Output feedback exponential stabilization for 1-D unstable wave equations with boundary control matched disturbance

Hua-Cheng Zhou, George Weiss

Abstract: We study the output feedback exponential stabilization of a one-dimensional unstable wave equation, where the boundary input, given by the Neumann trace at one end of the domain, is the sum of the control input and the total disturbance. The latter is composed of a nonlinear uncertain feedback term and an external bounded disturbance. Using the two boundary displacements as output signals, we design a disturbance estimator that does not use high gain. It is shown that the disturbance estimator can estimate the total disturbance in the sense that the estimation error signal is in $L^2[0, \infty)$. Using the estimated total disturbance, we design an observer whose state is exponentially convergent to the state of original system. Finally, we design an observer-based output feedback stabilizing controller. The total disturbance is approximately canceled in the feedback loop by its estimate. The closed-loop system is shown to be exponentially stable while guaranteeing that all the internal signals are uniformly bounded.

Keywords: Disturbance rejection, output feedback controller, unstable wave equation, exponential stabilization

AMS subject classifications: 37L15, 93D15, 93B51, 93B52.

1 Introduction

In this paper, we are concerned with the following one-dimensional wave equation:

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \\ w_x(0,t) = -qw(0,t), \\ w_x(1,t) = u(t) + f(w(\cdot, t), w_t(\cdot, t)) + d(t), \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \\ y_m(t) = (w(0,t), w(1,t)), \end{cases} \tag{1.1}$$

where $x \in (0,1), t \geq 0, (w, w_t)$ is the state, $u$ is the control input signal, and $y_m$ is the output signal, that is, the boundary traces $w(0,t)$ and $w(1,t)$ are measured. The equation containing the constant $q > 0$ creates a destabilizing boundary feedback at $x = 0$ that acts like a spring with negative spring constant. $f : H^1(0,1) \times L^2(0,1) \to \mathbb{R}$ is an unknown possibly nonlinear mapping that represents the internal uncertainty in the

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\[\text{H.-C. Zhou (hczhou@amss.ac.cn) and G. Weiss (gweiss@eng.tau.ac.il) are with the School of Electrical Engineering, Tel Aviv University, Ramat Aviv, Israel, 69978.}\]
model, and $d$ represents the unknown \textit{external disturbance}, which is only supposed to satisfy $d \in L^\infty[0,\infty)$. For the sake of simplicity, we denote

$$F(t) := f(w(\cdot,t), w_t(\cdot,t)) + d(t)$$

and we call this signal the \textit{total disturbance}. We often write $\dot{w}$ instead of $w_t$.

![Figure 1: Our plant, an unstable string system](image-url)

We consider system (1.1) in the state Hilbert space $\mathcal{H} = H^1(0,1) \times L^2(0,1)$ with the inner product given by

$$\langle (\phi_1, \psi_1), (\phi_2, \psi_2) \rangle_{\mathcal{H}} = \int_0^1 \phi_1'(x)\phi_2'(x) + \psi_1(x)\psi_2(x) \, dx + \phi_1(0)\phi_2(0).$$

The objective of this paper is to design a feedback controller which generates the control signal $u$, using only the measurements $y_m$, such that the state of the closed-loop system (that includes the state of the system (1.1)) converges to zero, exponentially. Later in the paper we shall also discuss a related problem, where the negative spring is replaced by a negative damper. More precisely, on the right hand-side of the equation in (1.1) containing $q$, we have $-qw_t(0,t)$ (instead of $-qw(0,t)$). We shall solve the exponential stabilization problem also for this alternative nonlinear wave system (5.1). These results have been announced (without proof) in the IFAC conference paper [37].

For simplicity of implementation, it is desirable to use a small number of input and output signals for output feedback stabilization. For the disturbance free situation (that is, $f \equiv 0$ and $d \equiv 0$), the stabilization of the system (1.1) was first investigated in [22], who used two measurement signals to obtain an exponentially stable closed-loop system. Using only one displacement signal as measurement, strong stability of the closed loop system was achieved in [15], using Lyapunov functionals. In the recent paper [12], the output signal is only one displacement signal and an exponentially stabilizing controller is designed by using a new “backstepping” method. However, when the total disturbance $F$ acts at the control end, the stabilization problem for (1.1) becomes much more difficult. Here we present a dynamic compensator which employs a disturbance estimator described by partial differential equations (PDEs) and full state feedback based on the observer state. Our compensator consists of two parts: the first part is to cancel the total disturbance by applying the active disturbance rejection control (ADRC) strategy, which is an unconventional design strategy first proposed by Han in 1998 [19]; the second part is to stabilize the system by using the classic backstepping approach. The stabilization problem of system (1.1) has been considered first in [14], where the vector of output measurement was taken to be $y_m(t) = (w(0,t), w_t(1,t))$ and the disturbance has the following form:

$$d(t) = \sum_{j=1}^m [\theta_j \sin \alpha_j t + \varphi_j \cos \alpha_j t], \quad t \geq 0,$$
with known frequencies $\alpha_j$ and unknown amplitudes $\theta_j, \vartheta_j, j = 1, 2, \ldots, m$, and the resulting closed-loop system is asymptotically stable. Obviously, the disturbance signal in this paper is more general than the one described above. Recently, the stabilization problem of system \((1.1)\) with $f \equiv 0, \ d \in L^\infty[0, \infty)$ has been investigated in \([10]\), where the output measurements are $\{w(0, t), w_t(0, t), w(1, t)\}$, their result is that the closed-loop system is asymptotically stable. The output feedback of \([10]\) uses one more measurement than \([14]\). Apart from the more general external disturbance, another point that is different here from \([14, 10]\) is that the closed-loop system in this paper is exponentially stable and we do not require to measure the velocity $w_t(0, t)$ (or $w_t(1, t)$) which is hard to measure \([9]\). In this paper, we only use two scalar signals (the components of $y_m$) and this is a minimal set of measurement signals. As shown in Figure \(1\) we apply the control force $u$ to deal with both the internal uncertainty $f$ and the unknown external disturbance $d$.

Many control methods have been applied to deal with uncertainties in PDE systems. The internal model principle, a classical method to cope with uncertainty, has been generalized to infinite-dimensional systems \([3, 23, 27, 24]\). In \([29]\), the tracking and disturbance rejection problems for infinite-dimensional linear systems, with reference and disturbance signals that are finite superpositions of sinusoids, are considered. The results are applied to some PDEs including the noise reduction in a structural acoustics model described by a two-dimensional PDE. An interesting PDE example in \([29]\) is disturbance rejection in a coupled beam where the disturbance and control are not matched. Very recently, the backstepping approach has been used to achieve output regulation for the one-dimensional heat equation in \([7, 8]\), and the one-dimensional Schrödinger equation in \([36]\). For a stochastic PDE, an optimal control problem constrained by uncertainties in system and control is addressed in \([30]\). An adaptive design is exploited in \([1, 21]\) for dealing with the anti-stable wave equation with unknown anti-damping coefficient. In \([13]\), a boundary control based on the Lyapunov method is designed for the one-dimensional Euler-Bernoulli beam equation with spatial and boundary disturbances. However, there are not so many works, to the best of our knowledge, on exponential stabilization (instead of reference tracking) of PDEs with disturbance by using output feedback. Sliding mode control that is inherently robust is the most popular approach that can achieve exponential stability for infinite-dimensional systems but most often, the literature considers state feedback controllers \([28, 5, 16, 34]\), while here we aim for output feedback.

Output feedback stabilization for one-dimensional anti-stable wave equation has been considered in \([17]\), where a new type of observer has been constructed by using three output signals to estimate the state first and then estimate the disturbance via the state of the observer through an extended state observer (ESO). However, the initial state is required to be smooth in \([17]\) and they obtain asymptotic stability (not exponential, like here). In the recent paper \([11]\) the authors continue to investigate this question and introduce a new disturbance estimator which is different from the traditional one, the smoothness requirement on the initial state being removed. In \([11]\), still three output signals are used as inputs to the controller and the controller achieves asymptotic stability of the closed-loop system. In this paper we consider the output feedback stabilization for a one-dimensional unstable (or anti-stable) wave equation by using two signals only, which is an improvement, and in addition we achieve exponential stability of the state of the controlled original systems, which is stronger than asymptotic stability.
Define the operators $\mathbf{A}: \mathcal{D}(\mathbf{A}) \to \mathbb{H}$, $\mathbf{B}_1, \mathbf{B}_2 : \mathbb{C} \to \mathcal{D}(\mathbf{A}^*)'$ by
\[
\mathbf{A}(\phi, \psi) = (\psi, \phi'') \quad \forall (\phi, \psi) \in \mathcal{D}(\mathbf{A}),
\]
\[
\mathcal{D}(\mathbf{A}) = \{ (\phi, \psi) \in H^2(0,1) \times H^1(0,1) \mid \phi'(0) = \phi(0), \phi'(1) = 0 \},
\]
where $\delta_a$ is the Dirac pulse at $x = a$, with a suitable interpretation. It can be shown (see [25] Example 5.2) that $\mathcal{D}(\mathbf{A}^*) = \mathcal{D}(\mathbf{A})$, $\mathbf{A}^* = -\mathbf{A}$ and
\[
\mathbf{B}_1^*(\phi, \psi) = -\psi(0), \quad \mathbf{B}_2^*(\phi, \psi) = \psi(1) \quad \forall (\phi, \psi) \in \mathcal{D}(\mathbf{A}^*). \tag{1.4}
\]

We often write a pair $(a, b)$ as a column vector $[^a \, b]$. The system (1.1) can be rewritten as
\[
\frac{d}{dt} \begin{bmatrix} w(\cdot, t) \\ w_1(\cdot, t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} w(\cdot, t) \\ w_1(\cdot, t) \end{bmatrix} - \mathbf{B}_1((q + 1)w(0, t)) + \mathbf{B}_2 \left[ f \left( \begin{bmatrix} w(\cdot, t) \\ w_1(\cdot, t) \end{bmatrix} \right) \right] + u(t) + d(t). \tag{1.5}
\]

The equivalence is meant in the algebraic sense, without any reference to existence or uniqueness of solutions, see Remark 10.1.4 in [33]. The proof of the equivalence between (1.1) and (1.5) uses the theory of boundary control systems in [33 Section 10.1], and the details (for a slightly different system) are in [25 Example 5.2], where the notation $B_N$ and $B$ is used in place of $\mathbf{B}_1$ and $\mathbf{B}_2$ (in this order). About existence and uniqueness of solutions we have the following proposition, whose proof is given in the Appendix.

