Reflection of conormal pulse solutions to large variable-coefficient semilinear hyperbolic systems

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Abstract

We provide a rigorous justification of nonlinear geometric optics expansions for reflecting pulses in space dimensions $n > 1$. The pulses arise as solutions to variable coefficient semilinear first-order hyperbolic systems. The justification applies to $N \times N$ systems with $N$ interacting pulses which depend on phases that may be nonlinear. The coherence assumption made in a number of earlier works is dropped. We consider problems in which incoming pulses are generated from pulse boundary data as well as problems in which a single outgoing pulse reflects off a possibly curved boundary to produce a number of incoming pulses. Although we focus here on boundary problems, it is clear that similar results hold by similar methods for the Cauchy problem for $N \times N$ systems in free space.

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1 Introduction and guide to the paper

The goal of this paper is to provide a rigorous justication of nonlinear geometric optics expansions for reflecting pulses in space dimensions $n > 1$. The pulses arise as solutions to variable coefficient semilinear first-order hyperbolic systems. The justification applies to $N \times N$ systems with $N$ interacting pulses which depend on phases that may be nonlinear. The coherence assumption made in earlier works such as [HMR86, JMR95, JMR93] is dropped. Although we focus here on boundary problems, it is clear that similar results hold by similar methods for the Cauchy problem for $N \times N$ systems in free space.

To explain some of these terms and place the problem in context, we consider first an $N \times N$ semilinear hyperbolic system in free space $\mathbb{R}_t \times \mathbb{R}^n_x$.

(1.1) \[ L(t, x, \partial_t, \partial_x)u^\epsilon := \partial_t u^\epsilon + \sum_{j=1}^n A_j(t, x)\partial_j u^\epsilon = f(t, x, u^\epsilon). \]

In nonlinear geometric optics one considers families of solutions that oscillate with wavelength proportional to $\epsilon > 0$, and attempts to show that they have asymptotic expansions of the form

(1.2) \[ u^\epsilon(t, x) = U_0(t, x, \theta)|_{\theta=\frac{\psi(t, x)}{\epsilon}} + o(1) \text{ as } \epsilon \to 0, \]

which are valid on a fixed space-time domain $U$ independent of $\epsilon$. Here $o(1)$ refers to some appropriate norm; in this paper we use the norm $L^\infty \cap N^m(U)$, where $N^m(U)$ is the conormal space defined in Definition 1.2; see also Remark 1.1. Here $\theta = (\theta_1, \ldots, \theta_N)$, the smooth component functions $\psi_j$ of $\psi = (\psi_1, \ldots, \psi_N)$ are called phases, and the explicitly constructed function $U_0(t, x, \theta)$ is called the leading profile. When $U_0$ is periodic in $\theta$ (or almost-periodic), we refer to the wave $u^\epsilon$ as a

$^1$We study here only the regime of “weakly nonlinear geometric optics”, where $u^\epsilon = \epsilon^p U_0(t, x, \frac{\psi}{\epsilon}) + o(\epsilon^p)$, with $p = 0$ chosen so that $U_0$ satisfies a nonlinear equation, while for larger $p$ the equation would be linear. In a quasilinear problem, we would replace (1.2) by $u^\epsilon(t, x) = u_0(t, x) + \epsilon U_0(t, x, \frac{\psi}{\epsilon}) + o(\epsilon)$, with $u_0$ a given background state.
wavetrain; when $U_0$ decays to 0 as $|\theta| \to \infty$, we refer to the wave as a pulse. This includes the case where $U_0$ has compact support in $\theta$.

Starting in the early 1990s rigorous justifications of expansions like (1.2) (and similar expansions in the quasilinear case) were provided in a series of papers by Joly, Métivier, and Rauch. In \textit{JMR95} the authors constructed a number of examples showing that in the case of wavetrains, without strong assumptions on the space of phases, the Cauchy problem is ill-posed; solutions $u^\epsilon$ may fail to exist on a domain independent of $\epsilon$ as $\epsilon \to 0$ because of focusing. The paper \textit{JMR95} justified expansions like (1.2) on a fixed domain independent of $\epsilon$ under the following coherence assumption on the space of phases.

If $\Omega \subset \mathbb{R}^{1+n}$ is open and $\Phi \subset C^\infty(\Omega, \mathbb{R})$ is a real vector space, then $\Phi$ is $\mathcal{L}$-coherent when for all $\phi \in \Phi \setminus \{0\}$ one of the following two conditions holds:

(i) at every point $(t,x) \in \Omega$ both $\det(\mathcal{L}(t,x,d\phi(t,x))) = 0$ and $d\phi(t,x) \neq 0$;
(ii) at every point $(t,x) \in \Omega$, $\det(\mathcal{L}(t,x,d\phi(t,x))) \neq 0$.

Obvious and important examples of $\mathcal{L}$-coherence occur when:

(A) $\mathcal{L}$ is a variable coefficient operator and $\Phi$ is a one-dimensional space of characteristic phases spanned by a single nondegenerate phase $\psi$.

(B) $\Phi$ is a space of linear phases $\phi(t,x) = \alpha \cdot (t,x)$, $\alpha \in \mathbb{R}^{1+n}$, and the operator $\mathcal{L}$ in (1.1) either has constant coefficients, or is replaced by a quasilinear operator where each $A_j$ is a function of $u$ alone and $u$ is taken close to a constant background state $u_0$.

It is quite difficult to find other examples of spaces of coherent phases, but a few examples do exist for problems in free space; see section 3 of \textit{JMR95}.

The coherence condition, which requires that linear combinations of characteristic phases satisfy $\det(\mathcal{L}(t,x,d\phi(t,x))) = 0$ either everywhere in $\Omega$ or nowhere in $\Omega$, is a natural condition for wavetrains, but not for pulses. To see this let $a(t,x)$ $b(t,x)$, $g(\theta)$, and $h(\theta)$ be compactly supported smooth functions, let $\psi_1(t,x)$ and $\psi_2(t,x)$ be characteristic phases, and consider the wavetrain and pulse interactions given respectively by

\begin{align*}
(i) & \quad a(t,x)e^{im_1 \psi_1 / \epsilon} \cdot b(t,x)e^{in_2 \psi_2 / \epsilon} = a(t,x)b(t,x)e^{im_1 \psi_1 / \epsilon} e^{in_2 \psi_2 / \epsilon} \\
(ii) & \quad a(t,x)g\left(\frac{\psi_1}{\epsilon}\right) \cdot b(t,x)h\left(\frac{\psi_2}{\epsilon}\right) = a(t,x)b(t,x)g\left(\frac{\psi_1}{\epsilon}\right) h\left(\frac{\psi_2}{\epsilon}\right).
\end{align*}

Even though the coherence condition appears to be less relevant in the case of pulses, most of the existing rigorous geometric optics results for pulses concern problems where either (A) holds (e.g., \textit{AR02} \textit{AR03} or (B) holds (e.g., \textit{CW13} \textit{CW14} \textit{Wil15}).

Without assuming coherence we now consider the variable-coefficient boundary problem that will be our focus in this paper, an $N \times N$ strictly hyperbolic system on the half-space $\mathbb{R}_t \times \mathbb{R}^n_+$.

\footnote{When $\phi$ satisfies $\det(\mathcal{L}(t,x,d\phi(t,x))) = 0$ on $\Omega$, we call it a characteristic phase. When $\phi$ satisfies $d\phi(t,x) \neq 0$ for all $(t,x) \in \Omega$, we say it is nondegenerate.}

\footnote{See also \textit{CR04} and its companion papers, which give a rigorous treatment of focusing spherical pulses, a case where coherence does not hold.}
\{(t, x) : x_n \geq 0\}\]

(a) \(L(t, x, \partial_t, \partial_x)u^\epsilon := \partial_t u^\epsilon + \sum_{j=1}^{n} A_j(t, x) \partial_j u^\epsilon = f(t, x, u^\epsilon)\) in \(x_n > 0\)

(1.4)

(b) \(B(t, x')u^\epsilon|_{x_n=0} = g(t, x', \theta_0)|_{\theta_0=\frac{\epsilon}{\epsilon}} := b^\epsilon(t, x')\),

(c) \(u^\epsilon = 0\) in \(t < 0\).

If we take any \(C^1\) function of the form \(U_0(t, x, \frac{\psi}{\epsilon})\) and substitute it for \(u^\epsilon\) in (1.4)(a), terms of order \(\frac{1}{\epsilon}\) will appear on the left. To make those terms vanish, the classical strategy is to choose the leading profile of the form

(1.5)

\[U_0(t, x, \theta) = \sum_{k=1}^{N} \sigma_k(t, x, \theta_k) r_k(t, x),\]

where each \(r_k(t, x) \in \mathbb{R}^N\) is an eigenvector associated to an eigenvalue \(\lambda_k(t, x, \partial_t, x')\psi_k\) of an appropriate operator depending on \(\psi_k\), and \(\psi_k\) solves an “eikonal problem” of the form

(1.6)

\[\partial_{x_n} \psi_k = -\lambda_k(t, x, \partial_t, x'\psi_k)\]
\[\psi_k|_{x_n=0} = t.\]

For \(r_k\) and \(\psi_k\) as constructed in section 3.1 the terms of order \(\frac{1}{\epsilon}\) vanish no matter how the scalar functions \(\sigma_k\) in (1.5) are chosen, and we obtain

(1.7) \(L(t, x, \partial_t, \partial_x) U_0 \left( t, x, \frac{\psi}{\epsilon} \right) = f \left( t, x, U_0 \left( t, x, \frac{\psi}{\epsilon} \right) \right) + R^\epsilon_a(t, x),\) where \(R^\epsilon_a = O(1)\) as \(\epsilon \to 0\).

The remainder \(R^\epsilon_a\) can be written as a sum of terms, some of which depend only on a single \(\psi_k\) (the “single-phase terms”) and some of which depend on at least two distinct \(\psi_k\) (the “multiphase-terms”). In order to prove (1.2) the part of \(R^\epsilon_a\) that is no smaller than \(O(1)\) must be solved away. The scalar profiles \(\sigma_k(t, x, \theta_k)\) are chosen to satisfy “transport equations” which effectively solve away the part of \(R^\epsilon_a\) consisting of terms of the form

(1.8)

\[q(t, x, \sigma_k, \partial_t, x'\sigma_k)|_{\theta_k=\frac{\epsilon}{\epsilon}} = r_k(t, x),\]

which depend on a single \(\psi_k\) for some \(k \in \{1, \ldots, N\}\) and are polarized in the direction of \(r_k\).

The transport equations (3.39) satisfied by the \(\sigma_k(x, \theta_k)\) are derived in sections 3.2, 3.4, and 3.5. These equations are completely decoupled. Provided (1.2) holds, this shows that unlike wavetrains, pulses do not interact at leading order, and “resonances” among phases have no effect on the leading profile \(U_0\). The decoupling reflects the fact that pulses interact on much smaller sets than

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4 We show in section 3 that (1.4) is local model to which more general continuation problems can be reduced by a change of variables; see also Remark 1.3. The assumptions on \((L, B), f(t, x, u)\) and \(g(t, x', \theta)\) are stated in Theorem 1.2.

5 Terms like (1.8) make up \(\mathcal{E} \mathcal{F}|_{\theta=\frac{\epsilon}{\epsilon}}\) for \(\mathcal{E} \mathcal{F}\) as in (3.24). The \(\sigma_k\) can be chosen so that \(U_0(t, x, \frac{\epsilon}{\epsilon})\) satisfies the boundary and initial conditions in (1.4) exactly.

6 A resonance occurs when an integer combination of characteristic phases is again a characteristic phase.
wavetrains, sets that get smaller, in fact, as $\epsilon$ decreases. For example, in (1.3) if we assume that $g$ and $h$ are supported in $|\theta| \leq 1$, we have

(1.9)

(i) $\text{supp } a(t,x)b(t,x)e^{\frac{n\psi_1 + n\psi_2}{\epsilon}} = \text{supp } a(t,x)b(t,x)$, but

(ii) $\text{supp } a(t,x)b(t,x) g\left(\frac{\psi_1}{\epsilon}\right) h\left(\frac{\psi_2}{\epsilon}\right) \subset \text{supp } a(t,x)b(t,x) \cap \{(t,x) : |\psi_1| \leq \epsilon \text{ and } |\psi_2| \leq \epsilon\}$.

To solve away the remaining parts of $R_{a,\epsilon}$ that are no smaller than $O(1)$, we construct a corrector $U_1^\epsilon(t,x)$ such that the approximate solution

(1.10)

$u_\epsilon(t,x) := U_0(t,x,\frac{\psi}{\epsilon}) + \epsilon U_1^\epsilon(t,x)$

gives a remainder that is $o(1)$ rather than $O(1)$:

(1.11)

$L(t,x,\partial_t,\partial_x)u_\epsilon^\epsilon = f(t,x,u_\epsilon^\epsilon) + r_\epsilon^\epsilon(t,x)$, where $r_\epsilon^\epsilon = o(1)$ as $\epsilon \to 0$.

In the single phase problem considered in [AR02], where $\theta$ and $\psi$ are scalar and the leading profile $U_0(t,x,\theta)$ is compactly supported in $\theta$, a corrector of the form $U_1^\epsilon(t,x) = V(t,x,\theta)|_{\theta = \frac{\psi}{\epsilon}}$ was constructed, where $V$ is bounded but no longer compactly supported in $\theta$. Considering that pulses do not interact at leading order, it is natural in the multiphase case, where $\theta = (\theta_1, \ldots, \theta_N)$, to look for a corrector of the form

(1.12)

$\sum_{k=1}^N V_k(t,x,\theta_k)|_{\theta_k = \frac{\psi_k}{\epsilon}} := V(t,x,\theta)|_{\theta = \frac{\psi}{\epsilon}}$

where the individual $V_k$ are constructed like the corrector $V_k$ in [AR02]. Such a corrector, constructed in section 3.6.1 is useful for solving away the single-phase part of $R_{a,\epsilon}$ that is left after all polarized terms of the form (1.8) are removed. But a corrector of the form (1.12) is useless for solving away the multiphase or interacting part of $R_{a,\epsilon}$ which may contain, for example, $O(1)$ products like (1.3)(ii), with factors depending on two or more linearly independent phases $\psi_k$. Thus, we look for a corrector of the form

(1.13)

$U_1^\epsilon(t,x) = V(t,x,\theta)|_{\theta = \frac{\psi}{\epsilon}} + W^\epsilon(t,x)$

where the noninteraction term $V$ solves away the unpolarized, single-phase part of $R_{a,\epsilon}$, and the interaction term $W^\epsilon$ solves away the multiphase or interaction part of $R_{a,\epsilon}$. In fact, the multiphase part of $R_{a,\epsilon}$ is not solved away completely, but there remains only a harmless piece, $r_\epsilon^\epsilon(t,x)$ in (1.11), of size $O(\sqrt{\epsilon}) = o(1)$ in $L^\infty \cap N^m$; see Proposition 1.25. A formula for $r_\epsilon^\epsilon$ is given just above (3.22).

Thus, even though pulses do not interact at leading order, their interactions determine the part of the corrector given by $W^\epsilon(t,x)$. The main novelties of this paper are contained in section 3.6.3 which constructs $W^\epsilon(t,x)$, and in sections 3.3.2 4.1.1 and 4.2 which give various conormal estimates involving $W^\epsilon(t,x)$. The construction in section 3.6.3 draws some inspiration from [CW13], which considered the corresponding quasilinear multiphase reflection problem with linear phases.

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This is the single-phase part of $(I - E)\mathcal{F}|_{\theta = \frac{\psi}{\epsilon}}$ in (3.24).
(case (B) of \(L\)—coherence). But the construction and estimation of \(W^s\) is more challenging here because the operator \(L(t, x, \partial_t, \partial_x)\) in (1.4) has variable coefficients, and the characteristic phases \(\psi_k\) obtained by solving the eikonal problems (1.6) are now nonlinear. The construction of \(W^s\) takes advantage of the fact that the phases \(\psi_k(t, x)\) are all nearly equal to \(t\) in the “interaction region” \(I_\epsilon := \{(t, x) : |t| \lesssim \epsilon, |x_n| \lesssim \epsilon\};\) see Remark 3.7.

We give the construction of the approximate solution \(u^\epsilon\) first for the case where the nonlinear function \(f(t, x, u)\) in (1.4) is at most quadratic in \(u\). This allows us to see the pulse interactions explicitly. In section 5 we explain how the construction and estimation of \(u^\epsilon\) can be extended to the case where \(f(t, x, u)\) in (1.4) is a general smooth function such that \(f(t, x, 0) = 0\).

The exact solution \(u \in L^\infty \cap N^m(U)\) to the boundary problem (1.3) is obtained in section 4.2 on a fixed open set \(U \ni 0\) independent of \(\epsilon\) by direct application of a result of \(\text{[Mé89]}\).

Remark 1.1 (Conormal spaces and progressing waves). The paper \(\text{[Mé89]}\) and its companion \(\text{[Mé86]}\) were concerned with a problem seemingly quite different from the one considered here. Building on earlier work of Bony \(\text{[Bon82, Bon80]}\) and Beals-Métiévier \(\text{[BM87]}\) which described the propagation, interaction, and reflection of smooth enough conormal progressing wave solutions to \(N \times N\) semilinear hyperbolic systems, Métiévier in the companion papers \(\text{[Mé86, Mé89]}\) was able to extend such results to the case of discontinuous progressing waves.

We now describe the spaces \(N^m(U)\) introduced in \(\text{[Mé89]}\), where \(U \ni 0\) is any small enough open set in \(\mathbb{R}^{1+n}_+\). Let \(\Sigma_j = \{\psi_j = 0\}\) be one of the \(N\) characteristic surfaces defined by \(\psi_j\) as in (1.6), and let \(\mathcal{M}_j\) denote the space of smooth vector fields on \(U \subset \mathbb{R}^{1+n}_+\) tangent to both \(\Sigma_j\) and \(\Delta := \Sigma_j \cap \{x_n = 0\}\). Define \(N^m(U, \mathcal{M}_j)\) as the set of \(u \in L^2(U)\) such that \(M^k u \in L^2(U)\), where \(M^k\) denotes any composition of \(\leq m\) elements of \(\mathcal{M}_j\). Let \(N^m(U)\) denote the set of \(u \in L^2(U)\) such that \(u = \sum_{j=1}^N u_j\) for some \(u_j \in N^m(U, \mathcal{M}_j)\). Under certain conditions involving \(U\) and \(\Delta\), the spaces \(L^\infty \cap N^m(U)\) are algebras for \(m > \frac{n+5}{2}\); see Proposition 4.22.

Clearly, elements of \(L^\infty \cap N^m(U, \mathcal{M}_j)\) or \(L^\infty \cap N^m(U)\) may be discontinuous across \(\Sigma_j\) even when \(m\) is large. The exact and approximate solutions we consider here are in \(C^\infty(U)\), but they depend on a small parameter \(\epsilon\). Instead of being discontinuous across \(\Sigma_j\), they exhibit a “fast transition region” near \(\Sigma_j\) that gets thinner as \(\epsilon \to 0\).

If \(h^\epsilon\) is a function that depends on the small parameter \(\epsilon > 0\) and \(p \geq 0\), we write

\[|h^\epsilon|_{L^\infty \cap N^m(U)} \lesssim \epsilon^p\]

if there exist positive constants \(\epsilon_0\) and \(C\) such that

\[|h^\epsilon|_{L^\infty \cap N^m(U)} \leq Ce^p\text{ for all } \epsilon \in (0, \epsilon_0].\]

The final step is to estimate the error \(u^\epsilon - u^\epsilon_0 := w^\epsilon\). We do this by applying the \(L^\infty \cap N^m(U)\) estimates of \(\text{[Mé89]}\) to the boundary problem (1.79) satisfied by \(w^\epsilon\). This yields

\[|w^\epsilon|_{L^\infty \cap N^m(U)} \lesssim \sqrt{\epsilon},\text{ where } m > \frac{n+5}{2}.\]

\[\text{[RR88]}\] had earlier studied discontinuous progressing waves for \(2 \times 2\) semilinear systems.

\[\text{These spaces along with their natural norms are discussed in more detail in section 4.4}\]
Since $|\epsilon U_1(t,x)|_{L^\infty\cap N^m(U)} \lesssim \epsilon$ (see Proposition 4.5), this completes the proof of (1.2) for the boundary problem (1.4) and provides the rate of convergence $\sqrt{\epsilon}$. This rate is faster than the rate found in [CWI13] by a quite different error analysis.

These arguments yield our first main result, which we state here for the model problem (1.4).

**Theorem 1.2** (Pulse generation at the boundary). Consider the boundary problem (1.4) on the half-space $\mathbb{R}_t \times \mathbb{R}^n_+$, where $\mathcal{L}(t,x,\partial_t,\partial_x)$ has real $C^\infty$ $N \times N$ matrix coefficients and is strictly hyperbolic with respect to $t$, the boundary is noncharacteristic, and $B(t,x')$ is a real $C^\infty$ matrix of size $p \times N$ for some $p \leq N$. Suppose $f(t,x,u)$ and $g(t,x,\theta_0)$ are $C^\infty$ functions valued in $\mathbb{R}^N$ and $\mathbb{R}^p$ respectively, where $f(t,x,0) = 0$ and $g(t,x',\theta_0)$ is supported in $\{(t,x,\theta_0) : t \geq 0$ and $|\theta_0| \leq 1\}$. Suppose also that $(\mathcal{L}, B)$ satisfies the uniform Lopatinski condition at $(t, x) = 0$.

Then the exact solution $u^\epsilon$ of (1.4) has the expansion (1.2). More precisely, if $U_0(t,x,\theta)$ as in (1.5) denotes the leading profile whose construction is outlined above, and $\psi = (\psi_1, \ldots, \psi_N)$ for $\psi_k$ satisfying (1.6), then for $m > \frac{n+5}{2}$ there exist $\epsilon_0 > 0$ and an open set $U \ni 0$ independent of $\epsilon$ such that

\begin{equation}
|u^\epsilon - U_0(t,x,\theta)|_{\theta - \frac{\epsilon}{\epsilon}} \lesssim \sqrt{\epsilon} \text{ for } \epsilon \in (0, \epsilon_0].
\end{equation}

In the expansion (1.5) of $U_0$, only the incoming profiles can be nonzero and, except for special choices of $g(t,x',\theta_0)$, they are all nonzero.

**Remark 1.3.** More general domains $D$ and initial surfaces $S$. Theorem 1.2 is local, so because we consider variable coefficient operators, it applies equally well to the following more general class of semilinear boundary problems on $\mathbb{R}^{n+1}_+$. Let

\begin{equation}
P(z, \partial_z) = \sum_{j=0}^n P_j(z)\partial_{z_j},
\end{equation}

where the $P_j$ are real $C^\infty$ $N \times N$ matrices. Suppose we are given a smooth hypersurface $S = \{z : \alpha(z) = 0\}$ that is spacelike at $z = 0$, and a smooth noncharacteristic hypersurface $bD = \{z : \beta = 0\}$ that is noncharacteristic at $0$ and intersects $S$ transversally at $0$.

Set $D = \{z : \beta \geq 0\}$ and for a small enough open set $O \ni 0$ let $\Delta := S \cap bD \cap O$, a smooth codimension-two manifold closed in $O$. Let $B(z)$ be a real $C^\infty$ $p \times N$ matrix, and consider the following pulse generation problem on $O$. The assumptions in Theorem 1.2 are explained in section 2.

The error analysis in [CWI13] did not use conormal estimates. In the quasilinear problem considered there, we constructed the exact solution as $u^\epsilon(t,x) = u_0 + \epsilon u^\epsilon(t,x,\theta_0)|_{\theta_0 = \frac{\beta(t,x')}{\epsilon^2}}$, and studied the “singular system” satisfied by $U^\epsilon(t,x,\theta_0)$. The error analysis was based on estimates for that singular system proved by using the pulse calculus of [CGW14] to construct singular Kreiss symmetrizers. This kind of analysis breaks down in the present setting.

This condition determines $p \leq N$. The number $p$ is equal to the number of positive eigenvalues of the matrix $A_n$ in (1.4), see Remark 3.10.

A careful definition of incoming/outgoing profiles is given in Remark 3.10. Roughly, a profile $\sigma_k(t,x,\theta_k)$ is incoming when its support propagates along curves that enter the domain $x_n \geq 0$ as $t$ increases, and is outgoing in the opposite case. When $\sigma_k$ is incoming, we call the associated phase $\psi_k$ and surface $\Sigma_k = \{\psi_k = 0\}$ incoming as well.

