LECTURES on EXOTIC ALGEBRAIC STRUCTURES on AFFINE SPACES

M. Zaidenberg

Abstract

These notes are based on the lecture courses given at the Ruhr-Universität-Bochum (03–08.02.1997) and at the Université Paul Sabatier (Toulouse, 08-12.01.1996).

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1 Introduction

Within the traditional algebraic geometry of quasi-projective varieties, the affine geometry occupies a special place, being known as a source of difficult problems. Let us recall the most famous ones. Hereafter we restrict the consideration to varieties defined over \( \mathbb{C} \).

1. The Zariski Cancellation Problem:
   \( \text{Is it true that an isomorphism } X \times \mathbb{C}^n \cong \mathbb{C}^{n+k} \text{ is only possible if } X \cong \mathbb{C}^k? \)

2. The Structure of the Automorphism Group (Nagata):
   \( \text{Given a polynomial automorphism of } \mathbb{C}^n, \text{ can it be presented as a product of linear and triangular ones?} \)

3. The Linearization Problem:
   \( \text{Is any regular } \mathbb{C}^* \text{-action on } \mathbb{C}^n \text{ conjugate with a linear one in the automorphism group } \text{Aut } \mathbb{C}^n? \)

4. The Embedding Problem (Abhyankar, Sathaye):
   \( \text{Is any regular embedding } \mathbb{C}^k \hookrightarrow \mathbb{C}^n \text{ equivalent to a linear one up to the actions of the groups } \text{Aut } \mathbb{C}^n \text{ on } \mathbb{C}^n \text{ resp. } \text{Aut } \mathbb{C}^k \text{ on } \mathbb{C}^k? \)

5. The Jacobian Problem (Keller):
   \( \text{Given a regular mapping } \mathbb{C}^n \hookrightarrow \mathbb{C}^n \text{ with a constant non-zero Jacobian, is it necessarily an automorphism of } \mathbb{C}^n? \)

To clarify the present day situation, some comments are in order.

1. The affirmative answer to the question (1) for \( k = 2 \) was the result of a series of papers by Miyanishi, Sugie and Fujita [MiySu, Fu 1] (see also [Kam 1]). In higher dimensions \( k \geq 3 \) there is no significant progress.

   An analog of the Zariski Cancellation Problem in birational setting was answered in negative by Beauville, Colliot-Thelene, Sansuc and Swinnerton-Dyer [BCTSS].

   There is the following more general Cancellation Problem (see e.g. [AEH, EH, Ho]):
   \( \text{Given an isomorphism of polynomial rings } A[x] \cong B[x] \text{ over two rings } A \text{ and } B, \text{ when does it follow that } A \cong B? \)

   In the corresponding counterexample by Danielewski [Dan] (see also Fieseler [Fi], tom Dieck [tD 3]) \( A \) and \( B \) are the rings of regular functions on any two of the smooth affine surfaces \( \{x^n y + z^2 = 1\} \subset \mathbb{C}^3, \ n \in \mathbb{N}. \)

2. The structure of the automorphism group \( \text{Aut } \mathbb{C}^n \) for \( n = 2 \) is classically known (Jung [Ju], van der Kulk [vdK]). Starting with \( n = 3 \) it is completely mysterious (see e.g. [AAS, Na 1, Na 2, Wr 1]). For instance, it is still unknown whether or not the Nagata automorphism \( g \in \text{Aut } \mathbb{C}^3, \ g : (x, y, z) \mapsto (x + z \Delta, y + 2x \Delta + z \Delta^2, z) \) where \( \Delta := x^2 - yz \), is ‘tame’, i.e. decomposable as it’s needed in the problem (2) [Na 1, Co, (6.8)].

3. To answer (2) it would be useful to describe the one-parameter subgroups of the automorphism group \( \text{Aut } \mathbb{C}^n \), that is, the regular \( \mathbb{C}^* \)-actions and \( \mathbb{C}^* \)-actions on \( \mathbb{C}^n \) where \( \mathbb{C}^* \), resp. \( \mathbb{C}^* \), denotes the additive resp. the multiplicative group of the complex number field. It seems rather natural to expect that any \( \mathbb{C}^* \)-action resp. any \( \mathbb{C}^* \)-action on \( \mathbb{C}^n \) is conjugate with a triangular one resp. with a linear one. The former one was shown to be false starting with
n = 3 (Bass [Ba]), while it is true for n = 2 (Rentschler [Re]). It is worthwhile noticing that giving a $C_+^n$-action on an affine variety $X$ is the same as giving a locally nilpotent derivation (LND for short) of the algebra $C[X]$ of regular functions on $X$ [Re].

As for the latter one, i.e. for the Linearization Problem, the positive answer is known for $n = 2$ (Gutwirth [Gut]). Below we say more about the recent positive solution for $n = 3$ and the role of exotic $C^3$-s in this solution (see Koras–Russell [KoRu 2, KoRu 3], Makar-Limanov [ML 3], Kaliman and Makar-Limanov [KaML 3], Kaliman, Koras, Makar-Limanov, Russell [KaKoMLRu 3]). The positive answer for $n = 3$ implies linearizability of any connected, reductive group action on the affine 3-space (Kraft, Popov [KrPo, Po]). An example of a non-linearizable action of the semi-direct product of $C^*$ and $Z/2Z$ on $C^4$ was constructed by G. Schwarz [Sch]. It is known that the Linearization Problem restricted to semi-stable $C^*-$actions, that is, the actions of the multiplicative semigroup $C^*$, is equivalent to the Zariski Cancellation Problem (Kambayashi-Russell [KamRu]; see also Bass-Haboush [BaHa]).

4. Any regular embedding $C \hookrightarrow C^2$ is equivalent to a linear one (Abhyankar-Moh, Suzuki [AM, Suz 3]). However, already for the embeddings $C \hookrightarrow C^3$ and $C^2 \hookrightarrow C^3$ it is unknown whether the analogous fact is true. But the embeddings $C^k \hookrightarrow C^n$ are linearizable as soon as $n \geq 2(k + 1)$ (Jelonek [Je], Kaliman [Ka 4, Ka 5], Nori, Srinivas [Sr]).

5. There is a number of equivalent formulations of the Jacobian Problem and partial results; see e.g. An [An], AAS [AAS], BCW [BCW], Dr [Dr], Kam 2, Or 1, Wa; see also Pj for the Jacobian Problem for $n = 2$ over $R$. False proofs which appear regularly definitely certify the difficulty of the problem.

Not so far ago, new unusual objects appeared in the affine geometry; they were called exotic $C^n$. These are smooth affine varieties diffeomorphic, but non-isomorphic to the affine spaces. Actually, the first example for $n = 3$ was constructed in a deep paper of Ramamujan [Ram] where also the non–existence of exotic $C^2$ was proven. Later on, many examples of smooth contractible affine $n$-folds for all $n \geq 3$ were found (Choudary–Dimca [ChoDi], tom Dieck [tD 1, tD 2], tom Dieck–Petrie [tDP 2], Dimca [Di 1], Kaliman [Ka 1, Ka 2], Kaliman-Zaidenberg [KaZa], Koras–Russell [KoRu 2], Russell [Ru 1], the author [Za 2, Za 3, Za 4]). Some of them occurred to be exotic $C^n-$ s, for the other ones this is still unknown. The main difficulty here is the absence of effective tools for recognition of exotics, or, in other words, for recognition of the affine spaces.

In the work of Koras and Russell on the Linearization Problem for $n = 3$ (see [KoRu 1, 2]) it was reduced to classification problems for a series of smooth contractible threefolds $X \subset C^4$ (the Koras–Russell threefolds), on one hand, and for a series of affine singular quotient surfaces, on the other one. As for the latter one, it was recently settled completely [KoRu 3].

As for the former one, it consists of clarification whether or not all the Koras-Russel threefolds $X \subset C^4$ are exotic $C^3$-s. The first partial results were obtained by Kaliman and Russell [Ka 3, Ru 2]. They succeeded to show that the logarithmic Kodaira dimension is non-negative for at least some of these threefolds.

A methods suggested by Kaliman and Makar-Limanov [KaML 1] allowed them to enlarge this class. Namely, it was shown that under certain restrictions on $X$ there is no dominant regular mapping $C^3 \rightarrow X$.

But all the above methods failed to distinguish from $C^3$ a certain subseries of the Koras-Russel threefolds. The Russell cubic threefold $X \subset C^4$ given by the equation $x^3 y^2 z + z^3 = 0$ is one of them. It looks especially simple, but in fact, this one is the most difficult to analyze. Its geometric structure can be described as follows. It contains ‘the book-surface’ $B := \{x = 0\} \subset X$ which is isomorphic to the product $C \times \Gamma_{2,3}$ where $\Gamma_{2,3} \subset C^2$ is the affine cuspidal cubic $z^2 + t^3 = 0$. The complement $X \setminus B$ is isomorphic to $C^* \times C^2$. Thus, $X$ is
obtained from $C^3$ after replacing $C^2 \subset C^3$ by the book-surface $B$. Notice also that there exists a dominant morphism $C^3 \to X$. Using the fact that $B$ is contractible, one can show that $X$ is contractible, too. It follows from the Smale h-Cobordism Theorem that, actually, $X$ is diffeomorphic to $R^6$.

Finally, Makar-Limanov [ML 2] succeeded to prove that the Russell cubic is an exotic $C^3$. Soon after, Kaliman and Makar-Limanov [KaML 3], along the same approach, showed that all the Koras-Russel threefolds are exotic $C^3$-s. Thus, the Linearization Problem for $C^3$ was answered in positive [KoRu 3, KoRu 4].

The proof of Makar-Limanov [ML 2] is based on the use of locally nilpotent derivations (LND, for short). The principal new ingredients suggested in [ML 2] consist in

1. using Jacobian derivations; in particular,
2. reducing the study of general LND-s to study of Jacobian LND-s;
3. introducing and systematically using generalized degree functions, and then
4. reducing the study of the LND-s of a filtered ring to those of the associated graded ring.

In section 7 below we present a simplified proof of the Makar-Limanov Theorem due to Derksen [De]. In section 2 we deal with contractible and more general acyclic surfaces. They serve as a base for constructing exotic $C^n$-s, but certainly merit being studied on their own right. Sections 3–6 are devoted to constructions of exotic $C^n$-s. Besides, in section 6 examples of computations of the logarithmic Kodaira dimension are given.

To simplify presentation we often restrict it to particularly interesting examples. We do not address at all, or say very little on closely related subjects such as analytically exotic structures (see e.g. [Ka 2, Za 3, Za 5]), deformations of exotic structures (see [FlZa 1, Za 3, Za 5]), $Q$-acyclic surfaces (see [FlZa 1, Fu 2, Miy 2, Or 2]), the positive characteristic case, etc. The interested reader can find additional information and open problems in [OPOV, Za 5].

It is my pleasure to thank Profs. H. Flenner and G. Schumacher, who suggested to give a lecture course on exotic structures at the Graduiertenkolleg of the Ruhr-Universität-Bochum, 03–07.02.1997, as well as the organizers of the school ‘Structures exotiques de $C^n$ ’ at the Université Paul Sabatier, Toulouse, 08-12.01.1996, and especially, Mme Laurence Fourrier, for an analogous suggestion. The author is grateful to Shulim Kaliman and Yuli Rudyak, who looked through the text and made many useful comments; to Konstantin Sonin for his help in editing the LaTeX-version of these notes.

2 Acyclic surfaces

2.1 The first acquaintance

By the Hironaka Resolution of Singularities Theorem, any smooth quasi-projective variety $X$ admits a smooth projective completion $V$ by a divisor $D$ with simple normal crossings. $X = V \setminus D$. We call $(V, D)$ an SNC-completion of $X$ or an SNC-pair. A variety $X$ is acyclic if $H_*(X, Z) \cong Z$.

**Lemma 2.1.** (Fujita [Fu 2]) Let $X$ be a smooth quasi-projective surface. If $X$ is acyclic, then it is affine.

\[1\text{.i.e. all the irreducible components of } D \text{ are smooth hypersurfaces in } V, \text{ and for any point } p \in D, \text{ the divisor } D \text{ can be given in appropriate local coordinates } (z_1, \ldots, z_n) \text{ on } V \text{ with center at } p \text{ by an equation of the form } z_1 \cdots z_k = 0 \text{ where } k \leq n.\]
Proof. Assume that the surface $X$ is acyclic. Let $V$ be a smooth completion of $X$ by a reduced divisor $D$ (not necessarily SNC). Let $D = \sum_{i=1}^{k} D_i$ where each $D_i$ is an irreducible component of $X$. We will show that there exists an effective ample divisor $A = \sum_{i=1}^{k} a_i D_i$ supported by $D$, i.e. such that $a_i > 0 \forall i = 1, \ldots, k$. Thus, $mA$ for $m$ large enough is a hyperplane section (for the embedding $\Phi_{|mA|} : V \hookrightarrow \mathbb{P}^N$). Hence,

$$X = V \setminus D = V \setminus \text{supp } (mA) \hookrightarrow \mathbb{P}^N \setminus H \simeq \mathbb{C}^N$$

is affine. By the Nakai-Moishezon criterion, it suffices to choose any $A$ as above such that $A^2 > 0$ and $AC > 0$ for any irreducible curve $C$ in $V$.

In view of acyclicity of $X$, from the standard topological dualities (see the proof of Proposition 2.3 below) it follows that the natural homomorphism $H_2(D) \rightarrow H_2(V)$ is surjective, and $D$ is connected. (In fact, to prove that $X$ is affine we use only these two conditions. It is well known that the boundary of an irreducible affine variety is connected, so, the second one is necessary.) Set $\sum = \{ A = \sum_{i \in I} a_i D_i \mid a_i > 0 \ \forall i \in I, AD_i > 0 \ \forall i \in I, \ I \subset \{1, \ldots, k\} \}. $ First, we show that $\sum$ is non-empty. Indeed, let $H \in \text{Div } V$ be any ample divisor. The classes of $D_i$, $i = 1, \ldots, k$, (which we denote by the same letters) generate the group $H_2(V, \mathbb{Z})$, and so $H = \sum_{i=1}^{k} h_i D_i = \sum_{i \in I} a_i D_i - \sum_{j \in J} a_j D_j = A_0 - B_0$, where $I, J \subseteq \{1, \ldots, k\}, I \neq \emptyset, I \cap J = \emptyset$, and $a_i > 0 \ \forall i \in I \cup J$. For any irreducible curve $C$ in $V$, we have $A_0 C - B_0 C = HC > 0$, whence $A_0 C > B_0 C$. Given $C = D_i, \ i \in I$, this implies $A_0 C > B_0 C \geq 0$. Therefore, $A_0 \in \sum$.

Suppose that $A \in \sum$, $supp A \neq D$, $D_j A > 0$. Then $m_i A + D_j \in \sum$ for some $m_j > 0$. Indeed, $(m_j A + D_j) D_i > 0$ for all $D_i \subset supp A$, and $(m_j A + D_j) D_j > 0$ when $m_j > -D_j^2/D_j A$.

Recall that the divisor $D$ is connected. Therefore, starting with $A_0$ and applying the procedure as above, in a finite number of steps one can find a divisor $A \in \sum$ with $supp A = D$.

Clearly, $A^2 > 0, AD_i > 0$ for all $i = 1, \ldots, k$, and $AC \geq DC$ for any irreducible curve $C$ such that $C$ is not contained in $D$. Since $A_0 C \geq HC > 0$, we have $DC > 0$, whence also $AC > 0$. Thus, $A$ is an ample divisor, and $supp A = D$. □

Remark 2.1. In higher dimensions the analogous statement is not true, in general, as an example of Winkelmann [Win] shows. In this example $X = \mathbb{Q} \setminus E$ is a contractible non-affine (and even non-Stein) quasi-projective variety where $\mathbb{Q}$ is a smooth projective quadric of dimension 4, and $E \subset \mathbb{Q}$ is a codimension 2 smooth subvariety.

Proposition 2.1. Let $X = V \setminus D$ where $V$ is a smooth projective surface, and $D$ is a curve in $V$. Then $X$ is acyclic if and only if the following conditions hold:

(i) $\pi_0(D) = \pi_1(V) = \pi_1(D) = 1$.

(ii) $i_* : H_2(D, \mathbb{Z}) \rightarrow H_2(V, \mathbb{Z})$ is an isomorphism.

Proof. By the Lefschetz duality [Do] we have

$$H^i(V, D) \simeq H_{n-i}(X), \ H_i(V, D) \simeq H^{n-i}(X), \ i = 0, \ldots, 4.$$ 

Assume that $X$ is acyclic; then the above groups are zero for $i = 0, \ldots, 3$. From the standard exact sequences of a pair (all the homology groups have coefficients in $\mathbb{Z}$):
C of is still open (for this and the related Hirzebruch problem on the description of compactifications

Is it true that every smooth contractible affine (or even quasi-projective) variety is rational?

is a rational tree on a smooth rational surface

\[ D \]

vertices, and \( \{ i \} \) is an edge of \( \Gamma_D \) iff \( D_i D_j > 0 \). Each vertex \( D_i \) of \( \Gamma_D \) is weighted by \( D_i^2 \).

If \( X = V \setminus D \) is acyclic, then, by Proposition \[ E \], \( \Gamma_D \) is a tree.

\textbf{Theorem 2.1 (Gurjar-Shastri \[ GuSha \], Gurjar, Pradeep, Shastri \[ GuPrSha \]). Every smooth acyclic surface is rational. Moreover, the same is true for the \textbf{Q}-acyclic surfaces \( X \) (i.e. such that \( H_*(X; \mathbb{Q}) = 0 \) ) with at most quotient singularities.}

\textbf{Remark 2.2.} The following general \textbf{Van de Ven Problem} \[ VdV \]:

Is it true that every smooth contractible affine (or even quasi-projective) variety is rational?

is still open (for this and the related Hirzebruch problem on the description of compactifications of \( \mathbb{C}^n \) see e.g. \[ MS \] \[ Fun \] \[ Pre \]).

\textbf{Corollary 2.2.} An SNC-pair \( (V, D) \) is a completion of an acyclic surface \( X = V \setminus D \) iff \( D \) is a rational tree on a smooth rational surface \( V \) such that the Picard group \( \text{Pic} V \) is freely generated over \( \mathbb{Z} \) by the irreducible components of \( D \), i.e. \( \text{Pic} V \simeq G(D) \simeq \mathbb{Z}^d \).

\textbf{Proof.} Indeed, in the case of a rational surface \( V \) we have \( \text{Pic} V \simeq H_2(V) \). \( \square \)

\textbf{Definition 2.2.} An SNC-pair \( (V, D) \) is called \textit{minimal} if no contraction of a component of \( D \) leads to a new SNC-pair. (Equivalently, \( \Gamma_D \) has neither linear nor end vertices weighted by \(-1\); recall the Castelnuovo criterion.)
Let \( S \) be a smooth compact real manifold with the connected boundary \( \partial S \) and with the interior \( X = S \setminus \partial S \). Then the fundamental group at infinity \( \pi_1^\infty(X) \) of \( X \) can be identified with the group \( \pi_1(\partial S) \). This group is known to be a topological invariant of the open manifold \( X \). Notice that smooth affine surface \( X \) can be presented as the interior of a compact real manifold with a connected boundary.

**Theorem 2.2 (Ramanujam)**. (a) Assume that \((V,D)\) is a minimal SNC-completion of a smooth acyclic surface \( X = V \setminus D \). Then \( X \simeq \mathbb{C}^2 \) iff the dual graph \( \Gamma_D \) is linear.

(b) Furthermore, a smooth contractible surface \( X \) is isomorphic to \( \mathbb{C}^2 \) iff it is simply connected at infinity, i.e. \( \pi_1^\infty(S) = 1 \).

**Example 2.1.** The Hirzebruch surface \( \sum_n \) where \( n > 0 \) is a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^1 \): \( \sum_n \xrightarrow{p} \mathbb{P}^1 \) such that there exists a unique section \( E_n \subset \sum_n \) with \( E_n^2 = -n \); for \( n = 0 \) one takes \( \sum_0 = \mathbb{P}^1 \times \mathbb{P}^1 \) with a constant section as \( E_0 \). If \( F_\infty \) is a fiber over the point \( \infty = (1 : 0) \in \mathbb{P}^1 \), then \( \sum_n \setminus (E_n \cup F_\infty) \simeq \mathbb{C}^2 \), and the dual graph of this completion of \( X = \mathbb{C}^2 \) looks like

\[
\begin{array}{ccc}
-3 & -2 & -2 \\
| & | & |
\end{array}
\]

Note that the standard completion \( (\mathbb{P}^2, \mathbb{P}^1) \) of \( \mathbb{C}^2 \) has the dual graph

\[
\begin{array}{c}
1 \\
| \\
0 \\
| \\
-1 \\
| \\
-2 \\
| \\
-3 \\
\end{array}
\]

**Example 2.2.** The Ramanujam surface \([\text{Ram}]\). There exists an arrangement of a smooth conic \( C_2 \) and a cuspidal cubic \( C_1 \), \( \{zx^2 - y^3 = 0\} \) say, in \( \mathbb{P}^2 \) such that \( (C_1C_2)_A = 1 \), and \( (C_1C_2)_B = 5 \) where \( A, B \) are the smooth intersection points.

Let \( \sigma_n : V \rightarrow \mathbb{P}^2 \) be the blow-up of \( \mathbb{P}^2 \) at \( A \) with the exceptional \((-1)\)-curve \( E \subset V \), and let \( C'_1, C'_2 \subset V \) be the proper transforms of \( C_1, C_2 \). Set \( D = C_1 \cup C_2 \). We have \( H_2(V) \simeq \text{Pic} V \simeq \mathbb{Z}H' + \mathbb{Z}E \) where \( H' \) is the proper transform of a generic line \( H \) in \( \mathbb{P}^2 \). Since \( C_1 \sim 3H \), and \( C_2 \sim 2H \) in \( \text{Pic} \mathbb{P}^2 \simeq \mathbb{Z} \), we get \( (C'_1, C'_2) = T(H', E) \) where \( T \) is the unimodular matrix

\[
\begin{pmatrix}
3 & 2 \\
-1 & -1 
\end{pmatrix}
\]

Thus, \( H_2(V) \simeq H_2(D) \), and so it follows from Proposition \( 2.1 \) that the surface \( X = V \setminus D \) is acyclic. By Fujita’s Lemma \( 2.1 \), \( X \) is affine. The resolution graph of \( D \subset V \) looks as follows:

\[
\begin{array}{ccccccccc}
-3 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\
| & | & | & | & | & | & | & |
\end{array}
\]

This graph is minimal and non-linear, so the Ramanujam Theorem \( 2.2(a) \) yields that \( X \) is not isomorphic to \( \mathbb{C}^2 \).

**Exercises.** (2.1.) Show that \( \pi_1(X) = 1 \), and so \( X \) is contractible.
The boundary $S$ of a ('tubular') neighborhood of $D$ in $V$ (= an attached boundary of $X$ ) is not simply connected; what is $\pi_1(S)$? (see e.g. [Mu, Hir] for an algorithm of computing $\pi_1(S)$ ). Hence, the surface $X$ is not homeomorphic to $\mathbb{R}^4$.

Next we give some more examples of contractible surfaces, following [Za 1, Za 4].

**Example 2.3.** Let $T$ be a matrix of non-negative integers of the form

$$
T = \begin{pmatrix}
m_{00} & 0 & n_{00} & 0 \\
m_{10} & 0 & 0 & n_{10} \\
0 & m_{01} & n_{01} & 0 \\
0 & m_{11} & 0 & n_{11}
\end{pmatrix}
$$

Consider the lines $l_{ij} \cong \mathbb{P}^1$ in the quadric $Q := \mathbb{P}^1 \times \mathbb{P}^1$ where $l_{ij} = \{ix + (1-i)y = j\}_{i,j=0,1}$ (i.e. $x = 0$, $x = 1$, $y = 0$, $y = 1$ ). Blow up over the points $z_{ij} = (i,j)$, $i,j = 0,1$, until the four rational functions

$$
\frac{(x-i)^{m_{ij}}}{(y-j)^{n_{ij}}}, \quad i,j = 0,1,
$$

become regular; denote the resulting surface by $V_T \xrightarrow{\pi} Q$. Set $D_0 = e_0 \cup e_1 \cup \{l_{ij}\} \subset Q$ where $e_0 = \{y = \infty\}$, $e_1 = \{x = \infty\} \subset Q$ (see Figure 1).

![Figure 1](image)

Set $\pi^{-1}(D_0) = D_T \cup \{v_{ij}\}$ where $v_{ij}$ are the only ($-1$) curves in the exceptional divisor of $\pi$. Then $D_T$ is a rational tree (deleting $v_{ij}$ is called ‘cutting cycles’ by tom Dieck and Petrie [DP 1, DP 3]). The proper transforms $e_0', e_1'$ of $e_0, e_1$ and the components of the exceptional divisor of the blowing up $\pi$ form a natural basis in $\text{Pic } V_T = H_2(V_T, \mathbb{Z})$. A new one is given by the components of $D_T$ provided that the decomposition matrix

$$
\begin{pmatrix}
T & 0 \\
B & I
\end{pmatrix}
$$

is unimodular, i.e. that $T$ is unimodular: $\det T = \pm 1$. Now Proposition [2.1] asserts that under this condition the surface $X_T := V_T \setminus D_T$ is acyclic.

**Remark 2.3.** Tom Dieck and Petrie [DP 1, DP 3] found all the basic line arrangements in $\mathbb{P}^2$ that lead to acyclic surfaces in a blowing up and cutting cycles process as above; there are seven of them. The first one (see Figure 2) depends on discrete and continuous parameters (cf. also the Classification Theorem 3.3(d) below); the other six are projectively rigid.
Then the group $\text{Ker}\{\pi\}$ whence $a_{\pi}$ we will see in Lemma 2.3 below that ing relations in $\pi$ is the minimal normal subgroup generated by these four products. Thus, we have the follow-

Figure 2

However, not every smooth acyclic surface arises from a line arrangement in $\mathbb{P}^2$; see [D 4].

Recall the following notion.

**Definition 2.3.** Let $D$ be a closed analytic hypersurface in a connected complex manifold $M$. By a *vanishing loop* of $D$ at a smooth point $e \in D$ we mean any loop $\delta_D = \delta_{D,e_0} (= \alpha^{-1} \beta \alpha)$ in $M \setminus D$ consisting of a path $\alpha$ which joins a base point $e_0 \in M \setminus D$ with a point $e' \in \omega \setminus D$ of a small complex disc $\omega \subset M$ transversal to $D$ at $e$ and a simple loop $\beta$ in positive direction in $\omega \setminus D$ with the base point $e'$. We also denote by $\delta_D$ the corresponding element of the fundamental group $\pi_1(M \setminus D, e_0)$. It is uniquely defined up to conjugation by the corresponding irreducible component of the hypersurface $D$.

The next lemma completes Example [2.3].

**Lemma 2.2.** If $T$ is unimodular then $X_T$ is a contractible surface.

**Proof.** The unimodularity condition $\det T = \pm 1$ implies that $X$ is an acyclic surface. Thus, in virtue of the Hurewicz and Whitehead Theorems it is enough to show that it also implies simply connectedness of $X$. Denote

$$X_0 := X_T \setminus \{v_{i,j}\} \simeq Q \setminus D_0$$

where, as before, $D_0$ is the union of six fixed generators of the quadric $Q$. The vanishing loops $a_i, b_j \in \pi_1(X_0)$, $i,j = 0,1$, of the lines $l_{i,j} \subset Q$ provide generators for the group $\pi_1(X_0)$, that is,

$$\pi_1(X_0) = \mathbb{F}_2 \times \mathbb{F}_2 = \langle a_0, a_1, b_0, b_1 \mid [a_i, b_j] = 1, i,j = 0,1 \rangle .$$

We will see in Lemma 2.3 below that $\pi_1(X) = \pi_1(X_0)/N$ where

$$N = \langle a_i^{n_{ij}} b_j^{m_{ij}} \mid i,j = 0,1 \rangle$$

is the minimal normal subgroup generated by these four products. Thus, we have the following relations in $\pi_1(X)$: $a_0^{n_{00}} b_0^{n_{00}} = 1$, $a_1^{n_{10}} b_0^{n_{10}} = 1$, and so $a_0^{m_{00} n_{10}} = a_1^{m_{10} n_{00}}$. Also, $a_0^{n_{01}} b_1^{n_{01}} = 1$, $a_1^{n_{11}} b_1^{n_{11}} = 1$, and hence $a_0^{m_{01} n_{11}} = a_1^{m_{11} n_{01}}$. It follows that

$$a_0^{m_{00} n_{10} m_{11} n_{01}} = a_0^{m_{01} n_{11} m_{10} n_{00}},$$

whence $a_0^{\det T} = 1$, that is $a_0 = 1$. In the same way we obtain $a_1 = b_0 = b_1 = 1$, and therefore $\pi_1(X) = 1$. $\square$

**Lemma 2.3.** (a) Let $D$ be a closed hypersurface in a complex manifold $M$, $\dim M \geq 2$. Then the group $\text{Ker}\{i_* : \pi_1(M \setminus D) \to \pi_1(M)\}$ is generated by the vanishing loops of $D$. In
Particular, if \( D \) is irreducible, this kernel is generated, as a normal subgroup, by any of these loops.

(b) (Fujita [Fu 2, (7.18)]) Let \( M \) be a surface, \( D_1, D_2 \) be two curves in \( M \), and \( p \) be an intersection point which is an ordinary double point of \( D_1 \cup D_2 \). Let \( \sigma_p : M' \rightarrow M \) be the blow-up at \( p \). Then (the class of) a vanishing loop \( \alpha_E \) of the exceptional \((-1)\)-curve \( E \subset M' \) of \( \sigma_p \) in the group \( \pi_1(M \setminus (D_1 \cup D_2)) = \pi_1((M' \setminus (E \cup D'_1 \cup D'_2)) \) can be represented as \( \alpha_E = \alpha_{D'_1} \alpha_{D'_2} \), where \( \alpha_{D'_i}, \alpha_{D'_2} \) are vanishing loops of the proper transforms \( D'_i \) of \( D_i \), \( i = 1, 2 \). Moreover, the classes \( \alpha_{D'_1}, \alpha_{D'_2} \) commute.

(c) (Fujita [Fu 2, (7.18)]) In the notation as in (b), let \( \sigma : \tilde{M} \rightarrow M \) be a sequence of blow-ups over \( p \) such that \( D'_1 = mE + \ldots, D'_2 = nE + \ldots \) for \( E \) being the exceptional \((-1)\)-curve of the last blow-up. Then we have \( \alpha_E = \alpha_{D'_1}^m \alpha_{D'_2}^n \).

Proof. (a) Denote \( \Delta \) the unit disc in \( \mathbb{C} \). Let \( \gamma : \partial \Delta = S^1 \rightarrow M \setminus D \). Observe that \( \gamma_* \in \text{Ker} i_* \) if there exists \( \tilde{\gamma} : \Delta \rightarrow M \) such that \( \tilde{\gamma} \circ \partial \Delta = \gamma \). After applying a small smooth deformation we may assume that \( \tilde{\gamma}(\Delta) \) meets \( D \) transversally at smooth points \( p_1, \ldots, p_k \in D \), let \( q_1, \ldots, q_k \in \Delta \) be the corresponding disc points.

Choose disjoint vanishing loops of \( q_1, \ldots, q_k \) in \( \Delta \), and contract the circle \( S^1 = \partial \Delta \) onto their union. Being composed with \( \gamma \) this yields a desired homotopy of \( \gamma \) to a product of vanishing loops of \( D \).

(b) Let \( D_i = \{z_i = 0\}, i = 1, 2 \), in a local chart \( (z_1, z_2) \) in \( M \) centered at \( p \). Representing the product \( \alpha_{D_1} \alpha_{D_2} \) on the torus \( |z_1| = \varepsilon, |z_2| = \varepsilon, \varepsilon > 0 \), as its diagonal section, after blowing up at the origin this loop becomes a vanishing one \( \alpha_E \).

(c) Apply induction on the number of blow-ups. \( \square \)

Exercises. (2.3.) (after Fujita [Fu 2]) Let \( X \) be a smooth acyclic (resp. contractible) surface, and let \( C \subset X \) be an irreducible simply connected curve. Consider the blow-up \( \sigma_p : \widehat{X} \rightarrow X \) at a smooth point \( p \in C \), and set \( X' = \widehat{X} \setminus C' \) where \( C' \subset \widehat{X} \) is the proper transform of \( C \). Show that the surface \( X' \) is also acyclic (resp. contractible).

