K-COWAIST ON COMPLETE FOLIATED MANIFOLDS

GUANGXIANG SU AND XIANGSHENG WANG

Abstract. Let \((M,F)\) be a connected (not necessarily compact) foliated manifold carrying a complete Riemannian metric \(g^{TM}\). We generalize Gromov’s K-cowaist using the coverings of \(M\), as well as defining a closely related concept called \(\hat{A}\)-cowaist. Let \(k^F\) be the associated leafwise scalar curvature of \(g^F = g^{TM}|_F\). We obtain some estimates on \(k^F\) using these two concepts. In particular, assuming that the generalized K-cowaist is infinity and either \(TM\) or \(F\) is spin, we show that \(\inf(k^F) \leq 0\).

1. Introduction

1.1. Main results. Let \(M\) be a closed connected oriented smooth Riemannian manifold. Let \(E \to M\) be a Hermitian vector bundle with a Hermitian connection \(\nabla^E\) and \(R^E\) be the curvature of \(\nabla^E\). If \(\dim M\) is even, Gromov ([6, §4] or [8, §4.1.4]) defines the K-cowaist\(^1\) of \(M\) by

\[
\text{K-cw}_2(M) = \sup_E (\|R^E\|^{-1}),
\]

where \(E \to M\) is a unitary bundle for which (at least) one characteristic (Chern) number of \(E\) does not vanish. Gromov also generalizes the definition of the K-cowaist to open manifolds by sticking to bundles \(E \to M\) trivialized at infinity and using the characteristic numbers coming from the cohomology with compact supports. If \(\dim M\) is odd, Gromov defines

\[
\text{K-cw}_2(M) = \sup_k \text{K-cw}_2(M \times \mathbb{R}^k),
\]

where one takes those \(k \geq 0\) such that \(\dim M + k\) is even.

In [6, §5\(\frac{1}{2}\)], Gromov proves that every complete Riemannian spin manifold of dimension \(n\) with the scalar curvature \(k^{g^{TM}} \geq \varepsilon^{-2}\) satisfies \(\text{K-cw}_2(M) \leq \text{const}_n \varepsilon^2\). In [6, §9\(\frac{2}{3}\)], Gromov also defines the K-cowaist for the foliated manifolds.

In this paper, for the case that \(M\) is a connected oriented (not necessarily compact) Riemannian manifold with a foliation, we generalize the definition of K-cowaist via considering the coverings of \(M\). As in [6], we also study the relation between the leafwise scalar curvature and this generalized K-cowaist.

We now explain it in detail. Let \(M\) be a connected oriented (not necessarily compact) manifold carrying a (not necessarily complete) Riemannian metric \(g^{TM}\). Let \(F \subset TM\) be an integrable subbundle of \(TM\) and \(g^F = g^{TM}|_F\) be the restricted metric on \(F\). In the following, we assume that both \(\dim M\) and \(\text{rk}(F)\) are even. If \(\dim M\) is odd and \(\text{rk}(F)\) is even, we replace \(M\) by \(M \times S^1\). If \(\text{rk}(F)\) is odd, we replace \(F\) by \(F \oplus TS^1\) and \(M\) by \(M \times S^1 \times S^1\) or \(M \times S^1\) depending on whether \(\dim M\) is even or odd.

\(^{1}\)In [6], K-cowaist was called K-area. But recently, Gromov [8] suggests that K-cowaist should a more proper name for this concept.
We take \( \tilde{\pi} : \tilde{M} \to M \) to be a covering of \( M \) and \( \tilde{F} \) to be the lifted foliation on \( \tilde{M} \). Then \( \tilde{M} \) and \( \tilde{F} \) carry the lifted metrics \( g^{\tilde{TM}} \) and \( g^{\tilde{F}} \).

Let \( (E, g^{E}, \nabla^{E}) \) be a Hermitian vector bundle over \( \tilde{M} \) with the Hermitian metric \( g^{E} \) and the Hermitian connection \( \nabla^{E} \). We assume that \( E \) is trivial at infinity.

Let \( R^{E} = (\nabla^{E})^{2} \) be the curvature of \( \nabla^{E} \). Hence, for any \( x \in \tilde{M} \) and \( \alpha, \beta \in T_{x} \tilde{M} \), \( R^{E}(\alpha \wedge \beta) \in \text{End}(T_{x} \tilde{M}) \). Recall that \( \| R^{E} \| \) is defined in the following way (cf. [6]),

\[
\| R^{E} \| = \sup_{x \in \tilde{M}} \sup_{\alpha, \beta \in \tilde{F} \at \tilde{M}} |R^{E}(\alpha \wedge \beta)|.
\]

Now, we can define a pair of closely related concepts.

**Definition 1.1.** With above notations, if the vector bundle \( E \) further satisfies

1. some Chern number of \( E \) is non-zero, we define the (covering) K-cowaist of \( (M, F) \) by

\[
\text{K-ccw}_{2}(M, F) = \sup_{\tilde{M}, E}(\| R^{E} \|^{-1});
\]

2. the inequality

\[
\int_{\tilde{M}} \hat{A}(\tilde{T}\tilde{M})(\text{ch}(E) - \text{rk}(E)) \neq 0,
\]

we define the (covering) \( \hat{A} \)-cowaist of \( (M, F) \) by

\[
\text{\hat{A}-ccw}_{2}(M, F) = \sup_{\tilde{M}, E}(\| R^{E} \|^{-1}).
\]

From the index theory viewpoint, among the two concepts given as above, \( \hat{A} \)-cowaist perhaps relates to the scalar curvature more directly. In fact, we can use \( \hat{A} \)-cowaist to give a quantitative estimate on leafwise scalar curvature.

**Theorem 1.2.** Let \( M \) be a connected oriented (not necessarily compact) manifold carrying a complete Riemannian metric \( g^{TM} \). Let \( F \subseteq TM \) be an integrable subbundle of \( TM \) with the restricted metric \( g^{F} = g^{TM}|_{F} \). Let \( k^{F} \) be the associated leafwise scalar curvature of \( F \). If either \( TM \) or \( F \) is spin, then one has

\[
\inf(k^{F}) \leq 2\text{rk}(F)(\text{rk}(F) - 1)/\text{\hat{A}-ccw}_{2}(M, F).
\]

Note that in the definition of \( \hat{A} \)-cowaist, we don’t need that \( g^{TM} \) is complete. But for the above theorem, the completeness is necessary. As in [13], we only give the proof of Theorem 1.2 for the \( TM \) spin case in detail. The \( F \) spin case can be proved similarly as in [13, §2.5].

To obtain a similar estimate using the K-cowaist, we notice the following reinterpretation of the result in [6, § 5.8]. One can also see [12] for a more detailed proof of this result.

**Proposition 1.3.** \( \text{K-ccw}_{2}(M, F) = +\infty \) implies \( \text{\hat{A}-ccw}_{2}(M, F) = +\infty \).

Due to this proposition, we have the following corollary of Theorem 1.2.
Theorem 1.4. Under the same assumption of Theorem 1.2, if we further assume that $K\text{-}cw_2(M, F) = +\infty$, then $\inf(k^F) \leq 0$.

Since $K\text{-}cw_2(M) = +\infty$ implies $K\text{-}ccw_2(M, F) = +\infty$, as a corollary of Theorem 1.4, one can show the following result.

Theorem 1.5. (Gromov, [8, p. 182, footnote 277]) Complete manifolds $X$ with infinite $K\text{-}cw_2(X)$, carry no spin foliations $F$, where the induced Riemannian metrics in the leaves satisfy $k^F \geq \sigma > 0$.

1.2. A discussion about the definitions of $\widehat{A}$-cowaist. Compared to other similar concept in the literature, a feature of the definition of $\widehat{A}$-cowaist is that the supremum is calculated using bundles over any coverings of $M$ rather than bundles over $M$ alone. Whether it is necessary to taking supremum on this large set turns out to be a delicate problem. For simplicity, we will assume $F = TM$ in this subsection.