**Proposition 1.1.** The above operator $\mathbf{A}$ generates a unitary group on $\mathbb{H}$ and $\mathbf{B}_1, \mathbf{B}_2$ are admissible control operators for it. Suppose that $f : \mathbb{H} \to \mathbb{R}$ satisfies a global Lipschitz condition on $\mathbb{H}$ and $f(0,0) = 0$. Then for any $(w_0, w_1) \in \mathbb{H}$ and $u, d \in L^2_{\text{loc}}[0, \infty)$, there exists a unique global solution to (1.1) such that $(w(\cdot, t), w_1(\cdot, t)) \in C(0, \infty; \mathbb{H})$.

The paper is organized as follows: We consider the exponential stabilization of the unstable wave equation (1.1) in Sections 2 to 4. More precisely, in Section 2 we design an infinite-dimensional total disturbance estimator that does not use high gain, for the system (1.1). We propose a state observer based on this estimator and develop an output feedback stabilizing controller by compensating the total disturbance in Section 3. The exponential stability of the resulting closed-loop system for (1.1) is proved in Section 4. Section 5 is devoted to the output feedback exponential stabilization of the alternative anti-stable wave equation mentioned earlier (with the negative damper).

## 2 Disturbance estimator design

In this section, our objective is to design a total disturbance estimator using the input and output signals of the system (1.1).

**Remark 2.1.** We explain the need for a disturbance estimator on a simple finite-dimensional example. Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^n$. Consider the system
\[
x(t) = Ax(t) + Bd(t) \tag{2.1}
\]
where $x(t) \in \mathbb{R}^n$ is the state trajectory at time $t$ and $d(t) \in \mathbb{R}$ is the disturbance signal at time $t$. Suppose that $A$ is stable (Hurwitz). The solution is given by
\[
x(t) - e^{At}x(0) = \int_0^t e^{A(t-s)}d(s)ds = e^{At} \int_0^1 e^{A(\frac{t}{2}-s)}d(s)ds + \int_1^t e^{A(t-s)}d(s)ds.
\]
From here, it is easy to verify that \( x(t) \to 0 \) as \( t \to \infty \) if \( d \in L^2[0, \infty) \). Therefore, to design a stabilizing control law for \( \dot{x}(t) = Ax(t) + Bu(t) + d(t) \), it suffices to find a control law that generates \( u \) such that \( u + d \in L^2[0, \infty) \).

For many boundary control systems, the control operator \( B \) is unbounded but admissible for the underlying operator semigroup. For more on the admissibility concept we refer for instance to [33]. When \( x \) takes values in a Hilbert space \( X \), \( A \) generates an exponentially stable operator semigroup on \( X \) and \( B \) is admissible, we still have a stability result similar to Remark 2.6; see the following lemma. For related results see [23, 20]. As is customary, we denote by \( X_{-1} \) the dual of \( D(A^*) \) with respect to the pivot space \( X \), see [33].

**Lemma 2.1.** Let \( A \) be the generator of an exponentially stable operator semigroup \( e^{At} \) on the Hilbert space \( X \). Assume that \( B_i \in \mathcal{L}(U_i, X_{-1}), \ i = 1, 2, \ldots, n \) are admissible control operators for \( e^{At} \) \((U_i \text{ are Hilbert spaces})\). Then the initial value problem

\[
\dot{x}(t) = Ax(t) + \sum_{i=1}^{n} B_i u_i(t), \quad x(0) = x_0, \quad u_i \in L^2_{t0\infty}([0, \infty), U_i),
\]

admits a unique solution \( x \in C([0, \infty); X) \), and if \( u_i \in L^\infty([0, \infty), U_i) \), \( i = 1, 2, \ldots, n \), then \( x \) is bounded. If for each index \( i \), either \( u_i \in L^2([0, \infty), U_i) \) or \( \lim_{t \to \infty} \|u(t)\|_{U_i} = 0 \) holds, then \( x(t) \to 0 \) as \( t \to \infty \). Moreover, if there exist two constants \( M_0, \mu_0 > 0 \) such that \( \|u\|_{U_i} \leq M_0 e^{-\mu_0 t}, \ i = 1, 2, \ldots, n \), then \( \|x(t)\|_X \leq M e^{-\mu t} \) for some \( M, \mu > 0 \).

**Proof.** Due to the admissibility, by [33] Proposition 4.2.5, the solution \( x \) is a continuous \( X \)-valued function of \( t \) given by

\[
x(t) = e^{At}x_0 + \sum_{i=1}^{n} \int_{0}^{t} e^{A(t-s)}B_i u_i(s)ds.
\]

By assumption, there exist constants \( M_1, \mu_1 > 0 \) such that \( \|e^{At}\| \leq M_1 e^{-\mu_1 t} \) for all \( t \geq 0 \). Thus, by superposition, we only have to prove the statements in the lemma for one of the integral terms in the above sum, \( x_i(t) = \int_{0}^{t} e^{A(t-s)}B_i u_i(s)ds \) (with \( i \) fixed).

Suppose that \( u_i \in L^\infty([0, \infty), U_i) \). Since \( B_i \) is \( L^\infty \)-admissible for \( e^{At} \) by virtue of [35] Remark 4.7, it follows from [35] Remark 2.6 that there exists a constant \( L_1 > 0 \) independent of \( u_i \) and of \( t \) such that \( x_i(t) \) is bounded: \( \|x_i(t)\|_{X} \leq L_1 \|u_i\|_{L^\infty([0, \infty), U_i)} \).

Now suppose that \( u_i \in L^2([0, \infty), U_i) \) or \( \lim_{t \to \infty} \|u_i(t)\|_{U_i} = 0 \). For any \( \sigma > 0 \), there exists \( t_0 > 0 \) such that

\[
\|u_i\|_{L^2([t_0, \infty), U_i)} \leq \sigma, \quad \text{or} \quad \|u_i\|_{L^\infty([t_0, \infty), U_i)} \leq \sigma.
\]

If \( u_i \in L^2([0, \infty), U_i) \) then it follows from [35] Remark 2.6 that for any \( t \geq t_0 \),

\[
\left\| \int_{t_0}^{t} e^{A(t-s)}B_i u_i(s)ds \right\|_{X} \leq L_2 \|u_i\|_{L^2([t_0, \infty), U_i)} \leq L_2 \sigma, \tag{2.2}
\]

where \( L_2 \) is a constant that is independent of \( u_i \) and of \( t \). If \( \lim_{t \to \infty} \|u_i(t)\|_{U_i} = 0 \), then by [35] Remark 2.6, the \( L^\infty \)-admissibility of \( B_i \) implies that for any \( t \geq t_0 \),

\[
\left\| \int_{t_0}^{t} e^{A(t-s)}B_i u_i(s)ds \right\|_{X} \leq L_1 \|u_i\|_{L^\infty([t_0, \infty), U_i)} \leq L_1 \sigma. \tag{2.3}
\]

Using the exponential stability of \( e^{At} \) again, we have that for any \( t \geq t_0 \),
This shows that \(\limsup_{t \to \infty} \|x(t)\|_X \leq \max\{L_1, L_2\} \sigma\). Since \(\sigma > 0\) was arbitrary, we conclude that the last limsup is 0, whence \(x(t) \to 0\) as \(t \to \infty\).

For the last part of the lemma, suppose that there exist \(M_0, \mu_0 > 0\) such that \(\|u_i\|_{V_i} \leq M_0 e^{-\mu_0 t}\). Choose a number \(\mu \in (0, \min\{\mu_0, \mu_1\})\), then \(A + \mu I\) still generates an exponentially stable operator semigroup. Define the functions \(x^\mu_i\) and \(u^\mu_i\) by

\[
x^\mu_i(t) = e^{\mu t} x_i(t), \quad u^\mu_i(t) = e^{\mu t} u_i(t),
\]

then it is easy to see that the differential equation \(\dot{x}_i^\mu = (A + \mu I)x_i^\mu + B_i u_i^\mu\) holds. Since \(u_i^\mu\) is bounded, by an argument used at the beginning of this proof (with \(x_i^\mu\) and \(u_i^\mu\) in place of \(x_i\) and \(u_i\)), there exists \(L_3 > 0\) such that \(\|x_i^\mu(t)\|_X \leq L_3\|u_i^\mu\|_L^\infty((0, \infty), U_i)\). Clearly this implies that \(x_i\) tends to zero at the exponential rate \(\mu\).

Now we design a total disturbance estimator for the system (1.1). This is an infinite dimensional system whose state consists of the functions \(v, v_t, z, z_t, W\) defined on \((0, 1)\):

\[
\begin{align*}
v_{tt}(x, t) &= v_{xx}(x, t), \\
v_x(0, t) &= -qw(0, t) + c_1[v(0, t) - w(0, t)], \quad v_x(1, t) = u(t) - W_x(1, t), \\
v(x, 0) &= v_0(x), \quad v_t(x, 0) = v_1(x), \\
z_{tt}(x, t) &= z_{xx}(x, t), \\
z_x(0, t) &= \frac{c_1}{1-c_0} z(0, t) + \frac{c_0}{1-c_0} z_t(0, t), \quad z(1, t) = -v(1, t) + w(1, t) - W(1, t), \\
z(x, 0) &= z_0(x), \quad z_t(x, 0) = z_1(x), \\
W_t(x, t) &= -W_x(x, t), \\
W(0, t) &= -c_0[v(0, t) - w(0, t)], \quad W(x, 0) = W_0(x),
\end{align*}
\]

(2.5)

where \(c_0\) and \(c_1\) are two positive design parameters, \(c_0 < 1\), \((v_0, v_1, z_0, z_1, W_0) \in \mathbb{H}^2 \times H^1(0, 1)\) is the initial state of the disturbance estimator and its input signals are \(u, w(0, t)\) and \(w(1, t)\). The output of this estimator is \(\hat{F}(t) = z_x(1, t)\).

*Remark 2.2.* Before going into the tedious technical details, we give an informal overview of how the total disturbance estimator (2.5) works. The \("\langle v, W \rangle\)\)-part of (2.5) is used to channel the total disturbance from the original system to an exponentially stable wave equation with state \((p, p_t)\), where \(p = w - v - W\), described in (2.9). (The equations (2.9) contain also a \(W\)-part, but from an input-output point of view, this \(W\)-part is irrelevant.) The effect of \(u\) is cancelled in the estimator, so that \(u\) has no influence on \(p\). The wave equation system with state \((p, p_t)\) has input \(F\) and output \(p(1, t)\) and it represents from an input-output view the linear part of the plant and the \("\langle v, W \rangle\)-part\) of (2.5), taken together, see Figure 2. This is a well-posed boundary control system (in the sense of [33 Definition 10.1.7]), with a bounded observation operator, so that for large \(\text{Re } s\), its transfer function \(G\) satisfies \(\|G(s)\| \leq m(\text{Re } s)^{-\frac{1}{2}}\), see for instance [33 Proposition 4.4.6].
The $z$-part of (2.5) is in fact the same boundary control system as the one just described, but with the roles of input and output reversed. This would be flow inversion in the sense of [32], except that the $z$-part is ill-posed. Indeed, its transfer function is $G^{-1}$, and from our estimate on $G$ it follows that $G^{-1}$ is not proper. Overall, the transfer function from $F$ to $\hat{F}$ is the constant 1. The difference $\hat{F} - F$ depends linearly on the deviation between the initial state of the $z$-part of (2.5) and the initial state of the $p$-part of (2.9). Since the $z$-part, in the absence of any input (i.e., when $p(1,t) \equiv 0$) is exponentially stable, and its observation operator giving $\hat{F}$ is admissible (as we shall see in Lemma 2.3), it follows that $\hat{F} - F \in L^2[0, \infty)$. The overall linear system shown in Figure 2 (with input $(F,u)$ and output $\hat{F}$) is well-posed. If $f$ is globally Lipschitz, then also the overall nonlinear system (with input $(d,u)$ and output $\hat{F}$) is well-posed (due to Proposition 1.1).

