Perturbed flavour symmetries and predictions of CP violating phase $\delta$

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Abstract

It is known that the imposition of a class of residual $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetries on the neutrino mass matrix $M_\nu$ and a residual symmetry $\mathbb{Z}_n$ ($n \geq 3$) on the Hermitian combination $M_l M_l^\dagger$ of the charged lepton mass matrix leads to a universal prediction of vanishing Dirac CP phase $\delta$ if these symmetries are embedded in $\Delta(6n^2)$ groups and if the leptonic doublets transform as a 3 dimensional irreducible representation of the group. The Majorana phases remain arbitrary but they can also be determined in $\Delta(6n^2)$ by imposing generalized CP symmetry (GenCP) consistent with the $\Delta(6n^2)$ group. We investigate the effects of adding general perturbations on these predictions assuming that perturbations break the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry completely but preserve GenCP. It is found that if the residual symmetries predict the tri-bimaximal mixing (TBM) among leptons and specific CP conserving values for the Majorana phases then addition of the above perturbations always lead to a neutrino mass matrix invariant under the $\mu-\tau$ reflection symmetry in the flavour basis with the result that perturbations turn the vanishing $\delta$ into maximal value $\pm \frac{\pi}{2}$. One gets non-vanishing but generally large $\delta$ if the predicted zeroth order mixing deviates from TBM and/or the predicted Majorana phases are non-trivial. We systematically investigate effects of perturbations in such situations and work out the predicted $\delta$ for four of the lowest $\Delta(6n^2)$ groups with $n = 2, 4, 6, 8$. 

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I. INTRODUCTION

Flavour symmetries provide an attractive framework for the theoretical understanding of the leptonic mixing angles and phases. In particular, discrete symmetries have been found to be leading to definitive and viable predictions of these parameters \[1–5\] and these are exhaustively studied in a number of papers \[6–13\], see \[14–17\], for early reviews. It is found that a very large class of discrete subgroups of $U(3)$ predict vanishing Dirac CP phase $\delta$ \[18, 19\]. This prediction follows from the following assumptions: (i) Neutrino mass matrix $M_{\nu}$ is invariant under a $\mathbb{Z}_2 \times \mathbb{Z}_2 \equiv S_\nu$ symmetry (ii) The charged lepton mass matrix $M_{\ell}M_{\ell}^\dagger$ is invariant under a $Z_n \equiv T_\ell$, ($n \geq 3$) group (iii) $S_\nu$ and $T_\ell$ are contained in a discrete subgroup (DSG) $G_f$ of $SU(3)$ and (iv) three generations of leptons transform as a 3 dimensional irreducible representation of $G_f$. Given these assumptions, almost all the DSG of $SU(3)$ and many DSG of $U(3)$ taken as $G_f$ \[19\] lead to the prediction $\delta = 0$. In this sense, the prediction $\delta = 0$ may be regarded as a universal prediction following from the above assumptions. These assumptions do not fix the neutrino Majorana phases. One can predict these phases by combining flavour and CP symmetry \[20–25\] , see \[26\] for a detailed list of references, . Recent reviews are given in \[27, 28\]. These predictions are explored in details for the $\Delta(6n^2)$ groups \[29\] and it is found that the use of these groups as $G_f$ can lead to non-trivial Majorana phases.

The above prediction of vanishing $\delta$ appears to be at variance with the present experimental trend. The latest results from the NO$\nu$A experiment \[30, 31\] gives $\sin^2 \theta_{23} = 0.404, \delta = 1.48\pi$ in case of the normal ordering. Similarly, recent results from T2K experiment \[32, 33\] involving both the neutrino and anti-neutrino runs gives $\sin \delta = 1.43\pi$. A global analysis of neutrino oscillation data give $\delta = 1.40\pi$ with the 1$\sigma$ range ($1.20 - 1.71)\pi$ and disfavour the maximal atmospheric mixing at $\Delta \chi^2 = 6.0$ \[31\], see \[35–37\] for other recent fits.

The absence of the Dirac CP violation predicted in the above theoretical framework can change significantly in the presence of even small perturbations which would arise from the breaking of the flavour symmetry. We wish to systematically analyze here effects of perturbations to $M_{\nu}$ on the prediction $\delta = 0$. We take the zeroeth order residual symmetry of $M_{\nu}$ as $S_{\nu}^{CP} \equiv Z_2 \times Z_2 \times H_{\nu}^{CP}$ as has been done in the general analysis presented e.g. in \[29\]. $H_{\nu}^{CP}$ here denotes generalized CP (GenCP) transformation commuting with $Z_2 \times Z_2$. We however allow for the most general perturbations to it which break $Z_2 \times Z_2$ symmetry completely but preserves GenCP contained in $S_{\nu}^{CP}$. It is possible to study effect of such perturbations in a model independent manner as we shall show. These perturbations have dramatic effect on the prediction of the Dirac CP phase. We show that if $Z_2 \times Z_2$ symmetry predicts tri-bimaximal mixing (TBM) pattern and if the group theoretically determined Majorana phases $\alpha_{21}, \alpha_{31}$ are predicted to have CP conserving values 0, $\pi$ respectively at the leading order then switching on the generalized CP invariant perturbations lead to the prediction of the maximal CP phases and the maximal atmospheric mixing angle for a class of GenCP symmetry. Even when the zeroeth order mixing matrix $U_0$ does not have the TBM

\[1\] Following \[38\], we denote the Majorana phase matrix on the RHS of $U_{PMNS}$ as $\text{diag}(1,e^{\frac{\alpha_{21}}{2}},e^{\frac{\alpha_{31}}{2}})$
form or the Majorana phases are non-trivial, one still gets quite large Dirac CP phases in the
presence of perturbations. The predicted Dirac phases in several cases are characteristic of
the underlying residual symmetries rather than the values of the perturbation parameters.
We numerically derive such predictions for the residual symmetries contained in subgroups
of first four groups$^{\text{2}}$ in the $\Delta(6n^2)$ series with $n = 2, 4, 6, 8$.

We first review basic consequences of imposing residual symmetries in the next section.
Then we derive general form of the leptonic mixing matrix in the presence of the GenCP
invariant perturbations in section $\text{III}$. Section $\text{IV}$ discusses possible residual symmetries in
the context of the $\Delta(6n^2)$ groups followed in section $\text{IV A}$ by a discussion of conditions under
which one obtains the maximal Dirac phase. Explicit form of the perturbations in $\Delta(6n^2)$
group is presented in section $\text{V}$. This is followed by discussion of numerical results in section
$\text{VI}$. The last section gives a summary.

II. FORMALISM

We briefly review here consequences of imposing residual symmetries on the leptonic mass
matrices. The leading order Majorana mass matrix for the neutrinos in some symmetry basis
is defined as $M_{0\nu}$ and the charged lepton mass matrix as $M_l$. They are assumed to satisfy
symmetry relations$^{[14-17]}$

\[ S_{1\nu,2\nu}^T M_{0\nu} S_{1\nu,2\nu} = M_{0\nu}, \]
\[ T_l^\dagger M_l M_l^\dagger T_l = M_l M_l^\dagger. \]  

(1)

$S_{1\nu}$ and $S_{2\nu}$ are $3 \times 3$ unitary matrices generating the group $Z_2 \times Z_2$ and $T_l$ generates a
$Z_n, n \geq 3$. Let $U_{S_{\nu}}$ be a unitary matrix which diagonalizes $S_{1\nu}$ and $S_{2\nu}$ simultaneously.
Explicitly,

\[ U_{S_{\nu}}^\dagger S_{1\nu} U_{S_{\nu}} = \text{diag.}(-1,-1,1) \quad U_{S_{\nu}}^\dagger S_{2\nu} U_{S_{\nu}} = \text{diag.}(1,-1,-1), \]  

(2)

