Waves propagating parallel to the magnetic field in relativistically hot plasmas: a hydrodynamic model with the average reverse gamma factor evolution

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The high-frequency part of spectrum of electromagnetic waves propagating parallel to the external magnetic field is considered for the macroscopically motionless plasmas with the relativistic temperatures $T \sim m_e c^2$, where $m_e$ is the mass of electron, $c$ is the speed of light. The analysis is based on the novel hydrodynamic model based on four equations for the material fields which can be combined in two four vectors. These material fields are the concentration and the velocity field and the average reverse relativistic $\gamma$ functor and the flux of the reverse relativistic $\gamma$ functor. In the nonrelativistic regime we have three waves (the ions are assumed to be motionless). Strong thermal effects lead to a coefficient in front of cyclotron frequency which decreases the effective contribution of the cyclotron frequency. At $T = 0.1 m_e c^2$ we have a decrease of area of existence of fast magneto-sound wave from the area of the large frequencies. While the area of existence of extraordinary waves becomes larger towards smaller frequencies. The strong magnetic field limit $| \Omega_e | > \omega_{Le}$ additional wave appears with frequency below thermally decreased cyclotron frequency, where $| \Omega_e |$ is the electron cyclotron frequency, and $\omega_{Le}$ is the Langmuir frequency. Further increase of temperature leads to the disappearance of fast magneto-sound wave and to the considerable increase of area of existence of extraordinary towards smaller frequencies.

Keywords: relativistic plasmas, hydrodynamics, microscopic model, arbitrary temperatures

I. INTRODUCTION

Relativistic effects in plasmas like the quantum effects in plasma-like mediums present interesting and important phenomena describing matter in the extreme conditions. Quantum effects are mostly important for the low temperatures [1], [2], [3], [4], [5]. While the relativistic effects can manifest in different regimes like propagation of relativistic beams in plasmas, relativistically hot plasmas, and cold high-density plasma, where the temperature Fermi is of order the rest-energy of electron.

Relativistic kinetic model is the most straightforward method of description of collective effects in the relativistically hot plasmas. However, technically, the kinetic model is rather complex even for nonrelativistic limit. Therefore, it is necessary to have the hydrodynamic model for relativistically hot plasmas. There is a widely used model consisting of two equations: the continuity equation and the Euler equation, where the Euler equation appears as the equation for the evolution of the four-momentum density $P^a$. Its structure resembles the structure of the nonrelativistic hydrodynamics. The continuity equation and the Maxwell equations contain the concentration of particles $n$ and the current of particles $v$, which are combined in the four-velocity vector $v^a$ [6], [7], [8]:

$$\partial_t n + \nabla \cdot (nv) = 0,$$  \hspace{1cm} (1)

and

$$\partial_t P^a + \nabla \cdot (vP^a) + \nabla^a P = q_n \left( E + \frac{1}{c} v \times B \right).$$  \hspace{1cm} (2)

where $P$ is the pressure. Hence, the described model requires the additional equation of state to get the relation between the momentum density $P^a$ and the four velocity $v^a$. Simple relation can be found for the "cold" plasmas, where all electrons have same velocity and the relativistic effects are related to the propagation of all electrons as the beam: $P^a = mn^2 v^a$, where $m$ is the mass of particle, and $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the relativistic factor, with $v^2 = v^2$. We cannot introduce such relation if there is nonzero temperature, but it is possible to calculate an approximate equation of state using the presentation of the momentum density $P^a$ and the current of particles $nv^a$ via the equilibrium distribution function. It leads to $P^a = (K_3(\beta)/K_2(\beta))mv^a\gamma v^a$, where parameter $\beta = mc^2/T$ is the reverse dimensionless temperature, $T$, the temperature itself $T$ is presented in the energy units, and functions $K_i(\beta)$ are the $i$-th order Macdonald functions.