Figure 2. The total disturbance estimator connected to the plant. The $z$-part of the disturbance estimator (2.5) is the (ill-posed) flow inverse of the wave system (2.9) (which has input $F$ and output $p(1,t)$). The system with input $(F,u)$ and output $\hat{F}$ is linear and its transfer function is [1 0].

Now we start providing the technical details for the operation of the total disturbance estimator. Consider the plant (1.1) coupled with the estimator (2.5) and denote $\tilde{v}(x,t) = v(x,t) - w(x,t)$.

Then it is easy to verify that the subsystem with state $(\tilde{v}(x,t), W(x,t))$ satisfies

$$
\begin{align*}
\tilde{v}_{tt}(x,t) &= \tilde{v}_{xx}(x,t), \\
\tilde{v}_x(0,t) &= \tilde{c}_1 \tilde{v}(0,t), \quad \tilde{v}_x(1,t) + W_x(1,t) = - F(t), \\
W_t(x,t) &= - W_x(x,t), \quad W(0,t) = - \tilde{c}_0 \tilde{v}(0,t),
\end{align*}
$$

where $F$ is the total disturbance from (1.2). It will be convenient to change variables once more, by introducing the notation

$$
\begin{align*}
p(x,t) = - \tilde{v}(x,t) - W(x,t), \quad \tilde{c}_0 = \frac{c_0}{1 - c_0}, \quad \tilde{c}_1 = \frac{c_1}{1 - c_0},
\end{align*}
$$

then from the last part of (2.7) we see that $p(0,t) = -(1 - \tilde{c}_0)\tilde{v}(0,t)$ and hence (using that $-W_x(0,t) = W_t(0,t)$) the subsystem with state $(p(\cdot,t), W(\cdot,t))$ is governed by

$$
\begin{align*}
p_{tt}(x,t) &= p_{xx}(x,t), \\
p_x(0,t) &= \tilde{c}_1 p(0,t) + \tilde{c}_0 p_t(0,t), \quad p_x(1,t) = F(t), \\
W_t(x,t) &= - W_x(x,t), \quad W(0,t) = \tilde{c}_0 p(0,t),
\end{align*}
$$

(2.9)
with the initial state \( p(x, 0) = -\hat{v}(x, 0) - W(x, 0) \), \( p_t(x, 0) = -\hat{v}_t(x, 0) + W_x(x, 0) \). The following lemma states some stability properties of the system (2.7).

**Lemma 2.2.** Suppose that \( d \in L^\infty[0, \infty) \) (or \( d \in L^2[0, \infty) \)), \( f : \mathbb{H} \to \mathbb{R} \) is continuous and that (3.1) admits a unique solution \((w, w_t) \in C(0, \infty; \mathbb{H})\) which is bounded. For any initial state \((\hat{v}_0, \hat{v}_t, W_0) \in \mathbb{H} \times H^1(0, 1)\) with the compatibility condition \( W_0(0) = -c_0\hat{v}_0(0) \), there exists a unique solution \((\hat{v}, \hat{v}_t, W) \in C(0, \infty; \mathbb{H} \times H^1(0, 1))\) to (2.7) and

\[
\sup_{t \geq 0} \|(\hat{v}(\cdot, t), \hat{v}_t(\cdot, t), W(\cdot, t))\|_{\mathbb{H} \times H^1(0, 1)} < \infty. \tag{2.10}
\]

If we assume further that \( \lim_{t \to \infty} |f(w, w_t)| = 0 \) and \( d \in L^2[0, \infty) \), then

\[
\lim_{t \to \infty} \|(\hat{v}(\cdot, t), \hat{v}_t(\cdot, t), W(\cdot, t))\|_{\mathbb{H} \times H^1(0, 1)} = 0. \tag{2.11}
\]

If we assume that \( f \equiv 0 \) and \( d \equiv 0 \), then there exist two constants \( M, \mu > 0 \) such that

\[
\|(\hat{v}(\cdot, t), \hat{v}_t(\cdot, t), W(\cdot, t))\|_{\mathbb{H} \times H^1(0, 1)} \leq M e^{-\mu t} \quad \forall t \geq 0. \tag{2.12}
\]

**Proof.** We shall use the equivalent system (2.9). We define the operators \( A \) and \( B \) (that resemble \( \hat{A} \) and \( \hat{B}_2 \) from (1.3)) by

\[
\begin{align*}
A(\phi, \psi) &= (\psi, \phi'') \quad \forall (\phi, \psi) \in D(A), \\
B &= (0, \delta_1), \\
D(A) &= \{(\phi, \psi) \in H^2(0, 1) \times H^1(0, 1) \mid \phi'(0) = \bar{c}_1\phi(0) + \bar{c}_0\psi(0), \phi'(1) = 0\}. \tag{2.13}
\end{align*}
\]

Then the “p-part” of (2.9) can be written in abstract form as

\[
\frac{d}{dt} \begin{bmatrix} p(\cdot, t) \\ p_t(\cdot, t) \end{bmatrix} = A \begin{bmatrix} p(\cdot, t) \\ p_t(\cdot, t) \end{bmatrix} + B \begin{bmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{bmatrix} + d(t). 
\]

It is well-known [15] Theorem 2.1] that \( A \) generates an exponentially stable operator semigroup \( e^{At} \) on \( \mathbb{H} \) and \( B \) is admissible for \( e^{At} \). Since \( f : \mathbb{H} \to \mathbb{R} \) is continuous and \((w, w_t) \in C(0, \infty; \mathbb{H}) \) is bounded, we have \( f(w, w_t) \in L^\infty[0, \infty) \). Thus, by \( d \in L^\infty[0, \infty) \) or by \( d \in L^2[0, \infty) \), it follows from Lemma 2.1 that the “p-part” of (2.9) admits a unique bounded solution, so that there exists a constant \( M_1 > 0 \) such that

\[
\sup_{t \geq 0} \|(p(\cdot, t), p_t(\cdot, t))\|_\mathbb{H} \leq M_1. \tag{2.14}
\]

We claim that \( \|W(\cdot, t)\|_{H^1(0, 1)} \) is uniformly bounded for all \( t \geq 0 \). To prove this, first we show that for all \( t \geq 1 \),

\[
\int_0^1 p^2_t(0, t - x)dx \leq 3 \max_{s \in [t-1, t]} \|(p(\cdot, t), p_t(\cdot, t))\|_\mathbb{H}^2. \tag{2.15}
\]

Indeed, define

\[
\rho(t) = 2 \int_0^1 (x - 1)p_t(x, t)p_x(x, t)dx.
\]

Then \( |\rho(t)| \leq 2\|p_t(\cdot, t)\|_{L^2} \|p_x(\cdot, t)\|_{L^2} \leq \|(p(\cdot, t), p_t(\cdot, t))\|_\mathbb{H}^2 \). Computing \( \frac{d}{dt} \rho(t) \) along the solution of the “p-part” of (2.9), using that \( 2 \frac{d}{dt} [p_x p_x] = \frac{d}{dx} [p_x^2 + p_t^2] \), yields

\[
\dot{\rho}(t) = p^2_x(0, t) + p^2_t(0, t) - \int_0^1 \left[p^2_x(x, t) + p^2_t(x, t)\right]dx \geq p^2(0, t) - \int_0^1 [p^2_x(x, t) + p^2_t(x, t)]dx,
\]
which implies that, for \( t \geq 1 \),
\[
\int_{t-1}^{t} p_s^2(0,s)ds \leq \int_{t-1}^{t} \|(p(\cdot,s),p_s(\cdot,s))\|_{L^2}^2 ds + \rho(t) - \rho(t-1) \leq 3 \max_{s \in [t-1,t]} \|(p(\cdot,t),p_t(\cdot,t))\|_{L^2}^2.
\]  
(2.16)

On the other hand, since for any \( t \geq 1 \), \( \int_{0}^{1} p_s^2(0,t-x)dx = \int_{t-1}^{t} p_s^2(0,s)ds \), we obtain \( (2.15) \).

Define the function
\[ W(x,t) = \begin{cases} 
\bar{c}_0 p(0,t-x), & t \geq x, \\
W_0(x-t), & x > t. 
\end{cases} \] 
(2.17)

Then a simple computation shows that \( W \) solves the “\( W \)-part” of \( (2.9) \). It follows from the Sobolev embedding theorem, the last part of \( (2.9) \) and \( (2.14) \) that
\[ |W(0,t)| = \bar{c}_0 |p(0,t)| \leq \bar{c}_0 \|p(\cdot,t)\|_{H^1(\mathbb{R})} \leq \bar{c}_0 \|p(\cdot,t),p_t(\cdot,t)\|_{\mathbb{H}} \leq \bar{c}_0 M_1. \] 
(2.18)

From \( (2.17) \) we derive that for \( t \geq 1 \), \( \int_{0}^{1} W_x^2(x,t)dx = \bar{c}_0^2 \int_{0}^{1} p_t^2(0,t-x)dx \). Then the boundedness of \( \|W(\cdot,t)\|_{H^1(\mathbb{R})} \) follows from here, using \( (2.14) \), \( (2.18) \) and \( (2.15) \).

Since \( \hat{v}(x,t) = -p(x,t) - W(x,t) \) and \( W_t(x,t) = -W_x(x,t) \), we have
\[ \sup_{t \geq 0} \|\hat{v}(\cdot,t), W(\cdot,t)\|_{\mathbb{H}} = \sup_{t \geq 0} \left[ \|\hat{v}(\cdot,t)\|_{\mathbb{H}} + \|W(\cdot,t)\|_{\mathbb{H}} \right]. \]

This with \( (2.14) \) and the boundedness of \( \|W(\cdot,t)\|_{H^1(\mathbb{R})} \) implies that \( (2.10) \) holds.

