Limits of Hodge structures in several variables, II

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Abstract

The aim of this article is to study degeneration of the variations of Hodge structure associated to a proper Kähler semistable morphism. We prove that the weight filtrations constructed in \cite{7} coincide with the monodromy weight filtrations on the relative log de Rham cohomology groups of a proper Kähler semistable morphism. Moreover, we show that the limiting mixed Hodge structures form admissible variations of mixed Hodge structure.

1 Introduction

1.1. Let $Y$ be a complex manifold and $E$ a simple normal crossing divisor on it. A morphism $f : X \to Y$ from a complex manifold $X$ to $Y$ is said to be semistable (along $E$) if $D = f^*E$ is a reduced simple normal crossing divisor on $X$ and if $f$ is log smooth in the sense of K. Kato \cite{13} (for the precise definition, see Definition 3.1). A proper Kähler semistable morphism $f : X \to Y$ along $E$ induces variations of Hodge structure $R^q f_* \mathbb{Q}_X|_{Y \setminus E}$ for all $q$, because $f$ is smooth over $Y \setminus E$. In this article, we study the degeneration of these variations of Hodge structure from the algebro-geometric viewpoint. Because the question is of local nature, we mainly treat the case of $Y = \Delta^k \times S$ and $E = \{t_1 t_2 \cdots t_k = 0\}$, where $\Delta^k$ is the $k$-dimensional polydisc with the coordinate functions $t_1, t_2, \ldots, t_k$ and $S$ is a complex manifold.

1.2. The degeneration of the variation of Hodge structure $R^q f_* \mathbb{Q}_X|_{Y \setminus E}$ is described by Deligne’s canonical extension of $\mathcal{O}_{Y \setminus E} \otimes R^q f_* \mathbb{Q}_X|_{Y \setminus E}$ and its decreasing filtration which is the extension of the Hodge filtration of $\mathcal{O}_{Y \setminus E} \otimes R^q f_* \mathbb{Q}_X|_{Y \setminus E}$.

For the case of $Y = \Delta$, Steenbrink \cite{18} proved that $R^q f_* \Omega_{X/\Delta}(\log D)$ is the canonical extension of $\mathcal{O}_\Delta \otimes R^q f_* \Omega_X|_{\Delta}$ and constructed a mixed Hodge structure on

$$H^q(D, \mathcal{O}_D \otimes \Omega_{X/\Delta}(\log D)) \simeq \mathbb{C}(0) \otimes R^q f_* \Omega_{X/\Delta}(\log D)$$
whose Hodge filtration is induced by the stupid filtration on $\Omega_{X/\Delta} (\log D)$. He also proved that the stupid filtration induces the desired filtration on $R^q f_* \Omega_{X/\Delta} (\log D)$. Moreover, he claimed that its weight filtration coincides with the monodromy weight filtration (see also [4, Remarque 3.18], [16, 4.2.5 Remarque], [21, (A.1)]).

For the case of $Y = \Delta^k$, Usui stated in [22] that $R^q f_* \Omega_{X/\Delta^k} (\log D)$ is the canonical extension of $\mathcal{O}_{\Delta^k \setminus E} \otimes R^q f_* \mathcal{Q}_X |_{\Delta^k \setminus E}$ (see also Theorem 7.7 below). Then, in [7], the author constructed a mixed Hodge structure on

$$H^q (X_0, \mathcal{O}_{X_0} \otimes \Omega_{X/\Delta} (\log D)) \simeq \mathbb{C}(0) \otimes R^q f_* \Omega_{X/\Delta^k} (\log D)$$

(1.2.1)

whose Hodge filtration is induced by the stupid filtration on $\Omega_{X/\Delta^k} (\log D)$, where $X_0 = f^{-1}(0)$, and proved that the stupid filtration on $\Omega_{X/\Delta^k} (\log D)$ induces the desired filtration on $R^q f_* \Omega_{X/\Delta^k} (\log D)$.

An aim of this article is to prove that the weight filtration of the mixed Hodge structure (1.2.1) coincides with the monodromy weight filtration as in the case of $k = 1$.

1.3. For $Y = \Delta^k \times S$, we set

$$Y[I] = \bigcap_{i \in I} \{ t_i = 0 \}$$

$$Y[I]^* = Y[I] \setminus (Y[I] \cap \bigcup_{j \not\in I} \{ t_j = 0 \})$$

for a subset $I \subset \{1, 2, \ldots, k\}$ as in 3.6 below. We note that we have $Y = \Delta^I \times Y[I]$, where $\Delta^I$ is the $|I|$-dimensional polydisc with the coordinate functions $(t_i)_{i \in I}$.

Now we consider the cohomology group

$$H^q (X_s, \mathcal{O}_{X_s} \otimes \Omega_{X/Y} (\log D)),$$

(1.3.1)

for any $s \in Y[I]^*$, where $X_s = f^{-1}(s)$. Because the restriction of the morphism $f$ over the closed subspace $\Delta^I \times \{ s \} \subset Y$ is again semistable, we have a mixed Hodge structure on the relative log de Rham cohomology groups (1.3.1) as in [7]. Thus we obtain a family of the mixed Hodge structures parametrized by the points $s \in Y[I]^*$. Then two questions arise naturally. The first is whether this family forms a variation of mixed Hodge structure. If this is the case, then the second is whether it is admissible. Another aim of this article is to give the affirmative answer to these questions.

1.4. For the case of $Y = \Delta^k$, the degeneration of abstract polarized variations of Hodge structure on $(\Delta^*)^k = \Delta^k \setminus E$ is intensively studied by Cattani, Kaplan, Kashiwara, Kawai, Schmid and others, apart from the morphism $f$. The existence of the limiting mixed Hodge structure of an abstract polarized variation of Hodge structure on $(\Delta^*)^k$ is proved in [17, (6.16) Theorem]. Combining results in [17] and [1] we can show that the limiting mixed Hodge structures form an admissible variation of mixed Hodge structure on $Y[I]^*$ (see also [6, Section 5]).

Coincidence of our weight filtration and the monodromy weight filtration on (1.2.1) implies that our mixed Hodge structure on (1.2.1) coincides with Schmid’s limiting mixed Hodge structure. Thus the results in this article are the algebro-geometric counterpart of [17] and [1].
1.5. It is expected that the results in this article give the alternative proof of the fact that $R^q f_*\Omega_{X/Y} (\log D)$ underlies a logarithmic Hodge structure of weight $q$ in the sense of Kato-Usui (cf. [14, Definition 2.6.5]). Remaining task is to compare the integral or rational structures. We will not discuss this question in this article.

1.6. This article is organized as follows. Section 2 presents some preliminaries. In Section 3, we give the precise definition of the semistable morphism. Moreover, we fix the notation which is constantly used in this article. In Section 4, we construct the complex $sB_I(f)$ which is the replacement for the relative log de Rham complex on the closed subspace $Y[I]$ for $I \subset \{1, 2, \ldots, k\}$, and study the properties of it in detail. Section 5 is devoted to the computation of the Gauss-Manin connection in terms of the complex $sB_I(f)$. In Section 6, we construct rational structures for the complex $sB_I(f)$, which play important roles in the following sections. Section 7 deals with the case where the semistable morphism $f$ is proper. By using the rational structures constructed in the last section, we prove that the higher direct image sheaves of $sB_I(f)$ are locally free of finite rank. In Section 8, we give the answer to the first question in 1.3: we show that the mixed Hodge structures on (1.3.1) form a variation of mixed Hodge structure on $Y[I]^*$. In the final section, Section 9, the main results of this article are proved. We prove that our weight filtrations coincide with the monodromy weight filtrations and that the variation of mixed Hodge structure on $Y[I]^*$ above is admissible.

Notation

1.7. The cardinality of a finite set $A$ is denoted by $|A|$.

1.8. Once we fix a set $A$, the complement of a subset $B \subset A$ is denoted by $\overline{B}$, that is, $\overline{B} = A \setminus B$.

1.9. For a finite set $I$,

$$\mathbb{Z}^I = \bigoplus_{i \in I} \mathbb{Z}e_i$$

is the free $\mathbb{Z}$-module of rank $|I|$ generated by $\{e_i\}_{i \in I}$. We set $e_I = \sum_{i \in I} e_i \in \mathbb{Z}^I$. For $q = \sum_{i \in I} q_i e_i \in \mathbb{Z}^I$, we set $|q| = \sum_{i \in I} q_i$. We also use the convention

$$\mathbb{Z}^I_{\geq 0} = \{q = \sum_{i \in I} q_i e_i \in \mathbb{Z}^I; q_i \geq 0 \text{ for all } i\}$$

$$\mathbb{Z}^I_{> 0} = \{q = \sum_{i \in I} q_i e_i \in \mathbb{Z}^I; q_i > 0 \text{ for all } i\}.$$

For a subset $J$ of $I$, we canonically have

$$\mathbb{Z}^I = \mathbb{Z}^J \oplus \mathbb{Z}^{I \setminus J}$$

which induces the canonical inclusion $\mathbb{Z}^J \hookrightarrow \mathbb{Z}^I$ and the canonical projection $\mathbb{Z}^I \to \mathbb{Z}^J$. For the case of $I = \{1, 2, \ldots, k\}$ we use $\mathbb{Z}^k$ instead of $\mathbb{Z}^I$. Similarly, we write $e$ instead of $e_I$. As usual, we write $q = (q_1, q_2, \ldots, q_k)$ for $q = \sum_{i=1}^k q_i e_i \in \mathbb{Z}^k$.

We use the notation $\mathbb{Q}^I$, $\mathbb{R}^I$, $\mathbb{R}^I_{> 0}$, $\mathbb{C}^I$ etc. in the same way as above.
1.10. Let $\Lambda$ be a set. We denote the set of all the subsets of $\Lambda$ is denoted by $S(\Lambda)$. Moreover, we set

$$S_n(\Lambda) = \{ \Gamma \in S(\Lambda); |\Gamma| = n \}$$

for $n \in \mathbb{Z}$. If a decomposition into disjoint union

$$\Lambda = \bigsqcup_{i=1}^{k} \Lambda_i$$

is given, we set

$$S^q(\Lambda) = \{ \Gamma \in S(\Lambda); |\Gamma \cap \Lambda_i| = q_i \text{ for any } i \in J \}$$
$$S^{\geq q}(\Lambda) = \{ \Gamma \in S(\Lambda); |\Gamma \cap \Lambda_i| \geq q_i \text{ for any } i \in J \}$$
$$S^{\leq q}(\Lambda) = \{ \Gamma \in S(\Lambda); |\Gamma \cap \Lambda_i| \leq q_i \text{ for any } i \in J \}$$

for $q \in \mathbb{Z}^J$ and for $J \subset I$. Moreover, we set

$$S^q_n(\Lambda) = S_n(\Lambda) \cap S^q(\Lambda), \quad S^{\geq q}_n(\Lambda) = S_n(\Lambda) \cap S^{\geq q}(\Lambda), \quad S^{\leq q}_n(\Lambda) = S_n(\Lambda) \cap S^{\leq q}(\Lambda)$$

for $n \in \mathbb{Z}$.

2 Preliminaries

2.1. In this section $(X, \mathcal{O}_X)$ denotes a commutative ringed space, i.e. $\mathcal{O}_X$ is a sheaf of commutative rings on $X$. We usually use the symbol $X$ instead of $(X, \mathcal{O}_X)$ if there is no danger of confusion.

Definition 2.2. Let $L$ and $M$ be $\mathcal{O}_X$-modules on $X$ and $F$ a finite decreasing filtration on $M$. Then we define a finite decreasing filtration $F$ on $L \otimes M$ by setting

$$F^p(L \otimes M) = \text{Image}(L \otimes F^p M \to L \otimes M)$$

for all $p$.

Remark 2.3. We have the canonical surjective morphisms

$$L \otimes F^p M \to F^p(L \otimes M) \quad (2.3.1)$$
$$L \otimes \text{Gr}^p_F M \to \text{Gr}^p_F(L \otimes M) \quad (2.3.2)$$

by definition.

Remark 2.4. If $\text{Gr}^p_F M$ is $\mathcal{O}_X$-flat for all $p$, then the canonical morphism $(2.3.1)$ is an isomorphism for all $p$. Then the canonical morphism $(2.3.2)$ is an isomorphism for all $p$.

Lemma 2.5. Let $L$, $M$ and $F$ be as above and $N$ an $\mathcal{O}_X$-submodule of $M$. We set

$$N \otimes \text{Gr}^p_F M = \text{Image}(N \cap F^p M \to \text{Gr}^p_F M)$$
$$N \otimes \text{Gr}^p_F M = \text{Image}(L \otimes N \to L \otimes M)$$
$$N \otimes \text{Gr}^p_F M = \text{Image}(L \otimes N \text{ Gr}^p_F M \to L \otimes \text{Gr}^p_F M)$$
$$N \otimes \text{Gr}^p_F(L \otimes M) = \text{Image}(N(L \otimes M) \cap F^p(L \otimes M) \to \text{Gr}^p_F(L \otimes M))$$

as before. Then the image of $N(L \otimes \text{Gr}^p_F M)$ by the canonical surjection $(2.3.2)$ is $N \otimes \text{Gr}^p_F(L \otimes M)$. If $\text{Gr}^p_F M$ is $\mathcal{O}_X$-flat for all $p$, then $N(L \otimes \text{Gr}^p_F M)$ is identified with $N \otimes \text{Gr}^p_F(L \otimes M)$ under the canonical isomorphism $(2.3.2)$. 

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Proof. Easy. □

Definition 2.6. Let $\mathcal{L}, \mathcal{M}$ be $\mathcal{O}_X$-modules and $F_1, F_2, \ldots, F_l$ finite decreasing filtrations on $\mathcal{M}$. Then the canonical morphism
\[
\mathcal{L} \otimes \text{Gr}^{p_1}_{F_1} \cdots \text{Gr}^{p_2}_{F_2} \text{Gr}^{p_1}_{F_1} \mathcal{M} \rightarrow \text{Gr}^{p_1}_{F_1} \cdots \text{Gr}^{p_2}_{F_2} \text{Gr}^{p_1}_{F_1} (\mathcal{L} \otimes \mathcal{M})
\] (2.6.1)
is obtained by the composition
\[
\mathcal{L} \otimes \text{Gr}^{p_1}_{F_1} \text{Gr}^{p_1-1}_{F_1} \text{Gr}^{p_1-2}_{F_1} \cdots \text{Gr}^{p_2}_{F_2} \text{Gr}^{p_1}_{F_1} \mathcal{M} \rightarrow \text{Gr}^{p_1}_{F_1} (\mathcal{L} \otimes \text{Gr}^{p_1-1}_{F_1} \text{Gr}^{p_1-2}_{F_1} \cdots \text{Gr}^{p_2}_{F_2} \text{Gr}^{p_1}_{F_1} \mathcal{M})
\rightarrow \text{Gr}^{p_1}_{F_1} \text{Gr}^{p_1-1}_{F_1} (\mathcal{L} \otimes \text{Gr}^{p_1-2}_{F_1} \cdots \text{Gr}^{p_2}_{F_2} \text{Gr}^{p_1}_{F_1} \mathcal{M})
\rightarrow \cdots
\rightarrow \text{Gr}^{p_1}_{F_1} \cdots \text{Gr}^{p_2}_{F_2} \text{Gr}^{p_1}_{F_1} (\mathcal{L} \otimes \mathcal{M}),
\]
where each step is the morphism induced by the canonical morphism (2.3.2). The canonical morphism (2.6.1) is surjective by Lemma 2.5.

Lemma 2.7. Let $\mathcal{L}$ and $\mathcal{M}$ be $\mathcal{O}_X$-modules and $F_1, F_2, \ldots, F_l$ finite decreasing filtrations on $\mathcal{M}$. If $\text{Gr}^{p_1}_{F_1} \cdots \text{Gr}^{p_2}_{F_2} \text{Gr}^{p_1}_{F_1} \mathcal{M}$ is $\mathcal{O}_X$-flat for all $p_1, p_2, \ldots, p_l$, then the canonical morphism (2.6.1) is an isomorphism for all $p_1, p_2, \ldots, p_l$. For a submodule $\mathcal{N}$ of $\mathcal{M}$, the canonical isomorphism (2.6.1) identifies $\mathcal{N}(\mathcal{L} \otimes \text{Gr}^{p_1}_{F_1} \cdots \text{Gr}^{p_2}_{F_2} \text{Gr}^{p_1}_{F_1} \mathcal{M})$ with $\mathcal{N} \text{Gr}^{p_1}_{F_1} \cdots \text{Gr}^{p_2}_{F_2} \text{Gr}^{p_1}_{F_1} (\mathcal{L} \otimes \mathcal{M})$.

Proof. Easy from Lemma 2.5. □

Definition 2.8. Let $K$ be a complex of $\mathcal{O}_X$-modules on $X$ and $F$ a finite decreasing filtration on $K$. We say that $(K, F)$ (or $F$ on $K$) is strict if the differential $d : K^p \rightarrow K^{p+1}$ is strictly compatible with $F$ for all $p$.

Lemma 2.9. Let $K$ be a complex of $\mathcal{O}_X$-modules and $F_1, F_2, \ldots, F_l$ finite decreasing filtrations on $K$. We assume that $(\text{Gr}^{p_1}_{F_1} \text{Gr}^{p_1-1}_{F_1} \cdots \text{Gr}^{p_1}_{F_1} K, F_j)$ is strict for $0 \leq i < j \leq l$. (For $i = 0$, we mean that $(K, F_j)$ is strict for all $j$.) Then there exists an isomorphism
\[
H^a(\text{Gr}^{p_1}_{F_1} \text{Gr}^{p_1-1}_{F_1} \cdots \text{Gr}^{p_1}_{F_1} K) \simeq \text{Gr}^{p_1}_{F_1} \text{Gr}^{p_1-1}_{F_1} \cdots \text{Gr}^{p_1}_{F_1} H^a(K)
\]
for any $i$ with $1 \leq i \leq l$ and for all $a, p_1, p_2, \ldots, p_i$, under which the filtrations $F_j$ on the both sides are identified for $j$ with $i < j \leq l$.

Proof. From the assumption that $(K, F_1)$ is strict, we have the isomorphism
\[
H^a(\text{Gr}^{p_1}_{F_1} K) \simeq \text{Gr}^{p_1}_{F_1} H^a(K)
\]
for all $a, p_1$, under which the filtrations $F_j$ on the both sides are identified by the lemma on two filtrations [3, Proposition (7.2.5)] and by the strictness of $F_j$ on $\text{Gr}^{p_1}_{F_1} K$ for $j = 2, 3, \ldots, l$. Thus we have an isomorphism
\[
H^a(\text{Gr}^{p_2}_{F_2} \text{Gr}^{p_1}_{F_1} K) \simeq \text{Gr}^{p_2}_{F_2} H^a(\text{Gr}^{p_1}_{F_1} K) \simeq \text{Gr}^{p_2}_{F_2} \text{Gr}^{p_1}_{F_1} H^a(K)
\]
for all $p_2$ by the strictness of $F_2$ on $\text{Gr}^{p_1}_{F_1} K$. Lemma on two filtrations implies that the filtration $F_j$ on the first term are identified with $F_j$ on the second term under the first isomorphism by the strictness of $F_j$ on $\text{Gr}^{p_2}_{F_2} \text{Gr}^{p_1}_{F_1} K$ for $j = 3, 4, \ldots, l$. Hence the filtrations $F_j$ on the first term are identified with $F_j$ on the third under these isomorphisms. By applying this procedure repeatedly, we obtain the conclusion. □
**Definition 2.10.** Let $K$ be a complex of $\mathcal{O}_X$-modules on $X$ and $F$ a finite decreasing filtration on $K$. We say that $(K, F)$ (or $F$ on $K$) is strongly strict if $(K, F)$ is strict and if $H^a(\text{Gr}_F^p K)$ is a locally free $\mathcal{O}_X$-module of finite rank for all $a, p$.

**Remark 2.11.** If $(K, F)$ is strongly strict, then we have the following:

(2.11.1) The sequence
\[
0 \longrightarrow H^a(F^{p-1}K) \longrightarrow H^a(F^p K) \longrightarrow H^a(\text{Gr}_F^p K) \longrightarrow 0
\]
is exact for all $a, p$. In particular, the canonical morphism
\[
H^a(F^p K) \longrightarrow F^p H^a(K)
\]
is an isomorphism for all $a, p$.

(2.11.2) The $\mathcal{O}_X$-module $H^a(F^p K)$ is locally free of finite rank for all $a, p$.

**Remark 2.12.** The notion of strong strictness is well-defined in the filtered derived category in the following sense. If $(K_1, F)$ and $(K_2, F)$ are isomorphic in the filtered derived category, then $(K_1, F)$ is strongly strict if and only if $(K_2, F)$ is strongly strict.

