Linearized modified gravity theories and gravitational waves physics in the GBD theory

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Using the method of the weak-field approximation, we explore the linearized physics in the generalized Brans-Dicke (abbreviated as GBD) theory. The GBD theory is obtained with replacing the Ricci scalar $R$ in the original Brans-Dicke (BD) action by an arbitrary function $f(R)$. The linearized equations of the gravitational field and the BD scalar field are given. We investigate their solutions in the linearized theory for a point mass. It is shown that the problem of the $\gamma$ value in the $f(R)$ theories can be solved in the GBD theory, where the $\gamma$ value in the GBD theory can be consistent with the observational results. At last, we study the gravitational waves physics in the vacuum for the GBD theory. It is found that the gravitational radiation in the GBD theory has the same form as that in the general relativity theory.

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I. Introduction

The several observations in the 1990s showed that the expansion of our universe is accelerating\textsuperscript{1,2}. According to the famous general relativity (GR) theory, this accelerating expansion is caused by the dark energy with the negative pressure. The most popular candidate of dark energy from the viewpoint of the observations is the cosmological constant with the equation of state $w = -1$, though it exists the fine-tuning and the coincidence problems in the theory. Another popular candidate of dark energy model is the quintessence scalar field. The problem for this scalar-filed model is that we have to introduce the scalar field and its potential by the hand. Also, other dark energy models\textsuperscript{3–13} exist the respective problems.

GR as the standard model of gravity is tested well, especially in the solar system. But it also confronts many unanswered questions. Several observational and theoretical motivations require us to investigate the modified or alternative theories of GR. For examples, it is hard to combine quantum physics with the principles of GR, or the GR does not lead to a renormalizable model, etc. Studies on the modified gravity theories of GR has been always a hot area. Several modified gravity theories have been widely studied\textsuperscript{14–19}, especially two simple modifications to GR: the $f(R)$ theory\textsuperscript{20,21} and the Brans-Dicke (BD) theory\textsuperscript{22}. In the BD theory, a scalar field $\phi$ can be introduced naturally by defining $\phi(t) = 1/G(t)$. $G$ is the Newton gravity constant. Obviously, BD theory is a time-variable $G(t)$ gravity theory which can be sustained by the recent observations\textsuperscript{23–28}. For observational and theoretical motivations, several extended versions of the BD theory have been investigated and developed, such as adding a

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potential term to the original BD theory [29], assuming the coupling constant \( \omega \) to be variable with respect to time [30, 31], etc. The applications of these extended BD theories have been investigated widely, such as at the aspects of cosmology [32–34], weak-field approximation [35], observational constraints [36, 37], and so on [38–40].

Recently, a different method is proposed to modify the BD theory (abbreviate as GBD) in Ref. [41] by generalizing the Ricci scalar \( R \) to be an arbitrary function \( f(R) \) in the BD action. As shown in Ref. [41], the GBD theory (just as the so-called \( f(\phi)R \) theory [42, 43]) can be considered as a special case of the more complex \( f(R, \phi) \) theory [44–46]. One knows that the more simple theory is usually more favored by researchers in physics. Given that the \( f(\phi)R \) theory have been widely studied [47–49], here we continue to explore the GBD theory.

Linearized theory is a weak-field approximation to gravity theory, which is a good method to test the gravity theories alternative to GR according to the current observations. In fact, this approximation is well applicative in nature except for phenomenon dealing with the large scale structure of the universe and phenomenon dealing with black holes and gravitational collapse. The weak-field approximation method has been studied in lots of gravity theories, such as the higher order gravity [50, 51], the conformal Weyl gravity [52], the Galilean gravity [53], the infinite derivative gravity [54], the \( f(R) \) gravity [55, 56], the \( f(T) \) theory [58], the scalar-tensor gravity [59, 60], the Horava-Lifshitz gravity [62], etc. In this paper, we discuss the linearized modified gravity theories of GBD. We derive to give the linearized GBD field equations, and find its solutions for a point mass. In addition, we perform the calculation on the post Newton parameter and compare the calculation result with the astrophysical observations. It is shown that the GBD theory can solve the problem about the variance between the theoretical value and the observational value of post Newton parameter \( \gamma \) in the popular \( f(R) \) modified gravity theory.

