A p-ROBUST POLYGONAL DISCONTINUOUS GALERKIN METHOD WITH MINUS ONE STABILIZATION

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Abstract. We introduce a new stabilization for discontinuous Galerkin methods for the Poisson problem on polygonal meshes, which induces optimal convergence rates in the polynomial approximation degree \( p \). In the setting of [S. Bertoluzza and D. Prada, A polygonal discontinuous Galerkin method with minus one stabilization, ESAIM Math. Mod. Numer. Anal. (DOI: 10.1051/m2an/2020059)], the stabilization is obtained by penalizing, in each mesh element \( K \), a residual in the norm of the dual of \( H^1(K) \). This negative norm is algebraically realized via the introduction of new auxiliary spaces. We carry out a \( p \)-explicit stability and error analysis, proving \( p \)-robustness of the overall method. The theoretical findings are demonstrated in a series of numerical experiments.

AMS subject classification:

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1. Introduction

Polytopal methods for the solution of partial differential equations have, in recent year, gained an increased popularity thanks to the flexibility inherently offered by the use of polytopal meshes. Indeed polygonal meshes allow to take into account the geometrical feature of the physical domain without resulting in an excess of degrees of freedom, they can be used, by agglomeration, as a transition step when dealing with triangular/tetrahedral and quadrilateral/hexahedral meshes, and they allow for simple refining and coarsening strategies in the framework of adaptive methods. Among the different approaches, besides Discontinuous Galerkin and its variants, such as the Hybridizable Discontinuous Galerkin method or the Discontinuous Petrov Galerkin method, we recall the Virtual Element method, the Hybrid High Order method and the Mimetic Finite Differences method (see [11], [12],...
A key feature in all such methods is the need to resort to some form of stabilization, which can lead to a loss of optimality of their convergence rates with respect to the mesh size $h$ (when the mesh contains very small edges, as compared to the element diameters) or to the polynomial degree $p$, or both.

In [8], the focus was on the first issue, namely the loss of optimality with respect to the mesh size $h$. In that paper, a stabilized discontinuous Galerkin (DG) method with negative norm stabilization was proposed, which allows to retrieve optimality in $h$ under quite weak conditions on the mesh (allowing for the presence of very small edges). The method there, an hybridized formulation of which is also presented and used in the implementation, discretizes the primal variable with polynomials of degree $k$, and the auxiliary variable associated with the flux with polynomials of degree $k' \in \{k, k-1\}$, discontinuous at the vertexes of the elements. As such choice does not satisfy the inf-sup condition needed for the stability of the discrete problem, a stabilization was introduced, the form of which constitutes the main novelty of such a method. More precisely, rather that measuring the residual term involved in a mesh dependent norm, as usually done, the proposed stabilization makes use of a negative norm, measuring such a residual in the space where it naturally “lives”. This allows to avoid the combined use of direct and inverse inequalities, which is the main source for the lack of optimality when mesh dependent norms are used. The negative scalar product is realized algebraically via the introduction of an auxiliary space of minimal dimension. The resulting formulation is shown, both theoretically and with numerical experiments, to yield quasi-optimal convergence in $h$ even in the presence of very small edges. However, the analysis therein is carried out for fixed $k$, and the constants involved in the different bounds depending on $k$. With the proposed stabilization, the method itself lacks robustness in $k$.

In a divide and conquer approach, in this paper, we address instead the issue of the optimality with respect to $k$. To this aim, we extend the theoretical analysis of the stabilized method of [8] by explicitly tracking the dependence on (or independence of) the polynomial degree, and we present an alternative construction of the negative norm stabilization, which allows to achieve quasi-optimality in $k$, this time under a stronger shape regularity assumption on the mesh, see Assumption 2.1 (ii) and Remark 6.3. Also in this case, the negative norm is algebraically realized via the introduction of a suitable auxiliary space. The auxiliary space is now constructed by suitably splitting the polygonal elements into triangles. On each triangle, the auxiliary space is defined as the push forward of a space of minimal dimension, which is constructed once and for all by numerically solving a set of Neumann problems on a sufficient fine mesh on a reference triangle.

Remark that, while we focus on a particular instance of the DG method, the idea of using a natural norm for the dual space in place of a mesh dependent norm used in constructing stabilization terms can be carried out to other polytopal formulations.

The paper is organized as follows. In Section 2 we recall the stabilized DG method from [8]. Then, in Section 3 we define the different norms and seminorms that we will use in the subsequent analysis, as well as some of their properties, and prove some inverse inequalities on polynomial spaces on the unit interval in Section 4. In Section 5 we carry out a $k$-explicit stability and convergence analysis of the method; a specific construction of a computable bilinear form, which yields an equivalent scalar product for $H^{-1}(K)$ on the discrete spaces used for the discretization in each element $K$, is presented in Section 6. After introducing a hybridization of the method in Section 7 which leads to an efficient
implementation of the method, we present in Section 8 some numerical result confirming the validity of the theoretical estimates.

In the following, we will employ the notation $A \lesssim B$ (resp. $A \gtrsim B$) to indicate that $A \leq cB$ (resp. $A \geq cB$), with $c$ positive constant independent on the mesh size parameters $h_K$ (the diameter of the polygon $K$), $h_e$ (the length of the edge $e$), $k$ (the polynomial degree), and possibly depending on the shape of the polygon $K$ only via the constant in the shape regularity Assumption 2.1. We will write $A \asymp B$ to signify that $A \lesssim B \lesssim A$. For $f \in V, V$ Hilbert space, and $F \in V'$, the notation $\langle F, f \rangle$ will stand for the action of $F$ on $f$ (the couple of dual spaces $V$ and $V'$ may vary, its identity will be clear from the context). Moreover, in order to avoid too cumbersome a notation, we will simply write $\sup_v$ instead of $\sup_{v : v \neq 0}$ when taking the supremum over a variable $v \neq 0$ of a quantity expressed as a fraction where $v$ appears at the denominator.

2. DG method with negative norm stabilization

As a model problem, we consider the Poisson equation with Dirichlet boundary conditions in a polygonal domain $\Omega \subset \mathbb{R}^2$:

\begin{equation}
- \Delta u = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial \Omega,
\end{equation}

with $f \in L^2(\Omega)$, $g \in H^{1/2}(\partial \Omega)$.

2.1. Assumptions on the meshes. We consider a family $\{T_h\}_h$ of meshes $T_h = \{K\}$ of the domain $\Omega$, each one containing a finite number of polygonal elements. The parameter $h$ is defined as $h = \max_{K \in T_h} h_K$, $h_K$ being the diameter of the polygonal element $K$. We denote by $\mathcal{E}_h$ the mesh skeleton, which is defined by $\mathcal{E}_h = \bigcup_{K \in T_h} \partial K$. For all edge $e \subset \mathcal{E}_h$, we let $h_e = |e|$ denote its length.

We assume that the family of meshes $\{T_h\}_h$ satisfies the following properties:

**Assumption 2.1.** There exists constants $\gamma_0, \gamma_1 > 0$ such that for all meshes $T_h$:

(i) each element $K \in T_h$ is star-shaped with respect to a ball of radius $\geq \gamma_0 h_K$;

(ii) for each element $K$ in $T_h$, the distance between any two vertices of $K$ is $\geq \gamma_1 h_K$.

Notice that (i) and (ii) imply that there exists a constant $N > 0$ such that, for all $K \in T_h$, the number of edges of $K$ is $\leq N$. Moreover, it is not difficult to realize that Assumption 2.1 implies that $T_h$ is graded, that is, that for all $K, K'$ sharing an edge it holds that $h_K \sim h_{K'}$.

We point out that, with Assumption 2.1 (ii), we are making a stronger shape regularity assumption than in [8].

Letting $\| \cdot \|_{0,D}, D \subset \mathbb{R}^d, d = 1, 2$, denote the standard $L^2(D)$ norm, we recall the following trace and Poincaré inequalities (see, e.g., [4, 9]).

**Trace inequality.** Under Assumption 2.1 (i), for all $u \in H^1(K)$, we have

\begin{equation}
\| u \|_{0, \partial K}^2 \leq C_{t_1}^2 \| u \|_{0, K} (h_K^{-1} \| u \|_{0, K} + \| \nabla u \|_{0, K}).
\end{equation}
Poincaré inequality: 1st version. Under Assumption 2.1 (i), for all $u \in H^1(K)$, we have

$$\inf_{q \in \mathbb{R}} \|u - q\|_{0,K} \leq C_p h_K \|\nabla u\|_{0,K}. \tag{2.3}$$

The following result is a straightforward consequence of (2.3).

**Corollary 2.1.** Let $\bar{u}^K = |K|^{-1} \int_K u$ denote the average of $u$ on $K$; then, under Assumption 2.1 (i), for all $u \in H^1(K)$, we have

$$\|u - \bar{u}^K\|_{0,K} \leq C_p h_K \|\nabla u\|_{0,K}. \tag{2.4}$$

Poincaré inequality: 2nd version. Under Assumption 2.1 (i), for all $u \in H^1(K)$, setting $\tilde{u} = |\partial K|^{-1} \int_{\partial K} u$, we have

$$\|u - \tilde{u}\|_{0,K} \leq \tilde{C}_p h_K \|\nabla u\|_{0,K}. \tag{2.5}$$

2.2. Continuous variational formulation on the mesh $T_h$. The DG methods we are going to introduce are based on the standard formulation of the primal hybrid method [18] on $T_h$.

Define $u^K = u|_K$. Multiplying the equation in (2.1) by discontinuous test functions $v \in \prod_{K \in T_h} H^1(K)$ (with an abuse of notation, we denote by $v$ also the function in $L^2(\Omega)$ such that $v|_K = v^K \in H^1(K)$) and integrating by parts elementwise give

$$\sum_{K \in T_h} \int_K \nabla u^K \cdot \nabla v^K - \sum_{K \in T_h} \int_{\partial K} \nabla u^K \cdot n_K v^K = \int_{\Omega} f v, \tag{2.6}$$

where $n_K$ denotes the outer unit normal to $\partial K$.

We define the following spaces on $T_h$:

$$V = \prod_{K \in T_h} H^1(K), \quad \Lambda = L^2(\mathcal{E}_h).$$

On $\mathcal{E}_h$, we choose a unit normal $n$, taking care that, on $\partial \Omega$, $n$ points outwards. Introduce $\lambda \in \Lambda$ defined as $\lambda = \nabla u|_{\partial \Omega} \cdot n$. The variational formulation (2.6) becomes: find $u \in V$, $\lambda \in \Lambda$ such that

$$\sum_{K \in T_h} \int_K \nabla u^K \cdot \nabla v^K - \sum_{K \in T_h} \int_{\partial K} \lambda(n_K \cdot n) v^K = \int_{\Omega} f v \quad \forall v \in V, \tag{2.7}$$

$$\sum_{K \in T_h} \int_{\partial K} u^K \mu(n_K \cdot n) = \int_{\partial \Omega} g \mu \quad \forall \mu \in \Lambda.$$

Notice that the second equation imposes the continuity of $u$ across $\mathcal{E}_h$, as well as the Dirichlet boundary condition on $\partial \Omega$.

Observe that the well-posedness of problem (2.7) relies on the validity of the following inf-sup condition:

$$\inf_{\mu \in \Lambda} \sup_{v \in V} \frac{\sum_{K \in T_h} \int_{\partial K} u^K \mu(n_K \cdot n)}{\|\mu\|_\Lambda \|v\|_V} \geq \beta,$$

with a positive constant $\beta$ that, for a suitable choice of the norms $\| \cdot \|_\Lambda$, $\| \cdot \|_V$, and under Assumption 2.1 we can show to be independent of $T_h$. As the well posedness of a corresponding discrete problem relies on the validity of an analogous inf-sup condition for the
discrete spaces, a direct discretization of problem \((2.7)\) would require excessively strong assumptions on the latter. Therefore, we write a stabilized version of problem \((2.7)\). We denote by \(D : H^1(K) \to (H^1(K))'\) the operator defined as

\[
\langle Du, v \rangle = \int_K \nabla u \cdot \nabla v \quad \text{for all } v \in H^1(K).
\]

Moreover, for all \(K \in \mathcal{T}_h\), we denote by \(\gamma^*_K : H^{-1/2}(\partial K) \to (H^1(K))'\) the adjoint of the trace operator \(\gamma_K : H^1(K) \to H^{1/2}(\partial K)\) and, by abuse of notation, also the operator \(\gamma^*_K : \Lambda \to (H^1(K))'\) defined as

\[
\langle \gamma^*_K \lambda, v^K \rangle = \int_{\partial K} \lambda (n_K \cdot \boldsymbol{n}) v^K \quad \forall v^K \in H^1(\Omega).
\]

The abuse of notation is justified by the fact that, for \(u \in H^2(\Omega)\), if we let \(\lambda \in L^2(\mathcal{E}_h)\) be defined as \(\lambda = \nabla u|_{\mathcal{E}_h} \cdot \boldsymbol{n}\), then, with the above definition, \(\gamma^*_K \lambda\) satisfies \(\langle \gamma^*_K \lambda, v \rangle = \int_{\partial K} (\partial u/\partial n_K) \gamma_K v\) for all \(v \in H^1(K)\). We define the jump \([u]\) of \(u \in V\) by setting, for every interior edge \(e\) shared by two elements \(K^+\) and \(K^-\),

\[
[u] = u^{K^+}_e n^{K^+}_e + u^{K^-}_e n^{K^-}_e,
\]

while for \(e \in \partial \Omega \cap \partial K\) we set

\[
[u] = u n_K.
\]

We observe that, for all \(u \in V, \lambda \in \Lambda\), we have the identity

\[
\sum_{K \in \mathcal{T}_h} \int_{\partial K} \lambda (n_K \cdot \boldsymbol{n}) u^K = \int_{\mathcal{E}_h} \lambda [u] \cdot \boldsymbol{n}
\]

We consider the following stabilized problem.

**Problem 2.1.** Find \(u = (u^K)_{K \in \mathcal{T}_h} \in V, \lambda \in \Lambda\) such that, for all \(v = (v^K)_{K \in \mathcal{T}_h} \in V, \mu \in \Lambda\), we have

\[
\sum_{K \in \mathcal{T}_h} \int_K \nabla u^K \cdot \nabla v^K - \int_{\mathcal{E}_h} \lambda [v] \cdot \boldsymbol{n} + t \alpha \sum_{K \in \mathcal{T}_h} (Du^K - \gamma^*_K \lambda, Dv^K)_{-1,K} = \int_{\Omega} f v + t \alpha \sum_{K \in \mathcal{T}_h} (f, Dv^K)_{-1,K}
\]

\[
\int_{\mathcal{E}_h} \mu [u] \cdot \boldsymbol{n} - \alpha (Dw^K - \gamma^*_K \lambda, \gamma^*_K \mu)_{-1,K} = \int_{\partial \Omega} g \mu - \alpha \sum_{K \in \mathcal{T}_h} (f, \gamma^*_K \mu)_{-1,K},
\]

where \(\alpha\) is a positive constant, \(t = \pm 1\), and the bilinear form \((\cdot, \cdot)_{-1,K}\) denotes the inner product in \(H^1(K)'\).

