GENERALIZED CESÀRO OPERATORS ON DIRICHLET-TYPE SPACES

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ABSTRACT. In this note, we introduce and study a new kind of generalized Cesàro operators $C_\mu$, induced by a positive Borel measure $\mu$ on $[0,1)$, between the Dirichlet-type spaces. We characterize the measures $\mu$ for which $C_\mu$ is bounded (compact) from one Dirichlet-type space $D_\alpha$ into another one $D_\beta$.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. We use $C, C_1, C_2, \cdots$ to denote universal positive constants that might change from one line to another. For two positive numbers $A, B$, we write $A \preceq B$, or $A \succeq B$, if there exists a positive constant $C$ independent of $A$ and $B$ such that $A \leq CB$, or $A \geq CB$, respectively. We will write $A \asymp B$ if both $A \preceq B$ and $A \succeq B$.

We denote by $\mathcal{H}(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let $0 < p < \infty$, the Hardy space $H^p$ is the class of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{r \in (0,1)} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{\frac{1}{p}}, \quad 0 < r < 1.$$

For $\alpha \in \mathbb{R}$, the Dirichlet-type space, denoted by $D_\alpha$, is defined as

$$D_\alpha = \{ f = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D}) : \|f\|_{D_\alpha} := \left( \sum_{n=0}^{\infty} (n+1)^{1-\alpha} |a_n|^2 \right)^{\frac{1}{2}} < \infty \}.$$

When $\alpha = 0$, $D_0$ coincides the classic Dirichlet space $D$, and when $\alpha = 1$, $D_1$ becomes the Hardy space $H^2$.

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The Cesàro operator, which is an operator on spaces of analytic functions by its action on the Taylor coefficients, is defined as, for $f = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D}),$

$$\mathcal{C}(f)(z) = \sum_{n=0}^{\infty} \left( \frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n, z \in \mathbb{D}. $$

The boundedness and compactness of the Cesàro operator and its generalizations defined on various spaces of analytic functions, like Hardy spaces, Bergman spaces and Dirichlet spaces, have attracted many attentions (see for example, [2], [3], [4], [5], [9], [6], [7], [8], [1] and the references therein).

In this note, we consider the boundedness and compactness of Cesàro operator between the Dirichlet-type spaces. We denote $N_0 = \mathbb{N} \cup \{0\}.$ When $0 < \alpha < 2.$ For $f = \sum_{n=0}^{\infty} a_n z^n \in D_\alpha,$ by Cauchy’s inequality, we obtain that, for $n \in N_0,$

\[
\left| \frac{1}{n+1} \sum_{k=0}^{n} a_k \right| = \frac{1}{n+1} \left| \sum_{k=0}^{n} (k+1)^{\frac{2-\alpha}{2}} a_k \right| \left[ (k+1)^{\frac{2-\alpha}{2}} \right] \leq \frac{1}{n+1} \left[ \sum_{k=0}^{n} (k+1)^{\frac{2-\alpha}{2}} a_k^2 \right]^{\frac{1}{2}} \left[ \sum_{k=0}^{n} (k+1)^{-\frac{2-\alpha}{2}} \right]^{\frac{1}{2}}.
\]

For $0 < \alpha < 2,$ it is easy to see that

\[
\sum_{k=0}^{n} (k+1)^{-\frac{2-\alpha}{2}} = \sum_{j=1}^{n+1} j^{-\frac{2-\alpha}{2}} \leq \int_{1}^{n+1} t^{-\frac{2-\alpha}{2}} dt = \frac{2}{\alpha} (n+1)^{\frac{\alpha}{2}}.
\]

