Fermions in odd space-time dimensions: back to basics

A. Bashir * 1, Ma. de Jesús Anguiano Galicia ** 1,2

1 Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Apartado Postal 2-82, Morelia, Michoacán 58040, México
2 Escuela de Ciencias Físico Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Apartado Postal 2-82, Morelia, Michoacán 58040, México

Abstract. It is a well known feature of odd space-time dimensions $d$ that there exist two inequivalent fundamental representations $A$ and $B$ of the Dirac gamma matrices. Moreover, the parity transformation swaps the fermion fields living in $A$ and $B$. As a consequence, a parity invariant Lagrangian can only be constructed by incorporating both the representations. Based upon these ideas and contrary to long held belief, we show that in addition to a discrete exchange symmetry for the massless case, we can also define chiral symmetry provided the Lagrangian contains fields corresponding to both the inequivalent representations. We also study the transformation properties of the corresponding chiral currents under parity and charge conjugation operations. We work explicitly in $2 + 1$ dimensions and later show how some of these ideas generalize to an arbitrary number of odd dimensions.

1 Introduction

There has been a growing interest in the last few years in theories with extra dimensions which have the capability of reducing the string mass scale $M_s$ several orders of magnitudes lower than the Planck mass scale of $1.9 \times 10^{16}$ TeV, see e.g., [1]. In extreme scenarios, [2], it can be as low as of the order of TeV. In such cases, we can nip the gauge hierarchy problem in the bud. In view of this exciting possibility, it is timely to revise various properties of field theories in higher dimensions. Theories in even space-time dimensions are rather similar to the ones in 4-dimensions. However, odd dimensions can have striking differences, [3] [4]. In this paper, we study the problem of chiral symmetry in odd space-time dimensions. We present a detailed discussion for the case of $d = 3$ and then extend some of the ideas in a straightforward fashion to arbitrarily
high number of odd $d$. In the context of dynamical chiral symmetry breaking, quantum electrodynamics in a plane (QED3) is interesting in its own right and this interest has been revived over the past several years specially because the lattice results and the ones obtained from the Shwinger-Dyson equation (SDE) studies, [5], have yet to arrive at a final consensus. An interesting and somewhat uncomfortable observation is that in the fundamental $2 \times 2$ representation of the gamma matrices, chiral symmetry cannot even be defined owing to the fact there does not exist any matrix which would anti-commute with all the three Pauli matrices, [6, 7]. This is in fact a feature of all odd dimensions $d$, [4]. It is for this reason that the dynamical chiral symmetry breaking in a plane is generally studied in the $4 \times 4$ representation of the gammas, e.g., [6, 7, 8]. There, we have two matrices anti-commuting with the gammas and thus the same number of chiral transformations can be defined. With an even (2N) number of 2-component fermions, one can combine two fermions to form two types of mass terms, one is the usual chiral symmetry violating and the other the chiral symmetry preserving term. However, requiring parity conservation filters out the latter and we end up having a Lagrangian similar in structure to the one in QED4. Although it is already known that the four component spinor is in fact composed of two two-component spinors, e.g., see the pioneering papers by Pisarski and Appelquist [6, 7], we do not find in literature explicit definition of chiral transformations in the context of two-component spinors alone without any reference to the four-component spinors.

In the present paper, we take up this problem of whether chiral symmetry can be defined in the fundamental representation of the gamma matrices or not. For QED3, this is the $2 \times 2$ representation. Naively, we can concentrate on any one set of gamma matrices specially because the corresponding spinors satisfy the well known completeness relation. However, parity conservation, transformation properties of the fields under charge conjugation operation (in a certain set of odd dimensions) [4], and chiral current conservation (for massless fermions) require taking into consideration the two inequivalent representations.

