A non incremental variational principle for brittle fracture

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Abstract

The aim of the paper is to propose a paradigm shift for the variational approach of brittle fracture. Both dynamics and the limit case of statics are treated in a same framework. By contrast with the usual incremental approach, we use a space-time principle covering the whole loading and crack evolution. The emphasis is given on the modelling of the crack extension by the internal variable formalism and a dissipation potential as in plasticity, rather than Griffith’s original approach based on the surface area. The new formulation appears to be more fruitful for generalization than the standard theory.

Keywords: Linear elastic fracture mechanics, brittle material, symplectic mechanics, calculus of variation

1 Introduction

Many dynamical systems are subjected to energy loss resulting from dissipation, for instance collisions, surface friction, viscosity, plasticity, fracture and damage. On the other hand, deformations of solids and motions of fluids are modeled through constitutive laws. Due to collisions, brittle fracture and threshold effects, most dissipative laws are non smooth and multivalued. Moreover, experimental testing suggests that convexity is a keystone property of these phenomenological laws. Continuum mechanics with internal variables provides a convenient foundation to develop constitutive models that describe the inelastic behavior of various materials, despite the large differences in their physical nature and the relevant scale. The phenomenological law link the vector of internal variable rates $\dot{\alpha}$ to the dual force vector $A$. The theory of Generalized Standard Materials [41] is based on an hypothesis of normal dissipation.

$\dot{\alpha} \in \partial \varphi(A)$
where occurs the subdifferential of a convex and lower semicontinuous function $\varphi$, not everywhere differentiable. Equivalently, the law reads

$$A = \text{argmax} (\langle \dot{\alpha}, A \rangle - \varphi(A))$$

or, introducing the Fenchel polar $\varphi^*$

$$(\dot{\alpha}, A) = \text{argmin} (\varphi(A) + \varphi^*(\dot{\alpha}) - \langle \dot{\alpha}, A \rangle)$$

that does not favour one of the two dual variables. In the present work, the latest formulation is chosen as starting point, leading to variational methods and unconstrained or constrained optimization problems. Among them, Brezis-Ekeland-Nayroles principle [13], or in short BEN principle, is based on the time integration of the sum of dissipation potential $\phi$ and its Fenchel polar (analogous of Legendre polar for convex functions). Although used a few in the literature, this principle is noteworthy in the sense that it allows covering the whole evolution of the dissipative system.

In a previous paper [15], Buliga proposed the formalism of Hamiltonian inclusions, able to model dynamical systems with 1-homogeneous dissipation potential (for laws such as brittle damage using Ambrosio-Tortorelli functional [3]). This formalism is a dynamical version of the quasi-static theory of rate-independent systems of Mielke (55, 56, 57).

Latter on, Buliga and the author merged in a symplectic framework this formalism with BEN principle to extend it to dynamics [16]. The key-idea is to decompose additively the time rate $\dot{z}$ into reversible part $\dot{z}_R$ (the symplectic gradient) and dissipative or irreversible one $\dot{z}_I$, next to define the symplectic subdifferential $\partial^\omega \phi(z)$ of the dissipative potential. To release the restrictive hypothesis of 1-homogeneity (in particular to address viscoplasticity), we introduce in this work the symplectic Fenchel polar $\phi^{\ast \omega}$, that allows to build theoretical methods to model and analyse dynamical dissipative systems in a consistent geometrical framework with the numerical approaches not very far in the background. Numerical simulation with the BEN principle were performed for elastoplastic structures in statics [17, 19] and in dynamics [18].

Closer to this approach, we can cite the contributions of Aubin [5], Aubin, Cellina and Nohel [4], Rockafellar [66], Stolz [70], which considered various extensions of Hamiltonian and Lagrangian mechanics. In the article [7] by Bloch, Krishnaprasad, Marsden and Ratiu, Hamiltonian systems are explored with an added Rayleigh dissipation. A theory of quasistatic rate-independent systems is proposed by Mielke and Theil [55], Mielke [56], and developed towards applications in many papers, among them Mielke and Roubíček [57], see also Visintin [75]. In (59, 61), Grmela and Öttinger proposed the framework GENERIC (General Equation for Non-Equilibrium Reversible-Irreversible Coupling), a systematic method to derive thermodynamically consistent evolution equations. A variational formulation of GENERIC is proposed in 53. In 58, Mielke proposed a GENERIC formulation for Generalized Standard Materials quite similar to the one in the present paper. It would be worth to make the present formalism thermodynamic. In 62, Stefanelli used Brezis-Ekeland-Nayroles variational principle to represent the quasistatic evolution of an elastoplastic material with hardening in order to prove the convergence of time and space-time discretizations.
as well as to provide some possible a posteriori error control. Finally, Ghoussoub and MacCann characterized the path of steepest descent of a non-convex potential as the global minimum of Brezis-Ekeland-Nayroles functional [37].

Moreover, another advantage of Brezis-Ekeland-Nayroles principle is the easiness to be generalized. Indeed, it is worth to know that many realistic dissipative laws, called non-associated, cannot be cast in the mould of the standard ones deriving of a dissipation potential. To skirt this pitfall, the author proposed in [24] a new theory based on a function called bipotential. It represents physically the dissipation and generalizes the sum of the dissipation potential and its Fenchel polar, reason for which extension of Brezis-Ekeland-Nayroles principle is natural. The applications of the bipotential approach to solid Mechanics are various: Coulomb’s friction law [25], non-associated Drücker-Prager [26] and Cam-Clay models [78] in Soil Mechanics, cyclic Plasticity ([25, 9, 10, 54, 11]) and Viscoplasticity [44] of metals with non linear kinematical hardening rule, Lemaitre’s damage law [8], the coaxial laws ([27, 73]). Such kind of materials are called implicit standard materials. A synthetic review of these laws can be found in the two later references. It is also worth to notice that monotone laws but which does not admit a convex potential can be represented by Fitzpatrick’s function [35] which is a bipotential.

There is an abundant literature on the crack propagation criteria and our intention is not to draw here an exhaustive picture. For a deep survey, the reader is referred to [61]. Let us cite only the most popular ones: the maximum tensile hoop stress criterion formulated in 1963 by Cherepanov [21], Erdogan and Sih [32], the criterion of minimum strain energy density by Sih [68] in 1973, the principle of local symmetry formulated in 1974 by Goldstein and Salganik [38], the maximum energy release rate criterion considered in 1974 by Hussain et al. [45] and in 1978 by Wu [76], and the generalized maximum energy release rate criterion [42, 40, 47, 48]. These criteria give predictions more or less closed to the experimental results and the possible link with the variational approach is puzzling, at least apparently. This is one of the challenge we have to address.

Our aim here is to study the propagation of already initiated cracks. Then we exclude of this paper the problem of the crack nucleation or initiation that would require extra information about stresses in the damaged region. Two models are used in the literature, the Coupled Criterion proposed par Leguillon [50] and the Cohesive Zone Model, originated in the pioneer works by Barenblatt [6] and Dugdale [30], next developed by Tvergaard and al [72], Xu [77] and, for the variational aspects, by Bourdin et al [12].

The paper is organized as follows.

• In Section 2, the initial crack and the loading history being known, the problem is to find the crack evolution. Instead of the usual modelling of the crack extension as a time-parameterized family of surfaces, we propose to introduce a flow of a vector field on the final crack manifold.

• Section 3 is devoted to the definition of the variable of the problem in the symplectic framework: the displacement and flow fields and the corresponding dynamic momenta.
In Section 4, the symplectic BEN principle is applied to the fracture mechanics, that leads to the definition of a driving force, dual of the crack flow rate.

In Section 5, we propose a constitutive law for brittle fracture in the form of a crack stability criterion and the normality law for the crack extension rule.

We prove in Section 6 for structure with uniform toughness the equivalence with the classical variational approach in the form of a Munford-Shah functional.

In Section 7, we deduce from the space-time principle an incremental version suitable for the usual step-by-step numerical approaches.

In Section 8, we show how a suitable interpretation of the experimental data prove the relevancy of the normality law and the link with the principle of local symmetry.

Section 9 is devoted to the calculation of the crack driving force using the calculus of variation on the jet space of order one in the general case of dynamics.

2 Modelling of the crack extension

2.1 Data

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set, with piecewise smooth boundary $\partial \Omega$ occupied by the uncracked body (Figure 2.1). Our aim is to study a crack extension during a time interval $[0, T]$.

