On Lipschitz functions on groups equipped with conjugation-invariant norms

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Abstract

We observe that a function on a group equipped with a bi-invariant word metric is Lipschitz if and only if it is a partial quasimorphism bounded on the generating set. We also show that an undistorted element is always detected by an antisymmetric homogeneous partial quasimorphism. We provide a general homogenisation procedure for Lipschitz functions and relate partial quasimorphisms on a group to ones on its asymptotic cones.

1 Introduction

Lipschitz functions vs partial quasimorphisms

Let \((G, d)\) be a metric group, where \(d\) is bi-invariant metric. Let \(\|g\| = d(g, 1)\) denote the corresponding norm. A function \(f : G \to \mathbb{R}\) is called:

- a **partial quasimorphism** relative to the norm \(\|\|\) if there exists a constant \(D \geq 0\) such that
  \[|\delta f(g, h)| = |f(g) - f(gh) + f(h)| \leq D \min\{\|g\|, \|h\|\},\]
  for all \(g, h \in G\);

- **homogeneous** if \(f(g^n) = nf(g)\) for all \(g \in G\) and \(n \in \mathbb{N}\);

- **antisymmetric** if \(f(g^{-1}) = -f(g)\) for all \(g \in G\).

An element \(g \in G\) is called **undistorted** with respect to the norm \(\|\|\) if \(\|g^n\| \geq Cn\) for some \(C > 0\) and all \(n \in \mathbb{N}\).

**Theorem 1.1.** Let a group \(G\) be equipped with a bi-invariant word metric \(d\). A function \(f : G \to \mathbb{R}\) is Lipschitz if and only if it is a partial quasimorphism relative to \(\|\|\) and it is bounded on the generating set. Moreover, an element \(g \in G\) is undistorted if and only if there exists an antisymmetric homogeneous partial quasimorphism \(f : G \to \mathbb{R}\) such that \(f(g) > 0\) and \(f\) is bounded on the generating set.

**Remark 1.2.** A word metric with respect to a symmetric generating set \(S \subseteq G\) is bi-invariant if and only if \(S\) is normal, that is, \(S = g^{-1}Sg\) for all \(g \in G\). Such generating sets are infinite unless \(G\) is virtually abelian and finitely generated.
Remark 1.3. A function \( f : G \to \mathbb{R} \) is antisymmetric and homogeneous if and only if \( f(g^n) = nf(g) \) for all \( n \in \mathbb{Z} \) and \( g \in G \). Since quasimorphisms (see Example 1.6 below) are nearly antisymmetric, their homogeneity is usually defined with integer exponents [4, Section 2.2.2]. This is not always the case for partial quasimorphisms. For example, the translation length (see Example 1.10) is a homogeneous partial quasimorphism. Since it is non-negative, it is not antisymmetric unless it is zero. See also [13, Theorem 1.3] for an example in symplectic geometry.

Remark 1.4. It is well known that partial quasimorphisms bounded on generating sets are Lipschitz. The above theorem shows the opposite. Homogeneous partial quasimorphisms are a common tool to detect undistorted elements and what we observe above is that every undistorted element is detected by an antisymmetric homogeneous partial quasimorphism.

Remark 1.5. Considering word norms is not very restrictive. We show in Lemma 2.5 that bi-invariant length metric is equivalent to a suitable word metric. Moreover, the most natural bi-invariant metrics are either word metrics (e.g., commutator length, verbal length, fragmentation norm, autonomous norm etc) or length metrics (e.g. Hofer’s norm).

Example 1.6. A function \( f : G \to \mathbb{R} \) is called a quasimorphism if there exists \( D \geq 0 \) such that

\[ |f(g) - f(gh) + f(h)| \leq D \]

for all \( g, h \in G \). It is, obviously, a partial quasimorphism. In [1], there are examples of groups \( G \) equipped with bi-invariant word metrics such that an element \( g \in G \) generates an unbounded cyclic subgroup if and only if it is detected by a homogeneous quasimorphism.

Example 1.7. The commutator subgroup \([B_\infty, B_\infty]\) of the infinite braid group is perfect and does not admit unbounded quasimorphisms [12]. On the other hand, it contains elements undistorted with respect to a bi-invariant word metric [2]. It follows from Theorem 1.1 that such elements are detected by homogeneous partial quasimorphisms. A concrete example was constructed by Kimura [11].