**Lemma 2.13.** Let $(K, F)$ be a filtered perfect complex of $\mathcal{O}_X$-modules and $F$ an $\mathcal{O}_X$-module. The filtered derived tensor product is denoted by $(F \otimes^L K, F)$. If $(K, F)$ is strongly strict, then $(F \otimes^L K, F)$ is strict and the canonical morphisms
\[
F \otimes H^a(F^p K) \longrightarrow H^a(F \otimes^L F^p K) \quad (2.13.1)
\]
\[
F \otimes H^a(\text{Gr}_F^p K) \longrightarrow H^a(F \otimes^L \text{Gr}_F^p K) \quad (2.13.2)
\]
are isomorphisms for all $a, p$. In particular, $(\mathcal{A} \otimes^L K, F)$ is strongly strict as a filtered complex of $\mathcal{A}$-modules for a commutative $\mathcal{O}_X$-algebra $\mathcal{A}$.

**Proof.** The canonical morphism (2.13.2) is an isomorphism because $H^a(\text{Gr}_F^p K)$ is locally free for all $a, p$. Then we have the commutative diagram
\[
\begin{array}{cccccc}
0 & \longrightarrow & F \otimes H^a(F^{p-1}K) & \longrightarrow & F \otimes H^a(F^p K) & \longrightarrow & F \otimes H^a(\text{Gr}_F^p K) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow \cong & \\
0 & \longrightarrow & H^a(F \otimes^L F^{p-1}K) & \longrightarrow & H^a(F \otimes^L F^p K) & \longrightarrow & H^a(F \otimes^L \text{Gr}_F^p K) & \longrightarrow & 0,
\end{array}
\]
in which the first row is exact because $(K, F)$ is strongly strict. Then we can see that the second row in the diagram above is exact and the canonical morphism (2.13.1) is an isomorphism for all $a, p$ by induction on $p$. The injectivity of the morphism
\[
H^a(F \otimes^L F^{p-1}K) \longrightarrow H^a(F \otimes^L F^p K)
\]
for all $p$ implies that $(F \otimes^L K, F)$ is strict. \hfill \Box

For the case where $X$ is a complex manifold, we have the following results.

**Lemma 2.14.** Let $X$ be a complex manifold and $(K, F)$ a filtered perfect complex such that $H^a(K)$ is locally free of finite rank for all $a$. Then $(K, F)$ is strongly strict if and only if $(\mathcal{C}(x) \otimes^L K, F)$ is strict for every point $x \in X$. 

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This lemma is a direct consequence of Lemma 3.4 (iv) in [6]. However, the proof of Lemma 3.4 (iv) in [6] is somewhat misleading. So, we restate the result and complete the proof.

**Lemma 2.15** (Lemma 3.4 (iv) of [6]). Let \((K, F)\) be a filtered perfect complex on a complex manifold \(X\). Assume that the function \(X \ni x \mapsto \dim H^q(C(x) \otimes L^F K)\) is locally constant. If the morphisms

\[
\begin{align*}
  d(x) : (C(x) \otimes L^F K)^{q-1} &\longrightarrow (C(x) \otimes L^F K)^q, \\
  d(x) : (C(x) \otimes L^F K)^{q} &\longrightarrow (C(x) \otimes L^F K)^{q+1}
\end{align*}
\]  

are strictly compatible with the filtration \(F(C(x) \otimes L^F K)\) for all \(x \in X\), then \(H^q(\text{Gr}_F^p K)\) and \(H^q(F^p K)\) are locally free of finite rank, the canonical morphisms

\[
\begin{align*}
  C(x) \otimes H^q(\text{Gr}_F^p K) &\longrightarrow H^q(C(x) \otimes L \text{Gr}_F^p K) \\
  C(x) \otimes H^q(F^p K) &\longrightarrow H^q(C(x) \otimes L F^p K)
\end{align*}
\]

are isomorphisms for all \(p\) and for all \(x \in X\), and the differentials

\[
\begin{align*}
  d : K^{q-1} &\longrightarrow K^q \\
  d : K^q &\longrightarrow K^{q+1}
\end{align*}
\]

are strictly compatible with the filtration \(F\).

**Proof.** Note that we have the canonical isomorphisms

\[
\begin{align*}
  C(x) \otimes L F^p K &\simeq F^p (C(x) \otimes L K) \\
  C(x) \otimes L \text{Gr}_F^p K &\simeq \text{Gr}_F^p (C(x) \otimes L K)
\end{align*}
\]

by the definition of filtered derived tensor product. We obtain the exact sequences

\[
\begin{array}{cccc}
0 & \longrightarrow & H^q(C(x) \otimes L F^{p+1} K) & \longrightarrow \\
& & H^q(C(x) \otimes L F^p K) & \longrightarrow \\
& & H^q(C(x) \otimes L \text{Gr}_F^p K) & \longrightarrow 0
\end{array}
\]

from the assumption that the differentials (2.15.1) are strictly compatible with the filtration \(F\). Thus we have the equality

\[
\dim H^q(C(x) \otimes L F^p K) = \sum_{p' \geq p} \dim H^q(C(x) \otimes L \text{Gr}_F^{p'} K)
\]  

(2.15.5)

for all \(p\) and for all \(x \in X\). Taking \(p\) sufficiently small, we obtain

\[
\dim H^q(C(x) \otimes L K) = \sum_p \dim H^q(C(x) \otimes L \text{Gr}_F^p K),
\]

which implies that the function \(X \ni x \mapsto \dim H^q(C(x) \otimes L \text{Gr}_F^p K)\) is locally constant as in the proof of Lemma 3.4 (iv) in [6]. Then the function \(X \ni x \mapsto \dim H^q(C(x) \otimes L F^p K)\) is locally constant from the equality (2.15.5). Therefore \(H^q(F^p K)\) and \(H^q(\text{Gr}_F^p K)\) are
locally free and the canonical morphisms (2.15.2) are isomorphisms for all \( p \) by Lemma 3.4 (iii) in [6]. From the exact sequence (2.15.4), we have the exact sequence

\[
0 \longrightarrow \mathbb{C}(x) \otimes H^q(F^p+1K) \longrightarrow \mathbb{C}(x) \otimes H^q(F^pK) \longrightarrow \mathbb{C}(x) \otimes H^q(\text{Gr}_F^p K) \longrightarrow 0,
\]

from which we obtain the exact sequence

\[
0 \longrightarrow H^q(F^p+1K) \longrightarrow H^q(F^pK) \longrightarrow H^q(\text{Gr}_F^p K) \longrightarrow 0
\]

by using the local freeness of \( H^q(F^p+1K) \) for all \( p \). Therefore the differentials (2.15.3) are strictly compatible with \( F \) as desired.

3 Semistable morphism

**Definition 3.1.** Let \( Y \) be a complex manifold and \( E \) a simple normal crossing divisor on \( Y \). A morphism of complex manifolds \( f : X \rightarrow Y \) is said to be semistable along \( E \) if the following two conditions are satisfied:

1. The divisor \( D = f^*E \) is a reduced simple normal crossing divisor on \( X \).
2. The morphism

\[
f^*\Omega^1_Y(\log E) \longrightarrow \Omega^1_X(\log D)
\]

is injective and the cokernel

\[
\Omega^1_{X/Y}(\log D) = \Omega^1_X(\log D)/f^*\Omega^1_Y(\log E)
\]

is locally free.

Once we fix the divisor \( E \), the words “along \( E \)” is omitted. Moreover, we use the notation

\[
\omega^p_Y = \Omega^p_Y(\log E), \quad \omega^p_X = \Omega^p_X(\log D), \quad \omega^p_{X/Y} = \Omega^p_{X/Y}(\log D)
\]

if there is no danger of confusion.

We set \( Y^* = Y \setminus E \) and \( X^* = X \setminus D = f^{-1}(Y^*) \).

**Example 3.2.** Let \( \Delta \) be the unit disc in \( \mathbb{C} \) with the coordinate function \( t \), and \( \Delta^n \) the polydisc in \( \mathbb{C}^n \) with the coordinate functions \( x_1, x_2, \ldots, x_n \). A morphism \( f : \Delta^n \rightarrow \Delta \) given by

\[
f^*t = x_1x_2 \cdots x_n
\]

is semistable along \( \{0\} \).

Let \( f_i : \Delta^{n_i} \rightarrow \Delta \) (\( i = 1, 2, \ldots, k \)) be morphisms given by the form (3.2.1) and \( g : T \rightarrow S \) a smooth morphism of complex manifolds. Set \( Y = \Delta^k \times S \) and \( E = \{t_1t_2 \cdots t_k = 0\} \) where \( t_1, t_2, \ldots, t_k \) is the coordinate functions of \( \Delta^k \). Then the morphism

\[
f_1 \times f_2 \times \cdots \times f_k \times g : \Delta^{n_1} \times \Delta^{n_2} \times \cdots \times \Delta^{n_k} \times T \rightarrow \Delta^k \times S
\]

is also semistable along \( E \).
Lemma 3.3. Locally on $X$ and $Y$, any semistable morphism $f : X \longrightarrow Y$ is isomorphic to the morphism of the form (3.2.2). In particular, any semistable morphism is equi-dimensional and flat.

Proof. Similar to the proof of Lemma (6.5) in [7].

Definition 3.4. Let $f : X \longrightarrow Y$ be a semistable morphism along $E$. A finite decreasing filtration $G$ on $\omega_X$ is defined by

$$G^p\omega_X^n = \text{Image}(f^{-1}\omega_Y^n \otimes_{f^{-1}\mathcal{O}_Y} \omega_X^{n-p} \xrightarrow{\Delta} \omega_X^n)$$

for all $n, p$ as in [15]. Then we have the canonical isomorphism

$$f^{-1}\omega_Y^n \otimes_{f^{-1}\mathcal{O}_Y} \omega_X^{-p} \xrightarrow{\cong} \text{Gr}_G^n \omega_X$$

for every $p$. Therefore the spectral sequence $E^{a,b}_r(Rf_*\omega_X, G)$ satisfies the property

$$E^{a,b}_1(Rf_*\omega_X, G) \simeq R^b f_* (f^{-1}\omega_Y^a \otimes_{f^{-1}\mathcal{O}_Y} \omega_X^{-p})$$

$$\simeq \omega_Y^a \otimes_{\mathcal{O}_Y} R^b f_* \omega_X$$

for all $a, b$ because $\omega_Y^a$ is a locally free $\mathcal{O}_Y$-module of finite rank. Then the morphism of $E_1$-terms induces the morphism

$$R^n f_* \omega_X \longrightarrow \omega_Y^1 \otimes_{\mathcal{O}_Y} R^n f_* \omega_X$$

(3.4.1)

which is called the Gauss-Manin connection and denoted by $\nabla$. The residue of the Gauss-Manin connection (3.4.1) along the divisor $E_i$ is denoted by

$$\text{Res}_{E_i}(\nabla) : \mathcal{O}_{E_i} \otimes_{\mathcal{O}_Y} R^n f_* \omega_X \longrightarrow \mathcal{O}_{E_i} \otimes_{\mathcal{O}_Y} R^n f_* \omega_X$$

(3.4.2)

for every $i = 1, 2, \ldots, k$.

Remark 3.5. As in [15], $\nabla$ is an integrable logarithmic connection, that is, $\nabla$ satisfies the Leibniz rule and $\nabla^2 = 0$. Moreover, it satisfies the Griffiths transversality

$$\nabla(F^p R^n f_* \omega_X) \subset \omega_Y^1 \otimes F^{p-1} R^n f_* \omega_X$$

where $F$ denotes the filtration induced by the stupid filtration $F$ on $\omega_X$. On the other hand, the restriction of $\nabla$ on $Y^* = Y \setminus E$ is identified with $d \otimes \text{id} : \mathcal{O}_{Y^*} \otimes_{\mathbb{C}} R^n f_* \mathcal{C}_X \longrightarrow \Omega^1_{Y^*} \otimes_{\mathbb{C}} R^n f_* \mathcal{C}_X$ via the isomorphisms

$$R^n f_* \omega_X |_{Y^*} = R^n f_* \Omega^* \mathcal{C}_X \simeq R^n f_* f^{-1} \mathcal{O}_Y \simeq \mathcal{O}_{Y^*} \otimes_{\mathbb{C}} R^b f_* \mathcal{C}_X$$

as in [15], [6, 4.3].
3.6. Let $Y$ be a complex manifold, $E$ a simple normal crossing divisor on $Y$ and

$$E = \sum_{i=1}^{k} E_i$$

the irreducible decomposition of $E$. We set

$$E_I = \sum_{i \in I} E_i, \quad Y[I] = \bigcap_{i \in I} E_i$$

for $I \subset \{1,2,\ldots,k\}$. (We set $E_\emptyset = 0, Y[\emptyset] = Y$.) We write $Y_0 = Y[\{1,2,\ldots,k\}]$ for short. Since $E$ is a simple normal crossing divisor, $Y[I]$ is a closed submanifold of $Y$ for any $I$. We denote the closed immersion $Y[I] \hookrightarrow Y$ by $\iota_I$ ($Y_0 = Y[\{1,2,\ldots,k\}] \hookrightarrow Y$ by $\iota$). By setting $\mathcal{T} = \{1,2,\ldots,k\} \setminus I$, we have the simple normal crossing divisor $E_\mathcal{T}$ on $Y$, whose restriction to $Y[I]$ is denoted by $E_\mathcal{T} \cap Y[I]$. We use the notation

$$\omega^p_{Y[I]} = \Omega^p_{Y[I]}(\log E_\mathcal{T} \cap Y[I])$$

for all $p$. We set $Y[I]^* = Y[I] \setminus (E_\mathcal{T} \cap Y[I])$ for $I \subset \{1,2,\ldots,k\}$.

We have a sequence of closed submanifolds

$$Y_0 = Y[\{1,2,\ldots,k\}] \subset Y[I] \subset Y[J] \subset Y[\emptyset] = Y$$

for $J \subset I \subset \{1,2,\ldots,k\}$.

3.7. Let $f : X \longrightarrow Y$ be a semistable morphism along $E$. We set $D = f^*E$ and $D_i = f^*E_i$ for $i = 1,2,\ldots,k$. Then $D$ and $D_i$’s are reduced simple normal crossing divisors on $X$ with $D = \sum_{i=1}^{k} D_i$. We set

$$D_I = \sum_{i \in I} D_i, \quad X[I] = \bigcap_{i \in I} D_i$$

for $I \subset \{1,2,\ldots,k\}$ ($D_\emptyset = 0, X[\emptyset] = X$ as above). We write $X_0 = X[\{1,2,\ldots,k\}]$ as before. The closed immersion $X[I] \hookrightarrow X$ (resp. $X_0 \hookrightarrow X$) is also denoted by $\iota_I$ (resp. $\iota$) if there is no danger of confusion. We have the cartesian square

$$\begin{array}{ccc}
X[I] & \xrightarrow{\iota_I} & X \\
\downarrow f_I & & \downarrow f \\
Y[I] & \xrightarrow{\iota_I} & Y
\end{array}$$

in which the left vertical arrow is denoted by $f_I$. We use the notation $f_{\{1,2,\ldots,k\}} = f_0$, which fits in the cartesian square

$$\begin{array}{ccc}
X_0 & \xrightarrow{\iota} & X \\
\downarrow f_0 & & \downarrow f \\
Y_0 & \xrightarrow{\iota} & Y
\end{array}$$
by definition. We set

$$X[I]^* = X[I] \setminus (D_I \cap X[I]) = f_I^{-1}(Y[I]^*)$$

for $I \subset \{1, 2, \ldots, k\}$.

Let $D = \sum_{\lambda \in \Lambda} D_\lambda$ be the irreducible decomposition of $D$. We set

$$X[\Gamma] = \bigcap_{\lambda \in \Gamma} D_\lambda$$

for $\Gamma \subset \Lambda$ as before. The closed immersion $X[\Gamma] \hookrightarrow X$ is denoted by $i_\Gamma$.

From the equality $D = \sum_{i=1}^k D_i$ and the fact that $D$ is reduced, the index set $\Lambda$ decomposes into a disjoint union

$$\Lambda = \coprod_{i=1}^k \Lambda_i$$

such that

$$D_i = \sum_{\lambda \in \Lambda_i} D_\lambda$$

for $i = 1, 2, \ldots, k$. We set

$$\Lambda_I = \coprod_{i \in I} \Lambda_i \subset \Lambda$$

for $I \subset \{1, 2, \ldots, k\}$. If $\Gamma \cap \Lambda_i \neq \emptyset$ for all $i \in I$, then $X[\Gamma] \subset X[I]$. Thus we have a commutative diagram

$$\begin{array}{ccc}
X[\Gamma] & \xrightarrow{f_\Gamma} & X \\
\downarrow f & & \downarrow f \\
Y[I] & \xrightarrow{i_I} & Y
\end{array} \tag{3.7.1}$$

in which the left vertical arrow is denoted by $f_\Gamma$, once we fix $I \subset \{1, 2, \ldots, k\}$. For $\Gamma \subset \Lambda_I$, the restriction $D_\Gamma \cap X[\Gamma]$ of the divisor $D_\Gamma$ to $X[\Gamma]$ is a simple normal crossing divisor on $X[\Gamma]$. The morphism $f_\Gamma : X[\Gamma] \longrightarrow Y[I]$ above is semistable along $E_\Gamma \cap Y[I]$ by Lemma 3.3.

4 Construction of $sB_I(f)$

4.1. From now on, we treat the case of $Y = \Delta^k \times S$ equipped with a simple normal crossing divisor $E = \{t_1 t_2 \cdots t_k = 0\}$, where $t_1, t_2, \ldots, t_k$ are the coordinate functions of $\Delta^k$. By setting $E_i = \{t_i = 0\}$ for $i = 1, 2, \ldots, k$, we have $E = \sum_{i=1}^k E_i$. We use the notation in 3.6. We have $S = \{0\} \times S = Y[\{1, 2, \ldots, k\}] = Y_0$ by definition.

There exists the canonical projection

$$\pi_I : Y \longrightarrow Y[I]$$

for every $I$. We trivially have $\pi_I i_I = \text{id}$. When we consider $Y[I]$ as a complex manifold below $Y$ by the projection $\pi_I$ above, we use the notation $Y_I$ instead of $Y[I]$. The simple
normal crossing divisor $E_T \cap Y[I]$ on $Y[I]$ defines a simple normal crossing divisor on $Y_I$, which we denote by $E_T \cap Y_I$. We always consider that $Y_I$ is equipped with the divisor $E_T \cap Y_I$, unless the otherwise is mentioned. Thus we use the notation

$$\omega_{Y_I}^p = \Omega_{Y_I}^p (\log E_T \cap Y_I)$$

as in Definition 3.1 and 3.6. We set $Y_I^* = Y_I \setminus (E_T \cap Y_I)$ for $I \subset \{1, 2, \ldots, k\}$. We trivially have $Y[I]^* = Y_I^*$ as complex manifolds.

For $J \subset I \subset \{1, 2, \ldots, k\}$, we have a sequence of the projections

$$Y = Y_0 \longrightarrow Y_J \longrightarrow Y_I \longrightarrow Y_{\{1, 2, \ldots, k\}} = S$$

by definition. We use $\pi : Y \longrightarrow S$ instead of $\pi_{\{1, 2, \ldots, k\}}$.

For $J \subset I \subset \{1, 2, \ldots, k\}$, the coordinate functions $t_1, t_2, \ldots, t_k$ induce the direct sum decomposition

$$\omega_{Y[I]}^1 \simeq \omega_{Y[J]/Y_I}^1 \oplus \omega_{Y[I]}^1$$

which induces the projection

$$\text{pr}_{J[I]} : \omega_{Y[J]}^p \longrightarrow \omega_{Y[I]}^p$$ \hspace{1cm} (4.1.1)

for every $p$.

**Remark 4.2.** For a semistable morphism $f : X \longrightarrow Y$ along $E$, $\pi f : X \longrightarrow Y_I$ and $\pi f_{\Gamma} : X[\Gamma] \longrightarrow Y_I$ are semistable along $E_T \cap Y_I$ for $I \subset \{1, 2, \ldots, k\}$ and for $\Gamma \subset \Lambda_I$. We can see these facts by Lemma 3.3. Moreover, we have $\pi f_{\Gamma} = f_{\Gamma}$ via the identification $Y_I = Y[I]$.

Now, we construct a complex $sB_I(f)$ which is a replacement for $f_I^{-1} O_{Y[I]} \otimes_{f_I^{-1} O_Y} \omega_{X/Y}$ for $I \subset \{1, 2, \ldots, k\}$.

**Definition 4.3.** Let $Y$ and $E$ be as above and $f : X \longrightarrow Y$ a semistable morphism along $E$. For a subset $I \subset \{1, 2, \ldots, k\}$, a complex $sB_I(f)$ is defined as follows.

(4.3.1) For $p \in \mathbb{Z}_{\geq 0}, q \in \mathbb{Z}_{\geq 0}$, we set

$$B_I(f)^{p,q} = \omega_{X/Y_I}^{p+|I|-1} \left/ \sum_{i \in I} W(D_i)q_i \right.$$ 

where $W(D_i)$ denotes the filtration induced by $W(D_i)$ on $\omega_X$.

