Thermodynamics of Schwarzschild-like black holes in bumblebee gravity models

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Abstract

Over the last decades, many methods were developed to prove Hawking radiation. Recently, a semiclassical method known as tunneling method, has been proposed as a simpler way of deriving black hole thermodynamical properties. This method has been widely applied to a huge sort of spacetimes with satisfactory results. In this work, we obtain the black hole thermodynamics in the presence of a Lorentz symmetry breaking (LSB). We apply the Hamilton-Jacobi method to Schwarzschild-like black holes and we investigate whether the LSB affects their thermodynamics. The results found show that the LSB changes the Schwarzschild black hole temperature, entropy and heat capacity by perturbative terms of the LSB parameter.

Keywords: Black Holes Thermodynamics, Quantum Tunneling Method, Lorentz Symmetry Breaking
I. INTRODUCTION

Although a black hole is classically defined as an object that can only absorb radiation, when it comes to a quantum mechanical approach it actually emits radiation. This intriguing result was shown by Hawking in 1975 using quantum field theory in curved space-time [1]. After that, black holes assumed an important role in the attempt of constructing a quantum theory of gravity.

The concept of the so-called Hawking radiation, puts the thermodynamical description of black holes in a more realistic basis. As a matter of fact, black hole mechanical properties, were earlier discussed by Bardeen, Carter and Hawking (1973) [2], where they formulated the four laws of black hole mechanics, after the ideas of Bekenstein (1972) [3]. These laws were proposed due to the close similarity of some mechanical properties of black hole to their entropy and temperature. Nowadays, black holes can be considered as thermal systems and this set of laws are called the four laws of black hole thermodynamics.

Besides the Hawking original derivation, many methods were developed in order to derive the black hole thermodynamical properties [4–8], including the so-called tunneling method [9, 10]. The tunneling method is a semiclassical approach and its basic idea is to interpretate the Hawking radiation as a process of quantum mechanical emission through the black hole horizon. This emission occurs due to spontaneous creation of particles just inside the black hole horizon. One of the particles tunnels the horizon towards the infinity emerging with positive energy, while the other one with negative energy remains inside the hole and contribute to the mass loss of the black hole. In this sense, it is possible to derive the tunneling probability and associate it with the black hole temperature.

The main advantages of using the tunneling method are: it is very useful to incorporate back-reaction effects since it provides a dynamical model of the radiation process; it is related only with the geometry of the spacetime, so it can be applied to a wide variety of spacetimes [11, 14]; the calculations involved are straightforward.

The tunneling process can be obtained by two ways, namely, the null geodesic method proposed by Parikh and Wilczek [9], and the Hamilton-Jacobi ansatz originated from the work of Padmanabhan et al. [15, 16] and used by Angheben et al [12]. However, the two approaches are basically equivalent in many cases. The Hamilton-Jacobi method is more direct and for this reason is the one used in this work [13]. In the Hamilton-Jacobi method
we use the so-called WKB approximation in order to obtain the imaginary part of the action. The self-gravitation effects of the particle are not considered.

In this work, we study the thermodynamical properties of Schwarzschild-like black hole in scenarios in which the so-called Lorentz symmetry breaking (LSB) occurs.

A recent example of LSB applied to modify black hole thermodynamics is found in the work of Li et al. [17], where the authors modified the thermodynamical quantities directly. In the present work, the tunneling method is applied to LSB modified space-times and we analyse how the LSB modifies the black hole thermodynamics.

This paper is organized as follows. In Sec. II we briefly review the LSB in the context of bumblebee gravity. In Sec. III the quantum tunneling method is also reviewed and its equivalence to Hawking’s method is confirmed. Then, we present our results for thermodynamical functions of the Schwarzchild black hole in two different kinds of bumblebee model in Sec. IV and Sec. V. The difference between both approaches is the metric proposed. For the sake of simplicity we name B-metric the proposal of Bertolami and Paramos [18], and we call C-metric the proposal of Casana et al. [19]. We also compare our results in the two scenarios in Sec. VI. Finally, we present further discussions and our conclusions in Sec. VII.

