SUBADDITIVITY OF KODAIRA DIMENSION DOES NOT HOLD IN POSITIVE CHARACTERISTIC

PAOLO CASCINI, SHO EJIRI, AND LEI ZHANG

Abstract. Over any algebraically closed field of positive characteristic, we construct examples of fibrations violating subadditivity of Kodaira dimension.

1. Introduction

Throughout this note, a fibration is a projective morphism $f : X \to Y$ between two normal varieties such that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. For fibrations between varieties defined over the field $\mathbb{C}$ of complex numbers, Iitaka [17] proposed the following conjecture:

Conjecture 1.1 (Iitaka conjecture [17]). Let $f : X \to Y$ be a fibration between smooth projective varieties over $\mathbb{C}$, with $\dim X = n$ and $\dim Y = m$. Let $X_{\eta}$ denote the geometric generic fibre of $f$ (which is automatically smooth over $k(\bar{\eta})$).

Then the following inequality holds:

$$(C_{n,m}) \quad \kappa(X) \geq \kappa(Y) + \kappa(X_{\eta}).$$

According to Mori [24, §6], before the work of Ueno [33] and Viehweg [34] appeared, Conjecture [17] was widely believed to be false, probably because of the existence of non-algebraic counterexamples (cf. [26, Remark 4]). On the other hand, by now, Conjecture [1.1] plays an important role in birational geometry. It is worthwhile to mention that, as Conjecture [1.1] predicts subadditivity of Kodaira dimension, it is closely related to the abundance conjecture in the minimal model program. Indeed, if $X$ is an irregular manifold, i.e. $q(X) = \dim \mathrm{Pic}^0(X) > 0$, then its Albanese map induces a natural fibration $f : X \to Y$ with $Y$ of maximal Albanese dimension. As varieties with maximal Albanese dimension are much better understood, Conjecture [1.1] turns out to be very useful to study the Abundance conjecture for irregular varieties.

Over the last few decades, many important cases of the conjecture were solved ([33, 35, 36, 19, 20, 21, 22, 14, 11, 7, 6, 5, 16]), and, in particular, it is known to hold true if (1) $\dim Y = 1$ or $2$ ([20, 5]); (2) $X_{\eta}$ has a good minimal model ([21]); (3) $X_{\eta}$ is of general type ([22]); (4) $Y$ is of maximal Albanese dimension ([7, 6, 16]).

Considering the recent developments of birational geometry over a field of positive characteristic, it is natural to ask if Conjecture [1.1] can be generalised to this setting. It is important to notice though that, given a fibration $f : X \to Y$, even assuming that both $X$ and $Y$ are smooth, the geometric generic fibre $X_{\eta}$ is possibly singular and not even reduced if $f$ is inseparable. Note that the generic fibre $X_{\eta}$ is regular.

2010 Mathematics Subject Classification. 14D06, 14E30.

Key words and phrases. Iitaka’s conjecture, positive characteristic, Raynaud’s surface.
and hence it is Gorenstein, i.e. $K_{X_{\eta}}$ is a Cartier divisor. It is therefore tempting to ask the following:

**Question 1.2.** Let $f: X \to Y$ be a fibration between smooth projective varieties over an algebraically closed field $k$ of positive characteristic, with $\dim X = n$ and $\dim Y = m$. Does the inequality

$$(C_{n,m}) \quad \kappa(X) \geq \kappa(Y) + \kappa(X_{\eta}, K_{X_{\eta}})$$

hold true?

This question has been answered affirmatively for fibrations with smooth geometric generic fibres with relative dimension one or with $\dim X = 3$ and $p > 5$ (8 13). One of the advantages of assuming that $X_{\eta}$ is smooth lies in the fact that, under some additional conditions, $f_\ast \omega_{X/Y}^n$ is weakly positive (27 11). Question 1.2 has also been studied without assuming that $X_{\eta}$ is smooth. Patakfalvi proved $C_{n,m}$ when $X$ is of general type and $X_{\eta}$ has non-nilpotent Hasse–Witt matrix (28). Assuming that $K_X$ is nef and some other additional conditions, the third author proved $f_\ast \omega_{X/Y}^n$ contains a non-zero weakly positive subsheaf. This plays a similar role as $f_\ast \omega_{X/Y}^n$ does when studying the Iitaka conjecture. The third author studied fibrations with singular geometric fibres and proved subadditivity of Kodaira dimension for three-folds over an algebraically closed field of characteristic $p > 5$, assuming that $Y$ is of maximal Albanese dimension and the abundance conjecture for minimal three-folds $X$ with $q(X) > 0$ (39 38). The arguments in 39 heavily rely on results of the minimal model program (15 12 4), among which, an important ingredient is the fact that the relative minimal model over an abelian variety is actually minimal, as a consequence of the cone theorem (41 Theorem 1.1).

The aim of this note is to construct examples of fibrations which provide a negative answer to Question 1.2.

**Theorem 1.3.** Let $k$ be an algebraically closed field of characteristic $p$ and let $n_2 = 8$, $n_3 = 5$ and $n_p = 2p + 2$ for $p \geq 5$. Assume that any variety over $k$ admits a smooth resolution of singularities.