(4.3.2) We set

$$d_0 = (-1)^{|I|} d : B_I(f)^{p,q} \longrightarrow B_I(f)^{p+1,q}$$

where $d$ on the right hand side of the equality is the morphism induced from $d$ on $\omega_{X/Y_I}$.

(4.3.3) We set

$$d_i = (-1)^{|I|} \text{dlog } t_i \wedge : B_I(f)^{p,q} \longrightarrow B_I(f)^{p,q+\epsilon_i}$$

where $\text{dlog } t_i \wedge$ denotes the morphism defined by

$$\omega \mapsto \text{dlog } t_i \wedge \omega = \frac{dt_i}{t_i} \wedge \omega$$

as in [10].

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By setting

\[ sB_I(f)^n = \bigoplus_{p+q=n} B_I(f)^{p,q} = \bigoplus_{q \in \mathbb{Z}_{\geq 0}^I} \left( \omega_{X/Y}^{n+[I]}/ \sum_{i \in I} W(D_i)_{q_i} \right) \]

with the differential \( d = d_0 + \sum_{i \in I} d_i \), we obtain the complex \( sB_I(f) \). We have

\[ (\text{Supp}(B_I(f)^{p,q}) \subset X[I]) \quad (4.3.4) \]

for any \( p, q \). Thus we identify \( B_I(f)^{p,q} \) and \( \iota^{-1}_I B_I(f)^{p,q} \) for all \( p, q \). Moreover \( B_I(f)^{p,q}(= \iota^{-1}_I B_I(f)^{p,q}) \) is an \( \mathcal{O}_X[I] \)-module for any \( p, q \). However \( sB_I(f) \) is considered as a complex of \( \mathcal{I}_I^{-1} \mathcal{O}_{Y[I]} \)-modules because \( d_0 \) is not a morphism of \( \mathcal{O}_X[I] \)-modules. We use the symbol \( sB(f) \) for the case of \( I = \{1, 2, \ldots, k\} \).

We define a finite decreasing filtration \( F \) on \( sB_I(f) \) by

\[ F^p sB_I(f)^n = \bigoplus_{p'+|q|=n} B_I(f)^{p',q} = \bigoplus_{q \in \mathbb{Z}_{\geq 0}^I} \left( \omega_{X/Y}^{n+[I]}/ \sum_{i \in I} W(D_i)_{q_i} \right) \quad (4.3.5) \]

for all \( n, p \).

By definition, \( sB_0(f) \) equipped with \( F \) is nothing but \( \omega_{X/Y} \) equipped with the stupid filtration \( F \).

**Remark 4.4.** In the case where \( S \) is a point and \( I = \{1, 2, \ldots, k\} \), the complex \( sB(f) \) is same as \( sB_X(D_1, \ldots, D_k) \) in \([7, (3.12)]\) except the sign of the differentials.

**Remark 4.5.** Because we have \( d_0(F^p sB_I(f)^n) \subset F^{p+1}sB_I(f)^{n+1} \) for all \( n, p \), the complex \( \text{Gr}_F^p sB_I(f) \) turns out to be a complex of \( \mathcal{O}_X[I] \)-modules for all \( p \). We trivially have

\[ \text{Gr}_F^p sB_I(f)^n = \bigoplus_{q \in \mathbb{Z}_{\geq 0}^I} B_I(f)^{p,q} = \bigoplus_{q \in \mathbb{Z}_{\geq 0}^I} \left( \omega_{X/Y}^{n+[I]}/ \sum_{i \in I} W(D_i)_{q_i} \right) \quad (4.5.1) \]

for all \( n, p \).

**Definition 4.6.** In the same situation as in Definition 4.3, a finite increasing filtration \( L(K) \) on \( B_I(f)^{p,q} \) for \( K \subset I \) is defined by

\[ L(K)_mB_I(f)^{p,q} = W(D_K)^{m+2|q_K|+|K|}[\omega_{X/Y}^{p+|q|+[I]}]/ \sum_{i \in I} W(D_i)_{q_i} \quad (4.6.1) \]

where \( q_K \) is the image of \( q \) by the projection \( \mathbb{Z}_{-}^I \rightarrow \mathbb{Z}_K \) as in 1.9. We can easily see that \( L(K) \) is preserved by every \( d_i \). Thus we obtain a finite increasing filtration \( L(K) \) on \( sB_I(f) \). We write \( L = L(\{1, 2, \ldots, k\}) \) on \( sB(f) \) for short. By definition, \( L(\emptyset) \) is the trivial filtration, that is, \( L(\emptyset)_{-1}sB_I(f) = 0 \) and \( L(\emptyset)_0sB_I(f) = sB_I(f) \).
Lemma 4.7. For any $K \subset I$, an isomorphism of complexes

$$\text{Gr}_m^{L(K)} sB_I(f) \simeq \bigoplus_{q \in \mathbb{Z}_{\geq 0}} \bigoplus_{\Gamma \in S_{m+2|q|+|K|}(\Lambda_K)} sB_{I \setminus K}(\pi_K f \iota \Gamma)[-m - 2|q|]$$

is induced from the residue morphism for $W(D_K)$, where $f \iota : X[\Gamma] \to Y[K]$ is the morphism in the commutative diagram (3.7.1). Under the isomorphism (4.7.1) above, we have

$$L(J) \text{Gr}_m^{L(K)} sB_I(f) \simeq \bigoplus L(J \setminus K \cap J)[|\Gamma \cap \Lambda_K \cap J| - 2|q_{K \cap J}| - |K \cap J|] sB_{I \setminus K}(f \iota \Gamma)[-m - 2|q|]$$

for any $J \subset I$ and

$$F \text{Gr}_m^{L(K)} sB_I(f) \simeq \bigoplus F[-m - |q|] sB_{I \setminus K}(f \iota \Gamma)[-m - 2|q|],$$

where the index sets of the direct sums above are the same as in (4.7.1). In particular, we have

$$L(J) \text{Gr}_m^{L(K)} sB_I(f) \simeq \bigoplus L(J \setminus K)[m] sB_{I \setminus K}(f \iota \Gamma)[-m - 2|q|]$$

for $J \supset K$ and

$$L(J) \text{Gr}_m^{L(K)} sB_I(f) \simeq \bigoplus L(J)sB_{I \setminus K}(f \iota \Gamma)[-m - 2|q|]$$

for $J$ with $J \cap K = \emptyset$.

Proof. We obtain the conclusion by Lemma 3.3 and by the local computation as in [2, Proposition 3.6].

Remark 4.8. In the lemma above, we use the residue morphism defined as in [2, II.3.7]. It is different by the sign from the residue morphism used in [7].

Lemma 4.9. For $K \subset I$, the $\mathcal{O}_{X[I]}$-sheaf $\text{Gr}_F^p \text{Gr}_m^{L(K)} sB_I(f)^n$ is $f^{-1}_I \mathcal{O}_Y[I]$-flat for all $m, n, p$.

Proof. First, we treat the case of $K = I$. We have an isomorphism

$$\text{Gr}_F^p \text{Gr}_m^{L(I)} sB_I(f)^n \simeq \bigoplus \omega_{X[I]/Y[I]}^{n-m-2|q|}$$

from Lemma 4.7, where the direct sum on the right hand side is taken over the index set

$$\{(q, \Gamma) | q \in \mathbb{Z}_{\geq 0} \text{ with } |q| = n - p, \Gamma \in S_{m+2|q|+|I|}(\Lambda_I)\}.$$  

Therefore $\text{Gr}_F^p \text{Gr}_m^{L(I)} sB_I(f)^n$ are $f^{-1}_I \mathcal{O}_Y[I]$-flat for all $m, n, p$ because the semistable morphism $f \iota : X[\Gamma] \to Y[I]$ is flat by Lemma 3.3. In particular,

$$\text{Gr}_F^p sB_I(f)^n, \quad \text{Gr}_m^{L(I)} sB_I(f)^n, \quad sB_I(f)^n$$

are $f^{-1}_I \mathcal{O}_Y[I]$-flat for all $m, n, p$. Hence we obtain the conclusion for general $K \subset I$ from Lemma 4.7 again. \qed
Definition 4.10. For $J \subset I \subset \{1, 2, \ldots, k\}$, a morphism of $\mathcal{O}_X$-modules

$$\omega_X^{n+|J|} \longrightarrow \omega_X^{n+|I|}$$

is defined by sending $\omega \in \omega_X^{n+|J|}$ to

$$d\log t_{i_1} \wedge d\log t_{i_2} \wedge \cdots \wedge d\log t_{i_l} \wedge \omega \in \omega_X^{n+|I|} \quad (4.10.1)$$

where $I \setminus J = \{i_1, i_2, \ldots, i_l\}$ with $1 \leq i_1 < i_2 < \cdots < i_l \leq k$. We can easily check that this morphism induces a morphism of $\mathcal{O}_{X|J}$-modules

$$\theta_{J|I}(f) : B_J(f)^{p,q} \longrightarrow B_I(f)^{p,q}$$

for $p \in \mathbb{Z}_{\geq 0}$ and for $q \in \mathbb{Z}_{\geq 0}^J \subset \mathbb{Z}_{\geq 0}^I$. Moreover, the equalities

$$d_0 \theta_{J|I}(f) = \theta_{J|I}(f)d_0$$

$$d_i \theta_{J|I}(f) = \theta_{J|I}(f)d_i \quad \text{for} \ i \in J$$

$$d_i \theta_{J|I}(f) = 0 \quad \text{for} \ i \in I \setminus J$$

are easily seen. Thus a morphism of complexes

$$\theta_{J|I}(f) : sB_J(f) \longrightarrow sB_I(f) \quad (4.10.2)$$

is obtained. It is easy to see that $\theta_{J|I}(f)$ preserves the filtrations $F$ and $L(K)$ for $K \subset J$. We use the notation $\theta_J(f) = \theta_{J\setminus\{1,2,\ldots,k\}}(f)$, $\theta_{\emptyset|I}(f) = \theta_{\emptyset|J}(f)$ and $\theta(f) = \theta_{\emptyset\setminus\{1,2,\ldots,k\}}(f)$ for short.

Lemma 4.11. For $K \subset J \subset I$, the diagram

$$\begin{array}{ccc}
sB_K(f) & \xrightarrow{\theta_{K|J}(f)} & sB_J(f) \\
\downarrow & & \downarrow \theta_{J|I}(f) \\
sB_K(f) & \xrightarrow{\theta_{K|J}(f)} & sB_I(f)
\end{array}$$

commutes up to sign.

Proof. Easy.

Lemma 4.12. For $K \subset J \subset I$, the morphism $\theta_{J|I}(f)$ induces a quasi-isomorphism

$$f_I^{-1}\mathcal{O}_{Y|I} \otimes_{f_J^{-1}\mathcal{O}_{Y|J}} \text{Gr}_F^p \text{Gr}_m^{L(K)} sB_J(f) \xrightarrow{\simeq} \text{Gr}_F^p \text{Gr}_m^{L(K)} sB_I(f)$$

for all $m, p$.

Proof. Once we obtain the conclusion for $K = \emptyset$, we obtain the conclusion for general $K$ by using Lemma 4.7. So, we assume that $K = \emptyset$. Namely, we will prove that the morphism

$$f_I^{-1}\mathcal{O}_{Y|I} \otimes_{f_J^{-1}\mathcal{O}_{Y|J}} \text{Gr}_F^p sB_J(f) \longrightarrow \text{Gr}_F^p sB_I(f) \quad (4.12.1)$$

induced by $\theta_{J|I}(f)$ is a quasi-isomorphism.
First, we treat the case where \( I = \{1, 2, \ldots, k\} \) and \( S \) is a point. If \( J = \emptyset \), the conclusion is already proved in \([7, \text{Corollary (5.14)}]\). Then the conclusion for \( J = \emptyset \) above and Lemma 4.7 imply that the morphism \( \theta_{J}(f) \) induces a filtered quasi-isomorphism

\[
\frac{f_{0}^{-1}O_{Y_{0}} \otimes f_{J}^{-1}O_{Y[J]}}{Gr_{F}^{p}Gr_{m}^{L(J)}sB_{J}(f)} \longrightarrow Gr_{F}^{p}Gr_{m}^{L(J)}sB(f)
\]

for all \( m, p \) and for any \( J \). Thus we conclude that the morphism (4.12.1) is a quasi-isomorphism for \( I = \{1, 2, \ldots, k\} \) and for \( Y = \Delta^{k} \) by using the \( f_{J}^{-1}O_{Y[J]} \)-flatness of \( Gr_{m}^{L(J)}sB_{J}(f)^{n} \) in Lemma 4.9.

Next, we treat the general case. By definition, we have \( Y = \Delta^{I} \times Y_{I} \), where \( \Delta^{I} \) denotes the polydisc of dimension \(|I|\) with the coordinate functions \((t_{i})_{i \in I}\). Since the problem is of local nature, we may assume that

\[
X = \Delta^{n} \times Z, \quad f = g \times h,
\]

where \( g : \Delta^{n} \rightarrow \Delta^{I} \) is a product of the morphisms of the form (3.2.1) and where \( h : Z \rightarrow Y_{I} \) is a semistable morphism along \( E_{T} \cap Y_{I} \). We denote the projections \( X \rightarrow \Delta^{n} \) and \( X \rightarrow Z \) by \( pr_{1} \) and \( pr_{2} \) and set \( D_{Z} = h^{*}(E_{T} \cap Y_{I}) \) for a while. From the equality (4.5.1), we have the identifications

\[
Gr_{F}^{p}sB_{J}(f) \simeq \bigoplus_{q \geq 0} pr_{1}^{*}Gr_{F}^{p-q}sB(g)[q] \otimes O_{X} pr_{2}^{*}O_{Z/Y_{I}},
\]

under which the morphism \( Gr_{F}^{p} \theta_{J}(f) \) is identified with the direct sum of the morphisms \( pr_{1}^{*}Gr_{F}^{p-q} \theta_{J}(g) \otimes \text{id} \) over all \( q \geq 0 \). From what we have proved above, \( pr_{1}^{*}Gr_{F}^{p-q} \theta_{J}(g) \otimes \text{id} \) induces a quasi-isomorphism because \( pr_{1} : X \rightarrow \Delta^{n} \) is flat and \( pr_{2}^{*}O_{Z/Y_{I}} \) is a locally free \( O_{X} \)-module. Thus we conclude that \( \theta_{J}(f) \) induces a quasi-isomorphism (4.12.1).

\[\square\]

**4.13.** Let \( S' \) be a complex manifold and \( g : S' \rightarrow S \) a morphism of complex manifolds. We set \( Y' = \Delta^{k} \times S' \), \( g = \text{id} \times g : Y' = \Delta^{k} \times S' \rightarrow Y = \Delta^{k} \times S \), \( E'_{i} = (\text{id} \times g)^{*}E_{i} \) and \( E' = (\text{id} \times g)^{*}E \). By the cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{(\text{id} \times g)} & Y
\end{array}
\]

we define \( f' : X' \rightarrow Y' \), which is semistable along \( E' \) by Lemma 3.3. We set \( D'_{i} = f'^{*}E'_{i} = g'^{*}D_{i} \) for \( i = 1, 2, \ldots, k \). We use the notation such as \( Y'[I], Y'_{I}, X'[I], X'[\Gamma] \) as in the case of \( f : X \rightarrow Y \). From the cartesian squares

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{(\text{id} \times g)} & Y
\end{array}
\]

\[
\begin{array}{ccc}
Y'_{I} & \xrightarrow{\pi_{I}} & Y_{I} \\
\downarrow (\text{id} \times g) & & \downarrow \pi_{I}
\end{array}
\]

\[
\begin{array}{ccc}
X'_{I} & \xrightarrow{g'_{I}} & X_{I} \\
\downarrow f'_{I} & & \downarrow f_{I} \\
Y'_{I} & \xrightarrow{(\text{id} \times g)_{I}} & Y_{I}
\end{array}
\]

\[
\begin{array}{ccc}
Y'_{I} & \xrightarrow{\pi_{I}} & Y_{I} \\
\downarrow (\text{id} \times g)_{I} & & \downarrow \pi_{I}
\end{array}
\]

\[
\begin{array}{ccc}
X'_{I} & \xrightarrow{g'_{I}} & X_{I} \\
\downarrow f'_{I} & & \downarrow f_{I} \\
Y'_{I} & \xrightarrow{(\text{id} \times g)_{I}} & Y_{I}
\end{array}
\]

\[
\begin{array}{ccc}
Y'_{I} & \xrightarrow{\pi_{I}} & Y_{I} \\
\downarrow (\text{id} \times g)_{I} & & \downarrow \pi_{I}
\end{array}
\]
we have the canonical morphisms
\[ g'^{-1}\omega^p_{X/Y} \longrightarrow \omega^p_{X'/Y'_i} \]
for every \( I \) and for all \( p \), which preserve the filtrations \( W(D_i) \) and \( W(D'_i) \) on the both sides for \( i \in I \). Thus we obtain the morphism of complexes
\[ f^{-1}\mathcal{O}_{Y'[I]} \otimes g^{-1}\mathcal{O}_{Y[I]} \overset{g'^{-1}sB_1(f)}\longrightarrow sB_1(f') \]
for all \( I \subset \{1, 2, \ldots, k\} \), which preserves the filtrations \( F \) and \( L(J) \) for all \( J \subset I \).

5 The Gauss-Manin connection

In this section, we construct the Gauss-Manin connection on \( R^a(f_t)sB_1(f) \) as in the case on \( R^a(f) \omega_{X/Y} \) in (3.4.1). For this purpose, we construct a complex \( s\tilde{B}_1(f) \) which is a replacement of \( f^{-1}\mathcal{O}_{Y[I]} \otimes \omega_X \) for \( I \subset \{1, 2, \ldots, k\} \) and follow the arguments in Definition 3.4. Then we describe its residues by a similar way to [18].

\textbf{Definition 5.1.} For a subset \( I \subset \{1, 2, \ldots, k\} \), we set
\[ \tilde{B}_1(f)^{p,q} = \omega^{p+|q|+|I|}_{X} \bigg/ \sum_{i \in I} W(D_i)_{q_i} \]
for \( p \in \mathbb{Z}_{\geq 0} \) and \( q \in \mathbb{Z}^I_{\geq 0} \), with the morphisms
\[ d_0 = (-1)^{|I|}d : \tilde{B}_1(f)^{p,q} \longrightarrow \tilde{B}_1(f)^{p+1,q} \]
\[ d_i = (-1)^{|I|}d \log t_i \wedge : \tilde{B}_1(f)^{p,q} \longrightarrow \tilde{B}_1(f)^{p,q+\epsilon_i} \]
as in Definition 4.3. Then we obtain the complex \( s\tilde{B}_1(f) \) by setting
\[ s\tilde{B}_1(f)^n = \bigoplus_{p+|q|=n} \tilde{B}_1(f)^{p,q} = \bigoplus_{q \in \mathbb{Z}^I_{\geq 0}} \left( \omega^{n+|I|}_{X} \bigg/ \sum_{i \in I} W(D_i)_{q_i} \right) \]
with the differential \( d = d_0 + \sum_{i \in I} d_i \) as in Definition 4.3. As in (4.3.4), we have \( \text{Supp}(\tilde{B}_1(f)^{p,q}) \subset X[I] \) for any \( p, q \). Thus we identify \( \tilde{B}_1(f)^{p,q} \) and \( \iota^{-1}_I \tilde{B}_1(f)^{p,q} \) for all \( p, q \) as before. Although \( \tilde{B}_1(f)^{p,q} \) is an \( \mathcal{O}_{X[I]} \)-module for all \( p, q \), we consider \( s\tilde{B}_1(f) \) as a complex of \( \mathbb{C} \)-sheaves because the morphism \( d_0 \) is not a morphism of \( \mathcal{O}_{X[I]} \)-modules. The filtration \( F \) on \( s\tilde{B}_1(f) \) is defined by
\[ F^p s\tilde{B}_1(f)^n = \bigoplus_{p'+|q|=n \atop p' \geq p} \tilde{B}_1(f)^{p',q} = \bigoplus_{q \in \mathbb{Z}^I_{\geq 0} \atop |q| \leq n-p} \left( \omega^{n+|I|}_{X} \bigg/ \sum_{i \in I} W(D_i)_{q_i} \right) \]
as in (4.3.5). For \( K \subset I \), the filtration \( L(K) \) on \( \tilde{B}_1(f) \) is defined by
\[ L(K)_m\tilde{B}_1(f)^{p,q} = W(D_K)_{m+2|q_K|+|K|} \omega^{p+|q|+|I|}_{X} \bigg/ \sum_{i \in I} W(D_i)_{q_i} \]
as in (4.6.1). The canonical projection $\omega^n_X \rightarrow \omega^n_{X/Y_I}$ induces a morphism of complexes

$$s\tilde{B}_I(f) \rightarrow sB_I(f),$$

which is denoted by $pr_I$ if there is no danger of confusion. It is easy to see that $pr_I$ preserves the filtrations $F$ and $L(K)$.

