THE FIBRES OF THE PRYM MAP OF 
ÉTALE CYCLIC COVERINGS OF DEGREE 7

HERBERT LANGE AND ANGELA ORTEGA

Abstract. We study the Prym varieties arising from étale cyclic coverings of degree 7 over a curve of genus 2. We prove that these Prym varieties are products of Jacobians $JY \times JY$ of genus 3 curves $Y$ with polarization type $D = (1, 1, 1, 1, 1, 7)$. We describe the fibers of the Prym map between the moduli space of such coverings and the moduli space of abelian sixfolds with polarization type $D$, admitting an automorphism of order 7.

1. Introduction

Given a finite covering $f : \tilde{C} \to C$ between smooth projective curves, one can associate to $f$ an abelian subvariety of the Jacobian $J\tilde{C}$ by taking the connected component containing the zero of the kernel of the norm map $Nm_f : J\tilde{C} \to JC$, $[D] \mapsto [f_*(D)]$. The resulting variety

$$P(f) := (\text{Ker } Nm_f)^0 \subset J\tilde{C}$$

is called the Prym variety of $f$. The restriction of the theta divisor of $J\tilde{C}$ to $P(f)$ defines a polarization on the Prym variety which is known to be twice a principal polarization when $f$ is an étale double covering (see [9]). The assignment $[f : \tilde{C} \to C] \mapsto P(f)$ yields a Prym map between the moduli space of the corresponding coverings and the moduli space of abelian varieties of suitable dimension and polarization (not necessarily principally polarized). In very few cases the Prym maps are generically finite over its image (see [1], [5], [6], [7] and [8]) and often the structure of the fibers can be understood in geometrical terms ([4], [6]).

We study the Prym varieties associated to a non-trivial 7 cyclic étale covering $f : \tilde{C} \to C$ over a curve $C$ of genus 2. The corresponding Prym variety $P(f)$ is a 6-dimensional abelian variety with polarization type $D = (1, 1, 1, 1, 1, 7)$. Let $R_{2,7}$ denote the moduli space of these coverings and $B_D$ the moduli space of abelian varieties of dimension 6 with a polarization of type $D$ and an automorphism of order 7 compatible with the polarization. In [8] we proved that the Prym map $Pr_{2,7} : R_{2,7} \to B_D$ is dominant and since both moduli spaces are 3-dimensional (see [8] for a proof), the map $Pr_{2,7}$ is generically finite and in fact, of degree 10. In this article we give a geometric description of the fibers of the Prym map $Pr_{2,7}$.

First we prove in Section 2 that for all $[f : \tilde{C} \to C] \in R_{2,7}$ the associated Prym variety $P(f) = JY \times JY$, where $Y$ is a curve of genus 3 whose Jacobian admits a multiplication

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in the totally real cubic subfield \( \mathbb{Q}(\zeta_7)^0 \) of the cyclotomic field \( \mathbb{Q}(\zeta_7) \) (Proposition 2.4); moreover \( Y \) is uniquely determined by the Prym variety. In Section 3 we show that the curve \( Y \) admits a degree 7-covering \( Y \to \mathbb{P}^1 \) with ramification type \((2, 2, 2, 1)\), that is, \( f \) is simply ramified over each branch point at 3 points and unramified at 1. Let \( \mathcal{C}_3 \) denote the locus in \( \mathcal{M}_2 \) of curves \( Y \) having a covering of this type. The main result is the following.

**Theorem 1.1.** The elements of a generic fiber of the Prym map \( \text{Pr}_{2,7} : \mathcal{R}_{2,7} \to \mathcal{B}_D \) are in bijection with the set of degree 7 coverings \( f : Y \to \mathbb{P}^1 \) with ramification type \((2, 2, 2, 1)\) over 6 general points that a curve \( Y \in \mathcal{C}_3 \) admits.

**Remark 1.2.** In [8] we considered a partial compactification of \( \text{Pr}_{2,7} \) which is proper and generically finite of degree 10. So, a finite fibre consists of 10 elements counted with multiplicities. This fact together with Theorem 1.1 shows that the number of degree 7 coverings \( f : Y \to \mathbb{P}^1 \) with ramification type \((2, 2, 2, 1)\) over each of the 6 ramification points is \( \leq 10 \).

