Several integral inequalities and an upper bound for the bidimensional Hermite-Hadamard inequality

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Abstract

In this paper we prove several integral inequalities and we find an upper bound of the Hermite-Hadamard inequality for a convex function on a bounded area from the plane in special cases.

1 Introduction

Let $f$ be a convex function on $[a,b]$. Then we have the following inequality, which is called Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a)+f(b)}{2}.$$  

(1.1)

There are many extensions, generalizations and similar results of inequality (1.1). In [1], Fejer established the following weighted generalization of inequality (1.1).

**Theorem 1.1** If $f : [a, b] \to \mathbb{R}$ is a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right) \int_a^b w(x) \, dx \leq \int_a^b f(x)w(x) \, dx \leq \frac{f(a)+f(b)}{2} \int_a^b w(x) \, dx,$$

(1.2)

holds, where $w : [a, b] \to \mathbb{R}$ is non-negative, integrable and symmetric about $\frac{a+b}{2}$.

In [2], Yang and Tseng proved the following theorem which refines inequality (1.2).

**Theorem 1.2** Let $f$ and $w$ be defined as in Theorem 1.1. If $P : [a, b] \to \mathbb{R}$ is defined by

$$P(t) = \int_a^b f \left( tx + (1-t) \frac{a+b}{2} \right) w(x) \, dx,$$

then $P$ is convex, increasing on $[0, 1]$ and for all $t \in [0, 1]$,

$$f\left(\frac{a+b}{2}\right) \int_a^b w(x) \, dx = P(0) \leq P(t) \leq P(1) = \int_a^b f(x)w(x) \, dx.$$
In this paper, we find an upper bound for \( \int_{a}^{b} f(x)g(x) \, dx \), where \( f \) is a convex function on \([a, b]\) and \( g \) is non-negative increasing (or decreasing) on \([a, b]\), and \( \int_{a}^{b} g(t) \, dt = 1 \). Finally, in Section 3 we find an upper bound for the following integral:

\[
\frac{1}{\int_{a}^{b} h(x) \, dx} \int_{a}^{b} \int_{h(x)}^{g(x)} F(x, y) \, dy \, dx.
\]

### 2 Integral inequalities

**Theorem 2.1** Let \( f : [a, b] \to \mathbb{R} \) be a differentiable convex function and \( g : [a, b] \to [0, \infty) \) be a continuous function.

(i) If \( g \) is decreasing on \([a, b]\), then

\[
\frac{1}{\int_{a}^{b} g(x) \, dx} \int_{a}^{b} f(x)g(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

(ii) If \( g \) is increasing on \([a, b]\), then

\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{\int_{a}^{b} g(x) \, dx} \int_{a}^{b} f(x)g(x) \, dx.
\]

**Proof** (i) Denote

\[
H(x) = \int_{a}^{x} f(t)g(t) \, dt - \frac{1}{2} (f(a) + f(x)) \int_{a}^{x} g(t) \, dt.
\]

We will show that \( H'(x) \leq 0 \). We have

\[
H'(x) = f(x)g(x) - \frac{1}{2} f'(x) \int_{a}^{x} g(t) \, dt - \frac{1}{2} (f(a) + f(x))g(x) - \frac{1}{2} f'(x) \int_{a}^{x} g(t) \, dt
\]

\[
= \frac{1}{2} \left[ g(x)(f(x) - f(a)) - f'(x) \int_{a}^{x} g(t) \, dt \right].
\]

By the extended mean value theorem (Cauchy's theorem), we have

\[
\frac{f(x) - f(a)}{\int_{a}^{x} g(t) \, dt} = \frac{f'(\xi)}{g(\xi)} \quad (a < \xi < x).
\]

On the other hand, by the convexity of \( f \) and decreasing of \( g \), we obtain

\[
\frac{f(x) - f(a)}{\int_{a}^{x} g(t) \, dt} = \frac{f'(\xi)}{g(\xi)} \leq \frac{f'(x)}{g(x)}.
\]

Since \( g \) is non-negative,

\[
H'(x) = \frac{1}{2} \left[ (f(x) - f(a))g(x) - f'(x) \int_{a}^{x} g(t) \, dt \right] \leq 0,
\]

which implies that \( H \) is decreasing. Hence, \( H(b) \leq H(a) = 0 \). The proof is complete.
(ii) Denote

\[ H(x) = f\left(\frac{a + x}{2}\right) \int_a^x g(t) \, dt - \int_a^x f(t)g(t) \, dt. \]

