Any closed 3-manifold supports A-flows with 2-dimensional expanding attractors

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Abstract

We prove that given any closed 3-manifold \( M^3 \), there is an A-flow \( f^t \) on \( M^3 \) such that the non-wandering set \( NW(f^t) \) consists of 2-dimensional non-orientable expanding attractor and trivial basic sets.

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1 Introduction

A-flows were introduced by Smale [15] (see basic definitions below). This class of flows contains structurally stable flows including Morse-Smale flows and Anosov flows. Recall that a Morse-Smale flow has a non-wandering set consisting of finitely many hyperbolic periodic trajectories and hyperbolic singularities, while any Anosov flow has hyperbolic structure on the whole supporting manifold. A-flows have hyperbolic non-wandering sets that are the topological closure of periodic trajectories. In a sense, A-flows with nontrivial and trivial pieces (basic sets) of non-wandering sets take an intermediate place between Morse-Smale and Anosov flows. We see that A-flows form an important class containing flows with regular and chaotic dynamics.

Due to Smale’s Spectral Theorem, a non-wandering set of A-flow is a disjoint union of closed transitive invariant pieces called basic sets. A basic set is called trivial if it is either an isolated fixed point or isolated periodic trajectory.

A nontrivial basic set \( \Omega \) is called expanding if its topological dimension coincides with the dimension of unstable manifold at each point of \( \Omega \). Due to Williams [16], an expanding attractor consists of unstable manifolds of its points. Moreover, the unstable manifolds of points of expanding attractor form a lamination whose leaves are planes and cylinders. In addition, this lamination is locally homeomorphic to the product of Cantor set and

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Euclidean space of dimension at least two. Thus, the minimal topological dimension of expanding attractor equals two. Therefore, the minimal dimension of manifold supporting two-dimension expanding attractors equals three. It is natural to study the following question: what manifolds admit $A$-flows with 2-dimensional expanding attractors?

In the paper, we consider closed 3-manifolds supporting $A$-flows with 2-dimensional expanding attractors. The main result of the paper is the following statement.

**Theorem 1.** Given any closed 3-manifold $M^3$, there is an $A$-flow $f^t$ on $M^3$ such that the non-wandering set $NW(f^t)$ consists of a two-dimensional non-orientable expanding attractor and trivial basic sets.

This result contrasts with the case for 3-dimensional $A$-diffeomorphisms. To be precise, it follows from [8, 9, 12, 17] that if a closed 3-manifold $M^3$ admits an $A$-diffeomorphism with 2-dimensional expanding attractor, then $\pi_1(M^3) \neq 0$. Note that though there are Anosov flows with 2-dimensional expanding attractors [1], the $A$-flow to be constructed in Theorem 1 will not be necessarily Anosov. In fact, Margulis [11] proved that the fundamental group $\pi_1(M^3)$ of closed 3-manifold $M^3$ supporting Anosov flows has an exponential growth. In the end of the paper, we discuss the result and formulate some conjectures.

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### 2 Basic definitions

Let $f^t$ be a smooth flow on a closed $n$-manifold $M^n$, $n \geq 3$. A subset $\Lambda \subset M^n = M$ is invariant provided $\Lambda$ consists of trajectories of $f^t$. An invariant nonsingular set $\Lambda \subset M$ is called hyperbolic if the sub-bundle $T\Lambda M$ of the tangent bundle $TM$ can be represented as a $Df^t$-invariant continuous splitting $E^{ss}_\Lambda \oplus E^t_\Lambda \oplus E^{uu}_\Lambda$ such that

1. $\dim E^{ss}_\Lambda + \dim E^t_\Lambda + \dim E^{uu}_\Lambda = n$;
2. $E^t_\Lambda$ is the line bundle tangent to the trajectories of the flow $f^t$;
3. there are $C_i > 0$, $C_n > 0$, $0 < \lambda < 1$ such that

$$
\|d f^t(v)\| \leq C_i \lambda^t \|v\|, \quad v \in E^{ss}_\Lambda, \quad t > 0, \quad \|d f^{-t}(v)\| \leq C_i \lambda^{-t} \|v\|, \quad v \in E^{uu}_\Lambda, \quad t > 0.
$$

If $x \in \Lambda$ is a fixed point of hyperbolic $\Lambda$, then $x$ is an isolated hyperbolic equilibrium state. The topological structure of flow near $x$ is described by Grobman-Hartman theorem, see for example [13]. In this case $E^t_\Lambda = 0$ and $\dim E^{ss}_\Lambda + \dim E^{uu}_\Lambda = n$.

