A Note on the Solutions for a Higher-Order Convective Cahn–Hilliard-Type Equation

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Abstract: The higher-order convective Cahn–Hilliard equation describes the evolution of crystal surfaces faceting through surface electromigration, the growing surface faceting, and the evolution of dynamics of phase transitions in ternary oil-water-surfactant systems. In this paper, we study the \(H^3\) solutions of the Cauchy problem and prove, under different assumptions on the constants appearing in the equation and on the mean of the initial datum, that they are well-posed.

Keywords: existence; uniqueness; stability; higher-order convective cahn-hilliard type equation; cauchy problem

MSC: 35G25; 35K55

1. Introduction

In this paper, we study the well-posedness of the Cauchy problem:

\[
\begin{align*}
&\partial_t u + \kappa \partial_x u^2 + \gamma \partial_x^2 u - \beta \partial_x^4 u + \alpha \partial_x^6 u + \delta \partial_x^8 (u^3) = 0, \quad t > 0, \quad x \in \mathbb{R}, \\
&u(0, x) = u_0(x), \quad x \in \mathbb{R},
\end{align*}
\]

with

\[
\begin{align*}
&\kappa, \gamma, \beta, \alpha, \delta \in \mathbb{R}, \quad \gamma < 0, \quad \alpha \geq 0, \quad \kappa, \beta, \delta \neq 0, \quad \text{or}, \\
&\beta, \alpha, \delta \in \mathbb{R}, \quad \kappa = \gamma = 0, \quad \alpha \geq 0, \quad \beta, \delta \neq 0, \quad \text{or}, \\
&\kappa, \gamma, \beta, \alpha, \delta \in \mathbb{R}, \quad \beta \neq 0, \quad \delta = 0, \quad \text{or}, \\
&\kappa, \gamma, \beta, \alpha, \delta \in \mathbb{R}, \quad \beta, \delta \neq 0.
\end{align*}
\]

On the initial datum, we assume

\[
\begin{align*}
&u_0 \in H^3(\mathbb{R}), \quad \text{or}, \\
&u_0 \in H^3(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \int_{\mathbb{R}} u_0(x) dx = 0.
\end{align*}
\]

Inspired by [1–12], in light of (7), we define the following function:

\[
P_0(x) = \int_{-\infty}^{x} u_0(y) dy,
\]
on which we assume
\[ \| P_0 \|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \left( \int_{-\infty}^{x} u_0(y) dy \right)^2 dx < \infty. \]  

(9)

The equation in (1) is derived in [13] to model the evolution of a crystalline surface with small slopes that undergoes faceting. The unknown \( u \) gives the surface slope, the constant \( \kappa \) is proportional to the atomic flux deposition strength and the convective term \( \kappa \partial_x u^2 \) arises from the deposited atoms normal impingement. The sixth-order linear term \( \partial_x^6 u \) regularizes the equation, taking into account the surface curvature and the anisotropy of the surface energy under the surface diffusion.

From a mathematical point of view, the existence and uniqueness of weak solutions of (1) with periodic boundary conditions is proven in [14], under the assumptions \( \kappa > 0 \) and \( \gamma = 0 \). In the same setting, a similar result is proven in two space dimensions in [15]. In [16], the authors derived the stationary solutions of (1), again assuming \( \kappa > 0 \) and \( \gamma = 0 \). In [17], the existence of a global-in-time attractor is studied, while the well-posedness of the classical solutions of (1) is proven in [18], requiring (7)–(9), and \( \gamma = 0 \). In this paper, we will prove that, if (2) or (3) hold, we have the well-posedness of the classical solutions of (1) assuming (6), while if (5) holds, we have the well-posedness of (1) assuming (7)–(9).

Taking \( \kappa = 0 \), (1) becomes
\[ \partial_t u + \gamma \partial_x^2 u - \beta \partial_x^6 u + a \partial_x^4 u + \delta^2 \partial_x^4 \left( u^3 \right) = 0, \]  

(10)

which is a Cahn-Hilliard type equation [19–21]. It was deduced in [22] to describe the evolution of crystal surfaces faceting through surface electromigration. It also describes the phase transition development in ternary oil-water-surfactant systems. One part of the surfactant is hydrophilic, and the other one (termed amphiphile) is lipophilic. Oil, water, and microemulsion (i.e., a homogeneous, isotropic mixture of oil and water) can coexist in equilibrium. The unknown \( u \), in (10), gives the local difference between oil and water concentrations.

From a mathematical point of view, in [23] the initial-boundary-value problem for (10) is analyzed, under appropriate assumptions on \( \gamma, \beta, a, \delta \). In [24], the authors analyze the existence of a global-in-time attractor. The existence of weak solutions for the initial-boundary-value problem for (10) is proven in the case of degenerate mobility in [25]. Finally, in [18], the well-posedness of the classical solution of the Cauchy problem of (10) is proven, assuming (7)–(9), with \( \gamma = 0 \). In this paper, we will show that the classical solutions of the Cauchy problem of (10) are well-posed, assuming (6), if \( \gamma \leq 0 \) and \( a > 0 \), while in the general case, we will prove the same result assuming (7)–(9).

Observe that in [13], it is proven that, as \( \kappa \to \infty \), (1) reduces
\[ \partial_t u + \partial_x u^2 + a \partial_x^2 u - b \partial_x^4 u + c \partial_x^4 u = 0, \]  

(11)

which is known as the Kuramoto-Sivashinsky equation (see [26–28]). In Section 4, we will prove the well-posedness of the Cauchy problem for (11), assuming (6).

When \( \beta = \delta = 0 \) and \( a = f^2 \neq 0 \), (1) reads
\[ \partial_t u + \kappa \partial_x u^2 + \gamma \partial_x^2 u + f^2 \partial_x^4 u = 0. \]  

(12)

(12) appears in several physical situations; for example, it models long waves on a viscous fluid flowing down an inclined plane [29] and drift waves in a plasma [30]. (12) was also independently deduced by Kuramoto [27,31,32] to describe the phase turbulence in reaction-diffusion systems, and by Sivashinsky [28] to describe plane flame propagation, taking into account the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front.

Equation (12) can be used to study incipient instabilities in several physical and chemical systems [33–35]. Moreover, (12) is also termed the Benney-Lin equation [36,37], and was deduced by Kuramoto as a model for phase turbulence in the Belousov-Zhabotinsky reaction [38].
The existence and the dynamical properties of the exact solutions for (12) can be found in [39–44]. The control problem for (12) with periodic boundary conditions, and on a bounded interval, are studied in [45–47]. The problem of global-in-time exponential stabilization of (12) with periodic boundary conditions is analyzed in [48]. A generalization of the optimal control theory for (12) is proposed in [49], while the global boundary controllability of (12) is considered in [50]. The existence of solitonic solutions for (12) is proven in [51]. The well-posedness of the Cauchy problem for (12) is proven in [52–54], using the energy space technique, a priori estimates together with an application of the Cauchy-Kovalevskaya and the fixed point method, respectively. Instead, the initial-boundary value problem for (12) is studied, using a priori estimates together with an application of the Cauchy-Kovalevskaya, and the energy space technique in [55–57]. Inspired by [58–60], the convergence of the solution of (12) to the unique entropy one of the Burgers equation is proven in [61].

Finally, due to its general structure, we conjecture that (1) can have a possible application in machine learning (see [62,63]).

2. Results and Organization of the Paper

In this paper, we improve and complete the results of [14,16–18] working with $H^3$ initial data and having general assumptions on the constants appearing in (1). The main result of this paper is the following theorem. We prove the global-in-time existence, uniqueness, and stability of the solutions of the Cauchy problem (1).

**Theorem 1.** Fix $T > 0$. Assuming one of the following

(i) (2) and (6),

(ii) (3) and (6),

(iii) (4) and (6),

(iv) (5) and (7),

and (9), there exists a unique solution $u$ of (1) such that

$$u \in H^1((0,T) \times \mathbb{R}) \cap L^\infty(0,T; H^3(\mathbb{R})).$$

(13)

In particular, under the Assumptions (7) and (9), we have that

$$\int_\mathbb{R} u(t,x)dx = 0.$$  

(14)

Moreover, if $u_1$ and $u_2$ are two solutions of (1), we have that

$$\|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^2(\mathbb{R})} \leq e^{C(T)t} \|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{R})},$$

(15)

for some suitable $C(T) > 0$, and every $0 \leq t \leq T$.

The well-posedness of (1) is guaranteed for a short time by the Cauchy-Kowaleskaya Theorem [64]. The solutions are indeed global, thanks to suitable a priori estimates.

The paper is organized as follows. In Section 3 we prove Theorem 1, assuming (i) or (ii). In Section 4 we prove Theorem 1, assuming (iii). In Section 5 we prove Theorem 1, assuming (iv).

3. Proof of Theorem 1 Assuming (i) or (ii)

In this section, we prove Theorem 1, assuming (i) or (ii). For the sake of notational simplicity, define

$$\gamma = -a^2, \quad \alpha = b^2,$$

(16)
and then (1) reads
\[
\begin{aligned}
\begin{cases}
\partial_t u + \kappa \partial_x u^2 - a^2 \partial_x^2 u - \beta^2 \partial_t^2 u + b^2 \partial_x^4 u + \delta^2 \partial_x^4 (u^3) = 0, & t > 0, \quad x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\end{aligned}
\] (17)

Since the short time well-posedness of (17) is guaranteed by the Cauchy-Kowaleskaya Theorem [64], here we need to prove some suitable global a priori estimates.

From now on, we denote with \( C_0 \) the constants which depend only on the initial data, and with \( C(T) \), the constants which depend also on \( T \).

Following [65] (Lemma 2.2), we begin with the following energy estimate in the space \( L^\infty(0, \infty; H^1(\mathbb{R})) \cap L^2(0, \infty; H^2(\mathbb{R})) \).

**Lemma 1.** Assuming (2), for each \( t > 0 \), we have that
\[
\frac{\beta^2}{2} \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4} \| u(t, \cdot) \|^4_{L^4(\mathbb{R})} + \frac{b^2}{2} \| u(t, \cdot) \|^2_{L^2(\mathbb{R})} + K_1^2 \int_0^t \| \partial_x^2 u(s, \cdot) \|^2_{L^2(\mathbb{R})} ds + K_2^2 \int_0^t \| u(s, \cdot) \|_{L^2(\mathbb{R})} (s, \cdot) \|^2_{L^2(\mathbb{R})} ds
\]
\[
+ a^2 \partial_x^2 u(t, \cdot) \|_{L^2(\mathbb{R})} ds
\]
\[
+ \int_0^t \int_{\mathbb{R}} \left[ b^2 \partial_x^2 u + \delta^2 \partial_x^2 (u^3) - \beta^2 \partial_x^4 u \right] ds dx \leq C_0.
\] (18)

where \( K_1^2, K_2^2 \) are two appropriate positive constants.

