A Generalized Fokker-Planck Equation for the Ratchet Problem: Memory Effects

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We study stochastic ratchets with inertia in the limit where the time correlations become important. We have developed a Fokker-Planck type equation for the ratchet problem which includes the memory effects. It is tested with comparison to the Langevin simulations. The effect of the memory in the system is analyzed extensively. A positive feedback regime which manifests itself as instabilities is observed. This may be regarded as an illustration of stochastic resonance.

The motion of a particle under the influence of random forces and an asymmetric periodic potential has been attracting a great deal of interest in recent years. The so-called ratchet models exhibit a variety of interesting phenomena stemming from nonequilibrium fluctuations. The most significant result of these investigations is the occurrence of a net macroscopic current which depends on the spatial asymmetry of the external potential as well as the statistical properties of the fluctuating forces. The stochastic ratchets are important in understanding biological systems such as molecular motors, and they also find application for electrons in superlattices, quantum dots, Josephson junctions, and atomic systems. Recent experiments making use of nanolithography techniques provide a wealth of possibilities in the study of quantum properties.

In this work we study the effects of memory friction for a Brownian particle subject to spatial and temporal forces. When the dissipative effects of the fluctuating (random) force are allowed to depend on the system’s past behavior, the memory function replaces the constant friction term and a generalized Langevin equation results. Previous studies of thermal ratchets making use of the memory function or correlated noise relied on the simulation of Langevin equation to obtain transport properties. It has been remarked that a corresponding Fokker-Planck equation (FPE) describing the dynamics of the system is difficult to construct owing to the non-Markovian nature of the underlying process. There has been several attempts to obtain a FPE with colored Gaussian noise. Our chief aim here is to obtain a FPE in the presence of memory effects in an approximate way. Introducing some simplifying assumptions we form an integro-differential equation satisfied by the probability density which may be construed as a generalized Fokker-Planck equation. We remark that the non-Markovian nature of the problem is preserved in the final FPE we obtain. It has been known that for a non-Markovian system, the is no generic way of obtaining a FPE if the system does not exhibit steady state solutions. In our case, we know from the outset that steady state solutions exist through Langevin simulations, even in the non-Markovian limit (i.e. with memory effects). Thus, deriving a FPE equation for this system does not contradict the well known features of the non-Markovian problems. We demonstrate by numerical calculations that the same qualitative and quantitative results can be obtained as those from the generalized Langevin equation.

Our application of the generalized FPE to the ratchet problem shows interesting properties resulting from the memory effects. In particular, we find a regime in the current characteristics exhibiting instabilities. In the following we first sketch a derivation of the generalized FPE in the form of an integro-differential equation which has general applicability to the ratchet problem. We then present and discuss our results emphasizing the memory effects.

Our aim is to construct a Fokker-Planck equation for stochastic ratchets including the memory effects. We begin with the basic equation of motion governing the dynamics of a Brownian particle, given by the generalized Langevin equation

\[ \frac{dv}{dt} = -\int_{-\infty}^{t} \mu(t-t') v(t') \, dt' + f(t) - \frac{dV(x)}{dx} + \xi(t), \]

where \( x(t) \) and \( v(t) = dx/dt \), respectively, are the position and velocity of the particle, \( V(x) \) is the asymmetric periodic potential, \( f(t) \) is a time-dependent external driving force, and \( \xi(t) \) is the stochastic force with zero mean. Each term in Eq. (1) is scaled with the particle mass \( m \). We note that the above description is quite general and encompasses a variety of situations already covered in previous studies. Assuming we know the solution at some time \( t \), it is possible to construct the solution at time \( t + dt \). Let the solution for the position (velocity) be \( x(v) \) and \( x'(v') \) at time \( t \) and \( t + dt \), respectively. Then it is straightforward to construct the following set of equations:

\[ v' = v - \left[ \int_{-\infty}^{t} \mu(t-t') v(t') \, dt' - f(t) + \frac{dV(x)}{dx} \right] \Delta t + \xi(t) (\Delta t)^{1/2}, \]

and \( x' = x + v \Delta t \).
The power 1/2 of $\Delta t$ multiplying the random force is a requirement imposed by the zero-mean property of $\xi(t)$, so that the lowest order contribution of the noise term is in second order.