A common construction of quasimorphisms is due to Brooks and it is well known that many groups admit an abundance of quasimorphisms [4]. For example, the space of quasimorphisms on a non-elementary hyperbolic group is infinite dimensional. On the other hand, systematic constructions of partial quasimorphisms are rare. One source is provided by Floer theoretic spectral invariants in symplectic geometry [8, 13] and another uses quasimorphisms [3, 10, 11]. It would be useful to know more constructions for finitely generated groups.

Homogeneous partial quasimorphisms

A homogenisation \( \hat{f} \) of a function \( f : G \to \mathbb{R} \) is defined by

\[ \hat{f}(g) = \lim_{n \to \infty} \frac{f(g^n)}{n} \] (1.2)

provided that the limit exists.
If \( f \) is a quasimorphism then it follows almost directly from Fekete’s Lemma that the above limit exists. The next result follows from a generalisation of Fekete’s Lemma due to de Bruijn-Erdős [6].

**Proposition 1.8.** Let \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) be an increasing function such that \( \int_1^\infty \frac{\varphi(t)}{t^2} \, dt < \infty \). Let \( f : G \to \mathbb{R} \) be such that
\[
|f(g) - f(gh) + f(h)| \leq \varphi(\|g\| + \|h\|),
\]
where \( \| \cdot \| \) is a norm on \( G \). Then the limit (1.2) exists and \( f \) admits a homogenisation.

**Example 1.9.** A standard walk is a function \( w : \mathbb{Z} \to \mathbb{Z} \) such that \( w(0) = 0 \) and \( |w(n) - w(n + 1)| = 1 \). It is a partial quasimorphism relative to the absolute value. It is easy to construct a walk such that the sequence \( \frac{w(n)}{n} \) has infinitely many convergent subsequences with pairwise distinct limits.

**Example 1.10.** Perhaps the most tautological example of a Lipschitz function is the norm, \( \| g \| = d(g, 1) \). Hence, it follows from Theorem 1.1 that the norm is a partial quasimorphism and its homogenisation, also known as the translation length is a homogeneous partial quasimorphism.

**Proposition 1.11.** Let \( f : G \to \mathbb{R} \) be a Lipschitz function with respect to a bi-invariant metric. Let \( \omega : 2^{\mathbb{N}} \to \{0, 1\} \) be a non-principal ultrafilter. Let \( f_\omega : G \to \mathbb{R} \) be defined by
\[
f_\omega(g) = \lim_\omega f(g^n).
\]
If \( \omega \) is a linear ultrafilter (see page 5 for definition) then \( f_\omega \) is a homogeneous partial quasimorphism relative to the associated norm. If \( f \) is antisymmetric then so is \( f_\omega \).

**Remark 1.12.** The homogenisation \( \hat{f} \) of a quasimorphism \( f \) has a very useful property that \( \sup_{g \in G} |\hat{f}(g) - f(g)| \leq D \). There is no such control in the general case of partial quasimorphisms or even for functions from Proposition 1.8.

**Relation to asymptotic cones**

Relations between partial quasimorphisms on verbal subgroups and their asymptotic cones have been studied first by Calegari and Zhuang [5]. The following observations are similar to their results.

Let \( \text{Cone}_\omega(G) \) be the asymptotic cone of \( G \) equipped with a bi-invariant metric \( d \) with respect to a non-principal ultrafilter \( \omega \). It is a complete metric group (see page 5 for more details). Let \( \eta : G \to \text{Cone}_\omega(G) \) be defined by \( \eta(g) = [g^n] \). Let \( f : G \to \mathbb{R} \) be a function for which there exists \( C > 0 \) such that \( |f(g)| \leq C\|g\| \) for all \( g \in G \). Let \( F_\omega : \text{Cone}_\omega(G) \to \mathbb{R} \) be defined by
\[
F_\omega[g_n] = \lim_\omega \frac{f(g_n)}{n}.
\]

**Proposition 1.13.** Let \( f : G \to \mathbb{R} \) be as above.
1. If $f$ is a (homogeneous) partial quasimorphism then $F_\omega$ is a (homogeneous) partial quasimorphism.

