General Treatment of All 2d Covariant Models

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Abstract

General matterless models of gravity include dilaton gravity, arbitrary powers in curvature, but also dynamical torsion. They are a special class of "Poisson–sigma–models" whose solutions are known completely, together with their general global structure. Beside the ordinary black hole, arbitrary singularity structures can be studied. It is also possible to derive an action "backwards", starting from a given manifold. The role of conservation laws, Noether charge and the quantization have been investigated. Scalar and fermionic matter fields may be included as well.

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1 Introduction

The interest in two dimensional diffeomorphism invariant theories has many roots. Presumably the most basic one is the central role played by spherically symmetric models in $d = 4$ General Relativity (GR) as a consequence of Birkhoff’s theorem. Two dimensional models with 'time' and 'radius' possess an impressive history with promising recent developments [1]. On the other hand, the fact that the Einstein–Hilbert action of pure gravity in 1+1 dimensions is trivial, also has spurred the development of models with additional nondynamical [2] and dynamical (dilaton and tachyon) scalar fields [3], besides higher powers of the curvature [3, 4]. Especially the study of 2d–dilaton theories turned out to be an important spin-off from string theory, and has led to novel insights into properties of black holes [3, 4, 5]. For a scalar field, coupled more generally than a dilaton field, even more complicated singularity structures have been found [3] than in the ordinary dilaton–black hole [3].

Actually such structures were known already before in another branch of completely integrable gravitational theories which modify Einstein–relativity in 1+1 dimensions by admitting nonvanishing dynamical torsion [7, 8]. Here the introduction of the light–cone (LC) gauge led to the expression of the full solution in terms of elementary functions [4] and to an understanding of quantum properties of such a theory in the topologically trivial [10] and nontrivial [11] case.

Recently important progress has been made by the insight [12] that all theories listed above are but special cases of a 'Poisson–sigma–model' (PSM) with action

$$ L = \int_M (A_B \wedge dX^B + \frac{1}{2} P^{BC} (X) A_B \wedge A_C) . $$

The zero forms $X^B$ are target space 'coordinates' with connection one forms $A_B$. $P$ expresses the (in general degenerate) Poisson–structure on the manifold $M$, it has to obey a Jacobi–type identity, generalizing the Yang–Mills case, where $P$ is linear in $X$ and proportional to the structure constants. For the subclass of models describing 2d–covariant theories the $A_B$ are identified with the zweibein $e^a$, with the connection $\omega^a {}_b = e^a {}_b \omega$, and may include possibly further Yang–Mills fields $A_i$. Introducing the Minkowskian frame metric $\eta_{ab} = \text{diag} (1, -1)$, target coordinates $X^A$ on the manifold will be denoted as
\{X^a, X, X^i\}.

Then the (matterless) dilaton, torsion, \(f(R)\)–gravity theories and even spherically symmetric gravity are obtained as special cases \([13]\) of an action of type (1), namely \((\epsilon = \frac{1}{2} \epsilon_{ab} e^a \wedge e^b)\)

\[L = \int_M \left( X_a D e^a + X d \omega - \epsilon V \right).\] (2)

Appropriate fixing of \(V = V(X^a X_a, X, Y)\) yields all the models listed above (and many more). It is possible in principle to write down the full solution for (2) in an arbitrary gauge (coordinate system). As seen below, the solution has much of the shape of the LC gauge solution \([9]\).

A crucial role for the integrability of theories \([1]\) in the general case play ‘Casimir–functions’ \(C_i(X^A)\) \([12, 13]\) which on–shell become constants and thus (gauge–independent) ‘observables’, also in the classical case \([14]\). These constants \(C_i\) together with other parameters in \(P\) (or \(V\), e.g. the cosmological constant) determine the almost limitless variety of Penrose diagrams, characterizing the singularities of such theories \([15]\). E.g. Schwarzschild or Reissner–Nordström black holes are just relatively simple members of that set. Also \(C_1\) for the special theory \([7]\) can be related to a global symmetry \([15]\). The relation to quasilocal energy \([17]\) on a 'surface' and to Noether charge \([18]\) has been clarified in \([19]\).

