Bosonic quantum communication across arbitrarily high loss channels

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A general attenuator $\Phi_{\lambda,\sigma}$ is a bosonic quantum channel that acts by combining the input with a fixed environment state $\sigma$ in a beam splitter of transmissivity $\lambda$. If $\sigma$ is a thermal state the resulting channel is a thermal attenuator, whose quantum capacity vanishes for $\lambda \leq 1/2$. We study the quantum capacity of these objects for generic $\sigma$, proving a number of unexpected results. Most notably, we show that for any arbitrary value of $\lambda > 0$ there exists a suitable single-mode state $\sigma(\lambda)$ such that the quantum capacity of $\Phi_{\lambda,\sigma(\lambda)}$ is larger than a universal constant $c > 0$. Our result holds even when we fix an energy constraint at the input of the channel, and implies that quantum communication at a constant rate is possible even in the limit of arbitrarily low transmissivity, provided that the environment state is appropriately controlled. We also find examples of states $\sigma$ such that the quantum capacity of $\Phi_{\lambda,\sigma}$ is not monotonic in $\lambda$. These findings may have implications for the study of communication lines running across integrated optical circuits, of which general attenuators provide natural models.

Introduction.— Quantum optics will likely play a major role in the future of quantum communication [1–4]. Indeed, practically all quantum communication in the foreseeable future will rely on optical platforms. For this reason, the study of quantum channels acting on continuous variable (CV) systems, that is, finite ensembles of electromagnetic modes, is a core area of the rapidly developing field of quantum information [5–7].

In the best studied models of optical communication, one represents an optical fibre as a thermal attenuator channel. Mathematically, its action can be thought of as that of a beam splitter with a certain transmissivity $0 \leq \lambda \leq 1$, where the input state is mixed with an environment state $\sigma$ that is assumed to be thermal. This approximation is well justified when the optical fibre is so long that the ‘effective’ environment state, resulting from averaging several elementary interactions that are effectively independent, due to the limited correlation length of the environment, is practically Gaussian and thermal. This phenomenon is a manifestation of the quantum central limit theorem [8, 9]. And indeed, an impressive amount of literature has been devoted to finding bounds on the quantum capacity of the thermal attenuator. We now have exact formulae for the zero-temperature case [10–13], and tight upper [14–18] and lower [19–21] bounds in all other cases.

However, the thermal noise approximation is challenged when the communication channel is so short that such an averaging process cannot possibly take place, or when the environment’s correlation length is comparatively large. This may be the case, for instance, in miniaturised quantum optical circuits, that the promising field of integrated quantum photonics aims to exploit to implement fault-tolerant quantum computation [22–24]. Further, a communication line connecting two sites of such a circuit may incur in noise that is far away from being thermal, as it comes from active quantum elements (e.g. single-photon sources). We are thus led to investigate general attenuator channels, hereafter denoted with $\Phi_{\lambda,\sigma}$, where the environment state $\sigma$ is no longer thermal. Unsurprisingly, such models have received increasing attention recently [9, 24–28].

Other motivations for considering general attenuators stem on the one hand from the need to go beyond the Gaussian formalism to accomplish several tasks that are critical to quantum information, e.g. universal quantum computation [29–32], entanglement distillation [31–33], entanglement swapping [34–36], error correction [36], and state transformations in general resource theories [37–39]. On the other hand, general attenuators are among the simplest examples of non-Gaussian channels that are nevertheless Gaussian dilatable, meaning that they can be Stinespring dilated [35, 40, 41] by means of a symplectic unitary [41]. This makes them amenable to a quantitative analysis in many respects. For example, it has been shown that making the environment state non-Gaussian, e.g. by means of a photon addition, can be advantageous when transmitting quantum or private information [28]. In spite of their increased complexity compared to Gaussian channels, the entanglement-assisted capacity of a general attenuator can nevertheless be upper bounded thanks to the conditional entropy power inequality [24, 25]. Similar bounds can be obtained for the quantum [28] and private [40] capacity as well, by making use of the solution to the minimum output entropy conjecture [11–13], combined with known extremality properties of Gaussian states [44–45]. Finally, we have mentioned that by concatenating a large number $n$ of general attenuators with a fixed total transmissivity one typically obtains an effective channel that resembles a thermal attenuator. In this regime of large but finite $n$, the associated quantum capacity can be bounded thanks to the...
quantum Berry–Esseen inequality [31 Corollary 13]. Here we investigate the quantum capacity of general attenuators $\Phi_{\lambda,\sigma}$, uncovering novel unexpected phenomena. It has been observed recently [6 Lemma 16] that output states of general attenuators with transmissivity $\lambda = 1/2$ always have non-negative Wigner functions [16, 47]. At first sight, this may suggest that such channels are ‘classical’ in some respect [48–50]. Indeed, we show that for all convex combinations of symmetric channels are ‘classical’ in some respect [46, 47]. At first sight, this may suggest that such states of general attenuators with transmissivity $\lambda > 0$ is universal (Theorem 2). As a corollary, we also see that the quantum capacity also vanishes for thermal attenuators and depolarising channels, which quantum capacity also vanishes for $\lambda < 1/2$. Indeed, this is exactly what happens for thermal attenuators and depolarising channels, and reveals that the phenomenon of general attenuators is much richer than perhaps expected. Our proof is fully analytical, and goes by analyising the environment to be in a thermal state $|\tau\rangle\langle\tau|$. All this marks a striking difference with the aforementioned behaviour of thermal attenuators and depolarising channels, whose quantum capacity also vanishes for $\lambda < 1/2$.

However, we establish the following surprising result: for all values of $\lambda > 0$ one can find suitable states $|\lambda\rangle$ that make $Q(\Phi_{\lambda,\sigma}(\lambda)) > c$, where the constant $c > 0$ is universal (Theorem 3). As a corollary, we also see that $Q(\Phi_{\lambda,\sigma})$ is in general not monotonic in $\lambda$ for fixed $\sigma$. All this marks a striking difference with the aforementioned behaviour of thermal attenuators and depolarising channels, and reveals that the phenomenon of general attenuators is much richer than perhaps expected.

The Wigner function $W_T$ of $T$ is the Fourier transform of $\chi_T$. Note that $W_T$ is typically not pointwise positive for a generic quantum state $\rho$.

A beam splitter of transmissivity $0 \leq \lambda \leq 1$ acting on two systems of $m$ modes each is represented by the unitary operator

$$U_\lambda := e^{i\arccos \sqrt{\lambda} \sum_j (a_j^\dagger a_j - a_j a_j^\dagger)},$$

where $a_j, b_j$ are the annihilation operators on the $j$-th modes of the first and second system, respectively. Our main object of study is the general attenuator channel $\Phi_{\lambda,\sigma}$, which acts on an $m$-mode system $B$ as

$$\Phi_{\lambda,\sigma}(\rho_B) := Tr_E \left[ U_{\lambda E}^{BE} (\rho_B \otimes \sigma_E) (U_{\lambda E}^{BE})^\dagger \right].$$

Dropping the system labels for simplicity, this can be cast in the language of characteristic functions as

$$\chi_{\Phi_{\lambda,\sigma}(\rho)}(\alpha) = \chi_\sigma \left( \sqrt{\lambda} \alpha \right) \chi_\lambda \left( \sqrt{1 - \lambda} \alpha \right).$$

The thermal attenuators $B_{\lambda,\nu} := \Phi_{\lambda,\tau_\nu}$ as well as the pure loss channels $\Phi_{\lambda,0} = \Phi_{\lambda,|0\rangle\langle 0|}$ are standard examples of single-mode attenuators, obtained by taking the environment to be in a thermal state $\tau_\nu := \frac{1}{\nu + 1} \sum_{n=0}^{\infty} \left( \frac{\nu}{\nu + 1} \right)^n |n\rangle\langle n|$, where $|n\rangle$ is the $n$-th Fock state.

Quantum channels are useful because they can transmit quantum information. The maximum rate at which independent copies of a channel $\Phi$ acting on a system $B$ can simulate instances of the noiseless qubit channel $I_2$ is called the quantum capacity of $\Phi$, and denoted with $Q(\Phi)$. For CV systems, it is often useful to account for an energy bound at the input of the channel. We shall assume that the relevant Hamiltonian is the total photon number: for an $m$-mode system, $H_m := \sum_{j=1}^m a_j^\dagger a_j$. The energy-constrained quantum capacity can be obtained

$$D(\alpha)D(\beta) = e^{\lambda^2 (\alpha^\dagger \beta - \alpha^\dagger \beta^\dagger)} D(\alpha + \beta) \text{ for all } \alpha, \beta \in \mathbb{C}^m.$$
thanks to the following modified version [14, Theorem 5] of the Lloyd–Shor–Devetak (LSD) theorem [53, 54, 105]:
\[
Q(\Phi, N) = \sup_k \frac{1}{k} Q_1(\Phi^\otimes k, kN),
\]
\[
Q_1(\Phi, N) := \sup_{\mathcal{F}([\Psi_B H_B] \leq N)} I_{\text{coh}}(A|B)(I_A \otimes \Phi_B)(\Psi_{AB}),
\]
where \(\Psi_{AB} := |\Psi\rangle\langle\Psi|_{AB}\) is pure, and \(I_{\text{coh}}(A|B)_\rho := S(\rho_B) - S(\rho_{AB})\) is the coherent information. The unconstrained quantum capacity is obtained as \(Q(\Phi) := \lim_{N \to \infty} Q(\Phi, N)\). In general, the expression in (5) is intractable. However, for the pure loss channel the regularisation is not needed, and the quantum capacity can be expressed in closed form as [10, 12, 13, 19].

**Results.**— Before expounding our findings, let us forge our intuition by looking at other channels that present some analogies with general attenuators. An obvious starting point is the thermal attenuator \(E_\lambda\). It is known that \(E_\lambda\) is anti-degradable – and hence its quantum capacity vanishes – when \(\lambda \leq 1/2\) [11, p. 3]. In fact, all \(k\)-extendibility regions have been precisely described [57]. On a different note, we can also consider a generalised depolarising channel in finite dimension \(d\), acting as \(\rho \mapsto \Delta_{\lambda,\sigma}(\rho) := \lambda \rho + (1-\lambda)\sigma\). As it turns out, its quantum capacity is again zero for \(\lambda \leq 1/2\). In fact, observe that \(\Delta_{\lambda,\sigma}\) can be obtained by processing the output of an erasure channel [58]. Since the quantum capacity of this latter object is known [59], by data processing we obtain that \(Q(\Delta_{\lambda,\sigma}) \leq \max \{(1-2\lambda) \log d, 0\}\) for all \(\sigma\). In particular, \(Q(\Delta_{\lambda,\sigma}) = 0\) for \(\lambda \leq 1/2\).

Our results show that the phenomenology of general attenuators is way richer than these considerations may have suggested. We start by looking at the role of the special point \(\lambda = 1/2\).

**Theorem 1.** Let \(\sigma\) be an \(m\)-mode states of the form \(\sigma = \int d\mu(\alpha) D(\alpha)\sigma(\alpha)D(\alpha)^\dagger\), where \(\alpha \in \mathbb{C}^m\), \(\mu\) is a probability measure on \(\mathbb{C}^m\), and the states \(\sigma(\alpha) = V\sigma(\alpha)V^\dagger\) are symmetric under the phase space inversion operation \(V := (-1)^{H_{s,\sigma}}\), with \(H_{s,\sigma}\) being the total photon number. Then the channel \(\Phi_{1/2,\sigma}\) is anti-degradable [52], and in particular \(Q(\Phi_{1/2,\sigma}) = 0\).

**Proof.** Under our assumptions it holds that \(\Phi_{1/2,\sigma} = \int d\mu(\alpha)\Phi_{1/2, D(\alpha)\sigma(\alpha)D(\alpha)^\dagger}\). Now, since the set of anti-degradable channels is convex [60, Appendix A.2], we can directly assume that \(\mu\) is a Dirac measure, i.e. \(\sigma = D(\alpha)\sigma(\alpha)D(\alpha)^\dagger\) with \(\sigma(\alpha)\) symmetric under phase space inversion. Acting on \(\rho \otimes \omega\) with the beam splitter unitary \(U_\lambda\) yields a global state with characteristic function
\[
\chi_\rho \left(\sqrt{\lambda} \alpha - \sqrt{1-\lambda} \beta\right) \chi_\omega \left(\sqrt{1-\lambda} \alpha + \sqrt{\lambda} \beta\right).
\]

While the reduced state on the first system is given by [41], that on the second system has characteristic function \(\chi_\rho \left(\sqrt{\lambda} \alpha - \sqrt{1-\lambda} \beta\right) \chi_\omega \left(\sqrt{1-\lambda} \alpha + \sqrt{\lambda} \beta\right)\), which coincides with that of \(V\Phi_{1/2,\sigma} V^\dagger\). Therefore, the weak complementary channel associated to \(\Phi_{\lambda,\sigma}\) via the representation [6] can be expressed as
\[
\Phi^\text{wc}_{\lambda,\sigma} = V \circ \Phi_{1/2,\sigma} V^\dagger,
\]
where \(V(\cdot) := V(\cdot)V^\dagger\).

Using the identity \(VD(\alpha)V^\dagger = D(-\alpha)\), we see that when \(\sigma = D(\alpha)\sigma(\alpha)D(\alpha)^\dagger\) we also have that \(V\Phi(\sigma) = D_{-2\alpha}(\sigma)\), where \(D_{\alpha}(\cdot) := D(z)(\cdot)D(z)^\dagger\). Noting that \(\Phi_{1-\lambda,\sigma} D_{\alpha}(\sigma) = D_{\sqrt{\lambda}_\alpha} \otimes \Phi_{1-\lambda,\sigma}\), we finally obtain that
\[
\Phi^\text{wc}_{\lambda,\sigma} = V \circ D_{-2\sqrt{\lambda}_\alpha} \circ \Phi_{1-\lambda,\sigma}.
\]

Thus, if \(\lambda = 1/2\) the channel is equivalent to its weak complementary up to a unitary post-processing.

The class of states \(\sigma\) to which Theorem 1 applies is invariant under symplectic unitaries and displacement operators, and it includes many states that are relevant for applications, for instance all conv combination of Gaussian states (e.g. classical states 61, 62) and all Fock-diagonal states. Remarkably, the above result no longer holds if we weaken the assumption on \(\sigma\). To see this, for \(0 \leq \eta \leq 1\) consider the family of single-mode states \(\xi(\eta) = (ξ(\eta))|ξ(\eta)\rangle\), with \(|ξ(\eta)\rangle = \sqrt{\eta}|0\rangle - \sqrt{1-\eta}|1\rangle\). A lower bound on the energy-constrained quantum capacity of the channels \(\Phi_{1/2,\xi(\eta)}\) can be obtained by setting \(\langle\Psi(\xi(\eta))\rangle_{AB} := \sqrt{\eta(1-\eta)}|00\rangle + (1-\eta)|01\rangle + \sqrt{\eta}|10\rangle\) and by considering that [63]
\[
Q(\Phi_{1/2,\xi(\eta)}, (1-\eta)^2) \geq I_{\text{coh}}(A|B)_{\zeta(\eta)}\),
\]
where \(\zeta_{AB}(1/2,\eta) := |1\otimes \Phi_{B}^{1/2,\xi(\eta)}(|\Psi_{AB}(\eta)\rangle\rangle\), and \(\Psi(\eta) := |\Psi(\eta)\rangle\langle\Psi(\eta)|\). The function on the r.h.s. of (8) is strictly positive for all \(0 < \eta < 1\) [63].