The detection of gravitational waves (GWs) by the LIGO Collaboration is a milestone in GW research and opens a new window to probe gravity theory and astrophysics [63–65]. Future GWs observations will offer more accurate data, so it is worthwhile to investigate GWs physics in alternative theories of gravity. GWs physics have been studied in several modified gravity theories, such as the scalar-tensor theories [66–68], the \( f(R) \) theories [69–73], the conformal gravity [75], the \( f(T) \) theories [76], etc. In this paper, we study the gravitational radiation in GBD theory. We derive to give the equations of gravitational radiation, which will be valuable to test gravity theories alternative to GR for future observations of GWs.

This paper is organised as follows. The gravity field equation and the weak-field approximation equation are given in Section II. Solutions to the linearized GBD field equations for a point mass and the calculations on the post Newton parameter are investigated in Section III. In Section IV, the gravitational waves physics are studied in this part. Section V is the conclusion and the discussion on our results.

II. Weak field equations in GBD theory

In the framework of time-variable gravitational constant, we study a generalized Brans-Dicke theory by using a function \( f(R) \) to replace the Ricci scalar \( R \) in the original BD action. The action of system is written as

\[
S = \frac{1}{2} \int d^4 x \mathcal{L}_T = \frac{1}{2} \int \sqrt{-g} \left[ f(R) - \frac{\omega}{2\phi} \partial_\mu \phi \partial^\mu \phi + \frac{16\pi}{c^4} L_m \right] d^4 x.
\]

(1)

Obviously, the system contains three dynamical variable: the gravitational field \( g_{\mu\nu} \), the matter field \( \psi \) and the scalar field \( \phi \). \( \omega \) is the couple constant. Varying the action (1), one can get the gravitational field equation and the BD
with the GBD theory, respectively, as follows

Thus, to the first order we obtain the linearized gravitational field equation and the linearized BD field equation in scalar field equation as follows

Obviously, the standard $f(R)$ for taking here $\eta$ spacetime metric is nearly flat and can be expanded as

where $f_R \equiv \partial f/\partial R$, $\nabla$ is the covariant derivative associated with the Levi-Civita connection of the metric, $\Box \equiv \nabla^\mu \nabla_\mu$, and $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}$ is the energy momentum tensor of the matter. The trace of Eq. (2) is

$$f_R R - 2 f(R) + \frac{3}{4} \Box (f_R) + \frac{\omega}{\phi^2} \partial_\mu \phi \partial^\mu \phi = \frac{8 \pi T}{\phi},$$

Obviously, the standard $f(R)$ modified gravity is recovered for $\phi=\text{constant}$, while the original BD theory is obtained for taking $f(R)=R$ in the above equations. Combining Eqs. (3) and (4), we get

$$\Box \phi - \frac{\partial_\mu \phi \partial^\mu \phi}{4 \phi} = \frac{1}{4 \omega} [8 \pi T - \phi f_R R - 3 \Box (f_R)].$$

Having the correct weak-field limit at the Newtonian and the post-Newtonian levels is a crucial issue that has to be addressed for any viable alternative gravitational theory of GR. In this weak-field limit method, it means that the spacetime metric is nearly flat and can be expanded as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

here $\eta_{\mu\nu}$ is the Minkowski metric, and $h_{\mu\nu}$ denotes a small deviate with respect to the flat spacetime. The inverse metric is $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, where the Minkowski metric is used to raise the indices. The trace $h$ is given by

$$h = \eta^{\mu\nu} h_{\mu\nu}.$$ For BD scalar field, the weak-field approximation is expressed as $\phi = \phi_0 + \varphi$ with $|\varphi| \ll \phi_0$. Considering $|h_{\mu\nu}| \ll 1$ and ignoring the second-order and the higher terms, we have the expressions of linearized connection and Ricci scalar under the approximation of weak field, respectively, as follows