**Definition 5.2.** For a subset $I \subset \{1, 2, \ldots, k\}$, a finite decreasing filtration $G(I)$ on $\omega_X$ is defined by

$$G(I)^p\omega^n_X = \text{Image}((\pi_I f)^{-1}\omega^n_{Y_I} \otimes (\pi_I f)^{-1}\omega^n_{Y_I} \rightarrow \omega^n_X)$$

for all $n, p$. By definition $G(\emptyset)$ is nothing but the filtration $G$ defined in Definition 3.4. Because we have

$$d\log t_i \land G(I)^p\omega_X \subset G(I)^p\omega_X$$

for every $i \in I$ and for every $p$, the filtration $G$ on $\tilde{B}_I(f)$ is defined by

$$G^p\tilde{B}_I(f)^{p,q} = G(I)^p\omega^{p+|q|+|I|}_X / \sum_{i \in I} W(D_i)q_i$$

for all $p, q, r$. Thus we obtain a finite decreasing filtration $G$ on $s\tilde{B}_I(f)$.

**Lemma 5.3.** We have the isomorphism

$$f_I^{-1}\omega^n_{Y[I]} \otimes f_I^{-1}\omega^n_{Y[I]} sB_I(f)[-p] \xrightarrow{\sim} \text{Gr}_G^p s\tilde{B}_I(f)$$

(5.3.1)

for every $p$, under which

$$f_I^{-1}\omega^n_{Y[I]} \otimes f_I^{-1}\omega^n_{Y[I]} F^{r-p} sB_I(f)[-p] \simeq F^r \text{Gr}_G^p s\tilde{B}_I(f)$$

and

$$f_I^{-1}\omega^n_{Y[I]} \otimes f_I^{-1}\omega^n_{Y[I]} L(K)_m \text{Gr}_G^p s\tilde{B}_I(f) \simeq L(K)_m \text{Gr}_G^p s\tilde{B}_I(f)$$

for $K \subset I$ and for all $m, r$.

**Proof.** Easy from $\iota_I^{-1}(\pi_I f)^{-1} = (\pi_I f \iota_I)^{-1} = f_I^{-1}$. \hfill $\square$

**Definition 5.4.** The morphism of $E_1$-terms

$$E_1^{p,n}(R(f_I)_* s\tilde{B}_I(f), G) \rightarrow E_1^{p+1,n}(R(f_I)_* s\tilde{B}_I(f), G)$$

induces a morphism of $\mathbb{C}$-sheaves

$$\nabla(I) : \omega^n_{Y[I]} \otimes \omega^n_{Y[I]} R^n(f_I)_* sB_I(f) \rightarrow \omega^{p+1}_{Y[I]} \otimes \omega^n_{Y[I]} R^n(f_I)_* sB_I(f)$$

via the identification

$$E_1^{p,n}(R(f_I)_* s\tilde{B}_I(f), G) \simeq \omega^n_{Y[I]} \otimes \omega^n_{Y[I]} \omega^p_{Y[I]} \otimes \omega^n_{Y[I]} R^n(f_I)_* sB_I(f)$$

which is given by Lemma 5.3 together with the local freeness of $\omega^n_{Y[I]}$. We have

$$\nabla(I)(\omega^n_{Y[I]} \otimes \omega^n_{Y[I]} L(K)_m R^n(f_I)_* sB_I(f)) \subset \omega^{p+1}_{Y[I]} \otimes \omega^n_{Y[I]} L(K)_m R^n(f_I)_* sB_I(f)$$

$$\nabla(I)(\omega^{p+1}_{Y[I]} \otimes \omega^n_{Y[I]} F^r R^n(f_I)_* sB_I(f)) \subset \omega^{p+1}_{Y[I]} \otimes \omega^n_{Y[I]} F^{r-1} R^n(f_I)_* sB_I(f)$$

for all $m, r$ by Lemma 5.3 again.
Remark 5.5. For the case of $I = \emptyset$, the filtered complex $(s\tilde{B}_0(f), G)$ coincides with $(\omega_X, G)$ in Definition 3.4. Therefore we have $\nabla(\emptyset) = \nabla$.

Remark 5.6. A morphism

$$(\pi_I f)^{-1}\omega^p_{Y_I} \otimes (\pi_I f)^{-1}\omega^n_X \longrightarrow \omega^{n+p}$$

is defined by sending $\omega \otimes \eta$ to $(-1)^{|I|}\omega \wedge \eta$. Direct computation shows that this morphism gives a morphism of complexes

$$i_I^{-1}(\pi_I f)^{-1}\omega_{Y_I} \otimes i_I^{-1}(\pi_I f)^{-1}\omega_{Y_I} sB_I(f) = f_I^{-1}\omega_{Y_I} \otimes f_I^{-1}\omega_{Y_I} sB_I(f) \longrightarrow s\tilde{B}_I(f).$$

Then we can prove the equality

$$\nabla(I)(\omega \otimes v) = (-1)^{|I|}d\omega \otimes v + (-1)^p\omega \wedge \nabla(I)(v)$$

for $\omega \in \omega^p_{Y_I}$ and $v \in R^n(f_I)_*sB_I(f)$ as in [15]. Therefore $(-1)^{|I|}\nabla(I)$ is an integrable logarithmic connection on $R^n(f_I)_*sB_I(f)$ for any $n$.

Definition 5.7. The filtration $G$ on $s\tilde{B}_I(f)$ induces a morphism

$$\gamma_I(f) : f_I^{-1}\omega_{Y_I} \otimes f_I^{-1}\omega_{Y_I} sB_I(f)[-p] \longrightarrow f_I^{-1}\omega_{Y_I} \otimes f_I^{-1}\omega_{Y_I} sB_I(f)[-p]$$

in the derived category for every $p$. Then we have

$$\nabla(I) = R^{p+n}(f_I)_*(\gamma_I(f))$$

by definition.

Definition 5.8. For $J \subset I \subset \{1, 2, \ldots, k\}$, the morphism defined by $(4.10.1)$ induces a morphism of complexes

$$\tilde{\theta}_{J\mid I}(f) : s\tilde{B}_J(f) \longrightarrow s\tilde{B}_I(f)$$

as in Definition 4.10.

Lemma 5.9. The morphism $\tilde{\theta}_{J\mid I}(f)$ preserves the filtration $G$ on the both sides. The morphism

$$Gr^p_G\tilde{\theta}_{J\mid I}(f) : Gr^p_G\tilde{B}_J(f) \longrightarrow Gr^p_G\tilde{B}_I(f)$$

is identified with the morphism

$$(-1)^{|I|+|J|}pr_{J\mid I} \otimes \theta_{J\mid I}(f)[-p] : f_J^{-1}\omega_{Y_J} \otimes f_J^{-1}\omega_{Y_J} sB_J(f)[-p] \longrightarrow f_I^{-1}\omega_{Y_I} \otimes f_I^{-1}\omega_{Y_I} sB_I(f)[-p]$$

for every $p$, where $pr_{J\mid I}$ is the morphism $(4.1.1)$.

Proof. Easy by definition.

Corollary 5.10. The diagram

$$\begin{array}{ccc}
\omega^p_{Y_J} \otimes \omega_{Y_J} & R^n(f_J)_*sB_J(f) & \longrightarrow \\
pr_{J\mid I} \otimes \theta_{J\mid I}(f) \downarrow & (-1)^{|I|}\nabla(J) & \downarrow pr_{J\mid I} \otimes \theta_{J\mid I}(f) \\
\omega^p_{Y_I} \otimes \omega_{Y_I} & R^n(f_I)_*sB_I(f) & \longrightarrow \\
\langle (-1)^{|I|}\nabla(I) \rangle & \langle (-1)^{|I|}\nabla(I) \rangle & \langle (-1)^{|I|}\nabla(I) \rangle
\end{array}$$

is commutative.
Definition 5.11. For \( j \in I \), the canonical projection
\[
B_I(f)^{p,q} = \omega_{X/Y}^{p+|q|+|I|} \bigg/ \sum_{i \in I} W(D_i)_{q_i}
\]
\[
\longrightarrow B_I(f)^{p-1,q+e_j} = \omega_{X/Y}^{p+|q|+|I|} \bigg/ \sum_{i \in I, i \neq j} W(D_i)_{q_i} + W(D_j)_{q_{j+1}}
\]
induces a morphism of complexes
\[
\nu_{j|I}(f) : sB_I(f) \longrightarrow sB_I(f)
\]
for \( j \in I \). We set
\[
\nu_{j|I}(f) = \sum_{j \in J} \nu_{j|I}(f) : sB_I(f) \longrightarrow sB_I(f)
\]
for \( J \subset I \). We use the convention \( \nu_I(f) = \nu_{I|I}(f) \) and \( \nu(f) = \nu_{\{1,2,\ldots,k\}}(f) \) as before.

On the other hand, we define a morphism of complexes \( \mu_{j|I}(f) \) by
\[
\mu_{j|I}(f) = \sum_{j \in I \setminus J} d \log t_j \otimes \nu_{j|I}(f) : sB_I(f) \longrightarrow (f|J)^{-1} \omega_{Y[J]}^{1} \otimes (f|J)^{-1} \omega_{Y[I]}^{1} sB_I(f)
\]
for \( J \subsetneq I \).

Similarly, the projection
\[
\tilde{B}_I(f)^{p,q} = \omega_{X}^{p+|q|+|I|} \bigg/ \sum_{i \in I} W(D_i)_{q_i}
\]
\[
\longrightarrow \tilde{B}_I(f)^{p-1,q+e_j} = \omega_{X}^{p+|q|+|I|} \bigg/ \sum_{i \in I, i \neq j} W(D_i)_{q_i} + W(D_j)_{q_{j+1}}
\]
induces a morphism of complexes
\[
\tilde{\nu}_{j|I}(f) : s\tilde{B}_I(f) \longrightarrow s\tilde{B}_I(f)
\]
for \( j \in I \).

Lemma 5.12. For \( K \subset I \subset \{1,2,\ldots,k\} \), we have
\[
\nu_{j|I}(f)(L(K)_m sB_I(f)) \subset L(K)_m sB_I(f),
\]
\[
\nu_{j|I}(f)(F^p sB_I(f)) \subset F^{p-1} sB_I(f)
\]
for all \( m, p \) and for every \( j \in I \). If \( j \in K \) in addition, we have
\[
\nu_{j|I}(f)(L(K)_m sB_I(f)) \subset L(K)_{m-2} sB_I(f)
\]
for every \( m \). Therefore we have
\[
\mu_{j|I}(f)(L(K)_m sB_I(f)) \subset (f|J)^{-1} \omega_{Y[J]}^{1} \otimes (f|J)^{-1} \omega_{Y[I]}^{1} L(K)_{m-2} sB_I(f)
\]
for every \( m \) if \( K \cup J = I \).
Lemma 5.13. The morphism $\tilde{\nu}_{j|I}(f)$ preserves the filtration $G$ on $s\tilde{B}_1(f)$ for every $j \in I$. Moreover $\text{Gr}^p_G \tilde{\nu}_{j|I}(f)$ coincides with $\text{id} \otimes \nu_{j|I}(f)[-p]$ for every $j \in I$ under the identification (5.3.1).

Proof. Easy by definition. \qed

Corollary 5.14. The diagram
\[
\begin{array}{ccc}
sB_I(f) & \xrightarrow{\gamma(f)} & f_I^{-1}\omega^1_Y \otimes_{f_I^{-1}\mathcal{O}_Y} sB_I(f) \\
\downarrow_{\nu_{j|I}(f)} & & \downarrow_{\text{id} \otimes \nu_{j|I}(f)} \\
sB_I(f) & \xrightarrow{\gamma(f)} & f_I^{-1}\omega^1_Y \otimes_{f_I^{-1}\mathcal{O}_Y} sB_I(f)
\end{array}
\]
is commutative in the derived category.

Lemma 5.15. For $J \subseteq I$, the diagram
\[
\begin{array}{ccc}
sB_I(f) & \xrightarrow{\gamma_J(f)} & f_J^{-1}\omega^1_Y \otimes_{f_J^{-1}\mathcal{O}_Y} sB_I(f) \\
\downarrow_{\theta_{j|I}(f)} & & \downarrow_{\text{id} \otimes \theta_{j|I}(f)} \\
sB_I(f) & \xrightarrow{(-1)^{|J|+1}\mu_{j|I}(f)} & f_J^{-1}\omega^1_Y \otimes_{f_J^{-1}\mathcal{O}_Y} sB_I(f)
\end{array}
\]
is commutative.

Proof. We define a complex of $\mathbb{C}$-sheaves $C_{j|I}(f)$ by
\[
C_{j|I}(f)^p = (f_J^{-1}\omega^1_Y \otimes_{f_J^{-1}\mathcal{O}_Y} sB_I(f)^{p-1}) \oplus sB_I(f)^p
\]
with the differentials
\[
d(\omega \otimes x, y) = (-\omega \otimes dx + (-1)^{|J|+1}\mu_{j|I}(f)(y), dy)
\]
for $\omega \in f_J^{-1}\omega^1_Y$, $x \in sB_I(f)^{p-1}$ and $y \in sB_I(f)^p$. We can easily check the equality $d^2 = 0$ from the fact that $\mu_{j|I}(f)$ is a morphism of complexes. Then a morphism of complexes
\[
C_{j|I}(f) \longrightarrow sB_I(f)
\]
is defined by the projection $C_{j|I}(f)^p \longrightarrow sB_I(f)^p$. Moreover, the injection
\[
f_J^{-1}\omega^1_Y \otimes_{f_J^{-1}\mathcal{O}_Y} sB_I(f)^{p-1} \longrightarrow C_{j|I}(f)^p
\]
induces a morphism of complexes
\[
f_J^{-1}\omega^1_Y \otimes_{f_J^{-1}\mathcal{O}_Y} sB_I(f)[-1] \longrightarrow C_{j|I}(f)
\]
which fit in the exact sequence
\[
0 \longrightarrow f_J^{-1}\omega^1_Y \otimes_{f_J^{-1}\mathcal{O}_Y} sB_I(f)[-1] \longrightarrow C_I(f) \longrightarrow sB_I(f) \longrightarrow 0
\]
by definition.

We denote by $I \setminus J = \{i_1, i_2, \ldots, i_l\}$ with $1 \leq i_1 < i_2 < \cdots < i_l \leq k$. The morphism

$$\omega^p_X \rightarrow \omega^{p-1+1}_X$$

sending $\omega \in \omega^p_X$ to

$$(-1)^{l-j} \, d \log t_{i_1} \wedge \cdots \wedge d \log t_{i_{j-1}} \wedge d \log t_{i_{j+1}} \wedge \cdots \wedge d \log t_{i_l} \wedge \omega$$

induces a morphism

$$\eta_j : \tilde{B}_J(f)^{p,q}/G^2 \tilde{B}_J(f)^{p,q} \rightarrow B_I(f)^{p-1,q}$$

for $p \in \mathbb{Z}_{\geq 0}$, $q \in \mathbb{Z}_{\geq 0}^J \subset \mathbb{Z}_{\geq 0}^I$. Thus we obtain a morphism

$$\eta_j : s \tilde{B}_J(f)^p/G^2 s \tilde{B}_J(f)^p \rightarrow sB_I(f)^{p-1}$$

for $p \in \mathbb{Z}$ for every $j = 1, 2, \ldots, l$.

Now a morphism

$$\eta : s \tilde{B}_J(f)^p/G^2 s \tilde{B}_J(f)^p \rightarrow C_{J|I}(f)^p$$

is defined by

$$\eta(\omega) = \left( \sum_{j=1}^l d \log t_{i_j} \otimes \eta_j(\omega), \text{pr}_I \tilde{\theta}_{J|I}(f)(\omega) \right)$$

for $\omega \in s \tilde{B}_J(f)^p$. It is easy to check that $\eta$ defines a morphism of complexes. Then the diagram

$$
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
& & \\
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\end{array}
$$

is commutative with exact columns. Thus we obtain the conclusion.

\[\square\]

**Definition 5.16.** The morphism

$$R^n(f_I)_* (\nu_{J|I}(f)) : R^n(f_I)_* sB_I(f) \rightarrow R^n(f_I)_* sB_I(f)$$
induced by $\nu_{j|I}(f)$ for every $n$ is denoted by $N_{j|I}(f)$ for short. We set

$$N_{j|I}(f; c) = \sum_{j \in J} c_j N_{j|I}(f) : R^n(f_1)sB_1(f) \to R^n(f_1)sB_1(f)$$

for $J \subset I$ and for $c = (c_j)_{j \in J} \in \mathbb{C}^J$. We simply denote $N_{j|I}(f) = N_{j|I}(f; e)$. We use the convention $N_I(f; c) = N_{I|I}(f; c)$, $N_I(f) = N_I(f; e_I)$, $N(f; c) = N_{(1, \ldots, k)}(f; c)$ and $N(f) = N(f; e)$.

**Lemma 5.17.** For $K \subset I$, we have

$$N_{j|I}(f)(F^pR^n(f_1)sB_1(f)) \subset F^{p-1}R^n(f_1)sB_1(f)$$

and for $$(c) = \sum_{j \in J} c_j N_{j|I}(f) : R^n(f_1)sB_1(f) \to R^n(f_1)sB_1(f)$$

for all $m, n, p$. Moreover, we have

$$N_{j|I}(f)(L(K)_mR^n(f_1)sB_1(f)) \subset L(K)_mR^n(f_1)sB_1(f)$$

for all $m, n$ if $j \in K$. Therefore

$$N_{j|I}(f; c)(L(K)_mR^n(f_1)sB_1(f)) \subset L(K)_{m-2}R^n(f_1)sB_1(f)$$

for $J \subset K$, for $c \in \mathbb{C}^J$ and for all $m, n, p$.

**Proof.** Easy by Lemma 5.12. \qed

**Lemma 5.18.** The diagram

$$
\begin{array}{ccc}
R^n(f_1)sB_1(f) & \xrightarrow{\nabla(I)} & \omega_Y \otimes_{\mathcal{O}_Y} R^n(f_1)sB_1(f) \\
N_{j|I}(f) \downarrow & & \downarrow \text{id} \otimes N_{j|I}(f) \\
R^n(f_1)sB_1(f) & \xrightarrow{\nabla(I)} & \omega_Y \otimes_{\mathcal{O}_Y} R^n(f_1)sB_1(f)
\end{array}
$$

is commutative for every $j \in I$. In particular, the diagram

$$
\begin{array}{ccc}
R^n(f_1)sB_1(f) & \xrightarrow{\nabla(I)} & \omega_Y \otimes_{\mathcal{O}_Y} R^n(f_1)sB_1(f) \\
N_{j|I}(f) \downarrow & & \downarrow \text{id} \otimes N_{j|I}(f) \\
R^n(f_1)sB_1(f) & \xrightarrow{\nabla(I)} & \omega_Y \otimes_{\mathcal{O}_Y} R^n(f_1)sB_1(f)
\end{array}
$$

is commutative for all $J \subset I$.

**Proof.** Easy by Corollary 5.14. \qed

**Theorem 5.19.** For the residue of $\nabla$ in (3.4.2), the diagram

$$
\begin{array}{ccc}
\mathcal{O}_{E_1} \otimes_{\mathcal{O}_Y} R^n f_1^* \omega_{X/Y} \xrightarrow{\theta_1(f)} & \mathcal{O}_{E_1} \otimes_{\mathcal{O}_Y} R^n f_1^* \omega_{X/Y} \\
\Res_{\mathcal{O}_X}(-\nabla) \downarrow & & \downarrow \theta_1(f) \\
R^n(f_1)sB_1(f) & \xrightarrow{-N_{j|I}(f)} & R^n(f_1)sB_1(f)
\end{array}
$$

is commutative for $j \in I$.\ 

\[ \text{23} \]
Proof. By Lemma 5.15 for $J = \emptyset$, we can easily obtain the conclusion.

Remark 5.20. Let $g : S' \rightarrow S$ be a morphism of complex manifolds. Then the construction of $\nu_{jI}(f)$ is compatible with the base change by $g$. More precisely, we have the commutative diagram

\[
\begin{array}{ccc}
  f'^{-1}\mathcal{O}_{Y'[I]} \otimes_{g'^{-1}\mathcal{O}_{Y'[I]}} g'^{-1}sB_I(f) & \longrightarrow & sB_I(f') \\
  \downarrow^\text{id \otimes \nu_{jI}(f')} & & \downarrow^\nu_{jI}(f') \\
  f^{-1}\mathcal{O}_{Y[I]} \otimes_{g^{-1}\mathcal{O}_{Y[I]}} g^{-1}sB_I(f) & \longrightarrow & sB_I(f')
\end{array}
\]

where the horizontal arrows are the morphism (4.13.1).

6 Rational structures

In this section, we construct a rational structure on $sB_I(f)$ as in [19], which is inspired by the log geometry of Fontaine-Illusie [13]. By using these rational structures, we prove the local freeness of $R^*(f_I)_*sB_I(f)$, $Gr^L(f_I)_*sB_I(f)$ etc. for a proper semistable morphism $f$ in the next section.

