EXTENDED DERDZINSKI-SHEN THEOREM
FOR THE RIEMANN TENSOR

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Abstract. We extend a classical result by Derdzinski and Shen, on the restrictions imposed on the Riemann tensor by the existence of a nontrivial Codazzi tensor. The new conditions of the theorem include Codazzi tensors (i.e. closed 1-forms) as well as tensors with gauged Codazzi condition (i.e. "recurrent 1-forms"), typical of some well known differential structures.

1. Introduction

Codazzi tensors are of great interest in the geometric literature and have been studied by several authors, as Berger and Ebin [11], Bourguignon [3], Derdzinski [9, 12], Derdzinski and Shen [7], Ferus [8], Simon [14]; a compendium of results is reported in Besse’s book [2].

In the following, \(M\) is a \(n \geq 4\) dimensional Riemannian manifold with metric \(g_{ij}\) and Riemannian connection \(\nabla\); the Ricci tensor is \(R_{kl} = -R_{mkl}^m\) and the scalar curvature is \(R = g^{ij}R_{ij}\) [16]. A \((0, 2)\) symmetric tensor is a Codazzi tensor if it satisfies the Codazzi equation:

\[
\nabla_j b_{kl} - \nabla_k b_{jl} = 0.
\]

A Codazzi tensor is trivial if it is a constant multiple of the metric tensor [7]. In terms of differential forms, the Codazzi equation is the condition for closedness of the 1-form \(B_j = b_{jk}dx^k\), with covariant exterior differential \(DB_j = \nabla_j b_{jk}dx^l \wedge dx^k\) [11, 9].

Codazzi tensors occur naturally in the study of harmonic Riemannian manifolds. For example, the Ricci tensor is a Codazzi tensor if and only if \(\nabla_m R_{jkl}^m = 0\) (by the contracted second Bianchi identity), i.e. the manifold has harmonic Riemann curvature [2]. The other way, the Weyl 1-form \(\Sigma_j = \left(R_{kj} - \frac{R}{2(n-1)}g_{kj}\right)dx^k\) is closed if and only if \(\nabla_m C_{jkl}^m = 0\), where \(C_{jkl}^m\) is the conformal curvature tensor [13], i.e. the manifold has harmonic conformal curvature [2].

Berger and Ebin [11] proved that in a compact Riemannian manifold the Codazzi tensors are parallel if the sectional curvature is non-negative and if there is a point where it is positive. In ref. [3] the important geometric and topological consequences of the existence of a non-trivial Codazzi tensor are examined, particularly the restrictions imposed on the structure of the curvature operator. Derdzinski and Shen improved these results and proved a remarkable theorem on the consequences of the existence of a non trivial Codazzi tensor [7]. The theorem is reported also in Besse’s book [2].

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Theorem 1.1 (Derdzinski-Shen). Let \( b_{ij} \) be a Codazzi tensor on a Riemannian manifold \( M \), \( x \) a point of \( M \), \( \lambda \) and \( \mu \) two eigenvalues of the operator \( b_{ij}(x) \), with eigenspaces \( V_\lambda \) and \( V_\mu \) in \( T_xM \). Then, the subspace \( V_\lambda \wedge V_\mu \) is invariant under the action of the curvature operator \( R_x \).

The theorem can be rephrased as follows: given eigenvalues \( \lambda, \mu, \nu \) of \( b_{ij}(x) \) and corresponding eigenvectors \( X, Y, Z \) in \( T_xM \), it is \( R(X,Y)Z = 0 \), provided that \( \lambda \) and \( \mu \) are different from \( \nu \).