To facilitate our discuss, we using the following definition which resembles the definition of $\widehat{A}$-cowaist except not using the covering space. Let $M$ be a manifold and $(E, g^E, \nabla^E)$ be a Hermitian vector bundle over $M$ with the Hermitian metric $g^E$ and the Hermitian connection $\nabla^E$. If $M$ is non-compact, we also assume that $E$ is trivial at infinity. Define

$$\widehat{A}\text{-area}(M) = \sup_E \{\|R^E\|^{-1} | \int_M \widehat{A}(TM)(\text{ch}(E) - \text{rk}(E)) \neq 0\}.$$ 

We discuss several cases separately.

$M$ is non-compact. In this case, $\widehat{A}\text{-}cw_2(M, TM)$ in strictly larger than $\widehat{A}\text{-area}(M)$ in general. As a example, let $M$ be a annulus in $\mathbb{R}^2$ and $F = TM$. By [7, p. 33 (c)], we have

$$K \text{-}cw_2(M) = \text{area}(M).$$

Since, $\text{dim } M = 2$, for any vector bundle $E$ over $M$, we have $\int_M \widehat{A}(TM)(\text{ch}(E) - \text{rk}(E)) = \int_M c_1(E)$. Hence, $K\text{-}cw_2(M)$ and $\widehat{A}\text{-area}(M)$ coincide in this case. Using (1.4), we can see that

$$\widehat{A}\text{-ccw}_2(M, TM) = \sup_{\widetilde{M}} \text{area}(\widetilde{M}) = +\infty > \widehat{A}\text{-area}(M).$$

$M$ is compact and the universal covering of $M$ is compact. Firstly, we note that if $M$ is compact, $\widehat{A}\text{-area}(M)$ is just a small variant of [4, Definition 1.6].

Since the universal covering of $M$ is compact, any covering space of $M$, $\widetilde{M}$, is also compact. Then by the same proof of [7, p. 33 (d)], we have the following result,

$$\widehat{A}\text{-area}(M) = \widehat{A}\text{-area}(\widetilde{M}).$$

Therefore, in this case, $\widehat{A}\text{-ccw}_2(M, TM)$ is equal to $\widehat{A}\text{-area}(M)$.

\footnote{In fact, if $M$ carries a metric with the positive scalar curvature, these two definitions are the same.}
is compact and the universal covering of \( M \) is non-compact. This case is the most
difficult and we do not have a definite answer at the moment. In fact, whether \( \hat{\Lambda} \)-area\((\hat{M}, TM)\)
in this case relates closely to Gromov’s following question \([7, p. 34, Question 23]\): is there a closed manifold \( M \) such that \( K-\text{cw}_2(\hat{M}) < \infty \) and the
universal covering of \( M \) satisfies \( K-\text{cw}_2(\hat{M}) = \infty \)?

The main difficulty in this case is that the pull-back and push-forward construction for
vector bundles do not work well for the non-compact spaces. If we put some restrictions
on the covering spaces, maybe some partial results are still possible. For example, if
we assume \( \pi_1(M) \) is residually finite, we have the following simple extension of \([6, p. 26, (v')]\).

**Proposition 1.6.** If \( M \) is a closed manifold and \( \pi_1(M) \) is residually finite, for the uni-
versal covering space \( \hat{M} \) of \( M \),

\[
K-\text{ccw}_2(M) \geq K-\text{cw}_2(M) \geq K-\text{cw}_2(\hat{M}) = K-\text{ccw}_2(\hat{M}).
\]

**Proof.** Clearly, we only need to show that

\[
(1.5) \quad K-\text{cw}_2(M) \geq K-\text{cw}_2(\hat{M})
\]

If \( \hat{M} \) is compact, (1.5) follows from the push-forward inequality \([6, \S 4\S 3]\). We try to show
that the push-forward argument also works for \( \hat{M} \) non-compact case.

Take a vector bundle \( E \) over \( \hat{M} \) which is trivial outside a compact set \( K \). Since \( \pi_1(M) \)
is residually finite, we can find a finite covering of \( M, N \), such that the covering map from
\( \hat{M} \) to \( N \) is injective on \( K \). As a result, we can push-forward the vector bundle \( E \) to a
vector bundle \( E_N \) over \( N \). Since \( N \) is a finite covering of \( M \), we have

\[
K-\text{cw}_2(N) = K-\text{cw}_2(M).
\]

Therefore,

\[
\|R^E\|^{-1} = \|R^{E_N}\|^{-1} \leq K-\text{cw}_2(N) = K-\text{cw}_2(M),
\]

from which we obtain (1.5). \( \square \)

We also note that the K-cowaist also generalizes \([3, Definition 5.1]\).

2. **Proof of Theorem 1.2**

In this section we prove Theorem 1.2. Our strategy to prove Theorem 1.2 follows the
proof of \([11, Theorem 1.2]\) closely. Note that in \([11]\), there is a map \( f : M \to S^n(1) \),
which enables us to construct suitable bundles over a compact manifold associated with
the non-compact manifold \( M \). Compared to \([11]\), the new idea in this paper is that
we will show that in the current situation, the bundles needed, as well as the bundle
endomorphisms, can be constructed \textit{without using the map }\( f \).

We argue by contradiction. Assume that

\[
\inf(k^F) > 2\text{rk}(F)(\text{rk}(F) - 1)/\hat{\Lambda}-\text{ccw}_2(M, F).
\]

Then, by the definition of \( \hat{\Lambda}-\text{ccw}_2(M, F) \), there exists

- a covering manifold \( \pi : \hat{M} \to M \) with the lifted foliation \( \hat{F} \) and the lifted metrics
  \( g^{TM} \) and \( g^{\hat{F}} \),
• a Hermitian vector bundle $E_0$ over $\widetilde{M}$ with the Hermitian metric $g^{E_0}$ and the Hermitian connection $\nabla^{E_0}$, which is trivial at infinity and satisfies

$$\int_{\widetilde{M}} \widehat{A}(T\widetilde{M}) \left( \text{ch}(E_0) - \text{rk}(E_0) \right) \neq 0, \tag{2.1}$$

• a constant $\kappa > 0$ such that

$$\widetilde{\pi}^*(k^F) - 2\text{rk}(F) (\text{rk}(F) - 1) \| R^E_F \| > \kappa \text{ on } \widetilde{M}. \tag{2.2}$$

As explained in Introduction, we only give the detail for the $TM$ spin case. In the following, we assume that $TM$ is spin.

If both $M$ and $\widetilde{M}$ are compact, by [14, Section 1.1], one gets a contradiction easily.

In the following, we assume that $\widetilde{M}$ is noncompact. For the rest of the proof we will only deal with quantities associated with $\widetilde{M}$ and $\widehat{F}$. To simplify the notations, we will denote the foliation ($\widetilde{M}$, $\widehat{F}$) by ($M$, $F$) and the metrics ($g^{TM}$, $g^{\widehat{F}}$) by ($g^{TM}$, $g^{F}$).

Roughly speaking, we will prove Theorem 1.2 in three steps.

(i) We construct a closed manifold $\widehat{M}_{H_{3m}}$ and a $\mathbb{Z}_2$-graded bundle $\widehat{E}$ over it. Besides, we construct a fiber bundle $\widehat{M}_{H_{3m},R}$ over $\widehat{M}_{H_{3m}}$ associated with the foliation $F$.

(ii) We construct a deformed Dirac operator on $\widehat{M}_{H_{3m},R}$ and obtain some estimates about it.

(iii) We construct a closed manifold using $\widehat{M}_{H_{3m},R}$ and an operator $P^{E_{3m,R}}_{R,\beta,\gamma,+]}$ using the deformed Dirac operator. We will show that (2.1) implies the index of $P^{E_{3m,R}}_{R,\beta,\gamma,+]}$ is not zero while (2.2) implies the index of $P^{E_{3m,R}}_{R,\beta,\gamma,+]}$ is zero. Thus we obtain a contradiction.



Step 1. Let $(E_1 = M \times \mathbb{C}^k, g^{E_1}, \nabla^{E_1})$, with $k = \text{rk}(E_0)$, be the trivial vector bundle on $M$. Then, let $E = E_0 \oplus E_1$ be a $\mathbb{Z}_2$-graded Hermitian vector bundle over $M$ with a $\mathbb{Z}_2$-graded metric $g^E = g^{E_0} \oplus g^{E_1}$ and a $\mathbb{Z}_2$-graded Hermitian connection $\nabla^E = \nabla^{E_0} \oplus \nabla^{E_1}$.