Then we have

\[ H'(x) = \frac{1}{2} f'\left(\frac{a + x}{2}\right) \int_a^x g(t) \, dt + f\left(\frac{a + x}{2}\right)g(x) - f(x)g(x) \]

\[ = \frac{1}{2} f'\left(\frac{a + x}{2}\right) \int_a^x g(t) \, dt + \left(f(x) - f\left(\frac{a + x}{2}\right)\right). \]

By the mean value theorem (Lagrange's theorem), there exist \( \zeta_1 \in \left(\frac{a + x}{2}, x\right) \) and \( \zeta_2 \in (a, x) \) such that

\[ f(x) - f\left(\frac{a + x}{2}\right) = \frac{f'\left(\zeta_1\right)}{x - \frac{a + x}{2}} \quad \text{and} \quad \int_a^x g(t) \, dt - \frac{f(a + x)}{x - a} = g(\zeta_2). \]

Hence,

\[ 2\left(f(x) - f\left(\frac{a + x}{2}\right)\right) \int_a^x g(t) \, dt = f'\left(\zeta_1\right) \quad \text{and} \quad \frac{\int_a^x g(t) \, dt - \frac{f(a + x)}{x - a}}{x - a} = g(\zeta_2). \]

By the convexity of \( f \) and increasing of \( g \), we obtain

\[ 2\left(f(x) - f\left(\frac{a + x}{2}\right)\right) \int_a^x g(t) \, dt \geq \frac{f'\left(\zeta_1\right)}{g(\zeta_2)}. \]

So,

\[ H'(x) = \frac{1}{2} f'\left(\frac{a + x}{2}\right) \int_a^x g(t) \, dt - g(x) \left(f(x) - f\left(\frac{a + x}{2}\right)\right) \leq 0. \]

Therefore, \( H \) is decreasing and \( H(b) \leq H(a) = 0 \). The proof is complete. \( \square \)

**Theorem 2.2** Let \( f : [a, b] \to \mathbb{R} \) be a convex function and \( P : [a, b] \to [0, \infty) \) be an integrable function such that \( \int_a^b P(x) \, dx = 1 \). Then

\[ \int_a^b f(x)P(x) \, dx \leq \frac{bf(a) - af(b)}{b-a} + \frac{f(b) - f(a)}{b-a} \int_a^b xP(x) \, dx. \]

**Proof** We have

\[
\frac{1}{b-a} \int_a^b f(x)P(x) \, dx = \int_0^1 f(tb + (1-t)a)P(tb + (1-t)a) \, dt
\[
\leq f(b) \int_0^1 tP(tb + (1-t)a) \, dt + f(b) \int_0^1 (1-t)P(tb + (1-t)a) \, dt
\[
= f(b) \int_a^b \frac{x-a}{b-a} P(x) \, dx + f(a) \int_a^b \frac{b-x}{b-a} P(x) \, dx
\]

\[
= \frac{bf(a) - af(b)}{b-a} + \frac{f(b) - f(a)}{b-a} \int_a^b xP(x) \, dx
\]
Let us consider the bidimensional interval \( \Delta = [a, b] \times [c, d] \) in \( \mathbb{R}^2 \). Recall that the mapping \( f : \Delta \to \mathbb{R} \) is convex on \( \Delta \) if

\[
 f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) \leq \lambda f(x, y) + (1 - \lambda)f(z, w)
\]

holds for all \((x, y), (z, w) \in \Delta \) and \( \lambda \in [0, 1] \). A function \( f : \Delta \to \mathbb{R} \) is called co-ordinated convex on \( \Delta \) if the partial mappings \( f_x : [a, b] \to \mathbb{R}, f_x(u) = f(u, y) \) and \( f_y : [c, d] \to \mathbb{R}, f_y(v) = f(x, v) \) are convex for all \( y \in [c, d] \) and \( x \in [a, b] \). Note that every convex function \( f : \Delta \to \mathbb{R} \) is co-ordinated convex, but the converse is not generally true; see [3].

Dragomir in [4] established the following similar inequality of the Hermite-Hadamard inequality for a co-ordinated convex function on a rectangle from the plane \( \mathbb{R}^2 \).