Furthermore it is possible to show how a theory quadratic in torsion and curvature may indeed be reformulated as an equivalent dilaton theory, by a local method starting from a first order formalism for the \(R^2 + T^2\)–theory \([20]\). Here the spin connection is eliminated in favor of the torsion which turns into a nondynamical field variable.

## 2 PSM–Gravitation

### 2.1 The General Model

With \(V\) in (2) depending linearly on \(X^a X_a\)

\[V = \frac{\alpha}{2} X^a X_a + v(X, Y),\] (3)
the equations of motion from (2) in a LC basis of the frame metric ( \( \varepsilon_{+-} = -1 \), \( X^\pm = (X^0 \pm X^1)/\sqrt{2}, \eta_{+-} = \eta_{-+} = 1 \)) are

\[
\begin{align*}
  dX^\pm \pm \omega X^\pm &= \pm e^\pm V \\
  dX + X^- e^+ - X^+ e^- &= 0
\end{align*}
\]  

(4)

and

\[
\begin{align*}
  de^\pm \pm \omega \wedge e^\pm &= -\alpha e^+ \wedge e^- X^\pm \\
  d\omega &= -e^+ \wedge e^- \frac{\partial v}{\partial X} .
\end{align*}
\]  

(5)

Multiplying the first pair of equations in (4) with \( X^- \) and \( X^+ \), respectively, the second one with \( V \) and adding yields

\[d(X^+ X^-) + V dX = 0,\]  

(6)

producing an absolutely conserved quantity (\( dC = 0 \))

\[C_1 = C = X^+ X^- e^{\alpha X} + w(X)\]  

(7)

\[w(X) = \int_{X_0}^X v(y)e^{\alpha y} dy.\]  

(8)

Clearly the lower limit \( X_0 = \text{const.} \) must be determined appropriately so that (inside a certain patch) the integral exists for a certain range of \( X \). Eq. (7) generalizes \([12, 13]\) the previously known \([7, 9]\) analogous quantity for 2d gravity with dynamical torsion. However, the limit \( \alpha \to 0 \) immediately also yields the conservation law for torsionless cases (\( F(R) \)-gravity, dilaton gravity \([3]\) etc.).

Setting one LC component of the torsion, e.g. \( X^+ \) identically zero, the first equation (5) (at \( e^+ \neq 0 \)) may yield a constant curvature. Such 'de–Sitter' solutions are related to \( C = 0 \) (within an appropriate convention for the integration constant in (8)). Representing discrete points in phase space they are notorious especially in the quantum case \([11]\). If \( X^+ \neq 0 \) the solution of (8) and (5) becomes

\[
\begin{align*}
  e^+ &= X^+ e^{\alpha X} df \\
  e^- &= \frac{dX}{X^+} + X^- e^{\alpha X} df \\
  \omega &= -\frac{dX^+}{X^+} + V e^{\alpha X} df .
\end{align*}
\]  

(9)
Of course, for $X^- \neq 0$ the analogous solution exists with the roles of $X^+ \leftrightarrow X^-$ exchanged. The first terms in the first three eqs. for $e^\pm \rightarrow \delta e^\pm, \omega \rightarrow \delta \omega, df \rightarrow \delta \gamma$ are the on-shell extension of a global nonlinear off-shell symmetry of (2). It is related to the conservation $\partial_\mu J^\mu = 0$ of a Noether current $J^\mu = C \delta^\mu \nu$ because under such a transformation the Lagrangian density in (2) changes by a total derivative only. Mathematically (3) coincides with the solution in the LC–gauge where the curvature $X$ is gauge–fixed to be linear in 'time'. But (4) has the big advantage that it is valid in an arbitrary gauge, whereas solutions obtained in the literature to such theories had to rely on special gauges and sometimes on sophisticated mathematical methods to solve the respective equations (cf. e.g. [4, 5, 6, 7]). The line element from (4) generally reads

$$(ds)^2 = 2e^{\alpha X} df \otimes (dX + X^+X^- e^{\alpha X} df),$$

with $X^+X^-$ to be expressed by (7) for fixed $C$. For the case with torsion a generic model may be chosen as

$$V = \alpha X^+X^- + \frac{\rho}{2} X^2 - \Lambda.$$  

(11)

