A new method is presented for the determination of Ricci Collineations (RC) and Matter Collineations (MC) of a given spacetime, in the cases where the Ricci tensor and the energy momentum tensor are non-degenerate and have a similar form with the metric. This method reduces the problem of finding the RCs and the MCs to that of determining the KVs whereas at the same time uses already known results on the motions of the metric.

We employ this method to determine all hypersurface homogeneous locally rotationally symmetric spacetimes, which admit proper RCs and MCs. We also give the corresponding collineation vectors. These results conclude a long due open problem, which has been considered many times partially in the literature.

KEY WORDS: Ricci Collineations; Matter Collineations; Locally Rotationally Symmetric spacetimes;

1 Introduction

The field equations of General Relativity are highly non-linear pdfs and their solution requires simplifying assumptions in the form of additional conditions/constraints. There are many ways to impose simplifying assumptions on the metric. These assumptions must satisfy various general rules one of them being the requirement that they must be consistent with the symmetry group of the metric and the geometric structures on the spacetime manifold. One class of assumptions, which satisfy the above demand, are the collineations or geometric symmetries.

A general collineation is defined $\mathcal{L}_\xi A = \Phi$ where $A$ is any of the quantities $g_{ab}, \Gamma^a_{bc}, R_{ab}, R^a_{bcd}$ and geometric objects constructed by them and $\Phi$ is a tensor with the same index symmetries as $A$. There are many types of collineations defined by the various forms of the tensors $A, \Phi$. For example $A_{ab} = g_{ab}$ and $\Phi_{ab} = 2\psi g_{ab}$ define a Conformal Killing vector (CKV), which specializes to a Special Conformal Killing vector (SCKV) when $\psi_{ab} = 0$, to a Homothetic vector field (HVF) when $\psi = \text{constant}$ and to a Killing vector (KV) when $\psi = 0$. When $A_{ab} = R_{ab}$ and $\Phi_{ab} = 2\psi R_{ab}$ the symmetry vector $\xi^a$ is called a Ricci Conformal Collineation (RCC) and specializes to a Ricci Collineation (RC) when $\Phi_{ab} = 0$. When $A_{ab} = T_{ab}$ and $\Phi_{ab} = 2\psi T_{ab}$, where
$T_{ab}$ is the energy momentum tensor, the vector $\xi^a$ is called a Matter Conformal Collineation (MCC) and specializes to a Matter collineation (MC) when $\Phi_{ab} = 0$. The function $\psi$ in the case of CKVs is called the conformal factor and in the case of conformal collineations the conformal function.

It is well known that two different collineations are not in general equivalent. For example a KV is a RC or a MC but the opposite does not hold. Collineations have been classified by means of their relative properness in [1, 2]. From this classification it is seen that the basic collineation is the KVs.

The role of the KVs (or symmetries) is to restrict (possibly with additional assumptions) the general form of the metric. This results in a reduction of the number of the independent field equations and (as a rule) in the simplification of their study. We should note that there are well known spacetime metrics which do not have KVs [3]. This does not exclude the possibility that they can admit higher collineations.

The role of a higher collineation is to supply the field equations with additional equations, which are the equations defining the collineation. These later equations involve the metric functions and the components of the vector field defining the collineation.

Obviously the constraints imposed by additional collineations do not guarantee that they will lead to a solution of the field equations. However if they do then these solutions are compatible with the general structure of the metric and the geometry resulting from it.

The standard method to deal with the augmented system of field equations is the direct solution of the system of partial differential equations. As expected, this procedure is in general difficult and, in many cases, it has the defect that one can loose solutions, especially those occurring as particular cases. As a result people have tried to find indirect methods of solution, which relay more on differential geometry and less on the solution of partial differential equations.

The purpose of this paper is twofold:

(a) to present a “practical” method, which reduces (whenever this is possible) the computation of the RCs and the MCs of a given metric to the computation of KVs and

(b) To apply this method and determine all hypersurface orthogonal locally rotationally symmetric (LRS) spacetime metrics, which admit proper MCs and proper RCs. We recall that a RC/MC is proper if it is not a KV or a HKV or a SCKV.

A first partial exposition of the method has been given previously in [4] and independently in [5] and has been applied in the determination of all Robertson-Walker metrics, which admit proper RCs and MCs.

The proposed method applies only when the Ricci tensor $R_{ab}$ and the energy momentum tensor $T_{ab}$ are non-degenerate and when the form of $R_{ab}$, $T_{ab}$ (equivalently $G_{ab}$) is similar to the form of the metric. Let us assume that this is the case. Then one can consider on the spacetime manifold the three metric elements

\[ ds_R^2 = R_{ab}dx^adx^b, \quad ds_T^2 = T_{ab}dx^adx^b \quad \text{and} \quad ds_G^2 = g_{ab}dx^adx^b, \]

which have in general different signature, all signatures being possible for the first two. Each of these ‘metrics’ has a symmetry group and because the ($C^\infty$) KVs of the metric are ($C^\infty$) RCs and ($C^\infty$) MCs these groups have a common subgroup. But this subgroup is (as a rule) the main factor, which defines the general form of the metric. Therefore it is logical to expect

\[ ds_R^2 = R_{ab}dx^adx^b, \quad ds_T^2 = T_{ab}dx^adx^b, \quad \text{and} \quad ds_G^2 = g_{ab}dx^adx^b, \]

\[ ds^2 = ds^2_R + ds^2_T - ds^2_G. \]

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that \( g_{ab}, R_{ab} \) and \( T_{ab} \) will have a ‘common form’, provided the definition of the metric does not involve other assumptions besides KVs, for example discrete symmetries. However this is not in general known or easy to prove and the best way to make sure that this is the case is to compute directly the tensors \( R_{ab}, T_{ab} \) and see if their form is or is not similar to the form of the metric.

If the answer is positive then it is possible to consider the ‘generic’ line element \( ds^2 = K_{ab}dx^a dx^b \), which reduces to the three line elements \( ds_g^2, ds_R^2, ds_T^2 \) for appropriate choices of the coefficients \( K_{ab} \). The gains from this consideration are twofold.

a. If one solves Killing’s equations for the generic metric then one has found simultaneously the KVs of the metric, the RCs and the MCs. That is, the problem of finding the RCs and MCs is reduced to that of calculating the KVs.

b. The generic metric is possible to have all signatures. Therefore Killing’s equations will have to be solved for all possible signatures. In case the KVs of the metric are known then only the signatures \((+,+,+,+)\) and \((+,+,−,−)\) need to be considered.

In case the form of the tensors \( R_{ab}, T_{ab} \) is different from the form of the metric \( g_{ab} \) the introduction of \( K_{ab} \) makes no sense and one has to follow the standard way, i.e. solve the pdfs resulting from the constraint. However we note, that in many cases one can still introduce \( K_{ab} \) for the tensors \( R_{ab}, T_{ab} \) only, and apply the same method.

It is useful to comment briefly when \( R_{ab}, T_{ab} \) are degenerate. In this case it has been shown [6] that, in general, there are infinitely many RCs and MCs, which must be found by the solution of the relevant pdfs. However, the RCs in the degenerate case are not as useful as the ones of the non-degenerate case. Indeed the assumption of degeneracy of \( R_{ab}, T_{ab} \) leads to differential equation(s), which fix the metric functions up to arbitrary constants of integration. Hence the form of the Ricci or the Matter tensor can be determined making the constraint imposed by the RC/MC redundant. For example if \( T_{ab} \) is degenerate it has been shown [7] that the only interesting case is when rank \( T_{ab} = 1 \) that is, a null Einstein-Maxwell field or a dust fluid. In most applications (including the LRS case as will be seen in the next section) the corresponding metrics are known and there is no need for an extra assumption to obtain the solution of the field equations.

The structure of the paper is as follows. In section 2 we consider the three possible classes of LRS metrics. In sections 3, 4 and 5 we determine the KVs of the generic metric (whenever it can be defined) for all possible signatures and give explicitly the proper RCs and the proper MCs. We also discuss examples which show how our general results can be applied in practice. Finally in section 6 we conclude the paper.

2 Hypersurface homogeneous LRS spacetimes

Hypersurface homogeneous spacetimes which are locally rotationally symmetric (LRS spacetimes) contain many well known and important solutions of Einstein field equations and have been studied extensively in the literature [8, 9, 10]. They admit a group of motions \( G_4 \) acting multiply transitively on three dimensional orbits spacelike \((S_3)\) or timelike \((T_3)\) the isotropy group being a spatial rotation. It is well known that there are three families of metrics describing these spacetimes [8, 9]:

\[
ds^2 = \varepsilon [dt^2 - A^2(t)dx^2] + B^2(t) \left[ dy^2 + \Sigma^2(y, k)dz^2 \right]
\] (1)
ds^2 = \varepsilon \left\{dt^2 - A^2(t) \left[dx + \Lambda(y,k)dz\right]^2 \right\} + B^2(t) \left[dy^2 + \Sigma^2(y,k)dz^2\right]  \tag{2}
\quad
\Rightarrow
\quad
\begin{align*}
\text{where } &\varepsilon = \pm 1, \Sigma(y,k) = \sin y, \sinh y, y \text{ and } \Lambda(y,k) = \cos y, \cosh y, y^2 \text{ for } k = 1, -1, 0 \text{ respectively. (The factor } \varepsilon = \pm 1 \text{ essentially distinguishes between the "static" and the "nonstatic" cases as it can be seen by interchanging the co-ordinates } t, x). \\
&\text{Indeed the Ricci tensor } R \text{ (1) we can consider the "generic" metric:}
\end{align*}

We note that the form of \( R \) acting on 3D spacelike or timelike orbits (\( \varepsilon = -1, 1 \) respectively).

