STATIONARY UNIVERSE

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ABSTRACT

If the Universe contains at least one inflationary domain with a sufficiently large and homogeneous scalar field $\phi$, then this domain permanently produces new inflationary domains of all possible types. We show that under certain conditions this process of the self-reproduction of the Universe can be described by a \textit{stationary} distribution of probability, which means that the fraction of the physical volume of the Universe in a state with given properties (with given values of fields, with a given density of matter, etc.) does not depend on time. This represents a strong deviation of inflationary cosmology from the standard Big Bang paradigm.

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The first models of inflation were based on the standard assumption of the Big Bang theory that the Universe was created at a single moment of time in a state with the Planck density, and that it was hot and large (much larger than the Planck scale $M_p^{-1}$) from the very beginning. The success of inflation in solving internal problems of the Big Bang theory apparently removed the last doubts concerning the Big Bang cosmology. It remained almost unnoticed that during the last ten years the inflationary theory has broken the umbilical cord connecting it with the old Big Bang theory, and acquired an independent life of its own. For the practical purposes of description of the observable part of our Universe one may still speak about the Big Bang. However, if one tries to understand the beginning of the Universe, or its end, or its global structure, then some of the notions of the Big Bang theory become inadequate.

For example, already in the first version of the chaotic inflation scenario [1] there was no need to assume that the whole Universe appeared from nothing at a single moment of time associated with the Big Bang, that the Universe was hot from the very beginning and that the inflaton scalar field $\phi$ which drives inflation originally occupied the minimum of its potential energy. Later it was found that if the Universe contains at least one inflationary domain of a size of horizon (‘$h$-region’) with a sufficiently large and homogeneous scalar field $\phi$, then this domain will permanently produce new $h$-regions of a similar type. In other words, instead of a single Big Bang producing a one-bubble Universe, we are speaking now about inflationary bubbles producing new bubbles, producing new bubbles, ad infinitum. In this sense, inflation is not a short intermediate stage of duration $\sim 10^{-35}$ seconds, but a self-regenerating process, which occurs in some parts of the Universe even now, and which will continue without end. The most striking realization of this scenario occurs in the context of chaotic inflation [2], but the basic features of this scenario remain valid in old inflation [3], new inflation [4, 5] and extended inflation as well [6].

Thus, recent developments of inflationary theory have considerably modified our cosmological paradigm [7]. Now we must learn how to formulate physical questions in the new context. For example, in a homogeneous part of the Universe there is a simple relation between the density of matter and time. However, on a very large scale the Universe becomes extremely inhomogeneous. Its density, at the same ‘cosmic time’, varies anywhere from zero to the Planck density. Therefore the question about the density of the Universe at the time $10^{10}$ years may not have any definite answer. Instead of addressing such questions we should study the distribution of probability of finding a part of the Universe with given properties, and find possible correlations between these properties.

It is extremely complicated to describe an inhomogeneous Universe and to find the corresponding probability distribution. Fortunately, there exists a particular kind of stationarity of the process of the Universe self-reproduction which makes things more regular. Due to the no-hair theorem for de Sitter space, the process of production of new inflationary domains occurs independently of any processes outside the horizon. This process depends only on the values of the fields inside each $h$-region of radius $H^{-1}$. Each time a new inflationary $h$-region is created during the Universe expansion, the physical processes inside this region will depend only on the properties of the fields inside it, but not on the ‘cosmic time’ at which it was created.
In addition to this most profound stationarity, there may also exist some simple stationary probability distributions which may allow us to say, for example, what the probability is of finding a given field \( \phi \) at a given point. To examine this possibility one should consider the probability distribution \( P_c(\phi, t|\chi) \), which describes the probability of finding the field \( \phi \) at a given point at a time \( t \), under the condition that at the time \( t = 0 \) the field \( \phi \) at this point was equal to \( \chi \) \[5, 8\]. The same function may also describe the probability that the scalar field which at time \( t \) was equal to \( \phi \), at some earlier time \( t = 0 \) was equal to \( \chi \).

