VERTEX PARTITIONS AND MAXIMUM $\mathcal{G}$-FREE SUBGRAPHS

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ABSTRACT. We define a $(V_1, V_2, \ldots, V_k)$-partition for a given graph $H$ and graphical properties $P_1, P_2, \ldots, P_k$ as a partition where each $V_i$ induces a subgraph of $H$ with property $P_i$. In 1979, Bollob’as and Manvel demonstrated that if a graph $H$ has a maximum degree $\Delta(H) \geq 3$ and clique number $\omega(H) \leq \Delta(H)$, with $\Delta(H) = p + q$, there exists a $(V_1, V_2)$-partition of $V(H)$. This partition ensures that $\Delta(H[V_1]) \leq p$, $\Delta(H[V_2]) \leq q$, $H[V_1]$ is $(p - 1)$-degenerate, and $H[V_2]$ is $(q - 1)$-degenerate. Matamala (2007) extended this result by showing that for any graph $H$ with $\Delta(H) = p + q$, there exists a $(V_1, V_2)$-partition of $V(H)$ where $H[V_1]$ is a maximum order $(p - 1)$-degenerate induced subgraph and $H[V_2]$ is $(q - 1)$-degenerate. Additionally, Catlin and Lai proved that if $\Delta(H) \geq 5$, $H$ has a $(V_1, V_2)$-partition such that $H[V_1]$ is a maximum order acyclic induced subgraph, $\omega(H[V_2]) \leq \Delta(H) - 2$, and $\Delta(H[V_2]) \leq \Delta(H) - 2$.

Rowshan and Taherkhani demonstrated that given a graph $G$ with a minimum degree $\delta(G)$ and for $k = \lceil \frac{\Delta(G)}{\delta(G)} \rceil$, there exists a $(V_1, V_2, \ldots, V_k)$-partition of the vertex set of $H$, such that each $H[V_i]$ is $G$-free, meaning it does not contain a subgraph isomorphic to $G$, and $H[V_1]$ is a maximum order $G$-free induced subgraph.

In our paper, we present a novel result for a connected graph $H$ with $\Delta(H) \geq 5$ and without $K_{\Delta(H)+1} \setminus e$ as a subgraph. We establish that when $p_1 \geq p_2 \geq \cdots \geq p_{k-1} \geq 2$, $p_k \geq 4$, $\sum_{i=1}^{k} p_i = \Delta(H) - 1 + k$, and $\mathcal{G}_i$ represents a family of graphs with a minimum degree at least $p_i - 1$ for each $i \in [k - 1]$, a $(V_1, V_2, \ldots, V_k)$-partition of $V(H)$ exists. This partition guarantees that $H[V_1]$ is a maximum order $\mathcal{G}_i$-free induced subgraph, $H[V_2]$ is $\mathcal{G}_i$-free for each $2 \leq i \leq k - 1$, $\Delta(H[V_k]) \leq p_k$, and either $H[V_k]$ is $K_{p_k}$-free or its $p_k$-cliques are disjoint.

1. INTRODUCTION

In this article, all graphs under consideration are finite, undirected, and simple. For a given graph $H = (V(H), E(H))$, the degree of a vertex $v \in V(H)$ is denoted as $\deg_H(v)$ (or simply $\deg(v)$), and its set of neighbors is represented by $N_H(v)$ (or $N(v)$). The maximum degree of graph $H$ is denoted as $\Delta(H)$, and the minimum degree as $\delta(H)$. When referring to a subset $W$ of $V(H)$, the induced subgraph on $W$ is denoted as $H[W]$. For two disjoint subsets $V_1$ and $V_2$ of $V(H)$, the set $E(V_1, V_2)$ represents all the edges $vv' \in E(H)$, where $v \in V_1$ and $v' \in V_2$. The clique number $\omega(H)$ of a graph $H$ is defined as the largest integer $k$ for which $H$ contains a complete subgraph of size $k$. The join of two graphs $G$ and $H$ is denoted as $G \oplus H$ and is obtained by connecting each vertex of $G$ to every vertex in $H$. Furthermore, when referring to an edge $e$ in graph $G$, we use $G \setminus e$ to denote the graph resulting from the removal of $e$ in $G$.

We define a $(V_1, V_2, \ldots, V_k)$-partition for a given graph $H$ and graphical properties $P_1, P_2, \ldots, P_k$ such that each subgraph induced on $V_i$ satisfies property $P_i$. Further references on $(V_1, \ldots, V_k)$-partition can be found in [24][9][14][15].

A graph $H$ is considered $k$-degenerate if every subgraph of $H$ contains a vertex with degree at most $k$. Specifically, when the property $P_i$ implies that $H[V_i]$ is $p_i$-degenerate for some positive integer $p_i$, refer to [3][4]. In the case where $k = 2$, Bollob’as and Manvel [5] have presented the following result concerning $(V_1, V_2)$-partition.

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Lemma A. [5] Assume that $H$ is a graph with a maximum degree $\Delta(H) \geq 3$ and a clique number $\omega(H) \leq \Delta(H)$. If $\Delta(H) = p + q$, it can be shown that there exists a $(V_1, V_2)$-partition of the vertex set $V(H)$. This partition satisfies the following properties: $\Delta(H[V_1]) \leq p$, $\Delta(H[V_2]) \leq q$, $H[V_1]$ is $(p - 1)$-degenerate, and $H[V_2]$ is $(q - 1)$-degenerate.

As an extension of Lemma A, Catlin and Lai [7] later proved the following theorem..

Theorem A. [7] Assuming that $H$ is a graph with $\Delta(H) = d \geq 3$ and a clique number $\omega(H) \leq \Delta(H)$, it can be shown that $H$ possesses a $(V_1, V_2)$-partition satisfying the following properties:

- For $d = 3$, $V_1$ is a maximum independent set and $H[V_2]$ is acyclic.
- For $d = 4$, $H[V_1]$ is a maximum acyclic induced subgraph and $H[V_2]$ is acyclic.
- For $d \geq 5$, $H[V_1]$ is a maximum acyclic induced subgraph, $\omega(H[V_2]) = d - 2$, and $\Delta(H[V_2]) \leq d - 2$.

The result presented below is closely related to Lemma A and Theorem A. It was established by Matamala in [10].

Theorem B. [10] Assume that $H$ is a graph with $\Delta(H) \geq 3$ and clique number $\omega(H) \leq \Delta(H)$. If $\Delta(H) = p + q$ then there exists a $(V_1, V_2)$-partition of $V(H)$, such that $H[V_1]$ is a maximum order $(p - 1)$-degenerate induced subgraph of $H$ and $H[V_2]$ is $(q - 1)$-degenerate subgraph.

In an extension of Brooks’ Theorem, Catlin demonstrated that any graph $H$ with $\Delta(H) \geq 3$ and without $K_{\Delta(H) + 1}$ as a subgraph can be colored with $\Delta(H)$ colors such that one of the color classes forms a maximum independent set [6]. Let $\mathcal{G}$ be a family of graphs. We define $H$ as $G$-free if it does not contain any subgraph isomorphic to $G$ for every $G \in \mathcal{G}$. As an analogy to Catlin’s result, the author and Taherkhani [13] established the following result:

Theorem C. [13] Let $d_1, \ldots, d_k$ be $k$ positive integers. Assume that $G_1, \ldots, G_k$ are connected graphs with minimum degrees $d_1, \ldots, d_k$, respectively, and $H$ is a connected graph with maximum degree $\Delta(H)$ where $\Delta(H) = \sum_{i=1}^{k} d_i$. Assume that $G_1, G_2, \ldots, G_k$, and $H$ satisfy the following conditions:

- If $k = 1$, then $H$ is not isomorphic to $G_1$.
- If $G_i$ is isomorphic to $K_{d_i + 1}$ for each $1 \leq i \leq k$, then $H$ is not isomorphic to $K_{\Delta(H) + 1}$.
- If $G_i$ is isomorphic to $K_2$ for each $1 \leq i \leq k$, then $H$ is neither an odd cycle nor a complete graph.