We point out that the Codazzi equation is a sufficient condition for the theorem to hold. A more general condition is suggested by the lemma:

Lemma 1.2. Any \((0,2)\) symmetric Codazzi tensor \( b_{kl} \) satisfies the algebraic identity:

\[
(2) \quad b_{lm} R_{jkl}^m + b_{jm} R_{kli}^m + b_{km} R_{ijl}^m = 0.
\]

Proof. The following condition involving commutators (indices are cyclically permuted) is true for a Codazzi tensor:

\[
(3) \quad [\nabla_j, \nabla_k] b_{il} + [\nabla_i, \nabla_k] b_{lj} + [\nabla_i, \nabla_j] b_{kl} = 0
\]

Each commutator is evaluated: \([\nabla_i, \nabla_j] b_{kl} = R_{ijk}^m b_{ml} + R_{ijl}^m b_{km} \). Three terms cancel by the first Bianchi identity, and the result is obtained. \( \square \)

We note in passing that also the following equation is solved by Codazzi tensors:

\[
(4) \quad [\nabla_i, \nabla_j] b_{kl} + [\nabla_j, \nabla_k] b_{li} + [\nabla_i, \nabla_l] b_{jk} = 0
\]

By evaluating the commutators and simplifying the result by means of the first Bianchi identity and the Codazzi property, the equation gives:

\[
(5) \quad R_{kij}^m b_{ml} + R_{jil}^m b_{mk} + R_{ijl}^m b_{km} = 0
\]

We are ready to formulate the main theorem of the paper: it states that if a symmetric tensor \( b_{kl} \) satisfies the algebraic identity \((2)\), then the same conclusions of the Derdzinski-Shen theorem are valid. Moreover, the proof is much simpler.

2. An extension of the Derdzinski-Shen theorem.

Definition 2.1. A \((0,4)\) tensor \( K_{ijklm} \) is a generalized curvature tensor \([10]\) if it has the symmetries of the Riemann curvature tensor:

a) \( K_{ijkl} = -K_{ijlk} = -K_{ijkl} \),

b) \( K_{ijkl} = K_{klij} \),

c) \( K_{ijkl} + K_{jikl} + K_{kijl} = 0 \) (first Bianchi identity).

Lemma 2.2. If a symmetric tensor \( b_{kl} \) satisfies eq.\((2)\), then \( K_{ijkl} = R_{ijrs} b_{kr}^* b_{ls}^* \) is a generalized curvature tensor.

Proof. Properties a) are shown easily. For example:

\( K_{ijlk} = R_{ijrs} b_{kr}^* b_{ls}^* = R_{ijrs} b_{ls}^* b_{kr}^* = -R_{ijrs} b_{ls}^* b_{kr}^* = -K_{ijkl} \).

Property c) follows from the condition \([2]\): \( K_{ijkl} + K_{jikl} + K_{kijl} = R_{ijrs} b_{kr}^* b_{ls}^* + R_{jirs} b_{kr}^* b_{ls}^* + R_{iks} b_{ir}^* b_{jr}^* + R_{jkr} b_{ir}^* b_{jr}^* + R_{iks} b_{ir}^* b_{jr}^* = 0 \).

Property b) follows from c): \( K_{ijkl} + K_{kijl} = 0 \). Sum the identity over cyclic permutations of all indices \( i, j, k, l \) and use the symmetries a) (this fact was pointed out in \([10]\)).
It is easy to see that a first Bianchi identity holds also for the last three indices:
\[ K_{ijkl} + K_{iklj} + K_{iljk} = 0. \]

**Theorem 2.3** (main theorem). Let \( M \) be a \( n \)-dimensional Riemannian manifold. If a \((0,2)\) symmetric tensor \( b_{kl} \) satisfies the algebraic equation
\[ b_{im} R_{jkl}^m + b_{jm} R_{kil}^m + b_{km} R_{ijl}^m = 0 \]
then, for any \( x \in M \), arbitrary eigenvalues \( \lambda \) and \( \mu \) of \( b_{ij}(x) \), with eigenspaces \( V_\lambda \) and \( V_\mu \) in \( T_x M \), the subspace \( V_\lambda \wedge V_\mu \) is invariant under the action of the curvature operator \( R_x \).