It is not difficult to check that the same arguments we will use in Section 5 to analyze the discrete problem actually also allows us to prove the well posedness of Problem 2.1.

### 2.3. Discontinuous Galerkin discretization.

We define the discrete spaces

\[
V_h = \prod_{K \in \mathcal{T}_h} \mathbb{P}_k(K) \subset V, \quad \Lambda_h = \{\lambda \in L^2(\mathcal{E}_h) : \lambda|_e \in \mathbb{P}_{k'}(e) \forall e \subset \mathcal{E}_h\},
\]

where \(k' \in \{k - 1, k\}\), and where \(\mathbb{P}_k(K)\) (resp. \(\mathbb{P}_{k'}(e)\)) denotes the space of polynomials of degree at most \(k\) (resp. \(k'\)) in two variables restricted to \(K\) (resp. \(e\)).

The discrete version of the stabilized problem (2.1) reads as follows.
Problem 2.2. Find $u_h = (u_h^K)_{K \in \mathcal{T}_h} \in V_h$, $\lambda_h \in \Lambda_h$ such that, for all $v = (v^K)_{K \in \mathcal{T}_h} \in V_h$ and $\mu \in \Lambda_h$, we have

\begin{align}
(2.10) \quad \sum_{K \in \mathcal{T}_h} \int_K \nabla u_h^K \cdot \nabla v^K - \sum_{E \in \mathcal{E}_h} \lambda_h[v] \cdot n + t\alpha \sum_{K \in \mathcal{T}_h} s_K(Du_h^K - \gamma_K^*\lambda_h, Du^K) \\
= \int_\Omega f v + t\alpha \sum_{K \in \mathcal{T}_h} s_K(f, Du^K)
\end{align}

\begin{align}
(2.11) \quad \sum_{E \in \mathcal{E}_h} \mu[u_h] \cdot n - \alpha \sum_{K \in \mathcal{T}_h} s_K(Du_h^K - \gamma_K^*\lambda_h, \gamma_K^*\mu) = \int_{\mathcal{E}_h} g\mu - \alpha \sum_{K \in \mathcal{T}_h} s_K(f, \gamma_K^*\mu).
\end{align}

Here, $s_K : (H^1(K))^\prime \times (H^1(K))^\prime \rightarrow \mathbb{R}$ is a continuous bilinear form that, when restricted to elements $F \in \gamma_K^*(\Lambda_h)$ with $\langle F, 1 \rangle = 0$, is spectrally equivalent to the $(H^1(K))^\prime$ inner product $(\cdot, \cdot)_{1, K}$.

More precisely, we define by duality the following norm and seminorm for elements $F \in (H^1(K))^\prime$ (see Section 3 for more details):

\begin{align}
(2.12) \quad \|F\|_{-1, K} = \sup_{g \in H^1(K)} \frac{\langle F, g \rangle}{\|g\|_{-1, K}} \quad \text{and} \quad \|F\|_{-1, K} = \sup_{g \in H^1(K)} \frac{\langle F, g \rangle}{\|g\|_{0, K}},
\end{align}

and we make the following assumptions on the stabilization forms $s_K(\cdot, \cdot)$.

Assumption 2.2. (Continuity) There exists a constant $M(k) > 0$, possibly depending on $k$, such that

$$s_K(F, G) \leq M(k)\|F\|_{-1, K}\|G\|_{-1, K} \quad \forall F, G \in (H^1(K))^\prime, \forall K \in \mathcal{T}_h.$$ 

Assumption 2.3. (Coercivity) There exists a constant $\rho(k) > 0$, possibly depending on $k$, such that

$$s_K(\gamma_K^*\lambda, \gamma_K^*\lambda) \geq \rho(k)\|\gamma_K^*\lambda\|_{-1, K}^2 \quad \forall \lambda \in \Lambda_h, \forall K \in \mathcal{T}_h.$$

Notice that $\rho(k) \leq M(k)$.

Computable stabilization forms $s_K(\cdot, \cdot) : (H^1(K))^\prime \times (H^1(K))^\prime \rightarrow \mathbb{R}$ satisfying Assumptions 2.2 and 2.3 will be introduced below in Section 6. Before presenting a stability and error analysis of the DG formulation in Problem 2.2 (see Section 5 below), we introduce some norms and seminorms, together with their properties (Section 3), and recall some properties of polynomial spaces (Section 4).

3. Norms and seminorms

We start by defining local norms and seminorms on $e \in \mathcal{E}_h$, on $K$ and $\partial K$, $K \in \mathcal{T}_h$, that are convenient in the application of scaling arguments, in particular when negative norms are concerned, and define global norms on $\mathcal{T}_h$ and $\mathcal{E}_h$. 
3.1. Norms and seminorms on elements. We define the following norms and seminorms for the Sobolev spaces $H^1(K)$ and its dual ($H^1(K))^\prime$.

For $u \in H^1(K)$, we let

$$\|u\|_{1,K}^2 = |\bar{u}_K|^2 + |u|_{1,K}^2,$$

with

$$\bar{u}_K = \frac{1}{|K|} \int_K u, \quad |u|_{1,K}^2 = \int_K |\nabla u|^2.$$

Observe that, if $\| \cdot \|_{-1,K}$ and $|\cdot|_{-1,K}$ are defined by duality with the norm $\| \cdot \|_{1,K}$ as in (2.12), then it holds (see [8])

$$\|F\|_{-1,K}^2 = |\langle F, 1 \rangle|^2 + |F|_{-1,K}^2.$$

Under Assumption (2.1), we have the following proposition.

**Proposition 3.1.** Let $\tau \subseteq \partial K$, $K \in \mathcal{T}_h$, be a connected subset of $\partial K$ with $|\tau| \geq \gamma_1 h_K$. Then, for all $u \in H^1(K)$, letting $\bar{u}_\tau = |\tau|^{-1} \int_\tau u$, we have

$$\|u\|_{1,K}^2 \simeq |\bar{u}_\tau|^2 + |u|_{1,K}^2.$$

*Proof.* We have, with $\bar{u}_K = |K|^{-1} \int_K u$,

$$\int_K |u(x)|^2 dx = \int_K |u(x) - \frac{1}{|\tau|} \int u(s) ds|^2 dx \leq \frac{1}{|\tau|} \int \int |u(x) - u(s)|^2 ds dx \leq \frac{2}{|\tau|} \int \int |u(s) - \bar{u}_K|^2 ds dx + \frac{2}{|\tau|} \int |u(s) - \bar{u}_K|^2 ds dx \leq 2 \|u - \bar{u}_K\|_{0,K}^2 + \frac{2|K|}{|\tau|} \|u - \bar{u}_K\|_{0,\partial K}^2.$$

Then, by applying the trace inequality (2.2) and the Poincaré inequality (2.3), for all $u \in H^1(K)$ with $\int_\tau u = 0$, we have

$$\|u\|_{0,K} \leq \bar{C}_p h_K |u|_{1,K}$$

with $\bar{C}_p$ only depending on the shape regularity constants $\gamma_0$ and $\gamma_1$. Conversely, by applying once again the trace inequality (2.2) and the Poincaré inequality (2.3), we can write

$$|\bar{u}_\tau|^2 \leq |\tau|^{-1} \|u\|_{0,\tau}^2 \leq h_K^{-1} \|u\|_{0,\partial K}^2 \leq h_K^{-2} \|u - \bar{u}_K\|_{0,K}^2 + |\bar{u}_K|^2 \leq |u|_{1,K}^2 \lesssim \|u\|_{1,K}^2.$$

\[\square\]

3.2. Norms and seminorms on edges. For every edge $e$ of the mesh, we define the following norms and seminorms for the Sobolev spaces $H^s(e)$ and their dual spaces ($H^s(e))^\prime$, $0 < s < 1$.

For $\varphi \in H^s(e)$, we let

$$\|\varphi\|_{s,e}^2 = |e|^{1-2s} |\varphi|^2 + |\varphi|_{s,e}^2,$$

with

$$\varphi = \frac{1}{|e|} \int_e \varphi, \quad |\varphi|_{s,e} = \int_e \int_e \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{2s+1}} dx dy.$$
On \((H^s(e))\)' we define
\[
\|\lambda\|_{-s,e}^2 = |e|^{2s-1} \langle \lambda, 1 \rangle^2 + |\lambda|_{-s,e}^2, \quad |\lambda|_{-s,e} = \sup_{\varphi \in H^s(e) \setminus \varphi = 0} \frac{\langle \lambda, \varphi \rangle}{|\varphi|_{s,e}}.
\]

The two norms defined by \((3.4)\) and \((3.5)\), respectively, satisfy the duality relations
\[
\|\lambda\|_{-s,e} = \sup_{\varphi \in H^s(e)} \int_e \lambda \varphi \, ds, \quad \|\varphi\|_{s,e} = \sup_{\lambda \in (H^s(e))'} \|\lambda\|_{-s,e},
\]
see [8, Lemma 2.1].

On \(e := (a, b)\), we will also consider the spaces \(H^s_0(e) (s \in [0, 1], s \neq 1/2)\) and \(H^{1/2}_0(e)\) of functions whose extension by zero is in \(H^s(\mathbb{R}) (s \neq 1/2)\) and \(H^{1/2}(\mathbb{R})\) respectively, which we will equip with the norms
\[
\|\varphi\|_{H^s_0(e)}^2 = |\varphi|^2_{H^s(e)} + \int_e \frac{|\varphi(x)|^2}{|x - a|^{2s}} ds(x) + \int_e \frac{|\varphi(x)|^2}{|x - b|^{2s}} ds(x), \quad s \neq 1/2,
\]
\[
\|\varphi\|_{H^{1/2}_0(e)}^2 = |\varphi|^2_{H^{1/2}(e)} + \int_e \frac{|\varphi(x)|^2}{|x - a|^2} ds(x) + \int_e \frac{|\varphi(x)|^2}{|x - b|} ds(x).
\]

For \(s = 1\), we set \(\|\varphi\|_{H^1_0(e)} = |\varphi|_{1,e}\).

We recall that, for \(s < 1/2\), the two spaces \(H^s(e)\) and \(H^s_0(e)\) coincide, and the two corresponding norms are equivalent. However, the constant in the equivalence depends on \(s\) and it explodes as \(s\) converges to \(1/2\). The behavior of such constant as \(s\) approaches the limit value \(1/2\) is given by the following bound, which holds for all \(\zeta \in H^{1/2-\varepsilon}(e)\), with \(\zeta = |e|^{-1} \int_e \zeta\) (see [5]):
\[
\|\zeta\|_{H^{1/2-\varepsilon}_0(e)} \lesssim \frac{1}{\varepsilon} |\zeta|_{1/2-\varepsilon,e} + \frac{|e|}{\sqrt{\varepsilon}} |\zeta|.
\]

By a simple duality argument, it is not difficult to check that we have
\[
\|\zeta\|_{-1/2+\varepsilon,e} \lesssim \frac{1}{\varepsilon} \|\zeta\|_{(H^{1/2}_0(e))'}.
\]

Observe that the seminorm \(|\cdot|_{1/2,e}\) and the norm \(|\cdot|_{H^{1/2}_0(e)}\) are scale invariant. In fact, letting \(I = [0, 1]\) and \(e = [0, |e|]\), and setting \(\hat{x} = |e|^{-1}x\), for \(\hat{\varphi} \in H^{1/2}(I)\) (resp. \(\hat{\varphi} \in H^{1/2}_0(I)\)) and \(\varphi(x) = \hat{\varphi}(\hat{x}) \in H^{1/2}(e)\) (resp. \(\varphi(x) = \hat{\varphi}(\hat{x}) \in H^{1/2}_0(e)\)), we have the identity
\[
|\hat{\varphi}|_{1/2,I} = |\varphi|_{1/2,e}, \quad \text{(resp. } \|\hat{\varphi}\|_{H^{1/2}_0(I)} = \|\varphi\|_{H^{1/2}_0(e)}\).
\]

For the \(|\cdot|_{-1/2,e}\) seminorm and the \(|\cdot|_{(H^{1/2}_0(e))'}\) norm, for \(\hat{\lambda} \in L^2(I)\) and \(\lambda(x) = \hat{\lambda}(\hat{x})\), we instead have
\[
|\lambda|_{-1/2,e} = |e| |\hat{\lambda}|_{-1/2,I}, \quad \|\lambda\|_{(H^{1/2}_0(e))'} = |e| \|\hat{\lambda}\|_{(H^{1/2}_0(I))'}.
\]

In fact,
\[
|\lambda|_{-1/2,e} = \sup_{\varphi \in H^{1/2}(e): \int_e \varphi = 0} \int_e \lambda(x) \varphi(x) \, dx = |e| \sup_{\hat{\varphi} \in H^{1/2}(I): \int_I \hat{\varphi} = 0} \int_I \hat{\lambda}(\hat{x}) \hat{\varphi}(\hat{x}) \, d\hat{x},
\]
We start by proving the second bound. Let

\[ \| \varphi \|_{1/2, e} \leq \| \lambda \|_{H^{1/2}(e)} \]

This readily follows from

\[ |\varphi|^2 \leq \frac{1}{|e|} \int_e |\varphi(x)|^2 \, dx \leq \int_e \frac{|\varphi(x)|^2}{|x|^2} \, dx. \]

By duality, we have that

\[ \| \lambda \|_{(H^{1/2}(e))'} \lesssim \| \lambda \|_{-1/2, e}. \]

### 3.3. Norms and seminorms on element boundaries

We define the norm in \( H^{1/2}(\partial K) \) as

\[ \| \varphi \|_{1/2, \partial K} = |\varphi|_{1/2, \partial K}, \]

with

\[ \varphi_{\partial K} = \frac{1}{|\partial K|} \int_{\partial K} \varphi, \quad |\varphi|_{1/2, \partial K} = \int_{\partial K} \int_{\partial K} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^2} \, dx \, dy. \]

We have the following equivalence between this norm and the norm obtained via the trace operator:

\[ \inf_{u \in H^1(K): u = \varphi \text{ on } \partial K} \| u \|_{1, K} \simeq \| \varphi \|_{1/2, \partial K}. \]

In fact, by using the definition of \( \| \cdot \|_{1/2, \partial K} \) and, recalling that

\[ |\varphi|_{1/2, \partial K} \simeq \inf_{u \in H^1(K): u = \varphi \text{ on } \partial K} |u|_{1, K}, \]

we can write

\[ \| \varphi \|_{1/2, \partial K} = |\varphi_{\partial K}|^2 + |\varphi|_{1/2, \partial K} \simeq \inf_{u \in H^1(K): u = \varphi \text{ on } \partial K} (|\bar{u}_{\partial K}|^2 + |u|_{1, K}^2), \]

and Proposition 3.1 implies the equivalence.

