A Dichotomy Theorem for the Approximate Counting of Complex-Weighted Bounded-Degree Boolean CSPs

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Abstract: We determine the computational complexity of approximately counting the total weight of variable assignments for every complex-weighted Boolean constraint satisfaction problem (or CSP) with any number of additional unary (i.e., arity 1) constraints, particularly, when degrees of input instances are bounded from above by a fixed constant. All degree-1 counting CSPs are obviously solvable in polynomial time. When the instance’s degree is more than two, we present a dichotomy theorem that classifies all counting CSPs admitting free unary constraints into exactly two categories. This classification theorem extends, to complex-weighted problems, an earlier result on the approximation complexity of unweighted counting Boolean CSPs of bounded degree. The framework of the proof of our theorem is based on a theory of signature developed from Valiant’s holographic algorithms that can efficiently solve seemingly intractable counting CSPs. Despite the use of arbitrary complex weight, our proof of the classification theorem is rather elementary and intuitive due to an extensive use of a novel notion of limited T-constructibility. For the remaining degree-2 problems, in contrast, they are as hard to approximate as Holant problems, which are a generalization of counting CSPs.

Keywords: constraint satisfaction problem, #CSP, bounded degree, approximate counting, dichotomy theorem, T-constructibility, signature, Holant problem

1 Bounded-Degree Boolean #CSPs

Our general objective is to determine the approximation complexity of constraint satisfaction problems (or CSPs) whose instances consist of variables (on certain domains) and constraints, which describe “relationships” among the variables. Such CSPs have found numerous applications in graph theory, database theory, and artificial intelligence as well as statistical physics. A decision CSP, for instance, asks whether or not, for two given sets of variables and of constraints, any assignment that assigns actual values in the domain to the variables satisfies all the constraints simultaneously. The satisfiability problem (SAT) of deciding the existence of a satisfying truth assignment for a given Boolean formula is a typical example of the decision CSPs. Since input instances are often restricted to particular types of constraints (where a set of these constraints is known as a constraint language), it seems natural to parameterize CSPs in terms of a given set of allowable constraints. Conventionally, such a parameterized CSP is expressed as CSP(\mathcal{F}). Schaefer’s dichotomy theorem classifies all such parameterized CSP(\mathcal{F})’s into exactly two categories: polynomial-time solvable problems (i.e., in P) and NP-complete problems, provided that NP is different from P. This situation highlights structural simplicity of the CSP(\mathcal{F})’s, because all NP problems, by contrast, fill up infinitely many categories.

In the course of a study of CSPs, various restrictions have been imposed on constraints as well as variables. Of all such restrictions, recently there has been a great interest in a particular type of restriction, of which each individual variable should not appear more than \(d\) times in the scope of all given constraints. The maximal number of such \(d\) on any instance is called the degree of the instance. This degree has played a key role in a discussion of the complexity of CSPs; for instance, the planar read-twice satisfiability problem, which comprised of logical formulas of degree at most three, is known to be NP-complete, while the planar read-twice satisfiability problem, whose degree is two, falls into P. Those CSPs whose instances have their degrees upper-bounded are referred to as bounded-degree CSPs. Under the assumption that unary constraints are freely available as part of input instances, Dalmau and Ford showed, for example, that, for certain cases of \(\mathcal{F}\), the complexity of solving CSP(\mathcal{F}) remains unchanged even if all instances are restricted to degree at most three. Notice that such a free use of unary constraints were frequently made in the past literature (see, e.g., [7] [11] [13] [14]) to draw stronger and more concise results.

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‡ We use this term to mean the computational complexity of approximately solving a given problem.
Apart from those decision CSPs, a counting CSP (or #CSP, in short) asks how many variable assignments satisfy the set of given constraints. In parallel to Schaefer’s theorem, Creignou and Herman [2] gave their dichotomy theorem on the computational complexity of Boolean #CSPs. Their result was later extended by Dyer, Goldberg, and Jerrum [12] to non-negative weighted Boolean #CSPs and then further extended to complex-weighted Boolean #CSPs by Cai, Lu, and Xia [3]. Cai et al. also studied the complexity of complex-weighted Boolean #CSPs whose maximal degree does not exceed three. Those remarkable results are meant for the computational complexity of “exact counting.” From a perspective of “approximate counting,” on the contrary, Dyer, Goldberg, and Jerrum [13] showed a classification theorem on the approximate counting of the number of variable assignments for unweighted Boolean CSPs, parametrized by the choice of constraint set $F$, under a notion of approximation-preserving (or AP) reducibility. This theorem, however, is quite different from the earlier dichotomy theorems for the exact-counting of Boolean CSPs; in fact, the theorem classifies all Boolean #CSPs into three categories, including an intermediate level located between a class of P-computable problems and a class of #P-complete problems.

The degree bound of input instances to #CSPs has drawn an unmistakable picture in a discussion on the approximation complexity of the #CSPs by Dyer, Goldberg, Jalsenius, and Richerby [11]. They discovered the following approximation-complexity classification of unweighted Boolean #CSPs when their degrees are further bounded. The succinct notation #CSP$^d_w(F)$ used below specifies a problem of counting the number of Boolean assignments satisfying a given CSP, provided that (i) any unary unweighted Boolean constraint is allowed to use for free of charge and (ii) each variable appears at most $d$ times among all given constraints, including free unary constraints.

Let $d \geq 3$ and let $F$ be any set of unweighted Boolean constraints. If every constraint in $F$ is affine, then #CSP$^d_w(F)$ is in FP. Otherwise, if $F$ is included in $IM$-$conj$, then #CSP$^d_w(F) \equiv_{AP}$ #BIS. Otherwise, if $F \subseteq OR$-$conj$ or $F \subseteq NAND$-$conj$, then #w-HIS$_d \leq_{AP}$ #CSP$^d_w(F) \leq_{AP}$ #w-HIS$_{2d}$. Otherwise, #CSP$^d_w(F) \equiv_{AP}$ #SAT, where $w$ is the width of $F$ and $k$ is a certain constant depending only on $F$. Here, $IM$-$conj$, $OR$-$conj$, $NAND$-$conj$ are three well-defined sets of unweighted Boolean constraints, #SAT is the counting satisfiability problem, #BIS is the bipartite independent set problem, and #w-HIS$_d$ denotes the hypergraph independent set problem with hyperedge degree at most $d$ and width at most $w$. The notations $\leq_{AP}$ and $\equiv_{AP}$ respectively refer to the AP-reducibility and AP-equivalence between two counting problems. As a special case, when $d \geq 25$ and $w \geq 2$, as shown in [8], there is no fully polynomial-time randomized approximation scheme (or FPRAS) for #w-HIS$_d$ unless NP = RP. This classification theorem heavily relies on the aforementioned work of Dyer et al. [13].

Toward our main theorem, we first introduce a set $\mathcal{ED}$ of complex-weighted constraints constructed from unary constraints, the equality constraint, and the disequality constraint. Similar to the above case of Dyer et al., we also allow a free use of arbitrary (complex-weighted) unary constraints. For notational convenience, we use the notation #CSP$^d_w(F)$ to emphasize that all complex-weighted unary constraints are freely given. The main purpose of this paper is to prove the following dichotomy theorem that classifies all #CSP$^d_w(F)$'s into exactly two categories.

**Theorem 1.1** Let $d \geq 3$ be any degree bound. If $F \subseteq \mathcal{ED}$, then #CSP$^d_w(F)$ belongs to FP$_C$; otherwise, #SAT$_C \leq_{AP}$ #CSP$^d_w(F)$ holds.

Here, #SAT$_C$ is a complex-weighted version of the counting satisfiability problem and FP$_C$ is the class of polynomial-time computable complex-valued functions (see Sections 2.4–2.5 and 3 for their precise definitions). In contrast to the result of Dyer et al. [13], Theorem 1.1 exhibits a stark difference between unweighted Boolean constraints and complex-weighted Boolean constraints, partly because of strong expressiveness of complex-weighted unary constraints even when the maximal degree of instances is upper-bounded.

Instead of relying on the result of Dyer et al., our proof is actually based on the following dichotomy theorem of Yamakami [21], who proved the theorem using a theory of signature [12] developed from Valiant’s holographic algorithms [19, 20].

If $F \subseteq \mathcal{ED}$, then #CSP$^*(F)$ is in FP$_C$. Otherwise, #SAT$_C \leq_{AP}$ #CSP$^*(F)$.

To appeal to this result, we wish to claim the following key proposition, which bridges between unbounded-degree #CSPs and bounded-degree #CSPs.

**Proposition 1.2** For any degree bound $d$ at least 3, #CSP$^d_w(F) \equiv_{AP}$ #CSP$^*(F)$ holds for any set $F$ of complex-weighted constraints.
From this proposition, Theorem 1.1 immediately follows. The most part of this paper will be therefore devoted to proving this key proposition. When the degree bound $d$ equals two, on the contrary, \#CSP$_2^*$($\mathcal{F}$) is equivalent in approximation complexity to Holant problems restricted to the set $\mathcal{F}$ of constraints, provided that all unary constraints are freely available, where Holant problems were introduced by Cai et al. \cite{Cai2005} to study a wider range of counting problems in a certain unified way. In the case of degree 1, however, every \#CSP$_1^*$($\mathcal{F}$) is solvable in polynomial time.

Our argument for complex-weighted constraints is obviously different from Dyer et al.’s argument for unweighted constraints and also from Cai et al.’s argument for exact counting using complex-valued signatures. While a key technique in \cite{Dyer2008} is “3-simulatability” as well as “ppp-definability,” our proof argument exploits a notion of limited T-constructibility—a restricted version of T-constructibility developed in \cite{Williams2008}. With its extensive use, the proof we will present in Section 8 becomes quite elementary and intuitive.

2 Preliminaries

Let $\mathbb{N}$ denote the set of all natural numbers (i.e., non-negative integers) and $\mathbb{N}^+$ means $\mathbb{N} - \{0\}$. Similarly, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the sets of all real numbers and of all complex numbers. For succinctness, the notation $[n]$ for a number $n \in \mathbb{N}^+$ expresses the integer set $\{1, 2, \ldots, n\}$. The notations $|\alpha|$ and $\arg(\alpha)$ for a complex number $\alpha$ denote the absolute value and the argument of $\alpha$. We always assume that $\arg(\alpha) \in (-\pi, \pi]$. To improve readability, we often identify the “name” of a node in a given undirected graph with the “label” of the same node although there might be more than one node with the same label. For instance, we may call a specific node $v$ whose label is $x$ by “node $x$” as far as the node in question is clear from the context.

Hereafter, we will give brief explanations to several important concepts and notations used in the rest of the paper.

2.1 Complex Numbers and Computability

Our core subject is the approximate computability of complex-weighted Boolean counting problems. Since such problems can be seen as complex-valued functions taking Boolean variables as input instances, we need to address a technical issue of how to handle arbitrary complex numbers and those complex-valued functions in an existing framework of string-based computation.

Our interest in this paper is not limited to so-called “polynomial-time computable” numbers, such as algebraic numbers, numbers expressed exactly by polynomially many bits, or numbers defined by efficiently generated Cauchy series \cite{Crandall2012}. Because there is no consensus of how to define “polynomial-time computability” of complex numbers, as done in the recent literature \cite{Cai2005} \cite{Barvinok2008} \cite{Williams2008} \cite{Williams2010}, we wish to make our arguments in this paper independent of the definition of “polynomial-time computable” numbers. To fulfill this ambitious purpose, although slightly unconventional, we rather treat the complex numbers as basic “objects” and perform natural “operations” (such as, multiplications, addition, division, subtraction, etc.) as well as simple “comparisons” (such as, equality checking, less-than-or-equal checking, etc.) as basic manipulations of those numbers. Each of such manipulations of one or more complex numbers is assumed to consume only constant time. We want to make this assumption on the constant execution time cause no harm in a later discussion on the computability of complex-valued functions. It is thus imperative to regulate all manipulations to perform only in a clearly described algorithmic way. This strict regulation guarantees that our arguments properly work in the scope of many choices of “polynomial-time computable” complex numbers.

From a practical viewpoint, the reader may ask how we will “describe” arbitrary complex-valued function or, when an input instance contains complex numbers, how we will “describe” those numbers as a part of the input given to an algorithm in question. Notice that, by running a randomized algorithm within a polynomial amount of execution time, we need to distinguish only exponentially many complex numbers. Hence, those numbers may be specified by appropriately designated “indices,” which may be expressed in polynomially many bits. In this way, all input complex numbers, for instance, can be properly indexed when they are given as a part of each input instance, and those numbers are referred to by those indices during an execution of the algorithm. The reader is referred to, e.g., \cite{Williams2010} Section 4 for a string-based treatment of arbitrary complex numbers. Indexing complex numbers also helps us view a complex-valued function as a “map” from Boolean variables to fixed indices of complex numbers.

In the rest of this paper, we assume a suitable method of indexing arbitrary complex numbers.
2.2 Constraints and \#CSPs

Given an undirected graph $G = (V, E)$ (where $V$ is a node set and $E$ is an edge set) and a node $v \in V$, an incident set $E(v)$ of $v$ is the set of all edges incident on $v$ (i.e., $E(v) = \{w \in V | (v, w) \in E \}$), and $\deg(v)$ is the degree of $v$ (i.e., $\deg(v) = |E(v)|$). A bipartite graph is described as a tripartite of the form $(V_1 | V_2, E)$, of which $V_1$ and $V_2$ respectively denote sets of nodes on the left-hand side and on the right-hand side of the graph and $E$ denotes a set of edges (i.e., $E \subseteq V_1 \times V_2$).

Each function $f$ from $\{0,1\}^k$ to $\mathbb{C}$ is called a $k$-ary constraint (or signature, in case of Holant problems), where $k$ is called the arity of $f$. Assuming the standard lexicographic order on $\{0,1\}^k$, we often express $f$ as a series of its output values, and thus it can be identified with an element in the space $\mathbb{C}^k$. For instance, when $k = 1$ and $k = 2$, $f$ can be written respectively as $(f(0), f(1))$ and $(f(00), f(01), f(10), f(11))$. A constraint $f$ is symmetric if $f$'s values depend only on the Hamming weight of inputs. When $f$ is a symmetric function of arity $k$, we also use a succinct notation $f_i = [f_0, f_1, \ldots, f_k]$, where each $f_i$ is the value of $f$ on any input of Hamming weight $i$. As a simple example, the equality function $EQ_k$ of arity $k$ is expressed as $[1, 0, \ldots, 0, 1]$ $(k-1$ zeros). In particular, $EQ_1$ equals $[1, 1]$. For convenience, let $\Delta_0 = [1, 0]$ and $\Delta_1 = [0, 1]$. For a later use, we reserve the notation $\mathcal{U}$ for the set of all unary (i.e., arity-1) constraints.