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2. Description of the Prym Variety as Product of Jacobians

Let \( C \) be a genus 2 curve and \( f : \widetilde{C} \to C \) a non-trivial étale cyclic covering of degree 7. We shall give an explicit description of the associated Prym variety \( P(f) \). Part of the results in this section are contained in [10]. It is known that the hyperelliptic involution \( \iota \) of \( C \) lifts, though non-canonically, to an involution \( j : \widetilde{C} \to \widetilde{C} \). If we denote by \( \sigma \) the automorphism of order 7 on the cover \( \widetilde{C} \), then \( j, \sigma \) generate the dihedral group

\[
D_7 = \langle j, \sigma \mid j^2 = \sigma^7 = 1, j\sigma j = \sigma^{-1} \rangle.
\]

Moreover, the composed map \( \widetilde{C} \to \mathbb{P}^1 \) is Galois with group \( D_7 \) ([3 Proposition 2.1]). Using the commutativity of the diagram

\[
\begin{array}{ccc}
\widetilde{C} & \xrightarrow{j} & \widetilde{C} \\
\downarrow f & & \downarrow f \\
C & \xrightarrow{\iota} & C
\end{array}
\]

one checks that the images of the fixed points of \( j \) are precisely the fixed points of \( \iota \) and, since \( f \) is étale, there is only one fixed point on each fiber of a fixed point of \( \iota \). Let

\[
g : \widetilde{C} \to Y := \widetilde{C}/\langle j \rangle
\]

be the double covering associated to the involution \( j \). Then \( g \) is ramified exactly at the 6 fixed points of \( j \), so by Hurwitz formula \( Y \) is of genus 3. Notice that since \( g \) is a ramified cover, the map \( g^*: JY \to J\widetilde{C} \) is injective, so we can regard \( JY \) as subvariety in \( J\widetilde{C} \). As before, let \( P = P(f) \) be the Prym variety associated to \( f \).

**Proposition 2.1.** The map

\[
\psi : JY \times JY \to P \\
(x, y) \mapsto x + \sigma(y)
\]

is an isomorphism of abelian varieties.

Certainly \( \psi \) does not respect the canonical polarizations, since \( JY \times JY \) is canonically principally polarized whereas the induced polarization \( \Xi \) on \( P \) is of type \( (1, \ldots, 1, 7) \).

**Proof.** First we check that \( JY \) and \( \sigma(JY) \) are contained in \( P \). Recall that \( P = \text{Ker}(1 + \sigma + \cdots + \sigma^6) \) and \( g^*(JY) = \text{Im}(1 + j) \). On \( \tilde{J}C \)

\[
(1 + \cdots + \sigma^6)(1 + j) = 1 + \cdots + \sigma^6 + j + \cdots + j\sigma^6 = 0
\]

since this map is the the norm map of \( \tilde{C} \to \mathbb{P}^1 \). This shows that \( JY \subset P \) and as \( \sigma \) acts on \( P \), we also have \( \sigma(JY) \subset P \).

Note that \( \dim(JY \times JY) = \dim P \), so in order to prove the proposition it suffices to show that \( \psi \) is injective. Recall that, since \( Y = \tilde{C}/(j) \), the elements of \( JY \cong g^*JY \) are fixed by \( j \). Let \((x, y) \in JY \times JY \) such that \( x + \sigma(y) = 0 \). Then

\[
x = j(x) = -j\sigma(y) = -\sigma^6(y) - j\sigma^6(y)
\]

and hence \( \sigma^2(x) = -\sigma(y) = x \). This shows that \( x \in \text{Fix}(j, \sigma) \). In particular \( x \in \text{Fix}(\sigma) \cap P \subset \tilde{J}C[7] \). Since \( \sigma(x) = x \) and \( f \) is étale there exists an element \( z \in JC \) such that \( x = f^*(z) \). Further, \( j(x) = x \) and the commutativity of diagram (2.1) implies that

\[
x = j(x) = jf^*(z) = f^*\epsilon(z) = -f^*(z) = -x
\]

since on \( JC \) we have \( \epsilon(z) = -z \), that is \( x \in \tilde{J}C[2] \). In conclusion \( x \in \tilde{J}C[2] \cap \tilde{J}C[7] = \{0\} \) and therefore \( \psi \) is injective.

In order to describe the polarization on the product let \( \Xi \) denote the polarization of type \( (1,1,1,1,1,1,7) \) on \( P \). Hence its pull back \( \psi^*\Xi \) is also of type \( (1,1,1,1,1,7) \). Now \( \Xi \) is the restriction of the canonical polarization of \( \tilde{J}C \) to \( P \). So we may consider the polarization \( \psi^*\Xi \) as the pullback of the canonical polarization of \( \tilde{J}C \) to \( JY \times JY \).

