Log canonical foliation singularities on surfaces

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Funding information
Division of Mathematical Sciences, Grant/Award Numbers: DMS-1801853, DMS-1840190; Simons Foundation, Grant/Award Number: 256202

Abstract
We give a classification of the dual graphs of the exceptional divisors on the minimal resolutions of log canonical foliation singularities on surfaces. As an application, we show the set of foliated minimal log discrepancies for foliated surface triples satisfies the ascending chain condition and a Grauert-Riemenschneider-type vanishing theorem for foliated surfaces with special log canonical foliation singularities.

KEYWORDS
foliation, singularity, surface

MSC (2020)
Primary 32S65, Secondary 32M25, 14B05, 14J99

1 | INTRODUCTION

In this paper, we always work over the complex numbers $\mathbb{C}$.

Singularities play an important role in many areas of algebraic geometry. For instance, singular varieties naturally appear in the minimal model program and in the study of moduli spaces, where it often happens that smooth objects degenerate to singular ones. However, many classes of singularities that naturally occur are mild and can sometimes be understood in detail. One of the natural classes of singularities is the class of (log) canonical singularities. These singularities have been extensively studied, especially in the case of surfaces. In particular, we have a full classification of log canonical surface singularities. (For a reference, see [12, 19, section 4.1], or [9, appendix].)

When studying foliations, one expects that similar results on their singularities may hold. In this direction, McQuillan introduces in [15] a notion of log canonical foliation singularities, which is a natural generalization of log canonical singularities to foliated varieties. Moreover, he also gives a classification for the canonical foliation singularities on surfaces. (See [15, Corollary I.2.2, Fact I.2.4, and Theorem III.3.2].)

In this paper, we first give a full list of all possibilities of the dual graphs of the exceptional divisors on the minimal resolutions of log canonical foliation singularities on surfaces. (See also Theorem 3.21.) This is achieved with the help of the work of Brunella, Camacho, and Sad on the indices and a theorem on separatrices.

Theorem 1.1. Let $(X, \mathcal{F}, p)$ be a germ of a foliated surface. Assume that $p$ is a log canonical singularity of $\mathcal{F}$. Let $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ be the minimal resolution (see Section 2.3) for $(X, \mathcal{F}, p)$ with exceptional divisors $E = \cup E_i$. Then, $E$ belongs to one of the following types:

1. A $\mathcal{G}$-chain.
2. A chain of three invariant curves $E_1 \cup E_2 \cup E_3$ where $E_1$ and $E_3$ are $(-1)$-$\mathcal{G}$-curves with self-intersection $-2$ and $E_2$ is a bad tail.
3. A chain of $(-2)$-$\mathcal{G}$-curves.
4. A dihedral singularity. More precisely, two \((-1)\)-\(\mathcal{G}\)-curves with self-intersection \(-2\) joined by a bad tail, which itself connects to a chain of \((-2)\)-\(\mathcal{G}\)-curves.

5. An elliptic Gorenstein leaf, which is either a rational curve with only one node or a cycle or \((-2)\)-\(\mathcal{G}\)-curves.

6. A chain \(E = \bigcup_{i=1}^{r} E_i\) with exactly one noninvariant curve \(E_\ell\) with \(1 \leq \ell \leq r\). Moreover, \(E_\ell\) has tangency order zero and \(\bigcup_{i=1}^{\ell-1} E_i\) and \(\bigcup_{i=\ell+1}^{r} E_i\) are \(\mathcal{G}\)-chains.

7. The dual graph is star-shaped with a noninvariant center \([E_0]\). Moreover, \(E_0\) has tangency order zero, all branches are \(\mathcal{G}\)-chains, and all first curves of \(\mathcal{G}\)-chains have intersection number one with \(E_0\).

Note that type 1 is terminal, and types 1–5 are canonical.

Inspired by the work in [2] on the ascending chain condition (ACC) for the set of the minimal log discrepancies of surface singularities, we show that the set of foliated minimal log discrepancies satisfies the ACC. (See also Definition 4.2 and Theorem 5.7.)

**Theorem 1.2.** For any set \(B\) satisfying the descending chain condition, the set

\[
\text{MLD}(2, B) := \{ \text{mld}_x(\mathcal{F}, \Delta) \mid (X, \mathcal{F}, \Delta) \text{ is a foliated triple with } x \in X \text{ and } \Delta \in B \}
\]

satisfies the ACC.

Finally, we prove a Grauert–Riemenschneider–type vanishing theorem for foliated surfaces with special log canonical foliation singularities by using the method in [11, Theorem 10.4]. (See Definition 6.1 and Theorem 6.4.) This is a generalization of [8, Theorem 6.1] in which \((X, \mathcal{F})\) is assumed to have only canonical foliation singularities.

**Theorem 1.3.** Let \(f : (Y, \mathcal{G}) \to (X, \mathcal{F})\) be a proper birational morphism where \((X, \mathcal{F})\) is a foliated surface with special log canonical foliation singularities and \((Y, \mathcal{G})\) is a foliated surface with only reduced singularities. Then, \(R^i f_\ast \mathcal{O}_Y(K_\mathcal{G}) = 0\) for \(i > 0\).

**Remark 1.4.** The condition that requires foliated surfaces with special log canonical singularities is not sharp. We set up the condition of special log canonical singularities in order to cover as many cases as possible. For example, if the exceptional divisor of \(f\) is an irreducible noninvariant divisor with self-intersection \(-1\), then \(R^1 f_\ast \mathcal{O}_Y(K_\mathcal{G}) = 0\) follows from the same arguments as in the proof of Theorem 6.4. However, the singularities on \((X, \mathcal{F})\) are not special log canonical. On the other hand, it is unclear whether the condition of the special log canonical singularity is necessary or not to obtain the vanishing.

## 2 | PRELIMINARIES

By a surface, we mean a normal algebraic space of dimension 2. In this section, we recall several definitions and results, which will be used later.

### 2.1 | Foliations on surfaces

A foliation \(\mathcal{F}\) on a surface \(X\) is a rank 1 saturated subsheaf \(\mathcal{F}\) of the tangent sheaf \(T_X\) of \(X\). So we have the following short exact sequence:

\[
0 \to \mathcal{F} \to T_X \to T_X/\mathcal{F} \to 0
\]

with \(T_X/\mathcal{F}\) torsion-free.

The point \(p\) on \(X\) is called a singular point of the foliation if either a singular point of \(X\) or a point at which the quotient \(T_X/\mathcal{F}\) is not locally free. Since \(X\) is normal and \(\mathcal{F}\) is saturated, the foliation singularities are isolated.
Definition 2.1. A foliated surface is a pair \((X, \mathcal{F})\) consisting of a surface \(X\) and a foliation \(\mathcal{F}\) on \(X\). A foliated triple is a triple \((X, \mathcal{F}, \Delta)\) consisting of a foliated surface \((X, \mathcal{F})\) and an \(\mathbb{R}\)-divisor \(\Delta = \sum a_i D_i\).

Notice that \(T_X \cong \text{Hom}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)\) is reflexive. Thus, \(\mathcal{F}\) is also reflexive and therefore we can define the canonical divisor \(K_\mathcal{F}\) of the foliation as a Weil divisor on \(X\) with \(\mathcal{O}_X(-K_\mathcal{F}) \cong \mathcal{F}\).

Definition 2.2. Let \((X, \mathcal{F})\) be a foliated surface. Given any birational map between normal surfaces \(f: Y \rightarrow X\) and a foliation \(\mathcal{F}\) on \(X\), then we define the pullback foliation \(f^* \mathcal{F}\) as follows:

Let \(U\) be an open subset such that \(V := f^{-1}(U) \rightarrow U\) is an isomorphism. Note that \(\mathcal{F}|_U \subset T_U \cong T_V\). By [10, Exercise II.5.15], we have a coherent subsheaf \(\mathcal{G}\) of \(T_Y\) such that \(\mathcal{G}|_V = \mathcal{F}|_U \subset T_V\). Then, the pullback foliation \(f^* \mathcal{F}\) is defined to be the saturation of \(\mathcal{G}\). By [8, Lemma 1.8], this definition is well defined.

Also, if \(\mathcal{G}\) is a foliation on \(Y\), then we can define the pushforward foliation \(f_* \mathcal{G}\) by taking the saturation of the image of the composition

\[
f_* \mathcal{G} \longrightarrow f_* T_Y \longrightarrow (f_* T_Y)^s = T_X.
\]

Definition 2.3 (Invariant curves). Let \((X, \mathcal{F})\) be a foliated surface and \(U\) be the nonsingular locus of \(X\). A curve \(C\) on \(X\) is called \(\mathcal{F}\)-invariant if the inclusion map

\[
T_{\mathcal{F}|_C} \longrightarrow T_{C|_U}
\]

factors through \(T_{C|_U}\).

Definition 2.4 [15, Definition I.1.5]. Let \((X, \mathcal{F}, \Delta)\) be a foliated triple and \(f: Y \rightarrow X\) be a proper birational morphism. For any divisor \(E\) on \(Y\), we define the discrepancy of \((\mathcal{F}, \Delta)\) along \(E\) to be \(a(E, \mathcal{F}, \Delta) = \text{ord}_E(K_{f^* \mathcal{F}} - f^*(K_\mathcal{F} + \Delta))\). We say

\((X, \mathcal{F}, \Delta)\) is terminal (resp. canonical) if \(a(E, \mathcal{F}, \Delta) > 0\) (resp. \(\geq 0\)) for every exceptional divisor \(E\) over \(X\)

and

\((X, \mathcal{F}, \Delta)\) is log terminal (resp. log canonical) if \(a(E, \mathcal{F}, \Delta) > -\varepsilon(E)\) (resp. \(\geq -\varepsilon(E)\)) for every divisor \(E\) over \(X\)

where \(\varepsilon(E)\) is defined to be 0 if \(E\) is \(f^* \mathcal{F}\)-invariant, and 1 otherwise.

2.2 Indices on foliated surfaces

Most definitions in this subsection follow from [5] with some generalizations.

Let \(p \in \text{Sing}(\mathcal{F}) \setminus \text{Sing}(X)\). That is, \(p\) is a smooth point on \(X\) but a singular point of the foliation. Let \(v\) be the vector field around \(p\) generating \(\mathcal{F}\). Since \(p \in \text{Sing}(\mathcal{F})\), we have \(v(p) = 0\). Then, we can consider the eigenvalues \(\lambda_1, \lambda_2\) of \((Dv)|_p\), which do not depend on the choice of \(v\).

Definition 2.5. If one of the eigenvalues is nonzero, say \(\lambda_2\), then we define the eigenvalue of the foliation \(\mathcal{F}\) at \(p\) to be

\[
\lambda := \frac{\lambda_1}{\lambda_2}.
\]

For \(\lambda \neq 0\), this definition is unique up to reciprocal \(\lambda \sim \frac{1}{\lambda}\).

If \(\lambda = 0\), then \(p\) is called a saddle-node; otherwise, we say \(p\) is nondegenerate. If \(\lambda \notin \mathbb{Q}^+\), then \(p\) is called a reduced singularity of \(\mathcal{F}\).
Reduced singularities arise naturally. Indeed, blowing up a smooth foliation point on surfaces will introduce a reduced singularity with $\lambda = -1$.

**Theorem 2.6** (Seidenberg’s theorem). Given any foliated surface $(X, \mathcal{F})$ with $X$ smooth. There is a sequence of blowups $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ such that $(Y, \mathcal{G})$ has only reduced singularities. More precisely, it is enough to perform blowups at smooth points.

**Proof.** See [14, 17, Appendix], or [5, Theorem 1.1].

By Seidenberg’s theorem, we can always resolve surface foliation singularities to the reduced singularities by a sequence of blowups. Besides, it is the best we can hope for the resolution of surface foliation singularities since blowing up a smooth foliation point on surfaces will introduce a reduced singularity with eigenvalue $-1$.

**Definition 2.7** (Separatrices). Let $p$ be a singular point of $\mathcal{F}$. A separatrix of $\mathcal{F}$ at $p$ is a holomorphic (possibly singular) irreducible $\mathcal{F}$-invariant curve $C$ on a neighborhood of $p$ which passes through $p$.

**Theorem 2.8** (Separatrix theorem). Let $\mathcal{F}$ be a foliation on a normal surface $X$ and $C \subset X$ be a connected, compact, and $\mathcal{F}$-invariant curve such that

1. all the singularities of $\mathcal{F}$ on $C$ are reduced and
2. the intersection matrix of $C$ is negative definite and the dual graph of the components of $C$ is a tree.

Then, there exists at least one point $p \in C \cap \text{Sing}(\mathcal{F})$ and a separatrix through $p$ not contained in $C$. In particular,

$$\# \{\text{irreducible components of } C\} \leq \# \{\text{reduced singularities on } C\}.$$  

**Proof.** See [6, 16], or [5, Theorem 3.4]. For the last statement, we suppose there are $n$ irreducible components of $C$. Since the dual graph is a tree, there are $n - 1$ points $p_1, p_2, \ldots, p_{n-1}$ on $C$ that are on exactly two irreducible components of $C$. Notice that each $p_i$ is not a smooth foliation point since there are two invariant curves passing through it. Thus, $p_i \in C \cap \text{Sing}(\mathcal{F})$, which is indeed a reduced singularity by assumptions. Moreover, all separatrices of these $p_i$s are contained in $C$. Now the conclusion of the theorem gives a point $p \in C \cap \text{Sing}(\mathcal{F})$ such that one separatrix through $p$ is not contained in $C$. This shows that $p$ is a reduced singularity on $C$ and different from $p_1, p_2, \ldots, p_{n-1}$. Hence, we have at least $n$ reduced singularities on $C$.

2.2.1 | Noninvariant curves

We first consider the noninvariant curves and define the tangency order for them.

**Definition 2.9.** Let $(X, \mathcal{F})$ be a foliated surface and $C$ be a noninvariant reduced curve. Let $p \in C \setminus \text{Sing}(X)$ and $v$ be the vector field generating $\mathcal{F}$ around $p$. Let $f$ be the local defining function of $C$ at $p$. We define the tangency order of $\mathcal{F}$ along $C$ at $p$ to be

$$\text{tang}(\mathcal{F}, C, p) := \dim \mathbb{C} \langle \mathcal{O}_{X, p}, f, v(f) \rangle.$$  

Note that $\text{tang}(\mathcal{F}, C, p) \geq 0$ and is independent of the choices of $v$ and $f$. Moreover, if $\mathcal{F}$ is transverse to $C$ at $p$, then $\text{tang}(\mathcal{F}, C, p) = 0$. Therefore, if $C$ is compact, then we can define

$$\text{tang}(\mathcal{F}, C) := \sum_{p \in C} \text{tang}(\mathcal{F}, C, p).$$
Theorem 2.10 (Adjunction for noninvariant divisors, [4, 5, Proposition 2.2]). Let $\mathcal{F}$ be a foliation on a smooth projective surface $X$. Let $C$ be a noninvariant irreducible curve on $X$ and $C'$ be the normalization of $C$. Then, there is an effective divisor $\Delta$ on $C'$ such that $(K_{\mathcal{F}} + C)|_{C'} = \Delta$ and $\deg \Delta = \text{tang}(\mathcal{F}, C)$.

Corollary 2.11. Let $C$ be a noninvariant curve on a foliated surface $(X, \mathcal{F})$. If $C$ is contained in the smooth locus of $X$, then we have $(K_{\mathcal{F}} + C) \cdot C \geq 0$.

Proposition 2.12. Let $(X, \mathcal{F})$ be a foliated surface with $X$ a smooth projective surface. Suppose $C$ is a noninvariant curve with the tangency order $\text{tang}(\mathcal{F}, C) = 0$. Then, $C$ is smooth.

Proof. Assume $C$ is singular at $p$. Let $f$ be the local defining function of $C$ around $p$ and $v$ be the vector field generating $\mathcal{F}$ around $p$. Note that $v(f)(p) = 0$ since $C$ is singular at $p$. This implies that $1 \notin \langle f, v(f) \rangle$ and thus $\mathcal{O}_{X, p} / \langle f, v(f) \rangle$ has a nonzero element. Therefore, $\text{tang}(\mathcal{F}, C, p) \geq 1$, which contradicts the assumption that $\text{tang}(\mathcal{F}, C) = 0$. □

2.2.2 Invariant curves

Now, we study the invariant curves.

Definition 2.13. Let $(X, \mathcal{F})$ be a foliated surface and $C$ be an invariant curve. Let $p \in C \setminus \text{Sing}(X)$ and $\omega$ be a 1-form generating $\mathcal{F}$ around $p$. If $C$ is an invariant curve and $f$ is the local defining function of $C$ at $p$, then we can write

$$g\omega = hdf + f\eta,$$

where $g$ and $h$ are holomorphic functions, $\eta$ is a holomorphic 1-form, and $h$, $f$ are relatively prime functions.

We define the index $Z(\mathcal{F}, C, p)$ to be the vanishing order of $\frac{h}{g}|_C$ at $p$. Also, we define the index $\text{CS}(\mathcal{F}, C, p)$ to be the residue of $-\frac{1}{h}\eta|_C$ at $p$. These two definitions are independent of the choices of $f$, $g$, $h$, $\omega$, and $\eta$. (For a reference, see [5, p. 15 in chapter 2 and p. 27 in chapter 3].)

Note that if $p \notin \text{Sing}(\mathcal{F})$, then $Z(\mathcal{F}, C, p) = 0 = \text{CS}(\mathcal{F}, C, p)$. Therefore, if $C$ is compact, then we can define

$$Z(\mathcal{F}, C) := \sum_{p \in C} Z(\mathcal{F}, C, p) \text{ and}$$

$$\text{CS}(\mathcal{F}, C) := \sum_{p \in C} \text{CS}(\mathcal{F}, C, p),$$

where the sums are taken over only finitely many points.

Theorem 2.14 (Adjunction for invariant divisors, [4]). Let $\mathcal{F}$ be a foliation on a smooth projective surface $X$. Let $C$ be an invariant irreducible curve on $X$ and $C'$ be its normalization. Then, there is an effective divisor $\Delta$ on $C'$ such that $K_{\mathcal{F}}|_{C'} = K_{C'} + \Delta$ and $\deg \Delta = Z(\mathcal{F}, C) + \deg \text{Diff}_C(0)$ where $\text{Diff}_C(0)$ is the different with $(K_X + C)|_{C'} = K_{C'} + \text{Diff}_C(0)$. In particular, we have $K_{\mathcal{F}} \cdot C = Z(\mathcal{F}, C) + 2p_a(C) - 2$ where $p_a(C)$ is the arithmetic genus of $C$.

The following formula was proved by Camacho and Sad in [7] when $C$ is nonsingular. Later it was successively generalized by Lins Neto [13] and Suwa [18].