As we have already remarked, the solution of the field equations for these metrics is possible only in special cases, that is, when the metric is required to satisfy additional constraints. In this paper we consider the extra constraint to be the requirement that the LRS metric admits a proper RC or a proper MC. To compute the subset of the LRS metrics selected by this constraint we apply - when it is possible - the method of generic metric described above. As it will be shown the method applies to the LRS metrics (1) and (2) and does not always applies to the metric (3), in which case we have to work differently.

3 Ricci and Matter Collineations of the LRS Ellis class II metrics (1.1)

The LRS metrics (1) admit the isometry group \( G_4 \) consisting of the four KVs \( \partial_x, X_\mu (\mu = 1, 2, 3) \):

\[ X_\mu = (\delta^1_\mu \cos z + \delta^2_\mu \sin z)\partial_y - \left[(\ln \Sigma)_y (\delta^1_\mu \sin z - \delta^2_\mu \cos z) - \delta^3_\mu \right] \partial_z \tag{4} \]

acting on 3D spacelike or timelike orbits (\( \varepsilon = -1, 1 \) respectively).

We only consider the case \( \varepsilon = -1 \) (3D spacelike orbits) because the results for \( \varepsilon = 1 \) follow from the interchange of the coordinates \( t, x \) (the Ricci and Matter tensor are identical up to a minus sign). Indeed the Ricci tensor \( R_{ab} \) and the Einstein tensor \( G_{ab} \) (i.e. the energy momentum tensor) are computed to be:

\[ R_{ab} = \varepsilon \cdot \text{diag} \left\{ \frac{2\dot{B}A + \dot{A}B}{AB}, -\frac{A(\dot{A}B + 2\dot{B}A)}{B}, -\frac{A(\dot{B}B + B^2 + k) + \dot{B}AB}{A} \right\} \tag{5} \]

\[ G_{ab} = \varepsilon \cdot \text{diag} \left\{ -\frac{A(\dot{B}^2 + k)}{AB^2}, \frac{A^2 (2\dot{B}B + \dot{B}^2 + k)}{B^2}, \frac{B(\dot{A}B + \dot{B}A + \dot{A}B)}{A} \right\} \tag{6} \]

We note that the form of \( R_{ab} \) and \( G_{ab} \) is similar to that of the metric, therefore for the metrics (1) we can consider the "generic" metric:

\[ ds^2 = K_0 dt^2 + K_1 dx^2 + K_2 \left[ dy^2 + \Sigma^2(y,k)dz^2\right] \tag{7} \]

which reduces to the metrics \( ds^2_y, ds^2_R, ds^2_I \) when \( K_a = \{g_a, R_a, G_a\} \) where:

\[ g_a = \{-1, A^2, B^2, B^2\} \tag{8} \]
\[ R_a = \left\{ \frac{-2\bar{B}A + \bar{A}B}{AB}, \frac{A(\bar{A}B + 2\bar{B}\bar{A})}{B}, \frac{A(\bar{B}B + B^2 + k) + \bar{B}\bar{A}B}{A} \right\}_{[1,1]} \]  \tag{9}

\[ G_a = \left\{ \frac{A(\bar{B}^2 + k)}{AB^2}, \frac{-A^2(2\bar{B}B + \bar{B}^2 + k)}{B^2}, \frac{-B(\bar{A}B + \bar{B}A + \bar{B}\bar{A})}{A} \right\}_{[1,1]} \]  \tag{10}

In order to compute the KVs of the generic metric we apply the transformation:

\[ d\bar{\tau} = |K_0|^{1/2} dt \]  \tag{11}

so that:

\[ ds^2 = \varepsilon_1(K_0) d\bar{\tau}^2 + K_1 dx^2 + K_2 \left[ dy^2 + \Sigma^2(y, k) dz^2 \right] \]  \tag{12}

where \( \varepsilon_1(K_0) \) is the sign of the component \( K_0 \).

The KVs \( X \) of \( ds^2 \) are computed from the solution of Killing’s equations \( \mathcal{L}_X K_{ab} = 0 \). Because the generic metric can have all possible signatures we have to consider three cases:

- Lorentzian case \((-1,1,1,1)\).
- Euclidean case \((1,1,1,1)\).
- The case \((-1,-1,1,1)\).

Killing’s equations for the case of the Lorentzian signature have been solved in [11] where it has been shown that there are only two possible cases to consider, i.e. non conformally flat metrics and the conformally flat metrics. This conclusion holds for the other two cases because it is independent from the signature of the generic metric.

The results of the (typical) calculations for the Lorentzian cases are collected in Table 1 and for the non-Lorentzian cases in Table 2. Concerning the explanation of Tables 1,2 we have the following. Classes \( A_1 - A_7 \) refer to the non-conformally flat cases and classes \( B_1 \) to \( B_8 \) to the conformally flat cases. The columns \( K_1, K_2 \) give the functional forms of the generic metric components in order the collineation(s) \( X \) to be admitted. \( \dim C \) is the dimension of the full isometry group of the “generic” metric element \( ds^2 \) including the four vectors (4). Last column gives the expression of the KV(s) in terms of the coordinates and the parameters entering the metric functions. It is worth noting that if we interchange \( t, x \) in the expressions for the vector fields we obtain the KVs for the static case.

Some of the collineations in Tables 1 and 2 have been found previously by various authors (see for example [12, 13] and references cited therein).

The RCs and the MCs we give in Tables 1,2 are proper because they cannot be reduced to the extra KV of the homogeneous or to the HVF of the self similar corresponding spacetime. By demanding this reduction we have found for each vector the values of the parameters, which should be excluded.
Table 1. KVs of the metrics (1) for the case $(K_0 K_1 K_2) < 0$, that is Lorentzian signature. $k$ is the curvature of the 2-space $y,z$. $\dim \mathcal{C}$ is the dimension of the full symmetry algebra of the generic metric $ds^2$.

| Class | $k$ | $K_1$ | $K_2$ | $\dim \mathcal{C}$ | $X$ |
|-------|-----|-------|-------|------------------|-----|
| $A_1$ | 0   | $\pm c^2 e^{-2\tau/\alpha_1 c}$ | $\pm c^2 e^{-2\tau/c}$ | 5 | $\alpha_1 c \partial_{\tilde{\tau}} + x \partial_x + \alpha_1 y \partial_y$ |
|       |     | $\pm c^2 e^{-2\tau/\alpha_1 c}$ | $\pm c^2 e^{-2\tau/c}$ |     | $c \neq -\sqrt{2 + \frac{1}{a_1^2}}$, $c \neq \frac{\text{sign}(b)}{2a_1 \sqrt{1+2a_1^2 - 3b}}$ |
| $A_2$ | $\pm 1$ | $\pm \frac{c^2}{1+e^{-2\tau/c^2}}$ | $\pm c^2$ | 6 | $\partial_{\tilde{\tau}}$ |
|       |     | $\pm \frac{c^2}{1+e^{-2\tau/c^2}}$ | $\pm c^2$ |     | $c_1 c_2 x \partial_{\tilde{\tau}} + \frac{\tilde{\tau}}{c_1 c_2} \partial_x$ |
| $A_3$ | 0, $\pm 1$ | $\pm \frac{c^2}{1+e^{-2\tau/c^2}}$ | $\pm c^2$ | 6 | $\frac{\partial_{\tilde{\tau}}}{2ac_2} + c \partial_x$ |
|       |     | $\pm \frac{c^2}{1+e^{-2\tau/c^2}}$ | $\pm c^2$ |     | $2ac_2 x \partial_{\tilde{\tau}} - \left( \frac{c^2}{1+e^{-2\tau/c^2}} \right) \partial_x$ |
| $A_4$ | 0, $\pm 1$ | $\pm c^2 \cosh^2 \frac{\tilde{\tau}}{ac}$ | $\pm c^2$ | 6 | $c \sin \frac{\tilde{\tau}}{ac} \partial_x + \tanh \frac{\tilde{\tau}}{ac} \cos \frac{\tilde{\tau}}{ac} \partial_x$ |
|       |     | $\pm c^2 \cosh^2 \frac{\tilde{\tau}}{ac}$ | $\pm c^2$ |     | $c \cos \frac{\tilde{\tau}}{ac} \partial_x - \tanh \frac{\tilde{\tau}}{ac} \sin \frac{\tilde{\tau}}{ac} \partial_x$ |
| $A_5$ | 0, $\pm 1$ | $\pm c^2 \sinh^2 \frac{\tilde{\tau}}{ac}$ | $\pm c^2$ | 6 | $c \sinh \frac{\tilde{\tau}}{ac} \partial_x - \coth \frac{\tilde{\tau}}{ac} \cosh \frac{\tilde{\tau}}{ac} \partial_x$ |
|       |     | $\pm c^2 \sinh^2 \frac{\tilde{\tau}}{ac}$ | $\pm c^2$ |     | $c \cosh \frac{\tilde{\tau}}{ac} \partial_x - \coth \frac{\tilde{\tau}}{ac} \sinh \frac{\tilde{\tau}}{ac} \partial_x$ |
| $A_6$ | 0, $\pm 1$ | $\pm c^2 \cos^2 \frac{\tilde{\tau}}{ac}$ | $\pm c^2$ | 6 | $c \sinh \frac{\tilde{\tau}}{ac} \partial_x + \tanh \frac{\tilde{\tau}}{ac} \cosh \frac{\tilde{\tau}}{ac} \partial_x$ |
|       |     | $\pm c^2 \cos^2 \frac{\tilde{\tau}}{ac}$ | $\pm c^2$ |     | $c \cosh \frac{\tilde{\tau}}{ac} \partial_x + \tanh \frac{\tilde{\tau}}{ac} \sinh \frac{\tilde{\tau}}{ac} \partial_x$ |
| $A_7$ | $\pm 1$ | $\pm \tilde{\tau}^2$ | $\pm c^2$ | 6 | $\cosh \tilde{\tau} - \tilde{\tau}^{-1} \sinh \tilde{\tau} \partial_x$ |
|       |     | $\pm \tilde{\tau}^2$ | $\pm c^2$ |     | $\sinh \tilde{\tau} \partial_x - \tilde{\tau}^{-1} \cosh \tilde{\tau} \partial_x$ |
Table 1 (continued). KVs of the metrics (1) for the case \((K_0K_1K_2) < 0\), that is Lorentzian signature. \(k\) is the curvature of the 2-space \(y, z\). \(\dim C\) is the dimension of the full symmetry algebra of the generic metric \(ds^2\).