We identify \( \tilde{J}C = \tilde{J}C \) and \( \tilde{J}Y = JY \) via the canonical polarizations. The induced split polarization on \( JY \times JY \) gives an identification \( (JY \times JY)^\wedge = JY \times JY \). Let \( \theta : JY \times JY \to P \leftarrow \tilde{J}C \) be the embedding. For \( i = 1, 2 \) denote by \( \theta_i \) the composition

\[
\theta_i : JY \to JY \times JY \xrightarrow{\theta} \tilde{J}C,
\]

where the first map is the natural embedding into the \( i \)-th factor. One checks that

\[
\theta_1 = g^* : JY \to \tilde{J}C \quad \text{and} \quad \theta_2 = \sigma \circ g^* : JY \to \tilde{J}C.
\]

Since the dual of \( g^* \) is the norm map

\[
N_g = 1 + j : \tilde{J}C \to JY,
\]

this implies that

\[
\hat{\theta}_1 = \hat{g}^* = N_g = 1 + j : \tilde{J}C \to JY \quad \text{and} \quad \hat{\theta}_2 = \hat{g}^* \circ \sigma^{-1} = \sigma^6 + \sigma j : \tilde{J}C \to JY,
\]

since \( \hat{\sigma} = \sigma^{-1} \).

\[\Box\]
Now note that \( \hat{\theta}_1 \theta_1 = N_g g^* = 2_{JY} \) and \( \hat{\theta}_2 \theta_2 = N_g \sigma^{-1} g^* = 2_{JY} \). Hence the matrix of \( \phi_{\psi \Xi} : JY \times JY \to JY \times JY \) is
\[
\phi_{\psi \Xi} = \begin{pmatrix}
\hat{\theta}_1 \theta_1 & \hat{\theta}_2 \theta_2 \\
\hat{\theta}_2 \theta_1 & \hat{\theta}_2 \theta_2
\end{pmatrix} = \begin{pmatrix}
2_{JY} & N_g \sigma g^* \\
N_g \sigma^{-1} g^* & 2_{JY}
\end{pmatrix}.
\]
So we have,

**Proposition 2.2.** Let \( g : \tilde{C} \to Y = \tilde{C}/\langle j \rangle \) be the natural map and suppose \( \tilde{JY} = JY \) and \( \hat{\theta}_i \) via the canonical principal polarizations. If we denote by \( \varphi \) the isogeny \( \varphi = N_g \sigma g^* : JY \to JY \), then the polarization \( \psi^* \Xi \) on \( JY \times JY \) is given by the matrix
\[
\phi_{\psi \Xi} = \begin{pmatrix}
2_{JY} & \varphi \\
\varphi & 2_{JY}
\end{pmatrix}.
\]

In order to study the polarizations of type \((1, \ldots, 1, 7)\) on the product \( P := JY \times JY \), we recall from \([2, \text{Section 5.2}]\) the description of the set of polarizations of degree 7 on \( P \). According to \([2, \text{Theorem 5.2.4}]\) the canonical principal polarization on \( JY \times JY \) induces a bijection between the sets of

(a) polarizations of degree 7 on \( JY \times JY \) and

(b) totally positive symmetric endomorphism of \( JY \times JY \) with analytic norm 7.

The endomorphisms of \( JY \times JY \) are given by square matrices of degree 2 with entries in \( \text{End}(JY) \). The symmetry in (b) is with respect to the Rosati involution. Since the Rosati involution with respect to the canonical polarization of \( JY \times JY \) is just transposition of the matrices, we are looking for the symmetric positive definite matrices
\[
A := \begin{pmatrix}
\rho_\alpha(\alpha) & \rho_\alpha(\beta) \\
\rho_\beta(\alpha) & \rho_\beta(\beta)
\end{pmatrix}
\]
with determinant 7, where \( \alpha, \beta, \delta \in \text{End}(JY) \).

**Proposition 2.3.** If \( P \) admits a polarization of degree 7, then \( \text{End}(JY) \not\supseteq \mathbb{Z} \).

**Proof.** Suppose \( \text{End}(JY) = \mathbb{Z} \). For a general \( Y \) the Rosati involution on \( JY \times JY \) is transposition composed with the Rosati involution of the pieces \( JY \), but for a general \( Y \) the Rosati involution on \( JY \) is the identity. Let \( A \) be an endomorphism of degree 7, given by the matrix \( A \). Then \( \alpha, \beta \) and \( \delta \) are integers and \( \rho_\alpha(\alpha) = \alpha_{CS} = \text{diag}(\alpha, \alpha, \alpha) \) etc. and we have
\[
7 = \det A = \det \begin{pmatrix}
\text{diag}(\alpha, \alpha, \alpha) & \text{diag}(\beta, \beta, \beta) \\
\text{diag}(\beta, \beta, \beta) & \text{diag}(\delta, \delta, \delta)
\end{pmatrix} = (\alpha \delta - \beta^2)^3.
\]
Since this equation does not admit an integer solution, this give a contradiction. \( \square \)