Any hydrodynamic and kinetic model requires a truncation. For instance, the nonrelativistic hydrodynamics of plasmas obtained in the meanfield approximation includes equation of state for the pressure. The relativistic hydrodynamic model presented within equations (1) and (2) requires two equations of state. One for the pressure. The second equation is for the momentum density, which is discussed above. The perturbation of pressure can be found within an extended hydrodynamic model [9], [10], [11], [12], [13]. However, some additional functions would appear, so they would require some equations of state anyway.
FIG. 1: The square of the refractive index square $n^2$ as the function of the frequency $n^2(\omega)$ is demonstrated to understand the spectrum of the electromagnetic waves in magnetized and relativistically hot plasmas. In this figure the relativistic effects are relatively small, so $\beta = mc^2/T = 10$. The refractive index square $n^2$ is demonstrated in compare with the nonrelativistic regime (it is presented within the dashed lines). The upper figure is made for small cyclotron frequency $b = |\Omega_e|/\omega_{Le} = 0.5$. The middle figure is made in the same regime, but it shows the high-frequency part of spectrum in more details. The lower figure is made for the large cyclotron frequency $b = |\Omega_e|/\omega_{Le} = 2$.

FIG. 2: The spectrum similar to figure 1 is demonstrated for the temperature equal to the rest energy of electron $\beta = mc^2/T = 1.0$. The upper figure is made for small cyclotron frequency $b = |\Omega_e|/\omega_{Le} = 0.5$. The middle figure is made in the same regime, but it shows the high-frequency part of spectrum in more details. The lower figure is made for the large cyclotron frequency $b = |\Omega_e|/\omega_{Le} = 2$.

It is well known that the application of the equation of state for the pressure for estimation of the perturbation of pressure gives incorrect coefficients. For instance, coefficient in corresponding term in the spectrum of Langmuir wave. In Maxwellian plasmas we have coefficient 1 instead of 3. For the degenerate plasmas we have coefficient $1/3$ instead of $3/5$. The application of the extended hydrodynamic model gives systematic corrections of these coefficients [9], [10].

One of the major collective plasma phenomena is the Langmuir wave. Its frequency is approximately equal to
FIG. 3: The spectrum similar to figure (1) is demonstrated for the temperature above the rest energy of electron $\beta = mc^2/T = 0.1$. The upper figure is made for small cyclotron frequency $b = |\Omega_e|/\omega_{Le} = 0.5$. The middle figure is made in the same regime, but it shows the high-frequency part of spectrum in more details. The lower figure is made for the large cyclotron frequency $b = |\Omega_e|/\omega_{Le} = 2$.

the Langmuir frequency $\omega_{Le} = \sqrt{4\pi e^2 n_0/m_e}$. There is increase of frequency with the increase of the wave vector $k$: $\omega = \sqrt{\omega_{Le}^2 + \alpha k^2 v_s^2}$, and this increase is related to the thermal effects (to the Pauli blocking in the degenerate plasmas), where coefficient $\alpha$ is discussed above. Hence, the coefficient in front of the Langmuir frequency, which is the major term, can be obtained from the momentum balance equation at the application of the first term on the left-hand side and the first term on the right-hand side (see eq. 2 for the illustration).

Some rough estimations in equation of state for the pressure would affect minor term. But the accuracy of the equation of state for the momentum density would directly affect the major term. The example of uncertainty of pressure discussed above gives us a hint that the application of the equation of state for the momentum density based on the equilibrium distribution is not completely reliable. Thus, we suggest and apply relativistic hydrodynamic model, where no equation of state is used for the perturbations of major functions.

The Euler equation is the equation of motion of the plasmas, it demonstrates the change of the four-momentum. Hence, it is questionable to make the approximate transition via equation of state in the major term in the model. Therefore, another model is suggested to get description of the plasmas with the relativistically large temperatures [14]. This model is derived from the microscopic motion of the relativistic charged particles [15], [16], [17], [18]. It consists of four equations. Derivation starts with the microscopic definition of the concentration. The evolution of concentration leads to the continuity equation and the definition of the current of particles. The velocity field is introduced as the ratio of the current of particles to the concentration. The example of uncertainty of pressure discussed above gives us a hint that the application of the equation of state for the momentum density based on the equilibrium distribution is not completely reliable. Thus, we suggest and apply relativistic hydrodynamic model, where no equation of state is used for the perturbations of major functions.
the additional function (the fourth rank tensor) requiring the equation of state. Necessary equations of state are obtained either \cite{14}.

