Outlier Robust and Sparse Estimation of Linear Regression Coefficients

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Abstract

We consider outlier-robust and sparse estimation of linear regression coefficients, when the covariates and the noises are contaminated by adversarial outliers and noises are sampled from a heavy-tailed distribution. Our results present sharper error bounds under weaker assumptions than prior studies that share similar interests with this study. Our analysis relies on some sharp concentration inequalities resulting from generic chaining.

1 Introduction

This study considers outlier-robust and sparse estimation of linear regression coefficients. Consider the following sparse linear regression model:

\[ y_i = x_i^\top \beta^* + \xi_i, \quad i = 1, \cdots, n, \]  

(1.1)

where \( \beta^* \in \mathbb{R}^d \) represents the true coefficient vector with \( s \) nonzero elements, \( \{x_i\}_{i=1}^n \) denotes a sequence of independent and identically distributed (i.i.d.) random covariate vectors, and \( \{\xi_i\}_{i=1}^n \) denotes a sequence of i.i.d. random noises. Throughout the present paper, we assume \( s \geq 1 \) and \( d/s \geq 3 \) for simplicity. There are many studies about estimation problems of \( \beta^* \) [70, 38, 77, 74, 11, 8, 61, 76, 7, 20, 65, 66, 39, 24, 6, 48, 41]. Let \( \|v\|_2 \) denote the \( \ell_2 \) norm for a vector \( v \). Especially, using the method in [6], with probability at least \( 1 - \delta \), we can construct an estimator \( \hat{\beta} \) such that

\[ \|\hat{\beta} - \beta^*\|_2 \lesssim \sqrt{\frac{s \log(d/s)}{n}} + \sqrt{\frac{\log(1/\delta)}{n}}, \]  

(1.2)

where \( \lesssim \) is the inequality up to an absolute constant factor, when, for simplicity, \( \{x_i\}_{i=1}^n \) and \( \{\xi_i\}_{i=1}^n \) are the sequences of i.i.d. random covariate vectors sampled from the multivariate Gaussian distribution with \( \mathbb{E}x_i = 0 \) and \( \mathbb{E}x_i x_i^\top = I \), and random noises sampled from the Gaussian distribution with \( \mathbb{E}\xi_i = 0 \) and \( \mathbb{E}\xi_i^2 = 1 \), respectively.

This paper considers the situation where \( \{x_i, y_i\}_{i=1}^n \) suffers from malicious outliers. We allow an adversary to inject outliers into (1.1), yielding

\[ y_i = X_i^\top \beta^* + \xi_i + \sqrt{n}\theta_i, \quad i = 1, \cdots, n, \]  

(1.3)

where \( X_i = x_i + \varrho_i \) for \( i = 1, \cdots, n \), and \( \{\varrho_i\}_{i=1}^n \) and \( \{\theta_i\}_{i=1}^n \) are outliers. Let \( O \) be the index set of outliers. We assume the following.

Assumption 1.1. Assume that
We note that, under Assumption 1.1, \{x_i\}_{i\in\mathcal{I}} and \{\xi_i\}_{i\in\mathcal{I}} are no longer sequences of i.i.d. random variables because \mathcal{O} is freely chosen by an adversary. This type of contamination by outliers is sometimes called the strong contamination, in contrast to the Huber contamination \cite{25}. The Huber contamination is more manageable to tame than strong contamination because outliers of Huber contamination are not correlated to the inliers and do not destroy the independence of the inliers. We consider a problem to estimate \(\beta^*\) in (1.3), and construct a computationally tractable estimator having a property similar to (1.2).

We briefly review recent developments in robust and computationally tractable estimators. \cite{12} derived optimal error bounds for the estimation of means and covariance (scatter) matrices under the presence of outliers and proposed estimators, which achieves the optimal error bounds. However, the estimators are computationally intractable. Subsequently, \cite{46} and \cite{30} considered tractable estimators for similar problem settings. After \cite{46} and \cite{30}, many outlier-robust tractable estimators have been developed in \cite{26, 45, 28, 15, 16, 37, 60, 23, 50, 22, 4, 31, 32, 3, 49, 18, 29, 55, 33, 34, 35, 69, 56, 14, 21, 17, 47, 19, 58, 54, 53}. These studies treated various estimating problems; e.g., estimation of mean, covariance, linear regression coefficients, half-spaces, parameters of Gaussian mixture models. Their primary interests are deriving sharp error bounds, deriving information-theoretical optimal error bounds, and reducing computational complexity. However, there are few studies on combining outlier-robust properties with sparsity \cite{14, 4, 31, 21, 49, 18, 42, 47, 19, 63, 54, 53}. Especially, \cite{14}, \cite{4} and \cite{49} dealt with the estimation problem of \(\beta^*\) from (1.3) under the assumption of Gaussian and subGaussian tails of \{x_i, \xi_i\}_{i=1}^n, with computationally tractable estimators. Our study can be considered as an extension of these prior studies from two perspectives: sharpening the error bound and relaxing the assumption with improved analyses. The overview of the analyses is described in Section 3.2.

The present paper is organized as follows. In Section 2, we present our main results of the estimation error in rough statements and describe some relationships to previous studies. In Sections 3 and 4, we describe our estimation methods and main results, without proofs. In Section 5, we describe the key propositions, lemma and corollary, without proofs. In Section 6, we provide proofs of a part of the propositions in Sections 3–5. In the Appendix, we provide the proofs that are omitted in Sections 3–6. In the remainder of this paper, we assume that Assumption 1.1 holds for outliers, and for simplicity, \(0 < \delta \leq 1/4\).

2 Our results and relationship to previous studies

2.1 Our results

In Section 2.1, we state our results. Before that, we introduce some definitions. First, we introduce the \(\psi_\alpha\)-norm and \(\mathcal{L}\)-subGaussian random vector, which is an extension of subGaussian random variables in a high dimension.

**Definition 2.1 (\(\psi_\alpha\)-norm).** For a random variable \(f\), let
\[
\|f\|_{\psi_\alpha} := \inf \{\eta > 0 : \mathbb{E}\exp(|f/\eta|^\alpha) \leq 2\} < \infty.
\] (2.1)

**Definition 2.2 (\(\mathcal{L}\)-subGaussian random vector).** A random vector \(x \in \mathbb{R}^d\) with mean \(\mathbb{E}x = 0\) is said to be an \(\mathcal{L}\)-subGaussian random vector if for any fixed \(v \in \mathbb{R}^d\),
\[
\|\langle x, v \rangle\|_{\psi_2} \leq \mathcal{L} \left(\mathbb{E}|\langle x, v \rangle|^2\right)^{\frac{1}{2}},
\] (2.2)
where the norm \(\|\cdot\|_{\psi_2}\) is defined in Definition 2.1 and \(\mathcal{L}\) is a numerical constant such that \(\mathcal{L} \geq 1\).
We note that, for example, the multivariate standard Gaussian random vector is an $\mathcal{S}$-subGaussian random vector. Second, we introduce the restricted eigenvalue condition for $\Sigma$ [10]. For a vector $v \in \mathbb{R}^d$, define $v_i$ as the $i$-th element of $v$, and define the $\ell_1$ norm of $v$ as $\|v\|_1$. For a vector $v \in \mathbb{R}^d$ and set $J$, define $v_J$ as a vector such that $v_J|_i = v_i$ for $i \in J$ and $v_i = 0$ for $i \notin J$. For a set $J$, define $J^c$ as a complement set of $J$. Additionally, for a vector $v$, define the number of non-zero elements of $v$ as $\|v\|_0$. For a set $S$, let $|S|$ be the number of the elements of $S$. Let $o = |O|$.

**Definition 2.3** (Restricted eigenvalue condition for $\Sigma$). The covariance matrix $\Sigma$ is said to satisfy the restricted eigenvalue condition $\text{RE}(s, c_{\text{RE}}, \tau)$ with some positive constants $c_{\text{RE}}, \tau$, if $\|\Sigma^\top v\|_2 \geq \tau \|v\|_2$ for any vector $v \in \mathbb{R}^p$ and any set $J$ such that $|J| \leq s$ and $\|v_{J^c}\|_1 \leq c_{\text{RE}} \|v_J\|_1$.

For simplicity, we redefine $\tau = \inf_{v \in \mathbb{R}^d} \frac{\|\Sigma^\top v\|_2}{\|v\|_2}$ for any vector $v \in \mathbb{R}^p$. Lastly, we introduce the following two quantities related to minimum/maximum eigenvalue:

$$\kappa_1 = \inf_{\|v\|_0 \leq s} \frac{\|\Sigma^\top v\|_2}{\|v\|_2}, \quad \kappa_2 = \sup_{\|v\|_0 \leq 2s^2} \frac{\|\Sigma^\top v\|_2}{\|v\|_2}. \tag{2.3}$$

We note that, from the definition, we see that the minimum eigenvalue of $\Sigma$ is smaller than $\kappa_1$. Define $\rho = \max_{i \in \{1, \ldots, d\}} \sqrt{\Sigma_{ii}}$ and the maximum eigenvalue of $\Sigma$ as $\Sigma_{\text{max}}$. Then, we use the following assumption for the sequence of the covariates $\{x_i\}_{i=1}^n$. We make the following assumption on $\{x_i, \xi_i\}_{i=1}^n$:

**Assumption 2.1.** Assume that

(i) $\{x_i\}_{i=1}^n$ is a sequence of i.i.d. random vectors sampled from an $\mathcal{S}$-subGaussian random vector with $\mathbb{E}x_i = 0, \mathbb{E}^2x_i = \Sigma$, $\rho \geq 1$ and $\kappa_1 > 0$. Assume that $\Sigma$ satisfies $\text{RE}(s, c_{\text{RE}}, \tau)$ with $c_{\text{RE}} > 1$ and $\tau \leq 1$;

(ii) $\{\xi_i\}_{i=1}^n$ is a sequence of i.i.d. random variables with $\mathbb{E}\xi_i^2 \leq \sigma^2$;

(iii) $\{x_i\}_{i=1}^n$ and $\{x_i, \xi_i\}_{i=1}^n$ are independent.

Define

$$R_{\text{lasso}} = \sigma \left( \frac{\rho}{\tau} \sqrt{\frac{s \log(d/s)}{n}} + \sqrt{\frac{\log(1/\delta)}{n}} \right),$$

$$R_{\text{outlier}} = \sigma \frac{\kappa_2}{\kappa_1} \left( \sqrt{\frac{\rho}{n}} \sqrt{s} \sqrt{\frac{\log(d/s)}{n}} + \sqrt{\frac{\log(1/\delta)}{n}} + \frac{\rho}{n} \sqrt{\frac{\log(n/\rho)}{n}} \right),$$

$$R_{\text{outlier}}' = \sigma \sqrt{\frac{\rho}{n}} \times \left( \frac{\kappa_2}{\kappa_1} \sqrt{s} \sqrt{\frac{\log(d/s)}{n}} + \sqrt{\frac{\log(1/\delta)}{n}} + \frac{\rho}{n} \sqrt{\frac{\log(n/\rho)}{n}} \right). \tag{2.4}$$

Our first result is as follows (for a precise statement, see Theorem 3.1 in Section 3.3).

**Theorem 2.1.** Suppose that Assumption 2.1 holds, and $\Sigma$ is allowed to be used in estimation. Define $C_{\text{cre}}$ is a constant that depends on $c_{\text{RE}}$. Then, with probability at least $1 - 4\delta$, we can construct an estimator $\hat{\beta}$ such that

$$\|\Sigma^\top (\hat{\beta} - \beta^*)\|_2 \leq C_{\text{cre}} \|\Sigma\| \left( R_{\text{lasso}} + R_{\text{outlier}} \right), \tag{2.5}$$

with a computationally tractable method, when $R_{\text{lasso}}$ and $R_{\text{outlier}}$ are sufficiently small.

Our second result is as follows (for a precise statement, see Theorem 4.1 in Section 4).
Theorem 2.2. Suppose that Assumption 2.1 holds, and $\Sigma$ is not allowed to be used in estimation. Define $C_{\text{cre}}'$ is a constant that depends on $\text{cre}$. Then, with probability at least $1 - 3\delta$, we can construct an estimator $\hat{\beta}$ such that

$$\|\Sigma^\frac{1}{2} (\hat{\beta} - \beta^*)\|_2 \leq C_{\text{cre}}' \delta^\frac{1}{2} (R_{\text{lasso}} + R_{\text{outlier}}'),$$

with a computationally tractable method, when $R_{\text{lasso}}$ and $R_{\text{outlier}}'$ are sufficiently small.

From (2.5) and (2.6), we see that the error bounds of our estimators match the ones of the normal lasso, up to $\mathcal{O}$ and numerical constant factors, when there is no outliers because $R_{\text{lasso}}$ is equivalent to the upper bound of (1.2) up to constant factors and $R_{\text{outlier}} = 0$ when there is no outliers. We see that $R_{\text{outlier}}'$ is larger than $R_{\text{outlier}}$. From the fact that $R_{\text{outlier}}'$ is larger than $R_{\text{outlier}}$, we see that there is a deterioration in error bounds, as a trade-off for not utilizing $\Sigma$ in estimation.

Remark 2.1. Conditions (ii) in Assumption 2.1 are provided to make the results simple, and condition (ii) can be weakens to a tail probability condition. For detail, see Assumption 3.1.

In Theorems 3.1 and 4.1, we explicitly describe the relationship between the tuning parameters used for estimation and the obtained error bounds in Theorems 2.1 and 2.2, respectively. Additionally, in Theorems 3.1 and 4.1, we derive error bounds not only about $\|\Sigma^\frac{1}{2}(\cdot)\|_2$ but also $\|\cdot\|_2$ and $\|\cdot\|_1$.

2.2 Relationship to previous studies

As we mentioned in Section 1, [14, 4, 49] dealt with the estimation problem of $\beta^*$ from (1.3) under Assumption 1.1 with computationally tractable estimators.

We note that [47, 19, 42] dealt with the estimation problem of $\beta^*$ from (1.3) under a stronger assumption about outlier than Assumption 1.1, with computationally intractable estimators deriving sharp error bounds. Additionally, we note that [63] dealt with a situation where $\{x_i\}_{i=1}^n$ is a sequence of i.i.d. random vectors sampled from a heavy-tailed distribution, and [53] dealt with more challenging situation weakening the assumptions for covariates than that of [63]. Their error bounds are looser than the results of the present paper because the weak assumptions restrict the techniques available. Therefore, these papers [47, 19, 42, 63, 53] treat computationally intractable estimators or suppose more weaker assumptions than our method, and hence we do not mention such papers further because the interests of such papers are different from that of our paper.

Therefore, we mainly discuss the results of [14, 4, 49] in the remainder of Section 2.2.

2.2.1 Case where $\Sigma$ is allowed to be used in estimation

The result of [4] and part of the result of [49] use $\Sigma$ in their estimation. [4] and [49] considered situations where the covariate vectors are sampled from the standard multivariate Gaussian distribution and the noises are sampled from a Gaussian distribution with mean 0 and variance $\sigma^2$. In contrast, our method works well for the case where the covariate vectors are sampled from an $\mathcal{L}$-subGaussian random variable with covariance which satisfies the restricted eigenvalue condition and the noises are sampled from a heavy-tailed distribution. [4] and [49] only considered the case where $o/n = \epsilon$. The $\ell_2$-norm error bound of [4] is

$$\leq \sigma \left( \sqrt{1 + \|\beta^*\|_2^2} \log^2 \frac{n}{\sigma} \right)$$

when $\frac{\sigma}{\epsilon} \log d + \frac{\sigma}{\epsilon} \log(1/\delta) \leq \sigma n$, where $\leq_o$ is the inequality up to an absolute constant factor and the standard deviation of the random noise $\sigma$. The $\ell_2$-norm error bound of [49] is

$$\leq \sigma \left( \frac{\epsilon}{o} \log \frac{1}{\delta} \right)$$

when $\frac{\sigma}{\epsilon} \log(d/s) + \frac{\sigma}{\epsilon} \log(1/\delta) \leq n$. We see that our result does not depend on $\beta^*$ and the error bound is sharper than the ones of [4] and [49]. Additionally, our sample complexity is smaller than the ones of [4] and [49] because, in our sample complexity, the term such that $\frac{\epsilon}{o} \times \log(1/\delta)$ does not appear.
We consider the optimality of the error bound in (2.5). From Theorem D.3 of [17], we see that the error bound can not avoid a term such that \( \text{constant} \times \sigma \sqrt{n \log \frac{m}{n}} \) even when \( d = 1 \). Detailed investigation of the influence of \( \Sigma \) in high dimension on information theoretical estimation limit is a task for future research.

When there is no outlier (\( o = 0 \)), our error bound coincides with the one of the normal lasso, up to numerical and \( L \) factors. The results in [4] and [49] do not have this property.

2.2.2 Case where \( \Sigma \) is not allowed to be used in estimation

[14], and a part of the result of [49] gives error bounds with tractable methods and do not require \( \Sigma \) in estimation. However, the method in [49] assumes a sparse structure of \( \Sigma \), and the sample complexity depends on not only \( s^2 \) but also the sparsity of \( \Sigma \). [14] propose some methods, however, the term in their error bounds containing \( o \) depends on \( \log d \) and \( s \) even when \( s^2 \log(d/s) \lesssim n \).

When there is no outlier (\( o = 0 \)), similarly to the case where \( \Sigma \) is allowed to be used in estimation, our error bound coincides with the one of the normal lasso, up to numerical and \( L \) factor. The results in [49] and [14] do not have this property.

2.2.3 Remaining problem

Our estimator and estimators in [4] and [49] require \( n \) to be proportional to \( s^2 \), which is not needed to derive (1.2) from 1.1. Similar phenomena can be observed in [72, 41, 49, 4, 31]. Some relationships between computational tractability and similar quadratic dependencies are unraveled [72, 32, 27]. We leave the analysis in our situation for future work. We note that [56, 21, 18, 69, 54] considered a simpler case in which only noise is contaminated by outliers (\( q_i = 0 \) for any \( i \in \{1, \cdots, d\} \)), with computationally tractable estimators, and the error bounds and sample complexities depend on \( s \), not \( s^2 \). In actual applications, it is necessary to consider an appropriate method after carefully clarifying the nature of data.

3 Method and result

Assume that \( x \) is a random vector drawn from the same distribution of \( \{x_i\}_{i=1}^n \). Hereafter, we often use the following simple notations to express error orders:

\[
\begin{align*}
    r_{d,s} &= \sqrt{\frac{\log(d/s)}{n}}, & r_{\delta} &= \sqrt{\frac{\log(1/\delta)}{n}}, & r_o &= \frac{o}{n} \sqrt{\frac{\log n}{o}}, & r_o' &= \frac{o}{\log o}.
\end{align*}
\] (3.1)

3.1 Some properties of \( \mathcal{L} \)-subGaussian random vector

We show some additional properties of \( \mathcal{L} \)-subGaussian random vector \( x \). We note that, from (2.2), we have

\[
\| \langle x, v \rangle \|_{\psi_2} \leq \mathcal{L} (\mathbb{E}|\langle x, v \rangle|^2)^{\frac{1}{2}} \leq \mathcal{L}\| \Sigma^{\frac{1}{2}} v \|_2,
\] (3.2)

and from (2.14) - (2.16) of [71], for any \( v \in \mathbb{R}^d \) and \( t \geq 0 \), we have

\[
\| v^\top x \|_{L_p} := \left\{ \mathbb{E}|v^\top x|^p \right\}^{\frac{1}{p}} \leq c_p \|v^\top x\|_{\psi_2} \leq c_p \|v\|_2 \|\Sigma^{\frac{1}{2}} v\|_2,
\] (3.3)

\[
\mathbb{E}\exp(v^\top x) \leq \exp(c_p \Sigma^{\frac{1}{2}} \|\Sigma^{\frac{1}{2}} v\|_2^2),
\] (3.4)

\[
\mathbb{E} \exp \left( \frac{(v^\top x)^2}{c_p^2 \Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2}} v\|_2^2} \right) \leq 2,
\] (3.5)

\[
P \left( |v^\top x| > t \right) \leq 2 \exp \left( -\frac{t^2}{c_p^2 \Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2}} v\|_2^2} \right),
\] (3.6)

where \( c_p \) is a numerical constant. Define \( L = \mathcal{L} \times \max\{1, c_p\} \).
3.2 Method

To estimate $\beta^*$ in (1.3), we propose the outlier-robust and sparse estimation (Algorithm 1). Algorithm 1 is similar to the ones used in [58] and [63]. However, [58] considered non-sparse case, and [63] considered heavy-tailed covariates. Therefore, to consider the sparsity of $\beta^*$ or to derive a sharper error bound than that in [63] taking advantage $\mathcal{L}$-subGaussian assumption, our analysis is more involved than [58] and [63]. Concretely, unlike the previous studies, extensions of Hanson–Wright inequalities, which appear later in Proposition 5.1 and Corollary 5.1 proved via generic chaining, play important roles. We will analyze an $\ell_1$-penalized Huber loss in step 3 in Algorithm 1. Our analysis of the $\ell_1$-penalized Huber loss is similar to the ones in [2] and [48], however, the analyses of [2] and [48] are mainly interested in the case $\mathbb{E}x^t = I$, and not very effective for a more general covariance. We modify their analysis and our analysis is effective for a more general covariance. In particular, Proposition 5.2 is important and a modified analysis method is described in Appendices C and D.

**Algorithm 1 Outlier-robust and sparse estimation**

**Input:** $\{y_i, X_i\}_{i=1}^n, \Sigma(= \mathbb{E}xx^t)$ and tuning parameters $\tau_{cut}, \varepsilon, r_1, r_2, \lambda_0, \lambda_s$

**Output:** $\hat{\beta}$

1. $\{\hat{w}_i\}_{i=1}^n \leftarrow$ WEIGHT($\{X_i\}_{i=1}^n, \tau_{cut}, \varepsilon, r_1, r_2, \Sigma$)
2. $\{\hat{w}'_i\}_{i=1}^n \leftarrow$ TRUNCATION($\{\hat{w}_i\}_{i=1}^n$)
3. $\hat{\beta} \leftarrow$ WEIGHTED-PENALIZED-HUBER-REGRESSION($\{y_i, X_i\}_{i=1}^n, \{\hat{w}'_i\}_{i=1}^n, \lambda_0, \lambda_s$)

Here we give simple explanations of the output steps. The details are provided in Sections 3.2.1, 3.2.2, and 3.2.3. The first step produces the weights $\{\hat{w}_i\}_{i=1}^n$ reducing adverse effects of covariate outliers. The second step is the truncation of the provided weights to zero or $1/n$, say $\{\hat{w}'_i\}_{i=1}^n$. The third step is the $\ell_1$-penalized Huber regression based on the weighted errors using the truncated weights $\{\hat{w}'_i\}_{i=1}^n$. The $\ell_1$-penalization addresses the high dimensional setting, and the Huber regression weakens the adverse effects of the response outliers.

3.2.1 WEIGHT

For a matrix $M = (m_{ij})_{1 \leq i \leq d_1, 1 \leq j \leq d_2} \in \mathbb{R}^{d_1 \times d_2}$, we define

$$\|M\|_1 = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} |m_{ij}|. \quad (3.7)$$

For a symmetric matrix $M$, we write $M \succeq 0$ if $M$ is positive semi-definite. Define $\text{Tr}(M)$ for a squared matrix $M$ as the trace of $M$. Define the following convex set:

$$\mathcal{M}_{r_1, r_2, d}^{\text{Tr}} = \{M \in \mathcal{S}(d) \mid \|M\|_1 \leq r_1^2, \text{Tr}(M) \leq r_2^2, \text{M} \succeq 0\}, \quad (3.8)$$

where $\mathcal{S}(d)$ is a set of symmetric matrices on $\mathbb{R}^d \times \mathbb{R}^d$. For a vector $\mathbf{v}$, we define the $\ell_\infty$ norm of $\mathbf{v}$ as $\|\mathbf{v}\|_\infty$ and define the probability simplex $\Delta_n^{-1}(\varepsilon)$ with $0 < \varepsilon < 1$ as follows:

$$\Delta_n^{-1}(\varepsilon) = \left\{ \mathbf{w} \in [0, 1]^n \mid \sum_{i=1}^n w_i = 1, \|\mathbf{w}\|_\infty \leq \frac{1}{n(1-\varepsilon)} \right\}. \quad (3.9)$$

The first step of Algorithm 1, WEIGHT, is stated as follows.
Algorithm 2 WEIGHT

Input: data \( \{X_i\}_{i=1}^n \), tuning parameters \( \tau_{\text{cut}}, \varepsilon, r_1, r_2 \).
Output: weight estimate \( \hat{w} = \{\hat{w}_1, \cdots, \hat{w}_n\} \).

Let \( \hat{w} \) be the solution to

\[
\min_{w \in \Delta^{n-1}(\varepsilon)} \max_{M \in \mathcal{W}_{r_1, r_2}} \sum_{i=1}^n w_i \langle X_i X_i^\top - \Sigma, M \rangle \tag{3.10}
\]

if the optimal value of (3.10) \( \leq \tau_{\text{cut}} \)

return \( \hat{w} \)
else
return \text{fail}

Algorithm 2 is a special case of Algorithm 3 of [4]. Therefore, as in Algorithm 3 of [4], Algorithm 2 can also be computed efficiently. An intuitive meaning of (3.10) is given in Section 3.2.4. For the detail of the value of \( \tau_{\text{cut}} \) and its validity, see Theorem 3.1 and Proposition 5.1, respectively.

3.2.2 TRUNCATION

The second step in Algorithm 1 is the discretized truncation of \( \{\hat{w}_i\}_{i=1}^n \), say \( \{\hat{w}_i'\}_{i=1}^n \), as in Algorithm 3. The discretized truncation, which makes it easy to analyze the estimator.

Algorithm 3 TRUNCATION

Input: weight vector \( \hat{w} = \{\hat{w}_i\}_{i=1}^n \).
Output: truncated weight vector \( \hat{w}' = \{\hat{w}_i'\}_{i=1}^n \).

For \( i = 1 : n \)

if \( \hat{w}_i \geq \frac{1}{n} \)

\( \hat{w}_i' = \frac{1}{n} \)
else

\( \hat{w}_i' = 0 \)

return \( \hat{w}' \).

3.2.3 WEIGHTED-PENALIZED-HUBER-REGRESSION

The Huber loss function \( H(t) \) is defined as follows:

\[
H(t) = \begin{cases} 
|t| - 1/2 & (|t| > 1) \\
\ell^2/2 & (|t| \leq 1)
\end{cases},
\]

and let

\[
h(t) = \frac{d}{dt} H(t) = \begin{cases} 
\text{sgn}(t) & (|t| > 1) \\
\frac{t}{2} & (|t| \leq 1)
\end{cases}.
\]

We consider the \( \ell_1 \)-penalized Huber regression with the weighted samples \( \{\hat{w}_i'y_i, \hat{w}_i'X_i\}_{i=1}^n \) in Algorithm 4. This is the third step in Algorithm 1.
Algorithm 4 WEIGHTED-PENALIZED-HUBER-REGRESSION

Input: data \( \{y_i, X_i\}_{i=1}^{n} \), truncated weight vector \( \hat{w}^t = \{\hat{w}_i^t\}_{i=1}^{n} \) and tuning parameters \( \lambda_o, \lambda_s \).

