STEADY 3D VISCOUS COMPRESSIBLE FLOWS WITH
ADIABATIC EXPONENT $\gamma \in (1, \infty)$

P.I. PLOTNIKOV AND W. WEIGANT

Abstract. The Navier-Stokes equations for compressible barotropic
flow in the stationary three dimensional case are considered. It is as-
sumed that a fluid occupies a bounded domain and satisfies the no-slip
boundary condition. The existence of a weak solution under the as-
sumption that the adiabatic exponent satisfies $\gamma > 1$ is proved. These
results cover the cases of monoatomic, diatomic, and polyatomic gases.

1. Introduction

We deal with a stationary boundary value problem for the compressible
Navier-Stokes equations. It is assumed that a compressible fluid occupies
a bounded domain $\Omega \subset \mathbb{R}^3$ with $C^2$ boundary. The state of the fluid is
characterized completely by the density $\rho(x) \geq 0$ and the velocity $u(x)$. The
governing equations represent two basic principles of fluid mechanics:
the mass balance
\[ \text{div} (\rho u) = 0 \quad \text{in} \quad \Omega, \quad (1.1a) \]
and the balance of momentum
\[ \text{div} (\rho u \otimes u) + \nabla p(\rho) = \text{div} S(u) + \rho f \quad \text{in} \quad \Omega. \quad (1.1b) \]
Here $f \in L^\infty(\Omega)$ is a given vector field, the viscous stress tensor $S$ and the
pressure $p$ are given by
\[ S(u) = \mu (\nabla u + \nabla u^\top - \frac{2}{3} \text{div} u I) + \nu \text{div} u I, \quad p = \rho^\gamma, \quad (1.1c) \]
the viscosity coefficients $\mu, \nu$ satisfy the inequalities $\mu + \frac{1}{3} \nu > 0$, $\mu > 0$. Throughout
the paper, we assume that $\gamma > 1$ is an arbitrary constant.

Notice that the standard values of the adiabatic exponent $\gamma$ are $5/3$ for the
monoatomic gas, between $9/11$ and $7/5$ for the diatomic gas, and between $1$
and $4/3$ for the polyatomic gas, see [8]. Equations (1.1a)–(1.1b) should be
supplemented with boundary conditions. We assume that the velocity field
satisfies the no-slip boundary condition on $\partial \Omega$, and that the total mass of
the fluid is prescribed:
\[ u = 0 \quad \text{on} \quad \partial \Omega, \quad (1.1d) \]
\[ \int_\Omega \rho \, dx = M. \quad (1.1e) \]

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Our goal is to prove that problem (1.1) has at least one weak solution. By a weak solution we mean a couple \((u, \rho) \in W^{1,2}_0(\Omega) \times L^\gamma(\Omega)\) such that the integral identities

\[
\int_\Omega \left( \rho u \otimes u : \nabla \xi + p(\rho) \text{div} \xi - S(u) : \nabla \xi + \rho f \cdot \xi \right) dx = 0,
\]

\[
\int_\Omega \rho u \cdot \nabla \zeta dx = 0
\]

hold for all vector fields \(\xi \in C^\infty_0(\Omega)\) and all \(\zeta \in C^\infty(\Omega)\).

The first nonlocal results concerning the mathematical theory of compressible Navier-Stokes equations are due to P.-L. Lions. In monograph [12] he proved the weak continuity of the viscous flux, which is the most important result in the mathematical theory of viscous compressible flows, and established the existence of a weak renormalized solution to problem (1.1) for all \(\gamma > 5/3\). More recently, Novotný & Straškraba, [13], employed the concept of oscillation defect measure developed in [3] to prove the existence result for all \(\gamma > 3/2\). Plotnikov & Sokolowski, [15], proved the existence of renormalized solutions to the Dirichlet boundary value problem for the compressible Navier-Stokes equations for all \(\gamma > 4/3\). However, they replaced mass conservation condition (1.1e) by a more restrictive integral condition. Finally, Frehse, Steinhauer, and Weigant, [5], proved the existence of weak renormalized solutions to problem (1.1) for all \(\gamma > 4/3\).

The better results were obtained for periodic structures and the Neumann boundary value problem. Jiang & Zhou, [10], [11], proved the existence of renormalized periodic solutions to the compressible Navier-Stokes equations assuming \(\gamma > 1\). In a paper [8], Jesslé & Novotný proved that for every \(\gamma > 1\), system (1.1b)- (1.1a) has a renormalized solutions satisfying the slip boundary conditions

\[
u \cdot n = 0, \quad (S(u)n) \times n = 0 \quad \text{on} \quad \partial \Omega.
\]

In the recent paper [9] by Jesslé, Novotný, and Pokorný these results were extended to the case of compressible heat conducting fluid.

At the present time, the existence of solutions to the classical no-slip problem (1.1) was proved under assumption \(\gamma > 4/3\). Our goal is to relax this restriction and to extend the existence theory to the range \(\gamma > 1\). We aim to prove the following

**Theorem 1.** Let \(\Omega\) be a bounded domain with \(C^2\) boundary and let \(\gamma > 1\). Then for every \(f \in L^\infty(\Omega)\) and \(M > 0\) problem (1.1) has a weak solution such that

\[
\|u\|_{W^{1,2}_0(\Omega)} + \|\rho\|_{L^\gamma(\Omega)} + \|\rho u\|_{L^s(\Omega)} \leq c,
\]

where the exponents \(q > 1\) and \(s > 1\) depend only on \(\gamma\), the constant \(c\) depends only on \(\Omega, f, \gamma, M, \mu, \) and \(\nu\).

Notice that the question of existence of solutions to problem (1.1) is closely related to the questions of boundedness and compactness of the set of solutions to compressible Navier-Stokes equations corresponding to various pressure functions. Indeed, we can approximate the pressure function \(p(\rho)\) with a fast growing artificial pressure \(p_*\). Then we obtain a sequence of solutions \((u_\epsilon, \rho_\epsilon)\) to the regularized problem with \(p\) replaced by \(p_\epsilon\). After
this we have to prove that the solutions \((u_\epsilon, \rho_\epsilon)\) have uniformly bounded energies. Hence, passing to a subsequence, if necessary, we can assume that \((u_\epsilon, \rho_\epsilon)\) converge weakly in the energy space to some limit functions \((u, \rho)\). Finally, we have to prove that the limit satisfies equations (1.1). To give a rigorous meaning to the above discussion, we take the approximation of the pressure function in the form

\[ p_\epsilon(\rho) = \rho^\gamma + \epsilon \rho^4, \tag{1.5} \]

and consider the regularized problem

\[
\begin{align*}
\text{div} (\rho u \otimes u) + \nabla p_\epsilon(\rho) &= \text{div} S(u) + \rho f \quad \text{in } \Omega, \\
\text{div} (\rho u) &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega, \\
\int_\Omega \rho \, dx &= M. 
\end{align*}
\tag{1.6}
\]

Without loss of generality we can assume that

\[ 1 < \gamma \leq 2. \tag{1.7} \]

It is known, see [12], that for every \(\epsilon \in (0, 1]\), problem (1.6) has at least one weak solution \((u_\epsilon, \rho_\epsilon) \in W^{1,2}_0(\Omega) \times L^8(\Omega)\) satisfying integral identities (1.2)-(1.3) with \(p\) replaced by \(p_\epsilon\).

The proof of Theorem 1 contains two contributions. The first states a priori estimates for solutions of the regularized equations. The second establishes compactness properties for these solutions. The question of compactness of solutions to regularized equations was investigated thoroughly in monographs [12, 13] and papers [8, 15]. It is known that if weak solutions \((u_\epsilon, \rho_\epsilon)\) to problem (1.6) satisfy the inequality

\[
\|u_\epsilon\|_{W^{1,2}_0(\Omega)} + \|p_\epsilon(\rho_\epsilon)\|_{L^\infty(\Omega)} + \|\rho_\epsilon|u_\epsilon|^2\|_{L^s(\Omega)} \leq c, \tag{1.8}
\]

then after passing to a subsequence we can assume that these solutions converge weakly in \(W^{1,2}_0(\Omega) \times L^\gamma(\Omega)\) to a weak solution to problem (1.1). Hence Theorem 1 will be proved if we prove the following

**Theorem 2.** Let \(\Omega\) be a bounded domain with \(C^2\) boundary, \(f \in L^\infty(\Omega)\), \(\gamma > 1\), and \(M > 0\). Furthermore assume that \((u_\epsilon, \rho_\epsilon) \in W^{1,2}_0(\Omega) \times L^8(\Omega), \epsilon \in (0, 1]\), are weak solutions to problem (1.6). Then there are exponents \(q > 1, s > 1,\) depending only on \(\gamma\), and a constant \(c,\) depending only on \(\gamma, f, M,\) \(\mu, \nu,\) such that these solutions satisfy inequality (1.8).

The rest of the paper is devoted to the proof of Theorem 2.

2. Notation and definitions. Auxiliary Propositions

**Notation.** For every \(\gamma > 1\) we denote by \(\theta, \beta, s,\) and \(q\) the quantities

\[
\theta = \frac{1}{8} (1 - \gamma^{-1}), \quad \beta = \frac{3(1 - 8\theta^2)}{2(3 - 8\theta^2)}, \quad s = 1 + 2\theta^2, \quad q = 1 + \frac{\beta(s - 1)}{\beta + (1 - \beta)s}. \tag{2.1}
\]

It is easily seen that

\[ 0 < \theta < 1/8, \quad 0 < \beta < 1/2, \quad 1 < s < 33/32. \tag{2.2} \]
Further the signed distance function $d(x)$ is given by
\[ d(x) = \text{dist} \left( x, \partial \Omega \right) \text{ for } x \in \Omega, \quad d(x) = -\text{dist} \left( x, \partial \Omega \right) \text{ for } x \in \mathbb{R}^3 \setminus \Omega. \] (2.3)

For every $c > 0$ denote by $A_c$ and $\Omega_c$ the annuluses
\[ A_c = \{ x \in \mathbb{R}^3 : \text{dist} \left( x, \partial \Omega \right) < c \}, \quad \Omega_c = A_c \cap \Omega. \] (2.4)

Since $\partial \Omega \in C^2(\Omega)$, there is $t > 0$, depending only on $\Omega$, such that
\[ d \in C^2(\bar{A}_{2t}), \quad |\nabla d(x)| = 1 \text{ in } \bar{A}_{2t}. \] (2.5)

See [7, chap. 14.6] for the proof.