This $V$ allows to produce $C$ by a simple integral according to (7). Integrating out $X$ and $X^\pm$ in (2) leads to the model quadratic in curvature and torsion of (4) which, in four dimensions, together with the Einstein–Hilbert term has been known as the 'Poincare–gauge theory' for some time [21]. It only contains second derivatives in the field equations for the variables $e^a$ and $\omega$. However, higher derivative theories are to be treated with equal ease, when polynomials of higher degree in $X$ and $X^+X^-$ are admitted in (3). Of course, $V$ could even be a nonpolynomial function. This would only make the integration harder which leads to (4). As we shall recall shortly below, the zeros of (4) determine the singularity structure of the theory. Thus one could design such a structure by prescribing $C(X^A)$. From (7) the corresponding $V$ can be read off by differentiation, and the action for that structure follows immediately.

Among the models with vanishing torsion ($\alpha = 0$ in (3)), the Jackiw–Teitelboim model obtains for $v = \Lambda X$. Witten’s black hole (4) represents a special case of a class of more general torsionless theories involving the curvature scalar $R$ and one additional scalar field $\Phi$ in a Lagrangian of the
type
\[ \mathcal{L} = \sqrt{-g}[\partial_\alpha \varphi \partial_\beta \varphi g^{\alpha\beta} + A(\varphi) + RB(\varphi)] \] (12)

with arbitrary functions A and B. Matterless dilaton–gravity [3] is the special case \( \varphi^2 = 4B = A/\lambda^2 = 4e^{-2\Phi} \)
\[ \mathcal{L}_{dil} = \sqrt{-\tilde{g}}[4\partial_\alpha \Phi \partial_\beta \Phi g^{\alpha\beta} + 4\lambda^2 + R] . \] (13)

Using the conformal identity for \( \tilde{g}_{\alpha\beta} = e^{-2\phi}g_{\alpha\beta} \) (or \( \tilde{e}^a = e^{-\phi}e^a \))
\[ \sqrt{-\tilde{g}} \tilde{R} = \sqrt{-g}R + 2\partial_\alpha(\sqrt{-gg^{\alpha\beta}}\partial_\beta \phi) . \] (14)

Eq. (14) allows the elimination of the kinetic term for \( \varphi \) in (12). The resulting action may be written readily in the first order form (2) for \( \tilde{e}^a = \tilde{e}^a/\sqrt{X/2} \), simply leads to
\[ e^+ = X^+e^\Phi df \]
\[ e^- = \frac{1}{X^+}[-4d\Phi e^{-\Phi} + df(C e^\Phi - 8\lambda^2 e^{-\Phi})] \] (15)
\[ \omega = -\frac{dX^+}{X^+} + 4\lambda^2 df . \]

Here \( \Phi, f \) and \( X^+ \) are arbitrary functions. E.g. the Kruskal form for the metric \((ds)^2 = 2e^+ \otimes e^- \) a dilaton black hole follows from the gauge–fixation \((X^+X^- = uv)\)
\[ 8\lambda^2 e^{-2\Phi} = C - uv \]
\[ 4\lambda^2 f = \ln u . \]

The mass of the dilaton black hole is related to \( C \) by \( C = 8\lambda M \). \( X^+ \neq 0 \) being still arbitrary, it may be used to gauge \( \omega = 0 \) which shows that the connection \( \omega \) in (13) really has nothing to do with a curvature belonging to the metric derived from that equation.

