ON FIELDS OF DIMENSION ONE THAT ARE GALOIS EXTENSIONS OF A GLOBAL OR LOCAL FIELD

IVAN D. CHIPCHAKOV

Abstract. Let $K$ be a global or local field, $E/K$ a Galois extension, and $\text{Br}(E)$ the Brauer group of $E$. This paper shows that if $K$ is a local field, $v$ is its natural discrete valuation, $v'$ is the valuation of $E$ extending $v$, and $q$ is the characteristic of the residue field $\bar{E}$ of $(E, v')$, then $\text{Br}(E) = \{0\}$ if and only if the following conditions hold:

1. $\bar{E}$ contains as a subfield the maximal $p$-extension of $\bar{K}$, for each prime $p \neq q$;
2. $\bar{E}$ is an algebraically closed field in case the value group $\text{v'}(E)$ is $q$-indivisible.

When $K$ is a global field, it characterizes the fields $E$ with $\text{Br}(E) = \{0\}$, which lie in the class of tame abelian extensions of $K$. We also give a criterion that, in the latter case, for any integer $n \geq 2$, there exists an $n$-variate $E$-form of degree $n$, which violates the Hasse principle.

1. Introduction

Let $F$ be a field, $F_{\text{sep}}$ a separable closure of $F$, $F_{\text{ab}}$ the maximal abelian extension of $F$ in $F_{\text{sep}}$, $\mathbb{P}$ the set of prime numbers, and $F(p)$ the maximal $p$-extension of $F$ in $F_{\text{sep}}$, for each $p \in \mathbb{P}$. We say that $F$ is a field of dimension $\leq 1$ if the Brauer groups $\text{Br}(F')$ are trivial, for all algebraic field extensions $F'/F$. It is known (cf. [19], Ch. II, 3.1) that $\dim(F) \leq 1$ if and only if $\text{Br}(F') = \{0\}$ when $F'$ runs across the set $F_{\text{F}}(F)$ of finite extensions of $F$ in $F_{\text{sep}}$. Note also that if $\dim(F) \leq 1$, then the absolute Galois group $\mathcal{G}_F := \mathcal{G}(F_{\text{sep}}/F)$ has cohomological dimension $\text{cd}(\mathcal{G}_F) \leq 1$ as a profinite group; the converse holds in case $F$ is a perfect field.

It is well-known (cf. [19], Ch. II, 3.1 and 3.2) that a field $F$ satisfies $\dim(F) \leq 1$ whenever it is of type $C_1$ (or a $C_1$-field), i.e. every $F$-form (a homogeneous nonzero polynomial with coefficients in $F$) $f$ of degree $\deg(f)$ in more than $\deg(f)$ variables has a nontrivial zero over $F$. The class of $C_1$-fields contains finite fields (by Chevalley-Warning’s theorem, see, e.g., [11], Theorem 6.2.6) as well as the extensions of transcendency degree 1 over any algebraically closed field, by Tsen’s theorem, and it is closed under taking algebraic extensions (cf. [14]). Note also that if $F$ is a $C_1$-field, then it is almost perfect, i.e. the following two equivalent conditions hold: (i) every finite extension of $F$ possesses a primitive element; (ii) $\text{char}(F) = q \geq 0$, and in case $q > 0$, the degree $[F: F^q]$ of $F$ as an extension of its subfield $F^q = \{\beta^q : \beta \in F\}$ is equal to 1 or $q$. At the same time, almost perfect fields of dimension $\leq 1$ need not be of type $C_1$. Indeed, the class $C$ contains a field $E$ with $\dim(E) \leq 1$ that is not of type $C_1$ in the following two cases: (i) $C$ consists of quasifinite fields of fixed characteristic $q$, where $q \in \mathbb{P}$ or $q = 0$ (see

Key words and phrases. Field of dimension $\leq 1$, field of type $C_1$, form
2020 MSC Classification: 11E76, 11R34, 12J10 (primary), 11D72, 11S15 (secondary).
Statement of the main results

For any field $K$ with a (nontrivial) Krull valuation $v$, $O_v(K) = \{ a \in K : v(a) \geq 0 \}$ denotes the valuation ring of $(K, v)$, $M_v(K) = \{ \mu \in K : v(\mu) > 0 \}$ the maximal ideal of $O_v(K)$, $O_v(K)^+ = \{ u \in K : v(u) = 0 \}$ the multiplicative group of $O_v(K)$, $v(K)$ the value group and $\hat{K}_v = O_v(K)/M_v(K)$ the residue field of $(K, v)$, respectively; $v(K)$ is a divisible hull of $v(K)$. We write for brevity $\hat{K}$ instead of $\hat{K}_v$ when there is no danger of ambiguity. As usual, $v$ is said to be discrete, if $v(K)$ is an infinite cyclic group. The valuation $v$ is called Henselian, or else, we say that $(K, v)$ is a Henselian field, if $v$ extends uniquely, up-to equivalence, to a valuation $v_L$ on each algebraic extension $L$ of $K$. The class of Henselian fields contains every complete nontrivially real-valued field (see page 5). When $v$ is Henselian, so is $v_L$, for any algebraic field extension $L/K$. In this case, we put $O_v(L) = O_{v_L}(L)$, $M_v(L) = M_{v_L}(L)$, $v(L) = v_L(L)$, and denote by $\hat{L}$ the residue field of $(L, v_L)$; also, we write $v$ instead of $v_L$ when the abbreviation is clear from the context. As shown in [4], if $(K, v)$ is a Henselian discrete valued field (abbr., an HDV-field) with $\hat{K}$ quasifinite, then algebraic extensions of $K$ that are fields of dimension $\leq 1$ are characterized as follows:

(2.1) In order that $\dim(L) \leq 1$ it is necessary and sufficient that the intersection $S(L) \cap \Sigma(L)$ be empty, where $S(L) = \{ p \in \mathbb{P} : \hat{L}(p) \neq \hat{L} \}$ and $\Sigma(L)$ consists of those $p \in \mathbb{P}$, for which $v(L)$ is a $p$-indivisible group.

The applicability of (2.1) is ensured by the former part of the following result (cf. [4], Lemma 4.1 and Corollary 4.3), which indicates that the existence of fields of dimension $\leq 1$ which are not of type $C_1$ is not uncommon among algebraic extensions of HDV-fields with finite residue fields:

(2.2) (a) For any HDV-field $(K, v)$ with $\hat{K}$ quasifinite, and any pair $S, \Sigma$ of subsets of $\mathbb{P}$, there exists an algebraic extension $E/K$ satisfying $S(E) = S$ and $\Sigma(E) = \Sigma$. The field $E$ has dimension $\leq 1$ but is not of type $C_1$, provided that $S \cap \Sigma = \emptyset$ and there is a pair of integers $m_j, j = 1, 2$, such
that $2 \leq m_1 \leq m_2$, $S$ contains all prime divisors of $m_1m_2$, and $\Sigma$ contains all prime divisors of $m_1 + m_2$ (this requires that $\gcd(m_1, m_2) = 1$). When $\dim(E) \leq 1$ and $E$ is not a $C_1$-field, $S$ contains at least 2 elements.

(b) For each pair of integers $k_1, k_2$ with $2 \leq k_1 < k_2$ and $\gcd(k_1, k_2) = 1$, there exist subsets $S_1$ and $\Sigma_1$ of $\mathbb{P}$, such that $S_1 \cap \Sigma_1 = \emptyset$, $S_1$ contains all prime divisors of $k_1k_2$, and $\Sigma_1$ contains all prime divisors of $k_1 + k_2$.

The main results of this paper are presented as two theorems. The former one concerns the special case where $(K, v)$ is an HDV-field with $\widehat{K}$ finite, and can be stated as follows:

**Theorem 2.1.** Let $(K, v)$ be an HDV-field with $\widehat{K}$ finite and $\text{char}(\widehat{K}) = q$, and suppose that $E$ is a Galois extension of $K$, such that $\dim(E) \leq 1$. Then $E(p) = \widehat{E}$, for every $p \in \mathbb{P}$, $p \neq q$. Moreover, if $v(E)$ is a $q$-indivisible group, then $E(q) = \widehat{E}$; in particular, $\widehat{E}$ is algebraically closed.

Theorem 2.1 shows that, for every Galois extension $E/K$ satisfying $\dim(E) \leq 1$, the set $S(E)$ is either empty or equal to $\{q\}$. Note that, in the former case, it can be deduced from Lang’s theorem [13], Theorem 10 (see also Corollary 1.1 below) that $E$ is of type $C_1$. On the other hand, it seems that the sets $S(E')$ contain at least 2 elements, for all presently known algebraic extensions $E'$ of $K$ with $\dim(E') \leq 1$, which are not $C_1$-fields. Therefore, any approach to Question 1 should rely on ideas and methods different from those used in [1]. It should also be based on a satisfactory information on basic algebraic and Diophantine properties of fields of dimension $\leq 1$ that are Galois extensions of $K$ belonging to suitably chosen special classes.

As a step in this direction, we consider fields of dimension $\leq 1$ that are tame Galois extensions of a global field $K$ with abelian Galois group. Frequently, we restrict to the special case where $\text{char}(K) = 0$, i.e. $K$ is a number field. Our second main result is contained in the following theorem.

**Theorem 2.2.** Let $K$ be a global field, $\Theta$ the compositum of tame finite extensions of $K$ in $K_{ab}$, and $E$ an intermediate field of $\Theta/K$. Then:

(a) $\dim(E) \leq 1$ if and only if $E$ is a nonreal field and the residue fields of its discrete valuations are algebraically closed; we have $\dim(\Theta) \leq 1$;

(b) If $K$ is a finite extension of the field $\mathbb{Q}$ of rational numbers, $\dim(E) \leq 1$, $E$ contains as subfields $\Theta \cap K(2)$ and $\Theta \cap K(3)$, and $n \geq 2$ is an integer, then there exist $n$-variate $E$-forms $N_m$, $m \in \mathbb{N}$, of degree $n$, which are pairwise nonequivalent and violate the Hasse principle.