**Definition 1.** Further, the notation $\varphi$ stands for the function $\varphi : \bar{A}_{2t} \cup \Omega \rightarrow \mathbb{R}$ with the properties:
1. $\varphi \in C^2(\bar{A}_{2t} \cup \Omega)$, $\varphi(x) = d(x)$ in $A_{2t}$,
2. there is $k > 0$ such that $\varphi > k$ in $\Omega \setminus \Omega_{2t}$.

Notice that $\varphi$ is positive in $\Omega$. The existence of such a function obviously follows from the Whitney extension theorem.

**Remark 1.** Recall that our goal is to obtain the a priori estimates of solutions to regularized equations (1.6). These estimates depend on the flow domain, the constitutive law, and the parameters in equations. Further, we denote by $c_e$ general constants depending only on $\Omega, \mu, \nu, \gamma, M$, and $\|f\|_{L^\infty(\Omega)}$.

**Auxiliary Lemmas.** In this section we prove two technical lemmas. The first constitutes the properties of solutions to the regularized problem.

**Lemma 3.** Let $\epsilon \in (0, 1]$. Then problem (1.6) has a weak renormalized solution $(u, \varrho) \in W^{1,2}_0(\Omega) \times L^8(\Omega)$ with the following properties. The integral identities
\[ \int_{\Omega} (\varrho u \otimes u : \nabla \xi + p_\epsilon(\varrho) \nabla \xi - S(u) : \nabla \xi + g f \cdot \xi) \, dx = 0, \] (2.6)
\[ \int_{\Omega} (\psi(\varrho) u \cdot \nabla \zeta + \zeta (\varrho \psi'(\varrho) - \psi(\varrho)) \text{div } u) \, dx = 0 \] (2.7)
hold for all $\xi \in W^{1,2}_0(\Omega), \zeta \in C^\infty(\Omega)$, and for all functions $\psi \in C^1[0, \infty)$ satisfying the condition
\[ |\psi(s)| + |s\psi'(s)| \leq c(1 + |s|^4), \quad s \in [0, \infty). \]

Moreover, we have
\[ \int_{\Omega} S(u) : \nabla u \, dx = \int_{\Omega} g f \cdot u \, dx. \] (2.8)

**Proof.** The existence of solutions satisfying (2.6) and (2.7) was proved in [12]. It remains to prove (2.8). Set
\[ \psi(\varrho) = \frac{1}{\gamma - 1} \varrho^\gamma + \frac{\epsilon}{3} \varrho^4. \]

Obviously we have $\varrho \psi'(\varrho) - \psi(\varrho) = p_\epsilon(\varrho)$. Substituting $\psi(\varrho)$ and $\zeta = 1$ into (2.7) we arrive at the identity
\[ \int_{\Omega} p_\epsilon(\varrho) \text{div } u \, dx = 0. \]
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It follows from this and identity (2.6) with $\xi = u$ that

$$\int_{\Omega} (\varrho u \otimes u : \nabla u - \mathbb{S}(u) : \nabla u + g f \cdot u) \, dx = 0. \quad (2.9)$$

Next we have

$$\int_{\Omega} \varrho u \otimes u : \nabla u \, dx = \frac{1}{2} \int_{\Omega} \varrho u \nabla (|u|^2) \, dx.$$ 

Since the embedding $W^{1,2}_0(\Omega) \hookrightarrow L^6(\Omega)$ is bounded, we have $|u|^2 \in W^{1,3/2}(\Omega)$ and $\varrho|u|^2 \in L^2(\Omega)$. Hence there is a sequence $\zeta_n \in C^1(\Omega)$ such that

$$\nabla \zeta_n \to \nabla (|u|^2) \text{ in } L^{3/2}(\Omega). \quad (2.10)$$

On the other hand, the Cauchy inequality implies

$$(\varrho|u|)^3 \leq \varrho^6 + |u|^6 \in L^1(\Omega).$$

From this we conclude that $\varrho u \in L^3(\Omega)$. Notice that $\varrho u$ satisfies the integral identity

$$\int_{\Omega} \varrho u \cdot \nabla \zeta_n \, dx = 0.$$ 

Letting $n \to \infty$ and using relation (2.10) we arrive at

$$\int_{\Omega} \varrho u \otimes u : \nabla u \, dx = \frac{1}{2} \int_{\Omega} \varrho u \nabla (|u|^2) \, dx = 0. \quad (2.11)$$

Inserting (2.11) into (2.9) we obtain the desired identity (2.8).

The second lemma is of general character. It is known, see [14, 15], that the boundedness of the Green potential of a Borel measure $\sigma$ implies the continuity of the embedding $W^{1,2}_0(\Omega) \hookrightarrow L^2(\Omega, d\sigma)$. This fact is a straightforward consequence of the Maz’ja-Adams embedding theorem, see [1]. We give an elementary proof of this result in a particular case.

**Lemma 4.** Let $\Omega \in \mathbb{R}^3$ be a bounded domain with $C^2$ boundary. Let $f \in L^2(\Omega)$ satisfy

$$f \geq 0, \quad \int_{\Omega} f(x)|x - x_0|^{-1} \, dx \leq E \text{ for all } x_0 \in \Omega.$$ 

Then there is $c > 0$, depending only on $\Omega$, such that

$$\int_{\Omega} |u|^2 f \, dx \leq cE \|u\|^2_{W^{1,2}_0(\Omega)} \text{ for all } u \in W^{1,2}_0(\Omega). \quad (2.12)$$

**Proof.** Let $h \in W^{2,2}(\Omega)$ be a solution to the boundary value problem

$$- \Delta h = f \text{ in } \Omega, \quad h = 0 \text{ on } \partial \Omega. \quad (2.13)$$

This solution has the representation

$$h(x_0) = \int_{\Omega} G(x, x_0) f(x) \, dx,$$

where the Green function $G(x, x_0)$ admits the estimate $G(x, x_0) \leq c|x - x_0|^{-1}$. We thus get

$$\|h\|_{L^\infty(\Omega)} \leq cE. \quad (2.14)$$

Set

$$K = \int_{\Omega} |u|^2 f \, dx.$$
We have
\[ K = \int_\Omega |u|^2 (-\Delta h) \, dx = 2 \int_\Omega (u \nabla u) \cdot \nabla h \, dx. \]

Applying the Cauchy inequality we obtain
\[ K \leq \|u\|_{W^{1,2}_0(\Omega)} \left( \int_\Omega |u|^2 |\nabla h|^2 \, dx \right)^{1/2}. \]  \tag{2.15}

Let us estimate the integral in the right hand side. Integration by parts gives
\[ \int_\Omega |u|^2 |\nabla h|^2 \, dx = -2 \int_\Omega (u \nabla u) \cdot \nabla h \, dx + \int_\Omega h (-\Delta h) |u|^2 \, dx \leq \|h\|_{L^\infty(\Omega)} \left( \int_\Omega |u||\nabla u||\nabla h| \, dx + K \right). \]  \tag{2.16}

In turn, the Cauchy inequality implies
\[ cE \int_\Omega |u||\nabla u||\nabla h| \, dx \leq \frac{1}{2} \int_\Omega |u|^2 |\nabla h|^2 \, dx + cE^2 \|u\|_{W^{1,2}_0(\Omega)}^2. \]

Inserting this inequality into (2.16) we arrive at
\[ \int_\Omega |u|^2 |\nabla h|^2 \, dx \leq E^2 \|u\|_{W^{1,2}_0(\Omega)}^2 + cEK. \]

Inserting this inequality into (2.15) we obtain
\[ K \leq c\|u\|_{W^{1,2}_0(\Omega)}^2 E + c\|u\|_{W^{1,2}_0(\Omega)} (EK)^{1/2}. \]

Noting that
\[ c\|u\|_{W^{1,2}_0(\Omega)} (EK)^{1/2} \leq \frac{1}{2} K + c\|u\|_{W^{1,2}_0(\Omega)}^2 E \]
we finally obtain
\[ K \leq c\|u\|_{W^{1,2}_0(\Omega)}^2 E. \]

\[ \square \]

3. Estimates near the boundary. Pressure estimates

The remarkable property of the compressible Navier-Stokes equations is that the normal component of the energy tensor \( 2^{-1} \rho u \otimes u + p_\epsilon (\rho) I \) is small by comparison to its tangent component in a neighborhood of \( \partial \Omega \), see [14, 17]. The following lemma is a refined version of this result.