Theorem 2.15 (Camacho–Sad formula). Let $\mathcal{F}$ be a foliation on a smooth projective surface $X$ and $C$ be an invariant curve on $X$, which is not necessarily smooth or irreducible. Then, we have $C^2 = \text{CS}(\mathcal{F}, C)$.

Lemma 2.16. Given a foliated surface $(X, \mathcal{F})$ and $p$ a reduced singularity.
1. If $p$ is nondegenerate, assume $\omega = \lambda y (1 + o(1)) dx - x (1 + o(1)) dy$ generates $\mathcal{F}$ around $p$. Then,

$$\text{CS}(\mathcal{F}, x = 0, p) = \frac{1}{\lambda}, \text{CS}(\mathcal{F}, y = 0, p) = \lambda, \text{ and } Z(\mathcal{F}, x = 0, p) = Z(\mathcal{F}, y = 0, p) = 1.$$ 

2. If $p$ is a saddle-node, assume $\omega = y^{k+1} dx - (x(1 + \nu y^k) + y o(k)) dy$ generates $\mathcal{F}$ around $p$ where $k \in \mathbb{N}$ and $\nu \in \mathbb{C}$. Then,

$$\text{CS}(\mathcal{F}, y = 0, p) = 0 \text{ and } Z(\mathcal{F}, y = 0, p) = 1.$$ 

Moreover, suppose there exists a weak separatrix, then

$$\text{CS}(\mathcal{F}, x = 0, p) = \nu \text{ and } Z(\mathcal{F}, x = 0, p) = k + 1.$$ 

Proof. This is done by direct computation. For a reference, see [5, pp. 30–31 in chapter 3].

\[ \square \]

2.3 Minimal resolutions of foliated surfaces

**Definition 2.17.** Let $(X, \mathcal{F})$ be a foliated surface with $X$ smooth. A curve $C \subset X$ is called $\mathcal{F}$-exceptional if

1. $C$ is a $(-1)$-curve, that is a smooth rational curve of self-intersection $-1$, and
2. the contraction of $C$ to a point $p$ introduces a foliated surface $(X', \mathcal{F}')$, which has a smooth foliation point or a reduced singularity at $p$.

**Proposition 2.18** [5, pp. 62–63 in chapter 5]. If $C$ is an $\mathcal{F}$-exceptional curve on $X$, then we have the following two possibilities:

1. $C$ is $\mathcal{F}$-invariant and consists of only one foliation singularity $q$ with $Z(\mathcal{F}, C, q) = 1$. The contraction of $C$ introduces a smooth foliation point.
2. $C$ is $\mathcal{F}$-invariant and consists of only two distinct foliation singularities $q_1$ and $q_2$ with $Z(\mathcal{F}, C, q_1) = Z(\mathcal{F}, C, q_2) = 1$. The contraction of $C$ introduces a reduced singularity.

**Definition 2.19.** A morphism $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ of foliated surfaces is called a resolution if $(Y, \mathcal{G})$ has only reduced foliation singularities.

A resolution $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ of a foliated surface $(X, \mathcal{F})$ is called minimal if any resolution $\phi : (Z, \mathcal{H}) \to (X, \mathcal{F})$ of the foliated surface $(X, \mathcal{F})$ factors through $\pi$. That is, there is a morphism $\psi : (Z, \mathcal{H}) \to (Y, \mathcal{G})$ with $\mathcal{H} = \psi^* \mathcal{G}$ such that $\phi = \pi \circ \psi$.

**Proposition 2.20.** Any foliated surface $(X, \mathcal{F})$ has a unique minimal resolution up to isomorphism.

Proof. Taking a resolution of $X$, and then by Seidenberg’s Theorem 2.6, we have a resolution $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ of the foliated surface $(X, \mathcal{F})$. After blowing down $\mathcal{G}$-exceptional curves over $X$, we may assume that $(Y, \mathcal{G})$ has no $\mathcal{G}$-exceptional curves over $X$. We will show that this $\pi$ gives a minimal resolution of the foliated surface $(X, \mathcal{F})$.

Given any resolution $\phi : (Z, \mathcal{H}) \to (X, \mathcal{F})$ of the foliated surface $(X, \mathcal{F})$, let $W$ be a minimal resolution of singularities of $Z \times_X Y$, and $\mathcal{E}$ be the pullback foliation on $W$. So we have the following diagram of foliated surfaces:

$$
\begin{array}{ccc}
(W, \mathcal{E}) & \xrightarrow{\pi} & (Z, \mathcal{H}) \\
\downarrow{\rho} & & \downarrow{\phi} \\
(Y, \mathcal{G}) & \xrightarrow{\pi} & (X, \mathcal{F}).
\end{array}
$$

We may assume $(W, \mathcal{E})$ is minimal in the sense that there is no birational morphism of foliated surfaces $\vartheta : (W, \mathcal{E}) \to (W', \mathcal{E}')$, which is not an isomorphism, such that both $\alpha$ and $\beta$ factor through $\vartheta$. 

Suppose $\alpha$ is not an isomorphism, then there is an (at last) $\alpha$-exceptional curve $\tilde{C}$ on $W$ with $\tilde{C}^2 = -1$, which is also $E$-exceptional. By the minimality of $(W, E')$, $\tilde{C}$ is not contracted by $\beta$. Let $C \subset Y$ be the curve $\beta(\tilde{C})$. Since $p_a(C) = 0$, we have that $p_a(C') = 0$. Note that $C$ is contracted by $\pi$ because $\tilde{C}$ is contracted by $\phi \circ \alpha = \pi \circ \beta$. Hence, $C^2 \leq -1$. Also, $C^2 \geq \tilde{C}^2 = -1$ and thus, $C^2 = -1$. Moreover, $\beta$ is isomorphic around $\tilde{C}$. Therefore, $C$ is $\mathcal{G}$-exceptional. Since there is no $\mathcal{G}$-exceptional curve over $X$ by assumption, $C$ is not contracted by $\pi$, which contradicts that $\tilde{C}$ is contracted by $\phi \circ \alpha = \pi \circ \beta$. \hfill \Box

**Remark 2.21.** In general, for any minimal resolution $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ of the foliated surface $(X, \mathcal{F})$, the morphism $\pi : Y \to X$ of surfaces is not the minimal resolution of $X$.

### 2.4 Dual graphs

**Definition 2.22.** Let $C = \bigcup C_i$ be a collection of proper curves on the smooth locus of a surface $X$. Then, the (weighted) dual graph $\Gamma = \Gamma(C)$ of $C$ is defined as follows:

1. The vertices of $\Gamma$ are the curves $C_i$. We will use $[C_i]$ to indicate the vertex of $\Gamma$ corresponding to the curve $C_i$.
2. Two vertices $[C_i]$ and $[C_j]$ for $i \neq j$ are connected with $C_i \cdot C_j$ edges.
3. The weight $w([C_i])$ of vertex $[C_i]$ is given by $-C_i^2$.

**Definition 2.23.** For any (dual) graph $\Gamma$, we have the following definitions:

1. A cycle is a graph whose vertices and edges can be ordered as $[C_1], \ldots, [C_m]$ and $e_1, \ldots, e_m$ where $m \in \mathbb{N}$, $[C_i] \neq [C_j]$, and $e_i \neq e_j$ for $1 \leq i < j \leq m$ such that edge $e_i$ connects vertices $[C_i]$ and $[C_{i+1}]$ for $i = 1, \ldots, m$, where $[C_{m+1}] := [C_1]$.
2. We say $\Gamma$ has simple edges if any two vertices are connected by at most one edge.
3. A tree is a connected graph, which has no cycle as its subgraph.
4. The degree, denoted by $\deg([C_i])$, of the vertex $[C_i]$ is the number of edges connecting to $[C_i]$.
5. The vertex $[C_i]$ is called a leaf of $\Gamma$ if the degree of $[C_i]$ is 1.
6. The vertex $[C_i]$ is called a fork of $\Gamma$ if the degree of $[C_i]$ is at least 3.
7. A tree without forks is called a chain.
8. The connected components of a tree minus a fork are called the branches of the fork.
9. A tree is star-shaped if there is exactly one fork.
10. We define $\Delta(\Gamma)$ to be the absolute value of the determinant of the intersection matrix for $\Gamma$. We set $\Delta(\emptyset) = 1$.

By direct computation, we have the following lemma.

**Lemma 2.24** [1, Lemma 3.1.8 and 3.1.9]. Suppose $\Gamma$ is a tree with simple edges.

1. For any vertex $[C]$ of $\Gamma$ of weight $w([C])$, we have

$$
\Delta(\Gamma) = w([C]) \Delta(\Gamma \setminus \{[C]\}) - \sum_{i=1}^{s} \Delta(\Gamma \setminus \{[C], [C_i]\})
$$

where $C_1, \ldots, C_s$ are all vertices adjacent to $C$.
2. For any two vertices $[C_i]$ and $[C_j]$ of $\Gamma$, the $(i, j)$-cofactor of the intersection matrix $A$ of $\Gamma$ is

$$
A_{i,j} = (-1)^{n+1} \Delta(\Gamma \setminus \{\text{path from } [C_i] \text{ to } [C_j]\}),
$$

where $n = |\Gamma|$. 

3 | CLASSIFICATION

In this section, we prove Theorem 3.21, which gives a full list of all possibilities of the dual graphs of the exceptional divisors on the minimal resolutions of log canonical foliation singularities on surfaces.

In order to describe the exceptional divisor over a foliation singularity, we recall the following definitions. (See also [5, Definition 5.1 and 8.1] and [15, Definition III.0.2 and III.2.3].)

**Definition 3.1.** A compact curve $C = \bigcup_{i=1}^{3} C_i$ is called a string if

1. each $C_i$ is a smooth rational irreducible curve and
2. $C_i \cdot C_j = 1$ if $|i - j| = 1$ and $0$ if $|i - j| \geq 2$.

If, moreover, $C_i^2 \leq -2$ for all $i$, then we call $C$ a Hirzebruch–Jung string.

**Definition 3.2.** Given a foliated surface $(X, \mathcal{F})$ with $X$ smooth.

1. $C$ is called a $(−1)$-$\mathcal{F}$-curve (resp. $(−2)$-$\mathcal{F}$-curve) if
   (a) $C$ is a smooth rational irreducible $\mathcal{F}$-invariant curve and
   (b) $Z(\mathcal{F}, C) = 1$ (resp. $Z(\mathcal{F}, C) = 2$).
2. We say $C = \bigcup C_i$ is an $\mathcal{F}$-chain if
   (a) $C$ is a Hirzebruch–Jung string,
   (b) each $C_i$ is $\mathcal{F}$-invariant,
   (c) $\text{Sing}(\mathcal{F}) \cap C$ are all reduced and nondegenerate, and
   (d) $Z(\mathcal{F}, C_i) = 1$ and $Z(\mathcal{F}, C_i) = 2$ for all $i \geq 2$.
3. If an irreducible $\mathcal{F}$-invariant curve $E$ is not contained in an $\mathcal{F}$-chain $C$ but meets the chain, then we call $E$ the tail of the chain $C$. Note that such a curve $E$ is unique since there are at most two separatrices passing through a reduced singularity.
4. $C$ is called a bad tail if
   (a) $C$ is a smooth rational irreducible $\mathcal{F}$-invariant curve with $Z(\mathcal{F}, C) = 3$ and $C^2 \leq -2$ and
   (b) $C$ intersects two $(−1)$-$\mathcal{F}$-curves whose self-intersections are both $−2$.

We recall the following theorem for classification of canonical singularities on foliated surfaces. The idea of the proof of the theorem is that we take Seidenberg’s resolution theorem, run foliated minimal model program, and study the contracted divisors.

**Theorem 3.3** [15, Theorem III.3.2]. Let $(X, \mathcal{F})$ be a foliated surface with at worst canonical singularities and $\pi : Y \to X$ be the minimal resolution of $X$ with exceptional divisors $E = \bigcup E_i$. Then, the connected components of $E$ belong to one of the following types:

1. A $\mathcal{G}$-chain. (See Figure 6.)
2. A chain of three invariant curves $E_1 \cup E_2 \cup E_3$ where $E_1$ and $E_3$ are $(−1)$-$\mathcal{G}$-curves with self-intersection $−2$ and $E_2$ is a bad tail. (See Figure 7.)
3. A chain of $(−2)$-$\mathcal{G}$-curves. (See Figure 8.)
4. A graph of $D_n$ type. More precisely, two $(−1)$-$\mathcal{G}$-curves with self-intersection $−2$ joined by a bad tail, which itself connects to a chain of $(−2)$-$\mathcal{G}$-curves. (See Figure 9.)
5. An elliptic Gorenstein leaf, which is either a rational curve with only one node or a cycle or $(−2)$-$\mathcal{G}$-curves. (See Figure 10.)

We recall the following well-known lemma, which is a key player in proving Proposition 3.14.

**Lemma 3.4.** Let $C = \bigcup_{i} C_i$ be a set of proper curves on a smooth surface. Assume that the intersection matrix $(C_i \cdot C_j)_{i,j}$ is negative definite. Let $A = \sum a_i C_i$ be an $\mathbb{R}$-linear combination of the curves $C_i$s. If $A \cdot C_j \geq 0$ for all $j$, then $a_i \leq 0$ for all $i$. If, moreover, $C$ is connected, then either $a_i = 0$ for all $i$ or $a_i < 0$ for all $i$. 
In particular, if $C$ is connected and one of the inequalities $A \cdot C_j \geq 0$ is strict, then $a_i < 0$ for all $i$.

**Proof.** For a reference, see [12, Lemma 3.41].

**Remark 3.5.** The exceptional divisor $E$ of a resolution of a singularity is connected by Zariski’s main theorem. Moreover, it is well-known that the intersection matrix of $E$ is negative definite. (For a reference, see [12, Lemma 3.40].)

### 3.1 Special leaves

In the following subsections, we fix the following notation.

$$(\star) \begin{cases} (X, \mathcal{F}, p) \text{ is a germ of a foliated surface with } p \text{ a log canonical singularity of } \mathcal{F}. \\
\pi : (Y, \mathcal{G}) \to (X, \mathcal{F}) \text{ is the minimal resolution for } (X, \mathcal{F}, p). \\
E = \bigcup E_i \text{ is the exceptional divisors of } \pi \text{ and } \Delta = \sum \varepsilon(E_i) E_i. \\
\Gamma \text{ is the dual graph of } E = \bigcup E_i. \end{cases}$$

In this subsection, we will investigate the special leaves (see Definition 3.7) and their important properties (see Propositions 3.14 and 3.15).

**Proposition 3.6.** Let notations be as in $(\star)$. Then we have $\langle K_G + \Delta \rangle \cdot E_i \geq \min \{ \deg [E_i] - 2, -1 \}$ for any vertex $[E_i]$ with $E_i$ invariant and $\langle K_G + \Delta \rangle \cdot E_i \geq 0$ if $E_i$ is noninvariant.

In particular, $\langle K_G + \Delta \rangle \cdot E_i < 0$ only if $\langle K_G + \Delta \rangle \cdot E_i = -1$ and $E_i$ is invariant with $\deg [E_i] \leq 1$.

**Proof.** Let $d = \deg [E_i]$. Suppose there are exactly $m$ edges of $[E_i]$ connecting to the vertices corresponding to invariant curves. Since the intersections of two invariant curves are reduced singularities and at most two separatrices pass through any reduced singularity, there are at least $m$ distinct foliation singularities on $E_i$. Thus, we have that $Z(\mathcal{G}, E_i) \geq m$.

The other $d - m$ edges of $[E_i]$ connect to the vertices corresponding to noninvariant curves, which are in the support of $\Delta$. Thus, we have that $\Delta \cdot E_i = d - m$ by the definition of edges. Therefore, by adjunction for invariant divisors (Theorem 2.14), we have

$$\langle K_G + \Delta \rangle \cdot E_i = Z(\mathcal{G}, E_i) + 2p_a(\mathcal{G}, E_i) - 2 + \Delta \cdot E_i \geq m - 2 + (d - m) = d - 2.$$ 

On the other hand, if $E_i$ is noninvariant, then we have $E_i \subset \Delta$ and

$$\langle K_G + \Delta \rangle \cdot E_i = \langle K_G + \Delta \rangle \cdot E_i \geq \langle K_G + E_i \rangle \cdot E_i = \tang(\mathcal{G}, E_i) \geq 0$$

by adjunction for noninvariant divisors (Theorem 2.10).

Thus, if $\langle K_G + \Delta \rangle \cdot E_i < 0$, then $\langle K_G + \Delta \rangle \cdot E_i = -1$ and $E_i$ is invariant. Hence, we have $-1 = \langle K_G + \Delta \rangle \cdot E_i \geq \deg [E_i] - 2$. Therefore, $\deg [E_i] \leq 1$. \qed

**Definition 3.7.** Let notations be as in $(\star)$. We call $[E_i]$ a special leaf if $\langle K_G + \Delta \rangle \cdot E_i = -1$.

Next, we investigate some properties of the special leaves.

**Proposition 3.8.** Let notations be as in $(\star)$. A special leaf $[L]$ itself is a $\mathcal{G}$-chain and disjoint from $\Delta$.

**Proof.** Note that, by adjunction of invariant divisors (Theorem 2.15), we have

$$-1 = \langle K_G + \Delta \rangle \cdot L = Z(\mathcal{G}, L) + 2p_a(L) - 2 + \Delta \cdot L \geq -2.$$
Hence, we have \( p_a(L) = 0 \), and thus \( L \) is a smooth rational curve. If \( Z(\mathcal{G}, L) = 0 \), then there is no singularity on \( L \) and thus \( L^2 = CS(\mathcal{G}, L) = 0 \) by Camacho–Sad formula (Theorem 2.15), which contradicts that \( L \) is an exceptional divisor of \( \pi \).

So \( Z(\mathcal{G}, L) = 1 \), and thus \( \Delta \cdot L = 0 \). By the minimality of \( \pi \), we have \( L^2 \leq -2 \). Thus, \( L \) itself is a \( \mathcal{G} \)-chain.