(2.4.) Draw the dual graph \( \Gamma_{D_T} \) where \( D_T \) is as in Example 2.3 above. Deduce that in many cases the surface \( X_T \) is not isomorphic to \( \mathbb{C}^2 \).

2.2 Elements of classification: the logarithmic Kodaira dimension

Definition 2.4. Let \( L \rightarrow V \) be a line bundle (i.e. a one-dimensional algebraic vector bundle) over a smooth projective variety \( V \), and let \( H^0(V, L) \) be the space of its regular sections. By the Cartan-Serre Theorem, \( h^0(V, L) := \dim H^0(V, L) < \infty \). Suppose that \( h^0(V, L) > 0 \), and fix a basis \( s_0, \ldots, s_n \) of the vector space \( H^0(V, L) \) where \( n = h^0(V, L) - 1 \). Then \( Z := \{z \in V \mid s_0(z) = \ldots = s_n(z) = 0\} \) is a proper subvariety of \( V \). For \( z \in V \setminus Z \) fix a vector space structure in the fiber \( L_z \cong \mathbb{C} \); then the point \( \Phi_L(z) := (s_0(z), \ldots, s_n(z)) \in \mathbb{P}^n \) is well-defined, and the rational map \( \Phi_L : V \rightarrow \mathbb{P}^n \) is regular in \( V \setminus Z \). The line bundle \( L \) is called very ample if \( \Phi_L \) is an embedding (assuming \( Z = \emptyset \)); ample if \( mL \) is very ample for some \( m > 0 \); big if \( \dim(V) := \dim \Phi_L(L) = n + 1 \); very big if \( mL \) is very ample for some \( m > 0 \); if \( mL \) is very ample for some \( m > 0 \) we have \( h^0(mL) \sim m^l \) where \( l = L \dim(V) \).

Theorem 2.3 (Serre-Siegel-Kodaira; see e.g. [4, Thm. 10.2]). For some \( m_0 > 0 \) we have \( h^0(mn_0L) \sim n! \) where \( L = L - \dim(V) \).
Definition 2.5. If \( L = K_V \) is the canonical line bundle (i.e. \( K_V = \Lambda^n T^* V \) where \( n = \dim c V \)), then \( k(V) := K - \dim V \) is called the Kodaira dimension of \( V \); \( k(V) \in \{-\infty, 0, 1, \ldots, \dim V\} \). If \( k(V) = \dim V \), then \( V \) is said to be of general type.

Thus, the projective variety \( V \) is of general type iff the canonical line bundle \( K \) of \( V \) is big, i.e. for some \( m > 0 \), \( \Phi_{mK} : V \to \mathbb{P}^n \) is a birational embedding.

Exercise (2.5.) A smooth irreducible projective curve \( V \) is of general type, i.e. \( k(V) = 1 \), iff \( g(V) \geq 2 \); \( k(V) = 0 \) iff \( g(V) = 1 \), i.e. if \( V = T_\Lambda := \mathcal{C}/\Lambda \) is an elliptic curve, where \( \Lambda = \mathbb{Z} + \tau \mathbb{Z} \subseteq \mathbb{C}, c \in \mathbb{C}, \Im \tau > 0 \), is a plane lattice; \( k(V) = -\infty \) iff \( g(V) = 0 \), i.e. if \( V \cong \mathbb{P}^1 \) is a rational curve.

Definition 2.6. Let \( (V,D) \) be an SNC-completion of a smooth quasi-projective variety \( X = V \setminus D \). The log-Kodaira dimension, or the Iitaka–Kodaira dimension of \( X \) is \( \overline{\kappa}(X) := \dim (V) \) where \( L = K + D \) (Iitaka [Ii 1], [Ii 3, Ch. 11]). \( X \) is said to be of log-general type if \( \overline{\kappa}(X) = \dim X \).

Sometimes \( K + D \) is called the log-canonical divisor; the holomorphic sections of \( O(K + D) \) correspond to the meromorphic forms regular in \( X \) which can be written as

\[
a \frac{dz_{i_1}}{z_{i_1}} \wedge \ldots \wedge \frac{dz_{i_k}}{z_{i_k}} \wedge d z_{i_{k+1}} \wedge \ldots \wedge z_{i_n}
\]

in local coordinates in \( V \) where \( D = \{z_{i_1} = \ldots = z_{i_k} = 0\} \).

Theorem 2.4 (Iitaka [Ii 1, Ii 3, Ch. 11]). \( \overline{\kappa}(X) \) is an invariant of \( X \) which does not depend on the choice of an SNC-completion \( (V,D) \) of \( X \).

Exercise (2.6.) Let \( X \) be a smooth irreducible quasi-projective curve. Show that \( \overline{\kappa}(X) = -\infty \) iff \( X = \mathbb{C} \) or \( X = \mathbb{P}^1 \); \( \overline{\kappa}(X) = 0 \) iff \( X = T_\Lambda \) or \( X = \mathbb{C}^* := \mathbb{C} \setminus \{0\} \); \( \overline{\kappa}(X) = 1 \) otherwise.

Theorem 2.5 (The main properties of the log-Kodaira dimension).

(a) ([Ii 1, Ii 3, Thm. 11.3]) \( \overline{\kappa}(X \times Y) = \overline{\kappa}(X) + \overline{\kappa}(Y) \).

(b) ([Ii 1, Ii 3, Prop. 11.5]) If \( Y \) is a Zariski open subset of \( X \), then \( \overline{\kappa}(Y) \geq \overline{\kappa}(X) \), and \( \overline{\kappa}(Y) = \overline{\kappa}(X) \) iff \( \text{codim } X \setminus Y \geq 2 \).

(c) (The Iitaka Easy Addition Theorem [Ii 1, Thm. 4], [Ii 3, Thm. 11.9]) If \( \pi : Y \to X \) is a surjective morphism of smooth quasi-projective varieties with a connected generic fiber \( F \), then \( \overline{\kappa}(Y) \leq \overline{\kappa}(F) + \dim X \).

(d) (The Kawamata–Viehweg Addition Theorem [Kaw 1, Vie]) If, in addition, \( \dim F = 1 \) then \( \overline{\kappa}(Y) \geq \overline{\kappa}(F) + \overline{\kappa}(X) \).

(e) (The Logarithmic Ramification Formula [Ii 1, Ii 3, Thm. 11.3]) Let \( \dim X = \dim Y \), and let \( f : Y \to X \) be a dominant morphism. By the Hironaka Resolution of Singularities Theorem, \( f \) can be extended to a morphism \( \overline{f} : V_Y \to V_X \) where \( (V_X,D_X) \) (resp. \( V_Y,D_Y \)) is an appropriate SNC-completion of \( X \) (resp. of \( Y \)). Then there exists an effective divisor \( R \) such that

\[
K_{V_Y} + D_{V_Y} = \overline{f}^*(K_{V_X} + D_{V_X}) + R \tag{R}
\]

\(^2\text{i.e. } f(Y) \text{ contains a Zariski open subset of } X \).
In particular,
\[ H^0(V_X, m(K_{V_X} + D_{V_X})) \hookrightarrow H^0(V_Y, \mathcal{I}^*m(K_{V_X} + D_{V_X})) \subset H^0(V_Y, m\mathcal{I}^*(K_{V_X} + D_{V_X}) + mR_\mathcal{I}) = H^0(V_Y, m(K_{V_Y} + D_{V_Y})). \]

Therefore, \( \overline{k}(X) \leq \overline{k}(Y). \)

(f) ([14][14] Prop. 1, Thm. 3), [15][15], Thms. 10.5, 11.10) If, in addition, \( f \) is either a proper birational morphism, or an étale covering, then we may assume \( R_\mathcal{I} \) being an \( f \)-exceptional divisor, i.e. \( \text{codim} R_\mathcal{I} \geq 2 \), and we have \( \overline{k}(Y) = \overline{k}(X). \)

**Theorem 2.6 (Classification of acyclic surfaces).** Let \( X \) be an acyclic surface. Then the following assertions hold.

(a) (Miyanishi-Sugie-Fujita [MiiSu, Fu 1]) \( \overline{k}(X) = -\infty \) iff \( X \cong \mathbb{C}^2 \).

(b) (Fujita [Fu 2]) If \( X \) is non-isomorphic to \( \mathbb{C}^2 \), then \( \overline{k}(X) \geq 1 \).

(c) (Iitaka-Kawamata [Ii 1, Thm. 5], [Kaw 2]) If \( \overline{k}(X) = 1 \), then there exists a morphism \( X \rightarrow \Gamma \) onto a smooth curve \( \Gamma \) with generic fibers isomorphic to \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) (called a \( \mathbb{C}^* \)-fibration).

(d) (Gurjar-Miyanishi [GuMiy]; cf. also [DP 1, FlZa 1]) There exists a complete list of acyclic surfaces with \( \overline{k}(X) = 1 \). Any such surface can be obtained from a tom Dieck–Petrie line configuration in \( \mathbb{P}^2 \) of the first kind (see Figure 2 above) by a composition \( \sigma : V \rightarrow \mathbb{P}^2 \) of blowing ups over the points \( p_0, \ldots, p_s \) to get, as \( \sigma^{-1}(p_i), i = 1, \ldots, s \), a linear chain of rational curves with only one \( (-1) \)-curve. All these curves except for the last \( (-1) \)-curve of each chain are components of the boundary divisor \( D \subset V \), as well as the proper preimage of the original line arrangement. In addition, all the blow ups over \( p_0 \) are outer, i.e. each of them is done at a smooth point of the total transform of \( l_0 \), whereas under the cutting cycle procedure over \( p_i, i = 1, \ldots, s \), the blow ups are inner, i.e. they are done only at double points of the preimage of the original line arrangement.

For each \( i = 1, \ldots, s \), we fix a rational number \( \frac{m_i}{n_i} \). The blow-up process over \( p_i \) is done according to the data \( (m_i, n_i) \); this means that locally at \( p_i \) it resolves the point of indeterminacy of the rational function \( x^{m_i}/y^{n_i}, i = 1, \ldots, s \). The numbers \( (m_i, n_i)_{i=1,\ldots,s} \) must satisfy a Diophantine equation of unimodularity which guarantees the acyclicity of the resulting open surface \( X = V \setminus D \).

In particular, all the contractible surfaces with \( \overline{k}(X) = 1 \) are obtained in this way for \( s = 2 \) and \( m_1n_2 + m_2n_1 - m_1m_2 = \pm 1, m_i > n_i, i = 1, 2 \). Their minimal dual graphs look as follows:

```
0 0 0
\cdots
\cdots
\cdots
```

The choice of centers of the blow-ups over \( p_0 \) (besides the first one), and the positions of the points \( p_3, \ldots, p_s \) on \( l_{s+1} \) (once the first three intersection points on \( l_{s+1} \) have been fixed) give

\[^3\text{Actually, any (not necessarily acyclic) affine surface } X \text{ with } \overline{k}(X) = 1 \text{ possesses a } \mathbb{C}^* \text{-- fibration.}\]
rise to the parameters of a versal deformation of the surface \( X \) or, more accurately, of its minimal SNC-completion \((V_{\text{min}}, D_{\text{min}})\), which is defined in a unique way \([\text{FlZa} 1]\).

**Theorem 2.7 (Simply connected curves on acyclic surfaces)** \([\text{AM}, \text{LiZa}, \text{GuMiy} 2, \text{Suz} 1, \text{Za} 1]\). Let \( X \) be a smooth acyclic surface, and let \( \Gamma \) be an irreducible simply connected curve in \( X \). Then either

\[ \ast \quad (X, \Gamma) \simeq (\mathbb{C}^2, \Gamma_{k,l}), \text{ where } \Gamma_{k,l} := \{ x^k - y^l = 0 \} \subset \mathbb{C}^2, \quad k \geq l \geq 1, \quad (k, l) = 1, \quad \text{or} \]

\[ \bar{k}(X) = 1 \quad \text{and} \quad \Gamma = E \setminus D \simeq \mathbb{C} \quad \text{where} \quad E \subset V \text{ is the last } (-1)-\text{curve over the point } p_0 \text{ in the reconstruction process as in Theorem 3.3(d) above, and} \ D \text{ is the boundary divisor of the corresponding SNC-completion of } X. \]

In particular, this theorem shows that, up to automorphisms of the affine plane, there is only a sequence of irreducible simply connected curves in \( \mathbb{C}^2 \) (namely, \( \{ \Gamma_{k,l} \} \)). Each smooth acyclic surface of logarithmic Kodaira dimension 1 contains exactly one such curve, and this curve is smooth. At last, there is no simply connected curves at all on acyclic surfaces of log-general type. See also \([\text{GuMiy} 2, \text{GuPa}]\) for some further information.

**Example 2.4. Tom Dieck-Petrie surfaces** \([\text{DP} 2]\). The surface \( X_{k,l} \subset \mathbb{C}^3 \) given by the equation

\[ \frac{(xz + 1)^{k} - (yz + 1)^{l}}{z} = 1 \]

where \( k > l \geq 2, \ (k, l) = 1 \) is a smooth contractible one with \( \bar{k}(X_{k,l}) = 1 \) (see Examples 4.4 and 4.6 below). The only simply connected curve in \( X_{k,l} \) is given by the equation \( z = 0 \).

In a similar way, any smooth contractible surface with \( \bar{k} = 1 \) can be properly embedded into \( \mathbb{C}^3 \) (Kaliman, Makar-Limanov \([\text{KaML} 2]\)).

**Example 2.5.** It can be shown (see \([\text{Za} 3, \text{Za} 4]\)) that \( \bar{k}(X_T) = 1 \) for a surface \( X_T \) as in Example 2.3 iff \( m_{ij} = n_{ij} = 1 \) for a pair of diagonal points from the square vertices \( (z_{ij} = (i, j))_{i,j=0,1} \) (see Figure 1 above). If so, then the only simply connected curve in \( X_T \) is the proper transform of the corresponding diagonal line. Otherwise, \( \bar{k}(X_T) = 2 \), i.e. \( X_T \) is of log-general type.

**Remark 2.4.** There is a number of examples of acyclic or even contractible surfaces of log-general type (see e.g. \([\text{ID} 2, \text{FlZa} 1, \text{GuMiy} 1, \text{Sug}]\)), but no classification is known. While acyclic surfaces of log-Kodaira dimension 1 admit deformations (see the Classification Theorem 3.3(d)), those of log-general type are rigid in all known examples \([\text{FlZa} 1, \text{FlZa} 2]\). So, the problem arises \([\text{OPOV, FlZa} 1]\):

Is it true that any smooth \( (\mathbb{Q}-) \)acyclic surface of log-general type is rigid?

### 3 Exotic product structures

We begin this section by recalling

**The Zariski Cancellation Problem.** Given an isomorphism \( X \times \mathbb{C}^k \overset{\Phi}{\sim} \mathbb{C}^{n+k} \), does it follow that \( X \simeq \mathbb{C}^n \)?

Take \( \mathbb{C}^n \) generic in \( \mathbb{C}^{n+k} \), and combine \( \Phi \) with the first projection. This yields a surjective morphism \( \mathbb{C}^n \rightarrow X \). Thus, by Theorem 2.5(e), \( \bar{k}(X) = -\infty \). Clearly, \( X \) is homotopically
trivial; in particular, for $n = 2$ $X$ is an acyclic surface $\bar{k}(X) = -\infty$. By the Miyanishi-Sugie-Fujita Theorem 2.3. (a), $X \simeq \mathbb{C}^2$. This provides the positive answer to the Zariski Cancellation Problem for $n = 1, 2$. For $n \geq 3$ the problem is open. In this respect the following fact could be useful.

**Theorem 3.1 (The Iitaka-Fujita Strong Cancellation Theorem [IiFu]).** Let $X, Y$ be smooth quasi-projective varieties of the same dimension, and let $\Phi : Y \times \mathbb{C}^n \rightarrow X \times \mathbb{C}^n$ be an isomorphism. Assume that $k(X) \geq 0$. Then there is a commutative diagram

$$
\begin{array}{ccc}
Y \times \mathbb{C}^k & \xrightarrow{\Phi} & X \times \mathbb{C}^k \\
\text{pr} & & \text{pr} \\
Y & \xrightarrow{\varphi} & X
\end{array}
$$

where $\varphi$ is an isomorphism.

We use below the following well known corollary of the Smale h-cobordism Theorem.

**Proposition 3.1 (see [Mil 1, §9]).** Let $D^n$ be a smooth simply connected manifold of (real) dimension $n \geq 5$ with a simply connected boundary. Then the following conditions are equivalent:

1) $D^n$ is diffeomorphic to the closed unit $n-$ball $\overline{B}^n$.
2) $D^n$ is homeomorphic to $\overline{B}^n$.
3) $D^n$ is contractible.
4) $D^n$ is acyclic.

**Theorem 3.2 (Dimca-Ramanujam [Di 1, Ram]).** Let $X$ be a contractible smooth affine algebraic variety. If $\dim \mathbb{C}X = n \geq 3$ then $X$ is diffeomorphic to $\mathbb{R}^{2n}$.

**Proof.** Fix a closed embedding $X \hookrightarrow \mathbb{C}^N$ such that the smooth function $\varphi := ||z||^2 | X$ on $X$ is a Morse function, i.e. it has only non-degenerate critical points (see [Mil 2, Thm. 6.6]). Since the smooth mapping $\varphi : X \rightarrow \mathbb{R}$ is proper and has only finite number of critical values, for $R > 0$ large enough the domain $X_R := \{ \varphi < R \}$ in $X$ is diffeomorphic to the whole manifold $X$. Denote $S_R = \partial X_R$; that is, $\overline{X_R}$ is a smooth manifold with the boundary $S_R$. By the Morse Theory applied to the Morse function $\psi := R - \varphi$ on $X$, the manifold $\overline{X_R}$ can be obtained, starting with the boundary $S_R$, by successively gluing handles of indices equal to those of the critical points of $\psi$ on $X_R$.

If $p \in X_R$ is a critical point of $\psi$, then $\ind_p \psi = 2n - \ind_p \varphi$. But $\ind_p \varphi \leq n$ [Mil 2, the proof of Thm. 7.2], Hence, $\ind_p \psi \geq n \geq 3$. Therefore, $\overline{X_R}$ is obtained from $S_R = \partial X_R$ by attaching handles of indices at least 3. Consequently, $\overline{X_R}$ is homotopically equivalent to a cell complex obtained from $S_R$ by successively attaching cells of dimension at least 3. It follows that the first two relative homotopy groups $\pi_i(\overline{X_R}, S_R)$, $i = 1, 2$, are trivial. Since $\overline{X_R}$ is contractible, applying the exact homotopy sequence of a pair

$$1 = \pi_2(\overline{X_R}, S_R) \xrightarrow{\partial} \pi_1(S_R) \xrightarrow{i} \pi_1(\overline{X_R}) = 1$$

we conclude that $\pi_1(S_R) = 1$. Now the theorem follows from Proposition 3.1. $\square$

\[4\text{Cf. the proof of the Lefschetz Hyperplane Section Theorem in [Mil 2].}\]
Remark 3.1. In section 2 above some examples have been given (see e.g. Examples 2.2, 2.3) of smooth contractible affine surfaces $S$ with non-simply connected attached boundaries $\partial S$ (in other words, $S$ is not simply connected at infinity: $\pi^1_\infty(S) \neq 1$). Therefore, these contractible surfaces are not homeomorphic to $\mathbf{R}^4$. This shows that the restriction $n \geq 3$ in the above theorem is crucial.

Corollary 3.1. Let $X$ be a smooth contractible surface. Then $X \times C$ is diffeomorphic to $C^3 \simeq R^6$, and so $X \times C^k$ is diffeomorphic to $R^{2k+4}$, $k \geq 1$.

Proof. We indicate, following Ramanujam [Ram], an alternative direct proof of this corollary. According to Proposition 3.1, it suffices to show that $X \times C$ is diffeomorphic to the interior of a smooth compact manifold $D$ with a simply connected boundary $\partial D$. There are two natural ways to compactify $X \times C$. First, consider any smooth affine variety $Z \hookrightarrow C^N$. Then the restriction $\varphi$ of the real polynomial $||z||^2$ to $Z$ has only a finite number of critical values, and hence, $\varphi^{-1}[R,\infty[$ for $R$ large enough is diffeomorphic to $[R,\infty[ \times T$ where $T := \varphi^{-1}(R)$. Thus, $Z$ is diffeomorphic to $Z_0 := \varphi^{-1}[0,R]$, the interior of the manifold with boundary $Z_0 := \varphi^{-1}[0,R]$, $\partial Z_0 = T$. Represent in this way $X \hookrightarrow C^n$ attaching the boundary $\partial X$, and $Y := X \times C \hookrightarrow C^{n+1}$ attaching the boundary $\partial_1 Y = \varphi^{-1}(R_1)$. Since $Y$ is diffeomorphic to $X \times \Delta$ where $\Delta = \{ |z| < 1 \}$, $Y$ can also be compactified by attaching the non-smooth boundary $\partial_2 Y = (\partial X \times \Delta) \cup (X \times S^1)$. In fact, $\partial_2 Y = \psi^{-1}(R_2)$ where $\psi(x, z) := \max \{ ||x||^2, |z|^2 \}$, and $R_2 > 0$ is large enough. By the Van Kampen Theorem, $\partial_2 Y$ is simply connected.

We may assume that sufficiently large $R_1', R_1'', R_2', R_2''$ are chosen in such a way that $\partial_1 Y = \varphi^{-1}(R_1) \subset \varphi^{-1}([R_2', R_2'']) \subset \varphi^{-1}([R_1', R_1''])$, and that $\varphi^{-1}([R_1', R_1'']) \approx \partial_1 Y \times [R_1', R_1'']$, $\psi^{-1}([R_2', R_2'']) \approx \partial_2 Y \times [R_1', R_1'']$. Thus, the composition of embeddings $\partial_1 Y \hookrightarrow \partial_2 Y \times [R_1', R_1''] \hookrightarrow \partial_1 Y \times [R_1', R_1'']$ provides a homotopy equivalence. Respectively, the induced isomorphism $\pi_1(\partial_1 Y) \approx \pi_1(\partial_1 Y \times [R_1', R_1''])$ factors through the trivial one $\pi_1(\partial_1 Y) \to \pi_1(\partial_1 Y \times [R_1', R_1'']) \approx 1$. This proves simply connectedness of the boundary $\partial_1 Y$, and the assertion follows.

Definition 3.1. By an exotic $C^n$ we mean a smooth affine variety diffeomorphic to $R^{2n}$ but non-isomorphic to $C^n$.

Theorem 3.3 (Exotic product structures).
(a) Let $S$ be a smooth contractible surface non-isomorphic to $C^2$. Then the product $S \times C^{n-2}$ $(n \geq 2)$ is an exotic $C^n$.
(b) Furthermore, if two smooth contractible surfaces $S_1$, $S_2$ are not isomorphic, then $S_1 \times C^{n-2}$, $S_2 \times C^{n-2}$ $(n \geq 2)$ are two non-isomorphic exotic $C^n$.

Proof. (a) By Lemma 3.1, $S \times C^{n-2}$ is diffeomorphic to $R^{2n}$ for $n \geq 3$. By the Miyanishi-Sugie-Fujita Theorem 2.3. (a), $\bar{\kappa}(S) \neq -\infty$ (otherwise $S \simeq C^2$), whence $\bar{\kappa}(S) \geq 0$. But if $S \times C^{n-2}$ were isomorphic to $C^n$, then by Theorem 2.3 (e) we would have $\bar{\kappa}(S) = -\infty$, a contradiction.

(b) We have to show that the classification of exotic product structures on $C^n$ of the type $S \times C^{n-2}$ where $S$ is a surface as above is reduced to the classification of surfaces $S$ themselves. Indeed, $S_1 \times C^{n-2} \simeq S_2 \times C^{n-2}$ and $\bar{\kappa}(S_1) \geq 0$ would imply that $S_1 \simeq S_2$. Since $S_1 \neq S_2$, and both surfaces are acyclic, by the Miyanishi-Sugie-Fujita Theorem 3.3 (a), $\bar{\kappa}(S_i) \geq 0$ for at least one value of $i$, say, for $i = 1$, and so, the assertion follows from the Iitaka-Fujita Strong Cancellation Theorem 3.1. □

Remarks. 3.2. For instance, a sequence of pairwise non-isomorphic surfaces $X_T$ of log-general type (see Example 2.3 above) yield a sequence of pairwise non-isomorphic exotic $C^n$-s
Since contractible surfaces $S$ with $\overline{k}(S) = 1$ admit deformations (see the Classification Theorem 3.3(d)), the corresponding exotic $C^n$-s of product type $S \times C^{n-2}$ admit deformations, too [FlZa 1].

3.3. Let $X = \prod_{i=1}^n S_i$ be a product of $n \geq 2$ smooth contractible surfaces. Then $X$ is diffeomorphic to the interior of a compact contractible variety with boundary. By the Van Kampen Theorem, the boundary $\partial X$ is simply connected. Therefore, by the h-cobordism Theorem, $X$ is diffeomorphic to $C^{2n}$. Also, $\overline{k}(X) = \sum_{i=1}^n \overline{k}(S_i)$. Hence, $X$ is of log-general type iff $S_i$ are so for all $i = 1, \ldots, n$; $\overline{k}(X) = -\infty$ if $\overline{k}(S_i) = -\infty$ for at least one value of $i$. This shows, in particular, that for any $n \geq 2$ there exist exotic $C^{2n}$-s of log-general type.

3.4. If $\overline{k}(S) = 2$, then the exotic $C^3$ $X = S \times C$ contains no copy of $C^2$, i.e. there is no embedding $C^2 \hookrightarrow S \times C$ [Za 3]. (This is based on the fact that the surface $S$ contains no simply connected curve; see [Za 1] and Theorem 2.7 above.) In the next section we present examples of exotic $C^3$ with many copies of $C^2$ (see Example 1.1).

3.5. Due to the Ramanujan Theorem 2.2(b), there is no exotic $C^2$.

3.6. Actually, the Zariski Cancellation Problem can be reformulated as follows: Given an exotic $C^n$, denote it $X$, should also the product $X \times C^k$ be an exotic $C^m$ $(m = n + k)$?

Exercise (3.1.) Verify that a smooth irreducible quadric hypersurface $X$ in $C^{n+1}$ is contractible if and only if it is isomorphic to $C^n$, and if so, then the embedding $X \hookrightarrow C^n$ is rectifiable.

4 Contractible affine modifications

4.1 The Kaliman modification

Definition 4.1. Consider a triple $(M, D, C)$, where $M \supset D \supset C$, $M$ and $C$ are smooth quasi-projective (or, more generally, complex) varieties, $D$ is an irreducible hypersurface in $M$, and $C$ is a proper subvariety in $D$ contained in the smooth part $\text{reg } D := D \setminus \text{sing } D$ of $D$, and so $	ext{codim } MD = 1$ and $\text{codim } MC \geq 2$. Let $\sigma_C : \tilde{M} \rightarrow M$ be the blow up of $M$ with center $C$ and with the exceptional divisor $E = \sigma^{-1}(C)$. Then $\sigma_C|E : E \hookrightarrow C$ is a fiber bundle with the fiber $P^k$, $k = \dim E - \dim C$; $E$ and the proper transform $D'$ of $D$ meet transversally, and $\sigma_C : E \cap D' \rightarrow C$ is a fiber bundle with the fiber $P^{k-1}$. The variety $M' := \tilde{M} \setminus D'$ is called the Kaliman modification of the triple $(M, D, C)$ along $D'$ with center $C$ and with the exceptional divisor $E' = E \setminus D'$ [Ka 2]. Clearly, the restriction $\sigma_C|E' : E' \hookrightarrow C$ is a fiber bundle with the fiber $C^k$.

One can show that the affine modification $X'$ of an affine variety $X$ is again an affine variety [Ka 2, Lemma 3.3].

In the proof of the next lemma we use the following notation.

Notation. Let $G$ be a group. For a subset $S$ in $G$ denote by $\langle \langle S \rangle \rangle$ the subgroup of $G$ generated by all the conjugacy classes of the elements of $S$, that is, $\langle \langle S \rangle \rangle$ is the minimal normal subgroup of the group $G$ which contains $S$. We also say that the subgroup $\langle \langle S \rangle \rangle$ is normally generated by $S$. 

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Lemma 4.1 (Kaliman [Ka 2], Lemma 3.4). The induced homomorphism \((\sigma_C)_* : \pi_1(M') \to \pi_1(M)\) is an isomorphism.

Proof. The restriction \(\sigma_C| (M' \setminus E') : M' \setminus E' \to M \setminus D\) is an isomorphism. Thus, we may consider the following commutative diagram (left) and the induced commutative triangle (right):

\[
\begin{array}{ccc}
M' & \xrightarrow{\sigma_C} & \pi_1(M') \\
\downarrow_i & & \downarrow_{(\sigma_C)_*} \\
M \setminus D & \xrightarrow{j} & \pi_1(M \setminus D)
\end{array}
\]

It is easily seen that both \(i_*\) and \(j_*\) are surjections (since a complex hypersurface has real codimension 2). Thus, \((\sigma_C)_*\) is also surjective. Denote by \(\alpha_D\) a vanishing loop of \(D\). By Lemma 2.3 (a), \(\ker j_* = \langle \langle \alpha_D \rangle \rangle\). We choose \(\alpha_D\) in such a way that near \(D\) it is a boundary circle of a small transversal disc \(\omega\) centered at a point \(c_0 \in C\). Then the proper transform \(\omega'\) of \(\omega\) in \(M'\) is a disc centered at a point of \(E' = E \setminus D'\). Thus, \(i_*(\alpha_D) = 1 \in \pi_1(M')\), i.e. \(\alpha_D \in \ker i_*\). This implies that \(\ker j_* \subset \ker i_*\). But since \(j_* = (\sigma_C)_* \circ i_*\), \(\ker i_* = \ker j_*\), and so \((\sigma_C)_* : \pi_1(M') \to \pi_1(M)\) is an isomorphism. \(\square\)

Remark 4.1. The proof can be word-by-word applied in a more general case when the center \(C\) of the blow up is not necessarily smooth and contained in the regular part \(\text{reg } D\) of \(D\); it is only important that it should not be contained in the singular locus \(\text{sing } D\) of the hypersurface \(D\).

Lemma 4.2. (cf. Kaliman [Ka 2], Proof of Thm. 3.5) Suppose that \(D\) is a topological manifold, and \((i)\) the varieties \(D\) and \(C\) are acyclic. Then the variety \(M'\) is acyclic iff \(M\) is.

Proof. As follows from Lemma 4.1, \((\sigma_C)_* : H_1(M') \to H_1(M)\) is an isomorphism (hereafter all the homology groups are with coefficients in \(\mathbb{Z}\)). Note that

- \(\sigma_C : E' \to C\) is a smooth fibration with a contractible fiber, and so, it yields a homotopy equivalence between \(E'\) and \(C\). Therefore, \((\sigma_C)_* : H_*(E') \to H_*(C)\) is an isomorphism. Hence, the exceptional divisor \(E'\) is also acyclic.
- Let \(\hat{X}\) be the one-point compactification of a manifold \(X\). Then we have
  \[
  \tilde{H}^i(\hat{X}) \simeq H^i(\hat{X}, \ast) \simeq H^i_c(X) \cong H_{m-i}(X)
  \]
  where \(\cong\) stands for the Lefschetz–Poincaré duality, and \(m = \dim \mathbb{R}X\). Thus, under our assumptions \(\hat{D}\) and \(\hat{E}'\) are homology spheres, and so is \(M\) resp. \(M'\) iff it is acyclic.