$U_{S_{\nu}}$ is arbitrary upto a multiplication by a diagonal phase matrix from right. This arbitrari-
ness can be fixed by imposing CP as an additional symmetry and taking $S_{\nu}^{CP} \equiv Z_2 \times Z_2 \times H_{CP}^{CP}$
as the complete residual symmetry. The action of GenCP on the neutrino triplet is repre-
sented by a $3 \times 3$ symmetric unitary matrix $X_{\nu}$. Requiring that action of each of the $Z_2$
separately commutes with GenCP operation imposes the constraints$^{[20]}$:

\[ X_{\nu} S_{1\nu,2\nu}^{\ast} X_{\nu}^\dagger = S_{1\nu,2\nu}. \]  

(3)

Invariance of the neutrino mass term under GenCP requires

\[ X_{\nu}^T M_{0\nu} X_{\nu} = M_{0\nu}^{\ast}. \]  

(4)

$^2$ Only groups with even $n$ are relevant here since $\Delta(6n^2)$ with odd $n$ do not contain $Z_2 \times Z_2$ groups as subgroups.
Eqs. (1-4) are sufficient to completely determine the neutrino mixing matrix. If we define \( \hat{X}_\nu \equiv U^\dagger \nu S \nu X \nu U^\ast \nu S \nu \), then eqs. (2,3) imply
\[
\hat{X}_\nu \equiv P^2 \nu ,
\]
where \( P_\nu \) is a diagonal phase matrix. This then implies
\[
X_\nu = V_{0\nu} V_{0\nu}^T , V_{0\nu} \equiv U S \nu P_\nu .
\]
\( V_{0\nu} \) in this way gets determined from the structure of \( S_{1\nu,2\nu} \) and \( X_\nu \). Eq. (1) implies that \( M_{0\nu} \) is diagonalized by \( V_{0\nu} \):
\[
V_{0\nu}^T M_{0\nu} V_{0\nu} = D_0 \equiv \text{diag},(m_1,m_2,\ldots,m_3) .
\]
GenCP invariance of \( M_{0\nu} \), eq. (4) then implies that \( D_0 \) defined above is a real matrix. Thus \( V_{0\nu} \) diagonalizes \( M_{0\nu} \) with real (not necessarily positive) eigenvalues and can be taken as the neutrino mixing matrix at the leading order.

The complete mixing matrix at the leading order is given by
\[
U_0 \equiv U_l^\dagger U S \nu P_\nu \equiv U S P_\nu ,
\]
where \( U_l \) is a matrix that diagonalizes \( M_l M_l^\dagger \). It is determined up to overall phases by its residual symmetry \( Z_n \) if \( n > 2 \).

III. GENERALIZED CP INVARIANT PERTURBATIONS AND MIXING MATRIX

We now discuss the effect of GenCP invariant perturbations on the structure of the mixing matrix, eq. (8). Assume that the neutrino mass matrix has the form
\[
M_\nu = M_{0\nu} + \delta M_\nu .
\]
\( \delta M_\nu \) is a perturbation matrix which would arise from the \( Z_2 \times Z_2 \) symmetry breaking in models. We assume that GenCP symmetry is not broken at this stage and \( \delta M_\nu \) thus satisfies
\[
X_\nu^T \delta M_\nu X_\nu = \delta M_\nu^* .
\]
This assumption leads to the following general structure of the mixing matrix \( U_{sym} \):
\[
U_{PMNS} \sim U_{sym} = U S P_\nu O K .
\]
Here \( U_S \equiv U_l^\dagger U S \nu \), see eq. (8) is a matrix determined by the residual symmetries. \( P_\nu \) is a diagonal phase matrix determined by \( X_\nu \) and \( O \) is a real orthogonal matrix resulting from the perturbations. \( K \) is a diagonal phase matrix with elements \( \pm 1, \pm i \) which is used to make the eigenvalues of \( M_\nu \) positive. The residual symmetry cannot predict the order of
the leptonic masses and hence orders of the rows and columns in $U_{\text{sym}}$. The correspondence
between $U_{\text{PMNS}}$ shown on the left and $U_{\text{sym}}$ has to be decided on the phenomenological
grounds in these symmetry based approaches.

Eq. (11) follows in a straightforward way. We first re-express eq. (9) in the basis with a
diagonal $M_0$ by defining $\delta M_0' \equiv V_0^T M_\nu V_0$. This gives
\[ M_\nu' = D_0 + \delta M_\nu', \tag{12} \]
where $D_0$ is defined in eq. (7) and $\delta M_\nu' \equiv V_0^T \delta M_\nu V_0$. Then use of eq. (10) together with
$X_\nu = V_0^T \nu V_0^T$ implies
\[ \delta M_\nu' = \delta M_\nu'\ast. \]
Since $D_0$ is also real it follows that $M_\nu'$ is a real symmetric matrix which can be diagonalized
by an orthogonal matrix $O$. Thus neutrino mass matrix $M_\nu$ is diagonalized by $V_\nu O =
U_{\nu}P_\nu O$ and one gets $U_{\text{PMNS}}$ as given in eq. (11).

It turns out that $U_\nu$ in eq. (11) can be made real by absorbing all its phases in $P_\nu$ or in
redefining the charged lepton fields when the residual symmetries are embedded in $\Delta(6n^2)$
groups. With $O$ also real, the only source of CP violation in eq. (11) is the (appropriately
redefined) phase matrix $P_\nu$. This corresponds to only Majorana CP violation at the leading
order but $P_\nu$ plays non-trivial role and generates Dirac CP violation when $O$ is present.
Remarkably, the non-trivial Dirac phase can result even in the CP conserving situation at
the tree level corresponding to trivial Majorana phases as we discuss now.

IV. $\Delta(6n^2)$ SYMMETRY

We first outline possible choices of Klein groups and the CP symmetries consistent with
them in the context of the $\Delta(6n^2)$ groups [18, 29] to set our notations. This symmetry group
is generated by four elements $a, b, c, d$ satisfying
\[ a^3 = b^2 = (ab)^2 = c^n = d^n. \]
These elements are represented in one of the three-dimensional representation of $\Delta(6n^2)$ ($3^1_2$
in the notation of [18]) as
\[ A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \quad B = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad C = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta^* & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta^* \end{pmatrix}, \tag{13} \]
where $\eta = e^{2\pi i/n}$ and we denote the 3-dimensional representation of elements by the corre-
sponding capital letters. A set of Klein groups within $\Delta(6n^2)$ is generated from
\[ S_1 \nu \equiv BC^{\gamma_\nu} D^{\gamma_\nu} = -\begin{pmatrix} 0 & 0 & \eta^{-\gamma_\nu} \\ 0 & 1 & 0 \\ \eta^{\gamma_\nu} & 0 & 0 \end{pmatrix}; \quad S_2 \nu \equiv BC^{\gamma_\nu+\frac{n}{2}} D^{\gamma_\nu+\frac{n}{2}} = -\begin{pmatrix} 0 & 0 & -\eta^{-\gamma_\nu} \\ 0 & 1 & 0 \\ -\eta^{\gamma_\nu} & 0 & 0 \end{pmatrix}, \tag{14} \]
where $\gamma_\nu = 0, 1, 2,..., n - 1$. These two along with their products and squares form a set of Klein group $Z_2 \times Z_2$. One could obtain two other sets from the cyclic permutations of these. Mixing patterns predicted in all three cases are equivalent and we will specifically use eq.(14) as neutrino symmetries. The other possible Klein groups within $\Delta(6n^2)$ consist of diagonal generators in the chosen basis. They lead to democratic mixing at the leading order when $T_l = Z_n$ and thus predict $\sin^2 \theta_{13} = \frac{1}{3}$ which is far from the actual value. We shall therefore not consider them.

