Spatio-temporal speckle correlations for imaging in turbid media

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Abstract
We discuss the far-field spatio-temporal cross-correlations of waves multiple-scattered in a turbid medium in which is embedded a hidden heterogeneous region (inclusion) characterized by a distinct scatterer dynamics (as compared to the rest of the medium). We show that the spatio-temporal correlation is affected by the inclusion which suggests a new method of imaging in turbid media. Our results allow qualitative interpretation in terms of diffraction theory: the cross-correlation of scattered waves behaves similarly to the intensity of a wave diffracted by an aperture.

A considerable progress has been made during the recent years in the understanding of wave transport in disordered media. Very similar phenomena are shown to exist in multiple scattering of electrons and classical waves (e.g., light) under particular circumstances. Some of the concepts developed first theoretically, and then studied in model experiments, are now very close to practical applications. One of the important fields where the physics of multiple-scattered waves is currently finding its applications is the (medical) imaging of disordered, turbid media. The light waves scattered inside a turbid medium (e.g., human tissue) carry information on the properties of the medium. The information can be considered as being “encoded” in the statistics of the waves. Analysis of the latter statistics allows one to reconstruct (or “image”) the scattering medium.

A simplified version of a typical geometry considered in connection with imaging problems is shown in Fig. 1. A slab of turbid medium occupies the space between the planes \( z = 0 \) and \( z = L \), and some region (a cylinder-shaped inclusion) inside the slab is assumed to have somewhat different properties as compared to the surrounding medium. If “different properties” means different scattering \( \mu_s \) and/or absorption \( \mu_a \) coefficients, one can image the inclusion by measuring the spatial distribution of the average intensity \( I(r) = \langle E(r, t) E^*(r, t) \rangle \) of transmitted (or reflected) wave. Here \( E(r, t) \) is the amplitude of scattered wave at spatial position \( r \) at time \( t \). If \( \mu_s \) and \( \mu_a \) are constant throughout the medium, and the contrast between the inclusion and surrounding medium is provided by the scatterer dynamics (different types and/or intensities of scatterer motion inside and outside the inclusion), the methods of diffusing-wave spectroscopy can be applied to visualize the inclusion.

In the latter case, one measures the time autocorrelation function \( C_1(r, \tau) = \langle E(r, t) E^*(r, t + \tau) \rangle \) of scattered wave field at multiple positions \( r \), which allows visualization of the inclusion.

FIG. 1. A cylindrical inclusion of height \( 2\Delta = z_2 - z_1 \gg \ell \) and radius \( a \gg \ell \) (if \( \ell \) is a photon transport mean free path) is embedded at \( z_0 = (z_1 + z_2)/2 \) inside a turbid slab of width \( L \) and surface area \( A = W^2, W \gg L, W \gg a \). \( k_a \) and \( k_b \) denote the wave vectors of incident waves, while \( k_a \) and \( k_b' \) are the wave vectors of transmitted waves. The scatterers in the medium undergo Brownian motion with diffusion coefficients \( D_{in} \) (inside the inclusion) and \( D_{out} \) (outside the inclusion).

In the present contribution, we propose to use the spatio-temporal cross-correlation function \( C_1(r, \Delta r, \tau) = \langle E(r, t) E^*(r + \Delta r, t + \tau) \rangle \) for the purpose of imaging in turbid media. Since the time autocorrelation function \( C_1(r, \tau) \) carries more information about the turbid medium than the average intensity \( I(r) \), we suggest that the information contents of the spatio-temporal correlation function \( C_1(r, \Delta r, \tau) \) should be even more rich. If the points \( r \) and \( r + \Delta r \) are taken far enough from the medium (in the far-field of scattered wave), \( C_1(r, \Delta r, \tau) \) is equivalent to the angular-temporal correlation func-

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tion \( \langle E(k, t) E^*(k', t + \tau) \rangle \) [where \( E(k, t) \) is the spatial Fourier transform of \( E(r', t) \) with \( r' \) taken at the plane where the scattered waves leave the medium].