The probability distribution \( P_c \) has been studied by many authors, see e.g. \[2\], \[7\]–\[9\]. Our investigation of this question has shown that in all realistic inflationary models the probability distribution \( P_c(\phi, t|\chi) \) is not stationary \[2, 7\]. The reason is very simple. The probability distribution \( P_c \) is in fact the probability distribution per unit volume in comoving coordinates (hence the index \( c \) in \( P_c \)), which do not change during expansion of the Universe. By considering this probability distribution we neglect the main source of the self-reproduction of inflationary domains, which is the exponential growth of their volume. Therefore, in addition to \( P_c \), we introduced the probability distribution \( P_p(\phi, t|\chi) \), which describes the probability to find a given field configuration in a unit physical volume \[2\]. In the present paper (see also \[10\] for a more detailed presentation) we will show that under certain conditions the stationary probability distribution \( P_p(\phi, t|\chi) \) does exist, and a typical relaxation time during which the distribution \( P_p(\phi, t|\chi) \) approaches the stationary regime is extremely small.

First of all we should remember some details of stochastic approach to inflation. Let us consider the simplest model of chaotic inflation based on the theory of a scalar field \( \phi \) minimally coupled to gravity, with the effective potential \( V(\phi) \). If the classical field \( \phi \) is sufficiently homogeneous in some domain of the Universe, then its behavior inside this domain is governed by the equations

\[
\ddot{\phi} + 3H\dot{\phi} = -dV/d\phi, \tag{1}
\]

\[
H^2 + \frac{k}{a^2} = \frac{8\pi}{3M_p} \left( \frac{1}{2}\dot{\phi}^2 + V(\phi) \right). \tag{2}
\]

Here \( H = \dot{a}/a, a(t) \) is the scale factor of the Universe, \( k = +1, -1, \) or 0 for a closed, open or flat Universe, respectively. \( M_p \) is the Planck mass, which we will put equal to one in the rest of the paper.

Investigation of these equations has shown that for many potentials \( V(\phi) \) (e.g., in all power-law \( V(\phi) \sim \phi^n \) and exponential \( V(\phi) \sim e^{\alpha\phi} \) potentials) there exists an intermediate asymptotic regime of slow rolling of the field \( \phi \) and quasi-exponential expansion (inflation) of the Universe \[7\]. At this stage the Hubble parameter is \( H(\phi) = \sqrt{8\pi V(\phi)/3} \). In the theories \( V(\phi) \sim \phi^n \) inflation ends at \( \phi_e \sim n/12 \). In the theory with \( V(\phi) \sim e^{\alpha\phi} \) inflation ends only if we bend the potential at some point \( \phi_e \); for definiteness we will take \( \phi_e = 0 \) in this theory.

Inflation stretches all initial inhomogeneities. Therefore, if the evolution of the Universe were governed solely by classical equations of motion, we would end up with an extremely smooth Universe with no primordial fluctuations to initiate the growth of galaxies. Fortunately, new density perturbations are generated during inflation due to quantum effects. The wavelengths of
all vacuum fluctuations of the scalar field \( \phi \) grow exponentially in the expanding Universe. When the wavelength of any particular fluctuation becomes greater than \( H^{-1} \), this fluctuation stops oscillating, and its amplitude freezes at some nonzero value \( \delta \phi(x) \) because of the large friction term \( 3H\dot{\phi} \) in the equation of motion of the field \( \phi \). The amplitude of this fluctuation then remains almost unchanged for a very long time, whereas its wavelength grows exponentially. Therefore, the appearance of such a frozen fluctuation is equivalent to the appearance of a classical field \( \delta \phi(x) \) that does not vanish after averaging over macroscopic intervals of space and time.

Because the vacuum contains fluctuations of all wavelengths, inflation leads to the creation of more and more perturbations of the classical field with wavelengths greater than \( H^{-1} \). The average amplitude of such perturbations generated during a time interval \( H^{-1} \) (in which the Universe expands by a factor of \( e \)) is given by

\[
|\delta \phi(x)| \approx \frac{H}{2\pi} .
\]

The phases of each wave are random. Therefore, the sum of all waves at a given point fluctuates and experiences Brownian jumps in all directions in the space of fields.