Assuming (3), for each \( t > 0 \), we have that
\[
\frac{\beta^2}{2} \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4} \| u(t, \cdot) \|^4_{L^4(\mathbb{R})} + \frac{b^2}{2} \| u(t, \cdot) \|^2_{L^2(\mathbb{R})} + \int_0^t \int_{\mathbb{R}} \left[ b^2 \partial_x^2 u + \delta^2 \partial_x^2 (u^3) - \beta^2 \partial_x^4 u \right] ds dx \leq C_0.
\] (19)

Moreover, there exists \( C_0 > 0 \), independent on \( \kappa, a, b \), such that, for each \( t \geq 0 \),
\[
\| u(t, \cdot) \|_{L^\infty(\mathbb{R})} \leq C_0.
\] (20)

**Proof.** We begin by proving (18). Assume (2). Multiplying (17) by \( -\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u \), we have that
\[
\begin{aligned}
&\left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u \right) \partial_t u + 2\kappa \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u \right) u \partial_x u \\
-& a^2 \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u \right) \partial_x^2 u - \beta^2 \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u \right) \partial_x^4 u \\
+ & b^2 \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u \right) \partial_x^4 u + \delta^2 \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u \right) \partial_x^4 (u^3) = 0.
\end{aligned}
\] (21)

Performing some integration by parts, we gain
\[
\int_{\mathbb{R}} \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u \right) \partial_t u dx = \frac{d}{dt} \left( \frac{\beta^2}{2} \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4} \| u(t, \cdot) \|_{L^4(\mathbb{R})}^4 + \frac{b^2}{2} \| u(t, \cdot) \|^2_{L^2(\mathbb{R})} \right),
\]
\[
2\kappa \int_{\mathbb{R}} \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u \right) u \partial_x u dx = -2\kappa \beta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx,
\]
\[
- a^2 \int_{\mathbb{R}} \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u \right) \partial_x^2 u dx
\]
\[
\begin{align*}
&= a^2 \beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 3a^2 \delta^2 \left\| u(t, \cdot) \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + a^2 \beta^2 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\
&\quad - \beta^2 \int_\mathbb{R} \left( - \beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u \right) \partial_x^2 u dx \\
&\quad = - \beta^4 \int_\mathbb{R} \partial_x^4 u_3 \partial_x^2 u dx + \beta^2 \delta^2 \int_\mathbb{R} \partial_x u \partial_x^2 u dx + \beta^2 b^2 \int_\mathbb{R} \partial_x u_3 \partial_x^2 u dx \\
&\quad = \beta^4 \int_\mathbb{R} (\partial_x^4 u)_3 dx - \beta^2 \delta^2 \int_\mathbb{R} \partial_x u_3 \partial_x^2 u dx - \beta^2 a^2 \int_\mathbb{R} \partial_x^2 u_3 \partial_x^2 u dx, \\
&\quad b^2 \int_\mathbb{R} \left( - \beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u \right) \partial_x^2 u dx \\
&\quad = - \beta^2 b^2 \int_\mathbb{R} \partial_x^2 u_3 \partial_x^2 u dx - \beta^2 b^2 \int_\mathbb{R} \partial_x u \partial_x^3 u dx - b^4 \int_\mathbb{R} \partial_x u_3 \partial_x^3 u dx \\
&\quad = - \beta^2 b^2 \int_\mathbb{R} \partial_x^2 u_3 \partial_x^2 u dx + \delta^2 b^2 \int_\mathbb{R} \partial_x^4 u_3 \partial_x^2 u dx + b^4 \int_\mathbb{R} \partial_x^4 u_3 \partial_x^2 u dx, \\
&\quad \delta^2 \int_\mathbb{R} \left( - \beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u \right) \partial_x^2 u dx \\
&\quad = \beta^2 \delta^2 \int_\mathbb{R} \partial_x^2 u_3 (u^3) dx - \delta^4 \int_\mathbb{R} \partial_x (u^3) \partial_x^3 (u^3) dx - b^2 \delta^2 \int_\mathbb{R} \partial_x u \partial_x^3 (u^3) dx \\
&\quad = - \beta^2 \delta^2 \int_\mathbb{R} \partial_x^2 u_3 (u^3) dx + \delta^4 \int_\mathbb{R} \left( \partial_x^2 (u^3) \right)^2 dx + b^2 \delta^2 \int_\mathbb{R} \partial_x^2 u_3 \partial_x^2 (u^3) dx.
\end{align*}
\]

Therefore, thanks to (22), an integration of (21) on \(\mathbb{R}\) gives

\[
\frac{d}{dt} \left( \frac{\beta^2}{2} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{b^2}{2} \left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
+ \alpha^2 \beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 3a^2 \delta^2 \left\| u(t, \cdot) \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + a^2 \beta^2 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
+ \beta^4 \int_\mathbb{R} (\partial_x^4 u)_3 dx - 2b^2 \delta^2 \int_\mathbb{R} \partial_x^2 (u^3) \partial_x^2 u dx - 2b^2 a^2 \int_\mathbb{R} \partial_x^2 u_3 \partial_x^2 u dx \\
+ 2b^2 \delta^2 \int_\mathbb{R} \partial_x^2 (u^3) \partial_x^2 u dx + b^4 \int_\mathbb{R} (\partial_x^2 u_3)^2 dx + \delta^4 \int_\mathbb{R} \left( \partial_x^2 (u^3) \right)^2 dx \\
= 2 \kappa \beta^2 \int_\mathbb{R} u \partial_x u \partial_x^2 u dx.
\]

Since

\[
\begin{align*}
&\quad b^4 \int_\mathbb{R} \partial_x^2 u_3^2 dx + \delta^4 \int_\mathbb{R} \left( \partial_x^2 (u^3) \right)^2 dx + \beta^4 \int_\mathbb{R} (\partial_x^4 u)_3 dx \\
&\quad + 2b^2 \delta^2 \int_\mathbb{R} \partial_x^2 (u^3) \partial_x^2 u dx - 2b^2 a^2 \int_\mathbb{R} \partial_x^2 u_3 \partial_x^2 u dx - 2b^2 \delta^2 \int_\mathbb{R} \partial_x^2 (u^3) \partial_x^4 u dx \\
&= \int_\mathbb{R} \left[ b^2 \partial_x^2 u + \delta^2 \partial_x^2 (u^3) - \beta^2 \partial_x^4 u \right]^2 dx,
\end{align*}
\]

(23) becomes

\[
\frac{d}{dt} \left( \frac{\beta^2}{2} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{b^2}{2} \left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
+ \alpha^2 \beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 3a^2 \delta^2 \left\| u(t, \cdot) \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + a^2 \beta^2 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
+ \int_\mathbb{R} \left[ b^2 \partial_x^2 u + \delta^2 \partial_x^2 (u^3) - \beta^2 \partial_x^4 u \right]^2 dx = 2 \kappa \beta^2 \int_\mathbb{R} u \partial_x u \partial_x^2 u dx.
\]

Due to the Young inequality,
where $D_1$ is a positive constant, which will be specified later. Consequently, by (24),

\[
\frac{d}{dt} \left( \frac{\beta^2}{2} \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4} \| u(t, \cdot) \|^4_{L^4(\mathbb{R})} + \frac{b^2}{2} \| u(t, \cdot) \|^2_{L^2(\mathbb{R})} \right) \\
+ \beta^2 \left( a^2 - D_1 \right) \frac{\| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})}}{\| u(t, \cdot) \|^2_{L^2(\mathbb{R})}} + \left( 3\alpha^2 \delta^2 - \frac{\beta^2 \kappa^2}{D_1} \right) \| u(t, \cdot) \|_{L^2(\mathbb{R})}^2 \\
+ a^2 \delta^2 \| \partial_x u(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \int_{\mathbb{R}} \left[ b^2 \partial_x^2 u + \delta^2 \partial_x^2 \left( u^3 \right) - \beta^2 \partial_x^4 u \right] dx \leq 0,
\]

(25)

We search $D_1$, such that

\[ a^2 - D_1 > 0, \quad 3\alpha^2 \delta^2 - \frac{\beta^2 \kappa^2}{D_1} > 0. \]

(26)

Since, after a rescaling, $|a|$ can be taken very big, $D_1$ does exist and (26) holds. Therefore, by (25) and (26),

\[
\frac{d}{dt} \left( \frac{\beta^2}{2} \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4} \| u(t, \cdot) \|^4_{L^4(\mathbb{R})} + \frac{b^2}{2} \| u(t, \cdot) \|^2_{L^2(\mathbb{R})} \right) \\
+ K_1^2 \| \partial_x^2 u(t, \cdot) \|_{L^2(\mathbb{R})}^2 + K_2^2 \| u(t, \cdot) \|_{L^2(\mathbb{R})}^2 + a^2 b^2 \| \partial_x u(t, \cdot) \|_{L^2(\mathbb{R})}^2 \\
+ \int_{\mathbb{R}} \left[ b^2 \partial_x^2 u + \delta^2 \partial_x^2 \left( u^3 \right) - \beta^2 \partial_x^4 u \right] dx \leq 0,
\]

where $K_1^2, K_2^2$ are two appropriate positive constants. Integrating on $(0, t)$, by (6), we have that

\[
\frac{\beta^2}{2} \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4} \| u(t, \cdot) \|^4_{L^4(\mathbb{R})} + \frac{b^2}{2} \| u(t, \cdot) \|^2_{L^2(\mathbb{R})} \\
+ K_1^2 \int_0^t \| \partial_x^2 u(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds + K_2^2 \int_0^t \| u(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \\
+ a^2 b^2 \int_0^t \| \partial_x u(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \\
+ \int_0^t \int_{\mathbb{R}} \left[ b^2 \partial_x^2 u + \delta^2 \partial_x^2 \left( u^3 \right) - \beta^2 \partial_x^4 u \right] \, dsdx \leq C_0,
\]

that is, (18).

We continue by proving (19). Assume (3). Since $\kappa = \gamma = 0$, (17) reads

\[ \partial_t u - \beta^2 \partial_x^6 u + b^2 \partial_x^4 u + \delta^2 \partial_x^4 \left( u^3 \right) = 0 \]

(27)

. Multiplying (27), by $-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u$, arguing as in the previous case, an integration on $\mathbb{R}$ gives

\[
\frac{d}{dt} \left( \frac{\beta^2}{2} \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4} \| u(t, \cdot) \|^4_{L^4(\mathbb{R})} + \frac{b^2}{2} \| u(t, \cdot) \|^2_{L^2(\mathbb{R})} \right) \\
+ \int_{\mathbb{R}} \left[ b^2 \partial_x^2 u + \delta^2 \partial_x^2 \left( u^3 \right) - \beta^2 \partial_x^4 u \right] dx = 0.
\]

(6) and an integration on $(0, t)$ give (19).

Finally, we prove (20). Thanks to (18) or (19) and the Hölder inequality,

\[ |u(t, x)|^3 = 3 \left| \int_{-\infty}^{x} u^2 \partial_x u dy \right| \leq 3 \int_{\mathbb{R}} u^2 |\partial_x u| \, dx \]
\[ \leq 3 \| u(t, \cdot) \|^2_{L^4(\mathbb{R})} \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} \leq C_0. \]

Hence,
\[ \| u(t, \cdot) \|^3_{L^8(\mathbb{R})} \leq C_0, \]
which gives (20). \( \square \)

We continue by proving an \( L^2 \)– estimate, which is independent on \( \kappa, a, b \).

**Lemma 2.** Fix \( T > 0 \) and assume (2) or (3). There exists a constant \( C(T) > 0 \), independent on \( \kappa, a, b \), such that
\[ \| u(t, \cdot) \|^2_{L^2(\mathbb{R})} + 2\alpha^2 \int_{\mathbb{R}} \| \partial_x u(s, \cdot) \|^2_{L^2(\mathbb{R})} \, ds + \beta^2 \int_{\mathbb{R}} \| \partial_x^3 u(s, \cdot) \|^2_{L^2(\mathbb{R})} \, ds \]
\[ + 2b^2 \int_{0}^{t} \| \partial_x^2 u(s, \cdot) \|^2_{L^2(\mathbb{R})} \, ds \leq C(T), \]
\[ \int_{0}^{t} \| u(s, \cdot) (\partial_x u(s, \cdot))^2 \|^2_{L^2(\mathbb{R})} \, ds \leq C(T), \]
\[ \int_{0}^{t} \| \partial_x u(s, \cdot) \|^4_{L^4(\mathbb{R})} \, ds \leq C(T), \]
for every \( 0 \leq t \leq T \).