We quickly review a set of useful notations used in \cite{21}. Let $k \in \mathbb{N}^*$, let $i, j \in [k]$, let $c \in \{0,1\}$, and let $f$ be any arity-$k$ constraint. Moreover, let $x_1, \ldots, x_k$ be $k$ Boolean variables. Pinning is a method of constructing a new constraint $f^{\text{pin}}_{\text{c}}$ from $f$, where $f^{\text{pin}}_{\text{c}}$ is the constraint defined as $f^{\text{pin}}_{\text{c}}(x_1, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_k) = f(x_1, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_k)$. In contrast, projection is a way of building a new constraint $f^{\text{proj}}_i$ that is defined as $f^{\text{proj}}_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) = \sum_{x_i \in \{0,1\}} f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_k)$. When $i \neq j$, the notation $f^{\text{proj}}_j$ denotes the constraint defined as $f^{\text{proj}}_j(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_k) = f(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_k)$. To normalize $f$ means that we choose an appropriate constant $\lambda \in \mathbb{C} - \{0\}$ and then construct a new constraint $\lambda \cdot f$ from $f$, where $\lambda \cdot f$ denotes the constraint $g$ defined as $g(x_1, \ldots, x_k) = \lambda \cdot f(x_1, \ldots, x_k)$. When $g_1$ and $g_2$ share the same input-variable sequence, $g_1 \cdot g_2$ denotes the constraint defined as $h(x_1, \ldots, x_k) = g_1(x_1, \ldots, x_k)g_2(x_1, \ldots, x_k)$. By extending these notations naturally, we abbreviate, e.g., $(f_{x_i = 0}^{\text{pin}}_{\text{c}})_{x_i^{\text{proj}}} = f_{x_i = 0}^{\text{pin}}_{\text{c}}$ and $(f_{x_i = 0}^{\text{pin}}_{\text{c}})_{x_i^{\text{proj}}} = f_{x_i = 0}^{\text{pin}}_{\text{c}}$.

For each set $F$ of constraints, a complex-weighted Boolean \#CSP, succinctly denoted \#CSP($F$), is a counting problem whose input instance is a finite set $\Omega$ of “elements” of the form $(h, (x_1, x_2, \ldots, x_k))$, where $h : \{0,1\}^k \to \mathbb{C}$ is in $F$ and $x_1, x_2, \ldots, x_k$ are some of $n$ Boolean variables $x_1, x_2, \ldots, x_n$ (i.e., $i_1, \ldots, i_k \in [n]$), and \#CSP($F$) asks to compute the value $c_{\text{CSP}}(\Omega) = \sum_{(h, (x_1, x_2, \ldots, x_k)) \in H} h(\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_k))$, where $x = (x_1, x_2, \ldots, x_n)$ and $\sigma : [n] \to [0, 1]$ ranges over the set of all variable assignments. To improve readability, we often omit the set notation and express, e.g., \#CSP($f, g, F, G$) to mean \#CSP($\{f, g\} \cup F \cup G$). Since we always admit arbitrary unary constraints for free of charge, we briefly write \#CSP($F$) instead of \#CSP($F, \mathcal{U}$).

From a different viewpoint, an input instance $\Omega$ to \#CSP($F$) can be stated as a triplet $(G, X | F', \pi)$, which consists of a finite undirected bipartite graph $G = (V_1 | V_2, E)$, a variable set $X = \{x_1, x_2, \ldots, x_n\}$, a finite subset $F'$ of $F$, and a labeling function $\pi : V_1 \cup V_2 \to X \cup F'$, where $\pi(V_1) = X$ and $\pi(V_2) \subseteq F'$. In this graph representation, the label of each node $v$ in $V_1$ is distinct variable $x_i$ in $X$, each node $v$ in $V_2$ has constraint $h$ in $F'$ as its label, and an edge $e$ in $E$ incident on both nodes $v$ and $e$ indicates that the constraint $h$ takes the variable $x_i$ (as part of its input variables). Such labeling of constraints is formally given by the labeling function $\pi$. For simplicity, $\pi(v)$ is written as $f_v$. To emphasize this graph representation, we intend to call $\Omega = (G, X | F', \pi)$ a constraint frame for \#CSP($F$) \cite{21}. The use of the notion of constraint frame makes it possible to discuss a counting problem in a general framework of Holant problem (on a Boolean domain) \cite{3}, which will be given in the next subsection.

For each input instance $\Omega = (G, X | F', \pi)$ given to \#CSP($F'$), the degree of the instance $\Omega$ is the greatest number of times that any variable appears among its constraints in $F'$; in other words, the maximum degree of any node that appears on the left-hand side of the bipartite graph $G$. For any positive integer $d$, we write \#CSP$_d(F)$ for the restriction of \#CSP($F$) to instances of degree at most $d$.

2.3 Holant Problems

In a Holant framework, “(complex-weighted) constraints” are always referred to as “signatures.” For our convenience, we often use these two words interchangeably. Now, we will follow the terminology developed in \cite{11}. A Holant problem Holant($F$) (on a Boolean domain) takes an input instance, called a signature grid $\Omega = (G, F', \pi)$, composed of a finite undirected graph $G = (V, E)$, a finite set $F' \subseteq F$, and a labeling function $\pi : V \to F'$, where each node $v \in V$ is labeled by a signature $\pi(v) : \{0,1\}^{|\deg(v)|} \to \mathbb{C}$. We often use the notation $f_v$ for $\pi(v)$. Instead of variable assignments used for \#CSP($F$)’s, here we use
"edge assignments." We denote by $\text{Asn}(E)$ the set of all edge assignments $\sigma : E \to \{0, 1\}$. The Holant problem asks to compute the value $\text{Holant}_\Omega = \sum_{\sigma \in \text{Asn}(E)} \prod_{e \in V} f_e(\sigma(E(e)))$, where $\sigma(E(e))$ denotes the sequence $(\sigma(w_1), \sigma(w_2), \ldots, \sigma(w_k))$ of bits if $E(e) = (w_1, w_2, \ldots, w_k)$, sorted in a certain pre-determined order (depending only on $f_e$). A bipartite Holant problem $\text{Holant}(F_1|F_2)$ is a variant of Holant problem, defined as follows. An input instance is a bipartite signature grid $\Omega = (G, F'_1|F'_2)$ consisting of a finite undirected bipartite graph $G = (V_1|V_2, E)$, two finite sets $F'_1 \subseteq F_1$ and $F'_2 \subseteq F_2$, and a labeling function $\pi : V_1 \cup V_2 \to F'_1 \cup F'_2$ satisfying that $\pi(V_1) \subseteq F'_1$ and $\pi(V_2) \subseteq F'_2$.

Exploiting a direct connection between #CSPs and Holant problems, it is useful to view #CSP($F$) as a special case of bipartite Holant problem by the following translation: a constraint frame $\Omega = (G, X|F', \pi)$ for #CSP($F$) with $G = (V_1|V_2, E)$ is modified into a signature grid $\Omega' = (G', \{EQ_k\}_{k \geq 1}|F', \pi')$ that is obtained as follows. The graph $G'$ is obtained from $G$ by replacing the variable label of any degree-$k$ node $v$ in $V_1$ by the arity-$k$ equality function $EQ_k$. It is not difficult to see that any edge assignment that assigns 0 (1, resp.) to all edges incident on this node $v$ uniquely substitutes a variable assignment giving 0 (1, resp.) to the node $v$ for #CSP($F$). The labeling function $\pi'$ is defined accordingly. In terms of Holant problems, #CSP($F$) is just another name for Holant($\{EQ_k\}_{k \geq 1}|F$). Similarly, #CSP$^*(F)$ coincides with Holant($\{EQ_k\}_{k \geq 1}|F, U$).

Moreover, for each degree bound $d \geq 1$, #CSP$^*_d(F)$ is identified with Holant($\{EQ_k\}_{k \leq d}|F, U$).

2.4 FP$_C$ and AP-Reductions

Following the way we handle complex numbers (see Section 2.1), a complex analogue of FP, denoted FP$_C$, is naturally defined as the set of all functions, mapping strings to $\mathbb{C}$, which can be computed deterministically in time polynomial in the lengths of input strings, where "strings" are finite sequences of symbols chosen from (nonempty finite) alphabets. Let $\Sigma$ be any alphabet and let $\mathbb{C}(\Sigma)$ be the set of all functions, mapping strings over $\Sigma$ to $\mathbb{C}$. A randomized approximation scheme (or RAS) for $F$ is a randomized algorithm equipped with a coin-flipping mechanism that takes a standard input $x \in \Sigma^*$ together with an error tolerance parameter $\varepsilon \in (0, 1)$ and outputs values $w \in \mathbb{C}$ with probability at least $(3/4)^{|x|}$ for which $2^{-\varepsilon} \leq |w/F(x)| \leq 2^\varepsilon$ and $|\arg(w/F(x))| \leq \varepsilon$, provided that, whenever $F(x) = 0$, we always demand $w = 0$. See [21, Lemma 9.2] for usefulness of this definition.

Given two functions $F$ and $G$, a polynomial-time approximation-preserving reduction (or AP-reduction) from $F$ to $G$ is a randomized algorithm $M$ that takes a pair $(x, \varepsilon) \in \Sigma^* \times (0, 1)$ as input instance, uses an arbitrary RAS $N$ for $G$ as oracle, and satisfies the following conditions: (i) $M$ is still a valid RAS for $F$; (ii) every oracle call made by $M$ is of the form $(w, \delta) \in \Sigma^* \times (0, 1)$ with $\delta^{-1} \leq \text{poly}(|x|, 1/\varepsilon)$ and its answer is the outcome of $N$ on $(w, \delta)$, provided that any complex number included in this string $w$ should be completely specified (see Section 2.1) by $M$; and (iii) the running time of $M$ is bounded from above by a certain polynomial in $(|x|, 1/\varepsilon)$, not depending on the choice of $N$. In this case, we write $F \leq_{AP} G$ and we also say that $F$ is AP-reducible to $G$ (or $F$ is AP-reduced to $G$). If both $F \leq_{AP} G$ and $G \leq_{AP} F$ hold, then $F$ and $G$ are said to be AP-equivalent and we write $F \equiv_{AP} G$.

The following lemma, whose proof is straightforward and left to the reader, is useful in later sections.

Lemma 2.1 Let $F, G, H$ be any three constraint sets and let $e, d \in \mathbb{N}^+$.  
1. If $d \leq e$, then $\#\text{CSP}_d^*(F) \leq_{AP} \#\text{CSP}_d^*(F)$.  
2. If $F \subseteq G$, then $\#\text{CSP}_d^*(F) \leq_{AP} \#\text{CSP}_d^*(G)$.  
3. If $\#\text{CSP}_d^*(F) \leq_{AP} \#\text{CSP}_d^*(G)$ and $\#\text{CSP}_d^*(G) \leq_{AP} \#\text{CSP}_d^*(H)$, then $\#\text{CSP}_d^*(F) \leq_{AP} \#\text{CSP}_d^*(H)$.

2.5 Counting Problem #SAT$^*_\mathbb{C}$

We briefly describe the counting problem #SAT$^*_\mathbb{C}$, introduced in [21], which has appeared in Section 1. For the proof of our main theorem, since our proof heavily relies on [21], there is in fact no need of knowing any structural property of this counting problem; however, the interested reader is referred to [21] for its properties and connections to other counting problems.

A complex-weighted version of the counting satisfiability problem (#SAT), denoted #SAT$^*_\mathbb{C}$, is induced naturally from #SAT as follows. Let $\phi$ be any propositional formula and let $V(\phi)$ denote the set of all Boolean variables appearing in $\phi$. In addition, let $\{w_x\}_{x \in V(\phi)}$ be any series of node-weight functions $w_x : \{0, 1\} \to \mathbb{C}$, provided that any variable $x$ in $V(\phi)$. Given the input pair $(\phi, \{w_x\}_{x \in V(\phi)})$, #SAT$^*_\mathbb{C}$ outputs the sum of all weights $w(\sigma)$ for truth assignments $\sigma$ satisfying $\phi$, where $w(\sigma)$ denotes the product of all values $w_x(\sigma(x))$ over all variables $x \in V(\phi)$. Since #SAT is a special case of #SAT$^*_\mathbb{C}$, it naturally holds that $\#\text{SAT} \leq_{AP} \#\text{SAT}^*_\mathbb{C}$.  

5
3 Special Constraint Sets

We treat a relation of arity \(k\) as both a subset of \([0,1]^k\) and a function mapping \(k\) Boolean variables to \([0,1]\). From this duality, we often utilize the following “functional” notation: for every \(x \in [0,1]^k\), \(R(x) = 1\) (\(R(x) = 0\), resp.) if \(x \in R\) (\(x \not\in R\), resp.). The underlying relation \(R_f\) of a constraint \(f\) of arity \(k\) is the set \(\{x \in [0,1]^k \mid f(x) \neq 0\}\). A constraint \(f\) is called non-zero if \(f(x) \neq 0\) for all inputs \(x \in [0,1]^k\). Note that, for any constraint \(f\), there exists a non-zero constraint \(g\) for which \(f = R_f \cdot g\), where \(R_f\) is viewed as a Boolean function. This fundamental property will be frequently used in the subsequent sections.

In this paper, we use the following special relations: \(\text{XOR} = [0,1,0]\), \(\text{Implies} = (1,1,0,1)\), \(OR_k = [0,1,\ldots,1] (k\ ones)\), \(NAND_k = [1,\ldots,1,0] (k\ ones)\), and \(EQ_k = [1,0,\ldots,0,1] (k-1\ zeros)\), where \(k \in \mathbb{N}^+\). Slightly abusing notations, we let the notation \(EQ\) (\(OR\) and \(NAND\), resp.) refer to the equality function (\(OR\)-function and \(NAND\)-function, resp.) of arbitrary arity larger than one. This notational convention is quite useful when we do not want to specify its arity.

Moreover, we use the following two sets of relations. A relation \(R\) is in \(\text{DISJ}\) (\(NAND\), resp.) if it equals a product of a positive number of relations of the forms \(\text{OR}_k\) (\(NAND_k\), resp.), \(\Delta_0\), and \(\Delta_1\), where \(k \geq 2\) (slightly different from \(\text{OR-conj}\) and \(\text{NAND-conj}\) in \([11]\)). Notice that the empty relation “\(\emptyset\)” is in \(\text{DISJ} \cup NAND\). Next, we introduce six sets of constraints, the first four of which were defined in \([21]\).