Then, for any positive integers $n$ and $m$ such that $n \geq n_p$ and $2 \leq m \leq n$, there exists a fibration $f: X \to Y$ between smooth projective varieties over $k$ with $\dim X = n$ and $\dim Y = m$ such that $C_{n,m}$ does not hold true.

As expected, such fibrations are over some uniruled variety and have singular geometric generic fibre. Our construction stems from a Raynaud’s surface $f: S \to C$ over a Tango curve $C$ and such that for any $l > 0$ the sheaf $f_\ast \omega_{S/C}^l$ is not nef and, instead, its dual is ample. It is easy to see that for any $m \gg 0$ and $l > 0$, the sheaf $((\bigotimes^m f_\ast \omega_{S/C}^l) \otimes \omega_C^l)^\vee$ is ample. Observe that $(\bigotimes^m f_\ast \omega_{S/C}^l) \otimes \omega_C^l \cong f_\ast (\omega_{X(m)}^l)$, where $f^{(m)}: X^{(m)} := S \times_C S \times_C \cdots \times_C S \to C$ is the induced fibration on the fibred product. Hence $\kappa(X^{(m)}, K_{X^{(m)}}) = -\infty$. If we assume that $X^{(m)}$ has a smooth resolution of singularities $Y^{(m)} \to X^{(m)}$, then $\kappa(Y^{(m)}) = -\infty$. If $m_0$ is the maximum integer such that $\kappa(Y^{(m)}) \geq 0$, then the projection $Y^{(m_0+1)} \to Y^{(m_0)}$ is the desired fibration. Finally, assuming the existence of a resolution of singularities, we get examples of dimension $2p + 2$ in characteristic $p > 3$, and we construct examples of dimension five in characteristic $p = 3$. 
Remark 1.4. The singularities of $X^{(m)}$ are explicitly described in the proof of Proposition 3.1. We show the existence of a resolution of singularities of $X^{(m)}$, for $m \leq 3$, by blowing up along the singular locus repeatedly. On the other hand, we were not able to show that, for $m \geq 4$, the same method terminates after finitely many steps.

It is still reasonable to expect that subadditivity of Kodaira dimension holds for fibrations in characteristic $p > 0$ if, either we assume that the base is not uniruled (cf. [12]), or we replace $\kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}})$ by the Kodaira dimension $\kappa(Z)$ of a smooth birational model $Z$ of $X_{\bar{\eta}}$ (cf. [8]). Note that $\kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}}) \geq \kappa(Z)$ (cf. [31]).

Acknowledgements. The authors are grateful to Professor Hiromu Tanaka for helpful comments and pointing out about the log version of Question 1.2 (Example 4.1). The first author is supported by EPSRC. The second author was supported by JSPS KAKENHI Grant Number 18J00171. He wishes to express his thanks to Professor Osamu Fujino for answering his questions and taking an interest in this work. The third author is supported by a NSFC grant (No. 11771260).

2. Counterexamples to subadditivity

We work over an algebraically closed field $k$ with char $k = p > 0$ and we assume the existence of a smooth resolution of singularities. We shall present our examples in this section. For the convenience of the reader, we include all the details.

2.1. Tango–Raynaud curves. We begin by recalling the notion of a Tango–Raynaud curve, introduced in [25]. A smooth projective curve $C$ is a Tango–Raynaud curve if there exists a line bundle $L$ on $C$ such that $L^p \cong \omega_C$ and the map

$$H^1(C, L^{-1}) \to H^1(C, L^{-p})$$

induced by the Frobenius morphism is not injective. As explained in [25 §1], [32] Lemma 12] implies that $C$ is a Tango–Raynaud curve if and only if there exists a rational function $f \in K(C)$ such that $df \neq 0$ and each coefficient in the divisor $(df)$ is divisible by $p$.

Example 2.1 ([25 Example 1.3]). Fix $e > 0$. Let $C \subset \mathbb{P}^2$ be the plane curve defined by

$$Y^{pe} - YX^{pe-1} = Z^{pe-1}X,$$

where $[X, Y, Z]$ are homogeneous coordinate of $\mathbb{P}^2$. Let $U = C \cap \{X \neq 0\} \subset \text{Spec } k[y, z]$ with $y = \frac{Y}{X}$, $z = \frac{Z}{X}$. Then $U$ is defined by $y^{pe} - y = z^{pe-1}$ and $C = U \cup \{\infty := [0, 0, 1]\}$. In particular, we have

$$-dy = -z^{pe-2}dz$$

on $U$. Thus, $\omega_C$ is generated by $dz$ on $U$. Since the degree of $C$ is $pe$, we have $\deg \omega_C = pe(pe - 3)$, and $(dz) = pe(pe - 3) \cdot (\infty)$. It follows that $C$ is a Tango–Raynaud curve.
2.2. Raynaud and Mukai’s construction of algebraic fibre spaces. We only use a special case of Mukai’s construction [25]. Fix a Tango–Raynaud curve $C$ as in Example 2.1 and let $D = e(pe - 3) \cdot (\infty)$ so that $K_C = (dz) = pD$. Assume $e$ is prime to $p$. We can regard $dz$ as an element of $H^0(C, \mathcal{B}^1(-D))$ where, if $F_C$ denotes the Frobenius morphism on $C$, then $\mathcal{B}^1 := dF_C \ast \mathcal{O}_C$. Thus, $dz$ induces a non-zero element $\xi \in H^1(C, \mathcal{O}_C(-D))$ lying in the kernel of the map

$$H^1(C, \mathcal{O}_C(-D)) \to H^1(C, \mathcal{O}_C(-pD))$$

induced by $F_C$.

