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MN-convergence and lim-inf_M-convergence in partially ordered sets

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Abstract: In this paper, we first introduce the notion of MN-convergence in posets as an unified form of O-convergence and O₂-convergence. Then, by studying the fundamental properties of MN-topology which is determined by MN-convergence according to the standard topological approach, an equivalent characterization to the MN-convergence being topological is established. Finally, the lim-inf_M-convergence in posets is further investigated, and a sufficient and necessary condition for lim-inf_M-convergence to be topological is obtained.

Keywords: MN-convergence, MN-topology, lim-inf_M-convergence, M-topology

MSC: 54A20, 06A06

1 Introduction, Notations and Preliminaries

The concept of O-convergence in partially ordered sets (posets, for short) was introduced by Birkhoff [1], Frink [2] and Mcshane [3]. It is defined as follows: a net \((x_i)_{i \in I}\) in a poset \(P\) is said to \(O\)-converge to \(x \in P\) if there exist subsets \(D\) and \(F\) of \(P\) such that
\(\text{(1)}\) \(D\) is directed and \(F\) is filtered;
\(\text{(2)}\) \(\sup D = x = \inf F;\)
\(\text{(3)}\) for every \(d \in D\) and \(e \in F\), \(d \leq x_i \leq e\) holds eventually, i.e., there exists \(i_0 \in I\) such that \(d \leq x_i \leq e\) for all \(i \geq i_0\).

As what has been showed in [4], the O-convergence (Note: in [4], the O-convergence is called order-convergence) in a general poset \(P\) may not be topological, i.e., it is possible that \(P\) can not be endowed with a topology such that the O-convergence and the associated topological convergence are consistent. Hence, much work has been done to characterize those special posets in which the O-convergence is topological. The most recent result in [5] shows that the O-convergence in a poset which satisfies Condition \((\triangle)\) is topological if and only if the poset is \(O\)-doubly continuous. This means that for a special class of posets, a sufficient and necessary condition for O-convergence being topological is obtained.

As a direct generalization of O-convergence, O₂-convergence in posets has been discussed in [11] from the order-theoretical point of view. It is defined as follows: a net \((x_i)_{i \in I}\) in a poset \(P\) is said to \(O_2\)-converge to \(x \in P\) if there exist subsets \(A\) and \(B\) of \(P\) such that

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(1) sup A = x = inf B;
(2) for every a ∈ A and b ∈ B, a ≦ x₁ ≦ b holds eventually.

In fact, the $O_2$-convergence is also not topological generally. To clarify those special posets in which the $O_2$-convergence is topological, Zhao and Li [6] showed that for any poset P satisfying Condition (*), $O_2$-convergence is topological if and only if P is a-doubly continuous. As a further result, Li and Zou [7] proved that the $O_2$-convergence in a poset P is topological if and only if P is $O_2$-doubly continuous. This result demonstrates the equivalence between the $O_2$-convergence being topological and the $O_2$-double continuity of a given poset.

On the other hand, Zhou and Zhao [8] have defined the lim-inf$_M$-convergence in posets to generalize lim-inf-convergence and lim-inf$_2$-convergence [4]. They also found that the lim-inf$_M$-convergence in a poset is topological if and only if the poset is α(M)-continuous when some additional conditions are satisfied (see [8], Theorem 3.1). This result clarified some special conditions of posets under which the lim-inf$_M$-convergence is topological. However, to the best of our knowledge, the equivalent characterization to the lim-inf$_M$-convergence in general posets being topological is still unknown.

One goal of this paper is to propose the notion of $MN$-convergence in posets which can unify $O$-convergence and $O_2$-convergence and search the equivalent characterization to the $MN$-convergence being topological. More precisely,

(G11) Given a general poset P, we hope to clarify the order-theoretical condition of P which is sufficient and necessary for the $MN$-convergence being topological.

(G12) Given a poset P satisfying such condition, we hope to provide a topology that can be equipped on P such that the $MN$-convergence and the associated topological convergence agree.

Another goal is to look for the equivalent characterization to the lim-inf$_M$-convergence being topological. More precisely,

(G21) Given a general poset P, we expect to present a sufficient and necessary condition of P which can precisely serve as an order-theoretical condition for the lim-inf$_M$-convergence being topological.

(G22) Given a poset P satisfying such condition, we expect to give a topology on P such that the lim-inf$_M$-convergence and the associated topological convergence are consistent.

To accomplish those goals, motivated by the ideal of introducing the Z-subsets system [9] for defining Z-continuous posets, we propose the notion of $MN$-doubly continuous posets and define the $MN$-topology on posets in Section 2. Based on the study of the basic properties of the $MN$-topology, it is proved that the $MN$-convergence in a poset P is topological if and only if P is an $MN$-doubly continuous poset if and only if the $MN$-convergence and the topological convergence with respect to $MN$-topology are consistent. In Section 3, by introducing the notion of α$^\infty$(M)-continuous posets and presenting the fundamental properties of $M$-topology which is induced by the lim-inf$_M$-convergence, we show that the lim-inf$_M$-convergence in a poset P is topological if and only if P is an α$^\infty$(M)-continuous poset if and only if the lim-inf$_M$-convergence and the topological convergence with respect to $M$-topology are consistent.

Some conventional notations will be used in the paper. Given a set $X$, $F \subseteq X$ means that $F$ is a finite subset of $X$. Given a topological space $(X, \tau)$ and a net $(x_i)_{i \in I}$ in $X$, we take $(x_i)_{i \in I} \rightarrow x$ to mean the net $(x_i)_{i \in I}$ converges to $x \in P$ with respect to the topology $\tau$.

Let $P$ be a poset and $x \in P$. $\uparrow x$ and $\downarrow x$ are always used to denote the principal filter $\{y \in P : y \supseteq x\}$ and the principal ideal $\{z \in P : z \subseteq x\}$ of $P$, respectively. Given a poset $P$ and $A \subseteq P$, by writing sup $A$ we mean that the least upper bound of $A$ in $P$ exists and equals to sup $A \in P$; dually, by writing inf $A$ we mean that the greatest lower bound of $A$ in $P$ exists and equals to inf $A \in P$. And the set $A$ is called an upper set if $A = \uparrow A = \{b \in P : (\exists a \in A) a \subseteq b\}$, the lower set is defined dually.

For a poset $P$, we succinctly denote
- $\mathcal{P}(P) = \{A : A \subseteq P\}; \mathcal{P}_0(P) = \mathcal{P}(P)/\emptyset$;
- $\mathcal{D}(P) = \{D \in \mathcal{P}(P) : D$ is a directed subset of $P\};$.

(1) $\sup A = x = \inf B$;
(2) for every $a \in A$ and $b \in B$, $a \leq x_1 \leq b$ holds eventually.
\[ \mathcal{F}(P) = \{ F \in \mathcal{P}(P) : F \text{ is a filtered subset of } P \}; \]
\[ \mathcal{L}(P) = \{ L \in \mathcal{P}(P) : L \subseteq P \}; \mathcal{L}_0(P) = \mathcal{L}(P)/\{ \emptyset \}; \]
\[ S_0(P) = \{ \{ x \} : x \in P \}. \]

To make this paper self-contained, we briefly review the following notions:

**Definition 1.1** ([5]). Let \( P \) be a poset and \( x, y, z \in P \). We say \( y \preceq_0 x \) if for every net \( (x_i)_{i \in I} \) in \( P \) which \( O \)-converges to \( x \in P \), \( x_i \geq y \) holds eventually; dually, we say \( z \triangleright_0 x \) if for every net \( (x_i)_{i \in I} \) in \( P \) which \( O \)-converges to \( x \in P \), \( x_i \leq z \) holds eventually.

**Definition 1.2** ([5]). A poset \( P \) is said to be \( \emptyset \)-doubly continuous if for every \( x \in P \), the set \( \{ a \in P : a \preceq_0 x \} \) is directed, the set \( \{ b \in P : b \triangleright_0 x \} \) is filtered and \( \sup \{ a \in P : a \preceq_0 x \} = x = \inf \{ b \in P : b \triangleright_0 x \} \).

**Condition (\( \Delta \)).** A poset \( P \) is said to satisfy Condition(\( \Delta \)) if
1. for any \( x, y, z \in P \), \( x \preceq_0 y \leq z \) implies \( x \preceq_0 z \);
2. for any \( w, s, t \in P \), \( w \triangleright_0 s \geq t \) implies \( w \triangleright_0 t \).

**Definition 1.3** ([6]). Let \( P \) be a poset and \( x, y, z \in P \). We say \( y \preceq_a x \) if for every net \( (x_i)_{i \in I} \) in \( P \) which \( O_2 \)-converges to \( x \in P \), \( x_i \geq y \) holds eventually; dually, we say \( z \triangleright_a x \) if for every net \( (x_i)_{i \in I} \) in \( P \) which \( O_2 \)-converges to \( x \in P \), \( x_i \leq z \) holds eventually.