Since $(E_0, g^{E_0}, \nabla^{E_0})$ is trivial at infinity, there exists a compact subset $K^3$ and an isomorphism $\psi$ between $(E_0|_{M\setminus K}, g^{E_0}, \nabla^{E_0})$ and $((M\setminus K) \times \mathbb{C}^k, g_{\text{st}}^m, \nabla_{\text{st}}) = (E_1|_{M\setminus K}, g^{E_1}, \nabla^{E_1})$.

Following [9, Theorem 1.17], we choose a fixed point $x_0 \in M$ and let $d : M \to \mathbb{R}^+$ be a regularization of the distance function $\text{dist}(x, x_0)$ such that

$$|\nabla d|(x) \leq 3/2,$$

for any $x \in M$.

Set

$$B_m = \{ x \in M : d(x) \leq m \}, \quad m \in \mathbb{N}$$

and choose a sufficiently large $m$ such that $K \subseteq B_m$.

To construct the desired closed manifold $\widehat{M}_{H_{3m}}$, following [9], we take a compact hypersurface $H_{3m} \subseteq M \setminus B_{3m}$, which cuts $M$ into two parts such that the compact part, denoted by $M_{H_{3m}}$, contains $B_{3m}$. Then $M_{H_{3m}}$ is a compact smooth manifold with boundary $H_{3m}$. Let $g^{TH_{3m}}$ be the induced metric on $H_{3m}$. For a sufficiently small $\varepsilon' > 0$, on the product manifold $H_{3m} \times [-\varepsilon', 1 + \varepsilon']$, we construct a metric as follows.

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3We can and will choose $K$ is a closure of an open subset.
Near the boundary $H_{3m} \times \{ -\varepsilon', \} \cup H_{3m} \times \{ -\varepsilon, 1 + \varepsilon' \}$, i.e., on $H_{3m} \times [-\varepsilon', 1 + \varepsilon']$, by using the geodesic normal coordinate of $H_{3m} \subseteq M$, for a small $\varepsilon'$, there is an isomorphism between $H_{3m} \times [-\varepsilon', \varepsilon']$ and a neighborhood of $H_{3m}$, denoted by $U$, in $M$ because $H_{3m}$ is compact. Moreover, we can require that under this isomorphism, $U \cap M_{H_{3m}}$ is mapped to $H_{3m} \times [-\varepsilon', 0]$.

Now, we define the metric on $H_{3m} \times [-\varepsilon', \varepsilon']$ to be the pull-back metric obtained from that of $U$. In the same way, we can construct a metric on $H_{3m} \times [1 - \varepsilon', 1 + \varepsilon']$. Meanwhile, the metric on $H_{3m} \times [1/3, 2/3]$ is defined to be the product metric of $g_{\mathcal{T}H_{3m}}$ and the standard metric on $[1/3, 2/3]$. Finally, the metric on $H_{3m} \times [-\varepsilon', 1 + \varepsilon']$ is a smooth extension of the metrics on the above three pieces.

Using the isometry between $H_{3m} \times [-\varepsilon', \varepsilon']$ and $U$, $M_{H_{3m}} \cup U$ and $H_{3m} \times [-\varepsilon', 1 + \varepsilon']$ can be glued into a smooth Riemannian manifold with boundary, $(M_{H_{3m}} \cup U) \cup_U (H_{3m} \times [-\varepsilon', 1 + \varepsilon'])$. Let $M'_{H_{3m}}$ be another copy of $M_{H_{3m}}$ with the same metric and the opposite orientation. In the similar way, we can glue $M'_{H_{3m}}$ and $(M_{H_{3m}} \cup U) \cup_U (H_{3m} \times [-\varepsilon', 1 + \varepsilon'])$ to obtain a closed manifold $\hat{M}_{H_{3m}}$. Note that $M_{H_{3m}} \cup U$, $M'_{H_{3m}}$ and $H_{3m} \times [1/3, 2/3]$ all have natural isometric embeddings into $\hat{M}_{H_{3m}}$. As a result, we will view these manifolds as submanifolds of $\hat{M}_{H_{3m}}$ in the following. Figure 1 is an illustration of this gluing construction.

Take $V_1 := M_{H_{3m}} \cup U^0$ and $V_2 = \hat{M}_{H_{3m}} \setminus M_{H_{3m}}$ to be two open subsets of $\hat{M}_{H_{3m}}$. Then $V_1 \cup V_2 = \hat{M}_{H_{3m}}$ and $V_1 \cap V_2 = Z := H_{3m} \times (0, \varepsilon')$. As we have said, we will treat $V_1 = M_{H_{3m}} \cup U^0$ as a submanifold of both $M$ and $\hat{M}_{H_{3m}}$. It means that, although $E$ is a $Z_2$-graded vector bundle defined over $M$, its restriction on $M_{H_{3m}} \cup U^0$, $E|_{M_{H_{3m}} \cup U^0} = E|_{V_1}$, is a $Z_2$-graded vector bundle defined over a submanifold of $\hat{M}_{H_{3m}}$. We are going to extend $E|_{V_1}$ to a $Z_2$-graded bundle $\hat{E} = \hat{E}_0 \oplus \hat{E}_1$ over $\hat{M}_{H_{3m}}$ which satisfies

$$
(\hat{E}_0|_{V_2}, \hat{g}^{\hat{E}_0}, \nabla^{\hat{E}_0}) \simeq (V_2 \times C^k, g_{st}, \nabla_{st}) \simeq (\hat{E}_1|_{V_2}, \hat{g}^{\hat{E}_1}, \nabla^{\hat{E}_1}).
$$

**Figure 1.** Gluing construction.

We will construct the $\hat{E}_0$ and $\hat{E}_1$ separately. The construction of $\hat{E}_1$ is straightforward. Since $E_1|_{M_{H_{3m}}} = M_{H_{3m}} \times C^k$, we can take $(\hat{E}_1, \hat{g}^{\hat{E}_1}, \nabla^{\hat{E}_1})$ to be $(\hat{M}_{H_{3m}} \times C^k, g_{st}, \nabla_{st})$, which satisfies (2.3).

To construct $(\hat{E}_0, g^{\hat{E}_0}, \nabla^{\hat{E}_0})$, we glue two vector bundles as in [1]. Choose the trivial bundle $(E_0', g^{E_0'}, \nabla^{E_0'}) := (V_2 \times C^k, g_{st}, \nabla_{st})$ over $V_2$. Recall that $Z \subset U$ can be viewed
as a submanifold of $M$ and $Z \cap K = \emptyset$. Hence, by the definition of $\psi$, we have an isomorphism between $(E_0|_Z, g^{E_0}, \nabla^{E_0})$ (i.e., $((E_0|_{V_1})|_Z, g^{E_0}, \nabla^{E_0})$ and $(Z \times C^k, g_{st}, \nabla_{st})$ (i.e., $(E'_0|_Z, g^{E'_0}, \nabla^{E'_0})$). In other words, the restriction of $\psi$ on $Z$, denoted by $\psi|_Z$, induces an isomorphism between $((E_0|_{V_1})|_Z, g^{E_0}, \nabla^{E_0})$ and $(E'_0|_Z, g^{E'_0}, \nabla^{E'_0})$. We define $\widehat{E}_0$ to be $E_0|_{V_1} \cup_{\psi|_Z} E'_0|_{V_2}$. By definition, $\psi$, thus $\psi|_Z$, preserves the metric and the connection. As a result, $\widehat{E}_0$ inherits a metric and a connection from those of $E_0|_{V_1}$ and $E'_0|_{V_2}$. Moreover, the property of gluing construction implies,

$$
(\widehat{E}_0|_{M^{3m}}, g^{\widehat{E}_0}, \nabla^{\widehat{E}_0}) \simeq (E_0|_{M^{3m}}, g^{E_0}, \nabla^{E_0}),
$$

$$
(\widehat{E}_0|_{V_2}, g^{\widehat{E}_0}, \nabla^{\widehat{E}_0}) \simeq (E'_0, g^{E'_0}, \nabla^{E'_0}) = (V_2 \times C^k, g_{st}, \nabla_{st}).
$$

Therefore, $\widehat{E}_0$ also satisfies (2.3).

