AN EXTENSION OF THE NORMED DUAL
FUNCTORS

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Abstract. By means of the direct limit technique, with every
normed space $X$ it is associated a bidualic (Banach) space $\hat{X}$
($D^2(\hat{X}) \cong \hat{X}$ - called the hyperdual of $X$) that contains (isometrically
embedded) $X$ as well as all the even (normed) duals $D^{2n}(X)$,
which make an increasing sequence of the category retracts. The
algebraic dimension $\dim \hat{X} = \dim X$ ($\dim \hat{X} = 2^{\aleph_0}$), whenever
$\dim X \neq \aleph_0$, ($\dim X = \aleph_0$). Furthermore, the correspondence
$X \mapsto \hat{X}$ extends to a faithful covariant functor (called the hyper-
dual functor) on the category of normed spaces.

1. Introduction

In several recent papers (the last two are [15, 16]) the author was
solving the problem of the quotient shape classification of normed vec-
torial spaces (especially, the finite quotient shape type classification),
which was initiated by a basic consideration in [14]. Since in a quotient
shape theory the main role play the infinite cardinal numbers, the usual
bipolar separation “finite-dimensional versus infinite-dimensional” of
normed spaces is quite unsatisfactory. Namely, the class of all infinite-
dimensional normed spaces had to be refined according to General Con-
tinuum Hypothesis ($GCH$), and it had become obvious that the special
bases (topological, Schauder, . . . ) cannot help in solving the problem.
The only way has led trough the strict division by the cardinalities of
algebraic (Hamel) bases. This was further leading to the normed dual
spaces and their algebraic dimensions. Surprisingly, the author discov-
ered that the inequality $\dim \hat{X} \leq \dim X^*$ was not refined in general.
Since this subproblem severely limited the study of the main one, the

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author focused his attention to its solution. In [16], Theorem 4 (by using the shape theory technique), the answer is given: \( \dim X^* = \dim X \), whenever \( \dim X \neq \aleph_0 \), while \( \dim X = \aleph_0 \) implies \( \dim X^* = 2^{\aleph_0} \). Consequently, every normed dual of every Banach space retains the algebraic dimension of the space. When, in addition, it became clear that every canonical embedding of a dual space into its second dual spaces is a categorical section ([16], Lemma 1 (i) and Theorem 1), the idea of a consistent embedding of all iterated even (odd) duals into the same Banach (“hyperdual”) space came by itself.

In the realization of the mentioned idea, a property rather close to the reflexivity (as much as possible) is desired and expected. According to the result and the example of [8], the first candidates was the somewhat reflexivity [1, 2]. However, that property (though rather suitable and useful for a local analysis) is little inappropriate for a global categorical consideration. Thus (keeping in mind the example of [8]), we had desired to get an isometric isomorphism between the associated space and its second dual space. That property is called the \textit{parareflexivity}. By dropping “isometric”, the notion of a \textit{bidualic} (originally, \textit{bidual-like}) normed space was introduced in [15], and it also has seemed to be an acceptable one for our final goal. By adding the somewhat reflexivity to parareflexivity, the obtained notion of \textit{almost reflexivity} is also considered.

By this work we have succeeded (Theorem 2) to associate with every normed space \( X \) a bidualic (Banach) space \( \tilde{X} \), i.e., \( D^2(\tilde{X}) \cong \tilde{X} \), called a \textit{hyperdual} space of \( X \), such that \( \tilde{X} \) contains (canonically embedded) \( X \) and all the iterated even duals \( D^{2n}(X) \). Moreover, those duals make a consistently increasing sequence of category retracts of \( \tilde{X} \) having the universal property (of a direct limit) with respect to the normed spaces and morphisms of norm \( \leq 1 \). Furthermore, the algebraic dimension \( \dim \tilde{X} = \dim X \) (\( \dim \tilde{X} = 2^{\aleph_0} \)), whenever \( \dim X \neq \aleph_0 \) (\( \dim X = \aleph_0 \)). Further (Theorem 3), the correspondence \( X \mapsto \tilde{X} \) extends to a faithful covariant functor (called the \textit{hyperdual functor}) on the category of normed spaces such that, for every \( k \in \{0\} \cup \mathbb{N} \), \( \tilde{D}D^{2k} = \tilde{D} \) and, for every \( X \), \( D^{2k}\tilde{D}(X) \cong \tilde{D}(X) \). Furthermore, \( \tilde{D} \) preserves the parareflexivity, quasi-reflexivity and reflexivity.

The main working technique is based on the direct limits of the direct sequences in \( iN_F \) (isometries of normed spaces) and the corresponding in-morphisms between such sequences that admit representatives having the terms of norm \( \leq 1 \).
Nevertheless, at least for technical reasons, we think that the very basic category theory terminology strictly follows [7].

- the fundamental facts concerning vectorial, normed and Banach spaces are learned from [9], [10] and [12];
- the “categorical Banach space theory” is that of [3] and [6];
- our category theory terminology strictly follows [7].

Nevertheless, at least for technical reasons, we think that the very basic of the categorical approach to normed and Banach spaces (see also [3, 6]) should be recalled.

Let $V_F$, denote the category of all vectorial spaces over a field $F$ and all the corresponding linear function. Let $N$ be the full subcategory of $F$ determined by all Banach (i.e., complete normed) spaces. Let $D : N_F \to N_F$

be the normed dual functor, i.e., the contravariant $Hom_F$ functor

$D(X) = X^* -$ the (normed) dual space of $X$,

$D(f : X \to Y) \equiv D(f) \equiv f^* : Y^* \to X^*, \quad D(f)(y^1) = y^1 f.$

Then $D[N_F] \subseteq B_F$ and, furthermore, for every ordered pair $X, Y \in Ob(N_F)$, the function

$D_1^X : N_F(X, Y) \equiv L(X, Y) \to L(Y^*, X^*) \equiv N_F(Y^*, X^*)$

is a linear isometry ($\|D(f)\| = \|f\|$), and hence, $D$ is a faithful functor.

Further, there exists a covariant Hom-functor

$Hom^2_F \equiv D^2 : N_F \to N_F$,

$D^2(X) = D(D(X)) \equiv X^{**} -$ the (normed) second dual space of $X$,

$D^2(f : X \to Y) \equiv D(D(f)) \equiv f^{**} : X^{**} \to Y^{**}$,

$D^2(f)(x^2) = x^2 D(f).$

Then, clearly, $D^2[N_F] \subseteq B_F$ and, for every ordered pair $X, Y \in Ob(N_F)$, the function

$(D^2)^X : N_F(X, Y) \equiv L(X, Y) \to L(X^{**}, Y^{**}) \equiv N_F(X^{**}, Y^{**})$

is a linear isometry ($\|D^2(f)\| = \|f\|$), and thus, $D^2$ is a faithful functor.

The most useful fact hereby is the existence of a certain natural transformation $j : 1_{N_F} \leadsto D^2$ of the functors, where, for every $X \in Ob(N_F)$, $j_X : X \to D^2(X)$ is an isometric embedding (the canonical embedding defined by $j_X(x) \equiv x^2_x \in D^2(X), \quad x \in X$, such that, for every $x^1 \in D(X), \quad x^2_x(x^1) = x^1(x) \in F$), and the closure $Cl(R(j_X)) \subseteq D^2(X)$ is the well known (Banach) completion of $X$. Namely, if $X, Y \in Ob(N_F)$, then
\[(\forall f \in \mathcal{N}_F(X,Y), j_Y f = D^2(f)j_X)\]
holds true. Clearly, if \(X\) is a Banach space, then the canonical embedding \(j_X\) is closed. Continuing by induction, for every \(k \in \mathbb{N}, k > 2\), there exists a faithful \(\text{Hom}_F\)-functor \(D^k \) of \(\mathcal{N}_F\) to \(\mathcal{N}_F\) such that \(D^k[\mathcal{N}_F] \subseteq \mathcal{B}_F\), \(D^k\) is contravariant (covariant) whenever \(k\) is odd (even), and for every ordered pair \(X, Y\) of normed spaces, the function \((D^k)_Y^X\) is an isometric linear morphism of the normed space \(L(X,Y)\) to the Banach space \(L(D^k(Y), D^k(X))\) whenever \(k\) is odd \((L(D^k(X), D^k(Y))\) whenever \(k\) is even). Further, for every \(k \in \{0\} \cup \mathbb{N}\), there exists a natural transformation of the functors \(j^k : D^k \rightsquigarrow D^{k+2}\), where \(j^0 \equiv j : 1_{\mathcal{N}_F} \rightsquigarrow D^2\) and, for every \(k > 0\), \(j^k\) is determined by the class \(\{j_{D^k(X)} \mid X \in \text{Ob}(\mathcal{N}_F)\}\). Consequently, there exist the composite natural transformations \(j^{2k-2} \cdots j^0 : 1_{\mathcal{N}_F} \rightsquigarrow D^{2k}\) as well.

3. Some special limits of normed spaces

Denote by \(i\mathcal{N}_F \subseteq \mathcal{N}_F (i\mathcal{B}_F \subseteq \mathcal{B}_F)\) the subcategory having \(\text{Ob}(i\mathcal{N}_F) = \text{Ob}(\mathcal{N}_F)\) \((\text{Ob}(i\mathcal{B}_F) = \text{Ob}(\mathcal{B}_F))\) and for the morphism class \(\text{Mor}(i\mathcal{N}_F) \) \((\text{Mor}(i\mathcal{B}_F)\) all the isometries of \(\text{Mor}(\mathcal{N}_F) \) \((\text{Mor}(\mathcal{B}_F)\). Further, we shall need their subcategories determined by all the contractive morphisms \(f\), i.e., for each \(x\), \(\|f(x)\| \leq \|x\|\), as well as those determined by all the morphisms having norm \(\|f\| \leq 1\). These are denoted by the subscript 1 and superscript 1 respectively. Clearly, \((\mathcal{N}_F)_1 \subseteq (\mathcal{N}_F)^1\) and \((\mathcal{B}_F)_1 \subseteq (\mathcal{B}_F)^1\). We shall also need the sequential in-categories \((\text{seq}-i\mathcal{N}_F)_1 \subseteq (\text{seq}-i\mathcal{N}_F)^1\) \((\text{subcategories of in-}\mathcal{N}_F = (\text{dir-}\mathcal{N}_F)/ \sim\) of all direct sequences in \(i\mathcal{N}_F\) and all the corresponding (in-)morphisms \(f\) admitting representatives \((\phi, f_n)\) such that all \(f_n\) belong to \((\mathcal{N}_F)_1 \subseteq (\mathcal{N}_F)^1\), respectively, and similarly, the sequential in-categories \((\text{seq}-i\mathcal{B}_F)_1 \subseteq (\text{seq}-i\mathcal{B}_F)^1\).