Output: estimator \( \hat{\beta} \).

\[
\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^d} \sum_{i=1}^{n} \lambda^2 H \left( \frac{\hat{w}_i^t y_i - X_i^\top \beta}{\lambda_o \sqrt{n}} \right) + \lambda_s \|\beta\|_1, \tag{3.13}
\]

return \( \hat{\beta} \).

Several studies, such as \([56, 64, 21, 67, 13, 18, 58, 63]\), have suggested that the Huber loss is effective for linear regression under heavy-tailed noise or the existence of outliers.

Lastly, we introduce the assumption on \( \{\xi_i\}_{i=1}^{n} \):

**Assumption 3.1.** (i) \( \{\xi_i\}_{i=1}^{n} \) is a sequence of i.i.d. random variables such that

\[
P \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \geq \frac{1}{2} \right) \leq \frac{1}{144L^4}, \tag{3.14}
\]

(ii) \( \mathbb{E} h \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right) \times x_i = 0. \)

**Remark 3.1.** For example, when \( \mathbb{E} \xi_i^2 \leq \sigma^2 \), from Markov’s inequality, we have

\[
P \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \geq \frac{1}{2} \right) \leq \frac{4}{\lambda_o^2 n} \mathbb{E} \xi_i^2 \leq \frac{4\sigma^2}{\lambda_o^2 n}. \tag{3.15}
\]

Therefore, to satisfy (3.14), it is sufficient to set

\[24L^2 \sigma \leq \lambda_o \sqrt{n}. \tag{3.16}\]

In this case, Condition (i) in Assumption 3.1 is weaker than \( \mathbb{E} \xi_i^2 \leq \sigma^2 \).

**Remark 3.2.** Condition (ii) in Assumption 3.1 is weaker than the independence between \( \{\xi_i\}_{i=1}^{n} \) and \( \{x_i\}_{i=1}^{n} \).

### 3.2.4 An intuitive meaning of Algorithm 2

For \( l = 0, 1, 2 \), we define \( d \)-dimensional \( \ell_l \)-ball with radius \( a \) as \( aB^d_l = \{v \in \mathbb{R}^d \mid \|v\|_l \leq a\} \), and for \( l = \Sigma \), we define \( aB^d_{\Sigma} = \{v \in \mathbb{R}^d \mid \|\Sigma^{1/2}v\|_2 \leq a\} \). We explain an intuitive meaning of WEIGHT. \([69]\) and \([54]\) considered the case where covariates are not contaminated by outliers, namely \( g_i = 0 \) for all \( i \in \{1, \cdots, n\} \). Proposition 4 of \([69]\) or Theorem 2.4 of \([54]\) played an important role in the theoretical analysis. Proposition 4 of \([69]\) or Theorem 2.4 of \([54]\) implies that, for any \( v \in r_1B^d_1 \cap r_\Sigma B^d_{\Sigma} \) with appropriate values of \( r_1, r_\Sigma \), and for any \( (u_1, \cdots, u_n) \in \mathbb{R}^n \) such that \( u_i = 0 \) for \( i \in \mathcal{I} \) and \( \|u\|_\infty \leq 2 \), we have

\[
P \left\{ \sum_{i=1}^{n} v^\top x_i u_i \lesssim \rho r_{d,s} r_1 + (\sqrt{r_{d,s}} + r_3 + r_o) r_\Sigma \right\} \geq 1 - \delta. \tag{3.17}
\]

We expect a property similar to (3.17) even when the covariates are contaminated by outliers, more precisely, when \( \{x_i\}_{i=1}^{n} \) is replaced by \( \{X_i\}_{i=1}^{n} \). In this case, the result (3.17) does not hold as it is, but we can have a similar property for a weighted type \( \sum_{i=1}^{n} v^\top w_i X_i u_i \) with a devised weight defined in Algorithm 2. To obtain a property similar to (3.17), \( \sum_{i=1}^{n} v^\top w_i X_i u_i \) must be
sufficiently small. Suppose \( w \in \Delta^{n-1}(\varepsilon) \). We see that, for any \( v \in r_1B_1^d \cap r_2B_2^d \), from Hölder’s inequality,
\[
\left( \sum_{i=1}^{n} v^\top w_i X_i u_i \right)^2 \leq \sum_{i \in \mathcal{O}} w_i u_i^2 \sum_{i \in \mathcal{O}} w_i (X_i^\top v)^2 \\
\leq \frac{4}{1 - \varepsilon} \frac{o}{n} \sum_{i \in \mathcal{O}} w_i (X_i^\top X_i^\top - \Sigma, vv^\top) \\
\leq \frac{4}{1 - \varepsilon} \left( \frac{o}{n} \sum_{i \in \mathcal{O}} w_i (X_i X_i^\top - \Sigma, vv^\top) + \frac{o^2}{n^2 (1 - \varepsilon)} (\Sigma, vv^\top) \right),
\]
where (a) follows from \( w_i \leq 1/(n(1 - \varepsilon)) \). Evaluation of \( \sum_{i \in \mathcal{O}} w_i (X_i X_i^\top - \Sigma, vv^\top) \) is difficult because it contains outlier. However, we see that it is sufficient to evaluate
\[
\sum_{i=1}^{n} w_i (X_i X_i^\top - \Sigma, vv^\top)
\]
in the proof of our results. Therefore, to make \( \sum_{i=1}^{n} v^\top w_i X_i u_i \) sufficiently small, we want to minimize (3.19) in \( w \in \Delta^{n-1}(\varepsilon) \) for any \( v \in r_1B_1^d \cap r_2B_2^d \), in other words, we want to consider
\[
\min_{w \in \Delta^{n-1}(\varepsilon)} \max_{v \in r_1B_1^d \cap r_2B_2^d} \sum_{i=1}^{n} w_i (X_i X_i^\top - \Sigma, vv^\top).
\]
A convex relaxation of (3.20) is (3.10), which is an essential part in Algorithm 2. We can see in Proposition 5.1 that the optimization (3.10) is enough to have a property similar to (3.17) for a weighted type \( \sum_{i=1}^{n} v^\top w_i X_i u_i \).

3.2.5 Approaches of the the previous studies

In this section, we explain the approaches of the previous studies [4] and [49]. To estimate \( \beta^* \) in (1.3), [4] used sparse and outlier-robust mean estimation on \( y_i X_i \), directly. [4] assumes that \( \{x_i\}_{i=1}^{n} \) and \( \{\xi_i\}_{i=1}^{n} \) are the sequences of i.i.d. random vectors sampled from the multivariate standard Gaussian distribution and random variables sampled from the standard Gaussian, respectively, and \( \{x_i\}_{i=1}^{n} \) and \( \{\xi_i\}_{i=1}^{n} \) are independent. Then, we have
\[
E y_i x_i = \beta^*, \quad V(y_i x_i)(y_i x_i)^\top = (\|\beta^*\|^2_2 + 1) I + \beta^* \beta^*^\top,
\]
where we use Isserlis’ theorem, which is a formula for the multivariate Gaussian distribution. From Remark 2.2 of [12], we see that the error bound of any outlier-robust estimator for mean vector can not avoid the effect of the operator norm of the covariance \( V(y_i x_i)(y_i x_i)^\top \). Consequently, the approach of [4] can not remove \( \|\beta^*\|^2_2 \) from their error bound. Additionally, we note that [49] proposed a gradient method that repeatedly updates the estimator of \( \beta^* \). Define the \( t \)-step of the update of the estimator of \( \beta^* \) as \( \beta^t \). The gradient for the next update is based on the result of an outlier-robust estimation of \( X_i(\beta^t Y_i - y_i) \). Therefore, for reason similar to the one of [4], the sample complexity of [49] is affected by \( \|\beta^*\|^2_2 \). On the other hand, our estimator is based on the \( \ell_1 \)-penalized Huber regression, and we can avoid this problem.

3.3 Result

We state our main theorem. Define
\[
R_{d,n,o} = \rho c_{r_1} \sqrt{s r_{d,s}} + r_{\delta} + c_{r_2} r_{a} \left( \sqrt{\frac{o}{n}} (s r_{d,s} + r_{\delta}) + r_{o} \right).
\]


Theorem 3.1. Suppose that (i) and (iii) of Assumption 2.1 and Assumption 3.1 hold. Suppose that the parameters \( \lambda_o, \lambda_s, \varepsilon, \tau_{cut}, r_1, r_2, r_{\Sigma} \) satisfy

\[
1 \geq 7c_o (4+c_s) c_{\text{max}}^2 \sqrt{1 + \log LL^2 R_{d,n,o}},
\]

\[
\lambda_s = c_s c_{\text{max}}^2 L \lambda_o \sqrt{n} \frac{1}{c_r \sqrt{s}} R_{d,n,o}, \quad \varepsilon = c_o \frac{\alpha}{n}, \quad \tau_{cut} = c_{cut}(L \kappa_1)^2 (s r_{d,s} + r_\delta + r_{\Sigma}) \frac{\varepsilon^2}{4},
\]

\[
r_1 = c_r \sqrt{s} r_{\Sigma}, \quad r_2 = c_r r_{\Sigma}, \quad r_{\Sigma} = 7 (4+c_s) c_{\text{max}}^2 L \lambda_o \sqrt{n} R_{d,n,o},
\]

where \( c_o, c_s, c_{cut}, c_r, c_{\Sigma} \), and \( c_{\text{max}} \) are sufficiently large numerical constants such that \( c_o \geq 4, c_s \geq 3(c_{\text{RE}} + 1)/(c_{\text{RE}} - 1), 2 > c_\varepsilon \geq 1, c_{\text{cut}} \geq c_2, c_r = c_{\text{num}}(1 + c_{\text{RE}})/\kappa_1, c_r = c_{\text{num}}(3 + c_{\text{RE}})/\kappa_1, \) \( c_r \geq 2 \) and \( c_{\text{num}}^2 / c_{\text{cut}} \leq 1 \). In Proposition 5.1 and Definition 5.1, \( c_2 \) and \( c_{\text{max}} \) are defined, respectively. Suppose that \( \max\{\sqrt{\Sigma} r_{\delta}, s r_{d,s}\} \leq 1 \) and \( 0 < o/n \leq 1/(5c) \) hold. Then, the optimal solution \( \hat{\beta} \) satisfies the following:

\[
\| \Sigma^* (\hat{\beta} - \beta^*) \|_2 \leq r_{\Sigma}, \quad \| \hat{\beta} - \beta^* \|_2 \leq r_2 \quad \text{and} \quad \| \hat{\beta} - \beta^* \|_1 \leq r_1,
\]

with probability at least \( 1 - \delta \).

We note that the conditions (3.23) and (3.25) in Theorem 3.1 imply

\[
\lambda_o \sqrt{n} \geq c_o L r_{\Sigma} \sqrt{1 + \log L}.
\]

Remark 3.3. We consider the results of (3.26) in details. Assume \( \mathbb{E} \xi_i^2 \leq \sigma^2 \) and the equality of (3.16) hold. Define \( C_{\text{RE},1}, C_{\text{RE},2} \) and \( C_{\text{RE},3} \) are constants depending on \( c_{\text{RE}} \). Then, we have

\[
\| \Sigma^* (\hat{\beta} - \beta^*) \|_2 \leq C_{\text{RE},1} \sigma^3 (R_{\text{lasso}} + R_{\text{outlier}}),
\]

\[
\| \hat{\beta} - \beta^* \|_2 \leq C_{\text{RE},2} \sigma^3 \frac{1}{r_1} (R_{\text{lasso}} + R_{\text{outlier}}),
\]

\[
\| \hat{\beta} - \beta^* \|_1 \leq C_{\text{RE},3} \sigma^3 \frac{1}{r} \sqrt{s} (R_{\text{lasso}} + R_{\text{outlier}}),
\]

and we see that (3.28) recovers (2.5).

4 Estimator without the covariance

In Theorem 3.1, we use the covariance of the covariates when we construct an estimator. On the other hand, especially in practical terms, the use of the covariance would be unfavorable. When we do not use the covariance in estimation, we need to modify algorithm WEIGHT as follows:

Algorithm 5 WEIGHT WITHOUT COVARIANCE

Input: data \( \{X_i\}_{i=1}^n \), tuning parameters \( \tau_{cut}, \varepsilon, r_1, r_2 \).

Output: weight estimate \( \hat{w} = \{\hat{w}_1, \ldots, \hat{w}_n\} \).

Let \( \hat{w} \) be the solution to

\[
\min_{w \in \mathcal{D}^{n-1}(\varepsilon)} \max_{M \in M_{\varepsilon, r_1, r_2, d}} \sum_{i=1}^n w_i (X_i, X_i^\top, M)
\]

if the optimal value of (3.10) \( \leq \tau_{cut} \)

return \( \hat{w} \)

else

return \( \text{fail} \)

In WEIGHT WITHOUT COVARIANCE, it is necessary to set the value of \( \tau_{cut} \) larger than that in WEIGHT. For detail, see Corollary 5.1. Then, Theorem 3.1 is changed as follows. Define

\[
R_{d,n,o}' = \rho c_r \sqrt{s} r_{d,s} + r_\delta + c_2 \sqrt{\frac{o}{n} \kappa_2^2 (s r_{d,s} + r_\delta) + \Gamma_{\text{max}}^2 r_2}.
\]
Theorem 4.1. Suppose that (i) and (iii) of Assumption 2.1 and Assumption 3.1 hold. Suppose that the parameters $\lambda_0, \lambda_s, \varepsilon, \tau_{\text{cut}}, r_1, r_2, \Sigma$ satisfy

$$1 \geq \tau_{\text{cut}} (4 + c_s) c_s^2 \sqrt{1 + \log LL^2 R'_{d,n,o}},$$

$$\lambda_s = c_s c_s^2 \max L_{\lambda_0} \frac{1}{c_{r_1} \sqrt{8}} R'_{d,n,o}, \quad \varepsilon = c_o \frac{c_o}{n}, \quad \tau_{\text{cut}} = c_{\text{cut}} ((L\kappa_u)^2 (s\tau_{d,s} + r_\delta) + \Sigma_{\text{max}}^2) r_2^2,$$

$$r_1 = c_{r_1} \sqrt{s} r_{\Sigma}, \quad r_2 = c_{r_2} r_{\Sigma}, \quad r_{\Sigma} = 7 (4 + c_s) c_{\text{max}} \sqrt{L_{\lambda_0}} \sqrt{n} R'_{d,n,o},$$

where $c_o, c_s, c_\varepsilon, c_{\text{cut}}, c_{r_1}, c_{r_2},$ and $c_{\max}$ are sufficiently large numerical constants such that $c_o \geq 4, c_s \geq 3(c_{\text{RE}} + 1)/(c_{\text{RE}} - 1), 2 > c_\varepsilon \geq 1, c_{\text{cut}} \geq c_\tau, c_{r_1} = c_{r_1}^{\text{num}} (1 + c_{\text{RE}})/\kappa_1, c_{r_2} = c_{r_2}^{\text{num}} (3 + c_{\text{RE}})/\kappa_1$, $\min\{c_{r_1}^{\text{num}}, c_{r_2}^{\text{num}}\} \geq 2$ and $c_{r_1}^{\text{num}}/c_{r_2}^{\text{num}} \leq 1$. In Corollary 5.1 and Definition 5.2, $c_7$ and $c_{\max}$ are defined, respectively. Suppose that $\max\{\sqrt{s}\tau_{d,s}\} \leq 1, 0 < o/n \leq 1/2$ hold. Then, the optimal solution $\hat{\beta}$ satisfies the following:

$$\|\Sigma_0^2 (\hat{\beta} - \beta^*)\|_2 \leq c_{\Sigma_0}^2 \Omega^2 (R_{\text{lasso}} + R'_{\text{outlier}}),$$

$$\|\hat{\beta} - \beta^*\|_2 \leq C_{\text{RE},4} \Omega^2 (R_{\text{lasso}} + R'_{\text{outlier}}),$$

$$\|\hat{\beta} - \beta^*\|_1 \leq C_{\text{RE},5} \Omega^2 \frac{1}{\tau} \sqrt{R_{\text{lasso}} + R'_{\text{outlier}}},$$

and we see that (4.7) recovers (2.6). Investigation of whether it is possible to achieve similar error bounds using an estimation method without covariance as in the case of with covariance, and the exploration of the trade-offs in such a scenario, are left as future research tasks.

5 Key techniques

5.1 Key propositions and lemma for Theorem 3.1

First, we introduce Proposition 5.1, that gives the condition on $\tau_{\text{cut}}$ when the covariance matrix $\Sigma$ is known. The proof is given in Section 6.

Proposition 5.1. Suppose that the assumptions in Theorem 3.1 hold. Define $c_1$ and $c_2$ as numerical constants. Then, with probability at least $1 - \delta$, we have

$$\max_{M \in \mathbb{M}^n_{r_1,r_2,d}} \frac{\sum_{i=1}^n \langle x_i, x_i^\top - \Sigma, M \rangle}{n} \leq c_1 (L\kappa_u)^2 (s\tau_{d,s} + r_\delta) r_2^2.$$

Additionally, with probability at least $1 - 2\delta$, we have

$$\max_{M \in \mathbb{M}^n_{r_1,r_2,d}} \frac{\sum_{i=1}^n \hat{w}_i \langle x_i, x_i^\top - \Sigma, M \rangle}{n} \leq c_2 (L\kappa_u)^2 (s\tau_{d,s} + r_\delta + r'_{\delta}) r_2^2.$$

Therefore, we see that, when $c_2 (L\kappa_u)^2 (s\tau_{d,s} + r_\delta + r'_{\delta}) r_2^2 \leq \tau_{\text{cut}}$, Algorithm 2 succeeds at returning $\hat{w}$ under (5.2). The key techniques of the proof of the proposition above are Corollary 2.8 of [75], that is the one of the variants of the Hanson–Wright inequality [43, 73, 62, 1, 44], and generic chaining for a subexponential random variable (Corollary 5.2 of [36]).
Next, we introduce a deterministic proposition related to Theorem 3.1. Let
\[ r_{v,i} = \frac{\hat{u}_i^T y_i - X_i^T v}{\lambda_n \sqrt{n}}, \quad X_{v,i} = \frac{X_i^T v}{\lambda_n \sqrt{n}}, \quad x_{v,i} = \frac{x_i^T v}{\lambda_n \sqrt{n}}, \quad \xi_{\lambda,v,i} = \frac{\xi_i}{\lambda_n \sqrt{n}}, \]
and for \( \eta \in (0, 1) \),
\[ \theta = \hat{\beta} - \beta^*, \quad \theta_\eta = (\hat{\beta} - \beta^*)\eta. \]

The following proposition is proved in a manner similar to the proof of Proposition 9.1 of [2], and the proof is given in the Appendix C and D.

**Proposition 5.2.** Suppose that, for any \( \theta_\eta \in r_1 \mathbb{B}^d \cap r_2 \mathbb{B}_2^d \cap r_2 \mathbb{B}_2^d \),
\[ \left| \lambda_n \sqrt{n} \sum_{i=1}^n \hat{u}_i^T h(r_{\beta^*,i}) X_i^T \theta_\eta \right| \leq r_{a,1} r_1 + r_{a,2} r_2 + r_{a,\Sigma} r_\Sigma, \]
where \( r_{a,1}, r_{a,2}, r_{a,\Sigma}, r_{\alpha}, r_{b,2}, r_{b,\Sigma} \geq 0, b > 0 \) are some numbers. Suppose that \( \text{RE}(s, c_{RE}, v) \), \( \kappa_1 > 0, \) and
\[ \lambda_s \left( r_{a,1} \frac{c_{r_2} r_{a,2} + r_{a,\Sigma}}{c_1 \sqrt{s}} \right) > 0, \quad \lambda_s + \left( r_{a,1} \frac{c_{r_2} r_{a,2} + r_{a,\Sigma}}{c_1 \sqrt{s}} \right) \leq c_{RE}, \]
\[ r_\Sigma \geq \frac{2}{b} \left( c_{r_1} \sqrt{s} r_{a,1} + c_{r_2} (r_{a,2} + r_{a,\Sigma}) + r_{a,\Sigma} + c_{r_1} \sqrt{s} \lambda_s \right), \quad r_1 = c_{r_1} \sqrt{s} r_\Sigma, \quad r_2 = c_{r_2} r_\Sigma \]
hold, where \( c_{r_1} = c_{r_1}^\text{num} (1 + c_{RE})/\kappa, c_{r_2} = c_{r_2}^\text{num} (3 + c_{RE})/\kappa_1, \) \( \min\{c_{r_1}^\text{num}, c_{r_2}^\text{num}\} \geq 2 \) and \( c_{r_1}/c_{r_2} \leq 1 \). Then, we have the following:
\[ \|\beta^* - \hat{\beta}\|_1 \leq r_1, \quad \|\beta^* - \hat{\beta}\|_2 \leq r_2, \quad \|\Sigma^2 (\beta^* - \hat{\beta})\|_2 \leq r_\Sigma. \]

In the remainder of Section 5, we introduce Propositions 5.3–5.6 and one lemma. In Section A, we prove Theorem 3.1 using the propositions. In the proof of Theorem 3.1, we prove that (5.5) - (5.8) are satisfied with high probability for appropriate values of \( r_{a,1}, r_{a,2}, r_{a,\Sigma}, r_{b,1}, r_{b,2}, r_{b,\Sigma} \) and \( b \) under the assumptions in Theorem 3.1, and we see that (5.9) is also satisfied. Then, we have the result (3.26) in Theorem 3.1. We note that, for Proposition 5.6, similar statements are found, for example, in [67, 13], and the proof of Proposition 5.6 basically follows the same line to the ones in [67, 13]. Propositions 5.4 and 5.5 are proved by relatively simple calculations based on the result of Proposition 5.1. Therefore, the proofs of Propositions 5.4–5.6 are given in the Appendix F.

**Proposition 5.3.** Suppose that the assumptions in Theorem 3.1 or Theorem 4.1 hold. Then, for any \( v \in r_1 \mathbb{B}_1^d \cap r_2 \mathbb{B}_2^d \), with probability at least \( 1 - \delta \), we have
\[ \left| \sum_{i=1}^n \frac{1}{n} h(\xi_{\lambda,v,i}) x_i^T v \right| \leq c_3 L \left( pr_d s r_1 + \sqrt{sr_d s} r_\Sigma + r_3 r_\Sigma \right), \]
where \( c_3 \) denotes a numerical constant.

**Proposition 5.4.** Suppose that the assumptions in Theorem 3.1 hold. Furthermore, suppose that (5.1) and (5.2) hold and that Algorithm 2 returns \( \tilde{w} \). For any \( u \in \mathbb{R}^n \) such that \( \|u\|_\infty \leq 2 \) and for any \( v \in r_1 \mathbb{B}_1^d \cap r_2 \mathbb{B}_2^d \), we have
\[ \left| \sum_{i \in \mathcal{O}} \hat{u}_i u_i^T x_i^T v \right| \leq c_4 L \sqrt{1 + c_{\text{cut}}} \left( \kappa_\delta \sqrt{\frac{o}{n}} \left( \sqrt{sr_d s} + \sqrt{\kappa_\delta} \right) + \kappa_3 r_3 \right) r_2, \]
where \( c_4 \) is a numerical constant that depends on \( c_1 \) and \( c_2 \).
Let \( I_m \) be an index set such that \(|I_m| = m\).

**Proposition 5.5.** Suppose that the assumptions in Theorem 3.1 hold. Furthermore, suppose that (5.1) holds. Then, for any \( m \in \mathbb{N} \) such that \( m \leq (2c_\varepsilon + 1)\alpha \), for any \( u \in \mathbb{R}^n \) such that \( |u|_\infty \leq 2 \) and for any \( v \in \mathbb{R}^d_1 \cap \mathbb{R}^d_2 \), we have the following:

\[
\left| \sum_{i \in I_m} \frac{1}{n} u_i x_i^T v \right| \leq c_5 L \left( \kappa_u \sqrt{\frac{1}{n} \left( \sqrt{\delta r_d, x} + \sqrt{\delta r_b} \right)} + \kappa_u r_\varepsilon \right) r_2, \tag{5.12}
\]

where \( c_5 \) is a numerical constant that depends on \( c_1 \) and \( c_\varepsilon \).

**Proposition 5.6.** Suppose that the assumptions in Theorem 3.1 or Theorem 4.1 hold. Then, for any \( v \in \mathbb{R}^d_1 \cap \mathbb{R}^d_2 \), with probability at least \( 1 - \delta \), we have

\[
\sum_{i=1}^n \frac{\lambda_i}{\sqrt{n}} (-h(\xi_{\omega,i} - x_{v,i}) + h(\xi_{\omega,i})) x_i^T v \geq \frac{\|\Sigma^2 v\|^2}{3} - c_6 L \lambda_\alpha \sqrt{n} \left( \rho r_d, r_1 + \sqrt{sr_d, r_\Sigma + r_b r_\Sigma} \right), \tag{5.13}
\]

where \( c_6 \) is a numerical constant.

We define \( c_{\text{max}} \).

**Definition 5.1.** Define

\[
c_{\text{max}} = \max \left( 1, c_3, c_4 \sqrt{1 + c_{\text{cut}}}, c_5, c_6 \right). \tag{5.14}
\]

Let \( I_- \) and \( I_+ \) be the sets of indices such that \( w_i < 1/(2n) \) and \( w_i \geq 1/(2n) \), respectively.

**Lemma 5.1.** Suppose that \( 0 < \varepsilon < 1 \). Then, for any \( w \in \Delta^{n-1}(\varepsilon) \), we have \( |I_-| \leq 2n \varepsilon \).

### 5.2 Key propositions and corollary for Theorem 4.1

First, we introduce Corollary 5.1, that gives the condition on \( \tau_{\text{cut}} \) when we do not use \( \Sigma \) in the estimator. The proof is given in Appendix.

