Not Call Me Cellular Any More: The Emergence of Scaling Law, Fractal Patterns and Small-World in Wireless Networks

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Abstract—In conventional cellular networks, for base stations (BSs) that are deployed far away from each other, it is general to assume them to be mutually independent. Nevertheless, after long-term evolution of cellular networks in various generations, the assumption no longer holds. Instead, the BSs, which seem to be gradually deployed by operators in a casual manner, have embed many fundamental features in their locations, coverage and traffic loading. Their features can be leveraged to analyze the intrinsic pattern in BSs and even human community. In this paper, according to large-scale measurement datasets, we build correlation model of BSs by utilizing one of the most important features, that is, traffic. Coupling with the theory of complex networks, we make further analysis on the structure and characteristics of this traffic load correlation model. Simulation results show that its degree distribution follows scale-free property. Moreover, the datasets also unveil the characteristics of fractality and small-world. Furthermore, we apply Collective Influence (CI) algorithm to localize the influential base stations and demonstrate that some low-degree BSs may outrank BSs with larger degree.

I. INTRODUCTION

As the theory of complex networks gets increasingly mature, it has been successfully applied to understand the embedded property in a variety of real-world complex systems from various fields, such as social, ecological, biological and public transport networks [1] [2] [3] [4]. In regard to these real-world complex networks, they are found to share several common properties despite their differences in nature. One interesting property that many networks possess is scale-free (SF) pattern, which can be depicted by a Power-law tail in the degree distribution, i.e. \( P(k) \sim k^{-\lambda} \), where \( \lambda \) is the degree exponent and the degree \( k \) denotes the links to a node [3]. Another crucial property is small-world, namely, although the size of network \( N \) (i.e., the number of nodes) is very large, the average distance \( d \) between two randomly chosen nodes is small, being well approximated by \( d \sim \ln N \) [4]. The most significant property that has opened a new perspective in complex network theory is fractality, which has been proved to exist in many networks [5] [6]. Basically, box-covering algorithm is used to validate this property [5]. Specifically, fractality refers to a Power-law relationship between the minimum number of boxes \( N_b \) required to tile the entire network and the size of the boxes \( L_b \), \( N_b \sim L_b^{-d_s} \), where \( d_s \) is the fractal dimension [3]. On the other hand, cellular networks have been undergoing a long history of evolution. Based on the above concepts and features of complex networks, we try to apply them to study the spatial distribution patterns of BSs in the long-term-evolved heterogeneous cellular networks. Commonly known, BSs are continually deployed to provision the ever-increasing mobile traffic in hotspots accompanied by the global popularity of smart phones and tablets. Consequently, much attention has been attached to the study of cellular traffic data and BSs deployment. While the monumental quantity of traffic continues to grow, we need to tap into the power of this data. It is naturally to ask what is the relationship between two BSs that are distant from each other? In this paper, we try to answer this question by using the traffic records from on-operating cellular networks. We first create a spatial load correlation model of BSs by regarding BSs as nodes and the traffic correlation between BSs as edges. Then, we analyze the structure and properties of this spatial load correlation model and surprisingly discover that it contains the three key properties (i.e., the SF pattern, fractality and small-world) above. It should be noted that fractality contradicts the small-world property in essence, since the former is mainly due to the repulsion between nodes of large degree (e.g., hubs) in disassortative networks while the latter is just the reverse [2]. In order to provide more evidences to prove our results, we further calculate the Pearson coefficient and the Correlation profile of the spatial load correlation model. Then, we extract its skeleton to search for the most compact pairs of BSs and the skeleton is found to be fractal as well.

In the aspect of identifying influential BSs in cellular networks, besides the two extensively used heuristic methods (i.e. high-degree and high-degree adaptive), we further leverage the Collective Influence (CI) [7] [8] algorithm to evaluate the influence of each BS and find the set of most influential BSs. Moreover, we extract the influential BSs found by CI to make sure whether high-degree BSs are more significant in our spatial load correlation model.

The remainder of this paper is organized as follows. In Section II, we briefly introduce the real measurement datasets and the box-covering algorithm. Section III describes the procedures to establish the spatial load correlation model.
Section IV studies the degree distribution of this model and uses three methods to identify the influential BSs. Section V focuses on the structural and characteristic analysis of this model. Finally, retrospective analysis of the significance of this work are included in Section VI.

