formulae (4.9) and (4.12) Section 4). The ODE approach gives a good handle on the local differential geometry on $M$ and allows expressing the components of intrinsic torsion of the $G_2$ structure (as well as the torsion of the associated Cartan connection) in terms of the contact invariants of the corresponding ODE (Theorems 5.1 and 5.2 formulated in Section 5 and proved in Section 10). Here we make an extensive use of the Tanaka–Morimoto theory of normal Cartan’s connection (Sections 8 and 9). These methods allow us show that if the component of the intrinsic $G_2$ torsion taking value in the 27-dimensional irreducible representation $\Lambda^3(\mathbb{R}^7^*)$ vanishes, then the resulting $G_2$ geometry admits a Riemannian real form and (up to diffeomorphisms) it is either flat, or is given by Bryant’s weak $G_2$ holonomy [2] on $SO(5)/SO(3)$, or is given by a seven-parameter family of curves

$$(y + Q(x))^2 + P(x)^3 = 0,$$

where the polynomials $(Q(x), P(x))$ are the general cubic and quadratic respectively. These curves have degree six, but we shall find that they are rational and form a complete analytic family. The corresponding 7th order ODE is

$$y^{(7)} = \frac{21}{5} \frac{y^{(6)}y^{(5)}}{y^{(4)}} - \frac{84}{25} (y^{(5)})^3 \frac{\partial^k y}{\partial x^k}$$

and the associated conformal structure is given by (4.9) and (6.22). There exists a choice of the conformal factor such that corresponding $G_2$ structure
is closed, i.e.
\[ d\phi = 0, \quad d^*\phi = \tau \wedge \phi \]
for some two–form \( \tau \) on \( M \).

Most calculations in the second half of the paper were performed using MAPLE. In particular proving Theorem 5.2 required solving a system of over 600 quadratic equations for components of curvature and torsion of Cartan’s normal connection. The resulting expressions are usually long and unilluminating and we have not included all of them in the manuscript. Readers who want to verify our calculations can obtain the MAPLE codes from us.

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2. \( GL(2, \mathbb{R}) \) STRUCTURES

Definition 2.1. A \( GL(2, \mathbb{R}) \) structure on a smooth \((n+1)\) dimensional manifold \( M \) is a smooth bundle isomorphism
\[ TM \cong \mathbb{S} \circ \mathbb{S} \circ \cdots \circ \mathbb{S} = \mathbb{S}^n, \]
where \( \mathbb{S} \to M \) is a real rank–two vector bundle, and \( \circ \) denotes symmetric tensor product.

The isomorphism (2.1) identifies each tangent space \( T_tM \) with the space of homogeneous \( n \)th order polynomials in two variables. The vectors corresponding to polynomials with repeated root of multiplicity \( n \) are called maximally null. A hyper-surface in \( M \) is maximally null if its normal vector is maximally null.

In practice the isomorphism (2.1) giving rise to a \( GL(2, \mathbb{R}) \) structure is specified by a binary quantic with values in \( T^*M \)
\[ Q(X_1, X_2) = \sum_{i=0}^{n} \binom{n}{i} \theta^{i+1}(X_1)^i(X_2)^{n-i}, \quad \binom{n}{i} = \frac{n(n-1)\ldots(n-i+1)}{i!}. \]

Here \( (X_1, X_2) \) are coordinates on \( \mathbb{R}^2 \), and the ‘coefficients’ in the quantic are given by linearly independent one–forms \( \theta^1, \theta^2, \ldots, \theta^{n+1} \) on \( M \). If \( V \) is a vector field on \( M \), then the corresponding polynomial is given by \( V \cdot Q \), where \( \cdot \) denotes the contraction of a one–form with a vector field. If \( V = \sum_i v^i \theta_i \) is expressed in a basis \( \theta_i \) of \( T^*M \) such that \( \theta_i \cdot \theta_j = \delta^{ij}_j \), the polynomial is
\[ \sum_{i=0}^{n} \binom{n}{i} v^{i+1}(X_1)^i(X_2)^{n-i}, \]
with the coefficients \( v^i \) being smooth functions on \( M \).
Consider a general ODE of order \((n + 1)\)

\[
\frac{d^{n+1}y}{dx^{n+1}} = F(x, y, y', \ldots, y^{(n)}),
\]

where \(y' = dy/dx\) etc, whose general solution is of the form \(y = Z(x, t)\) where \(t\) are constants of integration. Assume that the space of solutions to (2.4) is equipped with a \(GL(2, \mathbb{R})\) structure (2.1) such that the two–parameter family of hyper-surfaces given by fixing \((x, y)\) are maximally null. It has been shown in [9] that this imposes conditions on \(F\) which are expressed by vanishing of \((n - 1)\) expressions

\[
W_\alpha[F], \quad \alpha = 1, 2, \ldots, n - 1
\]

for the ODE (2.4). Each expression \(W_\alpha\) is a polynomial in the derivatives of \(F\). The simplest of these is the contact invariant

\[
W_1[F] = D^2F_n - \frac{6}{n+1}F_nDF_n + \frac{4}{(n+1)^2}(F_n)^3 - \frac{6}{n}DF_{n-1}
\]

\[
+ \frac{12}{n(n+1)}F_nF_{n-1} + \frac{12}{n(n-1)}F_{n-2},
\]

where

\[
F_k = \frac{\partial F}{\partial y^{(k)}}, \quad \text{and} \quad D = \frac{\partial}{\partial x} + \sum_{k=1}^n y^{(k)} \frac{\partial}{\partial y^{(k-1)}} + F \frac{\partial}{\partial y^{(n)}}.
\]

Moreover if \(W_1[F] = W_2[F] = \cdots = W_{n-1}[F] = 0\) then \(W_m[F]\) is a contact invariant of the ODE (2.4). The explicit expressions for \(W_\alpha\) are unilluminating, but for completeness we list the five invariants (in the form given in [12]) of 7th order ODEs in Appendix A.

The same invariants have also arisen in other related contexts [6, 7, 12]. The description given by Doubrov is particularly clear. First note that a linearisation of the ODE (2.4) around any of its solutions is a linear homogeneous ODE of the form

\[
(\delta y)^{(n+1)} = p_n(x)\delta y^{(n)} + \cdots + p_0(x)\delta y,
\]

where \(p_k = \partial F/\partial y^{(k)}\) is evaluated at the solution.

**Theorem 2.2.** [6, 7] The expressions (2.5) vanish if and only if the linear homogeneous ODE (2.6) can be brought to a form \(\delta y^{(n+1)} = 0\) by a coordinate transformation \((x, y) \rightarrow (\beta(x), \gamma(x)y)\) for some functions \(\beta\) and \(\gamma\). Vanishing of (2.5) is invariant under the contact transformations of the nonlinear ODE (2.4).

The linear homogeneous ODEs of the form (2.6) have been studied by Wilczynski [28] who gave explicit conditions for their trivialisability in terms of the functions \(p_k\) and their derivatives.

In the simplest nontrivial case \(n = 2\) the corresponding invariant was already known to Wünschmann [29]. In the case \(n = 3\) the invariants have been implicitly constructed by Bryant in his study of exotic holonomy [3] and developed by Nurowski [20].

One source of ODEs for which these contact invariants vanish comes from twistor theory [3, 9]. Let \(\mathcal{Y}\) be a complex contact three-fold with an embedded rational Legendrian curve with a normal bundle \(N = \mathcal{O}(n-1)\oplus\mathcal{O}(n-1)\).
The moduli space of such curves is \((n+1)\) dimensional and carries a natural (complexified) \(GL(2, \mathbb{R})\) structure.

The special case is \(\mathcal{Y} = P(T\mathbb{T})\), where \(\mathbb{T}\)’s complex two-fold \(\mathbb{T}\) containing embedded rational curve \(L\) with self-intersection number \(n\). Such curve has a natural lift \(\hat{L}\) to \(\mathcal{Y}\), given by \(z \in L \rightarrow (z, \bar{z} \in T_zL)\). The lifted curves are Legendrian with respect to the canonical contact structure on the projectivised tangent bundle. The ODE whose integral curves are given by holomorphic deformations of \(L\) satisfies the \(GL(2, \mathbb{R})\) conditions.

3. \(GL(2, \mathbb{R})\) CONFORMAL STRUCTURE

In this section we shall associate a conformal structure to a \(GL(2, \mathbb{R})\) structure. From now we assume that \(n = 2k\) is even. We shall first recall some classical theory of invariants [13]. Let \(V_n \subset \mathbb{R}[X_1, X_2]\) be the \((n+1)\) dimensional space of homogeneous polynomials of degree \(n\). Consider the linear action of \(GL(2, \mathbb{R})\) on \(\mathbb{R}^2\) given by

\[
\tilde{X}_1 = \alpha X_1 + \beta X_2, \quad \tilde{X}_2 = \gamma X_1 + \delta X_2, \quad \alpha \delta - \gamma \beta \neq 0.
\]

Given a binary quantic \(Q(X_1, X_2)\) (whose coefficients may be numbers, functions, one–forms, ...) let \(\tilde{Q}(\tilde{X}_1, \tilde{X}_2)\) be a binary quantic such that

\[
\tilde{Q}(\tilde{X}_1, \tilde{X}_2) = Q(X_1, X_2).
\]

This induces an embedding \(GL(2, \mathbb{R}) \subset GL(n+1, \mathbb{R})\), as the coefficients \(\tilde{\theta} = (\tilde{\theta}^1, \ldots, \tilde{\theta}^{n+1})\) are linear homogeneous functions of the coefficients of \(Q\).

Recall that an invariant of a binary quantic is a function \(I(\theta)\) depending on the coefficients \(\theta = (\theta^1, \theta^2, \ldots, \theta^{n+1})\) such that

\[
I(\theta) = (\det A)^w I(\tilde{\theta}), \quad \text{where} \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, \mathbb{R}).
\]

The number \(w\) is called the weight of the invariant. For example if \(n = 2\) the discriminant \(\theta^1 \theta^3 - (\theta^2)^2\) is an invariant with weight \(2\).

One of the classical results of the invariant theory is that all invariants arise from the transvectants [13].

**Definition 3.1.** For any homogeneous polynomials \(Q \in V_n, R \in V_m\) the \(p\)th transvectant is

\[
< Q, R >_p = \frac{1}{p!} \sum_{i=0}^{p} \binom{p}{i} \frac{\partial^p Q}{\partial (X_1)^{p-i} \partial (X_2)^i} \frac{\partial^p R}{\partial (X_1)^{p-i} \partial (X_2)^i} \in V_{n+m-2p}.
\]

In particular specifying \(Q = R\) the successive transvectant operations reduce to elements of \(V_0\) which are invariants. The simplest of these is

\[
I_0 = \equiv < Q, Q >_n.
\]

It vanishes if \(n\) is odd, and for even \(n = 2k\) it has weight \(n\) and is given by

\[
I_0 = \left\{ 2 \sum_{i=0}^{k-1} (-1)^i \binom{2k}{i} \theta^{i+1} \theta^{2k+1-i} \right\} + \binom{2k}{k} (-1)^k (\theta^{k+1})^2.
\]  

(3.7)

In particular if \(I_0\) is evaluated for the binary quantic (2.2) defining the \(GL(2, \mathbb{R})\) structure where \(\theta^i\) are one–forms, then \(I_0\) should be regarded as a section of \(S^2(T^*M)\). It is well known that a conformal structure \([g]\) is
determined by specifying null vectors, i.e. sections \( V \in \Gamma(TM) \) such that \( g(V, V) = 0 \) for \( g \in [g] \). This gives

**Proposition 3.2.** A \( GL(2, \mathbb{R}) \) structure on a \((2k+1)\)-dimensional manifold \( M \) induces a conformal structure \([g]\) of signature \((k+1, k)\) or \((k, k+1)\). A vector field is null w.r.t \([g]\) iff the corresponding polynomial has \( I_0(V) = 0 \).

**Proof.** The ‘nullness’ of a vector is a quadratic condition and thus leads to a quadratic bilinear form up to scale. Let a vector \( V \) correspond to a polynomial (2.3). The condition \( I_0(V) = 0 \), where \( I_0 \) is given by (3.7) is indeed a quadratic and leads to a symmetric bilinear form \( g(X, Y) = \langle X, Y \rangle_n \) of signature \((k+1, k)\) or \((k, k+1)\).

In general conformal structures induced by \( GL(2, \mathbb{R}) \) structures form a subclass of all conformal structures of signature \((k+1, k)\), except when \( k = 1 \) in which case the two notions are equivalent. For \( n \) odd (that is for even-dimensional \( M \)) the bilinear form (compare formula (3.7)) resulting from this definition is anti-symmetric, so does not lead to conformal structures.

**Example.** In three dimensions \( GL(2, \mathbb{R}) \) structures are the same as conformal structures of Lorentzian signature. This is related to the isomorphism \( SL(2, \mathbb{R})/\mathbb{Z}_2 \cong SO(2, 1) \) which underlies the existence of spinors. Let a conformal structure be represented by a metric \( g = \eta_{ij} e^i e^j \) where \( \eta = \text{diag}(1, -1, -1) \) and \( e^i, i = 1, 2, 3 \) is an orthonormal basis of one-forms. The \( GL(2, \mathbb{R}) \) structure is defined by (2.2) with \( \theta^1 = e^1 + e^3, \theta^2 = e^2, \theta^3 = e^1 - e^3 \). A vector \( V = v^i \theta_i \) corresponds to a polynomial

\[
v^1 + 2xv^2 + x^2v^3
\]

where \( x = X_2/X_1 \). The nullness condition

\[
I_0(V) = v^1v^3 - (v^2)^2 = 0
\]

is given by vanishing of the discriminant. Thus a vector is null iff the corresponding polynomial has a repeated root. In the standard approach to spinors in three dimensions one represents a vector by a symmetric two-by-two matrix \( V^{AB} \) where \( A, B = 1, 2 \), such that \( g(V, V) = \det(V^{AB}) \). The non-zero null vectors correspond to matrices with vanishing determinant, which therefore must have rank one. Any such matrix is of the form \( V^{AB} = p^A p^B \). In our approach the matrix \( V^{AB} \) gives rise to a homogeneous polynomial \( V^{AB}X_A X_B \) which, in case of null vectors, has a repeated root \( x = -p^1/p^2 \).