**Corollary 5.1.** Suppose that the assumptions in Theorem 4.1 hold. Define \( c'_1 \) and \( c_7 \) as numerical constants. Then, with probability at least \( 1 - \delta \), we have

\[
\max_{M \in \mathbb{M}_{r_1, r_2, d}} \sum_{i=1}^n \langle x_i, x_i^\top, M \rangle n \leq c'_1 L^2 \left( \kappa_u^2 (sr_d, r_\delta) + \Sigma_{\text{max}}^2 \right) r_2^2. \tag{5.15}
\]

Additionally, with probability at least \( 1 - \delta \), we have

\[
\max_{M \in \mathbb{M}_{r_1, r_2, d}} \sum_{i=1}^n w_i \langle x_i, x_i^\top, M \rangle \leq c_7 L^2 \left( \kappa_u^2 (sr_d, r_\delta) + \Sigma_{\text{max}}^2 \right) r_2^2. \tag{5.16}
\]

Therefore, under (5.16), when \( c_7 \left( (L\kappa_u)^2 (sr_d, r_\delta) + \Sigma_{\text{max}}^2 \right) r_2^2 \leq \tau_{\text{cut}} \), Algorithm 5 succeeds at returning \( \hat{w} \). We see that when the covariance is not used in the estimator, it is necessary to set the value of \( \tau_{\text{cut}} \) to a higher magnitude compared to the case when covariance is used. Using the propositions, in Section B, we can prove that (5.5) - (5.8) are satisfied with a high probability for appropriate values of \( r_{a,1}, r_{a,2}, r_{a,3}, r_{b,1}, r_{b,2}, r_{b,3} \) and \( b \) under the assumptions in Theorem 4.1, and we see that (5.9) is also satisfied. Then, we can have the result (4.6) in Theorem 4.1. When \( \Sigma \) is not used in the estimator, Propositions 5.2 and 5.6 and Lemma 5.1 are commonly used, and instead of Propositions 5.4 and 5.5, Propositions 5.7 and 5.8 are used, respectively. In Definition 5.2, \( c'_{\text{max}} \) is defined. The proofs of Propositions 5.7 and 5.8 are given in the Appendix F because Propositions 5.7 and 5.8 can be proved by simple calculations based on the result of Proposition 5.1.
Proposition 5.7. Suppose that the assumptions in Theorem 4.1 hold. Furthermore, suppose that (5.15) holds and that Algorithm 5 returns \( \mathbf{w} \). For any \( \mathbf{u} \in \mathbb{R}^n \) such that \( \| \mathbf{u} \|_{\infty} \leq 2 \) and for any \( \mathbf{v} \in r_1 \mathbb{B}_1^d \cap r_2 \mathbb{B}_2^d \), we have the following:

\[
\sum_{i \in \mathcal{O}} \langle \mathbf{u}_i, \mathbf{X}_i \rangle \mathbf{v} \leq c_8 \sqrt{c_{cut}} L \sqrt{\frac{D}{n}} \sqrt{\kappa_0^2 (sr_{d,s} + r_d) + \Sigma_{\max}^2 r_2},
\]

(5.17)

where \( c_8 \) is a numerical constant.

Proposition 5.8. Suppose that the assumptions in Theorem 4.1 hold. Furthermore, suppose that (5.16) holds. Then, for any \( m \in \mathbb{N} \) such that \( m \leq (2c_e + 1)\alpha \), for any \( \mathbf{u} \in \mathbb{R}^n \) such that \( \| \mathbf{u} \|_{\infty} \leq 2 \) and for any \( \mathbf{v} \in r_1 \mathbb{B}_1^d \cap r_2 \mathbb{B}_2^d \), we have the following:

\[
\sum_{i \in \mathcal{I}_m} \frac{1}{n} u_i \mathbf{x}_i^\top \mathbf{v} \leq c_9 L \sqrt{\frac{D}{n}} \sqrt{\kappa_0^2 (sr_{d,s} + r_d) + \Sigma_{\max}^2 r_2},
\]

(5.18)

where \( c_9 \) is a numerical constant that depends on \( c_1' \) and \( c_e \).

For Theorem 4.1, the following definition of \( c_{\max}' \) is used.

**Definition 5.2.** Define

\[
c_{\max}' = \max (1, c_3, c_6, c_8 \sqrt{c_{cut}}, c_9).
\]

(5.19)

6 Proofs of Propositions 5.1 and 5.3

The value of the numerical constant \( C \) shall be allowed to change from line to line. Define \( \gamma_{s_0,\alpha} \)-functional:

**Definition 6.1** \( (\gamma_{s_0,\alpha} \text{-functional,} \ [52]) \). Let \( (T, d) \) be a semi-metric space with \( d(x, z) \leq d(x, y) + d(y, z) \) and \( d(x, y) = d(y, x) \) for \( x, y, z \in T \). A sequence \( \mathcal{T} = \{T_m\}_{m \geq 0} \) with subsets of \( T \) is said to be admissible if \( |T_0| = 1 \) and \( |T_m| \leq 2^{2m} \) for all \( m \geq 1 \). For any \( \alpha \in (0, \infty) \), the \( \gamma_{s_0,\alpha} \)-functional of \( (T, d) \) is defined by

\[
\gamma_{s_0,\alpha}(T, d) = \inf_{\mathcal{T}} \sup_{t \in T} \sum_{m = s_0}^{\infty} 2^m \inf_{s \in T_m} d(t, s),
\]

(6.1)

where the infimum is taken over all admissible sequences \( \mathcal{T} = \{T_m\}_{m \geq 0} \).

In particular, when \( s_0 = 0 \), \( \gamma_{0,\alpha} \)-functional is called the \( \gamma_{\alpha} \)-functional. For a random variable \( z \) and \( p \geq 1 \), define \( \| \cdot \|_{(p)} \)-norm as

\[
\|z\|_{(p)} = \sup_{1 \leq q \leq p} \frac{\|z\|_{L_q}}{\sqrt[2]{q}}.
\]

(6.2)

Next, we define \( \Lambda_{s_0,\mathbf{u}} \) and \( \tilde{\Lambda}_{s_0,\mathbf{u}} \).

**Definition 6.2** \( (\Lambda_{s_0,\mathbf{u}} \text{ and} \ \tilde{\Lambda}_{s_0,\mathbf{u}}, \ [52]) \). Given a class of functions \( F \), \( u \geq 1 \), \( s_0 \geq 0 \), define

\[
\Lambda_{s_0,\mathbf{u}}(F) = \inf_{\mathcal{F}} \sup_{f \in \mathcal{F}} \sum_{m = s_0}^{\infty} 2^m \| f - \pi_m f \|_{(u^{2^m})},
\]

(6.3)

where the infimum is taken over all admissible sequences \( \mathcal{F} = \{F_m\}_{m \geq 0} \), and \( \pi_m f \) is the nearest point in \( F_m \) to \( f \) in \( \| \cdot \|_{(u^{2^m})} \)-norm. Additionally, define

\[
\tilde{\Lambda}_{s_0,\mathbf{u}}(F) = \Lambda_{s_0,\mathbf{u}}(F) + 2^m \sup_{f \in \mathcal{F}} \| \pi_{m_0} f \|_{(u^{2^{m_0})}}.
\]

(6.4)
We introduce Lemma 6.1, which is used in the proofs of Propositions 5.1 and 5.3. The proof of this lemma is given in the Appendix E.2. Let \( g = (g_1, \cdots, g_d)^\top \) be the \( d \)-dimensional standard normal Gaussian random vector.

**Lemma 6.1.** Suppose that the assumptions in Theorem 3.1 or Theorem 4.1 hold. We have
\[
\mathbb{E} \sup_{v \in \mathbb{R}^d \cap \mathbb{B}_E^d \setminus \mathbb{B}_E^d} \langle \Sigma \frac{v}{\|v\|_2}, v \rangle \leq C \rho_r 1 \sqrt{\log(d/s)}.
\] (6.5)

### 6.1 Preparation for the proof of Proposition 5.1

For a matrix \( M \), we define the operator norm and Frobenius norm of \( M \) as \( \|M\|_{\text{op}} \) and \( \|M\|_F \), respectively, and we define the number of non-zero elements of \( M \) as \( \|M\|_0 \). Additionally, we define \( \mathbb{B}_F^d \) = \{ \( M \in \mathbb{R}^{d \times d} \mid \|M\|_1 \leq a \} \), \( \mathbb{B}_F^d \) = \{ \( M \in \mathbb{R}^{d \times d} \mid \|M\|_F \leq a \} \), and \( \mathbb{B}_0^d \) = \{ \( M \in \mathbb{R}^{d \times d} \mid \|M\|_0 \leq a \} \).

**Lemma 6.2.** Suppose that (i) and (iii) of Assumptions 2.1. For any fixed \( M, M' \in s^2 \mathbb{B}_0^d \cap r_2^2 \mathbb{B}_E^d \), we have
\[
\mathbb{E} \|xx^\top - \Sigma, M\|_P^P \leq p! (C(L\kappa_a)^2 r_2^2)^p,
\] (6.6)
\[
\|xx^\top, M - M'\|_\psi_1 \leq C(L\kappa_a)^2 \|M - M'\|_F.
\] (6.7)

**Proof.** In this proof, we define \( c \) as some positive numerical constant. For any \( a \)-dimensional random variable \( a \), let the Luxemburg norm, that is an extension of the \( \psi_2 \)-norm from scalar to vector, be denoted by
\[
\|a\|_{\psi_2} = \inf \left\{ \eta > 0 : \sup_{v \in \mathbb{S}^{a-1}} \mathbb{E} \exp \left( \frac{\langle v, a \rangle^2}{\eta^2} \right) \leq 2 \right\}
\] (6.8)

From (3.5), for any index set \( J \) such that \( |J| \leq 2s^2 \) and \( v \in \mathbb{S}^{2s^2-1} \) we have
\[
\mathbb{E} \exp \left( \frac{\langle v, x_J \rangle^2}{c_2 \Sigma_k a} \right) \leq 2,
\] (6.9)
and we have
\[
\|x_J\|_{\psi_2} \leq c_2 \Sigma_k a \leq L\kappa_a.
\] (6.10)

For any matrix \( A \) and any index set \( J \subset \{1, \cdots, d\} \), define \( A|_{J,J} \) as the matrix such that all the elements of the \( i \)-th rows and columns are zero for \( i \in J^c \). Fix \( M \in s^2 \mathbb{B}_0^d \cap r_2^2 \mathbb{B}_E^d \). For \( M \), let \( K \subset \{1, \cdots, d\} \) be an index set such that \( M_{ij} = 0 \) for \( i \in K^c \) or \( j \in K^c \). We note that \( |K| \leq s^2 \).

From Corollary 2.8 of [75] and (6.10), for any \( t > 0 \), we have
\[
P \left( \|xx^\top - \Sigma, M\| > t \right) = P \left( \|x_K x_K^\top - \Sigma_{K,K}, M\| > t \right)
\leq 2 \exp \left\{ -c \min \left( \frac{t^2}{(L\kappa_a)^4 \|M\|_F^4}, \frac{t}{(L\kappa_a)^2 \|M\|_F} \right) \right\}.
\] (6.11)

From (6.11), for \( t \leq (L\kappa_a)^2 \|M\|_F \), we have
\[
P \left( \|xx^\top - \Sigma, M\| > t \right) \leq 2 \exp \left( -\frac{t^2}{(L\kappa_a)^4 \|M\|_F^4} \right),
\] (6.12)
and for \( t \geq (L\kappa_a)^2 \|M\|_F \), we have
\[
P \left( \|xx^\top - \Sigma, M\| > t \right) \leq 2 \exp \left( -\frac{t}{(L\kappa_a)^2 \|M\|_F} \right).
\] (6.13)
We follow almost the same argument of the proof of Proposition 2.5.2 of [71]. For any \( 1 \leq p < \infty \), we have

\[
\mathbb{E} \left( \langle xx^T - \Sigma, M \rangle \right)^p = \int_0^\infty \mathbb{P} \left( \left| \langle xx^T - \Sigma, M \rangle \right|^p > u \right) du \\
= \int_0^\infty \mathbb{P} \left( \left| \langle xx^T - \Sigma, M \rangle \right| \geq t \right) pt^{p-1} dt \\
\leq \int_0^\infty 2 \exp \left( -\frac{c}{(L_{K_u})^4 \|M\|_F^2} t^2 \right) pt^{p-1} dt + \int_0^\infty 2 \exp \left( -\frac{t}{(L_{K_u})^2 \|M\|_F} \right) pt^{p-1} dt \\
\leq p \left( \frac{(L_{K_u})^2 \|M\|_F^p}{\sqrt{c}} \Gamma(p/2) + 2p \left( \frac{(L_{K_u})^2 \|M\|_F^p}{c^p} \right) \Gamma(p) \right) \\
\leq C \left( \frac{1}{c^p} + \frac{1}{\sqrt{c}} \right)^2 \|M\|_F^p \left( \frac{p}{2} \right)^{p/2} + p^p, \quad (6.14)
\]

and we have

\[
\left( \mathbb{E} \langle x_i x_i^T - \Sigma, M \rangle \right)^{\frac{1}{p}} \leq C \left( \frac{1}{c^p} + \frac{1}{\sqrt{c}} \right)^2 \|M\|_F \left( \frac{p}{2} \right)^{p/2} + p^{p+1} \]

\[
\leq C \left( \frac{1}{c^p} + \frac{1}{\sqrt{c}} \right)^2 \|M\|_F \left( \frac{p}{2} \right)^{p+1} \]

\[
\leq C \left( \frac{1}{c^p} + \frac{1}{\sqrt{c}} \right)^2 \|M\|_{FP} \leq C \left( \frac{1}{c^p} \right)^2 \|M\|_{FP}. \quad (6.15)
\]

From Stirling’s formula \( p^p \leq p\text{e}p \), we have

\[
\max_{M \in \mathfrak{M}_{r_1, r_2, d}^{\text{tr}}} \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left| \langle x_i x_i^T - \Sigma, M \rangle \right|^p \leq C p \text{e}p \left( L_{K_u} \right)^{2p} r_2^{2p}, \quad (6.16)
\]

and the proof of (6.6) is complete.

For any fixed \( M, M' \in \mathfrak{M}_{r_1, r_2, d}^{\text{tr}} \), let \( K' \subset \{1, \cdots, d\} \) is the index set such that \( (M - M')_{ij} = 0 \) for \( i \in K' \) or \( j \in K' \). We note that \( |K'| \leq 2 \times s^2 \). From Proposition 2.6, Remark 2.7 of [75] and (6.10), we have

\[
\left\| \langle xx^T, M - M' \rangle \right\|_{\psi_1} = \left\| \langle x_{K'} x_{K'}^T, (M - M')_{K'K'} \rangle \right\|_{\psi_1} \leq C \left( L_{K_u} \right)^2 \| (M - M')_{K'K'} \|_F \\
\leq C \left( L_{K_u} \right)^2 \| M - M' \|_F, \quad (6.17)
\]

and the proof of (6.7) is complete.

Using the lemma above, we have the key lemma (Lemma 6.4) to prove Propositions 5.1. For a set \( K \), we define \( \text{conv}(K) \) as its convex hull. Before the statement and the proof of Lemma 6.4, we introduce the following lemma, which slightly generalizes Lemma 3.1 of [59] and the proof is in Appendix E.

**Lemma 6.3.** We have

\[
r_1^2 \mathbb{B}_1^{d \times d} \cap r_2^2 \mathbb{B}_F^{d \times d} \subset 2\text{conv} \left( \left( \frac{r_1}{r_2} \right) \mathbb{B}_0^{d \times d} \cap r_2^2 \mathbb{B}_F^{d \times d} \right), \quad (6.18)
\]

\[
r_1 \mathbb{B}_1^{d} \cap r_2 \mathbb{B}_2^{d} \subset 2\text{conv} \left( \left( \frac{r_1}{r_2} \right) \mathbb{B}_0^{d} \cap r_2 \mathbb{B}_2^{d} \right). \quad (6.19)
\]

Then, we introduce Lemma 6.4.
Lemma 6.4. Suppose that (i) and (iii) of Assumptions 2.1 hold. Then, with probability at least $1 - \delta$, we have

$$\max_{M \in \mathbb{M}^n_{r_1, r_2, d}} \frac{1}{n} \sum_{i=1}^{n} \langle x_i, x_i^T - \Sigma, M \rangle \leq C(L\kappa_u)^2 \left( \frac{\gamma_1(s^2 \mathbb{E}_0^n \cap r_2^2 \mathbb{B}_F^n, \| \cdot \|_F)}{n} + \frac{\gamma_2(s^2 \mathbb{E}_0^n \cap r_2^2 \mathbb{B}_F^n, \| \cdot \|_F)}{\sqrt{n}} \right).$$  

(6.20)

Proof. First, we have

$$\mathbb{M}^n_{r_1, r_2, d} \subset r_2^2 \mathbb{B}_F^n \cap r_2^2 \mathbb{B}_F^n \supseteq 2\text{conv}(s^2 \mathbb{B}_0^n \cap r_2^2 \mathbb{B}_F^n),$$  

(6.21)

where (a) follows from $r_i^4/r_i^4 \leq c_{r_1}^2 s^2/c_{r_2}^2$, $c_{r_1}^2/c_{r_2}^2 \leq 1$ and Lemma 6.3, identifying vectors with matrices. Then, we have

$$\max_{M \in \mathbb{M}^n_{r_1, r_2, d}} \frac{1}{n} \sum_{i=1}^{n} \langle x_i, x_i^T - \Sigma, M \rangle \leq \max_{M \in \text{2conv}(s^2 \mathbb{B}_0^n \cap r_2^2 \mathbb{B}_F^n)} \frac{1}{n} \sum_{i=1}^{n} \langle x_i, x_i^T - \Sigma, M \rangle. \quad (6.22)$$

Identifying vectors with matrices, from Lemma D.8 of [57], we have

$$\max_{M \in \text{2conv}(s^2 \mathbb{B}_0^n \cap r_2^2 \mathbb{B}_F^n)} \frac{1}{n} \sum_{i=1}^{n} \langle x_i, x_i^T - \Sigma, M \rangle \leq \max_{M \in s^2 \mathbb{B}_0^n \cap r_2^2 \mathbb{B}_F^n} \frac{2}{n} \sum_{i=1}^{n} \langle x_i, x_i^T - \Sigma, M \rangle. \quad (6.23)$$

Next, note that the upper bound of the first inequality in Lemma 6.2 is independent of $i$ and $M$. Hence, from Corollary 5.2 of [36], with probability at least $1 - \delta$, from the same argument used to have (6.17), we have

$$\max_{M \in \mathbb{M}^n_{r_1, r_2, d}} \frac{1}{n} \sum_{i=1}^{n} \langle x_i, x_i^T - \Sigma, M \rangle \leq C \left( \frac{\gamma_1(s^2 \mathbb{B}_0^n \cap r_2^2 \mathbb{B}_F^n, d_1)}{n} + \frac{\gamma_2(s^2 \mathbb{B}_0^n \cap r_2^2 \mathbb{B}_F^n, d_2)}{\sqrt{n}} + (L\kappa_u)^2 r_2^2 + (L\kappa_u^2 r_2^2 \right), \quad (6.24)$$

where the semi-metrics $d_1(M, M')$ and $d_2(M, M')$ for $M, M' \in s^2 \mathbb{B}_0^n \cap r_2^2 \mathbb{B}_F^n$ are defined in [36]. Since $\{x_i\}^n_{i=1}$ is an i.i.d. sequence, two semi-metrics are the same and then, for any $i \in \{1, \ldots, n\}$, we have

$$d_1(M, M') = d_2(M, M') = \|\langle x_i, x_i^T, M - M'\rangle\|_{\psi_1} \leq C(L\kappa_u)^2 \|M - M'\|_F, \quad (6.25)$$

where the last inequality follows from (6.7). We know that $\gamma_1(s^2 \mathbb{B}_0^n \cap r_2^2 \mathbb{B}_F^n, d_1)$ and $\gamma_2(s^2 \mathbb{B}_0^n \cap r_2^2 \mathbb{B}_F^n, d_2)$ is monotone increasing with respect to $d_1$ and $d_2$, and for some constants $c_1, c_2$, we have

$$\gamma_1(s^2 \mathbb{B}_0^n \cap r_2^2 \mathbb{B}_F^n, c_1 d_1) = c_1 \gamma_1(s^2 \mathbb{B}_0^n \cap r_2^2 \mathbb{B}_F^n, d_1),$$

$$\gamma_2(s^2 \mathbb{B}_0^n \cap r_2^2 \mathbb{B}_F^n, c_2 d_2) = c_2 \gamma_2(s^2 \mathbb{B}_0^n \cap r_2^2 \mathbb{B}_F^n, d_2). \quad (6.26)$$
Combining (6.24) with the above properties of the $\gamma_n$-functional, we have
\[
\max_{M \in \text{Sym}_{r_1, r_2, d}} \frac{1}{n} \sum_{i=1}^{n} \langle x_i, x_i^\top - \Sigma, M \rangle = \left( \frac{\gamma_1(s^2 B_0^{d \times d} \cap r_2 B_F^{d \times d}, \| \cdot \|_F)}{n} + \gamma_2(s^2 B_0^{d \times d} \cap r_2 B_F^{d \times d}, \| \cdot \|_F) \right) \frac{1}{\sqrt{n}} + \frac{r_3^2 r_2 + r_4^2}{n},
\]
and the proof is complete.

6.2 Proof of Proposition 5.1

First, we prove (5.1) in Proposition 5.1 via Lemma 6.5, that is necessary to calculate $\gamma_1(s^2 B_0^{d \times d} \cap r_2 B_F^{d \times d}, \| \cdot \|_F)$ and $\gamma_2(s^2 B_0^{d \times d} \cap r_2 B_F^{d \times d}, \| \cdot \|_F)$. They are provided in the following lemma, that is proved in the Appendix.

**Lemma 6.5.** Suppose that (i) and (ii) of Assumption 2.1 holds. Then, we have
\[
\gamma_1(s^2 B_0^{d \times d} \cap r_2 B_F^{d \times d}, \| \cdot \|_F) \leq C s^2 \delta r_2 \log(d/s),
\]
\[
\gamma_2(s^2 B_0^{d \times d} \cap r_2 B_F^{d \times d}, \| \cdot \|_F) \leq C s r_2 \sqrt{\log(d/s)},
\]
(6.28)

First, we prove (5.1). From Lemmas 6.4 and 6.5, we have, with probability at least $1 - \delta$,
\[
\max_{M \in \text{Sym}_{r_1, r_2, d}} \frac{1}{n} \sum_{i=1}^{n} \langle x_i, x_i^\top - \Sigma, M \rangle \leq C(L\kappa_n)^2 \left( \frac{s^2 \delta^2 \log(d/s)}{n} + \frac{s r_2 \sqrt{\log(d/s)}}{\sqrt{n}} + r_3^2 r_2 + r_4^2 \right)
\]
\[
\leq \frac{1}{(L\kappa_n)^2} (s r_4^2 + r_4)^2,
\]
(6.29)

where (a) follows from $s r_4, r_4 \leq 1$.

Next, we prove (5.2). Define
\[
w_i^o = \begin{cases} 
\frac{1}{n(1-o/n)} & i \in I \\
0 & i \in \mathcal{O}
\end{cases}
\]
(6.31)

From the optimality of $\{\hat{w}_i\}^{n}_{i=1}$ and the fact that $\{w_i^o\}^{n}_{i=1} \in \Delta^{n-1}(\varepsilon)$ and the definition of $X_i$, we have
\[
\max_{M \in \text{Sym}_{r_1, r_2, d}} \sum_{i=1}^{n} \hat{w}_i \langle x_i, x_i^\top - \Sigma, M \rangle \leq \max_{M \in \text{Sym}_{r_1, r_2, d}} \sum_{i=1}^{n} w_i^o \langle x_i, x_i^\top - \Sigma, M \rangle
\]
\[
= \max_{M \in \text{Sym}_{r_1, r_2, d}} \sum_{i=1}^{n} w_i^o \langle x_i, x_i^\top - \Sigma, M \rangle
\]
\[
\leq \max_{M \in \text{Sym}_{r_1, r_2, d}} \sum_{i=1}^{n} w_i^o \langle x_i, x_i^\top - \Sigma, M \rangle.
\]
(6.32)

From triangular inequality and the fact that $o/n \leq 1/2$, we have
\[
\sum_{i=1}^{n} w_i^o \langle x_i, x_i^\top - \Sigma, M \rangle = \frac{1}{1-o/n} \sum_{i=1}^{n} \frac{\langle x_i, x_i^\top - \Sigma, M \rangle}{n} - \sum_{i \in \mathcal{O}} \frac{\langle x_i, x_i^\top - \Sigma, M \rangle}{n}
\]
\[
\leq 2 \left| \sum_{i=1}^{n} \frac{\langle x_i, x_i^\top - \Sigma, M \rangle}{n} - \sum_{i \in \mathcal{O}} \frac{\langle x_i, x_i^\top - \Sigma, M \rangle}{n} \right|
\]
\[
\leq 2 \left| \sum_{i=1}^{n} \frac{\langle x_i, x_i^\top - \Sigma, M \rangle}{n} \right| + 2 \left| \sum_{i \in \mathcal{O}} \frac{\langle x_i, x_i^\top - \Sigma, M \rangle}{n} \right|.
\]
(6.33)
From (6.32) and (6.33), we have
\[
\max_{M \in \mathcal{M}_{1}^{TV_1}} \sum_{i=1}^{n} \hat{w}_i \langle \mathbf{X}_i, \mathbf{X}_i^\top - \Sigma, M \rangle 
\leq 2 \max_{M \in \mathcal{M}_{1}^{TV_1}} \left( \sum_{i=1}^{n} \frac{\langle \mathbf{X}_i, \mathbf{X}_i^\top - \Sigma, M \rangle}{n} + \left| \sum_{i \in \mathcal{O}} \langle \mathbf{x}, \mathbf{x}^\top - \Sigma, M \rangle \right| \right)
\leq 2 \left( \max_{M \in \mathcal{M}_{1}^{TV_1}} \sum_{i=1}^{n} \frac{\langle \mathbf{x}, \mathbf{x}^\top - \Sigma, M \rangle}{n} + \max_{M \in \mathcal{M}_{1}^{TV_1}} \left| \sum_{i \in \mathcal{O}} \langle \mathbf{x}, \mathbf{x}^\top - \Sigma, M \rangle \right| \right)
\leq 2 \left( \max_{M \in \mathcal{M}_{1}^{TV_1}} \sum_{i=1}^{n} \frac{\langle \mathbf{x}, \mathbf{x}^\top - \Sigma, M \rangle}{n} + \max_{|\mathcal{J}|=0} \max_{M \in \mathcal{M}_{1}^{TV_1}} \left| \sum_{i \in \mathcal{J}} \langle \mathbf{x}, \mathbf{x}^\top - \Sigma, M \rangle \right| \right),
\]
(6.34)

We evaluate the last term of (6.34) in a manner similar to the proof of Lemma 5 of [22]. From (6.29), we have
\[
\max_{|\mathcal{J}|=0} \max_{M \in \mathcal{M}_{1}^{TV_1}} \left| \sum_{i \in \mathcal{J}} \langle \mathbf{x}, \mathbf{x}^\top - \Sigma, M \rangle \right| \geq C(L_{\kappa_n})^2 \left( s^2 \frac{\log(d/s)}{o} + s \sqrt{\frac{\log(d/s)}{o}} + \sqrt{\frac{t}{o}} \right) r_2,
\]
(6.35)

with probability at least
\[
\leq \left( \frac{n}{o} \right) \times \nonumber
\]
\[
\mathbb{P} \left[ \max_{M \in \mathcal{M}_{1}^{TV_1}} \sum_{i=1}^{o} \frac{\langle \mathbf{z}, \mathbf{z}^\top - \Sigma, M \rangle}{o} \right] \geq C(L_{\kappa_n})^2 \left( s^2 \frac{\log(d/s)}{o} + s \sqrt{\frac{\log(d/s)}{o}} + \sqrt{\frac{t}{o}} \right)^2 \right]
\leq \left( \frac{n}{o} \right) e^{-t},
\]
(6.36)

where \(\{\mathbf{z}_i\}_{i=1}^{o}\) is a sequence of i.i.d. random vectors sampled from the same distribution as \(\{\mathbf{x}_i\}_{i=1}^{n}\). Let \(t = o \log(ne/o) + \log(1/\delta)\). We have
\[
\left( \frac{n}{o} \right) e^{-t} \leq \prod_{k=0}^{o-1} \left( \frac{n-k}{o} \right) \delta = \prod_{k=0}^{o-1} \frac{n-k}{o} \leq \delta,
\]
where the last inequality follows from Stirling’s formula \(o^o \leq o e^o\). From (6.35), with probability at least \(1 - \delta\), we have
\[
\max_{|\mathcal{J}|=0} \max_{M \in \mathcal{M}_{1}^{TV_1}} \left| \sum_{i \in \mathcal{J}} \frac{1}{n} \langle \mathbf{x}, \mathbf{x}^\top - \Sigma, M \rangle \right| 
\leq C \frac{o}{n} (L_{\kappa_n})^2 \left( s^2 \frac{\log(d/s)}{o} + s \sqrt{\frac{\log(d/s)}{o}} + \sqrt{\frac{o \log(ne/o) + \log(1/\delta)}{o}} \right) r_2.
\]
(6.37)

From \(\varepsilon \leq n/o\) and \(r_\delta \leq 1\), we have
\[
\frac{o \log(ne/o) + \log(1/\delta)}{n} \leq \frac{2}{n} \log(n/o) + \frac{1}{n} \log(1/\delta) = 2r_\delta + r_\delta \leq 2r_\delta + r_\delta,\]
\[
\sqrt{\frac{o}{n}} \frac{o \log(ne/o) + \log(1/\delta)}{n} \leq \sqrt{\frac{o}{n}} \left( \frac{2}{n} \log(n/o) + \sqrt{\frac{1}{n} \log(1/\delta)} \right) \leq 2r_\delta + r_\delta.\]
(6.38)
Combining (6.37) with the above two inequalities, with probability at least $1 - \delta$, we have

$$
\max_{|J| = o} \max_{M \in \mathbb{R}^{r_1, r_2, d}} \left| \sum_{i \in J} \frac{1}{n} (x_i, x_i^\top - \Sigma, M) \right| \leq C(L\kappa u)^2 \left( s^2 \frac{\log(d/s)}{n} + s \sqrt{\frac{\log(d/s)}{n}} \left( \frac{o}{n} + 4r_o' + 2r_\delta \right) \right) r_2^2
$$

$$
\leq C(L\kappa u)^2 \left( 2s^2 \frac{\log(d/s)}{n} + \frac{o}{n} + 4r_o' + 2r_\delta \right) r_2^2
$$

$$
\leq C(L\kappa u)^2 \left( s^2 \frac{\log(d/s)}{n} + 5r_o' + 2r_\delta \right) r_2^2,
$$

(6.39)

where (a) follows from Young’s inequality, and (b) follows from $0 < o/n \leq 1/(5e)$ and $o/n \leq r_o'$. Combining (6.29), (6.34) and (6.39), with probability at least $1 - 2\delta$, we have

$$
\max_{M \in \mathbb{R}^{r_1, r_2, d}} \sum_{i = 1}^{n} \hat{w}_i (x_i, x_i^\top - \Sigma, M) \leq C(L\kappa u)^2 \left( sr_{d,s} + r_\delta + s^2 \frac{\log(d/s)}{n} + 5r_o' + 2r_\delta \right) r_2^2
$$

$$
\leq C(L\kappa u)^2 (sr_{d,s} + r_o' + r_\delta) r_2^2,
$$

(6.40)

where (a) follows from $sr_{d,s} \leq 1$, and the proof is complete.