Further, given a functor \(F : \mathcal{C} \rightarrow \mathcal{D}\), the \(F\)-image \(F[\mathcal{C}]\) is a subcategory of \(\mathcal{D}\). We shall need in the sequel the \(D^{2k}\)-image and \(D^{2k+1}\)-image, \(k \in \{0\} \cup \mathbb{N}\), of the mentioned (sub)categories.

Recall that by the main result of [13] (see also [3], Section 4. (b), Theorem 4. 1), in the subcategory \((\mathcal{B}_F)_1 \subseteq \mathcal{B}_F\) there exist direct and inverse limits of the corresponding systems. However, we need a more special and somewhat more general results.

**Lemma 1.** There exist the direct limit functors
\[
\lim : (\text{seq}-i\mathcal{N}_F)_1 \rightarrow (\mathcal{N}_F)_1 \quad \text{and}
\lim : (\text{seq}-i\mathcal{N}_F)^1 \rightarrow (\mathcal{N}_F)^1
\]
such that, for every \(X = (X_n, i_{nn'}, \mathbb{N}) \in \text{Ob}(\text{seq}-i\mathcal{N}_F)\),
\[
\lim X \equiv (X, i_n)
\]
Furthermore, for every $f \in (\text{seq-}iN_F)^1(X, X')$, \(\lim f\) belongs to \((N_F)_1\). Furthermore, for every $f \in (\text{seq-}iN_F)(X, X')$, if $f$ admits a representative $(\phi, f_n)$ such that, for every $n \in \mathbb{N}$, $f_n$ is an isometry (isometric isomorphism), then

\[
\lim f : \lim X \to \lim X'
\]

is an isometry (isometric isomorphism).

Proof. Let a direct sequence $X = (X_n, i_{nn'}, \mathbb{N})$ in $iN_F$ be given. Consider the disjoint union $\sqcup_{n \in \mathbb{N}} X_n$ and the binary relation on it defined by

\[x_n \sim x'_{n'} \iff (i_{nn'}(x_n) = x'_{n'} \lor i_{n'n}(x'_{n'}) = x_n),\]

where $x_n \in X_n$ and $x'_{n'} \in X'_{n'}$. One readily verifies that $\sim$ is an equivalence relation on $\sqcup_{n \in \mathbb{N}} X_n$. Let

\[X = (\sqcup_{n \in \mathbb{N}} X_n) / \sim\]

be the corresponding quotient set. Since all $i_{nn'}$ are monomorphisms, for every $x = [x_n] \in X$, there exist a unique (minimal) $n(x) \in \mathbb{N}$ and a unique $x_{n(x)} \in X_{n(x)}$ (the grain of $x$) such that

\[x = [x_{n(x)}] \land x_{n(x)} \notin i_{n(x)-1,n(x)}[X_{n(x)-1}] \subseteq X_{n(x)}, (X_0 \equiv \emptyset).\]

Furthermore, for every $x \in X$ and every $n \geq n(x)$, there is a unique $x_n = i_{n(x)n}(x_{n(x)}) \in X_n$ such that $[x_n] = [x_{n(x)}] = x$. And conversely, for every $n$ and every $x_n \in X_n$, there is a unique $x = [x_n] \in X$ having the grain $x_{n(x)} \in X_{n(x)}$, $x_{n(x)} \sim x_n$ and $n(x) \leq n$. Consequently, every element $x \in X$ is a unique sequence $(i_{n(x)n}(x_{n(x)}))_{n \geq n(x)}$, which may be identified with the vector $x_{n(x)} \in X_{n(x)} \setminus R(i_{n(x)-1,n(x)})$ as well as with the vector $x_n \in X_n \setminus R(i_{n-1,n})$, $n \geq n(x)$, and vice versa. Given any $x' = [x'_{n'}], x'' = [x''_{n''}] \in X$, let us consider

\[x_{n2} = i_{n1n2}(y_{n1}) + y_{n2} \in X_{n2}\]

where $n_1 = \min\{n', n''\}$, $n_2 = \max\{n', n''\}$, \(\{y_{n1}, y_{n2}\} = \{x'_{n'}, x''_{n''}\}\) and “+” on the right side is the addition in $X_{n2}$. Then, for every $n \geq n_2$,

\[x_n = x'_{n} + x''_{n} \sim i_{n1n2}(y_{n1}) + y_{n2} = x_{n2}.\]

This shows that one can well define

\[+ : X \times X \to X, (+', x'') \equiv x' + x'' = x = [x_{n2}], \in X.\]

(Notice that $n(x) = n_{x'x''}$ formally depending on $x'$ and $x''$, actually depends on the $x' + x''$ only, i.e., it is the unique $n(x' + x'')$. Namely, if $x_1 + x_2 = x' + x'' = x$, then one readily sees that $n_{x_1x_2} = n_{x'x''} = n(x) \leq \max\{n_{x'}, n_{x''}\}$.) It is now a routine to verify that $(X, +)$ is an Abelian group. (For instance, in order to verify that $(x' + x'') + x''' = x' + (x'' + x''')$, consider $n = \max\{n_{x'}, n_{x''}, n_{x'''}\} \geq n_{x'' + x'''} + n_{x'' + x'''}$.) Further, given an $x = [x_n] = [x_{n(x)}] \in X$ and a $\lambda \in F$, then $\lambda x_n \in X_n$, $\lambda x_{n(x)} \in X_{n(x)}$ and $[\lambda x_n] = [\lambda x_{n(x)}]$. It allows us to define

\[\cdot : X \times F \to X, (x, \lambda) \equiv \lambda x = [\lambda x_{n(x)}].\]
One straightforwardly verifies that $X$ with so defined operations “$+$” and “$.$” is a vectorial space over $F$. (Notice that, $n_{\lambda x} \leq n_x$; in order to verify that $\lambda(x' + x'') = \lambda x' + \lambda x''$, consider $n = \max\{n_{x'}, n_{x''}\} \geq n_{x' + x''}, n_{\lambda x'}, n_{\lambda x''}$, while for $\mu(\lambda x) = (\mu\lambda)x$, consider $n = n_x \geq n_{\lambda x}, n_{(\mu\lambda)x'}$.)

Finally, let us define

$$ || \cdot || : X \to \mathbb{R}, \quad ||x|| = ||x_{n(x)}||_{n(x)}, $$

where $x = [x_{n(x)}]$ and $x_{n(x)} \in X_{n(x)}$ is the grain of $x$. Then, clearly,

$$ ||x|| = ||x_{n(x)}||_{n(x)} = ||x_n||, \quad n \geq n(x). $$

(The function $|| \cdot ||$ uniquely extends the sequence $(|| \cdot ||_n)$ of all the norms $|| \cdot ||_n$ on $X_n$ to $X$.) Again, one readily verifies that $|| \cdot ||$ is a well defined norm on $X$. For instance, given any $x', x'' \in X$, then (since all $i_{nm'}$ are isometries)

$$ \|x' + x''\| = \|i_{n1n2}(y_{n1}) + y_{n2}'\|_{n2} \leq \|i_{n1n2}(y_{n1})\|_{n2} + \|y_{n2}'\|_{n2} =$$

$$ = \|y_{n1}'\|_{n1} + \|y_{n2}'\|_{n2} = \|x_{n(x')}\|_{n(x')} + \|x_{n(x'')}\|_{n(x'')} = \|x'\| + \|x''\|, $$

that proves the triangle inequality. Thus, $X \equiv (X, || \cdot ||)$ is a normed space over $F$. Let us now define, for every $n \in \mathbb{N}$,

$$ i_n : X_n \to X, \quad i_n(x_n) = x = [x_n]. $$

Then each $i_n$ is linear and, by definition of $\sim$, for every related pair $n \leq n'$, $i_{nm'}i_{n''} = i_n$ holds. Further, $i_n$ is an isometry (and hence, continuous) because

$$ ||i_n(x_n)|| = ||x|| = ||x_{n(x)}||_{n(x)} = ||x||_n. $$

We have to prove the universal property of $(X, i_n)$ and $X$ with respect to $i_{N_F}$, $(N_F)_1$ and $(N_F)^1$. Let, for every $n \in \mathbb{N}$, an isometry $f_n : X_n \to Y$ (a morphism $f_n : X_n \to Y$ of $(N_F)_1$; of $(N_F)^1$) be given such that

$$ f_n : X \to Y, \quad f(x) = f_n(x_{n(x)}), $$

where $x_{n(x)} \in X_{n(x)}$ is the grain of $x$. Then, for every $n \geq n(x)$, $f(x) = f_n(x_{n(x)})$. Clearly, the function $f$ is well defined and linear, and, for every $n \in \mathbb{N}$, $fi_n = f_n$ holds. Further, for every $x \in X$,

$$ ||f(x)||_Y = ||f_n(x_{n(x)})||_Y = \|x_{n(x)}\|_{n(x)} = ||x|| $$

$$ (||f(x)||_Y = ||f_n(x_{n(x)})||_Y \leq ||x_{n(x)}||_{n(x)} = ||x||; $$

$$ ||f(x)||_Y = ||f_n(x_{n(x)})||_Y \leq ||f_n(x)|| \cdot ||x_{n(x)}||_{n(x)} \leq ||x_{n(x)}||_{n(x)} = ||x|| $$

