Conformal Compensators and Manifest Type IIB S-Duality

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Using the conformal compensator superfields of N=2 D=4 supergravity, the Type IIB S-duality transformations are expressed as a linear rotation which mixes the compensator and matter superfields. The classical superspace action for D=4 compactifications of Type IIB supergravity is manifestly invariant under this transformation. Furthermore, the introduction of conformal compensators allows a Fradkin-Tseytlin term to be added to the manifestly SL(2,Z)-covariant sigma model action of Townsend and Cederwall.
1. Introduction

Although the evidence for S-duality of the Type IIB superstring is continually growing, an explanation of this symmetry is still lacking. One cause of the difficulty is that the D=10 Type IIB supergravity action is poorly understood, both because of the chiral four-form and because of the lack of an off-shell D=10 superspace formalism. Another cause is that S-duality transformations take their simplest form in ‘Einstein gauge’, whereas the superstring is easiest to study in ‘string gauge’.

To avoid these problems, Type IIB S-duality will be studied in this paper for N=2 D=4 theories which are obtained by compactification of the Type IIB superstring on a Calabi-Yau manifold. Since off-shell N=2 D=4 superspace is well understood, it is straightforward to construct superspace actions for these N=2 D=4 supergravity theories. The superspace actions involve conformal compensators which will permit a conformally gauge-invariant definition of S-duality transformations.

The usual superspace procedure for coupling to supergravity is to first introduce conformal compensators which allow the action in a flat metric to be invariant under global conformal transformations. One then covariantly couples to conformal supergravity and finally, chooses a conformal-breaking condition which turns the conformal supergravity multiplet into a Poincaré supergravity multiplet. The choice of conformal-breaking condition determines if one is in ‘Einstein gauge’, ‘string gauge’, or some other gauge.

In the second section of this paper, it will be shown that Type IIB S-duality transformations take their simplest form before choosing a conformal-breaking condition, when they linearly rotate the conformal compensator hypermultiplet into the ‘universal’ hypermultiplet and leave all other multiplets unchanged. It is easy to prove that the classical superspace action obtained by compactifying the Type IIB superstring on a Calabi-Yau manifold is invariant under this transformation.

Recently, Townsend and Cederwall have proposed a manifestly SL(2,Z)-covariant sigma model action for the Type IIB Green-Schwarz superstring [1]. Like the usual Green-Schwarz sigma model, their action lacks a Fradkin-Tseytlin term which couples the spacetime dilaton to the worldsheet curvature [2]. But previously, a sigma model action which includes a Fradkin-Tseytlin term was constructed using a modified Green-Schwarz description of the Type IIB superstring compactified on a Calabi-Yau manifold [3][4]. In the third section of this paper, these two actions will be combined to form a manifestly SL(2,Z)-covariant sigma model action which includes a Fradkin-Tseytlin term.
2. Conformal Compensators

2.1. Calabi-Yau compactification of the Type IIB superstring

For compactifications of the Type IIB superstring on a six-dimensional Calabi-Yau manifold with $h_{2,1}$ complex moduli and $h_{1,1}$ Kahler moduli, the massless N=2 D=4 supersymmetry multiplets include an N=2 D=4 supergravity multiplet, $h_{2,1}$ vector multiplets, and $h_{1,1}+1$ hypermultiplets where the +1 comes from the ‘universal’ hypermultiplet. To construct covariant actions with manifest spacetime supersymmetry, it is convenient to split the supergravity multiplet into a conformal supergravity multiplet and conformal compensator multiplets.

If the action in a flat metric is invariant under global conformal transformations, one makes the action super-reparameterization invariant by covariantly coupling the action to conformal supergravity. If the action in a flat metric is not invariant under global conformal transformations, one first couples to the conformal compensators in such a way that the transformation of the compensators cancels the transformation of the action. One then couples the combined action to conformal supergravity. Gauge-fixing the conformal invariance turns the conformal supergravity multiplet into a Poincaré supergravity multiplet, but complicates the supersymmetry transformations.

Although there is some ambiguity in the choice of conformal compensator multiplets for N=2 D=4 supergravity [5], superstring field theory implies that these compensator multiplets consist of a vector multiplet and a tensor hypermultiplet [4,6]. Superstring field theory also implies that the $h_{1,1} + 1$ hypermultiplets are all tensor hypermultiplets (as opposed to scalar hypermultiplets). Conveniently, actions involving tensor hypermultiplets are much easier to construct in N=2 D=4 superspace than actions involving scalar hypermultiplets. Note that component versions of scalar hypermultiplet actions coming from Type IIB compactifications have been extensively studied in various important papers which include [7], [8], and [9].