6.1. Let $Y = \Delta^k \times S$ and $E$ be as in 4.1 and $f : X \rightarrow Y$ a semistable morphism along $E$ as before. We use the notation in 3.6, 3.7 and 4.1.

6.2. For $I \subset \{1, 2, \ldots, k\}$, the open immersion $X \setminus D_I \hookrightarrow X$ is denoted by $j_I$. Then the sheaf of monoids

$M_X(D_I) = (j_I)_*\mathcal{O}_{X \setminus D_I} \cap \mathcal{O}_X$

defines a log structure on $X$ (see [13]). We use the notation $M_X(D)$ for $I = \{1, 2, \ldots, k\}$ as before. We have

$\mathcal{O}_X^* = M_X(D_0) \subset M_X(D_J) \subset M_X(D_I) \subset M_X(D)$

for $J \subset I \subset \{1, 2, \ldots, k\}$. A morphism of abelian sheaves

$e(f) : \mathcal{O}_X \longrightarrow M_X(D)^{\gp}$

is the composite of the exponential map

$\mathcal{O}_X \ni f \mapsto \exp(2\pi\sqrt{-1}f) \in \mathcal{O}_X^*$

and the inclusion $\mathcal{O}_X^* \hookrightarrow M_X(D)$. We denote by

$e(f)_Q : \mathcal{O}_X \longrightarrow M_X(D)^{\gp}_Q = M_X(D)^{\gp} \otimes_\mathbb{Z} \mathbb{Q}$

the base extension of the morphism $e(f)$. For $I \subset \{1, 2, \ldots, k\}$, a morphism of abelian sheaves

$\mathbb{Z}^I \longrightarrow M_X(D)^{\gp}$

is defined by sending $e_i \in \mathbb{Z}^I$ to the global section $t_i$ of $M_X(D)$ for $i \in I$. Then a morphism of $\mathbb{Q}$-sheaves

$\mathbb{Q}^I \longrightarrow M_X(D)^{\gp}_Q$
is obtained by the base extension to \( \mathbb{Q} \). The direct sum of the morphism above and \( e(f)_\mathbb{Q} \) gives us the morphism

\[
\mathbb{Q}^I \oplus \mathcal{O}_X \rightarrow M_X(D)_{\mathbb{Q}}^{gp}
\]
denoted by \( \varphi^I(f) \). We write \( \varphi(f) \) for \( \varphi^{\{1,2,\ldots,k\}}(f) \). On the other hand, \( \varphi^0(f) \) is nothing but the morphism \( e(f)_\mathbb{Q} \).

A global section 1 of \( \mathcal{O}_X \) can be considered as a global section of \( \mathbb{Q}^I \oplus \mathcal{O}_X \) by the canonical inclusion \( \mathcal{O}_X \subset \mathbb{Q}^I \oplus \mathcal{O}_X \). Then the global section 1 is contained in \( \text{Ker}(\varphi^I) \) for every \( I \subset \{1,2,\ldots,k\} \). A complex of \( \mathbb{Q} \)-sheaves

\[
\text{Kos}(\varphi^I(f); \infty; 1)
\]
is defined in [9, Definition 1.8]. For \( J \subset \mathcal{T} = \{1,2,\ldots,k\} \setminus I \), we have

\[
\text{Image}(\varphi^I(f)) \subset M_X(D_{\mathcal{T}})_{\mathbb{Q}}^{gp}
\]
because \( I \subset \mathcal{T} = \{1,2,\ldots,k\} \setminus J \). Then an increasing filtration \( W(J) \) on \( \text{Kos}(\varphi^I(f); \infty; 1) \) is defined as \( W(M_X(D_{\mathcal{T}})_{\mathbb{Q}}^{gp}) \) in [9, Definition 1.8]. For the case of \( J = \{i\} \) with \( i \notin I \), we write \( W(i) \) instead of \( W(\{i\}) \).

**Definition 6.3.** For \( I \subset I' \subset \{1,2,\ldots,k\} \), we set

\[
A_{I|I'}(f)^{p,q} = \text{Kos}(\varphi^{I\setminus I'}(f); \infty; 1)^{p+|q|+|I|} \bigg/ \sum_{i \in I} W(i)_{q_i}
\]
for \( p \in \mathbb{Z}_{\geq 0}, q \in \mathbb{Z}_{\geq 0} \). We set

\[
d_0 = (-1)^{|I|} d : A_{I|I'}(f)^{p,q} \rightarrow A_{I|I'}(f)^{p+1,q}
\]
\[
d_i = (-1)^{|I|} t_i \wedge : A_{I|I'}(f)^{p,q} \rightarrow A_{I|I'}(f)^{p,q+e_i}, \quad i \in I,
\]
where \( d \) on the right hand side of the first equality denotes the differential of the complex

\[
\text{Kos}(\varphi^{I\setminus I'}(f); \infty; 1)
\]
and where \( t_i \wedge \) denotes the morphism of complexes

\[
\text{Kos}(\varphi^{I\setminus I'}(f); \infty; 1) \rightarrow \text{Kos}(\varphi^{I\setminus I'}(f); \infty; 1)[1]
\]
induced by the wedge product \( t_i \wedge \) as in [9, (1.11)]. By setting

\[
sA_{I|I'}(f)^n = \bigoplus_{p+|q|=n} A_{I|I'}(f)^{p,q}
\]
with the differential \( d = d_0 + \sum_{i \in I} d_i \), we obtain a complex \( sA_{I|I'}(f) \). By definition, \( sA_{I|I'}(f) \) is nothing but \( \text{Kos}(\varphi^I(f); \infty; 1) \). We write \( sA_I(f) \) for \( sA_{I|I'}(f) \). Moreover, \( sA(f) \) denotes \( sA_{\{1,2,\ldots,k\}}(f) \). We use \( sA_I(f) \) and \( sA_{I'}(f) \) instead of \( sA_{\{1,2,\ldots,k\}|I}(f) \) and \( sA_{\{1,2,\ldots,k\}|I'}(f) \) respectively.

An increasing filtration \( L(K) \) on \( A_{I|I'}(f)^{p,q} \) for \( K \subset I \) is defined by

\[
L(K)_m A_{I|I'}(f)^{p,q} = W(K)_{m+2|q|+|K|} \text{Kos}(\varphi^{I\setminus I'}(f); \infty; 1)^{p+|q|+|I|} \bigg/ \sum_{i \in I} W(i)_{q_i}
\]
for every \( m \). It is easy to check that this defines an increasing filtration \( L(K) \) on \( sA_{I|I'}(f) \).

We use the notation \( L = L(\{1,2,\ldots,k\}) \) on \( sA(f) \).
Remark 6.4. We can easily see the inclusion
\[ \text{Supp}(A_{I|I'}(f)^{p,q}) \subset X[I] \]
for all \( p, q \). Thus we identify \( A_{I|I'}(f)^{p,q} \) and \( \iota_I^{-1}A_{I|I'}(f)^{p,q} \) for all \( p, q \) as before.

Lemma 6.5. We have an isomorphism of complexes
\[ \text{Gr}^L_m(sA_{I|I'}(f)) \simeq \bigoplus_{q \in \mathbb{Z}} \bigoplus_{\Gamma \in S_{m+2|q|+|K|}(\Lambda K)} \iota_I^{-1}sA_{(I\setminus K)((I'\setminus K)(\pi_Kf))[-m-2|q|]} \]
for every integer \( m \), under which the filtration \( L(J)\text{Gr}^L_m(sA_{I|I'}(f)) \) coincides with the direct sum of
\[ L(J \setminus K \cap J)[|\Gamma \cap \Lambda K \cap J| - 2|q_K \cap J| - |K \cap J|] \iota_I^{-1}sA_{(I\setminus K)((I'\setminus K)(\pi_Kf))[-m-2|q|]} \]
over the same index set as in (6.5).

Proof. Easy from Proposition 1.10 in [9].

6.6. Under the situation in Lemma 6.5, we have the commutative diagram of the canonical morphisms
\[ \begin{array}{ccc}
\iota_I^{-1}(Q^{(I')} \oplus O_X) & \xrightarrow{\iota_I^{-1}\varphi^{(I')}\Gamma(\pi_Kf)} & \iota_I^{-1}M_X(D_{\Gamma})_{\mathbb{Q}}^\text{gp} \\
\downarrow & & \downarrow \\
Q^{(I')} \oplus O_X[\Gamma] & \xrightarrow{\varphi^{(I')}\Gamma(f\Gamma)} & M_X[\Gamma](D_{\Gamma \cap X[\Gamma]})_{\mathbb{Q}}^\text{gp}
\end{array} \]
which induces morphisms of complexes
\[ \iota_I^{-1}\text{Kos}(\varphi^{(I')}\Gamma(\pi_Kf); \infty; 1) \longrightarrow \text{Kos}(\varphi^{(I')}\Gamma(f\Gamma); \infty; 1) \]
and
\[ \iota_I^{-1}sA_{(I\setminus K)((I'\setminus K)(\pi_Kf))} \longrightarrow sA_{(I\setminus K)((I'\setminus K)(f\Gamma))} \]
for \( K \subset I \subset I' \subset \{1, 2, \ldots, k\} \).

Lemma 6.7. The morphism (6.6) is a filtered quasi-isomorphism with respect to \( L(J) \) for \( J \subset I \setminus K \).

Proof. Because the problem is of local nature, Lemma 3.3 enables us to assume the following:

- \( X = X[\Gamma] \times \Delta^\Gamma \) where \( \Delta^\Gamma \) denotes the polydisc with the coordinate functions \((x_\lambda)_{\lambda \in \Gamma}\).
- \( \pi_Kf \) coincides with the composite of the projection \( X = X[\Gamma] \times \Delta^\Gamma \longrightarrow X[\Gamma] \) and \( f\Gamma : X[\Gamma] \longrightarrow Y[K] \) via the identification \( Y[K] = Y_K \).
The morphism (6.6) is a quasi-isomorphism by Lemma (1.4) in [19] because
\[ \text{Ker}(\iota^{-1}_\Gamma \varphi' \cap (\pi_K f)) = \text{Ker}(\varphi' \cap (f_\Gamma)) \]
and
\[ \text{Coker}(\iota^{-1}_\Gamma \varphi' \cap (\pi_K f)) = \text{Coker}(\varphi' \cap (f_\Gamma)). \]
Therefore the morphism (6.6) is a quasi-isomorphism for the case of \( K = I \). Then the morphism (6.6) is a filtered quasi-isomorphism with respect to \( L(I \setminus K) \) by Lemma 6.5. In particular, the morphism (6.6) is a quasi-isomorphism. Then we obtain the conclusion by Lemma 6.5 again.

**6.8.** Let \( g : S' \rightarrow S \) be a morphism of complex manifolds. We set \( f' : X' \rightarrow Y' = \Delta^k \times S' \) etc. as in 4.13. Then we have the commutative diagram
\[
g'^{-1}(\mathbb{Q}' \cap I \oplus \mathcal{O}_X) \xrightarrow{g'^{-1} \varphi' \cap (f')} g'^{-1}M_X(D)^{\geq p}_\mathbb{Q} \]
\[
\downarrow \quad \downarrow
\]
\[
\mathbb{Q}' \cap I \oplus \mathcal{O}_{X'} \xrightarrow{\varphi' \cap (f')} M_{X'}(D)^{\geq p}_\mathbb{Q}
\]
which induces a morphism of complexes
\[ g'^{-1} \text{Kos}(\varphi' \cap (f); \infty; 1) \rightarrow \text{Kos}(\varphi' \cap (f'); \infty; 1) \]
for \( I \subset I' \subset \{1, 2, \ldots, k\} \). Then we obtain the morphism of complexes
\[ g'^{-1} sA_{I|I'}(f) \rightarrow sA_{I|I'}(f') \] (6.8.1)
which preserves the filtration \( L(K) \) for \( K \subset I \).

**Lemma 6.9.** The morphism (6.8.1) is a filtered quasi-isomorphism with respect to the filtration \( L(K) \) for \( K \subset I \).

**Proof.** Because we have
\[ \text{Ker}(g'^{-1} \varphi' \cap (f)) = \text{Ker}(\varphi' \cap (f')) \]
and
\[ \text{Coker}(g'^{-1} \varphi' \cap (f)) = \text{Coker}(\varphi' \cap (f')) \],
we can proceed the same way as in Lemma 6.7.

**Definition 6.10.** For \( I \subset \{1, 2, \ldots, k\} \), the projection
\[ \mathbb{Q}' \oplus \mathcal{O}_X \rightarrow \mathcal{O}_X \]
is denoted by \( \tau^I(f) \) for a while. Then a morphism
\[ \psi^I(f) : \text{Kos}(\varphi^I(f); n)^p \rightarrow \omega^p_X \]
is defined by sending
\[
\psi^I(f)(x_1^{[n_1]}x_2^{[n_2]} \cdots x_k^{[n_k]} \otimes y) = \frac{(2\pi \sqrt{-1})^{-p}}{n_1!n_2! \cdots n_k!} \tau^I(x_1)^{n_1} \tau^I(x_2)^{n_2} \cdots \tau^I(x_k)^{n_k} (\bigwedge^p \text{dlog})(y)
\]
for \( x_1, x_2, \ldots, x_k \in \mathbb{Q}^I \oplus \mathcal{O}_X, y \in \bigwedge^p M_X(D)^{\geq p} \) and \( n_1, n_2, \ldots, n_k \in \mathbb{Z}_{\geq 0} \) with \( n_1 + n_2 + \cdots + n_k = n - p \) as in [9, (2.4)]. Note that this does not define a morphism of complexes.
Lemma 6.11. The diagram

\[
\begin{array}{ccc}
\text{Kos}(\varphi^I(f); \infty; 1)^p & \xrightarrow{\psi^I(f)} & \omega^p_X \\
\downarrow t^\wedge & & \downarrow \text{dlog } t^\wedge \\
\text{Kos}(\varphi^I(f); \infty; 1)^{p+1} & \xrightarrow{(2\pi \sqrt{-1})\psi^I(f)} & \omega^{p+1}_X 
\end{array}
\]

is commutative for any global section \( t \) of \( M_X(D) \).

Proof. Easy by definition.

Definition 6.12. For \( I \subset I' \subset \{1, \ldots, k\} \) and for \( p \in \mathbb{Z}_{\geq 0}, q \in \mathbb{Z}_{\geq 0} \), the composite of the morphism

\[
(2\pi \sqrt{-1})^{q+|I|} \psi^I(f) : \text{Kos}(\varphi^{I'}(f); \infty; 1)^{p+q+|I|} \longrightarrow \omega^{p+q+|I|}_X
\]

and the projection

\[
\omega^{p+q+|I|}_X \longrightarrow \omega^{p+q+|I|}_{X/Y_i}
\]

induces a morphism

\[
A_{I,I'}(f)^{p,q} \longrightarrow B_I(f)^{p,q} \tag{6.12.1}
\]

because we have \( \psi^{I'}(f)(W(i)_m) \subset W(D_i)_m \) for all \( m \). It is easy to check that the morphism (6.12.1) is compatible with \( d_0 \). Moreover Lemma 6.11 implies that the morphism (6.12.1) is compatible with \( d_i \) for any \( i \in I \). Thus a morphism of complexes

\[
\alpha_{I,I'}(f) : sA_{I,I'}(f) \longrightarrow sB_I(f) \tag{6.12.2}
\]

is obtained. Clearly, \( \alpha_{I,I'}(f) \) preserves the filtration \( L(K) \) for \( K \subset I \). We use \( \alpha_I(f) = \alpha_{I,I}(f) \), \( \alpha_I(f) = \alpha_{\{1,2,\ldots,k\},I}(f) \), \( \alpha_I(f) = \alpha_{\emptyset,I'}(f) \), \( \alpha_0(f) = \alpha_{\emptyset,\emptyset}(f) \), \( \alpha_0(f) = \alpha_{\emptyset,\{1,2,\ldots,k\}}(f) \) and \( \alpha(f) = \alpha_{\{1,2,\ldots,k\},\emptyset}(f) = \alpha_{\{1,2,\ldots,k\},\{1,2,\ldots,k\}}(f) \) for short.

Lemma 6.13. The morphism \( \alpha_0(f) \) induces a quasi-isomorphism

\[
\mathcal{O}_{Y,f(x)} \otimes \mathbb{Q} \text{Kos}(\varphi(f); \infty; 1)_x \longrightarrow \omega_{X/Y,x}
\]

for any \( x \in X_0 \).

Proof. Since the question is of local nature, we may assume that \( f \) is a morphism of the form (3.2.2). By using Lemma (1.4) in [19], we can easily reduce the problem to the case of \( k = 1 \). Thus we obtain the conclusion by Lemma (1.4) in [19] and by Proposition (1.13) in [18].

Proposition 6.14. The morphism \( \alpha_{I,I'}(f) \) induces a filtered quasi-isomorphism

\[
(f_I^{-1}\mathcal{O}_{Y,I} \otimes \mathbb{Q} sA_{I,I'}(f))|_{X[I']} \longrightarrow sB_I(f)|_{X[I']},
\]

with respect to \( L(K) \) for \( K \subset I \).
Proof. We may assume \( I' = \{1, 2, \ldots, k\} \), because we consider the situation over \( X[I']^* \). Therefore it is sufficient to prove that the morphism \( \alpha_I(f) \) induces a filtered quasi-isomorphism

\[
(f_I^{-1}O_{Y[I]} \otimes_Q sA_I(f))|_{X_0} \rightarrow sB_I(f)|_{X_0}
\]

with respect to \( L(K) \) for \( K \subset I \).

By Lemma 4.7, Lemma 6.5 and Lemma 6.7 we obtain a commutative diagram

\[
\begin{array}{ccc}
\text{Gr}_{m}^{L(K)} sA_I(f) & \xrightarrow{\text{Gr}_{m}^{L(K)} \alpha_I(f)} & \text{Gr}_{m}^{L(K)} sB_I(f) \\
\cong & & \cong \\
\bigoplus i^{-1}_I sA_{(I \setminus K)}(\pi_K f)[-m - 2|q|] & \xrightarrow{\bigoplus i^{-1}_I sA_{(I \setminus K)}(f_I)[-m - 2|q|]} & \bigoplus sB_{I \setminus K}(f_I)[-m - 2|q|],
\end{array}
\]

(6.14.1)

where the direct sums are taken over the index set

\[
\{(q, \Gamma); q \in \mathbb{Z}^K_{\geq 0}, \Gamma \in S^{g + e_K}_{m + 2|q| + |K|}(\Lambda_K)\}
\]

(6.14.2)
as in (4.7.1), and where the bottom horizontal arrow is the direct sum of the morphisms

\[
(2\pi \sqrt{-1})^{-m - |q|} \alpha_{(I \setminus K)}(f_I)[-m - 2|q|]
\]

over the index set (6.14.2). Therefore Lemma 6.13 implies that \( \alpha_I(f) \) induces a filtered quasi-isomorphism with respect to \( L(I) \). In particular, \( \alpha_I(f) \) induces a quasi-isomorphism. Then we can conclude that \( \alpha_I(f) \) induces a filtered quasi-isomorphism with respect to \( L(K) \) for \( K \subset I \) by the commutative diagram (6.14.1) again.

**Corollary 6.15.** The morphism \( \alpha_{I[I']} \) induces a bifiltered quasi-isomorphism

\[
(f_I^{-1}O_{Y[I]} \otimes_Q sA_{I[I']}(f), L(J), L(K))|_{X[I']} \rightarrow (sB_I(f), L(J), L(K))|_{X[I']},
\]

for \( J, K \subset I \).

**Proof.** By the commutative diagram (6.14.1) we can easily obtain the conclusion.

**Definition 6.16.** From the morphism \( \alpha_{I[I']} : sA_{I[I']}(f) \rightarrow sB_I(f) \), we obtain morphisms

\[
R(f_I)_* sA_{I[I']}(f) \rightarrow R(f_I)_* sB_I(f)
\]

and

\[
O_{Y[I]} \otimes_Q R(f_I)_* sA_{I[I']}(f) \rightarrow R(f_I)_* sB_I(f)
\]

which we denote by the same letter \( \alpha_{I[I']}(f) \) if there is no danger of confusion. These morphisms preserves the filtration \( L(K) \) for \( K \subset I \).

**Lemma 6.17.** Under the situation in 4.13, the morphism (4.13.1) is a filtered quasi-isomorphism with respect to \( L(K) \) for all \( K \subset I \).
Proof. For any $I' \supset I$, we have the commutative diagram
\[
(f_I^{-1} \mathcal{O}_{Y[I]} \otimes \mathbb{Q} g_I^{-1} sA_I(f))|_{X[I]^*} \longrightarrow (f_I^{-1} \mathcal{O}_{Y[I]} \otimes g_I^{-1} \mathcal{O}_{Y[I]} g_I^{-1} sB_I(f))|_{X[I]^*}
\]
where the horizontal arrows and the left vertical arrow are filtered quasi-isomorphisms with respect to $L(K)$ by Proposition 6.14 and Lemma 6.9. Thus we obtain the conclusion.