1. Recall that \(U\) denotes the set of all unary constraints.
2. Let \(NZ\) denote the set of arbitrary non-zero constraints.
3. Let \(DG\) denote the set of all constraints \(f\) that are expressed as products of unary constraints, each of which is applied to a different variable of \(f\). Every constraint in \(DG\) is called degenerate. In particular, \(U\) is included in \(DG\). The underlying relation of any degenerate constraint is also degenerate; however, the converse is not true in general.
4. Let \(ED\) denote the set of constraints expressed as products of unary constraints, the binary equality \(EQ_2\), and the binary disequality \(\text{XOR}\). Clearly, \(DG \subseteq ED\) holds. The name “\(ED\)” refers to its key components of “equality” and “disequality.”
5. Let \(DISJ\) be the set of all constraints \(f\) for which \(R_f\) is in \(\text{DISJ}\).
6. Let \(NAND\) be the set of all constraints \(f\) for which \(R_f\) belongs to \(NAND\).

For later convenience, we list a simple characterization of binary constraints in \(DG\).

**Lemma 3.1** Let \(f\) be any binary constraint \(f = (a,b,c,d)\) with \(a,b,c,d \in C\). It holds that \(f \not\in DG\) iff \(ad \neq bc\).

**Proof.** Let \(f = (a,b,c,d)\) with \(a,b,c,d \in C\). First, assume that \(f\) is degenerate. Since \(f \in DG\), there are four constants \(x,y,\overline{z},\overline{w} \in C\) such that \(f(x_1,x_2) = [x,y](x_1) \cdot [\overline{z},\overline{w}](x_2)\) holds for every vector \((x_1,x_2) \in \{0,1\}^2\). This implies \(f = (xz,\overline{xw},\overline{y},\overline{w}y)\). Since \(f\) equals \((a,b,c,d)\), we obtain \(ad = \overline{zy}w = bc\), as required. Next, we assume that \(ad = bc\). There are three cases to examine separately.

(i) Consider the case where \(a = 0\). By our assumption, either \(b = 0\) or \(c = 0\) holds. If \(b = 0\), then it holds that \(f(x_1,x_2) = [0,1](x_1) \cdot [c,d](x_2)\); thus, \(f\) is degenerate. Similarly, when \(c = 0\), we obtain \(f(x_1,x_2) = [b,d](x_1) \cdot [0,1](x_2)\) and thus \(f\) is degenerate.

(ii) The case where \(d = 0\) is similar to Case (i).

(iii) Finally, assume that \(ad \neq 0\). Obviously, \(bc \neq 0\) holds since \(ad = bc\). Let us define \(y = \frac{b}{a} = \frac{d}{c}\). Since \(b = ay \text{ and } d = cy\), it instantly follows that \(f(x_1,x_2) = [a,c](x_1) \cdot [1,y](x_2)\). From this equality, we conclude that \(f\) is degenerate. \(\square\)

A key idea of \([21]\) is a certain form of “factorization” of a target constraint. For each constraint \(f\) in \(ED\), for instance, its underlying relation \(R_f\) can be expressed by a product \(R_f = g_1 \cdot g_2 \cdots g_m\), where each constraint \(g_i\) is one of the following forms: \(u(x)\), \(EQ_2(x,y)\), and \(\text{XOR}(x,y)\) where \(x\) and \(y\) may be the same, where \(u\) is an arbitrary unary constraint. This indicates that \(f\) is “factorized” into factors: \(g_1, g_2, \ldots, g_m\) (which always include the information on input variables). The list \(L = \{g_1, g_2, \ldots, g_m\}\) of all such factors is succinctly called a factor list for \(R_f\).

In our later argument, factor lists will play an essential role. Let us introduce a notion—an or-distinctive list—for each constraint in \(DISJ\). Associated with a constraint \(f\) in \(DISJ\), let us consider a list \(L\) of all factors of the form \(\Delta_0(x), \Delta_1(x), \text{and } OR_j(x_1,\ldots,x_j)\), that characterizes \(R_f\). This factor list \(L\) is called or-distinctive if (i) no variable appears more than once in each \(OR\) in \(L\), (ii) no two factors \(\Delta_c\) (\(c \in \{0,1\}\)) and \(OR\) in \(L\) share the same variable, (iii) no \(OR\)’s variables form a subset of any other’s (when ignoring the variable order), and (iv) every \(OR\) in \(L\) has at least two variables. For each constraint in \(NAND\), we obtain a similar notion of and-distinctive list by replacing \(ORs\) with \(NANDs\).
The following lemma is fundamentally the same as [11, Lemma 3.2] for Boolean constraints.

**Lemma 3.2** For any constraint \( f \) in \( \text{DISJ} \), there exists a unique or-distinctive list of all factors of \( R_f \). The same holds for \( \text{AND} \) distinguishes lists and \( \text{NAND} \).

**Proof.** Let \( f \) be any \( k \)-ary constraint in \( \text{DISJ} \) and let \( L \) be any factor list for \( R_f \) with the condition that each factor in \( L \) has one of the following forms: \( \Delta_0(x), \Delta_1(x) \), and \( OR_d(x_1, \ldots, x_d) \), where \( d \geq 2 \) and \( i_1, \ldots, i_d \in [k] \). Now, let us consider the following procedure that transforms \( L \) into another factor list, which becomes or-distinctive. For ease of the description of this procedure, we assume that, during the procedure, whenever all variables are completely deleted from an argument place of any factor \( g \) in \( L \), this \( g \) is automatically removed from the list \( L \), since \( g \) is no longer a valid constraint. Moreover, if there are two exactly the same factors (with the same series of input variables), then exactly one of them is automatically deleted from \( L \). Finally, since \( OR_1 \) equals \( \Delta_1 \), any factor \( OR_1(x) \) in \( L \) is automatically replaced by \( \Delta_1(x) \).

(i) For each factor \( OR_d \) in \( L \), if a variable \( x \) appears more than once in its argument place, then we delete the second occurrence of \( x \) from the argument place. This deletion causes this \( OR_d \) to shrink to an \( OR_{d-1} \). Now, we assume that every factor \( OR \) in \( L \) has no duplicated variables. (ii) If two factors \( OR_d \) and \( \Delta_1 \), in \( L \) share the same variable, say, \( x \), then we remove this \( OR_d \) from \( L \). This removal is legitimate because this \( OR_d \) is clearly redundant. (iii) If two factors \( OR_d \) and \( \Delta_0 \) in \( L \) share the same variable \( x \), then we delete \( x \) from any argument places of all \( OR_d \) in \( L \). This process is also legitimate, because \( x \) is pinned down to 0 by \( \Delta_0(x) \) and it does not contribute to the outcome of \( OR_d \). It is not difficult to show that the list obtained from \( L \) by executing this procedure is indeed or-distinctive.

To complete the proof, we will show the uniqueness of any or-distinctive list for \( R_f \). Assume that \( L_1 \) and \( L_2 \) are two different or-distinctive lists of all factors of \( R_f \). Henceforth, we intend to show that \( L_1 \subseteq L_2 \). For simplicity, let \( X_0 = \{x_1, \ldots, x_k\} \) denote the set of all variables that do not appear in any factor of the form \( \Delta_c \) \((c \in \{0, 1\})\) in \( L_1 \). We note that any factor \( \Delta_c \) in \( L_2 \) takes no variable in \( X_0 \) because, otherwise, \( L_1 \) and \( L_2 \) must define two different relations, a contradiction against our assumption that \( L_1 \) and \( L_2 \) are factor lists for the same relation \( R_f \). Toward our goal, we need to prove two claims.

First, we claim that all factors of the form \( \Delta_c \) \((c \in \{0, 1\})\) in \( L_1 \) belong to \( L_2 \). Assume otherwise; that is, there is a factor \( \Delta_c(x) \) that appears in \( L_1 \) but not in \( L_2 \). Notice that \( x \) should appear in a certain factor in \( L_2 \). If the factor \( \Delta_1-c \) is present in \( L_2 \), then \( L_1 \) and \( L_2 \) should define two different relations, a clear contradiction. Hence, \( L_2 \) does not contain \( \Delta_1-c \). Since \( x \) cannot appear in both \( \Delta_0 \) and \( \Delta_1 \) in \( L_2 \), \( x \) must appear in a certain \( OR \), say, \( h \) of arity \( m \) in \( L_2 \). Since \( L_2 \) is an or-distinctive list, \( m \geq 2 \) follows. Let us choose a variable assignment \( a \) to \( x \) satisfying \( \Delta_c(a) = 0 \). By choosing another assignment \( b \in \{0, 1\}^{k-1} \) appropriately, we can force \( h(a, b) = 1 \). This is a clear contradiction.

Next, we claim that all \( ORs \) in \( L_1 \) are also in \( L_2 \). Toward a contradiction, we assume that (after appropriately permuting variable indices) \( g(x_1, \ldots, x_d) \) is an \( OR_d \) in \( L_1 \) but not in \( L_2 \). Let \( X = \{x_1, \ldots, x_d\} \).

By the or-distinctiveness, any other \( OR \) in \( L_1 \) should contain at least one variable in \( X_0 - X \). We need to examine the following two cases separately. (1) Assume that there exist an index \( m \in [d-1] \) and a factor \( h \) of the form \( OR_m \) or \( \Delta_1 \) if \( m = 1 \) in \( L_2 \) satisfying that all variables of \( h \) are in \( X \). Since \( m < d \), we obtain both \( h(0^m) = 0 \) and \( g(0^m, 1^{d-m}) = 1 \). This is a contradiction. (2) Assume that every factor \( h \) of the form \( OR \) in \( L_2 \) contains at least one variable in \( X_0 - X \). Clearly, it holds that \( g(0^d) = 0 \) and \( h(a, b) = 1 \), where \( a \) and \( b \) are respectively appropriate nonempty portions of \( 0^d \) and \( 1^{k-d} \). This also leads to a contradiction. Therefore, \( g \) should belong to \( L_2 \).

In the end, we conclude that \( L_1 \subseteq L_2 \). Since we can prove by symmetry that \( L_2 \subseteq L_1 \), this yields the equality \( L_1 = L_2 \), and thus we establish the uniqueness of an or-distinctive list for \( R_f \). The case for \( \text{NAND} \) can be similarly treated. \( \square \)

4 Limited T-Constructibility

A technical tool used for an analysis of \#CSPs in [21] is the notion of \( T \)-constructibility, which asserts that a given constraint can be systematically “constructed” by applying certain specific operations recursively, starting from a finite set of target constraints. Such a construction directly corresponds to a modification of bipartite graphs in constraint frames. Since our target is bounded-degree \#CSPs, we rather use its weakened version.

Now, we introduce our key notion of limited \( T \)-constructibility, which will play a central role in our later arguments toward the proof of the main theorem. Let \( f \) be any constraint of arity \( k \geq 1 \) and let \( \mathcal{G} \) be any finite constraint set. We say that an undirected bipartite graph \( G = (V_1, V_2, E) \) (implicitly with
a labeling function $\pi$ represents $f$ if $V_1$ consists only of $k$ nodes labeled $x_1, \ldots, x_k$, which may have a certain number of dangling edges, and $V_2$ contains only a node labeled $f$, to whom every node $x_i$ is adjacent. As noted before, we write $f_w$ for $\pi(w)$. We also say that $G$ realizes $f$ by $G$ if the following four conditions are met: (i) $\pi(V_2) \subseteq G$, (ii) $G$ contains at least $k$ nodes labeled $x_1, \ldots, x_k$, possibly together with nodes associated with other variables, say, $y_1, \ldots, y_m$; namely, $V_1 = \{x_1, \ldots, x_k, y_1, \ldots, y_m\}$ (by identifying a node name with its variable label), (iii) only the nodes $x_1, \ldots, x_k$ may have dangling edges, and (iv) $f(x_1, \ldots, x_k) = \lambda \sum_{y_1, \ldots, y_m \in \{0, 1\}} \prod_{w \in V_2} f_w(x_1, \ldots, x_k, y_1, \ldots, y_m)$, where $\lambda \in \mathbb{C} - \{0\}$ and $\{z_1, \ldots, z_d\}$ is a subset of $V_1$.

**Example 4.1** Here, we give a useful example of an undirected bipartite graph that realizes a constraint $g$ of particular form: \((*) g(x_1, x_2) = \sum_{y \in \{0, 1\}} f(x_1, y)u(y)f(y, x_2).\) Corresponding to this equation \((*)\), we construct the following graph, denoted $G_{f,u}$. This graph is composed of three nodes labeled $x_1, x_2, y$ on its left-hand side and two nodes $v_1$ and $v_2$ labeled as well as a node $w$ labeled $u$ on the right-hand side. The graph has an edge set \(\{(x_1, v_1), (y, v_1), (y, w), (x_2, v_2), (y, v_2)\}\). Since this graph $G_{f,u}$ faithfully reflects the above equation \((*)\), it is not difficult to check that Condition (iv) of the definition of realizability is satisfied. Therefore, $G_{f,u}$ realizes $g$ by \(f, u\).

Let $d \in \mathbb{N}$ be any index. We write $f \leq_d G$ if the following conditions hold: for any number $m \geq 2$ and for any graph $G$ representing $f$ with distinct variables $x_1, \ldots, x_k$ whose node degrees are at most $m$, there exists another graph $G'$ such that (i) $G'$ realizes $f$ by $G$, (ii) $G'$ has the same dangling edges as $G$ does, (iii) the nodes labeled $x_1, \ldots, x_k$ have degree at most $m + d$, and (iv) all the other nodes on the left-hand side of $G'$ have degree at most $\max\{3, d\}$. In this case, we loosely say that $f$ is limited $T$-constructible from $G$. The constraint $g$ in Example 4.1 is limited $T$-constructible from $G$. More precisely, since $G_{f,u}$ contains the node $y$ of degree 3, $g \leq_{d} \{f, u\}$ holds. Although the above definition is general enough, in this paper, we are interested only in the case where $0 \leq d \leq 1$.

We will see another example.

**Example 4.2** Let $f$ and $g$ be any two constraints. If $f$ is obtained from $g$ by pinning $g$, then $f \leq_{d} \{g, \Delta_0, \Delta_1\}$ holds. To prove this statement, we here consider only a simple case where $f$ is obtained from $g$ by the equation $f(x_1, \ldots, x_k) = g^{x_1=c_1, x_2=c_2}(x_3, \ldots, x_k)$, where $k \geq 3$ and $c_1, c_2 \in \{0, 1\}$. A more general case can be treated similarly. Let $G$ be any undirected bipartite graph that represents $f$ with nodes having labels $x_3, \ldots, x_k$. We construct another bipartite graph $G'$ as follows. We prepare two “new” nodes whose labels are $x_1$ and $x_2$. Remember that these variables do not appear in the argument place of $f$. Add these new nodes into $G$, replace the node $f$ in $G$ by a “new” node labeled $g$ together with two extra edges incident on the nodes $x_1$ and $x_2$, and finally attach two “new” nodes with labels $\Delta_1$ and $\Delta_2$ to the nodes $x_1$ and $x_2$, respectively, by two “new” edges. Clearly, $G'$ realizes $f$ by $\{g, \Delta_1, \Delta_2\}$. Now, let us analyze the node degrees. Each node $x_i$ ($3 \leq i \leq k$) in $G'$ has the same degree as the original node $x_i$ in $G$ does. In contrast, the nodes $x_1$ and $x_2$ have only two incident edges. Therefore, we conclude that $f \leq_{d} \{g, \Delta_1, \Delta_2\}$.