2.2. N=2 D=4 superspace

The variables of N=2 D=4 superspace are $[x^\mu, \theta^\alpha_j, \bar{\theta}^{\dagger \dot{\alpha}}_j]$ where $\mu = 0$ to 3, $\alpha$ and $\dot{\alpha} = 1$ to 2, and $j = 1$ to 2 is an internal $SU(2)_R$ index which is raised and lowered using the anti-symmetric $\epsilon^{ijk}$ tensor. $\bar{\theta}^{j \dot{\alpha}}$ is the complex conjugate of $\theta^\alpha_j$ and under $U(1)_R$ transformations, $\theta^\alpha_j$ carries +1 charge and $\bar{\theta}^{j \dot{\alpha}}$ carries −1 charge. Under global conformal
transformations, $\theta^\alpha_j$ and $\bar{\theta}^i\dot{\alpha}$ carry scale-weight $-\frac{1}{2}$ and $x^\mu$ carries scale-weight $-1$. In a flat metric, supersymmetric derivatives are defined as
\[ D_\alpha^j = \frac{\partial}{\partial \theta^\alpha_j} + i \bar{\theta}^i\dot{\alpha} \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu, \quad \bar{D}_\dot{\alpha}^j = \frac{\partial}{\partial \bar{\theta}^j\dot{\alpha}} + i \theta^i_j \sigma^\mu_{\dot{\alpha}\alpha} \partial_\mu. \] (2.1)

There are three types of $\mathcal{N}=2$ $D=4$ multiplets which will be useful to review: the vector multiplet, the tensor hypermultiplet, and the conformal supergravity multiplet.

The field-strength of a vector multiplet is described by a restricted chiral superfield $W$ satisfying
\[ D_\alpha^j W = \bar{D}_{\dot{\alpha}}^j W = 0, \] (2.2)
\[ D_\alpha^{(j} D^{k)\alpha} W = \bar{D}^{(j}_{\dot{\alpha}} D^{k)\dot{\alpha}} \bar{W} \]
where the first constraint implies that $W$ is chiral/chiral, while the second constraint implies that $W$ is restricted. The physical bosonic components of $W$ appear as
\[ W = w(x) + \theta^j(\alpha \beta) \sigma^\mu_{\alpha\beta} F_{\mu \nu}(x) + ... \] (2.3)
where $w$ is a complex scalar and $F_{\mu \nu}$ is the vector field strength. Under $U(1)_R \times SU(2)_R$, $W$ transforms as $(+2, 1)$, so $w$ and $\bar{w}$ transform as $(+2, 1)$ and $(-2, 1)$ while $F_{\mu \nu}$ transforms as $(0, 1)$. Under conformal transformations, $w$ has scale-weight $+1$ and $F_{\mu \nu}$ has scale-weight $+2$.

The field-strength of a tensor hypermultiplet is described by a linear superfield $L_{jk}$ symmetric in its $SU(2)$ indices which satisfies the reality condition $L_{jk} = (L^{jk})^*$ and the linear constraint
\[ D_\alpha^{(j} L_{kl)} = 0, \quad \bar{D}^{(j}_{\dot{\alpha}} L_{kl)} = 0. \] (2.4)

The physical bosonic components of $L_{jk}$ appear as
\[ L_{jk} = l_{jk}(x) + \theta^\alpha_{(j} \bar{\theta}^{\dot{\alpha})_{k}} \epsilon_{\mu \nu \rho \kappa} \sigma^\mu_{\alpha\dot{\alpha}} H^{\nu \rho \kappa}(x) + ... \] (2.5)
where $l_{jk}$ is a triplet of scalars transforming as $(0, 3)$ under $U(1)_R \times SU(2)_R$ and $H^{\mu \nu \rho}$ is the tensor field-strength which transforms as $(0, 1)$. Under conformal transformations, $l_{jk}$ has scale-weight $+2$ and $H_{\mu \nu \rho}$ has scale-weight $+3$.

Although the constraints of (2.4) appear very different from the constraints of (2.2), they are actually closely related. This can be seen by noting that the constraints of (2.4) imply that $L_{++}$ is restricted twisted-chiral since it satisfies
\[ D_\alpha^+ L_{++} = \bar{D}^{\dot{\alpha}}_+ L_{++} = 0, \] (2.6)
\[ D^\alpha D_{-\alpha} L_{++} = D_+^\alpha D_{+\alpha} L_{--}, \quad \bar{D}^\dot{\alpha}\bar{D}_{-\dot{\alpha}} L_{++} = \bar{D}_+^\dot{\alpha}\bar{D}_{+\dot{\alpha}} L_{--}. \]

The first two constraints imply that \( L_{++} \) is chiral/anti-chiral, while the second two constraints imply that \( L_{++} \) is restricted.

Finally, the conformal supergravity multiplet is described by a supervierbein superfield \( E^M_A \) where \( A \) denotes tangent-space vector and spinor indices while \( M \) denotes curved-space vector and spinor indices. The superfield \( E^M_A \) is subject to various torsion constraints which will not be directly relevant for this paper.