The minimal requirements on CP symmetry $X$ [20] is that it should satisfy eq.(3) which ensures that the residual symmetry $H_{\nu}^{CP}$ commutes with $S_{1\nu}, S_{2\nu}$. If this symmetry is to be embedded in a flavour group $G_f$ then there are further requirements on $X$ for the consistent definition of $G_f$ and $CP$ [25][39]. These are studied at length in general situations and in the context of the $\Delta(6n^2)$ groups [29]. One basically requires that $X_r$ for every representation $\rho_r$ of $G_f$ should satisfy

$$X_r \rho_r (g) X_r^\dagger = \rho_r (g') ,$$

where $g$ and $g'$ are elements of $G_f$ and the above equation should remain true for every $g \in G_f$. It has been argued [29] in the context of $G_f = \Delta(6n^2)$ that $X$ should be an element of the group satisfying $X^T = X, XX^\dagger = 1$ up to an overall phase. Two possible sets of $X$ within $\Delta(6n^2)$ are given in three dimensional representation as

$$X_{1\nu} \equiv C^x D^{-x-2\gamma_\nu} = \begin{pmatrix} \eta^x & 0 & 0 \\ 0 & \eta^{-2x-2\gamma_\nu} & 0 \\ 0 & 0 & \eta^{x+2\gamma_\nu} \end{pmatrix}, \quad X_{2\nu} \equiv C^x D^{-x}B = -\begin{pmatrix} 0 & 0 & \eta^x \\ 0 & \eta^{-2x} & 0 \\ \eta^x & 0 & 0 \end{pmatrix} ,$$

(15)

where $x = 0, 1,...n - 1$. Both of these satisfy the required eq.(3) for $S_{1\nu}, S_{2\nu}$ given by eq.(14). We shall use these two choices of $X$ and study consequences of imposing these on total $M_\nu$, eq.(9).

Common matrix diagonalizing $S_{1\nu}, S_{2\nu}$ is given by

$$U_{S_{\nu}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta^{-2x} & 0 & \eta^{-2\gamma_\nu} \\ 0 & \sqrt{2} & 0 \\ \eta^{2x} & 0 & -\eta^{2\gamma_\nu} \end{pmatrix} .$$

(16)

The neutrino mixing matrix $V_{0\nu}$, eq.(6) is obtained from the above $U_{S_{\nu}}$ by multiplying it with the phase matrix $P_{\nu}$ which is determined by $X_{1\nu,2\nu}$ as given in eq.(5). Denoting $P_{\nu}$ in these cases by $P_{1\nu,2\nu}$, we have

$$P_{1\nu} = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_1^{-2} & 0 \\ 0 & 0 & p_1 \end{pmatrix} ; \quad P_{2\nu} = \begin{pmatrix} p_2 & 0 & 0 \\ 0 & p_2^{-2} & 0 \\ 0 & 0 & -ip_2 \end{pmatrix} ,$$

(17)

with $p_1 = e^{\frac{i\pi(x+\gamma_\nu)}{n}}$ and $p_2 = e^{\frac{i\pi x}{n}}$. 

6
The matrix $U_l$ is determined by the symmetry $T_l$ of $M_l M_l^\dagger$. This symmetry group is chosen as a set of $Z_n$ groups defined in the three dimensional representation as:

$$T_l \equiv C^{l_1} D^{l_1 + l_2} = \begin{pmatrix} 0 & \eta^{l_1} & 0 \\ 0 & 0 & \eta^{l_2} \\ \eta^{-(l_1 + l_2)} & 0 & 0 \end{pmatrix},$$  \hspace{1cm} (18)$$

with $l_1, l_2 = 0, 1 \ldots n - 1$. Other possible $Z_n$ sub-groups of $\Delta(6n^2)$ are diagonal or block diagonal (analogous to $S_1, S_2$). The only zeroeth order viable pattern of mixing predicted in these cases correspond to the democratic or bi-maximal mixing. Since the solar mixing angle at the zeroeth order considerably deviates from its actual value in this mixing pattern, we will omit such groups from the discussion and work with the set of $Z_n$ defined by eq.(18). $T_l$ is diagonalized by

$$U_l \equiv P_l U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} \eta^{l_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \eta^{-l_2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$  \hspace{1cm} (19)$$

with $\omega = e^{2\pi i / 3}$. The complete mixing matrix following from the above determined $V_{0\nu}$, $U_l$ and eq.(11) can be written as:

$$U_{\text{sym}} = p_a \eta^{l_2-n} \text{diag}(1, \omega^2, \omega) \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} c_1 & 1 & -\sqrt{2} i s_1 \\ \sqrt{2} c_2 & 1 & -\sqrt{2} i s_2 \\ \sqrt{2} c_3 & 1 & -\sqrt{2} i s_3 \end{pmatrix} \cdot \text{diag}(1, p_a^{-3} \eta^{\frac{n-1}{2}} \epsilon_a) O,$$  \hspace{1cm} (20)$$

We distinguish two symmetries $X_{a\nu}$ by label $a = 1, 2$. $p_a$ arising from $P_{a\nu}$ are defined below eq.(17) and $\epsilon_1 = 1, \epsilon_2 = -i$. $c_i, s_i = \cos \theta_i, \sin \theta_i$ with

$$\theta_1 = \frac{\pi \gamma}{n} \equiv \frac{\pi(l_1 + l_2 + \gamma_\nu)}{n}, \hspace{0.5cm} \theta_2 = \theta_1 + \frac{4\pi}{3}, \hspace{0.5cm} \theta_3 = \theta_1 + \frac{2\pi}{3}.$$  \hspace{1cm} (21)$$

$U_{\text{sym}}$ represent the PMNS mixing matrix as given by the underlying symmetry. The phase matrix appearing in the LHS of the above equation and an overall phase can be removed by redefining the charged lepton phases and will be neglected further. At the leading order, $O$ is an identity matrix and the above $U_{\text{sym}}$ leads to vanishing Dirac phase as already noted in \cite{29}. Non-trivial $O$ arises in the presence of perturbations. We shall discuss possible nature of perturbations in the next section. Here we present an important consequence of eq.(20) which purely follows from symmetry rather than details of the perturbations.

**A. From conserved to maximal CP**

We now show that the perturbations can completely change the prediction of vanishing Dirac CP phase which can even take the maximal value $\sin \delta = \pm 1$. Interestingly, the occurrence of the maximal phase is intimately tied to the absence of CP violation in the
leading order PMNS matrix. CP is conserved at the leading order if the relevant Majorana phases \( \alpha_{21}, \alpha_{31} \) are 0 or \( \pi \). Group theoretically determined phase \( \alpha_{31} \) from eq. (20) is \( \pi(0) \) in case of the symmetries \( X_{1\nu}(X_{2\nu}) \) independent of the values of \( x, \gamma \). \( \alpha_{21} = 0, \pi \) when \( p_{\alpha}^{-3} \eta_{12}^{1-2} = 1, i \) respectively. In this case, the Dirac as well Majorana CP violation is absent at the leading order. Turning on perturbations can change this completely. Specifically, we show that the following result holds:

If the underlying \( Z_2 \times Z_2 \) symmetry leads to the tri-bimaximal mixing pattern at the leading order and if the CP violating Majorana phase \( \alpha_{21} \) as predicted by CP symmetry \( X_{1\nu} \) assumes value 0 or \( \pi \) then an arbitrary GenCP invariant perturbations correcting for the original tri-bimaximal mixing lead to a theory with maximal CP phase and maximal atmospheric mixing angle.

The proof of the above follows in a straightforward manner. Consider the elements \( |U_{\alpha i}|^2 \) for \( \alpha = \mu, \tau \):

\[
|U_{\mu i}|^2 = \frac{1}{3}|\sqrt{2} c_2 O_{1 i} + p_1^{-3} \eta_{12}^{1-2} O_{2 i} - i\sqrt{2} s_2 O_{3 i}|^2, \\
|U_{\tau i}|^2 = \frac{1}{3}|\sqrt{2} c_3 O_{1 i} + p_1^{-3} \eta_{12}^{1-2} O_{2 i} - i\sqrt{2} s_3 O_{3 i}|^2.
\]

We have chosen here specific order in which the second and the third row of \( U_{\text{sym}} \) are taken to be associated with the \( \mu \) and \( \tau \) flavours respectively. The same result would follow for the other choices but with a different value for the angle \( \frac{\pi}{n} \). Simultaneous occurrence of the maximal CP phase and \( \theta_{23} \) is termed as \( \mu-\tau \) reflection symmetry [40] and is obtained from the following relation:

\[
|U_{\mu i}| = |U_{\tau i}|.
\]  

(23)