The above example shows that quantum communication can be possible on a general attenuator even for transmissivity \(\lambda = 1/2\). At this point, we may wonder whether at least for a fixed energy constraint at the input there exists a threshold value for \(\lambda\) below which quantum communication becomes impossible. Our main result states that this is not the case; on the contrary, the quantum capacity can be bounded away from 0 even when \(\lambda\) approaches 0, if the environment state \(\sigma\) is chosen appropriately. Note that the bounds by Lim et al. [28] cannot possibly be used to draw such a conclusion [58].

**Theorem 2.** For all \(0 < \lambda \leq 1\) there exists a single-mode state \(\sigma(\lambda)\) such that
\[
Q(\Phi_{\lambda,\sigma(\lambda)}) \geq Q(\Phi_{\lambda,\sigma(\lambda)}, 1/2) \geq c
\]
for some universal constant \(c > 0\). Both \(\sigma(\lambda)\) and \(c\) are explicitly given in the proof.
Sketch of the proof. When $1/2 < \lambda \leq 1$ the pure loss channel provides an example of an attenuator with positive quantum capacity [26]. Around $\lambda = 1/2$, we can draw the same conclusion by perturbing the lower bound in [8] thanks to the Alicki–Fannes–Winter inequality [64, 65]. It remains to establish the result for $0 < \lambda \leq 1/2 - \epsilon$, where $\epsilon > 0$ is fixed. We start by making an ansatz for a state $|\Psi\rangle_{AB}$ to be plugged into [6]. Let us set $|\Psi\rangle_{AB} := \frac{1}{\sqrt{2}} ((|01\rangle + |10\rangle)$ and $q(n) := |n\rangle|n\rangle$. The output state $\omega_{AB}(n, \lambda) := (I^A \otimes \Phi^B_{\lambda, q(n)})(|\Psi\rangle_{AB})$ can be computed e.g. thanks to the formulae derived by Saba-pathy and Winter [26, Section III.B]. One obtains that

$$Q(\Phi_{\lambda, q(n)}, 1/2) \geq J(n, \lambda) := I_{\text{coh}}(A|B)_{\omega_{AB}(n, \lambda)} - H(p(n, \lambda)) - H(q(n, \lambda)),$$

where the two probability distributions $p(n, \lambda)$ and $q(n, \lambda)$ over the alphabet $\{0, \ldots, n+1\}$ are defined by

$$p_\ell(n, \lambda) := \frac{1}{2(n+1)(1-\lambda)} \binom{n+1}{\ell} (1-\lambda)^\ell \lambda^{n-\ell} \times \left( (1-\lambda)(n-\ell+1) + ((n+1)(1-\lambda) - \ell)^2 \right),$$

$$q_\ell(n, \lambda) := \frac{1}{2(n+1)(1-\lambda)} \binom{n+1}{\ell} (1-\lambda)^\ell \lambda^{n-\ell} \times \left( \lambda + ((n+1)(1-\lambda) - \ell)^2 \right),$$

and $s^\dagger$ are obtained by sorting $r$ and $s$ in ascending order [66]. This definition captures the intuitive notion of $r$ being ‘more disordered’ than $s$. An immediate consequence is that the entropy of $r$ is never smaller than that of $s$. But more is true: a beautiful inequality recently established by Ho and Verdú [52, Theorem 3] allows us to lower bound the entropy difference as

$$H(s) - H(r) \geq D(\rho^\dagger || \rho^r),$$

where $D(\rho || \sigma) := \sum_x \rho(x) \log \frac{\rho(x)}{\sigma(x)}$ is the Kullback–Leibler divergence. This latter quantity can be in turn lower bounded as $D(\rho || \sigma) \geq \frac{1}{2 \log 2} \| \rho - \sigma \|_1^2$ in term of the total variation distance $\| u - v \|_1 := \sum_v |u_v - v_v|$ thanks to Pinsker’s inequality [67]. We find that

$$J(n, \lambda) = H(p(n, \lambda)) - H(q(n, \lambda)) \geq D(\rho^s || \rho^r) \geq \frac{1}{2 \log 2} \| \rho^s - \rho^r \|_1^2 \geq \frac{2}{\log 2} \left| q_{n+1}^\dagger(n, \lambda) - p_{n+1}^\dagger(n, \lambda) \right|^2 = \frac{2}{\log 2} \left| p_{n+1}(n, \lambda) - q_{n+1}(n, \lambda) \right|^2,$$

where in the last line we used the fact, proven in the SM [58], that $p_{n+1}(n, \lambda) = \max_{\epsilon \geq 0} p_\ell(n, \lambda)$ and $q_{n+1}(n, \lambda) = \max_{\epsilon \geq 0} q_\ell(n, \lambda)$ as long as $n \geq 2$ and all $\frac{1}{n+1} \leq \lambda \leq \frac{1}{\frac{1}{\lambda}}$. It remains to lower bound $k(n, \lambda) := |p_{n+1}(n, \lambda) - q_{n+1}(n, \lambda)|$, which can be done by inspection. We find that: (a) $k(2, \lambda) \geq \epsilon/4$ for all $1/3 \leq \lambda \leq 1/2 - \epsilon$ and (b) $k(n, \lambda) \geq c$ for some (explicitly given) universal constant $c > 0$ for all $n \geq 3$ and $\frac{1}{n+1} \leq \lambda \leq \frac{1}{c}$. This concludes the proof.

Note that $Q(\Phi_{1/2, |n\rangle|n\rangle}) \equiv 0$ for all $n$ by Theorem 1 while we have just shown that $Q(\Phi_{\lambda, |n\rangle|n\rangle}) > 0$ when $\frac{1}{n+1} \leq \lambda \leq \frac{1}{c}$. This illustrates the rather surprising fact that $Q(\Phi_{\lambda, \sigma})$ can happen not to be monotonic in $\lambda$ for a fixed $\sigma$. In the SM [58] we prove that monotonicity still holds under certain circumstances, e.g. when $\sigma = \sigma_\lambda$ is Gaussian. Combining this with Theorem 1 also shows that $Q(\Phi_{\lambda, \sigma}) \equiv 0$ for all $\lambda \leq 1/2$.

On a different note, we can ask for the optimal value of the constant $c$ in [6]. Our argument yields $c \geq 6 \times 10^{-6}$, while numerical investigations suggest that $c \approx 0.066$. If only sufficiently small values of $\lambda$ are taken into account, we can prove that $c \geq 0.0244$. To put this into perspective, elementary considerations show that $c \leq 1.377$ [58].

**Conclusions.**—We have studied the transmission of quantum information on general attenuator channels, which are among the simplest examples of non-Gaussian channels and may be relevant for applications. We have shown that their quantum capacity vanishes for transmissivity 1/2 and for a wide class of environment states.
At the same time, we have uncovered an unexpected phenomenon: namely, for any non-zero value of the transmissivity there exists an environment state that makes the quantum capacity of the corresponding general attenuator larger than a universal constant. This also implies that said quantum capacity is not necessarily monotonically increasing in the transmissivity for a fixed environment state.

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An erasure channel acts as $\rho \mapsto \lambda \rho + (1 - \lambda) |e\rangle \langle e|$, where $|e\rangle$ is an error flag that is orthogonal to every input state. Constructing the post-processing channel $\rho \mapsto \mathcal{M}_\sigma (\rho) := (1 - |e\rangle \langle e|) \rho (1 - |e\rangle \langle e|) + |e\rangle \langle e| \sigma$, we see that $\Delta_{\lambda, \sigma} = \mathcal{M}_\sigma \circ \mathcal{N}_\lambda$.
Supplemental Material

I. GENERALITIES

A. Quantum entropy

The (von Neumann) entropy of a quantum state $\rho$ can be defined as

$$S(\rho) := -\text{tr} [\rho \log \rho], \quad (S1)$$

which is well defined although possibly infinite. Indeed, one way to understand it is via the infinite sum

$$S(\rho) = \sum_i (-p_i \log p_i),$$

where $\rho = \sum_i p_i |e_i\rangle\langle e_i|$ is the spectral decomposition of $\rho$. Since all terms in the above sum are non-negative, the sum itself is well defined but possibly infinite.

Consider an $m$-mode system with Hilbert space $H_m = L^2(\mathbb{R}^m) \simeq H_1^\otimes m$. The total photon number is a densely defined operator on $H_m$ that takes the form

$$H_m := \sum_{j=1}^m a_j^\dagger a_j \quad (S2)$$

when written in terms of the creation and annihilation operators. It is well known to have a discrete spectrum of the form $\{\sum_{j=1}^m n_j : n_j \in \mathbb{N}\}$, with the eigenvector corresponding to $\sum_{j=1}^m n_j$ being given by the tensor product of Fock states $|n_1\rangle \ldots |n_m\rangle$. The single-mode thermal state with mean photon number $\nu \geq 0$ is given by

$$\tau_\nu := \frac{1}{\nu + 1} \sum_{n=0}^{\infty} \left(\frac{\nu}{\nu + 1}\right)^n |n\rangle\langle n| \quad (S3)$$

The thermal state over $m$ modes with total mean photon number $\nu$ can be easily obtained as the $m$-fold tensor product $\tau_\nu^\otimes m$. Thermal states are important because they are the maximisers of the entropy among all states with a fixed mean photon number. In formula

$$\max \{S(\rho) : \text{Tr} [\rho H_m] \leq \nu\} = S\left(\tau_\nu^\otimes m\right) = mg\left(\frac{\nu}{m}\right) \quad (S4)$$

for all $\nu \geq 0$, where the function $g$ defined by

$$g(x) := (x + 1) \log(x + 1) - x \log x, \quad (S5)$$

sometimes called the bosonic entropy, expresses the entropy of a single-mode thermal state in terms of its mean photon number. The function $g$ has many notable properties: (a) it is monotonically increasing; (b) it is subadditive, meaning that

$$g(x + y) \leq g(x) + g(y) \quad \forall \ x, y \geq 0; \quad (S6)$$

(c) it is concave; and (d) it has the asymptotic behaviour

$$g(x) = \log(ex) + o(1) \quad (x \to \infty). \quad (S7)$$

B. Beam splitters

A beam splitter is perhaps the simplest example of a passive unitary acting on an $(m + m)$-mode bipartite CV system. As reported in the main text [2], it is defined by $U_\lambda := e^{\arccos \sqrt{\lambda} \sum_j (a_j^\dagger b_j - a_j b_j^\dagger)}$, where $a_j, b_j$ are the annihilation operators on the $j$-th modes belonging to the first and second system, respectively. This exponential can be
decomposed thanks to a well-known trick. Consider the annihilation operators \( a_1, \ldots, a_m \) of \( m \) independent modes. The Jordan map \([S8]\)

\[ J : X \mapsto \sum_{j,k=1}^{m} X_{jk} a_j^\dagger a_k, \]  

(S8)

is a Lie algebra isomorphism between the set of \( m \times m \) matrices and that of the operators on the Hilbert space \( \mathcal{H}_m \) of \( m \) modes that are bilinear in the \( a_j^\dagger \) and \( a_k \). Let us note in passing that the Jordan map \([S8]\) can be extended so as to include all operators that can be expressed as polynomials of degree up to 2 in the creation and annihilation matrices. We obtain the explicit correspondence

\( a^\dagger b \leftrightarrow (0 1 0), \)  

(S10)

\( ab^\dagger \leftrightarrow (0 0 1), \)  

(S11)

\[ \frac{1}{2} (a^\dagger a - b^\dagger b) \leftrightarrow \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \]  

(S12)

\[ \frac{1}{2} (a^\dagger b + ab^\dagger) \leftrightarrow \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \]  

(S13)

\[ \frac{1}{2 i} (a^\dagger b - ab^\dagger) \leftrightarrow \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}. \]  

(S14)

By exploiting these formulae, performing the computations for \( 2 \times 2 \) matrices, and bringing back the result with the Jordan map, it is possible to prove that \([S11, Appendix 5]\)

\[ U_\lambda = e^{\sqrt{\frac{\lambda}{2}} a^\dagger b} e^{-\frac{1}{2} \log \lambda (a^\dagger a - b^\dagger b)} e^{-\sqrt{\frac{\lambda}{2}} ab^\dagger}. \]  

(S15)

This decomposition can be employed to derive an expression for the output state obtained by mixing in a beam splitter of arbitrary transmissivity the vacuum \( |0\rangle \) or the first Fock state \( |1\rangle \) with another Fock states \( |n\rangle \). Namely,

\[ U_\lambda |0\rangle |n\rangle = \sum_{\ell=0}^{n} \sqrt{\binom{n}{\ell} (1 - \lambda)^{\frac{\ell}{2}} \lambda^{\frac{n-\ell}{2}} |\ell\rangle |n-\ell\rangle,} \]  

(S16)

\[ U_\lambda |1\rangle |n\rangle = -\frac{1}{\sqrt{(n+1)(1-\lambda)}} \sum_{\ell=0}^{n+1} \sqrt{(n+1)_{\ell}} (1 - \lambda)^{\frac{\ell}{2}} \lambda^{\frac{n+1-\ell}{2}} ((n+1)(1-\lambda) - \ell) |\ell\rangle |n+1-\ell\rangle. \]  

(S17)