The work is organized as follows: We start by considering a Lagrangian containing field corresponding to only one representation. We take up the parity conserving Lagrangian, [4], and show that the massless theory has more symmetries than the massive one, a discrete exchange symmetry and two continuous symmetries. We can identify the continuous symmetries with the chiral symmetries after calculating the corresponding conserved currents and noting that they are the same as the conventional chiral currents obtained by working in the $4 \times 4$ representation of the gamma matrices. We go on to show that in an obvious way, the same chiral transformations can be defined for any number of odd dimensions in the fundamental representation of the gamma matrices. We also study the transformation properties of the chiral currents under the operation of parity and charge conjugation and make a comparison with results known earlier in the context of 4-dimensional representation of QED3, [9]. We then present our conclusions.
2 Symmetries

Starting from the Lagrangian \( \mathcal{L} = \bar{\psi} (i \hbar \mathbf{\not{D}} - mc) \psi \), we can choose the following representation of the gamma matrices in a plane: \( \gamma^0 = \sigma_3 \), \( \gamma^1 = i \sigma_1 \), and \( \gamma^2 = i \sigma_2 \) (the \( A \) representation hereafter), where \( \sigma_i \) are the Pauli matrices. We then readily obtain the solutions of the free Dirac equation:

\[
\psi^P_A (x) = \left( \frac{c(p_x - ip_y)}{E + mc^2} \right) e^{-\frac{i}{\hbar} \hat{r} \cdot \mathbf{p}} \equiv u_A (p) e^{-\frac{i}{\hbar} \hat{r} \cdot \mathbf{p}},
\]
\[
\psi^N_A (x) = \left( \frac{c(p_x + ip_y)}{E + mc^2} \right) e^{\frac{i}{\hbar} \hat{r} \cdot \mathbf{p}} \equiv v_A (p) e^{\frac{i}{\hbar} \hat{r} \cdot \mathbf{p}}.
\]

Choosing the normalization of the spinors to be such that there are \( 2E \) number of particles per unit volume, we have \( u_A \bar{u}_A = \hbar c + mc^2 \) and \( v_A \bar{v}_A = \hbar c - mc^2 \). This is to say that the completeness relations are not hampered by the fact there is just one particle spinor and one anti-particle spinor. The above relation permits us to define the projection operators \( \Lambda_\pm = (\pm \mathbf{p} + mc^2) / 2mc \) which project out the particle and ant-particle spinors respectively. Therefore, everything is apparently in order. However, there are reasons to believe that the above Lagrangian fails to incorporate various symmetries and their consequences:

- **Parity Invariance:** There are two independent solutions, one corresponding to a particle \((P)\) and the other to an anti-particle \((N)\). In a plane, there is just one orbital angular momentum which we can define as \( L = r_x p_y - r_y p_x \). It does not commute with the Hamiltonian \( H = \gamma^0 (\gamma \cdot \mathbf{p} + mc^2) \). However, if we define the spin operator as \( \Sigma = (\hbar / 2) \gamma^0 \), the total angular momentum \( J = L + \Sigma \) is a conserved quantity. It is easy to see that for the particle at rest, i.e., for \( p_x = p_y = 0 \), \( u_A \) and \( v_A \) are eigenfunctions of \( \Sigma \) with eigenvalues \( \hbar / 2 \) and \( -\hbar / 2 \) respectively. It implies a natural interpretation of the solution \( u_A \) as that of a particle with spin clockwise and of \( v_A \) as that of an anti-particle with spin anti-clockwise.

In a plane, just like any other odd dimensions, parity operation is defined by reversing the signs of all but one coordinate. Let us suppose that under parity transformation, \( r_x \rightarrow -r_x \) and \( r_y \rightarrow r_y \). Consequently, spin, being an angular momentum, changes sign. Therefore, particle with clockwise spin and anti-clockwise spin are related through the parity transformation. But one of these is not a solution of the Dirac equation. The Lagrangian and thus the particle spectrum are not parity invariant.