We now state some relations between dual norms on \( \partial K \).

**Proposition 3.2.** Let \( \lambda \in L^2(\mathcal{E}_h) \). Under Assumption 2.1, for all \( K \in \mathcal{T}_h \), we have

\[ \left( \sum_{e \in \partial K} \| \lambda \|_{(H^{1/2}(e))'}^2 \right)^{1/2} \lesssim \gamma^*_{K} \| \lambda \|_{-1/2, e} \lesssim \left( \sum_{e \in \partial K} \| \lambda \|_{2, e}^2 \right)^{1/2}. \]

**Proof.** We start by proving the second bound. Let \( \lambda \in L^2(\partial K) \). We have

\[ \sup_{u \in H^1(K)} \frac{\int_{\partial K} \lambda u}{\| u \|_{1, K}} = \sup_{u \in H^1(K)} \sum_{e \in \partial K} \frac{\int_{e} \lambda u}{\| u \|_{1, K}} \leq \sup_{u \in H^1(K)} \sum_{e \in \partial K} \frac{\| \lambda \|_{-1/2, e} \| u \|_{1/2, e}}{\| u \|_{1, K}}. \]

Let us now compare \( \| u \|_{1/2, e} \) with \( \| u \|_{1, K} \). From the definition of \( \| u \|_{1/2, e} \) and Proposition 3.1 using (3.13) we have, with \( \bar{u}^e = |e|^{-1} \int_e u \),

\[ \| u \|_{1/2, e} = |\bar{u}^e|^2 + |u|_{1/2, e}^2 \lesssim |\bar{u}^e|^2 + |u|_{1/2, \partial K} \lesssim |\bar{u}^e|^2 + |u|_{1, K}^2 \lesssim \| u \|_{1, K}. \]
yielding
\[ \sup_{u \in H^1(K)} \frac{\sum_{e \in \partial K} \lambda u |e|}{\|u\|_{1, K}} \lesssim \sum_{e \in \partial K} \|\lambda\|^{-1/2, e}_{-1/2, e} \lesssim \left( \sum_{e \in \partial K} \|\lambda\|^{-2, e}_{-1/2, e} \right)^{1/2}, \]
where we used that the number of edges of \( K \) is uniformly bounded thanks to Assumption 2.1.
This proves the second bound of the statement.

As far as the first bound is concerned, we remark that (3.12) implies that
\[ \|\varphi\|^{-1}_{1/2, \partial K} \lesssim \sup_{u \in H^1(K): \varphi|_{\partial K} = \varphi} \|u\|^{-1}_{1, K}. \]
Then we can write
\[ \|\lambda\|_{(H_0^{1/2}(e))'} = \sup_{\varphi \in H_0^{1/2}(e)} \frac{\sum_{e \in \partial K} \lambda \varphi |e|}{\|\varphi\|_{H_0^{1/2}(e)}} = \sup_{\varphi \in H^{1/2}(\partial K), \varphi|_{\partial K} = 0} \frac{\sum_{e \in \partial K} \lambda \varphi |e|}{\|\varphi\|_{1/2, \partial K}} \lesssim \sup_{\varphi \in H^{1/2}(\partial K), \varphi|_{\partial K} = \varphi} \sum_{e \in \partial K} \lambda u |e|, \]
and
\[ \|\lambda\|^{-1}_{1/2, \partial K} \lesssim \sup_{u \in H^1(K)} \frac{\sum_{e \in \partial K} \lambda u |e|}{\|u\|_{1, K}} = \sup_{u \in H^1(K)} \frac{\sum_{e \in \partial K} \lambda u |e|}{\|u\|_{1, K}} = \|\gamma^+_K \lambda\|_{-1, K}. \]
By squaring and adding up the contributions of the different edges, taking once again into account that the number of edges is uniformly bounded, we obtain the first bound, and the proof is complete. \( \square \)

3.4. Global norms and seminorms. We define the following global seminorms and norms on \( T_h \) and \( E_h \):

\[ |u|_{1, T_h}^2 = \sum_{K \in T_h} |u^K|_{1, K}^2, \quad \forall u \in \prod_{K \in T_h} H^1(K), \]
\[ |u|_{1, E_h}^2 = |u|_{1, T_h}^2 + \sum_{e \in E_h} |\vec{u}|^2, \quad \forall u \in \prod_{K \in T_h} H^1(K), \]
\[ |||u|||_{1, T_h}^2 = \sum_{K \in T_h} |||u^K|||_{1, K}^2, \quad \forall u \in \prod_{K \in T_h} H^1(K), \]
\[ |||u|||_{1, E_h}^2 = \sum_{e \in E_h} |||\vec{u}|||_{1/2, e}^2, \quad \forall u \in \prod_{K \in T_h} H^{1/2}(e)', \]
\[ \|\lambda\|_{-1/2, E_h}^2 = \sum_{e \in E_h} \|\lambda^e\|_{-1/2, e}^2, \quad \forall \lambda \in \prod_{e \in E_h} (H^{1/2}(e))', \]
\[ \|\lambda\|_{-1/2, E_h}^2 = \sum_{K \in T_h} \|\gamma^+_K \lambda^K\|_{-1, K}^2, \quad \forall \lambda \in \Lambda, \]
where the superscripts \( K \) and \( e \) denote the restrictions to \( K \) and \( e \), respectively, and \( \vec{u} \in \prod_{K \in T_h} H^1(K) \) is such that \( \vec{u}^K = \frac{1}{|K|} \int_K u \). Notice that \( |||\vec{u}||| = |\vec{u}^K - \vec{u}^-| \) on \( e \), if \( e \) is an interior edge shared by the elements \( K^+ \) and \( K^- \), or \( |||\vec{u}||| = |\vec{u}| \), if \( e \) is a boundary edge that belongs to the element \( K \).

For all \( u \in \prod_{K \in T_h} H^1(K) \), the following Poincaré-type inequality holds true (see [8, Lemma 2.6] with \( H_e = h_e = |e| \)):
\[ \|u\|_{0, \Omega} \lesssim \|u\|_{1, T_h}. \]
Moreover, it is easy to check that
\[ \|u\|_{1, T_h}^2 \lesssim \|u\|_{1, T_h}^2 + h^{-1} \|u\|_{0, \Omega}^2. \]
\[ (3.14) \]
4. Inverse inequalities in polynomial spaces on the unit interval

In this section, we recall some inverse inequalities for polynomials in positive Sobolev norms, and establish inverse inequalities in negative Sobolev norms. These results will be used in Section 6.

Assume that $I$ is an interval of unit length. We start by recalling that, for all $p \in \mathbb{P}_k(I)$, it holds that
\begin{equation}
\|p\|_{1,I} \lesssim k^2 \|p\|_{0,I},
\end{equation}
see, e.g., [19, Theorem 3.91].

We prove now inverse inequalities in negative Sobolev norms.

**Lemma 4.1.** Let $0 \leq r < s \leq 1$. If $r, s \neq 1/2$, for all $p \in \mathbb{P}_k(I)$, we have
\begin{equation}
\|p\|_{(H^s_0(I))^r} \lesssim k^{2(s-r)} \|p\|_{(H^r_0(I))^s}.
\end{equation}
Moreover, for $1/2 \leq s \leq 1$ and $0 \leq r \leq 1/2$, for all $p \in \mathbb{P}_k(I)$, we have
\begin{equation}
\|p\|_{(H^s_0(I))^r} \lesssim k^{2s-1} \|p\|_{(H^r_0(I))^s}, \\
\|p\|_{(H^r_0(I))^s} \lesssim k^{1-2r} \|p\|_{(H^s_0(I))^r}.
\end{equation}

**Proof.** We let $P^0_{k+2} : L^2(I) \rightarrow \mathbb{P}^0_{k+2}(I) = \mathbb{P}_{k+2}(I) \cap H^1_0(I)$ be defined as
\begin{equation}
\int_I (P^0_{k+2} \phi - \varphi)q = 0 \quad \text{for all } q \in \mathbb{P}_k(I).
\end{equation}
It is easy to see that $P^0_{k+2}$ is well defined. Indeed, for all $p \in \mathbb{P}^0_{k+2}(I)$, letting $q = -p'' \in \mathbb{P}_k(I)$, we have
\begin{equation}
\int_I pq = \int_I |p'|^2 \equiv 0.
\end{equation}
As $\dim(\mathbb{P}^0_{k+2}(I)) = \dim(\mathbb{P}_k(I))$, this implies that $P^0_{k+2}$ is well defined. We can write:
\begin{equation}
\|p\|^2_{0,I} = \int_I |p|^2 = \int_I p P^0_{k+2}(p) \leq \|p\|_{(H^s_0(I))^r} \|P^0_{k+2}(p)\|_{H^r_0(I)}.
\end{equation}
We then need to bound $\|P^0_{k+2}(p)\|_{H^r_0(I)}$. We have
\begin{equation}
\|P^0_{k+2}(p)\|_{H^r_0(I)} \lesssim \int_I |P^0_{k+2}(p')|^2 = - \int_I P^0_{k+2}(p'') P^0_{k+2}(p) = - \int_I P^0_{k+2}(p'') p,
\end{equation}
where the last identity stems from the definition of the projector $P^0_{k+2}$, as $P^0_{k+2}(p'') \in \mathbb{P}_k(I)$. Then, using (4.1), we have
\begin{equation}
\|P^0_{k+2}(p)\|_{H^r_0(I)} \lesssim \|P^0_{k+2}(p'')\|_{0,I} \|p\|_{0,I} \lesssim k^2 \|P^0_{k+2}(p)\|_{1,I} \|p\|_{0,I},
\end{equation}
whence by dividing both sides by $\|P^0_{k+2}(p)\|_{H^r_0(I)} = \|P^0_{k+2}(p)\|_{1,I}$ and substituting in (4.4), we obtain
\begin{equation}
\|p\|_{0,I} \lesssim k^2 \|p\|_{(H^r_0(I))^s}.
\end{equation}
By interpolating between $(H^1_0(I))^r$ and $L^2(I)$, we obtain that
\begin{equation}
\|p\|_{0,I} \lesssim k^2 \|p\|_{(H^r_0(I))^s}.
\end{equation}
Finally, by interpolating between $(H^s_0(I))^r$ and $L^2(I)$, we get (4.2). As the space $[(H^s_0(I))^r, L^2(I)]_{\theta}$ obtained by space interpolation between $(H^s_0(I))^r$ and $L^2(I)$ is, for $\theta = 1/2$, $(H^{1/2}_{00}(I))^r$ rather than $(H^{1/2}_{00}(I))^r$, for either $s = 1/2$ or $r = 1/2$ the above argument gives us (4.3). \qed
For polynomial functions, we also have the following lemma which, combined with the bound (3.11), states the equivalence of the norms for \((H^{1/2}(e))'\) and \((H_{00}^{1/2}(e))'\)

**Lemma 4.2.** For all \(p \in \mathbb{P}_k(I)\) we have

\[
\|\lambda\|_{-1/2,e} \lesssim \log k \|\lambda\|_{(H_{00}^{1/2}(e))'}.
\]

*Proof.* Denoting by \(\bar{\lambda}\) the average of \(\lambda\) on \(e\), we have

\[
\|\lambda\|_{-1/2,e}^2 = \left| \int_e \lambda \right|^2 + |\lambda|_{-1/2,e}^2.
\]

We bound the two terms on the right-hand side separately. For the first one, using (4.2) and (3.9), we have

\[
\left| \int_e \lambda \right| \lesssim \|\lambda\|_{H^{1/2-\varepsilon}(e)} \|\lambda\|_{(H_{00}^{1/2}(e))'} \lesssim \frac{k^{2\varepsilon}}{h_e^\varepsilon} \|\lambda\|_{(H_{00}^{1/2}(e))'} \lesssim \frac{k^{2\varepsilon}}{\varepsilon} \|\lambda\|_{(H_{00}^{1/2}(e))'}.
\]

For the second term, we have

\[
\|\lambda\|_{-1/2,e} = \sup_{\varphi \in H^{1/2}(e)} \left| \frac{\int_e \lambda \varphi}{\|\varphi\|_{1/2,e}} \right| \leq \sup_{\varphi \in H^{1/2}(e)} \frac{\|\lambda\|_{(H^{1/2-\varepsilon}(e))'}}{\|\varphi\|_{1/2,e}} \lesssim k^{2\varepsilon} \|\lambda\|_{(H_{00}^{1/2}(e))'},
\]

where we have used again (4.2), and (3.9). Therefore,

\[
\|\lambda\|_{-1/2,e} \lesssim k^{2\varepsilon} \|\lambda\|_{(H_{00}^{1/2}(e))'}.
\]

By taking \(\varepsilon = 1/(2 \log k)\), since \(k^{2\varepsilon} \lesssim 2 \exp(1) \log k\), we obtain (4.5). \(\square\)

5. **Stability and error analysis**

In this section, we prove well posedness of the DG formulation in Problem 2.2 and estimates of the error in the approximation of the solution to the continuous problem (2.1).

5.1. **Well posedness.** We prove the well posedness of Problem 2.2 by applying \([15, Theorem 2.2]\).

In order to do so, we specify the norms on the discrete spaces:

- \(V_h\) is endowed with the norm \(\| \cdot \|_{1,T_h}\),
- \(\Lambda_h\) is endowed with the norm \(\| \cdot \|_{-1/2,\mathcal{E}_h}\),

where the norms \(\| \cdot \|_{1,T_h}\) and \(\| \cdot \|_{-1/2,\mathcal{E}_h}\) are defined in Section 3.4. We also introduce the space \(\mathbb{V}_h = V_h \times \Lambda_h\) endowed with the product norm, which is denoted by \(\| \cdot \|_{\mathbb{V}_h}\).

