COMBINATORIAL DECOMPOSITION THEOREM FOR HITCHIN SYSTEMS VIA ZONOTOPES

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Abstract. We determine the summands of the decomposition theorem for the Hitchin system for $GL_n$, in arbitrary degree, over the locus of reduced spectral curves. The key ingredient is an equivariant formula for lattice point counts in graphical zonotopes.

1. Introduction

Intersection cohomology is a topological invariant well-adapted to investigate the symmetries of singular spaces. Nowadays, it is an established tool in Hodge theory, but also in representation theory and combinatorics. For instance, the symmetries of the $h$-polynomial of a polytope can be explained in terms of dualities of the intersection cohomology of the associated toric variety; see [16, 6]. Via the geometric Satake correspondence, the irreducible representations of a reductive group $G$ can be realized as the intersection cohomology of affine Schubert varieties for the Langlands dual group $^LG$; see [18, 31]. Another example is the calculation of the Frobenius character of the Orlik–Terao algebra by using the intersection cohomology ring of an hypertoric variety; see [7, 33, 37]. Viceversa, the intersection cohomology of toric varieties is controlled by the combinatorics of their moment polytopes, as one may expect given the combinatorial nature of these varieties; see again [16].

It may sound less obvious that the intersection cohomology of moduli spaces with no a priori combinatorial origin, like spaces of Higgs bundles, admits a combinatorial characterization. In this paper, we show how to obtain cohomological information for these spaces via lattice point count in rational polytopes called zonotopes.

1.1. Decomposition theorem for the Hitchin fibration. We consider the Dolbeault moduli space $M(n,d)$ parametrising semistable Higgs bundles $(\mathcal{E}, \phi)$ on a compact Riemann surface $C$ with $\text{rank}(\mathcal{E}) = n$, and $\text{deg}(\mathcal{E}) = d$; see [7]. One can argue that the entire study of non-abelian Hodge theory may be thought of as the study of the geometry of $M(n,d)$. Dolbeault moduli spaces play a central role in various other fields of mathematics like mirror symmetry, Teichmüller theory, knot theory and in the geometric Langlands program.
A key feature of the geometry of \(M(n,d)\) is that it is endowed with a projective fibration called \textit{Hitchin fibration} 

\[
\chi(n,d): M(n,d) \to A_n := \bigoplus_{i=1}^{\binom{n}{i}} H^0(C, \omega_C^{\otimes i})
\]

which assigns to \((\mathcal{E}, \phi)\) the characteristic polynomial \(\text{char}(\phi)\) of the Higgs field \(\phi\); see again §7.

Set \(A_n^{\text{red}} \subseteq A_n\) the locus of reduced characteristic polynomial, and \(M(n,d)^{\text{red}} := \chi^{-1}(n,d)(A_n^{\text{red}})\).

The intersection cohomology of \(M(n,d)\), denoted \(IH^*(M(n,d), \mathbb{Q})\), is the global cohomology of the intersection complex \(IC(M(n,d), \mathbb{Q})\) on \(M(n,d)\), or of the direct image complex \(R\chi(n,d)_* IC(M(n,d), \mathbb{Q})\) on the base \(A_n\). The advantage of the second complex is that we may decompose it in smaller complexes called \textit{Ngô strings}. On the reduced locus \(A_n^{\text{red}}\), the Ngô strings are indexed by a subset of partitions \(\underline{n}\) of \(n\), and they are denoted \(\mathcal{S}(\underline{L}_n(d))\). Any string \(\mathcal{S}(\underline{L}_n(d))\) is a combination of the cohomology of an abelian group scheme independent of \(d\), and a local system \(\underline{L}_n(d)\) depending on \(d\); see §9.

These local systems \(\underline{L}_n(d)\) should be considered as the combinatorial part of the intersection cohomology of \(M(n,d)\): a partition \(\underline{n}\) corresponds to a choice of a zonotope in \(\mathbb{R}^{\left|\underline{n}\right|} \cong \mathbb{Z}^{\left|\underline{n}\right|} \otimes \mathbb{R}\), and the degree \(d\) determines a translation of the zonotope. The local system \(\underline{L}_n(d)\) is a subrepresentation of a group of automorphisms of the zonotope on the lattice points contained in its translate.

The determination of these representations is a subtle task, and involves a delicate combination of combinatorics and algebraic geometry that we carried out in this paper. In §1.4 and [28, §1] we hint at the geometric significance of the result. Here we summarize the main output, and discuss it in detail in the next section.

\textbf{Theorem 1.1} (Effective decomposition theorem for the Hitchin fibration). For any partition \(\underline{n} = \{n_1, \ldots, n_r\}\) of \(n\), let \(S_{\underline{n}}^{\times} \subseteq A_n^{\text{red}}\) be the closure of the locus the characteristic polynomials \(\text{char}(\phi)\) whose irreducible factors have degree \(n_i\), and \(g_{\underline{n}} : \mathcal{S}_{\underline{n}}^{\times} \to S_{\underline{n}}^{\times}\) be the relative Jacobian of the simultaneous normalization of the spectral curves of equation \(\text{char}(\phi) = 0\), with \(\text{char}(\phi)\) in a dense open set \(S_{\underline{n}}^{\infty} \subseteq S_{\underline{n}}^{\circ} = S_{\underline{n}}^{\times} \setminus \bigcup_{m \leq n} S_m\). Denote by \(\text{Stab}(\underline{n})\) the subgroup of the symmetric group on \(r\) elements stabilizing \(\underline{n}\).

Then there is an isomorphism in the bounded derived category of algebraic mixed Hodge modules \(D^b MHM_{\text{alg}}(A_n^{\text{red}})\), or in the bounded derived category \(D^b(A_n^{\text{red}})\) of \(\mathbb{Q}\)-constructible complexes on \(A_n^{\text{red}}\) (ignoring the Tate shifts):

\[
R\chi(n,d)_* IC(M(n,d), \mathbb{Q})|_{A_n^{\text{red}}} \cong \bigoplus_{\underline{m} \leq \underline{n}} IC(S_{\underline{m}}, Rg_{\underline{m}}_* Q_{s_{\underline{m}}^{\times}} \otimes \underline{L}_n(d))[-2\text{codim} S_{\underline{m}}](-\text{codim} S_{\underline{m}}),
\]

(1)
where $\mathcal{L}_\mathfrak{n}(d)$ is a polarizable variation of pure Hodge structures of weight zero and of Hodge-Tate type on $S_\mathfrak{n}^\times$.

The monodromy of $\mathcal{L}_\mathfrak{n}(d)$ actually factors through a representation of the stabilizer $\text{Stab}(\mathfrak{n})$, i.e. for any $a \in S_\mathfrak{n}^\times$ we have

$$\pi_1(S_\mathfrak{n}^\times) \twoheadrightarrow \pi_1(S_\mathfrak{n}^\times) \twoheadrightarrow \text{Stab}(\mathfrak{n}) \twoheadrightarrow \text{Aut}(\mathcal{L}_\mathfrak{n}(d)_a);$$

see Remark 9.3. The $\text{Stab}(\mathfrak{n})$-representation admits a clear combinatorial characterization as described in Theorem 1.6, and its rank is computed in Theorem 1.8.

1.2. Supports and combinatorial local systems of the Hitchin fibration. In order to explain and prove Theorem 1.1, we first introduce the notion of $d$-integral partition.

Definition 1.2. A partition $\mathfrak{n} = \{n_i\}$ of $n = \sum_i n_i$ is $d$-integral if $dn_i/n \in \mathbb{Z}$.

Example 1.3. If $\gcd(n, e) = 1$, the only $e$-integral partition of $n$ is the trivial one, i.e. $\{n\}$. On the contrary, any partitions of $n$ is $0$-integral.

Such partitions arise naturally while studying the singularities of $M(n,d)^{\text{red}}$. Indeed, a singular point of $M(n,d)^{\text{red}}$ corresponds to a strictly polystable Higgs bundle $(E_1, \phi_1) \oplus \ldots \oplus (E_r, \phi_r)$ with $r > 1$, where $(E_i, \phi_i)$ are distinct stable Higgs bundles of slope $\text{deg} E_i/\text{rank} E_i = d/n$. In fact, $\{\text{rank} E_i\}$ is a $d$-integral partition of $n$. Let $M^0_\mathfrak{n}(d)$ be the locus of polystable Higgs bundles of multirank $\mathfrak{n}$. Then the space $M(n,d)^{\text{red}}$ admits a Whitney stratification by partition type

$$M(n,d)^{\text{red}} = \bigsqcup_\mathfrak{n} M^0_\mathfrak{n}(d),$$

where the sum runs over the $d$-integral partitions of $n$; see also [28, §1.1].

For any partition $\mathfrak{n} = \{n_i\}$ of $n$, let $S_\mathfrak{n} \subset A_n$ be the closure of the polynomials whose irreducible factors have degree $n_i$; see also §9. In [28, Cor. 3.8], the first and second authors showed that the supports of the Ngô strings of $R\chi(n,d)_* \text{IC}(M(n,d), \mathbb{Q})|_{A_n^{\text{red}}}$ ranges among the subvarieties $S_\mathfrak{n}$; see Theorem 9.1. Here we determine which of them actually supports a non-zero Ngô string.

Theorem 1.4 (Supports in arbitrary degree). The supports of $R\chi(n,d)_* \text{IC}(M(n,d), \mathbb{Q})|_{A_n^{\text{red}}}$ are all and only the subvarieties $S_\mathfrak{n}$ such that $\mathfrak{n}$ is not a $d$-integral partition of $n$.

Proof. It follows from Theorem 1.6, Proposition 3.11, and Proposition 3.12. □
Note that the closure of the images of $M_n^d(d)$ are all and only the subvarieties $S_n$ such that $n$ is indeed $d$-integral. Therefore, we have the following dichotomy: either the subvariety $S_n$ has a Hodge-theoretic interpretation as summand of $R\chi(n,d)\ast \text{IC}(M(n,d),\mathbb{Q})$, or it admits a description in terms of the singularities of $M(n,d)$.

Therefore, Example 1.3 implies that we have two extreme cases: the coprime and the zero case. If $\gcd(n,e) = 1$, $M(n,e)$ is smooth and $R\chi(n,e)\ast \text{IC}(M(n,e),\mathbb{Q})|_{A_{\text{red}}}^*$ has the maximal number of supports as the degree varies; see also [28, §1.1]. On the contrary, when $d = 0$, $M(n,0)$ is singular with the maximal number of strata in the stratification by partition type, but no proper summand of $R\chi(n,0)\ast \text{IC}(M(n,0),\mathbb{Q})|_{A_{\text{red}}}^*$. We could informally say that, despite their different origin, there exists a conservation law between summands of the decomposition theorem and singular strata as the degree $d$ varies. In particular, we obtain an alternative proof of [28, Thm 1.4], which is actually a key step in the proof of Theorem 1.4 and 1.8.

**Theorem 1.5** (Full support in degree zero). The complex $R\chi(n,0)\ast \text{IC}(M(n,0),\mathbb{Q})|_{A_{\text{red}}}^*$ has full support, i.e.

$$R\chi(n,0)\ast \text{IC}(M(n,0),\mathbb{Q})|_{A_{\text{red}}}^* \simeq \mathcal{S}(\mathbb{Q}_A)\big|_{A_{\text{red}}}.$$

If $\gcd(n,e) = 1$, the rank of $L_n^d(e)$ has been interpreted as the cohomology of a cographic matroid in [12, Thm 6.11], and as the top cohomology of a symplectic resolution of a normal slice of $M_n^d(0)$ in [28, Thm 1.1]. Here we propose a new combinatorial description that holds in arbitrary degree. The monodromy of the local system $L_n^d(d)$ is a representation of $\text{Stab}(n)$, i.e. the subgroup of the symmetric group on $\ell(n) = r$ elements stabilizing $n$. If $\underline{n} = \{n_1, \ldots, n_r\} \vdash n$, set

$$\omega_{\underline{n}}(d) := \left(\frac{d}{n_1}, \ldots, \frac{d}{n_r}\right) \in \mathbb{Q}_r^r. \tag{2}$$

Then the poset of non-$\omega_{\underline{n}}(d)$-integral partition of the set $\underline{n}$, denoted $\widehat{\Pi}_{\omega_{\underline{n}}(d)}$ (cf Definition 2.4 and 2.6), admits an action of $\text{Stab}(\underline{n})$.

**Theorem 1.6** (Combinatorial characterization of $L_n^d(d)$). Let $\underline{n}$ be a non-trivial partition of $n$. The local system $L_n^d(d)$ can be identified as $\text{Stab}(\underline{n})$-representation with the top reduced homology of the order complex of the poset $\Pi_{\omega_{\underline{n}}(d)}$, twisted by the sign representation

$$L_n^d(d) \simeq \text{sgn} \otimes \tilde{H}_{\ell(\underline{n})-3}(\Delta(\Pi_{\omega_{\underline{n}}(d)})).$$

In particular, the rank of $L_n^d(d)$ is the number of spheres of the order complex $\Delta(\Pi_{\omega_{\underline{n}}(d)})$.

Note that the order complex $\Delta(\Pi_{\omega_{\underline{n}}(d)})$ is homotopic equivalent to a wedge of spheres. This follows from the LEX-shellability of the poset $\Pi_{\omega_{\underline{n}}(d)}$; see Definition 3.2, Theorem 3.6 and Proposition 3.9.
Theorem 1.7 (Theorem 3.6; LEX-shellability). The poset $\hat{\Pi}_{\omega}(d)$ is LEX-shellable.

If $\gcd(n, e) = 1$, the poset $\Pi_{\omega}(e)$ is simply the poset of all partitions of $[r]$. Such poset appears in [12, Lemma 6.18] as the lattice of flats of a cographic matroid, and it was indeed used to compute the cohomology of the matroid. Note however that in arbitrary degree the poset $\Pi_{\omega}(d)$ may fail to be semimodular, so in general there is no associated matroid.

The Hodge-to-singular correspondence in [28] provides a recursive formula to determine the rank of $L_n(d)$ as explained in [28, §10.1]. Due to the new characterization Theorem 1.6, we can now provide a closed formula.

\begin{equation}
\sum_{\lambda = \{\lambda_i\} + [r] \atop \lambda_{\omega}(d)-integral} (\pm 1)^{\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} (|\lambda_i| - 1)!,
\end{equation}

where the sum runs over the $\omega_n(d)$-integral partitions of the set $n$; see Definition 2.4.

By direct inspection, we show that the sum in (3) vanishes when $\omega$ is an integral vector. Otherwise, we prove that $\Delta(\Pi_{\omega}(d))$ contains at least a sphere, which by Theorem 1.6 forces the sum in (3) to be positive; see Proposition 3.12. These facts together gives Theorem 1.4.

1.3. Strategy. The key input for the proof of the previous statements is that any local system $L_n(d)$ arises from a graphical zonotope.

According to the decomposition theorem for the Hitchin fibration (Theorem 9.1), the local system $L_n(d)$ are direct summands of $R^{2c(n,g)}\chi(n, d)_*Q_{M(n,d)}$. The stalk of $R^{2c(n,g)}\chi(n, d)_*Q_{M(n,d)}$ at the point $a \in A_n^{red}$ has a basis indexed by the irreducible components of $\chi(n, d)^{-1}(a)$. Via the BNR-correspondence, $\chi(n, d)^{-1}(a)$ is a compactified Jacobian parametrising semistable rank 1 torsion free sheaves on the spectral curve $C_a$; see [7]. In particular, the irreducible components of the compactified Jacobian of a reduced spectral curve are indexed by the degrees of the restrictions of stable line bundles to the irreducible components of $C_a$; see Proposition 7.6. The stability condition imposes numerical inequalities on the multidegree of a line bundle on $C_a$, which are defining equations for the graphical zonotope $Z_{\Gamma} + \omega$ for a suitable graph $\Gamma$ and translation vector $\omega$; see Theorem 7.8. A crucial ingredient is the following combinatorial version of the decomposition theorem.

Theorem 1.9 (Theorem 6.4; Combinatorial decomposition theorem). For any graph $\Gamma$ and for any $\text{Aut}(\Gamma)$-invariant vector $\omega \in \mathbb{Q}^r$, $\mathcal{C}(Z_{\Gamma} + \omega)$ is the rational vector space with basis the set
of integral points of the interior of the zonotope \( Z_{\Gamma} + \omega \). The following \( \text{Aut}(\Gamma) \)-representations are isomorphic

\[
\mathcal{C}(Z_{\Gamma} + \omega) = \mathcal{C}(Z_{\Gamma}) \oplus \bigoplus_{\Sigma \in L_{\omega}/\text{Aut}(\Gamma)} \text{Ind}^{\text{Aut}(\Gamma)}_{\text{Stab}(\Sigma)} (\alpha_{\Sigma} \otimes \overline{H}_{\ell}(\Sigma) - 3(\Delta(\mathcal{L}_{\omega}, \Sigma, \geq \Sigma)) \otimes \mathcal{C}(Z_{\Gamma_{\Sigma}})).
\]  

(4)

We refer to §6 for details about the notation. Here we emphasis the formal analogy between (1) and (4). Indeed, in the same way as \( R\chi(n, d)_{\ast} \text{IC}(M(n, d), \mathbb{Q})|_{A_{n}^{\text{ell}}} \) is a direct sum of Ngô strings \( \mathcal{H}(\mathcal{L}_{n}(d)) \) indexed by \( d \)-integral partitions \( n \) of \( n \), the number of lattice points in a translated zonotope is the sum of the lattice points contained in the faces of some non-translated faces of the zonotopes corresponding to \( d \)-integral partitions too.

1.4. Other work and broad context. Theorem 1.1 is a refinement of the celebrated Ngô support theorem [34, §7]. Let \( A_{n}^{\ell} \) be the locus of irreducible characteristic polynomials. In our context, Ngô support theorem asserts that \( R\chi(n, d)_{\ast} \text{IC}(M(n, d), \mathbb{Q})|_{A_{n}^{\text{ell}}} \) has full support, and it is independent of the degree \( d \). In hindsight, we can now say that \( A_{n}^{\ell} \) is the bigger open set of \( A_{n} \) avoiding proper summands of \( R\chi(n, d)_{\ast} \text{IC}(M(n, d), \mathbb{Q}) \) as the degree \( d \) varies. If \( \gcd(n, d) = 1 \), Theorem 1.1 is then the main result of [12]. It is not clear however how the
explicit computation of the Cattani–Kaplan–Schmidt complex in [12 §5] can be adapted in the singular setting, so here we pursue a different strategy.