The surface $S = \{\alpha = 0\}$ is spacelike at $0$ if $P(0, \partial_\nu)$ is strictly hyperbolic in the direction $d\alpha(0) \neq 0$. The hypersurface $bD$ is noncharacteristic at $0$ if $d\beta(0) \neq 0$. 

\[\text{7} \]
\( O \cap D: \)
\[
\mathcal{P}(z, \partial_z)u^\epsilon = f(z, u^\epsilon) \quad \text{in } \beta > 0
\]
\[
B(z)u^\epsilon|_{\beta=0} = g \left( z, \frac{\psi_0(z)}{\epsilon} \right) \quad \text{on } \beta = 0
\]
\[
u^\epsilon = 0 \quad \text{in } \alpha < 0.
\]
for \( \psi_0 \) as in (b) below. We show in section 4 that (1.19) can be reduced by a change of variables to a model problem (1.4) satisfying the hypotheses of Theorem 1.2 whenever the following conditions are satisfied:

(a) \( (\mathcal{P}, B) \) satisfies the uniform Lopatinski condition at 0 \( \in bD \cap S \) when \( \alpha \) is taken as a time variable; see Remark 2.4

(b) The map \( \psi_0 : bD \to \mathbb{R} \) is \( C^\infty \) and satisfies \( \Delta = \{ z \in bD : \psi_0(z) = 0 \} \) and \( d\psi_0(0) \neq 0 \).

(c) The functions \( f(z, u) \) and \( g(z, \theta_0) \) are \( C^\infty \) functions valued in \( \mathbb{R}^N \) and \( \mathbb{R}^P \) respectively, where \( f(z, 0) = 0 \) and \( g(z, \theta_0) \) is supported in \( \{(z, \theta_0) : \alpha \geq 0 \text{ and } |\theta_0| \leq 1\} \).

Next we state a closely related reflection result for the following problem on \( O \cap D: \)

\[
\mathcal{P}(z, \partial_z)u^\epsilon = f(z, u^\epsilon) \quad \text{in } \beta > 0
\]
\[
B(z)u^\epsilon|_{\beta=0} = 0 \quad \text{on } \beta = 0
\]
\[
u^\epsilon = u_o^\epsilon \quad \text{in } \alpha < -\gamma \quad \text{for some } \gamma > 0,
\]

where \( (\mathcal{P}, B), f, \alpha, \) and \( \beta \) are as in Remark 1.3, and \( u_o^\epsilon \) is a given outgoing pulse concentrated near a characteristic surface \( \Sigma = \{ z : \zeta = 0 \} \). We assume that \( \Sigma \ni 0 \) intersects \( bD = \{ \beta = 0 \} \ni 0 \) transversally starting at “time” \( \alpha = 0 \), where \( \alpha(0) = 0 \).

Let \( \mathcal{M}_\Sigma \) denote the set of smooth vector fields on \( O \) that are tangent to \( \Sigma \).

**Theorem 1.4** (Reflection of pulses). For \( m > \frac{n+5}{2} \) consider the problem (1.20), where the outgoing pulse is assumed to satisfy \( \mathcal{P}(z, \partial_z)u_o^\epsilon = f(z, u_o^\epsilon) \) on \( O \ni 0 \) and to have an expansion

\[
u_o^\epsilon(z) = \tau \left( z, \frac{\zeta}{\epsilon} \right) r(z) + u_o^\epsilon(z) \quad \text{with } |u_o^\epsilon(z)|_{L^\infty(N^\infty(O, \mathcal{M}_\Sigma))} = o(1) \quad \text{as } \epsilon \to 0
\]

for some smooth scalar outgoing profile \( \tau(z, \theta_{out}) \) supported in \( |\theta_{out}| \leq 1 \). We assume that \( \tau \) satisfies a transport equation, namely (1.9), on \( O \). To insure corner compatibility we assume that both \( u_o^\epsilon|_{\alpha < -\gamma} \) and \( \tau|_{\alpha < -\gamma} \) vanish on a neighborhood of \( \{ \beta = 0 \} \). We also assume that \( u_o^\epsilon \) is chosen so that the solution to (1.20) exists on some neighborhood of 0 independent of \( \epsilon \).

Then there exists an open set \( U \ni 0 \) such that the exact solution \( u^\epsilon \) of (1.20) satisfies

\[
|u^\epsilon(z) - \left[ \sum_{k=1}^P \sigma_k \left( z, \frac{\psi_k}{\epsilon} \right) r_k(z) + \tau \left( z, \frac{\zeta}{\epsilon} \right) r(z) \right]|_{L^\infty(N^\infty(U))} = o(1) \quad \text{as } \epsilon \to 0
\]

where the \( \psi_k \) and \( \sigma_k \) are smooth incoming characteristic phases and scalar profiles constructed as in section 7. If as in the construction of [AR02] we have \( |u_o^\epsilon|_{L^\infty(N^\infty(O, \mathcal{M}_\Sigma))} \leq \epsilon \) in (1.21), then we obtain the rate of convergence \( \epsilon \) in (1.22).

---

15We do not assume that \( \Sigma \cap \{ \beta = 0 \} = \Sigma \cap \{ \alpha = 0 \} \) or that \( \{ \alpha = 0 \} \cap \{ \beta = 0 \} = \Sigma \cap \{ \beta = 0 \} \) near 0.

16This can be arranged by an application of Theorem 2.1.5 of [Met89]. The point is to insure that the solution exists long enough for reflection to occur. The outgoing pulse (1.21) can be constructed as in [AR02].
Let \( \tilde{\Delta} \) be the closed codimension-two submanifold of \( U \) given by \( \Sigma \cap bD \), and set \( \Sigma_N := \Sigma = \{ \zeta = 0 \} \) and \( \Sigma_k = \{ \psi_k = 0 \} \) for \( k = 1, \ldots, p \). Then we have on \( U \):

\[
(1.23) \quad \Sigma_i \cap \Sigma_j = \tilde{\Delta} \quad \text{for} \quad i \neq j \quad \text{and} \quad \Sigma_i \cap \{ \beta = 0 \} = \tilde{\Delta} \quad \text{for all} \quad i,
\]

where all intersections are transversal.

It is not clear to us how to derive Theorem 1.4 as a strict corollary of Theorem 1.2. Instead, we show in section 7 that after some geometric preparation, Theorem 1.4 can be obtained as a corollary of the proof of Theorem 1.2.

**Remark 1.5. Some extensions.**

(1) *The Cauchy problem.* With slight modification the methods of this paper yield a rigorous construction of pulses with nonlinear phases for the Cauchy problem for variable-coefficient strictly hyperbolic \( N \times N \) systems in dimensions \( n > 1 \). Pulse initial data at \( t = 0 \) of the form \( u_0 |_{t=0} = g(x, \psi_0(x)) \) gives rise to pulses in \( t > 0 \) concentrated on the \( N \) characteristic hypersurfaces emanating from \( \Delta = \{(t, x) : t = \psi_0(x) = 0 \} \). Conormal estimates of [Mét86] can be used for the error analysis.

(2) *Quasilinear or nonstrictly hyperbolic problems.* We expect that the construction and estimation of the approximate solution \( u_\epsilon \) to the semilinear problem considered here extends in a straightforward manner to variable-coefficient, quasilinear hyperbolic systems satisfying our other assumptions. But to carry out a similar error analysis, one would need to perform the clearly nontrivial task of extending the results of [Mét89] from the semilinear to the quasilinear case. Modifying the strict hyperbolicity assumption made here would require a corresponding modification of [Mét89], which assumes strict hyperbolicity.

### 2 Assumptions and some notation

We now list with some discussion the structural assumptions made in Theorem 1.2 on the operators

\[
\mathcal{L}(t, x, \partial_t, \partial_x) = \partial_t + \sum_{j=1}^{n} A_j(t, x) \partial_j \quad \text{and} \quad B(t, x')
\]

appearing in (1.4).

**Assumption 2.1** (Strict hyperbolicity). Let \( (\tau, \xi) \in \mathbb{R}_+ \times \mathbb{R}^n_\xi \) denote variables dual to \( (t, x) \). The operator \( \mathcal{L}(t, x, \partial_t, \partial_x) \) is strictly hyperbolic with respect to \( t \) on an open neighborhood \( \mathcal{O} \subset \mathbb{R}^{1+n} \) of \( (t, x) = 0 \). That is, there exist functions \( \tau_i(t, x, \xi) : C^\infty(\mathcal{O} \times (\mathbb{R}^n_\xi \setminus 0)) \rightarrow \mathbb{R}, i = 1, \ldots, N \), positively homogeneous of degree one in \( \xi \), such that

\[
\tau_1 < \tau_2 < \cdots < \tau_N \quad \text{on} \quad \mathcal{O} \times (\mathbb{R}^n_\xi \setminus 0)
\]

\[
(2.2) \quad \det \left( \tau I + \sum_{j=1}^{n} A_j(t, x) \xi_j \right) = \prod_{i=1}^{N} (\tau + \tau_i(t, x, \xi)).
\]

Thus, for each \( (t, x, \xi) \in \mathcal{O} \times (\mathbb{R}^n_\xi \setminus 0) \), the matrix \( \sum_{j=1}^{n} A_j(t, x) \xi_j \) has \( N \) distinct real eigenvalues \( \tau_i(t, x, \xi) \).

---

\[17\] If \( \mathcal{L}(t, x, \partial_t, \partial_x) = A_0(t, x) \partial_t + \sum_{j=1}^{n} A_j(t, x) \partial_x, \) we say \( \mathcal{L} \) is strictly hyperbolic with respect to \( t \) if \( A_0 \) is invertible and the condition (2.2) is satisfied by \( A_0^{-1} \mathcal{L}. \)
Assumption 2.2 (Noncharacteristic boundary). The matrix $A_n(t,x)$ is invertible on $O$.

Assumption 2.3 (Uniform Lopatinski condition). The pair of operators $(\mathcal{L}, B)$ satisfies the uniform Lopatinski (a.k.a. uniform Kreiss) condition at $(t,x) = 0$.

We refer to [Kre70], to Definition 3.8, Chapter 7 of [CP82], or to Remark 9.7 of [BG07] for the exact statement of this condition, which guarantees that the $L^2$ norm of the solution $u$ of the linear boundary problem on the half-space $\mathbb{R} \times \mathbb{R}^n_+$

$$\mathcal{L}(0, \partial_t, \partial_x)u = f \text{ in } x_n > 0, \ B(0)u = g \text{ on } x_n = 0, \ u = 0 \text{ in } t < 0$$

can be estimated, along with $\langle u |_{x_n=0} \rangle_{L^2}$, in terms of the $L^2$ norms of $f$ and $g$. This condition is assumed in a few of the results of [M´et89] that we apply in this paper. We also need to use it at one point in the construction of the leading profile $U_0$; assumption 2.3 implies that the $p \times p$ boundary matrix $(B_{r1} \cdots B_{rp})$ in (3.38) is invertible for $y$ near 0.

**Remark 2.4.** The uniform Lopatinski condition at $(t,x) = 0$ is a coordinate invariant, “open” condition that depends, in the notation of Remark 1.3, only on $(\mathcal{L}, B)$, a noncharacteristic hypersurface $b\mathcal{D}$, and a spacelike hypersurface $S$ that intersects $b\mathcal{D}$ transversally at $(t,x) = 0$; see section 8.1 of Chapter 7 of [CP82].

We use this coordinate invariance in section 6 to reduce the problem (1.19) to our model problem.

In most of the paper instead of the $(t,x)$ notation for points in $\mathbb{R}^{1+n}$, we will use the following notation, which is more convenient for boundary problems. Set

$$y = (y_0, y', y_n), \text{ where } t = y_0, \ x' = y', \text{ and } x_n = y_n$$

$$y' = (y_0, y'').$$

Denote the corresponding dual variables by $\eta = (\eta_0, \eta'', \eta_n) = (\eta', \eta_n)$.

We now rewrite (1.4) after applying $A_n^{-1}$ on the left and redefining $A_n^{-1}f$ to be $f$ as

$$L(y, \partial_y)u^\epsilon = \partial_n u^\epsilon + \sum_{j=0}^{n-1} B_j(y) \partial_j u^\epsilon = f(y, u^\epsilon) \text{ in } y_n > 0, \text{ where } B_0 = A_n^{-1} \text{ and } \partial_0 = \partial_t$$

$$B(y')u^\epsilon |_{y_n=0} = g(y', \theta_0)|_{\theta_0=\eta_0} := b^\epsilon(y')$$

$$u^\epsilon = 0 \text{ in } y_0 < 0.$$  

We end this section by collecting some notation.

**Notations 2.5.** 1. Let $U \ni 0$ always denote some bounded connected open set in the half-space

$$\mathbb{R}^{1+n}_+ = \mathbb{R}_{y_0} \times \mathbb{R}_+^n = \{y : y_n \geq 0\}.$$  

Let $bU = \{y' : (y',0) \in U \cap \{y_n = 0\}\}$. The set $U$ may need to be taken sufficiently small (independently of $\epsilon$) for its use in a given context to make sense.

---

18 Here take $f$ and $g$ to be in $L^2$ on their respective domains and equal to zero in $t < 0$.

19 When the uniform Lopatinski condition holds at $(t,x) = 0$, it holds in an open neighborhood of 0.

10
2. If \( f^\epsilon: U \to \mathbb{R}^M \) and \( p \geq 0 \), we write \( |f^\epsilon(y)| \lesssim \epsilon^p \) if there exist positive constants \( \epsilon_0 < 1 \) and \( C \) such that \( |f^\epsilon(y)| \leq C\epsilon^p \) for all \( y \in U \) and \( \epsilon \in (0, \epsilon_0) \). Similarly, if \( \| \cdot \| \) is some norm, we write \( \|f^\epsilon\| \lesssim \epsilon^p \) if there exist positive constants \( \epsilon_0 < 1 \) and \( C \) such that \( \|f^\epsilon\| \leq C\epsilon^p \) for all \( \epsilon \in (0, \epsilon_0) \).

3. Denote by \( C_0^\infty(U, \mathbb{R}^M) \) the set of \( C^\infty \) functions on \( U \) and valued in \( \mathbb{R}^M \), which are bounded along with all their derivatives on \( U \).

4. The phases \( \psi_k \) constructed in section 3 satisfy \( \psi_k(y',0) = y_0 \) for all \( k \). We sometimes (for example, in \( \text{(3.18)} \)) write \( \psi_0(y') \) in place of \( y_0 \) as a reminder of this.

5. We often write \( d' = \partial_{y'} \).

6. Set \( \mathbb{N}_0 = \{0,1,2,3,\ldots \} \).

### 3 Construction of the approximate solution \( u_\epsilon^a \)

In this section we construct \( u_\epsilon^a = U_0(y, \theta)|_{\theta = \frac{x}{\epsilon}} + \epsilon U_1(y) \) such that \( \text{(1.11)} \) holds. The leading term is constructed as a sum of \( N \) pieces

\[
(3.1) \quad U_0 \left( y, \frac{\psi}{\epsilon} \right) = \sum_{k=1}^{N} \sigma_k \left( y, \frac{\psi_k(y)}{\epsilon} \right) r_k(y), \quad k = 1, \ldots, N,
\]

where each piece is a pulse concentrated near the codimension-one surface in spacetime defined by \( \Sigma_k = \{ y : \psi_k(y) = 0 \} \). The first step is to construct the characteristic phases \( \psi_k \) that define these \( N \) surfaces. The \( N \) surfaces all contain the codimension-two surface defined by \( \Delta := \{ y : y_0 = y_n = 0 \} \). The construction takes place on a small enough neighborhood of the origin \( y = 0 \).

#### 3.1 Eikonal equations for the phases

The functions \( \psi_k \) are constructed as solutions of a nonlinear Cauchy problem classically known as an eikonal problem. To formulate this problem we first define the eigenvectors \( r_k(y) \) that appear in \( \text{(3.1)} \) along with the associated eigenvalues. With \( L \) as in \( \text{(2.4)} \) we set

\[
(3.2) \quad L(y, \eta) = \eta_n I + \sum_{j=0}^{n-1} B_j(y) \eta_j := \eta_n I + A(y, \eta').
\]

We claim that if \( \eta' \) satisfies \( |\eta''| \leq \delta |\eta_0| \) for \( \delta > 0 \) small enough, then the matrix \( A(y, \eta') \) has \( N \) distinct real eigenvalues \( \lambda_m(y, \eta'), m = 1, \ldots, N \); see Remark \( \text{3.1} \). Let \( L_m(y, \eta'), R_m(y, \eta') \) denote associated left and right eigenvectors, which can be chosen positively homogeneous of degree zero in \( \eta' \) and such that \( L_k(y, \eta') R_j(y, \eta') = \delta_{kj} \).

**Remark 3.1.** It is not generally true that \( A(y, \eta') \) has \( N \) distinct real eigenvalues for all \( \eta' \neq 0 \). This is true, though, for the particular directions \( \eta' = (\pm 1, 0, \ldots, 0) \). Indeed, note that

\[
A(y, \pm 1, 0, \ldots, 0) = \pm B_0(y) = \pm A_n^{-1}(y),
\]

and observe that \( A_n(y) \) has \( N \) distinct real eigenvalues as a consequence of strict hyperbolicity, that is, Assumption \( \text{2.1} \). Moreover, since \( y_n = 0 \) is noncharacteristic, those eigenvalues must all

\[\text{20}\]
be nonzero. It follows that $\mathcal{A}(y, \eta')$ has $N$ distinct real nonzero eigenvalues whenever $\eta' = (\eta_0, \eta'')$ satisfies

$$\eta \in \Gamma := \{ \eta' : |\eta''| \leq \delta|\eta_0|\} \tag{3.3}$$

for $\delta > 0$ small enough.

We define the following projection operators for use in section 3.4:

**Definition 3.2.** (a) We define $\Pi_k(y, \eta')$ to be the projection on $\text{span} R_k(y, \eta')$ in the decomposition $\mathbb{R}^N = \bigoplus_{m=1}^N \text{span} R_m(y, \eta')$.

Explicitly, for $x \in \mathbb{R}^N$ we have

$$\Pi_k(y, \eta') x = (L_k(y, \eta') x) R_k(y, \eta').$$

(b) With $d' = \partial_y$ set $r_k(y) := R_k(y, d'\psi_k(y))$, $l_k(y) := L_k(y, d'\psi_k(y))$, and $\pi_k(y) = \Pi_k(y, d'\psi_k(y))$.

The next (classical) lemma [Lax57] and its corollary will be used in section 3.5 to determine “transport equations” satisfied by the leading profiles.

**Lemma 3.3.** For $j = 0, \ldots, n-1$ we have

$$L_m(y, \eta') B_j(y) R_m(y, \eta') = \partial_{\eta_j} \lambda_m(y, \eta'). \tag{3.4}$$

**Proof.** Differentiate the equation

$$0 = \left[ -\lambda_m(y, \eta') I + \sum_{j=0}^{n-1} B_j(y) \eta_j \right] R_m(y, \eta') = L(y, \eta', -\lambda_m) R_m$$

with respect to $\eta_j$ to obtain

$$\left[ -\partial_{\eta_j} \lambda_m(y, \eta') I + B_j(y) \eta_j \right] R_m(y, \eta') + L(y, \eta', -\lambda_m) \partial_{\eta_j} R_m(y, \eta') = 0.$$  

Apply $L_m(y, \eta')$ on the left to obtain (3.4). \hfill \square

The following immediate corollary is used in section 3.5.

**Corollary 3.4.** We have $l_m(y) B_j(y) r_m(y) = \partial_{\eta_j} \lambda_m(y, d'\psi_m)$.

We want to construct characteristic phases $\psi_k(y)$ satisfying the initial value problem near $y = 0$:

$$\begin{align*}
(a) \det L(y, d\psi_k) &= \det \left( \partial_n \psi_k I + \sum_{j=0}^{n-1} B_j(y) \partial_j \psi_k \right) = 0 \\
(b) \psi_k|_{y_n=0} &= y_0.
\end{align*} \tag{3.5}$$

We arrange for (3.5) to hold by solving the eikonal initial value problem for the unknown $\psi_k$:

$$\partial_n \psi_k = -\lambda_k(y, d'\psi_k), \ \psi_k|_{y_n=0} = y_0. \tag{3.6}$$

Since $L(y, \partial_y)$ has variable coefficients, the interior phases $\psi_k$ will be nonlinear.
The initial condition and the implicit function theorem imply that near $y = 0$ we can write

$$
\psi_k(y) = (y_0 - \phi_k(y'', y_n))\beta_k(y),
$$

for some smooth $\phi_k$ and $\beta_k$. Since

$$
\psi_k|_{y_n=0} = y_0 = (y_0 - \phi_k(y'', 0))\beta_k(y_0, y'', 0),
$$

we conclude $\phi_k(y'', 0) = 0$ by setting $y_0 = \phi_k(y'', 0)$ in (3.8), and thus (3.8) implies $\beta_k(y_0, y'', 0) = 1$. Using the positive homogeneity of $\lambda_k$ of degree one, we see that (3.6) implies $\lambda_k(y_0, y'', 0) = 1$.

\begin{equation}
\partial_n \phi_k(y'', y_n) = \lambda_k(\phi_k, y'', y_n, 1, -d_{y''}\phi_k), \quad \phi_k(y'', 0) = 0.
\end{equation}

**Remark 3.5.** We use the functions $\phi_k$ in section 3.6.2 to show that near $y = 0$ the characteristic surfaces $\Sigma_j = \{\psi_j = 0\}$ satisfy

$$
\Sigma_j \cap \Sigma_k = \Delta \text{ for } j \neq k, \quad \Sigma_k \cap \{x_n = 0\} = \Delta, \quad \Sigma_k \cap \{y_0 = 0\} = \Delta
$$

where all intersections are transversal. Observe that while the surface $x_0 - \phi_j = 0$ is characteristic, the surface $x_0 - \phi_j = c$ need not be characteristic for $c \neq 0$. The functions $\psi_j$, however, have the property that $\psi_j = c$ is characteristic for all $c$ near 0, a property that is needed in the construction of the approximate solution $u_a^r$.

**Remark 3.6 (Caution).** By Definition 3.2 for a given $k$ we have

$$
\mathbb{R}^N = \oplus_{m=1}^N \text{span } R_m(y, d\psi_k(y)),
$$

where $R_k(y, d\psi_k(y)) = r_k(y)$, but $R_m(y, d\psi_k(y)) \neq r_m(y)$ for $m \neq k$. However, (3.11) implies that at $y = (y', 0)$ we have

$$
\mathbb{R}^N = \oplus_{m=1}^N \text{span } r_m(y', 0),
$$

since $d\psi_k(y', 0) = d'y_0$ for all $k$. For $y$ near 0 (3.12) implies by continuity

$$
\mathbb{R}^N = \oplus_{m=1}^N \text{span } r_m(y),
$$

*but* the projection on span $r_k(y)$ in the decomposition (3.13) is *not* $\pi_k(y)$, unless $y = (y', 0)$. Similarly, while we have $I = \sum_k \pi_k(y', 0)$, this equality fails for general $y$ near 0. While $l_k(y', 0)r_j(y', 0) = \delta_{jk}$, this equality also fails for general $y$ near 0.

### 3.2 Preliminary computations

In this section we carry out some preliminary computations needed for the construction of the approximate solution $u_a^r$ to the system (1.4). We look for an approximate solution of the form

$$
u_a^r = U_0(y, \theta)|_{\theta = \omega} + \epsilon U_a^r(y), \quad \text{where } U_0(y, \theta) = \sum_{k=1}^N \sigma_k(y, \theta_k)r_k(y).$$

\[\text{To see this substitute } \partial_n \psi_k = (-\partial_n \phi_k)\beta_k + (y_0 - \phi_k)\partial_n \beta_k \text{ in (3.30) and set } y_0 = \phi_k.\]

\[\text{The projection } \pi_k(y) := \Pi_k(y, d\psi_k(y)) \text{ is, rather, the projection on the } k-\text{th summand in the decomposition (3.11).}\]
The $\sigma_k(y, \theta_k)$ are scalar functions with compact support in $\theta_k$ that will be determined by solving “transport equations”.

Below we let $\theta_0$ and $\xi_n$ be placeholders for $\frac{\psi_0}{\epsilon}$ and $\frac{\psi_n}{\epsilon}$ respectively. The “corrector” $U_1^\eps$ has the form

$$U_1^\eps(y) = U_1^\eps(\theta, \theta_0, \xi_n)|_{\theta=\Psi_0, \theta_0=\frac{\psi_0}{\epsilon}, \xi_n=\frac{\psi_n}{\epsilon}},$$

with

$$U_1^\eps(y, \theta, \theta_0, \xi_n) := V(y, \theta) + W^\eps(y, \theta_0, \xi_n),$$

$$V(y, \theta) = \sum_{k=1}^N V_k(y, \theta_k)$$

and

$$W^\eps(y, \theta_0, \xi_n) = \chi^\eps(y_0, y) W(y, \theta_0, \xi_n) \text{ with } W(y, \theta_0, \xi_n) = \sum_{k=1}^N t_k(y, \theta_0, \xi_n) r_k(y', 0)$$

for a cutoff function $\chi^\eps$, vector functions $V_k$, and scalar functions $t_k$ to be constructed.