**Definition 3.9 (Construction of \( \mathcal{G}_L \)).** Let notations be as in (\( \star \)). We construct the maximal \( \mathcal{G} \)-chain \( \mathcal{G}_L \) which starts from a special leaf \( L \) and is disjoint from \( \Delta \). We have showed that \( L \) itself is a \( \mathcal{G} \)-chain and disjoint from \( \Delta \). Note that any \( \mathcal{G} \)-chain \( G \) has at most one tail. If there is no tail of \( L \), then we set \( C_L = L \). Otherwise, let \( T_1 \) be the tail of \( L \). If \( G_1 := L \cup T_1 \) is not a \( \mathcal{G} \)-chain or meets \( \Delta \), then we set \( C_L = L \). Otherwise, that is \( G_1 \) is a \( \mathcal{G} \)-chain and disjoint from \( \Delta \), then we may consider the tail \( T_2 \) of the \( \mathcal{G} \)-chain \( G_1 \). If \( T_2 \) does not exist or \( T_2 \) exists but \( G_2 := G_1 \cup T_2 \) is not a \( \mathcal{G} \)-chain or meets \( \Delta \), then we set \( C_L = G_1 \). Otherwise, we have a \( \mathcal{G} \)-chain \( G_2 \), which starts from \( L \) and is disjoint from \( \Delta \). Since \( \Gamma \) is a finite graph, we iterate this process, which will end up with the maximal \( \mathcal{G} \)-chain \( \mathcal{G}_L \), which starts from \( L \) and is disjoint from \( \Delta \). We will write \( C_L = \bigcup_{i=1}^{N_L} \tilde{C}_{L,i} \) where \( \tilde{C}_{L,i} \)'s are all irreducible with \( \tilde{C}_{L,1} = L \) and \( \tilde{C}_{L,j} \cdot \tilde{C}_{L,j+1} = 1 \) for \( j = 1, \ldots, N_L - 1 \).

**Proposition 3.10.** Let notations be as in (\( \star \)). Suppose there is a special leaf \([L]\) and the dual graph of \( \mathcal{G}_L \) is not \( \Gamma \). Then the tail of \( \mathcal{G}_L \) exists, which will be denoted by \( C_{L,N_L+1} \). Moreover, it is invariant with

\[
Z(\mathcal{G}, C_{L,N_L+1}) \geq 2 \quad \text{and} \quad (K_G + \Delta) \cdot C_{L,N_L+1} \geq 1.
\]

**Proof.** Since the dual graph of \( \mathcal{G}_L \) is not \( \Gamma \), there is a curve \( C_{L,N_L+1} \) meeting \( C_L \) but not in \( C_{L} \). Notice that \( C_{L,N_L+1} \) is invariant, otherwise \( C_{L,N_L+1} \) is contained in \( \Delta \) and \( C_L \) intersects \( \Delta \), which contradicts the construction of \( C_L \). Since \( C_{L,N_L+1} \) intersects \( C_L \), we have \( Z(\mathcal{G}, C_{L,N_L+1}) \geq 1 \). By applying separatrix theorem (Theorem 2.8) to \( \tilde{C}_L := \bigcup_{j=1}^{N_L} \tilde{C}_{L,j} \), there is a foliation singularity \( p \) on \( \tilde{C}_L \) with a separatrix through \( p \) but not in \( \tilde{C}_L \). Thus, \( p \) must be on \( C_{L,N_L+1} \) only, which implies that \( Z(\mathcal{G}, C_{L,N_L+1}) \geq 2 \). By adjunction for invariant divisors (Theorem 2.14), we have

\[
(K_G + \Delta) \cdot C_{L,N_L+1} = Z(\mathcal{G}, C_{L,N_L+1}) + 2p_a(C_{L,N_L+1}) - 2 + \Delta \cdot C_{L,N_L+1}.
\]

Hence, if \( Z(\mathcal{G}, C_{L,N_L+1}) \geq 3 \) or \( p_a(C_{L,N_L+1}) \geq 1 \), then \( (K_G + \Delta) \cdot C_{L,N_L+1} \geq 1 \). Otherwise, \( \tilde{C}_L \) is a \( \mathcal{G} \)-chain and thus \( \Delta \cdot C_{L,N_L+1} \geq 1 \) by the maximality of \( \mathcal{G}_L \). Therefore, we still have \( (K_G + \Delta) \cdot C_{L,N_L+1} \geq 1 \).

**Proposition 3.11.** Let notations be as in (\( \star \)). For any two distinct special leaves \([L]\) and \([L']\), two chains \( C_L \) and \( C_{L'} \) are disjoint.

**Proof.** Suppose \( C_L \) and \( C_{L'} \) are not disjoint. We assume first that they have no common components. Then, they must intersect at their last curves (see Figure 1). Then, \( C_L \cup C_{L'} \) has \( N_L + N_{L'} \) irreducible components on which there are only \( N_L + N_{L'} - 1 \) reduced singularities, which contradicts separatrix theorem (Theorem 2.8).

Now, if they have some common components, then we define \( j_0 \) to be the smallest integer such that \( C_{L,j_0} \) is a component of \( C_{L'} \), say \( C_{L',j_0'} \). Then, \( D := \left( \bigcup_{i=1}^{j_0} C_{L,i} \right) \cup \left( \bigcup_{i=1}^{j_0'} C_{L',i} \right) \) forms a similar graph as Figure 1 with \( C_{L,j_0} = C_{L',j_0'} \). Thus, \( D \) has \( j_0 + j_0' - 2 \) irreducible components because of one (and only one) common component. However, there are only \( j_0 + j_0' - 2 \) reduced singularities on \( D \) because of two common singularities on the common component \( C_{L,j_0} = C_{L',j_0'} \). This contradicts separatrix theorem (Theorem 2.8).
Definition 3.12. Let notations be as in (∗). For any special leaf \([L]\) with its maximal \(\mathcal{G}\)-chain \(C_L = \bigcup_{i=1}^{N_L} C_{L,i}\), we define an associated divisor as

\[
D_L := \sum_{j=1}^{N_L} \frac{N_L + 1 - j}{N_L + 1} C_{L,j}.
\]

Also we define \(D := \sum D_L\) where the sum is over all special leaves \([L]\).

From now on, we will fix more notation as follows.

\[
\begin{align*}
(X, \mathcal{F}, p) & \text{ is a germ of a foliated surface with } p \text{ a log canonical singularity of } \mathcal{F}. \\
\pi : (Y, \mathcal{G}) & \to (X, \mathcal{F}) \text{ is the minimal resolution for } (X, \mathcal{F}, p). \\
E & = \bigcup E_i \text{ is the exceptional divisors of } \pi \text{ and } \Delta = \sum_i \varepsilon(E_i)E_i. \\
\Gamma & \text{ is the dual graph of } E = \bigcup E_i. \\
D & \text{ is a divisor defined in Definition 3.12.}
\end{align*}
\]

Proposition 3.13. Let notations be as in (†). Suppose \(\Gamma\) is not the dual graph of a \(\mathcal{G}\)-chain. Then, there is a vertex \([V]\) of \(\Gamma\) such that the coefficient of \(V\) in \(D\) is zero.

Proof. Suppose not, then the support of \(D\), which is a union of \(C_{L,i}\)s where \([L]\)s are special leaves, is the same as the support of \(E\). Since \(\Gamma\) is connected and any two distinct \(C_{L,i}\) and \(C_{L,j}\) are disjoint by Proposition 3.11, we conclude that \(\Gamma\) is the dual graph of the \(\mathcal{G}\)-chain \(C_L\) for one special leaf \([L]\), which contradicts our assumption. □

Proposition 3.14. Let notations be as in (†). Suppose \(\Gamma\) is not the dual graph of a \(\mathcal{G}\)-chain. We have \((K_\mathcal{G} + \Delta) \cdot V = D \cdot V\) for all vertices \([V]\) of \(\Gamma\). Moreover, all special leaves \([L]\) have self-intersections \(L^2 = -2\).

Proof. It suffices to show that \(D \cdot V \leq (K_\mathcal{G} + \Delta) \cdot V = \sum a_i E_i \cdot V\) for all vertices \([V]\) of \(\Gamma\). In fact, assume we have the inequalities for all vertices \([V]\) and one of them is strict, then by Lemma 3.4, we have \(a_i < \text{ord}_{E_i} D\) for all \(i\). However, by Proposition 3.13, there is a vertex, say \([E_j]\) for some \(j\), of \(\Gamma\) such that \(\text{ord}_{E_i} D = 0\), which implies that the corresponding coefficient \(a_j\) is strictly less than zero, contradicting our assumption in (†).

We have the following cases:

1. If \(V\) is noninvariant, then \(D \cdot F = 0\) since \(C_{L,i}\)s are disjoint from noninvariant divisors for any special leaves \([L]\). Moreover, \((K_\mathcal{G} + \Delta) \cdot V \geq 0\) by Proposition 3.6.
2. If \(V = L\) is a special leaf with \(N_L = 1\), then by Proposition 3.11, we have

\[
D \cdot L = \frac{1}{2} L^2 \leq -1 = (K_\mathcal{G} + \Delta) \cdot L
\]

and the equality holds if and only if \(L^2 = -2\).
3. If \(V = L\) is a special leaf with \(N_L \geq 2\), then

\[
D \cdot L = \frac{N_L}{N_L + 1} L^2 + \frac{N_L - 1}{N_L + 1} C_{L,2} \cdot L \leq \frac{-2N_L}{N_L + 1} + \frac{N_L - 1}{N_L + 1} = -1 = (K_\mathcal{G} + \Delta) \cdot L
\]

and the equality holds if and only if \(L^2 = -2\).
4. If \(V = C_{L,j}\) for some special leaf \([L]\) and \(j \geq 2\), then we have

\[
D \cdot C_{L,j} = \frac{N_L + 2 - j}{N_L + 1} + \frac{N_L + 1 - j}{N_L + 1} C_{L,j}^2 \leq \frac{N_L + 2 - j}{N_L + 1} + \frac{N_L + 1 - j}{N_L + 1} (-2) + \frac{N_L - j}{N_L + 1} = 0 = (K_\mathcal{G} + \Delta) \cdot L,
\]

where the last equality holds since \(C_{L,j}\) is a \((-2)\)-\(\mathcal{G}\)-curve and \(C_L\) is disjoint from \(\Delta\).
5. If \( V = F \) with \([F]\) a fork of \( \Gamma \) and with \( F \) invariant, then

\[
D \cdot F = \sum \frac{1}{N_L + 1} \leq \frac{m}{2},
\]

where the sum is over all special leaves \([L]\) such that \( C_{L,N_L+1} = F \) and \( m \) is the number of special leaves \( L \) such that \( C_{L,N_L+1} = F \). Also, by Theorem 2.14, we have

\[
(K_g + \Delta) \cdot F \geq K_g \cdot F \geq Z(\mathcal{G}, F) - 2 \geq m - 2.
\]

Moreover, by Proposition 3.6, we have \((K_g + \Delta) \cdot F \geq \deg[F] - 2 \geq 1 \) since \([F]\) is a fork, which is the only place I use the assumption that \([F]\) is a fork. Hence, we have

\[
(K_g + \Delta) \cdot F \geq \max\{m - 2, 1\} \geq \frac{m}{2} \geq D \cdot F.
\]

6. If \( V = C_{L,N_L+1} \) for some special leaf \([L]\) and has \( \deg[V] \leq 2 \), then we have

\[
D \cdot V = \sum \frac{1}{N_L + 1} \leq 1,
\]

where the sum is over all special leaves \([L]\) such that \( C_{L,N_L+1} = V \). By Proposition 3.10, we have \((K_g + \Delta) \cdot V \geq 1\).  

7. The remaining case is when \( V \) has \( \deg[V] \leq 2 \) and is not \( C_{L,j} \) where \([L]\) is a special leaf and \( 1 \leq j \leq N_L + 1 \), then \( D \cdot V = 0 \) and \((K_g + \Delta) \cdot V \geq 0 \) since \([V]\) is not a special leaf.

\[\square\]

**Proposition 3.15.** Let notations be as in (†). Suppose \( \Gamma \) is not the dual graph of a \( \mathcal{G} \)-chain. For any special leaf \([L]\), we have \( N_L = 1 \) and \( C_{L,2} \), if exists, must be a bad tail (see Definition 3.2 (4)) which is disjoint from \( \Delta \) and has \( \deg[C_{L,2}] = 2 \) or 3.

**Proof.** Let \( V := C_{L,N_L+1} \) and \( L_1, ..., L_s \) be all special leaves such that \( C_{L_i,N_{L_i}+1} = V \) for \( i = 1, ..., s \). So we have \( Z(\mathcal{G}, V) \geq s \).

If \( Z(\mathcal{G}, V) = s \), then \( V \cup \bigcup_{i=1}^{s} C_{L_i} \) has \( \sum_{i=1}^{s} N_{L_i} + 1 \) irreducible components on which there are only \( \sum_{i=1}^{s} N_{L_i} \) reduced singularities, which contradicts separatrix theorem (Theorem 2.8). Thus, we have

\[
Z(\mathcal{G}, V) \geq s + 1 \tag{3.1}
\]

and then \((K_g + \Delta) \cdot V \geq s - 1 \) by adjunction for invariant divisors (Theorem 2.14). For the case when \( s = 1 \), we have \( D \cdot V = \frac{1}{N_L + 1} \leq \frac{1}{2} \). However, \((K_g + \Delta) \cdot V \geq 1 \) by Proposition 3.10, which is a contradiction to Proposition 3.14.

For the case when \( s \geq 2 \), we have

\[
s - 1 \leq (K_g + \Delta) \cdot V = D \cdot V = \sum_{i=1}^{s} \frac{1}{N_{L_i} + 1} \leq \frac{s}{2}.
\]

This inequality holds only when \( s = 2, N_{L_1} = N_{L_2} = 1, \) and \((K_g + \Delta) \cdot V = 1\). Since \( V = C_{L_1,2} = C_{L_2,2} \) is invariant, by Proposition 3.6, we have \( \deg[V] \leq 3 \). Note that \( V \) meets \( L_1 \) and \( L_2 \), then \( \deg[V] \geq 2 \) and \( Z(\mathcal{G}, V) \geq 2 \).

Suppose \( Z(\mathcal{G}, V) = 2 \), then \( L_1 \cup V \cup L_2 \) (see Figure 2) has three irreducible components on which there are only two reduced singularities, which contradicts separatrix theorem (Theorem 2.8).
So we have $Z(\mathcal{G}, V) \geq 3$. Then, by adjunction for invariant divisors (Theorem 2.14), we have

$$1 = (K_{\mathcal{G}} + \Delta) \cdot V = Z(\mathcal{G}, V) + 2p_a(V) - 2 + \Delta \cdot V \geq 3 + 0 - 2 + 0 = 1.$$ 

Therefore, all inequalities are equalities. In particular, $Z(\mathcal{G}, V) = 3$ and $p_a(V) = 0$. By Proposition 3.14, $L_1^2 = L_2^2 = -2$. To show $V$ is a bad tail, it remains to show $V^2 \leq -2$. In fact, if $V^2 = -1$, then the intersection matrix for $L_1, V,$ and $L_2$ is

$$\begin{pmatrix}
-2 & 1 & 0 \\
1 & -1 & 1 \\
0 & 1 & -2
\end{pmatrix}$$

whose determinant is zero. This implies that this matrix is not negative definite, which contradicts Artin’s criterion [3] for contractibility of curves. Thus, $V^2 \leq -2$. 

3.2 | Nonfork

In this subsection, we study the vertices of degree $\leq 2$.

**Proposition 3.16.** Let notations be as in (†). All irreducible $\pi$-exceptional noninvariant divisors have tangency order zero and are pairwise disjoint.

**Proof.** Let $V$ be any irreducible $\pi$-exceptional noninvariant divisor. Then, $\Gamma$ is not the dual graph of a $\mathcal{G}$-chain. So by Proposition 3.14 and adjunction for noninvariant divisors (Theorem 2.10), we have

$$0 = D \cdot V = (K_{\mathcal{G}} + \Delta) \cdot V \geq (K_{\mathcal{G}} + V) \cdot V + (\Delta - V) \cdot V = \text{tang}(\mathcal{G}, V) + (\Delta - V) \cdot V \geq 0.$$ 

Thus, $V$ has tangency order zero and is disjoint from other irreducible $\pi$-exceptional noninvariant divisors. 

**Proposition 3.17.** Let notations be as in (†). All vertices $[V]$ of degree $\leq 2$ with $V$ invariant belong to one of the following types:

1. The first curve of a $\mathcal{G}$-chain ($Z(\mathcal{G}, V) = 1$ and $p_a(V) = 0$) with $\Delta \cdot V = 0$ or 1. In particular, it is a $(-1)$-$\mathcal{G}$-curve.
2. A $(-2)$-$\mathcal{G}$-curve ($Z(\mathcal{G}, V) = 2$ and $p_a(V) = 0$) with $\Delta \cdot V = 0$.
3. A rational curve with only one node and with $\Delta \cdot V = 0$.
4. A bad tail (see Definition 3.2 (4)) with $\Delta \cdot V = 0$.

**Proof.** Suppose $\deg [V] = 0$. By Zariski’s main theorem, $\Gamma$ is connected and thus $\Gamma$ is a one-vertex graph. In fact, $\Gamma = \{[V]\}$. So $\Delta = 0$. Now we may write $K_{\mathcal{G}} = \pi^*K_{\mathcal{G}} + aV$ for some $a \geq 0$. By adjunction for invariant divisors (Theorem 2.14), we have

$$0 \geq aV^2 = K_{\mathcal{G}} \cdot V = Z(\mathcal{G}, V) + 2p_a(V) - 2 \geq -2.$$ 

- If $K_{\mathcal{G}} \cdot V = -2$, then $Z(\mathcal{G}, V) = 0$ and $p_a(V) = 0$. So $E_1$ is smooth with no foliation singularity. By Camacho–Sad formula (Theorem 2.15), $V^2 = 0$, which is impossible.
- If $K_{\mathcal{G}} \cdot V = -1$, then $Z(\mathcal{G}, V) = 1$ and $p_a(V) = 0$. So $V$ is a smooth rational curve with exactly one reduced singularity. By the minimality of $\pi$, we have $V^2 \leq -2$, and thus $V$ is the first curve of a $\mathcal{G}$-chain.
- If $K_{\mathcal{G}} \cdot V = 0$, then $a = 0$. Thus, we have $p_a(V) = 0$ or 1. If $p_a(V) = 0$, then $Z(\mathcal{G}, V) = 2$, which is a $(-2)$-$\mathcal{G}$-curve. If $p_a(V) = 1$, then $Z(\mathcal{G}, V) = 0$. So $V$ is not smooth, otherwise $V$ is smooth with no foliation singularity. By Camacho–Sad formula (Theorem 2.15), $V^2 = 0$, which is impossible. Thus, $V$ is a rational curve with only one node.
Now, we assume that \( \deg[V] \geq 1 \). If \( \Gamma \) is the dual graph of a \( \mathcal{G} \)-chain, then all vertices \([V]\) have the property that \( V \) is an invariant curve with \( p_a(V) = 0 \), \( \Delta \cdot V = 0 \), and \( Z(\mathcal{G}, V) = 1 \) or \( 2 \). So we may assume that \( \Gamma \) is not the dual graph of a \( \mathcal{G} \)-chain.