  Assume first that \(M\) is acyclic. Then by the exact cohomology sequence of a pair

\[
\begin{array}{ccc}
H^*(\hat{M}) & \xrightarrow{r^*} & H^*(\hat{M}, \hat{D}) \\
\downarrow{r^*} & & \downarrow{\delta^*} \\
H^*(\hat{D}) & \xrightarrow{\delta^*} & H^*(\hat{M}, \hat{D})
\end{array}
\]

where \(\deg r^* = \deg i^* = 0\), \(\deg \delta^* = 1\), we have \(H^{2n-1}(\hat{M}, \hat{D}) \simeq H^{2n}(\hat{M}, \hat{D}) \simeq \mathbb{Z}\), where \(n = \dim \mathbb{C}M\), and \(\tilde{H}^j(\hat{M}, \hat{D}) = 0\) for \(j \leq 2n - 2\). Since \(\hat{M} \setminus \hat{D} = M \setminus D \approx M' \setminus E'\), we have
Hence,
\[ H^* (\tilde{M}', \tilde{E}') \simeq \tilde{H}^* (\tilde{M}/\tilde{E}') \simeq \tilde{H}^* (\tilde{M}/\tilde{D}) \simeq H^* (\tilde{M}, \tilde{D}) . \]
Thus, \( H^{2n-1} (\tilde{M}', \tilde{E}') \simeq H^{2n} (\tilde{M}', \tilde{E}') \simeq \mathbb{Z} \), and the other groups are zero. From the exact cohomology sequence of the pair \((\tilde{M}', \tilde{E}')\) we obtain \( H^j (\tilde{M}') \simeq H^j (\tilde{E}) = 0 , \quad 1 \leq j \leq 2n - 3 \), and
\[
0 = H^{2n-2} (\tilde{M}', \tilde{E}') \longrightarrow H^{2n-2} (\tilde{M}') \longrightarrow H^{2n-2} (\tilde{E}') \simeq \mathbb{Z} \overset{\partial^r} \longrightarrow \\
\longrightarrow H^{2n-1} (\tilde{M}', \tilde{E}') \simeq \mathbb{Z} \longrightarrow H^{2n-1} (\tilde{M}') \longrightarrow H^{2n-1} (\tilde{E}') = 0 . \tag{*}
\]
By the Poincaré duality, we have
\[ H_{2n-j} (\tilde{M}') = \tilde{H}^j (\tilde{M}') . \]
Hence, \( H_i (\tilde{M}') = 0 \) for \( i \geq 3 \), and \( H_1 (\tilde{M}') \simeq H_1 (M) = 0 \). Thus, by the Poincaré duality, \( H^{2n-1} (\tilde{M}') = 0 \) in \((*)\), whence \( \partial^r : H^{2n-2} (\tilde{E}') \simeq \mathbb{Z} \longrightarrow H^{2n-1} (\tilde{M}', \tilde{E}') \simeq \mathbb{Z} \) is onto, and so, it is an isomorphism. This implies that \( H^{2n-2} (\tilde{M}') = 0 \), and also, by the Poincaré duality, \( H_2 (\tilde{M}') = 0 \). Finally, we have that \( \tilde{H}_s (\tilde{M}') = 0 \), which means that the variety \( M' \) is acyclic.

Vice versa, assuming that \( M' \) is an acyclic manifold, one can prove that so is \( M \) repeating word-in-word the above arguments, but exchanging the roles of the pairs \((M, D)\) and \((M', E')\). This completes the proof. \( \square \)

**Theorem 4.1 (Kaliman [Ka 2, Thm. 3.5]).** Suppose that (i) \( D \) is a topological manifold, and (ii) \( D \) and \( C \) are acyclic. Then \( M' \) is contractible iff \( M \) is.

**Proof.** By the Theorems of Hurewicz and Whitehead, \( M \) resp. \( M' \) is contractible iff it is acyclic and simply connected. Thus, the statement follows immediately from Lemmas 1.1 and 1.2. \( \square \)

**Lemma 4.3 (Kaliman [Ka 2]).** \( \bar{k}(M') \geq \bar{k}(M) \).

**Proof.** Indeed, \( M' = \tilde{M} \setminus D' \) implies \( \bar{k}(M') \geq \bar{k}(\tilde{M}) \) (see Theorem 2.3 (a)). Since \( \sigma_C : \tilde{M} \longrightarrow M \) is a proper birational morphism, by Theorem 2.3(e), we get \( \bar{k}(M') \geq \bar{k}(M) \), as claimed. \( \square \)

**Example 4.1. (Kaliman [Ka 2])** Let \( X = S \times C \) be an exotic \( C^3 \) where \( S \) is a contractible surface of log-general type. Choose a finite sequence of points \( \{ (s_i, z_i) \} \subset X \) (where the complex numbers \( (z_i) \) are pairwise distinct) as the center \( C \) of the Kaliman modification \( \sigma_C : X' \longrightarrow X \) along the union of the fibers \( H_i = S \times \{ z_i \} \) of the second projection \( X \longrightarrow C \), \( i = 1, \ldots , n \). Then \( E'_i \simeq C^2 \), and one can show that \( E'_i \), \( i = 1, \ldots , n \), are the only copies of \( C^2 \) in \( X' \). Thus, by Kaliman’s Theorem 4.1, \( X' \) is an exotic \( C^3 \). It contains precisely \( n \) copies \( E'_i \), \( i = 1, \ldots , n \), of \( C^2 \), and their positions in \( X' \) or, what is the same, the positions of the points \( \{ (s_i, z_i) \} \subset X \) up to automorphisms of \( X \) provide deformation parameters.

\(^5\)More generally, one can show that if \( D \) is a non-compact connected closed subspace of a smooth connected manifold \( M \), then the identity mapping of the complement \( M \setminus D \) extends to a homeomorphism of Hausdorff compact spaces \( \tilde{M}/\tilde{D} \xrightarrow{\sim} (M \setminus D) \).
4.2 Affine modifications

Here we follow the recent paper [KnZa].

**Definition 4.2.** More generally, consider a triple \((A, I, f)\) where \(A\) is an affine domain, i.e. a finitely generated integral domain over \(C\), \(I\) is an ideal of \(A\), and \(f \in I\) is a nonzero element. By the *affine modification* of the domain \(A\) along the principal divisor \((f)\) with center \(I\) we mean the affine domain

\[
A' = \Sigma_{I,f}(A) := A[I]/(1-ft)
\]

where

\[
A[I] := A \oplus \bigoplus_{n=1}^{\infty} (I^n) \simeq A \oplus I \oplus I^2 \oplus \ldots = Bl_I(A)
\]

is the blow up algebra, or the Rees algebra of the ideal \(I\) (see e.g. [Ei, §5.2]).

The affine algebra \(A\) resp. \(A'\) coincides with the algebra of regular functions \(C[X]\) (resp. \(C[X']\)) on its spectrum \(X = \text{spec } A\) resp. \(X' = \text{spec } A'\) which is an irreducible affine variety. The variety \(X' = \Sigma_{I,f}(X)\) is also called the affine modification of the variety \(X\) along the principal divisor \(D_f\) with center \(I\).

**Remark 4.2.** It is easily seen that \(X' = \hat{X} \setminus D_f'\) where \(\hat{X} := Bl_I X\) is the blow up of the variety \(X\) with center \(I\), and \(D_f'\) is the proper transform in \(\hat{X}\) of the divisor \(D_f\), defined in an appropriate way [KnZa, Proposition 1.1.a]. Thus, the Kaliman modification is a particular case of the affine modification. Moreover, we have the following theorem.

**Theorem 4.2 [KnZa, Theorem 1.1].** Any birational morphism \(X' \to X\) of reduced irreducible affine varieties is an affine modification.

**Remark 4.3.** A choice of a system of generators \(a_1, \ldots, a_r\) of the algebra \(A\) defines the proper embedding \(X \hookrightarrow C^r_{(\bar{x}, \bar{y})}, x_i = a_i(x), i = 1, \ldots, r\). If also a system of generators \(b_0 = f, b_1, \ldots, b_s\) of the ideal \(I\) is given, then the formulas

\[
 x_i = a_i, \quad i = 1, \ldots, r, \quad y_j = b_j t, \quad j = 1, \ldots, s,
\]

yield a proper embedding \(X' \hookrightarrow C^{r+s}_{(\bar{x}, \bar{y})}\). The blowup morphism \(\sigma_I : X' \to X\) coincides with the restriction to \(X'\) of the natural projection \(C^{r+s}_{(\bar{x}, \bar{y})} \to C^r_{\bar{x}}\).

Recall such a notion.

**Definition 4.3.** A system of generators \(b_0 = f, b_1, \ldots, b_s\) of the ideal \(I\) is called regular if for any \(j = 0, \ldots, s-1\) the image of the element \(b_{j+1}\) in the quotient algebra \(A/(b_0, \ldots, b_j)\) is not a zero divisor.

**Proposition 4.1 (Davis [Dav]).** Let \(b_0 = f, b_1, \ldots, b_s\) be a regular system of generators of the ideal \(I\). Then the image of the variety \(X'\) under the embedding \(X' \hookrightarrow X \times C^s_{\bar{y}}\) coincides with the subvariety given by the equations \(f(x)y_j = b_j(x), j = 1, \ldots, s\).

In Examples 4.2 - 4.5 below we write down explicit equations of certain affine modifications by making use of Proposition 4.1.

Let \(E \subset \hat{X}\) be the exceptional divisor of the blow up \(\sigma_I : \hat{X} \to X\). The divisor \(E' := E \setminus D_f' = E \cap X'\) on \(X'\) is also called the exceptional divisor of the affine modification \(\sigma_I : X' \to X\). Its image \(\sigma_I(E')\) is contained in the subvariety \(C := V(I) \subset \text{supp } D_f\). Denote \(\tau = \sigma_I | E' : E' \to C \hookrightarrow \text{supp } D_f\).
The next proposition is a generalization of Kaliman’s Lemmas 4.1 and 4.2 above. It provides a control on preservation of the topology under affine modifications.

**Proposition 4.2** (see [KaZa, Prop. 3.1 and Thm. 3.1]). Suppose that the following conditions (i) – (iii) are fulfilled:

(i) the affine varieties \( X \) and \( X' = \Sigma_{I,f}(X) \) are smooth;
(ii) the divisors \( E' \) and \( \text{supp} D_f \) are irreducible and \( E' = \sigma_f^*(\text{supp} D_f) \);
(iii) these divisors \( E' \) and \( \text{supp} D_f \) are topological manifolds.

Then the following statements hold:

(a) the induced homomorphism \( \sigma_1^* : \pi_1(X') \to \pi_1(X) \) is an isomorphism, and
(b) the homomorphism \( \sigma_1^* : H_*(X'; \mathbb{Z}) \to H_*(X; \mathbb{Z}) \) is an isomorphism iff the homomorphism \( \tau_* : H_*(E'; \mathbb{Z}) \to H_*(\text{supp} D_f; \mathbb{Z}) \) is.

**Remark 4.4.** Actually, under the condition (ii) we have that \( \sigma_f(E') \cap \text{reg} D_f \neq \emptyset \). This allows us to apply Remark 4.1 above to prove (a).

The next statement follows from Proposition 4.2 in the same way as Theorem 4.1 follows from Kaliman’s Lemmas 4.1, 4.2.

**Theorem 4.3** [KaZa, Corollary 3.1]. Under the conditions (i) – (iii) of Proposition 4.2 the variety \( X' \) is contractible (resp. acyclic) iff the variety \( X \) is.

We give below several examples of application of this theorem.

**Example 4.2.** An affine modification of the affine space along a divisor with center at a codimension two complete intersection. Let \( A = \mathbb{C}[X] = \mathbb{C}[r] \) be a polynomial algebra, i.e. \( X = \mathbb{C}^r \) is an affine space, and set \( I = (f, g) \) where \( f, g \in A \) are two non-constant relatively prime polynomials. Then \( \{f, g\} \) is a regular system of generators of the ideal \( I \). In virtue of Proposition 4.1, the affine modification \( X' = \Sigma_{I,f}(X) \) is a hypersurface in \( \mathbb{C}^{r+1} \) given by the equation \( f(\mathfrak{r})y - g(\mathfrak{r}) = 0 \) where \( \mathfrak{r} = (x_1, \ldots, x_r) \). The blowup morphism \( \sigma_f : X' \to X \) coincides with the restriction to \( X \) of the projection \( \mathbb{C}^{r+1} \to \mathbb{C}^r \), \( (\mathfrak{r}, y) \to \mathfrak{r} \). The exceptional divisor \( E' \subset X' \) is given in \( \mathbb{C}^{r+1} \) by the equations \( f(\mathfrak{r}) = g(\mathfrak{r}) = 0 \); thus, \( E' \cong C \times C \) where \( C = V(I) \) is the center of the blow up.

**Example 4.3.** The Russell cubic threefold as affine modification. In particular, set \( A = \mathbb{C}[x, z, t] \) (i.e. \( X = \mathbb{C}^3 \)), \( f = -x^2 \) and \( I = (f, g) \) where \( g = x + z^2 + t^3 \). Consider the affine modification \( X' = \Sigma_{I,f}(X) \) along the divisor \( D_f = 2D_x \) with center at the ideal \( I = (-x^2, x + z^2 + t^3) \subset \mathbb{C}[3] \) supported by the affine plane curve \( C = V(I) = \Sigma_{2,3} := \{x = z^2 + t^3 = 0\} \subset \mathbb{C}^2 \). Then \( X' \) is the smooth 3-fold \( x + x^2y + z^2 + t^3 = 0 \in \mathbb{C}^4 \), which has been called in the Introduction the Russell cubic (see also [Ru 1] and Examples 3.1 and 3.3 below). It birationally dominates the affine space \( X = \mathbb{C}^3 \) under the blowup morphism \( \sigma_f : X' \to X = \mathbb{C}^3 \), \( \sigma_f : (x, y, z, t) \mapsto (x, z, t) \). The exceptional divisor \( E' \) coincides with the ‘book-surface’ \( B := \{x = 0\} \subset X' \), \( B \cong C \times \Sigma_{2,3} \). It is easily seen that the conditions (i) – (iii) of Proposition 4.2 are fulfilled. Therefore, by Theorem 4.3, the Russell cubic \( X' \) is contractible. Moreover, by the Dimca–Ramanujan Theorem 5.2, it is diffeomorphic to \( \mathbb{R}^6 \). However, as we will see in §7 below, the Russell cubic is not isomorphic to the affine space \( \mathbb{C}^3 \), and whence it is an exotic \( \mathbb{C}^3 \).
Example 4.4. The tom Dieck–Petrie surfaces as affine modifications. Recall (see Example 2.4 above) that these are smooth surfaces $X_{k,l}$ in $\mathbb{C}^3$ defined by the polynomials
\[
p_{k,l} = \frac{(xz + 1)^k - (yz + 1)^l - z}{z} \in \mathbb{C}[x, y, z]
\]
where $k, l \geq 2$, $(k, l) = 1$. Actually, one can see that $X_{k,l} = \Sigma_{p, \Gamma_{k,l}}(\mathbb{C}^2)$ where $\Gamma_{k,l} := \{x^k - y^l = 0\} \subset \mathbb{C}^2$ and $p = (1, 1) \in \Gamma_{k,l}$ (cf. [KaZa, Example 2.2]). Moreover, the conditions (i) $- (iii)$ of Proposition 4.2 hold. In view of Theorem 4.3, the affine modification $X_{k,l}$ is contractible.

Example 4.5. Let $M = f^*(0)$, $f \in \mathbb{C}^n$, be a smooth reduced irreducible hypersurface in $\mathbb{C}^n$, and let $f_1, \ldots, f_k \in \mathbb{C}^n$ be polynomials without common zeros on $M$. Consider the smooth affine varieties
\[
D := M \times \mathbb{C}^k \subset \mathbb{C}^{n+k}(x, u) = \mathbb{C}_x^n \times \mathbb{C}^k_u \quad \text{and} \quad C := \{f(\bar{x}) = 0 = g(\bar{x}, \bar{u})\} \subset D \subset \mathbb{C}^{n+k}(x, u, v)
\]
where $g(\bar{x}, \bar{u}) := \sum_{i=1}^k u_i f_i \in \mathbb{C}^{n+k}$. It is easily seen that the natural embeddings $M \times \bar{0} \hookrightarrow C \hookrightarrow D$ provide homotopy equivalences. Therefore, by Theorem 4.3, the affine modification
\[
X' := \Sigma_{C, D}(\mathbb{C}^{n+k}) = \{f(\bar{x})v - g(\bar{x}, \bar{u}) = 0\} \subset \mathbb{C}^{n+k+1}(x, u, v)
\]
of the affine space $\mathbb{C}^{n+k}$ along the hypersurface $D$ with center $C$ is a smooth contractible hypersurface in $\mathbb{C}^{n+k+1}$.

Since the hypersurface $M$ is assumed being smooth one may take e.g. $k = n$ and $f_i = \frac{\partial f}{\partial x_i}$, $i = 1, \ldots, n$; then we have $C \simeq TM$ and $D \simeq TC^n | M$.

4.3 The hyperbolic modification

Here we follow, up to minor changes, tom Dieck (cf. another treatment in Petrie [Pe]). We restrict the consideration to the simplest possible case.

Definition 4.4. Let $h \in \mathbb{C}[x_1, \ldots, x_n]$ be an irreducible polynomial such that $h(\bar{0}) = 0$. Suppose that $\text{grad } \nabla h \neq \bar{0}$, and so the hypersurface $X = \{h = 0\} \subset \mathbb{C}^n$ is smooth at the origin. Define the hyperbolic modification $q$ of $h$ as follows:
\[
q(\bar{\tau}, u) = \frac{h(u\bar{\tau})}{u} \in \mathbb{C}[x_1, \ldots, x_n, u].
\]

Since $h(u\bar{\tau}) = uq(\bar{\tau}, u)$, we have the equalities
\[
u \frac{\partial q}{\partial u}(\bar{\tau}, u) + q(\bar{\tau}, u) = \sum_{i=1}^n x_i \frac{\partial h(u\bar{\tau})}{\partial x_i},
\]
\[
\frac{\partial q}{\partial x_i}(\bar{\tau}, u) = \frac{\partial h(u\bar{\tau})}{\partial x_i}, \quad i = 1, \ldots, n.
\]
It follows that, once $(\bar{x}_0, u_0)$ is a critical point of $q$, i.e. $\text{grad } \nabla q = \bar{0}$, then also $\nabla q(u_0, \bar{x}_0) = \bar{0} = h(u_0\bar{x}_0)$, that is, $u_0\bar{x}_0 \in X$ is a singular point of the hypersurface $X$, and $(\bar{x}_0, u_0) \in Y_0 := \{q = 0\} \subset \mathbb{C}^{n+1}$. Thus, all the fibers $Y_c = \{q = c\}$, $c \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, of the polynomial $q$ are smooth hypersurfaces, and the fiber $Y_0 = \{q = 0\}$ is smooth iff so is $X$, which will be assumed in the sequel.
Lemma 4.4. The restriction $q \mid (\mathbb{C}^{n+1} \setminus Y) : \mathbb{C}^{n+1} \setminus Y \longrightarrow \mathbb{C}^*$ is a trivial algebraic fiber bundle with the fiber $Y_1 := \{q = 1\}$.

Proof. Consider the commutative triangle

$$
\begin{array}{ccc}
\mathbb{C}^{n+1} \setminus Y_0 & \xrightarrow{q} & \mathbb{C}^* \\
\Phi \downarrow & & \downarrow \Phi \\
Y_1 \times \mathbb{C}^* & \xrightarrow{pr} & \mathbb{C}^*
\end{array}
$$

where the mapping $\Phi$ is defined as follows:

$$(\overline{y}, \lambda) := ((\overline{x}, u), \lambda) \mapsto (\lambda \overline{x}, \lambda^{-1} u) := \overline{y}_\lambda \in Y_\lambda.$$ 

It is easy to check that $\Phi$ is a fibrewise (biregular) isomorphism, so we are done. □

Define a $\mathbb{C}^*$-action on $\mathbb{C}^{n+1} : (\lambda, (\overline{x}, u)) \mapsto G_\lambda (\lambda \overline{x}, \lambda^{-1} u)$, $\lambda \in \mathbb{C}^*$. Then

$$q(G_\lambda (\overline{x}, u)) = \frac{h(u \overline{x})}{\lambda^{-1} u} = \lambda q(\overline{x}, u).$$

That means that $q$ is a quasi-invariant of weight 1 of the $\mathbb{C}^*$-action $G$. In particular, the hypersurface $Y_0$ is invariant with respect to $G$, and $G_\lambda (Y_e) = Y_e$. In the above diagram to the action $G$ there corresponds the canonical $\mathbb{C}^*$-action on the direct product, whence $\Phi$ is an equivariant morphism.

The monomials $ux_1, \ldots, ux_n \in \mathbb{C}[x_1, \ldots, x_n, u]$ are $G$-invariants. Moreover, they generate the algebra of $G$-invariants $\mathbb{C}[x_1, \ldots, x_n, u]^G = \mathbb{C}[ux_1, \ldots, ux_n]$. Hence, the algebraic quotient of $\mathbb{C}^{n+1}$ by this $\mathbb{C}^*$-action is isomorphic to the affine space $\mathbb{C}^n$:

$$\mathbb{C}^{n+1} // G \simeq \mathbb{C}^n = \text{spec } \mathbb{C}[x_1, \ldots, x_n, u]^G.$$

The $\mathbb{C}^*$-action $G$ on $\mathbb{C}^{n+1}$ is hyperbolic, that is, it has only one fixed point (the origin $\overline{0} \in \mathbb{C}^{n+1}$), and the weights $(1, \ldots, 1, -1)$ of the action $G$ at the origin are of different signs. The origin belongs to the closure of each $G$-orbit which is contained either in the hyperplane $\{u = 0\}$ or in the axis $OU := \{0\} \times \mathbb{C}$; all the other $G$-orbits are closed.

Denote by $M$ the complement of the axis $OU$ in $\mathbb{C}^{n+1}$. Then the $\mathbb{C}^*$-action $G$ restricts to $M$ with closed orbits only. Let $\pi : M \longrightarrow T$ be the canonical morphism onto the orbit space, or the geometric quotient, $T = M/G$. Also, consider the morphism $\tau : M \longrightarrow \mathbb{C}^n$, $(\overline{x}, u) \longmapsto u \overline{x}$. Since $\tau$ is constant on any orbit, it factors as $\tau = \sigma \circ \pi$:

$$
\begin{array}{ccc}
T & \xrightarrow{\sigma} & \mathbb{C}^n \\
\pi \downarrow & & \\
M & \xrightarrow{\tau} & \mathbb{C}^n
\end{array}
$$

The restriction of the morphism $\pi$ to the hypersurface $M \cap \{u = 0\} := \tilde{E} \simeq \mathbb{C}^n \setminus \{\overline{u}\}$ coincides with the standard projection $\mathbb{C}^n \setminus \{0\} \longrightarrow \mathbb{P}^{n-1}$, $\overline{x} \longmapsto \{\lambda \overline{x}\}_{\lambda \in \mathbb{C}^*}$. Set $\pi(\tilde{E}) = E \subset T$. Then

This is why the modification was called hyperbolic.
\[ \sigma(E) = \{ \overline{0} \} , \text{ i.e. } E \text{ is the exceptional divisor of } \sigma; \text{ it is straightforward that } \sigma|_{(T \setminus E)} : T \setminus E \to \mathbb{C}^n \setminus \{0\} \text{ is an isomorphism. It is easily seen that } \sigma : T \to \mathbb{C}^n \text{ is the blow up of the origin.} \]

Furthermore, \( \pi(Y_0 \cap M) := X' \) is the proper transform of \( X \) in \( T \). Indeed, \( Y_0 \) is saturated by the orbits, whence \( \pi(Y_0 \cap M) \) is an irreducible closed hypersurface in \( T \) which contains the proper transform \( \sigma'(X) \). Therefore, \( T' \) is the Kaliman transform of the affine space \( \mathbb{C}^n \) along \( X \) with center at the origin \( \overline{0} \in \mathbb{C}^n \).

**Lemma 4.5.** There is an isomorphism \( Y_1 \simeq T \setminus X' \).

**Proof.** Fix a point \( y = (\overline{x}, u) \in Y_1 \). Since \( q \) is a \( G \)-quasi-invariant of weight \( 1 \), \( G_\lambda(y) \in Y_\lambda \), and hence, the orbit \( G_y \) of \( y \) meets the hypersurface \( Y_1 \) at the point \( y \) only. This means that the morphism \( \pi|_1 : Y_1 \to T \setminus X' \) is injective. On the other hand, any \( G \)-orbit outside the hypersurface \( Y_0 \) meets the \( q \)-fibre \( Y_1 \); thus, this morphism is also surjective. Finally, a bijective morphism of smooth varieties is an isomorphism. \( \square \)

**Corollary 4.1.** The hypersurface \( Y_1 \subset \mathbb{C}^{n+1} \) is isomorphic to the Kaliman modification of the affine space \( \mathbb{C}^n \) along the hypersurface \( X \subset \mathbb{C}^n \) with center at the origin.

**Lemma 4.6.** The hypersurface \( Y_0 \subset \mathbb{C}^{n+1} \) is isomorphic to the Kaliman modification of the product \( X \times \mathbb{C} \) along the hypersurface \( X \times \{0\} \) with center at the point \((\overline{0}, 0) \in X \times \mathbb{C} \).

**Proof.** The morphism
\[
Z' := \mathbb{C}^{n+1} \xrightarrow{\sigma} \mathbb{C}^{n+1} =: Z, \quad (y_1, \ldots, y_n, u) \mapsto (uy_1, \ldots, uy_n, u),
\]
is nothing but the affine modification of the variety \( Z \) along the hyperplane \( H_0 := \{ u = 0 \} \) with center at the origin, and with the exceptional divisor \( E' = \{ u = 0 \} \subset Z' \). Consider the natural embedding \( i : X \times \mathbb{C} \to Z \). Set \( \hat{h}(\overline{x}, u) = h(\overline{x}) \); then the image of \( i \) is the hypersurface \( \hat{h} = 0 \) in \( Z \simeq \mathbb{C}^{n+1} \).

We have
\[
\hat{h} \circ \sigma(\overline{y}, u) = h(u\overline{y}) = uq(\overline{y}, u).
\]
Hence, \( \hat{h} \circ \sigma(\overline{y}, u) = 0 \) for any point \( (\overline{y}, u) \in Y_0 \) (i.e. such that \( q(\overline{y}, u) = 0 \)). Thus, \( \sigma(\overline{y}, u) \in X \times \mathbb{C} \), and so \( \sigma(Y_0) \subset X \times \mathbb{C} \). Furthermore, the total preimage of the product \( X \times \mathbb{C} \) in \( Z' \) is the union of the hypersurface \( Y_0 \) and of the exceptional divisor \( E' = \{ u = 0 \} \). Therefore, \( Y_0 \) is the proper transform of the variety \( X \times \mathbb{C} \) in \( Z' \), and the assertion follows. \( \square \)

**Remark 4.5.** The \( \mathbb{C}^* \)-action \( \lambda(y_1, \ldots, y_n, u) = (\lambda y_1, \ldots, \lambda y_n, \lambda^{-1} u) \) on \( Z' \) provides the \( \mathbb{C}^* \)-action \( \lambda(x_1, \ldots, x_n, u) = (x_1, \ldots, x_n, \lambda^{-1} u) \) on the affine space \( Z \) and on the product \( X \times \mathbb{C} \).

**Exercise (4.1.)** Show that under the embedding \( \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1} \) given as \( \overline{x} :\to (\overline{x}, 1) \) the variety \( X \) is naturally isomorphic to the hyperplane section \( Y_0 \cap H_1 \) where \( H_1 := \{ u = 1 \} \subset \mathbb{C}^{n+1} \). Furthermore, show that the exceptional divisor \( E' \subset Y_0 \) of the Kaliman transform \( \sigma : Y_0 \to X \times \mathbb{C} \) coincides with the linear subspace \( Y_0 \cap \{ u = 0 \} \) in \( \mathbb{C}^{n+1} \). Let \( \sigma' : Y_0 \to X \) be composed of the contraction \( \sigma \) and the first projection. Verify that
\[
\sigma' : (\overline{x}, u) \mapsto (\overline{x}, u, 1) \in Y_0 \cap H_1 \simeq X
\]
outside the exceptional divisor \( E' \), and \( \sigma'(\overline{0}, 0) = (0, 1) \) on \( E' \). Deduce that the hypersurface \( Y_0 \) is the closure in \( \mathbb{C}^{n+1} \) of the \( \mathbb{C}^* \)-orbit of the subvariety \( Y_0 \cap H_1 \simeq X \).
Theorem 4.4 (tom Dieck [D 1]). Let \( X \subset \mathbb{C}^n \) be a smooth contractible hypersurface given by an irreducible polynomial \( h \in \mathbb{C}[x_1, \ldots, x_n] \) where \( h(0) = 0 \). Then any fiber \( Y_c = q^{-1}(c) \), \( c \in \mathbb{C} \), of the hyperbolic modification \( q(\overline{x}, u) = \frac{h(\overline{x})}{u} \in \mathbb{C}[x_1, \ldots, x_n, u] \) of the polynomial \( h \) is a smooth contractible hypersurface in \( \mathbb{C}^{n+1} \). Thus, \( q : \mathbb{C}^{n+1} \rightarrow \mathbb{C} \) yields a foliation of \( \mathbb{C}^{n+1} \) by smooth contractible hypersurfaces.

Proof. Indeed, by Lemma 4.1, \( Y_c \simeq Y_1 \) for any \( c \neq 0 \). By Corollary 4.1 and Lemma 4.6, the hypersurfaces \( Y_0 \) and \( Y_1 \) are both Kaliman modifications of triples of smooth contractible varieties. By Kaliman’s Theorem 4.1, these hypersurfaces \( Y_0 \) and \( Y_1 \) are contractible.

Remarks. 4.6. The inequality \( \overline{\kappa}(M') \geq \overline{\kappa}(M) \) of Lemma 4.3 does not help to answer to the question whether the contractible affine varieties \( Y_0 \) and \( Y_1 \) are exotic \( \mathbb{C}^n \)-s. Indeed, we have \( \overline{\kappa}(\mathbb{C}^n) = \overline{\kappa}(\mathbb{C} \times \mathbb{C}) = -\infty \) (see Theorem 2.3 (a)). However, in certain cases the intermediate Eisenman-Kobayashi intrinsic measures serve as appropriate analytic invariants (see Kaliman [Ka 2]).

4.7. The Kaliman Theorem 4.1 is still applied if \( X \) is only assumed being a contractible topological manifold smooth at the origin. In that case we still have that all the hypersurfaces \( Y_c, c \neq 0 \), are smooth and contractible, but the central fiber \( Y_0 \) can be singular, as it is in the following example.

Example 4.6. The foliations of the affine spaces with contractible leaves arising from the tom Dieck-Petrie surfaces [DP 3] (see Examples 2.3, 4.4 above). Recall (see [LiZa] and Theorem 2.7 above) that up to automorphisms of \( \mathbb{C}^2 \) the only irreducible simply connected singular affine plane curves are the curves \( \Gamma_{k,l} := \{x^k - y^l = 0\} \subset \mathbb{C}^2 \), \( (k, l) = 1 \), \( k > l \geq 2 \). Starting with \( \Gamma_{k,l} \) perform the hyperbolic modification at the smooth point \( (1, 0) \in \Gamma_{k,l} \). We obtain a foliation \( p_{k,l} : \mathbb{C}^3 \rightarrow \mathbb{C} \) of \( \mathbb{C}^3 \) by the fibers of the polynomial

\[
p_{k,l} := \frac{(xz + 1)^k - (yz + 1)^l}{z} \in \mathbb{C}[x, y, z].
\]

All of them are irreducible contractible surfaces; all but the central one \( p_{k,l}^{-1}(0) \) are smooth. One can see that \( \overline{\kappa}(X_{k,l}) = 1 \) where \( X_{k,l} := p_{k,l}^{-1}(1) \) (see Exercise 4.6 below). Next, starting with \( X_{k,l} \), by means of hyperbolic modifications one can construct non-trivial foliations of \( \mathbb{C}^4 \), \( \mathbb{C}^5 \), etc. by smooth contractible hypersurfaces. Moreover, the corresponding polynomials are quasi-invariants of hyperbolic \( \mathbb{C}^* \)-actions on \( \mathbb{C}^n \). In particular, for \( n \geq 4 \) the zero fiber of such a polynomial is a smooth contractible hypersurface in \( \mathbb{C}^n \) endowed with a hyperbolic \( \mathbb{C}^* \)-action. Furthermore, new contractible affine hypersurfaces can be obtained by passing to cyclic \( \mathbb{C}^* \)-coverings over these ones (see the next section).

Exercise (4.2.) Verify that \( \overline{\kappa}(X_{k,l}) = 1 \) when \( (k, l) = 1 \), \( k > l \geq 2 \).