2.3. S-duality in superspace

The F-theory conjecture states that the Type IIB superstring compactified on \( \mathcal{M} \) is equivalent to F-theory compactified on \( T_2 \times \mathcal{M} \) with the complex modulus of \( T_2 \) parameterized by \( \tau = a - ie^{-\phi} \) where \( a \) is the axion and \( e^{-\phi} \) is the dilaton. So choosing \( \mathcal{M} \) to be the Calabi-Yau manifold, modular invariance of \( T_2 \) implies that the compactified theory is invariant under the S-duality \( SL(2,\mathbb{Z}) \) transformation

\[ \tau \to \frac{A\tau + B}{C\tau + D} \quad (2.7) \]

where \( A,B,C,D \) are integers satisfying \( AD - BC = 1 \).

A natural question is how do the supersymmetry multiplets transform under (2.7). For compactification on a Calabi-Yau manifold, the massless \( N=2 \) \( D=4 \) superfields include a compensating vector multiplet described by \( W^{(0)} \), physical vector multiplets described by \( W^{(X)} \) where \( X = 1 \) to \( h_{2,1} \), a compensating tensor hypermultiplet described by \( L^{(0)}_{jk} \), physical tensor hypermultiplets described by \( L^{(Y)}_{jk} \) for \( Y=1 \) to \( h_{1,1} \), a physical ‘universal’ hypermultiplet described by \( L'_{jk} \), and the conformal supergravity multiplet described by \( E^M_A \). As will be shown below, the S-duality transformations of \( 2.7 \) transform these superfields as

\[ E^M_A \rightarrow E^M_A, \quad W^{(0)} \rightarrow W^{(0)}, \quad W^{(X)} \rightarrow W^{(X)}, \quad L^{(Y)}_{jk} \rightarrow L^{(Y)}_{jk}, \quad (2.8) \]

\[ L^{(0)}_{jk} \rightarrow AL^{(0)}_{jk} + BL'_{jk}, \quad L'_{jk} \rightarrow CL^{(0)}_{jk} + DL'_{jk}. \]

To verify (2.8), one first needs to determine how the components of the various superfields depend on \( a \) and \( \phi \). Since the \( N=2 \) \( D=4 \) superconformal group includes local \( SU(2)_R \times U(1)_R \) rotations, one can gauge to zero the component fields \( \text{Im}(w^{(0)}) \), \( \text{Im}(l^{(0)}_{jk}) \),
\( l'_{++} \) and \( l'_{--} \). This still leaves local conformal transformations which can be used to gauge-fix \( l'_{+-} = 1 \).

In this gauge, the equation of motion for the scalar in the conformal supergravity multiplet implies that \( \text{Re}(w^{(0)}) \) is on-shell a function of the other fields. So besides the scalars coming from \( W^{(X)} \) and \( L^{(Y)}_{jk} \), there are four independent scalars which can be defined in terms of \( l^{(0)}_+ \), \( \text{Re}(l^{(0)}_{++}) \), and the duals of \( H^{(0)}_{\mu\nu\rho} \) and \( H'_{\mu\nu\rho} \). In terms of the original bosonic fields of D=10 Type IIB supergravity,

\[
\text{Re}(l^{(0)}_{++}) = e^{-\phi}, \quad l^{(0)}_{+-} = a, \quad H^{(0)}_{\mu\nu\rho} = \partial_{[\mu}b_{\nu\rho]}, \quad (2.9)
\]

\[
l^{(Y)}_+ = e^{-\phi}(G^{(Y)}J\bar{K}g_{J\bar{K}} + iB^{(Y)}J\bar{K}b_{J\bar{K}}), \quad l^{(Y)}_{+-} = B^{(Y)\bar{J}\bar{K}}(\bar{b}_{J\bar{K}} - a b_{J\bar{K}}),
\]

\[
H^{(0)}_{\mu\nu\rho} = B^{(Y)\bar{J}\bar{K}}\partial_{[\mu}A_{\nu\rho]J\bar{K}}, \quad H'_{\mu\nu\rho} = \partial_{[\mu}b_{\nu\rho]},
\]

\[
F^{(0)}_{\mu\nu} = \omega^{J\bar{K}L}\partial_{[\mu}A_{\nu]J\bar{K}L} + \text{c.c.},
\]

\[
w^{(X)} = h^{(X)J\bar{K}}g_{J\bar{K}}, \quad F^{(X)}_{\mu\nu} = h^{(X)J\bar{K}}\omega^{J\bar{K}L\bar{M}}\partial_{[\mu}A_{\nu]J\bar{K}L\bar{M}} + \text{c.c.},
\]

