Self-Duality in $D \leq 8$-dimensional Euclidean Gravity

B.S. Acharya

Queen Mary and Westfield College,
Mile End Rd., London. E1 4NS, UK.

and

M. O’Loughlin

ICTP, PO Box 586, Trieste, 34014, Italy.

Abstract

In the context of $D$-dimensional Euclidean gravity, we define the natural generalisation to $D$-dimensions of the self-dual Yang-Mills equations, as duality conditions on the curvature 2-form of a Riemannian manifold. Solutions to these self-duality equations are provided by manifolds of $SU(2), SU(3), G_2$ and $Spin(7)$ holonomy. The equations in eight dimensions are a master set for those in lower dimensions. By considering gauge fields propagating on these self-dual manifolds and embedding the spin connection in the gauge connection, solutions to the $D$-dimensional equations for self-dual Yang-Mills fields are found. We show that the Yang-Mills action on such manifolds is topologically bounded from below, with the bound saturated precisely when the Yang-Mills field is self-dual. These results have a natural interpretation in supersymmetric string theory.

---

1 e-mail: r.acharya@qmw.ac.uk, Work supported by PPARC
2 e-mail: mjol@ictp.trieste.it
1 Introduction.

Self-duality in four dimensional Yang-Mills theory has had a remarkable impact in physics and mathematics. It is natural to ask if the four-dimensional equations for self-duality have an analogue in higher dimensions. This question was considered in [1] and a natural set of equations for self-duality in dimensions four through eight were found (the eight dimensional equations were independently discovered in [2]). These equations are:

$$F_{\mu\nu} = \frac{1}{2} \phi_{\mu\nu\lambda\rho} F_{\lambda\rho}$$  \hspace{1cm} (1)

Here, the duality operator $\phi_{\mu\nu\lambda\rho}$ is related to the structure constants of the octonions [1, 2]. The self-dual Yang-Mills equations were considered in Euclidean gravity by replacing the Yang-Mills curvature with the curvature of the Riemannian 4-manifold. These equations imply Ricci flatness and thus their solutions obey Einstein’s equations with zero cosmological constant. It is thus natural to consider the higher dimensional equations for self-duality [1, 2] in the context of Euclidean gravity. That is the purpose of this note.

In the next section we will consider the above self-duality equations in Euclidean gravity, by replacing the curvature of the gauge connection with the components of the curvature 2-form of a $D$-dimensional Riemannian manifold. This is in analogy with the ansatz for self-dual gravity in four dimensions [3]. In the four dimensional case (both in Yang-Mills and Euclidean gravity) the equations for self-duality turn out to be first order equations. We show that this is also true for the equations considered herein. We will find that the $D$-dimensional Riemannian manifold must have holonomy $SU(2), SU(3), G_2$ and $Spin(7)$ in $D=4, 6, 7$ and 8 respectively. In fact the equations in 4, 6 and 7 dimensions are derivable from those in $D=8$ via dimensional reduction. Following this we consider gauge fields propagating on the self-dual manifolds and by embedding the spin connection in the gauge connection, we show that solutions to the self-duality equations in Yang-Mills theory (1) can always be found. Once again, in analogy with the four dimensional case,

---

3Equations similar to (1), but with a nonzero constant different from $\frac{1}{2}$ were also considered in [1] in their investigations of generalisations of self-duality to higher dimensions. However, these equations do not have such a simple interpretation in higher dimensional Euclidean gravity or Riemannian geometry and for this reason we do not consider them here. These equations have recently been considered in [3].
the existence of these solutions is directly related to a topological bound on the Yang-Mills action. Finally we discuss a natural interpretation of these results in the context of supersymmetry and superstring theory.

2 Self-Duality in $D$-dimensional Euclidean Gravity.

The purpose of this section is to study the $D$-dimensional equations for self-duality in the context of Riemannian geometry. We will see that self-duality has some remarkable consequences: namely that Einstein’s equations are obeyed with zero cosmological constant.

