Maxwell problem about thermal sliding of rarefied gas along plate plane

A. V. Latyshev¹, A. A. Yushkanov² and E. E. Korneeva³

Faculty of Physics and Mathematics,
Moscow State Regional University, 105005,
Moscow, Radio str., 10–A

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One of classical boundary problems of the kinetic theory (a problem about thermal sliding) of the rarefied gas along a flat firm surface is considered. Kinetic Boltzmann equation with model integral of collisions BGK (Bhatnagar, Gross, Krook) is used. As boundary conditions the boundary Maxwell conditions (mirror-diffuse) are used. The generalized method of a source is applied to the problem decision. Comparison with earlier received results is spent.

Key words: thermal sliding, rarefied gas, accommodation coefficient, kinetic equation, Fredholm equation, distribution function, Neumann series.

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Introduction

Maxwell was the first who has paid attention to the movement of the rarefied gas under the influence of heterogeneous temperature distribution [1]-[3]. A problem about the thermal sliding of gas along a surface

¹avlatshev@mail.ru
²yushkanov@inbox.ru
³ee – korneeva@yandex.ru
(not necessarily flat) causes constant interest (see, for example, [4]-[12]). It is related both to rarely theoretical interest and with numerous applications in area of aerodynamics and physics of the aero dispersible systems. The most complete review of works in this direction is presented in work [4] and the monograph [5]. We note a number of works [6]-[9], mirror-boundary conditions were examined in that. A great contribution to the study of thermal sliding brought Loyalka S. K. [7]-[9]. In our works [10]-[15] were worked out approximation [10]-[12] and exact [13, 14] methods of solution of boundary value problems for model kinetic equations. In works [10], [11] the coefficient of thermal sliding is approximated by fractional rational functions. Analytical solution of the problem on thermal sliding for a gas with frequency of collisions proportional to the modulus of the speed of molecules, and diffuse boundary value problems, it was got in [15]. Then in works [16]-[18] thermal sliding was considered for quantum gases.

1. Statement problem

Let the rarefied gas fills a half-space \( x > 0 \) and moves along an axis \( y \). Far from a surface \((y, z)\) the logarithmic gradient of temperature

\[
\ln T(y) = \left( \frac{d \ln T(y)}{dy} \right)_{x=\infty}
\]

is set.

It is required to find the speed of thermal sliding \( u_{sl} = u_y(+\infty) \) and distribution of mass speed of gas \( u_y = u_y(x) \) in a half-space.

In work [12] it is shown that if to search the function of distribution in the form

\[
f(t, r, v) = f_M(v)(1 + h(t, r, v)),
\]

where

\[
f_M(r, v, t) = n(y)\left(\frac{m}{2\pi kT(y)}\right)^{3/2} \exp \left[ -\frac{m}{2kT(y)}v^2 \right],
\]
\[ T(y) = T_0 (1 + gry), \]
then function \( h(x, \mu) \) \((\mu = C_x)\) satisfies to the nonhomogeneous kinetic equation
\[
\mu \frac{\partial h}{\partial x_1} + G_T \left( \mu^2 - \frac{1}{2} \right) + h(x_1, \mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu'^2} h(x_1, \mu') d\mu. \tag{1.1}
\]

Here and below
\[
G_T = \left( \frac{d \ln T}{dy_1} \right)_{x_1=+\infty}, \quad x_1 = \nu \sqrt{\beta x}, \quad y_1 = \nu \sqrt{\beta y},
\]
\[
\beta = \frac{m}{2kT_0}; \quad U_{sl} = \sqrt{\beta u_{sl}},
\]
\(\nu\) is the frequency of collisions, \(x_1, y_1\) is the dimensionless coordinates, \(k\) is the Boltzmann constant.

Further we will consider that \( x_1 \equiv x, y_1 \equiv y \).

We will consider that molecules are reflected from a wall mirror-diffuse
\[
f(t, +0, \mathbf{v}) = q f_0(v) + (1 - q) f(t, +0, -v_x, v_y, v_z), \quad v_x > 0, \tag{1.2}
\]
where \(q\) is the accommodation coefficient (coefficient of diffusively), \(0 \leq q \leq 1\).

At the \(q = 1\) condition (1.2) is a condition of diffuse reflection, at \(q = 0\) is the condition of a specular reflection.