Thus, $\xi$ corresponds to an extension

$$0 \to \mathcal{O}_C(-D) \to \mathcal{E} \xrightarrow{\alpha} \mathcal{O}_C \to 0. \tag{1}$$

We define $P = \mathbb{P}(\mathcal{E}) := \text{Proj}_{\mathcal{O}_C} \bigoplus_{i=0}^{\infty} S^i \mathcal{E}$ and let $g : P \to C$ denote the natural projection. Let $T$ be a divisor corresponding to $\mathcal{O}_P(1)$ and let $F \subset P$ be the section of $g$ corresponding to $\alpha$. Then $T|_F \sim 0$. Since $T - F$ is relatively trivial over $C$, applying $g_\ast$ to the exact sequence

$$0 \to \mathcal{O}_P(T - F) \to \mathcal{O}_P(T) \to \mathcal{O}_F \to 0, \tag{2}$$

we obtain (1), which implies that $T - F \sim -g^* D$. Thus,$\quad F \sim T + g^* D$ and $F^2 = \deg D. \tag{3}$

If we set $P_1 := \mathbb{P}(F_C \ast \mathcal{E})$, then we have the following commutative diagram:

$$\begin{array}{ccc}
P^1 & \xrightarrow{F_P} & P \\
\downarrow F_{P/C} & & \downarrow F_P \\
P_1 & \xrightarrow{g_1} & P \\
\downarrow g_1 & & \downarrow g \\
C^1 & \xrightarrow{g} & C
\end{array}$$

where $F_{P/C}$ denotes the relative Frobenius morphism. By the choice of $\xi$, we see that the pull-back

$$0 \to \mathcal{O}_C(-pD) \to F_C^* \mathcal{E} \to \mathcal{O}_C \to 0. \tag{5}$$

of (1) by $F_C$ splits. The splitting map $F_C^* \mathcal{E} \rightarrow \mathcal{O}_C(-pD)$ induces a section $G' \subset P_1$ of $g_1 : P_1 \to C^1$. We then have

$$G' \sim W^* F - pg_1^* D. \tag{6}$$

Set $G := F_{P/C}^* G'$. Then we get

$$G \sim pF - pg^* D \sim pT. \tag{6}$$

We may choose $e$ so that there exists a positive integer $l$ which divides both $e$ and $p + 1$. Let $e' := \frac{e}{l}$, $r := \frac{p + 1}{l}$ and $D' := \frac{D}{l} = e'(pe - 3) \cdot (\infty)$. Then

$$G + F \sim (p + 1)T + g^* D = l \cdot (rT + g^* D') = lM,$$
where $M := rT + g^*D'$. The equivalence yields a cyclic $l$-cover $\pi: S \to P$ branched over $G + F$ such that $S$ is smooth. We use the following notation:

$$S \xrightarrow{\pi} P \xrightarrow{g} C.$$ 

By an easy calculation we have

$$K_{S/C} \sim \pi^*(K_{P/C} + (l - 1)M)$$

(7)

$$\sim \pi^*(-2T - g^*D + (l - 1)M)$$

$$\sim \pi^*((p - 1 - r)T) - f^*D'.$$

Let $q := p - 1 - r$. Then, for all positive integer $n$ such that $nq \geq r(l - 1)$, we have

$$f_*\omega^n_{S/C} \cong g_* \left( \mathcal{O}_P \left( (n(p - 1 - r)T - nq^*D') \otimes \bigoplus_{i=0}^{l-1} \mathcal{O}_P(-iM) \right) \right)$$

$$\cong \bigoplus_{i=0}^{l-1} g_* \mathcal{O}_P \left( (nq - ri)T - (n + i)g^*D' \right)$$

$$\cong \bigoplus_{i=0}^{l-1} S^{nq-ri} \mathcal{E}(-(n + i)D').$$

Lemma 2.2. If $\{l, p\} \neq \{2, 3\}$, then $K_S$ is ample, otherwise $\kappa(S) = 1$.