$$\Gamma_\mu^{\lambda\nu} = \frac{1}{2} \eta^{\lambda\sigma} (h_{\mu\nu,\sigma} + h_{\rho\nu,\mu} - h_{\mu\nu,\rho}).$$

$$R_{\mu\nu} = \frac{1}{2} (\Box h_{\mu\nu} - h_{\nu,\mu,\lambda} h_\lambda - h_{\mu,\nu,\lambda} + h_{\lambda,\mu,\nu}),$$

with $\Box h = \partial^\sigma \partial_\sigma$. Choosing a gauge, as called the Lorenz gauge or the Harmonic gauge in GR, with a form

$$(h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h - \eta^{\mu\nu} \frac{\varphi}{\phi_0})_{,\nu} = h^{\mu\nu} - \frac{1}{2} h^{\mu\nu} - \frac{\varphi_{,\mu}}{\phi_0} = 0,$$

we get the Ricci tensor and the Ricci scalar

$$R_{\mu\nu} = -\frac{1}{2} \Box h_{\mu\nu} + \frac{\partial_\nu \varphi}{\phi_0},$$

$$R = -\frac{1}{2} \Box h + \frac{1}{\phi_0} \Box \varphi.$$ Thus, to the first order we obtain the linearized gravitational field equation and the linearized BD field equation in the GBD theory, respectively, as follows

$$-\frac{1}{2} f_R \Box h_{\mu\nu} - \frac{1}{2} f(R) \eta_{\mu\nu} + \frac{f_R}{\phi_0} \eta_{\mu\nu} \Box \varphi$$

$$= \frac{8 \pi T}{\phi_0} - \frac{3}{4} \eta_{\mu\nu} \Box f_R - \frac{1}{\phi_0} [\partial_\mu \varphi \partial \nu f_R + \partial_\nu \varphi \partial_\mu f_R - \eta_{\mu\nu} (\partial^\sigma \varphi \partial_\sigma f_R + \partial^\sigma f_R \partial_\sigma \varphi)],$$

(12)
\[ \Box \eta \varphi = \frac{1}{4\omega + 3f_R} (8\pi T + \frac{1}{2} \phi_0 f_R \Box \eta h - \Box \eta \varphi f_R) - \frac{3}{4\omega + 3f_R} (\nabla^\sigma \varphi \nabla_\sigma f_R + \nabla_\sigma \varphi \nabla^\sigma f_R + \phi_0 \Box f_R). \] (13)

In the following, we solve these two linearized field equations for case of a static point mass. And then we calculate the theoretical value of the parametrized post-Newtonian (PPN) parameter \( \gamma \), and compare the theoretical result with the observations. At last, on the basis of the Eqs. (12) and (13), we consider the vacuum GWs physics in the GBD theory.

III. Solutions to the linearized field equations for a point mass and discussions on PPN parameter in GBD

A. Solutions to the linearized field equations for a point mass in GBD theory

In this part, we investigate to obtain a physically relevant solution to the linearized GBD theory. For consistency, we keep the first-order terms and ignore the second-order or the higher terms in the following calculations. To calculation, we have to take a concrete form of \( f(R) \) function. As an example, we consider a simple model \( f(R) = R - \beta R_s (1 - e^{-R/R_s}) \) [77–79]. Here \( \beta \) and \( R_s \) are two model parameters. We can expand \( f(R) \) and \( f_R \) to get

\[ f(R) = (1 + \beta) \left( \frac{1}{2} \Box \eta h - \frac{\Box \varphi}{\phi_0} \right) \] (14)

\[ f_R = 1 + \beta + \frac{\beta}{R_s} \left( \frac{1}{2} \Box \eta h - \frac{\Box \varphi}{\phi_0} \right). \] (15)