Our goal is to construct an equation for the probability density function $P(v, x, t)$. Since we can express $v'$ and $x'$ at time $t + \Delta t$ when we know $v$ and $x$ at time $t$, it should also be possible to find a solution for $P(v', x', t + \Delta t)$ when we know $P(v, x, t)$. In other words, by using the fact that the total probability is conserved, we aim at finding how $P(v, x, t)$ transforms to $P(v', x', t + \Delta t)$. To this end, we first discretize the integral in Eq. (2), so that

$$
-\int_{-\infty}^{t} \mu(t - t') v(t') \, dt' = -\sum_{n=0}^{\infty} \mu(n\Delta t') v(t - n\Delta t') \Delta t',
$$

with $\mu(0)\Delta t'$ approaching $\gamma$ as $\Delta t' \to 0$. Here $\gamma$ is the friction constant in the absence of memory correlations in the dissipation term. In order to simplify the notation we define $v_n = v(t - n\Delta t')$, $v$ as velocity at time $t$, and $v'$ as velocity at time $t + \Delta t$. At this point, remembering that all $v_n$'s, $v$, $v'$, $x$, and $x'$ are random variables, we write

$$
P(v', x', t + \Delta t) = \int dx \int dv \int d\xi \int \prod_{n} dv_{n} \left[ \prod_{n} P_{n}(v_{n}) \right] P(v, x, t) P_{\xi} \delta(x' - x - v \Delta t)
$$

$$
\times \delta \left( v' - v + \sum_{n=0}^{\infty} \mu(n\Delta t')v_{n}\Delta t' - F(x) - f(t) \right) \Delta t - \xi(t)(\Delta t)^{1/2}.
$$

Here $P_{n}$, is the probability density function for the random variable $v_{n}$ and similarly $P_{\xi}$ is the probability density function for $\xi$. The product over $P_{n}(v_{n})$ indicates an assumption of statistical independence of the velocities $v_{n}$ at different times, although such velocities are clearly expected to be correlated. Nevertheless, this approximation is expected to be valid for velocity distributions near steady state, and enables us to proceed with the algebra. Performing the integrations over $v$ and $x$, keeping only terms of order $\Delta t$, and considering the limits $\Delta t' \to 0$ and $\Delta t \to 0$, we find

$$
\frac{\partial P}{\partial t} = \gamma P(v, x, t) + \frac{\partial P}{\partial v} \left[ \int_{-\infty}^{t} \mu(t - t') \bar{v}(t') \, dt' - F(x) - f(t) + \gamma(v - \bar{v}) \right] - \frac{\partial P}{\partial x} v + D \frac{\partial^{2} P}{\partial v^{2}}.
$$

Here $\bar{v}(t)$ is the average velocity at a given time, $\gamma$ is equal to $\mu(0)\Delta t'$ as $\Delta t' \to 0$ as defined earlier, and $D$ is the diffusion constant. Equation (5) is the main result of this paper. Under the assumptions set out in the preceding paragraphs (i.e. statistical independence), it describes the dynamics of stochastic ratchets when the memory effects are included. We recover the conventional FPE if the memory function is chosen to be a delta function. In the following we demonstrate that the solutions of Eq. (5) are in good agreement with the Langevin simulation results.

We have solved the Fokker-Planck and Langevin equations numerically to test how well our proposed FPE describes the phenomenology of thermal ratchets. We have found that Eq. (5) reproduces all the established results such as current reversal and noise rectification rather well. In the results to be discussed below, we have specifically used $f(t) = A \sin(\omega t)$ with $A = 1.0$ and $\omega = 0.2$ for the driving force. The ratchet potential is taken to be $V(x) = b_{0} \sin(2\pi x / L) + b_{1} \sin(4\pi x / L)$, with $b_{0}$, $b_{1}$, and $L$ are constants as has been used by others. We model the memory function in the form $\mu(t) = \langle \gamma / \tau \rangle \exp(-|t| / \tau)$, where $\tau$ is the correlation time. Finally, $\xi(t)$ is a Gaussian random variable whose average is zero and time correlation is given by the fluctuation-dissipation theorem $\langle \xi(t)\xi(t') \rangle = 2D\mu(t - t')$, for $t > t'$, and approaches $\langle \xi(t)\xi(t') \rangle = 2\gamma D\delta(t - t')$ as $\tau$ approaches zero.