We want $L(y, \partial_y) u^\eps_a - f(y, u_0^\eps)$ to be “small”\(^{23}\) To compute $L(y, \partial_y) u^\eps_a$ we begin by computing (set $B_n(y) = I$) for a fixed $k \in \{1, \ldots, N\}$:

$$B_j \partial_j[\sigma_k(y, \frac{\psi_k(y)}{\epsilon}) r_k(y)] = \left[ \frac{1}{\epsilon} (\partial_j \psi_k(y)(\partial \theta_k \sigma_k(y, \theta_k)) B_j r_k + (\partial_j \sigma_k) B_j r_k + \sigma_k B_j \partial_j r_k \right]_{\theta=\Psi_0},$$

so

$$L(y, \partial_y)[\sigma_k(y, \frac{\psi_k(y)}{\epsilon}) r_k(y)] := \left[ \frac{1}{\epsilon} L(y, d\psi_k)(\partial \theta_k \sigma_k) r_k + \sum_{j=0}^n (\partial_j \sigma_k) B_j r_k + \sum_{j=0}^n \sigma_k B_j \partial_j r_k \right]_{\theta=\Psi_0} =$$

$$\left[ \frac{1}{\epsilon} L(y, d\psi_k)(\partial \theta_k \sigma_k) r_k + (L(y, \partial_y) \sigma_k) r_k + \sigma_k L(y, \partial_y) r_k \right]_{\theta=\Psi_0}.$$  

Define (with $\psi_0(y') = y_0$)

$$\mathcal{L}_1(y, d\psi, \partial_\theta) = \sum_{k=1}^N L(y, d\psi_k) \partial \theta_k \text{ and } \mathcal{L}_2(y, d\psi_0, \partial \theta_0, \xi_n) = L(y, d\psi_0) \partial \theta_0 + \partial \xi_n.$$  

and observe that (3.17) implies in view of the polarization of $U_0$:

$$L(y, \partial_y) U_0(y, \frac{d\psi}{\epsilon}) = \left[ \frac{1}{\epsilon} \mathcal{L}_1(y, d\psi, \partial_\theta) U_0(y, \theta) + L(y, \partial_y) U_0(y, \theta) \right]_{\theta=\Psi_0} =$$

$$\left[ \frac{1}{\epsilon} \mathcal{L}_1(y, d\psi, \partial_\theta) U_0(y, \theta) + \sum_{k=1}^N (L(y, \partial_y) \sigma_k(y, \theta_k)) r_k(y) + \sum_{k=1}^3 \sigma_k(y, \theta_k) L(y, \partial_y) r_k(y) \right]_{\theta=\Psi_0} =$$

$$\left[ \sum_{k=1}^N (L(y, \partial_y) \sigma_k(y, \theta_k)) r_k(y) + \sum_{k=1}^N \sigma_k(y, \theta_k) L(y, \partial_y) r_k(y) \right]_{\theta=\Psi_0}.$$  

\(^{23}\)This means $o(1)$ in $L^\infty \cap N^m(U)$. 

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Similarly we have

\begin{equation}
(3.20) \\
L(y, \partial_y)V(y, \frac{\psi}{\epsilon}) = \left[ \frac{1}{\epsilon}L_1(y, d\psi, \partial_\theta)V(y, \theta) + L(y, \partial_y)V(y, \theta) \right] \bigg|_{\theta = \frac{\psi}{\epsilon}} \\
L(y, \partial_y)W^\epsilon(y, \frac{\psi_0}{\epsilon}, \frac{y_0}{\epsilon}) = \left[ \frac{1}{\epsilon}L_2(y, d\psi_0, \partial_{\theta_0}, \xi_n)W^\epsilon(y, \theta_0, \xi_n) + L(y, \partial_y)W^\epsilon(y, \theta_0, \xi_n) \right] \bigg|_{\theta_0 = \frac{\psi_0}{\epsilon}, \xi_n = \frac{\xi_n}{\epsilon}}.
\end{equation}

We can expand\(^{24}\)

\begin{equation}
(3.21) \\
f(y, u^\epsilon_a) = \left[ f(y, U_0(y, \theta)) + \epsilon K(y, U_0, \partial U^\epsilon_1) \right] \bigg|_{\theta = \frac{\psi}{\epsilon}, \theta_0 = \frac{\psi_0}{\epsilon}, \xi_n = \frac{\xi_n}{\epsilon}}.
\end{equation}

Thus, we have

\begin{equation}
(3.22) \\
\mathcal{E}_0^\epsilon = 0, \\
\mathcal{E}_0^\epsilon = \sum_{k=1}^N (L(y, \partial_y)\sigma_k) r_k + \sum_{k=1}^N \sigma_k L(y, \partial_y) r_k - f(y, U_0) + \mathcal{L}_1(y, d\psi, \partial_\theta)V + \mathcal{L}_2(y, d\psi_0, \partial_{\theta_0}, \xi_n)W^\epsilon \bigg|_{\theta = \frac{\psi}{\epsilon}, \theta_0 = \frac{\psi_0}{\epsilon}, \xi_n = \frac{\xi_n}{\epsilon}} \\
\mathcal{E}_1^\epsilon = -K(y, U_0, \partial U^\epsilon_1) \partial U^\epsilon_1 + L(y, \partial_y)V + L(y, \partial_y)W^\epsilon \bigg|_{\theta = \frac{\psi}{\epsilon}, \theta_0 = \frac{\psi_0}{\epsilon}, \xi_n = \frac{\xi_n}{\epsilon}}.
\end{equation}

Our next task is to choose \(U_0, V\), and \(W^\epsilon\) to make \(\mathcal{E}_0^\epsilon\) as small as possible.

### 3.3 Temporary choice of a quadratic nonlinearity

We now make a preliminary choice of \(f(y, u)\) as an at most quadratic function of \(u\) in order to make all pulse interactions readily visible in the construction of the corrector. We show in section \(^5\) that everything generalizes to the case where \(f\) is any smooth function as in Theorem \(^1.2\).

Any function \(u : \mathbb{R}^{1+n} \rightarrow \mathbb{R}^N\) can be written

\[ u(y) = \sum_k s_k(y) r_k(y) \]

for \(r_k\) as in Definition \(3.2\) and some scalar functions \(s_k\). Set \(s = (s_1, \ldots, s_N)\). We take \(f\) for now to have only linear and a quadratic parts with respect to \(s_k\):

\begin{equation}
(3.23) \\
f(y, u) := f^l(y, s) + f^q(y, s), \text{ where} \\
f^l(y, s) = \sum_k f^l_k(y, s) r_k = \sum_k \left( \sum_m f^l_{km}(y) s_m \right) r_k \\
f^q(y, s) = \sum_k f^q_k(y, s) r_k = \sum_k \left( \sum_{m \leq p} f^q_{kmp}(y) s_m s_p \right) r_k.
\end{equation}

In the case where \(u\) is a function of \((y, \theta)\), the same definition of \(f(y, u)\) applies with \(s_k = s_k(y, \theta)\).

\(^{24}\)Here \(K(y, U_0, \partial U^\epsilon_1) = \int_0^1 f_u(y, U_0 + s\partial U^\epsilon_1) ds\).
3.4 Definition of the projection operator $E$

In this section we define a projection operator $E = P \circ S$ (or $PS$) that we will use as follows in our effort to make $\mathcal{E}_0'$ small. We first write (with $U_0 = U_0(y, \theta)$):

$$
\mathcal{F}(y, U_0, \partial_y U_0) := L(y, \partial_y)U_0 - f(y, U_0) = 
\sum_{k=1}^{N} (L(y, \partial_y)\sigma_k)r_k + \sum_{k=1}^{N} \sigma_k L(y, \partial_y)r_k - f(y, U_0) = EF + (1 - E)F.
$$

(3.24)

We will construct $U_0$ to satisfy

$$
EF = PSF = 0.
$$

(3.25)

Next we write

$$
(1 - E)F = H_1 + H_2,
$$

(3.26)

where $H_1 = (1 - E)SF = (1 - P)SF$ and $H_2 = (1 - E)(1 - S)F = (1 - S)F$.

We will construct $V$ and $W$ to satisfy

$$
(a)\mathcal{L}_1(y, d\psi, \partial_0)V + H_1 = 0
\quad (b)\mathcal{L}_2(y', 0, d\psi_0, \partial_{\theta_0, \xi_n})W + H_2 = 0,
$$

(3.27)

where $H_2(y, \theta_0, \xi_n)$ is a slightly modified version of $H_2(y, \theta)$ specified later; see (3.62).

Remark 3.7. We use the operator $\mathcal{L}_{2,0} := \mathcal{L}_2(y', 0, d\psi_0, \partial_{\theta_0, \xi_n})$ in (3.27) (b) rather than $\mathcal{L}_2 = \mathcal{L}_2(y, d\psi_0, \partial_{\theta_0, \xi_n})$, because $\mathcal{L}_{2,0}$ is diagonalizable by the matrix $(r_1(y', 0) \cdots r_N(y', 0))$, while we do not see how to diagonalize $\mathcal{L}_2$; see (3.74). This property of the $r_k(y', 0)$ also explains their use, rather than the $r_k(y)$, in (3.74).

The use of $\mathcal{L}_{2,0}$ here introduces an error (see (3.86)) that is estimated in section 4.4.1.

To define $E = PS$ we first define the action of $S$ on $f(y, U_0)$ using (3.25), except that now we write $\sigma = (\sigma_1, \ldots, \sigma_N)$ instead of $s = (s_1, \ldots, s_N)$. It is convenient to introduce the following notation:

**Definition 3.8.** If $f(y, \sigma)$ is any element of $C^\infty(\mathbb{R}_+^{1+n} \times \mathbb{R}^N, \mathbb{R})$ (or $C^\infty(\mathbb{R}_+^{1+n} \times \mathbb{R}^N, \mathbb{R})$), set $f^k(y, \sigma_k) := f(y, \sigma_k e_k)$, where $e_k$ is the $k$–th standard basis vector of $\mathbb{R}^N$. When $f$ is either $f^l$ or $f^q$, we write $f^{l,k}$ or $f^{q,k}$ for $f^k$.

Roughly, $S$ selects the part of $f(y, U_0)$ that can be written as a sum of terms involving just one of the $\sigma_k$. We set

$$
Sf(y, U_0) := \sum_{k=1}^{N} f^{l,k}(y, \sigma_k) + \sum_{k=1}^{N} f^{q,k}(y, \sigma_k) = f^l(y, \sigma) + \sum_{k=1}^{N} f^{q,k}(y, \sigma_k).
$$

(3.28)

\[\text{A belated motivation for the decomposition of } \mathcal{F} \text{ given by (3.25), (3.26) is given in Remark 3.20.}\]

\[\text{See Remark 3.30.}\]
The operator $P$ then polarizes these terms by applying the projection $\pi_k(y)$ (as in Definition 3.2(b)) to any term that depends only on $\sigma_k$. Thus, we obtain

\[Ef(y, U_0) = PSf(y, U_0) = \sum_{k=1}^{N} \pi_k(y) [f^{l,k}(y, \sigma_k) + f^{a,k}(y, \sigma_k)].\]  

(3.29)

The other terms in $F$, namely

\[\sum_{k=1}^{N} (L(y, \partial_y) \sigma_k) r_k := \ell_1(y, \partial_y U_0) \quad \text{and} \quad \sum_{k=1}^{N} \sigma_k L(y, \partial_y) r_k := \ell_2(y, U_0),\]

(3.30)

are linear functions of $\sigma_y$ and $\sigma$ respectively. Parallel to (3.29) we define

\[E\ell_1(y, \partial_y U_0) := \sum_{k=1}^{N} \pi_k(y) [(L(y, \partial_y) \sigma_k(y, \theta_k)) r_k(y)] \]

(3.31)

\[E\ell_2(y, U_0) \sum_{k=1}^{N} \pi_k(y) [\sigma_k(y, \theta_k) L(y, \partial_y) r_k(y)].\]

Remark 3.9. (a) From (3.25) and (3.27) we see that the choice of $U_0$ makes the polarized part of $SF$ vanish, and $V$ “solves away” the unpolarized part of $SF$, namely $H_1 = (1 - P)SF$. Similarly, $W$ solves away “most of” the multiphase part of $F$, namely $H_2 = (1 - S)F$.

(b) The operators $L_i$, $i = 1, 2$ in (3.27) are singular. The equation (3.25) both helps to determine the $\sigma_k$ and acts as a solvability condition that will allow us to solve the equations (3.27) using bounded profiles $V$ and $W$; see (3.41).

3.5 Construction of the leading profile $U_0(y, \theta)$

We now construct $U_0(y, \theta)$ to satisfy $EF = 0$ (3.25) with a boundary condition corresponding to

\[BU_0(y, \frac{\psi}{c})|_{y_n=0} = g(y', \frac{y_0}{c}) \quad \text{(recall } \psi_k|_{y_n=0} = y_0).\]

(3.32)

That is, we want $U_0(y, \theta)$ to satisfy

\[(a) E [L(y, \partial_y) U_0 - f(y, U_0)] = 0 \text{ in } y_n > 0,\]

(3.33)

\[(b) BU_0(y', 0, \theta_0, \ldots, \theta_0) = g(y', \theta_0)\]

\[(c) U_0 = 0 \text{ in } y_0 < 0.\]
To solve (3.33) we first use (3.29) to compute

\[ EL(y, \partial_y)U_0 = E \left( \sum_{k=1}^{N} (L(y, \partial_y)\sigma_k(y, \theta_k)) r_k(y) + \sum_{k=1}^{N} \sigma_k(y, \theta_k) L(y, \partial_y) r_k(y) \right) = \]

\[ \sum_{k=1}^{N} \pi_k(y) [(L(y, \partial_y)\sigma_k(y, \theta_k)) r_k(y)] + \sum_{k=1}^{N} \pi_k(y) [\sigma_k(y, \theta_k) L(y, \partial_y) r_k(y)] = \]

(3.34)

\[ \sum_{k=1}^{N} l_k(y) \left( \sum_{j=0}^{n-1} (\partial_j \sigma_k) B_j(y) r_k(y) \right) r_k(y) + \sum_{k=1}^{N} l_k(y) [\sigma_k L(y, \partial_y) r_k] r_k(y) = \]

\[ \sum_{k=1}^{N} [X_k(y, \partial_y) \sigma_k + c_k(y) \sigma_k] r_k(y). \]

In the last line \( X_k(y, \partial_y) \), the characteristic vector field associated to the phase \( \psi_k \), and the coefficients \( c_k(y) \) are defined by

(3.35)

\[ X_k(y, \partial_y) = \partial_n + \sum_{j=0}^{n-1} \partial_n \lambda_k(y, d' \psi_k) \partial_j \]

\[ c_k(y) = l_k(y) L(y, \partial_y) r_k(y). \]

We have used Corollary 3.4 here to obtain \( X_k \).

**Remark 3.10** (Incoming vs. outgoing). (a) A phase \( \psi_k \) is said to be incoming (resp., outgoing) when the group velocity described by the corresponding vector field \( X_k \) is incoming (resp., outgoing). We say \( X_k \) (or its group velocity) is incoming (resp. outgoing) when the coefficients of \( \partial_n \) and \( \partial_0 = \partial_t \) in \( X_k \) have the same (resp. opposite) signs. Corollary 3.4 implies that \( p \) is equal to the number of positive eigenvalues of \( A_n \) in (1.4). Relabeling if necessary, we take the vector fields \( X_k, k = 1, \ldots, p \) to be incoming and the others to be outgoing.

(b) Observe that \( X_k(y, \partial_y) \) is tangent to the characteristic surfaces \( \psi_k(y) = c \).  

Next we use (3.29) to compute

(3.36)

\[ Ef(y, U_0) = \sum_{k=1}^{N} l_k(y) \left[ f^{l,k}(y, \sigma_k) + f^{q,k}(y, \sigma_k) \right] r_k := \sum_{k=1}^{N} [d_k(y) \sigma_k + e_k(y) \sigma_k^2] r_k. \]

With (3.34) and (3.36) we see that (3.33)(a) is equivalent to

(3.37)

\[ X_k(y, \partial_y) \sigma_k + c_k(y) \sigma_k - (d_k(y) \sigma_k + e_k(y) \sigma_k^2) = 0, \quad k = 1, \ldots, N. \]

The boundary condition (3.33)(b) is satisfied provided

\[ BU_0(y', 0, \theta_0, \ldots, \theta_0) = g(y', \theta_0) \iff \sum_{k=1}^{N} \sigma_k Br_k = g \iff \]

(3.38)

\[ (Br_1 \quad \ldots \quad Br_p) \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_p \end{pmatrix} = g - \sum_{k=p+1}^{N} \sigma_k Br_k. \]

^27 Use the eikonal equation and the Euler identity for positively homogeneous functions to see this.
We claim \( \sigma_k(y, \theta_k) = 0 \) for \( k \geq p + 1 \). This follows from \( \mathbf{3.37} \) since \( X_k(y, \partial_y) \) is outgoing for such \( k \) and \( \sigma_k = 0 \) in \( y_0 < 0 \). The uniform Lopatinski condition implies \( (Br_1 \ldots Br_p) \) is invertible, so \( \mathbf{3.38} \) determines explicit functions \( \sigma^1(y', \theta_0), \ldots, \sigma^p(y', \theta_0) \) supported in \( y_0 \geq 0 \) as boundary values of \( \sigma_1, \ldots, \sigma_p \). Except for special choices of \( g(y', \theta_0) \), all these boundary values will be nonzero.

Summarizing, we have shown that outgoing profiles \( \sigma_k = 0 \) for \( k \geq p + 1 \), and that incoming profiles \( \sigma_1, \ldots, \sigma_p \) are determined by solving:

\[
X_k(y, \partial_y)\sigma_k + c_k(y)\sigma_k - (d_k\sigma_k + e_k\sigma_k^2) = 0
\]

\[
\sigma_k(y', 0, \theta_k) = \sigma^*_k(y', \theta_k) \quad \text{for} \quad k = 1, \ldots, p.
\]

This problem, in which \( \theta_k \) is just a parameter, has \( C^\infty \) solutions on a fixed time interval, so we now have \( U_0(y, \theta) \) defined and satisfying \( \mathbf{3.33} \) for \( y \) in some open set \( U \ni 0 \).

3.6 Construction of the corrector \( \epsilon U_1^\epsilon(y) \).

We begin by constructing the part of the corrector \( U_1^\epsilon \) determined by \( V(y, \theta) = \sum_{k=1}^N V_k(y, \theta_k) \) as in \( \mathbf{3.16} \).

3.6.1 The noninteraction term \( V(y, \theta) \).

The construction of each \( V_k \) is essentially the same as the construction of the corrector in \( \mathbf{AR02} \) for the one phase case. Recall (Definition 3.22) that for \( \eta' \) near \( (1, 0, \ldots, 0) \) the matrix \( A(y, \eta') \) has \( N \) eigenvalues \( \lambda_k(y, \eta') \), and corresponding left and right eigenvectors \( L_k(y, \eta'), R_k(y, \eta') \) such that \( L_j R_k = \delta_{jk} \). We have denoted by \( \Pi_k(y, \eta') \) the projection on span \( R_k(y, \eta') \) in the decomposition

\[
\mathbb{R}^N = \bigoplus_{m=1}^N \text{span} \ R_m(y, \eta').
\]

Thus, for \( x \in \mathbb{R}^N \) we have \( \Pi_k(y, \eta') x = (L_k(y, \eta') x) R_k(y, \eta') \).

We want \( V(y, \theta) \) to satisfy

\[
\mathcal{L}_1(y, d\psi, \partial_\theta)V = -H_1,
\]

for \( H_1 \) as in \( \mathbf{3.26} \) and \( \mathcal{L}_1(y, d\psi, \partial_\theta) = \sum_{k=1}^N L(y, d\psi_k) \partial_{\theta_k} \).

First we define a partial inverse \( Q_k(y) \) for each matrix \( L(y, d\psi_k) \) such that

\[
L(y, d\psi_k)Q_k(y) = Q_k(y)L(y, \psi_k) = 1 - \Pi_k(y, d'\psi_k) = 1 - \pi_k(y).
\]

For each \( k \) we have

\[
A(y, \eta') = \sum_{j=1}^N \lambda_j(y, \eta') \Pi_j(y, \eta') \Rightarrow A(y, d'\psi_k) = \sum_{j=1}^N \lambda_j(y, d'\psi_k) \Pi_j(y, d'\psi_k),
\]

so using the eikonal equation \( \mathbf{3.6} \) we obtain

\[
L(y, d\psi_k) = \partial_n \psi_k I + A(y, d'\psi_k) = \sum_{j=1}^N \left( \lambda_j(y, d'\psi_k) - \lambda_k(y, d'\psi_k) \right) \Pi_j(y, d'\psi_k).
\]

\[\text{The method of characteristics can be used to solve } \mathbf{3.39}.\]
Thus,
\begin{equation}
Q_k(y) := \sum_{j \neq k} \left( \lambda_j(y, d' \psi_k) - \lambda_k(y, d' \psi_k) \right)^{-1} \Pi_j(y, d' \psi_k)
\end{equation}
satisfies (3.41).

Since $H_1(y, \theta) = (1 - P)SF$, it has the form
\begin{equation}
H_1(y, \theta) = \sum_{k=1}^{N} H_{1k}(y, \theta_k).
\end{equation}

With $f^{l,k}$ and $f^{q,k}$ as in Definition 3.8 we have
\begin{equation}
H_{1k}(y, \theta_k) = (1 - \pi_k(y)) \left[ (L(y, \partial_y)\sigma_k) r_k + \sigma_k L(y, \partial_y) r_k - \left( f^{l,k}(y, \sigma_k) + f^{q,k}(y, \sigma_k) \right) \right].
\end{equation}

Using this and (3.41), we see that if we define
\begin{equation}
V_k(y, \theta_k) := - \int_{-\infty}^{\theta_k} Q_k(y) H_{1k}(y, s) ds,
\end{equation}
then $V(y, \theta) = \sum_k V_k(y, \theta_k)$ satisfies (3.40).

**Remark 3.11.** (a) For each $k$, $H_{1k}(y, \theta_k)$ is a finite sum of terms of the form
\begin{equation}
(\partial_y \sigma_k(y, \theta_k)) C(y), \quad \sigma_k(y, \theta_k) C(y), \quad \text{or} \quad \sigma_k^2(y, \theta_k) C(y),
\end{equation}
where $C(y)$ is an $\mathbb{R}^N$-valued $C^{\infty}$ function such that $\pi_k(y) C(y) = 0$, and which varies from term to term.

(b) Since each profile $\sigma_k$ has compact support in $\theta_k$, the integral in (3.47) is finite. The functions $V_k(y, \theta_k)$ vanish in $\theta_k < -M$ for $M > 0$ large enough, but have nonzero finite limits as $\theta_k \to +\infty$.

### 3.6.2 Choice of $\Omega^0$.

The remaining part of the construction of $u_\epsilon^n$ and the error analysis will take place within a small neighborhood, and in particular bounded, connected open neighborhood of $0$, $\Omega^0 \subset \{ y : y_n \geq 0 \}$, that is independent of $\epsilon$. Here we specify the choice of $\Omega^0$. First we require that the structural assumptions (Assumptions 2.1, 2.2, 2.3) all hold on $\Omega^0$. We also require that the solutions $\psi_k$ to the eikonal equation exist on $\Omega^0$ and satisfy (3.6)-(3.9) there. Moreover, we require
\begin{equation}
\mathbb{R}^N = \oplus_{k=1}^{N} \text{span} r_k(y) = \oplus_{k=1}^{N} \text{span} r_k(y', 0) \text{ for } y \in \Omega^0.
\end{equation}

The next lemma collects some obvious relations satisfied by the $\psi_j$, $\phi_j$ and needed below.