**Example.** The five-dimensional \( GL(2, \mathbb{R}) \) structures correspond to special conformal structure in signature \((3, 2)\). The nullness condition can also be described geometrically in this case and the following interpretation is well known in the context of classical invariant theory [13, 22]. In the five-dimensional case vectors correspond to binary quartics. A generic quartic will have four distinct roots, and the nullness condition \( I_0(V) = 0 \) implies that their cross ratio is a cube root of unity. This is the equianharmonic condition. The roots of the quartic, when viewed as points on the Riemann sphere, can in this case be transformed into vertices of a regular tetrahedron.
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by Möbius transformation. Riemannian analogues of such geometries have been studied in [1].

We have been unable to find a geometric interpretation of the null condition $I_0(V) = 0$ in the case of seven–dimensional $GL(2, \mathbb{R})$ conformal structures which will play a role in the rest of the paper. The vectors correspond to binary sextics which generically admit six distinct roots $z_1, z_2, \ldots, z_6$. In this case one can also form an $SL(2, \mathbb{C})$ invariant multi cross-ratio

$$
\frac{(z_1 - z_2)(z_3 - z_4)(z_5 - z_6)}{(z_2 - z_3)(z_4 - z_5)(z_6 - z_1)}.
$$

Let $z_1, z_2, \ldots, z_6$ denote positions of six points on a plane. Given a triangle with vertices $(z_1, z_3, z_5)$, and three points $(z_2, z_4, z_6)$ on the lines $(z_3z_5), (z_5z_1)$ and $(z_1z_3)$ respectively, the lines $(z_1z_2), (z_3z_4)$ and $(z_5z_6)$ are concurrent iff the multi cross-ratio is equal to 1. This is the Ceva theorem.

The theorem of Menelaus states that the points $(z_2, z_4, z_6)$ are colinear if the multi cross-ratio is equal to $-1$.

We have expressed the invariant $I_0$ in terms of the roots, hoping to characterise its vanishing it by the Menelaus/Ceva conditions, but found that the invariant does not vanish in neither of these two cases.

3.1. Twistor theory. If the $GL(2, \mathbb{R})$ structure comes from an ODE, then induced conformal structure (3.7) arises from the twistor correspondence described at the end of Section 2. Here we shall concentrate on the special case when the Legendrian curves on a complex three–fold are lifts of rational curves from a two–fold.

Let

$$
x \rightarrow (x, y = Z(x, t_1, t_2, \ldots, t_{2k+1})
$$

be a graph of a rational curve $L$ in a complex surface $\mathbb{T}$ with a normal bundle $N(L) = \mathcal{O}(2k)$. The cohomological obstruction group $H^1(L, N(L))$ vanishes, and therefore the Kodaira theorems [16] imply that the curve belongs to a $2k + 1$ dimensional complete family $\{L_t, t \in M\}$ parametrised by points in a $(2k + 1)$–dimensional complex manifold (the space of solutions to (2.4)). Moreover there exists a canonical isomorphism

$$
T_t M \cong H^0(L_t, N(L_t))
$$

which associates a tangent vector at $t \in M$ to a global holomorphic section of a normal bundle $N(L_t) = \mathcal{O}(2k)$. Such sections are given by homogeneous polynomials of degree $2k$ which establishes the existence of a $GL(2, \mathbb{R})$ structure.

The curve $L_t$ has self-intersection number $2k$, i.e.

$$
d\delta y = \partial Z \delta t
$$

vanishes at the zeros of a polynomial of degree $2k$ in $x = X_2/X_1$. In its homogeneous form this polynomial is a binary form (2.3) with coefficients $v^\alpha, \alpha = 1, \ldots, 2k + 1$ which depend on $t_\alpha$ and are linear in $\delta t_\alpha$. A vector at a point in $t \in M$ corresponds to a normal vector field to the rational curve $L_t$, i.e. a section of $N(L_t) = \mathcal{O}(2k)$ which is the same as a homogeneous polynomial of degree $2k$. The corresponding invariant $I_0$ gives a quadratic
form on $M$ up to a multiple and its vanishing selects the null vectors. This determines the conformal structure.

In practice one proceeds as follows: If the rational curve is given by
\[ F(x, y, t_\alpha) = 0 \]
and its rational parametrisation is
\[ x = p(\lambda, t_\alpha), \quad y = q(\lambda, t_\alpha) \]
where $p, q$ are functions rational in $\lambda \in \mathbb{C}P^1$, then the polynomial in $\lambda$ giving rise to a null vector is given by the polynomial part of
\[ \sum_\alpha \frac{\partial F}{\partial t_\alpha}|_{\{x=p, y=q\}} \delta t_\alpha. \quad (3.8) \]

4. $G_2$ Structures from $GL(2, \mathbb{R})$ Conformal Structures

We shall now restrict to the case $n = 6$, and demonstrate that the seven-dimensional $GL(2, \mathbb{R})$ manifolds admit a conformal structure with a compatible $G_2$ structure. If the associated three-form is closed and co-closed, then the conformal structure is necessarily flat. We will however find examples of non-trivial $G_2$ structures where some components of the torsion vanish. In particular there is a non-trivial example of weak $G_2$ holonomy compatible with the $GL(2, \mathbb{R})$ structure. This example is originally due to Bryant [2]. In Theorem 5.2 we shall show that this example is essentially unique.

Consider a $GL(2, \mathbb{R})$ structure given by the binary form
\[ Q(x) = \theta^1 x^6 + 6 \theta^2 x^5 + 15 \theta^3 x^4 + 20 \theta^4 x^3 + 15 \theta^5 x^2 + 6 \theta^6 x + \theta^7, \]
with the corresponding quadratic invariant (conformal structure) (3.7)
\[ I_0 = \theta^1 \theta^7 - 6 \theta^2 \theta^6 + 15 \theta^3 \theta^5 - 10(\theta^4)^2. \quad (4.9) \]
Here $x = X_2/X_1$ is an inhomogeneous coordinate on the projective line $\mathbb{R}P^1$.

Use a combination of transvectants to construct a three-form
\[ \phi(X, Y, Z) = \text{tr}(X, Y), Z) >_3, Z >_6, \quad (4.10) \]

**Proposition 4.1.** The three-form $\phi$ is compatible with the conformal structure $I_0$. The vector $V$ is null with respect to $I_0$ iff
\[ (V \cdot \phi) \wedge (V \cdot \phi) \wedge \phi = 0 \quad (4.11) \]
where $(V \cdot \phi)(X, Y) := \phi(V, X, Y)$.

\[ ^1 \text{Some readers may prefer the two component spinor notation [22]. The capital letter indices } A, B, ... \text{ take values 1, 2. They are raised and lowered by a symplectic form represented by an anti-symmetric matrix } \varepsilon_{AB} \text{ on } \mathbb{R}^2 \text{ such that } \varepsilon_{12} = 1. \text{ The homogeneous polynomials are of the form } Q = Q_{AB}...C \pi^A \pi^B ... \pi^C, \text{ where } \pi^A = (X_1, X_2). \text{ Then } < Q, P >_n = Q_{AB}...C P^{AB}...^C. \text{ The conformal structure and the three form are given by}\]
\[ I_0 = e_{ABCD\cdot EF} \oplus e^{ABCD\cdot EF}, \]
\[ \phi = e^{AB} \cdot DEF \wedge e^{DEF} \cdot GHI \wedge e^{GHI} \cdot ABC, \]
where $e_{ABCD\cdot EF}$ is $\mathbb{R}^7$ valued one-form such that
\[ e^{111111} = \theta^1, e^{111112} = \theta^2, e^{111122} = \theta^3, e^{112222} = \theta^4, e^{112222} = \theta^5, e^{122222} = \theta^6, e^{222222} = \theta^7. \]
Proof. Consider the conformal structure induced by the vanishing of (3.7). Calculating the components of the three–form $\phi$ given by (4.10) and its dual with respect to $I_0$ gives

\[
\phi = 3(\theta^2 \wedge \theta^3 \wedge \theta^7 + \theta^1 \wedge \theta^5 \wedge \theta^6) + \theta^4 \wedge (\theta^1 \wedge \theta^7 + 6 \theta^2 \wedge \theta^6 - 15 \theta^3 \wedge \theta^5),
\]

\[
\ast \phi = -20 \theta^1 \wedge \theta^4 \wedge \theta^5 \wedge \theta^6 + 5 \theta^1 \wedge \theta^3 \wedge \theta^5 \wedge \theta^7 - 20 \theta^2 \wedge \theta^3 \wedge \theta^4 \wedge \theta^7
\]

\[
-2 \theta^1 \wedge \theta^2 \wedge \theta^6 \wedge \theta^7 + 30 \theta^2 \wedge \theta^3 \wedge \theta^5 \wedge \theta^6.
\]

(4.12)

This is in fact the non-compact form $G_2^{\text{split}}$ of the $G_2$ structure, as these forms agree with the more usual orthonormal frame formulae (see e.g. [2])

\[
I_0 = (e^1)^2 + (e^2)^2 + (e^3)^2 - (e^4)^2 - (e^5)^2 - (e^6)^2 - (e^7)^2,
\]

\[
\phi = e^{123} - e^{145} - e^{167} - e^{246} + e^{257} + e^{347} + e^{356},
\]

\[
\ast \phi = e^{4567} - e^{2367} - e^{2345} - e^{1357} + e^{1247} + e^{1256},
\]

provided that

\[
e^1 = \frac{1}{2}(\theta^1 + \theta^7), \quad e^5 = \frac{1}{2}(-\theta^1 + \theta^7), \quad e^2 = \frac{\sqrt{6}}{2}(\theta^2 - \theta^6)
\]

\[
e^6 = \frac{\sqrt{6}}{2}(\theta^2 + \theta^6), \quad e^3 = \frac{\sqrt{10}}{2}(\theta^3 + \theta^5),
\]

\[
e^7 = \frac{\sqrt{10}}{2}(-\theta^3 + \theta^5), \quad e^4 = \sqrt{10} \theta^4.
\]

(here $e^{ijk} = e^i \wedge e^j \wedge e^k$ etc). The condition (4.11) can now be verified directly. Conversely, given a three–form $\phi$, the conformal structure defined by (4.11) is represented by $I_0$ as shown in [2].

\[\square\]

We have therefore explicitly demonstrated that $\mathfrak{gl}(2, \mathbb{C})$ can be embedded in the complexification $\mathfrak{g}_2^{\mathbb{C}} \oplus \mathbb{C}$ of $\mathfrak{g}_2 \oplus \mathbb{R}$, or equivalently that $\mathfrak{sl}(2, \mathbb{C})$ can be embedded in $\mathfrak{g}_2^{\mathbb{C}}$. This follows more abstractly from a theorem of Morozov [26] which says that for any nilpotent element $e$ of a complex semi–simple Lie algebra $\mathfrak{g}$ there exist $f, h \in \mathfrak{g}$ and a homomorphism $\rho : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{g}$ such that $\rho(e) = e, \rho(f) = f, \rho(h) = h$, where $e, f, h$, is the basis of $\mathfrak{sl}(2, \mathbb{C})$ such that

\[
[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.
\]

4.1. Fernandez–Gray types. In this paper we follow the standard terminology of $G$–structures and define a $G_2$ structure on a seven–dimensional manifold to be a reduction of the frame bundle from $GL(7, \mathbb{R})$ to $G_2$ (or its non–compact analogue $G_2^{\text{split}}$). This structure is represented by a three–form $\phi$ in the open orbit of $GL(7, \mathbb{R})$ in $\Lambda^3(M)$. The three–form induces a metric [2] on $M$. If $\phi$ is given by (4.13) then the metric is given by $I_0$. It is Riemannian if the forms $e^1, \ldots, e^3$ are real and $e^4, \ldots, e^7$ are imaginary and has signature $(3, 4)$ if all one–forms are real\(^2\). The latter case corresponds to

\(^2\) Another equivalent definition [11] is to start form a Riemannian (respectively signature (3, 4)) metric $g$ and define a $G_2$ structure to be a cross product $P : T_tM \times T_tM \rightarrow T_tM$ on each tangent space which varies smoothly with $t \in M$ and such that $P$ is a bilinear map satisfying

\[
g(P(X, Y), X) = 0, \quad |P(X, Y)|^2 = |X|^2|Y|^2 - g(X, Y)^2, \quad \forall X, Y,
\]
Proposition 4.1 shows that seven-dimensional $GL(2, \mathbb{R})$ structure is equivalent to a further reduction of the frame bundle from $\mathbb{R}^+ \times G_2 \subset \mathbb{R}^+ \times SO(3,4)$ to $GL(2, \mathbb{R})$.