**Theorem 3.1** Suppose that \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is co-ordinated convex on \( \Delta \). Then one has the inequalities

\[
 f\left(\frac{a + b}{2}, \frac{c + d}{2}\right) \leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \\
\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\]

Now, let \( \Delta \) be a convex area from the plane \( \mathbb{R}^2 \), bounded by a convex function \( y = h(x) \) and a concave function \( y = g(x) \) and \( x = a, x = b, \) such that for any \( x \in [a, b], g(x) \geq h(x) \).
Also, let $F$ be a two-variable convex function on $\Delta$. In [5] and [6], the following inequality is proved:

$$
\int_a^b \frac{1}{g(x) - h(x)} \int_a^b F(x, y) \, dy \, dx
\geq F\left( \int_a^b t(g(t) - h(t)) \, dt \right) - \frac{1}{2} \int_a^b (g^2(t) - h^2(t)) \, dt
\geq F\left( \int_a^b t(g(t) - h(t)) \, dt \right) - \frac{1}{2} \int_a^b (g(t) - h(t)) \, dt.
$$

In this paper, we want to find an upper bound for the integral

$$
\int_a^b \frac{1}{g(x) - h(x)} \int_a^b F(x, y) \, dy \, dx.
$$

For this purpose, we reach to the following integral:

$$
\int_a^b \frac{1}{g(x) - h(x)} \left[ F(x, g(x)) + F(x, h(x)) \right] (g(x) - h(x)) \, dx.
$$

It is well known that if $F(x, y)$ is increasing relative to $y$ and $y = h(x)$ is convex on $[a, b]$, then $F(x, h(x))$ is convex on $[a, b]$, but we have no information about the convexity of $F(x, h(x))$ generally. So, in special cases, we will find an upper bound for the integral (3.1).

**Theorem 3.2** Let $\Delta$ be a bounded area by a convex function $y = h(x)$ and a concave function $y = g(x)$ on $[a, b]$ such that for any $x \in [a, b]$, $g(x) \geq h(x)$ and $g - h$ is increasing on $[a, b]$. Also, let $F$ be a two-variable convex function on $\Delta$ such that $F(x, g(x))$ and $F(x, h(x))$ are convex on $[a, b]$. Then one has the inequality

$$
\int_a^b \frac{1}{g(t) - h(t)} \int_a^b F(x, y) \, dy \, dx
\leq \frac{1}{4} \left[ F(a, g(a)) + F(a, h(a)) + F(b, g(b)) + F(b, h(b)) \right].
$$

**Proof** Since $F$ is convex on $\Delta$, hence $F$ is co-ordinated convex on $\Delta$. So, $F_x : [h(x), g(x)] \rightarrow \mathbb{R}$, $F_x(y) = F(x, y)$ is convex on $[h(x), g(x)]$ for all $x \in [a, b]$. By the right-hand side of Hermite-Hadamard inequality (1.1), we have

$$
\int_{h(x)}^{g(x)} F(x, y) \, dy \leq (g(x) - h(x)) \left[ \frac{F(x, g(x)) + F(x, h(x))}{2} \right].
$$

Integrating this inequality on $[a, b]$, we obtain

$$
\int_a^b \frac{1}{g(t) - h(t)} \int_a^b F(x, y) \, dy \, dx
\leq \frac{1}{2} \int_a^b (g(x) - h(x)) \left( F(x, g(x)) + F(x, h(x)) \right) \, dx.
$$
Since \( g - h \) is increasing and \( F(x,g(x)), F(x,h(x)) \) are convex on \([a,b]\), by Theorem 2.1(i), we have

\[
\frac{1}{\int_{a}^{b} (g(t) - h(t)) \, dt} \int_{a}^{b} \left( g(x) - h(x) \right) \left( F(x,g(x)) + F(x,h(x)) \right) \, dx \\
\leq \frac{1}{2} \left[ F(a,g(a)) + F(a,h(a)) + F(b,g(b)) + F(b,h(b)) \right].
\]

The proof is complete. \(\square\)