For function \( h(x, \mu) \) the condition (1.2) passes into the condition
\[
h(0, \mu) = (1 - q) h(0, -\mu), \quad \mu > 0. \tag{1.3}
\]

Far from a wall function of distribution turns into the Champen–Enskog distribution
\[
h(\infty, \mu) = 2U_{sl} - G_T \left( \mu^2 - \frac{1}{2} \right). \tag{1.4}
\]

We will designate further
\[
h(x, \mu) + G_T \left( \mu^2 - \frac{1}{2} \right) = \psi(x, \mu).
\]
Then the equation (1.1) passes into the homogeneous equation
\[ \mu \frac{\partial \psi}{\partial x} + \psi(x, \mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \psi(x, t) \, dt, \quad (1.5) \]

Boundary conditions (1.3) and (1.4) pass into the following
\[ \psi(0, \mu) = qG_T (\mu^2 - \frac{1}{2}) + (1 - q)\psi(0, -\mu), \quad \mu > 0, \quad (1.6) \]
and
\[ \psi(0, \mu) = 2U_{sl}. \quad (1.7) \]

Further we will put
\[ \psi(x, \mu) + 2U_{sl} + h_c(x, \mu). \]

Thus function satisfies to the equation (1.5)
\[ \mu \frac{\partial h_c}{\partial x} + h_c(x, \mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} h_c(x, t) \, dt, \quad (1.8) \]
and to boundary conditions
\[ h_c(0, \mu) = -2qU_{sl} + qG_T (\mu^2 - \frac{1}{2}) + (1 - q)h_c(0, -\mu), \quad \mu > 0, \quad (1.9) \]
and
\[ h_c(\infty, \mu) = 0. \quad (1.10) \]

2. Inclusion of boundary conditions in the kinetic equation

We will continue function of distribution to the interfaced half-space \( x < 0 \) by symmetric character
\[ h(x, \mu) = h(-x, -\mu), \quad \mu > 0. \]

At such extension the function \( h_c(x, \mu) \) in negative half-space \( x < 0 \) satisfies to equation (1.8) again and to the boundary conditions
\[ h_c(-0, \mu) = \]
\[= -2qU_{sl} + qG_T \left( \mu^2 - \frac{1}{2} \right) + (1 - q)h_c(-0, -\mu), \quad \mu < 0, \quad (2.1)\]

and

\[h_c(-\infty, \mu) = 0. \quad (2.2)\]

We will include boundary conditions (1.9), (1.10) and (2.1) (2.2) in the kinetic equation in the form of a source, which member contains Dirac’s delta function

\[\mu \frac{\partial h_c}{\partial x} + h_c(x, \mu) =
\]

\[= 2U_c(x) + |\mu| \left[ - 2qU_{sl} + qG_T \left( \mu^2 - \frac{1}{2} \right) - qh_c(\mp 0, -\mu) \right] \delta(x), \quad (2.3)\]

where \(\delta(x)\) is Dirac’s delta function, and

\[2U_c(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu'^2} h_c(x, \mu') d\mu'. \quad (2.4)\]

We will notice that \(U_c(x) = 0\) satisfies to the boundary conditions

\[U_c(\pm \infty) = 0. \quad (2.5)\]

For the equation (2.3) at \(x > 0, \mu < 0\), we get the solution satisfying to the boundary conditions (1.9) and (2.5)

\[h^+_c(x, \mu) = -\frac{1}{\mu} \exp\left( -\frac{x}{\mu} \right) \int_{x}^{+\infty} \exp\left( +\frac{t}{\mu} \right) 2U_c(t) \, dt. \quad (2.6)\]

Similarly, at \(x < 0, \mu > 0\) we find

\[h^-_c(x, \mu) = \frac{1}{\mu} \exp\left( -\frac{x}{\mu} \right) \int_{-\infty}^{x} \exp\left( +\frac{t}{\mu} \right) 2U_c(t) \, dt. \quad (2.7)\]

We will rewrite the equation (2.3), and we will replace in it the last member, using continuation of function \(h(x, \mu)\) on the interfaced half-space and equalities (2.6) and (2.7). On this way we come to the following equation

\[\mu \frac{\partial h_c}{\partial x} + h_c(x, \mu) = \]
\[ 2U_c(x) + |\mu| \left[ -2qU_{sl} + qG_T \left( \mu^2 - \frac{1}{2} \right) - qh_c^\pm(0, -\mu) \right] \delta(x). \quad (2.8) \]