where \( J, K \) and \( \bar{J}, \bar{K} \) are the complex coordinates of the Calabi-Yau manifold \((J, K = 1 \text{ to } 3)\), \( G^{(Y)J\bar{K}} \) and \( B^{(Y)J\bar{K}} \) are the Kahler moduli and torsion of the Calabi-Yau manifold, \( h^{(X)J\bar{K}} \) are the complex moduli of the Calabi-Yau manifold, \( a \) and \( e^{-\phi} \) are the D=10 axion and dilaton, \( g_{mn} \) is the D=10 metric \((m, n \text{ can point either in the spacetime directions or in the Calabi-Yau directions})\), \( b_{mn} \) and \( \bar{b}_{mn} \) are the D=10 NS-NS and R-R two-forms, and \( A_{mnpq} \) is the D=10 self-dual four-form. Note that \( l^{(Y)}_+ \) has \( e^{-\phi} \) dependence since the Kahler moduli are give by the conformally-invariant quantities \( l^{(Y)}_++/l^{(0)}_{++} \).

To prove that (2.8) correctly defines the S-duality transformation, first consider the shift transformation when \( A = B = D = 1 \) and \( C = 0 \). Under this transformation, \( L^{(0)}_{jk} \rightarrow L^{(0)}_{jk} + L'_{jk} \) and all other superfields remain unchanged. Comparing with (2.9), this implies that

\[
a \rightarrow a + 1, \quad \bar{b}_{mn} \rightarrow \bar{b}_{mn} + b_{mn}, \quad (2.10)
\]

which is the desired transformation.

Now consider the strong/weak transformation when \( A = D = 0 \), \( B = -1 \) and \( C = 1 \). Under this transformation, \( L^{(0)}_{jk} \rightarrow -L'_{jk} \) and \( L'_{jk} \rightarrow L^{(0)}_{jk} \), which does not preserve the gauge-fixing condition \( l'_{jk} = \delta_{jk} \) (i.e. \( l'_{++} = l'_{--} = 0 \), \( l'_{+-} = 1 \)).

So to obtain the transformations of the component fields, one needs to perform a local \( SU(2)_R \) and conformal transformation to return to the original gauge choice. Alternatively,
one can express the component fields in a form which is invariant under \( SU(2)_R \) and conformal transformations, e.g.

\[
a = \frac{l(0) \cdot l'}{l' \cdot l} \quad , \quad e^{-2\phi} + a^2 = \frac{\bar{l}(0) \cdot l(0)}{l' \cdot l'}
\]

where \( A \cdot B \equiv A_{jk}B^{jk} \). Under the strong/weak transformation,

\[
a \to -l' \cdot l(0) \quad , \quad e^{-2\phi} + a^2 \to \frac{l' \cdot l}{\bar{l}(0) \cdot l(0)} = \frac{1}{e^{-2\phi} + a^2},
\]

which implies that \( a + ie^{-\phi} \to -(a + ie^{-\phi})^{-1} \) as desired.

Similarly, one can show that the other component fields transform appropriately, e.g.

\[
g_{mn} \to \sqrt{e^{-2\phi} + a^2} \quad , \quad b_{mn} \to -\tilde{b}_{mn} \quad , \quad \tilde{b}_{mn} \to b_{mn} \quad , \quad A_{mnpq} \to A_{mnpq}.
\]

Note that there are no \((e^{-2\phi} + a^2)\) factors in the transformations of \(b_{mn}\) and \(A_{mnpq}\) since, in a curved background, the component fields appearing in \(L_{jk}\) and \(W\) carry tangent-space indices and the transformation of the vierbein absorbs the \((e^{-2\phi} + a^2)\) factors coming from the conformal rescaling.

Since any S-duality transformation can be described by a product of shift and strong/weak transformations, the transformation of \(2.8\) correctly reproduces \(2.7\). It will now be shown that the classical superspace action for the Type IIB compactification is invariant under \(2.8\).

### 2.4. N=2 D=4 superspace actions

Two-derivative actions for the vector multiplets and tensor hypermultiplets can be written in manifestly supersymmetric notation as \[10\]

\[
\int d^4x|_{\theta_\alpha = \bar{\theta}_\dot{\alpha} = 0} [(D_+)^2(D_-)^2 f_V(W^{(I)}) + \oint_C \frac{d\zeta}{2\pi i} (D_-)^2 (\tilde{D}_-)^2 f_T(\tilde{L}^{(J)}, \tilde{L}') + c.c.] \quad (2.14)
\]

where \(I = 0\) to \(h_2,1\), \(J = 0\) to \(h_{1,1}\), \(f_V\) and \(f_T\) are arbitrary functions, \(\oint_C \frac{d\zeta}{2\pi i}\) is some contour integration, and

\[
\tilde{L} = L_{++} + \zeta L_{+-} + \zeta^2 L_{--}.
\]

\[6\]
The hypermultiplet contribution to (2.14) is supersymmetric where

\[ \delta_Q f_T = [\xi_j^\alpha D_\alpha^j + \bar{\xi}_\dot{\alpha} \bar{D}_{\dot{\alpha}}^j - 2i(\xi_\sigma^\mu \theta + \bar{\xi}_{\dot{\sigma}}^\mu \bar{\theta}) \partial_\mu] f_T, \]  

