Time parameterization and stationary distributions in a relativistic gas

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In this paper we consider the effect of different time parameterizations on the stationary velocity distribution function for a relativistic gas. We clarify the distinction between two such distributions, namely the Jüttner and the modified Jüttner distributions. Using a recently proposed model of a relativistic gas, we show that the obtained results for the proper-time averaging does not lead to modified Jüttner distribution (as recently conjectured), but introduces only a Lorentz factor $\gamma$ to the well-known Jüttner function which results from observer-time averaging. We obtain results for rest frame as well as moving frame in order to support our claim.

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Introduction—Following a maximum entropy principle, Ferencz Jüttner [1] presented the first relativistic generalization of Maxwell-Boltzmann (MB) distribution in 1911. Although Jüttner distribution has been generally accepted and used in high energy and astrophysics [2, 3, 4, 5, 6, 7, 8], some authors suggested several alternatives [9, 10, 11, 12] that can be summarized in terms of the following $\eta$-parameterized probability distribution functions (PDFs),

$$f_\eta(p) = \frac{\exp(-\beta E)}{Z \eta^\gamma},$$

(1)

in which $E = (m^2 + p^2)^{1/2} = m \gamma(v)$ is the relativistic single-particle energy with $c = 1$, rest mass $m$ and Lorentz factor $\gamma(v) = (1 - v^2)^{1/2}$. $Z$ is the normalization constant and $\beta$ is the temperature parameter. For $\eta = 0$ and $\eta = 1$, the above PDFs reduce to the Jüttner function and the so called modified Jüttner function, respectively.

The lack of rigorous microscopic derivations and experimental evidences, made it difficult to decide which of the proposed relativistic distributions is the correct generalization of MB distribution. To resolve the uncertainty, semi-relativistic [13] and fully relativistic [14, 15] molecular dynamics simulations as well as Monte Carlo studies [16] have been performed by different groups in recent years that unequivocally favored Jüttner distribution. However, some recent investigations on relativistic Brownian motions [17, 18] have revealed that stationary distributions can differ depending on the underlying time-parameterizations, a problem which never arises in Newtonian physics due to the existence of a universal time for all inertial observers. On the other hand, the maximum relative entropy principle (MREP) [19, 20, 21] depicts how symmetry considerations lead to different stationary distributions, each with its own merits [22]. Arguments of this kind suggest that one may possibly seek multiple relativistic generalizations of MB distribution. Using molecular dynamics simulations [23] and theoretical analysis [17], some authors have recently proposed that it is possible to establish a connection between the time-parameter and the kind of symmetry that lead to a special distribution. Consequently, relativistic distributions fall into two classes that are distinguished by their associated symmetry and time-parameter. We, however, believe that the entire issue calls for a more careful reconsideration.

In this article, we will first review how the choice of time parameters or reference measures affect the resulting stationary distribution. We further clarify the resulting consequence of these different choices. Using our recently proposed two dimensional hard disk model [15], we perform simultaneous measurements at fixed observer’s time ‘$t$’ and particle’s proper time ‘$\tau$’ with respect to a rest as well as a moving observer. The obtained results are then compared with stationary distributions consistent with different symmetry considerations. $t$-parametrization is shown to lead to Jüttner distribution, as is well-known. However, we find that $\tau$-parametrization, leads to a PDF that is consistent with a Jüttner function divided by a $\gamma$ term, which is decidedly different from the original modified Jüttner distribution obtained from MREP. This distinction comes more to light when one considers a moving frame.

Stationary distributions and symmetry considerations—The idea of maximizing relative entropy with respect to a pre-specified measure was a major step forward to clarify the underlying mathematical differences of the two mostly cited generalizations of MB distribution, i.e., Jüttner and modified Jüttner functions [22].

Relative entropy [21] characterizes probability distributions with respect to a specific reference density,

$$S[f|\rho] = -\int d^d p f(p) \log \frac{f(p)}{\rho(p)},$$

(2)

in which, $\rho(p) > 0$ plays the role of a reference density. It is then possible [22] to develop a general, coordinate invariant form of maximum entropy principle, under the constraints

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The resulting stationary distribution is then given by,
\[ f(p) = \rho(p) \exp(-\beta E(p)) / Z, \]
Choosing a constant reference density, \( \rho(p) = \rho_0 \), the above equation reduces to the well known Juttner function. This choice correspond to a Lebesgue integrating measure, \( d\mu = dp \), which is the only translational invariant measure in the momentum space \[22\]. Hence, it implies that Juttner distribution is associated with a reference measure that has translational symmetry. Since this kind of symmetry is not relevant in relativistic mechanics, it is interpreted as a momentum conservation law during collisions.