Here

\[ h_c^\pm(0, \mu) = -\frac{1}{\mu} e^{-x/\mu} \int_0^{\pm\infty} e^{t/\mu} 2U_c(t) dt. \]

We find the solution of the equations of (2.8) and (2.4) in the form of Fourier’s integrals

\[ 2U_c(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} E(k) \, dk, \quad \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, dk, \quad (2.9) \]

\[ h_c(x, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \Phi(k, \mu) \, dk. \quad (2.10) \]

Thus the function \( h_c^+(x, \mu) \) is expressed through the spectral density \( E(k) \) of mass speed as follows

\[
\begin{align*}
    h_c^+(x, \mu) &= -\frac{1}{\mu} \exp\left(-\frac{x}{\mu}\right) \int_x^{+\infty} \exp\left(\frac{t}{\mu}\right) dt \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikt} E(k, \mu) \, dk = \\
    &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx} E(k, \mu)}{1 + ik\mu} \, dk.
\end{align*}
\]

Similarly we receive that

\[ h_c^-(x, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx} E(k, \mu)}{1 + ik\mu} \, dk. \]

Therefore

\[ h_c^\pm(x, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx} E(k, \mu)}{1 + ik\mu} \, dk. \]

Further we will get with the glance odd parity \( E(k) \)

\[ h_c^\pm(0, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E(k, \mu)}{1 + i\mu k} \, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E(k) \, dk}{1 + k^2\mu^2} = \]
$$= \frac{1}{\pi} \int_0^\infty \frac{E(k)}{1 + k^2 \mu^2} \, dk. \tag{2.11}$$

3. The characteristic equation

Using the Fourier integral (2.9) and (2.10) the equation (2.4) and (2.11) is transformed into the following system of equations

$$E(k) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \Phi(k, t) \, dt, \tag{3.1}$$

$$\Phi(k, \mu)(1 + ik\mu) =$$

$$= E(k) + |\mu| \left[ -2qU_{sl}(q) + G_T \left( \mu^2 - \frac{1}{2} \right) - \frac{q}{\pi} \int_0^\infty \frac{E(k) \, dk}{1 + k^2 \mu^2} \right] \tag{3.2}$$

We will express the function \( \Phi(k, \mu) \) from equation (3.2) and we will substitute it into the equation (3.1). We get the following characteristic equation

$$E(k) L(k) = -2qU_{sl}(q) T_1(k) +$$

$$+ qG_T \left( T_3(k) - \frac{1}{2} T_1(k) \right) - \frac{q}{\pi} \int_0^\infty K(k, k_1) E(k_1) \, dk_1 \tag{3.3}$$

with the kernel

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2 |t|} \, dt}{(1 + ikt)(1 + k_1^2 t^2)} =$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2} \, dt}{(1 + k^2 t^2)(1 + k_1^2 t^2)} = K(k, k_1). \tag{3.4}$$

The equation (3.3) is the Fredholm’s integral equation of the second kind. In addition, the notation is entered in (3.3)

$$T_n(k) = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2} t^n \, dt}{1 + k^2 t^2}, \quad n = 1, 2, 3, \cdots, \tag{3.5}$$
$$L(k) = 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2} dt}{1 + ikt}. \quad (3.6)$$

It is easy to see that

$$L(k) = 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2} dt}{1 + k^2 t^2} =$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-t^2} dt}{1 + k^2 t^2} = k^2 \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-t^2} dt}{1 + k^2 t^2},$$

or, in short,

$$L(k) = k^2 T_2(k).$$

4. The Neumann Series

The solution of equation (3.3) we are looking for in the form

$$E(k) = qG_T \left[ E_0(k) + q E_1(k) + q^2 E_2(k) + \cdots \right], \quad (4.1)$$

$$U_{sl}(q) = \frac{1}{2} G_T \left[ V_0 + V_1 q + V_2 q^2 + \cdots + V_n q^n + \cdots \right]. \quad (4.2)$$

Let us substitute the decomposition (4.1) and (4.2) in equation (3.3). Using the equalities (3.4)-(3.6), we obtain the countable system of equations

$$E_0(k) L(k) = -V_0 T_1(k) + T_3(k) - \frac{1}{2} T_1(k), \quad (4.3)$$

$$E_1(k) L(k) = -V_1 T_1(k) - \frac{1}{\pi} \int_{0}^{\infty} K(k, k_1) E_0(k_1) dk_1, \quad (4.4)$$

$$E_2(k) L(k) = -V_2 T_1(k) - \frac{1}{\pi} \int_{0}^{\infty} K(k, k_1) E_1(k_1) dk_1, \quad (4.5)$$