Extended sets of hydrodynamic equations allows to calculate equations of state for the hydrodynamic models applying smaller number of functions. However, to some extend, it is possible to avoid the derivation and the application of the extended hydrodynamic models. Equilibrium distribution functions can be used to calculate necessary equations of state. For the relativistic plasmas we can use following distribution function

\[ f_0(p) = Z e^{-\epsilon/T}, \]  

where

\[ Z = \frac{n}{4\pi m^2 c T K_2(\frac{mc^2}{T})}, \]

with \( T \) is the equilibrium temperature in the energy units, \( p \) is the momentum, \( K_2(\xi) \) is the second order MacDonald function, and \( \epsilon = \sqrt{m^2 c^4 + p^2 c^2} \). Hence, we use it below to get a number of equations of state. We consider the magnetized plasmas, therefore, the application of the isotropic distribution function does not reflect all major physical phenomena. The phenomena related to the two temperature regimes, one (another) temperature for the motion parallel (perpendicular) to the magnetic field. However, the isotropic limit can show some essential effects related to the relativistic phenomena.

This paper is organized as follows. In Sec. II the relativistic hydrodynamic equations are presented and discussed. In Sec. III the spectrum of collective excitations is considered analytically. In Sec. IV numerical analysis of obtained spectra is demonstrated. In Sec. V a brief summary of obtained results is presented.

II. RELATIVISTIC HYDRODYNAMIC MODEL

Here we follow Ref. \cite{14}, where the following set of hydrodynamic equations is derived for the relativistic plasmas with the relativistic temperature. It consists of four equations (in three-vector notations). First equation is the continuity equation

\[ \partial_t n + \nabla \cdot (n \mathbf{v}) = 0. \]

Next, the velocity field evolution equation is

\[ n \partial_t \mathbf{v} + n (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\partial \mathbf{p}}{m} = \frac{e}{m} \Gamma E^a + \frac{e}{mc^2} \varepsilon^{abc}(\Gamma \mathbf{v}_b + t_b) B_c \]

\[ - \frac{e}{mc^2} (\Gamma v^a v^b + v^a t^b + v^b t^a) E_b - \frac{e}{mc^2} t^a E^a. \]

It includes the flux of the thermal velocities \( p \) which is similar to the pressure, but the pressure is the flux of momentum. The interaction of electrons via the electromagnetic field is presented within four terms placed on the right-hand side. It is found in the mean-field approximation (the self-consistent field approximation). Parameters \( m \) and \( e \) are the mass and charge of particle, \( c \) is the speed of light, \( \delta^{ab} \) is the three-dimensional Kronecker symbol, \( \varepsilon^{abc} \) is the three-dimensional Levi-Civita symbol.

In equation (3) and below we assume the summation on the repeating index \( v^b E_b = \sum_{b=x,y,z} v^b E_b \). Moreover, the metric tensor has diagonal form corresponding to the Minkovskii space, it has the following sings \( \gamma^{a \beta} = \{-1, +1, +1, +1\} \). Hence, we can change covariant and contrvariant indexes for the three-vector indexes: \( v^s_a = v_b s \). The Latin indexes like \( a, b, c \) etc describe the three-vectors, while the Greek indexes are deposited for the four-vector notations. The Latin indexes can refer to the species \( s = e \) for electrons or \( s = i \) for ions. The Latin indexes can refer to the number of particle \( j \) at the microscopic description. However, the indexes related to coordinates are chosen from the beginning of the alphabet, while other indexes are chosen in accordance with their physical meaning.