### 6.3 Proof of Proposition 5.3

In this proof, we define $c_1$ and $c_2$ as the numerical constants that are the same ones of Theorem 4.4 of [52]. Applying Theorem 4.4 of [52], with probability at least $1 - 2\exp(-c_1 u^2 2^{s_0}) - 2\exp(-c_1 w^2 n)$, we have

$$
\sup_{v \in r_1 \mathbb{B}_1 \cap r_2 \mathbb{B}_2} \left| \frac{1}{n} \sum_{i = 1}^{n} \langle h(\xi_{\lambda,i}), x_i, v \rangle \right| \leq c_2 u w \frac{\| h(\xi_{\lambda,i}) \|_{\psi_2}}{\sqrt{n}} \hat{A}_{s_0,u} \left( \langle \cdot, v \rangle, v \in r_1 \mathbb{B}_1^d \cap r_2 \mathbb{B}_2^d \right)
$$

$$
\leq c_2 u w \frac{1}{\sqrt{n}} \hat{A}_{s_0,u} \left( \langle \cdot, v \rangle, v \in r_1 \mathbb{B}_1^d \cap r_2 \mathbb{B}_2^d \right)
$$

(6.41)

$$
\leq C L \frac{c_2 u w}{\sqrt{n}} \sup_{v \in r_1 \mathbb{B}_1^d \cap r_2 \mathbb{B}_2^d} \| \Sigma^{1/2} g, v \rangle + 2^{s_0} \| v \|_{L_2} \sup_{v \in r_1 \mathbb{B}_1^d \cap r_2 \mathbb{B}_2^d} \| v \|_{L_2}
$$

$$
\leq C L \frac{1}{\sqrt{n}} \left( \mathbb{E} \sup_{v \in r_1 \mathbb{B}_1^d \cap r_2 \mathbb{B}_2^d} \langle \Sigma^{1/2} g, v \rangle + 2^{s_0} r_\Sigma \right),
$$

where (a) follows from $\| h(\xi_{\lambda,i}) \|_{\psi_2} \leq 1$, and (b) follows from the same argument of Section 1 of [52].

Define $s'$ as the minimum integer such that $2^{s'} \geq 2 \log(1/\delta)$. Set $s_0, w$ and $u$ such that $s_0 = s'$ and $c_1 u^2, c_1 w^2 \geq 1$. Then, from Lemma 6.1, we have

$$
\sup_{v \in r_1 \mathbb{B}_1^d \cap r_2 \mathbb{B}_2^d} \left| \frac{1}{n} \sum_{i = 1}^{n} \langle h(\xi_{\lambda,i}), x_i, v \rangle \right| \leq C \frac{L}{\sqrt{n}} \left( pr_1 \sqrt{\log(d/s)} + \sqrt{\log(1/\delta)} r_\Sigma \right),
$$

(6.42)

with probability at least

$$
1 - 2\exp(-c_1 u^2 2^{s_0}) - 2\exp(-c_1 w^2 n) \geq 1 - 2\exp(-2^{s_0}) - 2\exp(-n) \geq 1 - \delta,
$$

(6.43)

where we use $\sqrt{2} r_\delta \leq 1$.

### 7 Acknowledgement

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References

[1] R. Adamczak. A note on the hanson-wright inequality for random vectors with dependencies. *Electronic Communications in Probability*, 20:1–13, 2015.

[2] P. Alquier, V. Cottet, and G. Lecué. Estimation bounds and sharp oracle inequalities of regularized procedures with lipschitz loss functions. *The Annals of Statistics*, 47(4):2117–2144, 2019.

[3] A. Bakshi and A. Prasad. Robust linear regression: Optimal rates in polynomial time. In *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*, pages 102–115, 2021.

[4] S. Balakrishnan, S. S. Du, J. Li, and A. Singh. Computationally efficient robust sparse estimation in high dimensions. In *Conference on Learning Theory*, pages 169–212. PMLR, 2017.

[5] P. C. Bellec. Localized gaussian width of \( m \)-convex hulls with applications to lasso and convex aggregation. *Bernoulli*, 25(4A):3016–3040, 2019.

[6] P. C. Bellec, G. Lecué, and A. B. Tsybakov. Slope meets lasso: improved oracle bounds and optimality. *The Annals of Statistics*, 46(6B):3603–3642, 2018.

[7] A. Belloni, V. Chernozhukov, and L. Wang. Square-root lasso: pivotal recovery of sparse signals via conic programming. *Biometrika*, 98(4):791–806, 2011.

[8] P. J. Bickel, Y. Ritov, and A. B. Tsybakov. Simultaneous analysis of lasso and dantzig selector. *The Annals of Statistics*, 37(4):1705–1732, 2009.

[9] S. Boucheron, G. Lugosi, and P. Massart. *Concentration inequalities: A nonasymptotic theory of independence*. Oxford university press, 2013.

[10] P. Bühlmann and S. Van De Geer. *Statistics for high-dimensional data: methods, theory and applications*. Springer Science & Business Media, 2011.

[11] E. Candes and T. Tao. The dantzig selector: Statistical estimation when \( p \) is much larger than \( n \). *The Annals of Statistics*, 35(6):2313–2351, 2007.

[12] M. Chen, C. Gao, and Z. Ren. Robust covariance and scatter matrix estimation under Huber’s contamination model. *The Annals of Statistics*, 46(5):1932–1960, 2018.

[13] X. Chen and W.-X. Zhou. Robust inference via multiplier bootstrap. *The Annals of Statistics*, 48(3):1665–1691, 2020.

[14] Y. Chen, C. Caramanis, and S. Mannor. Robust sparse regression under adversarial corruption. In *International Conference on Machine Learning*, pages 774–782. PMLR, 2013.

[15] Y. Cheng, I. Diakonikolas, and R. Ge. High-dimensional robust mean estimation in nearly-linear time. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2755–2771. SIAM, 2019.

[16] Y. Cheng, I. Diakonikolas, R. Ge, and D. Woodruff. Faster algorithms for high-dimensional robust covariance estimation. *arXiv preprint arXiv:1906.04661*, 2019.

[17] Y. Cherapanamjeri, E. Aras, N. Tripuraneni, M. I. Jordan, N. Flammarion, and P. L. Bartlett. Optimal robust linear regression in nearly linear time. *arXiv preprint arXiv:2007.08137*, 2020.

[18] G. Chinot. Erm and rerm are optimal estimators for regression problems when malicious outliers corrupt the labels. *Electronic Journal of Statistics*, 14(2):3563–3605, 2020.
[19] G. Chinot, G. Lecuê, and M. Lerasle. Robust high dimensional learning for lipschitz and convex losses. *Journal of Machine Learning Research*, 21:233, 2020.

[20] A. Dalalyan and Y. Chen. Fused sparsity and robust estimation for linear models with unknown variance. *Advances in Neural Information Processing Systems*, 25, 2012.

[21] A. Dalalyan and P. Thompson. Outlier-robust estimation of a sparse linear model using \( \ell_1 \)-penalized Huber’s m-estimator. In H. Wallach, H. Larochelle, A. Beygelzimer, F. d’Alché Buc, E. Fox, and R. Garnett, editors, *Advances in Neural Information Processing Systems 32*, pages 13188–13198. Curran Associates, Inc., 2019.

[22] A. S. Dalalyan and A. Minasyan. All-in-one robust estimator of the gaussian mean. *The Annals of Statistics*, 50(2):1193–1219, 2022.

[23] J. Depersin and G. Lecué. Robust sub-gaussian estimation of a mean vector in nearly linear time. *The Annals of Statistics*, 50(1):511–536, 2022.

[24] A. Derumigny. Improved bounds for square-root lasso and square-root slope. *Electronic Journal of Statistics*, 12(1):741–766, 2018.

[25] I. Diakonikolas and D. M. Kane. Recent advances in algorithmic high-dimensional robust statistics. *arXiv preprint arXiv:1911.05911*, 2019.

[26] I. Diakonikolas, G. Kamath, D. M. Kane, J. Li, A. Moitra, and A. Stewart. Being robust (in high dimensions) can be practical. In *Proceedings of the 34th International Conference on Machine Learning—Volume 70*, pages 999–1008. JMLR. org, 2017.

[27] I. Diakonikolas, D. M. Kane, and A. Stewart. Statistical query lower bounds for robust estimation of high-dimensional gaussians and gaussian mixtures. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 73–84. IEEE, 2017.

[28] I. Diakonikolas, G. Kamath, D. M. Kane, J. Li, A. Moitra, and A. Stewart. Robustly learning a gaussian: Getting optimal error, efficiently. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2683–2702. Society for Industrial and Applied Mathematics, 2018.

[29] I. Diakonikolas, T. Gouleakis, and C. Tzamos. Distribution-independent pac learning of halfspaces with massart noise. *Advances in Neural Information Processing Systems*, 32, 2019.

[30] I. Diakonikolas, G. Kamath, D. Kane, J. Li, A. Moitra, and A. Stewart. Robust estimators in high-dimensions without the computational intractability. *SIAM Journal on Computing*, 48(2):742–864, 2019.

[31] I. Diakonikolas, D. Kane, S. Karmalkar, E. Price, and A. Stewart. Outlier-robust high-dimensional sparse estimation via iterative filtering. In *Advances in Neural Information Processing Systems*, pages 10689–10700, 2019.

[32] I. Diakonikolas, W. Kong, and A. Stewart. Efficient algorithms and lower bounds for robust linear regression. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2745–2754. SIAM, 2019.

[33] I. Diakonikolas, D. M. Kane, and P. Manurangsi. The complexity of adversarially robust proper learning of halfspaces with agnostic noise. *Advances in Neural Information Processing Systems*, 33:20449–20461, 2020.

[34] I. Diakonikolas, V. Kontonis, C. Tzamos, and N. Zarifis. Learning halfspaces with massart noise under structured distributions. In *Conference on Learning Theory*, pages 1486–1513. PMLR, 2020.
[35] I. Diakonikolas, D. M. Kane, V. Kontonis, C. Tzamos, and N. Zarifis. Efficiently learning halfspaces withtsybakov noise. In Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing, pages 88–101, 2021.

[36] S. Dirksen. Tail bounds via generic chaining. Electronic Journal of Probability, 20:1–29, 2015.

[37] Y. Dong, S. Hopkins, and J. Li. Quantum entropy scoring for fast robust mean estimation and improved outlier detection. In Advances in Neural Information Processing Systems, pages 6067–6077, 2019.

[38] J. Fan and R. Li. Variable selection via nonconcave penalized likelihood and its oracle properties. Journal of the American statistical Association, 96(456):1348–1360, 2001.

[39] J. Fan, Q. Li, and Y. Wang. Estimation of high dimensional mean regression in the absence of symmetry and light tail assumptions. Journal of the Royal Statistical Society. Series B, Statistical methodology, 79(1):247, 2017.

[40] J. Fan, H. Liu, Q. Sun, and T. Zhang. I-lamm for sparse learning: Simultaneous control of algorithmic complexity and statistical error. Annals of statistics, 46(2):814, 2018.

[41] J. Fan, W. Wang, and Z. Zhu. A shrinkage principle for heavy-tailed data: High-dimensional robust low-rank matrix recovery. Annals of statistics, 49(3):1239, 2021.

[42] C. Gao. Robust regression via mutivariate regression depth. Bernoulli, 26(2):1139–1170, 2020.

[43] D. L. Hanson and F. T. Wright. A bound on tail probabilities for quadratic forms in independent random variables. The Annals of Mathematical Statistics, 42(3):1079–1083, 1971.

[44] D. Hsu, S. Kakade, and T. Zhang. A tail inequality for quadratic forms of subgaussian random vectors. Electronic Communications in Probability, 17:1–6, 2012.

[45] P. K. Kothari, J. Steinhardt, and D. Steurer. Robust moment estimation and improved clustering via sum of squares. In Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, pages 1035–1046. ACM, 2018.

[46] K. A. Lai, A. B. Rao, and S. Vempala. Agnostic estimation of mean and covariance. In Foundations of Computer Science (FOCS), 2016 IEEE 57th Annual Symposium on, pages 665–674. IEEE, 2016.

[47] G. Lecué and M. Lerasle. Robust machine learning by median-of-means: theory and practice. The Annals of Statistics, 48(2):906–931, 2020.

[48] G. Lecué and S. Mendelson. Regularization and the small-ball method i: sparse recovery. The Annals of Statistics, 46(2):611–641, 2018.

[49] L. Liu, Y. Shen, T. Li, and C. Caramanis. High dimensional robust sparse regression. In International Conference on Artificial Intelligence and Statistics, pages 411–421. PMLR, 2020.

[50] G. Lugosi and S. Mendelson. Robust multivariate mean estimation: the optimality of trimmed mean. The Annals of Statistics, 49(1):393–410, 2021.

[51] P. Massart. About the constants in talagrand’s concentration inequalities for empirical processes. The Annals of Probability, 28(2):863–884, 2000.

[52] S. Mendelson. Upper bounds on product and multiplier empirical processes. Stochastic Processes and their Applications, 126(12):3652–3680, 2016.

[53] I. Merad and S. Gaifas. Robust methods for high-dimensional linear learning. Journal of Machine Learning Research, 24(165):1–44, 2023.
[54] S. Minsker, M. Ndaoud, and L. Wang. Robust and tuning-free sparse linear regression via square-root slope. *arXiv preprint arXiv:2210.16808*, 2022.

[55] O. Montasser, S. Goel, I. Diakonikolas, and N. Srebro. Efficiently learning adversarially robust halfspaces with noise. In *International Conference on Machine Learning*, pages 7010–7021. PMLR, 2020.

[56] N. H. Nguyen and T. D. Tran. Robust lasso with missing and grossly corrupted observations. *IEEE transactions on information theory*, 59(4):2036–2058, 2012.

[57] S. Oymak. Learning compact neural networks with regularization. In *International Conference on Machine Learning*, pages 3966–3975. PMLR, 2018.

[58] A. Pensia, V. Jog, and P.-L. Loh. Robust regression with covariate filtering: Heavy tails and adversarial contamination. *arXiv preprint arXiv:2009.12976*, 2020.

[59] Y. Plan and R. Vershynin. One-bit compressed sensing by linear programming. *Communications on Pure and Applied Mathematics*, 66(8):1275–1297, 2013.

[60] A. Prasad, S. Balakrishnan, and P. Ravikumar. A robust univariate mean estimator is all you need. In *International Conference on Artificial Intelligence and Statistics*, pages 4034–4044. PMLR, 2020.

[61] G. Raskutti, M. J. Wainwright, and B. Yu. Restricted eigenvalue properties for correlated gaussian designs. *The Journal of Machine Learning Research*, 11:2241–2259, 2010.

[62] M. Rudelson and R. Vershynin. Hanson-wright inequality and sub-gaussian concentration. *Electronic Communications in Probability*, 18:1–9, 2013.

[63] T. Sasai. Robust and sparse estimation of linear regression coefficients with heavy-tailed noises and covariates. *arXiv preprint arXiv:2206.07594*, 2022.

[64] Y. She and A. B. Owen. Outlier detection using nonconvex penalized regression. *Journal of the American Statistical Association*, 106(494):626–639, 2011.

[65] V. Sivakumar, A. Banerjee, and P. K. Ravikumar. Beyond sub-gaussian measurements: High-dimensional structured estimation with sub-exponential designs. *Advances in neural information processing systems*, 28, 2015.

[66] W. Su and E. Candes. Slope is adaptive to unknown sparsity and asymptotically minimax. *The Annals of Statistics*, 44(3):1038–1068, 2016.

[67] Q. Sun, W.-X. Zhou, and J. Fan. Adaptive Huber regression. *Journal of the American Statistical Association*, 115(529):254–265, 2020.

[68] M. Talagrand. *Upper and lower bounds for stochastic processes*, volume 60. Springer, 2014.

[69] P. Thompson. Outlier-robust sparse/low-rank least-squares regression and robust matrix completion. *arXiv preprint arXiv:2012.06750*, 2020.

[70] R. Tibshirani. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society: Series B*, 58(1):267–288, 1996.

[71] R. Vershynin. *High-dimensional probability: An introduction with applications in data science*, volume 47. Cambridge university press, 2018.

[72] T. Wang, Q. Berthet, and R. J. Samworth. Statistical and computational trade-offs in estimation of sparse principal components. *The Annals of Statistics*, 44(5):1896–1930, 2016.
[73] F. T. Wright. A bound on tail probabilities for quadratic forms in independent random variables whose distributions are not necessarily symmetric. *The Annals of Probability*, 1(6):1068–1070, 1973.

[74] M. Yuan and Y. Lin. Model selection and estimation in regression with grouped variables. *Journal of the Royal Statistical Society: Series B*, 68(1):49–67, 2006.

[75] K. Zajkowski. Bounds on tail probabilities for quadratic forms in dependent sub-gaussian random variables. *Statistics & Probability Letters*, 167:108898, 2020.

[76] C.-H. Zhang. Nearly unbiased variable selection under minimax concave penalty. *The Annals of Statistics*, 38(2):894–942, 2010.

[77] H. Zou and T. Hastie. Regularization and variable selection via the elastic net. *Journal of the Royal Statistical Society: Series B*, 67(2):301–320, 2005.
Appendix

In the appendices, the value of the numerical constant $C$ be allowed to change from line to line.

A Proof of Theorem 3.1

Suppose that the assumptions in Theorem 3.1 hold. To prove Theorem 3.1, it is sufficient that we confirm (5.5) - (5.8) in Proposition 5.2 hold with high probability, as described already, and then we obtain the result (3.26) with concrete values of $r_{a,1}, r_{a,2}, r_{a,\Sigma}, r_{b,1}, r_{b,2}, r_{b,\Sigma}, b$. Obviously, we have $r_1 = c_1 \sqrt{s r_{\Sigma}}$ and $r_2 = c_2 r_{\Sigma}$ in (5.8) from (3.25). In the following, we show the remaining. We see that, under the assumptions in Theorem 3.1, Propositions 5.1, 5.3 and 5.6 hold immediately with probability at least $1 - 4\delta$, and then, Propositions 5.4 and 5.5 also hold because (5.1) and (5.2) in Proposition 5.1 hold and Algorithm 2 returns $\hat{w}$ from the definition of $\tau_{\text{cut}}$.

A.1 Confirmation of (5.5)

From the following lemma, we can confirm (5.5). The proofs are given in the Appendix G.

Lemma A.1. Assume that Propositions 5.1, 5.3-5.6 hold. For any $\theta, \eta \in r_{1,1} B^d_1 \cap r_{2,2} B^d_2 \cap r_{\Sigma} B^d_\Sigma$, we have

$$\left| \sum_{i=1}^{n} \hat{w}_i h(\beta_{\theta, \eta}) X_{i}^T \theta \right| \leq 3c_{\text{max}}^2 L \left( \rho r_{d,s} r_{1} + r_{\delta} r_{\Sigma} + \frac{\kappa u}{n} (s r_{d,s} + r_{\delta}) r_{2} + \kappa u r_{\alpha} r_{2} \right). \quad (A.1)$$

From Lemma A.1, we see that (5.5) holds with

$$r_{a,1} = 3c_{\text{max}}^2 L \lambda_o \sqrt{n} r_{d,s}, \quad r_{a,2} = 3c_{\text{max}}^2 L \lambda_o \sqrt{n} \left( \frac{\kappa u}{n} (s r_{d,s} + r_{\delta}) + \kappa u r_{\alpha} r_{2} \right), \quad r_{a,\Sigma} = 3c_{\text{max}}^2 L \lambda_o \sqrt{n} r_{\delta}. \quad (A.2)$$

A.2 Confirmation of (5.7)

From (A.2),

$$(C_s :=) r_{a,1} + \frac{c_{\text{re}} r_{a,2} + r_{a,\Sigma}}{c_{\text{re}} \sqrt{s}} = 3c_{\text{max}}^2 L \lambda_o \sqrt{n} \times \frac{1}{c_{\text{re}} \sqrt{s}} R_{d,n,o}. \quad (A.3)$$

From the definition of $\lambda_s$,

$$\frac{\lambda_s}{C_s} \geq \frac{c_{\text{re}}^2 c_{\text{max}}^2 L \lambda_o \sqrt{n} \frac{1}{c_{\text{re}} \sqrt{s}} R_{d,n,o}}{3c_{\text{max}}^2 L \lambda_o \sqrt{n} \frac{1}{c_{\text{re}} \sqrt{s}} R_{d,n,o}} = \frac{c_{\text{re}} + 1}{c_{\text{re}} - 1} > 0. \quad (A.4)$$

Hence, we have $\lambda_s - C_s > 0$ and

$$\frac{\lambda_s + C_s}{\lambda_s - C_s} = \frac{1 + \frac{C_s}{\lambda_s}}{1 - \frac{C_s}{\lambda_s}} \leq 1 + \frac{c_{\text{re}} - 1}{c_{\text{re}} + 1} = c_{\text{re}}. \quad (A.5)$$

Therefore, we see that (5.7) holds.

A.3 Confirmation of (5.6)

From the following lemma, we can confirm (5.6). The proofs are given in the Appendix G.
Lemma A.2. Assume that Propositions 5.1, 5.3-5.6 hold. For any $\theta_\eta \in r_1 \mathbb{B}^d_1 \cap r_2 \mathbb{B}^d_2 \cap r_\Sigma \mathbb{B}^d_\Sigma$, we have

$$\sum_{i=1}^{n} \lambda_\eta \sqrt{n} \hat{w}_i \left( -h(r_\beta + \theta_\eta, i) + h(r_\beta, i) \right) X_i \theta_\eta \geq \frac{\| \Sigma^2 v \|^2}{3} - c_2^2 \lambda_\eta \sqrt{n} \left( \rho r_{d,s} r_1 + r_\delta r_\Sigma + \kappa_u \sqrt{\frac{\theta}{n} (sr_{d,s} + r_\delta)} r_2 + \kappa_u r_\rho r_2 \right).$$  \hfill (A.6)

From Lemma A.2, we see that (5.6) holds with

$$b = \frac{1}{3}, \quad r_{b,1} = \frac{r_{a,1}}{3}, \quad r_{b,2} = \frac{r_{a,2}}{3}, \quad r_{b,\Sigma} = \frac{r_{a,\Sigma}}{3}. \hfill (A.7)$$

### A.4 Confirmation of (5.8)

As we mentioned at the beginning of Section A, $r_1 = c_{r_1} \sqrt{s} r_\Sigma$ and $r_2 = c_{r_2} r_\Sigma$ is clear from their definitions. We confirm $r_\Sigma$. We see that

$$\frac{2}{b} \left( c_{r_1} \sqrt{s} (r_{a,1} + r_{b,1}) + c_{r_2} (r_{a,2} + r_{b,2}) + r_{a,\Sigma} + r_{b,\Sigma} + c_{r_1} \sqrt{s} \lambda_s \right) \leq 6c_{\max}^2 \lambda_\eta \sqrt{n} (4 + c_d) R_{d,n,o} \leq r_\Sigma,$$  \hfill (A.8)

and the proof is complete.