Hint. One can proceed, for instance, as follows. Lifting the meromorphic function \( x^k/y^l \) on \( \mathbb{C}^2 \) to the function \( f := (xz + 1)^k/(yz + 1)^l \) on \( X_{k,l} \) on \( X_{k,l} \) (see Exercise 4.4) we obtain a \( \mathbb{C}^* \)-fibration \( f : X_{k,l} \rightarrow \mathbb{P}^1 \). Hence, by Iitaka’s Easy Addition Theorem 2.3 (c), \( \overline{\kappa}(X_{k,l}) \leq 1 \). Since \( X_{k,l} \) is acyclic, by the Classification Theorem 3 (b), \( \overline{\kappa}(X_{k,l}) = 1 \) as soon as \( X_{k,l} \not\subset \mathbb{C}^2 \). Recall that the surface \( X_{k,l} \) is the Kaliman modification of \( \mathbb{C}^2 \) along the curve \( \Gamma_{k,l} \subset \mathbb{C}^2 \) with center at the point \( (1, 1) \in \Gamma_{k,l} \). Resolving singularities of the plane projective curve \( \Gamma_{k,l} \cup \mathbb{C} \subset \mathbb{P}^2 \) and blowing up at the point \( (1, 1) \in \Gamma_{k,l} \), we obtain an SNC-completion \( (V_{k,l}, D_{k,l}) \) of \( X_{k,l} \). Contracting, if necessary, the \((-1)\)-components of the boundary divisor \( D_{k,l} \) of valence
at most two of the dual graph $\Gamma_{D_{k,l}}$ we come to a minimal completion $(V_{\min}^{m}, D_{\min}^{k,l})$ of the open surface $X_{k,l}$. The dual graph of its boundary divisor $D_{\min}^{k,l}$ is non-linear (what is this graph?). Therefore, by the Ramanujam Theorem 2.2, $X_{k,l} \neq C^2$.

5 Cyclic $C^*$-coverings

**Definition 5.1** (cf. [KoRu 2, Prop. 2.11]). Let $X$ be an affine variety, and let $q \in C[X]$ be a regular function on $X$. For an integer $s > 1$ set $Y_s = \{(x, u) \in X \times C \mid q(x) = u^s\}$. The projection $\varphi_s : Y_s \rightarrow X$, $(x, u)^{\varphi_s} \rightarrow x$, yields a cyclic covering of $X$ branched to order $s$ along the principal divisor $F_0 = q^s(0)$. We suppose that $X$ is a smooth affine variety and $F_0$ is a smooth reduced divisor on $X$; then the variety $Y_s$ is also smooth (indeed, $\text{grad} (x,u)(q(x) - u^s) = (\text{grad}_x q, -su^{s-1})$), as well as the hypersurface $F_{s,0} := \varphi_{s}^{-1}(F_0)$ in $Y_s$.

Let $X$ be endowed with a regular $C^*$-action $t : C^* \times X \rightarrow X$. Suppose that the regular function $q$ is a quasi-invariant of $t$ of weight $d$, i.e.

$$q(t, x, u) = \lambda^d q(x)$$

where $d \in \mathbb{Z}$. Then the $C^*$-action $\lambda(x, u) := (\lambda^s(x), \lambda^d u)$ on $X \times C$ restricts to $Y_s$ making the following commutative diagram equivariant

\[
\begin{array}{ccc}
Y_s & \xrightarrow{(x, u) \mapsto x} & X \\
pr_2 \downarrow & & \downarrow q \\
C & \xrightarrow{u \mapsto u^s} & C
\end{array}
\]

where the original $C^*$-action $G$ on $X$ is replaced by its `$s$-th power’ $(\lambda, x) \mapsto \lambda^s(x) := t(\lambda^s, x)$. Indeed, for $(x, u) \in Y_s$ we have:

$$q(\lambda^s(x)) = \lambda^sdq(x) = \lambda^sd\lambda^s = (\lambda^d u)^s,$$

whence $(\lambda^s(x), \lambda^d u) \in Y_s$, which shows that the above diagram is equivariant.

If, in addition, $(d, s) = 1$, then the monodromy of the cyclic covering $\varphi_s : Y_s \rightarrow X$ is represented via the action on $Y_s$ of the subgroup $\omega_s \subset C^*$ of the $s$-th roots of unity. Indeed, since $(s, d) = 1$ the $\omega_s$-orbit of a point $(x, u)$ in $Y_s$ is

$$\omega_s (x, u) = \{(x, \lambda^d u) \mid \lambda^s = 1\} = \varphi^{-1}_s(x).$$

The fixed point set $Y_s^{\omega_s} = \{(x, u) \in Y_s \mid u = 0\} = F_{s,0} \subset Y_s$ of the monodromy action on the variety $Y_s$ can be identified with the hypersurface $F_0 \subset X$. Thus, we get $X = Y_s/\omega_s$ with the quotient action of $C^*/\omega_s \simeq C^*$ on $X$.

The equivariant covering $Y_s \rightarrow X$ as above is called a cyclic $C^*$-covering.

**Remarks. 5.1.** The action of the monodromy group $\omega_s \simeq \mathbb{Z}/s\mathbb{Z}$ on $Y_s$ is homologically trivial. Indeed, this is so for the action on $Y_s$ of the connected group $C^* \supset \omega_s$.

5.2. The above observations are equally applied in the more general setting when the regular $C^*$-action is only given on the Zariski open subset $X^* := X \setminus F_0$ of $X$. In particular, if $(d, s) = 1$, then the monodromy group $\omega_s$ of the cyclic covering $\varphi_s : Y_s^* \rightarrow X^*$ where $Y_s^* := Y_s \setminus \varphi_s^{-1}(F_0)$, acts trivially in the homology $H_s(Y_s^*; \mathbb{Z})$. 

25
The following result provides a generalization of Theorem A in \cite{Ka1}.

**Theorem 5.1 (Kaliman).** Let $X$ be a smooth contractible affine variety, and let the principal divisor $F_0 = q^*(0) \subset X$ where $q \in \mathbb{C}[X]$ be smooth, reduced and irreducible. Denote $G = \pi_1(X \setminus F_0)$, and fix a vanishing loop $\alpha = \alpha_{F_0} \in G$ of the divisor $F_0$. Suppose that the following conditions are fulfilled.

1. The regular function $q$ is a quasi-invariant of weight $d \neq 0$ of a regular $\mathbb{C}^*$-action defined on $X \setminus F_0$.
2. For some integer $c \neq 0$, $\alpha^c$ is an element of the center $Z(G)$ of the group $G$.
3. For an integer $s > 0$ such that $(s,c) = (s,d) = 1$, the hypersurface $F_0$ is $\mathbb{Z}_p$-acyclic for each prime divisor $p$ of $s$.

Consider the cyclic covering $\varphi_s : Y_s \to X$ branched to order $s$ along $F_0$. Then $Y_s$ is a smooth contractible affine variety.

Due to the Theorems of Hurewicz and Whitehead, it is enough to show that $Y_s$ is acyclic and simply connected. This is done, respectively, in Theorems 5.2 and 5.3 below. Notice that the conditions 1 and 2 guarantee acyclicity of $Y_s$ whereas the condition 1 provides its simply connectedness.

### 5.1 Acyclicity of cyclic $\mathbb{C}^*$-coverings

**Theorem 5.2 (Kaliman \cite{Ka1}; tom Dieck \cite{D2}).** Let $X$ be an acyclic smooth affine variety, and $F_0 = q^*(0)$ where $q \in \mathbb{C}[X]$ be a smooth reduced irreducible principal divisor on $X$. Consider a cyclic covering $\varphi_s : Y_s \to X$ branched to order $s$ along $F_0$ where $(s,d) = 1$. Suppose that the following conditions hold.

1. The regular function $q$ is a quasi-invariant of weight $d \neq 0$ of a regular $\mathbb{C}^*$-action defined on $X \setminus F_0$.
2. The hypersurface $F_0$ is $\mathbb{Z}_p$-acyclic for each prime divisor $p$ of $s$.

Then $Y_s$ is acyclic, too.

Before proving Theorem 5.2 we recall the Smith theory (see \cite[Ch.III]{Br2}).

**Elements of Smith’s Theory.** Consider a finite simplicial polyhedron $Y$ endowed with a simplicial action of a finite group $\omega$. Usually, passing to the second barycentric subdivision one obtains certain additional regularity properties of the action, which are always to be assumed (see \cite[III.1]{Br2}). Let $k$ be a field, and let $\mathbb{Z}[\omega], \mathbb{Z}_p[\omega], k[\omega]$ be the group rings of $\omega$ (e.g. $\mathbb{Z}[\omega] = \{\sum_{g \in \omega} n_g g \mid n_g \in \mathbb{Z}\}$ with natural ring operations). The simplicial chain complexes $C(Y), C(Y) \otimes \mathbb{Z}_p, C(Y) \otimes k$ are, respectively, $\mathbb{Z}[\omega]$-, $\mathbb{Z}_p[\omega]$-, $k[\omega]$-modules (indeed, given a simplex $\delta$ of $Y$ we have

\[
(\sum_{g \in \omega} n_g g)(\delta) = \sum_{g \in \omega} n_g g(\delta) \in C(Y).
\]

In the sequel $\omega$ is assumed to be a finite cyclic group $\mathbb{Z}_s = \mathbb{Z}/s\mathbb{Z}$ acting on $Y$ in such a way that the fixed point set $Y^\omega$ of the $\omega$-action on $Y$ coincides with the individual fixed point set $Y^g$ for every $g \in \omega$, $g \neq e$. In particular, the $\omega$-action on the complement $Y \setminus Y^\omega$ is free. We denote by $X = Y/\omega$ the orbit space, by $\pi : Y \to X$ the natural projection, and

\footnote{Exposing this result in \cite[Thm. 6.9]{Za3}, the condition (2) below has been missed. In the proof of Theorem 5.3 below it replaces Lemma 6.8 in \cite{Za3} which is wrong.}

\footnote{hereafter $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$.}
we identify $Y^\omega$ with its image in $X$. Consider the following three homomorphisms of chain complexes:

$$
\pi_* : C(Y) \longrightarrow C(X),
$$

$$
\sigma : C(Y) \longrightarrow C(Y), \quad \sigma = \sum_{g \in \omega} g \in \mathbb{Z}[\omega],
$$

$$
\mu = \pi^* : C(X) \longrightarrow C(Y), \quad \mu(\delta) = \pi^{-1}(\delta) \quad \text{if} \quad \delta \cap Y^\omega = \emptyset; \quad \mu(\delta) = \sigma(c) \quad \text{if} \quad \pi^*(c) = \delta.
$$

Note that the induced homomorphism $\pi^*$ is surjective. Then we have \[\text{Ker } \pi^* = \text{Ker } \sigma,\] and there is an isomorphism

$$
\sigma C(Y) \simeq C(Y)/\text{Ker } \sigma \simeq C(X)/\text{Ker } \pi^* = C(X),
$$

whence $\mu \pi^* = \sigma$. But $\pi^* \mu(c) = |\omega| \pi^*(c)$. On the homology level, this leads to the following assertions.

**Lemma 5.1.** [Bre, III(2.2), (2.3)]

$$
\pi_* \mu_* = |\omega| : H_*(X) \longrightarrow H_*(X),
$$

$$
\mu_* \pi_* = \sigma_* = \sum_{g \in \omega} g_* : H_*(Y) \longrightarrow H_*(Y).
$$

Here $\mu_*$ is called a transfer. On the invariant part of homology we have

$$
\mu_* \pi_* | H_*(Y)^\omega = |\omega| : H_*(Y)^\omega \longrightarrow H_*(Y)^\omega.
$$

This implies such a corollary.

**Corollary 5.1.** [Bre, III(2.4)]. If $k$ is a field of characteristic $\text{char } k = q$ with $(q, |\omega|) = 1$, then the restriction

$$
\pi_*|H_*(Y;k)^\omega : H_*(Y;k)^\omega \longrightarrow H_*(X;k)
$$

is an isomorphism, and its inverse is the transfer $\mu_*$. Moreover,

$$
H_*(Y;k) = \mu_* H_*(X;k) \oplus \text{Ker } \pi_*,
$$

where $\text{Ker } \pi_* = \text{Ker } \sigma_*$. 

**Corollary 5.2.** Suppose that $\omega$ acts trivially in the homology, i.e.

$$
\omega_* | H_*(Y) = \text{id}.
$$

Then for any field $k$ with $(\text{char } k, |\omega|) = 1$ we have the isomorphism of transfer

$$
\pi_* = \pi_*^{-1} : H_*(Y;k) \xrightarrow{\sim} H_*(X;k).
$$

In particular, if $\omega \simeq \mathbb{Z}_q$ where $q$ is a prime number, then the elements of the kernel and of the cokernel of the homomorphism $\pi_* : H_*(Y;\mathbb{Z}) \longrightarrow H_*(X;\mathbb{Z})$ are torsions of order $q$.

The last assertion follows by the Universal Coefficient Formula:

$$
\tilde{H}_j(Y;\mathbb{Z}_q) = \tilde{H}_j(Y;\mathbb{Z}) \otimes \mathbb{Z}_q \oplus \text{Tor } (H_{j-1}(Y;\mathbb{Z});\mathbb{Z}_q) \quad \forall j.
$$
Smith’s exact homology sequences and the following two

Proposition 5.1 [Bre, III(3.3),(3.4),(3.8)]. For the homology groups with $\mathbb{Z}_p$ coefficients, one has

(a) an isomorphism $H^p_*(Y) \simeq H_*(\hat{X}; Y^\omega)$,

and the following two Smith’s exact homology sequences:

(b)

$$
\begin{align*}
0 & \longrightarrow \sigma C(Y) \oplus C(Y^\omega) \xrightarrow{i} C(Y) \xrightarrow{\rho} \rho C(Y) \longrightarrow 0, \\
0 & \longrightarrow \sigma C(Y) \xrightarrow{i} \tau^j C(Y) \xrightarrow{\tau} \tau^{j+1} C(Y) \longrightarrow 0, \quad j = 1, \ldots, p - 1.
\end{align*}
$$

Besides, the kernel of the homomorphism $\sigma : C(Y; \mathbb{Z}_p) \to C(Y; \mathbb{Z}_p)$ and those of the composition $C(Y; \mathbb{Z}_p) \to C(Y, Y^\omega; \mathbb{Z}_p) \to C(X, Y^\omega; \mathbb{Z}_p)$ where $Y^\omega$ is identified with its image in $X$, are the same [Bre, p. 124]. These observations lead to the following

(c)

Proposition 5.2. Suppose that

(ii) the fixed point set $Y^\omega$ is non-empty and $\mathbb{Z}_p$ -acyclic: $\tilde{H}_*(Y^\omega; \mathbb{Z}_p) = 0$, and

(iii) the quotient $X = Y/\omega$ is $\mathbb{Z}_p$ -acyclic: $\tilde{H}_*(X; \mathbb{Z}_p) = 0$.

Then also $Y$ is $\mathbb{Z}_p$ -acyclic: $\tilde{H}_*(Y; \mathbb{Z}_p) = 0$.

Proof. In view of the vanishing

$$\tilde{H}_*(X; \mathbb{Z}_p) = \tilde{H}_*(Y^\omega; \mathbb{Z}_p) = 0,$$

9 The numeration of the conditions that we use here agrees with those in the next Corollary 5.3 and Exercise 5.1.
from the exact homology sequence of a pair

\[ \cdots \xrightarrow{i} H_j(X; \mathbb{Z}_p) \xrightarrow{r} H_j(X, Y^\omega; \mathbb{Z}_p) \xrightarrow{\delta} H_{j-1}(Y^\omega; \mathbb{Z}_p) \xrightarrow{i} \cdots \]

it follows that \( H_s(X, Y^\omega; \mathbb{Z}_p) = 0 \), and thus, by Proposition \ref{prop:isomorphisms}(a), also \( H^q(Y; \mathbb{Z}_p) = 0 \). Therefore, by Smith’s exact sequence (c), \( H^q(Y; \mathbb{Z}_p) = 0 \) \( \forall \rho = \rho_j, j = 1, \ldots, p - 1 \). Now, by Smith’s exact sequence (b), \( H_s(Y; \mathbb{Z}_p) \simeq H^s_\ast(Y; \mathbb{Z}_p) = 0 \). □

**Corollary 5.3.** Suppose that the following conditions hold:

(i) \( \omega \simeq \mathbb{Z}_p \) acts trivially in homology: \( \omega \vert H_s(Y) = \text{id} \),

(ii) the fixed point set \( Y^\omega \) is non-empty and \( \mathbb{Z}_p \)-acyclic: \( \tilde{H}_s(Y^\omega; \mathbb{Z}_p) = 0 \), and

(iii) the quotient \( X = Y/\omega \) is acyclic: \( \tilde{H}_s(X; \mathbb{Z}) = 0 \).

Then also \( Y \) is acyclic: \( \tilde{H}_s(Y; \mathbb{Z}) = 0 \).

**Proof.** By Corollary 5.2, \( H_s(Y; \mathbb{Z}_q) \simeq H_s(X; \mathbb{Z}_q) = 0 \) for any prime \( q \neq p \). By Proposition 5.2, also \( H_s(Y; \mathbb{Z}_p) = 0 \). Thus, by the Universal Coefficient Formula, \( \tilde{H}_s(Y; \mathbb{Z}) \otimes \mathbb{Z}_q = 0 \) for all prime \( q \). Then \( H_s(Y; \mathbb{Z}) = 0 \).

**Exercise (5.1.)** Assume that \( \omega \simeq \mathbb{Z}_s \) acts on \( Y \) in such a way that

(0) \( Y^g = Y^\omega \neq \emptyset \) for every \( g \in \omega, \ g \neq e \);

(i) the action is homologically trivial, i.e. \( \omega \vert H_s(Y; \mathbb{Z}) = \text{id} \);

(ii) the fixed point set \( Y^\omega \) is \( \mathbb{Z}_p \)-acyclic for any prime divisor \( p \) of \( s \);

(iii) the quotient \( X = Y/\omega \) is acyclic: \( \tilde{H}_s(X; \mathbb{Z}) = 0 \).

Show that \( Y \) is acyclic, too: \( \tilde{H}_s(Y; \mathbb{Z}) = 0 \).

**Remark 5.3.** Assume that the \( C^\ast \) - action in Theorem 5.2 is regular on the whole variety \( X \). Then the monodromy group action on the covering variety \( Y_s \) is homologically trivial (see Remark 5.1 above), that is, the above condition (i) is fulfilled. This provides a proof of Theorem 5.2 in that particular case. Notice that the assumptions of smoothness of \( X \) and \( F_0 \) are not used in this proof.

In general case, following tom Dieck \cite{tD}, we need to consider branched coverings over smooth varieties and to make use of the Thom classes. We recall below their definition and some properties (see e.g. \cite[VIII.11]{Do}, \cite[§9, §10]{MilSta}).

**Thom’s classes and Thom’s isomorphisms.** Consider an oriented connected smooth real manifold \( X \) and a codimension 2 closed oriented submanifold \( F_0 \) of \( X \). Let \( N \to F_0 \) be the (oriented) normal bundle of \( F_0 \) in \( X \) with the zero section \( Z_0 \simeq F_0 \). Fix a tubular neighborhood \( U \subset X \) be of the submanifold \( F_0 \) in \( X \) such that the pair \( (U, F_0) \) is diffeomorphic to the pair \( (N, Z_0) \). Denote \( U^\ast := U \setminus F_0 \) and \( N^\ast := N \setminus Z_0 \). By excision, we have the isomorphisms \( \tilde{H}_s(X, X^\ast; \mathbb{Z}) \simeq \tilde{H}_s(U, U^\ast; \mathbb{Z}) \simeq \tilde{H}_s(N, N^\ast; \mathbb{Z}) \), and similarly for the cohomology groups. The *Thom class* \( t(F_0) \in H^2(X, X^\ast; \mathbb{Z}) \simeq H^2(N, N^\ast; \mathbb{Z}) \) is a unique cohomology class which takes the value 1 on any oriented relative two-cycle \( (F, F^\ast) \in H_2(N, N^\ast; \mathbb{Z}) \) defined by a fiber \( F \) of the normal bundle \( N \).

The cap-product with the Thom class \( t(F_0) \in H^2(X, X^\ast; \mathbb{Z}_q) \) yields the *Thom isomorphism*

\[ \tilde{H}_i(X, X^\ast; \mathbb{Z}_q) \simeq H_{i-2}(F_0; \mathbb{Z}_q), \quad i = 0, 1, \ldots. \]

\[^{10}\text{the homology groups with negative indices are considered being zero.}\]
Let \( \varphi_s : Y_s \to X \) be a smooth cyclic ramified covering of \( X \) branched to order \( s \) along \( F_0 \), i.e. \( Y_s \) is an oriented manifold equipped with an action of a group \( \omega \simeq \mathbb{Z}_s \) of orientation preserving diffeomorphisms; the fixed point set \( Y_s^\omega \subset Y_s \) is a codimension 2 closed oriented submanifold; \( \omega \) acts freely in the complement \( Y_s \setminus Y_s^\omega \), and \( \varphi_s : Y_s \to X \) is the orbit map which provides a natural identification of \( Y_s^\omega \) with the submanifold \( F_0 \subset X \).

Note that under the assumption (\#) of Theorem 5.2 the monodromy group \( \omega \simeq \mathbb{Z}_s \) acts trivially in the homology \( H_*(Y_s \setminus F_0; \mathbb{Z}) \). Thus, the next proposition yields Theorem 5.2 (cf. Corollary 5.3).

**Proposition 5.3** (see [1D2, Thm. 2.9]). Let in the notation as above \( \varphi_s : Y_s \to X \) be a smooth cyclic ramified covering of a smooth manifold \( X \) branched to order \( s \) along a codimension 2 submanifold \( F_0 \subset X \). Suppose that

(i) the covering group \( \omega \simeq \mathbb{Z}_s \) acts trivially in the homology of the complement \( Y_s \setminus Y_s^\omega : \omega_*|H_*(Y_s \setminus Y_s^\omega; \mathbb{Z}) = \text{id} \);

(ii) the fixed point set \( Y_s^\omega \) is \( \mathbb{Z}_p \)-acyclic for any prime divisor \( p \) of \( s \);

(iii) the quotient \( X = Y_s/\omega \) is acyclic: \( \tilde{H}_*(X; \mathbb{Z}) = 0 \).

Then the manifold \( Y_s \) is acyclic, too: \( \tilde{H}_*(Y_s; \mathbb{Z}) = 0 \).

**Proof.** Assume, for simplicity, that \( s = p \) is a prime number; like in Exercise 5.1 above the general case can be reduced to this one. By Proposition 5.2, we have \( \tilde{H}_*(Y_p; \mathbb{Z}_p) = 0 \). By the Universal Coefficient Formula, it suffices to prove that \( \tilde{H}_*(Y_p; \mathbb{Z}_q) = 0 \) (i.e. \( Y_p \) is \( \mathbb{Z}_q \)-acyclic) for any prime \( q \neq p \).

Denote \( Y_p^* = Y_p \setminus Y_p^\omega \) and \( X^* = X \setminus F_0 \). The restriction \( \pi : Y_p^* \to X^* \) is a non-ramified cyclic covering of order \( p \). By Corollary 5.1, \( (\varphi_s)_* : H_*(X^*; \mathbb{Z}_q) \to H_*(X; \mathbb{Z}_q) \) is an isomorphism for any prime \( q \neq p \).

We have the Thom isomorphisms

\[
\tilde{H}_i(X, X^*; \mathbb{Z}_q) \simeq H_{i-2}(F_0; \mathbb{Z}_q), \quad \tilde{H}_i(Y_p, Y_p^*; \mathbb{Z}_q) \simeq H_{i-2}(Y_p^\omega; \mathbb{Z}_q),
\]

given by cap-products with the Thom classes \( t(F_0) \in H^2(X, X^*; \mathbb{Z}_q) \) resp. \( t(Y_p^\omega) \in H^2(Y_p, Y_p^*; \mathbb{Z}_q) \). It is easily seen that \( (\varphi_s)^*(t(F_0)) = p \cdot t(Y_p^\omega) \). Since the multiplication by \( p \) is an invertible operation in \( \mathbb{Z}_q \)- (co)homology for \( q \neq p \), it follows that \( (\varphi_s)_* : H_*(Y_p, Y_p^*; \mathbb{Z}_q) \to H_*(X, X^*; \mathbb{Z}_q) \) is an isomorphism.

Consider the following commutative diagram where the horizontal lines are exact homology sequences of pairs with \( \mathbb{Z}_q \)-coefficients:

\[
\cdots \to H_{j+1}(Y_p, Y_p^*) \to H_j(Y_p^*) \to H_j(Y_p) \to H_j(Y_p, Y_p^*) \to H_{j-1}(Y_p^*) \to \cdots \]

\[
\begin{array}{cccc}
\simeq & \simeq & \simeq & \simeq \\
\end{array}
\]

\[
\cdots \to H_{i+1}(X, X^*) \to H_i(X^*) \to H_j(X) \to H_j(X, X^*) \to H_{j-1}(X^*) \to \cdots
\]

By the above observations, we may conclude that the four indicated vertical arrows are isomorphisms induced by the projection \( \varphi_s \). By the 5-lemma, the middle vertical arrow is an isomorphism, too. Hence, since \( X \) is acyclic, \( \tilde{H}_*(Y_p; \mathbb{Z}_q) \simeq \tilde{H}_*(X; \mathbb{Z}_q) = 0 \) for any prime \( q \). This yields the assertion. \( \square \)

Thus, the proof of Theorem 5.2 is completed.
Example 5.1. The acyclic surfaces $Y_{k,l,s}$ in $\mathbb{C}^3$. Let $X$ be a smooth acyclic surface, $F_0 = q^*(0)$ be a smooth reduced irreducible simply connected curve in $X$ where $q \in \mathbb{C}[X]$ is a quasi-invariant of weight $d \neq 0$ of a regular $\mathbb{C}^*$--action on $X \setminus F_0$. Then by Theorem 5.2, $Y_s := \{ z^s = q(x) \} \subset X \times \mathbb{C}$, where $(d, s) = 1$, is a smooth acyclic surface, too.

For instance, for $k, l, s$ pairwise relatively prime the surface $Y_{k,l,s} \subset \mathbb{C}^3$ given by the equation

$$\frac{(xz^s + 1)^k - (yz^s + 1)^l}{z^s} = 1$$

is a smooth acyclic one, and $\overline{k}(Y_{k,l,s}) = 1$. Indeed, there is a cyclic $\mathbb{C}^*$--covering $Y_{k,l,s} \to X_{k,l}$ over the tom Dieck-Petrie surface $X_{k,l}$ (see Examples 4.4, 4.6 and Exercise 4.6 above) branched to order $s$ along the curve $L_{k,l} := X_{k,l} \cap \{ z = 0 \}$ in $X_{k,l}$. The $\mathbb{C}^*$--action in $X_{k,l} \setminus L_{k,l}$ is induced via the isomorphism $X_{k,l} \setminus L_{k,l} \simeq \mathbb{C}^2 \setminus \Gamma_{k,l}$ by the linear $\mathbb{C}^*$--action $(\lambda, (x, y)) \mapsto (\lambda^k x, \lambda^s y)$ on $\mathbb{C}^2$. Thus, we may apply Theorem 5.2 to show that the surface $Y_{k,l,s}$ is acyclic.

5.2 Simply connectedness of cyclic $\mathbb{C}^*$--coverings

Theorem 5.3 (Kaliman). Let $X$ be a simply connected smooth affine variety, and let $F_0 = q^*(0)$, where $q \in \mathbb{C}[X]$, be a smooth reduced irreducible principal divisor in $X$. Fix a vanishing loop $\alpha = \alpha_{F_0} \in G := \pi_1(X \setminus F_0)$ of the divisor $F_0$. Consider a cyclic covering $\varphi_s : Y_s \to X$ branched to order $s$ along $F_0$. Assume that

$(\#_1)$ For an integer $c \neq 0$ such that $(s, c) = 1$, $\alpha^c$ is an element of the center $Z(G)$ of the group $G$.

Then $Y_s$ is simply connected, too.

Remark 5.4. In [Kaliman], Lemmas 7 and 8] conditions on a polynomial $q \in \mathbb{C}[x_1, \ldots, x_n]$ are given which ensure that $\pi_1(\mathbb{C}^n \setminus F_0) \simeq \mathbb{Z}$. In particular, repeating word-in-word the proof of Lemma 8 in [Kaliman] (based on the Seifert-van Kampen Theorem) one can easily see that $\pi_1(X \setminus F_0) \simeq \mathbb{Z}$ if $F_0$ is a generic fibre of a regular function $q$ on a simply connected smooth affine variety $X$, that is, the restriction of $q$ onto a preimage $q^{-1}(\Delta_\epsilon)$ of a small disc $\Delta_\epsilon \subset \mathbb{C}$ centered at the origin yields a (trivial) smooth fibre bundle over $\Delta_\epsilon$. Thus, in this case also the assumption $(\#_1)$ of Theorem 5.3 holds.

We need the following definition.

Definition 5.3. We say that a subgroup $H$ of a group $G$ is normally generated by elements $a_1, \ldots, a_n \in H$ if it is generated by the set of all elements conjugate with $a_1, \ldots, a_n$, i.e. if $H$ is the minimal normal subgroup of $G$ which contains $a_1, \ldots, a_n$. We denote it by $<< a_1, \ldots, a_n >>$. $G$ is said to be normally one-generated if $G = << a >>$ for some element $a \in G$.

Lemma 5.2. Let $X$ be a smooth irreducible affine variety, and let $F_0 = q^*(0)$, where $q \in \mathbb{C}[X]$, be a reduced irreducible principal divisor in $X$. Fix a vanishing loop $\alpha = \alpha_{F_0} \in G := \pi_1(X \setminus F_0)$ of the divisor $F_0$. Then the following statements hold.

(a) $\pi_1(X) = \mathbb{Z}$ iff $G = << \alpha >>$.

(b) Let $\varphi : Y_s \to X$ be a cyclic covering branched to order $s$ along $F_0$. Set $\tilde{G}_s = << \alpha^s >>$. Assume that $F_0$ is a smooth divisor. Then $\pi_1(Y_s) = \mathbb{Z}$ iff $G/\tilde{G}_s \simeq \mathbb{Z}/s\mathbb{Z}$.

\[1^{1}\text{see Theorem 2.7 above for a description of such pairs $(X, F_0)$}.\]
**Proof.** (a) By Lemma 2.3 (a), we have that Ker \( (i_s : \pi_1(X \setminus F_0) \to \pi_1(X)) = \langle \langle \alpha \rangle \rangle \), and the assertion follows.

(b) Set \( F_{s,0} = \varphi_s^{-1}(F_0) \subset Y_s \). \( X^* = X \setminus F_0 \) and \( Y_s^* = Y_s \setminus F_{s,0} \). Then \( \varphi_s : Y_s^* \to X^* \) is a non-ramified cyclic covering of order \( s \). The induced homomorphism

\[
(\varphi_s)_* : \pi_1(Y_s^*) \to \pi_1(X^*) =: G
\]

is an injection onto a normal subgroup \( G_s \) of \( G \) of index \( s \), and \( G/G_s \simeq \mathbb{Z}/s\mathbb{Z} \). Observe that by (a), \( G = \langle \langle \alpha \rangle \rangle \), and that \( \alpha^s \in G_s \) is the image of a vanishing loop \( \beta \in \pi_1(Y_s^*) \) of the smooth irreducible divisor \( F_{s,0} \subset Y_s \), i.e. \( (\varphi_s)_*(\beta) = \alpha^s \). Therefore, \( \widehat{G}_s : = \langle \langle \alpha^s \rangle \rangle \subset G_s \), and \( G_s = G \) iff \( G/G_s \simeq G/G_s \simeq \mathbb{Z}/s\mathbb{Z} \).

Denote also \( \widehat{G}_s = \langle \langle \alpha^s \rangle \rangle_{G_s} \) the subgroup of \( G_s \) normally generated (in \( G_s \)) by the element \( \alpha^s \in G_s \).