II. LORENTZ SYMMETRY BREAKING IN A BUMBLEBEE GRAVITY MODEL

In bumblebee models, the Lorentz Symmetry Breaking (LSB) occurs spontaneously due the presence of dynamical terms of the vectorial field $B_\mu$, which is known as bumblebee field. The bumblebee action is given by [20]

\[ S = \int d^4x \mathcal{L}_B, \]

where the bumblebee Lagrangian $\mathcal{L}$ is

\[ \mathcal{L}_B = \mathcal{L}_g + \mathcal{L}_{gB} + \mathcal{L}_K + \mathcal{L}_V + \mathcal{L}_J. \]

The terms in (2) are, respectively: the gravitational Lagrangian, the gravity-bumblebee coupling Lagrangian, the dynamical Lagrangian of the field $B_\mu$, the Lagrangian which contains a potential $V$ responsible for the LSB and the Lagrangian which contains the interaction terms of $B_\mu$ to the matter or other sections of the model.
The dynamical Lagrangian $\mathcal{L}_K$ contains the kinetic terms of $B_\mu$, expressed through the field strength for $B_\mu$, namely
\[
B_{\mu\nu} = D_\mu B_\nu - D_\nu B_\mu, \tag{3}
\]
where $D_\mu$ is the covariant derivative, defined according to the spacetime curvature.

The potential contained in $\mathcal{L}_V$ is responsible for the LSB and it is chosen to have a minimum in the bumblebee Vacuum Expectation Value (VEV), denoted by $b_\mu$. For this reason, we usually refer to $b_\mu$ as the LSB parameter. Thus, it must have the following functional form:
\[
V = V(B^\mu B_\mu \pm b^2), \tag{4}
\]
where $b^2$, the norm squared of $b_\mu$, is a positive constant. The signs in (4) determine whether the VEV $b_\mu$ is timelike or spacelike. If $V$ has a polynomial form, like $V(x) = \lambda x^2/2$, the constant $\lambda$ is a coupling constant. The potential $V$ can also have the form $V(x) = \lambda x$ in which case $\lambda$ is a Lagrangian-multiplier field, responsible for the constraint $B^\mu B_\mu \pm b^2$; this form is interesting for sigma models, since it preserves only the Nambu-Goldstone modes \[20\].

Since we focus in vacuum solutions, we are permitted to set matter-bumblebee coupling term as $\mathcal{L}_J = 0$. Therefore, the bumblebee action is given by
\[
S_B = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa}(R + \xi B^\mu B_\nu R_{\mu\nu}) - \frac{1}{4}B_{\mu\nu}B_{\mu\nu} - V(B^\mu B_\mu \pm b^2) \right], \tag{5}
\]
where $\kappa = 8\pi G$. The potential form is not relevant, since it is assumed that the bumblebee field is fixed at its nonzero VEV, which makes $V = 0$ and $V' = 0$. In particular, the VEV $b_\mu$ is assumed to take the form $b_\mu = (0, b(r), 0, 0)$, with $b^2 = b^\mu b_\mu = \text{const.}$, which means that the LSB is purely radial. The spacetime is considered to have no torsion, in such a way that the bumblebee field strength is given by $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. It is also considered that the spacetime is static and has spherical symmetry; thus, we can describe it with a Birkhoff metric $g_{\mu\nu} = \text{diag}(-e^{2\phi}, e^{2\rho}, r^2, r^2 \sin^2 \theta)$, where $\phi$ and $\rho$ are function of $r$.

In this work, we will use the black hole solutions in LSB scenarios presented in Ref. \[18, 19\] and find their corresponding thermodynamical properties. In Ref. \[18\], Bertolami and Paramos imposed the condition $\nabla_\mu b_\mu = 0$ in place of the usual prescription $\partial_\mu b_\mu = 0$. By the other hand, Casana et al. \[19\] proposed that $b(r)$ has the explicit form $|b| e^\rho$, which yields to $\nabla_\mu b_\mu \neq 0$. As a result, the equation of motion found in \[19\] leads to the Schwarzschild-like...
metric
\[ ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + (1 + \ell) \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\Omega^2, \]
where \( \ell = \xi b^2 \), \( M = G_N m \) is the geometrical mass and \( d\Omega^2 \) is the solid angle.