The equation of evolution of the averaged reverse relativistic gamma factor, called here the hydrodynamic Gamma function, is

\[ \partial_t \Gamma + \partial_b (\Gamma v^b + t^b) = - \frac{e}{mc^2} n \mathbf{v} \cdot \mathbf{E} \left( 1 - \frac{1}{c^2} \left( \mathbf{v}^2 + \frac{5p}{n} \right) \right). \]

This function appears on the right-hand side of the velocity field evolution equation together with two other functions. They are the vector of current of the reverse relativistic gamma factor and the second rank tensor describing the flux of current of the reverse relativistic gamma factor.

The final equation in this set of hydrodynamic equations is the equation of evolution of current of the reverse relativistic gamma factor (the hydrodynamic Theta function). Actually it is equation for its thermal part \( t^a \):

\[ (\partial_t + \nabla \cdot \nabla) t^a + \partial^a \tilde{\mathbf{v}} + (\mathbf{t} \cdot \nabla) v^a + t^a (\nabla \cdot \mathbf{v}) \]

\[ + \Gamma (\partial_t + \nabla \cdot \mathbf{v}) v^a = \frac{e}{n} m E^a \left( 1 - \frac{\mathbf{v}^2}{c^2} - \frac{3p}{nc^2} \right) \]

\[ + \frac{e}{mc^2} \varepsilon^{abc} n v_b B_c \left( 1 - \frac{\mathbf{v}^2}{c^2} - \frac{5p}{nc^2} \right) - \frac{2e}{mc^2} E^a p \left( 1 - \frac{\mathbf{v}^2}{c^2} \right) \]

\[ - \frac{e}{mc^2} n v^a v_b E_b \left( 1 - \frac{\mathbf{v}^2}{c^2} - \frac{9p}{nc^2} \right) + \frac{10e}{3mc^2} M E^a. \]

We use equation of state for the second rank tensor describing the flux of current of the reverse relativistic gamma factor, which enters equation (3).

The equations for the material fields \( B, E \) are coupled to the Maxwell equations \( \nabla \times \mathbf{E} = - \frac{1}{c} \partial_t \mathbf{B}, \quad \nabla \cdot \mathbf{E} = 4\pi (e n_i - e n_e), \)

\[ \nabla \times \mathbf{E} = - \frac{1}{c} \partial_t \mathbf{B}, \quad \nabla \cdot \mathbf{E} = 4\pi (e n_i - e n_e), \]
The ions are considered as the motionless positively charged background.

### III. ELECTROMAGNETIC WAVES IN THE RELATIVISTIC MAGNETIZED PLASMAS

The small amplitude waves are considered. The macroscopically motionless plasmas is considered in the equilibrium state. Its equilibrium state can be described by the relativistic Maxwellian distribution. Hence, required equilibrium equations of state can be gained from this distribution. The equilibrium concentration $n_0$ is nonzero. The equilibrium velocity field $v_0$ and the equilibrium electric field $E_0$ are equal to zero. The external magnetic field is constant and uniform. It is directed along Oz axis $B_0 = B_0\hat{e}_z$. The pressure tensor $p^{ab}$ and tensor $\tau^{ab}$ are assumed to be diagonal tensors: $p^{ab} = p\delta^{ab}$ and $\tau^{ab} = \tau\delta^{ab}$. The "diagonal" form is assumed for tensor $M^{abcd}$ as well: $M^{abcd} = M_0(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc})/3$. Functions $\Gamma_0$, $\tau_0$, $p_0$, $q_0$, $M_0$ describe the equilibrium state. The perturbations $\delta p$, $\delta t$ require some equations of state. The equilibrium expressions for functions $p$, $\tau$, $q$, $M$ are used as the equations of state for the non-equilibrium functions. Approximate calculation of functions $p^{ab}$, $\tau^{ab}$, $q$, and $M^{abcd}$ gives to the following representations $p^{ab} = c^2\delta^{ab}Zf_1(\beta)/3$, $\tau^{ab} = c^2\delta^{ab}Zf_2(\beta)/3$, $M^{abcd} = c^4(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc})Zf_3(\beta)/15$, and $q = 0$, where $\beta = mc^2/T$, $Z = 4\pi ZZ(m^3) = n\beta K_z^2(\beta)$,