Consequently, we get from (1.1) and (1.2) that

\[
\| \mathcal{C}(f) \|_{D_\alpha} = \left[ \sum_{n=0}^{\infty} (n+1)^{1-\alpha} \left| \frac{1}{n+1} \sum_{k=0}^{n} a_k \right|^2 \right]^{\frac{1}{2}} \leq \left( \frac{2}{\alpha} \right)^{\frac{1}{2}} \left[ \sum_{n=0}^{\infty} \frac{1}{(n+1)^{\frac{2-\alpha}{2}}} \sum_{k=0}^{n} (k+1)^{\frac{2-\alpha}{2}} a_k^2 \right]^{\frac{1}{2}} = \left( \frac{2}{\alpha} \right)^{\frac{1}{2}} \left[ \sum_{k=0}^{\infty} (k+1)^{1-\alpha} \sum_{n=k}^{\infty} \frac{(k+1)^{\frac{\alpha}{2}}}{(n+1)^{\frac{2-\alpha}{2}}} a_k^2 \right]^{\frac{1}{2}}.
\]

We notice that, for $k \in N_0,$

\[
\sum_{n=k}^{\infty} \frac{(k+1)^{\frac{\alpha}{2}}}{(n+1)^{\frac{2-\alpha}{2}}} = \frac{1}{k+1} + \sum_{n=k+1}^{\infty} \frac{(k+1)^{\frac{\alpha}{2}}}{(n+1)^{\frac{2-\alpha}{2}}} \leq \frac{1}{k+1} + \int_{k}^{\infty} \frac{(k+1)^{\frac{\alpha}{2}}}{(t+1)^{\frac{2-\alpha}{2}}} dt \leq \frac{1}{k+1} + \frac{2}{\alpha} \leq \frac{2 + \alpha}{\alpha}.
\]
It follows from (1.3) and (1.4) that
\[ \|C(f)\|_{D_\alpha} \leq \frac{\sqrt{2(2+\alpha)}}{\alpha} \|f\|_{D_\alpha}. \]

This means that \( C : D_\alpha \rightarrow D_\alpha \) is bounded for \( 0 < \alpha < 2 \). We have proved the following

**Proposition 1.1.** Let \( 0 < \alpha < 2 \), then the Cesàro operator \( C \) is bounded from \( D_\alpha \) into itself.

It is natural to ask whether the Cesàro operator is still bounded from \( D_\alpha \) into \( D_\beta \), when \( \alpha \neq \beta \).

We observe that the Cesàro operator \( C \) is not bounded from \( D_\alpha \) into \( D_\beta \), if \( \alpha > \beta \) and \( 0 < \alpha < 2 \). Actually, if \( 0 < \alpha < 2 \) and \( \alpha > \beta \), let \( 0 < \varepsilon < \alpha \) and set \( f = \sum_{n=0}^{\infty} a_n z^n \) with
\[ a_n = \sqrt{\frac{\varepsilon}{1+\varepsilon}} (n+1)^{-\frac{2-\alpha+\varepsilon}{2}}, \quad n \in \mathbb{N}_0. \]

It is easy to see that
\[ \|f\|_{D_\alpha} = \sqrt{\frac{\varepsilon}{1+\varepsilon}} \sum_{n=0}^{\infty} (n+1)^{-1-\varepsilon} \leq \sqrt{\frac{\varepsilon}{1+\varepsilon}} [1 + \int_1^{\infty} t^{-1-\varepsilon} dt]^{\frac{1}{2}} = 1. \]

Since \( 0 < \alpha < 2 \) and \( 0 < \varepsilon < \alpha \), we see that
\begin{equation}
\|C(f)\|^2_{D_\beta} = \frac{\varepsilon}{1+\varepsilon} \sum_{n=0}^{\infty} (n+1)^{1-\beta} \left[ \frac{1}{n+1} \sum_{k=0}^{n} (k+1)^{-\frac{2-\alpha+\varepsilon}{2}} \right]^2
\end{equation}
\[ = \frac{\varepsilon}{1+\varepsilon} \sum_{n=0}^{\infty} (n+1)^{\alpha-\beta-1-\varepsilon} \left[ \sum_{k=0}^{n} (k+1)^{-\frac{2-\alpha+\varepsilon}{2}} \right]^2
\geq \frac{\varepsilon}{1+\varepsilon} \sum_{n=0}^{\infty} (n+1)^{\alpha-\beta-1-\varepsilon} \left[ \left( n+1 \right)^{-\frac{\alpha+\varepsilon}{2}} \int_1^{n+2} \frac{dt}{t^{\frac{\alpha+\varepsilon}{2}}} \right]^2
\end{equation}
\[ = \frac{\varepsilon}{1+\varepsilon} \left( \frac{2}{\alpha-\varepsilon} \right)^2 \sum_{n=0}^{\infty} (n+1)^{\alpha-\beta-1-\varepsilon} \left[ \left( \frac{n+2}{n+1} \right)^{\frac{\alpha+\varepsilon}{2}} - \frac{1}{(n+1)^{\frac{\alpha+\varepsilon}{2}}} \right]^2.
\]