implying that $f$ is an isometry (a morphism of $(N_F)_1$, of $(N_F)_1 \subseteq (N_F)^1$). Further, assume that $f' : X \to Y$ is any morphism of $N_F$ such that, for every $n$, $f'i_n = f_n$. Then, for every $x \in X$,

$$ f'(x) = f'(i_n(x_{n(x)})) = f_n(x_{n(x)}) = f(x), $$

implying that $f' = f$. Therefore, $(X, i_n) = \varinjlim X$ in $i_{N_F}$, in $(N_F)_1$ and in $(N_F)^1$ (up to isomorphisms of the category $i_{N_F}$). The constructed
direct limit \((X, i_n)\) of \(X\) is said to be the canonical one. In order to extend this \(\lim\) to a functor, let firstly an
\[
f = [(\phi, f_n)] \in (\text{seq-i}\mathcal{N}_F)_1(X, X')
\]
be given. We may assume that \((\phi, f_n) : X \to X'\) is a special representative of \(f\) (the dual of [11], Lemma I. 1. 2), i.e., that \(\phi\) is increasing and
\[
f_n' i_{n'} = i_{\phi(n)\phi(n')} f_n, \quad n \leq n'.
\]
Let \((X, i_n) = \lim X\) and \((X', i'_n) = \lim X'\) be the canonical limits. We define
\[
f : X \to X', \quad f(x) = i'_{\phi(n)(x)} f_n(x_{\phi(n)}(x))
\]
(equivalently, \(f(x) = i'_{\phi(n)} f_n(x_n), \quad x = [x_n]\)).

Then \(f\) is a well defined linear function satisfying
\[
f i_n = i'_{\phi(n)} f_n, \quad n \in \mathbb{N}.
\]
Further, since all the \(i'_n\) are isometries, and for all \(n\) and all \(x_n \in X_n\),
\[
\|f_n(x_n)\|_n = \|x_n\|_n\]
holds, it follows that, for every \(x = [x_n] \in X\),
\[
\|f(x)\|' = \|i'_{\phi(n)} f_n(x_n)\|' = \|f_n(x_n)\|'_{\phi(n)} \leq \|x_n\|_n = \|x\|.
\]
Hence, \(f \in (\mathcal{N}_F)_1(X, X')\). Let now an \(f = [(\phi, f_n)] \in (\text{seq-i}\mathcal{N}_F)_1(X, X')\) be given. By assuming that \((\phi, f_n)\) is a special representative as before with \(\|f_n\| \leq 1\) for all \(n\), it follows that
\[
\|f(x)\|' = \|i'_{\phi(n)} f_n(x_n)\|' = \|f_n(x_n)\|'_{\phi(n)} \leq \|f_n\| \cdot \|x_n\|_n \leq \|x_n\|_n = \|x\|
\]
implies also that \(f\) belongs to \((\mathcal{N}_F)_1 \subseteq (\mathcal{N}_F)^1\). Now, by putting \(\lim f = f\), one straightforwardly shows that
\[
\lim : (\text{seq-i}\mathcal{N}_F)_1 \to (\mathcal{N}_F)_1 \quad \text{and}
\lim : (\text{seq-i}\mathcal{N}_F)^1 \to (\mathcal{N}_F)^1
\]
are functors, i.e., that \(\lim 1_X = 1_X = 1\lim X\) and \(\lim (gf) = (\lim g)(\lim f)\) hold true. Since we have already proven, by the very construction, that \(\lim X \equiv (X, i_n)\) belongs to \(i\mathcal{N}_F\), it remains to verify the last statement. Let an
\[
f = [(\phi, f_n)] \in (\text{seq-i}\mathcal{N}_F)(X, X')
\]
be given such that all
\[
f_n : X_n \to X'_{\phi(n)}
\]
are isometries. Then, for every \(x = [x_n] \in X\),
\[
\|f(x)\|' = \|i'_{\phi(n)} f_n(x_n)\|' = \|f_n(x_n)\|'_{\phi(n)} = \|x_n\|_n = \|x\|.
\]
Therefore, \(f\) is an isometry. Finally, if all \(f_n\) are isometric isomorphisms, then \(f\) belongs to \((\text{seq-i}\mathcal{N}_F)_1\), implying that \(f \equiv \lim f\) is an isomorphism of \((\mathcal{N}_F)_1\). Therefore, \(f\) is an isometric isomorphism, and the proof of the lemma is finished. \(\square\)
Lemma 2. Let $X = (X_n, i_{nn'}, N)$ be a direct sequence in $iN_F$ such that, for every $n$, the bonding morphism $i_{nn+1}$ is a section of $(N_F)_1$. Then every limit morphism $i_n : X_n \to \lim X$ is a section of $(N_F)_1$. If, in addition, $X_n = X$ for all $n$, the $X$ is dominated by $\lim X$ in $(N_F)_1$.

Proof. Given such an $X = (X_n, i_{nn'}, N)$ in $iN_F$, every $i_{nn+1}$ admits a retraction

$$r_{n,n+1} : X_{n+1} \to X_n$$

of $(N_F)_1$, i.e., $r_{nn+1}i_{nn+1} = 1_X$ and $\|r_{nn+1}(x_{n+1})\| \leq \|x_{n+1}\|$ (having $\|r_{n+1}\| = 1$ whenever $X_n \neq \{0\}$). Denote, for every related pair $n \leq n'$,

$$r_{nn'} ≡ r_{nn+1} \cdots r_{n'-1,n'} : X_{n'} \to X_n \quad (r_{nn} = 1_X).$$

Let $n_0 \in N$ be chosen arbitrarily. Put

$$r_n^{n_0} : X_n \to X_{n_0}, \quad r_n^{n_0} = \begin{cases} i_{n_0n}, & n \leq n_0 \\ r_{n_0n}, & n > n_0 \end{cases}.$$

Then, for every $n$,

$$r_{n+1}^{n+1} = r_n^{n_0} \circ r_n^{n_0} : X_{n+1} \to X_n$$

holds. By Lemma 1, there exists (in $(N_F)_1$) an

$$r_{n_0} : \lim X \to X_{n_0}, \quad r_{n_0}i_n = r_n^{n_0}, \quad n \in N.$$

Then especially, $r_{n_0}i_{n_0} = i_{n_0n_0} = 1_{X_{n_0}}$, and the conclusion follows. \(\square\)

Lemma 3. Let $X = (X_n', i_{nn'}, N)$ and $X'' = (X'_n, i''_{nn'}, N)$ be direct sequences in $iN_F$ and let $f = [(\phi, f_n)] \in (\text{seq}-iN_F)^1(X', X'')$ such that all $f_n$ are isomorphisms of $N_F$. Then $\lim f$ is an isomorphism of $N_F$ and there exists $\lim (f^{-1})$ such that

$$\lim (f^{-1}) = (\lim f)^{-1}.$$

Proof. We may assume, without loss of generality, that $(\phi, f_n)$ is a special representative of $f$ with $\|f_n\| \leq 1$ for all $n$. By Lemma 1, there exists

$$\lim f \equiv f : \lim X' \to \lim X''$$

and $f$ belongs to $(N_F)_1$. Further, since, for every $n$, $fi_n' = i''_{n}f_n$ and $i_n'$, $i''_n$ are the isometries, it readily follows that $\|f_n\| \leq \|f\| \leq 1$. We are to prove that $f$ is an isomorphism of $N_F$. Since all $i_{nn'}'$, $i_{nn''}$, and $f_n$ are monomorphisms, the construction of the canonical limit implies that $\lim f$ is a monomorphism. Let $x'' \in X'' = \lim X''$. Then there exists a unique $x''_{n(x'')} \in X''_{n(x'')} \supseteq X'_n$ such that $x'' = [x''_{n(x'')}].$ Choose an $n \in N$ such that $\phi(n) \geq n(x'')$. Since $f_n$ is an epimorphism, there exists an $x'_n \in X'_n$ such that $f_n(x'_n) = x''_{\phi(n)}$. Now, there exists a unique $x' = [x'_n] = i'(x'_n) \in X'$, and it follows, by the very definition of
\[ \lim f_n \rightarrow X' \] that \( f(x') = x'' \). Hence, \( f \) is an epimorphism. and consequently, an isomorphism. (If, especially, \( \|f_n\| = 1, n \geq n_0 \), then \( \|f\| = 1 \).) Let \( f^{-1} : X'' \rightarrow X' \) be the inverse of \( f \). Notice that the sequence \( (f_{n^{-1}}) \) induces the in-morphism \( f^{-1} = [\psi, f^{-1}_{i''_{\phi(n)}}] : X'' \rightarrow X' \).

Let \( f^{-1} : X'' \rightarrow X' \) be the inverse of \( f \). (Caution: In general, \( f^{-1} \) does not belong to \( (N_F)^{1} \supseteq (N_F)_1 \)) One readily verifies (by our construction of the direct limit) that, for every \( n \),

\[ f^{-1}_{i''_{\phi(n)}} = i'_{n}f_{n^{-1}} \]

holds true. Hence, \( \lim_j(f^{-1}) = f^{-1} \). (Notice that \( i''_{\phi(n)\phi(n+1)}f_n = f_{n+1}i''_{nn+1} \) implies that the sequence \( (\|f_n\|) \) in \( [\|f_1\|, 1] \subseteq \mathbb{R} \) is increasing and bounded, and, further, that every “restriction \( f\big|_{X''} \) carries the norm of \( f_n \). Therefore, one may say that \( \|\lim_j f\| = \|f\| = \lim(\|f_n\|) \)

Further, we show that the functors \( D^{2k} \) preserve the direct limits of direct sequences in \( iN_F \).