Next, suppose that \( \lim_{t \to \infty} |f(w,w_t)| = 0 \) and \( d \in L^2[0,\infty) \). It follows from Lemma \ref{lem:uniq} that the “\( p \)-part” of \( (2.9) \) admits a unique solution satisfying
\[ \lim_{t \to \infty} \|\hat{v}(\cdot,t), p(\cdot,t)\|_{\mathbb{H}} = 0. \] 
(2.19)

By \( (2.15) \) and \( (2.19) \), we get \( \int_{0}^{1} p_t^2(0,t-x)dx \to 0 \) as \( t \to \infty \). Then from
\[ \|W(\cdot,t)\|_{H^1(\mathbb{R})}^2 = |W(0,t)|^2 + \int_{0}^{1} W_x^2(x,t)dx = \bar{c}_0^2 \left[p^2(0,t) + \int_{0}^{1} p_t^2(0,t-x)dx \right] \] 
(2.20)

we see that \( \lim_{t \to \infty} \|W(\cdot,t)\|_{H^1(\mathbb{R})} = 0 \). This, with \( \hat{v}(x,t) = -p(x,t) - W(x,t) \), \( W_t(x,t) = -W_x(x,t) \) and \( (2.19) \), gives \( (2.11) \).

Next, suppose that \( f \equiv 0 \) and \( d \equiv 0 \). Since \( A \) generates an exponentially stable operator semigroup \( e^{At} \) on \( \mathbb{H} \), there exist two constants \( M_3, \mu_3 > 0 \) such that
\[ \|\hat{v}(\cdot,t), p(\cdot,t)\|_{\mathbb{H}} \leq M_3 e^{-\mu_3 t} \quad \forall t \geq 0. \] 
(2.21)

Since by \( (2.15) \) and \( (2.21) \) we have \( \int_{0}^{1} p_t^2(0,t-x)dx \leq 3M_3^2 e^{-2\mu_3 t} \), it follows from \( (2.18) \), \( (2.20) \) and \( (2.21) \) that \( \|W(\cdot,t)\|_{H^1(\mathbb{R})} \) also converges to zero exponentially. Combining this with \( \hat{v}(x,t) = -p(x,t) - W(x,t) \) and \( W_t(x,t) = -W_x(x,t) \), we get \( (2.12) \). \( \square \)

To understand that the “\( z \)-part” of \( (2.5) \) is used to invert the system \( (2.9) \), denote
\[ \beta(x,t) = z(x,t) - p(x,t). \] 
(2.22)

Still using the notation \( (2.8) \), we can see that \( \beta(x,t) \) is governed by
\[
\left\{ \begin{array}{l}
\beta_{tt}(x,t) = \beta_{xx}(x,t), \\
\beta_x(0,t) = \bar{c}_1 \beta(0,t) + \bar{c}_0 \beta_t(0,t), \quad \beta(1,t) = 0.
\end{array} \right.
\] 
(2.23)
We consider the system (2.23) in the energy Hilbert state space \( \mathbb{H}_0 = H^1_x(0,1) \times L^2(0,1) \), where \( H^1_R(0,1) = \{ \phi \in H^1(0,1) : \phi(1) = 0 \} \), with the usual inner product from (1.3), so that \( \mathbb{H}_0 \) is a closed subspace of \( \mathbb{H} \). The system (2.23) can be rewritten as
\[
\frac{d}{dt}(\beta(\cdot,t), \beta_t(\cdot,t)) = A_0(\beta(\cdot,t), \beta_t(\cdot,t)),
\]
where
\[
A_0(\phi, \psi) = (\psi, \phi''), \quad \forall (\phi, \psi) \in D(A_0), \quad D(A_0) = \left\{ (\phi, \psi) \in H^2(0,1) \times H^1(0,1) : \phi(1) = 0, \psi(1) = 0, \phi'(0) = \tilde{c}_1 \phi(0) + \tilde{c}_0 \psi(0) \right\}.
\]

It is well-known (\cite{H} Theorem 3) that \( A_0 \) generates an exponentially stable operator semigroup \( e^{A_0 t} \) on \( \mathbb{H}_0 \). Thus, for any initial state \( (\beta_0, \beta_1) \in \mathbb{H}_0 \), (2.23) has a unique solution \( (\beta(\cdot, t), \beta_t(\cdot, t)) = e^{A_0 t}(\beta_0, \beta_1) \in C(0, \infty; \mathbb{H}_0) \), and this decays exponentially.

**Lemma 2.3.** The observation operator \( C : D(A_0) \to \mathbb{R} \) defined by \( C(\beta_0, \beta_1) = (\frac{d}{d x} \beta_0)(1) \) is admissible for the operator semigroup \( e^{A_0 t} \).

**Proof.** Consider the semigroup generator \( A_1 \) on \( X = H^1_0(0,1) \times L^2(0,1) \) by the same formula as \( A_0 \), but with domain \( D(A_1) = [H^2(0,1) \cap H^1_0(0,1)] \times H^1_0(0,1) \). It is well-known that \( C \) is admissible for \( e^{A_1 t} \), see for instance \cite{SS} Proposition 6.2.1. Take \( (\beta_0, \beta_1) \in D(A_0) \cap D(A_1) \), which is dense in \( \mathbb{H}_0 \). By the result just mentioned, the function \( y : [0, \frac{1}{2}] \to \mathbb{R} \) defined by \( y(t) = Ce^{A_1 t}(\beta_0, \beta_1) \) is in \( L^2[0, \frac{1}{2}] \) and there is a \( k \geq 0 \) (independent of \( (\beta_0, \beta_1) \)) such that \( \|y\|_{L^2} \leq k(\|\beta_0\|_{\mathbb{H}_0}) \). Notice that \( \|\beta(\cdot,t)\|_X = \|\beta(0,1)\|_{\mathbb{H}_0} \).

Because information in solutions of the wave equation propagates with speed at most 1, the left boundary condition has no influence on \( y \), so that we have \( y(t) = Ce^{A_0 t}(\beta_0, \beta_1) \). This fact, together with our estimate on \( \|y\|_{L^2} \), proves that \( C \) is admissible also for \( A_0 \). \( \square \)

**Remark 2.3.** Since \( C \) is admissible for \( e^{A_0 t} \) and this operator semigroup is exponentially stable, it follows (see \cite{SS} Remark 4.3.5) that the function \( y(t) = Ce^{A_0 t}(\beta_0, \beta_1) \) is in \( L^2[0, \infty) \), for any \( (\beta_0, \beta_1) \in \mathbb{H}_0 \). In terms of solutions of (2.23), \( y(t) = \beta_x(1,t) \). From (2.22) \( \beta_x(1,t) = z_x(1,t) - p_x(1,t) \). Now using the third equation in (2.22), we get \( \beta_x(1,t) = F(t) - F(t) \). Thus, \( F \) can be regarded as an estimate of \( F \), because \( F - F \in L^2[0, \infty) \).

### 3 Controller and observer design

In this section, based on our disturbance estimator, we design a state observer for the system (1.1) as follows:
\[
\begin{align*}
\dot{\hat{w}}(t, x) &= \hat{w}_{xx}(t, x), \\
\hat{w}_x(0, t) &= -qw(0, t) + c_1[\hat{w}(0, t) - w(0, t)], \\
\hat{w}_x(1, t) &= u(t) + \hat{F}(t) - Y_x(1, t), \\
\hat{w}(x, 0) &= \hat{w}_0(x), \quad \hat{w}_t(x, 0) = \hat{w}_1(x), \\
Y(t, x) &= -Y_x(x, t), \\
Y(0, t) &= -c_0[\hat{w}(0, t) - w(0, t)], \quad Y(x, 0) = Y_0(x),
\end{align*}
\]

where \( c_0 \) and \( c_1 \) are the same as in (2.5) and \( \hat{F}(t) = z_x(1, t) \) is generated by the total disturbance estimator (2.5). The system (3.1) is a “natural observer” \( \hat{x} \) after canceling
the disturbance, in the sense that it employs a copy of the plant plus output injection (in this case, only at the boundary). Note that the observer (3.1) is different from the one in [22], where the signal \( w_j(1, t) \) (that is unavailable here) is used.

To show the asymptotic convergence of the above observer, we introduce the observer error variable

\[
\varepsilon(x, t) = \hat{w}(x, t) - w(x, t).
\]

Then, using the notation \( \beta \) from [22], \((\varepsilon(x, t), Y(x, t))\) satisfies

\[
\begin{align*}
\varepsilon_h(x, t) &= \varepsilon_{xx}(x, t), \\
\varepsilon_x(0, t) &= c_1\varepsilon(0, t), \quad \varepsilon_x(1, t) = \beta_x(1, t) - Y_x(1, t), \\
Y_t(x, t) &= -Y_x(x, t), \quad Y(0, t) = -c_0\varepsilon(0, t).
\end{align*}
\]

We have the following lemma to show that (3.3) is asymptotically stable.

**Lemma 3.1.** For any initial state \((\varepsilon(\cdot, 0), \varepsilon_t(\cdot, 0), Y(\cdot, 0)) \in \mathbb{H} \times H^1(0, 1)\) with the compatibility condition \(Y(0, 0) = -c_0\varepsilon(0, 0)\), there exists a unique solution to (3.3) such that \((\varepsilon, \varepsilon_t, Y) \in C(0, \infty; \mathbb{H} \times H^1(0, 1))\) and it satisfies

\[
\lim_{t \to \infty} \| (\varepsilon(\cdot, t), \varepsilon_t(\cdot, t), Y(\cdot, t)) \|_{\mathbb{H} \times H^1(0, 1)} = 0.
\]

**Proof.** We introduce a new variable

\[
\tilde{\varepsilon}(x, t) = \varepsilon(x, t) + Y(x, t).
\]

Then it is easy to check that \((\tilde{\varepsilon}(x, t), Y(x, t))\) is governed by

\[
\begin{align*}
\tilde{\varepsilon}_h(x, t) &= \tilde{\varepsilon}_{xx}(x, t), \\
\tilde{\varepsilon}_x(0, t) &= \tilde{c}_1\tilde{\varepsilon}(0, t) + \tilde{c}_0\tilde{\varepsilon}_x(0, t), \quad \tilde{\varepsilon}_x(1, t) = \beta_x(1, t), \\
Y_t(x, t) &= -Y_x(x, t), \quad Y(0, t) = -\tilde{c}_0\tilde{\varepsilon}(0, t),
\end{align*}
\]

with the initial state \(\tilde{\varepsilon}(x, 0) = \varepsilon(x, 0) + Y(x, 0), \tilde{\varepsilon}_t(x, 0) = \varepsilon_t(x, 0) - Y_x(x, 0), Y(x, 0) = Y(x, 0)\). The \(\varepsilon\)-part of the system (3.6) can be rewritten as

\[
\frac{d}{dt} \begin{bmatrix} \tilde{\varepsilon}(\cdot, t) \\ \tilde{\varepsilon}_t(\cdot, t) \end{bmatrix} = \mathcal{A} \begin{bmatrix} \tilde{\varepsilon}(\cdot, t) \\ \tilde{\varepsilon}_t(\cdot, t) \end{bmatrix} + \mathcal{B}\beta_x(1, t),
\]

where \(\mathcal{A}\) and \(\mathcal{B}\) are defined by (2.13). As already mentioned, we know from [18] that \(\mathcal{A}\) is an exponentially stable semigroup generator on \(\mathbb{H}\) and \(\mathcal{B}\) is admissible for it. By Remark 2.3 \(\beta_x(1, t) \in L^2[0, \infty)\). It follows from Lemma 2.1 that for any initial state in \(\mathbb{H}\), (3.6) has a unique solution that satisfies

\[
\lim_{t \to \infty} \| (\tilde{\varepsilon}(\cdot, t), \tilde{\varepsilon}_t(\cdot, t)) \|_{\mathbb{H}} = 0.
\]

The remaining part of the proof is very similar to the proof of (2.11), just replace \(\hat{\nu}, W, p\) and \(F\) used there with \(\varepsilon, Y, -\tilde{\varepsilon}\) and \(-\beta_x\) used here. 