II. BACKGROUND

A. Data Acquisition and Preparation

We acquire the real measurement data from one commercial mobile operator in China, which contains the information of traffic and base stations from second-generation cellular network in City A and the counterpart from third-generation cellular network in City B. Specifically, the traffic data is measured in the unit of bytes that each BS transmits to the covered users. The available data spans 1 week and 1 day, and the temporal granularity is one-hour for City A and half hour for City B, respectively. Therefore, each base station load is a time series of length 168 covering 7 days or length 48 covering 1 day. Meanwhile, the BS related information such as BS type, location area and geographic location is included as well and more details are summarized in Table I.

| Attributes          | BS information | Traffic Information |
|---------------------|----------------|---------------------|
| Network Type        | City A         | City B             |
| BS Type             | 1441 microcells| 3G cellular network |
| Location            | Longitude, latitude | Longitude, latitude |
| No. of BSs          | 5573           | 2053               |
| Traffic Resolution  | One hour       | Half an hour       |
| Duration            | 7 days         | 1 day              |

B. Box-covering Algorithm

As a widely used technique for characterizing fractal networks and calculating their fractal dimensions, box-covering algorithm has experienced lots of distinct versions since the generalized box-covering algorithm was introduced by Song et al. [5]. The random sequential (RS) box-covering algorithm [5] is not accepted in our work due to its low efficiency in finding the minimum number of boxes among all the possible tiling configurations. Therefore, we adopt a slightly improved algorithm in [9].

- Label all vertices as "not burned" (NB).
- Select a vertex randomly, which serves as a seed.
- Search the network by distance \( L_b \) from the seed and label the vertices newly found as "burned", which are later assigned to the new box.
- Select a vertex randomly from those labeled NB; this vertex serves as the new seed.
- Repeat Step 3 and Step 4 until all vertices are labeled "burned".

- Change the size \( L_b \), repeat the steps above until \( L_b \) equals the network diameter.

Given that the box number \( N_b \) derived from this algorithm may not be the minimum number of the corresponding size \( L_b \). In order to solve this problem, we repeat the process 1000 times and obtain 1000 values for each \( L_b \), expecting that the maximum of them can approximate the desired value.

III. MODELING PROCESS

A. Basics

In graph theory, networks and many real-world complex systems can be depicted as sets of nodes connected by edges. Denote an undirected network as \( G(V,E) \), where \( V \) is the set of nodes, \( E \) is the set of edges and \( e_{ij} \in E \) represents the link between node \( i \) and node \( j \). The degree of node \( i \) is defined as the number of its linked neighbours [10]. In theory, any undirected network can be described by a corresponding adjacency square matrix \( W \) with the dimension of \( N \), where \( N \) is the size of the network. Each element \( w_{ij} \in W \) equals one if there exists a link between node \( i \) and node \( j \), and zero otherwise [11].

With the collected datasets, we model undirected networks with BSs regarding as nodes, whose edge \( e_{ij} \) represents the load correlation between BS \( i \) and BS \( j \). Here, each base station load is a simple time series covering the 7 day or 1 day period, thus base station \( i \) can be expressed by a vector \( x_i(1), \ldots, x_i(T) \). With the temporal granularity of one-hour for City A and half hour for City B, we have \( T \) equals 168 and 48, respectively. Afterwards, we use each vector to calculate Pearson correlation coefficients with the remaining vectors until all BSs are traversed [?]. For BS \( i \) and BS \( j \), the Pearson correlation coefficient is defined as:

\[
\rho_{ij} = \frac{\sum x_i x_j - \sum x_i \sum x_j}{\sqrt{\sum x_i^2 - (\sum x_i)^2} \sqrt{\sum x_j^2 - (\sum x_j)^2}}
\]

(1)

Here, we set a threshold \( Z \) to judge the existence of one link between two BSs by comparing \( \rho_{ij} \) and \( Z \), if \( \rho_{ij} \) is larger than \( Z \), we consider there exist a link between BS \( i \) and BS \( j \). Hence, the adjacency matrix \( W \) can be derived from the Pearson coefficient matrix. After constructing the spatial load correlation model based on all base station loads, the model can be represented as a graph and we find some BSs that are isolated from the model. In the following analysis, BSs with degree being equal to zero are deleted from datasets owing to their weak load correlation with all the other BSs.