The data of the problem is:

- the initial crack $\Gamma_0$
- the imposed displacements $\bar{u}$ on the part $\partial \Omega_0$ of the boundary (the supports)
- the surface forces $\bar{f}$ on the remaining part $\partial \Omega_1$
- the volume forces $f$

All the external actions, $\bar{u}$, $\bar{f}$ and $f$, are prescribed during the time interval. No stress is transmitted through the crack.

2.2 Crack

A crack bifurcates when its shape suffers a change of topology. The most common example is a crack which develops in time new branches. During this phenomenon the number of crack fronts increases. In this paper we do not study the bifurcation of an existing crack.

We denote $S$ the set of admissible surfaces, i.e. closed countably 2 rectifiable subsets of $\Omega$ without change of topology. The crack evolution is described by a time-parameterized family of surfaces (Figure 2.1)

$$ \Gamma : [0, T] \rightarrow S : t \mapsto \Gamma_t = \Gamma(t) $$
such that $\Gamma(0) = \Gamma_0$ and the map $\Gamma$ is monotone increasing, $S$ being equipped with the inclusion order. The crack front $c_t$ at time $t$ is parameterized by the arc length $s$.

The cornerstone of the formulation is to model the crack extension by a flow on the cracked surface $\Gamma_T \setminus \Gamma_0$ during the time interval

$$[0, T] \times c_0 \to \Gamma_T \setminus \Gamma_0 : (t, x_0) \mapsto x = \psi(t, x_0)$$

such that

$$\psi(\{t\} \times c_0) = c_t, \quad \psi([0, t] \times c_0) = \Gamma_t \setminus \Gamma_0$$

It is worth to remark that:

- From a calculus viewpoint, the field $\psi$ represents both the crack and its evolution. The main advantage is that it is easier working with fields living in a functional space than with surfaces.

- In particular, our approach is well suited for numerical applications. $s \mapsto x_0 = f(s)$ being the parameterization of the initial crack $c_0$ by the arc length $s$,
the cracked surface can be parameterized according to \((t, s) \mapsto x = \tilde{\psi}(t, s) = \psi(t, f(s))\). The node \(x_{jk} = \tilde{\psi}(t_j, s_k)\) being the position at time \(t_j\) of the node \(x_{0k} = f(s_k) \in c_0\) and \((s, t) \mapsto N_{jk}(t, s)\) the corresponding shape function, the cracked surface \(\Gamma_t\) is parameterized by
\[
x = \tilde{\psi}(t, s) = \sum_j \sum_k N_{jk}(t, s) \ x_{jk}
\] (2)

In the forthcoming variational formulation, the nodal values \(x_{jk} = \tilde{\psi}(t_j, s_k)\) will be unknowns of the discretized problem.

• A streamline \(t \mapsto x = \psi(t, x_0)\) does not correspond to the motion of a material particle initially at the position \(x_0 \in c_0\). The flow is not Lagrangian but Eulerian in the sense it represents just the evolution of the crack front.

If the cracked surface is smooth enough, it is generated by a vector field \(\dot{\psi}\), solution on \(\Gamma_t \setminus \Gamma_0\) of the ODE (Figure 2.1)
\[
\frac{d}{dt}(\psi(t, x_0)) = \dot{\psi}(\psi(t, x_0)) \quad \text{with} \quad \psi(0, x_0) = x_0
\]

As in the FEM method, it is easy to compute the vector field \(\dot{\psi}\) as the partial derivative of (2) with respect to \(t\).

3 Variables of the problem

To do not enter into too complex formulations, we consider the case of an elastic body in small strains, although the generalization to the elastoplasticity and the finite strains is rather straightforward. The cracked solid at time \(t\) is denoted \(\Omega_t = \Omega \setminus \Gamma_t\). Within the body, the displacement field at time \(t\) is
\[
\Omega_t \rightarrow \mathbb{R}^3 : x \mapsto u(t, x)
\]

According to the symplectic BEN principle [16], the variables of the problem of the elastodynamics are:

• the field couple \(\xi = (u, \psi)\) of the displacement and the flow

• the momenta \(\eta = (p, \pi)\) where \(p\) is the classical linear momentum and \(\pi\) is the momentum associated to the flow

The system evolution is given by a curve
\[
t \mapsto z(t) = (\xi(t), \eta(t))
\]
4 The symplectic BEN variational principle

The duality between the space $X$ of degrees of freedom $\xi$ and the space $Y$ of momenta $\eta$ has the form

$$\langle \dot{\xi}, \dot{\eta} \rangle = \langle \dot{u}, \dot{p} \rangle + \langle \dot{\psi}, \dot{\pi} \rangle = \int_{\Omega} \dot{u} \cdot \dot{p} \, d^3x + \int_{\partial \Omega} \dot{\psi} \cdot \dot{\pi} \, ds$$

The symplectic form is the 2-form field on $X \times Y$ defined by

$$\omega(\dot{z}, \dot{z}') = \omega((\dot{\xi}, \dot{\eta}), (\dot{\xi}', \dot{\eta}')) = \langle \dot{\xi}, \dot{\eta}' \rangle - \langle \dot{\xi}', \dot{\eta} \rangle$$

The total Hamiltonian of the structure is taken of the integral form

$$H_t = H(t, z) = \int_{\Omega} \left\{ \frac{1}{2\rho} \| p \|^2 + w(\nabla_s u) - f(t) \cdot u \right\} \, d^3x - \int_{\partial \Omega_1} f(t) \cdot u \, d(\partial \Omega_1) \quad (3)$$

The first term is the kinetic energy. The second one is the elastic strain energy $w$ depending on the strain tensor $\varepsilon = \nabla_s u$ and the two latter terms are the works of the external forces. As usual, the stress tensor is

$$\sigma = \frac{\partial w}{\partial \varepsilon}$$

A crack being a material discontinuity, we must distinguish two material surfaces $\Gamma^+_t$ and $\Gamma^-_t$ that occupy the same position as $\Gamma_t$ but are the two sides of the crack and have opposite unit normal vectors, exterior to $\Omega_t$: $n^+ = -n^-$. A curve is said admissible if it satisfies

- the boundary conditions:
  \[\forall t \in [0, T], \quad u = \bar{u} \text{ on } \partial \Omega_0, \quad \sigma \cdot n = \bar{f} \text{ on } \partial \Omega_1, \quad \sigma \cdot n^\pm = 0 \text{ on } \Gamma^\pm_t\]

- the initial conditions: $z(0) = z_0$

The SBEN formalism of dissipative media is based on the additive decomposition of the velocity into reversible and irreversible parts

$$\dot{z} = \dot{z}_R + \dot{z}_I$$

As the variables of the problem are fields, we use functional derivatives

$$\langle \delta \xi, D_\xi H + D_\eta H, \delta \eta \rangle = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (H(t, z + \epsilon \delta z) - H(t, z))$$

The reversible part of the velocity is given by the symplectic gradient of the Hamiltonian (or Hamiltonian vector field)

$$\dot{z}_R = \nabla^\omega H = (D_\eta H, -D_\xi H) = ((D_p H, D_\pi H), (-D_u H, -D_\psi H))$$
For the Hamiltonian (3), we obtain

\[ D_p H = \frac{p}{\rho}, \quad D_\pi H = 0 \]

\[ -D_u H = \nabla \cdot \sigma + f, \quad -D_\psi H = G \]

The last equation is just a definition of the driving force \( G \) of which the explicit expression will be discussed afterwards.

In [16], we formulated a general variational principle for the dynamical dissipative systems which claims that

**Symplectic BEN principle 1.** *The natural evolution of the system minimizes the functional*

\[ \Pi(z) = \int_0^T \{ \phi(\dot{z}) + \phi^*\omega(\dot{z}_I) - \omega(\dot{z}_I, \dot{z}) \} \, dt \]

among the admissible curves \( t \mapsto z(t) \) and the minimum is zero.