### B Proof of Theorem 4.1

The strategy of this proof is the same as the one of Theorem 3.1. Suppose that the assumptions in Theorem 4.1 hold. To prove Theorem 4.1, it is sufficient that we confirm (5.5) - (5.8) in Proposition 5.2 hold with high probability, as described already, and then we obtain the result (4.6) with concrete values of $r_{a,1}$, $r_{a,2}$, $r_{a,\Sigma}$, $r_{b,1}$, $r_{b,2}$, $r_{b,\Sigma}$, $b$. Obviously, we have $r_1 = c_{r_1} \sqrt{s} r_\Sigma$ and $r_2 = c_{r_2} r_\Sigma$ in (5.8) from (3.25). In the following, we show the remaining. We see that, under the assumptions in Theorem 4.1, Corollary 5.1, Propositions 5.3 and 5.6 hold immediately with probability at least $1 - 3\delta$, and then, Propositions 5.7 and 5.8 also hold because (5.15) and (5.16) in Corollary 5.1 hold, and Algorithm 5 returns $\hat{w}$ from the definition of $\tau_{\text{cut}}$.

#### B.1 Confirmation of (5.5)

From the following lemma, we can confirm (5.5). The proofs are given by Appendix G.

**Lemma B.1.** Assume that Corollary 5.1, Propositions 5.3, 5.6, 5.7 and 5.8 hold. For any $\theta_\eta \in r_1 \mathbb{B}^d_1 \cap r_2 \mathbb{B}^d_2 \cap r_\Sigma \mathbb{B}^d_\Sigma$, we have

$$\left| \sum_{i=1}^{n} \hat{w}_i h(r_\beta, i) X_i \theta_\eta \right| \leq 3c_{\max}^2 \lambda_\eta \sqrt{n} \rho r_{d,s} + \frac{\lambda_\eta^2}{n} \sqrt{\frac{\theta}{n} (sr_{d,s} + r_\delta)} + \lambda_\eta \sqrt{n} \Sigma_{\max}^2 r_2.$$  \hfill (B.1)

From Lemma A.1, we see that (5.5) holds with

$$r_{a,1} = 3c_{\max}^2 \lambda_\eta \sqrt{n} \rho r_{d,s}, \quad r_{a,2} = 3c_{\max}^2 \lambda_\eta \sqrt{n} \lambda_\eta \sqrt{n} \sqrt{\frac{\theta}{n} (sr_{d,s} + r_\delta)} + \lambda_\eta \sqrt{n} \Sigma_{\max}^2,$$

$$r_{a,\Sigma} = 3c_{\max}^2 \lambda_\eta \sqrt{n} r_\delta.$$  \hfill (B.2)
B.2 Confirmation of (5.7)

From (B.2),

\[(C_s :=) r_{a,1} + \frac{c_{r_2} r_{a,2} + r_{a,\Sigma}}{c_{r_3}} = 3 \sqrt[n]{\lambda} \sqrt{n} \times \frac{1}{c_{r_1}} \sqrt{r_{d,n,o}}. \]  

(B.3)

From the definition of \(\lambda_s\),

\[\frac{\lambda_s}{C_s} \geq \frac{c_s c_{\lambda}^2 L \sqrt{n}}{3 \epsilon_{\lambda}^2 L \sqrt{n}} \frac{1}{c_{r_1}} \sqrt{r_{d,n,o}} = \frac{c_s}{3} \geq \frac{c_{\text{RE}} + 1}{c_{\text{RE}} - 1} > 0. \]  

(B.4)

Hence, we have \(\lambda_s - C_s > 0\) and

\[\frac{\lambda_s + C_s}{\lambda_s - C_s} = 1 + \frac{C_s}{\lambda_s} \leq 1 - \frac{c_{\text{RE}} - 1}{c_{\text{RE}} + 1} = c_{\text{RE}}. \]  

(B.5)

Therefore, we see that (5.7) holds.

B.3 Confirmation of (5.6)

From the following lemma, we can confirm (5.6). The proofs are given by Appendix G.

Lemma B.2. Assume that Corollary 5.1, Propositions 5.3, 5.6, 5.7 and 5.8 hold. For any \(\theta_{\eta} \in r_1 E_1 \cap r_2 E_2 \cap r_{\Sigma} E_{\Sigma}, \) we have

\[\sum_{i=1}^{n} \lambda_{i} \sqrt{n} h_{i}^2(-h(r_{\beta} \cdot + \theta_{\eta,i}) + h(r_{\beta} \cdot - i)) X_i \theta_{\eta} \geq \frac{\|\Sigma \|^2}{3} - c_{\lambda}^2 L \lambda \sqrt{n} \left(\rho r_{d,s} r + r_{d,s} + \sqrt{\frac{\rho}{n}} \sqrt{\nu_{\Sigma}^2 (sr_{d,s} + r_{d,s}) + \Sigma_{\max}^2} \right). \]  

(B.6)

From Lemma B.2, we see that (5.6) holds with

\[b = \frac{1}{3}, \quad r_{b,1} = \frac{r_{a,1}}{3}, \quad r_{b,2} = \frac{r_{a,2}}{3}, \quad r_{b,\Sigma} = \frac{r_{a,\Sigma}}{3}. \]  

(B.7)

B.4 Confirmation of (5.8)

As we mentioned at the beginning of Appendix B, \(r_1 = c_{r_1} \sqrt{r_{\Sigma}}\) and \(r_2 = c_{r_2} r_{\Sigma}\) is clear from their definitions. We confirm \(r_{\Sigma}\). We see that

\[\frac{2}{b} \left( c_{r_1} \sqrt{s} (r_{a,1} + r_{b,1}) + c_{r_2} (r_{a,2} + r_{b,2}) + r_{a,\Sigma} + r_{b,\Sigma} + c_{r_1} \sqrt{s} \lambda_s \right) \leq 6 c_{\lambda}^2 L \lambda \sqrt{n} (4 + c_s) \left(\rho c_{r_1} \sqrt{s} r_{d,s} + r_{d,s} + c_{r_2} \sqrt{\frac{\rho}{n}} \sqrt{\nu_{\Sigma}^2 (sr_{d,s} + r_{d,s}) + \Sigma_{\max}^2} \right) \leq r_{\Sigma}, \]  

(B.8)

and the proof is complete.

C Preparation of proof of Proposition 5.2

We introduce the following four lemmas, that are used in the proof of Proposition 5.2.

Lemma C.1. Suppose that (5.5), (5.7), \(\|\theta_{\eta}\|_1 \leq r_1\), \(\|\theta_{\eta}\|_2 = r_2\) and \(\|\Sigma^{\frac{1}{2}} \theta_{\eta}\|_2 \leq r_{\Sigma}\), where \(r_1 = c_{r_1} \sqrt{s} r_{\Sigma}\) and \(r_2 = c_{r_2} r_{\Sigma}\). Then, for any fixed \(\eta \in (0, 1)\), we have

\[\|\theta_{\eta}\|_2 \leq \frac{3 + c_{\text{RE}}}{\kappa_1} \|\Sigma^{\frac{1}{2}} \theta_{\eta}\|_2. \]  

(C.1)
Lemma C.2. Suppose that \((5.5), (5.7)\), \(||\theta_\eta||_1 = r_1, ||\theta_\eta||_2 \leq r_2\) and \(||\Sigma^{1/2} \theta_\eta||_2 \leq r_\Sigma\), where \(r_1 = c_r \sqrt{sr_\Sigma}\) and \(r_2 = c_r r_\Sigma\) hold. Then, for any fixed \(\eta \in (0, 1)\), we have
\[
||\theta_\eta||_1 \leq \frac{1+ \text{CRE}}{r} \sqrt{s} ||\Sigma^{1/2} \theta_\eta||_2. \tag{C.2}
\]

Lemma C.3. Suppose that \((5.5) - (5.6)\), \(||\theta_\eta||_1 \leq r_1\), \(||\theta_\eta||_2 \leq r_2\) and \(||\Sigma^{1/2} \theta_\eta||_2 = r_\Sigma\), where \(r_1 = c_r \sqrt{sr_\Sigma}\) and \(r_2 = c_r r_\Sigma\) hold. Then, for any \(\eta \in (0, 1)\), we have
\[
||\Sigma^{1/2} \theta_\eta||_2 \leq \frac{1}{b} (c_r \sqrt{s}(r_{a,1} + r_{b,1}) + c_r (r_{a,2} + r_{b,2}) + r_{a,\Sigma} + r_{b,\Sigma} + c_r \sqrt{s} \lambda_\eta). \tag{C.3}
\]

C.1 Proof of Lemma C.1

For a vector \(v = (v_1, \ldots, v_d)\), define \(\{v^1_1, \ldots, v^d_d\}\) as a non-increasing rearrangement of \(\{|v_1|, \ldots, |v_d|\}\), and \(v^\sharp \in \mathbb{R}^d\) as a vector such that \(v^\sharp_i = v^i_i\). For the sets \(S_1 = \{1, \ldots, s\}\) and \(S_2 = \{s+1, \ldots, d\}\), let \(v^{S_1} = v^s_1\) and \(v^{S_2} = v^s_2\).

In Section C.1.1, we have
\[
||\theta_\eta||_2 \leq \frac{\sqrt{1 + \text{CRE}}}{\kappa_1} ||\Sigma^{1/2} \theta_\eta||_2 \tag{C.4}
\]
assuming \(||\theta_\eta||_2 \leq ||\theta_\eta||_1/\sqrt{s}||\), and in Section C.1.2, we have
\[
||\theta_\eta||_2 \leq 2 ||\theta^S\eta||_2 \leq \frac{2}{\kappa_1} ||\Sigma^{1/2} \theta_\eta||_2 \tag{C.5}
\]
assuming \(||\theta_\eta||_2 \geq ||\theta_\eta||_1/\sqrt{s}||\). From the above two inequalities, we have
\[
||\theta_\eta||_2 \leq \frac{3 + \text{CRE}}{\kappa_1} ||\Sigma^{1/2} \theta_\eta||_2. \tag{C.6}
\]

C.1.1 Case I

In Section C.1.1, suppose that \(||\theta_\eta||_2 \leq ||\theta_\eta||_1/\sqrt{s}||\). Let
\[
Q'(\eta) = \lambda_\alpha \sqrt{n} \hat{w}_i' \sum_{i=1}^n (-h(r_{\beta^*,\theta,\eta,i}) + h(r_{\beta^*,\eta,i})) X_{\eta}^\top \theta. \tag{C.7}
\]

From the proof of Lemma F.2. of [40], we have \(\eta Q'(\eta) \leq \eta Q'(1)\) and this means
\[
\sum_{i=1}^n \lambda_\alpha \sqrt{n} \hat{w}_i' (-h(r_{\beta^*,\eta,i}) + h(r_{\beta^*,\eta,i})) X_{\eta}^\top \theta \leq \sum_{i=1}^n \lambda_\alpha \sqrt{n} \hat{w}_i' \eta (-h(r_{\beta^*,\eta,i}) + h(r_{\beta^*,\eta,i})) X_{\eta}^\top \theta. \tag{C.8}
\]

Let \(\partial v\) be the sub-differential of \(||v||_1\). Adding \(\eta \lambda_\alpha (||\bar{\theta}||_1 - ||\bar{\theta}^*||_1)\) to both sides of (C.8), we have
\[
\sum_{i=1}^n \lambda_\alpha \sqrt{n} \hat{w}_i' (-h(r_{\beta^*,\eta,i}) + h(r_{\beta^*,\eta,i})) X_{\eta}^\top \theta + \eta \lambda_\alpha (||\bar{\theta}||_1 - ||\bar{\theta}^*||_1) \leq \sum_{i=1}^n \lambda_\alpha \sqrt{n} \hat{w}_i' \eta (-h(r_{\beta^*,\eta,i}) + h(r_{\beta^*,\eta,i})) X_{\eta}^\top \theta + \eta \lambda_\alpha (||\bar{\theta}||_1 - ||\bar{\theta}^*||_1) \tag{C.9}
\]
where (a) follows from \(||\bar{\theta}||_1 - ||\bar{\theta}^*||_1 \leq (\partial \bar{\theta}, \theta)\), which is the definition of the sub-differential, and
(b) follows from the optimality of \(\bar{\theta}\).
From the convexity of Huber loss, the left-hand side of (C.9) is positive and we have

\[ 0 \leq \sum_{i=1}^{n} \lambda_i^\eta \sqrt{n} w_i^\eta h(r_{\beta^*,i} X_i^\top \theta^*_\eta + \eta \lambda_s (\|\beta^*\|_1 - \|\hat{\beta}\|_1)). \]  

(C.10)

From (5.5), the first term of the right-hand side of (C.10) is evaluated as

\[ \sum_{i=1}^{n} \lambda_i^\eta \sqrt{n} w_i^\eta h(r_{\beta^*,i} X_i^\top \theta^*_\eta) \leq r_{a,1} r_1 + r_{a,2} r_2 + r_{a,\Sigma} r_{\Sigma} \]

\[ \leq \left( \frac{c_{r_2}}{c_{r_2}} \sqrt{s r_{a,1}} + r_{a,2} + \frac{1}{c_{r_2}} r_{a,\Sigma} \right) \|\theta^*_\eta\|_2 \]

\[ \leq \frac{1}{\sqrt{s}} \left( \frac{c_{r_2}}{c_{r_2}} \sqrt{s r_{a,1}} + r_{a,2} + \frac{1}{c_{r_2}} r_{a,\Sigma} \right) \|\theta^*_\eta\|_1, \]  

(C.11)

where (a) follows from \( r_1 = c_{r_2} \sqrt{s r_{\Sigma}} \) and \( r_2 = c_{r_2} r_{\Sigma} \), and (b) follows from the assumption of this section, \( \|\theta^*_\eta\|_2 \leq \|\theta^*_\eta\|_1/\sqrt{s} \). From (C.10), (C.11) and the assumption \( \|\theta^*_\eta\|_2 \leq \|\theta^*_\eta\|_1/\sqrt{s} \), we have

\[ 0 \leq \left( \frac{r_{a,2}}{\sqrt{s}} + \frac{c_{r_2} \sqrt{s r_{a,1}} + r_{a,\Sigma}}{c_{r_2} \sqrt{s}} \right) \|\theta^*_\eta\|_1 + \eta \lambda_s (\|\beta^*\|_1 - \|\hat{\beta}\|_1). \]

(C.12)

Define \( J_a \) as the index set of non-zero entries of \( a \), and \( \theta^*_\eta \), \( J_a \) as a vector such that \( \theta^*_\eta, J_a \| = \theta^*_\eta \| \) for \( i \in J_a \) and \( \theta^*_\eta, J_a \| = 0 \) for \( i \notin J_a \). Furthermore, we see

\[ 0 \leq \left( \frac{r_{a,2}}{\sqrt{s}} + \frac{c_{r_2} \sqrt{s r_{a,1}} + r_{a,\Sigma}}{c_{r_2} \sqrt{s}} \right) \|\theta^*_\eta\|_1 + \eta \lambda_s (\|\beta^*\|_1 - \|\hat{\beta}\|_1) \]

\[ \leq \left( \frac{r_{a,2}}{\sqrt{s}} + \frac{c_{r_2} \sqrt{s r_{a,1}} + r_{a,\Sigma}}{c_{r_2} \sqrt{s}} \right) \|\theta^*_\eta, J_a^\ast\|_1 + \eta \lambda_s (\|\beta^*\|_1 - \|\hat{\beta}\|_1) \]

\[ = \left( \lambda_s + \left( \frac{r_{a,2}}{\sqrt{s}} + \frac{c_{r_2} \sqrt{s r_{a,1}} + r_{a,\Sigma}}{c_{r_2} \sqrt{s}} \right) \|\theta^*_\eta, J_a^\ast\|_1 \right) \]

\[ \|\theta^*_\eta, J_a^\ast\|_1 \leq \|\theta^*_\eta, J_a^\ast\|_1 - \]  

\[ \left( \lambda_s + \left( \frac{r_{a,2}}{\sqrt{s}} + \frac{c_{r_2} \sqrt{s r_{a,1}} + r_{a,\Sigma}}{c_{r_2} \sqrt{s}} \right) \right) \|\theta^*_\eta, J_a^\ast\|_1 \]  

\[ \leq \left( \lambda_s + \left( \frac{r_{a,2}}{\sqrt{s}} + \frac{c_{r_2} \sqrt{s r_{a,1}} + r_{a,\Sigma}}{c_{r_2} \sqrt{s}} \right) \right) \|\theta^*_\eta, J_a^\ast\|_1. \]

(C.13)

Then, we have

\[ \|\theta^*_\eta, J_a^\ast\|_1 \leq \lambda_s + \left( \frac{r_{a,2}}{\sqrt{s}} + \frac{c_{r_2} \sqrt{s r_{a,1}} + r_{a,\Sigma}}{c_{r_2} \sqrt{s}} \right) \|\theta^*_\eta, J_a^\ast\|_1 \]

\[ \leq \lambda_s + \left( \frac{r_{a,1} + c_{r_2} \sqrt{s r_{a,1}} + r_{a,\Sigma}}{c_{r_2} \sqrt{s}} \right) \|\theta^*_\eta, J_a^\ast\|_1 \]

\[ \leq \lambda_s + \left( \frac{r_{a,1} + c_{r_2} \sqrt{s r_{a,1}} + r_{a,\Sigma}}{c_{r_2} \sqrt{s}} \right) \|\theta^*_\eta, J_a^\ast\|_1 \]

\[ \leq c_{RE} \|\theta^*_\eta, J_a^\ast\|_1. \]

(C.14)

where (a) follows from the fact that \( c_{r_2} \geq c_{r_1} \) and (b) follows from (5.7), and from the definition of \( \|\theta^*\eta^2\|_1 \) and \( \|\theta^*\eta^1\|_1 \), we have

\[ \|\theta^*\eta^2\|_1 \leq c_{RE} \|\theta^*\eta^1\|_1. \]

(C.15)

Then, from the standard shelling argument, we have

\[ \|\theta^*\eta^2\|_2 \leq \sum_{i=s+1}^{d} (\theta^*_\eta^i)^2 \leq \sum_{i=s+1}^{d} (\theta^*_\eta^i)^2 \left( \frac{1}{s} \sum_{j=1}^{s} (\theta^*_\eta^j)^2 \right) \]

\[ \leq \frac{1}{s} \|\theta^*\eta^2\|_1 \|\theta^*\eta^2\|_1 \leq \frac{c_{RE} \|\theta^*\eta^1\|_2^2}{s} \leq c_{RE} \|\theta^*\eta^2\|_2. \]

(C.16)

and from the definition of \( \kappa_1 \), we have

\[ \kappa_1^2 \|\theta^*\eta^2\|_2 \leq \kappa_1^2 \left( \|\theta^*\eta^1\|_2^2 + \|\theta^*\eta^2\|_2^2 \right) \leq \kappa_1^2 (1 + c_{RE}) \|\theta^*\eta^2\|_2^2 \leq (1 + c_{RE}) \|\Sigma^2 \theta^*\eta^2\|_2^2. \]

(C.17)
C.1.2 Case II

In Section C.1.2, suppose that \( \| \mathbf{\theta}_\eta \|_2 \geq \| \mathbf{\theta}_\eta \|_1 / \sqrt{s} \).

\[
\| \mathbf{\theta}^{(2)}_\eta \|_2^2 = \sum_{i=s+1}^{d} (\mathbf{\theta}^{(2)}_\eta | i)^2 \leq \sum_{i=s+1}^{d} | \mathbf{\theta}^{(2)}_\eta | i \left( \frac{1}{s} \sum_{j=1}^{s} | \mathbf{\theta}^{(2)}_\eta | j \right) \leq \frac{1}{s} \| \mathbf{\theta}^{(2)}_\eta \|_1 \| \mathbf{\theta}^{(2)}_\eta \|_1 \leq \| \mathbf{\theta}^{(2)}_\eta \|_2 \| \mathbf{\theta}_\eta \|_2. \quad (C.18)
\]

Then, we have

\[
\| \mathbf{\theta}_\eta \|_2^2 \leq \| \mathbf{\theta}^{(2)}_\eta \|_2^2 + \| \mathbf{\theta}^{(2)}_\eta \|_2^2 \leq \| \mathbf{\theta}^{(2)}_\eta \|_2 \| \mathbf{\theta}_\eta \|_2 + \| \mathbf{\theta}^{(2)}_\eta \|_2 \| \mathbf{\theta}_\eta \|_2 \Rightarrow \| \mathbf{\theta}_\eta \|_2 \leq 2 \| \mathbf{\theta}^{(2)}_\eta \|_2,
\]

and we have

\[
\| \mathbf{\theta}_\eta \|_2 \leq 2 \| \mathbf{\theta}^{(2)}_\eta \|_2 \leq \frac{2}{k_1} \| \Sigma^{\frac{1}{2}} \mathbf{\theta}^{(2)}_\eta \|_2 \leq \frac{2}{k_1} \| \Sigma^{\frac{1}{2}} \mathbf{\theta} \|_2.
\]

(C.20)

C.2 Proof of Lemma C.2

From the same argument of the proof of Lemma C.1, we have (C.10). From (5.5), the first term of the right-hand side of (C.10) is evaluated as

\[
\sum_{i=1}^{n} \lambda_i \sqrt{\eta_i} \mathbf{h}(r_{\beta^*}, \mathbf{x})^\top \mathbf{\theta}_\eta \leq r_{a,1} r_1 + r_{a,2} r_2 + r_{a,\Sigma} \Sigma
\]

\[
\leq \left( r_{a,1} + \frac{c_{r_2}}{c_{r_1} \sqrt{s}} r_{a,2} + \frac{1}{c_{r_1} \sqrt{s}} r_{a,\Sigma} \right) \| \mathbf{\theta}_\eta \|_1.
\]

where (a) follows from \( r_1 = c_{r_1} \sqrt{s} \Sigma \) and \( r_2 = c_{r_2} \Sigma \). From (C.10) and (C.21), we have

\[
0 \leq \left( r_{a,1} + \frac{c_{r_2} r_{a,2} + r_{a,\Sigma}}{c_{r_1} \sqrt{s}} \right) \| \mathbf{\theta}_\eta \|_1 + \eta \lambda_\alpha (\| \mathbf{\beta}^* \|_1 - \| \hat{\mathbf{\beta}} \|_1).
\]

(C.22)

Furthermore, we see

\[
0 \leq \left( r_{a,1} + \frac{c_{r_2} r_{a,2} + r_{a,\Sigma}}{c_{r_1} \sqrt{s}} \right) \| \mathbf{\theta}_\eta \|_1 + \eta \lambda_\alpha (\| \mathbf{\beta}^* \|_1 - \| \hat{\mathbf{\beta}} \|_1)
\]

\[
\leq \left( r_{a,1} + \frac{c_{r_2} r_{a,2} + r_{a,\Sigma}}{c_{r_1} \sqrt{s}} \right) (\| \mathbf{\theta}_{\eta,\mathbf{J}_{\mathbf{\beta}^*}} \|_1 + \| \mathbf{\theta}_{\eta,\mathbf{J}_{\hat{\mathbf{\beta}}}} \|_1) + \eta \lambda_\alpha (\| \mathbf{\beta}^*_{\mathbf{J}_{\mathbf{\beta}^*}} - \| \hat{\mathbf{\beta}}_{\mathbf{J}_{\hat{\mathbf{\beta}}}} \|_1)
\]

\[
= \left( \lambda_\alpha + \left( r_{a,1} + \frac{c_{r_2} r_{a,2} + r_{a,\Sigma}}{c_{r_1} \sqrt{s}} \right) \right) \| \mathbf{\theta}_{\eta,\mathbf{J}_{\mathbf{\beta}^*}} \|_1 + \left( -\lambda_\alpha + \left( r_{a,1} + \frac{c_{r_2} r_{a,2} + r_{a,\Sigma}}{c_{r_1} \sqrt{s}} \right) \right) \| \mathbf{\theta}_{\eta,\mathbf{J}_{\hat{\mathbf{\beta}}}} \|_1.
\]

(C.23)

Then, we have

\[
\| \mathbf{\theta}_{\eta,\mathbf{J}_{\mathbf{\beta}^*}} \|_1 \leq \frac{\lambda_\alpha + \left( r_{a,1} + \frac{c_{r_2} r_{a,2} + r_{a,\Sigma}}{c_{r_1} \sqrt{s}} \right)}{\lambda_\alpha - \left( r_{a,1} + \frac{c_{r_2} r_{a,2} + r_{a,\Sigma}}{c_{r_1} \sqrt{s}} \right)} \| \mathbf{\theta}_{\eta,\mathbf{J}_{\mathbf{\beta}^*}} \|_1 \leq \| \mathbf{\theta}_{\eta,\mathbf{J}_{\mathbf{\beta}^*}} \|_1, \quad (C.24)
\]

where (a) follows from (5.7), and we have

\[
\| \mathbf{\theta}_\eta \|_1 = \| \mathbf{\theta}_{\eta,\mathbf{J}_{\mathbf{\beta}^*}} \|_1 + \| \mathbf{\theta}_{\eta,\mathbf{J}_{\hat{\mathbf{\beta}}}} \|_1 \leq (1 + c_{\text{RE}}) \| \mathbf{\theta}_{\eta,\mathbf{J}_{\mathbf{\beta}^*}} \|_1 \leq (1 + c_{\text{RE}}) \sqrt{s} \| \mathbf{\theta}_{\eta,\mathbf{J}_{\mathbf{\beta}^*}} \|_2.
\]

(C.25)

From (C.24) and the restricted eigenvalue condition, we have

\[
\| \mathbf{\theta}_\eta \|_1 \leq (1 + c_{\text{RE}}) \sqrt{s} \| \mathbf{\theta}_{\eta,\mathbf{J}_{\mathbf{\beta}^*}} \|_2 \leq \frac{1 + c_{\text{RE}}}{\sqrt{s}} \| \Sigma^{\frac{1}{2}} \mathbf{\theta}_\eta \|_2.
\]

(C.26)
C.3 Proof of Lemma C.3

From the same argument of the proof of Lemma C.1, we have (C.10). From (C.10), we have

$$\sum_{i=1}^{n} \lambda_{i} \sqrt{n \hat{w}_{i}} (-h(r_{\beta_{i}+\theta_{i},i}) + h(r_{\beta_{i},i})) X_{i}^{\top} \theta_{\eta} \leq \sum_{i=1}^{n} \lambda_{i} \sqrt{n \hat{w}_{i}} h(r_{\beta_{i},i}) X_{i}^{\top} \theta_{\eta} + \eta \lambda_{s}(\|\beta^{*}\|_{1} - \|\hat{\beta}\|_{1}).$$

(C.27)

We evaluate each term of (C.27). From (5.6), the left-hand side of (C.27) is evaluated as

$$\sum_{i=1}^{n} \lambda_{i} \sqrt{n \hat{w}_{i}} h(r_{\beta_{i}+\theta_{i},i}) X_{i}^{\top} \theta_{\eta} \geq b \|\Sigma \hat{\theta}_{\eta}\|_{2}^{2} - r_{b,1} r_{1} - r_{b,2} r_{2} - r_{b,\Sigma} r_{\Sigma}$$

where (a) follows from $r_{1} = c_{r_{1}} \sqrt{s_{r}}$ and $r_{2} = c_{r_{2}} r_{\Sigma}$. From (5.5), the first term of the right-hand side of (C.27) is evaluated as

$$\sum_{i=1}^{n} \lambda_{i} \sqrt{n \hat{w}_{i}} h(r_{\beta_{i},i}) X_{i}^{\top} \theta_{\eta} \leq r_{a,1} r_{1} + r_{a,2} r_{2} + r_{a,\Sigma} r_{\Sigma}$$

where (a) follows from $r_{1} = c_{r_{1}} \sqrt{s_{r}}$ and $r_{2} = c_{r_{2}} r_{\Sigma}$. The second term of the right-hand side of (C.27) is evaluated as

$$\eta \lambda_{s}(\|\beta^{*}\|_{1} - \|\hat{\beta}\|_{1}) \leq \lambda_{s}(\|\theta_{\eta}\|_{1} \leq c_{r_{1}} \sqrt{s_{r}} \lambda_{s}(\|\Sigma \hat{\theta}_{\eta}\|_{2}),$$

(C.30)

where (a) follows from $r_{1} = c_{r_{1}} \sqrt{s_{r}}$. Combining the inequalities above, we have

$$b \|\Sigma \hat{\theta}_{\eta}\|_{2}^{2} \leq (c_{r_{1}} \sqrt{s_{r}}(r_{a,1} + r_{b,1}) + c_{r_{2}}(r_{a,2} + r_{b,2}) + r_{a,\Sigma} + c_{r_{1}} \sqrt{s_{r}} \lambda_{s}) \|\Sigma \hat{\theta}_{\eta}\|_{2},$$

(C.31)

and from $\|\Sigma \hat{\theta}_{\eta}\|_{2} \geq 0$, we have

$$\|\Sigma \hat{\theta}_{\eta}\|_{2} \leq \frac{1}{b} (c_{r_{1}} \sqrt{s_{r}}(r_{a,1} + r_{b,1}) + c_{r_{2}}(r_{a,2} + r_{b,2}) + r_{a,\Sigma} + c_{r_{1}} \sqrt{s_{r}} \lambda_{s}),$$

(C.32)

and the proof is complete.