**Lemma 3.12 (Relations between phases).** Let $\omega_k(y') := \partial_n \psi_k(y', 0)$. For $y \in \Omega^0$ we have for all $k$:
\begin{align}
(a)\psi_k(y) &= (y_0 - \phi_k(y'', y_n)) \beta_k(y) \quad \text{where} \quad \phi_k(y'', 0) = 0 \ \text{and} \ \beta_k(y', 0) = 1
\end{align}
\begin{align}
(b)\psi_k(y) &= y_0 - \phi_k(y'', y_n) + O(y_0 y_n) + O(y_n^2)
\end{align}
\begin{align}
(c)\omega_k(y') y_n &= -\phi_k(y'', y_n) + O(y_0 y_n) + O(y_n^2) = -\partial_n \phi(y'', 0) y_n + O(y_0 y_n) + O(y_n^2)
\end{align}
\begin{align}
(d)\phi_k(y'', y_n) &= \partial_n \phi_k(y'', 0) y_n + O(y_n^2).
\end{align}

\text{29Observe that a vector $Y$ is in the range of $L(y, d\psi_k)$ if and only if $\pi_k(y) Y = 0$.}
Proof. For (a) recall (3.7) and the discussion just after it. For (b) write $\beta_k(y) = 1 + O(y_n)$ and use part (a). For (c) write
\begin{equation}
\psi_k(y) = \psi_k(y', 0) + \partial_n \psi_k(y', 0)y_n + O(y_n^2) = y_0 + \omega_k(y')y_n + O(y_n^2)
\end{equation}
and use part (b). Part (d) is immediate.

Let $\gamma_k, k = 1, \ldots, N$ denote the distinct real eigenvalues of $B_0(0) = A(0, 1, 0) \ (1 = \eta_0$ here). They are nonzero because $y_n = 0$ is noncharacteristic. We have
\begin{equation}
\partial_n \phi_k(0) = \lambda_k(0, 1, 0) = \gamma_k,
\end{equation}
so if $\Omega^0$ is small enough, the characteristic surfaces
\begin{equation}
\Sigma_j = \{y : y_0 = \phi_j(y'', y_n)\}
\end{equation}
will be disjoint on $y_n > 0$. To see this, first observe that we can choose $\Omega^0$ so that there exist positive constants $\delta, \kappa$ such that on $\Omega^0$:
\begin{equation}
(a) \partial_{y''} \phi_k < \kappa \text{ and } |\partial_n \phi_k - \gamma_k| < \frac{\delta}{3}, \text{ where } \delta < \frac{1}{4} \inf_{k \neq j} |\gamma_j - \gamma_k|
\end{equation}
\begin{equation}
(b) |\lambda_k(y, 1, \eta'') - \gamma_k| < \frac{\delta}{3} \text{ for } |\eta''| < \kappa \eta_0 = \kappa \cdot 1.
\end{equation}
In view of Lemma 3.12 we can choose $\Omega^0$ so that additionally:
\begin{equation}
(c) |\phi_k(y'', y_n) - \gamma_k y_n| < \frac{\delta}{2} y_n
\end{equation}
\begin{equation}
(d) | - \omega_k(y') y_n - \gamma_k y_n| < \frac{\delta}{2} y_n.
\end{equation}
The disjointness of the surfaces $\Sigma_j$ in $\Omega^0$ is obvious from (3.55)(c) and the choice of $\delta$. We now complete the specification of $\Omega^0$ by shrinking it if necessary so that both $U_0(y, \theta)$ and $V(y, \theta)$ lie in $C^\infty_b(\Omega^0 \times \mathbb{R}_0^N)$.

We conclude this section by describing a partition of unity subordinate to a cover of $y_n \geq 0$ by wedges that we need for the error analysis. For $\delta$ as above let $I_j = (\gamma_j - \delta, \gamma_j + \delta)$. There exist functions $\tilde{\chi}_j \in C^\infty_c(\mathbb{R}), j = 1, \ldots, N$ and $\tilde{\chi}_0 \in C^\infty(\mathbb{R})$ such that
\begin{equation}
\sum_{j=0}^N \tilde{\chi}_j(t) = 1 \text{ and } \tilde{\chi}_j = 1 \text{ on } \bar{T}_j
\end{equation}
\begin{equation}
\text{supp } \tilde{\chi}_j \cap \bar{T}_k = \emptyset \text{ if } j \neq k,
\end{equation}

\[\text{The condition (3.55)(d) is a slight variant of (3.54)(a). It does not appear in \text{[Met89]} but is needed here for the error analysis.}\]

\[\text{A similar partition is used in \text{[Met89]}.}\]
Set $\chi_j(y) := \tilde{\chi}_j \left( \frac{y_0}{y_n} \right)$ and define wedges

$$W'_j := \{ y : \frac{y_0}{y_n} \in I_j, y_n > 0 \}, \ j = 1, \ldots, N$$

(3.57)

$$W_j := \{ y : \frac{y_0}{y_n} \in \text{supp} \, \chi_j, y_n > 0 \}, \ j = 0, 1, \ldots, N.$$  

(3.58)

So in $y_n > 0$ we have $\Sigma_j \subset W'_j \subset W_j$ for $j = 1, \ldots, N$.

The next definition is formulated in [Méta89].

**Definition 3.13.**

(a) Let $\mathcal{M}_0$ denote the space of vector fields on $\Omega^0$ with $C_b^\infty(\Omega^0, \mathbb{R})$ coefficients that are tangent to $\Delta = \{ y : y_0 = y_n = 0 \}$.

(b) Denote by $\Lambda(\Omega^0)$ the space of functions $a \in C^\infty(\Omega^0 \cap \{ y_n > 0 \})$ such that $a \in L^\infty(\Omega^0)$ and $M_1 M_2 \cdots M_n a \in L^\infty(\Omega^0)$ for all finite sequences $M_1, M_2, \ldots$ of vector fields in $\mathcal{M}_0$.

Observe that the functions $\chi_j$ defined above satisfy

$$\chi_j \in \Lambda(\Omega^0).$$

**3.6.3 The interaction term $W^\epsilon(y, \theta_0, \xi_n).$**

We now construct $W(\theta_0, \xi_n)$ and $W^\epsilon(\theta_0, \xi_n) = \chi^\epsilon(y_0, y_n)W$, where the cutoff $\chi^\epsilon$ is chosen below. We use these profiles to define the piece of the corrector denoted by $H^\epsilon$ in (1.33) as follows:

(3.59)

$$W^\epsilon(y) := W^\epsilon(y, \theta_0, \xi_n)|_{\theta_0 = \frac{\psi_0}{\epsilon}, \xi_n = \frac{\psi_n}{\epsilon}}.$$  

Recall from (3.27) that the profile $W$ is constructed to satisfy a problem of the form

$$L_2(y', 0, d\psi_0, \partial_{\theta_0}, \xi_n)W = -H_2.$$  

(3.60)

In order to define $H_2$ we first expand

(3.61)

$$\psi_k(y) = \psi_k(y', 0) + \partial_n \psi_k(y', 0) y_n + r_{\psi_k}(y) = y_0 + \omega_k(y') y_n + r_{\psi_k}(y),$$

where $\omega_k(y') := \partial_n \psi_k(y', 0)$ and $r_{\psi_k}(y) = O(y_n^2)$ near $y = 0$. Using $H_2(y, \theta) = H_2(y, \theta_1, \ldots, \theta_N)$ we set

$$H_2(\theta_0, \xi_n) := H_2(y_0, \theta_0 + \omega_1(y') \xi_n, \ldots, \theta_0 + \omega_N(y') \xi_n).$$

**Remark 3.14.** Although

$$H_2(y, \theta)|_{\theta = \frac{\psi_0}{\epsilon}} \neq H_2(y, \theta_0, \xi_n)|_{\theta_0 = \frac{\psi_0}{\epsilon}, \xi_n = \frac{\psi_n}{\epsilon}}$$

for all $y$, (3.61) implies that equality holds at $y_n = 0$.

To see why $H_2$ might be a useful substitute for $H_2$, we first write out $H_2$. For $F$ as in (3.24) we have $H_2(y, \theta) = (1 - S)F$ from (3.26). Thus, the parts of $F$ that are linear in $(\sigma, \sigma_y)$ or quadratic in a single $\sigma_k$ cancel out in $(1 - S)F$. Explicitly, we compute (with $f^{q,k}(y, \sigma_k)$ as in (3.46) and using (3.23)):

$$H_2 = (1 - S)F = L(y, \partial_y) U_0 - f(y, U_0) - \left[ L(y, \partial_y) U_0 - \left( f^l(y, \sigma) + \sum_k f^{q,k}(y, \sigma_k) \right) \right] =$$

$$- \left( f^q(y, \sigma) - \sum_k f^{q,k}(y, \sigma_k) \right) = - \sum_k \left( \sum_{m < p} f^q_{kmp}(y) \sigma_m \sigma_p \right) \epsilon_k.$$  

(3.63)
Remark 3.15. We can write \( H_2(y, \theta) = \sum_k H_{2,k}(y, \theta)r_k(y) \), where each \( H_{2,k}(y, \theta) \) is a finite sum of terms of the form

\[
\sigma_m(y, \theta_m)\sigma_p(y, \theta_p)c(y), \quad m \neq p
\]

with \( c(y) \) a smooth real-valued function that varies from term to term. Thus, \( H_2(y, \theta_0, \xi_n) = \sum_k H_{2,k}(y, \theta_0, \xi_n)r_k(y) \), where each \( H_{2,k} \) is a finite sum of terms of the form

\[
\sigma_m(y, \theta_0 + \omega_m(y')\xi_n)\sigma_p(y, \theta_0 + \omega_p(y')\xi_n)c(y), \quad m \neq p
\]

with \( c(y) \) a smooth real-valued function that varies from term to term.

Since \( g(y', \theta_0) \) has support in \( |\theta_0| \leq 1 \), the equations (3.39) imply that the \( \sigma_k(y, \theta_k) \) have support in \( |\theta_k| \leq 1 \). We show in Proposition 3.16 that this implies products like

\[
\sigma_m(y, \frac{\psi_m}{\epsilon})\sigma_p(y, \frac{\psi_p}{\epsilon}), \quad m \neq p
\]

are supported in a small neighborhood of \( \Delta \).

Proposition 3.16. Let \( \Omega^0 \) be as chosen in section 3.6.2. There exists \( M > 0 \) such that the following statements hold on \( \Omega^0 \):

(a) In \( y_n \geq M\epsilon \) we have

\[
\text{supp } \sigma_m \left( y, \frac{\psi_m}{\epsilon} \right) \cap \text{supp } \sigma_p \left( y, \frac{\psi_p}{\epsilon} \right) = \emptyset, \quad \text{when } m \neq p.
\]

(b) All products

\[
\sigma_m \left( y, \frac{\psi_m}{\epsilon} \right) \sigma_p \left( y, \frac{\psi_p}{\epsilon} \right), \quad m \neq p,
\]

and thus \( H_2 \left( y, \frac{\psi}{\epsilon} \right) \), are supported in the region \( I^\epsilon := \{ y : |y_n| \leq M\epsilon, |y_0| \leq M\epsilon \} \). The same applies to the products

\[
\sigma_m \left( y, \frac{y_0 + \omega_m(y')y_n}{\epsilon} \right) \sigma_p \left( y, \frac{y_0 + \omega_p(y')y_n}{\epsilon} \right), \quad m \neq p
\]

and to \( H_2(y, \frac{\omega m}{\epsilon}, \frac{\omega m}{\epsilon}) \).

(c) In \( y_n \leq M\epsilon \) we have for each \( m \)

\[
\left| \sigma_m \left( y, \frac{\psi}{\epsilon} \right) - \sigma_m \left( y, \frac{y_0 + \omega_m(y')y_n}{\epsilon} \right) \right| \lesssim \epsilon.
\]

(d) In \( y_n \leq M\epsilon \) we have

\[
\left| H_2 \left( y, \frac{\psi}{\epsilon} \right) - H_2 \left( y, \frac{y_0}{\epsilon}, \frac{y_n}{\epsilon} \right) \right| \lesssim \epsilon.
\]

Proof. a. Recall that \( \psi_k(y) = (y_0 - \phi_k(y', y_n))\beta_k(y) \), where \( \beta_k \) is smooth and positive near \( y = 0 \). Since \( |\psi_k| \lesssim \epsilon \) on \( \text{supp } \sigma_k \left( y, \frac{\psi_k}{\epsilon} \right) \) for each \( k \), we have

\[
|y_0 - \phi_k(y'', y_n)| \lesssim \epsilon \text{ on } \text{supp } \sigma_k \left( y, \frac{\psi_k}{\epsilon} \right) \text{ for each } k.
\]
By (3.55)(c) we can write
\[(3.70) \quad \left(\gamma_k - \frac{\delta}{2}\right) y_n < \phi_k < \left(\gamma_k + \frac{\delta}{2}\right) y_n \text{ for all } k \Rightarrow |\phi_m - \phi_p| \geq 5\delta y_n \text{ for } m \neq p,\]
and this implies
\[(3.71) \quad |\phi_m(y) - \phi_p(y)| \geq 5\delta M \epsilon \text{ for } y_n \geq M \epsilon.\]

If \(y \in \text{supp} \sigma_m \left(y, \frac{\omega_m}{\epsilon}\right) \cap \text{supp} \sigma_p \left(y, \frac{\omega_p}{\epsilon}\right)\) and \(y_n \geq M \epsilon\) for a large enough choice of \(M\), then (3.71) contradicts the fact that \(y\) satisfies (3.69) for \(m \neq p\).

b. From (3.69) and (3.70) we see that for all \(k\) we have:
\[|\phi_m(y) - \phi_p(y)| \geq 5\delta M \epsilon \text{ for } y_n \geq M \epsilon.\]

Using (3.61) we have
\[(3.72) \quad |\sigma_m \left(y, \frac{\psi_m}{\epsilon}\right) - \sigma_m \left(y, \frac{\psi_0 + \omega_m(y') y_n}{\epsilon}\right)| = \partial \theta \sigma_m \left(y, \frac{\psi_0 + \omega_m(y') y_n}{\epsilon}\right) \frac{r_{\psi_m}(y)}{\epsilon} + O \left(\frac{r_{\psi_m}(y)}{\epsilon}\right)^2.\]

This implies (3.67) since \(|r_{\psi_m}(y)| = O(|y_n|^2) \leq \epsilon^2\) in \(y_n \leq \epsilon\).

d. Part (d) follows immediately from Remark 3.15 and (3.67).

**Definition 3.17.** In view of Proposition 3.16 (b) we refer to \(I^\epsilon := \{y : |y_n| \leq M \epsilon, |y_0| \leq M \epsilon\}\) as the interaction region.

Proposition 3.16 (d) indicates that \(H_2\) may be an acceptable substitute for \(H_2\) in the construction of \(W\). We can obtain a solution of
\[(3.73) \quad \mathcal{L}_2(y', 0, d\psi_0, \partial \theta_{0} \xi_n) W = -\mathcal{H}_2,\]
for \(H_2\) as in (3.62) as follows. First write
\[(3.74) \quad \mathcal{H}_2(y, \theta_0, \xi_n) = \sum_k \mathcal{H}_2^k(y, \theta_0, \xi_n) r_k(y', 0).\]

Define \(W\), formally at first, by
\[(3.75) \quad W(y, \theta_0, \xi_n) = \sum_k t_k(y, \theta_0, \xi_n) r_k(y', 0), \text{ where } t_k(y, \theta_0, \xi_n) = -\int_{+\infty}^{\xi_n} \mathcal{H}_2^k(y, \theta_0 + \omega_k(y') (\xi_n - s), s) \, ds.\]

\(^{32}\)In this proof the constant \(M\) may increase from part to part.

\(^{33}\)Recall (3.49) and Remark 3.7
Since
\begin{equation}
\mathcal{L}_2(y', 0, d\psi_0, \partial_0, \xi_n) = \partial_{\xi_n} + \mathcal{A}(y', 0, d'\psi_0)\partial_0 \quad \text{and} \quad \mathcal{A}(y', 0, d'\psi_0)\partial_k(y', 0) = -\omega_k(y')r_k(y', 0),
\end{equation}
we see that for any \(k\)
\begin{equation}
\mathcal{L}_2(y', 0, d\psi_0, \partial_0, \xi_n) t_k(y, \theta_0, \xi_n) r_k(y', 0) = [\partial_{\xi_n} - \omega_k(y')\partial_0] t_k r_k(y', 0).
\end{equation}
Thus, \(W\) as in \((3.75)\) is by inspection a formal solution of \((3.73)\).

It remains to examine the integral in \((3.75)\). Although \(\mathcal{H}_2^k \neq \mathcal{H}_{2,k}\) for \(\mathcal{H}_{2,k}\) as in Remark 3.15 it is still true that \(\mathcal{H}_2^k\) is a finite sum of terms of the form \((3.65)\); the only difference is that the smooth coefficients \(c(y)\) change. For a particular \(k\) the contribution of a term \((3.65)\) to the integral in \((3.75)\) is
\begin{equation}
c(y) \int_{-\infty}^{\infty} \sigma_m (y, \theta_0 + \omega_k(y')\xi_n + s(\omega_m(y') - \omega_k(y'))) \sigma_p (y, \theta_0 + \omega_k(y')\xi_n + s(\omega_p(y') - \omega_k(y'))) \, ds := A.
\end{equation}
At most one of \(m, p\) can equal \(k\) and the \(\sigma_i\) are bounded with compact support in \(\theta_i\). Suppose \(m \neq k\). Then
\begin{equation}
|A| \lesssim \int_{-\infty}^{\infty} |\sigma_m(y, \theta_0 + \omega_k\xi_n + s(\omega_m - \omega_k))| \, ds = \int_{-\infty}^{\infty} |\sigma_m(y, t)| \, \frac{dt}{|\omega_m - \omega_k|} \leq K,
\end{equation}
where \(K\) is independent of \((y, \theta_0, \xi_n)\). Here we used that \(|\omega_m(y') - \omega_k(y')| \geq 3\delta > 0\) for \(y'\) near 0.\(^{34}\) So \(W\) is indeed a bounded solution of \((3.73)\).

**Remark 3.18.** Similar estimates show that \(t_k(y, \theta_0, \xi_n)\) is \(C^\infty\) on \(\Omega^0 \times \mathbb{R}_0^2, \xi_n\). Although \(t_k\) is bounded on this set, that is not true of its derivatives. For example, applying \(\partial_{y'}\) to the integral in \((3.78)\) pulls out a factor of \((\partial_{y'}\omega_k)\xi_n\). The derivatives are bounded on bounded subsets of \(\Omega^0 \times \mathbb{R}_0^2, \xi_n\) though. That will be useful because factors like \(\frac{y_n}{\epsilon}\) are bounded uniformly with respect to \(\epsilon \in (0, \epsilon_0]\) on the interaction region \(I_\epsilon\).

Finally, we modify \(W\) to obtain \(W^\epsilon\) as follows.
\begin{equation}
W^\epsilon(y, \theta_0, \xi_n) := \chi \left( \frac{y_0}{\sqrt{\epsilon}} \right) \chi \left( \frac{y_n}{\sqrt{\epsilon}} \right) W(y, \theta_0, \xi_n) := \chi^\epsilon(y_0, y_n) W(y, \theta_0, \xi_n),
\end{equation}
where \(\chi(s) \in C^\infty(\mathbb{R})\) is 1 for \(|s| \leq 1\), 0 for \(|s| \geq 2\). This enforces \(|y_0| \lesssim \sqrt{\epsilon}, |y_n| \lesssim \sqrt{\epsilon}\) on \(\text{supp} \, W^\epsilon\).

Now
\begin{equation}
\mathcal{L}_2(y', 0, d\psi_0, \partial_0, \xi_n) W^\epsilon = -\chi \left( \frac{y_0}{\sqrt{\epsilon}} \right) \chi \left( \frac{y_n}{\sqrt{\epsilon}} \right) \mathcal{H}_2(y, \theta_0, \xi_n),
\end{equation}
but note that because of Proposition 3.16(b):
\begin{equation}
\left[ -\chi \left( \frac{y_0}{\sqrt{\epsilon}} \right) \chi \left( \frac{y_n}{\sqrt{\epsilon}} \right) \mathcal{H}_2(y, \theta_0, \xi_n) \right] |_{\theta_0 = \frac{m}{\epsilon}, \xi_n = \frac{m}{\epsilon}} = -\mathcal{H}_2(y, \theta_0, \xi_n) |_{\theta_0 = \frac{m}{\epsilon}, \xi_n = \frac{m}{\epsilon}}.
\end{equation}
for \(\epsilon_0\) small. Thus, we have
\begin{equation}
[\mathcal{L}_2(y', 0, d\psi_0, \partial_0, \xi_n) W^\epsilon] |_{\theta_0 = \frac{m}{\epsilon}, \xi_n = \frac{m}{\epsilon}} = [\mathcal{L}_2(y', 0, d\psi_0, \partial_0, \xi_n) W] |_{\theta_0 = \frac{m}{\epsilon}, \xi_n = \frac{m}{\epsilon}}.
\end{equation}\(^{34}\) Here we use the fact that for all \(k\) we have \(d'\psi_k(y', 0) = d'\psi_0(y') = (1, 0)\). Also recall \((3.2)\) and Definition \(3.2\)
\(^{35}\) See \((3.54)\) and \((3.55)\).
Remark 3.19. The cutoff $\chi^\epsilon(y_0,y_n)$ is introduced in (3.80) for later use in the error analysis. It is needed for the estimates of $W^\epsilon$ and the part of $\mathcal{E}_0^\epsilon$ given by $(L_2 - L_{2,0})W^\epsilon$; see (4.32) (a) and the footnote there. The cutoff has no effect on the estimate of the part of $\mathcal{E}_0^\epsilon$ given by $H_2 - \mathcal{H}_2$, because of (3.82). We will show $\chi^\epsilon$ is harmless in the estimate of $\epsilon \mathcal{E}_1^\epsilon$, even though we must take into account the effect of applying vector fields to it.

This completes the construction of the approximate solution $u_a^\epsilon$ as in (3.11).

Remark 3.20. Observe that if any term in the expansion of $H_2$ had the form $\sigma_k^2(y, \theta_0 + \omega_k(y')) c(y)$, then the contribution of that term to the integral in (3.75) would not be finite (set $m = p = k$ in Remark 3.21).

Scalar factors of the form $\sigma_k^2(y, \theta_0 + \omega_k(y')) c(y)$ do appear in terms of $E,F$ and in $H_1$ (3.16), but those contributions to $F$ are solved away, respectively, by the leading profile equations (3.39) and by the choice of $V$ (3.40). This provides a belated motivation for the decomposition of $F$ given in (3.25), (3.26); see also Remark 3.11.

3.7 Summary

We recall from (3.22) that $u_a^\epsilon(y) = U_0(y,\theta)|_{\theta = \frac{y}{\epsilon}} + \epsilon U_1^\epsilon(y)$ satisfies

$$L(y, \partial_y) u_a^\epsilon - f(y, u_a^\epsilon) = \epsilon^{-1} \mathcal{E}_{-1}^\epsilon + \mathcal{E}_0^\epsilon + \epsilon \mathcal{E}_1^\epsilon.$$

We saw that $\mathcal{E}_{-1}^\epsilon = 0$ by virtue of the polarization of the terms of $U_0$. With $F = L(y, \partial_y) U_0 - f(y, U_0)$ we have

$$\mathcal{E}_0^\epsilon(y) = \frac{b(y, U_0) + \epsilon B(y') U_1^\epsilon(y', 0)}{\epsilon}. $$

The profile $U_0$ was constructed to make $E F = 0$. We wrote $(1 - E) F = H_1 + H_2$ and constructed $V_1$ such that $L_1 V_1 + H_1 = 0$. Denoting $L_2(y', 0, d\psi_0, \partial \theta_0, \xi_n)$ by $L_{2,0}$ we can therefore write

$$\mathcal{E}_0^\epsilon(y) = \left[ H_2 + L_2 W^\epsilon \right] |_{\theta = \frac{y_0}{\epsilon}, \partial \theta_0 = \frac{\psi_0}{\epsilon}, \xi_n = \frac{\xi_n}{\epsilon}} = \left[ (H_2 - H_2) + (L_2 - L_{2,0}) W^\epsilon \right] |_{\theta = \frac{y_0}{\epsilon}, \partial \theta_0 = \frac{\psi_0}{\epsilon}, \xi_n = \frac{\xi_n}{\epsilon}}.$$ 

Hence on the open set $\Omega^0 \ni 0$, chosen as in section 3.6.2, the approximate solution $u_a$ satisfies

$$L(y, \partial_y) u_a^\epsilon - f(y, u_a^\epsilon) + r_a^\epsilon in y_n > 0$$

$$B(y') u_a^\epsilon |_{y_n = 0} = b'(y') + \epsilon B(y') U_1^\epsilon(y', 0),$$

$$u_a^\epsilon = 0 in y_0 < 0,$$

where

$$r_a^\epsilon(y) = \mathcal{E}_0^\epsilon(y) + \epsilon \mathcal{E}_1^\epsilon(y)$$

with

$$\mathcal{E}_0^\epsilon(y) = \left[ (H_2 - H_2) + (L_2 - L_{2,0}) W^\epsilon \right] |_{\theta = \frac{y_0}{\epsilon}, \partial \theta_0 = \frac{\psi_0}{\epsilon}, \xi_n = \frac{\xi_n}{\epsilon}}$$

$$\mathcal{E}_1^\epsilon(y) = \left[ -K(y, U_0, \partial U_1^\epsilon) U_1^\epsilon + L(y, \partial_y) V + L(y, \partial_y) W^\epsilon \right] |_{\theta = \frac{y_0}{\epsilon}, \partial \theta_0 = \frac{\psi_0}{\epsilon}, \xi_n = \frac{\xi_n}{\epsilon}}.$$ 

Remark 3.21. At this point it is clear that for each $\epsilon$, $u_a^\epsilon \in C^\infty_0(\Omega^0)$, and that $u_a^\epsilon \in L^\infty(\Omega^0)$ uniformly with respect to small $\epsilon$. Indeed, the same is true for the individual pieces

$$U_0 \left( y, \frac{\psi}{\epsilon} \right), V \left( y, \frac{\psi}{\epsilon} \right), and W^\epsilon \left( y, \frac{y_0}{\epsilon}, \frac{y_n}{\epsilon} \right).$$

We show in section 4.3 that for any $r \in \mathbb{N}_0$, $u_a^\epsilon \in N^m(\Omega^0)$ uniformly with respect to small $\epsilon$. 26
4 Exact solution and error analysis

To complete the proof of Theorem 1.2 it remains to estimate the exact solution \( u^\epsilon \) to (1.4) and the difference \( u^\epsilon - u^\epsilon_0 \) in \( L^\infty \cap N^m(U) \), where \( m > \frac{n+5}{2} \) and \( U \) is some neighborhood of 0.