Let $R_{ab}$ be the components of the curvature two-form $R$ for the $D$-dimensional oriented Riemannian manifold $M_D$. $R$ is constructed from the connection one-form coefficients $\omega_{ab}$ through:

$$R_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega_{cb} \quad (2)$$

We are interested in studying the self-duality equations (1) with the gauge field strength replaced with the curvature $R_{ab}$. These equations are simply:

$$R_{ab} = \frac{1}{2} \phi_{abcd} R_{cd} \equiv \phi R_{ab} \quad (3)$$

We will take these equations (with the duality operator $\phi_{abcd}$ to be specified shortly) to be the defining conditions for self-duality in $D$-dimensional Euclidean gravity, with $4 \leq D \leq 8$.

We begin in $D = 8$, because from here, the equations in $D = 7, 6, 5, 4$ can be derived by dimensional reduction. In $D = 8$, the natural choice, $\Box$, for the duality operator, $\phi_{abcd}$, is the unique $Spin(7)$ invariant four-index antisymmetric tensor which is Hodge self-dual. In fact, this tensor can be identified with the coefficients of the self-dual 4-form (denoted in what follows by $\phi$) which encodes the $Spin(7)$ structure of an 8-manifold, $M_8$, with holonomy $Spin(7)$.

$\Box$In fact, it is straightforward to show that this second order equation on the metric is actually equivalent to a first order equation. Specifically the duality condition is equivalent to $\omega = \phi \omega$. 


With this choice, the equations that the curvature be self-dual in the sense of (3) are a set of 7 non-trivial constraints among the 28 components $R_{ab}$. Thus, if the curvature is self-dual, the holonomy group of $M_8$ is 21 dimensional. From Bergers list [4], we see that self-duality implies that $M_8$ has holonomy $Spin(7)$. In fact, the equations for self-duality are equivalent to those discussed by Bonan [5], who pointed out that $M_8$ is Ricci flat, and admits a unique, nowhere vanishing 4-form, whose coefficients are precisely $\phi_{abcd}$.

2.1 $4 \leq D < 8$

It was pointed out in [1] that by deleting one index, say the eighth, in the $D = 8$ case, one finds equations for $D = 7$ self-duality. Repeating the process, one gets equations in $D = 6, 5, 4$. The $D = 4$ equations are precisely the instanton equations. In this sense, the eight-dimensional equations are a master set in which the lower dimensional ones are imbedded.

In the context of Euclidean gravity that we are discussing here, being able to delete $n$-indices in the eight dimensional equations requires that the eight-manifold is the product of an $(8-n)$-manifold, $M_{8-n}$, with an $n$-torus or $R^n$. The equations then reduce to a set of self-duality equations on the $(8-n)$-manifold, which will have non-trivial holonomy. The fact that the original eight-manifold is Ricci flat automatically implies that $M_{8-n}$ is Ricci flat also.

For $n = 1$ we find that self-duality is a set of 7 conditions for self duality among the 21 components of the curvature 2-form of the 7-manifold, $M_7$. These equations imply that $M_7$ has holonomy $G_2$ and were first discovered by Bonan [3]. In fact in this case, $\phi_{abcd}$ is identified with the components of the coassociative 4-form, which together with its 3-form Hodge dual encodes the $G_2$ structure of the manifold.

For $n = 2$ we find a set of 7 self-duality constraints among the 15 components of the curvature of $M_6$. These equations imply that $M_6$ has holonomy $SU(3)$. The duality operator in this case is identified with the components of the Hodge dual of the Kahler form.

The case $n = 3$ is essentially trivial as was also noted in the gauge theory case in [1]. This is because there exists a direction, say the fifth coordinate, for which all components of the curvature with indices in this direction vanish under self-duality. Thus, $M_5$ is the product of a four-manifold with a circle.
or the real line. In fact, the four manifold has holonomy $SU(2)$.

The case $n = 4$ leads to the usual equations for self-dual Euclidean gravity in four dimensions.

Explicit (non-compact) examples of metrics with the holonomy groups above are known [6, 7, 8, 9]. All of these may now be interpreted as gravitational instantons which satisfy equation (3).

