The $\sigma$ and $\rho$ in $D$ and $B$ decays

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Abstract

We study the $D^+ \rightarrow \sigma \pi^+$, $D^+ \rightarrow \rho^0 \pi^+$, $B^- \rightarrow \sigma \pi^-$, $B^- \rightarrow \rho^0 \pi^-$ and $\bar{B}^0 \rightarrow \rho^\mp \pi^\pm$ decays in a valence quark triangle model, incorporating chiral symmetries. We find a good agreement with recent experimental data for $D^+ \rightarrow \rho^0 \pi^+$ and for $D^+ \rightarrow \rho^0 \pi^+$. We point out that a long-distance contribution due to the axial vector $a_1$ meson pole, calculated by using chiral symmetry, can be relevant to explain $D^+ \rightarrow \rho^0 \pi^+$ and for lowering the ratio

$$R = \frac{\mathcal{B}(\bar{B}^0 \rightarrow \rho^\mp \pi^\pm)}{\mathcal{B}(B^- \rightarrow \rho^0 \pi^-)}$$

to be consistent with its phenomenological determination, within the large experimental uncertainty.
1 Introduction

Recently there has been a revival of interest \[1\text{–}6\] in a broad scalar-isoscalar light $\pi\pi$ resonance, the $\sigma$ meson, which has been controversial for a long time. It has appeared in the Reviews of Particle Physics \[7\], as a broad resonance under the entry $f_0\,(400 - 1200)$ or $\sigma$. The E791 collaboration measurement of the $D^+ \rightarrow 3\pi$ rate provides an evidence for a scalar resonance $\sigma$ having mass $m_\sigma = 478 \pm 24$ MeV and width $\Gamma_\sigma = 324 \pm 41$ MeV; the $\sigma$ is seen as a dominant peak leading to a fit in which 46% of the rate occurs via $D^+ \rightarrow \sigma\pi^+$ while 33% of the rate occurs via $D^+ \rightarrow \rho^0\pi^+$ \[8\]. There has been considerable interest in explaining these rates \[1\].

The effective weak Hamiltonian for the above decays can be written as \[9\]

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{ud}^* V_{ud} \left\{ a_1 (\bar{d}c)_{V-A} (\bar{u}d)_{V-A} + a_2 (\bar{d}d)_{V-A} (\bar{u}c)_{V-A} \right\},$$

(1)

where, in the factorization ansatz, $a_1 = 1.10 \pm 0.05$ and $a_2 = -0.49 \pm 0.04$ fitted with $D$-decays \[9\] \[10\]. In this ansatz the relevant transition matrix elements are given as $[A_\mu = d\gamma_\mu \gamma_5 c, V_\mu = \bar{u}\gamma_\mu c]$:

$$\langle \sigma (k) \pi^+ (q) | H_{\text{eff}} | D^+ (p) \rangle = \frac{G_F}{\sqrt{2}} V_{ud}^* V_{ud} a_1 f_\pi (-iq^\mu) \langle \sigma (k) | A_\mu | D^+ (p) \rangle,$$

(2)

$$\langle \rho (k) \pi^+ (q) | H_{\text{eff}} | D^+ (p) \rangle = \frac{G_F}{\sqrt{2}} V_{ud}^* V_{ud} \left[ a_1 f_\pi (-iq^\mu) \langle \rho (k) | A_\mu | D^+ (p) \rangle \right.$$

$$\left. + a_2 \left( - \frac{f_\rho}{\sqrt{2}} \right) \epsilon^{* \mu} \langle \pi (q) | V_\mu | D^+ (p) \rangle \right].$$

(3)

The problem thus reduces to evaluating the form factors

$$A_{\mu}^{D\sigma} = \langle \sigma (k) | A_\mu | D^+ (p) \rangle$$

$$= i \left[ G_+ (q^2) (p + k)_\mu + G_- (q^2) (p - k)_\mu \right]$$

$$= i \left[ \left( \frac{m_D^2 - m_\sigma^2}{q^2} \right) q_\mu G_0 (q^2) + \left( (p + k)_\mu - \frac{m_D^2 - m_\sigma^2}{q^2} q_\mu \right) G_1 (q^2) \right],$$