Thus, we can rewrite the linearized gravitational field equation and the linearized BD field equation, respectively, as follows

\[ \Box \eta \theta_{\mu\nu} \equiv \Box \eta (h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h - \eta_{\mu\nu} \frac{\varphi}{\phi_0}) = - \frac{16\pi T_{\mu\nu}}{(1 + \beta) \phi_0}, \] (16)

\[ \Box \varphi = \frac{8\pi T}{2\omega + 3(1 + \beta)}. \] (17)

Here \( \theta_{\mu\nu} \) is a new defined tensor. Next we consider a point mass term as a source. The energy momentum tensor of the point particle is described by

\[ T_{\mu\nu} = m \delta(\vec{r}) \text{diag}(1, 0, 0, 0). \] (18)

Obviously, here point particle is located at \( \vec{r} = 0 \) with \( r^2 = x^2 + y^2 + z^2 \). Substituting the \( T = \eta^{\mu\nu} T_{\mu\nu} = -m\delta(\vec{r}) \) into Eq. (17) and considering a static state, we obtain the expression of perturbation variable of the BD scalar field

\[ \varphi(\vec{r}) = \frac{2m}{2\omega + 3(1 + \beta)} \frac{1}{r}. \] (19)

Next, we derive the expressions of the perturbation variable of the metric field in the GBD theory. For the "00" component in Eq. (16), we get the solution

\[ \theta_{00} = \frac{4m}{(1 + \beta) \phi_0} \frac{1}{r}. \] (20)

Using the relation \( \theta = \eta^{\mu\nu} h_{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} \eta_{\mu\nu} h - \eta^{\mu\nu} \eta_{\mu\nu} \frac{\varphi}{\phi_0} = -h - \frac{4\varphi}{\phi_0} \), we have

\[ h_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \theta - \eta_{\mu\nu} \frac{\varphi}{\phi_0}. \] (21)
According to Eqs. (19-21) and using the relation \( \theta = \eta^\mu\nu \theta_{\mu\nu} = -\frac{4m}{(1+\beta)\phi_0} \), we obtain the non-vanishing components of the metric perturbation term as follows

\[
h_{00} = \frac{2m}{(1+\beta)\phi_0 \vec{r}} + \frac{2m}{[2\omega + 3(1+\beta)]\phi_0 \vec{r}},
\]

\[
h_{ij} = \left[ \frac{2m}{(1+\beta)\phi_0 \vec{r}} - \frac{2m}{[2\omega + 3(1+\beta)]\phi_0 \vec{r}} \right] \delta_{ij},
\]

Here \( i, j = 1, 2, 3 \) denote the space index, and the Greece letters denote the spacetime index. The second terms of the right hand in Eqs. (22) and (23) describe the effects of BD scalar field in the metric perturbation tensor by relating to the Eq. (19). So, we can lastly receive the expressions of the non-zero field variables as follows

\[
g_{00} = -1 + \frac{2m}{(1+\beta)\phi_0 \vec{r}} + \frac{\varphi}{\phi_0},
\]

\[
g_{ij} = [1 + \frac{2m}{(1+\beta)\phi_0 \vec{r}} - \frac{\varphi}{\phi_0}] \delta_{ij}.
\]

\[
\phi = \phi_0 [1 + \frac{2m}{2\omega + 3(1+\beta)\phi_0 \vec{r}}].
\]

Here the expression of the current BD scalar field in the GBD theory can be gained as follows

\[
\phi_0 = \frac{2\omega + 4(1+\beta)}{[2\omega + 3(1+\beta)](1+\beta)},
\]

by comparing the metric component \( g_{00} \) in the GBD with the weak-field GR or Newton potential of a point mass.