The standard way to describe the stochastic behavior of the inflaton field during the slow-rolling stage is to coarse-grain it over \( h \)-regions and consider the effective equation of motion of the long-wavelength field \( [8] \):

\[
\frac{d\phi}{dt} = -\frac{V'(\phi)}{3H(\phi)} + \frac{H^{3/2}(\phi)}{2\pi} \xi(t) .
\]

Here \( H = \sqrt{8\pi V/3} \), \( \xi(t) \) is the effective white noise generated by quantum fluctuations, which leads to the Brownian motion of the classical field \( \phi \).

This Langevin equation leads to two stochastic equations for the probability distribution \( P_c(\phi,t|\chi) \). The first one is called the backward Kolmogorov equation,

\[
\frac{\partial P_c(\phi,t|\chi)}{\partial t} = \frac{1}{2} \frac{H^{3/2}(\chi)}{2\pi} \frac{\partial}{\partial \chi} \left( \frac{H^{3/2}(\chi)}{2\pi} \frac{\partial}{\partial \chi} P_c(\phi,t|\chi) \right) - \frac{V'(\chi)}{3H(\chi)} \frac{\partial}{\partial \chi} P_c(\phi,t|\chi) .
\]

In this equation one considers the value of the field \( \phi \) at the time \( t \) as a constant, and finds the time dependence of the probability that this value was reached during the time \( t \) as a result of diffusion of the scalar field from different possible initial values \( \chi \equiv \phi(0) \).

The second equation is the adjoint to the first one; it is called the forward Kolmogorov equation, or the Fokker-Planck equation,

\[
\frac{\partial P_c(\phi,t|\chi)}{\partial t} = \frac{1}{2} \frac{\partial}{\partial \phi} \left( \frac{H^{3/2}(\phi)}{2\pi} \frac{\partial}{\partial \phi} \left( \frac{H^{3/2}(\phi)}{2\pi} P_c(\phi,t|\chi) \right) + \frac{V'(\phi)}{3H(\phi)} P_c(\phi,t|\chi) \right) .
\]

This equation was derived in \([8]\), see also \([5]\).
One may try to find a stationary solution of equations (5), (6), assuming that \( \frac{\partial P_c(\phi,t|\chi)}{\partial t} = 0 \). The simplest stationary solution (subexponential factors being omitted) would be

\[
P_c(\phi,t|\chi) \sim \exp \left( \frac{3}{8V(\phi)} \right) \cdot \exp \left( -\frac{3}{8V(\chi)} \right). \tag{7}
\]

This function is extremely interesting. Indeed, the first term in (7) is equal to the square of the Hartle-Hawking wave function of the Universe \([11]\), whereas the second one gives the square of the tunneling wave function \([12]\)!

At first glance, this result gives a direct confirmation and a simple physical interpretation of both the Hartle-Hawking wave function of the Universe and the tunneling wave function. However, in all realistic cosmological theories, in which \( V(\phi) = 0 \) at its minimum, the Hartle-Hawking distribution \( \exp \left( \frac{3}{8V(\phi)} \right) \) is not normalizable. The source of this difficulty can be easily understood: any stationary distribution may exist only due to compensation of the classical flow of the field \( \phi \) downwards to the minimum of \( V(\phi) \) by the diffusion motion upwards. However, diffusion of the field \( \phi \) discussed above exists only during inflation. Thus, there is no diffusion motion upwards from the region \( \phi < \phi_c \). Therefore all solutions of equation (6) with the proper boundary conditions at \( \phi = \phi_c \) (i.e. at the end of inflation) are non-stationary (decaying) \([2]\).

The situation with the probability distribution \( P_p \) is much more interesting and complicated. As was shown in \([2]\) its behavior depends strongly on initial conditions. If the distribution \( P_p \) was initially concentrated at \( \phi < \phi^* \), where \( \phi^* \) is some critical value of the field, then it moves towards small \( \phi \) in the same way as \( P_c \), i.e. it cannot become stationary. On the other hand, if the initial value of the field \( \phi \) is larger than \( \phi^* \), the distribution moves towards larger and larger values of the field \( \phi^* \), until it reaches the field \( \phi_p \), at which the effective potential of the field becomes of the order of Planck density \( M_p^4 \) (we will assume \( M_p = 1 \) hereafter), where the standard methods of quantum field theory in a curved classical space are no longer valid.