(4)

$$A_{\mu}^{D\rho} = \langle \rho (k) | A_\mu | D^+ (p) \rangle$$

$$= i \left[ \left( \epsilon^{* \mu} - q_\mu \frac{\epsilon^* \cdot q}{q^2} \right) (m_\rho + m_D) A_1 (q^2) - \left( (p + k)_\mu - \frac{m_D^2 - m_\rho^2}{q^2} q_\mu \right) \epsilon^{* \mu} \cdot q \frac{A_2 (q^2)}{m_\rho + m_D} \right.$$

$$\left. + q_\mu \epsilon^{* \mu} \cdot q \frac{2m_\rho}{q^2} A_0 (q^2) \right],$$

(5)

$$V_{\mu}^{D\pi} = \langle \pi^+ (q) | V_\mu | D^+ (p) \rangle$$

$$= F_+ (k^2) (p + q)_\mu + F_- (k^2) (p - q)_\mu.$$ 

(6)

Thus we obtain:

$$\langle \sigma (k) \pi^+ (q) | H_{\text{eff}} | D^+ (p) \rangle = \frac{G_F}{\sqrt{2}} V_{ud}^* V_{ud} a_1 f_\pi \left( m_D^2 - m_\sigma^2 \right) \left[ G_0^{D\sigma} (m_\pi^2) \right],$$

(7)
\[ \langle \rho (k) \pi^+ (q) | H_{\text{eff}} | D^+ (p) \rangle = \frac{G_F}{\sqrt{2}} V_{cd}^* V_{ud} a_1 f_{\pi} (2m_\rho) \left( e^* \cdot q \right) \left[ A_0^{D\rho} \left( m_\pi^2 \right) - \frac{a_2}{a_1} \frac{f_\rho}{\sqrt{2} f_\pi m_\rho} F_+^{D\pi} \left( m_\pi^2 \right) \right]. \] (8)

We evaluate the above form factors \( G_0^{D\sigma} \) and \( A_0^{D\rho} \) in the model based on the constituent quark “triangle” graph of Fig. 1. It is in this respect that we differ from the calculation in Ref. [4]. Moreover, we take into account the “long-distance” contribution coming through the \( a_1^- \)-pole shown in Fig. 2, which has not been previously considered. Here, the weak vertex in the factorization ansatz can be expressed as

\[ \langle a_1 | H_w | D^+ \rangle = \frac{G_F}{\sqrt{2}} V_{cd}^* V_{ud} a_1 f_{a_1} (i f_D p^\mu), \] (9)

with \( f_D \) the leptonic constant of the \( D \) meson, while the strong vertices are defined by

\[ \langle \sigma (k) \pi^+ (q) | a_1 (p) \rangle = \frac{i}{2} \gamma_{a_1 \sigma \pi} \eta \cdot (q - k), \] (10)

corresponding to the Lagrangian \( (\sigma \partial_{\mu} \pi - \pi \partial_{\mu} \sigma) \cdot a_1^\mu \). Moreover,

\[ \langle \rho (k) \pi^+ (q) | a_1 (p) \rangle = i \left( m_{a_1}^2 - m_\rho^2 \right) \eta \cdot e^* f_{a_1 \rho \pi}, \] (11)

where we have neglected the \( D \)-wave coupling \( g_{a_1 \rho \pi} \), for which there is evidence to be negligible [7]:

\[ \frac{\text{D-wave amplitude}}{\text{S-wave amplitude}} = -0.107 \pm 0.016. \] (12)

The above considerations can be easily extended to \( B^- \to \rho^0 \pi^- \) and \( \bar{B}^0 \to \rho^\pm \pi^\mp \), where the \( a_1^- \) contributes to \( B^- \to \rho^0 \pi^- \) and only in a negligible way (being proportional to \( a_2 \)) to \( \bar{B}^0 \to \rho^\pm \pi^\mp \). In principle, this provides a mechanism to lower the value of the ratio

\[ \mathcal{R} = \frac{\mathcal{B} (\bar{B}^0 \to \rho^\pm \pi^\mp)}{\mathcal{B} (B^- \to \rho^0 \pi^-)}. \]

Previous theoretical estimates computed in the simple factorization ansatz of Ref. [9] tend to give this ratio much larger than its experimental value: \( 2.65 \pm 1.8 \) or \( 2.0 \pm 1.3 \) determined, respectively, from the measured indicated branching ratios in [11] and [12]. Recent efforts to understand the size of this ratio have been published, e.g., in Refs. [5] and [13–16].