B. Threshold Selection

In the process of modeling, choosing appropriate threshold \( Z \) is of great significance for our study. Thus, two aspects should be taken into consideration to choose the threshold \( Z \). On one hand, \( Z \) should be large enough, so as to avoid mistakenly assuming weakly correlated BSs to be linked. On the other hand, the chosen threshold needs to ensure the proportion of isolated BSs is relatively low. In our work, we
TABLE II
DATA REPROCESSING AND DEGREE DISTRIBUTION ANALYSIS OF TWO CITIES.

| City | Threshold | Number Of Isolated BSs | Network Size N | Rate Of Isolated BSs | Degree Exponent \( \lambda \) |
|------|-----------|------------------------|----------------|----------------------|-----------------|
| A    | 0.50      | 527                    | 5046           | 0.0946               | 1.0394          |
|      | 0.52      | 773                    | 4800           | 0.1387               | 1.0593          |
|      | 0.54      | 1079                   | 4494           | 0.1936               | 1.1298          |
|      | 0.56      | 1453                   | 4120           | 0.2607               | 1.1489          |
|      | 0.58      | 1862                   | 3711           | 0.3341               | 1.2049          |
|      | 0.60      | 2236                   | 3287           | 0.4102               | 1.2581          |
|      | 0.65      | 3345                   | 2228           | 0.6002               | 1.4088          |
| B    | 0.50      | 3                      | 2050           | 0.0015               | 2.2790          |
|      | 0.54      | 11                     | 2042           | 0.0054               | 2.5951          |
|      | 0.56      | 22                     | 2031           | 0.0077               | 2.5955          |
|      | 0.58      | 51                     | 2002           | 0.0248               | 2.6544          |
|      | 0.60      | 79                     | 1974           | 0.0385               | 2.6191          |
|      | 0.65      | 246                    | 1807           | 0.1198               | 2.5835          |
|      | 0.70      | 532                    | 1521           | 0.2591               | 2.6483          |

would like to model our datasets of two cities with various thresholds \( Z \), ranging from 0.5 to 0.7. Afterwards, we are going to analyze some properties of the models we just learnt. To our surprise, no matter how threshold \( Z \) changes, the properties of our model remain the same and more detailed information is shown in table II and table III. Without loss of generality, we build the spatial load correlation model and study its properties in Section IV and Section V with threshold \( Z \) being equal to 0.54.

IV. ANALYSIS AND APPLICATION OF DEGREE DISTRIBUTION

A. Degree Distribution

As depicted in Section III, the spatial load correlation model is built in terms of the traffic loads and contributes to understanding the underlying relationship of BSs, which can not be directly discovered from brief information as locations (e.g., longitude and latitude) or BS types.

After modeling the spatial load correlation model, an adjacency matrix \( W \) can be acquired and the definition of degree refers to the number of BSs that are highly correlated with a chosen BS. In other words, the degree of BS \( i \) can be expressed by

\[
\begin{align*}
  k_i &= \sum_j w_{ij} \\
  P(k) &= \text{probability that a randomly chosen vertex has degree } k \quad [10].
\end{align*}
\]

Meanwhile, \( P(k) \) is defined as the probability that a randomly chosen vertex has degree \( k \) [10]. Since the number of BSs in City A is 5573, an adjacency square matrix with the dimension of 5573 can be obtained. It is verified that the obtained degree distributions are scale-free and obviously satisfy Power-law with different exponents \( \lambda \) for various thresholds \( Z \) that are larger than 0.5 in table II. Recalling the statements in Section III and uniting the simulation results in table II, although our conclusion remains the same under different thresholds \( Z \), we advise that the optimal value of \( Z \) can be set within the range from 0.5 to 0.6 to make sure the proportion of removed BSs is acceptable. Therefore, in Fig. 1, we provide the results of the degree distributions of City A and City B. In general, the spatial load correlation model has a property of scale-free and what is crucial for us is to know which BSs have high degree values. Scale-free property of the load correlation model demonstrates that the minority of BSs are highly correlated with plenty of other BSs while the remaining BSs are only correlated with a few number of BSs.