$$f_1(\beta) = \int_1^{+\infty} \frac{dx}{x^2} (x^2 - 1)^{3/2} e^{-\beta x},$$

$$f_2(\beta) = \int_1^{+\infty} \frac{dx}{x^2} (x^2 - 1)^{3/2} e^{-\beta x},$$

and

$$f_3(\beta) = \int_1^{+\infty} \frac{dx}{x^2} (x^2 - 1)^{5/2} e^{-\beta x}.$$  

The dynamics of the x- and y-projections of the velocity and the x- and y-projections of the electric field require the equations for evolution of the x- and y-projections of the flux of the reverse relativistic factor:

$$\partial_t \delta x + \Gamma_0 \partial_t \delta v_x = 0$$

$$\partial_t \delta y + \Gamma_0 \partial_t \delta v_y = 0,$$

describe the longitudinal waves described in Ref. [14].

The Maxwell equations lead to

$$(\omega^2 - k_z^2c^2)\delta E + 4\pi q_n \omega n_0 \delta v = 0.$$  

The z-projection of the velocity field together with the z-projection of the electric field and the concentration satisfying linearized continuity equation (13):

$$\partial_t \delta n + n_0 \partial_z \delta v_z = 0$$

describe the longitudinal waves described in Ref. [14].

The evolution of the average reverse relativistic factor is not involved in the equations presented above. Its equation

$$\partial_t \delta \Gamma + \Gamma_0 \partial_z \delta v_x + \partial_z \delta t_z = 0,$$

together with evolution of $\delta t_z$ do not give any additional solutions either.

To consider various temperature regimes we choose three regimes $T_1 = 0.1mc^2$, $T_2 = mc^2$, and $T_3 = 10mc^2$. It leads to the following values of the dimensionless parameter $\beta = mc^2/T$: $\beta_1 = 10$, $\beta_2 = 1$, and $\beta_3 = 0.1$. The following values of parameters defining the equilibrium values of parameters and equations of state in these regimes are calculated. For $\beta_2 = 1$, we obtain $K_1/K_2 = 0.38$, $U_{Ez}^2/c^2 = \beta f_2/(3K_2) = 0.1$, $U_p/E_z^2 = \beta f_1/(3K_2) = 0.28$, and $U_m/E_z^2 = \beta f_3/(5K_2) = 0.15$, where $K_1(1) = 0.6$, $K_2(1) = 1.6$, $f_1(1) = 1.35$, $f_2(1) = 0.46$, $f_3(1) = 1.17$. Next, for $\beta_3 = 0.1$, we find $K_1/K_2 = 0.05$, $U_{Ez}^2/c^3 = \beta f_2/(3K_2) = 0.02$, $U_p/E_z^2 = 0.33$, and $U_m/E_z^2 = 0.2$, where $K_1(0.1) = 10$, $K_2(0.1) = 200$, $f_1(0.1) = 2 \times 10^4$, $f_2(0.1) = 100$, $f_3(0.1) = 2 \times 10^3$. Finally, for $\beta_1 = 10$, we have $K_1/K_2 = 0.91$, $U_{Ez}^2/c^2 = 0.07$, $U_p/E_z^2 = 0.08$, and $U_m/E_z^2 = 0.02$, where $K_1(10) = 2 \times 10^{-5}$, $K_2(10) = 2.2 \times 10^{-5}$, $f_1(10) = 5 \times 10^{-7}$, $f_2(10) = 4.2 \times 10^{-7}$, $f_3(10) = 1.7 \times 10^{-7}$.