(2.16)

since \( D_\alpha^+ f_T = \zeta D_\alpha^+ f_T \) and \( \bar{D}_{\dot{\alpha}}^+ f_T = \zeta \bar{D}_{\dot{\alpha}}^+ f_T \), so \( (D_\alpha^-)^2(\bar{D}_{\dot{\alpha}}^-)^2 \delta_Q f_T \) is a total derivative in \( x^\mu \).

These actions are invariant under global \( SU(2)_R \times U(1)_R \) and conformal transformations when \( f_V \) is of degree 2 and \( f_T \) is of degree 1 (i.e. \( f_V(\lambda W(I)) = \lambda^2 f_V(W(I)) \) and \( f_T(\lambda \tilde{L}(J), \lambda \tilde{L}') = \lambda f_V(\tilde{L}(J), \tilde{L}') \)). Although \( SU(2)_R \) invariance is not manifest, it can be made manifest by writing the hypermultiplet action as [10]

\[ \int d^4x |\theta^\alpha_j = \bar{\theta}^\dot{\alpha}_j = 0 \oint_{\mathcal{C}} \frac{\epsilon_{jk}}{2\pi i} \zeta^j d\zeta^k (v \cdot D)^2(v \cdot \bar{D})^2 (\zeta \cdot v)^4 f_T(\tilde{L}(J), \tilde{L}') + c.c. \]  

(2.17)

where \( \tilde{L} = \zeta^j \zeta^k L_{jk} \) and \( v^j \) is a fixed two-component constant. Note that (2.17) is invariant under

\[ v^j \rightarrow c_1 v^j + c_2 \zeta^j, \quad \zeta^j \rightarrow c_3 \zeta^j \]  

(2.18)

for arbitrary constants \( c_1, c_2 \) and \( c_3 \), so one can choose \( v^+ = 0 \) and \( v^- = \zeta^+ = 1 \). With this gauge choice, (2.17) becomes the hypermultiplet action of (2.14).

For generic choices of \( f_T \), the action of (2.14) is not invariant under (2.8). However, it will now be argued that for classical actions coming from Type IIB compactifications, \( f_T \) takes the form

\[ f_T(\tilde{L}(Y), \tilde{L}') = \frac{i d_{Y_1 Y_2 Y_3} \tilde{L}(Y_1) \tilde{L}(Y_2) \tilde{L}(Y_3)}{\tilde{L}(0) \tilde{L}'} \]  

(2.19)

where \( d_{Y_1 Y_2 Y_3} \) are real symmetric constants and the contour \( \mathcal{C} \) goes clockwise around the two values of \( \zeta \) for which \( \tilde{L}' = 0 \). It will then be shown that this choice of \( f_T \) produces action invariant under (2.8).

2.5. Type IIB hypermultiplet action

The M-theory conjecture states that the Type IIA superstring compactified on a manifold \( \mathcal{M} \) is equivalent to M-theory compactified on \( S_1 \times \mathcal{M} \). So when \( \mathcal{M} \) is the Calabi-Yau manifold, validity of this conjecture implies that the Type IIA action can be obtained from dimensional reduction on a circle of a D=5 action. In such an action, one of the scalars in the N=2 D=4 vector multiplets comes from the fifth component of a D=5 vector. Therefore, gauge invariance of the D=5 action implies that the zero mode of these scalars decouples in the D=4 action.
As shown in [7], this implies that \( f \) in the action for the Type IIA compactification must have the form

\[
f_V(W^{(0)}, W^{(Y)}) = \frac{i d_{Y_1Y_2Y_3} W^{(Y_1)} W^{(Y_2)} W^{(Y_3)}}{W^{(0)}}
\]

(2.20)

where \( d_{Y_1Y_2Y_3} \) is a real symmetric constant and \( Y = 1 \) to \( h_{1,1} \). Note that the zero modes of the relevant scalars are given by \( Re(w^{(Y)}/w^{(0)}) \). These zero modes decouple since, under \( \delta W^{(Y)} = c^{(Y)}/c^{(0)} \) for real constants \( c^{(Y)} \), the action changes by

\[
-2 \int d^4x |_{\theta_j^{\alpha}=\bar{\theta}_j^{\alpha}=0} Im[(D_+)^2(D_-)^2 d_{Y_1Y_2Y_3} c^{(Y_1)} W^{(Y_2)} W^{(Y_3)}]
\]

(2.21)

which is a surface term.