The proof of Theorem 2.2 (b) relies on the following two facts concerning tame abelian extensions $E$ of an arbitrary global field $K$ (see Corollary 5.2 and Proposition 5.3 (b)):

(2.3) (a) Nontrivial Krull valuations of $E$ are discrete, and in case $\dim(E) \leq 1$, their residue fields are algebraically closed;

(b) If $\dim(E) \leq 1$, then for each $p \in \mathbb{P}$, there exist infinitely many degree $p$ extensions $\Theta_{p,\nu}$, $\nu \in \mathbb{N}$, of $K$ in $Y$.

Under the hypothesis of (2.3) (a), an algebraic extension $E'/E$ is said to be unramified, if $v'(E') = v(E)$ whenever $v$ is a nontrivial valuation of $E$.
and \(v'\) is a valuation of \(E'\) extending \(v\). When \(\dim(E) \leq 1\), \(E'/E\) is an unramified finite extension of degree \(n\), \(X_1, \ldots, X_n\) is a set of algebraically independent variables over \(E\), and \(B\) is a basis of \(E'\) as a vector space over \(E\), (2.3) implies the following:

(2.4) The norm form \(N = N(X_1, \ldots, X_n)\) of \(E'/E\) associated with \(B\) is an \(n\)-variate \(E\)-form of degree \(n\), which has a nontrivial zero over \(E_v\), for each nontrivial valuation \(v\) of \(E\), but does not possess a nontrivial \(E\)-zero. In particular, \(N\) violates the local-to-global principle over \(E\).

Assuming that \(\dim(E) \leq 1\) and \(E\) is a tame extension of \(K\) in \(K_{ab}\), where \(K/Q\) is a finite Galois extension, one formulates the main step towards the proof of Theorem 2.2 (b) as follows:

(2.5) For any integer \(n \geq 2\), there exist unramified extensions \(E_{n,\nu}, \nu \in \mathbb{N}\), of \(E\) in \(E_{ab}\), such that \([E_{n,\nu}: E] = n\) and \(E_{n,\nu} \cap E_{n',\nu} = E\), for each index \(\nu\), where \(E'_{n,\nu}\) is the compositum of the fields \(E_{n,\nu}, \nu' \neq \nu\); hence, by (2.3), every \(E_{n,\nu}\) is embeddable as an \(E\)-subalgebra in \(E_w\), for any nontrivial valuation \(v\) of \(E\). Moreover, if \(N_{n,\nu}\) is a norm form of \(E_{n,\nu}/E\), for each \(\nu \in \mathbb{N}\), then \(N_{n,\nu}, \nu \in \mathbb{N}\), are \(n\)-variate and pairwise nonequivalent \(E\)-forms of degree \(n\), which violate the Hasse principle.

The former part of (2.5) indicates that, for each \(\nu \in \mathbb{N}\), \(N_{n,\nu}\) is a norm form of the field extension \(E_{n,\nu}E_{n',\nu}/E_{n,\nu}\), whereas \(N_{n,\nu'}, \nu' \neq \nu\), decompose over \(E_{n,\nu}\) into products of \(n\) linear forms in \(n\) variables; in particular, this implies the latter part of (2.5).

The basic notation, terminology and conventions kept in this paper are standard and virtually the same as in [20], [16] and [2]. Throughout, Brauer and value groups are written additively, Galois groups are viewed as profinite with respect to the Krull topology, and by a profinite group homomorphism, we mean a continuous one. An additively written abelian group \(A\) is called reduced, if it does not possess a nonzero divisible subgroup. When \(A\) is also a torsion group, \(A_p\) denotes its \(p\)-component, for every \(p \in \mathbb{P}\). For any field extension \(E'/E\), we write \(I(E'/E)\) for the set of intermediate fields of \(E'/E\), \(\pi_{E/E'}\) for the scalar extension map \(\text{Br}(E) \to \text{Br}(E')\), and \(\text{Br}(E'/E)\) for the relative Brauer group of \(E'/E\) (the kernel of \(\pi_{E/E'}\)). When \(E'/E\) is a Galois extension, \(G(E'/E)\) denotes its Galois group and \(X(E'/E)\) is the continuous character group of \(G(E'/E)\). As usual, \(\mathbb{Z}(p^\infty)\) stands for a quasi-cyclic \(p\)-group, \(\mathbb{Z}_p\) is the ring of \(p\)-adic integers, and a \(\mathbb{Z}_p\)-extension means a Galois extension \(\Psi'/\Psi\) with \(G(\Psi'/\Psi)\) isomorphic to the additive group of \(\mathbb{Z}_p\). For any field \(E\), \(E^*\) is its multiplicative group, and \(E^{*n} = \{a^n : a \in E^*\}\), for each \(n \in \mathbb{N}\). The field \(E\) is said to be formally real, if \(-1\) is not presentable as a finite sum of elements of \(E^{*2}\); \(E\) is called a nonreal field, otherwise. The value group of any discrete valued field is assumed to be an ordered subgroup of the additive group of the field \(\mathbb{Q}\); this is done without loss of generality, in view of [7], Theorem 15.3.5, and the fact that \(\mathbb{Q}\) is a divisible hull of its infinite subgroups (see page 3).

Here is an overview of this paper: Section 3 includes valuation-theoretic preliminaries used in the sequel. Theorem 2.1 is proved in Section 4. It is also shown there that if \(E\) is a field with \(\dim(E) \leq 1\), which is an abelian
extension of a local field $K$, then the valuation of $E$ extending the natural valuation of $K$ is discrete, and its residue field $\hat{E}$ is algebraically closed.

Theorem 2.2 (a) and statements (2.3) are proved in Section 5, where we consider fields $\Delta$ of dimension $\leq 1$ that are abelian tame extensions of a global field $K$. We prove (2.3) (b) by showing that, in characteristic zero, the groups $X(\Delta/K)$ are reduced with finitely many elements of infinite height, and with infinitely many elements of order $p$, for each $p \in \mathbb{P}$. The latter property of $X(\Delta/K)$ is used in Section 6 for proving Theorem 2.2 (b).

In Section 5, we also characterize the maximal abelian tame extension of $\mathbb{Q}$ in $\mathbb{Q}_{\text{sep}}$ as the extension of $\mathbb{Q}$ obtained by adjunction of the primitive $p$-th root of unity $\epsilon_p \in \mathbb{Q}_{\text{sep}}$, for all $p \in \mathbb{P}$: its analog in characteristic $q > 0$ is the rational function field $\mathbb{F}_{q,\text{sep}}(X)$.

3. Preliminaries and characterizations of algebraic extensions $E$ of local or global fields with $\text{Br}(E)_p \neq \{0\}$, for a given prime $p$

Let $K$ be a field with a (nontrivial) Krull valuation $v$. It is known (cf. [10], Ch. XII) that $v$ is Henselian in case $v(K)$ embeds as an ordered subgroup in the additive group $\mathbb{R}$ of real numbers and $K$ has no proper separable extensions in its completion $K_v$ (with respect to the topology of $K$ induced by $v$). For an arbitrary $v$, the Henselian condition has the following two equivalent forms (cf. [7], Sect. 18.1, and [10], Ch. XII, Sect. 4):

(3.1) (a) Given a polynomial $f(X) \in O_v(K)[X]$ and an element $a \in O_v(K)$, such that $2v(f'(a)) < v(f(a))$, where $f'$ is the formal derivative of $f$, there is a zero $c \in O_v(K)$ of $f$ satisfying the equality $v(c - a) = v(f(a)/f'(a))$;

(b) For each normal extension $\Omega/K$, $v'(\tau(\mu)) = v'(\mu)$ whenever $\mu \in \Omega$, $v'$ is a valuation of $\Omega$ extending $v$, and $\tau$ is a $K$-automorphism of $\Omega$.

When $v$ is Henselian, so is $v_L$, for any algebraic field extension $L/K$. In this case, we put $O_v(L) = O_{v_L}(L)$, $M_v(L) = M_{v_L}(L)$, $v(L) = v_{L}(L)$, and denote by $\hat{L}$ the residue field of $(L, v_L)$; also, we write $v$ instead of $v_L$ when there is no danger of ambiguity. Clearly, $\hat{L}/\hat{K}$ is an algebraic extension and $v(K)$ is an ordered subgroup of $v(L)$, such that $v(L)/v(K)$ is a torsion group; hence, one may assume without loss of generality that $v(L)$ is an ordered subgroup of $\overline{v(K)}$. By Ostrowski’s theorem (cf. [7], Theorem 17.2.1), if $[L: K]$ is finite, then it is divisible by $[\hat{L}: \hat{K}]\epsilon(L/K)$ and $[L: K][\hat{L}: \hat{K}]^{-1}e(L/K)^{-1}$ has no divisor $p \in \mathbb{P}$, $p \neq \text{char}(\hat{K})$; here $e(L/K)$ is the index of $v(K)$ in $v(L)$. Ostrowski’s theorem implies the following:

(3.2) The quotient groups $v(K)/pv(K)$ and $v(L)/pv(L)$ are isomorphic, if $p \in \mathbb{P}$ and $[L: K] < \infty$. When $\text{char}(\hat{K}) \nmid [L: K]$, the natural embedding of $K$ into $L$ induces canonically an isomorphism $v(K)/pv(K) \cong v(L)/pv(L)$.

The extension $L/K$ is defectless, i.e. $[L: K] = [\hat{L}: \hat{K}]\epsilon(L/K)$, in the following three cases:

(3.3) (a) If $\text{char}(\hat{K}) \nmid [L: K]$ (apply Ostrowski’s theorem).

(b) If $(K, v)$ is HDV and $L/K$ is separable (see [7], Sect. 17.4).
(c) If \((K, v)\) is HDV and \(K\) is almost perfect. In particular, this holds in the following two cases: when \((K, v)\) is a complete discrete valued field with \(\hat{K}\) perfect; if \((K, v)\) is HDV and \(K\) is an algebraic extension of a global field.

Assume that \((K, v)\) is a nontrivially valued field and \(R\) is a finite extension of \(K\), which has a unique, up-to equivalence, valuation \(v_R\) extending \(v\). The extension \(R/K\) is called inertial relative to \(v\), if the residue field \(\bar{R}\) of \((R, v_R)\) is separable over \(\bar{K}\), and \([R: K] = [R: \bar{K}]\); we say that \(R/K\) is totally ramified with respect to \(v\), if the index of \(v(K)\) in \(v_R(R)\) equals \([R: K]\). When \(v\) is Henselian, \(R/K\) is totally ramified, if \(e(R/K) = [R: K]\).