**Lemma 4.** For each \( k \in \{0\} \cup \mathbb{N} \), there exist the direct limit functors

\[ \lim : (\text{seq}-D^{2k}[iN_F])_1 \rightarrow D^{2k}[(N_F)_1] \quad \text{and} \quad \lim : (\text{seq}-D^{2k}[iN_F])^1 \rightarrow D^{2k}[(N_F)^1] \]

such that, for every \( X = (X_n, i_{nn'}, N) \in \text{Ob(seq-iN_F)} \),

\[ \lim D^{2k}[X] \equiv (X', i'_{n}) \cong (D^{2k}(X), D^{2k}(i_{n})) \]

belongs to \( D^{2k}[iN_F] \). Furthermore, for every \( f \in (\text{seq}-D^{2k}[iN_F])(X, X') \), if \( f \) admits a representative \( (\phi, f_n) \) such that, for every \( n \in \mathbb{N} \), \( f_n \) is an isometry (isometric isomorphism), then

\[ \lim_j f : \lim_j X \rightarrow \lim_j X' \]

is an isometry (isometric isomorphism).

**Proof.** Clearly, every direct sequence in \( D^{2k}[iN_F] \) is of the form \( D^{2k}[X] = (D^{2k}(X_n), D^{2k}(i_{nn'}), N) \), where \( X = (X_n, i_{nn'}, N) \) is a direct sequence in \( iN_F \). Since, by Lemma 1 (i) of [16], all \( D^{2k}(i_{nn'}) \) are isometries, every such direct sequence \( D^{2k}[X] \) belongs to \( iN_F \) as well. By Lemma 1, the direct limit

\[ \lim D^{2k}[X] = (X', i'_{n}), X' = ((\cup_{n \in \mathbb{N}} D^{2k}(X_n))/\sim, \|\cdot\|') \]

exists in \( iN_F \) and has the universal property with respect to \( iiN_F \), \( (N_F)_1 \) and \( (N_F)^1 \). We are to prove that \( (D^{2k}(X), D^{2k}(i_n)) \) is a direct limit of \( D^{2k}[X] \) in \( D^{2k}[iN_F] \), in \( D^{2k}[(N_F)_1] \) and in \( D^{2k}[(N_F)^1] \) (implying that \( X' \) is isomorphic to \( D^{2k}(X) \) in \( iN_F \), and hence, a Banach space). Firstly, since \( j^{2k} : 1_{N_F} \sim D^{2k} \) is a natural transformation of the
functors, by applying \( D^2k \) to \( X \) and \( \lim X \), the following commutative diagram

\[
\begin{array}{ccccccc}
X_1 & \xrightarrow{i_{12}} & \cdots & \xrightarrow{i_{nn'}} & X_n & \xrightarrow{i_{nn'+1}} & X_{n+1} & \xrightarrow{\cdots} & X \\
\downarrow j_{X_1}^{2k} & \cdots & \downarrow j_{X_n}^{2k} & \downarrow j_{X_{n+1}}^{2k} & \cdots & \downarrow j_{X}^{2k} & \\
D^2k(X_1) & \xrightarrow{D^2k(i_{12})} & \cdots & \xrightarrow{D^2k(i_{nn'})} & D^2k(X_n) & \xrightarrow{D^2k(i_{nn'+1})} & D^2k(X_{n+1}) & \xrightarrow{\cdots} & D^2k(X) \\
\end{array}
\]

in \( iN_F \) occurs, and also

\[
D^2k(i_{n'})D^2k(i_{nn'}) = D^2k(i_n), \quad D^2k(i_n)j_{X_n}^{2k} = j_{X}^{2k}i_n,
\]

whenever \( n \leq n' \). Secondly, we are verifying the universal property of \( (D^2(X), D^2(i_n)) \) and \( D^2([X]) \) with respect to the categories \( D^2k[iN_F] \), \( D^2k([N_F]) \) and \( D^2k([N_F]) \). Let, for every \( n \in \mathbb{N} \), a morphism \( D^2k(f_n) : D^2kX_n \to D^2k(Y) \) of \( D^2k[iN_F] \) (of \( D^2k([N_F]) \)); of \( D^2k([N_F]) \) be given such that

\[
D^2k(f_n')D^2k(i_{nn'}) = D^2k(f_n), \quad n \leq n',
\]

holds. Since each \( D^2k \) is a faithful functor, it follows that \( f_n' = f_n' \), \( n \leq n' \). By Lemma 1 (the case \( k = 0 \)), there exists a unique \( f : X \to Y \) of \( iN_F \) of \( (N_F)_1 \); of \( (N_F)_1 \) such that, for every \( n \in \mathbb{N} \), \( f_{i_n} = f_n \). Then \( D^2k(f) : D^2k(X) \to D^2k(Y) \) belongs to \( D^2k[iN_F] \) (of \( D^2k([N_F]) \); to \( D^2k([N_F]) \)) and, for every \( n \),

\[
D^2k(f_n)D^2k(i_n) = D^2k(f_n).
\]

If \( D^2(f') : D^2(X) \to D^2(Y) \) is any morphism of \( D^2k[iN_F] \) (of \( D^2k([N_F]) \); of \( D^2k([N_F]) \)) such that \( D^2(f')D^2(i_n) = D^2(f_n), \quad n \in \mathbb{N} \), then

\[
D^2(f'i_n) = D^2(f_n) = D^2(f_n)
\]

implying

\[
f'i_n = f_n = f_{i_n}, \quad n \in \mathbb{N}.
\]

Since \( f \) is unique having that property, it follows that \( f' = f \), and thus, \( D^2k(f') = D^2k(f) \), implying the uniqueness of \( D^2k(f) \) in \( D^2k[iN_F] \) (in \( D^2k([N_F]) \)), in \( D^2k([N_F]) \)). Therefore, \( (D^2k(X), D^2k(i_n)) \) is a direct limit of \( D^2k([X]) \) in \( D^2k[iN_F] \), in \( D^2k([N_F]) \); in \( D^2k([N_F]) \). Consequently, by construction of the object of the canonical direct limit of a direct sequence in \( iN_F \), it follows that \( X' \cong D^2k(X) \) in \( D^2k[iN_F] \subseteq iB_F \), and the statement for objects follows in general. Concerning the morphisms, let firstly an

\[
f = ((\phi, f_n) \in (\text{seq } D^2k[iN_F])_1(D^2k([X]), D^2k([X']))\]

be given. Then we define

\[
f = \lim_{\longrightarrow} f : \lim_{\longrightarrow} D^2k([X]) \to \lim_{\longrightarrow} D^2k([X'])
\]

as in the proof of Lemma 1, and the functoriality of \( \lim_{\longrightarrow} \) follows straightforwardly. The same holds true for an

\[
f = ((\phi, f_n) \in (\text{seq } D^2k([N_F])_1(D^2k([X]), D^2k([X']))\]

Finally, since \( D^2k \) preserves isometries and isomorphisms, if every \( f_n \) is an isometry (isometric isomorphism) then, as in the proof of Lemma 1, \( \lim_{\longrightarrow} f \) is an isometry (isometric isomorphism) as well.
Theorem 1. (i) Each restriction functor
\[ D^{2k} : i\mathcal{N}_F \to D^{2k}[i\mathcal{N}_F] \subseteq i\mathcal{B}_F, \ k \in \mathbb{N}, \]
preserves directedness of direct sequences and it is continuous, i.e., it commutes with the direct limit:
\[ D^{2k}(\lim_{\to} X) \cong \lim_{\to} D^{2k}[X] \text{ isometrically}; \]
(ii) Each restriction functor
\[ D^{2k-1} : i\mathcal{N}_F \to D^{2k-1}[i\mathcal{N}_F] \subseteq \mathcal{B}_F, \ k \in \mathbb{N}, \]
turns direct sequences into inverse sequences and their direct limits into the corresponding inverse limits, i.e.,
\[ D^{2k-1}(\lim_{\to} X) \cong \lim_{\to} D^{2k-1}[X] \text{ (isometrically in } D^{2k-1}[i\mathcal{N}_F]); \]
(iii) Each restriction functor
\[ D^{2k} : D^{2l-1}[i\mathcal{N}_F] \to D^{2k+2l-1}[i\mathcal{N}_F] \subseteq \mathcal{B}_F, \ k, l \in \mathbb{N}, \]
preserves inverseness of inverse sequences and commutes with inverse limits, i.e.,
\[ D^{2k}(\lim_{\to} D^{2l-1}[X]) \cong \lim_{\to} D^{2k}[D^{2l-1}[X]] = \lim_{\to} D^{2k+2l-1}[X] \]
(isometrically in \( D^{2k+2l-1}[i\mathcal{N}_F] \)).

Proof. (i). Firstly, by Lemma 1, the needed direct limits exist. Furthermore, by Lemma 4 and its proof, if \( X \) is a direct sequence in \( i\mathcal{N}_F \) and \( X \equiv \lim_{\to} X \) in \( i\mathcal{N}_F \), then, for every \( k \in \mathbb{N}, \)
\[ D^{2k}(\lim_{\to} X) \cong \lim_{\to} D^{2k}[X] \]
in \( D^{2k}[i\mathcal{N}_F] \) holds. Consequently, \( D^{2k}(\lim_{\to} X) \cong \lim_{\to} D^{2k}[X] \) isometrically.