If \( V \) is a bad tail, then \( \Delta \cdot V = 0 \) since \( \deg[V] \leq 2 \). Thus, we may assume further that \( V \) is not a bad tail and therefore \( D \cdot V = 0 \). By Proposition 3.14 and adjunction for invariant divisors (Theorem 2.14), we have

\[
0 = D \cdot V = (K_\mathcal{G} + \Delta) \cdot V = Z(\mathcal{G}, V) + 2p_a(V) - 2 + \Delta \cdot V.
\]

Note that we have \( 0 \leq \Delta \cdot V \leq 2 \) since \( \deg[V] \leq 2 \).

(1) If \( \Delta \cdot V = 0 \), then either \( Z(\mathcal{G}, V) = 2 \) and \( p_a(V) = 0 \), which is a \((-2)\)-\( \mathcal{G} \)-curve, or \( Z(\mathcal{G}, V) = 0 \) and \( p_a(V) = 1 \), which is a rational curve with only one node as we have already seen above.

(2) If \( \Delta \cdot V = 1 \), then \( Z(\mathcal{G}, V) = 1 \) and \( p_a(V) = 0 \), which is a \((-1)\)-\( \mathcal{G} \)-curve, that is the first curve of a \( \mathcal{G} \)-chain.

(3) If \( \Delta \cdot V = 2 \), then we have \( Z(\mathcal{G}, V) = 0 = p_a(V) \). So \( V \) is a smooth rational curve and has no foliation singularity. By Camacho–Sad formula (Theorem 2.15), \( V^2 = 0 \), which is impossible. \( \square \)

### 3.3 Fork

In this subsection, we study the case when \( \Gamma \) has a fork.

**Proposition 3.18.** Let notations be as in \((†)\). Let \( [F] \) be a fork. Then, \( F \) is either a bad tail or a noninvariant curve with tangency order zero.

**Proof.** Assume that \( F \) is not a bad tail. By Proposition 3.15, we have \( F \neq C_{L,2} \) for any special leaf \([L]\). Then, \( \Gamma \) is not the dual graph of a \( \mathcal{G} \)-chain. By Proposition 3.14, we have \( D \cdot F = 0 \).

If \( F \) is invariant, then by Proposition 3.6, we have

\[
0 = D \cdot F = (K_\mathcal{G} + \Delta) \cdot F \geq \deg[F] - 2 \geq 1,
\]

where the last inequality follows since \([F]\) is a fork. This is impossible and hence \( F \) is noninvariant. By adjunction for noninvariant divisors (Theorem 2.10), we have

\[
0 = D \cdot F = (K_\mathcal{G} + \Delta) \cdot F \geq (K_\mathcal{G} + F) \cdot F = \text{tang}(\mathcal{G}, F) \geq 0
\]

and thus all inequalities are equalities. Therefore, \( \text{tang}(\mathcal{G}, F) = 0 \). \( \square \)

**Proposition 3.19.** Let notations be as in \((†)\). \( \Gamma \) has at most one fork.

**Proof.** Suppose there are at least two forks. Let \([F_1]\) be one of the forks. In particular, \( \deg[F_1] \geq 3 \). If \( F_1 \) is invariant, then it is a bad tail by Proposition 3.18, and the degree of \([F_1]\) must be 3 by Proposition 3.15. Also, two of the three branches of \([F_1]\) are special leaves \([L_1]\) and \([L_2]\).

Let \([F_2]\) be the fork connecting to \([F_1]\) by a (ordered) chain of vertices \([V_1], ..., [V_r]\) of degree 2 where \( V_1 \) intersects \( F_1 \). Let \( V_{r+1} := F_2 \). Suppose at least one \( V_i \) is noninvariant for \( 1 \leq i \leq r + 1 \), then we put \( i_0 \) as the minimum of \( i \) such that \( V_i \) is noninvariant. By Proposition 3.15, we have \( F_1 \cdot \Delta = 0 \), and thus \( V_1 \) is invariant. In particular, \( i_0 \geq 2 \). Also by Proposition 3.17, \([V_j]\) a \((-2)\)-\( \mathcal{G} \)-curve with \( \Delta \cdot V_j = 0 \) for \( 1 \leq j \leq i_0 - 2 \) and \( V_{i_0-1} \) is a \((-1)\)-\( \mathcal{G} \)-curve with \( \Delta \cdot V_{i_0-1} = 1 \). Thus, \( L_1 \cup L_2 \cup F_1 \cup \bigcup_{j=1}^{i_0-1} V_j \) (see Figure 3) has \( i_0 + 2 \) irreducible components on which there are only \( i_0 + 1 \) reduced singularities, which contradicts separatrix theorem (Theorem 2.8).

Therefore, all \( C_i \) are invariant for \( 1 \leq i \leq r + 1 \). In particular, \( F_2 = [V_{r+1}] \) is an invariant fork, which is a bad tail by Proposition 3.18. We put \( L_3 \) and \( L_4 \) the two special leaves connecting to \( F_2 \). However, \( \Gamma \) (see Figure 4) has \( r + 6 \) irreducible components on which there are only \( r + 5 \) reduced singularities, which contradicts separatrix theorem (Theorem 2.8).
Hence, we may assume that all forks are noninvariant. By Proposition 3.16, they are disjoint. Then, there are two (non-invariant) forks connected by a (ordered) chain of vertices \([V_1], \ldots, [V_r]\) of degree 2 where \(r \geq 1\). Let \([V_0]\) and \([V_{r+1}]\) be these two (non-invariant) forks.

Let \(i_0\) be the minimum of positive \(i\) such that \(V_i\) is noninvariant. Note that \(i_0 \geq 2\). Thus, by Proposition 3.17, we have that \([V_1]\) and \([V_{i_0-1}]\) are \((-1)-\mathcal{G}\)-curves with \(\Delta \cdot V_1 = 0 = \Delta \cdot V_r\) and \([V_j]\)'s are \((-2)-\mathcal{G}\)-curves with \(\Delta \cdot V_j = 0\) for \(2 \leq j \leq i_0 - 2\). However, \(\bigcup_{i=1}^{i_0-1} V_i\) (see Figure 5) has \(i_0 - 1\) irreducible components on which there are only \(i_0 - 2\) reduced singularities, which contradicts separatrix theorem (Theorem 2.8).

\[\square\]

Remark 3.20. Let notations be as in (†). In the proof above, we also show that all curves \(E_i\)'s are invariant if there is an invariant fork.

### 3.4 Proof

**Theorem 3.21.** Let \((X, \mathcal{F}, p)\) be a germ of a foliated surface. Assume that \(p\) is a log canonical singularity of \(\mathcal{F}\). Let \(\pi : (Y, \mathcal{E}) \to (X, \mathcal{F})\) be the minimal resolution (see Section 2.3) for \((X, \mathcal{F}, p)\) with exceptional divisors \(E = \bigcup E_i\). Then \(E\) belongs to one of the following types:

1. A \(\mathcal{G}\)-chain. (See Figure 6.)
2. A chain of three invariant curves \(E_1 \cup E_2 \cup E_3\) where \(E_1\) and \(E_3\) are \((-1)-\mathcal{G}\)-curves with self-intersection \(-2\) and \(E_2\) is a bad tail. (See Figure 7.)

**Figure 6** A \(\mathcal{G}\)-chain. The straight lines indicate invariant divisors and the solid circles indicate all reduced singularities on these divisors.

**Figure 7** A graph of \(A_3\) type with a bad tail. The straight lines indicate invariant divisors and the solid circles indicate all reduced singularities on these divisors.
3. A chain of \((−2)\)-\(\mathcal{G}\)-curves. (See Figure 8.)

4. A dihedral singularity. More precisely, two \((-1)\)-\(\mathcal{G}\)-curves with self-intersection \(-2\) joined by a bad tail, which itself connects to a chain of \((−2)\)-\(\mathcal{G}\)-curves. (See Figure 9.)

5. An elliptic Gorenstein leaf, which is either a rational curve with only one node or a cycle or \((−2)\)-\(\mathcal{G}\)-curves. (See Figure 10.)

6. A chain \(E = \bigcup_{i=1}^{r} E_i\) with exactly one noninvariant curve \(E_\ell\) with \(1 \leq \ell \leq r\). Moreover, \(E_\ell\) has tangency order zero and \(\bigcup_{i=1}^{\ell-1} E_i\) and \(\bigcup_{i=\ell+1}^{r} E_i\) are \(\mathcal{G}\)-chains. (See Figure 11.)

7. The dual graph is star-shaped with a noninvariant center \([E_0]\). Moreover, \(E_0\) has tangency order zero, all branches are \(\mathcal{G}\)-chains, and all first curves of \(\mathcal{G}\)-chains have intersection number one with \(E_0\). (See Figure 12.)
In this section, we define the foliated minimal log discrepancy and show some of its properties.

Proof. We fix the notations as in (†).

Suppose \( \Gamma = \{[E_1]\} \) is a one-vertex graph. If \( E_1 \) is noninvariant, then by Proposition 3.16, \( E_1 \) has tangency order zero so that \( \Gamma \) is the sixth type with \( r = \ell = 1 \). On the other hand, if \( E_1 \) is invariant, then by Proposition 3.17, we have the following possibilities:

1. If there is no fork in \( \Gamma \), then \( \Gamma \) is either a chain or a cycle.
   - Assume \( \Gamma \) is a chain with a special leaf \([L_1]\), then by Proposition 3.15, the tail \( C_{L_{1,2}} \) of this special leaf is actually a bad tail, which will also connect to another special leaf \([L_2]\). We claim that \( \Gamma = \{[L_1],[C_{L_{1,2}}],[L_2]\} \), which is the second type. If not, there is no vertex \([V]\) such that \( V \) intersects \( L_1 \cup L_2 \cup C_{L_{1,2}} \) nontrivially. Notice that \( L_1 \) and \( L_2 \) are disjoint from \( \Delta \) since they are special leaves. Also \( C_{L_{1,2}} \) is disjoint from \( \Delta \) by Proposition 3.15. Thus, \( V \) must be invariant. Hence, \( V \) must meet the bad tail \( C_{L_{1,2}} \), which implies that \( C_{L_{1,2}} \) is a fork. This contradicts our assumption.
   - Assume \( \Gamma \) is a chain without special leaves. Then, by Proposition 3.16 and 3.17, all \( E_i \)s are noninvariant with tangency order zero, \((−1)\)-\( \mathcal{G} \)-curves with \( \Delta \cdot E_i = 1 \), or \((−2)\)-\( \mathcal{G} \)-curves with \( \Delta \cdot E_i = 0 \). We claim that there are at most one non-invariant divisor among \( E_i \)s. Suppose there are more than two non-invariant divisors among \( E_i \)s. By Proposition 3.16, they are pairwise disjoint. Let \( [C_1],..., [C_r] \) be a (ordered) chain of vertices connecting two non-invariant divisors with all \( C_i \)s invariant. Note that \( \cup_{i=1}^r C_i \) has \( r \) irreducible components on which there are only \( r − 1 \) reduced singularities, which contradicts separatrix theorem (Theorem 2.8).

   Thus, if there is no noninvariant divisor among \( E_i \)s, then \( \Gamma \) is the third type. On the other hand, if there is exactly one non-invariant divisor among \( E_i \)s, then \( \Gamma \) is the sixth type.
   - Assume \( \Gamma \) is a cycle. By the similar arguments as above, there is no noninvariant divisor among \( E_i \)s and thus \( \Gamma \) is the fifth type.

2. If there is exactly one fork \([F]\) in \( \Gamma \), then we study \( \Gamma \) when \( F \) is invariant or noninvariant, respectively.
   - Assume \( F \) is invariant. Then, \( F \) must be a bad tail by Proposition 3.18. Note that the bad tail can intersect with two or three invariant divisors. So \( \deg [F] = 3 \) since \([F]\) is a fork. Let \([L_1]\) and \([L_2]\) be two special leaves connected to \([F]\). Since \([F]\) is the only fork, all vertices other than \([F]\) have degree at most two. Thus, \( \Gamma' := \Gamma \setminus \{[F],[L_1],[L_2]\} \) is a chain. Let \( \Gamma' = \{[C_i]\}_{i=1}^r \) where \( C_1 \) intersects \( F \) and \( C_i \cdot C_{i+1} = 1 \) for all \( i = 1, ..., r − 1 \). By Remark 3.20, all \( C_i \)s are invariant. Then, by Proposition 3.17, all \( C_i \)s are \((−2)\)-\( \mathcal{G} \)-curves. Hence, \( \Gamma \) is the fourth type.
   - Assume \( F \) is noninvariant, then there is no special leaf, otherwise, there is a bad tail, which has degree 3 by Proposition 3.15 since \( \Gamma \) is connected. This is impossible by Remark 3.20.

   Note that each branch of \( F \) is a chain. Let \( B = \{[B_i]\}_{i=1}^r \) be one of the branches of \([F]\) with \([B_i]\) intersecting \([F]\) and \( B_i \cdot B_{i+1} = 1 \) for \( i = 1, ..., r − 1 \). We claim that all \( B_i \)s are invariant. Suppose that there is a noninvariant curve \( B_i \). Let \( i_0 \) be the minimal \( i \) such that \([B_i]\) is noninvariant. Note that \( i_0 \geq 2 \) by Proposition 3.16. Then, \( B_1 \) and \( B_{i_0−1} \) are \((−1)\)-\( \mathcal{G} \)-curves with \( \Delta \cdot B_1 = 1 = \Delta \cdot B_{i_0−1} \) and \( B_i \)s are \((−2)\)-\( \mathcal{G} \)-curves with \( \Delta \cdot B_i = 0 \) for \( i = 1, ..., i_0 − 2 \). Thus, \( \bigcup_{i=1}^{i_0−1} B_i \) (see Figure 5) has \( i_0 − 1 \) irreducible components on which there are only \( i_0 − 2 \) reduced singularities, which contradicts separatrix theorem (Theorem 2.8). Hence, all \( B_i \)s are invariant and \( B_r \) is not a \((−1)\)-\( \mathcal{G} \)-curve. Therefore, \( \Gamma \) is the seventh type.

\[ \square \]

4  |  FOLIATED MINIMAL LOG DISCREPANCY

In this section, we define the foliated minimal log discrepancy and show some of its properties.
**Definition 4.1.** Let \((X, \mathcal{F}, \Delta)\) be a foliated triple. For any divisor \(E\) over \(X\), we define the foliated log discrepancy of \((X, \mathcal{F}, \Delta)\) along \(E\) to be \(a(E, \mathcal{F}, \Delta) + \varepsilon(E)\) where \(\varepsilon(E) = 0\) if \(E\) is invariant under the pullback foliation and \(\varepsilon(E) = 1\) otherwise.

**Definition 4.2.** Given \((X, \mathcal{F}, \Delta)\) a foliated triple, let \((Y, \mathcal{G})\) be the minimal resolution of \((X, \mathcal{F})\). We define the foliated minimal log discrepancy of \((X, \mathcal{F}, \Delta)\) as

\[
\text{mld}(\mathcal{F}, \Delta) := \inf \{ a(E, \mathcal{F}, \Delta) + \varepsilon(E) \mid E \text{ is a divisor over } X \}
\]

and the partial log discrepancy as

\[
\text{pld}(\mathcal{F}, \Delta) := \min \{ a(E, \mathcal{F}, \Delta) + \varepsilon(E) \mid E \text{ is a divisor on } Y \}.
\]

Also, for any fixed \(x \in X\), we define

\[
\text{mld}_x(\mathcal{F}, \Delta) := \inf \{ a(E, \mathcal{F}, \Delta) + \varepsilon(E) \mid \text{the center of } E \text{ on } X \text{ is } x \} \text{ and }
\]

\[
\text{pld}_x(\mathcal{F}, \Delta) := \min \{ a(E, \mathcal{F}, \Delta) + \varepsilon(E) \mid E \text{ is a divisor on } Y \text{ over } x \}.
\]

From now on, we make the convention that \(\min \emptyset = \inf \emptyset = 0\).

**Remark 4.3.** By Corollary 4.7, if \(\text{mld}_x(\mathcal{F}, \Delta) \geq 0\), then it is indeed a minimum, that is, there is a divisor \(E\) over \(X\) that computes the \(\text{mld}_x(\mathcal{F}, \Delta)\).

**Proposition 4.4.** If \(\text{mld}(\mathcal{F}, \Delta) < 0\), then \(\text{mld}(\mathcal{F}, \Delta) = -\infty\).

**Proof.** Suppose \(\text{mld}(\mathcal{F}, \Delta) < 0\), then there is a divisor \(E\) over \(X\) such that

\[
a(E, \mathcal{F}, \Delta) + \varepsilon(E) < 0.
\]

If \(E\) is noninvariant, that is, \(\varepsilon(E) = 1\), then blowing up a general point \(p\) on \(E\) introduces an invariant exceptional divisor \(E'\) with 

\[
a(E', \mathcal{F}, \Delta) + \varepsilon(E') \leq a(E, \mathcal{F}, \Delta) + 1 < 0.
\]

Therefore, we may assume that \(E\) is invariant.

Let \(a(E, \mathcal{F}, \Delta) = -c\) for some positive number \(c\). Now blowing up a general point \(p\) on \(E\) introduces an invariant exceptional divisor \(E_1\) with

\[
a(E_1, \mathcal{F}, \Delta) \leq 1 + a(E, \mathcal{F}, \Delta) = 1 - c.
\]

Notice that the proper transform of the support of \(\Delta\) does not contain the intersection of \(E_1\) and the proper transform of \(E\).