| Class | \(k\) | \(K_1\) | \(K_2\) | \(\dim C\) | \(X\) |
|-------|-------|------|------|--------|------|
| \(B_1\) | 1 | \(\pm c_1^2 c^2\) | \(\pm c^2 \cosh^2 \frac{\tau}{c}\) | 7 | \(X_{\mu+\nu+3} = -f(\mu) [f'(\nu)]_{,\tau} \left(c \cosh \frac{\tau}{c}\right)^2 \partial_{\tau} + \frac{f'_{(\mu)} f'_{(\nu)}}{c_1^2 \tanh \frac{\tau}{c}} \partial_x - f'_{(\mu)} [f_{(\nu)}]_{,y} \partial_y - \frac{f''_{(\nu)} [f_{(\mu)}]}{\sin^2 y} \partial_z\) |
| \(B_2\) | -1 | \(\pm c_1^2 c^2\) | \(\pm c^2 \sinh^2 \frac{\tau}{c}\) | 7 | \(X_{\mu+\nu+3} = -f(\mu) [f'(\nu)]_{,\tau} \left(c \sinh \frac{\tau}{c}\right)^2 \partial_{\tau} + \frac{f'_{(\mu)} f'_{(\nu)}}{c_1^2 \coth \frac{\tau}{c}} \partial_x - f'_{(\mu)} [f_{(\nu)}]_{,y} \partial_y - \frac{f''_{(\nu)} [f_{(\mu)}]}{\sinh^2 y} \partial_z\) |
| \(B_3\) | -1 | \(\pm c_1^2 c^2\) | \(\pm c^2 \sin^2 \frac{\tau}{c}\) | 7 | \(X_{\mu+\nu+3} = -f(\mu) [f'(\nu)]_{,\tau} \left(c \sin \frac{\tau}{c}\right)^2 \partial_{\tau} + \frac{f'_{(\mu)} f'_{(\nu)}}{c_1^2 \cot \frac{\tau}{c}} \partial_x - f'_{(\mu)} [f_{(\nu)}]_{,y} \partial_y - \frac{f''_{(\nu)} [f_{(\mu)}]}{\sin^2 y} \partial_z\) |
| \(B_4\) | 1 | \(\pm c_1^2 c^2 \sinh^2 \frac{\tau}{c}\) | \(\pm c^2 \cosh^2 \frac{\tau}{c}\) | 10 | \(X_{2(\mu+1)+\nu} = -f(\mu) [f'(\nu)]_{,\tau} \left(c \cosh \frac{\tau}{c}\right)^2 \partial_{\tau} + \frac{f'_{(\mu)} f'_{(\nu)}}{c_1^2 \tanh \frac{\tau}{c}} \partial_x - f'_{(\mu)} [f_{(\nu)}]_{,y} \partial_y - \frac{f''_{(\nu)} [f_{(\mu)}]}{\sin^2 y} \partial_z\) |
| \(B_5\) | -1 | \(\pm c_1^2 c^2 \cosh^2 \frac{\tau}{c}\) | \(\pm c^2 \sinh^2 \frac{\tau}{c}\) | 10 | \(X_{2(\mu+1)+\nu} = -f(\mu) [f'(\nu)]_{,\tau} \left(c \sinh \frac{\tau}{c}\right)^2 \partial_{\tau} + \frac{f'_{(\mu)} f'_{(\nu)}}{c_1^2 \coth \frac{\tau}{c}} \partial_x - f'_{(\mu)} [f_{(\nu)}]_{,y} \partial_y - \frac{f''_{(\nu)} [f_{(\mu)}]}{\sinh^2 y} \partial_z\) |
| \(B_6\) | -1 | \(\pm c_1^2 c^2 \cos^2 \frac{\tau}{c}\) | \(\pm c^2 \sin^2 \frac{\tau}{c}\) | 10 | \(X_{2(\mu+1)+\nu} = -f(\mu) [f'(\nu)]_{,\tau} \left(c \sin \frac{\tau}{c}\right)^2 \partial_{\tau} + \frac{f'_{(\mu)} f'_{(\nu)}}{c_1^2 \cot \frac{\tau}{c}} \partial_x - f'_{(\mu)} [f_{(\nu)}]_{,y} \partial_y - \frac{f''_{(\nu)} [f_{(\mu)}]}{\sin^2 y} \partial_z\) |
| \(B_7\) | 0 | \(\pm c_1^2\) | \(\pm c_2^2\) | 10 | \(X_{2(\mu+1)+\nu} = -c_2 f(\mu) [f'(\nu)]_{,\tau} \partial_{\tau} + \frac{c_2 f(\mu) f'(\nu)}{c_1^2} \partial_x - \frac{c_2 f(\mu) f'(\nu)}{c_1^2} \partial_y - \frac{f''_{(\nu)} [f_{(\mu)}]}{y^2 c_2} \partial_z\) |
| \(X_9\) | | | | | \(X_{10} = c_1 x \partial_{\tau} + \frac{\tau}{c_1} \partial_x\) |
| \(B_8\) | 0 | \(\pm c_1^2 \tau^2\) | \(\pm c_2^2\) | 10 | \(X_{2(\mu+1)+\nu} = -c_2 f(\mu) f'(\nu) \partial_{\tau} + \frac{c_2 f(\mu) f'(\nu)}{c_1^2} \partial_x - \frac{c_2 f(\mu) f'(\nu)}{c_1^2} \partial_y - \frac{f''_{(\nu)} [f_{(\mu)}]}{y^2 c_2} \partial_z\) |
| \(X_9\) | | | | | \(X_{10} = c_1 x \partial_{\tau} - \frac{1}{c_1^2} \sinh c_1 x \partial_x\) |
| \(X_{10}\) | | | | | \(X_{10} = \sinh c_1 x \partial_{\tau} - \frac{1}{c_1^2} \cosh c_1 x \partial_x\) |
Table 2. KVs of the metrics (1) for the case $(K_0 K_1 K_2) > 0$, that is the signatures $(+, +, +)$ and $(-, -, +, +)$. $k$ is the curvature of the 2-space $y, z$. $\dim \mathcal{C}$ is the dimension of the full symmetry algebra of the generic metric $ds^2$.