Let $i : E_0|_{V_1} \hookrightarrow \widehat{E}_0$ and $j : E'_0|_{V_2} \hookrightarrow \widehat{E}_0$ be the canonical embeddings in the gluing construction. The definition of gluing construction implies the composition of following maps

$$
j|_{Z}^{-1} \circ i|_{Z} : (E_0|_{Z}, g^{E_0}, \nabla^{E_0}) \xrightarrow{i|_{Z}} (\widehat{E}_0|_{Z}, g^{\widehat{E}_0}, \nabla^{\widehat{E}_0}) \xrightarrow{j|_{Z}} (E'_0|_{Z}, g_{st}, \nabla_{st}) = (Z \times C^k, g_{st}, \nabla_{st})
$$

is just $\psi|_Z$. Since $\widehat{E}_1|_{V_2} = V_2 \times C^k$, we can define

$$
\nu := j^{-1} : (\widehat{E}_0|_{V_2}, g^{\widehat{E}_0}, \nabla^{\widehat{E}_0}) \rightarrow (E'_0, g^{E'_0}, \nabla^{E'_0}) = (V_2 \times C^k, g_{st}, \nabla_{st}) = (\widehat{E}_1|_{V_2}, g^{\widehat{E}_1}, \nabla^{\widehat{E}_1}).
$$

At the same time, $\psi$ induces the following map

$$
\psi|_{V_1} \circ i^{-1} : (\widehat{E}_0|_{V_1}, g^{\widehat{E}_0}, \nabla^{\widehat{E}_0}) \rightarrow (E_0|_{V_1}, g^{E_0}, \nabla^{E_0}) \rightarrow (E_1|_{V_1}, g^{E_1}, \nabla^{E_1})
$$

$$
= (V_1 \times C^k, g_{st}, \nabla_{st}) = (\widehat{E}_1|_{V_1}, g^{\widehat{E}_1}, \nabla^{\widehat{E}_1}).
$$

Therefore, we have

$$
(\psi|_{V_1} \circ i^{-1})|_Z = \psi|_Z \circ i^{-1}|_Z = j|^{-1}_Z \circ i|_Z \circ i^{-1}|_Z = j|^{-1}_Z = \nu|_Z.
$$

Consequently, by gluing $\psi|_{V_1} \circ i^{-1}$ and $\nu$, we can construct a smooth section $\omega \in \Gamma(\text{Hom}(\widehat{E}_0, \widehat{E}_1))$. Besides, by the property of $\psi$ and $\nu$, we know that $\omega$ preserves the metrics and connections on $\widehat{M}_{3m} \setminus K$.

Take $\omega^*$ to be the adjoint of $\omega$ with respect to $g^{\widehat{E}_0}$ and $g^{\widehat{E}_1}$. Set

$$
W = \omega + \omega^* : \Gamma(\widehat{E}) \rightarrow \Gamma(\widehat{E}),
$$

which is an odd and self-adjoint bundle endomorphism of $\widehat{E}$. There exists a constant $\delta > 0$ such that

$$
W^2 \geq \delta \text{ on } \widehat{M}_{3m} \setminus K. \tag{2.4}
$$

Let $F^\perp$ be the orthogonal complement to $F$, i.e., we have the orthogonal splitting

$$
TM = F \oplus F^\perp, \quad g^{TM} = g^F \oplus g^{F^\perp}. \tag{2.5}
$$

Following [5, §5] (also cf. [13, §2.1]), let $\pi : \mathcal{M} \rightarrow M$ be the Connes fibration over $M$ such that for any $x \in M$, $\mathcal{M}_x = \pi^{-1}(x)$ is the space of Euclidean metrics on the linear space $T_x M/F_x$. Let $T^V \mathcal{M}$ denote the vertical tangent bundle of the fibration $\pi : \mathcal{M} \rightarrow M$. Then it carries a natural metric $g^{T^V \mathcal{M}}$ such that any two points $p, q \in \mathcal{M}_x$
with \( x \in M \) can be joined by a unique geodesic along \( \mathcal{M}_x \). Let \( d^{M_x}(p, q) \) denote the length of this geodesic.

By using the Bott connection on \( TM/F \) (cf. [13, (1.2)]) , which is leafwise flat, one lifts \( F \) to an integrable subbundle \( \mathcal{F} \) of \( TM \). Then \( g^F \) lifts to a Euclidean metric \( g^F = \pi^*g^F \) on \( F \).

Let \( \mathcal{F}_1^\perp \subset TM \) be a subbundle, which is transversal to \( \mathcal{F} \oplus T^V M \), such that we have a splitting \( TM = (\mathcal{F} \oplus T^V M) \oplus \mathcal{F}_1^\perp \). Then \( \mathcal{F}_1^\perp \) can be identified with \( TM/(\mathcal{F} \oplus T^V M) \) and carries a canonically induced metric \( g_{\mathcal{F}_1^\perp} \). We denote \( \mathcal{F}_2^\perp \) to be \( T^V M \).

The metric \( g_{\mathcal{F}_1^\perp} \) in (2.5) determines a canonical embedded section \( s : M \hookrightarrow \mathcal{M} \). For any \( p \in \mathcal{M} \), set \( \rho(p) = d^{M_\pi(p)}(p, s(\pi(p))) \).

For any \( \beta, \gamma > 0 \), following [13, (2.15)], let \( g_{\beta,\gamma}^{TM} \) be the metric on \( TM \) defined by the orthogonal splitting,

\[
\begin{align*}
TM &= \mathcal{F} \oplus \mathcal{F}_1^\perp \oplus \mathcal{F}_2^\perp, & g_{\beta,\gamma}^{TM} &= \beta^2 g^F + \frac{g_{\mathcal{F}_1^\perp}}{\gamma^2} + g_{\mathcal{F}_2^\perp}.
\end{align*}
\]

For any \( R > 0 \), let \( \mathcal{M}_R \) be the smooth manifold with boundary defined by

\[
\mathcal{M}_R = \{ p \in \mathcal{M} : \rho(p) \leq R \}.
\]

Set \( \mathcal{H}_{3m} = \pi^{-1}(H_{3m}) \) and

\[
\mathcal{M}_{\mathcal{H}_{3m},R} = \left( \pi^{-1}(M_{H_{3m}}) \right) \cap \mathcal{M}_R, \mathcal{H}_{3m,R} = \mathcal{H}_{3m} \cap \mathcal{M}_R.
\]

Consider another copy \( \mathcal{M}'_{\mathcal{H}_{3m},R} \) of \( \mathcal{M}_{\mathcal{H}_{3m},R} \) carrying the metric \( g_{\mathcal{M}'_{\mathcal{H}_{3m},R}}^{TM} \) defined by (2.6) with \( \beta = \gamma = 1 \). Meanwhile, let \( g_{\mathcal{H}_{3m},R}^{TM} \) be the induced metric on \( \mathcal{H}_{3m,R} \) by (2.6) with \( \beta = \gamma = 1 \) and \( dt^2 \) be the standard metric on \([1/3, 2/3] \). As we have done for \( \hat{M}_{H_{3m}} \), we can glue \( \mathcal{M}_{\mathcal{H}_{3m},R} \), \( \mathcal{M}'_{\mathcal{H}_{3m},R} \) and \( \mathcal{H}_{3m,R} \times [-\varepsilon', 1 + \varepsilon'] \) together to get a manifold \( \hat{\mathcal{M}}_{3m,R} \), cf. [11, Section 2.2]. But, unlike \( \hat{M}_{H_{3m}} \), \( \hat{\mathcal{M}}_{3m,R} \) is a smooth manifold with boundary. Moreover, we can define a smooth metric \( g_{\mathcal{H}_{3m},R}^{TM} \) on \( \hat{\mathcal{M}}_{3m,R} \) such that

\[
\begin{align*}
g_{\mathcal{M}_{\mathcal{H}_{3m},R}}^{TM} &= g_{\mathcal{M}_{\mathcal{H}_{3m},R}}, & g_{\mathcal{M}'_{\mathcal{H}_{3m},R}}^{TM} &= g_{\mathcal{M}'_{\mathcal{H}_{3m},R}}, & g_{\mathcal{H}_{3m,R} \times [1/3, 2/3]}^{TM} &= g_{\mathcal{H}_{3m,R} \times [1/3, 2/3]} \oplus dt^2.
\end{align*}
\]