Then, there is a partition of vertices of $H$ to $V_1, \ldots, V_k$ such that each $H[V_i]$ is $G_i$-free and moreover one of $V_i$s can be chosen in a way that $V_i$ has maximum possible size such that for which we have $H[V_i]$ be a $G_i$-free subgraph in $H$.

Also the author and Taherkhani [13] established the following result:

Theorem D. [13] Suppose that $H$ is a graph $\omega(H) \leq \Delta(H) - 1$. Let $k \geq 2$ be a positive integer. Assume that $p_1, p_2, \ldots, p_k$ are $k$ positive integers and $p_1 + p_2 + \cdots + p_k = \Delta(H) - 1 + k$. If $p_1 \geq p_2 \geq \cdots \geq p_k \geq 2$ and $\sum_{i=1}^{k} p_i = \Delta(H) - 1 + k$, then there exists a partition of $V(H)$ into $V_1, V_2, \ldots, V_k$ such that for each $1 \leq i \leq k$, $H[V_i]$ is $K_{p_i}$-free.

Theorem E. [13] Assume that $H$ is a graph with $\Delta(H) \geq 6$ and clique number $\omega(H)$ where $4 \leq \omega(H) \leq \Delta(H) - 2$. Denote $\omega(H) = p$ and $\Delta(H) + 1 - p = q$. Then there exists $V_1 \subseteq V(H)$ such that $V_1$ is a maximum $K_p$-free subset of $H$, and $H[V \setminus V_1]$ is $K_q$-free.

Consider a connected graph $H = (V, E)$ with a maximum degree $d \geq 3$, which is distinct from $K_{d+1}$. Let $k \geq 2$ be a positive integer, and let $p_1, \ldots, p_k \geq 0$ be $k$ integers. We define $H$ as ($p_1, \ldots, p_k$)-partitionable if there exists a partition of $V(H)$ into sets $V_1, \ldots, V_k$ such that $H[V_i]$ is $p_i$-degenerate for $i \in [k]$. Abu-Khzam, Feghali, and Hegghernes have established the following two results concerning ($p_1, \ldots, p_k$)-partitionable graphs.
Theorem F. Suppose that $H = (V, E)$ is a connected graph with $\Delta(H) = d \geq 3$ distinct from $K_{d+1}$. For all integers $k \geq 2$ and $1 \leq p_1, \ldots, p_k \geq 0$, such that $\sum_{i=1}^{k} p_i \geq d - k$, a $(p_1, \ldots, p_k)$-partition of $H$ can be found in $O(|V| + |E|)$-time.

Theorem G. For every integer $d \geq 5$ and every pair of non-negative integers $(p, q)$, so that $(p, q) \neq (1, 1)$ and $p + q = d - 3$, deciding whether a graph with maximum degree $d$ is $(p, q)$-partitionable is NP-complete.

As a related result of Lemma A, Theorem A, Theorem B, and Theorem C in this article, we prove the following theorem.

Theorem 1.1 (Main result). Suppose that $H = (V, E)$ is a connected graph with maximum degree $\Delta(H) \geq 5$ and $H$ is $K_{\Delta(H)+1} \setminus e$-free. Suppose that $p$ and $q$ are two positive integers, such that $p \geq 2, q \geq 4$ and $\Delta(H) + 1 = p + q$. Set $\mathcal{G}$ as a collection of graphs with minimum degree at least $p - 1$. Then there exists a $(V_1, V_2)$-partition of $V(H)$, such that $H[V_1]$ is a maximum order $\mathcal{G}$-free induced subgraph of $H$, $\Delta(H[V_2]) \leq q$, and either $H[V_2]$ is $K_q$-free subgraph or its $q$-cliques are disjoint.

By utilizing induction on $k$, we can demonstrate that the following result holds as a generalization of Theorem 1.1.

Corollary 1.2. Assume that $H = (V, E)$ is a connected graph with a maximum degree $\Delta(H) \geq 5$. Let $p_1 \geq p_2 \geq \cdots \geq p_k \geq 2$ and $p_k \geq 4$ be $k$ positive integers such that $\sum_{i=1}^{k} p_i = \Delta(H) - 1 + k$. Furthermore, let $\mathcal{G}_1$ denote a collection of graphs with a minimum degree of at least $p_i - 1$ for each $i \in [k - 1]$. If $H$ is free of $K_{\Delta(H)+1} \setminus e$, then there exists a $(V_1, V_2, \ldots, V_k)$-partition of $V(H)$ satisfying the following properties: $H[V_1]$ is a maximum order $\mathcal{G}_1$-free graph, $H[V_i]$ is $\mathcal{G}_i$-free for each $2 \leq i \leq k - 1$, $\Delta(H[V_k]) \leq p_k$, and either $H[V_k]$ is $K_{p_k}$-free or its $p_k$-cliques are disjoint.

Theorem C implies that for every graph $H$ with a maximum degree $\Delta(H) \geq 5$ and without $K_{\Delta(H)+1} \setminus e$ as a subgraph, if $\Delta + 1 = 2p$ for some positive integer $p$, then there exists a $K_p$-free $\lceil \frac{\Delta}{p-1} \rceil$-coloring of $H$ such that one of its color classes is a maximum induced $K_p$-free subgraph in $H$. For instance, for $\Delta(H) = 9$ and $p = 5$, Theorem C guarantees a $K_5$-free 3-coloring of $H$ such that one of its color classes is a maximum order induced $K_5$-free subgraph in $H$. However, Theorem 1.1 states that there is a 2-coloring of $H$ such that one of its color classes is a maximum order induced $K_5$-free subgraph in $H$ and $H[V_2]$ is either $K_5$-free or the 5-cliques of $H[V_2]$ are disjoint.

Clearly, the result of Theorem 1.1 is stronger than that of Theorem C. Moreover, if we consider $\mathcal{G} = \mathcal{C} = C_n$, $n \geq 3$, $p = 3$, and $q = \Delta(H) - 2$, then Theorem 1.1 coincides with Theorem A. It is worth noting that by incorporating some results from Demetres Christofides, Katherine Edwards, and Andrew D. King in [8], if we replace the assumption $q \geq 4$ with $q \geq 2$ in Theorem 1.1, we obtain the same result, but we lose the maximality of $|V_1|$ in Theorem 1.1. In other words, we have the following result.

Lemma 1.3. Consider a connected graph $H$ with $\Delta(H) \geq 7$ that is $K_{\Delta(H)} \setminus e$-free, and let $p$ and $q$ be two integers satisfying $p, q \geq 2$ and $\Delta(H) + 1 = p + q$. Then there exists a $(V_1, V_2)$-partition of $V(H)$ such that $H[V_1]$ is $K_p$-free, and either $H[V_2]$ is $K_q$-free or its $q$-cliques are disjoint.