On the other hand, a momentum dependent reference density, \( \rho(p) = 1/E(p) \), leads to the modified Juttner distribution. The integration measure consistent with this choice assigns to any subset \( \mathcal{A} \) in momentum space, the measure number
\[ \chi(\mathcal{A}) = \int_{\mathcal{A}} d\mu = \int_{\mathcal{A}} dp / E(p) \]
Considering the fact that \( dp/E(p) \) is an invariant quantity under Lorentz transformation, the modified Juttner distribution is associated with a reference measure that is invariant under the fundamental symmetry group of relativity.

It should be noted that we have no rigorous theoretical analysis or experimental evidence to favor one reference density/measure to the other. Numerical studies are also not decisive due to the arbitrariness in the choice of time parameters \[17, 23\] or discretization rules \[24, 22, 20\]. Therefore, at this stage we are obliged to accept the arbitrariness in the choice of reference density. However, once the reference density is chosen, other parameters like \( Z \) and \( \beta \) are deterministically obtained using the pre-specified constraints (Eqs. 3 and 4). We will show how misleading conclusions can arise if one fails to appreciate this important point.

Stationary distributions and time parameters-- Discarding the classic notion of universal, absolute time introduces complexity to almost all time-dependent subjects in physics. Among these, are the evolution of dynamical systems towards equilibrium and more specifically their equilibrium probability distribution function (PDF). As reported recently \[17, 18, 23\], the description of the motion of relativistic particles with respect to the observer’s time, \( t \), and the particle’s proper time, \( \tau \) are completely different. To elucidate, consider the \( t \)-averaged PDF of a Brownian particle in terms of the observer’s time
\[ f_t(v) = \frac{1}{t} \int_0^t dt' \delta(v - \mathbf{V}(t')). \]
The velocity of the particle, \( \mathbf{V}(t) \), can also be parameterized in terms of the particle’s proper time denoted by \( \mathbf{V}(\tau) \). The corresponding \( \tau \)-averaged PDF is then defined as
\[ \hat{f}_\tau(v) = \frac{1}{\tau} \int_0^\tau d\tau' \delta(v - \mathbf{V}(\tau')). \]

By simply changing the integration variable, it can be shown \[22\] that the relation between stationary \((t, \tau \to \infty)\) distributions is
\[ \hat{f}_\infty(v) \propto \frac{f_\infty(v)}{\gamma(v)} \]
The appearance of the factor \( \gamma(v) \), in the denominator of Eq. (9) suggests that the \( \tau \)-averaged distribution is a modified Juttner, if the \( t \)-averaged distribution turns out to be Juttner function \[17, 23\]. Although the similarity is very suggestive, in this paper we show that a Juttner function which is simply divided by \( E(p) = m\gamma(v) \) is not necessarily equivalent to a modified Juttner function.

In order to distinguish the correct interpretation of Eq. (9), we will check the validity of this equation by comparing the numerically obtained equilibrium distributions with respect to different time parameters, in the rest frame. Next, the results are further discussed by considering the problem from the viewpoint of a moving observer. In the light of these numerical simulations, one can decide whether the \( \tau \)-averaged distribution is truly described by a modified Juttner function.

The model we have used here is the previously proposed two dimensional relativistic hard-disk gas \[13\]. In this model, the disk-like particles move in straight lines at constant speed and change their momenta instantaneously when they touch at distance \( \sigma \). The exchange of energy and momenta is governed by the relativistic energy-momentum conservation laws. In our simulation we have used \( N \) particles of equal rest masses \( m \) that are constrained to move in a square box of linear size \( L \) with periodic boundary condition. In order to simulate a stationary system in the rest frame, the center-of-mass momentum is put to zero manually. This condition would automatically be satisfied (if not at each instant but at least on average) if fixed reflecting walls were used \[27\]. The density is chosen to correspond to a dilute gas.

In the next two sections we obtain \( t \)-averaged as well as \( \tau \)-averaged PDFs directly from simulations of such a model and compare them with the various proposed PDFs in order to clarify the relevance of various PDFs.