We can notice that, as in the case with no LSB, we have the singularities \( r = 2M \) and \( r = 0 \). We have to find out whether this singularities are physical or not. Thus, we have to calculate the Kretschmann scalar given by \( K = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \) for the metric (6), which is given as
\[ K_C = \frac{4(12M^2 + 4\ell Mr + \ell^2 r^2)}{r^6(\ell + 1)^2} = \frac{K_S}{(\ell + 1)^2} + \frac{4(4\ell Mr + \ell^2 r^2)}{r^6(\ell + 1)^2}. \]
where \( K_S = 48M^2r^{-6} \) is the usual Kretschmann scalar for the Schwarzschild black hole with no LSB. We can notice from (7) that, for \( r = r_S = 2M \), the Kretschmann scalar is finite, which implies that this singularity is removable. For \( r = 0 \), however, the Kretschmann scalar is infinite, which implies that this is a physical singularity. Therefore, the nature of the singularities is not modified by this radial LSB.

As it was said before, Bertolami and Paramos impose \( \nabla_\mu b_\mu = 0 \), which yields, after some calculation, the following Schwarzschild-like metric
\[ ds^2 = - \left( 1 - \frac{2M}{r} \frac{r^L}{r_0^L} \right) dt^2 + \left( 1 - \frac{2M}{r} \frac{r^L}{r_0^L} \right)^{-1} dr^2 + r^2 d\Omega^2, \]
where \( L = \xi b^2 / 2 \), \( M = G_L m \) is the geometrical mass, \( r_0 \) is an arbitrary distance and \( d\Omega^2 \) is the solid angle.

The event horizon is given by the condition \( g_{00} = 0 \), i.e.,
\[ 1 - \frac{2M}{r} \frac{r^L}{r_0^L} = 0, \]
which gives \( r_B = (2Mr_0^{-L})^{1/(1-L)} \). The corresponding Kretschmann scalar is given by
\[ K_B = 48 \left[ 1 - \frac{5}{3} L + \frac{17}{12} L^2 - \frac{1}{2} L^3 + \frac{1}{12} L^4 \right] M^2 \left( \frac{r}{r_0} \right)^{2L} r^{-6} \approx \left( 1 - \frac{5}{3} L \right) \left( \frac{r}{r_0} \right)^{2L} K_S. \]
It can be seen from (10) that \( K(r = r_B) \) is finite and, therefore, this singularity is removable. Since \( b \) is very small, \( r^{2L-6} \to \infty \) as \( r \to 0 \), which means that the singularity \( r = 0 \) is intrinsic as in the usual case. We can notice that, unless \( r_0 = 2M \), the nature of the singularities for the metric (9) is modified.

We can now verify whether the radial LSB modifies the Schwarzschild black hole thermodynamics. For this purpose, we will use the quantum tunneling formalism.
III. QUANTUM TUNNELING METHOD

Is this section, for illustrative purposes, we review the tunnelling approach \[21, 22\] to derive the black hole thermodynamical properties. Near the black hole horizon, we have only the temporal and the radial terms of metric, since the angular part is red-shifted away. The metric becomes 2-dimensional and can be rewritten as

\[ ds^2 = -f(r)dt^2 + g(r)^{-1}dr^2. \] (11)

The Klein-Gordon equation for the field \( \phi \) is given by

\[ \hbar^2 g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - m^2 \phi = 0. \] (12)

The last equation with aid of Eq. (11) leads to

\[- \partial_t^2 \phi + \Lambda(r) \partial_r^2 \phi + \frac{1}{2} \partial_r \Lambda(r) \partial_r \phi - \frac{m^2}{\hbar^2} f(r) \phi = 0, \] (13)

where we have defined

\[ \Lambda(r) \equiv f(r)g(r). \] (14)

Using the so called WKB method \[23\], we have the following solution for Eq. (13)

\[ \phi(t, r) = \exp \left[ -\frac{i}{\hbar} \mathcal{I}(t, r) \right]. \] (15)