**Theorem 3.3** Let \( \triangle \) be a bounded area by a convex function \( h \) and a concave function \( g \) on \([a,b]\) such that for any \( x \in [a,b] \), \( g(x) \geq h(x) \). Also, let \( F \) be a two-variable convex function on \( \triangle \) such that \( F(x,g(x)) \) and \( F(x,h(x)) \) are convex on \([a,b]\). Then one has the inequality

\[
\frac{1}{\int_{a}^{b} (g(t) - h(t)) \, dt} \int_{a}^{b} \int_{h(x)}^{g(x)} F(x,y) \, dy \, dx \\
\leq \frac{1}{2} \left[ \frac{b-a}{b-a} \left( F(a,g(a)) + F(a,h(a)) \right) + \frac{\alpha(b) - a}{b - a} \left( F(b,g(b)) + F(b,h(b)) \right) \right]
\]

where \( \alpha(b) = \frac{\int_{a}^{b} (g(t) - h(t)) \, dt}{\int_{a}^{b} (g(t) - h(t)) \, dt} \).

**Proof** By a similar way to the proof of Theorem 3.2, we have

\[
\frac{1}{\int_{a}^{b} (g(t) - h(t)) \, dt} \int_{a}^{b} \int_{h(x)}^{g(x)} F(x,y) \, dy \, dx \\
\leq \frac{1}{2} \left[ \frac{b-a}{b-a} \left( F(a,g(a)) + F(a,h(a)) \right) + \frac{\alpha(b) - a}{b - a} \left( F(b,g(b)) + F(b,h(b)) \right) \right]
\]

Since \( F(x,g(x)) + F(x,h(x)) \) is convex, by Theorem 2.2 (\( P(x) = \frac{g(x) - h(x)}{\int_{a}^{b} (g(x) - h(x)) \, dx} \)), we obtain

\[
\frac{1}{2} \left[ \frac{b-a}{b-a} \left( F(a,g(a)) + F(a,h(a)) \right) + \frac{\alpha(b) - a}{b - a} \left( F(b,g(b)) + F(b,h(b)) \right) \right]
\]

The proof is complete. \(\square\)

In the following theorem, we prove the assertion of Theorem 3.3 with weak conditions.
Theorem 3.4 Let $\Delta$, $g$ and $h$ be defined as in Theorem 3.3. Also, let $F$ be a two-variable convex function on $\Delta$ such that

$$\frac{\partial F(x,g(x))}{\partial g} \left( \frac{g(x) - g(a)}{x - a} - g'(x) \right) + \frac{\partial F(x,h(x))}{\partial h} \left( \frac{h(x) - h(a)}{x - a} - h'(x) \right) \leq 0,$$

then we have

$$\frac{1}{b-a} \int_a^b (g(x) - h(x)) \frac{dx}{x} \int_a^b F(x,y) dy dx \leq \frac{1}{2} \left[ (b - \alpha(b))(F(a,g(a)) + F(a,h(a))) + (\alpha(b) - a)(F(b,g(b)) + F(b,h(b))) \right] \frac{b-a}{b-a},$$

where $\alpha(b) = \frac{\int_a^b \frac{dg(t)}{dt} dt}{\int_a^b \frac{dh(t)}{dt} dt}$.

Proof Denote

$$H(x) = \int_a^x \frac{g(s)}{h(s)} f(t,y) dy dt - \frac{1}{2} K(x) \int_a^x (g(t) - h(t)) dt,$$

where

$$K(x) = \left( \frac{x - \alpha(x)}{x - a} \right) \left[ F(a,g(a)) + F(a,h(a)) \right] + \left( \frac{\alpha(x) - a}{x - a} \right) \left[ F(x,g(x)) + F(x,h(x)) \right].$$

Then we have

$$H'(x) = \int_a^x \frac{g(s)}{h(s)} F(x,y) dy - \frac{1}{2} K(x) (g(x) - h(x)) - \frac{1}{2} K'(x) \int_a^x (g(t) - h(t)) dt.$$

Since $F$ is convex, so it is co-ordinated convex. Hence, by the right-hand side of the Hermite-Hadamard inequality, we obtain

$$H'(x) \leq \frac{1}{2} (g(x) - h(x))(F(x,g(x)) + F(x,h(x)))$$

$$- \frac{1}{2} K(x) (g(x) - h(x)) - \frac{1}{2} K'(x) \int_a^x (g(t) - h(t)) dt.$$

So,

$$H'(x) \leq \frac{1}{2} \left[ (g(x) - h(x))(F(x,g(x)) + F(x,h(x)) - K(x)) - K'(x) \int_a^x (g(t) - h(t)) dt \right].$$