We rewrite Problem 2.2 as follows: find \((u, \lambda) \in \mathbb{V}_h\) such that, for all \((v, \mu) \in \mathbb{V}_h\), it holds that

\[
a(u, \lambda; v, \mu) = F(v, \mu),
\]

where

\[
a(u, \lambda; v, \mu) = \sum_{K \in \mathcal{T}_h} \int_K \nabla u^K \cdot \nabla v^K - \int_{\mathcal{E}_h} \lambda [v] \cdot n + \int_{\mathcal{E}_h} \mu [u] \cdot n
\]

\[
+ \alpha \sum_K s_K (Du^K - \gamma^*_K \lambda; tDv^K - \gamma^*_K \mu),
\]
and
\[ F(v, \mu) = \int_{\Omega} f v + \int_{\partial \Omega} g \mu + \alpha \sum_{K} s_K (f, t Dv^K - \gamma^*_K \mu). \]

In order to prove the well posedness of Problem (5.1), we need to prove continuity of the bilinear form \( a(\cdot, \cdot) \) and of the linear functional \( F(\cdot) \), as well as an inf-sup condition for \( a(\cdot, \cdot) \) in \( \mathcal{V}_h \times \mathcal{V}_h \).

Remark that, for \( u \in H^1(K) \), we have
\[ \| Du \|_{-1,K} = \sup_{0 \neq v \in H^1(K)} \frac{\int_K \nabla u \cdot \nabla v}{\| v \|_{1,K}} \leq \| u \|_{1,K}. \]

Then, the following continuity property for the linear functional \( F(\cdot) \) is not difficult to prove:
\[ |F(v, \mu)| \lesssim C(f, g; k) \| (v, \mu) \|_{\mathcal{V}_h} \quad \forall (v, \mu) \in \mathcal{V}_h, \]
where
\[ C(f, g; k) = \log(\kappa') \inf_{v \in V : v|_{\partial \Omega} = g} \| v \|_{1, \mathcal{T}_h} + \| f \|_{0, \Omega} + \alpha M(k) \sqrt{\sum_{K} |f|^2_{-1,K}}. \]

Indeed, for \( v \in V \) with \( v|_{\partial \Omega} = g \), letting \( \tilde{\mu} \) denote the function coinciding with \( \mu \) on \( \partial \Omega \) and vanishing on \( \mathcal{E}_h \setminus \partial \Omega \), we have
\[ \int_{\partial \Omega} g \mu \lesssim \log(\kappa') \inf_{v \in V : v|_{\partial \Omega} = g \text{ on } \partial \Omega} \| v \|_{1, \mathcal{T}_h} \| \mu \|_{-1/2, \mathcal{E}_h}. \]

The arbitrariness of \( v \) yields
\[ \int_{\partial \Omega} g \mu \lesssim \log(\kappa') \inf_{v \in V : v|_{\partial \Omega} = g \text{ on } \partial \Omega} \| v \|_{1, \mathcal{T}_h} \| \mu \|_{-1/2, \mathcal{E}_h}. \]

For the continuity of the bilinear form \( a(\cdot, \cdot) \), we start by observing that we can write
\[ \int_{\mathcal{E}_h} \lambda \| v \| \cdot n = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \langle \gamma^*_K \lambda, v^K \rangle = \sum_{K \in \mathcal{T}_h} \langle \gamma^*_K \lambda, v^K - \bar{v}^K \rangle + \sum_{K \in \mathcal{T}_h} \langle \gamma^*_K \lambda, \bar{v}^K \rangle \lesssim \sum_{K \in \mathcal{T}_h} |\gamma^*_K \lambda|_{-1,K} |v|_{-1,K} + \sum_{K \in \mathcal{T}_h} |\bar{v}^K| \| \langle \gamma^*_K \lambda, 1 \rangle \|. \]

If \( \lambda \) and \( v \) satisfy the following condition
\[ \forall K \in \mathcal{T}_h \quad \text{either} \quad \langle \gamma^*_K \lambda, 1 \rangle = 0 \quad \text{or} \quad \bar{v}^K = 0, \]
then
\[ \int_{\mathcal{E}_h} \lambda \| v \| \cdot n \lesssim |v|_{1, \mathcal{T}_h} \| \lambda \|_{-1/2, \mathcal{E}_h} \]
easily follows. In the general case, we get the suboptimal bound
\[ \int_{\mathcal{E}_h} \lambda \| v \| \cdot n \lesssim \| v \|_{1, \mathcal{T}_h} \| \lambda \|_{-1/2, \mathcal{E}_h}. \]
Therefore, in the general case (without loss of generality we can assume that $\alpha \lesssim 1$), thanks to (5.2), using Assumption 2.2 we have

$$a(u, \lambda; v, \mu) \lesssim \left( \|u\|^2_{1, \mathcal{T}_h} + \| \lambda \|^2_{-1/2, \mathcal{E}_h} \right)^{1/2} \left( \|v\|^2_{1, \mathcal{T}_h} + \| \mu \|^2_{-1/2, \mathcal{E}_h} \right)^{1/2} + M(k) \left( |u|_{1, \mathcal{T}_h} + \| \lambda \|_{-1/2, \mathcal{E}_h} \right) \left( |v|_{1, \mathcal{T}_h} + \| \mu \|_{-1/2, \mathcal{E}_h} \right).$$

If $(u, \lambda)$ and $(v, \mu)$ are such that both $\lambda$ and $v$, and $\mu$ and $u$ satisfy condition (5.6), then we have

$$a(u, \lambda; v, \mu) \lesssim |u|_{1, \mathcal{T}_h} |v|_{1, \mathcal{T}_h} + |u|_{1, \mathcal{T}_h} \| \mu \|_{-1/2, \mathcal{E}_h} + |v|_{1, \mathcal{T}_h} \| \lambda \|_{-1/2, \mathcal{E}_h} + M(k) \left( |u|_{1, \mathcal{T}_h} + \| \lambda \|_{-1/2, \mathcal{E}_h} \right) \left( |v|_{1, \mathcal{T}_h} + \| \mu \|_{-1/2, \mathcal{E}_h} \right),$$

that yields

$$|a(u, \lambda; v, \mu)| \leq C \left( 1 + M(k) \right) \|u, \lambda\|_{\mathcal{V}_h} \|(v, \mu)\|_{\mathcal{V}_h}. \tag{5.7}$$

We prove the following proposition (inf-sup condition).

**Proposition 5.1.** (Inf-sup for $a(\cdot, \cdot)$) We have

$$\inf_{(u, \lambda) \in \mathcal{V}_h} \sup_{(v, \mu) \in \mathcal{V}_h} \frac{a(u, \lambda; v, \mu)}{\|u, \lambda\|_{\mathcal{V}_h} \|(v, \mu)\|_{\mathcal{V}_h}} \gtrsim \left( \frac{\rho(k)}{M(k)} \right)^2, \tag{5.8}$$

where $M(k)$ and $\rho(k)$ are as in Assumptions 2.2 and 2.3, respectively.

**Proof.** Let $(u, \lambda) \in \mathcal{V}_h$, and let

$$v = u - \hat{v}, \quad \text{with} \quad \hat{v}|_K = \hat{v}^K, \quad \hat{v}^K = \int_{\partial K} \lambda(n_K \cdot n) = \langle \gamma^*_K \lambda, 1 \rangle, \quad K \in \mathcal{T}_h,$$

and

$$\mu = \lambda + \beta \hat{\mu}, \quad \text{with} \quad \hat{\mu}|_e = h_e^{-1} \hat{u} \cdot n, \quad e \in \mathcal{E}_h,$$

where $\hat{u}$ denote the piecewise constant function that assumes on each $K$ the value $\hat{u}^K$ of the average of $u$ on $K$, and where $\beta$ is a positive constant whose choice will be made later on. We observe that

$$\|\hat{v}\|^2_{1, \ast} = \sum_{e \in \mathcal{E}_h} \sum_{K: e \subset K} \langle \gamma^{\ast}_K \lambda, 1 \rangle^2 \lesssim \sum_{K} |\langle \gamma^{\ast}_K \lambda, 1 \rangle|^2,$$

where the last bound is obtained by using the fact that any edge belongs to at most two elements, and that the number of edges per element is uniformly bounded by a constant, thanks to Assumption 2.1. Moreover, we have that

$$\|\hat{\mu}\|_{-1/2, e}^2 = \|\hat{u}\|^2, \tag{5.9}$$

which, thanks to Proposition 3.2 yields

$$\|\hat{\mu}\|^2_{-1/2, \mathcal{E}_h} \lesssim \sum_{e \in \mathcal{E}_h} \|\hat{u}\|^2. \tag{5.10}$$

By combining the previous two bounds and applying a triangular inequality, we get

$$\|v, \mu\|_{\mathcal{V}_h} \lesssim \|u, \lambda\|_{\mathcal{V}_h}. \tag{5.11}$$
We can write
\[ a(u, \lambda; v, \mu) = \sum_{K \in T_h} |u^K|^2_{1, K} + \sum_{K \in T_h} |(\gamma^*_K \lambda, 1)|^2 + \beta \sum_{e \in \mathcal{E}_h} h_e^{-1} \int_e [u] \cdot [\bar{u}] \]
\[ + \alpha \sum_{K \in T_h} s_K(Du^K - \gamma^*_K \lambda, tDu^K - \gamma^*_K (\lambda + \beta \bar{\mu})). \]

By adding and subtracting \( \bar{u} \), and using a Young inequality, we have
\[ h_e^{-1} \int_e [u] \cdot [\bar{u}] = h_e^{-1} \int_e [[\bar{u}]]^2 + h_e^{-1} \int_e [u - \bar{u}] \cdot [\bar{u}] \geq \frac{1}{2} [[\bar{u}]]^2 - \frac{1}{2} h_e^{-1} \int_e [[u - \bar{u}]]^2. \]

We can bound the last term as
\[ \frac{1}{2} h_e^{-1} \int_e [[u - \bar{u}]]^2 \leq h_e^{-1} \left( \int_e |u^{K^+} - \bar{u}^{K^+}|^2 + \int_e |u^{K^-} - \bar{u}^{K^-}|^2 \right) \]
\[ \leq h_e^{-1} \left( \|u^{K^+} - \bar{u}^{K^+}\|_{0, \partial K^+}^2 + \|u^{K^-} - \bar{u}^{K^-}\|_{0, \partial K^-}^2 \right) \lesssim |u^{K^+}|^2_{1, K^+} + |u^{K^-}|^2_{1, K^-}, \]

where in the last inequality we have used the trace inequality (2.2), the shape regularity assumption Assumption 2.1 (ii), and the Poincaré inequality (2.3). Therefore, we obtain, with some positive constant \( c \),
\[ h_e^{-1} \int_e [u] \cdot [\bar{u}] \geq \frac{1}{2} [[\bar{u}]]^2 - \frac{1}{2} c (|u^{K^+}|^2_{1, K^+} + |u^{K^-}|^2_{1, K^-}). \]

Then we can write, again for some positive constant \( c \),
\[ a(u, \lambda; v, \mu) \geq \sum_{K \in T_h} |u^K|^2_{1, K} + \sum_{K \in T_h} |(\gamma^*_K \lambda, 1)|^2 + \beta \sum_{e \in \mathcal{E}_h} \frac{1}{2} [[\bar{u}]]^2 \]
\[ - c\beta \sum_{K \in T_h} |u^K|^2_{1, K} + \alpha \sum_{K \in T_h} s_K(Du^K - \gamma^*_K \lambda, tDu^K - \gamma^*_K (\lambda + \beta \bar{\mu})) \]
\[ = \sum_{K \in T_h} |u^K|^2_{1, K} + \sum_{K \in T_h} |(\gamma^*_K \lambda, 1)|^2 + \frac{\beta}{2} \sum_{e \in \mathcal{E}_h} [[\bar{u}]]^2 \]
\[ - c\beta \sum_{K \in T_h} |u^K|^2_{1, K} - \alpha \sum_{K \in T_h} s_K(Du^K, \gamma^*_K \lambda) - \alpha \beta \sum_{K \in T_h} s_K(Du^K, \gamma^*_K \lambda) \]
\[ - \alpha \sum_{K \in T_h} t s_K(\gamma^*_K \lambda, Du^K) + \alpha \sum_{K \in T_h} s_K(\gamma^*_K \lambda, \gamma^*_K \lambda) + \alpha \beta \sum_{K \in T_h} s_K(\gamma^*_K \lambda, \gamma^*_K \lambda) \]
\[ = (1 - c\beta) \sum_{K \in T_h} |u^K|^2_{1, K} + \sum_{K \in T_h} |(\gamma^*_K \lambda, 1)|^2 + \frac{\beta}{2} \sum_{e \in \mathcal{E}_h} [[\bar{u}]]^2 + T_1 + T_2 + T_3 + T_4 + T_5 + T_6. \]

We bound separately the terms \( T_1 \) to \( T_6 \) on the right-hand side. We immediately observe that \( T_5 \) yields control on \( \gamma^*_K \lambda \). In fact, by Assumption 2.2 we have
\[ T_5 = \alpha \sum_{K \in T_h} s_K(\gamma^*_K \lambda, \gamma^*_K \lambda) \geq \alpha \rho(k) \sum_{K \in T_h} |\gamma^*_K \lambda|^2_{-1, K}. \]
On the other hand, by Assumption 2.2, we have

\[
|T_1| = \left| \alpha \sum_{K \in T_h} t s_K(Du^K, Du^K) \right| 
\leq |t| \alpha M(k) \sum_{K \in T_h} |u^K|_{1,K}^2,
\]
as well as

\[
|T_2| = \left| \alpha \sum_{K \in T_h} s_K(Du^K, \gamma_K^* \lambda) \right| 
\leq \alpha M(k) \sum_{K \in T_h} |u^K|_{1,K} \gamma_K^* \lambda \gamma_K \lambda_{-1,K} 
\leq \alpha M(k) \frac{\varepsilon_3}{2} \sum_{K \in T_h} |\gamma_K^* \lambda|_{-1,K}^2 + \alpha M(k) \frac{1}{2\varepsilon_3} \sum_{K \in T_h} |u^K|_{1,K}^2,
\]
and

\[
|T_4| = \left| \alpha \sum_{K \in T_h} t s_K(\gamma_K^* \lambda, Du^K) \right| 
\leq |t| \alpha M(k) \sum_{K \in T_h} |\gamma_K^* \lambda|_{-1,K} |u^K|_{1,K} 
\leq |t| \alpha M(k) \frac{\varepsilon_1}{2} \sum_{K \in T_h} |\gamma_K^* \lambda|_{-1,K}^2 + |t| \alpha M(k) \frac{1}{2\varepsilon_1} \sum_{K \in T_h} |u^K|_{1,K}^2.
\]