We do not assume anything about the closure of the three–form $\phi$ or its dual. Various types of $G_2$ structures (or their non–compact analogues) are characterised by a representation theoretic decomposition of $\nabla \phi$, where $\nabla$ is the Levi–Civita connection of the metric induced by $\phi$. Following [11, 4] we have

\begin{align*}
\text{d}\phi &= \lambda \ast \phi + \frac{3}{4} \Theta \wedge \phi + \ast \tau_3 \tag{4.14} \\
\text{d} * \phi &= \Theta \wedge * \phi - \tau_2 \wedge \phi,
\end{align*}

where $\lambda$ is a scalar, $\Theta$ is a one–form, $\tau_2$ is a two–form such that $\tau_2 \wedge \phi = - * \tau_2$ and $\tau_3$ is a three–form such that $\tau_3 \wedge \phi = \tau_3 \wedge * \phi = 0$. The forms $(\lambda, \Theta, \tau_2, \tau_3)$ can be interpreted as components of intrinsic torsion of a natural connection of $G_2$ structure. To define this connection apply the canonical decomposition $so(7) = g_2 \oplus \mathbb{R}^7$ (or its $so(3,4)$ analog) to the Levi–Civita connection. If $\gamma$ is the $so(7)$–valued connection one–form, then writing $\gamma = \hat{\gamma} + \tau$ defines a connection with torsion (not to be confused with the torsion $T$ of the $gl(2, \mathbb{R})$–valued connection $\Gamma$ studied in Sections 7 and 10) represented by a one–form $\hat{\gamma}$ with values in $g_2$. See e.g. [4] for details. If the three–form is only defined up to a multiple by a non–zero function (as it is the case in this paper) then $\lambda, \tau_2$ and $\tau_3$ scale with appropriate weights and $\Theta$ transforms like a Maxwell field. More precisely conformal rescalling $g \rightarrow e^{2f}g$ leaves (4.14) invariant if

\begin{align*}
\phi \rightarrow e^{3f} \phi, \quad \lambda \rightarrow e^{-f} \lambda, \quad \Theta \rightarrow \Theta + 4df, \quad \tau_2 \rightarrow e^f \tau_2, \quad \tau_3 \rightarrow e^{2f} \tau_3. \tag{4.15}
\end{align*}

If all components of the torsion vanish, then the $G_2$ structure gives rise to $G_2$ holonomy and the resulting metric is Ricci–flat. Such $G_2$ structures are sometimes called integrable or more correctly torsion–free. If $\Theta = \tau_2 = \tau_3 = 0$ then the metric is Einstein with non–zero Ricci scalar and one speaks of weak $G_2$ holonomy. If $\lambda = \Theta = \tau_3 = 0$ then the $G_2$ structure is closed (see e.g. [5]).

The representation theoretic decomposition of the torsion is as follows:

- $\lambda$ is a function and $\lambda \phi$ belongs to the 1-dimensional irreducible representation $W_1 \subset \Lambda^3 \mathbb{R}^{*7}$ of $G_2$.
- The 2-form $\tau_2$ belongs to the 14-dimensional irreducible representation $W_2 \subset \Lambda^2 \mathbb{R}^{*7}$.
- The 3-form $\tau_3$ belongs to the 27-dimensional irreducible representation $W_3 \subset \Lambda^3 \mathbb{R}^{*7}$.
- The Lee 1-form $\Theta$ belongs to the 7-dimensional representation $W_4 = \mathbb{R}^{7*}$.

Equations (4.14) uniquely define $\lambda$, $\tau_2$, $\tau_3$ and $\Theta$. Vanishing of these objects defines the Fernandez–Gray $W$ type of $G_2$ geometry: if none of them
vanishes the geometry is of generic type $W_1 + W_2 + W_3 + W_4$, if $\lambda = 0$ then the geometry is of type $W_2 + W_3 + W_4$, when $\tau_2 = 0$ we have the type $W_1 + W_3 + W_4$ and so on. There are sixteen $W$ types.

Proposition 4.1 demonstrates that the seven-dimensional $GL(2, \mathbb{R})$ geometry is a special case of conformal split $G_2$ geometry as $GL(2, \mathbb{R}) \subset \mathbb{R} \times G_2^{split}$. The representations of $G_2^{split}$ decompose into irreducible representations of $GL(2, \mathbb{R})$ as follows.

\[
\begin{align*}
W_1 &= V^1, \\
W_2 &= V^3 \oplus V^{11}, \\
W_3 &= V^5 \oplus V^9 \oplus V^{13}, \\
W_4 &= V^7,
\end{align*}
\]

where $V^k$ is the $k$-dimensional representation space $S^{k-1}(\mathbb{S})$. Hence $\tau_2$ and $\tau_3$ have a priori two and three irreducible components under action of $GL(2, \mathbb{R})$.

A $GL(2, \mathbb{R})$ geometry defines a whole conformal class of $G_2^{split}$ geometries, hence we may only talk about those $W$ types which are invariant with respect to conformal rescalings (4.15). In particular vanishing of $\Theta$ is not conformally invariant. However, $d\Theta$ is a well defined 2-form, in particular the condition $d\Theta = 0$ means that in some conformal gauge $\Theta$ vanishes locally. In general $d\Theta$ is a 2-form decomposing according to

\[
\Lambda^2 \mathbb{R}^7 = V^3 \oplus V^7 \oplus V^{11}.
\]

5. $G_2$ structures from ODEs

Now we are ready to give the relations between the the intrinsic torsion of the split $G_2$ structure and the contact invariants of the 7th order ODE

\[
y^{(7)} = F(x, y, y', \ldots, y^{(6)}).
\]

In the following theorems (which will be established in Section 10) we shall assume the vanishing of the conditions $W_\alpha$ (Appendix A) which are necessary and sufficient for an ODE to give rise to a $GL(2, \mathbb{R})$ geometry.

**Theorem 5.1.** Let the 7th order ODE (5.18) admit the $GL(2, \mathbb{R})$ geometry on the solution space. The following conditions hold

- **no $W_1$ component** $(\lambda = 0) \iff F_{66} (9DF_6 - \frac{9}{7}F_6^2 - 15F_5) + 12F_6F_6 + 14F_6F_5 - \frac{84}{5}F_{66} = 0.$

- **no $W_2$ component** $(\tau_2 = 0) \iff 21DF_{66} + 14F_{65} + 15F_6F_{66} = 0.$

- **no $W_3$ component** $(\tau_3 = 0) \iff F_{66} = 0.$

The 2-form $d\Theta$ falls into components in irreducible representations $V^3$, $V^7$ and $V^{11}$. The $V^3$-part is expressed algebraically by $\lambda$, $\tau_2$ and $\tau_3$. In
particular it vanishes if $\tau_2$ vanishes. The $V^7$-part of $d\Theta$ vanishes iff
\[
(DF)_{66}F_{66} + \frac{3}{2}(DF)_6F_{666} - \frac{12}{7}F_{666}F_6^2 - 4F_{666}F_5
\]
\[+ 2F_{665}F_6 - \frac{14}{5}F_{664} + \frac{7}{3}F_{665} - \frac{4}{3}F_{66}F_{65} - \frac{16}{7}F_{66}F_6 = 0.
\]
The $V^{11}$-part of $d\Theta$ vanishes iff
\[F_{666} = 0.
\]

The non-generic $W$ types are characterised by the following result

**Theorem 5.2.** There are only three conformal split $G_2$ geometries from ODEs of type $W_1 + W_2 + W_4$.

1. The flat geometry of $y^{(7)} = 0$, which is the only case admitting holonomy $G_2^{\text{split}}$.

2. The geometry of
\[y^{(7)} = 7\frac{y^{(6)}y^{(4)}}{y^{(3)}} + \frac{49}{10}\frac{(y^{(5)})^2}{y^{(3)}} - 28\frac{y^{(5)}(y^{(4)})^2}{(y^{(3)})^2} + \frac{35}{2}\frac{(y^{(4)})^4}{(y^{(3)})^3}, \quad (5.19)
\]
This is the only geometry of type $W_1 + W_4$. The Lee form is closed, so that in certain conformal gauge it is the nearly-parallel ($W_1$) geometry of $SO(3,2)/SO(2,1)$.

3. The geometry of
\[y^{(7)} = \frac{21}{5}\frac{y^{(6)}y^{(5)}}{y^{(4)}} - \frac{84}{25}\frac{(y^{(5)})^3}{(y^{(4)})^2}, \quad (5.20)
\]
which is of type $W_2 + W_4$. The Lee form is closed, so in certain conformal gauge it is a closed $G_2$ structure ($W_2$).

The $G_2$ geometry associated with (5.19) has two real forms: The homogeneous space $SO(3,2)/SO(2,1)$ which yields a weak $G_2$ metric in signature $(3,4)$ and $SO(5)/SO(3)$ which gives a Riemannian metric. The later metric was first constructed by Bryant in his seminal paper [2] without using the ODE or twistor techniques. Theorem 5.2 implies that up to diffeomorphisms of $M$ this is the only weak $G_2$ metric arising from an ODE. The ODE (5.19) has appeared is several other contexts. See [23, 21].

The ODE (5.20) has an elementary solution given by certain rational curve which will be analysed in the next section. The solution to (5.19) can also be constructed in terms of rational curves, but explicit description in this case is more involved [10].

We note that there exists at least one more connection between differential equations and non-compact $G_2$: The holonomy of an ambient metric associated to Nurowski’s $(3,2)$ conformal structure [19] is contained in $G_2^{\text{split}}$.

### 6. Examples

In this section we give some examples. The first three arise on moduli spaces of rational curves by twistor theoretic techniques. The last one comes from an ODE satisfying the Wünshmann conditions (2.5).
We shall write the general 7th order ODE (5.18) as
\[ y^{(7)} = F(x, y, p, q, r, s, t, u), \]
where \( p = y', q = y'', r = y^{(3)}, s = y^{(4)}, t = y^{(5)}, u = y^{(6)}. \)

The examples below can be partially classified by the dimension of the group of contact symmetries (recall that the maximal symmetry group of the trivial ODE \( y^{(7)} = 0 \) is eleven dimensional and given by \( GL(2, \mathbb{R}) \times \mathbb{R}^7 \)).

6.0.1. Example 1. Consider a hyperelliptic curve of degree 6 with 2 cusps. The general sextic has genus 10 and so is not rational, but in our case the genus is zero and a rational parametrisation exist. To see it write the curve as
\[ (y + Q(x))^2 + P(x)^3 = 0, \]
where \((Q, P)\) are general cubic and quadratic respectively which we write as
\[ Q(x) = q_0 + q_1 x + q_2 x^2 + q_3 x^3, \quad P(x) = p_3(x - p_2)(x - p_1). \]
This has three singular points. Two double points at \((p_1, -Q(p_1))\) and \((p_2, -Q(p_2))\) of type [2, 1, 1] (see e. g. [27]) and one point of order 4 at infinity, of type [4, 8, 2] which can be seen by writing (6.21) in the homogeneous coordinates. Calculating the genus yields
\[ g = \frac{5 \cdot 4}{2} - 1 - 1 - 8 = 0, \]
as the quadruple point at infinity is not ordinary and has the \( \delta \)-invariant equal to 8. The rational parametrisation can now be found
\[ x(\lambda) = \frac{p_1 + p_2 \lambda^2}{\lambda^2 + 1}, \]
\[ y(\lambda) = p_3^{3/2}(p_1 - p_2)^3 \frac{\lambda^3}{(\lambda^2 + 1)^3} - Q(x(\lambda)). \]
Eliminating the parameters \((p_1, p_2, p_3, q_0, \ldots, q_3)\) between (6.21) and its six derivatives yields the 7th order ODE characterising the sextic (6.21)
\[ \frac{d^7 y}{dx^7} = \frac{21 ut}{5 s} - \frac{84 t^3}{25 s^2}, \]
which is the ODE (5.20) from Theorem 5.2.

Using the prescription (3.8) we find that the conformal structure and the associated three–form are represented by (4.9) and (4.12) with
\[ \theta^1 = -2\Omega \sum_{\alpha=0}^{3} (p_2)^\alpha dq_\alpha, \quad \theta^7 = -2\Omega \sum_{\alpha=0}^{3} (p_1)^\alpha dq_\alpha, \]
\[ \theta^2 = -\frac{\Omega}{2} (p_2 - p_1)^2(p_3)^{3/2} dp_2, \quad \theta^0 = \frac{\Omega}{2} (p_2 - p_1)(p_3)^{3/2} dp_1, \]
\[ \theta^3 = -\frac{\Omega}{15} (3dq_0 + (2p_2 + p_1) dq_1 + (2p_1 p_2 + (p_2)^2) dq_2 + 3p_1(p_2)^2 dq_3), \]
\[ \theta^5 = -\frac{\Omega}{15} (3dq_0 + (2p_1 + p_2) dq_1 + (2p_1 p_2 + (p_1)^2) dq_2 + 3p_2(p_1)^2 dq_3), \]
\[ \theta^4 = -\frac{3\Omega}{20} (p_2 - p_1)^2 \sqrt{p_3} d(p_3(p_2 - p_1)), \]
where \( \Omega = (p_1 - p_2)^{-12/5}(p_3)^{-9/10}. \)
This conformal $G_2$ structure can be analytically continued to Riemannian signature: Setting $p_2 = p, p_1 = \overline{p}$ where $p \in \mathbb{C}$ and keeping $(q_0, q_1, q_2, q_3, p_3)$ real gives purely imaginary $\theta^4$ and

$$\theta^7 = \overline{\theta^1}, \quad \theta^6 = -\overline{\theta^2}, \quad \theta^3 = \overline{\theta^5}.$$ 

The corresponding conformal structure is positive definite and the three form $\phi$ is real. It gives rise to a closed (in a sense of decomposition (4.14)) $G_2$ structure as $d\phi = 0, \ast d\phi = -\tau_2 \wedge \phi$ in agreement with Theorem 5.2. This Theorem also implies that up to diffeomorphisms this is the only closed $G_2$ structure arising from ODEs.