As pointed out in [28 §1.2], Theorem 1.1 implies that $R\chi(n,d)_* IC(M(n,d),\mathbb{Q})|_{A^\text{red}_n}$ depends only on the gcd$(n,d)$, and so $IH^*(M(n,d),\mathbb{Q})$ does. The independence of $H^*(M(n,e),\mathbb{Q})$ for gcd$(n,e) = 1$ has already been proved using vanishing cycle techniques in [21 Theorem 1.1] or positive characteristic methods in [13 Theorem 0.1]. It is worth remarking that the authors of [13] actually prove the stronger statement that the cohomology ring itself is independent of the degree.

In the singular case, the dependence of $IH^*(M(n,d),\mathbb{Q})$ on the degree has been investigated in [28 Corollary 1.9], and settled unconditionally in [10 Corollary 14.9], by studying the geometry and representation theory of local singularities of $M(n,d)$. The viewpoint of our paper is instead global and combinatorial. Observe also that [10] does not prescribe the shape of the summands of $R\chi(n,d)_* IC(M(n,d),\mathbb{Q})$. After Theorem 1.1, the main open conjecture is whether there are summands supported on the complement of $A^\text{red}_n$. As a corollary of [10], it suffices to check it in degree zero, and this is indeed equivalent to the following generalization of Theorem 1.5.

**Conjecture 1.10** (Full support conjecture). The complex $R\chi(n,0)_* IC(M(n,0),\mathbb{Q})$ has full support on the whole basis of the Hitchin fibration.

In other words, we ask whether the complex $R\chi(n,0)_* IC(M(n,0),\mathbb{Q})$ is determined by its restriction to any open subsets of $A_n$, in particular the locus over which the Hitchin fibration is smooth. A positive answer to the full support conjecture would imply that $IH^*(M(n,0),\mathbb{Q})$ should be regarded as the primitive indecomposable building block of the cohomology of any other Dolbeault moduli space $M(n,d)$.

The key geometric fact used in this paper is the characterization of the fibres of the Hitchin fibration as compactified Jacobian. In fact, the same proof of Theorem 1.1 works if we replace the Hitchin system with other families of compactified Jacobian, e.g. the Mukai systems on moduli space of sheaves on K3 or abelian surfaces of Picard rank one; cf [28 §10.3]. In particular, we obtain the following equivariant description of the irreducible components of compactified Jacobians, which is a result of independent interest, and generalizes a classical result by Raynaud [40 Prop. 8.1.2] and Melo–Rapagnetta–Viviani [29 Cor. 5.14] for fine compactified Jacobian. We refer to §7 for details on the terminology.

**Definition 1.11.** Let $X = \bigcup_{i \in I} X_i$ be a nodal curve with smooth irreducible components and dual graph $\Gamma(X)$, and $X' \to X$ be a partial normalization of $X$. Note that the restriction $\text{Pic}(X) \to \text{Pic}(X')$ is always surjective, and the kernel is an algebraic torus of dimension
$H^1(\Gamma(X), \Gamma(X'), \mathbb{Z})$. We say that $X$ has no tails if the dual graph $\Gamma(X)$ does not collapse onto any proper subgraph, or equivalently $\text{Pic}(X) \not\cong \text{Pic}(X')$ for any partial normalization $X'$ of $X$.

**Theorem 1.12** (Irreducible components of compactified Jacobians). Let $X = \bigcup_{i \in I} X_i$ be a nodal curve with no tails and dual graph $\Gamma$. The top cohomology of the compactified Jacobian $\overline{J}_g(X)$ is the $\text{Aut}(\Gamma)$-representation $\mathcal{C}(Z_\Gamma + g)$, i.e.

$$H^{\text{top}}(\overline{J}_g(X), \mathbb{Q}) \cong \text{Aut}(\Gamma)\mathcal{C}(Z_\Gamma + g).$$

**Proof.** As $X$ is nodal, rank 1 torsion free sheaves are pushforward of line bundles from partial normalization of $X$; see for instance [35, Proposition 10.1] or [8, Theorem]. Since $X$ has no tails, the locus of pushforward of line bundles from non-trivial partial normalizations of $X$ is a proper subvariety of $\overline{J}_g(X)$, or equivalently line bundles are dense in $\overline{J}_g(X)$. Then the same proof of Proposition 7.6.(1) and (2) and Theorem 7.8 gives the result. \hfill \Box

1.5. **Outline.** The paper is divided into two parts: a combinatorial half (from §3 to §6) and an algebro-geometric half (from §7 to §10).

- In §2 we fix the notation of partitions, flats and posets used all over the text.
- In §3 we prove the LEX-shellability of some posets of flats, see Theorem 3.6 and compute the Betti numbers of the order complex $\Delta(\Pi_\omega)$; see Proposition 3.11. This are combinatorial results of independent interest.
- In §4 we recall a formula due to Ardila, Beck and McWhirter to compute the number of lattice points in the interior of a translated zonotope; see [1, Proposition 3.1]. We apply Möbius inversion to obtain a numerical version of the combinatorial decomposition theorem, namely Theorem 4.5.
- In §5 we specialize the previous count to graphical zonotopes of a complete graph with multiple edges, and realize that the same combinatorial pattern govern both the coefficients of the lattice point count formula and the numbers of spheres of the order complex $\Delta(\Pi_\omega)$; see Corollary 5.3.
- In §6 we promote the previous numerical identities to an isomorphism of representations. This is the most technical and delicate part of the paper. We achieve this by comparing characters, which in turns boils down to compute lattice points of fixed loci of automorphisms in graphical zonotope. In general, the fixed loci are no longer graphical zonotopes, and this is why in §4 we work with arbitrary zonotopes.
• In §7 we establish the relation between irreducible components of the fibers of the Hitchin fibrations and lattice point count in graphical zonotopes as described in the Strategy 1.3.

• After an analysis of the stalks of Ngô strings in §9, we can finally prove Theorem 1.5 and Theorem 1.6 in §10.

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2. Notation

2.1. Partitions of a set or of an integer.

Definition 2.1. Let $I$ be a finite set. A partition of $I$ of length $\ell(S) = l$ is a set $S := \{S_1, \ldots, S_l\}$ of $l$ disjoint subsets $S_i \subseteq I$ such that $I = \bigsqcup_{i=1}^l S_i$.

Let $S'$ be another partition of $I$. We say that $S \leq S'$ if any $S'_j$ is the union of some $S_i$.

The poset of partitions of $I$ is denoted $(\Pi(I), <)$, or simply by $\Pi$. The minimum $\hat{0}$ of $\Pi$ is the collection of the singletons of $I$, while the maximum $\hat{1}$ is the trivial partition $\{I\}$ consisting of the single block $I$.

Definition 2.2. Let $n \in \mathbb{Z}_{>0}$. A partition $\underline{n}$ of $n$ of length $\ell(\underline{n}) = r$ is a set $\{n_1, \ldots, n_r\}$ of positive integers such that $\sum_{i=1}^r n_i = n$.

Let $\underline{m}$ be another partition of $n$ of length $s$. We say that $\underline{n} \geq \underline{m}$, if there exists a partition $\underline{S}$ of length $r$ of the set $\underline{m}$ such that

$$\underline{n} = \{n_1, \ldots, n_r\}, \quad \underline{m} = \{m_1, \ldots, m_s\}, \quad n_i = \sum_{j \in S_i} m_j.$$

In particular, denote $n_i$ the partition $\{m_j : j \in S_i\}$ of $n_i$. 
2.2. Flats of a graph. Let $\Gamma$ be a graph on the vertex set $V(\Gamma) = [r]$. Denote $E(\Gamma)$ the set of edges of $\Gamma$.

**Definition 2.3.** A flat $S = \{S_1, \ldots, S_s\}$ of $\Gamma$ is a partition of $V(\Gamma)$ with the property that $S_i$ are the vertices of a complete subgraph $\Gamma(S_i)$ of $\Gamma$. By abuse of notation we continue to call flat the set of edges of the subgraphs $\Gamma_i$, i.e. $\bigcup_{i=1}^s E(\Gamma(S_i))$.

The **poset of flats** of $\Gamma$, denoted $\mathcal{L} = \mathcal{L}(\Gamma)$, is the subposet of flats in the partition poset $(\Pi(V(\Gamma)), <)$.

Given a flat $S$ of $\Gamma$, we can construct two associated graphs:

- $\Gamma_S := \bigsqcup_{i=1}^s \Gamma(S_i)$ is the subgraph of $\Gamma$ obtained by removing all edges whose vertices do not lie in $S$;
- $\Gamma^S$ is the **contracted (loopless) graph**, i.e. the graph obtained from $\Gamma$ by contracting all the edges whose vertices lie in $S$ and then deleting all loops.

We introduce an arithmetic condition on the flats of a graph that we call $\omega$-integrality.

**Definition 2.4.** Let $\omega = (\omega_1, \ldots, \omega_r) \in \mathbb{R}^r$. A partition $S = \{S_1, \ldots, S_l\}$ of the set $[r]$ is $\omega$-**integral** if for all $j = 1, \ldots, l$ we have

$$\sum_{i \in S_j} \omega_i \in \mathbb{Z}.$$ 

The poset $\mathcal{L}_\omega = \mathcal{L}_\omega(\Gamma) \subset \mathcal{L}(\Gamma)$ is the subposet of all non-$\omega$-integral flats of $[r]$.

**Remark 2.5.** Any partition $S = \{S_1, \ldots, S_r\}$ of $m$ is $\omega_m(d)$-**integral** (cf eq. [2]) if and only if the corresponding partition $\underline{\omega} := \{\sum_{j \in S_i} m_j \mid j = 1, \ldots, r\}$ is $d$-integral.
If the graph $\Gamma$ contains a complete graph on the vertices, then the poset of flats $L(\Gamma)$ is the full poset of partitions of $V(\Gamma)$, denoted $(\Pi(V(\Gamma)),\prec)$. We simply write $L = \Pi$. In particular, if $\gcd(d,n) = 1$, then $L_{\omega_k}(d) = L = \Pi$; see also Definition 2.6. On the other hand, if $d = 0$, then $L = \emptyset$.

2.3. Posets.

Definition 2.6. A poset is **bounded** if it contains a least and greatest element $\hat{0}$ and $\hat{1}$. For a bounded poset $\mathcal{P}$ we denote by $\bar{\mathcal{P}} = \mathcal{P} \setminus \{\hat{0},\hat{1}\}$ and $\hat{\mathcal{P}} = \mathcal{P} \cup \{\hat{0},\hat{1}\}$. A poset is **pure** if all maximal chains have the same length. A poset is **graded** if it is finite, bounded and pure.

Definition 2.7. Given a poset $(\mathcal{P},\prec)$ and $x,y \in \mathcal{P}$, the symbol $x \lesssim y$ denotes a **covering relation** and it means that $x < y$ and there is no $z \in \mathcal{P}$ such that $x < z < y$.

Recall that the **Möbius function** $\mu$ of a finite poset $\mathcal{P}$ assigns an integer to each interval $[x,y] \subseteq \mathcal{P}$ such that for any $x,y \in \mathcal{P}$ we have

$$
\sum_{x \leq z \leq y} \mu(z) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise}. \end{cases}
$$

3. Shellability of certain posets of partitions

3.1. LEX-shellability. Let $\omega = (\omega_1,\ldots,\omega_r) \in \mathbb{R}^r$ such that $\sum_{i=1}^r \omega_i \in \mathbb{Z}$. The main result of this section is the LEX-shellability of the poset $\hat{L}_\omega$ of non-$\omega$-integral partitions; see Definition 2.4. The notion of LEX-shellability was first introduced in [23]; see also [24, §12.2.3]. For the convenience of the reader, we recall here the definition.

Definition 3.1. Let $\mathcal{P}$ be a graded poset, and $E(\mathcal{P}) = \{(x,y) \in \mathcal{P} \times \mathcal{P} | x \prec y\}$ be the set of covering relations of $\mathcal{P}$.

An **edge labelling** of $\mathcal{P}$ is a function $l : E(\mathcal{P}) \to \Lambda$, where $\Lambda$ is a poset. For any maximal chain $c = (x = x_0 \prec x_1 \prec \ldots \prec x_{k-1} \prec x_k = y)$ we consider the associated $k$-uple $l(c) := (l(x_0 \prec x_1), l(x_1 \prec x_2), \ldots, l(x_{k-1} \prec x_k)) \in \Lambda^k$. A chain $c$ is less than a chain $d$ if $l(c)$ lexicographically precedes $l(d)$.

Definition 3.2. An edge labelling is a **LEX-labelling** if the following condition holds: for any interval $[x,t]$, any maximal chain $c$ in $[x,t]$, and any $y,z \in c$, with $x < y < z < t$, if $c|_{[x,z]}$ and
$c_{[u,t]}$ are lexicographically least in $[x,z]$ and $[y,t]$ respectively, then $c$ is lexicographical least in $[x,t]$.\\

A poset that admits a LEX-labelling is **LEX-shellable**.

In Definition 3.4 we define an edge labelling of $\hat{L}_\omega(\Gamma)$, and then prove that it is indeed a LEX-labelling.

**Lemma 3.3.** *If the graph $\Gamma$ is a forest, then there exists a unique minimal $\omega$-integral flat of $\Gamma$.***

**Proof.** We may assume that $\Gamma$ is a tree. The removal of an edge $a \in E(\Gamma)$ disconnects the tree $\Gamma$ into two connected components $\Gamma_a \sqcup \Gamma^c_a$. The flat $F := \{a \in E(\Gamma) \mid \sum_{i \in V(\Gamma_a)} \omega_i \notin \mathbb{Z}\}$ is $\omega$-integral, minimal and unique with these properties, since any $\omega$-integral flat contains the edges $a \in E(\Gamma)$ such that $\sum_{i \in V(\Gamma_a)} \omega_i \notin \mathbb{Z}$. $\square$

Let us fix a total order $\prec$ on the edges of the graph $\Gamma$. For any flat $S$ we will define a total order $\prec_S$ on the edges of $\Gamma_S$. Let $T := T(S)$ be the spanning forest of $\Gamma_S$ which is lexicographically minimal with respect to the order $\prec$, and $F := F(S)$ be the unique minimal $\omega$-integral flat of $L(T(S)) \subseteq L_{\geq S}$. We define the total order $\prec_S$ on the edges of $\Gamma_S$ such that

1. For any $e_1 \in E(T) \nleq_F$, $e_2 \in E(T) \leq_F$, and $e_3 \in E(\Gamma_S) \setminus E(T)$,
   $$e_1 \prec_S e_2 \prec_S e_3;$$

2. For any $e_4, e_5 \in E(T) \nleq_F$ (respectively in $E(T) \leq_F$ or $E(\Gamma_S) \setminus E(T)$),
   $$e_4 \prec_S e_5 \quad \text{if and only if} \quad e_4 \prec e_5.$$

**Definition 3.4.** *The poset $\hat{L}_\omega = \hat{L}_\omega(\Gamma)$ is endowed with the labelling

$$l : E(\hat{L}_\omega) \to \bigsqcup_{S \in L_\omega} (E(\Gamma_S), \prec_S)$$

which sends any covering relation $S = \{S_1, S_2, S_3, \ldots, S_s\} \lessdot T = \{S_1 \cup S_2, S_3, \ldots, S_s\}$ to the minimum with respect to the order $\prec_S$ among the edges between the set of vertices $S_1$ and $S_2$, i.e.

$$l(S \lessdot T) = \min_{\prec_S} \{e \in E(\Gamma_S) \mid e \in E(\Gamma^{S_1}_{\leq S})\}.$$

$\Gamma = \Gamma^0$

$F(\hat{0}) = ac \in T(\hat{0}) = abc$

$\Gamma^\omega$

$F(d) = b \in T(d) = ab$

$\hat{a} \prec \hat{b} \prec \hat{c} \prec \hat{d}$

$\hat{b} \prec_0 \hat{a} \prec_0 \hat{c} \prec_0 \hat{d}$

$\hat{a} \prec_\omega \hat{b} \prec_\omega \hat{c}$

Figure 3. Let $\Gamma$ be a square whose edges are ordered by $a \prec b \prec c \prec d$ where $a$ and $c$ are opposite edges. Set $\omega = (1,\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Consider the flat $\hat{0} = 1|2|3|4$ and $\hat{d} = 14|2|3$. Then $\hat{b} \prec_0 \hat{a} \prec_0 \hat{c} \prec_0 \hat{d}$ and $\hat{a} \prec_\omega \hat{b} \prec_\omega \hat{c}$.

Lemma 3.5. Let $\Gamma$ be a graph with $r$ vertices, and $S < U$ be two flats. Then $E(T(U)) \subset E(T(S))$. Moreover, if $a \in E(T(S))$ then $E(T(S \cup a)) = E(T(S)) \setminus \{a\}$ and $E(S \cup a) = E(S) \cup a$.

Proof. The image of $E(T(S))$ under the contraction $\Gamma^S \to \Gamma^U$ is a spanning subgraph. In particular, the minimal spanning forest of $\Gamma^U$ can be identified with a subset of $E(T(S))$. If $a \in E(T(S))$, then $E(T(S)) \setminus \{a\}$ is a spanning forest in $\Gamma^{S\cup a}$, and the equality $E(T(S \cup a)) = E(T(S)) \setminus \{a\}$ holds. Notice that $L(T(S \cup a)) = L(T(S)) \ge a$, hence $E(S \cup a) \ge E(S) \cup a$. On the other hand, $E(S) \cup a$ is a $\omega$-integral flat in $L(T(S \cup a))$, therefore by definition $E(S \cup a) \le E(S) \cup a$. \hfill $\square$

Theorem 3.6. The labelling $l$ is a LEX-labelling for $\hat{L}_{\omega}$. In particular, $\hat{L}_{\omega}$ is LEX-shellable.

Proof. By [24, Proposition 12.13], it is enough to check the LEX-labelling condition for any $x \prec y \prec z \prec t$. Let $c$ be a maximal chain in $[x,t]$ such that $c^{[x,z]}$ and $c^{[y,t]}$ are lexicographically least. Replacing $\Gamma$ with $\Gamma^x$ we may assume $t = \hat{1}$ and $x = \hat{0}$.