**Definition 1.4** ([7]). A poset \( P \) is said to be \( O_2 \)-doubly continuous if for every \( x \in P \),
1. \( \sup \{ a \in P : a \preceq_a x \} = x = \inf \{ b \in P : b \triangleright_a x \} \);
2. for any \( y, x \in P \) with \( y \preceq_a x \) and \( z \triangleright_a x \), there exist \( A \subseteq \{ a \in P : a \preceq_a x \} \) and \( B \subseteq \{ b \in P : b \triangleright_a x \} \) such that \( y \preceq_a c \) and \( z \triangleright_a c \) for each \( c \in \bigcap \{ a \cap b : a \in A \& b \in B \} \).

## 2 MN-topology on posets

Based on the introduction of \( MN \)-convergence in posets, the \( MN \)-topology can be defined on posets. In this section, we first define the \( MN \)-double continuity for posets. Then, we show the equivalence between the \( MN \)-convergence being topological and the \( MN \)-double continuity of a given poset.

A \( PMN \)-space is a triplet \( (P, M, N) \) which consists of a poset \( P \) and two subfamily \( M, N \subseteq \mathcal{P}(P) \).

All \( PMN \)-spaces \( (P, M, N) \) considered in this section are assumed to satisfy the following conditions:

(C1) If \( P \) has the least element \( \bot \), then \( \{ \bot \} \in M; \)
(C2) If \( P \) has the greatest element \( \top \), then \( \{ \top \} \in N; \)
(C3) \( \emptyset \notin M \) and \( \emptyset \notin N. \)

**Definition 2.1.** Let \( (P, M, N) \) be a \( PMN \)-space. A net \( (x_i)_{i \in I} \) in \( P \) is said to \( MN \)-converge to \( x \in P \) if there exist \( M \in M \) and \( N \in N \) satisfying:

\( MN \text{(1)} \) sup \( M = x = \inf N; \)
\( MN \text{(2)} \) \( x \in \uparrow m \cap \downarrow n \) eventually for every \( m \in M \) and every \( n \in N. \)

In this case, we will write \( (x_i)_{i \in I} \xrightarrow{MN} x. \)

**Remark 2.2.** Let \( (P, M, N) \) be a \( PMN \)-space.

(1) If \( M = \mathcal{D}(P) \) and \( N = \mathcal{F}(P) \), then a net \( (x_i)_{i \in I} \xrightarrow{MN} x \in P \) if and only if it \( O \)-converges to \( x. \) That is to say, \( O \)-convergence is a particular case of \( MN \)-convergence.

(2) If \( M = N = \mathcal{P}(P) \), then a net \( (x_i)_{i \in I} \xrightarrow{MN} x \in P \) if and only if it \( O_2 \)-converges to \( x. \) That is to say, \( O_2 \)-convergence is a special case of \( MN \)-convergence.
(3) If $M = N = \mathcal{L}_0(P)$, then a net $(x_i)_{i \in I} \rightarrow x \in P$ if and only if $x_1 = x$ holds eventually.

(4) The $\mathcal{MN}$-convergent point of a net $(x_i)_{i \in I}$ in $P$, if it exists, is unique. Indeed, suppose that $(x_i)_{i \in I} \rightarrow x_1$ and $(x_i)_{i \in I} \rightarrow x_2$. Then there exist $A_k \subseteq M$ and $B_k \subseteq N$ such that $\sup A_k = x_k = \inf B_k$ and $a_k \leq x_i \leq b_k$ holds eventually for every $a_k \in A_k$ and $b_k \in B_k$ ($k = 1, 2$). This implies that for any $a_1 \in A_1, a_2 \in A_2, b_1 \in B_1$ and $b_2 \in B_2$, there exists $i_0 \in I$ such that $a_1 \leq x_{i_0} \leq b_2$ and $a_2 \leq x_{i_0} \leq b_1$. Thus we have $sup A_1 = x_1 \leq \inf B_2 = x_2$ and $sup A_2 = x_2 \leq \inf B_1 = x_1$. Therefore $x_1 = x_2$.

(5) For any $A \subseteq M$ and $B \subseteq N$ with $sup A = \inf B = x \in P$, we denote $F^x_{(A,B)} = \{\bigcap\{a \cap b : a \in A_0 & b \in B_0\} : A_0 \subseteq A & B_0 \subseteq B\}$. Let $D^x_{(A,B)} = \{(d, D) \in P \times F^x_{(A,B)} : d \in D\}$ and let the preorder $\triangleright$ on $D^x_{(A,B)}$ be defined by

$$(\forall (d_1, D_1), (d_2, D_2) \in D^x_{(A,B)})(d_1, D_1) \triangleright (d_2, D_2) \iff D_2 \subseteq D_1.$$ 

One can readily check that $(D^x_{(A,B)}, \triangleright)$ is directed. Now if we take $(x_{(d,D)})_{(d,D) \in D^x_{(A,B)}}$ be the net defined in (5) for any $A \subseteq M$ and $B \subseteq N$ with $sup A = \inf B = x \in P$. If $(x_{(d,D)})_{(d,D) \in D^x_{(A,B)}}$ converges to $p \in P$ with respect to some topology $\tau$ on the poset $P$, then for every open neighborhood $U_p$ of $p$, there exist $A_0 \subseteq A$ and $B_0 \subseteq B$ such that

$$\bigcap\{a \cap b : a \in A_0 & b \in B_0\} \subseteq U_p.$$ 

Indeed, suppose that $(x_{(d,D)})_{(d,D) \in D^x_{(A,B)}} \rightarrow p$. Then for every open neighborhood $U_p$ of $p$, there exists $(d_0, D_0) \in D^x_{(A,B)}$ such that $x_{(d,D)} = d \in U_p$ for all $(d, D) \geq (d_0, D_0)$. Since $(d, D) \geq (d_0, D_0)$ for every $d \in D_0, x_{(d,D)} = d \in U_p$ for every $d \in D_0$. This shows $D_0 \subseteq U_p$. So, there exist $A_0 \subseteq A$ and $B_0 \subseteq B$ such that

$$D_0 = \bigcap\{a \cap b : a \in A_0 & b \in B_0\} \subseteq U_p.$$ 

Given a $PMN$-space $(P, M, N)$, we can define two new approximate relations $\ll_{M}^{N}$ and $\gg_{M}^{N}$ on the poset $P$ in the following definition.

**Definition 2.3.** Let $(P, M, N)$ be a $PMN$-space and $x, y, z \in P$.

1. We define $y \ll_{M}^{N} x$ if for any $A \subseteq M$ and $B \subseteq N$ with $sup A = x = \inf B$, there exist $A_0 \subseteq A$ and $B_0 \subseteq B$ such that

$$\bigcap\{a \cap b : a \in A_0 & b \in B_0\} \subseteq \uparrow y.$$ 

2. Dually, we define $z \gg_{M}^{N} x$ if for any $M \subseteq M$ and $N \subseteq N$ with $sup M = x = \inf N$, there exist $M_0 \subseteq M$ and $N_0 \subseteq N$ such that

$$\bigcap\{m \cap n : m \in M_0 & n \in N_0\} \subseteq \downarrow z.$$ 

For convenience, given a $PMN$-space $(P, M, N)$ and $x \in P$, we will briefly denote

- $\nabla_{M}^{N} x = \{y \in P : y \ll_{M}^{N} x\};$
- $\Delta_{M}^{N} x = \{z \in P : x \ll_{M}^{N} z\};$
- $\nabla_{M}^{\nabla_{M}^{N}} x = \{a \in P : x \gg_{M}^{N} a\};$
- $\Delta_{M}^{\nabla_{M}^{N}} x = \{b \in P : b \gg_{M}^{N} x\}.$

**Remark 2.4.** Let $(P, M, N)$ be a $PMN$-space and $x, y, z \in P$.

1. If there is no $A \subseteq M$ such that $sup A = x$, then $p \ll_{M}^{N} x$ and $p \gg_{M}^{N} x$ for all $p \in P$; similarly, if there is no $B \subseteq N$ such that $inf B = x$, then $p \ll_{M}^{N} x$ and $p \gg_{M}^{N} x$ for all $p \in P$.

2. By Definition 2.3, one can easily check that if $P$ has the least element $\bot$, then $\bot \ll_{M}^{N} p$ for every $p \in P$, and if $P$ has the greatest element $\top$, then $\top \gg_{M}^{N} p$ for every $p \in P$.

1 From the logical point of view, we stipulate $\bigcap\{a \cap b : a \in A_0 & b \in B_0\} = P$ if $A_0 = \emptyset$ or $B_0 = \emptyset$. 
(3) The implications \( y \leq M_N \Rightarrow x \leq y \) and \( z > M_N \Rightarrow z \geq x \) are not true necessarily. See the following example:

let \( \mathbb{R} \) be the set of all real numbers, in its ordinal order, and \( M = \mathbb{N} = \{ n \in \mathbb{Z} \} \), where \( \mathbb{Z} \) is the set of all integers. Then, by (1), we have \( 1 \leq M_N 1/2 \) and \( 0 > M_N 1/2 \). But \( 1 \leq 1/2 \) and \( 0 > 1/2 \).