To summarise the results of this section: self-duality in $D$-dimensional Euclidean gravity as defined by equation (3), leads to the conclusion that the space-time manifold $M_D$ has holonomy $SU(2)$, $SU(2) \times \{1\}$, $SU(3)$, $G_2$ and $Spin(7)$ for $D = 4, 5, 6, 7, 8$ respectively. Further the duality operator $\phi_{abcd}$ is identified with the components of some fundamental, nowhere vanishing 4-form $\phi$ on the manifold. All of these manifolds obey Einstein’s equations with zero cosmological constant.

### 3 Self-dual Yang-Mills fields

Using the technique of embedding the spin connection [10] in the gauge connection we are able to construct a self-dual gauge field directly from the self-dual metric of the previous section.

Let $G_{ab}$ be the generators of one of $SU(2)$, $SU(3)$, $G_2$ or $Spin(7)$. The ansatz for the gauge field (embedding the spin connection in the gauge group) is $A = \gamma G_{ab} \omega_{ab}$. The form index of $A$ comes from the form index of $\omega$ while the Lie algebra structure of $A$ comes from that of $G$. We easily see that

\[
F = dA + A \wedge A
\]

\[
= \gamma G_{ab} d\omega_{ab} + \kappa \gamma^2 G_{ab} \omega_{ac} \wedge \omega_{cb}
\]

where $\kappa$ is a constant that depends upon the group generated by $G_{ab}$. In each case it is trivial to solve for $\gamma$ giving $F = \gamma G_{ab} R_{ab}$. Duality of $F$ follows from that of $R$ and the symmetry of $R_{\mu\nu\lambda\rho}$ between the first pair and second pair of indices.

As noted in the introduction, self-duality of the gauge field in four-dimensions has a deeper meaning. Self-dual gauge fields minimize the Yang-Mills action and the action at the minimum is a topological invariant. The

---

5 The 4-form in eight dimensions can be written as $\bar{\eta} \gamma_{\mu\nu\rho\sigma} \eta$, where $\eta$ is a covariantly constant unit spinor.
field configuration is that of an instanton. It is a remarkable result and further confirmation of the naturalness of our construction that an entirely analogous topological bound appears in eight dimensions for these instantons. The topological bounds for Yang-Mills field configurations in lower dimensions are unified in the discussion of the eight dimensional bound.

From the definition of Hodge duality and $\phi$-duality in eight dimensions we can easily derive the following.

\[
S_{YM} = \frac{1}{8!} \int_{M_8} Tr(\mathcal{F} \wedge \ast \mathcal{F})
\]

(6)

\[
= \frac{1}{16} \int_{M_8} Tr(F - \phi F)^2 - \frac{1}{8!} \int_{M_8} Tr(F \wedge F) \wedge \phi
\]

(7)

\[
\geq \frac{1}{8!} \int_{M_8} Tr(F \wedge F) \wedge \phi
\]

(8)

We have used the identity $\phi_{ijkl} \phi_{kl} = 6(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{kj}) - 4 \phi_{ijkl}$ to prove the equality in (7). The final line is proportional to the evaluated first Pontrjagin class of the gauge bundle for which $F$ is the curvature. Furthermore, when $F = \phi F$ the inequality is saturated. The key point is that the $\phi$-self-dual action is a topological invariant.

In line with our previous observations on the reduction of this eight-dimensional duality to lower-dimensional dualities we have corresponding reductions of this topological bound. The process of reduction leads us finally to the well-known formula in four dimensions. In general we have,

\[
S_{YM}^D = \frac{1}{4} \int_{M_D} Tr F^2 \geq - \int_{M_D} Tr(F \wedge F) \wedge \chi_{D-4}
\]

(9)

where $\chi_{D-4}$ is proportional to the Hodge dual of the appropriate 4-form $\phi$ on $M_D$. That is, $\chi_{D-4}$ is proportional to: 1 in four dimensions; a trivial one-form in five dimensions; the Kahler form in six dimensions; the Hodge dual of the coassociative 4-form in seven dimensions; and the 4-form itself in eight dimensions. In each case the Yang-Mills action is bounded by a topological quantity and the bound is saturated when $F$ satisfies the relevant duality equation in $D$-dimensions.
4 Discussions

We have shown that in eight dimensions there exists a set of master duality equations for self-dual Yang-Mills and self-dual Euclidean gravity in the sense defined by equations (1) and (3). This master equation incorporates self-duality in lower dimensions four, five, six and seven.