Lemma 3.1 shows that (3.1) is indeed an observer for the system (1.1). Now, by the observer-based feedback control law of [22], we propose the following observer-based feedback controller (the motivation behind it will be clear from (3.10) to (3.13)):
\[ u(t) = -\tilde{F}(t) + Y_x(1,t) - c_3\tilde{w}_t(1,t) - (c_2 + q)\tilde{w}(1,t) \]
\[ - (c_2 + q) \int_0^1 e^{q(1-\xi)} [c_3\tilde{w}_t(\xi,t) + q\tilde{w}(\xi,t)]d\xi, \]  \tag{3.8}

where \( c_2, c_3 \) are positive design parameters. The term \(-\tilde{F}(t)\) is used to essentially cancel the total disturbance \( F(t) \) in \( (1.1) \), which is the estimation/cancellation strategy, and the remaining terms are used to stabilize the system \( (3.1) \). The closed-loop system formed of the observer \( (3.1) \) and the controller \( (3.8) \) is

\[
\begin{align*}
\tilde{w}_{tt}(x,t) &= \tilde{w}_{xx}(x,t), \\
\tilde{w}_x(0,t) &= -(c_2 + q)\tilde{w}(0,t) + c_1[\tilde{w}(0,t) - w(0,t)], \\
\tilde{w}_x(1,t) &= -(c_2 + q)\tilde{w}(1,t) - (c_2 + q) \int_0^1 e^{q(1-\xi)} [c_3\tilde{w}_t(\xi,t) + q\tilde{w}(\xi,t)]d\xi, \\
Y_t(x,t) &= -Y_x(x,t), \quad Y(0,t) = -c_0[\tilde{w}(0,t) - w(0,t)].
\end{align*}
\]
\tag{3.9}

Consider the invertible change of variable

\[
\tilde{w}(x,t) = [(I + \mathbb{P})\tilde{w}](x,t) = \tilde{w}(x,t) + (c_2 + q) \int_0^x e^{q(x-\xi)} \tilde{w}(\xi,t)d\xi,
\]
\tag{3.10}

where \( \mathbb{P} \) is a Volterra transformation \cite{22}. The inverse \((I + \mathbb{P})^{-1}\) is given by

\[
\tilde{w}(x,t) = [(I + \mathbb{P})^{-1}\tilde{w}](x,t) = \tilde{w}(x,t) - (c_2 + q) \int_0^x e^{-c_2(x-\xi)} \tilde{w}(\xi,t)d\xi.
\]
\tag{3.11}

It can be shown that the transformation \( (3.10) \) converts system \( (3.9) \) into

\[
\begin{align*}
\tilde{w}_{tt}(x,t) &= \tilde{w}_{xx}(x,t) - (c_1 + q)(c_2 + q)e^{q\varepsilon}(0,t), \\
\tilde{w}_x(0,t) &= c_2\tilde{w}(0,t) + (c_1 + q)\varepsilon(0,t), \quad \tilde{w}_x(1,t) = -c_3\tilde{w}_t(1,t), \\
Y_t(x,t) &= -Y_x(x,t), \quad Y(0,t) = -c_0\varepsilon(0,t).
\end{align*}
\]
\tag{3.12}

Thus, the overall system is a cascade of the exponentially stable \( \tilde{w}(0,t) \)-part" subsystem and the asymptotical stable \( \varepsilon \)-part" subsystem. For \( \varepsilon(0,t) = 0 \), the resulting system \( (3.12) \) is exponentially stable:

\[
\begin{align*}
\tilde{w}_{tt}(x,t) &= \tilde{w}_{xx}(x,t), \\
\tilde{w}_x(0,t) &= c_2\tilde{w}(0,t), \quad \tilde{w}_x(1,t) = -c_3\tilde{w}_t(1,t), \\
Y_t(x,t) &= -Y_x(x,t), \quad Y(0,t) = 0.
\end{align*}
\]
\tag{3.13}

This is a familiar form of a wave equation with a “passive damper” boundary condition coupled with a finite time stable transport equation. The solution of the “\( \tilde{w} \)-part” is exponentially stable and the solution of the “\( Y \)-part” satisfies \( Y(x,t) \equiv 0 \) for \( t \geq 1 \). The idea of the transformation \( (3.10) \) is that it makes the closed-loop system \( (3.12) \) behave like the system \( (3.13) \) (in the absence of an observer) by propagating the destabilizing \( q \)-term from the boundary \( x = 0 \), through the entire domain, to the boundary \( x = 1 \), where it gets cancelled by the feedback.

**Lemma 3.2.** Suppose that the signal \( \varepsilon(0,t) \) is determined by the system \( (3.8) \). Then for any initial state \( (\tilde{w}(\cdot, 0), \tilde{w}_t(\cdot, 0), Y(\cdot, 0)) \in \mathbb{H} \times H^1(0,1) \) satisfying the compatibility condition \( Y(0,0) = -c_0\varepsilon_0(0,0) \), there exists a unique solution to \( (3.12) \) such that \( (\tilde{w}, \tilde{w}_t) \in C(0,\infty; \mathbb{H}) \) and this solution satisfies

\[
\lim_{t \to \infty} \| (\tilde{w}(\cdot, t), \tilde{w}_t(\cdot, t), Y(\cdot, t)) \|_{\mathbb{H} \times H^1(0,1)} = 0.
\]  \tag{3.14}
Proof. The convergence of “Y-part” of (3.12) follows from Lemma 3.1. We can write the “w-part” of system (3.12) into abstract operator form as follows:
\[
\frac{d}{dt} \begin{bmatrix} \bar{w}(\cdot, t) \\ \bar{w}_t(\cdot, t) \end{bmatrix} = A_{\bar{w}} \begin{bmatrix} \bar{w}(\cdot, t) \\ \bar{w}_t(\cdot, t) \end{bmatrix} + B_1 \bar{e}(0, t) + B_2 \bar{e}(0, t),
\]
where the operators \(A_{\bar{w}} : \mathcal{D}(A_{\bar{w}})(\subset \mathbb{H}) \to \mathbb{H}, B_1 \) and \(B_2\) are given by
\[
\begin{cases}
A_{\bar{w}}(\phi, \psi) = (\psi, \phi'') & \forall \ (\phi, \psi) \in \mathcal{D}(A_{\bar{w}}),
\mathcal{D}(A_{\bar{w}}) = \{ (\phi, \psi) \in H^2(0, 1) \times H^1(0, 1) \mid \phi'(0) = c_2 \phi(0), \ \phi'(1) = -c_3 \psi(1) \}, \\
B_1 = (c_1 + q)(0, -\delta_0), & B_2 = -(c_1 + q)(c_2 + q)(0, -e^{qx}).
\end{cases}
\]

It is well known ([16, Proposition 2]) that \(A_{\bar{w}}\) generates an exponentially stable operator semigroup \(\exp(A_{\bar{w}}t)\) on \(\mathbb{H}\) and \(B_1\) is admissible for it. On the other hand, since the operator \(B_2\) is bounded, it is also admissible for \(\exp(A_{\bar{w}}t)\). By the Sobolev embedding theorem and Lemma 3.1, we obtain
\[
|\bar{e}(0, t)| \leq \|\bar{e}(0, t)\|_{H^1(0, 1)} \to 0, \quad \text{as } t \to \infty.
\]
It follows from Lemma 2.1 that \(\lim_{t \to \infty} \|(\bar{w}(\cdot, t), \bar{w}_t(\cdot, t))\|_\mathbb{H} = 0\). 

4 Well-posedness and stability of the closed-loop system

In this section we show the well-posedness and exponential stability of the closed-loop system of (1.1). First we claim that the system (3.12) is exponentially stable. To this end, we consider the overall system (2.23), (3.12) and (3.12) as follows:
\[
\begin{align*}
\varepsilon_{tt}(x, t) &= \varepsilon_{xx}(x, t), \\
\varepsilon_x(0, t) &= c_1 \varepsilon(0, t), \quad \varepsilon_x(1, t) = \beta_x(1, t) - Y_x(1, t), \\
\beta_{tt}(x, t) &= \beta_{xx}(x, t), \\
\beta_x(0, t) &= c_1 \beta(0, t) + c_0 \beta_t(0, t), \quad \beta(1, t) = 0, \\
\bar{w}_{tt}(x, t) &= \bar{w}_{xx}(x, t) - (c_1 + q)(c_2 + q)e^{q_{\bar{x}}} \varepsilon(0, t), \\
\bar{w}_x(0, t) &= c_2 \bar{w}(0, t) + (c_1 + q) \varepsilon(0, t), \quad \bar{w}_x(1, t) = -c_3 \bar{w}_t(1, t), \\
Y_t(x, t) &= -Y_x(x, t), \quad Y(0, t) = -c_0 \varepsilon(0, t),
\end{align*}
\]
in the space \(\mathcal{X} = \mathbb{H} \times H^1(0, 1) \times H^1_H(0, 1) \times L^2(0, 1) \times \mathbb{H}\), with the normal inner product.