Recall that \( R_{2,7} \) is an irreducible 3-dimensional variety and the Prym map is generically finite onto \( B_D \). So also \( B_D \) is of dimension 3. Since every element of \( B_D \) is isomorphic to \( JY \times JY \) for some curve \( Y \) of genus 3, and since the number of decompositions \( P = JY \times JY \) is at most countable, we get a 3-dimensional algebraic set, say \( V \), of curves \( Y \) such that \( JY \times JY \) admits a polarization of degree 7. In fact, as a consequence of the results in the paper \( V \) is even a variety.
Proposition 2.4. The Jacobians $J_Y$ of all curves $Y \in \mathcal{V}$ admit real multiplication in the totally real cubic number subfield $\mathbb{Q}(\zeta_7)^0$ of the cyclotomic field $\mathbb{Q}(\zeta_7)$.

Proof. Since by Proposition 2.3 $\text{End}_{\mathbb{Q}}(J_Y) \neq \mathbb{Q}$, the Jacobian admits either real, quaternion or complex multiplication (in the more general sense of [2, Section 9.6]). But an abelian variety with quaternion multiplication is even dimensional ([2, 9.4 and 9.5]). If $J_Y$ admits complex multiplication by a skew field of degree $d^2$ over a totally complex quadratic extension of a totally real number field of degree $e_0$, we would have (see [2, 9.6]) $3 = d^2 e_0 m$ for some integer $m \geq 1$. So $d = 1$ and $e_0 = 3$ (by Proposition 2.3). Then $J_Y$ admits complex multiplication by a number field. Since there are only countably many such abelian varieties, we conclude that $J_Y$ admits multiplication by a totally real number field $F$ of degree say $e$.

Then we have $3 = en$ for some positive integer $e$ [2, §9.2] and Proposition 2.3 implies $e = 3$. So $J_Y \times J_Y$ admits multiplication in $\text{SL}_2(F)$, with $F$ a totally real cubic number field. On the other hand $P$ admits a multiplication in the cyclotomic field $\mathbb{Q}(\zeta_7)$ and this is a subfield of $\text{SL}_2(F)$ only if $F$ is the totally real cubic subfield $\mathbb{Q}(\zeta_7)^0$ in $\mathbb{Q}(\zeta_7)$. Therefore $F = \mathbb{Q}(\zeta_7)^0$.

As a consequence of the description of the polarization in Proposition 2.1 we have

Proposition 2.5. Let $\mathcal{O}$ denote the maximal order of the totally real cubic field $\text{End}_{\mathbb{Q}}(J_Y)$ and $\varphi : J_Y \rightarrow J_Y$ the isogeny of Proposition 2.2. The following equation admits a solution in $\mathcal{O}$

$$7 = \det(4 \cdot 1 - \rho_\alpha(\varphi \hat{\varphi})).$$

Theorem 2.6. For a general covering $f$ the decomposition $P(f) \simeq J_Y \times J_Y$ is unique up to automorphisms.

Proof. We may assume that the the ring of endomorphisms of $J_Y$ is the maximal order $\mathcal{O}$ of the field $F$, since both families, the family of Prym varieties $P(f)$ and the family of Jacobians with multiplication in $F$ are irreducible of the same dimension 3. So

$$\text{End}(P(f)) \simeq M_2(\mathcal{O}),$$

the ring of matrices of degree 2 with entries in $\mathcal{O}$. Let $A$ be any direct factor of $P(f)$. So there is an abelian subvariety $B$ of $P(f)$ (necessarily isogenous to $A$) with

$$P(f) \simeq A \times B.$$

We have to show that there is an automorphism $\alpha$ of $P(f)$ with $\alpha(J_Y) = A$.

Recall that an element $\epsilon \in \text{End}(P(f))$ is idempotent if $\epsilon^2 = \epsilon$. Now the direct factors of $P(f)$ correspond bijectively to the non-trivial idempotents of the endomorphism ring of $P(f)$, i.e. of $M_2(\mathcal{O})$. Namely, if $A$ is a direct factor, then the composition $\epsilon$ of the maps

$$P(f) \simeq A \times B \xrightarrow{p_1} A \xrightarrow{i_1} A \times B \simeq P(f)$$

is an idempotent of $\text{End}(P(f))$. Here $p_1$ and $i_1$ are the natural projection and inclusion. Conversely, if $\epsilon$ is an idempotent, the factor $A$ is given by the kernel of the endomorphism $1 - \epsilon$. Moreover, if $\epsilon' = \alpha \epsilon \alpha^{-1}$ with an automorphism $\alpha$ of $P(f)$, then

$$\alpha(1 - \epsilon)\alpha^{-1} = 1 - \epsilon'.$$
Hence $\epsilon$ and $\epsilon'$ correspond to isomorphic direct factors. It suffices to show that all nontrivial idempotents (i.e. different from 0 and 1) are conjugate to each other, which is the content of the following lemma.