In references [7] and [8], a relation was found connecting the perturbative effective action for Type IIA and Type IIB compactifications on the same Calabi-Yau manifold. This relation was later extended to superspace in [4] and states that in the string gauge \( L'_{jk} = \delta_{jk} \), the perturbative Type IIA action is obtained from the Type IIB action by replacing the superfields \( W^{(I)} \) with \( L^{(I)}_{++} \), \( W^{(j)} \) with \( L^{(j)}_{++} \) with \( W^{(j)} \), \( L^{(j)}_{--} \) with \( \bar{W}^{(j)} \), and by swapping \( \theta^+ \) with \( \bar{\theta}^- \) and \( D_+ \) with \( \bar{D}_- \).

So in the gauge \( L'_{jk} = \delta_{jk} \), the hypermultiplet action for the Type IIB compactification must have the form

\[
-2 \Im\left[ \int d^4x |_{\theta_j^{\alpha}=\bar{\theta}_j^{\alpha}=0} (D_-)^2 (\bar{D}_+)^2 \frac{d_{Y_1Y_2Y_3} L^{(Y_1)} L^{(Y_2)} L^{(Y_3)}}{L^{(0)}_{++}} \right].
\]

(2.22)

It will now be shown that this comes from gauge-fixing a hypermultiplet action with \( f_T \) and \( C \) defined as in (2.19).

In the string gauge \( \tilde{L}' = \zeta \) (i.e. \( L'_{jk} = \delta_{jk} \)), the contour \( C \) should go clockwise around the origin and counter-clockwise at \( \infty \). Note that the zero of \( \tilde{L}' \) at \( \zeta = \infty \) can be understood by taking the limit as \( c \to 0 \) of \( \tilde{L}' = c(\zeta - c)(\zeta + \frac{1}{c}) \). So the hypermultiplet action defined with \( f_T \) and \( C \) as in (2.19) is

\[
\int d^4x |_{\theta_j^{\alpha}=\bar{\theta}_j^{\alpha}=0} \left( \oint_0 - \oint_\infty \right) \frac{d\zeta}{2\pi i} (D_-)^2 (\bar{D}_+)^2 i d_{Y_1Y_2Y_3} \tilde{L}^{(Y_1)} \tilde{L}^{(Y_2)} \tilde{L}^{(Y_3)} \tilde{L}^{(0)} \zeta
\]

(2.23)

\(^1\) This symmetry relation is broken non-perturbatively and was mistakenly called mirror symmetry in reference [4]. Mirror symmetry is believed to be preserved non-perturbatively and relates Type IIB compactification on a Calabi-Yau manifold with Type IIA compactification on the mirror Calabi-Yau manifold.
\[
\begin{align*}
&= \int d^4x|_{\theta_j^\alpha = \tilde{\theta}_j^\alpha} = \left((D_-)^2(\tilde{D}_-)\right)^2 \frac{i \bar{d}_{Y_1 Y_2 Y_3} L^{(Y_1)}_+ L^{(Y_2)}_+ L^{(Y_3)}_+}{L^{(0)}_+} \\
&\quad - \int \frac{d\zeta}{2\pi i\zeta} (D_+)^2 (\bar{D}_+)^2 \frac{i \bar{d}_{Y_1 Y_2 Y_3} \bar{L}^{(Y_1)}_+ \bar{L}^{(Y_2)}_+ \bar{L}^{(Y_3)}_+}{L^{(0)}} \\
&= \int d^4x|_{\theta_j^\alpha = \tilde{\theta}_j^\alpha} \left[(D_-)^2(\tilde{D}_-)\right)^2 \frac{i \bar{d}_{Y_1 Y_2 Y_3} L^{(Y_1)}_+ L^{(Y_2)}_+ L^{(Y_3)}_+}{L^{(0)}_+} \\
&\quad - (D_+)^2 (\bar{D}_-)^2 \frac{i \bar{d}_{Y_1 Y_2 Y_3} L^{(Y_1)}_- L^{(Y_2)}_- L^{(Y_3)}_-}{L^{(0)}_-},
\end{align*}
\]

which agrees with (2.22).

Note that when the Calabi-Yau manifold is the mirror of another Calabi-Yau manifold, the effective action can be obtained from either a Type IIA or Type IIB compactifications, so it must be of the form

\[
\int d^4x|_{\theta_j^\alpha = \tilde{\theta}_j^\alpha} = \left((D_-)^2(D_-)\right)^2 \frac{i \bar{d}_{X_1 X_2 X_3} W^{(X_1)} W^{(X_2)} W^{(X_3)}}{W^{(0)}} \tag{2.24}
\]

where \( \bar{d}_{X_1 X_2 X_3} \) are the symmetric constants on the mirror manifold and \( \mathcal{C} \) goes clockwise around the two zeros of \( \bar{L}' \).