Some further steps towards the solution of this problem were made by Nambu and Sasaki \([13]\) and Mijić \([14]\). Their papers contain many important results and insights. However, Mijić \([14]\) did not have a purpose to obtain a complete expression for the stationary distribution \( P_p(\phi,t|\chi) \). The corresponding expressions were obtained for various types of potentials \( V(\phi) \) in \([13]\). Unfortunately, according to \([13]\), the stationary distribution \( P_p(\phi,t|\chi) \) is almost entirely concentrated at \( \phi \gg \phi_p \), i.e. at \( V(\phi) \gg 1 \), where the methods used in \([13]\) are inapplicable.

We will continue this investigation by writing the system of stochastic equations for \( P_p \). These equations can be obtained from eqs. (5), (6) by adding the term \( 3HP_p \), which appears due to the growth of physical volume of the Universe by the factor \( 1 + 3H(\phi)dt \) during each time interval \( dt \) \([13]\): \([16]\), \([14]\):

\[
\frac{\partial P_p}{\partial t} = \frac{1}{2} \frac{H^{3/2}(\chi)}{2\pi} \frac{\partial}{\partial \chi} \left( \frac{H^{3/2}(\chi)}{2\pi} \frac{\partial P_p}{\partial \chi} \right) - \frac{V'(\chi)}{3H(\chi)} \frac{\partial P_p}{\partial \chi} + 3H(\chi)P_p. \tag{8}
\]

\[
\frac{\partial P_p}{\partial t} = \frac{1}{2} \frac{H^{3/2}(\phi)}{2\pi} \frac{\partial}{\partial \phi} \left( \frac{H^{3/2}(\phi)}{2\pi} \frac{\partial P_p}{\partial \phi} \right) + \frac{V'(\phi)}{3H(\phi)} P_p + 3H(\phi)P_p. \tag{9}
\]
To find solutions of these equations one must specify boundary conditions. The boundary conditions at the end of inflation follow from the conservation of the probability flux [10]:

$$\frac{\partial}{\partial \phi} P_p(\phi_e) = -\frac{3}{4} \frac{V'}{V} P_p(\phi_e), \quad \frac{\partial}{\partial \chi} \left( P_p \exp\left( \frac{3}{8V(\chi)} \right) \right) \bigg|_{\chi=\phi_e} = 0.$$  

(10)

The last boundary condition is especially interesting, since it indicates that at least in the vicinity of $\chi = \phi_e$ the probability distribution with respect to $\chi$ looks as a square of the tunneling wave function. However, we have found solutions of (8), (9) to be rather stable with respect to modification of these boundary conditions, whereas the conditions at the Planck boundary $\phi = \phi_p$ do play a very important role.

Investigation of these conditions poses many difficult problems. First of all, our diffusion equations are based on the semiclassical approach to quantum gravity, which breaks down at super-Planckian densities. Secondly, the shape of the effective potential may be strongly modified by quantum effects at $\phi \sim \phi_p$. Finally, our standard interpretation of the probability distribution $P_c$ and $P_p$ breaks down at $V(\phi) > 1$, since at super-Planckian densities the notion of a classical scalar field in classical space-time does not make much sense.

However, these problems by themselves suggest a possible answer. Inflation happens only in theories with very flat effective potentials. At the Planck density nothing can protect the effective potential from becoming steep. Hence, one may expect that inflation ceases to exist at $\phi > \phi_p$, which leads to the boundary condition

$$P_p(\phi_p, t|\chi) = P_p(\phi, t|\chi_p) = 0,$$  

(11)

where $V(\phi_p) \equiv V(\chi_p) = O(1)$.

There is also another, much more general reason to expect that inflation kills itself as the potential energy density approaches the Planck density $V \sim 1$. Indeed, the amplitude of fluctuations of the scalar field generated during the typical time $\delta t = H^{-1}$ is given by $H/2\pi$, and their typical wavelength at that time is $O(H^{-1})$. This means that the energy density associated with the gradients of these perturbations is of the order of $H^4 \sim V^2$. Thus, in the domains with $V > 1$ the gradient energy density $\sim V^2$ becomes larger than the potential energy density $V$. This violates one of the basic assumptions necessary for inflation in domains with $V \gtrsim 1$. One may expect also that large gradients of energy on the scale comparable to the scale of the horizon $H^{-1}$ lead to creation of black holes rather than to the permanent self-reproduction of inflationary $h$-regions.