2. Vertex Partitions and Maximum $R$-free Subgraphs

In this section, we prove Theorem 1.1 for the case that $\mathcal{G}$ has only one member, say $R$, where $R$ is a graph with minimum degree at least $p - 1$ ($p \geq 2$).

Theorem 2.1. Assume that $H$ is a connected graph with $\Delta(H) = d \geq 5$ and $p$ and $q$ be two positive integers, where $p \geq 2, q \geq 4$ and $d + 1 = p + q$. Also, let $G \cong K_{q+1} \setminus e$. Suppose that $F$ consists of $S \subseteq V(H)$ for which:
\* S has the maximum possible size (M-P-size) such that H[S] is R-free, in other words S is a maximum order R-free induced subgraph of H.

Now, for each S ∈ F the following statements hold:

(I) : For each vertex v ∈ S, we have |N(v) ∩ S| ≥ p − 1, Δ(H[S]) ≤ q, and the induced subgraph H[S ∪ {v}] has at least one copy of R.

(II) : Every vertex v ∈ S lies in either at most one copy of G in H \ S or a copy of K_{q+1} which is a connected component of H[S].

(III) : For any member S of F if S has a copy of G, then K_{d+1} \ e ⊆ H.

Proof. Suppose that S be an arbitrary member of F.

Proof (I): Since the size of the set S is maximal, for every vertex v ∈ S, the graph H[S ∪ v] contains at least one copy R that includes the vertex v. Therefore, we can conclude that |N(v) ∩ S| ≥ p − 1 and |N(v) ∩ S| ≤ q for every vertex v.

Proof (II): Assume by contradiction that there exists a vertex v in S that is present in at least two copies of G but does not belong to any copy of K_{q+1} in H[S]. Let G_1 and G_2 be two copies of G in H[S] that contain v. Let \( X = \{x_1, x_2, \ldots, x_{q-1}\} \) and \( X' = \{x_1', x_2', \ldots, x_{q-1}'\} \) be the sets of \( (q - 1) \) vertices from \( V(G_1) \) and \( V(G_2) \), respectively, such that for each \( 1 \leq i \leq q - 1 \), \( \deg_{H[S]}(x_i) = \deg_{H[S]}(x_i') = q \).

If \( |X ∩ X'| \leq q - 3 \), then we can conclude that \( |N(v) ∩ S| \geq q + 1 \), which implies \( |N(v) ∩ S| ≤ p - 2 \). However, this contradicts part (I) of the result. Therefore, we assume that \( |X ∩ X'| \geq q - 2 \geq 2 \). Let \( v' \) be a vertex in \( X ∩ X' \).

Since \( G_1 \neq G_2 \), we can deduce that \( |N(v') ∩ S| \geq q + 1 \), which implies \( |N(v') ∩ S| ≤ p - 2 \). However, this contradicts part (I) of the result. Thus, our initial assumption that there exists such a vertex v is false, and the claim holds.

Proof (III): Suppose that \( p \geq 2 \) and \( q \geq 4 \). We take a \( S ∈ F \) for which

- (P1): \( H[S] \) has the least possible number of copies G, and, subject to that,
- (P2): \( E(H[S]) \) has the least possible size.

Let’s assume that \( G' \) is a copy of G in H[S]. We define \( V' = V(G') = \{v_0', v_1', \ldots, v_q\} \) such that \( G'[V' \setminus \{v_{q-1}', v_q\}] \cong K_{q-1} \) and \( G' \cong K_{q-1} \oplus H[v_{q-1}', v_q] \). It should be noted that \( v_{q-1}' \) and \( v_q \) can be adjacent in H, in which case \( H[V(G')] \) is isomorphic to \( K_{q+1} \). Now, let’s define \( B' \) as follows:

\[
B'_{\text{def}} = \{v_0', v_1', \ldots, v_{q-2}'\} = V' \setminus \{v_{q-1}', v_q\}.
\]

Since \( q \geq 4 \), we have \( |B'| \geq 2 \). By the maximality of S and using properties (I) and (II), we can conclude that for each \( v ∈ B' \), \( H[S ∪ v] \) contains at least one copy of R. Since \( v ∈ B' \), we can easily observe that \( |N(v) ∩ S| \geq q \). Consequently, \( |N(v) ∩ S| \leq p - 1 \). The fact that \( v \) lies in at least one copy of R in \( H[S ∪ \{v\}] \) implies that \( |N(v) ∩ S| = p - 1 \). Therefore, we have \( |N(v) ∩ S| = q \). If this were not the case, then \( \deg(v) ≥ p + q = d + 1 \), which would be a contradiction to the assumption that \( Δ(H) = d \).

Claim 2.2. Suppose that C is a connected component of \( H[S ∪ \{v\}] \) containing v. Then C is \((p - 1)\)-regular graph and isomorphic to R.

Proof of Claim 2.2 C contains at least one copy of R, denoted as \( R' \). We want to show that \( C = R' \).

Suppose, for the sake of contradiction, that \( C \neq R' \). Without loss of generality, assume that \( |V(C)| ≥ |V(R')| + 1 \), which implies that there exists at least one vertex \( v' ∈ C \) such that \( v' \notin R' \).

As C is a connected component of \( H[S ∪ \{v\}] \), let us consider the distance \( d_{C}(v, v') \) between \( v \) and \( v' \) in C. Note that since \( |N(v) ∩ S| = p - 1 \), all copies of R in C must contain the neighbourhood \( N(v) ∩ S \).
Now, let \( i \geq 1 \) be the largest integer such that for every vertex \( u \in C \) with \( d_C(v, u) = i \), \( u \) is included in all copies of \( R \) in \( C \). Since \( v' \notin R' \), there exists at least one vertex \( w \) in \( R' \cap C \) such that \( d_C(v, w) = i + 1 \leq d_C(v, v') \), and \( w \notin V(R') \). Consequently, there exists at least one neighbour \( y \) of \( w \) in \( C \) such that \( d(v, y) = d(v, w) - 1 = i \).

As \( d(v, y) = i \), vertex \( y \) is included in all copies of \( R \) in \( C \). Let \( S_1 = (S \cup \{v\}) \setminus \{y\} \). Note that \( |S_1| = |S| \) and \( H[S_1] \) is \( R \)-free because \( y \) lies in all copies of \( R \) in \( H[S \cup \{v\}] \). Since \( y \) is included in at least one copy of \( R \) in \( H[S \cup \{v\}] \), and at least one of them does not include \( w \), we have \( |N(y) \cap S_1| = |N(y) \cap (S \cup \{v\})| \geq p \). Therefore, \( |N(y) \cap S_1| \leq q - 1 \), which implies that \( y \) lies in at most one copy of \( G \) in \( H \setminus S_1 \). If \( y \) is not contained in any copy of \( G \) in \( H \setminus S_1 \), it contradicts property (P1).

Therefore, we can assume that \( y \) lies in one copy of \( G \), and the number of copies of \( G \) in \( H \setminus S_1 \) is equal to the number of copies of \( G \) in \( H \setminus S \). Since \( |N(v) \cap S| = q \) and \( |N(y) \cap S_1| = q - 1 \), it can be verified that \( |E(H[S_1])| \leq |E(H[S])| - 1 \). This contradicts property (P2). Hence, we conclude that \( V(C) = V(R) \).

Assume that all copies of \( R \) in \( C \) have the same vertex set \( V(C) \). If there are at least two distinct copies of \( R \) in a common vertex set, then there exists a vertex \( u \in R \subset C \) such that \( |N(u) \cap V(C)| \geq p \). Define \( S_1 = (S \cup \{v\}) \setminus \{u\} \), and the proof follows similarly as in the previous paragraph. Therefore, \( C \equiv R \).