Lemma 4.1. Suppose that \(c_i > 0, i = 1, 2, 3\). For any initial value \((\bar{e}_0, \bar{e}_1, \bar{Y}_0, \beta_0, \beta_t, \bar{w}_0, \bar{w}_1) \in \mathcal{X}\), with the compatibility condition \(\bar{Y}_0(0) = -c_0 \bar{e}_0(0)\), the system (1.1) admits a unique solution \((\varepsilon, \varepsilon_t, Y, \beta, \beta_t, \bar{w}, \bar{w}_t) \in C(0, \infty; \mathcal{X})\) and there exist two constants \(M, \mu > 0\) such that
\[
\| (\varepsilon(\cdot, t), \varepsilon_t(\cdot, t), Y(\cdot, t), \beta(\cdot, t), \beta_t(\cdot, t), \bar{w}(\cdot, t), \bar{w}_t(\cdot, t)) \|_{\mathcal{X}} \leq M e^{-\mu t}.
\]

Proof. Let \(\bar{e}(x, t)\) be given by (3.13). Introduce a new variable \(\eta(x, t) = \bar{e}(x, t) - \beta(x, t)\). We convert the system (4.1) into the following equivalent system:
\[
\begin{align*}
\eta_t(x, t) &= \eta_{xx}(x, t), \\
\eta_x(0, t) &= \bar{c}_1 \eta(0, t) + \bar{c}_0 \eta_t(0, t), \quad \eta_x(1, t) = 0, \\
Y_t(x, t) &= -Y_x(x, t), \quad Y(0, t) = -\bar{c}_0 [\eta(0, t) + \beta(0, t)], \\
\beta_t(x, t) &= \beta_{xx}(x, t), \\
\beta_x(0, t) &= \bar{c}_1 \beta(0, t) + \bar{c}_0 \beta_t(0, t), \quad \beta(1, t) = 0, \\
\tilde{w}_{tt}(x, t) &= \tilde{w}_{xx}(x, t) - \frac{(c_1 + q)(c_2 + q)}{1 - c_0} e^{q x} [\eta(0, t) + \beta(0, t)], \\
\tilde{w}_x(0, t) &= c_2 \tilde{w}(0, t) + \frac{c_1 + q}{1 - c_0} [\eta(0, t) + \beta(0, t)], \quad \tilde{w}_x(1, t) = -c_3 \tilde{w}_t(1, t).
\end{align*}
\] (4.3)

We see that the “\((\eta, \beta)\)-part” of (4.3) is independent of the “\((Y, \tilde{w})\)-part” of (4.3). It is well-known (\[18\] Theorem 2.1 and [4] Theorem 3) that the subsystem \((\eta, \beta)\) is exponentially stable, i.e., there exist two constants \(M_1, \mu_1 > 0\) such that

\[\|(\eta(\cdot, t), \eta_t(\cdot, t), \beta(\cdot, t), \beta_t(\cdot, t))\|_{H^1 \times H^1(0,1) \times L^2(0,1)} \leq M_1 e^{-\mu_1 t}.\] (4.4)

By the Sobolev embedding theorem we have

\[
\begin{align*}
|\eta(0, t) + \beta(0, t)| &\leq \|\eta(0, t) + \beta(0, t)\|_{H^1(0,1)} \\
&\leq 2 \|(\eta(\cdot, t), \eta_t(\cdot, t), \beta(\cdot, t), \beta_t(\cdot, t))\|_{H^1 \times H^1(0,1) \times L^2(0,1)} \\
&\leq 2M_1 e^{-\mu_1 t}.
\end{align*}
\] (4.5)

We can write the “\(\tilde{w}\)-part” of (4.3) in operator form as follows:

\[
\frac{d}{dt} \begin{bmatrix} \tilde{w}(\cdot, t) \\ \tilde{w}_t(\cdot, t) \end{bmatrix} = A_{\tilde{w}} \begin{bmatrix} \tilde{w}(\cdot, t) \\ \tilde{w}_t(\cdot, t) \end{bmatrix} + B_1 [\eta(0, t) + \beta(0, t)] + B_2 [\eta(0, t) + \beta(0, t)],
\] (4.6)

where the operators \(A_{\tilde{w}}\) is given by (3.16) and \(B_1 = (c_1 + q)/(1 - c_0)(0, -\delta_0), B_2 = -(c_1 + q)(c_2 + q)/(1 - c_0)(0, -e^{q x}).\) Since \(A_{\tilde{w}}\) generates an exponentially stable operator semigroup \(e^{A_{\tilde{w}} t}\) on \(H\) and \(B_1, B_2\) are admissible for this semigroup, it follows from (4.5) and Lemma 2.1 that there exist two constants \(M_2, \mu_2 > 0\) such that

\[\|(\tilde{w}(\cdot, t), \tilde{w}_t(\cdot, t))\| \leq M_2 e^{-\mu_2 t}.\] (4.7)

Next, we claim that the solution of the “\(Y\)-part” of (4.3) is exponentially stable. Define the function

\[
Y(x, t) = \begin{cases} 
-\bar{c}_0 [\eta(0, t - x) + \beta(0, t - x)], & t \geq x, \\
Y_0(x - t), & x > t.
\end{cases}
\] (4.8)

Then it is straightforward to verify that \(Y\) solves the “\(Y\)-part” of (4.3). Based on the proof of the exponential stability of \(W(\cdot, t)\) on \(H^1(0,1)\) in Lemma 2.2 it suffices to show that there exist two constants \(M_3, \mu_3 > 0\) such that

\[\int_0^1 [\eta(0, t - x) + \beta_t(0, t - x)]^2 dx \leq M_3 e^{-\mu_3 t}.\] (4.9)

Indeed, define

\[
\rho(t) = 2 \int_0^1 (x - 1) \eta_t(x, t) \eta_x(x, t) dx + 2 \int_0^1 (x - 1) \beta_t(x, t) \beta_x(x, t) dx.
\]
Then \( |\rho(t)| \leq \| (\eta(\cdot, t), \eta_t(\cdot, t), \beta(\cdot, t), \beta_t(\cdot, t)) \|_{\mathbb{H}^1 \times H^1_{\rho}(0,1) \times L^2(0,1)}^2 \). Computing the derivative of \( \rho(t) \) along the solution of (4.3) gives (we suppress the arguments \((x, t)\) that appear within integrals)

\[
\dot{\rho}(t) = \eta_x^2(0, t) + \eta_t^2(0, t) - \int_0^t [\eta_x^2 + \eta_t^2] \, dx + \beta_x^2(0, t) + \beta_t^2(0, t) - \int_0^t [\beta_x^2 + \beta_t^2] \, dx \\
\geq \eta_t^2(0, t) - \int_0^t [\eta_x^2 + \eta_t^2] \, dx + \beta_t^2(0, t) - \int_0^t [\beta_x^2 + \beta_t^2] \, dx,
\]

which, combining with (4.4), implies that

\[
\int_{t-1}^t [\eta_x^2(s) + \beta_x^2(s)] \, ds \\
\leq \int_{t-1}^t \| (\eta(\cdot, s), \eta_t(\cdot, s), \beta(\cdot, s), \beta_t(\cdot, s)) \|_{\mathbb{H}^1 \times H^1_{\rho}(0,1) \times L^2(0,1)}^2 \, ds + \rho(t) - \rho(t - 1) \\
\leq M_1^2 \int_{t-1}^t e^{-2\mu_1 s} \, ds + M_1^2 e^{-2\mu_1 t} + M_1^2 e^{-2\mu_1 (t-1)} \leq \left( \frac{e^{2\mu_1}}{2\mu_1} + e^{2\mu_1} + 1 \right) M_1^2 e^{-2\mu_1 t}.
\]

On the other hand, since for all \( t \geq 1 \),

\[
\int_0^1 |\eta_t(0, t - x) + \beta_t(0, t - x)|^2 \, dx \\
\leq 2 \int_0^1 [\eta_t^2(0, t - x) + \beta_t^2(0, t - x)] \, dx = 2 \int_{t-1}^t [\eta_s^2(0, s) + \beta_s^2(0, s)] \, ds,
\]

we obtain (4.9) with \( M_3 = 2\left( \frac{e^{2\mu_1}}{2\mu_1} + e^{2\mu_1} + 1 \right) M_1^2 \) and \( \mu_3 = 2\mu_2 \). Combining with \( \varepsilon(x, t) = \bar{\varepsilon}(x, t) - Y(x, t), \ Y_t(x, t) = -Y_x(x, t), \) (4.3) and (4.7), we get (4.12).

**Remark 4.1.** In the proof of Theorem 4.1 below, we introduce the new variable \( \eta(x, t) = \bar{\varepsilon}(x, t) - \beta(x, t) \) which is a useful trick in proving the exponential stability of the subsystem \( \bar{\varepsilon}(x, t) \) and the subsystem \( \varepsilon(x, t) \). This is because we are not able to prove that \( \beta_x(1, t) \) decays exponentially, only that \( \beta_x(1, t) \in L^2[0, \infty) \). So, the exponential stabilities mentioned cannot follow from Lemma 2.1.

Now we go back to the closed-loop system (1.1) under the feedback (3.8):

\[
\begin{align*}
\left\{ \begin{array}{l}
w_t(x, t) = w_{xx}(x, t), \\
w_x(0, t) = -qw(0, t), \\
w_x(1, t) = -z_x(1, t) + Y_x(1, t) - c_3 \bar{w}_t(1, t) - (c_2 + q) \bar{w}(1, t) + f(w(\cdot, t), w_t(\cdot, t)) \\
\quad + d(t) - (c_2 + q) \int_0^1 e^{q(1-\xi)} [c_3 \bar{w}_t(\xi, t) + q \bar{w}(\xi, t)] \, d\xi, \\
v_t(x, t) = v_{xx}(x, t), \\
v_x(0, t) = -qw(0, t) + c_1 [v(0, t) - w(0, t)], \\
v_x(1, t) = -z_x(1, t) + Y_x(1, t) - W_x(1, t) - c_3 \bar{w}_t(1, t) - (c_2 + q) \bar{w}(1, t) \\
\quad - (c_2 + q) \int_0^1 e^{q(1-\xi)} [c_3 \bar{w}_t(\xi, t) + q \bar{w}(\xi, t)] \, d\xi,
\end{array} \right.
\end{align*}
\]

(4.10)
Theorem 4.1. Suppose that \( c_i > 0, i = 1, 2, 3, f : H^1(0,1) \times L^2(0,1) \to \mathbb{R} \) is continuous, and \( d \in L^\infty[0,\infty) \) or \( d \in L^2[0,\infty) \). For any initial state \((w_0, w_1, v_0, v_1, z_0, z_1, W_0, \tilde{w}_0, \tilde{w}_1, Y_0) \in \mathcal{H}\) satisfying the compatibility conditions

\[
-z_0(1) - v_0(1) - W_0(1) + w_0(1) = 0,
W_0(0) + c_0[v_0(0) - w_0(0)] = 0,
Y_0(0) + c_0[\tilde{w}_0(0) - w_0(0)] = 0,
\]

there exists a unique solution to \((4.10)-(4.11)\) such that

\[
(w, w_t, v, v_t, W, z, z_t, \tilde{w}_0, \tilde{w}_1, Y) \in C(0,\infty; \mathcal{H}),
\]

\[
\|(w(\cdot,t), w_t(\cdot,t), \tilde{w}(\cdot,t), \tilde{w}_t(\cdot,t), Y(\cdot,t))\|_{\mathcal{H}^2 \times H^1(0,1)} \leq M e^{-\mu t}, \quad \forall t \geq 0,
\]

with some \( M, \mu > 0 \) independent of the initial state, and

\[
\sup_{t \geq 0} \|(v(\cdot,t), v_t(\cdot,t), z(\cdot,t), z_t(\cdot,t), W(\cdot,t))\|_{\mathcal{H}^2 \times H^1(0,1)} < \infty.
\]