Proof. By (7), we have that

$$K_S \sim \pi^*(qT) - f^*D' + f^*K_C$$

$$\sim \pi^*(qT) - f^*D' + f^*(pD)$$

$$= \pi^*(q(T + g^*D) + g^*((p - q)D - D'))$$

$$\sim \pi^*(qF + g^*((p + l)D')).$$

Note that $F$ is nef and big because $F^2 > 0$. Therefore, if $\{l, p\} \neq \{2, 3\}$ then $q > 0$ and $K_S$ is ample; otherwise $q = 0$ and then $\kappa(S) = 1$. \qed

2.3. Fibre products of Tango–Raynaud–Mukai surface. We use the same notation as in Section 2.2. Pick a positive integer $m$. Let $X^{(m)} := S \times_C \cdots \times_C S$ be the $m$-th fibre product of $S$ over $C$. Then $X^{(m)}$ is a Gorenstein integral scheme, as each $X^{(i)} \to X^{(i-1)}$ is a flat morphism whose every fibre is a Gorenstein (integral) curve. Denote by $f^{(m)}: X^{(m)} \to C$ the natural fibration and by $p_i: X^{(m)} \to S$ the projection to the $i$-th factor. We then have

$$\omega_{X^{(m)}/C} \cong \bigotimes_{i=1}^{m} p_i^* \omega_{S/C}.$$
Using the projection formula, for any positive integer \( n \) such that \( nq \geq r(l - 1) \), we get

\[
(8) \quad f_\ast^{(m)} \omega^{n}_{X(m)} \cong f_\ast^{(m)} \omega^{n}_{X(m)/C} \otimes \omega^{n}_{C}
\]

\[
\cong \left( \bigotimes_{0 \leq i_1, \ldots, i_m \leq l - 1} \left( \bigotimes_{1 \leq j \leq m} S^{nq - ri_j} \mathcal{E} \right) \right) \otimes \mathcal{O}_{C} \left( \left( npl - mn - \sum_{j=1}^{m} i_{j} \right) D' \right).
\]

It follows that

**Lemma 2.3.** For \( m > pl \), the dual \( (f_\ast^{(m)} \omega^{n}_{X(m)})^\ast \) is ample. In particular,

\[
\kappa(X^{(m)}), K_{X^{(m)}} = -\infty.
\]

Note that for \( m \geq 2 \), the variety \( X^{(m)} \) is singular and, if \( Z \subset S \) is the non-smooth locus of \( f \) (i.e., \( Z = \text{Supp}(G) \)), then \( (X^{(m)})_{\text{sing}} = \bigcup_{1 \leq i < j \leq m} p_i^{-1}(Z) \cap p_j^{-1}(Z) \), which is of codimension 2. In particular, \( X^{(m)} \) is normal. Assuming resolution of singularities, we have

**Theorem 2.4.** If \( m > pl \) and if \( \sigma: Y^{(m)} \rightarrow X^{(m)} \) is a resolution of singularities, then \( \kappa(Y^{(m)}) = -\infty \).

**Proof.** If \( \kappa(Y^{(m)}) \geq 0 \), then there is a \( \mathbb{Q} \)-Cartier divisor \( E \geq 0 \) with \( E \sim_{\mathbb{Q}} K_{Y^{(m)}} \). Since \( X^{(m)} \) is normal, we have \( K_{X} \sim \sigma_{\ast} K_{Y^{(m)}} \sim_{\mathbb{Q}} \sigma_{\ast} E \geq 0 \). But this contradicts Lemma 2.3.

\[ \square \]

### 2.4. Counterexamples to subadditivity of Kodaira dimension

We use the same notation as in the previous sections. Let \( m_0(p) \) be the maximal integer such that \( \kappa(Y^{(m_0(p))}) \geq 0 \). Lemma 2.2 and Lemma 2.3 imply that \( 1 \leq m_0(p) \leq pl \). We assume varieties over \( k \) admit a smooth resolution of singularities.

For any \( 1 \leq m \leq m_0(p) \), let \( Y^{(m)} \) and \( Y^{(m_0(p)+1-m)} \) be smooth resolutions of \( X^{(m)} \) and \( X^{(m_0(p)+1-m)} \), respectively. Then \( Y^{(m)} \times_{C} Y^{(m_0(p)+1-m)} \) is birational to \( X^{(m_0(p)+1)} \). Notice that since \( Y^{(m)} \rightarrow X^{(m)} \rightarrow C \) is separable and \( Y^{(m_0(p)+1-m)} \) is regular, the generic fibre of \( p_1: Y^{(m)} \times_{C} Y^{(m_0(p)+1-m)} \rightarrow Y^{(m)} \), which is isomorphic to \( Y^{(m)}_{\eta} \otimes_{K(C)} K(Y^{(m)}) \), is regular and has nonnegative Kodaira dimension. We have the following commutative diagram

\[ \begin{array}{ccc}
Y^{(m_0(p)+1)} & \rightarrow & Y^{(m)} \times_{C} Y^{(m_0(p)+1-m)} \\
\downarrow^{h} & & \downarrow^{p_{1}} \\
Y^{(m)} & \rightarrow & X^{(m_0(p)+1)} \rightarrow X^{(m_0(p)+1-m)} \\
\downarrow & & \downarrow \\
Y^{(m)} & \rightarrow & X^{(m)} \rightarrow C
\end{array} \]
where $Y^{(m_0(p)+1)} \to Y^{(m)} \times_C Y^{(m_0(p)+m-1)}$ is a resolution of singularities such that it is isomorphic on the generic fibre of $p_1$. We see that the fibration $h: Y^{(m_0(p)+1)} \to Y^{(m)}$ does not satisfy subadditivity of the Kodaira dimension.