Using (5.10), we also have

\[
|T_3| = \left| \alpha \beta \sum_{K \in T_h} s_K(Du^K, \gamma_K^* \hat{\mu}) \right| 
\leq \alpha \beta M(k) \sum_{K \in T_h} |u^K|_{1,K} |\gamma_K^* \hat{\mu}|_{-1,K} 
\leq \frac{1}{2} \alpha \beta M(k) \sum_{e \in E_h} \|\tilde{u}\|^2 + \frac{1}{2} \alpha \beta M(k) \sum_{K \in T_h} |u^K|_{1,K}^2,
\]
and

\[
|T_6| = \left| \alpha \beta \sum_{K \in T_h} s_K(\gamma_K^* \lambda, \gamma_K^* \hat{\mu}) \right| 
\leq \alpha \beta M(k) \sum_{K \in T_h} |\gamma_K^* \lambda|_{-1,K} |\gamma_K^* \hat{\mu}|_{-1,K} 
\leq \alpha \beta M(k) \left( \sum_{K \in T_h} |\gamma_K^* \lambda|_{-1,K}^2 \right)^{1/2} \left( \sum_{e \in E_h} \|\tilde{u}\|^2 \right)^{1/2} 
\leq \alpha \beta M(k) \frac{\varepsilon_2}{2} \sum_{K \in T_h} |\gamma_K^* \lambda|_{-1,K}^2 + \alpha \beta M(k) \frac{1}{2\varepsilon_2} \sum_{e \in E_h} \|\tilde{u}\|^2.
\]

By combining everything, we obtain

\[
a(u, \lambda; v, \mu) \geq \left( 1 - c\beta - \frac{\alpha M(k)}{2\varepsilon_3} - \frac{\alpha \beta M(k)}{2} - |t| \alpha M(k) - \frac{|t| \alpha M(k)}{2\varepsilon_1} \right) \sum_{K \in T_h} |u^K|_{1,K}^2 
+ \frac{\varepsilon_1}{2} \sum_{K \in T_h} |\gamma_K^* \lambda|_{-1,K}^2 
+ \beta \left( 1 - \frac{\alpha M(k)}{\varepsilon_2} - \alpha M(k) \right) \sum_{e \in E_h} \|\tilde{u}\|^2 
+ \alpha \left( \rho(k) - \frac{\beta M(k)\varepsilon_2}{2} - \frac{M(k)\varepsilon_3}{2} - \frac{|t|M(k)\varepsilon_1}{2} \right) \sum_{K \in T_h} |\gamma_K^* \lambda|_{-1,K}^2.
\]
We now set \( \beta = 1/(2c) \), and we choose \( \epsilon_1, \epsilon_2, \) and \( \epsilon_3 \) as \( C\rho(k)/M(k) \), with \( C \) sufficiently small so that \( \left( \rho(k) - \frac{\beta M(k)\epsilon_2}{2} - \frac{M(k)\epsilon_3}{2} - \frac{|\ell|M(k)\epsilon_1}{2} \right) \leq \frac{\rho(k)}{2} \). Recalling that \( \rho(k) \leq M(k) \), because \( \rho(k) \) and \( M(k) \) are coercivity and continuity constants, respectively, we choose \( \alpha = C\rho(k)/(M(k))^2 \), with \( C \) sufficiently small so that all the constants are bounded from below by \( (\rho(k)/M(k))^2 \) (up to a constant).

Observe that neither \( \beta \) nor \( \alpha \) depend on \( h \); \( \beta \) is also independent of \( k \), but \( \alpha \) depends on \( k \) and behaves as \( \rho(k)/(M(k))^2 \) for increasing \( k \).

With such a choice, for a constant \( c_0 \) independent of \( h \) but dependent on \( k \) as \( (\rho(k)/M(k))^2 \), we have

\[
a(u, \lambda; v, \mu) \geq c_0(k) \left( \sum_{K \in T_h} |u_K|^2_{1,K} + \sum_{e \in E_h} \|\bar{u}\|^2_{e} + \sum_{K \in T_h} |\gamma_{K}^* \lambda|^2_{-1,K} + \sum_{K \in T_h} \langle \gamma_{K}^* \lambda, 1 \rangle^2 \right).
\]

Therefore, using (5.11), we conclude that

\[
\sup_{(v, \mu) \in V_h} \frac{a(u, \lambda; v, \mu)}{\|v\|_{V_h}} \geq a(u, \lambda; u, \lambda + \hat{\lambda}) \geq \left( \frac{\rho(k)}{M(k)} \right)^2 \frac{\|u\|_{V_h}^2}{\|\lambda\|_{V_h}^2}.
\]

Owing to the continuity properties (5.3) and (5.7), and the inf-sup condition in Proposition 5.8, we apply [15, Theorem 2.2] and conclude with the following result.

**Theorem 5.2.** Under Assumptions 2.1 2.2 and 2.3 Problem 2.2 admits a unique solution \((u_h, \lambda_h)\). Moreover, the following stability bound for \((u_h, \lambda_h)\) is satisfied:

\[
\|u_h, \lambda_h\|_{V_h} \lesssim \left( \frac{M(k)}{\rho(k)} \right)^2 \left( \log(k') \inf_{v \in V, v = g \text{ on } \partial \Omega} \|v\|_{1, T_h} + |f|_{0, \Omega} + \alpha M(k) \sqrt{\sum K |f|^2_{-1,K}} \right).
\]

5.2. **Error estimate.** We have the following theorem.

**Theorem 5.3.** Under Assumptions 2.1 2.2 and 2.3, letting \( u \) denote the solution of (2.1), \( \lambda = \nabla u \cdot n \), and \((u_h, \lambda_h)\) the solution of Problem 2.2, and assuming that \( u \in H^\ell(\Omega), \ell \geq 2 \), then the following bound holds:

\[
\|u - u_h, \lambda - \lambda_h\|_{V_h} \lesssim (1 + M(k)) \left( \frac{M(k)}{\rho(k)} \right)^2 \frac{h^{s-1}}{k^{s-1}} |u|_{\ell, \Omega}.
\]

where \( s = \min\{\ell, k + 1\} \).

**Proof.** As we are interested in a \( k \)-robust estimate, for the sake of simplicity we can assume that \( k' \neq 0 \), which is always the case except when \( k = 1, k' = k - 1 \). The latter case, which has little interest in our framework, can be treated with minor modifications to the following arguments. Let us start by observing that, letting \((u, \lambda) \in V \times \Lambda\) denote the solution to (2.7), for any \((w, \zeta) \in V_h\) it holds that

\[
a(u, \lambda; w, \zeta) = F(w, \zeta).
\]
Then, using (5.8), for any \((v, \mu) \in \mathbb{V}_h\) with, for all \(K \in \mathcal{T}_h\) and \(e \in \mathcal{E}_h\), \(\int_K v^K = \int_K u\) and \(\int_e \mu = \int_e \lambda\), we can write:

\[
\left( \frac{\rho(k)}{M(k)} \right)^2 \|(u_h - v, \lambda_h - \mu)\|_{\mathbb{V}_h} \lesssim \sup_{(w, \zeta) \in \mathbb{V}_h} \frac{a(u_h - v, \lambda_h - \mu; w, \zeta)}{\|w, \zeta\|_{\mathbb{V}_h}} = \sup_{(w, \zeta) \in \mathbb{V}_h} \frac{a(u - v, \lambda - \mu; w, \zeta)}{\|w, \zeta\|_{\mathbb{V}_h}} \lesssim (1 + M(k)) \|(u - v, \lambda - \mu)\|_{\mathbb{V}_h}.
\]

Using a triangular inequality and the arbitrariness of \(\mu\) we obtain

\[
\|(u - u_h, \lambda - \lambda_h)\|_{\mathbb{V}_h} \lesssim (1 + M(k)) \left( \frac{M(k)}{\rho(k)} \right)^2 \inf_{(v, \mu) \in \mathbb{V}_h} \|(u - v, \lambda - \mu)\|_{\mathbb{V}_h}.
\]

In order to bound the right hand side, we recall (see Lemma 23 of [11]) that, for all \(K\) and for all \(u \in H^\ell(K)\), with \(\ell \geq 1\), there exist a polynomial \(\tilde{\Pi}_K u \in \mathbb{P}_k(K)\) such that, under Assumption 2.1, we have that

\[
h_K^{-1}\|u - \tilde{\Pi}_K u\|_{0,K} + \|u - \tilde{\Pi}_K u\|_{1,K} \lesssim \frac{h_K^{s-1}}{k^\ell-1}\|u\|_{H^\ell(K)},
\]

where \(s = \min\{\ell, k + 1\}\), \(| \cdot \|_{H^\ell(K)}\) denotes the standard, unscaled norm for \(H^\ell(K)\):

\[
\|u\|^2_{H^\ell(K)} = \sum_{j=0}^\ell \sum_{|\alpha| = j} \int_K \left( \frac{\partial^{|\alpha|} u}{\partial x^\alpha} \right)^2.
\]

Moreover, for \(\lambda \in H^\ell(e)\) and \(\pi_e \lambda\) its \(L^2(e)\) projection into \(\mathbb{P}_k(e)\), we have

\[
\|\lambda - \pi_e \lambda\|_{0,e} \lesssim \frac{h_e^{s'}}{(k')^{1/2}} |\lambda|_{\ell,e},
\]

with \(s' = \min\{\ell', k' + 1\}\). Using an Aubin-Nitsche duality argument, we can write

\[
\|\lambda - \pi_e \lambda\|_{-1/2,e} = \|\lambda - \pi_e \lambda\|_{-1/2,e} \equiv \sup_{\varphi \in H^{1/2}(e)} \int_e (\lambda - \pi_e \lambda) \varphi \int_{1/2,e} \varphi = \sup_{\varphi \in H^{1/2}(e)} \frac{\int_e (\lambda - \pi_e \lambda) \varphi}{\|\varphi\|_{1/2,e}} \lesssim \|\lambda - \pi_e \lambda\|_{0,e} \sup_{\varphi \in H^{1/2}(e)} \frac{\|\varphi - \pi_e \varphi\|_{0,e}}{\|\varphi\|_{1/2,e}} \lesssim \frac{h_e^{1/2}}{(k')^{1/2}} |\lambda - \pi_e \lambda|_{0,e},
\]

finally yielding

\[
\|\lambda - \pi_e \lambda\|_{-1/2,e} \lesssim \frac{h_e^{s'+1/2}}{(k')^{1/2+1/2}} |\lambda|_{e',e}.
\]

Let now the solution of (2.1) satisfy \(u \in H^\ell(\Omega)\) with \(\ell \geq 2\). As \(\nabla u \in H^{\ell-1}(\Omega) \subseteq H^1(\Omega)\) we have that for edge of \(K\), \(\nabla u \cdot n \in H^{\ell-3/2}(e) \subseteq H^{1/2}(e)\) and \(\|\nabla u \cdot n\|_{\ell-3/2,e} \lesssim \|\nabla u\|_{\ell-1,K}\).
Letting then \( u^K = \tilde{\Pi}_h^K(u) \) and \( \mu|_e = \pi_e(\nabla u \cdot n) \), and observing that for \( k > 1 \) we have \( k' \gtrless k \), thanks to (3.14) we have
\[
\| (u - v, \lambda - \mu) \|^2_{\mathbb{V}_h} \lesssim \frac{h_{K}^{2(s-1)}}{k^{2\ell-1}} \sum_{K \in \mathcal{T}_h} \left( \| u \|^2_{L^2(K)} + \sum_{e \subset \partial K} \| \nabla u \cdot n \|^2_{L^{3/2}(e)} \right) \lesssim \sum_{K \in \mathcal{T}_h} \frac{h_{K}^{2(s-1)}}{k^{2\ell-1}} \| u \|^2_{L^2(K)},
\]
which concludes the proof. \( \square \)

6. Stabilization forms

In order for the proposed method to be practically feasible, we need to construct computable bilinear forms \( s_K(\cdot, \cdot) \) satisfying Assumptions 2.2 and 2.3. We follow the approach of [6]. Let
\[
(6.1) \quad \Lambda_K = \Lambda_k|_{\partial K} = \{ \lambda \in L^2(\partial K) : \lambda|_e \in P_{k'}(e), \ e \subset \partial K \}
\]
we recall that \( k' \in \{ k, k - 1 \} \), and introduce an auxiliary space \( W^K \subseteq H^1(K) \) with \( W^K \cap P_0(K) = \{ 0 \} \), and with \( \dim(W^K) = n_K = \dim(\Lambda_K) - 1 \), satisfying, for some positive constant \( \rho(k') \), an inf-sup condition of the form
\[
(6.2) \quad \inf_{\lambda \in \Lambda_K} \sup_{\lambda \neq 0, w \in W^K} \frac{\int_K \lambda w}{|\lambda|_{1,K} |w|_{1,K}} \geq \rho(k).
\]
The choice of the subspace \( W^K \) that characterize our method is specified below.

Let \( \varphi_i, i = 1, \ldots, n_K \), denote a basis for \( W^K \). Consider the operator \( \sigma : H^1(K) \times H^1(K) \to \mathbb{R} \) given by
\[
\sigma(w, v) = \int_K \nabla w \cdot \nabla v.
\]
We observe that we have
\[
|\sigma(w, v)| \leq |w|_{1,K} |v|_{1,K}, \quad \sigma(w, w) = |w|^2_{1,K}.
\]
We also observe that, as \( W^K \cap P_0(K) = \{ 0 \} \), the seminorm \( |\cdot|_{1,K} \) is a norm on \( W^K \). We let \( \Sigma \) denote the stiffness matrix associated with the restriction of \( \sigma(\cdot, \cdot) \) to \( W^K \), i.e.
\[
\Sigma_{ij} = \sigma(\varphi_j, \varphi_i), \quad i, j = 1, \ldots, n_K.
\]

We can now introduce the following bilinear form \( s_K : (H^1(K))' \times (H^1(K))' \to \mathbb{R} \) defined as
\[
s_K(F, G) = \bar{F}^T \Sigma^{-1} \bar{G}, \quad \text{with} \quad \bar{F} = (\langle F, \varphi_i \rangle)_{i=1}^{n_K}, \quad \bar{G} = (\langle G, \varphi_i \rangle)_{i=1}^{n_K}.
\]
It is not difficult to prove that the bilinear form \( s_K(\cdot, \cdot) \) satisfies Assumption 2.2 with \( M(k) = 1 \). Moreover, it is possible to prove (see [6]) that, provided that (6.2) holds, \( s_K(\cdot, \cdot) \) satisfies also Assumption 2.3 (actually, (6.2) is a necessary and sufficient condition for Assumption 2.3 to hold).