\textbf{Lemma 2.7.} Any 2 nontrivial idempotents of $M_2(\mathcal{O})$ are conjugate to each other.

\textbf{Proof.} Let $\epsilon \in M_2(\mathcal{O})$ be an idempotent. Its minimal polynomial $p(x)$ is different from $x$ and $x-1$, since $\epsilon$ is nontrivial.

Let $M := \mathcal{O}^2$ with $\epsilon$ acting by multiplication in the natural way. Let $N_0$ be its 0-eigenspace, i.e. the kernel of $\epsilon$ and $N_1$ its 1-eigenspace, i.e. the kernel of $\epsilon - 1$. We have

$$\epsilon M \subset N_1 \quad \text{and} \quad (\epsilon - 1)M \subset N_0,$$

since $\epsilon^2 - \epsilon$ annihilates $M$. Moreover,

\begin{align*}
N_0 \cap N_1 &= 0 \\
N_0 + N_1 &= M.
\end{align*}

The first equation is clear. For the second equation note that every $m \in M$ can be expressed as $m = \epsilon m - (\epsilon - 1)m$.

Neither $N_0$ nor $N_1$ can equal $M$, since otherwise we would have $p(x) = x$ or $x - 1$. Hence by (2.3) neither can be zero, so $N_0$ and $N_1$ are both $\mathcal{O}$-modules of rank 1. Since $\mathcal{O}$ is a principal ideal domain, there are $m_0, m_1 \subset M$ such that

$$N_0 = \mathcal{O}m_0 \quad \text{and} \quad N_1 = \mathcal{O}m_1.$$ 

It follows from (2.2) and (2.3) that $\{m_0, m_1\}$ is an $\mathcal{O}$-basis of $M$. With respect to this basis $\epsilon$ has the form

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

This implies the assertion. \hfill \Box

\textbf{Remark 2.8.} The proof works more generally for $M_2(\mathcal{O})$ with $\mathcal{O}$ any principal ideal domain.

3. The number of isomorphism classes of coverings $\mathcal{F}$

Let $f : \tilde{C} \to C$ be an étale cyclic covering of degree 7 of a smooth curve of genus 2. As we saw in Section 2, the lifting $j$ of the hyperelliptic involution $\iota$ of $C$ is not unique and in fact there are exactly 7 liftings $j_0, \ldots, j_6$ which are the 7 involutions of $D_7$. If $g_i : \tilde{C} \to Y_i := \tilde{C}/\langle j_i \rangle$ denotes the quotient map given by the involution $j_i := j\sigma^i$ (so $g_0 = g$) we have the following cartesian diagram.

\begin{align*}
\begin{array}{c}
\tilde{C} \\
| f \\
\downarrow g_i \\
C \\
| h \\
\downarrow g_i \\
\mathbb{P}^1 \\
| \mathcal{F}_i \end{array}
\end{align*}
In the previous section we saw that \( Y_i \) is of genus 3 and \( g_i \) is ramified exactly at one point over each Weierstrass point of \( C \). Since \( f \) is étale, the commutativity of (3.1) implies that the branch points of \( \overline{f}_i : Y_i \to \mathbb{P}^1 \) coincide with the branch points of \( h \) and each branch point is of ramification type \((2, 2, 2, 1)\), i.e. \( \overline{f}_i \) is simply ramified at 3 points and unramified at 1 point over the branch point. So the branch points of \( \overline{f}_i \) coincide for all \( i \). If \( p_1, \ldots, p_6 \in \mathbb{P}^1 \) are the branch points, then the permutations corresponding to the fibers \((\overline{f}_i)^{-1}(p_j)\) are pairwise conjugate within \( D_7 \) for all \( i = 0, \ldots, 6 \) and fixed \( j \) with the same conjugation for \( j = 1, \ldots, 6 \) and such that the non-ramified points of \( f_j \) over each \( p_i \) are pairwise different for \( j = 0, \ldots, 6 \).

Note that the coverings \( \overline{f}_i \) correspond to conjugate subgroups of \( D_7 \). Hence the \( \overline{f}_i \) are isomorphic coverings.