We start the discussion of the spectrum with the representation of the spectrum in nonrelativistic regime found
from the standard hydrodynamic model based on the continuity and Euler equations:

\[ \omega^2 - k^2c^2 - \frac{\omega^2_{Le}}{\omega + \Omega_e} = 0, \]  

(22)

where \( \omega^2_{Le} = 4\pi e^2n_0/m \) is the Langmuir frequency, \( \Omega_e = -|\Omega_e| = q_eB_0/mc \).

Next, we present corresponding spectrum found from equations (14), (15), (17), (19), and (20):

\[
\frac{\omega^2 - k^2c^2}{\omega^2 - \Omega_e^2(1 - 5\frac{u_p^2}{c^2})} \times \left[ \omega^2 \left( \frac{\Gamma_0}{n_0} - \frac{u_t^2}{c^2} \right) \pm \omega \Omega_e \left( 1 - \frac{u_p^2}{c^2} + \frac{10u_M^4}{3c^4} \right) \right] = 0. \]  

(23)

If the temperature is rather small we have that the characteristic velocities \( u_p, u_t, u_M \) are small in compare with the speed of light. So, we have \( u_p \ll c, u_t \ll c, u_M \ll c \) and \( \Gamma_0 \rightarrow n_0 \). Hence, the thermal effects can be neglected in (23), appearing ratio \((\omega \pm \Omega_e)/\omega^2 - \Omega_e^2 \) simplifies to \( \omega/\omega \pm \Omega_e \), and we obtain equation (22). We have \( 1/\omega \pm \Omega_e = 1/\omega \pm |\Omega_e| \) in equation (22). The upper sign corresponds to the slower extraordinary wave. The lower sign leads to the pole in this fraction at \( \omega = \Omega_e \). This frequency divides range of frequencies on two areas. Each of them contains the wave solution. Fast magneto-sound wave in the small frequency area and the fast extraordinary wave in the large frequency area.

Presence of the temperature effects does not allow to cancel parts of numerator and denominator in the last term in equation (22). Hence, both branches of the spectrum have pole at \( \omega = \Omega_e \sqrt{1 - 5u_p^2/c^2} \). While this pole is shifted from the cyclotron frequency in the area of smaller frequencies.

We present the analysis of spectrum based on the refractive index. Hence, we start with the representation of equation (22) via corresponding expression for the refractive index:

\[ n^2 = 1 - \frac{1}{\xi(\xi \pm b)}, \]  

(24)

where we use the definition of the refractive index \( n = kc/\omega \) and dimensionless cyclotron frequency \( b = \Omega_e/\omega_{Le} \).

Let us represent the dispersion equation (22) as the expression for the refractive index which gives the generalization of equation (22):

\[ n^2 = 1 - \frac{[\xi(g - \xi^2)] \pm b(1 - 5p^2 + 10M^4/3)]}{\xi(\xi^2 - b^2(1 - 5p^2))}, \]  

(25)

where \( g = \Gamma_0/n_0, \xi = u_t/c, p = u_p/c, \) and \( M = u_M/c \). Physical effects entering equation (25) reflects all described after equation (23). So we do not repeat this discussion.

A. Analysis of the spectrum

Let us remind the properties of the electromagnetic waves propagation parallel to the magnetic field in the limit of small nonrelativistic temperatures and motionless ions. We have three wave solutions: the fast magneto-sound wave, the slow extraordinary wave, and the fast extraordinary wave. Some times notion "extraordinary wave" is reserved for the waves propagating perpendicular to the magnetic field, but we use it for the high-frequency circularly polarized waves propagating parallel to the magnetic field either. Frequency of the fast extraordinary wave (the slow extraordinary wave) is above of \( \omega^{(1)}_0 = \sqrt{\omega^2_{Le} + \Omega_e^2/4} \) \( \Omega_e \) \( /2 \) (of \( \omega^{(3)}_0 = \sqrt{\omega^2_{Le} + \Omega_e^2/4} \) \( \Omega_e \) \( /2 \)), it goes steady with the increase of the wave vector. For its refractive index we find that it goes to \(-\infty \) at \( \omega \rightarrow +|\Omega_e| \) \( \Omega_e \) \( + \infty \). It grows up to zero value \( n \rightarrow 0 \) at \( \omega^{(1)}_0 \) (at \( \omega^{(3)}_0 \)).