**Lemma 5.** Let a solution \( (u, \rho) \in W^{1,2}_0(\Omega) \times L^8(\Omega) \) to problem (1.6) be defined by Lemma 3. Then
\[ \rho \epsilon (1 + \|u\|_{W^{1,2}_0(\Omega)} + \|p_\epsilon (\rho) \|_{L^1(\Omega)} + \|\epsilon \|_{L^1(\Omega)} + \|\epsilon u \|_{L^1(\Omega)} \leq \frac{1}{2} K + c\|u\|_{W^{1,2}_0(\Omega)}^2 E, \]  \tag{3.1}

where \( \beta \) is given by (2.1) and \( \phi \) is given by Definition 1.
Proof. Recall that $\beta \in (0,1/2)$. Introduce the vector field
\[ \xi(x) = \varphi^{1-\beta}(x) \nabla \varphi(x), \quad x \in \Omega. \] (3.2)

Obviously we have
\[ \nabla \xi = \varphi^{1-\beta} \nabla^2 \varphi + (1-\beta) \varphi^{-\beta} \nabla \varphi \otimes \nabla \varphi. \]
It follows from (2.2) that
\[ |\varphi^{1-\beta} \nabla^2 \varphi| \leq c, \quad |\varphi^{-\beta} \nabla \varphi \otimes \nabla \varphi| \leq c \varphi^{-\beta}, \]
and hence $\xi \in W_0^{1,r}(\Omega)$ for all $r \in [1,1/\beta)$. Substituting $\xi$ in identity (2.6) we obtain
\[ \int_\Omega (\rho u \otimes u : \nabla \xi + p_\epsilon(\rho) \text{div} \xi) \, dx = \int_\Omega (S(u) : \nabla \xi - \rho f \cdot \xi) \, dx \leq \]
\[ c \left( \int_\Omega |S(u)|^2 \, dx \right)^{1/2} + c \int_\Omega \rho \, dx \leq c(1 + \|u\|_{W_0^{1,2}(\Omega)}). \] (3.3)

On the other hand, we have
\[ p_\epsilon \text{div} \xi = (1-\beta)p_\epsilon \varphi^{-\beta} |\nabla \varphi|^2 + p_\epsilon \varphi^{1-\beta} \Delta \varphi \geq 2^{-1} \varphi^{-\beta} p_\epsilon - c \varphi^{1-\beta} p_\epsilon \]
and
\[ \rho u \otimes u : \nabla \xi = (1-\beta)\varphi^{-\beta} \rho (u \cdot \nabla \varphi)^2 + \varphi^{1-\beta} \rho u_i u_j \partial_i \partial_j \varphi \geq (1-\beta)\varphi^{-\beta} \rho (u \cdot \nabla \varphi)^2 - c \varphi^{-\beta} \rho |u|^2. \]
Inserting these inequalities into (3.3) we obtain (3.1).

The proof that the pressure function is integrable with some exponent greater than one is the essential part of the mathematical analysis of compressible viscous flows. The following lemma establishes this result for a weighted pressure function.

Lemma 6. Let a solution $(u, \rho) \in W_0^{1,2}(\Omega) \times L^8(\Omega)$ to problem (1.6) be given by Lemma 5 and $(\beta,s)$ be given by (2.1). Then
\[ \|p_\epsilon(\rho)\varphi^{1-\beta}\|_{L^s(\Omega)} \leq \]
\[ c_\epsilon \left( 1 + \|p_\epsilon(\rho)\varphi^{-\beta}\|_{L^1(\Omega)} + \|\rho\|_{L^1(\Omega)} \|u\|_{W_0^{1,2}(\Omega)} \right). \] (3.4)

Proof. Choose an arbitrary function $g \in L^{8/(s-1)}(\Omega)$. It follows from the Bogovskii lemma that the problem
\[ \text{div} \, \omega = g - \frac{1}{|\Omega|} \int_\Omega g \, dx \text{ in } \Omega, \quad \omega = 0 \text{ on } \partial \Omega, \]
has a solution $\omega \in W_0^{1,s/(s-1)}(\Omega)$ satisfying the inequality
\[ \|\omega\|_{W_1^{s/(s-1)}(\Omega)} \leq c \|g\|_{L^{8/(s-1)}(\Omega)}. \] (3.5)
Since $s/(s-1) > 3$, the embedding $W_0^{1,s/(s-1)}(\Omega) \hookrightarrow C(\bar{\Omega})$ is bounded. It follows from this and general properties of $W_0^{1,s/(s-1)}(\Omega)$ that
\[ \|\omega\|_{L^\infty(\Omega)} + \|\varphi^{-1} \omega\|_{L^{s/(s-1)}(\Omega)} \leq c \|g\|_{L^{s/(s-1)}(\Omega)}. \] (3.6)
Introduce the vector field
\[ \xi(x) = \varphi(x) \omega(x), \quad x \in \Omega. \] (3.7)
It is easily seen that
\[ \partial_j \xi_i = \varphi^{1-\beta} H_{ij}, \quad \text{where} \quad H_{ij} = \partial_j \omega_i + (1-\beta) \varphi^{-1} \partial_j \varphi \omega_i. \]
Inequalities (3.6) and (3.7) imply
\[ \|H_{ij}\|_{L^s/(s-1)(\Omega)} \leq c \|g\|_{L^s/(s-1)(\Omega)}. \tag{3.9} \]
In particular, we have
\[ \|\xi\|_{W^{1,s/(s-1)}(\Omega)} + \|\xi\|_{L^\infty(\Omega)} \leq c \|g\|_{L^s/(s-1)(\Omega)}. \tag{3.10} \]
Substituting \( \xi \) into (2.6) we arrive at
\[ \int_{\Omega} p_\gamma(g) \, \text{div} \, \xi \, dx = \int_{\Omega} (\mathcal{S}(u) : \nabla \varphi - \varphi f : \xi) \, dx - \int_{\Omega} \varphi^{1-\beta} g H_{ij} u_i u_j \, dx \]
\[ \leq c \|u\|_{W_0^{1,2}(\Omega)} \|\xi\|_{W_0^{1,2}(\Omega)} + c \|\xi\|_{L^\infty(\Omega)} + c \|\varphi^{1-\beta} g\|_{L^s(\Omega)} \|\xi\|_{W_0^{1,2}(\Omega)} \]
\[ \leq c (1 + \|u\|_{W_0^{1,2}(\Omega)} + \|\varphi^{1-\beta} g\|_{L^s(\Omega)}) \|g\|_{L^s/(s-1)(\Omega)}. \tag{3.11} \]
On the other hand, we have
\[ \int_{\Omega} p_\gamma(g) \, \text{div} \, \xi \, dx = \int_{\Omega} \varphi^{1-\beta} p_\gamma(g) g \, dx - \frac{1}{|\Omega|} \int_{\Omega} g \, dx \int_{\Omega} \varphi^{1-\beta} p_\gamma(g) \, dx + \]
\[ (1 - \beta) \int_{\Omega} \varphi^{-\beta} p_\gamma(g) \varphi \cdot \varphi \, dx. \]
Next, inequality (3.7) implies
\[ \left| \int_{\Omega} \varphi^{-\beta} p_\gamma(g) \varphi \cdot \varphi \, dx \right| \leq c \|g\|_{L^s/(s-1)(\Omega)} \int_{\Omega} \varphi^{-\beta} p_\gamma(g) \, dx, \]
which yields
\[ \int_{\Omega} p_\gamma(g) \, \text{div} \, \xi \, dx \geq \int_{\Omega} \varphi^{1-\beta} p_\gamma(g) g \, dx - \]
\[ c \left( \int_{\Omega} \varphi^{1-\beta} p_\gamma(g) \, dx + \int_{\Omega} \varphi^{-\beta} p_\gamma(g) \, dx \right) \|g\|_{L^s/(s-1)(\Omega)} \]
\[ \geq \int_{\Omega} \varphi^{1-\beta} p_\gamma(g) \, dx - c \left( \int_{\Omega} \varphi^{-\beta} p_\gamma(g) \, dx \right) \|g\|_{L^s/(s-1)(\Omega)}. \]
Combining this result with (3.11) we finally arrive at the inequality
\[ \int_{\Omega} \varphi^{1-\beta} p_\gamma(g) g \, dx \leq c (1 + \|u\|_{W_0^{1,2}(\Omega)} + \|\varphi^{1-\beta} g\|_{L^s(\Omega)} \|g\|_{L^s/(s-1)(\Omega)} + \]
\[ c \|p_\gamma(g)\|_{L^1(\Omega)} \|\xi\|_{W_0^1(\Omega)} \|g\|_{L^s/(s-1)(\Omega)}, \]
which yields (3.11).

4. Quantities \( A \) and \( B \)

Introduce the quantities
\[ A = \int_{\Omega} g |u|^{2(2-\theta)} \varphi^{2\beta} \, dx, \quad B = \int_{\Omega} g^\gamma \varphi^{-\beta} \, dx. \tag{4.12} \]
Lemma 7. Let \((u, \mathbf{u}) \in W^{1,2}_0(\Omega) \times L^8(\Omega)\) be a weak solution to problem (1.6), and let \((\theta, \beta, s)\) be given by relations (2.1). Then

\[
\|u\|_{W^{1,2}_0(\Omega)} \leq c e A^{1/(2-\theta)} B^{1/(2+\theta)(2\gamma-1)},
\]

\[
\|\phi\mathbf{u}\|^{2\beta - 1}_{L^\infty(\Omega)} \leq c e A^{1/(2-\theta)} B^{1/(2+\theta)(2\gamma-1)}.
\]

Proof. Integral identity \((2.8)\) implies

\[
\int_{\Omega} (\mu|\nabla u|^2 + (\nu + \frac{\mu}{3})|\text{div} \mathbf{u}|^2) \, dx = \int_{\Omega} \phi \mathbf{u} \cdot \mathbf{f} \, dx \leq c e \|\phi\mathbf{u}\|_{L^1(\Omega)}.
\]

Since \(\mu > 0\) and \(\mu/3 + \nu > 0\), we have

\[
\|u\|_{W^{1,2}_0(\Omega)} \leq c e \|\phi\mathbf{u}\|_{L^1(\Omega)}.
\]

Introduce the quantities

\[
\alpha_1 = \frac{1}{2(2 - \theta)}, \quad \alpha_2 = \frac{1}{(2 - \theta)(2\gamma - 1)},
\]

\[
\alpha_3 = \frac{\gamma - 1}{(2 - \theta)(2\gamma - 1)}, \quad \alpha_4 = \frac{4\gamma - 3 - 2\theta(2\gamma - 1)}{2(2 - \theta)(2\gamma - 1)}.
\]

It follows from \((2.1)\) that \(\alpha_i\) are positive and

\[
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1, \quad \alpha_1 + \gamma \alpha_2 + \alpha_4 = 1, \quad 2\beta \alpha_1 - \beta \alpha_2 - 2\beta \alpha_3 = 0.
\]

We thus get

\[
\|\phi\mathbf{u}\| = (\phi\mathbf{u})^{2(2 - \theta)} \varphi_2^b \alpha_1 (\varphi^\gamma \varphi^{-\beta} \alpha_2 (\varphi^{-2\beta}) \alpha_3 \phi \alpha_4.
\]