Now precisely the same procedure may be applied to the generalized theories of type (12). Here \( \varphi \) can be eliminated [4] using (14)
\[ \tilde{\mathcal{L}} = \sqrt{-\tilde{g}}[A(\varphi)/F(\varphi) + \tilde{R}B(\varphi)] \] (16)
with
\[ g_{\alpha\beta} = \tilde{g}_{\alpha\beta}/F(\varphi) \]
\[ \ln F(\varphi) = \int dy/(dB/dy) \]  \hspace{1cm} (17)

The corresponding first order action \( \tilde{L} \) becomes
\[ \tilde{L} = \int (X_a D\tilde{e}^a + 2Bd\omega - \tilde{\epsilon}A/F) \] .

In (2) for \( \alpha = 0 \) we have as a consequence
\[ X = 2B \]
\[ V = v = \frac{A(B^{-1}(X/2))}{F(B^{-1}(X/2))} \] ,  \hspace{1cm} (18)
and the conserved quantity for any theory of type (12) is (7) with \( \alpha = 0 \) and
\[ w(X) = \frac{B^{-1}(X/2)}{B^{-1}(X_0/2)} \int_{B^{-1}(X_0/2)} A(y)dy/F(y) \] .  \hspace{1cm} (19)

The line–element in terms of coordinates \( (f, X) = Y^\alpha \) for any such theory and in any gauge reads
\[ (ds)^2 = F^{-1}2df \otimes [dX + df(C - w(X))] \] .  \hspace{1cm} (20)

Of course, in each application to a particular model a careful analysis of the range of validity of the mathematical manipulations is required in order to determine a patch, where those steps are justified: allowed ranges for transformations of fields, inversions of functions like \( B^{-1} \), integrability of \( F \), admissible gauges for \( f \) and \( X \) etc.

Another example is the action for the Schwarzschild black hole in 4d GR. \( v(X) = -1/(2X^2) \) in (4) is found to yield the correct line–element.
2.2 Killing Vector and Singularities

In our very general class of models the Killing vector can be found without fixing the gauge (coordinate–system). Using (9) we rewrite the line element (10) in a theory (2) as

$$(ds)^2 = df \otimes [2e^{\alpha X} dX + l df]$$ (21)

where

$$l = 2X^+ X^- e^{2\alpha X} = 2e^{\alpha X}(C - w(X, Y)) .$$ (22)

In terms of the variables $Y^\alpha = (f, X)$, resp. $\partial/\partial Y^\alpha$, $k^\alpha$ is the Killing vector with norm (22)

$$
k^\alpha = (1, 0)

k^2 = k^\alpha k^\beta g_{\alpha\beta} = l .$$ (23)

For the discussion of the singularity structure of (9) a (partial) gauge fixing is useful. If $l > 0$ in (21) we choose coordinates time ($t$) and space ($r$) in $f = f(t, r), X = X(r)$ with $\dot{f} = T(t)$ and

$$X' e^{\alpha X} + f' l(X) = 0 ,$$ (24)

where $f' = \partial f/\partial r , \dot{f} = \partial f/\partial t$. Introducing

$$K(z) = - \int^z_{z_0} dy e^{\alpha y} l^{-1}(y) ,$$ (25)

(24) implies

$$f = \int^t_0 T(t') dt' + K(X(r)) .$$ (26)

In such a gauge $g_{tr}$, the off–diagonal part of the metric vanishes, so that ‘space’ and ‘time’ are separated. In order to avoid zeros in the norm of the Killing–vector field $k$ it is obvious to restrict $z$ and $z_0$ to a suitable interval of $y = X(r)$ where $k$ exists. The remaining elements of $g_{\alpha\beta}$ are:

$$g_{tt} = \dot{f}^2 l$$

$$g_{rr} = -(f')^2 l$$ (27)
Requiring a 'Schwarzschild'–form of the metric, i.e. $\det g = -1$, eliminates the arbitrary functions $T(t)$ and $X(r)$ altogether,

\[
\begin{align*}
T &= 1 \\
\alpha X &= \ln (\alpha r) \ (\alpha \neq 0) \\
X &= r \ (\alpha = 0),
\end{align*}
\]

dropping a multiplicative constant $a$ in $f$, and $1/a$ together with $r$, and two further constants for the zero points of $t$ and $r$. Now

\[
g_{tt} = -g_{rr}^{-1} = l(X(r))
\] (29)

follows with $X(r)$ from (28). Especially (29) clarifies the remark above, how an action may be reconstructed for a given singularity in the metric, proceeding backwards through (22) to (2).