Let $a = l(x \le y)$ and $b = l(y \le z)$, so that $l(c) = (a,b,\ldots)$. Since $c^{[y,t]}$ is lexicographically least then $b \in E(T(y)) \subset E(T(x))$. We have $x \le (x \vee b) \le z$ and since $c^{[x,z]}$ is lexicographically least then $a \prec x l(x \le x \vee b) \le x b$. In particular $a \in E(T(x))$.

By contradiction, we assume that the chain $c$ is not lexicographically least in $[x,t]$. Hence there exists $d \prec x a$ and $d \in E(T(x))$. Notice that $\{a,b,d\}$ is an independent set of distinct edges in $\Gamma^x$ because contained in $T(x)$, i.e. the poset $[x,x \vee a \vee b \vee d]$ is Boolean of rank 3.
Note that \( x \lor a \lor d \) cannot be \( \omega \)-integral. Otherwise, \( F(x) \leq x \lor a \lor d \), and \( d \leq F(x) \) since \( y = x \lor a \) is not \( \omega \)-integral. The conditions \( d \in E(T(x)) \leq F \), \( b \in E(T(x)) \) and \( d \prec_x b \) would imply \( b \in E(T(x)) \geq F \), so \( x \lor b \leq F(x) \leq x \lor a \lor d \), which is a contradiction since \( [x, x \lor a \lor b \lor d] \) is Boolean.

Therefore, \( y \lor d = x \lor a \lor d \in L_\omega \) and we are assuming that it is different from \( z = x \lor a \lor b \). Since \( c_{[y,t]} \) is lexicographically least, we have \( b \prec_y l(y \leq y \lor d) \geq_y d \). Moreover, \( b \leq F(y) = F(x \lor a) = F(x) \lor a \) if and only if \( b \leq F(x) \), and the same holds for \( d \). This fact, together the inequalities \( d \prec_x b \) and \( b \prec_y d \), gives a contradiction. \( \square \)

Suppose that \( \Gamma \) contains a complete graph on the vertices. Then one can show that the poset \( \hat{L}_\omega = \hat{\Pi}_\omega \) associated to the complete graph satisfied the stronger condition of EL-shellability for any \( \omega \in \mathbb{R}_r \); see for instance \([24, \S 2.1.2]\) for a definition of EL-shellability.

**Question 3.7.** Is the poset \( \hat{L}_\omega \) EL-shellable for any graph \( \Gamma \) and any vector \( \omega \)?

### 3.2. Homotopy type of the order complex of non-\( \omega \)-integral partitions.

Recall that the order complex \( \Delta(P) \) associated to a poset \( P \) is a simplicial complex whose vertices are in correspondence with the set underlying \( P \), and whose faces corresponds to finite chains in \( P \).

Recall that \( \overline{L}_\omega = L_\omega \setminus \{\hat{0}\} \). We denote by \( \tilde{H}_k(T) \) the reduced \( k \)-homology group of the topological space \( T \) and by \( \tilde{\chi}(T) \) the reduced Euler characteristic.

**Definition 3.8.** Given an edge labelling of a poset, a saturated chain \( c \) is mediocre if for any \( x, y, z \in c \) such that \( x \lessdot y \lessdot z \), the chain \( c_{[x,z]} \) is not lexicographically least in \( [x,z] \).

**Proposition 3.9.** Let \( r > 2 \). The order complex \( \Delta(\overline{L}_\omega) \) has the homotopy type of the wedge of \( (r-3) \)-spheres. The number of mediocre maximal chains in \( \Pi_\omega \) equals the number of \( (r-3) \)-spheres in \( \Delta(\overline{L}_\omega) \), i.e. the rank of the reduced homology group \( \tilde{H}_{r-3}(\Delta(\overline{L}_\omega)) \).

**Proof.** It follows from Theorem 3.6 and the general properties of LEX-shellability \([24, \text{Theorem 12.15}]\). \( \square \)

**Remark 3.10.** If \( r = 2 \), \( \Delta(\overline{L}_\omega) \) is the empty set, but since \( \tilde{H}_{-1}(\emptyset) = \mathbb{Z} \) we still have \( -\mu(\hat{0}, \hat{1}) = 1 = \text{rk} \tilde{H}_{-1}(\emptyset) \).
Proposition 3.11. The following equalities hold for $r > 1$:

$$\text{rk } \tilde{H}_{r-3}(\Delta(\mathcal{Z}_\omega)) = |\chi(\Delta(\mathcal{Z}_\omega))| = (-1)^{r-1} \sum_{\lambda \in \mathcal{L} \setminus \mathcal{L}_\omega} \mu_{\mathcal{L}}(\hat{0}, \lambda)$$

Moreover, if $\Gamma$ contains a complete graph on the vertices, then

$$\text{rk } \tilde{H}_{r-3}(\Delta(\mathcal{P}_\omega)) = \sum_{\lambda \vdash [r]} (-1)^{|\lambda|-1} \prod_{i=1}^{\lambda} (|\lambda_i| - 1)!.$$

Proof. The first equality follows from Proposition 3.9 and Remark 3.10. For the second equality, we use that $\mu_{\mathcal{L}}(\hat{0}, \lambda) = \mu_{\mathcal{L}}(\hat{0}, \hat{1})$ for all flats $\lambda \in \mathcal{L}_\omega$. Therefore, we obtain

$$(-1)^{r-1} \sum_{\lambda \in \mathcal{L} \setminus \mathcal{L}_\omega} \mu_{\mathcal{L}}(\hat{0}, \lambda) = (-1)^{r} \sum_{\lambda \in \mathcal{L}_\omega} \mu_{\mathcal{L}}(\hat{0}, \lambda)$$

$$= (-1)^{r-1} \mu_{\mathcal{L}_\omega}(\hat{0}, \hat{1})$$

$$= (-1)^{r-1} \chi(\Delta(\mathcal{Z}_\omega)).$$

The last equality is the Philip Hall’s theorem [45, Proposition 3.8.6]; the other inequalities follow directly from the definition of Möbius function.

In the case of the complete graph, the Möbius function of the partition poset $\mathcal{P}$ is

$$\mu_{\mathcal{P}}(\hat{0}, \lambda) = (-1)^{r-|\lambda|} \prod_{i=1}^{\lambda} (|\lambda_i| - 1)!$$

for $\lambda \vdash [r]$; see [45, Example 2.9] or [45, Example 3.10.4].

Proposition 3.12. The sum in (3) is positive if and only if $\omega \notin \mathbb{Z}^r$ and $r > 1$.

Proof. If $\omega \in \mathbb{Z}^r$, then the vanishing of the sum in (3) follows from Remark 5.4(ii). Alternatively, by definition of Möbius function and Proposition 3.11 we have

$$\sum_{\lambda \vdash [r]} (-1)^{\ell(\lambda)} \prod_{i=1}^{\lambda} (|\lambda_i| - 1)! = (-1)^{r-1} \sum_{\lambda \vdash [r]} \mu_{\mathcal{P}}(\hat{0}, \lambda) = 0 \quad \text{when } r > 1.$$

If $\omega \notin \mathbb{Z}^r$, we must verify the positivity of (3). By Proposition 3.9 and Proposition 3.11 it suffices to exhibit a mediocre maximal chain $c$ in $\mathcal{P}_\omega$. We construct $c$ inductively by using a greedy algorithm: $\hat{0}$ is in $c$, and for any $x \in c$ we add the unique $y > x$ such that $l(y > x)$ is maximal with respect to $\prec x$.

Given $x, t \in c$ with $x < t$ and $x \not\prec t$, we want to show that $c_{[x,t]}$ is not lexicographically least in $[x, t]$. To this end, choose $y, z \in c$ such that $x \ll y < z$. As $\Gamma$ contains a complete graph on the
vertices, $\Gamma_x^z$ contains a complete subgraph over 3 vertices with edges $\{a, b, d\}$. In particular, the poset $\mathcal{L}(\Gamma_x^z)$ contains the 3 atoms $y = x \lor a$, $x \lor b$, $x \lor d$. If two of them are $\omega$-integral, then the third must be so too, but since $y$ is non-$\omega$ integral, then at least one of $x \lor b$ and $x \lor d$ is non $\omega$-integral too, say $x \lor b$. Then the chain $(c_{[x,t]} \setminus y) \cup (x \lor b)$ obtained substituting $y$ with $x \lor b$ lexicographically precedes $c_{[x,t]}$ by construction of the chain $c$. Therefore the chain $c$ is mediocre, and the sum in (3) is positive. □

4. Lattice points in zonotopes

In this section, we recall the definition of zonotope, and we provide the key formula for the number of lattice points in the interior of its translates; see Theorem 4.5.

Recall that the Minkowski sum of several subsets $A,...,B \in \mathbb{R}^r$ is the locus of sums of vectors that belong to these subsets

$$A + \ldots + B := \{a + \ldots + b | a \in A, \ldots, b \in B\}.$$ 

If $A,\ldots,B$ are convex and/or lattice polytopes, then so is their Minkowski sum.

**Definition 4.1.** Let $U$ be a finite set of points in $\mathbb{R}^r \setminus \{0\}$. The **zonotope** generated by $U$ is the Minkowski sum of line segments $[0,u]$ with $u \in U$, i.e.

$$Z(U) := \sum_{u \in U} [0,u].$$

The zonotope $Z(U)$ decomposes in the disjoint union of translates of the half-open parallelepipeds $\sum_{u \in W} [0,u]$ spanned by the linearly independent subsets $W \subseteq U$; see [43, Theorem 54]. This observation allows to compute the number of lattice points in the interior of a translate of $Z(U)$ in Theorem 4.4, as proved in [1, Proposition 3.1].

To state the result, we first adapt the definitions of §2.2 to the present context, and recall the definition of Ehrhart quasi-polynomial.

**Definition 4.2.** A **flat** of $U$ of rank $k$ is a maximal subset of $U$ spanning a vector subspace of rank $k$. The poset of flats of $U$ is denoted $\mathcal{L} = \mathcal{L}(U)$.

Let $Z^r \subset \mathbb{R}^r$ be the standard lattice, and $\omega$ be a vector in $\mathbb{R}^r$. Then a flat $S \subseteq U$ is $\omega$-**integral** if the affine subspace obtained by translating the vector space spanned by $S$ by the translation vector $\omega$ contains lattice points, i.e. $(\omega + \langle S \rangle) \cap Z^r \neq \emptyset$. The poset of $\omega$-integral flats is denoted $\mathcal{L}_\omega = \mathcal{L}_\omega(U)$.

Given a set $W$ of independent vectors in $Z^r \subset \mathbb{R}^r$, the **relative volume** of the parallelepiped generated by $W$, denoted $\text{vol}(W)$, is the volume of the parallelepiped in $\mathbb{R}^r$ normalized by the
volume of a basis of the lattice \((W) \cap \mathbb{Z}^r\), where \((W)\) is the linear span of \(W\) in \(\mathbb{Q}^r\). Alternatively, it is the index of the sublattice generated by \(W\) in \((W) \cap \mathbb{Z}^r\).

**Definition 4.3.** The **Ehrhart quasi-polynomial**

\[
L_P(t) := |tP \cap \mathbb{Z}^r|
\]

counts the number of lattice points in the dilated polytope \(tP\).

**Theorem 4.4.** [1, Proposition 3.1] Let \(U \subset \mathbb{Z}^r\) be a finite set of integer vectors, \(\omega \in \mathbb{Q}^r\) a rational vector, and \(Z(U)\) be the zonotope generated by \(U\). Then the Ehrhart quasi-polynomial of \(Z(U) + \omega\) is

\[
L_{Z(U)+\omega}(t) = \sum_{\substack{W \subseteq U \text{ independent} \quad \langle W \rangle = \langle S \rangle \text{ for some } S \in L}} \delta_{W,\omega}(t) \operatorname{vol}(W) |W| t^{|W|};
\]

where

\[
\delta_{W,\omega}(t) = \begin{cases} 
1 & \text{if } (t \omega + \langle W \rangle) \cap \mathbb{Z}^r \neq \emptyset, \\
0 & \text{otherwise}.
\end{cases}
\]

Given any polytope \(P\) we denote by \(\text{Int}(P)\) the interior of \(P\) in its linear span. The number of lattice points contained in the interior of the polytope \(P\) is denoted

\[
C(P) := |\text{Int}(P) \cap \mathbb{Z}^r|.
\]

By Ehrhart-Macdonald reciprocity [25], \(C(P)\) is a value of the Ehrhart quasi-polynomial \(L_P(t)\) up to sign, namely

\[
C(P) = (-1)^{\dim(P)} L_P(-1).
\]

If \(P\) is a translated zonotope \(Z(U) + \omega\), then Möbius inversion applied to Theorem 4.4 gives the combinatorial identity (5) and (9).

**Theorem 4.5.** Let \(U \subset \mathbb{Z}^r\) be a finite set of integer vectors, \(\omega \in \mathbb{Q}^r\) a rational vector, and \(Z(U)\) be the zonotope generated by \(U\). Then we have

\[
C(Z(U) + \omega) = \sum_{S \in L} \left( \sum_{T \in L \geq S \setminus L_{\omega}} \mu_L(S, T) \right) C(Z(S)).
\] (5)

**Proof.** From Theorem 4.4 we write

\[
L_{Z(U)+\omega}(t) = \sum_{S \in L} \delta_{S,\omega}(t) \left( \sum_{\substack{W \subseteq S \text{ independent} \quad \langle W \rangle = \langle S \rangle \text{ for some } S \in L}} \operatorname{vol}(W) t^{|W|} \right) t^{|k(S)|}
\]
where
\[
\delta_{S,\omega}(t) = \begin{cases} 
1 & \text{if } (t\omega + \langle S \rangle) \cap \mathbb{Z}^r \neq \emptyset, \\
0 & \text{otherwise},
\end{cases}
\]
because the function \(\delta_{W,\omega}(t)\) does not depend on \(W\) but only on the flat \(\langle W \rangle = \langle S \rangle\).

Let us define
\[
Q_S(t) := \ell^k(S) \sum_{W \subseteq S \text{ independent}} \frac{\text{vol}(W)}{\langle W \rangle = \langle S \rangle}
\]
and notice that it is independent on \(\omega\). For any flat \(T \in \mathcal{L}\) we write
\[
L_{Z(T)+\omega}(t) = \sum_{S \leq T} \delta_{S,\omega}(t) Q_S(t).
\]
We specialize to the case \(\omega = 0\) so that \(\delta_{S,0} = 1\), and we apply the Möbius inversion formula to obtain
\[
Q_T(t) = \sum_{S \leq T} \mu_{\mathcal{L}}(S,T) L_{Z(S)}(t).
\]
Therefore, for any \(\omega \in \mathbb{R}^r\) we have
\[
L_{Z(U)+\omega}(t) = \sum_{T \in \mathcal{L}} \delta_{T,\omega}(t) \left( \sum_{S \leq T} \mu_{\mathcal{L}}(S,T) L_{Z(S)}(t) \right) = \sum_{S \in \mathcal{L}} \left( \sum_{T \geq S} \mu_{\mathcal{L}}(S,T) \delta_{T,\omega}(t) \right) L_{Z(S)}(t).
\]
Specializing at \(t = -1\) we have \(\delta_{T,\omega}(-1) = 1\) if and only if \(T\) is \(\omega\)-integral, and we obtain
\[
C(Z(U) + \omega) = \sum_{S \in \mathcal{L}} \left( \sum_{T \geq S} \mu_{\mathcal{L}}(S,T) \frac{\text{vol}(T_{\omega\text{-integral}})}{\langle T \rangle = \langle S \rangle} \right) C(Z(S)).
\]
This completes the proof.

5. Graphical zonotopes

In this section, we specialize the lattice point count formula in Theorem 4.5 to the case of graphical zonotopes of a complete graph with multiple edges; see Corollary 5.3.

5.1. Definition and equations of graphical zonotope. Denote by \(x_1, \ldots, x_r\) the real coordinates of the affine space \(\mathbb{R}^r\) with respect to an integer basis \(v_1, \ldots, v_r\) of the standard lattice \(\mathbb{Z}^r \subset \mathbb{R}^r\).

Let \(\Gamma\) be a graph on the vertex set \(I \subset [r]\). Denote \(E(\Gamma)\) the set of edges of \(\Gamma\). To every \(e \in E(\Gamma)\) one can associate the pair of endpoints \(\{s(e), t(e)\}\). For any pair of vertices \(J \subseteq I\), set
\[
y_J := |\{e \in E(\Gamma) \mid \{s(e), t(e)\} = J\}|.
\]
Definition 5.1. Denote $\Delta_K$ the convex hull of the points $v_k$ for all $k \in K \subseteq [r]$. The graphical zonotope $Z_\Gamma$ of $\Gamma$ in $\mathbb{R}^r$ is the Minkowski sum of the line segments $[s(e), t(e)]$, with $e \in E(\Gamma)$, i.e.

$$Z_\Gamma := \sum_{e \in E(\Gamma)} [s(e), t(e)] = \sum_{J \subseteq I, |J|=2} y_J \Delta_J.$$  

In case $I = \{i\}$, we set $Z_\Gamma$ to be $v_i$.

Equivalently, for $|I| > 1$, the zonotope $Z_\Gamma$ is the lattice polytope in $\mathbb{R}^I := \text{Span}\{v_i : i \in I\} \subseteq \mathbb{R}^r$ defined by the system of equality and inequalities

$$\sum_{i \in I} x_i = z_I, \quad \sum_{i \in K} x_i \geq z_K \quad \forall K \subset I,$$

where $z_K := \sum_{J \subseteq K} y_J$; see [39, Proposition 6.3].

5.2. Flats and faces. Fix an orientation for the graph $\Gamma$. The finite set of vectors $U_\Gamma := \{t(e) - s(e) : e \subseteq E(\Gamma)\} \subset \mathbb{R}^r$ is the image of the basis $E(\Gamma)$ of $\mathbb{Z}[E(\Gamma)]$ via the boundary map

$$\delta: \mathbb{Z}[E(\Gamma)] \to \mathbb{Z}[V(\Gamma)],$$

$$e \mapsto t(e) - s(e).$$

By construction, the graphical zonotope $Z_\Gamma$ is isomorphic to the zonotope $Z(U_\Gamma) \subseteq \mathbb{R}[V(\Gamma)]$ via a translation by an integral vector (depending on the orientation).