The map \( \pi : \mathcal{M}_{\mathcal{H}_{3m},R} \rightarrow M_{H_{3m}} \) can be extended to \( \hat{\mathcal{M}}_{\mathcal{H}_{3m},R} \rightarrow \hat{M}_{H_{3m}} \) and we still denote the extended map by \( \pi \). As before, we pull the bundles on \( \hat{M}_{H_{3m}} \) back to \( \hat{\mathcal{M}}_{3m,R} \), that is, we take \( (E_{3m,R}, \nabla^{E_{3m,R}}, g^{E_{3m,R}}) = \pi^*(\hat{E}, \nabla^\hat{E}, g^\hat{E}) \). As usual, \( R^{E_{3m,R}} = (\nabla^{E_{3m,R}})^2 \) is the curvature of \( \nabla^{E_{3m,R}} \).

**Step 2.** Recall that we have assumed that \( TM \) is oriented and spin. Without loss of generality, we assume further that \( F \) is oriented and \( \text{rk}(F^\perp) \) is divisible by 4. Then \( F^\perp \) is also oriented and \( \text{dim} \mathcal{M} \) is even.
It is clear that $F \oplus F_1^\perp, F_2^\perp$ over $\mathcal{M}_{3m,R}$ can be extended to $(\mathcal{H}_{3m,R} \times [-\varepsilon', 1 + \varepsilon']) \cup \mathcal{M}'_{3m,R}$ such that we have the orthogonal splitting

$$T\hat{\mathcal{M}}_{3m,R} = (F \oplus F_1^\perp) \oplus F_2^\perp$$
onumber

(2.7) on $\hat{\mathcal{M}}_{3m,R}$.

Let $S_{\beta,\gamma}(F \oplus F_1^\perp)$ denote the spinor bundle over $\hat{\mathcal{M}}_{3m,R}$ with respect to the metric $g^{T\hat{\mathcal{M}}_{3m,R}|_{F\oplus F_1^\perp}}$ (thus with respect to $\beta^2 g^F \oplus \frac{g^F}{\beta}$ on $\mathcal{M}_{3m,R}$). Let $\Lambda^*(F_2^\perp)$ denote the exterior algebra bundle of $F_2^\perp$, with the $\mathbb{Z}_2$-grading given by the natural even/odd parity.

Let

$$D_{F\oplus F_1^\perp, \beta, \gamma} : \Gamma(S_{\beta,\gamma}(F \oplus F_1^\perp) \otimes \Lambda^*(F_2^\perp)) \to \Gamma(S_{\beta,\gamma}(F \oplus F_1^\perp) \otimes \Lambda^*(F_2^\perp))$$

be the sub-Dirac operator on $\hat{\mathcal{M}}_{3m,R}$ constructed as in [13, (2.16)]. It is clear that one can define canonically the twisted sub-Dirac operator (twisted by $\mathcal{E}_{3m,R}$) on $\hat{\mathcal{M}}_{3m,R}$,

$$D_{F\oplus F_1^\perp, \beta, \gamma}^\mathcal{E}_{3m,R} : \Gamma(S_{\beta,\gamma}(F \oplus F_1^\perp) \otimes \Lambda^*(F_2^\perp) \otimes \mathcal{E}_{3m,R}) \to \Gamma(S_{\beta,\gamma}(F \oplus F_1^\perp) \otimes \Lambda^*(F_2^\perp) \otimes \mathcal{E}_{3m,R}).$$

(2.9)

Let $\tilde{f} : [0, 1] \to [0, 1]$ be a smooth function such that $\tilde{f}(t) = 0$ for $0 \leq t \leq \frac{1}{2}$, while $\tilde{f}(t) = 1$ for $\frac{1}{2} \leq t \leq 1$. For any $p \in \mathcal{M}_{3m,R}$, we connect $p$ and $s(p) \in \mathcal{M}_{3m,R}$ by the unique geodesic in $\mathcal{M}_{3m,R}$. Let $\sigma(p) \in F_2^\perp \mid_p$ denote the unit vector tangent to this geodesic. Then

$$\tilde{\sigma} = \tilde{f} \left( \frac{\rho}{R} \right) \sigma$$

(2.10)

is a smooth section of $F_2^\perp \mid_{\mathcal{M}_{3m,R}}$. It extends to a smooth section of $F_2^\perp \mid_{\hat{\mathcal{M}}_{3m,R}}$, which we still denote by $\tilde{\sigma}$. It is easy to see that we may and will assume that $\tilde{\sigma}$ is transversal to (and thus nowhere zero on) $\partial \hat{\mathcal{M}}_{3m,R}$. Note that the Clifford action $\overline{c}(\tilde{\sigma})$ (cf. [13, (1.47)]) now acts on $S_{\beta,\gamma}(F \oplus F_1^\perp) \otimes \Lambda^*(F_2^\perp) \otimes \mathcal{E}_{3m,R}$ over $\hat{\mathcal{M}}_{3m,R}$.

With $\overline{c}(\tilde{\sigma})$ and $W$, for $\varepsilon > 0$, we introduce the following deformation of $D_{F\oplus F_1^\perp, \beta, \gamma}^\mathcal{E}_{3m,R}$ on $\hat{\mathcal{M}}_{3m,R}$, which combines the deformations in [13, (2.21)] and [14, (1.11)],

$$D_{F\oplus F_1^\perp, \beta, \gamma}^\mathcal{E}_{3m,R} + \frac{\overline{c}(\tilde{\sigma})}{\beta} + \frac{\varepsilon \pi^* W}{\beta}.$$ 

(2.11)

About the deformed operator (2.11), the following estimate holds, which is an analog of [11, Lemma 2.1]. Let $h : [0, 1] \to [0, 1]$ be a smooth function such that $h(t) = 1$ for $0 \leq t \leq \frac{3}{4}$, while $h(t) = 0$ for $\frac{7}{8} \leq t \leq 1$.

**Lemma 2.1.** There exist $c_0 > 0$, $\varepsilon > 0$, $m > 0$ and $R > 0$ such that when $\beta, \gamma > 0$ are small enough (which may depend on $m$ and $R$),

\[ c_0 \leq | \partial \tilde{\sigma} | \leq \frac{1}{\varepsilon} \]
(i) for any \( s \in \Gamma(S_{\beta,\gamma}(\mathcal{F} \oplus \mathcal{F}_1) \otimes \Lambda^*(\mathcal{F}_2^\perp) \otimes \mathcal{E}_{3m,R}) \) supported in the interior of \( \bar{M}_{H_{3m},R} \), one has\(^5\)
\[
\| (D^\mathcal{E}_{3m,R}_{\mathcal{F} \oplus \mathcal{F}_1^\perp,\beta,\gamma} + \frac{\hat{c}(\hat{\sigma})}{\beta} + \frac{\varepsilon \pi^* W}{\beta} ) s \| \geq \frac{c_0}{\beta} \| s \|;
\]
(ii) for any \( s \in \Gamma(S_{\beta,\gamma}(\mathcal{F} \oplus \mathcal{F}_1) \otimes \Lambda^*(\mathcal{F}_2^\perp) \otimes \mathcal{E}_{3m,R}) \) supported in the interior of \( \mathcal{M}_{H_{3m},R} \setminus \bar{M}_{H_{3m},R} \), one has
\[
\| (h \left( \frac{\rho}{R} \right) D^\mathcal{E}_{3m,R}_{\mathcal{F} \oplus \mathcal{F}_1^\perp,\beta,\gamma} h \left( \frac{\rho}{R} \right) + \frac{\hat{c}(\hat{\sigma})}{\beta} + \frac{\varepsilon \pi^* W}{\beta} ) s \| \geq \frac{c_0}{\beta} \| s \|.
\]

**Proof.** We follow the same strategy of [11, Lemma 2.1] to prove this lemma. Especially, the proof of (ii) is the same with the proof of [11, Lemma 2.1 (ii)]. Here, we only show how to modify the arguments in [11, Lemma 2.1] to prove (i).