We note that
\[ \left( \frac{n+2}{n+1} \right)^{\frac{\alpha+\varepsilon}{2}} \rightarrow 1, \quad n \rightarrow \infty, \]
and
\[ \frac{1}{(n+1)^{\frac{\alpha+\varepsilon}{2}}} \rightarrow 0, \quad n \rightarrow \infty. \]

Then we conclude from (1.5) that there is a constant \( \mathcal{N} \in \mathbb{N} \) such that
\[ \|C(f)\|^2_{D_\beta} \geq \frac{\varepsilon}{2(1+\varepsilon)} \left( \frac{2}{\alpha-\varepsilon} \right)^2 \sum_{n=\mathcal{N}}^{\infty} (n+1)^{\alpha-\beta-1-\varepsilon}. \]
If $C : D_\alpha \to D_\beta$ is bounded, then there exists a constant $C_1 > 0$ such that

$$
C_1 \geq \frac{\|C(f)\|_{D_\beta}^2}{\|f\|_{D_\alpha}^2} \geq \frac{\varepsilon}{2(1 + \varepsilon)} \left( \frac{2}{\alpha - \varepsilon} \right)^2 \sum_{n=N}^\infty (n + 1)^{\alpha - \beta - 1 - \varepsilon}.
$$

However, when $\varepsilon < \min\{\alpha - \beta, \alpha\}$, we see that

$$
\sum_{n=N}^\infty (n + 1)^{\alpha - \beta - 1 - \varepsilon} = +\infty.
$$

Hence we get that (1.6) is a contradiction. This means that the Cesàro operator $C$ is not bounded from $D_\alpha$ into $D_\beta$, if $\alpha > \beta$ and $0 < \alpha < 2$.

We note that

$$
\frac{1}{n + 1} = \int_0^1 t^n \, dt, \quad n \in \mathbb{N}_0.
$$

Let $\mu$ be a positive Borel measure on $[0, 1)$. For $f = \sum_{n=0}^\infty a_n z^n \in \mathcal{H}(\mathbb{D})$, we define the generalized Cesàro operators $C_\mu$ as

$$
C_\mu(f)(z) := \sum_{n=0}^\infty \left( \mu[n] \sum_{k=0}^n a_k \right) z^n, \quad z \in \mathbb{D},
$$

where

$$
\mu[n] = \int_0^1 t^n \, d\mu(t), \quad n \in \mathbb{N}_0.
$$

In this paper, we first study the question of characterizing measures $\mu$ such that $C_\mu : D_\alpha \to D_\beta$ is bounded. We obtain a sufficient and necessary condition of $\mu$ for which $C_\mu : D_\alpha \to D_\beta$ is bounded.

To state our first result, we introduce the notation of generalized Carleson measure on $[0, 1)$. Let $s > 0$ and $\mu$ be a positive Borel measure on $[0, 1)$. We say $\mu$ is an $s$-Carleson measure if there is a constant $C_2 > 0$ such that

$$
\mu([t, 1)) \leq C_2 (1 - t)^s,
$$

for all $t \in [0, 1)$.

Now we can state the first main result of this paper.