Unlike the case of $T$-constructibility, the property of transitivity does not hold for limited $T$-constructibility. Nonetheless, the following restricted form of transitivity is sufficient for our later arguments.

**Lemma 4.3** Let $f$ and $g$ be any two constraints and let $G_1$ and $G_2$ be any two finite constraint sets. Moreover, let $d$ be any number in $\mathbb{N}$. If $f \leq_{d} G_1 \cup \{g\}$ and $g \leq_{d} G_2$, then $f \leq_{d} G_1 \cup G_2$.

**Proof.** If $g$ is already in $G_1 \cup G_2$, then the lemma is trivially true; henceforth, we assume that $g \notin G_1 \cup G_2$. Now, let $f(x_1, x_2, \ldots, x_k)$ be any constraint of arity $k \geq 1$ and let $G_f$ be any undirected bipartite graph, comprised of $k$ nodes labeled $x_1, \ldots, x_k$ and a node labeled $f$, that represents $f$. Assume that $m \geq 2$ and each node $x_i$ ($i \in [k]$) on the left-hand side of $G_f$ has degree at most $m$. Since $f \leq_{d} G_1 \cup \{g\}$, there exists another undirected bipartite graph $G'_f = (V'_1 \cup V'_2, E)$ that realizes $f$ by $G_1 \cup \{g\}$. For simplicity, let $V_1 = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_m\}$ with $m$ variables $y_1, \ldots, y_m$ not appearing in $G_f$. Note that, by the degree requirement of limited $T$-constructibility, every node $x_i$ ($i \in [k]$) has degree at most $m + d$ and every node $y_j$ ($j \in [m]$) has degree at least $\max\{3, d\}$.

Since there may be one or more nodes in $G'_f$ whose labels are $g$, we want to eliminate recursively those nodes one by one. Choose any such node, say, $w$. We first remove from $G'_f$ all nodes in $V_1 \cup V_2$ that are not adjacent to $w$ and also remove their incident edges; however, we keep dangling edges, all edges between the remaining nodes in $V_1$ and the nodes other than $w$ in $V_2$. Let $G = (V'_1 \cup V'_2, E')$ be the resulting graph from $G'_f$. Since $V'_1$ is the set of remaining nodes in $V_1$, without loss of generality, we assume that

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\(^{5}\)A dangling edge is obtained from an edge by deleting exactly one end of the edge. These dangling edges are treated as "normal" edges. Therefore, the degree of a node should count dangling edges as well.
$V' = \{x_1, \ldots, x_a, y_1, \ldots, y_b\}$, where $0 \leq a \leq k$ and $0 \leq b \leq m$. Since $g$ takes all those variables, $\tilde{G}$ obviously represents $g$. In this graph $\tilde{G}$, since $f \leq^{+d}_{\text{con}} G_1 \cup \{g\}$, every node $x_i$ must have degree at most $m + d$ while each node $y_j$ has degree at most $\max\{3, d\}$. Since $f \leq^{+d}_{\text{con}} G_2$, there is another bipartite graph $\tilde{G}' = (V''_1 \cup V''_2, E'')$ that realizes $g$ by $G_2$. Now, assume that $V''_1 = \{x_1, \ldots, x_a, y_1, \ldots, y_b, z_1, \ldots, z_c\}$ with “fresh” variables $z_1, \ldots, z_c$. Note that the degrees of the nodes $x_i$ and $y_j$ in $\tilde{G}'$ are the same as that in $\tilde{G}$, and the degree of any other node $z_i$ in $V''_1$ is at most three. Inside $G_f$, we then replace the subgraph $G_f$ by $\tilde{G}'$. Clearly, the resulting graph has fewer nodes with the label $g$ than $\tilde{G}_f$ does. We continue this elimination process until the nodes labeled $g$ are all removed.

In the end, let $G_*$ be the obtained bipartite graph. On the right-hand side of $G_*$, there are only nodes whose labels are taken from $G_1 \cup G_2$. By its definition, $G_*$ realizes $f$ by $G_1 \cup G_2$. Moreover, in this graph $G_*$, the degree of every node $x_i$ is still at most $m + d$ whereas any other node has degree at most $\max\{3, d\}$. Therefore, we conclude that $f \leq^{+d}_{\text{con}} G_1 \cup G_2$, as requested. \hfill $\square$

## 5 Constructing AP-Reductions to the Equality

Dyer et al. \[11\] analyzed the complexity of approximately solving unweighted bounded-degree Boolean #CSPs and proved the first approximation-complexity classification theorem for those #CSPs using notions of “3-simulatability” and “ppp-definability.” In their classification theorem, stated in Section \[1\] they recognized four fundamental categories of counting problems. We intend to extend their theorem from unweighted #CSPs to complex-weighted #CSPs by employing the notion of limited T-constructibility described in Section \[4\]. Our goal is therefore to prove our main theorem, Theorem \[11\].

We start with a brief discussion on the polynomial-time computability of bounded-degree Boolean #CSPs. For any constraint set $F$, it is already known from \[21\] that, when $F \subseteq \mathcal{ED}$, #CSP$^*$($F$) is solvable in polynomial time and thus belongs to FP$_C$. From this computability result, since #CSP$^*_d(F) \leq_{\text{AP}}$ #CSP$^*$($F$), the following statement is immediate.

**Lemma 5.1** For any constraint set $F$ and any index $d \geq 2$, if $F \subseteq \mathcal{ED}$, then #CSP$^*_d(F)$ belongs to FP$_C$.

The remaining case where $F \not\subseteq \mathcal{ED}$ is the most challenging one in this paper. In what follows, we are focused on this difficult case. At this point, we are ready to describe an outline of our proof of the main theorem. For notational convenience, we write $\mathcal{EQ}$ for the set $\{EQ_k\}_{k \geq 2}$, where we do not include the equality of arity 1, because it is in $\mathcal{U}$ and is always available for free of charge. Cai et al. \[3\] first laid out a basic scheme of how to prove a classification theorem for complex-weighted degree-3 Boolean #CSPs. Later, this scheme was modified by Dyer et al. \[11\] to prove their classification theorem for unweighted degree-$d$ Boolean #CSPs for any $d \geq 3$. Our proof strategy closely follows theirs even though we deal with weighted degree-$d$ #CSPs.

For a technical reason, it is better for us to introduce a notation #CSP$^*_d(\mathcal{EQ}, F)$, which is induced from #CSP$^*_d(\mathcal{EQ}, F)$, by imposing the following extra condition (assuming $F \cap \mathcal{EQ} = \emptyset$):

(*) In each constraint frame $\Omega = (G, X|F', \pi)$ given as input instance instance, no two nodes labeled $EQ_k$s in $\mathcal{EQ}$ (possibly having different arities) on the right-hand side of the undirected bipartite graph $G$ are adjacent to the same node having a variable label on the left-hand side of the graph.

In other words, any two nodes with labels from $\mathcal{EQ}$ on the right-hand side of $G$ are not linked directly by any single node. This artificial condition (*) is necessary in the proof of Lemma 5.3. Similarly, we define #CSP$^*_d(\mathcal{EQk}|F)$ using the singleton $\{EQ_k\}$ instead of $\mathcal{EQ}$. Our proof strategy comprises the following four steps.

1. First, for any constraint set $F$, we will claim that #CSP$^*(F) \equiv_{\text{AP}}$ #CSP$^*(F')$, where $F' = F - \mathcal{EQ}$. Meanwhile, we will focus on this set $F'$. Second, we will add the equality of various arity and then reduce the original #CSPs to bounded-degree #CSPs with the above-mentioned condition (*). More precisely, we will AP-reduce #CSP$^*(F')$ to #CSP$^*_2(\mathcal{EQ}, F')$.

2. For any index $d \geq 2$ and for any constraint $f \in F$, we will AP-reduce #CSP$^*_d(\mathcal{EQ}_d|F')$ to #CSP$^*_2(f, F')$, which is clearly AP-reducible to #CSP$^*_d(F)$ since $\{f\} \cup F' \subseteq F$. In addition, we will demand that this reduction be algorithmically “generic” and “efficient” so that if we can AP-reduce #CSP$^*_2(\mathcal{EQ}_d|F')$ to #CSP$^*_3(f, F')$ for every index $d \geq 3$, then we immediately obtain #CSP$^*_d(\mathcal{EQ}|F') \leq_{\text{AP}}$ #CSP$^*_3(f, F')$.
3. Combining the above two AP-reductions, we obtain the AP-reduction #CSP*(F) \leq_{AP} #CSP∗_2(F) by Lemma 2.1. Since #CSP∗_2(F) \leq_{AP} #CSP∗_3(F) \leq_{AP} #CSP∗(F) for any index d \geq 3, we conclude that #CSP∗(F) \equiv_{AP} #CSP∗_2(F). This becomes our key claim, Proposition 1.2.

4. Finally, we will apply the dichotomy theorem [21] for #CSP∗(F)'s to determine the approximation complexity of #CSP∗_2(F)'s using the key claim stated in Step 3.

The first step of our proof strategy described above is quite easy and we intend to present it here.

**Lemma 5.2** Let F be any constraint set and define F' = F - ΕQ.
1. #CSP*(F) \equiv_{AP} #CSP*(F').
2. #CSP*(F') \leq_{AP} #CSP∗_2(ΕQ\parallel F').

**Proof.** (1) Obviously, it holds that #CSP*(F) \leq_{AP} #CSP*(F) because F' \subseteq F. What still remains is to build the opposite AP-reduction. Now, let Ω be any constraint frame given to #CSP*(F) with an undirected bipartite graph G = (V_1|V_2,E), where all nodes in V_1 have variable labels. Note that, whenever there is a node \(v\) labeled EQ_q (d \geq 2) in V_2 that has two or more edges incident on the same node in V_1, we can delete all but one such edge without changing the outcome of CSP. To keep the node labeling valid, we need to replace the label EQ_q by EQ_q', where d' equals deg(v) in the modified graph. In the following argument, we assume that any node with label EQ_q in V_2 is always adjacent to d distinct nodes in V_1.

Choose any node, say, v whose label is EQ_q (d \geq 2) in V_2. Let us consider a subgraph G_v consisting only of the node v and of all nodes labeled, say, x_1, . . . , x_d adjacent to v, together with all edges between v and those d nodes. The graph G_v is also composed of, as dangling edges, all edges that have been linked between any node x_i (i \in [d]) and any node in V_2 - \{v\}. We first observe that all values of the variables x_1, . . . , x_d should coincide in order to make EQ_q(x_1, . . . , x_d) non-zero. From this property, we merge all the nodes x_1, . . . , x_d into a single node w with a “new” variable label, say, w' and then delete all edges but one that become incident on both w and v, while we keep the dangling edges as all distinct edges. Finally, we label the node v by EQ_q. Let G' be the graph induced from G_v by the above modification. Now, we replace G_v that appears as a subgraph inside G by G'. This replacement process is repeated until all nodes labeled EQ_q (d \geq 2) are removed. The obtained graph G' has no node whose label is taken from ΕQ. Let Ω' be the constraint frame associated with G'. Since the replacement does not change the value of CSP, CSP_{Ω'} = CSP_{Ω} follows, and thus we obtain #CSP*(F') \leq_{AP} #CSP*(F').

(2) Given an input instance Ω = (G, X/F, π) to #CSP*(F') with G = (V_1|V_2,E), we will construct another instance Ω' to #CSP∗_2(ΕQ\parallel F) by applying the following recursive procedure. Choose any node of degree d (d \geq 2) in V_1 and assume that this node has label x. Let e_1, . . . , e_d be the d distinct edges incident on this node x and assume that each e_i (i \in [d]) bridges the node x and a node labeled, say, y_i in F'. Delete this node x and replace it with d “new” nodes having variable labels, say, y_1, y_2, . . . , y_d that do not appear in G. Introduce an additional “new” node, say, v labeled EQ_q to V_2. For each index i \in [d], we re-attach to node y_i each edge e_i from the node y_i and then make all the nodes y_1, . . . , y_d adjacent to the node v by d “new” edges. Notice that each node y_i (i \in [d]) is now adjacent to two nodes v and y_i. We continue this procedure until all original nodes of degree at least two in V_1 are replaced.

To the end, let G' denote the obtained bipartite graph from G and let Ω' be its associated constraint frame. By our construction, any node on the left-hand side of G' has degree exactly two. In addition, no two nodes labeled EQ_q share the same variables. Since CSP_{Ω'} = CSP_{Ω} obviously holds, the lemma thus follows.

The reader might wonder why we have used ΕQ, instead of \{EQ_2\}, in the above lemma although any EQ_q can be expressed by a finite chain of EQ_q's; for instance, \(EQ_2(x_1,x_2,x_3) = EQ_2(x_1,2)EQ_2(x_2,x_3).\) The reason we have not used EQ_2 alone in (2) of the above proof is that, after running the construction procedure in (2), any node with a variable label that directly connects two EQ_q's becomes degree three instead of two, and thus this fact proves #CSP*(F') \leq_{AP} #CSP∗_2(ΕQ\parallel F'), from which we deduce #CSP*(F) \equiv_{AP} #CSP∗_2(F). This consequence is clearly weaker than what we wish to establish.

In the second step of our strategy, we plan to define an AP-reduction from #CSP∗_2(EQ\parallel F) to #CSP∗_{2+m}(G, F). For this purpose, it suffices to prove, as a special case of the following lemma, that EQ_q \leq_{+m} G by a generic and efficient algorithm.

**Lemma 5.3** Let d, m \in \mathbb{N} with d \geq 2. Let F and G be any two constraint sets and assume that F \cap ΕQ = \emptyset and G is finite. If EQ_q \leq_{+m} G, then #CSP∗_2(EQ_q\parallel F) \leq_{AP} #CSP∗_{2+m}(G, F). In addition, assume that there exists a procedure of transforming any graph G representing EQ_q into another graph G' realizing EQ_q by G in time polynomial in the size of d and the size of the graph G. It therefore holds that #CSP∗_{2+m}(G, F) \leq_{AP} #CSP∗_{2+m}(G, F).
Proof. Let \( \Omega \) be any constraint frame given as an input instance to \( \#\text{CSP}^*\{EQ\|F\} \), including an undirected bipartite graph \( G = (V_1|V_2,E) \). Similarly to the proof of Lemma 5.2(1), we hereafter assume that any node with label \( EQ \) in \( V_2 \) is adjacent to \( d \) distinct nodes in \( V_1 \).