Since $\ker \delta = H^1(\Gamma, \mathbb{Z})$ and $\text{coker} \delta = H^0(\Gamma, \mathbb{Z})$, there exist bijective correspondences between

- independent subsets of $U_\Gamma$ $\iff$ forests of $\Gamma$, i.e. graphs $F$ with $H^1(F, \mathbb{Z}) = 0$,
- $\mathcal{L}(U_\Gamma) := \text{flats of } U_\Gamma$ $\iff$ $\mathcal{L}(\Gamma) := \text{flats of } \Gamma$;

![Graphical zonotopes of the complete graph $\Gamma$ with $r = 3$ or $4$ vertices respectively. Each face of $Z_\Gamma$ is a translate of $Z_{\Gamma_S}$ for some flat $S \in \mathcal{L}(\Gamma)$.](image)
see Definitions 2.3 and 4.2. In particular, the latter correspondence assigns the partition $\mathcal{S} \in \mathcal{L}(\Gamma)$ to the flat in $\mathcal{L}(U\Gamma)$ cut by the linear subspace

$$W_\mathcal{S} := \{ \mathbf{x} \in \mathbb{R}^r \mid \sum_{j \in S_i} x_j = 0 \text{ for all } S_i \in \mathcal{S} \}.$$  

(7)

Note that the affine span of any face of the zonotope $Z(U\Gamma)$ is an integral translation of the linear space spanned by a flat $S \in \mathcal{L}(U\Gamma)$, and the given face is isomorphic to $Z(\mathcal{S})$, up to an integral translation. In particular, any face of a graphical zonotope $Z_\Gamma$ is isomorphic to $Z_\Gamma(\mathcal{S}) = Z_\Gamma(S_1) \times \ldots \times Z_\Gamma(S_l)$ (8) for some flat $\mathcal{S} = \{S_1, \ldots, S_l\} \in \mathcal{L}(\Gamma)$ via an integral translation. Viceversa, a flat $\mathcal{S} \in \mathcal{L}(\Gamma)$ with an acyclic orientation of the contracted graph $\Gamma_\mathcal{S}$ determines uniquely a face of $Z_\Gamma$; see [36, Proposition 1.3].

Remark 5.2. The factorization (8) of the faces of $Z_\Gamma$ explains the occurrence of the factor $\prod_{j=1}^{\ell(S)} C(Z_\Gamma(S_j))$ in (9), and it is a combinatorial analogue of the splitting of the normal slice in (26).

5.3. Lattice point counts in graphical zonotopes. We now specialize Theorem 4.5 to the case of the graphical zonotopes of a complete graph with multiple edges.

Corollary 5.3. If $\Gamma$ contains a complete graph on the vertices, then for any $\omega \in \mathbb{Q}^r$ we have

$$C(Z_\Gamma + \omega) = \sum_{\mathcal{S} \vdash [r]} \left( \sum_{\lambda \vdash \mathcal{S}, \omega_\mathcal{S} \text{ integral}} (-1)^{\ell(\lambda)} \frac{\ell(\lambda)!}{\prod_{i=1}^{\ell(\lambda)} (|\lambda_i| - 1)!} \prod_{j=1}^{\ell(S)} C(Z_\Gamma(S_j)) \right) \prod_{j=1}^{\ell(S)} C(Z_\Gamma(S_j))$$

(9)

where $\omega_\mathcal{S} = (\omega_{S_1}, \ldots, \omega_{S_l}) \in \mathbb{R}^{\ell(\mathcal{S})}$, and $\omega_{S_i} = \sum_{j \in S_i} \omega_j$. Moreover, if $\sum_{i=1}^r \omega_i \in \mathbb{Z}$, then

$$C(Z_\Gamma + \omega) = C(Z_\Gamma) + \sum_{\mathcal{S} \in \Pi_\omega} \text{rk } \tilde{H}_{\ell(\mathcal{S})-3}(\Delta(\Pi_{\omega_\mathcal{S}})) \prod_{j=1}^{\ell(\mathcal{S})} C(Z_\Gamma(S_j)).$$

(10)

Proof. It follows from Theorem 4.5, Section 5.2 and Proposition 3.11.

Remark 5.4.

(i) If $\omega \in \Delta_{[r]}$, then the only $\omega$-integral partition is the trivial one. Hence

$$C(Z_\Gamma + \omega) = \sum_{\mathcal{S} \vdash [r]} (\ell(\mathcal{S}) - 1)! \prod_{j=1}^{\ell(\mathcal{S})} C(Z_\Gamma(S_j)).$$
(ii) If $\omega \in \mathbb{Z}^r$, then any partition is $\omega$-integral. Therefore, for any partition $S \vdash \{r\}$, the coefficient of $\prod_{j=1}^{\ell(S)} C(\Gamma(S_j))$ in (9) is

$$\sum_{\lambda \vdash \mathbb{N}} (-1)^{\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} (|\lambda_i| - 1)! = \sum_{\sigma \in \mathcal{S}_k} (-1)^{k-1-\ell(\sigma)} \prod_{i=1}^k (|\sigma_i| - 1)!$$

where $\ell(\sigma)$ is the length of the permutation $\sigma \in \mathcal{S}_k$, i.e. the smaller number of transpositions whose product gives $\sigma$. This coefficient vanishes for $\ell(S) > 1$, so we obtain the trivial identity

$$C(Z_{\mathcal{G}} + \omega) = C(Z_{\Gamma([r])}) = C(Z_{\mathcal{G}}).$$

(iii) Let $\omega = d(n_1, n_2, \ldots, n_r)$ and $n = \sum_{i=1}^r n_i$. Assuming the claim, then

$$C(Z_{\mathcal{G}} + \omega) = C\left(Z_{\mathcal{G}} + \gcd(d, n)\left(\frac{n_1}{n}, \frac{n_2}{n}, \ldots, \frac{n_r}{n}\right)\right).$$

In other words, $C(Z_{\mathcal{G}} + \omega)$ depends only on $\gcd(d, n)$ and $n_i$, for $i = 1, \ldots, r$.

Example 5.5. Let $K(n, e)$ be the graph on $n$-vertices with $e$ edges between any two distinct vertices. The graph $K(n, 1)$ is the complete graph $K_n$. For any partition $\underline{n} \vdash n$, we denoted by $b(\underline{n})$ the number of partitions $\underline{S} = \{S_1, \ldots, S_k\}$ of the set $[n]$ whose partition type is $\underline{n}$, i.e. $\underline{n} = \{\left|S_1\right|, \ldots, \left|S_k\right|\}$.

Set

$$l(\underline{n}, \omega) := \sum_{\lambda \vdash \underline{n}} (-1)^{\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} (|\lambda_i| - 1)!. $$

Hence, we can rewrite (9) as follows

$$C(Z_{K(n,e)} + \omega) = \sum_{\underline{n} \vdash \{n_1, \ldots, n_k\} \vdash n} b(\underline{n}) \cdot l(\underline{n}, \omega) \cdot \prod_{j=1}^k C(Z_{K(n_j,e)}).$$

(11)

Fix for instance $n = 4$, and let $\omega(d) = d(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \in \mathbb{R}^r$. We collect the values of $b(\underline{n})$ and $l(\underline{n}, \omega(d))$ in the Table 1.

---

1 The integer $b(\underline{n})$ has a clear geometric meaning: the stratum $S_\underline{n}$ defined in (9) is not normal in general, and $b(\underline{n})$ is the number of branches of $S_\underline{n}$ passing through the general point of $S_{\vdash [1]}^\Gamma$. 

Table 1. Values of $b(n)$ and $l(n, \omega(d))$ for $n = 4$. The third row equals $(\ell(n) - 1)!$, as explained in Remark 5.4.(i). To fill the fifth row, note that the only $\omega(2)$-integral partitions are $12|34$, $13|24$, $14|23$, $1234$. The coefficients $l(n, \omega(d))$ have been computed also in [28, Table 1] using a different algebro-geometric approach.

To determine $C(K(n,e) + \omega)$, it is then sufficient to count the number of lattice points in the interior of the non-translated zonotope $Z_{K(m,e)}$ for $m \leq 4$. To this end, observe that the Tutte polynomial of the complete graph $K_{m+1}$ can be obtained recursively from the following formula [17, 38]

$$T_m(x,y) = \sum_{k=1}^m \binom{m-1}{k-1} (x+y+y^2+\cdots+y^{k-1})T_{k-1}(1,y)T_{m-k}(x,y).$$

By [32, 39], the Tutte polynomial determines the Ehrhart polynomial for the zonotope $Z_{K_m}$ as follows

$$E_{K_{m+1}}(q) = \left| qZ_{K_{m+1}} \cap \mathbb{Z}^m \right| = q^m T_m(1 + \frac{1}{q}, 1).$$

Now by Ehrhart–Macdonald reciprocity we have

$$C(K(m,e)) = (-1)^m E_{K_m}(-e).$$

We report these polynomial for $m \leq 4$ in Table 2. Finally, we obtain the desired lattice point count

- $C(K(4,e), \omega(0)) = 16e^3 - 15e^2 + 6e - 13$,
- $C(K(4,e), \omega(1)) = 16e^3$,
- $C(K(4,e), \omega(2)) = 16e^3 - 3e^2$.

6. Permutation representations on lattice points in graphical zonotopes

Let $\text{Aut}(\Gamma) < \mathfrak{S}_r$ be the group of automorphisms of the graph $\Gamma$ with $r$ vertices, i.e. the permutations of the vertices of $\Gamma$ mapping edges of $\Gamma$ in edges of $\Gamma$. In this section, we promote the numerical identity (9) to an isomorphism of $\text{Aut}(\Gamma)$-representations; see Theorem 6.4.

From now on assume that $\omega \in \mathbb{Q}^r$ is an $\text{Aut}(\Gamma)$-invariant vector. The group $\text{Aut}(\Gamma)$ acts by permutations on the graphical zonotope $Z_{\Gamma} \subset \mathbb{R}^r$, and so on the vector space with basis the set of integral points in $\text{Int}(Z_{\Gamma} + \omega)$. We denote the latter $\text{Aut}(\Gamma)$-representation by $C(\Gamma, \omega)$.
Table 2. First values of Tutte polynomials, Ehrhart polynomials for the complete graphs, and $C(Z_{K(m,e)})$.

| $m$ | $T_{m-1}(x,y)$ | $E_{K_m}$ | $C(K(m,e))$ |
|-----|----------------|-----------|-------------|
| 1   | 1              | 1         | 1           |
| 2   | $x$            | $q + 1$   | $e - 1$     |
| 3   | $x^2 + x + y$  | $3q^2 + 3q + 1$ | $3e^2 - 3e + 1$ |
| 4   | $x^3 + 3x^2 + 2x + 4xy + 2y + 3y^2 + y^3$ | $1 + 6q + 15q^2 + 16q^3$ | $16e^3 - 15e^2 + 6e - 1$ |

Figure 5. Let $K_4$ be the complete graph with 4 vertices. The pictures represent the 3-dimensional polytope $Z_{K_4} + \omega(d) \subset \mathbb{R}^4$, i.e. the graphical zonotope of $Z_{K_4}$ translated by the vector $\omega(d)$. The lattice points in the interior of $Z_{K_4} + \omega(d)$ are the circles filled in black, and their number is $C(Z_{K_4} + \omega(d))$. The visible faces of the zonotope, i.e. those that are not contained in the closure of faces whose outward normal unit vector $n$ in $\mathbb{R}^3$ pairs negatively with $v := (1, 1, 1)$, in symbols $n \cdot v < 0$, can be enumerated as follows: 4 hexagons, 3 rectangles, 12 edges and 6 points. The same multiplicities appear in (11), namely

$$C(K_4 + \omega(1)) = C(K_4) + 4C(K_3)C(K_1) + 3C(K_2)^2 + 12C(K_2)C(K_1)^2 + 6C(K_1)^4.$$
Furthermore, the group $\text{Aut}(\Gamma) < S_r$ acts by permutations on the posets $\mathcal{L}$ and $\mathcal{L}_\omega$ as $\omega$ is $\text{Aut}(\Gamma)$-invariant. The stabilizer of $S \in \mathcal{L}$, denoted $\text{Stab}(S) < \text{Aut}(\Gamma)$, acts on $\mathcal{L}_{\omega \geq S}$. Therefore, $\tilde{H}_{\ell}(\mathcal{L})$ admits a right action of $\text{Stab}(S)$.

**Definition 6.1.** Let $\Gamma$ be an oriented graph. Define the **orientation character**

$$ o_\Gamma: \text{Aut}(\Gamma) \to \mathbb{Z}/2\mathbb{Z} \quad \text{by} \quad o_\Gamma(\sigma) := \prod_{e \in E(\Gamma)} o(e, \sigma), $$

where $o(e, \sigma)$ is 1 if the map $\sigma|_e: e \to \sigma(e)$ is orientation-preserving and $-1$ otherwise.

Note that the definition of $o_\Gamma$ does not depend on the chosen orientation and it is indeed multiplicative, i.e.

$$ o_\Gamma(\sigma) o_\Gamma(\tau) = o_\Gamma(\sigma \tau). \quad (12) $$

Indeed, let $\Gamma^d$ be the double of $\Gamma$, i.e. the oriented graph obtained from $\Gamma$ by replacing each edge $e$ of $\Gamma$ with a pair of edges $e^+$ and $e^-$ having the same endpoints as $e$ but opposite orientations. Any $\sigma \in \text{Aut}(\Gamma)$ define an automorphism $\sigma^d \in \text{Aut}(\Gamma^d)$ given by

$$ \sigma^d(e^\pm) = \begin{cases} 
\sigma(e)^\pm & \text{if } o(e, \sigma) = 1, \\
\sigma(e)^\mp & \text{if } o(e, \sigma) = -1.
\end{cases} $$

Thus we can identify $\text{Aut}(\Gamma)$ and $\text{Aut}(\Gamma^d)$ as subgroups of $S_r$. Now, $o_\Gamma$ is the sign representation of the action by permutation of $\text{Aut}(\Gamma)$ on the set of edges $E(\Gamma^d)$ of $\Gamma^d$

$$ \text{Aut}(\Gamma) \simeq \text{Aut}(\Gamma^d) \to S(E(\Gamma^d)). $$

In the following, we also consider the sign representation $\text{sgn}(\sigma: V(\Gamma) \to V(\Gamma))$ with respect to the action on the vertex set.

**Definition 6.2.** Let $S \in \mathcal{L}$ be a flat of $\Gamma$, $\text{Stab}(S) < \text{Aut}(\Gamma)$ be its stabilizer and $\Gamma^S$ be the contracted (loopless) graph defined in Definition 2.3. Define $\alpha_S$ the character of $\text{Stab}(S)$ given by

$$ \alpha_S = \text{sgn}(\sigma: V(\Gamma^S) \to V(\Gamma^S)) \otimes o_{\Gamma^S}(\sigma). $$

**Example 6.3.** Consider the graph $\Gamma$
where \( e \) and \( f \) are the number of edges between two vertices. If \( e \neq f \), then \( \Aut(\Gamma) = \mathbb{Z}/2\mathbb{Z} = \langle (12) \rangle \) and \( a_\Gamma(\sigma) = (-1)^{e+1} \).

**Theorem 6.4.** For any graph \( \Gamma \) and for any \( \Aut(\Gamma) \)-invariant vector \( \omega \in \mathbb{Q}^r \), we have

\[
C(Z_\Gamma + \omega) = C(Z_\Gamma) \oplus \bigoplus_{S \in L_\omega / \Aut(\Gamma)} \Ind_{\text{Stab}(S)}^{\Aut(\Gamma)} (\alpha_S \otimes \bar{H}_{\ell(S)} - 3(\Delta(Z_\omega, S))) \otimes C(Z_{\Gamma_S}),
\]

(13)

with \( \omega_S = (\omega_{S_1}, \ldots, \omega_{S_{\ell(S)}}) \in \mathbb{R}^{\ell(S)} \) and \( \omega_S = \sum_{j \in S} \omega_j \).

**Remark 6.5.** The isomorphism type of the induced representation

\[
\Ind_{\text{Stab}(S)}^{\Aut(\Gamma)} (\alpha_S \otimes \bar{H}_{\ell(S)} - 3(\Delta(Z_\omega, S)))
\]

is independent of the choice of a representative \( S \) in its \( \Aut(\Gamma) \)-coset of \( L_\omega \).

**Example 6.6.** Fix \( \omega = (\frac{1}{2}, \frac{1}{2}, 0) \). Consider the graph \( \Gamma \) below and its poset of non-\( \omega \)-integral flats \( L_\omega \)

\[
\begin{array}{c|c|c}
\text{(13|2)} & \text{23|1} \\
\hline
\text{1|2|3} & \\
\end{array}
\]

Then there exists an isomorphism of \( \Aut(\Gamma) \)-representations

\[
C(Z_\Gamma + \omega) = C(Z_\Gamma) \oplus \text{Reg}^{\oplus 3} \oplus (\text{sgn} \otimes \text{sgn} \otimes 1),
\]

where \( \text{Reg} \) and \( \text{sgn} \) denote respectively the regular and sign representation of \( \Aut(\Gamma) = \langle (12) \rangle = \mathbb{Z}/2\mathbb{Z} \).