(4) Assume that \( \sup A_0 = x = \inf B_0 \) for some \( A_0 \in M \) and \( B_0 \in N \). Then it follows from Definition 2.3 that \( y \leq M_N x \) implies \( y \leq x \) and \( z > M_N x \) implies \( z \geq x \). In particular, if \( 0 \leq M_N 0 \), then \( b \leq a \) and \( c \geq a \) imply \( b \leq a \) and \( c \geq a \) for all \( a, b, c \in P \). More particularly, for any \( p_1, p_2, p_3 \in P \), we have \( p_1 < M_N p_2 \) if and only if \( p_1 \leq p_2 \) and \( p_3 > M_N p_2 \) if and only if \( p_3 > p_2 \).

Proposition 2.5. Let \( (P, M, N) \) be a PMN-space and \( x, y, z \in P \). Then

(1) \( y \leq M_N x \) if and only if for every net \( (x_i)_{i \in I} \) that \( M_N \)-converges to \( x \), \( x_i \geq y \) holds eventually.

(2) \( z > M_N x \) if and only if for every net \( (x_i)_{i \in I} \) that \( M_N \)-converges to \( x \), \( x_i \leq z \) holds eventually.

Proof. (1) Suppose \( y \leq M_N x \). If a net \( (x_i)_{i \in I} \rightarrow x \), then there exist \( A \in M \) and \( B \in N \) such that \( \sup A = x = \inf B \), and for any \( a \in A \) and \( b \in B \), there exists \( i \in I \) such that \( a \leq x_i \leq b \) for all \( i \in I \). According to Definition 2.3 (1), it follows that there exist \( A_0 = \{ a_1, a_2, \ldots, a_n \} \subseteq A \) and \( B_0 = \{ b_1, b_2, \ldots, b_m \} \subseteq B \) such that \( x \in \bigcap \{ a_k \cap b_j : 1 \leq k \leq n \text{ and } 1 \leq j \leq m \} \subseteq \uparrow y \). Take \( i_0 \in I \) with that \( i_0 \geq b_i \) for every \( k \in \{ 1, 2, \ldots, n \} \) and every \( j \in \{ 1, 2, \ldots, m \} \). Then \( x_i \in \bigcap \{ a_k \cap b_j : 1 \leq k \leq n \text{ and } 1 \leq j \leq m \} \subseteq \uparrow y \) for all \( i \geq i_0 \). This means \( x_i \geq y \) holds eventually.

Conversely, suppose that for every net \( (x_i)_{i \in I} \) that \( M_N \)-converges to \( x \), \( x_i \geq y \) holds eventually. For every \( A \in M \) and \( B \in N \) with \( \sup A = x = \inf B \), consider the net \( (x_i)_{(x_i, i)} \in D_{(a, b)}^{\uparrow y} \) defined in Remark 2.2 (5). By Remark 2.2 (5), the net \( (x_i, i)_{D_{(a, b)}^{\uparrow y}} \rightarrow x \). So, there exists \( (d_0, D_0) \in D_{(a, b)}^{\uparrow y} \) such that \( x(d, D) = y \) for all \( (d, D) \geq (d_0, D_0) \). Since \( (d, D) \geq (d_0, D_0) \) for all \( d \in D_0 \), \( x(d) = y \) for all \( d \in D_0 \). Thus, \( D_0 \subseteq \uparrow y \) if \( D_0 \subseteq \uparrow y \). It follows from the definition of \( D_{(a, b)}^{\uparrow y} \) that there exist \( A_0 \subseteq A \) and \( B_0 \subseteq B \) such that \( D_0 = \bigcap \{ a_k \cap b_j : a \in A_0 \text{ and } b \in B_0 \} \subseteq \uparrow y \). This shows \( y \leq M_N x \).

The proof of (2) can be processed similarly. \( \square \)

Remark 2.6. Let \( (P, M, N) \) be a PMN-space.

(1) If \( M = D(P) \) and \( N = \uparrow \uparrow P \), then \( \leq M_N = \leq \uparrow \uparrow P = \uparrow \uparrow D(P) \).

(2) If \( M = N = \uparrow \uparrow P \), then \( \leq \uparrow \uparrow P = \leq \uparrow \uparrow P = \uparrow \uparrow \uparrow D(P) \).

Given a PMN-space \( (P, M, N) \), depending on the approximate relations \( \leq M_N \) and \( \uparrow \uparrow \uparrow D(P) \) on \( P \), we can define the \( M_N \)-double continuity for the poset \( P \).

Definition 2.7. Let \( (P, M, N) \) be a PMN-space. The poset \( P \) is called an \( M_N \)-doubly continuous poset if for every \( x \in P \), there exist \( M_x \in M \) and \( N_x \in N \) such that

\[
\begin{align*}
(1) & M_x \subseteq \uparrow \uparrow M_N x, N_x \subseteq \downarrow \downarrow M_N x \text{ and } x = \inf N_x. \\
(2) & \text{For any } y \in \uparrow \uparrow M_N x \text{ and } z \in \downarrow \downarrow M_N x, \bigcap \{ \uparrow m \cap \downarrow n : m \in M_0 \text{ and } n \in N_0 \} \subseteq \uparrow \uparrow M_N y \cap \downarrow \downarrow M_N z \text{ for some } M_0 \subseteq M_x \text{ and } N_0 \subseteq N_x.
\end{align*}
\]

By Remark 2.4 (4) and Definition 2.7, we have the following basic property about \( M_N \)-doubly continuous posets:

Proposition 2.8. Let \( (P, M, N) \) be a PMN-space and \( x, y, z \in P \). If the poset \( P \) is an \( M_N \)-doubly continuous poset, then \( y \leq M_N x \) implies \( y \leq x \) and \( z > M_N x \) implies \( z \geq x \).

Example 2.9. Let \( (P, M, N) \) be a PMN-space.

(1) If \( M = N = 0 \), then by Remark 2.4 (4), we have \( \leq 0 = \leq 0 = \uparrow \uparrow 0 = \uparrow \uparrow 0 \). By Definition 2.7, one can easily check that \( P \) is an \( 0 \)-doubly continuous poset.

(2) If \( M = N = L_0 \), then by Definition 2.3, we have \( \leq L_0 = \leq L_0 = \uparrow \uparrow L_0 = \uparrow \uparrow L_0 \). It can be easily checked from Definition 2.7 that \( P \) is an \( L_0 \)-doubly continuous poset.
(3) Let \(M = \mathcal{D}(P)\) and \(N = \mathcal{F}(P)\). Then it is easy to check that if \(P\) is an \(\bigwedge\)-doubly continuous poset which satisfies Condition (\(\diamond\)), then it is a \(\mathcal{DF}\)-doubly continuous poset. Particularly, finite posets, chains and anti-chains, completely distributive lattices are all \(\mathcal{DF}\)-doubly continuous posets.

(4) Let \(M = N = \mathcal{P}_0(P)\). Then the poset \(P\) is \(\mathcal{P}_0\mathcal{P}_0\)-double continuous if and only if it is \(O_2\)-double continuous.

Thus, chains and finite posets are all \(\mathcal{P}_0\mathcal{P}_0\)-doubly continuous posets.

Next, we are going to consider the \(MN\)-topology on posets, which is induced by the \(MN\)-convergence.

**Definition 2.10.** Given a PMN-space \((P, M, N)\), a subset \(U\) of \(P\) is called an \(MN\)-open set if for every net \((x_i)_{i \in I}\) with that \((x_i)_{i \in I} \rightarrow x \in U\), \(x_1 \in U\) holds eventually.

Clearly, the family \(\mathcal{O}_M^N(P)\) consisting of all \(MN\)-open subsets of \(P\) forms a topology on \(P\). And this topology is called the \(MN\)-topology.

**Theorem 2.11.** Let \((P, M, N)\) be a PMN-space. Then a subset \(U\) of \(P\) is an \(MN\)-open set if and only if for every \(M \in M\) and \(N \in N\) with \(\sup M = x = \inf N \in U\), we have

\[
\bigcap \{ \uparrow m \cap \downarrow n : m \in M_0 \land n \in N_0 \} \subseteq U
\]

for some \(M_0 \subseteq M\) and \(N_0 \subseteq N\).