We start with a macroscopically homogeneous turbid medium (no inclusion), and assume that scatterers in the medium undergo Brownian motion with a diffusion coefficient \( D \). In addition, we assume a weak-scattering limit \( k\ell \gg 1 \) (where \( \ell = 1/\mu_0 \) is the photon transport mean free path) and neglect the absorption of light in the medium \( (\mu_a = 0) \). As depicted in Fig. 1, a plane wave is incident upon a turbid slab at time \( t \) with a wave vector \( k_0 \). The transmitted wave leaves the slab with a wave vector \( k_0' \). Similarly, for an incident wave with a wave vector \( k_0' \) at time \( t + \tau \), the transmitted wave has a wave vector \( k_0'' \). Assuming unit amplitudes of incident waves, we calculate the correlation function of transmitted fields \( C_1(k_0, k_0'; k_0', k_0''; \tau) = \langle E(k_0, t) E^*(k_0', t + \tau) \rangle \) using the standard diagrammatic techniques following the general calculation scheme developed in Ref. 14. In the leading order in a small parameter \( 1/k\ell \), we obtain:

\[
C_1 = \frac{\ell^2}{4k_l^2A^2} \int \int d^2R_1 \ d^2R_2 \times \exp(-i\Delta q_1 R_1 + i\Delta q_2 R_2) \times P((R_1, \ell), (R_2, L - \ell), \tau),
\]

(1)

where \( q_1 \) and \( q_2 \) denote projections of \( k \)'s onto the plane \( z = \text{const} \), \( \Delta q_1 = q_1 - q_1', \Delta q_2 = q_2 - q_2' \), \( r = (R, z) \) with \( R \) being a two-dimensional vector perpendicular to the \( z \)-axis, and we assume the first and the last scattering events to occur at \( z = \ell \) and \( z = L - \ell \), respectively. If \( \ell \gg \ell \) and \( |r_1 - r_2| \gg \ell \), the reduced ladder propagator \( P \) entering into Eq. (1), obeys the diffusion equation:

\[
\left[ \nabla^2 - \alpha^2(\tau) \right] P(r_1, r_2, \tau) = -\frac{3}{\ell^3} \delta(r_1 - r_2),
\]

(2)

where \( \alpha^2(\tau) = 3\tau/2(\tau_0\ell^2) \) with \( \tau_0 = (4k_l^2D)^{-1} \). The solution of Eq. (2) with Dirichlet boundary conditions at \( z = 0 \) and \( z = L \) \( (P = 0 \text{ if } z_1 = 0, L \text{ or } z_2 = 0, L) \) is readily found:

\[
P_0(r_1, r_2, \tau) = \frac{12\pi}{\ell^3} \int d^2p \times \frac{\sinh[\beta_0(L - z_2)]}{\sinh[\beta_0L]} \frac{\sinh(\beta_0z_1)}{\sinh(\beta_0L)} \times \exp[i(p_1 - p_2)\ell].
\]

(3)

Here \( \beta_0^2 = p^2 + \alpha^2(\tau) \), \( z_2 = \max\{z_1, z_2\} \), \( z_1 = \min\{z_1, z_2\} \), and the subscript “0” of \( P_0 \) denotes macroscopically homogeneous case. Inserting Eq. (3) into Eq. (1), we get

\[
C_1^{(0)}(\Delta q_0, \Delta q_0', \tau) = \delta(\Delta q_0) \frac{3\pi}{k_l^2A} \frac{\sin^2(\beta_0\ell)}{\beta_0\ell} \frac{\sinh(\beta_0L)}{\sinh(\beta_0L)}
\]

(4)

with \( \beta_0^2 = \Delta q_0^2 + \alpha^2(\tau) \). For \( \alpha^2(\tau) = 0 \), Eq. (4) reduces to the angular correlation function while for \( \Delta q_0 = 0 \) the time autocorrelation function of transmitted light is recovered. The Kronecker delta symbol in Eq. (4) describes the memory effect.

Now we turn to the case of macroscopically heterogeneous medium, assuming that the scatterer diffusion coefficient \( D_{in} \) inside a cylindrical region depicted in Fig. 1 is not the same as \( D_{out} \) in the surrounding medium (while \( \ell \) is assumed to be constant throughout the whole sample). The correlation function of transmitted waves can be again described by Eqs. (1), (2) but with \( \alpha^2(\tau) = \alpha_{in}^2(\tau) + \alpha_{out}^2(\tau) = 3\tau/(2\tau_{in}\ell^2) \) inside the inclusion and \( \alpha^2(\tau) = \alpha_{out}^2(\tau) = 3\tau/(2\tau_{out}\ell^2) \) outside it. Here \( \tau_{in, out} = (4k_l^2D_{in, out})^{-1} \) and \( \alpha_{in}^2(\tau) = 3\tau/(2\ell^2)[1/\tau_{in} - 1/\tau_{out}] \).