### 6.1. Fixed loci of zonotopes.

The cycle type of a partition \( \sigma \in \Aut(\Gamma) \times \mathfrak{S}_r \) is the partition of \( r \) consisting of the lengths \( l_1, l_2, \ldots, l_{|\sigma|} \) of the cycles \( \sigma_1, \sigma_2, \ldots, \sigma_{|\sigma|} \) of \( \sigma \). Define the quotient graph \( \Gamma/\sigma \) as the loopless graph whose vertices are orbits of vertices of \( \Gamma \) under the action of \( \langle \sigma \rangle \) and whose edges are orbits of edges of \( \Gamma \) with endpoints in different orbits of vertices. Explicitly, the set of vertices is \([\ell(\sigma)]\) and the number of edges between \( i \) and \( j \) is

\[
x_{ij} := \frac{1}{\text{lcm}(l_i, l_j)} \sum_{a \in \sigma_i, b \in \sigma_j} y_{ab} \in \mathbb{Z}.
\]

Define the translation vectors \( \omega_\sigma \) and \( t_\sigma = t_\sigma(\Gamma) \) in coordinates by

\[
\omega_\sigma, i := \frac{1}{l_i} \sum_{a \in \sigma_i} \omega_a, \quad t_\sigma, i := \frac{1}{l_i} \sum_{\{a, b\} \in \sigma_i} y_{ab}.
\]
Finally, by linearity it is enough to compute

\[ t_{\sigma,i} = \frac{1}{l_i} \sum_{\{a,b\} \subseteq \sigma_i} y_{ab} = \sum_{\{a,b\} \subseteq \sigma_i} \left( \frac{1}{l_i} \sum_{b \neq \sigma_i/2(a)} y_{ab} + \frac{1}{l_i} \sum_{b = \sigma_i/2(a)} y_{ab} \right) \]  

(14)

is an half-integer. Moreover, \( t_{\sigma,i} \) is an integer if and only if \( l_i \) is odd, or if \( l_i \) and \( y_{ab\sigma_i/2(a)} \) are both even.

A permutahedron is endowed with a standard action of the symmetric group. The fixed locus of any such permutation is described in [2, Proposition 2.11]. In the next lemma, we extend it to arbitrary graphical zonotopes.

**Lemma 6.8.** Let \( \sigma \in \text{Aut}(\Gamma) \) be a permutation with cycle lengths \( l_1, l_2, \ldots, l_{|\sigma|} \). Then the fixed locus \( Z_\sigma^\Gamma \) of the graphical zonotope \( Z_\Gamma \) by the action of \( \sigma \) is the zonotope

\[ \pi(Z_\sigma^\Gamma) = \sum_{i < j \in [\sigma]} x_{ij} \left[ \frac{l_j}{\gcd(l_i, l_j)} e_i, \frac{l_i}{\gcd(l_i, l_j)} e_j \right] + \sum_{i=1}^{\sigma} t_i a_i \]

where \( \{e_i\}_i \) is the standard basis of \( \mathbb{R}^{[\sigma]} \).

**Proof.** Consider the averaging map \( \pi : \mathbb{R}^r \to \mathbb{R}^{[\sigma]} \) defined by

\[ \pi(x) = \left( \frac{1}{l_i} \sum_{a \in \sigma_i} x_a \right)_{i=1, \ldots, |\sigma|} \]

and the map \( \iota : \mathbb{R}^{[\sigma]} \to \mathbb{R}^r \) defined by \( y \mapsto x \) where \( x_a = y_i \) if \( a \in \sigma_i \). The set of \( \sigma \)-fixed points is \( \text{Im} \iota \) and \( \pi \circ \iota = \text{id}_{\mathbb{R}^{[\sigma]}} \). Hence \( Z_\sigma^\Gamma \subseteq \iota(\pi(Z_\Gamma)) \). The other inclusion \( Z_\sigma^\Gamma \supseteq \iota(\pi(Z_\Gamma)) \) follows from the convexity of \( Z_\Gamma \) and the \( \sigma \)-invariance \( \sigma(Z_\Gamma) = Z_\Gamma \).

Finally, by linearity it is enough to compute \( \pi([e_a, e_b]) \). If \( a, b \in \sigma_i \) then \( \pi([e_a, e_b]) = e_i/\Gamma \), otherwise \( a \in \sigma_i, b \in \sigma_j \) and \( \pi([e_a, e_b]) = [e_i/\Gamma, e_j/\Gamma] \). Notice that

\[ x_{ij} \left[ \frac{l_j}{\gcd(l_i, l_j)} e_i, \frac{l_i}{\gcd(l_i, l_j)} e_j \right] = \sum_{a \in \sigma_i, b \in \sigma_j} y_{ab} \left[ e_i/l_i, e_j/l_j \right]. \]

\[ ^2\text{Observe that the description of the permutahedron chosen in [2, Proposition 2.1] differs from Definition 5.1 by a translation by an integral vector.} \]
Let $K_4$ be the complete graph with 4 vertices, and $\sigma = (12) \in S_4$. The fixed locus $Z^\sigma_{K_4}$ of the graphical zonotope $Z_{K_4}$ by the action of $\sigma$ is the zonotope
\[ \pi(Z^\sigma_{K_4}) = [e_1, 2e_2] + [e_1, 2e_3] + [e_2, e_3] + \frac{1}{2} e_1. \]
Note that $C(Z_{K_4}) = 6$ and $C(Z^\sigma_{K_4}) = 2$. 

\[ \square \]

**Lemma 6.9.** For any vector $\omega \in \mathbb{Q}^r$ and any $\sigma \in S_r$ such that $\sigma(\omega) = \omega$, we have
\[ \chi_{\tilde{H}_{r-3}(\Delta(\mathcal{L}_\omega))}(\sigma) = (-1)^{r-3} \tilde{\chi}(\Delta(\mathcal{L}^\sigma_\omega)) = (-1)^{r-3} \mu_{\mathcal{L}^\sigma_\omega}(\hat{0}, \hat{1}), \]
where $\chi_V(\sigma)$ is the character of the representation $V$ evaluated at $\sigma$, and $\tilde{\chi}$ is the reduced Euler characteristic of a topological space.

**Proof.** Since the order complex $\Delta(\mathcal{L}_\omega)$ has the homotopy type of a wedge of $(r - 3)$-spheres by Proposition 3.9, we apply the Lefschetz fixed-point theorem for simplicial complexes:
\[ \chi_{\tilde{H}_{r-3}(\Delta(\mathcal{L}_\omega))}(\sigma) = (-1)^{r-3} \sum_i (-1)^i \chi_{\tilde{H}_i(\Delta(\mathcal{L}_\omega))}(\sigma) = (-1)^{r-3} \tilde{\chi}(\Delta(\mathcal{L}_\omega)^\sigma). \]
We are left to prove that $\Delta(\mathcal{L}_\omega)^\sigma = \Delta(\mathcal{L}^\sigma_\omega)$. Consider a $\sigma$-fixed point $p \in \Delta(\mathcal{L}_\omega)$ and let $F$ be the unique face that contains $p$ in its relative interior. Since the $\sigma$-action is simplicial then $\sigma(F) = F$. Moreover, $\sigma$ acts on $\mathcal{L}_\omega$ by a rank-preserving automorphism and the vertices of $F$ have different rank as elements in $\mathcal{L}_\omega$. Hence, $F \in \Delta(\mathcal{L}_\omega)^\sigma$, and this proves that the set of fixed points is the full subcomplex generated by $\sigma$-fixed vertices. The third equation in (15) follows from the Hall’s theorem [15] Proposition 3.8.6]. \[ \square \]

6.2. **Combinatorial lemmas.** Fix a graph $\Gamma$, an automorphism $\sigma \in \text{Aut}(\Gamma)$ and a forest $F$ of $\Gamma/\sigma$. The quotient map $q: \Gamma \to \Gamma/\sigma$ induces a morphism between the posets of flats
\[ \pi: \mathcal{L}(\Gamma)^\sigma \to \mathcal{L}(\Gamma/\sigma), \quad \mathcal{S} = \{S_i\} \mapsto \mathcal{T} = \{T_j\}, \]
where \( T_j = q(S_i) \) for some \( i \). Define
\[
\mathcal{B} = \mathcal{B}(F) \in \mathcal{L}(\Gamma/\sigma)
\]
as the finest flat refining \( F \), or equivalently as the partition of \( V(\Gamma/\sigma) \) whose blocks are the sets of vertices of the connected components of \( F \). For each block \( B_j \in \mathcal{B} \) define
\[
m_j := \min_{i \in B_j} v_2(l_i)
\]
where \( v_2 \) is the 2-adic valuation of the length \( l_i \) of the cycle \( \sigma_i \). If \( m_j > 0 \), then define the sets
\[
A_{B,\sigma,j} = \{ \text{cycle type of the permutation } \sigma_i^{l_i/2} \mid i \in B_j, v_2(l_i) = m_j, y_{\sigma_i^{l_i/2}(a)} \text{ odd} \} \subseteq \mathcal{L}^\sigma
\]
for any \( a \in \sigma_i \). The set \( A_{B,\sigma,j} \) does not depend on the choice of \( a \). Define also
\[
A_{B,\sigma} = \{ \forall A_{B,\sigma,j} \mid m_j > 0, |A_{B,\sigma,j}| \text{ is odd} \} \subseteq \mathcal{L}^\sigma.
\]
We write \( a \sim_S b \) if the vertices \( a, b \in V(\Gamma) \) lies in the same block of the flat \( S \).

**Lemma 6.10.** Let \( S \in \mathcal{L}^\sigma \) be a flat such that \( \pi(S) \geq \mathcal{B} \). Then the set of flats in \( A_{B,\sigma,j} \) smaller or equal than \( S \), denoted \( A^{\leq S}_{B,\sigma,j} \), is empty or equal to \( A_{B,\sigma,j} \).

**Proof.** Consider two vertices \( i, k \) in \( F \) which belongs to the same connected component \( B_j \), and such that the type of \( \sigma_i^{l_i/2} \) and of \( \sigma_k^{l_k/2} \) are in \( A_{B,\sigma,j} \). Note that:

1. For any \( b \in \sigma_k \), there exists \( a \in \sigma_i \) such that \( a \sim_S b \), since \( \pi(S) \geq \mathcal{B} \);
2. \( \sigma_i^{l_i/2}(a) = \sigma_k^{l_k/2}(a) \) for any \( a \in \sigma_i \), since \( v_2(l_i) = m_j = v_2(l_k) \);
3. \( \sigma_k^{l_k/2}(b) = \sigma_k^{l_k/2}(b) \) for any \( b \in \sigma_k \), since \( v_2(l_i) = m_j = v_2(l_k) \);
4. \( \sigma_k^{l_k/2}(a) = \sigma_k^{l_k/2}(b) \), since \( S \) is \( \sigma \)-invariant.

If the type of \( \sigma_i^{l_i/2} \) is smaller or equal than \( S \), equivalently \( a \sim_S \sigma_i^{l_i/2}(a) \) for any \( a \in \sigma_i \), then
\[
b \sim_S a \sim_S \sigma_i^{l_i/2}(a) = \sigma_k^{l_k/2}(a) \sim_S \sigma_k^{l_k/2}(b) = \sigma_k^{l_k/2}(b)
\]
for any \( b \in \sigma_k \), i.e. the type of \( \sigma_k^{l_k/2} \) is smaller or equal than \( S \). We conclude that either all types of \( \sigma_i^{l_i/2} \) in \( A_{B,\sigma,j} \) are smaller or equal than \( S \), or none of them is. \( \square \)

By Lemma 6.8, the affine transformation \( \mathbb{R}^{[\sigma]} \to \mathbb{R}^{[\sigma]} \), given by \( x_i \mapsto l_i(x_i - t_i) \), sends the zonotope \( \pi(Z_{\Gamma,S}^\sigma) \simeq Z_{\Gamma,S}^\sigma \) to the graphical zonotope \( Z_{\Gamma,S(\sigma)} \), where \( S(\sigma) \) is the cycle type of the permutation \( \sigma \) seen as partition of \( V(\Gamma) \). In particular, the affine transformation exchanges the affine spaces supporting the faces of the two zonotopes. Equivalently, any flat \( \mathcal{B} \in \mathcal{L}(\Gamma/\sigma) \)
determines a linear subspace $W_B$ as in [17], which via the affine transformation above corresponds to the affine subspace

$$W_B := \{ x \in \mathbb{R}^{|\sigma|} | \sum_{i \in B_j} l_i x_i = \sum_{i \in B_j} l_i t_i \text{ for all } B_j \in \mathcal{B} \}.$$ 

Up to an integral translation, $W_B$ supports faces of $Z^\sigma_\Gamma$, and so the affine span of any such face contains a lattice point if and only if $W_B$ does. The following lemma generalizes [3, Lemma 4.5] and [15, Proposition 3.12].

**Lemma 6.11.** Let $S \in \mathcal{L}^\sigma$ be a flat such that $\pi(S) \geq B$. The affine subspace supporting a face of the zonotope $Z^\sigma_{\Gamma_S}$ and corresponding to the flat $B$ has a lattice point if and only if $A_{B,\sigma}^\leq S = \emptyset$.

**Proof.** A face of the zonotope $\pi(Z^\sigma_\Gamma) \simeq Z^\sigma_\Gamma$ is determined by a flat $B \in \mathcal{L}(\Gamma_S/\sigma)$ and an acyclic orientation of $\Gamma_{\Sigma_S}$. Up to an integral translation, its supporting affine space depends only on the flat $B$, and it satisfies the equations

$$\sum_{i \in B_j} l_i x_i = \sum_{i \in B_j} \sum_{\{a,b\} \in \sigma_i \atop a \sim_S b} y_{ab}$$

for all $B_j \in \mathcal{B}$. This linear system has integer solution if and only if

$$\gcd(l_i) \bigg| \sum_{i \in B_j \atop \{a,b\} \in \sigma_i \atop a \sim_S b} y_{ab}$$

for all $j$. By [14], this is equivalent to

$$\gcd(l_i) \bigg| \sum_{i \in B_j \atop l_i \text{ even} \atop a \sim_S \sigma_{i}^{1/2}(a)} \frac{l_i}{2} y_{a \sigma_{i}^{1/2}(a)}$$

for all $j$ and any $a \in \sigma_i$. This happens if and only if

$$m_j = v_2(\gcd(l_i)) < v_2 \left( \sum_{i \in B_j \atop l_i \text{ even} \atop a \sim_S \sigma_{i}^{1/2}(a)} l_i y_{a \sigma_{i}^{1/2}(a)} \right),$$

which is equivalent to $|A_{B,\sigma,j}^\leq S|$ being even. Using Lemma 6.10 and the definition of $A_{B,\sigma}$, we obtain the thesis. \hfill $\square$

**Lemma 6.12.** Let $S \in \mathcal{L}^\sigma$ be a flat such that $\pi(S) \geq B$. If $A_{B,\sigma}^\leq S = \emptyset$, then:

- $A_{\pi(S),\sigma}^\leq S = \emptyset$;
Lemma 6.13. For any \( \sigma \in \text{Aut}(\Gamma) \), we write
\[
\sigma_{|\Gamma}(\sigma) = (-1)^{\sum_i t_i \sum_{a \in \pi_1} y_{v_i}^{s_i/2}} = \begin{cases} 
-1 & \text{if } \sum_i t_i \in (1/2 \mathbb{Z}) \setminus \mathbb{Z}, \\
1 & \text{if } \sum_i t_i \in \mathbb{Z}.
\end{cases}
\]
In particular, $o_\Gamma(\sigma) = 1$ if $\gcd(l_i)$ is odd.

Proof. Orient the edges between $\sigma_i$ and $\sigma_j$ (with $i < j$) from $\sigma_i$ to $\sigma_j$. Since $\sigma$ preserves the orientation of all edges between different cycles, we have

$$o_\Gamma(\sigma) = \prod_i o_{\sigma_i}(\sigma). \quad (19)$$

By the multiplicativity of $o_\Gamma$, we can further suppose that $\sigma$ acts transitively on $V(\Gamma)$. The stabilizers of a pair of vertices $V(\Gamma)$ for the action $\sigma \cdot \{a, b\} = \{\sigma(a), \sigma(b)\}$ is either trivial, or $\mathbb{Z}/2\mathbb{Z} \simeq \langle \sigma^{l/2} \rangle$, in which case $l$ is even and $b = \sigma^{l/2}(a)$. Edges whose pairs of endpoints have trivial stabilizer can be oriented in such a way that $\sigma$ is orientation-preserving: fix the orientation of a representative of the orbit, and induce the orientation on the other elements in the orbit by imposing that the map $\sigma|_{\sigma^k(e)}: \sigma^k(e) \to \sigma^{k+1}(e)$ is orientation-preserving. Orient the remaining edges as follows: identify $V(\Gamma)$ and $\mathbb{Z}/l\mathbb{Z}$ as $\langle \sigma \rangle = \mathbb{Z}/l\mathbb{Z}$-modules, and orient the edges between $j$ and $j + l/2$, with $0 \leq j < l/2$, from $j$ to $j + l/2$. In this way, the map $\sigma|_e: e \to \sigma(e)$ is orientation-preserving for all edges $e \in E(\Gamma)$ except for the edges from $l/2 - 1$ to $l - 1$. Therefore, we obtain

$$o_{\sigma_i}(\sigma_i) = (-1)^{y_{l/2-1,l-1}} = (-1)^{\frac{1}{l} \sum_{a \in \sigma_i} y_{a^{l/2}(a)}}.$$

The second equality follows from Remark 6.7. \qed

Lemma 6.14. Let $S \in \mathcal{L}^\sigma$ be a flat such that $\pi(S) \geq B$. If $A_{B,\sigma}^{\leq S} = \emptyset$, then

$$\alpha_S(\sigma) = (-1)^{\ell(\sigma; S) + |A_{\sigma}(S)|}.$$
Proof. By Lemma 6.12 \( A^{\leq \Sigma}_{\pi(S), \sigma} = \emptyset \). We first reduce to the case \( \pi(S) = \hat{1} \) and \( \Gamma/\sigma \) connected. This follows from the definition of \( o_{\Sigma} \), and the fact that

\[
o_{\Sigma}(\sigma) = \prod_i o_{q^{-1}(\Gamma_{T_i})}^{\Sigma_{T_i}}(\sigma), \tag{20}\]

where \( \pi(S) = (T_1, \ldots, T_k) \), \( q: \Gamma \to \Gamma/\sigma \) is the quotient map, and \( S_{T_i} := q^{-1}(\Gamma_{T_i}) \cap \Sigma \). The proof of (20) is identical to that of (19).

Assume then that \( \pi(S) = \hat{1} \) consists of a single connected block. We want to prove the equality

\[
o_{\Sigma}(\sigma) = (-1)^{|A_{1,\sigma}|} = (-1)^{|A_{1,\sigma}|}.
\]

Note that \( \sigma \) acts transitively on the blocks \( S_i \) of \( S \) which intersect all cycles \( \sigma_i \) (equivalently the fibers of \( q \)) in subsets of the same cardinality, since \( S \) is \( \sigma \)-invariant, and \( q(S) = q(S_i) = \hat{1} \).