6.0.2. Example 2. Consider a rational curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ of bidegree $(1, k)$

$$y = \frac{r_0 + r_1 x + \cdots + r_k x^k}{s_0 + s_1 x + \cdots + s_k x^k}.$$ 

It has self-intersection number $2k$, and the enumerator of the perturbed curve (section of a normal bundle) $\delta y$ defines the conformal structure (3.7) with

$$\hat{\theta}^{i+1} = \binom{2k}{i}^{-1} \sum_{\alpha + \beta = i} (r_\alpha ds_\beta - s_\beta dr_\alpha), \quad i = 0, \ldots, 2k.$$ 

This conformal structure is defined on a hypersurface where the resultant of the denominator and enumerator in $y$ has a non–zero fixed value. Alternatively we can fix the ambiguity by choosing affine coordinates, say $r_k = 1$. Now restrict to the seven–dimensional case $k = 3$. This also gives $\phi \wedge d\phi = 0$. The corresponding 7th order ODE is

$$\frac{d^7 y}{dx^7} = \frac{P}{Q},$$

where

$$P = 420 q^2 u^2 + 2520 q s^2 t^2 - 1680 q r s u - 504 p t^3 + 1680 r^2 t^2 - 6300 t r s^2 + 840 t u p s + 2625 s^4 - 280 u^2 r p + 2800 u r^2 s$$

$$Q = 360 q^2 t - 1200 q s t - 240 r t p + 800 r^3 + 300 s^2 p.$$ 

This example has six–dimensional group of point symmetries, given by the Möbius transformations of $x$ and $y$.

6.0.3. Example 3. We can construct less trivial conformal structures and the associated three forms by generalising the last example, and taking a double covering of a neighbourhood of a non–singular curve of bidegree $(1, 6)$ branched along a fixed curve. Consider a $(1, 6)$ curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$

$$y = \frac{R(x)}{S(x)}$$

where

$$S = s_0 + s_1 x + \cdots + s_6 x^6, \quad R = r_0 + r_1 x + \cdots + r_6 x^6.$$ 

This curve has normal bundle $\mathcal{O}(12)$, and is parametrised by $\mathbb{CP}^1$ minus a hypersurface where both polynomials have common factor. We take the branch locus to be the $(1, 6)$ curve

$$y = x^6.$$
The curves in a covering space we are constructing project to those curves (6.23) which meet the branch locus in seven points to second order. Thus
\[ x^6S(x) - R(x) = (t_0 + t_1 x + \ldots + t_6 x^6)^2. \]
This gives 13 conditions on 20 coefficients \((s, r, t)\), leaving the seven dimensional moduli space of curves.

6.0.4. Example 4. In [9] it was shown that the moduli space of solutions to the ODE
\[ \frac{d^{n+1}y}{dx^{n+1}} = \left( \frac{d^n y}{dx^n} \right)^{\frac{n+1}{n}} \]
admits the \( GL(2, \mathbb{R}) \) structure. Consider a solution curve \( x \rightarrow (x, y(x)) \) and its perturbation
\[ \delta y = \frac{1}{x + t_{n+1}} \left( \left( - \frac{n^n}{(n-1)!} \delta t_{n+1} + t_{n+1} \delta t_1 \right) + \sum_{i=1}^{n-1} (\delta t_i + t_{n+1} \delta t_{i+1}) x^i + \delta t_n x^n \right). \]
The enumerator of the polynomial \( \delta y \) defines a conformal structure (3.7) with
\[ \theta^1 = - \frac{n^n}{(n-1)!} dt_{n+1} + t_{n+1} dt_1, \quad \theta^{i+1} = \left( \frac{2k}{i} \right)^{-1} \left( dt_i + t_{n+1} dt_{i+1} \right), \quad \theta^{2k+1} = dt_{2k}, \]
where \( i = 1, \ldots, 2k-1 \). We can now specify \( 2k = 6 \) and construct the three–form. We find that \( \phi \wedge d\phi = 0 \) so that \( \lambda = 0 \) but there is no conformal scale which makes \( \phi \) closed.

7. Construction of the Cartan Connection

We shall now describe the \( GL(2, \mathbb{R}) \) and conformal \( G_2^{\text{split}} \) structures arising from 7th order ODEs by constructing a \( gl(2, \mathbb{R}) \)-valued linear connection on \( M \). The basic object in this description is the torsion, which contains lowest order invariants of the \( GL(2, \mathbb{R}) \) geometry, identifies the Fernandez–Gray types of the associated conformal \( G_2^{\text{split}} \) geometry, and expresses these quantities by contact invariants of the underlying ODE. This approach will give us a better handle on the various torsion components. Our aim is to express these components in terms of invariants of the 7th order ODE and eventually prove Theorems 5.1 and 5.2 Our treatment of Cartan’s connection follows closely that of [12].

We shall use an equivalent form of Definition 2.1 and regard a \( GL(2, \mathbb{R}) \) geometry on a manifold \( M \) as a reduction of the frame bundle \( FM \) to a \( GL(2, \mathbb{R}) \)-sub-bundle, where \( GL(2, \mathbb{R}) \subset GL(n, \mathbb{R}) \) acts irreducibly in each tangent space [12]. We shall focus on the case \( n = 6 \) where
\[ GL(2, \mathbb{R}) \subset \mathbb{R}^+ \times G_2^{\text{split}} \subset \mathbb{R}^+ \times SO(3, 4) \]
holds (see Proposition 4.1). The central role will be played by the six-jet space \( J^6 \) and its description via the Tanaka–Morimoto theory, [25, 18], which is a special version of Cartan’s method of equivalence. We shall first construct a \( gl(2, \mathbb{R}) \oplus \mathbb{R}^7 \)-valued Cartan connection \( \Omega \) on a bundle over \( J^6 \) and then re-interpret \( \Omega \) from the point of view of the \( GL(2, \mathbb{R}) \) structure.
The conditions for the existence of the geometry appear to be certain linear conditions for the curvature of $\Omega$. If they are satisfied, then the $\mathfrak{gl}(2, \mathbb{R})$-part of $\Omega$ is the desired linear connection on $M$.

7.1. Jet space. Let us consider the space $J^6$ of six-jets of functions from $\mathbb{R}$ to $\mathbb{R}$. It is an eight-dimensional real manifold, locally parametrised by $(x, y, y_1, \ldots, y_6)$, and such that each curve $x \mapsto (x, f(x))$ in the $xy$-space has a unique lift to a curve in $J^6$ given by $x \mapsto (x, f(x), f'(x), \ldots, f^{(6)}(x))$ in the above coordinate system. This gives a distinguished family of all curves lifted from the $xy$-space in $J^6$. One may encode this family in a coordinate-free language of distributions. Let us fix a point $w \in J^6$ and consider all lifted curves through $w$. The linear span of their tangent vectors at $w$ is a two-dimensional subspace $C_w$ in $T_wJ^6$. The collection $C = \cup_w C_w$ is by definition the contact distribution on $J^6$. It is generated by two vector fields

$$
\mathcal{D} = \partial_x + y_1 \partial_y + y_2 \partial_{y_1} + \ldots + y_6 \partial_{y_5} \quad \text{and} \quad \partial_{y_6}.
$$

Given the distribution $C$ we define

$$
\partial C = [C, C], \quad \partial^2 C = [\partial C, C], \quad \ldots \quad \partial^5 C = [\partial^4 C, C], \quad \partial^6 C = TJ^6.
$$

The distributions constitute a filtration, that is

$$
C \subset \partial C \subset \cdots \subset \partial^5 C \subset \partial^6 C = TJ^6
$$

(7.24)

and

$$
[\partial^i C, \partial^j C] = \partial^{i+j+1} C.
$$

(7.25)

The diffeomorphisms of $J^6$ which preserve $C$ are called contact transformations. The well-known Lie-Bäcklund theorem states that all contact transformations of $J^6$ are uniquely defined by the contact transformations of $J^1$ and have the following form

$$
x \mapsto \bar{x}(x, y, y_1), \quad y \mapsto \bar{y}(x, y, y_1), \quad y_1 \mapsto \bar{y}_1(x, y, y_1),
$$

(7.26)

and for higher order jet coordinates

$$
y_{k+1} \mapsto \frac{\mathcal{D} y_k}{\mathcal{D} \bar{x}}, \quad k = 1, 2, \ldots, n.
$$

The functions $\bar{x}, \bar{y}$ and $\bar{y}_1$ in (7.26) are not arbitrary but subject to the condition

$$
\bar{y}_1 = \frac{\mathcal{D} \bar{y}}{\mathcal{D} \bar{x}}.
$$

The contact transformation preserve the whole filtration (7.24).

Now consider the 7th order ODE (5.18). Any solution $y = f(x)$ of the equation is uniquely defined by a choice of $f(x_0), f'(x_0), \ldots, f^{(6)}(x_0)$ at some $x_0$. Since this choice of initial data is equivalent to a choice of a point in $J^6$ there exists exactly one lifted curve $x \mapsto (x, f(x), f'(x), \ldots, f^{(6)}(x))$ through any point of $J^6$. Therefore the solutions form a one-dimensional foliation in $J^6$. The corresponding tangent distribution is spanned by

$$
\mathcal{D} = \partial_x + y_1 \partial_y + y_2 \partial_{y_1} + \ldots + y_6 \partial_{y_5} + F \partial_{y_6}.
$$

An important consequence of this is that $J^6 \to M$ is locally a line bundle, where $M$ is the solution space of the 7th order ODE (5.18).

**Definition 7.1.** The contact geometry of 7th order ODEs is the jet space $J^6$ equipped with
The filtration $C \subset \ldots \subset \partial^5 C \subset \partial^6 C = T J^6$.

The foliation by the solutions, tangent to the field $D$.

One may associate to the contact geometry of the ODEs a sub-bundle
\[ \tilde{G} \to \tilde{P} \to J^6 \] (7.27)
of the frame bundle $F J^6$: The structure group $\tilde{G}$ is the lower triangular group preserving the filtration and the 1-distribution span$\{D\} \subset C$ tangent to solutions.

7.2. Cartan connection. The main object we use in the construction is Cartan connection defined here as in [15].

Definition 7.2. Let $M$ be a manifold of dimension $n$, $G$ a Lie group, $H$ a closed subgroup of $G$ with $\dim G/H = n$ and $H \to P \xrightarrow{\pi} M$ a principal bundle. A Cartan connection of type $(G, H)$ on $P$ is a one–form $\Omega$ with values in the Lie algebra $\mathfrak{g}$ of $G$ satisfying the following conditions:

i) $\Omega_u : T_u P \to \mathfrak{g}$ for every $u \in P$ is an isomorphism of vector spaces.

ii) $\Omega(A^*) = A$ for every $A \in \mathfrak{h}$ and the corresponding fundamental field $A^*$.

iii) $R_h^* \Omega = \text{Ad}(h^{-1}) \Omega$ for every $h \in H$.

The curvature of a Cartan connection is a $\mathfrak{g}$-valued 2-form on $P$ defined by
\[ K(X, Y) = d\Omega(X, Y) + \frac{1}{2}[\Omega(X), \Omega(Y)]. \]
If $\Omega$ is given in a matrix representation then
\[ K = d\Omega + \Omega \wedge \Omega. \] (7.28)
The curvature is horizontal, that is it vanishes on each vertical vector field:
\[ K(X, \cdot) = 0 \quad \text{if} \quad \pi_*(X) = 0. \] (7.29)
Horizontality of the curvature is locally equivalent to the property iii) in Definition 7.2. Cartan connections with vanishing curvature are called flat.

We are now in position to describe the construction of $GL(2, \mathbb{R})$ geometry on the solution space. We start from the bundle $\tilde{P}$ of the contact geometry of ODEs. We are interested in invariants of this geometry. The filtration is preserved by the contact transformations but the foliation of solutions is not, and generates the contact invariants of the underlying ODE. However, the situation further is complicated by the fact that the object generating the invariants – a Cartan connection – exists on a sub-bundle $P \subset \tilde{P}$ rather than $\tilde{P}$ itself. Using the Tanaka–Morimoto theory we shall construct the sub-bundle $H \to P \to J^6$ together with a Cartan connection $\Omega$ of type $(GL(2, \mathbb{R}) \times \mathbb{R}^7, H)$, where $H$ isomorphic to the group of triangular $2 \times 2$ matrices. The curvature $K$ of $\Omega$ contains all the local information about the contact geometry of the ODEs. The contact invariants are either components of $K$ or certain combinations of their derivatives of sufficiently high order.

The jet space $J^6$ is a bundle over the solution space $M$ and $P \to M$ is also a principal bundle with the structure group $GL(2, \mathbb{R})$. That $\Omega$ generates the $GL(2, \mathbb{R})$ geometry on $M$ only if certain conditions (which we will determine)
First of all, we ask whether $\Omega$ (which is a Cartan connection on $P \to J^6$) satisfies the conditions for the Cartan connection of $P \to M$. It holds if and only if

$$K(X, \cdot) = 0$$

for all $X$ vertical with respect to $P \to M$.

This condition is not satisfied automatically but only holds for the ODEs with vanishing Wünschmann invariants (Appendix A).

The Cartan connection $\Omega$ on $P \to M$ is of type $(GL(2, \mathbb{R}) \times \mathbb{R}^7, GL(2, \mathbb{R}))$. It naturally decomposes into the $\mathbb{R}^7$-part and the $\mathfrak{gl}(2, \mathbb{R})$-part. The former behaves like a canonical form $\theta$ on a principal bundle and turns $P$ into a sub-bundle of the frame bundle $FM$. The latter is a linear $\mathfrak{gl}(2, \mathbb{R})$-valued connection $\Gamma$ on $P$. Together $\theta$ and $\Gamma$ define a $GL(2, \mathbb{R})$ geometry on $M$.

The torsion $T$ and curvature of $\Gamma$ contain the information about local invariants of the geometry, which are in turn expressed by contact invariants of the underlying ODE, since $\Omega$ also describes the contact geometry of the ODEs.

Example. For the trivial equation $y^{(7)} = 0$, all the objects may be immediately constructed by means of the symmetry group. The full group of contact symmetries is $GL(2, \mathbb{R}) \ltimes \mathbb{R}^7$. Its action on $J^6$ is transitive and turns it into a homogeneous space $GL(2, \mathbb{R}) \ltimes \mathbb{R}^7/H$, where $H$ is isomorphic to the group of triangular $2 \times 2$ matrices. Thus we have the bundle $H \to P \to J^6$ and $P = GL(2, \mathbb{R}) \ltimes \mathbb{R}^7$ locally. The connection $\Omega$, flat in this case, is given by the Maurer-Cartan form on $P$.