The third wave is the fast magneto-sound wave. Its frequency is restricted by \( \omega \in (0, |\Omega_e|) \). The square of the refractive index goes to infinity at \( \omega \rightarrow 0 \) and \( \omega \rightarrow |\Omega_e| \) \( -\infty \).

Next, we consider the spectrum at the relatively small relativistic temperatures \( \beta = 10 \). We start with the discussion of the small magnetic field regime \( b = |\Omega_e| \) \( /\omega_{Le} = 0.5 \). It is presented in the upper and middle figures in Fig. 1. The asymptotic of fast extraordinary wave is shifted in area of smaller frequencies from \( |\Omega_e| \) \( |\Omega_e| \) \( \beta = 10 \). Hence, the minimal frequency of the fast extraordinary wave is shifted in area of smaller frequencies as well \( \omega^{(3)}_{min,FEx} < \omega^{(1)}_0 \) \( \beta = 10 \). Corresponding \( n^2 \) grows from \( 0 \) to \( 1 \) at the increase of \( \omega \) from \( \omega^{(1)}_0 \) \( \beta = 10 \) up to \( +\infty \).

The third wave is the fast magneto-sound wave. Its frequency is restricted by \( \omega \in (0, |\Omega_e|) \). The square of the refractive index goes to infinity at \( \omega \rightarrow 0 \) and \( \omega \rightarrow |\Omega_e| \) \( -\infty \).
is located in area \( \omega_{up,\text{min},SEx} < \omega_{\text{min},FEx} < |\Omega_c| \). The small frequency wave exists in area \( \omega \in (\omega_{l,\text{up,\text{min},SEx}}, |\Omega_c| (\sqrt{1 - 5p^2}), \) while \( \omega_{l,\text{up,\text{min},SEx}} < \omega_0^3 \).

We consider further modification of spectrum at the increase of temperature up to \( \beta = 1 \). It is represented in Fig. 2. General picture of spectrum is same for the large and small magnetic fields. We have two wave solutions, which are associated with slow and fast extraordinary waves. If we compare this result with nonrelativistic limit we see the disappearance of the fast magneto-sound wave. If we compare it with the regime of small relativistic temperatures we obtain that two wave solutions disappear. The fast extraordinary wave starts at \( \omega_{\text{min},FEx}: \omega_{\text{min},FEx} < \omega_0^3 < \omega_0^1 \) if \( b < 1 \), and \( \omega_{\text{min},FEx} \ll \omega_0^3 \ll \omega_0^1 \) if \( b > 1 \). The spectrum of slow extraordinary wave starts at \( \omega = 0 \) (let us remind that ions are motionless, otherwise this value changes). The frequency increases with the increase of the wave vector, it goes up to \( \infty \) in accordance with \( \omega \rightarrow kc \) at \( k \rightarrow \infty \). All features related to the cyclotron frequency disappear since \( \omega^2 - \Omega^2 (1 - 5p^2) \neq 0 \) due to \( 5p^2 > 1 \) at \( p^2 < 1 \).

Further increase of temperature does not show any dramatic changes (see Fig. 3). The minimal frequency of the fast extraordinary wave becomes smaller \( \omega_{\text{min},FEx} \ll \omega_0^3 < \omega_0^1 \) (for all \( b \)) and the phase velocities of both waves becomes larger, but these changes are rather small.

A relativistic hydrodynamics obtained from the kinetic model with the focus on the low frequency excitations for the magnetized plasmas is presented in Refs. 10, 20 with the corresponding analysis of the spectrum.