Rest frame-- To obtain the \( t \)-averaged PDF we let the system equilibrate (typically after \( 10^2 N \) collisions) and
simultaneously measure velocities of all particles at a given instant of time $t$. To obtain the $\tau$-averaged PDF the proper-time of each particle, $\tau_i (i = 1, ..., N)$, is computed during the simulation and velocities are recorded at a fixed proper-time value, $\tau_i = \tau$. That is, velocities are measured when the particles have the same lifetime. To collect more data in the former (latter) scenario, we may either repeat the procedure for different initial conditions or perform measurements at several equally-separated time (proper time) instances.

Simulation results for $N = 100$ particles are presented in Fig.1. Particles are initially placed on a square lattice of constant $L/\sqrt{N}$ and velocities are chosen randomly and the mean energy per particle is $\varepsilon = 3.06m$. Velocities are all measured with respect to a rest observer at $2 \times 10^7$ instants using time intervals $10^9T$, where $T$ is the system’s mean free time. As shown in Fig.1, the $t$-averaged (+) and $\tau$-averaged (●) PDFs agree well, respectively, with Jüttner function with $\beta = 7.62$ (dotted red line) and Jüttner function with the same $\beta$, divided by $\gamma$ (dashed blue line). This result confirms the relation described in equation (9). The more important question, however, is the correct interpretation of the right hand side of this equation.

Considering the fact that the modified Jüttner function differs from Jüttner function by a factor $1/\gamma$, one may conclude that $\tau$-averaged PDFs are described by modified Jüttner function in the same manner that $t$-averaged PDFs are depicted to be Jüttner function [23]. Before drawing such a conclusion, we note that by substituting modified Jüttner function in Eq. (4), one obtains a relation between the temperature parameter, $\beta$, and the conserved quantities of the system ($\epsilon, m$), which is different from the one obtained if one substitutes Jüttner function instead. For the two dimensional momentum space we have:

$$\epsilon = \frac{2}{\beta_J} + \frac{m^2 \beta_J}{1 + m \beta_J} \quad (10)$$

$$\epsilon = m + \frac{1}{\beta_{MJ}}, \quad (11)$$

where the indexes $J$ and $MJ$ refers to Jüttner and modified Jüttner, respectively. In our simulation, this leads to $\beta_J = 7.62$ and $\beta_{MJ} = 4.86$. Therefore, simply dividing a Jüttner function by $\gamma$ does not lead to a modified Jüttner function. The functional form as well as the temperature parameter are important in distinguishing a modified Jüttner function from its counterpart. This point is clearly demonstrated in Fig.1 as the obtained PDF from $\tau$-averaging (●) does not fit well with a modified Jüttner function (solid green line) but fits well with a Jüttner function divided by $\gamma$ (dashed blue line).

Note that the difference of temperature parameters of Jüttner and modified Jüttner distributions in the above argument is a result of using same expectation values in the energy constraints in maximum relative entropy principle. It may be argued that this assumption is incorrect due to the fact that the two distributions refer to two different hyper surfaces in space-time [23]. Undoubtedly, a suitable choice of energy expectation value in Eq.(11) gives the correct temperature parameter consistent with simulation data (i.e., $\beta = \beta_J$). The important question, however, is how should one obtain the new energy expectation value, consistent with Lorentz invariant measure, based on fundamental laws of relativity or statistical mechanics? As far as maximum entropy principles are considered, the relevant constraints should be specified with respect to the accessible information. Does changing the reference measure introduce new information that must be adapted as a constraint?

At this stage we make no effort to answer these open questions and accept (at least with regard to numerical data) that the temperature parameter of both distribution equals $\beta_J$. By this choice we still have two scenarios, first the $\tau$-averaged PDF is described by a modified Jüttner with $\beta = \beta_J$ and second, it is a rescaled Jüttner function. These two are indistinguishable in the rest frame. However, because of different transformational properties of Jüttner and modified Jüttner functions under Lorentz boost, they take on distinctly different functional forms. We next consider our model in a moving frame in order to bias this distinction.