To prove (S16), write

\[ U_\lambda |0\rangle |n\rangle = e^{\sqrt{\frac{\lambda}{2}} a^\dagger b} e^{-\frac{1}{2} \log \lambda (a^\dagger a - b^\dagger b)} e^{-\sqrt{\frac{\lambda}{2}} ab^\dagger} |0\rangle |n\rangle \]

\[ = e^{\sqrt{\frac{\lambda}{2}} a^\dagger b} e^{-\frac{1}{2} \log \lambda (a^\dagger a - b^\dagger b)} |0\rangle |n\rangle \]

\[ = e^{\sqrt{\frac{\lambda}{2}} a^\dagger b} \lambda^{\frac{n}{2}} |0\rangle |n\rangle \]

\[ = \lambda^{\frac{n}{2}} \sum_{\ell=0}^{n} \frac{1}{\ell!} \left( \frac{1 - \lambda}{\lambda} \right)^{\frac{\ell}{2}} \left( \sqrt{\ell!} |\ell\rangle \right) \left( \sqrt{\frac{n!}{(n-\ell)!}} |n-\ell\rangle \right) \]

\[ = \sum_{\ell=0}^{n} \sqrt{\binom{n}{\ell} (1 - \lambda)^{\frac{\ell}{2}} \lambda^{\frac{n-\ell}{2}} |\ell\rangle |n-\ell\rangle}. \]
In the same spirit, one can compute

\[
U_\lambda |1\rangle |n\rangle = e^{\sqrt{\frac{2\pi}{\lambda}} a^\dagger b} e^{-\frac{1}{2} \log \lambda (a^\dagger a - b^\dagger b)} e^{-\sqrt{\frac{2\pi}{\lambda}} a b^\dagger} |1\rangle |n\rangle \\
= e^{\sqrt{\frac{2\pi}{\lambda}} a^\dagger b} e^{-\frac{1}{2} \log \lambda (a^\dagger a - b^\dagger b)} \left( |1\rangle |n\rangle - \sqrt{\frac{1-\lambda}{\lambda}} \sqrt{n+1} |0\rangle |n+1\rangle \right) \\
= e^{\sqrt{\frac{2\pi}{\lambda}} a^\dagger b} \left( \frac{n^{-1}}{\lambda} |1\rangle |n\rangle - \sqrt{\frac{1-\lambda}{\lambda}} \sqrt{n+1} \frac{n^{-1}}{\lambda} |0\rangle |n+1\rangle \right) \\
= \lambda^{-\frac{n-1}{\lambda}} e^{\sqrt{\frac{2\pi}{\lambda}} a^\dagger b} |1\rangle |n\rangle - \lambda^\frac{2}{\lambda} \sqrt{(1-\lambda)(n+1)} e^{\sqrt{\frac{2\pi}{\lambda}} a^\dagger b} |0\rangle |n+1\rangle \\
= \lambda^{-\frac{n-1}{\lambda}} \sum_{\ell=0}^n \frac{1}{\ell!} \left( \frac{1-\lambda}{\lambda} \right)^{\frac{\ell}{2}} \sqrt{(\ell+1)! |\ell+1\rangle} \sqrt{n! (n-\ell)!} |n-\ell\rangle \\
- \lambda^\frac{2}{\lambda} \sqrt{(1-\lambda)(n+1)} \sum_{\ell=0}^{n+1} \frac{1}{\ell!} \left( \frac{1-\lambda}{\lambda} \right)^{\frac{\ell}{2}} \sqrt{\ell! |\ell\rangle} \sqrt{(n+1)! (n+1-\ell)!} |n+1-\ell\rangle \\
= \frac{1}{\sqrt{n+1}} \sum_{\ell=1}^{n+1} \sqrt{n+1 \ell} (1-\lambda)^{\frac{\ell}{2}} \lambda^{-\frac{n-\ell}{\lambda}} |\ell\rangle |n+1-\ell\rangle \\
- \sqrt{(n+1)(1-\lambda)} \sum_{\ell=0}^n \sqrt{n+1 \ell} (1-\lambda)^{\frac{\ell}{2}} \lambda^{-\frac{n-\ell}{\lambda}} |\ell\rangle |n+1-\ell\rangle \\
= \frac{1}{\sqrt{(n+1)(1-\lambda)}} \sum_{\ell=0}^{n+1} \sqrt{n+1 \ell} (1-\lambda)^{\frac{\ell}{2}} \lambda^{-\frac{n-\ell}{\lambda}} (n+1)(1-\lambda) |\ell\rangle |n+1-\ell\rangle,
\]

which proves (S17). If instead of (S15) one employs the alternative decomposition

\[
U_\lambda = e^{-\sqrt{\frac{2\pi}{\lambda}} a b^\dagger} e^{\frac{1}{2} \log \lambda (a^\dagger a - b^\dagger b)} e^{\sqrt{\frac{2\pi}{\lambda}} a^\dagger b},
\]

one finds that

\[
U_\lambda |n\rangle |0\rangle = \sum_{\ell=0}^n (-1)^{\ell} \sqrt{n \ell} (1-\lambda)^{\frac{\ell}{2}} \lambda^{-\frac{n-\ell}{\lambda}} |\ell\rangle |n-\ell\rangle,
\]

\[
U_\lambda |n\rangle |1\rangle = - \frac{1}{\sqrt{(n+1)(1-\lambda)}} \sum_{\ell=0}^{n+1} (-1)^{\ell} \sqrt{n+1 \ell} (1-\lambda)^{\frac{\ell}{2}} \lambda^{-\frac{n-\ell}{\lambda}} ((n+1)(1-\lambda) - \ell) |\ell\rangle |n+1-\ell\rangle.
\]

Note that (S19) and (S20) can also be derived from (S16) and (S17) by applying the swap operator to both sides of the equations.

Finally, for future convenience we report the expressions of the matrices that represent \(U_\lambda\) on subspaces with low total photon number. By applying (S15) or (S18) one can verify that

\[
U_\lambda \big|_{\text{span}\{ |0\rangle |1\rangle, |1\rangle |0\rangle \}} = \begin{pmatrix} \sqrt{\lambda} & -\sqrt{1-\lambda} \\ 1-\lambda & \sqrt{\lambda} \end{pmatrix},
\]

\[
U_\lambda \big|_{\text{span}\{ |0\rangle |2\rangle, |1\rangle |1\rangle, |2\rangle |0\rangle \}} = \begin{pmatrix} \lambda & -\sqrt{2\lambda(1-\lambda)} & 1-\lambda \\ \sqrt{2\lambda(1-\lambda)} & 2\lambda - 1 & -\sqrt{2\lambda(1-\lambda)} \\ 1-\lambda & \sqrt{2\lambda(1-\lambda)} & \lambda \end{pmatrix}.
\]

C. General attenuators

The family of channels that we consider here is that of general attenuators \[24][26][28\], sometimes called additive noise channels \[24\]. They are parametrised by a generic \(m\)-mode quantum state \(\sigma\) and by a value of the associated
transmissivity $0 \leq \lambda \leq 1$. As reported in the main text \textsuperscript{3}–\textsuperscript{4}, the action of a general attenuator $\Phi_{\lambda,\sigma}$ on a system $B$ is defined by $\Phi_{\lambda,\sigma}^B (\rho_B) := \operatorname{Tr}_E \left[ U_{\lambda}^{BE} (\rho_B \otimes \sigma_E) \left( U_{\lambda}^{BE} \right)^\dagger \right]$, which – dropping the system labels for simplicity – translates to $\chi_{\Phi_{\lambda,\sigma}} (\rho) = \chi_\rho \left( \sqrt{\lambda} \alpha \right) \chi_\sigma \left( \sqrt{1 - \lambda} \alpha \right)$ at the level of characteristic functions. This particularly simple expression can be used in conjunction with the composition rule for displacement operators to prove the covariance formulae
\begin{align}
\Phi_{\lambda,\sigma} \circ D_z = D_{\sqrt{\lambda} z} \circ \Phi_{\lambda,\sigma},
\Phi_{\lambda,\sigma} \circ D_{\lambda z} = D_{\sqrt{\lambda} z} \circ \Phi_{\lambda,\sigma},
\end{align}
where the displacement channel is defined as
\begin{align}
D_z (\rho) := D(z) \rho D(z)^\dagger.
\end{align}

Note that the identity \textsuperscript{24} has been used in the proof of Theorem \textsuperscript{1}.

The canonical example of a general attenuator channel $\Phi_{\lambda,\sigma}$ – say, in the single-mode case – is obtained by setting $\sigma = \tau_\nu$, where the thermal state with mean photon number $\nu$ is defined by \textsuperscript{3}. The resulting map $\mathcal{E}_{\lambda,\nu} := \Phi_{\lambda,\tau_\nu}$ is usually referred to as a thermal attenuator. An even simpler yet extremely important channel, called the quantum-limited attenuator (or the pure loss channel) and usually denoted with $\mathcal{E}_\lambda := \mathcal{E}_{\lambda,0} = \Phi_{\lambda,|0\rangle\langle 0|}$, is obtained by setting the temperature of the environment equal to zero.

The energy-constrained quantum capacity of the pure loss channel has been determined exactly. It reads
\begin{align}
Q (\mathcal{E}_\lambda, N) = \max \{ g(\lambda N) - g ((1 - \lambda) N), \}
\end{align}
The decisive step towards establishing \textsuperscript{26} has been done by Wolf et al. \textsuperscript{12} Eq. (12)], who proved that for this particular channel the regularisation in \textsuperscript{5} is not needed. This implies that the quantum capacity is simply given by the coherent information \textsuperscript{5}, which had been previously computed by Holevo and Werner \textsuperscript{10} Eq. (5.9)]. A more complete discussion of these latter calculations, and in particular of why it suffices to consider thermal states at the input, can be found in Holevo’s monograph \textsuperscript{72} Propositions 12.38 and 12.47] (see also the more recent version \textsuperscript{7} Propositions 12.40 and 12.62]). The problem of completeness of the original argument was recently raised by Wilde and Qi \textsuperscript{14} Remark 4], and further elaborated on by Noh et al. \textsuperscript{15} Theorem 9]. An alternative derivation of the formula \textsuperscript{26} has been put forward by Wilde et al. \textsuperscript{13]}. 

We do not yet have an exact expression for the energy-constrained capacity of all thermal attenuators. However, many upper \textsuperscript{15} as well as lower \textsuperscript{10} bounds have been discovered so far. We do not report the corresponding formulae here, as we do not need them. What we will need, instead, is a much simpler observation due to Caruso and Giovannetti \textsuperscript{11].

Lemma S1 \textsuperscript{11} p. 3]. For all $0 \leq \lambda \leq 1/2$ and all $\nu \geq 0$, the thermal attenuator $\mathcal{E}_{\lambda,\nu}$ is anti-degradable, and thus $Q (\mathcal{E}_{\lambda,\nu}) = 0$.

The above result can be further generalised thanks to the concept of channel $k$-extendibility. Here, anti-degradable channels are precisely those that are 2-extendible. The complete characterisation of the $k$-extendibility regions of all thermal attenuators has been put forward recently \textsuperscript{57].

We now turn to the problem of estimating the quantum capacity of general attenuators. We start by recalling the following elementary fact, that is part of the folklore.

Lemma S2. Let $\Phi$ be a quantum channel acting on a system of $m$ modes. For all $N \geq 0$, its energy-constrained quantum capacity satisfies
\begin{align}
Q (\Phi, N) \leq mg \left( \frac{N}{m} \right),
\end{align}
where the bosonic entropy is given by \textsuperscript{5}.

Proof. For any bipartite quantum system $AB$ we have that $H(AB) \geq |H(A) - H(B)|$. From this we deduce that the coherent information in \textsuperscript{5} satisfies
\begin{align}
I_{\text{coh}}(A)I_{(I_A \otimes \Phi_B)(\Psi_{AB})} = (H(B) - H(AB))_{(I_A \otimes \Phi_B)(\Psi_{AB})} \leq H(A)_{\Psi_A} = S(\Psi_A) = S(\Psi_B) \leq mg \left( \frac{N}{m} \right),
\end{align}
where we used (i) the fact that the initial state $\Psi_{AB} = |\Psi\rangle \langle \Psi|_{AB}$ is pure; and (ii) the fact that the thermal state maximises the entropy for a given mean photon number, as stated in \textsuperscript{4}. Since the above upper bound is additive, applying the LSD theorem \textsuperscript{7] yields the claim.\hfill \square


Exploiting known extremality properties of Gaussian states [14, 45], the recent solution of the minimum output entropy conjecture [43] (see also [11, 42]), and the even more recently established conditional entropy power inequality [24, 25], Lim et al. [28] were able to prove the following more sophisticated bounds.

Lemma S3 [28] Sections III and IV. Let \( \sigma \) be a single-mode state with mean photon number \( \nu_\sigma \) and entropy \( S(\sigma) \). Then, for all \( 0 \leq \lambda \leq 1 \) and \( N \geq 0 \) the energy-constrained quantum capacity of the corresponding general attenuator satisfies

\[
g \left( (1-\lambda)g^{-1}(S(\sigma)) + \lambda N \right) - S(\sigma) - g(\lambda \nu_\sigma + (1-\lambda)N) \leq Q(\Phi_{\lambda,\sigma}, N) \leq g(\lambda N + (1-\lambda)\nu_\sigma) - \ln \left( \lambda + (1-\lambda)e^{S(\sigma)} \right),
\]

where \( g^{-1} \) is the inverse function of the bosonic entropy defined by (S5).

Remark S4. Note that the lower bound in (S27) always vanishes when \( \lambda \leq 1/2 \). Indeed, using the subadditivity (S6) and monotonicity of the bosonic entropy yields

\[
g \left( (1-\lambda)g^{-1}(S(\sigma)) + \lambda N \right) - S(\sigma) - g(\lambda \nu_\sigma + (1-\lambda)N)
\leq g \left( g^{-1}(S(\sigma)) \right) + g(\lambda N) - S(\sigma) - g((1-\lambda)N)
\leq g(\lambda N) - g((1-\lambda)N)
\leq 0,
\]

where the last inequality holds provided that \( \lambda \leq 1/2 \), again using the monotonicity of \( g \). It follows that the recent results by Lim et al. [28] cannot be possibly used to detect a positive quantum capacity below the threshold value \( \lambda = 1/2 \).

Remark S5. The upper bound in (S27) diverges for every fixed \( N \) and \( \lambda > 0 \) when \( \nu_\sigma \to \infty \). However, we have already seen in Lemma S2 that the maximum capacity \( Q(\Phi_{\lambda,\sigma}, N) \) stays finite in the same limit.

II. CONVEX COMBINATIONS OF GAUSSIAN STATES

Throughout this section, we look at general attenuators whose environment state is a convex combination of Gaussian states. Note that this family of states encompasses the so-called classical states [01, 02], which by definition can be written as convex combinations of coherent states, i.e.

\[
\sigma = \int d\mu(\alpha) |\alpha\rangle\langle\alpha|,
\]

where \( \mu \) is a probability measure on \( \mathbb{C}^m \).

We start by showing how to apply the data processing bound to constrain the quantum capacity of general attenuators.

Lemma S6. Let \( 0 \leq \lambda, \mu \leq 1 \), and let \( \sigma, \omega \) be \( m \)-mode states. Then we have the composition rule

\[
\Phi_{\lambda,\sigma} \circ \Phi_{\mu,\omega} = \Phi_{\lambda\mu, \tau},
\]

where

\[
\tau := \Phi_{\lambda(1-\mu)}(\sigma) \Phi_{\mu(1-\lambda)}(\omega).
\]

Proof. The easiest way to verify (S29) is by looking at the transformation rules for characteristic functions. For an arbitrary input state \( \rho \), using [4] multiple times we obtain that

\[
\chi(\Phi_{\lambda,\sigma} \circ \Phi_{\mu,\omega})(\rho)(\alpha) = \chi_{\Phi_{\mu,\omega}(\rho)} \left( \sqrt{\lambda} \alpha \right) \chi_\sigma \left( \sqrt{1-\lambda} \alpha \right)
\]
\[
= \chi_{\rho} \left( \sqrt{\lambda \mu} \alpha \right) \chi_\omega \left( \sqrt{1-\lambda \mu} \alpha \right)
\]
\[
= \chi_{\rho} \left( \sqrt{\lambda \mu \alpha} \right) \chi_\eta \left( \sqrt{1-\lambda \mu} \alpha \right)
\]
\[
= \chi_{\Phi_{\lambda,\eta}(\rho)}(\alpha).
\]

Since quantum states are in one-to-one correspondence with characteristic functions, this implies that \( \Phi_{\lambda,\eta}(\rho) = (\Phi_{\lambda,\sigma} \circ \Phi_{\mu,\omega})(\rho) \). Given that \( \rho \) was arbitrary, the proof is complete. \( \square \)
A first immediate corollary is as follows.