In section 4.2 we apply a result from [Méth89] to obtain \( u^\epsilon \in L^\infty \cap N^m(\Omega_T') \) on a certain domain of determinacy \( \Omega_T' \subset \Omega^0 \) for that problem. We define the error problem to be the problem satisfied by the difference \( w^\epsilon := u^\epsilon - u^\epsilon_0 \). We have

\[
(4.1) \quad f(y, u^\epsilon) - f(y, u^\epsilon_0) = \left( \int_0^1 \partial_a f(y, u^\epsilon_a + s(u^\epsilon - u^\epsilon_0)) ds \right) w^\epsilon := D(y, u^\epsilon, u^\epsilon_0)w^\epsilon,
\]

so the error problem, a linear problem for \( w^\epsilon \), is

\[
(4.2) \quad \begin{align*}
L(y, \partial_y)w^\epsilon &= D(y, u^\epsilon, u^\epsilon_0)w^\epsilon - r^\epsilon_0 \text{ in } y_n > 0 \\
B(y')w^\epsilon|_{y_n=0} &= -\epsilon B(y')U_1(y', 0) \\
w^\epsilon &= 0 \text{ in } y_0 < 0.
\end{align*}
\]

We will solve this problem and estimate \( w^\epsilon \) on \( U := \Omega_T \subset \Omega_T' \subset \Omega^0 \) for some small enough \( T > 0 \) independent of \( \epsilon \). We have

\[
(4.3) \quad u^\epsilon = u^\epsilon_0 + w^\epsilon = U_0(y, \theta)|_{\theta=\frac{\epsilon}{\epsilon_0}} + \epsilon U^\epsilon_1(y) + w^\epsilon,
\]

where \( U^\epsilon_1 \) is bounded in \( L^\infty(\Omega^0) \) uniformly for \( \epsilon \) small by the above construction. Thus, if we can show that \( w^\epsilon \) is small in \( L^\infty(\Omega_T) \) uniformly for \( \epsilon \) small, we will have shown that the exact solution is close to \( U_0(y, \theta)|_{\theta=\frac{\epsilon}{\epsilon_0}} \) in a useful sense. To control the \( L^\infty \) norm of \( w^\epsilon \), we need to control the \( L^\infty \cap N^m \) norms of the interior and boundary forcing terms in (4.2); see Proposition 4.26. Thus, we need to estimate \( u^\epsilon_0 \) (section 4.3) and \( r^\epsilon_0 \) (section 4.4) in the conormal spaces \( N^m(\Omega^0) \). The error \( w^\epsilon \) is estimated in section 4.5.

4.1 Spaces of conormal distributions

Here we give a precise description, based on [Méth89], of the conormal spaces \( N^m(\Omega^0) \). In this section the characteristic surfaces \( \Sigma_j \) are as in (3.3) and the codimension two surface \( \Delta \) is \( \{y : y_0 = y_n = 0\} \).

Definition 4.1. (a) For each \( j = 1, \ldots, N \) let \( \mathcal{M}_j \) denote the space of vector fields on \( \Omega^0 \) with coefficients in \( C^\infty_b(\Omega^0) \) that are tangent to \( \Sigma_j \) and \( \Delta \). A set of generators of \( \mathcal{M}_j \) over \( C^\infty_b(\Omega^0) \) is given by

\[
\begin{align*}
M_0 &= (y_0 - \phi_j)\partial_0 \\
M_\nu &= \partial_\nu + (\partial_\nu \phi_j)\partial_0 \quad (\nu = 1, \ldots, n - 1) \\
M_n &= y_n(\partial_n + (\partial_n \phi_j)\partial_0) \\
M_{n+1} &= (y_0 - \phi_j)\partial_n.
\end{align*}
\]

36 The sets \( \Omega_T \) are defined in (1.8).
37 The spaces \( \mathcal{M}_1, \mathcal{M}_0, \mathcal{M} \) are \( C^\infty_b(\Omega^0) \) (or \( C^\infty(\Omega^0) \)) modules which are closed under multiplication on the left by elements of \( C^\infty_b \) and closed under the Lie bracket \([V, W] = VW - WV\).
Let $\mathcal{M}_0$ denote the space of vector fields on $\Omega^0$ with coefficients in $C^\infty_b(\Omega^0)$ that are tangent to $\Delta$. A set of generators is given by
\[ y_0 \partial_0, \ y_0 \partial_n, \ y_n \partial_0, \ y_n \partial_n, \ \partial_l, \ l = 1 \ldots, n - 1. \]

Let $\mathcal{M}'$ denote the space of vector fields on $b\Omega^0$ with coefficients in $C^\infty_b(b\Omega^0)$ that are tangent to $\Delta$. A set of generators of $\mathcal{M}'$ is given by
\[ m_0 = y_0 \partial_0, \ m_\nu = \partial_\nu, \ \nu = 1 \ldots, n - 1. \]

**Definition 4.2** (Conormal spaces). Let $U \subset \Omega^0$ be some neighborhood of $0$.

(a) For $m \in \mathbb{N}_0$ let $N^m(U, \mathcal{M}_j)$ denote the set of $u \in L^2(U)$ such that $V_1 V_2 \cdots V_k u \in L^2(U)$ for any choice of $V_i \in \mathcal{M}_j$ and $k \leq m$. To define a norm on $N^m(U, \mathcal{M}_j)$ let $M_0, \ldots, M_{n+1}$ denote the generators of $\mathcal{M}_j$ given in Definition 4.1. Then set
\begin{equation}
|u|_{N^m(U, \mathcal{M}_j)}^2 = \sum_{|\alpha| \leq m} |(M_0, \ldots, M_{n+1})^\alpha u|_{L^2(U)}^2.
\end{equation}

(b) The spaces $N^m(U, \mathcal{M}_0)$ and $N^m(bU, \mathcal{M}')$ and their norms $|u|_{N^m(U, \mathcal{M}_0)}$ and $\langle u \rangle_{N^m(U, \mathcal{M}_0)}$ are defined analogously using the generators of $\mathcal{M}_0$ and $\mathcal{M}'$ given in Definition 4.1. Usually, we will write simply $\langle u \rangle_{N^m(U, \mathcal{M}_0)}$ in place of $\langle u \rangle_{N^m(U, \mathcal{M}_0)}$.

(c) Let $N^m(U)$ be the set of $u \in L^2(U)$ such that $u = \sum_{j=1}^N u_j$ for some choice of $u_j \in N^m(U, \mathcal{M}_j)$. Define
\begin{equation}
|u|_{N^m(U)} = \inf \sum_{j=1}^N |u_j|_{N^m(U, \mathcal{M}_j)},
\end{equation}
where the inf is taken over all decompositions $u = \sum_{j=1}^N u_j$ as above.

(d) For $m \in \mathbb{N}_0$ and $j \in \{0, 1, \ldots, N\}$ let $N^m_\infty(U, \mathcal{M}_j)$ denote the set of $u \in L^\infty(U)$ such that $V_1 V_2 \cdots V_k u \in L^\infty(U)$ for any choice of $V_i \in \mathcal{M}_j$ and $k \leq m$. With generators $M_0, \ldots, M_{n+1}$ as in (a), define
\begin{equation}
|u|_{N^m_\infty(U, \mathcal{M}_j)}^2 = \sum_{|\alpha| \leq m} |(M_0, \ldots, M_{n+1})^\alpha u|_{L^\infty(U)}^2.
\end{equation}
The space $N^m_\infty(bU, \mathcal{M}')$ and its norm $\langle u \rangle_{N^m_\infty(bU, \mathcal{M}')}^\infty$ (or simply $\langle u \rangle_{N^m_\infty(bU)}^\infty$) are defined similarly.

(e) If $u \in L^\infty \cap N^m(U, \mathcal{M}_j)$, we define
\[ |u|_{L^\infty \cap N^m(U, \mathcal{M}_j)} = |u|_{L^\infty(U)} + |u|_{N^m(U, \mathcal{M}_j)}, \]
and do similarly for the other $N^m$ norms.

The set $\Omega^0$ is bounded, so for $U \subset \Omega^0$ we have the obvious estimate
\begin{equation}
|u|_{N^m(U, \mathcal{M}_j)} \leq \sqrt{|\Omega^0|} \ |u|_{N^m_\infty(U, \mathcal{M}_j)},
\end{equation}
where $|\Omega^0|$ is the Lebesgue measure of $\Omega^0$. Later we often use (4.7) to estimate $N^m(U, \mathcal{M}_j)$ norms.

We will need to use the following lemma, which is stated in [Met89] and proved in [Met86]. The lemma uses the partition of unity $\sum_{j=0}^N \chi_j(y)$ on $\{y_n > 0\}$. The $\chi_j$ are defined just below (8.35).

**Lemma 4.3.** Let $u \in L^2(U)$. Then $u \in N^m(U)$ if and only if $\chi_j u \in N^m(U, \mathcal{M}_j)$ for $j = 0, \ldots, N$. Moreover, the norm $|u|_{N^m(U)}$ (4.5) is equivalent to the norm $\sum_{j=0}^N |\chi_j u|_{N^m(U, \mathcal{M}_j)}$ with constants independent of $U$. 28
4.2 Exact solution \( u^\varepsilon \)

In this section we apply a result of \[\text{M}^\varepsilon t89\] to obtain an exact solution of (1.4) on an appropriate domain of determinacy for that problem. First, for positive constants \( T_0, \alpha, \text{and } T < T_0 \) we define

\[ \Omega = \{ y : -T_0 < y_0 < T_0 - \alpha |y''|, y_n \} \cap \{ y_n \geq 0 \} \text{ and } b\Omega = \{ y' : (y', 0) \in \Omega \cap \{ y_n = 0 \} \} \]

\[ \Omega_T = \Omega \cap \{ y_0 < T \} \text{ and } b\Omega_T = \{ y' : (y', 0) \in \Omega_T \cap \{ y_n = 0 \} \}. \]

The constants \( T_0 \) and \( \alpha \) are chosen so that \( \Omega \), and thus \( \Omega_T \), is a domain of determinacy for the problem (1.4). We also require \( \Omega \) to be a domain of determinacy for that problem. First, for positive constants \( \alpha \) and \( \Omega \), we have

\[ \| b^\varepsilon \|_{L^\infty \cap N^m(\Omega)} \lesssim 1. \]

The next proposition is just a rephrasing of Theorem 2.1.1 of \[\text{M}^\varepsilon t89\] in a form suitable for the problem (1.4).

**Proposition 4.4 (Exact solution).** Suppose \( m > \frac{n+5}{2} \). Then (4.9) implies that if \( T_0 \) and \( \alpha \) in (4.8) are small enough (depending on \( L, B \)), the problem (1.4) has an exact solution \( u^\varepsilon \) in \( L^\infty \cap N^m(\Omega_T) \) for some \( 0 < T_1 < T_0 \) that satisfies

(4.10) \[ |u^\varepsilon|_{L^\infty \cap N^m(\Omega_T)} \lesssim 1. \]

4.3 Estimate of the approximate solution \( u^\varepsilon \).

At the moment we know \( u^\varepsilon \) has the properties described in Remark 3.21. We will show:

**Proposition 4.5.** For all \( r \in \mathbb{N}_0 := \{ 0, 1, 2, 3, \ldots \} \) we have \( |u^\varepsilon(y)|_{N^r(\Omega^\varepsilon)} \lesssim 1 \). Moreover,

\[ |\lambda_0 \left( y, \frac{\psi}{\varepsilon} \right) |_{N^r(\Omega^\varepsilon)} \lesssim 1, \quad |V \left( y, \frac{\psi}{\varepsilon} \right) |_{N^r(\Omega^\varepsilon)} \lesssim 1, \quad \text{and } |W^\varepsilon \left( y, \frac{y_0}{\varepsilon}, \frac{y_0}{\varepsilon} \right) |_{N^r(\Omega^\varepsilon)} \lesssim 1, \]

and

\[ \langle V \left( y', 0, \frac{y_0}{\varepsilon}, \ldots, \frac{y_0}{\varepsilon} \right) \rangle_{N^r(\Omega^\varepsilon)} \lesssim 1, \quad \text{and } \langle W^\varepsilon \left( y', 0, \frac{y_0}{\varepsilon}, 0 \right) \rangle_{N^r(\Omega^\varepsilon)} \lesssim 1. \]

The proof is contained in Propositions 4.6, 4.8, and 4.11 below.

**Proposition 4.6.** For all \( r \in \mathbb{N}_0 := \{ 0, 1, 2, 3, \ldots \} \) we have

(4.12) \[ |\lambda_0 \left( y, \frac{\psi}{\varepsilon} \right) |_{N^r(\Omega^\varepsilon)} \lesssim 1. \]

**Proof.** 1. We have \( \lambda_0 \left( y, \frac{\psi}{\varepsilon} \right) = \sum_k \sigma_k \left( y, \frac{\psi k}{\varepsilon} \right) r_k(y) \), so it suffices to show for each \( k \) that

(4.13) \[ |\sigma_k \left( y, \frac{\psi k}{\varepsilon} \right) |_{N^r(\Omega^\varepsilon, M_k)} \lesssim 1. \]

2. **Preparation.** Recall that a set of generators of \( M_k \) is given by \( M_0, \ldots, M_{n+1} \), where

---

\[ \text{The proofs show that in each case the } N^r \text{ norm can be replaced by an } N^r_\infty \text{ norm.} \]
\[ M_0 = (y_0 - \phi_k)\partial_0 \]
\[ M_\nu = \partial_\nu + (\partial_\nu \phi_k)\partial_0, \; \nu = 1, \ldots, n - 1 \]
\[ M_n = y_n(\partial_n + (\partial_n \phi_k)\partial_0) \]
\[ M_{n+1} = (y_0 - \phi_k)\partial_n. \]

Also, generators of \( \mathcal{M} \) are given by \( m_0 = y_0\partial_0, \; m_\nu = \partial_\nu, \; \nu = 1, \ldots, n - 1. \)

For a fixed \( k \) let
\[
\Gamma(\Omega^0) = \{ G^\epsilon \in C^\infty(\Omega^0, \mathbb{R}) \colon |(M_0, \ldots, M_{n+1})\gamma G^\epsilon|_{L^\infty(\Omega^0 \cap \bar{J}_k)} \lesssim C_\gamma \text{ for } \epsilon \in (0, 1], \; \gamma \in \mathbb{N}_0^{n+2} \}
\]
\[
b\Gamma^\epsilon = \{ g^\epsilon \in C^\infty(b\Omega^0, \mathbb{R}) \colon |(m_0, \ldots, m_{n-1})\gamma g^\epsilon|_{L^\infty(M\mathcal{P}\cap bJ_k)} \lesssim C_\gamma \text{ for } \epsilon \in (0, 1], \; \gamma \in \mathbb{N}_0^n \}. \]

**Remark 4.7.** Observe that smooth \( \epsilon \)-independent functions \( c(y) \) lie in \( \Gamma^\epsilon(\Omega^0) \) and \( \frac{\partial c}{\partial \epsilon} \in \Gamma^\epsilon(\Omega^0) \). Moreover, \( \Gamma^\epsilon(\Omega^0) \) is mapped to itself by elements of \( \mathcal{M}_k \) and is closed under products.

For the next step we define the set of “profile-type” functions
\[
P_k := \{ P \in C^\infty(\Omega^0 \times \mathbb{R}_{\theta_k}, \mathbb{R}) \colon P(y, \theta_k) \text{ has support in } |\theta_k| \leq 1 \}
\]
\[
bP_k := \{ p \in C^\infty(b\Omega^0 \times \mathbb{R}_{\theta_0}, \mathbb{R}) \colon p(y', \theta_0) \text{ has support in } |\theta_0| \leq 1 \}. \]

3. Using \( \psi_k = (y_0 - \phi_k)\beta_k \) we compute, for example,
\[
M_\nu \sigma_k \left( y, \frac{\psi_k}{\epsilon} \right) = (M_\nu \sigma_k(y, \theta_k))|_{\theta_k = \frac{\psi_k}{\epsilon}} + \partial_{\theta_k} \sigma_k \left( y, \frac{\psi_k}{\epsilon} \right) \cdot \frac{y_0 - \phi_k}{\epsilon} M_\nu \beta_k(y). \]

This has the form \( P(y, \frac{\psi_k}{\epsilon}) + P(y, \frac{\psi_k}{\epsilon})G^\epsilon(y) \), where \( P \in P_k, \; G^\epsilon \in \Gamma^\epsilon(\Omega^0) \) here and below can change from term to term. A similar result is obtained for any choice of generator \( M_j \) in (4.17). Thus, with Remark 4.7 an induction on \( \gamma \) shows that \( (M_0, \ldots, M_{n+1})\gamma \sigma_k \left( y, \frac{\psi_k}{\epsilon} \right) \) is a finite sum of terms of the form \( P(y, \frac{\psi_k}{\epsilon})G^\epsilon(y) \). We have
\[
\left| P(y, \frac{\psi_k}{\epsilon})G^\epsilon(y) \right|_{L^\infty(\Omega^0)} \lesssim 1,
\]

since \( P(y, \frac{\psi_k}{\epsilon}) \) has support in \( J_\epsilon \), so by the estimate (4.7), this concludes the proof.

**4.3.1 Estimate of \( V(y, \frac{\psi_k}{\epsilon}) \).**

Next we estimate the noninteraction term of \( U_\epsilon \).

---

39 Here \( C_\gamma \) is independent of \( \epsilon \). Both \( J_\epsilon \) and \( \Gamma^\epsilon(\Omega^0) \) depend on \( k \), but we suppress that in the notation.

40 Henceforth, we will usually apply (4.7) without comment.
Proposition 4.8. For all \( r \in \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} \) we have

\[
\begin{align*}
(a) \quad & \left| V \left( y, \frac{\psi}{\epsilon} \right) \right|_{N^r(\Omega^0)} \lesssim 1 \\
(b) \quad & \left| V \left( y', 0, \frac{y_0}{\epsilon}, \ldots, \frac{y_0}{\epsilon} \right) \right|_{N^r(\Omega^0)} \lesssim 1.
\end{align*}
\]

Proof. 1. We have \( V(y, \theta) = \sum_k V_k(y, \theta_k) \) and

\[
V_k \left( y, \frac{\psi_k}{\epsilon} \right) = -\int_{-\infty}^{\psi_k} Q_k(y) H_{1k}(y, s) ds.
\]

By Remark 3.11 this integral can be written as a finite sum of terms of the form

\[
\left( \int_{-\infty}^{\psi_k} P(y, s) ds \right) C_k(y),
\]

where \( P(y, \theta_k) \in \mathcal{P}_k \) and \( C_k(y) \) is a smooth \( \epsilon \)-independent function of \( y \). We can ignore \( C_k(y) \) in proving (4.19), so it will suffice to show for each \( k \) that

\[
\left| \int_{-\infty}^{\psi_k} P(y, s) ds \right|_{N^r(\Omega^0, M_k)} \lesssim 1.
\]

2. We compute for example

\[
M_0 \int_{-\infty}^{\psi_k} P(y, s) ds = \int_{-\infty}^{\psi_k} M_0 P(y, s) ds + \left( M_0 \frac{\psi_k}{\epsilon} \right) P \left( y, \frac{\psi_k}{\epsilon} \right),
\]

which has the form \( \int_{-\infty}^{\psi_k} P(y, s) ds + G(x) P \left( y, \frac{\psi_k}{\epsilon} \right) \), where \( G(x) \in \Gamma^e(\Omega^0) \). A similar result for the other choices of \( M_j \) and an induction on \( |\gamma| \) show that \( (M_0, \ldots, M_{n+1})^\gamma \int_{-\infty}^{\psi_k} P(y, s) ds \) is a finite sum of terms of the form\(^{41}\)

\[
\int_{-\infty}^{\psi_k} P(y, s) ds \text{ or } G(x) P \left( y, \frac{\psi_k}{\epsilon} \right)
\]

An obvious estimate of the integral and (4.18) yields (4.20).

A parallel argument using \( b\Gamma^e \) (4.15) and \( b\mathcal{P}_k \) (4.16) proves (4.19)(b).

As an immediate corollary of the preceding proof we give an estimate of a piece of \( E_1^e \).

Corollary 4.9. For all \( r \in \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} \) we have

\[
\left| (L(y, \partial_y) V(y, \theta)) \right|_{\Omega^0, \frac{\psi_k}{\epsilon}} \lesssim 1.
\]

\(^{41}\)Recall (4.17).
4.3.2 Estimates of \((H_2 - H_2)|_{\theta = \frac{y}{\varepsilon}, \theta_0 = \frac{y_0}{\varepsilon}, \xi_n = \frac{y_n}{\varepsilon}} \) \text{ and } W^\varepsilon|_{\theta_0 = \frac{y_0}{\varepsilon}, \xi_n = \frac{y_n}{\varepsilon}}.

We give the estimate of \(W^\varepsilon\) in Proposition 4.11 below. First we do a simpler, related estimate of the contribution to \(\mathcal{E}_0^\varepsilon\) given by the \(H_2 - H_2\) term.

By Proposition 3.19, the term \((H_2 - H_2)|_{\theta = \frac{y}{\varepsilon}, \theta_0 = \frac{y_0}{\varepsilon}, \xi_n = \frac{y_n}{\varepsilon}}\) is supported in the interaction region \(I_\varepsilon\), a fact that simplifies the analysis and makes possible an estimate of the \(N^k(\Omega^\varepsilon, M_0)\) norm.\(^{42}\)

Recall that a set of generators of \(M_0\) is given by \(\{V_j, j = 1, \ldots, n + 3\}\), which are respectively

\[
y_0 \partial_0, \quad y_0 \partial_n, \quad y_n \partial_0, \quad \partial_\ell, l = 1 \ldots, n - 1.
\]

With \(\psi_{m,0}(y) = y_0 + \omega_m(y')y_n\), by (3.61) we can write

\[
\sigma_m(y, \frac{\psi_m}{\varepsilon}) - \sigma_m(y, \frac{\psi_{m,0}}{\varepsilon}) = \left( \int_0^1 \partial_{\theta_0} \sigma_m(y, \frac{\psi_{m,0}}{\varepsilon}) + s(\frac{\psi_m}{\varepsilon} - \frac{\psi_{m,0}}{\varepsilon}) \right) c(y) \frac{y_n^2}{\varepsilon} = f\left(y, \frac{y_0}{\varepsilon}, \frac{y_n}{\varepsilon}\right),
\]

where \(f(y, \theta_0, \xi_n)\) is a smooth function of its arguments which is bounded along with its derivatives on bounded subsets of \(\Omega^0 \times \mathbb{R}^2_{\theta_0, \xi_n}\).\(^{43}\) Let \(\gamma\) be a multi-index with \(|\gamma| = k \in \mathbb{N}_0\). Then for some constant \(C\) (possibly 0),

\[
(V_1, \ldots, V_{n+3})^\gamma \left( \frac{y_n^2}{\varepsilon} \right) = C\left(\frac{y_0, y_n}{\varepsilon}\right)^k \text{ for some multi-index } |\gamma| = 2.
\]

Observe for example that

\[
y_0 \partial_0 \left( f\left(y, \frac{y_0}{\varepsilon}, \frac{y_n}{\varepsilon}\right) \right) = (y_0 \partial_0 f + \partial_{\theta_0} f \theta_0)|_{\theta_0 = \frac{y_0}{\varepsilon}, \xi_n = \frac{y_n}{\varepsilon}}.
\]

Similar computations with the other generators of \(M_0\) and an easy induction on \(k = |\gamma|\) yield

\[
(V_1, \ldots, V_{n+3})^\gamma \left[ f\left(y, \frac{y_0}{\varepsilon}, \frac{y_n}{\varepsilon}\right) \right] = \sum_{L=0}^k \sum_{|\alpha| = L} f_{L,\alpha}\left(y, \frac{y_0}{\varepsilon}, \frac{y_n}{\varepsilon}\right) \cdot \frac{(y_0, y_n)^\alpha}{\varepsilon^L},
\]

where \(f_{L,\alpha}(y, \theta_0, \xi_n)\) has the same properties as \(f\). Thus\(^{44}\)

\[
\left| (V_1, \ldots, V_{n+3})^\gamma \left[ f\left(y, \frac{y_0}{\varepsilon}, \frac{y_n}{\varepsilon}\right) \right] \right|_{N^k(\Omega^\varepsilon \cap I_\varepsilon, M_0)} \lesssim 1, \quad \text{since } \left| \frac{(y_0, y_n)^\alpha}{\varepsilon^L} \right| \lesssim 1 \text{ on } I_\varepsilon \text{ when } |\alpha| = L.
\]

We also have \(\left| \frac{(y_0, y_n)}{\varepsilon} \right| \lesssim \epsilon\) on \(I_\varepsilon\) when \(|\gamma| = 2\). With (4.28) this implies

\[
\left| \sigma_m(y, \frac{\psi_m}{\varepsilon}) - \sigma_m(y, \frac{\psi_{m,0}}{\varepsilon}) \right|_{N^k(\Omega^\varepsilon \cap I_\varepsilon, M_0)} \lesssim \epsilon.
\]

\(^{42}\)This dominates the \(N^k(\Omega^\varepsilon)\) norm since \(M_k \subset M_0\) for all \(k\).