**Proof.** Let \( 0 \leq t \leq T \). We begin by observing that
\[ \partial_x u^3 = 3u^2 \partial_x u, \quad \partial_x^2 \left( u^3 \right) = 6u(\partial_x u)^2 + 3u^2 \partial_x^2 u. \]

Multiplying (17) by \( 2u \), an integration on \( \mathbb{R} \) and several integrations by part give
\[ \frac{d}{dt} \| u(t, \cdot) \|^2_{L^2(\mathbb{R})} = 2 \int_{\mathbb{R}} u \partial_t u \, dx \]
\[ = -4\kappa \int_{\mathbb{R}} u^2 \partial_x^2 u \, dx + 2\alpha^2 \int_{\mathbb{R}} u \partial_x^2 u \, dx + 2\beta^2 \int_{\mathbb{R}} u \partial_x^4 u \, dx \]
\[ - 2b^2 \int_{\mathbb{R}} u \partial_x^6 u \, dx - 2\delta^2 \int_{\mathbb{R}} u \partial_x^4 \left( u^3 \right) \, dx \]
\[ = -2\alpha^2 \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} - 2\beta^2 \int_{\mathbb{R}} \partial_x u \partial_x^2 u \, dx + 2b^2 \int_{\mathbb{R}} \partial_x u \partial_x^4 u \, dx \]
\[ + 2\delta^2 \int_{\mathbb{R}} \partial_x u \partial_x^4 \left( u^3 \right) \, dx \]
\[ = -2\alpha^2 \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + 2\beta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^2 u \, dx - 2b^2 \| \partial_x^2 u(t, \cdot) \|^2_{L^2(\mathbb{R})} \]
\[ - 2\delta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^2 \left( u^3 \right) \, dx \]
\[ = -2\alpha^2 \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} - 2\beta^2 \| \partial_x^2 u(t, \cdot) \|^2_{L^2(\mathbb{R})} - 2b^2 \| \partial_x^2 u(t, \cdot) \|^2_{L^2(\mathbb{R})} \]
\[ - 2\delta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^2 \left( u^3 \right) \, dx, \]
that is,
\[
\frac{d}{dt} \|u(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2\alpha^2 \|\partial_x u(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2 \|\partial_x^3 u(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2\delta^2 \|\partial_x^2 u(t, \cdot)\|^2_{L^2(\mathbb{R})} = -2\delta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^3 (u^3) \, dx.
\] (33)

Due to the Young inequality, we can estimate the right-hand side of (33), as follows:
\[
2\delta^2 \int_{\mathbb{R}} |\partial_x^2 u| |\partial_x^3 (u^3)| \, dx \leq \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \delta^4 \int_{\mathbb{R}} |\partial_x^2 (u^3)|^2 \, dx.
\]

Consequently, (33) becomes
\[
\frac{d}{dt} \|u(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2\alpha^2 \|\partial_x u(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2 \|\partial_x^3 u(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2\delta^2 \|\partial_x^2 u(t, \cdot)\|^2_{L^2(\mathbb{R})} \leq \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \delta^4 \int_{\mathbb{R}} |\partial_x^2 (u^3)|^2 \, dx.
\] (34)

Observe that, by (32),
\[
\int_{\mathbb{R}} \frac{\partial_x^2 (u^3)}{2} \, dx = 36 \int_{\mathbb{R}} u^4 (\partial_x u)^2 \, dx + 9 \int_{\mathbb{R}} u^2 (\partial_x u)^2 \partial_x^2 u \, dx
\]
\[
= 36 \int_{\mathbb{R}} u^2 (\partial_x u)^4 \, dx + 9 \int_{\mathbb{R}} u^2 (\partial_x u)^2 \partial_x^2 u \, dx
\]
\[
= 36 \int_{\mathbb{R}} u^2 (\partial_x u)^4 \, dx + 9 \int_{\mathbb{R}} u^4 (\partial_x^2 u)^2 \, dx.
\] (35)

Using (20),
\[
\int_{\mathbb{R}} \frac{\partial_x (u^3)}{2} \, dx \leq 9 \int_{\mathbb{R}} u^4 (\partial_x u)^4 \, dx + 9 \int_{\mathbb{R}} u^4 (\partial_x^2 u)^2 \, dx.
\] (36)

It follows from (34) and (36) that
\[
\frac{d}{dt} \|u(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2\alpha^2 \|\partial_x u(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2 \|\partial_x^3 u(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2\delta^2 \|\partial_x^2 u(t, \cdot)\|^2_{L^2(\mathbb{R})} \leq C_0 \|\partial_x^2 u(t, \cdot)\|^2_{L^2(\mathbb{R})}.
\] (37)

Since
\[
C_0 \|\partial_x^2 u(t, \cdot)\|^2_{L^2(\mathbb{R})} = C_0 \int_{\mathbb{R}} \partial_x^2 u \partial_x^3 u \, dx = -C_0 \int_{\mathbb{R}} \partial_x u \partial_x^4 u \, dx,
\]
Lemma 1 and the Young inequality give
\[
C_0 \|\partial_x^2 u(t, \cdot)\|^2_{L^2(\mathbb{R})} \leq 2 \int_{\mathbb{R}} \left| \frac{C_0 \partial_x u}{2\beta} \right| \beta \partial_x^3 u \, dx
\]
\[
\leq C_0 \|\partial_x u(t, \cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \|\partial_x^3 u(t, \cdot)\|^2_{L^2(\mathbb{R})}
\]
\[
\leq C_0 + \beta^2 \|\partial_x^3 u(t, \cdot)\|^2_{L^2(\mathbb{R})}.
\] (38)

Consequently, by (37),
\[
\frac{d}{dt} \|u(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2\alpha^2 \|\partial_x u(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2 \|\partial_x^3 u(t, \cdot)\|^2_{L^2(\mathbb{R})} \leq C_0 + \beta^2 \|\partial_x^3 u(t, \cdot)\|^2_{L^2(\mathbb{R})}.
\]
\[
+ 2b^2 \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \leq C(T).
\]

Integrating on \((0, t)\), by \((6)\), we have that
\[
\left\| u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + 2a \int_0^t \left\| \partial_x u(s, \cdot) \right\|^2_{L^2(\mathbb{R})} ds + \beta^2 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|^2_{L^2(\mathbb{R})} ds
+ 2b^2 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|^2_{L^2(\mathbb{R})} ds \leq C_0 + C(T)t \leq C(T),
\]

which gives \((28)\).

\((29)\) follows from \((28), (38)\) and an integration on \((0, t)\).

We prove \((30)\). We begin by observing that, by \((35)\) and \((36)\), we have that
\[
36 \int_\mathbb{R} u^2(\partial_x u)^4 dx \leq C_0 \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} - 9 \left\| u^2(t, \cdot) \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}
- 36 \int_\mathbb{R} u^2(\partial_x u)^2 \partial_x^2 u dx.
\]

Thanks to the Young inequality,
\[
36 \int_\mathbb{R} |u|^3(\partial_x u)^2 \partial_x^2 u |dx \leq 18 \int_\mathbb{R} u^2(\partial_x u)^4 dx + 18 \left\| u^2(t, \cdot) \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}.
\]

Consequently, by \((39)\),
\[
18 \int_\mathbb{R} u^2(\partial_x u)^4 dx \leq C_0 \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + 9 \left\| u^2(t, \cdot) \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}. \tag{40}
\]

Observe that, by \((20)\),
\[
9 \left\| u^2(t, \cdot) \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} = 9 \int_\mathbb{R} u^4(\partial_x^2 u)^2 dx \leq C_0 \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}.
\]

Therefore, by \((40)\),
\[
18 \left\| u(t, \cdot)(\partial_x u(t, \cdot))^2 \right\|^2_{L^2(\mathbb{R})} \leq C_0 \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}. \tag{41}
\]

Integrating \((41)\) on \((0, t)\), by \((29)\), we have \((30)\).

Finally, we prove \((31)\). We begin by observing that \([66] (Lemma 2.3)\) says that
\[
\left\| \partial_x u(t, \cdot) \right\|^4_{L^4(\mathbb{R})} \leq 6 \left( \left\| u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \left\| \partial_x u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \right) \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}.
\]

Therefore, by Lemma 1 and \((28)\), we have that
\[
\left\| \partial_x u(t, \cdot) \right\|^4_{L^4(\mathbb{R})} \leq C(T) \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}.
\]

Integrating on \((0, t)\), by \((29)\), we have \((31)\).

We continue with an a priori estimate in the space \(L^2(0, \infty; H^4(\mathbb{R}))\).
Lemma 3. Fix $T > 0$ and assume (2) or (3). There exists a constant $C(T) > 0$, independent on $\kappa, a, b$, such that

$$\left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2a^2 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds + \beta^2 \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds + 2b^2 \int_0^t \left\| \partial_x^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T)$$

(42)

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying (17) by $-2\partial_x^2 u$, an integration on $\mathbb{R}$ gives

$$\frac{d}{dt} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = -2 \int_{\mathbb{R}} \partial_x^2 u \partial_x u \, dx$$

$$= 4R \int_{\mathbb{R}} \partial_x u \partial_x^2 u \, dx - 2a^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\beta^2 \int_{\mathbb{R}} \partial_x^4 u \partial_x^2 u \, dx$$

$$+ 2\delta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u \, dx + 2\delta^2 \int_{\mathbb{R}} \partial_x^4 u \partial_x^2 u \left( u^3 \right) \, dx$$

$$= -2a^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \int_{\mathbb{R}} \partial_x^4 u \partial_x^2 u \, dx$$

$$- 2\beta \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\delta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u \left( u^3 \right) \, dx$$

$$= -2a^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\beta \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\delta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u \left( u^3 \right) \, dx,$$

that is,

$$\frac{d}{dt} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2a^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2$$

$$+ 2\delta^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = 4\kappa \int_{\mathbb{R}} \left( \partial_x u \right)^3 \, dx + 2\delta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u \left( u^3 \right) \, dx.$$

Due to Lemma 1, (36) and the Young inequality,

$$2\kappa \int_{\mathbb{R}} \left( \partial_x u \right)^3 \, dx \leq \kappa^2 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \left\| \partial_x u(t, \cdot) \right\|_{L^4(\mathbb{R})}^4$$

$$\leq C_0 + \left\| \partial_x u(t, \cdot) \right\|_{L^4(\mathbb{R})}^4,$$

$$2\delta^2 \int_{\mathbb{R}} \left| \partial_x^2 u \right| \left| \partial_x^4 u \right| \left( u^3 \right) \, dx = 2 \int_{\mathbb{R}} \left| \partial_x^2 u \right| \left| \frac{\partial_x^2 u}{\beta} \right| \left( u^3 \right) \, dx$$

$$\leq \beta \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\delta^4}{\beta^2} \int_{\mathbb{R}} \left| \partial_x^2 u \right| \left( u^3 \right) \, dx$$

$$\leq \beta \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C_0 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.$$

Therefore, by (43),

$$\frac{d}{dt} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2a^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2$$

$$+ 2b^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq \left\| \partial_x u(t, \cdot) \right\|_{L^4(\mathbb{R})}^4 + C_0 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.$$
Integrating on \((0, t)\), by (6), (29) and (31),
\[
\|\partial_x u(t, \cdot)\|_{L^2(R)}^2 + 2a^2 \int_0^t \|\partial_x^2 u(s, \cdot)\|_{L^2(R)}^2 \, ds + \beta^2 \int_0^t \|\partial_x^4 u(s, \cdot)\|_{L^2(R)}^2 \, ds
+ 2b^2 \int_0^t \|\partial_x^3 u(s, \cdot)\|_{L^2(R)}^2 \, ds
\leq C_0 + \int_0^t \|\partial_x u(s, \cdot)\|_{H^4(R)}^4 \, ds + C_0 \int_0^t \|\partial_x^2 u(s, \cdot)\|_{L^2(R)}^2 \, ds \leq C(T),
\]
which gives (42). \(\square\)

We continue with an a priori estimate in the space \(L^2(0, \infty; H^5(R))\).