FIG. 2. Normalized angular-temporal correlation of a wave transmitted through a macroscopically homogeneous (no inclusion) slab of width \( L = 10\ell \) for three different time delays \( \tau/\tau_0 = 0, 0.01, 0.1 \). This correlation function vanishes identically for \( \Delta q_0 = \Delta q_0', \) which corresponds to the memory effect.

Assuming \( |\alpha_{in}^2(\tau)| \ll |\alpha_{out}^2(\tau)| \), we can write an approximate solution of Eq. (2) as a sum of \( P_0 \) corresponding to the macroscopically homogeneous medium with \( \tau_0 = \tau_{out} \) [see Eq. (3)], and \( P_1 \) which describes the influence of inclusion:

\[
P_1((R_1, \ell), (R_2, L - \ell), \tau) \simeq -\alpha_{in}^2(\tau) \frac{\ell^3}{3} \int d^3p P_0((R_1, \ell), \tau) \times P_0((R_2, L - \ell), \tau) \times \exp[ipR_1 - isR_2],
\]

(5)

where the integration of the second line is taken over the volume of inclusion, and \( F_T \) is a form factor:

\[
F_T(p, s, \tau) = \frac{2J_1(a|p - s|)}{a|p - s|}
\]

(6)
is not necessarily zero for $\Delta q$ function $J_p$ region (inclusion). As follows from Eq. (4), from the presence of a dynamically heterogeneous re-

\[
\frac{\sinh(\beta_a \ell) \sinh(\beta_b \ell)}{(\beta_a \ell) (\beta_b \ell) \sinh(\beta_a L) \sinh(\beta_b L)} \times \left\{ \frac{\sinh[(\beta_a - \beta_b) \Delta]}{(\beta_a - \beta_b) \Delta} \cosh[\beta_a (L - z_0) + \beta_b z_0] \right. \\
- \frac{\sinh[(\beta_a + \beta_b) \Delta]}{(\beta_a + \beta_b) \Delta} \cosh[\beta_a (L - z_0) - \beta_b z_0] \right\}.
\]

(6)

Here $J_1$ is the Bessel function of the first order, $\beta_1^2 = p^2 + a_0^2(\tau)$, $\beta_2^2 = s^2 + a_0^2(\tau)$, $\Delta = (z_2 - z_1)/2$, and $z_0 = (z_1 + z_2)/2$ (see Fig. 4).

Inserting $P = P_0 + P_1$ into Eq. (1), we obtain the angular-temporal correlation function corresponding to the macroscopically heterogeneous medium as a sum of two contributions: $C_1 = C_1^{(0)} + C_1^{(1)}$, where $C_1^{(0)}$ is given by Eq. (4) with $\alpha(\tau) = \alpha_0(\tau)$, and

\[
C_1^{(1)}(\Delta q_a, \Delta q_b, \tau) = -4\pi a_0^2(\tau) \ell^2 \frac{3\pi}{k^2 A} \pi a_0^2 \Delta F_T(\Delta q_a, \Delta q_b, \tau).
\]

(7)

Let us consider the simplest and practically important case of a single incident plane wave ($\Delta q_a = 0$). For a macroscopically homogeneous slab, the angular-temporal cross-correlation vanishes if $\Delta q_b \neq 0$, i.e. the waves scattered in different directions are uncorrelated. If a heterogeneous region is embedded inside the slab, the $C_1^{(1)}$ term appears and correlation between the waves scattered along different directions is not necessarily zero.

The $C_1^{(1)}$ term is plotted in Fig. 3 for three different radii $a$ of the inclusion (solid lines). As is seen from the figure, the correlation range can be estimated as $\Delta q_b \sim 1/a$. It is worthwhile to note that the correlation between the waves scattered along different directions ($\Delta q_b \neq 0$), introduced by the inclusion, exists only for $\tau \neq 0$. If $\tau = 0$, $C_1^{(1)} = 0$ and the correlation function is given by $C_1^{(0)}$ which is identically zero for $\Delta q_b \neq \Delta q_a = 0$.