2 Form factors and \( D^+ \to \sigma \pi^+ \) and \( D^+ \to \rho^0 \pi^+ \) decays

The valence quark contribution shown in Fig. 1 gives

\[ J^{(\sigma)}_\mu = \int \frac{d^3 K}{(2\pi)^3} K^{(\sigma)}_\mu, \] (13)
where $F^{(\sigma)}_\mu$ is the matrix element

$$F^{(\sigma)}_\mu = -ig_{\sigma q} \sqrt{\frac{m_d}{p_{do}}} \bar{v}^i(p_d) \left( \frac{p_d^i + m_d^k}{p_d^2 - m_d^2} \right) (\gamma_\mu \gamma_5)^k \gamma_i (p_c) \sqrt{\frac{m_c}{p_{co}}} \times \left( \sqrt{2m_D} \frac{1}{\sqrt{2}} \sqrt{3} \bar{m} (p_c) (\gamma_5)^n v_n (p_d) \right) \phi_D (K).$$

(14)

Here, the term within the parenthesis is the bound state wave function of the $D$-meson, $\sqrt{3}$ being the color factor. We define the kinematical variables $K = p_c - p_d$ and $P = p_c + p_d$, so that $K$ is the relative momentum and $P$ is the center of mass momentum of the $c\bar{d}$ system.

The evaluation of the trace implied in Eq. (14) gives:

$$F^{(\sigma)}_\mu = -4iC(K) g_{\sigma q} \{ (p_c \cdot p_d + m_c m_d) p_d^\mu \}
-(p_c' \cdot p_c + m_c m_d) p_d^\mu + (p_c' \cdot p_d - m_d^2) p_c^\mu \}
\frac{1}{p_d^2 - m_d^2},$$

(15)

where

$$C(K) = \sqrt{2m_D} \frac{1}{\sqrt{2}} \sqrt{3} \bar{m} (p_c) (\gamma_5)^n v_n (p_d) \phi_D (K).$$

(16)

Working in the $D$-meson rest frame ($P = 0$), where

$$p_d^2 - m_d^2 = \frac{m_c^2 - m_d^2 + m_d^2}{2} \left( 1 - \frac{q^2}{m_D^2} \right) + \frac{m_c^2 + m_d^2 - m_d^2}{2} \frac{k^2}{m_D^2} + q \cdot K,$$

(17)

and noting that, if $\phi_D (K)$ is of Gaussian type, $K \approx 0$ dominates in the integration [17], one obtains [18]

$$J^{(\sigma)}_\mu = 4iC(0) g_{\sigma q} \frac{m_c^2 - (m_c - m_d)^2}{2m_D^2} \frac{1}{m_D^2 - m_c^2 + m_d^2} \left\{ \frac{1}{1 - \frac{q^2}{m_D^2} - a \frac{k^2}{m_D^2}} \right\} \times \left\{ (m_D^2 - 2m_d (m_c + m_d)) (p + k)_\mu - (m_D^2 + 2m_d (m_c + m_d)) q_\mu \right\},$$

(18)

where

$$a = \frac{m_c^2 + m_d^2 - m_d^2}{m_D^2 - m_c^2 + m_d^2}.$$

Note that, in the above approximation, $4\pi \int K^2 dK \phi_D (K)$ becomes $\int d^3 K \phi_D (K)$, which is the Fourier transform of the wave function at the origin, and we write it as $\phi_D (0)$ or equivalently $C(0)$.
To eliminate $4C(0)$, we consider the matrix element

$$\langle 0 | A_\mu | D(p) \rangle = i f_D p_\lambda$$

which, when evaluated in the same valence quark approximation employed for the calculation of $J^{(\sigma)}_\mu$, gives:

$$f_D = \frac{4C(0)}{2m_D^2} (m_c + m_d) \left[ m_D^2 - (m_c - m_d)^2 \right].$$

(20)

Thus, we finally obtain the valence quark triangle contribution

$$G_+ (q^2) = f_D \sigma q \bar{q} \left( \frac{m_D^2 - 2m_d (m_c + m_d)}{m_D^2 - m_c^2 + m_d^2} \right) \frac{1}{1 - \frac{q^2}{m_D^2} - \frac{k^2}{m_D^2}},$$