Now we shall show that \( C \) is \((p - 1)\)-regular. Assume that there exists a vertex \( y \) in \( C \) with more than \( p - 1 \) neighbour in \( C \). Then, \( |N(y) \cap S_1| = |N(y) \cap (S \cup \{v\})| \geq p \). This implies that \( |N(y) \cap S_1| \leq q - 1 \), which means that \( y \) lies in at most one copy of \( G \) in \( H \setminus S_1 \). If \( y \) is not contained in any copy of \( G \) in \( H \setminus S_1 \), it contradicts property (P1). Therefore, we can assume that \( y \) lies in one copy of \( G \), and the number of copies of \( G \) in \( H \setminus S_1 \) is equal to the number of copies of \( G \) in \( H \setminus S \). However, it can be shown that \( |E(H[S_1])| \leq |E(H[S])| - 1 \), which contradicts property (P2).

Hence, we conclude that \( C \) is a \((p - 1)\)-regular graph. 

Claim 2.3. There is a copy of \( G \) in \( H[(S \setminus \{v\}) \cup \{y\}] \), which contains \( y \) and \( |N(y) \cap (S \setminus \{v\})| = q \).

Proof of Claim 2.3. According to Claim 2.2, \( R' \) is a \((p - 1)\)-regular graph and one of the connected components in \( H[S \cup \{v\}] \). Consequently, for any vertex \( y \) in \( V(R') \), we have \( |N(y) \cap (S \cup \{v\})| = p - 1 \), which implies \( |N(y) \cap (S \setminus \{v\})| \leq q \). Let’s define \( S' = (S \cup \{v\}) \setminus \{y\} \). Since \( |S'| = |S| \) and \( H[S'] \) is \( R \)-free, we can conclude that \( S' \in F \). As \( v \) belongs to a copy of \( G \) in \( H[S] \) and \( v \in B' \), we have \( |N(v) \cap S| = q \). According to (P1), \( y \) must be present in at least one copy of \( G \) in \( H[S] \). By applying (P2), it can be easily verified that \( |N(y) \cap (S \setminus \{v\})| = q \).

Assume that \( S = S_0 \) and \( G_0 \) is a copy of \( G \) in \( H[S_0] \). Assume that \( V_0 = V(G_0) = \{v_0^0, v_1^0, \ldots, v_q^0\} \), where \( G_0[V_0 \setminus \{v_{q-1}^0, v_q^0\}] \cong K_{q-1} \) and \( G_0 \cong K_{q-1} \oplus H[\{v_{q-1}^0, v_q^0\}] \). Now define \( B_0 \) as follows

\[
B_0 \overset{\text{def}}{=} \{v_0^0, v_1^0, \ldots, v_{q-2}^0\} = V_0 \setminus \{v_{q-1}^0, v_q^0\}.
\]

Since \( q \geq 4 \), we can conclude that \( |B_0| \geq 2 \). Let’s consider a vertex \( v_0 \) in \( B_0 \). Based on Claim 2.2, \( H[S_0 \cup \{v_0\}] \) contains a unique copy of \( R \), denoted as \( R_0 \), which is \((p - 1)\)-regular and one of the connected components of \( H[S_0 \cup \{v_0\}] \).

Let’s choose a vertex \( y_0 \) from \( V(R_0) \) such that \( y_0 \) is not a cut vertex in \( R_0 \). Please note that we will use the assumption that \( y_0 \) is not a cut vertex in the remaining part of the proof.

Define \( S_1 = (S_0 \cup \{v_0\}) \setminus \{y_0\} \). It can be verified that \( H[S_1] \) is \( R \)-free, and \( |S_1| = |S_0| \). Hence, \( S_1 \in F \). According to (P1), and considering the fact that \( v_0 \) lies in at least one copy of \( G \) in \( S_0 \), we can deduce that \( y_0 \) must lie in a copy of \( G \) in \( H[S_1] \), denoted as \( G_1 \).
As \( v_0 \in B_0 \), we have \(|N(v_0) \cap \overline{S}_0| = q\). Therefore, utilizing (P2), we can conclude that \(|N(y_0) \cap (\overline{S}_0 \setminus \{v_0\})| = q\).

Let’s assume that \( V_1 = V(G_1) = \{v_1^0, v_1^1, \ldots, v_q^1\} \), and \( G_1 \cong K_{q-1} \oplus H[\{v_{q-1}^1, v_q^1\}] \), where \( G_1[V_1 \setminus \{v_{q-1}^1, v_q^1\}] \cong K_{q-1} \). Now, let’s define \( B_1 \) as follows: if \( v_{q-1}^1 v_q^1 \not\in E(H) \), then we define

\[
B_1 \overset{\text{def}}{=} \{v_1^0, v_1^1, \ldots, v_{q-2}^1\} = V_1 \setminus \{v_{q-1}^1, v_q^1\}
\]

If \( v_{q-1}^1 v_q^1 \in E(H) \), we proceed as follows: Let’s assume that \( W \) is a subset of \( q - 1 \) elements from \( V(G_1) \) such that \(|W \cap B_0|\) has the maximum cardinality among all subsets of \( q - 1 \) elements from \( V(G_1) \). In this case, we define \( B_1 = W \).

Now, let’s present the following two claims based on the above construction:

**Claim 2.4.** If \(|(B_0 \setminus \{v_0\}) \cap (B_1 \setminus \{y_0\})| \neq 0\), then \( B_0 \setminus \{v_0\} = B_1 \setminus \{y_0\} \).

**Proof of Claim 2.4.** Let’s consider the case where \( v_{q-1}^0 v_q^0 \not\in E(H) \). In this situation, it can be observed that \( \{v_{q-1}^0, v_q^0\} \not\subseteq B_1 \) and \( v_{q-1}^0 v_q^0 \not\in B_0 \). Suppose there exists a vertex \( z \in (B_0 \setminus \{v_0\}) \cap (B_1 \setminus \{y_0\}) \). Since \( z \in B_0 \setminus \{v_0\} \), we can deduce that \(|N(z) \cap \overline{S}_0 \setminus \{v_0\})| = q - 1\). Considering the fact that \( z \in B_1 \) and based on Part (I), we have \(|N(z) \cap \overline{S}_0| = q\). As \( y_0 \) is adjacent to \( z \) and \( y_0 \not\in \{v_{q-1}^0, v_q^0\} \), it must be the case that \( y_0 \in B_1 \). Consequently, we have \(|N(z) \cap \overline{S}_1 \setminus \{y_0\})| = N(y_0) \cap (\overline{S}_1 \setminus \{z\}) | \).

Now, let’s assume by contradiction that there exists a vertex \( z' \in B_0 \setminus \{v_0, z\} \) such that \( z' \not\in B_1 \setminus \{y_0\} \). As \( z' \) is adjacent to \( v_0, z \), \( v_{q-1}^0, v_q^0 \in V(G_0) \setminus B_0 \), we have \( z' v_{q-1}^0, v_q^0 \in E(H) \). This implies that \(|N(z) \cap \overline{S}_1| \geq q + 1\), which is not possible. Hence, we can conclude that \( B_0 \setminus \{v_0\} = B_1 \setminus \{y_0\} \).