\(^{43}\)With \(\psi_m - \psi_{m,0} = c(y)y_n^2\), we have \(f(y, \theta_0, \xi_n) = c(y) \int_0^1 \partial_{\theta_0} \sigma_m(y, \theta_0 + \omega_m(y')\xi_n + sc(y)y_n\xi_n) ds\).

\(^{44}\)Here we use that \(\frac{y_n}{\varepsilon}, \frac{y_n}{\varepsilon}\) are bounded on \(I_\varepsilon\).
Proposition 4.10. We have

\( |(H_2 - \mathcal{H}_2)|_{\theta_0 = \bar{\theta}_0, \frac{\psi_m}{\epsilon}, \xi_n = \bar{\xi}_n, \epsilon} \leq \epsilon. \)

Proof. Using Remark 3.15 we reduce to estimating in \( \Omega_T \cap I_\epsilon \) products like

\[
\left[ \sigma_m \left( y, \frac{\psi_m}{\epsilon}, \xi_n \right) - \sigma_m \left( y, \frac{\psi_m, 0}{\epsilon} \right) \right] \cdot \sigma_p \left( y, \frac{\psi_p}{\epsilon} \right) c(y) = \\
\int f \left( y, \frac{y_0}{\epsilon}, \frac{y_n}{\epsilon} \right) \frac{y_0^2}{\epsilon} \cdot \left( g \left( y, \frac{y_0}{\epsilon}, \frac{y_n}{\epsilon} \right) = h \left( y, \frac{y_0}{\epsilon}, \frac{y_n}{\epsilon} \right) \frac{y_0^2}{\epsilon}, \right.
\]

where \( g(y, \theta_0, \xi_n) \) and \( h(y, \theta_0, \xi_n) \) have the same properties as \( f \). So essentially the same estimate as the one that gave (4.29) yields the result.

Next we estimate

\( W^\epsilon(y, \theta_0, \xi_n) |_{\theta_0 = \bar{\theta}_0, \xi_n = \bar{\xi}_n, \epsilon} = \chi^\epsilon(y_0, y_n) \sum_k t_k(y, \theta_0, \xi_n) r_k(y', 0) |_{\theta_0 = \bar{\theta}_0, \xi_n = \bar{\xi}_n, \epsilon}, \)

where each \( t_k(y, \theta_0, \xi_n) \) is \( C^\infty \) with derivatives that are bounded on bounded subsets of \( \Omega^0 \times \mathbb{R}^2_{\theta_0, \xi_n} \) by Remark 3.18.

Proposition 4.11. For all \( r \in \mathbb{N}_0 := \{0, 1, 2, 3, \ldots \} \) we have

\[
(a) \left| W^\epsilon \left( y, \frac{y_0}{\epsilon}, \frac{y_n}{\epsilon} \right) \right|_{N^r(\Omega^0)} \lesssim 1 \\
(b) \left| W^\epsilon \left( y', 0, \frac{y_0}{\epsilon}, 0 \right) \right|_{N^r(\Omega^0)} \lesssim 1.
\]

This will be an immediate consequence of Propositions 4.14, 4.15, and 4.18 below. In preparation for the proof note that by (3.78) each \( t_k(y, \theta_0, \xi_n) \) is a finite sum of terms of the form

\[
c(y) \int_{+\infty}^{\xi_n} \sigma_m \left( y, \theta_0 + \omega_k(y') \xi_n + s(\omega_m(y') - \omega_k(y')) \right) \sigma_p \left( y, \theta_0 + \omega_k(y') \xi_n + s(\omega_p(y') - \omega_k(y')) \right) ds,
\]

where \( m \neq p \). Unfortunately, \( t_k(y, \theta_0, \xi_n) |_{\theta_0 = \bar{\theta}_0, \xi_n = \bar{\xi}_n, \epsilon} \) is not supported in the interaction region \( I_\epsilon \).

But we do have:

Lemma 4.12. For \( y \in \Omega^0 \) we have for every \( k \):

\[
|y_0 + \omega_k(y') y_n| \leq \epsilon \quad \text{on the support of} \quad t_k(y, \theta_0, \xi_n) |_{\theta_0 = \bar{\theta}_0, \xi_n = \bar{\xi}_n, \epsilon}.
\]

In fact, there exists \( M > 0 \) such that if \( |y_0 + \omega_k(y') y_n| \geq M \epsilon \), then the integrand in (4.34) vanishes for all \( s \in \mathbb{R} \) when \( \theta_0 = \frac{y_0}{\epsilon}, \xi_n = \frac{y_n}{\epsilon} \).

\[\text{If it were, we could essentially repeat the estimates of section 4.1 here.}\]
Proof. It suffices to show there exists $M > 0$ such that if $|y_0 + \omega_k(y') y_n| \geq M \epsilon$, then no $s \in \mathbb{R}$ lies in the $s-$support of both factors of the integrand in (4.34) when they are evaluated at $\theta_0 = \frac{y_0}{\epsilon}, \xi_n = \frac{\omega_k y_n}{\epsilon}$.

**Case 1: both $m$ and $p$ are $\neq k$.** As before let $\psi_{k,0} = y_0 + \omega_k(y') y_n$. Suppose both $\omega_m$ and $\omega_p$ are $> \omega_k$. On the $s-$support of the first factor we have

\[
|\psi_{k,0} + s(\omega_m - \omega_k)\epsilon| \leq 1 \Rightarrow |s - \frac{\psi_{k,0}/\epsilon}{\omega_k - \omega_m}| \lesssim 1.
\]

Similarly, on the $s-$support of the second factor we have

\[
|s - \frac{\psi_{k,0}/\epsilon}{\omega_k - \omega_p}| \lesssim 1.
\]

These conditions on $s$ are incompatible if $M$ is chosen sufficiently large and $|\psi_{k,0}| \geq M \epsilon$.

**Case 2: $m \neq k$, $p = k$.** In this case we can rewrite (4.34) as

\[
c(y)\sigma_p(y, \theta_0 + \omega_p(y') \xi_n) \int_{-\infty}^{\infty} \sigma_m(y, \theta_0 + \omega_k(y') \xi_n + s(\omega_m(y') - \omega_k(y'))) ds,
\]

The result follows since the factor $\sigma_p(y, \frac{\psi_{k,0}}{\epsilon})$ is supported in $|\psi_{p,0}| \leq \epsilon$.

In the next lemma we use the wedge partition of unity $\{\chi_j\}$ defined in section 3.6.2.

**Lemma 4.13.** For $y \in \Omega^0$ we have when $j \neq k$:

\[
|y_0 + \omega_k(y') y_n| \leq M \epsilon \text{ and } y \in \text{supp } \chi_j \Rightarrow |y_0| \lesssim \epsilon \text{ and } |y_n| \lesssim \epsilon.
\]

**Proof.** 1. Recall the covering of $\{y_n > 0\}$ by overlapping wedges $W_j$, $j = 0, 1, \ldots, N$ defined in section 3.6.2. There we also defined smaller disjoint wedges $W'_j$ such that

\[
(a) \Sigma_j \subset W'_j \subset\subset W_j \text{ for } j = 1, \ldots, N
\]

\[
(b) W_j \cap W'_k = \emptyset \text{ for } j \neq k, \ j = 0, \ldots, N.
\]

2. Let $\delta > 0$ be as chosen in section 3.6.2 and choose $P > 0$ such that $\frac{M}{P} < \frac{\delta}{\frac{\epsilon}{2}}$. Then

\[
|y_0 + \omega_k(y') y_n| \leq M \epsilon \text{ and } |y_n| \geq P \epsilon \Rightarrow \left| \frac{y_0}{y_n} + \frac{\omega_k(y')}{y_n} \right| < \frac{\delta}{2} \Rightarrow
\]

\[
|y_0 - \gamma_k| = \left| \frac{y_0}{y_n} + \omega_k(y') - \gamma_k - \omega_k(y') \right| \leq \left| \frac{y_0}{y_n} + \omega_k(y') \right| + |\gamma_k + \omega_k(y')| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\]

For the last inequality we used (3.55)(d). By definition of the wedges $W'_k$ this implies $y \in W'_k$, so by (4.40)(b) we have $y \notin W_j = \text{supp } \chi_j$. Thus, if (4.39) holds we must have $|y_n| \leq P \epsilon$, and thus also $|y_0| \lesssim \epsilon$.

---

\[^{46}\text{It will be clear that the other possible orderings can be handled the same way.}\]

\[^{47}\text{The } \omega_k \text{ are well separated on } \Omega^0 \text{ by (4.34) and (4.35).}\]

\[^{48}\text{The remaining cases are handled just like cases 1 and 2.}\]

\[^{49}\text{The plausibility of Lemma 4.13 is clear from a simple picture.}\]
In view of (4.32) and (4.34), to estimate the $N^r(\Omega^0)$ norm of $W^{\epsilon}|_{\theta_0=\frac{m_0}{\epsilon}, \xi_n=\frac{m_n}{\epsilon}}$ it suffices to estimate the $N^r(\Omega^0)$ norm of terms of the form

\[ (4.42) \quad T_{k,m,p}^{\epsilon}(y) := \chi^{\epsilon}(y_0, y_n) \left( \int_{I_{k,m,p}} I_{k,m,p}(s; y, \theta_0, \xi_n) ds \right) \big|_{\theta_0=\frac{m_0}{\epsilon}, \xi_n=\frac{m_n}{\epsilon}}, \quad m \neq p, \]

where $\int_{I_{k,m,p}} I_{k,m,p}(s; y, \theta_0, \xi_n) ds$ denotes the integral (4.34). Here and below we ignore smooth functions like $c(y)$ in (4.34), which can have no significant effect on the estimate. For a fixed choice of $k$ and $m \neq p$ we use the wedge partition of unity \( \{ \chi_j \} \), $\chi_j \in \Lambda(\Omega^0)$, constructed in section 3.6.2 to write

\[ (4.43) \quad T_{k,m,p}^{\epsilon}(y) = \sum_{j=0}^{N} \chi_j(y) T_{k,m,p}^{\epsilon}(y) := \sum_{j} T_{k,j}^{\epsilon}(y) \]

and proceed to estimate each $T_{k,j}^{\epsilon}$ in $N^r(\Omega^0, \mathcal{M}_j)$. In fact, for $j \neq k$ Lemma 4.13 shows $T_{k,j}^{\epsilon}$ is supported in the interaction region $I_{\epsilon}$, and that will allow us to estimate $T_{k,j}^{\epsilon}$ in the stronger norm of $N^r(\Omega^0, \mathcal{M}_0)$. We will do that first, and then estimate $T_{k,k}^{\epsilon}$ in $N^r(\Omega^0, \mathcal{M}_k)$.

**Proposition 4.14.** Consider $T_{j,k}^{\epsilon}$ as in (4.43) when $j \neq k$. For $r \in \mathbb{N}_0$ we have

\[ (4.44) \quad |T_{j,k}^{\epsilon}|_{N^r(\Omega^0, \mathcal{M}_0)} \lesssim 1. \]

**Proof.** Since $T_{k,j}^{\epsilon}$ has support in $I_{\epsilon}$, it is unchanged when the cutoff $\chi^{\epsilon}(y_0, y_n)$ is removed, so we will ignore that cutoff in this proof.

We can write

\[ (4.45) \quad T_{k,j}^{\epsilon}(y) = \chi_j(y) f(y, y_0 \frac{\epsilon}{\epsilon}, y_n \frac{\epsilon}{\epsilon}), \]

where $\chi_j(y) \in \Lambda(\Omega^0)$ and by Remark 3.18, $f(y, \theta_0, \xi_n)$ is a smooth function of its arguments with derivatives that are bounded on bounded subsets of $\Omega^0 \times \mathbb{R}_{\theta_0, \xi_n}^2$. We now argue as in the proof of Proposition 4.10. In fact, we can use (4.28) directly:

\[ (4.46) \quad |(V_1, \ldots, V_{n+3})^\gamma [f \left( y, \frac{y_0}{\epsilon}, \frac{y_n}{\epsilon} \right)]|_{N^r(\Omega^0 \cap I_{\epsilon}, \mathcal{M}_0)} \lesssim 1, \quad |\gamma| = r. \]

The estimate (4.44) now follows from (4.46) and the definition of $\Lambda(\Omega^0)$, Definition 3.13.

The most challenging term to estimate is the $T_{k,k}^{\epsilon}$ term.

**Proposition 4.15.** Consider $T_{k,k}^{\epsilon}$ as in (4.43). For $r \in \mathbb{N}_0$ we have

\[ (4.47) \quad |T_{k,k}^{\epsilon}|_{N^r(\Omega^0, \mathcal{M}_k)} \lesssim 1. \]

\[ ^{50} \text{The term } T_{k,k}^{\epsilon} \text{ is not supported in the interaction region } I_{\epsilon}. \]

\[ ^{51} \text{Here we use } \epsilon \in (0, \epsilon_0] \text{ for some small enough } \epsilon_0 < 1. \]
In preparation for the proof we recall:

\[ T_{k,k}^{\epsilon}(y) = \chi^{\epsilon}(y_0, y_n)\chi_k(y). \]  
(4.48) 

\[ \int_{-\infty}^{\infty} \sigma_m \left( y, \frac{\psi_{k,0}}{\epsilon} + s(\omega_m(y') - \omega_k(y')) \right) \sigma_p \left( y, \frac{\psi_{k,0}}{\epsilon} + s(\omega_p(y') - \omega_k(y')) \right) ds \]

\[ := \chi^{\epsilon}(y_0, y_n)\chi_k(y) \cdot t_{k,k}^{\epsilon}(y). \]

As before we denote the generators of \( M_k \) by \( M_0, \ldots, M_{n+1} \). They are:

- \( M_0 = (y_0 - \phi_k)\partial_0 \)
- \( M_\nu = \partial_\nu + (\partial_\nu \phi_k)\partial_0, \nu = 1, \ldots, n - 1 \)
- \( M_n = y_n(\partial_n + (\partial_n \phi_k)\partial_0) \)
- \( M_{n+1} = (y_0 - \phi_k)\partial_n. \)

From Lemma 4.12 and the definition of \( \chi^{\epsilon} \), we obtain that\( ^{52} \)

\[ \text{supp } T_{k,k}^{\epsilon}(y) \subset I_{\sqrt{\epsilon}} := \{ y : |y - \phi_k| \leq \epsilon, |y_0| \leq \sqrt{\epsilon}, 0 \leq y_n \leq \sqrt{\epsilon} \} \text{ and thus } \]

\[ \text{supp } (M_0, \ldots, M_{n+1})^\gamma T_{k,k}^{\epsilon}(y) \subset I_{\sqrt{\epsilon}} \text{ for any multi-index } \gamma. \]  
(4.49)

**Definition 4.16.** Let

(a) \( \Lambda^{\epsilon}(\Omega^0) = \{ A^\epsilon \in C^\infty(\Omega^0, \mathbb{R}) : |\langle M_0, \ldots, M_{n+1} \rangle^\gamma A^\epsilon \lambda_{L^\infty(\Omega^0 \cap I_{\sqrt{\epsilon}})} \leq C_\gamma \text{ for } \epsilon \in (0, 1], \gamma \in \mathbb{N}_{0}^{n+2} \} \)

(b) \( \Lambda^{\sqrt{\epsilon}}(\Omega^0) = \{ B^\epsilon \in C^\infty(\Omega^0, \mathbb{R}) : |\langle M_0, \ldots, M_{n+1} \rangle^\gamma B^\epsilon \lambda_{L^\infty(\Omega^0 \cap I_{\sqrt{\epsilon}})} \leq C_\gamma \text{ for } \epsilon \in (0, 1], \gamma \in \mathbb{N}_{0}^{n+2} \}. \)

**Remark 4.17.** The following properties of these spaces are easy to check.

1) Each of the spaces in Definition 4.16 is mapped to itself by the generators \( V_j, j = 0, \ldots, n+1 \).

2) We have \( \Lambda^{\sqrt{\epsilon}}(\Omega^0) \subset \Lambda^{\epsilon}(\Omega^0) \).

3) Smooth functions \( c(y) \) and functions of the forms \( \frac{(y_0, y_n)^\kappa}{\epsilon} \) with \( |\kappa| = 2 \) or \( \frac{y_0 - \phi_k}{\epsilon} \) belong to \( \Lambda^{\sqrt{\epsilon}}(\Omega^0) \). The cutoff \( \chi^{\epsilon}(y_0, y_n) \in \Lambda^{\sqrt{\epsilon}}(\Omega^0) \).

4) The functions \( \frac{y_0}{\epsilon}, \frac{y_n}{\epsilon} \in \Lambda^{\epsilon}(\Omega^0) \setminus \Lambda^{\sqrt{\epsilon}}(\Omega^0) \).

5) If \( A^\epsilon_1, A^\epsilon_2 \in \Lambda^{\epsilon}(\Omega^0) \), then the product \( A^\epsilon_1 \cdot A^\epsilon_2 \in \Lambda^{\epsilon}(\Omega^0) \). Similarly, if \( B^\epsilon_1, B^\epsilon_2 \in \Lambda^{\sqrt{\epsilon}}(\Omega^0) \), then \( B^\epsilon_1 \cdot B^\epsilon_2 \in \Lambda^{\sqrt{\epsilon}}(\Omega^0) \).

**Proof of Proposition 4.15.** 1. Since \( \chi_k(y) \in \Lambda(\Omega^0), \chi^{\epsilon}(y_0, y_n) \in \Lambda^{\sqrt{\epsilon}}(\Omega^0) \), and (4.49) holds, in order to prove (4.47) it will suffice to prove for \( t_{k,k}^{\epsilon} \) as in (4.48) that

\[ |\langle M_0, \ldots, M_{n+1} \rangle^\gamma t_{k,k}^{\epsilon}|_{L^\infty(\Omega^0 \cap I_{\sqrt{\epsilon}})} \lesssim 1 \text{ for any } \gamma. \]  
(4.50)

2. To motivate the rest of the argument we examine the action of \( M_\nu \) and \( M_0 \) on the second argument of \( \sigma_m \) in (4.48). Below we let \( c(y) \) denote a smooth function that can change from term to term. With \( \psi_{k,0}(y) := y_0 + \omega_k(y')y_n \) we see from (3.50)(c) that

\[ \psi_{k,0}(y) = y_0 - \phi_k(y'', y_n) + c(y)y_n^2 + c(y)y_0y_n. \]

\( ^{52} \)Use (3.50)(c) to see that \( |y - \phi_k| \lesssim \epsilon \) on the support of \( T_{k,k}^{\epsilon}. \)

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Using (4.51) and \( \partial_y \phi_k(y''), y_n = O(y_n) \), we compute\(^{53}\)

\[
\begin{align*}
(a) (\partial_y + \partial_y \phi_k \partial_0) \left[ \frac{\psi_{k,0}}{\epsilon} + s(\omega_m(y') - \omega_k(y')) \right] &= c(y) \frac{y_n^2}{\epsilon} + c(y) \frac{y_0 y_n}{\epsilon} + sc(y) \\
(b) (y_0 - \phi_k) \partial_0 \left[ \frac{\psi_{k,0}}{\epsilon} + s(\omega_m(y') - \omega_k(y')) \right] &= c(y) \frac{y_0 - \phi_k}{\epsilon} + sc(y).
\end{align*}
\]

(4.52)

In each case we obtain a function of the form \( B^c + sB^c \), where here and below we let \( B^c \) denote an element of \( \Lambda^c(\Omega^0) \) that may change from term to term. We get results of the same form if \( M_n \) or \( M_{n+1} \) are used here in place of \( M_0 \) and \( M_\nu \).

3. Suppose first that we let \((M_0, \ldots, M_{n+1})^\gamma\) act only under the integral sign in the expression for \( t^*_{k,k} \). Step 2 and an induction on \(|\gamma|\) show that we obtain a finite sum of terms of the form.

\[
J^c(y) := \int_{+\infty}^{y_n} \left( (\partial_{y,\theta, m}^\alpha \sigma_m) \left( y, \frac{\psi_{k,0}}{\epsilon} + s(\omega_m(y') - \omega_k(y')) \right) (\partial_{y,\theta, p}^\beta \sigma_p) \left( y, \frac{\psi_{k,0}}{\epsilon} + s(\omega_p(y') - \omega_k(y')) \right) \right) B^c(y) s^l \, ds
\]

for some \( l \in \mathbb{N}_0 \) and multi-indices \( \alpha, \beta \). To see that \(|J^c(y)| \lesssim 1\) make the change of variable\(^{54}\)

\[
t = \frac{\psi_{k,0}}{\epsilon} + s(\omega_m(y') - \omega_k(y')), \quad s = \frac{t - \frac{\psi_{k,0}}{\epsilon}}{\omega_m - \omega_k} \quad \text{and} \quad ds = \frac{dt}{\omega_m - \omega_k}.
\]

(4.54)

Since \(|t| \leq 1\) on the support of \((\partial_{y,\theta, m}^\alpha \sigma_m)(y,t)\) and the integrand of (4.53) is nonvanishing only when \( |\frac{\psi_{k,0}}{\epsilon}| \leq 1\) (by Lemma 4.12), we have \(|s| \leq 1\) on the support of that integrand.

4. Only the vector fields \( M_n, M_{n+1} \) differentiate the upper limit of integration \( \frac{\psi_{k,0}}{\epsilon} \) in (4.53). For example, when \( M_n = y_n(\partial_{\theta} + \partial_0 \phi_k \partial_0) \), applied to an integral of the form (4.53), we get an additional term of the form

\[
K^c(y) := (\partial_{y,\theta, m}^\alpha \sigma_m) \left( y, \frac{\psi_{m,0}}{\epsilon} \right) (\partial_{y,\theta, p}^\beta \sigma_p) \left( y, \frac{\psi_{p,0}}{\epsilon} \right) B^c(y) \left( \frac{y_n}{\epsilon} \right)^{l+1}.
\]

(4.55)

Observe that the product of the first two factors has support in \( I_c \) by Prop. 3.16(b) and \( \frac{\psi_{k,0}}{\epsilon} \in \Lambda^c(\Omega^0) \). The arguments used in the proof of Proposition 4.10 in particular (4.28), show that

\[
(M_0, \ldots, M_{n+1})^\gamma \left[ (\partial_{y,\theta, m}^\alpha \sigma_m) \left( y, \frac{\psi_{m,0}}{\epsilon} \right) (\partial_{y,\theta, p}^\beta \sigma_p) \left( y, \frac{\psi_{p,0}}{\epsilon} \right) \right] \right|_{L^\infty(\Omega^0 \cap I_c)} \lesssim 1,
\]

(4.56)

and Remark 4.17 implies\(^{55}\)

\[
(M_0, \ldots, M_{n+1})^\gamma \left( B^c(y) \left( \frac{y_n}{\epsilon} \right)^{l+1} \right) \right|_{L^\infty(\Omega^0 \cap I_c)} \lesssim 1.
\]

(4.57)

\(^{53}\)Note that \( M_c(y_0 - \phi_k) = (\partial_{\theta} + \partial_0 \phi_k \partial_0)(y_0 - \phi_k) = 0\). Having \( \frac{\psi_{k,0}}{\epsilon} \) on the right in 4.52(a) would not be good enough. The cutoff \( \chi^c(y_n, y_n) \) enforces boundedness of \( \frac{\psi_{k,0}}{\epsilon} \) and \( \frac{\psi_{k,0}}{\epsilon} \).