(21)

and, for $k^2 = m_D^2$, this gives:

$$G_{0D}^\sigma (m_\pi^2) \simeq G_+ (0) = f_D \sigma q \bar{q} \left( \frac{m_D^2 - 2m_d (m_c + m_d)}{m_D^2 - m_c^2 + m_d^2} \right) \frac{1}{1 - \frac{m_D^2}{m_D^2}}.$$

(22)

An exactly similar calculation for the case of the $\rho^0$ in the $\rho$-dominance approximation ($k^2 = 0$), so that $g_{\rho d} f_\rho = -\frac{1}{2}$, gives:

$$-i q^\mu J^{(\rho)}_{\mu} \equiv (q \cdot \epsilon^*) (2m_\rho) A_0 (q^2) = -\frac{\sqrt{2} m_\rho^2}{2 f_\rho} \frac{1 + \frac{q^2}{m_\rho^2} q \cdot \epsilon^*}{1 - \frac{q^2}{m_\rho^2}}.$$

(23)

Thus, for $q^2 \simeq m_\pi^2 \simeq 0$, on using the KSRF relation $f_\rho = \sqrt{2} f_\pi m_\rho$ [19], we find:

$$A_{0D}^{\rho} (0) = -\frac{1}{4} \left( \frac{f_D}{f_\pi} \right).$$

(24)

Note that this result is independent of quark masses in contrast to Eq. (22). It is, however, subject to a suppression factor $F_\rho (0)$ due to the off-mass-shellness of the $\rho$-meson [$F_\rho (m_\rho^2) = 1$]. From the $\rho$-dominance of the pion form factor, the experimental determination $\gamma_{\rho\pi\pi} \frac{F_\rho}{m_\rho^2} = 1.22 \pm 0.03$ indicates $F_\rho (0) \simeq 0.8$ [20]. Accordingly, we rewrite Eq. (24) as:

$$A_{0D}^{\rho} (0) = -\frac{1}{4} \left( \frac{f_D}{f_\pi} \right) F_\rho (0) = -1.52 f_D \text{ GeV}^{-1}. $$

(25)

To account for the effect of the $a_2$-term in Eq. (8), we use the KSRF relation and the numerical value [21],

$$F_{0D}^{\rho} (m_\rho^2) \simeq \frac{1}{1 - \frac{m_\rho^2}{m_D^2}} \frac{F_{0D}^{\rho} (0)}{1 - \frac{m_\rho^2}{m_{D'}^2}} \simeq \frac{1}{4} \left( \frac{1.62}{f_\pi} \right).$$

(26)
with \( F^D_\pi(0) \simeq 0.3 f_D, \frac{m_\rho}{m_D} = 1.14, \) \( D' \) being the radial excitation of the \( D \). Using \( f_D = 0.23 \text{ GeV} \) \cite{22}, \( F^D_\pi(0) = 0.53 \), not inconsistent with its other estimates \cite{23}. With \(-a_2/a_1 = 0.44\), the square bracket on the right-hand side of Eq. (8) has the value

\[
\left[ A^D_0(0) + 0.44 F^D_\pi \left( m_\rho^2 \right) \right] \simeq -\left[ F_\rho(0) - 0.71 \right] \frac{1}{4} \frac{f_D}{f_\pi} = -0.09 \times \frac{1}{4} \frac{f_D}{f_\pi}. \tag{27}
\]

This indicates that, in the framework used here, the \( a_2 \)-term of Eq. (1) can give a significant contribution to the \( D \to \rho \pi \) channel.

To obtain the numerical estimate for \( G^D_\rho(0) \) from Eq. (22), we have to first fix \( g_{\sigma q q} \).