Finally, it will be shown that the Type IIB action is invariant under the S-duality transformations of (2.8) when \( f_T \) and \( \mathcal{C} \) are defined as in (2.13).

Under the shift transformation \( L^{(0)}_{jk} \rightarrow L^{(0)}_{jk} + L'_{jk} \),

\[
f_T \rightarrow f_T + \frac{d_{Y_1 Y_2 Y_3} \bar{L}^{(Y_1)} \bar{L}^{(Y_2)} \bar{L}^{(Y_3)}}{(\bar{L}^{(0)})^2} \left(-1 - \frac{\bar{L}'}{\bar{L}^{(0)}} - \left(\frac{\bar{L}'}{\bar{L}^{(0)}}\right)^2 - \ldots\right) \tag{2.25}
\]

and since \( \delta f_T \) has no poles when \( \bar{L}' = 0 \), the contour integral \( \oint_{\mathcal{C}} d\zeta (D_-)^2(D_-)^2 \delta f_T \) vanishes.

Under the strong/weak transformation defined by \( L^{(0)}_{jk} \rightarrow -L'_{jk} \) and \( L'_{jk} \rightarrow L^{(0)}_{jk} \), \( f_T \rightarrow -f_T \) where the contour \( \mathcal{C} \) now goes clockwise around the two values of \( \zeta \) for which \( \bar{L}^{(0)} = 0 \). But since the only poles of \( f_T \) occur when \( \bar{L}^{(0)} = 0 \) or when \( \bar{L}' = 0 \), one can deform the contour off the zeros of \( \bar{L}^{(0)} \) until they go counter-clockwise around the zeros of \( \bar{L}' \). Finally, reversing the direction of the contour cancels the minus sign to give the original expression.

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3. S-Dual Fradkin-Tseytlin Term

Just as compactification to D=4 simplifies the analysis of S-duality transformations, it also simplifies quantization of the superstring. For Calabi-Yau compactifications of the Type II superstring to D=4, the four fermionic $\kappa$-symmetries can be interpreted as N=(2,2) worldsheet superconformal invariances \[11\]. After slightly modifying the usual Green-Schwarz worldsheet variables, the superstring can be quantized in worldsheet conformal gauge with manifest N=2 D=4 super-Poincaré invariance \[3\]. Unlike the standard Green-Schwarz sigma model, the sigma model action for this modified Green-Schwarz superstring includes a Fradkin-Tseytlin term which couples the spacetime dilaton to the worldsheet supercurvature \[12\] [4].

For compactifications of the Type IIB superstring, this sigma model action in worldsheet conformal gauge is given by \[4\]

\[
\frac{1}{\alpha'} \int dzd\bar{z}\left[\eta_{ab} E^{a}_{M} E^{b}_{N} \partial_{z} Y^{M} \partial_{\bar{z}} Y^{N} + b'_{MN} \partial_{z} Y^{M} \partial_{\bar{z}} Y^{N} + \ldots \right] (3.1)
\]

\[
+ \int dzd\bar{z}\left[d\kappa_{+} d\bar{\kappa}_{+} \Sigma_{c} \log(W) + d\kappa_{+} d\bar{\kappa}_{-} \Sigma_{tc} \log(L^{(0)}_{++}) + c.c.\right]
\]

where $Y^{M} = [x^{m}, \theta_{j}^{a}, \bar{\theta}_{j}^{\dot{a}}]$, $b'_{MN}$ is the two-form potential whose field-strength is $L'_{jk}$ of the previous section, $\kappa_{\pm}$ and $\bar{\kappa}_{\pm}$ are the anti-commuting variables of D=2 N=(2,2) superspace, $\Sigma_{c}$ and $\Sigma_{tc}$ are the chiral and twisted-chiral N=(2,2) worldsheet superfields whose top component is the worldsheet curvature, and ... refers to terms (written explicitly in \[4\]) which will not be relevant to the discussion.

As argued in \[4\], the first line in (3.1) is invariant under classical N=(2,2) worldsheet superconformal invariance if $E^{M}_{A}$ satisfies the torsion constraints of N=2 D=4 supergravity and the field-strength for $b'_{MN}$ satisfies $L'_{jk} = \delta_{jk}$. The second line of (3.1) is not invariant under classical worldsheet superconformal transformations, and as usual for a Fradkin-Tseytlin term, its classical variation is expected to cancel the quantum variation of the first line when the background superfields are on-shell. This has been explicitly checked for the heterotic version of (3.1) in reference \[13\].