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$$E_n(k) L(k) =$$
\[ -V_n T_1(k) - \frac{1}{\pi} \int_0^\infty K(k, k_1) E_{n-1}(k_1) dk_1, \quad n = 1, 2, 3, \ldots \] \hspace{1cm} (4.6)

From the equation (4.3) we find

\[ E_0(k) = \frac{-\left(V_0 + \frac{1}{2}\right) T_1(k) + T_3(k)}{k^2 T_2(k)}. \hspace{1cm} (4.7) \]

We will eliminate the pole of the second order in right part (4.7). We will notice that

\[ T_1(k) = \frac{1}{\sqrt{\pi}} - k^2 T_3(k), \quad T_3(k) = \frac{1}{\sqrt{\pi}} - k^2 T_5(k). \]

Then

\[ E_0(k) = \frac{-\left(V_0 + \frac{1}{2}\right) \frac{1}{\sqrt{\pi}} + \left(V_0 + \frac{1}{2}\right) k^2 T_3(k) + \frac{1}{\sqrt{\pi}} - k^2 T_5(k)}{k^2 T_2(k)}. \]

For elimination of the pole we will demand that

\[ -\left(V_0 + \frac{1}{2}\right) \frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi}} = 0, \]

from which we have \( V_0 = \frac{1}{2} \). Then

\[ E_0(k) = \frac{T_3(k) - T_5(k)}{T_2(k)}. \]

By means of the last equality from equation (4.4) we find

\[ V_1 T_1(k) + \frac{1}{\pi} \int_0^\infty K(k, k_1) E_0(k_1) dk_1 \]

\[ E_1(k) = -\frac{V_1 T_1(k)}{k^2 T_2(k)}. \hspace{1cm} (4.8) \]

For the eliminating of the pole in the right part (4.8) we choose \( V_1 \) in the form

\[ V_1 = -\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{T_1(k_1)}{T_1(0)} E_0(k_1) dk_1 = 0.28566. \]
We will find the numerator of right part (4.8). We have

\[ V_1 T_1(k) + \frac{1}{\pi} \int_0^\infty K(k, k_1) E_0(k_1) dk_1 = \]

\[ = \frac{1}{\pi} \int_0^\infty \left[ K(k, k_1) - \frac{T_1(k) T_1(k_1)}{T_1(0)} \right] E_0(k_1) dk_1. \]

We notice that

\[ K(k, k_1) = T_1(0) - k_1^2 T_3(k_1) - k^2 T_3(k) + k^2 k_1^2 K_3(k, k_1). \]

We will find the difference

\[ K(k, k_1) - \frac{T_1(k) T_1(k_1)}{T_1(0)} = \]

\[ = K(k, k_1) - \frac{(T_1(0) - k^2 T_3(k))(T_1(0) - k_1^2 T_3(k_1))}{T_1(0)} = \]

\[ = k^2 k_1^2 \left[ K_5(k, k_1) - \sqrt{\pi} T_3(k) T_3(k_1) \right] = k^2 S(k, k_1), \]

where

\[ S(k, k_1) = k_1^2 \left[ K_5(k, k_1) - \sqrt{\pi} T_3(k) T_3(k_1) \right]. \]

According to (4.8) we get now

\[ E_1(k) = -\frac{1}{\pi T_2(k)} \int_0^\infty S(k, k_1) E_0(k_1) dk_1. \]

From equation (4.5) we find

\[ V_2 T_1(k) + \frac{1}{\pi} \int_0^\infty K(k, k_1) E_1(k_1) dk_1 \]

\[ E_2(k) = -\frac{1}{k^2 T_2(k)} \int_0^\infty K(k, k_1) E_1(k_1) dk_1. \]

From here we find

\[ V_2 = -\frac{1}{\sqrt{\pi}} \int_0^\infty T_1(k_1) E_1(k_1) dk_1 = -0.021135. \]
By means of this correlation we will transform the previous equality to the form

\[ E_2(k) = -\frac{1}{\pi T_2(k)} \int_0^\infty S(k, k_1) E_1(k_1) dk_1. \]