Now we compare the canonical connection $d \otimes \text{id}$ on $\mathcal{O}_{Y[I]} \otimes R^\alpha(f)_* sA_I(f)$ and the connection $(-1)^{|I|} \nabla(I)$ on $R^\alpha(f)_* sB_I(f)$ via the morphism $\alpha_I(f)$.

**Definition 6.18.** For $I \subset \{1, 2, \ldots, k\}$, the morphism
\[
(2\pi \sqrt{-1})^{|I|} \epsilon(f)_\otimes (f) : \text{Kos}(e(f)_\otimes, 1)_{p+|I|} \longrightarrow \omega^p X_{p+|I|}
\]
induces a morphism $A_I(f)^{p,q} \longrightarrow \tilde{B}_I(f)^{p,q}$ for $p \in \mathbb{Z}_{\geq 0}, q \in \mathbb{Z}_{\geq 0}$. We define a morphism
\[
f_I^{-1} \omega^p_I \otimes Q A_I(f)^{p,q} \longrightarrow \tilde{B}_I(f)^{p+r,q}
\]
by sending $\omega \otimes \eta$ to $(-1)^{|I|} \omega \wedge \alpha_I(f)(\eta)$ for $p \in \mathbb{Z}_{\geq 0}, q \in \mathbb{Z}_{\geq 0}, r \in \mathbb{Z}_{\geq 0}$ as in Remark 5.6. It is easy to check that these morphisms induce a morphism of complexes
\[
f_I^{-1} \omega^p_I \otimes Q sA_I(f) \longrightarrow s\tilde{B}_I(f)
\]
which is denoted by $\tilde{\alpha}_I(f)$. A finite decreasing filtration $G$ on $f_I^{-1} \omega_I^p \otimes Q sA_I(f)$ is induced from the stupid filtration on $\omega_I^p$. Then $\tilde{\alpha}_I(f)$ preserves the filtrations $G$ on the both sides.

**Remark 6.19.** From the definition above, the diagram
\[
\begin{array}{ccc}
\omega^p_I \otimes R^\alpha(f)_* sA_I(f) & \xrightarrow{id \otimes \alpha_I(f)} & \omega^p_I \otimes R^\alpha(f)_* sB_I(f) \\
\downarrow{d \otimes \text{id}} & & \downarrow{(-1)^{|I|} \nabla(I)} \\
\omega^{p+1}_I \otimes R^\alpha(f)_* sA_I(f) & \xrightarrow{id \otimes \alpha_I(f)} & \omega^{p+1}_I \otimes R^\alpha(f)_* sB_I(f)
\end{array}
\]
is commutative for all $p$.

**Lemma 6.20.** The restriction
\[
\tilde{\alpha}_I(f) \mid_{X[I]^*} : (f_I^{-1} \omega_I^p \otimes Q sA_I(f)) \mid_{X[I]^*} \longrightarrow s\tilde{B}_I(f) \mid_{X[I]^*}
\]
of the morphism $\tilde{\alpha}_I(f)$ on $X[I]^*$ is a filtered quasi-isomorphism with respect to the filtrations $G$ on the both sides.

**Proof.** We have isomorphisms of complexes
\[
\text{Gr}_G^p(f_I^{-1} \omega_I^p \otimes Q sA_I(f)) \simeq f_I^{-1} \omega_I^{p} \otimes sA_I(f)[-p]
\]
\[
\text{Gr}_G^p s\tilde{B}_I(f) \simeq f_I^{-1} \omega_I^{p} \otimes sB_I(f)[-p]
\]
for every $p$. Under these identification, $\text{Gr}_G^p \tilde{\alpha}_I(f)$ is identified with $(-1)^p \text{id} \otimes \alpha_I(f)[-p]$ by definition. Thus we obtain the conclusion by Proposition 6.14.
7 Proper case

From now on, we assume that the semistable morphism \( f : X \to Y = \Delta^k \times S \) is proper.

**Lemma 7.1.** For \( K, J \subset I \subset I' \), the morphism \( \alpha_{I'I'}(f) \) in (6.12.2) induces an isomorphism

\[
(\mathcal{O}_Y)[ \otimes R(f)_*sA_{I'I'}(f), L(J), L(K)]_{Y[I']} \xrightarrow{\simeq} (R(f)_*sB_I(f), L(J), L(K))_{Y[I']}.
\]

in the bifiltered derived category.

**Proof.** It suffices to prove that the morphism \( \alpha_{I'}(f) \) induces a quasi-isomorphism

\[
\mathcal{O}_Y[l] \otimes_{\mathcal{O}_Y} \mathbb{G}_{L}^{(l,j)} \mathbb{G}_{m}^{L(K)} R(f)_*sA_{I'I'}(f) \mid_{Y[I']} \xrightarrow{\simeq} \mathbb{G}_{L}^{(l,j)} \mathbb{G}_{m}^{L(K)} R(f)_*sB_I(f) \mid_{Y[I']}
\]

for all \( l, m \). Since \( f \) is proper, we obtain the conclusion from Corollary 6.15, by applying the projection formula (cf. [12, Proposition 2.6.6]).

**Lemma 7.2.** For \( K, J \subset I \), the morphism \( \theta_{J|I}(f) \) in (4.10.2) induces an isomorphism

\[
(\mathcal{O}_Y[l] \otimes_{\mathcal{O}_Y} \mathbb{G}_{L}^{(l,j)} R(f)_*sB_J(f), F, L(K)) \xrightarrow{\simeq} (R(f)_*sB_I(f), F, L(K))
\]

in the bifiltered derived category, where \( \otimes_{\mathcal{O}_Y}^{L} \) stands for the bifiltered derived tensor product over \( \mathcal{O}_Y[I] \).

**Proof.** It suffices to prove that the morphism \( \theta_{J|I}(f) \) induces an isomorphism

\[
\mathcal{O}_Y[l] \otimes_{\mathcal{O}_Y} \mathbb{G}_{F}^{p} \mathbb{G}_{m}^{L(K)} R(f)_*sB_I(f) \to \mathbb{G}_{F}^{p} \mathbb{G}_{m}^{L(K)} R(f)_*sB_I(f)
\]

in the derived category, where \( \otimes_{\mathcal{O}_Y}^{L} \) stands for the derived tensor product over \( \mathcal{O}_Y[I] \). By Lemma 4.9 and Lemma 4.12, we have the isomorphisms

\[
f^{-1}_{I} \mathcal{O}_Y[l] \otimes_{(f)_*}^{L} \mathbb{G}_{F}^{p} \mathbb{G}_{m}^{L(K)} sB_J(f) \xrightarrow{\simeq} f^{-1}_{I} \mathcal{O}_Y[l] \otimes_{(f)_*}^{L} \mathbb{G}_{F}^{p} \mathbb{G}_{m}^{L(K)} sB_J(f)
\]

for all \( m, p \) in the derived category. Then we obtain the conclusion by the projection formula and the proper base change theorem (cf. [12, Proposition 2.6.7]).

**Lemma 7.3.** For \( K, J \subset I \subset I' \), the morphism \( \alpha_{I'I'}(f) \) induces an isomorphism

\[
(\mathcal{O}_Y[l] \otimes_{\mathcal{O}_Y} E^{a,b}_r(R(f)_*sA_{I'I'}(f), L(K))_{Y[I']} \xrightarrow{\simeq} E^{a,b}_r(R(f)_*sB_I(f), L(K))_{Y[I']}, \tag{7.3.1}
\]

for all \( a, b, r \), under which we have

\[
(\mathcal{O}_Y[l] \otimes_{\mathcal{O}_Y} (L(J)_{rec})_{m} E^{a,b}_r(R(f)_*sA_{I'I'}(f), L(K))_{Y[I']} \xrightarrow{\simeq} (L(J)_{rec})_{m} E^{a,b}_r(R(f)_*sB_I(f), L(K))_{Y[I']}
\]

for every \( m \). In particular, we have

\[
(\mathcal{O}_Y[l] \otimes_{\mathcal{O}_Y} \mathbb{G}_{m}^{L(J)_{rec}} \cdots \mathbb{G}_{m}^{L(J)_{rec}} E^{a,b}_r(R(f)_*sA_{I'I'}(f), L(K))_{Y[I']} \xrightarrow{\simeq} \mathbb{G}_{m}^{L(J)_{rec}} \cdots \mathbb{G}_{m}^{L(J)_{rec}} E^{a,b}_r(R(f)_*sB_I(f), L(K))_{Y[I']}
\]

for \( K, J_1, J_2, \ldots, J_l \subset I \) and for all \( a, b, m_1, m_2, \ldots, m_l, r \).
Proof. Easy from Lemma 7.1 and Lemma 2.7.

**Proposition 7.4.** For \( J \subset I \subset I' \subset \{1, 2, \ldots, k\} \), the morphism \( \alpha_{I'|I}(f) \) induces an isomorphism

\[
(O_Y[I] \otimes_{Q} R^n(f_I)_* sA_{I'|I}(f))|_{Y[I']} \cong R^n(f_{I'})_* sB_I(f) |_{Y[I']} \tag{7.4.1}
\]

for all \( n \), under which we have

\[
(O_Y[I] \otimes_{Q} L(J)_m R^n(f_I)_* sA_{I'|I}(f))|_{Y[I']} \cong L(J)_m R^n(f_{I'})_* sB_I(f) |_{Y[I']} \tag{7.4.1}
\]

for all \( m \). In particular, the morphism \( \alpha_{I'|I}(f) \) induces an isomorphism

\[
(O_Y[I] \otimes_{Q} Gr_{m_1}^{L(J_1)} \cdots Gr_{m_2}^{L(J_2)} Gr_{m_1}^{L(J_1)} R^n(f_I)_* sA_{I'|I}(f))|_{Y[I']} \cong Gr_{m_1}^{L(J_1)} \cdots Gr_{m_2}^{L(J_2)} Gr_{m_1}^{L(J_1)} R^n(f_{I'})_* sB_I(f) |_{Y[I']} \tag{7.4.1}
\]

for \( J_1, J_2, \ldots, J_l \subset I \) and for all \( m_1, m_2, \ldots, m_l, n \).

**Proof.** Take \( K = \emptyset \) and \( r = 1 \) in Lemma 7.3.

**Proposition 7.5.** For \( K, J_1, J_2, \ldots, J_l \subset I \), the coherent \( O_{Y[I]} \)-module

\[
Gr_{m_1}^{L(J_1)} \cdots Gr_{m_2}^{L(J_2)} Gr_{m_1}^{L(J_1)} E_{r}^{a,b}(R(f_I)_* sB_I(f), L(K))
\]

is locally free of finite rank for all \( a, b, m_1, m_2, \ldots, m_l, r \). In particular,

\[
E_{r}^{a,b}(R(f_I)_* sB_I(f), L(K))
\]

is locally free \( O_{Y[I]} \)-module of finite rank for \( K \subset I \) and for all \( a, b, r \).

**Proof.** For any \( y \in Y[I] \), there exists \( I' \supset I \) such that \( y \in Y[I']^* \). Then the stalk

\[
Gr_{m_1}^{L(J_1)} \cdots Gr_{m_2}^{L(J_2)} Gr_{m_1}^{L(J_1)} E_{r}^{a,b}(R(f_{I'})_* sB_I(f), L(K))_y
\]

is a free \( O_{Y[I],y} \)-module from Lemma 7.3.

**Proposition 7.6.** For \( J_1, J_2, \ldots, J_l \subset I \), the coherent \( O_{Y[I]} \)-module

\[
Gr_{m_1}^{L(J_1)} \cdots Gr_{m_2}^{L(J_2)} Gr_{m_1}^{L(J_1)} R^n(f_I)_* sB_I(f)
\]

is locally free of finite rank for all \( m_1, m_2, \ldots, m_l, n \). In particular,

\[
R^n(f_I)_* sB_I(f)
\]

is locally free of finite rank for all \( n \).

**Proof.** Take \( K = \emptyset \) and \( r = 1 \) in Proposition 7.5.

**Theorem 7.7.** If the semistable morphism \( f : X \longrightarrow Y = \Delta^k \times S \) is proper, then \( R^n(f_I)_* sB_I(f) \) is the canonical extension of \((O_Y[I] \otimes R^n(f_I)_* sA_I(f))|_{Y[I']} \) via the isomorphisms (7.4.1) for \( I = I' \).
Proof. Considering Proposition 7.6 and Remark 6.19, it suffices to check that the residues of the connection $(-1)^{|I|}\nabla(I)$ are nilpotent. By Corollary 5.10, these residues coincide with the residues of $\nabla$ on $R^nf_*\omega_{X/Y}$. Therefore Theorem 5.19 shows that the residues of $(-1)^{|I|}\nabla(I)$ along $E_j \cap Y[I]$ for $j \notin I$ coincide with $-N_{j|I}(f)$, which is nilpotent by definition. Thus $R^m(f_*)_sB_f(f)$ equipped with $(-1)^{|I|}\nabla(I)$ is the canonical extension as desired.

Remark 7.8. For $I = \emptyset$, the theorem above tells us that $R^nf_*\omega_{X/Y}$ is the canonical extension of its restriction on $Y^*$. This has been already claimed by Usui [22, p.138].

Lemma 7.9. For $K \subset I$, the locally free $\mathcal{O}_{Y[I]}$-module

$$E_r^{a,b}(R(f_*)_sB_f(f), L(K))$$

is the canonical extension of

$$(\mathcal{O}_{Y[I]} \otimes_{\mathcal{Q}} E_r^{a,b}(R(f_*)_sA_f(f), L(K))) \mid_{Y[I]}$$

via the isomorphism (7.3.1) for $I' = I$.

Proof. From the commutative diagram (6.14.1), Theorem 7.7 implies the conclusion for $r = 1$. Then we obtain the conclusion for general $r$ because taking the canonical extension is the exact functor.

Remark 7.10. For $K, J \subset I$, the locally free $\mathcal{O}_{Y[I]}$-module

$$(L(J)_{\text{rec}})_mE_r^{a,b}(R(f_*)_sB_f(f), L(K))$$

is the canonical extension of its restriction on $Y[I]^*$.

Lemma 7.11. For $K \subset J \subset I$, the morphism $\theta_{J|I}(f)$ in (4.10.2) induces an isomorphism

$$\mathcal{O}_{Y[I]} \otimes_{\mathcal{O}_{Y[I]}} E_r^{a,b}(R(f_*)_sB_J(f), L(K)) \xrightarrow{\cong} E_r^{a,b}(R(f_*)_sB_J(f), L(K)) \quad (7.11.1)$$

for all $a, b, r$.

Proof. Lemma 7.2 and the local freeness of $E_r^{a,b}(R(f_*)_sB_J(f), L(K))$ implies the conclusion from the base change theorem.

Proposition 7.12. For $K \subset J \subset I$, the morphism $\theta_{J|I}(f)$ induces isomorphisms

$$\mathcal{O}_{Y[I]} \otimes_{\mathcal{O}_{Y[I]}} L(K)_mR^n(f_*)_sB_J(f) \xrightarrow{\cong} L(K)_mR^n(f_*)_sB_J(f) \quad (7.12.1)$$

$$\mathcal{O}_{Y[I]} \otimes_{\mathcal{O}_{Y[I]}} \text{Gr}^L_m(L(K)) R^n(f_*)_sB_J(f) \xrightarrow{\cong} \text{Gr}^L_m(L(K)) R^n(f_*)_sB_J(f) \quad (7.12.2)$$

for all $m, n$.

Proof. Taking $r$ sufficiently large in (7.11.1), we obtain the isomorphism (7.12.2) for all $m, n$. The local freeness of $\text{Gr}^L_m(L(K)) R^n(f_*)_sB_J(f)$ implies (7.12.1).
Corollary 7.13. For $J_1, J_2, \ldots, J_l \subset J \subset I$, the morphism $\theta_{J|I}(f)$ induces an isomorphism

$$O_{Y[l]} \otimes_{O_{Y[l]}} \text{Gr}^{L(J_1)}_{m_1} \ldots \text{Gr}^{L(J_l)}_{m_l} \text{Gr}^{L(Y)}_{r} R^n(f_J)_*sB_J(f) \xrightarrow{\sim} \text{Gr}^{L(Y)}_{r} \text{Gr}^{L(J_1)}_{m_1} \ldots \text{Gr}^{L(J_l)}_{m_l} R^n(f_I)_*sB_I(f) \quad (7.13.1)$$

for all $m_1, m_2, \ldots, m_l, n$.

Proof. From the local freeness of $\text{Gr}^{L(J_1)}_{m_1} \ldots \text{Gr}^{L(J_l)}_{m_l} \text{Gr}^{L(Y)}_{r} R^n(f_J)_*sB_J(f)$ in Proposition 7.6, we obtain the conclusion by Lemma 2.7. □

8 Degeneration of Hodge structures

In this section, we assume that the semistable morphism $f : X \rightarrow Y = \Delta^k \times S$ is proper and Kähler.

Lemma 8.1. If the semistable morphism $f : X \rightarrow Y = \Delta^k \times S$ is proper and Kähler, $(R(f_I)_*sB_I(f), F)$ is strongly strict for all $I \subset \{1, 2, \ldots, k\}$. In particular, the coherent $O_{Y[l]}$-module $\text{Gr}^{p}_F R^n(f_I)_*sB_I(f)$ is locally free of finite rank for all $n, p$.

Proof. For the case of $I = \emptyset$, the strong strictness of $(Rf_*\omega_{X/Y}, F)$ is proved in [7, Theorem (6.10)]. The morphism $\theta_{J}(f)$ induces an isomorphism

$$(O_{Y[l]} \otimes^{L}_{O_Y} Rf_*\omega_{X/Y}, F) \simeq (R(f_I)_*sB_I(f), F)$$

in the filtered derived category by Lemma 7.2. Therefore we obtain the conclusion by Lemma 2.13. □

Proposition 8.2. The morphism $\theta_{J|I}(f)$ induces an isomorphism

$$O_{Y[l]} \otimes_{O_{Y[l]}} R^n(f_J)_*sB_J(f) \simeq R^n(f_I)_*sB_I(f)$$

for every $n$, under which we have

$$O_{Y[l]} \otimes_{O_{Y[l]}} F^p R^n(f_J)_*sB_J(f) \simeq F^p R^n(f_I)_*sB_I(f)$$

for all $p$.

Proof. Lemma 7.2 and the strong strictness of $(R(f_I)_*sB_J(f), F)$ implies the conclusion as in Lemma 2.13. □

Proposition 8.3. The isomorphism (7.13.1) induces an isomorphism

$$F^p(O_{Y[l]} \otimes \text{Gr}^{L(J_1)}_{m_1} \ldots \text{Gr}^{L(J_l)}_{m_l} \text{Gr}^{L(Y)}_{r} R^n(f_J)_*sB_J(f)) \xrightarrow{\sim} F^p \text{Gr}^{L(Y)}_{r} \text{Gr}^{L(J_1)}_{m_1} \ldots \text{Gr}^{L(J_l)}_{m_l} R^n(f_I)_*sB_I(f)$$

for all $p$.

Proof. Easy by Lemma 2.7. □
Lemma 8.4. If the semistable morphism \( f : X \longrightarrow Y = \Delta^k \times S \) is proper and Kähler, then the data
\[
((R^n f_\ast L(J)_m sA(f), L(I)[n]), (R^n f_\ast L(J)_m sB(f), L(I)[n], F), \alpha_I(f))|_{Y[I]}
\]
is a graded polarizable variation of \( \mathbb{Q} \)-mixed Hodge structure on \( Y[I]^* \) for \( J \subset I \) and for all \( m, n \).

Proof. We may assume that \( I = \{1, 2, \ldots, k\} \). Then
\[
Y[\{1, 2, \ldots, k\}]^* = Y[\{1, 2, \ldots, k\}] = \{0\} \times S = S
\]
by definition. As in the proof of Proposition 6.14, we have the isomorphism
\[
(\text{Gr}_{-a}^L L(J)_m sB(f), F) \simeq \bigoplus (\Omega_{X[I]/S}[a - 2|q|], F[a - |q|])
\]
and the quasi-isomorphism
\[
\text{Gr}_{-a}^L L(J)_m sA(f) \simeq \bigoplus \text{Kos}(e(f_I); \infty; 1)[a - 2|q|]
\]
by Lemma 4.7, Lemma 6.5 and Lemma 6.7, where the direct sums are taken over the same index set
\[
\{(q, \Gamma); q \in \mathbb{Z}_{\geq 0}^k, \Gamma \in S_{\leq a + 2|q| + k}^2 (\Lambda) \text{ with } |\Gamma \cap \Lambda_J| \leq m + 2|q_J| + |J|\}.
\]
Under these identifications the morphism \( \text{Gr}_{-a}^L \alpha(f) \) is identified with the direct sum of the morphisms \((2\pi \sqrt{-1})^{a - |q|} \psi(f_I)[a - 2|q|]\). Because
\[
f_I : X[I] \longrightarrow S
\]
is a proper Kähler smooth morphism, we have the following:

(8.4.1) The data
\[
(R^{a+b} f_\ast, \text{Gr}_{-a}^L L(J)_m sA(f), (R^{a+b} f_\ast, \text{Gr}_{-a}^L L(J)_m sB(f), F), R^{a+b} f_\ast, \text{Gr}_{-a}^L \alpha(f))
\]
is a polarizable variation of \( \mathbb{Q} \)-Hodge structure of weight \( b \) on \( S \).