Next, we blow up the intersection of \(E_1\) and the proper transform of \(E\), then we have an invariant exceptional divisor \(E_2\) with

\[
a(E_2, \mathcal{F}, \Delta) = a(E, \mathcal{F}, \Delta) + a(E_1, \mathcal{F}, \Delta) \leq 1 + 2a(E, \mathcal{F}, \Delta) = 1 - 2c.
\]

Then, we blow up the intersection of the proper transform of \(E\) and \(E_2\) to have an invariant exceptional divisor \(E_3\) with

\[
a(E_3, \mathcal{F}, \Delta) \leq 1 + 3a(E, \mathcal{F}, \Delta) = 1 - 3c.
\]

Repeating the process, by inductively, we have an invariant exceptional divisor \(E_n\) over \(X\) with \(a(E_n, \mathcal{F}, \Delta) \leq 1 - nc\). This shows that \(\text{mld}(\mathcal{F}, \Delta) = -\infty\). \(\Box\)

**Proposition 4.5.** Let \((X, \mathcal{F}, \Delta)\) be a foliated triple with only log canonical foliation singularities, that is, \(\text{mld}(\mathcal{F}, \Delta) \geq 0\). Assume that \(\Delta = \sum a_iD_i\) is an \(\mathbb{R}\)-divisor, which has the simple normal crossing support where \(a_i \leq 1\). Suppose \(X\) is smooth, \(\mathcal{F}\) has only reduced singularities, and any separatrix \(C\) through a nonsmooth foliation point on noninvariant \(D_i\) has \(C \cdot D_i = 1\).
then we have

\[
\text{mld}(\mathcal{F}, \Delta) = \min \left\{ \min_{(i,j) \in S_1} \{1 - a_i - a_j\}, \min_{(i,j) \in S_0} \{-a_i - a_j\}, \min_{i \in I} \{1 - a_i\}, \min_{i \notin I} \{-a_i\}, 1 \right\},
\]

where

\[
S_1 = \{(i, j) | i \neq j, D_i \cap D_j \neq \emptyset, \text{and } D_i \cap D_j \text{ supports on smooth foliation points}\},
\]

\[
S_0 = \{(i, j) | i \neq j \text{ and } D_i \cap D_j \neq \emptyset \} \setminus S_1, \text{ and}
\]

\[
I = \{i | D_i \text{ is noninvariant and has only smooth foliation points}\}.
\]

\[\text{Proof.}\] Let \(r(\mathcal{F}, \Delta)\) be the right-hand side of the equality. It is clear that \(\text{mld}(\mathcal{F}, \Delta) \leq r(\mathcal{F}, \Delta)\).

Let \(E\) be the exceptional divisor for some birational morphism \(f : Y \to X\). It is well known that \(f\) is a composition of \(t\) blowups at regular points for some \(t \in \mathbb{N}\). Without loss of generality, we assume that \(t\) is minimal, which will be denoted as \(t(E)\). Then, it suffices to show the following claim.

\[\text{Claim.}\] For any divisor \(E\) over \(X\), if \(a(E, \mathcal{F}, \Delta) + \varepsilon(E) > 0\), then \(a(E, \mathcal{F}, \Delta) + \varepsilon(E) \geq r(\mathcal{F}, \Delta)\).

\[\text{Proof (Claim).}\] We will prove by induction on \(t(E)\). Note that, for \(t(E) = 0\), we have that \(E\) is a divisor on \(X\) and

\[
a(E, \mathcal{F}, \Delta) + \varepsilon(E) = -\text{ord}_E \Delta + \varepsilon(E) \geq r(\mathcal{F}, \Delta).
\]

When \(t(E) = 1\), we have \(\varepsilon(E) = 0\) and \(a(E, \mathcal{F}, \Delta) \geq r(\mathcal{F}, \Delta)\).

Suppose \(t(E) \geq 2\). Note that \(\varepsilon(E) = 0\). Let \(g_1 : X_1 \to X\) be the blowup at \(p := f(E)\) and \(E_1\) be the (invariant) exceptional divisor for \(g_1\). Let \(K_{\mathcal{F}_1} + \Delta_1 \equiv g_1^*(K_{\mathcal{F}} + \Delta)\), where \(\mathcal{F}_1\) is the pullback foliation on \(X_1\). Notice that \(\text{ord}_{E_1} \Delta_1 = -a(E_1, \mathcal{F}, \Delta)\) and \(-a_i \geq 0\) (resp. \(1 - a_i \geq 0\)) for those \(i\) with \(p \in D_i\) and \(i \notin I\) (resp. \(i \in I\)). Moreover, for those \(i\) with \(p \in D_i\) and \(i \in I\), \(E_1 \cap (g_1)_*^{-1}D_i\) is nonempty and supports on smooth foliation points since we require that any separatrix \(C\) through a nonsmooth foliation point on noninvariant \(D_i\) has \(C \cdot D_i = 1\). Thus,

\[
r(\mathcal{F}_1, \Delta_1) \geq \min\{r(\mathcal{F}, \Delta), -\text{ord}_{E_1} \Delta_1\} = \min\{r(\mathcal{F}, \Delta), a(E_1, \mathcal{F}, \Delta)\} \geq r(\mathcal{F}, \Delta),
\]

where the last inequality follows since \(t(E_1) = 1\) and the first inequality follows because, for those \(i\) with \(p \in D_i\) and \(i \in I\), \(E_1 \cap (g_1)_*^{-1}D_i\) is nonempty and supports on smooth foliation points, which is a consequence of the assumption that any separatrix \(C\) through a nonsmooth foliation point on noninvariant \(D_i\) has \(C \cdot D_i = 1\). Therefore, we have

\[
a(E, \mathcal{F}, \Delta) = a(E, \mathcal{F}_1, \Delta_1) \geq r(\mathcal{F}_1, \Delta_1) \geq r(\mathcal{F}, \Delta),
\]

where the first inequality comes from the induction hypothesis.

\[\text{Remark 4.6.}\] Suppose we do not require the condition that any separatrix \(C\) through a nonsmooth foliation point on noninvariant \(D_i\) has \(C \cdot D_i = 1\). Then, for those \(i\) with \(p \in D_i\) and \(i \in I\), \(E_1 \cap (g_1)_*^{-1}D_i\) may support on a reduced singularity. In such a case, \(a(E_1, \mathcal{F}, \Delta) - a_i\) will contribute to \(r(\mathcal{F}_1, \Delta_1)\). However, for those \(i \in I\), we only have \(1 - a_i \geq 0\) from the inequality \(r(\mathcal{F}, \Delta) \geq \text{mld}(\mathcal{F}, \Delta) \geq 0\). Hence, the inequality \(r(\mathcal{F}_1, \Delta_1) \geq r(\mathcal{F}, \Delta)\) may not hold.

Now we show that the minimal log discrepancy is indeed obtained as a minimum.

\[\text{Corollary 4.7.}\] Given \((X, \mathcal{F}, \Delta = \sum a_i D_i)\) a foliated triple with only log canonical foliation singularities where \(a_i \leq 1\). Then, there is a \((log)\) resolution \(f : (Y, \mathcal{G}, \Theta) \to (X, \mathcal{F}, \Delta)\) such that

1. \(K_\mathcal{G} + \Theta = f^*(K_{\mathcal{F}} + \Delta)\).
2. \(Y\) is smooth,
3. \(\mathcal{G}\) has only reduced singularities.
4. $\Theta$ has the simple normal crossing support,
5. the proper transform $\Delta_Y$ on $Y$ of $\Delta$ is smooth,
6. the union of the support of $\Theta$ and the exceptional divisor $\cup_j E_j$ of $f$ is $\cup_k T_k$,
7. all noninvariant irreducible components of $\cup_k T_k$ are pairwise disjoint, and
8. any separatrix $C$ through a nonsmooth foliation point on a noninvariant prime divisor $T_k$ has $C \cdot T_k = 1$.

Therefore, we have

$$\mld(\mathcal{F}, \Delta) = \min \left\{ \min_{E_j \in \tilde{I}} \{ 1 + a(E_j, \mathcal{F}, \Delta) \}, \min_{E_j \notin \tilde{I}} \{ a(E_j, \mathcal{F}, \Delta) \}, \min_{f^{-1}_* D_i \in \tilde{I}} \{ 1 - a_i \}, \min_{f^{-1}_* D_i \notin \tilde{I}} \{ -a_i \} \right\}$$

where $\tilde{I} = \{ T_k \mid T_k$ in noninvariant and has only smooth foliation points$\}$.

**Proof.** First we take $\pi : (Y, \mathcal{F}_Y) \to (X, \mathcal{F})$ to be the minimal resolution of $(X, \mathcal{F})$. Then, we take $\phi : Z \to Y$ to be a log resolution of $(Y, (\pi^*)^{-1} \Delta + \text{Exc}(\pi))$. After some further blowups, this gives the existence of such $f$.

Let $b_j := -a(E_j, \mathcal{F}, \Delta) \leq \varepsilon(E_j)$. By Proposition 4.5, we have that $\mld(\mathcal{G}, \Theta)$ is a minimum of numbers of the following forms:

1. $1$.
2. $-a_i$ if $f^{-1}_* D_i \notin \tilde{I}$.
3. $1 - a_i$ if $f^{-1}_* D_i \in \tilde{I}$.
4. $-b_j$ if $E_j \notin \tilde{I}$.
5. $1 - b_j$ if $E_j \in \tilde{I}$.
6. $-a_i - b_j$ where $f^{-1}_* D_i \cap E_j \neq \emptyset$ has some nonsmooth foliation singularities.
7. $1 - a_i - b_j$ where $f^{-1}_* D_i \cap E_j \neq \emptyset$ has only smooth foliation singularities.
8. $-b_j_1 - b_j_2$ for $E_{j_1} \cap E_{j_2} \neq \emptyset$ has some nonsmooth foliation singularities.
9. $1 - b_j_1 - b_j_2$ for $E_{j_1} \cap E_{j_2} \neq \emptyset$ has only smooth foliation singularities.

Then, it suffices to show the last four terms above are irrelevant when taking the minimum. Indeed, it is clear if one of $a_i$ and $b_j$ (resp. $b_j_1$ and $b_j_2$) is nonpositive. So we may assume both numbers are positive. Moreover, the associated divisors are contained in $\tilde{I}$; otherwise, either $-a_i$ or $-b_j$ (resp. either $-b_j_1$ or $-b_j_2$) is strictly less than zero, which gives a contradiction. Thus, both associated divisors are noninvariant, and hence they have empty intersections. This completes the proof. 

**5 | ACC FOR FOLIATED MINIMAL LOG DISCREPANCY**

In this section, we fix the following notations.

The set $B \subset [0, 1]$ is always assumed to satisfy the descending chain condition. Let $(\mathcal{F}, \Delta, x)$ be the germ of a log canonical foliation singularity where $\Delta$ is an $\mathbb{R}$-divisor whose coefficients are in $B$. Let $\Gamma = \{ E_j \}_{j=1}^s$ be the dual graph of exceptional divisors of a resolution $\pi : (Y, \mathcal{F}_Y) \to (X, \mathcal{F}, x)$ and $\Theta = \sum_{i=1}^s b_i B_i$ be the proper transform of $\Delta$ on $Y$. Let $a_j := a(E_j, \mathcal{F}, \Delta) \geq 0$ for all $j = 1, \ldots, r$.

**Lemma 5.1.** Assume that $\Gamma$ is a tree, all $E_j$ are invariant, and $a_j \leq 1$ for all $j = 1, \ldots, r$. Then, we have the following:

(a) If $a_j \geq \varepsilon$ for some positive real number $\varepsilon$, then $-E_j^2 \leq \left\lfloor \frac{2}{\varepsilon} \right\rfloor$.

(b) If $-E_j^2 \geq 2$ for some $j$, then $1 - a_j \geq \frac{1-a_k}{2}$ and $2a_j \leq a_k + a_{k'}$ for any $k \neq k'$ such that $E_j \cdot E_k = E_j \cdot E_{k'} = 1$.

(c) Suppose the vertex $[E_{j_0}]$ is a fork and the vertices $[E_{j_k}]$ are connected to $[E_{j_0}]$ for $k = 1, 2, 3$. If $-E_{j_k}^2 \geq 2$ for $k = 0, 1, 2$, then $a_{j_0} \leq a_{j_k}$.

(d) If $-E_{j_k}^2 = 1$ for some $j$, $-E_{j_k}^2 \geq 2$, $a_j < a_k$ for all $k \neq j$, and the vertex $[E_j]$ is not a fork, then $\Gamma$ is a chain.
(e) Given a sequence of vertices \([E_1], \ldots, [E_m]\) where \(E_i \cdot E_{i+1} = 1\) for all \(i = 1, \ldots, m - 1\). Suppose \(-E_i^2 \geq 2\) for \(i = 2, \ldots, m - 1\) and \(a_1 \leq a_2\). If either \(-E_2^2 \geq 3\) and \(a_2 \geq \varepsilon\) or \(\Theta \cdot E_2 \geq \varepsilon\), then \(m \leq \lfloor \frac{1}{\varepsilon} \rfloor + 2\).

**Proof.** Notice that

\[
0 = \left( K_G + \Theta - \sum_{i=1}^{r} a_i E_i \right) \cdot E_j
\]

\[
= K_G \cdot E_j + \sum_{i=1}^{s} t_{i,j} b_i - a_j E_j^2 - \sum_{E_i E_j = 1} a_i
\]

\[
\geq \#\{ i \mid E_i \cdot E_j = 1 \} - 2 + \sum_{i=1}^{s} t_{i,j} b_i - a_j E_j^2 - \sum_{E_i E_j = 1} a_i
\]

\[
= -2 + \sum_{i=1}^{s} t_{i,j} b_i - a_j E_j^2 + \sum_{E_i E_j = 1} (1 - a_i)
\]

(5.1)

where \(t_{i,j} = B_i \cdot E_j \geq 0\).

(a) From the inequality (5.1), we have that \(0 \geq -2 - a_j E_j^2\). Thus,

\[-E_j^2 \leq \frac{2}{a_j} \leq \frac{2}{\varepsilon}.
\]

Since \(-E_j^2 \in \mathbb{Z}\), we have that \(-E_j^2 \leq \lfloor \frac{2}{\varepsilon} \rfloor\).

(b) From the inequality (5.1), we have that \(0 \geq -2 - a_j E_j^2 + 1 - a_k \geq -1 + 2a_j - a_k\) and \(0 \geq -2 - a_j E_j^2 + 1 - a_k + 1 - a_\ell \geq 2a_j - a_k - a_\ell\). Then, we get \(1 - a_j \geq \frac{1 - a_k}{2}\) and \(2a_j \leq a_k + a_\ell\).

(c) By (b), we have

\[1 - a_{j_k} \geq \frac{1 - a_{j_0}}{2}\]

for \(k = 1, 2\). Thus,

\[1 - a_{j_1} + 1 - a_{j_2} \geq 1 - a_{j_0}\]

(5.2)

Also from the inequality (5.1), we have

\[0 \geq -2 + 2a_{j_0} + \sum_{k=1}^{3} (1 - a_{j_k}) = 1 + 2a_{j_0} - \sum_{k=1}^{3} a_{j_k}.
\]

Combining with the inequality (5.2), we get

\[a_{j_3} \geq 1 + 2a_{j_0} - a_{j_1} - a_{j_2} \geq 2a_{j_0} - a_{j_0} = a_{j_0}.
\]

(d) Assume \(\Gamma\) is not a chain. Let the vertex \([E_k]\) be a fork. Let \([E_k] = [E_{e_0}], [E_{e_1}], \ldots, [E_{e_m}] = [E_j]\) be a sequence of vertex connecting from \([E_k]\) to \([E_j]\). By (c), we know that \(a_{e_0} \leq a_{e_1}\). By (b), we get \(a_{e_1} \leq a_{e_2} \leq \ldots \leq a_{e_m}\). Thus, \(a_k \leq a_j\), which gives a contradiction.
(e) From the inequality (5.1) with \( j = 2 \), we have that

\[
0 \geq -2 + \sum_{i=1}^{s} t_i b_i - a_2 E_2^2 + (1 - a_1) + (1 - a_3)
\]

\[
\geq 2a_2 - a_1 - a_3 + \varepsilon
\]

\[
\geq a_2 - a_3 + \varepsilon.
\]

Therefore, \( a_3 \geq a_2 + \varepsilon \).

\[\square\]

**Proof (Claim).** \( a_{j+1} - a_j \geq \varepsilon \) for all \( j = 2, \ldots, m-1 \).

**Claim.** We have seen the claim holds true when \( j = 2 \). Then by (b), we have \( a_{j+1} + a_{j-1} \geq 2a_j \). Thus, \( a_{j+1} - a_j \geq a_j - a_{j-1} \geq \varepsilon \) by induction on \( j \).

Therefore, we have

\[
1 \geq a_m \geq a_{m-1} + \varepsilon \geq \cdots \geq a_2 + (m-2)\varepsilon \geq (m-2)\varepsilon.
\]

Hence, \( m \leq \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 2 \).

\[\square\]

**Lemma 5.2.** Assume that \((Y, \mathcal{G})\) is the minimal resolution of the germ \((X, \mathcal{F}, x)\). Then, we have the followings:

(a) \( \text{pld}_x(\mathcal{F}) = 0 \) if \( \mathcal{F} \) is not terminal at \( x \), that is, if \( \Gamma \) is not of the type 1 in Theorem 3.21.

(b) If \( a_j > 0 \) for all \( j \), then \( a_j \leq a(E_j, \mathcal{F}) \leq 1 \) for all \( E_j \).

(c) Fix a number \( \delta \), there are only finitely many sequences \( \{t_1, \ldots, t_r\} \), where \( t_i \in \mathbb{N} \) such that \( \sum_{i=1}^{r} t_i b_i \leq \delta \) for some \( b_i \in B \setminus \{0\} \).

**Proof.**

(a) This is straightforward.

(b) Note that

\[
\left( \sum_{i=1}^{r} a_i E_i \right) \cdot E_j = (K_{\mathcal{G}} + \Theta) \cdot E_j \geq K_{\mathcal{G}} \cdot E_j = \left( \sum_{i=1}^{r} a(E_i, \mathcal{F}) E_i \right) \cdot E_j
\]

for all \( j \). By Lemma 3.4, we have \( 0 < a_j \leq a(E_j, \mathcal{F}) \) for all \( j \). Then, \( \Gamma \) is of type 1 in Theorem 3.21. Also, we notice that

\[
K_{G} \cdot E_j \geq \left( \sum_{i=1}^{r} E_i \right) \cdot E_j.
\]

Thus, we have \( a(E_j, \mathcal{F}) \leq 1 \) by Lemma 3.4.

(c) Since \( B \) satisfies the descending chain condition, there is a positive number \( \varepsilon \) such that \( b_i \geq \varepsilon \) for all \( b_i \in B \setminus \{0\} \). Note that

\[
\delta \geq \sum_{i=1}^{r} t_i b_i \geq \sum_{i=1}^{r} t_i \varepsilon \quad \text{and thus} \quad \sum_{i=1}^{r} t_i \leq \frac{\delta}{\varepsilon}.
\]

This shows that there are only finitely many possible \( r \) and, for any fixed \( r \), there are only finitely many sequences \( \{t_1, \ldots, t_r\} \) such that \( \sum t_i \leq \frac{\delta}{\varepsilon} \). This proves (c).

\[\square\]

**Lemma 5.3.** Fix \( \varepsilon > 0 \). Suppose \((Y, \mathcal{G})\) is the minimal resolution of the germ \((X, \mathcal{F}, x)\) with \( \text{pld}_x(\mathcal{F}, \Delta) \geq \varepsilon \) and \( b_j \geq \varepsilon \) for all \( j \). Then, \( \Gamma \) belongs to one of the following cases:
FIGURE 13 Reindexed vertices for the dual graph of a ⫿-chain. The solid circles indicate the vertices for the curves whose self-intersections are −2 and which are disjoint from Δ. The labels of square and triangle vertices are just vertices other than the circle ones.