On the other hand, we have

$$\left[ \frac{x - \alpha(x)}{x - a} F(a,g(a)) + \frac{\alpha(x) - a}{x - a} F(x,h(x)) \right]'$$

$$= \frac{(1 - \alpha'(x))(x - a) - x + \alpha(x)}{(x - a)^2} F(a,g(a))$$

$$+ \frac{\alpha'(x)(x - a) - \alpha(x) + a}{(x - a)^2} F(x,g(x)) + F'(x,g(x)) \frac{\alpha(x) - a}{x - a}.$$
Now, multiplying each term by
\[
\int_a^x (g(t) - h(t)) \, dt
\]
and using the fact
\[
\int_a^x (g(t) - h(t)) \, dt \alpha(x) = \int_a^x t(g(t) - h(t)) \, dt,
\]
we obtain
\[
\int_a^x (g(t) - h(t)) \, dt \alpha'(x) = (g(x) - h(x))(x - \alpha(x)).
\]

Therefore,
\[
\int_a^x (g(t) - h(t)) \, dt \left[ \frac{x - \alpha(x)}{x - a} F(a, g(a)) + \frac{\alpha(x) - a}{x - a} F(x, h(x)) \right]
\]
\[
= \left[ -\frac{(g(x) - h(x))(x - \alpha(x))}{x - a} + \frac{\alpha(x) - a}{(x - a)^2} \int_a^x (g(t) - h(t)) \, dt \right] F(a, g(a))
\]
\[
+ \left[ \frac{(g(x) - h(x))(x - \alpha(x))}{x - a} + \frac{a - \alpha(x)}{(x - a)^2} \int_a^x (g(t) - h(t)) \, dt \right] F(x, h(x))
\]
\[
+ \int_a^x (g(t) - h(t)) \, dt F'(x, g(x)) \frac{\alpha(x) - a}{x - a}.
\]

By a similar way, we obtain
\[
\int_a^x (g(t) - h(t)) \, dt \left[ \frac{x - \alpha(x)}{x - a} F(a, h(a)) + \frac{\alpha(x) - a}{x - a} F(x, h(x)) \right]
\]
\[
= \left[ -\frac{(g(x) - h(x))(x - \alpha(x))}{x - a} + \frac{\alpha(x) - a}{(x - a)^2} \int_a^x (g(t) - h(t)) \, dt \right] F(a, h(a))
\]
\[
+ \left[ \frac{(g(x) - h(x))(x - \alpha(x))}{x - a} + \frac{a - \alpha(x)}{(x - a)^2} \int_a^x (g(t) - h(t)) \, dt \right] F(x, h(x))
\]
\[
+ \int_a^x (g(t) - h(t)) \, dt F'(x, h(x)) \left( \frac{\alpha(x) - a}{x - a} \right).
\]

Thus,
\[
\int_a^x (g(t) - h(t)) \, dt K'(x)
\]
\[
= \frac{(g(x) - h(x))(x - \alpha(x))}{x - a} \left[ F(x, g(x)) - F(a, g(a)) + F(x, h(x)) - F(a, h(a)) \right]
\]
\[
- \left( \frac{\alpha(x) - a}{(x - a)^2} \right) \int_a^x (g(t) - h(t)) \, dt \left[ F(x, g(x)) - F(a, g(a)) + F(x, h(x)) - F(a, h(a)) \right]
\]
\[
+ \left( \frac{\alpha(x) - a}{x - a} \right) \int_a^x (g(t) - h(t)) \, dt \left[ F'(x, g(x)) + F'(x, h(x)) \right].
\]
So,

\[
H'(x) \leq \frac{1}{2} \left[ (g(x) - h(x)) (F(x, g(x)) + F(x, h(x)) - K(x)) - K'(x) \int_0^x (g(t) - h(t)) \, dt \right] \\
= \frac{1}{2} \left[ (g(x) - h(x)) \frac{\alpha(x)}{x - a} \frac{x - \alpha(x)}{x - a} \right] \\
- \frac{1}{2} \left[ \frac{(g(x) - h(x))(x - \alpha(x))}{x - a} - \frac{\alpha(x) - a}{(x - a)^2} \int_0^x (g(t) - h(t)) \, dt \right] \\
\times \left[ F(x, g(x)) - F(x, h(x)) - F(a, g(a)) - F(a, h(a)) \right] \\
- \frac{1}{2} \int_a^x (g(t) - h(t)) \, dt \left( \frac{\alpha(x) - a}{x - a} \right) (F'(x, g(x)) + F'(x, h(x))).
\]