Applying the Hölder inequality and recalling \((1.12)\) and \((1.6c)\) we arrive at

\[
\|\phi\mathbf{u}\|_{L^1(\Omega)} \leq A^{\alpha_1} B^{\alpha_2} \|\varphi^{-\beta}\|_{L^\infty(\Omega)} M^{\alpha_4} \leq c e A^{1/(2-\theta)} B^{1/(2+\theta)(2\gamma-1)}.
\]

Inserting this inequalities into \((1.14)\) we obtain \((1.13a)\). In order to prove \((1.13b)\), notice that

\[
(\phi\mathbf{u})^2 \varphi^{1 - \beta} = (\phi\mathbf{u})^{2(2 - \theta)} \varphi_2^b \beta_1 (\varphi^\gamma \varphi^{-\beta} \beta_2 (\varphi^{-2\beta}) \beta_3 (\phi \beta_4 \varphi^{-\gamma}),
\]

where

\[
\beta_1 = \frac{s}{2 - \theta}, \quad \beta_2 = \frac{s\theta}{(2 - \theta)(2\gamma - 1)}, \quad \beta_3 = 1 - s\left(1 - \frac{\theta(\gamma - 1)}{(2 - \theta)(2\gamma - 1)}\right),
\]

\[
\beta_4 = s\left(1 - \frac{(2\gamma - 1) + \gamma \theta}{(2 - \theta)(2\gamma - 1)}\right), \quad \kappa = s(1 - \beta + \beta_2 + 2\beta_3 - 2\beta_1).
\]

It follows from \((2.1)\) that \(\beta_i\) are positive and

\[
\beta_1 + \beta_2 + \beta_3 + \beta_4 = 1.
\]

Moreover, we have

\[
2\beta_3 + \beta_2 - 2\beta_1 = 2 - 2s + \frac{2s\theta(\gamma - 1)}{(2 - \theta)(2\gamma - 1)} + \frac{s\theta}{(2 - \theta)(2\gamma - 1)} - \frac{2s(2\gamma - 1)}{(2 - \theta)(2\gamma - 1)} = 2 - 3s.
\]

It follows that

\[
\kappa = s + \beta(2 - 4s) = \frac{1}{3 - 8\theta^2} (2\theta^2 + 8\theta^4) > 0.
\]

In particular, we have

\[
(\phi\mathbf{u})^2 \varphi^{1 - \beta} \leq c e (\phi\mathbf{u})^{2(2 - \theta)} \varphi_2^b \beta_1 (\varphi^\gamma \varphi^{-\beta} \beta_2 (\varphi^{-2\beta}) \beta_3 (\phi) \beta_4.
\]
Applying the Hölder inequality we obtain
\[ \|p(u)\|_{L^s(\Omega)} \leq c_0 A^{1/4} B^{\theta/4} M^{1/4} \left( \int_\Omega \varphi^{-2\beta} \, dx \right)^{\beta/s} \leq c_0 A^{1/4} B^{\theta/4} M^{1/4}, \]
which yields \((4.13a)\).

The following lemma gives the estimates of the pressure function and the energy density in terms of the quantity \(A\).

**Lemma 8.** Let a solution \((u, p) \in W^{1,2}_0(\Omega) \times L^8(\Omega)\) to problem \((1.6)\) be given by Lemma 3. Furthermore assume that \(\theta, \beta, \) and \(s\) are given by \((2.1)\). Then
\[
\|p_\varepsilon(\varphi)\varphi^{-\beta}\|_{L^s(\Omega)} + \|\varphi^{-\beta} p_\varepsilon(\varphi)\|_{L^1(\Omega)} + \|\varphi^{-\beta} g(\nabla \varphi \cdot u)^2\|_{L^s(\Omega)} \leq c_0(1 + A^{(1+\theta)/2}),
\]
\[
\|\varphi^{-\beta} g(\nabla \varphi \cdot u)^2\|_{L^s(\Omega)} \leq c_0(1 + A^{(1-2\theta)/4}),
\]
\[
\|\varphi^{-\beta} g|u|^2\|_{L^s(\Omega)} \leq c_0(1 + A^{(1+\theta)/2}).
\]

**Proof.** Combining estimates \((3.1)\) and \((3.4)\) we obtain
\[
\|p_\varepsilon(\varphi)\varphi^{-\beta}\|_{L^s(\Omega)} + \|\varphi^{-\beta} p_\varepsilon(\varphi)\|_{L^1(\Omega)} + \|\varphi^{-\beta} g(\nabla \varphi \cdot u)^2\|_{L^s(\Omega)} \leq c_0(1 + \|p_\varepsilon(\varphi)\varphi^{-\beta}\|_{L^s(\Omega)} + \|\varphi^{-\beta} g(\nabla \varphi \cdot u)^2\|_{L^s(\Omega)} + \|u\|_{W^{1,2}_0(\Omega)}).
\]

Let us estimate \(\|p_\varepsilon(\varphi)\varphi^{-\beta}\|_{L^s(\Omega)}\). Recall that for every integrable function \(g\), for every \(1 \leq \sigma \leq \tau \leq \infty\), and for every \(v \in (0, 1)\), we have
\[
\|g\|_{L^r(\Omega)} \leq \|g\|_{L^r(\Omega)}^{(1-v)\tau/r} \|g\|_{L^r(\Omega)}^{(1-v)\sigma/r} \text{ where } r = u\sigma + (1 - v)\tau.
\]

Now set
\[ g = (p_\varepsilon(\varphi)\varphi^{-\beta})^{1/4}, \quad r = 4, \quad \sigma = 1, \quad \tau = 4s, \quad v = \frac{4s - 4}{4s - 1} \]
Obviously we have
\[
\|g\|_{L^s(\Omega)} = \|p_\varepsilon(\varphi)\varphi^{-\beta}\|_{L^{1/4}(\Omega)}^{1/4}, \quad \|g\|_{L^s(\Omega)} = \|p_\varepsilon(\varphi)\varphi^{-\beta}\|_{L^{1/4}(\Omega)}^{1/4},
\]
On the other hand, relations \((1.5), (1.6a), \) and \((1.7)\) imply
\[
\|g\|_{L^1(\Omega)} = \|p_\varepsilon^{1/4}\varphi^{-\beta/4}\|_{L^1(\Omega)} \leq c_0\|\varphi^{-\beta}\|_{L^1(\Omega)} \leq c_0.
\]
Inserting these estimates into \((4.20)\) we arrive at
\[
\|p_\varepsilon(\varphi)\varphi^{-\beta}\|_{L^s(\Omega)} \leq \|p_\varepsilon(\varphi)\varphi^{-\beta}\|_{L^{s(1-v)\tau/r}}^{(1-v)\tau/r},
\]
Substituting this inequality into \((4.19)\) and noting that \((1 - v)\tau/r < 1\) we get
\[
\|p_\varepsilon(\varphi)\varphi^{-\beta}\|_{L^s(\Omega)} + \|\varphi^{-\beta} p_\varepsilon(\varphi)\|_{L^1(\Omega)} + \|\varphi^{-\beta} g(\nabla \varphi \cdot u)^2\|_{L^s(\Omega)} \leq c_0(1 + \|g\|_{L^{s(1-v)\tau/r}}^{(1-v)\tau/r} + \|u\|_{W^{1,2}_0(\Omega)}).
\]
From this and \((4.13a)\) we obtain
\[
\|p_\varepsilon(\varphi)\varphi^{-\beta}\|_{L^s(\Omega)} + \|\varphi^{-\beta} p_\varepsilon(\varphi)\|_{L^1(\Omega)} + \|\varphi^{-\beta} g(\nabla \varphi \cdot u)^2\|_{L^s(\Omega)} \leq c_0(1 + A^{(1+\theta)/2} B^{\theta/4} M^{1/4}).
\]
Next, it follows from the Young inequality that for every $\delta > 0$, 
\[
A^{\frac{1}{\gamma-\delta}} B^{\frac{1}{1-\gamma+\delta}} \leq \delta B + c(\delta) A^{\kappa_1},
\]
where 
\[
\kappa_1 = \frac{1}{2} \frac{(2\gamma-1)}{2(2-\theta)(2\gamma-1)} = \frac{1}{2} \frac{1 + 8\theta}{2(2-\theta)(1+8\theta) - (1-8\theta)} < 1/2.
\]
Similarly, we have 
\[
A^{\frac{1}{\gamma-\delta}} B^{\frac{\theta}{1-\gamma+\delta}} \leq \delta B + c(\delta) A^{\kappa_2},
\]
where 
\[
\kappa_2 = \frac{2\gamma-1}{(2-\theta)(2\gamma-1)} = \frac{1}{2} \frac{\theta}{2+1+8\theta} < \frac{\theta}{2}.
\]
We thus get 
\[
A^{\frac{1}{\gamma-\delta}} B^{\frac{1}{(2-\theta)(2\gamma-1)}} \leq \delta B + c(\delta) A^{1/2}, \quad A^{\frac{1}{\gamma-\delta}} B^{\frac{\theta}{(2-\theta)(2\gamma-1)}} \leq \delta B + c(\delta) A^{1/2+\theta/2}.
\]
Inserting these inequalities into (4.21) we obtain 
\[
\|p_r(\varphi)^{-\beta}\|_{L^1(\Omega)} + \|\varphi^{-\beta} p_r(\varphi)\|_{L^1(\Omega)} + \|\varphi^{-\beta} \varphi (\nabla \varphi \cdot \mathbf{u})^2\|_{L^1(\Omega)} \leq c\delta B + c(\delta)(1 + A^{1/2+\theta/2}).
\]
Noting that 
\[
B = \|\varphi^{-\beta} \varphi\|_{L^1(\Omega)} \leq c\|\varphi^{-\beta} p_r(\varphi)\|_{L^1(\Omega)}, \quad (4.22)
\]
we arrive at 
\[
\|p_r(\varphi)^{-\beta}\|_{L^1(\Omega)} + (1 - c\delta)\|\varphi^{-\beta} p_r(\varphi)\|_{L^1(\Omega)} + \|\varphi^{-\beta} \varphi (\nabla \varphi \cdot \mathbf{u})^2\|_{L^1(\Omega)} \leq c(\delta)(1 + A^{1/2+\theta/2}).
\]
Choosing $\delta > 0$ sufficiently small we obtain the desired estimate (4.13).