IV. CONCLUSION

Propagation of the electromagnetic waves parallel to the external magnetic field in the nonrelativistic plasmas is not affected by the thermal effects. The relativistically hot magnetized plasmas has been considered. It has been found that the thermal effects reveal themselves at the propagation of the electromagnetic waves parallel to the external magnetic field. Three regimes have been found.

1. Small relativistic temperatures \( (T = 0.1mc^2) \) and the small magnetic fields \( (|\Omega_c| / \omega_{Le} = 0.5) \).
2. Small relativistic temperatures \( (T = 0.1mc^2) \) and the large magnetic fields \( (|\Omega_c| / \omega_{Le} = 2) \).
3. Large relativistic temperatures \( T \geq mc^2 \) and arbitrary magnetic field.

Consider the first regime. Three waves exist in the first regime like in the nonrelativistic limit. Two high frequency waves extend area of their existence toward smaller frequencies. However, the small frequency fast magneto-sound wave exists in the smaller interval of frequencies, the maximal frequency is decreased by the relativistic thermal effects.

Consider the second regime. Four waves exist in the second regime. The interval of frequencies for the fast magneto-sound wave is described by the thermal effects from the high frequency side. The interval of frequencies for the fast extraordinary wave is increased from the area of small frequencies. In the nonrelativistic limit they areas of existence are separated by nonzero interval \( (|\Omega_c|, \omega_0^1) \). The length of this interval changes and its position shifts in the area of small frequencies. The slow extraordinary wave is splitted on two waves. They areas of existence are also separated by an interval. The low frequency part exists from nonzero minimal frequency to the thermally shifted cyclotron frequency. The high frequency part exists from minimal frequency located above the thermally shifted cyclotron frequency up to infinity.

Consider the third regime. The fast magneto-sound wave does not exist. Fast and slow extraordinary wave extended the area of existence down to the rather small frequencies. Fast extraordinary wave exists at frequencies above zero value. Slow extraordinary wave exists at the frequency above minimal frequency.

The analysis is based on the model constructed of four hydrodynamic functions, which reduces to two traditional functions: the concentration and the velocity field, in the nonrelativistic limit. These functions are the concentration, the velocity field, the density of reverse relativistic factor, and the current of the reverse relativistic factor. These set of functions and equations of their evolution make up the microscopically justified background for the hydrodynamic study of the collective phenomena in the relativistically hot plasmas.

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VI. DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study, which is a purely theoretical one.

Appendix A: Definitions of the hydrodynamic functions

All microscopic definitions presented below are expressed via the same operator:

\[
\langle ... \rangle \equiv \frac{1}{\Delta} \int_{\Delta} d\xi \sum_{i=1}^{N} \delta(r + \xi - r_i(t)), \quad (A1)
\]

which is replaced by symbol \( \langle ... \rangle \) to express equations in shorter form.

Once we use averaging \( [A1] \) in explicit to present the concentration of particles \( n(r, t) \) in the arbitrary inertial
The current of particles can be written in a short form:

\[ j = \langle v_i(t) \rangle \]  

where \( v_i(t) = dr_i(t)/dt \). Presented definition of current \( j \) gives total velocity of all particles being in the delta vicinity over the volume of the vicinity \( \Delta \):

\[ j = \sum_{i \in \Delta} v_i(t)/\Delta. \]

The average velocity can be introduced at the usage of the current \( v = j/n \). The average velocity is found as arithmetic mean.

Similar definitions are found for other functions [14]:

\[ \Gamma = \langle (1/\gamma_i) t^a \rangle, \quad t^a = \langle (1/\gamma_i) v_i^a \rangle - \Gamma v^a, \quad p^{ab} = \langle v_i^a v_i^b \rangle - nv^av^b, \quad t^{ab} = \langle (1/\gamma_i) v_i^a v_i^b \rangle - \Gamma v^av^b - t^a v^b - t^b v^a, \]

for \( M^{abcd} \) see equation (17) of Ref. [14], where \( \gamma_i = 1/\sqrt{1 - v_i(t)^2/c^2} \).

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