Now, we will describe a procedure of how to generate a new instance \( \tilde{\Omega} \) to \( \#\text{CSP}^*\{EQ\|F\} \). Let \( D \) be the collection of all nodes in \( V_2 \) whose labels are \( EQ \). The following procedure will remove all nodes in \( D \) recursively. Let us pick an arbitrary node \( v \) in \( D \) and consider any subgraph \( G' \) of \( G \) satisfying that \( G' \) consists only of the node \( v \) and \( d \) different nodes labeled, say, \( x_{i1}, \ldots, x_{id} \) in \( V_1 \) that are all adjacent to \( v \). Because of the degree bound of \( \#\text{CSP}^*\{EQ\|F\} \), each of those \( d \) nodes on the left-hand side of \( G' \) should contain at most one dangling edge, which is originally incident on a certain other node in \( V_2 \). Clearly, \( G' \) represents \( EQ_d \). Since \( EQ_d \leq_{\text{con}} G \), there exists another undirected bipartite graph \( G'' \) that realizes \( EQ_d \) by \( G \). Inside the original graph \( G \), we replace this subgraph \( G' \) by \( G'' \). Note that, in this replacement, any node other than \( x_{i1}, \ldots, x_{id} \) in \( G'' \) are treated as “new” nodes; thus, those new nodes are not adjacent to any node outside of \( G'' \). Furthermore, for each dangling edge appearing in \( G'' \), we restore its original edge connection to a certain node in \( V_2 \). Clearly, the resulting graph contains less nodes having the label \( EQ_d \). The above process is repeated until all nodes in \( D \) are removed.

Let \( \tilde{G} \) be the bipartite graph obtained by applying the aforementioned procedure and let \( \tilde{\Omega} \) be the new constraint frame associated with \( \tilde{G} \). The degree of each node \( x_i \) in \( \tilde{G} \) is at most \( m \) plus the original degree in \( G \) since no two nodes labeled \( EQ \) in \( G \) share the same variables. By the realizability notion, it is not difficult to show that \( \text{csp}_p = \text{csp}_{\tilde{G}} \). This implies that \( \#\text{CSP}^*\{EQ_d\|F\} \leq_{\text{AP}} \#\text{CSP}^*\{EQ\|\tilde{F}\} \).

By examining the proofs of each lemma given below, it is easy to check that the procedure of showing a limited T-constructibility relation \( EQ_d \leq_{\text{con}} G \cup \{f\} \) for each index \( d \geq 3 \) is indeed “generic” and “efficient,” as requested by Lemma 5.3. Therefore, we will finally conclude that \( \#\text{CSP}^*\{EQ\|F\} \leq_{\text{AP}} \#\text{CSP}^*\{f,\tilde{F}\} \).

This section deals only with non-degenerate constraints of arity two, because degenerate constraints have been already handled by Lemma 5.1. The first case to discuss is a constraint \( f \) of the form \( (0, a, b, 0) \) with \( ab \neq 0 \), whose underlying relation \( R_f \) is XOR.

\begin{lemma}
Let \( d \) be any index at least two. Let \( f = (0, a, b, 0) \) with \( a, b \in \mathbb{C} \). If \( ab \neq 0 \), then \( EQ_d \leq_{con} f \) holds.
\end{lemma}

Proof. From a given constraint \( f = (0, a, b, 0) \), we define another constraint \( g \) as \( g(x_1, x_2) = \sum_{y \in \{0, 1\}} f(x_1, y) f(y, x_2) \). A direct calculation shows that \( g = (ab, 0, 0, ab) \). From this definition of \( g \), we note that \( \ast \) the value of \( y \) is uniquely determined from \( (x_1, x_2) \) if \( g(x_1, x_2) \neq 0 \). More generally, for each index \( d \geq 2 \), we define \( h(x_1, \ldots, x_d) = \sum_{y_1, \ldots, y_{d-1} \in \{0, 1\}} \prod_{i=1}^{d-1} f(x_i, y_i) f(y_i, x_{i+1}) \). Clearly, when \( d = 2 \), \( h \) coincides with \( g \). Because of the uniqueness property of \( g \) stated in \( \ast \), \( h(x_1, \ldots, x_d) = \prod_{i=1}^{d-1} g(x_i, x_{i+1}) \). This implies that \( h(0, \ldots, 0) = h(1, \ldots, 1) = (ab)^{d-1} \) and \( h(e) = 0 \) for any other variable assignment \( e \in \{0, 1\}^d \). It therefore follows that \( h = (ab)^{d-1} \cdot EQ_d \). Since \( ab \neq 0 \), by normalizing \( h \) appropriately, we then obtain \( EQ_d \leq_{con} f \).

Next, we will show that \( EQ_d \leq_{con} f \). Let \( G \) be any undirected bipartite graph representing \( EQ_d \) with \( d \) nodes whose labels are \( x_1, \ldots, x_d \). Consider a new graph \( G' \) obtained from \( G \), using the above equation of \( h \), by adding \( d - 1 \) “new” nodes labeled \( y_1, \ldots, y_{d-1} \) and by replacing the node \( EQ_d \) in \( G \) with \( 2(d - 1) \) “new” nodes labeled \( f \), each of which is adjacent to two nodes \( x_i \) and \( y_i \) (\( i \in \{d - 1\} \) or two nodes \( y_i \) and \( x_{i+1} \). This bipartite graph \( G' \) clearly realizes \( EQ_d \) by \( f \). Two special nodes \( x_1 \) and \( x_2 \) in \( G' \) maintain their original degree in \( G \), whereas each node \( x_i \), except for \( x_1 \) and \( x_2 \) has one more than its original degree in \( G \). In addition, all nodes with the labels \( y_1, \ldots, y_{d-1} \) are of degree exactly two. Therefore, we conclude that
EQ\textsubscript{d} \leq_{\text{con}}^{+1} f$, as requested.

As the second case, we will handle a constraint $f = (a, 0, 0, b)$ satisfying $ab \neq 0$. Since its underlying relation is precisely $EQ_2$, the proof of its limited T-constructibility is rather simple.

**Lemma 6.2** Let $d \geq 2$ and let $f = (a, 0, 0, b)$ with $a, b \in \mathbb{C}$. If $ab \neq 0$, there exists a constraint $u \in \mathcal{U} \cap \mathcal{N}\mathbb{Z}$ such that $EQ_d \leq_{\text{con}}^{+1} \{f, u\}$.

**Proof.** Let $f = (a, 0, 0, b)$ with $ab \neq 0$. First, we consider the base case of $d = 2$. By setting $u = [1/a, 1/b]$, we define a constraint $g$ as $g(x_1, x_2) = u(x_1)f(x_1, x_2)$. Clearly, $g$ equals $EQ_2$. For a degree analysis, let us consider any undirected bipartite graph $G$ that represents $EQ_d$. Since $g$ is $EQ_2$, a new bipartite graph $G'$ is obtained from $G$ by replacing the existing node $EQ_d$ and its associated edges in $G$ with two “new” nodes labeled $u$ and $f$ together with three “new” edges $\{(x_1, u), (x_1, f), (x_2, f)\}$. From this construction, the node $x_1$ in $G'$ has more than its original degree in $G$; however, the degree of the node $x_2$ in $G'$ remains the same as that in $G$. We therefore obtain $EQ_d \leq_{\text{con}}^{+1} \{f, u\}$. This argument will be extended to the general case of $d \geq 2$.

For each fixed index $d \geq 2$, we set $u' = [1/a^d, 1/b^d]$ and define $h(x_1, \ldots, x_d) = u'(x_1)\prod_{i=1}^{d-1} f(x_i, x_{i+1})$. It is not difficult to show that $h$ equals $EQ_d$. Similarly, to the base case, from the definition of $h$, we can build a bipartite graph $G'$ that realizes $h$ by $\{f, u'\}$. In this graph $G'$, each node $x_i (1 \leq i < d)$ has one more than its original degree in $G$, while the node $x_d$ keeps the same degree as that in $G$. This fact helps us conclude that $EQ_d \leq_{\text{con}}^{+1} \{f, u'\}$. 

Our next target is a constraint $f$ of the form $(a, b, 0, c)$ with $abc \neq 0$. The underlying relation of $f$ is exactly $\text{Implies}$.

**Lemma 6.3** Let $d \geq 2$. Let $f = (a, b, 0, c)$ with $a, b, c \in \mathbb{C}$. If $abc \neq 0$, then there exist two constraints $u_1, u_2 \in \mathcal{U} \cap \mathcal{N}\mathbb{Z}$ for which $EQ_d \leq_{\text{con}}^{+1} \{f, u_1, u_2\}$. By permuting variable indices, the case of $(a, 0, b, c)$ is similar.

**Proof.** First, we set $f = (a, b, 0, c)$ and assume that $abc \neq 0$. For this constraint $f$, we prepare the following two unary constraints: $u = [1/a^2, 1/c^2]$ and $u' = [1/a^3, 1/c^3]$. Let us begin with the base case of $d = 2$. In this case, we define $g(x_1, x_2) = f(x_2, x_3)\sum_{y \in \{0, 1\}} f(x_1, y)u(y)\left(f(y, x_2) + u'(y)\right)$. Since $u'$ cancels out the effect of both terms $f(x_2, x_1)$ and $f(x_1, y)f(y, x_2)$, we immediately obtain $g = (1, 0, 0, 1)$. Let $G$ be any undirected bipartite graph representing $EQ_2$ with two variables $x_1$ and $x_2$. To obtain another bipartite graph $G'$ realizing $EQ_2$, we first build a graph $G'[f, u']$ (using $u'$ instead of $u$), introduced in Example 1.2 which is equipped with all the original dangling edges in $G$. We next add an extra “new” node with label $f$ that becomes adjacent to the two nodes $x_2$ and $x_3$. This newly constructed graph $G'$ obviously realizes $EQ_2$ by $\{f, u'\}$. Since $G'$ contains two edges from each node $x_i (i \in \{2, 3\})$, the degree of the node $x_i$ in $G'$ thus increases by one, and therefore $EQ_2 \leq_{\text{con}}^{+1} \{f, u'\}$ follows.

In the case of $d \geq 3$, by extending the base case, we naturally define a constraint $h$ as $h(x_1, \ldots, x_d) = f(x_2, x_3)\sum_{y \in \{0, 1\}} u_{d-1} f(x_1, y)u(y)f(y, x_2)x_i(i \in \{2, 3\})$, where $u_{d-1} = u'$ and $u_i = u$ for each $i \in \{d-2\}$. Note that $u$ and $u'$ bring the same effect as $u'$ does in the base case. The analysis of the node degrees in the corresponding graph is similar in essence to the degree analysis of the base case. Therefore, it immediately follows that $EQ_d \leq_{\text{con}}^{+1} \{f, u, u'\}$. 

Unlike the constraints we have discussed so far, the non-degenerate non-zero constraints $f = (1, a, b, c)$ with $a, b, c \in \mathbb{C}$ are quite special, because they appear only in the case of complex-weighted #CSPs. When $f$ is limited to be a Boolean relation, by contrast, it never becomes both non-degenerate and non-zero. Notice that, by Lemma 5.1, $f \notin \mathcal{D}G$ is equivalent to $ab \neq c$.

**Lemma 6.4** Let $d \geq 2$ and let $f = (1, a, b, c)$ with $abc \neq 0$. If $ab \neq c$, then there exist two constraints $u_1, u_2 \in \mathcal{U} \cap \mathcal{N}\mathbb{Z}$ satisfying that $EQ_d \leq_{\text{con}}^{+1} \{f, u_1, u_2\}$.

**Proof.** Let $f = (1, a, b, c)$ be any binary constraint satisfying that $abc \neq 0$ and $ab \neq c$. Now, we set $u_1 = [1, z]$ and define $g$ as $g(x_1, x_2) = \sum_{y \in \{0, 1\}} f(x_1, y)u_1(y)f(y, x_2)$. This gives $g = (1 + abz, a(1 + cz), b(1 + cz), ab + c^2z)$. If we choose $z = -1/c$, then the constraint $g$ becomes of the form $(1 - ab/c, 0, 0, ab - c)$. Note that, since $ab \neq c$, the first and last entries of $g$ are non-zero. By appealing to (the proof of) Lemma 6.2 which requires another non-zero unary constraint $u_2$, the new constraint $g'(x_1, x_2) = u_2(x_1)g(x_1, x_2)$ equals $EQ_2(x_1, x_2)$.

To show $EQ_2 \leq_{\text{con}}^{+1} \{f, u_1, u_2\}$, from any undirected bipartite graph $G$ representing $EQ_2$ with variables $x_1$
and \(x_2\), we construct another graph \(G'\) by taking \(G[f,u_1]\) (stated in Example 4.1) with the original dangling edges in \(G\) and further by adding a “new” node labeled \(u_2\) that is adjacent to the node \(x_1\). Overall, the degree of any node on the left-hand side of \(G'\) increases by at most one in comparison with the degree of the same node in \(G\).

In a more general case of \(d \geq 3\), with a series \(x = (x_1, \ldots, x_d)\) of \(d\) variables, we define \(g(x) = \sum_{y_1, \ldots, y_d \in \{0,1\}} \prod_{i=1}^{d} f(x_i, y_i) u_1(y_i) g(y_i, x_{i+1})\). Since \(g\) has the form \((a', 0, \ldots, 0, b')\), with an appropriate constraint \(u_2' \in U \cap N \mathbb{Z}\), the constraint \(g'(x) = u_2'(x_1) g(x)\) coincides with \(EQ_d\). A degree analysis of a graph realizing \(EQ_d\) is similar to the base case. We therefore obtain \(EQ_d \leq_{\text{con}} \{f, u_1, u_2'\} \).

As a summary of Lemmas 6.1–6.4 we wish to make a general claim on binary constraints that do not belong to \(DISJ \cup N \wedge \wedge \cup U_{\text{disj}}\). This claim will be a basis of the proof of Proposition 7.3.

**Proposition 6.5** Let \(d \geq 2\). For any non-degenerate binary constraint \(f\), if \(f \notin DISJ \cup N \wedge \wedge \cup U_{\text{disj}}\), then there exists a constraint set \(G \subseteq U \cap N \mathbb{Z}\) with \(|G| \leq 2\) such that \(EQ_d \leq_{\text{con}} G \cup \{f\}\).