2. If $|f(g) - f(gh) + f(h)| \leq \varphi(\|g\| + \|h\|)$, where $\frac{\varphi(n)}{n} \to 0$, then $F_\omega$ is a homomorphism.

3. $F_\omega \circ \eta = f_\omega$.

**Proposition 1.14.** Let $G$ be equipped with a bi-invariant word metric. If $F: \text{Cone}_\omega(G) \to \mathbb{R}$ is a Lipschitz function such that $F(1) = 0$ then $F \circ \eta: G \to \mathbb{R}$ is a partial quasimorphism.

**Remark 1.15.** If $F: \text{Cone}_\omega(G) \to \mathbb{R}$ is a Lipschitz homomorphism then $F \circ \eta$ is also a partial quasimorphism. That is, $F$ being a homomorphism does not seem to imply a stronger statement on $F \circ \eta$. The situation is a bit different for the commutator length.

**Example 1.16.** If $F: \text{Cone}_\omega([G, G], \text{cl}) \to \mathbb{R}$ is a Lipschitz homomorphism then $F \circ \eta$ is a quasimorphism. This is a special case of a result by Calegari-Zhuang [5, Section 3]. The proof is similar to the proof of Proposition 1.14.

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## 2 Definitions and supporting results

**Norms.** A function $\| \|: G \to \mathbb{R}$ is called a norm if it satisfies the following conditions for all $g, h \in G$:

1. $\|g\| \geq 0$ and $\|g\| = 0$ if and only if $g = 1_G$;
2. $\|g^{-1}\| = \|g\|$;
3. $\|gh\| \leq \|g\| + \|h\|$;

A norm $\| \|$ is called conjugation-invariant if moreover

4. $\|h^{-1}gh\| = \|g\|$.

The associated metric defined by $d(g, h) = \|gh^{-1}\|$ is right-invariant. If $\| \|$ is conjugation-invariant then $d$ is bi-invariant, that is, both left- and right-invariant.

**Example 2.1.** If $S = S^{-1} \subseteq G$ is a symmetric generating set then

$$\|g\|_S = \min\{n \in \mathbb{N} \mid g = s_1 \ldots s_n, \ s_i \in S\}$$

is a norm called the word norm associated with the generating set $S$. If $S$ is normal, that is, $S = g^{-1}Sg$ for all $g \in G$ then the associated word norm is conjugation-invariant. We will usually omit the subscript when it does not lead to confusion.
Example 2.2. If $S$ is the set of all commutators $[g, h] \in [G, G]$, where $g, h \in G$ then the corresponding word norm on the commutator subgroup is called **commutator length** and it is relatively well understood [4]. It is invariant under conjugations by elements of $G$. ♦

Example 2.3. If $S$ is a union of finitely many conjugacy classes of $G$ then the corresponding word norm is **maximal** in the sense that the identity homomorphism is Lipschitz from $\| \|$ to any other bi-invariant norm. We will refer to this word norm as the **maximal word norm** on $G$. ♦

Lipschitz functions. A function $f: G \to \mathbb{R}$ is **Lipschitz** with constant $C > 0$ if $|f(g) - f(h)| \leq C d(g, h)$ holds for all $g, h \in G$. It is Lipschitz with respect to the norm if $|f(g)| \leq C \|g\|$ holds for all $g \in G$.

Ultrafilters. An **ultrafilter** $\omega$ on the set $\mathbb{N}$ of natural numbers is a maximal (with respect to inclusion) filter on the power set $2^\mathbb{N}$ (with the partial order given by inclusion of sets). Equivalently it is a finitely additive measure $\omega: 2^\mathbb{N} \to \{0, 1\}$. The equivalence is given by saying that sets of full measure belong to the filter. An ultrafilter is called non-principal if every finite set is of measure zero. If $a: \mathbb{N} \to X$ is a bounded sequence in a metric space $(X, d)$ then the **ultralimit** with respect to an ultrafilter $\omega$ is defined by the following condition: $\lim_{\omega} a_n = a$ if and only if for every $\varepsilon > 0$ $\omega\{n \in \mathbb{N} \mid d(a_n, a) < \varepsilon\} = 1$.