The theorem can be rephrased as follows:
If \( b_{rs} \) is a symmetric tensor with the property (2) and \( X, Y \) and \( Z \) are three eigenvectors of the matrix \( b_{rs} \) at a point \( x \) of the manifold, with eigenvalues \( \lambda, \mu \) and \( \nu \), then
\[ X^i Y^j Z^k R_{ijkl} = 0 \]
provided that \( \lambda \) and \( \mu \) are different from \( \nu \).

**Proof.** Consider the first Bianchi identity for the Riemann tensor, the condition eq.(2) and the third Bianchi identity for the curvature\( K_{\ellijk} = R_{\ellirs} b_{jr} b_{ks} \), and apply them to the three eigenvectors. The resulting algebraic relations can be put in matrix form:
\[
\begin{bmatrix}
1 & 1 & 1 \\
\lambda & \mu & \nu \\
\mu \nu & \lambda \nu & \lambda \mu
\end{bmatrix}
\begin{bmatrix}
R_{\ellijk} X^i Y^j Z^k \\
R_{\elljki} X^i Y^j Z^k \\
R_{\ellkij} X^i Y^j Z^k
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]
The determinant of the matrix is \( (\lambda - \mu)(\lambda - \nu)(\nu - \mu) \). If the eigenvalues are all different then \( R_{\ellijk} X^i Y^j Z^k = 0 \); by the symmetries of the Riemann tensor the statement is true for the contraction of any three indices.
Suppose now that \( \lambda = \mu \neq \nu \), i.e. \( X \) and \( Y \) belong to the same eigenspace; the system of equations implies that \( R_{\ellkij} X^i Y^j Z^k = 0 \).

**Proposition 2.4.** The hypothesis (2) of the main theorem is fulfilled if the symmetric tensor \( b_{ij} \) is a Codazzi tensor or, more generally, if it solves a gauged Codazzi equation
\[ (\nabla_k - \beta_k) b_{ij} = (\nabla_i - \beta_i) b_{kj} \]
where the covariant gauge field \( \beta_k \) is closed: \( \nabla_k \beta_j - \nabla_j \beta_k = 0 \).

**Proof.** The proof is straightforward and may start from eq.(3) by noting that the closedness condition for the gauge field ensures that \([\nabla_i - \beta_i, \nabla_j - \beta_j] = [\nabla_i, \nabla_j]\). Then eq.(3) is again true and eq.(2) follows.

The gauged Codazzi equation (8) can be interpreted in terms of differential forms.
3. Recurrent tensors forms

Definition 3.1. A 1-form $B_j = b_{kj} dx^k$ is recurrent if there is a nonzero scalar 1-form $\beta = \beta_i dx^i$ such that

\begin{equation}
DB_j = \beta \wedge B_j.
\end{equation}

In local components it is \((\nabla_i - \beta_i)b_{kl}(dx^i \wedge dx^k) = 0\). Therefore, the 1-form $B_j = b_{kj} dx^k$ is recurrent if and only if

\begin{equation}
(\nabla_i - \beta_i)b_{kl} = (\nabla_k - \beta_k)b_{il}.
\end{equation}

If $\beta = 0$ the closedness of the 1-form $B_j = b_{kj} dx^k$ and the Codazzi equation are recovered. Eq.\((10)\) enlarges the ordinary notion of recurrence, by which a tensor is recurrent if

\begin{equation}
\nabla_i b_{kl} = \beta_i b_{kl}.
\end{equation}

The extended recurrence eq.\((10)\) includes well known differential structures, as the Weakly $b$ symmetric manifolds, defined by the condition

\begin{equation}
\nabla_i b_{kl} = A_i b_{kl} + B_k b_{il} + D_l b_{ik}.
\end{equation}

For these manifolds, eq.\((10)\) is satisfied with the choice $\beta_i = A_i - B_i$ and, if the covector $\beta_i$ is a closed 1-form, the main theorem applies. Weakly Ricci symmetric manifolds (see [4] and [12] for a compendium) are of this sort:

\begin{equation}
\nabla_i R_{kl} = A_i R_{kl} + B_k R_{il} + D_l R_{ik}.
\end{equation}

They were introduced by Tamassy and Binh [15].

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