(ii). Let \( k \in \mathbb{N} \), and let \( (X, i_n) \) be a direct limit (not necessarily canonical) of a direct sequence \( X = (X_n, i_{nn'}, \mathbb{N}) \) in \( i\mathcal{N}_F \). Then \( D^{2k-1}X \equiv (D^{2k-1}(X_n), D^{2k-1}(i_{nn'}), \mathbb{N}) \) is an inverse sequence in \( D^{2k-1}[i\mathcal{N}_F] \subseteq \mathcal{B}_F \) and there exist morphisms
\[ D^{2k-1}(i_n) : D^{2k-1}(X) \to D^{2k-1}(X_n), \ n \in \mathbb{N}, \]
of \( D^{2k-1}[i\mathcal{N}_F] \) such that
\[ D^{2k-1}(i_{nn'})D^{2k-1}(i_{n'}) = D^{2k-1}(i_n), \ n \leq n'. \]
We are to verify the universal property of \( (D^{2k-1}(X), D^{2k-1}(i_n)) \) and \( D^{2k-1}[X] \) with respect to the category \( D^{2k-1}[i\mathcal{N}_F] \). Let, for every \( n \in \mathbb{N} \), a morphism \( D^{2k-1}(f_n) : D^{2k-1}(Y) \to D^{2k-1}(X_n) \) of \( D^{2k-1}[i\mathcal{N}_F] \) be given such that
\[ D^{2k-1}(i_{nn'})D^{2k-1}(f_{n'}) = D^{2k-1}(f_n), \ n \leq n'. \]
Then, for every \( n \in \mathbb{N} \),
\[ D^{2k-1}(f_{n'i_{nn'}}) = D^{2k-1}(f_n) : D^{2k-1}(Y) \to D^{2k-1}(X_n), \]
and it follows that \( f_{n'i_{nn'}} = f_n \), because the functor \( D^{2k-1} \) is faithful. By the universal property of \( (X, i_n) \) and \( X \) with respect to \( i\mathcal{N}_F \), there exists a unique \( f : X \to Y \) of \( i\mathcal{N}_F \) such that \( fi_n = f_n, \ n \in \mathbb{N} \). Then \( D^{2k-1}(f) : D^{2k-1}(Y) \to D^{2k-1}(X) \) belongs to \( D^{2k-1}[i\mathcal{N}_F] \) and
Finally, let \( D^{2k-1}(i_n)D^{2k-1}(f) = D^{2k-1}(f_n), \) \( n \in \mathbb{N}. \)

Finally, let \( D^{2k-1}(f') : D^{2k-1}(Y) \to D^{2k-1}(X) \) be any morphism of \( D^{2k-1}(i_N) \) such that, for every \( n, D^{2k-1}(i_n)D^{2k-1}(f') = D^{2k-1}(f_n) \) holds. Then

\[
D^{2k-1}(f'_i) = D^{2k-1}(f_n) = D^{2k-1}(f) \quad \text{implying}
\]

\[
f'_i = f_n, \quad n \in \mathbb{N}.
\]

Since \( f \) is unique having that property, it follows that \( f' = f \), and thus,

\[
D^{2k-1}(f') = D^{2k-1}(f),
\]

implying the uniqueness of \( D^{2k-1}(f) \) in \( D^{2k-1}[i_N] \). Therefore, \( (D^{2k-1}(X), D^{2k-1}(i_n)) = \lim D^{2k-1}[X] \) in \( D^{2k-1}[i_N] \) (up to an isomorphism of \( D^{2k-1}[i_N] \)), and the conclusion follows.

(iii). Consider the simplest case, i.e. \( l = k = 1 \), i.e., the restriction functor

\[
D^2 : D[i_N] \to D^3[i_N] = D^2[D[i_N]].
\]

Clearly, every inverse sequence in \( D[i_N] \) is of the form \( D[X] = (D(X_n), D(i_{nn'}), \mathbb{N}), \) where \( X = (X_n, i_{nn'}, \mathbb{N}) \) is a direct sequence in \( i_N \). By (ii),

\[
\prod D[X] \cong \prod D[i_N].
\]

Then, by (i) and (ii),

\[
D(\prod D[X]) \cong D^2(\prod D[i_N]) = D^3(\prod D[X]) \cong \lim D^3[X]
\]

in \( D^3[i_N] \). The general case follows in a quite similar way. \( \square \)

We shall also need a special case of the following general fact.

**Lemma 5.** Let \( i' \in i_B(X', Y') \) and \( i'' \in i_B(X'', Y'') \) yield the closed direct-sum presentations \( Y' = R(i') + Z' \) and \( Y'' = R(i'') + Z'', \) respectively, such that \( Z' \) continuously linearly embeds into \( Z''. \) Then every \( f \in B_{i_B}(X', X'') \) with \( \|f\| < 1 \) admits an extension \( g \in B_{i_B}(Y', Y''), \)

\[
g \equiv i''f, \quad \|g\| < 1.
\]

In addition, if \( f \) is an isomorphism and \( Z' \cong Z'', \) then there exists an extending isomorphism \( g. \)

**Proof.** Since \( i' \) and \( i'' \) are isometries, the morphism

\[
u \equiv i''f(i')^{-1} : R(i') \to R(i'')
\]

of \( B_{i_B} \) is well defined, and \( \|u\| = \|f\|. \) By the assumptions on the isometries \( i' \) and \( i'' \), each \( y' \in Y' \) \((y'' \in Y'') \) admits a unique presentation

\[
y' = i'(x') + z', \quad x' \in X', \quad z' \in Z'
\]

\[
y'' = i''(x'') + z'', \quad x'' \in X'', \quad z'' \in Z''.
\]

Since \( \|f\| < 1 \) and \( Z' \) admits a continuous linear embedding into \( Z'', \) there exists a continuous linear embedding

\[
v : Z' \to Z'', \quad \|v\| < 1 - \|f\|.
\]

Then by

\[
y = i'(x') + z' \mapsto u(i'(x')) + v(z') \equiv g(y)
\]

a function \( g : Y' \to Y'' \) is well defined. One readily verifies that \( g \) is linear. Since \( g = u + v, \) the Inverse Mapping Theorem (applied to the identity functions on the both direct-sums and the corresponding
direct products with the norm $\|\cdot\|_1$ implies that $g$ is continuous, i.e.,
$g \in B_F(Y', Y'')$. The extension property (commutativity) $gi'' = i''f$
holds obviously. Finally,
$$\|g\| = \|u + v\| \leq \|(u, v)\|_1 = \|u\| + \|v\| < \|f\| + 1 - \|f\| = 1.$$ 
If, in addition, $f$ is an isomorphism and $Z' \cong Z''$, then one can choose $v$
to be an appropriate isomorphism with $\|v\| < 1 - \|f\|$, and the conclusion follows.
\[\square\]

4. The hyperdual functor

Let $X$ be a normed vectorial space over $F \in \{\mathbb{R}, \mathbb{C}\}$ and let $k \in$
$\{0\} \cup \mathbb{N}$. By simplifying notations, let
$$j_{2k} : D^{2k}(X) \rightarrow D^{2k+2}(X)$$
denote the canonical embedding $j_{D^{2k}(X)}$. Since every $j_{2k}$ is an isometry,
the direct sequence
$$\tilde{X} \equiv D^{2k}(X) = (D^{2k}(X), j_{2k}, \{0\} \cup \mathbb{N}), \ i.e.,
X \xrightarrow{\hat{h}_{2k}} D^{2k+2}(X) \rightarrow \cdots \xrightarrow{\hat{h}_{2k+2}} D^{2k+4}(X) \rightarrow \cdots,
$$
in $\mathcal{N}_F$ occurs.

**Definition 1.** Given a normed space $X$, a normed space $\tilde{X}$ is said to
be a hyperdual of $X$ if
(i) $(\forall k \in \{0\} \cup \mathbb{N})$ there exists an isometry $i_{2k} : D^{2k}(X) \rightarrow \tilde{X}$;
(ii) for every normed space $Y$ and every sequence $(f_{2k}, f_{2k} \in (\mathcal{N}_F)^1(D^{2k}(X), Y)$
satisfying $f_{2k+2}j_{D^{2k}(X)} = f_{2k}$, there exists a unique $f \in (\mathcal{N}_F)^1(\tilde{X}, Y)$
(equivalently, $f \in (\mathcal{N}_F)^1(\tilde{X}, Y)$) such that $fj_{2k} = f_{2k}$.

According to Lemma 1, every normed space has a hyperdual, and
moreover, all hyperduals of an $\tilde{X}$ are mutually isometrically isomorphic.

Recall that a normed space $X$ is said to be reflexive, if the canonical
embedding $j_X : X \rightarrow D^2(X)$ is an epimorphism, i.e., if $j_X$ is an
isometric isomorphism (isomorphism of $(\mathcal{N}_F)^1$). Then, clearly, $X$ itself
must be a Banach space. It is well known that $X$ is reflexive if
and only if $D^n(X)$ (for some, equivalently, every $n$) is reflexive. Obviously, $X$ is reflexive if
and only if, it is isomorphic to a reflexive space.

In [15], Lemma 4, the notion of a bidual-likeness was introduced by
$D^2(X) \cong X$ in $\mathcal{N}_F$. We shall hereby repeat and strengthen the definition.
Before that, for the sake of completeness, recall briefly (see [1, 2, 4]) that a normed space $X$ is said to be somewhat reflexive
(quasi-reflexive (of order $n$)) if, for every infinite-dimensional closed subspace
$W \subseteq X$, there exists a reflexive infinite-dimensional closed subspace
of $Z \subseteq W$ (if the quotient space $D^2(X)/R(j_X)$ is finite-dimensional
($\dim(D^2(X)/R(j_X)) = n$)). Clearly, the quasi-reflexivity of order 0
means reflexivity.
Definition 2. A normed space $X$ is said to be bidualic (parareflexive), if $X \cong D^2(X)$ (isometrically). $X$ is said to be almost reflexive if it is parareflexive and somewhat reflexive.

Example 1. All the spaces $l_p$ and $L_p(n)$, $1 < p < \infty$, are reflexive separable Banach spaces; James’ space $J$ of [8] is a non-reflexive almost reflexive separable Banach space; the spaces $l_1$ and $c_0 \subseteq l_\infty$ are non-bidualic separable Banach spaces, while $l_\infty$ is a non-bidualic and non-separable Banach space.

One easily sees that a normed space $X$ is bidualic (parareflexive) if and only if, it is (isometrically) isomorphic to a bidualic (parareflexive) space. The following facts are almost obvious.