- **Chiral Symmetry:** In odd dimensions, all the anti-commuting matrices in the fundamental representation are consumed by the Dirac gamma matrices. We are left with no extra \( \gamma_5 \) which will anti-commute with all the gamma matrices. Therefore, chiral transformations (and chiral symmetry) cannot be defined. Apparently, the massless Lagrangian has no more symmetry than the massive one.

If we wish to incorporate these symmetries within the framework of two-component spinors, we can do this thanks to the well-known fact that for odd \( d \),
there exist two inequivalent representations, \[4\.\] In the planar case, we can choose \( B \) to be \( \gamma^0 = \sigma_3, \gamma^1 = i\sigma_1, \gamma^{2'} = -\gamma^2 = -i\sigma_2 \). We transform the corresponding solutions \( \phi^P_B \) and \( \phi^N_B \) of the Dirac equation to \( \psi^P_B \) and \( \psi^N_B \) for obvious particle identification as follows:

\[
\psi^P_B(x) = i\gamma^2 \phi^P_B = \left( \frac{c(p_0 + ip_x)}{E + mc^2} \right) e^{-\frac{1}{c}p^x} = u_B(p) e^{-\frac{1}{c}p^x},
\]

\[
\psi^N_B(x) = i\gamma^2 \phi^N_B = \left( \frac{1}{c(p_0 - ip_x)} \right) e^{\frac{1}{c}p^x} = v_B(p) e^{\frac{1}{c}p^x}.
\]

Looking at the stationary case, \( p_x = p_y = 0 \), by applying the spin operator, we can see that \( \psi^P_A \) and \( \psi^P_B \) correspond to particles with opposite spins. Similarly, \( \psi^N_A \) and \( \psi^N_B \) correspond to anti-particles with opposite spins. The parity transformation \( P : (\psi_A)^P = -i\gamma^1 \psi_B e^{i\phi_1} \) and \( (\psi_B)^P = -i\gamma^1 \psi_A e^{i\phi_2} \) swaps the spinors in the two inequivalent representations. It converts particle of one spin to the particle of opposite spin, and the same is true for the anti-particle. The Lagrangian which takes into account both the representations is, \[4\.\]

\[
\mathcal{L} = \bar{\psi}_A(ih \not\partial - mc)\psi_A + \bar{\psi}_B(ih \not\partial + mc)\psi_B.
\]

It is parity invariant, \[4\.\]. The transformation properties of the fields under the charge conjugation operation \( C \) are\(^1\) : \((\psi_A)^C = \gamma^2(\bar{\psi}_A)^T e^{i\phi_1} \) and \((\psi_B)^C = \gamma^2(\bar{\psi}_B)^T e^{i\phi_2} \). Most importantly, we shall show in the next section that this Lagrangian also permits us to define chiral symmetry.

### 3 Chiral Symmetry

We now pose ourselves the question whether the Lagrangian of Eq. (3) has any more symmetries for the massless case. An immediate look reveals there is an exchange symmetry \( \psi_A \leftrightarrow \psi_B \) which leaves the massless Lagrangian invariant. As it is only a discrete symmetry, it cannot be considered a serious candidate for the chiral symmetry. However, one can define following sets of simultaneous continuous transformations:

**Set 1**

\[
\psi_A \rightarrow \psi'_A = \psi_A + \alpha \psi_B,
\]

\[
\psi_B \rightarrow \psi'_B = \psi_B - \alpha \psi_A.
\]

**Set 2**

\[
\psi_A \rightarrow \psi'_A = \psi_A + i\alpha \psi_B,
\]

\[
\psi_B \rightarrow \psi'_B = \psi_B + i\alpha \psi_A.
\]

\(^1\)Note that the charge conjugation operation does not mix the fields corresponding to the inequivalent representations for \( d = 3, 7, 11, \cdots \) but does so for \( d = 1, 5, 9, \cdots \), \[4\.\].
Correspondingly, the Lagrangian (3) transforms in the following manner respectively:

\[ L'_{1} = L - 2mc\alpha (\bar{\psi}_{A}\psi_{B} + \bar{\psi}_{B}\psi_{A}) , \]
\[ L'_{2} = L - 2imc\alpha (\bar{\psi}_{A}\psi_{B} - \bar{\psi}_{B}\psi_{A}) , \]