Now, let’s consider the case where \( v_{q-1}^0 v_q^0 \in E(H) \), which implies \( H[V(G_0)] \cong K_{q+1} \). According to (II), \( H[V(G_0)] \) is a connected component of \( H[\overline{S}] \). Additionally, based on (P1), we can assume that \( H[V(G_1)] \cong K_{q+1} \). Since \(|(B_0 \setminus \{v_0\}) \cap (B_1 \setminus \{y_0\})| \neq 0\), it can be verified that \( V(G_0) \setminus \{v_0\} = V(G_1) \setminus \{y_0\} \). Therefore, considering this fact and the definition of \( B_1 \), we have \( (B_0 \setminus \{v_0\}) = (B_0 \setminus \{v_0\}) \).

**Claim 2.5.** If \(|(B_0 \setminus \{v_0\}) \cap (B_1 \setminus \{y_0\})| \neq 0\), then \( K_{d+1} \setminus e \subseteq H \).

**Proof of Claim 2.5.** Let’s assume that \(|(B_0 \setminus \{v_0\}) \cap (B_1 \setminus \{y_0\})| \neq 0\). According to Claim 2.4, we have \( B_0 \setminus \{v_0\} = B_1 \setminus \{y_0\} \). Now, let \( z \) be a vertex in \( B_1 \setminus \{y_0\} \). It follows that \(|N(z) \cap \overline{S}_1| = q\) and, consequently, \(|N(z) \cap S_1| = p - 1\). Based on Claim 2.2, we know that \( S_1 \cup \{z\} \) contains a unique copy of \( R \), denoted as \( R_z \), such that \( z \) lies in \( R_z \), and \( R_z \) is one of the connected components of \( H[S_1 \cup \{z\}] \). Considering the fact that \( z \) is adjacent to \( v_0 \) in \( H[S_0 \cup \{v_0\}] \), \( y_0 \) is not a cut vertex in \( H[S_0 \cup \{v_0\}] \), and \( R_0 \) is \((p - 1)\)-regular and one of the connected components of \( H[S_0 \cup \{v_0\}] \), we can conclude that \( N(y_0) \cap (S_0 \cup \{v_0\}) = N(y_0) \cap V(R_0) = N(z) \cap S_1 \). Therefore, we can deduce that \( y_0 \) is adjacent to \( y_0 \).

We need to show that \( H[N(y_0) \cap V(R_0)] \cong K_{p-1} \). Let’s consider an arbitrary vertex \( y \in N(y_0) \cap V(R_0) \). From the previous reasoning, we know that \( N(z) \cap S_0 = (N(y_0) \cap S_0) \cup \{y_0\} \) for each \( z \in B_1 \setminus \{y_0\} \).

Now, let’s define \( S'' \overset{\text{def}}{=} S_0 \cup \{v_0, z\} \cup \{y\} \). Since \( S_0 \) is maximal, \( H[S''] \) must contain at least one copy of \( R \), denoted as \( R' \). It’s easy to see that \( R' \) must contain \( z \). As \( R_0 \) is \((p - 1)\)-regular and one of the connected components of \( H[S_0 \cup \{v_0\}] \), and \( z \) has exactly \( p - 1 \) neighbours in \( S'' \), we can conclude that \( N(z) \cap V(R'_0) = (N(y) \cap V(R_0)) \).

Considering that \( N(z) \cap S_0 = (N(y_0) \cap S_0) \cup \{y\} \), we have \( N(y) \cap (V(R_0) \setminus \{y\}) = N(y_0) \cap (V(R_0) \setminus \{y\}) \). Since \( y \) is an arbitrary vertex in \( N(y_0) \cap V(R_0) \), we can conclude that \( H[N(y_0) \cap V(R_0)] \cong K_{p-1} \). Therefore, \( R_0 \) is isomorphic to \( K_p \).

To summarize the argument:
We want to show that $K_{d-1} = K_{p+q-2} \subseteq H$. We already know that $R_0 \cong K_p$ and that $N(y_0) \cap R_0 = N(z) \cap S_1$ for each $z \in (B_0 \setminus \{v_0\}) \cap (B_1 \setminus \{y_0\})$. Therefore, it is sufficient to prove that each vertex $y \in V(R_0)$ is adjacent to both $v_{q-1}^0$ and $v_q^0$.

Assume by contradiction that there exists a vertex $y \in V(R_0)$ that is not adjacent to either $v_{q-1}^0$ or $v_q^0$. In this case, we define $S' = (S_0 \cup \{v_0\}) \setminus \{y\}$. Since $q \geq 4$, if $v_{q-1}^0$ and $v_q^0$ are not adjacent, then $H[S']$ does not contain any copy of $G$, which contradicts (P1). Otherwise assume that $G'$ is a copy of $G$ in $H[S']$, which contains $y$. Hence one can say that each vertex $x \in B' \cap B_0$, has at least $q + 1$ neighbours in $S'$, which is not possible.

Therefore, we conclude that $v_{q-1}^0$ and $v_q^0$ must be adjacent.

Now, as $v_{q-1}^0$ and $v_q^0$ are adjacent, we have $H[V(G_0)] \cong K_{q+1}$. Since $y$ is not adjacent to at least one of $v_{q-1}^0$ and $v_q^0$, it follows that $H[S']$ does not contain any copy of $K_{q+1}$, which contradicts (P1).

Therefore, in either case, we have shown that $K_{d+1} \subseteq H[N[v_0]] \subseteq H$. Combining this with the previous result that $K_{d-1} \subseteq H$, we conclude that $K_{d+1} \setminus \{e\} \subseteq H$.

We can assume that $(B_0 \setminus v_0) \cap (B_1 \setminus y_0) = \emptyset$ based on Claim 2.3.

Now, let’s consider the case when $i \geq 2$. We will assume that we have already defined $S_{i-1}$, $G_{i-1}$, $V_{i-1}$, $B_{i-1}$, $v_{i-1}$, $R_{i-1}$, and $y_{i-1}$, where:

- $S_{i-1} \in \mathcal{F}$,
- $G_{i-1}$ is a copy of $G$ in $\overline{S}_{i-1}$,
- $V_{i-1} = V(G_{i-1}) = \{v_{0}^{i-1}, v_{1}^{i-1}, \ldots, v_{q}^{i-1}\}$ and $H[V_{i-1} \setminus \{v_{q}^{i-1}\}] \cong K_{q-1} = K'$,
- $G_{i-1} \cong K' \oplus H[\{v_{q}^{i-1}\}]$,
- If $v_{q}^{i-1} \notin E(H)$, then: $B_{i-1} = \{v_{0}^{i-1}, v_{1}^{i-1}, \ldots, v_{q-1}^{i-1}\}$,
- If $v_{q}^{i-1} \in E(H)$, we proceed as follows: Let’s assume that $W$ is a subset of $q-1$ elements from $V(G_{i-1})$ such that $|W \cap B_{j}|$ has the maximum cardinality among all subsets of $q-1$ elements from $V(G_{i-1})$, for each $j \leq i - 2$. In this case, we define $B_{i-1} = W$.
- $v_{i-1}$ in $B_{i-1}$,
- $R_{i-1}$ is a copy of $R$ in $H[S_{i-1} \cup \{v_{i-1}\}]$.

As $S_{i-1} \in \mathcal{F}$, we can conclude that $H[S_{i-1} \cup \{v_{i-1}\}]$ contains a unique copy of $R$, denoted as $R_{i-1}$. This copy $R_{i-1}$ is one of the connected components in $H[S_{i-1} \cup \{v_{i-1}\}]$, as stated in Claim 2.4.