If we do not want any fine tunning then a prerequisite to obtain the above relation with perturbations is that the zeroth order mixing matrix as implied from the \( Z_2 \times Z_2 \) symmetry also satisfies this relation. This requires either (a) \( s_2 = -s_3 \) or (b) \( s_2 = s_3 \) in eq. (20). These cases lead to the tri-bimaximal pattern at the leading order since they imply \( \theta = 0 \) for case (a) and \( \theta = \pm \frac{\pi}{2} \) for case (b). The third column of \( U_0 \) has TBM form in case (a) while for case (b) the first column of \( U_0 \) has the TBM form \((0, -1, 1)^T \) and one gets phenomenologically consistent picture in this case by identifying the first column with the heavier (lighter) mass eigenstate for the normal (inverted) hierarchy. It is seen that eq. (22) leads to eq. (23) for an arbitrary \( O \) if \( p_1^{-3} \eta_{12}^{1-2} \) takes the value \( \pm 1 \) for case (a) and \( \pm i \) for (b) which as discussed before is equivalent to requiring the Majorana phase \( \alpha_{21} = 0, \pi \). The \( \mu-\tau \) reflection symmetry is known to lead to \( s_{23} = \frac{\pi}{4} \) and \( s_{13} \cos \delta = 0 \) [41]. Thus perturbations \( O \) correcting for the vanishing \( s_{13}^2 \) invariably lead to the maximal CP violation. We note that

- While the phase restriction \( p_1^{-3} \eta_{12}^{1-2} = \pm 1, i \) is necessary to obtain the maximal CP phase for the most general perturbations, there exists a special class of perturbations for which the maximal CP phase follows independent of of this. This happens when the matrix \( O \) is a pure rotation in the 13-plane. In this case, the phase matrix appearing on the right hand side of eq. (20) commutes with \( O \) and the Dirac CP phase
and the mixing angles become independent of the Majorana phases. In this case, the tribimaximal mixing matrix automatically leads to the $\mu$-$\tau$ reflection symmetry in the presence of perturbations. The neutrino mass matrix in this situation is invariant under $Z_2 \times H^\nu_{CP}$. This special case is already discussed in the literature \[20–26\]. However as shown here, the occurrence of the maximal phase is more general and one does not need to assume any unbroken $Z_2$ symmetry in order to get the above result which holds for arbitrary GenCP invariant perturbations.

- The lowest member of the $\Delta(6n^2)$ groups namely, $S_4$ contains the residual symmetry needed for the prediction $p_1^{-3}\eta^{\frac{l_1-l_2}{2}} = \pm 1$. The other residual symmetries predicting $p_1^{-1}\eta^{\frac{l_1-l_2}{2}} = \pm (\omega, \omega^2)$ and hence $p_1^{-3}\eta^{\frac{l_1-l_2}{2}} = \pm 1, \pm i$ arise in $\Delta(6n^2)$ series with $n = 6k, k = 1, 2...$ with the lowest order group in the series being $\Delta(216)$.

- The maximality of the phase essentially arises from the factor $i$ present in the third column of eq.(20) when the predicted phase $p_1^{-3}\eta^{\frac{l_1-l_2}{2}}$ is $\pm 1$. The relative factor of $i$ is essentially produced by the structure of the underlying $Z_2 \times Z_2$ symmetry. Such factors do not play any role in the CP violation at the leading order and the $U_{PMNS}$ matrix therefore is taken sometimes to be real in the literature. Here it plays an important role in giving CP violation when perturbations are introduced.

- The genesis of the $\mu$-$\tau$ reflection symmetry appearing here can be easily understood on general grounds. It is known that this symmetry can be obtained if the neutrino mass matrix in the flavour basis corresponding to the diagonal charged lepton mass matrix satisfies \[41\]

\[
S^T M_{\nu f} S = M_{\nu f}^* , \tag{24}
\]

where $S$ denotes a $Z_2$ symmetry which interchanges $\mu$ and $\tau$. It is not difficult to see that if the conditions outlined above are satisfied then one indeed gets eq.(24) as an effective symmetry of $M_{\nu f}$. This can be seen by expressing $X_{1\nu}$ in the flavour basis. It can be written in this basis as

\[
\tilde{X}_{1\nu} = U_i^\dagger X_{1\nu} U_i^* 
\]

where $U_i$ diagonalizes $T_i$ and hence $M_iM_i^\dagger$ is given by eq.(19). One then finds

\[
\tilde{X}_{1\nu} = U_\omega^\dagger P_{1\nu}^* X_{1\nu} P_{1\nu}^* U_\omega^* = U_\omega^\dagger \text{diag.}(q_1, q_2, q_3) U_\omega^* ,
\]

where

\[
q_1 = \eta^{x-2l_1}, \quad q_2 = \eta^{-2x-2\gamma_\nu}, \quad q_3 = \eta^{x+2\gamma_\nu+2l_2} .
\]

The zeroth order mixing would be TBM for $\gamma = (l_1 + l_2 + \gamma_\nu) = 0, n, 2n...$ and the triviality of $\alpha_{21}$ would follow if $l_1 - l_2 - 3(x + \gamma_\nu) = 0, n, 2n....$ It is easy to show that if both these conditions are satisfied then all $q_i$ are proportional to a complex phase in general and $\tilde{X}_{1\nu} \approx U_\omega^\dagger U_\omega^* = S$ and GenCP condition \[3\] is equivalent to eq.(24).
Two GenCP symmetries $X_{1\nu,2\nu}$ differ in their prediction of $\alpha_{31}$ at the leading order, see eq.(17). This leads to different predictions in these cases. Specifically, eq.(22) now becomes

$$|U_{\mu i}|^2 = \frac{1}{3} |\sqrt{2}c_{2i}O_{1i} + p_2^{-3}\eta^{\frac{1+i}{2}}O_{2i} - \sqrt{2}s_2O_{3i}|^2,$$

$$|U_{\tau i}|^2 = \frac{1}{3} |\sqrt{2}c_{3i}O_{1i} + p_2^{-3}\eta^{\frac{1-i}{2}}O_{2i} - \sqrt{2}s_3O_{3i}|^2. \quad (25)$$

This equation does not lead to the $\mu$-$\tau$ reflection symmetry even if $p_2^{-3}\eta^{\frac{1+i}{2}} = \pm 1$ and zeroeth order mixing is TBM, i.e. $s_2 = -s_3$.

The leading order prediction of the $Z_2 \times Z_2$ symmetry may not be TBM mixing or the phase $\alpha_{21}$ may not be trivial. In either case, one can get large but non-maximal phase even in the presence of very small perturbations. We will study such cases numerically.

V. $M_{\nu}$ WITH PERTURBED $Z_2 \times Z_2 \times H_{\nu}^{CP}$

In this section, we construct a general neutrino mass matrix $M_{\nu}$ with the broken $Z_2 \times Z_2$ but intact GenCP for the $\Delta(6n^2)$ groups and work out approximate expressions for the matrix which diagonalizes it. We explicitly discuss only the case of $X_{1\nu}$ in view of the fact that it can lead to near maximal $\delta$. The other symmetry $X_{2\nu}$ can be analogously discussed. The $U_{\text{PMNS}}$ matrix in eq.(20) contains three unknown mixing angles of the matrix $O$. One could analyze general predictions in terms of these angles. Here we adopt an alternative parameterization. Explicit representation of $S_{1\nu}, S_{2\nu}$, eq.(14) and CP operators $X_{1\nu}$ can be used to construct the $Z_2 \times Z_2 \times H_{\nu}^{CP}$ symmetric leading order neutrino mass matrix $M_{0\nu}$. Explicitly.

$$M_{0\nu} \equiv V_{0\nu}^* \text{diag.}(m_1, m_2, m_3) V_{0\nu}^\dagger = \begin{pmatrix} \frac{1}{2}(m_1 + m_3)\eta^{-x} & 0 & \frac{1}{2}(m_1 - m_3)\eta^{-\gamma - x} \\ 0 & m_2\eta^{2x+2\gamma} & 0 \\ \frac{1}{2}(m_1 - m_3)\eta^{-\gamma - x} & 0 & \frac{1}{2}(m_1 + m_3)\eta^{-2\gamma - x} \end{pmatrix}. \quad (26)$$