Under the same condition, inertial extensions of \(K\) (with respect to \(v\)) have the following useful properties (see \cite{20}, Theorem A.23):

(3.4) (a) An inertial extension \(R'/K\) is Galois if and only if so is \(\bar{R'}/\bar{K}\).

When this holds, \(\mathcal{G}(R'/K)\) and \(\mathcal{G}(\bar{R'}/\bar{K})\) are canonically isomorphic.

(b) The compositum \(K_{ur}\) of inertial extensions of \(K\) in \(K_{sep}\) is a Galois extension of \(K\) with \(v(K_{ur}) = v(K)\) and \(\mathcal{G}(K_{ur}/K) \cong \mathcal{G}_{\bar{R}}\).

(c) Finite extensions of \(K\) in \(K_{ur}\) are inertial, and the natural mapping of \(I(K_{ur}/K)\) into \(I(\bar{K}_{sep}/\bar{K})\), by the rule \(L \to \bar{L}\), is bijective.

Returning to the case of \((K, v)\) Henselian, we denote by \(K_{tr}\) the compositum of those \(\Lambda \in \text{Fe}(K)\), which are tamely ramified extensions of \(K\), i.e. \(e(\Lambda/K)\) is not divisible by \(\text{char}(\hat{K})\). It is known that \(K_{tr}/K\) is a Galois extension with the following properties (see \cite{20}, Appendix A2):

(3.5) (a) \(K_{ur} \cap I(K_{tr}/K)\) and \(K_{tr}/K_{ur}\) is Galois with \(\mathcal{G}(K_{tr}/K_{ur})\) isomorphic to the topological group product \(\prod_{p \in \mathbb{P}} \mathbb{Z}^T_p\), where \(\mathbb{P}\) is the set of primes \(p\) such that \(v(K) \neq pv(K)\) and for each \(p \in \mathbb{P}\), \(\mathbb{Z}_{T_p}^T\) is the topological product of \(\mathbb{Z}_p\) valued copies of \(\mathbb{Z}_p\), indexed by a set of cardinality equal to the dimension \(T_p\) of the group \(\mathcal{G}v(K)/pv(K)\), viewed as a vector space over \(\mathbb{F}_p\) (with respect to the operations naturally induced by the addition in \(v(K)\));

(b) If \(K_{sep} \neq K_{tr}\), then \(\text{char}(\hat{K}) = q > 0\) and \(K_{sep}/K_{tr}\) is a \(q\)-extension.

Lemma 3.1. Let \((K, v)\) be a real-valued field, \((K_v, \bar{v})\) its completion, and \((K', v')\) an intermediate valued field of \((K_v, \bar{v})/(K, v)\). Assume that \((K', v')\) is Henselian and identify \(K'_{tr}\) with its \(K'\)-isomorphic copy in \(K'_{v, sep}\). Then:

(a) \(K'_{sep} \cap K_v = K'\), and each \(\Lambda \in \text{Fe}(K_v)\) contains a primitive element \(\lambda \in K'_{sep}\) over \(K_v\), such that \([K_v(\lambda) : K_v] = [K'(\lambda) : K']\);

(b) \(K'_{sep}K_v = K_{v, sep}\) and \(\mathcal{G}_{K_v} \cong \mathcal{G}_{K'_{v, sep}}\);

(c) The mapping, say \(f\), of \(\text{Fe}(K')\) into \(\text{Fe}(K_v)\), by the rule \(\Lambda' \to \Lambda'_{K_v}\), is bijective and degree-preserving. Moreover, \(f\) and the inverse mapping \(f^{-1} : \text{Fe}(K_v) \to \text{Fe}(K')\), preserve the Galois property and the isomorphism class of the corresponding Galois groups;

(d) For each \(\nu \in \mathbb{N}\) not divisible by \(\text{char}(K)\), \(K_v^\nu = K'_{v, sep}K'_{v, sep}^\nu\).

Lemma 3.2. In the setting of Lemma 3.1, suppose that \((K, v)\) is Henselian, identify \(K_{sep}\) with its \(K\)-isomorphic copy in \(K_{v, sep}\), fix an extension \(R\) of \(K\) in \(K_{sep}\), and put \(R' = K'\bar{R}\). Then \((R', v'_R)\) is an intermediate valued field of \((R_v, \bar{v}_R)/(R, v_R)\).
Our next lemma characterizes those fields of dimension $\leq 1$, which lie in the class of algebraic extensions of any HDV-field $(K, v)$ with $\hat{K}$ quasifinite. The lemma has been proved in [1] (see [19], Ch. II, for a proof in case $(K, v)$ is a local field with $\text{char}(K) = 0$). In particular, it shows that if $(K, v)$ is HDV, $\hat{K}$ is quasifinite, and $E/K$ is an algebraic extension, then $\dim(E) \leq 1$ if and only if the intersection $S(E) \cap \Sigma(E)$ is empty, where $S(E) = \{ p \in \mathbb{P}: G(\hat{E}(p)/\hat{E}) \cong \mathbb{Z}_p \}$ and $\Sigma(E) = \{ p \in \mathbb{P}: v(E) \neq pv(E) \}$.

**Lemma 3.3.** Assume that $(K, v)$ is an HDV-field with $\hat{K}$ quasifinite, fix some $p \in \mathbb{P}$, and take an algebraic field extension $R/K$. Then $\text{Br}(R)_p = \{0\}$ if and only if each of the following three equivalent conditions is fulfilled:

- (a) $\text{Br}(R')_p = \{0\}$, for every algebraic extension $R'/R$; this holds if and only if $\text{Br}(R')_p = \{0\}$, when $R'$ runs across the set $\text{Fe}(R)$;
- (b) $p$ does not divide the period of the quotient group $R^*/N(R'_1/R')$, for any $R'_1 \in \text{Fe}(R)$ and every $R' \in I(R'_1/R)$;
- (c) $v(R) = pv(R)$ or $\hat{R}(p) = \hat{R}$.

In addition, if $R/K$ is separable or $K$ is almost perfect, or $p \neq \text{char}(K)$, then $\text{Br}(R)_p = \{0\}$ if and only if there are finite extensions $R_n$, $n \in \mathbb{N}$, of $K$ in $R$, such that $p^n$ divides $[R_n: K]$, for each index $n$.

The following lemma characterizes fields of dimension $\leq 1$ that are algebraic extensions of a global field as follows:

**Lemma 3.4.** Let $K$ be a global field, $E/K$ an algebraic extension, $\mathcal{M}_E$ the set of equivalence classes of nontrivial Krull valuations of $E$, and $\mathcal{R}_E$ a system of representatives of $\mathcal{M}_E$ consisting of $\mathbb{Q}$-valued valuations. For each $v' \in \mathcal{R}_E$, fix a completion $E_{v'}$ of $E$ with respect to the topology of $v'$, denote by $v$ the valuation of $K$ induced by $v'$, and identifying $K_v$ with the closure of $K$ in $E_{v'}$, put $E(v') = E_1 K_v$. Then $\text{Br}(E')_p = \{0\}$, for some $p \in \mathbb{P}$, if and only if $\text{Br}(E(v'))_p = \{0\}$, $v' \in \mathcal{R}_E$, and in case $p = 2$, $E$ is a nonreal field.

Lemma 3.4 is implied by Lemma 3.3 and [19], Ch. II, Proposition 9.

**Lemma 3.5.** Assume that $K$ is a global field, $\mathcal{M}_K$ is the set of equivalence classes of discrete valuations of $K$, and $\mathcal{R}_K$ is a system of representatives of $\mathcal{M}_K$. Let $E$ be a Galois extension of $K$, and for each $v \in R_K$, let $E(v)$ be a $K$-isomorphic copy of $E$ in $K_v$.sep. Then $\text{Br}(E)_p = \{0\}$, for some $p \in \mathbb{P}$, if and only if $E$ is a nonreal field with $\text{Br}(E(v)K_v)_p = \{0\}$, for every $v \in R_K$.

For any field $R$, it is known that conditions (a) and (b) of Lemma 3.3 are equivalent, and if they hold, then $\mathcal{G}_R$ is a profinite group of cohomological $p$-dimension $\text{cd}_p(\mathcal{G}_R) \leq 1$; this implication is an equivalence in case $R$ is perfect or $p \neq \text{char}(\hat{R})$ (cf. [19], Ch. II, 3.1, and [11], Theorem 6.1.8). Generally, the condition that $\text{Br}(R)_p = \{0\}$ is weaker than conditions (a) and (b) of Lemma 3.3. It is known, however, that if $E \in I(\mathbb{Q}_{\text{sep}}/\mathbb{Q})$ and $\text{Br}(E)_p = \{0\}$, for some $p \in \mathbb{P}$, then $\text{Br}(E')_p = \{0\}$ whenever $E' \in I(\mathbb{Q}_{\text{sep}}/E)$ [8], Theorem 4; this also follows from Lemma 3.3 and [19], Ch. II, Proposition 9.)
4. Proof of Theorem \([2.1]\) and applications to fields of dimension \(\leq 1\) that are tamely ramified abelian extensions of a local field