D Proof of Proposition 5.2

D.1 Step 1

We derive a contradiction if $\|\theta\|_{1} > r_{1}$, $\|\theta\|_{2} > r_{2}$ and $\|\Sigma \hat{\theta}\|_{2} > r_{\Sigma}$ hold. Assume that $\|\theta\|_{1} > r_{1}$, $\|\theta\|_{2} > r_{2}$ and $\|\Sigma \hat{\theta}\|_{2} > r_{\Sigma}$. Then we can find $\eta_{1}, \eta_{2}, \eta_{3} \in (0,1)$ such that $\|\theta_{\eta_{1}}\|_{1} = r_{1}$, $\|\theta_{\eta_{2}}\|_{2} = r_{2}$ and $\|\Sigma \hat{\theta}_{\eta_{3}}\|_{2} = r_{\Sigma}$ hold. Define $\eta_{3} = \min\{\eta_{1}, \eta_{2}\}$. We consider the case $\eta_{3} = \eta_{2}$ in Section D.1.1, the case $\eta_{3} = \eta_{2}$ in Section D.1.2, and the case $\eta_{3} = \eta_{1}$ in Section D.1.3.

D.1.1 Step 1(a)

Assume that $\eta_{3} = \eta_{2}$. We see that $\|\Sigma \hat{\theta}_{\eta_{3}}\|_{2} = r_{\Sigma}$, $\|\theta_{\eta_{1}}\|_{1} = r_{1}$ and $\|\theta_{\eta_{3}}\|_{2} \leq r_{2}$ hold. Then, from Lemma C.3, we have

$$\|\Sigma \hat{\theta}_{\eta_{3}}\|_{2} \leq \frac{1}{b} (c_{r_{1}} \sqrt{s_{r}}(r_{a,1} + r_{b,1}) + c_{r_{2}}(r_{a,2} + r_{b,2}) + r_{a,\Sigma} + c_{r_{1}} \sqrt{s_{r}} \lambda_{s}).$$

(D.1)

The case $\eta_{3} = \eta_{2}$ is a contradiction from $\|\Sigma \hat{\theta}_{\eta_{3}}\|_{2} = r_{\Sigma}$ and (5.8).
D.1.2 Step 1(b)

Assume that $\eta_3 = \eta_2$. We see that $\|\Sigma\hat{\theta}_{\eta_3}\|_2 \leq r_\Sigma$, $\|\theta_{\eta_3}\|_1 \leq r_1$ and $\|\theta_{\eta_3}\|_2 = r_2$ hold. Then, from Lemma C.1, we have

$$\|\theta_{\eta_3}\|_2 \leq \frac{3 + cr_2 e}{\eta_1} \|\Sigma\hat{\theta}_{\eta_3}\|_2 \leq \frac{3 + cr_2 e}{\eta_1} r_\Sigma.$$  \hspace{2cm} (D.2)

The case $\eta_3 = \eta_2$ is a contradiction from $\|\theta_{\eta_3}\|_2 = r_2$ and (5.8).

D.1.3 Step 1(c)

Assume that $\eta_3 = \eta_1$. We see that $\|\Sigma\hat{\theta}_{\eta_3}\|_2 \leq r_\Sigma$, $\|\theta_{\eta_3}\|_1 = r_1$ and $\|\theta_{\eta_3}\|_2 \leq r_2$ hold. Then, from Lemma C.2, for $\eta = \eta_3$, we have

$$\|\theta_{\eta_3}\|_1 \leq \frac{1 + cr_2 e}{\tau} \sqrt{\|\Sigma\hat{\theta}_{\eta_3}\|_2} \leq \frac{1 + cr_2 e}{\tau} \sqrt{sr_\Sigma}.$$  \hspace{2cm} (D.3)

The case $\eta_3 = \eta_1$ is a contradiction from $\|\theta_{\eta_3}\|_1 = r_1$ and (5.8).

D.2 Step 2

From the arguments in Section D.1, we have $\|\Sigma\hat{\theta}\|_2 \leq r_\Sigma$, or $\|\theta\|_1 \leq r_1$ or $\|\theta\|_2 \leq r_2$ holds.

(a) In Section D.2.1, assume that $\|\Sigma\hat{\theta}\|_2 \leq r_\Sigma$ and $\|\theta\|_1 > r_1$ and $\|\theta\|_2 > r_2$ hold and then derive a contradiction.

(b) In Section D.2.2, assume that $\|\Sigma\hat{\theta}\|_2 > r_\Sigma$ and $\|\theta\|_1 \leq r_1$ and $\|\theta\|_2 > r_2$ hold and then derive a contradiction.

(c) In Section D.2.3, assume that $\|\Sigma\hat{\theta}\|_2 > r_\Sigma$ and $\|\theta\|_1 > r_1$ and $\|\theta\|_2 \leq r_2$ hold and then derive a contradiction.

(d) In Section D.2.4, assume that $\|\Sigma\hat{\theta}\|_2 > r_\Sigma$ and $\|\theta\|_1 \leq r_1$ and $\|\theta\|_2 \leq r_2$ hold and then derive a contradiction.

(e) In Section D.2.5, assume that $\|\Sigma\hat{\theta}\|_2 \leq r_\Sigma$ and $\|\theta\|_1 > r_1$ and $\|\theta\|_2 \leq r_2$ hold and then derive a contradiction.

(f) In Section D.2.6, assume that $\|\Sigma\hat{\theta}\|_2 \leq r_\Sigma$ and $\|\theta\|_1 \leq r_1$ and $\|\theta\|_2 > r_2$ hold and then derive a contradiction.

Finally, we have

$$\|\Sigma\hat{\theta}\|_2 \leq r_\Sigma, \|\hat{\beta} - \theta^*\|_2 \leq r_2, \text{ and } \|\hat{\beta} - \beta^*\|_1 \leq r_1,$$ \hspace{2cm} (D.4)

and the proof is complete.

D.2.1 Step 2(a)

Assume that $\|\Sigma\hat{\theta}\|_2 \leq r_\Sigma$, and $\|\theta\|_1 > r_1$ and $\|\theta\|_2 > r_2$ hold, and then we can find $\eta_4, \eta'_4 \in (0,1)$ such that $\|\theta_{\eta_4}\|_1 = r_1$ and $\|\theta_{\eta'_4}\|_2 = r_2$ hold. We note that $\|\Sigma\hat{\theta}_{\eta_4}\|_2 \leq r_\Sigma$ and $\|\Sigma\hat{\theta}_{\eta'_4}\|_2 \leq r_\Sigma$ also hold. Then, from the same arguments of Sections D.1.2 and D.1.3, we have a contradiction.

D.2.2 Step 2(b)

Assume that $\|\Sigma\hat{\theta}\|_2 > r_\Sigma$ and $\|\theta\|_1 \leq r_1$ and $\|\theta\|_2 > r_2$ hold, and then we can find $\eta_5, \eta'_5 \in (0,1)$ such that $\|\Sigma\hat{\theta}_{\eta_5}\|_2 = r_\Sigma$ and $\|\theta_{\eta'_5}\|_2 = r_2$ hold. We note that $\|\theta_{\eta_5}\|_1 \leq r_1$ and $\|\theta_{\eta'_5}\|_1 \leq r_1$ also hold. Then, from the same arguments of Sections D.1.1 and D.1.2, we have a contradiction.
D.2.3 Step 2(c)
Assume that $\|\Sigma \hat{\theta}\|_2 > r_\Sigma$ and $\|\hat{\theta}\|_1 > r_1$ and $\|\hat{\theta}\|_2 \leq r_2$ hold and, then we can find $\eta_6, \eta_6' \in (0, 1)$ such that $\|\hat{\theta}_{\eta_6}\|_1 = r_1$ and $\|\Sigma \hat{\theta}_{\eta_6'}\|_2 = r_\Sigma$ hold. We note that $\|\hat{\theta}_{\eta_6}\|_2 \leq r_2$ and $\|\hat{\theta}_{\eta_6'}\|_2 \leq r_2$ also hold. Then, from the same arguments of Sections D.1.1 and D.1.3, we have a contradiction.

D.2.4 Step 2(d)
Assume that $\|\Sigma \hat{\theta}\|_2 > r_\Sigma$ and $\|\hat{\theta}\|_1 \leq r_1$ and $\|\hat{\theta}\|_2 \leq r_2$ hold and, then we can find $\eta_7 \in (0, 1)$ such that $\|\Sigma \hat{\theta}_{\eta_7}\|_2 = r_\Sigma$ holds. We note that $\|\hat{\theta}_{\eta_7}\|_1 \leq r_1$ and $\|\hat{\theta}_{\eta_7}\|_2 \leq r_2$ also hold. Then, from the same arguments of Section D.1.1, we have a contradiction.

D.2.5 Step 2(e)
Assume that $\|\Sigma \hat{\theta}\|_2 \leq r_\Sigma$ and $\|\hat{\theta}\|_1 > r_1$ and $\|\hat{\theta}\|_2 \leq r_2$ hold and, then we can find $\eta_8 \in (0, 1)$ such that $\|\hat{\theta}_{\eta_8}\|_1 = r_1$ holds. We note that $\|\Sigma \hat{\theta}_{\eta_8}\|_2 \leq r_\Sigma$ and $\|\hat{\theta}_{\eta_8}\|_2 \leq r_2$ also hold. Then, from the same arguments of Section D.1.3, we have a contradiction.

D.2.6 Step 2(f)
Assume that $\|\Sigma \hat{\theta}\|_2 \leq r_\Sigma$ and $\|\hat{\theta}\|_1 \leq r_1$ and $\|\hat{\theta}\|_2 > r_2$ hold and, then we can find $\eta_9 \in (0, 1)$ such that $\|\hat{\theta}_{\eta_9}\|_2 = r_2$ holds. We note that $\|\Sigma \hat{\theta}_{\eta_9}\|_2 \leq r_\Sigma$ and $\|\hat{\theta}_{\eta_9}\|_1 \leq r_1$ also hold. Then, from the same arguments of Section D.1.2, we have a contradiction.

E Proofs of Lemmas 5.1, 6.1, 6.3 and 6.5

E.1 Proof of Lemma 5.1
We assume $|I_<| > 2\varepsilon n$, and then we derive a contradiction. From the constraint about $w_i$, we have $0 \leq w_i \leq \frac{1}{(1-\varepsilon)n}$ for any $i \in \{1, \cdots, n\}$ and we have

$$\sum_{i=1}^{n} w_i = \sum_{i \in I_<} w_i + \sum_{i \in I_>} w_i \leq |I_<| \times \frac{1+\varepsilon}{2n} + (n - |I_<|) \times \frac{1}{(1-\varepsilon)n} = 2\varepsilon n \times \frac{1}{2n} + (|I_<| - 2\varepsilon n) \times \frac{1+\varepsilon}{2n} + (n - 2\varepsilon n) \times \frac{1}{(1-\varepsilon)n} + (2\varepsilon n - |I_<|) \times \frac{1}{(1-\varepsilon)n} \leq \varepsilon + \frac{n - 2\varepsilon n}{(1-\varepsilon)n} = \varepsilon + \frac{1 - 2\varepsilon}{1-\varepsilon} = \frac{1-\varepsilon - \varepsilon^2}{1-\varepsilon} < 1. \quad (E.1)$$

This is a contradiction because $\sum_{i=1}^{n} w_i = 1$. Then, we have $|I_<| \leq 2\varepsilon n$. 

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E.2 Proofs of Lemmas 6.1, 6.3 and 6.5

E.2.1 Proof of Lemma 6.1

The following argument is essentially identical to a part of the proof of Proposition 9 of \( [5] \). We note that

\[
r_1 \mathbb{B}_1^d \cap r_2 \mathbb{B}_2^d \subset 2r_1 \mathbb{B}_1^d \cap r_2 \mathbb{B}_2^d = 2r_1 \left( 1 \times \mathbb{B}_1^d \cap \frac{r_2}{2r_1} \mathbb{B}_2^d \right),
\]

and we have

\[
E \sup_{v \in r_1 \mathbb{B}_1^d \cap r_2 \mathbb{B}_2^d} \langle \Sigma \frac{\mathbf{g}}{2}, v \rangle \leq E \sup_{v \in 2r_1 \left( 1 \times \mathbb{B}_1^d \cap \frac{r_2}{2r_1} \mathbb{B}_2^d \right)} \langle \Sigma \frac{\mathbf{g}}{2}, v \rangle.
\]

Let \( \{u_1, \cdots, u_d\} \) be the columns of \( \Sigma \frac{\mathbf{g}}{2} \). Then, we have

\[
E \sup_{v \in 2r_1 \left( 1 \times \mathbb{B}_1^d \cap \frac{r_2}{2r_1} \mathbb{B}_2^d \right)} \langle \Sigma \frac{\mathbf{g}}{2}, v \rangle \leq E \sup_{v \in 2r_1 \left( \frac{\mathbf{g}}{2} \left( \pm u_1, \cdots, \pm u_d \right) \cap \frac{r_2}{2r_1} \mathbb{B}_2^d \right)} \langle \mathbf{g}, v \rangle.
\]

From Proposition 1 of \( [5] \), we have

\[
E \sup_{v \in 2r_1 \left( \frac{\mathbf{g}}{2} \left( \pm u_1, \cdots, \pm u_d \right) \cap \frac{r_2}{2r_1} \mathbb{B}_2^d \right)} \langle \mathbf{g}, v \rangle \leq 4r_1 \sqrt{\log \left( \frac{cd}{S^2} \right)} \leq 4r_1 \sqrt{\log \left( \frac{cd}{S^2} \right)} \leq C r_1 \sqrt{\log(d/s)},
\]

where (a) follows from \( r \leq 1 \).

E.2.2 Proof of Lemma 6.3

First, we prove (6.19). For any \( v \in r_1 \mathbb{B}_1^d \cap r_2 \mathbb{B}_2^d \), we see that

\[
\frac{v}{r_2} \in \left\{ v \in \mathbb{R}^d \mid \|v\|_2 \leq 1, \|v\|_1 \leq \frac{r_1}{r_2} \right\} = \frac{r_1}{r_2} \mathbb{B}_1^d \cap \mathbb{B}_2^d.
\]

From Lemma 3.1 of \( [59] \), we have

\[
\frac{r_1}{r_2} \mathbb{B}_1^d \cap \mathbb{B}_2^d \subset 2\text{conv} \left( \left( \frac{r_1}{r_2} \right)^2 \mathbb{B}_0^d \cap \mathbb{B}_2^d \right),
\]

and

\[
\frac{v}{r_2} \in 2\text{conv} \left( \left( \frac{r_1}{r_2} \right)^2 \mathbb{B}_0^d \cap \mathbb{B}_2^d \right).
\]

From the definition of the convex hull, this means

\[
\frac{v}{r_2} = 2 \sum_{i=1}^{a} \lambda_i a_i.
\]
for some \( \{ \lambda_i \}_{i=1}^a \) and \( \{ a_i \}_{i=1}^a \) such that \( \sum_{i=1}^a \lambda_i = 1, \lambda_i \geq 0 \) and \( a_i \in \left( \frac{r_2}{r_1} \right)^2 \mathbb{B}_0 \cap \mathbb{B}_2^d \) for any \( i \in \{1, \ldots, a\} \). We note that

\[
\| r_2 a_i \|_0 \leq \left( \frac{r_1}{r_2} \right)^2, \quad \| r_2 a_i \|_2 \leq r_2. \tag{E.10}
\]

From (E.8) and (E.10), we see

\[
v \in 2\text{conv} \left( \left( \frac{r_1}{r_2} \right)^2 \mathbb{B}_0^d \cap r_2 \mathbb{B}_2^d \right). \tag{E.11}
\]

Then the proof of (6.19) is complete. Identifying the vectors and the matrices, the proof of (6.18) is also complete.

### E.2.3 Proof of Lemma 6.5

We note that

\[
\int_0^\infty \frac{x}{e^x} dx = 1, \quad \int_0^\infty \frac{\sqrt{e^x}}{e^x} dx \leq \int_0^1 \frac{1}{e^x} dx + \int_0^\infty \frac{x}{e^x} dx \leq \left[ \frac{-1}{e^x} \right]_0^\infty + 1 = 2. \tag{E.12}
\]

First, we evaluate \( \gamma_1(s^2 \mathbb{B}_0^{d \times d} \cap r_2^2 \mathbb{B}_F^{d \times d}, \| \cdot \|_F) \). From a standard entropy bound from chaining theory Lemma D.17 of [57], we have

\[
\gamma_1(s^2 \mathbb{B}_0^{d \times d} \cap r_2^2 \mathbb{B}_F^{d \times d}, \| \cdot \|_F) \leq C \int_0^{r_2^2} \log N(s^2 \mathbb{B}_0^{d \times d} \cap r_2^2 \mathbb{B}_F^{d \times d}, \epsilon) d\epsilon, \tag{E.13}
\]

where \( N(s^2 \mathbb{B}_0^{d \times d} \cap r_2^2 \mathbb{B}_F^{d \times d}, \epsilon) \) is the covering number of \( s^2 \mathbb{B}_0^{d \times d} \cap r_2^2 \mathbb{B}_F^{d \times d} \), that is the minimal cardinality of an \( \epsilon \)-net of \( s^2 \mathbb{B}_0^{d \times d} \cap r_2^2 \mathbb{B}_F^{d \times d} \). From Lemma Lemma 6.3 and \( d/s \geq 3 \), we have

\[
\int_0^{r_2^2} \log N(s^2 \mathbb{B}_0^{d \times d} \cap r_2^2 \mathbb{B}_F^{d \times d}, \epsilon) d\epsilon \leq C \int_0^{r_2^2} s^2 \log r_2^2 \frac{9d^2}{\epsilon x^2} d\epsilon
\]

\[
= C s^2 \int_0^{r_2^2} \left( \log(9d^2/s^2) + \log(r_2^2/\epsilon) \right) d\epsilon
\]

\[
= C s^2 \left( r_2^2 \log(9d^2/s^2) + r_2^2 \int_1^\infty \frac{x}{e^x} dx \right)
\]

\[
\leq C s^2 \left( r_2^2 \log(9d^2/s^2) + r_2^2 \int_0^\infty \frac{x}{e^x} dx \right)
\]

\[
\overset{(a)}{=} C s^2 \left( r_2^2 \log(9d^2/s^2) + r_2^2 \right)
\]

\[
\overset{(b)}{\leq} C s^2 r_2^2 \log(d/s), \tag{E.14}
\]

where (a) follows from (E.12), and (b) follows from \( 3 \leq d/s \). Consequently, we have

\[
\gamma_1(s^2 \mathbb{B}_0^{d \times d} \cap r_2^2 \mathbb{B}_F^{d \times d}, \| \cdot \|_F) \leq C s^2 r_2^2 \log(d/s). \tag{E.15}
\]

Second, we evaluate \( \gamma_2(s^2 \mathbb{B}_0^{d \times d} \cap r_2^2 \mathbb{B}_F^{d \times d}, \| \cdot \|_F) \). From similar argument of the case \( \gamma_1(s^2 \mathbb{B}_0^{d \times d} \cap \mathbb{B}_F^{d \times d}) \).
$r_2^2 \mathbb{B}_F^{d \times d}, \| \cdot \|_F$, we have
\[
\int_0^{r_2^2} \sqrt{\log N(s^2 \mathbb{B}_F^{d \times d} \cap r_2^2 \mathbb{B}_F^{d \times d}, \epsilon)} \, d\epsilon \leq C \int_0^{r_2^2} s \sqrt{\log \frac{r_2^2 \epsilon d^2}{\epsilon^2}} \, d\epsilon.
\]
where (a) follows from triangular inequality, and (b) follows from (F.2) and $d/s \geq 3$.

F Proofs of Corollary 5.1 and Propositions 5.4, 5.5, 5.6, 5.7 and 5.8

Define
\[\mathcal{M}_{r_1, r_2, d, v} = \{ M \in S(d) : M = vv^\top, v \in \mathbb{R}^d, \|v\|_1 \leq r_1, \|v\|_2 \leq r_2\}. \tag{F.1}\]

F.1 Proof of Corollary 5.1

First, we prove (5.15). We see that, for any $M \in \mathcal{M}_{r_1, r_2, d}$,
\[
\left| \frac{1}{n} \sum_{i=1}^n \langle x_i, x_i^\top, M \rangle \right| \leq \left| \frac{1}{n} \sum_{i=1}^n \langle x_i, x_i^\top - \Sigma, M \rangle \right| + |\langle \Sigma, M \rangle| \leq \frac{1}{n} \sum_{i=1}^n \langle x_i, x_i^\top - \Sigma, M \rangle + \Sigma_{\text{max}}^2 r_2^2, \tag{F.2}
\]
where we use Hölder’s inequality. From (6.30) and (F.2), with probability at least $1 - \delta$, we have
\[
\max_{M \in \mathcal{M}_{r_1, r_2, d}} \left| \frac{1}{n} \sum_{i=1}^n \langle x_i, x_i^\top, M \rangle \right| \leq C (Lk_n)^2 (sr_{d,s} + r_3) r_2^2 + \Sigma_{\text{max}}^2 r_2^2. \tag{F.3}
\]

Next, we prove (5.16). Remember (6.31), we have
\[
\max_{M \in \mathcal{M}_{r_1, r_2, d}} \sum_{i=1}^n \hat{w}_i (x_i x_i^\top, M) \overset{(a)}{=} \max_{M \in \mathcal{M}_{r_1, r_2, d}} \sum_{i=1}^n w_i^\circ (x_i x_i^\top, M)
\]
\[
= \frac{1}{1 - o/n} \max_{M \in \mathcal{M}_{r_1, r_2, d}} \sum_{i=1}^n \frac{1}{n} \langle x_i, x_i^\top, M \rangle, \tag{F.4}
\]
where (a) follows from the optimality of $\{\hat{w}_i\}_{i=1}^n$ and $\{w_i^\circ\}_{i=1}^n \in \Delta^{n-1}(\epsilon)$. From (5.15), (F.4) and $1/(1 - o/n) \leq 2$, with probability at least $1 - \delta$, we have
\[
\max_{M \in \mathcal{M}_{r_1, r_2, d}} \sum_{i=1}^n \hat{w}_i (x_i x_i^\top, M) \leq C ((Lk_n)^2 (sr_{d,s} + r_3) + \Sigma_{\text{max}}^2) r_2^2, \tag{F.5}
\]
and from $L \geq 1$, the proof is complete.
F.2 Proof of Proposition 5.4

We have

\[
\max_{\mathbf{v} \in r_1B_1^d \cap r_2B_2^d} \left| \sum_{i \in \mathcal{I}} \hat{w}_i u_i \mathbf{x}_i^T \mathbf{v} \right|^2 \leq \max_{\mathbf{v} \in \text{conv} \left( \left( \frac{\ell}{2} \right)^2 B_1^d \cap r_2B_2^d \right)} 2 \left| \sum_{i \in \mathcal{I}} \hat{w}_i u_i \mathbf{x}_i^T \mathbf{v} \right|^2 \leq 2 \max_{\mathbf{v} \in \left( \frac{\ell}{2} \right)^2 B_1^d \cap r_2B_2^d} \left| \sum_{i \in \mathcal{I}} \hat{w}_i u_i \mathbf{x}_i^T \mathbf{v} \right|^2, \quad (F.6)
\]

where (a) follows from Lemma 6.3, and (b) follows from Lemma D.8 of [57]. We note that, for any \( \mathbf{v} \in \mathbb{R}^d \),

\[
\left| \sum_{i \in \mathcal{I}} \hat{w}_i u_i \mathbf{x}_i^T \mathbf{v} \right|^2 \leq \frac{4}{n} \sum_{i \in \mathcal{I}} \hat{w}_i | \mathbf{x}_i^T \mathbf{v}|^2 \leq \frac{8}{n} \sum_{i \in \mathcal{I}} \hat{w}_i | \mathbf{x}_i^T \mathbf{v}|^2, \quad (F.7)
\]

where (a) follows from Hölder’s inequality and \( \sum_{i \in \mathcal{I}} u_i^2 \leq 4o \), and (b) follows from the fact that \( \hat{w}_i^2 \leq 2\hat{w}_i \) for any \( i \in (1, \ldots, n) \). We focus on \( \sum_{i \in \mathcal{I}} \hat{w}_i | \mathbf{x}_i^T \mathbf{v}|^2 \).