If we assume further that \( f(0,0) = 0 \) and \( d \in L^2[0,\infty) \), then

\[
\lim_{t \to \infty} \|(v(\cdot,t), v_t(\cdot,t), z(\cdot,t), z_t(\cdot,t), W(\cdot,t))\|_{\mathcal{H}^2 \times H^1(0,1)} = 0.
\]

If we assume that \( f \equiv 0 \) and \( d \equiv 0 \), then there exist two constants \( M', \mu' > 0 \) such that

\[
\|(v(\cdot,t), v_t(\cdot,t), z(\cdot,t), z_t(\cdot,t), W(\cdot,t))\|_{\mathcal{H}^2 \times H^1(0,1)} \leq M' e^{-\mu' t}, \quad \forall t \geq 0.
\]

Proof. Using the variables \( \varepsilon(x,t), \beta(x,t) \) and \( \tilde{\varepsilon}(x,t) \) given by \((3.2), (2.22)\) and \((2.6)\), respectively, and the invertible transformation \((3.10)\), we can rewrite \((4.10)-(4.11)\) as follows:

\[
\begin{aligned}
\varepsilon_{tt}(x,t) &= \varepsilon_{xx}(x,t), \\
\varepsilon_x(0,t) &= c_1\varepsilon_0(0,t), \quad \varepsilon_x(1,t) = \beta_x(1,t) - Y_x(1,t), \\
\beta_{tt}(x,t) &= \beta_{xx}(x,t), \\
\beta_x(0,t) &= \frac{c_1}{1 - c_0}\beta_0(0,t) + \frac{c_0}{1 - c_0}\beta_1(0,t), \quad \beta(1,t) = 0, \\
\tilde{\varepsilon}_{tt}(x,t) &= \tilde{\varepsilon}_{xx}(x,t) - (c_1 + q)(c_2 + q)e^{\gamma x}\varepsilon(0,t), \\
\tilde{\varepsilon}_x(0,t) &= c_2\tilde{\varepsilon}_0(0,t) + (c_1 + q)\varepsilon(0,t), \quad \tilde{\varepsilon}_x(1,t) = -c_3\tilde{\varepsilon}_1(1,t),
\end{aligned}
\]
It is clear that (4.10)- (4.11) is well-posed if and only if (4.16)-(4.17) is well-posed. We see that the “\((\varepsilon, \beta, \tilde{w}, Y)\)-part” of (4.10)- (4.11) is independent of the “\((\tilde{v}, W)\)-part” of this system. By Lemma 4.1, there exist two constants \(M_1, \mu_1 > 0\) such that the solution \((\varepsilon, \varepsilon_t, Y, \beta, \beta_t, \tilde{w}, \tilde{w}_t) \in C(0, \infty; \mathcal{X})\) satisfies
\[
\|(\varepsilon(\cdot, t), \varepsilon_t(\cdot, t), Y(\cdot, t), \beta(\cdot, t), \beta_t(\cdot, t), \tilde{w}(\cdot, t), \tilde{w}_t(\cdot, t))\|_{\mathcal{X}} \leq M_1 e^{-\mu_1 t}.
\] (4.18)

Owing to the invertibility of the transformation

\[
\begin{pmatrix}
\begin{pmatrix} w(x,t) \\
\varepsilon_t(x,t) \\
\tilde{w}(x,t) \\
\tilde{w}_t(x,t)
\end{pmatrix}
\end{pmatrix} = \begin{pmatrix}
\begin{pmatrix} -I & 0 & (I + \mathbb{P})^{-1} & 0 \\
0 & -I & 0 & (I + \mathbb{P})^{-1} \\
0 & 0 & (I + \mathbb{P})^{-1} & 0 \\
0 & 0 & 0 & (I + \mathbb{P})^{-1}
\end{pmatrix}
\begin{pmatrix} \varepsilon(x,t) \\
\varepsilon_t(x,t) \\
\tilde{w}(x,t) \\
\tilde{w}_t(x,t)
\end{pmatrix}
\end{pmatrix},
\]

where \(I + \mathbb{P}\) is defined by (3.10), \((w(\cdot,t), \varepsilon_t(\cdot,t), \tilde{w}(\cdot,t), \tilde{w}_t(\cdot,t)) \in C(0, \infty; \mathbb{H}^2)\) is well-defined and satisfies
\[
\|(w(\cdot,t), \varepsilon_t(\cdot,t), \tilde{w}(\cdot,t), \tilde{w}_t(\cdot,t))\|_{\mathbb{H}^2} \leq 2[1 + \|(I + \mathbb{P})^{-1}\|]M_1 e^{-\mu_1 t},
\] (4.19)

which, combined with (4.18), implies that (4.12) holds with \(M = 3[1 + \|(I + \mathbb{P})^{-1}\|]M_1\) and \(\mu = \mu_1\). Now we consider the “\((\tilde{v}, W)\)-part”:
\[
\begin{align}
\tilde{v}_H(x,t) &= \tilde{v}(x,t), \\
\tilde{v}_x(0,t) &= c_1 \tilde{v}(0,t), \\
\tilde{v}_x(1,t) &= -f(w(\cdot,t), \varepsilon_t(\cdot,t)) - d(t) - W_x(1, t), \\
W_t(x,t) &= -W_x(x,t), \quad W(0,t) = -c_0 \tilde{v}(0,t).
\end{align}
\] (4.20)

Since \(f : H^1(0,1) \times L^2(0,1) \to \mathbb{R}\) is continuous and \((w, \tilde{w})\) is bounded, due to the convergence \(\|(w(\cdot,t), \varepsilon_t(\cdot,t))\|_{\mathbb{H}} \to 0\), we conclude that \(f(w(\cdot,t), \varepsilon_t(\cdot,t)) \in L^\infty[0,\infty)\). Since \(d \in L^\infty[0, \infty)\) or \(d \in L^2[0, \infty)\), it follows from Lemma 2.2 that the system (4.20) admits a unique bounded solution, i.e.,
\[
\sup_{t \geq 0} \|(\tilde{v}(\cdot,t), \tilde{v}_t(\cdot,t), W(\cdot,t))\|_{\mathbb{H} \times H^1(0,1)} < \infty.
\] (4.21)

Noting that \(W_t(x,t) = -W_x(x,t)\), it follows from (2.6), (2.22) and (4.21) that
\[
\begin{align}
\|(v(\cdot,t), \varepsilon_t(\cdot,t))\|_{\mathbb{H}} &\leq \|(\tilde{v}(\cdot,t), \tilde{v}_t(\cdot,t))\|_{\mathbb{H}} + \|(w(\cdot,t), \varepsilon_t(\cdot,t))\|_{\mathbb{H}}, \\
\|(z(\cdot,t), \varepsilon_t(\cdot,t))\|_{\mathbb{H}} &\leq \|(\beta(\cdot,t), \beta_t(\cdot,t))\|_{\mathbb{H}} + \|(\tilde{v}(\cdot,t), \tilde{v}_t(\cdot,t))\|_{\mathbb{H}} + \|(W(\cdot,t), W_x(\cdot,t))\|_{\mathbb{H}}.
\end{align}
\]

The right-hand sides above are finite, which gives (4.13).

Now suppose that \(f(0,0) = 0\) and \(d \in L^2[0, \infty)\). By (4.19) and the continuity of \(f\), we have \(\lim_{t \to \infty} |f(w, \varepsilon_t)| = 0\). By Lemma 2.2 we obtain
\[
\lim_{t \to \infty} \|(\tilde{v}(\cdot,t), \tilde{v}_t(\cdot,t), W(\cdot,t))\|_{\mathbb{H} \times H^1(0,1)} = 0.
\] (4.22)
By (4.18), (4.19) and (4.22), we derive
\[ \| (v(\cdot, t), v_t(\cdot, t)) \|_H \leq \| (\hat{v}(\cdot, t), \hat{v}_t(\cdot, t)) \|_H + \| (w(\cdot, t), w_t(\cdot, t)) \|_H \to 0 \text{ as } t \to \infty, \]
\[ \| (z(\cdot, t), z_t(\cdot, t)) \|_H \leq \| (\beta(\cdot, t), \beta_t(\cdot, t)) \|_H + \| (\hat{v}(\cdot, t), \hat{v}_t(\cdot, t)) \|_H + \| (W(\cdot, t), W_x(\cdot, t)) \|_H. \]

Next, suppose that \( f \equiv 0 \) and \( d \equiv 0 \). It follows from Lemma 2.2 that there exist two constants \( M_2, \mu_2 > 0 \) such that for all \( t \geq 0 \),
\[ \| (\hat{v}(\cdot, t), \hat{v}_t(\cdot, t), W(\cdot, t)) \|_{H^1(0, 1)} \leq M_2 e^{-\mu_2 t}. \tag{4.23} \]

By (4.18), (4.19) and (4.23), we obtain that for all \( t \geq 0 \),
\[ \| (v(\cdot, t), v_t(\cdot, t)) \|_H \leq \| (\hat{v}(\cdot, t), \hat{v}_t(\cdot, t)) \|_H + \| (w(\cdot, t), w_t(\cdot, t)) \|_H \]
\[ \leq M_2 e^{-\mu_2 t} + 2(1 + \| (I + P) \|_2) \| M_1 e^{-\mu_1 t}, \]
\[ \| (z(\cdot, t), z_t(\cdot, t)) \|_H \leq \| (\beta(\cdot, t), \beta_t(\cdot, t)) \|_H + \| (\hat{v}(\cdot, t), \hat{v}_t(\cdot, t)) \|_H + \| (W(\cdot, t), W_x(\cdot, t)) \|_H \]
\[ \leq M_1 e^{-\mu_1 t} + 3M_2 e^{-\mu_2 t}, \]
which, combined with (4.23), implies that (4.15) holds.

**Remark 4.2.** The signals \( \{w(0, t), w(1, t)\} \) are almost a minimal set of measurement signals to exponentially stabilize the system (1.1). Indeed, from Theorem 3.1 we see that we can design disturbance estimator and state observer by using \( \{w(0, t), w(1, t)\} \) only. Based on this disturbance estimator and state observer, the system (1.1) can be exponentially stabilized by using \( \{w(0, t), w(1, t)\} \) only. However,
(a). Each of the observations \( \{w(0, t), w(1, t)\} \) alone is not enough for exact observability, i.e., for any \( T > 0 \), there is no constant \( C_T > 0 \) such that
\[ \int_0^T w^2(0, s) \, ds \geq C_T \| w(\cdot, 0), w_t(\cdot, 0) \|_H, \quad \int_0^T w^2(1, s) \, ds \geq C_T \| w(\cdot, 0), w_t(\cdot, 0) \|_H. \]
(b). The signal \( y(t) = w(1, t) \) is also not enough for exponential stabilization. Actually, let \( f(w, w_t) \equiv 0 \), and let \( d = q \). Then the system (1.1) admits a solution \( (w, w_t) = (q(x-1), 0) \) which makes the output \( y(t) = w(1, t) \equiv 0 \).