In conclusion,

- For $p \geq 3$, we take $l = 2$ and $m_0(p) \leq 2p$. The fibration $h: Y^{(m_0(p)+1)} \to Y^{(m)}$ with $\dim Y^{(m_0(p)+1)} \leq 2p + 2$ violates subadditivity of Kodaira dimension $C_{m_0(p)+2,m+1}$.
- For $p = 2$, we take $l = 3$ and $m_0(p) \leq 6$. The fibration $h: Y^{(m_0(p)+1)} \to Y^{(m)}$ with $\dim Y^{(m_0(p)+1)} \leq 8$ violates subadditivity of Kodaira dimension $C_{m_0(p)+2,m+1}$.

3. Singularity

We now study the singularities of some of the varieties we constructed in Section 2.3 in order to get examples of fibrations violating subadditivity of Kodaira dimension in lower dimension. We work over an algebraically closed field $k$ with $\text{char } k = p > 0$. Let $S$ be the Raynaud surface constructed in Section 2.2 by setting the covering degree $l = 2$. Recall that, for any positive integer $m$, the variety $X^{(m)}$, obtained as the $m$-th fibre product of $f: S \to C$, has dimension $m + 1$ (cf. Section 2.3).

**Proposition 3.1.** The variety $X^{(4)}$ has non-canonical singularities, and the non-canonical locus of $X^{(4)}$ is dominant over $C$.

**Proof.** Recall that $g: P = \mathbb{P}(\mathcal{E}) \to C$ has a smooth multiple section $G$ such that $g|_G : G \to C$ coincides with the Frobenius map. Take an affine open set $B = \text{Spec } A$ of $C$. We may assume that $\mathcal{E}|_B \cong \mathcal{O}_B \cdot X \oplus \mathcal{O}_B \cdot Y$, and that $G$ is given by $aX^p = bY^p$ for some $a, b \in A$. After possibly shrinking $B$, we may also assume $G$ is contained in the affine piece

$$V := \{y \neq 0\} \cong B \times \mathbb{A}^1 = \text{Spec } A[x], \text{ where } x = \frac{X}{Y}$$

and $G$ is defined by the equation $t = x^p$ where $t = \frac{b}{a} \in A$. In fact since $G$ is smooth, $t$ is a local parameter at every point $Q \in B$. In a natural way we get étale morphisms

$$B = \text{Spec } A \to \text{Spec } k[t] \text{ and } V = B \times \mathbb{A}^1 \to \text{Spec } k[t, x].$$

The double cover $S_V \to V$ is locally described as

$$\text{Spec } A[x, y]/(y^2 + x^p - t) \to V,$$

and there is an étale morphism $S_V \to U = \text{Spec } k[t, x, y]/(y^2 + x^p - t)$. In turn, we get étale morphisms

$$X_B^{(m)} \to U^{(m)} = \times_{\text{Spec } k[t]}^m \text{Spec } k[t, x, y]/(y^2 + x^p - t = 0) \cong \text{Spec } k[t; x_1, y_1; \ldots; x_m, y_m]/(y_1^2 + x_1^p - t, \ldots, y_m^2 + x_m^p - t).$$

Notice that $U^{(m)}$ is the complete intersection in $\mathbb{A}^{2m+1}$ of $m$ equations

$$t = y_1^2 + x_1^p, t = y_2^2 + x_2^p, \ldots, t = y_m^2 + x_m^p.$$
By eliminating $t$ and applying the substitutions
\[ z_1 = x_m - x_1, \quad z_2 = x_m - x_2, \ldots, \quad z_m = x_m - x_m, \]
we see that $U^{(m)}$ is isomorphic to $W^{(m)} \times \text{Spec } k[x_m] \subset \mathbb{A}^{2m-1} \times \text{Spec } k[x_m]$, where $W^{(m)}$ is the complete intersection in $\mathbb{A}^{2m-1}$ of the $m-1$ equations
\[ y_1^2 - y_2^2 = z_1^p, \quad y_2^2 - y_3^2 = z_2^p, \ldots, \quad y_m^2 - y_{m-1}^2 = z_{m-1}^p. \]

We see that when $m \geq 2$ then $X^{(m)}$ is singular and the singular locus is exactly the union of $p^{-1}_{ij}(G \times_C G) \subset S \times_C S$, where $p_{ij}$ denote the projection from $X^{(m)}$ to the product of the $i$-th and $j$-th factors.