Now our task is to derive estimates (4.17) and (4.18). We begin with the observations that inequalities (4.16) and (4.22) imply 
\[
B \leq c_e(1 + A^{1/2+\theta/2}), \quad (4.23)
\]
Inserting this estimate into (4.13a) we obtain 
\[
\|\mathbf{u}\|_{W^{1,2}_0(\Omega)} \leq c_e A^{\frac{1}{2-\theta}} (1 + A^{1/2+\theta/2}) A^{\frac{1}{2-\theta}(2\gamma-1)} \leq c_e(1 + A^{\kappa_3}), \quad (4.24)
\]
where 
\[
\kappa_3 = \frac{1}{4(2-\theta)} + \frac{1 + \theta}{4(2-\theta)(2\gamma-1)} = \frac{1}{4(2-\theta)} \frac{2\gamma + \theta}{2\gamma - 1} = \frac{1}{4(2-\theta)} \frac{2 + \theta}{1 + 8\theta}.
\]
Here we use the relations $2 - \gamma^{-1} = 1 + 8\theta$ and $\gamma^{-1} < 1$. It follows that 
\[
\kappa_3 = \frac{1}{4(2-\theta)} h(\theta), \quad \text{where} \quad h(\theta) = (2 - \theta)(1 - 2\theta) - \frac{2 + \theta}{1 + 8\theta}.
\]
It is easily seen that 
\[
(1 + 8\theta)h(\theta) = 2\theta(5 - 19\theta + 8\theta^2) > 0,
\]
since $\theta \in (0, 1/8)$. Hence $\kappa_3 \leq 1/4 - \theta/2$, which along with (4.24) yields (4.17). It remains to prove (4.18). Combining estimates (4.13b) and (4.23) we obtain 
\[
\|\varphi\|_{L^q(\Omega)} \leq c_e A^{\frac{1}{2-\theta}} B^{\frac{\theta}{(2-\theta)(2\gamma-1)}} \leq c_e(1 + A^{\kappa_4}), \quad (4.25)
\]
where
\[ \kappa_4 = \frac{1}{2 - \theta} + \frac{1 + \theta}{2} \frac{\theta}{(2 - \theta)(2\gamma - 1)}. \]

It is easy to check that
\[ \frac{1 + \theta}{2} - \kappa_4 = \frac{\theta(1 + \theta)}{2(2 - \theta)} \left( \frac{1 - \theta}{2\gamma - 1} - \frac{1}{2\gamma - 1} \right) = \frac{\theta(1 + \theta)}{2(2 - \theta)} \left( \frac{1 - \theta}{2\gamma - 1} - \frac{1 - 8\theta}{2\gamma - 1} \right) > 0. \]
Hence \( \kappa_4 \leq (1 + \theta)/2 \), which along with (4.25) yields (4.18).

The following lemma gives the weighted estimates for the energy density in terms of \( A \).

**Lemma 9.** Let a solution \( (u, \varrho) \in W^{1,2}_0(\Omega) \times L^8(\Omega) \) to problem (1.6) be given by Lemma 3. Then for every \( \alpha \in (0,1) \) and \( x_0 \in \Omega \), we have
\[ \int_{\Omega} \left( p_{\varrho}(\varrho) + \varrho|u|^2 \right)(x) \phi(x)^{3/2-\beta} |x - x_0|^{-\alpha} dx \leq c(1 + A(1+\theta)/2), \] (4.26)
where \( c \) depends only on \( c_e \) and \( \alpha \).

**Proof.** Fix an arbitrary \( \alpha \in (0,1) \) and \( x_0 \in \Omega \). Introduce the vector field
\[ \xi(x) = \phi(x)^{3/2-\beta} |x - x_0|^{-\alpha}(x - x_0). \] (4.27)

It is easily seen that
\[ \nabla \xi = \frac{\phi(x)^{3/2-\beta}}{|x - x_0|^{\alpha}} \left( I - \frac{\alpha}{|x - x_0|^2} (x - x_0) \otimes (x - x_0) + \frac{3 - 2\beta}{2\varrho} (x - x_0) \otimes \nabla \phi \right). \]

Recall that \( \beta \in (0,1/2) \). From this we conclude that
\[ |\nabla \xi(x)| \leq c|x - x_0|^{-\alpha}. \]
Hence the vector field \( \xi \) belongs to the class \( W^{1,r}_0(\Omega) \) for all \( r \in [1,3/\alpha) \). In particular, we have
\[ \|\xi\|_{W^{1,2}_0(\Omega)} \leq c, \quad \|\xi\|_{L^\infty(\Omega)} \leq c. \]

Substituting \( \xi \) into integral identity (2.6) leads to
\[ \int_{\Omega} \left( \varrho u \otimes u : \nabla \xi + p_{\varrho}(\varrho) \text{div} \xi \right) dx = \int_{\Omega} \left( S(u) : \nabla \xi + \varrho f \cdot \xi \right) dx \leq c\|u\|_{W^{1,2}_0(\Omega)} \|\xi\|_{W^{1,2}_0(\Omega)} + cM\|\xi\|_{L^\infty(\Omega)}, \]
which yields the estimate
\[ \int_{\Omega} \left( \varrho u \otimes u : \nabla \xi + p_{\varrho}(\varrho) \text{div} \xi \right) dx \leq c(1 + \|u\|_{W^{1,2}_0(\Omega)}). \] (4.28)

We have
\[ \text{div} \xi = (3 - \alpha)\phi^{3/2-\beta}/|x - x_0|^{\alpha} - (3 - 2\beta)\phi^{1/2-\beta}/2|x - x_0|^{\alpha} (x - x_0) \cdot \nabla \varphi \geq \frac{(3 - \alpha)\phi^{3/2-\beta}/|x - x_0|^{\alpha}}{c}, \] (4.29)
and

\[ \rho u \otimes u : \nabla \xi = \frac{\varphi^{3/2-\beta}}{|x-x_0|^\alpha} \rho \left( |u|^2 - \frac{\alpha}{|x-x_0|^2} (u \cdot (x-x_0))^2 \right) + \frac{\varphi^{(3-2\beta)/2}}{2|x-x_0|^\alpha} \left( (x-x_0) \cdot u \right) \left( \nabla \varphi \cdot u \right) \geq \frac{(1-\alpha)\varphi^{3/2-\beta}}{|x-x_0|^\alpha} \rho |u|^2 - c\varphi^{1/2-\beta} \rho |u| \left( |\nabla \varphi \cdot u| \right) \geq \frac{(1-\alpha)\varphi^{3/2-\beta}}{|x-x_0|^\alpha} \rho |u|^2 - c\varphi^{1-\beta} \rho |u|^2 - c\varphi^{-\beta} \rho \left( \nabla \varphi \cdot u \right)^2. \]

Inserting (4.29) and (4.30) into (4.28) we arrive at the inequality

\[
\int_{\Omega} \frac{\varphi^{3/2-\beta} (p_\epsilon + \rho |u|^2)}{|x-x_0|^\alpha} \, dx \leq c \left( 1 + \int_{\Omega} p_\epsilon \varphi^{-\beta} \, dx + \int_{\Omega} \varphi^{1-\beta} \rho |u|^2 \, dx + \int_{\Omega} \varphi^{-\beta} \rho \left( \nabla \varphi \cdot u \right)^2 \, dx + \|u\|_{W^{1,2}_0(\Omega)} \right).
\]

From this and (4.16)-(4.18) we finally obtain

\[
\int_{\Omega} \frac{\varphi^{3/2-\beta} (p_\epsilon + \rho |u|^2)}{|x-x_0|^\alpha} \, dx \leq c(1 + A^{(1+\theta)/2} + A^{(1-2\theta)/4}) \leq c(1 + A^{(1+\theta)/2}).
\]

5. Estimates for \( u \) and \( p_\epsilon \)

In this section we establish the a priori estimates for \( \|u\|_{W^{1,2}_0(\Omega)} \) and the pressure function. The result is given by the following

**Proposition 10.** Let a solution \( (u, \rho) \in W^{1,2}_0(\Omega) \times L^8(\Omega) \) to problem (1.6) be given by Lemma 3 and \( q \) be given by (2.1). Then

\[ \|u\|_{W^{1,2}_0(\Omega)} + \|p_\epsilon\|_{L^2(\Omega)} \leq c_\epsilon, \]

where the constant \( c_\epsilon \) is specified by Remark 7.

**Proof.** The proof is based on the following technical lemmas.

**Lemma 11.** Let a solution \( (u, \rho) \in W^{1,2}_0(\Omega) \times L^8(\Omega) \) to problem (1.6) be given by Lemma 3 and \( (\theta, \beta) \) be given by (2.1). Then for every \( x_0 \in \Omega \), we have

\[
\int_{\Omega} \frac{\varphi(x) |u|^{2(1-\theta)}(x) \varphi(x)^{2\beta}}{|x-x_0|^\alpha} \, dx \leq c_\epsilon(1 + A^{(1+\theta)/2}).
\]

**Proof.** Formulae (2.1) imply

\[ 2\beta = \frac{\theta}{\gamma} \left( \frac{3}{2} - \beta \right) + (1-\theta) \left( \frac{3}{2} - \beta \right). \]

Next we set

\[ \alpha = (1-16\theta)/(1-8\theta) \in (0,1). \]

We have

\[
\frac{\varphi^{3/2-\beta}}{|x-x_0|^\alpha} = \left( \frac{\varphi^{\gamma \frac{3}{2}-\beta}}{|x-x_0|^\alpha} \right)^{\theta/\gamma} \left( \frac{\varphi^{\gamma \frac{3}{2}-\beta}}{|x-x_0|^\alpha} \right)^{1-\theta} \left( \frac{1}{|x-x_0|^\gamma} \right)^{\theta-\theta/\gamma}.
\]
Applying the Young inequality and noting that \( \varphi^\gamma \leq c p(\varphi) \) we arrive at
\[
\frac{\varrho|u|^{2(1-\theta)}\varphi^{2\beta}}{|x-x_0|} \leq c p(\varphi)^{3/(2-\beta)}\varphi^{3/(2-\beta)} + \frac{\varrho|u|^2\varphi^{3/(2-\beta)}}{|x-x_0|^{\alpha}} + \frac{1}{|x-x_0|^2}
\]
Integrating both sides of this inequality over \( \Omega \) and using estimates (5.26), we obtain (5.2).