We note that for a (generalized) dilaton theory, besides $\alpha = 0$, because of the additional factor $1/F$ in (20) there is a corresponding change to $l$ in (27) etc. Thus the singularity structure is determined by $l/F$.

The Katanaev–Volovich model (9) is sufficiently general to show the intrinsic singularity structure by an analysis of completeness of geodesics. $C^2$ global completeness was first shown in [8] within the conformal gauge. The more suitable present approach allows the extension to $C^\infty$ completeness and a discussion of possible compactifications [15]. In that model altogether 11 types of Penrose diagrams appear (G1, . . . G11 in the classification of [8]). Some show similarities to Schwarzschild and to Reissner–Nordström types, but there are many more. In the more complicated cases they are obtained by the possibility to successively gluing together solutions for patches, each given by (15). The diffeomorphism for doing that is essentially (25) again. For further details we refer to [15]. It is sufficient for our present purposes to note that for all types of singularities (including also e.g. naked ones) there are space–like directions allowing the study of surfaces (points) between such singularities at finite (incomplete case) or infinite (complete) distances. Also a second point is obvious from this section: In all two dimensional theories the conserved quantity $C$ never has a well-defined sign. Thus any hope to find a positive 'energy' must be in vain. Therefore, also adding matter to the theories (2) is not likely to improve this situation.
3 Equivalence of Generalized Dilaton Theory

We now show that in a theory of type (2), i.e. in a 'Poincaré–gauge theory' \[21\] the torsion can be eliminated. In the definition

\[ T^\pm = (\partial_\mu \pm \omega_\mu)e^\pm_\nu \tilde{\epsilon}^{\mu\nu} \] (30)

we introduce light cone coordinates \( T^\pm = \frac{1}{\sqrt{2}} (T^0 \pm T^1) \) in the Lorentz–indices. It should be noticed that the subsequent steps will even remain correct for \( \alpha = \alpha(X) \) and general \( v = v(X) \), i.e. a theory quadratic in torsion but with arbitrary higher powers in curvature.

Now instead of \( \omega_\mu \) the \( T^\pm \) in (30) are introduced as new variables:

\[ \tilde{\epsilon}^{\mu\nu} \partial_\mu \omega_\nu = \tilde{\epsilon}^{\mu\nu} \partial_\mu \tilde{\omega}_\nu + \tilde{\epsilon}^{\mu\nu} \partial_\mu \left[ \frac{(e_\nu^- T^+ + e_\nu^+ T^-)}{e} \right] \] (31)

The first term on the r.h.s. of (31) is proportional to a torsionless curvature \( \tilde{R} \),

\[ \tilde{\epsilon}^{\mu\nu} \partial_\mu \tilde{\omega}_\nu = - \frac{\tilde{R} e}{2} \] . (32)

Inserting (31) into (2), after shifting the derivatives in the second term of (31) onto \( X \) exhibits the nondynamical nature of \( T^\pm \) which may be 'integrated out' by solving their (algebraic) equations of motion. With a definition of the dilaton field

\[ \frac{X}{2} = e^{-2\phi} \] (33)

and after reexpressing the factors \( e_\nu^\pm/e \) from the square bracket of (31) in terms of the inverse zweibeins combining them into \( g^{\alpha\beta} = e^{+\alpha} e^{-\beta} + e^{+\beta} e^{-\alpha} \), the Lagrangian \( \mathcal{L}^{(1)} \) in (2) is found to be equivalent to

\[ \mathcal{L}^{(2)} = \sqrt{-\hat{g}} \left[ -e^{-2\phi} \hat{R} + 8\alpha \cdot e^{-4\phi} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - v(2e^{-2\phi}) \right] \] (34)