Let’s assume that $y_{i-1}$ is a vertex in $R_{i-1} \setminus \{v_{i-1}\}$ and it is not a cut vertex.

Next, we will define $S_i$, $B_i$, $G_i$, $V_i$, $v_i$, $R_i$, and $y_i$ for the next iteration $i$.

We define $S_i$ as $S_i = (S_{i-1} \cup \{v_{i-1}\}) \setminus \{y_{i-1}\}$. Since $y_{i-1} \in R_{i-1}$ and $R_{i-1}$ is one of the connected components in $H[S_{i-1} \cup \{v_{i-1}\}]$, it follows that $S_i$ belongs to $\mathcal{F}$.

Based on Claim 2.3 let’s assume that $G_i$ is a copy of $G$ in $H[S_i]$ that contains $y_{i-1}$. We can denote the vertex set of $G_i$ as $V_i = V(G_i) = \{v_0^{i}, v_1^{i}, \ldots, v_q^{i}\}$. Furthermore, we have $G_i \cong K_{q-1} \oplus H[v_{q-1}^{i}, v_q^{i}]$.

Now, we define $B_i$ as follows:

- If $v_{q-1}^{i}, v_q^{i} \notin E(H)$, then $B_i$ is given by $B_i = \{v_0^{i}, v_1^{i}, \ldots, v_{q-2}^{i}\} = V_i \setminus \{v_{q-1}^{i}, v_q^{i}\}$.
- If $v_{q-1}^{i}, v_q^{i} \in E(H)$, we proceed as follows:

  We assume that $W$ is a $(q-1)$-element subset of $V(G_i)$ such that $W$ has the maximum possible intersection with one of the sets $B_j$ for $0 \leq j \leq i - 1$. We define $B_i$ as $B_i = W$.

  Since $H$ is a finite graph, there exists a minimum number $\ell \geq 2$ such that $(B_\ell \setminus \{y_\ell-1\}) \cap (B_j \setminus \{v_j\}) \neq \emptyset$ for some $j \leq \ell - 1$. Without loss of generality, we may assume that $j = 0$.

Claim 2.6. We have $B_{\ell} \setminus \{y_{\ell-1}\} \subseteq V(G_0) \setminus \{v_0\}$. In particular, if $v_{q-1}^0 v_q^0 \notin E(H)$, then $B_{\ell} \setminus \{y_{\ell-1}\} = B_0 \setminus \{v_0\}$.

Proof of Claim 2.6. By the minimality of $\ell$, we have $B_0 \setminus \{v_0\} \subseteq \overline{S}_\ell$. We also aim to show that $\{v_{q-1}^0, v_q^0\} \subset \overline{S}_\ell$. 7
Assume, to the contrary, that \( v_q^0 \not\in S^\ell \). Therefore, there exists some \( 1 < i_0 < \ell \) such that \( v_q^0 = v_{i_0} \in B_{i_0} \). Since \( q \geq 4 \), we have \( B_{i_0} \cap B_0 \not= \emptyset \), which contradicts the minimality of \( \ell \).

Suppose \( z \in (B_0 \setminus \{v_0\}) \cap B_\ell \). Vertex \( z \) has \( q \) neighbour in \( G_\ell \). Now, assume to the contrary that there exists \( w \in (B_0 \setminus \{v_0\}) \setminus B_\ell \).

First, consider the case where \( v_{q-1}^0 v_q^0 \not\in E(H) \). Since the induced subgraph of \( H \) on \( B_\ell \) is isomorphic to \( K_{q-1} \), we have \( \{v_{q-1}^0, v_q^0\} \not\subseteq B_\ell \). We aim to show that \( w \) does not belong to \( N(v_q) \setminus B_\ell \). Assume, by contradiction, that \( w \) belongs to \( N(v_q) \setminus B_\ell \). Therefore, at least one of the vertices in \( \{v_{q-1}^0, v_q^0\} = N(v_0) \setminus B_0 \) does not belong to \( N(v_q) \setminus \overline{S^\ell} \). Consequently, \( |N(z) \setminus \overline{S^\ell}| \geq q + 1 \), which is not possible. Hence, \( w \) does not belong to \( N(v_q) \setminus B_\ell \). As a result, \( |N(z) \setminus \overline{S^\ell}| \geq q + 1 \), which is not possible.

Now, let’s consider the case where \( v_{q-1}^0 v_q^0 \in E(H) \). In this case, \( H[V(G_0)] \cong K_{q+1} \) by condition (II). We can assume that \( H[V(G_0)] \setminus \{v_0\} \) is a connected component of \( H[S^\ell] \). Since \( z \in B_\ell \cap B_0 \) and \( y_{\ell-1} \in V(G_\ell) \), \( y_{\ell-1} \) must be adjacent to all neighbors of \( z \) in \( \overline{S^\ell} \), except possibly one of them. Therefore, we have \( V(G_0) \setminus \{v_0\} = V(G_\ell) \setminus \{y_{\ell-1}\} \). If \( y_{\ell-1} \) is not adjacent to one of the neighbour of \( z \) in \( \overline{S^\ell} \), denoted as \( v_i^0 \), then we can set \( B_\ell = V(G_0) \setminus \{v_0, v_i^0\} \). If \( y_{\ell-1} \) is adjacent to all neighbours of \( z \), then \( G_\ell \) is isomorphic to \( K_{q+1} \). In this case, we define \( B_\ell = (B_0 \setminus \{v_0\}) \cup \{y_{\ell-1}\} \), and the proof is complete.

Now we are in a position to complete the proof of Theorem 2.1.

Recall that we have defined a sequence of sets \( S_0, S_1, \ldots, S_\ell \) and corresponding subgraphs \( G_0, G_1, \ldots, G_\ell \) in the graph \( H \).

We have shown that each set \( S_i \) belongs to \( F \) and that \( G_i \) is a copy of \( G \) in \( H[S_i] \). Furthermore, we have defined sets \( B_i \) for \( 0 \leq i \leq \ell \).

By the construction of \( B_i \), it follows that for \( 0 \leq i \leq \ell \), the subgraph induced by \( B_i \) is isomorphic to \( K_{q-1} \), except for the last step where it may be isomorphic to \( K_{q} \) if \( y_{\ell-1} \) is adjacent to all neighbours of \( z \) in \( \overline{S^\ell} \).

We have also shown that \( B_\ell \setminus \{y_{\ell-1}\} \subseteq \overline{S^\ell} \). Additionally, if \( v_{q-1}^0 v_q^0 \in E(H) \), then \( H[V(G_0)] \setminus \{v_0\} \) is a connected component of \( H[S^\ell] \), and we have \( V(G_0) \setminus \{v_0\} = V(G_\ell) \setminus \{y_{\ell-1}\} \).

Therefore, we have successfully constructed a sequence of sets \( S_0, S_1, \ldots, S_\ell \) and subgraphs \( G_0, G_1, \ldots, G_\ell \) that satisfy the properties stated in Theorem 2.1. Therefore, the theorem is proven.

Claim 2.7. The statement of part (III) is true.

Proof of Claim 2.7. By the maximality of \( S_\ell \), for each \( w \in B_\ell \cap (B_0 \setminus \{v_0\}) \), \( S_\ell \cup \{w\} \) contains a unique copy of \( R \) that contains \( w \), denoted as \( R_w \). This fact, combined with Claim 2.3 implies that \( v_0 \) has exactly \( p - 1 \) neighbour in \( S_\ell \cup \{w\} \).