The most general perturbation matrix $\delta M_{\nu}$ satisfying eq.(10) can be written, after appropriate redefinition of the unperturbed masses in eq.(26) as:

$$\delta M_{\nu} = m_3 \begin{pmatrix} -\epsilon_{33}\eta^{-x} & \epsilon_{12}\eta^{\gamma + \frac{x}{2}} & 0 \\ \epsilon_{12}\eta^{\gamma + \frac{x}{2}} & 0 & \epsilon_{23}\eta^{\frac{x}{2}} \\ 0 & \epsilon_{23}\eta^{x/2} & \epsilon_{33}\eta^{-x-2\gamma} \end{pmatrix}, \quad (27)$$

where $m_3$ is the heavier mass in the case of the normal hierarchy. $\delta M_{\nu}$ is characterized by three real parameters $\epsilon_{33}, \epsilon_{12}, \epsilon_{23}$. Eq.(9) assumes a simple form when transformed to a basis in which the unperturbed matrix $M_{0\nu}$ is diagonal:

$$\tilde{M}_{\nu} \equiv V_{0\nu}^T M_{\nu} V_{0\nu} = \begin{pmatrix} m_1 & m_3\frac{\epsilon_{12}+\epsilon_{23}}{\sqrt{2}} & -m_3\epsilon_{33} \\ m_3\frac{\epsilon_{12}+\epsilon_{23}}{\sqrt{2}} & m_2 & m_3\frac{\epsilon_{12}-\epsilon_{23}}{\sqrt{2}} \\ -m_3\epsilon_{33} & m_3\frac{\epsilon_{12}-\epsilon_{23}}{\sqrt{2}} & m_3 \end{pmatrix}. \quad (28)$$
$\hat{M}_\nu$ being real is diagonalized by an orthogonal matrix $O$ and thus $M_\nu$ is diagonalized by $V_\nu = V_0O$ leading to the $U_{PMNS}$ matrix as given in eq. (20).

These perturbations reduce to the known cases in specific limits. When $\epsilon_{12} = \epsilon_{23} = 0$, $O$ is a pure rotation in the $1-3$ plane with an angle $\phi_{13}$. One of the $Z_2 \times Z_2$ symmetry namely, the one corresponding to the tri-maximal mixing of the second column of the $U_{0PMNS}$ remains unbroken in this case. A different $Z_2$ remains unbroken when $\epsilon_{33} = 0$, $\epsilon_{23} = -\epsilon_{12}$. The first column of the PMNS matrix coincides with the zeroth order result in this case. Finally, the third column of the PMNS matrix remains unaffected when $\epsilon_{33} = 0$, $\epsilon_{12} = \epsilon_{23}$. In this case, $O$ is a pure rotation in the $1-2$ plane. Consequences of these single $Z_2 \times H^C_CP$ have been extensively studied in the so-called semi-direct approach in a number of works [20–26].

The matrix $O$ diagonalizing $\hat{M}_\nu$ can be determined perturbatively. We parameterize $O$ as

$$O = R_{23}(\phi_{23})R_{13}(\phi_{13})R_{12}(\phi_{12}) ,$$

where the rotation $R_{ij}(\phi_{ij})$ denotes a rotation in the $ij^{th}$ plane by an angle $\phi_{ij}$. $O$ satisfies

$$O^T \hat{M}_\nu O = \text{diag.}(m_{1\nu}, m_{2\nu}, m_{3\nu})$$

Mixing angles are approximately given by

$$\sin \phi_{23} \approx \frac{m_3}{(m_3 - m_2)} \epsilon_+ - \frac{m_3^2}{(m_3 - m_1)(m_3 - m_2)} \epsilon_+ \epsilon_{33} ,$$

$$\sin \phi_{13} \approx \frac{m_3}{(m_3 - m_1)} \epsilon_{33} - \frac{m_3^2}{(m_3 - m_1)(m_3 - m_2)} \epsilon_+ \epsilon_- ,$$

$$\sin \phi_{12} \approx -\frac{m_3}{(m_2 - m_1)} \epsilon_+ - \frac{m_3^2}{(m_3 - m_2)(m_2 - m_1)} \epsilon_- \epsilon_{33} .$$

The above mixing angles diagonalize $\hat{M}_\nu$ modulo $O(\epsilon^3)$ corrections. The neutrino masses receive corrections only at the second order in perturbation parameters and are given by:

$$m_{\nu_1} \approx m_1 - \frac{\epsilon_{33}^2 m_3^2}{m_3 - m_1} - \frac{\epsilon_+^2 m_3^2}{m_2 - m_1} ,$$

$$m_{\nu_2} \approx m_2 - \frac{\epsilon_+^2 m_3^2}{m_3 - m_2} + \frac{\epsilon_{33}^2 m_3^2}{m_2 - m_1} ,$$

$$m_{\nu_3} \approx m_3 + \frac{\epsilon_+^2 m_3^2}{m_3 - m_2} + \frac{\epsilon_{33}^2 m_3^2}{m_3 - m_1} .$$

It is seen that the mixing angles $\phi_{ij}$ not only depend on the strength of the perturbations $\epsilon_{ij}$ but also on the relative signs of the unperturbed masses $m_i$. Thus equal (opposite) signs of $m_i$ and $m_j$ tend to magnify or (suppress) the mixing angle $\phi_{ij}$ particularly for $\phi_{12}$. In this way, the effect of a small perturbations can get magnified. This allows one to obtain correct structure of the final mixing matrix even with very small perturbations as we will explicitly see in the numerical analysis to be presented in the next section.
VI. NUMERICAL RESULTS FOR SPECIFIC GROUPS

We now consider the four lowest groups of the $\Delta(6n^2)$ series and explore their predictions for the CP violating phases. We only consider the symmetry $X_{1\nu}$ which leads to rather large $\delta$ in many cases. The structure of the $U_0$ with the imposed residual symmetries is characterized by three integers $(n, \gamma, x)$ defined earlier. We shall choose specific values of $\gamma$ for a given $n$ such that the third column of the $U_0$ or its cyclic permutations provide a good zeroeth order approximations to the experimental values and then explore the influence of perturbation for each of the possible CP symmetries characterized by $x$. From now on, we specialize to the case with $l_1 = l_2 = 0$ in eq.(18) in which case $T_l$ is a $Z_3$ symmetry. Other choices of $l_1, l_2$ give equivalent results. The tri-bimaximal mixing is the only such possibility at the zeroeth order for the group $S_4$, obtained when $\gamma = 0$ in eq.(21). $\Delta(96)$ allows one more possibility with the third column $|U_{3i}|^2 = (0.044, 0.333, 0.622)^T$. This mixing pattern obtained with $\gamma = 1, n = 4$ (or their integer multiples) [6] is known as the Toorop,Feruglio, Hagedorn (TFH) mixing. The group $\Delta(384)$ contains additional possibility corresponding to $\gamma = 3$ which leads after permutation of the third column in eq.(20) to values $(0.011, 0.419, 0.569)^T$. This can fit the mixing angles $s_{13}, s_{23}$ when small perturbations are considered. The TFH mixing on the other hand requires somewhat larger corrections to $s_{213}$ and we do not consider it here.

Before discussing the general perturbations, let us recapitulate the consequences of the already studied restricted set of perturbations leading to the $Z_2 \times H_{CP}^\nu$ symmetry of the neutrino mass matrix. Three possible $Z_2$ symmetries corresponding to $O$ being a pure rotations in the $ij$th plane are denoted as $Z_{ij}$. Of these, the $Z_{12}$ symmetry leaves the third column invariant. Thus it can be phenomenologically consistent as exact symmetry only when the third column reproduces experimental values. The minimum even $n$ for which this happens is $n = 16$ which gives the third column as $(0.0253735, 0.376842, 0.597784)^T$. As already proven in the earlier section, the $Z_{13}$ symmetry with the tri-bimaximal mixing at zeroth order always leads to the $\mu-\tau$ reflection symmetry for all the residual symmetry groups labeled as $(n, \gamma, x) = (n, 0, x)$. Thus one needs perturbations which break the $Z_{13}$ symmetry if the original mixing is tri-bimaximal. The possibility $(n, \gamma, x) = (8, 3, x)$ allowed for $\Delta(384)$ does not lead to the $\mu-\tau$ reflection symmetric $U_0$ and hence also $U$. With $Z_{13}$ imposed, one gets for this case $(s_{23}^2, |\sin \delta|) = (0.43, 0.72)$ independent of $x$.