If hyperbolic $\Lambda$ does not contain fixed points, then the bundles

$$
E^{uu}_\Lambda \oplus E^t_\Lambda = E^u_\Lambda, \quad E^{ss}_\Lambda \oplus E^t_\Lambda = E^s_\Lambda, \quad E^{uu}_\Lambda, \quad E^{ss}_\Lambda
$$

are uniquely integrable [5], [15]. The corresponding leaves

$$
W^u(x), \quad W^s(x), \quad W^{uu}(x), \quad W^{ss}(x)
$$

through a point $x \in \Lambda$ are called unstable, stable, strongly unstable, and strongly stable manifolds.

Given a set $U \subset M^n$, denote by $f^0(U)$ the shift of $U$ along the trajectories of $f^t$ on the time $t_0$. Recall that a point $x$ is non-wandering if given any neighborhood $U$ of $x$ and a number $T_0$, there is $t_0 \geq T_0$ such that $U \cap f^{t_0}(U) \neq \emptyset$. The non-wandering set $NW(f^t)$ of $f^t$ is the union of all non-wandering point.

Denote by $Fix(f^t)$ the set of fixed points of flow $f^t$. Following Smale [15], we call $f^t$ an $A$-flow provided its non-wandering set $NW(f^t)$ is hyperbolic and the periodic trajectories are dense in $NW(f^t) \setminus Fix(f^t)$. It is well known [10, 15] that the non-wandering set $NW(f^t)$ of $A$-flow $f^t$ is a disjoint union of closed, and invariant, and transitive sets called basic sets. Following Williams [16], we’ll call a basic set $\Omega$ an expanding attractor.
provided $\Omega$ is an attractor and its topological dimension equals the dimension of unstable manifold $W^u(x)$ for every points $x \in \Omega$. A basic set $\Lambda$ is called `orientable' provided the fiber bundles $E_{\Lambda}^{ss}$ and $E_{\Lambda}^{uu}$ are orientable. Note that if $E_{\Lambda}^{SS}$ and $E_{\Lambda}^{uu}$ are one-dimensional, then the orientability of $\Lambda$ means that the both $E_{\Lambda}^{ss}$ and $E_{\Lambda}^{uu}$ can be embedded in vector fields on $M^n$.

Recall that an A-flow is a Morse-Smale flow provided its non-wandering set is the union of finitely many singularities and periodic trajectories. Clearly that all basic sets of Morse-Smale flow are trivial.

### 3 Proof of the main result

We begin with previous results which are interesting itself. Recall that any closed manifold admits a Morse-Smale flow with a source that is a repelling fixed point. In particular, any closed manifold admits a gradient-like Morse-Smale flow having at least one source and sink [14]. The following result says that any closed 3-manifold embedded in vector fields on $M$, every points $x \in E$.

#### Lemma 2. Given any closed 3-manifolds $M^3$, there is a Morse-Smale flow $f^t$ with repelling isolated periodic trajectory on $M^3$.

**Sketch of the proof.** Take a gradient-like Morse-Smale flow $f_0^t$ with a sink, say $\omega$, on $M^3$. Let $U(\omega)$ be a neighborhood of $\omega$. Without loss of generality, we can suppose that $U(\omega)$ is a ball. Since $\omega$ is a hyperbolic fixed point, one can assume that the boundary $\partial U(\omega)$ is a smooth sphere that is transversal to the trajectories. This means that the vector field inducing the flow $f_0^t$ is directed inside of $U(\omega)$. Let us introduce coordinates $(x, y, z)$ and the corresponding cylinder coordinates $(\rho, \phi, z)$ in $U(\omega)$ smoothly connected with the original coordinates in $U(\omega)$. Consider the system

$$\begin{align*}
\dot{\rho} &= \rho \cdot (1 - \rho), \quad \dot{\phi} = 1, \quad \dot{z} = -z.
\end{align*}$$