Of course, one may argue that all our considerations do not make sense at densities larger than the Planck density. When the energy density in any $h$-region approaches the Planck density, it may no longer be described in terms of classical space-time and should be just thrown away from our consideration. In particular, its volume should not be considered as contributing to the total volume of the Universe. Thus, such domains should be neglected in our definition of $P_p$. In this case the distribution $P_p$ for the field $\phi$ will stop moving towards higher values of $\phi$ and will approach a stationary regime when this distribution will be shifted towards $\phi \sim \phi_p$. 
What we are saying is even stronger. Even if one makes an attempt to consider the domains with \( V > 1 \) as a part of classical space-time, many parts of these domains drop out from the process of inflation. This means that the total volume of inflationary \( h \)-regions cannot grow as fast as \( e^{3Ht} \).

We do not know which of these arguments, if any, will survive in the future theory of all fundamental interactions. However, all these arguments point out in the same direction: For a phenomenological description of stochastic processes in classical space-time one should impose boundary conditions of the type of (11) which do not permit penetration of inflation deep into the realm of super-Planckian densities. As we will show in [10], the exact form of these boundary conditions is not very important; most of them allow the same class of solutions as the boundary condition (11). Therefore in this paper we will use these boundary conditions, assuming for definiteness that they are imposed at \( V(\phi_p) \equiv V(\chi_p) = 1 \).

One may try to obtain solutions of equations (8), (9) in the form of the following series of biorthonormal system of eigenfunctions of the pair of adjoint linear operators (defined by the left hand sides of the equations below):

\[
P_p(\phi,t|\chi) = \sum_{s=1}^{\infty} e^{\lambda_st} \psi_s(\chi) \pi_s(\phi). \tag{12}
\]

Indeed, this gives us a solution of eq. (11) if

\[
\frac{1}{2} \frac{H^{3/2}(\chi)}{2\pi} \frac{\partial}{\partial \chi} \left( \frac{H^{3/2}(\chi)}{2\pi} \frac{\partial}{\partial \chi} \psi_s(\chi) \right) - \frac{V'(\chi)}{3H(\chi)} \frac{\partial}{\partial \chi} \psi_s(\chi) + 3H(\chi) \cdot \psi_s(\chi) = \lambda_s \psi_s(\chi). \tag{13}
\]

and

\[
\frac{1}{2} \frac{\partial}{\partial \phi} \left( \frac{H^{3/2}(\phi)}{2\pi} \frac{\partial}{\partial \phi} \left( \frac{H^{3/2}(\phi)}{2\pi} \pi_j(\phi) \right) \right) + \frac{\partial}{\partial \phi} \left( \frac{V'(\phi)}{3H(\phi)} \pi_j(\phi) \right) + 3H(\phi) \cdot \pi_j(\phi) = \lambda_j \pi_j(\phi). \tag{14}
\]

The orthonormality condition reads

\[
\int_{\phi_p}^{\phi_\chi} \psi_s(\chi) \pi_j(\chi) d\chi = \delta_{sj}. \tag{15}
\]

In our case (with regular boundary conditions) one can easily show that the spectrum of \( \lambda_j \) is discrete and bounded from above. Therefore the asymptotic solution for \( P_p(\phi,t|\chi) \) (in the limit \( t \to \infty \)) is given by

\[
P_p(\phi,t|\chi) = e^{\lambda_1 t} \psi_1(\chi) \pi_1(\phi) \cdot \left( 1 + O(e^{-\lambda_2 t}) \right). \tag{16}
\]