**Lemma 12.** Let a solution \( (u, \varrho) \in W_0^{1,2}(\Omega) \times L^8(\Omega) \) to problem (1.6) be given by Lemma 3. Then
\[
A \leq c_e. \tag{5.3}
\]

**Proof.** Estimate (5.2) and Lemma 4 imply
\[
A = \int_\Omega |u|^2(\varrho|u|^{2(1-\theta)}\varphi^{2\beta}) \, dx \leq c\|u\|^2_{L^2(\Omega)} \sup_{x_0 \in \Omega} \int_\Omega \varrho|u|^{2(1-\theta)}\varphi^{2\beta}|x-x_0|^{-1} \, dx \leq c_e(1 + A^{(1+\theta)/2})\|u\|^2_{L^2(\Omega)}
\]
From this, the inequalities \( 0 < \theta < 1/8 \), and (4.17) we get the inequality
\[
A \leq c_e(1 + A^{1-\theta})(1 + A^{(1+\theta)/2}) \leq c_e(1 + A^{1-\theta/2}),
\]
which obviously yields (5.3).

Let us turn to the proof of Proposition 10. Estimate (6.1) for \( \|u\|_{L^2(\Omega)} \) obviously follows from Lemmas 3 and 12. It remains to estimate \( \|p\|_{L^1(\Omega)} \). Recall formula (2.1) for \( q \). Next, the Young inequality implies
\[
p^s = \{ (p^s \varphi^{1-\beta})^s \}^{(1+\theta)/(1-\beta)} \leq (p^s \varphi^{1-\beta})^s + p^s \varphi^{-\beta}.
\]
Integrating both sides of this relation over \( \Omega \) and applying estimates (4.16) and (5.2), we obtain the desired estimate
\[
\int_\Omega p^s \, dx \leq \int_\Omega (p^s \varphi^{1-\beta})^s \, dx + \int_\Omega p^s \varphi^{-\beta} \, dx \leq c_e(1 + A^{(1+\theta)/2}) \leq c_e.
\]

6. **Kinetic energy estimate. Proof of Theorem 2**

In this section we establish a priori estimate of the kinetic energy density, and by doing so we complete the proof of Theorem 2.

**Proposition 13.** Let a solution \( (u, \varrho) \in W_0^{1,2}(\Omega) \times L^8(\Omega) \) be a weak renormalized solution to problem (1.6) and \( s \) be given by (2.1). Then
\[
\|\varrho|u|^2\|_{L^1(\Omega)} \leq c_e. \tag{6.1}
\]

We divide the proof into a sequence of lemmas. Let us consider the function \( \varphi \in C^2(\Omega) \) given by Definition 1. In view of this definition, there is \( t > 0 \) such that \( \varphi(x) \) equals the signed distance function in the annulus
\[
A_{2t} = \{ x \in \mathbb{R}^3 : \text{dist} (x, \partial \Omega) < 2t \}.
\]
Introduce the vector field \( n(x) = \nabla \varphi(x), \quad n \in C^1(\bar{A}_{2t}), \quad |n(x)| = 1 \).
Fix an arbitrary $\alpha \in (0, 1)$ and $x_0 \in A_t$. Define the vector field
\[
\xi(x) = \left\{ \frac{\varphi(x) - \varphi(x_0)}{\Delta_-(x, x_0)^\alpha} + \frac{\varphi(x) + \varphi(x_0)}{\Delta_+(x, x_0)^\alpha} \right\} n(x),
\]
where
\[
\Delta_{\pm}(x, x_0) = |\varphi(x) \pm \varphi(x_0)| + |x - x_0|^2.
\]
The following two lemmas, which proofs are given in the appendix, constitute the basic properties of $\xi$.

**Lemma 14.** There is a constant $c$, depending only on $\alpha$ and $\Omega$, such that for every $x, x_0 \in A_t$ and for every $u \in \mathbb{R}^3$,
\[
|\xi(x)| \leq c, \quad |\nabla \xi(x)| \leq c \left( \frac{1}{\Delta_-(x, x_0)^\alpha} + \frac{1}{\Delta_+(x, x_0)^\alpha} + 1 \right),
\]
(6.3a)
\[
\frac{\partial \xi_i}{\partial x_j}(x) u_i u_j \geq \frac{1 - \alpha}{2} \left( \frac{1}{\Delta_-(x, x_0)^\alpha} + \frac{1}{\Delta_+(x, x_0)^\alpha} \right) |u \cdot n(x)|^2 - c|u|^2,
\]
(6.3b)
\[
\text{div } \xi \geq \frac{1 - \alpha}{2} \left( \frac{1}{\Delta_-(x, x_0)^\alpha} + \frac{1}{\Delta_+(x, x_0)^\alpha} \right) - c.
\]
(6.3c)

**Proof.** The proof is in the Appendix.

**Lemma 15.** Let $\Omega_t = \Omega \cap A_t$. Then there is a constant $c$, depending only on $\alpha$ and $\Omega$ such that
\[
\|\nabla \xi\|_{L^2(\Omega_t)} \leq c \text{ for every } x_0 \in \Omega_t.
\]
(6.4)

**Proof.** The proof is in the Appendix.

The next lemma gives the weighted pressure estimate near $\partial \Omega$.

**Lemma 16.** Let a solution $(u, p) \in W^{1,2}_0(\Omega) \times L^8(\Omega)$ to problem (1.6) be given by Lemma 3. Let $\alpha \in (0, 1)$ and $x_0 \in \Omega$. Furthermore, assume that $\zeta \in C^\infty(\Omega)$ satisfies the conditions
\[
\zeta \geq 0 \text{ in } \Omega, \quad \zeta = 0 \text{ in } \Omega \setminus \Omega_t/2.
\]
(6.5)

Then
\[
\int_\Omega \frac{\zeta p_\epsilon(x)}{|x - x_0|^{\alpha}} \, dx \leq c(\|p_\epsilon\|_{L^1(\Omega)} + \|p\|_{L^2(\Omega)}^2 + \|u\|_{W^{1,2}_0(\Omega)} + 1),
\]
(6.6)
where $c$ depends on $c_\epsilon$, $\alpha$, and $\zeta$.

**Proof.** We first consider the case of $x_0 \in \Omega_t$. In this case, the vector field $\xi$ meets all requirements of Lemmas 14 and 15. It follows from Lemma 15 that $\zeta \xi \in W^{1,2}_0(\Omega)$. Integral identity (1.2) with $\xi$ replaced by $\zeta \xi$ implies
\[
\int_\Omega \left( \zeta g u \otimes u : \nabla \xi + \zeta p_\epsilon \text{div } \xi \right) \, dx =
\int_\Omega (S(u) : \nabla (\zeta \xi) - g f \cdot \zeta \xi) \, dx - \int_\Omega (\rho(\nabla \zeta \cdot u)(u \cdot \xi) + p_\epsilon \nabla \zeta \cdot \xi) \, dx.
\]
(6.7)
Notice that $|\xi| \leq 4(\text{diam } \Omega)^{1-\alpha}$. From this and Lemma 15 we conclude that $\|\xi\|_{W^{1,2}_0(\Omega)} \leq c$. It follows that
\[\int_\Omega (\Sigma(u) : \nabla(x, \xi) - \varphi \cdot \xi) \, dx \leq c(\|u\|_{W^{1,2}_0(\Omega)} + 1)\]
and
\[\int_\Omega \left( q(\nabla \cdot u)(u : \xi) + p(\varphi \nabla \cdot \xi) \right) \, dx \leq c\left( \|p(\varphi)\|_{L^1(\Omega)} + \|q\|_{L^1(\Omega)} \right).
\]
Combining these results with (6.11) we arrive at the estimate
\[\int_\Omega (\zeta q u \otimes u : \nabla \xi + \zeta p(\varphi) \text{div } \xi) \, dx \leq c\left( \|p(\varphi)\|_{L^1(\Omega)} + \|q\|_{L^1(\Omega)} \|u\|_{W^{1,2}_0(\Omega)} + 1 \right)\]
for all $x \in A_t$. Next, Lemma 14 implies that
\[q u \otimes u : \nabla \xi = q \frac{\partial \xi_i}{\partial x_j} u_i u_j \geq \frac{1 - \alpha}{\frac{1}{\Delta_-(x, x_0) \alpha} + \frac{1}{\Delta_+(x, x_0) \alpha}} \frac{|u \cdot n(x)|^2 - c\varphi |u|^2}{2}\]
in $A_t$. From this, (6.3c), and (6.8) we obtain
\[\int_\Omega \zeta p(\varphi) \left( \frac{1}{\Delta_-(x, x_0) \alpha} + \frac{1}{\Delta_+(x, x_0) \alpha} \right) \, dx \leq c\left( \|p(\varphi)\|_{L^1(\Omega)} + \|q\|_{L^1(\Omega)} \|u\|_{W^{1,2}_0(\Omega)} + 1 \right)\]
for all $x \in A_t$. Next, the inequality $|\varphi(x) - \varphi(x_0)| \leq |x - x_0|$ yields $\Delta_-(x, x_0) \leq c|x - x_0|$, where $c > 0$ depends only on $\Omega$. Combining this result with (6.9) we arrive at the estimate
\[\int_\Omega \zeta p(\varphi) \varphi |x - x_0|^{-\alpha} \, dx \leq c\left( \|p(\varphi)\|_{L^1(\Omega)} + \|q\|_{L^1(\Omega)} \|u\|_{W^{1,2}_0(\Omega)} \right)\]
for all $x \in A_t$. Let us consider the case of $x_0 \in \Omega \setminus A_t$. Since $\zeta$ vanishes in $\Omega \setminus A_t$, the inequality $2|x - x_0| \geq t$ holds for all $x \in \zeta$ and $x_0 \in \Omega \setminus A_t$, and hence
\[\int_\Omega \zeta p(\varphi) |x - x_0|^{-\alpha} \, dx \leq c\|p(\varphi)\|_{L^1(\Omega)}\]
for all $x \in \Omega \setminus A_t$. Combining this inequality with (6.10) we obtain the desired estimate (6.6).