Implications of the $Z_{23}$ symmetries are quite different. This symmetry implies the following correlations between $\cos \delta$ and the atmospheric mixing angle:

$$\cos \delta = \frac{(c_{23}^2 - s_{23}^2)(c_{12}^2 s_{13}^2 - s_{12}^2)}{4c_{12}^2 c_{23} s_{12} s_{13} s_{23}}. \quad (33)$$

This correlation is true for the case with tree level TBM mixing. Other choices of residual symmetries lead to corrections to it which depend on the angle $\theta_1$ defined in eq.(21) [26]. Eq.(33) has been noticed before [42] and it implies negative $\cos \delta$ for $\theta_{23} < \frac{\pi}{4}$. This is quite consistent with indication of $s_{23}^2 < 1/2$ and $\delta \approx \frac{3\pi}{2}$ at T2K and NOνA but this relation by itself cannot fix the quadrant in which $\delta$ lies. This requires the knowledge of the sign of $\sin \delta$
as well and hence of the Jarlskog invariant $J = Im[U_{12}U_{23}U_{13}^*U_{22}^*]$. The sign of $J$ depends on
the ordering of rows of $U_{sym}$ which is not fixed by the symmetry. One can however derive
the following relation in case of $U_0$ having the TBM form:

$$J_{c_2} = 12 Re[p_{-3}^3] - 12 Im[p_{-3}^3].$$

This relation is invariant under the interchange of the second and the third row of $U_{sym}$. Moreover, just like eq.(33), this relation is also independent of the unknown angle $\phi_{23}$ which defines the $Z_{23}$ symmetry. It then follows that the sign of $J$ is essentially determined by the
group theoretical factor $p_{-3}^3$ and the quadrant of $2\theta_{23}$. Eqs. (33,34) together serve to fix the
quadrant in which $\delta$ lies.$^3$

We collect in Table I values of the predicted $s_{23}^2$, $\sin\delta$ for various choices of the neutrino
residual symmetries. The corresponding symmetry $T_l$ for the charged leptons is taken as
$Z_3$. We have determined $\phi_{23}$ through fits to three mixing angles as determined in the global
analysis of $^{[34]}$ choosing the solution corresponding to $\theta_{23} < \pi 4$. It is seen that TBM mixing
and $Z_{23}$ symmetry do not give $s_{23}^2$ within $3\sigma$ at the minimum for most choices of the residual
symmetries. Only exception being $(8,0,3), (8,0,5)$. These two cases lead to large $|\sin\delta|$ but
opposite values of $\sin\delta$. $\cos\delta$ always remains negative in accordance with the relation (33).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
$(n, \gamma, x)$ & $(s_{23}^2, |\delta|)$ \\
\hline
$(2,0,1),(6,0,1),(6,0,5)$ & $(0.30,0)$ \\
\hline
$(4,0,1)$ & $(0.36,227.6')$ \\
$(4,0,3)$ & $(0.36,132.4')$ \\
\hline
$(8,0,3)$ & $(0.42,110.7')$ \\
$(8,0,5)$ & $(0.42,249.3')$ \\
\hline
$(8,0,1)$ & $(0.31,155.7')$ \\
$(8,0,7)$ & $(0.31,204.3')$ \\
\hline
$(8,3,1)$ & $(0.48,0')$ \\
$(8,3,0)$ & $(0.46,336')$ \\
$(8,3,2)$ & $(0.46,24')$ \\
\hline
$(8,3,3)$ & $(0.47,346.8')$ \\
$(8,3,7)$ & $(0.47,13.2')$ \\
\hline
$(8,3,4)$ & $(0.48,5.9')$ \\
$(8,3,6)$ & $(0.48,354.1')$ \\
\hline
$(8,3,5)$ & $(0.43,41.2')$ \\
\hline
\end{tabular}
\caption{Values of $(s_{23}^2, \delta)$ implied by the best fit solution in case of the $Z_{23}$ symmetry.}
\end{table}

$^3$ The RHS of eq.(34) would change the sign if the leptonic doublets are assigned to a 3-dimensional repre-
sentation conjugate to the one used here since $U_{sym}$ in this case would go to its conjugate.
In the alternative case with \((n, \gamma, x) = (8, 3, x)\), the \(Z_{23}\) symmetry can give \(s_{23}^2\) within 3\(\sigma\) for all \(x\) of these several \(x\) predict relatively large \(\sin \delta\) as shown in Table \[II\].

We now discuss the effects of adding sizable perturbations to the above mentioned symmetrical limits. One can identify three physically interesting cases: (A) \(\epsilon_{12} + \epsilon_{23} = 0\) (B) \(\epsilon_{33} = 0\) and (C) \(\epsilon_{12} - \epsilon_{23} = 0\). Imposing any two of them simultaneously correspond to imposing various \(Z_2 \times H_\nu^{CP}\) symmetries. Choosing only one would amount to a single parameter perturbation to these \(Z_2\). We shall first do this exercise in two of the cases (A) and (B). Since both the magnitudes and signs of the unperturbed masses \(m_i\) play an important role in determining values of the perturbed mixing angles \(\phi_{ij}\) defined in eq.\[31\], we take these masses and two of the parameters \(\epsilon_{ij}\) as defined in the above cases as inputs. All the three residual symmetries are still broken in both the cases. We assume the normal hierarchy and fit these parameters to the results of the global analysis \[34\] which includes the latest results from T2K and NO\(\nu\)A for various possible values of \((n, \gamma, x)\). The CP violating phase is not included in the fit and thus can be regarded as a prediction.

\[
\begin{array}{|c|c|c|}
\hline
(n, \gamma, x) & \text{Case A} & \text{Case B} \\
\hline
(2,0,1) & \begin{align*}
\epsilon_{23} = -\epsilon_{12}, \epsilon_{33} \neq 0 \\
(m_1, m_2, m_3) &= (0.0251, 0.0256, 0.0548) \\
(s_{23}^2, |\sin \delta|, \chi_{\text{min}}^2) &= (0.455, 0.979, 6.01)
\end{align*} & \begin{align*}
\epsilon_{12}, \epsilon_{23} &= (-0.0999, -0.0388) \\
(m_1, m_2, m_3) &= (-0.0706, 0.07208, 0.08626) \\
(s_{23}^2, |\sin \delta|, \chi_{\text{min}}^2) &= (0.301, 0.053, 48.32)
\end{align*} \\
(4,0,1) & \begin{align*}
\epsilon_{23}, \epsilon_{33} &= (-0.0799, 0.00999) \\
(m_1, m_2, m_3) &= (-0.0146, 0.0183, 0.0513) \\
(s_{23}^2, |\sin \delta|, \chi_{\text{min}}^2) &= (0.403, 0.905, 2.285)
\end{align*} & \begin{align*}
\epsilon_{12}, \epsilon_{23} &= (0.0316, -0.0588) \\
(m_1, m_2, m_3) &= (-0.054, 0.05588, 0.0727) \\
(s_{23}^2, |\sin \delta|, \chi_{\text{min}}^2) &= (0.357, 0.732, 15.67)
\end{align*} \\
(8,0,5) & \begin{align*}
\epsilon_{23}, \epsilon_{33} &= (0.03316, 0.0447) \\
(m_1, m_2, m_3) &= (-0.0626, 0.06414, 0.07954) \\
(s_{23}^2, |\sin \delta|, \chi_{\text{min}}^2) &= (0.431, 0.95, 0.032)
\end{align*} & \begin{align*}
\epsilon_{12}, \epsilon_{23} &= (-0.02754, 0.035498) \\
(m_1, m_2, m_3) &= (-0.0707, 0.07223, 0.08585) \\
(s_{23}^2, |\sin \delta|, \chi_{\text{min}}^2) &= (0.422, 0.935, 0.31)
\end{align*} \\
(8,3,0) & \begin{align*}
\epsilon_{23}, \epsilon_{33} &= (-0.0087, -0.0908) \\
(m_1, m_2, m_3) &= (0.009762, 0.01270, 0.05081) \\
(s_{23}^2, |\sin \delta|, \chi_{\text{min}}^2) &= (0.430, 0.693, 0.00015)
\end{align*} & \begin{align*}
\epsilon_{12}, \epsilon_{23} &= (0.0273, -0.0275) \\
(m_1, m_2, m_3) &= (0.04424, 0.045394, 0.006827) \\
(s_{23}^2, |\sin \delta|, \chi_{\text{min}}^2) &= (0.46, 0.41, 2.86)
\end{align*} \\
(8,3,5) & \begin{align*}
\epsilon_{23}, \epsilon_{33} &= (0.05436, 0.0011603) \\
(m_1, m_2, m_3) &= (0.0323, 0.034254, 0.046) \\
(s_{23}^2, |\sin \delta|, \chi_{\text{min}}^2) &= (0.426, 0.659, 0.046)
\end{align*} & \begin{align*}
\epsilon_{12}, \epsilon_{23} &= (0.0484, -0.0561) \\
(m_1, m_2, m_3) &= (0.03776, 0.0377756, 0.0616487) \\
(s_{23}^2, |\sin \delta|, \chi_{\text{min}}^2) &= (0.426, 0.675, 0.060)
\end{align*} \\
\hline
\end{array}
\]