**Claim.** \( \widehat{G}_s = \widehat{G}_s \).

**Proof of the claim.** Clearly, \( \widehat{G}_s \subset \widehat{G}_s \subset G_s \). Since the quotient \( G/G_s \simeq \mathbb{Z}/s\mathbb{Z} \) is Abelian we have that \( K := [G, G] \subset G_s \). Since \( G = \langle \langle \alpha \rangle \rangle \) the abelianization \( G_{ab} := G/K \) is a cyclic group generated by the class \( K\alpha \) of the vanishing loop \( \alpha \). Hence, any element \( g \in G \) can be written as \( g = g'\alpha^t \) where \( g' \in K \subset G_s \) and \( t \in \mathbb{Z} \). Thus, we have \( g\alpha^s g^{-1} = g'\alpha^s g'^{-1} \in \widehat{G}_s \) for any \( g \in G \). Therefore, \( \widehat{G}_s \subset \widehat{G}_s \), and the claim follows. \( \square \)

By (a), \( \pi_1(Y_s^*) = 1 \) iff \( \pi_1(Y_s^*) = \langle \langle \beta \rangle \rangle \), or, what is the same, iff \( \widehat{G}_s = G_s \). Due to the above Claim, the latter holds iff \( \widehat{G}_s = G_s \), or, equivalently, iff \( G/G_s \simeq \mathbb{Z}/s\mathbb{Z} \). This proves (b). \( \square \)

**Proof of Theorem 5.3.** Since \( G = \langle \langle \alpha \rangle \rangle \), any element \( g \in G \) can be written as\( g = \prod_{i=1}^n g_i \alpha^{r_i} g_i^{-1} \), where \( g_i \in G \) and \( r_i \in \mathbb{Z} \), \( i = 1, \ldots, n \). Let \( \rho : G \to G/K \simeq H_1(X \setminus F_0; \mathbb{Z}) \simeq \mathbb{Z} \) be the canonical surjection. Then, clearly, \( \rho(\alpha) = 1 \), and so, \( \rho(g) = \sum_{i=1}^n r_i \in \mathbb{Z} \).

Since \( K \subset G_s := \langle \langle \pi_1(Y_s^*) \rangle \rangle \) and \( G/G_s \simeq \mathbb{Z}/s\mathbb{Z} \), we have that \( \rho(G_s) = s\mathbb{Z} \). That is, \( g = \prod_{i=1}^n g_i \alpha^{r_i} g_i^{-1} \in G_s \) iff \( \rho(g) = \sum_{i=1}^n r_i \equiv 0(\text{mod } s) \).

Using the assumption \((s, c) = 1\) write \( r_i = k_i s + l_i c \) where \( k_i, l_i \in \mathbb{Z} \), \( i = 1, \ldots, n \). By our assumption \((\ast_1)\), \( \alpha^c \in Z(G) \). Hence, \( g_i \alpha^{s k_i} g_i^{-1} = g_i \alpha^{s k_i} g_i^{-1} \alpha^{c l_i} \), \( i = 1, \ldots, n \), and furthermore,

\[
g = \prod_{i=1}^n g_i \alpha^{r_i} g_i^{-1} = \left( \prod_{i=1}^n g_i \alpha^{s k_i} g_i^{-1} \right)^{mc}
\]

where \( m = \sum_{i=1}^n l_i \). For an element \( g \in G_s \) it follows that \( \rho(g) = s \sum_{i=1}^n k_i + mc \equiv 0(\text{mod } s) \), or, equivalently, \( m \equiv 0(\text{mod } s) \). Set \( m = ls, l \in \mathbb{Z} \). Whence, we have \( g = \left( \prod_{i=1}^n g_i \alpha^{s k_i} g_i^{-1} \right)^{mc} \in \widehat{G}_s \). Therefore, \( \widehat{G}_s \subset G_s \subset \widehat{G}_s \), and so, \( \widehat{G}_s = G_s \), as required (see Lemma 5.2 (b)). \( \square \)

Now the proof of Kaliman’s Theorem 5.1 is completed. In Exercises 5.2 - 5.7 below we expose some additional properties of the fundamental group \( G = \pi_1(X \setminus F_0) \) in the situation where the variety \( X \setminus F_0 \) is equipped with a \( \mathbb{C}^* \) - action. After that, in Example 5.2 we show that without the assumption \((\ast_1)\) (or, perhaps, a weaker one which has to be precised) the fundamental group of a cyclic \( \mathbb{C}^* \) - covering \( Y_s \) of a contractible smooth affine variety (even surface) \( X \) can be quite big.

\[\text{[12]}\] These exercises and example were elaborated in [KuZa].
Exercises. (5.2.) Let $G = \langle\langle \alpha \rangle\rangle$ be a normally one-generated group. Denote by $K = [G, G]$ the commutator subgroup of the group $G$. Show that $\alpha^c \in Z(G)$ iff $[G, K] = 1$, and that under this condition $K \subset G_s := \langle\langle \alpha^s \rangle\rangle$ for any $s \in Z$ prime to $c$.

(5.3.) Let $X$ and $q \in C[X]$ be as in Theorem 5.3; in particular, $\pi_1(X) = 1$. Assume that \((\gamma')\) the restriction $q|_{(X \setminus F_0)} : X \setminus F_0 \to C^*$ is a smooth fiber bundle with a connected fiber $F_1 := q^{-1}(1)$. Show that $q_*(\alpha) = 1 \in Z$, $\iota, \pi_1(F_1) = K := [G, G]$, $G_{\text{ab}} := G/K \simeq H_1(X \setminus F_0; Z) \simeq Z$, and $q_\ast = \rho : G \to G/K = Z$ is the canonical surjection. Deduce that $G = \pi_1(X \setminus F_0) \simeq Z$ if and only if $\pi_1(F_1) = 1$, which in turn implies the condition \((\gamma_1')\).

(5.4.) Show, furthermore, that under the condition \((\gamma')\) of Theorem 5.3 the above assumption \((\gamma')\) is fulfilled, and, furthermore, the group $G$ contains a normal subgroup $G_d$ of index $d$ with the cyclic quotient $G/G_d \simeq \mathbb{Z}/d\mathbb{Z}$ such that $(G_d \simeq K \times Z \simeq \pi_1(F_1) \times Z$. Let an element $\gamma \in G_d$ correspond to a generator of the second factor $Z$ of this decomposition. Verify that $\gamma^d \in K$, and that the centralizer subgroup $C_\gamma$ of $\gamma$ in $G$ contains $G_d$. Hint. Put $G_d = (\varphi_d)_* \pi_1(Y_d^*)$ where $\varphi_d : Y_d \to X$ is the $d$-fold branched cyclic covering, $X^* = X \setminus F_0$ and $Y_d^* = Y_d^* \setminus \varphi_d^{-1}(F_0)$. The induced $C^*$-action on $Y_d^*$ yields an equivariant isomorphism $F_1 \times C^* \to Y_d^*$: $(x, \mu) \to (t_\mu(x), \mu)$, which provides, in turn, the desired decomposition of the subgroup $G_d$ of $G$. The element $\gamma$ corresponds to the image of a generator of the group $\pi_1(C^*) \simeq Z$ under the homomorphism induced by the mapping $C^* \to O_{x_0} \subset X^*$ onto the $C^*$-orbit $O_{x_0}$ of a base point $x_0 \in F_1$.

(5.5.) Set $X = C^2$ and $q(x, y) = x^2 - y^3 \in C[x, y]$. Show that the group $G = \pi_1(X \setminus F_0)$ can be identified with the 3-braid group

$$B_3 := \langle \sigma_1, \sigma_2 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle,$$

the generators $\sigma_1, \sigma_2 \in G$ being vanishing loops of the divisor $F_0 = \Gamma_{2, 3} := q^*(0) \subset X$. Describe the subgroups $G_d$ and $K = [G, G]$ in this example (see the preceding exercise). Verify that one can take for $\gamma$ the element $\sigma_1\sigma_2$ which generates the center $Z(G) \simeq Z(B_3) \simeq Z$ of the braid group $B_3$. Putting $\alpha = \sigma_1$ check that $G = \langle\langle \alpha \rangle\rangle$, and $\alpha^c \notin Z(G)$ whatever $c \in Z \setminus \{0\}$ is. Hint. One can use the presentation of $-q$ as the discriminant of the cubic polynomial $t^3 - (y/\sqrt{3})t + x/\sqrt{27} \in C[t]$. Consider, further, the Vieta covering $C^2 \to C^2$ which is a branched Galois covering ramified over $\Gamma_{2, 3}$ with the symmetric group $S_3$ as the Galois group.

(5.6.) Let $X$ be an irreducible quasi-projective variety, $D \subset X$ be an irreducible hypersurface which contains the singularity locus $\text{sing } X$ of $X$, and let $C \subset D$ be a non-empty smooth subvariety such that $C \cap (\text{sing } X \cup \text{sing } D) = \emptyset$. Consider the Kaliman modification $\sigma_C : X' \to X$ of the variety $X$ along $D$ with center $C$. Show that $\pi_1(X') \simeq \pi_1(X \setminus \text{sing } X)$. Hint. Replace the triple $(X, D, C)$ by the triple $(X \setminus \text{sing } X, D \setminus \text{sing } X, C)$, and then apply Lemma 4.1.

(5.7.) For a polynomial $q \in C[x_1, \ldots, x_n]$ such that $q(\overline{0}) = 0$ denote $h_q = q \circ \sigma_n(x_1, \ldots, x_n)/x_n$ where $\sigma_n : C^n \to C^n$, $\sigma_n(x_1, \ldots, x_n) := (x_1x_n, \ldots, x_{n-1}x_n, x_n)$ is the affine modification of the affine space $C^n$ along the hyperplane $\{x_n = 0\}$ with center at the origin (cf. the proof of Lemma 4.6). Put $X = q^{-1}(0) \subset C^n$ and $X' = h_q^{-1}(0) \subset C^n$.

---

13See [Za 3], Appendix. A shorter proof was suggested by H. Flennet.

14For a group $G$ and two subsets $A, B \subset G$ we denote by $[A, B]$ the subgroup generated by all the commutators $[a, b] = aba^{-1}b^{-1}$ where $a \in A, b \in B$.

15In the proof of Lemma 6.8 in [Za 4] it was taken $\gamma = \alpha^d$. In general, this is not true; see the next exercise.
Suppose that \( \overline{S} \in X \) is a smooth point. Verify that \( X' \) is the affine modification of the variety \( X \) along the divisor \( D := X \cap \{ x_n = 0 \} \) with center at the origin. Wright down an explicit equation of the hypersurface \( X' := \Sigma\pi D(X) \subset \mathbb{C}^n \) in the general case (see [KaZa, Example 2.1]).

**Example 5.2.** [16] Consider the smooth surface \( X_{k,l,s,m} := p^{-1}_{k,l,s,m}(0) \subset \mathbb{C}^3 \) defined by the polynomial

\[
 p_{k,l,s,m} := \frac{(xz^m + 1)^k - (yz^m + 1)^l - z^s}{z^m} \in \mathbb{C}[x, y, z]
\]

where \( 0 \leq m \leq s \). For \( m > 0 \) this surface is smooth, and for \( m = 0 \) it is smooth outside of the point \( P_0 = (1, 1, 0) \in X_{k,l,s,0} \). Suppose that \((k, l) = (k, s) = (l, s) = 1\). Then the surface \( X_{k,l,s,m} = Y_{k,l,s} \) is acyclic (see Example 5.1). Moreover, it can be presented as a cyclic \( \mathbb{C}^* \) - covering over the smooth contractible tori Dieck-Petrie surface \( X_{k,l} = X_{k,l,1,1} \subset \mathbb{C}^3 \) (see Examples 2.4, 4.4 and 6.6 above) branched to order \( s \) along the curve \( L_{k,l} := X_{k,l} \cap \{ z = 0 \} \cong \mathbb{C} \) in \( X_{k,l} \). However, the smooth acyclic surface \( Y_{k,l,s} \) is not contractible and possesses quite a big fundamental group which we describe below. [18]

Indeed, Exercise 5.7 above shows that \( \sigma_3 \mid X_{k,l,s,m} : X_{k,l,s,m} \rightarrow X_{k,l,s,m-1} \) is the Kaliman transform of the surface \( X_{k,l,s,m-1} \) along the curve \( D := X_{k,l,s,m-1} \cap \{ z = 0 \} \) with center at the origin. The repeated application of Lemma 4.1 and Exercise 5.6 yields the isomorphisms

\[
 \pi_1(Y_{k,l,s}) = \pi_1(X_{k,l,s}) \cong \pi_1(X_{k,l,s,s-1}) \cong \ldots \cong \pi_1(X_{k,l,s,1}) \cong \pi_1(X_{k,l,s,0} \setminus \{ P_0 \})
\]

where \( X_{k,l,s,0} \cong X_{k,l,s} := \{ x_k^l = y^l - z^s = 0 \} \subset \mathbb{C}^3 \). Whence, the surface \( X_{k,l,s,0} \cong X_{k,l,s} \) is homotopically equivalent to the cone over the Pham–Brieskorn 3-manifold \( M_{k,l,s} := X_{k,l,s} \cap S^5 \) (the link of the surface singularity of \( X_{k,l,s} \) in the sphere \( S^5 \)). In turn, \( X_{k,l,s} \setminus \{ \overline{0} \} \) is homotopically equivalent to the link \( M_{k,l,s} \), and thus \( \pi_1(Y_{k,l,s}) \cong \pi_1(M_{k,l,s}) \). We denote the latter group as \( G_{k,l,s}^\prime \).

The structure of these groups is well known (see [Mil 3]). The groups \( G_{k,l,s}^\prime \) are finite iff \( 1/k + 1/l + 1/s > 1 \), infinite nilpotent iff \( 1/k + 1/l + 1/s = 1 \). If \( 1/k + 1/l + 1/s < 1 \), then \( G_{k,l,s}^\prime = [G_{k,l,s}, G_{k,l,s}] \), where

\[
 G_{k,l,s} := \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1^k = \gamma_2^l = \gamma_3^s = \gamma_1 \gamma_2 \gamma_3 >
\]

is a central extension of the **Schwarz triangular group**

\[
 T_{k,l,s} := \langle b_1, b_2, b_3 \mid b_1^2 = b_2^2 = b_3^2 = 1, (b_1 b_2)^k = (b_2 b_3)^l = (b_3 b_1)^s = 1 >
\]

which is a discrete group of isometries of the non-Euclidean plane generated by reflections in the sides of an appropriate triangle.

Note that for \( 1/k + 1/l + 1/s < 1 \) the triangular group \( T_{k,l,s} \) contains a free subgroup with two generators. Therefore, the group \( G_{k,l,s}^\prime \) also contains such a subgroup; in particular, it is not solvable. Observe that this group is perfect, i.e. it coincides with its commutator subgroup; indeed, its abelianization \( H_1(Y_{k,l,s}; \mathbb{Z}) \) is trivial. On the other hand, it is known that for \((k, l) = (k, s) = (l, s) = 1\) and only under this condition the Pham–Brieskorn manifold \( M_{k,l,s} \) is a homology 3-sphere; see [Bri, HNK, Appendix I.8].

---

[16] We are thankful to V. Sergiescu for useful discussions related to this example.
[17] Actually, by Lemma 4.2, all the surfaces \( X_{k,l,s,m}, m = 1, \ldots, s \), are acyclic, too.
[18] More generally, see [GuMiy 3] for a description of the fundamental groups of acyclic surfaces with log-Kodaira dimension \( 1 \).
Recall that \( X_{k,l} \setminus L_{k,l} \simeq \mathbb{C}^2 \setminus \Gamma_{k,l} \) where \( \Gamma_{k,l} := \{ x^k - y^l = 0 \} \subset \mathbb{C}^2 \) (see Example 4.6). The group \( B_{k,l} := \pi_1(X_{k,l} \setminus L_{k,l}) \simeq \pi_1(\mathbb{C}^2 \setminus \Gamma_{k,l}) \) has the presentation \( B_{k,l} = \langle a, b \mid a^k = b^l \rangle \) (see e.g. [Di 3]). In turn, the group \( \pi_1(Y_{k,l,s} \setminus \{ z = 0 \}) \) is isomorphic to an index \( s \) subgroup of the group \( B_{k,l} \) with a cyclic quotient. By Lemma 2.3 (a), \( \text{Ker} (i_* : \pi_1(Y_{k,l,s} \setminus \{ z = 0 \}) \to \pi_1(Y_{k,l,s})) = \langle \langle \alpha^s \rangle \rangle \) where \( \alpha \in B_{k,l} \) represents a vanishing loop of the line \( L_{k,l} \subset X_{k,l} \). Let \( p, q \in \mathbb{Z} \) be such that \( kp + lq = 1 \). Then one may take \( \alpha = a^p b^q \in B_{k,l} \).

Therefore, for \( 1/k + 1/l + 1/s < 1 \) and \( (k, l) = (k, s) = (l, s) = 1 \) the group \( G'_{k,l,s} \simeq \pi_1(Y_{k,l,s}) \) is isomorphic to an index \( s \) subgroup of the quotient
\[
B_{k,l,s} := B_{k,l}/\langle\langle \alpha^s \rangle\rangle = \langle a, b \mid a^k = b^l, (a^p b^q)^s = 1 \rangle.
\]
In particular, for \( k = 2, l = 3, \) and \( s \geq 7 \) we have that \( B_{2,3} = B_3 \) is the 3-braid group with generators \( \sigma_1, \sigma_2 \in B_3 \) being vanishing loops of \( L_{2,3} \) in \( X_{2,3} \) (see Exercise 5.3 above), \( a = \sigma_1 \sigma_2 \sigma_1, \ b = \sigma_1 \sigma_2, \) and \( G'_{2,3,s} \) is isomorphic to an index \( s \) subgroup of the group
\[
B_{2,3,s} = B_3/\langle\langle \sigma_1^s \rangle\rangle = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2, \ \sigma_1^s = \sigma_2^s = 1 \rangle
\]
which consists of the words in the generators \( \sigma_1, \sigma_2 \) with the sum of exponents divisible by \( s \).

## 6 Multicyclic \( C^* \)– coverings

### 6.1 Contractibility of multicyclic \( C^* \)– coverings. Examples

To clarify the very idea of the construction of contractible multicyclic \( C^* \)– coverings due to Koras and Russell [KoRu 1, KoRu 2], let us start with simple examples. We exhibit two different approaches (see Examples 6.1 and 6.3 below). Recall first the following definitions.

**Definition 6.1.** The suspension over a topological space \( X \) is the cylinder \( X \times [0, 1] \) with the bases \( X \times \{ 0 \}, X \times \{ 1 \} \) being contracted each one to a point. The join \( X \star Y \) of two topological spaces \( X, Y \) is the cylinder \( (X \times Y) \times [0, 1] \) with the base \( (X \times Y) \times \{ 0 \} \) resp. \( (X \times Y) \times \{ 1 \} \) being contracted to \( X \times \{ 0 \} \) resp. to \( Y \times \{ 1 \} \). Clearly, the join \( X \star \mathbb{Z}/2\mathbb{Z} \) is nothing but the suspension over \( X \).

We also use the following fact.

**Theorem 6.1 (Némethi–Sebastiani–Thom [Ne, Di 4]).** A generic fibre of a polynomial \( p(x) + q(y), p \in \mathbb{C}[x_1, \ldots, x_k], q \in \mathbb{C}[y_1, \ldots, y_l], \) is homotopically equivalent to the join of generic fibres of the polynomials \( p \) and \( q \).

**Example 6.1 (The Russell cubic threefold; see also Examples 4.3, 4.3).** The polynomial \( q_0 = x(xy + 1) \in \mathbb{C}[x, y] \) is the hyperbolic modification of the polynomial \( h = x + x^2 \in \mathbb{C}[x] \). Thus, it is a quasi-invariant of weight \( 1 \) of the \( C^* \)– action \( \lambda \mapsto (\lambda, (x, y)) \) on \( \mathbb{C}^2 \). The zero fiber \( \Gamma_0 = q_0^{-1}(0) \) is a disjoint union of two affine curves isomorphic to \( \mathbb{C} \) and to \( \mathbb{C}^* \), respectively. Consider the two-fold \( C^* \)– covering \( F_0 \to \mathbb{C}^2 \) branched along \( \Gamma_0 \), given as the surface \( F_0 = \{ x + x^2 y + z^2 = 0 \} \subset \mathbb{C}^3 \) with the projection \( \varphi_2 : F_0 \to \mathbb{C}^2 \), \( \lambda \mapsto (\lambda^2 x, \lambda^{-2} y, \lambda z) \). Thus, \( \varphi_2 \) is \( C^* \)– equivariant with respect to the actions \( (\lambda, (x, y, z)) \mapsto (\lambda^2 x, \lambda^{-2} y, \lambda z) \) on \( F_0 \) and \( (\lambda, (x, y)) \mapsto (\lambda^2 x, \lambda^2 y, \lambda z) \) on \( \mathbb{C}^2 \). The restriction of the above \( C^* \)– action on \( F_0 \) to the subgroup \( \omega_2 = \{ \lambda^2 = 1 \} \simeq \mathbb{Z}/2\mathbb{Z} \) of \( C^* \) yields the monodromy of the covering \( F_0 \to \mathbb{C}^2 \). Since this monodromy acts trivially in the homology of \( F_0 \), by Corollary 5.2, \( F_0 \) is \( \mathbb{Z}/3 \)– acyclic.
By the Némethi–Sebastiani–Thom Theorem \cite{KoRu2}, the generic fibre $F_1 = q^{-1}(1)$ of the polynomial $q = (x + x^2 y) + z^2 \in \mathbb{C}[x, y, z]$ is homotopically equivalent to the join $\Gamma_1 \ast \mathbb{Z}/2\mathbb{Z}$ where $\Gamma_1 := q_0^{-1}(1) \subset \mathbb{C}^3$, i.e. to the suspension over $\Gamma_1$. Since the curve $\Gamma_1 \cong \mathbb{C}^*$ is connected, the fibre $F_1$ is simply connected, and hence, $G := \pi_1(\mathbb{C}^3 \setminus F_0) \cong \mathbb{Z}$ (see Exercise 5.3).

The Russell cubic threefold

$$X = \{x + x^2 y + z^2 + t^3 = 0\} \subset \mathbb{C}^4$$

can be regarded as a three-sheeted cyclic $\mathbb{C}^*-$ covering over $\mathbb{C}^3$ branched along the surface $F_0$ under the projection $\varphi_3 : (x, y, z, t) \mapsto (x, y, z)$ onto $\mathbb{C}^3$. Since the polynomial $q$ is a quasi-invariant of weight 2 of the above $\mathbb{C}^*-$ action on $\mathbb{C}^3$, we are under the assumptions of Kaliman’s Theorem 5.1. Due to this theorem, the Russell cubic $X$ is a contractible smooth affine variety.

**Exercises. (6.1.)** Verify that the smooth cubic threefold

$$X' = \{x + x^2 y + z^2 + t^2 = 0\} \subset \mathbb{C}^4$$

is simply connected, but not acyclic (what are the homology groups of $X'$?).

(6.2.) Show that a generic fiber $p^{-1}(c), c \in \mathbb{C}^*$, of the Russell polynomial $p = x + x^2 y + z^2 + t^3$ is not contractible.

*Hint.* Apply the Némethi–Sebastiani–Thom Theorem \cite{KoRu2} to find the homotopical type of this fiber.

**Example 6.2** (see tom Dieck \cite{tD2} Thm. B). More generally, let $X$ be a smooth contractible affine variety equipped with a regular $\mathbb{C}^*-$ action $t$, and let $q \in \mathbb{C}[X]$ be a quasi-invariant of $t$ of weight $d \neq 0$ such that $F_0 := q^t(0)$ is a smooth reduced (not necessarily irreducible) principal divisor in $X$. Fix $s_1, s_2 \in \mathbb{N}$ such that $d, s_1, s_2$ are pairwise relatively prime. Consider the smooth affine hypersurface

$$Y_{s_1, s_2} := \{q(x) + z^{s_1} + t^{s_2} = 0\} \subset X \times \mathbb{C}^2.$$

We assert that this hypersurface is contractible.

Indeed, consider first the cyclic $\mathbb{C}^*-$ covering $Y_{s_1} \to X$, $Y_{s_1} = \{q(x) + z^{s_1} = 0\} \subset X \times \mathbb{C}$, branched to order $s_1$ along $F_0$. Then $Y_{s_1} \subset X \times \mathbb{C}$ is a smooth reduced irreducible divisor defined by the quasi-invariant $q_1(x, z) := q(x) + z^{s_1} \in \mathbb{C}[X \times \mathbb{C}]$ of weight $ds_1$ of the $\mathbb{C}^*-$ action $\lambda(x, z) \mapsto (t^\lambda x, \lambda^d z)$ on $X \times \mathbb{C}$. Since the monodromy group $G \simeq \mathbb{Z}_{s_1}$ of the covering acts trivially in the homology of $Y_{s_1}$, by Corollary 5.2, $Y_{s_1}$ is $\mathbb{Z}_p-$ acyclic for any prime $p$ which is prime to $s_1$, and hence, for any prime divisor $p$ of $s_2$.

Besides, the fibre $F_1 = q_1^{-1}(1)$ of the regular function $q \in \mathbb{C}[X]$ is connected (see Exercise 7.3). As above, it follows from the Némethi–Sebastiani–Thom Theorem \cite{KoRu2} that the fibre $q_1^{-1}(1) = \{q(x) + z^{s_1} = 1\}$ of the function $q_1 \in \mathbb{C}[X \times \mathbb{C}]$ is simply connected. Hence, we have $\pi_1((X \times \mathbb{C}) \setminus Y_{s_1}) \cong \mathbb{Z}$.

Therefore, by Kaliman’s Theorem 7.1, the total space of the cyclic $\mathbb{C}^*-$ covering $Y_{s_1, s_2} \to X \times \mathbb{C}$ branched to order $s_2$ over $Y_{s_1}$ is a smooth contractible affine variety.

Applying Kaliman’s Theorem \cite{KoRu2} successively in the same way as above, one can derive the following result (cf. Koras and Russell \cite{KoRu2} (7.14)).
Theorem 6.2. Let $X$ be a smooth contractible affine variety equipped with an effective $\mathbb{C}^*-$ action. Let $q_i \in \mathbb{C}[X], \ i = 1, \ldots, k$, be a sequence of quasi-invariants of positive weights $d_1, \ldots, d_k$, respectively, and let $s_1, \ldots, s_k$ be a sequence of positive integers. Suppose that the following conditions are fulfilled:

(i) For each $i = 1, \ldots, k$, $F_i := q^+_i(0)$ is a smooth reduced irreducible divisor, the union $\bigcup F_i$ is a divisor with normal crossings, and the group $\pi_1(X \setminus \bigcup F_i)$ is Abelian;

(ii) $(d_i, s_i) = (s_j, s_j) = 1$ for all $i, j = 1, \ldots, k$;

(iii) $F_i$ is a divisor with normal crossings, and the group $\pi_1(X \setminus \bigcup F_i)$ is Abelian;

Let $Y \rightarrow X$ be a multicyclic covering branched to order $s_i$ along $F_i, \ i = 1, \ldots, k$, i.e. $Y = Y_{s_1 \ldots s_k}$ is the last one in the tower of cyclic $\mathbb{C}^*-$ coverings

$$Y_{s_1 \ldots s_k} \rightarrow Y_{s_1 \ldots s_{k-1}} \rightarrow \ldots \rightarrow Y_{s_1 s_2} \rightarrow Y_{s_1} \rightarrow X,$$

where $Y_{s_1 \ldots s_i} \rightarrow Y_{s_1 \ldots s_{i-1}}$ is a $\mathbb{C}^*$- covering branched to order $s_i$ over the preimage of the divisor $F_i$ in $Y_{s_1 \ldots s_{i-1}}$.

Then $Y$ is a smooth contractible affine variety given in $X \times \mathbb{C}^k$ by the equations $z_i^{s_i} = q_i(x), \ i = 1, \ldots, k$.

Remarks. 6.1. In the case when $X \subset \mathbb{C}^n$, and $q_j$ is a variable, $q_j = x_j$ say, the equations of the cyclic covering $Y_{s_j} \rightarrow X$ can be obtained from the equations $P_i(x_1, \ldots, x_n) = 0, \ i = 1, \ldots, m$, which define $X$, by the substitution $x_j \rightarrow x_j^{s_j}$, i.e. $Y_{s_j} = \{P_i(x_1, \ldots, x_j^{s_j}, \ldots, x_n) = 0, \ i = 1, \ldots, m\} \subset \mathbb{C}^n$. In particular, if $X$ is a hypersurface in $\mathbb{C}^n$, so is $Y_{s_j}$.

6.2. To construct contractible $\mathbb{C}^*-$ invariant hypersurfaces one can use the hyperbolic modification in the same way as in Example 6.1 above. Recall that if $X = \{h = 0\} \subset \mathbb{C}^n$ is a smooth contractible hypersurface, then the hypersurface $Y_0 := q^{-1}(0) \subset \mathbb{C}^{n+1}$, where $q(x, y) = h(x, y)/u$, is isomorphic to the Kaliman modification of the product $X \times \mathbb{C}$ along the divisor $X$ with center at the origin (Lemma 6.6), and so, by Theorem 6.3, $Y_0$ is a smooth contractible hypersurface, too. Moreover, $q$ is a quasi-invariant of weight 1 of the regular $\mathbb{C}^*-$ action $(\lambda, (x, y)) \mapsto (\lambda x, \lambda^{-1} y)$ on $\mathbb{C}^{n+1}$ with the only fixed point at the origin (of hyperbolic type with the weights $(1, \ldots, 1, -1)$).

Let smooth hypersurfaces $H_i = \{h_i(x) = 0\} \cap X, \ i = 1, \ldots, k$, in $X$ satisfy the condition (i) of Theorem 5.2. Put $q_i(x, u) = h_i(x)/u$. Then the $q_i$ are $\mathbb{C}^*-$ invariants of weight 1 of the above $\mathbb{C}^*-$ action on $Y_0$. The hypersurfaces $F_i := q_i^{-1}(0) \cap Y$, $i = 1, \ldots, k$, in $Y$ also satisfy the condition (i) (note that $F_i$ is the closure of the $\mathbb{C}^*-$ orbit of the subvariety $H_i \subset X \simeq Y_0 \cap H \subset Y$, $i = 1, \ldots, k$, where $H := \{u = 1\}$; see Exercise 4.1). This construction can be illustrated by the following simple example.

Example 6.3. The Russell cubic threefold once again (see Koras-Russell \cite{KoRu2}; cf. Examples 4.3, 6.1). Starting with $X = \mathbb{C}^2$, fix two smooth curves $(f)$ and $(g)$ where $f, g \in \mathbb{C}[x, z]$, isomorphic to $\mathbb{C}$ and such that $(f), (g)$ meet transversally at the origin and in $k$ other points, $k \geq 1$. For instance, take $f = z, g = z + x + x^2$. Then the Kaliman modification $Y$ of $X \times \mathbb{C}$ along $X$ with center at the origin is nothing but $\mathbb{C}^3$. The plane curves $(f)$ and $(g)$ give rise, respectively, to the surfaces $(F)$ and $(G)$ in $\mathbb{C}^3$ where $F = f(yg-zy), g = g(yg-zy)$. Observe that $(F)$ and $(G)$ are isomorphic to $\mathbb{C}^2$, meet transversally, and $\pi_1(\mathbb{C}^3 \setminus (F \cup (G))) \approx \mathbb{Z}$.  

\(^{19}\)In other words, if $A = \mathbb{C}[X]$, then $Y = \text{spec } A[\sqrt[3]{a_1}, \ldots, \sqrt[3]{a_k}]$.  

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In our particular example $F = z$ and $G = z + x + x^2y$. The polynomials $F$ and $G$ are $\mathbb{C}^*$-quasi-invariants of weight 1 with respect to the action $(\lambda, (x, y, z)) \mapsto (\lambda x, \lambda^{-1} y, \lambda z)$ on $\mathbb{C}^3$. We may also take the plane $H_0 = \{ y = 0 \}$ for the third surface transversal to the first two $(F)$ and $(G)$.

Fix two relatively prime positive integers $s_1, s_2$. Passing to the bicyclic $\mathbb{C}^*$-covering of $Y \cong \mathbb{C}^3$ branched to order $s_1$ along $(F)$ and to order $s_2$ along $(G)$ we obtain a hypersurface $Y_{s_1,s_2} \subset \mathbb{C}^4$ given by the equation

$$x + x^2y + z^{s_1} + t^{s_2} = 0$$

which is a smooth contractible threefold. For $s_1 = 2, s_2 = 3$ we get once again the Russell cubic. More generally, passing to the tricyclic covering of $\mathbb{C}^3$ branched to order $s_0$ resp. $s_1, s_2$ along the surface $H_0$ resp. $(F), (G)$ where $(s_i, s_j) = 1$, $i \neq j$ yields the smooth contractible hypersurface $\{ x + x^2y^s + z^{s_1} + t^{s_2} = 0 \}$ in $\mathbb{C}^4$.