Our first objective in this Section is to prove Theorem \([2.1]\). One may assume without loss of generality that \(E \in I(K_{\text{sep}}/K)\). As in Section 4, we put \(S(L) = \{ p \in \mathfrak{P}; \, \hat{L}(p) \neq \hat{L} \} \) and \(\Sigma(L) = \{ p \in \mathfrak{P}; \, v(L) \neq pv(L) \}\), for every \(L \in I(K_{\text{sep}}/K)\). Since \(\dim(E) \leq 1\), we have \(S(E) \cap \Sigma(E) = \emptyset\), so it suffices to prove that \(\hat{E}(p) = \hat{E}\), for an arbitrary fixed \(p \in \Sigma(E), \quad p \neq \text{char}(\hat{K})\). Fix a primitive \(p\)-th root of unity \(\varepsilon \in K_{\text{sep}}\). It is well-known that \(K(\varepsilon)/K\) is a Galois extension, \(\mathcal{G}(K(\varepsilon)/K)\) is cyclic and \([K(\varepsilon): K] = p - 1\) (cf. [15], Ch. VI, Sect. 3). This implies \(E(\varepsilon)/K\) is Galois and \([E(\varepsilon): E] = [K(\varepsilon): K]\), so it follows from (3.2) that \(\Sigma(E(\varepsilon)) = \Sigma(E)\). Observing also that \(L_{\text{ur}} = K_{\text{ur}}L\) and \(\mathcal{G}_K \cong \prod_{\ell \in \mathfrak{P}} \mathbb{Z}/\ell\), for each \(L \in I(K_{\text{sep}}/K)\), one obtains from (3.4) (b) and Galois theory that \(\hat{L}(\ell)/\hat{L}\) is a \(\mathbb{Z}/\ell\)-extension, for every \(\ell \in S(L)\). It is therefore clear that \(\Sigma(E(\varepsilon)) = S(E)\) and it suffices to prove that \(p \notin S(E)\) in the special case where \(\varepsilon \in E\). Consider now the fields \(E_0 = E \cap K_{\text{ur}}\) and \(E_0' = E \cap K_{\text{tr}}\), and fix a system \(\varepsilon_n, \, n \in \mathbb{N}\), of elements of \(K_{\text{sep}}\), chosen so that \(\varepsilon_0 = \varepsilon\) and \(\varepsilon_n = \varepsilon_{n-1}\), for every index \(n \geq 2\). It is easily verified that \(\varepsilon_n \notin K\) whenever \(n\) is sufficiently large, which ensures that the extension \(K'\) of \(K\) generated by the set \(\{ \varepsilon_n; \, n \in \mathbb{N} \}\) is a \(\mathbb{Z}/p^n\)-extension of \(K\) and equals the field \(K(p) \cap K_{\text{ur}}\). Thus it turns out that the equality \(\hat{E}(p) = \hat{E}\) will follow, if we show that \(K'\) is included in \(E\), i.e. \(\varepsilon_n \in E\), for all \(n \in \mathbb{N}\). Note also that \(p \in s(E)\) if and only if \(p \in S(E_0')\). This can be deduced from (3.2) and the well-known fact that \(\mathcal{G}(E/E_0)\) is a pro-\(q\)-subgroup of \(\mathcal{G}(E/K)\), where \(q = \text{char}(\hat{K})\). In other words, one may assume for the rest of our proof that \(E = E_0\), i.e. \(E \in I(K_{\text{tr}}/K)\).

Our argument also relies on the following two facts: (i) finite extensions of \(E_0\) in \(E\) are tamely and totally ramified; (ii) \(v(E_0) = v(K)\), by (3.4) (b), i.e. \((E_0, v)\) is an HDV-field. Note further that \(\mathcal{G}(E/E_0)\) is an abelian group. Indeed, it follows from Galois theory and the equality \(E_0 = E \cap K_{\text{ur}}\) that \(E.K_{\text{ur}}/K_{\text{ur}}\) is a Galois extension with \(\mathcal{G}(E.K_{\text{ur}}/K_{\text{ur}}) \cong \mathcal{G}(E/E_0)\). Since \(E.K_{\text{ur}} \in I(K_{\text{tr}}/K_{\text{ur}})\), one also sees that \(\mathcal{G}(E/E_0)\) is a homomorphic image of \(\mathcal{G}(K_{\text{tr}}/K_{\text{ur}})\), so it follows from (3.5) (a) that \(\mathcal{G}(E/E_0)\) is abelian. Furthermore, (3.5) (a) and the cyclicity of the group \(v(E_0) = v(K)\) imply \(\mathcal{G}(E/E_0)\) is a procyclic group. Hence, by (3.2) and the assumption that \(v(E) = pv(E)\), \(E\) contains as a subfield a \(\mathbb{Z}/p^n\)-extension \(\Lambda_\infty\) of \(E_0\). This means that \(E_0\) has (Galois) extensions \(\Lambda_n, \, n \in \mathbb{N}\), such that \(\Lambda_\infty = \cup_{n=0}^{\infty} \Lambda_n\), and for each index \(n\), \(\Lambda_n\) is a subfield of \(\Lambda_{n+1}\), and \(\mathcal{G}(\Lambda_n/E_0)\) is cyclic of order \(p^n\). Using the fact that \(\Lambda_n\) is tamely and totally ramified over \(E_0\), one obtains from Lemma 3.1 and [15], Ch. II, Proposition 12, that \(\Lambda_n/E_0\) possesses a primitive element which is a \(p^n\)-th root of some element \(\pi_n \in E_0^{p^n}\), such that \(v(\pi_n) > 0\) and \(v(\pi_n)\) generates the cyclic group \(v(E_0)\).

In order to complete the proof of Theorem \([2.1]\) we use the fact that \(\Lambda_n/E_0\) is a Galois extension. Therefore, the preceding observation shows that \(\Lambda_n\) is a root field over \(E_0\) of the polynomial \(X^{p^n} - \pi_n\). This implies \(\varepsilon_n \in \Lambda_n\) and the extension \(E_0(\varepsilon_n)/E_0\) is totally ramified. However, it has already been proved that \(\varepsilon_n \in E_0(\varepsilon_n) = K_{\text{ur}}\), so it follows from (3.3) (a) that \(\varepsilon_n \in E_0\), for all \(n \in \mathbb{N}\). Theorem \([2.1]\) is proved.
Corollary 4.1. Let \((K, v)\) be an HDV-field and \(E/K\) a Galois extension satisfying the conditions of Theorem 2.1. Assume that \(K\) is an almost perfect field, \(\text{char}(\hat{K}) = q\), and \(v(E) \neq qv(E)\). Then \(E\) is a \(C_1\)-field.

Proof. Under the conditions of Theorem 2.1, \(\hat{K}\) is finite and we have \(\text{dim}(E) \leq 1\), so it follows from the inequality \(v(E) \neq qv(E)\) that \(\hat{E}\) is algebraically closed. This implies \(K_{ur}\) is \(K\)-isomorphic to a subfield of \(E\). At the same time, the assumption that \(K\) is almost perfect allows us to deduce from Lang’s theorem that \(K_{ur}\) is a \(C_1\)-field. Therefore, algebraic extensions of \(K_{ur}\) are also \(C_1\)-fields, which proves our assertion.

Remark 4.2. It is known that \(\mathbb{Q}\) has a unique \(\mathbb{Z}_p\)-extension \(\Gamma_p\) in \(\mathbb{Q}_{sep}\), for each \(p \in \mathbb{P}\) (and \(\Gamma_p\) is included in \(\mathbb{Q}(\Theta_p)\), where \(\Theta_p\) is the set of all roots of unity in \(\mathbb{Q}_{sep}\) of \(p\)-primary degrees). Also, it is clear from Galois theory that the compositum \(\Gamma\) of fields \(\Gamma_p, p \in \mathbb{P}\), is a Galois extension of \(\mathbb{Q}\) with \(\mathcal{G}(\Gamma/\mathbb{Q}) \cong \prod_{p \in \mathbb{P}} \mathbb{Z}_p\). Identifying \(\mathbb{Q}_{sep}\) with its isomorphic copy in \(\mathbb{Q}_q\), for an arbitrary fixed \(q \in \mathbb{P}\), and using the decomposition law in cyclotomic extensions of \(\mathbb{Q}\) (see [2], Ch. III, Lemmas 1.3, 1.4), one obtains by the method of proving Theorem 2.1 that \(\Gamma \cap \mathbb{Q}_q = \mathbb{Q}\). \(\mathbb{Q}_q \Gamma/\mathbb{Q}_q\) is a Galois extension with \(\mathcal{G}(\mathbb{Q}_q \Gamma/\mathbb{Q}_q) \cong \mathcal{G}(\Gamma/\mathbb{Q}),\) \(\text{dim}(\mathbb{Q}_q \Gamma) \leq 1\) and \(s(\mathbb{Q}_q \Gamma) = \{q\}\). In addition, if \(v_q\) is a valuation of \(\Gamma\) extending the standard \(q\)-adic valuation, say \(\omega_q\), of \(\mathbb{Q}\), then \(v_q(\Gamma) = \{r \in \mathbb{Q} : q^r \in \omega_q(\mathbb{Q})\}\), for some \(r = \rho(r) \in \mathbb{N} \cup \{0\}\).

Proposition 4.3. Let \((K, v)\) be an HDV-field with \(\hat{K}\) finite, and let \(T\) be the compositum of tamely ramified finite extensions of \(K\) in \(K_{ab}\). Then \(T/K\) is a Galois extension with \(\mathcal{G}(T/K)\) isomorphic to \(\mathcal{G}_{\hat{K}} \times \hat{K}^*\); hence, the valuation \(v_T\) of \(T\) extending \(v\) is discrete.

Proof. Clearly, \(K_{ab}/K\) is a Galois extension with \(\mathcal{G}(K_{ab}/K)\) which ensures that so is \(L/K\) whenever \(L \in I(K_{ab}/K)\). Specifically, \(T/K\) is Galois, so it remains to be proved that \(\mathcal{G}(T/K)\) and \(v_T\) have the properties claimed by Proposition 4.3. It is well-known that \(\mathcal{G}_{\hat{K}}\) is isomorphic to the topological group product \(\prod_{p \in \mathbb{P}} \mathbb{Z}_p\), whence, it is a projective profinite group, in the sense of [19]. Since, by (3.4) (b), \(\mathcal{G}(K_{ur}/K) \cong \mathcal{G}_{\hat{K}}\), this enables one to deduce from Galois theory that \(T\) equals the compositum \(K_{ur} T_0\), for some \(T_0 \in I(T/K)\), such that \(K_{ur} \cap T_0 = K\). Hence, by (3.3) (b) and [20], Proposition A.17, finite extensions of \(K\) in \(T_0\) are tamely and totally ramified, which implies \(\widehat{T_0} = \hat{K}\) and \(v(T_0)/v(K)\) is an abelian torsion group without elements of order \(p\). Taking now into account that \((K, v)\) is an HDV-field, and using Lemma 3.1 and [15], Ch. II, Proposition 12, one obtains that, for each finite extension \(K_1\) of \(K\) in \(T_0\), \(K\) and \(\hat{K}\) contain primitive roots of unity of degree \([K_1 : K]\). Also, it becomes clear that \(K_1/K\) is Galois, \(\mathcal{G}(K_1/K)\) is cyclic of order \(n\), and \(\mathcal{G}(K_1/K) \cong v(K_1)/v(K)\). As \(\hat{K}\) is finite, this leads to the conclusion that \(v(T_0)/v(K)\) is cyclic of order \(q - 1\), and \(T_0/K\) is a Galois extension with \(\mathcal{G}(T_0/K) \cong v(T_0)/v(K)\), where \(q\) is the cardinality of \(\hat{K}\). Thus it follows that \([T : K_{ur}] = [T_0 : K] = q - 1\), which ensures that \((T, v_T)\)
is an HDV-field (cf. [7], Corollary 14.2.2). Note finally that \( T = K_{ur}T_0 \) is a Galois extension of \( K \) with \( G(T/K) \) isomorphic to the topological direct products \( G(K_{ur}/K) \times G(T_0/K) \) and \( \hat{G}_R \times \hat{K}^* \) (\( G(T_0/K) \) and \( \hat{K}^* \) are viewed as discrete topological group), so Proposition 4.3 is proved.