The linear \( \sigma \)-model gives \cite{3, 24, 25}:

\[
v = \langle \sigma \rangle = \frac{f_\pi}{\sqrt{2}}; \quad g = g_{\sigma q q} = g_{\pi q q}; \quad g_{\sigma \pi \pi} = 2 \lambda v = 2g'; \tag{28}
\]

\[
m^2_\sigma = 2 \lambda v^2; \quad m_q = gv = g \frac{f_\pi}{\sqrt{2}}; \quad g' = 2gm_q = \sqrt{2g^2 f_\pi}. \tag{29}
\]

From these relations one finds:

\[
g_{\sigma \pi \pi} = \frac{\sqrt{2}m_\rho^2}{f_\pi} = 2g' \tag{30}
\]

\[
g_{\sigma q q} = g = \left( \frac{g'}{\sqrt{2}f_\pi} \right)^{1/2} = \left( \frac{g_{\sigma \pi \pi}}{2\sqrt{2}f_\pi} \right)^{1/2} = \frac{m_\sigma}{\sqrt{2}f_\pi} \simeq 2.57 \tag{31}
\]

Using Eqs. (30) and (31), \( m_D = 1.87 \text{ GeV} \) and \( m_c = 1.45 \text{ GeV} \), we obtain

\[
G^D_\rho(0) = 3.7 f_D \text{ GeV}^{-1} \tag{32}
\]

The \( a_1 \)-pole contribution from Fig. 2 gives, on using Eqs. (33):

\[
\langle \sigma (k) \pi^+(q) | H_{\text{eff}} | D^+(p) \rangle = \frac{G_F}{\sqrt{2}} V^*_{cd} V^*_{ud} a_1 \left( if_D f_{a_1} \right) p_\mu \\
\times \left[ \frac{-g^{\mu \lambda} + p^\mu p^\lambda}{m_{a_1}^2} \frac{-1}{m_{a_1}^2 - p^2} \frac{i}{2} \gamma_{a_1 \pi \sigma} (q - k) \right]_\mu \\
= \frac{-G_F}{\sqrt{2}} V^*_{cd} V^*_{ud} a_1 f_D f_{a_1} \gamma_{a_1 \pi \sigma} \frac{p \cdot (q - k)}{2m_{a_1}^2}, \tag{33}
\]

\[
\langle \rho (k) \pi^+(q) | H_{\text{eff}} | D^+(p) \rangle = \frac{G_F}{\sqrt{2}} V^*_{cd} V^*_{ud} a_1 f_D f_{a_1} i p_\mu \\
\times \left[ \frac{-g^{\mu \lambda} + p^\mu p^\lambda}{m_{a_1}^2} \frac{-1}{m_{a_1}^2 - p^2} - \frac{i}{2} \frac{\left( m_{a_1}^2 - m_\rho^2 \right) f_{a_1 \rho \sigma} \epsilon^*_\lambda}{f_{a_1 \rho \sigma} q \cdot \epsilon^*} \right] \tag{34}
\]

\]

5
Now \( p \cdot (q - k) = m_\pi^2 - m_\sigma^2 \) independent of \( p^2 \), and the above equations give, in the square brackets on the right-hand sides of Eqs. (7) and (8), the additional contributions to \( G_{a_1}^{D\sigma} \) and \( A_{a_1}^{D\rho} \), respectively:

\[
G_{a_1}^{D\sigma} = -\frac{f_D f_{a_1} m_\pi^2 - m_\sigma^2}{f_\pi^2 m_D^2 - m_\sigma^2} \frac{1}{2m_{a_1}^2} \gamma_{a_1\sigma\pi} \quad (35)
\]

\[
A_{a_1}^{D\rho} = -\frac{f_D f_{a_1} m_\sigma^2 - m_\rho^2}{2m_\rho m_{a_1}^2} f_{a_1\rho\pi}. \quad (36)
\]

Moreover, the effective Lagrangian approach to Chiral symmetry gives \[26\]:

\[
g_{a_1\rho\pi} = 0, \quad f_{a_1\rho\pi} = \frac{1}{\sqrt{2} f_\pi}, \quad m_{a_1} = \sqrt{2} m_\rho,
\]

\[
f_{a_1} = f_\rho = \sqrt{2} f_\pi m_\rho, \quad \gamma_{a_1\sigma\pi} = \sqrt{2} \gamma_{\rho\pi\pi} = \sqrt{2} \frac{m_\rho}{f_\pi}.
\]