For the lowest order in \( \hbar \), we have

\[ (\partial_t \mathcal{I})^2 - \Lambda(r)(\partial_r \mathcal{I})^2 - m^2 f(r) = 0, \] (16)

with

\[ \mathcal{I}(t, r) = -\omega t + W(r), \] (17)

as solution. The explicit form for \( W(r) \) is

\[ W(r) = \int \frac{dr}{\sqrt{f(r)g(r)}} \sqrt{\omega^2 - m^2 f(r)}. \] (18)

Now we take the approximation of the functions \( f(r) \) and \( g(r) \) near the event horizon \( r_+ \),

\[ f(r) = f(r_+) + f'(r_+)(r - r_+) + \cdots, \] (19)
\[ g(r) = g(r_+) + g'(r_+)(r - r_+) + \cdots. \] (20)

The Eq. (18) then becomes

\[ W(r) = \int \frac{dr}{\sqrt{f'(r_+)g'(r_+)}} \sqrt{\omega^2 - m^2f'(r_+)(r - r_+)}, \] (21)

where the prime denotes derivative with respect to the radial coordinate.

The last integral can be made using the residue theorem, which results in

\[ W = \frac{2\pi i\omega}{\sqrt{f'(r_+)g'(r_+)}}. \] (22)

The particles tunneling rate is given by \( \Gamma \approx \exp \left[-\frac{2}{\hbar} Im \mathcal{I}\right] \), so

\[ \Gamma \approx \exp \left[-2Im \mathcal{I}\right] = \exp \left[-\frac{4\pi \omega}{\sqrt{f'(r_+)g'(r_+)}}\right] \] (23)

Comparing Eq. (23) with the Boltzmann factor, namely \( e^{-\omega/T} \), we can obtain the Bekenstein-Hawking temperature, which is given by

\[ T_{\text{BH}} = \frac{\omega}{2Im \mathcal{I}} = \frac{\sqrt{f'(r_+)g'(r_+)}}{4\pi}. \] (24)

From the temperature above, we can obtain the black hole entropy by making use of the thermodynamical relation \( TdS = dM \), which yields to

\[ S_{\text{BH}} = \int \frac{dM}{T(M)}. \] (25)

We can now test whether the tunneling method gives the same results as the Hawking’s method. For this purpose, we apply the Hamilton-Jacobi method for a Schwarzschild black hole. In this case, the metric is given by

\[ ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2, \] (26)

where \( f(r) = 1 - \frac{2M}{r} \). Using in Eq. (24) the metric given in Eq. (26), we can obtain \( T = (8\pi M)^{-1} \) and \( S = 4\pi M^2 \), which are the same results found by Hawking [1].

Hawking’s original derivation gives us

\[ T = \frac{\kappa}{2\pi}, \] (27)
where $\kappa$ is the surface gravity, given by

$$k^\mu \nabla_\mu k_\nu = \kappa k_\nu,$$  \hspace{1cm} (28)

with $k_\nu$ being the Killing vector. For a metric given by (11), we have $k_\nu = (1, 0, 0)$ and

$$\kappa = \frac{\sqrt{f'(r_+)g'(r_+)} }{2},$$  \hspace{1cm} (29)

which makes clear that Hawking’s method is equivalent to quantum tunneling method.

IV. THERMODYNAMICAL PROPERTIES FOR THE BUMBLEBEE MODEL WITH C-METRIC

We first consider the purely radial LSB metric obtained in Ref. [19], namely

$$ds^2 = -f(r)dt^2 + g(r)^{-1}dr^2 + d\Omega^2,$$  \hspace{1cm} (30)

where

$$f(r) = 1 - \frac{2M}{r},$$  \hspace{1cm} (31)

$$g(r) = (1 + \ell)^{-1} \left( 1 - \frac{2M}{r} \right).$$  \hspace{1cm} (32)

The surface gravity is given by

$$\kappa_C = \frac{\kappa_0}{\sqrt{1 + \ell}},$$  \hspace{1cm} (33)

where $\kappa_0 = 1/4M$ is the Schwarzschild surface gravity, while the temperature is given by

$$T_C = \frac{\sqrt{f'(rs)g'(rs)}}{4\pi} = \frac{1}{\sqrt{\ell + 1}} \frac{1}{8\pi M} = \frac{T_0}{\sqrt{1 + \ell}},$$  \hspace{1cm} (34)

where $T_0 = 1/8\pi M$ is the Hawking temperature for a typical Schwarzschild black hole.