**Lemma 4.** Fix \(T > 0\) and assume (2) or (3). There exists a constant \(C(T) > 0\), independent on \(\kappa, a, b\),
\[
\|\partial_x u\|_{L^\infty((0, T)\times R)} \leq C(T),
\]
\[
\|\partial_x^2 u(t, \cdot)\|_{L^2(R)}^2 + 2a^2 \int_0^t \|\partial_x^4 u(s, \cdot)\|_{L^2(R)}^2 \, ds
+ \beta^2 \int_0^t \|\partial_x^6 u(s, \cdot)\|_{L^2(R)}^2 \, ds
\leq C(T),
\]
for every \(0 \leq t \leq T\), where \(C(T)\) is independent on \(\kappa, a, b\).

**Proof.** Let \(0 \leq t \leq T\). We begin by observing that, by (32), we have that
\[
\partial_x^3 (u^3) = 6(\partial_x u)^3 + 18u\partial_x u \partial_x^2 u + 3u^2 \partial_x^3 u.
\]

Multiplying (17) by \(2\partial_x^4 u\), thanks to (46), an integration on \(R\) gives
\[
\frac{d}{dt} \|\partial_x^2 u(t, \cdot)\|_{L^2(R)}^2 = 2 \int_R \partial_x^4 u \partial_x^4 u dx
- 2 \int_R u \partial_x u \partial_x^4 u dx + 2a^2 \int_R \partial_x^2 u \partial_x^4 u dx + 2\beta^2 \int_R \partial_x^4 u \partial_x^6 u dx
- 2b^2 \|\partial_x^4 u(t, \cdot)\|_{L^2(R)}^2
- 2 \int_R u \partial_x^2 u \partial_x^4 u dx + 2\beta^2 \int_R \partial_x^4 u \partial_x^6 u dx
- 2 \int_R u \partial_x u \partial_x^4 u dx - 2\beta^2 \|\partial_x^4 u(t, \cdot)\|_{L^2(R)}^2
- 2 \int_R u \partial_x u \partial_x^4 u dx + 2\beta^2 \int_R \partial_x^4 u \partial_x^6 u dx
- 12 \int_R (\partial_x u)^3 \partial_x^3 u dx + 36 \int_R u \partial_x u \partial_x^3 u \partial_x^5 u dx
- 6 \int_R u^2 \partial_x^3 u \partial_x^5 u dx,
\]
that is,
\[
\frac{d}{dt} \|\partial_x^2 u(t, \cdot)\|_{L^2(R)}^2 + 2a^2 \|\partial_x^4 u(t, \cdot)\|_{L^2(R)}^2
+ 2\beta^2 \|\partial_x^6 u(t, \cdot)\|_{L^2(R)}^2
- 2 \int_R u \partial_x u \partial_x^4 u dx + 2\beta^2 \int_R \partial_x^4 u \partial_x^6 u dx
= -2 \int_R u \partial_x u \partial_x^4 u dx - 12 \int_R (\partial_x u)^3 \partial_x^3 u dx
\]
\[
-36\delta^2 \int_{\mathbb{R}} u \partial_u \omega_0^2 u \partial_u^3 u \, dx - 6\delta^2 \int_{\mathbb{R}} u^2 \partial_u^2 \omega_0^2 u \, dx.
\]

Due to Lemma 1 and the Young inequality,

\[
2|x| \int_{\mathbb{R}} |u| \left| \partial_x u \right| \left| \partial_x^4 u \right| \, dx \leq \kappa^2 \int_{\mathbb{R}} u^2 (\partial_x u)^2 \, dx + \left\| \partial_x^4 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]

\[
\leq \kappa^2 \left\| u (t, \cdot) \right\|_{L^\infty(0, T) \times \mathbb{R}} \left\| \partial_x u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \left\| \partial_x^4 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]

\[
\leq C_0 \left\| \partial_x u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \left\| \partial_x^4 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C_0 + \left\| \partial_x^4 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]

\[
12\delta^2 \int_{\mathbb{R}} |\partial_x u|^3 |\partial_x^3 u| \, dx = 2 \int_{\mathbb{R}} \frac{6\delta^2 (\partial_x u)^3}{\beta \sqrt{D_2}} \left| \beta \sqrt{D_2} \partial_x^3 u \right| \, dx
\]

\[
\leq \frac{36\delta^4}{\beta^2 D_2^2} \int_{\mathbb{R}} (\partial_x u)^4 \, dx + \beta^2 D_2 \left\| \partial_x^3 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]

\[
\leq \frac{36\delta^4}{\beta^2 D_2^2} \left\| \partial_x u \right\|_{L^\infty(0, T) \times \mathbb{R}} \left\| \partial_x^3 u (t, \cdot) \right\|_{L^4(\mathbb{R})}^4 + \beta^2 D_2 \left\| \partial_x^3 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]

\[
36\delta^2 \int_{\mathbb{R}} |u| |\partial_x u| \left| \partial_x^2 u \right| \, dx \leq 36\delta^2 \left\| u (t, \cdot) \right\|_{L^\infty(0, T) \times \mathbb{R}} \int_{\mathbb{R}} |\partial_x u \partial_x^2 u| |\partial_x^3 u| \, dx
\]

\[
\leq 2C_0 \int_{\mathbb{R}} |\partial_x u \partial_x^2 u| |\partial_x^3 u| \, dx = 2 \int_{\mathbb{R}} \left| C_0 \partial_x u \partial_x^2 u \right| \left\| \beta \sqrt{D_2} \partial_x^3 u \right| \, dx
\]

\[
\leq \frac{C_0}{D_2} \int_{\mathbb{R}} (\partial_x u)^2 (\partial_x^2 u)^2 \, dx + \beta^2 D_2 \left\| \partial_x^3 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]

\[
\leq \frac{C_0}{D_2} \left\| \partial_x u \right\|_{L^\infty(0, T) \times \mathbb{R}} \left\| \partial_x^3 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 D_2 \left\| \partial_x^3 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]

\[
6\delta^2 \int_{\mathbb{R}} u^2 |\partial_x^4 u| |\partial_x^3 u| \, dx \leq 6\delta^2 \left\| u (t, \cdot) \right\|_{L^\infty(0, T) \times \mathbb{R}} \int_{\mathbb{R}} |\partial_x^3 u| |\partial_x^3 u| \, dx
\]

\[
\leq 2C_0 \int_{\mathbb{R}} |\partial_x^3 u| |\partial_x^3 u| \, dx = 2 \int_{\mathbb{R}} \left| C_0 \partial_x^3 u \right| \left\| \beta \sqrt{D_2} \partial_x^3 u \right| \, dx
\]

\[
\leq \frac{C_0}{D_2} \left\| \partial_x^3 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 D_2 \left\| \partial_x^3 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
\]

where \(D_2\) is a positive constant, which will be specified later. Therefore, by (47),

\[
\frac{d}{dt} \left\| \partial_x^4 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2a^2 \left\| \partial_x^4 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]

\[
+ \beta^2 (2 - 3D_2) \left\| \partial_x^3 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\delta^2 \left\| \partial_x^4 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C_0 + \left\| \partial_x^4 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]

\[
+ \frac{36\delta^4}{\beta^2 D_2^2} \left\| \partial_x u \right\|_{L^\infty(0, T) \times \mathbb{R}} \left\| \partial_x u (t, \cdot) \right\|_{L^4(\mathbb{R})}^4 + \frac{C_0}{D_2} \left\| \partial_x^3 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
\]

and taking \(D_2 = \frac{1}{4}\)

\[
\frac{d}{dt} \left\| \partial_x^3 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2a^2 \left\| \partial_x^4 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]

\[
+ \beta^2 \left\| \partial_x^3 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\delta^2 \left\| \partial_x^4 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C_0 + \left\| \partial_x^4 u (t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{108\delta^4}{\beta^2} \left\| \partial_x u \right\|_{L^\infty(0, T) \times \mathbb{R}} \left\| \partial_x u (t, \cdot) \right\|_{L^4(\mathbb{R})}^4.
\]
Integrating on $(0,t)$, by (6), (29), (31) and (42), we obtain that
\[
\left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\alpha^2 \int_0^t \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, dt \leq C_0 + C_0 t + \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \\
+ \beta^2 \int_0^t \left\| \partial_x^5 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds + 2\beta^2 \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds.
\]
Hence,
\[
\left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T) \left( 1 + \left\| \partial_x^2 u \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 \right).
\]
which gives (44).

Finally, (45) follows from (44) and (48).

We prove (44). Thanks to Lemma 1, (48) and the Hölder inequality,
\[
(\partial_x u(t,x))^2 \leq 2 \int_{-\infty}^x |\partial_x^2 u|^2 dx \leq 2 \int_{\mathbb{R}} |\partial_x^2 u|^2 dx \leq 2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}
\]
\[
\leq C(T) \sqrt{\left( 1 + \left\| \partial_x^2 u \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 \right)}.
\]
Hence,
\[
\left\| \partial_x^4 u \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 \leq C(T) \left\| \partial_x^2 u \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 - C(T) \leq 0.
\]

We continue with an a priori estimate in the space $L^4(0,\infty;W^{2,4}(\mathbb{R}))$.

**Lemma 5.** Fix $T > 0$ and assume (2) or (3). There exists a constant $C(T) > 0$, independent on $\kappa$, $a$, $b$, such that
\[
\int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^4(\mathbb{R})}^4 \, ds \leq C(T),
\]
for every $0 \leq t \leq T$.

**Proof.** Let $0 \leq t \leq T$. We begin by observing that
\[
\int_{\mathbb{R}} (\partial_x^2 u)^4 \, dx = \int_{\mathbb{R}} (\partial_x^2 u)^3 (\partial_x^2 u) \, dx = -3 \int_{\mathbb{R}} \partial_x u (\partial_x^2 u)^2 \partial_x^3 u \, dx.
\]
Due to the Young inequality,
\[
3 \int_{\mathbb{R}} |u| |\partial_x u (\partial_x^2 u)^2 | \partial_x^3 u | \, dx \leq \frac{9}{2} \int_{\mathbb{R}} (\partial_x u)^2 (\partial_x^3 u)^2 \, dx + \frac{1}{2} \int_{\mathbb{R}} (\partial_x^2 u)^4 \, dx.
\]
It follows from (50) that
\[
\int_{\mathbb{R}} (\partial_x^2 u)^4 \, dx \leq 9 \int_{\mathbb{R}} (\partial_x u)^2 (\partial_x^3 u)^2 \, dx.
\]
By (44), we have that
\[
\left\| \partial_x^2 u(t, \cdot) \right\|_{L^4(\mathbb{R})}^4 \leq 9 \left\| \partial_x u \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T) \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]
Integrating on \((0, t)\), by (28), we have that
\[
\int_0^t \left\| \partial_x^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^4 \, ds \leq C(T) \int_0^t \left\| \partial_x^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T),
\]
which gives (49). □

We continue with an a priori estimate in the space \(L^2(0, \infty; H^6(\mathbb{R}))\).