**Theorem 1.2.** Let $0 < \alpha, \beta < 2$. Then the following statements are equivalent:

1. $C_\mu : D_\alpha \to D_\beta$ is bounded.
2. $\mu$ is a $[1 + \frac{1}{2}(\alpha - \beta)]$-Carleson measure on $[0, 1)$.
3. There is a constant $C_3 > 0$ such that

$$
\mu[n] \leq \frac{C_3}{(n + 1)^{1 + \frac{1}{2}(\alpha - \beta)}},
$$

for all $n \in \mathbb{N}_0$.

The proof of Theorem 1.2 will be given in the next section. We shall characterize a measure $\mu$ such that $C_\mu : D_\alpha \to D_\beta$ is compact in the last section.
2. Proof of Theorem 1.2

In our proof of Theorem 1.2, we need the Beta function defined as follows.

\[ B(u, v) = \int_0^1 \frac{t^{u-1}(1-t)^{v-1}}{} \, dt, \quad u > 0, v > 0. \]

It is known that

\[ B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \]

where \( \Gamma \) is the Gamma function, defined as

\[ \Gamma(x) = \int_0^\infty e^{-t}t^{x-1} \, dt, \quad x > 0. \]

For more detailed introduction to the Beta function and Gamma function, see [10].

\((2)\Rightarrow(3)\). We note that \((3)\) is obvious when \(n = 0\). We get from integration by parts that, for \(n \geq 1\) \(\in \mathbb{N}\),

\[ \mu[n] = \int_0^1 t^n \, d\mu(t) = \mu([0,1)) - n \int_0^1 t^{n-1} \mu([0,t)) \, dt \]

\[ = n \int_0^1 t^{n-1} \mu([t,1)) \, dt. \]

Since \(\mu\) is a \([1+\frac{1}{2}(\alpha-\beta)]\)-Carleson measure on \([0,1)\), then we see that there is a constant \(C_4 > 0\) such that

\[ \mu([t,1)) \leq C_4(1-t)^{1+\frac{1}{2}(\alpha-\beta)}, \]

for all \(t \in [0,1)\).

It follows that

\[ \mu[n] \leq C_4n \int_0^1 t^{n-1}(1-t)^{1+\frac{1}{2}(\alpha-\beta)} \, dt \]

\[ = C_4n \cdot \frac{\Gamma(n)\Gamma(2+\frac{1}{2}(\alpha-\beta))}{\Gamma(n+2+\frac{1}{2}(\alpha-\beta))} \]

\[ \leq \frac{1}{(n+1)^{1+\frac{1}{2}(\alpha-\beta)}}. \]

Here we have used the fact that

\[ \Gamma(x) = \sqrt{2\pi}x^{\frac{1}{2}}e^{-x}[1 + r(x)], \quad x > 0, \]

where \(|r(x)| \leq e^{-\frac{1}{2x}} - 1\). Hence \((2)\Rightarrow(3)\) is true.

\((3)\Rightarrow(1)\). Let \(f = \sum_{n=0}^\infty a_n z^n \in D_\alpha\). By Cauchy’s inequality, we see from

\[ \mu[n] \leq \frac{C_5}{(n+1)^{1+\frac{1}{2}(\alpha-\beta)}}. \]
that, for } n \in \mathbb{N}_0, \\
|\mu[n] \sum_{k=0}^{n} a_k| \leq \frac{C_5}{(n + 1)^{1 + \frac{1}{2}(\alpha - \beta)}} \left| \sum_{k=0}^{n} a_k \right| \\
= \frac{C_5}{(n + 1)^{1 + \frac{1}{2}(\alpha - \beta)}} \left| \sum_{k=0}^{n} \left[ (k + 1)^{2 - \alpha} a_k \right] \left[ (k + 1)^{- \frac{2 - \alpha}{2}} \right] \right| \\
\leq \frac{C_5}{(n + 1)^{1 + \frac{1}{2}(\alpha - \beta)}} \left[ \sum_{k=0}^{n} (k + 1)^{2 - \alpha} a_k^2 \right]^\frac{1}{2} \left[ \sum_{k=0}^{n} (k + 1)^{- \frac{2 - \alpha}{2}} \right]^\frac{1}{2}.