| Class | $k$ | $K_1$ | $K_2$ | $\dim \mathcal{C}$ | $X$ |
|-------|-----|-------|-------|----------------|-----|
| $A_1$ | 0   | $\pm c^2 e^{-2\tau/a_1c}$ | $\pm c^2 e^{-2\tau/c}$ | 5   | $\alpha_1 c \partial \tau + x \partial_x + \alpha_1 y \partial_y$ |
| $A_2$ | $\pm 1$ | $\pm c_1^2 c_2^2$ | $\pm c_2^2$ | 6   | $c_1 c_2 x \partial \tau - \frac{\partial \tau}{c_1 c_2} \partial_x$ |
| $A_3$ | $0, \pm 1$ | $\pm c_1^2 e^{\frac{2\tau}{a_1 c}}$ | $\pm c_2^2$ | 6   | $-ac_2 \partial \tau + x \partial_x$ |
| $A_4$ | $0, \pm 1$ | $\pm c^2 \cos^2 \frac{\tau}{a c}$ | $\pm c^2$ | 6   | $c \sin \frac{\tau}{a} \partial \tau - \tan \frac{\tau}{a c} \cos \frac{\tau}{a} \partial_x$ |
|       |      |       |       |     | $c \cos \frac{\tau}{a} \partial \tau + \tan \frac{\tau}{a c} \sin \frac{\tau}{a} \partial_x$ |
| $A_5$ | $0, \pm 1$ | $\pm c^2 \sinh^2 \frac{\tau}{a c}$ | $\pm c^2$ | 6   | $c \sin \frac{\tau}{a} \partial \tau + \coth \frac{\tau}{a c} \cos \frac{\tau}{a} \partial_x$ |
|       |      |       |       |     | $c \cos \frac{\tau}{a} \partial \tau - \coth \frac{\tau}{a c} \sin \frac{\tau}{a} \partial_x$ |
| $A_6$ | $0, \pm 1$ | $\pm c^2 \cosh^2 \frac{\tau}{a c}$ | $\pm c^2$ | 6   | $c \sinh \frac{\tau}{a} \partial \tau - \tanh \frac{\tau}{a c} \cosh \frac{\tau}{a} \partial_x$ |
|       |      |       |       |     | $c \cosh \frac{\tau}{a} \partial \tau - \tanh \frac{\tau}{a c} \sinh \frac{\tau}{a} \partial_x$ |
| $A_7$ | $\pm 1$ | $\pm \tau^2$ | $\pm c^2$ | 6   | $\cos x \partial \tau - \tau^{-1} \sin x \partial_x$ |
|       |      |       |       |     | $\sin x \partial \tau + \tau^{-1} \cos x \partial_x$ |
Table 2 (continued) KVs of the metrics (1) for the case \((K_0 K_1 K_2) > 0\), that is the signatures 
\(+, +, +, +\) and \((-,-,+,+)\). \(k\) is the curvature of the 2-space \(y, z\). \(\dim \mathcal{C}\) is the dimension of the 
full symmetry algebra of the generic metric \(ds^2\).

| Class | \(k\) | \(K_1\) | \(K_2\) | \(\dim \mathcal{C}\) | \(\mathbf{X}\) |
|-------|-------|-------|-------|------------|------------|
| \(B_1\) | 1 | \(\pm c_1^2 c^2\) | \(\pm c^2 \cos^2 \frac{\xi}{c}\) | 7 | \(\mathbf{X}_{\mu+\nu+3} = f_{(\mu)} \left[ f'_{(\nu)} \right]_{,\xi} \left( c \cos \frac{\xi}{c} \right)^2 \partial_\tau + \) \(\frac{f_{(\mu)}}{c_1^2} \left[ f'_{(\nu)} \right]_{,\xi} \partial_x - f'_{(\mu)} \partial_y - \frac{f'_{(\nu)} [f_{(\mu)}]}{\sin^2 y} \partial_z\) |
| \(B_2\) | -1 | \(\pm c_1^2 c^2\) | \(\pm c^2 \cosh^2 \frac{\xi}{c}\) | 7 | \(\mathbf{X}_{\mu+\nu+3} = f_{(\mu)} \left[ f'_{(\nu)} \right]_{,\xi} \left( c \cosh \frac{\xi}{c} \right)^2 \partial_\tau + \) \(\frac{f_{(\mu)}}{c_1^2} \left[ f'_{(\nu)} \right]_{,\xi} \partial_x - f'_{(\mu)} \partial_y - \frac{f'_{(\nu)} [f_{(\mu)}]}{\sinh^2 y} \partial_z\) |
| \(B_3\) | 1 | \(\pm c_1^2 c^2\) | \(\pm c^2 \sinh^2 \frac{\xi}{c}\) | 7 | \(\mathbf{X}_{\mu+\nu+3} = f_{(\mu)} \left[ f'_{(\nu)} \right]_{,\xi} \left( c \sinh \frac{\xi}{c} \right)^2 \partial_\tau + \) \(\frac{f_{(\mu)}}{c_1^2} \left[ f'_{(\nu)} \right]_{,\xi} \partial_x - f'_{(\mu)} \partial_y - \frac{f'_{(\nu)} [f_{(\mu)}]}{\sinh^2 y} \partial_z\) |
| \(B_4\) | 1 | \(\pm c_1^2 c^2 \sin^2 \frac{\xi}{c}\) | \(\pm c^2 \cos^2 \frac{\xi}{c}\) | 10 | \(\mathbf{X}_{2(\mu+1)+\nu} = f_{(\mu)} \left[ f'_{(\nu)} \right]_{,\xi} \left( c \cos \frac{\xi}{c} \right)^2 \partial_\tau + \) \(\frac{f_{(\mu)}}{c_1^2} \left[ f'_{(\nu)} \right]_{,\xi} \partial_x - f'_{(\mu)} \partial_y - \frac{f'_{(\nu)} [f_{(\mu)}]}{\sin^2 y} \partial_z\) |
| \(B_5\) | -1 | \(\pm c_1^2 c^2 \sinh^2 \frac{\xi}{c}\) | \(\pm c^2 \cosh^2 \frac{\xi}{c}\) | 10 | \(\mathbf{X}_{2(\mu+1)+\nu} = f_{(\mu)} \left[ f'_{(\nu)} \right]_{,\xi} \left( c \cosh \frac{\xi}{c} \right)^2 \partial_\tau + \) \(\frac{f_{(\mu)}}{c_1^2} \left[ f'_{(\nu)} \right]_{,\xi} \partial_x - f'_{(\mu)} \partial_y - \frac{f'_{(\nu)} [f_{(\mu)}]}{\sinh^2 y} \partial_z\) |
| \(B_6\) | 1 | \(\pm c_1^2 c^2 \cosh^2 \frac{\xi}{c}\) | \(\pm c^2 \sinh^2 \frac{\xi}{c}\) | 10 | \(\mathbf{X}_{2(\mu+1)+\nu} = f_{(\mu)} \left[ f'_{(\nu)} \right]_{,\xi} \left( c \sinh \frac{\xi}{c} \right)^2 \partial_\tau + \) \(\frac{f_{(\mu)}}{c_1^2} \left[ f'_{(\nu)} \right]_{,\xi} \partial_x - f'_{(\mu)} \partial_y - \frac{f'_{(\nu)} [f_{(\mu)}]}{\sinh^2 y} \partial_z\) |
| \(B_7\) | 0 | \(\pm c_1^2\) | \(\pm c_2\) | 10 | \(\mathbf{X}_{2(\mu+1)+\nu} = c_2 f_{(\mu)} \left[ f'_{(\nu)} \right]_{,\xi} \partial_\tau + \frac{c_2 f_{(\mu)} f'_{(\nu)}}{c_1} \partial_x - \frac{f_{(\mu)} f_{(\nu)}}{c_2} \partial_y - \frac{f'_{(\nu)} [f_{(\mu)}]}{y c_2} \partial_z\) |
| \(B_8\) | 0 | \(\pm c_1^2 \frac{\xi}{c}\) | \(\pm c_2\) | 10 | \(\mathbf{X}_{2(\mu+1)+\nu} = c_2 f_{(\mu)} f'_{(\nu)} \partial_\tau + \frac{c_2 f_{(\mu)} f'_{(\nu)}}{c_1} \partial_x - \frac{f_{(\mu)} f_{(\nu)}}{c_2} \partial_y - \frac{f'_{(\nu)} [f_{(\mu)}]}{y c_2} \partial_z\) |

\[ X_{9} = \cos c_1 x \partial_\xi - \frac{1}{c_1} \sin c_1 x \partial_x \]
\[ X_{10} = \sin c_1 x \partial_\xi + \frac{1}{c_1} \cos c_1 x \partial_x \]
metrics of type (1), which have been determined by Sintes [14]. These spacetimes do not admit completeness of the present study. For the same reason we give in Table 6 the self similar calculations are given in Table 5 (see also Table 3 p.3780 [11]), which we include for convenience terms of.

4.1 The homogeneous and self-similar LRS metrics (1.1)

LRS metric of the type (1). To demonstrate this and to show their usefulness we consider the following examples.

4 Examples

The results of the Tables 1,2 are general and can give the proper RCs and MCs of any given LRS metric of the type (1). To demonstrate this and to show their usefulness we consider the following examples.