Lemma 6. Let $X$ be a normed space $X$. If
(i) $X$ is bidualic (parareflexive), then $X$ is a Banach space and, for every $n \in \mathbb{N}$, $D^{2n}(X) \cong X$ and $D^{2n+1}(X) \cong D(X)$ (isometrically) and $D^n(X)$ is bidualic (parareflexive);
(ii) $X$ is almost reflexive, then $X$ is a Banach space and, for every $n \in \mathbb{N}$, $D^{2n}(X) \cong X$ and $D^{2n+1}(X) \cong D(X)$ isometrically, and $D^{2n}(X)$ is almost reflexive.
(iii) None parareflexive non-reflexive space can be isometrically embedded into any reflexive space.

Proof. Concerning statement (i), recall that every continuous linear function is uniformly continuous, and thus, it preserves Cauchy sequences. The rest is obvious. Concerning statement (ii), one has to verify that $D^{2n}(X)$ is somewhat reflexive, whenever $X$ is almost reflexive. However, it is an immediate consequence of $D^2(X) \cong X$ isometrically and [16], Lemma 1 (i).

(iii). Let $X$ be a parareflexive space that is not reflexive (such is, for instance, James’ space $J$ of [8], as well as all the $l_p$ and $L_p$ spaces, $1 < p < \infty$). Let $Y$ be any normed space that admits an isometry $f : X \to Y$. We have to prove that $Y$ cannot be reflexive. Assume to the contrary, and consider the following commutative diagram

\[
\begin{array}{cccc}
X & \xrightarrow{f'} & R(f) \\
D^2(X) & \downarrow{j_X} & \downarrow{j_{R(f)}} & D^2(R(f)) \\
& \downarrow{D^2(f')} & & \\
& D^2(R(f)) & &
\end{array}
\]

in $\mathcal{NF}$, where $f' : X \to R(f)$ is the restriction of $f$. By (i), $f'$ is an isometric isomorphism of Banach spaces. Since $R(f)$, being closed in $Y$, is reflexive, it follows that $j_{R(f)} f$ is an isometric isomorphism. Then
$D^2(f')j_X = j_{R(f')}f$ is an isometric isomorphism as well, implying that so is $j_X$ - a contradiction.

Let $\rho N_F, \alpha N_F, \pi N_F, \sigma N_F, \beta N_F$ and $\chi N_F (\chi, N_F)$ denote the full subcategories of $N_F$ (actually, of $B_F$) determined by all the reflexive, almost reflexive, parareflexive, somewhat reflexive, bidualic and quasi-reflexiv (of order $n$) spaces, respectively. Clearly, $\rho N_F$ is a full subcategory of all the mentioned subcategories and $\alpha N_F \subseteq \pi N_F \subseteq \beta N_F$ as well. Further, one readily sees that $\chi, N_F \subseteq \chi N_F \subseteq \beta N_F$ also holds.

**Theorem 2.** For every $X \in Ob(N_F)$, every hyperdual $\tilde{X}$ of $X$ has the following properties:

(i) $\tilde{X}$ is a bidualic Banach space, i.e., $D^2(\tilde{X}) \cong \tilde{X}$ in $N_F$;

(ii) all $D^{2k}(X), k \in \mathbb{N}$, embed isometrically into $\tilde{X}$ making an increasing sequence of retracts and retracts of $\tilde{X}$ in $(B_F)_1$, implying that, for each $k \in \mathbb{N}$,

$$\tilde{X} \cong D^{2k}(X) + N(r_{2k})$$

isometrically;

(iii) $dim \tilde{X} = \begin{cases} 
\dim X, & \text{dim } X \neq \aleph_0 \\
2^{\aleph_0}, & \text{dim } X = \aleph_0 
\end{cases}$

**Proof.** Let an $X \in Ob(N_F)$ be given. According to Definition 2 and Lemma 1, it suffices to prove the statements for the canonical direct limit space $\tilde{X}$ of $X = (D^{2k}(X), j_{D^{2k}(X)} \equiv j_{2k}, \{0\} \cup \mathbb{N})$, i.e.,

$$\lim \tilde{X} = (X, i_{2k}) = (\sqcup_{k \geq 0} D^{2k}(X))/\sim, \|\cdot\|, i_{2k}),$$

where $\sim$ is induced by $(j_{2k})$ and the norm $\|\cdot\|$ uniquely extends the sequence $(\|\cdot\|_{2k})$ of norms $\|\cdot\|_{2k}$ on $D^{2k}(X)$ to $\tilde{X}$, while the limit morphisms into $\tilde{X}$ are the isometries $i_{2k} : D^{2k}(X) \to \tilde{X}$. Since $j : 1_N \to D^2$ is a natural transformation of the functors, by applying $D^2$ to $\tilde{X}$ and $\lim \tilde{X}$, the following commutative diagram

$$\begin{array}{ccccccc}
X & j_0 & \to & \cdots & \to & D^{2k}(X) & j_{2k} & D^{2k+2}(X) & \to & \cdots & \tilde{X} \\
\downarrow & j_0 & \cdots & \downarrow j_{2k} & \downarrow j_{2k+2} & \cdots & j_{\tilde{X}} \\
D^2(X) & D^2(j_0) & \to & \cdots & \to D^{2k+2}(X) & D^2(j_{2k}) & D^{2k+4}(X) & \to & \cdots & D^2(\tilde{X})
\end{array}$$

in $i_{N_F}$ occurs and $D^2(i_{2k})j_{2k} = j_{\tilde{X}}i_{2k}$. By Lemma 1 and its proof, the canonical direct limit of the direct sequence $D^2[\tilde{X}] \equiv (D^{2k+2}(X), D^2(j_{2k}), \{0\} \cup \mathbb{N})$ is

$$\lim D^2[\tilde{X}] = (X', i'_{2k+2}) \equiv ((\sqcup_{k \in \{0\}} D^{2k+2}(X))/\sim', \|\cdot\|', i'_{2k+2}).$$

By Theorem 1 (i), there exists an (isometric) isomorphism

$$g : \lim D^2[\tilde{X}] = X' \to D^2(\tilde{X}) = D^2(\lim X).$$

We are to prove that $\tilde{X}$ is (in-)isomorphic to $D^2[\tilde{X}]$ in $(\text{seq-i}_{N_F})^1$. Since all $j_{2k}$ are the canonical embeddings, Lemma 1 (i) of [16] assures that all $D^2(j_{2k})$ are closed isometric embeddings. Notice that by
excluding (including) $j_0$ off $\tilde{X}$ (into $D^2[\tilde{X}]$) nothing relevant for this consideration is changing. Let us exclude $j_0$ off $\tilde{X}$. By [16], Corollary 1, for every $k \in \mathbb{N}$, there exist the closed direct-sum presentations of $D^{2k+2}(X)$, induced by sections $j_{2k}$ and $D^2(j_{2k-1})$ (having $D(j_{2k-1})$ for a common retraction), with the same closed complementary subspace. More precisely,

$$D^{2k+2}(X) = \mathcal{R}(j_{2k}) + \mathcal{N}(D(j_{2k-1})) = \mathcal{R}(D^2(j_{2k-2}) + \mathcal{N}(D(j_{2k-1})).$$

Therefore, by starting with an isomorphism

$$f_2 : D^2(X) \to D^2(X), \|f_2\| < 1,$$

(for instance, $f_2 = \lambda l_{D^2(X)}$, $0 < \lambda < 1$), we may apply Lemma 5

$$(X' = X'' = D^2(X), Y' = Y'' = D^4(X), i' = j_2, i'' = D^2(j_0), \text{ and } Z' = Z'' = N(D(j_1))$$

and obtain an isomorphism

$$f_4 : D^4(X) \to D^4(X), \|f_4\| < 1,$$

which extends $f_2$, i.e.,

$$f_4 j_2 = D^2(j_0) f_2.$$

Continuing by induction, we obtain a sequence $(f_{2k})$ of isomorphisms

$$f_{2k} : D^{2k}(X) \to D^{2k}(X), \|f_{2k}\| < 1,$$

such that

$$f_{2k+2} j_{2k} = D^2(j_{2k-2}) f_{2k}$$

(commutating with the bonding morphisms of $\tilde{X}$ and $D^2[\tilde{X}]$). Then the sequence $(f_{2k})$ determines an in-(iso)morphism

$$f = [(1_N, f_n)] \in (\text{seq-}i_{\mathcal{N}_F})^1(\tilde{X}, D^2[\tilde{X}]), 2k \mapsto k \equiv n,$$

having all $f_n$ to be isomorphisms with $\|f_n\| < 1$. Now, by Lemma 3, the existing limit morphism

$$f \equiv \lim f : \lim \tilde{X} = X \to X' = \lim D^2[\tilde{X}].$$

is an isomorphism. Consequently, the composite $gf$ is an isomorphism of $\tilde{X}$ onto $D^2(\tilde{X})$ (which, in general, is not the limit morphism $\lim (j_{2k})!$, and property (i) follows by Lemma 6 (i).

(ii). By [16], Theorem 1, for every $k \in \mathbb{N}$, the canonical embedding $j_{2k}$ is a section of $(\mathcal{B}_F)_1$ having for an appropriate retraction

$$D(j_{2k-1}) : D^{2k+2}(X) \to D^{2k}(X), D(j_{2k-1}) j_{2k} = 1_{D^{2k}(X)}.$$

Then the conclusion follows by Lemma 2.

(iii). This property follows by Theorem 5 of [16]. Namely, if $X$ is finite-dimensional then one may choose $\tilde{X} = X$, while if $\dim X = \infty$, the normed dual functor $D$ rises the countable algebraic dimension ($\aleph_0$ to $2^{\aleph_0}$) only, and (a quotient of) a countable union in $\mathcal{V}_F$ cannot rise an infinite algebraic dimension. □

**Remark 1.** (a) By Theorem 2 and its proof, the constructed bidualic hyperdual $\tilde{X}$ of a normed space $X$ is also a bidualic hyperdual of every
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even normed dual space $D^{2k}(X)$, $k \in \{0\} \cup \mathbb{N}$, as well. Further, by applying
the same construction to the direct sequence $(D(X)_{2k+1}^{2k+1}, j_{2k+1}, \{0\} \cup \mathbb{N})$, one
obtains a bidualic hyperdual $D(X)$ of every odd normed dual $D^{2k+1}(X)$ of $X$.