In this principle occurs the symplectic form \( \omega \), the Hamiltonian through the irreversible part of the velocity \( \dot{z}_I = \dot{z} - \nabla \cdot \sigma H \), a convex dissipation potential \( \phi \) and its symplectic polar \( \phi^*\omega \). We calculate now the detailed expression of this ingredients for the brittle fracture. We start with a simplifying hypothesis similar to the one introduced in [16] to recover the classical elastoplasticity. We claim that in the dissipation potential, the variables other than \( \dot{\pi} \) are ignorable

\[ \phi(\dot{z}) = \varphi(\dot{\pi}) \]

The symplectic polar is defined in [16] as

\[ \phi^*\omega(\dot{z}') = \sup \left[ \omega(\dot{z}', \dot{z}) - \phi(\dot{z}) : \dot{z} \right] \]

Under the previous hypothesis, one has

\[ \phi^*\omega(\dot{z}) = \sup \left[ \langle \dot{u}', \dot{p} \rangle + \langle \dot{\psi}', \dot{\pi} \rangle - \langle \dot{u}, \dot{p}' \rangle - \langle \dot{\psi}, \dot{\pi}' \rangle - \varphi(\dot{\pi}) : \dot{z} \right] \]

that gives

\[ \phi^*\omega(\dot{z}') = \chi_{\{0\}}(\dot{u}') + \varphi^*(\dot{\psi}') + \chi_{\{0\}}(\dot{p}') + \chi_{\{0\}}(\dot{\pi}') \]

where \( \chi_K \) is the indicatory function of the set \( K \), equal to 0 in \( K \) and \( +\infty \) otherwise, and \( \varphi^* \) is the Fenchel polar of \( \varphi \). The second term in the functional (4) becomes:

\[ \phi^*\omega(\dot{z}_I) = \chi_{\{0\}} \left( \dot{u} - \frac{p}{\rho} \right) + \varphi^*(\dot{\psi}) + \chi_{\{0\}}(\dot{p} - \nabla \cdot \sigma - f) + \chi_{\{0\}}(\dot{\pi} - G) \]

As the minimum of the functional is the finite value zero, it will be reached when the arguments of the indicatory functions vanish. Then we may take the zero value of the indicatory functions in the functional while we introduce extra corresponding constraints

\[ p = \rho \dot{u}, \quad \nabla \cdot \sigma + f = \dot{p}, \quad \dot{\pi} = G \]
Next we can transform the last term in the functional, taking into account the velocity decomposition, the linearity and the antisymmetry of the symplectic form

$$-\omega(\dot{z}_I, \dot{z}) = -\omega(\dot{z} - \dot{z}_R, \dot{z}) = -\omega(\dot{z}, \dot{z}) + \omega(\dot{z}_R, \dot{z}) = \omega(\dot{z}_R, \dot{z}) \tag{6}$$

or in detail

$$-\omega(\dot{z}_I, \dot{z}) = \left\langle \frac{\partial}{\partial \rho}, \dot{p} \right\rangle - \left\langle \dot{u}, \nabla \cdot \sigma + f \right\rangle - \left\langle \dot{\psi}, G \right\rangle$$

Owing to the constraints (5), it holds

$$-\omega(\dot{z}_I, \dot{z}) = \left\langle \dot{u}, \dot{p} - \nabla \cdot \sigma - f \right\rangle - \left\langle \dot{\psi}, G \right\rangle = - \left\langle \dot{\psi}, G \right\rangle$$

Moreover, thanks to the former and latter constraints in (5), the momenta can be eliminated from the functional and the intermediate constraint becomes

$$\nabla \cdot \sigma + f = \rho \ddot{u}$$

while the initial condition on the linear momentum can be transformed into an initial condition on the velocity because \( p(0) = \rho \dot{u}(0) \). Taking into account these transformations, the minimum can be searched only on the space of the degrees of freedom \( u \) and \( \psi \). Then we obtain a second version of the variational principle

**Symplectic BEN principle 2.** The natural evolution of the system minimizes

$$\Pi(\xi) = \int_0^T \left\{ \varphi(G) + \varphi^*(\dot{\psi}) - \left\langle \dot{\psi}, G \right\rangle \right\} \, dt \tag{7}$$

among the admissible curves \( t \mapsto \xi(t) \) such that \( \nabla \cdot \sigma + f = \rho \ddot{u} \) and the minimum is zero.

It could seem puzzling that the displacement does not appear explicitly in the expression of the functional but in fact \( u \) and \( \psi \) are coupled in the minimization problem:

- \( \psi \) is controlled by \( G \) which depends on \( u \) as derivative of the Hamiltonian.
- \( u \) must satisfy the constraint and the admissibility conditions that are defined on \( \Omega_t = \Omega \setminus \Gamma_t \) controlled by \( \psi \).

5 Constitutive laws for brittle fracture

As discussed in the Introduction, most of the crack stability criteria in the literature, in particular the most popular, are of local nature and not variational. Nevertheless, Strifors proposed in [71] to predict the onset of brittle fracture and the extension direction thanks to the crack extension force that can be identified to the driving force \( G \). Now we show how to cast it into the mold of our variational approach. Based on the works by Strifors [71] and Hellen et al. [43], we propose a simple constitutive law. At least for plane problems, it seems reasonable to think that the influence on the fracture
extension of the projection of $G$ onto the tangent to the crack front can be neglected. Then we introduce the deviatoric force

$$G_\perp = G - (G \cdot \tau) \tau$$

where $\tau$ is the unit tangent vector to the crack front in the direction of increasing arc length (Figure 2.1). As in the FEM method, it is easy to compute $\tau$ from the partial derivative of $\mathbf{2}$ with respect to the arc length $s$ of $c_0$. If we assume that the toughness of the material is isotropic, the critical energy release rate $G_c$ being a parameter measuring it, we introduce the convex crack stability domain

$$K = \{ G \text{ such that } \| G_\perp \| \leq G_c \}$$

and the crack extension rule

$$\dot{\psi} \in \partial \varphi(G) = \partial \chi_K(G)$$

that is the normality law

$$\begin{align*}
\text{if } & G \in \bar{K} \text{ then } \\
& \dot{\psi} = 0 \\
\text{else } & G \in \partial K \text{ and } \exists \lambda \geq 0, \quad \dot{\psi} = \lambda G_\perp \quad \text{! crack extension}
\end{align*}$$

and $\dot{\psi}$ is perpendicular to $\tau$ when the crack extends. The Fenchel polar is the support function

$$\varphi^*(\dot{\psi}) = \int_{c_t} G_c \| \dot{\psi} \| \, ds$$

where $s$ is the arc length of $c_t$. With this choice of constitutive law, we particularize the variational principle in the form

**Symplectic BEN principle 3.** The natural evolution of the system minimizes

$$\Pi(\xi) = \int_0^T \left\{ \int_{c_t} G_c \| \dot{\psi} \| \, ds - \left\langle \dot{\psi}, G \right\rangle \right\} \, dt$$

among the admissible curves $t \mapsto \xi(t)$ such that $\nabla \cdot \sigma + f = \rho \ddot{u}, \| G_\perp \| \leq G_c$ and the minimum is zero.

6 Link with the classical variational approach to fracture

First we recall that the last term in (8) is in fact the last one of the integrand of (4). Owing to (6), we have

$$\omega(\dot{z}_I, \ddot{z}) = \omega(\dot{z}_R, \ddot{z}) = \langle D_{\eta}H, \dot{\eta} \rangle - \langle \dot{\xi}, -D_{\xi}H \rangle = \dot{H} - D_tH$$
Besides the crack surface element comprised between the crack fronts $c_t$ and $c_{t+dt}$, and the streamlines of $x_0$ and $x_0 + dx_0$ is an infinitesimal parallelogram with two sides supported by the vectors $d_1x = \tau \, ds$ and $d_2x = \dot{\psi} \, dt$. Its area is

$$d\Gamma = \parallel \tau \times \dot{\psi} \parallel \, ds \, dt = \parallel \tau \parallel \parallel \dot{\psi} \parallel \, ds \, dt = \parallel \dot{\psi} \parallel \, ds \, dt$$

because $\tau$ is a unit vector and $\dot{\psi}$ is perpendicular to $\tau$.

If we assume that the toughness properties of the material are homogeneous, the critical energy release rate is uniform

$$G_c(x) = G_c$$

The area of a surface will be denoted by $\parallel \cdot \parallel$. The variational principle can be recast as

**Symplectic BEN principle 4.** The natural evolution of the system minimizes

$$\Pi(\Gamma, u) = G_c \parallel \Gamma_T \setminus \Gamma_0 \parallel - \int_0^T D_t H \, dt + H_T - H_0$$

among the admissible curves $t \mapsto (\Gamma(t), u(t))$ such that $\nabla \cdot \sigma + f = \rho \ddot{u}$, $\parallel G_\bot \parallel \leq G_c$ and the minimum is zero.

Two cases must be considered in the applications:

- If the loading is controlled by the forces, the structural evolution is in general dynamic and the functional to minimize is

$$\Pi(\Gamma, u) = G_c \parallel \Gamma_T \setminus \Gamma_0 \parallel - \int_0^T D_t H \, dt + H_T - H_0$$

- If the loading is slow and controlled by the displacements (for instance, $\bar{f}$ is null, the support is divided into two disjoint parts, one of them is fixed to the foundation, the other one is loaded), the evolution is quasi-static and the functional to minimize is reduced to

$$\Pi(\Gamma, u) = G_c \parallel \Gamma_T \setminus \Gamma_0 \parallel + H_T - H_0$$

Regardless of the constant $H_0$, we recover the Munford-Shah functional [62, 36, 14, 12].