8. The Tanaka–Morimoto theory

We turn to detailed description of the construction. First of all, we briefly describe the general pattern, next we apply it to our case. The references for this subsection are [24, 25, 17, 18] and [8]. The contact geometry of ODEs contains the filtration (7.24) which is encoded by the graded tangent bundle $gr T_{J^6}$, denoted here by $gr$ for short. Its fibre over $w \in J^6$ is $gr(w) = \bigoplus_{i=1}^7 gr^{-i}(w)$, where

$$gr^{-1}(w) = C_w, \quad gr^{-2}(w) = \partial C_w/C_w, \ldots$$

$$\ldots gr^{-6}(w) = \partial^5 C_w/\partial^4 C_w \quad gr^{-7}(w) = T_w J^6/\partial^6 C_w.$$

The relation (7.25) implies that $gr(w)$ carries the structure of a nilpotent graded Lie algebra, that is

$$[gr^{-i}(w), gr^{-j}(w)] \subset gr^{-i-j}(w), \quad \text{and} \quad gr(w) \text{ is generated by } gr^{-1}(w).$$

Let

$$m = g_{-1} \oplus \ldots \oplus g_{-7},$$

where $gr^{-i} \cong g_{-i}$. We have $\dim m = \dim gr(w) = \dim T_w J^6 = 8$ and

$$[g_{-i}, g_{-j}] \subset g_{-i-j}.$$

The additional piece of structure — the distribution span $\{D\} —$ is encoded in the following manner. One defines a weighted frame $z_w$ at $w \in J^6$ to be an isomorphism of graded Lie algebras $z_w: m \to gr(w)$. The bundle of weighted frames $\mathcal{R} J^6$ is a principal bundle over $J^6$ with the structure group $G_0(m)$ being the group of all grading preserving algebra automorphisms of $m$. 


The vector $D_w$ at any $w$ belongs to $C_w$ and is complementary to the 1-dimensional subspace of $C_w$ which is vertical with respect to $J^6 \to J^5$ and spanned by $\partial y_6$. At the level of $m$ it is reflected by a decomposition of $g_{-1}$ into two 1-dimensional subspaces. These subspaces, call them $D$ and $V$ for short, are then encoded by reducing $RJ^6$ to a $G_0$-sub-bundle, where $G_0$ is the 2-dimensional subgroup of $G_0(m)$ preserving the decomposition $g_{-1} = D \oplus V$.

However, the $G_0$-sub-bundle still is not the bundle $P$ where the connection $\Omega$ exists. In order to construct $P$ one needs to prolong the $G_0$-bundle. The procedure of prolongation is quite involved and the reader is referred to the original paper [24]. The underlying idea it is however simple. One aims to extend the $G_0$-bundle so that it is large enough to contain all the symmetries in the most symmetric homogeneous case. From the example of the trivial ODE we know that the total space $P$ must be 11-dimensional, so one dimension is lacking. After the prolongation one obtains the desired $H \to P \to J^6$ with the structural group $H$ being a product of $G_0$ and the 1-dimensional prolongation, isomorphic to the group of triangular 2 by 2 matrices.

At the algebraic level the filtration is encoded by $m$ and the full (prolonged) structural group $H$ is encoded by its algebra $h = g_0 \oplus g_1$. Since commutators $[m, h]$ are known from the construction, we obtain a graded algebra

$$ g = m \oplus h = g_{-7} \oplus g_{-6} \oplus \ldots \oplus g_0 \oplus g_1 = gl(2, \mathbb{R}) \oplus \mathbb{R}^7. \quad (8.30) $$

The Cartan connection $\Omega$ takes values in this algebra.

The next step is constructing the form $\Omega$ using the normality conditions of Tanaka and Morimoto. The normality conditions, which are certain linear constraints for the curvature, were originally introduced by E. Cartan in the context of conformal and projective geometries. The purpose was fixing ambiguity in the choice of Cartan connections and providing canonical connections for these geometries in a sense analogous to the Levi–Civita connection in Riemannian geometry. Later, these conditions were generalised to the case of the filtered manifolds. We discuss them below.

The connection 1-form at $p \in P$ is a vector space isomorphism $\Omega_p : T_pP \to g$. We define $\nu_p = \Omega_p^{-1}(h)$ and $\mathcal{H}_p = \Omega_p^{-1}(m)$, hence $T_pP = \nu_p \oplus \mathcal{H}_p$. The curvature $K_p$ is then characterised by a tensor $\kappa_p \in \text{Hom}(\wedge^2 m, g)$ given by

$$ \kappa_p(A, B) = K_p(\Omega_p^{-1}(A), \Omega_p^{-1}(B)), \quad A, B \in m. \quad (8.31) $$

In the space $\text{Hom}(\wedge^2 m, g)$ let us define $\text{Hom}^1(\wedge^2 m, g)$ to be the space of all $\alpha \in \text{Hom}(\wedge^2 m, g)$ fulfilling

$$ \alpha(g_i, g_j) \subset g_{i+j+1} \oplus \ldots \oplus g_k \quad \text{for} \quad i, j < 0. $$

The algebra $g$ is equipped with the following complex

$$ \ldots \xrightarrow{\partial} \text{Hom}(\wedge^q m, g) \xrightarrow{\partial} \text{Hom}(\wedge^{q+1} m, g) \xrightarrow{\partial} \ldots $$
with $\partial$: $\text{Hom}(\wedge^q \mathfrak{m}, \mathfrak{g}) \to \text{Hom}(\wedge^{q+1} \mathfrak{m}, \mathfrak{g})$ given by

$$(\partial^* \alpha)(A_1 \wedge \ldots \wedge A_{q+1}) = \sum_i (-1)^{i+1} [A_i, \alpha(A_1 \wedge \ldots \wedge \hat{A}_i \wedge \ldots \wedge A_{q+1})]$$

$$+ \sum_{i<j} (-1)^{i+j} \alpha([A_i, A_j] \wedge A_1 \wedge \ldots \wedge \hat{A}_i \wedge \ldots \wedge \hat{A}_j \wedge \ldots \wedge A_{q+1}),$$

where $\alpha \in \text{Hom}(\wedge^q \mathfrak{m}, \mathfrak{g})$ and $A_1, \ldots, A_{q+1} \in \mathfrak{m}$.

Consider a positive definite scalar product $(\cdot, \cdot)$ in $\mathfrak{g}$ satisfying three conditions:

i) $(g_i, g_j) = 0$ for $i \neq j$.

ii) There exists a mapping $\tau: \mathfrak{h} \to \mathfrak{g}$ such that

$$(\tau(g_i), g_j) = 0 \quad \text{for } i \geq 0,$$

$$([A, X], Y) = (X, [\tau(A), Y]) \quad \text{for } X, Y \in \mathfrak{g}, A \in \mathfrak{h}.$$ (8.32)

iii) There exists a mapping $\tau_0: \mathfrak{g}_0 \to \mathfrak{g}_0$ such that

$$(aX, Y) = (X, \tau_0(a)Y) \quad \text{for } X, Y \in \mathfrak{g}, a \in \mathfrak{g}_0.$$ (8.31)

This product extends to $\text{Hom}(\wedge^q \mathfrak{m}, \mathfrak{g})$ through

$$(\alpha, \beta) = \frac{1}{q!} \sum_{i_1, \ldots, i_q} \left( \alpha(v_{i_1} \wedge \ldots \wedge v_{i_q}), \beta(v_{i_1} \wedge \ldots \wedge v_{i_q}) \right),$$

where $\alpha, \beta \in \text{Hom}(\wedge^q \mathfrak{m}, \mathfrak{g})$ and $(v_i)$ is any orthonormal basis of $\mathfrak{g}$. Given $\partial$ and $(\cdot, \cdot)$ the formal adjoint operator

$$\ldots \to \partial^* \text{Hom}(\wedge^{q+1} \mathfrak{m}, \mathfrak{g}) \to \partial^* \text{Hom}(\wedge^q \mathfrak{m}, \mathfrak{g}) \to \partial^* \ldots$$

is defined by

$$(\partial^* \alpha, \beta) = (\alpha, \partial \beta).$$

A normal connection is defined as follows.

**Definition 8.1.** A Cartan connection $\Omega$ is normal if $\kappa$ given by (8.31) satisfies conditions

i) $\kappa \in \text{Hom}^1(\wedge^2 \mathfrak{m}, \mathfrak{g})$,

ii) $\partial^* \kappa = 0$.

By a general result of Morimoto [18] (Theorem 2.3. and Proposition 2.10 in this reference), given an inner product satisfying the three properties (8.32) one can construct the normal Cartan connection which preserves the contact equivalence of the underlying ODEs.

9. **Application to 7th order ODEs**

Define seven 1-forms on $J^6$ by:

$$\omega^i = dy_{i-1} - y_i dx, \quad \omega^7 = dy_6 - F dx, \quad i = 1, \ldots, 6.$$ (9.33)

These forms encode the geometry of an ODE, and in particular $C$ is annihilated by the ideal span$\{\omega^1, \omega^2, \ldots, \omega^6\}$ and the foliation by solutions is annihilated by span$\{\omega^1, \omega^2, \ldots, \omega^7\}$. On $\tilde{P}$ there is the fundamental $\mathbb{R}^8$-valued 1-form, whose components are denoted by $\theta^1, \ldots, \theta^7$ and $\Gamma_+$. (The notation $\Gamma_+$ instead of $\theta^8$ will be useful later on.) One may introduce a
coordinate system \((x, y, y_1, \ldots, y_6, u_1, u_2, \ldots, u_{36})\) compatible with the local trivialisation \(\mathcal{P} \cong J^6 \times \mathcal{G}\) and such that locally

\[
\begin{pmatrix}
\theta^1 \\
\theta^2 \\
\theta^3 \\
\theta^4 \\
\theta^5 \\
\theta^6 \\
\theta^7 \\
\Gamma_+
\end{pmatrix} =
\begin{pmatrix}
u_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
u_2 & u_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
u_4 & u_5 & u_6 & 0 & 0 & 0 & 0 & 0 \\
u_7 & u_8 & u_9 & u_{10} & 0 & 0 & 0 & 0 \\
u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & 0 & 0 & 0 \\
u_{16} & u_{17} & u_{18} & u_{19} & u_{20} & u_{21} & 0 & 0 \\
u_{22} & u_{23} & u_{24} & u_{25} & u_{26} & u_{27} & u_{28} & 0 \\
u_{29} & u_{30} & u_{31} & u_{32} & u_{33} & u_{34} & u_{35} & u_{36}
\end{pmatrix}
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5 \\
\omega^6 \\
\omega^7 \\
dx
\end{pmatrix}.
\]

(9.34)

The structural group \(\mathcal{G}\) is the group of the lower triangular matrices as above.

We choose a representation of \(\mathfrak{gl}(2, \mathbb{R}) \oplus \mathbb{R}^7\) and write down \(\Omega\) in the following matrix form

\[
\Omega =
\begin{pmatrix}
-6\Gamma_0 - 6\Gamma_1 & 6\Gamma_+ & 0 & 0 & 0 & 0 & 0 & \theta^1 \\
\Gamma_- & -4\Gamma_0 - 6\Gamma_1 & 5\Gamma_+ & 0 & 0 & 0 & 0 & \theta^2 \\
0 & 2\Gamma_- & -2\Gamma_0 - 6\Gamma_1 & 4\Gamma_+ & 0 & 0 & 0 & \theta^3 \\
0 & 0 & 3\Gamma_- & -6\Gamma_1 & 3\Gamma_+ & 0 & 0 & \theta^4 \\
0 & 0 & 0 & 4\Gamma_- & 2\Gamma_0 - 6\Gamma_1 & 2\Gamma_+ & 0 & \theta^5 \\
0 & 0 & 0 & 0 & 0 & 5\Gamma_- & 4\Gamma_0 - 6\Gamma_1 & \Gamma_+ & \theta^6 \\
0 & 0 & 0 & 0 & 0 & 0 & 6\Gamma_- & 6\Gamma_0 - 6\Gamma_1 & \theta^7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(9.35)

Here \(\theta^1, \ldots, \theta^7, \Gamma_+, \Gamma_0, \Gamma_1\) and \(\Gamma_-\) are 1-forms on \(\mathcal{P}\).

Starting from this representation we construct a basis \((e_\mu), \mu = 1, \ldots, 11\) of \(\mathfrak{gl}(2, \mathbb{R}) \oplus \mathbb{R}^7\). To get the element \(e_1\) we formally set \(\theta^1 = 1\) and the remaining 1-forms equal to zero. All the remaining elements of the basis can be obtained in an analogous way, so that (9.35) may be written as

\[
\Omega = \sum_{i=1}^{7} \theta^i e_i + \Gamma_+ e_8 + \Gamma_0 e_9 + \Gamma_1 e_{10} + \Gamma_- e_{11}.
\]

(9.36)

The basis satisfies

\[
\begin{align*}
\mathfrak{g}_{-7} &= \text{span}\{e_1\}, & \mathfrak{g}_{-6} &= \text{span}\{e_2\}, & \mathfrak{g}_{-5} &= \text{span}\{e_3\}, \\
\mathfrak{g}_{-4} &= \text{span}\{e_4\}, & \mathfrak{g}_{-3} &= \text{span}\{e_5\}, & \mathfrak{g}_{-2} &= \text{span}\{e_6\}, \\
\mathfrak{g}_{-1} &= \text{span}\{e_7, e_8\}, & \mathfrak{g}_0 &= \text{span}\{e_9, e_{10}\}, & \mathfrak{g}_1 &= \text{span}\{e_{11}\},
\end{align*}
\]

and moreover

\[
\mathfrak{gl}(2, \mathbb{R}) = \text{span}\{e_8, \ldots, e_{11}\}, \quad \mathbb{R}^7 = \text{span}\{e_1, \ldots, e_7\}.
\]

To construct \(\mathcal{P}\) and \(\Omega\) we need to
i) Find a scalar product satisfying the conditions (8.32).

ii) Find formulae of $\mathcal{P} \mapsto \tilde{\mathcal{P}}$ by expressing $u_4, \ldots, u_{36}$ as certain functions of $u_1, u_2, u_3, x, y, y_1, \ldots, y_6$. Then $(u_1, u_2, u_3, x, y, y_1, \ldots, y_6)$ is a local coordinate system in $\mathcal{P}$ and the forms $\theta^1, \ldots, \theta^7, \Gamma_+$ of (9.35) are given by the pull-back of (9.34).

iii) Find formulae for $\Gamma_-, \Gamma_0$ and $\Gamma_1$.