1. finitely many graphs (that include the way how $B_i$ intersects $E_j$).
2. The chain $∪_j E_j$ (see Figure 13) given by the ordered curves $L_{𝓁1}$, $M_1$, $M_2$, $R_1$, ..., $R_{𝓁2}$, where $K_G \cdot L_{𝓁1} = −1$, the weights of $M_k$s are 2, and each $B_j$ does not meet any $M_k$s. Moreover, the partial log discrepancy $\text{pld}_x(ℱ, Δ)$ is achieved at either $L_1$ or $R_1$ and there are only finitely many possibilities (independent of $n$) for the dual graphs $\{L_1, ..., L_{𝓁1}\}$ and $\{R_1, ..., R_{𝓁2}\}$ and the way how $B_i$ intersects $L_α$ and $R_β$.

Proof. By Lemma 5.2, we have that $Γ$ is of type 1, and therefore $Γ$ satisfies the assumption in Lemma 5.1. So from the inequality (5.1), we have that $0 \geq −2 + \sum t_{i,j} b_i − a_j E_j^2$ and hence $\sum t_{i,j} b_i \leq 2 + a_j E_j^2 \leq 2$. Therefore, there are only finitely many possibilities for $t_{i,j}$ for any $i, j$ by Lemma 5.2(c).

Let the chain be $∪_j E_j$. By Lemma 5.1(b), the function $f : \{1, ..., r\} \to \mathbb{R}$ that maps $j$ to $a_j$ is convex. Let

$S = \{j | a_j = \min_k \{a_k\}\}, N = \#S$, and $j_0 = \min S$.

If $N \geq 2$, then $S = \{j_0, ..., j_0 + N - 1\}$ by convexity of $f$. Note that, by inequality (5.1) with fixed $j$, we have

$0 \geq −2 + \sum_{i=1}^{s} t_{i,j} b_i − a_j E_j^2 + (1 − a_{j−1}) + (1 − a_{j+1})$

and thus

$a_{j−1} − 2a_j + a_{j+1} \geq \sum_{i=1}^{s} t_{i,j} b_i − a_j (E_j^2 + 2) \geq \left(\sum_{i=1}^{s} t_{i,j} − E_j^2 − 2\right) \varepsilon \geq \varepsilon$

(5.3)

whenever $−E_j^2 \geq 3$ or $t_{i,j} \geq 1$ for some $i$. So we can choose $L_1 = E_{j_0}$ and $R_1 = E_{j_0 + N−1}$. By Lemma 5.1(e), we have $ℓ_1 \leq \left\lfloor \frac{2}{\varepsilon} \right\rfloor + 1$ for $i = 1, 2$.

If $N = 1$, then $S = \{j_0\}$. Let $T = \{j | E_j^2 = −2$ and $E_j \cdot Θ = 0\}$. We have the following three cases:

(i) If both $j_0 − 1$ and $j_0 + 1$ are not in $T$, then we choose $L_1 = E_{j_0}$ and $R_1 = E_{j_0+1}$.
(ii) If $j_0 − 1 \in T$, then we choose $R_1 = E_{j_0}$ and $L_1 = E_u$ where $u$ is the maximal integer strictly less than $j_0$ and not in $T$.
(iii) If $j_0 + 1 \in T$, then we choose $L_1 = E_{j_0}$ and $R_1 = E_u$ where $u$ is the minimal integer strictly greater than $j_0$ and not in $T$.

By Lemma 5.1(e), we also have $ℓ_1 \leq \left\lfloor \frac{2}{\varepsilon} \right\rfloor + 1$ for $i = 1, 2$. To complete the proof, it remains to show the following claim.

Claim. There are only finitely many possibilities for the dual graphs $\{L_1, ..., L_{ℓ_1}\}$ and $\{R_1, ..., R_{ℓ_2}\}$ and the way how $B_i$ intersects $L_α$ and $R_β$.

Proof (Claim). To simplify some notations, we assume that we have a chain $C = \cup_{j=1}^{ℓ} C_j$ with increasing associated discrepancies $a_j \geq \varepsilon$. By inequality (5.3), we have
1 ≥ \alpha_{\ell}
≥ \alpha_{\ell-1} + \left( \sum_{i=1}^{s} t_{i,\ell-1} - E_{\ell-1}^2 - 2 \right) \varepsilon
:
≥ \alpha_1 + \sum_{j=1}^{\ell-1} \left( \sum_{i=1}^{s} t_{i,j} - E_j^2 - 2 \right) \varepsilon
≥ \varepsilon + \sum_{j=1}^{\ell-1} \left( \sum_{i=1}^{s} t_{i,j} - E_j^2 - 2 \right) \varepsilon.

Thus, we have
\sum_{j=1}^{\ell-1} \left( \sum_{i=1}^{s} t_{i,j} - E_j^2 - 2 \right) \leq \frac{1}{\varepsilon} - 1.

Therefore, there are only finitely many possibilities for \(E_j^2\) for any \(j\) since we have seen that there are only finitely many possibilities for \(t_{i,j}\) for any \(i, j\).

Lemma 5.4. For the second case in Lemma 5.3, if the chain has length \(r = \ell_1 + n + \ell_2 > 2n_0 + 2\) where \(n_0 = \lfloor \frac{1}{\varepsilon} \rfloor\), then
\[
\text{md}_{\chi}(\mathcal{F}, \Delta) = \text{pld}_{\chi}(\mathcal{F}, \Delta).
\]

Moreover, for the fixed graphs \(\{L_1, \ldots, L_{\ell_1}\}\) and \(\{R_1, \ldots, R_{\ell_2}\}\), the fixed number \(s\) of irreducible components of \(\Theta\), and the fixed way how \(B_i\) intersects \(E_j\), if \(\lim_{n \to \infty} b_j = \overline{b}_j\) exists where \((b_1, \ldots, b_s)\) is some ordering of coefficients of \(\Theta\), then
\[
\text{pld}(\mathcal{F}, \Delta) \geq \min \left\{ \frac{\alpha^L}{m_1 - q_1}, \frac{\alpha^R}{m_2 - q_2} \right\}
\]
and
\[
\lim_{n \to \infty} \text{pld}(\mathcal{F}, \Delta) = \min \left\{ \frac{\overline{\alpha}^L}{m_1 - q_1}, \frac{\overline{\alpha}^R}{m_2 - q_2} \right\},
\]

where

1. \(c^L_i = \sum_j (B_i \cdot L_j) g^L_j\) and \(c^R_i = \sum_j (B_i \cdot R_j) g^R_j\),
2. \(g^L_j\) (resp. \(g^R_j\)) is the determinant of the dual graph of the chain \(L_{j+1}, \ldots, L_{\ell_1}\) (resp. \(R_{j+1}, \ldots, R_{\ell_2}\)),
3. \(m_1 = g^L_0, q_1 = g^L_1, m_2 = g^R_0, q_2 = g^R_1\),
4. \(\alpha^L = 1 - \sum_i b_i c^L_i\) and \(\alpha^R = -\sum_i b_i c^R_i\), and
5. \(\overline{\alpha}^L = 1 - \sum_i \overline{b}_i c^L_i\) and \(\overline{\alpha}^R = -\sum_i \overline{b}_i c^R_i\).

Proof. We first show that \(\text{md}_{\chi}(\mathcal{F}, \Delta) = \text{pld}_{\chi}(\mathcal{F}, \Delta)\) if \(r > 2n_0 + 2\). Suppose not, then there is an irreducible divisor \(F\) over \(Y\) such that \(a(F, \mathcal{F}, \Delta) < \text{pld}(\mathcal{F}, \Delta)\). Then, \(F\) is an exceptional divisor for the composition of \(t\) blowups. Let \(F_1, \ldots, F_t := F\) be the exceptional divisors for the corresponding blowups. We may assume that \(a(F, \mathcal{F}, \Delta) < a(F_j, \mathcal{F}, \Delta)\) for all \(1 \leq j \leq t - 1\). Note that \(-F_j^2 \geq 2\) for all \(1 \leq j \leq t - 1\) and \([F]\) is not a fork since \(\sum_{i=1}^{t-1} F_i\) has the simple normal crossing support. Then, by Lemma 5.1(d), the graph \(\{E_j, F_j\} = \{G_k\}_{k=1}^N\) is a chain with \(F = F_t = G_u\) for some \(u\). Notice that exactly one of \(-G^2_{u-1}\) or \(-G^2_{u+1}\) is 2. Without loss of generality, we assume that \(-G^2_{u+1} = 2\) and \(-G^2_{u-1} = 3\). Then, \(u \leq n_0 + 2\) by Lemma 5.1(e).
Put \( \gamma = \min \{ k > u | -G_k^2 \geq 3 \} \). Then, by Lemma 5.1(e) again, we have \( N - \gamma \leq n_0 \). By blowing down \( G_u, ..., G_{\gamma - 1} \) successively, we get a new foliated surface with only reduced singularities and with at most \((u - 1) + (N - \gamma + 1) \leq 2n_0 + 2 < r \) exceptional divisors over \( X \). However, since this new foliated surface factors through the minimal resolution, the number of exceptional divisors over \( X \) is at least \( r \), which gives a contradiction.

Now, we compute the partial log discrepancy \( \text{pld}_x (\mathcal{F}, \Delta) \). It is known that the graph \( \Gamma \) is uniquely determined by these five numbers \( m_1, q_1, m_2, q_2, \) and \( n \). By Lemma 5.5, we have that both \( a(L_1, \mathcal{F}, \Delta) \) and \( a(R_1, \mathcal{F}, \Delta) \) are between

\[
\min \left\{ \frac{\alpha^L}{m_1 - q_1}, \frac{\alpha^R}{m_2 - q_2} \right\} \text{ and } \max \left\{ \frac{\alpha^L}{m_1 - q_1}, \frac{\alpha^R}{m_2 - q_2} \right\}.
\]

Moreover, we have that

\[
\lim_{n \to \infty} a(L_1, \mathcal{F}, \Delta) = \frac{\alpha^L}{m_1 - q_1} \text{ and } \lim_{n \to \infty} a(R_1, \mathcal{F}, \Delta) = \frac{\alpha^R}{m_2 - q_2}.
\]

**Lemma 5.5.** Notation as in Lemma 5.4, we have

\[
a(L_1, \mathcal{F}, \Delta) = \frac{\alpha^L(n(m_2 - q_2) + m_2) + \alpha^R q_1}{n(m_1 - q_1)(m_2 - q_2) + m_2(m_1 - q_1) + q_1(m_2 - q_2)}
= \frac{\alpha^L}{m_1 - q_1}(n + \frac{m_2}{m_2 - q_2}) + \frac{\alpha^R}{m_2 - q_2} \frac{q_1}{m_1 - q_1}
= \frac{\alpha^L(n + \frac{m_1}{m_2 - q_2}) + \frac{\alpha^R}{m_1 - q_1} q_2}{n + \frac{m_1}{m_1 - q_1} + \frac{q_2}{m_2 - q_2}}.
\]

and

\[
a(R_1, \mathcal{F}, \Delta) = \frac{\alpha^R(n(m_1 - q_1) + m_1) + \alpha^L q_2}{n(m_1 - q_1)(m_2 - q_2) + m_1(m_2 - q_2) + q_2(m_1 - q_1)}
= \frac{\alpha^R}{m_2 - q_2}(n + \frac{m_1}{m_2 - q_2}) + \frac{\alpha^L}{m_1 - q_1} \frac{q_2}{m_1 - q_1}
= \frac{\alpha^R(n + \frac{m_1}{m_2 - q_2}) + \alpha^L q_2}{n + \frac{m_1}{m_1 - q_1} + \frac{q_2}{m_2 - q_2}}.
\]

**Proof.** We denote \( \Gamma \) as \( \Gamma_{m_1,q_1,m_2,q_2,n} \) and its intersection matrix \( A \) as \( A_{m_1,q_1,m_2,q_2,n} \). Let \( S_n = \det(-A_{1,1,m_2,q_2,n}) \). Then, by Lemma 2.24 (1), we have that \( \det(-A_{m_1,q_1,m_2,q_2,n}) = 2m_1S_{n-1} - a_1S_{n-1} - m_1S_{n-2} = (2m_1 - q_1)S_{n-1} - m_1S_{n-2} \) and \( S_n = 2S_{n-1} - S_{n-2} \). Thus, \( S_{i+1} - S_i = S_i - S_{i-1} \) for all \( i \). Since \( S_0 = m_2 \) and \( S_1 = 2S_0 - q_2 = 2m_2 - q_2 \), we have

\[
S_n = \sum_{i=1}^{n} (S_i - S_{i-1}) + S_0 = n(S_1 - S_0) + S_0 = n(m_2 - q_2) + m_2
\]

and thus

\[
\det(-A) = \det(-A_{m_1,q_1,m_2,q_2,n})
= (2m_1 - q_1)(n - 1)(m_2 - q_2) + m_2 - m_1((n - 2)(m_2 - q_2) + m_2)
= n(m_1 - q_1)(m_2 - q_2) + m_2(m_1 - q_1) + q_1(m_2 - q_2).
\]

Also by Lemma 2.24(2), we have

\[
(L_1, L_j)\text{-cofactor} = (-1)^{|I|+1} g_j^L S_n,
(L_1, M_j)\text{-cofactor} = (-1)^{|I|+1} a_1 S_{n-j}, \text{ and}
(L_1, R_j)\text{-cofactor} = (-1)^{|I|+1} g_j^R q_1.
\]
Put \( d^L_j = (K_G + \Theta) \cdot L_j \), \( d^R_j = (K_G + \Theta) \cdot R_j \), and \( d^M_j = (K_G + \Theta) \cdot M_j \). By assumption, we know that \( d^M_j = 0 \) and \( d^R_j = \Theta \cdot R_j \) for all \( j \). Moreover, \( d^L_j = \Theta \cdot L_j \) if \( j \neq \ell_1 \) and \( d^L_{\ell_1} = -1 + \Theta \cdot L_{\ell_1} \). Then, we have

\[
\det(A)a(L_1, \mathcal{F}, \Delta) = \sum_{j=1}^{\ell_1} d^L_j \cdot (L_1, L_j) \text{-cofactor} + \sum_{j=1}^{\ell_2} d^R_j \cdot (L_1, R_j) \text{-cofactor}.
\]

Thus,

\[
-\det(-A)a(L_1, \mathcal{F}, \Delta) = \sum_{j=1}^{\ell_1} d^L_j g^L_j S_n + \sum_{j=1}^{\ell_2} d^R_j g^R_j q_1
\]

\[
= \left( \sum_{j=1}^{\ell_1} \left( \sum_{i=1}^{s} b_i \cdot (B_i \cdot L_j) \right) g^L_j S_n - g^L_{\ell_1} S_n + \sum_{j=1}^{\ell_2} \left( \sum_{i=1}^{s} b_i \cdot (B_i \cdot R_j) \right) g^R_j q_1 \right)
\]

\[
= \left( \sum_{i=1}^{s} b_i c^L_j - 1 \right) S_n + \left( \sum_{i=1}^{s} b_i c^R_j \right) q_1
\]

\[
= -\alpha^L S_n - \alpha^R q_1.
\]

Therefore,

\[
a(L_1, \mathcal{F}, \Delta) = \frac{\alpha^L (n(m_2 - q_2) + m_2) + \alpha^R q_1}{n(m_1 - q_1)(m_2 - q_2) + m_1(m_1 - q_1) + q_1(m_2 - q_2)}
\]

\[
= \frac{\alpha^L}{m_1 - q_1} \left( n + \frac{m_2}{m_2 - q_2} \right) + \frac{\alpha^R}{m_2 - q_2} \frac{q_1}{m_1 - q_1}.
\]

By the similar way, we also have

\[
a(R_1, \mathcal{F}, \Delta) = \frac{\alpha^R (n(m_1 - q_1) + m_1) + \alpha^L q_2}{n(m_1 - q_1)(m_2 - q_2) + m_1(m_2 - q_2) + q_2(m_1 - q_1)}
\]

\[
= \frac{\alpha^R}{m_2 - q_2} \left( n + \frac{m_1}{m_1 - q_1} \right) + \frac{\alpha^L}{m_1 - q_1} \frac{q_2}{m_2 - q_2}.
\]

**Theorem 5.6.** For any DCC set \( B \), the set

\[
\text{PLD}(2, B) := \{ \text{pld}_x(\mathcal{F}, \Delta) | (X, \mathcal{F}, \Delta) \text{ is a foliated triple with } x \in X \text{ and } \Delta \in B \}
\]

satisfies the ACC.

**Proof.** Given any nondecreasing sequence \( \{ \text{pld}_{x_k}(\mathcal{F}, \Delta_k) \}_{k=1}^{\infty} \) in the set \( \text{PLD}(2, B) \), where \( (X_k, \mathcal{F}_k, \Delta_k) \) is a germ of foliated triple around \( x_k \) and \( \Delta_k \in B \) for all \( k \). We may assume that

\[
\text{pld}_{x_k}(\mathcal{F}_k, \Delta_k) > 0
\]

for all \( k \), otherwise the sequence \( \{ \text{pld}_{x_k}(\mathcal{F}_k, \Delta_k) \}_{k=1}^{\infty} \) stabilizes. Now let \( \varepsilon > 0 \) be a number such that \( \text{pld}_{x_k}(\mathcal{F}_k, \Delta_k) \geq \varepsilon \) for all \( k \). Since \( B \) satisfies the descending chain condition, we may assume that \( \min(B \setminus \{ 0 \}) \geq \varepsilon \).
Claim. We may also assume that the number \( s \) of irreducible components of \( \Theta_k \) is independent of \( k \).

Proof (Claim). For ease of our notation, we will drop the subscription \( k \).

By Lemma 5.3, the number of irreducible components of exceptional divisors \( E_i \), which meet \( \Theta \), is bounded by \( N \) for some \( N > 0 \). Moreover, by the inequality (5.1), we have \( \sum_{i=1}^{s} t_{i,j} b_i \leq \sum_{i=1}^{s} t_{i,j} \varepsilon \geq \sum_{i=1}^{s} \varepsilon = s\varepsilon \),

where the third inequality follows from \( t_{i,j} \geq 1 \) for some \( j \) for any fixed \( i \). Therefore, \( s \leq \frac{2N}{\varepsilon} \). Hence, after taking a subsequence, we may assume that the number of irreducible components of \( \Theta_k \) is independent of \( k \). This completes the proof of claim.