### Incremental method

Our goal now is to find an incremental or step-by-step formulation as a by-product of our general principle. As usual, the time interval $[0, T]$ is divided into $N$ small sub-intervals $[t_n, t_{n+1}]$ with $t_0 = 0$ and $t_N = T$. The value of a variable $a$ at time $t_n$ is denoted $a_n$. The time step is $\Delta t = t_{n+1} - t_n$ and $\Delta a = a_{n+1} - a_n$ is the increment of $a$. $\xi_n = (u_n, \psi_n)$ being known from the initial conditions for $n = 0$ or from the previous increment otherwise and the time step being prescribed, the problem is to calculate $\xi_{n+1}$.
or equivalently $\Delta \xi$. The idea is to apply the symplectic BEN principle 3 to the sub-interval $[t_n, t_{n+1}]$. The time step being small, the functional can be approximated by considering the integrand is constant. As for the radial return algorithm in plasticity, we use the implicit scheme by evaluating the crack extension rate $\psi$ and the driving force $G$ at the end of the step. The crack extension increment is

$$\Delta \psi = \dot{\psi}_{n+1} \Delta t = \lambda G_{n+1} \Delta t$$ \hspace{1cm} (10)

Hence the incremental principle reads

**Symplectic BEN principle 3*. The natural evolution of the system minimizes**

$$\Delta \Pi(\xi_{n+1}) = \int_{c_{t_{n+1}}} G_c \parallel \Delta \psi \parallel ds - \langle \Delta \psi, G_{n+1} \rangle$$ \hspace{1cm} (11)

among the $\xi_{n+1}$ such that $\nabla \cdot \sigma_{n+1} + f_{n+1} = \rho \ddot{u}_{n+1}$, $\parallel (G_{n+1})_\perp \parallel \leq G_c$ and the minimum is zero.

Likewise, the last version of the principle in the previous section gives rise to the incremental form

**Symplectic BEN principle 4*. The natural evolution of the system minimizes**

$$\Delta \Pi(\Gamma_{n+1}, u_{n+1}) = G_c \parallel \Delta \Gamma \parallel - (D_t H)_{n+1} \Delta t + \Delta H$$ \hspace{1cm} (12)

among the couples $(\Gamma_{n+1}, u_{n+1})$ such that $\nabla \cdot \sigma_{n+1} + f_{n+1} = \rho \ddot{u}_{n+1}$, $\parallel (G_{n+1})_\perp \parallel \leq G_c$ and the minimum is zero.

If the loading is slow and controlled by the displacements, the evolution is quasi-static and the functional to minimize is reduced to

$$\Delta \Pi(\Gamma_{n+1}, u_{n+1}) = G_c \parallel \Delta \Gamma \parallel + \Delta H$$

and we recover Buliga’s incremental formulation [14].

### 8 Comparison of the constitutive law with experiments

The aim of this section is to check the validity of the normality law with respect to the experimental results on PMMA specimens [65]. A good test is to consider an initial straight crack in a plate which, under mixed mode loading, may extend suddenly in a direction deviating of an angle $\beta$ from the original one by kinking (Figure 8.1). The length of the kink crack is taken very small. The idea is to use the incremental formulation of the previous section with only one step. To simplify the notations, the initial value $a_0$ of a given quantity is denoted simply $a$ while the value $a_1$ at the end of the step is denoted $a^*$. In Hellen et al. [43], Euler explicit scheme is used that can read with the simplified notations

$$\Delta \psi = \dot{\psi} \Delta t = \lambda G \Delta t$$

In the frame of origin $O$ at the initial crack tip, the axis $Ox_1$ in the direction ahead the initial crack and the axis $Ox_2$ perpendicular to $Ox_1$ within the plane (Figure 8.1), the driven force is given in terms of stress intensity factors (SIFs) by

$$G_1 = \frac{1}{E} (K_I^2 + K_{II}^2), \hspace{1cm} G_2 = -\frac{1}{E} 2 K_I K_{II}$$
where \( \kappa = 3 - 4 \nu \) in plane strain, \( \kappa = (3 - \nu)/(1 + \nu) \) in plane stress and

\[
\frac{1}{E} = \frac{(1 + \nu) (1 + \kappa)}{4 E}
\]

The deviation angle \( \beta \) of the kink crack is given by the slope of the driven force

\[
\beta = \tan^{-1} \left( -\frac{2 K_I K_{II}}{K_I^2 + K_{II}^2} \right)
\]

In mode \( I \), the formula gives \( \beta = 0^\circ \), according to the experience. For \( K_I = K_{II} \), \( \beta = -45^\circ \) but the experimental values are above \(-49^\circ\). In mode \( II \), \( \beta = 0^\circ \) while the experimental values are in the range from \(-70^\circ\) to \(-74^\circ\). Clearly, the predictions are disastrous. Our opinion is that the constitutive law must not be \textit{a priori} rejected but it is the explicit scheme which is problematic.

The problem of the kinked crack were studied by many authors. It is an awkward problem of Elasticity and only approximated formula were proposed to express the SIFs at the kink crack tip in terms of the SIFs at the initial crack tip. At the limit of vanishing length of the kink crack, we adopt in the sequel the expression proposed in [23]

\[
K_I^*(\beta) = \cos \frac{\beta}{2} K_I - 3 \sin \frac{\beta}{2} \cos \frac{\beta}{2} K_{II}
\]

\[
K_{II}^*(\beta) = \sin \frac{\beta}{2} \cos \frac{\beta}{2} K_I + \cos \frac{\beta}{2} \left( 1 - 3 \sin^2 \frac{\beta}{2} \right) K_{II}
\]

However, it is noticeable that a correction is proposed in [34] for the largest values of the \( |\beta| \) and the dependency with respect to the length of the kink crack is taken into account in [49, 2] but these improvements will be not considered here. With the simplified notations, the implicit scheme (10) reads

\[
\Delta \psi = \dot{\psi}^* \Delta t = \lambda G^* \Delta t
\]
In the frame of origin $K$ at the kink crack tip, the axis $Kx_1^*$ in the direction ahead the kink crack and the axis $Kx_2^*$ perpendicular to $Kx_1^*$ within the plane (Figure 8.1), the driven force is given in terms of stress intensity factors (SIFs) by

$$G_1^* = \frac{1}{E} [(K_1^*)^2 + (K_{II}^*)^2], \quad G_2^* = -\frac{1}{2} K_1^* K_{II}^*$$

The normality law (13) entails

$$E G_2^* = -2 K_1^* K_{II}^* = 0, \quad E \parallel G^* = (K_1^*)^2 + (K_{II}^*)^2, \quad (14)$$

To satisfy the former condition, two scenarios may be considered for the crack extension:

1. **Scenario 1.** The kink crack inclination is $\beta_1$, solution of $K_{II}^*(\beta_1) = 0$, and $E \parallel G^* = (K_1^*(\beta_1))^2$

2. **Scenario 2.** The kink crack inclination is $\beta_2$, solution of $K_1^*(\beta_2) = 0$, and $E \parallel G^* = (K_1^*(\beta_2))^2$

According to the stability criterion, the crack extends for the inclination $\beta_i$ with the maximum driven force magnitude. Then scenario 1 is realized if $(K_1^*(\beta_1))^2 > (K_{II}^*(\beta_2))^2$ and scenario 2 otherwise.

The admissible solutions must be searched in the interval $]-\pi, \pi[$. The mode $I$ singularity disappearing in compression, $K_1$ is positive. In mode $II$, the SIF $K_{II}$ may be positive or negative, depending on the sign of the shear loading. Before tackling the mixed mode, let us consider the limit case:

- **Mode I.** For scenario 1, there is a unique admissible solution $\beta_1 = 0^\circ$ and $(K_1^*(\beta_1))^2 = K_1^2$. For scenario 2, there is no admissible solution. Then the scenario 1 occurs, according to the experimental observations.