**Lemma 6.** Fix \(T > 0\) and assume (2) or (3). There exists a constant \(C(T) > 0\), independent on \(\kappa, a, b\), such that
\[
\left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2a^2 \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds
+ \beta^2 \int_0^t \left\| \partial_x^5 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds + 2b^2 \int_0^t \left\| \partial_x^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T)
\]
(51)

In particular, we have that
\[
\left\| \partial_x^2 u \right\|_{L^\infty((0, T) \times \mathbb{R})} \leq C(T),
\]
(52)
where \(C(T)\) is independent on \(\kappa, a, b\).

**Proof.** Let \(0 \leq t \leq T\). We begin by observing that, by (46),
\[
\partial_x^4 \left( u^3 \right) = 36(\partial_x u)^2 \partial_x^3 u + 18u(\partial_x^2 u)^2 + 24u\partial_x u \partial_x^3 u + 3u^2 \partial_x^4 u
\]
(53)

Multiplying (17) by \(-2\partial_x^3 u\), thanks to (46), an integration on \(\mathbb{R}\) gives
\[
\frac{d}{dt} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = -2 \int_{\mathbb{R}} \partial_x^3 u \partial_x^4 u \, dx
\]
\[
= 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^3 u \, dx - 2a^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^3 u \, dx - 2\beta^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
+ 2b^2 \int_{\mathbb{R}} \partial_x^5 u \partial_x^3 u \, dx
+ 36\delta^2 \int_{\mathbb{R}} (\partial_x^2 u)^2 \partial_x^3 u \, dx + 48\delta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^3 u \partial_x^3 u \, dx
+ 6\delta^2 \int_{\mathbb{R}} u^2 \partial_x^4 u \partial_x^3 u \, dx
\]
\[
= 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^3 u \, dx + 2 \left( \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\beta^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right)
- 2a^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
+ 2b^2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
+ 36\delta^2 \int_{\mathbb{R}} (\partial_x^2 u)^2 \partial_x^3 u \, dx + 48\delta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^3 u \partial_x^3 u \, dx
+ 6\delta^2 \int_{\mathbb{R}} u^2 \partial_x^4 u \partial_x^3 u \, dx.
\]
Therefore, we have that
\[
\frac{d}{dt} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2a^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2b^2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]
(54)
\[
4\|x\| \int_{\mathbb{R}} |u_\delta u| |\partial_x^2 u| dx \leq 4\|x\| \|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\partial_x u| |\partial_x^2 u| dx
\]
\[
\leq 2C_0 \int_{\mathbb{R}} |\partial_x u| |\partial_x^2 u| dx = 2 \int_{\mathbb{R}} \left| \frac{C_0 \partial_x u}{\beta \sqrt{D^3}} \right| \beta \sqrt{D^3} \partial_x^2 u dx
\]
\[
\leq \frac{C_0}{D^3} \|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 D^3 \|\partial_x^2 u(t,\cdot)\|_{L^2(\mathbb{R})}^2
\]
\[
\leq \frac{C_0}{D^3} + \beta^2 D^3 \|\partial_x^2 u(t,\cdot)\|_{L^2(\mathbb{R})}^2
\]

Due to Lemma 1, (44), (45) and the Young inequality,

\[
72\delta^2 \int_{\mathbb{R}} (\partial_x u)^2 |\partial_x^2 u| |\partial_x^2 u| dx \leq 72\delta^2 \|\partial_x u\|_{L^\infty((0,T) \times \mathbb{R})}^2 \int_{\mathbb{R}} |\partial_x^2 u| |\partial_x^2 u| dx
\]
\[
\leq 2C(T) \int_{\mathbb{R}} (\partial_x^2 u)^2 |\partial_x^2 u| dx = 2 \int_{\mathbb{R}} \left| \frac{C(T) \partial_x^2 u}{\beta \sqrt{D^3}} \right| \beta \sqrt{D^3} \partial_x^2 u dx
\]
\[
\leq \frac{C(T)}{D^3} \|\partial_x^2 u(t,\cdot)\|_{L^2(\mathbb{R})}^4 + \beta^2 D^3 \|\partial_x^2 u(t,\cdot)\|_{L^2(\mathbb{R})}^2
\]

where \(D^3\) is a positive constant, which will be specified later. It follows from (54) that

\[
\frac{d}{dt} \|\partial_x^2 u(t,\cdot)\|_{L^2(\mathbb{R})}^2 + 2\delta^2 \|\partial_x^4 u(t,\cdot)\|_{L^2(\mathbb{R})}^2
\]
\[
+ \beta^2 (2 - 5D^3) \|\partial_x^4 u(t,\cdot)\|_{L^2(\mathbb{R})}^2 + 2\delta^2 \|\partial_x^6 u(t,\cdot)\|_{L^2(\mathbb{R})}^2
\]
\[
\leq \frac{C(T)}{D^3} \|\partial_x^2 u(t,\cdot)\|_{L^2(\mathbb{R})}^4 + \frac{C(T)}{D^3} \|\partial_x^4 u(t,\cdot)\|_{L^2(\mathbb{R})}^2 + \frac{C_0}{D^3} \|\partial_x^2 u(t,\cdot)\|_{L^2(\mathbb{R})}^2.
\]
Taking $D_3 = \frac{1}{2}$, we have that
\[
\frac{d}{dt} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2a^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2b^2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
+ \beta^2 \left\| \partial_x^6 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2b^2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
\leq C(T) + C(T) \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^4 + C(T) \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C_0 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]
Integrating on $(0, t)$, by (6), (28), (42), (49), we have that
\[
\left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2a^2 \int_0^t \left\| \partial_x^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\
+ \beta^2 \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + 2b^2 \int_0^t \left\| \partial_x^5 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\
\leq C_0 + C(T) \int_0^t \left\| \partial_x^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^4 ds + C(T) \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + C_0 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\
\leq C(T),
\]
which gives (51).

Finally, we prove (52). Thanks to (45), (51) and the Hölder inequality,
\[
(\partial_x^2 u(t, x))^2 = 2 \int_{-\infty}^x \partial_x^2 u \partial_x^3 u dy \leq 2 \int_{\mathbb{R}} |\partial_x^2 u| |\partial_x^3 u| dx \\
\leq 2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})} \leq C(T).
\]
Hence,
\[
\left\| \partial_x^2 u \right\|_{L^\infty((0, T) \times \mathbb{R})} \leq C(T),
\]
which gives (52). \( \Box \)

We continue with an a priori estimate in the space $H^1((0, \infty) \times \mathbb{R})$.

**Lemma 7.** Fix $T > 0$ and assume (2) or (3). There exists a constant $C(T) > 0$, independent on $\kappa, a, b$, such that
\[
2a^2 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + b^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \int_0^t \left\| \partial_t u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T),
\]
for every $0 \leq t \leq T$.

**Proof.** Let $0 \leq t \leq T$. Multiplying (17) by $2 \partial_t u$, we have that
\[
2(\partial_t u)^2 + 4\kappa u \partial_x u \partial_t u - 2a^2 \partial_x^2 u \partial_t u - 2b^2 \partial_x^4 u \partial_t u + 2b^2 \partial_x^4 u \partial_t u + 2\delta^2 \partial_t u \partial_x^4 \left( u^3 \right) = 0.
\]
Since,
\[
-2a^2 \int_{\mathbb{R}} \partial_x^2 u \partial_t u = a^2 \frac{d}{dt} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\
-2b^2 \int_{\mathbb{R}} \partial_x^4 u \partial_t u dx = b^2 \frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\
2b^2 \int_{\mathbb{R}} \partial_x^4 u \partial_t u dx = b^2 \frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
\]
thanks to (53), an integration of (56) on \(\mathbb{R}\) gives
\[
\frac{d}{dt} \left( a^2 \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + \beta^2 \| \partial_x^3 u(t, \cdot) \|^2_{L^2(\mathbb{R})} \right) + \nu^2 \frac{d}{dt} \| \partial_x^2 u(t, \cdot) \|^2_{L^2(\mathbb{R})} + 2 \| \partial_t u(t, \cdot) \|^2_{L^2(\mathbb{R})} = -4\kappa \int_{\mathbb{R}} u \partial_x u \partial_t u dx - 72\delta^2 \int_{\mathbb{R}} \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} \partial_x^2 u \partial_t u dx
\]
\[+ 36\delta^2 \int_{\mathbb{R}} u \| \partial_x^2 u(t, \cdot) \|^2_{L^2(\mathbb{R})} \partial_t u dx - 48\delta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^2 u \partial_t u dx - 6\delta^2 \int_{\mathbb{R}} u^2 \partial_x^4 u \partial_t u dx.
\]
Due to Lemma 1, (44), (45), (52) and the Young inequality,
\[
4|\kappa| \int_{\mathbb{R}} |u| \| \partial_x u \| \partial_t u dx \leq 4|\kappa| \| u(t, \cdot) \|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \| \partial_x u \| \partial_t u dx
\]
\[\leq 2C_0 \int_{\mathbb{R}} \| \partial_x u \| \partial_t u dx = 2 \int_{\mathbb{R}} \left| \frac{C_0 \partial_x u}{\sqrt{D_3}} \right| \sqrt{D_3} \partial_t u dx
\]
\[\leq \frac{C_0}{D_3} \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + D_3 \| \partial_t u(t, \cdot) \|^2_{L^2(\mathbb{R})} \leq \frac{C_0}{D_3} + D_3 \| \partial_t u(t, \cdot) \|^2_{L^2(\mathbb{R})},
\]
\[
72\delta^2 \int_{\mathbb{R}} \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} \partial_x^2 u \partial_t u dx \leq 72\delta^2 \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}} \| \partial_x^2 u \| \partial_t u dx
\]
\[\leq 2C(T) \int_{\mathbb{R}} \| \partial_x^2 u \| \partial_t u dx = 2 \int_{\mathbb{R}} \left| \frac{C(T) \partial_x^2 u}{\sqrt{D_3}} \right| \sqrt{D_3} \partial_t u dx
\]
\[\leq \frac{C(T)}{D_3} \| \partial_x^2 u(t, \cdot) \|^2_{L^2(\mathbb{R})} + D_3 \| \partial_t u(t, \cdot) \|^2_{L^2(\mathbb{R})} \leq \frac{C(T)}{D_3} + D_3 \| \partial_t u(t, \cdot) \|^2_{L^2(\mathbb{R})},
\]
\[
36\delta^2 \int_{\mathbb{R}} |u| \| \partial_x^2 u \|^2_{L^2(\mathbb{R})} \partial_t u dx \leq 36\delta^2 \| u(t, \cdot) \|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \| \partial_x^2 u \|^2_{L^2(\mathbb{R})} \partial_t u dx
\]
\[\leq 2C_0 \int_{\mathbb{R}} \| \partial_x^2 u \|^2_{L^2(\mathbb{R})} \partial_t u dx = 2C_0 \int_{\mathbb{R}} \| \partial_x^2 u \|^2_{L^\infty(0,T) \times \mathbb{R}} \int_{\mathbb{R}} \| \partial_x^2 u \| \partial_t u dx
\]
\[\leq 2C(T) \int_{\mathbb{R}} \| \partial_x^2 u \| \partial_t u dx = 2 \int_{\mathbb{R}} \left| \frac{C(T) \partial_x^2 u}{\sqrt{D_3}} \right| \sqrt{D_3} \partial_t u dx
\]
\[\leq \frac{C(T)}{D_3} \| \partial_x^2 u(t, \cdot) \|^2_{L^2(\mathbb{R})} + D_3 \| \partial_t u(t, \cdot) \|^2_{L^2(\mathbb{R})} \leq \frac{C(T)}{D_3} + D_3 \| \partial_t u(t, \cdot) \|^2_{L^2(\mathbb{R})},
\]
\[
48\delta^2 \int_{\mathbb{R}} |u| \| \partial_x u \|^2_{L^2(\mathbb{R})} \partial_t u dx \leq 48\delta^2 \| u(t, \cdot) \|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \| \partial_x u \|^2_{L^\infty(0,T) \times \mathbb{R}} \int_{\mathbb{R}} \| \partial_x^2 u \| \partial_t u dx
\]
\[\leq 2C_0 \int_{\mathbb{R}} \| \partial_x u \|^2_{L^\infty(0,T) \times \mathbb{R}} \int_{\mathbb{R}} \| \partial_x^2 u \| \partial_t u dx
\]
\[\leq 2C(T) \int_{\mathbb{R}} \| \partial_x^2 u \| \partial_t u dx = 2 \int_{\mathbb{R}} \left| \frac{C(T) \partial_x^2 u}{\sqrt{D_3}} \right| \sqrt{D_3} \partial_t u dx
\]
\[\leq \frac{C(T)}{D_3} \| \partial_x^2 u(t, \cdot) \|^2_{L^2(\mathbb{R})} + D_3 \| \partial_t u(t, \cdot) \|^2_{L^2(\mathbb{R})} \leq \frac{C(T)}{D_3} + D_3 \| \partial_t u(t, \cdot) \|^2_{L^2(\mathbb{R})},
\]
\[
6\delta^2 \int_{\mathbb{R}} u^2 \| \partial_x^4 u \| \partial_t u dx = 6\delta^2 \| u(t, \cdot) \|^2_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \| \partial_x^4 u \| \partial_t u dx
\]
\[\leq 2C_0 \int_{\mathbb{R}} \| \partial_x^4 u \|^2_{L^2(\mathbb{R})} \partial_t u dx = 2 \int_{\mathbb{R}} \left| \frac{C_0 \partial_x^4 u}{\sqrt{D_3}} \right| \sqrt{D_3} \partial_t u dx
\]
\[\leq \frac{C_0}{D_3} \| \partial_x^4 u(t, \cdot) \|^2_{L^2(\mathbb{R})} + D_3 \| \partial_t u(t, \cdot) \|^2_{L^2(\mathbb{R})},
\]
where \(D_3\) is a positive constant, which will be specified later. As a consequence, (57) becomes
\[
\frac{d}{dt} \left( a^2 \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + \beta^2 \| \partial_x^3 u(t, \cdot) \|^2_{L^2(\mathbb{R})} \right)
\[+ \nu^2 \frac{d}{dt} \| \partial_x^2 u(t, \cdot) \|^2_{L^2(\mathbb{R})} + 2 \| \partial_t u(t, \cdot) \|^2_{L^2(\mathbb{R})} = -4\kappa \int_{\mathbb{R}} u \partial_x u \partial_t u dx - 72\delta^2 \int_{\mathbb{R}} \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} \partial_x^2 u \partial_t u dx
\]
\[+ 36\delta^2 \int_{\mathbb{R}} u \| \partial_x^2 u(t, \cdot) \|^2_{L^2(\mathbb{R})} \partial_t u dx - 48\delta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^2 u \partial_t u dx - 6\delta^2 \int_{\mathbb{R}} u^2 \partial_x^4 u \partial_t u dx.
\]
we have that the global-in-time existence of a is the solution of (1) that satisfies (13), which gives (55).