We choose a scalar product on $\mathfrak{g}$ so that the basis $(e_1, \ldots, e_{11})$ is orthogonal and

$$(e_1, e_1) = 1, \quad (e_2, e_2) = 6, \quad (e_3, e_3) = 15,$$

$$(e_4, e_4) = 20, \quad (e_5, e_5) = 15, \quad (e_6, e_6) = 6,$$

$$(e_7, e_7) = 1, \quad (e_8, e_8) = 1, \quad (e_9, e_9) = 2,$$

$$(e_{10}, e_{10}) = 1, \quad (e_{11}, e_{11}) = 1.$$

The product satisfies the conditions (8.32) if we set $\tau_0 = \text{id}$, $\tau(e_9) = e_9$, $\tau(e_{10}) = e_{10}$ and $\tau(e_{11}) = e_8$.

Both ii) and iii) are obtained from the horizontality condition (7.29) and the normality conditions of Definition 8.1 with the scalar product as above. The 1-forms $\Gamma_0, \Gamma_1$ and $\Gamma_+$ on $\mathcal{P}$ are a priori arbitrary

$$\Gamma_A = \sum_{j=1}^3 a^j_A \, du_j + \sum_{i=1}^7 b^i_A \theta^i + b^+_A \, \Gamma_+ \quad A = -, 0, 1.$$  

The functions $a$ and $b$ are arbitrary but sufficiently smooth on $\mathcal{P}$, so they depend on the jet coordinates and $u_1, u_2, u_3$.

The curvature (7.28) becomes

$$K = \sum_{\mu=1}^{11} \sum_{j=1}^7 K^\mu_{ij} \, \Gamma_+ \wedge \theta^j \otimes e_\mu + \frac{1}{2} \sum_{\mu=1}^{11} \sum_{i,j=1}^7 K^\mu_{ij} \theta^i \wedge \theta^j \otimes e_\mu.$$  

and $K^{\rho}_{\mu\nu} = -K^{\rho}_{\nu\mu}$. The terms proportional to $\Gamma_0, \Gamma_1$ and $\Gamma_-$ must be absent since $K$ is horizontal. This produces a set of first order differential equations for the functions $a$, which may be determined without ambiguity giving

$$\Gamma_0 = \frac{1}{2} \frac{du_1}{u_1} - \frac{1}{2} \frac{du_3}{u_3} + \sum_i b^i_0 \theta^i + b^+_0 \Gamma_+,$$

$$\Gamma_1 = -\frac{1}{3} \frac{du_1}{u_1} + \frac{1}{2} \frac{du_3}{u_3} + \sum_i b^i_1 \theta^i + b^+_1 \Gamma_+,$$

$$\Gamma_- = \frac{du_2}{u_1} + \frac{u_2 \, du_3}{u_1 u_3} + \sum_i b^-_i \theta^i + b^+_\Gamma_+.$$  

The tensor $\kappa$ is equal to

$$\kappa = \frac{1}{2} \sum_{\mu=1}^{11} \sum_{i,j=1}^8 K^\mu_{ij} \mathcal{A}_i \wedge \mathcal{A}_j \otimes e_\mu.$$
The condition $\kappa \in \text{Hom}^1(\wedge^2 m, g)$ is equivalent to vanishing of the following components of $K$.

\[
\begin{align*}
K_{67}^{1,2,3,4,5} & \quad K_{57}^{1,2,3,4} & \quad K_{47}^{1,2,3} \\
K_{37}^{1,2} & \quad K_{27}^{1} & \quad K_{56}^{1,2,3} \\
K_{46}^{1,2} & \quad K_{36}^{1} & \quad K_{45}^{1} \\
K_{87}^{1,2,3,4,5,6} & \quad K_{86}^{1,2,3,4,5} & \quad K_{85}^{1,2,3,4} \\
K_{84}^{1,2,3} & \quad K_{83}^{1,2} & \quad K_{82}^{1} 
\end{align*}
\]

(9.39)

where $K_{37}^{1,2}$ is an abbreviation for $K_{37}^{1,1}$, $K_{37}^{2,2}$ and so on.

In order to evaluate the condition $\partial^* \kappa = 0$ we introduce the notation $(e_\mu, e_\nu) = p_{\mu\mu}$ (no summation) and $[e_\mu, e_\nu] = c_{\rho\mu\nu} e_\rho$. The explicit form of $\partial^* \kappa = 0$ is

\[
4 \sum_{\nu=1}^{11} \sum_{j=1}^{8} \frac{p_{\mu\nu}}{p_{ii}p_{jj}} K_{ij}^{\nu} c_{\nu j}^{\mu} + \sum_{j,k=1}^{8} \frac{p_{\mu\mu}}{p_{jj}p_{kk}} K_{jk}^{\mu} c_{i j k}^{\mu} = 0,
\]

(9.40)

where $\mu = 1, \ldots, 11$ and $i = 1, \ldots, 8$.

We compute $K$ via (9.33), (9.35) and (9.38). The conditions (9.37) and (9.39) become a set of easy algebraic and differential equations for the functions $u_4, \ldots, u_{36}$ and $b$. By solving these equations we obtain $u_4, \ldots, u_{36}$ and $b$ as rational functions of $u_1, u_2, u_3$ with coefficients given by arbitrary functions of the jet coordinates. After these substitutions the normality condition (9.40) becomes a set of algebraic and differential equations on the coefficients. The equations, although complicated, are overdetermined and may be solved without integration. It is enough to perform usual algebraic elimination of the functions, provided it is done in an appropriate order. The elimination also assures us that the solution — the Cartan connection — is unique. We have therefore proved

**Proposition 9.1.** Given a 7th order ODE $y^{(7)} = F(x, y, y', \ldots, y^{(6)})$ one can construct

i) A principal fibre bundle $H \to P \to J^6$, where $H = \mathbb{R} \times (\mathbb{R} \ltimes \mathbb{R})$.
ii) A Cartan connection $\Omega$ on $P$ of type $(GL(2, \mathbb{R}) \ltimes \mathbb{R}^7, H)$.

Two ODEs

\[ y^{(7)} = F(x, y, y', \ldots, y^{(6)}) \]

and

\[ \tilde{y}^{(7)} = \tilde{F}(\tilde{x}, \tilde{y}, \tilde{y}', \ldots, \tilde{y}^{(6)}) \]

are locally contact equivalent if and only if there exists a local bundle diffeomorphism $\Phi: \tilde{P} \to P$ such that $\Phi^* \Omega = \tilde{\Omega}$. The connection is given by (9.35), where
\[ \theta^1 = u_1 \omega^1, \]
\[ \theta^2 = u_2 \omega^1 + u_3 \omega^2, \]
\[ \theta^3 = \frac{u_2}{u_1} \omega^1 + 2 \frac{u_2 u_3}{u_1} \omega^2 + \frac{u_2^2}{u_1} \left( \left( \frac{3}{14} (DF) - \frac{12}{35} F_5 - \frac{13}{49} F_6 \right) \omega^1 - \frac{2}{35} F_6 \omega^2 + \frac{6}{5} \omega^3 \right), \]
\[ \theta^4 = \ldots \]

The explicit formulae for \( \theta^i, i = 4, 5, 7 \) and \( \Gamma_+, \Gamma_-, \Gamma_0, \Gamma_1 \) are omitted since they are complicated and unilluminating.

10. \( GL(2, \mathbb{R}) \) geometry from Cartan connection

The manifold \( P \) is endowed with two structures of a principal bundle: \( H \to P \to J^6 \) given by construction, and \( GL(2, \mathbb{R}) \to P \to M \) over the solution space which is generated by the connection \( \Omega \). Let \( X_\mu, \mu = 1, \ldots, 11 \) denote the frame dual to the coframe \( (\theta^2, \theta^3, \theta^4, \theta^5, \theta^6, \theta^7, \Gamma_+, \Gamma_0, \Gamma_1, \Gamma_-) \) of (9.35). The curvature \( K \) written in the form

\[ d\Omega = -\Omega \wedge \Omega + K, \quad (10.41) \]

and split into scalar-valued equations reads

\[ d\theta^1 = 6(\Gamma_1 + \Gamma_0) \wedge \theta^1 - 6 \Gamma_+ \wedge \theta^2 + \frac{1}{2} K^1_{ij} \theta^i \wedge \theta^j, \quad (10.42) \]
\[ d\theta^2 = -\Gamma_- \wedge \theta^1 + (6 \Gamma_1 + 4 \Gamma_0) \wedge \theta^2 - 5 \Gamma_+ \wedge \theta^3 + \frac{1}{2} K^2_{ij} \theta^i \wedge \theta^j, \]
\[ d\theta^3 = -2 \Gamma_- \wedge \theta^2 + (6 \Gamma_1 + 2 \Gamma_0) \wedge \theta^3 - 4 \Gamma_+ \wedge \theta^4 + \frac{1}{2} K^3_{ij} \theta^i \wedge \theta^j + K^3_{18} \theta^1 \wedge \Gamma_+, \]
\[ d\theta^4 = -3 \Gamma_- \wedge \theta^3 + 6 \Gamma_1 \wedge \theta^4 - 3 \Gamma_+ \wedge \theta^5 + \frac{1}{2} K^4_{ij} \theta^i \wedge \theta^j + (K^4_{18} \theta^1 + K^4_{28} \theta^2) \wedge \Gamma_+, \]
\[ d\theta^5 = -4 \Gamma_- \wedge \theta^4 + (6 \Gamma_1 - 2 \Gamma_0) \wedge \theta^5 - 2 \Gamma_+ \wedge \theta^6 + \frac{1}{2} K^5_{ij} \theta^i \wedge \theta^j + (K^5_{18} \theta^1 + K^5_{28} \theta^2 + K^5_{38} \theta^3) \wedge \Gamma_+, \]
\[ d\theta^6 = -5 \Gamma_- \wedge \theta^5 + (6 \Gamma_1 - 4 \Gamma_0) \wedge \theta^6 - \Gamma_+ \wedge \theta^7 + \frac{1}{2} K^6_{ij} \theta^i \wedge \theta^j + (K^6_{18} \theta^1 + K^6_{28} \theta^2 + K^6_{38} \theta^3 + K^6_{48} \theta^4) \wedge \Gamma_+, \]
\[ d\theta^7 = -6 \Gamma_- \wedge \theta^6 + (6 \Gamma_1 - 6 \Gamma_0) \wedge \theta^7 + \frac{1}{2} K^7_{ij} \theta^i \wedge \theta^j + (K^7_{18} \theta^1 + K^7_{28} \theta^2 + K^7_{38} \theta^3 + K^7_{48} \theta^4 + K^7_{58} \theta^5) \wedge \Gamma_+, \]
\[ d\Gamma_+ = 2 \Gamma_0 \wedge \Gamma_+ + \frac{1}{2} K^8_{ij} \theta^i \wedge \theta^j + K^8_{ij} \theta^i \wedge \Gamma_+, \]
\[ d\Gamma_0 = \Gamma_+ \wedge \Gamma_- + \frac{1}{2} K^9_{ij} \theta^i \wedge \theta^j + K^9_{ij} \theta^i \wedge \Gamma_+, \]
\[ d\Gamma_1 = \frac{1}{2} K^{10}_{ij} \theta^i \wedge \theta^j + K^{10}_{ij} \theta^i \wedge \Gamma_+, \]
\[ d\Gamma_- = -2 \Gamma_0 \wedge \Gamma_- + \frac{1}{2} K^{11}_{ij} \theta^i \wedge \theta^j + K^{11}_{ij} \theta^i \wedge \Gamma_+. \]

Since \( J^6 \) is a bundle over \( M \) then so is \( P \) and Theorem 9.1 together with Eqs. (9.33) guarantees that the fibres of the projection \( P \to M \) are annihilated by the simple ideal span\{\( \theta^1, \ldots, \theta^7 \)\}. The relation (10.42) implies that
this ideal is closed
\[ d\theta^i \wedge \theta^1 \wedge \ldots \wedge \theta^7 = 0 \quad \text{for} \quad i = 1, \ldots, 7. \]

It is annihilated by an integrable distribution span\{X_8, X_9, X_{10}, X_{11}\} and the maximal integral leaves of this distribution are locally the fibres of the projection \( P \to M \). Moreover, by (10.42) the commutation relations of the vector fields are isomorphic to the commutation relations of the algebra \( \mathfrak{gl}(2, \mathbb{R}) \). This allows us to define an action of \( GL(2, \mathbb{R}) \) on \( P \) by defining \( X_8, X_9, X_{10}, X_{11} \) to be the associated fundamental vector fields.