Let us examine now the general case ($K_1 K_{II} \neq 0$). For scenario 1, there are two admissible solutions

$$\beta_{1\pm} = \sin^{-1} \left( \frac{K_{II}^*}{K_1^*} ± \frac{3 K_{II}^*}{K_1^*} \sqrt{8 \left( \frac{K_{II}^*}{K_1^*} \right)^2 + 1} \right)$$

For scenario 2, there is a unique admissible solution

$$\beta_2 = 2 \tan^{-1} \left( \frac{K_1^*}{3 K_{II}^*} \right)$$

In particular, let us discuss the case:

- **Mixed mode $K_1 = K_{II} > 0$.** For scenario 1, there are two admissible solutions: $\beta_{1-} = \sin^{-1}(-4/5) = -53.13^\circ$ then $(K_1^*(\beta_{1-}))^2 = (1.847 K_1)^2$ and $\beta_{1+} = \sin^{-1}(1) = 90^\circ$ then $(K_1^*(\beta_{1+}))^2 = (-1.14 K_1)^2$. For scenario 2, there is a unique admissible solution $\beta_2 = 36.87^\circ$ and $(K_{II}^*(\beta_2))^2 = (0.948 K_I)^2$. Comparing the values of $(K_1^*(\beta_{1-}))^2$, $(K_1^*(\beta_{1+}))^2$ and $(K_{II}^*(\beta_2))^2$, we conclude that the scenario 1 occurs with the angle $\beta = \beta_{1-} = -53.13^\circ$ closed to the experimental values ($-49^\circ$ for $K_{II} = 0.874 K_I$ and values in the range from $-56^\circ$ to $-59^\circ$ for $K_{II} = 1.142 K_I$).
In this respect, it is worth to remark that, if the sign of $K_I$ is reversed, the sign of the kink angles are reversed too: $\beta_1^- = 53.13^\circ$, $\beta_1^+ = -90^\circ$ and $\beta_2 = -36.87^\circ$ but the corresponding values of $(K_I^*(\beta_1^-))^2$, $(K_I^*(\beta_1^+))^2$ and $(K_{II}^*(\beta_2))^2$ are the same. We conclude that the sign of the kink angle is reversed: $\beta = 53.13^\circ$.

Finally, let us deal with the other limit case:

- **Mode II.** For scenario 1, there are two admissible solutions $\beta_1^\pm = 2 \sin^{-1}(\pm 1/\sqrt{3}) = \pm 70.47^\circ$ and $(K_I^*(\beta_1^\pm))^2 = 4 K_{II}^2/3$. For scenario 2, there is a unique admissible solution $\beta_2 = 0^\circ$ and $(K_I^*(\beta_2))^2 = K_{II}^2$. Then the scenario 1 occurs and the sign of the kink angle depends of the one of mode $II$. For $K_{II} > 0$, the prediction $\beta = -70.47^\circ$ is just above the experimental values.

**Comment 1.** On the basis of the experimental data, Richard proposed an empirical criterion in terms of the SIFs at the original crack tip

$$\frac{K_I}{K_{Ic}} + \left(\frac{K_{II}}{K_{IIc}}\right)^2 = 1$$

According to (14), the crack extends when

$$(K_I^*)^2 + (K_{II}^*)^2 = \bar{E} G_c$$

that gives, for the scenario 1 of mode I, $K_I^2 = \bar{E} G_c$ then

$$K_{Ic}^2 = \bar{E} G_c$$

and, for the scenario 1 of mode II, $4 K_{IIc}^2/3 = \bar{E} G_c = K_{Ic}^2$ then

$$\frac{K_{IIc}}{K_{Ic}} = \frac{\sqrt{3}}{2} = 0.86$$

a value better than $\sqrt{2/3} = 0.81$ proposed in [34], by comparison with the experimental data covering the range from 0.88 to 0.95. Moreover, our prediction for the mixed mode $K_I = K_{II}$ leads to values $\frac{K_I}{K_{Ic}} = 0.54$ and $\frac{K_{IIc}}{K_{Ic}} = 0.62$ close to the curve of Richard criterion.

**Comment 2.** At least in this example, the relevant scenario corresponds always to the **Principle of Local Symmetry** $K_I^*(\beta) = 0$ proposed in 1974 by Goldstein and Salganik [38]. This principle is valid at least for a kinked straight crack with vanishing kink crack but there can be no assurance that it is valid for arbitrary cracks and we recommend to replace it by the normality law proposed in this paper.

**Comment 3.** The present criterion should not be confused with the **Maximum Energy Release Rate criterion**, proposed in 1974 by Hussain et al. [45], for which the maximum must be searched among all the directions of the crack extension while with our criterion the maximum is found only among the directions satisfying the normality law, according to the different scenarios. As argued in [20], the questions of when and how a crack propagates should be simultaneously investigated and the energy conservation is not sufficient for such a task.

**Comment 4.** This example shows also that the implicit scheme must be preferred to the explicit scheme in fracture mechanics.
9 Calculation of the crack driving force

As we are working at time $t$ in this Section, the dependency with respect to the time will be omitted. In order to obtain the expression of $G$, we use a special form of the calculus of variation performed on the jet space of order one ([1], [31], [52]). For more details about the jet spaces, the reader is referred for instance to [67]. The first jet prolongation of the smooth function $u : \Omega_t \rightarrow \mathbb{R}^3$ is the function $j^1 u$ from $\Omega_t$ into the jet bundle $J^1(\Omega_t, \mathbb{R}^3)$ such that

$$j^1 u (x) = (x, u(x), \nabla u(x))$$

The Hamiltonian (3) at time $t$ has the form

$$H = \int_{\Omega_t} h(x, u, \nabla u, p) \, d^3x - \int_{\partial \Omega_1} \bar{f} \cdot u \, d(\partial \Omega_1)$$

$$H = \int_{\Omega_t} h(j^1 u, p) \, d^3x - \int_{\partial \Omega_1} f \cdot u \, d(\partial \Omega_1)$$

The new viewpoint which consists in replacing the original field $u$ by its first jet prolongation $j^1 u$ leads to perform variations not only on the field and its derivatives but also on the variable $x$. We want to calculate the functional derivative of $H$ in the direction $\delta \psi$ defined on the crack front $c_t$. The idea is to prolong it locally by a field $\delta x$ on an open cylindrical neighbourhood $C_\eta$ of radius $\eta$ around the crack front, the field vanishing at radius $\eta$ and beyond (Figure 9.1). To avoid the singularity of the stress

![Diagram](image-url)
field, we replace $C_\eta$ by $C_{\eta,r} = C_\eta \setminus C_r$ where $C_r$ is an open cylindrical neighbourhood of radius $r < \eta$ around the crack front. Next, we pass to the limit $\eta \to r \to 0$.

We consider now a new parameterization given by a regular map $x = \theta(y)$ of class $C^1$ on $C_{\eta,r} = \theta^{-1}(C_{\eta,r})$ and we perform the variation of the function $\theta$, the new variable being $y$. The variation of the action is

$$\delta H = \delta \int_{C'_{\eta,r}} h(\theta(y), u, \nabla_y u \cdot \nabla_x y, (\nabla_x y)^T \cdot p') \det(\nabla_y x) \, d^3y$$

where $p'$ is the linear momentum in $y$ coordinates. We do not give here the details of the calculations that can be found in Appendix A but only indicate the sketch of the method and the result. As usual in calculus of variation, we integrate by part. Next we consider the limit case where the function $\theta$ approaches the identity of $\Omega_t$, $y$ approaches $x$, $p'$ approaches $p$ and $C'_{\eta,r}$ approaches $C_{\eta,r}$. Finally, the variation of the Hamiltonian has the form

$$\delta H = \int_{\partial C_{\eta,r}} n \cdot T \cdot \delta x \, d(\partial C_{\eta,\rho}) - \int_{C_{\eta,r}} (\nabla \cdot T + f) \cdot \delta x \, d^3x$$

where

$$T = h \mathbf{1}_{\mathbb{R}^3} - \sigma \cdot \nabla u - \dot{u} \otimes p$$

The contribution on the cylindrical part $\partial C_{\eta,r}$ of radius $\eta$ vanishes as $\delta x$. When $\eta$ approaches $r$, the contributions on the remaining parts of $\partial C_{\eta,r}$ excepted the cylindrical part of radius $r$, vanish too. Let $n'$ the outward unit vector to $\partial C_r$, opposite to the outward unit vector $n$ to $\partial C_{\eta,r}$ (Figure 9.1)

$$n' = -n$$

Then using the polar coordinates $r, \theta$, one has

$$\delta H = - \lim_{r \to 0} \int_{\partial C_{r} \cap \partial C_{\eta,r}} n' \cdot T \cdot \delta x \, r \, d\theta \, ds$$