4.1 The homogeneous and self-similar LRS metrics (1.1)

One application of the results of Table 1 is the determination of all LRS metrics of type (1) which are homogeneous, that is they accept additional KVs. These spacetimes have been determined previously in [11] using the reduction of a CKV to a KV by vanishing the conformal factor. However here the procedure is different, that is, one uses the component \( g_0 \) to compute \( \tilde{\tau} \) in terms of \( t \) and then replace the result in the expressions for \( K_1, K_2 \), and \( X \). The results of the calculations are given in Table 5 (see also Table 3 p.3780 [11]), which we include for convenience and completeness of the present study. For the same reason we give in Table 6 the self similar metrics of type (1), which have been determined by Sintes [14]. These spacetimes do not admit proper RCs and MCs.

| Class | \( k \) | \( f'(\mu) \) | \( f(\mu) \) |
|-------|-------|----------------|---------------|
| \( B_1 \) | 1 | \((- \tanh \frac{\mu}{c}, 0, 0)\) | \((- \cos y, \sin y \cos z, \sin y \sin z)\) |
| \( B_2 \) | -1 | \((\coth \frac{\mu}{c}, 0, 0)\) | \((\cosh y, \sinh y \cos z, \sinh y \sin z)\) |
| \( B_3 \) | -1 | \((- \cot \frac{\mu}{c}, 0, 0)\) | \((\cosh y, \sinh y \cos z, \sinh y \sin z)\) |
| \( B_4 \) | 1 | \((\tanh \frac{\mu}{c} \cos c_1 x, \tanh \frac{\mu}{c} \sin c_1 x, 0)\) | \((- \cos y, \sin y \cos z, \sin y \sin z)\) |
| \( B_5 \) | -1 | \((- \coth \frac{\mu}{c} \cos c_1 x, - \coth \frac{\mu}{c} \sin c_1 x, 0)\) | \((\cosh y, \sinh y \cos z, \sinh y \sin z)\) |
| \( B_6 \) | -1 | \((- \cot \frac{\mu}{c} \cos c_1 x, - \cot \frac{\mu}{c} \sin c_1 x, 0)\) | \((\cosh y, \sinh y \cos z, \sinh y \sin z)\) |
| \( B_7 \) | 0 | \(- (\tilde{\tau}, c_1 x, 0)\) | \((y \cos z, y \sin z, 0)\) |
| \( B_8 \) | 0 | \(- (\cos c_1 x, \sin c_1 x, 0)\) | \((y \cos z, y \sin z, 0)\) |

Table 3. Explanations for the quantities \( f(\mu), f'(\mu) \) appearing in Table 1. Note that \( \mu, \nu = 1, 2, 3 \).

| Class | \( k \) | \( f'(\mu) \) | \( f(\mu) \) |
|-------|-------|----------------|---------------|
| \( B_1 \) | 1 | \((- \tan \frac{\mu}{c}, 0, 0)\) | \((- \cos y, \sin y \cos z, \sin y \sin z)\) |
| \( B_2 \) | -1 | \((\tan \frac{\mu}{c}, 0, 0)\) | \((\cosh y, \sinh y \cos z, \sinh y \sin z)\) |
| \( B_3 \) | 1 | \((- \coth \frac{\mu}{c}, 0, 0)\) | \((\cosh y, \sinh y \cos z, \sinh y \sin z)\) |
| \( B_4 \) | 1 | \((\tan \frac{\mu}{c} \cos c_1 x, \tan \frac{\mu}{c} \sin c_1 x, 0)\) | \((- \cos y, \sin y \cos z, \sin y \sin z)\) |
| \( B_5 \) | -1 | \((- \tan \frac{\mu}{c} \cos c_1 x, - \tan \frac{\mu}{c} \sin c_1 x, 0)\) | \((\cosh y, \sinh y \cos z, \sinh y \sin z)\) |
| \( B_6 \) | 1 | \((- \coth \frac{\mu}{c} \cos c_1 x, - \coth \frac{\mu}{c} \sin c_1 x, 0)\) | \((\cosh y, \sinh y \cos z, \sinh y \sin z)\) |
| \( B_7 \) | 0 | \(- (\tilde{\tau}, c_1 x, 0)\) | \((y \cos z, y \sin z, 0)\) |
| \( B_8 \) | 0 | \(- (\cos c_1 x, \sin c_1 x, 0)\) | \((y \cos z, y \sin z, 0)\) |

Table 4. Explanations for the quantities \( f(\mu), f'(\mu) \) appeared in Table 2. Note that \( \mu, \nu = 1, 2, 3 \).
Table 5. This Table contains all homogeneous LRS spacetimes with metric (1.1). The indices α, µ = 2, 3 and the constants a, c, ε₁ satisfy the constraints ac ≠ 0 and ε₁ = ±1.

| Case | k | A(t) | B(t) | KVs | Type of the metric |
|------|---|------|------|-----|-------------------|
| A₁   | 0 | ce⁻ᵗ/c | ce⁻ᵗ/c | ξ   | LRS               |
| A₂   | ±1 | c     | c     | ξ_µ | 1+1+2             |
| A₃   | 0, ±1 | ce²ᵗ/ac | c     | ξ_µ | 2+2               |
| A₄   | 0, ±1 | c cosh t/c | c     | ξ_µ | 2+2               |
| A₅   | 0, ±1 | c sinh t/c | c     | ξ_µ | 2+2               |
| A₆   | 0, ±1 | c cos cot t/c | c     | ξ_µ | 2+2               |
| A₇   | ±1 | ct     | c     | ξ_µ | 1+1+2             |

| Case | k | A(t) | B(t) | HKVs | Conformal Factor |
|------|---|------|------|------|------------------|
| B₁   | 1 | c cot τ | c cot τ | X₃(α+1)+µ | Constant Curvature (Type a) |
| B₂   | −1 | c coth τ | c coth τ | X₃(α+1)+µ | Constant Curvature (Type a) |
| B₃   | −1 | c tanh τ | c tanh τ | X₃(α+1)+µ | Constant Curvature (Type a) |
| B₄   | 1 | c cos f/ε | c | X₆+µ | 1+3 (Type b) |
| B₅   | −1 | c cos f/ε | c | X₆+µ | 1+3 (Type b) |
| B₆   | −1 | c sinh f/ε | c | X₆+µ | 1+3 (Type b) |

Table 6. LRS spacetimes (1) with transitive homothety group H₅ and ε₁ = ±1.

| Case | k | A(t) | B(t) | KVs | Conformal Factor |
|------|---|------|------|-----|------------------|
| A₁   | 0 | t⁻¹⁺² | c₁ t⁻¹⁺² | ξ   | b                |
| A₂   | ±1 | α₁ t | α₁ t | ξ   | α₁               |
| A₃   | 0, ±1 | (α₁ t)⁺¹⁺² | α₁ t | ξ = −ε₁ aα₁ t∂_t + x∂_x | −ε₁ aα₁ |

4.2 The Datta solution

The Einstein-Maxwell spacetimes admitting a G₃I on spacelike 3D hypersurfaces have been given explicitly by Datta [15]. Physically these solutions can be used to model cosmologies with a cosmic magnetic field. Geometrically they contain - among other - plane symmetric LRS models with symmetry group G₄ ⊃ G₃I and metric:

\[ ds^2 = -\frac{dt^2}{bt^{-1} - at^{-2}} + \left(bt^{-1} - at^{-2}\right) dx^2 + t^2 \left(dy^2 + y^2 dz^2\right) \]  (13)

Using the results of Tables 1,2 we shall determine the proper RCs and the proper MCs of these LRS solutions.

We compute the Ricci tensor (which is identical for these metrics with the Einstein tensor):

\[ R_{ab} = \frac{a}{t^2 (bt - a)} dt^2 + \frac{a (a - bt)}{t^6} dx^2 + \frac{a}{t^2} \left(dy^2 + y^2 dz^2\right). \]  (14)

An examination of Tables 1,2 shows that the only possible case for the spacetime (13) to admit a proper RC/MC is case A₁ with collineation vector \( X = α₁ c∂_t + x∂_x + α₁ y∂_y \). Taking the
Lie derivative of \( R_{ab} \) (14) w.r.t. \( X \) and setting equal to zero we find \( b = 0, c = 1, \alpha_1 = 1/3 \). But for these values \( X \) becomes a HVF with homothetic factor \( \psi = 2/3 \). We conclude that the Einstein-Maxwell plane symmetric models (13) do not admit proper RCs and MCs.

### 4.3 The stiff perfect fluid LRS solution

Another example of LRS spacetime (1) is the stiff fluid model \((\gamma = 2)\) [16] with metric:

\[
ds^2 = -dt^2 + \frac{t^{2/(1+2\lambda)}}{2\lambda+1} dx^2 + \frac{t^{2\lambda/(1+2\lambda)}}{(2\lambda+1)^2} (dy^2 + y^2 dz^2).
\]

(15)

The Einstein tensor is computed to be (the Ricci tensor is degenerate!):

\[
G = \frac{\lambda(\lambda+2)}{t^2(2\lambda+1)^2} dt^2 + t^{-4\lambda/(2\lambda+1)} \frac{\lambda(\lambda+2)}{(2\lambda+1)^2} dx^2 + t^{-2(\lambda+1)/(2\lambda+1)} \frac{\lambda(\lambda+2)}{(2\lambda+1)^2} (dy^2 + y^2 dz^2).
\]

(16)

From Tables 1,2 we find that the only possible case for a proper MC to be admitted is case \( A_1 \), the collineation vector being \( X = \alpha_1 c \partial_t + x \partial_x + \alpha_1 y \partial_y \). By demanding \( L_X G_{ab} = 0 \) (and recalling that \( d\tilde{\tau} = |G_0|^{1/2} dt \)) we find \( \alpha_1 = \frac{\lambda+1}{2\lambda} \) and \( c = \frac{\sqrt{\lambda(\lambda+2)}}{\lambda} \). However for these values of the parameters it is easy to show that the collineation \( X \) reduces to a HVF with homothetic factor \( \psi = \frac{2\lambda+1}{\lambda} \). We conclude that the \( \gamma = 2 \) (plane symmetric) model (15) does not admit proper MCs.