**Corollary S7.** Let $0 \leq \lambda, \mu \leq 1$, and let $\sigma, \omega$ be $m$-mode states. Define $\tau$ as in (S30). Then: (a) if $\Phi_{\mu,\omega}$ is anti-degradable, then so is $\Phi_{\lambda\mu,\tau}$; (b) it holds that

$$Q(\Phi_{\lambda\mu,\tau}) \leq \min \{Q(\Phi_{\lambda,\sigma}), Q(\Phi_{\mu,\omega})\}. \tag{S31}$$

**Proof.** Claim (a) is a consequence of the fact that set of anti-degradable channels is invariant by post-processing. Claim (b), instead, follows from the observation that in the definition of quantum capacity any pre- or post-processing can be included into the encoding or decoding transformations.

We now show that the phenomenon illustrated in Theorem 2 does not occur for general attenuators whose environment state is a convex combination of Gaussian states.

**Corollary S8.** Let $\sigma$ be a state in the convex hull of all Gaussian states. Then $\Phi_{\lambda,\sigma}$ is anti-degradable for all $0 \leq \lambda \leq 1/2$, and in particular

$$Q(\Phi_{\lambda,\sigma}) \equiv 0 \quad \forall \ 0 \leq \lambda \leq \frac{1}{2}. \tag{S32}$$

**Proof.** For $\sigma$ satisfying the hypothesis, we have that $\Phi_{\lambda,\sigma}$ is a convex combination of channels of the form $\Phi_{\lambda,\sigma_G}$, where $\sigma_G$ is Gaussian. Since the set of anti-degradable channels is convex [60, Appendix A.2], it suffices to prove that $\Phi_{\lambda,\sigma_G}$ is anti-degradable for all $0 \leq \lambda \leq 1/2$.

Moreover, we can assume without loss of generality that $\sigma_G$ is centred, i.e. that $\text{Tr}[\sigma_G a_j] \equiv 0$ for all $j = 1, \ldots, m$, where $a_j$ are the annihilation operators. In fact, $\sigma_G$ can always be displaced by an arbitrary amount by means of a unitary post-processing as in (S24). Note that unitary post-processing does not affect anti-degradability, and that $D(z)a_jD(z)\dagger = a_j - z_j$. Thus, we can make sure that $D_z(\sigma_G)$ is centred by choosing $z$ appropriately.

In light of the above reasoning, from now on we shall assume that $\sigma_G$ is centred. The characteristic function of $\sigma_G$ then is a centred Gaussian, entailing that

$$\chi_{\sigma_G}(\sqrt{\eta} \alpha) \chi_{\sigma_G}(\sqrt{1-\eta} \alpha) \equiv \chi_{\sigma_G}(\alpha) \quad \forall \alpha \in \mathbb{C}^m, \quad \forall \ 0 \leq \eta \leq 1.$$ Using (4), this translates to

$$\Phi_{\eta,\sigma_G}(\alpha) \equiv \sigma_G \quad \forall \ 0 \leq \eta \leq 1. \tag{S33}$$

Leveraging (S29)–(S30), we see that

$$\Phi_{\lambda,\sigma_G} = \Phi_{2\lambda,\sigma_G} \circ \Phi_{1/2,\sigma_G}$$

for all $0 \leq \lambda \leq 1/2$. Note that $\Phi_{1/2,\sigma_G}$ is anti-degradable by Theorem 1. Since anti-degradable channels remain such upon post-processing [60, Lemma 17], we conclude that also $\Phi_{\lambda,\sigma_G}$ is anti-degradable, completing the proof.

Another consequence of Lemma S6 is that the quantum capacity of a general attenuator is monotonically increasing as a function of the transmissivity for a fixed Gaussian environment state. By comparison, remember that in the main text we have instead shown that monotonicity fails to hold when the environment state is a Fock state.

**Corollary S9.** Let $\sigma = \sigma_G$ be an arbitrary $m$-mode Gaussian state. Then the function

$$\lambda \mapsto Q(\Phi_{\lambda,\sigma_G}) \tag{S34}$$

is monotonically increasing for all $0 \leq \lambda \leq 1$, and strictly zero for $0 \leq \lambda \leq 1/2$.

**Proof.** The proof is along the same lines as that of Corollary S8. We can assume without loss of generality that $\sigma_G$ is centred, which in turn implies that (S33) holds. Picking $0 \leq \lambda' \leq \lambda \leq 1$ and setting $\mu := \frac{\lambda}{\lambda'}$ and $\eta := \frac{1-\lambda}{1-\lambda'} = \frac{1-\lambda}{1-\lambda'}$ in (S30)–(S31), we deduce that

$$\Phi_{\lambda',\sigma_G} = \Phi_{\lambda\mu,\sigma_G} = \Phi_{\lambda\mu,\Phi_{\eta,\sigma_G}(\sigma_G)} = \Phi_{\lambda,\sigma_G} \circ \Phi_{\mu,\sigma_G}.$$ Then, applying (S31) we conclude that $Q(\Phi_{\lambda',\sigma_G}) \leq Q(\Phi_{\lambda,\sigma_G})$, completing the proof.
III. POSITIVE CAPACITY AT $\lambda = 1/2$

Given the fact that general attenuator channels of the form $\Phi_{1/2, \sigma}$ always output states with positive Wigner functions [9, Lemma 16], one may be tempted to conjecture that their quantum capacities vanish. Interestingly, this is not the case, as the next example shows.

Example S10. For $0 \leq \eta \leq 1$, set $\xi(\eta) = |\xi(\eta)|/|\xi(\eta)|$, with

$$|\xi(\eta)| := \sqrt{\eta} |0\rangle - \sqrt{1-\eta} |1\rangle .$$

We will see (numerically) that for all $\eta \in (0,1)$ the channel $\Phi_{1/2, \xi(\eta)}$ has nonzero quantum capacity. A lower bound on $Q(\Phi_{1/2, \xi(\eta)})$ is plotted in Figure 2.

To estimate the quantum capacity of $\Phi_{1/2, \xi(\eta)}$ from below, we apply the achievability part of the LSD theorem. This is done by finding a suitable ansatz for the state $|\Psi\rangle_{AB}$ to be plugged into (6). Let us define the family of two-mode states

$$|\Psi(\eta)\rangle_{AB} := \sqrt{\eta}(1-\eta) |0\rangle_A |0\rangle_B + (1-\eta) |0\rangle_A |1\rangle_B + \sqrt{\eta} |1\rangle_A |0\rangle_B .$$

Upon re-ordering the terms, the joint state reads

$$|\Psi(\eta)\rangle_{AB} |\xi(\eta)\rangle_E = \eta \left( |1\rangle_A + \sqrt{1-\eta} |0\rangle_A \right) |0\rangle_B |0\rangle_E - \sqrt{\eta}(1-\eta) \left( \sqrt{1-\eta} |0\rangle_A + |1\rangle_A \right) |0\rangle_B |1\rangle_E$$

$$+ \sqrt{\eta}(1-\eta) |0\rangle_A |1\rangle_B |0\rangle_E - (1-\eta)^{3/2} |0\rangle_A |2\rangle_B |1\rangle_E + \sqrt{\eta}(1-\eta) |0\rangle_A |1\rangle_B |0\rangle_E$$

$$+ \sqrt{\eta}(1-\eta) |0\rangle_A |1\rangle_B |0\rangle_E - \sqrt{\eta}(1-\eta) \left( \sqrt{1-\eta} |0\rangle_A + |1\rangle_A \right) |0\rangle_B |1\rangle_E$$

$$+ \sqrt{\eta}(1-\eta) |0\rangle_A |1\rangle_B |0\rangle_E - \sqrt{\eta}(1-\eta) \left( \sqrt{1-\eta} |0\rangle_A + |1\rangle_A \right) |0\rangle_B |1\rangle_E$$

$$+ (1-\eta)^{3/2} \sqrt{2}\lambda \sqrt{1-\lambda} |0\rangle_A |0\rangle_B |2\rangle_E .$$

Using the explicit representations (S21) and (S22) of the action of the beam splitter unitary on the low photon number subspaces, it is not difficult to see that the tripartite output state, which we denote as

$$|\zeta(\lambda, \eta)\rangle_{ABE} := U^{BE}_{\lambda} |\Psi(\eta)\rangle_{AB} |\xi(\eta)\rangle_E ,$$

reads

$$|\zeta(\lambda, \eta)\rangle_{ABE} = \eta \left( |1\rangle_A + \sqrt{1-\eta} |0\rangle_A \right) |0\rangle_B |0\rangle_E - \sqrt{\eta}(1-\eta) \left( \sqrt{1-\eta} |0\rangle_A + |1\rangle_A \right) |0\rangle_B |1\rangle_E$$

$$+ \sqrt{\eta}(1-\eta) |0\rangle_A |1\rangle_B |0\rangle_E - (1-\eta)^{3/2} \sqrt{1-\lambda} |0\rangle_A |2\rangle_B |1\rangle_E$$

From now on, we focus only on the case $\lambda = 1/2$. Upon tedious yet straightforward calculations, we find that with respect to the lexicographically ordered product basis $\{ |0\rangle_A |1\rangle_A \} \otimes \{ |0\rangle_B |1\rangle_B |2\rangle_B \}$ we have that

$$\zeta_{AB}(1/2, \eta) = \left( I^A \otimes \Phi_{1/2, \xi(\eta)}^B \right) |\Psi_{AB}(\eta)\rangle =$$

$$\begin{pmatrix}
0 & 0 & -\frac{(1-\eta)^2 \eta}{\sqrt{2}} & \frac{\eta \sqrt{1-\eta}}{\sqrt{2}} & -\frac{(1-\eta)^{3/2}}{\sqrt{2}} \\
-\frac{(1-\eta)^2 \eta}{\sqrt{2}} & 0 & \frac{1}{2} (1-\eta)^2 & -\frac{(1-\eta)^{3/2}}{\sqrt{2}} & \frac{1}{2} (1-\eta)^2 \sqrt{\eta} \\
\eta \sqrt{1-\eta} & 0 & -\frac{(1-\eta)^{3/2}}{\sqrt{2}} & \frac{1}{2} \eta (1+\eta) & -\eta^{3/2} \sqrt{1-\eta} \sqrt{2} \\
-\frac{(1-\eta)^{3/2}}{\sqrt{2}} & 0 & \frac{1}{2} (1-\eta)^2 \sqrt{\eta} & -\eta^{3/2} \sqrt{1-\eta} \sqrt{2} & \frac{1}{2} (1-\eta)^2 \eta \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

(S39)
and that
\[ \zeta_B(1/2, \eta) = \Phi^{B}_{1/2, \xi(\eta)}(\Psi_B(\eta)) = \begin{pmatrix} \frac{1}{2} (1 + 2\eta - 2\eta^2 + \eta^3) & -\frac{\sqrt{1-\eta}}{\sqrt{2}} - \frac{(1-\eta)^2\eta}{\sqrt{2}} \\ -\frac{\sqrt{1-\eta}}{\sqrt{2}} & 0 \end{pmatrix} \] (S40)

We are now ready to apply the LSD theorem to our case. Note that the mean photon number of \( \phi_B(\eta) \) is precisely \((1-\eta)^2\). Then, employing (34)–(36) we find that
\[ Q(\Phi_{1/2, \xi(\eta)}) \geq Q(\Phi_{1/2, \xi(\eta)}; (1-\eta)^2) \geq I_{coh}(A)B_{\zeta_{AB}(1/2, \eta)}. \] (S41)

The coherent information \( I_{coh}(A)B_{\zeta_{AB}(1/2, \eta)} \) is plotted in Figure 2. The numerics shows clearly that this is strictly positive for all \( \eta \in (0, 1) \). We do not provide an analytical proof of this claim, because it is not necessary for what follows. In our proof of Theorem 2 we will only use the easily verified fact that \( I_{coh}(A)B_{\zeta_{AB}(1/2, \eta)} > 0 \) for some values of \( \eta \).

![Figure 1](image)

**FIG. 1.** The coherent information lower bound \( [S35] \) on the quantum capacity of the channel \( \Phi_{1/2, \xi(\eta)} \) defined by the environment state \( [S35] \). The maximum can be numerically evaluated, yielding \( \max_{0 \leq \eta \leq 1} I_{coh}(A)B_{\zeta_{AB}(1/2, \eta)} \approx 0.0748 \).

Incidentally, general attenuators can have a substantially larger transmission capacity if one allows for a higher input power to be deployed.

**Example S11.** For \( n \geq 3 \) to be fixed, consider the environment state \( \xi'(n) = |\xi'(n)\rangle\langle \xi'(n)| \), with
\[ |\xi'(n)\rangle := \frac{|n-1\rangle + |n\rangle}{\sqrt{2}}. \] (S42)

We look at the transmission scheme identified by an initial state
\[ |\Psi'(n)\rangle_{AB} := \frac{1}{2} (|0\rangle_A (|n-1\rangle_B + |n\rangle_B) + |1\rangle_A (|n-3\rangle_B + |n-2\rangle_B)). \] (S43)

Note that the mean photon number of \( \Psi'_B(n) \) is \( n - \frac{3}{2} \). Applying the LSD theorem in the form of (34)–(36) then yields
\[ Q(\Phi_{1/2, \xi'(n)}) \geq Q(\Phi_{1/2, \xi'(n)}; n - \frac{3}{2}) \geq I_{coh}(A)B_{\eta_{AB}(1/2, \eta)} \] (S44)

with \( \zeta_{AB}(1/2, n) := (I^A \otimes \Phi_{1/2, \xi(n)}(\Psi'_{AB}(n))) \). The values of the right-hand side of (S44) for \( n = 3, \ldots, 35 \) are reported in Figure 2. For \( n = 34 \) the lower bound evaluates to around 0.3389.
IV. POSITIVE CAPACITY AT ARBITRARY TRANSMISSIVITY

Theorem 2 For all $0 < \lambda \leq 1$ there exists a single-mode state $\sigma(\lambda)$ such that

$$Q(\Phi_{\lambda, \sigma(\lambda)}) \geq Q(\Phi_{\lambda, \sigma(\lambda)}, 1/2) \geq c$$

for some universal constant $c > 0$. Both $\sigma(\lambda)$ and $c$ are explicitly given in the proof.

Throughout this section we will provide a complete proof of the above result. In light of its complexity, we will break it down into several elementary steps, corresponding to the various subsections. Here is a brief account of their content:

IV.A Here we fix a transmission scheme, that is, a family of environment states $\sigma(n)$ (for $n = 1, 2, \ldots$) and an ansatz $|\Psi\rangle_{AB}$ to be plugged into the coherent information (6). The result will be a lower bound of the form $Q(\Phi_{\lambda, \sigma(\lambda)}) \geq H(p(n, \lambda)) - H(q(n, \lambda))$, where $p(n, \lambda)$ and $q(n, \lambda)$ are appropriate probability distributions over some index $\ell \in \{0, \ldots, n + 1\}$ (Proposition S12).

IV.B We will then proceed to identify a range of values of $\lambda$ (depending on $n$) for which $q(n, \lambda)$ can be sorted in ascending order by a fixed permutation (luckily enough, this turns out to be the identity). The result is contained in Proposition S13.

IV.C The same is then done for $p(n, \lambda)$, with considerably more effort and by keeping three distinct possibilities on the table (Proposition S17).

IV.D The crux of the argument is to verify that for a sufficiently large range of values of $\lambda$ (for varying $n$) the probability distribution $q(n, \lambda)$ actually majorises $p(n, \lambda)$.