We first demonstrate that our proposed Fokker-Planck equation [Eq. (5)] and Langevin equation [Eq. (1)] yield the same current value for the ratchet problem. As seen from these two equations, we can construct a solution for both of them if we know the solution at some initial time. We have solved both equations using a simple finite element algorithm. In Fig. 1 we illustrate our results for the time dependence of the velocity. Figure 1a shows the instantaneous velocities in both approaches and we observe that the solutions are similar. The effective or average velocity of the particle corresponding to long time behavior is shown in Fig. 1b. A simple running average algorithm is used to find the effective (average) velocities. In fact, when properly averaged over the periods of oscillations, the long-time behaviors are just straight lines. Nevertheless, it is seen that the effective velocity, or equivalently the current which is the physical observable, is the same for both methods. It is evident in Fig. 1 that the FPE results, approach the Langevin equation values with increasing time.

We now turn to the effects of memory correlation in the ratchet problem. In this regard, we have analyzed the ratchet problem defined in the previous section for various values of the correlation time $\tau$ and strength of the
correlations $\gamma$. In Fig. 2 we depict $x(t)$ for various values of $\tau$. It is evident that as we increase $\tau$, because of negative feedback of the correlation to the system, the current is decreased (current is proportional to the slope of the curve). When we explore the large $\tau$ regime such that it becomes comparable to the period of oscillations in $x(t)$, we find a different behavior. In Fig. 3 we again show $x(t)$ for different values of $\tau$. The topmost curve ($\tau = 0$) is enlarged many times in order to show the differences between the case when $\tau = 0$ and when $\tau$ is large. It is observed that as $\tau$ becomes comparable to the period of oscillations of the $\tau = 0$ case, there appears some instabilities in the system stemming from the positive feedback of the memory. The memory term in the definition of the friction now behaves as if it is not a friction term but rather as a driving term in resonance with the actual frequency of the oscillations. Stochastic resonances in ratchet problems or in a class of Fokker-Planck equations have been reported. In the absence of a time-dependent driving force, no instabilities are observed, consistent with the known results. We have checked that similar instabilities are also obtained by direct Langevin simulations. A further question is how the system behaves as the strength of the total friction changes. In Fig. 4 the time development of the position $x(t)$ for various values of $\gamma$ when an instability is encountered ($\tau = 30$) is shown. We observe that as the magnitude of $\gamma$ increases, the period of oscillations decreases. In the inset of Fig. 4 we show the period of oscillations as a function of $\gamma$ which indicates an exponential decrease. This again illustrates the positive feedback in our ratchet system.

In summary, we have developed a Fokker-Planck equation corresponding to a generalized Langevin equation to describe the dynamics of a Brownian particle under the influence of external potentials and memory effects. The proposed FPE reproduces the established results and reduces to the known case in the white-noise limit when the memory effect is absent. We have demonstrated the adequacy of generalized FPE description by comparing the solutions to that of the Langevin equation simulations. The correlated fluctuating force gives rise to resonances for certain choice of the parameters. Finally, since Eq. (1) can also be interpreted as the Heisenberg equation for a quantum particle coupled to a heat bath, our corresponding FPE may be explored to study the quantum ratchets.

This work was partially supported by the Scientific and Technical Research Council of Turkey (TUBITAK) under Grant No. TBAG-1662. We thank Ö. Türel for discussions.

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FIG. 1. (a) The instantaneous velocity calculated from the Langevin simulation (thin lines) and from the solution of generalized Fokker-Planck equation (thick lines). The solid and dashed lines indicate the amplitude $A = 1$ and $A = 0.5$, respectively, of the driving force $f(t)$. (b) The corresponding average velocities for the same parameters.

FIG. 2. The position variable $x(t)$ for various values of the correlation time $\tau$ at fixed $\gamma = 1$. The dotted, dashed, dot-dashed, and solid lines indicate $\tau = 0$, 1, 5, and 10, respectively.

FIG. 3. The position variable $x(t)$ for various values of the correlation time $\tau$ at $\gamma = 1$. The dotted and solid lines indicate $\tau = 18$ and $\tau = 30$, respectively. $\tau = 0$ (dashed line) case is also shown for reference.

FIG. 4. The position variable $x(t)$ at $\tau = 30$. The solid, dashed and dotted lines indicate $\gamma = 0.01$, $\gamma = 0.1$, and $\gamma = 1.0$, respectively. The inset shows the dependence of the oscillation period on $\gamma$.
Figure 4