Assuming the approximation $\ell << 1$, the temperature becomes

$$T_C \approx \frac{1}{8\pi M} - \frac{\ell}{16\pi M} = \left( 1 - \frac{\ell}{2} \right) T_0$$  \hspace{1cm} (35)

From the right side of Eq. (35), we can notice that the LSB modified metric given in Eq. (6) contributes to decrease the temperature of the Schwarzschild black hole.

The entropy is then given by

$$S_C = \int \frac{dM}{T(M)} = 4\pi M^2 \sqrt{1 + \ell} = \sqrt{1 + \ell} S_0,$$  \hspace{1cm} (36)
where we used $T$ as given in Eq. (34), $r_C = 2M$ and the Schwarzschild black hole entropy given by $S_0 = A/4 = 4\pi M^2$, which is provided by the area law. This means that, while the black hole temperature obtained by the metric given in Eq. (6) is smaller than the Schwarzschild one, its entropy is bigger than the usual Schwarzschild entropy. On the other hand, since the black hole surface area is given by $A = 4\pi r_+^2 \sqrt{\ell + 1}$, we can obtain, by using the area law, the same result obtained by the tunneling method.

At this point, it is interesting to calculate the heat capacity for metric (6) in order to verify under what conditions we have instability. In this case, we will consider the heat capacity at constant volume \[24\], namely

$$C_C = T \frac{\partial S}{\partial T} = T \left( \frac{\partial T}{\partial r} \right)^{-1} = \sqrt{1 + \ell C_0}, \quad (37)$$

where $C_0 = -8\pi M^2$ is the usual Schwarzschild heat capacity at constant volume. In spite of the term $\sqrt{1 + \ell}$, the heat capacity for the black hole with C-metric has the same features as the Schwarzschild one. For example, the negative sign indicates that the LSB does not modify the Schwarzschild black hole instability. This occurs due the fact that the Schwarzschild temperature decreases as the black hole absorbs mass \[25\].

Using the same approximation as in Eq. (35), we can find

$$S_C = \left( 1 + \frac{\ell}{2} \right) S_0,$$

$$C_C = \left( 1 + \frac{\ell}{2} \right) C_0, \quad (38)$$

which will be used later.

V. THERMODYNAMICAL PROPERTIES FOR THE BUMBLEBEE MODEL WITH B-METRIC

Now we consider the purely radial LSB metric obtained by Bertolami and Páramos \[18\], given by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + d\Omega^2, \quad (39)$$

where

$$f(r) = 1 - \frac{2M}{r} \frac{r^L}{r_0^L}. \quad (40)$$
By performing the same calculation as before, we can determine the temperature and the entropy for this system. As the event horizon radius is now $r_B = \left(2Mr_0^{-L}\right)^{1/1-L}$, where $r_0$ is a arbitrary distance, we have

$$T_B = \frac{\sqrt{f'(r_S)g'(r_S)}}{4\pi} = \left(1 - L\right)\left(2Mr_0^{-1}\right)^{-L/(1-L)}T_0.$$  \hspace{1cm} (41)

This result is the same that Bertolami and Páramos obtained by calculating the surface gravity, unless the factor $(1 - L)$. Here, we must point out that Bertolami’s derivation was not the most accurate, since it used the Schwarzschild surface gravity without LSB. However, a more careful derivation should consider the surface gravity of B-metric, which is given by

$$\kappa_B = (1 - L)\left(2Mr_0^{-1}\right)^{-L/(1-L)}\kappa_0,$$  \hspace{1cm} (42)

where $\kappa_0 = 1/4M$ is the Schwarzschild surface gravity. Again, Hawking’s method is in agreement with tunneling method.