(8.4.2) The filtered complex \((R f_\ast, \text{Gr}_{-a}^L L(J)_m sB(f), F)\) is strongly strict for all \( a \).

The morphism
\[
d_0 : E_0^{a,b}(R f_\ast L(J)_m sB(f), L) \longrightarrow E_0^{a,b+1}(R f_\ast L(J)_m sB(f), L)
\]
is strictly compatible with respect to the filtration \( F = F_{rec} = F_d = F_{d'} \) by (8.4.2). Then we have \( F_{rec} = F_d = F_{d'} \) on \( E_1^{a,b}(R f_\ast L(J)_m sB(f), L) \) and on \( E_2^{a,b}(R f_\ast L(J)_m sB(f), L) \) by the lemma on two filtration [3, Proposition (7.2.5)]. Moreover
\[
(E_1^{a,b}(R f_\ast L(J)_m sA(f), L), (E_1^{a,b}(R f_\ast L(J)_m sB(f), L), F_{rec}), E_1^{a,b}(\alpha(f)))
\]
is a polarizable variation of \( \mathbb{Q} \)-Hodge structure of weight \( b \) on \( S \) by (8.4.1). Because the morphism
\[
d_1 : E_1^{a,b}(R f_\ast L(J)_m sB(f), L) \longrightarrow E_1^{a+1,b}(R f_\ast L(J)_m sB(f), L)
\]
underlies a morphism of variations of \( \mathbb{Q} \)-Hodge structure, the morphism \( d_1 \) is strictly compatible with the filtration \( F_{\text{rec}} \) and

\[
(E_2^{a,b}(Rf_{0*}L(J)_msA(f), L), (E_2^{a,b}(Rf_{0*}L(J)_msB(f), L), F_{\text{rec}}), E_2^{a,b}(\alpha(f)))
\]

is a polarizable variation of \( \mathbb{Q} \)-Hodge structures of weight \( b \) on \( S \). Because the morphism

\[
d_2 : E_2^{a,b}(Rf_{0*}L(J)_msB(f), L) \rightarrow E_2^{a+2,b-1}(Rf_{0*}L(J)_msB(f), L)
\]

underlies a morphism of variations of \( \mathbb{Q} \)-Hodge structure again, we obtain \( d_2 = 0 \). Inductively, we obtain \( d_r = 0 \) for \( r \geq 2 \). Then we have

\[
E_2^{a,b}(Rf_{0*}L(J)_msA(f), L) \simeq E_2^{a,b}(Rf_{0*}L(J)_msA(f), L)
\]

and

\[
(F = F_{\text{rec}} = F_d = F_{d^*} \text{ on } E_2^{a,b}(Rf_{0*}L(J)_msB(f), L) \text{ for all } a, b. \text{ Thus the data}
\]

\[
(Gr^{L}_a R^{a+b}f_{0*}L(J)_msA(f), (Gr^{L}_a R^{a+b}f_{0*}L(J)_msB(f), L), Gr^{L}_a \alpha_0(f))
\]

is a polarizable variation of Hodge structures of weight \( b \) on \( S \). By Remark 6.19 the canonical connection \( d \otimes \text{id} \) on \( \mathcal{O}_S \otimes R^if_{0*}sa(f) \) is identified with the Gauss-Manin connection \( (-1)^k\nabla(\{1, 2, \ldots, k\}) \), which satisfies the Griffiths transversality. Thus we obtain the conclusion.

The lemma above implies the following theorem.

**Theorem 8.5.** In the situation above, the data

\[
((R^n(f_1)_sA_1(f), L(I)[n]), (R^n(f_1)_sB_1(f), L(I)[n], F), \alpha(f))|_{Y[I]}, \quad (8.5.1)
\]

is a graded polarizable variation of \( \mathbb{Q} \)-mixed Hodge structure on \( Y[I]^* \). Moreover, \( L(J)_m \) gives a subvariation of \( \mathbb{Q} \)-mixed Hodge structure for every \( J \subset I \) and for every \( m \).

**Theorem 8.6.** If the semistable morphism \( f : X \rightarrow Y = \Delta^k \times S \) is proper and Kähler, the spectral sequence \( E_r^{a,b}(R(f)_sB_1(f), L(K)) \) degenerates at \( E_2 \)-terms for \( K \subset I \). In other words, we have the canonical isomorphism

\[
Gr^{L(K)}_m R^n(f_1)_sB_1(f) \xrightarrow{\sim} E_2^{-m,n+m}(R(f_1)_sB_1(f), L(K)) \quad (8.6.1)
\]

for all \( m, n \).

**Proof.** We first treat the case of \( I = K \). As in the proof of Theorem 8.5, the morphisms of \( E_r \)-terms

\[
E_r^{a,b}(R(f_K)_sB_K(f), L(K)) \rightarrow E_r^{a+r,b-r+1}(R(f_K)_sB_K(f), L(K))
\]

are the zero morphisms over \( Y[K]^* \) for all \( a, b \) and for all \( r \geq 2 \). Therefore the morphisms of \( E_r \)-terms are the zero morphisms over the whole \( Y[K] \) for \( r \geq 2 \) because \( E_r^{a,b}(R(f_K)_sB_K(f), L(K)) \) are locally free over \( Y[K] \) by Proposition 7.5. Thus we obtain the desired \( E_2 \)-degeneracy.

For the general case, we have the isomorphism (7.11.1) for \( J = K \)

\[
\mathcal{O}_{Y[I]} \otimes \mathcal{O}_{Y[K]} E_r^{a,b}(R(f_K)_sB_K(f), L(K)) \xrightarrow{\sim} E_r^{a,b}(R(f_1)_sB_1(f), L(K))
\]

for all \( a, b, r \), which is compatible with the morphisms \( d_r \). Then the \( E_2 \)-degeneracy of \( E_r^{a,b}(R(f_K)_sB_K(f), L(K)) \) implies that of \( E_r^{a,b}(R(f_1)_sB_1(f), L(K)) \).
Remark 8.7. By taking \( I = \{1, 2, \ldots, k\} \) in Theorem 8.6, we obtain Theorem (4.1) in [8].

Lemma 8.8. Under the identification (8.6.1), the filtration \( F \) on the left hand side coincides with the filtration \( F_{\text{rec}} \) on the right.

Proof. From Lemma 8.1, the filtration \( F \) on the complex \( R(f_\cdot)_* \text{Gr}_m^{L(K)} sB_1(f) \) is strongly strict for all \( m \). Therefore the lemma on two filtrations [3, Proposition (7.2.5)] implies that \( F_{\text{rec}} = F_d = F_d^- \) on \( E_2^{p,b}(R(f_\cdot)_*sB_1(f), L(K)) \) for all \( a, b \) as in the proof of Lemma 8.4. Then the \( E_2 \)-degeneracy above implies the conclusion. \( \square \)

Lemma 8.9. For \( K \subset I \), the data

\[
\begin{align*}
((E_1^{a,b}(R(f_\cdot)_*sA_1(f), L(K)), L(I \setminus K)[b]), \\
(E_1^{a,b}(R(f_\cdot)_*sB_1(f), L(K)), L(I \setminus K)[b], F), E_1^{a,b}(\alpha_1(f))) \big|_{Y[I]^*}
\end{align*}
\]

is a variation of \( \mathbb{Q} \)-mixed Hodge structure on \( Y[I]^* \). For \( J \subset I \), \( L(J)_m \) gives us a subvariation of \( \mathbb{Q} \)-mixed Hodge structure for all \( m \).

Proof. By the identifications

\[
\begin{align*}
E_1^{a,b}(R(f_\cdot)_*sA_1(f), L(K)) &\simeq R^{a+b}(f_\cdot)_* \text{Gr}_m^{L(K)} sA_1(f) \quad (8.9.1) \\
E_1^{a,b}(R(f_\cdot)_*sB_1(f), L(K)) &\simeq R^{a+b}(f_\cdot)_* \text{Gr}_m^{L(K)} sB_1(f) \quad (8.9.2)
\end{align*}
\]

we obtain the conclusion by Lemma 4.7, Lemma 6.5, Lemma 6.7 and Theorem 8.5. \( \square \)

8.10. Now we fix an integer \( b \) and consider the complex

\[
(E_1^{a,b}(R(f_\cdot)_*sB_1(f), L(K)), d_1),
\]

where

\[
d_1 : E_1^{a,b}(R(f_\cdot)_*sB_1(f), L(K)) \rightarrow E_1^{a+1,b}(R(f_\cdot)_*sB_1(f), L(K))
\]

is the morphism of \( E_1 \)-terms for all \( a \). For any \( J \subset I \), the filtration \( L(J)_{\text{rec}} = L(J)_d = L(J)_d^- \) on \( E_1^{a,b}(R(f_\cdot)_*sB_1(f), L(K)) \) is simply denoted by \( L(J) \) for a while, which forms an increasing filtration of the complex \( E_1^{a,b}(R(f_\cdot)_*sB_1(f), L(K)) \).

Lemma 8.11. For \( K, J_1, J_2, \ldots, J_l \subset J \subset I \), the morphism \( \theta_{J[I]}(f) \) induces an isomorphism of complexes

\[
\begin{align*}
\mathcal{O}_{Y[I]} \otimes \mathcal{O}_{Y[J]} \text{Gr}_m^{L(J_1)} \cdots \text{Gr}_m^{L(J_2)} \text{Gr}_m^{L(J_1)} E_1^{a,b}(R(f_\cdot)_*sB_1(f), L(K)) \\
\simeq \text{Gr}_m^{L(J_1)} \cdots \text{Gr}_m^{L(J_2)} \text{Gr}_m^{L(J_1)} E_1^{a,b}(R(f_\cdot)_*sB_1(f), L(K))
\end{align*}
\]

for all \( m_1, m_2, \ldots, m_l \), under which we have the identification

\[
\begin{align*}
F^p(\mathcal{O}_{Y[I]} \otimes \mathcal{O}_{Y[J]} \text{Gr}_m^{L(J_1)} \cdots \text{Gr}_m^{L(J_2)} \text{Gr}_m^{L(J_1)} E_1^{a,b}(R(f_\cdot)_*sB_1(f), L(K))) \\
\simeq \mathcal{O}_{Y[I]} \otimes \mathcal{O}_{Y[J]} \text{Gr}_m^{L(J_1)} \cdots \text{Gr}_m^{L(J_2)} \text{Gr}_m^{L(J_1)} E_1^{a,b}(R(f_\cdot)_*sB_1(f), L(K))
\end{align*}
\]

for all \( p \).
Proof. From the identification (8.9.2), we easily obtain the conclusion by Lemma 4.7, Corollary 7.13 and Proposition 8.3.

**Lemma 8.12.** For \( K, J_1, J_2, \ldots, J_t \subset J \subset I \), the filtration \( L(J \setminus K) \) on the complex

\[
\text{Gr}_{m_1}^{L(J_1)} \cdots \text{Gr}_{m_2}^{L(J_2)} \text{Gr}_{m_1}^{L(J_1)} E_1^{\bullet,b}(R(f_1)_* sB_1(f), L(K))
\]

is strongly strict for all \( m_1, m_2, \ldots, m_t \).

**Proof.** By Lemma 7.3, we have the isomorphism

\[
\mathcal{O}_{Y[I]} \otimes_{\mathbb{Q}} H^a(\text{Gr}_{m_1}^{L(J_1) K} \text{Gr}_{m_2}^{L(J_2)} \text{Gr}_{m_1}^{L(J_1)} E_1^{\bullet,b}(R(f_1)_* sA_{I'K}(f), L(K))) |_{Y[I]^*} \\
\simeq H^a(\text{Gr}_{m_1}^{L(J_1) K} \text{Gr}_{m_2}^{L(J_2)} \text{Gr}_{m_1}^{L(J_1)} E_1^{\bullet,b}(R(f_1)_* sB_1(f), L(K))) |_{Y[I]^*}
\]

for all \( m, m_1, m_2, \ldots, m_t \) and for any \( I' \supset I \). Therefore the coherent \( \mathcal{O}_{Y[I]} \)-module

\[
H^a(\text{Gr}_{m_1}^{L(J_1) K} \text{Gr}_{m_2}^{L(J_2)} \text{Gr}_{m_1}^{L(J_1)} E_1^{\bullet,b}(R(f_1)_* sB_1(f), L(K)))
\]

is locally free for all \( a, m_1, m_2, \ldots, m_t \) as Proposition 7.6. Hence it suffices to prove the strictness of \( L(J \setminus K) \).

First we treat the case of \( J = I \). On the open subset \( Y[I]^* \),

\[
(\text{Gr}_{m_1}^{L(J_1)} \cdots \text{Gr}_{m_2}^{L(J_2)} \text{Gr}_{m_1}^{L(J_1)} E_1^{\bullet,b}(R(f_1)_* sB_1(f), L(K)), L(I \setminus K)[b], F) |_{Y[I]^*}
\]

underlies a variation of \( \mathbb{Q} \)-mixed Hodge structures for all \( a \) and the morphism \( d_1 \) underlies a morphism of variations from Lemma 8.9. Therefore the filtration \( L(I \setminus K) \) on the complex

\[
\text{Gr}_{m_1}^{L(J_1)} \cdots \text{Gr}_{m_2}^{L(J_2)} \text{Gr}_{m_1}^{L(J_1)} E_1^{\bullet,b}(R(f_1)_* sB_1(f), L(K)) |_{Y[I]^*}
\]

is strict, that is, the canonical morphism

\[
H^a(L(I \setminus K)_m \text{Gr}_{m_1}^{L(J_1)} \cdots \text{Gr}_{m_2}^{L(J_2)} \text{Gr}_{m_1}^{L(J_1)} E_1^{\bullet,b}(R(f_1)_* sB_1(f), L(K))) \\
\longrightarrow H^a(\text{Gr}_{m_1}^{L(J_1)} \cdots \text{Gr}_{m_2}^{L(J_2)} \text{Gr}_{m_1}^{L(J_1)} E_1^{\bullet,b}(R(f_1)_* sB_1(f), L(K)))
\]

is injective over \( Y[I]^* \). Since

\[
\text{Gr}_{m_1}^{L(J_1)} \cdots \text{Gr}_{m_2}^{L(J_2)} \text{Gr}_{m_1}^{L(J_1)} E_1^{a,b}(R(f_1)_* sB_1(f), L(K))
\]

and

\[
L(I \setminus K)_m \text{Gr}_{m_1}^{L(J_1)} \cdots \text{Gr}_{m_2}^{L(J_2)} \text{Gr}_{m_1}^{L(J_1)} E_1^{a,b}(R(f_1)_* sB_1(f), L(K))
\]

are the canonical extensions of their restrictions on \( Y[I]^* \) by Lemma 7.9 (see also Remark 7.10) for all \( a \), the morphism (8.12.1) is injective over the whole \( Y[I] \). Thus we obtain the strictness of \( L(I \setminus K) \) over \( Y[I] \).

From the case of \( J = I \) above, the filtration \( L(J \setminus K) \) on the complex

\[
\text{Gr}_{m_1}^{L(J_1)} \cdots \text{Gr}_{m_2}^{L(J_2)} \text{Gr}_{m_1}^{L(J_1)} E_1^{\bullet,b}(R(f_1)_* sB_1(f), L(K))
\]

is strongly strict. On the other hand, the morphism \( \theta_{J|I}(f) \) induces the isomorphism

\[
(\mathcal{O}_{Y[I]} \otimes_{\mathcal{O}_{Y[I]}} \text{Gr}_{m_1}^{L(J_1)} \cdots \text{Gr}_{m_2}^{L(J_2)} \text{Gr}_{m_1}^{L(J_1)} E_1^{\bullet,b}(R(f_1)_* sB_1(f), L(K)), L(J \setminus K)) \\
\simeq (\text{Gr}_{m_1}^{L(J_1)} \cdots \text{Gr}_{m_2}^{L(J_2)} \text{Gr}_{m_1}^{L(J_1)} E_1^{\bullet,b}(R(f_1)_* sB_1(f), L(K)), L(J \setminus K))
\]

in the filtered derived category from Lemma 8.11. Therefore we obtain the conclusion from Lemma 2.13.
Lemma 8.13. For $K \subset J \subset I$, the morphism $\theta_{J/I}(f)$ induces an isomorphism

\[
O_Y[l] \otimes_{O_{Y[J]}} \text{Gr}^{L(J,K)}_m E_2^{a,b}(R(f)_s B_J(f), L(K)) 
\cong \text{Gr}^{L(J,K)}_m E_2^{a,b}(R(f)_s B_I(f), L(K))
\]

for all $a, b, m$.

Proof. From Lemma 8.11, we have the isomorphism

\[
(O_Y[l] \otimes_{O_{Y[J]}} E_1^{a,b}(R(f)_s B_J(f), L(K)), L(J \setminus K)) \cong E_1^{a,b}(R(f)_s B_I(f), L(K)), L(J \setminus K))
\]

in the filtered derived category. Because $L(J \setminus K)$ is strongly strict on the complex $E_1^{a,b}(R(f)_s B_J(f), L(K))$, we obtain the conclusion by Lemma 2.13.

Proposition 8.14. We assume that the semistable morphism $f : X \to Y = \Delta^k \times S$ is proper and Kähler. Under the identification

\[
\text{Gr}^{L(K)}_m R^n f_0 s B(f) \cong E_2^{-m,n+m}(R f_0 s B(f), L(K))
\]

in (8.6.1), the filtration $L(J)[-m]$ on the left hand side coincides with the filtration $L(J \setminus K)_{\text{rec}}$ on the right for all $K \subset J \subset I$.

Proof. First we treat the case $J = I$. It suffices to prove the desired coincidence over $Y[I]^*$ because $\text{Gr}^{L(J)}_i \text{Gr}^{L(K)}_m R^n f_0 s B_I(f)$ and $\text{Gr}^{L(J,K)}_l E_2^{-m,n+m}(R(f)_s B_I(f), L(K))$ are local free for all $l$. Therefore we may assume that $J = I = \{1, 2, \ldots, k\}$. Thus what we have to prove is the coincidence of $L[-m]$ and $L(K)_{\text{rec}}$ under the identification

\[
\text{Gr}^{L(K)}_m R^n f_0 s B(f) \cong E_2^{-m,n+m}(R f_0 s B(f), L(K))
\]

in Theorem 8.6, where $\overline{K} = \{1, 2, \ldots, k\} \setminus K$ as in the proof of Lemma 8.9. The following argument is similar to that of El Zein [5, 6.1.12 Théorème]. So we only give a sketch of the proof here.

From the $E_2$-degeneracy in Theorem 8.6, the identification (8.14.1) fits in the commutative diagram

\[
R^n f_0 s B(f) \longrightarrow \text{Ker}(d_1)
\]

\[
\text{Gr}^{L(K)}_m R^n f_0 s B(f) \cong E_2^{-m,n+m}(R f_0 s B(f), L(K))
\]

where $d_1$ in the right hand side of the top horizontal arrow is the morphism of $E_1$-terms

\[
d_1 : E_1^{-m,n+m}(R f_0 s B(f), L(K)) \to E_1^{-m+1,n+m}(R f_0 s B(f), L(K))
\]

and the top horizontal arrow is induced from the projection

\[
L(K)_m s B(f) \longrightarrow \text{Gr}^{L(K)}_m s B(f)
\]
by the identification \( R^n f_{0*} \operatorname{Gr}_m^{L(K)} sB(f) \simeq E^{−m,n+m}_1(Rf_{0*} sB(f), L(K)) \). It is easy to see that the projection (8.14.3) sends \( L_l \) to \( L(\overline{K})_{l−m} = L(\overline{K})[m]_l \) for all \( l \). Lemma 8.4, Theorem 8.5 and Lemma 8.9 imply that

\[
\begin{align*}
(R^n f_{0*} L(K) m sB(f), L[n], F),
(\operatorname{Gr}_m^{L(K)} R^n f_{0*} sB(f), L[n], F),
(\operatorname{Ker}(d_1), L(\overline{K})[n + m], F),
(E^{−m,n+m}_2(Rf_{0*} sB(f), L(K)), L(\overline{K})_{rec}[n + m], F_{rec})
\end{align*}
\]

underlie variations of \( \mathbb{Q} \)-mixed Hodge structure on \( Y[\{1, 2, \ldots, k\}] = S \). Moreover, the top horizontal arrow and two vertical arrows in the commutative diagram (8.14.2) underlie morphisms of variations of \( \mathbb{Q} \)-mixed Hodge structure by definition. Thus we can easily see that the identification (8.14.1), the bottom horizontal arrow in the commutative diagram (8.14.2), underlies a morphism of variations of \( \mathbb{Q} \)-mixed Hodge structure by using the fact that the left vertical arrow is surjective and strictly compatible with the filtrations \( L[n] \) and \( F \). Therefore we obtain the desired coincidence.