**B. Discussions on PPN parameter in GBD theory**

A gravity theory alternative to GR should be tested by the well-founded experimental results. As well known, the observational results can be directly applied to constrain the value of the parametrized post-Newtonian (PPN) parameter \( \gamma \). In addition, several debates can also be found in the \( f(R) \) modified gravity theories. For example, the value of PPN parameter in \( f(R) \) theories can be calculated to give \( \gamma = \frac{1}{2} \) by using the weak-field approximation method [21] or other methods [80, 81], which is a gross violation of the experimental bound \( |\gamma - 1| < 2.3 \times 10^{-5} \) [82]. Or Ref. [83] originally claimed that all \( f(R) \) theories should be ruled out according to the fact that metric \( f(R) \) gravity is equivalent to an \( \omega = 0 \) BD theory, since this theoretical predict value contradicts with the observational constraint \( |\omega| > 40000 \) [82]. The discussions on the equivalence between \( f(R) \) and scalar-tensor gravity theories can be found in Refs. [21, 83, 85]. The solutions to above contradictions were usually considered by the following aspects: the scalar is explained to be short-ranged, or there is even the possibility that the effective mass of the scalar field itself is actually scale-dependent [21], i.e. the so-called chameleon mechanism—where the scalar has a large effective mass at terrestrial and Solar System scales, while being effectively light at cosmological scales. So, exploring the solution to the problem of the PPN parameter is valuable. Here we discuss the theoretical value of the PPN parameter \( \gamma \) in the GBD theory and compare its value with the observation.

The concrete form of the PPN parameter \( \gamma \) in the GBD theory can be derived as follows

\[
\gamma = \frac{h_{ii}}{h_{00}} = \frac{\omega + \beta + 1}{\omega + 2\beta + 2}.
\]
From Eq. (28), one can see the dependence of the PPN parameter \( \gamma \) with respect to model parameters: \( \omega \) and \( \beta \). Obviously, for large value of \( |\omega| \), the value of PPN parameter is near to one, \( \gamma \approx 1 \). Also, it can be calculated to obtain that the theoretical predict value of \( \gamma \) in GBD theory can be consistent with the observational value \( |\gamma - 1| < 2.3 \times 10^{-5} \) for cases: \( \omega > 43476 \) or \( \omega < -43480.3 \) with \( \beta = 0 \), and \( \omega > 86952.5 \) or \( \omega < -86960.5 \) with \( \beta = 1 \), etc. In Fig. 1, we plot the pictures of \( \gamma(\omega) \) and \( \gamma(\beta) \) by taking the different values of model parameters: \( \omega \) and \( \beta \). We can see from Fig. 1 that the larger value of \( \gamma \) indicates the less value of \( \beta \) (or the larger value of \( |\omega| \)). In addition, it is shown that the problem about the \( \gamma \) value in the \( f(R) \) theories is not existence in the GBD theory, even if we do not introduce the chameleon mechanism in the latter theory.

**FIG. 1:** The variation of theoretical value \( \gamma \) with respect to \( \omega \) and \( \beta \).

### IV. Gravitational waves physics in the GBD theory

Studies on the GWs in the gravity theory are very important. The linearized framework from the weak-field approximation method provides a natural way to study the GWs. Lots of references paid attention to the studies on the GWs in the different aspects \[86-93\]. In this paper, we are interested in the vacuum GWs of the GBD theory. Considering an infinitesimal coordinate transformation, \( x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu \), we obtain

\[
\begin{align*}
\theta_{\mu\nu}' &= h_{\mu\nu}' - \partial_{\mu} \xi_{\nu} - \partial_{\nu} \xi_{\mu}, \\
h' &= h - 2 \partial_{\sigma} \xi^{\sigma}, \\
\theta_{\mu\nu}' &= h_{\mu\nu}' - \frac{1}{2} \eta_{\mu\nu} h' - \eta_{\mu\nu} \frac{\varphi'}{\varphi_0} = \theta_{\mu\nu} + \eta_{\mu\nu} \partial_{\sigma} \xi^{\sigma} - \partial_{\mu} \xi_{\nu} - \partial_{\nu} \xi_{\mu}, \\
\theta' &= \theta + 2 \partial_{\sigma} \xi^{\sigma},
\end{align*}
\]

here \( \xi^\mu \) is an arbitrary infinitesimal vector field with \( |\xi^\mu| \ll 1 \). If we choose \( \xi^\mu \) to satisfy \( \partial^\mu \theta_{\mu\nu} = \Box \eta_{\xi^\nu} \), then we again preserve and get the gauge condition \[4\]

\[
\partial^\mu \theta_{\mu\nu}' = 0.
\]