**Lemma 3.1.** For any general smooth curve \( C \) of genus 2 with double covering \( h : C \to \mathbb{P}^1 \), there is a canonical bijection between the sets of

(a) isomorphism classes of étale cyclic coverings \( f : \tilde{C} \to C \) of degree 7,

(b) isomorphism classes of degree-7 coverings \( \overline{f} : Y \to \mathbb{P}^1 \) of ramification type \((2, 2, 2, 1)\) over each of the 6 ramification points.

**Proof.** We saw above that any covering \( f \) in (a) gives 7 coverings in (b), which are all isomorphic to each other. Conversely, let \( \overline{f} : Y \to \mathbb{P}^1 \) be one of the coverings in (b). Since \( C \) is general, the monodromy group \( G \) of \( \overline{f} \) coincides with the Galois group of the Galois closure of \( Y \to \mathbb{P}^1 \). Define

\[
\tilde{C} := C \times_{\mathbb{P}^1} Y.
\]

According to the Lemma of Abhyankar [11, Lemma 2.14] the projection \( \tilde{C} \to C \) is étale. So \( \tilde{C} \) is smooth and it is easy to see that it is Galois over \( \mathbb{P}^1 \) with Galois group \( \simeq D_7 \). Hence \( \tilde{C} \) is the Galois closure of \( \overline{f} \). In particular \( \tilde{C} \to C \) is cyclic étale of degree 7. Clearly both constructions are inverse to each other and isomorphic coverings in (a) correspond to isomorphic coverings in (b). \( \square \)

Given a covering \( f : \tilde{C} \to C \) corresponding to a general element in \( R_{2,7} \), by Lemma 3.1 there is a corresponding degree 7 map \( Y \to \mathbb{P}^1 \) of ramification type \((2, 2, 2, 1)\) and, according to Proposition 2.1 the Prym variety \( P(f) \) decomposes as

\[
P(f) = JY \times JY.
\]

Now there are \( 7^4 - 1 = 2400 \) non-trivial 7-torsion points of \( C \). Since a cyclic étale cover \( \tilde{C} \to C \) of degree 7 is given by a cyclic subgroup of order 7 of \( JC[7] \), there are exactly

\[
\frac{2400}{6} = 400
\]

isomorphism classes of cyclic étale coverings \( \tilde{C} \to C \).

Let \( p_1, \ldots, p_6 \) be the branch points of the hyperelliptic covering \( C \to \mathbb{P}^1 \). Let \( N \) be the number of isomorphism classes of coverings \( \overline{f} : Y \to \mathbb{P}^1 \) ramified exactly over \( p_1, \ldots, p_6 \) of ramification type \((2, 2, 2, 1)\). If we denote by \( \beta : R_{2,7} \to M_2 \) the forgetful map onto the moduli space of smooth curves of genus 2 and the intersection \( Pr_{2,7}^{-1}(P(f)) \cap \beta^{-1}([C]) \) consists of \( d \) elements, then Lemma 3.1 implies that

\[
(3.2) \quad 400 = d \cdot N.
\]
Let \( p_1, \ldots, p_6 \) denote 6 points of \( \mathbb{P}^1 \) in general position. We want to compute the number \( N \) of isomorphism classes of degree 7 coverings ramified of ramification type \( (2, 2, 2, 1) \) exactly over the points \( p_1, \ldots, p_6 \).

Let \( \pi_1 \) denote the monodromy group of \( \tilde{f} \) with base point \( p \neq p_i \) for all \( i = 1, \ldots, 6 \). We consider \( \pi_1 \) as a subgroup of the symmetric group \( S_7 \) of degree 7 acting on the set \( \{1, \ldots, 7\} \). We need the following 2 trivial lemmas (which can of course be formulated for any odd prime \( p \) instead of 7).

**Lemma 3.2.** Let \( a \) and \( b \) be involution of \( S_7 \) such that

\[
s := ab
\]

is a cycle of order 7, then the group generated by \( a \) and \( b \) is isomorphic to \( D_7 \). Conversely, any subgroup of \( S_7 \) isomorphic to \( D_7 \) is of this type.

**Proof.** For the first assertion we have to show that \( asa = s^{-1} \). But

\[
as = aab = b = b^{-1} = b^{-1}a^{-1}a = s^{-1}a.
\]

The second assertion is obvious. \( \square \)

**Lemma 3.3.** Let \( a \) and \( b \) be involutions of \( S_7 \) such that \( \langle a, b \rangle \simeq D_7 \). Then \( a \) and \( b \) are products of 3 disjoint transpositions such that no transposition in \( a \) occurs in \( b \). Equivalently \( a \), respectively \( b \), fixes exactly one number \( i \), respectively \( j \), and \( i \neq j \). Each of the numbers \( 1, \ldots, 7 \) is fixed by exactly one of the 7 involutions.