An ultrafilter on $\mathbb{N}$ is called **linear** if it contains all sets of the form $k\mathbb{N}$, where $0 < k \in \mathbb{N}$. Such ultrafilter exists. Indeed, let $\mathcal{F}$ be a collection containing all sets of the form $0 < k\mathbb{N}$, where $k \in \mathbb{N}$, and all their supersets. If $A, B \in \mathcal{F}$ then $k_A \mathbb{N} \subseteq A$ and $k_B \mathbb{N} \subseteq B$, for some $k_A, k_B \in \mathbb{N} \setminus \{0\}$. Then $\text{lcm}(k_A, k_B)\mathbb{N} = k_A \mathbb{N} \cap k_B \mathbb{N} \subseteq A \cap B$, which shows that $A \cap B \in \mathcal{F}$ and hence it is a filter since it is nonempty and upward closed by definition. It is thus contained in some non-principal ultrafilter.

Asymptotic cones. Let $\omega$ be a non-principal ultrafilter. Let $G$ be equipped with a bi-invariant metric $d$ and let $\| \|$ denote the associated norm. Let $\prod_0 G = \{(g_n) \in G^\mathbb{N} \mid \|g_n\| = O(n)\}$ be the set of sequences of elements of $G$ such that their norms grow at most linearly. It is a group with pointwise multiplication and $\|(g_n)\|_\omega = \lim_{\omega} \frac{\|g_n\|}{n}$ defines a degenerate norm, where degenerate means that some elements can have norm equal to zero. These elements form a normal subgroup and the corresponding quotient group $\text{Cone}_\omega(G, d)$ is called the **asymptotic cone** of $(G, d)$ [5]. It is a complete metric group with the metric $d_\omega$ associated to the norm $\| \|_\omega$ defined above. See Drutu-Kapovich [7] for a general and systematic approach to asymptotic cones of metric spaces. See also the thesis of Jakob Schneider [14] for a metric ultraproduct approach.
Example 2.4. In general, very little is known about the topological or algebraic structure of asymptotic cones of groups with bi-invariant metrics. Here is a sample of relatively easy facts.

1. The asymptotic cone of a free abelian group $\mathbb{Z}^n$ equipped with its standard word metric is isometric to $\mathbb{R}^n$ with the $L^1$-metric.

2. Let $G$ be a finitely generated nilpotent group equipped with the maximal bi-invariant word metric. The abelianisation $G \to G/[G,G]$ is a quasi-isometry and hence the asymptotic cone of $G$ is isometrically isomorphic to $\mathbb{R}^n$, where $n$ is the rank of the abelianisation.

3. The asymptotic cone of the infinite symmetric group $S_\infty$ equipped with a bi-invariant word metric associated with any finite normally generating set is a simple contractible metric group [9, Theorem 5.1].

4. The asymptotic cone $\text{Cone}(F_2)$ of the free group $F_2 = \langle a, b \rangle$ on two generators with respect to the maximal bi-invariant word metric is non-separable. Indeed, there is a quasi-isometric embedding of a regular tree $T \to \text{Cay}(F_2)$ and hence the asymptotic cone contains an isometrically embedded $\mathbb{R}$-tree. Since the section $\mathbb{Z}^2 \to F_2$ of the abelianisation given by $(m, n) \mapsto a^m b^n$ is an isometric embedding the cone $\text{Cone}_\omega(F_2)$ contains many flats, i.e., isometrically embedded copies of $\mathbb{R}^2$.

5. The asymptotic cone $\text{Cone}_\omega([G,G], \text{cl})$ of $[G,G]$ equipped with the commutator length is abelian.

Word norms on length groups.

Lemma 2.5. Let $G$ be a metric group with a bi-invariant metric $d$. Assume that $(G,d)$ is a length space. Let $S = B(1) \subseteq G$ be the ball of radius 1 centred at the identity and let $d_S$ denote the corresponding word metric. Then the identity is a quasi-isometry between $d$ and $d_S$. More precisely, we have

$$\|g\| \leq \|g\|_S \leq \|g\| + 1.$$ 

Proof. Let $\|g\| = d(g,1)$ and let $\|g\|_S = d_S(g,1)$ be the corresponding norms. Let $\mathcal{L}$ denote the length of rectifiable paths. Since the identity is a homomorphism, it is enough to verify the statement for the norms.