Consequently, we obtain from (1.2) that

\[ \|C_\mu(f)\|_{D^\beta} \leq \left[ \sum_{n=0}^{\infty} (n + 1)^{1 - \beta} \left| \sum_{k=0}^{n} \mu[n] a_k \right|^2 \right]^\frac{1}{2} \]

\leq C_5 \left( \frac{2}{\alpha} \right)^\frac{1}{2} \left[ \sum_{n=0}^{\infty} \frac{1}{(n + 1)^{\alpha} \sum_{k=0}^{n} (k + 1)^{2 - \alpha} a_k^2} \sum_{k=0}^{n} \left( k + 1 \right)^{2 - \alpha} a_k^2 \right]^\frac{1}{2} \\
= C_5 \left( \frac{2}{\alpha} \right)^\frac{1}{2} \left( \sum_{k=0}^{\infty} (k + 1)^{1 - \alpha} \sum_{n=k}^{\infty} (n + 1)^{\alpha} \frac{1}{(n + 1)^{\alpha} a_k^2} \right)^\frac{1}{2}.

Then it follows from (1.4) that

\[ \|C_\mu(f)\|_{D^\beta} \leq C_5 \frac{\sqrt{2(2 + \alpha)}}{\alpha} \|f\|_{D^\alpha}. \]

This proves (3)⇒(1).

(1)⇒(2). We need the following estimate presented in [11]. Let 0 < t < 1. For any } c > 0, we have

\[ \sum_{n=1}^{\infty} n^{c - l} l^{2n} \asymp \frac{1}{(1 - t^2)^c}. \]

For 0 < b < 1, let } N \text{ be a natural number. We set } \tilde{f} = \sum_{n=0}^{\infty} \tilde{a}_n z^n \text{ with}

\[ \tilde{a}_n = \begin{cases} 
[\Omega_N]^{- \frac{1}{2}} b^{n+1}, & \text{if } n \in [0, N], \\
0, & \text{if } n \geq N + 1,
\end{cases} \]

where

\[ \Omega_N = \sum_{k=0}^{N} (k + 1)^{1 - \alpha} b^{2(k+1)}. \]
Then it is easy to see \( \|\tilde{f}\|_{D_\alpha} = 1 \). We set \( S_N = \{ k \in \mathbb{N}_0 : k \leq N \} \). In view of the boundedness of \( C_\mu : D_\alpha \to D_\beta \), we obtain that

\[
1 \supseteq \|C_\mu(\tilde{f})\|_{D_\beta}^2 = \sum_{n=0}^{N} (n+1)^{1-\beta} \left| \sum_{k=0}^{n} \tilde{a}_k \int_0^1 t^n d\mu(t) \right|^2.
\]

On the other hand, we note that, when \( n \leq N \),

\[
\sum_{k=0}^{n} \chi_{S_N}(k) b^{k+1} \geq (n+1) b^{n+1}.
\]

Then we get that

\[
\sum_{n=0}^{\infty} (n+1)^{1-\beta} b^{2n} \cdot \left[ \sum_{k=0}^{n} \chi_{S_N}(k) b^{k+1} \right]^2 \geq \sum_{n=0}^{N} (n+1)^{3-\beta} b^{4n+2}.
\]

It follows from (2.2) that

\[
1 \supseteq [\Omega_N]^{-1} [\mu([b,1])]^2 \sum_{n=0}^{\infty} (n+1)^{3-\beta} b^{4n+2}.
\]

Taking \( N \to \infty \) in (2.3), we see that

\[
[\mu([b,1])]^2 \sum_{n=0}^{\infty} (n+1)^{3-\beta} b^{4n+2} \leq \sum_{n=0}^{\infty} (n+1)^{1-\alpha} b^{2(n+1)},
\]

for all \( b \in [0,1) \). Then we conclude from (2.1) that

\[
[\mu([b,1])]^2 \frac{1}{(1-b^2)^{4-\beta}} \leq \frac{1}{(1-b^2)^{2-\alpha}}.
\]