**Proof.** Let \(f = (a, b, c, d)\) be any non-degenerate constraint. It is important to note that \(f \notin DISJ \cup N \wedge \wedge \cup U_{\text{disj}}\) iff \(f\) is one of the following forms: \((0, b, c, 0), (a, 0, 0, d), (a, 0, c, d), (a, b, 0, d), (a, b, c, d)\), provided that \(abcd \neq 0\). In particular, for the last form \((a, b, c, d)\), since \(f \notin U_{\text{disj}}\), Lemma 6.1 yields the inequality \(ad \neq bc\). All the above five forms have been already dealt with in Lemmas 6.1–6.4, and therefore the lemma should hold.

The most notable case is where \(f = (0, a, b, c)\) or \(f = (a, b, c, 0)\) with \(abc \neq 0\). These two constraints respectively extend \(OR_2\) and \(NAND_2\) from Boolean values to complex values. Our result below contrasts complex-weighted constraints with unweighted constraints, because this result is not known to hold for the Boolean constraints.

**Proposition 6.6** Let \(d \geq 2\). If \(f = (0, a, b, c)\) with \(abc \neq 0\), then there exists a constraint \(u \in U \cap N \mathbb{Z}\) such that \(EQ_d \leq_{\text{con}} \{f, u\}\). A similar statement holds for \(f = (a, b, c, 0)\) with \(abc \neq 0\).

The proof of this proposition utilizes two useful lemmas, Lemmas 6.7 and 6.8 which are described below. In the first lemma, we want to show that two constraints whose underlying relations are \(OR_2\) and \(NAND_2\) together help compute \(EQ_d\) for any index \(d \geq 2\).

**Lemma 6.7** Let \(d \geq 2\). Let \(f_1 = (0, a, b, c)\) and \(f_2 = (a', b', c', 0)\) with \(a, b, c, a', b', c' \in \mathbb{C}\). If \(ab \neq 0\) and \(b'c' \neq 0\), then \(EQ_d \leq_{\text{con}} \{f_1, f_2\}\).

**Proof.** Let \(f_1 = (0, a, b, c)\) and \(f_2 = (a', b', c', 0)\) with \(ab'c' \neq 0\). First, we explain our construction for the base case of \(d = 2\). By defining \(g(x_1, x_2) = \sum_{y_1, y_2 \in \{0,1\}} f_1(x_1, y_1) f_2(y_2, x_2) g(y_2, x_{2+1})\), \(g\) becomes of the form \((ab'c', 0, 0, ab'c')\), from which we immediately obtain \(EQ_2 = (1, 0, 0, 1)\) by normalizing it since \(ab'c' \neq 0\). Let \(G = (V_1, V_2, E)\) be any undirected bipartite graph representing \(EQ_2\). Based on the definition of \(g\), we will construct an appropriate bipartite graph \(G'\) as follows. We first introduce two additional nodes labeled \(y_1\) and \(y_2\) into \(V_1\). In place of the node labeled \(EQ_2\) in \(V_2\), we next add two “fresh” nodes with the same label \(f_1\), which respectively become adjacent to the two nodes \(x_1\) and \(y_1\) and to the two nodes \(y_2\) and \(x_2\), and we also add two “fresh” nodes having the same label \(f_2\), which are respectively adjacent to the nodes \(y_1\) and \(x_2\) and to the nodes \(x_1\) and \(y_2\). The degree of each node \(x_i\) \((i \in \{2\})\) in \(G'\) increases by one from its original degree in \(G\), because each node \(x_i\) is linked in \(G'\) to the two nodes with labels \(f_1\) and \(f_2\). Moreover, the new nodes \(y_1\) and \(y_2\) have degree exactly two. It therefore holds that \(EQ_2 \leq_{\text{con}} \{f_1, f_2\}\).

In what follows, we assume \(d \geq 3\) and focus on the case where \(d\) is even. We will extend the argument used in the base case. Let \(x = (x_1, \ldots, x_d)\) and \(y = (y_1, \ldots, y_d)\) be two series of distinct variables. We then introduce two useful constraints \(g_1\) and \(g_2\) defined by \(g_1(x, y) = \prod_{i=0}^{d-2} f_1(x_{2i+1}, y_{2i+1}) f_1(y_{2i+2}, x_{2i+2})\) and \(g_2(x, y) = \left(\prod_{i=0}^{d-1} f_2(x_{2i+1}, y_{2i+1})\right) f_2(x_{2d+2}, y_{2d+2})\). With these new constraints, we define \(h(x) = \sum_{y_1, \ldots, y_d \in \{0,1\}} g_1(x, y) g_2(x, y) f_2(y_{2d}, x_{2d})\). By a straightforward calculation, it is not difficult to check that \(h\) truly computes \(\lambda \cdot EQ_d\) for a certain constant \(\lambda \in \mathbb{C} \setminus \{0\}\). Similar to the construction of the base case, from a graph \(G\) representing \(EQ_d\), we can construct a new bipartite graph \(G'\) that realizes \(EQ_d\) by \(\{f_1, f_2\}\). The degree of every node \(x_i\) \((i \in \{d\})\) in \(G'\) is one more than its original degree in \(G\), whereas all nodes \(y_j\) \((j \in \{d\})\) in \(G'\) are of degree two. Thus, we conclude that \(EQ_d \leq_{\text{con}} \{f_1, f_2\}\).

When \(d\) is odd, we initially introduce a fresh variable called \(x_{d+1}\) as a “dummy.” After defining \(h(x_1, \ldots, x_{d+1})\) as done before, we need to define \(h' = h^{x_{d+1}=\ast}\), which turns out to equal \(\lambda' \cdot EQ_d\) for an appropriate non-zero constant \(\lambda'\). The degree analysis of \(G'\) is similar to the even case. Therefore, the
Lemma 7.2 Let \( f = OR_2 \cdot h \) for a given constraint \( h \in \mathbb{N} \mathbb{Z} \) of arity two. By normalizing \( f \) appropriately, we assume, without loss of generality, that \( f \) is of the form \((a, b, c) \in \mathbb{N} \mathbb{Z}, \) where \( ab \neq 0 \). With a use of an extra constraint \( u = [1, 2] \), let us define \( g(x_1, x_2) = \sum_{y \in \{0, 1\}} f(x_1, y)u(y)f(y, x_2) \), which implies \( g = (abz, az, bz, ab + z) \). Hence, if we set \( z = -ab \), then \( g \) equals \((-\cdot)^2, -ab^2, -ab^2, 0\). We then define the desired \( h' \) as \((-\cdot)^2, -ab^2, -ab^2, 1\), which is obviously a non-zero constraint. Obviously, \( g(x_1, x_2) \) coincides with \( NAND_2(x_1, x_2) x_1'(x_1, x_2) \); thus, we obtain \( g = NAND_2 \cdot h' \).

Next, we want to show that \( g \leq_{\text{equiv}} f \). Against any graph \( G \) representing \( g \), we define \( G' \) to be the graph \( G[f, u] \), stated in Example 4.1, together with all dangling edges appearing in \( G \). Recall that \( G[f, u] \) is comprised of nodes labeled \( x_1, x_2, \) and \( y \). The degree of the node \( y \) in \( G' \) is three and the other variable nodes have the same degree as their original ones in \( G \). It therefore follows that \( g \leq_{\text{equiv}} f \).

Finally, we are ready to give the proof of Proposition 6.6

The second lemma ensures that, with a help of unary constraint, we can transform a constraint in \( \text{DISJ} \) into another in \( \text{NAND} \lor \text{DG} \) without increasing the degree of its realizing graph. This is a special phenomenon not seen for Boolean constraints and it clearly exemplifies a power of the \( \text{weighted} \) unary constraints.

Lemma 6.8 For any binary constraint \( h \in \mathbb{N} \mathbb{Z} \), there exist a binary constraint \( h' \in \mathbb{N} \mathbb{Z} \) and a unary constraint \( u \in \mathbb{N} \mathbb{Z} \) such that \( NAND_2 \cdot h' \leq_{\text{equiv}} [OR_2 \cdot h, u] \). A similar statement holds if we exchange the roles of \( OR_2 \) and \( NAND_2 \).

Proof. Let \( f = OR_2 \cdot h \) for a given constraint \( h \in \mathbb{N} \mathbb{Z} \) of arity two. By normalizing \( f \) appropriately, we assume, without loss of generality, that \( f \) is of the from \((0, a, b, 1) \), where \( ab \neq 0 \). With a use of an extra constraint \( u = [1, 2] \), let us define \( g(x_1, x_2) = \sum_{y \in \{0, 1\}} f(x_1, y)u(y)f(y, x_2) \), which implies \( g = (abz, az, bz, ab + z) \). Hence, if we set \( z = -ab \), then \( g \) equals \((-\cdot)^2, -ab^2, -ab^2, 0\). We then define the desired \( h' \) as \((-\cdot)^2, -ab^2, -ab^2, 1\), which is obviously a non-zero constraint. Obviously, \( g(x_1, x_2) \) coincides with \( NAND_2(x_1, x_2) h'(x_1, x_2) \); thus, we obtain \( g = NAND_2 \cdot h' \).

Finally, we are ready to give the proof of Proposition 6.6

Proof of Proposition 6.6. Let \( d \geq 2 \) and let \( f = (0, a, b, c) \) with \( abc \neq 0 \). By setting \( h = (1, a, b, c) \in \mathbb{N} \mathbb{Z} \), we obtain \( f(x_1, x_2) = OR_2(x_1, x_2)h(x_1, x_2) \). By Lemma 6.3 there are two constraints \( u \in U \cap \mathbb{N} \mathbb{Z} \) and \( h' \in \mathbb{N} \mathbb{Z} \) of arity two for which \( g \leq_{\text{equiv}} [f, u] \) and \( g = NAND_2 \cdot h' \). Note that, since \( h' \in \mathbb{N} \mathbb{Z} \), \( f \) should have the form \((a', b', c', 0) \) for certain constants \( a', b', c' \in \mathbb{C} \) with \( a'b'c' \neq 0 \). Now, we apply Lemma 6.7 to \( f \) and \( g \) and then obtain \( EQ_d \leq_{\text{equiv}} [f, u] \). Combining this with \( g \leq_{\text{equiv}} [f, u] \), Lemma 4.3 draws the desired conclusion that \( EQ_d \leq_{\text{equiv}} [f, u] \).

7 Constraints of Higher Arity

We have shown in Section 6 that the equality \( EQ \) of arbitrary arity can be limited T-constructible from non-degenerate binary constraints. Here, we want to prove a similar result for constraints of three or higher arities. Since constraints in \( \mathbb{E} \mathbb{D} \) already fall into \( \text{FP}_C \), it suffices for us to concentrate on the following two types of constraints: (i) constraints within \( \text{DISJ} \lor \text{NAND} \lor \text{DG} \) and (ii) constraints outside of \( \text{DISJ} \lor \text{NAND} \lor \text{DG} \). These types will be discussed in two separate subsections.

7.1 Constraints in \( \text{DISJ} \lor \text{NAND} \lor \text{DG} \)

First, we will focus our attention on constraints residing in \( \text{DISJ} \lor \text{NAND} \lor \text{DG} \). Proposition 6.4 has already handled binary constraints chosen from \( \text{DISJ} \lor \text{NAND} \lor \text{DG} \) with an argument that looks quite different from the unweighted case of Dyer et al. [11]. We will show that this result can be extended to constraints of arbitrary high arity.

Proposition 7.1 Let \( k \geq 2 \) and \( d \geq 2 \). Let \( f \) be any \( k \)-ary constraint in \( \text{DISJ} \lor \text{NAND} \). If \( f \notin \text{DG} \), then there exists a non-zero unary constraint \( u \) such that \( EQ_d \leq_{\text{equiv}} [f, u, \Delta_0, \Delta_1] \). Moreover, it holds that \( \#\text{CSP}_d^*(EQ)[F] \leq_{\text{AP}} \#\text{CSP}_d^*(f, 2) \) for any constraint set \( F \) satisfying \( F \cap EQ = \emptyset \).

Before proving this proposition, we will show below a useful lemma, which requires the following terminology. The width of a constraint \( f \) in \( \text{DISJ} (\text{NAND}, \text{resp.}) \) is the maximal arity of any factor that appears in a unique or-distinctive (nand-distinctive, resp.) factor list for the underlying relation \( R_f \). For each index \( w \geq 2 \), we denote \( DISJ_w (\text{NAND}_w, \text{resp.}) \) the set of all constraints in \( \text{DISJ} (\text{NAND}, \text{resp.}) \) of width exactly \( w \). Note that \( DISJ = \bigcup_{w \geq 2} DISJ_w \) and \( \text{NAND} = \bigcup_{w \geq 2} \text{NAND}_w \).

Lemma 7.2 Let \( w \geq 2 \) be any width index. For any constraint \( f \in DISJ_w (\text{NAND}_w, \text{resp.}) \), there exists a non-zero constraint \( h \) of arity \( w \) satisfying that \( OR_w \cdot h \leq_{\text{equiv}} [f, \Delta_1] \) (\( NAND_w \cdot h \leq_{\text{equiv}} [f, \Delta_0] \), resp.).
Proof. In this proof, we will show the lemma only for $\text{DISJ}_w$ because the other case, $\text{NAND}_w$, is similar. Assume that $w \geq 2$. Let $k \geq 2$ and let $f \in \text{DISJ}_w$ be any arity-$k$ constraint with $k$ variables $x_1, \ldots, x_k$. Notice that the arity of $f$ should be more than or equal to $w$. We can express $f$ as $R_f \cdot h$ using an appropriate $k$-ary constraint $h \in \mathcal{N}$. Hereafter, we look into the underlying relation $R_f$. Let us consider a unique or-distinctive factor list $L$ for $R_f$. Since $L$ should contain at least one OR of arity $w$, $f$ does not belong to $\mathcal{DG}$. By pinning $f$, we want to construct a constraint $g$ whose underlying relation equals a factor $OR_{w}$ in $L$. For this purpose, we describe below a two-step procedure of how to build such a constraint $g$.

(1) If there exists a factor of the form $\Delta_c(x)$ ($c \in \{0, 1\}$) in $L$, then, by assigning the value $c$ to the variable $x$, we obtain a pinned constraint $g' = f^{\tau_{c,c}}$. Since the or-distinctiveness forbids both factors $\Delta_c$ and $OR$ in $L$ to share the same variables, this pinning operation makes $g'$ becomes neither an all-0 function nor an all-1 function.