**Proof.** Suppose that \(U\) is an \(MN\)-open subset of \(P\). For every \(M \in M\) and \(N \in N\) with \(\sup M = x = \inf N \in U\), let \((x_{(d,d)})_{(d,d) \in D_{(d,d)}^{(d,d)}} \rightarrow x\) be the net defined in Remark 2.2 (5). Then the net \((x_{(d,d)})_{(d,d) \in D_{(d,d)}^{(d,d)}} \rightarrow x\). By the definition of \(MN\)-open set, the exists \((d_0, D_0) \in D_{(M,N)}^x\) such that \(x_{(d,d)} = d \in U\) for all \((d, D) \geq (d_0, D_0)\). Since \((d, D) \geq (d_0, D_0)\) for all \(d \in D_0\), \(x_{(d,D)} = d \in U\) for every \(d \in D_0\), and thus \(D_0 \subseteq U\). It follows from the definition of the directed set \(D_{(M,N)}^x\) that 

\[
D_0 = \bigcap \{ \uparrow m \cap \downarrow n : m \in M_0 \land n \in N_0 \} \subseteq U
\]

for some \(M_0 \subseteq M\) and \(N_0 \subseteq N\).

Conversely, assume that \(U\) is a subset of \(P\) with the property that for any \(M \in M\) and \(N \in N\) with \(\sup M = x = \inf N \in U\), there exist \(M_0 = \{ m_1, m_2, \ldots, m_k \} \subseteq M\) and \(N_0 = \{ n_1, n_2, \ldots, n_l \} \subseteq N\) such that \(\bigcap \{ \uparrow m \cap \downarrow n : 1 \leq h \leq k \land 1 \leq j \leq l \} \subseteq U\). Let \((x_i)_{i \in I}\) be a net that \(MN\)-converges to \(x \in U\). Then there exist \(M \in M\) and \(N \in N\) such that \(\sup M = x = \inf N \in U\), and for every \(m \in M\) and \(n \in N\), \(m \leq x \leq n\) holds eventually. This means that for every \(m_h \in M_0\) and \(n_j \in N_0\), there exists \(i_{h,j} \in I\) such that \(m_h \leq x_i \leq n_j\) for all \(i \geq i_{h,j}\). Take \(i_0 \in I\) such that \(i_0 \geq i_{h,j}\) for all \(h \in \{ 1, 2, \ldots, k \}\) and \(j \in \{ 1, 2, \ldots, l \}\). Then \(x_i \in \bigcap \{ \uparrow m_h \cap \downarrow n_j : 1 \leq h \leq k \land 1 \leq j \leq l \}\) for all \(i \geq i_0\). Therefore, \(U\) is an \(MN\)-open subset of \(P\).

**Proposition 2.12.** Let \((P, M, N)\) be a PMN-space in which \(P\) is an \(MN\)-doubly continuous poset, and \(y, z \in P\). Then \(\Delta^N_{MN} y \cap \nabla^N_{MN} z \in \mathcal{O}_M^N(P)\).

**Proof.** Suppose that \(M \in M\) and \(N \in N\) with \(\sup M = \inf N = x \in \Delta^N_{MN} y \cap \nabla^N_{MN} z\). Since \(P\) is an \(MN\)-doubly continuous poset, there exist \(M_x \in M\) and \(N_x \in N\) satisfying condition (A1) and (A2) in Definition 2.7. This means that there exist \(M_0 \subseteq M_x \subseteq \nabla^N_{MN} x\) and \(N_0 \subseteq N_x \subseteq \Delta^N_{MN} x\) such that \(\bigcap \{ \uparrow m_0 \cap \downarrow n_0 : m_0 \in M_0 \land n_0 \in N_0 \} \subseteq \Delta^N_{MN} y \cap \nabla^N_{MN} z\). As \(M_0 \subseteq M_x \subseteq \nabla^N_{MN} x\) and \(N_0 \subseteq N_x \subseteq \Delta^N_{MN} x\), by Definition 2.3, there exist \(M_{m_0} \subseteq M\) and \(N_{n_0} \subseteq N\) such that \(\bigcap \{ \uparrow m \cap \downarrow n : m \in M_{m_0} \land n \in N_{n_0} \} \subseteq \uparrow m_0 \cap \downarrow n_0\) for every \(m_0 \in M_0\) and \(n_0 \in N_0\). Take \(M_F = \bigcup \{ M_{m_0} : m_0 \in M_0 \}\) and \(N_F = \bigcup \{ N_{n_0} : n_0 \in N_0 \}\). Then it is easy to check that \(M_F \subseteq M\) and \(N_F \subseteq N\) and

\[
x \in \bigcap \{ \uparrow a \cap \downarrow b : a \in M_F \land b \in N_F \}
\]

\[
\subseteq \bigcap \{ \uparrow m_0 \cap \downarrow n_0 : m_0 \in M_0 \land n_0 \in N_0 \}
\]

\[
\subseteq \Delta^N_{MN} y \cap \nabla^N_{MN} z.
\]

So, it follows from Theorem 2.11 that \(\Delta^N_{MN} y \cap \nabla^N_{MN} z \in \mathcal{O}_M^N(P)\).

**Lemma 2.13.** Let \((P, M, N)\) be a PMN-space in which \(P\) is an \(MN\)-doubly continuous poset. Then a net \((x_i)_{i \in I} \rightarrow x \in P \iff (x_i)_{i \in I} \rightarrow x\).
Proof. From the definition of $\bigtriangleup_{\mathcal{M}}^N(P)$, it is easy to see that a net

$$(x_i)_{i \in I} \rightarrow x \in P \Rightarrow (x_i)_{i \in I} \rightarrow x.$$ \hspace{1cm} (5.1)

To prove the Lemma, it suffices to show that a net $(x_i)_{i \in I} \rightarrow x$ implies $(x_i)_{i \in I} \rightarrow x$. Suppose a net $(x_i)_{i \in I} \rightarrow x$. Since $P$ is an $\mathcal{MN}$-doubly continuous poset, there exist $M_\varepsilon \in \mathcal{M}$ and $N_\varepsilon \in \mathcal{N}$ such that $M_\varepsilon \subseteq \Delta_{\mathcal{M}}^N x$ and $N_\varepsilon \subseteq \Delta_{\mathcal{M}}^N x$ and hence $x_i \in \bigtriangleup_{\mathcal{M}}^N x$ holds eventually for every $y \in M_\varepsilon \subseteq \bigtriangleup_{\mathcal{M}}^N x$ and every $z \in N_\varepsilon \subseteq \bigtriangleup_{\mathcal{M}}^N x$. It follows from Proposition 2.8 that $y \subseteq x_i \subseteq z$ holds eventually for every $y \in M_\varepsilon$ and $z \in N_\varepsilon$. Thus $(x_i)_{i \in I} \rightarrow x$. \hfill \Box

Lemma 2.14. Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space. If the $\mathcal{MN}$-convergence in $P$ is topological, then $P$ is $\mathcal{MN}$-doubly continuous.

Proof. Suppose that the $\mathcal{MN}$-convergence in $P$ is topological. Then there exists a topology $\mathcal{T}$ on $P$ such that for every $x \in P$, a net $(x_i)_{i \in I} \rightarrow x$ if and only if $(x_i)_{i \in I} \rightarrow x$. Define $I_x = \{(p, U) \in P \times \mathcal{N}(x) : p \in U\}$, where $\mathcal{N}(x)$ denotes the set of all open neighbourhoods of $x$ in the topological space $(P, \mathcal{T})$, i.e., $\mathcal{N}(x) = \{U \in \mathcal{T} : x \in U\}$. Define the preorder $\leq$ on $I_x$ as follows:

$$(p_1, U_1) \leq (p_2, U_2) \iff U_2 \subseteq U_1.$$ \hspace{1cm} (5.2)

Now one can easily see that $I_x$ is directed. Let $x_{(p, U)} = p$ for every $(p, U) \in I_x$. Then it is straightforward to check that the net $(x_{(p, U)})_{(p, U) \in I_x} \rightarrow x$, and hence $(x_{(p, U)})_{(p, U) \in I_x} \rightarrow x$. By Definition 2.1, there exist $M_\varepsilon \in \mathcal{M}$ and $N_\varepsilon \in \mathcal{N}$ such that sup $M_\varepsilon = x = \inf N_\varepsilon$, and for every $m \in M_\varepsilon$ and $n \in N_\varepsilon$, there exists $(p_m^n, U_m^n) \in I_x$ such that $x_{(p_m^n, U_m^n)} = p \in \uparrow m \cap \downarrow n$ for all $(p, U) \ni (p_m^n, U_m^n)$. Since $(p, U_m^n) \ni (p_m^n, U_m^n)$ for every $p \in U_m^n$, $x_{(p, U_m^n)} = p \in \uparrow m \cap \downarrow n$ for every $p \in U_m^n$. This shows

$$(\forall m \in M_\varepsilon, n \in N_\varepsilon)(\exists U_m^n \in \mathcal{N}(x)) x \in U_m^n \subseteq \uparrow m \cap \downarrow n.$$ \hspace{1cm} (5.3)

For any $A \in \mathcal{M}$ and $B \in \mathcal{N}$ with sup $A = x = \inf B$, let $(x_{(d, D)})_{(d, D) \in D_{\mathcal{M}, \mathcal{N}}} \rightarrow x$, and hence $(x_{(d, D)})_{(d, D) \in D_{\mathcal{M}, \mathcal{N}}} \rightarrow x$. This implies, by Remark 2.2 (6), that there exist $A_0 \subseteq A$ and $B_0 \subseteq B$ satisfying

$x \in \bigcap\{\uparrow a \cap \downarrow b : a \in A_0 \& b \in B_0\}$

\hspace{1cm} $\subseteq (\uparrow a \cap \downarrow b : a \in A_0 \& b \in B_0)$

$\subseteq \uparrow a \cap \downarrow b.$

Therefore, $m \in \bigtriangledown_{\mathcal{M}}^N x$ and $n \in \bigtriangledown_{\mathcal{N}}^N x$, and hence $M_\varepsilon \subseteq \bigtriangleup_{\mathcal{M}}^N x$ and $N_\varepsilon \subseteq \Delta_{\mathcal{M}}^N x$.