We obtain that \( \ell(S) \mid l_i \) for all \( i = 1, \ldots, |\sigma| \), and so \( v_2(\ell(S)) \leq m := \min(m_i) \).

We distinguish the following cases.

\( m = v_2(\ell(S)) = 0 \): then \( A_{1,\sigma} = \emptyset \), and \( o_{\Sigma}(\sigma) = 1 \) by Lemma 6.13.

\( m > v_2(\ell(S)) = 0 \): then \( o_{\Sigma}(\sigma) = 1 \) by Lemma 6.13, since \( \ell(S) \) is odd. Moreover, the pairs of vertices in \( V(\Gamma) \) of the form \( \{a, \sigma_i/2(a)\} \), with \( a \in \sigma_i \), are all contained in the fibers of the contraction map \( \Gamma \to \Gamma_{\Sigma} \), since they are \( \sigma_i/2 \)-orbits and \( \ell(S) \) is odd. Therefore, the type of \( \sigma_i/2 \) is smaller or equal than \( S \) for all \( i = 1, \ldots, |\sigma| \), and so \( A_{1,\sigma} = A_{1,\sigma}^{\leq S} \), which is empty by assumption. Hence, \( o_{\Sigma}(\sigma) = 1 = (-1)^{|A_{1,\sigma}|} \).

\( m > v_2(\ell(S)) > 0 \): the number of edges in \( \Gamma_{\Sigma} \) between \( S_0 \) and \( S_{\ell(S)/2} := \sigma^{\ell(S)/2}(S_0) \) is even, which implies \( o_{\Sigma}(\sigma) = 1 \) by Lemma 6.13. Indeed, let \( e_{ab} \) be an edge whose endpoints \( a, b \) lie respectively in \( \sigma_i \) and \( \sigma_j \). Recall that the length of the orbit of \( e_{ab} \) is

\[
|\langle \sigma \rangle \cdot e_{ab}| = \begin{cases} \text{lcm}(l_i, l_j) & \text{if } i \neq j, \\ l_i & \text{if } i = j, b \neq \sigma_i/2(a), \\ l_i/2 & \text{if } i = j, b = \sigma_i/2(a), \end{cases}
\]

while the length of the orbit of a edge between \( S_0 \) and \( S_{\ell(S)/2} \) in \( \Gamma_{\Sigma} \) is \( \ell(S)/2 \). Thus we write

\[
\frac{\ell(S)}{2} y_{S_0 S_{\ell(S)/2}} = \sum_{0 \leq i < \ell(S)/2} y_{S_i,\sigma^{\ell(S)/2}(S_i)} = \sum_{\{a, b\} \subseteq V(\Gamma)} \sum_{q(b) = \sigma^{\ell(S)/2}(q(a))} y_{ab} |\langle \sigma \rangle \cdot e_{ab}| y_{ab},
\]

where \( \mathcal{O} \) is the set of orbits of pairs \( \{a, b\} \subseteq V(\Gamma) \) such that \( q(b) = \sigma^{\ell(S)/2}(q(a)) \). In all cases, the ratio \( |\langle \sigma \rangle \cdot e_{ab}|/(\ell(S)/2) \) is even, and \( y_{S_0 S_{\ell(S)/2}} \) is even too, as we wanted.
Further, since $2\ell(\mathcal{S})l_i, \sigma^i/2$ acts trivially on $\mathcal{S}$. As above, the type of $\sigma^i/2$ is smaller or equal than $\mathcal{S}$ for all $i$, and $A_{1,\sigma} = A_{1,\sigma}^{\leq \mathcal{S}} = 0$. Hence, $o_{\frac{\mathcal{S}}{2}}(\sigma) = 1 = (-1)^{|A_{1,\sigma}|}$.

$m = v_2(\ell(\mathcal{S})) > 0$: the parity of the number of edges between $S_0$ and $S_{\ell(\mathcal{S})/2}$ coincides with the parity of

$$\sum_{v_2(l_i) = v_2(\ell(\mathcal{S}))} y_{a^i/2(a)}$$

for any $a \in \sigma_i$. Hence

$$o_{\frac{\mathcal{S}}{2}}(\sigma) = (-1)^{\sum_{v_2(l_i) = v_2(\ell(\mathcal{S}))} y_{a^i/2(a)}} = (-1)^{|A_{1,\sigma}|}.$$ 

This completes the proof. \hfill \qed

Let us define

$$\omega_B = \left(\sum_{i \in B_j} l_i \omega_i\right)_j = \left(\sum_{k \in \bigcup_{l \in B_j} \sigma_i} \omega_k\right)_j \in \mathbb{R}^{\ell(B)}$$

and

$$\text{Half}_\omega = \left\{ \forall A_{B,\sigma,j} | \frac{\omega_B_j}{\gcd_{i \in B_j}(l_i)} \in \frac{1}{2} \mathbb{Z} \setminus \mathbb{Z} \right\} \subseteq \mathcal{L}^\sigma.$$

Define $R = R(F) \in \mathcal{L}^\sigma(\Gamma)$ as the finest flat containing all the lifts of the edges of the forest $F$ via the quotient map $q: \Gamma \to \Gamma/\sigma$.

**Lemma 6.15.** The inverse image $\pi^{-1}(B_j)$ of any block $B_j$ of $\mathcal{B}(F)$ under the map $\pi: \mathcal{L}^\sigma(\Gamma) \to \mathcal{L}(\Gamma/\sigma)$ is the disjoint union of $\gcd_{i \in B_j}(l_i)$ blocks of $R$ over which $\sigma$ acts transitively.

**Proof.** Working block-wise, we can assume $B = \{B_i\} = \hat{1}$ and the statement becomes $\ell(R) = \gcd_{i \in B_1}(l_i)$. Since $\sigma$ acts transitively on the blocks of $R$, then $\ell(R) | \gcd_{i \in B_1}(l_i)$.

By contradiction suppose that $\ell(R) < \gcd_{i \in B_1}(l_i)$. For each $i$, each vertex $a \in \sigma_i$ and each integer $k$ coprime with $\gcd_{i \in B_1}(l_i)/\ell(R)$ there is a path in $\pi^{-1}(F)$ from $a$ to $\sigma^{k\ell(R)}(a)$. Choose $i, a \in \sigma_i$ and $k$ such that the path has minimal length among all considered paths. The assumption $\ell(R) < \gcd_{i \in B_1}(l_i)$ implies that the path is non-constant. Since $F$ is a forest, the path starts with an edge with endpoints $\{a, b\}$, and it terminates with an edge with endpoints $\{\sigma^h(a), \sigma^h(b)\}$ in the same orbit of the pair $\{a, b\}$. Notice that $h \equiv k\ell(R) \mod l_i$, and so $h = k'\ell(R)$ with $k'$ coprime to $\gcd_{i \in B_1}(l_i)/\ell(R)$. Eliminating the first and last edges of the path, we obtain a shorter path from $b$ to $\sigma^{k'\ell(R)}(b)$ contrary to the minimality assumption on $a$. \hfill \qed

**Lemma 6.16.** Let $H \subseteq A_{B,\sigma}$. The flat $\mathcal{R} \vee H$ is $\omega$-integral if and only if $\gcd_{i \in B_j}(l_i) \mid 2(\omega_B)_j$ for all $j = 1, \ldots, \ell(B)$ and $H \supseteq \text{Half}_\omega$. 

Proof. We can reduce to a single block of \( B = \pi(R) \), hence we assume \( B = \hat{1} \). By Lemma 6.15 we have \( \ell(R) = \gcd_i(l_i) \) and \( \omega_B = \ell(R)(\omega_R)_1 = \gcd_i(l_i)(\omega_R)_1 \). There are two cases.

\( H = \emptyset \): then \( (\omega_R)_1 \in \mathbb{Z}, \gcd_i(l_i) \mid \omega_B \) and \( \text{Half}_\omega = \emptyset \).

\( H = \lor A_{1,\sigma,1} \): then \( R \lor H \) has \( \gcd_i(l_i)/2 \) blocks and \( (\omega_R \lor H)_1 = 2(\omega_R)_1 \). Therefore, \( 2(\omega_R)_1 \in \mathbb{Z}, \gcd_i(l_i) \mid 2\omega_B \) and \( H \supset \text{Half}_\omega \).

\( \square \)

Lemma 6.17. The affine subspace supporting a face of the zonotope \( Z_\Gamma^\sigma + \omega \) corresponding to the flat \( B \) has a lattice point if and only if \( \gcd_i \in B_j(l_i) \mid 2(\omega_B)_j \) for all \( j = 1, \ldots, \ell(B) \) and \( A_{B,\sigma} = \text{Half}_\omega \).

Proof. The proof is similar to the one of Lemma 6.11. The affine subspace corresponding to \( B \) has a lattice point if and only if the linear system

\[
\sum_{i \in B_j} l_i x_i = \sum_{i \in B_j} \left( l_i \omega_i + \sum_{a,b \in \sigma} y_{ab} \right)
\]

for \( j = 1, \ldots, \ell(B) \) has integral solutions. This is equivalent to

\[
\gcd_i \in B_j(l_i) \left| \sum_{i \in B_j} l_i \omega_i + \sum_{l_i \text{ even}} \frac{l_i}{2} y_{a\sigma_i l_i}(a) \right.
\]

for all \( j = 1, \ldots, \ell(B) \). In particular \( \gcd_i \in B_j(l_i) \mid 2(\omega_B)_j \).

In addition, \( \gcd_i \in B_j(l_i) \mid (\omega_B)_j \) if and only if \( \gcd_i \in B_j(l_i) \mid \sum_{l_i \text{ even}} \frac{l_i}{2} y_{a\sigma_i l_i}(a) \). The former is equivalent to \( \lor A_{B,\sigma,j} \in \text{Half}_\omega \) and the latter to \( \lor A_{B,\sigma,j} \in A_{B,\sigma} \).

\( \square \)

6.3. Proof of Theorem 6.4.

Proof of Theorem 6.4. Denote the \( \text{Stab}(\Sigma) \)-representation \( \alpha_{\Sigma} \otimes \overline{H}_\sigma(\Sigma_{\omega,\geq \Sigma}) \otimes C(\Sigma) \) simply by \( V_{\Sigma} \). Then the character of the right hand side of (13) is given by:

\[
\chi_{\text{RHS}}(\sigma) = \sum_{\Sigma \in \mathcal{L}_\omega/\text{Aut}(\Gamma)} \chi_{\text{Ind}_{\text{Stab}(\Sigma)}(\Gamma)} V_{\Sigma}(\sigma)
\]

\[
= \sum_{\Sigma \in \mathcal{L}_\omega} \frac{1}{|O_\Sigma|} \chi_{\text{Ind}_{\text{Stab}(\Sigma)}(\Gamma)} V_{\Sigma}(\sigma)
\]

\[
= \sum_{\Sigma \in \mathcal{L}_\omega} \frac{1}{|O_\Sigma|} \text{Stab}(\Sigma) \sum_{g \in \text{Aut}(\Gamma)} \chi_{V_{\Sigma}}(g^{-1}\sigma g)
\]

\[
\chi_{V_{\Sigma}}(g^{-1}\sigma g)
\]
By Lemma 6.9, the identity (13) is equivalent to
\[
C(Z^\sigma_\Gamma + \omega) - C(Z^\sigma_\Gamma) = \sum_{S \in L^\omega} a_S(\sigma)(-1)^{\ell(S)-3} \mu L^\omega(S, \hat{1}) C(Z^\sigma_{\Gamma S})
\]  
(21)

Following Theorem 4.4, we rewrite the LHS of (21) as follows
\[
C(Z^\sigma_\Gamma + \omega) - C(Z^\sigma_\Gamma) = \sum_{F \text{ forest of } \Gamma/\sigma} (-1)^{\dim(Z^\sigma_\Gamma)+|F|} (\delta_B(F) (\omega_\sigma + t_\sigma)\text{-integral} - \delta_B(F) t_\sigma\text{-integral}) Q_F
\]

where
\[
Q_F := \sum_{W \text{ spanning forest of } F} \text{vol}(W);
\]

of §5.2 and Definition 4.2.

Note that if $R(F)$ is $\omega$-integral, then $\omega_B \in \mathbb{Z}^{|E|}$ and $\gcd_{i \in B_j} (l_i) | (\omega_B)_j$ by Lemma 6.15. In this case, $\delta_B(F) (\omega_\sigma + t_\sigma)\text{-integral} = \delta_B(F) t_\sigma\text{-integral}$. Therefore, we obtain
\[
\text{LHS} (21) = \sum_{F \text{ forest of } \Gamma/\sigma} (-1)^{\dim(Z^\sigma_\Gamma)+|F|} (\delta_B(F) (\omega_\sigma + t_\sigma)\text{-integral} - \delta_B(F) t_\sigma\text{-integral}) Q_F
\]

Observe that $\ell(S) = \ell(\pi(S)) + \ell(\sigma: S \to \hat{S})$, and $\dim(Z^\sigma_{\Gamma S}) = \dim(Z^\sigma_\Gamma) - (\ell(\pi(S)) - 1)$. 

On the other hand, Lemma 6.14 implies that

\[ \text{RHS}(\ref{21}) = \sum_{S \in \mathcal{L}^\sigma_\omega} (-1)^{\ell(S) - 3 + |A_S, \sigma|} \mu_{\mathcal{L}^\sigma_\omega}(S, \hat{1}) C(\mathcal{Z}^\sigma_{\Gamma_S}) \]

\[ = \sum_{S \in \mathcal{L}^\sigma_\omega} (-1)^{\ell(S) - 3 + |A_S, \sigma|} \mu_{\mathcal{L}^\sigma_\omega}(S, \hat{1}) \sum_{F \text{ forest of } \Gamma_S/\sigma} (-1)^{\dim(\mathcal{Z}^\sigma_{\Gamma_S})} + |F| Q_F \delta_{\mathcal{B}(F)} t_\sigma(\Gamma_S) \text{-integral} \]

So it is enough to prove that

\[ \delta_{\mathcal{B}(F)}(\omega + t_\sigma) \text{-integral} - \delta_{\mathcal{B}(F)} t_\sigma \text{-integral} = \sum_{S \in \mathcal{L}^\sigma_\omega, S \supset R(F)} (-1)^{|A_S, \sigma|} \mu_{\mathcal{L}^\sigma_\omega}(S, \hat{1}) \delta_{\mathcal{B}(F)} t_\sigma(\Gamma_S) \text{-integral} \tag{22} \]

for all \( F \) such that \( R(F) \in \mathcal{L}^\sigma_\omega \).

Applying lemmas from the previous section we have:

\[ \text{RHS}(\ref{22}) = \sum_{S \in \mathcal{L}^\sigma_\omega} (-1)^{|A_S, \sigma|} \mu_{\mathcal{L}^\sigma_\omega}(S, \hat{1}) \delta_{\mathcal{A}_{\mathcal{B}_\sigma}^\omega = \emptyset} \]

\[ = \sum_{S \in \mathcal{L}^\sigma_\omega, S \supset R(F)} (-1)^{|A_S, \sigma|} \mu_{\mathcal{L}^\sigma_\omega}(S, \hat{1}) \delta_{\mathcal{A}_{\mathcal{B}_\sigma}^\omega = \emptyset} \]

\[ = (-1)^{|A_S, \sigma|} \sum_{S \in \mathcal{L}^\sigma_\omega, S \supset R(F)} \mu_{\mathcal{L}^\sigma_\omega}(S, \hat{1}) \sum_{H \subseteq A_{\mathcal{B}_\sigma}^\omega} (-1)^{|H|} \]

\[ = (-1)^{|A_S, \sigma|} \sum_{H \subseteq A_{\mathcal{B}_\sigma}^\omega} (-1)^{|H|} \sum_{S \in \mathcal{L}^\sigma_\omega, S \supset R \vee H} \mu_{\mathcal{L}^\sigma_\omega}(S, \hat{1}) \]

\[ = -\delta_{A_{\mathcal{B}_\sigma} = \emptyset} + (-1)^{|A_{\mathcal{B}_\sigma}|} \sum_{H \subseteq A_{\mathcal{B}_\sigma}} (-1)^{|H|} \left( 1 + \sum_{S \in \mathcal{L}^\sigma_\omega, S \supset R \vee H} \mu_{\mathcal{L}^\sigma_\omega}(S, \hat{1}) \right) \]

\[ = -\delta_{A_{\mathcal{B}_\sigma} = \emptyset} + (-1)^{|A_{\mathcal{B}_\sigma}|} \sum_{H \subseteq A_{\mathcal{B}_\sigma}} (-1)^{|H|} \delta_{R \vee H} \omega \text{-integral} \]

\[ = -\delta_{A_{\mathcal{B}_\sigma} = \emptyset} + (-1)^{|A_{\mathcal{B}_\sigma}|} \prod_{j=1}^{\ell(B)} \delta_{\text{gcd}_i \in B_j (t_i)} 2(\omega_{B_j}) \sum_{\text{Half}_\omega \subseteq H \subseteq A_{\mathcal{B}_\sigma}} (-1)^{|H|} \]
\[
= -\delta A,\sigma = 0 + (-1)^{|A,\sigma|}(-1)^{|\text{Half}_C|}\delta_{\text{Half}_C = A,\sigma} \prod_{j=1}^{l(B)} \delta_{g_c(d_i, \langle t_i \rangle |2(\omega_B))}, 
\]
Lemma 6.17 and 6.11
\[
\text{Lemma 6.17 and 6.11}
\]
where (†) accounts for the elementary property that \( \sum_{C \subseteq B \subseteq A} (-1)^{|B|} = (-1)^{|C|} \delta_{A = C} \) for any finite sets \( A, B, C \). This concludes the proof. 

7. Hitchin systems and compactified Jacobians

The goal of this section is to determine a bijection between the irreducible components of the fibers of the Hitchin fibration and the lattice points of certain graphical zonotopes; see Theorem 7.8.

Let \( C \) be a compact Riemann surface of genus \( g \) with canonical bundle \( \omega_C \).

**Definition 7.1.** The Dolbeault moduli space \( M(C, n, d) \) is the coarse moduli space which parametrises \( S \)-equivalence classes of semistable Higgs bundles on \( C \) of rank \( n \) and degree \( d \), i.e. polystable pairs \((E, \phi)\) consisting of a vector bundle \( E \) of rank \( n \) and degree \( d \) and a section \( \phi \in H^0(C, \text{End}(E) \otimes \omega_C) \); see [44].