IV.E The existence of a majorisation relation between $q(n, \lambda)$ and $p(n, \lambda)$ allows us to exploit a beautiful inequality due to Ho and Verdú [52, Theorem 3] to lower bound their entropy difference by means of the relative entropy distance. In turn, this can be lower bounded in terms of their total variation distance thanks to Pinsker’s inequality (see also [67] p. 58 and references therein). We look at the resulting bounds and draw our conclusions.
A. A transmission scheme

Proposition S12. Set $|\Psi_{AB}\rangle := \frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B)$ and $\sigma(n) := |n\rangle\langle n|$. Then for all $\lambda \in (0, 1)$ it holds that

$$Q(\Phi_{\lambda, \sigma(n)}, 1/2) \geq J(n, \lambda) := I_{\text{coh}}(A)B(1^{\lambda} \otimes \Phi^{B}_{\lambda, \sigma(n)})(\Psi_{AB}) = H(p(n, \lambda)) - H(q(n, \lambda)),$$

where $H$ denotes the Shannon entropy, and the two probability distributions $p(n, \lambda) = (p_0(n, \lambda), \ldots, p_{n+1}(n, \lambda))$ and $q(n, \lambda) = (q_0(n, \lambda), \ldots, q_{n+1}(n, \lambda))$ are defined by

$$p_\ell(n, \lambda) := \frac{1}{2(n+1)(1-\lambda)} \binom{n+1}{\ell} (1-\lambda)^{\ell} \lambda^{n-\ell} \left((1-\lambda)(n-\ell+1) + ((n+1)(1-\lambda) - \ell)^2\right)$$

$$q_\ell(n, \lambda) := \frac{1}{2(n+1)(1-\lambda)} \binom{n+1}{\ell} (1-\lambda)^{\ell} \lambda^{n-\ell} \left(\lambda \ell + ((n+1)(1-\lambda) - \ell)^2\right)$$

Proof. Thanks to [S16] and [S17], the action of the beam splitter on the $BE$ system can be expressed as

$$U_{AB}^{BE} |\Psi_{AB}\rangle |n\rangle_E$$

$$= -\frac{1}{\sqrt{2}} |0\rangle_A \left(\frac{1}{\sqrt{(n+1)(1-\lambda)}} \sum_{\ell=0}^{n+1} \sqrt{\binom{n+1}{\ell}} (1-\lambda)^{\ell} \lambda^{n-\ell} \left((n+1)(1-\lambda) - \ell\right) |\ell\rangle_B |n+1-\ell\rangle_E\right)$$

$$+ \frac{1}{\sqrt{2}} |1\rangle_A \left(\sum_{\ell=0}^{n} \sqrt{\binom{n}{\ell}} (1-\lambda)^{\ell} \lambda^{n-\ell} \left(-((n+1)(1-\lambda) - \ell) |0\rangle_A |\ell\rangle_B + \sqrt{(1-\lambda)(n-\ell+1)} |1\rangle_A |\ell\rangle_B |n-\ell\rangle_E\right)\right)$$

$$= -\frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{(n+1)(1-\lambda)}} \sum_{\ell=0}^{n+1} \sqrt{\binom{n+1}{\ell}} (1-\lambda)^{\ell} \lambda^{n-\ell} \left(-((n+1)(1-\lambda) - \ell) |0\rangle_A |\ell\rangle_B + \sqrt{(n+1)(1-\lambda)} |1\rangle_A |\ell\rangle_B |n-\ell\rangle_E\right)\right)$$

with the convention that $|\ell\rangle_E := 0$. Introducing the normalised vectors

$$|\zeta_\ell(n, \lambda)\rangle := \frac{1}{\sqrt{(1-\lambda)(n-\ell+1) + ((n+1)(1-\lambda) - \ell)^2}} \left(-((n+1)(1-\lambda) - \ell) |0\rangle |n+1-\ell\rangle + \sqrt{(1-\lambda)(n-\ell+1)} |1\rangle |n-\ell\rangle\right)$$

$$|\eta_\ell(n, \lambda)\rangle := \frac{1}{\sqrt{\lambda \ell + ((n+1)(1-\lambda) - \ell)^2}} \left(-((n+1)(1-\lambda) - \ell) |0\rangle |\ell\rangle + \sqrt{\lambda \ell} |1\rangle |\ell-1\rangle\right),$$

for $\ell = 0, \ldots, n+1$, we finally arrive at

$$U_{AB}^{BE} |\Psi_{AB}\rangle |n\rangle_E$$

$$= -\frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{(n+1)(1-\lambda)}} \sum_{\ell=0}^{n+1} \sqrt{\binom{n+1}{\ell}} (1-\lambda)^{\ell} \lambda^{n-\ell} \sqrt{(1-\lambda)(n-\ell+1) + ((n+1)(1-\lambda) - \ell)^2} |\zeta_\ell(n, \lambda)\rangle_B |\ell\rangle_E\right).$$

$$= -\frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{(n+1)(1-\lambda)}} \sum_{\ell=0}^{n+1} \sqrt{\binom{n+1}{\ell}} (1-\lambda)^{\ell} \lambda^{n-\ell} \sqrt{\lambda \ell + ((n+1)(1-\lambda) - \ell)^2} |\eta_\ell(n, \lambda)\rangle_B |n+1-\ell\rangle_E\right).$$

Tracing away the subsystem $E$ from [S52] yields the output state of the channel as

$$\omega_{AB}(n, \lambda) := (I^{A} \otimes \Phi^{B}_{\lambda, \sigma(n)}) \langle \Psi_{AB} \rangle$$

$$= \frac{1}{2(n+1)(1-\lambda)} \sum_{\ell=0}^{n+1} \binom{n+1}{\ell} (1-\lambda)^{\ell} \lambda^{n-\ell} \left(\lambda \ell + ((n+1)(1-\lambda) - \ell)^2\right) |\eta_\ell(n, \lambda)\rangle |\eta_\ell(n, \lambda)\rangle_{AB}. $$

(S53)
Note that the total photon number of the state $|\eta_\ell\rangle$ is exactly $\ell$, for all $\ell = 0, \ldots, n + 1$:

$$ (a^\dagger a + b^\dagger b) |\eta_\ell\rangle_{AB} = \ell |\eta_\ell\rangle_{AB}. \quad (S54) $$

Hence, the vectors $|\eta_\ell\rangle$ are all orthogonal to each other. This allows us to immediately deduce the spectrum of $\omega_{AB}(n, \lambda)$. We obtain that

$$ \text{sp} (\omega_{AB}(n, \lambda)) = \{ q_0(n, \lambda), \ldots, q_{n+1}(n, \lambda) \}, \quad (S55) $$

where the probability distribution $q(n, \lambda)$ is given by $\{S48\}$.

To derive an expression for $\omega_B(n, \lambda) = \Phi_{\lambda, \sigma(n)}^{B}(\Psi_B)$ we could trace away $A$ from $\{S55\}$. However, it is slightly more convenient to read off the result directly from $\{S51\}$. We obtain that

$$ \omega_B(n, \lambda) = \Phi_{\lambda, \sigma(n)}^{B}(\Psi_B) = \frac{1}{2 (n+1)(1-\lambda)} \sum_{\ell=0}^{n+1} \binom{n+1}{\ell} (1-\lambda)^\ell \lambda^{n-\ell} \left( (1-\lambda)(n-\ell+1) + ((n+1)(1-\lambda) - \ell)^2 \right) |\ell\rangle \langle \ell|_B \quad (S56) $$

The above decomposition allows us to write down the spectrum of the reduced output state on the $B$ system immediately. We obtain that

$$ \text{sp} (\omega_B(n, \lambda)) = \{ p_0(n, \lambda), \ldots, p_{n+1}(n, \lambda) \}, \quad (S57) $$

where the probability distribution $p(n, \lambda)$ is given by $\{S47\}$.

Since the reduced input state $\Psi_B$ on the $B$ system has mean photon number $1/2$, the (energy-constrained) LSD theorem $\{S5\}$ yields the estimate in $\{S46\}$, thus concluding the proof.

### B. Sorting $q(n, \lambda)$

In the following, for a given probability distribution $r = (r_0, \ldots, r_N)$, we denote with $\tau^\uparrow = (\tau^\uparrow_0, \ldots, \tau^\uparrow_N)$ the distribution obtained by sorting it in ascending order, so that e.g. $\tau^\uparrow_0 = \min_{\ell=0,\ldots,N} r_\ell$. Our first result tells us that for a wide range of values of $\lambda$ the distribution $q(n, \lambda)$ is actually already sorted. It is useful to define the two functions

$$ \lambda_+(n) := \frac{3}{n+2} \left( 1 - \sqrt{\frac{n-1}{3(n+1)}} \right), \quad (S58) $$

$$ \lambda_-(n) := \frac{2}{n+2} \left( 1 - \sqrt{\frac{n}{2(n+1)}} \right). \quad (S59) $$

We are now ready to state and prove our first result.

**Proposition S13.** For all $n \geq 2$,

$$ q^\uparrow(n, \lambda) = q(n, \lambda) \quad \forall \quad \frac{1}{n+1} \leq \lambda \leq \lambda_+(n). \quad (S60) $$

**Proof.** For $\ell = 0, \ldots, n$, leveraging the fact that

$$ \binom{n+1}{\ell+1} = \frac{n-\ell+1}{\ell+1} \binom{n+1}{\ell}, \quad (S61) $$

the formula $\{S48\}$ yields

$$ \lambda(\ell+1) \left( \lambda(\ell+1) + ((n+1)(1-\lambda) - \ell)^2 \right) \left( q_{\ell+1}(n, \lambda) / q_\ell(n, \lambda) - 1 \right) $$

$$ = (n-\ell+1)(1-\lambda) \left( \lambda(\ell+1) + ((n+1)(1-\lambda) - \ell-1)^2 \right) - \lambda(\ell+1) \left( \lambda(\ell+1) + ((n+1)(1-\lambda) - \ell)^2 \right) $$

$$ = \lambda(\ell+1) \left( (n-\ell+1)(1-\lambda) - \ell \right) - \lambda(\ell+1) \left( (n-\ell+1)(1-\lambda) - \ell - 1 \right)^2 + (n-\ell+1)(1-\lambda) \left( (n+1)(1-\lambda) - \ell - 1 \right)^2 $$

$$ = -\lambda(\ell+1) \left( (n+1)(1-\lambda) - \ell - 1 \right) \left( (n+1)(1-\lambda) - \ell \right) + (n-\ell+1)(1-\lambda) \left( (n+1)(1-\lambda) - \ell - 1 \right)^2 $$

$$ = ((n+1)(1-\lambda) - \ell - 1) \left( \lambda(\ell+1) \left( (n+1)(1-\lambda) - \ell \right) + (n-\ell+1)(1-\lambda) \left( (n+1)(1-\lambda) - \ell - 1 \right) \right) $$

$$ = ((n+1)(1-\lambda) - \ell - 1) \left( \ell^2 - 2 \left( (n+1)(1-\lambda) - \frac{1}{2} \right) \ell + (n+1)(1-\lambda) (n-(n+2)\lambda) \right). $$


Setting
\[ f_{n,\lambda}(\ell) := \ell^2 - 2 \left( (n+1)(1-\lambda) - \frac{1}{2} \right) \ell + (n+1)(1-\lambda)(n-(n+2)\lambda), \] (S62)
we arrive at the identity
\[ \lambda(\ell + 1) \left( \lambda \ell + ((n+1)(1-\lambda) - \ell)^2 \right) \left( \frac{q_{\ell+1}(n,\lambda)}{q_{\ell}(n,\lambda)} - 1 \right) = ((n+1)(1-\lambda) - \ell - 1) f_{n,\lambda}(\ell). \] (S63)

Now, the function \( f_{n,\lambda}(\ell) \) is a second-degree polynomial in the variable \( \ell \). By finding its roots we can determine its sign on the whole real line. We see that
\[ f_{n,\lambda}(\ell) \leq 0 \quad \text{if} \quad \ell - (n,\lambda) \leq \ell \leq \ell + (n,\lambda), \]
\[ f_{n,\lambda}(\ell) \geq 0 \quad \text{otherwise}, \]
where
\[ \ell_{\pm}(n,\lambda) := n + \frac{1}{2} - (n+1)(1-\lambda) \pm \sqrt{\frac{1}{4} + (n+1)\lambda(1-\lambda)}. \]

One can show that
\[ \ell_{-}(n,\lambda) \geq n - 2 \quad \forall \quad 0 \leq \lambda \leq \min \left\{ \frac{5}{2(n+1)}, \lambda_{+}(n) \right\} = \lambda_{+}(n). \]

Moreover,
\[ \ell_{-}(n,\lambda) \leq n - 1 \quad \forall \quad \lambda \geq \lambda_{-}(n). \]

Putting all together, we find that
\[ n - 2 \leq \ell_{-}(n,\lambda) \leq n - 1 \quad \forall \quad \lambda_{-}(n) \leq \lambda \leq \lambda_{+}(n). \]

It is also easy to verify that
\[ \ell_{+}(n,\lambda) \geq n \quad \forall \quad 0 \leq \lambda \leq \frac{2}{n+2}. \]

Since \( \frac{2}{n+2} \geq \lambda_{+}(n) \) for all \( n \geq 2 \), we deduce that
\[ \ell_{+}(n,\lambda) \geq n \quad \forall \quad 0 \leq \lambda \leq \lambda_{+}(n). \]

Going back to the function \( f_{n,\lambda}(\ell) \), the above discussion implies that
\[ f_{n,\lambda}(\ell) \leq 0 \quad \text{if} \quad \ell = n-1, \quad \forall \quad \lambda_{-}(n) \leq \lambda \leq \lambda_{+}(n). \] (S64)

Also, it is not difficult to verify that
\[ n - 2 \leq (n+1)(1-\lambda) - 1 \leq n - 1 \quad \forall \quad \frac{1}{n+1} \leq \lambda \leq \frac{2}{n+1}; \]
we infer that
\[ (n+1)(1-\lambda) - \ell - 1 \leq 0 \quad \text{if} \quad \ell = n-1, \quad \forall \quad \frac{1}{n+1} \leq \lambda \leq \frac{2}{n+1}. \] (S65)

Using the fact that \( \frac{1}{n+1} \geq \lambda_{-}(n) \) and \( \lambda_{+}(n) \leq \frac{2}{n+1} \) for all \( n \), and combining (S63) on the one hand with (S64)–(S65) on the other, we finally see that
\[ q_{\ell+1}(n,\lambda) \geq q_{\ell}(n,\lambda) \quad \forall \quad \frac{1}{n+1} \leq \lambda \leq \lambda_{+}(n), \]
which proves the claim.
C. Sorting $p(n, \lambda)$

As it turns out, for an analogous range of values of $\lambda$ the probability distribution $p(n, \lambda)$, unlike $q(n, \lambda)$, is not automatically sorted in ascending order. The next lemma represents a first step in the direction of ascertaining how $p(n, \lambda)$ can be sorted.