First, we have

\[
\max_{\mathbf{v} \in \left( \frac{\ell}{2} \right)^2 B_1^d \cap r_2B_2^d} \sum_{i \in \mathcal{I}} \hat{w}_i | \mathbf{x}_i^T \mathbf{v}|^2 = \max_{\mathbf{v} \in \left( \frac{\ell}{2} \right)^2 B_1^d \cap r_2B_2^d} \sum_{i \in \mathcal{I}} \hat{w}_i | \mathbf{x}_i^T \mathbf{x}_i^T, \mathbf{v} \mathbf{v}^T \rangle
\]

\[
\leq \max_{\mathbf{v} \in \left( \frac{\ell}{2} \right)^2 B_1^d \cap r_2B_2^d} \sum_{i=1}^n \hat{w}_i | \mathbf{x}_i^T \mathbf{x}_i^T, \mathbf{v} \mathbf{v}^T \rangle + \max_{\mathbf{v} \in \left( \frac{\ell}{2} \right)^2 B_1^d \cap r_2B_2^d} \sum_{i \in \mathcal{I}} \hat{w}_i | \mathbf{x}_i^T \mathbf{x}_i^T, \mathbf{v} \mathbf{v}^T \rangle
\]

\[
= \max_{\mathbf{v} \in \left( \frac{\ell}{2} \right)^2 B_1^d \cap r_2B_2^d} \sum_{i=1}^n \hat{w}_i | \mathbf{x}_i^T \mathbf{x}_i^T, \mathbf{v} \mathbf{v}^T \rangle + \max_{\mathbf{v} \in \left( \frac{\ell}{2} \right)^2 B_1^d \cap r_2B_2^d} \sum_{i \in \mathcal{I}} \hat{w}_i | \mathbf{x}_i^T \mathbf{x}_i^T, \mathbf{v} \mathbf{v}^T \rangle
\]

\[
\leq \max_{\mathbf{v} \in \left( \frac{\ell}{2} \right)^2 B_1^d \cap r_2B_2^d} \sum_{i=1}^n \hat{w}_i | \mathbf{x}_i^T \mathbf{x}_i^T - \Sigma, \mathbf{v} \mathbf{v}^T \rangle + \max_{\mathbf{v} \in \left( \frac{\ell}{2} \right)^2 B_1^d \cap r_2B_2^d} \sum_{i \in \mathcal{I}} \hat{w}_i | \mathbf{x}_i^T \mathbf{x}_i^T - \Sigma, \mathbf{v} \mathbf{v}^T \rangle
\]

\[
+ \max_{\mathbf{v} \in \left( \frac{\ell}{2} \right)^2 B_1^d \cap r_2B_2^d} \sum_{i \in \mathcal{I}} \hat{w}_i | \Sigma, \mathbf{v} \mathbf{v}^T \rangle
\]

\[
\leq \max_{\mathbf{v} \in \text{conv} \left( \left( \frac{\ell}{2} \right)^2 B_1^d \cap r_2B_2^d \right)} \sum_{i=1}^n \hat{w}_i | \mathbf{x}_i^T \mathbf{x}_i^T - \Sigma, \mathbf{v} \mathbf{v}^T \rangle + \max_{\mathbf{v} \in \text{conv} \left( \left( \frac{\ell}{2} \right)^2 B_1^d \cap r_2B_2^d \right)} \sum_{i \in \mathcal{I}} \hat{w}_i | \mathbf{x}_i^T \mathbf{x}_i^T - \Sigma, \mathbf{v} \mathbf{v}^T \rangle
\]

\[
+ \max_{\mathbf{v} \in \left( \frac{\ell}{2} \right)^2 B_1^d \cap r_2B_2^d} \sum_{i \in \mathcal{I}} \hat{w}_i | \Sigma, \mathbf{v} \mathbf{v}^T \rangle
\]

\[
\leq \max_{\mathbf{v} \in r_1B_1^d \cap r_2B_2^d} \sum_{i=1}^n \hat{w}_i | \mathbf{x}_i^T \mathbf{x}_i^T - \Sigma, \mathbf{v} \mathbf{v}^T \rangle + \max_{\mathbf{v} \in r_1B_1^d \cap r_2B_2^d} \sum_{i \in \mathcal{I}} \hat{w}_i | \mathbf{x}_i^T \mathbf{x}_i^T - \Sigma, \mathbf{v} \mathbf{v}^T \rangle + \max_{\mathbf{v} \in \mathbb{R}^d \cap r_2B_2^d} \sum_{i \in \mathcal{I}} \hat{w}_i | \Sigma, \mathbf{v} \mathbf{v}^T \rangle
\]

\[
\leq \max_{\mathbf{M} \in \text{conv} \left( \left( \frac{\ell}{2} \right)^2 r_1B_1^d \cap r_2B_2^d \right)} \sum_{i=1}^n \hat{w}_i | \mathbf{x}_i^T \mathbf{x}_i^T - \Sigma, \mathbf{v} \mathbf{v}^T \rangle + \max_{\mathbf{M} \in \text{conv} \left( \left( \frac{\ell}{2} \right)^2 r_1B_1^d \cap r_2B_2^d \right)} \sum_{i \in \mathcal{I}} \hat{w}_i | \mathbf{x}_i^T \mathbf{x}_i^T - \Sigma, \mathbf{v} \mathbf{v}^T \rangle + \max_{\mathbf{v} \in \mathbb{R}^d \cap r_2B_2^d} \sum_{i \in \mathcal{I}} \hat{w}_i | \Sigma, \mathbf{v} \mathbf{v}^T \rangle,
\quad (F.8)
\]
where (a) follows from Lemma 6.3 and \( r_1/r_2 \leq \sqrt{\frac{2}{\pi} + \frac{2}{\pi}} \), and (b) follows from \( \mathcal{M}^{f_1,f_2}_{r_1,r_2,d,v} \subseteq \mathcal{M}^{f_1,T}_{r_1,r_2,d,v} \) because \( \text{Tr}(M) = \text{Tr}(\mathbf{vv}^T) = \|\mathbf{v}\|^2 \) for \( M \in \mathcal{M}^{f_1,f_2}_{r_1,r_2,d,v} \). We note that, from \( 1/(1 - \varepsilon) \leq 2 \) and matrix Hölder’s inequality,

\[
\max_{\mathbf{v} \in \mathbb{S}^{2} \cap \mathbb{B}^2} \sum_{i \in \mathcal{O}} \tilde{w}_i (\Sigma, \mathbf{vv}^T) \leq \frac{1}{\varepsilon} \mathbf{v} \in \mathbb{S}^{2} \cap \mathbb{B}^2 \sum_{i \in \mathcal{O}} (\Sigma, \mathbf{vv}^T) \leq 2 \kappa_u^2 r_o r_2^2. \tag{F.9}
\]

where (a) follows from \( \tilde{w}_i' \leq 1/(1 - \varepsilon) \leq 2 \), and (b) follows from \( o/n \leq 1/(5e) \) and the definition of \( \kappa_u \). From (F.8) and (F.9), we have

\[
\sum_{i \in \mathcal{O}} \tilde{w}_i |\mathbf{X}_i^\top \mathbf{v}|^2 \leq \max_{M \in \mathcal{M}^{f_1,f_2}_{r_1,r_2,d,v}} \sum_{i \in \mathcal{I}} \tilde{w}_i (\mathbf{X}_i \mathbf{X}_i^\top - \Sigma, M) + \max_{M \in \mathcal{M}^{f_1,f_2}_{r_1,r_2,d,v}} \sum_{i \in \mathcal{I}} \tilde{w}_i ((\mathbf{x}_i \mathbf{x}_i^\top - \Sigma, M) + 2 \kappa_u^2 r_o r_2^2 \leq \tau_{\text{cut}} + \max_{M \in \mathcal{M}^{f_1,f_2}_{r_1,r_2,d,v}} \sum_{i \in \mathcal{I}} \tilde{w}_i ((\mathbf{x}_i \mathbf{x}_i^\top - \Sigma, M) + 2 \kappa_u^2 r_o r_2^2, \tag{F.10}
\]

where the last line follows from the assumption of success of Algorithm 2.

Next, we consider the second term of right-hand side of (F.10). We have \( \frac{1 - o/n}{1 - \varepsilon} \leq 2 \) because \( \varepsilon = c_\varepsilon \times \frac{2}{n} \) with \( 1 \leq c_\varepsilon < 2 \) and \( o/n \leq 1/(5e)(1/3) \), and we have

\[
\max_{M \in \mathcal{M}^{f_1,f_2}_{r_1,r_2,d,v}} \left| \sum_{i \in \mathcal{I}} \tilde{w}_i ((\mathbf{x}_i \mathbf{x}_i^\top - \Sigma, M) \right| \leq \sum_{j \in \mathcal{I}} \tilde{w}_j \max_{M \in \mathcal{M}^{f_1,f_2}_{r_1,r_2,d,v}} \left| \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \tilde{w}_j ((\mathbf{x}_i \mathbf{x}_i^\top - \Sigma, M) \right| \leq \frac{1 - o/n}{1 - \varepsilon} \max_{M \in \mathcal{M}^{f_1,f_2}_{r_1,r_2,d,v}} \left| \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \tilde{w}_j ((\mathbf{x}_i \mathbf{x}_i^\top - \Sigma, M) \right| \leq 2 \max_{M \in \mathcal{M}^{f_1,f_2}_{r_1,r_2,d,v}} \left| \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \tilde{w}_j ((\mathbf{x}_i \mathbf{x}_i^\top - \Sigma, M) \right|. \tag{F.11}
\]

Define the following three sets:

\[
\Delta^2(\varepsilon + o/n) = \left\{ (w_1, \ldots, w_n) \mid 0 \leq w_i \leq \frac{1}{n(1 - (\varepsilon + o/n))}, \sum_{i \in \mathcal{I}} w_i = \sum_{i \in \mathcal{I}} w_i = 1 \right\},
\]

\[
\Delta^2(3o/n) = \left\{ (w_1, \ldots, w_n) \mid 0 \leq w_i \leq \frac{1}{n(1 - 3o/n)}, \sum_{i \in \mathcal{I}} w_i = \sum_{i \in \mathcal{I}} w_i = 1 \right\},
\]

\[
\Delta^{n-1}(3o/n) = \left\{ (w_1, \ldots, w_n) \mid 0 \leq w_i \leq \frac{1}{n(1 - 3o/n)}, \sum_{i \in \mathcal{I}} w_i = \sum_{i \in \mathcal{I}} w_i = 1 \right\}.
\]

From \( \sum_{j \in \mathcal{I}} \tilde{w}_j \geq 1 - \frac{o}{n(1 - o/n)} = \frac{1 - o/n}{1 - \varepsilon} \), for any \( i \in \mathcal{I} \), we have \( 0 \leq \frac{w_i}{\sum_{j \in \mathcal{I}} \tilde{w}_j} \leq \frac{1}{n(1 - (\varepsilon + o/n))} \), and from \( \varepsilon = c_\varepsilon \times o/n \) with \( 1 \leq c_\varepsilon < 2 \), we have \( \Delta^2(\varepsilon + o/n) \subseteq \Delta^2(3o/n) \subseteq \Delta^{n-1}(3o/n) \). Therefore, we have

\[
\max_{M \in \mathcal{M}^{f_1,f_2}_{r_1,r_2,d,v}} \left| \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \tilde{w}_j ((\mathbf{x}_i \mathbf{x}_i^\top - \Sigma, M) \right| \leq \max_{w \in \Delta^2(\varepsilon + o/n)} \max_{M \in \mathcal{M}^{f_1,f_2}_{r_1,r_2,d,v}} \left| \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \tilde{w}_j ((\mathbf{x}_i \mathbf{x}_i^\top - \Sigma, M) \right| \leq \max_{w \in \Delta^{n-1}(3o/n)} \max_{M \in \mathcal{M}^{f_1,f_2}_{r_1,r_2,d,v}} \left| \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \tilde{w}_j ((\mathbf{x}_i \mathbf{x}_i^\top - \Sigma, M) \right|. \tag{F.12}
\]
From Lemma 1 of [22] and $1/(1 - 3\alpha/n) \leq 1/(1 - 3/(5e)) \leq 2$, we have

\[
\max_{\mathcal{W} \subseteq \Delta^{n-1}(3\alpha/n)} \max_{M \in \mathcal{G}_{r_1, r_2, d}} \left| \sum_{i=1}^{n} w_i \langle x_i, x_i^T - \Sigma, M \rangle \right| \\
\leq \max_{|\mathcal{J}| = n - 3\alpha} \max_{M \in \mathcal{G}_{r_1, r_2, d}} \left| \frac{1}{n(1 - 3\alpha/n)} \sum_{i \in J} (x_i, x_i^T - \Sigma, M) \right| \\
\leq 2 \max_{|\mathcal{J}| = n - 3\alpha} \max_{M \in \mathcal{G}_{r_1, r_2, d}} \left| \sum_{i \in J} \frac{1}{n} (x_i, x_i^T - \Sigma, M) \right| \\
\leq 2 \left( \max_{M \in \mathcal{G}_{r_1, r_2, d}} \left| \sum_{i=1}^{n} (x_i, x_i^T - \Sigma, M) \right| + \max_{M \in \mathcal{G}_{r_1, r_2, d}} \left| \sum_{i \in J} \frac{1}{n} (x_i, x_i^T - \Sigma, M) \right| \right). \tag{F.13}
\]

Consequently, from almost the same calculation for (6.34), we have

\[
\max_{M \in \mathcal{G}_{r_1, r_2, d}} \left| \sum_{i \in J} \hat{w}_i (x_i, x_i^T - \Sigma, M) \right| \leq CL \kappa_u^2 (s_{r_d, s} + r_\delta + r_o') r_2^2. \tag{F.14}
\]

Lastly, from (F.7), (F.10) and (F.14), we have

\[
\left| \sum_{i \in \mathcal{O}} \hat{w}_i u_i x_i^T \right| \leq C \sqrt{\frac{\alpha}{n}} \sqrt{\tau_{\text{cut}}} + (L \kappa)_u^2 (s_{r_d, s} + r_\delta + r_o') r_2^2 + \kappa_u^2 r_o' r_2^2 \\
\leq (a) \quad CL \sqrt{1 + c_{\text{cut}}} \sqrt{\frac{\alpha}{n}} \sqrt{\kappa_u^2 s_{r_d, s} r_2^2 + \kappa_u^2 r_\delta r_2^2 + \kappa_u^2 r_o' r_2^2} \\
\leq (b) \quad CL \sqrt{1 + c_{\text{cut}} \kappa_u} \left( \sqrt{\frac{\alpha}{n}} \left( \sqrt{s_{r_d, s} + r_\delta} + r_o \right) r_2, \tag{F.15}\right)
\]

where (a) follows from the definition of $\tau_{\text{cut}}$, and (b) follows from triangular inequality, and the proof is complete.

**F.3 Proof of Proposition 5.5**

From $\log \frac{n}{m} \geq 1$ and the arguments very similar to the proof of Proposition 5.4, we have

\[
\sum_{i \in \mathcal{O}_m} u_i x_i^T \mathbf{v} \leq CL \sqrt{\frac{m}{n}} \sqrt{\kappa_u^2 \left( s_{r_d, s} r_2^2 + \frac{m}{n} \log \frac{n}{r_2^2} \right) + \kappa_u^2 \frac{m}{n} r_2^2} \\
\leq (a) \quad CL \sqrt{\frac{(1 + 2c_{\varepsilon}) \alpha}{n}} \sqrt{s_{r_d, s} r_2^2 + \frac{m}{n} \log \frac{n}{r_2^2} + \frac{(1 + 2c_{\varepsilon}) \alpha}{n} \log \frac{n}{r_2^2}} \\
\leq (b) \quad CL \sqrt{1 + 2c_{\varepsilon} \kappa_u} \sqrt{\frac{\alpha}{n}} \sqrt{s_{r_d, s} r_2^2 + \frac{m}{n} \log \frac{n}{r_2^2}}. \tag{F.16} \]

where (a) follows from the fact that $0 < x < 1/e$, $x \log(1/x)$ is increasing, and (b) follows from $\log \frac{1}{1 + x} \leq 1 \leq \log \frac{1}{x}$. From triangular inequality, the proof is complete.

**F.4 Proof of Proposition 5.6**

First, we introduce Lemma F.1, which is used in the proof of Proposition 5.6.

**Lemma F.1.** Suppose that (i) and (iii) of Assumption 2.1 holds. Define $\{a_i\}_{i=1}^n$ as a sequence of i.i.d. Rademacher random variables which are independent of $\{x_i\}_{i=1}^n$. Then, we have

\[
\mathbb{E} \sup_{\mathcal{V} \in \mathcal{B}_{(r_1, r_2, d)}^T} \left| \frac{1}{n} \sum_{i=1}^{n} a_i x_i^T \mathbf{v} \right| \leq Lr_{d, s} r_1. \tag{F.17}\]

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Proof. Define $E^*$ as the conditional expectation of $\{a_i\}_{i=1}^n$ given $\{x_i\}_{i=1}^n$. From Exercise 2.2.2 of [68], for any $v_0 \in r_1B^d \cap r_2B^d$, we have

$$E^* \sup_{v \in r_1B^d \cap r_2B^d} \left| \frac{1}{n} \sum_{i=1}^{n} a_i x_i^T v \right| \leq 2E^* \sup_{v \in r_1B^d \cap r_2B^d} \left| \frac{1}{n} \sum_{i=1}^{n} a_i x_i^T v + E \right| \left| \frac{1}{n} \sum_{i=1}^{n} a_i x_i^T v_0 \right|, \quad (F.18)$$
and taking $v_0 = 0$, we have

$$E^* \sup_{v \in r_1B^d \cap r_2B^d} \left| \frac{1}{n} \sum_{i=1}^{n} a_i x_i^T v \right| \leq 2E^* \sup_{v \in r_1B^d \cap r_2B^d} \frac{1}{n} \sum_{i=1}^{n} a_i x_i^T v. \quad (F.19)$$

Taking the expectation with respect to $\{x_i\}_{i=1}^n$ on both sides of $(F.19)$, we have

$$E \sup_{v \in r_1B^d \cap r_2B^d} \left| \frac{1}{n} \sum_{i=1}^{n} a_i x_i^T v \right| \leq 2E \sup_{v \in r_1B^d \cap r_2B^d} \frac{1}{n} \sum_{i=1}^{n} a_i x_i^T v. \quad (F.20)$$

For any $i$ and any $v_1, v_2 \in r_1B^d \cap r_2B^d$, we have

$$\| \langle a_i x_i, v_1 - v_2 \rangle \|_{\psi_2} \leq \| \langle x_i, v_1 - v_2 \rangle \|_{\psi_2} \leq \mathcal{E} \| \langle x, v_1 - v_2 \rangle \|_{L_2}, \quad (F.21)$$
where (a) follows from $|a_i| = 1$ and the definition of $\| \cdot \|_{\psi_2}$ and (b) follows from (3.2), and we see that $\langle a_i x_i, v \rangle$ is a subGaussian random variable. Then, from Proposition 2.6.1 of [71], we have

$$\left\| \frac{1}{n} \sum_{i=1}^{n} a_i x_i, v_1 - v_2 \right\|_{\psi_2}^2 \leq C \frac{1}{n} \left\| \left( \frac{a_i x_i}{n} \right), v_1 - v_2 \right\|_{\psi_2}^2 \leq C \frac{\delta^2}{n} \| \langle x, v_1 - v_2 \rangle \|_{L_2}^2, \quad (F.22)$$

where (a) follows from (F.21). From the assumption on $x$, we have

$$\| \langle x, v_1 - v_2 \rangle \|_{L_2}^2 = \| \langle \Sigma x, v_1 - v_2 \rangle \|_{L_2}^2 = \| \langle \Sigma x, v_1 - v_2 \rangle \|_{L_2}^2. \quad (F.23)$$

From (F.22) and (F.23), we have

$$\left\| \frac{1}{n} \sum_{i=1}^{n} a_i x_i, v_1 - v_2 \right\|_{\psi_2} \leq C \frac{\delta}{\sqrt{n}} \| \langle \Sigma x, v_1 - v_2 \rangle \|_{L_2}. \quad (F.24)$$

Then, from Corollary 8.6.2 of [71], we have

$$E \sup_{v \in r_1B^d \cap r_2B^d} \left\| \frac{1}{n} \sum_{i=1}^{n} a_i x_i \right\|_{\psi_2} \leq C \frac{\delta}{\sqrt{n}} E \sup_{v \in r_1B^d \cap r_2B^d} \langle \Sigma x, v \rangle. \quad (F.25)$$

Then, from Lemma 6.1 and (F.20), the proof is complete. \qed

Then, we proceed the proof of Proposition 5.6. The left-hand side of (5.13) divided by $\lambda_n^2$ can be expressed as

$$\sum_{i=1}^{n} \left( -h(\xi_{n,i} - x_v, i) + h(\xi_{n,i}) \right) x_v, i. \quad (F.26)$$

From the convexity of Huber loss, for any $a, b \in \mathbb{R}$, we have

$$H(a) - H(b) \geq h(b)(a - b) \quad \text{and} \quad H(b) - H(a) \geq h(a)(b - a), \quad (F.27)$$
and we have

$$0 \leq (h(a) - h(b))(a - b). \quad (F.28)$$
Let $I_{E_i}$ be the indicator function of the event

$$E_i := (|\xi_{\alpha,i}| \leq 1/2) \cap (|x_{\nu,i}| \leq 1/2).$$  \hfill (F.29)

From (F.28), we have

$$\sum_{i=1}^{n} (-h(\xi_{\alpha,i} - x_{\nu,i}) + h (\xi_{\alpha,i})) x_{\nu,i} \geq \sum_{i=1}^{n} (-h(\xi_{\alpha,i} - x_{\nu,i}) + h (\xi_{\alpha,i})) x_{\nu,i} I_{E_i} = \sum_{i=1}^{n} x_{\nu,i}^2 I_{E_i}. \hfill (F.30)$$

Define the functions

$$\varphi(x) = \begin{cases} x^2 & \text{if } |x| \leq 1/4 \\ (x - 1/4)^2 & \text{if } 1/4 \leq x \leq 1/2 \\ (x + 1/4)^2 & \text{if } -1/2 \leq x \leq -1/4 \\ 0 & \text{if } |x| > 1/2 \end{cases} \quad \text{and } \psi(x) = I_{(|x| \leq 1/2)}, \hfill (F.31)$$

where $I_{(|x| \leq 1/2)}$ is the indicator function of the event $|x| \leq 1/2$. Let $f_i(v) = \varphi(x_{\nu,i}) \psi(\xi_{\alpha,i})$ and we have

$$\sum_{i=1}^{n} x_{\nu,i}^2 I_{E_i} \overset{(a)}{\geq} \sum_{i=1}^{n} \varphi(x_{\nu,i}) \psi(\xi_{\alpha,i}) = \sum_{i=1}^{n} f_i(v), \hfill (F.32)$$

where (a) follows from $\varphi(v) \leq v^2$ for $|v| \leq 1/2$. We note that

$$f_i(v) \leq \varphi(x_{\nu,i}) \leq |\varphi(x_{\nu,i})| \overset{(b)}{\leq} \min(x_{\nu,i}^2, 1). \hfill (F.33)$$

To bound $\sum_{i=1}^{n} f_i(v)$ from below, we have

$$\inf_{v \in r_1 B_{\nu}^{d} \cap r_2 B_{\alpha}^{d}} \sum_{i=1}^{n} f_i(v) \geq \inf_{v \in r_1 B_{\nu}^{d} \cap r_2 B_{\alpha}^{d}} \mathbb{E} \sum_{i=1}^{n} f_i(v) - \sup_{v \in r_1 B_{\nu}^{d} \cap r_2 B_{\alpha}^{d}} \left| \sum_{i=1}^{n} f_i(v) - \mathbb{E} \sum_{i=1}^{n} f_i(v) \right|. \hfill (F.34)$$

Define the supremum of a random process indexed by $r_1 B_{\nu}^{d} \cap r_2 B_{\alpha}^{d}$:

$$\Delta := \sup_{v \in r_1 B_{\nu}^{d} \cap r_2 B_{\alpha}^{d}} \left| \sum_{i=1}^{n} f_i(v) - \mathbb{E} \sum_{i=1}^{n} f_i(v) \right|. \hfill (F.35)$$

Define

$$I_{|x_{\nu,i}| \geq 1/2} \text{ and } I_{|\xi_{\alpha,i}| \geq 1/2} \hfill (F.36)$$

as the indicator functions of the events $|x_{\nu,i}| \geq 1/2$ and $|\xi_{\alpha,i}| \geq 1/2$, respectively. From $I_{E_i} = 1 - I_{|x_{\nu,i}| \geq 1/2} - I_{|\xi_{\alpha,i}| \geq 1/2}$ and (F.32), we have

$$\sum_{i=1}^{n} x_{\nu,i}^2 I_{E_i} \geq \mathbb{E} \sum_{i=1}^{n} f_i(v) \geq \sum_{i=1}^{n} \mathbb{E} x_{\nu,i}^2 - \sum_{i=1}^{n} \mathbb{E} x_{\nu,i}^2 I_{|x_{\nu,i}| \geq 1/2} - \sum_{i=1}^{n} \sum_{i=1}^{n} \mathbb{E} x_{\nu,i}^2 I_{|\xi_{\alpha,i}| \geq 1/2} \overset{(a)}{\geq} \|\Sigma^2 v\|_2^2 - \sum_{i=1}^{n} \mathbb{E} x_{\nu,i}^2 I_{|x_{\nu,i}| \geq 1/2} - \sum_{i=1}^{n} \sum_{i=1}^{n} \mathbb{E} x_{\nu,i}^2 I_{|\xi_{\alpha,i}| \geq 1/2}. \hfill (F.37)$$

We note that, from (3.3)

$$\mathbb{E}(x^\top v)^4 \leq 16 L^4 \|\Sigma^2 v\|_2^4. \hfill (F.38)$$
We evaluate the right-hand side of (F.37) at each term. First, for any \( v \in r_1 \mathbb{B}_1^d \cap r_\Sigma \mathbb{B}_\Sigma^d \), we have
\[
\sum_{i=1}^{n} \mathbb{E} x_{v,i}^2 I_{|x_{v,i}| \geq 1/2} \leq \sum_{i=1}^{n} \mathbb{E} x_{v,i}^4 \mathbb{E} I_{|x_{v,i}| \geq 1/2}
\]
\[
= \sum_{i=1}^{n} \sqrt{\mathbb{E} x_{v,i}^4} \sqrt{\mathbb{E} I_{|x_{v,i}| \geq 1/2}}
\]
\[
\leq \sum_{i=1}^{n} \sqrt{\mathbb{E} x_{v,i}^4} \sqrt{\mathbb{P}(|x_{v,i}| \geq 1/2)}
\]
\[
\leq \sum_{i=1}^{n} \sqrt{\mathbb{E} x_{v,i}^4} \left( 2 \exp \left( -\frac{\lambda_o^2 n}{c_o^2 \|\Sigma \|_2^2} \right) \right)
\]
\[
\leq \frac{4}{\lambda_o^2} L^2 \|\Sigma \|_2^2 \left( 2 \exp \left( -\frac{\lambda_o^2 n}{L^2 \|\Sigma \|_2^2} \right) \right)
\]
\[
\leq \frac{1}{3\lambda_o^2} \|\Sigma \|_2^2,
\]  
(F.39)
where (a) follows from Hölder’s inequality, (b) follows from the relation between indicator function and expectation, (c) follows from (3.6), (d) follows from (F.38) and the definition of \( L \), and (e) follows from (3.27). Second, for any \( v \in r_1 \mathbb{B}_1^d \cap r_\Sigma \mathbb{B}_\Sigma^d \), we have
\[
\sum_{i=1}^{n} \mathbb{E} x_{v,i}^2 I_{|\xi_{\lambda_o,i}| \geq 1/2} \leq \sum_{i=1}^{n} \mathbb{E} x_{v,i}^4 \mathbb{E} I_{|\xi_{\lambda_o,i}| \geq 1/2}
\]
\[
= \sum_{i=1}^{n} \sqrt{\mathbb{E} x_{v,i}^4} \sqrt{\mathbb{E} I_{|\xi_{\lambda_o,i}| \geq 1/2}}
\]
\[
\leq \sum_{i=1}^{n} \sqrt{\mathbb{E} x_{v,i}^4} \sqrt{\mathbb{P}(|\xi_{\lambda_o,i}| \geq 1/2)}
\]
\[
\leq \frac{4L^2}{\lambda_o^2} \sqrt{\mathbb{P}(|\xi_{\lambda_o,i}| \geq 1/2) \|\Sigma \|_2^2}
\]
\[
\leq \frac{1}{3\lambda_o^2} \|\Sigma \|_2^2,
\]  
(F.40)
where (a) follows from Hölder’s inequality, (b) follows from relation between indicator function and expectation, and (c) follows from (F.38), and (d) follows from (3.14). Consequently, from (F.39) and (F.40) we have
\[
\frac{\|\Sigma \|_2^2}{3\lambda_o^2} - \Delta \leq \sum_{i=1}^{n} (-h(\xi_{\lambda_o,i} - x_{v,i}) + h(\xi_{\lambda_o,i})) x_{v,i}.
\]  
(F.41)