6.3. A theorem due to Koras and Russell [KoRu 2, Thm. 4.1] says that any smooth contractible affine threefold with a ‘good’ hyperbolic $\mathbb{C}^*$-action appears in the same way as in the above example.

### 6.2 The logarithmic Kodaira dimension of multicyclic coverings

Lemma 6.1. Let $V$ be a smooth projective variety, and let $L$ be a line bundle on $V$.

(a) (Mori [Mo, Prop. 1.9]) The line bundle $L$ is big (i.e. $k(V,L) = \dim \mathcal{O}_V$) iff for some $k \in \mathbb{N}$ the multiple $kL$ can be written as $kL = A + E$ where $A$ is an ample line bundle on $V$, and $E$ is an effective one (that is, $E$ admits a non-zero holomorphic section).

(b) (Kleiman-Kodaira; see e.g. [Wil, (2.3)], [KMM, Lemma 0-3-3]) If the line bundle $L$ is ample (resp. big), then for any line bundle $L'$ on $V$ and for any $k \in \mathbb{N}$ large enough the line bundle $kL - L'$ is ample (resp. big), too.

**Proof of (b).** Denote by $NE_1(V)$ the cone of numerically effective 1-cycles modulo numerical equivalence on a projective variety $V$. Recall the Kleiman criterion of ampleness [K]: a line bundle $L$ on $V$ is ample iff it is positive on the cone $NE_1(V)$ with the origin being deleted.

This finite dimensional cone is closed, and hence, it has a compact intersection with the unit sphere. Thus, the openness of ampleness follows.

Let $L$ be a big line bundle, and let $k_0L = A + E$ be a decomposition as in (a). Then for $n_0 \in \mathbb{N}$ large enough we have $n_0k_0L - L' = (n_0A - L') + n_0E$ where $n_0A - L'$ is ample. Therefore, by (a), the line bundle $n_0k_0L - L'$ is big. It follows that for any $k \geq n_0k_0$ the line bundle $kL - L' = (n_0k_0L - L') + (k - n_0k_0)L$ is also big.

**Proposition 6.1 (Kaliman [Ka 1, Lemma 11]).** Let $X$ be a quasi-projective variety, $(V, D)$ be an SNC-completion of $X$, and $Z = \sum_i Z_i$ be an SNC-divisor on $V$ such that $D \cup Z$ is also an SNC-divisor, and $D$ and $Z$ have no irreducible component in common. Let $Y = V \overleftarrow{\rightarrow} X$, $\pi := (s_1, \ldots, s_k)$, be a ramified covering branched to order $s_i$ over $Z_i \cap X$, $i = 1, \ldots, k$. Then

$$\mathcal{K}(Y) = k \left( V, K_V + D + \sum_{i=1}^{k} \left( 1 - \frac{1}{s_i} \right) Z_i \right).$$

---

*See section 2.2 for the terminology.*

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Proof. One can compactify \( Y \) by an SNC-divisor \( D' \) (i.e. \( Y = V' \setminus D' \)) to obtain a commutative diagram of morphisms

\[
\begin{array}{c}
Y & \xrightarrow{\varphi} & V' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\varphi} & V.
\end{array}
\]

Then \( \varphi^*(Z_i) = s_iZ'_i + E_i \) where each divisor \( E_i \) is \( \varphi \)-exceptional, i.e. \( \text{codim}_V \varphi(E_i) \geq 2 \). The restriction \( \varphi \mid (V' \setminus (D' \cup Z')) : V' \setminus (D' \cup Z') \to V \setminus (D \cup Z) \) is an étale covering. Therefore, by the Logarithmic Ramification Formula (R), we have

\[
K_{V'} + D' + Z' = \varphi^*(K_V + D + Z) + R
\]

where \( R \) is an effective \( \varphi \)-exceptional divisor in \( V' \) (see Theorem 2.5. (e), (f)). Hence,

\[
K_{V'} + D' = \varphi^*(K_V + D + Z) + R - Z' = \varphi^*(K_V + D + Z) +
\]

\[
+ \varphi^*\left( \sum_{i=1}^{k} \left( -\frac{1}{s_i} \right) Z_i \right) + R + \sum_{i=1}^{k} \frac{1}{s_i} E_i = \varphi^*\left( K_V + D + \sum_{i=1}^{k} (1 - \frac{1}{s_i}) Z_i \right) + E
\]

where \( E := \sum_{i=1}^{k} \frac{1}{s_i} E_i + R \) is a \( \varphi \)-exceptional \( \mathbb{Q} \)-divisor. By \([1],[1], \text{Lemma 1}\) or \([1], \text{Thm.} 10.5\], \( k(V', \varphi^*(D_1) + E) = k(V, D_1) \) for any \( \mathbb{Q} \)-Cartier divisor \( D_1 \) on \( V \) where \( E \) is a \( \varphi \)-exceptional divisor in \( V' \) (indeed, a meromorphic section of the associated line bundle \([\varphi^*(D_1)]\) with poles at most along \( E \) has no pole). Thus, the assertion follows. \( \square \)

Corollary 6.1 ([1], [2, Cor.6.2]). If \( \mathfrak{s} = (s_1', \ldots, s_k') \) and \( s_i' \geq s_i \), \( i = 1, \ldots, k \), then \( k(Y_{\mathfrak{s}}) \geq k(Y_{\mathfrak{s}}) \).

Corollary 6.2. If \( X \setminus Z \) is a variety of log-general type, that is, \( k(X \setminus Z) = \dim_{\mathbb{C}} X \), then for \( s_i \), \( i = 1, \ldots, k \), large enough \( Y_{\mathfrak{s}} \) is a variety of log-general type, too.

Proof. Indeed, by Lemma 2.4 \((b)\), for \( s_i >> 1 \), \( i = 1, \ldots, k \), we have

\[
k(Y_{\mathfrak{s}}) = k\left( V, K_V + (D + Z) - \sum_{i=1}^{k} \frac{1}{s_i} Z_i \right) = k(V, K_V + D + Z) = k(X \setminus Z) = \dim_{\mathbb{C}} X.
\]

\( \square \)

Proposition 6.2 (see [1], [2, Prop. 6.5]). Consider the Koras-Russell threefolds \( Y_{\mathfrak{s}} \subset \mathbb{C}^4 \) where \( \mathfrak{s} = (s_1, s_2, s_3) \) and \( (s_i, s_j) = 1 \), \( i \neq j \) given as\(^{21}\)

\[Y_{\mathfrak{s}} = \{ x + x^2 y^{s_1} + z^{s_2} + t^{s_3} = 0 \}.\]

If \( s_1, s_2, s_3 >> 1 \), then \( Y_{\mathfrak{s}} \) is an exotic \( \mathbb{C}^3 \), and \( k(Y_{\mathfrak{s}}) = 2 \).

Proof. Set

\[X = \{ x + x^2 u_1 + u_2 + u_3 = 0 \} \subset \mathbb{C}^4 \quad \text{and} \quad Z_i = \{ u_i = 0 \} \subset X, \ i = 1, 2, 3.\]

Evidently, \( X \cong \mathbb{C}^3 \), \( Z_i \cong \mathbb{C}^2, i = 1, 2, 3 \), and \( Z := Z_1 \cup Z_2 \cup Z_3 \) is an SNC-divisor in \( X \). The threefold \( Y_{\mathfrak{s}} \) is a tricyclic covering of \( X \) branched to order \( s_i \) along \( Z_i, i = 1, 2, 3 \), with

\(^{21}\)This is a particular kind of the Koras-Russell threefolds; see Example 6.3 above.
the covering morphism \( \varphi : (x, y, z, t) \mapsto (x, u_1, u_2, u_3) := (x, y^{s_1}, z^{s_2}, t^{s_3}) \). By Theorem 1.2, it follows that \( Y_{\varphi} \subset \mathbb{C}^4 \) is a smooth contractible affine hypersurface. Due to the Dimca-Ramanujam Theorem 3.2, the variety \( Y_{\varphi} \) is diffeomorphic to \( \mathbb{R}^5 \). It remains to show that \( \overline{k(Y_{\varphi})} = 2 \) when \( s_1, s_2, s_3 \) are large enough.

Due to Corollary 3.1, \( \overline{k(Y_{\varphi})} \geq 2 \) for sufficiently large \( s_1, s_2, s_3 \) if it is so for a particular choice of \( \varphi = (s_1, s_2, s_3) \) (even without the assumption of relative primeness, which guarantees the contractibility).

Note that the hypersurface \( Y_{\varphi} \subset \mathbb{C}^4 \) is invariant under the hyperbolic linear \( \mathbb{C}^* \)– action on \( \mathbb{C}^4 \)

\[ G : (\lambda, (x, y, z, t)) \mapsto (\lambda^a x, \lambda^b y, \lambda^c z, \lambda^d t) \]

where \( a = s_1 s_2 s_3, b = s_2 s_3, c = s_1 s_3, d = s_1 s_2 \).

The morphism \( \varphi_\varphi : Y_{\varphi} \rightarrow X \) is a \( \mathbb{C}^* \)– covering with respect to the \( \mathbb{C}^* \)– action \( G \) on \( Y_{\varphi} \) and the \( \mathbb{C}^* \)– action

\[ \overline{G} : (\lambda, (x, u_1, u_2, u_3)) \mapsto (\lambda^a x, \lambda^{-a} u_1, \lambda^b u_2, \lambda^c u_3) \]

on \( X \). We have: \( \text{spec} (\mathbb{C}[X])^{\overline{G}} \subset X/\overline{\mathbb{G}} \approx S \) where \( S : = \{ u + u^2 + v + w = 0 \} \subset \mathbb{C}^3 \); clearly, \( S \approx \mathbb{C}^2 \). Indeed, \( (\mathbb{C}[X])^{\overline{G}} = \mathbb{C}[u, v, w] \) where \( u := u_1 x, v := u_1 u_2, w := u_1 u_3 \in (\mathbb{C}[X])^{\overline{G}} \) are the basic \( \overline{G} \)– invariants. This yields the following commutative diagram of morphisms:

\[ Y_{\varphi} \xrightarrow{\varphi_{\varphi}} Y_{\varphi}/G = S_{\varphi} \]

\[ X \xrightarrow{\rho} X/\overline{\mathbb{G}} = S \]

where \( S_{\varphi} := Y_{\varphi}/G = \text{spec} (\mathbb{C}[Y_{\varphi}])^{\overline{G}} \) is a normal surface. A generic fiber of the quotient morphism \( \rho_{\varphi} : Y_{\varphi} \rightarrow S_{\varphi} \) (i.e. a generic orbit) is isomorphic to \( \mathbb{C}^* \). Since \( \overline{k(\mathbb{C}^*)} = 0 \), from the Addition Theorems 2.2 (c), (d) \[22\] we obtain

\[ 2 = \dim S_{\varphi} \geq \overline{k(Y_{\varphi})} \geq \overline{k(S_{\varphi})}. \]

Thus, it remains to find a particular triple \( \varphi = (s_1, s_2, s_3) \) such that \( \overline{k(S_{\varphi})} = 2 \).

Note that the threefold \( Y_{\varphi} \) is the closure of the \( \mathbb{G} \)– orbit of the surface \( T_{\varphi} := Y_{\varphi} \cap H \) where \( H := \{ y = 1 \} \subset \mathbb{C}^4 \) (see Exercise 4.1). The surface \( T_{\varphi} \) is invariant under the induced action of the cyclic subgroup \( \omega_b \subset \mathbb{C}^* \) on \( Y_{\varphi} \), and \( S_{\varphi} = Y_{\varphi}/G \approx T_{\varphi}/\omega_b \).

Take \( s_1 = pq, s_2 = p, s_3 = q \) where \( p, q \in \mathbb{N} \) are prime and distinct. Then we have \( Y_{\varphi} = \{ x + x^2 y^{pq} + z^p + t^q = 0 \} \), and

\[ G(\lambda, (x, y, z, t)) = (\lambda^{pq^2} x, \lambda^{-pq} y, \lambda^{pq} z, \lambda^{pq^2} t) \].

Therefore, the subgroup \( \omega_b = \omega_{pq} \subset \mathbb{C}^* \) which coincides with the non-effectiveness kernel of the \( \mathbb{C}^* \)– action \( G \) on \( Y_{\varphi} \) acts trivially on \( T_{\varphi} = Y_{\varphi} \cap H = \{ x + x^2 + z^p + t^q = 0 \} \subset \mathbb{C}^3 \). Hence, \( S_{\varphi} = Y_{\varphi}/G \approx T_{\varphi} \subset Y_{\varphi} \). The projection

\[ \rho \circ \varphi_{\varphi} : T_{\varphi} \rightarrow S, \quad (x, z, t) \mapsto (u, v, w) = (x, z^p, t^q) \],

\[22\] They are still available, although the quotient surface \( S_{\varphi} \) might be singular.
is a bicyclic covering branched to order $p$ resp. $q$ over the curve $C_1 := \{v = 0\} \subset S$ resp. $C_2 := \{w = 0\} \subset S$. By Corollary 6.2 above, we have $\mathbb{k}(S_\tau) = \mathbb{k}(T_\tau) = \mathbb{k}(S \setminus (C_1 \cup C_2))$, if $p$ and $q$ are sufficiently large. Thus, the proof is completed by the following simple exercises. □

**Exercises.** Show that

(6.3.) $(S, C_1 \cup C_2) \simeq (C^2, D_1 \cup D_2)$, where $D_1 := \{y = 1\}$, $D_2 := \{y = x^2\} \subset C^2$; and that

(6.4.) $\mathbb{k}(C^2 \setminus (D_1 \cup D_2)) = 2$.

**Remark 6.4.** (see [KoRu 2, Prop. 7.8.]) However, for $s_1 = 1$ the threefold $Y_1^\tau \subset C^4$ is dominated by $C^3$; in particular, it has the log-Kodaira dimension $\kappa = -\infty$. Indeed, if $s_1 = 1$, then for any $x \neq 0$, $y$ is expressed in terms of $z$ and $t$, whence the part $\{x \neq 0\}$ of the threefold $Y_1^\tau$ is isomorphic to the cylinder $C^2 \times C^*$. The ‘book-surface’ $B := \{x = 0\} \subset Y$ is the product $C \times \Gamma_{s_2,s_3}$ where $\Gamma_{s_2,s_3} := \{z^{s_2} + t^{s_3} = 0\} \subset C^2$. Fix a smooth point $\rho \in \Gamma_{s_2,s_3}$, and perform the Kaliman modification $\sigma : Y_1^\tau \rightarrow Y_1^\tau$ of the 3-fold $Y_1^\tau$ along the divisor $B$ with the center $C := C \times \{\rho\}$. In this way, we replace the singular book-surface $B$ by a smooth surface $E' \simeq C^2$, and replace the function $x$ by a function $f : Y_1^\tau \rightarrow C$ such that all the fibers of $f$ are smooth reduced surfaces isomorphic to $C^2$. By the Miyanishi Theorem 6.3 below we have: $Y_1^\tau \simeq C^3$. So, $\sigma : C^3 \simeq Y_1^\tau \rightarrow Y_1^\tau$ is a birational (whence, dominant) morphism.

In the case of Russell’s cubic $X_0 = \{x + x^2y + z^2 + t^3 = 0\} \subset C^4$, a dominant morphism $C^3 \rightarrow X_0$ can be given explicitly as $(u, v, w) \mapsto (x, y, z, t)$ where

$$(x, y, z, t) = \left(-u, \frac{u - (u^2v + 1)^2 - (u^2w + u/3 - 1)^3}{u^2}, u^2v + 1, u^2w + u/3 - 1\right).$$

We conclude this section by the following characterization of the affine 3-space $C^3$ due to Miyanishi [Miy 1].

**Theorem 6.3 (Miyanishi).** Let $X$ be a smooth affine threefold. Then $X$ is isomorphic to the affine 3-space $C^3$ iff the following conditions hold:

(i) the Euler characteristic $e(X)$ of $X$ is equal to 1;

(ii) the algebra $C[X]$ of regular functions on $X$ is UFD, and all its invertible elements are constants;

(iii) there exists a non-empty Zariski open subset $\Omega \subset X$ isomorphic to a cylinder $\Gamma \times C^2$ where $\Gamma$ is an affine curve;

(iv) the algebra of regular functions on each irreducible component of the divisor $X \setminus \Omega$ is UFD.

**Remark 6.5.** Observe that for the Russell cubic threefold the conditions (i) – (iii) are fulfilled, and only the last condition (iv) does not hold.

7 The Makar-Limanov invariant of the Russell cubic threefold

Let $X$ be an affine variety. We assume in what follows that $X$ is irreducible, so that the algebra $A = C[X]$ of regular functions on $X$ is an integral domain. Makar-Limanov [ML 3] (see also [KaML 3]) introduced a subring ML$(A)$ of a ring $A$ such that ML$(A)$ is invariant.

\[^{23}\text{Cf. [Ka 1 Lemma 16], [KoRu 2 Lemma 6.3].}\]
under ring isomorphisms; that is, if \( B \cong A \), then \( \text{ML}(B) \cong \text{ML}(A) \). He proved the following theorem.

**Theorem 7.1** (Makar-Limanov \([\text{ML} 2, \text{ML} 3]\)). Set \( A_0 = \mathbb{C}[X_0] \) where \( X_0 = \{ x + x^2y + z^2 + t^3 = 0 \} \subset \mathbb{C}^4 \) is the Russell cubic threefold. Then \( \text{ML}(A_0) \) is not isomorphic to \( \mathbb{C} \). Thus, \( X_0 \) is not isomorphic to \( \mathbb{C}^3 \), and hence, \( X_0 \) is an exotic \( \mathbb{C}^3 \).

Later on, Kaliman and Makar-Limanov \([\text{KaML} 3]\) extended this result to all the Koras-Russell threefolds. This was one of the crucial steps in the recent proof of the Linearization Conjecture for \( n = 3 \) (Koras and Russell \([\text{KoRu 2, KoRu 3, KaKoMLRu, KrPo, Po}]\). Let, as above, \( A \) be an affine variety and \( \lambda : C^+ \rightarrow X. \) Deduce that the action \( \lambda : \mathbb{C}^+ \rightarrow X \) induces an algebra homomorphism

\[ A \rightarrow \mathbb{C}[C^+ \times X], \quad p \in A \rightarrow p(\lambda(t, x)) \in A[t]. \]

Set \( \partial p = \frac{d}{dt}|_{t=0} (p \circ \lambda) \). Then \( \partial \in \text{LND}(A) \). Vice versa, any locally nilpotent derivation \( \partial \in \text{LND}(A) \) corresponds to the algebra homomorphism \( \varphi_{\partial} : A \rightarrow A[t] \) given by

\[ \varphi_{\partial}(a) = \exp(t\partial)(a) = \sum_{i=0}^{\infty} \frac{t^i \partial^i a}{i!}, \quad a \in A, \]

and thus, to a regular \( \mathbb{C}_+ \) action on the variety \( X \).

**Exercises.** (7.1.) Prove the equality \( A^{A^\partial} = A^{\partial} \) where \( A^\partial := \text{Ker} \partial, \) and \( A^{A^\partial} \) is the subalgebra of invariants of the \( \mathbb{C}_+ \) action \( \varphi_{\partial} \) on \( X \). Deduce that the action \( \varphi_{\partial} \) is trivial iff \( \partial = 0 \). Verify that the subalgebra \( A^\partial \) is algebraically closed in the algebra \( A \).

(7.2.) Let \( \partial \in \text{LND}(A) \setminus \{0\} \). Verify that the transcendence degree of the algebra extension \([A : A^\partial]\) is 1. More precisely, let \( r_0 \) be any element of \( A \) such that \( \partial r_0 \in A^\partial \) and \( r_0 \notin A^\partial \). Show that the subalgebra \( A^\partial[r_0] \subset A \) is a free \( A^\partial \) module, and for any element \( a \in A \) there exists an element \( b \in A^\partial \setminus \{0\} \) such that \( ba \in A^\partial[r_0] \).

(7.3.) Given a linear representation \( \varphi : \mathbb{C}_+ \rightarrow \text{GL}_n(\mathbb{C}), \) \( t \rightarrow \varphi \) \( e^{tB} \) where \( B \in L_n(\mathbb{C}) \), verify that it provides a regular \( \mathbb{C}_+ \) action on \( \mathbb{C}^n \) iff it is unipotent, i.e. iff \( B \) is a nilpotent matrix. Or, equivalently, iff the associated derivation \( \partial_{\varphi}(p) = \langle Bx, \text{grad} p \rangle \) of the polynomial algebra \( \mathbb{C}[x_1, \ldots, x_n] \) is locally nilpotent.

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(7.4.) Given a locally nilpotent derivation \( \partial \in \text{LND}(A) \) show that \( a\partial \in \text{LND}(A) \) for any \( \partial \) constant \( a \in A^0 \). Thus, \( A^0\partial \subset \text{LND}(A) \). Conclude that if \( A \) is an algebra over \( \mathbf{C} \) of \( \text{tr.deg~} A \geq 2 \) such that \( \text{LND}(A) \neq \{0\} \), then the automorphism group \( \text{Aut~} A \) is of infinite dimension. In particular, if an affine variety \( X \) with \( \dim~X \geq 2 \) admits a non-trivial regular \( \mathbf{C}_+ - \) action, then\(^2\) \( \dim \text{Aut}~X = \infty \).

(7.5.) Let \( \Gamma \) be an irreducible affine algebraic curve. Show that it admits a non-trivial regular \( \mathbf{C}_+ - \) action iff \( \Gamma \simeq \mathbf{C} \).

**Definition 7.1.** Let \( A \) be an algebra over \( \mathbf{C} \). The *Makar-Limanov invariant* \( \text{ML}(A) \) of the algebra \( A \) is the subalgebra \( \text{ML}(A) := \bigcap_{\partial \in \text{LND}(A)} A^0 \subset A \).

The *Derksen invariant* \( \text{Dk}(A) \) of the algebra \( A \) is the smallest subalgebra of \( A \) which contains \( A^0 \) for all \( \partial \in \text{LND}(A) \setminus \{0\} \).

Clearly, \( \text{ML}(\mathbf{C}^{[n]}) = \mathbf{C} \), and \( \text{Dk}(\mathbf{C}^{[n]}) = \mathbf{C}^{[n]} \).

**Theorem 7.3 (Derksen)\((\text{Dd})\).** Let, as above, \( A_0 \) denotes the algebra of regular functions on the Russell cubic threefold \( X_0 = \{x + x^2 y + z^2 + t^3 = 0\} \subset \mathbf{C}^4 \). Then \( \text{Dk}(A_0) \neq A_0 \). Hence, the algebra \( A_0 \) is not isomorphic to \( \mathbf{C}^{[3]} \). In turn, the Russell cubic \( X_0 \) is not isomorphic to \( \mathbf{C}^3 \), i.e. \( X_0 \) is an exotic \( \mathbf{C}^3 \).

Before proceeding with the proof, we recall the following notions (see e.g. \[\text{Bou}\]).

### 7.2 Degree functions, filtrations and the associated graded algebras

Let \( A \) be an integral domain (usually, it will be also an algebra over \( \mathbf{C} \)).

**Definition 7.2.** A *degree function* \( \deg : A \to \mathbf{Z} \cup \{-\infty\} \) on \( A \) is a mapping which satisfies the following axioms:

(d1) \( \deg 0 = -\infty \), and \( \deg a \in \mathbf{Z} \) for all \( a \neq 0 \); \( \deg 1 = 0 \).

(d2) \( \deg fg = \deg f + \deg g \) for all \( f, g \in A \).

(d3) \( \deg (f + g) \leq \max\{\deg f, \deg g\} \) for all \( f, g \in A \).

**Definition 7.3.** A degree function \( \deg \) determines an *ascending filtration* \( F = \{F_i A\} \) on \( A \) where \( F^0 A := \{a \in A \mid \deg a \leq 0\} \). This filtration satisfies the following conditions:

(f1) \( F^i A \) is a \( \mathbf{C} \)-linear subspace of \( A \), and \( F^i A \subset F^{i+1} A \) (ascending).

(f2) \( A = \bigcup_{i \in \mathbf{Z}} F^i A \) (exhaustive); \( \bigcap_{i \in \mathbf{Z}} F^i A = \{0\} \) (separated); \( 1 \in F^0 A \setminus F^{-1} A \).

(f3) \( (F^i A \setminus F^{i-1} A)(F^j A \setminus F^{j-1} A) \subset (F^{i+j} A \setminus F^{i+j-1} A) \).

Clearly, \( F^0 A \subset A \) is a subring (resp. subalgebra), and the ring \( A \) represents as an \( F^0 A \)-module.

Vice versa, given a filtered domain \( (A, F) \) which satisfies the conditions (f1)-(f3) one can define a degree function \( d_F \) on \( A \) as follows: \( d_F(0) = -\infty \) and \( d_F(a) = i \) iff \( a \in F^i A \setminus F^{i-1} A \).

**Definition 4.** The associated graded algebra \( \text{Gr} A = \bigoplus_{i \in \mathbf{Z}} \text{Gr}^i A \) of a filtered algebra \( (A, F) \) where \( \text{Gr}^i A := F^i A / F^{i-1} A \) can be identified with the algebra of the Laurent polynomials \( \{\sum_{i=k}^{k+i} f_i u^i\} \) where \( f_i \) is either zero or is equal to \( \text{gr} f_i := f_i + F^{i-1} A \in \text{Gr}^i A \) for

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\(^2\)This observation was communicated to us by L. Makar-Limanov.

\(^3\)In a similar way, one may define a degree function with values in arbitrary ordered semigroup.
some \( f_i \in F^i A \setminus F^{i-1} A \). Due to the property (f3) of filtrations, the mapping \( \text{gr} : A \to \text{Gr} A \), \( \text{gr} f = \hat{f} \), is a homomorphism of multiplicative semigroups.

**Definition 7.5.** A weight degree function on the polynomial algebra \( C^{[n]} \) is a degree function \( d \) such that \( d(p) = \max_i \{d(m_i)\} \) where \( p \in C^{[n]} \) is a non-zero polynomial, and \( m_i \) runs over the set \( M(p) \) of all the monomials of \( p \). Clearly, \( d \) is uniquely determined by the weights \( d_i := d(x_i), \ i = 1, \ldots, n \). A weight degree function \( d \) defines a grading \( C^{[n]} = \oplus_{j \in \mathbb{Z}} C^{[n]}_{d,j} \) where \( C^{[n]}_{d,j} \setminus \{0\} \) consists of all the \( d \)-quasihomogeneous polynomials of \( d \)-degree \( j \). Accordingly, for any \( p \in C^{[n]} \setminus \{0\} \) we have a unique decomposition \( p = \sum_{i=m(p)}^{d(p)} p_i \) into a sum of \( d \)-quasihomogeneous components; here \( p_d := p_d(p) \) is called the principal \( d \)-quasihomogeneous component of \( p \). It is clear that \( (pq)_d = p_aq_d \).

Let \( X = (I) \subset C^n \) be a reduced irreducible affine variety defined by a prime ideal \( I \subset C^{[n]} \). Denote \( A = C[X] = C^{[n]} / I \), and let \( \hat{I} \) be the (graded) ideal in \( C^{[n]} \) generated by the principal \( d \)-quasihomogeneous components \( p_d \) where \( p \) runs over \( I \). We say that the weight degree function \( d \) is appropriate for the ideal \( I \) if the following conditions hold:

(\*) \( \overline{I} \in X \), i.e. \( I \subset \alpha := (x_1, \ldots, x_n) \);
(\**\) the ideal \( \hat{I} \) is also prime, and \( x_i \notin \hat{I} \forall i = 1, \ldots, n \).

For \( f \in A \setminus \{0\} \) set

\[
 d_A(f) = \min_{p \in [f]} \{d(p)\} \quad \text{where} \quad [f] := \{p \in C^{[n]} \mid p \mid X = f\}.
\]

**Exercises.** (7.6.) Show that \( d_A(f) = d(p) \) for a polynomial \( p \in [f] \) iff \( p_d \notin \hat{I} \).

(7.7.) Assume that a weight degree function \( d \) on the polynomial algebra \( C^{[n]} \) is appropriate for an ideal \( I \subset C^{[n]} \). Deduce that \( d_A \) is a degree function on \( A \), and that \( d_A(\hat{x}_i) = d(x_i) = d_i \) where \( \hat{x}_i := x_i \mid X = x_i + I \in A, i = 1, \ldots, n \). Hence, due to the property (d2) of a degree function, \( d_A(m \mid X) = d(m) \) for any monomial \( m \in C^{[n]} \).

**Hint.** Suppose that \( f \in A \) and \( d_A(f) = -\infty \), that is, there exists a sequence of polynomials \( p_j \in C^{[n]}, j = 1, \ldots, \), such that \( p_j \mid X = f \) and \( \lim_{j \to \infty} d(p_j) = -\infty \). For \( p \in C^{[n]} \) set \( \mu(p) = \min_{m \in M(p)} \{\text{deg } m\} \) where \( \text{deg} \) is the usual degree. Then \( \mu \binom{[n]}{\alpha} \) where, as above, \( \alpha \subset C^{[n]} \) denotes the maximal ideal which corresponds to the origin of \( C^n \). By the condition (\*) from Definition 7.3, \( \overline{\alpha} := (\hat{x}_1, \ldots, \hat{x}_n) \subset A \) is a proper ideal, and we have \( f = p_j \mid X \in \overline{\alpha}^{\mu(p_j)}, j = 1, \ldots, \). Thus, by the Krull Theorem, \( f \in \bigcap_{n \in \mathbb{N}} \overline{\alpha}^n = \{0\} \), and so, \( f = 0 \). Hence, \( d_A(f) > -\infty \) for any \( f \in A \setminus \{0\} \).

The rest of the exercise, including checking of the other properties of a degree function, can be done without difficulty.

(7.8.) let \( F = \{F^i A\} \) be the filtration on \( A \) determined by the above degree function \( d_A \), and let \( \hat{A} = \text{Gr} A \) be the associated graded algebra. Verify that the elements \( \hat{x}_1, \ldots, \hat{x}_n \in \hat{A} \) where \( \hat{x}_i := \text{gr} \hat{x}_i \in \hat{A} \) generate the graded algebra \( \hat{A} \).

**Lemma 7.1 (Kaliman, Makar-Limanov\[^{26}\](KaML \[4\], Prop. 4.1)).** Keeping the same notation and assumptions as in the above exercises we have

\[
\hat{A} \simeq C^{[n]} / \overline{I} = C[\hat{X}]
\]

\[^{26}\]We place here this lemma and the preceding definition and exercises with a kind permission of Sh. Kaliman and L. Makar-Limanov.
where \( \hat{X} = (\hat{I}) \subset C^n \) is the affine variety defined by the prime ideal \( \hat{I} \).

**Proof.** According to Exercise 7.8, the elements \( \hat{x}_1, \ldots, \hat{x}_n \in \hat{A} \) generate the graded algebra \( \hat{A} \). Henceforth, \( \hat{A} = C[\hat{x}_1, \ldots, \hat{x}_n] \) is the ideal of relations between the generators \( \hat{x}_1, \ldots, \hat{x}_n \) in \( \hat{A} \). Thus, we must show that \( J = \hat{I} \).

Fix an arbitrary polynomial \( p = \sum_{i=m(p)}^{d(p)} p_i \in I \). Then \( p \equiv 0 \mod I \), i.e. \( p_d \equiv - \sum_{i=m(p)}^{d(p)-1} p_i \mod I \), and hence

\[
d_{A}(p_d) \leq \max_{m(p) \leq d(p)-1} \{d_{A}(p_i) \} \leq \max_{m(p) \leq i < d(p)-1} \{d(p_i) \} < d(p) = d(p_d).
\]

Therefore, \( p_d \mid X \in F^{d(p)-1}A \).

Since the weight degree function \( d \) is appropriate for the ideal \( I \), by Exercise 7.7, we have \( d_A(m_j \mid X) = d(m_j) = d(p) \) for any monomial \( m_j \in M(p_d) \). Thus, \( m_j \mid X \in F^{d(p)}A \setminus F^{d(p)-1}A \) for any \( m_j \in M(p_d) \), and \( p_d \mid X = \sum_{m_j \in M(p_d)} \{m_j \mid X \} \in F^{d(p)-1}A \). It follows that \( (m_j \mid X)^{\wedge} := \text{gr} (m_j \mid X) = m_j(\hat{x}_1, \ldots, \hat{x}_n) \in \hat{A}^{d(p)} \), and \( \sum_{m_j \in M(p_d)} (m_j \mid X)^{\wedge} = 0 \) in \( \hat{A}^{d(p)} \), i.e. \( p_d(\hat{x}_1, \ldots, \hat{x}_n) = 0 \) in \( \hat{A}^{d(p)} \). Whence, \( p_d \mid J \), and so, \( \hat{I} \subset J \).