Then it follows that

\[
H'(x) \leq \frac{1}{2} \left[ F(x, g(x)) + F(x, h(x)) - F(a, g(a)) - F(a, h(a)) \right] \\
\times \left[ \frac{(g(x) - h(x)) - \alpha(x)}{x - a} \frac{x - \alpha(x)}{x - a} \right] \\
+ \frac{\alpha(x) - a}{(x - a)^2} \int_a^x (g(t) - h(t)) \, dt \\
- \frac{1}{2} \int_a^x (g(t) - h(t)) \, dt \left( \frac{\alpha(x) - a}{x - a} \right) (F'(x, g(x)) + F'(x, h(x))).
\]

Thus,

\[
H'(x) \leq \frac{1}{2} \left( \frac{\alpha(x) - a}{x - a} \right) \int_a^x (g(t) - h(t)) \, dt \\
\times \left[ \frac{F(x, g(x)) - F(a, g(a))}{x - a} + \frac{F(x, h(x)) - F(x, g(x))}{x - a} - F'(x, g(x)) - F'(x, h(x)) \right].
\]

Now, notice that if \( F(x, g(x)), F(x, h(x)) \) were convex on \([a, b]\), we can deduce the assertion of Theorem 3.3. Since \( F \) is convex on \( \triangle \), we have

\[
F(x, g(x)) - F(a, g(a)) \leq \frac{\partial F(x, g(x))}{\partial x} (x - a) + \frac{\partial F(x, g(x))}{\partial g} (g(x) - g(a))
\]
or

\[
\frac{F(x, g(x)) - F(a, g(a))}{x - a} \leq \frac{\partial F(x, g(x))}{\partial x} + \frac{\partial F(x, g(x))}{\partial g} \frac{(g(x) - g(a))}{x - a}.
\]

By a similar way, we have

\[
\frac{F(x, h(x)) - F(a, h(a))}{x - a} \leq \frac{\partial F(x, h(x))}{\partial x} + \frac{\partial F(x, h(x))}{\partial h} \frac{h(x) - h(a)}{x - a}.
\]
Note that
\[ F'(x, g(x)) = \frac{\partial F(x, g(x))}{\partial x} + \frac{\partial F(x, g(x))}{\partial g} g'(x) \]
and
\[ F'(x, h(x)) = \frac{\partial F(x, h(x))}{\partial x} + \frac{\partial F(x, h(x))}{\partial h} h'(x). \]

So,
\[
H'(x) \leq \frac{1}{2} \frac{\alpha(x) - a}{x - a} \int_a^x (g(t) - h(t)) \, dt \left[ \frac{\partial F(x, g(x))}{\partial x} \left( g(x) - g(a) \right) \right.
+ \frac{\partial F(x, h(x))}{\partial g} \left( x - a \right) - g'(x)
+ \frac{\partial F(x, h(x))}{\partial x} \left( h(x) - h(a) \right) - \frac{\partial F(x, h(x))}{\partial h} h'(x) - \left. \frac{\partial F(x, g(x))}{\partial g} g'(x) - \frac{\partial F(x, h(x))}{\partial x} - \frac{\partial F(x, h(x))}{\partial h} h'(x) \right].
\]

Thus,
\[
H'(x) \leq \frac{1}{2} \frac{\alpha(x) - a}{x - a} \int_a^x (g(t) - h(t)) \, dt \left[ \frac{\partial F(x, g(x))}{\partial x} \left( g(x) - g(a) \right) \right.
+ \frac{\partial F(x, h(x))}{\partial g} \left( x - a \right) - g'(x)
+ \frac{\partial F(x, h(x))}{\partial x} \left( h(x) - h(a) \right) - \frac{\partial F(x, h(x))}{\partial h} h'(x) \right] \leq 0.
\]

Note that \( \alpha(x) \geq a \). Therefore, \( H \) is decreasing and
\[
H(b) \leq H(a) = 0.
\]

The proof is complete. \( \square \)

**Remark 3.1** Notice that since \( g \) is concave and \( h \) is convex on \([a, b]\), so \( g' \) is decreasing and \( h' \) is increasing on \([a, b]\). By the mean value theorem, we have

\[
\frac{g(x) - g(a)}{x - a} - g'(x) \geq 0 \quad \text{and} \quad \frac{h(x) - h(a)}{x - a} - h'(x) \leq 0.
\]

