FINITE GROUPS AND QUANTUM YANG-BAXTER EQUATION

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Abstract. We construct integrable modifications of 2d lattice gauge theories with finite gauge groups.

The solvability of many 2d lattice statistical models is closely connected to the Quantum Yang-Baxter equation (QYBE) [1, 2]. Solutions of the QYBE are equivalent to weight functions of vertex models.

Probably the most simple 2d integrable system is (lattice) gauge theory. The weights of the field configurations around a plaquette satisfy the QYBE Fig.1a. (The gauge group is assumed to be finite.)

\[ w(a, b, c, d) = R_{a,b}^{d\leftarrow c\leftarrow 1}\chi_r(abcd) \]

\[ \sum_{g,h,i} R_{a,b}^{d\leftarrow c\leftarrow 1}R_{c,d}^{e\leftarrow f\leftarrow 1}(\mu)\chi_r(ab) = \sum_{g,h,i} R_{b,c}^{f\leftarrow e\leftarrow 1}R_{h,i}^{g\leftarrow f\leftarrow 1}(\nu)\chi_r(abcd). \]

\((a, b, c, \ldots \text{ are elements of the group } G, \chi_r \text{ is a character of some irreducible representation } r \in R(G). [3])\)

Equation (0.2) is satisfied since both sides depend on only the holonomy \(abcdef\) around the three plaquettes, which is the same on Fig.1b and Fig.1c. A more direct proof is based on the character identity

\[ \sum_{x \in G} \chi_r(ax)\chi_s(x^{-1}b) = \frac{|G|}{d_r}\delta_{rs}\chi_r(ab), \]

Key words and phrases. Finite groups, group representation, Yang-Baxter equation, lattice models.

This research was partially supported by OTKA grant F-015470.
where $d_r$ is the dimension of the representation $r$. Using (0.3) three times, both sides of (0.2) evaluate to

$$
(0.4) \quad \sum_{r \in G} \chi_r(abcdef). \quad \sum_{r \in G} \chi_r(abcd) + \chi_r(bd). \quad \sum_{r \in G} \chi_r(ac).
$$

A slight modification of (0.4) also satisfies QYBE:

$$
(0.5) \quad w(a, b, c, d) = R_{a,b}^{d-1, c-1}(\lambda) = \\
\sum_{r \in R(G)} \left( \lambda^r \chi_r(abcd) + \chi_r(bd) \right).
$$

At this choice of weights $w$ both sides of (0.2) equal to

$$
(0.6) \quad |G|^3 \sum_{r \in G} d_r^{-2} \left\{ \lambda^r \chi_r(abcd) + \chi_r(bd) \right\}.
$$

If $G$ is a compact Lie-group, (0.2) can be obtained as the weights of the lattice version of the continuous 2d Lie-algebra valued vector field model with action

$$
(0.7) \quad S = \sum_{a=1}^{\dim G} \left\{ \frac{1}{2} F_{x}^{a} + \alpha \left[ \left( \partial_{x} A_{q}^{a} \right)^{2} + \left( \partial_{t} A_{q}^{a} \right)^{2} \right] \right\},
$$

where $A_{q}^{a}, A_{t}^{a}$ are the spatial and temporal components of the vector field. (We assumed that the Killing-metric on $G$ is $\delta_{a,b}$. Since (0.7) is a continuous version of an integrable quantum system, it is reasonable to believe that the classical Euler-Lagrange equation are also integrable ones. The transfer matrices of the lattice system commute for different $\alpha$ and $\beta$ parameters. However, the classical system is constrained since $A_{t}$ is not dynamical. The allowable initial conditions live on different constraint surfaces for different parameters, so it is not quite clear to us that the integrability of the lattice version really implies the integrability of the classical version.

After this short digression we return to the lattice world and present another modification of lattice gauge theory. The weight of a field configuration around a plaquette is given by

$$
(0.8) \quad w(a, b, c, d) = R_{a,b}^{d-1, c-1}(\lambda) = \\
\sum_{r \in R(G)} \chi_r(abcd).
$$

As it does not matter if the variable assigned to a link is $g$ or $g^{-1}$ the set of link variables are the equivalence classes $\hat{G} = G/\{g \sim g^{-1}\}$. For this weight system QYBE does not hold automatically. The summation over the variables $g, h, i$ generate terms like (on Figure 1b):

$$
(0.9) \quad \sum_{g \in G} \chi_r(ab^{g}c^{g}d^{g}) \chi_s(g h^{\sigma} e^{\tau} f^{\rho}) = \frac{G}{d_r} \chi_r(ab^{g}c^{g}d^{g} f^{-\sigma} e^{-\tau} h^{-\rho}).
$$

($s$ is the complex conjugate of the representation $s$.) Since the cyclic order of the variables $a, b, f, c, e$ would change a different way on Fig.1c, the QYBE is not necessarily satisfied. There are two ways to avoid this problem. The first is to require that if $\lambda_r \neq 0$ then $\lambda_r = 0$. Unfortunately, in this case the weights of
the variable configurations are not real, so they are not the weights of a statistical mechanical system. The second method is to use abelian groups, so the order of the group elements is irrelevant.

At last we investigate the ground state structure of these models. Two dimensional discrete lattice gauge theory has infinitely degenerate ground state structure, which prevents the occurrence of of phase transitions. The ground state degeneracy is somewhat lifted in the model (0.5), however, the number of ground state configurations is still infinite, since the configuration on Fig.2 has the same weight as the configuration where all the link variables are equal to $e$ if $a_i$ and $b_i$ are in an abelian subgroup $G_A \subset G$.

|   |   |   |   |
|---|---|---|---|
| $b_4$ | $a_4$ | $b_3$ | $a_3$ |
| $b_2$ | $a_2$ | $b_2$ | $a_2$ |
| $b_1$ | $a_1$ | $b_1$ | $a_1$ |
| $a_1$ | $a_2$ | $a_3$ | $a_3$ |

Figure 2

In contrast to the previous cases, a system with weights (1.8) has unique ground state if and only if there is no involution $p = p^{-1}$ in $G$, where we assume that $G$ is abelian and $\tilde{w}(g) = \sum r \chi_r(g)$ is minimal at $g = e$. If there is an involution $p \in G$, then the configurations where $a_i$ and $b_i$ are either $e$ or $p$ have the same weight as the configuration where all the link variables are equal to $e$. The absence of involutions in an abelian group implies to that $|G|$ is odd. If $\tilde{w}(g)$ has minimum at some $g \neq e$, then the ground state is infinitely degenerate. Indeed, let us mark a subset of the links so that each plaquette has exactly one marked side, and set the marked link variables to $g$ (or $g^{-1}$) and set to $e$ the rest of the links. Such configuration is ground state. Since the marking of the links can be done in many ways, the ground state is highly degenerate. If $\tilde{w}(g)$ has two minimums at $g_1$ and $g_2$, then the marked links can be set either to $g_1$ or $g_2$ in a completely random manner, so the Gibbs-state remains unique even in this case. Consequently the Phase structure is trivial.

References

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