**Lemma S14.** For all $n \geq 2$,

$$
p_0(n, \lambda) \leq p_1(n, \lambda) \leq \ldots \leq p_{n-1}(n, \lambda) \geq p_n(n, \lambda) \leq p_{n+1}(n, \lambda) \quad \forall \frac{1}{n+1} \leq \lambda \leq \lambda_+(n). \quad (S66)
$$

**Proof.** For all $\ell = 0, \ldots, n$, employing (S47) and (S61) we compute

\[
\begin{align*}
\lambda(\ell + 1) \left( (1 - \lambda)(n - \ell + 1) + ((n + 1)(1 - \lambda) - \ell)^2 \right) \left( \frac{p_{\ell+1}(n, \lambda)}{p_\ell(n, \lambda)} - 1 \right) \\
= (n - \ell + 1)(1 - \lambda) \left( (1 - \lambda)(n - \ell) + ((n + 1)(1 - \lambda) - \ell - 1)^2 \right) \\
- \lambda(\ell + 1) \left( (1 - \lambda)(n - \ell + 1) + ((n + 1)(1 - \lambda) - \ell)^2 \right) \\
= (n - \ell + 1)(1 - \lambda) \left( (1 - \lambda)(n - \ell) + ((n + 1)(1 - \lambda) - \ell - 1)^2 - \lambda(\ell + 1) \right) \\
= (n - \ell + 1)(1 - \lambda) \left( (n + 1)(1 - \lambda) - \ell \right) \left( (n - \ell + 1)(1 - \lambda) - \ell - 1 \right) - \lambda(\ell + 1) \left( (n + 1)(1 - \lambda) - \ell \right) \\
= (n - 1)(1 - \lambda) - \ell \left( \ell^2 - 2 \left( (n + 1)(1 - \lambda) - \frac{1}{2} \right) \ell + (n + 1)(1 - \lambda) (n - (n + 2)\lambda) \right).
\end{align*}
\]

Thus,

\[
\lambda(\ell + 1) \left( (1 - \lambda)(n - \ell + 1) + ((n + 1)(1 - \lambda) - \ell)^2 \right) \left( \frac{p_{\ell+1}(n, \lambda)}{p_\ell(n, \lambda)} - 1 \right) = ((n + 1)(1 - \lambda) - \ell) f_{n, \lambda}(\ell), \quad (S67)
\]

where $f_{n, \lambda}(\ell)$, defined by (S62), is – luckily enough – the same function that we already encountered in the proof of Proposition S13 which makes (S64) available. Since

\[
n - 1 \leq (n + 1)(1 - \lambda) \leq n \quad \forall \frac{1}{n+1} \leq \lambda \leq \frac{2}{n+1},
\]

we obtain that

\[
\begin{align*}
(n + 1)(1 - \lambda) - \ell &\leq 0 \quad \text{if } \ell = n, \\
(n + 1)(1 - \lambda) - \ell - 1 &\geq 0 \quad \text{if } \ell = 0, \ldots, n - 1, \quad \forall \frac{1}{n+1} \leq \lambda \leq \frac{2}{n+1}. \quad (S68)
\end{align*}
\]

Combining (S67) with (S64) and (S68) shows that for all $\frac{1}{n+1} \leq \lambda \leq \lambda_+(n)$ the inequalities $p_\ell(n, \lambda) \leq p_{\ell+1}(n, \lambda)$ hold true for $\ell = 0, \ldots, n - 2$ or $\ell = n$, while for $\ell = n - 1$ we have the opposite relation $p_{n-1}(n, \lambda) \geq p_n(n, \lambda)$. This completes the proof.

**Lemma S15.** For all $n \geq 3$,

\[
p_{n-3}(n, \lambda) \leq p_n(n, \lambda) \quad \forall \frac{1}{n+1} \leq \lambda \leq \lambda_+(n). \quad (S69)
\]

where

\[
\lambda_+(n) := \frac{3^{1/3}}{2^{1/3}n + 3^{1/3} - 2^{1/3}}. \quad (S70)
\]

**Proof.** Using the explicit formulae (S47) and (S48), we compute

\[
\frac{p_{n-3}(n, \lambda)}{p_n(n, \lambda)} = \frac{1}{24} n(n - 1)(n - 2) \left( \frac{\lambda}{1 - \lambda} \right)^3 \left( \frac{4(1 - \lambda) + (4 - (n + 1)\lambda)^2}{1 - \lambda + (1 - (n + 1)\lambda)^2} \right) \\
= \left( \frac{2}{3} n(n - 1)(n - 2) \left( \frac{\lambda}{1 - \lambda} \right)^3 \right) \left( \frac{1}{16} \frac{4(1 - \lambda) + (4 - (n + 1)\lambda)^2}{1 - \lambda + (1 - (n + 1)\lambda)^2} \right).
\]
We now evaluate separately the above two factors, and show that they are both upper bounded by 1. The first one can be estimated by resorting to the elementary inequality $n(n - 2) \leq (n - 1)^2$; one obtains that
\[
\frac{2}{3} n(n - 1)(n - 2) \left( \frac{\lambda}{1 - \lambda} \right)^3 \leq \frac{2}{3} (n - 1)^3 \left( \frac{\lambda}{1 - \lambda} \right)^3 \leq 1 \quad \forall \quad 0 \leq \lambda \leq \bar{\lambda} + (n),
\]
where the last inequality can be easily proved by taking the cubic root of both sides. Upon simple algebraic manipulations, the inequality
\[
\frac{1}{16} \frac{4(1 - \lambda) + 4 - (n + 1)\lambda^2}{1 - \lambda + (1 - (n + 1)\lambda)^2} \leq 1,
\]
which is to be proved, becomes
\[
\frac{5}{4} (1 + n)^2\lambda^2 - (3 + 2n)\lambda - 1 \geq 0.
\]
The discriminant of the second-degree polynomial on the left-hand side is $-n^2 + 2n + 4$. This is negative for all $n \geq 4$, and hence in this case the above inequality is satisfied for all $0 \leq \lambda \leq 1$ and a fortiori in the prescribed range. If $n = 3$, an explicit calculation shows that the inequality holds true for $\lambda \leq \frac{1}{2}$ or $\lambda \geq \frac{1}{4}$, i.e. in particular for all $\lambda \geq \frac{1}{n+1} = \frac{1}{4}$. This completes the proof.

**Lemma S16.** For all $n \geq 2$,
\[
p_{n-1}(n, \lambda) = \max_{\ell=0, \ldots, n+1} p_{\ell}(n, \lambda) \quad \forall \quad \frac{1}{n+1} \leq \lambda \leq \bar{\lambda} + (n).
\]

**Proof.** Since Lemma S14 holds in the prescribed interval in $\lambda$, we need only to prove that $p_{n-1}(n, \lambda) \geq p_{n+1}(n, \lambda)$. Indeed, one verifies that
\[
(n + 1)^2(1 - \lambda)^2 \left( \frac{p_{n-1}(n, \lambda)}{p_{n+1}(n, \lambda)} - 1 \right) = \frac{1}{2} (n - 1)(n + 1)(n + 2)\lambda^2 - (2n^2 + n - 2)\lambda + 2n - 1 \geq 0,
\]
where the last inequality holds because the above second-degree polynomial in $\lambda$ has discriminant $n(-2n^2 + n + 2) < 0$ as soon as $n \geq 2$.

**Proposition S17.** For all $n \geq 2$ and all $\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}$,
\[
\begin{align*}
\text{either} \quad &p^+(n, \lambda) = \left( p_0(n, \lambda), p_1(n, \lambda), \ldots, p_{n-2}(n, \lambda), p_{n-1}(n, \lambda), p_{n+1}(n, \lambda), p_{n-2}(n, \lambda), p_{n-1}(n, \lambda) \right), \\
\text{or} \quad &p^+(n, \lambda) = \left( p_0(n, \lambda), p_1(n, \lambda), \ldots, p_{n-2}(n, \lambda), p_{n-1}(n, \lambda), p_{n+1}(n, \lambda), p_{n-1}(n, \lambda) \right), \\
\text{or} \quad &p^+(n, \lambda) = \left( p_0(n, \lambda), p_1(n, \lambda), \ldots, p_{n-2}(n, \lambda), p_{n-1}(n, \lambda), p_{n+1}(n, \lambda), p_{n-1}(n, \lambda) \right).
\end{align*}
\]

When $n = 2$, it is understood that only the last 4 entries are to be taken into account.

**Proof.** It suffices to combine Lemmata S14, S15 and S16. Note that $\frac{1}{n} \leq \min \left\{ \lambda_+(n), \bar{\lambda}_+(n) \right\}$ for all $n \geq 2$.

**D. Majorisation**

Let $r = (r_0, \ldots, r_N)$ and $s = (s_0, \ldots, s_N)$ be two probability distributions. We remind the reader that $r$ is said to be majorised by $s$, and we write $r \prec s$, if
\[
\sum_{\ell=0}^{k} r_\ell^+ \geq \sum_{\ell=0}^{k} s_\ell^+ \quad \forall \; k = 0, \ldots, N.
\]
Of course, the above inequality becomes an equality for $k = N$, since the elements of both distributions add up to 1. For a complete introduction to the theory of majorisation, we refer the reader to the excellent monograph by Marshall and Olkin [66].

The goal of this subsection is to show that the two probability distributions $p(n, \lambda)$ and $q(n, \lambda)$ obey precisely a majorisation relation $p(n, \lambda) \prec q(n, \lambda)$. Our first step in this direction is a simple lemma.
Lemma S18. For all $n \geq 2$,
\[ q_\ell(n, \lambda) \leq p_\ell(n, \lambda) \quad \forall \quad \ell = 0, \ldots, n-1, \quad \forall \quad 0 \leq \lambda \leq \frac{2}{n+1}. \quad (S73) \]

Proof. Using the expressions (S47) and (S48), one verifies that
\[
\left( \lambda \ell + ((n+1)(1-\lambda) - \ell)^2 \right) \left( \frac{p_\ell(n, \lambda)}{q_\ell(n, \lambda)} - 1 \right) \\
= (1-\lambda)(n - \ell + 1) + ((n+1)(1-\lambda) - \ell)^2 - \lambda \ell - ((n+1)(1-\lambda) - \ell)^2 \\
= n - \ell + 1 - (n+1)\lambda \\
\geq 0,
\]
where the last inequality holds provided that $\lambda \leq \frac{2}{n+1}$ and $\ell \leq n-1$. \( \square \)

Lemma S19. For all $n \geq 3$,
\[ q_{n+1}(n, \lambda) - q_{n-1}(n, \lambda) \geq \frac{(n+1)(n-2)}{4n(n-1)} \left( 1 - \frac{1}{n} \right)^n \quad \forall \quad 0 \leq \lambda \leq \frac{1}{n}. \quad (S74) \]

When $n = 2$, we have instead that
\[ q_3(2, \lambda) - q_1(2, \lambda) \geq \frac{\epsilon}{4} \quad \forall \quad 0 \leq \lambda \leq \frac{1}{2} - \epsilon \quad (S75) \]
for any fixed $\epsilon > 0$.

Proof. For all $n \geq 2$, one verifies that
\[
\frac{\partial}{\partial \lambda} (q_{n+1}(n, \lambda) - q_{n-1}(n, \lambda)) \\
= \frac{1}{4} (1-\lambda)^{n-3} \left( 2 - 6n + 2(6n^2 + n - 3) \lambda - (n+1)^2(7n-6)\lambda^2 + (n+1)^2(n^2 + n - 2)\lambda^3 \right) \\
= \frac{1}{4} (1-\lambda)^{n-3} g_n(\lambda).
\]

Now, since
\[
\frac{d^2 g_n(\lambda)}{d\lambda^2} = 2(n+1)^2 \left( 6 - 7n + 3(n^2 + n - 2) \lambda \right) \\
\leq 2(n+1)^2 \left( 6 - 7n + 3(n^2 + n - 2) \frac{1}{n} \right) \\
= -\frac{2}{n}(n+1)^2 \left( 4n^2 - 9n + 6 \right) \leq 0
\]
for $0 \leq \lambda \leq \frac{1}{n}$, the first derivative $\frac{dg_n(\lambda)}{d\lambda}$ of $g_n(\lambda)$ is a decreasing function of $\lambda$ in the same interval $[0, \frac{1}{n}]$. Hence,
\[
\min_{\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}} \frac{dg_n(\lambda)}{d\lambda} = \frac{dg_n(\lambda)}{d\lambda}\bigg|_{\lambda = \frac{1}{n}} = \frac{n+1}{n^2} \left( 6 + n(n-1)(n-3) \right) \geq 0.
\]
In turn, this implies that
\[
\max_{\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}} g_n(\lambda) = g_n \left( \frac{1}{n} \right) = -\frac{(n-1)^2(n+2)}{n^3} \leq 0.
\]
Thus, $q_{n+1}(n, \lambda) - q_{n-1}(n, \lambda)$ is decreasing in $\lambda$. Finally, we conclude from this that
\[
\min_{\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}} \{q_{n+1}(n, \lambda) - q_{n-1}(n, \lambda)\} = q_{n+1} \left( \frac{1}{n} \right) - q_{n-1} \left( \frac{1}{n} \right) = \frac{(n+1)(n-2)}{4n(n-1)} \left( 1 - \frac{1}{n} \right)^n.
\]
When $n = 2$, we have instead that
\[
\min_{\frac{1}{2} \leq \lambda \leq \frac{1}{2} - \epsilon} \{q_3(2, \lambda) - q_1(2, \lambda)\} = q_3 \left( \frac{2}{2} - \epsilon \right) - q_1 \left( \frac{2}{2} - \epsilon \right) = \frac{\epsilon}{4} + 3\epsilon^3 \geq \frac{\epsilon}{4}.
\]
This concludes the proof. \( \square \)
Proposition S20. For all $n \geq 2$,

$$p(n, \lambda) \prec q(n, \lambda) \quad \forall \quad \frac{1}{n+1} \leq \lambda \leq \frac{1}{n}.$$  \hfill (S76)

Proof. According to (S72), we need to verify that

$$\sum_{\ell=0}^{k} p_{\ell}^+(n, \lambda) \geq \sum_{\ell=0}^{k} q_{\ell}^+(n, \lambda) \quad \forall \quad k = 0, \ldots, n, \quad \forall \quad \frac{1}{n+1} \leq \lambda \leq \frac{1}{n},$$  \hfill (S77)

where we used the fact that the inequality corresponding to $k = n + 1$ is in fact an equality, by normalisation. Using Proposition S17 and Lemma S18, and observing that $k = 0, \ldots, n - 3$ of (S77) are automatically satisfied. Exploiting again normalisation, we recast the difference of the two sides of (S77) (for arbitrary $k$) as

$$\sum_{\ell=0}^{k} p_{\ell}^+(n, \lambda) - \sum_{\ell=0}^{k} q_{\ell}^+(n, \lambda) = \sum_{\ell=k+1}^{n+1} q_{\ell}^+(n, \lambda) - \sum_{\ell=k+1}^{n+1} p_{\ell}^+(n, \lambda),$$  \hfill (S78)

where the last identity follows from Proposition S13 once one observes that $\frac{1}{n} \leq \lambda(n)$ as long as $n \geq 2$. It remains to check the positivity of (S78) for $k = n, n - 1, n - 2$ and for $\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}$. When $k = n - 1$ and $k = 2$ we have to reckon the (two) distinct possibilities offered by Proposition S17. This makes a total of 5 different cases to vet. We break down the proof into the separate analysis of each of these cases.