It is easy to check that this system has an attractive hyperbolic trajectory and the saddle fixed point at the origin. Reversing time, one gets the repelling trajectory. \(\square\)

#### Lemma 3. There is an $A$-flow on $S^2 \times S^1$ such that the spectral decomposition of $f^t$ consists of two-dimensional (non-orientable) expanding attractor $A_0$ and four isolated hyperbolic repelling trajectories. Moreover, there is a neighborhood $P$ of $A_0$ homeomorphic to the solid torus $S^1 \times D^2$ such that the boundary $\partial P = S^1 \times S^1$ is transversal to the trajectories which enter inside of $P$ as the time parameter increases.

**Proof.** Take an A-diffeomorphism $f: S^2 \to S^2$ whose the spectral decomposition consists of a Plykin attractor $A_0$ and four hyperbolic sources. Due to Plykin [12], such diffeomorphism exists. Contemporary construction of Plykin attractor can be found in [7]. Since Plykin attractor is one-dimensional, $A_0$ is an expanding attractor. Without loss of generality, one can assume that $f$ is a preserving orientation diffeomorphism (otherwise, one takes $f^2$).

Let $sus^t(f)$ be the dynamical suspension over $f$. Since $f$ is an A-diffeomorphism, $sus^t(f)$ is an A-flow. Obviously, the spectral decomposition of $f$ corresponds to the spectral decomposition of $sus^t(f)$. Because of $f$ preserves orientation, the supporting manifold for $sus^t(f)$ is homeomorphic to $S^2 \times S^1$. Since $A_0$ is a one-dimensional expanding attractor, $sus^t(f)$ has a two dimensional expanding attractor denoted by $A_0$.

Take an isolated hyperbolic repelling trajectory $\gamma$ that corresponds to some source of $f$. Since $\gamma$ is a repelling trajectory, there is a neighborhood $V(\gamma)$ homeomorphic to a solid torus such that the boundary $\partial V(\gamma)$ is transversal to the trajectories of $sus^t(f)$, so that the trajectories move outside of $V(\gamma)$ as the time parameter increases. This follows that the interior of $(S^2 \times S^1) \setminus V(\gamma)$ is the neighborhood, say $P$, of $A_0$ such that the boundary $\partial P = S^1 \times S^1$ is transversal to the trajectories which enter inside of $P$ as a time parameter increases.

Since $f$ is an orientation preserving diffeomorphism of the sphere $S^2$, $f$ is homotopic to the identity. This implies that $P$ is homeomorphic to the solid torus $S^1 \times D^2$. Hence, $sus^t(f) = f^t$ is a desired flow. \(\square\)
Proof of Theorem 1. Let $M^3$ be a closed 3-manifold. According to Lemma 2, there is a Morse-Smale flow $\phi^t$ with repelling isolated periodic trajectory $I$ on $M^3$. Hence, there is a neighborhood $U(I)$ of $I$ such that $U(I)$ is homeomorphic to the interior of the solid torus $S^1 \times D^2$, and the boundary $\partial U(I)$ is transversal to the trajectories of $\phi^t$, so that the trajectories move outside of $U(I)$ as a time parameter increases. This follows that $U(I)$ can be replaced by the solid torus $P$ satisfying Lemma 3. As a consequence, one gets the A-flow with two-dimensional (non-orientable) expanding attractor $\Lambda_a$. This completes the proof. □

4 Conclusions

It follows from the proof of the main result that the 2-dimensional expanding attractor satisfying Theorem 1 is non-orientable. We suggest the following conjecture.

Conjecture 4. Given any closed 3-manifold $M^3$, there is an A-flow $f^t$ on $M^3$ such that the non-wandering set $NW(f^t)$ contains an orientable two-dimensional expanding attractor.

Two-dimensional expanding attractors are evidence of chaotic dynamics. However, it seems that the following conjecture holds because of Plykin diffeomorphism is structurally stable.

Conjecture 5. Given any closed 3-manifold $M^3$, there is a structurally stable flow $f^t$ on $M^3$ such that the non-wandering set $NW(f^t)$ consists of a two-dimensional expanding attractor and trivial basic sets.

Note that a structurally stable flow automatically is an A-flow.

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