Here \( \psi_1(\chi) \) is the only positive eigenfunction of eq. (13), \( \lambda_1 \) is the corresponding (real) eigenvalue, and \( \pi_1(\phi) \) is the eigenfunction of the conjugate operator (14) with the same eigenvalue \( \lambda_1 \). Note, that \( \lambda_1 \) is the highest eigenvalue, \( \text{Re} (\lambda_1 - \lambda_2) > 0 \). This is the reason why the asymptotic equation (16) is valid at large \( t \). We have found [10] that in realistic theories of inflation a typical
time of relaxing to the asymptotic regime, $\Delta t \sim (\lambda_1 - \lambda_2)^{-1}$, is extremely small. Typically it is only about a few thousands Planck times, i.e. about $10^{-40}$ sec. This means that the normalized distribution

$$\tilde{P}_p(\phi, t|\chi) = e^{-\lambda_1 t} P_p(\phi, t|\chi)$$

rapidly converges to the time-independent normalized distribution

$$\tilde{P}_p(\phi|\chi) \equiv \tilde{P}_p(\phi, t \to \infty|\chi) = \psi_1(\chi) \pi_1(\phi).$$

It is this stationary distribution that we were looking for. Because the growing factor $e^{-\lambda_1 t}$ is the same for all $\phi$ and $\chi$, one can use $\tilde{P}_p$ instead of $P_p$ for calculation of all relative probabilities.

In particular, $\tilde{P}_p(\phi|\chi)$ gives us the fraction of the volume of the Universe occupied by the field $\phi$, under the condition that the corresponding part of the Universe at some time in the past contained the field $\chi$. The remaining problem is to find the functions $\psi_1(\chi)$ and $\pi_1(\phi)$, and to check that all assumptions about the boundary conditions which we made on the way to eq. (16) are actually satisfied.

We have solved this problem for chaotic inflation in a wide class of theories including the theories with polynomial and exponential effective potentials $V(\phi)$ and found the corresponding stationary distributions [10]. Here we will present some of our results for the theories $\lambda_4 \phi^4$ and $V_o e^{\alpha \phi}$.

Solution of equations (13) and (14) for $\psi_1(\chi)$ and $\pi_1(\phi)$ in the theory $\lambda_4 \phi^4$ shows that these functions are extremely small at $\phi \sim \phi_e$ and $\chi \sim \chi_e$. They grow at large $\phi$ and $\chi$, then rapidly decrease, and vanish at $\phi = \chi = \phi_p$. With a decrease of $\lambda$ the solutions become more and more sharply peaked near the Planck boundary. (The functions $\psi_1$ and $\pi_1$ for the exponential potential have a similar behavior, but they are less sharply peaked near $\phi_p$.) A detailed discussion of these solutions will be contained in [10]. The eigenvalues $\lambda_1$ corresponding to different coupling constants $\lambda$ are given by the following table:

| $\lambda$ | $10^{-1}$ | $10^{-2}$ | $10^{-3}$ | $10^{-4}$ | $10^{-5}$ | $10^{-6}$ |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $\lambda_1$ | 2.813 | 4.418 | 5.543 | 6.405 | 7.057 | 7.538 | 7.885 |

One can find also the second eigenvalue $\lambda_2$. For example, for $\lambda = 10^{-4}$ one gets $\lambda_2 = 6.789$. This means that for $\lambda = 10^{-4}$ the time of relaxation to the stationary distribution is $\Delta t \sim (\lambda_1 - \lambda_2)^{-1} \sim 4M_p^{-1} \sim 10^{-42}$ seconds — a very short time indeed.

Note that the parameter $\lambda_1$ shows the speed of exponential expansion of the volume filled by a given field $\phi$. This speed does not depend on the field $\phi$, and has the same order of magnitude as the speed of expansion at the Planck density. Indeed, $\lambda_1$ should be compared to $3H(\phi) = 2\sqrt{6\pi V(\phi)}$, which is equal to $2\sqrt{6\pi}$ at the Planck density. It can be shown [10] that in the limit $\lambda \to 0$ the eigenvalue $\lambda_1$ also becomes equal to $2\sqrt{6\pi} \approx 8.681$. The meaning of this result is very simple: in the limit $\lambda \to 0$ our solution becomes completely concentrated near the Planck boundary, and $\lambda_1$ becomes equal to $3H(\phi_p)$.

At first glance, independence of the speed of expansion of volume $e^{\lambda_1 t}$ on the value of the field

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φ may seem counterintuitive. The meaning of this result is that the domain filled with the field φ gives the largest contribution to the growing volume of the Universe if it first diffuses towards the Planckian densities, spends there as long time as possible expanding with nearly Planckian speed, and then diffuses back to its original value φ.