We now describe the non-canonical locus of $W^{(4)}$. After suitable substitutions, we can identify $W^{(4)}$ with the complete intersection in $Z_0 = \mathbb{A}^7$ of the equations
\[ y_1^2 - y_2^2 = z_1^p, \quad y_3^2 - y_4^2 = z_2^p, \quad y_1^2 - y_3^2 = z_3^p. \]
Remark that $W^{(3)}$ is singular along the locus $S_{i,j} : y_i = y_j = 0$ for $1 \leq i, j \leq 4$, and the singular locus $S_{i,j} \times \text{Spec } k[x_4] \subset U^{(4)}$ is dominant over $\text{Spec } k[t]$. We proceed with the following steps:

Step 1: Blow-up $Z_0 = \mathbb{A}^7$ along the ideals $I_1 = (y_1, y_2, z_1^r)$ and $I_2 = (y_3, y_4, z_2^r)$ independently where $r = \frac{\mathbb{Z}}{2}$. On the affine chart
\[ Z_1 \cong \mathbb{A}^7 = \text{Spec } k[y_1^r, y_2^r, y_3^r, y_4^r, z_1^r, z_2^r, z_3], \]
the strict transform $W_1 \subset Z_1$ of $W_0$ is defined by the equations
\[ y_1^2 - y_2^2 = z_1, \quad y_3^2 - y_4^2 = z_2, \quad y_1^2 - y_3^2 = z_3. \]
The morphism $\rho_1 : Z_1 \to Z_0$ has two exceptional divisors $E_1 = \{z_1 = 0\}$ and $E_2 = \{z_2 = 0\}$. Note that $K_{Z_1} = \rho_1^*K_{Z_0} + 2rE_1 + 2rE_3$. Applying the adjunction formula, it follows that $\rho_1|_{W_1} : W_1 \to W_0$ is crepant.

Step 2: Note that under the projection $Z_1 \to Z'_1 \cong \mathbb{A}^5 = \text{Spec } k[y_1^r, y_2^r, y_3^r, y_4^r, z_3]$, we get a closed embedding
\[ W_1 \hookrightarrow W'_1 = V((y_1^2 - y_2^2)^2 - (y_3^2 - y_4^2)^2) \subset Z'_1 \cong \mathbb{A}^5. \]
Blow-up $Z'_1$ along the ideal $I' = (y_1^2, y_2^2, y_3^2, y_4^2, z_3)$ and do the normalization (equivalent to do a weighted blow up with weight $(1, 1, 1, 1, 2)$). On the $y_1'$-chart
\[ Z_2 \cong \mathbb{A}^5 = \text{Spec } k[y_1', y_3', y_4', z_3') \]
the strict transform $W_2 \subset Z_2$ of $W'_1$ is defined by the equations
\[ (1 - y_1'^2)^2 - (y_3'^2 - y_4'^2)^2 = z_3'^p. \]
The morphism $\rho_2 : Z_2 \to Z'_1$ has unique exceptional divisors $E = \{y_1' = 0\}$. Note that
\[ K_{Z_2} = \rho_2^*K_{Z'_1} + 5E \quad \text{and} \quad W_2 \sim \rho_2^*W'_1 - 2pE. \]
Thus, by the adjunction formula we have
\[ K_{W_2} = \rho_2^*K_{W'_1} + (5 - 2p)E. \]

In summary, we get a birational morphism $\rho : W_2 \to W_0 = W^{(4)}$ and an exceptional divisor $E$ with discrepancy $5 - 2p$, hence $W^{(4)}$ is not canonical when $p = 3$.
and not log canonical when $p \geq 5$. Tracing back these morphisms, it is easy to check that $E \times \text{Spec } k[x_4] \subset U^{(4)}$ is dominant over $\text{Spec } k[t]$. Therefore, the result follows.

Remark 3.2. By the arguments in the proof of Proposition 3.1, it follows that the singular locus of $X^{(m)}$ admits a neighbourhood which is isomorphic to the product of an $m$-dimensional variety and a smooth curve, up to some étale morphism. In positive characteristic, a resolution of singularities is known to exist in dimension three [9, 10], hence $X^{(2)}$ and $X^{(3)}$ admit a smooth resolution. Moreover, if $p = 3$, it is easy to show that $X^{(2)}$ and $X^{(3)}$ have canonical singularity by describing an explicit resolution of singularities for the varieties

\[ W^{(2)} = V(y_1^2 - y_3^2 - z_1^3) \subset \mathbb{A}^3 \quad \text{and} \quad W^{(3)} = V(y_1^2 - y_3^2 - z_1^3, y_2^2 - y_3^2 - z_2^3) \subset \mathbb{A}^5. \]

Hence, formula (8) in Section 2.3 implies that $\kappa(Y^{(m)}) = 1$, for $m \leq 3$.

Proposition 3.3. Let $k$ be an algebraically closed field of characteristic $p = 3$. Then, for any $m \in \{2, 3, 4\}$, there exists a fibration between smooth projective varieties defined over $k$ violating subadditivity of Kodaira dimensions $C_{5,m}$.

Proof. Using the same notation as above, the generic fibre $X^{(4)}_{\eta}$ of $X^{(4)} \to C$ has trivial dualizing sheaf. By Proposition 3.1, there exists a resolution $\sigma: Y^{(4)} \to X^{(4)}$ such that the generic fibre over $C$ has negative Kodaira dimension, as a consequence $\kappa(Y^{(4)}) = -\infty$. Thus, using the same argument as in Section 2.4, the claim follows.