Using the above relations, we obtain for Eqs. (35) and (36) the numerical values

\[
G_{a_1}^{D\sigma} = -\frac{1}{2} \frac{f_D m_\pi^2 - m_\sigma^2}{f_\pi^2 m_D^2 - m_\sigma^2} = 0.27 f_D \text{ GeV}^{-1} \quad (37)
\]

\[
A_{a_1}^{D\rho} = -\frac{1}{4} \frac{f_D}{f_\pi} = -1.9 f_D \text{ GeV}^{-1} \quad (38)
\]

and finally, using Eqs. (27), (32), (37) and (38), the total contributions to the square brackets in the right-hand sides of Eqs. (7) and (8) become:

\[
\left[ G_0^{D\sigma} + G_{a_1}^{D\sigma} \right] \simeq [1 + 0.073] 3.7 f_D \text{ GeV}^{-1}, \quad (39)
\]

\[
\left[ A_0^{D\rho} + 0.44 F_+^{D\sigma} (m_\rho^2) + A_{a_1}^{D\rho} \right] \simeq [-0.09 + 1] (1.9) f_D \text{ GeV}^{-1}. \quad (40)
\]

For \( f_D \simeq 230 \text{ MeV} \), one gets

\[
\left[ G_0^{D\sigma} + G_{a_1}^{D\sigma} \right] \simeq 0.91, \quad (41)
\]

to be compared with \( 0.79 \pm 0.15 \) needed \[2, 3\] to explain the experimental branching ratio for \( D^+ \to \sigma \pi^+ \). Clearly, predicted branching ratios depend on the actual values of \( f_D \) (and \( f_B \)) which, hopefully, will be experimentally determined in the near future \[27\]. With the same values we obtain, from Eq. (40), the width \( \Gamma (D^+ \to \rho^0 \pi^+) = 10.39 \times 10^{-16} \text{ GeV} \) giving the branching ratio \( 1.66 \times 10^{-3} \) to be compared with its experimental value \( (1.05 \pm 0.31) \times 10^{-3} \) \[7\].

If we extend the previous analysis to \( D_s \to \phi \pi \) where \( \phi (1020) \) is treated as a pure \( \bar{s}s \) state, we obtain \( \langle 0 | \bar{s} \gamma_{\mu} s | \phi \rangle = f_\phi \epsilon_\mu \):

\[
A_{a_1}^{D\phi} \simeq \frac{f_D m_\phi^2}{2m_\rho f_\phi}. \quad (42)
\]
In this case the intermediate $a_1$-exchange should be absent and, in the factorization approximation, the $a_2$-term in $H_{\text{eff}}$ should not contribute. Using $f_\phi \simeq 0.23$ GeV$^2$ from $\Gamma(\phi \to e^+e^-)$, we would obtain

$$A_{0}^{D\phi} \simeq 2.2 f_D, \text{ GeV}^{-1} \simeq 0.62.$$  \hfill (43)

This leads to $\Gamma(D_s \to \phi \pi) \approx 2.8 \times 10^{-14}$ GeV and $B(D_s \to \phi \pi) \approx 2.1\%$, compatible with the experimentally measured value $3.6 \pm 0.9\%$ \cite{7} and the theoretical estimate of Ref. \cite{28}.

### 3 $B \to \sigma \pi$, $B \to \rho \pi$ decays

The effective weak Hamiltonian is given by \cite{9}

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{ud} \left\{ a_1 (\bar{u}b)_{V-A} (\bar{d}u)_{V-A} + a_2 (\bar{d}b)_{V-A} (\bar{u}u)_{V-A} \right\},$$ \hfill (44)

where the Wilson coefficients $c_1$ and $c_2$, fitted for $B$-decays, are $c_1 (m_b) = 1.105$ and $c_2 (m_b) = -0.228$ so that $a_1 = c_1 + \frac{1}{3} c_2 = 1.03$ and $a_2 = c_2 + \frac{1}{3} c_1 = 0.14$. The factorization ansatz gives for the decay $B^- \to \sigma \pi^-$ the analogue of Eqs. (21) and (22). With $m_B = 5.28$ GeV, $m_b = 4.757$ GeV, $m_d = 0.240$ GeV, one obtains:

$$\langle \sigma (k) \pi^- (q) | H_{\text{eff}} | B^- (p) \rangle = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{ud} a_1 f_\pi \left( m_B^2 - m_\sigma^2 \right) \left[ G_0^{B\sigma} (m_\pi^2) \right],$$ \hfill (45)

and the valence quark triangle contribution

$$G_0^{B\sigma} = 2.67 f_B \text{ GeV}^{-1}. $$ \hfill (46)

With $f_B = 0.150$ GeV, this gives [the $a_1$-pole contribution is negligible because of the factor $(m_\sigma^2/m_B^2) / (1 - m_\sigma^2/m_B^2)$ in Eq. (37)]:

$$G_0^{B\sigma} = 0.4,$$ \hfill (47)

consistent with the value found in \cite{5}.