10.1. \textbf{Existence of} \( GL(2, \mathbb{R}) \) \textbf{geometry.} Does the bundle \( GL(2, \mathbb{R}) \to P \to M \) define a \( GL(2, \mathbb{R}) \) geometry on \( M \)? The answer to this question is positive only if \( P \) may be identified with a sub-bundle of the frame bundle \( F \mathbf{M} \). However, this may only be done if the original ODE satisfies additional conditions. An object that turns \( P \) into a sub-bundle of \( F \mathbf{M} \) is the canonical \( \mathbb{R}^7 \)-valued 1-form. The \( \mathbb{R}^7 \)-part of \( \Omega \) — the 1-forms \( \theta^1, \ldots, \theta^7 \) arranged into a column — is the natural candidate for it here, but one must still check whether the canonical 1-form has the property \( R^g_\theta \theta = g^{-1} \theta \) under the actions of \( GL(2, \mathbb{R}) \) in \( P \) and \( \mathbb{R}^7 \). In the language of the curvature \( K \) this is equivalent to the horizontality with respect to the projection \( P \to M \). Since \( K \) is already horizontal with respect to \( P \to J^0 \) we must only impose \( K(X_8, \cdot) = 0 \) which amounts to
\[ K^\mu_{\nu 8} = 0, \quad \mu, \nu = 1, \ldots, 11. \]

Due to algebraic and differential relations among the curvature components this condition may be further reduced to
\[ K^3_{18} = 0, \quad K^4_{18} = 0, \quad K^5_{18} = 0, \quad K^6_{18} = 0, \quad K^7_{18} = 0 \quad (10.43) \]
where
\[ K^{\alpha + 2}_{18} = \sum_{\beta=1}^{\alpha} c_{\alpha \beta} W_\beta, \quad \alpha = 1, \ldots, 5. \]

The expressions \( W_1, W_2, \ldots, W_5 \) are the Wünschmann conditions discussed in the Introduction and given by (Appendix A), and \( c_{\alpha \beta} \) are rational functions of \( u_1, u_2, u_3 \). The condition (10.43) is therefore equivalent to the vanishing of \( W_\alpha \). Simultaneous vanishing of these expressions is a property of a 7th order ODE invariant under contact transformations. It is also equivalent to the conditions for trivial linearizations obtained in [6, 7].

From now on we restrict our considerations to those ODEs which satisfy all five conditions in Appendix A. Then the curvature contains no \( \Gamma^i_{\mu \nu} \theta^i \) terms, and the equations (10.42) may be written as the structural equations for a \( \mathfrak{gl}(2, \mathbb{R}) \)-connection. We have proven
Theorem 10.1. Consider a 7th order ODE satisfying the conditions $W_\alpha = 0$, $\alpha = 1, \ldots, 5$. Then its solution space $M$ is equipped with a $\mathfrak{gl}(2, \mathbb{R})$ geometry. Let

\[
\Gamma = \begin{pmatrix}
-6\Gamma_0 - 6\Gamma_1 & 6\Gamma_- & 0 & 0 & 0 & 0 & 0 \\
\Gamma_- & -4\Gamma_0 - 6\Gamma_1 & 5\Gamma_+ & 0 & 0 & 0 & 0 \\
0 & 2\Gamma_- & -2\Gamma_0 - 6\Gamma_1 & 4\Gamma_+ & 0 & 0 & 0 \\
0 & 0 & 3\Gamma_- & -6\Gamma_1 & 3\Gamma_+ & 0 & 0 \\
0 & 0 & 0 & 4\Gamma_- & 2\Gamma_0 - 6\Gamma_1 & 2\Gamma_+ & 0 \\
0 & 0 & 0 & 0 & 5\Gamma_- & 4\Gamma_0 - 6\Gamma_1 & \Gamma_+ \\
0 & 0 & 0 & 0 & 0 & 0 & 6\Gamma_- \\
\end{pmatrix}. 
\]

be the $\mathfrak{gl}(2, \mathbb{R})$-part of the Cartan connection $\Omega$ of Proposition 9.1 and $\theta = (\theta^i)$ be its $\mathbb{R}^7$ part. Then $\Gamma$ is a $\mathfrak{gl}(2, \mathbb{R})$ linear connection on $P$ compatible with the $GL(2, \mathbb{R})$ geometry and the equations (7.28), (10.42) read

\[
d\theta^i + \Gamma^j_{ij} \wedge \theta^j = \frac{1}{2} T^i_{kl} \theta^k \wedge \theta^l, \quad i, j = 1, \ldots, 7, \tag{10.45}
\]

\[
d\Gamma^i_{ij} + \Gamma^i_{kj} \Gamma^k_{mj} = \frac{1}{2} R^i_{jlm} \theta^l \wedge \theta^m, \tag{10.46}
\]

where $T$ and $R$ are the torsion and curvature of $\Gamma$ respectively.

We will construct two tensor fields on $M$ preserved by the $GL(2, \mathbb{R})$ geometry: the conformal metric $g$ and the conformal 3-form $\phi$. This is done as follows. The action of $\mathfrak{gl}(2, \mathbb{R})$ on $\mathbb{R}^7$ is given by the matrix representation (10.44), in particular it defines two the conformal classes of tensors represented by $g \in S^2 \mathbb{R}^7^*$ and $\phi \in \Lambda^3 \mathbb{R}^7^*$. Next we transport these tensors to $T^*P$. The connection $\Omega$ gives the identification $e^i \leftrightarrow \theta^i$, $i = 1, \ldots, 7$ where $(e^i)$ is dual of the basis $(e_i)$ of (9.35) and (9.36). By this identification we get the tensor fields on $P$:

\[
g = \theta^1 \theta^7 - 6 \theta^2 \theta^6 + 15 \theta^3 \theta^5 - 10 (\theta^4)^2
\]

and

\[
\phi = 3 \theta^2 \wedge \theta^3 \wedge \theta^7 - 6 \theta^2 \wedge \theta^4 \wedge \theta^6 - \theta^1 \wedge \theta^4 \wedge \theta^7 + 3 \theta^1 \wedge \theta^5 \wedge \theta^6 + 15 \theta^3 \wedge \theta^4 \wedge \theta^5. \tag{10.47}
\]

Finally, we project these fields to conformal fields on $M$. This projection is well-defined because $g$ and $\phi$ satisfy two conditions: i) the vertical directions of $P \to M$ are degenerate for $g$ and $\phi$, and ii) the vertical directions are conformal symmetries of $g$ and $\phi$, that is

\[
\mathcal{L}_{X_0} g = 0, \quad \mathcal{L}_{X_1} g = 0, \quad \mathcal{L}_{X_8} g = 12 g, \quad \mathcal{L}_{X_0} \phi = 0, \quad \mathcal{L}_{X_1} \phi = 0.
\]
It is worth noting that $\mathcal{L}_{X_4}g = \mathcal{L}_{X_4}\phi = 0$ are equivalent to conditions listed in Appendix A. The conformal fields on $M$ will be also denoted by $g$ and $\phi$ — on solutions to the ODE they coincide with (4.9) and (4.12) respectively.

The following fact is an immediate consequence of Theorem 10.1.

**Proposition 10.2.** Let $\nabla$ denote the covariant derivative on $M$ associated to $\Gamma$. We have

\[
\nabla_X g = -A(X)g, \\
\nabla_X \phi = -\frac{2}{3}A(X)\phi,
\]

where the 1-form $A$ is proportional to the trace of the connection matrix:

\[
A = \frac{2}{7} \sum \Gamma^j_j = \frac{2}{7} \sum_{i,j} \langle \nabla_i X_j, \xi^j \rangle \xi^i,
\]

for any frame $(X_i)$ and the dual coframe $(\xi^i)$ such that $g = g_{ij} \xi^i \otimes \xi^j$ with constant $g_{ij}$.

Of course, $g$ and $\phi$ do not reduce $GL(7, \mathbb{R})$ to $GL(2, \mathbb{R})$, since their conformal stabilisers are $CO(3, 4)$ and $\mathbb{R}^+ \times G_2^{\text{split}}$ respectively. The object whose conformal stabiliser is precisely the irreducible $GL(2, \mathbb{R})$ is a certain totally symmetric 4-tensor $\Upsilon_{ijkl}$, which is however irrelevant in our approach.

10.2. **Torsion.** In this section we consider only those ODEs which admit the $GL(2, \mathbb{R})$ geometry on the solution space. First we shall characterise the torsion $T$ of $\Gamma$. Let $V^k$ denote the $k$-dimensional irreducible representation of $GL(2, \mathbb{R})$ as before. Torsion of any $\text{gl}(2, \mathbb{R})$-connection at $p \in P$ belongs to the representation $\Lambda^2 V^7 \otimes V^7$ which decomposes as

\[
\Lambda^2 V^7 \otimes V^7 = V^1 \oplus V^3 \oplus 3V^5 \oplus 3V^7 \oplus 3V^9 \oplus 2V^{11} \oplus 2V^{13} \oplus V^{15} \oplus V^{17}.
\]

**Proposition 10.3.** The only non-vanishing components of the torsion $T$ of the connection $\Gamma$ in Theorem 10.1 are in the 1-dimensional, the 3-dimensional, and a fixed 5-dimensional representation in the above decomposition.

\[
T = T_1 + T_3 + T_5.
\]

Explicit form of $T$ in (10.45) is given in Appendix B, where $\lambda$ spans $T_1$; $a_1, a_2, a_3$ span $T_3$, and $b_1, b_2, b_3, b_4, b_5$ span $T_5$.

**Proof.** To prove the formula of Appendix B we use Proposition 9.1 to explicitly calculate (10.45). Next we check that $T$ only occupies the irreducible representations as above.

We are now ready to prove Theorems 5.1 and 5.2.

**Proof of Theorem 5.1.** Using (10.47), (10.45) and (4.14) we calculate $\lambda, \tau_2, \tau_3$ and $\Theta$ in terms of the torsion coefficients and the forms $\theta^i$. We find that $\tau_2 = 0$ if $T_3 = 0$, $\tau_3 = 0$ if $T_5 = 0$, and also $\Theta = 24 \Gamma_1$. Next we calculate explicitly $\lambda, a_i$ and $b_i$ using formulae for $\Omega$ given in Theorem 9.1. Since the components $T_3$ and $T_5$ lie in irreducible representations they vanish iff any of the components $a_i$ or $b_i$ vanishes. In the Theorem we gave the simplest ones.
Proof of Theorem 5.2. In order to prove this result we need to extensively use the Bianchi identities. First, we suppose that $\tau_3 = 0$ which is equivalent to vanishing of $b_1, b_2, \ldots, b_5$. Then from $d^2\theta^i = 0$ we find that either i) $a_i = 0$ (equivalently $\tau_2 = 0$) or ii) $\lambda = 0$.

Suppose i). Then the torsion is reduced to $\lambda$ and it makes all the curvature except the Ricci scalar vanish. In particular the Lee form $\Theta = 24\Gamma_1$ is closed. Therefore there exists a conformal gauge in which locally $\Theta = 0$ and $\lambda = \text{const}$, and which defines a 10-dimensional sub-bundle $P'$ of $P$. Eqs. (10.45), (10.46) pulled-back to $P'$ become the structural equations of $SO(3,2)$ while the integrable distribution on $P'$ annihilated by $\theta^1, \ldots, \theta^7$ defines the action of $SO(2,1)$ on $P'$, which is vertical w.r.t. $P' \to M$. This also means that the maximal symmetry group of an underlying ODE is $SO(3,2)$. We find the ODE of point 2. by integration of the conditions from Appendix A and the conditions of Theorem 5.1.

Suppose ii). Lengthy but straightforward calculations show that the condition $\lambda = 0$ specifies curvature in Eqs. (10.46); all torsion and curvature coefficients and their coframe derivatives are polynomials of $a_1, a_2$ and $a_3$, which span $T_3$. Again, we have $d\Theta = 0$. Since $T_3$ belongs to the 3-dimensional representation $V^3$ of $GL(2,\mathbb{R})$ we may classify it by the orbit which sweeps out. $T_3$ is a tensor field on $P$, which is a $\mathfrak{gl}(2,\mathbb{R})$ bundle over $M$. If we fix $x \in M$ and sweep out the fibre $P_x$ then $T_3$ at points $p \in P_x$ sweeps a $GL(2,\mathbb{R})$-orbit in $V^3$. These orbits are labelled by the sign of $\langle \cdot, \cdot \rangle$, the conformal product in $V^3$ preserved by $GL(2,\mathbb{R})$. The case $\langle T_3, T_3 \rangle = 0$ is forbidden by the Bianchi identities. The only remaining possibilities are $\langle T_3, T_3 \rangle > 0$ and $\langle T_3, T_3 \rangle < 0$. The ODE (5.20) generates both cases in two disjoint areas of $M$, depending on the sign of $\langle T_3, T_3 \rangle = \text{const} \cdot (5y^{(6)}y^{(4)} - 6(y^{(5)})^2)$. 