- $k = n$. Thanks to Lemma S16 (or Proposition S17) and Lemma S19

$$q_{n+1}(n, \lambda) - p_{n+1}^+(n, \lambda) = q_{n+1}(n, \lambda) - p_{n-1}(n, \lambda) \geq 0.$$  

- $k = n - 1$ and $p_{n}^+(n, \lambda) = p_{n-2}(n, \lambda)$. Let us write

$$q_{n+1}(n, \lambda) + q_{n}(n, \lambda) - p_{n-1}(n, \lambda) - p_{n-2}(n, \lambda)$$

$$= \frac{1}{2} (1 - \lambda)^{-n} \left[ 2 - 3(n + 2)\lambda + (6 + 9n + n^2) \lambda^2 + \frac{1}{2} (n + 1) (n^2 - 10n - 4) \lambda^3 - \frac{1}{6} n(n + 1)(n + 2)(n - 5) \lambda^4 \right]$$

$$= \frac{1}{2} (1 - \lambda)^{-n} h_n(\lambda).$$

Now, since

$$\frac{d^3 h_n(\lambda)}{d \lambda^3} = (n + 1) \left[ 3 (n^2 - 10n - 4) - 4n(n + 2)(n - 5) \lambda \right]$$

is a linear function of $\lambda$, we have that

$$\max_{\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}} \left\{ \frac{d^3 h_n(\lambda)}{d \lambda^3} \right\} = \max \left\{ \frac{d^3 h_n(\lambda)}{d \lambda^3} \bigg|_{\lambda = \frac{1}{n+1}}, \frac{d^3 h_n(\lambda)}{d \lambda^3} \bigg|_{\lambda = \frac{1}{n}} \right\}$$

$$\leq 0.$$  

That is to say, the function $\frac{d^2 h_n(\lambda)}{d \lambda^2}$ is non-increasing on $\left[ \frac{1}{n+1}, \frac{1}{n} \right]$. Therefore,

$$\min_{\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}} \left\{ \frac{d^2 h_n(\lambda)}{d \lambda^2} \right\} = \frac{d^2 h_n(\lambda)}{d \lambda^2} \bigg|_{\lambda = \frac{1}{n+1}} = 3n^2 - 5n - 4 + \frac{8}{n} \geq n - 4 + \frac{8}{n} \geq 4 \left( \sqrt{2} - 1 \right) \geq 0,$$

where we exploited the fact that $n \geq 2$. This shows that the function $\frac{dh_n(\lambda)}{d \lambda}$ is non-decreasing on $\left[ \frac{1}{n+1}, \frac{1}{n} \right]$. We infer that

$$\max_{\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}} \left\{ \frac{dh_n(\lambda)}{d \lambda} \right\} = \frac{dh_n(\lambda)}{d \lambda} \bigg|_{\lambda = \frac{1}{n+1}} = -\frac{n + 1}{6n^2} \left( n^2 + 2n + 4 \right) \leq 0.$$
Finally, given that $h_n(\lambda)$ has been shown to be non-increasing on $\left[\frac{1}{n+1}, \frac{1}{n}\right]$, we have that
\[
\min_{\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}} h_n(\lambda) = h_n\left(\frac{1}{n}\right) = \frac{1}{6n^3} (n-1)(n-2)(2n-1) \geq 0,
\]
which shows that $q_{n+1}(n, \lambda) + q_n(n, \lambda) - p_{n-1}(n, \lambda) - p_{n-2}(n, \lambda)$ for all $\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}$ and concludes the analysis of this case.

- $k = n - 1$ and $p_n^1(n, \lambda) = p_{n+1}(n, \lambda)$. We compute
  \[
  q_{n+1}(n, \lambda) + q_n(n, \lambda) - p_{n-1}(n, \lambda) - p_{n-2}(n, \lambda) = (1 - \lambda)^{n-2} \left(1 - \frac{1}{2} (4n + 5) \lambda + \frac{1}{2} (3n^2 + 6n + 4) \lambda^2 - \frac{1}{4} (n + 1)^2 (n + 2) \lambda^3\right).
  \]

Let us first deal with the case $n = 2$; note that $s_2(\lambda) = 1 - \frac{13}{2} \lambda + 14 \lambda^2 - 9 \lambda^3$. Now, $\frac{d s_2(\lambda)}{d \lambda} = -\frac{13}{2} + 28 \lambda - 27 \lambda^2 \geq 0$ for $0.351 \approx \frac{28 - \sqrt{82}}{54} \leq \lambda \leq \frac{28 + \sqrt{82}}{54} \approx 0.686$, and $\frac{d s_2(\lambda)}{d \lambda} \leq 0$ outside of that interval. Hence,
\[
\min_{\frac{1}{2} \leq \lambda \leq \frac{1}{2}} s_2(\lambda) = s_2\left(\frac{28 - \sqrt{82}}{54}\right) \approx 0.054 \geq 0.
\]

We now consider the case where $n \geq 3$. Since
\[
\frac{d^2 s_n(\lambda)}{d \lambda^2} = 3n^2 + 6n + 4 - \frac{3}{2} (n + 1)^2 (n + 2) \lambda
\]
is decreasing in $\lambda$, we obtain that
\[
\min_{\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}} \frac{d^2 s_n(\lambda)}{d \lambda^2} \left|_{\lambda = \frac{1}{n+1}} = \frac{3}{2} n^2 - \frac{3}{2} n - \frac{27}{2} \geq -\frac{1}{2} = 9 \geq 0,
\]
where we used the fact that $n \geq 3$. This proves that $\frac{ds_n(\lambda)}{d \lambda}$ is non-decreasing on $\left[\frac{1}{n+1}, \frac{1}{n}\right]$. Hence,
\[
\min_{\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}} \frac{ds_n(\lambda)}{d \lambda} = \frac{ds_n(\lambda)}{d \lambda} \left|_{\lambda = \frac{1}{n+1}} = \frac{n(n-3)}{4(n+1)} \geq 0,
\]
where the last estimate holds because $n \geq 3$. We have just shown that $s_n(\lambda)$ is non-decreasing in the interval $\left[\frac{1}{n+1}, \frac{1}{n}\right]$. We infer that
\[
\min_{\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}} s_n(\lambda) = s_n\left(\frac{1}{n+1}\right) = \frac{n(n-1)}{4(n+1)^2} \geq 0,
\]
which shows that $q_{n+1}(n, \lambda) + q_n(n, \lambda) - p_{n-1}(n, \lambda) - p_{n-2}(n, \lambda) \geq 0$ for all $n \geq 2$ and all $\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}$, thus completing the argument for this case.

- $k = n - 2$ and $\left\{p_{n-1}^1(n, \lambda), p_n^1(n, \lambda)\right\} = \left\{p_{n+1}(n, \lambda), p_{n-2}(n, \lambda)\right\}$. The relevant quantity is now
  \[
  q_{n+1}(n, \lambda) + q_n(n, \lambda) - p_{n-1}(n, \lambda) - p_{n-2}(n, \lambda) = (1 - \lambda)^{n-3} \left(1 - \left(\frac{7}{2} + \frac{1}{4} \lambda^2\right) (n^2 - 15n - 18) \lambda^2 + \frac{1}{2} (n^3 - 2n^2 - 7n - 5) \lambda^3 - \frac{1}{12} (n + 1)^2 (n + 2) (n - 3) \lambda^4\right).
  \]

To study the polynomial $t_n(\lambda)$, let us treat separately the cases $n = 2$ and $n \geq 3$. Note that
\[
t_2(\lambda) = (1 - \lambda)^2 \left(1 - \frac{1}{2} \lambda (7 - 6 \lambda)\right) \geq 0 \quad \forall \frac{1}{3} \leq \lambda \leq \frac{1}{2},
\]
where the last inequality is a consequence of the fact that the function $\lambda \mapsto 1 - \frac{1}{2} \lambda (7 - 6 \lambda)$ is decreasing on $(-\infty, \frac{7}{12}] \supset [\frac{1}{3}, \frac{1}{2}]$ and vanishes for $\lambda = \frac{1}{2}$.

We now look at the case where $n \geq 3$. Since $\frac{d^2 t_n(\lambda)}{d\lambda^2} = -2(n+1)^2(n+2)(n-3) \leq 0$, the function $\frac{d^2 t_n(\lambda)}{d\lambda^2}$ is concave. Hence,

$$\min_{\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}} \frac{d^2 t_n(\lambda)}{d\lambda^2} = \min \left\{ \left. \frac{d^2 t_n(\lambda)}{d\lambda^2} \right|_{\lambda = \frac{1}{n+1}}, \left. \frac{d^2 t_n(\lambda)}{d\lambda^2} \right|_{\lambda = \frac{1}{n}} \right\}$$

$$= \min \left\{ \frac{n}{2(n+1)} (3n^2 + 2n + 5), \frac{1}{2n^2} (3n^4 + n^3 - 10n^2 - 4n + 12) \right\} \geq 0,$$

where in the last step we used the fact that $n \geq 3$. We deduce that $\frac{dt_n(\lambda)}{d\lambda}$ is non-decreasing on $[\frac{1}{n+1}, \frac{1}{n}]$, in turn implying that

$$\max_{\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}} \frac{dt_n(\lambda)}{d\lambda} = \left. \frac{dt_n(\lambda)}{d\lambda} \right|_{\lambda = 1/n} = - \frac{(n-1)}{n^4} (2n^3 - 2n^2 - 7n + 12) \leq 0,$$

where the last inequality holds because $2n^3 - 2n^2 - 7n + 12 \geq 4n^2 - 7n + 12 \geq 9 \geq 0$ for $n \geq 3$. Since we have just shown that $t_n(\lambda)$ is non-increasing on $[\frac{1}{n+1}, \frac{1}{n}]$, we conclude that

$$\min_{\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}} t_n(\lambda) = t_n \left( \frac{1}{n} \right) = \frac{(n-2)(n-1)(2n^2 - 2n + 3)}{12n^4} \geq 0,$$

concluding the argument.

- $k = n - 2$ and $\left( p_{n-1}^{\downarrow}(n, \lambda), p_n^{\uparrow}(n, \lambda) \right) = (p_n(n, \lambda), p_{n+1}(n, \lambda))$. The analysis of this last case is much simpler. It suffices to verify that

$$q_{n+1}(n, \lambda) + q_n(n, \lambda) + q_{n-1}(n, \lambda) - p_{n-1}(n, \lambda) - p_{n+1}(n, \lambda) - p_n(n, \lambda) = \frac{1}{4} n(n-1) \lambda^2 (1 - \lambda)^{n-2} \geq 0.$$

This completes the proof. 

\[\blacksquare\]

### E. Concluding the proof

**Proof of Theorem 3** Let us partition the $(0,1)$ into the three regions

$$(0,1) = \left( 0, \frac{1}{2} - \epsilon \right] \cup \left[ \frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon \right] \cup \left[ \frac{1}{2} + \epsilon, 1 \right], \quad (S79)$$

where $\epsilon > 0$ is a small constant to be determined later. In the third region, i.e. for $\frac{1}{2} + \epsilon \leq \lambda \leq 1$, the claim follows elementarily from the ansatz $\sigma = |0\rangle \langle 0|$, which brings us back to the case of the pure loss channel. Thanks to (S26), we know that

$$Q \left( \Phi_\lambda, |0\rangle \langle 0|, \frac{1}{2} \right) = Q \left( \mathcal{F}_\lambda, \frac{1}{2} \right) = \max \left\{ g \left( \frac{\lambda}{2} \right) - g \left( \frac{1 - \lambda}{2} \right), 0 \right\} \geq g \left( \frac{1}{2} \left( \frac{1}{2} + \epsilon \right) \right) - g \left( \frac{1}{2} \left( \frac{1}{2} - \epsilon \right) \right) > 0$$

as long as $\epsilon > 0$. In the second region, that is, for $\frac{1}{2} - \epsilon \leq \lambda \leq \frac{1}{2} + \epsilon$, one can use Example S10 and some standard continuity arguments. Namely, consider the state $\sigma = \xi(1/3)$ as defined by (S35); specialising (S41) we find that

$$Q \left( \Phi_1/2, \xi(1/3), \frac{1}{2} \right) \geq I_{coh}(A)B_{\zeta_{AB}(1/2,1/3)} \approx 0.07392 > 0,$$

where $\zeta_{AB}(\lambda, \eta)$ is the reduced state on $AB$ corresponding to (S38). The density matrices $\zeta_{AB}(\lambda, 1/3)$ clearly depend continuously on $\lambda$; moreover, they live in a qubit–qutrit system for all values of $\lambda$. Hence, the Alicki–Fannes–Winter
inequality \[63\] \[65\] implies that \( I_{\text{coh}}(A)B)_{\lambda, 1/3} \) is a continuous function of \( \lambda \). By choosing \( \epsilon > 0 \) small enough, we can therefore insure that
\[
Q \left( \Phi_{\lambda, \xi(1/3)}, \frac{1}{2} \right) \geq I_{\text{coh}}(A)B)_{\lambda, 1/3} \geq c_1 \quad \forall \quad \frac{1}{2} - \epsilon \leq \lambda \leq \frac{1}{2} + \epsilon,
\]
where \( c_1 > 0 \) is a universal constant.

We are thus left with the first region, corresponding to \( 0 < \lambda \leq \frac{1}{2} - \epsilon \). We further split it according to
\[
\left( 0, \frac{1}{2} - \epsilon \right] = \left[ \frac{1}{3}, \frac{1}{2} - \epsilon \right] \cup \bigcup_{n=3}^{\infty} \left[ \frac{1}{n+1}, \frac{1}{n} \right].
\]

Thanks to Proposition \[S12\] we need only to show that
\[
\mathcal{J}(n, \lambda) = H(p(n, \lambda)) - H(q(n, \lambda)) \geq c \quad \forall \quad \frac{1}{n+1} \leq \lambda \leq \frac{1}{n}
\]
for all \( n \geq 3 \) and for some universal constant \( c_2 > 0 \), and also that
\[
\mathcal{J}(2, \lambda) = H(p(2, \lambda)) - H(q(2, \lambda)) \geq c_3 \quad \forall \quad \frac{1}{3} \leq \lambda \leq \frac{1}{2} - \epsilon
\]
for some other constant \( c_3 > 0 \).