**Corollary 4.4.** Under the hypotheses of Proposition 4.3, let \( E \) be an extension of \( K \) in \( T \). Then \( \dim(E) \leq 1 \) if and only if \( E \in I(T/K_{ur}) \); when this holds and char(\( K \)) > 0, \( E \) is a \( C_1 \)-field.

**Proof.** Proposition 4.3 and our assumptions show that \((E, v_E)\) is an HDV-field, so the former part of our conclusion follows from the former one and Corollary 4.1. The noted property of \((E, v_E)\) also enables one to deduce the former part of Corollary 4.4 from Theorem 2.1 and Lemma 3.1.

**Corollary 4.5.** Let \( K \) be a finite extension of \( \mathbb{Q} \) and \( L_1 \) an extension of \( K \) in \( K_{ab} \), of degree \( p \in \mathbb{P} \). Suppose that \( L_1/K \) is of infinite height, i.e. there are fields \( L_n \in I(K_{ab}/L_1), n \in \mathbb{N} \), such that \( L_n/K \) is a cyclic extension of degree \( p^n \), for each index \( n \). Then one of the following conditions holds:

(a) \( L_1/K \) is wildly ramified; more precisely, if \( L_1/K \) is totally ramified relative to a valuation \( v \) of \( K \), then \( v(p) > 0 \);

(b) \( L_1 \in I(H(K)/K) \), where \( H(K) \) is the Hilbert class field of \( K \); in this case, \( p \) divides the class number of \( K \).

**Proof.** Let \( w \) be an arbitrary discrete valuation of \( K \), such that \( w(p) = 0 \), i.e. the characteristic of the residue field of \((K, w)\) is different from \( p \). Identifying \( K_{ab} \) with its \( K \)-isomorphic copy in \( K_{w,sep} \), put \( L_n' = L_nK_w \), for each \( n \in \mathbb{P} \). It is easy to see that if \( L_1/K \) is totally ramified relative to \( w \), then so is \( L_1'/K_w \) (relative to the continuous prolongation \( \bar{w} \) of \( w \) on \( K_w \)). In view of Galois theory and [20], Proposition A.17, these observations ensure that the extensions \( L_n'/K_w \), \( n \in \mathbb{N} \), are both cyclic of degree \( p^n \) and tamely and totally ramified. Therefore, \( \bar{w}(T) \) must include \( \bar{w}(K_w) = w(K) \) as a subgroup of infinite index, \( T \) being the compositum of tame finite extensions of \( K_{w,ab} \). Since \( w(K) \neq \{0\} \), and by Proposition 4.3 \( \bar{w}(T) \) is cyclic, this is a contradiction due to the assumption that \( L_1/K \) is totally ramified relative to \( w \), so Corollary 4.5 (a) is proved.

We turn to the proof of Corollary 4.5 (b). The latter part of our assertion follows from the former one and well-known general properties of \( H(K) \). Also, it is clearly sufficient to prove the former part of Corollary 4.5 (b), under the hypothesis that \( L_1/K \) is unramified relative to \( v \), for any discrete valuation \( v \) of \( K \) (and each valuation \( v_1 \) of \( L_1 \) extending \( v \)). When \( p > 2 \) or \( K \) is a nonreal field, our assertion follows from the definition of \( H(K) \), so we consider only the case where \( p = 2 \) and \( K \) is formally real. Fix an Archimedean absolute value \( \omega \) of \( K \) so that the completion \( K_\omega \) of \( K \) with respect to the topology induced by \( \omega \) be isomorphic to \( \mathbb{R} \). It is easily obtained from Galois theory that if \( L_1 \otimes_K K_\omega \) is a field, then so must be \( L_2 \otimes_K K_\omega \). This, however, means that \((L_1 \otimes_K K_\omega)/K_\omega \) must be a quartic extension, which is impossible. Therefore, \( L_1 \otimes_K K_\omega \) is not a field, which
implies that if $\omega_1$ is an absolute value of $L_1$ extending $\omega$, then the completion of $L_1$ with respect to the topology of $\omega_1$ is isomorphic to $\mathbb{R}$ (see [2], Ch. II, Theorem 10.2). Hence, $L_1/K$ is unramified, and since $L_1 \in I(K_{ab}/K)$, it follows that $L_1 \in I(H(K)/K)$, so Corollary 1.5 is proved. \[\square\]

**Proposition 4.6.** Let $(K, v)$ be an HDV-field and $E/K$ a Galois extension. Suppose that $\text{char}(K) = 0$, $\hat{K}$ is finite, $\dim(E) \leq 1$, and $\mathcal{G}(E/K)$ is abelian. Then there exists $\Lambda \in I(E/K)$ with $\mathcal{G}(\Lambda/K)$ isomorphic to the topological group product $\prod_{p \in \mathbb{P}} \mathbb{Z}_p$.

**Proof.** In view of Galois theory, it suffices to prove that $K$ has a $\mathbb{Z}_p$-extension $\Lambda_p$ in $E$, for each $p \in \mathbb{P}$. The existence of $\Lambda_p$, $p \in \mathbb{P} \setminus \{q\}$, where $q = \text{char}(\hat{K})$, is implied by (3.4), the structure of $G_{\hat{K}}$, and Theorem 2.1. It remains to be seen that $K$ has a $\mathbb{Z}_q$-extension $\Lambda_q$ in $E$. Clearly $p$ does not divide the degree of any finite extension of $E \cap K(p)$ in $E$. Hence, by Lemma 3.3, $\left|(E \cap K(p)) : K\right| = \infty$. Moreover, it follows from Krasner’s lemma (cf. [13], Ch. II, Propositions 3, 4) and Galois theory that $X(E/K)_q$ is an infinite group with finitely many elements of order $q^n$, for each $n \in \mathbb{N}$. Using Galois theory and applying the following lemma, one proves the existence of $\Lambda_q$. \[\square\]

**Lemma 4.7.** Assume that $A$ is an infinite abelian torsion $p$-group, for some $p \in \mathbb{P}$, such that the subgroup $pA = \{a \in A : pa = 0\}$ is finite. Then $A$ possesses a subgroup isomorphic to $\mathbb{Z}(p^\infty)$.

**Proof.** Arguing by induction on $n$, one obtains first that $A$ contains finitely many elements of order $p^n$, for each $n \in \mathbb{N}$. Consider a sequence $\overline{C} = C_n$, $n \in \mathbb{N}$, of cyclic subgroups of $A$, chosen so that $C_n$ be of order $p^n$, for each index $n$. It follows from the finitude of $pA$ that there is a subsequence $C_1 = C_{1,n}$, $n \in \mathbb{N}$, of $\overline{C}$, such that the groups $C_{1,n}$, $n \in \mathbb{N}$, share a common subgroup $H_1$ of order $p$. Similarly, there exist subsequences $\overline{C}_k$, $k \in \mathbb{N}$, of $\overline{C}$, such that $\overline{C}_k = C_{k,n}$, $n \in \mathbb{N}$, the groups $C_{k,n}$, $n \in \mathbb{N}$, have a common subgroup $H_k$ of order $p^k$, and $\overline{C}_{k+1}$ is a subsequence of $\overline{C}_k$, for each $k$. Observing that $H_k$ is a subgroup of $H_{k+1}$, one concludes that the union $H = \bigcup_{k=1}^\infty H_k$ is a subgroup of $A$ and there is a group isomorphism $H \cong \mathbb{Z}(p^\infty)$. \[\square\]

Given an HDV-field $(K, v)$ with $\hat{K}$ finite and a Galois extension $E/K$, the conclusion of Proposition 4.6 need not be true without the assumptions that $\text{char}(K) = 0$ and $\mathcal{G}(E/K)$ is abelian. For example, if $\text{char}(K) = q > 0$, then $K(q)$ includes infinitely many subfields $\Theta_n$, $n \in \mathbb{N}$, that are Artin-Schreier extensions of $K$ of degree $q$. This implies the compositum $\Lambda_q$ of the fields $\Theta_n$, $n \in \mathbb{N}$, is a Galois extension of $K$ with $\mathcal{G}(\Lambda_q/E)$ infinite abelian of period $q$. Assuming that $\Lambda_q$ is the $\mathbb{Z}_q$-extension of $K$ in $K_{ur}$, for each $p \in \mathbb{P} \setminus \{q\}$, one obtains that the compositum $\Lambda$ of the fields $\Lambda_p$, $p \in \mathbb{P}$, is a Galois extension of $K$ with $\mathcal{G}(\Lambda/K) \cong \prod_{p \in \mathbb{P}} \mathcal{G}(\Lambda_p/K)$. Hence, $\dim(\Lambda) \leq 1$ and $\mathcal{G}(\Lambda/K)$ is abelian without a quotient group isomorphic to $\mathbb{Z}_q$. Therefore, there is no $\mathbb{Z}_q$-extension of $K$ in $\Lambda$, proving that the condition on $\text{char}(K)$ in Proposition 4.6 is essential. Similarly, if $(K, v)$ is an HDV-field with $\text{char}(K) = 0$ and...
char(\(\bar{K}\)) = q, then the conclusion of Proposition 4.6 need not be true when \(\mathcal{G}(E/K)\) is nonabelian. Omitting the details, note that a counter-example \(E\) can be chosen so as to satisfy the following: (i) \(E \cap K_{\text{ab}} \subset K_{\text{ur}}\) and \(K\) does not possess a degree \(q\) extension in \(E \cap K_{\text{ab}}\); (ii) \(E/K\) is a Galois extension with \(\mathcal{G}(E/(E \cap K_{\text{ab}}))\) infinite abelian of period \(q\).