Let $\varepsilon > 0$. Let $g \in G$ and let $\{g_t\}$ be a path from the identity to $g$ such that $\mathcal{L}\{g_t\} \leq \|g\| + \varepsilon$. By subdividing the path into segments of length 1 and the last segment of possibly smaller length we see that $\|g\|_S \leq \|g\| + 1$.

If $\|g\|_S = n$ then $g = s_1 \ldots s_n$, where $\|s_i\| \leq 1$. Thus there is a path from the identity to $g$ of length at most $n + \varepsilon$ which shows that $\|g\| \leq n = \|g\|_S$. Thus the identity is a quasi-isometry.

Corollary 2.6. The metrics $d_\omega$ and the word metric $d_\omega S$ on the asymptotic cone $\text{Cone}_\omega(G)$ are quasi-isometric.
A useful identity. The following identity will be used several times in subsequent proofs. Its straightforward proof is left to the reader.

**Lemma 2.7.** Let $G$ be equipped with a conjugation-invariant norm $\| \|$. Then for all $g, h \in G$ and $n \in \mathbb{N}$

$$g^n h^n = (gh)^n c_1 \ldots c_{n-1},$$

(2.1)

where all $c_i$’s are conjugates of $[g, x_i]$ for some $x_i \in G$ or all $c_i$’s are conjugates of $[h, x_i]$ for some $x_i \in G$. In particular,

$$\|c_1 \ldots c_n\| \leq 2(n - 1) \min\{\|g\|, \|h\|\}.$$ 

\[ \square \]

3 Proofs

3.1 Proof of Theorem 1.1

Suppose that $f : G \to \mathbb{R}$ is a partial quasimorphism bounded on the generating set. Let $\sigma = \sup_{s \in S} f(s)$. Let $\|h\| = n$ and let $h = s_1 \ldots s_n$ for some $s_i \in S$. First observe that $f$ is Lipschitz with respect to the norm.

$$|f(h)| = |f(s_1 \ldots s_n)|$$

$$\leq \sum_{i=1}^{n} |f(s_i)| + (n - 1)D$$

$$\leq (\sigma + D)n = (\sigma + D)\|h\|.$$

Now we use the above to show that $f$ is Lipschitz.

$$|f(g) - f(gh) + f(h)| \leq |f(g) - f(gh)| + |f(h)|$$

$$\leq \sigma + D\|h\| = (\sigma + 2D)\|h\| = (\sigma + 2D)d(g, gh)$$

Conversely, assume that $f$ is Lipschitz with constant $C > 0$. Let $s \in S$ be a generator.

$$|f(s) - f(1)| \leq |f(s) - f(1)| \leq Cd(s, 1) = C$$

$$|f(s)| \leq C + |f(1)|$$

which shows that $f$ is bounded on the generating set.

$$|f(g) - f(gh) + f(h)| \leq |f(g) - f(gh)| + |f(h) - f(1)| + |f(1)|$$

$$\leq C\|h\| + C\|h\| + |f(1)|$$

$$\leq (2C + |f(1)|)\|h\|.$$ 

In the last inequality we use the fact that $d$ is a word metric, hence, $\|h\| \geq 1$ if $h \neq 1_G$. A similar computation and the assumption that $d$ is bi-invariant shows the bound by
Moreover, let $g$ be undistorted, that is, $\|g^n\| \geq c|n|$ for some $c > 0$. Let $f': \langle g \rangle \to \mathbb{R}$ be defined by $f'(g^n) = cn$ for all $n \in \mathbb{Z}$. Then $f'$ is antisymmetric homogeneous quasimorphism on $\langle g \rangle$, the subgroup generated by $g$, equipped with the induced metric. Indeed,

$$|f'(g^n) - f'(g^m)| = |cn - cm| \leq \|g^{n-m}\| = d(g^n, g^m),$$

which shows that $f'$ is Lipschitz with constant 1. Antisymmetry and homogeneity are clear. In the following part of the proof we construct an extension $f$ of $f'$ that is also Lipschitz. Let $f : G \to \mathbb{R}$ be defined by

$$f(h) = \inf \{ f'(g^n) + d(h, g^n) \mid n \in \mathbb{Z} \}$$

First we show that $f$ is well defined.