In addition, using the identity

\[ g_{\mu\nu} = e^{2\phi} \hat{g}_{\mu\nu} \]

\[ \sqrt{-gR} = \sqrt{-\hat{g}} \hat{R} - 2\partial_\alpha \left[ \sqrt{-\hat{g}} \hat{g}^{\alpha\beta} \partial_\beta \phi \right] \] (35)
which also represents a local transformation, allows to write down the most
general dilaton theory equivalent to the $R^2 + T^2$–theory:

$$\mathcal{L}^{(3)} = \sqrt{-\hat{g}} \left[ -e^{-2\phi} \hat{R} + 4\hat{g}^{\alpha\beta} (\partial_\alpha \phi) \left( e^{-2\phi} \partial_\beta \phi + 2\alpha e^{-4\phi} \partial_\beta \phi \right) - e^{2\varphi} v(2e^{-2\phi}) \right]$$

(36)

For $\varphi = -\phi$

$$\mathcal{L}^{(4)} = -\sqrt{-\hat{g}} \left[ \hat{R} + 4(1 - 2\alpha e^{-2\phi})(\nabla \phi)^2 + v(2e^{-2\phi}) \right]$$

(37)

the deviation from ordinary dilaton theory ($\alpha = 0, v = 4\lambda^2$) is most obvious.
Of course, the dilaton field may be eliminated altogether as well, if in (36) (for constant $\alpha$)

$$\varphi = \varphi(\phi) = \alpha e^{-2\phi} = \frac{\alpha X}{2}$$

(38)

is chosen. In that case it seems more useful to retain the variable $X$ instead of $\phi$:

$$\mathcal{L}^{(5)} = -\sqrt{-\hat{g}} \left[ \frac{X \hat{R}}{2} + e^{\alpha X} v(X) \right]$$

(39)

Comparing (39) to a torsionless theory with $\alpha = 0$ but modified $v$, the
difference now just resides in the additional exponential $e^{\alpha X}$.

4 Global Solutions for Dilatonic Versions of Theories with Torsion

The study of global properties for 2d theories is based upon the extension of
the solution which is known at first only in local patches, continued maxi-
mally to global ones. The analysis uses null–directions which become the co-
ordinates of Penrose diagrams which are sewed together appropriately. The
continuation across horizons and the determination of singularities can be
based upon extremals or geodesics. The physical interpretation of an ex-
tremal is the interaction of the space–time manifold with a point like test
particle, 'feeling' the metric $g_{\alpha\beta}$ through the Christoffel symbol [9]. After
torsion has been eliminated, there is no ambiguity for our analysis which
only has extremals at its disposal. That interaction with extremals, how-
ever, crucially depends on the choice of the 'physical' metric to be used: the
one computed from the $e_{\mu}^{\alpha}$ of (1), or any $\hat{g}_{\alpha\beta}$ which is a result of different field transformations involving the dilaton field? Clearly the torsionless dilaton theory (34) has the same metric as (1), e.g. the global analysis of [8, 15] applies directly and the different types of solutions are exhausted by those studied there. However, from the point of view of a ‘true’ dilaton theory, one could argue as in Section 2 that with a redefined matrix as in (37), $\hat{g}_{\alpha\beta} = e^{2\phi}g_{\alpha\beta} = 2g_{\alpha\beta}/X$ has some physical justification as well. In fact, for Witten’s black hole $g_{\alpha\beta}$ is flat and the interesting (black hole) singularity structure just results from the factor $2/X$. Now, in the original $R^2 + T^2$--theory [7] there are solutions (G3) resembling e.g. the black hole but not completely: Their singularity resides at light–like distances and they are not asymptotically flat in the Schwarzschild sense. Thus the factor $2/X$ may well yield improvements on that situation.