In particular, if we have \( g(x) = mx + n \), then \( \frac{g(x) - g(a)}{x - a} - g'(x) = 0 \). So, if \( \frac{\partial F(x, h(x))}{\partial h} \geq 0 \), then
\[
\frac{\partial F(x, g(x))}{\partial g} \left[ \frac{g(x) - g(a)}{x - a} - g'(x) \right] + \frac{\partial F(x, h(x))}{\partial x} \left[ \frac{h(x) - h(a)}{x - a} - h'(x) \right]
= \frac{\partial F(x, h(x))}{\partial x} \left[ \frac{h(x) - h(a)}{x - a} - h'(x) \right] \leq 0.
\]

In the following theorem, we find an upper bound of the Hermite-Hadamard inequality for a co-ordinated convex function.
Theorem 3.5 Let $\triangle, g$ and $h$ be defined as in Theorem 3.3. Also, let $F$ be a convex function only relative to $y$, that is, $F_x : \{h(x), g(x)\} \to \mathbb{R}$, $F_x(v) = F(x, v)$ is convex for all $x \in [a, b]$. If $F''(x, g(x)) + F'(x, h(x)) \geq 0$, then

\[ \frac{1}{b-a} \int_a^b (g(t) - h(t)) \, dt \int_a^b F(x, y) \, dy \, dx \leq \frac{1}{2} [F(b, g(b)) + F(b, h(b))]. \]

Proof Denote

\[ H(x) = \int_a^x \int_{h(x)}^{g(x)} F(t, y) \, dy \, dt - \frac{1}{2} \int_a^b (g(t) - h(t)) \, dt [F(x, g(x)) + F(x, h(x))]. \]

We have

\[ H'(x) = \int_{g(x)}^{h(x)} F(x, y) \, dy - \frac{1}{2} (g(x) - h(x)) (F(x, g(x)) + F(x, h(x))) \]
\[ - \frac{1}{2} \int_a^b (g(t) - h(t)) \, dt (F'(x, g(x)) + F'(x, h(x))). \]

Since $F$ is convex relative to $y$, by the right-hand side of the Hermite-Hadamard inequality, we obtain

\[ H'(x) \leq \frac{1}{2} (g(x) - h(x)) (F(x, g(x)) + F(x, h(x))) \]
\[ - \frac{1}{2} (g(x) - h(x)) (F(x, g(x)) + F(x, h(x))) \]
\[ - \frac{1}{2} \int_a^b (g(t) - h(t)) \, dt (F'(x, g(x)) + F'(x, h(x))) \]
\[ = - \frac{1}{2} \int_a^b (g(t) - h(t)) \, dt (F'(x, g(x)) + F'(x, h(x))) \]
\[ \leq 0. \]

So, $H$ is decreasing on $[a, b]$. That is, $H(b) \leq H(a) = 0$. \[ \square \]

4 Examples

Example 4.1 Let $F(x, y) = x^2 + y^2$ and $\triangle$ be bounded by $g(x) = \sqrt{1-x^2}$, $h(x) = x - 1$ on $[0, 1]$. Then $g(x) - h(x) = \sqrt{1-x^2} - x + 1$ is decreasing on $[0, 1]$ and $F(x, g(x)) = 1$, $F(x, h(x)) = x^2 + (x - 1)^2$ are convex on $[0, 1]$. By Theorem 3.2, we have

\[ \frac{1}{\int_0^1 (\sqrt{1-x^2} - x + 1) \, dx} \int_0^1 \int_{x-1}^{\sqrt{1-x^2}} (x^2 + y^2) \, dy \, dx \]
\[ \leq \frac{1}{4} [F(0, g(0)) + F(0, h(0)) + F(1, g(1)) + F(1, h(1))]. \]

By easy calculation, we see that

\[ \int_0^1 (\sqrt{1-x^2} - x + 1) \, dx = \frac{\pi}{4} + \frac{1}{2} = \frac{\pi + 2}{4}. \]
and
\[
\frac{4}{\pi + 2} \int_0^1 \int_{x-1}^{x+1} (x^2 + y^2) \, dy \, dx \leq 1.
\]