(b) In the proof of Theorem 2 (i), the application of Lemma 5 has been essential. If it could hold an
appropriate analogue of Lemma 5 for the isometries (in the very special case of the proof of
Theorem 2 (ii)), then $\tilde{X}$ would be a parareflexive space.

We now want to extend the direct limit construction $X \mapsto \tilde{X} \mapsto \lim_{\rightarrow} \tilde{X}$ \equiv \tilde{X}
in $i\mathscr{N}_F$ to a functor on $\mathscr{N}_F$, which is closely related to all $D^{2k}$ functors.

**Theorem 3.** There exists a covariant functor (the normed hyperdual functor)

\[ \hat{D} : \mathscr{N}_F \rightarrow \mathscr{N}_F, \quad X \mapsto \hat{D}(X), \; f \mapsto \hat{D}(f), \]

such that $\hat{D}[\mathscr{N}_F] \subseteq \beta \mathscr{B}_F$, and $\hat{D}$ does not rise the algebraic dimension
but $\mathfrak{R}_0$ (to $2^{\mathfrak{R}_0}$). Moreover,

(i) $\hat{D}$ is faithful;

(ii) $\hat{D}(f)$ is an isometry if and only if $f$ is an isometry;

(iii) $\hat{D}$ is continuous, i.e., it commutes with the direct limit:

\[ \hat{D}(\lim_{\rightarrow} X) \cong \lim_{\rightarrow} \hat{D}[X] \text{ isometrically}; \]

(iv) $(\forall k \in \{0\} \cup \mathbb{N}) \hat{D}D^{2k} = \hat{D};$

(v) $(\forall k \in \{0\} \cup \mathbb{N})(\forall X \in \text{Ob}(\mathscr{N}_F)) D^{2k} \hat{D}(X) \cong \hat{D}(X);$

(vi) for each $k \in \{0\} \cup \mathbb{N}$, there exist an isometric natural transformation $\iota^{2k} : D^{2k} \rightarrow \hat{D}$ of the functors;

(vii) $\hat{D}[\pi \mathscr{N}_F] \subseteq \pi \mathscr{N}_F$, $\hat{D}[\chi \mathscr{N}_F] \subseteq \chi \mathscr{B}_F$, $\hat{D}[\chi_n \mathscr{N}_F] \subseteq \chi_n \mathscr{B}_F$, $\hat{D}[\rho \mathscr{N}_F] \subseteq \rho \mathscr{B}_F$.

**Proof.** According to Theorem 2, $\hat{D}$ is well defined on the object class
$\text{Ob}(\mathscr{N}_F)$ by putting $\hat{D}(X) = \tilde{X}$, where $\tilde{X}$ is the object of the canonical
direct limit $\lim_{\rightarrow} \tilde{X} = (\tilde{X}, i_{2k})$, and $\tilde{X} = (D^{2k}(X), j_{D^{2k}(X)}, \{0\} \cup \mathbb{N})$. Let
$f \in \mathscr{N}_F(X, Y)$. Since, for every $k \in \{0\} \cup \mathbb{N}$, $j^{2k} : D^{2k} \sim D^2(D^{2k}) = D^{k+2}$ is a
natural transformation of the covariant functors, the following diagram

\[
\begin{array}{ccc}
X & \cdots & \mapsto D^{2k}(X) & \cdots & \mapsto D^{2k}(X) & \cdots & \tilde{X} \\
\downarrow & \cdots & \downarrow j^{2k} & \cdots & \downarrow D^{2k} & \cdots \\
D^{2k}(f) & \cdots & \mapsto D^{2k}(f) & \cdots & \mapsto D^{2k}(f) & \cdots \\
Y & \cdots & \mapsto D^{2k}(Y) & \cdots & \mapsto D^{2k}(Y) & \cdots & \tilde{Y}
\end{array}
\]

in $\mathscr{N}_F$ commutes. Then the equivalence class $[(1_{\{0\} \cup \mathbb{N}}, D^{2k}(f))]$ of
$(1_{\{0\} \cup \mathbb{N}}, D^{2k}(f))$ is an in-morphism $\tilde{j} : \tilde{X} \rightarrow \tilde{Y}$ of the direct sequences
in \( iN_F \). If \( f \) belongs to \((N_F)^1 \), then \( \tilde{f} \in (\text{seq}-iN_F)^1 \), and \( \lim \tilde{f} \) exists by Lemma 1. In general, we have to construct (in this special case of direct sequences in \( iN_F \)) a limit morphism

\[
\tilde{f} : \tilde{X} = \lim \tilde{X} \rightarrow \lim \tilde{Y} = \tilde{Y}
\]

explicitly. Given an \( \tilde{x} = [x_{2k(\tilde{x})}] \in \tilde{X} \), put

\[
\tilde{f}(\tilde{x}) = i_{2k(\tilde{x}),Y}D^{2k(\tilde{x})}(f)(x_{2k(\tilde{x})}),
\]

where \( x_{2k(\tilde{x})} \in D^{2k(\tilde{x})}(X) \) is the grain of \( \tilde{x} \). Then \( \tilde{f} \) is a well defined linear function. Furthermore, \( \tilde{f} \) is continuous because, for every \( \tilde{x} \in \tilde{X} \),

\[
\left\| \tilde{f}(\tilde{x}) \right\| \leq \|f\| \cdot \|\tilde{x}\|
\]

holds. Indeed, each \( \tilde{x} \in \tilde{X} \) has a unique grain \( x_{2k(\tilde{x})} \in D^{2k(\tilde{x})}(X) \) (and conversely), and thus (by definitions of the norms on \( \tilde{X} \) and \( \tilde{Y} \)), it follows (recall that the elements of the terms are continuous functionals)

\[
\left\| \tilde{f}(\tilde{x}) \right\| = \left\| i_{2k(\tilde{x}),Y}D^{2k(\tilde{x})}(f)(x_{2k(\tilde{x})}) \right\| = \left\| D^{2k(\tilde{x})}(f)(x_{2k(\tilde{x})}) \right\|_{2k(\tilde{x}),Y} = \left\{ \begin{array}{l}
\|f(x_0)\|_Y, \quad k(\tilde{x}) = 0 \\
\|x_{2k(\tilde{x})}D^{2k(\tilde{x})-1}(f)\|_{2k(\tilde{x}),Y}, \quad k(\tilde{x}) \in \mathbb{N} 
\end{array} \right. \leq \left\{ \begin{array}{l}
\|f\| \cdot \|x_0\|_X, \quad k(\tilde{x}) = 0 \\
\|x_{2k(\tilde{x})}\|_{2k(\tilde{x})} \cdot \|D^{2k(\tilde{x})-1}(f)\|, \quad k(\tilde{x}) \in \mathbb{N} 
\end{array} \right. = \left\{ \begin{array}{l}
\|f\| \cdot \|x_0\|_X, \quad k(\tilde{x}) = 0 \\
\|x_{2k(\tilde{x})}\|_{2k(\tilde{x})} \cdot \|f\|, \quad k(\tilde{x}) \in \mathbb{N} 
\end{array} \right. = \|f\| \cdot \|\tilde{x}\|.
\]

Moreover, if \( \|f\| \leq 1 \), then \( \left\| \tilde{f} \right\| \leq 1 \). Further, since \( \tilde{f}i_{2k,Y} = i_{2k,Y}D^{2k}(f) \)
(by the very definition), it follows that \( \tilde{f} \) is an isometry whenever \( f \) is an isometry. We finally define

\[
\tilde{D}(f) \equiv \tilde{f} : \tilde{X} \equiv \tilde{D}(X) \rightarrow \tilde{D}(Y) \equiv \tilde{Y}.
\]

Then \( \tilde{D}(1_X) = 1_{\tilde{D}(X)} \) obviously holds. Further, given an \( f \in N_F(X,Y) \) and a \( g \in N_F(Y,Z) \), then, since each \( D^{2k} \) is a covariant functor, the definition from above (see also the diagram) implies that

\[
\tilde{D}(gf) = \tilde{g}\tilde{f} = \tilde{g}\tilde{f} = \tilde{D}(g)\tilde{D}(f).
\]

Therefore, \( \tilde{D} : N_F \rightarrow N_F \) is a covariant functor. By Theorem 2 (i), \( \tilde{X} \) is a bidualic Banach space, hence, \( \tilde{D}[N_F] \subseteq \beta B_F \), while Theorem 2 (iii) assures the statement about algebraic dimension. Let us now verify the additional properties.

(i). Let \( \tilde{D}(f) = \tilde{D}(f') : \tilde{D}(X) \rightarrow \tilde{D}(Y) \). Assume to the contrary, i.e., that \( f \neq f' \). Then there is an \( x \in X \) such that \( f(x) \neq f'(x) \). Since \( i_{1X} \) and \( i_{1Y} \) are monomorphism, it follows that

\[
\tilde{D}(f)i_{1X}(x) = i_{1Y}f(x) \neq i_{1Y}f'(x) = \tilde{D}(f')i_{1X}(x),
\]
implying that \( \tilde{D}(f)(\tilde{x}) \neq \tilde{D}(f')(\tilde{x}) \) - a contradiction.

(ii). It suffices to verify the sufficiency. Let \( \tilde{D}(f) : \tilde{D}(X) \to \tilde{D}(Y) \) be an isometry. Since \( i_{1X} \) and \( i_{1Y} \) are isometries, it follows that, for every \( x \in X \),

\[
\|f(x)\| = \|i_{1Y}(f(x))\| = \|\tilde{D}(f)(i_{1X}(x))\| = \|i_{1X}(x)\| = \|x\|.
\]

(iii). Since, by (i) and (ii), \( \tilde{D} \) is faithful and preserves isometries, we may apply the proof of Lemma 4 (for \( D^{2k} \)) to \( \tilde{D} \) as well, and the statement follows.