This implies that

\[
\mu([b,1]) \leq (1-b^2)^{1+\frac{1}{2}(\alpha-\beta)},
\]

for all \( 0 < b < 1 \). It follows that \( \mu \) is a \([1 + \frac{1}{2}(\alpha-\beta)]\)-Carleson measure on \([0,1)\) and (1) \( \Rightarrow \) (2) is proved. The proof of Theorem 1.2 is now finished.
3. COMPACTNESS OF THE GENERALIZED CESÀRO OPERATORS ON DIRICHLET-TYPE SPACES

For $0 < s < \infty$, we say a positive Borel measure $\mu$ on $[0, 1)$ is a vanishing $s$-Carleson measure, if $\mu$ is an $s$-Carleson measure and satisfies that

$$\lim_{t \to 1^-} \frac{\mu([t, 1))}{(1 - t)^s} = 0.$$ 

The following theorem is the main result of this section.

**Theorem 3.1.** Let $0 < \alpha, \beta < 2$. Then the following statements are equivalent:

1. $\mathcal{C}_\mu : \mathcal{D}_\alpha \to \mathcal{D}_\beta$ is compact.
2. $\mu$ is a vanishing $[1 + \frac{1}{2}(\alpha - \beta)]$-Carleson measure on $[0, 1)$.
3. 
   $$\mu[n] = o \left( (n + 1)^{1 - \frac{1}{2}(\alpha - \beta)} \right), \quad n \to \infty.$$ 

**Proof of Theorem 3.1.** First note that, by minor modifications of the arguments of (2) $\Rightarrow$ (3) in the proof of Theorem 1.2, we can similarly show (2) $\Rightarrow$ (3) of Theorem 3.1.

We proceed to prove (3) $\Rightarrow$ (1). For any $f = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{D}_\alpha$. Let $N \in \mathbb{N}$, we consider

$$\mathcal{C}_\mu^{[N]}(f)(z) := \sum_{n=0}^{N_1} \left( \mu[n] \sum_{k=0}^{n} a_k \right) z^n, \quad z \in \mathbb{D}.$$ 

Then we see that $\mathcal{C}_\mu^{[N]}$ is a finite rank operator, hence $\mathcal{C}_\mu^{[N]}$ is compact from $\mathcal{D}_\alpha$ into $\mathcal{D}_\beta$.

In view of

$$\mu[n] = o \left( (n + 1)^{1 - \frac{1}{2}(\alpha - \beta)} \right), \quad n \to \infty,$$

we see that, for any $\epsilon > 0$, there is an $N_0 \in \mathbb{N}$ such that

$$\mu[n] \leq \epsilon (n + 1)^{1 - \frac{1}{2}(\alpha - \beta)},$$

for all $n > N_0$.

Note that

$$\| (\mathcal{C}_\mu - \mathcal{C}_\mu^{[N]}) (f) \|_{\mathcal{D}_\beta}^2 = \sum_{n=N+1}^{\infty} (n + 1)^{1 - \beta} \left| \mu[n] \sum_{k=0}^{n} a_k \right|^2.$$ 

When $N > N_0$, we get that

$$\| (\mathcal{C}_\mu - \mathcal{C}_\mu^{[N]}) (f) \|_{\mathcal{D}_\beta}^2 \leq \epsilon^2 \sum_{n=N+1}^{\infty} (n + 1)^{1 - \beta} \left| \frac{1}{(n + 1)^{1 + \frac{1}{2}(\alpha - \beta)}} \sum_{k=0}^{n} a_k \right|^2.$$ 