$$f'(g^n) + d(h, g^n) = f'(g) - f'(g) + f'(g^n) + d(h, g^n) \geq f'(g) - d(g, g^n) + d(h, g^n) \geq f'(g) - d(g, h) > -\infty.$$

Moreover $f(g^n) = f'(g^n)$, so $f$ is an extension of $f'$.

Let $\varepsilon > 0$ and let $h, h' \in G$ be any two elements. Without loss of generality, we can assume that $f(h') - f(h) \geq 0$. Let $g_h \in \langle g \rangle$ be an element such that

$$f(h) \geq f(g_h) + d(h, g_h) - \varepsilon.$$

Then by definition of $f$ we have also that $f(h') \leq f(g_h) + d(h', g_h)$. Consequently,

$$0 \leq f(h') - f(h) \leq f(g_h) + d(h', g_h) - f(g_h) - d(h, g_h) + \varepsilon \leq d(h', g_h) + d(g_h, h) + \varepsilon \leq d(h', h) + \varepsilon,$$

which shows that $f$ is Lipschitz (with constant 1) and bounded on the generating set.

Let $\overline{f}(g) = \frac{1}{2} (f(g) - f(g^{-1}))$ be the antsymmetrisation of $f$. Homogenising $\overline{f}$ with respect to a linear ultrafilter $\omega$ as in Proposition 1.11 does not change $f'$ and hence $\overline{f}_\omega$ is the required antisymmetric homogeneous partial quasimorphism. Boundedness on the generating set is clearly preserved by anti-simetrisation and homogenisation. 

**3.2 Proof of Proposition 1.8**

Theorem 23 in [6] states that if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function such that

$$\int_1^\infty \frac{\varphi(t)}{t^2} dt < \infty$$

(3.1)
and $a : \mathbb{N} \to \mathbb{R}$ is a sequence such that

$$a(m + n) \leq a(m) + a(n) + \varphi(m + n)$$

then $-\infty \leq \lim_{n \to \infty} \frac{a(n)}{n} < \infty$. Let $f : G \to \mathbb{R}$ be a function satisfying the hypothesis of Proposition 1.8 and let $g \in G$. It follows that

$$\left| f(g^m) - f(g^{m+n}) + f(g^n) \right| \leq \varphi(\|g^m\| + \|g^n\|)$$

$$f(g^{m+n}) \leq f(g^m) + f(g^n) + \varphi(\|g^m\| + \|g^n\|)$$

$$\leq f(g^m) + f(g^n) + \varphi((m + n)\|g\|).$$

Since the function $n \mapsto \varphi(n\|g\|)$ also satisfies the integral condition (3.1) the statement of the de Bruijn-Erdös applies. If we use only the above subadditivity then can only conclude that the limit $\lim_{n \to \infty} f(g^n) n$ either exists or it is $-\infty$. However, we also have that

$$f(g^m) + f(g^n) \leq f(g^{m+n}) + \varphi(\|g^m\| + \|g^n\|)$$

and the modification of de Bruijn-Erdös proof with the use of Fekete’s Lemma for superadditive sequences shows that the limit is bigger than $-\infty$.

### 3.3 Proof of Proposition 1.11

Since $f$ is Lipschitz and the associated norm satisfies the triangle inequality, we have that $|f(g^n)| \leq C\|g^n\| \leq C\|g\|n$. Consequently, the sequence $f(g^n)_n$ is bounded and its $\omega$-limit exists. The following computation shows that $f_\omega$ is a partial quasimorphism.