Observe that, for \( u, v \in (H^1(K))' \) and \( \lambda, \mu \in H^{-1/2}(\partial K) \), we have
\[
(6.3) \quad s_K(Du - \gamma^*_K \lambda, tDv - \gamma^*_K \mu) = \bar{F}^T \Sigma^{-1} \bar{\zeta}, \quad s_K(F, tDv - \gamma^*_K \mu) = \bar{F}^T \Sigma^{-1} \bar{\zeta}
\]
with
\[
(6.4) \quad \eta_i = \int_K \nabla u \cdot \nabla \varphi_i - \int_{\partial K} \lambda_i \varphi_i, \quad \zeta_i = t \int_K \nabla v \cdot \nabla \varphi_i - \int_{\partial K} \mu_i \varphi_i, \quad f_i = \int_K f \varphi_i.
\]
In order to complete the definition of our method, we only need to choose the subspace $W^K$ of $H^1(K)$. In order to do that, we subdivide the polygonal element $K$ into $N_K$ triangles $T_i$, $1 \leq i \leq N_K$, each having one edge, denoted by $e_i$, coinciding with one edge of $K$, and the opposite vertex coinciding with $x_K$, the center of the ball in Assumption 2.1 (i). Due to Assumption 2.1 all the triangles $T_i$ are shape regular. Let us consider a reference triangle $\hat{T}$ and denote by $F_i$ the affine maps from $\hat{T}$ to $\hat{T}_i$, defined in such a way that the edge $\hat{e}$ is mapped onto $e_i$. We construct a finite dimensional space $\hat{W} \subseteq H^1(\hat{T})$ as follows.

Let $\hat{V}_\delta \subseteq H^1(\hat{T})$ be a family of finite dimensional approximation spaces, whose elements vanish on $\partial \hat{T} \setminus \hat{e}$, each constructed on a quasi uniform mesh of $\hat{T}$ of mesh size $\delta$. The approximation assumptions $\hat{V}_\delta$ needs to satisfy are stated in Lemma 6.1 and Theorem 6.2 below; a specific choice will be given in Section 8.

We define the operator $\mathcal{G} : \mathbb{P}_{k'}(\hat{e}) \to \hat{V}_\delta$ that maps $\hat{\lambda} \in \mathbb{P}_{k'}(\hat{e})$ to the (unique) function $\hat{\varphi} \in \hat{V}_\delta$ that satisfies
\begin{equation}
\int_{\hat{T}} \nabla \hat{\varphi} \cdot \nabla \hat{v} = \int_{\hat{e}} \hat{\lambda} \hat{v} \quad \text{for all } \hat{v} \in \hat{V}_\delta.
\end{equation}
Notice that $\mathcal{G}(\hat{\lambda})$ is a discretized harmonic lifting in $\hat{T}$ of the Neumann datum $\hat{\lambda}$ on $\hat{e}$; see the proof of Lemma 6.1 below. We then define $\hat{W}$ as
\[ \hat{W} = \mathcal{G}(\mathbb{P}_{k'}(\hat{e})). \]

We set
\begin{equation}
W_i = \{ \hat{w} \circ F_i^{-1} \text{ with } \hat{w} \in \hat{W} \},
\end{equation}
and
\begin{equation}
W^K = \{ w \in L^2(K) : w|_{T_i} \in W_i, 1 \leq i \leq N_K \}.
\end{equation}

Notice that, as the functions in $W_i$ have zero Dirichlet traces along the edges of each $T_i$ interior to $K$, the functions of $W^K$ are continuous; therefore $W^K \subset H^1(K)$. We also remark that, in order to construct $W^K$ for any $K \in \mathcal{T}_h$, one needs to solve (6.5) for each function of a basis of $\mathbb{P}_{k'}(\hat{e})$ on the reference element. This can be done offline once and for all; for more details, see Section 8.

We prove the following inf-sup condition on the reference triangle $\hat{T}$.

**Lemma 6.1.** Assume that the space $\hat{V}_\delta$ is such that, for all $v \in H^2(\hat{T})$ with $v = 0$ on $\partial \hat{T} \setminus \hat{e}$, it holds
\begin{equation}
\inf_{v_\delta \in \hat{V}_\delta} \| v - v_\delta \|_{1,\hat{T}} \lesssim \delta |v|_{2,\hat{T}}.
\end{equation}
Then there exists a constant $c_0 > 0$ independent of $k'$ such that, provided that $\delta < c_0(k')^{-2}$, we have
\[ \inf_{\lambda \in \mathbb{P}_{k'}(\hat{e})} \sup_{\varphi \in \hat{W}} \frac{\int_{\hat{e}} \lambda \hat{\varphi}}{\| \lambda \|_{(H^1(\hat{T}))'}} |\hat{\varphi}|_{1,\hat{T}} \gtrsim 1. \]

**Proof.** Fix $\lambda \in \mathbb{P}_{k'}(\hat{e})$. We let $u_\lambda$ denote the solution to
\begin{equation}
-\Delta u_\lambda = 0 \text{ in } \hat{T}, \quad u_\lambda = 0 \text{ on } \partial \hat{T} \setminus \hat{e}, \quad \nabla u_\lambda \cdot n = \lambda \text{ on } \hat{e}.
\end{equation}
Writing (6.9) in variational form, we easily see that \( u_\lambda \) satisfies

\[
\int_{\hat{T}} \nabla u_\lambda \cdot \nabla v = \int_{\tilde{e}} \lambda v \quad \text{for all } v \in H^1(\hat{T}) \text{ with } v = 0 \text{ on } \partial \hat{T} \setminus \tilde{e}.
\]

We can then write

\[
\int_{\tilde{e}} u_\lambda \lambda = \int_{\hat{T}} |\nabla u_\lambda|^2 = |u_\lambda|_{1,\hat{T}}^2 \gtrsim |u_\lambda|_{1,\hat{T}} \|\lambda\|_{(H^{1/2}_0(\tilde{e}))'},
\]

where, using (6.9), the last bound follows from

\[
(6.11) \quad \|\lambda\|_{(H^{1/2}_0(\tilde{e}))'} = \sup_{\varphi \in H^{1/2}_0(\tilde{e})} \frac{\int_{\tilde{e}} \lambda \varphi}{\|\varphi\|_{H^{1/2}_0(\tilde{e})}} \simeq \sup_{\varphi \in H^{1/2}(\partial T), \varphi_{|\partial T} = 0} \frac{\int_{\hat{T}} |\nabla u_\lambda \cdot \nabla \varphi|}{\|\varphi\|_{H^{1/2}(\partial T)}} \lesssim |u_\lambda|_{1,\hat{T}},
\]

\( \tilde{\varphi} \in H^1(\hat{T}) \) denoting the harmonic lifting of \( \varphi \).

Let now \( \hat{\varphi} = \mathcal{G}(\lambda) \in \hat{W} \). It is easily seen that \( \hat{\varphi} \) is the Galerkin projection of \( u_\lambda \) onto \( \hat{\delta} \). Then it holds

\[
(6.12) \quad |\hat{\varphi}|_{1,\hat{T}} \leq |u_\lambda|_{1,\hat{T}}.
\]

By using (4.3), we have

\[
(6.13) \quad \int_{\tilde{e}} \lambda \hat{\varphi} = \int_{\tilde{e}} \lambda (\hat{\varphi} - u_\lambda) + \int_{\tilde{e}} \lambda u_\lambda \geq -\|\lambda\|_{0,\tilde{e}} \|u_\lambda - \hat{\varphi}\|_{0,\partial \hat{T}} + |u_\lambda|_{1,\hat{T}} \|\lambda\|_{(H^{1/2}_0(\tilde{e}))'}
\]

\[
\geq -k' \|\lambda\|_{(H^{1/2}_0(\tilde{e}))'} \|u_\lambda - \hat{\varphi}\|_{0,\partial \hat{T}} + |u_\lambda|_{1,\hat{T}} \|\lambda\|_{(H^{1/2}_0(\tilde{e}))'}
\]

\[
\gtrsim -k' \|\lambda\|_{(H^{1/2}_0(\tilde{e}))'} \|u_\lambda - \hat{\varphi}\|_{1,\hat{T}}^{1/2} + |u_\lambda|_{1,\hat{T}} \|\lambda\|_{(H^{1/2}_0(\tilde{e}))'}
\]

\[
\geq -\delta^{1/2} k' \|\lambda\|_{(H^{1/2}_0(\tilde{e}))'} |u_\lambda|_{1,\hat{T}} + |u_\lambda|_{1,\hat{T}} \|\lambda\|_{(H^{1/2}_0(\tilde{e}))'}
\]

where we have used the trace bound (2.2), the Aubin-Nitsche duality argument, which, thanks to (6.8), allows us to estimate \( \|u_\lambda - \hat{\varphi}\|_{0,\partial \hat{T}} \) with \( \delta |u_\lambda - \hat{\varphi}|_{1,\hat{T}} \), and (6.12). Then we have

\[
\int_{\tilde{e}} \lambda \hat{\varphi} \geq |\hat{\varphi}|_{1,\hat{T}} \|\lambda\|_{(H^{1/2}_0(\tilde{e}))'} (c_1 - c_2 \delta^{1/2} k').
\]

Therefore, by choosing \( \delta \) in such a way that \( (c_1 - c_2 \delta^{1/2} k') > 0 \) (i.e. \( \delta < c_1/(c_2(k')^2) \)), and inserting (6.12) into (6.13), after dividing by \( |\hat{\varphi}|_{1,\hat{T}} \), we get the thesis.

Due to a scaling argument, thanks to the shape regularity of the triangles \( T_i \), the inf-sup condition of Lemma 6.1 implies the following inf-sup condition on \( T_i \):

\[
(6.14) \quad \inf_{\lambda \in \mathbb{R}} \sup_{\varphi \in W_i} \frac{\int_{\tilde{e}} \lambda \varphi}{\|\lambda\|_{(H^{1/2}_0(\tilde{e}))'} |\varphi|_{1,T_i}} \gtrsim 1,
\]

provided that the parameter \( \delta \) entering the definition of \( \hat{\delta} \) in the construction of \( \hat{W} \) satisfies \( \delta < c_0(k')^{-2} \), with \( c_0 > 0 \) given in Lemma 6.1.

By mapping and assembling on all subtriangles \( T_i \) of \( K \), we obtain the following inf-sup condition on \( K \).
Theorem 6.2. Let the space $W^K$ be defined as in (6.7) with $\delta < c_0(k')^{-2}$, where the constant $c_0 > 0$ is given in Lemma 6.1. Then, under Assumption 2.1, we have, for all $K \in T_h$,
\[
\inf_{\lambda \in \Lambda^K} \sup_{\varphi \in W^K} \frac{\int_{\partial K} \lambda \varphi}{|\lambda|_{-1/2, \partial K} |\varphi|_{1,K}} \gtrsim (\log k')^{-1}.
\]

Proof. Let $\{e_i\}_{i=1}^{N_K}$ denote the set of edges of $K$. Recall that $\Lambda^K = \prod_{1 \leq i \leq N_K} \mathbb{P}_i(e_i)$ and $W^K \sim \Pi_{1 \leq i \leq N_K} W_i$.

By a standard argument as in [7], from the local inf-sup conditions (6.14), we have
\[
\inf_{\lambda \in \Lambda^K} \sup_{\varphi \in W^K} \sum_{i=1}^{N_K} \int_{e_i} \lambda \varphi \leq \left( \sum_{i=1}^{N_K} |\lambda|_{(H^{1/2}_0(e_i))'}^2 \right)^{1/2} \left( \sum_{i=1}^{N_K} |\varphi|_{1,T_i}^2 \right)^{1/2} \gtrsim 1.
\]

As $\sum_{i=1}^{N_K} \int_{e_i} \lambda \varphi = \int_{\partial K} \lambda \varphi$ and $\left( \sum_{i=1}^{N_K} |\varphi|_{1,T_i}^2 \right)^{1/2} = |\varphi|_{1,K}$, we only need to prove the bound
\[
(6.15) \quad |\lambda|_{-1/2, \partial K} = \sup_{\varphi \in H^{1/2}(\partial K)} \frac{\int_{\partial K} \lambda \varphi}{|\varphi|_{1/2, \partial K}} \lesssim \log k' \left( \sum_{i=1}^{N_K} |\lambda|_{(H^{1/2}_0(e_i))'}^2 \right)^{1/2}.
\]

From the Cauchy-Schwarz inequality, we have
\[
\int_{\partial K} \lambda \varphi = \sum_{i=1}^{N_K} \int_{e_i} \lambda \varphi \leq \sum_{i=1}^{N_K} \|\lambda\|_{-1/2, e_i} \|\varphi\|_{1/2, e_i} \leq \left( \sum_{i=1}^{N_K} \|\lambda\|_{-1/2, e_i}^2 \right)^{1/2} \left( \sum_{i=1}^{N_K} \|\varphi\|_{1/2, e_i}^2 \right)^{1/2}.
\]

Now, on the one hand, denoting by $\bar{\varphi}^e_i$ the average of $\varphi$ on $e_i$, i.e., $\bar{\varphi}^e_i = |e_i|^{-1} \int_{e_i} \varphi$, we have
\[
\sum_{i=1}^{N_K} \|\varphi\|_{1/2, e_i}^2 = \sum_{i=1}^{N_K} (|\bar{\varphi}^e_i|^2 + |\varphi|_{1/2, e_i}^2) \leq \sum_{i=1}^{N_K} |\bar{\varphi}^e_i|^2 + |\varphi|_{1/2, \partial K}^2.
\]

For the first term on the right-hand side, as $\varphi$ has zero mean value on $\partial K$, we obtain
\[
\sum_{i=1}^{N_K} |\bar{\varphi}^e_i|^2 = \sum_{i=1}^{N_K} |e_i|^{-2} \int_{e_i} \varphi^2 \leq \sum_{i=1}^{N_K} |e_i|^{-1} \int_{e_i} \varphi^2 \lesssim h_K^{-1} \int_{\partial K} \varphi^2 \lesssim |\varphi|_{1/2, \partial K}^2,
\]

where we also have used $|e_i| \simeq h_K$ (due to shape regularity), and the Poincaré inequality (2.5). Therefore,
\[
\frac{\int_{\partial K} \lambda \varphi}{|\varphi|_{1/2, \partial K}} \lesssim \left( \sum_{i=1}^{N_K} \|\lambda\|_{-1/2, e_i}^2 \right)^{1/2}.
\]

In order to prove (6.15), we only need to apply inequality (4.5) \hfill □

As $k' \leq k$, Theorem 6.2 yields (6.2) with $\rho(k) = (\log k)^{-1}$.

Remark 6.3. Assumption 2.1 (ii) is needed in the proof of Theorem 6.2 as, under such an assumption, we manage to bound the $H^{-1/2}(\partial K)$ semi norm of $\lambda$ with the sum of its $(H^{1/2}_0(e_i))'$ norms over all edges $e_i$ of $\partial K$. This bound is not generally valid if $K$ has very small edges. Therefore, the stabilization that we propose here is not proven to be robust with respect to decreasing edge length.
However we are confident that suitably combining the present approach with the approach used in [8] will allow us to obtain a method that is simultaneously robust with respect to decreasing edge length and increasing polynomial degree.