Our results have an explicit link with supersymmetry. The list of self-dual manifolds that we have presented are precisely those which admit covariantly constant spinors. These are the only manifolds which provide supersymmetric vacuum solutions of theories with local supersymmetry. This is an intriguing and unexpected surprise.

As well as being important for Einstein and Yang-Mills theories and their supersymmetric counterparts in eight dimensions and lower, our construction turns out to be very natural in string theory. In the low energy effective field theory of the heterotic string there is an anomaly cancellation condition,

\[ dH = trR \wedge R - \frac{1}{30} TrF \wedge F \quad (10) \]

which is automatically satisfied by our ansatz. Our solutions are in fact solutions to all the equations of low energy heterotic string theory in 9 + 1 dimensions when one takes the dilaton to be constant and the torsion to be zero. Hence the noncompact examples of such metrics are solitonic solutions to heterotic string theory \(^6\). More general classes of string soliton may easily be constructed from this starting point \(^8\) if we now relax the restrictions on the dilaton \( \phi \) and torsion \( H \) and only partially embed the spin connection in the gauge connection\(^6\). New classes of string solitons arise in these cases and one finds a variety of these solitons entirely analogous to the varieties of heterotic 5-brane solitons as discussed in \(^13\).

This new construction in field theory and string theory is intimately linked with the existence of octonions or Cayley numbers. \( SO(8) \) may be decomposed into \( G_2 \times S_L^7 \times S_R^7 \) where the left and right seven spheres are left and right multiplication by octonions\(^2\). The eight dimensional duality condition is related to this decomposition in precisely the way that the four-dimensional

\(^6\)Note that by embedding the spin connection in the gauge connection, each metric on \( M_D \) automatically gives a solution of (1) in flat space.

\(^7\)Simple examples of solitonic solutions with non-zero torsion are given in \(^1\).
duality is related to the decomposition of $SO(4)$ into $S^3_{L} \times S^3_{R}$ where the left and right $S^3$’s are multiplication by quaternions.

Could it be that octonions have a more fundamental role to play in the current attempts to reformulate string theory? There are now several pieces of interesting work pointing in this direction [14].

Acknowledgement

B.A. would like to thank the ICTP for inviting him for the visit during which this work commenced and M. O’L. would like to thank QMW college for a reciprocal invitation during which this paper was completed.
References

[1] E. Corrigan, C. Devchand, D. Fairlie and J. Nuyts, Nucl. Phys. B214 (1983), 452.

[2] R. Dündarer, F. Gürsey and C-H. Tze, J. Math. Phys. 25 (1984), 1496.

[3] A. H. Bilge, T. Dereli and S. Koçak, Lett. Math. Phys. 36 (1996), 301.

[4] M. Berger, Bull. Soc. Math. France 83 (1955), 279.

[5] E. Bonan, C. R. Acad. Sci. Paris 262 (1966), 127.

[6] T. Eguchi and A.J. Hanson, Phys. Lett. B74 (1978), 249.

[7] E. Calabi, Ann. scient. Éc. norm. sup. 12 (1979), 269.

[8] R. Bryant, Ann. Math. 126 (1987), 525.

[9] G. Gibbons, D. Page and C. Pope, Comm. Math. Phys. 127 (1990), 529.

[10] J. Charap and M. Duff, Phys. Lett. B69 (1977), 445.

[11] J. Harvey and A. Strominger, Phys. Rev. Lett. 66 (1991), 549. M. Günaydin and H. Nicolai, Phys. Lett. 351 (1995), 169.

[12] B. Acharya and M. O’Loughlin, work in progress.

[13] C. Callan, J. Harvey and A. Strominger, Nuc. Phys. B359 (1991), 611.

[14] D.B. Fairlie and C.A. Manogue, Phys. Rev. D36 (1987), 475-489 and E. Martinec, hep-th/9608017.