The next step is to calculate the entropy of this system. We can write the mass dependence of the temperature as $M^{-1/1-L}$. Therefore,

$$S_B = \int \frac{dM}{T(M)} = \frac{2\left(2Mr_0^{-1}\right)^{L/1-L}}{2 - L}S_0,$$  \hspace{1cm} (43)

where $S_0$ is the usual entropy for the Schwarzschild black hole. Using the area law, we find

$$S'_B = \left(2Mr_0^{-1}\right)^{\frac{2L}{1-L}}S_0,$$  \hspace{1cm} (44)

which differs from the quantum tunneling result. To solve this problem, we need to modify the first law in the following way \cite{26, 27}

$$T_BdS_B = F(M, r_0, L)dM,$$  \hspace{1cm} (45)

where $F$ is a function to be determined and we have chosen it to be function of $M, r_0$ and $L$ for convenience. In order to determine $F$ we can notice that

$$F(M, r_0, L) = T_B \frac{dS'_B}{dM}.$$  \hspace{1cm} (46)

We can substitute Eq. (41) and (44) in the last equation and then find

$$F(M, r_0, L) = \left(2Mr_0^{-1}\right)^{\frac{L}{1-L}}.$$  \hspace{1cm} (47)
Thus, if we consider the temperature obtained by the tunneling method and modification in the first law (Eq. (45)), we can obtain the entropy given by (44), which we assumed to be the correct one.

We now calculate the heat capacity for the bumblebee model with B-metric. To do this, we should rewrite the temperature and the entropy in terms of the radius $r_B = (2Mr_0^{-L})^{1/1-L}$ such that
\begin{align*}
T_B &= \frac{1 - L}{4\pi} r_B^{-1}, \\
S_B &= \pi r_B^2.
\end{align*}

Therefore, the heat capacity for the bumblebee model with B-metric is given by
\begin{equation}
C_B = T_B \frac{\partial S_B}{\partial r_B} \left( \frac{\partial T_B}{\partial r_B} \right)^{-1} = (2Mr_0^{-1})^{2L/\pi} C_0.
\end{equation}

We can see that, as in the case of C-metric, the radial LSB does not modify the main features of the Schwarzschild heat capacity.

Since $L \ll 1$, the temperature becomes
\begin{equation}
T_B \approx \left\{ 1 - L[\ln(2Mr_0^{-1}) + 1] \right\} T_0,
\end{equation}
while the entropy and the heat capacity become, respectively,
\begin{equation}
S'_B \approx [1 + 2L \ln(2Mr_0^{-1})] S_0,
\end{equation}
and
\begin{equation}
C_B \approx [1 + 2L \ln(2Mr_0^{-1})] C_0.
\end{equation}

We can notice that, due the presence of the logarithmic term in the three expressions above, it is interesting to verify three different cases for the value of the arbitrary distance:

- $r_0 < 2M$: In this case, we have
\begin{equation}
r_B = (2Mr_0^{-L})^{1/(1-L)} > 2M > r_0.
\end{equation}

The arbitrary distance $r_0$ can be interpreted as the distance from the source for which the LSB effects are detected as due a Yukawa potential [18]. This distance should be greater than the event horizon radius, which does not occur in the present case.
• $r_0 = 2M$: In this case, the arbitrary radius is equal to the event horizon radius $r_B = r_0 = 2M$. Then, we have $\ln(2Mr_0^{-1}) = 0$ and

\[
T_B = (1 - L)T_0, \quad (55)
\]

\[
S_B' \approx S_0, \quad (56)
\]

\[
C_B \approx C_0. \quad (57)
\]

We can see that the temperature is greater than the usual Schwarzschild temperature. The entropy and the heat capacity, however, are approximately equal to their Schwarzschild correspondents and the horizon radius is equal to the Schwarzschild radius. In this way, the Lorentz symmetry is partially recovered in the limit $r_0 \to 2M$, but we still have some LSB remnants.