Here we indicate the analysis of a generalized dilaton gravity [20]

$$\mathcal{L} = \sqrt{-\hat{g}}e^{-2\phi} \left[ \hat{R} + 4(1 - 2\alpha e^{-2\phi})(\nabla\phi)^2 + 2\beta e^{-4\phi} + 4\lambda^2 \right]$$  \hspace{1cm} (40)

which is obtained from (2) by taking

$$X = 2e^{-2\phi} \quad g_{\mu\nu} = \hat{g}_{\mu\nu}e^{-2\phi} \quad \Lambda = -4\lambda^2$$  \hspace{1cm} (41)

and omitting an overall minus sign. We need to consider only the cases for $\beta = \rho$ positive, negative or 0. The absolute value of a nonvanishing $\beta$ may always be absorbed by rescaling $X$ and $\omega$ to $X \to \sqrt{\beta}X$ and $\omega \to \frac{\omega}{\sqrt{|\beta|}}$.

Let us start with a positive value for $\beta$ e.g. +2. All global solutions are most easily obtained by the known general solution (9). Using (41) and defining coordinates by $v = -4f, u = \phi$ in (10) yields the line element of the generalized dilaton Lagrangian (41)

$$(ds)^2 = g(u) \left( 2dvdu + l(u)dv^2 \right)$$  \hspace{1cm} (42)

with

$$l(u) = \frac{e^{2u}}{8} \left( C - g(u) \left( \frac{4e^{-4u}}{\alpha} - \frac{4e^{-2u}}{\alpha^2} + C_0 \right) \right)$$  \hspace{1cm} (43)

$$g(u) = e^{2\alpha e^{-2u}}$$  \hspace{1cm} (44)

$$C_0 = \frac{2}{\alpha^3} + \frac{4\lambda^2}{\alpha}$$  \hspace{1cm} (45)
which automatically implies the convention for the constant of integration in (7) to be used in the following. The conformal gauge $(ds)^2 = F(u') d\tilde{u}' d\tilde{v}'$ in (12) is obtained by ‘straightening’ the null extremals

$$v = \text{const.}$$  \hspace{1cm} (46)

$$\frac{dv}{du} = -\frac{2}{l} \text{ for all } u \text{ with } l(u) \neq 0$$  \hspace{1cm} (47)

$$u = u_0 = \text{const} \text{ for } l(u_0) = 0$$  \hspace{1cm} (48)

by means of a diffeomorphism

$$\tilde{u}' = v + f(u), \quad \tilde{v}' = v$$  \hspace{1cm} (49)

$$f(u) \equiv \int^u \frac{2dy}{l(y)}. \quad (50)$$

A subsequent one $\tilde{v}' \to \tan \tilde{v}'$ and another appropriately chosen one for $\tilde{u}'$ produce the Penrose diagram. It is valid for a certain patch where (50) is well defined. Clearly the shape of those diagrams depends crucially on the (number and kind of) zeros and on the asymptotic behavior of $l(u)$. The analysis of all possible cases as described by the ranges of parameters $\alpha, C$ and $\lambda^2$ is straightforward, but tedious. Apart from $C_0$, defined in (15), also

$$C_1 = \frac{2}{\alpha^2} e^{2\alpha\sqrt{-\lambda^2}} \left( \frac{1}{\alpha} - 2 \sqrt{-\lambda^2} \right)$$  \hspace{1cm} (51)

plays a role for $C < C_0$ and $\lambda^2 < 0$, discriminating the possible cases with two zeros, with one double–zero and without zero in $l$, i.e. the presence of two nondegenerate or one degenerate killing–horizon. The qualitatively distinct cases for $\alpha > 0$ and $\alpha < 0$ are listed in (52) and (53):

$$\alpha > 0:$$

D1\textsuperscript{+} : $C > C_0$

D2\textsuperscript{+} : $C = C_0, \quad \lambda^2 < 0$

D3\textsuperscript{+} : $C = C_0, \quad \lambda^2 \geq 0$

D4\textsuperscript{+} : $C < C_0, \quad C > C_1, \quad \lambda^2 < 0$

D5\textsuperscript{+} : $C < C_0, \quad C = C_1, \quad \lambda^2 < 0$

D6\textsuperscript{+} : $C < C_0, \quad C < C_1$.  \hspace{1cm} (52)
\[ \alpha < 0 : \]