\[\Box\]
The five Wunschmann conditions for the 7th order ODE

\[ W_1 = 245D^2 F_6 - 245DF_5 + 98F_4 - 210DFF_6 + 70F_5F_6 + 20F_6^3 \]

\[ W_2 = 6860D^2 F_5 - 10976DF_4 + 6615(DF_6)^2 + 6860F_3 - 8330DF_6F_5 + 1715F_5^2 - 1960DF_5F_6 + 1568F_4F_6 - 1890DF_6F_6^2 + 1190F_5F_6^2 + 135F_6^4 \]

\[ W_3 = 9604D^2 F_4 - 24010DF_5 + 15435DF_5DF_6 + 24010F_2 - 14749DF_6F_4 - 5145DF_6F_5 + 4459F_4F_5 - 2744DF_4F_6 + 6615(DF_6)^2 F_6 + 3430F_3F_6 - 6615F_6F_6F_6 - 1470F_2F_6^2 - 2205DF_6F_6^2 - 2107F_4F_6^2 - 1890DF_6F_6^3 + 945F_6^2F_6^3 + 135F_6^5 \]

\[ W_4 = 336140D^2 F_3 - 1344560DF_2 + 180075(DF_5)^2 + 432180DF_4DF_6 + 2352980F_1 - 624260DF_5F_5 - 216090DF_5F_4 + 64827F_4^2 - 14406D_4F_5 + 154350(DF_6)^2F_5 + 192080F_3F_5 - 102900DF_6F_5^2 + 17150F_5^2 - 9604DF_3F_6 + 308700DF_5DF_6F_6 + 192080F_2F_6 - 246960DF_6F_4F_6 - 154350DF_5F_5F_6 + 113190F_4F_5F_6 - 61740DF_4F_6^2 + 132300(DF_6)^2F_6^2 + 89180F_3F_6^2 - 176400DF_5F_6F_6^2 + 47775F_5F_6^2 - 44100DF_5F_6^3 + 35280F_4F_6^3 - 37800DF_6F_6^4 + 22050F_5F_6^4 + 2700F_6^6 \]

\[ W_5 = 2352980D^2 F_2 - 16470860DF_1 + 1512630DF_4DF_5 + 2268945DF_3DF_6 - 5126135DF_6F_2 - 1512630DF_5F_3 - 907578DF_4F_4 + 648270(DF_6)^2F_4 + 907578F_3F_4 - 756315DF_3F_6 + 108450DF_3DF_6F_5 + 1596665F_2F_5 - 108450DF_6F_4F_5 - 360150DF_5F_5^2 + 288120F_4F_5^2 - 672280DF_2F_6 + 540225(DF_5)^2F_6 + 1296540DF_4DF_6F_6 + 2352980F_1F_6 - 1620675DF_6F_3F_6 - 864360DF_5F_4F_6 + 324135F_4^2F_6 - 648270DF_4F_5F_6 + 926100(DF_6)^2F_5F_6 + 756315F_3F_6F_6 - 771750DF_6F_5^2F_6 + 154350F_3^2F_6 - 324135DF_3F_6^2 + 926100DF_5DF_6F_6^2 + 732305F_2F_6^2 - 926100DF_6F_4F_6^2 - 617400DF_5F_6F_6^2 + 524790F_4F_5F_6^2 - 185220DF_4F_5^2F_6 + 396900(DF_6)^2F_6^2 + 231525F_3F_6^3 - 661500DF_6F_5F_6^3 + 209475F_5F_6^3 - 132300DF_5F_6^3 + 119070F_4F_6^4 - 113400DF_6F_6^5 + 75600F_5F_6^5 + 8100F_7^7 + 65883440F_6. \]
Appendix B

The torsion components in Proposition 10.3

\[ T^1 = \frac{55}{36} b_1 \theta^1 \wedge \theta^2 + \frac{55}{36} b_4 \theta^1 \wedge \theta^3 + \left( \frac{55}{18} b_3 - \frac{10}{3} \lambda - 3a_3 \right) \theta^1 \wedge \theta^4 \\
+ \left( \frac{55}{16} b_5 + \frac{3}{2} a_2 \right) \theta^1 \wedge \theta^5 - \frac{77}{36} b_2 \theta^1 \wedge \theta^6 + \left( \frac{55}{2} b_3 + 10 \lambda + 9a_3 \right) \theta^2 \wedge \theta^3 \\
+ \left( \frac{55}{16} b_5 - 3a_2 \right) \theta^2 \wedge \theta^4 + \frac{55}{12} b_2 \theta^2 \wedge \theta^5, \]

\[ T^2 = \frac{55}{36} b_1 \theta^1 \wedge \theta^3 + \left( \frac{22}{23} b_4 + \frac{1}{2} a_1 \right) \theta^1 \wedge \theta^4 + \left( \frac{55}{36} b_3 - \frac{5}{2} \lambda - a_3 \right) \theta^1 \wedge \theta^5 \\
+ \left( \frac{11}{16} b_5 + \frac{1}{2} a_2 \right) \theta^1 \wedge \theta^6 - \frac{77}{216} b_2 \theta^1 \wedge \theta^7 + \left( \frac{55}{18} b_4 - \frac{3}{2} a_1 \right) \theta^2 \wedge \theta^3 \\
+ \left( \frac{55}{9} b_5 + 2a_3 + \frac{10}{3} \lambda \right) \theta^2 \wedge \theta^4 - \frac{11}{18} b_2 \theta^2 \wedge \theta^6 + \left( \frac{275}{18} b_5 - \frac{5}{2} a_2 \right) \theta^3 \wedge \theta^4 \\
+ \frac{275}{72} b_2 \theta^3 \wedge \theta^5, \]

\[ T^3 = \frac{11}{16} b_1 \theta^1 \wedge \theta^5 + \left( \frac{22}{23} b_4 + \frac{1}{2} a_1 \right) \theta^1 \wedge \theta^6 + \left( \frac{44}{15} b_3 - \frac{1}{5} a_3 - \frac{1}{3} \lambda \right) \theta^1 \wedge \theta^7 \\
+ \frac{22}{35} b_5 + \frac{1}{16} a_2 \right) \theta^1 \wedge \theta^8 + \frac{22}{35} b_1 \theta^2 \wedge \theta^3 + \left( \frac{11}{12} b_4 - a_1 \right) \theta^2 \wedge \theta^4 \\
+ \left( \frac{11}{16} b_5 + \frac{3}{2} a_2 \right) \theta^2 \wedge \theta^5 + \frac{22}{23} b_2 \theta^2 \wedge \theta^7 + \left( \frac{55}{18} b_3 + \frac{10}{3} \lambda + a_3 \right) \theta^3 \wedge \theta^4 \\
+ \left( \frac{55}{9} b_5 - \frac{3}{2} a_2 \right) \theta^3 \wedge \theta^5 + \frac{22}{23} b_2 \theta^3 \wedge \theta^6 + \frac{22}{23} b_1 \theta^2 \wedge \theta^4, \]

\[ T^4 = -\frac{11}{20} b_1 \theta^1 \wedge \theta^7 + \left( \frac{22}{23} b_4 + \frac{3}{10} a_1 \right) \theta^1 \wedge \theta^8 + \left( \frac{11}{18} b_3 - \frac{1}{3} \lambda \right) \theta^1 \wedge \theta^7 \\
+ \frac{11}{6} b_4 \theta^2 \wedge \theta^5 + \left( -\frac{22}{5} b_3 - \lambda \right) \theta^2 \wedge \theta^6 + \left( \frac{11}{18} b_5 + \frac{3}{10} a_2 \right) \theta^2 \wedge \theta^7 \\
+ \left( \frac{55}{9} b_5 - \frac{3}{2} a_2 \right) \theta^2 \wedge \theta^6 + \frac{22}{23} b_2 \theta^2 \wedge \theta^7 + \frac{22}{23} b_1 \theta^2 \wedge \theta^4, \]

\[ T^5 = -\frac{11}{20} b_1 \theta^1 \wedge \theta^6 + \left( \frac{22}{23} b_4 + \frac{3}{10} a_1 \right) \theta^1 \wedge \theta^7 + \frac{11}{12} b_1 \theta^2 \wedge \theta^5 \\
+ \left( -\frac{11}{18} b_4 + \frac{3}{5} a_1 \right) \theta^2 \wedge \theta^6 + \left( -\frac{44}{15} b_3 - \frac{2}{3} \lambda + \frac{1}{3} a_3 \right) \theta^2 \wedge \theta^7 \\
+ \frac{55}{18} b_5 \theta^3 \wedge \theta^4 + \left( \frac{55}{9} b_4 - \frac{3}{2} a_1 \right) \theta^3 \wedge \theta^5 - \frac{11}{2} b_3 \theta^3 \wedge \theta^6 \\
+ \left( \frac{55}{9} b_5 + \frac{1}{2} a_2 \right) \theta^3 \wedge \theta^7 + \left( \frac{55}{18} b_3 + \frac{10}{3} \lambda - a_3 \right) \theta^4 \wedge \theta^5 \\
+ \left( \frac{11}{18} b_5 - a_2 \right) \theta^4 \wedge \theta^6 + \frac{11}{14} b_2 \theta^4 \wedge \theta^7 + \frac{22}{23} b_2 \theta^5 \wedge \theta^6, \]

\[ T^6 = -\frac{11}{20} b_1 \theta^1 \wedge \theta^7 - \frac{11}{18} b_3 \theta^2 \wedge \theta^6 + \left( -\frac{11}{18} b_4 + \frac{1}{2} a_1 \right) \theta^2 \wedge \theta^7 \\
+ \left( -\frac{55}{36} b_4 - \frac{5}{3} \lambda + a_3 \right) \theta^3 \wedge \theta^7 + \left( \frac{275}{18} b_4 - \frac{5}{2} a_1 \right) \theta^4 \wedge \theta^6 \\
+ \left( -\frac{55}{9} b_3 + \frac{10}{3} \lambda - 2a_3 \right) \theta^4 \wedge \theta^6 + \left( \frac{275}{54} b_5 + \frac{1}{2} a_2 \right) \theta^4 \wedge \theta^7 \\
+ \left( -\frac{55}{18} b_5 - \frac{3}{2} a_2 \right) \theta^5 \wedge \theta^6 + \frac{55}{18} b_3 \theta^5 \wedge \theta^7 + \frac{275}{72} b_1 \theta^5 \wedge \theta^6, \]

\[ T^7 = -\frac{275}{72} b_1 \theta^2 \wedge \theta^7 + \frac{55}{9} b_1 \theta^3 \wedge \theta^6 + \left( -\frac{55}{9} b_4 + \frac{3}{2} a_1 \right) \theta^3 \wedge \theta^7 \\
+ \left( \frac{55}{9} b_4 - 3a_1 \right) \theta^4 \wedge \theta^6 + \left( \frac{55}{18} b_3 - \frac{10}{3} \lambda + 3a_3 \right) \theta^4 \wedge \theta^7 \\
+ \left( -\frac{55}{9} b_1 + 10 \lambda - 9a_3 \right) \theta^5 \wedge \theta^6 + \frac{55}{9} b_5 \theta^5 \wedge \theta^7 + \frac{55}{18} b_2 \theta^6 \wedge \theta^7. \]

References

[1] Bobiński, M. and Nurowski, P. (2007) Irreducible SO(3) geometries in dimension five. J. reine angew Math. 605.
[2] Bryant, R. L. (1987) Metrics with exceptional holonomy. Ann. of Math. (2) 126, no. 3, 525–576.
[3] Bryant, R. L. (1991) Two exotic holonomies in dimension four, path geometries, and twistor theory. Proc. Symp. Pure. Maths. Vol. 53, 33–88.
[4] Bryant, R. L. (2003) Some remarks on $G_2$-structures, arXiv:math/0305124v4.
[5] Cleyton, R. & Ivanov, S. (2007) On the geometry of closed $G_2$-structures. Comm. Math. Phys. 270, 53–67.
[6] Doubrov, B. (2001) Contact trivialization of ordinary differential equations. Differential geometry and its applications, 73–84.
[7] Doubrov, B. (2008) Generalized Wilczynski invariants for non-linear ordinary differential equations, The IMA Volumes in Mathematics and its Applications 144, 25-40.
[8] Doubrov, B., Komrakov, B. & Morimoto, T. (1999) Equivalence of holonomic differential equations, Lobachevskij Journal of Mathematics 3, 39.
[9] Dunajski, M. & Tod, K. P. (2006) Paraconformal geometry of nth order ODEs, and exotic holonomy in dimension four. J. Geom. Phys. 56, 1790-1809.
[10] Dunajski, M. & Sokolov, V. V. (2010) On 7th order ODE with submaximal symmetry. Preprint arXiv:1002.1620.
[11] Fernandez, M., Gray, A. (1982) Riemannian manifolds with structure group $G_2$, Ann. Mat. Pura Appl. 132 19-45.
[12] Godliński, M. & Nurowski, P. (2010) GL(2, R) geometry of ODEs, J. Geom. Phys. 60, 991-1027.
[13] Grace, J. H. and Young, Alfred (1903). The algebra of invariants. Cambridge: Cambridge University Press.
[14] Hitchin, N. (1982) Complex manifolds and Einstein’s equations, in Twistor Geometry and Non-Linear systems, Springer LNM 970, Doebner, H. & Palev. T.
[15] Kobayashi, S. Transformation Groups in Differential Geometry.
[16] Kodaira, K. (1963) On stability of compact submanifolds of complex manifolds, Am. J. Math. 85, 79-94.
[17] Morimoto, T. (1993) Geometric structures on filtrred manifolds, Hokkaido Math. J. 22 263.
[18] Morimoto, T. Lie Algebras, Geometric Structures and Differential Equations on Filtrred Manifolds, in: Advanced Studies in Pure Mathematics 37, 2002, Lie Groups, Geometric Structures and Differential Equations — One Hundred Years after Sophus Lie, 205-252.
[19] Nurowski, P. (2005) Differential equations and conformal structures. J. Geom. Phys. 55, 19–49.
[20] Nurowski, P. (2009) Comment on GL(2,R) geometry of 4th order ODEs, J. Geom. Phys. 59.
[21] Olver, P. J. (1995) Equivalence, invariants, and symmetry. CUP, Cambridge.
[22] Penrose, R. & Rindler, W. (1987, 1988) Spinors and space-time. Two-spinor calculus and relativistic fields. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge.
[23] Sokolov, V. V. (1988) Symmetries of evolution equations. Russian Math. Surveys 43 165–204.
[24] Tanaka, N. (1970) On differential systems, graded Lie algebras and pseudo-groups, J. Math. Kyoto Univ 10 1.
[25] Tanaka, N. (1979) On the equivalence problems associated with simple graded Lie algebras, Hokkaido Math. J. 8 23.
[26] Vinberg, E. B., Gorbatsevich, V. V. & Onishchik, A. L. (1994) Structure of Lie groups and Lie algebras. Lie groups and Lie algebras, III. Encyclopaedia of Mathematical Sciences, 41. Springer-Verlag, Berlin.
[27] Walker, R (1950) Algebraic Curves, Princeton.
[28] Wilczynski, E. J. (1905) Projective differential geometry of curves and ruled surfaces, Leipzig, Teubner.
[29] Wünschmann, K.W. (1905) Inaug. Dissert., Teubner, Leipzig.
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