Equation (4) is the main result of the paper. We now compare the $C_1^{(0)}$ correlation function, corresponding to a macroscopically homogeneous slab [Eq. (4)], and the $C_1^{(1)}$ correlation function [Eq. (7)], originating from the presence of a dynamically heterogeneous region (inclusion). As follows from Eq. (4), $C_1^{(0)}$ vanishes identically if $\Delta q_a \neq \Delta q_b$, which is a manifestation of the memory effect. If $\Delta q_a = \Delta q_b$, $C_1^{(0)}$ decays to zero for $\Delta q_a > 1/L$ (see Fig. 3). In contrast, $C_1^{(1)}$ correlation is not necessarily zero for $\Delta q_a \neq \Delta q_b$ (see Fig. 3). The memory effect is still present for the $C_1^{(1)}$ correlation function, as it is peaked near $\Delta q_a = \Delta q_b$ due to $J_1(a |\Delta q_a - \Delta q_b|)/(a |\Delta q_a - \Delta q_b|)$ term in $F_T(\Delta q_a, \Delta q_b, \tau)$. The memory effect for $C_1^{(1)}$ is considerably less sharp than for $C_1^{(0)}$, as one can see from Fig. 3.

![FIG. 3. Normalized angular-temporal cross-correlation function $C_1^{(1)}$ corresponding to the setup of Fig. 1. We assume $\Delta q_a \parallel \Delta q_b$ for this plot. In contrast to $C_1^{(0)}$ (see Fig. 3), $C_1^{(1)}$ is not necessarily zero for $\Delta q_a \neq \Delta q_b$. $C_1^{(1)} \neq 0$ only for $\tau \neq 0$.](image)

![FIG. 4. Normalized angular-temporal correlation functions of transmitted waves for a single plane wave ($\Delta q_a = 0$) incident upon a slab with a cylindrical inclusion inside (solid lines). Dashed lines show the (normalized) angular distribution of the wave field $E$ diffracted by a circular aperture of radius $b$ ($b = 5\Lambda$ and $b = 20\Lambda$ with $\Lambda$ being the wavelength). For $a = 20\ell \gg \Delta$ and $b = 20\Lambda$, correlation and diffraction curves are remarkably close, suggesting that correlation is “diffracted” by the inclusion.](image)
where \( K_\perp \) is the projection of \( K \) onto the plane of the aperture. In the case of \( a \gg \Delta \), which corresponds to a “pill-shaped” inclusion, the correlation function given by our Eq. (2) and the diffraction pattern of Eq. (3) are remarkably close (see, e.g., the curves corresponding to \( a = 20\ell \) and \( b = 20\Lambda \) in Fig. 3). In this case, one can explain the appearance of correlation between the waves scattered in different directions in a macroscopically heterogeneous medium using the classical diffraction theory and assuming \( \lambda = \ell \). As a consequence, some theorems known for diffraction of waves (e.g., the Babinet’s principle), apply directly to the angular-temporal correlation function of light transmitted through a macroscopically heterogeneous turbid medium. It is worthwhile to note that this holds for any shape of inclusion, provided that the transverse extent of inclusion is significantly greater than its extent along the \( z \)-axis (\( a \gg \Delta \) in our notation). For \( a \sim \Delta \), the quantitative agreement between Eqs. (3) and (4) is absent, although their overall behavior is similar (see, e.g., the curves corresponding to \( a = 5\ell \) and \( b = 5\Lambda \) in Fig. 4).

Up to now, our analysis has been devoted to the correlation functions of transmitted light. In experiments, however, it could be more convenient to work with diffusely reflected waves. Calculation of the angular-temporal correlation function of reflected waves is performed similarly to that of transmitted ones. If \(|q_a + q_b|, |q'_a + q'_b|, |q_a + q'_b|, |q'_a + q_b| \gg 1/\ell \), we can ignore the time-reversal symmetry and obtain:

\[
C_1^{(0)}(\Delta q_a, \Delta q_b, \tau) = \delta_{\Delta q_a, \Delta q_b} \frac{3\pi}{k^2 A} \times \frac{\sinh[\beta_a(L - \ell)] \sinh(\beta_a \ell)}{\beta_a \ell \sinh(\beta_a L)},
\]