7. Hybridization

As in [8], in order to efficiently implement the method, we perform an hybridization procedure by introducing an auxiliary unknown \( \varphi \) approximating the trace on \( \mathcal{E}_h \) of the solution and by using independent unknowns \( \hat{\lambda}^K \in \Lambda_K \) (\( \Lambda_K \) defined by (6.1)) to approximate \( \nabla u \cdot n_K \). To this aim, we introduce the following discrete spaces

\[
\hat{\Lambda}_h = \prod_{K \in T_h} \Lambda_K, \quad \Phi_h = \{ \varphi \in L^2(\mathcal{E}_h) : \varphi|_e \in P_{k'}(e), \ e \subset \mathcal{E}_h \}.
\]

Letting \( b : \hat{\Lambda}_h \times \Phi_h \to \mathbb{R} \) be defined by

\[
b(\hat{\lambda}, \psi) = \sum_K \int_{\partial K} \hat{\lambda}^K \psi,
\]

it is easy to check that \( \Lambda_h \) is isomorphic to

\[
\ker b = \{ \hat{\lambda} \in \hat{\Lambda}_h : b(\hat{\lambda}, \psi) = 0 \ \forall \psi \in \Phi_h \} \subset \hat{\Lambda}_h.
\]

More precisely, \( \hat{\lambda} \in \ker b \) if and only if \( \hat{\lambda}^K = \lambda(u \cdot n_K) \) for some \( \lambda \in \Lambda_h \). Introducing the bilinear forms \( \hat{a}^K : P_k(K) \times \Lambda_K \to \mathbb{R} \) given by

\[
\hat{a}^K(u^K, \hat{\lambda}^K, v^K) = \int_K \nabla u^K \cdot \nabla v^K - \int_{\partial K} \hat{\lambda}^K v^K + \int_{\partial K} \hat{\mu}^K u^K + \alpha s_K(Du^K - \gamma^*_K \hat{\lambda}^K; tDu^K - \gamma^*_K \hat{\mu}^K),
\]

and letting

\[
\hat{a}(u, \hat{\lambda}; v, \hat{\mu}) = \sum_K \hat{a}^K(u^K, \hat{\lambda}^K, v^K, \hat{\mu}^K), \quad F(v, \hat{\mu}) = \int_{\Omega} fv + \int_{\partial \Omega} g + \alpha \sum_{K \in T_h} s_K(f; tDu^K - \gamma^*_K \mu),
\]

we can then consider the following hybridized problem.

**Problem 7.1.** Find \( u_h = (u_h^K)_{K \in T_h} \in V_h, \hat{\lambda}_h = (\hat{\lambda}_h^K) \in \hat{\Lambda}_h, \varphi \in \Phi_h \) with \( \varphi|_{\partial \Omega} = g \) such that, for all \( v = (v^K)_{K \in T_h} \in V_h, \hat{\mu} = (\hat{\mu}^K) \in \hat{\Lambda}_h, \psi \in \Phi_h \) with \( \psi|_{\Omega} = 0 \),

\[
\begin{align*}
(7.1) & \quad \hat{a}(u_h, \hat{\lambda}_h; v, \hat{\mu}) - b(\hat{\mu}, \varphi) = F(v, \hat{\mu}) \\
(7.2) & \quad b(\hat{\lambda}_h, \psi) = 0.
\end{align*}
\]

The well posedness of Problem 7.1 and its equivalence to Problem 2.2 are proven in [8]. Observe that (7.1) reduces to independent Dirichlet problems in each \( K \), with boundary condition \( u^K = \varphi \) on \( \partial K \), and with non standard stabilization given by the bilinear form \( s_K \). The local unknown can then be eliminated by static condensation, reducing the solution to a problem on the unknown \( \varphi \).
Figure 1. Meshes used in experiment ii). From left to right: meshes made of regular hexagons, Central Voronoi Tessellation, and random Voronoi cells.

Table 1. Meshes of regular hexagons used in experiment i). For all these meshes, $\gamma_0 \approx 5.33$, $\gamma_1 \approx 3.16$.

| Mesh      | $N_p$ | $N_e$ | $h$    | $h_{\min}$ |
|-----------|-------|-------|--------|-------------|
| r-hexa_1  | 4193  | 12580 | 2.08·10^{-2} | 5.21·10^{-3} |
| r-hexa_2  | 10151 | 30454 | 1.33·10^{-2} | 3.33·10^{-3} |
| r-hexa_3  | 22726 | 68179 | 8.89·10^{-3} | 2.22·10^{-3} |
| r-hexa_4  | 40301 | 120904| 6.67·10^{-3} | 1.67·10^{-3} |
| r-hexa_5  | 62876 | 188629| 5.33·10^{-3} | 1.33·10^{-3} |
| r-hexa_6  | 90451 | 271354| 4.44·10^{-3} | 1.11·10^{-3} |

Table 2. Central Voronoi Tessellations used in experiment i).

| Mesh   | $N_p$ | $N_e$ | $h$          | $h_{\min}$ | $\gamma_0$ | $\gamma_1$ |
|--------|-------|-------|--------------|-------------|-------------|-------------|
| cvt_1  | 2048  | 6124  | 3.40·10^{-2} | 1.51·10^{-3} | 3.18        | 1.88·10^{1} |
| cvt_2  | 4096  | 12250 | 2.70·10^{-2} | 1.01·10^{-3} | 3.50        | 1.98·10^{1} |
| cvt_3  | 8192  | 24516 | 1.74·10^{-2} | 7.23·10^{-4} | 3.38        | 2.00·10^{1} |
| cvt_4  | 16384 | 49019 | 1.21·10^{-2} | 5.04·10^{-4} | 3.35        | 2.17·10^{1} |
| cvt_5  | 32768 | 98064 | 8.70·10^{-3} | 3.40·10^{-4} | 3.45        | 2.14·10^{1} |
| cvt_6  | 65536 | 196067| 6.11·10^{-3} | 2.33·10^{-4} | 3.68        | 2.28·10^{1} |

Table 3. Random Voronoi cells used in experiment i).

| Mesh     | $N_p$ | $N_e$ | $h$          | $h_{\min}$ | $\gamma_0$ | $\gamma_1$ |
|----------|-------|-------|--------------|-------------|-------------|-------------|
| voro_1   | 2500  | 7505  | 6.38·10^{-2} | 6.19·10^{-6} | 1.42·10^{1} | 5.85·10^{3} |
| voro_2   | 5000  | 15007 | 4.34·10^{-2} | 5.85·10^{-7} | 1.46·10^{1} | 3.40·10^{4} |
| voro_3   | 10000 | 30006 | 3.47·10^{-2} | 1.73·10^{-7} | 2.53·10^{1} | 9.49·10^{4} |
| voro_4   | 20000 | 60010 | 2.41·10^{-2} | 2.14·10^{-7} | 2.09·10^{1} | 7.25·10^{4} |
| voro_5   | 40000 | 120006| 1.73·10^{-2} | 8.26·10^{-8} | 2.68·10^{1} | 7.18·10^{4} |
| voro_6   | 80000 | 240027| 1.14·10^{-2} | 6.00·10^{-9} | 2.88·10^{1} | 1.10·10^{6} |
8. Numerical Results

The goal of this section is to discuss in greater detail the numerical implementation of our method and to provide evidence of the theoretical estimates proven in Section 5.

As a basis for \( \mathbb{P}_k(K) \), for each \( K \in \mathcal{T}_h \), we use the scaled monomials of degree less then or equal to \( k \)

\[
m_\alpha(x, y) = \left( \frac{x - x_K}{h_K} \right)^{\alpha_1} \left( \frac{y - y_K}{h_K} \right)^{\alpha_2} \quad \forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2, \quad \alpha_1 + \alpha_2 \leq k,
\]

where \((x_K, y_K)\) are the coordinates of the barycenter of \( K \). Moreover, as a basis for \( \mathbb{P}_k(e) \), for each \( e \in \mathcal{E}_h \), we use Legendre polynomials of degree \( \leq k \).

In order to construct the stabilization form described in Section 6, we let \( \hat{T} \) be the unit triangle \( \{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1, x + y \leq 1 \} \) and \( \hat{V}_\delta \) be the conforming finite element space of polynomial order 1 constructed on a mesh \( \hat{T}_h \) of \( \hat{T} \) of mesh size \( \delta = (k')^{-2} \), whose elements vanish on \( \partial \hat{T} \setminus \hat{e} \)

\[
\hat{V}_\delta = \left\{ v \in C^0(\hat{T}) \mid v|_E \in \mathbb{P}_1(E) \quad \forall E \in \hat{T}_h, v|_{\partial \hat{T} \setminus \hat{e}} = 0 \right\}.
\]

Note that the results shown in the following suggest that the choice \( \delta = (k')^{-2} \) may be too conservative. With this definition of \( \hat{V}_\delta \), a basis for the space \( \hat{V} \subseteq H^1(\hat{T}) \) is built once for all during the pre-processing phase of Problem 2.2 by solving 6.5, for \( \lambda = \lambda_i, i = 0, \ldots, k \), being \( \lambda_i \) a basis of \( \mathbb{P}_k(\hat{e}) \). Then, a basis \( \{ \varphi_i \}_{i=1}^{n_K} \) for the auxiliary space \( W^K \subset H^1(K) \) is computed as indicated by equations (6.6) and (6.7). In order to assemble the stabilization term on \( K \), the next steps are:

1. Assemble the stiffness matrix \( S \) associated with \( \varphi_i \in W^K \), i.e.,

   \[
   S_{i,j} = (\nabla \varphi_j, \nabla \varphi_i)_K, \quad \text{for } i, j = 1, \ldots, n_K.
   \]

2. For \( u, v \in \mathbb{P}_k(K) \), \( \lambda, \mu \in \Lambda^K \), \( \varphi_i \in W^K \), compute \( \eta_i, \zeta_i \), and \( f_i \) from 6.4

3. Solve \( S\tilde{\varphi} = \zeta \), and compute \( \tilde{\tau}^T \tilde{\varphi}, f^T \tilde{\varphi} \) from 6.3

Since functions in \( W_i \subset W_K \) have zero Dirichlet traces along the edges interior to \( K \), the stiffness matrix \( S \) is block diagonal, with blocks of size \((k+1) \times (k+1)\), since \( \dim(W_i) = k+1 \), thereby decoupling the contribution of the \( n_K \) triangles \( T_i \) to the stabilization term \( s_K \). Thus, at step 2 one has to solve \( n_K \) small systems of dimension \( k+1 \), rather than a single big system of dimension \( n_K \cdot (k+1) \times n_K \cdot (k+1) \). Moreover, for each \( T_i \), computing the terms

\[
S|_{T_i}, \quad \int_{T_i} \nabla v \cdot \nabla \varphi_i, \quad \int_{\partial T_i \cap \partial K} \mu \varphi_i, \quad \int_{T_i} f \varphi_i
\]

coming from (6.4) does not require much effort: one can store the stiffness matrix, the right hand side, and the nodal values of the basis \( \varphi_i \) computed only once during the pre-processing phase, and then apply proper push-back operations between \( T_i \) and \( \hat{T} \), which amount to matrix-matrix, or matrix-vector multiplications, efficiently performed in our code using PETSc interfaces to BLAS/LAPACK software [2].

After computing the stabilization term, locally for each \( K \), problem 2.2 is solved using static condensation: for each \( K \), equation (7.1) yields a local discrete Dirichlet problem, thereby allowing to express \( u|_K, \lambda|_K \) as a function of the sole variable \( \varphi|_{\partial K} \). At this point,
we use (7.2), which imposes continuity of the fluxes λ, to glue all the local problems together and obtain a global system of equations where only ϕ appears as unknown. The global system is solved with the direct solver STRUMPACK [16]. Reconstruction of u, λ is done by solving local problems in parallel.

We performed a series of experiments in order to investigate the performance of our method with regards to: i) optimal order of convergence of |u − uh|_{H^1}; ii) robustness for increasing polynomial degree k; iii) sensitivity to the choice of the mesh size δ; iv) robustness with respect to collapsing minimum edge length.

In all the experiments, we let the domain Ω be the unit square [0, 1] × [0, 1]. Problem 2.2 is solved with Neumann boundary conditions on Γ_N = {(x, y) | 0 ≤ x ≤ 1, y = 1}, Dirichlet boundary conditions on Γ_D = ∂Ω \ Γ_N, and load term chosen in such a way that

\[ u = \frac{1}{128\pi^2} \cos(8\pi x) \cos(8\pi y) \]

is the exact solution. The stabilization parameters are chosen to be α = t = 1. For the first three experiments, we consider three types of meshes: meshes made mainly of regular hexagons (see, e.g., Figure 1a), Central Voronoi Tessellations (see, e.g., Figure 1b), and random Voronoi meshes (see, e.g., Figure 1c). Geometrical data for the meshes used in experiment i) are shown in Tables 1, 2, 3, respectively. For each mesh, we provide: N_p, the number of elements of T_h; N_e, the number of edges of T_h; h = max_{K \in \mathcal{T}_h} h_K; h_{min} = min_{K \in \mathcal{T}_h} h_{min,K}, where h_{min,K} is the minimum distance between any two vertices of K; γ_0 = max_{K \in \mathcal{T}_h} ρ_K, where ρ_K is the radius of the largest ball that is contained inside K; γ_1 = max_{K \in \mathcal{T}_h} h_{K,h_{min,K}}^2.

i) Optimal order of convergence in h: Tables 4[6] show the relative errors \[ e_h^n = |u - u_h|_1/|u|_1, e_h^u = |u - u_h|_2/|u|_2 \] and the estimated convergence rates (ecr) for several values of the polynomial degree k on the random Voronoi cells versus the total number of the degrees of freedom dofs = dim(V_h) (notice that dofs behaves like \( O(h^{-2}) \)). Analogous results are plotted in Figure 2 for both hexagonal and CVT meshes. We note that the results confirm the theoretical estimate, with the correct order of convergence for the \( H^1 \) norm of the error, i.e. \( O(h^k) \), as h tends to zero.