Let $y \in \bigtriangledown_{\mathcal{M}}^N x$ and $z \in \Delta_{\mathcal{N}}^N x$. Since sup $M_\varepsilon = x = \inf N_\varepsilon$, by Definition 2.3, $\bigcap\{\uparrow m \cap \downarrow n : m \in M_1 \& n \in N_1\} \subseteq \uparrow y \cap \downarrow z$ for some $M_1 \subseteq M_\varepsilon$ and $N_1 \subseteq N_\varepsilon$. This concludes by Condition (5.3) and the finiteness of sets $M_1$ and $N_1$ that $\bigcap\{U_m^n : m \in M_1 \& n \in N_1\} \subseteq N(x)$ and

$x \in \bigcap\{U_m^n : m \in M_1 \& n \in N_1\}$

\hspace{1cm} $\subseteq \bigcap\{\uparrow m \cap \downarrow n : m \in M_1 \& n \in N_1\}$

\hspace{1cm} $\subseteq \uparrow y \cap \downarrow z.$

Considering the net $(x_{(d, D)})_{(d, D) \in D_{\mathcal{M}, \mathcal{N}}} \rightarrow x$, defined in Remark 2.2 (5), we have $(x_{(d, D)})_{(d, D) \in D_{\mathcal{M}, \mathcal{N}}} \rightarrow x$, and hence $(x_{(d, D)})_{(d, D) \in D_{\mathcal{M}, \mathcal{N}}} \rightarrow x$. So, by Remark 2.2 (6), there exist $M_2 \subseteq M_\varepsilon$ and $N_2 \subseteq N_\varepsilon$ such that

$x \in \bigcap\{\uparrow m \cap \downarrow n : m \in M_2 \& n \in N_2\}$

\hspace{1cm} $\subseteq \bigcap\{U_m^n : m \in M_2 \& n \in N_2\}$

\hspace{1cm} $\subseteq \uparrow y \cap \downarrow z.$
Finally, we show \( \bigcap \{ \uparrow m \cap \downarrow n : m \in M_2 \& n \in N_2 \} \subseteq \bigwedge_{M}^N \bigvee_{M}^N z \). Let \((x_{(d,D)})_{(d,D) \in D_{M,N}}\) be the net defined in 2.2 (5) for any \( M \in M \) and \( N' \in N \) with \( \sup M = \inf N' = x' \in \bigcap \{ \uparrow m \cap \downarrow n : m \in M_2 \& n \in N_2 \} \). Then \((x_{(d,D)})_{(d,D) \in D_{M,N}} \rightarrow x' \), and thus \((x_{(d,D)})_{(d,D) \in D_{M,N}} \rightarrow \) \( x' \). This implies by Remark 2.2 (6) that there exist \( M_0 \subseteq M \) and \( N_0 \subseteq N \) satisfying

\[
\exists \in \bigcap \{ \uparrow m \cap \downarrow n : m \in M_0 \& n \in N_0 \}
\subseteq \bigcap \{ U_m^n : m \in M_1 \& n \in N_1 \}
\subseteq \bigcap \{ m \in M_1 \& n \in N_1 \}
\subseteq \bigcap \{ m \in M_1 \& n \in N_1 \}.
\]

Hence, we have \( x' \in \bigwedge_{M}^N \bigvee_{M}^N z \) by Definition 2.3. This shows \( \bigcap \{ \uparrow m \cap \downarrow n : m \in M_2 \& n \in N_2 \} \subseteq \bigwedge_{M}^N \bigvee_{M}^N z \).

Therefore, it follows from Definition 2.7 that \( P \) is \( M \& N \)-doubly continuous.

Combining Lemma 2.13 and Lemma 2.14, we obtain the following theorem.

**Theorem 2.15.** Let \((P, M, N)\) be a \( PM \& N \)-space. Then the following statements are equivalent:

1. \( P \) is an \( MN \)-doubly continuous poset.
2. For any net \((x_i)_{i \in I} \in P \), \((x_i)_{i \in I} \rightarrow x \) if and only if \((x_i)_{i \in I} \rightarrow x \).
3. The \( MN \)-convergence in \( P \) is topological.

**Proof.** (1) \( \Rightarrow \) (2): By Lemma 2.13.

(2) \( \Rightarrow \) (3): It is clear.

(3) \( \Rightarrow \) (1): By Lemma 2.14.

\[\square\]

### 3 \( M \)-topology induced by \( \liminf_M \)-convergence

In this section, the notion of \( \liminf_M \)-convergence is reviewed and the \( M \)-topology on posets is defined. By exploring the fundamental properties of the \( M \)-topology, those posets under which the \( \liminf_M \)-convergence is topological are precisely characterized.

By saying a \( PM \)-space, we mean a pair \((P, M)\) that contains a poset \( P \) and a subfamily \( M \) of \( P(P) \).

**Definition 3.1 (8).** Let \((P, M)\) be a \( PM \)-space. A net \((x_i)_{i \in I} \) in \( P \) is said to \( \liminf_M \)-converge to \( x \in P \) if there exists \( M \in M \) such that

\begin{itemize}
  \item [(M1)] \( x \leq \sup M; \)
  \item [(M2)] for every \( m \in M \), \( x_i \geq m \) holds eventually.
\end{itemize}

In this case, we write \((x_i)_{i \in I} \rightarrow_M x \).

It is worth noting that both \( \liminf \)-convergence and \( \liminf_{2} \)-convergence \([4]\) in posets are particular cases of \( \liminf_M \)-convergence.

**Remark 3.2.** Let \((P, M)\) be a \( PM \)-space and \( x, y \in P \).

1. Suppose that a net \((x_i)_{i \in I} \rightarrow_M x \) and \( y \leq x \). Then \((x_i)_{i \in I} \rightarrow y \) by Definition 3.1. This concludes that the set of all \( \liminf_M \)-convergent points of the net \((x_i)_{i \in I} \) in \( P \) is a lower subset of \( P \). Thus, the \( \liminf_M \)-convergent points of the net \((x_i)_{i \in I} \) need not be unique.

2. If \( P \) has the least element \( \bot \) and \( \emptyset \in M \), then we have \((x_i)_{i \in I} \rightarrow_M \bot \) for every net \((x_i)_{i \in I} \) in \( P \).
(3) For every $M \in \mathcal{M}$ with $\text{sup} M \geq x$, we denote $F_M^x = \{ \{ \uparrow m : m \in M_0 \} : M_0 \sqsubseteq M \}$. Let $D_M^x = \{(d, D) \in P \times F_M^x : d \in D \}$ be the preorder $\leq$ defined by
\[
(\forall (d_1, D_1), (d_2, D_2) \in D_M^x)(d_1, D_1) \leq (d_2, D_2) \iff D_2 \subseteq D_1.
\]

It is easy to see that the set $D_M^x$ is directed. Take $x_{(d, D)} = d$ for every $(d, D) \in D_M^x$. Then, by Definition 3.1, one can straightforwardly check that the net $x_{(d, D)}(d, D) \in D_M^x$ defined in (3) converges to $p \in P$ with respect to some topology $\mathcal{T}$ on $P$, then for every open neighbourhood $U_p$ of $p$, there exists $M_0 \sqsubseteq M$ such that $\bigcap \{ \{ \uparrow m : m \in M_0 \} : M_0 \sqsubseteq M \} \subseteq U_p$.

**Definition 3.3** ([8]). Let $(P, \mathcal{M})$ be a PM-space.

(1) For $x, y \in P$, define $y \ll_{a(M)} x$ if for every net $(x_i)_{i \in I}$ that $\text{lim-inf}_{a(M)}$-converges to $x$, $x_i \geq y$ holds eventually.

(2) The poset $P$ is said to be $a(M)$-continuous if for every $x \in P : x \ll_{a(M)} a$ and $a = \text{sup} \{ x \in P : x \ll_{a(M)} a \}$ holds for every $a \in P$.

Given a PM-space $(P, \mathcal{M})$, the approximate relation $\ll_{a(M)}$ on the poset $P$ can be equivalently characterized in the following proposition.