We are finally ready to prove Theorem 1, assuming (i) or (ii).

**Proof of Theorem 1 assuming (i) or (ii).** The well-posedness of (1) is guaranteed for a short time by the Cauchy-Kowalevskaya Theorem [64]. Thanks to the a priori estimates proved in Lemmas 1–7, we have that the global-in-time existence of a is the solution of (1) that satisfies (13).

The stability estimates (15) can be proved using the same arguments of [18] (Theorem 1).

**4. Proof of Theorem 1 Assuming (iii)**

In this section, we prove Theorem 1 assuming (iii). Due to (4), here, (1) becomes

\[
\begin{align*}
\partial_t u + \kappa \partial_x u^2 + \gamma \partial_x^2 u - \beta^2 \partial_x^4 u + a \partial_x^4 u &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R}.
\end{align*}
\]

The argument of this section is analogous to that of the previous one. We deduce the local-in-time well-posedness from the Cauchy-Kowalevskaya Theorem [64], and we improve the local-in-time existence to the global-in-time one, proving some suitable a priori estimates on u.

We begin with an energy estimate in the space \(L^\infty_{\text{loc}}(0, \infty; L^2(\mathbb{R})) \cap L^2_{\text{loc}}(0, \infty; H^3(\mathbb{R}))\).

**Lemma 8.** Fix \(T > 0\). There exists a constant \(C(T) > 0\), such that

\[
\|u(t, \cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 e^{C_t} \int_0^t e^{-C_s} \|\partial_x^4 u(s, \cdot)\|^2_{L^2(\mathbb{R})} \, ds \leq C(T),
\]

for every \(0 \leq t \leq T\). In particular, (29) holds. Moreover, we have that

\[
\int_0^t \|\partial_x u(s, \cdot)\|^2_{L^2(\mathbb{R})} \, ds \leq C(T),
\]

for every \(0 \leq t \leq T\).

**Proof.** Let \(0 \leq t \leq T\). Multiplying (58) by 2\(u\), an integration on \(\mathbb{R}\) gives

\[
\frac{d}{dt} \|u(t, \cdot)\|^2_{L^2(\mathbb{R})} = 2 \int_\mathbb{R} u \partial_t u \, dx
\]

\[
= 4\kappa \int_\mathbb{R} u^2 \partial_x u \, dx - 2\gamma \int_\mathbb{R} u \partial_x^2 u \, dx + 2\beta^2 \int_\mathbb{R} u \partial_x^4 u \, dx - 2\kappa \int_\mathbb{R} u \partial_x^4 u \, dx
\]
we have that
that is,
Mathematics
Since, using the Young inequality,
Choosing
that is

\[
\frac{d}{dt} \|u(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2 \|\partial_x^3 u(t, \cdot)\|^2_{L^2(\mathbb{R})} \leq C_0 \|u(t, \cdot)\|^2_{L^2(\mathbb{R})} + C_0 \|\partial_x^5 u(t, \cdot)\|^2_{L^2(\mathbb{R})},
\]
we can pass from (61) to

\[
\frac{d}{dt} \|u(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2 \|\partial_x^3 u(t, \cdot)\|^2_{L^2(\mathbb{R})} \leq C_0 \|u(t, \cdot)\|^2_{L^2(\mathbb{R})} + C_0 \|\partial_x^5 u(t, \cdot)\|^2_{L^2(\mathbb{R})},
\]
we have that
that is

\[
C_0 \|\partial_x^5 u(t, \cdot)\|^2_{L^2(\mathbb{R})} = C_0 \int \partial_x^3 u \partial_x^2 u dx = -C_0 \int \partial_x \partial_x^3 u dx.
\]
Therefore, by the Young inequality,

\[
\frac{C_0}{D_4} \|\partial_x u(t, \cdot)\|^2_{L^2(\mathbb{R})} \leq C_0 D_4 \int \partial_x \partial_x^3 u dx \leq \frac{C_0}{D_4} \|\partial_x u(t, \cdot)\|^2_{L^2(\mathbb{R})} + D_4 \|\partial_x^3 u(t, \cdot)\|^2_{L^2(\mathbb{R})},
\]
where \(D_4\) is a positive constant, which will be specified later. Observe again that

\[
\frac{C_0}{D_4} \|\partial_x u(t, \cdot)\|^2_{L^2(\mathbb{R})} = \frac{C_0}{D_4} \int \partial_x \partial_x^3 u dx = -\frac{C_0}{D_4} \int \partial_x^3 u dx.
\]
Consequently, by the Young inequality,

\[
\frac{C_0}{D_5} \|\partial_x u(t, \cdot)\|^2_{L^2(\mathbb{R})} \leq \frac{C_0 D_5^2}{D_5^2} \|\partial_x^3 u(t, \cdot)\|^2_{L^2(\mathbb{R})},
\]
where \(D_5\) is a positive constant, which will be specified later. It follows from (63) and (64) that

\[
(C_0 - D_5) \|\partial_x^3 u(t, \cdot)\|^2_{L^2(\mathbb{R})} \leq \frac{C_0 D_5^2}{D_5^2} \|u(t, \cdot)\|^2_{L^2(\mathbb{R})} + D_4 \|\partial_x^5 u(t, \cdot)\|^2_{L^2(\mathbb{R})}.
\]
Choosing

\[
D_5 = \frac{C_0}{2},
\]
we have that

\[
\frac{C_0}{2} \|\partial_x^3 u(t, \cdot)\|^2_{L^2(\mathbb{R})} \leq 2D_4 \|u(t, \cdot)\|^2_{L^2(\mathbb{R})} + D_4 \|\partial_x^5 u(t, \cdot)\|^2_{L^2(\mathbb{R})},
\]
that is

\[
C_0 \|\partial_x^3 u(t, \cdot)\|^2_{L^2(\mathbb{R})} \leq 4D_4 \|u(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2D_4 \|\partial_x^5 u(t, \cdot)\|^2_{L^2(\mathbb{R})}.
\]
It follows from (62) and (66) that
\[
\frac{d}{dt} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2 \left( \beta^2 - D_4 \right) \left\| \partial_t^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq \left( C_0 + 4D_4^2 \right) \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\]
Choosing
\[
D_4 = \frac{\beta^2}{2},
\]
we have that
\[
\frac{d}{dt} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \left\| \partial_t^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C_0 \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\]
By the Gronwall Lemma and (6), we get
\[
\|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq e^{C_0 t} \|u(0, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C_0 e^{C_0 t} \leq C(T),
\]
which gives (59).

We prove (29). Thanks to (59), (66) and (67),
\[
C_0 \left\| \partial_t^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T) + \frac{\beta^2}{2} \left\| \partial_t^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]
Integrating on \((0, t)\), by (59), we have that
\[
C_0 \int_0^t \left\| \partial_t^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T) t + \frac{\beta^2}{2} \int_0^t \left\| \partial_t^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T),
\]
which gives (29).

Finally, we prove (60). Thanks to (59), (64) and (67),
\[
C_0 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T) + C_0 \left\| \partial_t^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]
Integrating on \((0, t)\), by (29), we have that
\[
C_0 \int_0^t \left\| \partial_x u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T) t + C_0 \int_0^t \left\| \partial_t^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T),
\]
which gives (60).

We continue with an energy estimate in the space \(L^\infty_{loc}(0, \infty; H^1(\mathbb{R})) \cap L^2_{loc}(0, \infty; H^4(\mathbb{R}))\).

**Lemma 9.** Fix \(T > 0\). There exists a constant \(C(T) > 0\), such that
\[
\|u\|_{L^\infty((0, T) \times \mathbb{R})} \leq C(T),
\]
\[
\left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2 \beta^2 \int_0^t \left\| \partial_t^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T),
\]
for every \(0 \leq t \leq T\).

**Proof.** Let \(0 \leq t \leq T\). Multiplying (58) by \(-2\partial_x^2 u\), an integration on \(\mathbb{R}\) gives
\[
\frac{d}{dt} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = -2 \int_\mathbb{R} \partial_x^2 u \partial_t u dx
\]
\[
-4\gamma \int_\mathbb{R} \partial_x u \partial_x^2 u dx + 2 \gamma \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2 \beta^2 \int_\mathbb{R} \partial_t^3 u \partial_x^2 u dx + 2 \alpha \int_\mathbb{R} \partial_t^4 u \partial_x^2 u dx.
\]
We prove (68). Thanks to (59), (69), and the Hölder inequality, which gives (68).