Finally, recalling that the functional derivative is a density on the crack front, the value of the driving force corresponding to the arc length $s$ is

$$G = -D_{\delta x} H = \lim_{r \to 0} \int_{c_{r,s}} n' \cdot T \, r \, d\theta$$

where $c_{r,s}$ is the line at the intersection between the cylindrical-shaped surface $\partial C_r \cap \partial C_{\eta,r}$ and the plane normal to the crack front $c_t$ at the arc length $s$. This limit is called by Gurtin et al. [40] a tip integral and denoted

$$G = \oint_{\text{tip}} n' \cdot T$$

but we do not recover the last term of (17) in Gurtin’s and Stolz’s [70] expressions of $T$. In this form, the driving force appears as a generalization of Rice-Eshelby $J$ integral.
In statics, this quantity has the well-known property of path-independence and it is relevant to wonder whether this property remains true in dynamics. The elastic stress field being singular at the crack front, the volume force field may be neglected when $r$ approaches zero. In Appendix B, it is shown that for the natural evolution of the system

$$\nabla \cdot T = (\nabla u) \cdot \dot{p} - (\nabla \cdot \dot{u}) p - 2(\nabla \cdot \dot{u} p) \cdot \dot{u}$$

(19)

Unfortunately, this expression does not vanish in general but only in the limit case of statics where we recover the classical Rice-Eshelby identity

$$\nabla \cdot T = \nabla \cdot [\omega 1_{R^3} - \sigma \nabla u] = 0$$

Then except the particular case of statics, there is no path-independence of the integral in dynamics and passing to the limit is required in (18).

10 Conclusions and perspectives

In this work, we presented a particular version of the non incremental symplectic BEN principle. It can be generalized in various ways:

- The Hamiltonian $H$ depends on $u, p$ and $\psi$ (through $\Omega_t = \Omega \setminus \Gamma_t$) but not on $\pi$. It was a bias motivated by the wish to recover classical formulations of the fracture mechanics but we could give up this restrictive assumption by introducing an explicit dependence of the Hamiltonian on $\pi$.

- For the same motivation, the dissipation potential $\phi$ depends on $\dot{\pi}$ but not on $\dot{u}, \dot{p}$ and $\dot{\psi}$. Nevertheless there is nothing preventing us from lifting this restriction, in particular concerning $\dot{\psi}$.

- The potential $\varphi$ could fully depend on $G$, not only on $G_{\perp}$. It could be also generalized to anisotropic behaviours with symmetry group techniques.

- For easiness, we considered only 1-homogeneous potential $\varphi^*$ in this work to illustrate the method but we can add extra terms to represent for instance the dependence of the critical energy release rate with respect to the crack extension vector as in [22, 46, 47, 48]. The crack extension by fatigue can be also modelled thanks to homogeneous potentials of degree linked to the crack growth law ([51], [12]).

- We could easily take into account more sources of dissipation such as damping, plasticity, damage and so on, by considering in the symplectic formalism new internal variables and the corresponding momenta, and by simply adding terms in the dissipation potential $\phi$.

- Sometimes there is no getting away from experimental facts and the kinetic law is non-associative. For such atypical events, the present principle can be generalized, replacing the sum of the dissipation potential and its Fenchel polar by a bipotential.
We are convinced of the interest of this non incremental principle in computational structural mechanics because the error can be controlled uniformly on the whole evolution, in contrast to incremental methods which accumulate the errors and degrade the accuracy over time.

We hope that our application of the normality law to the kinked crack allows to take forward the discussion in the literature on the choice of the crack stability criterion.

In the future, we plan to develop numerical methods based on the present approach. We already have given a hint about the discretization of the crack flow at the end of Section 2.2. In order to avoid a cumbersome remeshing, an efficient technique is the XFEM where the standard displacement-based approximation is enriched by incorporating both discontinuous fields and the near tip asymptotic fields ([59], [28], [29], [60]).

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Appendix A

To calculate the expression of the crack driven force, we start from the variation of the Hamiltonian (15)

$$\delta H = \delta \int_{C_{\eta,r}} h(\theta(y), u, \nabla_y u \cdot \nabla_x y, (\nabla_x y)^T \cdot p') \det(\nabla_y x) \; d^3 y$$

and we take the variation with respect to the fields $u$ and $\theta$

$$\delta H = \int_{C_{\eta,r}} \left\{ \left[ Tr \left( \sigma \delta(\nabla_y u \cdot \nabla_x y) \right) + \dot{u} \cdot \delta((\nabla_x y)^T \cdot p') - f \cdot \delta x \right] \det(\nabla_y x) \right.$$  

$$+ h \delta(\det(\nabla_y x)) \right\} \; d^3 y \tag{20}$$

Next we calculate the variation of the field derivative in terms of the derivative of the variation of the field

$$\delta(\nabla_y u \cdot \nabla_x y) = -\nabla_y u \cdot \nabla_x y \cdot \nabla_y(\delta x) \cdot \nabla_x y$$

$$\delta(\det(\nabla_y x)) = Tr \left( \nabla_y(\delta x) \cdot \text{adj}(\nabla_y x) \right)$$

where $\text{adj}(A)$ is the transposed of the matrix of cofactors of $A$

$$\text{adj}(A) = \det(A) A^{-1}$$

Likewise, one has

$$\delta((\nabla_x y)^T \cdot p') = -(\nabla_x y \cdot \nabla_y(\delta x) \cdot \nabla_x y)^T \cdot p' = -(\nabla_x y)^T \cdot (\nabla_y(\delta x))^T \cdot (\nabla_x y)^T \cdot p'$$
Inserting the three previous expression into (20) and after simple manipulations, we obtain
\[ \delta H = \int_{C'_{\eta,\rho}} \{ \text{Tr} \left( \text{adj}(\nabla y x) \cdot \left( h 1_{\mathbb{R}^3} - \sigma \cdot \nabla y u \cdot \nabla x y - (\dot{u} \otimes p') \cdot \nabla x y \right) \cdot \nabla y (\delta x) \right) \} \]
\[ - \det(\nabla y x) f \cdot \delta x \} \, d^3 y \]

Integrating by parts leads to
\[ \delta H = \int_{\partial C'_{\eta,\rho}} n \cdot \text{adj}(\nabla y x) \cdot \left( h 1_{\mathbb{R}^3} - \sigma \cdot \nabla y u \cdot \nabla x y - (\dot{u} \otimes p') \cdot \nabla x y \right) \cdot \delta x \, d(\partial C'_{\eta,\rho}) \]
\[ - \int_{C'_{\eta,\rho}} \{ \nabla \cdot \left( \text{adj}(\nabla y x) \cdot \left( h 1_{\mathbb{R}^3} - \sigma \cdot \nabla y u \cdot \nabla x y - (\dot{u} \otimes p') \cdot \nabla x y \right) \right) \}
\[ + \det(\nabla y x) f \} \cdot \delta x \, d^3 y \]

Considering the limit case where the function \( \theta \) approaches the identity of \( C_{\eta,\rho} \), \( y \) approaches \( x \), \( p' \) approaches \( p \) and \( C'_{\eta,\rho} \) approaches \( C_{\eta,\rho} \), we obtain (16)
\[ \delta H = \int_{\partial C_{\eta,\rho}} n \cdot T \cdot \delta x \, d(\partial C_{\eta,\rho}) - \int_{C_{\eta,\rho}} (\nabla \cdot T + f) \cdot \delta x \, d^3 x \]
where occurs the tensor (17)
\[ T = h 1_{\mathbb{R}^3} - \sigma \cdot \nabla u - \dot{u} \otimes p \]

**Appendix B**

Because of the stress singularity, the volume force that are regular may be consider as uniform near the crack front. In one hand, the variation of the Hamiltonian density \( h \) in the direction \( \delta x \) is
\[ \delta h = \nabla h \cdot \delta x = \nabla \cdot (h 1_{\mathbb{R}^3}) \cdot \delta x \] (21)

In the other hand, according to (3), \( h \) is a function of \( x \) through \( j^3 u \) and \( p \)
\[ h = \frac{1}{2p} \| p \|^2 + w(\nabla s u) - f(t) \cdot u \]

Using the chain rule and assuming the volume force are uniform, one has
\[ \delta h = \text{Tr}(\sigma \cdot \nabla (\delta u)) + \dot{u} \cdot \delta p - f \cdot u \]