### 4.4 RCs and MCs of static spherically symmetric spacetimes

The static spherically symmetric spacetimes are a special and interesting class of LRS spacetimes of type (1). The problem of finding all static spherically symmetric spacetimes admitting RCs has been considered many times in the literature [17, 18, 19, 20, 21]. The results of all these works follow immediately from Tables 1,2 by considering the \( k = 1, \epsilon = 1 \) cases and interchanging \( x \leftrightarrow t \). Table 7 shows the correspondence between the cases resulting from Tables 1,2 with the most recent and complete work on this topic [21].

Concerning the determination of the static spherically symmetric spacetimes which admit MCs these follow immediately from Tables 1,2 without any further calculations. These results are new. Indeed the results in the current literature [22] concern very special cases of LRS spacetimes ( (anti) de Sitter spacetime,Bertotti-Robinson spacetime, anti-Bertotti-Robinson spacetime, (anti) Einstein spacetime, Schwarzschild spacetime and Reissner-Nordstrom spacetime) in which either there no proper MCs or their Lie algebra is infinite dimensional (degenerate case).

Table 7. Comparison of the results found in the present paper concerning RCs in static spherically symmetric spacetimes with known results from the literature.

| Case of the present paper | Reference | Result |
|---------------------------|-----------|--------|
| \( A_2, A_3, A_4, A_5, A_6, A_7 \) | [21]      | Theorem 3 |
| \( B_1, B_3 \)            | [21]      | Theorem 5 |
| \( B_4, B_6 \)            | [21]      | Theorem 6 |

2The others are easily excluded. E.g. \( A_4 \) implies that \( G_2 = t^{-2(\lambda+1)/(2\lambda+1)} \frac{\lambda(\lambda+2)}{(2\lambda+1)^2} = const \), which in turn gives \( \lambda + 1 = 0 \) which is not acceptable because the solution (15) is defined for \( \lambda > 0 \) or \( \lambda < -2 \).
5 Ricci and Matter Collineations of the LRS metrics (1.2)

Concerning the metrics (2) working as previously we compute the Ricci tensor and the Einstein tensor. We find:

\[
\begin{align*}
R_{00} &= R_0 = -\frac{\ddot{A}B + 2\dot{B}A}{AB} \\
R_{11} &= R_1 = \frac{A^4}{2B^4} + \frac{2\dot{B}\dot{A}}{B} + \ddot{A}A \\
R_{13} &= R_{1A} \\
R_{22} &= R_2 = -\frac{A^2(\Lambda_y)^2}{2B^2\Sigma^2} + \frac{\dot{B}\dot{A}B}{A} + \ddot{B}B + \dot{B}^2 + k \\
R_{33} &= \left( R_1\Lambda^2 + R_2\Sigma^2 \right)
\end{align*}
\]

\[
\begin{align*}
G_{00} &= G_0 = -\frac{A^2(\Lambda_y)^2}{4B^4\Sigma^2} + \frac{2\dot{B}\dot{A}}{AB} + \frac{\ddot{B}^2}{B^2} + \frac{k}{B^2} \\
G_{11} &= G_1 = \frac{3A^4(\Lambda_y)^2}{4B^4\Sigma^2} - \frac{2\dot{B}A^2}{B} \ddot{B} - \frac{\dot{A}}{B^2} - \frac{kA^2}{B^2} \\
G_{13} &= G_{1A} \\
G_{22} &= G_2 = -\frac{A^2(\Lambda_y)^2}{4B^2\Sigma^2} - \frac{\ddot{A}}{A} - \frac{\dot{B}\dot{A}B}{A} - \ddot{B}B \\
G_{33} &= \left( G_1\Lambda^2 + G_2\Sigma^2 \right)
\end{align*}
\]

We observe that the form of \(R_{ab}, G_{ab}\) is similar to that of the metric \(g_{ab}\), therefore it is possible to consider the “generic” metric:

\[
ds^2 = \varepsilon_1d\tilde{\tau}^2 + \varepsilon_2\left[ K_1\left[ dx + \Lambda(y,k)d\tilde{z} \right]^2 + \varepsilon_3\left[ K_2\left[ dy^2 + \Sigma^2(y,k)d\tilde{z} \right]^2 \right] \right]
\]

where \(K_\alpha = \{ g_\alpha, R_\alpha, G_\alpha \}\) and \(\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1\) are the signs of \(K_0, K_1, K_2\) components respectively.

As in the case of LRS metrics (1) we have to consider two cases, i.e. the non-conformally flat and the conformally flat “generic” metrics (19).

Concerning the non-conformally flat we have the following result (see the similar result in [11]; the proof is given in Appendix 1):

The LRS spacetimes (2) admit at most one proper RC or MC given by:

\[
X = \partial_\tilde{\tau} + 2ax\partial_x + ay\partial_y
\]

in which case the components of the Ricci and the Einstein tensor satisfy the constraints:

\[
K_1 = \pm \left( c_1e^{-2a\tilde{\tau}} \right)^2 \quad K_2 = \pm \left( c_1e^{-a\tilde{\tau}} \right)^2
\]

where \(a = 0\) when \(k \neq 0\) and \(c_1, c_2\) are non-vanishing constants. Furthermore \(c_1 \neq 1\) for \(a = 0\) in order to avoid the conformally flat case.
In the conformally flat cases we show in Appendix 2 that the functions $K_1, K_2$ satisfy
$K_1 = k K_2$ ($k \neq 0$) and the “generic” metric becomes conformally related to a 1+3 decomposable
metric. The KVs are then determined easily (see for example the method developed in [23], or
[24]). The calculations are standard and there is no need to be referred explicitly. The result is
that in this case (i.e $K_1 = k K_2$ ($k \neq 0$)) there are two proper RCs and two proper MCs given
by the following vectors:

\[ X_1 = -k \frac{N'}{\Lambda} \cos x \partial_x + \sin x \partial_y - \Lambda^{-1} \cos x \partial_z \]  
\[ X_2 = k \frac{N'}{\Lambda} \sin x \partial_x + \cos x \partial_y + \Lambda^{-1} \sin x \partial_z. \]  

In case the spacetime admits extra RCs and MCs the components $K_1$ (or $K_2$) satisfy ad-
ditional restrictions. The analysis shows that in this case the component $K_1$ takes one of the
following forms:

\[ K_1 = \sinh^2 \frac{\tau}{2}, \quad K_1 = \cosh^2 \frac{\tau}{2}, \quad K_1 = \sin^2 \frac{\tau}{2} \]  

and, furthermore, that there are four extra RCs or MCs (proper or not) as follows:

\[ X_{(n)} = 2 \lambda_{(n)} \partial_\tau + 4 k \cdot (\ln K_1)_\tau \lambda_{(n)\alpha} F^{\alpha\beta}_{(n)} \partial_\beta \]  

where the quantities $\lambda_{(n)}$, $F^{\alpha\beta}$ are given by:

\[ \lambda_1 = \left[ \Lambda(y, k) + 1 \right]^{1/2} \sin \left( \frac{\tau}{2} + \frac{\tau}{2} \right) \]
\[ \lambda_2 = \left[ \Lambda(y, k) + 1 \right]^{1/2} \cos \left( \frac{\tau}{2} + \frac{\tau}{2} \right) \]
\[ \lambda_3 = \left[ 1 - \Lambda(y, k) \right]^{1/2} \sin \left( \frac{\tau}{2} - \frac{\tau}{2} \right) \]
\[ \lambda_4 = \left[ 1 - \Lambda(y, k) \right]^{1/2} \cos \left( \frac{\tau}{2} - \frac{\tau}{2} \right) \]  

\[ F^{\alpha\beta}_{(1,2)} = \text{diag} \left( \frac{1}{\Lambda(y, k) + 1} k, \frac{1}{\Lambda(y, k) + 1} \right) \]  
\[ F^{\alpha\beta}_{(3,4)} = \text{diag} \left( \frac{1}{1 - \Lambda(y, k)} k, \frac{1}{1 - \Lambda(y, k)} \right). \]

6 Ricci and Matter Collineations of the LRS metrics (1.3)

In this section we continue with the remaining LRS metrics (3). We compute the Ricci and the
Einstein tensor and we find:

\[ R_{00} = -\frac{2\dot{B}A + \dot{A}B}{AB} = R_0 \]
\[ R_{01} = 2 \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) \]  
\[ R_{11} = \frac{A}{B} \left( \dot{A}B + 2\dot{B}\dot{A} \right) - 2 = R_1 \]
\[ R_{22} = R_{33} = e^{2x} \left( \frac{\ddot{B}A\dot{B}}{A} - 2 \frac{\dot{B}^2}{A^2} + \ddot{B}B + \dot{B}^2 \right) = e^{2x} R_2 = e^{2x} R_3 \]
\[
G_{00} = \frac{\dot{B}^2A^2 + 2\dot{B}\dot{A}AB - 3B^2}{A^2B^2} = G_0
\]
\[
G_{01} = 2\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)
\]
\[
G_{11} = \frac{B^2 - A^2 (2\ddot{B}B + \dot{B}^2)}{B^2} = G_1
\]
\[
G_{22} = G_{33} = -e^{2x} \frac{B}{A^2} \left(\dot{B}A^2 + A\dot{A}B + A\dot{B}A - B\right) = e^{2x} G_2 = e^{2x} G_3
\]

We observe that the tensors $R_{ab}, G_{ab}$ are not (in general) diagonal therefore we cannot consider (in general) a generic metric as we did for the previous cases and we have to work differently.