Vice versa, fix an element \( f = \sum_{i=m(f)}^{d(f)} f_i \in J \). It is clear that \( f_i(\hat{x}_1, \ldots, \hat{x}_n) \in \hat{A}^i \) (indeed, as above, this is true for any monomial \( m \in M(f_i) \)). Since

\[
\sum_{i=m(f)}^{d(f)} f_i(\hat{x}_1, \ldots, \hat{x}_n) = f(\hat{x}_1, \ldots, \hat{x}_n) = 0
\]

we have \( f_i(\hat{x}_1, \ldots, \hat{x}_n) = 0 \) for each \( i = m(f), \ldots, d(f) \). Thus, \( J \) is a homogeneous ideal of the \( d \)-graded algebra \( C[\hat{X}] \) (see Definition 7.5 above). Hence, it is enough to show that \( J_r \subset \hat{I} \) for any \( d \)-homogeneous component \( J_r \) of \( J \).

Let \( f \in J_r \) be a \( d \)-quasihomogeneous polynomial of \( d \)-degree \( r = d(f) \). For any monomial \( m \in M(f) \) we have, as above, that \( m \mid X \in F^r A \setminus F^{r-1}A \), and so, \( m(\hat{x}_1, \ldots, \hat{x}_n) \in \hat{A}^r \). Since \( \sum_{m \in M(f)} m(\hat{x}_1, \ldots, \hat{x}_n) = f(\hat{x}_1, \ldots, \hat{x}_n) = 0 \), it follows that \( f \mid X \in F^{r-1}A \), i.e. \( d_A(f \mid X) < r = d(f) \). By Exercise 7.8, this implies that \( f_d \in \hat{I} \). But \( f = f_d \), and so, we are done. \( \square \)

**Gradings and \( C^* \)-actions** (see e.g. [KamRu], [Ru 3]). Let \( \hat{X} \) be an affine variety endowed with a \( C^* \)-action \( t \). Then \( t \) induces a grading \( \hat{A} = \oplus_{n \in Z} \hat{A}^n \) on the algebra \( \hat{A} = C[\hat{X}] \) of regular functions on \( \hat{X} \) where \( \hat{A}^n := \{ f \in \hat{A} \mid f \circ t_{\lambda} = \lambda^n f \} \) consists of the quasi-invariants of weight \( n \) of \( t \).

Vice versa, given a grading \( \hat{A} = \oplus_{n \in Z} \hat{A}^n \) of \( \hat{A} = C[\hat{X}] \), one can define a \( C^* \)-action on \( \hat{A} \) by setting \( t_\lambda(f_n) = \lambda^n f_n \) for \( f_n \in \hat{A}^n \), \( n \in Z \), and extending it to the whole \( \hat{A} \) in a natural way. If \( \hat{A} \) is finitely generated, then it also has a finite system of homogeneous generators \( (f_{n_1}, \ldots, f_{n_k}) \), \( f_{n_i} \in \hat{A}^{n_i} \). The morphism \( F = (f_{n_1}, \ldots, f_{n_k}) : \hat{X} \hookrightarrow C^k \) is an embedding equivariant with respect to the linear \( C^* \)-action \( t_{\lambda}(x_1, \ldots, x_k) = (\lambda^{n_1}x_{n_1}, \ldots, \lambda^{n_k}x_{n_k}) \) on \( C^k \) and the induced \( C^* \)-action on \( \hat{X} \).

**Gradings and locally nilpotent derivations** (see e.g. [ML 2], [KaML 3], [Dr]).

**Definition 7.6.** Let \( \partial \in LND(A) \setminus \{0\} \) where \( (A, F) \) is a filtered domain. Suppose that (*) there exists \( k \in Z \) such that \( \partial F^i A \subset F^{i+k} A \) for all \( i \in Z \).
Denote by \( \deg \partial = k_0 \) the minimal such \( k \). Define \( \hat{\partial} = \gr \partial : \Gr A \rightarrow \Gr A \) as follows: for \( f \in F^i A \setminus F^{i-1} A \), set \( \hat{\partial} f = \partial f + F^{i+k_0-1} A \), and then naturally extend \( \hat{\partial} \) to the whole algebra \( \Gr A \).

**Exercises. (7.9.)** Given \( \partial \in \LND(A) \setminus \{0\} \), verify that \( \hat{\partial} \in \LND_{\gr}(\Gr A) \setminus \{0\} \) where \( \LND_{\gr}(A) \) denotes the set of all homogeneous locally nilpotent derivations of a graded algebra \( A = \oplus_{n \in \mathbb{Z}} A^n \).

(7.10.) Suppose that a filtered domain \( (A, F) \) is finitely generated. Show that, given \( \partial \in \LND(A) \setminus \{0\} \), the condition \((*)\) of Definition 7.4 above is fulfilled.

(7.11.) Let \( \hat{A} = \oplus_{n \in \mathbb{Z}} \hat{A}^n \) be a graded algebra. Show that, given any locally nilpotent derivation \( \hat{\partial} \in \LND_{\gr}(\hat{A}) \), there exists \( k_0 = k_0(\hat{\partial}) \in \mathbb{Z} \) called the *degree* of \( \hat{\partial} \) such that \( \hat{\partial}(\hat{A}^n) \subset \hat{A}^{n+k_0} \). Furthermore, show that, if \( a = \sum_{i=k}^{l} a_i \in \Ker \hat{\partial} = \hat{A}^0 \) where \( a_i \in \hat{A}^i \), then \( a_i \in \hat{A}^0, \ i = k, \ldots, k + l. \) Therefore, \( \hat{A}^0 \) is a graded subalgebra of the graded algebra \( \hat{A} \).

(7.12.) Let \( A \) be an integral domain. Given \( \partial \in \LND(A) \), set \( \deg_{\partial} a = n \) if \( \partial^{n+1}a = 0 \) and \( \partial^n a \neq 0; \ deg_{\partial} 0 = -\infty \). Verify that:

(a) \( \deg_{\partial} \) is a degree function on \( A \) over \( \mathbb{N} \);
(b) if \( A \) is a \( \mathbb{C} \)-algebra, then \( \partial \lambda = 0 \) for any \( \lambda \in \mathbb{C} \);
(c) the equality \( \partial(ab) = 0 \) where \( a, b \in A \setminus \{0\} \) implies that \( \partial a = \partial b = 0 \);
(d) the equality \( \partial(a^k + b^l) = 0 \) where \( a^k + b^l \neq 0 \) and \( k, l \geq 2 \) implies that \( \partial a = \partial b = 0 \).

(7.13.) Let \( A = \mathbb{C}[x, y] \). Consider the \( \mathbb{C} \)-action \( \varphi_x : (x, y) \mapsto (x, y + \lambda x^2) \) on \( \mathbb{C}^2 \). Let \( \partial_{\varphi} \) be the locally nilpotent derivation which corresponds to \( \varphi \). Show that \( \partial_{\varphi}(x) = 0, \partial_{\varphi}(y) = x^2 \) and \( \partial_{\varphi}^2(y) = 0 \). Deduce that \( \deg_{\partial_{\varphi}} x = 0, \ deg_{\partial_{\varphi}} y = 1 \) for the associated degree function \( \deg_{\partial_{\varphi}} \) on \( A \), and so, that \( \deg_{\partial_{\varphi}} f = \deg_y f \) for any polynomial \( f = f(x, y) \in A \).

**The Brody hyperbolicity.** Recall that a complex manifold \( M \) which does not contain entire curves, that is, does not admit non-constant holomorphic mappings \( \mathbb{C} \rightarrow M \), is called *Brody hyperbolic*. By analogy with this, let us introduce the following notion.

**Definition 7.7.** We say that a quasi-projective variety \( X \) is *algebraically Brody hyperbolic* if any morphism \( \mathbb{C} \rightarrow X \) is constant. A regular function \( f \) on \( X \) is *algebraically Brody hyperbolic* if so are its generic fibres \( f^{-1}(c), \ c \in \mathbb{C} \).

**Exercises. (7.14.)** Show that any non-constant polynomial of one variable is Brody hyperbolic. Verify that the polynomials \( xy \) and \( x^k + y^l \in \mathbb{C}^2 \) where \( k, l \geq 2 \) are Brody hyperbolic.

(7.15.) More generally, show that a polynomial \( p \in \mathbb{C}^2 \) fails to be Brody hyperbolic iff it can be linearized, that is, \( p \circ \alpha = x \) for some automorphism \( \alpha \in \Aut \mathbb{C}^2 \).

The next lemma is inspired by Exercises 7.3 and 7.12 (c), (d). It will be used in the proof of Derksen’s Theorem 7.3 in section 7.3 below.

**Lemma 7.2.** (a) Let \( X \) be a quasi-projective variety, and let \( f \in A := \mathbb{C}[X] \) be a Brody hyperbolic regular function on \( X \). Then \( f \) cannot be a constant of a locally nilpotent derivation \( \partial \in \LND(A) \).

(b) Let \( A \) be an integral domain over \( \mathbb{C} \) with the unit element \( e \in A \), and let \( p \in \mathbb{C}^2 \) be a Brody hyperbolic polynomial. If for some elements \( a_1, \ldots, a_k \in A \) we have \( b := p(a_1, \ldots, a_k) \in A^0 \setminus e A \), then \( a_1, \ldots, a_k \in A^0 \).

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27This observation is due to L. Makar-Limanov; it is essential in his new approach to the Theorem 7.1 ML 3.
Proof. (a) Assume the contrary. Being an invariant of the associated \( C_+ \) action \( \varphi_0 \) on \( X \) (see Exercise \ref{exercise:7.1} above) the function \( f \) is constant on the orbits of this action. Thus, generic \( \varphi_0 \)– orbits are \( C^- \)– curves contained in the fibres of \( f \), in contradiction with the assumption that the function \( f \) is Brody hyperbolic.

(b) Let \( A' \) be the subalgebra of the algebra \( A \) generated by the elements \( a_1, \ldots, a_k \) and all its successive \( \partial \)– derivatives. It is finitely generated (henceforth, this is an affine domain) and \( \partial \)– invariant. Consider the associated \( C_+ \)– action \( \varphi_0 \) on the affine variety \( X' := \text{spec} A' \). By the assumption, the nonconstant regular function \( b = p(a_1, \ldots, a_k) \in A' = C[X'] \) is \( \varphi_0 \)– invariant, and so, it is constant along the \( \varphi_0 \)– orbits. Therefore, the image \( \alpha(O_z) \) of the \( \varphi_0 \)– orbit of a generic point \( x \in X' \) under the regular mapping \( \alpha : X' \to C^k \), \( x \mapsto (a_1(x), \ldots, a_k(x)) \), is contained in a generic fibre \( p^{-1}(c) \) of the polynomial \( p \). Since this polynomial is assumed to be Brody hyperbolic, the mapping \( \alpha \) is constant along the generic \( \varphi_0 \)– orbits. It follows that its coordinate functions \( a_1, \ldots, a_k \) are \( \varphi_0 \)– invariants. Whence, \( a_1, \ldots, a_k \in A^0 \) (see again Exercise \ref{exercise:7.1}). \( \square \)

Remarks. 7.1. Suppose that under assumptions of Lemma \ref{lemma:7.2} (b), all the fibres of the polynomial \( p \) except the zero one \( p^{-1}(0) \) are Brody hyperbolic. Then the conclusion of the lemma holds if one replaces the condition \( b \in A^0 \setminus \text{Ce} \) by the weaker one \( b \in A^0 \setminus \{0\} \).

7.2. In Exercises \ref{exercise:7.1}, \ref{exercise:7.12} (c), (d) one can make use of Lemma \ref{lemma:7.2} (b), of the previous remark and of Exercise \ref{exercise:7.14}.

7.3 Gradings and LND’s on Russell’s cubic

In this section we provide a proof of Theorem \ref{thm:7.3}. We use the following notation.

**Notation.** From now on \( A = A_0 = C[x, y, z, t]/(p_0) \) where \( p_0 = x + x^2y + z^3 + t^2 \), will be the algebra of regular functions on the Russell cubic threefold \( X_0 \subset \mathbb{C}^4 \). Consider the weight degree function \( \deg x = -1, \deg y = 2, \deg z = \deg t = 0 \) on the polynomial ring \( C^4 \). It is easily seen that \( \deg \) is appropriate for the principal ideal \( I := (p_0) \) (see Definition \ref{def:7.5}). Hence, by Exercise \ref{exercise:7.7} it induces a degree function \( d_A \) on \( A \) which, in turn, defines a filtration on \( A \) (in what follows we use the notation \( \deg \) instead of \( d_A \)). Let \( \hat{A} := \text{Gr} A \) be the associated graded algebra. From Lemma \ref{lemma:7.2} we obtain such a corollary.

**Corollary 7.1.** \( \hat{A} \cong C[\bar{x}, \bar{y}, \bar{z}, \bar{t}]/(q_0) \) where \( q_0(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \bar{x}^2\bar{y} + \bar{z}^3 + \bar{t}^2 \), i.e. \( \hat{A} = C[\bar{X}_0] \), where \( \bar{X}_0 = \{\bar{x}^2\bar{y} + \bar{z}^3 + \bar{t}^2 = 0\} \).

The next lemma exploits the specific equation of the Russell cubic.

**Lemma 7.3** \ref{lemma:7.3}. (a) Any element \( f \in A \) has a unique presentation of the form

\[
f = a(x, z, t) + yb(y, z, t) + xc(y, z, t) | X
\]

where \( a, b, c \) are polynomials.

(b) We have:

(i) \( \hat{f} = \hat{x}^rh(\bar{z}, \bar{t}) \) iff \( \deg f = r \leq 0 \);

(ii) \( \hat{f} = \hat{y}^rh(\bar{z}, \bar{t}) \) iff \( \deg f = 2r > 0 \);

(iii) \( \hat{f} = \hat{x}\hat{y}^rh(\bar{z}, \bar{t}) \) iff \( \deg f = 2r - 1 > 0 \),

where \( h(\bar{z}, \bar{t}) \in C[\bar{z}, \bar{t}] \).

**Proof of** (b). Since the degree \( \deg [yb(y, z, t)] \) is even and positive when \( b \neq 0 \), \( \deg [xc(y, z, t)] \) is odd and positive when \( c \neq 0 \), and \( \deg a(x, z, t) \leq 0 \), we have that in the case (i)
\( f = a(x, z, t) \) and hence \( \hat{f} = \hat{x}^r h(\hat{z}, \hat{t}) \); in the case (ii) \( \hat{f} = \text{gr} [yb(y, z, t)] = \hat{y}^r h(\hat{z}, \hat{t}) \); finally, in the case (iii) \( \hat{f} = \text{gr} [yc(y, z, t)] = \hat{x}\hat{y}^r h(\hat{z}, \hat{t}) \) for some \( h(\hat{z}, \hat{t}) \in \mathbb{C}[\hat{z}, \hat{t}] \).

\[ \square \]

**Corollary 7.2.** We have:

\( \hat{A}^0 = \mathbb{C}[\hat{z}, \hat{t}] \), and thus \( \hat{A} \) is a \( \mathbb{C}[\hat{z}, \hat{t}] \)-module;

\( \hat{A}^{i} = \hat{x}^{-i} \mathbb{C}[\hat{z}, \hat{t}] \) for \( i \leq 0 \);

\( \hat{A}^{2r} = \hat{y}^r \mathbb{C}[\hat{z}, \hat{t}] \) for \( r > 0 \), and

\( \hat{A}^{2r-1} = \hat{x}\hat{y}^r \mathbb{C}[\hat{z}, \hat{t}] \) for \( r > 0 \).

**Lemma 7.4.** Consider a surface \( S_c = \{ c\hat{x}^2 + \hat{z}^2 + \hat{t}^3 = 0 \} \subset \mathbb{C}^3 \) where \( c \in \mathbb{C}^* \). Then for any nonconstant polynomial \( h \in \mathbb{C}[z, t] \) the regular function \( f := h|_{S_c} \in \mathbb{C}[S_c] \) on the surface \( S_c \) is Brody hyperbolic.

**Proof.** We may assume that the generic fibres of the polynomial \( h \) are irreducible plane curves. Indeed, otherwise the polynomial \( h \) has the Stein factorization \( h = \varphi \circ h_1 \) where \( \varphi \in \mathbb{C}[z] \), \( \deg \varphi \geq 2 \), and \( h_1 \in \mathbb{C}[z, t] \) is a polynomial with irreducible generic fibres, and we may replace \( h \) by the polynomial \( h_1 \).

Assume, on the contrary, that for some \( c \neq 0 \) the regular function \( f = h|_{S_c} \) is not Brody hyperbolic, that is, its generic fibre \( F_\lambda := f^{-1}(\lambda) \subset S_c \) admits a nonconstant morphism \( \mathbb{C} \to F_\lambda \). It follows that an irreducible component, say, \( F'_\lambda \) of the curve \( F_\lambda \) is a smooth affine curve isomorphic to \( \mathbb{C} \) (see Exercise 7.3 above).

The curve \( F'_\lambda \) is represented as a two-sheeted ramified covering of \( C_\lambda \) under the projection \( \pi : S_c \to \mathbb{C}^2 \), \( (\hat{x}, \hat{z}, \hat{t}) \mapsto (\hat{z}, \hat{t}) \). Thus, the irreducible affine plane curve \( C_\lambda \) admits a dominant morphism \( \mathbb{C} \simeq F'_\lambda \to C_\lambda \), and hence, it is isomorphic to \( \mathbb{C} \), too.

We have the following alternative: either

- The generic fibre \( F_\lambda \) of the regular function \( f \) is irreducible, i.e. \( F_\lambda = F'_\lambda \simeq \mathbb{C} \), and then \( p := \pi \mid F'_\lambda : F'_\lambda \simeq \mathbb{C} \to C_\lambda \simeq \mathbb{C} \) is a two-sheeted branched covering; or

- The generic fibre \( F_\lambda \) is reducible, and then the mapping \( p \) as above is univalent, and hence, isomorphic.

In the first case a generic curve \( C_\lambda \) meets the ramification locus \( \Gamma_{2,3} = \{ \hat{z}^2 + \hat{t}^3 = 0 \} \subset \mathbb{C}^2 \) of the projection \( \pi \) at one point which corresponds to the only critical value of the quadratic polynomial \( p \in \mathbb{C}[u] \). Therefore, the restriction \( h_1 \mid_{\Gamma_{2,3}} \to \mathbb{C} \) is generically one-to-one, which is impossible.

In the second case a generic curve \( C_\lambda \) does not meet the ramification locus \( \Gamma_{2,3} \) at all, and then it should be contained in an elliptic curve \( \hat{z}^2 + \hat{t}^3 = \text{const} \neq 0 \), which is impossible either. This completes the proof.

\[ \Box \]

Lemmas 7.2 (a) and 7.4 yield the following corollary.

**Corollary 7.3.** The surface \( S_c \) where \( c \neq 0 \) does not admit a non-trivial \( \mathbb{C}^* \)-action with a non-constant invariant function \( h(\hat{z}, \hat{t}) \).

**Lemma 7.5.** For any nonzero locally nilpotent derivation \( \hat{\partial} \in \text{LND}_A(\hat{A}) \) and for any homogeneous element \( \hat{f} \in \hat{A}^n \) the equality \( \hat{\partial}\hat{f} = 0 \) implies that \( n := \deg \hat{f} \leq 0 \).

**Proof.** Assume, on the contrary, that \( n > 0 \). Suppose first that \( n \) is odd, i.e. \( n = 2r - 1 \) for some \( r \in \mathbb{N} \). Then, by Lemma 7.3 (b), we have that \( \hat{f} = \hat{x}\hat{y}^r h(\hat{z}, \hat{t}) \) for some non-zero

\[ \text{We give a simplified proof suggested by Sh. Kaliman.} \]
polynomial \( h(\hat{z}, \hat{t}) \in \mathbb{C}[\hat{z}, \hat{t}] \), and the equality \( \hat{\partial} \hat{f} = 0 \) implies that \( \hat{\partial} \hat{x} = \hat{\partial} \hat{y} = \hat{\partial} h = 0 \) (see Exercise 7.12 (c) above). Thus, \( \hat{x}, \hat{y} \in \hat{A}^\partial \) are invariants of the associated \( C_+ \) action \( \varphi = \varphi_{\hat{y}} \) on \( \hat{X}_0 = \{ \hat{z}^2 \hat{y} + \hat{z}^3 + \hat{t}^2 = 0 \} \) (see Exercise 7.1 above). Therefore, each orbit of the \( C_+ \) action \( \varphi \) is contained in a curve \( \Gamma_{c_1,c_2} = \{ \hat{x} = c_1, \hat{y} = c_2 \} \subset \hat{X}_0 \). Conversely, any such curve in \( \hat{X}_0 \) consists of \( C_+ \) orbits. Since a generic curve \( \Gamma_{c_1,c_2} = \{ \hat{z}^3 + \hat{t}^2 = -c_2^3 c_2 \} \) is elliptic, it does not admit an embedding of \( C \) (see Exercise 7.3 above), and hence all the points of \( \Gamma_{c_1,c_2} \) are fixed by the \( C_+ \) action \( \varphi \). It follows that the \( C_+ \) action \( \varphi \) on \( \hat{X}_0 \) is trivial, i.e. \( \hat{\partial} = 0 \), a contradiction.

Now consider the case when \( n > 0 \) is even, i.e. \( n = 2r \) for some \( r \in \mathbb{N} \). In this case by Lemma 7.3 (b), we have \( \hat{f} = \hat{g}^r h(\hat{z}, \hat{t}) \) where \( h(\hat{z}, \hat{t}) \in \mathbb{C}[\hat{z}, \hat{t}] \) and \( h \neq 0 \). Hence, \( \hat{\partial} \hat{g} = 0 \), and so \( \hat{g} \in \hat{A}^\partial \) is an invariant of the \( C_+ \) action \( \varphi \) on \( \hat{X}_0 \) (see Exercise 7.4). Thus, \( \hat{X}_0 \) is foliated by the \( \varphi \)-invariant surfaces \( S_c = \{ \hat{y} = c \}, c \in \mathbb{C} \). Denote by \( \hat{\partial}_c \in \text{LND}(A_c) \) the corresponding locally nilpotent derivation on \( A_c = \mathbb{C}[S_c] \), that is, the infinitesimal generator of the \( C_+ \) action \( \varphi_c := \varphi |_{S_c} \) on \( S_c \). For a generic \( c \in \mathbb{C} \), \( \varphi_c \) is a non-trivial \( C_+ \) action on \( S_c \), whence \( \hat{\partial}_c \neq 0 \).

Next we show that there exists a non-constant \( \varphi \)-invariant function \( h_1(\hat{z}, \hat{t}) \in \hat{A}^\partial \). Indeed, since \( \text{tr.deg} \hat{A}^\partial = \text{tr.deg} \hat{A} - 1 = 2 \) (see Exercise 7.2 above), the subalgebra \( \hat{A}^\partial \) contains a function \( g \) such that \( \hat{g} \) and \( \hat{g}^r \) are algebraically independent; in particular, \( \hat{g} \notin \mathbb{C}[\hat{y}] \). Furthermore, \( \hat{g} \) and \( \hat{y} \) are both \( \varphi \)-invariants, and so, for \( s \in \mathbb{N} \) sufficiently large \( \hat{g}^s \hat{y}^s \) is a \( \varphi \)-invariant of a positive degree. We have proven above that the equality \( \hat{g} \hat{y}^r = \hat{x} \hat{y}^r h_1(\hat{z}, \hat{t}) \) is impossible. Hence, we get \( \hat{A}^\partial \ni \hat{g} \hat{y}^r = \hat{g} h_1(\hat{z}, \hat{t}) \) for some \( r > 0 \) where \( h_1 \in \hat{A}^\partial \) is non-constant. Finally, the restriction of the polynomial \( h_1 \) onto a generic surface \( S_c \) is a nonconstant invariant of the non-trivial \( C_+ \) action \( \varphi_c \) on \( S_c \), which contradicts to Corollary 7.3. The proof is completed.

**Corollary 7.4.** (a) \( A^\partial \subset F^0 A \) for any non-zero locally nilpotent derivation \( \partial \in \text{LND}(A) \).
(b) \( \text{Dk}(A) \subset F^0 A \neq A \).

**Proof.** The statement of (b) follows from (a) in virtue of Definition 7.1. To prove (a) assume the contrary, i.e. that for some \( f \in A \) where \( \deg f > 0 \) and for some \( \partial \in \text{LND} \setminus \{0\} \) we have \( \partial f = 0 \). Then \( \deg \hat{f} > 0 \) as well, and \( \hat{\partial} \hat{f} = 0 \) (see Definition 7.6 and Exercise 7.3). It remains to apply Lemma 7.5. \( \square \)

Now the proof of Theorem 7.3 is completed.

## 8 Concluding remarks

In this section as in section 4.2 above we mainly follow [KaZa]. To begin with, remind the following problem.

**Generalized Serre Problem:**

Let \( X \) be a smooth contractible affine variety. Is any algebraic vector bundle on \( X \) trivial?

Due to the Quillen-Suslin Theorem, this is true for the affine spaces.

The next question suggested by A. Beilinson generalizes the Zariski Cancellation. Following [BaWa], sect. 4], we say that a closed subvariety \( Y \) of an affine variety \( X \) is a *retract* of \( X \) if there exists a morphism \( f : X \to Y \) such that \( f | Y = \text{id}_Y \). An abstract variety \( Y \)
is called a retract of $X$ if the image of $Y$ under some proper embedding $Y \hookrightarrow X$ is a retract of $X$.

Is it true that an affine variety which is a retract of an affine space is isomorphic to an affine space?

More generally, let $R$ be a commutative ring and $A$ be an $R$-algebra such that any surjective map of $R$-algebras $B \longrightarrow A$ admits a right inverse. Is it true that $A$ is isomorphic to $\text{Sym}_P$ where $P$ is a projective $R$-module?

**Exercises.** (8.1.) Show that if $X$ is a retract of an affine space, then it is a smooth contractible affine variety, $\overline{k}(X) = -\infty$, and, moreover, any algebraic vector bundle on $X$ is trivial.

(8.2.) Assume that a variety $X$ is a retract of an affine space. Show that then $X$ is a retract of any affine variety which contains a copy of $X$. Deduce that an affine space is a retract of any affine variety which contains a copy of it. In particular, if a subvariety $X \subset C^n$ is isomorphic to $C^{k}$, then $X$ is a retract of $C^n$ (under the given embedding).

(8.3.) Let $f : X \hookrightarrow C^n$ and $g : X \hookrightarrow C^k$ be two closed embeddings. Verify that if $f(X)$ is a retract of $C^n$, then also $g(X)$ is a retract of $C^k$.

**Hint.** Apply the Jelonek-Kaliman-Nori-Srinivas Theorem on equivalence of embeddings cited in the introduction.

(8.4.) Find an example of a pair of affine varieties $X$ and $Y$ and a pair of proper embeddings $f, g : Y \hookrightarrow X$ such that the image $f(Y)$ is a retract of $X$ whereas the image $g(Y)$ is not.

**Notation.** Let $X$ be a smooth affine $n$-fold, and $A = C[X]$ be the algebra of regular functions on $X$. Set $ML(X) = ML(A)$ and $Dk(X) = Dk(A)$ (see Definition 7.1).

We would like to know when these invariants are nontrivial. For instance,

Is it true that $ML(X_0 \times C) \neq C$ where $X_0$ is the Russell cubic?

It is unknown whether this product is, indeed, an exotic $C^4$.

**Remark 8.1.** In general, the Makar-Limanov invariant is not invariant under cancellation. Indeed, consider the Danielewski surfaces $D_n = \{x^3y + z^2 = 1\}$ in $C^3$ (see Introduction), and set $X_n = D_n \times C$. Then we have: $X_1 = \cong X_2 \text{ [Dan]}$, and so, $ML(X_1) = ML(X_2)$, whereas $ML(D_2) \neq ML(D_1) = C$ (see [ML 1, ML 4], and also Corollary 8.1 below).

**Exercises.** (8.5.) Verify that $D_1 \cong (P^1 \times P^1) \setminus \Delta$ where $\Delta \subset P^1 \times P^1$ is the diagonal. Find the minimal SNC-compactifications for the other Danielewski surfaces $D_n$ and their dual graphs.

(8.6.) Deduce that the fundamental groups at infinity $\pi_1^{\infty}(D_n)$ are pairwise non-isomorphic, and hence the surfaces $D_n$ are pairwise non-homeomorphic. What are the homology groups $H_*(D_n; Z)$? (see Fieseler [F], tom Dieck [D 3].)

(8.7.) Show that $\overline{k}(X) = -\infty$ for any smooth quasiprojective variety $X$ which admits a non-trivial regular $C_+-$ action. Deduce that $\overline{k}(D_n) = -\infty$ for all $n$. Observe that the latter conclusion also follows by the Iitaka-Fujita Strong Cancellation Theorem [F].

**Notation.** To any regular $C_+-$ action $g$ on $X$ there corresponds a one-parameter algebraic subgroup $G$ of the automorphism group $\text{Aut} X$. Denote $\text{Aut}_+ X$ the subgroup of the group $\text{Aut} X$ generated by all such $C_+-$ subgroups.

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29 This observation is due to T. Bandman and L. Makar-Limanov.
The hypersurface $H$, over $Y$ the variety $X$ and let $A$ Suppose that an affine algebra $A$ the group $\text{Aut}_+ \mathbb{C}^n$ acts $m-$ transitively on $\mathbb{C}^n$, i.e. it is transitive on the set of all $m-$ tuples of distinct points in $\mathbb{C}^n$, whatever $m \in \mathbb{N}$ is. In such a case we say that the action is infinitely transitive.

Clearly, the Makar-Limanov invariant is trivial, i.e. $ML(X) = \mathbb{C}$, if the group $\text{Aut}_+ X$ is transitive on a variety $X$ or, at least, has a dense orbit. The next surprising examples were found in [ML 1] for $k = 1$ and in [KaZa] for $k > 1$.

**Theorem 8.1** [ML 1] [KaZa] Thm. 6.1. Let $p \in \mathbb{C}^{[k]}$, $k \geq 1$, be a non-constant polynomial, and let $X_p \subset \mathbb{C}^k$, resp. $Y_p \subset \mathbb{C}^{k+2}$, be the hypersurface given by the equation $p(\bar{x}) = 0$ resp. $uv - p(\bar{x}) = 0$ where $\bar{x} = (x_1, \ldots, x_k)$. Then the following statements hold.

(a) The hypersurface $Y_p$ is smooth iff $X_p$ is a smooth reduced hypersurface. If so, then the variety $Y_p$ is simply connected, and there is an isomorphism of the reduced homology groups $H_*(Y_p; \mathbb{Z}) \simeq H_{-2}(X_p; \mathbb{Z})$.

(b) The group $\text{Aut}_+ Y_p$ is transitive on the smooth part $\text{reg} Y_p$ of the variety $Y_p$; moreover, for $k > 1$ this group is infinitely transitive on $\text{reg} Y_p$.

**Corollary 8.1.** (a) Suppose that the hypersurface $X_p \subset \mathbb{C}^k$ is smooth and reduced. Then the hypersurface $Y_p \subset \mathbb{C}^{k+2}$ is contractible iff the variety $X_p$ is acyclic.

(b) We have $\text{ML}(Y_p) = \mathbb{C}$.

**Remarks.** 8.2. There is such a question:

Suppose that an affine algebra $A$ over $\mathbb{C}$ possesses $n = \text{tr.deg} A$ locally nilpotent derivations which are linearly independent over $A$. Does it follow that $A \simeq \mathbb{C}^{[n]}$?

The answer is negative. Indeed, following the lines of the proof of Theorem 6.1 in [KaZa] it is easily seen that smooth non-contractible varieties $Y_p$ as above provide corresponding counterexamples.

However, we do not know what is the answer under a stronger assumption that the corresponding regular vector fields on the variety $X := \text{spec} A$ are linearly independent at any point of $X$. It is positive provided that, in addition, the given locally nilpotent derivations pairwise commute. Indeed, then the variety $X$ is the unique orbit of the associated regular free action of the additive group $\mathbb{C}^n_+$ on $X$.