B. Identifying Influential BSs

Influential nodes usually play a decisive role on maintaining the network connectivity, enhancing network stability and improving the information transmission efficiency [12] [13] [19]. Similarly, the influential BSs can take more important roles in cellular networks. For example, compared with the conventional method that employs macrocell as signaling BS, those important BSs are more suitable due to their greater influence and more easily to predict the tendency of BS traffic.
loads. As a result, it is imperative to pick out the most important BSs so as to give them more functions such as signaling control. Given that it is critical to investigate how to find out the most influential BSs, we apply two heuristic strategies to identify influential BSs in our correlation model. Moreover, based on the theory of influence maximization in complex networks, we introduced Collective Influence (CI) algorithm for localizing the most influential BSs [7] [8].

Influential nodes are defined as a set of nodes, which is much smaller than the total size, which, if removed, would break down the network into many disconnected components. At a general level, we use the size of the giant connected component to measure the remaining network structure after the influential nodes are removed from the network. In this paper, we aim at finding out the minimal set that guarantees a global connection of the network and the size of this minimal set is \( q_c \), the existence of the giant connected component can be expressed by \( G(q) \) after removing a certain fraction \( q \) of the network size. Then, our problem corresponds to find the optimal set whose removal would dismantle the network:

\[
q_c = \min\{q \in [0, 1] : G(q) = 0\} \tag{3}
\]

CI is an effective algorithm in terms of finding the most influential nodes, which removes nodes one-by-one according to their CI value:

\[
CI_i = (k_i - 1) \sum_{j \in \partial Ball(i, l)} (k_j - 1) \tag{4}
\]

where \( Ball(i, l) \) is the ball of radius \( l \) centered on node \( i \), and \( \partial Ball(i, l) \) is the frontier of the ball, which is the set of nodes at distance \( l \) from node \( i \). CI algorithm removes the node with the highest \( CI_i \) value at each step, and the process is repeated until the giant component is destroyed.

Fig. 2 shows the optimal threshold \( q_c \) for the traffic load correlation model of City A and City B. In the same figure, we compare the optimal threshold against two best heuristic methods: high-degree (HD), high-degree adaptive (HDA). In the two cases, CI produces a smaller threshold, which represents a better performance of this algorithm.

Afterwards, according to the optimal set of nodes found by CI algorithm, we display the locations of the most influential 500 base stations in the map and color codes its degree in Fig. 3 and Fig. 4. From the two figures, we learn that among the most influential base stations extracted by CI, lots of low-degree BSs even exhibit a greater influence than some high-degree BSs. That is to say, we should pay more attention to those low-degree influential BSs compared with part of high-degree BSs and degree is not a better criteria in measure influence to some extent.
Fig. 6. Box-covering analysis of the original network (○), its skeleton (▽) and random spanning tree (△) of City A.

Fig. 7. Box-covering analysis of the original network (○), its skeleton (▽) and random spanning tree (△) of City B.

V. STRUCTURAL PROPERTIES OF THE LOAD CORRELATION MODEL

A. Fractal Patterns

One important property that exists in many complex and real-world networks is fractality [14]. In fractal geometry [15], box-covering [5] is widely used to approximately evaluate the fractal dimension of a fractal object. Based on this method, fractal networks can be characterized by the following scaling relations:

\[ N_b/L^d_b \sim \log N_b(L_b) - \log L_b \]  

where \( L_b \) denotes the size of boxes used to cover the network and \( N_b(L_b) \) is the minimum number of boxes among all the possible tiling configurations with the box size equaling to \( L_b \). Therefore, the fractal dimension can be calculated through the following equation:

\[ d_b \sim \lim_{L_b \to 0} \frac{\log N_b(L_b)}{-\log L_b} \]  

In reality, the value of \( d_b \) can be obtained by fitting the slope between \( \log N_b(L_b) \) and \( \log L_b \). After these early-stage preparations, we employ box-covering method to investigate the traffic load correlation model of BSs. In this paper, we start the fractal pattern analysis with the size of box varying from 1 to the diameter of network, namely, 17 for City A and 7 for City B. This means that when \( L_b \) is no less than the network diameter, the value of \( N_b \) must be 1. Fig. 5 shows the results from the box-covering algorithm applied in City A and City B.