Therefore, by (70),

\[
\frac{d}{dt} \| \partial_x u(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2\gamma \| \partial_x^2 u(t, \cdot) \|_{L^2(\mathbb{R})}^2 - 2\beta^2 \| \partial_x^4 u(t, \cdot) \|_{L^2(\mathbb{R})}^2 - 2\alpha \| \partial_x^3 u(t, \cdot) \|_{L^2(\mathbb{R})}^2.
\]

Consequently, we have that

\[
\frac{d}{dt} \| \partial_x u(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2\beta^2 \| \partial_x^4 u(t, \cdot) \|_{L^2(\mathbb{R})}^2
= 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx + 2\gamma \| \partial_x^2 u(t, \cdot) \|_{L^2(\mathbb{R})}^2 - 2\alpha \| \partial_x^3 u(t, \cdot) \|_{L^2(\mathbb{R})}^2.
\]

Due to the Young inequality,

\[
4\kappa \int_{\mathbb{R}} |u \partial_x u| \| \partial_x^2 u \| dx \leq 2\kappa^2 \int_{\mathbb{R}} u^2 (\partial_x u)^2 dx + 2 \| \partial_x^2 u(t, \cdot) \|_{L^2(\mathbb{R})}^2
\]

\[
\leq 2\kappa^2 \| u \|_{L^\infty(0, T) \times \mathbb{R}}^2 \| \partial_x u(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2 \| \partial_x^2 u(t, \cdot) \|_{L^2(\mathbb{R})}^2.
\]

Therefore, by (70),

\[
\frac{d}{dt} \| \partial_x u(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2\beta^2 \| \partial_x^4 u(t, \cdot) \|_{L^2(\mathbb{R})}^2
\]

\[
\leq 2\kappa^2 \| u \|_{L^\infty(0, T) \times \mathbb{R}}^2 \| \partial_x u(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2(1 + |\gamma|) \| \partial_x^2 u(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2|\alpha| \| \partial_x^3 u(t, \cdot) \|_{L^2(\mathbb{R})}^2
\]

\[
\leq C_0 \| u \|_{L^\infty(0, T) \times \mathbb{R}}^2 \| \partial_x u(t, \cdot) \|_{L^2(\mathbb{R})}^2 + C_0 \| \partial_x^2 u(t, \cdot) \|_{L^2(\mathbb{R})}^2 + C_0 \| \partial_x^3 u(t, \cdot) \|_{L^2(\mathbb{R})}^2.
\]

(6), (29), (59), (60) and an integration on (0, 1) give

\[
\| \partial_x u(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2\beta^2 \int_0^t \| \partial_x^4 u(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds
\]

\[
\leq C_0 + C_0 \| u \|_{L^\infty(0, T) \times \mathbb{R}}^2 \| \partial_x u(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds + C_0 \| \partial_x^2 u(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds + C_0 \| \partial_x^3 u(t, \cdot) \|_{L^2(\mathbb{R})}^2
\]

(71)

\[
\leq C(T) \left( 1 + \| u \|_{L^\infty(0, T) \times \mathbb{R}}^2 \right).
\]

We prove (68). Thanks to (59), (69), and the Hölder inequality,

\[
u^2(t, x) = 2 \int_{-\infty}^x u \partial_x u dy \leq 2 \int_{\mathbb{R}} |u| \| \partial_x u \| dx
\]

\[
\leq \| u(t, \cdot) \|_{L^2(\mathbb{R})} \| \partial_x u(t, \cdot) \|_{L^2(\mathbb{R})} \leq C(T) \sqrt{1 + \| u \|_{L^\infty(0, T) \times \mathbb{R}}^2}.
\]

Hence,

\[
\| u \|_{L^\infty(0, T) \times \mathbb{R}}^4 - C(T) \| u \|_{L^\infty(0, T) \times \mathbb{R}}^2 - C(T) \leq 0,
\]

which gives (68).

Finally, (69) follows from (68) and (71). \(\square\)

We continue with an energy estimate in the space \(L^\infty_{loc}(0, \infty; H^2(\mathbb{R})) \cap L^2_{loc}(0, \infty; H^5(\mathbb{R}))\).

**Lemma 10.** Fix \(T > 0\). There exists a constant \(C(T) > 0\), such that

\[
\| \partial_x^2 u(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \beta^2 \int_0^t \| \partial_x^4 u(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds \leq C(T),
\]

(72)
for every $0 \leq t \leq T$. In particular, (44) holds.

**Proof.** Let $0 \leq t \leq T$. Multiplying (58) by $2\partial_t^4 u$, an integration on $\mathbb{R}$ gives

\[
\frac{d}{dt} \left\| \partial_t^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = 2 \int_{\mathbb{R}} \partial_t^4 u \partial_t^3 u u dx
\]

\[
= -4\kappa \int_{\mathbb{R}} u \partial_x u \partial_t^4 u u dx + 2\gamma \int_{\mathbb{R}} \partial_x^2 u \partial_t^4 u u dx + 2\beta^2 \int_{\mathbb{R}} \partial_x^4 u \partial_t^4 u u dx - 2\alpha \left\| \partial_t^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]

\[
= -4\kappa \int_{\mathbb{R}} u \partial_x u \partial_t^4 u u dx + 2\gamma \left\| \partial_t^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\beta^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\alpha \left\| \partial_t^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]

Therefore, we have that

\[
\frac{d}{dt} \left\| \partial_t^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]

\[
= -4\kappa \int_{\mathbb{R}} u \partial_x u \partial_t^4 u u dx + 2\gamma \left\| \partial_t^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\alpha \left\| \partial_t^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]

Due to (68), (69) and the Young inequality,

\[
4|\kappa| \int_{\mathbb{R}} |u\partial_x u| |\partial_t^4 u| dx \leq 4|\kappa| \left\| u \right\|_{L^\infty((0,T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x u| |\partial_t^4 u| dx
\]

\[
\leq C(T) \int_{\mathbb{R}} |\partial_x u| |\partial_t^4 u| dx \leq C(T) \left\| \partial_x u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + C(T) \left\| \partial_t^4 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}
\]

\[
\leq C(T) + C(T) \left\| \partial_t^4 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}.
\]

It follows from (73) that

\[
\frac{d}{dt} \left\| \partial_t^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]

\[
\leq C(T) + C(T) \left\| \partial_t^4 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + 2|\gamma| \left\| \partial_x^3 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + 2|\kappa| \left\| \partial_t^4 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}
\]

\[
\leq C(T) + C(T) \left\| \partial_t^4 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + C(T) \left\| \partial_t^4 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}.
\]

Integrating on $(0, t)$ by (6), (59) and (69), we have

\[
\left\| \partial_t^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds
\]

\[
\leq C_0 + C(T) t + C(T) \int_0^t \left\| \partial_t^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + C(T) \int_0^t \left\| \partial_t^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds
\]

\[
\leq C(T),
\]

which gives (72).

Finally, by (69), (72) and the H"older inequality, we have (44). \qed

We continue with an energy estimate in the space $L^\infty_{loc}(0, \infty; H^3(\mathbb{R})) \cap L^2_{loc}(0, \infty; H^6(\mathbb{R}))$.

**Lemma 11.** Fix $T > 0$. There exists a constant $C(T) > 0$, such that

\[
\left\| \partial_t^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T),
\]

for every $0 \leq t \leq T$. In particular, (52) holds.
Proof. Let $0 \leq t \leq T$. Multiplying (58) by $-2\partial^3_x u$, an integration on $\mathbb{R}$ gives

$$
\frac{d}{dt} \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = -2 \int_{\mathbb{R}} \partial^3_x u \partial_t u \, dx
$$

$$
= 4\gamma \int_{\mathbb{R}} u \partial_x u \partial_t \partial^3_x u \, dx - 2\beta \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\alpha \int_{\mathbb{R}} \partial^3_x u \partial^3_x u \, dx
$$

$$
= 4\gamma \int_{\mathbb{R}} u \partial_x u \partial_t \partial^3_x u \, dx - 2\beta \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\alpha \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
$$

Therefore, we have that

$$
\frac{d}{dt} \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
$$

$$
= 4\gamma \int_{\mathbb{R}} u \partial_x u \partial_t \partial^3_x u \, dx - 2\beta \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\alpha \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
$$

(75)

Due to (68), (69) and the Young inequality,

$$
4|\gamma| \int_{\mathbb{R}} |u| |\partial_x u| |\partial^3_x u| \, dx \leq 2|\gamma| \left\| u \right\|_{L^2(0, T) \times \mathbb{R}} \int_{\mathbb{R}} |\partial_x u| |\partial^3_x u| \, dx
$$

$$
\leq 2C(T) \int_{\mathbb{R}} |\partial_x u| |\partial^3_x u| \, dx = 2 \int_{\mathbb{R}} \left| \frac{C(T)\partial_x u}{\beta} \right| |\beta \partial^3_x u| \, dx
$$

$$
\leq C(T) \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
$$

$$
\leq C(T) + \beta^2 \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
$$

It follows from (75) that

$$
\frac{d}{dt} \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
$$

$$
\leq C(T) + 2|\gamma| \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2|\alpha| \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
$$

$$
\leq C(T) + C(T) \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T) \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
$$

By (6), (69), (72), and an integration on $(0, t)$,

$$
\left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \int_0^t \left\| \partial^3_x u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds
$$

$$
\leq C_0 + C(T)t + C(T) \int_0^t \left\| \partial^3_x u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds + C(T) \int_0^t \left\| \partial^3_x u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds
$$

$$
\leq C(T),
$$

which gives (74).

Finally, arguing as in Lemma 6, we have (52). \qed

We continue with an energy estimate in the space $L^\infty_{loc} (0, \infty; H^3(\mathbb{R})) \cap H^1_{loc} ((0, \infty) \times \mathbb{R})$.

Lemma 12. Fix $T > 0$. There exists a constant $C(T) > 0$, such that

$$
\beta^2 \left\| \partial^3_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \int_0^t \left\| \partial_t u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T).
$$

(76)
for every \(0 \leq t \leq T\).

**Proof.** Let \(0 \leq t \leq T\). Multiplying \((58)\) by \(2\partial_t u\), an integration on \(\mathbb{R}\) gives

\[
\frac{d}{dt} \left( \beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \alpha \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - \gamma \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) = -2\beta^2 \int_{\mathbb{R}} \partial_t u \partial_x^4 u \, dx + 2\alpha \int_{\mathbb{R}} \partial_t u \partial_x^3 u \, dx + 2\gamma \int_{\mathbb{R}} \partial_t \partial_x^2 u \, dx
\]
\[
= -2 \left\| \partial_t u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 4\gamma \int_{\mathbb{R}} \partial_x \partial_t u \partial_t u \, dx.
\]

Therefore, we have that

\[
\frac{d}{dt} \left( \beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \alpha \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - \gamma \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) + 2 \left\| \partial_t u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = -4\gamma \int_{\mathbb{R}} \partial_x \partial_t u \partial_t u \, dx.
\]

(77)

Due to (68), (69), and the Young inequality,

\[
4|\kappa| \int_{\mathbb{R}} |u| \left\| \partial_x u \right\| \partial_t u \, dx \leq 4|\kappa| \left\| u \right\|_{L^\infty((0,T) \times \mathbb{R})} \int_{\mathbb{R}} \left\| \partial_x u \right\| \partial_t u \, dx
\]
\[
\leq 2C(T) \int_{\mathbb{R}} \left\| \partial_x u \right\| \partial_t u \, dx \leq C(T) \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \left\| \partial_t u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
\leq C(T) + \left\| \partial_t u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]

Consequently, by (77),

\[
\frac{d}{dt} \left( \beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \alpha \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - \gamma \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) + \left\| \partial_t u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T).
\]

(6) and an integration on \((0,t)\) give

\[
\beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \alpha \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - \gamma \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \int_0^t \left\| \partial_t u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \leq C_0 + C(T) t \leq C(T).
\]

Therefore, by (69), (72), we have that

\[
\beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \int_0^t \left\| \partial_x u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T) + |\alpha| \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + |\gamma| \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T),
\]

which gives (76). \(\square\)

We are finally ready to prove Theorem 1 assuming (iii).