For $B \to \rho \pi$ decays, using the factorization ansatz:

$$\langle \rho^0 (k) \pi^- (q) | H_{\text{eff}} | B^- (p) \rangle = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{ud} \left[ a_1 f_\pi \left( -i q^\mu \right) \langle \rho^0 (k) | A_\mu | B^- (p) \rangle + a_2 \left( \frac{f_\rho}{\sqrt{2}} \right) e^{\nu \mu} \langle \pi^- (q) | V_\mu | B^- (p) \rangle \right]$$

1 Treating the $f_0(980)$ as a pure $s\bar{s}$ state we would obtain from the analogous quark triangle diagram, with $m_s \approx 1.6 m_q$, a value for $B(D_s \to f_0 \pi)$ substantially larger than the experimental one (and the result of \cite{28}). To have agreement we would require a mixing angle with the nonstrange scalar-isoscalar component of the order of 10 - 20 degrees for $m_q = (0.24 - 0.31)$ GeV. Thus, our model does not favour the description of $f_0$ as a pure $s\bar{s}$ state.
the quark triangle diagrams give:

\[
\langle \rho^+ (k) \pi^- (q) | H_{\text{eff}} | B^0 (p) \rangle = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{ud} a_1 f_\pi (2m_\rho) \epsilon^* \cdot q A_0^{B\rho^0} \left( m_\pi^2 \right) + a_2 \left( \frac{f_\rho}{\sqrt{2}} \right) (2\epsilon^* \cdot q) F_+^{B-\pi^-} \left( m_\rho^2 \right),
\]

(48)

\[
\langle \rho^- (k) \pi^+ (q) | H_{\text{eff}} | \bar{B}^0 (p) \rangle = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{ud} a_1 f_\pi (2m_\rho) \epsilon^* \cdot q \left[ A_0^{B\rho^+} \left( m_\pi^2 \right) \right],
\]

(49)

\[
\langle \pi^+ (k) | V_\mu | \bar{B}^0 (p) \rangle = (p + q)_\mu F_+ \left( k^2 \right) + (p - q)_\mu F_- \left( k^2 \right).
\]

(50)

Here: \( A_\mu = \bar{u} \gamma_\mu \gamma_5 b \), \( V_\mu = \bar{u} \gamma_\mu b \), and

\[
\langle \pi^+ (k) | V_\mu | \bar{B}^0 (p) \rangle = (p + q)_\mu F_+ \left( k^2 \right) + (p - q)_\mu F_- \left( k^2 \right).
\]

(51)

Noting the relations

\[
g_{\rho^+ ud} = \sqrt{2} g_{\rho^0 ub} = \frac{m_\rho^2}{f_\rho} = \frac{m_\rho}{2f_\pi},
\]

the quark triangle diagrams give:

\[
A_0^{B\rho^+} = \sqrt{2} A_0^{B-\rho^0} = \frac{\sqrt{2}}{4} \frac{f_B}{f_\pi} = \sqrt{2} (0.25) \frac{f_B}{f_\pi}.
\]

(52)

The form factor \( F_{\rho^0 \pi^-} \) introduced in Eq. (51) has been found to be about 0.30 [23, 18], so that, with \( f_B = 0.150 \) GeV [notice that, here, \( F_{\rho^0 \pi^-}(m_\rho^2) \approx F_{\rho^0 \pi^-}(0) \) to a very good approximation as \( m_\rho^2/m_B^2 \) corrections are negligible):

\[
F_{\rho^0 \pi^-}(0) = F_{\rho^0 \pi^-}(0) \approx 0.26 \frac{f_B}{f_\pi}.
\]

(53)