Let \( 2c(n, g) := 2(g - 1)n^2 + 2 \) be the dimension of \( M(C, n, d) \). In the following we drop the dependence from \( C, n, d \), or some of them when it does not cause any confusion. The Dolbeault moduli space is endowed with a proper map called Hitchin fibration.

**Definition 7.2.** The Hitchin fibration

\[
\chi(n, d) : M(n, d) \to A_n := \bigoplus_{i=1}^{n} H^0(C, \omega_C^{\otimes i}), \quad (E, \phi) \mapsto \text{char}(\phi) = (\text{tr}(\phi), \text{tr}(\Lambda^2 \phi), \ldots, \text{tr}(\Lambda^i \phi), \ldots, \det(\phi)),
\]

assigns to \((E, \phi)\) the characteristic polynomial \( \text{char}(\phi) \) of the Higgs field \( \phi \).

The fibers of the Hitchin fibration have a classical modular interpretation that we recall here. Any point \( a \in A_n \) is the characteristic polynomial of some Higgs field which can be seen as the defining equation of a curve \( C_a \subset \text{Tot}(\omega_C) \) of degree \( n \) over \( C \), called spectral curve. The fiber \( \chi(n, d)^{-1}(a) \) is a compactified Jacobian of the spectral curve \( C_a \) with respect to the canonical polarization \( q_a \), defined as follows.
Definition 7.3. A polarization on a connected curve $X = \bigcup_{i \in I} X_i$ is a tuple of rational numbers $q = (q_{X_i})$, one for each irreducible component $X_i$ of $X$, such that $\sum_i q_{X_i} \in \mathbb{Z}$.

Given any proper subscheme $Y \subseteq X$ of pure dimension 1, we set $q_Y := \sum_j q_{X_j} \in \mathbb{Q}$, where the sum runs over all the irreducible components $X_j$ of $Y$.

A polarization is general if $q_Y \notin \mathbb{Z}$ for any proper subscheme $Y \subseteq X$ of pure dimension 1 such that $Y$ and the complementary subcurve $Y^c := X \setminus Y$ are connected.

Definition 7.4. Let $q$ be a polarization of $X$. A rank 1 torsion free sheaves $I$ on a curve $X$ with $\chi(I) = q_X$ is $q$-semistable (resp. $q$-stable) if for any proper subscheme $Y \subset X$ of pure dimension 1 we have that

$$\chi(Y, I_Y) \geq q_Y \quad (\text{resp. } \chi(Y, I_Y) > q_Y)$$

where $I_Y$ is the biggest torsion free quotient of the restriction $I|_Y$ of $I$ to $Y$.

Then the compactified Jacobian $\overline{J}_q(X)$ is the (projective) coarse moduli space of $q$-semistable rank 1 torsion free sheaves; see for instance [42].

Proposition 7.5 (BNR correspondence). Let $a \in A_n$ and consider the associated spectral curve $\pi_a : C_a = \bigcup_i C_i \to C$. The canonical polarization assigns to any proper subscheme $Y \subseteq C_b$ of pure dimension 1 the rational number

$$q^a_Y := \chi(I) \frac{\deg(\pi_a|_Y)}{n},$$

where $\deg(\pi_a|_Y)$ is the degree of the restriction $\pi_a|_Y : Y \to C$.

There exists an algebraic isomorphism

$$\overline{J}_q(C_a) \to \chi^{-1}(n, d)(a),$$

given by $(C_a, I) \mapsto (E := \pi_a*\mathcal{I}, \phi := t \cdot)$,

where the Higgs field $\phi$ is induced by multiplication by the tautological section $t$ of the line bundle $\pi_a^*\omega_C$ on $C_a \subset \text{Tot}(\omega_C)$.

From now on, we restrict our attention to reduced spectral curve. We denote by $A^n_{n} \subset A_n$ the locus of reduced spectral curves. For any partition $n = \{n_i \} \vdash n$, $S^n_{n}$ is the locus of characteristic polynomials whose irreducible factors have degree $n_i$, and they gives a stratification of $A^n_{n}$, see also §10.

Proposition 7.6. Let $a \in A^n_{n}$. The following facts hold.
(1) The subvariety parametrising \( q^a \)-stable line bundles on the spectral curve \( C_a \) is dense in \( \chi(n, d)^{-1}(a) \);

(2) The irreducible components of \( \chi(n, d)^{-1}(a) \) are indexed by the degrees of the restrictions of the \( q^a \)-stable line bundles to the irreducible components of \( C_a \).

(3) The number of irreducible components of \( \chi(n, d)^{-1}(a) \) is constant for \( a \in S^0_n \).

**Proof.** The statement (1) holds for reduced curves with respect to a general polarization by [29, Cor. 2.20]. We show (1) for the (not necessarily general) canonical polarization \( q^a \), via a small perturbation of the polarization \( q^a \) to a general one, denoted \( q' \). Indeed, \( q' \)-stable sheaves are \( q^a \)-semistable, and by the formation of GIT quotients there exists a specialization map

\[
\mathcal{J}_{\bar{q}}(C_a) \to \mathcal{J}_{\bar{q}'}(C_a)
\]

which is an isomorphism over the locus of \( q^a \)-stable sheaves. Since \( q' \) is general, the locus of \( q' \)-stable line bundles is open and dense in \( \mathcal{J}_{\bar{q}'}(C_a) \) by [29, Cor. 2.20]. It is then sufficient to prove that the \( q^a \)-stable locus is dense in \( \mathcal{J}_{\bar{q}}(C_a) \). To this end, note that its complement \( \mathcal{J}_{\bar{q}}(C_a)^{\text{sing}} \) is an irreducible variety. Since any \( q^a \)-stable sheaf is \( q^a \)-stable, the statement (1) follows from the fact that the stability of a line bundle is determined by numerical inequalities involving the degree of the irreducible components of \( C_a \), and the sheaf of these irreducible components is locally constant on \( S^0_n \).

Now let \( L \) be a \( q^a \)-stable line bundle on \( C_a \), and \( J(C_a, \deg_Y(L)) \) be the Jacobian of line bundles \( L' \) on \( C_a \) with \( \deg_Y(L') = \deg_Y(L) \) for all \( Y \subseteq C_a \), which is an irreducible variety. Since any line bundle \( L' \) deformation equivalent to \( L \) is \( q^a \)-stable, we have \( J(C_a, \deg_Y(L)) \subset \mathcal{J}_{\bar{q}}(C_a) \). Because the two spaces have the same dimension, the closure of \( J(C_a, \deg_Y(L)) \) is an irreducible component of \( \mathcal{J}_{\bar{q}}(C_a) \). This implies (2).

Statement (3) now follows from (2) and the fact that the stability of a line bundle is determined by numerical inequalities involving the degree of the irreducible components of \( C_a \), and the sheaf of these irreducible components is locally constant on \( S^0_n \). \( \square \)

**Notation 7.7.** Recall that the dual graph of a nodal curve \( X = \bigcup_{i \in I} X_i \) with smooth irreducible components is a graph with \( |I| \) vertices corresponding to the irreducible components \( X_i \) of \( X \), and \( |X_i \cap X_j| \) edges between the vertices \( i \) and \( j \). In particular, for any partition \( \underline{n} \) of \( n \) we denote by \( \Gamma[\underline{n}] \) the dual graph of the general spectral curve in \( S^0_n \), i.e. \( \Gamma[\underline{n}] \) is the graph with
|n| vertices corresponding to the irreducible components of the curve and \( n_i n_j (2g - 2) \) edges between the vertices \( i, j \), corresponding to the intersection points of the components.

Recall that \( \omega_n(d) \) is the rational vector in \( \mathbb{Q}^{|n|} \) whose \( i \)-th component is \( dn_i/n \). As a corollary of Proposition 7.6, the automorphism group \( \text{Aut}(\Gamma[n]) \) of the graph \( \Gamma[n] \) exchanges the irreducible components of the fibre \( \chi(n,d)^{-1}(a) \), and acts by permutation on the translated graphical zonotope \( Z_{\Gamma[n]} + \omega_n(d) \).

**Theorem 7.8** (Hitchin system and zonotopes). Let \( a \in S^0_n \). The following \( \text{Aut}(n) \)-representations are isomorphic

\[
(R^{2\chi(n,g)} \chi(n,d) \cdot \mathbb{Q}_M(n,d))_a \simeq C(Z_{\Gamma[n]} + \omega_n(d)).
\]

In particular, the number of irreducible components of the fibre \( \chi(n,d)^{-1}(a) \) equals the number of lattice points contained in the zonotope \( Z_{\Gamma[n]} + \omega_n(d) \), in symbols

\[
\dim(R^{2\chi(n,g)} \chi(n,d) \cdot \mathbb{Q}_M(n,d))_a = C(Z_{\Gamma[n]} + \omega_n(d)).
\]

**Proof.** By Proposition 7.6 we can suppose that the spectral curve \( C_a = \bigcup_{i \in I} C_i \) is nodal with smooth irreducible components. Denote by \( C_K = \bigcup_{i \in K} C_i \subseteq C_a \) the subcurve corresponding to \( K \subseteq I \). Pick a \( q^a \)-stable line bundle \( \mathcal{I} \) in \( \overline{J}_{q^a}(C_a) \), and let \( \mathcal{I}_K \) be its restriction to \( C_K \). By pushing forward from the normalization of \( C_a \), we have the standard short exact sequence

\[
0 \to \mathcal{I}_K \to \bigoplus_{k \in K} \mathcal{I}_k \to \bigoplus_{J=\{i,j\} \subseteq K, |J|=2} \mathcal{O}_{C_i \cap C_j} \to 0.
\]

The stability condition reads

\[
\sum_{i \in I} \chi(\mathcal{I}_i) - \sum_{J=\{i,j\} \subseteq I, |J|=2} |C_i \cap C_j| = \chi(\mathcal{I}) = d + n(1 - g),
\]

\[
\sum_{i \in K} \chi(\mathcal{I}_i) - \sum_{J=\{i,j\} \subseteq K, |J|=2} |C_i \cap C_j| = \chi(\mathcal{I}_K) > [d + n(1 - g)] \sum_{k \in K} \frac{n_k}{n}.
\]

For any pair \( J = \{i,j\} \subseteq I \), the number of edges of \( \Gamma(C_a) \) between the vertices corresponding to \( C_i \) and \( C_j \), denoted \( y_J \), equals \( |C_i \cap C_j| \). Set \( x_i := \chi(\mathcal{I}_i) - (1 - g)n_i \), and \( z_K = \sum_{J \subseteq K} y_J \).

We can rewrite the previous system of equality and inequalities as

\[
\sum_{i \in I} x_i = z_I + d,
\]

\[
\sum_{i \in K} x_i > z_K + \sum_{k \in K} \frac{dn_k}{n} \quad \forall K \subseteq I.
\]

As a result, the possible multidegrees of \( \mathcal{I} \) are in correspondence with the lattice points of the zonotope \( Z_{\Gamma(C_a)} + \omega \) by \([\mathcal{I}]\), and the bijection is \( \text{Aut}(\Gamma[n]) \)-equivariant. \( \square \)
8. Notation: intersection cohomology

Given a \(\mathbb{Q}\)-local system \(L\) on a Zariski-open subset of a complex variety \(X\), we denote the intersection cohomology of \(X\) with coefficient in \(L\) by \(IH^*(X,L) := H^*(IC(X,L))\), where \(IC(X,L)\) is the perverse intersection cohomology complex of \(X\) with coefficient in \(L\) shifted by \(- \dim X\). See [19, 14, 22] for an account. We will consider \(IC(X,L)\) as an element of the derived category \(D^b(X,\mathbb{Q})\) of sheaves of \(\mathbb{Q}\)-vector spaces on \(X\) with bounded algebraically constructible cohomology, or as an element of the bounded derived category of algebraic mixed Hodge modules \(D^bMHM_{\text{alg}}(X)\).

9. Ngô strings for the Hitchin system

In this section we recall how the cohomology of the Dolbeault moduli space \(M(n,d)\) decomposes in Ngô strings, and we carefully study the stalk of these strings at the general point of their support.

For any partition \(\underline{n}\) of \(n\) of length \(r\), we define \(S_{\underline{n}}\) as the image of the multiplication map

\[
\text{mult}_{\underline{n}} : A_{\underline{n}} := \prod_{i=1}^{r} A_{n_i} \rightarrow A_n
\]

given by \(\text{mult}_{\underline{n}}(a_1, \ldots, a_r) = \prod_{i=1}^{n} a_i\). There exists a partition \(S(\underline{n}) = \{S(n)_k\}_{k=1}^{r}\) of \([r]\), maximal with respect to the order \(<\), with the property that there exists \(i(k) \in S(n)_k\) such that \(n_j = n_{i(k)}\) for all \(j \in S(n)_k\). In other words, we group together the \(n_i\) which are equal. Then the multiplication \(\text{mult}_{\underline{n}}\) factors as follows

\[
A_{\underline{n}} \xrightarrow{\eta_{\underline{n}}} A_{S(\underline{n})} := \prod_{k=1}^{r} \text{Sym}^{|S(n)_k|} A_{n_{i(k)}} \xrightarrow{\nu_{\underline{n}}} S_{\underline{n}}.
\]

The map \(\nu_{\underline{n}}\) is the normalization of \(S_{\underline{n}}\), since \(\text{mult}_{\underline{n}}\) is a finite map and \(\nu_{\underline{n}}\) is generically 1:1 with normal domain. In particular, the composition is étale over

\[
S_{\underline{n}}^\circ := (S_{\underline{n}} \cap A^{\text{red}}) \setminus \bigcup_{\underline{m} < \underline{n}} S_{\underline{m}}
\]

with Galois group the automorphism \(\text{Stab}(\underline{n}) := \prod_{i=1}^{n} \mathfrak{S}_{\alpha_i}\) of the partition \(\underline{n} = 1^{\alpha_1}2^{\alpha_2} \cdots n^{\alpha_n}\), i.e. the subgroup of the symmetric group \(\mathfrak{S}_r\) stabilizing \(\underline{n}\), or equivalently the automorphism group \(\text{Aut}(\Gamma[\underline{n}])\) of the general spectral curve in \(S_{\underline{n}}\).

We define also the following spaces:

(i) \(S_{\underline{n}}^x \subseteq S_{\underline{n}}^\circ\) is the locus parametrising reducible nodal curves having smooth irreducible components of degree \(n_i\) over \(C\);

(ii) \(\pi_{\underline{n}} : C_{\underline{n}}^x \rightarrow S_{\underline{n}}^x\) is the universal spectral curve over \(S_{\underline{n}}^x\);
is the (relative) Jacobian of the universal spectral curve over $S^X_{\underline{a}}$;

(iv) $g_{\underline{a}} : S^X_{\underline{a}} \to S^X_{\underline{n}}$ is the maximal abelian proper quotient of $\Pic^0(C_{\underline{a}})$, or equivalently the Jacobian of the normalization of $C_{\underline{a}}$;

(v) $\Lambda^l_{\underline{a}} := R^l g_{\underline{a}}_* Q_{\text{aff}X_{\underline{a}}} = \Lambda^l R^1 g_{\underline{a}}_* Q_{\text{aff}X_{\underline{a}}} = \Lambda^l R^1 \pi_{\underline{a}}_* Q_{\text{aff}X_{\underline{a}}}$; see [11] Lemma 1.3.5.

**Theorem 9.1** (Ngô strings from Hitchin system). [28 Thm 3.6, Thm 3.11] There exists an isomorphism in $D^b M_{\text{MHM}}(A^\text{red}_{\underline{n}})$ (resp. $D^b (A^\text{red}_{\underline{n}}, \mathbb{Q})$)

$$R\chi(n,d)_* IC(M(n,d), \mathbb{Q})|_{A^\text{red}_{\underline{n}}} \simeq \bigoplus_{n^2-n} \mathcal{J}(\mathcal{L}_{\underline{n}}(d))|_{A^\text{red}_{\underline{n}}}$$

where the Ngô string $\mathcal{J}(\mathcal{L}_{\underline{n}}(d))$ is the mixed Hodge module supported on $S_{\underline{n}}$ given by

$$\mathcal{J}(\mathcal{L}_{\underline{n}}(d)) := \bigoplus_{l=0}^{2 \dim S_{\underline{n}}} IC(S_{\underline{n}}, \Lambda^l_{\underline{n}} \otimes \mathcal{L}_{\underline{n}}(d))[-l - 2 \text{codim } S_{\underline{n}}](-\text{codim } S_{\underline{n}}),$$

where $\mathcal{L}_{\underline{n}}(d)$ is a polarizable variation of pure Hodge structures of weight zero and of Hodge-Tate type supported on an open set of $S_{\underline{n}}$. Moreover, $R^{2\dim(n,d)} \chi(n,d)_* \mathbb{Q}_{M(n,d)}$ admits the following direct summand

$$\bigoplus_{n \in I} i_{\underline{n}_*} \mathcal{L}_{\underline{n}}(d),$$

where $i_{\underline{n}} : S^X_{\underline{n}} \to A_{\underline{n}}$ is the natural immersion.

**Remark 9.2.** If $\underline{n} = \{n\}$, then $\mathcal{L}_{\{n\}}(d)$ is the trivial local system, and $\mathcal{J}(\mathcal{L}_{\{n\}}(d))$ is independent of $d$. In this case we simply write $\mathcal{J}_n$ for $\mathcal{J}(\mathcal{L}_{\{n\}}(d))$.

**Remark 9.3.** The pullback of $\mathcal{L}_{\underline{n}}(d)$ to the preimage of $S^X_{\underline{n}}$ under the map $A_{\underline{n}} \to S_{\underline{n}}$ has trivial monodromy. This is proved in [12] Cor. 6.14 for $\gcd(n,d) = 1$, but the same proof of [12] Cor. 6.14 works in arbitrary degree using Proposition 7.6. This means that the monodromy of the local system $\mathcal{L}_{\underline{n}}(d)$ is a representation of $\text{Aut}(\Gamma[\underline{n}]) \simeq \text{Stab}(\underline{n})$.

From now on, let $\underline{m}$ be a partition of $n$, and $a$ be a general closed point of the stratum $S_{\underline{m}}$.