(2) After recursively applying (1), we now assume that there is no factor of the from $\Delta_c$ in $L$. Let us choose an $OR_w$ in $L$. For simplicity, by permuting variable indices, we assume that this $OR_w$ takes $w$ distinct variables $x_1, x_2, \ldots, x_w$. By assigning 1 to all the other variables $x_{w+1}, \ldots, x_k$, we obtain $g = f^{x_{w+1}=1,\ldots,x_k=1}$, which obviously implies $R_g = R_f^{x_{w+1}=1,\ldots,x_k=1}$. By Example 1.2, it holds that $g \leq_{\text{con}} {\{f, \Delta_1\}}$. Since no variable set of any other OR in $L$ becomes a subset of $\{x_1, \ldots, x_w\}$, $R_g$ actually coincides with the given $OR_w$.

To end the proof, we set $h' = h^{x_{w+1}=1,\ldots,x_k=1}$, implying that $h'$ is of arity $w$. With this $h'$, the constraint $g$ can be expressed as $g = R_g \cdot h'$, and thus $g$ equals $OR_w \cdot h'$ since $R_g = OR_w$. Notice that $h' \in \mathcal{N}$ since $h \in \mathcal{N}$ and $w \geq 2$. Moreover, since $g \leq_{\text{con}} {\{f, \Delta_1\}}$, the constraint $OR_w \cdot h'$ is limited T-constructible from $\{f, \Delta_1\}$. This completes the proof of the lemma.

Proposition 7.4 follows directly from Lemma 7.2 together with Proposition 6.6.

Proof of Proposition 7.1 Assume that $f \in \text{DISJ}$ and $f$ has arity $k$. In addition, we assume that $f$ has width $w$ for a certain number $w \geq 2$; namely, $f \in \text{DISJ}_w$. Notice that $k \geq w$. Lemma 7.2 ensures the existence of a constraint $h \in \mathcal{N}$ of arity $w$ for which $OR_w \cdot h \leq_{\text{con}} \{f, \Delta_1\}$.

Assume that this relation $OR_w$ takes $w$ distinct variables, say, $x_1, \ldots, x_w$. We then choose two specific variables, $x_1$ and $x_2$, and assign 0 to all the other variables. Let $f'$ be the constraint obtained from $OR_w \cdot h$ by performing these pinning operations. By the construction of $f'$, Example 1.2 implies $f' \leq_{\text{con}} \{f, \Delta_0, \Delta_1\}$. It is not difficult to show that, since $h \in \mathcal{N}$, $R_f(x_1, x_2)$ equals $OR_2(x_1, x_2)$; in other words, $f'$ is of the form $(0, a, b, c)$ with $abc \neq 0$.

Finally, we apply Proposition 6.3 and then obtain a constraint $u \in \mathcal{U} \cap \mathcal{N}$ satisfying that $EQu \leq_{\text{con}} \{f', u\}$. We combine this with $f' \leq_{\text{con}} \{f, \Delta_0, \Delta_1\}$ to conclude by Lemma 1.3 that $EQu \leq_{\text{con}} \{f, u, \Delta_0, \Delta_1\}$. The case where $f \in \text{NANDD}$ is similarly treated.

The second part of the proposition follows by Lemma 6.3 from the fact that the above procedure is indeed generic and efficient.

7.2 Constraints Outside of $\text{DISJ} \cup \text{NANDD} \cup \mathcal{DG}$

The remaining type of constraints to discuss is ones that sit outside of $\text{DISJ} \cup \text{NANDD} \cup \mathcal{DG}$. As a key claim for those constraints, we will prove the following proposition.

Proposition 7.3 Let $d$ and $k$ be any two indices at least two. For any constraint $f$ of arity $k$, if $f \not\in \text{DISJ} \cup \text{NANDD} \cup \mathcal{DG}$, then there exists a finite subset $\mathcal{G}$ of $\mathcal{U}$ such that $EQu \leq_{\text{con}} \mathcal{G} \cup \{f\}$. In addition, it holds that $\#\text{CSP}^*_u(\mathcal{E}, \mathcal{Q}, \mathcal{F}) \leq_{\text{AP}} \#\text{CSP}^*_u(f, \mathcal{F})$ for any constraint set $\mathcal{F}$ satisfying $\mathcal{F} \cap \mathcal{E} = \emptyset$.

This proposition will be proven by induction on the arity of a given constraint $f$. As our starting point, we want to prove a useful lemma regarding non-degenerate constraints of particular form.

Lemma 7.4 Let $k \geq 3$. Let $f$ be any non-degenerate constraint of arity $k$. If $f^{x_1=0}, f^{x_1=1} \in \mathcal{DG}$, then there exists a non-degenerate constraint $h$ of arity $k-1$ for which $h \leq_{\text{con}} \mathcal{G} \cup \{f\}$ for a certain finite subset $\mathcal{G}$ of $\mathcal{U} \cap \mathcal{N}$.

Proof. For any fixed index $k \geq 3$, let us choose any arity-$k$ constraint $f$ not in $\mathcal{DG}$ and set $g_b = f^{x_1=b}$ for every index $b \in \{0, 1\}$. Assume that $g_0$ and $g_1$ are degenerate. First, we define a “factor list” for $g_b$. Since $g_b \in \mathcal{DG}$, $g_b(x_2, x_3, \ldots, x_k)$ can be expressed as $\alpha'g_{b,2}(x_2)g_{b,3}(x_3)\cdots g_{b,k}(x_k)$, where $\alpha'$ is an appropriate constant in $C = \{0\}$ and each $g_{b,i}$ has one of the following forms: $\Delta_0(x_i)$, $\Delta_1(x_i)$, and $[1, a](x_i)$ with $a \neq 0$. 


We call the set \( L_b = \{g_{b,2}(x_2), g_{b,3}(x_3), \ldots, g_{b,k}(x_k)\} \) (ignoring the global constant \( a' \)) a factor list for \( g_b \).

Such a factor list is obviously unique.

(1) If \( L_0 \) and \( L_1 \) share the same factor of the form, \( \Delta_0(x_i) \) or \( [1,1](x_i) \) for a certain index \( i \) with \( 1 \leq i \leq k \), then we define \( h = f^{x_i=0} \). In case of \( \Delta_0(x_i) \), for example, it holds that \( f(x_1, x_2, \ldots, x_k) = \Delta_0(x_i)h(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \). From this equation, if \( h \) is degenerate, then \( f \) should be degenerate, contradicting our assumption. Thus, \( h \) cannot be degenerate. The other cases are similar. Obviously, the arity of \( h \) is exactly \( k-1 \). Since \( h \leq cong f \), we immediately obtain the lemma.

(2) Hereafter, we assume that Case (1) never occurs; namely, \( L_0 \cap L_1 = \emptyset \). Let us discuss several cases separately.

(i) Assume that, for a certain index \( i \), \( L_1 \) contains a factor \( \Delta_0(x_i) \) and \( L_2 \) contains \( \Delta_1(x_i) \). For case of the description below, we set \( i = 2 \). By the definition of \( g_0 \), there exists a degenerate constraint \( g'_0 \) such that \( g_0(x_2, x_3, \ldots, x_k) = \Delta_0(x_2)g'_0(x_3, \ldots, x_k) \). Similarly, \( g_0(x_2, x_3, \ldots, x_k) \) is of the form \( \Delta_1(x_2)g'_1(x_3, \ldots, x_k) \) for a certain \( g'_1 \in DG \). For the desired \( h \), we define \( h = f^{x_2=0} \), which implies that \( h^{x_2=0} = g'_0 \) and \( h^{x_2=1} = g'_1 \). Obviously, \( g \leq cong h \) holds. Now, we want to claim that \( h \not\in DG \). Toward a contradiction, we assume otherwise. This yields an equation \( h^{x_2=0} = \gamma \cdot h^{x_2=1} \) for a certain non-zero constant \( \gamma \); in other words, \( g'_0 = \gamma \cdot g'_1 \) holds. Let us consider two factor lists \( L_0' \) and \( L_1' \) for \( g'_0 \) and \( g'_1 \), respectively. Since \( g'_0 = \gamma \cdot g'_1 \), those two factor lists must coincide. Since \( L_0' \subseteq L_0 \) and \( L_1' \subseteq L_1 \), we conclude that \( L_0 \cap L_1 = \emptyset \). This is a contradiction against \( L_0 \cap L_1 = \emptyset \). Therefore, \( h \not\in DG \) follows. This \( h \) satisfies the lemma since \( h \)'s arity is \( k-1 \).

(ii) Consider the case where \( L_1 \) contains \( \Delta_0(x_i) \) and \( L_2 \) contains \( [1,0](x_i) \). As before, we set \( i = 2 \). Assume that \( g_0(x_2, x_3, \ldots, x_k) = \Delta_0(x_2)g'_0(x_3, \ldots, x_k) \) and \( g_1(x_2, x_3, \ldots, x_k) = [1,0](x_2)g'_0(x_3, \ldots, x_k) \) for two degenerate constraints \( g'_0 \) and \( g'_1 \). First, we select a non-zero constant \( \xi \) for which \( 1 + \alpha \xi \neq 0 \). With this constant, we then define \( h(x_1, x_3, \ldots, x_k) = \sum_{\xi \in \{0,1\}} f(x_1, y, x_3, \ldots, x_k)1, \xi[1](y) \). A simple calculation shows that \( h^{x_1=0} = g'_0 \) and \( h^{x_1=1} = (1 + \alpha \xi) \cdot g'_1 \). Note that \( [1,\xi] \in U \cap \mathbb{N}Z \) and \( h \leq cong f, [1,\xi] \). If \( h \not\in DG \), then an argument similar to (i) proves that \( L_0 \cap L_1 \neq \emptyset \), a contradiction. Hence, we conclude that \( h \not\in DG \), ensuring the lemma.

(iii) Let us assume that \( L_1 \) contains \( [1,0](x_i) \) and \( L_2 \) contains \( [1,0](x_i) \). Set \( i = 2 \) for simplicity. Assume that \( g_0 \) and \( g_1 \) are of the form: \( g_0(x_2, x_3, \ldots, x_k) = [1,0](x_2)g'_0(x_3, \ldots, x_k) \) and \( g_1(x_2, x_3, \ldots, x_k) = [1,0](x_2)g'_1(x_3, \ldots, x_k) \) for certain constraints \( g'_0 \) and \( g'_1 \). To obtain the lemma, here we first choose a non-zero constant \( \xi \) to satisfy that \( \xi + a \neq 0 \) and \( \xi + b \neq 0 \). The desired \( h \) is now defined as \( h(x_1, x_3, \ldots, x_k) = \sum_{\xi \in \{0,1\}} f(x_1, y, x_3, \ldots, x_k)1, \xi[1](y) \). Then it holds that \( h^{x_1=0} = (\xi + a) \cdot g'_0 \) and \( h^{x_1=1} = (\xi + b) \cdot g'_1 \). When \( h \not\in DG \), \( (\xi + a) \cdot g'_0 = \xi(\xi + b) \cdot g'_1 \) holds for a non-zero constant \( \gamma \). Since both values \( \xi + a \) and \( \xi(\xi + b) \) are not zero, a similar argument to (i) leads to a contradiction. Therefore, we obtain \( h \not\in DG \), as required.

(iv) The other cases are similar to (i)–(iii).

The second step for the proof of Proposition 7.3 is made by the following lemma.

**Lemma 7.5** Let \( d \geq 2 \) and \( k \geq 3 \). For any \( k \)-ary constraint \( f \not\in DISJ \cup \wedge \cup \vee \cup DG \), if \( EQ_d \not\leq cong f \cup \{f\} \) for any finite set \( G \subseteq U \), then there exists another constraint \( g \) of arity \( k-1 \) such that \( g \not\in DISJ \cup \wedge \cup \vee \cup DG \) and \( g \leq cong G \cup \{f\} \) for a certain finite set \( G \subseteq U \cap \mathbb{N}Z \).

**Proof.** Let \( f \not\in DISJ \cup \wedge \cup \vee \cup DG \) be any \( k \)-ary constraint. Assume that \( EQ_d \not\leq cong G \cup \{f\} \) for any finite set \( G \subseteq U \). With constraints \( g_0 = f^{x_i=0} \) for two values \( b \in \{0,1\} \), it holds that \( f(x_1, x_2, \ldots, x_k) = \sum_{b \in \{0,1\}} \Delta_0(x_1)g_0(x_2, x_3, x_4, \ldots, x_k) \). Obviously, both \( g_0 \) and \( g_I \) have arity \( k-1 \) and \( g \leq cong \{f, \Delta_0\} \) holds by Example 7.2 for any \( b \in \{0,1\} \). If either \( g_0 \) or \( g_I \) stays out of \( DISJ \cup \wedge \cup \vee \cup DG \), then we immediately obtain the lemma. Henceforth, we assume that \( g_0, g_I \in DISJ \cup \wedge \cup \vee \cup DG \).

Let us consider \( g_0 \) first. If \( g_0 \) belongs to \( DISJ \cup \wedge \cup \vee \cup DG \), then Proposition 7.1 yields \( EQ_d \leq cong \{g_0, u, \Delta_0, \Delta_1\} \) for a certain constraint \( u \subseteq U \cap \mathbb{N}Z \). Since \( g \leq cong \{f, \Delta_0\} \), we reach the conclusion that \( EQ_d \not\leq cong \{f, u, \Delta_0, \Delta_1\} \) by Lemma 1.3. This obviously contradicts our assumption. A similar contradiction is drawn if we exchange the roles of \( g_0 \) and \( g_I \). Therefore, there is only one remaining case \( g_0, g_I \in DG \) to examine. By Lemma 7.3, since \( f \not\in DG \), we immediately obtain a non-degenerate constraint \( g \) of arity \( k-1 \) such that \( g \leq cong G \cup \{f\} \) for a certain finite set \( G \subseteq U \cap \mathbb{N}Z \). If this \( g \) is actually in \( DISJ \cup \wedge \cup \vee \cup DG \), then we conclude, as before, that \( EQ_d \leq cong G' \cup \{f\} \) for another finite subset \( G' \) of \( U \). Since this is a contradiction,
it thus follows that \( g \notin \text{DISJ} \cup \text{NAND} \cup \text{DG} \). The constraint \( g \) certainly satisfies the lemma. \( \square \)

In the end, we will prove Proposition 7.3 by combining Proposition 6.5 and Lemma 7.6.

**Proof of Proposition 7.3.** Let \( k \geq 2 \) and let \( f \) be any \( k \)-ary constraint not in \( \text{DISJ} \cup \text{NAND} \cup \text{DG} \). Our proof proceeds by induction on the airy of \( f \).