8.3. For $k = 1$ the only example of a smooth contractible surface of type $Y_p$ is the affine plane $\mathbb{C}^2$. Furthermore, as follows from the Abhyankar-Moh and Suzuki Theorem [AM] [Suz 1] (see Theorem 2.7 (a) above), for $k = 2$ every smooth contractible $3-$fold in $\mathbb{C}^4$ of type $Y_p$ is isomorphic to the affine space $\mathbb{C}^3$, and the embedding $\mathbb{C}^3 \simeq Y_p \hookrightarrow \mathbb{C}^4$ is rectifiable (see [KaZa], Prop. 5.2 (b) ).

Starting with $k = 3$ one can obtain a number of new examples of smooth contractible hypersurfaces of type $Y_p$ in $\mathbb{C}^{k+2}$ choosing for $X_p$ smooth acyclic (not necessarily contractible) hypersurfaces in $\mathbb{C}^k$ (see e.g. Example 5.2 above of such a surface $X_p = Y_{k,t,s} \subset \mathbb{C}^3$). Presumably, among these hypersurfaces $Y_p$ there are examples of exotic $\mathbb{C}^n$, $n \geq 4$, with infinitely transitive automorphism groups (for instance, the hypersurface $uv - p_{k,t}(x, y, z) = 0$ in $\mathbb{C}^5$ where $p_{k,t} \in \mathbb{C}^[[3]]$ is a tom Dieck-Petrie polynomial should be such one). This would provide

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[30] At the same time, by Theorem 8.1 (b), the variety $Y_p$ may have rather non-trivial topology.
For which polynomials \( p \in \mathbb{C}[k] \) the variety \( Y_p \) is an exotic \( \mathbb{C}^{k+1} \)?

Corollary 8.1 (b) above shows that the Makar-Limanov invariant does not supply an answer to this question.

8.4. Furthermore, these examples are interesting in connection with the Abhyankar-Sathaye Embedding Problem (see Introduction). Indeed, consider the regular function \( f := u \mid Y_p \in \mathbb{C}[Y_p] \). For \( c \neq 0 \) the level hypersurface \( U_c = f^{-1}(c) \) is isomorphic to \( \mathbb{C}^k \); further, the complement \( Y_p \setminus U_0 \) is isomorphic to the cylinder \( \mathbb{C}^* \times \mathbb{C}^k \). At last, the hypersurface \( U_0 \) is isomorphic to the product \( X_p \times \mathbb{C} \). Actually, the hypersurface \( U_0 \) coincides with the exceptional divisor \( E' \) of the affine modification

\[
\sigma : Y_p \to \mathbb{C}^{k+1}, \quad (\bar{x}, u, v) \mapsto (\bar{x}, v),
\]

of the affine space \( \mathbb{C}^{k+1} \) along the hyperplane \( D := \{ v = 0 \} \) with center \( X_p \subset D \) (see Example 4.2). The following question arises:

Is it true that \( Y_p \simeq \mathbb{C}^{k+1} \) iff \( X_p \simeq \mathbb{C}^{k-1} \)?

It has a direct relation to the Zariski Cancellation Problem. Namely, it would be useful to know

Whether the implication \( X \times \mathbb{C} \simeq \mathbb{C}^k \implies X \simeq \mathbb{C}^{k-1} \) holds at least for hypersurfaces \( X \) in \( \mathbb{C}^k \)?

8.5. Assume that for some polynomial \( p \in \mathbb{C}[k] \) we have \( U_0 = X_p \times \mathbb{C} \not\simeq \mathbb{C}^k \) whereas \( Y_p \simeq \mathbb{C}^{k+1} \). Then, evidently, the embedding \( \mathbb{C}^k \simeq U_1 \hookrightarrow Y_p \simeq \mathbb{C}^{k+1} \) is not rectifiable, thus providing a counterexample to the Abhyankar-Sathaye Embedding Problem. Otherwise, i.e. in the case where \( U_0 \not\simeq \mathbb{C}^k \) and \( Y_p \not\simeq \mathbb{C}^{k+1} \), we obtain an example showing that the analog of the Miyanishi Theorem 6.3 does not hold in dimension \( n = k + 1 \) (clearly, here \( n \geq 4 \)).

The following fact, related to the Abhyankar-Sathaye Embedding Problem, was established in Sathaye [Sath] and Wright [Wr 2] in the case where \( X \simeq \mathbb{C}^2 \), and in [KaZa] in the present more general form.

**Theorem 8.2** [Sath, Wr 2, KaZa]. Let \( X = X_{n,f,g} \) be an irreducible smooth surface in \( \mathbb{C}^3 \) given by the equation \( f(x, y)z^n + g(x, y) = 0 \) where \( f, g \in \mathbb{C}[x, y], n \in \mathbb{N} \). Then the following conditions (i) – (iv) are equivalent:

(i) \( e(X) = 1 \) and \( H_1(X; \mathbb{Z}) = 0 \) where \( e(X) \) denotes the Euler characteristic of the variety \( X \).

(ii) The surface \( X \) is acyclic, i.e. \( \tilde{H}_*(X; \mathbb{Z}) = 0 \).

(iii) \( X \simeq \mathbb{C}^2 \).

(iv) The surface \( X \) is rectifiable, i.e. it can be transformed into a plane by an automorphism of \( \mathbb{C}^3 \).

For \( n > 1 \) these conditions are equivalent also to the following one:

(v) The pair \( (f, g) \) is rectifiable, i.e. it can be transformed into a pair \( (\alpha(f), \alpha(g)) = (p(x), y) \) by an automorphism \( \alpha \in \text{Aut} \mathbb{C}^2 \).

Quite recently, following the same idea as in the proof of Theorem 8.2 in [KaZa], S. Venereau has obtained such a result.

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31 We place it here with his kind permission.
Proposition 8.1 (Venereau). Let \( q \in \mathbb{C}[\bar{x}, u] \) (where \( \bar{x} = (x_1, \ldots, x_k) \)) be a polynomial such that \( p(\bar{x}) := q(\bar{x}, 0) \in \mathbb{C}[\bar{x}] \) is a non-constant polynomial. Consider the hypersurfaces \( X_p := \{ p(\bar{x}) = 0 \} \subset \mathbb{C}^k \) and \( Y_{q, n} := \{ u^n v - q(\bar{x}, u) = 0 \} \subset \mathbb{C}^{k+2} \). If the hypersurface \( X_p \) is rectifiable in \( \mathbb{C}^k \), then also the hypersurface \( Y_{q, n} \) is rectifiable in \( \mathbb{C}^{k+2} \).

We do not know whether the converse is true. Notice that under the conditions of Proposition 8.1 an analog of Theorem 8.1 (a) above holds. See also [KaZa, Prop. 5.3, 5.4] on extensions of Theorem 8.1 (a) to complete intersections of higher codimensions.

Remark 8.6. Every tom Dieck-Petrie surface \( X_{k,l} \) (see Examples 2.4, 4.4 and 4.6 above) admits at least two non-equivalent embeddings into \( \mathbb{C}^3 \) [KaZa, Example 6.3]. The question arises:

What is the cardinality of the set of all pairwise non-equivalent proper embeddings \( X_{k,l} \hookrightarrow \mathbb{C}^3 \)? Are those embeddings rigid or, conversely, do admit non-trivial deformations up to the action of the automorphism group \( \text{Aut} \mathbb{C}^3 \) on \( \mathbb{C}^3 \)?

The same question has sense for any contractible or acyclic surface embeddable into \( \mathbb{C}^3 \).

References

[AEH] S. Abhyankar, P. Eakin, W. Heinzer, On the uniqueness of the coefficient ring in a polynomial ring, J. Algebra, 23 (1972), 310–342.

[AM] S.S. Abhyankar, T.T. Moh, Embedding of the line in the plane, J. Reine Angew. Math. 276 (1975), 148–166.

[AF] A. Andreotti, T. Frenkel, The Lefschetz theorem on hyperplane sections, Ann. Math. 69 (1959), 713–717.

[An] G. Angermüller, Connectedness properties of polynomial maps between affine spaces, Manuscr. Math. 54 (1986), 349-359.

[AAS] Automorphisms of affine spaces, Proc. Conf., July 4-8, 1994, Curaçao, Van den Essen (ed.), Kluwer Acad. Publ., Dordrecht e.a., 1995.

[BD] G. Barthel, A. Dimca, On complex projective hypersurfaces which are homology \( P_n \)'s, In: Singularities, Proc Conf. ‘Singularities in geometry and topology’, Lille (France), 3-8 June, 1991, J.-P. Brasselet (ed.), Cambridge: Cambridge University Press, Lond. Math. Soc. Lect. Note Ser. 201 (1994), 1-27.

[Ba] H. Bass, A non-triangular action of \( G_a \) on \( \mathbb{A}^3 \), J. Pure Appl. Algebra 33 (1984), 1–5.

[BCW] H. Bass, E. Connell, D. Wright, The Jacobian Conjecture: reduction of degree and formal expansion of the inverse, Bull. Amer. Math. Soc. 7 (1982), 287-330.

[BaHa] H. Bass, W. Haboush, Linearizing certain reductive group actions, Trans. Amer. Math. Soc. 292 (1985), 463–482.

[BaWr] H. Bass, D. Wright, Localization in the K-theory of invertible algebras, J. Pure Appl. Algebra 9 (1976), 89-105.
BCTSSD | A. Beauville, J.-L. Colliot-Thelene, J.-J. Sansuc, P. Swinnerton-Dyer, \textit{Variétés stablye rationnelles non rationnelles}, Ann. Math. 121 (1985), 283-318.

Bia 1 | A. Białynicki-Birula, \textit{Remarks on the action of an algebraic torus on } \( k^n \), \textit{I, II}, Bull. Acad. Polon. Sci. Sér. Sci. Math. 14 (1966), 177–181; 15 (1967), 123–125.

Bia 2 | A. Białynicki-Birula, \textit{Some theorems on action of algebraic groups}, Ann. Math. 98 (1973), 480–497.

Bou | N. Bourbaki, \textit{Algèbre Commutative, Ch. 3}, Hermann, Paris, 1961.

Bre | G.E. Bredon, \textit{Introduction to compact transformation groups}, Ac. Press, N.Y., 1972.

Bri | E. Brieskorn, \textit{Beispiele zur Differentialtopologie von Singularitäten}, Invent. Math. 2 (1966), 1-14.

ChoDi | A.D.R. Choudary, A. Dimca, \textit{Complex hypersurfaces diffeomorphic to affine spaces}, Kodai Math. J. 17 (1994), 171–178.

Co | P.H. Cohn, \textit{Free rings and their relations}, Second edition, Acad. Press, London e.a., 1985.

Dan | W. Danielewski, \textit{On the cancellation problem and automorphism group of affine algebraic varieties}, preprint, 1989.

Dav | E.D. Davis, \textit{Ideals of the principal class, R-sequences and a certain monoidal transformation}, Pacific J. Math. 20 (1967), 197–205.

De | H. Derksen, \textit{Constructive Invariant Theory and the Linearization Problem}, Ph.D. thesis, Basel, 1997.

tDP 1 | T. tom Dieck, T. Petrie, \textit{Contractible affine surfaces of Kodaira dimension one}, Japan J. Math. 16 (1990), 147–169.

TDP 2 | T. tom Dieck, T. Petrie, \textit{The Abhyankar–Moh problem in dimension 3}, Lect. Notes Math. 1375, 1989, 48–59.

tDP 3 | T. tom Dieck, T. Petrie, \textit{Homology planes. An announcement and survey}, Topological Methods in Algebraic Transformation Groups, Progress in Mathem., 80, Birkhauser, Boston, 1989, 27–48.

Di 1 | A. Dimca, \textit{Hypersurfaces in } \( \mathbb{C}^{2n} \) \textit{diffeomorphic to } \( \mathbb{R}^{4n-2} (n \geq 2) \), Max-Planck Institute, preprint, 1991.

Di 2 | A. Dimca, \textit{Singularities and Topology of Hypersurfaces}, Universitext, Springer, 1992.

Do | A. Dold, \textit{Lectures on algebraic topology}, Springer, Berlin e.a., 1974.

Dr | L. M. Druzkowski, \textit{The Jacobian Conjecture: survey of some results}, in: Topics in Complex Analysis, Banach Center Publications, 31, Warszawa 1995, 163–171.

EH | P. Eakin, W. Heinzer, \textit{A cancellation problem for rings}, In: Conference on Commutative Algebra (J.W. Brewer, E.A. Rutter, eds.), Lect. Notes in Mathematics, Springer, Berlin e.a. 311 (1973), 61–77.
[Ei] D. Eisenbud, *Commutative algebra with a view towards algebraic geometry*, Graduate Texts in Math., Springer, N.Y. e.a., 1994

[Fi] K.-H. Fieseler, *On complex affine surfaces with $\mathbb{C}_+$ action*, Comment. Math. Helv. 69:1 (1994), 5-27.

[FlZa 1] H. Flenner, M. Zaidenberg, *Q-acyclic surfaces and their deformations*, Proc. Conf. "Classification of Algebraic Varieties", Mai 22–30, 1992, Univ. of l’Aquila, L’Aquila, Italy, Livorni (ed.) Contempor. Mathem. 162, Providence, RI, 1994, 143–208.

[FlZa 2] H. Flenner, M. Zaidenberg, *On a class of rational cuspidal plane curves*, Manuscr. Mathem. 89 (1996), 439-460.

[For] F. Forstnerič, *Holomorphic automorphisms of $\mathbb{C}^n$: A survey*, Complex Analysis and Geometry (Trento, 1993), Ancona e.a. (eds.) Lect. Notes in Pure and Applied Math. 173 Marcel Dekker, N.Y., 1996, 173-200.

[Fu 1] T. Fujita, *On Zariski problem*, Proc. Japan Acad. Ser. A Math. Sci. 55 (1979), 106–110.

[Fu 2] T. Fujita, *On the topology of non complete algebraic surfaces*, J. Fac. Sci. Univ. Tokyo, Sect.IA, 29 (1982), 503–566.

[Fur] M. Furushima, *The complete classification of compactifications of $\mathbb{C}^3$ which are projective manifolds with second Betti number equal to one*, Math. Ann. 297 (1993), 627–662.

[GuMiy 1] R.V. Gurjar, M. Miyanishi, *Affine surfaces with $\tilde{k} \leq 1$*, Algebraic Geometry and Commutative Algebra, in honor of M. Nagata, 1987, 99–124.

[GuMiy 2] R.V. Gurjar, M. Miyanishi, *Affine lines on logarithmic $\mathbb{Q}$– homology planes*, Math. Ann. 294 (1992), 463–482.

[GuPa] R.V. Gurjar, A.J. Parameswaran, *Affine lines on $\mathbb{Q}$– homology planes*, J. Math. Kyoto Univ. 35:1 (1995), 63–77.

[GuPraSha] R.V. Gurjar, C.R. Pradeep, A.R. Shastri, *On rationality of logarithmic $\mathbb{Q}$– homology planes: I, II, III*, preprints, 1997.

[GuSha] R.V. Gurjar, A.R. Shastri, *On rationality of complex homology 2–cells: I, II*, J. Math. Soc. Japan 41 (1989), 37–56, 175–212.

[Gut] A. Gutwirth, *The action of an algebraic torus on the affine plane*, Trans. Amer. Math. Soc. 105 (1962), 407 - 414.

[Ha] H. A. Hamm, *Lefschetz theorems for singular varieties*, Proceedings of Symposia in Pure Mathematics, Part I (Arcata Singularities Conference), 40, 1983, 547–557.

[Hir] F. Hirzebruch, *The topology of normal singularities of an algebraic surface*, Séminaire Bourbaki 15 (1962/1963), No. 250, 9p. (1964).

[HNK] F. Hirzebruch, W.D. Neumann, and S.S. Koh, *Differentiable manifolds and quadratic forms*, Lect. Notes Pure Appl. Math. 4, M. Dekker Inc., New York, 1971

[Ho] M. Hochster, *Non-uniqueness of coefficient rings in a polynomial ring*, Proc. Amer. Math. Soc. 34 (1972), 81–82.
[Ii 1] S. Iitaka, *On logarithmic Kodaira dimensions of algebraic varieties*, Complex Analysis and Algebraic geometry, Iwanami, Tokyo, 1977, 175–189.

[Ii 2] S. Iitaka, *Some applications of logarithmic Kodaira dimensions*, Algebraic Geometry (Proc. Intern. Symp. Kyoto 1977), Kinokuniya, Tokyo, 1978, 185–206.

[Ii 3] S. Iitaka, *Algebraic Geometry: An introduction to birational geometry of algebraic varieties*, Graduate Texts in Mathematics, 76, Springer Verlag, Berlin–Heidelberg–New York, 1982.

[IiFu] S. Iitaka, T. Fujita, *Cancellation theorem for algebraic varieties*, J. Fac. Sci. Univ. Tokyo, Sect.IA, 24 (1977), 123–127.

[Je] Z. Jelonek, *The extension of regular and rational embeddings*, Math. Ann. 113 (1987) 113–120.

[Ju] H. W. E. Jung, *¨Uber ganze birationale Transformationen der Ebene*, J. reine und angew. Math., 184 (1942), 161–174.

[Ka 1] S. Kaliman, *Smooth contractible hypersurfaces in $\mathbb{C}^n$ and exotic algebraic structures on $\mathbb{C}^3$*, Math. Zeitschrift 214 (1993), 499–510.

[Ka 2] S. Kaliman, *Exotic analytic structures and Eisenman intrinsic measures*, Israel Math. J. 88 (1994), 411–423.

[Ka 3] S. Kaliman, *Exotic structures on $\mathbb{C}^n$ and $\mathbb{C}^*\text{-action on } \mathbb{C}^3$*, Proc. Conf. ”Complex Analysis and Geometry”, Lect. Notes in Pure and Appl. Math., Marcel Dekker Inc. 173 (1996), 299–300.

[Ka 4] S. Kaliman, *Isotopic embeddings of affine algebraic varieties into $\mathbb{C}^n$*, Contempor. Mathem. 137 (1992), 291–295.

[Ka 5] S. Kaliman, *Extensions of isomorphisms between affine algebraic subvarieties of $k^n$ to automorphisms of $k^n$*, Proc. Amer. Math. Soc. 113 (1991), 325–334.

[KaML 1] S. Kaliman, L. Makar-Limanov, *On some family of contractible hypersurfaces in $\mathbb{C}^4$*, Séminaire d’algèbre. Journées Singulières et Jacobiennes, 26–28 mai 1993, Prépublication de l’Institut Fourier, Grenoble, 1994, 57–75.

[KaML 2] S. Kaliman, L. Makar-Limanov, *Affine algebraic manifolds without dominant morphisms from Euclidean spaces*, Rocky Mount. J. Math. 27:2 (1997), 601–609.

[KaML 3] S. Kaliman, L. Makar-Limanov, *On Russell–Koras contractible threefolds*, J. of Algebraic Geom. 6 (1997), 247-268.

[KaML 4] S. Kaliman, L. Makar-Limanov, *Locally nilpotent derivations of Jacobian type*, Preprint, 1998, 16p.

[KaKoMLRu] S. Kaliman, M. Koras, L. Makar-Limanov, P. Russell, *$\mathbb{C}^*$ -actions on $\mathbb{C}^3$ are linearizable*, Electronic Research Announcement Journal of the AMS, 3 (1997), 63–71.

[KaZa] S. Kaliman, M. Zaidenberg, *Affine modifications and affine varieties with a very transitive automorphism group*, Prépublication de l’Institut Fourier des Mathématiques, 406, Grenoble 1998, 46p. E-print: math.AG/9801076
[Kam 1] T. Kambayashi, *On Fujita’s strong cancellation theorem for the affine space*, J. Fac. Sci. Univ. Tokyo **23** (1980), 535–548.

[Kam 2] T. Kambayashi, *Pro-affine groups, Ind-affine groups and the Jacobian Problem*, J. Algebra **185** (1996), 481-501.

[KamRu] T. Kambayashi, P. Russell, *On linearizing algebraic torus actions*, J. Pure Appl. Algebra **23** (1982), 243–250.

[Kaw 1] Y. Kawamata, *Addition formula of logarithmic Kodaira dimension for morphisms of relative dimension one*, Proc. Intern. Sympos. Algebraic Geom., Kyoto, 1977. Kinokuniya, Tokyo, 1978, 207–217.

[Kaw 2] Y. Kawamata, *On the classification of non-complete algebraic surfaces*, Algebraic Geom., Proc. Summer Meeting, Copenhagen, 1978, Lect. Notes Math. **732**, 1979, 215–232.

[KMM] Y. Kawamata, K. Matsuda, K. Matsuki, *Introduction to the minimal model problem*, Alg. Geom. Sendai, 1985, Adv. St. in Pure Math. **10** (1987), 283-360.

[Kl] S. Kleiman, *Toward a numerical theory of ampleness*, Ann. Math. **84** (1966), 293-344.

[Ko] M. Koras, *A characterization of $A^2/\mathbb{Z}_a$*, Compositio Math. **87** (1993), 241–267.

[KoRu 1] M. Koras, P. Russell, *On linearizing ”good” $C^*$-action on $\mathbb{C}^3$*, Canadian Math. Society Conference Proceedings, **10** (1989), 93–102.

[KoRu 2] M. Koras, P. Russell, *Contractible threefolds and $C^*$-actions on $\mathbb{C}^3$*, J. of Algebraic Geom. **6** (1997), 671-695.

[KoRu 3] M. Koras, P. Russell, *$C^*$-actions on $\mathbb{C}^3$: the smooth locus of the quotient is not of hyperbolic type*, preprint, CICMA Reports, Concordia-Laval-McGill, 1996-06, 93p.

[Kr 1] H. Kraft, *Algebraic automorphisms of affine space*, Topological Methods in Algebraic Transformation Groups, Birkhäuser, Boston e.a., 1989, 81–105.

[Kr 2] H. Kraft, *$C^*$–actions on affine space*, Operator Algebras etc., Progress in Mathem. **92**, 1990, Birkhäuser, Boston e.a., 561–579.

[Kr 3] H. Kraft, *Challenging problems on affine n-space*, Séminaire Bourbaki **802** (1994/1995), Astérisque **237** (1996), 295–318.

[KrPeRun] H. Kraft, T. Petrie, J.D. Rundall, *Quotient varieties*, Advances in Math. **74** (1989), 145-162.

[KrPo] H. Kraft, V. L. Popov, *Semisimple group actions on the three dimensional affine space are linear*, Comment. Math. Helv. **60** (1985), 466–479.

[Lef] S. Lefschetz, *L’analysis situs et la géométrie algébrique*, Paris, 1924.

[Lib] A. Libgober, *A geometric procedure for killing the middle dimensional homology groups of algebraic hypersurfaces*, Proc. Amer. Math. Soc. **63** (1977), 198–202.

[LiZa] V. Lin, M. Zaidenberg, *An irreducible simply connected curve in $\mathbb{C}^2$ is equivalent to a quasihomogeneous curve*, Soviet Math. Dokl., **28** (1983), 200-204.
[ML 1] L. Makar-Limanov, *On groups of automorphisms of a class of surfaces*, Israel J. Math. 69 (1990), 250-256.

[ML 2] L. Makar-Limanov, *On the hypersurface* $x + x^2y + z^2 + t^3 = 0$ *in* $\mathbb{C}^4$ *or a* $\mathbb{C}^3$-*like threefold which is not* $\mathbb{C}^3$, Israel J. Math. 96 (1996), 419–429.

[ML 3] L. Makar-Limanov, *Again* $x + x^2y + z^2 + t^3 = 0$, Preprint, 1998, 3p.

[ML 4] L. Makar-Limanov, *On the group of automorphisms of a surface* $x^n y = P(z)$, Preprint, 1997, 11p.

[Mil 1] J. Milnor, *Lectures on the h-cobordism Theorem*, Princeton Univ. Press, Princeton, NJ, 1965.

[Mil 2] J. Milnor, *Morse Theory*, Princeton Univ. Press, Princeton, NJ, 1963.

[Mil 3] J. Milnor, *On the 3-dimensional Brieskorn manifolds* $M(p, q, r)$, *in: Knots, groups, and 3-manifolds*, L. P. Neuwirth, ed. Annals of Math. Stud., Princeton Univ. Press, Princeton, NJ, 1975, 175–225.

[MilSta] J. Milnor, J. Stasheff, *Characteristic classes*, Annals of Mathem. Studies 76, Princeton Univ. Press and Univ. of Tokyo Press, Princeton, NJ, 1974.

[Miy 1] M. Miyanishi, *Algebraic characterization of the affine 3–space*, Proc. Algebraic Geom. Seminar, Singapore, World Scientific, 1987, 53–67.

[Miy 2] M. Miyanishi, *Recent topics on open algebraic surfaces*, Amer. Math. Soc. Transl. 172 (1996), 61-75.

[MiySu] M. Miyanishi, T. Sugie, *Affine surfaces containing cylinderlike open sets*, J. Math. Kyoto Univ., 20 (1980), 11–42.

[Mo] S. Mori, *Classification of higher-dimensional varieties*, Proc. Symp. in Pure Math. 46 (1987), 269-331

[MS] S. Müller–Stach, *Projective compactifications of complex affine varieties*, London Math. Soc. Lect. Notes Ser. 179 (1991), 277–283.

[Mu] D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Public. Math. IHES 9 (1961), 229–246.

[Na 1] M. Nagata, *On an automorphism group of* $k[x, y]$, Kinokuniya, Tokyo, 1972.

[Na 2] M. Nagata, *Commutative algebra and algebraic geometry*, Proc. Intern. Mathem. Conf. L.H.Y. Chen, T.B.Ng, M.J.Wicks (eds.), North-Holland Publ. Co., 1982, 125-154.

[Ne] A. Némethi, *Global Sebastiani–Thom theorem for polynomial maps*, J. Math. Soc. Japan, 43 (1991), 213–218.

[OPOV] *Open problems on open varieties (Montreal 1994 problems)*, P. Russell (ed.), Prépublication de l’Institut Fourier des Mathématiques, 311, Grenoble 1995, 23p. E-print alg-geom/9506006.

[Or 1] S. Orevkov, *On three-sheeted polynomial mappings of* $\mathbb{C}^2$, Math. USSR Izvestiya, 29 (1987), 587-598.
[Or 2] S. Orevkov, *Acyclic algebraic surfaces bounded by Seifert spheres*, Osaka J. Math. **34**:2 (1997), 457–480.

[Pe] T. Petrie, *Topology, representations and equivariant algebraic geometry*, Contemporary Math. **158**, 1994, 203–215.

[Pi] S. Pinchuk, *A countereexample to the strong real Jacobian conjecture*, Math. Z. **217**:1 (1994), 1-4.

[Po] V.L. Popov, *Algebraic actions of connected reductive algebraic groups on $A^n$ are linearizable*, preprint, 1996, 3p.

[Pro] Y.G. Prokhorov, *Compactifications of $C^4$ of index 3*, Algebraic Geometry and its Applications, Proc. 8th Algebraic Geometry Conf., Yaroslavl’ 1992, Tikhomirov, Tyurin (eds.) Vieweg, Braunschweig/Wiesbaden, 1994, 159-169.

[Ram] C.P. Ramanujam, *A topological characterization of the affine plane as an algebraic variety*, Ann. Math., **94** (1971), 69-88.

[Re] R. Rentschler, *Opérations du groupe additif sur le plane affine*, C.R. Acad. Sci. Paris, **267** (1968), 384–387.

[Ru 1] P. Russell, *On a class of $C^3$ -like threefolds*, Preliminary Report, 1992.

[Ru 2] P. Russell, *On affine-ruled rational surfaces*, Math. Ann., **255** (1981), 287–302.

[Ru 3] P. Russell, *Gradings of polynomial rings*, Algebraic Geometry and its Applications (C. L. Bajaj ed.), Springer, 1994.

[Ru 4] P. Russell, *Sufficiently homogeneous closed embeddings of $A^n-1$ into $A^n$ are linear*, Preprint CICMA, 1997, 13p.

[Sak] F. Sakai, *Kodaira dimension of complement of divisor*, Complex Analysis and Algebraic geometry, Iwanami, Tokyo, 1977, 239–257.

[Sat] A. Sathaye, *On linear planes*, Proc. Amer. Math. Soc. **56** (1976), 1–7.

[Sch] G. W. Schwarz, *Exotic algebraic group actions*, C. R. Acad. Sci. Paris **309** (1989), 89–94.

[Sn] D. Snow, *Unipotent actions on affine space*, Topological methods in algebraic transformation groups, Proc. Conf., New Brunswick/NJ (USA) 1988, Prog. Math. **80** (1989), 165-176.

[Sr] V. Srinivas, *On the embedding dimension of an affine variety*, Math. Ann. **289** (1991), 125-132.

[Sug] T. Sugie, *On Petrie’s problem concerning homology planes*, J. Math. Kyoto Univ. **30** (1990), 317-324.

[Suz 1] M. Suzuki, *Propiétés topologiques des polynômes de deux variables complexes, et automorphismes algébrique de l’espace $C^2$*, J. Math. Soc. Japan, **26** (1974), 241-257.

[Suz 2] M. Suzuki, *Sur les opération holomorphes du groupe additif complexe sur l’espace de deux variables complexes*, Ann. Sci. École Norm. Sup. **10** (1977), 517–546.
[tD 1] T. tom Dieck, *Hyperbolic modifications and acyclic affine foliations*, preprint, Mathemat-ica Gottingensis, Göttingen, H. 27 (1992), 1–19.

[tD 2] T. tom Dieck, *Ramified coverings of acyclic varieties*, preprint, Mathemat-ica Gottingen-sis, Göttingen, H. 26 (1992), 1–20.

[tD 3] T. tom Dieck, *Homology planes without cancellation property*, Arch. Math. 59 (1992), 105–114.

[tD 4] T. tom Dieck, *Symmetric homology planes*, Math. Ann. 286 (1990), 143-152.

[VdV] A. Van de Ven, *Analytic compactifications of complex homology cells*, Math. Ann. 147 (1962), 189–204.

[vdK] W. van der Kulk, *On polynomial rings in two variables*, Nieuw Arch. Wisk. (3) 1 (1953), 33–41.

[Vie] E. Viehweg, *Canonical divisors and the additivity of the Kodaira dimension for morphisms of relative dimension one*, Compos. Math. 35 (1977), 197-223, *Correction*, ibid. 336.

[Wa] S. S.-S. Wang, *Extension of derivations*, J. Alg. 69 (1981), 240-246.

[Wil] P.M.H. Willson, *Towards birational classification of algebraic varieties*, Bull. London Math. Soc. 19 (1987), 1-48.

[Win] J. Winkelmann, *On free holomorphic $\mathbb{C}^*$-actions on $\mathbb{C}^n$ and homogeneous Stein manifolds*, Math. Ann. 286 (1990), 593–612.

[Wr 1] D. Wright, *Abelian subgroups of $\text{Aut}_k(k[X,Y])$ and applications to actions on the affine plane*, Ill. J. Math. 23 (1979), 579–634.

[Wr 2] D. Wright, *Cancellation of variables of the form $bT^n - a$*, J. Algebra 52 (1978), 94–100.

[Za 1] M. Zaidenberg, *Isotrivial families of curves on affine surfaces and characterization of the affine plane*, Math. USSR Izvestiya 30 (1988), 503-531. Addendum: ibid, 38 (1992), 435–437.

[Za 2] M. Zaidenberg, *Ramanujam surfaces and exotic algebraic structures on $\mathbb{C}^n$*, Soviet Math. Doklady 42 (1991), 636–640.

[Za 3] M. Zaidenberg, *An analytic cancellation theorem and exotic algebraic structures on $\mathbb{C}^n$, $n \geq 3$*, Astérisque 217 (1993), 251–282.

[Za 4] M. Zaidenberg, *On Ramanujam surfaces, $\mathbb{C}^*$-families and exotic algebraic structures on $\mathbb{C}^n$, $n \geq 3$*, Trans. Moscow Math. Soc. 55 (1994), 1–56.

[Za 5] M. Zaidenberg, *On exotic algebraic structures on affine spaces*, Geometric Complex Analysis, J. Noguchi e.a. eds. World Scientific Publ. Co., Singapore 1996, 691–714; E-print alg-geom/9506003.

Mikhail Zaidenberg
Université Grenoble I
Institut Fourier
UMR 5582 CNRS-UJF
BP 74
38402 St. Martin d’Hères–cedex
France
e-mail: zaidenbe@mozart.ujf-grenoble.fr