**Example 4.2** Let \( F, g \) and \( h \) be defined as in Example 4.1. By Theorem 3.3, we have
\[
\alpha(1) = \frac{\int_0^1 t(\sqrt{1-t^2} - t + 1) \, dt}{\int_0^1 (\sqrt{1-t^2} - t + 1) \, dt} = \frac{5}{16} + \frac{x+1}{2}
\]
\[
= \frac{10}{3(\pi + 2)} \frac{1}{\int_0^1 (\sqrt{1-t^2} - t + 1) \, dt} \int_0^1 \int_{x-1}^{x+1} (x^2 + y^2) \, dy \, dx
\]
\[
\leq \frac{1}{2} \left[ \frac{3\pi - 4}{3(\pi + 2)} (F(0, g(0)) + F(0, h(0))) + \frac{10}{3(\pi + 2)} (F(1, g(1)) + F(1, h(1))) \right].
\]
So,
\[
\frac{4}{\pi + 2} \int_0^1 \int_{x-1}^{x+1} (x^2 + y^2) \, dy \, dx \leq 1.
\]

**Example 4.3** Let \( F(x, y) = x^2 + y^2 \) and \( \triangle \) be bounded by \( g(x) = x + 2, h(x) = x^2 \) on \([-1, 2]\). Then \( g - h \) is not decreasing on \([-1, 2]\) and also \( F(x, h(x)) = x^2 + x^4 \) is not convex on \([-1, 2]\). So, \( g, h \) and \( F \) do not hold in the hypothesis of Theorems 3.2 and 3.3. But we have
\[
g(x) - g(-1) \quad x + 1 - x - 1 = 0, \quad \frac{h(x) - h(-1)}{x + 1} - h'(x) = -x - 1 \leq 0
\]
and
\[
\frac{\partial F(x, g(x))}{\partial g} = 2(x+2), \quad \frac{\partial F(x, h(x))}{\partial h} = 2x^2.
\]
So,
\[
\frac{\partial F(x, g(x))}{\partial g} \left[ \frac{g(x) - g(-1)}{x + 1} - g'(x) \right] + \frac{\partial F(x, h(x))}{\partial h} \left[ \frac{h(x) - h(-1)}{x + 1} - h'(x) \right]
\]
\[
= 2x^2(-x-1) = 2x^3(x+1) \leq 0.
\]
Thus, we can apply Theorem 3.4
\[
\int_0^1 \int_{x+1}^{x+2} (x^2 + y^2) \, dy \, dx = \frac{1}{2} \left[ (2 - \alpha(2))(F(-1, h(-1)) + F(-1, g(-1))) + (2 + 1)(F(2, g(2)) + F(2, h(2))) \right],
\]
\[
\alpha(2) = \frac{\int_0^1 t(t+2-t^2) \, dt}{\int_0^1 (t+2-t^2) \, dt} = \frac{9}{10} = \frac{1}{2},
\]
\[
g(-1) = h(-1) = 1, \quad g(2) = h(2) = 4.
\]
Hence,
\[
\frac{2}{9} \int_{-1}^{2} \int_{x^2}^{x+y^2} (x^2 + y^2) \, dy \, dx \leq 11.
\]

**Example 4.4** Let \( F(x, y) = xy \) and \( \Delta \) be bounded by \( g(x) = x + 2 \), and \( h(x) = x^2 \) on \([-1, 2]\). Then \( F \) is not convex on \( \Delta \), but it is convex relative to \( y \), we have
\[
F(x, g(x)) = x^2 + 2x \quad \text{and} \quad F(x, h(x)) = x^3.
\]
So,
\[
F'(x, g(x)) + F'(x, h(x)) = 2x + 2 + 3x^2 > 0.
\]
Hence, by Theorem 3.5, we have
\[
\frac{1}{\int_{-1}^{2} (x + 2 - x^2) \, dx} \int_{-1}^{2} \int_{x^2}^{x+y^2} xy \, dy \, dx \leq \frac{1}{2} \left[ F(2, g(2)) + F(2, h(2)) \right].
\]
Hence,
\[
\frac{2}{9} \int_{-1}^{2} \int_{x^2}^{x+y^2} xy \, dy \, dx \leq 8.
\]

**Competing interests**
Authors declare that they have no competing interest.

**Authors' contributions**
Both the authors contributed equally in preparation as well as in typing and further both authors read the proof and approved the modifications.

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