\( \text{(9)} \)

\[
C_1^{(1)}(\Delta q_a, \Delta q_b, \tau) = -4\pi a_1^2(\tau)^2 \frac{3\pi}{k^2 A} \frac{\pi a^2 \Delta}{A} \times F_R(\Delta q_a, \Delta q_b, \tau),
\]

\( \text{(10)} \)

\[
F_R(p, s, \tau) = \frac{2J_1(a |p - s|)}{a |p - s|} \times \frac{\sinh(\beta_a \ell) \sinh(\beta_b \ell)}{(\beta_a \ell)(\beta_b \ell) \sinh(\beta_a L) \sinh(\beta_b L)} \times \left\{ \frac{\sinh[(\beta_a + \beta_b)\Delta]}{(\beta_a + \beta_b)\Delta} \cosh[(\beta_a + \beta_b)(L - z_0)] - \frac{\sinh[(\beta_a - \beta_b)\Delta]}{(\beta_a - \beta_b)\Delta} \cosh[(\beta_a - \beta_b)(L - z_0)] \right\}.
\]

\( \text{(11)} \)

The \( C_1^{(0)} \) correlation function given by Eq. (3) reduces to the result of Ref. 14 for \( \tau = 0 \), \( \beta_a L \to \infty \) and \( \beta_b \ell \to 0 \). The \( C_1^{(1)} \) term [Eq. (10)] has the same qualitative features as the \( C_1^{(1)} \) correlation of transmitted light [Eq. (6)]. In reflection, however, low-order scattering events become important for \( \Delta q_a, \Delta q_b \sim 1/\ell \), and thus our results (7)–(11) make sense only for \( \Delta q_a \ell, \Delta q_b \ell \ll 1 \). If \(|q_a + q_b|, |q'_a + q'_b|, |q_a + q'_b|, |q'_a + q_b| \) are of order of or smaller than \( 1/\ell \), the time-reversal symmetry of the problem cannot be ignored any more. This significantly complicates calculations even in the case of macroscopically homogeneous medium.

In conclusion, we have calculated and discussed the angular-temporal cross-correlation functions of waves scattered in a turbid, dynamically heterogeneous medium. Our analysis demonstrates that the considered correlation functions can be used to image a hidden dynamic inclusion embedded in an otherwise homogeneous medium. Comparison of our results with diffraction patterns obtained for a wave diffracted by an aperture suggests that the angular-temporal correlation function can be considered as being “diffracted” by inclusion. Such an interpretation is particularly successful for “pill-shaped” inclusions which are much more extended in the transverse directions (i.e., in the directions parallel to the surfaces of the slab where they are embedded) than in the longitudinal one.

1. M.C.W. van Rossum and Th. M. Nieuwenhuizen, Rev. Mod. Phys. 71, 313 (1999); see also E. Akkermans (this volume).
2. A. Lagendijk and B.A. van Tiggelen, Phys. Rep. 270, 143 (1996).
3. A. Yodh and B. Chance, Phys. Today 10, No. 3, 34 (1995).
4. M.A. O’Leary, D.A. Boas, B. Chance, and A.G. Yodh, Phys. Rev. Lett. 69, 2658 (1992).
5. P.N. den Outer, T.M. Nieuwenhuizen, and A. Lagendijk, J. Opt. Soc. Am. A 10, 1209 (1993).
6. G. Maret and P.E. Wolf, Z. Phys. B 65, 409 (1987).
7. D.J. Pine, D.A. Weitz, P.M. Chaikin, and E. Herbolzheimer, Phys. Rev. Lett. 60, 1134 (1988).
8. See G. Maret and M. Heckmeier (this volume) and references therein.
9. U. Frisch, in: Probabilistic Methods in Applied Mathematics, ed. A.T. Bharucha-Reid (Academic, New York, 1968).
10. R. Berkovits and S. Feng, Phys. Rep. 238, 135 (1994).
11. S.E. Skripov, Europhys. Lett. 40, 382 (1997).
12. I. Freund, M. Rosenbluh, S. Feng, Phys. Rev. Lett. 61, 2328 (1988).
13. M. Born and E. Wolf, Principles of Optics (Pergamon Press, Oxford, 1965).
14. L. Wang and S. Feng, Phys. Rev. B 40, 8284 (1989).
15. R. Berkovits and M. Kaveh, Phys. Rev. B 41, 2635 (1990); R. Berkovits, Phys. Rev. B 42, 10750 (1990).