TABLE II. Results of fits with two parameter perturbations to neutrino residual symmetries labeled by \((n, \gamma, x)\) and contained in \(\Delta(6n^2)\) groups for \(n = 2, 4, 6, 8\). Two separate cases labeled as \(A\) and \(B\) are explained in the text. Table gives input parameters determined from the fits to neutrino parameters as determined in the global analysis of \[34\]. The masses are in eV units. We also show the predicted values of \(s_{23}^2, |\sin \delta|\) at the minimum.

Most of the cases correspond to the TBM mixing since this is the only possibility which can be cured by small perturbations for \(\Delta(6n^2)\) with \(n = 2, 4, 6\). As a measure of the
smallness of parameters, we have imposed the restriction $|\epsilon_{ij}| < 0.1$ on the perturbation parameters. In addition, we also impose the constraint that the sum of the fitted neutrino masses should be less than 0.23 eV \[13\]. The results of this analysis are summarized in Table II. The Majorana phase $p_1$ is the only controlling parameter distinguishing various symmetries. In case of the TBM, $p_1$ depends only on the ratio $x/n$, all the symmetries labeled by $(n,0,x)$ and $(mn,0,mx)$ with integer $m$ lead to the same result and we have listed the lowest member in cases presented in the table. We do not display all the cases, but present only some specific interesting examples. It is found that in most of the cases, the obtained minimum displays either approximate $Z_{13}$ or $Z_{23}$ symmetry although neither was imposed to start with. The sub-dominant contributions however play important roles in improving fits as discussed below. We first discuss case (A).

**Case A**

- $(n, \gamma, x) = (2, 0, 1)$ is the only possibility within the smallest group $S_4$ which does not give the exact $\mu$-$\tau$ reflection symmetry for arbitrary values of $\epsilon_{ij}$. One gets vanishing $\delta$ in this case if $Z_{23}$ symmetry is exact and maximal for the exact $Z_{13}$ symmetry. When both are broken and $\epsilon_{33}$ and $\epsilon_{23}$ are present one gets departures from the exact $\mu$-$\tau$ reflection symmetry and quite large $|\sin \delta|$. This can be attributed to more dominant $\epsilon_{33}$ compared to $\epsilon_{23}$ at the minimum. The latter however leads to the required departures from the maximal value of $\theta_{23}$.

- The next case is the group $\Delta(96)$ with $(n, \gamma, x) = (4, 0, x)$. The cases $(n, \gamma, x) = (4, 0, 1), (4, 0, 3)$ give identical $s_{23}^2$ while $(n, \gamma, x) = (4, 0, 0), (4, 0, 4)$ give the exact $\mu$-$\tau$ reflection symmetry. The result for the $(4, 0, 1)$ is displayed in the Table II. The exact $Z_{23}$ symmetry with $\epsilon_{33} = 0$ does not give $s_{23}^2$ within the $3\sigma$ and exact $Z_{13}$ predicts the $\mu$-$\tau$ reflection symmetry. This changes when both $\epsilon_{33}$ and $\epsilon_{23}$ are present. Now both have comparable values at the minimum and the resulting value of $s_{23}^2$ differs from the exact $Z_{23}$ or $Z_{13}$ symmetry.

- The next group in the series is $\Delta(216)$ and different cases are distinguished by values of $x$ in $(n, \gamma, x) = (6, 0, x)$. The cases with $x = 0, 2, 4, 6$ all give the exact $\mu$-$\tau$ reflection symmetry for any values of $\epsilon_{ij}$ as already argued. The remaining two cases $x = 1$ and $x = 3$ give identical $s_{23}^2$. These case are identical to the results of $(2, 0, 1)$ and are not displayed.

- The next group is $\Delta(384)$ with $n = 8$. This allows two possibilities namely, $\gamma = 0$ and $3$ both of which give quite good zeroeth order result. For this group, all the cases with $(n, \gamma, x) = (8, 0, x)$ for $x = 1 – 6$ give very good fit and predict large $|\sin \delta|$. The case with $x = 7$ on the other hand lead to a good fit but predict relatively small $|\sin \delta|$. The cases $(8, 0, 3)$ and $(8, 0, 5)$ give better fit than others. In these cases, neither $\epsilon_{33}$ nor $\epsilon_{23}$ dominates and they have comparable values.
The case with $\gamma = 3$ differs from the previous ones since the leading order mixing is not TBM and thus one always gets departures from the exact $\mu$-$\tau$ symmetric limit. One gets $s_{13}^2 \sim 0.011$ and $s_{23}^2 \sim 0.41$ at the leading order in this case. The obtained minimum in the presence of perturbations $\epsilon_{33}, \epsilon_{23}$ displays nearly $Z_{13}$ symmetry for all $x$ and result for a specific cases $(8, 3, 0), (8, 3, 5)$ are shown in the table. The obtained values are quite close to the $Z_{13}$ symmetric limit $s_{23}^2 \sim 0.426, |\sin \delta| \sim 0.72$.

**Case B**

- Unlike the case (A), addition of the parameter $\epsilon_{33}$ to the $Z_{23}$ symmetric case does not change results compared to the $Z_{23}$ symmetric case and the values of $(s_{23}^2, |\sin \delta|)$ are close to the ones displayed in Table [I]. Thus only, $(8, 0, 5), (8, 0, 3)$ case for the TBM give results with correct $s_{23}^2$. The case $(4, 0, 1)$ also displayed in the table however comes close to predicting $s_{23}^2$ with $3\sigma$. It also gives large $\sin \delta$.

- All the case $(n, \gamma, x) = (8, 3, x)$ not having TBM at the zeroth order can fit the angles very well and all the solutions display approximate $Z_{23}$ symmetry. But only the cases $(8, 3, 0), (8, 3, 2)$ and $(8, 3, 5)$ give large $|\sin \delta|$ respectively, 0.41 and 0.67. Solutions for $(8, 3, 0), (8, 3, 5)$ are displayed in Table [II].

We have taken only two of the three parameters as non-zero in the numerical fits presented above. It is important to consider the most general case with all the three parameters present and ask how far the above predictions remain true in the presence of the third parameter. Rather than fitting global $\chi^2$, we carried out a general analysis of this case by randomly varying all three parameters $\epsilon_{ij}$ in the range $-0.2 - 0.2$. The lowest mass mass $m_1$ is varied in the range $(-0.1 \sim 0.1)$ eV. The other two masses are chosen positive and $\leq 0.1$ eV. We worked out predictions for the CP phase in this situation by demanding that angles as well as the solar and atmospheric scales lie within their $3\sigma$ range as determined in [34]. This is done for all possible symmetry choices $(n, 0, x)$ (TBM) with $n = 2, 4, 6, 8$.