The form of the metric indicates that we must consider two cases, that is, $R_{ab}, G_{ab}$ diagonal and not diagonal.

**Case I: $G_{ab}, R_{ab}$ diagonal**

The requirement $G_{ab}, R_{ab}$ diagonal gives the condition $A(t) = cB(t)$ from which it follows easily that the metric (3) is conformally flat. Although we cannot consider a generic metric element for all three metrics $ds^2_g, ds^2_t, ds^2_R$ it is still possible to consider one such for the two metric elements\(^3\) $ds^2_R$ and $ds^2_T$ as follows:

\[
ds^2_{R-T} = K_0 dt^2 + K_1 dx^2 + K_2 e^{2x} \left(dy^2 + dz^2\right)
\]

where $K_a = \{R_a, G_a\}$ ($R_a, T_a$) being defined in (29) and (30). We note that $ds^2_{R-T}$ need not be conformally flat. By means of the transformation:

\[
d\tilde{\tau} = |K_0|^{1/2} dt
\]

the generic metric takes the form:

\[
d\tilde{s}^2 = \varepsilon_1 (K_0) d\tilde{\tau}^2 + K_1 dx^2 + K_2 e^{2x} \left(dy^2 + dz^2\right).
\]

For the determination of the KVs of the generic metric $ds^2_{R-T}$ we have to consider, as previously, two cases, that is, the non-conformally flat and the conformally flat case.

**Case IA. $ds^2_{R-T}$ non-conformally flat**

For the Lorentzian signature $\varepsilon = -1$ it has been shown [11] that there exist two KVs\(^4\). For the remaining Euclidean signature we determine the KVs in the standard way in which there exists at most one KV for the generic metric. The results for all cases are collected in Table 8.

**Table 8. Proper RCs and Proper MCs for the case IA**

| Class | $K_1$ | $K_2$ | sign $K_0 K_1$ | $X$ |
|-------|-------|-------|---------------|-----|
| $A_1$ | $\sinh^2 \tilde{\tau}$ | $\sinh^{-2} \tilde{\tau}$ | $-1$ | $e^{-x} (\partial_{\tilde{\tau}} + \coth \tilde{\tau} \partial_x)$ |
| $A_2$ | $\cos^2 \tilde{\tau}$ | $\cos^{-2} \tilde{\tau}$ | $-1$ | $e^{-x} (\partial_{\tilde{\tau}} - \tan \tilde{\tau} \partial_x)$ |
| $A_3$ | $\cosh^2 \tilde{\tau}$ | $\cosh^{-2} \tilde{\tau}$ | $1$ | $e^{-x} (\partial_{\tilde{\tau}} + \tanh \tilde{\tau} \partial_x)$ |

\(^3\)Assuming that both $R_{ab}$ and $G_{ab}$ are non-degenerate.

\(^4\)We note that in [11] a second KV has been omitted.
Case II: Non Diagonal $G_{ab}, R_{ab}$

In this case there are no shortcuts to apply and we must solve directly the collineation equations. We introduce the functions $v(t), w(t)$ as follows:

$$B(t) = e^{v(t)}, \quad \frac{A(t)}{B(t)} = e^{w(t)}$$

where we assume that $w \cdot w, t \neq 0$ in order to avoid the case $G_{ab}, R_{ab}$ diagonal. Next we introduce the new coordinate $\tilde{\tau}$ by the relation:

$$dt = \frac{d\tilde{\tau}}{2w, t} \leftrightarrow \tilde{\tau} = w(t).$$

Using equations (38) and (39) the metric (3) becomes:

$$ds^2 = -\frac{d\tilde{\tau}^2}{(2w, t)^2} + e^{2[\tilde{\tau} + v(t)]} dx^2 + e^{2[x + v(t)]} \left(dy^2 + dz^2\right).$$

The reason for introducing the new coordinate $\tilde{\tau}$ is that in the coordinate system $\{\tilde{\tau}, x, y, z\}$ the non-diagonal component of $R_{ab}$ and $G_{ab}$ is constant. Indeed the non-vanishing components of the Ricci and Einstein tensor in these coordinates are:

\[
\begin{align*}
R_{00} &= -\left[3 \left(\dot{v} + \ddot{w} + \frac{\dot{w}}{w}\right) + 2\dot{v} + \frac{\ddot{w}}{w} - 1\right] = R_0 \\
R_{01} &= 2 \\
R_{11} &= 4e^{2[\tilde{\tau} + v(t)]} \left(\ddot{v}w^2 + 3\dot{w}^2\dot{v}^2 + \dot{v}\dot{w}\ddot{w} + 4\dot{w}^2 + \ddot{w}^2 + \ddot{w}^2\right) - 2 = R_1 \\
R_{22} &= R_{33} = 4e^{2[x + v(t)]} \left(\ddot{v}w^2 + 3\dot{w}^2\dot{v}^2 + \dot{v}\dot{w}\ddot{w} + 4\dot{w}^2\right) - 2e^{2(x - \tilde{\tau})} = e^{2x}R_2 \\
G_{00} &= -\frac{3e^{-2[\tilde{\tau} + v(t)]}}{\dot{w}^2} + 3\dot{v}^2 + 2\dot{v} = G_0 \\
G_{01} &= 2 \\
G_{11} &= -4e^{2[\tilde{\tau} + v(t)]} \left(2\ddot{v}\dot{w}^2 + 3\dot{w}^2\dot{v}^2 + 2\dot{v}\dot{w}\ddot{w}\right) + 1 = G_1 \\
G_{22} &= G_{33} = -4e^{2[x + v(t)]} \left(\ddot{v}w^2 + 3\dot{w}^2\dot{v}^2 + 2\dot{v}\dot{w}\ddot{w} + 3\dot{w}^2\dot{v} + \ddot{w}^2\right) + 1 = e^{2x}G_2
\end{align*}
\]
where a dot denotes differentiation w.r.t. the new coordinate $\tilde{\tau}$.

As in the last case the Ricci and the Einstein tensor (but not the metric $g_{ab}$!) follow as particular cases of a new “generic” metric element:

$$ds^2_{R-T} = K_0 d\tilde{\tau}^2 + 4 dx d\tilde{\tau} + K_1 dx^2 + K_2 e^{2x} \left(dy^2 + dz^2\right). \quad (43)$$

whose KV's will produce all proper RCs and MCs, if there exist. In order to solve Killing's equations for the metric $ds^2_{R-T}$ we consider again subcases according to whether the metric $ds^2_{R-T}$ is conformally flat or not.

**Case IIA:** $ds^2_{R-T}$ non-conformally flat

In this case the metric element $ds^2_{R-T}$ is written:

$$ds^2_{R-T} = K_2 e^{2x} \left[K_2^{-1} e^{-2x} \left(K_0 d\tilde{\tau}^2 + 4 dx d\tilde{\tau} + K_1 dx^2\right) + \left(dy^2 + dz^2\right)\right] \quad (44)$$

that is, it becomes conformal to a 2+2 decomposable metric. It is well known [11, 24, 25] that the KV's of a 2+2 decomposable metric are identical with the KV's of the constituent 2-metrics. Therefore the only possible KV's can come from the KV's of the 2-space $\tilde{\tau}, x$. It is also well known that if a 2-metric admits 2 KV's then it admits 3 and it is a metric of constant curvature. However if the 2-space is of constant curvature then it must be flat (because the scalar curvature contains the factor $e^{2x}$). If this is the case the 4-metric is conformally flat, which contradicts our assumption. We conclude that there exists at most one KV (in addition to the four KV's given in (4), which give rise to trivial RCs and MCs and do not interest us), which must be of the form:

$$X = X^0(\tilde{\tau}, x) \partial_{\tilde{\tau}} + X^1(\tilde{\tau}, x) \partial_x \quad (45)$$

where $X^0(\tilde{\tau}, x), X^1(\tilde{\tau}, x)$ are smooth functions of their arguments and $X^1 \neq 0$ because it leads to a degenerate $K_{ab}$.