\(^{54}\)Here we use that at least one of \( m, p \) is different from \( k \), say \( m \).

\(^{55}\)Note that \( B^c \) and \( \left( \frac{\psi_{k,0}}{\epsilon} \right)^{l+1} \) lie in \( \Lambda^c(\Omega^0) \).
Proposition 4.19. Proof. With the estimate of $W^r$ recall from (3.88) that (4.63)

The same kind of argument treats the additional term that arises when $M_{n+1}$ is applied to the upper limit of integration in (4.53). With the estimate of $J^r$ in step 3, this finishes the proof of (4.50) and Proposition 4.15.

Proposition 4.18. For any $r \in \mathbb{N}_0$ we have

(4.59) \[ \left\langle W^r \left( y', 0, \frac{y_0}{\epsilon}, 0 \right) \right\rangle_{\mathcal{N}^r(\mathcal{M}^r)} \lesssim 1. \]

Proof. We have $W^r(y', 0, \frac{y_0}{\epsilon}, 0) = \chi(\frac{y_0}{\epsilon^2}) \sum_k t_k(y', 0, \frac{y_0}{\epsilon}, 0) r_k(y', 0)$, and by Lemma 4.12 $\|y_0\|_0 \lesssim \epsilon$ on supp $t_k(y', 0, \frac{y_0}{\epsilon}, 0)$. Thus, the cutoff $\chi(\frac{y_0}{\epsilon^2})$ may be ignored here. An argument parallel to the proof of Proposition 4.14 but simpler, now gives (4.59). There is no need to use the cutoffs $\chi_j$, the set $bJ \in (4.11)$ is used in place of $I_e$ in (4.46), and $\mathcal{M}^r$ is used in place of $\mathcal{M}_0$.

4.4 Estimate of the remainder $r^r_\alpha$.

Recall from (3.88) that $r^r_\alpha = \mathcal{E}^r_0 + \mathcal{E}^r_1$. In Proposition 4.10 we have already estimated the contribution to $\mathcal{E}^r_0$ given by the $H_2 - H_2$ term. Next we estimate the other contribution to $\mathcal{E}^r_0$.

4.4.1 Estimate of \[\|(\mathcal{L}_2 - \mathcal{L}_2, 0)W^r\|_{\theta_0=\frac{y_0}{\epsilon}, \xi_n=\frac{y_n}{\epsilon}}.\]

With the estimate of $W^r$ in hand it is now not hard to prove:

Proposition 4.19. For every $r \in \mathbb{N}_0$ we have

(4.60) \[ \|(\mathcal{L}_2 - \mathcal{L}_2, 0)W^r\|_{\theta_0=\frac{y_0}{\epsilon}, \xi_n=\frac{y_n}{\epsilon}} \big|_{\mathcal{N}^r(\mathcal{M}^r)} \lesssim \sqrt{\epsilon}. \]

Proof. 1. Since $\psi_0(y') = y_0$, we have

(4.61) \[ (\mathcal{L}_2 - \mathcal{L}_2, 0)W^r = [\mathcal{A}(y, d' \psi_0) - \mathcal{A}(y', 0, d' \psi_0)] \partial_{\theta_0} W^r = [B_0(y) - B_0(y', 0)] \partial_{\theta_0} W^r, \]

where $W^r(y, \theta_0, \xi_n) = \chi(y_0, y_n) \sum_k t_k(y, \theta_0, \xi_n) r_k(y', 0)$ and $t_k(y, \theta_0, \xi_n)$ is $C^\infty$ with derivatives that are bounded on bounded subsets of $\Omega^0 \times \mathbb{R}_y^2$; see Remark 3.18. Thus,

(4.62) \[ [(\mathcal{L}_2 - \mathcal{L}_2, 0)W^r]|_{\theta_0=\frac{y_0}{\epsilon}, \xi_n=\frac{y_n}{\epsilon}} = \chi(y_0, y_n) [B_0(y) - B_0(y', 0)] \sum_k \partial_{\theta_0} t_k(y, \theta_0, \xi_n) r_k(y', 0)|_{\theta_0=\frac{y_0}{\epsilon}, \xi_n=\frac{y_n}{\epsilon}}. \]

In view of (4.62) in place of (4.42) we should now estimate terms of the form

(4.63) \[ \mathcal{T}_{k, m,p}(y) := \chi(y_0, y_n) y_n \left( \int_{-\infty}^{\xi_n} I_{k, m,p}(s; y, \theta_0, \xi_n) ds \right) \bigg|_{\theta_0=\frac{y_0}{\epsilon}, \xi_n=\frac{y_n}{\epsilon}, m \neq p}, \]

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where \( \int_{t_n}^{t_n} T_{k,m,p}(s; y, \theta_0, \xi_n)ds \) denotes the integral (4.33) with one of \( \sigma_m \) or \( \sigma_p \) replaced by \( \partial \theta_0, \sigma_m \) or \( \partial \theta_0, \sigma_p \).

2. Parallel to (4.43) we now write

\[
T_{k,m,p}(y) = \sum_j \chi_j(y) T_{k,m,p}^j(y) = \sum_j T_{k,j}^\epsilon(y).
\]

We claim that

\[
|T_{k,j}^\epsilon(y)|_{N^r(\Omega^0, M_0)} \lesssim \epsilon \quad \text{for } j \neq k. \tag{4.64}
\]

To see this use the argument of Proposition 4.14, the observation that

\[
|\gamma| \leq r \quad (V_1, \ldots, V_{n+3})^\gamma y_n = C(y_0, y_n)^\kappa \quad \text{where } |\kappa| = 1,
\]

and the fact \(|(y_0, y_n)^\kappa| \leq \epsilon \) on \( \text{supp } T_{k,j}^\epsilon \subset I_\epsilon \).

3. Finally, we claim that

\[
|T_{k,k}^\epsilon(y)|_{N^r(\Omega^0, M_k)} \lesssim \sqrt{\epsilon}. \tag{4.66}
\]

To see this, first use an induction on \(|\gamma|\) to show

\[
(M_0, \ldots, M_{n+1})^\gamma y_n = (c_\gamma(y)(y_0 - \phi_k), d_\gamma(y)y_n)^\kappa
\]

for some smooth functions \( c_\gamma, d_\gamma \) and multi-index \( \kappa \) such that \(|\kappa| = 1\) \(^57\) The estimate (4.66) then follows from the argument of Proposition 4.15 and the fact that

\[
|(c_\gamma(y)(y_0 - \phi_k), d_\gamma(y)y_n)^\kappa|_{L^\infty(\Omega^0 \cap I_\sqrt{\epsilon})} \lesssim \sqrt{\epsilon},
\]

\[\Box\]

### 4.4.2 Estimate of \( \epsilon \mathcal{E}_1^\epsilon(y) \).

In this section we prove the following proposition as a consequence of Corollary 4.9 and Propositions 4.21 and 4.24 below:

**Proposition 4.20.** For every \( r \in \mathbb{N}_0 \) and \( \mathcal{E}_1^\epsilon \) as in (3.88) we have

\[
|\epsilon \mathcal{E}_1^\epsilon(y)|_{N^r(\Omega^0)} \lesssim \sqrt{\epsilon}. \tag{4.67}
\]

Consider the term \( \epsilon [L(y, \partial_y)W^\epsilon(y, \theta_0, \xi_n)] |_{\theta_0 = \frac{m}{r}, \xi_n = \frac{m}{r}} \), where

\[
L(y, \partial_y)W^\epsilon = \sum_{j=0}^{n} B_j(y)\partial_j \left[ \chi^\epsilon(y_0, y_n) \sum_{k} \tau_k(y, \theta_0, \xi_n) r_k(y', 0) \right]. \tag{4.68}
\]

**Proposition 4.21.** For all \( r \in \mathbb{N}_0 \) we have

\[
\left| [L(y, \partial_y)W^\epsilon(y, \theta_0, \xi_n)] |_{\theta_0 = \frac{m}{r}, \xi_n = \frac{m}{r}} \right|_{N^r(\Omega^0)} \lesssim \frac{1}{\sqrt{\epsilon}}. \tag{4.69}
\]

\(^{56}\) Here the \( V_j \) are the generators of \( \mathcal{M}_0 \).

\(^{57}\) The \( M_j \) are the generators of \( \mathcal{M}_k \).
Proof. As usual we are free to ignore smooth \( \epsilon \)-independent functions of \( y \) in the estimates, so we focus on estimating for example:

\[
\begin{aligned}
\left[ \partial_y \left( \frac{y_0}{\sqrt{\epsilon}} \right) \chi \left( \frac{y_n}{\sqrt{\epsilon}} \right) t_k(y, \theta_0, \xi_n) \right] |_{\theta_0 = \frac{y_0}{\sqrt{\epsilon}}, \xi_n = \frac{y_n}{\sqrt{\epsilon}} = \left[ \frac{1}{\sqrt{\epsilon}} \chi' \left( \frac{y_0}{\sqrt{\epsilon}} \right) \chi \left( \frac{y_n}{\sqrt{\epsilon}} \right) t_k(y, \theta_0, \xi_n) \right] |_{\theta_0 = \frac{y_0}{\sqrt{\epsilon}}, \xi_n = \frac{y_n}{\sqrt{\epsilon}} } + \\
\left[ \chi \left( \frac{y_0}{\sqrt{\epsilon}} \right) \chi \left( \frac{y_n}{\sqrt{\epsilon}} \right) \partial_y t_k(y, \theta_0, \xi_n) \right] |_{\theta_0 = \frac{y_0}{\sqrt{\epsilon}}, \xi_n = \frac{y_n}{\sqrt{\epsilon}} } := P_k(y) + Q_k(y).
\end{aligned}
\]

We claim

\[
|P_k|_{N^\tau(\Omega^p)} \lesssim \frac{1}{\sqrt{\epsilon}} \quad \text{and} \quad |Q_k|_{N^\tau(\Omega^p)} \lesssim \frac{1}{\sqrt{\epsilon}}.
\]

Each of \( P_k, Q_k \) is similar to \( T_{k,m,p} \) in (4.32), and so can be estimated by the argument used to prove Proposition 4.11. This is clear for \( P_k \), since \( \chi' \left( \frac{y_0}{\sqrt{\epsilon}} \right) \chi \left( \frac{y_n}{\sqrt{\epsilon}} \right) \) \( \in \Lambda^\tau(\Omega^0) \). To treat \( Q_k \) first use the formula (3.78) to see that \( \partial_{\theta} t_k(y, \theta_0, \xi_n) \) is a sum of terms of the form

\[
\int_{+\infty}^{\xi_n} \left[ \partial_0 \sigma_m(\ldots) + \partial_m \sigma_m(\ldots) \left( (\partial_0 \omega_k) \xi_n + s(\partial_0 \omega_m - \partial_0 \omega_k) \right) \right] \sigma_p(\ldots) ds + II.
\]

Here the derivatives of \( \sigma_m \) are evaluated at \( (y, \theta_0 + \omega_k(y')) \xi_n + s(\omega_m(y') - \omega_k(y')) \), and \( II \) is a similar term where \( \sigma_p \) is differentiated and not \( \sigma_m \). When (4.72) is evaluated at \( \theta_0 = \frac{y_0}{\sqrt{\epsilon}}, \xi_n = \frac{y_n}{\sqrt{\epsilon}} \), the term \( \partial_0 \omega_k(y') \frac{y_n}{\sqrt{\epsilon}} \) appears. The support of \( \chi_j(y)Q_k \) when \( j \neq k \) is contained in \( I \) by Lemma 4.13 and we have \( \frac{y_n}{\sqrt{\epsilon}} \in \Lambda(\Omega^0) \). But the support of \( \chi_k(y)Q_k \) is contained in \( I_{\sqrt{\tau}} \), and we only have \( \frac{y_n}{\sqrt{\epsilon}} = \frac{1}{\sqrt{\epsilon}} \frac{y_n}{\sqrt{\epsilon}} \) with \( \frac{y_n}{\sqrt{\epsilon}} \in \Lambda_{\sqrt{\tau}}(\Omega^0) \).

The estimates (4.71) still hold when \( \partial_0 \) is replaced by \( \partial_i, i \neq 0 \) in (4.70). \( \square \)

To estimate the term of \( \epsilon \mathcal{E}_1 \) given by \( \epsilon K(y, U_0, \epsilon \delta r_1') \mathcal{U}_r' \), we can apply the following general result from [Met80]. In Proposition 4.22 \( T_0 \) need not be small and for any \( 0 < T < T_0, \Omega_T \) is a truncated backward cone as defined in (4.8).

**Proposition 4.22.** (a) For \( M \in \mathbb{N} \) let \( F(y, Z) \in C^\infty(\mathbb{R}^{1+n} \times \mathbb{R}^M, \mathbb{R}^N) \). For any \( 0 < T_1 < T_0 \) and \( r \in \mathbb{N}_0 \) there exist a constant \( C_0 > 0 \) and an increasing function \( h: \mathbb{R}_+ \to \mathbb{R}_+ \) such that for any \( 0 < T \leq T_1 \), if \( Z \in L^\infty \cap N^\tau(\Omega_T) \), then \( f(y) := F(y, Z(y)) \) belongs to \( L^\infty \cap N^\tau(\Omega_T) \) and satisfies

\[
(i) |f|_{L^\infty(\Omega_T)} \leq h(|Z|_{L^\infty(\Omega_T)}) \\
(ii) |f|_{N^\tau(\Omega_T)} \leq C_0 + h(|Z|_{L^\infty(\Omega_T)})|Z|_{N^\tau(\Omega_T)}.
\]

(b) For \( T_1 \) as above and any \( r \in \mathbb{N}_0 \) there exists \( C > 0 \) such that for any \( 0 < T \leq T_1 \), if \( u, v \) are real valued functions in \( L^\infty \cap N^\tau(\Omega_T) \) then

\[
|uv|_{L^\infty \cap N^\tau(\Omega_T)} \leq C |u|_{L^\infty \cap N^\tau(\Omega_T)} |v|_{L^\infty \cap N^\tau(\Omega_T)}.
\]

\(^{58}\) Recall Definition 4.16

\(^{59}\) The constant \( C_0 \) and function \( h \) depend on \( T_1 \) but not on \( T \).
Remark 4.23. The estimate \((4.73)\)(i) is trivial. Estimate \((4.73)\)(ii) is stated in Proposition 3.2.2 of \cite{Met89}. The corresponding estimate for functions on \(\mathbb{R}^{1+n}_+\) is a consequence of Lemmas 3.3.1 and 3.3.3 of the companion paper \cite{Met86}. The estimate on \(\Omega_T\) is then deduced using the bounded extension operators \(R_T : L^\infty \cap N^m(\Omega_T) \rightarrow L^\infty \cap N^m(\mathbb{R}^{1+n}_+)\) constructed in Proposition 3.2.1 of \cite{Met89}. The estimate \((4.74)\) is not explicitly stated in \cite{Met89} or \cite{Met86}, but it too is a direct consequence of Lemmas 3.3.1 and 3.3.3 of \cite{Met86} and Proposition 3.2.1 of \cite{Met89}.

**Proposition 4.24.** Let \(\Omega_T\) be as in Proposition 4.22 and suppose \(\Omega_T \subset \Omega^0\) for \(\Omega^0\) as chosen in section 3.6.3. For any \(r \in \mathbb{N}_0\) we have

\[
\left| \epsilon K(y, U_0, aU_1', \xi) \right|_{y=\psi, \theta=\nu, \xi_n=\nu_n}^{1, \infty(N_r(\Omega_T))} \lesssim \epsilon. \tag{4.75}
\]

**Proof.** By construction we have

\[
\left| (U_0, U_1')\right|_{y=\psi, \theta=\nu, \xi_n=\nu_n}^{1, \infty(N_r(\Omega_T))} \lesssim 1,
\]

so the result follows directly from Propositions 4.15 and 4.22.

The formulas \((3.88)\) and the estimates of this section imply:

**Proposition 4.25.** Let \(\Omega_T\) be as in Proposition 4.22 and suppose \(\Omega_T \subset \Omega^0\). For any \(r \in \mathbb{N}_0\) we have

\[
|r_a^c|_{L^\infty \cap N^r(\Omega_T)} \lesssim \sqrt{\epsilon}, \tag{4.76}
\]

where the implied constant is independent of \(0 < T < T_1\).

### 4.5 Estimate of the error \(w^c = u^c - u_0^c\).

In this section we finish the proof of Theorem 1.2.

For \((L(y, \partial_y), B)\) as in section 2 consider the linear boundary problem on \(\mathbb{R}^{1+n}_+\):

\[
L(y, \partial_y)v = f \text{ in } y_n > 0,
\]

\[
B(y')v|_{y_n=0} = g,
\]

\[
v = 0 \text{ in } y_0 < 0,
\]

where \(f\) and \(g\) are zero in \(y_0 < 0\). We will use the following estimate from \cite{Met89}.

In this general result we let \(\Omega_T\) denote any domain of determinacy as in section 4.2.

**Proposition 4.26.** If \(T_0\) as in \((1.8)\) is small enough, for \(m > \frac{n+5}{2}\) and \(0 < T_1 < T_0\) there exists \(C > 0\) such that for any \(0 < T \leq T_1\) the solution \(u\) of \((4.77)\) satisfies

\[
|v|_{L^\infty \cap N^m(\Omega_T)} \leq C \left[ \sqrt{T} |f|_{L^\infty \cap N^m(\Omega_T)} + |g|_{L^\infty(\Omega_T)} + \sqrt{T} |g|_{N^m(\Omega_T)} \right]. \tag{4.78}
\]

We will apply this proposition to the error problem \((4.2)\) satisfied by \(w^c = u^c - u_0^c\):

\[
L(y, \partial_y)w^c = D(y, u^c, u_0^c)w^c - r_a^c := f^c \text{ in } y_n > 0,
\]

\[
B(y')w^c|_{y_n=0} = -\epsilon B(y')U_1'(y',0) := g^c,
\]

\[
w^c = 0 \text{ in } y_0 < 0, \tag{4.79}
\]

\[\text{This estimate combines the estimates of Corollary 4.1.4 and Proposition 5.1.2 of } \cite{Met89}.\]
Suppose now that the parameters $T_0$ and $\alpha$ as in (4.8) are chosen small enough so that Propositions 4.4 and 4.20 apply, and fix some $T_1$ with $0 < T_1 < T_0$ such that the exact solution $u^e$ of (1.4) satisfies $\|u^e\|_{L^\infty \cap N^m(\Omega_{T_1})} \lesssim 1$. We also require $\Omega_{T_1} \subset \Omega^0$. We have

$$
\|u^e\|_{L^\infty \cap N^m(\Omega_{T_1})} \lesssim 1 \quad \text{and} \quad |r^e_a|_{L^\infty \cap N^m(\Omega_{T_1})} \lesssim \sqrt{\epsilon}
$$

by Propositions 4.3 and 4.25 and

$$
\langle g^e \rangle_{L^\infty \cap N^m(\Omega_{T_1})} \lesssim \epsilon
$$

by Propositions 4.8 and 4.11. We use Proposition 4.22(a),(b) to estimate for $0 < T \leq T_1$:

$$
|D(y, u^e, u^e_0)w^e|_{L^\infty \cap N^m(\Omega_T)} \lesssim \|w^e\|_{L^\infty \cap N^m(\Omega_T)} \Rightarrow |f^e|_{L^\infty \cap N^m(\Omega_T)} \lesssim \|w^e\|_{L^\infty \cap N^m(\Omega_T)} + \sqrt{\epsilon}.
$$

Applying the estimate (4.78) we obtain:

$$
|w^e|_{L^\infty \cap N^m(\Omega_T)} \leq C \left[ \sqrt{T}(|w^e|_{L^\infty \cap N^m(\Omega_T)} + \sqrt{\epsilon}) + \epsilon + \sqrt{T}\epsilon \right].
$$

Decreasing $T$ if necessary so that $C\sqrt{T} \leq \frac{1}{2}$, we thus obtain

$$
|w^e|_{L^\infty \cap N^m(\Omega_T)} \leq 2C(\sqrt{T}\sqrt{\epsilon} + \epsilon + \sqrt{T}\epsilon) = O(\sqrt{\epsilon}).
$$

5 Extension to general nonlinear terms $f(y, u)$.

In the preceding sections we worked with a function $f(y, u)$ that was quadratic in $u$, mainly in order to be able to see pulse interactions explicitly. We discuss here how the results of this paper can be extended to the case where $f$ is any element of $C^\infty(\mathbb{R}_+^{1+n} \times \mathbb{R}^N, \mathbb{R}^N)$ such that $f(y, 0) = 0$.

**Construction of $U_0$.** With $U_0(y, \theta) = \sum_k \sigma_k(y, \theta_k)r_k(y)$ we write with slight abuse $f(y, U_0) = f(y, \sigma)$. Recall the following definition:

**Definition 5.1.** If $f(y, \sigma)$ is any element of $C^\infty(\mathbb{R}_+^{1+n} \times \mathbb{R}^N, \mathbb{R}^N)$ (or $C^\infty(\mathbb{R}_+^{1+n} \times \mathbb{R}^N, \mathbb{R})$) such that $f(y, 0) = 0$, set $f^m(y, \sigma_m) := f(y, \sigma_m e_m)$, where $e_m$ is the $m$-th standard basis vector of $\mathbb{R}^N$.

We define $Ef = PSf$, where

$$
Sf = \sum_m f^m(y, \sigma_m)
$$

$$
PSf = \sum_m \pi_m f^m(y, \sigma_m).
$$

Writing $f(y, \sigma) = \sum_k f_k(y, \sigma)r_k$ and $f^m(y, \sigma_m) = \sum_k f^m_k(y, \sigma_m)r_k$, we have

$$
(I - S)f = \sum_k \left[ f_k(y, \sigma) - \sum_m f^m_k(y, \sigma_m) \right] r_k.
$$

With the above definition of the operator $E$, the construction of $U_0$ in section 3.5 can be repeated with no significant change. The nonlinear term in the interior profile equation (3.39) for $\sigma_k$
is now \(-l_k f^k(y, \sigma_k)\).

**Construction of \(V\).** With \(F = L(y, \partial_y)U_0 - f(y, U_0)\) we write as before:

\[
(I - E)F = H_1(y, \theta) + H_2(y, \theta)
\]

where

\[
H_1(y, \theta) = (I - P)SF = \sum_k H_{1k}(y, \theta_k)
\]

\[
H_2(y, \theta) = (I - S)F = -(I - S)f(y, U_0) = \sum_k H_{2k}(y, \theta)r_k.
\]

We have

\[
H_{1k}(y, \theta_k) = (1 - \pi_k) \left[ (L(y, \partial_y)\sigma_k)r_k + \sigma_k L(y, \partial_y)r_k - f^k(y, \sigma_k) \right].
\]

Since \(f^k(y, 0) = 0\), it follows that for any \(\theta_k, \sigma_k(y, \theta_k) = 0 \Rightarrow H_{1k}(y, \theta_k) = 0\). Thus, Remark 3.11(b) still applies, and the construction of \(V\) (in \(\mathcal{U}_1^* = V + W^*\)) goes through unchanged.