$$|f_\omega(g) - f_\omega(gh) + f_\omega(h)| = \lim_{\omega} \frac{|f(g^n) - f((gh)^n) + f(h^n)|}{n}$$

$$= \lim_{\omega} \frac{|f(g^n) - f(g^n h c_1 \ldots c_{n-1}) + f(h^n)|}{n}$$

$$\leq \lim_{\omega} \frac{|f(g^n) - f(g^n h)^n + f(h^n)| + |f(c_1 \ldots c_{n-1})|}{n}$$

$$\leq \lim_{\omega} \frac{D \min\{\|g^n\|, \|h^n\|\} + C\|c_1 \ldots c_{n-1}\|}{n}$$

$$\leq \lim_{\omega} \frac{D \min\{|n\|g\|, n\|h\|\} + 2C \min\{(n - 1)\|g\|, (n - 1)\|h\|\}}{n}$$

$$\leq (2C + D) \min\{|\|g\|, \|h\|\}.$$ 

If $\omega$ is linear and $k \in \mathbb{N}$ then we have that

$$f_\omega(g^k) = \lim_{\omega} \frac{f(g^{kn})}{n} = k \lim_{\omega} \frac{f(g^{kn})}{kn} = k \lim_{\omega} \frac{f(g^n)}{n}$$

since $\omega(k\mathbb{N}) = 1$. This shows that $f_\omega$ is homogeneous.
3.4 Proof of Proposition 1.13

If $f : G \to \mathbf{R}$ is a partial quasimorphism then the following computation shows that $F_\omega$ is also a partial quasimorphism.

\[
|F_\omega[g_n] - F_\omega[g_nh_n] + F_\omega[h_n]| \leq \lim_{\omega} \frac{|f(g_n) - f(g_nh_n) + f(h_n)|}{n} \\
\leq \lim_{\omega} D \min\{\|g_n\|, \|h_n\|\} \\
\leq D \min\{\|g_n\|, \|h_n\|\}.
\]

It is straightforward to see that if $f$ is homogeneous then so is $F_\omega$. This proves the first item. The second item also follows from the above computation with the assumed estimate. The third item is immediate. \qed

3.5 Proof of Proposition 1.14

Let $F : \text{Cone}_\omega(G) \to \mathbf{R}$ be a Lipschitz function with constant $C > 0$ and such that $F(1) = 0$. If follows from Lemma 2.5 that it is also Lipschitz with respect to the word metric $d_{\omega S}$ with the same constant $C$. Furthermore, Theorem 1.1 implies that $F$ is a partial quasimorphism relative to $d_{\omega S}$ with constant $2C + F(1) = 2C$.

Assume that neither $g$ nor $h$ is equal to the identity. Otherwise the computation below is trivial. Moreover, since the metric on $G$ is a word metric, we have that $\|g\| + 1 \leq 2\|g\|$ for $g \neq 1_G$, which is used below. Since $C = \sup_{s \in S} F(s)$, we have that $|F(g)| \leq 3C\|g\|_\omega$, (see the beginning of the proof of Theorem 1.1), which we also use below. The following computation finishes the proof.

\[
|F(\eta(g)) - F(\eta(\eta(h)))| + F(\eta(h))| \\
= |F[g^n] - F[(gh)^n] + F[h^n]| \\
= |F[g^n] - F[g^n h^n c_1 \ldots c_{n-1}] + F[h^n]| \\
\leq |F[g^n] - F[g^n h^n] + F[h^n]| + |F[c_1 \ldots c_{n-1}]| + 2C \|c_1 \ldots c_{n-1}\|_\omega \\
\leq 2C \min\{\|g^n\|_\omega, \|h^n\|_\omega\} + 5C \|c_1 \ldots c_{n-1}\|_\omega \\
\leq 2C \min\{\|g^n\|_\omega + 1, \|h^n\|_\omega + 1\} + 5C (\|c_1 \ldots c_{n-1}\|_\omega + 1) \\
\leq 2C \min\{\|g\| + 1, \|h\| + 1\} + 5C \left(\lim_{\omega} \frac{\|c_1 \ldots c_{n-1}\|_\omega}{n} + 1\right) \\
\leq 4C \min\{\|g\|, \|h\|\} + 5C \left(\lim_{\omega} \frac{2n \min\{\|g\|, \|h\|\}}{n} + 1\right) \\
\leq 4C \min\{\|g\|, \|h\|\} + 20C \min\{\|g\|, \|h\|\} \\
\leq 24C \min\{\|g\|, \|h\|\}.
\]

\qed
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