To determine the functions $X^0, X^1$ we use Killing's equations. They read:

$$C_{00} : \quad \dot{K}_0 X^0 + 2K_0 \dot{X}^0 + 4 \dot{X}^1 = 0$$
$$C_{01} : \quad K_0 \left(X^0\right)_x + 2 \dot{X}^0 + 2 \left(X^1\right)_x + K_1 \dot{X}^1 = 0 \quad (46)$$
$$C_{11} : \quad \dot{K}_1 X^0 + 4 \left(X^0\right)_x + 2K_1 \left(X^1\right)_x = 0$$
$$C_{22} : \quad \dot{K}_2 X^0 + 2K_2 X^1 = 0$$

We consider two subcases depending on the vanishing of $(X^0)_x, (X^1)_x$.

**Subcase IIA,1:** $(X^0)_x = (X^1)_x = 0$.

Equation $C_{11}$ gives $\dot{K}_1 = 0 \Rightarrow K_1 = \text{const}$. Then from $C_{01}$ we obtain:

$$X^0 = \frac{c_1 - X^1 K_1}{2} \quad (47)$$

where $c_1$ is a constant of integration.

Replacing $X^0$ back to $C_{22}$ and demanding $\frac{\dot{K}_2}{K_2} \frac{K_1}{2} - 2 \neq 0$ (in order to avoid the the conformal flatness of $ds^2_{R-G}$) we find:

$$X^1 = \frac{\dot{K}_2 c_1}{K_2} \left(\frac{K_2 K_1}{2} - 2\right)^{-1} \quad (48)$$
Finally using $C_{00}$ and (47) we obtain $X$ as well as the constraints on the metric components for $X$ to be admitted.  

\[ X = \frac{c_2}{|K_0 - 4/K_1|^{1/2}} \partial_t + \frac{\dot{K}_2 c_1}{K_2} \left( \frac{\dot{K}_2 K_1}{K_2} - 2 \right)^{-1} \partial_x \]  

(49)

\[ K_1 = \text{const.}, \quad \frac{c_2}{|K_0 - 4/K_1|^{1/2}} = \frac{c_1}{2} - \frac{\dot{K}_2 c_1}{K_2} \left( \frac{\dot{K}_2 K_1}{K_2} - 2 \right)^{-1}. \]  

(50)

where $c_2$ is a constant of integration.

Subcase IIA.2: $(X^0)_x$, $(X^1)_x \neq 0$

In this case from $C_{22}$ we obtain $K_2 = D_2 e^{c_2 \tau}$ and $X^1 = -\frac{c_2}{2} X^0$. Then $C_{11}$ gives $K_1 = \frac{4 + e^{-c_1 c_2 \tau}}{c_2}$ (note that $K_1 \neq 4/c_2$ in order to avoid the conformally flat case) and $X^0 = e^{-c_1 x} + f(\tau)$ where $f(\tau)$ is a smooth function. From $C_{01}$ we compute the function $f(\tau) = \ln \left( \frac{D_0}{D_0 K_0 - c_2} \right)$ in terms of $K_0$. It remains equation $C_{00}$ which gives $K_0 = c_2 - \frac{1}{2} e^{c_1 c_2 \tau + D_3}$ where $D_0, D_3, c_1, c_2$ are constants of integration.

Therefore in this case we have the collineation:

\[ X = \left( e^{-c_1 x} + \ln \left( \frac{1}{D_0 |K_0(\tau) - c_2|^{1/2}} \right) \right) \left( \partial_t - \frac{c_2}{2} \partial_x \right) \]  

(51)

under the conditions:

\[ K_0 = c_2 - \frac{1}{2} e^{c_1 c_2 \tau + D_3}, \quad K_1 = \frac{4 + e^{-c_1 c_2 \tau}}{c_2}, \quad K_2 = D_2 e^{c_2 \tau}. \]  

(52)

Case IIB: $ds^2_{R-G}$ conformally flat

In order the metric $ds^2_{R-G}$ to be conformally flat the 2-dimensional metric:

\[ ds^2 = e^{-2x} K^{-1} \left[ K_0 dt^2 + 4 dtdx + K_2 dx^2 \right] \]  

(53)

where $K_0(t), K_1(t), K_2(t)$ are smooth functions of $t$ must be flat. This implies the condition:

\[ K_2 K_1 \dot{K}_0 \left( K_2 \dot{K}_1 - K_1 \dot{K}_2 + 4 K_2 \right) + 2 \ddot{K}_1 K_2^2 (4 - K_1 K_0) + K_0 K_2^2 \dot{K}_1^2 + K_2 \dot{K}_1 \left[ K_2 (K_1 K_0 - 8) + 4 K_2 K_0 \right] + 2 K_1 (K_1 K_0 - 4) \left( K_2 \dddot{K}_2 - \dot{K}_2^2 \right) = 0 \]  

(54)

Furthermore the 2-metric must admit 3 independent KVs of the form:

\[ X = X^0(t, x) \partial_t + X^1(t, x) \partial_x \]  

(55)

where $X^0(t, x), X^1(t, x)$ are at least $C^1$ functions. To determine the the functions $X^0(t, x), X^1(t, x)$ we use Killing’s equations $\mathcal{L}_X ds^2 = 0$. We compute:

---

\[ 5\text{We note that } K_0 \neq 4/K_1 \iff \det K_{ab} \neq 0. \]
\[
C_{00} : - \left( \dot{K}_0 K_2 - K_0 \dot{K}_2 \right) X^0 + 2K_2 K_0 X^0_t + 4K_2 X^1_t - 2K_2 K_0 X^1 = 0 \quad (56)
\]

\[
C_{01} : K_2 \left( [K_0 X^0_x + 2X^0_t + K_1 X^1_t] - 2X^0 = 0 \quad (57)
\]

\[
C_{11} : X^0 \left( [K_2 \dot{K}_1 - K_1 \dot{K}_2] + 4K_2 X^0_x + 4K_2 K_1 X^1_x - 2K_2 K_1 X^1 = 0 \quad (58)
\]

where a dot denotes differentiation w.r.t. \( t \). Unfortunately we have not been able to solve the system of pdf’s (54), (56)-(58).

7 Conclusions

Following the method of the generic metric, proposed in the introduction, we have been able to compute (whenever the method is possible to be applied) explicitly all LRS spacetimes which admit proper RCs and MCs as well as the collineation vector itself. The class (1) is the richer in admitting these higher collineations and for this reason we have considered various examples which show on the one hand the usefulness and the generality of the results and on the other the way one should follow in the exploitation of Tables 1, 2. Perhaps we should remark that we have obtained without any effort all homogeneous LRS spacetimes and also all static spherically symmetric spacetimes admitting RCs and MCs a subject which has been considered many times in the literature the first part of it answered completely only very recently [21].

The physical applications we have considered are the most immediate and the simplest ones and they do not really show the importance of the results or their power, which could be used in many ways and directions. This will be the subject of a future work.

Appendix 1

The KVs which span the \( G_4 \) are [11]:

\[
\begin{align*}
K_1 &= \partial_x, & K_2 &= \partial_z \\
K_3 &= f(y, k) \cos z \partial_x + \sin z \partial_y + [\ln \Sigma(y, k)]_y \cos z \partial_z \\
K_4 &= f(y, k) \sin z \partial_x - \cos z \partial_y + [\ln \Sigma(y, k)]_y \sin z \partial_z
\end{align*}
\]

where:

\[
f(y, k) = \Lambda(y, k) \left[ \ln \frac{\Lambda(y, k)}{\Sigma(y, k)} \right]_y
\]

They have the Lie brackets:

\[
\begin{align*}
[K_1, K_2] &= 0, & [K_1, K_3] &= 0, & [K_1, K_4] &= 0, & [K_2, K_3] &= -K_4 \\
[K_2, K_4] &= K_3, & [K_3, K_4] &= -kK_2 + 2(1 - k^2)K_1.
\end{align*}
\]
Let us assume that the "metrics" (19) admit exactly one more KV, the $X_I$ say, which together with the $G_4$, generates an isometry group $G_5$. Considering the commutator of $X_I$ with the KVs $K_1, K_2, K_3, K_4$ and using Jacobi identities and Killing equations we compute easily the form of $X_I$ and the "metric" (19). The cases with 6 or 7 KVs are excluded because they lead to conformally flat "metrics".

**Appendix 2**

We compute the Weyl tensor for the "generic" metric (19) and find that conformal flatness implies the condition:

$$K_1 = cK_2$$

$$c^2\varepsilon_2\Lambda^2 + \varepsilon_3\Sigma\Sigma'' = 0$$

where $c$ is a constant of integration. Hence $c^2 = 1$ and $\varepsilon_2 = k\varepsilon_3$ from which it turns out that $c = k$ i.e. the essential constant is the curvature of the 2-dimensional space.

In terms of the "generic" metric the condition (A2) means that the 3-dimensional "metric" $K_{\alpha\beta}$ has either Euclidean or Lorentzian signature and the overall signature of the "generic" metric is Euclidean, Lorentzian and (+ + --).

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