5. Characterization of fields of dimension \(\leq 1\) within the class of abelian and tame extensions of a global field

Our main goal in this Section is to characterize the fields pointed out in its title and thereby to prove Theorem 2.2 (a). First, we present some basic properties of maximal abelian tame extensions of global fields, as follows:

**Proposition 5.1.** Let \(K\) be a global field and \(\Theta\) the compositum of tame finite extensions of \(K\) in \(K_{\text{ab}}\). Then:

(a) All nontrivial valuations of \(\Theta\) are discrete with algebraically closed residue fields; in particular, \(\dim(\Theta) \leq 1\);

(b) For each nontrivial valuation \(v\) of \(K\), the compositum \(T\) of tamely ramified finite extensions of \(K_v\) in \(K_{v,\text{ab}}\) contains as a subfield a \(K\)-isomorphic copy \(\Theta'\) of \(\Theta\); in addition, \(\Theta'K_v = T\).

**Proof.** Identifying \(K_{\text{sep}}\) with its \(K\)-isomorphic copy in \(K_{v,\text{sep}}\), one may put \(\Theta' = \Theta\). Using also Galois theory and the tameness of \(\Theta/K\), one obtains further that \(K_{\text{ab}},K_v \subseteq K_{v,\text{ab}}\) and \(\Theta.K_v \subseteq T\). Conversely, it follows from Grunwald-Wang’s theorem (see [17]) that each cyclic extension \(\Lambda\) of \(K_v\) in \(K_{v,\text{ab}}\) equals \(\Lambda_0 K_v\), for some cyclic extension \(\Lambda_0\) of \(K\) in \(K_{\text{ab}}\) satisfying the divisibility conditions \([\Lambda: K_v] \mid [\Lambda_0: K] \mid 2[\Lambda: K_v]\). Moreover, Grunwald-Wang’s theorem guarantees that if \(\Lambda \in I(T/K)\), then \(\Lambda_0\) can be chosen so as to lie in \(I(\Theta/K)\). This implies the inclusions \(T \subseteq \Theta K_v\) and \(K_{v,\text{ab}} \subseteq K_{\text{ab}},\Theta K_v\). These observations prove Proposition 5.1 (b). Consider now the fields \(\Theta\) and \(\Theta_1 = \Theta.K_1\), where \(K_1\) is the maximal separable (algebraic) extension of \(K\) in \(K_v\). Denote by \(\theta\) and \(\theta_1\) the valuations of \(\Theta\) and \(\Theta_1\), respectively, induced by the valuation of \(T = \Theta K_v\), extending the continuous prolongation \(\bar{v}_T\) of \(v\) on \(K_v\). Clearly, \(\theta\) and \(\theta_1\) are discrete (\(\bar{v}_T\) is discrete, by Proposition 4.3 and \(\{0\} \neq v(K) \leq \theta(\Theta) \leq \theta_1(\Theta_1) \leq \bar{v}(T)\)), and \((K_1,v_1)\) is a Henselization of \((k,v)\), where \(v_1\) is the valuation of \(K_1\) induced by \(\bar{v}\). The latter ensures that \((\Theta_1,\theta_1)\) is a Henselization of \((\Theta,\theta)\) (cf. [20], Proposition A.30). Taking also into account that \(\Theta_1 K_v = \Theta K_v = T\), one deduces from Lemma 3.2 and Proposition 4.3 that \(\Theta_1 = \Theta \cong \bar{K}_{\text{sep}}\). In view of the normality of \(\Theta/K\), every valuation of \(\Theta\) possesses the obtained properties of \(v\), so the former part of Proposition 5.1 (a) is proved. Note finally that \(\Theta\) is a nonreal field, i.e. \(-1\) is presentable as a sum of the squares of finitely many elements of \(\Theta\). This is obvious in case \(\text{char}(K) > 0\), and if \(\text{char}(K) = 0\), then \(\Theta\) contains a square root \(\sqrt{-p}\), for any \(p \in \mathbb{P}\) satisfying \(p \equiv 3\pmod{4}\) (whence, \(-1\) is presentable as a sum of squares of four elements of \(\Theta\)). Since, Corollary 4.4 and the former part of Proposition 5.1 (a) yield \(\text{Br}(T) = \{0\}\), for each nontrivial valuation \(v\) of \(\Theta\), this allows to deduce from Lemma 3.3 that \(\text{Br}(\Theta) = \{0\}\). Hence, by [8], Theorem 4, \(\dim(\Theta) \leq 1\), so Proposition 5.1 is proved. □
Theorem 2.2 (a) is contained in Proposition 5.1 (a) and the following:

**Corollary 5.2.** Let $K$ be a global field and $E$ a tame extension of $K$ in $K_{ab}$. Then all nontrivial Krull valuations of $E$ are discrete. In addition, $\dim(E) \leq 1$ if and only if $E$ is a nonreal field and the residue fields of its discrete valuations are algebraically closed.

**Proof.** Let $\Theta$ be the compositum of tame finite extensions of $K$ in $K_{ab}$. Then $E \in \mathcal{I}(\Theta/K)$ and every nontrivial valuation $w$ of $E$ extends to a valuation $w'$ of $\Theta$, so the former conclusion of Corollary 5.2 follows from Proposition 5.1 (a). Note further that if $E$ is a formally real field, i.e. $-1$ is not presentable as a sum of the squares of finitely many elements of $E$, then the Hamiltonian quaternion $E$-algebra is a division one, whence, $\text{Br}(E)$ contains an element of order 2. When $E$ is nonreal, the latter assertion of the corollary can be deduced following the concluding part of the proof of Proposition 5.1. □

The following result shows that if $K$ is a number field with $\text{Cl}(K) = 1$, then $X(\Delta/K)$ decomposes into a direct sum of finite cyclic groups, for each tame extension $\Delta$ of $K$ in $K_{ab}$. This need not be valid without the tameness condition; one may take as a counter-example any nonreal field $\Delta' \in \mathcal{I}(K_{ab}/\Gamma K)$ with $[\Delta': \Gamma K] < \infty$, where $\Gamma$ is defined in Remark 4.2.

**Proposition 5.3.** Let $K$ be a number field, $\text{Cl}(K)$ its class number, $\Delta$ a tame extension of $K$ in $K_{ab}$, and $X(\Delta/K)_p$ the $p$-component of $X(\Delta/K)$, for each $p \in \mathbb{P}$. Then:

(a) $X(\Delta/K)$ is a reduced abelian torsion group with finitely many elements of infinite height;

(b) $X(\Delta/K)_p$ contains infinitely many elements of order $p$ unless it is a finite group; $X(\Delta/K)_p$ is infinite in case $\text{Br}(\Delta)_p = \{0\}$;

(c) $X(\Delta/K)$ decomposes into a direct sum of cyclic $p$-groups, for every $p \in \mathbb{P}$ not dividing $\text{Cl}(K)$.

**Proof.** Clearly, $X(\Delta/K)$ is a countable abelian torsion group, and Corollary 4.5 shows that $X(\Delta/K)_p$ does not contain nonzero elements of infinite height, for any $p \in \mathbb{P}$, $p \nmid \text{Cl}(K)$, so Proposition 5.3 (c) follows from Prüfer’s theorem (see [10], Theorem 5.3) and Proposition 5.3 (a). For the proof of former part of Proposition 5.3 (b), one may clearly assume that $X(\Delta/K)_p$ is infinite. Suppose for a moment that $X(\Delta/K)_p$ contains only finitely many elements of order $p$. Then, by Lemma 4.7, $X(\Delta/K)_p$ must have a subgroup isomorphic to $\mathbb{Z}(p^\infty)$. In view of Galois theory, this requires the existence of a $\mathbb{Z}_p$-extension $\Lambda_p$ of $K$ in $\Delta$. Clearly, $\Lambda_p/K$ must preserve the tameness of $\Delta/K$, whence, using repeatedly Corollary 4.5, one concludes that $\Lambda_p$ must be a subfield of $H(K)$. On the other hand, by class field theory (Furtwängler’s theorem, see [2], Chs. IX and XI), $H(K)/K$ is a finite extension with $[H(K): K] = \text{Cl}(K)$. The obtained contradiction proves the former part of Proposition 5.3 (b).

For the rest of the proof of Proposition 5.3 (b), note that $X(\Delta/K)_p$ is finite if and only if $\Delta \cap K(p)$ is a finite extension of $K$ (this is a well-known consequence of Galois theory and Pontrjagin’s duality, see ). Since, by class
field theory (cf. 23, Ch. XIII, Sects. 3 and 6), \( \text{Br}(Y)_{p'} \neq \{0\}, \) for every global field \( Y, \) this implies the latter part of Proposition 5.3 (b).

It remains for us to prove Proposition 5.3 (a). Clearly, the set \( X_0(\Delta/K) \) of elements of \( X(\Delta/K) \) of infinite height forms a subgroup of \( X(\Delta/K) \), so it follows from Proposition 5.3 (c) that it suffices to show that if \( \text{Cl}(K) > 1 \), then the \( p \)-component \( X_0(\Delta/K)_{p} \) of \( X_0(\Delta/K) \) is a finite group, for an arbitrary \( p \in \mathbb{P} \) dividing \( \text{Cl}(K) \). Since \( K \) has finitely many valuations, up-to equivalence, with residue fields of characteristic \( p \), it follows from Proposition 5.3 (c) that it suffices to show that if \( \text{Cl}(\Delta) \) implies the existence of finitely many degree \( p \) extensions of \( K \) in \( K_{\text{ab}} \) of infinite height. In view of Galois theory, this means that \( X_0(\Delta/K)_{p} \) contains finitely many elements of order \( p \). Using Galois theory, one also concludes there is \( I_p \in I(\Delta/K) \) with \( X(I_p/K) \cong X_0(\Delta/K)_{p} \). This allows to obtain by the method of proving Proposition 5.3 (b) that \( \mathbb{Z}(p^\infty) \) does not embed as a subgroup of \( X_0(\Delta/K)_{p} \). Hence, by Lemma 4.7, \( X_0(\Delta/K)_{p} \) is finite, so Proposition 5.3 is proved.