**Proof.** It is easy to check that if \( a \) or \( b \) consist of less than 3 disjoint transpositions or if one transposition occurring in \( a \) occurs also in \( b \), then \( s = ab \) is not of order 7, which contradicts Lemma 3.2. For the last assertion note that if 2 involutions would fix the same number \( i \), then all involutions and hence all elements of the group would fix \( i \). \( \square \)

We give an example, although this is not necessary for the sequel, but makes the following proposition perhaps clearer.

**Example 3.4.** Consider the following involutions of \( S_7 \) (with right action, i.e. the product is from left to right)

\[
a = (12)(34)(56) \quad \text{and} \quad b = (23)(45)(67),
\]

with product

\[
s := ab = (1357642).
\]

According to Lemma 3.2 the group \( \langle a, b \rangle \) is isomorphic to \( D_7 \).

**Proposition 3.5.** All dihedral subgroups \( D_7 \) of \( S_7 \) are conjugate to each other.

**Proof.** Different labelings of the set on which \( S_7 \) acts give conjugate subgroups of \( S_7 \). We label this set in such a way that

\[
a = (12)(34)(56).
\]

The labelling is unique up to the transpositions \( (12), (34) \) and \( (56) \). Moreover, according to Lemma 3.3 we may choose \( b \) such that it fixes 1. The involution \( b \) cannot contain the
transposition (27), since otherwise $s = ab$ would contain the cycle (172). So we remain with the following possibilities $b_i$ for $b$:

$$b_1 = (23)(45)(67), \quad b_2 = (23)(46)(57), \quad b_3 = (23)(47)(56),$$

$$b_4 = (24)(35)(67), \quad b_5 = (24)(36)(57), \quad b_6 = (24)(37)(56),$$

$$b_7 = (25)(34)(67), \quad b_8 = (25)(36)(47), \quad b_9 = (25)(37)(46),$$

$$b_{10} = (26)(34)(57), \quad b_{11} = (26)(35)(47), \quad b_{12} = (26)(37)(45).$$

We compute

$$s_1 = ab_1 = (1357642), \quad s_2 = ab_2 = (1367542), \quad s_3 = ab_3 = (13742),$$

$$s_4 = ab_4 = (1457632), \quad s_5 = ab_5 = (1467532), \quad s_6 = ab_3 = (14732),$$

$$s_7 = ab_7 = (15762), \quad s_8 = ab_8 = (1537642), \quad s_9 = ab_9 = (1547362),$$

$$s_{10} = ab_{10} = (16752), \quad s_{11} = ab_{11} = (1637452), \quad s_{12} = ab_{12} = (1647352).$$

So exactly the groups $G_i := \langle a, b_i \rangle$ with $i = 1, 2, 4, 5, 8, 9, 11$ and 12 are isomorphic to $D_7$.

We may still conjugate the subgroups with the transpositions $(12)$, $(34)$ and $(56)$. One checks that

$$(12)s_1(12) = s_5^6 \in G_6, \quad (34)s_1(34) = s_4 \in G_4, \quad (56)s_1(56) = s_2 \in G_2,$$

$$(12)s_8(12) = s_6^2 \in G_{12}, \quad (34)s_8(34) = s_9 \in G_9, \quad (56)s_8(56) = s_{11} \in G_{11}.$$  

This implies that $G_1, G_2, G_4$ and $G_5$ as well as $G_8, G_9, G_{11}$ and $G_{12}$ are pairwise conjugate. Hence it suffices to show that $G_1$ is conjugate to $G_8$. But one easily checks that with the permutation $p := (2463)$ we have

$$pap^{-1} = a \quad \text{and} \quad pb_1p^{-1} = b_8.$$

which gives the assertion. \hfill \Box

**Proposition 3.6.** Given 6 points of $\mathbb{P}^1$ in general position. There are exactly 400 isomorphism classes of degree-7 coverings $\bar{f} : Y \to \mathbb{P}^1$ with monodromy group $D_7$ ramified of type $(2,2,2,1)$ over each of the 6 points.