Recently, Townsend and Cederwall \[1\] have constructed a manifestly SL(2,Z)-covariant action for the superstring by introducing two worldsheet $U(1)$ gauge fields, $A_{i}$ and $\tilde{A}_{i}$. For the $(p, q)$ superstring, the constants $p$ and $q$ are replaced by worldsheet fields, $S$ and $\tilde{S}$, which are the conjugate momenta to these worldsheet gauge fields.
Like the standard Green-Schwarz sigma model action, their sigma model lacks a Fradkin-Tseytlin term. But using the methods of [1] and the results of the previous section, it is straightforward to generalize (3.1) to an SL(2,Z)-covariant action in worldsheet conformal gauge. The appropriate generalization in conformal gauge is

\[
\frac{1}{\alpha'} \int dzd\bar{z}[ SF + \tilde{S} \tilde{F} + \eta_{ab} E^a_M E^b_N \partial_\bar{z} Y^M \partial_\bar{z} Y^N + (S b'_{MN} + \tilde{S} \tilde{b}^{(0)}_{MN}) \partial_\bar{z} Y^M \partial_\bar{z} Y^N + \ldots ] \quad (3.2)
\]

\[+ \int dzd\bar{z}[d\kappa_+ d\bar{\kappa}_+ \Sigma_c \log(W) + d\kappa_+ d\bar{\kappa}_- \Sigma_{tc} \log(L^{(0)}_{++} L'_{+-} - L^{(0)}_{+-} L'_{++}) + c.c.] \]

where \(F\) and \(\tilde{F}\) are the field-strengths for \(A_i\) and \(\tilde{A}_i\), \(b^{(0)}_{MN}\) is the potential whose field-strength is \(L^{(0)}_{jk}\), and \(L^{(0)}_{jk}\) is replaced with \((L^{(0)}_{jk} L'_{+-} - L^{(0)}_{+-} L'_{jk})\) everywhere it appears in \(\ldots\). It is easy to check that (3.2) is invariant under the SL(2,Z) transformations

\[S \rightarrow AS - B\tilde{S}, \quad \tilde{S} \rightarrow -CS + D\tilde{S}, \quad A_i \rightarrow CA_i + D\tilde{A}_i, \quad \tilde{A}_i \rightarrow A\tilde{A}_i + B\tilde{A}_i, \quad (3.3)\]

\[L^{(0)}_{jk} \rightarrow AL^{(0)}_{jk} + BL'_{jk}, \quad L'_{jk} \rightarrow CL^{(0)}_{jk} + DL'_{jk}, \]

where \(AD - BC = 1\).

The equations of motion for \(A_i\) and \(\tilde{A}_i\) imply that \(S\) and \(\tilde{S}\) are constants on-shell, and since the gauge fields are \(U(1)\), these constants must be integer-valued and can be identified with \(p\) and \(q\) [14]. The transformations of the spacetime superfields in (3.3) are the same as in (2.8), so the action of (3.2) correctly describes the \((p, q)\) superstring.

When \(S\) and \(\tilde{S}\) take background values \(p\) and \(q\), classical worldsheet superconformal invariance of the first line in (3.2) implies that \(pL'_{jk} + qL^{(0)}_{jk} = \delta_{jk}\). So to compare with the usual \((p, q)\) sigma model action, one needs to perform a local conformal and \(SU(2)_R\) transformation on all background superfields in order to recover the string gauge \(L'_{jk} = \delta_{jk}\). For example, when written in terms of the string-gauge metric \(\hat{g}_{\mu\nu}\),

\[g_{\mu\nu} \partial_\bar{z} x^\mu \partial_\bar{z} x^\nu = \sqrt{(qe^{-\phi})^2 + (p + qa)^2} \hat{g}_{\mu\nu} \partial_\bar{z} x^\mu \partial_\bar{z} x^\nu, \quad (3.4)\]

reproducing the \((p, q)\) tension formula of [13].

Although it might seem surprising that the Fradkin-Tseytlin term can be written in manifestly SL(2,Z)-invariant form, one should remember that the on-shell values of \(S\) and \(\tilde{S}\) spontaneously break this SL(2,Z)-invariance. So the spacetime equations of motion coming from the \(\beta\)-functions of the sigma model are not expected to be SL(2,Z) invariant.
Note that the terms $S F - \tilde{S} \tilde{F}$ and $(S b_{MN}' - \tilde{S} b_{MN}'^{(0)}) \partial z Y^M \partial \bar{z} Y^N$ do not couple to the worldsheet metric components $h_{zz}$ and $h_{\bar{z}\bar{z}}$, so $S$ and $\tilde{S}$ do not appear in the Virasoro generators. It seems reasonable to assume that they are absent also from the super-Virasoro generators, which would imply that $S$ and $\tilde{S}$ are inert under worldsheet superconformal transformations.

A curious feature of (3.2) is that it depends on two types of worldsheet gauge fields, one type coming from D=2 $\tilde{N}=(2,2)$ worldsheet supergravity and the other type coming from the action of [1]. It would be very interesting to find a relation between these two types of gauge fields.

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