Using the linear momentum balance (intermediate constraints in (5) which is satisfied for the natural evolution of the system), the first term of the right hand member becomes
\[ \text{Tr}(\sigma \cdot \nabla (\delta u)) = \nabla \cdot (\sigma \cdot \delta u) - (\nabla \cdot (\sigma \cdot \delta u) \cdot \delta u = \nabla \cdot (\sigma \cdot \delta u) + (f - \dot{p}) \cdot \delta u \]
Hence
\[ \delta h = \nabla \cdot (\sigma \cdot \delta u) - \dot{\mathbf{p}} \cdot \delta u + \dot{u} \cdot \delta \mathbf{p} \]
The infinitesimal variations of the displacement and linear momentum fields resulting from an infinitesimal arbitrary variation \( \delta \mathbf{x} \) being
\[ \delta u = \nabla u \cdot \delta x, \quad \delta \mathbf{p} = \nabla \mathbf{p} \cdot \delta x \]
and in particular if the field \( \delta \mathbf{x} \) is uniform
\[ \delta h = [\nabla \cdot (\sigma \cdot \nabla u) - \dot{\mathbf{p}} \cdot \nabla u + \dot{u} \cdot \nabla \mathbf{p}] \cdot \delta x \] (22)
The expressions (21) and (22) being equal for every \( \delta \mathbf{x} \), we obtain
\[ \nabla \cdot (h_{1_{\mathbb{R}^3}} - \sigma \cdot \nabla u) = (\nabla u)^T \cdot \dot{\mathbf{p}} - (\nabla \mathbf{p})^T \cdot \dot{u} \]
Taking into account the identity
\[ \nabla \cdot (\dot{\mathbf{u}} \otimes \mathbf{p}) = (\nabla \cdot \dot{\mathbf{u}}) \mathbf{p} + (\nabla \mathbf{p}) \cdot \dot{\mathbf{u}} \]
we obtain (19)
\[ \nabla \cdot \mathbf{T} = (\nabla u) \cdot \dot{\mathbf{p}} - (\nabla \cdot \dot{\mathbf{u}}) \mathbf{p} - 2 (\nabla_\mathbf{s} \mathbf{p}) \cdot \dot{\mathbf{u}} \]
where occurs the tensor (17)
\[ \mathbf{T} = h_{1_{\mathbb{R}^3}} - \sigma \cdot \nabla u - \dot{\mathbf{u}} \otimes \mathbf{p} \]

References

[1] Aldaya, V., Azcárraga, J. A., 1978. Variational principles on rth order jets of fibre bundles in field theory, J. Math. Phys. 19(9), 1869-1975.

[2] Amestoy, M., Leblond, J.B., 1992. Crack path in plane situation – II. Detailed form of the expansion of the stress intensity factors. Int J Solids Struct. 29, 465–501.

[3] Ambrosio, L., Tortorelli, V., 1992. On the approximation of free discontinuity problems. Bollett UMI. 7(6B), 105–123.

[4] Aubin, J.P., Cellina, A.,Nohel, J., 1997. Monotone trajectories of multivalued dynamical systems. Annali di Matematica Pura ed Appl. 115, 99-117.

[5] Aubin, J.P., 2002. Boundary-Value Problems for Systems of Hamilton-Jacobi-Bellman Inclusions with Constraints. SIAM J. Control. 41, 425-456.

[6] Barenblatt, G.I., 1959. The formation of equilibrium cracks during brittle fracture: general ideas and hypotheses, Axially-symmetric Cracks. Prikladnaya Matematika i Mekhanika. 23, 622-636.

[7] Bloch, A.M., Krishnaprasad, P.S., Marsden, J.E., Ratiu, T.S., 1994. Dissipation induced instabilities. Ann. de l'Institut Henri Poincaré. Analyse non linéaire. 11(1), 37-90.
[8] Bodovillé, G., 1999. On damage and implicit standard materials. C. R. Acad. Sci. Paris Série IIB. 327(8), 715-720.

[9] Bobovillé, G. de Saxcé, G., 2001. Plasticity with nonlinear kinematic hardening : modelling and shakedown analysis by the bipotential approach. Eur. J. Mech. A/Solids. 20, 99-112.

[10] Bouby, C., de Saxcé, G., Tritsch, J.B., 2009. On shakedown of structures under variable loads with a kinematic nonlinear and non associated hardening rule, in: Weichert, D., Ponter, A. (Eds.), Limit States of Materials and Structures: Direct Methods. Springer.

[11] Bouby, C., Kondo, D., de Saxcé, G., 2015. A comparative analysis of two formulations for nonlinear hardening plasticity models: Application to shakedown analysis. European Journal of Mechanics A/Solids. 53, 48-61.

[12] Bourdin, B., Francfort, G., Marigo, J.J., 2008. The variational approach to fracture. J of Elasticity. 91, 5-148.

[13] Brezis, H., Ekeland, I., 1976. Un principe variationnel associé à certaines équations paraboliques. I. Le cas indépendant du temps, II. Le cas dépendant du temps. C. R. Acad. Sci. Paris Série A-B. 282, 971-974, and 282, 1197-1198.

[14] Buliga, M., 1999. Energy minimizing brittle fracture propagation. J. of Elasticity. 52, 201-238.

[15] Buliga, M., 2009. Hamiltonian inclusions with convex dissipation with a view towards applications. Math Appl 1(2), 228–225.

[16] Buliga, M., de Saxcé, G., 2016. A symplectic Brezis-Ekeland-Nayroles principle. Mathematics and Mechanics of Solids. 22(6), 1-15.

[17] Cao, X., Oueslati, A., Nguyen, A.D., de Saxcé, G., 2020. Numerical simulation of elastoplastic problems by Brezis-Ekeland-Nayroles non-incremental variational principle. Computational Mechanics. 65(4), 1006-1018

[18] Cao, X., Oueslati, A., Shirafkan, N., Bamer, F., Markert, B., de Saxcé, G., 2021. A non-incremental numerical method for dynamic elastoplastic problems by the symplectic Brezis-Ekeland-Nayroles principle. Computer Methods in Applied Mechanics and Engineering. 384, 11908.

[19] Cao, X., Oueslati, A., de Saxcé, G., 2021. A non-incremental approach for elastoplastic plates basing on the Brezis-Ekeland-Nayroles principle. Applied Mathematical Modelling. 99, 359-379.

[20] Chambolle, A., Francfort, G.A., Marigo, J.J., 2009. When and how do cracks propagate? J of the Mechanics and Physics of Solids. 57, 1614-1622.

[21] Cherepanov, G.P., 1963. One problem of indentor testing with the formation of cracks. Prikladnaya Matematika i Mekhanika. 27(1) (in Russian).
[22] Chopin, J., Bhaskar, A., Jog, A., Ponson, L., 2018. Depinning dynamics of crack fronts. Phys. Rev. Lett. 121, 235501.

[23] Cotterell, B., Rice, J.R., 1980. Some remarks on elastic crack-tip stress fields. Int. J. of Fracture. 16, 155-169

[24] de Saxcé, G., Feng, Z.Q., 1991. New inequation and functional for contact with friction: the implicit standard material approach. Int. J. Mech. of Struct. and Machines. 19(3), 301-325.

[25] de Saxcé, G., 1992. Une généralisation de l’inégalité de Fenchel et ses applications aux lois constitutives. C. R. Acad. Sci. Paris Série II. 314, 125-129.

[26] de Saxcé, G., Bousshine, L., 1998. Limit Analysis Theorems for the Implicit Standard Materials: Application to the Unilateral Contact with Dry Friction and the Non Associated Flow Rules in Soils and Rocks. Int. J. Mech. Sci. 40(4), 387-398.

[27] de Saxcé, G., 2002. Implicit standard materials, in: Weichert, D., Maier, G. (Eds.), Inelastic Behaviour of Structures under Variable Repeated Loads, CISM International Centre for Mechanical Sciences, Courses and Lectures No 432. Springer.

[28] Dolbow, J., Moës, N., Belytschko, T., 2000. Discontinuous enrichment in finite elements with a partition of unity method. Finite Elements in Analysis and Design. 36, 235-260.

[29] Dolbow, J., Moës, N., Belytschko, T., 2000. Modeling fracture in Mindlin-Reissner plates with the extended finite element method. Int. J. of Solids and Structures. 37, 7161-7183.

[30] Dugdale, D.S., 1960. Yielding of steel sheets containing slits. J. Mech. Phys. Sol. 8, 100-104.

[31] Edelen, D.G.B, 1985. Applied Exterior Calculus. John Wiley and sons.

[32] Erdogan, F., Sih, G.C., 1963. On the crack extension in plates under plane loading and transverse shear. Trans. ASME, ser. D. 85(4), 519-527.

[33] Fenchel, W., 1949. On conjugate convex functions. Canadian Journal of Mathematics. 1, 1105.