\[ D1^- : \quad C > C_0, \quad C \geq 0 \]
\[ D2^- : \quad C > C_0, \quad C < 0 \]
\[ D3^- : \quad C = C_0, \quad C \geq 0 \]
\[ D4^- : \quad C = C_0, \quad C < 0, \quad \lambda^2 \geq 0 \]
\[ D5^- : \quad C = C_0, \quad C < 0, \quad \lambda^2 < 0 \]
\[ D6^- : \quad C < C_0, \quad C < C_1 \]
\[ D7^- : \quad C < C_0, \quad C = C_1 \]
\[ D8^- : \quad C < C_0, \quad C > C_1, \quad C < 0 \]
\[ D9^- : \quad C < C_0, \quad C > C_1, \quad C \geq 0 \]

(53)

For the corresponding Penrose diagrams we refer to [20]. Except for the cases \( D2^+, D3^\pm, D4^-, D5^-, D6^- \) (where \( R \to \pm \alpha C_0 \)) the scalar curvature diverges at \( u \to \pm \infty \). For each set of the parameters as summarized in (52), and (53) another solution in conformal coordinates is obtained by interchanging the role of the null–directions. The transformation

\[
\ddot{u}'' = u \\
\ddot{v}'' = -f(u) - w
\]

(54)

with \( f(u) \) from (50) may be easily verified to do this job.

Extremals obey simple first order differential equations. All cases (time–like, space–like, null) must be checked, especially at the boundaries. With these tools patches may be glued together. For \( D2^+ \) this leads to the well–known shape of the 'classical' black hole in the corresponding global solution. It seems instructive to compare our present global structure to the one studied for other theories. The original \( R^2 + T^2 \)–theory contains one solution resembling the 'real' Schwarzschild black hole only in a very approximate sense. In the notation of [8] the solution \( G3 \) exhibits an (incomplete) singularity, but into null–directions, the 'asymptotically flat' direction is replaced there by a singularity of the curvature, albeit at an infinite distance (complete case). Here precisely the example \( D2^+ \) is completely Schwarzschild–like. Other solutions with similar properties, but more complicated singularity structure are \( D3^-, D4^- \) (naked singularities) and \( D5^- \). Among the remaining diagrams the absence of manifolds with two dimensional (infinite) periodicity as in the \( R^2 + T^2 \)–case can be emphasized. On the other hand, the 'eye' diagram \( D6^+ \) appears here, as well as the square diagrams \( D1^+ \) of \( R^2 \)–gravity.
$D5^-$ represents an interesting variety of a manifold where the ordinary black hole is replaced by a 'light'-like singularity.

5 Conclusion and Outlook

In the quantum case a genuine field theory only arises in interaction with matter. Without that only on a suitable compactified space isotopic to $S^1$ the finite number of zero modes precisely of the $C$-s covers a quantum mechanical theory with a finite number of degrees of freedom.

Although $C_1$ (in our generic case) turns into a 'energy density', not necessarily constant in space and time anymore when matter is present [22], it retains its physical aspects related to the geometrical part of the action — very much like the mass parameter in the so far very most prominent case, the example of the dilaton black hole interacting with matter: E.g. generalizing to a matter dependent (minimally coupled) action $L^{(m)}(e_a^\mu)$ with a contribution to the r.h.s. of the first eq. (4), containing a one–form $S^{(m)}\pm$, the steps leading to impose a relation

$$dC + W^{(m)} = 0$$ (55)

Poincaré’s lemma or the other e.o.m-s (3) require that

$$W^{(m)} = X^+S^{(m)-} + X^-S^{(m)+} = dC^{(m)}$$ (56)

Thus a generalization of the absolute conservation law for $C$ emerges. $C$ now will vary in space and time, in general.

All our results, however, point into the direction that in 1 + 1 dimensions, including spherically symmetric gravity and thus also the Schwarzschild black hole, the geometrical part of the action does not acquire quantum corrections [10], such quantum effects being restricted to compactified topologies [11].

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