**Proposition 3.4.** Let $(P, \mathcal{M})$ be a PM-space and $x, y \in P$. Then $y \ll_{a(M)} x$ if and only if for every $M \in \mathcal{M}$ with $\text{sup} M \geq x$, there exists $M_0 \sqsubseteq M$ such that
\[
\{ \{ \uparrow m : m \in M_0 \} : M_0 \sqsubseteq M \} \subseteq \uparrow y.
\]

**Proof.** Suppose $y \ll_{a(M)} x$. Let $(x_{(d, D)})_{(d, D) \in D_M^x}$ be the net defined in Remark 3.2 (3) for every $M \in \mathcal{M}$ with $\text{sup} M = p \geq x$. Then the net $(x_{(d, D)})_{(d, D) \in D_M^x} \xrightarrow{M} x$. By Definition 3.3 (1), there exists $(d_0, D_0) \in D_M^x$ such that $x_{(d_0, D_0)} = d \geq y$ for all $(d, D) \geq (d_0, D_0)$. Since $(d, D_0) \geq (d_0, D_0)$ for every $d \in D_0$, $x_{(d, D_0)} = d \geq y$ for every $d \in D_0$. So $D_0 \subseteq \uparrow y$. This shows that there exists $M_0 \sqsubseteq M$ such that $D_0 = \bigcap \{ \{ \uparrow m : m \in M_0 \} : M_0 \sqsubseteq M \} \subseteq \uparrow y$.

Conversely, suppose that for every $M \in \mathcal{M}$ with $\text{sup} M \geq x$, there exists $M_0 \sqsubseteq M$ such that $\bigcap \{ \{ \uparrow m : m \in M_0 \} : M_0 \sqsubseteq M \} \subseteq \uparrow y$. Let $(x_i)_{i \in I}$ be a net that $\text{lim-inf}_{a(M)}$-converges to $x$. Then, by Definition 3.1, there exists $M \in \mathcal{M}$ such that $\text{sup} M = p \geq x$, and for every $m \in M$, there exists $i_m \in I$ such that $x_i \geq m$ for all $i \geq i_m$. Take $i_0 \in I$ with that $i_0 \geq i_m$ for every $m \in M_0 \subseteq M$, we have that $x_i \in \bigcap \{ \{ \uparrow m : m \in M_0 \} \} \subseteq \uparrow y$ for all $i \geq i_0$. This shows that $x_i \geq y$ holds eventually. Thus, by Definition 3.3 (1), we have $y \ll_{a(M)} x$.

**Remark 3.5.** Let $(P, \mathcal{M})$ be a PM-space and $x, y \in P$.

(1) If there is no $M \in \mathcal{M}$ such that $\text{sup} M \geq x$, then $p \ll_{a(M)} x$ for every $p \in P$. And, if the poset $P$ has the least element $\bot$, then $\bot \ll_{a(M)} p$ for every $p \in P$.

(2) The implication $y \ll_{a(M)} x \implies y \leq x$ may not be true. For example, let $P = \{0, 1, 2, \ldots \}$ be in the discrete order $\leq$ defined by
\[
(\forall i, j \in P) \quad i \leq j \iff i = j.
\]

And let $\mathcal{M} = \{ \{2\} \}$. Then, it is easy to see from Remark 3.5 (1) that $0 \ll_{a(M)} 1$ and $\emptyset \ll_{a(M)} 1$.

(3) Assume the PM-space $(P, \mathcal{M})$ has the property that for every $p \in P$, there exists $M_p \in \mathcal{M}$ such that $\text{sup} M_p = p$. Then, by Proposition 3.4, we have
\[
(\forall q, r \in P) \quad q \ll_{a(M)} r \implies q \leq r.
\]

For more interpretations of the approximate relation $\ll_{a(M)}$ on posets, the readers can refer to Example 3.2 and Remark 3.3 in [8].

For simplicity, given a PM-space $(P, \mathcal{M})$ and $x \in P$, we will denote
\[
\nabla_{M} x = \{ y \in P : y \ll_{a(M)} x \};
\]

2 From the logical point of view, we stipulate $\bigcap \{ \{ \uparrow m : m \in M_0 \} : M_0 \sqsubseteq M \} = P$ if $M_0 = \emptyset$. 

Based on the approximate relation \( \preceq_{\text{at}} \) on posets, the \( \alpha'({\mathcal{M}}) \)-continuity can be defined for posets in the following:

**Definition 3.6.** Let \((P, \mathcal{M})\) be a PM-space. The poset \(P\) is called an \( \alpha'({\mathcal{M}}) \)-continuous poset if for every \(x \in P\), there exists \(M_x \in \mathcal{M}\) such that

\[
\text{(O1)} \sup M_x = x \quad \text{and} \quad M_x \subseteq \downarrow_{\mathcal{M}} x.
\]

And,

\[
\text{(O2)} \text{for every } y \in \downarrow_{\mathcal{M}} x, \text{ there exists } F \subseteq M_x \text{ such that } \bigcap \{\uparrow f : f \in F\} \subseteq \downarrow_{\mathcal{M}} y.
\]

Noticing Remark 3.5 (3), we have the following proposition about \( \alpha'({\mathcal{M}}) \)-continuous posets.

**Proposition 3.7.** Let \((P, \mathcal{M})\) be a PM-space in which the poset \(P\) is \( \alpha'({\mathcal{M}}) \)-continuous. Then

\[
(\forall x, y \in P) \ y \preceq_{\text{at}} x \Rightarrow y \preceq x.
\]

The following examples of \( \alpha'({\mathcal{M}}) \)-continuous posets can be formally checked by Definition 3.6.

**Example 3.8.** Let \((P, \mathcal{M})\) be a PM-space.

1. If \(P\) is a finite poset, then \(P\) is an \( \alpha'({\mathcal{M}}) \)-continuous poset if and only if for every \(x \in P\), there exists \(M_x \in \mathcal{M}\) such that \(\sup M_x = x\).
2. Let \(\mathcal{M} = \mathcal{L}(P)\). Then \(P\) is an \( \alpha'({\mathcal{L}}) \)-continuous poset. This means that every poset is \( \alpha'({\mathcal{L}}) \)-continuous.
3. Let \(\mathcal{M} = \mathcal{D}(P)\). Then we have \(\preceq = \preceq_{\text{at}}\) (see Example 3.2 (1) in [8]). The poset \(P\) is a continuous poset if and only if it is an \( \alpha'({\mathcal{D}}) \)-continuous poset. In particular, finite posets, chains, anti-chains and completely distributive lattices are all \( \alpha'({\mathcal{D}}) \)-continuous.
4. Let \(\mathcal{M} = \mathcal{D}(P)\). If \(P\) is a finite poset (resp. chain, anti-chain), then \(P\) is an \( \alpha'({\mathcal{D}}) \)-continuous poset.

**Proposition 3.9.** Let \((P, \mathcal{M})\) be a PM-space. If \(P\) is an \( a({\mathcal{M}}) \)-continuous poset, and \(\{y \in P : (\exists z \in P) \ y \preceq_{\text{at}} z \preceq_{\text{at}} a\} \in \mathcal{M}\) for every \(a \in P\), then \(P\) is an \( \alpha'({\mathcal{M}}) \)-continuous poset.

**Proof.** Suppose that \(P\) is an \( a({\mathcal{M}}) \)-continuous poset, and \(\{y \in P : (\exists z \in P) \ y \preceq_{\text{at}} z \preceq_{\text{at}} a\} \in \mathcal{M}\) for every \(a \in P\). Take \(M_a = \downarrow_{\mathcal{M}} a\). Then it is easy to see that \(\sup M_a = a\) and \(M_a \subseteq \downarrow_{\mathcal{M}} a\). By Remark 3.3 (4) in [8], we have \(\sup \{y \in P : (\exists z \in P) \ y \preceq_{\text{at}} z \preceq_{\text{at}} a\} = a\). This implies, by Proposition 3.4 and Remark 3.5 (2), that for every \(y \in \downarrow_{\mathcal{M}} a\), there exist \(\{y_1, y_2, \ldots, y_n\}, \{z_1, z_2, \ldots, z_n\} \subseteq M_a \subseteq \downarrow_{\mathcal{M}} a\) such that

\[
\bigcap \{\uparrow z_i : i \in \{1, 2, \ldots, n\}\} \subseteq \bigcap \{\uparrow y_i : i \in \{1, 2, \ldots, n\}\}
\]

and

\[
y_i \preceq_{\text{at}} z_i \preceq_{\text{at}} a\text{ for every } i \in \{1, 2, \ldots, n\}.
\]