**Lemma 17.** Let a solution $(u, \varphi) \in W^{1,2}_0(\Omega) \times L^8(\Omega)$ to problem (1.6) be given by Lemma 3. Then for every nonnegative function $\eta \in C_0^\infty(\Omega) and every $x_0 \in \Omega$,
\[\int_\Omega \frac{\eta p(\varphi)(x)}{|x - x_0|} \, dx \leq c\left( \|p(\varphi)\|_{L^1(\Omega)} + \|\eta\|_{L^1(\Omega)} + \|u\|_{W^{1,2}_0(\Omega)} + 1 \right),\]
where $c$ depends only on $\eta$ and $c_\epsilon$.

**Proof.** Fix an arbitrary $x_0 \in \Omega$ and introduce the vector field
\[\xi_{int}(x) = |x - x_0|^{-1}(x - x_0).
\]
Obviously $|\nabla \xi_{int}(x)| \leq c|x - x_0|^{-1}$. Hence $\eta \xi_{int} \in W^{1,2}_0(\Omega)$ and
\[\|\eta \xi_{int}\|_{W^{1,2}_0(\Omega)} \leq c.\]
Integral identity (1.2) with $\xi$ replaced by $\eta \xi_{\text{int}}$ implies

$$
\int_{\Omega} \left( \eta \mathbf{u} \otimes \mathbf{u} : \nabla \xi_{\text{int}} + \eta p_\epsilon (\phi) \text{div} \xi_{\text{int}} \right) dx =
$$

$$
\int_{\Omega} \left( \mathbb{S}(\mathbf{u}) : \nabla (\eta \xi_{\text{int}}) - \phi \mathbf{f} \cdot \eta \xi_{\text{int}} \right) dx - \int_{\Omega} \left( \phi (\nabla \eta \cdot \mathbf{u}) (\mathbf{u} \cdot \xi_{\text{int}}) + p_\epsilon (\phi) \nabla \eta \cdot \xi_{\text{int}} \right) dx
$$

From this, (6.12) and the obvious equality $|\xi_{\text{int}}| = 1$ we obtain

$$
\int_{\Omega} \left( \eta \mathbf{u} \otimes \mathbf{u} : \nabla \xi_{\text{int}} + \eta p_\epsilon (\phi) \text{div} \xi_{\text{int}} \right) dx \leq c(\|p_\epsilon (\phi)\|_{L^1(\Omega)} + \|\phi \|^2_{L^1(\Omega)} + \|\mathbf{u}\|_{W^{1,2}_0(\Omega)} + 1)
$$

Straightforward calculations give

$$
\mathbf{u} \otimes \mathbf{u} : \nabla \xi_{\text{int}} = |x - x_0|^{-1}(1 - \xi_{\text{int}} \otimes \xi_{\text{int}}) \mathbf{u} \cdot \mathbf{u} \geq 0,
$$

and

$$\text{div} \xi_{\text{int}} = 2|x - x_0|^{-1}.$$

Combining these results with (6.13) we obtain (6.11).

**Lemma 18.** Let a solution $(\mathbf{u}, \phi) \in W^{1,2}_0(\Omega) \times L^8(\Omega)$ to problem (1.6) be given by Lemma 3. Let $\alpha \in (0, 1)$. Then for every $x_0 \in \Omega$,

$$
\int_{\Omega} \frac{p_\epsilon (\phi)^\alpha}{|x - x_0|^\alpha} dx \leq c(\|\phi \|^2_{L^1(\Omega)} + 1),
$$

(6.14)

where $c$ depends only on $c_\epsilon$ and $\alpha$.

**Proof.** Choose a nonnegative function $\zeta \in C^\infty(\Omega)$ such that $\zeta$ equals 1 in a neighborhood of $\partial \Omega$ and $\zeta$ vanishes in $\Omega \setminus \Omega_{1/2}$. In particular, we have $1 - \zeta \in C_0^\infty(\Omega)$. Applying Lemmas 17 and 18 we obtain

$$
\int_{\Omega} \frac{p_\epsilon (\phi) \zeta p_\epsilon (\phi)}{|x - x_0|^\alpha} dx + c \int_{\Omega} \frac{(1 - \zeta)p_\epsilon (\phi)}{|x - x_0|^\alpha} dx \leq c(\|p_\epsilon (\phi)\|_{L^1(\Omega)} + \|\phi \|^2_{L^1(\Omega)} + \|\mathbf{u}\|_{W^{1,2}_0(\Omega)} + 1).
$$

(6.15)

On the other hand, the Young inequality implies

$$
\frac{p_\epsilon (\phi)^\alpha}{|x - x_0|} = \left( \frac{p_\epsilon (\phi)}{|x - x_0|} \right)^\alpha \left( \frac{1}{|x - x_0|^{1+\alpha}} \right)^{1-\alpha} \leq \frac{cp_\epsilon (\phi)}{|x - x_0|^\alpha} + \frac{c}{|x - x_0|^{1+\alpha}}.
$$

Integrating both sides over $\Omega$ and noting that $1 + \alpha \leq 2$ we obtain

$$
\int_{\Omega} \frac{p_\epsilon (\phi)^\alpha}{|x - x_0|} dx \leq c \int_{\Omega} \frac{p_\epsilon (\phi)}{|x - x_0|^\alpha} dx + c.
$$

Combining this result with (6.15) we arrive at

$$
\int_{\Omega} \frac{p_\epsilon (\phi)^\alpha}{|x - x_0|} dx \leq c(\|p_\epsilon (\phi)\|_{L^1(\Omega)} + \|\phi \|^2_{L^1(\Omega)} + \|\mathbf{u}\|_{W^{1,2}_0(\Omega)} + 1).
$$

It remains to note that

$$
\|p_\epsilon (\phi)\|_{L^1(\Omega)} + \|\mathbf{u}\|_{W^{1,2}_0(\Omega)} \leq c_\epsilon,
$$

and the lemma follows. □
We are now in a position to complete the proof of Proposition 13. In view of (2.1) we have

\[ 2\gamma^{-1}s - (3 - s) = 2(1 - 8\theta)(1 + 2\theta^2) - 2 + 2\theta^2 = 2\theta(3\theta + 16\theta^2 - 8) < 0 \]

since \( \theta < 1/8 \). Set \( \alpha := 2\gamma^{-1}s/(3 - s) \) \( \in (0, 1) \). It follows from Lemma 4, inequality (6.14), and estimate (5.1) that

\[ \int_\Omega p_\epsilon(\theta)^{\alpha} |u|^2 dx \leq c(\|p|u|^2\|_{L^1(\Omega)} + 1) \|u\|^2_{W^{1,2}_0(\Omega)} \leq c(\|p|u|^2\|_{L^1(\Omega)} + 1). \]

On the other hand, we have

\[ p_\epsilon^{2s/(3-s)} = \|u|^{2s} = \left( p_\epsilon^{2s/(3-s)} \right)^{(3-s)/2} \left( |u|^6 \right)^{(s-1)/2}. \]

Applying the Hölder inequality and using (6.16) we obtain

\[ \int_\Omega p_\epsilon^{s} |u|^{2s} dx \leq \left( \int_\Omega p_\epsilon^{2s/(3-s)} |u|^2 dx \right)^{(3-s)/2} \left( \int_\Omega |u|^6 dx \right)^{(s-1)/2}. \]

It follows from the embedding theorem that

\[ \|u\|_{L^6(\Omega)} \leq c\|u\|_{W^{1,2}_0(\Omega)} \leq c, \]

which yields

\[ \int_\Omega p_\epsilon^{s} |u|^{2s} dx \leq c \left( \int_\Omega p_\epsilon^{2s/(3-s)} |u|^2 dx \right)^{(3-s)/2}. \]

Inserting (6.16) into (6.17) we arrive at the inequality

\[ \int_\Omega p_\epsilon^{s} |u|^{2s} dx \leq \left( \|p|u|^2\|_{L^1(\Omega)} + 1 \right)^{(3-s)/2}. \]

Notice that

\[ \|p|u|^2\|_{L^1(\Omega)} + 1 \leq c \left( \int_\Omega p_\epsilon^{s} |u|^{2s} dx \right)^{1/s} + 1 \leq c \left( \int_\Omega p_\epsilon^{s} |u|^{2s} dx + 1 \right)^{1/s}. \]

Substituting this inequality into (6.18) we finally obtain

\[ \int_\Omega p_\epsilon^{s} |u|^{2s} dx \leq c \left( \int_\Omega p_\epsilon^{2s} dx + 1 \right)^{(3-s)/(2s)}. \]

It remains to note that \( 3 - s < 2s \) and the proposition follows.