**Lemma 9.4.** Let $r(\underline{n}, d)$ be the rank of $\mathcal{L}_{\underline{n}}(d)$. For any $\underline{n} \geq \underline{m}$, then $(i_{\underline{n}*} \mathcal{L}_{\underline{n}}(d))_a \simeq \mathbb{Q}_{\underline{a}}^{r(\underline{n}, d)}$.

**Proof.** By Remark 9.3, the monodromy of $\mathcal{L}_{\underline{n}}(d)$ is non-trivial only along the loops winding around the branch locus $B_{\underline{n}}$ of $A_{\underline{n}} \to S_{\underline{n}}$, i.e. the locus of polynomials with coefficients in $H^0(C, K^*_{C^\text{aff}_{\underline{n}}})$ with non-distinct irreducible factors. But the point $a \in S_{\underline{n}}$ does not lay on $B_{\underline{n}}$ for any $\underline{n} \geq \underline{m}$. So $\mathcal{L}_{\underline{n}}(d)$ extends through $a$ as a local system, and $i_{\underline{n}*} \mathcal{L}_{\underline{n}}(d)_a \simeq \mathbb{Q}_{\underline{a}}^{r(\underline{n}, d)}$. □
Proposition 9.5. Let $N_{n/m}$ be a general slice transverse to $S_m$ in $S_n$ passing through $a$. Set $N^o_{n/m} := N_{n/m} \cap S^o_m$. Then we have

\[
(R^{2c}(n,1)_a \otimes Q_{M(n,1)})_a \simeq \bigoplus_{n \geq m} \bigoplus_{l=0}^{\text{codim}_n S_{n/m} - 1} \mathcal{H}_a^l(\text{IC}(N_{n/m} \mid N^o_{n/m}))[r(n,d)] = \bigoplus_{n \geq m} \bigoplus_{l=0}^{\text{codim}_n S_{n/m} - 1} \mathcal{H}_a^l(\text{IC}(N_{n/m} \mid N^o_{n/m}))[r(n,d)].
\]

Proof. By Theorem 9.1 we have

\[
\mathcal{H}_a^{2c}(\mathcal{F}(\mathcal{L}_n(d))) = \bigoplus_{l=0}^{2\dim S_n} \mathcal{H}_a^{2\dim(S_n) - l}(\text{IC}(S_n, \Lambda^l_n \otimes \mathcal{L}_n(d))).
\]

Relative Hard Lefschetz gives $\Lambda^l_n \simeq L_n^{2\dim(S_n) - l}$. By Lemma 9.4 we also have $\mathcal{H}_a^0(\text{IC}(S_n, \mathcal{L}_n(d))) = (i_* \circ L_n)_a \simeq Q_a^{r(n,d)}$. All together we obtain

\[
\mathcal{H}_a^{2c}(\mathcal{F}_n) = Q_a^{r(n,d)} \oplus \bigoplus_{l=1}^{2\dim S_n} \mathcal{H}_a^l(\text{IC}(S_n, \Lambda^l_n))[r(n,d)].
\]

By stratification theory it is clear that

\[
\mathcal{H}_a^l(\text{IC}(S_n, \Lambda^l_n)) \simeq \mathcal{H}_a^l(\text{IC}(N_{n/m} \mid N^o_{n/m})).
\]

Further, by the strong support condition for intersection cohomology we obtain

\[
\mathcal{H}_a^l(\text{IC}(N_{n/m} \mid N^o_{n/m}))[r(n,d)) = 0 \quad \text{for all } l > \dim N_{n/m} - 1 = \text{codim}_n S_{n/m} - 1.
\]

\[\square\]

Corollary 9.6. [12, Lemma 6.9] $\mathcal{H}_a^l(\text{IC}(S_n, \Lambda^k_n)) = 0$ for $l > k$.

Proof. The same proof of Proposition 9.5 gives that $\mathcal{H}_a^l(\text{IC}(S_n, \Lambda^k_n))$ is a direct summand of $(R^{2c+(-l-k)}(n,1)_a \otimes Q_{M(n,1)})_a = 0$.

\[\square\]
Let \( \tilde{a} \) be a lift in \( A_m \) of a general point \( a \) in \( S_m \). Since \( A_n \) and \( A_m \) are smooth, a slice of \( A_n \) transverse to \( j_S(A_m) \) passing through \( \tilde{a} \) is smooth, and it can be identified with the product

\[
N_{S_1} \times \ldots \times N_{S_r} := N_{(n_1)/\mathbb{Z}_1} \times \ldots \times N_{(n_r)/\mathbb{Z}_r},
\]

Since the map \( \eta_m \) in (25) is étale at \( \tilde{a} \), we can identify the normalization of a branch of \( N_{\mathbb{Z}/m} \), denoted \( N_{\bar{S}} \), with the product above. To summarize, there exists a bijection between the branches of \( N_{\mathbb{Z}/m} \) and the partition of \( [s] \).

For brevity we denote \( \Lambda_{\bar{S}}^l := \Lambda_{\mathbb{Z},N_{\bar{S}}}^l \cap S_{m}^o \) and \( \Lambda_{S_i}^l := \Lambda_{a_i}^l \cap N_{(n_i)/\mathbb{Z}_i} \).

**Proposition 9.7 (Reduction to lower rank).** Under the identification \( N_{\bar{S}} \cong N_{S_1} \times \ldots \times N_{S_r} \) above, we have

\[
H^2_c(n,g)(\mathcal{S}(L^o_n(d))) \cong \bigoplus_{N_{\bar{S}} \subseteq N_{a/m}} H^2_c(n,g)(\mathcal{S}(L^o_n(d)|_{N_{\bar{S}}})
\]

(27)

\[
H^2_c(n,g)(\mathcal{S}(L^o_n(d)|_{N_{\bar{S}}}) \cong \left( \bigotimes_{i=1}^r H^2_c(n_i,g)(\mathcal{S}(a_i|_{N_{(n_i)/\mathbb{Z}_i}}) \right) \otimes \mathbb{Q}^r(\mathbb{Z},d).
\]

(28)

**Proof.** Eq. (27) follows from the proof of Proposition 9.5 and from the behaviour of intersection cohomology under normalization

\[
H^l_a(\text{IC}(N_{a/m}, \Lambda^l_{\mathbb{Z},a/m})) \cong \bigoplus_{N_{\bar{S}} \subseteq N_{a/m}} H^l_a(\text{IC}(N_{\bar{S}}, \Lambda^l_{\bar{S}})).
\]

Eq. (28) boils down to the Künneth decomposition for intersection cohomology. In order to prove this identity, consider the spectral curve \( \pi_b : C_b = \bigcup_{i=1}^r C_i \to C \times b \) over the generic point \( b \) of \( N_{\bar{S}} \). The map \( \pi_b \) restricts to a finite map \( C_i \to C \times b \) of degree \( n_i \) on the smooth irreducible component \( C_i \) of genus \( g_i \). In particular, over the general point \( b \) we have

\[
\text{Pic}^0(C^o_{b, \nu}) \cong \text{Pic}^0(C^o_{1,b}) \times \ldots \times \text{Pic}^0(C^o_{r,b}).
\]

Since the point \( a \) does not lay on the branch locus of \( A_m \to S_m \), the monodromy of \( \pi_b \) over \( N_{\bar{S}} \) preserves the irreducible components \( C_i \times \bar{b} \), and this implies

\[
\Lambda^l_{\bar{S}} \cong (\Lambda^l_{S_1} \boxtimes \ldots \boxtimes \Lambda^l_{S_r})^l = \bigoplus_{l \vdash n: l(i) = r} \Lambda^l_{S_1} \boxtimes \ldots \boxtimes \Lambda^l_{S_r}.
\]

where the summation index runs over the partition \( l \) of \( l \) of length \( r \). Then the Künneth decomposition for intersection cohomology gives

\[
H^l_a(\text{IC}(N_{\bar{S}}, \Lambda^l_{\bar{S}})) \cong \bigoplus_{k \vdash l: k(i) = r} H^l_a(\text{IC}(N_{S_1} \times \ldots \times N_{S_r}, \Lambda^{k_1}_{S_1} \boxtimes \ldots \boxtimes \Lambda^{k_r}_{S_r}))
\]

\[
\cong \bigoplus_{k \vdash l: k(i) = r} \bigotimes_{l \vdash: l(i) = r} H^l_a(\text{IC}(N_{S_1}, \Lambda^{k_1}_{S_1})) \otimes \ldots \otimes H^l_a(\text{IC}(N_{S_r}, \Lambda^{k_r}_{S_r})),
\]
with \( l = k \) (i.e. the two summation indexes both running over the same set of partition of \( l \) of length \( r \)). For each pair \( \{i, l\} \) of partitions of \( l \) there are two options: either \( l_i = k_i \) for all \( i \in [r] \), or there exists \( i_0 \in [r] \) such that \( l_{i_0} > k_{i_0} \). In the latter case, \( \mathcal{H}_a^{l_0}(IC(N_{S_{i_0}}, \Lambda_{S_{i_0}}^{l_0})) = 0 \) by Corollary 9.6. Thus we obtain
\[
\mathcal{H}_a^l(IC(N_{\Xi}, \Lambda_{\Xi}^l)) \approx \bigoplus_{p-l: \ell(l) = r} \mathcal{H}_a^l(IC(N_{S_1}, \Lambda_{S_1}^l)) \oplus \cdots \oplus \mathcal{H}_a^l(IC(N_{S_r}, \Lambda_{S_r}^l)).
\]

We conclude that
\[
\mathcal{H}_a^{2c(n,g)}(\mathcal{F}(A(d)|N_{\Xi})) \approx \bigoplus_{l \geq 0} \mathcal{H}_a^l(IC(N_{\Xi}, \Lambda_{\Xi}^l)) \otimes Q^r(n,d)
\]
\[
\approx \left( \bigotimes_{i=1}^r \mathcal{H}_a^{l_i}(IC(N_{S_i}, \Lambda_{S_i}^{l_i})) \right) \otimes Q^r(n,d)
\]
\[
\approx \left( \bigotimes_{i=1}^r \mathcal{H}_a^{2c(n,g)}(\mathcal{F}(\{n_i\}|N_{n_i/d_i})) \right) \otimes Q^r(n,d).
\]
\[
\square
\]

10. Proof of Theorem 1.5 and Theorem 1.6

Proof of Theorem 1.5. By Theorem 9.1 it suffices to check that \( (R^{2c(n,g)} \chi(n,0)_a Q M(n,0))_a \) has no proper support at the general point \( a \in S_m \) for any \( m \). We prove it by induction on the length of \( m \). If \( m \) has length 1, then \( a \in A^\text{reg} \) avoids any potential proper support of the decomposition theorem, so there is nothing to prove.

Suppose now that \( m \) is a permutation of length \( s > 1 \), and no summands of \( R^{2c(n,g)} \chi(n,0)_a Q M(n,0)|A^\text{reg} \) is supported on \( S_m \cap A^\text{red} \), for any \( n \in \mathbb{Z}_{>0} \) and for any partition \( n \) of \( \mathbb{N} \) such that \( m < n < \{n\} \).

We show that \( (R^{2c(n,g)} \chi(n,0)_a Q M(n,0))_a \) has no summands at \( a \in S_m \). Otherwise, there exists a non-zero \( \mathbb{Q} \)-vector space \( \mathcal{F}_a \) such that
\[
(R^{2c(n,g)} \chi(n,0)_a Q M(n,0))_a \approx \mathcal{H}_a^{2c(n,g)}(\mathcal{F}(\{n\})) \oplus \mathcal{F}_a.
\]

Let \( e \) be an integer with \( \gcd(e, n) = 1 \). Proposition 9.7 gives
\[
(R^{2c(n,g)} \chi(n,e)_a Q M(n,e))_a \approx \bigoplus_{n \geq m} \bigotimes_{S^{N_{S_{i}}}} \mathcal{H}_a^{2c(n,g)}(\mathcal{F}(\{\sigma_{\ell(S)}\}_{N_{S_{i}}})) \otimes Q^r(\ell(S), e)
\]
\[
\approx \mathcal{H}_a^{2c(n,g)}(\mathcal{F}(\{n\}) \oplus \bigoplus_{S^{S_{i}} \neq \{S_{i}\}} \bigotimes_{S^{N_{S_{i}}}} \mathcal{H}_a^{2c(n,g)}(\mathcal{F}(\{\sigma_{\ell(S)}\}_{N_{S_{i}}})) \otimes Q^r(\ell(S), e),
\]
\[\]
where we set \( \sigma_i := \sum_{j \in S_i} m_j \). By induction hypothesis we have that for any \( S \vdash [s] \)
\[
\mathcal{H}^{2c(\sigma,g)}_a(\mathcal{J}_{\{\sigma_i\}}|_{N_{S_i}}) \simeq \mathcal{H}^{2c(\sigma,g)}_a(\mathcal{J}_{\{\sigma_i\}}) \simeq (R^{2c(\sigma,g)}(\chi(\sigma_i, 0)_* \mathbb{Q}_{M(\sigma_i, 0)})_a).
\]

Recall that the vertices of the dual graph \( \Gamma[m] \) are labelled by \([s]\). Let \( \Gamma(S_i) \) be the maximal subgraph of \( \Gamma[m] \) on the vertex set \( S_i \). Theorem 7.8 gives
\[
\dim \mathcal{H}^{2c(\sigma,g)}_a(\mathcal{J}_{\{\sigma_i\}}|_{N_{S_i}}) = C(\Gamma(S_i)).
\]
Now set \( \omega_m(e) = (m_1 e/n, \ldots, m_s e/n) \in \mathbb{Q}^s \) with \( m = \{m_1, \ldots, m_s\} \), and write
\[
C(\Gamma[m] + \omega_m(e)) = \dim (R^{2c(n,g)}(\chi(n, e)_* \mathbb{Q}_{M(n,e)})_a
\]
\[
= \dim \mathcal{H}^{2c(n,g)}_a(\mathcal{J}_{\{n\}}) + \sum_{S \vdash [s]} r(\ell|S), e \cdot \prod_{i=1}^{[S]} \dim \mathcal{H}^{2c(\sigma_i,g)}_a(\mathcal{J}_{\{\sigma_i\}}|_{N_{S_i}})
\]
\[
= \dim \mathcal{H}^{2c(n,g)}_a(\mathcal{J}_{\{n\}}) + \sum_{S \vdash [s]} r(\ell|S), e \cdot \prod_{i=1}^{[S]} C(\Gamma(S_i)).
\]

Note that \( r(\ell|S), 1 \geq (\ell|S) - 1)! \) by the Hodge-to-singular correspondence \([28]\) Thm 1.1 and Prop. 4.11.(8). Together with the combinatorial identity of Remark 5.4.(i)
\[
C(\Gamma[m] + \omega_m(e)) = \sum_{S \vdash [s]} |S| \prod_{j=1}^{[S]} C(\Gamma(S_j)),
\]
we conclude that \( r(\ell|S), 1 \) actually equals \((\ell|S) - 1)! \) and so
\[
\dim \mathcal{H}^{2c(n,g)}_a(\mathcal{J}_{\{n\}}) = C(\Gamma[m]) = \dim (R^{2c(n,g)}(\chi(n, 0)_* \mathbb{Q}_{M(n,0)})_a,
\]
which implies \( \mathcal{F}_a = 0 \), i.e. \( S_m \) is not a support of \( R^{2c(n,g)}(\chi(n, 0)_* \mathbb{Q}_{M(n,0)}) \).

**Proof of Theorem 1.6.** Let \( a \) be a general point in \( S_m \) for any partition \( m = \{m_1, \ldots, m_s\} \). Proposition 9.7 gives
\[
(R^{2c(n,g)}(\chi(n,d)_* \mathbb{Q}_{M(n,d)})_a \simeq \bigoplus_{S \vdash [s]} \mathcal{L}_{|\ell|S_i}}(d)_a \otimes \left( \bigotimes_{i=1}^{[S]} \mathcal{H}^{2c(\sigma_i,g)}_a(\mathcal{J}_{\{\sigma_i\}}|_{N_{S_i}}) \right).
\]
The LHS of \([29]\) is an \( \Aut(m) \)-representation, while the summands of the RHS are \( \Aut(m) \)-representations induced by the injection \( \text{Stab}(S) \hookrightarrow \Aut(m) \). For convenience, write \( \omega \) for the vector \( \omega_m(d) \) defined in \([2]\). By Theorem 7.8 and Theorem 6.4 the following \( \Aut(m) \)-representations are isomorphic
\[
(R^{2c(n,g)}(\chi(n,d)_* \mathbb{Q}_{M(n,d)})_a \simeq C(\Gamma[m] + \omega_m(d))
\]
\[\text{[3] Alternatively, the equality } r(\ell|S), 1 = (\ell|S) - 1)! \text{ follows via the explicit computation of the Cattani-Kaplan-Schmid complex for a versal family of } C_a \text{ in [12] Cor. 6.20.}\]
\[ \simeq C(Z_{\Gamma [m]}) \oplus \bigoplus_{S \in \Pi_m / \text{Aut}(\Gamma [m])} \text{Ind}_{\text{Stab}(\Gamma [m])}^{\text{Aut}(\Gamma [m])} \left( \text{sgn} \otimes \tilde{H}_{\ell(S)} - 3(\Delta(\Pi_{\omega_S})) \otimes C(Z_{\Gamma [m]}) \right), \]
\[ \simeq \left( R^{2c(n,g)}(\chi(n,0) + Q \mathcal{M}(n,0)) \right)_{\text{a}} \oplus \bigoplus_{S \in \Pi_m / \text{Aut}(\Gamma [m])} \text{Ind}_{\text{Stab}(\Gamma [m])}^{\text{Aut}(\Gamma [m])} \left( \text{sgn} \otimes \tilde{H}_{\ell(S)} - 3(\Delta(\Pi_{\omega_S})) \otimes \left( \bigotimes_{i=1}^{\ell(S)} \mathcal{L}_{\sigma_i}^{2c(\sigma_i,g)}(\mathcal{M}_{\sigma_i} | N_{\sigma_i}) \right) \right). \]

Comparing with (29), we conclude that the Aut(\Gamma [m])-representations
\[ \mathcal{L}_{\alpha}^{\ell}(d) \simeq \text{sgn} \otimes \tilde{H}_{\ell(d)} - 3(\Delta(\Pi_{\omega_{\alpha}}(d))) \]
are isomorphic by induction on the length of m. \qed

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