Next, we show \(\bigcap \{\uparrow z_i : i \in \{1, 2, \ldots, n\}\} \subseteq \downarrow_{\mathcal{M}} y\). For every \(M \in \mathcal{M}\) with \(\sup M > b \in \bigcap \{\uparrow z_i : i \in \{1, 2, \ldots, n\}\}\), by Proposition 3.4, there exists \(M_i \subseteq M\) such that \(\bigcap \{\uparrow m : m \in M_i\} \subseteq \uparrow y\) for every \(i \in \{1, 2, \ldots, n\}\). Take \(M_0 = \bigcup \{M_i : i \in \{1, 2, \ldots, n\}\}\). Then \(M_0 \subseteq M\) and

\[
\bigcap \{\uparrow m : m \in M_0\} \subseteq \bigcap \{\uparrow y_i : i \in \{1, 2, \ldots, n\}\} \subseteq \uparrow y.
\]

This shows \(\preceq_{\text{at}} b\) for every \(b \in \bigcap \{\uparrow z_i : i \in \{1, 2, \ldots, n\}\}\). Hence, \(\bigcap \{\uparrow z_i : i \in \{1, 2, \ldots, n\}\} \subseteq \downarrow_{\mathcal{M}} y\). Thus \(P\) is an \( \alpha'({\mathcal{M}}) \)-continuous poset. \(\square\)

The fact that an \( \alpha'({\mathcal{M}}) \)-continuous poset \(P\) in a PM-space \((P, \mathcal{M})\) may not be \( a({\mathcal{M}}) \)-continuous can be demonstrated in the following example.
Example 3.10. Let \((P, Μ)\) be the PM-space in which the poset \(P = ℝ\) is the set of all real number with its usual order \(≤\) and \(Μ = Σ₀(ℝ)\). Then we have \(≪_{α(δ₀)} = ≪\) by Proposition 3.4. It is easy to check, by Definition 3.6, that \(ℝ\) is an \(α(δ₀)\)-continuous poset. But \(ℝ\) is not an \(α(δ₀)\)-continuous poset because \(\downarrow_{δ₀}x = \downarrow x ∈ Σ₀(P)\) for every \(x ∈ ℝ\).

We turn to consider the topology induced by the \(\liminf_M^*\)-convergence in posets.

Definition 3.11. Let \((P, Μ)\) be a PM-space. A subset \(V\) of \(P\) is said to be \(Μ\)-open if for every net \((x_i)_{i∈I} \rightarrow x \in V\), \(x_i ∈ V\) holds eventually.

Given a PM-space \((P, Μ)\), one can formally verify that the set of all \(Μ\)-open subsets of \(P\) forms a topology on \(P\). This topology is called the \(Μ\)-topology, and denoted by \(\mathcal{O}_Μ(P)\).

The following Theorem is an order-theoretical characterization of \(Μ\)-open sets.

Theorem 3.12. Let \((P, Μ)\) be a PM-space. Then a subset \(V\) of \(P\) is \(Μ\)-open if and only if it satisfies the following two conditions:

(V1) For every \(M ∈ Μ\) with sup \(M \in V\), there exists \(M₀ ⊆ M\) such that \(\bigcap\{↑m : m ∈ M₀\} ⊆ V\).

(V2) For every \(M ∈ Μ\) with sup \(M \in V\), there exists \(M₀ ← M\) such that \(\bigcap\{↓m : m ∈ M₀\} ⊆ V\).

Proof. Suppose that \(V\) is an \(M\)-open subset of \(P\). By Remark 3.2 (1), it is easy to see that \(V\) is an upper set. Let \((x_{(d),D})_{(d,D) ∈ D'_M}^{M} \in V\) be the net defined in Remark 3.2 (3) for every \(M ∈ Μ\) with sup \(M = x ∈ V\). Then \(x_{(d),D})_{(d,D) ∈ D'_M}^{M} \in V\). This implies, by Definition 3.11, that there exists \((d₀, D₀) ∈ D'_M\) such that \(x_{(d),D})_{(d,D) ∈ D'_M}^{M} = d ∈ V\) for all \((d, D) ≥ (d₀, D₀)\). Since \((d, D) ≥ (d₀, D₀)\) for all \(d ∈ D₀\), \(x_{(d),D})_{(d,D) ∈ D'_M}^{M} = d ∈ V\) for all \(d ∈ D₀\). This shows \(D₀ ⊆ V\). Thus there exists \(M₀ ⊆ M\) such that \(D₀ = \bigcap\{↑m : m ∈ M₀\} ⊆ V\).

Conversely, suppose \(V\) is a subset of \(P\) which satisfies Condition (V1) and (V2). Let \((x_i)_{i∈I} \rightarrow x \in V\) be a net that \(\liminf_M^*\)-converges to \(x \in V\). Then there exists \(M \in Μ\) such that sup \(M = y \geq x \in V\) (hence, \(y \in V\)), and for every \(m \in M\), there exists \(i_m \in I\) such that \(x_i ≳ m\) for all \(i ≥ i_m\). By Condition (V2), we have that \(\bigcap\{↑m : m ∈ M₀\} ⊆ V\) for some \(M₀ ⊆ M\). Take \(i_0 \in I\) with that \(i_0 ≥ i_m\) for all \(m ∈ M₀\). Then \(x_i \in \bigcap\{↑m : m ∈ M₀\} ⊆ V\) for all \(i ≥ i_0\). This shows that \(V\) is an \(M\)-open set.

Recall that given a topological space \((X, T)\) and a point \(x ∈ P\), a family \(B(x)\) of open neighbourhoods of \(x\) is called a base for the topological space \((X, T)\) at the point \(x\) if for every neighbourhood \(V\) of \(x\) there exists an \(U ∈ B(x)\) such that \(x ∈ U ⊆ V\).

If the poset \(P\) in a PM-space \((P, Μ)\) is an \(α(Μ)\)-continuous poset, we provide a base for the topological space \((P, \mathcal{O}_Μ(P))\) at a point \(x ∈ P\).

Proposition 3.13. Let \((P, Μ)\) be a PM-space in which the poset \(P\) is \(α(Μ)\)-continuous. Then \(\uparrow_Μx ∈ \mathcal{O}_Μ(P)\) for every \(x ∈ P\).

Proof. One can readily see, by Proposition 3.4, that \(\uparrow_Μx\) is an upper subset of \(P\) for every \(x ∈ P\). For every \(M ∈ Μ\) with sup \(M = y \in \uparrow_Мx\), by Definition 3.6 (01) there exists \(M_y ∈ Μ\) such that \(M_y \subseteq \uparrow_Μy\) and sup \(M_y = y\). Since \(x ≪_{α(Μ)y}\), by Definition 3.6 (02), we have \(\bigcap\{↑m_i : i ∈ \{1, 2, ..., n\}\} ⊆ \uparrow_Мx\) for some \(\{m₁, m₂, ..., mₙ\} ⊆ M_y\). Observing \(\{m₁, m₂, ..., mₙ\} ⊆ M_y\), we can conclude that there exists \(M_i ⊆ M\) such that \(\bigcap\{↑a : a ∈ M_i\} ⊆ \uparrow m_i\) for every \(i ∈ \{1, 2, ..., n\}\). Let \(M₀ = \bigcup\{M_i : i ∈ \{1, 2, ..., n\}\}\). Then \(M₀ ⊆ M\) and

\[
\bigcap\{↑m : m ∈ M₀\} ⊆ \bigcap\{↑m_i : i ∈ \{1, 2, ..., n\}\} ⊆ \uparrow_Мx.
\]

This shows, by Theorem 3.12, that \(\uparrow_Мx ∈ \mathcal{O}_М(P)\) for every \(x ∈ P\).
Proposition 3.14. Let $(P, M)$ be a PM-space in which the poset $P$ is $\alpha^*(M)$-continuous and $x \in P$. Then \( \bigcap \{ \bigtriangleup_M a : a \in A \} : A \subseteq \bigtriangledown_M x \) is a base for the topological space $(P, O_M(P))$ at the point $x$.

Proof. Clearly, by Proposition 3.13, we have \( \bigcap \{ \bigtriangleup_M a : a \in A \} \subseteq O_M(P) \) for every $A \subseteq \bigtriangledown_M x$. Let $U \in O_M(P)$ and $x \in U$. Since $P$ is a $\alpha^*(M)$-continuous poset, there exists $M_x \in M$ such that $M_x \subseteq \bigtriangledown_M x$ and $\sup M_x = x \in U$. By Theorem 3.12, it follows that \( \bigcap \{ \uparrow m : m \in M_0 \} \subseteq U \) for some $M_0 \subseteq M_x \subseteq \bigtriangledown_M x$. So, from Proposition 3.7, we have

\[
x \in \bigcap \{ \bigtriangleup_M m : m \in M_0 \} \subseteq \bigcap \{ \uparrow m : m \in M_0 \} \subseteq U.
\]

Thus, \( \bigcap \{ \bigtriangleup_M a : a \in A \} : A \subseteq \bigtriangledown_M x \) is a base for the topological space $(P, O_M(P))$ at the point $x$.