But what about the field φ which is already at the Planck boundary? Why do the corresponding domains not grow exactly with the Planckian Hubble constant $H(φ_p) = 2\sqrt{6π}/3$? It happens partially due to diffusion and slow rolling of the field towards smaller φ. However, the leading effect is the destructive diffusion towards the space-time foam with φ > φp. One may visualize this process by painting white all domains with $V(φ) < 1$, and by painting black domains filled by space-time foam with $V(φ) > 1$. Then each time $H^{-1}(φ_p)$ the volume of white domains with φ ∼ φp grows approximately $e^3$ times, but some ‘black holes’ appear in these domains, and, as a result, the total volume of white domains increases only $e^{3λ_1/2\sqrt{6π}}$ times. This suggests (by analogy with [14]) calling the factor $df = 3λ_1/2\sqrt{6π}$ ‘the fractal dimension of classical space-time’, or ‘the fractal dimension of the inflationary Universe’. (Note that $df < 3$ for $λ ≠ 0$; for example, $df = 2.6$ for $λ = 10^{-5}$.) However, one should keep in mind that the fractal structure of the inflationary Universe in the chaotic inflation scenario in general is more complicated than in the new or old inflation and cannot be completely specified just by one fractal dimension [10].

The distribution $\tilde{P}_p(φ|χ) = ψ_1(χ) π_1(φ)$ which we have obtained does not depend on time t. However, in general relativity one may use many different time parametrizations, and the same physics can be described differently in different ‘times’. One of the most natural choices of time in the context of stochastic approach to inflation is the time $τ = \ln \frac{α(x,t)}{α(x,0)} = \int H(φ(x,t),t) dt$ [8, 9]. Here a(x, t) is a local value of the scale factor in the inflationary Universe. By using this time variable, we were able to obtain not only numerical solutions to the stochastic equations, but also simple asymptotic expressions describing these solutions. For example, for the theory $\frac{λ}{4}φ^4$ both the eigenvalue $λ_1$ and the ‘fractal dimension’ $df$ (which in this case refers both to the Planck boundary at $φ_p$ and to the end of inflation at $φ_e$) are given by $df = λ_1 ∼ 3 − 1.1 \sqrt{λ}$, and the stationary distribution is

$$\tilde{P}_p(φ, τ|χ) \sim \exp\left(-\frac{3}{8V(χ)}\right)\left(\frac{1}{V(χ) + 0.4} - \frac{1}{1.4}\right) \cdot φ \exp\left(-π(3 − λ_1)φ^2\right)$$

$$\sim \exp\left(-\frac{3}{2λχ^4}\right)\left(\frac{4}{λχ^4 + 1.6} - \frac{1}{1.4}\right) \cdot φ \exp\left(-3.5\sqrt{λχφ^2}\right). \quad (19)$$

Note that the first factor coincides with the square of the tunneling wave function [12]! This expression is valid in the whole interval from $φ_e$ to $φ_p$ and it correctly describes asymptotic behavior of $\tilde{P}_p(φ, τ|χ)$ both at $χ ∼ χ_e$ and at $χ ∼ χ_p$.

A similar investigation can be carried out for the theory $V(φ) = V_o e^{αφ}$. The corresponding solution is

$$\tilde{P}_p(φ, τ|χ) \sim \exp\left(-\frac{3}{8V(χ)}\right)\left(\frac{1}{V(χ) − 1}\right) \cdot \left(\frac{1}{V(φ) − 1}\right) V^{-1/2}(φ). \quad (20)$$

This expression gives a rather good approximation for $\tilde{P}_p(φ, τ|χ)$ for all φ and χ.
The main result of our work is that under certain conditions the properties of our Universe can be described by a time-independent probability distribution, which we have found for theories with polynomial and exponential effective potentials. A lot of work still has to be done to verify this conclusion, see [10]. However, once this result is taken seriously, one should consider its interpretation and rather unusual implications.