Our main tool here will be a beautiful inequality proved by Ho and Verdú \[52, \text{Theorem 3}\]. This states that whenever \( r \) and \( s \) are two probability distributions such that \( r \prec s \), it holds that
\[
H(s) - H(r) \geq D\left( s^\uparrow \| r^\uparrow \right),
\]
where \( D(u\|v) := \sum u_i \log \frac{u_i}{v_i} \) is the Kullback–Leibler divergence, i.e. the relative entropy. Let us first deal with the case \( n \geq 3 \). We obtain that
\[
\mathcal{J}(n, \lambda) = H(p(n, \lambda)) - H(q(n, \lambda))
\]
\[
\begin{align*}
&\geq 1 \geq D(q^\uparrow(n, \lambda)\|p^\uparrow(n, \lambda)) \\
&\geq \frac{1}{2 \log 2} \|q^\uparrow(n, \lambda) - p^\uparrow(n, \lambda)\|_1 \\
&\geq \frac{3}{\log 2} \left| q_{n+1}^\uparrow(n, \lambda) - p_{n+1}^\uparrow(n, \lambda) \right|^2 \\
&= \frac{4}{\log 2} \left| p_{n-1}(n, \lambda) - q_{n+1}(n, \lambda) \right|^2 \\
&\geq \frac{5}{2} \left( \frac{(n+1)(n-2)}{4n(n-1)} \right) \left( 1 - \frac{1}{n} \right)^{2n} \\
&\geq 6 \frac{32}{6561 \log 2} > 0.
\end{align*}
\]

Here, 1 comes from applying the Ho–Verdú inequality \[S83\] to the case of \( r = p(n, \lambda) \) and \( s = q(n, \lambda) \), which is possible by Proposition \[S20\]. The estimate in 2 is just Pinsker’s inequality (see \[67, \text{p.58}\] and references therein). In 3 we estimated the total variation or \( L_1 \) distance between \( p^\uparrow(n, \lambda) \) and \( q^\uparrow(n, \lambda) \) from below as twice their \( L_\infty \) distance, namely
\[
\|q^\uparrow(n, \lambda) - p^\uparrow(n)\|_1 \geq 2 \max_{\ell=0, \ldots, n-1} \left| q_{\ell+1}^\uparrow(n, \lambda) - p_{\ell+1}^\uparrow(n) \right| \geq \left| q_{n+1}^\uparrow(n, \lambda) - p_{n+1}^\uparrow(n) \right|.
\]

Then, in 4 we used Proposition \[S13\] and Lemma \[S16\] together with the observation that \( \lambda_+(n) \geq \frac{1}{n} \) for all \( n \geq 2 \). The estimate in 5 follows from \[S74\], while in 6 we noted that both \( n \mapsto \frac{(n+1)(n-2)}{4n(n-1)} \) and \( n \mapsto \left( 1 - \frac{1}{n} \right)^n \) are increasing function of \( n \) for \( n \geq 3 \), and therefore their product can be lower bounded by evaluating it for \( n = 3 \). Thus, \( \text{(S81)} \) holds with \( c_3 = \frac{32}{6561 \log 2} \).
It remains to deal with the \( n = 2 \) case. We can repeat the same reasoning as above all the way until step 5, where we have to use instead the estimate in (S75), thus obtaining
\[
\mathcal{F}(2, \lambda) \geq \frac{c^2}{8 \log 2} =: c_2 > 0,
\]
which proves (S82). Setting \( c := \min\{c_1, c_2, c_3\} \) completes the argument.

**Remark S21.** The optimal constant in Theorem 2 can be expressed as a function of the energy constraint \( N \) as
\[
c(N) := \inf_{0 < \lambda \leq 1} \sup_{0 < \lambda' \leq \lambda} Q(\Phi_{\lambda', \sigma}, N),
\]
where the supremum is over all single-mode states \( \sigma \). Using the explicit form of the Alicki–Fannes–Winter inequality [41, 65] could yield an explicit yet very small lower bound on \( c(1/2) \), something along the lines of \( c(1/2) \geq 6 \times 10^{-6} \). Numerical investigations suggest that this is very far away from the truth, and that one could take at least \( c(1/2) \geq 0.066 \), which is four orders of magnitude larger than the former estimate. This must be confronted with the ‘trivial’ upper bound descending from Lemma S22 which reads \( c(1/2) \leq g(1/2) \approx 1.377 \).

**Remark S22.** It is perhaps more interesting to look at the slightly different quantities
\[
c_0(N) := \lim_{\lambda \to 0^+} \sup_{0 < \lambda' \leq \lambda} Q(\Phi_{\lambda', \sigma}, N),
\]
which represent the best-case-scenario quantum communication rates when the transmissivity approaches 0 but the single-mode environment state \( \sigma \) is chosen optimally. Since
\[
\lim_{n \to \infty} \frac{(n + 1)(n - 2)}{4n(n - 1)} \left( 1 - \frac{1}{n} \right)^n = \frac{1}{4e},
\]
it can be seen that our argument yields
\[
c_0(1/2) \geq \frac{1}{8e^2 \log 2} \approx 0.0244.
\]
Numerical investigations produce a substantially higher estimate \( c_0(1/2) \geq 0.133 \), which again must be confronted with the upper bound \( c_0(1/2) \leq g(1/2) \approx 1.377 \).

**F. Further considerations**

It turns out that one can get rid of the multiple options in Proposition S17 if one is willing to exclude the special cases \( n = 2 \) and \( n = 3 \). When this is done something more happens. Namely, the majorisation \( p(n, \lambda) \prec q(n, \lambda) \) of Proposition S20 is of a very special type. It actually holds that \( p^*_n(n, \lambda) \geq q^*_n(n, \lambda) \) for all \( n \geq 4 \) and \( \frac{1}{n+1} \leq \lambda \leq \frac{1}{n} \). Throughout this section we prove these claims.

**Lemma S23.** For all \( n \geq 4 \),
\[
p_{n+1}(n, \lambda) \leq p_{n-2}(n, \lambda) \quad \forall \quad \frac{1}{n+1} \leq \lambda \leq 1.
\]

**Proof.** Employing the expressions (S47), we see that
\[
\frac{p_{n-2}(n, \lambda)}{p_{n+1}(n, \lambda)} = \frac{n(n-1)}{6(n+1)} \frac{\lambda}{(1-\lambda)^2} \left( 3 + \frac{1}{1-\lambda}(3-(n+1)\lambda)^2 \right) =: \frac{n(n-1)}{6(n+1)} g_n(\lambda).
\]
It is not difficult to see that
\[
\frac{dg_n(\lambda)}{d\lambda} = \frac{6}{(1-\lambda)^4} \left( \frac{1}{2}(n^2 - 2)\lambda^2 - (2n - 1)\lambda + 2 \right) \geq 0 \quad \forall \quad 0 \leq \lambda \leq 1,
\]
because the discriminant of the second-degree polynomial on the right-hand side equals \( 9 - 4n \) and is therefore negative as long as \( n \geq 3 \). Thus,
\[
\min_{\frac{1}{n+2} \leq \lambda \leq 1} \left\{ \frac{p_{n-2}(n, \lambda)}{p_{n+1}(n, \lambda)} - 1 \right\} = \frac{n(n-1)}{6(n+1)} \frac{1}{g_n} \left( \frac{1}{n+1} \right) - 1 = \frac{1}{6n^2} (n+1)(n-4) \geq 0
\]
for all \( n \geq 4 \).
Proposition S24. For all \( n \geq 4 \),

\[
p^1(n, \lambda) = \left( p_0(n, \lambda), p_1(n, \lambda), \ldots, p_{n-3}(n, \lambda), p_n(n, \lambda), p_{n+1}(n, \lambda), p_{n-2}(n, \lambda), p_{n-1}(n, \lambda) \right) \quad \forall \frac{1}{n+1} \leq \lambda \leq \tilde{\lambda}_+(n),
\]

where \( \tilde{\lambda}_+(n) \) is defined by (S70). In other words, for the stated range of values of \( \lambda \) the probability vector \( p(n, \lambda) \) can be sorted in ascending order by exchanging the last two pairs of entries.

Proof. It suffices to combine Lemmata S14, S15 and S23. Note that \( \tilde{\lambda}_+(n) \leq \lambda_+(n) \) for all \( n \geq 4 \). This can be shown e.g. by noting that

\[
\tilde{\lambda}_+(n) \leq \frac{3 - \sqrt{3}}{n + 2} \leq \lambda_+(n) \quad \forall n \geq 18,
\]

where the first relation is equivalent to a linear inequality upon elementary algebraic manipulations, while the second is easily seen to hold for all \( n \geq 1 \) by direct inspection of (S58). In the remaining cases \( n = 4, \ldots, 17 \), the fact that \( \lambda_+(n) \leq \lambda_+(n) \) can be checked numerically.

Now that the probability distribution \( p(n, \lambda) \) has been sorted in ascending order by a fixed permutation, we proceed to check that indeed \( p_i^1(n, \lambda) \geq q_i^1(n, \lambda) \) for all \( n \geq 4 \) and \( \frac{1}{n+1} \leq \lambda \leq \frac{1}{n} \).

Lemma S25. For all \( n \geq 2 \),

\[
q_{n-2}(n, \lambda) \leq p_n(n, \lambda) \quad \forall \frac{1}{n+1} \leq \lambda \leq \frac{1}{n}.
\]

Proof. One verifies that

\[
\frac{\partial}{\partial \lambda} (p_n(n, \lambda) - q_{n-2}(n, \lambda)) = \frac{1}{12} (1 - \lambda)^{n-3} (2 - (n+1)\lambda) (6 - 3(n+6)\lambda - 3(n^2 - 4n - 6)\lambda^2 + (n+1)(n+2)(n-3)\lambda^3)
\]

\[
= \frac{1}{12} (1 - \lambda)^{n-3} (2 - (n+1)\lambda) h_n(\lambda).
\]

We will now show that \( h_n(\lambda) \geq 0 \) for all \( \frac{1}{n+1} \leq \lambda \leq \frac{1}{n} \). To this end, compute

\[
\frac{d^2 h_n(\lambda)}{d\lambda^2} = -(n^2 - 4n - 6) + (n+1)(n+2)(n-3)\lambda \geq 0 \quad \forall \frac{1}{n+1} \leq \lambda \leq \frac{1}{n}.
\]

where the last inequality holds because: (i) it can be verified explicitly for \( n = 2 \) and \( n = 3 \); (ii) for \( n \geq 4 \), one has that

\[
\frac{n^2 - 4n - 6}{(n+1)(n+2)(n-3)} \leq \frac{1}{n+1} \quad \forall n \geq 4,
\]

with equality for \( n = 4 \). Since we have shown that \( \frac{dh_n(\lambda)}{d\lambda} \) is increasing in \( \lambda \) on \( \left[ \frac{1}{n+1}, \frac{1}{n} \right] \), there it holds that

\[
\frac{1}{3} \frac{dh_n(\lambda)}{d\lambda} \leq \left. \frac{1}{3} \frac{dh_n(\lambda)}{d\lambda} \right|_{\lambda=1/n} = \left. \frac{1}{3} \frac{dh_n(\lambda)}{d\lambda} \right|_{\lambda=1/n} = -2(n-1) + \frac{5}{n} - \frac{6}{n^2} \leq -(n-1) \left( 2 - \frac{5}{n^2} \right) \leq 0.
\]

Thus, \( h_n(\lambda) \) is decreasing in \( \lambda \) on \( \left[ \frac{1}{n+1}, \frac{1}{n} \right] \). From this we deduce that

\[
h_n(\lambda) \geq h_n(\frac{1}{n}) = \frac{(n-1)(n-2)(n-3)}{n^3} \geq 0
\]

for all \( n = 2, 3, 4, \ldots \).
Lemma S26. For all $n \geq 2$,
\[ q_{n-1}(n, \lambda) \leq p_{n+1}(n, \lambda) \quad \forall \quad \frac{1}{n+1} \leq \lambda \leq \frac{2}{n+2}. \]  
(S89)

Proof. A simple calculation shows that
\[ 2(n+1)(1-\lambda)^2 \left(1 - \frac{q_{n-1}(n, \lambda)}{p_{n+1}(n, \lambda)}\right) = (n-1) ((n+1)\lambda - 1) (2 - (n+2)\lambda) \geq 0 \quad \forall \quad \frac{1}{n+1} \leq \lambda \leq \frac{2}{n+2}, \]
completing the proof.

Lemma S27. For all $n \geq 4$,
\[ q_n(n, \lambda) \leq p_{n-2}(n, \lambda) \quad \forall \quad \frac{1}{n+1} \leq \lambda \leq \frac{1}{n}. \]  
(S90)

Proof. One finds that
\[ \frac{n(n-1)}{6} \lambda^2 (3(1-\lambda) + (3 - (n+1)\lambda)^2) \left(1 - \frac{q_n(n, \lambda)}{p_{n-2}(n, \lambda)}\right) = -1 + (n+4)\lambda + (n^2 - 6n - 6) \lambda^2 - (n+1) \left(n^2 - \frac{5}{2} \right) \lambda^3 + \frac{1}{6} (n+1)^2(n+2)(n-3)\lambda^4 \]
\[ =: r_n(\lambda). \]
We look at the polynomial $r_n(\lambda)$ and its derivatives in the interval \( \left[ \frac{1}{n+1}, \frac{1}{n} \right] \). Since \( \frac{d^2r_n(\lambda)}{d\lambda} = 4(n+1)^2(n+2)(n-3) \geq 0 \), the function \( \frac{d^2r_n(\lambda)}{d\lambda} \) is convex. Therefore, on the larger interval \( \left[ \frac{1}{n+1}, \frac{2}{n+1} \right] \supset \left[ \frac{1}{n+1}, \frac{1}{n} \right] \) it holds that
\[ \frac{d^2r_n(\lambda)}{d\lambda} \leq \max \left\{ \frac{d^2r_n(\lambda)}{d\lambda} \bigg|_{\lambda=1/(n+1)}, \frac{d^2r_n(\lambda)}{d\lambda} \bigg|_{\lambda=2/(n+1)} \right\} \]
\[ = \max \{ -n(2n-1), -2(n-2)(n-3) \} \]
\[ \leq 0. \]
In turn, this tells us that $r_n(\lambda)$ is concave. Thus, on \( \left[ \frac{1}{n+1}, \frac{1}{n} \right] \) it holds that
\[ r_n(\lambda) \leq \min \left\{ r_n \left( \frac{1}{n+1} \right), r_n \left( \frac{1}{n} \right) \right\} \]
\[ = \min \left\{ \frac{n(n-4)}{6(n+1)^2}, \frac{(n-1)(n-2)(n^2 + n - 3)}{6n^4} \right\} \]
\[ \geq 0. \]
This proves the claim.

We are finally ready to prove our last claim.

Proposition S28. Let $n \geq 4$ be an integer. Then
\[ p_\ell^\dagger(n, \lambda) \geq q_\ell^\dagger(n, \lambda) \quad \forall \quad \ell = 0, \ldots, n, \quad \forall \quad \frac{1}{n+1} \leq \lambda \leq \frac{1}{n}. \]  
(S91)

with the reverse inequality holding instead for $\ell = n+1$. In particular, $p(n, \lambda) \prec q(n, \lambda)$ for all $\frac{1}{n+1} \leq \lambda \leq \frac{1}{n}$.

Proof. Since $\tilde{\lambda}_*(n) \geq \frac{1}{n}$ for all $n \geq 1$, Proposition S24 applies and tell us that the ordering of $p(n, \lambda)$ is as in (S87). Now, the cases $\ell = 0, \ldots, n-3$ of (S91) follow from Lemma S18 as usual. When $\ell = n-2$, we have instead to verify that $p_n(n, \lambda) \geq q_{n-2}(n, \lambda)$, which is a consequence of Lemma S25. For $\ell = n-1$, the claim amounts to the inequality $p_{n+1}(n, \lambda) \geq q_{n-1}(n, \lambda)$, which holds by Lemma S26 because $\frac{n-1}{n+2} \geq \frac{1}{n}$ whenever $n \geq 2$. The last case is $\ell = n$, for which we have to show that $p_{n-2}(n, \lambda) \geq q_n(n, \lambda)$; this is guaranteed to hold by Lemma S27. The reverse inequality holds for $\ell = n+1$ by normalisation:
\[ p_{n+1}(n, \lambda) = 1 - \sum_{\ell=0}^{n} p_\ell^\dagger(n, \lambda) \geq 1 - \sum_{\ell=0}^{n} q_\ell^\dagger(n, \lambda) = q_{n+1}(n, \lambda). \]
Finally, majorisation follows by direct inspection.