Let \( (b_{1,k}, \ldots, b_{s,k}) \) be some ordering of coefficients of \( \Theta_k \). After taking a further subsequence, we may assume the sequence \( \{b_{j,k}\}_{k=1}^{\infty} \) is nondecreasing for any \( j = 1, \ldots, s \). If there are infinitely many dual graph \( \Gamma_k \) belonging to the first case of Lemma 5.3, then, by Lemma 3.4 and after taking a further subsequence, we have that \( \text{pld}_{\chi_k}(\mathcal{F}_k, \Delta_k) \) is nonincreasing. Since it is also nondecreasing, the sequence \( \{\text{pld}_{\chi_k}(\mathcal{F}_k, \Delta_k)\}_{k=1}^{\infty} \) stabilizes.

Thus, there are infinitely many dual graphs \( \Gamma_k \) belonging to the second case of Lemma 5.3. After taking a subsequence, we may assume that the dual graphs of \( \{L_1, \ldots, L_{\varepsilon_1}\} \) and \( \{R_1, \ldots, R_{\varepsilon_2}\} \) are fixed, the way how \( B_i \) intersects \( E_j \) is fixed, and \( n \) is sufficiently large. Since \( b_{j,k} \leq 1 \), the limit \( \lim_{k \to \infty} b_{j,k} = \overline{b}_j \) exists for any \( j = 1, \ldots, s \).

By Lemma 5.4, we have, for all \( k \), that

\[
\text{pld}_{\chi_k}(\mathcal{F}_k, \Delta_k) \geq \min \left\{ \frac{\alpha^L}{m_1 - q_1}, \frac{\alpha^R}{m_2 - q_2} \right\} \geq \min \left\{ \frac{\alpha^L}{m_1 - q_1}, \frac{\alpha^R}{m_2 - q_2} \right\}
\]

and

\[
\lim_{k \to \infty} \text{pld}_{\chi_k}(\mathcal{F}_k, \Delta_k) = \min \left\{ \frac{\alpha^L}{m_1 - q_1}, \frac{\alpha^R}{m_2 - q_2} \right\}.
\]

Thus, the sequence \( \{\text{pld}_{\chi_k}(\mathcal{F}_k, \Delta_k)\}_{k=1}^{\infty} \) stabilizes.

□

Theorem 5.7. For any DCC set \( B \), the set

\[
\text{MLD}(2, B) := \{\text{mld}_{\chi}(\mathcal{F}, \Delta) \mid (X, \mathcal{F}, \Delta) \text{ is a foliated triple with } x \in X \text{ and } \Delta \in B\}
\]

satisfies the ACC.

Proof. Given any nondecreasing sequence \( \{\text{mld}_{\chi_k}(\mathcal{F}_k, \Delta_k)\}_{k=1}^{\infty} \) in the set MLD(2, \( B \)) where \( (X_k, \mathcal{F}_k, \Delta_k) \) is a germ of foliated triple around \( x_k \) and \( \Delta_k \in B \) for all \( k \). We may assume that

\[
\text{mld}_{\chi_k}(\mathcal{F}_k, \Delta_k) > 0
\]

for all \( k \), otherwise the sequence \( \{\text{mld}_{\chi_k}(\mathcal{F}_k, \Delta_k)\}_{k=1}^{\infty} \) stabilizes. Now let \( \varepsilon > 0 \) be a number such that \( \text{mld}_{\chi_k}(\mathcal{F}_k, \Delta_k) \geq \varepsilon \) for all \( k \). Since \( B \) satisfies the descending chain condition, we may also assume that \( \min(B \setminus \{0\}) \geq \varepsilon \). If there are infinitely many \( k \) such that \( \text{pld}_{\chi_k}(\mathcal{F}_k, \Delta_k) = \text{mld}_{\chi_k}(\mathcal{F}_k, \Delta_k) \), then by Theorem 5.6, we get that the sequence \( \{\text{mld}_{\chi_k}(\mathcal{F}_k, \Delta_k)\}_{k=1}^{\infty} \) stabilizes.

Thus, by Lemmas 5.3 and 5.4, after taking a subsequence, we may assume \( (X_k, \mathcal{F}_k, \Delta_k) \) have the same weighted dual graph for the minimal resolution \( (Y_k, \mathcal{G}_k, \Theta_k) \), the ways how \( B_i \) intersects \( E_j \) are the same, and \( \text{pld}_{\chi_k}(\mathcal{F}_k, \Delta_k) > \text{mld}_{\chi_k}(\mathcal{F}_k, \Delta_k) \).
We may assume that the number of irreducible components of $\Theta_k$ is fixed as the claim in the proof of Theorem 5.6. Let $(b_{1,k}, ..., b_{s,k})$ be some ordering of the coefficients of $\Theta_k$. Then, by taking a further subsequence, we may assume that the sequence $\{b_{j,k}\}_{k=1}^{\infty}$ is nondecreasing for any $j = 1, ..., s$.

Since $\text{pld}_{x_k}(\mathcal{F}_k, \Delta_k) > \text{mld}_{x_k}(\mathcal{F}_k, \Delta_k)$, there exists an exceptional divisor $F_k$ over $Y_k$ such that $a(F_k, \mathcal{F}_k, \Delta_k) = \text{mld}_{x_k}(\mathcal{F}_k, \Delta_k)$. Then, $F_k$ is an exceptional divisor of the birational morphism $\pi_k : Z_k \rightarrow Y_k$, which is the composition of $N_k$ blowups. Let $F_{1,k}, ..., F_{N_k,k} = : F_k$ be all $\pi_k$-exceptional divisors. We may assume that $a(F_{j,k}, \mathcal{F}_k, \Delta_k) > a(F_k, \mathcal{F}_k, \Delta_k)$ for all $j \leq N_k - 1$. Notice that $-(F_{j,k})^2 \geq 2$ for all $j \leq N_k - 1$ and $[F_k]$ is not a fork since $\bigcup_{i=1}^{r} E_i \cup \bigcup_{j=1}^{N_k-1} F_{j,k}$ has the simple normal crossings support. By Lemma 5.1(d), the dual graph of $\{E_i, F_{j,k}\}_{i,j}$ is a chain.

By Lemma 5.8, the length of the chain of the dual graph of $\{E_i, F_{j,k}\}_{i,j}$ is bounded. After taking a subsequence, we may assume that they have the same dual graphs, and the ways $B_i$ intersecting $E_j$ and $F_{j,k}$ are the same.

Therefore, by Lemma 3.4, we have that

$$\text{mld}_{x_k}(\mathcal{F}_k, \Delta_k) = a(F_k, \mathcal{F}_k, \Delta_k) \geq a(F_{k+1}, \mathcal{F}_{k+1}, \Delta_{k+1}) = \text{mld}_{x}(\mathcal{F}_{k+1}, \Delta_{k+1})$$

for all $k$. This shows that the sequence $\{\text{mld}_{x_k}(\mathcal{F}_k, \Delta_k)\}_{k=1}^{\infty}$ stabilizes.\[□\]

Lemma 5.8. Fix an $\varepsilon > 0$. Suppose the sequence $(X_k, \mathcal{F}_k, \Delta_k)$ with $\text{pld}_{x_k}(\mathcal{F}_k, \Delta_k) > \text{mld}_{x_k}(\mathcal{F}_k, \Delta_k) \geq \varepsilon$ and $b_i \geq \varepsilon$ have the following properties:

(i) The dual graph for the minimal resolution $(Y_k, \mathcal{G}_k, \Theta_k)$ are the same.
(ii) The numbers of irreducible components of $\Theta_k$ are the same.
(iii) $(b_{1,k}, ..., b_{s,k})$ is some ordering of the coefficients of $\Theta_k$.
(iv) $\{b_{j,k}\}_{k=1}^{\infty}$ is nondecreasing for any $j = 1, ..., s$.

Then, there is a positive integer $N$ independent of $k$ such that, for each $(Y_k, \mathcal{G}_k, \Theta_k)$, there exists a birational morphism $\pi_k : Z_k \rightarrow Y_k$ such that

1. the relative Picard number $\rho(Z_k/Y_k) \leq N$ and
2. one of the exceptional divisors on $Z_k$ over $X_k$ computes the minimal log discrepancy.

Proof. To simplify our notation, we will drop the subscription $k$ and, for any fixed divisor $D$, denote all of the proper transforms of $D$ by $D$. Since the set $B$ satisfies DCC, there is a $\delta_a > 0$ for any $a \in \mathbb{R}$ such that $\sum_{i=1}^{s} m_i \beta_i - a \geq \delta_a$ if $\sum_{i=1}^{s} m_i \beta_i - a > 0$ for some $\beta_i \in B$.

We have seen in the proof of Theorem 5.7 that the dual graph of $\{G_p\} = \{E_i, F_j\}$ is a chain. This implies that $\pi$ is the composition of blowups with center $p$ either at the foliation singularities or on two curves at the ends. We may also assume that $N$ is minimal for all $k$. We will proceed by induction on $\sum_{i=1}^{s} \text{mult}_p B_i$. Note that

$$\sum_{i=1}^{s} \text{mult}_p B_i \leq \frac{1}{\varepsilon} \sum_{j=1}^{r} \Theta \cdot E_j \leq \frac{1}{\varepsilon} \sum_{j=1}^{r} 2 = \frac{2r}{\varepsilon}$$

by the inequality (5.1), and $\frac{2r}{\varepsilon}$ is independent of $k$.

Let $F_1$ be the exceptional divisor of blowup at $p$. We have the following three cases:

1. Suppose $p$ is a smooth foliation point on the curve at the end, say $E_r$. Note that

$$a(F_1, \mathcal{F}, \Delta) = a(E_r, \mathcal{F}, \Delta) + 1 - \sum_{i=1}^{s} (\text{mult}_p B_i) b_i.$$
If all $B_i$s meet $E_r$ at $p$ transversally but $\sum_{i=1}^s (\text{mult}_p B_i) b_i > 1$, then

$$\sum_{i=1}^s (\text{mult}_p B_i) b_i - 1 \geq \delta_1 > 0$$

and $a(F_1, \mathcal{F}, \Delta) \leq a(E_r, \mathcal{F}, \Delta) - \delta_1 \leq 1 - \delta_1$. If $\sum_{i=1}^s \text{mult}_q B_i < \sum_{i=1}^s \text{mult}_p B_i$ for all $q \in F_1$, then we are done by induction. If not, then the exceptional divisor from blowing up at such $q$ has discrepancy $\leq a(F_1, \mathcal{F}, \Delta) - \delta_1 \leq 1 - 2\delta_1$. Since the minimal log discrepancy $\text{mld}_x(\mathcal{F}, \Delta)$ is positive, after finitely many blowups, the quantity $\sum_{i=1}^s \text{mult}_p B_i$ will strictly decrease.

If some of $B_i$s meet $E_r$ at $p$ transversally and some do not, then the quantity $\sum_{i=1}^s \text{mult}_p B_i$ strictly decreases.

If each $B_i$ does not meet $E_r$ at $p$ transversally, then all $B_i$s meet $F_1$ at the reduced singularity $q$ on $F_1$. Notice that we have $\sum_{i=1}^s \text{mult}_q B_i \leq \sum_{i=1}^s \text{mult}_p B_i$. If it is not an equality, then we are done by induction. Suppose it is an equality, then we may assume that $p$ is contained in two exceptional divisors, say $E_j$ and $E_{j+1}$ for some $j$.

2. Suppose $p$ is contained in two exceptional divisors, say $E_j$ and $E_{j+1}$ for some $j$. Note that

$$a(F_1, \mathcal{F}, \Delta) = a(E_j, \mathcal{F}, \Delta) + a(E_{j+1}, \mathcal{F}, \Delta) - \sum_{i=1}^s (\text{mult}_p B_i) b_i.$$

Assume $a(E_j, \mathcal{F}, \Delta) \leq a(E_{j+1}, \mathcal{F}, \Delta)$. If $a(F_1, \mathcal{F}, \Delta) > a(E_j, \mathcal{F}, \Delta)$, then all exceptional divisors from further blowups have discrepancies at least $a(E_j, \mathcal{F}, \Delta)$. Thus, we may assume that $a(F_1, \mathcal{F}, \Delta) < a(E_j, \mathcal{F}, \Delta)$. Notice that

$$a(E_j, \mathcal{F}, \Delta) - a(F_1, \mathcal{F}, \Delta) = \sum_{i=1}^s (\text{mult}_p B_i) b_i - a(E_{j+1}, \mathcal{F}, \Delta) \geq \delta_\alpha,$$

where $\alpha = a(E_{j+1}, \mathcal{F}, \Delta)$. Therefore, $a(F_1, \mathcal{F}, \Delta) \leq a(E_j, \mathcal{F}, \Delta) - \delta_\alpha \leq 1 - \delta_\alpha$. There are two reduced singularities $q_1$ and $q_2$ on $F_1$. Note that the quantities $\sum_{i=1}^s \text{mult}_{q_1} B_i$ and $\sum_{i=1}^s \text{mult}_{q_2} B_i$ are at most $\sum_{i=1}^s \text{mult}_p B_i$.

If both are strictly inequalities, then we are done by induction.

If one of them is an equality, say $q_1$, then the exceptional divisor $F_2$ from blowing up at $q_1$ has discrepancy

$$a(F_2, \mathcal{F}, \Delta) \leq a(F_1, \mathcal{F}, \Delta) - \delta_\alpha \leq 1 - 2\delta_\alpha.$$

Since the minimal log discrepancy $\text{mld}_x(\mathcal{F}, \Delta)$ is positive, after finitely many blowups, the quantity $\sum_{i=1}^s \text{mult}_p B_i$ will strictly decrease.

3. Suppose $p$ is a reduced singularity contained in precisely one exceptional divisor $E_r$. Note that

$$a(F_1, \mathcal{F}, \Delta) = a(E_r, \mathcal{F}, \Delta) - \sum_{i=1}^s (\text{mult}_p B_i) b_i \leq a(E_r, \mathcal{F}, \Delta) - \varepsilon \sum_{i=1}^s \text{mult}_p B_i.$$

If $\sum_{i=1}^s \text{mult}_q B_i < \sum_{i=1}^s \text{mult}_p B_i$ for all $q \in F_1$, then we are done by induction.

When $\sum_{i=1}^s \text{mult}_q B_i = \sum_{i=1}^s \text{mult}_p B_i$ for some $q$, if $q$ is either a smooth foliation point or contained in two exceptional divisors, then we are done. Otherwise, $q$ is a reduced singularity contained in precisely one exceptional divisor. Then, the exceptional divisor $F_2$ from blowing up at $q$ has discrepancy

$$a(F_2, \mathcal{F}, \Delta) \leq a(F_1, \mathcal{F}, \Delta) - \sum_{i=1}^s (\text{mult}_q B_i) b_i \leq a(E_r, \mathcal{F}, \Delta) - 2\varepsilon \sum_{i=1}^s \text{mult}_p B_i.$$


Since the minimal log discrepancy \( \text{mld}_x(\mathcal{F}, \Delta) \) is positive, after finitely many blowups, the quantity \( \sum_{i=1}^s \text{mult}_p B_i \) will strictly decrease.

\[ \square \]

**Remark 5.9.** When \( B = \emptyset \), we have

\[
\text{MLD}(2, \emptyset) = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \cup \{0, -\infty\}.
\]

### 6 | VANISHING THEOREM FOR FOLIATIONS

In this section, we prove a Grauert–Riemenschneider–type vanishing theorem for foliated surfaces with special log canonical foliation singularities (see Definition 6.1), which generalizes [8, Theorem 6.1] in which \((X, \mathcal{F})\) is assumed to have only canonical foliation singularities.

**Definition 6.1.** In Theorem 3.21, we say a log canonical foliation singularity is **special** if it is either of the types (1)–(5), or of the type (6) and (7) such that all invariant curves are \((-1)\)\(\cdot\mathcal{G}\)-curves and the noninvariant curve \(E\) satisfying

\[
-E^2 \geq \max\{2\pi_0(E) - 1 + \deg[E], 2 - 2\pi_0(E)\}.
\]

**Remark 6.2.** All canonical foliation singularities are special log canonical foliation singularities.

**Lemma 6.3.** Let \((X, \mathcal{F}, p)\) be a germ of a foliated surface with at worst reduced singularity at \(p\). Let \(\pi : (Y, E) \to (X, p)\) be a blowup at \(p\) and \(\mathcal{G}\) be the pullback foliation of \(\mathcal{F}\). Then, we have \(R^1\pi_* \mathcal{O}_Y(K_{\mathcal{G}}) = 0\).

**Proof.** Note that \(K_{\mathcal{G}} = \pi^* K_{\mathcal{F}} + a(E)E\) where \(a(E)\) is either 0 or 1. Also, since \(p\) is a smooth point of \(X\), we have \(R^1\pi_* \mathcal{O}_Y = 0\).

If \(a(E) = 0\), then, by the projection formula, we have

\[
R^1\pi_* \mathcal{O}_Y(K_{\mathcal{G}}) = R^1\pi_* \pi^* \mathcal{O}_X(K_{\mathcal{F}}) = R^1\pi_* \mathcal{O}_Y \otimes \mathcal{O}_X(K_{\mathcal{F}}) = 0.
\]

If \(a(E) = 1\), then we consider the following short exact sequence:

\[
0 \longrightarrow \pi^* \mathcal{O}_X(K_{\mathcal{F}}) \longrightarrow \mathcal{O}_Y(K_{\mathcal{G}}) \longrightarrow \mathcal{O}_E(E) \longrightarrow 0.
\]

Pushing forward via \(\pi\), we obtain the exact sequence

\[
0 = R^1\pi_* \pi^* \mathcal{O}_X(K_{\mathcal{F}}) \longrightarrow R^1\pi_* \mathcal{O}_Y(K_{\mathcal{G}}) \longrightarrow R^1\pi_* \mathcal{O}_E(E) \longrightarrow 0.
\]

Since \(R^1\pi_* \mathcal{O}_E(E) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0\), we have \(R^1\pi_* \mathcal{O}_Y(K_{\mathcal{G}}) = 0\). \(\square\)

**Theorem 6.4.** Let \(f : (Y, \mathcal{G}) \to (X, \mathcal{F})\) be a proper birational morphism of foliated pairs where \((X, \mathcal{F})\) is a foliated surface with special log canonical foliation singularities and \((Y, \mathcal{G})\) is a foliated surface with only reduced singularities. Then, \(R^i f_* \mathcal{O}_Y(K_{\mathcal{G}}) = 0\) for \(i > 0\).

**Proof.** We divide the proof into several steps.

1. We first consider \((Y, \mathcal{G})\) to be the **minimal resolution** of \((X, \mathcal{F})\) with exceptional divisors \(E_1, \ldots, E_r\). Let \(Z = \sum_{i=1}^r a_i E_i\) where \(a_i\)s are nonnegative integers. By the theorem on formal functions, it suffices to show that

\[
h^1(Z, \mathcal{O}_Y(K_{\mathcal{G}}) \otimes \mathcal{O}_Z) = 0
\]

for any effective (nonzero) divisor \(Z\). Let \(A := \sum_{i=1}^r a_i\). We will show the vanishing by induction on \(A\).
2. When $A = 1$, there is exactly one $i$ such that $a_i$ is positive and equals to 1. Without loss of generality, we assume $a_1 = 1$ and $a_i = 0$ for $i \geq 2$. \hfill \Box

Claim. $\deg(K_{E_1}) - K_G \cdot E_1 < 0$ when $E_1$ is smooth.