• $r_0 > 2M$: Now, we have the arbitrary radius $r_0$ is greater than the event horizon radius

\[
r_B = (2Mr_0^{-1-L})^{1/(1-L)} < 2M < r_0. \quad (58)
\]

In this case, $\ln(2Mr_0^{-1}) < 0$, but this condition is not enough for determine whether the temperature and entropy are greater or smaller than the usual ones. However, if we suppose that $2Mr_0^{-1} < e^{-1}$, or $r_0 > e(2M)$, where $e = 2.718$ is the Euler number, we have $\ln(2Mr_0^{-1}) + 1 < 0$. Thus,

\[
T_B > T_0. \quad (59)
\]

In this case, the LSB increases the black hole temperature. When we have $r_0 \to e(2M)$ the temperature $T_B$ approaches the Schwarzschild temperature. On the other hand, the entropy and the heat capacity become, as $r_0 \to e(2M)$,

\[
S_B' \approx (1 - 2L)S_0, \quad (60)
\]

and

\[
C_B \approx (1 - 2L)C_0. \quad (61)
\]

Under these conditions, we can see that the black hole entropy and heat capacity are decreased by the LSB even though the temperature is approximately equal to the Schwarzschild one. This case is similar to the last one in the sense that some of the usual results are recovered but we still have LSB remnants.
VI. COMPARISON BETWEEN THE TWO MODELS

Although the two approaches used in this work are different [18, 19], we should point out some similarities in their respective thermodynamical properties. In the last section, we analyzed the behaviour of the temperature, entropy and heat capacity with B-metric of the bumblebee model for different values of the arbitrary distance $r_0$. The case $r_0 = 2M$ is of special interest since the temperature found is very similar to the one obtained in Sec. IV.

On the other hand, when we use the C-metric in the bumblebee model, we have $\ell = \xi b^2$ with $b_r(r) = |b|e^\rho$ and thus, $b^2 = const$. In turn, with B-metric, we have $L = \xi \tilde{b}^2 / 2$ where $\tilde{b}_r = \xi^{-1/2} b_0 e^\rho$ and, therefore, $\tilde{b}^2 = b_0^2 \xi^{-1}$. If we could make the correspondence $|b| = \xi^{-1/2} b_0$, then we would have $\frac{\ell}{2} = L$. This would allow us to say that the temperatures in both cases are the same:

$$T_C = \left(1 - \frac{\ell}{2}\right) T_0 \Leftrightarrow T_B = (1 - L) T_0.$$  

We can see in Fig. 1 the temperature behaviour for the different cases investigated in the last two sections. Figure 2 is specially interesting because it shows exactly what is expressed in eq (62).

The correspondence made above is not valid for the heat capacity and it is not an identity for the entropy:

$$S_C = \left(1 + \frac{\ell}{2}\right) S_0 \Leftrightarrow S_B' \approx S_0.$$  

We can also achieve these conclusions from Fig. 3.
The heat capacities $C_C/C_0$ and $C_B/C_0$ behave as the entropies. From Figs. 3 and 4, we can see that, the entropy (heat capacity) for B-metric is not bigger than the Schwarzschild entropy (heat capacity). However, the entropy and heat capacity for C-metric is bigger than the cases without LSB. We can also notice that, the thermodynamical properties studied depend linearly on the LSB, when this parameter is small.

VII. DISCUSSIONS AND CONCLUSIONS

In this work we consider black holes thermodynamical properties for black hole scenarios where the Lorentz symmetry is not preserved. We use two different metrics based in the work of Bertolami and Paramos [18] and in the work of Casana et al. [19]; both metric presented in the cited works are obtained from a radial LSB using the bumblebee formalism. The methodology used was the quantum tunneling and, more specifically, the Hamilton-Jacobi ansatz. We obtained the temperature, entropy and heat capacity for both scenarios and highlighted their similarities.

We showed that the Schwarzschild black hole thermodynamical properties are modified when a Lorentz symmetry violation is taken into account. The temperature change, for example, happens due the surface gravity modification, which is a consequence of the geometry modification caused by the LSB. The geometry modification is also responsible for changing the black hole surface area and, consequently, the black hole entropy. The temperature and entropy modifications lead to the heat capacity modification. However, the black hole
instability is not modified by the LSB.

It was possible to relate the temperatures for the two scenarios, but not the entropies and heat capacities. The reason behind this is that the temperature is proportional to the black hole surface gravity, which is similar for both cases when we consider the LSB parameter to be small and the the arbitrary distance to be \( r_0 = 2M \). On the other hand, the entropy is proportional to the black hole surface area. Thus, since the black hole areas for the two approaches are different under the conditions above, the entropies are also different.

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