Proof of Theorem 1.3. The Theorem follows immediately from the construction in Section 2.4 and Proposition 3.3.

4. Counterexamples to a log version of $C_{2,1}$ over imperfect fields

We conclude this paper by introducing an example of a fibration that violates a log version of Question 1.2 over an imperfect field. The authors learned about this example from Hiromu Tanaka.

Example 4.1. Let $k$ be an imperfect field of characteristic $p \in \{2, 3\}$. Tanaka [30, Theorem 1.4] constructed a $k$-morphism $\rho: S \to C$ with the following properties:

- $S$ is a projective regular surface over $k$;
- there exists a prime divisor $C_2$ on $S$ such that if $\Delta_S := \left(\frac{2}{p} - \varepsilon\right) C_2$, where $\varepsilon \in \mathbb{Q}$ is such that $0 < \varepsilon \ll \frac{2}{p}$, then $(S, \Delta_S)$ is Kawamata log terminal and $-(K_S + \Delta_S)$ is ample;
- $\rho$ is a $\mathbb{P}^1$-bundle;
- $C$ is a projective regular curve over $k$ with $K_C \sim 0$.

These properties can be viewed as pathologies in birational geometry in positive characteristic. Indeed, it is known that the image of a variety of log Fano type in characteristic zero is again of log Fano type.

Set $\Gamma := \frac{2}{p} C_2$. Then the pair $(S, \Gamma)$ is log canonical and [30, Lemma 3.4] implies that $-(K_S + \Gamma)$ is $\mathbb{Q}$-linearly equivalent to the pullback of an ample $\mathbb{Q}$-divisor on
Hence, if $S_\eta$ is the generic fibre of $\rho$, then
\[-\infty = \kappa(S, K_S + \Gamma) < \kappa(S_\eta, K_{S_\eta} + \Gamma|_{S_\eta}) + \kappa(C, K_C) = 0.\]

\textbf{Remark 4.2.} We use the same notation as in Example 4.1.

(1) The pair $(S_\eta, \Gamma|_{S_\eta})$ is not $F$-pure, where $S_\eta$ is the geometric generic fibre of $\rho$. Indeed, by the construction in [30, §3], there exists a closed point $Q \in S_\eta$ such that $C_2|_{S_\eta} = pQ$ as divisors.

(2) Roughly speaking, the curve $C$ is unirational. More precisely, [30, Proposition 3.6] implies that there is an extension $k'/k$ of degree $p$ such that the normalisation of $C \times_k k'$ is $k'$-isomorphic to the projective line $\mathbb{P}^1_{k'}$.

\begin{thebibliography}{18}

[1] C. Birkar, \textit{The Iitaka conjecture $C_{n,m}$ in dimension six}, Compos. Math. \textbf{145} (2009), no. 6, 1442–1446.

[2] C. Birkar, \textit{Existence of flips and minimal models for 3-folds in char $p$}, Ann. Sci. Éc. Norm. Supér. \textbf{49} (2016), 169–212.

[3] C. Birkar, Y. Chen and L. Zhang, \textit{Iitaka’s $C_{n,m}$ conjecture for 3-folds over finite fields}, Nagoya Math. J. \textbf{229} (2018), 21–51.

[4] C. Birkar and J. Waldron, \textit{Existence of Mori fibre spaces for 3-folds in char $p$}, Adv. Math. \textbf{313} (2017), 62–101.

[5] J. Cao, \textit{Kodaira dimension of algebraic fiber spaces over surfaces}, Algebr. Geom. \textbf{5} (2018), no. 6, 728–741.

[6] J. Cao and M. Păun, \textit{Kodaira dimension of algebraic fiber spaces over abelian varieties}, Invent. Math. \textbf{207} (2017), no. 1, 345–387.

[7] J.A. Chen and C.D. Hacon, \textit{Kodaira dimension of irregular varieties}, Invent. Math. \textbf{186} (2011), no. 3, 481–500.

[8] Y. Chen and L. Zhang, \textit{The subadditivity of the Kodaira dimension for fibrations of relative dimension one in positive characteristics}, Math. Res. Lett. \textbf{22} (2015), no. 3, 675–696.

[9] V. Cossart and O. Piltant, \textit{Resolution of singularities of threefolds in positive characteristic I}, J. Algebra \textbf{320} (2008), 1051–1082.

[10] V. Cossart and O. Piltant, \textit{Resolution of singularities of threefolds in positive characteristic II}, J. Algebra \textbf{321} (2009), 1836–1976.

[11] S. Ejiri, \textit{Weak positivity theorem and Frobenius stable canonical rings of geometric generic fibers}, J. Algebraic Geom. \textbf{26} (2017), 691–734.

[12] S. Ejiri, \textit{Direct images of pluricanonical bundles and Frobenius stable canonical rings of fibers}, preprint, available at \texttt{arXiv:1909.07000} (2019).