**[Basis Case: \( k = 2 \).]** For this basis case, Proposition 6.3 gives the desired conclusion of the proposition.

**[Induction Case: \( k \geq 3 \).]** Our goal is to show that \( \text{EQ}_d \leq_{\text{conv}}^+ G \cup \{f\} \) for a certain set \( G \subseteq U \). Toward a contradiction, we assume on the contrary that \( \text{EQ}_d \not\leq_{\text{conv}}^+ G \cup \{f\} \) for any finite set \( G \) of \( U \). By Lemma 7.5, there is a constraint \( g \) of arity \( < k \) for which \( g \not\in \text{DISJ} \cup \text{NAND} \cup \text{DG} \) and \( g \leq_{\text{conv}}^+ G \cup \{f\} \) for a certain finite set \( G' \subseteq U \). We apply the induction hypothesis to this \( g \) and then obtain \( \text{EQ}_d \leq_{\text{conv}}^+ G' \cup \{g\} \) for another finite set \( G'' \subseteq U \). Since \( g \leq_{\text{conv}}^+ G' \cup \{f\}, \text{EQ}_d \leq_{\text{conv}}^+ G'' \cup \{f\} \) follows from Lemma 7.3. This is clearly a contradiction; therefore, the proposition holds for \( f \).

Moreover, we obtain the second part of the proposition by appealing to Lemma 5.3 because the above proof can be efficiently simulated. \( \square \)

## 8 The Dichotomy Theorem

Throughout the previous sections, we have already established all necessary foundations for our main theorem—Theorem 1.1—on the approximation complexity of complex-weighted bounded-degree Boolean \#CSPs. Here, we re-state this theorem, which has appeared first in Section 1.

**Theorem 1.1 (rephrased)** Let \( d \geq 3 \) be any degree bound and let \( F \) be any set of constraints. If \( F \subseteq \mathcal{E} \mathcal{D} \), then \( \#\text{CSP}_d(F) \) belongs to \( \mathcal{F} \mathcal{P}_C \). Otherwise, \( \#\text{SAT}_C \leq_{\text{AP}} \#\text{CSP}_d(F) \).

This theorem is an immediate consequence of our key claim, Proposition 1.2, which directly bridges between unbounded-degree \#CSPs and bounded-degree \#CSPs, when unary constraints are freely available. Once the claim is proven, the theorem follows from the dichotomy theorem (stated in Section 1) of Yamakami [21]. Now, we aim at proving Proposition 1.2.

**Proposition 1.2 (rephrased)** For any index \( d \geq 3 \) and for any constraint set \( F \), \( \#\text{CSP}_d(F) \equiv_{\text{AP}} \#\text{CSP}_d^*(F) \).

**Proof.** Let \( d \) be any index at least 3. Obviously, it holds that \( \#\text{CSP}_d^*(F) \leq_{\text{AP}} \#\text{CSP}_d(F) \). It thus suffices to show the opposite direction of this AP-reduction. For convenience, set \( F' = F - \mathcal{E} \mathcal{Q} \).

Let us consider the case where \( F \) satisfies \( F \subseteq \mathcal{E} \mathcal{D} \). Lemma 5.1 directly shows that \( \#\text{CSP}_d(F) \in \mathcal{F} \mathcal{P}_C \). Since \( \#\text{CSP}_d^*(F) \) is also in \( \mathcal{F} \mathcal{P}_C \) [21], \( \#\text{CSP}_d^*(F) \equiv_{\text{AP}} \#\text{CSP}_d(F) \) follows immediately. Hereafter, let us assume that \( F \not\subseteq \mathcal{E} \mathcal{D} \). Note that Lemma 5.2 helps us AP-reduce \( \#\text{CSP}_d(F) \) to \( \#\text{CSP}_d^*(\mathcal{E} \mathcal{Q}|F') \). Now, we want to prove that \( \#\text{CSP}_d^*(\mathcal{E} \mathcal{Q}|F') \) is AP-reducible to \( \#\text{CSP}_d^*(f,F') \) for an appropriate constraint \( f \in F \). This leads us to the conclusion that \( \#\text{CSP}_d^*(\mathcal{E} \mathcal{Q}|F') \leq_{\text{AP}} \#\text{CSP}_d(F) \) since \( \{f\} \cup F' \subseteq F \).

Let us consider the case where either \( F \notin \text{DISJ} \cup \text{NAND} \cup \text{DG} \). Since \( F \not\subseteq \mathcal{E} \mathcal{D} \) implies \( F \not\subseteq \mathcal{D} \mathcal{Q} \), there exists a constraint \( f \) in \( \text{DISJ} \cup \text{NAND} \cup \text{DG} \). The arity of \( f \) should be at least 2 since \( f \not\subseteq \mathcal{D} \mathcal{Q} \). To this \( f \), we apply Proposition 7.1 and then obtain \( \#\text{CSP}_d^*(\mathcal{E} \mathcal{Q}|F') \leq_{\text{AP}} \#\text{CSP}_d(f,F') \). Since \( d \geq 3 \), we conclude that \( \#\text{CSP}_d^*(\mathcal{E} \mathcal{Q}|F') \leq_{\text{AP}} \#\text{CSP}_d(f,F') \). The remaining case is that \( F \) is not included in \( \text{DISJ} \cup \text{NAND} \cup \mathcal{D} \mathcal{Q} \). Now, we choose a constraint \( f \in F \) that does not belong to \( \text{DISJ} \cup \text{NAND} \cup \mathcal{D} \mathcal{Q} \). Such a constraint can be handled by Proposition 7.3. We thus obtain \( \#\text{CSP}_d^*(\mathcal{E} \mathcal{Q}|F') \leq_{\text{AP}} \#\text{CSP}_d^*(f,F') \), which immediately implies \( \#\text{CSP}_d^*(\mathcal{E} \mathcal{Q}|F') \leq_{\text{AP}} \#\text{CSP}_d(f,F') \). This completes the proof.

Proposition 1.2 is a consequence of the powerful expressiveness of complex-weighted free unary constraints. When free unary constraints are limited to Boolean, Dyer et al. [11] showed a similar proposition only under the assumption that every Boolean constraint in \( F \) is “3-simulatable.”

Now, Theorem 1.1 is immediate from Proposition 1.2.

**Proof of Theorem 1.1.** Let \( d \geq 3 \). If \( F \subseteq \mathcal{E} \mathcal{D} \) holds, then \( \#\text{CSP}_d(F) \) belongs to \( \mathcal{F} \mathcal{P}_C \) by Lemma 5.1. When \( F \not\subseteq \mathcal{E} \mathcal{D} \), as noted in Section 1, it was shown in [21] that \( \#\text{SAT}_C \leq_{\text{AP}} \#\text{CSP}_d(F) \). Since Proposition 1.2 establishes the AP-equivalence between \( \#\text{CSP}_d(F) \) and \( \#\text{CSP}_d^*(F) \), we can replace \( \#\text{CSP}_d(F) \) in the above result by \( \#\text{CSP}_d^*(F) \). This clearly gives the desired consequence of the theorem. \( \square \)

Another immediate consequence of Proposition 1.2 is an AP-equivalence between \( \#\text{CSP}_d(F) \) and a bipartite Holant problem \( \text{Holant}(\mathcal{E} \mathcal{Q}|F, U) \). This immediately follows from the proposition and also a known
fact that degree-3 #CSPs are essentially identical to bipartite Holant problems whose node labels appearing on the left-hand side of input graphs are always restricted to EQ3. To make this paper self-contained, we will include the detailed proof of the AP-equivalence between #CSP*(F) and Holant(EQ3(F,U)).

**Corollary 8.1** For any set $F$ of constraints, it holds that $\#CSP^*(F) \equiv_{AP} \text{Holant}(EQ3(F,U))$.

**Proof.** Let $F$ be an arbitrary set of constraints. Since $\#CSP^*_d(F)$ is shorthand for $\#CSP^*_d(F,U)$, by Proposition 8.2, it is enough to prove that $\#CSP^*_d(F,U)$ and $\text{Holant}(EQ3(F,U))$ are AP-equivalent. Recall from Section 2 that $\#CSP(G)$ always coincides with $\text{Holant}(|E|=1|G|)$ for any constraint set $G$. In particular, $\#CSP^*_d(F,U)$ coincides with $\text{Holant}(EQ1, EQ2, EQ3(F,U))$. Our goal is therefore to show that $\text{Holant}(EQ1, EQ2, EQ3(F,U)) \leq_{AP} \text{Holant}(EQ3(F,U))$.

Let us consider any bipartite signature grid $\Omega = (G(F_1,F_2,\pi))$ as an input instance to $\text{Holant}(EQ1, EQ2, EQ3(F,U))$, where $F_1 \subseteq \{EQ1, EQ2, EQ3\}$ and $F_2 \subseteq \{U\}$. Assume that $G = (V_1|V_2, E)$. Now, we will describe how to replace every node labeled EQ1 with another node whose label is EQ3. For any node $v$ having the label $EQ1$ that appears in $V_1$, let $w$ denote any node, adjacent to $v$, whose label is, say, $g \in F_2$. Take any bipartite subgraph $G' = (\{v\}|\{w\}, E')$, where $E'$ consists of the edge $(v,w)$ and of all dangling edges obtained from the edges linking between the node $g$ and any other node other than $v$. This value can be easily computed from all constraints in $F_1$. Note that the aforementioned replacement of two subgraphs does not change the value of $\text{Holant}_{\Omega}$, and therefore we obtain $\text{Holant}_{\Omega'} = \text{Holant}_{\Omega}$. Similarly, we can replace $EQ2$ by $EQ3$. When all nodes labeled $EQ1$ and $EQ2$ are replaced, then the desired AP-reduction from $\text{Holant}(EQ1, EQ2, EQ3(F,U))$ to $\text{Holant}(EQ3(F,U))$. \hfill $\Box$

### 9 Cases of Degree 1 and Degree 2

When the degree bound $d$ is more than two, our main theorem—Theorem 11—has given a complete characterization of the approximation complexity of counting problems $\#CSP^*_d(F)$ for any constraint set $F$. This has left a question of what the approximation complexity of $\#CSP^*_d(F)$ is, when $d$ is less than three. We briefly discuss this issue in this section. Let us consider the trivial case of degree one.

**Lemma 9.1** For any constraint set $F$, $\#CSP^*_1(F)$ is in $\text{FP}_C$.

**Proof.** Let $\Omega = (G,X,F',\pi)$ be any given constraint frame for $\#CSP^*_1(F)$. Note that all nodes on the left-hand side of the undirected bipartite graph $G$ have degree at most one. By this degree requirement, no two edges in $G$ are incident on the same node on the left-hand side of $G$. In other words, any two constraints in $F'$ share no single variable. This makes $\text{csp}_{\Omega}$ equal to a product of all values $\sum f(\sigma(x_1), \sigma'(x_2), \ldots, \sigma'(x_n))$ for any constraint $f \in F'$ that takes a variable series $(x_1, x_2, \ldots, x_n)$, where “sum” is taken over all variable assignments $\sigma : \{x_1, x_2, \ldots, x_n\} \rightarrow \{0, 1\}$. This value can be easily computed from all constraints in $F'$ in polynomial time. Therefore, $\#CSP^*_1(F)$ belongs to $\text{FP}_C$. \hfill $\Box$

Next, we consider the case of degree two. Earlier, Dyer et al. [4] left this case unanswered for unweighted Boolean $\#CSP$s. For a complex-weighted case, however, it is possible to obtain a precise characterization of $\#CSP^*_2(F)$’s using a known transformation between degree-2 $\#CSP$s and Holant problems. For completeness, we will formally prove that $\#CSP^*_2(F)$ is indeed AP-equivalent to $\text{Holant}(F,U)$. To simplify the description of Holant problems, similar to the notation $\#CSP^*(F)$, we succinctly write $\text{Holant}^*(F)$ for $\text{Holant}(F,U)$.

**Proposition 9.2** For any constraint set $F$, it holds that $\#CSP^*_2(F) \equiv_{AP} \text{Holant}^*(F)$.

**Proof.** Firstly, we will claim that $\#CSP^*_2(F)$ is AP-equivalent to $\text{Holant}(EQ2(F,U))$. Secondly, we will claim that $\text{Holant}(F) \equiv_{AP} \text{Holant}(EQ2(F,U))$. By combining these two claims, the proposition clearly follows.

1. The first claim is proven as follows. In the proof of Corollary 8.1 we have actually proven that $\text{Holant}(EQ1, EQ2, EQ3(F,U)) \equiv_{AP} \text{Holant}(EQ3(F,U))$. A similar argument shows that $\text{Holant}(EQ1, EQ2, EQ3(F,U))$ and $\text{Holant}(EQ2(F,U))$ are AP-equivalent. Since $\#CSP^*_2(F)$ is, as shown in Section
essentially the same as Holant($EQ_1, EQ_2, F, \mathcal{U}$), we immediately obtain the desired claim.

(2) For the second claim, we want to establish two AP-reductions between Holant($\mathcal{F}$) and Holant($EQ_2, \mathcal{F}$).

(i) In the first step, we prove that Holant($\mathcal{F}$) is AP-reducible to Holant($EQ_2, \mathcal{F}$). Let $\Omega = (G, F', \pi)$ be any signature grid given as an input instance to Holant($\mathcal{F}$) with $G = (V, E)$. Let us define a new bipartite signature grid $\Omega' = (G', \{EQ_2\}[F', \pi'])$ as follows. For each edge $(v, w)$ incident on both nodes $v$ and $w$ in $G$, we add a new node $u$ labeled $EQ_2$ and replace $(v, w)$ by an edge pair $\{(u, v), (u, w)\}$. Let $V'_1$ denote the set of all such newly added nodes and let $V'_2$ equal $V$. Let $\pi'$ be obtained from $\pi$ by assigning $EQ_2$ to all the new nodes. A new edge set $E'$ is obtained from $E$ by the above replacement. Clearly, $G' = (V'_1 V'_2, E')$ forms an undirected bipartite graph. It is not difficult to show that Holant$_{\Omega'} = \text{Holant}_\Omega$. Therefore, it holds that Holant($\mathcal{F}$) is AP-reducible to Holant($EQ_2, \mathcal{F}$).

(ii) In the second step, we will show that Holant($EQ_2, \mathcal{F}$) is AP-reducible to Holant($\mathcal{F}$). Fundamentally, we do the opposite of (i), starting from a bipartite signature grid $\Omega'$. More precisely, for any node in $V'_1$, which is labeled $EQ_2$, we delete it and replace each edge pair $\{(u, v), (u, w)\}$ by a new edge $(v, w)$. This defines a new signature grid $\Omega$. Since Holant$_\Omega = \text{Holant}_{\Omega'}$ holds, we obtain an AP-reduction: Holant($EQ_2, \mathcal{F}$) is AP-reducible to Holant($\mathcal{F}$).

The computational complexity of exactly solving Holant problems Holant$(\mathcal{F})$ was completely classified by Cai et al. [3, 4] under polynomial-time Turing reductions; on the contrary, it is not known that a similar classification holds in the case of approximate counting under AP-reductions.

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