**Proof of Theorem 1 assuming (iii).** The well-posedness of (1) is guaranteed for a short time by the Cauchy-Kowalevskaya Theorem [64]. Thanks to the a priori estimates proved in Lemmas 8–12, we have that the global-in-time existence of a is solution of (1) that satisfies (13).

The stability estimates (15) can be proved using the same arguments of [18] (Theorem 1). \(\square\)
5. Proof of Theorem 1 Assuming (iv)

In this final section, we prove Theorem 1 assuming (iv).

The argument is again analogous to the one of the previous sectionss. We deduce the local-in-time well-posedness from the Cauchy-Kowaleskaya Theorem \[64\], and we improve the local-in-time existence to the global-in-time one proving some suitable a priori estimates on \(u\).

We begin with the zero mean estimate.

**Lemma 13.** For each \(t > 0\), we have (14).

**Proof.** Integrating (1) on \(\mathbb{R}\), we have that
\[
\int_{\mathbb{R}} \partial_t u(t, x) dx = \frac{d}{dt} \int_{\mathbb{R}} u(t, x) dx = 0
\]
(78)
(14) follows from (6) and (78). \(\square\)

**Remark 1.** In light of (14), we can consider the following equation:
\[
P(t, x) = \int_{-\infty}^x u(t, y) dy.
\]
Moreover, again by (14), we have that
\[
P(t, -\infty) = P(t, \infty) = 0.
\]

We continue by proving some energy estimates on the function \(P\).

**Lemma 14.** Let \(T > 0\). There exists a constant \(C(T) > 0\), such that
\[
\|P(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\beta^2 e^{2t} \int_0^t e^{-2s} \left\|\partial_x^2 u(s, \cdot)\right\|_{L^2(\mathbb{R})}^2 ds + 2\delta^2 e^{2t} \int_0^t e^{-2s} \|u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
\leq C(T) + C(T) \int_0^t \|u(s, \cdot)\|_{L^4(\mathbb{R})}^4 ds + C(T) \int_0^t \|\partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds,
\]
for every \(0 \leq t \leq T\).

**Proof.** Let \(0 \leq t \leq T\). Integrating (1) on \((-\infty, x)\), we have that
\[
\int_{-\infty}^x \partial_t u dx + \kappa u^2 + \gamma \partial_x u - \beta^2 \partial_x^3 u + \alpha \partial_x^3 u + \delta^2 \partial_x^3 \left(u^3\right) = 0.
\]
(82)
Differentiating (79) with respect to \(t\), we obtain that
\[
\partial_t P(t, x) = \frac{d}{dt} \int_{-\infty}^x u(t, y) dy = \int_{-\infty}^x \partial_t u(t, y) dy.
\]
(83)
It follows from (82) and (83) that
\[
\partial_t P + \kappa u^2 + \gamma \partial_x u - \beta^2 \partial_x^3 u + \alpha \partial_x^3 u + \delta^2 \partial_x^3 \left(u^3\right) = 0.
\]
(84)
Arguing as in [18] (Lemma 2), we have that
\[-2\beta^2 \int P\partial_t^4 u dx = 2\beta^2 \left\| \partial_t^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,\]
\[2\alpha \int P\partial_t^3 u dx = 2\alpha \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,\]
\[2\delta^2 \int P\partial_t^3 \left( u^3 \right) dx = 6\delta^2 \left\| u(t, \cdot) \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.\]
Therefore, multiplying (84) by 2P, thanks to (85), an integration on \( \mathbb{R} \) gives
\[\frac{d}{dt} \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_t^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 6\delta^2 \left\| u(t, \cdot) \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = -2\gamma \int_{\mathbb{R}} P\partial_x u dx - 2\alpha \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.\]
Due to the Young inequality,
\[2|\alpha| \int_{\mathbb{R}} |P| u^2 dx \leq \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \kappa^2 \left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})}^4,\]
\[2|\gamma| \int_{\mathbb{R}} |P| \partial_x u dx \leq \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \gamma^2 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.\]
It follows from (86) that
\[\frac{d}{dt} \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_t^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 6\delta^2 \left\| u(t, \cdot) \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq 2 \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C_0 \left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})}^4 + C_0 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,\]
Therefore, by the Gronwall Lemma and (9), we have that
\[\left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 e^{2t} \int_0^t e^{-2s} \left\| \partial_t^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + 6\delta^2 e^{2t} \int_0^t e^{-2s} \left\| u(s, \cdot) \partial_x u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C_0 + C_0 e^{2t} \int_0^t e^{-2s} \left\| u(s, \cdot) \right\|_{L^2(\mathbb{R})}^4 ds + C_0 e^{2t} \int_0^t e^{-2s} \left\| \partial_x u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T) + C(T) \int_0^t \left\| u(s, \cdot) \right\|_{L^2(\mathbb{R})}^4 ds + C(T) \int_0^t \left\| \partial_x u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds,
which gives (81). \( \square \)

**Lemma 15.** Let \( T > 0 \). There exists a constant \( C(T) > 0 \), such that
\[\frac{\beta^2}{6} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T),\]
for every \( 0 \leq t \leq T \). In particular, we have (29), (31),
\[\left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})} \leq C(T),\]
\[\left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})} \leq C(T),\]
\[\int_0^t \left\| u(s, \cdot) \partial_x u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T),\]
\[\int_0^t \left\| \beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^2 \left( u^3 \right)^2 \right\| ds dx \leq C(T),\]
for every $0 \leq t \leq T$. Moreover,
\[
\|P\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T),
\]
\[
\|u\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T).
\]

**Proof.** Let $0 \leq t \leq T$. Consider an real constant $A$, which will be specified later. Observe that
\[
\delta^2 \gamma \int_\mathbb{R} u^3 \partial_x^2 u dx = -3 \delta^2 \gamma \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 2 A \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\]

Multiplying (1) by
\[-\beta^2 \partial_x^2 u + \delta^2 u^3 + Au,
\]
thanks to (88) and arguing as in [18] (Lemma 3), an integration on $\mathbb{R}$ gives
\[
\frac{d}{dt} \left( \frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{A}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right)
\]
\[
+ \int_\mathbb{R} \left[ \beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^2 \left( u^3 \right) \right] dx
\]
\[
= -\beta^2 (A + \alpha) \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - \delta^2 (A + \alpha) \int_\mathbb{R} \partial_x^2 u \partial_x^2 \left( u^3 \right) dx
\]
\[
+ \left( \alpha + \gamma \beta^2 \right) \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \delta^2 \gamma \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2 \beta^2 \kappa \int_\mathbb{R} u \partial_x u \partial_x^2 u dx.
\]

Taking $A = -\alpha$,
we have that
\[
\frac{d}{dt} \left( \frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 - \frac{\alpha}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right)
\]
\[
+ \int_\mathbb{R} \left[ \beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^2 \left( u^3 \right) \right] dx
\]
\[
= \left( \alpha^2 + \gamma \beta^2 \right) \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \delta^2 \gamma \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2 \beta^2 \kappa \int_\mathbb{R} u \partial_x u \partial_x^2 u dx.
\]

Due to the Young inequality,
\[
2 \beta^2 |\kappa| \int_\mathbb{R} |u \partial_x u| \|\partial_x^2 u| dx \leq \beta^2 \kappa^2 \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]

It follows from (89) that
\[
\frac{d}{dt} \left( \frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 - \frac{\alpha}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right)
\]
\[
+ \int_\mathbb{R} \left[ \beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^2 \left( u^3 \right) \right] dx
\]
\[
\leq C_0 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C_0 \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\]

Integrating on $(0, t)$, by (7), we have that
\[
\frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 - \frac{\alpha}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2
\]
Therefore, by the Young inequality,

\[ C_0 \| u(t, \cdot) \|_{2L^2(\mathbb{R})}^2 \leq 2 \int_{\mathbb{R}} \left| \frac{\sqrt{3} C_0 P}{2 \beta} \right| |\partial_4^2 u| \sqrt{3} \, dx \]

\[ \leq C_0 \| P(t, \cdot) \|_{2L^2(\mathbb{R})}^2 + \frac{\beta^2}{3} \| \partial_4 u(t, \cdot) \|_{2L^2(\mathbb{R})}^2. \]  (91)

It follows from (81), (90) and (91) that

\[ \frac{\beta^2}{6} \| \partial_4 u(t, \cdot) \|_{2L^2(\mathbb{R})}^2 + \frac{\beta^2}{4} \| u(t, \cdot) \|_{4L^4(\mathbb{R})}^4 + \int_{0}^{t} \left[ \beta^2 (\partial_4^2 u)^2 - \beta^2 \partial_4^2 (u^3) \right] \, ds \]

\[ \leq C_0 + C_0 \int_{0}^{t} \| \partial_4^2 u(s, \cdot) \|_{2L^2(\mathbb{R})}^2 \, ds + C_0 \int_{0}^{t} \| u(s, \cdot) \|_{2L^2(\mathbb{R})}^2 \, ds + C_0 \| P(t, \cdot) \|_{2L^2(\mathbb{R})}^2 \]

\[ \leq C(T) + C(T) \int_{0}^{t} \| u(s, \cdot) \|_{4L^4(\mathbb{R})}^4, \]

\[ \leq C(T) \left( \frac{\beta^2}{6} \int_{0}^{t} \| \partial_4 u(s, \cdot) \|_{2L^2(\mathbb{R})}^2 \, ds + \frac{\beta^2}{4} \int_{0}^{t} \| u(s, \cdot) \|_{4L^4(\mathbb{R})}^4 \, ds \right). \]

Therefore, we have that

\[ \frac{\beta^2}{6} \| \partial_4 u(t, \cdot) \|_{2L^2(\mathbb{R})}^2 + \frac{\beta^2}{4} \| u(t, \cdot) \|_{4L^4(\mathbb{R})}^4 \]

\[ \leq C(T) + C(T) \left( \frac{\beta^2}{6} \int_{0}^{t} \| \partial_4 u(s, \cdot) \|_{2L^2(\mathbb{R})}^2 \, ds + \frac{\beta^2}{4} \int_{0}^{t} \| u(s, \cdot) \|_{4L^4(\mathbb{R})}^4 \, ds \right). \]

The Gronwall Lemma and (7) give (87).

Finally, arguing as in [18] (Lemma 3), the proof is concluded. \( \square \)

Arguing as in [18] (Theorem 1), we have Theorem 1.

6. Conclusions

This paper is dedicated to the well-posedness of a solution to the Cauchy problem for a higher-order convective Cahn-Hilliard equation. Such an equation models the evolution of crystal surfaces faceting through surface electromigration, the growing surface faceting, and the evolution of dynamics of phase transitions in ternary oil-water-surfactant systems. The well-posedness of (1) is
proved for a short time by the Cauchy-Kowaleskaya Theorem [64]. The global-in-time well-posedness is thus proved, proving several a priori estimates.

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