[34] Fett, T., Munz, D., 2002. Kinked cracks and Richard fracture criterion. Int. J. of Fracture. 115, L69-L73

[35] Fitzpatrick, S., 1988. Representing monotone operators by convex functions, in: Workshop/Miniconference on Functional Analysis and Optimization. Canberra, Proc. Centre Math. Anal. Austral. Nat. Univ., 20, Austral. Nat. Univ, pp. 59-65.

[36] Francfort, G.A., Marigo, J.J., 1998. Revisiting brittle fracture as an energy minimization problem. J. Mech. Phys. Solids. 46, 1319–1342.
[37] Ghoussoub, N., MacCann, R.J., 2004. A least action principle for steepest descent in non-convex landscape. Contemporary Mathematics. 362, 177-187.

[38] Goldstein, R.V., Salganik, R.L., 1974. Brittle fracture of solids with arbitrary cracks. Int. J. of Fracture. 10, 507–23.

[39] Grmela, M., Öttinger, H.C., 1997. Dynamics and thermodynamics of complex fluids. I. Development of a general formalism. Phys. Rev. E 56(6), 6620-6632.

[40] Gurtin, M., Podio-Guidugli, P., 1998. Configurational forces and a constitutive theory for crack propagation that allows for kinking and curving. J. Mech. Phys. Solids. 46, 1343–1378.

[41] Halphen, B., Nguyen Quoc, S., 1975. Sur les matériaux standard généralisés. Journal de Mécanique. 14, 39-63.

[42] He, M., Hutchinson, J., 1989. Crack deflection at an interface between dissimilar elastic materials. Int. J. Solids Struct. 25, 1053–1067.

[43] Hellen, T.K., Blackburn, W.S., 1975. The calculation of stress intensity factors for combined tensile and shear loading. Int J Fract. Mech. 11, 605–17.

[44] Hjiaj, M., Bobovillé, G. de Saxcé, G., 2000. Matériaux viscoplastiques et loi de normalité implicites. C. R. Acad. Sci. Paris Série IIb. 328, 519-524.

[45] Hussain, M.A., Pu, S.L., Underwood, J.H., 1974. Strain energy release rate for a crack under combined mode I and mode II. Fract Anal ASTM STP. 560, 2–28.

[46] Kolvin, I., Cohen, G., Fineberg, J., 2015. Crack front dynamics: the interplay of singular geometry and crack instabilities. Phys. Rev. Lett. 114, 175501.

[47] Lebihain, M., Leblond, J.B., Ponson, L., 2020. Effective toughness of periodic heterogeneous materials: the effect of out-of-plane excursions of cracks. J of the Mechanics and Physics of Solids. 137, 103876.

[48] Lebihain, M., Ponson, L., Leblond, J.B., Kondo, D., 2021. Effective toughness of disordered brittle solids: A homogenization framework. J of the Mechanics and Physics of Solids. 153, 104463

[49] Leblond, J.B., 1989. Crack paths in plan situation – I. General form of the expansion of the stress intensity factors. Int J Solids Struct. 25, 1311–25.

[50] Leguillon, D., 2002. Strength or toughness? A criterion for crack onset at a notch. Eur. J. Mech. A/Solids. 21, 61-72.

[51] Lemaître, J., Chaboche, J.L., 2012. Mechanics of solid materials, Cambridge University Press.

[52] Mangiarotti, L., Modugno, M., 1983. Some results on the calculus of variation on jet spaces, Ann. Inst. H. Poincaré. 23(1), 29-43.
[53] Manh Hong D., Peletier, M.A., Zimmer, J., 2013. GENERIC formalism of a Vlasov-Fokker-Planck equation and connection to large-deviation principles. Nonlinearity. 26, 2951-2971.

[54] Magnier, V., Charkaluk, E., Bouby, C., de Saxcé, G., 2006. Bipotential Versus Return Mapping Algorithms: Implementation of Non-Associated Flow Rules, in: Topping, B.H.V., Montero, G., Montenegro, R. (Eds.), Proceedings of The Eighth International Conference on Computational Structures Technology (las Palmas de Gran Canaria, sept. 12-15, 2006). Civil-Comp Press, Stirlingshire, United Kingdom.

[55] Mielke, A., Theil, F., 1999. A mathematical model for rate-independent phase transformations with hysteresis, in: Alber, H.D., Balean, R., Farwig, R. (Eds.), Workshop on Models of Continuum Mechanics in Analysis and Engineering, Shaker-Verlag, pp.117-129.

[56] Mielke, A., 2005, in: Feireisl, E. (Ed.), Handbook of Differential Equations, Evolutionary Equations, vol. 2. Elsevier, pp.461-559.

[57] Mielke, A., Roubíček, T., 2006. Rate-independent damage processes in nonlinear elasticity. Mathematical Models and Methods in Applied Sciences (M3AS). 16(2), 177-209.

[58] Mielke, A., 2011. Formulation of thermo-elastic dissipative material behavior using GENERIC. Contin. Mech.Thermodyn. 23, 233-256.

[59] Moës, N., Dolbow, J., Belytschko, T., 1999. A finite element method for crack growth without remeshing. Int. J. for Numerical Methods in Engineering 46:131-150.

[60] Moës, N., Belytschko, T., 2002. Extended finite element method for cohesive crack growth. Engineering Fracture Mechanics. 69, 813-833.

[61] Mróz, K.P., Mróz, Z., 2010. On crack path evolution rules. Engineering Fracture Mechanics. 77, 1781-1807

[62] Mumford, D., Shah, J., 1989. Optimal approximation by piecewise smooth functions and associated variational problems. Comm. on Pure and Appl. Math. 42(5), 577-685.

[63] Nayroles, B., 1976. Deux théorèmes de minimum pour certains systèmes dissipatifs. C. R. Acad. Sci. Paris Série A-B. 282, A1035-A1038.

[64] Öttinger, H.C., Grmela, M., 1997. Dynamics and thermodynamics of complex fluids. II. Illustrations of a general formalism. Phys. Rev. E. 56(6), 6633-6655.

[65] Richard, H.A., 1984. Examination of brittle fracture criteria for overlapping mode I and II loading applied to cracks, in: Sih, G.C. et al. (Eds.), Applications of Fracture Mechanics to Materials and Structures. Nijhoff Publ., The Hague, pp. 309-316.

25
[66] Rockafellar, R.T., 1970. Generalized Hamiltonian equations for convex problems of Lagrange. Pacific J. of Math. 33(2), 411-427.

[67] Saunders, D.J., 1989. The Geometry of Jet Bundles, Cambridge University Press.

[68] Sih, G.C., 1973. Some basic problems in fracture mechanics and new concepts. Engineering Fracture Mechanics. 5, 365-377

[69] Stefanelli, U., 2008. A variational principle for hardening elasto-plasticity. SIAM J. Math. Anal. 40(2), 623-652.

[70] Stolz, C., 1995. Functional approach in nonlinear dynamics. Arch. Mech. 47, 421-435.

[71] Strifors, H.C., 1973. A generalized force measure of conditions at crack tips. Int. J. Solids Struct. 10,1389–404.

[72] Tvergaard, V., Hutchinson, J.W., 1992. The relation between crack growth resistance and fracture process parameters in elastic-plastic solids. J. Mech. Phys. Sol. 40, 1377-1397.

[73] Vallé, V., Lerintiu, C., Fortuné, D., Ban, M., de Saxcé, G., 2005. A bipotential expressing simultaneous ordered spectral decomposition between stress and strain rate tensor, in: International conference New Trends in Continuum Mechanics (ed Theta), Constanța (Romania), 8-12 September 2003. Published under the title "Hill's bipotential", New Trends in Continuum Mechanics, pp.339-351.

[74] Visintin, A., 2008. Extension of the Brezis-Ekeland-Nayroles principle to monotone operators. Adv. math. Sci. Appl. 18, 633-650.

[75] Visintin, A., 2013. Structural stability of rate-independent nonpotential flows. Discrete and Continuous Dynamical Systems Series S. 6, 257-275.

[76] Wu, C.H., 1978. Fracture under combined loads by maximum-energy-release-rate-criterion. ASME J Appl Mech. 45, 553-8.

[77] Xu, X., Needleman, A., 1994. Numerical simulations of fast crack growth in brittle solids. J. Mech. Phys. Sol. 42, 1397-1434.

[78] Zouain, N., Pontes Filho, I., Vaunat, J., 2010. Potentials for the modified Cam-Clay model. European Journal of Mechanics A/Solids. 29, 327-336.