Next, we evaluate the stochastic term \( \Delta \) defined in (F.35). From (F.33) and Theorem 3 of [51], with probability at least \( 1 - \delta \), we have
\[
\Delta \leq 2\mathbb{E} \Delta + \sigma_f \sqrt{8 \log(1/\delta)} + 18 \log(1/\delta),
\]  
(F.42)
where
\[
\sigma_f^2 = \sup_{v \in r_1 \mathbb{B}_1^d \cap r_\Sigma \mathbb{B}_\Sigma^d} \sum_{i=1}^{n} \mathbb{E} (f_i(v) - \mathbb{E} f_i(v))^2.
\]  
(F.43)
From (F.33) and (F.38), we have
\[
\mathbb{E} (f_i(v) - \mathbb{E} f_i(v))^2 \leq \mathbb{E} f_i^2(v) \leq \mathbb{E} f_i(v) \leq \mathbb{E} x_{v,i}^2 \leq \frac{L^2 \|\Sigma \|_2^2}{\lambda_o^2 n}.
\]  
(F.44)
Combining this and (F.42), with probability at least $1 - \delta$, we have
\[
\Delta \leq 2E\Delta + \frac{L}{\lambda_0} \sqrt{8 \log(1/\delta)} \| \Sigma^{1/2} \psi \|_2 + 18 \log(1/\delta). \tag{F.45}
\]

From symmetrization inequality (Theorem 11.4 of [9]), we have $E\Delta \leq 2E \sup_{v \in r_1B_1^d \cap r_2B_2^d} |G_v| \leq 2 \sup_{v \in r_1B_1^d \cap r_2B_2^d} |G_v|$, where
\[
G_v := \sum_{i=1}^{n} a_i \varphi(x_{i,v}) \psi(\xi_{\lambda_{n,i}}), \tag{F.46}
\]
and $\{a_i\}_{i=1}^{n}$ is a sequence of i.i.d. Rademacher random variables which are independent of $\{x_i, \xi_i\}_{i=1}^{n}$. We denote $E^*$ as a conditional expectation of $\{a_i\}_{i=1}^{n}$ given $\{x_i, \xi_i\}_{i=1}^{n}$. Since $\varphi$ is 1-Lipschitz and $\varphi(0) = 0$, from contraction principle (Theorem 11.5 of [9]), we have
\[
E^* \sup_{v \in r_1B_1^d \cap r_2B_2^d} \left| \sum_{i=1}^{n} a_i \varphi(x_{i,v}) \psi(\xi_{\lambda_{n,i}}) \right| \leq \frac{1}{2\lambda_0 \sqrt{n}} \left( E \sup_{v \in r_1B_1^d \cap r_2B_2^d} \left| \sum_{i=1}^{n} a_i x_i^\top v \right| \right). \tag{F.47}
\]
and from the basic property of the expectation, we have
\[
E \sup_{v \in r_1B_1^d \cap r_2B_2^d} \left| \sum_{i=1}^{n} a_i \varphi(x_{i,v}) \psi(\xi_{\lambda_{n,i}}) \right| \leq \frac{1}{2\lambda_0 \sqrt{n}} E \sup_{v \in r_1B_1^d \cap r_2B_2^d} \left| \sum_{i=1}^{n} a_i x_i^\top v \right|. \tag{F.48}
\]
From Lemma F.1, we have
\[
\lambda_0^2 \Delta \leq \lambda_0^2 \sup_{v \in r_1B_1^d \cap r_2B_2^d} \left| \sum_{i=1}^{n} a_i \varphi(x_{i,v}) \psi(\xi_{\lambda_{n,i}}) \right| \leq C L \lambda_0 \sqrt{n} \rho_{r_1} r_{d,s}. \tag{F.49}
\]
From (F.45) and (F.49), we have
\[
\lambda_0^2 \Delta \leq C L \lambda_0 \sqrt{n} \rho_{r_1} r_{d,s} + C L \lambda_0 \sqrt{\log(1/\delta)} r_\Sigma + C \lambda_0^2 n r_\delta^2 \leq (a) C L \lambda_0 \sqrt{n} (\rho_{r_1} r_{d,s} + r_\delta r_\Sigma), \tag{F.50}
\]
where (a) follows from $\lambda_0 \sqrt{n} r_\delta \leq r_\Sigma$ and $L \geq 1$. From (F.41) and (F.50), we have
\[
\inf_{v \in r_1B_1^d \cap r_2B_2^d} \sum_{i=1}^{n} \left( -h(\xi_{\lambda_{n,i}} - x_{i,v}) + h(\xi_{\lambda_{n,i}}) \right) x_{i,v} \geq \frac{\| \Sigma^{1/2} \psi \|_2^2}{3} - C L \lambda_0 \sqrt{n} (\rho_{r_1} r_{d,s} + r_\delta r_\Sigma), \tag{F.51}
\]
and the proof is complete.

F.5 Proof of Proposition 5.7
We note that
\[
\left| \sum_{i \in \mathcal{O}} \tilde{w}_i u_i X_{i,v} \right|^2 \overset{(a)}{=} \frac{4}{n} \sum_{i \in \mathcal{O}} \tilde{w}_i^2 |X_{i,v}|^2 \leq \sum_{i=1}^{n} \tilde{w}_i^2 |X_{i,v}|^2 \overset{(b)}{=}\frac{4}{n} \sum_{i=1}^{n} \tilde{w}_i |X_{i,v}|^2, \tag{F.52}
\]
where (a) follows from Hölder’s inequality and $\sum_{i \in \mathcal{O}} \tilde{w}_i^2 \leq 4$, and (b) follows from the fact that $\tilde{w}_i \leq 2 \tilde{w}_1$ for any $i \in (1, \cdots, n)$. We focus on $\sum_{i=1}^{n} \tilde{w}_i |X_{i,v}|^2$. First, for any $v \in r_1B_1^d \cap r_2B_2^d$, we have
\[
\sum_{i=1}^{n} \tilde{w}_i |X_{i,v}|^2 \overset{(a)}{=} \max_{M \in \mathcal{M}_{r_1}^{r_2}} \sum_{i=1}^{n} \tilde{w}_i (X_i X_{i,v}^\top, M) \leq \tau_{cut}, \tag{F.53}
\]
where (a) follows from Hölder’s inequality and $\sum_{i \in \mathcal{O}} \tilde{w}_i^2 \leq 4\tilde{w}_1$. And (b) follows from the fact that $\tilde{w}_i \leq 2 \tilde{w}_1$ for any $i \in (1, \cdots, n)$. We focus on $\sum_{i=1}^{n} \tilde{w}_i |X_{i,v}|^2$. First, for any $v \in r_1B_1^d \cap r_2B_2^d$, we have
\[
\sum_{i=1}^{n} \tilde{w}_i |X_{i,v}|^2 \overset{(a)}{=} \max_{M \in \mathcal{M}_{r_1}^{r_2}} \sum_{i=1}^{n} \tilde{w}_i (X_i X_{i,v}^\top, M) \leq \tau_{cut}, \tag{F.53}
\]
where (a) follows from the fact that $\mathcal{M}_{r_1,r_2,d,v}^{\ell_1,\ell_2} \subseteq \mathcal{M}_{r_1,r_2}^{\ell_1,\ell_2}$ and (b) follows from the succeeding condition of Algorithm 5.

Combining the arguments above, we have

$$\sum_{i \in \mathcal{O}} \hat{w}_i x_i^T v \leq CL \sqrt{\frac{\alpha}{n}} \sqrt{\frac{\kappa_0^2 (s r_{d,s} + r_3)}{n}} + \Sigma_{\max}^2 r_2,$$

where we use the definition of $\tau_{\text{cut}}$ and the proof is complete.

### F.6 Proof of Proposition 5.8

We note that, from Hölder’s inequality, we have

$$\left| \sum_{i \in \mathcal{I}_m} \frac{u_i x_i^T v}{n} \right|^2 \leq \sum_{i \in \mathcal{I}_m} \frac{1}{n} u_i^2 \sum_{i=1}^n |x_i^T v|^2 \leq 4 \frac{m}{n} \sum_{i=1}^n |x_i^T v|^2,$$

where (a) follows from Hölder’s inequality, and (b) follows from the fact that $\|u\|_2^2 \leq 4m$. From the fact that $\mathcal{M}_{r_1,r_2,d,v}^{\ell_1,\ell_2} \subseteq \mathcal{M}_{r_1,r_2}^{\ell_1,\ell_2}$, we have

$$\sum_{i=1}^n \frac{(x_i^T v)^2}{n} \leq \sup_{M \in \mathcal{M}_{r_1,r_2}^{\ell_1,\ell_2}} \sum_{i=1}^n \frac{(x_i x_i^T, M)}{n}.$$  \hfill (F.56)

From Corollary 5.1 and triangular inequality, we have

$$\sum_{i \in \mathcal{I}_m} u_i x_i^T v \leq \sqrt{c_1 \frac{m}{n} \left( (L \kappa_0)^2 (s r_{d,s} + r_3) + \Sigma_{\max}^2 r_2 \right)}$$

$$\leq CL \sqrt{\frac{\alpha}{n}} \sqrt{\frac{\kappa_0^2 (s r_{d,s} + r_3)}{n}} + \Sigma_{\max}^2 r_2,$$

where (a) follows from $m \leq (2c_0 + 1)o$ and $L \geq 1$, and the proof is complete.

### G Proof of Lemmas A.1, A.2, B.1 and B.2

#### Proof of Lemma A.1.

From the triangular inequality, we have

$$\sum_{i=1}^n \hat{w}_i h(r_{\beta,i}) x_i^T \theta \leq \sum_{i \in \mathcal{I}} \hat{w}_i h(r_{\beta,i}) x_i^T \theta + \sum_{i \in \mathcal{O}} \hat{w}_i h(r_{\beta,i}) x_i^T \theta$$

$$\leq \sum_{i \in \mathcal{I}} \hat{w}_i h(r_{\beta,i}) x_i^T \theta + \sum_{i \in \mathcal{O}} \hat{w}_i h(r_{\beta,i}) x_i^T \theta$$

$$= \sum_{i=1}^n \frac{1}{n} h(r_{\beta,i}) x_i^T \theta - \sum_{i \in \mathcal{O}} \frac{1}{n} h(r_{\beta,i}) x_i^T \theta + \sum_{i \in \mathcal{O}} \sum_{i \in \mathcal{I}} \frac{1}{n} h(r_{\beta,i}) x_i^T \theta$$

$$\leq \sum_{i=1}^n \frac{1}{n} h(r_{\beta,i}) x_i^T \theta + \sum_{i \in \mathcal{O}} \hat{w}_i h(r_{\beta,i}) x_i^T \theta + \sum_{i \in \mathcal{O}} \sum_{i \in \mathcal{I}} \frac{1}{n} h(r_{\beta,i}) x_i^T \theta.$$  \hfill (G.1)
We note that \(|h(\cdot)| \leq 1\) and from Lemma 5.1, \(|\mathcal{O} \cup (\mathcal{I} \cap I_<)| \leq (1 + 2c_c)\alpha\). Therefore, from Propositions 5.3 - 5.5, we have

\[
\sum_{i=1}^{n} \frac{1}{n} h(r_{\beta^*,i}) x_i^\top \theta_{\eta} \leq c_2 L (\rho d_s r_1 + r_\delta r_\Sigma)
\]

\[
\sum_{i \in \mathcal{O}} \frac{1}{n} h(r_{\beta^*,i}) x_i^\top \theta_{\eta} \leq c_2 L \sqrt{1 + c_{\text{cut}}} \left( \kappa_u \sqrt{\frac{O}{n}} \left( \sqrt{sr_{d_s}} + \sqrt{r_\delta} \right) r_2 + \kappa_u r_ar_2 \right)
\]

\[
\sum_{i \in \mathcal{O} \cup (\mathcal{I} \cap I_<)} \frac{1}{n} h(r_{\beta^*,i}) x_i^\top \theta_{\eta} \leq c_2 L \left( \kappa_u \sqrt{\frac{O}{n}} \left( \sqrt{sr_{d_s}} + \sqrt{r_\delta} \right) r_2 + \kappa_u r_ar_2 \right),
\]

(G.2)

and we see that

\[
\left| \sum_{i=1}^{n} \hat{w}_i^* h(r_{\beta^*,i}) x_i^\top \theta_{\eta} \right| \leq 3\epsilon_{\text{max}}^2 L \left( \rho d_s r_1 + r_\delta r_\Sigma + \kappa_u \sqrt{\frac{O}{n}} \left( \sqrt{sr_{d_s}} + \sqrt{r_\delta} \right) r_2 + \kappa_u r_ar_2 \right). \tag{G.3}
\]

Proof of Lemma A.2. We have

\[
\sum_{i=1}^{n} \lambda_\alpha \sqrt{n} \hat{w}_i^* (-h(r_{\beta^*,i} + \theta_{\eta,i}) + h(r_{\beta^*,i})) x_i^\top \theta_{\eta}
\]

\[
= \sum_{i \in \mathcal{I}} \lambda_\alpha \sqrt{n} \hat{w}_i^* (-h(r_{\beta^*,i} + \theta_{\eta,i}) + h(r_{\beta^*,i})) x_i^\top \theta_{\eta} + \sum_{i \in \mathcal{O}} \lambda_\alpha \sqrt{n} \hat{w}_i^* (-h(r_{\beta^*,i} + \theta_{\eta,i}) + h(r_{\beta^*,i})) x_i^\top \theta_{\eta}
\]

\[
= \sum_{i \in \mathcal{I}} \lambda_\alpha \sqrt{n} \hat{w}_i^* (-h(\xi_{\lambda_{\alpha,i}} - x_{\theta_{\eta,i}}) + h(\xi_{\lambda_{\alpha,i}})) x_i^\top \theta_{\eta} + \sum_{i \in \mathcal{O}} \lambda_\alpha \sqrt{n} \hat{w}_i^* (-h(r_{\beta^*,i} + \theta_{\eta,i}) + h(r_{\beta^*,i})) x_i^\top \theta_{\eta}
\]

\[
= \left( \sum_{i=1}^{n} \lambda_\alpha \sqrt{n} \hat{w}_i^* (-h(\xi_{\lambda_{\alpha,i}} - x_{\theta_{\eta,i}}) + h(\xi_{\lambda_{\alpha,i}})) x_i^\top \theta_{\eta} \right)
\]

\[
+ \sum_{i \in \mathcal{O}} \lambda_\alpha \sqrt{n} \hat{w}_i^* (-h(r_{\beta^*,i} + \theta_{\eta,i}) + h(r_{\beta^*,i})) x_i^\top \theta_{\eta}
\]

\[
\geq \sum_{i=1}^{n} \lambda_\alpha \sqrt{n} \hat{w}_i^* (-h(\xi_{\lambda_{\alpha,i}} - x_{\theta_{\eta,i}}) + h(\xi_{\lambda_{\alpha,i}})) x_i^\top \theta_{\eta} - \sum_{i \in \mathcal{O} \cup (\mathcal{I} \cap I_<)} \lambda_\alpha \sqrt{n} \hat{w}_i^* (-h(\xi_{\lambda_{\alpha,i}} - x_{\theta_{\eta,i}}) + h(\xi_{\lambda_{\alpha,i}})) x_i^\top \theta_{\eta}
\]

\[
\geq \sum_{i \in \mathcal{O}} \lambda_\alpha \sqrt{n} \hat{w}_i^* (-h(r_{\beta^*,i} + \theta_{\eta,i}) + h(r_{\beta^*,i})) x_i^\top \theta_{\eta}. \tag{G.4}
\]

We note that \(|h(\cdot)| \leq 1\) and from Lemma 5.1, \(|\mathcal{O} \cup (\mathcal{I} \cap I_<)| \leq (1 + 2c_c)\alpha\). From Propositions 5.6, Propositions 5.4 and 5.5, we have

\[
\sum_{i=1}^{n} \frac{\lambda_\alpha}{\sqrt{n}} (-h(\xi_{\lambda_{\alpha,i}} - x_{\theta_{\eta,i}}) + h(\xi_{\lambda_{\alpha,i}})) x_i^\top \theta_{\eta} \geq \frac{\|\Sigma^\frac{1}{2}\|_2^2}{3} r_2 - c_{\text{max}}^2 L \lambda_\alpha \sqrt{n} (\rho d_s r_1 + r_\delta r_\Sigma)
\]

\[
\left| \sum_{i \in \mathcal{O}} \lambda_\alpha \sqrt{n} \hat{w}_i^* (-h(r_{\beta^*,i} + \theta_{\eta,i}) + h(r_{\beta^*,i})) x_i^\top \theta_{\eta} \right| \leq \epsilon_{\text{max}}^2 L \lambda_\alpha \sqrt{n} \left( \kappa_u \sqrt{\frac{O}{n}} \left( \sqrt{sr_{d_s}} + \sqrt{r_\delta} \right) r_2 + \kappa_u r_ar_2 \right)
\]

\[
\left| \sum_{i \in \mathcal{O} \cup (\mathcal{I} \cap I_<)} \lambda_\alpha \sqrt{n} \hat{w}_i^* (-h(\xi_{\lambda_{\alpha,i}} - x_{\theta_{\eta,i}}) + h(\xi_{\lambda_{\alpha,i}})) x_i^\top \theta_{\eta} \right| \leq \epsilon_{\text{max}}^2 L \lambda_\alpha \sqrt{n} \left( \kappa_u \sqrt{\frac{O}{n}} \left( \sqrt{sr_{d_s}} + \sqrt{r_\delta} \right) r_2 + \kappa_u r_ar_2 \right), \tag{G.5}
\]
Combining the arguments above, we see that

\[
\sum_{i=1}^{n} \lambda_{i} \sqrt{n} \hat{w}_{i}^r \left( \tilde{h}(r_{\beta^{r_i} + \theta_{r_i}^{s_i}}) + h(r_{\beta^{r_i} - \theta_{r_i}^{s_i}}) \right) X_i^\top \theta_\eta \\
\geq \frac{\| \sum \nabla v \|_2^2}{3} - c_{\max}^2 L \lambda_{\max} \sqrt{n} \left( \rho r_{d,s} r_1 + r_3 r_2 + \kappa u \sqrt{\frac{\alpha}{n}} \left( \sqrt{\rho r_{d,s}^2 + \kappa u} r_2 + \kappa u r_2 \right) \right),
\]

and the proof is complete. \( \square \)

**Proof of Lemma B.1.** From the triangular inequality, we have

\[
\left| \sum_{i=1}^{n} \hat{w}_i^r h(r_{\beta^{r_i}}) X_i^\top \theta_\eta \right| \\
\leq \left| \sum_{i \in I} \hat{w}_i^r h(r_{\beta^{r_i}}) X_i^\top \theta_\eta \right| + \left| \sum_{i \in \mathcal{O}} \hat{w}_i^r h(r_{\beta^{r_i}}) X_i^\top \theta_\eta \right| \\
\leq \left| \sum_{i \in I} \frac{1}{n} h(r_{\beta^{r_i}}) X_i^\top \theta_\eta \right| + \left| \sum_{i \in \mathcal{O}} \frac{1}{n} h(r_{\beta^{r_i}}) X_i^\top \theta_\eta \right| + \left| \sum_{i \in \mathcal{O} \cup (\mathcal{I} \cap \mathcal{I}_c)} \frac{1}{n} h(r_{\beta^{r_i}}) X_i^\top \theta_\eta \right|.
\]

We note that \(|h(\cdot)| \leq 1\) and from Lemma 5.1, \(|\mathcal{O} \cup (\mathcal{I} \cap \mathcal{I}_c)| \leq (1 + 2c'_d) \sigma_o\). Therefore, from Propositions 5.3, 5.7 and 5.8, we have

\[
\left| \sum_{i=1}^{n} \frac{1}{n} h(r_{\beta^{r_i}}) X_i^\top \theta_\eta \right| \leq c_3 L \left( \rho r_{d,s} r_1 + r_3 r_2 \right), \\
\left| \sum_{i \in \mathcal{O}} \frac{1}{n} h(r_{\beta^{r_i}}) X_i^\top \theta_\eta \right| \leq c_8 \sqrt{\text{cut}_L} \sqrt{\frac{\alpha}{n}} \sqrt{\frac{\kappa_{d,s}^2 (sr_{d,s} + r_3) + \kappa_u^2 r_2}{n}},
\]

\[
\left| \sum_{i \in \mathcal{O} \cup (\mathcal{I} \cap \mathcal{I}_c)} \frac{1}{n} h(r_{\beta^{r_i}}) X_i^\top \theta_\eta \right| \leq c_9 L \sqrt{\frac{\alpha}{n}} \sqrt{\frac{\kappa_{d,s}^2 (sr_{d,s} + r_3) + \kappa_u^2 r_2}{n}}.
\]

and we see that

\[
\left| \sum_{i=1}^{n} \hat{w}_i^r h(r_{\beta^{r_i}}) X_i^\top \theta_\eta \right| \leq 3c_{\max}^2 L \left( \rho r_{d,s} r_1 + r_3 r_2 + \sqrt{\frac{\alpha}{n}} \sqrt{\frac{\kappa_{d,s}^2 (sr_{d,s} + r_3) + \kappa_u^2 r_2}{n}} \right).
\]

\( \square \)
Proof of Lemma B.2. We have

\[
\sum_{i=1}^{n} \lambda_o \sqrt{n} \sqrt{\hat{w}_i^2} \left( -h(r_{\beta^*, \theta_{n,i}}) + h(r_{\beta^*, i}) \right) X_i^T \theta_{\eta} = \sum_{i \in \mathcal{I}} \lambda_o \sqrt{n} \sqrt{\hat{w}_i^2} \left( -h(r_{\beta^*, \theta_{n,i}}) + h(r_{\beta^*, i}) \right) X_i^T \theta_{\eta} + \sum_{i \in \mathcal{O}} \lambda_o \sqrt{n} \sqrt{\hat{w}_i^2} \left( -h(r_{\beta^*, \theta_{n,i}}) + h(r_{\beta^*, i}) \right) X_i^T \theta_{\eta}
\]

\[
= \sum_{i \in \mathcal{I}} \lambda_o \sqrt{n} \sqrt{\hat{w}_i^2} \left( -h(\xi_{\lambda^*, i} - x_{\theta_{n,i}}) + h(\xi_{\lambda^*, i}) \right) X_i^T \theta_{\eta} + \sum_{i \in \mathcal{O}} \lambda_o \sqrt{n} \sqrt{\hat{w}_i^2} \left( -h(r_{\beta^*, \theta_{n,i}}) + h(r_{\beta^*, i}) \right) X_i^T \theta_{\eta}
\]

\[
\geq \sum_{i=1}^{n} \lambda_o \sqrt{n} \sqrt{\hat{w}_i^2} \left( -h(\xi_{\lambda^*, i} - x_{\theta_{n,i}}) + h(\xi_{\lambda^*, i}) \right) X_i^T \theta_{\eta} - \sum_{i \in \mathcal{O} \cup (\mathcal{I} \cap \mathcal{I}_<)} \lambda_o \sqrt{n} \sqrt{\hat{w}_i^2} \left( -h(\xi_{\lambda^*, i} - x_{\theta_{n,i}}) + h(\xi_{\lambda^*, i}) \right) X_i^T \theta_{\eta} \quad \text{(G.10)}
\]

We note that \(|h(\cdot)| \leq 1\) and from Lemma 5.1, \(|\mathcal{O} \cup (\mathcal{I} \cap \mathcal{I}_<)| \leq (1 + 2c) \mathcal{O}\). Therefore, from Proposition 5.6, we have

\[
\sum_{i=1}^{n} \lambda_o \sqrt{n} \sqrt{\hat{w}_i^2} \left( -h(\xi_{\lambda^*, i} - x_{\theta_{n,i}}) + h(\xi_{\lambda^*, i}) \right) X_i^T \theta_{\eta} \geq \frac{\|\Sigma^2 v\|^2}{3} r_\eta - c_{max} L \lambda_o \sqrt{n} (\rho r_{d,s} r_1 + r_\delta r_\Sigma)
\]

\[
\sum_{i \in \mathcal{O}} \lambda_o \sqrt{n} \sqrt{\hat{w}_i^2} \left( -h(r_{\beta^*, \theta_{n,i}}) + h(r_{\beta^*, i}) \right) X_i^T \theta_{\eta} \leq c_8 \sqrt{c_{cut}} L \sqrt{n} \sqrt{\alpha^2 \kappa^2_{d,s} (sr_{d,s} + r_\delta) + \kappa^2_{r} r_2},
\]

\[
\sum_{i \in \mathcal{O} \cup (\mathcal{I} \cap \mathcal{I}_<)} \lambda_o \sqrt{n} \sqrt{\hat{w}_i^2} \left( -h(r_{\beta^*, \theta_{n,i}}) + h(r_{\beta^*, i}) \right) X_i^T \theta_{\eta} \leq c_8 \sqrt{c_{cut}} L \sqrt{n} \sqrt{\alpha^2 \kappa^2_{d,s} (sr_{d,s} + r_\delta) + \kappa^2_{r} r_2}.
\]

\[
\text{G.11}
\]

Combining the arguments above, we see that

\[
\sum_{i=1}^{n} \lambda_o \sqrt{n} \sqrt{\hat{w}_i^2} \left( -h(r_{\beta^*, \theta_{n,i}}) + h(r_{\beta^*, i}) \right) X_i^T \theta_{\eta} \geq \frac{\|\Sigma^2 v\|^2}{3} - c_{max} L \lambda_o \sqrt{n} \left( \rho r_{d,s} r_1 + r_\delta r_\Sigma + \sqrt{n} \sqrt{\alpha^2 \kappa^2_{d,s} (sr_{d,s} + r_\delta) + \kappa^2_{r} r_2} \right),
\]

\[
\text{G.12}
\]

and the proof is complete. \(\square\)