Similarly, from equation (4.6), we find

\[ v_n T_1(k) + \frac{1}{\pi} \int_0^\infty K(k, k_1) E_{n-1}(k_1) dk_1 \]

\[ E_n(k) = -\frac{1}{k^2 T_2(k)} \int_0^\infty S(k, k_1) E_{n-1}(k_1) dk_1. \]

It follows that

\[ V_n = -\frac{1}{\sqrt{\pi}} \int_0^\infty T_1(k_1) E_{n-1}(k_1) dk_1, \quad n = 1, 2, 3, \ldots. \]

In this case

\[ E_n(k) = -\frac{1}{\pi T_2(k)} \int_0^\infty S(k, k_1) E_{n-1}(k_1) dk_1, \quad n = 1, 2, 3, \ldots. \]

We will write out formulas for distribution of mass speed in the second approximation

\[ U_c(x) = U_{sl}(q) + \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{ikx}[E_0(k) + E_1(k)q + E_2(k)q^2 + \cdots] dk. \]

For the function \( h(x, \mu) \) we get

\[ h(x, \mu) = 2U_{sl}(q) - G_T\left(\mu^2 - \frac{1}{2}\right) + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \Phi(k, \mu) dk, \]

where

\[ \Phi(k, \mu) = \frac{1 - ik\mu}{1 + k^2 \mu^2} \left[ E_0(k) + q \left( E_1(k) + |\mu| G_T \left( \mu^2 - \frac{1}{2} - V_0 \right) - V_0 \right) - |\mu| \int_0^\infty \frac{E_0(k_1) dk_1}{1 + k_1^2 \mu^2} \right] + q^2 \left( E_2(k) - |\mu| G_T V_1 - |\mu| \int_0^\infty \frac{E_1(k_1) dk_1}{1 + k_1^2 \mu^2} \right). \]
5. Discussion of results and conclusions

We will compare the got results to previous. In particular, with the precise decision for the speed of thermal sliding at $q = 1$: $U_{sl} = 0.38316 G_T$ (see [12]).

We will rewrite this formula in the dimensional form: $u_{sl} = 1.1495 \zeta g_T$.

Here $\zeta$ is the kinematic viscosity of gas.

According to the got results the dimensionless speed of the thermal sliding is equal

$$U_{sl}(q) = \frac{1}{2}(V_0 + V_1 q + V_2 q^2)G_T = 0.5(0.5 + 0.2857q - 0.0211q^2)G_T.$$

We will bring this formula over to the dimensional form

$$u_{sl}(q) = 0.75(1 + 0.5714q - 0.0422q^2)\zeta g_T. \quad (5.1)$$

In the zero approximation from a formula (5.1) it is visible that we received exact result of Maxwell for mirror boundary conditions: $u_{sl} = 0.75 \zeta g_T$.

We will introduce the relative error

$$O_n(q) = \frac{u_{sl} - u_{sl}(q)}{u_{sl}} \cdot 100\%, \quad (5.2)$$

where $u_{sl}(q)$ is determined by the equality (5.1).

In the first approximation the speed of the thermal sliding is equal

$$u_{sl}^{(1)}(q) = 0.75(0.5 + 0.2857q)\zeta g_T,$$

in the second approximation it is determined by equality (5.1)

$$u_{sl}^{(2)}(q) = 0.75(1 + 0.5714q - 0.0422q^2)\zeta g_T.$$

It is easy to see that the relative error is equal in a zero approximation $O_0(1) = 34.48\%$, in the first it is equal $O_1(1) = 2.52\%$, in the second it is equal $O_2(1) = 0.23\%$. 
Comparing the brought estimations over, we come to the conclusion that exactly the nonlinear analysis leads to an efficient approximating formula for the calculation of the speed of the thermal sliding.

We will rewrite a formula (5.1) to the form \( u_{sl}(q) = K(q)\zeta g_T \), where \( K(q) \) is the coefficient of the thermal sliding. Let us compare our coefficient from this work with the coefficients found in [6]-[8]. The same coefficient was found in works [6] and [7]: \( K(q) = 0.75(1 + 0.5q) \). In work [8] it was obtained that \( K(q) = 0.75(0.5321 + q) \). The comparison of this coefficient with (5.1) according to (5.2) shows that at \( q = 1 \) the coefficient deviation from [6] and [7] does not exceed 0.2% from the coefficient received in work.

Thus, in work efficient approximating formulas for the solution of the problem on thermal sliding with mirror-diffuse boundary conditions are removed. Further authors intend to consider the problem of thermal sliding for quantum gases.

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