**Remark 5.4.** Let \( K \) be a number field. Then Corollary 4.3 and 12, Theorem 1.48, imply that for any \( p \in \mathbb{P}, K \) has finitely many extensions in \( K_{\text{ab}} \) of degree \( p \) and infinite height. Hence, by Galois theory and the proof of Lemma 14.4, \( X(K_{\text{ab}}/K) \) contains finitely many elements of order \( p^n \) and infinite height, for each \( n \in \mathbb{N} \). In view of the structure and injectivity of divisible abelian torsion groups (cf. 10, Theorems 2.6, 3.1), this recovers the proof of 9, Theorem 1, in the case of a number ground field. It is not known whether the quotient group of \( X(K_{\text{ab}}/K) \) by its maximal divisible subgroup \( DX(K_{\text{ab}}/K) \) contains only finitely many elements of infinite height. By 9, Theorem 14, this holds for global function fields \( K' \).

The maximal tame extension of \( \mathbb{Q} \) in \( \mathbb{Q}_{\text{ab}} \) is described explicitly as follows:

**Proposition 5.5.** The maximal tame extension \( \Theta \) of \( \mathbb{Q} \) in \( \mathbb{Q}_{\text{ab}} \) is generated over \( \mathbb{Q} \) by the set \( \Psi \) of primitive \( p \)-th roots of unity \( \varepsilon_p \in \mathbb{Q}_{\text{sep}}, p \in \mathbb{P} \).

**Proof.** Denote by \( E \) the extension of \( \mathbb{Q} \) generated by \( \Psi \), and let \( \Gamma \) be the field defined in Remark 1.2. It is well-known (see 2, Ch. II, Theorem 3.2) that the natural \( p \)-adic valuations \( v_p, p \in \mathbb{P}, \) form a system of representatives of the equivalence classes of nontrivial Krull valuations of \( \mathbb{Q} \). Assume now that \( \Phi \) is the set of square-free odd integers \( \geq 3 \), and for each \( n \in \Phi \), put \( \Lambda_n = \mathbb{Q}(\varepsilon_n) \), where \( \varepsilon_n \in \mathbb{Q}_{\text{sep}} \) is a primitive \( n \)-th root of unity; also, let \( \lambda_{n,p} \) be a valuation of \( \Lambda_n \) extending \( v_p, \) for each \( p \in \mathbb{P} \). Then \( \Lambda_n/\mathbb{Q} \) is a Galois extension with \( \mathcal{G}(\Lambda_n/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^* \) (cf. 2, Ch. III, Lemma 1.1). As \( n \in \Phi \), this means that \( \mathcal{G}(\Lambda_n/\mathbb{Q}) \) is isomorphic to the direct product \( \prod_{p' \neq p}^\prime F_{p'}^{\nu_p} \), indexed by the set of prime divisors of \( n \). It is therefore clear that, for any pair \( \nu_1, \nu_2 \in \Phi, \Lambda_{\nu_1} \Lambda_{\nu_2} = \Lambda_{\nu}, \) where \( \nu = \text{lcm}[\nu_1, \nu_2]; \) in particular \( E \) equals the union of \( \Lambda_n, n \in \Phi. \) To prove the inclusion \( E \subseteq \Theta, \) it is sufficient to show that \( \Lambda_n/\mathbb{Q} \) is tamely ramified relative to \( \lambda_{n,p}/v_p, \) for arbitrary fixed \( n \in \Phi, p \in \mathbb{P} \). This is implied by the following facts: \( \lambda_{n,p}(\Lambda_n) = v_p(\mathbb{Q}), \) provided that \( p \nmid n; v_p(\mathbb{Q}) \) is a subgroup of \( \lambda_{n,p}(\Lambda_n) \) of index \( p - 1 \) in case \( p | n \) (cf. 2, Ch. III, Lemmas 1.3 and 1.4). Thus the assertion that \( E \subseteq \Theta \)
becomes obvious as well as the fact that every valuation \( w_p \) of \( E \) extending \( \nu_p \) is discrete with \( w_p(E) \) including \( v_p(\mathbb{Q}) \) as a subgroup of index \( p - 1 \).

It remains to prove that \( \Theta = E \). It follows from the Kronecker-Weber theorem, Galois theory and well-known basic properties of cyclotomic extensions of \( \mathbb{Q} \) that \( \mathbb{Q}_{ab} = ET(\sqrt{-1}) \) and \( E \cap \Gamma(\sqrt{-1}) = \mathbb{Q} \). This implies \( G(\mathbb{Q}_{ab}/E) \cong G(\Gamma(\sqrt{-1})/\mathbb{Q}) \) and the map of \( I(\Gamma(\sqrt{-1})/\mathbb{Q}) \) into \( I(\mathbb{Q}_{ab}/E) \), by the rule \( Y \to YE \), is bijective. Also, it is clear that \( \{Y_1 : \mathbb{Q} \} = [Y_1E : E] \), for any finite extension \( Y_1 \) of \( \mathbb{Q} \) in \( \Gamma(\sqrt{-1}) \). We show that if \( Y_1 \neq \mathbb{Q} \), then \( Y_1/\mathbb{Q} \) is not tame. Indeed, for any divisor \( p \in \mathbb{P} \) of \( \{Y_1 : \mathbb{Q} \} \), \( Y_1/\mathbb{Q} \) is wildly ramified relative to \( y_p/v_p \), where \( y_p \) is a valuation of \( Y_1 \) extending \( v_p \) (apply \[2\], Ch. II, Lemmas 1.3, 1.4). Since the set of tame extensions of \( \mathbb{Q} \) in \( \mathbb{Q}_{ab} \) is closed under taking intermediate fields (cf. \[14\], Ch. II, Propositions 8, 13, and \[2\], Theorem 10.2), this proves that \( E = \Theta \), as required.

Let us note that Proposition \[5.5\] allows us to give an alternative proof of Proposition \[5.1\] (b). Indeed, Theorem 4 of \[8\], reduces our considerations to the special case where \( K = \mathbb{Q} \). Then our assertion can be obtained by applying the following statement (which in turn is implied by the Lemma in \[3\], page 131): for any fixed pair \((l, n) \in \mathbb{P} \times \mathbb{N}\), there are infinitely many \( q \in \mathbb{P} \setminus \{p\} \), \( n \in \mathbb{F} \), \( \{p\} \), such that the extension \( \mathbb{F}_p(\gamma_n)/\mathbb{F}_p \), where \( \gamma_n \) is a primitive \( q_n \)-th root of unity in \( \mathbb{F}_p )\), has degree \( [\mathbb{F}_p(\gamma_n) : \mathbb{F}_p] \) divisible by \( l^n \).

**Corollary 5.6.** Assume that \( \Theta_{\Pi} \) is an extension of \( \mathbb{Q} \) obtained by adjunction of primitive \( p \)-th roots of unity \( \varepsilon_\Pi \in \mathbb{Q}_{sep} \), where \( p \) runs across the complement \( \mathbb{P} \setminus \Pi \), for an arbitrary finite subset \( \Pi \) of \( \mathbb{P} \). Then \( dim(\Theta_{\Pi}) \leq 1 \).

**Proof.** Let \( \Theta \) be the field defined in Proposition \[5.5\]. Then \( \Theta/\Theta_{\Pi} \) is a finite extension. More precisely, if \( \Pi = \{p_1, \ldots, p_s\} \), then \( [\Theta : \Theta_{\Pi}] = \prod_{i=1}^{s} (p_i - 1) \). This means that \( G_{\Theta} \) is an open subgroup of \( G_{\Theta_{\Pi}} \) of index \( \prod_{i=1}^{s} (p_i - 1) \). Note also that \( \Theta_{\Pi} \) is a nonreal field, because an ordered field does not contain a primitive root of unity of any degree greater than 2. It is therefore clear from Galois theory that \( G_{\Theta_{\Pi}} \) is a torsion-free group. Consider now an arbitrary nontrivial valuation \( w \) of \( \Theta_{\Pi} \), fix a valuation \( w' \) of \( \Theta \) extending \( w \), and denote by \( \Theta_{\Pi} \) and \( \hat{\Theta} \) the residue fields of \((\Theta_{\Pi}, w)\) and \((E, w')\), respectively. Then \( w \) is discrete and \( \hat{\Theta}/\Theta_{\Pi} \) is a finite extension. Moreover, it follows from Propositions \[5.1\] (a) and \[5.5\] that \( \hat{\Theta} \) is algebraically closed. In view of Galois theory, this yields \( \Theta = \Theta_{\Pi} \). It is now easy to see that \( dim(\Theta_{\Pi}) \leq 1 \). \( \square \)

In the setting of Propositions \[5.1\] (a) and \[5.5\] the question of whether a field \( \Delta \in I(\mathbb{Q}/\mathbb{Q}) \) with \( dim(\Delta) \leq 1 \) is of type \( C_1 \) is widely open. Proposition \[5.1\] and \[11\], Theorem 10, show that the completion \( \Delta_\nu \) is a \( C_1 \)-field, for any nontrivial valuation \( \nu \) of \( \Delta \). Thus our question concerning \( \Delta \) is equivalent to the problem of finding whether the Hasse principle applies to \( \Delta \)-forms in more variables than their degrees. It is worth mentioning that, for each \( q \in \mathbb{F} \), \( \Theta \) can be viewed as an analog in characteristic \( q \) to the rational function field \( \mathbb{F}_{q,sep}(X) \) in one variable over \( \mathbb{F}_{q,sep} \) (which is of type \( C_1 \), by Tsen’s theorem). In contrast to the extension \( \Theta/\mathbb{Q} \), however, \( \mathbb{F}_{q,sep}(X)/\mathbb{F}_q(X) \) has no intermediate field \( \Delta_q \neq \mathbb{F}_{q,sep}(X) \) with \( dim(\Delta_q) \leq 1 \), for any \( q \).
6. Proof of Theorem 2.2

Assume that $E$ and $K$ are Galois extensions of $\mathbb{Q}$ in $\mathbb{Q}_{\text{sep}}$, such that $[K: \mathbb{Q}] < \infty$, $E \in I(K_{\text{ab}}/K)$, $\dim(E) \leq 1$, and $E/K$ is tame. We prove the existence of an $n$-variate $E$-form $N_n$ of degree $n$, violating the Hasse principle, by showing that there is an abelian unramified Galois extension $E_n$ of $E$ over $\mathbb{Q}$, whence $N_n$ has no nontrivial zero over $E$, whereas Propositions 5.1 (a) and 5.5 imply $E_n/E$ is split relative to $v$, whence $N_n$ decomposes over $E_v$ into a product of $n$ linear forms in $n$ variables and so it has a nontrivial zero over $E_v$. The existence of the extension $E_n/E$ is ensured by the following result.