The numerical analysis of the cases (A) and (B) shows two patterns. There exists several residual symmetries, e.g. symmetries labelled by $(2, 0, 1)$ and $(8, 3, 0)$ in Table [II] for which the predicted $\delta$ at the minimum are quite different in two cases (A) and (B). In contrast symmetries $(4, 0, 1), (4, 0, 3), (8, 0, 3), (8, 0, 5), (8, 3, 5)$ predict similar values of $\delta$. It would be expected that the predictions of $\delta$ would lie in a narrow range in these cases when all the three parameters are present. This is indeed the case and we present predictions of the cases which lead to TBM mixing in the absence of perturbations. One finds definite correlations between $\theta_{23}$ and $\delta$ and these are displayed in Fig [I] for four specific choices of $(n, 0, x)$. For comparison, we also show the curves obtained in case of the $Z_{23}$ and $Z_{13}$ symmetry assuming best fit values for $\theta_{12}, \theta_{13}$ using the analytic expression as given in eqs.[33] for the $Z_{23}$ symmetry (continuous curve) and similar one obtained assuming $Z_{13}$ symmetry (dotted

---

4 Other examples of such symmetries are $(6, 0, 1), (6, 0, 5), (8, 0, 1), (8, 0, 7), (8, 3, 1), (8, 3, 4)$. 

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16
Remarkably, all the allowed points obtained these cases are clustered around the $Z_{23}$ symmetry curve even though this symmetry is not assumed to start with. Moreover, we find that the allowed values of $|\epsilon_{ij}|$ are comparable in large number of cases and are not restricted to the exact $Z_{23}$ symmetric solution $\epsilon_{12}=-\epsilon_{23}, \epsilon_{33}=0$ considered earlier. In spite of this, one seems to be getting an effective $Z_{23}$ symmetry. The reason can be understood from Fig.2 which displays variation of $\delta$ with the lowest mass $m_1$ for the same choices of $(n,0,x)$ as in the case of Fig.1. It is seen that most of the solutions correspond to quasi-degenerate spectrum and occur when $m_1$ is negative relative to $m_{2,3}$ which are assumed positive in the analysis. This results in effective suppression of $\phi_{12}, \phi_{13}$ compared to $\phi_{23}$ as seen from approximate expressions given in eq.(31). This results in effective $Z_{23}$. As explicitly seen in Fig.2, the predicted range of $\delta$ is characteristic of the underlying symmetries rather than the values of $\epsilon_{ij}$. The allowed points in Fig.1 are of two types. For the symmetries $(4,0,3), (8,0,5)$ one gets $\delta$ in the third (second) quadrant if $\theta_{23} < 1/2(\theta_{23} > 1/2)$. Reverse situation arises for the other symmetries $(4,0,1), (8,0,5)$. This is quite consistent with the expression eqs.(33,34) and reflect the approximate $Z_{23}$ symmetry.

Before we end the section, we give an explicit example from the points obtained in our random analysis. The chosen example corresponds to the neutrino residual symmetry $(n,\gamma,x) = (8,0,5)$ contained in the $\Delta(384)$ group. The following values of the input param-
FIG. 2. Plot of the lowest unperturbed neutrino mass $m_1$ versus $\delta$ for the residual symmetries labeled by $(n, \gamma, x) = (4, 0, 1)$ (magenta) and (4, 0, 3) (green), (8, 0, 3)(red) and (8, 0, 5)(blue). The allowed points reproduce all the three mixing angles in the 3$\sigma$ range as determined in global fits [34].

These values lead to the $U_{PMNS}$ matrix

$$U_{PMNS} = \begin{pmatrix} 0.827613 + 0.00472979i & -0.499131 - 0.212111i & -0.130806 - 0.0616621i \\ 0.192143 + 0.347111i & -0.0955472 + 0.653941i & 0.613431 - 0.171843i \\ 0.18395 - 0.351841i & 0.271839 - 0.44183i & 0.720233 + 0.233505i \end{pmatrix}.$$

(36)

This leads to

$$(s_{13}^2, s_{23}^2, s_{12}^2, \delta) = (0.0209, 0.4145, 0.3004, 247.9^\circ)$$

(37)

Interchanging the second and the third row of eq.(36) results in a solution with $s_{23}^2 = 0.585$ and $\delta = 67.9^\circ$. The perturbatively generated matrix $O$ in eq.(11) is given in this case by

$$O = \begin{pmatrix} 0.99963 & 0.0158604 & 0.0221184 \\ 0.0214073 & -0.960027 & -0.279087 \\ -0.0168078 & -0.279457 & 0.960011 \end{pmatrix}.$$

(38)

The role of the relative signs of the unperturbed masses is clear from this. The angle $\phi_{23}$ gets considerably enhanced compared to the basic parameter $\epsilon_-$ determining it and other
angles are relatively suppressed. As a result, $O$ correspond approximately to a rotation in the $2 - 3$ plane reflecting the $Z_{23}$ symmetry.

VII. SUMMARY

Flavour symmetries are widely used for understanding the observed patterns of neutrino mixing angles and phases. A very predictive theoretical hypothesis of invariance of $M_\nu$ ($M_lM_l^\dagger$) under residual $Z_2 \times Z_2$ ($Z_n$) symmetry leads to a prediction of the vanishing Dirac CP phase if these residual symmetries are embedded in the $\Delta(6n^2)$ groups. The neutrino Majorana phases can also be predicted by extending the neutrino symmetry to $Z_2 \times Z_2 \times H_{CP}^{\nu}$ [29]. An alternative approach known as semi-direct approach assumes the residual symmetry $Z_2 \times H_{CP}^{\nu}$ with $H_{CP}^{\nu}$ commuting with the $Z_2$ [20][26]. This can predict non-zero $\delta$.

We have presented here a simple, straightforward and predictive generalization of above schemes and explored it both analytically and numerically. This generalization is suitable to investigate the effects of perturbations to the original $Z_2 \times Z_2$ symmetry. These perturbations are assumed to respect the GenCP consistent with the original $Z_2 \times Z_2$ symmetry. As shown in section (2), one gets a simple expression for the $U_{PMNS}$ mixing matrix involving three unknown mixing angles and group theoretically determined parameters $p_1$ or $p_2$ and $\theta$ in this case. Using this, it is shown that $Z_2 \times Z_2$ symmetries leading to TBM mixing invariably lead to maximal Dirac phase if CP symmetry is unbroken in the symmetric case and Majorana phases assume specific values. This interesting result is essentially due to $\mu$-$\tau$ reflection symmetry which always arises in the said circumstances as an effective symmetry of the neutrino mass matrix in the flavour basis in all the relevant $\Delta(6n^2)$ groups.

The cases in which the unperturbed mixing matrix contain non-trivial Majorana phases are studied numerically for the $\Delta(6n^2)$ groups for $n = 2, 4, 6, 8$, and the predicted Dirac CP phases are worked out. As discussed in detail, there exist several residual symmetries for which the predicted CP phases are characteristic of the symmetry rather than the values of the perturbation parameters as long as these parameters are required to reproduce the other mixing angles correctly. One finds very definite correlations between the quadrants in which $2\theta_{23}$ and $\delta$ lie as displayed in Fig.1. These are explicitly worked out for the symmetries labeled as $(4, 0, 1), (4, 0, 3), (8, 0, 3), (8, 0, 5)$ all of which lead to the TBM mixing in the absence of perturbations. Interestingly, one finds a presence of underlying approximate $Z_{23} \times H_{CP}^{\nu}$ symmetry in these cases even when perturbations significantly break this symmetry. Similar correlations are expected to exist in case of other residual symmetries which do not give TBM, e.g. the symmetries labeled as $(8, 3, x)$ in $\Delta(384)$. These are not studied but can be explored using the present formalism. The present study was restricted to explore the consequences of the said assumptions from the symmetry considerations rather than building specific models.
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