**Proof.** Let $p_1, \ldots, p_6 \in \mathbb{P}^1$ be 6 points in general position. According to Proposition 3.5 all subgroups isomorphic to $D_7$ are conjugate. For example, we may take the monodromy subgroup of the covering to be

$$\pi_1 = \langle a, b_1 \rangle = \{1, s_1, \ldots, s_6, a, as_1, \ldots, as_6 \}.$$  

One has to compute the number of conjugacy classes of 6 involutions $c_i$ of type $(2,2,2,1)$ (associating $c_i$ to the point $p_i$) such that

$$c_1c_2c_3c_4c_5c_6 = 1.$$  

The elements of $D_7$ are either an involution (an odd permutation) or an element of order 7 (an even permutation). Hence the product of 5 involutions in $D_7$ gives an odd permutation, so it is an involution. Thus, for any involutions $c_1, \ldots, c_5$ of $\pi_1$, $c_6 := c_1 \cdots c_5$ is an involution satisfying (3.3). Since any 2 involutions of $\pi_1$ generate the group, only 7 of the corresponding coverings are not connected, namely those with $c_1 = \cdots = c_6$. This gives $7^5 - 7$ coverings. Two 6-tuples of involutions give isomorphic coverings if and only if they are conjugate.
In order to compute the set of conjugate classes of 6-tuples of involutions, recall that the only transitive subgroups of $S_7$ which are not contained in $A_7$ are subgroups isomorphic to

- the dihedral group $D_7$,
- the group $L_7 = AGL_1(\mathbb{F}_7)$ of affine transformations of the line with 7 points,
- $S_7$ itself.

(For the convenience of the reader we give a proof of this statement: Let $G$ be a proper subgroup of $S_7$ of this type. Clearly $G$ is a soluble group, since it is not contained in $A_7$. It is primitive, being a permutation group of degree 7. Hence, according to [12, 7.2.7], it is a subgroup of $L_7$. Since $D_7$ is the only transitive subgroup of $L_7$, this gives the statement.)

The group $D_7$ is not normal in $S_7$, but it is normal in $L_7$. Since on the other hand $L_7$ is isomorphic to the group of outer automorphisms of $D_7$, two 6-tuples of involutions give isomorphic coverings if and only if they are conjugate under an element of $L_7$. Since $L_7$ is of order 42, we finally get

$$\frac{7^5 - 7}{42} = 400$$

isomorphism classes of degree 7 coverings $Y \to \mathbb{P}^1$ of ramification type $(2,2,2,1)$.

Together with (3.2) we get

**Corollary 3.7.** Let $\overline{f} : Y \to \mathbb{P}^1$ be a degree 7 covering with monodromy group $D_7$ ramified of type $(2,2,2,1)$ over 6 general points. There is exactly one cyclic étale cover $\tilde{f} : \tilde{C} \to C$ such that the diagram (3.1) is commutative. In particular, $P(\overline{f}) \simeq JY \times JY$ as polarized abelian varieties of type $(1, \ldots, 1, 7)$.

4. **Proof of the main theorem**

Let $C_3$ denote the locus in $M_3$ of genus 3 curves admitting a degree 7 covering $\overline{f} : Y \to \mathbb{P}^1$ with ramification type $(2,2,2,1)$ over 6 general points. It is an irreducible variety of dimension 3 since it is an image of the moduli space of pairs $(Y, \overline{f})$, which according to Lemma 3.3 are in bijection with the elements in $R_{2,7}$.

**Theorem 4.1.** The elements of a generic fiber of the Prym map $Pr_{2,7} : R_{2,7} \to B_D$ are in bijection with the set of degree 7 coverings $\overline{f} : Y \to \mathbb{P}^1$ with ramification type $(2,2,2,1)$ over 6 general points that a curve $Y \in C_3$ admits.

**Proof.** By Corollary 3.7 the elements in the fiber of $P(\overline{f}) \in B_D$ are necessarily in different fibers of the forgetful map $\beta : R_{2,7} \to M_2$. Together with uniqueness of the decomposition $P(\overline{f}) \simeq JY \times JY$ as polarized abelian varieties (see Theorem 2.5) this implies that the covering $\tilde{f} : \tilde{C} \to C$ is completely determined by the pair $(Y, \overline{f})$, since the branch points of $\overline{f}$ determine the genus 2 curve $C$. So the elements in $R_{2,7}$ with Prym variety isomorphic to $JY \times JY$ are in bijection with the coverings $\overline{f} : Y \to \mathbb{P}^1$ with ramification type $(2,2,2,1)$ for a fixed curve $Y \in C_3$.

It has been shown in [8, Theorem 1.1] that the degree of the Prym map $Pr_{2,7}$ is 10, so we get as immediate corollary

**Corollary 4.2.** There are at most 10 coverings $\overline{f} : Y \to \mathbb{P}^1$ of degree 7 with ramification type $(2,2,2,1)$ for a curve $Y \in C_3$. 

THE FIBERS OF THE PRYM MAP

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H. Lange, Department Mathematik der Universität Erlangen, Germany
E-mail address: lange@mi.uni-erlangen.de

A. Ortega, Institut für Mathematik, Humboldt Universität zu Berlin, Germany
E-mail address: ortega@math.hu-berlin.de