As in [11, (2.21)–(2.22)], on \( \bar{M}_{H_{3m}} \), we can find cut-off functions \( \psi_{m,1}, \psi_{m,2} : \bar{M}_{H_{3m}} \to [0, 1] \) satisfying for \( i = 1, 2 \),
\[
(2.12) \quad \psi_{m,1}^2 + \psi_{m,2}^2 = 1,
\]
\[
(2.13) \quad |\nabla \psi_{m,i}(x)| \leq C/m \text{ for any } x \in \bar{M}_{H_{3m}}.
\]
Then we pull \( \psi_{m,1}, \psi_{m,2} \) back to cut-off functions \( \varphi_{m,1}, \varphi_{m,2} \) defined on \( \bar{M}_{H_{3m},R} \). Due to [11, (2.23)], we know that
\[
(2.14) \quad \varphi_{m,1} = 1 \text{ if } x \in \pi^{-1}(B_m) \text{ and } \varphi_{m,1} = 0 \text{ if } x \in \bar{M}_{H_{3m},R} \setminus \pi^{-1}(B_m),
\]
\[
\varphi_{m,2} = 0 \text{ if } x \in \pi^{-1}(B_m) \text{ and } \varphi_{m,2} = 1 \text{ if } x \in \bar{M}_{H_{3m},R} \setminus \pi^{-1}(B_m).
\]

Using \( \varphi_{m,i} \), for any \( s \in \Gamma(S_{\beta,\gamma}(\mathcal{F} \oplus \mathcal{F}_1) \otimes \Lambda^*(\mathcal{F}_2^\perp) \otimes \mathcal{E}_{3m,R}) \) supported in the interior of \( \bar{M}_{H_{3m},R} \), by (2.12), we have the following estimate, cf. [11, (2.25)].
\[
(2.15) \quad \sqrt{2} \| (D^\mathcal{E}_{3m,R}_{\mathcal{F} \oplus \mathcal{F}_1^\perp,\beta,\gamma} + \frac{\hat{c}(\hat{\sigma})}{\beta} + \frac{\varepsilon \pi^* W}{\beta} ) s \| \geq \| (D^\mathcal{E}_{3m,R}_{\mathcal{F} \oplus \mathcal{F}_1^\perp,\beta,\gamma} + \frac{\hat{c}(\hat{\sigma})}{\beta} + \frac{\varepsilon \pi^* W}{\beta} ) (\varphi_{m,1}s) \|
\]
\[
+ \| (D^\mathcal{E}_{3m,R}_{\mathcal{F} \oplus \mathcal{F}_1^\perp,\beta,\gamma} + \frac{\hat{c}(\hat{\sigma})}{\beta} + \frac{\varepsilon \pi^* W}{\beta} ) (\varphi_{m,2}s) \| - \| c_{\beta,\gamma} (d\varphi_{m,1}) s \| - \| c_{\beta,\gamma} (d\varphi_{m,2}) s \|,
\]
where for each \( i \in \{1, 2\} \), we identify \( d\varphi_{m,i} \) with the gradient of \( \varphi_{m,i} \) and \( c_{\beta,\gamma}(\cdot) \) means the Clifford action with respect to the metric (2.6).

For the last two terms in the r.h.s. of (2.15), we can use the estimate [11, (2.29)],
\[
(2.16) \quad |c_{\beta,\gamma}(d\varphi_{m,i})s|(x) = \left( O\left( \frac{1}{\beta m} \right) + O_{m,R}(\gamma) \right) s\|s\|, \quad x \in \mathcal{M}_{H_{3m},R},
\]
where the subscripts in \( O_{m,R}(\cdot) \) mean that the big O constant may depend on \( m \) and \( R \).

For the first two terms in the r.h.s. of (2.15), by a direct computation, we have
\[
(2.17) \quad (D^\mathcal{E}_{3m,R}_{\mathcal{F} \oplus \mathcal{F}_1^\perp,\beta,\gamma} + \frac{\hat{c}(\hat{\sigma})}{\beta} + \frac{\varepsilon \pi^* W}{\beta} )^2 = (D^\mathcal{E}_{3m,R}_{\mathcal{F} \oplus \mathcal{F}_1^\perp,\beta,\gamma} + \frac{\hat{c}(\hat{\sigma})}{\beta})^2
\]
\[
+ \left[ D^\mathcal{E}_{3m,R}_{\mathcal{F} \oplus \mathcal{F}_1^\perp,\beta,\gamma} \frac{\varepsilon \pi^* W}{\beta} \right] + \frac{\varepsilon^2 (\pi^* W)^2}{\beta^2}.
\]

---

\(^5\) The norms below depend on \( \beta \) and \( \gamma \). In case of no confusion, we omit the subscripts for simplicity.
Meanwhile, since \( W \) is a constant endomorphism outside \( K \), we know

\[
(D^\varepsilon_{3m,R} F_{\partial F^1_+,\beta,\gamma} + \frac{\varepsilon \pi^* W}{\beta} ) = 0 \text{ on } \mathcal{M}_{H_{3m,R}} \setminus \pi^{-1}(K).
\]

Therefore, for the second term on the r.h.s. of (2.15), by (2.4), (2.14), (2.17) and (2.18), one has

\[
(2.19) \quad \|(D^\varepsilon_{3m,R} F_{\partial F^1_+,\beta,\gamma} + \frac{\varepsilon \pi^* W}{\beta} ) (\varphi_{m, 2s})\|^2 = \|(D^\varepsilon_{3m,R} F_{\partial F^1_+,\beta,\gamma} + \frac{\varepsilon \pi^* W}{\beta} ) (\varphi_{m, 2s})\|^2 + \|\frac{\varepsilon \pi^* W}{\beta} (\varphi_{m, 2s})\|^2 \geq \frac{\delta \varepsilon^2}{\beta^2} \|\varphi_{m, 2s}\|^2.
\]

Then main difference between the proof of (i) and [11, Lemma 2.1 (i)] lies in the estimate of the first term in the r.h.s. of (2.15). Let \( \text{rk}(F) = \text{rk}(F_i) = q_1 \) and \( \text{rk}(F_2) = q_2 \). Since on \( \mathcal{M}_{H_{3m,R}} \), \( g^F = \pi^g F \), for a local orthonormal basis \( \{f_1, \ldots, f_q\} \) of \( (F, g^F) \), we can choose it to be lifted from a local orthonormal basis of \( (F^+_1, g^{F^+_1}) \) (resp. \( (F^+_2, g^{F^+_2}) \)). Then,

\[
\{f_1, \ldots, f_q, h_1, \ldots, h_{q_1}, e_1, \ldots, e_{q_2}\}
\]

is a local orthonormal frame for \( TM_{H_{3m,R}} \). Then, by [11, (2.39)], we have

\[
(2.20) \quad \|(D^\varepsilon_{3m,R} F_{\partial F^1_+,\beta,\gamma} + \frac{\varepsilon \pi^* W}{\beta} ) (\varphi_{m, 1s})\|^2 \geq \left( \frac{\pi^* k_F}{4\beta^2} \varphi_{m, 1s}, \varphi_{m, 1s} \right)
\]

\[
+ \left( \frac{1}{2\beta^2} \sum_{i,j=1}^{q} R^E_{3m,R}(f_i, f_j)c_{\beta,\gamma}(\beta^{-1} f_i)c_{\beta,\gamma}(\beta^{-1} f_j)\varphi_{m, 1s}, \varphi_{m, 1s} \right)
\]

\[
+ \left( \|D^\varepsilon_{3m,R} F_{\partial F^1_+,\beta,\gamma} + \frac{\varepsilon \pi^* W}{\beta} \| \varphi_{m, 1s}, \varphi_{m, 1s} \right)
\]

\[
+ \left( \frac{\varepsilon^2 (\pi^* W)^2}{\beta^2} \varphi_{m, 1s}, \varphi_{m, 1s} \right) + (O_{m,R} \left( \frac{1}{\beta} + \frac{\gamma^2}{\beta^2} \right) \varphi_{m, 1s}, \varphi_{m, 1s}).
\]

To estimate the r.h.s. of (2.20), we proceed term by term.