As illustrated in Fig. 5, for City A, the relation between \( \log(N_b) \) and \( \log(L_b) \) can be well-fitted by a straight line, which implies a clear fractal property of the network. Moreover, the fractal dimension \( d_b \) approximates 3.0348 with the \( R \) square value being 0.9460 denoting the good fitness of the curve. Meanwhile, for City B, the fractal dimension approaches 3.5027 with the \( R \) square value being 0.9532.

B. Skeleton Features

Regardless of the entanglement, a network always possesses a "skeleton" to simply represent the network structure and understand the topological organization[16]. The skeleton is a particular spanning tree consisting of edges with the highest betweenness centralities [9]. Plenty of researches have elaborated the importance of skeleton in understanding the topological organization of a network.

Basically, skeleton is thought to be a maximum spanning tree. Thus, the skeleton of our model is a spanning tree connected by the most compact links, whose topology can be regarded as the core of our model. Inspired by the classical Prim and Kruskal algorithms for building the minimum spanning tree, we propose a modified algorithm to find out the skeleton of the load correlation model (i.e., Algorithm 1).

Algorithm 1 Modified algorithm

\textbf{input:} \( G = (V, E) \), adjacency matrix \( W \); betweenness centralities matrix \( EC \);
\textbf{output:} \( P, Q \);
\textbf{initialization:}
set \( P = \{v_1\}, Q = \emptyset \).
\textbf{repeat}
find the maximum value \( EC(p, v), p \in P, v \in V \);
set \( P = P + \{v\} \);
set \( Q = Q + \{pv\} \);
\textbf{until} \( (P = V) \)

Following Algorithm 1, we extract the skeletons of the spatial load correlation models and study their degree distribution along with fractality. Simulation results verify that skeletons are also scale-free with exponent values \( \lambda \) equaling...
According to the figures, the relevant results express that although the random spanning tree possesses a different statistics of $N_i$, the fractal dimensions of the random spanning tree and the original network are just the same. Meanwhile, the fractality of the skeleton matches the fractality of the original correlation model very well. Hence, understanding the properties of the skeleton is of great importance for analyzing the original model.

**C. Further Exploration On small-world**

The small-world property usually coexists with scale-free networks [17]. Specifically, small-world property refers to the average distance $d$ scales logarithmically with the network size as $d \sim \ln N$. Another indispensable characteristic of small-world networks is their high clustering coefficient [10]. Structural analysis of the traffic load correlation model tells us that its size is 4494 for City A and the average distance $d$ equals to 4.5257. The relationship between $d$ and $N$ conforms to the above equation. City B with size 2042 and $d$ equaling to 3.3947 also meets this mathematical expression. Then, the clustering coefficients can be obtained, being equal to 0.5144 and 0.5177, respectively, which represents a highly clustering property. We do ensure that the spatial load correlation model of BSs possesses the small-world property as well and more supporting evidences are given in the following.

1) Pearson Coefficient

Degree of assortativity is one of the important features to describe network. Degree-degree correlations can be characterized by Pearson coefficient, which is defined as:

$$
r = \frac{M^{-1} \sum_{e_{ij}} k_i k_j - [M^{-1} \sum_{e_{ij}} \frac{1}{2}(k_i + k_j)]^2}{M^{-1} \sum_{e_{ij}} \frac{1}{2}(k_i^2 + k_j^2) - [M^{-1} \sum_{e_{ij}} \frac{1}{2}(k_i + k_j)]^2}
$$

(7)

where M denotes the total number of edges, $k_i$ and $k_j$ is the degrees of the two vertices at the ends of edge $e_{ij}$. The Pearson coefficient $r$ ranges from -1 to 1, it is positive for assortative networks and negative for disassortative ones. The Pearson coefficients of our models are 0.1535 and 0.5362 for City A and City B, respectively. In other words, the spatial correlation models are assortative [18].