(iv). The equality \( \tilde{D}D^{2k} = \tilde{D} \), \( k \in \{0\} \cup \mathbb{N} \), follows by the definition of \( \tilde{D} \). Namely, in the (defining) direct sequence \( \tilde{X} \) for \( \tilde{D}(X) = \tilde{X} \) one may drop any initial part obtaining the same direct limit space. The same argument keeps valid for an \( f \in \mathcal{N}_F(X,Y) \), i.e.,

\[
\tilde{D}(f) = \tilde{D}(D^{2k}(f)) \in \mathcal{N}_F(\tilde{D}D^{2k}(X), \tilde{D}D^{2k}(Y)) = \mathcal{N}_F(\tilde{D}(X), \tilde{D}(Y)).
\]

(v). This property is a consequence of \( \tilde{D}[\mathcal{N}_F] \subseteq \beta\mathcal{N}_F \) and and [16], Lemma 1 (i), i.e., the inductive consequence of \( D^2\tilde{D}(X) \cong \tilde{D}(X) \).

(vi). Observe that, for given \( X, Y \in OB(\mathcal{N}_F) \) and each \( k \in \{0\} \cup \mathbb{N} \), the relation

\[
\tilde{D}(f)i_{2k,X} = i_{2k,Y}D^{2k}(f)
\]

follows straightforwardly by the construction of \( \tilde{X} \) and \( \tilde{Y} \) and by the definition of \( \tilde{f} \). Hence, for each \( k \in \{0\} \cup \mathbb{N} \), the class

\[
\{i_{2k,X} : D^{2k}(X) \to \tilde{X} \mid X \in Ob(\mathcal{N}_F)\}
\]

determines an isometric natural transformation \( \iota^{2k} : D^{2k} \hookrightarrow \tilde{D} \) of the functors.

(vii) Let \( X \) be a parareflexive space, i.e., \( X \in Ob(\pi\mathcal{N}_F) \). Then there exists an isometric isomorphism \( f : X \to D^2(X) \). By applying \( D^2 \) to \( \tilde{X} \) and \( \varprojlim \tilde{X} = \tilde{D}(X) \), one readily obtains an in-morphism

\[
f = [(1_n, f_n)] : \tilde{X} \to D^2[\tilde{X}], f_1 = f, f_n = D^2(f),
\]

with all the \( f_n \) isometric isomorphisms. Then, by Lemma 1,

\[
\varprojlim f : \tilde{D}(X) \to D^2(\tilde{D}(X))
\]

is an isometric isomorphism, and thus, \( \tilde{D}(X) \in Ob(\pi\mathcal{N}_F) \). Let \( X \) be a quasi-reflexive space of order \( n \in \{0\} \cup \mathbb{N} \), i.e., \( X \in Ob(\chi_n\mathcal{N}_F) \). Then \( D^2(X) \cong R(j_X) + F^n \) in \( \mathcal{B}_F \). Since the functors \( D^{2k} \) are exact ([6], Proposition 6. 5. 20, or [16], Lemma 3), the construction of \( \tilde{D}(X) \) and property (iv) straightforwardly imply that

\[
D^2(\tilde{D}(X)) \cong R(j_{\tilde{D}(X)}) + F^n.
\]

Consequently, \( \tilde{D}[\chi_n\mathcal{N}_F] \subseteq \chi_n\mathcal{B}_F \), \( \tilde{D}[\chi\mathcal{N}_F] \subseteq \chi\mathcal{B}_F \) and \( \tilde{D}[\rho\mathcal{N}_F] \subseteq \rho\mathcal{B}_F \) (the case \( n = 0 \)). This completes the proof of the theorem. \( \square \)
Concerning the somewhat reflexivity and, posteriori, the almost reflexivity of $\tilde{D}(X)$, we have established the following characterizations.

**Theorem 4.** For every normed space $X$, $\tilde{D}(X)$ is somewhat reflexive if and only if, for every $n \in \mathbb{N}$, $D^{2n}(X)$ is somewhat reflexive. Consequently, for every paranreflexive space $X$, the following properties are equivalent:

(i) $\tilde{D}(X)$ is almost reflexive;
(ii) $\tilde{D}(X)$ is somewhat reflexive;
(iii) $(\forall n \in \mathbb{N})$ $D^{2n}(X)$ is somewhat reflexive.

**Proof.** Let $X$ be a normed space such that $\tilde{D}(X)$ is somewhat reflexive. Let a $n \in \mathbb{N}$ be given. Notice that $D^{2n}(X)$ is somewhat reflexive if and only if $R(i_{2n}) \leq \tilde{D}(X)$ is somewhat reflexive, where $i_{2n} : D^{2n}(X) \to \tilde{D}(X)$ is the (isometric) limit morphism. If $R(i_{2n})$ is finite-dimensional, there is nothing to prove. If $R(i_{2n})$ is infinite-dimensional, then the conclusion follows because it is a closed subspace of $\tilde{D}(X)$. Conversely, let $X$ be a normed space such that, for every $n \in \mathbb{N}$, $D^{2n}(X)$ is somewhat reflexive. If $\tilde{D}(X)$ is finite-dimensional, then there is nothing to prove. Let $\tilde{D}(X)$ be infinite-dimensional. Since $\tilde{D}(X)$ is a Banach space, it follows that dim $\tilde{D}(X) \geq 2^{\aleph_0}$ ($CH$ accepted). Let $W \leq \tilde{D}(X)$ be a closed infinite-dimensional subspace. Then $W$ is a Banach space and dim $W \geq 2^{\aleph_0}$. Denote $W_{2n} \equiv W \cap R(i_{2n}) \leq \tilde{D}(X)$, $n \in \{0\} \cup \mathbb{N}$.

Then every $W_{2n}$ is a closed subspace of $W$, hence, a Banach space. Observe that there exists a $n_0 \in \{0\} \cup \mathbb{N}$ such that $W_{2n_0}$ is infinite-dimensional. Indeed, if all $W_{2n}$ were finite-dimensional, then $W$ would be at most countably infinite-dimensional - a contradiction. Let $W'_{2n} \leq D^{2n}(X)$ be the inverse image of $W_{2n}$ by $i_{2n}$, i.e., $i_{2n}[W'_{2n}] = W_{2n}$. Since all $i_{2n}$ are isometries of Banach spaces, it follows that every $W'_{2n}$ is a closed subspaces of $D^{2n}(X)$ and dim $W'_{2n} = \dim W_{2n}$. Especially, $W'_{2n_0}$ is a closed infinite-dimensional subspace of $D^{2n_0}(X)$. Since $D^{2n_0}(X)$ is somewhat reflexive, there exists a closed infinite-dimensional subspace $Z_{2n_0} \leq W'_{2n_0}$ that is reflexive. Since $i_{2n_0}$ is an isometry, the subspace $Z \equiv i_{2n_0}[Z_{2n_0}] \leq W_{2n_0}$ is closed, infinite-dimensional and reflexive. Consequently, $Z$ is closed, infinite-dimensional and reflexive subspace of $W$, hence, $\tilde{D}(X)$ is somewhat reflexive. Then the characterizations of the almost reflexivity follows by $\tilde{D}[\pi\mathcal{N}_F] \subseteq \pi\mathcal{N}_F$ of Theorem 3 (vii). □

By Theorem 4, $\tilde{D}(X)$ cannot be extended towards the almost reflexivity. Observe that Theorem 3 (vii) can be slightly refined as follows.
**Corollary 1.** Let $C \in \{\chi_{\rho N_F}, \chi N_F, \beta N_F, \pi N_F, \rho N_F\}$. Then the restrictions $\tilde{D} : C \rightarrow C$, $\tilde{D} : C_1 \rightarrow C_1$ and $\tilde{D} : iC \rightarrow iC$ are covariant functors retaining all the properties of $\tilde{D} : N_F \rightarrow N_F$. Furthermore, the restriction functor $\tilde{D} : \rho N_F \rightarrow \rho N_F$ is naturally isometrically isomorphic to the identity functor $1_{\rho N_F} : \rho N_F \rightarrow \rho N_F$.

**Proof.** The last statement only asks for a proof. Let $X$ and $Y$ be reflexive spaces, i.e., $X, Y \in Ob(\rho N_F)$. Then the canonical embeddings $j_X$ and $j_Y$ are isometric isomorphisms, as well as all the $j_{D^{2k}(X)}$ and $j_{D^{2k}(Y)}$. Consequently, all the limit morphisms $i_{2k,X} : D^{2k}(X) \rightarrow \tilde{D}(X)$ and $i_{2k,Y} : D^{2k}(Y) \rightarrow \tilde{D}(Y)$ are isometric isomorphisms. Especially, $i_{0,X} : X \rightarrow \tilde{D}(X)$ and $i_{0,Y} : Y \rightarrow \tilde{D}(Y)$ are isometric isomorphisms. Then, for every $f \in \rho N_F(X, Y)$, the following diagram

$$
\begin{align*}
X & \xrightarrow{f} Y \\
i_{0,X} & \downarrow \quad \downarrow i_{0,Y} \\
\tilde{D}(X) & \xrightarrow{\tilde{D}(f)} \tilde{D}(Y)
\end{align*}
$$

in $\rho N_F$ commutes. Therefore, the class $\{i_{0,X} \mid X \in Ob(\rho N_F)\}$ determines a natural transformation $\eta : 1_{\rho N_F} \Rightarrow \tilde{D}$, that is an isometric isomorphism of the functors. \hfill $\square$

At the end, concerning the dual space of a hyperdual, Theorem 3, Theorem 1 (ii) and [16], Theorem 1, imply the following result:

**Corollary 2.** For every normed space $X$,

$$(D(\tilde{D}(X), D(i_{2k})) \cong \varinjlim(D^{2k+1}(X), D(j_{D^{2k+1}(X)}), \{0\} \cup \mathbb{N}))$$

in $(\mathcal{B}_F)_1$, where all $D(j_{D^{2k+1}(X)})$ and all $D(i_{2k})$ are the category retractions.

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