(1) For the first two terms, by (1.1) and (2.2), we have

\[
(2.21) \quad \left( \frac{1}{2\beta^2} \sum_{i,j=1}^{q} R^E_{3m,R}(f_i, f_j)c_{\beta,\gamma}(\beta^{-1} f_i)c_{\beta,\gamma}(\beta^{-1} f_j)\varphi_{m, 1s}, \varphi_{m, 1s} \right)
\]

\[
+ \left( \frac{\pi^* k_F}{4\beta^2} \varphi_{m, 1s}, \varphi_{m, 1s} \right) \geq \frac{\kappa}{4\beta^2} \|\varphi_{m, 1s}\|^2.
\]

Note that (2.21) is the counterpart of [11, (2.36) and (2.37)] in our situation.

(2) For the third term, by [13, Lemma 2.1], on \( \mathcal{M}_{H_{3m,R}} \setminus s(M_{H_{3m}}) \), we have

\[
(2.22) \quad \left[ D^\varepsilon_{3m,R} F_{\partial F^1_+,\beta,\gamma} \frac{\varepsilon \pi^* W}{\beta} \right] = O_{m,R} \left( \frac{1}{\beta^2 R} \right) + O_{m,R} \left( \frac{1}{\beta} \right).
\]
(3) For the fourth term, since $\nabla^{E_{3,m,R}}$ (resp. $\pi^* W$) is a pull-back connection (resp. bundle endomorphism) via $\pi$, by [11, (2.33)] and (2.18), we have

$$[D_{F \oplus F_1^+}^{E_{3,m,R}} \frac{\varepsilon \pi^* W}{\beta}, \frac{\varepsilon \pi^* W}{\beta}] = \begin{cases} O \left( \frac{\varepsilon}{\beta^2} \right) + O_R \left( \frac{\varepsilon^2}{\beta} \right) & \text{on } \pi^{-1}(K), \\
o & \text{on } \hat{M}_{H_{3,m,R}} \setminus \pi^{-1}(K). \end{cases}$$

(2.23)

(4) For the fifth term, by (2.4), we have

$$\left( \frac{\varepsilon^2 (\pi^* W)^2}{\beta^2} \varphi_{m,1s}, \varphi_{m,1s} \right) \geq \frac{\varepsilon^2 \delta}{\beta^2} \left\| \varphi_{m,1s} \right\|_{\pi^{-1}(B_{2m} \setminus K)},$$

where the subscript on the norm means the integral on $\pi^{-1}(B_{2m} \setminus K)$.

Now, as in [11, Lemma 2.1], we split every term on the r.h.s. of (2.20) into integrals on $\pi^{-1}(K)$ and $\pi^{-1}(B_{2m} \setminus K)$ separately. By (2.21)–(2.24), we have

$$\left\| (D_{F \oplus F_1^+}^{E_{3,m,R}} + \frac{\bar{c}(\bar{\sigma})}{\beta} + \frac{\varepsilon \pi^* W}{\beta}) (\varphi_{m,1s}) \right\|^2 \geq \frac{K}{8 \beta^2} \left\| \varphi_{m,1s} \right\|^2$$

$$\quad + O_R \left( \frac{\varepsilon^2}{\beta} \right) \left\| \varphi_{m,1s} \right\|^2_{\pi^{-1}(K)} + O_{m,R} \left( \frac{\varepsilon^2}{\beta^2} \right) \left\| \varphi_{m,1s} \right\|^2$$

$$\quad + O_{m,R} \left( \frac{1}{\beta^2} \right) \left\| \varphi_{m,1s} \right\|^2 + O_m \left( \frac{1}{\beta^2 R} \right) \left\| \varphi_{m,1s} \right\|^2.$$

(2.25)

By (2.16), (2.19) and (2.25), for $m$ sufficiently large and suitably chosen $\varepsilon, \beta, \gamma, R$, we obtain the estimate in (i). \qed

Step 3. Let $\partial \hat{M}_{H_{3,m,R}}$ bound another oriented manifold $N_{3,m,R}$ so that $\hat{N}_{3,m,R} = \hat{M}_{H_{3,m,R}} \cup N_{3,m,R}$ is an oriented closed manifold. Let $g^{TN_{3,m,R}}$ be a smooth metric on $TN_{3,m,R}$ so that $g^{TN_{3,m,R}}|_{\hat{M}_{H_{3,m,R}}} = g^{T\hat{M}_{H_{3,m,R}}}$. The existence of $g^{T\hat{N}_{3,m,R}}$ is clear.

Let $Q$ be a Hermitian vector bundle over $\hat{M}_{H_{3,m,R}}$ such that

$$(S_{\beta,\gamma}(F \oplus F_1^+) \hat{\otimes} \Lambda^* (F_2^+) \hat{\otimes} \mathcal{E}_{m,R})_- \oplus Q$$

is a trivial vector bundle over $\hat{M}_{H_{3,m,R}}$. Then

$$(S_{\beta,\gamma}(F \oplus F_1^+) \hat{\otimes} \Lambda^* (F_2^+) \hat{\otimes} \mathcal{E}_{m,R})_+ \oplus Q$$

is also a trivial vector bundle near $\partial \hat{M}_{H_{3,m,R}}$ under the identification $\bar{c}(\bar{\sigma}) + \pi^* \omega + \text{Id}_Q$.

Since the above two vector bundles are both trivial near $\partial \hat{M}_{H_{3,m,R}}$, by extending them via the trivial bundle over $\hat{N}_{3,m,R} \setminus \hat{M}_{H_{3,m,R}}$, we get a $\mathbb{Z}_2$-graded Hermitian vector bundle $\xi = \xi_+ \oplus \xi_-$ over $\hat{N}_{3,m,R}$ and an odd self-adjoint endomorphism $\mathcal{W}' = \omega' + \omega'^* \in \Gamma(\text{End}(\xi))$ (with $\omega' : \Gamma(\xi_+) \to \Gamma(\xi_-)$, $\omega'^*$ being the adjoint of $\omega'$) such that

$$\xi_\pm = (S_{\beta,\gamma}(F \oplus F_1^+) \hat{\otimes} \Lambda^* (F_2^+) \hat{\otimes} \mathcal{E}_{m,R})_\pm \oplus Q$$

over $\hat{M}_{H_{3,m,R}}$, $\mathcal{W}'$ is invertible on $N_{3,m,R}$ and

$$\mathcal{W}' = \bar{c}(\bar{\sigma}) + \pi^* W + \begin{pmatrix} 0 & \text{Id}_Q \\ \text{Id}_Q & 0 \end{pmatrix}$$

on $\hat{M}_{H_{3,m,R}}$, which is invertible on $\hat{M}_{H_{3,m,R}} \setminus \mathcal{M}_{H_{3,m,R}}$. 

(2.26)
Recall that \( h(\frac{t}{R^2}) \) vanishes near \( \mathcal{M}_{H_{3m,R}} \cap \partial \mathcal{M}_R \). We extend it to a function on \( \widetilde{N}_{3m,R} \) which equals to zero on \( N_{3m,R} \) and an open neighborhood of \( \partial \mathcal{M}_{H_{3m,R}} \) in \( \widetilde{N}_{3m,R} \), and we denote the resulting function on \( \widetilde{N}_{3m,R} \) by \( \tilde{h}_R \).