**Construction of \(W^*\).** As before we define real-valued functions \(\mathcal{H}_2(y, \theta_0, \xi_n)\) and set

\[
\mathcal{H}_2(y, \theta_0, \xi_n) := H_2(\theta_0 + \omega_n(y)\xi_n, \ldots, \theta_0 + \omega_N(y)\xi_n) = \sum_k \mathcal{H}_{2k}(y, \theta_0, \xi_n)r_k(y).
\]

Using (5.2) and (5.3), we see that

\[
(5.5)
\]

(a) \(- \mathcal{H}_{2k}(y, \theta) = f_k(y, \sigma(y, \theta)) - \sum_m f^m_k(y, \sigma_m(y, \theta_m)), \text{ so}
\]

\[
(b) - \mathcal{H}_{2k}(y, \theta_0, \xi_n) = f_k(y, \sigma_1(y, \theta_0 + \omega_1(y)\xi_n), \ldots, \sigma_N(y, \theta_0 + \omega_N(y)\xi_n)) - \sum_m f^m_k(y, \sigma_m(y, \theta_0 + \omega_m(y)\xi_n)).
\]

As before we define real-valued functions \(\mathcal{H}^m_{2k}(y, \theta_0, \xi_n)\) by

\[
(5.6)
\]

(a) \(\mathcal{H}_2(y, \theta_0, \xi_n) = \sum_m \mathcal{H}^m_{2k}(y, \theta_0, \xi_n)r_m(y', 0), \text{ so from (5.5)}
\]

\[
(b) \mathcal{H}^m_{2k}(y, \theta_0, \xi_n) = \sum_k \mathcal{H}_{2k}(y, \theta_0, \xi_n)l_m(y', 0)r_k(y).
\]

We now define \(W(y, \theta_0, \xi_n)\) and \(l_k(y, \theta_0, \xi_n)\) exactly as in (3.75). Using (5.6)(b) and (5.7)(b), we see that \(\mathcal{H}^m_{2k}(y, \theta_0, \xi_n)\) is a finite sum of terms of the form

\[
(5.8)
\]

\[
h(y, \sigma_1(y, \theta_0 + \omega_1(y)\xi_n), \ldots, \sigma_N(y, \theta_0 + \omega_N(y)\xi_n)) - \sum_m h^m(y, \sigma_m(y, \theta_0 + \omega_m(y)\xi_n)),
\]

where \(h(y, \sigma) \in C^\infty\) satisfies \(h(y, 0) = 0\). Similarly, (5.6)(a) implies that \(H_{2k}(y, \theta)\) has the same form where each \(\sigma_j(y, \theta_0 + \omega_j(y)\xi_n)\) is replaced by \(\sigma_j(y, \theta_j)\).

The expression (5.8) is our replacement in this general setting for the products in (3.65). Define \(b \in C^\infty(\mathbb{R}^{1+n}_+ \times \mathbb{R}^N, \mathbb{R})\) by

\[
(5.9)
\]

\[
b(y, \sigma) = h(y, \sigma) - \sum_m h^m(y, \sigma_m).
\]

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Then for a given $k$, $t_k(y, \theta_0, \xi_n)$ is a finite sum of terms of the form

\begin{equation}
\int_{+\infty}^{\xi_n} b(y, \sigma_1(y, \theta_0 + \omega_k(y') \xi_n + s(\omega_1(y') - \omega_k(y'))), \ldots, \sigma_N(y, \theta_0 + \omega_k(y') \xi_n + s(\omega_N(y') - \omega_k(y'))))ds.
\end{equation}

We claim that each $t_k(y, \theta_0, \xi_n)$ continues to satisfy Remark 3.18. To see this, observe first that

\begin{equation}
b^k(y, \sigma_k) = 0 \text{ for all } k.
\end{equation}

That is, $b(y, \sigma) = 0$ whenever all but one component of $\sigma$ are zero. For any $m \neq k$ the $s$–support of $\sigma_m(y, \theta_0 + \omega_k(y') \xi_n + s(\omega_m(y') - \omega_k(y')))$ is contained in

$$K_m := \{s : |\theta_0 + \omega_k(y') \xi_n + s(\omega_m(y') - \omega_k(y'))| \leq 1\},$$

so by (5.11) the $s$–support of the integrand in (5.10) is contained in $K := \cup_{m \neq k}K_m$. This is a compact set whose length is bounded independently of $(y, \theta_0, \xi_n) \in \Omega^0 \times \mathbb{R}^2_0$. This completes the construction of $W^\epsilon(y, \theta_0, \xi_n) = \chi^\epsilon(y_0, y_n)W(y, \theta_0, \xi_n)$.

\textbf{Error analysis.} Using (5.11) again, we can apply Proposition 3.16 (a),(b) to conclude as before that for every $k$

\begin{equation}
H_{2k}(y, \theta)|_{\theta_0 = \frac{\beta}{\epsilon}} \text{ and } H^k_2(y, \theta_0, \xi_n)|_{\theta_0 = \frac{\beta}{\epsilon}, \xi_n = \frac{\beta}{\epsilon}} \text{ are supported in the interaction region } I_\epsilon.
\end{equation}

Moreover, the proof of Lemma 4.12 shows that there exists $M > 0$ such that for each $k$, if $|y_0 + \omega_k(y')| \geq M\epsilon$, then no $s \in \mathbb{R}$ lies in the $s$–support of supp $\sigma_j\left(y, \frac{y_0 + \omega_k(y') y_n}{\epsilon} + s(\omega_j(y') - \omega_k(y'))\right)$ for two distinct $j$. With this (5.11) implies that for each $k$

\begin{equation}
|y_0 + \omega_k(y') y_n| \lesssim \epsilon \text{ on the support of } t_k(y, \theta_0, \xi_n)|_{\theta_0 = \frac{\beta}{\epsilon}, \xi_n = \frac{\beta}{\epsilon}}.
\end{equation}

That is, Lemma 4.12 continues to hold. Having the support conditions (5.12) and (5.13), it is now straightforward to repeat the remaining arguments of section 4 letting $b(y, \sigma)$ play the role of the products (3.65).

\section{A class of problems locally reducible to the model system (1.4).}

Here we explain how to reduce a general class of pulse generation problems to the the model problem (1.4).

On a small enough neighborhood $U \ni 0$ in $\mathbb{R}^{1+n}$ consider the pulse generation problem (1.19) on $\mathcal{D} \cap U$ described in Remark 1.3:

\begin{equation}
\begin{align*}
\mathcal{P}(z, \partial_z)u^\epsilon &= f(z, u^\epsilon) \text{ in } \beta > 0 \\
B(z)u^\epsilon|_{\beta = 0} &= g\left(z, \frac{\psi_0(z)}{\epsilon}\right) \text{ on } \beta = 0 \\
u^\epsilon &= 0 \text{ in } \alpha < 0.
\end{align*}
\end{equation}
Let $S = \{ \alpha = 0 \}$ and $bD = \{ \beta = 0 \}$ be the given smooth spacelike surface and noncharacteristic boundary described there. The transversal intersection assumption allows us to suppose $d\alpha$ and $d\beta$ are linearly independent on $U$. Choosing new coordinates $y = (y_0, y''_n, y_n)$ so that $y_0 = \alpha$, $y_n = \beta$, we arrange so that

$$S = \{ y_0 = 0 \}, \quad bD = \{ y_n = 0 \}, \quad \Delta = \{ y_0 = y_n = 0 \}.$$  

(6.2) Since $y_n = 0$ is noncharacteristic, the problem (6.1) can be written in the new coordinates as

$$L(y, \partial_y) u^\ell = \partial_n u^\ell + \sum_{j=0}^{n-1} B_j(y) \partial_j u^\ell = f(y, u^\ell) \text{ in } y_n > 0,$$

(6.3) $$B(y') u^\ell|_{y_n=0} = g(y', \theta_0)|_{\theta_0=\psi_0(y')} \quad \quad u^\ell = 0 \text{ in } y_0 < 0.$$  

for slightly modified $f$, $g$, and $\psi_0$.

The boundary phase $\psi_0$ is assumed to be a smooth defining function for $\Delta \subset \{ y_n = 0 \}$, so in the new coordinates it has the form

$$\psi_0(y') = y_0 q(y'),$$

(6.4) for some smooth function $q \neq 0$. Replacing $\psi_0$ by $-\psi_0$ if necessary, we arrange so that $q > 0$. Now consider the eikonal problem

$$\partial_n \psi_k = -\lambda_k(y, d'' \psi_k) \quad \quad \psi_k|_{y_n=0} = y_0 q(y'),$$

(6.5) for $\lambda_k(y, \eta')$, $k = 1, \ldots, N$ as defined in section 3.1.

If we change coordinates again, modifying only the time-variable (or $y_0$-coordinate) by taking it to be $y_0 q(y')$, then the pulse continuation problem (6.1) takes the form (1.4). In particular, the phase in the boundary datum is now $\frac{\psi_0}{\psi}$.

Moreover, coordinate invariance of the uniform Lopatinski condition implies that this condition is satisfied by (6.3) at 0. Thus, we have verified that the transformed problem (6.3) satisfies all the hypotheses of Theorem 1.2.

As in (3.7) for each $k = 1, \ldots, N$ we can write

$$\psi_k(y) = (y_0 - \phi_k(y'', y_n)) \beta_k(y)$$

(6.6) for some smooth $\beta_k \neq 0$, where $\phi_k$ satisfies the eikonal problem (3.9). The argument of section 3.6.2 in particular the fact that the numbers $\gamma_k$ in (3.54) are distinct, real, and nonzero, shows that near 0 the surfaces $\Sigma_j = \{ \psi_j = 0 \} = \{ y_0 - \phi_j = 0 \}$ satisfy

$$\Sigma_j \cap \Sigma_k = \Delta \text{ for } j \neq k, \quad \Sigma_j \cap \{ y_n = 0 \} = \Delta, \quad \Sigma_j \cap \{ y_0 = 0 \} = \Delta,$$

(6.7) where all intersections are transversal.

\[\text{In defining the } \lambda_k(y, \eta') \text{ we work with the matrix symbol } L(y, \eta) = \eta_0 I + A(y, \eta') \text{ of } L(y, \partial_y) \text{ as in (6.3). Since } y_n = 0 \text{ is noncharacteristic, Remark 6.1 applies to yield } N \text{ distinct real nonzero eigenvalues } \lambda_k(y, \eta') \text{ of } A(y, \eta'), \text{ which are defined for } \eta' \text{ such that } |\eta''| \leq \delta|\eta_0| \text{ when } \delta > 0 \text{ is small enough.}\]
7 Application to pulse reflection

Consider the pulse reflection problem (1.20), where a single outgoing pulse $u_0^\prime$ concentrated on a characteristic surface $S = \{ \zeta = 0 \} \ni 0$ reflects transversally off a boundary $bD = \{ \beta = 0 \} \ni 0$ starting at “time” $t = 0$, where $\alpha(0) = 0$. In order to define the characteristic phases $\psi_k$ and surfaces $\Sigma_k$ that appear in Theorem 1.4, we begin by showing that near the first time of intersection, the codimension two manifold $\Delta := \Sigma \cap bD$ is contained in a new spacelike surface $\tilde{S} \ni 0$ transverse to $bD$ such that

$$\tilde{\Delta} := \Sigma \cap bD = \tilde{S} \cap bD.$$  

Moreover, there exist $N$ characteristic phases $\psi_k$ and surfaces $\Sigma_k = \{ \psi_k = 0 \}$ such that

$$\Sigma_j \cap \Sigma_k = \tilde{\Delta} \text{ for } j \neq k, \Sigma_j \cap bD = \tilde{\Delta}, \Sigma_j \cap \tilde{S} = \tilde{\Delta},$$

where all intersections are transversal and $\Sigma$ equals one of the $\Sigma_k$. Recall that in formulating (1.20) we assumed that $S = \{ \alpha = 0 \}$ was spacelike at 0, but we did not assume that $S \cap bD = \Sigma \cap bD$. After this geometric preparation we prove the pulse reflection theorem, Theorem 1.4.

In new coordinates such that $y_0 = \alpha$ and $y_n = \beta$ the problem (1.20) takes the form

$$L(y, \partial_y)u^\epsilon = \partial_n u^\epsilon + \sum_{j=0}^{n-1} B_j(y) \partial_j u^\epsilon = f(y, u^\epsilon) \text{ in } y_n > 0,$$

$$B(y') u^\epsilon |_{y_n = 0} = 0$$

$$u^\epsilon = u_0^\prime \text{ in } y_0 < -\gamma.$$  

By assumption $L(y, \partial_y)$ is strictly hyperbolic with respect to $y_0$ and $B_n$ is invertible. We are given that $y = 0$ is a point in the “first-reflection set” $\Sigma \cap \{ y_n = 0 \} \cap \{ y_0 = 0 \}$. Since the intersection $\tilde{\Delta} := \Sigma \cap \{ y_n = 0 \}$ is transversal, some component of $\partial_y \zeta(0)$ is nonzero. If $\partial_j \zeta(0) \neq 0$ for some $j \in \{ 1, \ldots, n-1 \}$, we can write $\zeta(y) = (y_j - \phi(\tilde{y})) h(y)$ near 0 for some smooth $\phi$ and $h$ such $h(0) \neq 0$, where $\tilde{y}$ includes all the components of $y$ except $y_j$. But then

$$\tilde{\Delta} = \{ y : y_n = 0, y_j = \phi(\tilde{y}) \}$$

near 0. This contradicts the fact that the intersection is empty in $y_0 < 0$, because $y_0$ is a component of $\tilde{y}$. Thus, we must have $\partial_y \zeta(0) \neq 0$, which implies $\zeta(y) = (y_0 - \phi(y''', y_n)) h(y)$ so

$$\tilde{\Delta} = \{ y : y_n = 0, y_0 = \phi(y''', 0) \}.$$  

Moreover, $\partial_y \zeta'(0, 0) = 0$, since $\phi(y''', 0)$ has a minimum at $y''' = 0$, and we can suppose $h(0) > 0$ after replacing $\zeta$ by $-\zeta$ if necessary.

Now let $\tilde{S} = \{ y : t(y) := \zeta(y') = 0 \}$. We have $\partial_y t(0) = (1, 0, \ldots, 0) h(0)$, so $\tilde{S}$ is spacelike at 0 and we can take $t(y)$ as a new time (i.e., $y_0$) coordinate. In the new coordinates $u^\epsilon$ again satisfies a problem of the form (7.3) near 0, but with $\gamma$ replaced, for example, by $2\gamma h(0)$.

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62 This was stated in [Met89]; here we provide some detail. In general, the surface $\tilde{S}$ is not equal to $S = \{ \alpha = 0 \}$.

63 We have $t(y_0, 0) = y_0 h(0) < -\gamma h(0) \Leftrightarrow y_0 < -\gamma$.  

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\( \zeta(y', 0) = y_0 \). Defining \( A(y, \eta') \) as in (3.2), we see by Remark 3.1 that \( A(y, \eta') \) has \( N \) distinct real nonzero eigenvalues \( \lambda_j(y, \eta') \) defined for \( |\eta''| \leq \delta|y_0| \) for \( \delta \) small enough. Since \( \zeta \) is a characteristic phase, for some index, say \( \lambda \), it satisfies

\[
\partial_\alpha \zeta = -\lambda_N(y, d'\zeta)
\]

\[
\zeta|_{y,\theta=0} = y_0
\]

near 0. If we define \( \psi_k \) to be the solution of

\[
\partial_\alpha \psi_k = -\lambda_k(y, d'\psi_k)
\]

\[
\psi_k|_{y,\theta=0} = y_0,
\]

we see that \( \zeta = \psi_N \), and the surfaces \( \Sigma_k := \{ \psi_k = 0 \} \) satisfy (7.2) by the argument given at the end of section 6.

The assumption that (1.20) satisfies the uniform Lopatinski condition at \( z = 0 \), when \( \alpha \) is taken as the time coordinate, implies that \( (L(y, \partial_y), B(\eta')) \), when written in the final \( y \)-coordinates constructed above, satisfies the uniform Lopatinski condition at \( y = 0 \) when \( y_0 \) is the time coordinate. In particular, the number of incoming phases \( \psi_k \) is the same as the number of positive eigenvalues of \( B_0(y) \), namely \( p \). After relabeling if necessary, we take \( \psi_1, \ldots, \psi_p \) as the incoming phases.

**Proof of Theorem 1.4.** 1. We consider the problem (7.3) in the final coordinates where (7.4) and (7.5) hold, \( \zeta = \psi_N \), the phases \( \psi_1, \ldots, \psi_p \) are incoming, and the remaining \( \psi_j \) are outgoing. We will write out the proof for the case of a quadratic nonlinearity \( f(y, u) \) as in (3.22). The extension to general nonlinear functions \( f(y, u) \) is then obtained as in section 3.

Parallel to the proof of Theorem 1.2 we look for an approximate solution to (7.3) of the form

\[
u^\epsilon(y) = [U_0(y, \theta) + \epsilon U_1(y, \theta, \theta_0, \xi_n)]|_{\theta = \frac{N}{2}, \theta_0 = \frac{\alpha_N}{2}, \xi_n = \frac{\pi}{2}} = U_0 \left( y, \frac{\psi}{\epsilon} \right) + \epsilon U_1(y), \quad \text{where}
\]

(7.6)

\[
U_0(y, \theta) = \sum_{k=1}^N \sigma_k(y, \theta) r_k(y)
\]

for profiles \( \sigma_k \) to be determined and \( U_1 \) as in (7.13). Now we set \( \sigma_N(y, \theta_N) = s_N(y, \theta_N) + \tau(y, \theta_N) \), where \( s_N \) is unknown and \( \tau \) is the given leading profile of the outgoing pulse \( u^\epsilon \). We have the initial conditions

\[
\sigma_1, \ldots, \sigma_{N-1} = 0 = y_0 < -\gamma, \quad s_N = 0 = y_0 < -\gamma, \quad \text{and } \tau|_{y,\theta=0} = 0 = y_0 < -\gamma.
\]

2. The computations of section 3.3 yield the interior transport equations (3.37), where the coefficients \( c_k, d_k, e_k \) are just as before. As before we conclude that the outgoing profiles \( \sigma_{p+1}, \ldots, \sigma_{N-1} \) are zero. The transport equation for \( \sigma_N \) can be written

\[
X_N(y, \partial_y)s_N + c_N s_N - \left[ d_N s_N + e_N(s_N^2 + 2\tau s_N) \right] = - \left[ X_N(y, \partial_y)\tau + c_N \tau - (d_N \tau + e_N \tau^2) \right].
\]

The equation

\[
X_N(y, \partial_y)\tau + c_N \tau - (d_N \tau + e_N \tau^2) = 0
\]

\[\text{Here we have renamed } -2\gamma h(0) \text{ as } -\gamma.\]
is exactly the interior equation that the (nonzero) profile $\tau$ is constructed to satisfy so that (1.21) holds, so we conclude from (7.7) and (7.8) that $s_N$ is zero. Henceforth we will sometimes write $\sigma_N$ in place of $\tau$. The boundary condition $B(y', 0)U_0(y', 0, \theta_0, \ldots, \theta_0) = 0$ yields in place of (3.38):

\[
(Br_1 \ldots Br_p) \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_p \end{pmatrix} = -\tau Br_N.
\]

As before this determines the boundary values of the incoming profiles. With the transport equations (3.37) and initial conditions (7.7), the $\sigma_j$, $j \leq p$ are now uniquely determined and we have

\[
U_0(y, \theta) = \sum_{k=1}^p \sigma_k(y, \theta_k) r_k(y) + \tau(y, \theta_N) r_N(y).
\]

Observe that

\[
\text{supp } \sigma_j \subset \{ y_0 \geq -\gamma \} \text{ for } j = 1, \ldots, p \text{ and supp } \sigma_N|_{y_n=0} \subset \{ y_0 \geq -\gamma \}.
\]

3. The profile of the corrector

\[
U_0^\epsilon(y, \theta, \theta_0, \xi_n) = V(y, \theta) + W^\epsilon(y, \theta_0, \xi_n)
\]

is constructed exactly as in sections 3.6.1 and 3.6.3. We now have

\[
V(y, \theta) = \sum_{k=1}^p V_k(y, \theta_k) + V_N(y, \theta_N)
\]

\[
W^\epsilon(y, \theta_0, \xi_n) = \chi^\epsilon(y_0, y_n) W(y, \theta_0, \xi_n),
\]

where $V_k$ (resp. $W$) is given by the formula (3.47) (resp. (3.75) and (3.78)). Inspection of these formulas shows

\[
\text{supp } V_k \subset \{ y_0 \geq -\gamma \} \text{ for } k = 1, \ldots, p \text{ and supp } V_N|_{y_n=0} \subset \{ y_0 \geq -\gamma \}
\]

\[
\text{supp } W \subset \{ y_0 \geq -\gamma \},
\]

where the support condition on $W$ reflects the fact that products appearing in the integrand of terms like (3.78) always involve at least one incoming profile.

4. This yields an approximate solution $u_0^\epsilon$ as in (7.6) defined on a set $\Omega^0 \ni 0$ chosen to satisfy $\Omega^0 \subset O$ and all the conditions in section 3.6.2. If $r_0^\epsilon$ given by (3.88) we have

\[
L(y, \partial_y) u_0^\epsilon = f(y, u_0^\epsilon) + r_0^\epsilon
\]

\[
B(y') u_0^\epsilon|_{y_n=0} = \epsilon B(y') U_0^\epsilon(y', 0) := b_0^\epsilon(y')
\]

\[
u_0^\epsilon = \text{supp } \sigma_N \left( y, \frac{\psi_N}{\epsilon} \right) r_N(y) + \epsilon V_N \left( y, \frac{\psi_N}{\epsilon} \right) \text{ in } y_0 < -\gamma.
\]

The estimates of Propositions 4.5 and 12.2 give for any $r \in \mathbb{N}_0$:

\[
|u_0^\epsilon|_{L^\infty \cap N^r(\Omega^0)} \lesssim 1, \quad \langle b_0^\epsilon \rangle_{L^\infty \cap N^r(\delta \Omega^0)} \lesssim \epsilon, \quad |r_0^\epsilon|_{L^\infty \cap N^r(\Omega_T)} \lesssim \sqrt{\epsilon},
\]
where \( \Omega_T \subset \Omega^0 \) is as in Proposition 4.22.

5. Exact solution. By (1.21) we have
\[
u_0^\varepsilon = \sigma_N \left( y, \frac{\psi_N}{\varepsilon} \right) r_N(y) + w_0^\varepsilon, \quad \sigma_N \left( y, \frac{\psi_N}{\varepsilon} \right) r_N(y) + \varepsilon V_N \left( y, \frac{\psi_N}{\varepsilon} \right)
\]
(7.18)

Thus, as in Proposition 4.4 we can use Theorem 2.1.1 of [Mét89] to conclude that (7.3) has an exact solution \( u^\varepsilon \) such that for some \( \Omega_T^1 \subset \Omega^0 \) with \( 0 < T_1 < T_0 \):
\[
|u^\varepsilon|_{L^\infty \cap N^m(\Omega^0 \cap \{y_0 < -\gamma\})} \leq 1.
\]
(7.19)

6. Error problem. In \( y_0 < -\gamma \) we compute
\[
u^\varepsilon - \nu_a^\varepsilon = \sigma_N \left( y, \frac{\psi_N}{\varepsilon} \right) r_N(y) + w_0^\varepsilon - \left[ \sigma_N \left( y, \frac{\psi_N}{\varepsilon} \right) r_N(y) + \varepsilon V_N \left( y, \frac{\psi_N}{\varepsilon} \right) \right] =
\]
(7.20)
\[
\begin{align*}
\Delta_N^1(y, \varepsilon) &:= w_1^\varepsilon, \quad \text{where } |w_1^\varepsilon|_{L^\infty \cap N^m(\Omega^0 \cap \{y_0 < -\gamma\})} = O(1).
\end{align*}
\]

Let \( w^\varepsilon := u^\varepsilon - u_a^\varepsilon \). Parallel to (4.79) we have
\[
L(y, \partial_y)w^\varepsilon = D(y, u^\varepsilon, u_a^\varepsilon)w^\varepsilon - r_a^\varepsilon
\]
(7.21)
\[
B(y', 0)w^\varepsilon|_{y_n=0} = -b^\varepsilon(y')
\]
\[
w^\varepsilon = w_1^\varepsilon \text{ in } y_0 < -\gamma.
\]

The estimates of \( r_a^\varepsilon, b^\varepsilon, \) and \( w_1^\varepsilon \) in (7.17) and (7.20) imply, by an argument similar to that of section 4.5, that
\[
|w^\varepsilon|_{L^\infty \cap N^m(\Omega_T)} = O(1) \text{ as } \varepsilon \to 0.
\]
(7.22)

for some \( 0 < T \leq T_1 \). If as in the construction of [AR02] we have
\[
|w_0^\varepsilon|_{L^\infty \cap N^m(\Omega^0 \cap \{y_0 < -\gamma\}, \mathcal{M}_N)} \leq \varepsilon,
\]
then we obtain the rate of convergence
\[
|w^\varepsilon|_{L^\infty \cap N^m(\Omega_T)} \lesssim \sqrt{\varepsilon} \text{ for } \varepsilon \in (0, \epsilon_0]
\]
(7.23)

for some \( \epsilon_0 > 0 \).

\[\square\]

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\[\text{For the estimate (7.18) we used } |w_0^\varepsilon|_{L^\infty \cap N^m(\Omega^0 \cap \{y_0 < -\gamma\})} \leq |w_0^\varepsilon|_{L^\infty \cap N^m(\Omega^0 \cap \{y_0 < -\gamma\}, \mathcal{M}_N)}, \text{ where } m > \frac{n+2}{2} \text{ is as in Theorem 1.3.}\]
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