In the vacuum, we consider the gauge condition \[4\] and then solve wave Eq. \[16\] to get

\[
\theta_{\mu\nu} = A_{\mu\nu} \exp(ik_{\alpha}x^\alpha).
\]
Where \( k_\alpha \) denotes the four-wavevector, and it is a null vector with \( \eta_{\mu\nu}k^\mu k^\nu = 0 \). For GWs that propagate along the \( z \)-direction, \( k^\alpha = \varpi(1,0,0,1) \) with \( \varpi \) the angular frequency. From the gauge condition \( \partial_\mu \theta^\mu = 0 \), we can see that the amplitude tensor \( A_{\mu\nu} \) is orthogonal to the direction of propagation of the waves \( k^\mu A_{\mu\nu} = 0 \), which implies that the gauge freedom can not be fixed completely. If \( \xi^\alpha \) satisfies the equation \( \theta^\prime = 2 \partial_\sigma \xi^\sigma \), we then have \( \theta = 0 \). To uniquely specify the perturbation, we have to search for four additional constraints on \( A_{\mu\nu} \). Let us consider an observer detecting the gravitational radiation with describing by a unit timelike vector \( u^\alpha = (1,0,0,0) \). We can impose constraints \( A_{0\nu} = 0 \), and obtain the components of the metric perturbation as follows

\[
A_{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & A_+ & A_\times & 0 \\
0 & A_\times & -A_+ & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

(35)

where \( A_+ \) and \( A_\times \) represent the amplitudes of the two independent polarization states of propagating gravitational radiation, just like electromagnetic waves. Obviously, in the above system of reference there are four non-zero components for the GBD gravity theory and they have the relations: \( A_{11} = -A_{22} = A_+ \) and \( A_{12} = -A_{21} = A_\times \). So, the perturbed line element, due to the passing of a GW, can be described by

\[
ds^2 = -dt^2 + (1 + A_+)dx^2 + (1 - A_+)dy^2 + 2A_\times dxdy + dz^2.
\]

(36)

From Eqs. (34) and (35), it is clear to see that \( \theta_{\mu\nu} \) has the same form of the gravitational radiation as in GR, while with the different expression of \( \theta_{\mu\nu} \). Also, we can furthermore gain the solutions of the perturbation components \( h_{\mu\nu} \) via the relation, \( h_{\mu\nu} = \theta_{\mu\nu} - \eta_{\mu\nu} \phi \). Here \( \theta_{\mu\nu} \) have been received by Eq. (34), while the plane-wave solutions of the BD scalar-field perturbation can be easily given by solving the wave equation \( \Box \eta \phi = 0 \), according to the Eq. (17) with \( T = 0 \) in the vacuum.

V. Conclusion

Several observational and theoretical motivations require us to investigate the modified or alternative theories of GR. Lots of modified gravity theories have been proposed and widely studied, especially two simple modified gravity of GR: the \( f(R) \) theory and the Brans-Dicke theory. In the BD theory, the BD scalar field can be introduced naturally by considering a time-variable Newton gravity constant. Many extended versions of the BD theory have been explored and developed. In this paper, we explore a modified Brans-Dicke theory by generalizing the Ricci scalar \( R \) in the original BD action to an arbitrary function \( f(R) \). Using the method of the weak-field approximation, we have derived to obtain the linearized equations of the gravitational field and the BD scalar field in the GBD theory. We have investigated their solutions in the linearized theory for a point mass. It is shown that the problem about the \( \gamma \) value in the \( f(R) \) theories can be solved in the GBD theory, where the \( \gamma \) value in the GBD theory is consistent with the observational results. At last, we study the gravitational waves physics in the vacuum for the GBD theory. It is found that the gravitational radiation in the GBD theory has the same form as that in the general relativity theory.

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