Let \( \pi_{\tilde{N}_{3m,R}} : T\tilde{N}_{3m,R} \to \tilde{N}_{3m,R} \) be the projection of the tangent bundle of \( \tilde{N}_{3m,R} \). Let \( \gamma_{\tilde{N}_{3m,R}} \in \text{Hom}(\pi_{\tilde{N}_{3m,R}}^*, \pi_{\tilde{N}_{3m,R}}^*) \) be the symbol defined by

\[
(2.27) \quad \gamma_{\tilde{N}_{3m,R}}(p, u) = \pi_{\tilde{N}_{3m,R}}^* \left( \sqrt{-\tilde{h}_R^2 c_{\beta, \gamma}(u) + \omega'(p)} \right) \quad \text{for} \quad p \in \tilde{N}_{3m,R}, \quad u \in T_p \tilde{N}_{3m,R}.
\]

By (2.26) and (2.27), \( \gamma_{\tilde{N}_{3m,R}} \) is singular only if \( u = 0 \) and \( p \in \mathcal{M}_{H_{3m,R}} \). Thus \( \gamma_{\tilde{N}_{3m,R}} \) is an elliptic symbol.

On the other hand, it is clear that \( \tilde{h}_R D_{\mathcal{E}_{3m,R}}^{\xi_+ \oplus \xi_-} \tilde{h}_R \) is well defined on \( \tilde{N}_{3m,R} \) if we define it to be zero on \( \tilde{N}_{3m,R} \setminus \mathcal{M}_{H_{3m,R}} \).

Let \( A : L^2(\xi) \to L^2(\xi) \) be a second order positive elliptic differential operator on \( N_{m,R} \) preserving the \( \mathbb{Z}_2 \)-grading of \( \xi = \xi_+ \oplus \xi_- \), such that its symbol equals to \( |\eta|^2 \) at \( \eta \in T\tilde{N}_{3m,R} \). As in [13, (2.33)], let \( F_{R, \beta, \gamma}^{\xi_\pm} : L^2(\xi) \to L^2(\xi) \) be the zeroth order pseudodifferential operator on \( \tilde{N}_{3m,R} \) defined by

\[
(2.28) \quad F_{R, \beta, \gamma}^{\xi_\pm} = A^{-\frac{1}{4}} \tilde{h}_R D_{F_{\xi_\pm}^{F_1, \beta, \gamma}} \tilde{h}_R A^{-\frac{1}{4}} + \frac{\psi'}{\beta}.
\]

Let \( F_{R, \beta, \gamma, \beta_+}^{\xi_\pm} : L^2(\xi_+) \to L^2(\xi_-) \) be the obvious restriction. Then the principal symbol of \( F_{R, \beta, \gamma, \beta_+}^{\xi_\pm} \), which we denote by \( \gamma(F_{R, \beta, \gamma, \beta_+}^{\xi_\pm}) \), is homotopic through elliptic symbols to \( \gamma_{\tilde{N}_{3m,R}} \).

Thus \( F_{R, \beta, \gamma, \beta_+}^{\xi_\pm} \) is a Fredholm operator. Moreover, the index of the symbol \( \gamma_{\tilde{N}_{3m,R}} \) can be calculated by the Atiyah-Singer index theorem directly (cf. [2] and [10, Proposition III. 11.24]). Therefore,

\[
(2.29) \quad \text{ind}(F_{R, \beta, \gamma, \beta_+}^{\xi_\pm}) = \text{ind}\left( \gamma(F_{R, \beta, \gamma, \beta_+}^{\xi_\pm}) \right) = \text{ind}\left( \gamma_{\tilde{N}_{3m,R}} \right) = (\tilde{A}(TM)(\text{ch}(E_0) - \text{ch}(E_1)), [\tilde{M}_{H_{3m,R}}]) = \langle \tilde{A}(TM)(\text{ch}(E_0) - \text{ch}(E_1)), [M] \rangle \neq 0,
\]

where the inequality comes from (2.1).

For any \( 0 \leq t \leq 1 \), set

\[
(2.30) \quad F_{R, \beta, \gamma, \beta_+}^{\xi_\pm}(t) = F_{R, \beta, \gamma, \beta_+}^{\xi_\pm} + \frac{(t - 1)\omega'}{\beta} + \frac{1}{2}(1 - t)\omega' A^{-\frac{1}{4}}.
\]

Then \( F_{R, \beta, \gamma, \beta_+}^{\xi_\pm}(t) \) is a smooth family of zeroth order pseudodifferential operators such that the corresponding symbol \( \gamma(F_{R, \beta, \gamma, \beta_+}^{\xi_\pm}(t)) \) is elliptic for \( 0 < t \leq 1 \). Thus \( F_{R, \beta, \gamma, \beta_+}^{\xi_\pm}(t) \) is a continuous family of Fredholm operators for \( 0 < t \leq 1 \) with \( F_{R, \beta, \gamma, \beta_+}^{\xi_\pm}(1) = F_{R, \beta, \gamma, \beta_+}^{\xi_\pm} \).

Then, by using Lemma 2.1, the exactly same arguments in [11, Proposition 2.2] show that for suitable \( \varepsilon, m, R, \beta, \gamma > 0 \),

\[
\dim(\ker(F_{R, \beta, \gamma, \beta_+}^{\xi_\pm}(0))) = \dim(\ker(F_{R, \beta, \gamma, \beta_+}^{\xi_\pm}(0)^*)) = 0.
\]

---

\(^6\)To be more precise, here \( A \) also depends on the defining metric. We omit the corresponding subscript/superscript only for convenience.
As a result, when \( t = 0 \), \( P_{R,\beta,\gamma,+}^{E_{3m}, R}(0) \) is also Fredholm and has a vanishing index. By the property of Fredholm index, we have
\[
\text{ind}(P_{R,\beta,\gamma,+}^{E_{3m}, R}) = \text{ind}(P_{R,\beta,\gamma,+}^{E_{3m}, R}(1)) = \text{ind}(P_{R,\beta,\gamma,+}^{E_{3m}, R}(0)) = 0,
\]
which contradicts to (2.29) and the proof of Theorem 1.2 is completed.

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References

[1] M. F. Atiyah, *K-Theory*. Notes by D. W. Anderson. Second edition. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989. xx+216 pp.

[2] M. F. Atiyah and I. M. Singer, The index of elliptic operators I. Ann. of Math. 87 (1968), 484-530.

[3] M.-T. Benameur and J. L. Heitsch, Enlargeability, foliations, and positive scalar curvature. Invent. Math. 215 (2019), 367–382.

[4] S. Cecchini and R. Zeidler, Scalar and mean curvature comparison via the Dirac operator. Preprint, arXiv: 2103.06833v2.

[5] A. Connes, Cyclic cohomology and the transverse fundamental class of a foliation. In Geometric Methods in Operator Algebras. H. Araki eds., 52–144, Pitman Res. Notes in Math. Series, vol. 123, 1986.

[6] M. Gromov, Positive curvature, macroscopic dimension, spectral gaps and higher signatures. In Functional Analysis on the Eve of the 21st Century, Gindikin, Simon (ed.) et al., 1–213, Progr. Math., vol. 132, Birkhäuser, Basel, 1996.

[7] M. Gromov, 101 questions, problems and conjectures around scalar curvature. Preprint, https://www.ihes.fr/~gromov/positivescalarcurvature/596/598/.

[8] M. Gromov, Four lectures on scalar curvature. Preprint, arXiv: 1908.10612v6.

[9] M. Gromov and H. B. Lawson, Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. Inst. Hautes Études Sci. Publ. Math. 58 (1983), 295–408.

[10] H. B. Lawson and M.-L. Michelsohn, *Spin Geometry*. Princeton Univ. Press, Princeton, NJ, 1989. xii+427 pp.

[11] G. Su, X. Wang and W. Zhang, Nonnegative scalar curvature and area decreasing maps on complete foliated manifolds. Preprint, arXiv: 2104.03472v1.

[12] X. Wang, On a relation between the K-cowait and the \( \hat{A} \)-cowait. Preprint, in preparation.

[13] W. Zhang, Positive scalar curvature on foliations. Ann. of Math. 185 (2017), 1035–1068.

[14] W. Zhang, Positive scalar curvature on foliations: the enlargeability. In Geometric Analysis, in Honor of Gang Tian’s 60th Birthday, 537–544, Progr. Math., vol. 333, Birkhäuser, Cham, 2020.