**Proof of Theorem 2.** It suffices to note that estimate (1.8) is a straightforward consequence of Propositions 10 and 13.
Appendix A. Proof of Lemmas 14 and 15

Proof of Lemma 14. The first estimate in (6.3a) is obvious. In order to prove the second estimate notice that

\[
\frac{\partial \xi_i}{\partial x_j}(x) = P(x, x_0)n_i(x)n_j(x) + Q_j(x, x_0)n_i(x) + R(x, x_0)\frac{\partial n_i}{\partial x_j}(x),
\]

where

\[
P(x, x_0) = \frac{(1 - \alpha)|\varphi(x) - \varphi(x_0)| + |x - x_0|^2}{\Delta_-(x, x_0)^{1+\alpha}} + \frac{(1 - \alpha)|\varphi(x) + \varphi(x_0)| + |x - x_0|^2}{\Delta_+(x, x_0)^{1+\alpha}},
\]

\[
Q_j(x, x_0) = -2\alpha\left(\frac{\varphi(x) - \varphi(x_0)}{\Delta_-(x, x_0)^{1+\alpha}} + \frac{\varphi(x) + \varphi(x_0)}{\Delta_+(x, x_0)^{1+\alpha}}\right)(x - x_0)_j,
\]

\[
R(x, x_0) = \frac{\varphi(x) - \varphi(x_0)}{\Delta_-(x, x_0)^{\alpha}} + \frac{\varphi(x) + \varphi(x_0)}{\Delta_+(x, x_0)^{\alpha}}.
\]

It follows that

\[
(1 - \alpha)\left(\frac{1}{\Delta_-} + \frac{1}{\Delta_+}\right) \leq P(x, x_0) \leq \left(\frac{1}{\Delta_-} + \frac{1}{\Delta_+}\right).
\]

Next we have

\[
|\varphi(x) \pm \varphi(x_0)|/(x - x_0)_j \leq c|\varphi(x) \pm \varphi(x_0)|^{1/2}|x - x_0| \leq c|\varphi(x) \pm \varphi(x_0)| + c|x - x_0|^2 \leq c\Delta_\pm(x, x_0).
\]

From this and (7.2b) we obtain

\[
|Q_j| \leq c\left(\frac{1}{\Delta_-} + \frac{1}{\Delta_+}\right).
\]

Inserting this inequality and inequalities (7.3), (7.4) into (7.1) we arrive at (6.3a). Next, (7.1) and (7.2) imply

\[
\frac{\partial \xi_i}{\partial x_j}u_iu_j = P(u \cdot n)^2 - (N_- + N_+)((x - x_0) \cdot u)(n \cdot u) + R\frac{\partial n_i}{\partial x_j}u_iu_j,
\]

where

\[
N_\pm(x, x_0) = 2\alpha \left(\varphi(x) \pm \varphi(x_0)\right)\Delta_\pm(x, x_0)^{-1-\alpha}.
\]

It follows from this and (7.3), (7.4) that

\[
\frac{\partial \xi_i}{\partial x_j}u_iu_j \geq (1 - \alpha)\left(\frac{1}{\Delta_-} + \frac{1}{\Delta_+}\right)(n \cdot u)^2 -
\]

\[
(|N_-| + |N_+|)|x - x_0||u||n \cdot u| - c|u|^2.
\]
It follows from the expression for \( N_\pm \) and the Cauchy inequality that
\[
\left| \varphi(x) - \varphi(x_0) \right| |x - x_0| |n \cdot u| \leq c \left( \frac{|\varphi(x) - \varphi(x_0)| |x - x_0|}{\Delta_{-1+\alpha}^{-\alpha}} + \frac{|\varphi(x) + \varphi(x_0)| |x - x_0|}{\Delta_{+1+\alpha}^{1+\alpha}} \right) |n \cdot u| \leq \frac{1 - \alpha}{2} \left( \frac{|x - x_0|^2}{\Delta_{-1+\alpha}^{-\alpha}} + \frac{|x - x_0|^2}{\Delta_{+1+\alpha}^{1+\alpha}} \right) (n \cdot u)^2 + c \frac{|\varphi(x) - \varphi(x_0)|^2}{\Delta_{-1+\alpha}^{-\alpha}} + \frac{|\varphi(x) + \varphi(x_0)|^2}{\Delta_{+1+\alpha}^{1+\alpha}} |u|^2.
\]

Notice that
\[
\frac{|x - x_0|^2}{\Delta_{-1}(x, x_0)} \leq 1, \quad \frac{|x - x_0|^2}{\Delta_{+1}(x, x_0)} \leq 1
\]
and
\[
\frac{|\varphi(x) - \varphi(x_0)|^2}{\Delta_{-1}(x, x_0)^{1+\alpha}} + \frac{|\varphi(x) + \varphi(x_0)|^2}{\Delta_{+1}(x, x_0)^{1+\alpha}} \leq c.
\]

We thus get
\[
(|N_-| + |N_+|)|x - x_0||u||n \cdot u| \leq \frac{1 - \alpha}{2} \left( \frac{1}{\Delta_{-1}^{-\alpha}} + \frac{1}{\Delta_{+1}^{1+\alpha}} \right) (n \cdot u)^2 + c |u|^2.
\]

Inserting this inequality into (7.6) we arrive at the desired inequality (6.3b). It remains to prove (6.3c). To this end set \( u = e_k \), where \( e_k \) is a vector of the canonical basis in \( \mathbb{R}^3 \). Substituting \( u \) into (6.3b) we obtain
\[
\frac{\partial \xi_k}{\partial x_k}(x) \geq \frac{1 - \alpha}{2} \left( \frac{1}{\Delta_{-1}^{-\alpha}} + \frac{1}{\Delta_{+1}^{1+\alpha}} \right) n_k^2 - c |u|^2.
\]

Summing both sides over \( k \) and noting that \( |n| = 1 \) we obtain (6.3c). This completes the proof.

**Proof of Lemma 15.** We begin with the observation that \( \Delta_{-1}(x, x_0) \leq \Delta_{+1}(x, x_0) \) for all \( x, x_0 \in \Omega_t \). From this and (6.3a) we conclude that
\[
|\nabla \xi(x)|^2 \leq c \left( \frac{1}{\Delta_{-1}(x, x_0)^{2\alpha}} + 1 \right).
\]

Recall that \( \Omega_t \subset A_t \). Hence it suffices to prove that
\[
\int_{A_t} \Delta_{-1}(x, x_0)^{-2\alpha} \, dx \leq c \text{ for all } x_0 \in A_t. \tag{7.7}
\]

To this end fix an arbitrary \( x_0 \in A_t \) and denote by \( B_t \) the ball \( \{|x - x_0| < t\} \). We have
\[
\int_{A_t} \Delta_{-1}(x, x_0)^{-2\alpha} \, dx \leq \int_{B_t} \Delta_{-1}(x, x_0)^{-2\alpha} \, dx + \int_{A_t \setminus B_t} \Delta_{-1}(x, x_0)^{-2\alpha} \, dx. \tag{7.8}
\]

It is easily seen that \( \Delta_{-1}(x, x_0) \geq t^2 \) for all \( x \in A_t \setminus B_t \), which leads to the estimate
\[
\int_{A_t \setminus B_t} \Delta_{-1}(x, x_0)^{-2\alpha} \, dx \leq ct^{-4\alpha} \text{ meas } A_t \leq c. \tag{7.9}
\]

Recall that \( B_t \subset A_{2t} \) and that the function \( \varphi \) belongs to the class \( C^2(A_{2t}) \). From this and the Taylor formula we obtain
\[
\varphi(x) - \varphi(x_0) = n_0 \cdot (x - x_0) + D(x, x_0) \text{ for } x \in B_t,
\]
where $n_0 = n(x_0) = \nabla \varphi(x_0)$ and the remainder admits the estimate
\[ |D(x, x_0)| \leq m|x - x_0|^2, \text{ where } m = \sup_{x \in A_{2t}} |\nabla^2 \varphi(x)|. \]

It follows that
\[ (m + 1)\Delta_n(x, x_0) \geq |\varphi(x) - \varphi(x_0)| + (m + 1)|x - x_0|^2 \geq |n_0 \cdot (x - x_0)| - |D(x, x_0)| + (m + 1)|x - x_0|^2 \geq |n_0 \cdot (x - x_0)| + |x - x_0|^2. \]

Introduce the orthogonal projection $P_0 = I - n_0 \otimes n_0$. The trivial relation $|P_0(x - x_0)| \leq |x - x_0|$ leads to the inequality
\[ \Delta_n(x, x_0) \geq (m + 1)^{-1} (|n_0 \cdot (x - x_0)| + |P_0(x - x_0)|^2). \]

We thus get
\[
\int_{B_t} \Delta_n(x, x_0)^{-2\alpha} dx \leq c \int_{B_t} \left( |n_0 \cdot (x - x_0)| + |P_0(x - x_0)|^2 \right)^{-2\alpha} dx. \tag{7.10}
\]

Now choose an orthogonal basis $(b_i)$, $i = 1, 2, 3$, in $\mathbb{R}^3$ such that $b_3 = n_0$.

We have
\[ x - x_0 = \sum_i y_i b_i, \quad y_3 = n_0 \cdot (x - x_0), \quad |P_0(x - x_0)|^2 = y_1^2 + y_2^2, \]

which yields
\[
\int_{B_t} \left( |n_0 \cdot (x - x_0)| + |P_0(x - x_0)|^2 \right)^{-2\alpha} dx = \int_{|y_3| \leq t} \left( |y_3| + y_1^2 + y_2^2 \right)^{-2\alpha} dy \leq \int_{-t}^{t} \int_{y_1^2 + y_2^2 < t^2} \left( |y_3| + y_1^2 + y_2^2 \right)^{-2\alpha} dy = 4\pi \int_{0}^{t} \int_{0}^{t} (z + r^2)^{-2\alpha} r dr dz = 2\pi \int_{0}^{t} \int_{0}^{t} (z + v)^{-2\alpha} dz dv \leq c.
\]

From this and (7.10) we obtain the inequality
\[
\int_{B_t} \Delta_n(x, x_0)^{-2\alpha} dx \leq c,
\]

which, being combined with (7.8) and (7.9), implies the desired estimate (7.7).

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LAVERTEYEV INSTITUTE OF HYDRODYNAMICS, LAVERTEYEV PR. 15, 630090 NOVOSIBIRSK, RUSSIA

E-mail address: plotnikov@hydro.nsc.ru

UNIVERSITAT BONN, ENDENICHER ALLEE 60, D-53115 BONN, GERMANY

E-mail address: w-weigant@t-online.de