Therefore, we conclude that under the continuous transformations (4,5), the massless Lagrangian is invariant and the corresponding conserved currents are:

\[ j_{\mu}^{1} = c (\bar{\psi}_{A} \gamma^{\mu} \psi_{B} - \bar{\psi}_{B} \gamma^{\mu} \psi_{A}) , \]
\[ j_{\mu}^{2} = c (\bar{\psi}_{A} \gamma^{\mu} \psi_{B} + \bar{\psi}_{B} \gamma^{\mu} \psi_{A}) . \]

To be able to identify continuous transformations (4,5) as the chiral transformation, we resort to address the issue of chiral symmetry in the often studied 4-dimensional representation of the gamma matrices in the Lagrangian

\[ L = \bar{\psi} (i\hbar \slashed{\partial} - mc) \psi : \]

\[ \gamma^{0} = \begin{pmatrix} \sigma_{3} & 0 \\ 0 & -\sigma_{3} \end{pmatrix} , \quad \gamma = \begin{pmatrix} i\sigma & 0 \\ 0 & -i\sigma \end{pmatrix} , \]

where the vector \( \sigma \) has only two components, namely, \( \sigma_{1} \) and \( \sigma_{2} \). In the 4-dimensional representation, we have sufficient freedom to define two matrices which anti-commute with the Dirac gamma matrices:

\[ \gamma^{5} = i \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} , \quad \gamma^{3} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} , \]

and hence we have two types of chiral transformations which yield the following chiral current in the massless limit:

\[ j_{\mu}^{5} \equiv c\bar{\psi} \gamma^{\mu} \gamma^{5} \psi , \quad j_{\mu}^{3} \equiv c\bar{\psi} \gamma^{\mu} \gamma^{3} \psi . \]

Interestingly, if we write out

\[ \psi = \begin{pmatrix} \psi_{A} \\ \psi_{B} \end{pmatrix} , \]

the Lagrangian \( L = \bar{\psi} (i\hbar \slashed{\partial} - mc) \psi \) reduces to the one in Eq. (3). Therefore, the components \( \psi_{A} \) and \( \psi_{B} \) themselves satisfy the Dirac equation in the 2-dimensional representation of the gamma matrices justifying the use of this notation. The chiral currents \( j_{5}^{\mu} \) and \( j_{3}^{\mu} \) of Eq. (10) can then be written as:

\[ j_{5}^{\mu} = c (\bar{\psi}_{A} \gamma^{\mu} \psi_{B} - \bar{\psi}_{B} \gamma^{\mu} \psi_{A}) , \]
\[ j_{3}^{\mu} = c (\bar{\psi}_{A} \gamma^{\mu} \psi_{B} + \bar{\psi}_{B} \gamma^{\mu} \psi_{A}) . \]
Comparing Eqs. (8,12), we see that the currents obtained from the conventional definition of chiral symmetry in the 4-dimensional representation and the ones we propose in the 2-dimensional representation, Eq. (4,5), are identical. Therefore, Eq. (4,5) are indeed chiral transformations.

We know that the 4-component massless Dirac Lagrangian has a global $U(2)$ symmetry generated by $\{1,\gamma_5,\gamma_3,i\gamma_3\gamma_5 = \gamma\}$. With the dynamical generation of fermion masses, this symmetry is broken down to a $U(1) \times U(1)$ symmetry generated by $\{1, \gamma\}$. One may ask how does this latter symmetry manifests itself in the $(\psi_A, \psi_B)$ space? We have already seen that the symmetry corresponding to $\gamma_5$ and $\gamma_3$ mixes the fields $\psi_A$ and $\psi_B$. We do not expect the same for the residual symmetry and indeed such is the case. The following sets of transformations correspond to the symmetry related to the generators 1 and $\gamma$, respectively:

**Set I**

\[
\psi_A' = \psi_A + i\alpha \psi_A, \\
\psi_B' = \psi_B + i\alpha \psi_B. \tag{14}
\]

**Set II**

\[
\psi_A' = \psi_A + i\alpha \psi_A, \\
\psi_B' = \psi_B - i\alpha \psi_B. \tag{15}
\]

The corresponding conserved currents are

\[
j^\mu_I = c \left( \bar{\psi}_A \gamma^\mu \psi_A + \bar{\psi}_B \gamma^\mu \psi_B \right), \tag{16}
\]

\[
j^\mu_{II} = c \left( \bar{\psi}_A \gamma^\mu \psi_A - \bar{\psi}_B \gamma^\mu \psi_B \right). \tag{17}
\]

It is not hard to see that the line of reasoning similar to the one just presented can be transported to higher odd dimensions smoothly. Let $\mathcal{L} = \psi(i\hbar \partial - mc)\psi$ be the Lagrangian with doubly higher-dimensional representation of the gamma matrices where chiral symmetry can be conventionally defined. We denote the gamma matrices as $\gamma_{2(d+1)/2}$. We can always choose them to be

\[
\gamma_{2(d+1)/2}^\mu = \begin{pmatrix}
\gamma_{2(d-1)/2}^\mu & 0 \\
0 & -\gamma_{2(d-1)/2}^\mu
\end{pmatrix}, \tag{18}
\]

where $\gamma_{2(d-1)/2}$ are the gamma matrices in the fundamental representation. It is easy to show that

\[
\{\gamma_{2(d+1)/2}^\mu, \gamma_{2(d+1)/2}^\nu\} = \{\gamma_{2(d-1)/2}^\mu, \gamma_{2(d-1)/2}^\nu\} = 2g^{\mu\nu}.
\]

Therefore, again expressing $\psi$ as in Eq. (11), we get the Lagrangian in Eq. (3). Note that in this generalized case, $\psi_A$ and $\psi_B$ are $2^{(d-1)/2}$-dimensional, whereas, $\psi$ is consequently $2^{(d+1)/2}$-dimensional. $\gamma_3$ and $\gamma_5$ retain their definitions in the $2^{(d-1)/2}$ dimensions. Therefore, the transformations of Eq. (4,5) are in fact chiral transformations in any number $d$ of odd dimensions.
4 Transformation Properties of Currents

In connection with the consequences of the advertised transformations in the \((\psi_A, \psi_B)\) space to the positronium bound states, it may be interesting to look at the transformation properties of the currents \(j_I^\mu\), \(j_{II}^\mu\), \(j_1^\mu\) and \(j_2^\mu\). One can easily show that the currents \(j_I^\mu\), \(j_{II}^\mu\) transform as follows:

\[
(j_I^\mu)^P = A^\mu_\nu j_I^\nu,
\]

\[
(j_{II}^\mu)^P = -A^\mu_\nu j_{II}^\nu,
\]

where \(A^\mu_\nu = \text{diag}(1, -1, 1)\). To obtain the reality of currents, we should define \(j_1^\mu \rightarrow i j_1^\mu\). On doing so and writing \(\phi_1 - \phi_2 \equiv \phi\), we have

\[
\begin{pmatrix} j_1^\mu \\ j_2^\mu \end{pmatrix}^P = \begin{pmatrix} -\cos\phi & -\sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} A^\mu_\nu \begin{pmatrix} j_1^\nu \\ j_2^\nu \end{pmatrix} = R_p A^\mu_\nu \begin{pmatrix} j_1^\nu \\ j_2^\nu \end{pmatrix}.
\]