When making cosmological observations, we study our part of the Universe and find that in this part inflation ended about $t_e \sim 10^{10}$ years ago. The standard assumption of the first models of inflation was that the total duration of the inflationary stage was $\Delta t \sim 10^{-35}$ seconds. Thus one could come to an obvious conclusion that our part of the Universe was created in the Big Bang, at the time $t_e + \Delta t \sim 10^{10}$ years ago. However, in our scenario the answer is quite different.

Let us consider an inflationary domain which gave rise to the process of self-reproduction of new inflationary domains. For illustrative purposes, one can visualize self-reproduction of inflationary domains as a branching process, which gives a qualitatively correct description of the actual physical process we consider. During this process, the first inflationary domain of initial radius $\sim H^{-1}(\phi)$ within the time $H^{-1}(\phi)$ splits into $e^3 \sim 20$ independent inflationary domains of similar size. Each of them contains a slightly different field $\phi$, modified both by classical motion down to the minimum of $V(\phi)$ and by long-wavelength quantum fluctuations of amplitude $\sim H/2\pi$. After the next time step $H^{-1}(\phi)$, which will be slightly different for each of these domains, they split again, etc. The whole process now looks like a branching tree growing from the first (root) domain. The radius of each branch is given by $H^{-1}$; the total volume of all domains at any given time $t$ corresponds to the ‘cross-section’ of all branches of the tree at that time, and is proportional to the number of branches. This volume rapidly grows, but when calculating it, one should take into account that those branches, in which the field becomes larger than $\phi_p$, die and fall down from the tree, and each branch in which the field becomes smaller than $\phi_e$, ends on an apple (a part of the Universe where inflation ended and life became possible).

One of our results is that even after we discard at each given moment the dead branches and the branches ended with apples, the total volume of live (inflationary) domains will continue growing exponentially, as $e^{\lambda t}$. What is even more interesting, we have found that very soon the portion of branches with given properties (with given values of scalar fields, etc.) becomes time-independent. Thus, by observing any finite part of a tree at any given time $t$ one cannot tell how old the tree is.

To give a most dramatic representation of our conclusions, let us see where most of the apples grow. This can be done simply by integrating $e^{\lambda t}$ from $t = 0$ to $t = T$ and taking the limit as $T \to \infty$. The result obviously diverges at large $T$ as $\lambda^{-1} e^{\lambda T}$, which means that most apples grow at an indefinitely large distance from the root. In other words, if we ask what is the total duration of inflation which produced a typical apple, the answer is that it is indefinitely large.

This conclusion may seem very strange. Indeed, if one takes a typical point in the root domain, one can show that inflation at this point ends within a finite time $\Delta t \sim 10^{-35}$ seconds. This is a correct (though model-dependent) result which can be confirmed by stochastic methods, using the distribution $P_c(\phi, \Delta t | \chi)$ [4]. How could it happen that the duration of inflation was any
longer than $10^{-35}$ seconds?

The answer is related to the choice between $P_c$ and $P_p$, or between roots and fruits. Typical points in the root domain drop out from the process of inflation within $10^{-35}$ seconds. The number of those points which drop out from inflation at a much later stage is exponentially suppressed, but they produce the main part of the total volume of the Universe. Note that the length of each particular branch continued back in time may well be finite [18]. However, there is no upper limit to the length of each branch, and, as we have seen, the longest branches produce almost all parts of the Universe with properties similar to the properties of the part where we live now. Since by local observations we can tell nothing about our distance in time from the root domain, our probabilistic arguments suggest that the root domain is, perhaps, indefinitely far away from us. Moreover, nothing in our part of the Universe depends on the distance from the root domain, and, consequently, on the distance from the Big Bang.

Thus, inflation solves many problems of the Big Bang theory and ensures that this theory provides an excellent description of the local structure of the Universe. However, after making all kinds of improvements of this theory, we are now winding up with a model of a stationary Universe, in which the notion of the Big Bang loses its dominant position, being removed to the indefinite past.

The stochastic approach to inflation used in our work has an intermediate position between purely classical and purely quantum mechanical approaches to cosmology. In particular, stationarity of our probability distribution is closely reminiscent of the time-independence of the wave function of the Universe in the Wheeler-DeWitt equation [19]. We hope that stochastic methods may show us a way towards a complete quantum mechanical description of the stationary state of the Universe.

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