[13] S. Ejiri and L. Zhang, \textit{Iitaka’s $C_{n,m}$ conjecture for 3-folds in positive characteristic}, Math. Res. Lett. \textbf{25} (2018), no. 3, 783–802.

[14] O. Fujino, \textit{Algebraic fiber spaces whose general fibers are of maximal Albanese dimension}, Nagoya Math. J. \textbf{172} (2003), 111–127.

[15] C.D. Hacon and C. Xu, \textit{On the three dimensional minimal model program in positive characteristic}, J. Amer. Math. Soc. \textbf{28} (2015), 711–744.

[16] C.D. Hacon, M. Popa and C. Schnell, \textit{Algebraic fiber spaces over abelian varieties: Around a recent theorem by Cao and Păun}, in the proceedings held in honor of Lawrence Ein’s 60th birthday, Contemp. Math. \textbf{712} (2018), 143–195.

[17] S. Iitaka, \textit{Genera and classification of algebraic varieties. 1}, Sūgaku \textbf{24} (1972), 14–27. (Japanese)

[18] S. Iitaka, \textit{Algebraic geometry}, An introduction to birational geometry of algebraic varieties, Graduate Texts in Mathematics, \textbf{76} (1982).

[19] Y. Kawamata, \textit{Characterization of abelian varieties}, Compos. Math. \textbf{43} (1981), no. 2, 253–276.

\end{thebibliography}
Y. Kawamata, *Kodaira dimension of algebraic fiber spaces over curves*, Invent. Math. **66** (1982), no. 1, 57–71.

Y. Kawamata, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. Reine Angew. Math. **363** (1985), 1–46.

J. Kollár, *Subadditivity of the Kodaira dimension: fibers of general type*, Algebraic geometry, Sendai, 1985. Adv. Stud. Pure Math., **10** (1987), 361–398.

H. Matsumura, *Commutative Ring Theory*, Cambridge studies in advanced mathematics, **8** (1986).

S. Mori, *Classification of higher-dimensional varieties*, Proc. Sympos. Pure Math. **46** (1987), 269–331.

S. Mukai, *Counterexamples to Kodaira’s vanishing and Yau’s inequality in positive characteristics*, Kyoto J. Math. **53** (2013), no. 2, 515–532.

I. Nakamura and K. Ueno, *An addition formula for Kodaira dimensions of analytic fibre bundles whose fibres are Moishezon manifolds*, J. Math. Soc. Japan, **25** (1973), no. 3, 363–371.

Z. Patakfalvi, *Semi-positivity in positive characteristics*, Ann. Sci. École Norm. Sup. **47** (2014), 993–1025.

Z. Patakfalvi, *On subadditivity of Kodaira dimension in positive characteristic over a general type base*, J. Algebraic Geom. **27** (2018), 21–53.

M. Raynaud, *Contre-exemple au “vanishing theorem” en caractéristique p > 0*, C.P. Ramanujam—a tribute, Tata Inst. Fund. Res. Studies in Math. **8** (1978), 273–278.

H. Tanaka, *Pathologies on Mori fibre spaces in positive characteristic*, to appear in Ann. Sc. Norm. Super. Pisa Cl. Sci. arXiv:1609.00574.

H. Tanaka, *Behavior of canonical divisors under purely inseparable base changes*, J. Reine Angew. Math. **2018** (2018), no. 744, 237–264.

H. Tango, *On the behavior of extensions of vector bundles under the Frobenius map*, Nagoya Math. J. **48** (1972), 73–89.

K. Ueno, *Kodaira dimension of certain fibre spaces*, Complex Analysis and Algebraic Geometry, Iwanami Shoten, Tokyo (1977), 279–292.

E. Viehweg, *Canonical divisors and the additivity of the Kodaira dimension for morphisms of relative dimension one*, Compos. Math. **35** (1977), 197–233.

E. Viehweg, *Die Additivität der Kodaira Dimension für projektive Faserräume über Varietäten des allgemeinen Typs*, J. Reine Angew. Math. **330** (1982), 132–142.

E. Viehweg, *Weak positivity and the additivity of the Kodaira dimension for certain fiber spaces*, Algebraic varieties and analytic varieties, Adv. Stud. in Pure Math. **1** (1983), 329–353.

L. Zhang, *Subadditivity of Kodaira dimensions for fibrations of three-folds in positive characteristics*, Adv. Math. **354** (2019), arXiv: 1601.06907.

L. Zhang, *Abundance for non-uniruled 3-fold with non-trivial Albanese map in positive characteristics*, J. Lond. Math. Soc. **99** (2019), no. 2, 332–348.

L. Zhang, *Abundance for 3-folds with non-trivial Albanese maps in positive characteristic*, to appear in J. Eur. Math. Soc. arXiv: 1705.00847.

Department of Mathematics, Imperial College London, 180 Queen’s Gate, London SW7 2AZ, UK

E-mail address: p.cascini@imperial.ac.uk

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail address: shoejiri.math@gmail.com, s-ejiri@cr.math.sci.osaka-u.ac.jp

School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, P. R. of China

E-mail address: zhlei18@ustc.edu.cn