On the other hand, since \( R_0 \) is \((p-1)\)-regular and contains \( v_0 \), \( v_0 \) has exactly \( p-1 \) neighbour in \( S_0 \). Since \( w \) is adjacent to \( v_0 \) in \( H[S_\ell \cup \{w\}] \), there must exist \( 0 \leq i_0 < \ell \) such that \( y_{i_0} \) is adjacent to \( v_0 \) in \( R_{i_0} \). If there is no such \( i_0 \), then \( |N(v_0) \cap (S_\ell \cup \{w\})| \geq p \), which contradicts the fact that \( R_w \) is \((p-1)\)-regular component of \( H[S_\ell \cup \{w\}] \). Note that the minimality of \( \ell \) and \( \{v_0^{q-1}, v_0^q\} \subset S^\ell \) imply that there is only one such \( i_0 \).

For each \( w \in B_0 \setminus \{v_0\} \), since \( w \) is adjacent to \( v_0 \) in \( H[S_\ell \cup \{w\}] \) and \( R_0 \setminus \{y_{i_0}\} \) is one of the connected components of \( H[S_\ell \cup \{w\}] \), we have \( N(w) \cap S_\ell = N(w) \cap (S_{i_0} \cup \{v_0\}) = N(y_{i_0}) \cap (S_{i_0} \cup \{v_0\}) \).

Now, consider an arbitrary vertex \( y \in N(w) \cap S_\ell = N(y_{i_0}) \cap S_{i_0} \), and let \( w' \) be a vertex in \( B_0 \setminus \{v_0, w\} \). Since \( H[(S_{i_0} \cup \{w\}) \setminus \{y\}] \) does not contain any copy of \( R \), and \( H[(S_{i_0} \cup \{w', w\}) \setminus \{y\}] \) contains a copy of \( R \), it can be shown that \( N(w') \cap ((S_{i_0} \cup \{w\}) \setminus \{y\}) = N(y) \cap (S_{i_0} \cup \{w\}) \). Therefore, the subgraph induced by \( N(w) \cap S_\ell \) is isomorphic to \( K_{p-1} \).
Since $H[S_t \cup \{w\}]$ contains a copy of $R$ and $w \in B_t$, by Claim 2.3, every vertex $y \in N(w) \cap S_t$ must lie in at least one copy of $G$ in $H[(S_t \setminus \{w\}) \cup y]$, and consequently, $y$ must be adjacent to all vertices of $G_t \setminus \{w\}$, except possibly $y_{t-1}$.

Now, consider $w$ and its neighbour. It can be verified that $K_{p+q-1} \subseteq H[N(w) \cup \{w\}]$, which completes the proof.

Note that if $v_{q-1}^0 u_q^0 \notin E(H)$, then $y$ must be adjacent to $y_{t-1}$, and as a result, a copy of $K_{p+q} \setminus e$ is contained in $H[N(w) \cup \{w\}]$.

3. Proof the Main result

**Proof Theorem 1.1.** Suppose that $H = (V, E)$ is a connected graph with maximum degree $\Delta(H) \geq 5$ and $H$ is $K_{\Delta(H)+1} \setminus e$-free. Suppose that $p$ and $q$ are two positive integers, such that $p \geq 2, q \geq 4$ and $\Delta(H) + 1 = p + q$. Set $G$ as a collection of graphs with minimum degree at least $p - 1$. Consider a $(V_1, V_2)$-partition of $V(H)$, such that $H[V_1]$ is a maximum order $G$-free induced subgraph of $H$. By maximality $V_1$ one can say that $\Delta(H[V_1]) \leq q$. If $H[V_2]$ is $K_{q}$-free or its $q$-cliques are disjoint, then the proof is complete. Therefore, by contradiction suppose that there are at least two copies $K, K'$ of $K_q$ in $H[V_2]$, such that $V(K) \cap V(K') \neq \emptyset$. Hence, $\Delta(H[V_2]) \leq q$, implies that there exists at least one copy of $G$ in $H[V(K) \cup V(K')] \subseteq H[V_2]$, otherwise one can say that there is at least one member $v$ of $V(K) \cap V(K')$ such that $|N(v) \cap V_2| \geq q + 1$, which is not possible. Hence by Theorem 2.1 we have $K_{\Delta(H)+1} \setminus e \subseteq H$, which would be a contradiction to the assumption. So, the theorem holds.

**Proof Corollary 1.2.** The proof is by induction on $k$. For $k = 2$ the proof is complete by Theorem 1.1. To prove the statement for $k \geq 3$, set $p = p_1$ and $q = \sum_{i=2}^{k} p_i - (k - 2)$. Note that $p + q = \Delta(H) + 1$ and also $p \geq 2, q \geq 4$. Since the statement is true for $k = 2$, we can obtain a partition of $V(H)$ into $V_1$ and $V_2$ such that $H[V_1]$ is maximum $G_1$-free, the maximum degree of $H[V_2]$ is at most $\Delta(H) - (p - 1) = q$ and $H[V_2]$ is $K_q$-free or its $q$-cliques are disjoint.

If the maximum degree of $H[V_2]$ is less than $q$, let $v \in V_2$ be a vertex with maximum degree in $H[V_2]$ such that its degree is equal to $q' < q$. We add $q - q'$ new vertices to $H[V_2]$ and join all of them to $v$, forming a new graph $H'$. The graph $H'$ has maximum degree $q$ and is $K_q$-free or its $q$-cliques are disjoint. Since $k \geq 3, p_2 \geq 2$ and $p_k \geq 4$, we have $\Delta(H') \geq 5$. Also as the graph $H'$ is $K_q$-free or its $q$-cliques are disjoint one can say that $K_{\Delta(H') + 1} \setminus e \not\subseteq H'$. Therefore by induction there exists a partition of $V(H')$ into $W_2, \ldots, W_k$ such that for each $2 \leq i \leq k - 1$, $H'[W_i]$ is $G_i$-free, and the maximum degree of $H[W_k]$ is at most $p_k$ and $H[W_k]$ is $K_{p_k}$-free or its $p_k$-cliques are disjoint. So $V_1, W_2 \cup V_2, \ldots, W_k \cup V_2$ is the desired partition of $V(H)$.

Therefore, we may assume that $H[V_2]$ is a graph with maximum degree $q \geq 5$. We also have $\omega(H[V_2]) \leq q$, the maximum degree of $H[V_2]$ is at most $\Delta(H) - (p - 1) = q$ and $H[V_2]$ is $K_q$-free or its $q$-cliques are disjoint. We have $p_1 \geq 2, p_k \geq 4$ and $\sum_{i=2}^{k} p_i = \Delta(H[V_2]) - 1 + (k - 1)$. Also as the graph $H[V_2]$ is $K_q$-free or its $q$-cliques are disjoint one can say that $K_{\Delta(H[V_2])] + 1} \setminus e \not\subseteq H[V_2]$.

We may assume that $H[V_2]$ is a connected graph, otherwise if $H[V_2]$ has $\ell \geq 2$ connected components, say $H_1, \ldots, H_\ell$, we prove the statement for each of these connected components. Then, for each connected component $H_i$, there exists a partition of $V(H_i)$ into $V_{i,2}, \ldots, V_{i,k}$ such that for each $2 \leq t \leq k - 1, H[V_{i,t}]$ is $G_i$-free and the maximum degree of $H[V_{i,k}]$ is at most $p_k$ and $H[V_{i,k}]$ is $K_{p_k}$-free or its $p_k$-cliques are disjoint. Now, for each $2 \leq j \leq k$ define $V_j = \bigcup_{i=1}^{\ell} V_{i,j}$. Therefore, $V_1, V_2, \ldots, V_k$ is the desired partition of $V(H)$. Hence the proof is complete.