Consequently, by using the arguments of (3) $\Rightarrow$ (1) in the proof of Theorem 1.2, we see that

$$\| (\mathcal{C}_\mu - \mathcal{C}_\mu^{[N]}) (f) \|_{\mathcal{D}_\beta}^2 \leq \epsilon^2 \| f \|_{\mathcal{D}_\alpha}^2,$$

holds for any $f \in \mathcal{D}_\alpha$. Hence we see

$$\| \mathcal{C}_\mu - \mathcal{C}_\mu^{[N]} \|_{\mathcal{D}_\alpha \to \mathcal{D}_\beta} \leq \epsilon,$$
when \( n > N_0 \). Here

\[
\| T \|_{\mathcal{D}_\alpha \rightarrow \mathcal{D}_\beta} = \sup_{f \neq 0 \in \mathcal{D}_\alpha} \frac{\| T(f) \|_{\mathcal{D}_\beta}}{\| f \|_{\mathcal{D}_\alpha}},
\]

where \( T \) is a linear bounded operator from \( \mathcal{D}_\alpha \) into \( \mathcal{D}_\beta \). This means that \( C_\mu \) is compact from \( \mathcal{D}_\alpha \) into \( \mathcal{D}_\beta \) and \((3) \Rightarrow (1)\) is proved.

Finally, we show that \((1) \Rightarrow (2)\). For \( 0 < b < 1 \). We set \( \hat{f}_b = \sum_{n=0}^{\infty} \hat{a}_n z^n \) with \( \hat{a}_n = (1 - b^2)^{\frac{2-a}{2}} b^{n-1}, n \in \mathbb{N}_0 \).

We see from \((2.1)\) that \( \| \hat{f}_b \|_{\mathcal{D}_\alpha} \asymp 1 \). By the fact that \( \lim_{b \to 1^-} \hat{f}_b = 0 \) for any \( z \in \mathbb{D} \), we conclude that \( \hat{f}_b \) is convergent weakly to 0 in \( \mathcal{D}_\alpha \) as \( b \to 1^- \). Since \( C_\mu : \mathcal{D}_\alpha \rightarrow \mathcal{D}_\beta \) is compact, we see

\[
\lim_{b \to 1^-} \| C_\mu(\hat{f}_b) \|_{\mathcal{D}_\beta} = 0.
\]

On the other hand, we have

\[
\| C_\mu(\hat{f}_b) \|_{\mathcal{D}_\beta}^2 = \sum_{n=0}^{\infty} (n+1)^{1-\beta} \left| \sum_{k=0}^{n} \hat{a}_k \int_0^1 t^n d\mu(t) \right|^2 \geq (1 - b^2)^{2-\alpha} \sum_{n=0}^{\infty} (n+1)^{1-\beta} \left[ \sum_{k=0}^{n} b^{k+1} \int_0^1 t^n d\mu(t) \right]^2 \geq (1 - b^2)^{2-\alpha} [\mu([b, 1])]^2 \sum_{n=0}^{\infty} (n+1)^{1-\beta} \left[ \sum_{k=0}^{n} b^{k+1} \cdot b^n \right]^2 = (1 - b^2)^{2-\alpha} [\mu([b, 1])]^2 \sum_{n=0}^{\infty} (n+1)^{1-\beta} b^{2n} \cdot \left( \sum_{k=0}^{n} b^{k+1} \right)^2.
\]

Also, we have

\[
\sum_{n=0}^{\infty} (n+1)^{1-\beta} b^{2n} \cdot \left( \sum_{k=0}^{n} b^{k+1} \right)^2 \geq \sum_{n=0}^{\infty} (n+1)^{1-\beta} b^{2n} \cdot [(n+1)b^{n+1}]^2 \geq \sum_{n=0}^{\infty} (n+1)^{3-\beta} b^{4n+2} \asymp \frac{1}{(1 - b^2)^{1-\beta}}.
\]

Combining \((3.3)\) and \((3.4)\), we see

\[
\mu([b, 1]) \leq \| C_\mu(\hat{f}_b) \|_{\mathcal{D}_\beta} (1 - b^2)^{1+\frac{1}{2}(\alpha - \beta)}.
\]
It follows from (3.2) that
\[
\lim_{b \to 1^{-}} \frac{\mu([b,1))}{(1-b)^{1+\frac{1}{2}(\alpha-\beta)}} = 0.
\]
This proves (1)⇒(2) and the proof of Theorem 3.1 is complete. □

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