**Lemma 6.1.** Let $K$ be a finite extension of $\mathbb{Q}$ in $\mathbb{Q}_{\text{ab}}$, $H(K)$ the Hilbert class field of $K$, and $p \in \mathbb{P}$ a divisor of $[\langle H(K) \cap \mathbb{Q}_{\text{ab}} \rangle : \mathbb{Q}]$. Then $p | [K: \mathbb{Q}]$.

**Proof.** It follows from the normality of $K/\mathbb{Q}$ and the definition of $H(K)/\mathbb{Q}$ that $H(K)/\mathbb{Q}$ is a finite Galois extension. Note further that our assumptions ensure the existence of a Galois extension $L_p$ of $\mathbb{Q}$ in $H(K)$ of degree $[L_p: \mathbb{Q}] = p$; this is clear from Galois theory and the general structure of finite abelian groups. Hence, by the Hermite-Minkowski theorem (cf. [13], Ch. V, Sect. 4), there exists $\lambda \in \mathbb{P}$, such that $L_p/\mathbb{Q}$ is totally ramified at the $\lambda$-adic valuation $\omega_\lambda$ of $\mathbb{Q}$. This implies $\lambda | e(H(K)/\mathbb{Q})\omega_\lambda$, which indicates that $p | [K: \mathbb{Q}]$ (since $H(K)/K$ is unramified).

**Lemma 6.2.** Let $K_1$ and $K_2$ be finite Galois extensions of $\mathbb{Q}$ in $\mathbb{Q}_{\text{sep}}$ of $p$-primary degrees, for a given $p \in \mathbb{P}$, and $H(K_2) \in I(\mathbb{Q}_{\text{sep}}/\mathbb{Q})$, be the Hilbert class fields of $K_1$ and $K_2$, respectively. Suppose that $K_1 \cap K_2 = \mathbb{Q}$, and for $j = 1, 2$, denote by $H_j$ the maximal subfield of $H(K_j)$ with respect to the property that $[H_j : K_j]$ is not divisible by $p$. Then $H_1$, $H_2$ and $H_1H_2$ are Galois extensions of $\mathbb{Q}$, such that $H_1 \cap H_2 = \mathbb{Q}$, $H_1H_2 \cap \mathbb{Q}_{\text{ab}} = K_1K_2$, and $\mathcal{G}(H_1H_2/\mathbb{Q})$ is isomorphic to the direct product $\mathcal{G}(H_1/\mathbb{Q}) \times \mathcal{G}(H_2/\mathbb{Q})$.

**Proof.** Denote for brevity by $H_0$ and $H_3$ the fields $H_1 \cap H_2$ and $H_1H_2$, respectively, and put $K_3 = K_1K_2$. Our assumptions, the normality of $K_1$ and $K_2$ over $\mathbb{Q}$, and properties of $H(K_1)/K_1$, $H(K_2)/K_2$ available by definition, ensure that $H_1/\mathbb{Q}$ and $H_2/\mathbb{Q}$ are Galois extensions, so it follows from Galois theory that $H_3/\mathbb{Q}$ is Galois as well. They also show that $\mathcal{G}(H_j/K_j)$ is abelian and $\mathcal{G}(H_j/\mathbb{Q})$, $\mathcal{G}(H_2/\mathbb{Q})$ is metabelian, for $j = 1, 2, 3$. Taking further into account $p \nmid [H_j : K_j]$, $j = 1, 2, 3$, one concludes that, for each $j$, $K_j$ contains as a subfield every extension of $\mathbb{Q}$ in $H_j$ of $p$-primary degree. At the same time, it is clear from Galois theory that $H_0/\mathbb{Q}$ is a Galois extension and $\mathcal{G}(H_0/\mathbb{Q})$ is a homomorphic image of $\mathcal{G}(H_j/\mathbb{Q})$, $j = 1, 2$; in particular, $\mathcal{G}(H_0/\mathbb{Q})$ is metabelian. These facts and the equality $K_1 \cap K_2 = \mathbb{Q}$ indicate that if $H_0 \neq \mathbb{Q}$, then $\mathbb{Q}$ has a Galois extension $K_0$ in $H_0$ of prime degree $[K_0: \mathbb{Q}] = l \neq p$. This, however, contradicts Lemma 6.1 and thereby proves that $H_0 = \mathbb{Q}$. Now it follows from Galois theory that
\[G(H_3/\mathbb{Q}) \cong G(H_1/\mathbb{Q}) \times G(H_2/\mathbb{Q})\]
which implies \(G(H_3/K_3)\) equals the commutator subgroup of \(G(H_3/\mathbb{Q})\) and \(K_3 = H_3 \cap \mathbb{Q}_{ab}\). Lemma 6.2 is proved. \(\square\)

We are now in a position to prove (2.5) (and Theorem 2.2 (b)), for a given integer \(n \geq 2\). In view of the general structure of finite abelian groups, one may consider only the special case where \(n\) is odd or \(n = 2^\mu\), for some \(\mu \in \mathbb{N}\). Our argument relies on the existence of Galois extensions \(Q_\nu, Y_\nu, \nu \in \mathbb{N}\), with class numbers \(\text{Cl}(Q_\nu)\) and \(\text{Cl}(Y_\nu)\) divisible by \(n\), and with \([Q_\nu: \mathbb{Q}] = 2\) and \([Y_\nu: \mathbb{Q}] = 3\), for every index \(\nu\) (see 24 and 21, respectively). Clearly, the sequence \(Q_\nu, Y_\nu, m \in \mathbb{N}\), can be chosen so that \([Q_\nu \cdot \ldots \cdot Q_\nu : \mathbb{Q}] = 2^k\) and \([Y_\nu \cdot \ldots \cdot Y_\nu : \mathbb{Q}] = 3^k\) whenever \(k \geq 2\) is an integer and \(\nu_1 < \cdots < \nu_k\). For each \(\nu \in \mathbb{N}\), denote by \(C_\nu\) the field \(Q_\nu\) if \(2 \nmid n\), and put \(C_\nu = Y_\nu\) when \(n = 2^\mu\); also, let \(H_\nu\) be an extension of \(C_\nu\) in the Hilbert class field \(H(C_\nu) \in I(\mathbb{Q}_{sep}/C_\nu)\) of degree \([H_\nu : C_\nu]\) not divisible by \([C_\nu: \mathbb{Q}]\). Proceeding by induction on \(k \in \mathbb{N} \setminus \{1\}\), and using Galois theory, Lemma 6.2 and properties of Hilbert class fields available by definition, one obtains that \(H_{\nu_1} \cdot \ldots \cdot H_{\nu_k} \cap \mathbb{Q}_{ab} = C_{\nu_1} \cdot \ldots \cdot C_{\nu_k}\), \([H_{\nu_1} \cdot \ldots \cdot H_{\nu_k} : \mathbb{Q}] = \prod_{j=1}^k [H_{\nu_j} : \mathbb{Q}]\) and \([H_{\nu_1} \cdot \ldots \cdot H_{\nu_k} : C_{\nu_1} \cdot \ldots \cdot C_{\nu_k}] = \prod_{j=1}^k [H_{\nu_j} : C_{\nu_j}]\), provided that \(k \geq 2\) and \(\nu_1 < \cdots < \nu_k\). It is now easy to see that if \(H_\nu, \nu \in \mathbb{N}\), are chosen so that \([H_\nu : C_\nu] = n\), for every \(\nu\) (which is possible, since \(n \mid \text{Cl}(C_\nu)\)), then the fields \(E_\nu = H_\mu E, \nu \in \mathbb{N}\), have the properties required by (2.5), so the \(E\)-norms \(N_\nu\) of \(E_\nu/E, \nu \in \mathbb{N}\), are pairwise nonequivalent.

**Corollary 6.3.** Assume that \(K\) and \(E\) satisfy the conditions of Theorem 2.2 (b), and let \(\Delta\) be an extension of \(E\) in \(K_{ab}\). Then there exist abelian extensions \(\Delta_\nu, \nu \in \mathbb{N}\), of \(\Delta\) in \(\mathbb{Q}_{sep}\) with the following properties:

(a) \([\Delta_\nu : \Delta] = n\), for every index \(\nu\);

(b) The tensor product of the \(\Delta\)-algebras \(\Delta_\nu, \nu \in \mathbb{N}\), is a field;

(c) If \(N_\nu\) is a norm form of \(\Delta_\nu/\Delta\), for each \(\nu\), then \(N_\nu\), \(\nu \in \mathbb{N}\), are pairwise nonequivalent \(\Delta\)-forms of degree \(n\), violating the Hasse principle.

**Proof.** Clearly, one may take \(E_\nu, \nu \in \mathbb{N}\), as in the proof of Theorem 2.2 (b) and put \(\Delta_\nu = \Delta E_\nu\), for each index \(\nu\). \(\square\)

Corollary 6.3 applied to the case of \(K = \mathbb{Q}\), shows that the latter part of the conclusion of Theorem 2.2 (b), remains valid, for any number field \(K\), when the assumption that \(E = \Theta\) is replaced by the one that \(E = \Theta_{\Pi} K\), where \(\Pi\) is a finite subset of \(\mathbb{P}\) and \(\Theta_{\Pi}\) is defined as in Corollary 5.6.

**Acknowledgements.** This research has partially been supported by Grant KP-06 N 32/1 of 07.12. 2019 of the Bulgarian National Science Fund.

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Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria: E-mail address: chipchak@math.bas.bg