For the case \( J \subset I \), we have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_{Y[I]} \otimes \operatorname{Gr}_m^{L(K)} R^l(f_I)_* sB_I(f) & \longrightarrow & \operatorname{Gr}_m^{L(K)} R^l(f_I)_* sB_I(f) \\
\downarrow & & \downarrow \\
\mathcal{O}_{Y[I]} \otimes E^{−m,q+m}_2(R(f_I)_* sB_I(f), L(K)) & \longrightarrow & E^{−m,q+m}_2(R(f_I)_* sB_I(f), L(K)).
\end{array}
\]

where the vertical arrows are the identifications (8.6.1) for \( J \) and \( I \), and the horizontal arrows are the morphisms induced by \( \theta_{f/J}(f) \), which are filtered isomorphisms with respect to \( L(J) \) and \( L(J \setminus K)_{rec} \) by Corollary 7.13 and Lemma 8.13. Then the coincidence of \( L(J)[m] \) and \( L(J \setminus K)_{rec} \) under the left vertical arrow implies the desired coincidence under the right vertical arrow.

\[ \square \]

**Theorem 8.15.** If the semistable morphism \( f : X \longrightarrow Y = \Delta^k \times S \) is proper and Kähler, we have the following:

(A) For \( J_1 \subset J_2 \subset \cdots \subset J_l \subset I \subset \{1, 2, \ldots, k\} \), the coherent \( \mathcal{O}_{Y[I]} \)-module

\[
\operatorname{Gr}_F^p \operatorname{Gr}_m^{L(J_1)} \cdots \operatorname{Gr}_m^{L(J_2)} \operatorname{Gr}_m^{L(J_l)} R^n(f_I)_* sB_I(f)
\]

is locally free of finite rank for all \( m_1, m_2, \ldots, m_l, n, p \).

(B) For \( K \subset J_1 \subset J_2 \subset \cdots \subset J_l \subset I \subset \{1, 2, \ldots, k\} \), the filtration \( F \) on the complex

\[
\operatorname{Gr}_m^{L(J_1 \setminus K)} \cdots \operatorname{Gr}_m^{L(J_2 \setminus K)} \operatorname{Gr}_m^{L(J_l \setminus K)} E^b_1(R(f_I)_* sB_I(f), L(K))
\]

is strongly strict for all \( b, m_1, m_2, \ldots, m_l \).

**Proof.** We already proved (A0) in Lemma 8.1. So it suffices to prove the implications (A) \( \Rightarrow \) (B) and (B0) \( + \) (B) \( \Rightarrow \) (A1).

First, we prove (A) \( \Rightarrow \) (B). From the identification (8.9.2), the assumption (A) implies that

\[
\operatorname{Gr}_F^p \operatorname{Gr}_m^{L(J_1 \setminus K)} \cdots \operatorname{Gr}_m^{L(J_2 \setminus K)} \operatorname{Gr}_m^{L(J_l \setminus K)} E^a_1(R(f_I)_* sB_I(f), L(K))
\]

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is locally free of finite rank. Thus the filtered complex

$$(\text{Gr}_{m_1}^{L(J_i \setminus K)} \cdots \text{Gr}_{m_2}^{L(J_i \setminus K)} \text{Gr}_{m_1}^{L(J_i \setminus K)} E_1^b(R(f_i)_* sB_I(f), L(K)), F)$$

is filtered perfect. Therefore it is sufficient to prove that the filtered complex

$$(\mathbb{C}(x) \otimes_{\mathcal{O}_{Y[I]}} \text{Gr}_{m_1}^{L(J_i \setminus K)} \cdots \text{Gr}_{m_2}^{L(J_i \setminus K)} \text{Gr}_{m_1}^{L(J_i \setminus K)} E_1^b(R(f_i)_* sB_I(f), L(K)), F)$$

is strict for all $x \in Y[I]$ by Lemma 2.14 because

$$H^a(\text{Gr}_{m_1}^{L(J_i \setminus K)} \cdots \text{Gr}_{m_2}^{L(J_i \setminus K)} \text{Gr}_{m_1}^{L(J_i \setminus K)} E_1^b(R(f_i)_* sB_I(f), L(K)))$$

is locally free as in the proof of Lemma 8.12. Because the filtration $F$ on the complex

$$\text{Gr}_{m_1}^{L(J_i \setminus K)} \cdots \text{Gr}_{m_2}^{L(J_i \setminus K)} \text{Gr}_{m_1}^{L(J_i \setminus K)} E_1^b(R(f_i)_* sB_I(f), L(K))$$

is strongly strict as in the proof of Lemma 8.9, the filtered complex (8.15.1) is strict for $x \in Y[I]^*$ by Lemma 2.14. For a point $x \in Y[I] \setminus Y[I]^*$, we can find $I' \supset I$ such that $x \in Y[I']^*$. By Lemma 8.11, we have the isomorphism in the filtered derived category

$$(\mathcal{O}_{Y[I']} \otimes_{\mathcal{O}_{Y[I]}} \text{Gr}_{m_1}^{L(J_i \setminus K)} \cdots \text{Gr}_{m_2}^{L(J_i \setminus K)} \text{Gr}_{m_1}^{L(J_i \setminus K)} E_1^b(R(f_i)_* sB_I(f), L(K)), F)$$

$$\cong (\text{Gr}_{m_1}^{L(J_i \setminus K)} \cdots \text{Gr}_{m_2}^{L(J_i \setminus K)} \text{Gr}_{m_1}^{L(J_i \setminus K)} E_1^b(R(f_i)_* sB_I'(f), L(K)), F)$$

induced by the morphism $\theta_{I'I}(f)$. Then we have an isomorphism

$$(\mathbb{C}(x) \otimes_{\mathcal{O}_{Y[I]}} \text{Gr}_{m_1}^{L(J_i \setminus K)} \cdots \text{Gr}_{m_2}^{L(J_i \setminus K)} \text{Gr}_{m_1}^{L(J_i \setminus K)} E_1^b(R(f_i)_* sB_I(f), L(K)), F)$$

$$\cong (\mathbb{C}(x) \otimes_{\mathcal{O}_{Y[I]}} \text{Gr}_{m_1}^{L(J_i \setminus K)} \cdots \text{Gr}_{m_2}^{L(J_i \setminus K)} \text{Gr}_{m_1}^{L(J_i \setminus K)} E_1^b(R(f_i)_* sB_I'(f), L(K)), F)$$

in the filtered derived category. Therefore the complex (8.15.1) is strict as desired.

Next, we prove $(B_0) + (B_1) \Rightarrow (A_{l+1})$. We set $K = J_1$. Then the complex

$$E_1^b(R(f_i)_* sB_I(f), L(K))$$

equipped with the filtrations

$$F, L(J_2 \setminus K), \ldots, L(J_I \setminus K), L(J_{l+1} \setminus K)$$

satisfies the assumption in Lemma 2.9 by the assumptions $(B_0)$, $(B_1)$ and Lemma 8.12. Therefore we have the isomorphism

$$H^a(\text{Gr}_F^{p} \text{Gr}_{m_{l+1}}^{L(J_{l+1} \setminus K)} \text{Gr}_F^{p} \text{Gr}_{m_1}^{L(J_i \setminus K)} \cdots \text{Gr}_{m_2}^{L(J_i \setminus K)} E_1^b(R(f_i)_* sB_I(f), L(K)))$$

$$\cong \text{Gr}_F^{p} \text{Gr}_{m_{l+1}}^{L(J_{l+1} \setminus K)} \text{Gr}_F^{p} \text{Gr}_{m_1}^{L(J_i \setminus K)} \cdots \text{Gr}_{m_2}^{L(J_i \setminus K)} E_2^b(R(f_i)_* sB_I(f), L(K)))$$

for all $a, b, m_2, \ldots, m_l, m_{l+1}, p$. Moreover, it is locally free of finite rank by the strong strictness of $F$ in the assumption $(B_l)$. Then the identification (8.6.1), Lemma 8.8 and Proposition 8.14 give us $(A_{l+1})$. 

**Corollary 8.16.** For $J_1 \subset J_2 \subset \cdots \subset J_l \subset J \subset I$, the morphism $\theta_{I'I}(f)$ induces an isomorphism

$$\mathcal{O}_{Y[I]} \otimes \text{Gr}_F^{p} \text{Gr}_{m_{l}}^{L(J_l)} \cdots \text{Gr}_{m_2}^{L(J_2)} \text{Gr}_{m_1}^{L(J_1)} R^n(f_j)_* sB_J(f)$$

$$\cong \text{Gr}_F^{p} \text{Gr}_{m_{l+1}}^{L(J_{l+1})} \cdots \text{Gr}_{m_2}^{L(J_2)} \text{Gr}_{m_1}^{L(J_1)} R^n(f_i)_* sB_I(f)$$

for all $m_1, m_2, \ldots, m_l, n, p$.

**Proof.** Easy from Lemma 2.7.
9 Main result

9.1. Let \( f : X \to Y = \Delta^k \times S \) be a proper Kähler semistable morphism as in the last section. For \( i = 1, 2, \ldots, k \), we write \( Y[i] = Y[\{i\}] (\equiv E_i) \) for short. We use the similar convention for other objects, \( X[i] = X[\{i\}] \), \( sB_i(f) = sB_{(i)}(f) \), \( L(i) = L(\{i\}) \) and so on.

**Lemma 9.2.** The morphism \( N_{j|I}(f)^l \) induces an isomorphism

\[
\text{Gr}^{L(j)}_{l+l} R^n(f_{l+i})_{*}sB_I(f) \cong \text{Gr}^{L(j)}_{l-l} R^n(f_{l+i})_{*}sB_I(f)
\]

for all \( l \geq 0 \) and for all \( j \in I \). In other words, the filtration \( L(j) \) coincides with the monodromy weight filtration \( W(N_{j|I}(f)) \) on \( R^n(f_{l+i})_{*}sB_I(f) \) for \( j \in I \).

**Proof.** The morphism \( \theta_{j|I}(f) \) induces the isomorphism

\[
\mathcal{O}_{Y[j]} \otimes \text{Gr}^{L(j)}_{l+i} R^n(f_{l+i})_{*}sB_J(f) \cong \text{Gr}^{L(j)}_{l+i} R^n(f_{l+i})_{*}sB_I(f)
\]

for all \( l \). The morphism \( \theta_{j|I}(f) \) is compatible with the morphism \( N_{j|I}(f) \) and \( N_{j}(f) \) for all \( j \in J \). Therefore we may assume \( I = \{j\} \). Moreover, the morphism \( N_{j}(f) \) is compatible with the logarithmic connection \( -\nabla(j) \) on \( R^n(f_{l+i})_{*}sB_I(f) \), it is sufficient to prove that the restriction \( N_{j}(f)|_{Y[j]} \) induces the isomorphism (9.2.1) over \( Y[j]^* \) because \( (R^n(f_{l+i})_{*}sB_I(f), -\nabla(j)) \) is the canonical extension of its restriction over \( Y[j]^* \). Thus we may assume that \( k = 1 \). Because \( N_{j}(f) \) is compatible with the base change as in Remark 5.20, we can reduce the problem to the case that \( S \) is a point by Lemma 6.17. Hence we obtain the conclusion by [16], [11] and [21].

The following theorem is the main result of this article.

**Theorem 9.3.** Let \( f : X \to Y = \Delta^k \times S \) be a proper Kähler semistable morphism. For all \( c \in \mathbb{R}_{>0}^k \) and for a non-negative integer \( l \), the morphism \( N_{(j,K)|I}(f;c)^l \) induces an isomorphism

\[
\text{Gr}^{L(j,K)}_{l+m} R^n(f_{l+i})_{*}sB_I(f) \cong \text{Gr}^{L(j,K)}_{l-l} \text{Gr}^{L(K)}_{m} R^n(f_{l+i})_{*}sB_I(f)
\]

for all \( m, n \in \mathbb{Z} \) and for all \( K \subset J \subset I \subset \{1, 2, \ldots, k\} \). For the case of \( K = \emptyset \), \( N_{j|I}(f;c)^l \) induces the isomorphism

\[
\text{Gr}^{L(j)}_{l} R^n(f_{l+i})_{*}sB_I(f) \cong \text{Gr}^{L(j)}_{l-l} R^n(f_{l+i})_{*}sB_I(f)
\]

for all \( J \subset I \).

**Proof.** The morphism \( \theta_{j|I}(f) \) induces the isomorphism

\[
\mathcal{O}_{Y[j]} \otimes \text{Gr}^{L(j,K)}_{l+m} \text{Gr}^{L(K)}_{m} R^n(f_{l+i})_{*}sB_J(f) \cong \text{Gr}^{L(j,K)}_{l+l} \text{Gr}^{L(K)}_{m} R^n(f_{l+i})_{*}sB_I(f)
\]

for all \( l, m, n \). Moreover the morphism \( \theta_{j|I}(f) \) is compatible with the morphism \( N_{j|I}(f) \) for all \( j \in J \). Therefore we may assume \( J = I \). Moreover, we may assume that \( I = \{1, 2, \ldots, k\} \) as before by using the fact that \( (R^n(f_{l+i})_{*}sB_I(f), (-1)^{l+i}\nabla(I)) \) is the
canonical extension of its restriction over \( Y[I]^* \). Hence it suffices to prove that the morphism \( N_{\overline{K}}(f; c)^l \) induces an isomorphism

\[
\text{Gr}_{l+m}^{L} \text{Gr}_{m}^{L(K)} R^n f_0 sB(f) \xrightarrow{\simeq} \text{Gr}_{l-m}^{L} \text{Gr}_{m}^{L(K)} R^n f_0 sB(f)
\]

for all \( l, m, n \) with \( l \geq 0 \) and for \( c \in \mathbb{P}_K \), where \( \overline{K} = \{1, \ldots, k\} \setminus K \).

Now we proceed by induction on \( k \). The case of \( k = 1 \) is already proved in Lemma 9.2. So we assume \( k \geq 2 \).

For a non-empty subset \( K \subset \{1, \ldots, k\} \), we have the identification

\[
(\text{Gr}_{m}^{L(K)} sB(f), L(K)) \simeq \bigoplus_{(q, \Gamma)} (sB(f_{\Gamma})[-m - 2|q|], L)
\]

by Lemma 4.7, where the index \((q, \Gamma)\) runs through the set

\[
\{ (q, \Gamma); q \in \mathbb{Z}^K_{\geq 0}, \Gamma \in S_{m+2|q|+|K|}(\Lambda_K) \}
\]

as in Lemma 4.7. Under the identification (9.3.1) above, the morphism

\[
\text{Gr}_{m}^{L(K)} \nu_{j}(f) : \text{Gr}_{m}^{L(K)} sB(f) \longrightarrow \text{Gr}_{m}^{L(K)} sB(f)
\]

is identified with

\[
\bigoplus \nu_{j}(f_{\Gamma})[-m - 2|q|]
\]

for any \( j \in \overline{K} \), where the direct sum is taken over the same index set as (9.3.2). Thus the morphism \( \text{Gr}_{m}^{L(K)} N_{\overline{K}}(f; c)^l \) induces an isomorphism

\[
\text{Gr}_{l}^{L(\overline{K})} E_{1}^{-m,n+m} (R f_0 sB(f), L(K)) \xrightarrow{\simeq} \text{Gr}_{l}^{L(\overline{K})} E_{1}^{-m,n+m} (R f_0 sB(f), L(K))
\]

for all \( l, m, n \) with \( l \geq 0 \) because of the identifications (9.3.1) and

\[
E_{1}^{-m,n+m} (R f_0 sB(f), L(K)) \simeq R^n f_0 \text{Gr}_{m}^{L(K)} sB(f),
\]

together with the induction hypothesis. Then \( \text{Gr}_{m}^{L(K)} N_{\overline{K}}(f; c)^l \) induces an isomorphism

\[
\text{Gr}_{l}^{L(\overline{K})_{\text{rec}}} E_{2}^{-m,n+m} (R f_0 sB(f), L(K)) \xrightarrow{\simeq} \text{Gr}_{l}^{L(\overline{K})_{\text{rec}}} E_{2}^{-m,n+m} (R f_0 sB(f), L(K))
\]

because \( L(\overline{K}) \) is strictly compatible with the morphism \( d_1 \) of the \( E_1 \)-terms by Lemma 8.12. Therefore we obtain the desired isomorphism by Proposition 8.14.

Now the case \( K = \emptyset \) is remained. The morphism \( N_{(j)}(f)^l \) induces an isomorphism

\[
\text{Gr}_{l+m}^{L} \text{Gr}_{m}^{L(j)} R^n f_0 sB(f) \xrightarrow{\simeq} \text{Gr}_{l+m}^{L} \text{Gr}_{m}^{L(j)} R^n f_0 sB(f)
\]

for all \( l, m, n \) with \( l \geq 0 \) as proved above. Since we know the coincidence \( L(j) = W(N_{(j)}(f)) \) by Lemma 9.2, the monodromy weight filtration \( W(N(f; c)) \) satisfies the same property, that is, the morphism \( N_{(j)}(f)^l \) induces an isomorphism

\[
\text{Gr}_{l+m}^{W(N(f; c))} \text{Gr}_{m}^{L(j)} R^n f_0 sB(f) \xrightarrow{\simeq} \text{Gr}_{l+m}^{W(N(f; c))} \text{Gr}_{m}^{L(j)} R^n f_0 sB(f)
\]
for all $l, m, n$ with $l \geq 0$ by [1, (3.3) Theorem] (see also [20, (3.12) Theorem]). Then the uniqueness of the relative monodromy weight filtrations implies the coincidence $L = W(N(f; c))$. Then we conclude that the morphism $N(f; c)^l$ induces an isomorphism

$$\text{Gr}^I_l R^n f_* sB(f) \xrightarrow{\sim} \text{Gr}^I_{l-1} R^n f_* sB(f)$$

for all $l \geq 0$ by the definition of $W(N(f; c))$.

\[ \square \]

**Theorem 9.4.** The variation of $\mathbb{Q}$-mixed Hodge structure (8.5.1) on $Y[I]$* is admissible in $Y[I]$.

**Proof.** Let $\varphi : \Delta \rightarrow Y[I]$ be a holomorphic map such that $\varphi^{-1}(Y[I]^*) = \Delta^*$. Take $I' \supset I$ such that $\varphi(0) \in Y[I']^*$. For every $j \in I' \setminus I$, $c_j$ denotes the order of zero of the holomorphic function $\varphi^* t_j$ at the origin $0 \in \Delta$. Then $c_j$ is a positive integer for all $j \in I' \setminus I$.

Now we consider a locally free $\mathcal{O}_\Delta$-module of finite rank with an integrable logarithmic connection $(\varphi^* R^n(f_\ast)s B_I(f), (-1)^{|I|} \varphi^* \nabla(I))$ on $\Delta$. Since $\text{Gr}_F \text{Gr}^{L(J)}_m R^n(f_\ast)s B_I(f)$ is locally free of finite rank, we have

$$\text{Gr}_F \text{Gr}^{L(J)}_m \varphi^* R^n(f_\ast)s B_I(f) \simeq \varphi^* \text{Gr}_F \text{Gr}^{L(J)}_m R^n(f_\ast)s B_I(f),$$

which is locally free $\mathcal{O}_\Delta$-module of finite rank. By using the morphism $\theta_{I'|I}(f)$, we have the identification

$$\mathbb{C}(0) \otimes \varphi^* R^n(f_\ast)s B_I(f) \simeq \mathbb{C}(\varphi(0)) \otimes R^n(f_\ast)s B_I(f) \simeq \mathbb{C}(\varphi(0)) \otimes R^n(f_{I'}\ast)s B_{I'}(f)$$

under which the equality

$$\text{Res}_0((-1)^{|I|} \varphi^* \nabla(I)) = \sum_{j \in I' \setminus I} c_j \text{Res}_j((-1)^{|I|} \varphi^* \nabla(I)) = (-1)^{|I|} N_{I'\setminus I} f(c)$$

is easily seen. Thus $\text{Res}_0((-1)^{|I|} \varphi^* \nabla(I))$ is nilpotent. Therefore

$$(\varphi^* R^n(f_\ast)s B_I(f), (-1)^{|I|} \varphi^* \nabla(I))$$

is of unipotent monodromy around $0 \in \Delta$ and the canonical extension of its restriction over $\Delta^*$. Moreover, $L(I')$ gives us the relative monodromy weight filtration of $\text{Res}_0((-1)^{|I|} \varphi^* \nabla(I))$ with respect to $L(I)$ on

$$\mathbb{C}(0) \otimes \varphi^* R^n(f_\ast)s B_I(f) \simeq \mathbb{C}(\varphi(0)) \otimes R^n(f_{I'}\ast)s B_{I'}(f)$$

by Theorem 9.3. \[ \square \]

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