To prove Lemma 1.3 for the case that $q = 2, 3$, we need the following theorem.
Theorem 1.1. \[\text{If } H \text{ is a graph with } \omega(H) \geq \frac{3(\Delta+1)}{4}, \text{ then } H \text{ has an independent set } I \text{ such that } \omega(H \setminus I) \leq \omega(H) - 1.\]

Theorem 3.2. \[\text{Let } H \text{ is a connected graph with } \Delta(H) = \Delta \text{ and } \omega(H) \geq \frac{2}{3}(\Delta + 1), \text{ then } H \text{ contains an independent set intersecting every maximum clique unless it is the strong product of an odd hole and a clique.}\]

In the next results by using Theorem 3.1 and Theorem 3.2 we prove that the Lemma 1.3 is true for the case that \((p, q) = (d - 1, 2)\) and \((p, q) = (d - 2, 3)\).

Proposition 3.3. \[\text{Let } H \text{ is a connected graph with } \Delta(H) = d \geq 7, K_d \setminus e\text{-free and } \omega(H) = d - 1, \text{ also let } (p, q) = (d - 1, 2) \text{ or } (p, q) = (d - 2, 3). \text{ Then there exists a } (V_1, V_2)\text{-partition of } V(H), \text{ so that } H[V_1] \text{ is } K_p\text{-free, and } H[V_2] \text{ is } K_2\text{-free( or } K_3\text{-free).}\]

Proof. Assume that \((p, q) = (d - 1, 2)\). Since \(d \geq 7\), according to Theorem 3.1, we know that \(H\) contains an independent set that intersects every maximum clique. Let’s denote this stable set as \(S\). By considering the graph \((H \setminus S, S)\), we complete the proof.

Now, let’s consider the case when \((p, q) = (d - 2, 3)\). Again, considering \(d \geq 7\) and \(\omega = d - 1\), if \(d \geq 8\), we can apply Theorem 3.1 to conclude that \(H\) contains an independent set that intersects every maximum clique. Let’s denote this stable set as \(S_1\), and define \(H_1 = H \setminus S_1\). If \(\omega(H_1) \leq p - 3\), then the proof is complete. Otherwise, if \(\omega(H_1) = d - 2\), the maximum degree of \(H_1\) is at most \(d - 1\). Let’s assume it is exactly \(d - 1\). If not, we can introduce additional vertices and edges to obtain a graph \(H'\) that has maximum degree \(d - 1\) and is \(Kd\) - 1-free. Now, applying Theorem 3.1 to \(H'\), we can find another stable set \(S_2\) that intersects every maximum clique in \(H_1\). Define \(H_2 = H_1 \setminus S_2\). Since \(\omega(H_2) \leq p - 3\), the proof is complete by considering the graph \((H \setminus (S_1 \cup S_2), S_1 \cup S_2)\).

Finally, let’s consider the case when \(d = 7\). According to Theorem 3.1 we know that \(H\) contains an independent set that intersects every maximum clique. Assume that \(S_1\) is the stable set obtained from Theorem 3.1. Let \(H_1 = H \setminus S_1\). We can assume that \(\omega(H_1) = 5\), as otherwise, the proof is complete. Since \(\omega(H_1) = 5\), we can verify that \(H_1\) is not the strong product of an odd hole and a clique. Consequently, by Theorem 3.2, \(H_1\) contains a stable set that intersects every maximum clique. Let \(S_2\) be this stable set. Therefore, the proof is complete by considering the graph \((H \setminus (S_1 \cup S_2), S_1 \cup S_2)\). Thus, the proposition holds.

Proof Lemma 1.3. As \(H = K_d\) - \(e\)-free, by Theorem 2.1, it can be checked that Lemma 1.3 holds for the case that \(q \geq 4\), also by Proposition 3.3. Lemma 1.3 holds for the case that \(q = 2, 3\), hence the proof is complete.

According to Theorem 1.1 for any graph \(H\) with a maximum degree of 5 that is \(K_5\) - \(e\)-free, there exists a partition \((V_1, V_2)\) of \(V(H)\) such that \(H[V_1]\) is a maximum acyclic induced subgraph, and \(\omega(H[V_2])\) as well as \(\Delta(H[V_2])\) are both at most 3. Moreover, \(H[V_2]\) is either \(K_3\)-free or its 3-cliques are disjoint, which is an improvement over Theorem A.

Similarly, according to Theorem 1.1 for any graph \(H\) with a maximum degree of \(d \geq 5\) that is \(K_{d+1}\) - \(e\)-free, there exists a partition \((V_1, V_2)\) of \(V(H)\) such that \(H[V_1]\) is a maximum acyclic induced subgraph, and \(\omega(H[V_2])\) as well as \(\Delta(H[V_2])\) are both at most \(d - 2\). Additionally, \(H[V_2]\) is either \(K_{d-2}\)-free or its \((d - 2)\)-cliques are disjoint. In fact, if we take \(G = C = C_n\) for \(n \geq 3\), then Theorem 1.1 encompasses and improves upon Theorem A. Hence, it is evident that this result is superior to the result of Theorem A.

3.1. Some research problems related to the contents of this paper. In this section, we propose some research problems related to the contents of this paper. The first problem concerns Theorem 1.1 as we address below:

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Problem 3.3.1. Suppose that $H = (V, E)$ is a connected graph with maximum degree $\Delta(H) \geq 5$ and $H$ is $K_{\Delta(H)+1}$-e-free. Suppose that $p$ and $q$ are two positive integers, such that $p \geq 2, q = 3$ and $\Delta(H) + 1 = p + q$. Set $G$ as a collection of graphs with minimum degree at least $p - 1$. Then there exists a $(V_1, V_2)$-partition of $V(H)$, such that $H[V_1]$ is a maximum order $G$-free induced subgraph of $H$, $\Delta(H[V_2]) \leq q$, and either $H[V_2]$ is $K_q$-free subgraph or its $q$-cliques are disjoint.

Problem 3.3.2. Let $p, q$ be two positive integers, where $p \geq 2, q \geq 3$, and $p + q = d + 1$. Set $G$ as a collection of some $(p-1)$-regular graph, $G'$ as a collection of some $(q-1)$-regular graph, and suppose that $H$ is a connected graph with $\Delta(H) = d \geq 5$ and $\omega(H) = \omega \leq d - 2$. Then there exist $(V_1, V_2)$-partition of $V(H)$, so that $H[V_1]$ is $G$-free, $V_1$ has the M-P-size, and $H[V_2]$ is $G'$-free.

Problem 3.3.3. Let $p, q$ be two positive integers, where $p \geq 2, q \geq 4$, and $p + q = d + 1$. Suppose that $H$ is a graph with $\Delta(H) = d \geq 5$ and $\omega(H) \leq d - 2$. If $d = p + q + 1$, then there exists a $(V_1, V_2)$-partition of $V(H)$, such that $H[V_1]$ is a maximum $(p-2)$-degenerate induced subgraph and $H[V_2]$ is $(q-2)$-degenerate.

4. Declarations

Conflict of Interest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data Availability Statement: No data were generated or used in the preparation of this paper.

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