These are exactly the transformation properties given in \([9]\) in the context of 4-dimensional representation of the gamma matrices. In the spectrum of the \(e^+ - e^-\) bound states for the dynamically broken chiral symmetry, as discussed in \([9]\), in addition to scalar, pseudoscalar, vector and pseudovector “mesons”, one also encounters axi-scalar, axi-pseudoscalar, axi-vector and axi-pseudovector states in accordance with the appearance of the parity matrix \(R_p\). Here we see that this matrix appears naturally in the \((\psi_A, \psi_B)\) space. Defining \(\psi_1 - \psi_2 \equiv \psi\), under the charge conjugation transformations

\[
\begin{pmatrix} j_1^\mu \\ j_2^\mu \end{pmatrix}^C = \begin{pmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{pmatrix} \begin{pmatrix} j_1^\mu \\ j_2^\mu \end{pmatrix} = R_c \begin{pmatrix} j_1^\mu \\ j_2^\mu \end{pmatrix}.
\]

which also agree with the transformation properties of the corresponding currents in the 4-dimensional representation tabulated in \([9]\). Hence, the transformation properties of the currents studied in the 4-dimensional representation of the gamma matrices can be mapped nicely onto the \((\psi_A, \psi_B)\) space.

5 Conclusions

We demonstrate that by taking into account both the inequivalent fundamental representations of the gamma matrices for odd number of space-time dimensions, the resulting Lagrangian is not only parity invariant but also that we can write out chiral transformations within the two-component description of the fermion spinors which mix the fields belonging to the different inequivalent representations. Interestingly, these transformations reproduce the same chiral currents yielded by the conventional chiral symmetries defined in the doubly higher dimensional representation of the gamma matrices. In connection with the dynamically broken chiral symmetry and the existence of the \(e^+ - e^-\) bound states in a plane, we also study the transformation properties of the currents under the operations of parity and charge conjugation and find the expected correspondence with the studies carried out in the 4-dimensional representation of the gamma-matrices.
Acknowledgements

Support for this work has been received in part by CIC under grant number 4.10 and CONACyT under grant number 32395-E.

References

1. I. Antoniadis, Phys. Lett. B246 377 (1990); J.D. Lykken, Phys. Rev. D54 R3693 (1996); P. Horava and E. Witten, Nucl. Phys. B460 506 (1996); E. Witten, Nucl. Phys. B471 135 (1996); P. Horava and E. Witten, Nucl. Phys. B475 94 (1996).

2. N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B429 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali Phys. Lett. B436 257 (1998); N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Rev. D59 086004 (1999).

3. S. Weinberg, Phys. Lett. B143 97 (1984); S. Deser, R. Jackiw and S. Templeton, Ann. Phys. 140 37 (1982).

4. K. Shimizu, Prog. Theor. Phys. 74 610 (1985).

5. S.J. Hands, J.B. Kogut, L. Scorzato, and C.G. Strouthos, Phys. Rev. B70 104501 (2004); S.J. Hands, J.B. Kogut and C.G. Strouthos, Nucl. Phys. B645 321 (2002); S.J. Hands, J.B. Kogut, L. Scorzato and C.G. Strouthos, Nucl. Phys. Proc. Suppl. 119 974 (2003); V.P. Gusynin and M. Reenders, Phys. Rev. D68 025017 (2003); M.R. Pennington and D. Walsh, Phys. Lett. B253 246 (1991).

6. R.D. Pisarski, Phys. Rev. D29 2423 (1984).

7. T.W. Appelquist, M. Bowick, D. Karabali and L.C.R. Wijewardhana, Phys. Rev. D33 3704 (1986); T.W. Appelquist, M. Bowick, D. Karabali and L.C.R. Wijewardhana, Phys. Rev. D33 3774 (1986).

8. C.J. Burden and C.D. Roberts, Phys. Rev. D44 540 (1991); C.J. Burden and C.D. Roberts, Phys. Rev. D46 2695 (1992); C.S. Fischer, R. Alkofer, T. Dahm and P. Maris, Phys. Rev. 70 073007 (2004).

9. C.J. Burden, Nucl. Phys. B387 419 (1992).