Moving frame—To this end, we examine the system for an observer who is moving with a uniform velocity $u$ in the negative $x$-direction with respect to the rest frame. Using the relative entropy maximization principle, the stationary distribution will be determined by an additional constraint on the system, namely, that of a definite total momentum $P'$ [23]. We therefore obtain:

$$f_J'(p') = \frac{1}{\gamma(u)Z_J} \exp[-\beta_J \gamma(u)(E' - u.p')] \quad (12)$$
efficients is det($a_\mu$) described in Fig. 1 with relative velocity $u = 0.1$. The system parameters are the same as Fig. 1. In particular, $\beta_J = 7.62$ and $\beta_{M,J} = 4.86$. Part (a) shows the $x$ component and (b) shows the $y$ component distributions. The solid lines associated with modified J"uttner function with either $\beta_J$ and $\beta_{M,J}$ do not fit the data well, while J"uttner function divided by $E'$ with $\beta_J$ fits perfectly well.

\[
f_{M,J}'(p') = \frac{1}{\gamma(u)Z_{M,J}} \frac{\exp[-\beta_{M,J} \gamma(u)(E' - u.p')]}{\gamma(u)(E' - u.p')}, \quad (13)
\]

The primed quantities are measured in the moving frame and the additional $\gamma(u)$ term in the denominator, is due to the contraction of the moving box that encloses the system [23]. Note that the reference density in the moving frame is obtained by $\rho'(p') = \mathcal{J} \rho(p)$, in which $\mathcal{J} = [\partial p^\mu / \partial p'^\nu] = \det(a'^\mu_\nu)$ is the Jacobian of the coordinate transformation, $p'^\mu = a'^\mu_\nu p^\nu$. Here, $\mathcal{J}$ equals unity since for all proper Lorentz transformations, like boost and rotation, the determinant of the transformation coefficients is $\det(a'^\mu_\nu) = 1$ [30].

Figure 2 shows simulation results for a system described in Fig. 1 with $u = 0.1$. Here, $t$-averaged PDF is obtained by measuring velocities simultaneously with respect to the moving observer as described in [14, 15]. Considering the Lorentz invariance of proper-time it is not difficult to find the $\gamma$-averaged PDF in the moving frame. In this case, the appropriate instant of measurement is the same as that of the rest frame, however, the velocities must be recorded as seen by the moving observer. To provide a better understanding of the effect of motion on PDFs we have shown the $x(y)$ component of velocities separately. The dotted (red) and solid (green and pink) lines are the $x(y)$ component of velocity distribution obtained, respectively, by integrating Eqs. (12) and (13) over $v_y (v_x)$. We have also used the fact that the temperature parameters in the moving frame are the same as that of the rest frame [14, 15]. As clearly seen from numerical results, our two dimensional model shows that $t$-averaged PDF (+) is fitted to the expected $x(y)$ component of J"uttner function (dotted red line). The behavior of $\gamma$-averaged PDF (•), however, shows an evident deviation from the modified J"uttner function (solid green and pink lines). This result shows that no form of modified J"uttner function can fit the numeric data obtained by proper time averaging. But how is a $\gamma$-averaged PDF described in the moving frame?

According to the arguments presented in the previous section, we consider a J"uttner function in the moving frame divided by $E'$, i.e.,

\[
f_{M,J}'(p') = \frac{1}{\gamma(u)Z} \exp[-\beta_J \gamma(u)(E' - u.p')] / E', \quad (14)
\]

where $Z$ is the normalization constant. As shown in Fig. 2, the $x(y)$ component of the above distribution (dashed blue line), fits to simulation data as expected. These results again confirm that $\gamma$-averaged PDFs are described by J"uttner functions which are simply divided by $E$. The similarity of such functional form to the modified J"uttner function, especially in the rest frame, has recently led to misleading conclusions [14, 23].

**Concluding remarks**—In the light of the above discussions one might now ask what is the correspondence between symmetries that lead to different choice of reference density in relative entropy with the symmetries that lead to the choice of time parameterization? In [23] it is argued that the choice of coordinate time, $t$, and constant reference density results in a J"uttner distribution while a relativistically invariant measure along with a relativistically invariant time (i.e., proper time $\gamma$) leads to a modified J"uttner distribution. We, however, believe that the connection between various reference densities and time parameterization has not been well-established yet. Certainly a MREP should directly and uniquely lead to the equilibrium properties (i.e., PDF) once the reference density and constraints have been specified. To the best of our knowledge, there is no proof that choice of coordinate time leads to a constant reference density, just as no proof exists that choice of proper time leads to $\rho = 1/E$. Here, we have shown that if one calls what results from choosing different $\rho$'s in the MREP as J"uttner and modified J"uttner functions, the “modification” that results (due to the choice of $\rho$) in this method is different from the modification that results from a mere reparameterization of time (which is in accordance with Eq.(9)).

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