Proof (Claim).

(a) If $E_1$ is invariant, then $E_1$ is a smooth rational curve on which there is at least one reduced singularity. Thus, by adjunction for invariant divisors (Theorem 2.14), we have $K_G \cdot E_1 \geq -1$ and

$$\deg(K_{E_1}) - K_G \cdot E_1 = -2 - K_G \cdot E_1 \leq -1 < 0.$$ 

(b) If $E_1$ is noninvariant, then the tangency order of $E_1$ is zero. Thus, we have

$$\deg(K_{E_1}) - K_G \cdot E_1 = 2p_a(E_1) - 2 + E_1^2 \leq 2p_a(E_1) - 2 - (2p_a(E_1) - 1 + \deg[E_1]) = -1 - \deg[E_1] < 0,$$

where the first equality comes from adjunction for noninvariant divisors (Theorem 2.10) and the inequality follows from the assumption that $-E_1^2 \geq 2p_a(E_1) - 1 + \deg[E_1]$.

Thus, by Serre duality, we have $h^1(E_1, \mathcal{O}_Y(K_G) \otimes \mathcal{O}_{E_1}) = h^0(E_1, \omega_{E_1} \otimes \mathcal{O}_{E_1}(-K_G)) = 0$ when $E_1$ is smooth.

If $E_1$ is not smooth, then, by Proposition 2.12, $E_1$ is an invariant rational curve with exactly one node. In particular, $E_1$ is an elliptic Gorenstein leaf. By [15, Fact III.0.4 and Theorem IV.2.2], we have $\mathcal{O}_{E_1}(K_G)$ is not torsion and has degree 0. Also, $E_1$ is Cohen–Macaulay with trivial dualizing sheaf, then by Serre duality, we have

$$h^1(E_1, \mathcal{O}_Y(K_G) \otimes \mathcal{O}_{E_1}) = h^0(E_1, \mathcal{O}_Y(-K_G) \otimes \mathcal{O}_{E_1}) = 0.$$

3. Let $Z_\ell = Z - E_\ell$ for some $\ell$ with positive $a_\ell$, which will be determined later. Then, we combine the following two short exact sequences:

$$0 \longrightarrow \mathcal{O}_Y(-Z) \longrightarrow \mathcal{O}_Y(-Z_\ell) \longrightarrow \mathcal{O}_{E_\ell} \otimes \mathcal{O}_Y(-Z_\ell) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_Y(-Z_\ell) / \mathcal{O}_Y(-Z) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_{E_\ell} \longrightarrow 0$$

into a short exact sequence

$$0 \longrightarrow \mathcal{O}_{E_\ell} \otimes \mathcal{O}_Y(-Z_\ell) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_{E_\ell} \longrightarrow 0.$$

Tensoring with $\mathcal{O}_Y(K_G)$, we get

$$0 \longrightarrow \mathcal{O}_{E_\ell} \otimes \mathcal{O}_Y(K_G - Z_\ell) \longrightarrow \mathcal{O}_Y(K_G) \otimes \mathcal{O}_Z \longrightarrow \mathcal{O}_Y(K_G) \otimes \mathcal{O}_{E_\ell} \longrightarrow 0 \quad (6.1).$$

Claim. To show $h^1(Z, \mathcal{O}_Y(K_G) \otimes \mathcal{O}_Z) = 0$, it suffices to show that $(K_G - K_Y - Z) \cdot E_\ell > 0$. 

Proof (Claim). Suppose \((K_G - K_Y - Z) \cdot E_\ell > 0\). By Serre duality, we have

\[
h^1(E_\ell, \mathcal{O}_{E_\ell} \otimes \mathcal{O}_Y(K_G - Z_\ell)) = h^0(E_\ell, \omega_{E_\ell} \otimes \mathcal{O}_{E_\ell}(Z_\ell - K_G)) = 0,
\]

where the last equality follows from

\[
\text{deg}(K_{E_\ell}) + (Z_\ell - K_G) \cdot E_\ell = (K_Y + E_\ell) \cdot E_\ell + (Z_\ell - K_G) \cdot E_\ell
\]

\[
= (K_Y - K_G + Z) \cdot E_\ell < 0.
\]

By induction hypothesis, we have \(h^1(Z_\ell, \mathcal{O}_Y(K_G) \otimes \mathcal{O}_{Z_\ell}) = 0\). Therefore, by the long exact sequence from the short exact sequence (6.1), we get \(h^1(Z, \mathcal{O}_Y(K_G) \otimes \mathcal{O}_Z) = 0\). This completes the proof of the claim.

So we reduce the proof to find an \(\ell\) with \(a_\ell > 0\) such that \((K_G - K_Y - Z) \cdot E_\ell > 0\). The strategy is as follows. We first write \(K_G - K_Y \equiv \sum_{i=1}^r b(E_i)E_i\) for some \(b(E_i) \in \mathbb{R}\) and then compare \(K_G - K_Y - Z \equiv \sum_{i=1}^r (b(E_i) - a_i)E_i\) with an \(f\)-anti-ample effective divisor \(\sum_{i=1}^r t_iE_i\). Suppose \(t_i + b(E_i) - a_i \geq 0\) for all \(i\). If there is an index \(\ell\) with \(a_\ell > 0\) such that \(t_\ell + b(E_\ell) - a_\ell = 0\), then we have \((K_G - K_Y - Z) \cdot E_\ell = -\sum_{i=1}^r t_iE_i \cdot E_\ell > 0\) as desired. We will elaborate in the following steps.

4. Let \(K_G - K_Y \equiv \sum_{i=1}^r b(E_i)E_i\) for some \(b(E_i) \in \mathbb{R}\). We will denote \(b(E_i)\) by \(b_i\) as well.

Claim. Either \(b(E_i) = 1\) for all \(i\) or \(b(E_i) < 1\) for all \(i\). Moreover, \(b(E_i) = 1\) for all \(i\) only when \(\bigcup_{i=1}^r E_i\) is an elliptic Gorenstein leaf.

Proof (Claim).

(a) If \(E_j\) is invariant, then we have

\[
(K_G - K_Y) \cdot E_j = Z(\mathcal{G}, E_j) - 2 + E_j^2 + 2
\]

\[
= Z(\mathcal{G}, E_j) + E_j^2 \geq \left( \sum_{i=1}^r E_i \right) \cdot E_j,
\]

where the inequality is clear for the case of canonical singularities and follows from the assumption that all invariant curves on the minimal resolution are \((-1)\)-\(\mathcal{G}\)-curves for the case of noncanonical singularities. Moreover, except when \(\bigcup_{i=1}^r E_i\) is an elliptic Gorenstein leaf, the inequality (6.2) is strict for some \(j\).

(b) If \(E_j\) is noninvariant, then we have

\[
(K_G - K_Y) \cdot E_j = -E_j^2 + E_j^2 + 2 - 2p_a(E_j)
\]

\[
> \deg[E_j] + E_j^2 = \left( \sum_{i=1}^r E_i \right) \cdot E_j,
\]

where the inequality follows from the assumption that \(-E_j^2 \geq 2p_a(E_i) - 1 + \deg[E_1]\).

By Lemma 3.4, we have that \(b_i \leq 1\) for all \(i\). Moreover, \(\bigcup_{i=1}^r E_i\) is connected, we have either \(b_i = 1\) for all \(i\) or \(b_i < 1\) for all \(i\). In particular, we have \(b_i < 1\) for all \(i\) except when \(\bigcup_{i=1}^r E_i\) is an elliptic Gorenstein leaf. This is because there is an index \(j\) such that the inequality (6.2) or (6.3) is strict.

Claim. We have \(b(E_i) \geq 0\) for all \(i\) except for the noninvariant \(E_0\) of (6) and (7) in Theorem 3.21.
Proof (Claim). In the case of canonical singularities, we put the divisor $D$ the half sum of all $(-1)$-curves whose self-intersections are $-2$. Then, we have

$$(K_G - K_Y) \cdot E_i \leq D \cdot E_i$$

for all $i$ by adjunction for invariant divisors (Theorem 2.14). Thus, by Lemma 3.4, we have that $b_i \geq 0$ for all $i$.

In the case of noncanonical singularities, we put $E_0$ the noninvariant exceptional divisor and note that

$$(K_G - K_Y) \cdot E_0 = 2 - 2p_a(E_0) \leq -E_0^2$$

and

$$(K_G - K_Y) \cdot E_j = 1 + E_j^2 \leq -1 = -E_0 \cdot E_j$$

for any invariant curve $E_j$ by Proposition 2.12 and adjunction for noninvariant divisors (Theorem 2.10). Then, by Lemma 3.4, we have $b(E_j) \geq 0$ for all invariant curves $E_j$.

5. Let $W = \sum_{i=1}^r c_i E_i$ be an $f$-anti-ample effective divisor where $c_i > 0$ for all $i$. We have following two cases:

(a) If $a_i - b_i$ is positive for some $i$, then we define $\alpha \in \mathbb{R}_{\geq 0}$ to be the maximal number such that $\alpha(a_i - b_i) \leq c_i$ for all $i$. Then, there is an index $\ell$ such that $c_{\ell} - \alpha(a_{\ell} - b_{\ell}) = 0$, and therefore

$$(W + \alpha(K_G - K_Y - Z)) \cdot E_{\ell} = \left(\sum_{i=1}^r (c_i - \alpha(a_i - b_i))E_i\right) \cdot E_{\ell} \geq 0.$$  

Hence, we have

$$(K_G - K_Y - Z) \cdot E_{\ell} \geq \frac{-1}{\alpha} W \cdot E_{\ell} > 0.$$  

(b) If $a_i - b_i \leq 0$ for all $i$, then $a_i \leq b_i \leq 1$ for all $i$. Thus, $b_i = 1$ for all $i$, otherwise, we have $b_i < 1$ for all $i$, and then $a_i = 0$ for all $i$ since $a_i$s are nonnegative integers.

6. Suppose $\bigcup_{i=1}^r E_i$ is the exceptional divisor of the minimal resolution of a canonical foliation singularity. Then, we have $b(E_i) \geq 0$ for all $i$. If $b_i < 1$ for all $i$, then by Step 5, there is an index $\ell$ such that $c_{\ell} - \alpha(a_{\ell} - b_{\ell}) = 0$ and thus $a_{\ell} = b_{\ell} + \frac{c_{\ell}}{\alpha} > 0$.

If $b_i = 1$ for all $i$, then $a_i \leq 1$ for all $i$ and $\bigcup_{i=1}^r E_i$ is an elliptic Gorenstein leaf $\Gamma$. By [15, Fact III.0.4 and Theorem IV.2.2], we have $\mathcal{O}_\Gamma(K_G)$ is not torsion and has degree $0$. Also $\Gamma$ is Cohen–Macaulay with trivial dualizing sheaf, then by Serre duality, we have

$$h^1(\Gamma, \mathcal{O}_\Gamma(K_G) \otimes \mathcal{O}_\Gamma) = h^0(\Gamma, \mathcal{O}_\Gamma(-K_G) \otimes \mathcal{O}_\Gamma) = 0.$$  

Thus, we have shown the case when $a_i = 1$ for all $i$. Therefore, when $a_i = 0$ for some $i$, we have an index $j$ with $a_j = 1$ such that

$$(K_G - K_Y - Z) \cdot E_j = \left(\sum_i (b_i - a_i)E_i\right) \cdot E_j > 0.$$  

7. Now suppose $\bigcup_{i=1}^r E_i$ is the exceptional divisor of the minimal resolution of a noncanonical foliation singularity. Note that we have $b_i < 1$ for all $i$. If $E_{\ell}$ in step 5 is invariant, then from $c_{\ell} - \alpha(a_{\ell} - b_{\ell}) = 0$, we have

$$a_{\ell} = b_{\ell} + \frac{c_{\ell}}{\alpha} > 0.$$
Thus, it suffices to consider the case when \( E_f \) is noninvariant and \( a_f = 0 \). Then, the support of \( Z = \sum_i Z_i = \sum_i a_i E_i \) is the disjoint union of \((-1)\)-\( S \)-curves \( E_i \) where \( Z_i = a_i E_i \). Note that

\[
H^1(Z, \mathcal{O}_Y(K_S) \otimes \mathcal{O}_Z) = \bigoplus H^1(Z_i, \mathcal{O}_Y(K_S) \otimes \mathcal{O}_{Z_i}).
\]

Since the contraction of \( E_i \) introduces a terminal foliation singularity, we have \( h^1(Z_i, \mathcal{O}_Y(K_S) \otimes \mathcal{O}_{Z_i}) = 0 \) for all \( i \), and thus \( h^1(Z, \mathcal{O}_Y(K_S) \otimes \mathcal{O}_Z) = 0 \).

8. Now we consider the general resolution. Note that any general resolution \( f : (Y, S) \to (X, F) \) factors through the minimal resolution \( h : (Z, H) \to (X, F) \). Let \( f = h \circ g \). So \( g \) is the composition of blowups.

We have shown that \( R^1h_*\mathcal{O}_Z(K_{\mathcal{F}}) = 0 \). Notice that \( g_*\mathcal{O}_Y(K_S) = \mathcal{O}_Z(K_{\mathcal{F}}) \). By Grothendieck spectral sequence, we have the following exact sequence:

\[
\begin{align*}
0 & \longrightarrow R^1h_*(g_*\mathcal{O}_Y(K_S)) \\
& \longrightarrow R^1f_*\mathcal{O}_Y(K_S) \\
& \longrightarrow h_* R^1g_*\mathcal{O}_Y(K_S).
\end{align*}
\]

Since \( R^1h_*(g_*\mathcal{O}_Y(K_S)) = R^1h_*\mathcal{O}_Z(K_{\mathcal{F}}) = 0 \) and \( R^1g_*\mathcal{O}_Y(K_S) = 0 \) by a repeated use of Lemma 6.3, we have \( R^1f_*\mathcal{O}_Y(K_S) = 0 \).

\( \square \)

ACKNOWLEDGMENTS
A large part of the paper was written when the author was at the University of Utah. He would like to thank the University of Utah for wonderful research conditions. The author would also like to thank Christopher D. Hacon for his insightful suggestions and encouragements and the referees for their valuable comments and suggestions. This research was supported by the NSF research grants no: DMS-1801851, DMS-1840190 and by a grant from the Simons Foundation; Award Number: 256202.

REFERENCES

[1] V. Alexeev, Classification of log-canonical surface singularities: arithmetical proof, in: Flips and Abundance for Algebraic Threefolds, vol. 211, Société Mathématique de France, Paris, Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991 [Astérisque 211 (1992), 47–58.]

[2] V. Alexeev, Two two-dimensional terminations, Duke Math. J. 69 (1993), no. 3, 527–545. https://doi.org/10.1215/S0012-7094-93-06922-0

[3] M. Artin, Algebraization of formal moduli. II. Existence of modifications, Ann. of Math. (2) 91 (1970), 88–135. https://doi.org/10.2307/1970602

[4] M. Brunella, Feuilletages holomorphes sur les surfaces complexes compactes, Ann. Sci. École Norm. Sup. (4) 30 (1997), no. 5, 569–594. https://doi.org/10.1016/S0012-9593(97)89932-6.

[5] M. Brunella, Birational geometry of foliations, IMPA Monographs, vol. 1, Springer, Cham, 2015, pp. xiv+130. https://doi.org/10.1007/978-3-319-14310-1

[6] C. Camacho, Quadratic forms and holomorphic foliations on singular surfaces, Math. Ann. 282 (1988), no. 2, 177–184. https://doi.org/10.1007/BF01456970

[7] C. Camacho and P. Sad, Invariant varieties through singularities of holomorphic vector fields, Ann. of Math. (2) 115 (1982), no. 3, 579–595. https://doi.org/10.2307/2007013

[8] C. D. Hacon and A. Langer, On birational boundedness of foliated surfaces, J. Reine Angew. Math. 770 (2021), 205–229. https://doi.org/10.1515/crelle-2020-0009

[9] N. Hara, Classification of two-dimensional \( F \)-regular and \( F \)-pure singularities, Adv. Math. 133 (1998), no. 1, 33–53. https://doi.org/10.1006/aima.1997.1682

[10] R. Hartshorne, Algebraic geometry, Graduate Texts in Mathematics, No. 52, Springer, New York-Heidelberg, 1977, pp. xvi+496.

[11] J. Kollár, Singularities of the minimal model program, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013, pp. x+370. https://doi.org/10.1017/CBO9781139547895, with a collaboration of Sándor Kovács.

[12] J. Kollár and S. Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, pp. viii+254, https://doi.org/10.1017/CBO9780511662560, with the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[13] A. Lins Neto, Algebraic solutions of polynomial differential equations and foliations in dimension two, Holomorphic dynamics (Mexico, 1986), Lecture Notes in Math., vol. 1345, Springer, Berlin, 1988, pp. 192–232. https://doi.org/10.1007/BFb0081403

[14] J.-F. Mattei and R. Moussu, Holonomie et intégrales premières, Ann. Sci. École Norm. Sup. 13 (1980), no. 4, 469–523. http://www.numdam.org/item?id=ASENS_1980_4_13_4_469_0

[15] M. McQuillan, Canonical models of foliations, Pure Appl. Math. Q. 4 (2008), no. 3, Special Issue: In honor of Fedor Bogomolov. Part 2, 877–1012. https://doi.org/10.4310/PAMQ.2008.v4.n3.a9

[16] M. Sebastiani, Sur l’existence de séparatrices locales des feuilletages des surfaces, An. Acad. Brasil. Ciênc. 69 (1997), no. 2, 159–162.
[17] A. Seidenberg, Reduction of singularities of the differential equation $A \, dy = B \, dx$, Amer. J. Math. 90 (1968), 248–269. https://doi.org/10.2307/2373435

[18] T. Suwa, Indices of holomorphic vector fields relative to invariant curves on surfaces, Proc. Amer. Math. Soc. 123 (1995), no. 10, 2989–2997. https://doi.org/10.2307/2160652

[19] K. Watanabe, On plurigenera of normal isolated singularities. I, Math. Ann. 250 (1980), no. 1, 65–94. https://doi.org/10.1007/BF01422185

**How to cite this article:** Y.-A. Chen, Log canonical foliation singularities on surfaces, Math. Nachr. 296 (2023), 3222–3256. https://doi.org/10.1002/mana.202100215