In the rest, we are going to establish a characterization theorem which demonstrates the equivalence between the lim-inf$_M$-convergence being topological and the $\alpha^*(M)$-continuity of the poset in a given PM-space.

Lemma 3.15. Let $(P, M)$ be a PM-space. If $P$ is an $\alpha^*(M)$-continuous poset, then a net

\[
(x_i)_{i \in I} \overset{\mathcal{M}}{\rightarrow} x \in P \iff (x_i)_{i \in I} \overset{O_M(P)}{\rightarrow} x.
\]

Proof. By the definition of $O_M(P)$, it is easy to see that a net

\[
(x_i)_{i \in I} \overset{\mathcal{M}}{\rightarrow} x \in P \implies (x_i)_{i \in I} \overset{O_M(P)}{\rightarrow} x.
\]

To prove the Lemma, we only need to show that a net \( (x_i)_{i \in I} \overset{O_M(P)}{\rightarrow} x \in P \) implies \( (x_i)_{i \in I} \overset{\mathcal{M}}{\rightarrow} x \). Suppose \( (x_i)_{i \in I} \overset{O_M(P)}{\rightarrow} x \). As $P$ is an $\alpha^*(M)$-continuous poset, there exists $M_x \in M$ such that $M_x \subseteq \bigtriangledown_M x$ and $\sup M_x = x$. By Proposition 3.13, we have \( x \in \bigtriangleup_M y \in O_M(P) \) for every $y \in M_x \subseteq \bigtriangledown_M x$. Hence, \( x_i \in \bigtriangleup_M y \) holds eventually. This implies, by Proposition 3.7, that \( x_i \in \bigtriangleup_M y \subseteq \uparrow y \) holds eventually. By the definition of lim-inf$_M$-convergence, we have \( (x_i)_{i \in I} \overset{\mathcal{M}}{\rightarrow} x \).

In the converse direction, we have the following Lemma.

Lemma 3.16. Let $(P, M)$ be a PM-space. If the lim-inf$_M$-convergence in $P$ is topological, then $P$ is an $\alpha^*(M)$-continuous poset.

Proof. Suppose that the lim-inf$_M$-convergence in $P$ is topological. Then there exists a topology $\mathcal{T}$ such that for every $x \in P$, a net

\[
(x_i)_{i \in I} \overset{\mathcal{T}}{\rightarrow} x \iff (x_i)_{i \in I} \overset{\mathcal{M}}{\rightarrow} x.
\]

Define $I_x = \{ (p, V) \in P \times N(x) : p \in V \}$, where $N(x)$ is the set of all open neighbourhoods of $x$, namely, $N(x) = \{ V \in \mathcal{T} : x \in V \}$. Define also the preorder $\preceq$ on $I_x$ as follows:

\[
(\forall (p_1, V_1), (p_2, V_2) \in I_x) (p_1, V_1) \preceq (p_2, V_2) \iff V_2 \subseteq V_1.
\]

It is easy to see that $I_x$ is directed. Now, let $x_{(p, V)} = p$ for every $(p, V) \in I_x$. Then one can readily check that the net \( (x_{(p, V)})_{(p, V) \in I_x} \overset{\mathcal{T}}{\rightarrow} x \), and hence \( (x_{(p, V)})_{(p, V) \in I_x} \overset{\mathcal{M}}{\rightarrow} x \). This means that there exists $M_x \in M$ such that $\sup M_x = x$, and for every $m \in M_x$, there exists $(p_m, V_m) \in I_x$ with that $x_{(p, V)} = p \geq m$ for all $(p, V) \geq (p_m, V_m)$. Since $(p, V) \geq (p_m, V_m)$ for all $p \in V_m$, we have $x_{(p, V_m)} = p \geq m$ for all $p \in V_m$. This shows

\[
(\forall m \in M_x) (\exists V_m \in N(x)) x \in V_m \subseteq \uparrow m. \quad (**)
\]

Next we prove $M_x \subseteq \bigtriangledown_M x$. For every $m \in M_x$ and every $M \in M$ with $\sup M > x$, let $(x_{(d, D)})_{(d, D) \in D_M}$ be the net defined in Remark 3.2 (3). Then the net $(x_{(d, D)})_{(d, D) \in D_M} \overset{\mathcal{T}}{\rightarrow} x$, and thus $(x_{(d, D)})_{(d, D) \in D_M} \overset{\mathcal{M}}{\rightarrow} x$. It follows from
Remark 3.2 (4) that there exists $M_0 \subseteq M$ such that $x \in \bigcap\{\uparrow a : a \in M_0\} \subseteq V_m$. By Condition (**), we have $x \in \bigcap\{\uparrow a : a \in M_0\} \subseteq V_m \subseteq \uparrow m$. So, $m \ll_{\langle a(M) \rangle} x$. This shows $M_x \subseteq \nabla_M x$.

Let $y \in \nabla_M x$. Then there exists $\{m_1, m_2, ..., m_n\} \subseteq M_x$ such that $\bigcap\{\uparrow m_i : i \in \{1, 2, ..., n\}\} \subseteq \uparrow y$ as $M_x \subseteq M$ and sup $M_x \geq x$. By Condition (**), it follows that $\bigcap\{V_{m_i} : i \in \{1, 2, ..., n\}\} \subseteq \bigcap\{\uparrow m_i : i \in \{1, 2, ..., n\}\} \subseteq \uparrow y$. Considering the net $(x_{(d,D)})_{(d,D) \in D'_m}$ defined in Remark 3.2 (3), we have $(x_{(d,D)})_{(d,D) \in D'_m} \downarrow x$, and hence $(x_{(d,D)})_{(d,D) \in D'_m} \downarrow x$. This implies, by Remark 3.2 (4), that

$$\bigcap\{\uparrow b : b \in M_{00}\} \subseteq \bigcap\{V_{m_i} : i \in \{1, 2, ..., n\}\} \subseteq \bigcap\{\uparrow m_i : i \in \{1, 2, ..., n\}\} \subseteq \uparrow y$$

for some $M_{00} \subseteq M_x$. Finally, we show $\bigcap\{\uparrow b : b \in M_{00}\} \subseteq \nabla_M y$. For every $x' \in \bigcap\{b : b \in M_{00}\}$ and every $M' \in M$ with sup $M' \geq x'$, let $(x_{(d,D)})_{(d,D) \in D'_M}$ be the net defined in Remark 3.2 (3). Then $(x_{(d,D)})_{(d,D) \in D'_M} \nabla_M x'$, and thus $(x_{(d,D)})_{(d,D) \in D'_m} \downarrow x'$. It follows from Condition (*** ) and Remark 3.2 (4) that there exists $M_0 \subseteq M'$ such that

$$\bigcap\{\uparrow a' : a' \in M_0\} \subseteq \bigcap\{V_{m_i} : i \in \{1, 2, ..., n\}\} \subseteq \bigcap\{\uparrow m_i : i \in \{1, 2, ..., n\}\} \subseteq \uparrow y.$$ 

This shows $x' \in \nabla_M y$, and thus $\bigcap\{\uparrow b : b \in M_{00}\} \subseteq \nabla_M y$. Therefore, $P$ is an $a^*(M)$-continuous poset.

Combining Lemma 3.15 and Lemma 3.16, we deduce the following result.

**Theorem 3.17.** Let $(P, M)$ be a PM-space. The following statements are equivalent:

1. $P$ is an $a^*(M)$-continuous poset.
2. For any net $(x_i)_{i \in I}$ in $P$, $(x_i)_{i \in I} \nabla_M x \in P \iff (x_i)_{i \in I} \nabla_{\langle a(M) \rangle} x$.
3. The lim-inf$_M$-convergence in $P$ is topological.

**Proof.** (1) $\Rightarrow$ (2): By Lemma 3.15.
(2) $\Rightarrow$ (3): Clear.
(3) $\Rightarrow$ (1): By Lemma 3.16.

**Corollary 3.18 ([8]).** Let $(P, M)$ be a PM-space with $S_0(P) \subseteq M \subseteq \mathcal{P}(P)$. Suppose $\nabla_M a \in M$ and $\{y \in P : (\exists z \in P) y \ll_{< a(M)} z \ll_{< a(M)} a\} \in M$ holds for every $a \in P$. Then the lim-inf$_M$-convergence in $P$ is topological if and only if $P$ is an $a^*(M)$-continuous poset.

**Proof.** ($\Rightarrow$): To show the $a(M)$-continuity of $P$, it suffices to prove sup $\nabla_M a = a$ for every $a \in P$. Since the lim-inf$_M$-convergence in $P$ is topological, by Theorem 3.17, $P$ is an $a^*(M)$-continuous poset. This implies that there exists $M_a \in M$ such that sup $M_a \subseteq \nabla_M a$ and sup $M_a = a$ for every $a \in P$. By Proposition 3.7, we have $\nabla_M a \subseteq \downarrow a$. So sup $\nabla_M a = a$.
($\Leftarrow$): By Proposition 3.9 and Theorem 3.17.

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