2) Correlation Profile

Correlation profile is a metric of great importance to explain the structural information and the statistical property of correlation between the nodes within a network configuration [19]. The correlation profile is defined as:

$$
R(k_1, k_2) = P(k_1, k_2) / P_r(k_1, k_2)
$$

(8)

where $P(k_1, k_2)$ is the joint probability distribution representing the probability of finding a node with $k_1$ links connected to a node with $k_2$ links. While $P_r(k_1, k_2)$ is acquired by randomly swapping of the links with the degree distribution remaining unchanged. The plot of the ratio $R(k_1, k_2)$ demonstrates a correlated structure that deviates from the random uncorrelated case. We apply this metric to depict the correlation models and results are shown in Fig. 8 and Fig. 9.

From Fig. 8 and Fig. 9, we observe that the models exhibit a higher degree of correlation, namely, nodes with large degree tend to be connected with nodes of large degree and vice versa, which is the primary cause that contributes to the small
world behavior. While the emergence of scale-free fractal networks is due to the repulsion between nodes of large degree, and fractality seems to be contradicted with small-world behavior. Nevertheless, empirical results suggest that there exist networks with the simultaneous appearance of both fractal and small-world properties, for which a mathematical generation model was given in [2].

We have demonstrated the spatial load correlation model of BSs shows scale-free, fractal and small-world properties simultaneously, which will further facilitate the performance analysis of cellular networks as well as the design of efficient networking protocols. Firstly, scale-free behavior signifies the large-scale feature of the load correlation model, the traffic discovery of small-world property means that, despite the resource management based on base stations. Finally, the topology and irregularity, which contributes to more effective with fractality, we can find some regularities from its special network self-similarity. Moreover, for a topological structure explaining the possibility that the degree distribution might remain unchanged under scale transformation and leads to network self-similarity. Moreover, for a topological structure with fractality, we can find some regularities from its special topology and irregularity, which contributes to more effective resource management based on base stations. Finally, the discovery of small-world property means that, despite the large-scale feature of the load correlation model, the traffic association on base stations is very compact.

VI. CONCLUSION AND DISCUSSIONS

In this paper, we have proposed a unique approach to establish the spatial load correlation model for the base stations in cellular networks, leveraging a traffic vector with the elements being the traffic data crossing each BS in a certain interval. We first create spatial load correlation model according to the Pearson coefficient values between BSs. Afterwards, based on the correlation model, we discovered that the spatial correlation model is scale-free along with the coexistence of fractality and small-world feature, after careful verification in terms of common metrics in the literature. Additionally, we extracted the skeleton of the spatial correlation model in order to search for the most compact pairs of BSs to obtain the most significant links in our model. At last, we conducted some comparisons between CI algorithm and two best heuristic methods to pick out the set of the most influential base stations. Meanwhile, detailed results of various threshold $Z$ are shown in Table III to explain the same properties of our model and some suggestions on the potential main applications in real scenarios were provided in Section IV.

TABLE III

| City | Threshold | Fractal Dimension $d_f$ | Network Size $N$ | Average Distance $d$ | Clustering Coefficient | Pearson Coefficient $r$ |
|------|-----------|-------------------------|-----------------|----------------------|------------------------|-------------------------|
| A    | 0.50      | 3.7944                  | 5046            | 3.7944               | 0.5146                 | 0.1461                  |
|      | 0.54      | 3.0348                  | 4494            | 4.5257               | 0.5144                 | 0.1535                  |
|      | 0.56      | 3.5422                  | 4120            | 4.9610               | 0.5094                 | 0.1577                  |
|      | 0.58      | 2.7259                  | 3711            | 5.5517               | 0.5095                 | 0.1603                  |
|      | 0.60      | 2.4134                  | 3287            | 6.5524               | 0.4843                 | 0.1761                  |
|      | 0.65      | 2.3755                  | 2288            | 6.8535               | 0.5214                 | 0.1593                  |
| B    | 0.50      | 3.7120                  | 2050            | 2.9881               | 0.4972                 | 0.5378                  |
|      | 0.54      | 3.5027                  | 2042            | 3.3947               | 0.5177                 | 0.5362                  |
|      | 0.58      | 3.1010                  | 2002            | 3.9098               | 0.5342                 | 0.5291                  |
|      | 0.60      | 3.0007                  | 1974            | 4.2315               | 0.5378                 | 0.5221                  |
|      | 0.65      | 3.0206                  | 1907            | 5.2704               | 0.5470                 | 0.5143                  |
|      | 0.70      | 3.3045                  | 1521            | 6.5584               | 0.5439                 | 0.5042                  |

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