Now, the \( a_1^- \) -pole contributes to \( B^- \to \rho^0 \pi^- \) and, in vacuum saturation, negligibly to \( \bar{B}^0 \to \rho^\pm \pi^\mp \), the latter contribution being controlled by the small \( a_2 \) coefficient. This can enhance the branching ratio for \( B^- \to \rho^0 \pi^- \) and, as such, provide a mechanism (in addition to the \( \sigma \)-contribution to \( B^- \to \rho^0 \pi^- \) decay [5]) to lower the ratio \( R \). The additional, intermediate \( a_1^- \)-contribution to be included in the square brackets on the right-hand sides of Eqs. (48)-(50), see Eq. (38), is given by

\[
A_{a_1}^{B-\rho^0} = (0.25) \frac{f_B}{f_\pi}.
\]

(54)

One can note the change of sign since the \( a_1^- \to \rho^0 \pi^- \) coupling has sign opposite to \( a_1^+ \to \rho^0 \pi^+ \), and similar is the case for the relative signs of \( a_1^0 \to \rho^+ \pi^- \) and \( a_1^0 \to \rho^- \pi^+ \).
Thus, on using Eqs. (48)-(54), and the suppression factor $F_\rho(0) \simeq 0.8$ to take care of the off-mass-shellness of the $\rho$-meson in Eq. (52), one finds [we also include the small contribution controlled by $a_2/a_1 \simeq 0.13$ of the $a_1$ meson to the $\rho^\pm\pi^\mp$ modes]:

$$R = \left(\sqrt{2}\right)^2 \frac{[0.20 + 0.25 \cdot 0.13/\sqrt{2}]^2 + [0.26 - 0.25 \cdot 0.13/\sqrt{2}]^2}{[0.20 + 0.26 \cdot 0.13 + 0.25]^2} \approx 0.91,$$

(55)
in the lower range, but still consistent with the interval allowed by the experimental determination. Note that this ratio is almost independent of the value of $f_B/f_\pi$, and that the effect of the $a_2$-term of Eq. (44) is almost negligible\(^2\). The individual branching ratio is

$$B(B^- \to \rho^0\pi^-) = 1.99 |V_{ub}|^2 = (2.43 \pm 2.08) \times 10^{-5}$$

for $|V_{ub}| = (3.5 \pm 1.5) \times 10^{-3}$, that is compatible, within the uncertainty, with the experimental upper limit $B < (1.0 \pm 0.4) \times 10^{-5}$ [7].

\section{Conclusions}

Our analysis of the decays $D^+ \to \sigma\pi^+$, $D^+ \to \rho^0\pi^+$, $B^- \to \sigma\pi^-$, $B^- \to \rho^0\pi^-$ and $\bar{B}^0 \to \rho^\pm\pi^\mp$ show that the valence quark "triangle" graph, supplemented by the long distance $a_1$-exchange, is in reasonable agreement with the available branching ratios, in particular with $D^+ \to \rho^0\pi^+$ and that of $D^+ \to \sigma\pi^+$ recently measured. The contribution from the $a_1$-pole has also been found important. In particular, the inclusion of this contribution gives the ratio

$$R = \frac{B(\bar{B}^0 \to \rho^\pm\pi^\mp)}{B(B^- \to \rho^0\pi^-)} \approx 0.9,$$

consistent with the experimental values within the large experimental uncertainties. More accurate determinations of this ratio would provide a stringent test of the model presented here.

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\(^2\) Actually, in principle the "long-distance" $a_1$-meson contribution could be subject to a suppression factor taking into account the $a_1$ off-mass-shellness. This effect does not relate to the $a_1$-meson propagator, that is cancelled by a corresponding numerator, see Eqs. (33)-(36), but may reside in the coupling $f_{a_1\rho\pi}$. Such correction might be taken into account by introducing a $B$-factor $B_{a_1}$. Assuming $B_{a_1} \simeq 0.7 - 0.8$, i.e., the same order of magnitude found for $K - \bar{K}$ and $B - \bar{B}$ mixing [23], the correction would slightly increase the numerical result for $R$ in Eq. (55), thus improving the agreement with the experimental value.
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Figure 1: Quark triangle graph for $\langle \sigma, \rho^0 | d\gamma_\mu \gamma_5 c | D^+ \rangle$

Figure 2: $a_1$-pole contribution to $D^+ \to \sigma (\rho^0) \pi^+$