| k = 1 | k = 2 |
|---|---|
| dofs | e_1^u | ecr | e_0^u | ecr | dofs | e_1^u | ecr | e_0^u | ecr |
| 7500 | 2.62·10^{-1} | - | 8.57·10^{-2} | - | 15000 | 3.84·10^{-2} | - | 1.08·10^{-2} | - |
| 15000 | 1.87·10^{-1} | 0.97 | 4.48·10^{-2} | 1.87 | 30000 | 1.93·10^{-2} | 1.98 | 4.02·10^{-3} | 2.85 |
| 30000 | 1.30·10^{-1} | 1.04 | 2.29·10^{-2} | 1.93 | 60000 | 9.66·10^{-3} | 2.00 | 1.52·10^{-3} | 2.80 |
| 60000 | 9.19·10^{-2} | 1.00 | 1.13·10^{-2} | 2.04 | 120000 | 4.87·10^{-3} | 1.98 | 6.37·10^{-4} | 2.52 |
| 120000 | 6.49·10^{-2} | 1.00 | 5.70·10^{-3} | 1.98 | 240000 | 2.46·10^{-3} | 1.97 | 2.91·10^{-4} | 2.26 |
| 240000 | 4.59·10^{-2} | 1.00 | 2.93·10^{-3} | 1.92 | 480000 | 1.22·10^{-3} | 2.02 | 1.30·10^{-4} | 2.33 |
Table 5. Errors and estimated convergence rates (ecr) for experiment i) on random Voronoi cells, $k = 3, 4$.

| dofs  | $\varepsilon_1^u$ | ecr  | $\varepsilon_0^u$ | ecr  | dofs  | $\varepsilon_1^u$ | ecr  | $\varepsilon_0^u$ | ecr  |
|-------|-------------------|------|-------------------|------|-------|-------------------|------|-------------------|------|
| 25000 | 3.99·10^{-3}      | -    | 7.05·10^{-4}      | -    | 37500 | 3.69·10^{-4}      | -    | 5.71·10^{-5}      | -    |
| 50000 | 1.47·10^{-3}      | 2.89 | 1.91·10^{-4}      | 3.77 | 75000 | 8.90·10^{-5}      | 4.10 | 1.01·10^{-5}      | 4.99 |
| 100000| 5.09·10^{-4}      | 3.06 | 4.64·10^{-5}      | 4.09 | 150000| 2.22·10^{-5}      | 4.01 | 1.88·10^{-6}      | 4.86 |
| 200000| 1.78·10^{-4}      | 3.03 | 1.16·10^{-5}      | 3.99 | 300000| 5.55·10^{-6}      | 3.99 | 3.86·10^{-7}      | 4.56 |
| 400000| 6.41·10^{-5}      | 2.95 | 3.04·10^{-6}      | 3.87 | 600000| 1.41·10^{-6}      | 3.95 | 7.71·10^{-8}      | 4.65 |
| 800000| 2.25·10^{-5}      | 3.02 | 7.65·10^{-7}      | 3.98 | 200000| 3.49·10^{-7}      | 4.04 | 1.67·10^{-8}      | 4.42 |

Table 6. Errors and estimated convergence rates (ecr) for experiment i) on random Voronoi cells, $k = 5, 6$.

| dofs  | $\varepsilon_1^u$ | ecr  | $\varepsilon_0^u$ | ecr  | dofs  | $\varepsilon_1^u$ | ecr  | $\varepsilon_0^u$ | ecr  |
|-------|-------------------|------|-------------------|------|-------|-------------------|------|-------------------|------|
| 52500 | 2.70·10^{-5}      | -    | 3.32·10^{-6}      | -    | 70000 | 2.18·10^{-6}      | -    | 2.35·10^{-7}      | -    |
| 105000| 4.85·10^{-6}      | 4.96 | 4.17·10^{-7}      | 5.99 | 140000| 2.29·10^{-7}      | 6.51 | 1.83·10^{-8}      | 7.37 |
| 210000| 8.34·10^{-7}      | 5.08 | 5.27·10^{-8}      | 5.97 | 280000| 2.89·10^{-8}      | 5.97 | 1.68·10^{-9}      | 6.88 |
| 420000| 1.41·10^{-7}      | 5.12 | 6.37·10^{-9}      | 6.10 | 560000| 3.66·10^{-9}      | 5.97 | 1.67·10^{-10}     | 6.67 |
| 840000| 2.66·10^{-8}      | 4.82 | 8.69·10^{-10}     | 5.75 | 1120000| 4.58·10^{-10}     | 5.99 | 1.63·10^{-11}     | 6.70 |
| 1680000| 4.56·10^{-9}     | 5.09 | 1.02·10^{-10}     | 6.18 | 2240000| 5.68·10^{-11}     | 6.02 | 1.66·10^{-12}     | 6.59 |

Figure 2. Experiment i): relative errors $\varepsilon_1^u$ and convergence rates (·) on hexagonal meshes (left) and CVT meshes (right).
Table 7. History of convergence for increasing polynomial degrees.

| k   | r-hexa | cvt | voro |
|-----|--------|-----|------|
|     | $e_1^u$ | ecr | $e_1^u$ | ecr | $e_1^u$ | ecr |
| 1   | 9.03-10^{-1} | -  | 9.76-10^{-1} | -  | 9.83-10^{-1} | -  |
| 2   | 6.28-10^{-1} | -  | 6.31-10^{-1} | -  | 8.37-10^{-1} | -  |
| 3   | 2.41-10^{-1} | 0.38 | 3.25-10^{-1} | 0.66 | 5.41-10^{-1} | 0.37 |
| 4   | 8.82-10^{-2} | 0.96 | 1.01-10^{-1} | 0.57 | 3.35-10^{-1} | 0.91 |
| 5   | 2.02-10^{-2} | 0.68 | 3.61-10^{-2} | 1.14 | 1.65-10^{-1} | 0.68 |
| 6   | 5.25-10^{-3} | 1.10 | 6.86-10^{-3} | 0.62 | 7.85-10^{-2} | 0.95 |
| 7   | 8.08-10^{-4} | 0.72 | 1.88-10^{-3} | 1.28 | 2.99-10^{-2} | 0.77 |
| 8   | 1.57-10^{-4} | 1.14 | 2.56-10^{-4} | 0.65 | 1.10-10^{-2} | 0.96 |
| 9   | 1.87-10^{-5} | 0.77 | 5.98-10^{-5} | 1.37 | 3.26-10^{-3} | 0.83 |
| 10  | 2.99-10^{-6} | 1.16 | 6.05-10^{-6} | 0.63 | 9.63-10^{-4} | 0.99 |
| 11  | 2.92-10^{-7} | 0.79 | 1.24-10^{-6} | 1.44 | 2.34-10^{-4} | 0.86 |
| 12  | 3.85-10^{-8} | 1.15 | 9.99-10^{-8} | 0.63 | 5.75-10^{-5} | 1.01 |
| 13  | 3.10-10^{-9} | 0.80 | 1.86-10^{-8} | 1.50 | 1.18-10^{-5} | 0.88 |
| 14  | 3.64-10^{-10} | 1.18 | 1.23-10^{-9} | 0.62 | 2.49-10^{-6} | 1.02 |
| 15  | 2.71-10^{-10} | 7.21 | 2.83-10^{-10} | 1.85 | 5.00-10^{-6} | -2.23 |
| 16  | 1.23-10^{-9} | -0.20 | 6.40-10^{-10} | -1.80 | 1.57-10^{-6} | -0.60 |

Table 8. Experiment iii): relative errors $e_1^u$ on the hexagonal mesh shown in Figure [1] for different choices of the mesh size $\delta$.

| k   | $\delta = k^{-2}$ | $\delta = k^{-1}$ | $\delta = 1/4$ | $\delta = 1/8$ |
|-----|--------------------|--------------------|----------------|----------------|
| 2   | 6.28-10^{-1}       | 1.26-10^{2}        | 6.28-10^{-1}   | 6.28-10^{-1}   |
| 3   | 2.41-10^{-1}       | 7.77-10^{-1}       | 1.68-10^{1}    | 2.41-10^{-1}   |
| 4   | 8.82-10^{-2}       | 1.30-10^{1}        | 1.30-10^{1}    | 8.80-10^{-2}   |
| 5   | 2.02-10^{-2}       | 2.00-10^{-2}       | 4.16-10^{1}    | 2.01-10^{-2}   |
| 6   | 5.25-10^{-3}       | 5.12-10^{-3}       | 5.15-10^{1}    | 5.13-10^{-3}   |
| 7   | 8.08-10^{-4}       | 8.77-10^{-4}       | 5.72-10^{2}    | 8.35-10^{-4}   |
| 8   | 1.57-10^{-4}       | 1.59-10^{-4}       | 7.97-10^{1}    | 1.59-10^{-4}   |
| 9   | 1.87-10^{-5}       | 2.52-10^{-5}       | 3.53-10^{1}    | 2.52-10^{-5}   |
| 10  | 2.99-10^{-6}       | 2.96-10^{-6}       | 6.15-10^{2}    | 3.19-10^{-6}   |
| 11  | 2.92-10^{-7}       | 3.00-10^{-7}       | 2.18-10^{2}    | 5.77-10^{-7}   |
| 12  | 3.85-10^{-8}       | 3.80-10^{-8}       | 7.75-10^{3}    | 3.99-10^{-8}   |
| 13  | 3.10-10^{-9}       | 3.53-10^{-9}       | 4.30-10^{2}    | 9.34-10^{-9}   |
| 14  | 3.64-10^{-10}      | 3.61-10^{-10}      | 2.61-10^{3}    | 1.24-10^{-8}   |
| 15  | 2.71-10^{-10}      | 2.14-10^{-10}      | 2.99-10^{4}    | 3.25-10^{-8}   |
| 16  | 1.23-10^{-9}       | 9.60-10^{-10}      | 3.78-10^{5}    | 4.03-10^{-7}   |
ii) **Validity as a k-method**: we test the validity of our method as a k-method, by fixing the mesh (one of those depicted in Figure 1) and increasing \( k \) from 1 to 16. We compute the relative errors \( e_k^u \) as functions of \( k \) and check whether the rates

\[
\frac{\log(e_k/e_{k+1})}{\log(e_{k+1}/e_{k+2})} \approx 1,
\]

as would be expected. Table 7 shows that this is indeed the case. The loss of accuracy at high order, i.e., \( k = 15, 16 \), is most probably a consequence of the ill-conditioning due to the choice of the monomial basis \([5,1]\).

iii) **Sensitivity with respect to the mesh size \( \delta \)**: Table 8 shows that taking \( \delta = k^{-2} \) is a conservative choice ensuring that the error decreases with increasing \( k \). However, more permissive choices, e.g., \( \delta = k^{-1} \), might be enough to compute the stabilization, provided that \( \delta \) is small enough when \( k \) is also small, say \( k = 2, 3 \). Letting \( \delta \) being a constant, even if small, has a detrimental effect for increasing \( k \); see columns corresponding to \( \delta = 1/4 \) and \( \delta = 1/8 \) in Table 8.

iv) **Robustness with respect to collapsing minimum edge length**: for this experiment, we consider a mesh and two non-nested refinements as reference meshes, and then progressively shrink the length of their vertical edges by a factor of \( s = 1 \) (original mesh), \( 2^{-1}, 2^{-2}, \ldots, 2^{-32} \). Convergence is severely and abruptly affected only starting with \( k = 8 \), on the finest mesh, for the smallest shrinking factor \( s = 2^{-32} \) (\( h_{\text{min}} \approx 2.43 \times 10^{-12} \)) (see Table 9). Although Assumption 2.1 (ii) is not satisfied, the method seems quite robust with respect to the minimal edge length, at least for low degrees \( k \), in the approximation of \( u \). On the other hand, for \( k \geq 8 \), the loss of robustness could also be caused by round-off errors.

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Table 9. Experiment iv): History of convergence for shrinking minimum edge length and at different polynomial degrees.

| k   | s = 1         | s = 2⁻⁴      | s = 2⁻⁸      | s = 2⁻¹⁶     | s = 2⁻³²     |
|-----|---------------|--------------|--------------|--------------|--------------|
|     | e₁^u | ecr | e₁^u | ecr | e₁^u | ecr | e₁^u | ecr | e₁^u | ecr |
| 5.25×10⁻³ | - 1.14×10⁻² | - 1.27×10⁻² | - 1.28×10⁻² | - 1.28×10⁻² | - 1.28×10⁻² |
| 6    | 9.10×10⁻⁵   | 6.26 2.10×10⁻⁴ | 6.18 2.35×10⁻⁴ | 6.16 2.37×10⁻⁴ | 6.16 2.37×10⁻⁴ | 6.16 |
| 1.49×10⁻⁶ | 6.14 3.46×10⁻⁶ | 6.13 3.87×10⁻⁶ | 6.13 3.90×10⁻⁶ | 6.13 3.91×10⁻⁶ | 6.13 |
| 8.08×10⁻⁴ | - 2.12×10⁻³ | - 2.40×10⁻³ | - 2.42×10⁻³ | - 2.42×10⁻³ | - 2.42×10⁻³ |
| 7    | 8.19×10⁻⁶   | 7.09 2.23×10⁻⁵ | 7.04 2.55×10⁻⁵ | 7.02 2.57×10⁻⁵ | 7.02 2.57×10⁻⁵ | 7.02 |
| 6.85×10⁻⁸ | 7.14 1.86×10⁻⁷ | 7.15 2.13×10⁻⁷ | 7.14 2.14×10⁻⁷ | 7.14 2.78×10⁻⁷ | 6.76 |
| 1.57×10⁻⁴ | - 4.81×10⁻⁴ | - 5.59×10⁻⁴ | - 5.64×10⁻⁴ | - 5.65×10⁻⁴ | - 5.65×10⁻⁴ |
| 8    | 6.77×10⁻⁷   | 8.42 2.13×10⁻⁶ | 8.38 2.48×10⁻⁶ | 8.37 2.51×10⁻⁶ | 8.37 2.51×10⁻⁶ | 8.37 |
| 2.73×10⁻⁹ | 8.23 8.80×10⁻⁹ | 8.19 1.03×10⁻⁸ | 8.19 1.04×10⁻⁸ | 8.19 7.78×10⁻⁸ | 5.18 |
| 1.87×10⁻⁵ | - 6.93×10⁻⁵ | - 8.18×10⁻⁵ | - 8.27×10⁻⁵ | - 8.27×10⁻⁵ | - 8.27×10⁻⁵ |
| 9    | 4.71×10⁻⁸   | 9.24 1.78×10⁻⁷ | 9.22 2.11×10⁻⁷ | 9.21 2.14×10⁻⁷ | 9.21 2.15×10⁻⁷ | 9.19 |
| 9.81×10⁻¹¹ | 9.22 3.69×10⁻¹⁰ | 9.22 4.41×10⁻¹⁰ | 9.21 4.46×10⁻¹⁰ | 9.21 7.14×10⁻⁸ | 1.65 |

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