From (a), (b), \( w(0, t) \) seems to be necessary for stabilization. We leave two open question here: (I): Can we design a state observer for system (1.1) using only \( y(t) = w(0, t) \)?
(II): Is \( y(t) = w(0, t) \) enough to make the system (1.1) stabilizable?

## 5 An anti-stable wave equation with negative damper

In this section we consider the output feedback exponential stabilization for a new system, where the “negative spring” from (1.1) is replaced with a “negative damper”, so that only the second equation in (1.1) is changed:

\[
\begin{align*}
  w_{tt}(x, t) & = w_{xx}(x, t), \\
  w_x(0, t) & = -qw_0(0, t), \\
  w_x(1, t) & = u(t) + f(w(\cdot, t), w_t(\cdot, t)) + d(t), \\
  w(x, 0) & = w_0(x), \quad w_t(x, 0) = w_1(x), \\
  y_m(t) & = (w(0, t), w(1, t)),
\end{align*}
\tag{5.1}
\]
where \((w, w_1)\) is the state, \(u\) is the control input signal, \(y_m\) is the output signal, that is, the boundary traces \(w(0,t)\) and \(w(1,t)\) are measured. The equations containing the parameter \(q > 0, q \neq 1\) creates a destabilizing feedback, it is like the equation of a damper but with the reversed sign. The function \(f : H^1(0,1) \times L^2(0,1) \to \mathbb{R}\) is a possibly unknown nonlinear mapping that represents the internal uncertainty, and \(d\) represents the unknown external disturbance which is only supposed to satisfy \(d \in L^\infty[0, \infty)\).

We consider system (5.1) in the state Hilbert space \(\mathbb{H} = H^1(0,1) \times L^2(0,1)\). The intuitive representation is as in Figure 1, but with a damper in place of the spring. The following result is similar to Proposition 1.1 and can be proved along the same lines.

**Proposition 5.1.** Suppose that \(f : H^1(0,1) \times L^2(0,1) \to \mathbb{R}\) is continuous with \(f(0,0) = 0\) and satisfies a global Lipschitz condition in \(H^1(0,1) \times L^2(0,1)\). Then, for any \((w_0, w_1) \in \mathbb{H}, u \in L^2_{loc}[0, \infty), d \in L^2_{loc}[0, \infty)\), there exists a unique global solution to (5.1) such that \((w(\cdot, t), \hat{w}(\cdot, t)) \in C(0, \infty; \mathbb{H})\).

### 5.1 The disturbance estimator

We design a disturbance estimator for the system (5.1), that uses the signal \(y_m(t) = (w(0,t), w(1,t))\), as follows:

\[
\begin{align*}
    v_{tt}(x, t) &= v_{xx}(x, t), \\
    v_x(0,t) &= -qv_t(0,t) + c_1[v(0,t) - w(0,t)], \quad v_x(1,t) = u(t) - W_x(1,t), \\
    v(x,0) &= v_0(x), \quad v_t(x,0) = v_1(x), \\
    z_{tt}(x, t) &= z_{xx}(x, t), \\
    z_x(0,t) &= \frac{c_1}{1-c_0}z(0,t) + \frac{c_0-q}{1-c_0}z_t(0,t), \quad z(1,t) = -v(1,t) - W(1,t) + w(1,t), \\
    z(x,0) &= z_0(x), \quad z_t(x,0) = z_1(x), \\
    W_t(x, t) &= -W_x(x, t), \\
    W(0,t) &= -c_0[v(0,t) - w(0,t)], \quad W(x,0) = W_0(x),
\end{align*}
\]

(5.2)

where \(c_0\) and \(c_1\) are two design parameters so that \(\frac{c_1}{1-c_0} > 0\) and \(\frac{c_0-q}{1-c_0} > 0\). The initial state of the disturbance estimator (5.2) is \((v_0, v_1, z_0, z_1, W_0) \in \mathbb{H}^2 \times H^1(0,1)\). It is clear that the above disturbance estimator receives as inputs the control input \(u\) of the original system and the two measurement signals \(w(0,t)\) and \(w(1,t)\). The \((v, W)\)-subsystem is an auxiliary system which is used to separate the total disturbance from the original system (5.1) to an exponential system. Indeed, let

\[
    \hat{v}(x, t) = v(x, t) - w(x, t).
\]

(5.3)

Then it is easy to verify that \((\hat{v}(x, t), W(x, t))\) satisfies

\[
\begin{align*}
    \hat{v}_{tt}(x, t) &= \hat{v}_{xx}(x, t), \\
    \hat{v}_x(0,t) &= -q\hat{v}_t(0,t) + c_1\hat{v}(0,t), \\
    \hat{v}_x(1,t) &= -f(w(\cdot, t), w_t(\cdot, t)) - d(t) - W_x(1,t), \\
    W_t(x, t) &= -W_x(x, t), \quad W(0,t) = -c_0\hat{v}(0,t).
\end{align*}
\]

(5.4)

It is seen that the inhomogeneous part of (5.4) is just the total disturbance.
Lemma 5.1. Suppose that $\frac{c_1}{1-c_0} > 0$, $\frac{c_0-q}{1-c_0} > 0$; $d \in L^\infty[0, \infty)$, (or $d \in L^2[0, \infty)$), $f : H^1(0,1) \times L^2(0,1) \to \mathbb{R}$ is continuous and that (5.1) admits a unique solution $(w, \hat{w}) \in C(0,\infty; \mathbb{H})$ which is bounded. For any initial value $(\hat{v}_0, \tilde{v}_1, W_0) \in \mathbb{H} \times H^1(0,1)$ with the compatibility condition $W_0(0) = -c_0\hat{v}_0(0)$, there exists a unique solution $(\tilde{v}, \hat{v}_t, W) \in C(0,\infty; \mathbb{H} \times H^1(0,1))$ such that

$$\sup_{t \geq 0} \|\left(\tilde{v}(\cdot, t), \hat{v}_t(\cdot, t), W(\cdot, t)\right)\|_{\mathbb{H} \times H^1(0,1)} < \infty. \quad (5.5)$$

If we assume further that $\lim_{t \to \infty} |f(w, w_t)| = 0$ and $d \in L^2[0, \infty)$, then

$$\lim_{t \to \infty} \|\left(\tilde{v}(\cdot, t), \hat{v}_t(\cdot, t), W(\cdot, t)\right)\|_{\mathbb{H} \times H^1(0,1)} = 0. \quad (5.6)$$

If we assume that $f \equiv 0$ and $d \equiv 0$, then there exist two constants $M, \mu > 0$ such that

$$\|\left(\tilde{v}(\cdot, t), \hat{v}_t(\cdot, t), W(\cdot, t)\right)\|_{\mathbb{H} \times H^1(0,1)} \leq M \mu^t \quad \forall t \geq 0. \quad (5.7)$$

Proof. First we introduce a new variable $p(x, t) = -\tilde{v}(x, t) - W(x, t)$, then $(p(x, t), W(x, t))$ satisfies

$$\begin{align*}
p_{tt}(x, t) &= p_{xx}(x, t), \\
p_x(0, t) &= \frac{c_1}{1-c_0} p(0, t) + \frac{c_0-q}{1-c_0} p_t(0, t), \\
p_x(1, t) &= f(w(\cdot, t), w_t(\cdot, t)) + d(t), \\
W_t(x, t) &= -W_x(x, t), \quad W(0, t) = \frac{c_0}{1-c_0} p(0, t),
\end{align*} \quad (5.8)$$

with the initial state $p(x, 0) = -\tilde{v}(x, 0) - W(x, 0)$, $p_t(x, 0) = -\hat{v}_t(x, 0) + W_x(x, 0)$ and $W(x, 0) = W(x, 0)$. Comparing (5.8) with (2.9), it is seen that (5.8) is exactly the same as the system (2.23) by replacing $\frac{c_0}{1-c_0}$ with $\frac{c_0-q}{1-c_0}$. Thus, the rest of the proof of this lemma is exactly the same as for Lemma 2.2.

Let

$$\beta(x, t) = z(x, t) - p(x, t). \quad (5.9)$$

Then we can see that $\beta(x, t)$ is governed by

$$\begin{align*}
\beta_{tt}(x, t) &= \beta_{xx}(x, t), \\
\beta_x(0, t) &= \frac{c_1}{1-c_0} \beta(0, t) + \frac{c_0-q}{1-c_0} \beta_t(0, t), \quad \beta(1, t) = 0, \quad (5.10)
\end{align*}$$

which is exactly the same as the system (2.23) by replacing $\frac{c_0}{1-c_0}$ with $\frac{c_0-q}{1-c_0}$. Therefore, it follows from Lemma 2.3 and Remark 2.3 that $z_x(1, t)$ can be regarded as an estimate of the total disturbance $F(t) = f(w(\cdot, t), w_t(\cdot, t)) + d(t)$, that is, $z_x(1, t) \approx F(t)$.

5.2 Controller and observer design

In this subsection we investigate the following state observer for the system (5.1):

$$\begin{align*}
\hat{w}_t(x, t) &= \hat{w}_{xx}(x, t), \\
\hat{w}_x(0, t) &= -q\hat{w}_t(0, t) + c_1[\hat{w}(0, t) - w(0, t)], \\
\hat{w}_x(1, t) &= u(t) + z_x(1, t) - Y_x(1, t), \\
\hat{w}(x, 0) &= \hat{w}_0(x), \quad \hat{w}_t(x, 0) = \hat{w}_1(x), \\
Y_t(x, t) &= -Y_x(x, t), \\
Y(0, t) &= -c_0[\hat{w}(0, t) - w(0, t)], \quad Y(x, 0) = Y_0(x),
\end{align*} \quad (5.11)$$
Now we introduce the new variable \( \tilde{\epsilon}(x, t) \) that 

\[
\tilde{\epsilon}(x, t) = \hat{\omega}(x, t) - \omega(x, t).
\]  

Then it is easy to see that \((\tilde{\epsilon}(x, t), Y(x, t))\) is governed by

\[
\begin{align*}
\tilde{\epsilon}_t(x, t) &= \tilde{\epsilon}_{xx}(x, t), \\
\tilde{\epsilon}_x(0, t) &= -q\epsilon_t(0, t) + c_1\epsilon(0, t), \\
\tilde{\epsilon}_x(1, t) &= \beta_x(1, t) - Y_x(1, t), \\
Y_t(x, t) &= -Y_x(x, t), \\
Y(0, t) &= -c_0\epsilon(0, t).
\end{align*}
\]  

**Lemma 5.2.** Suppose that \( \frac{c_0 - q}{1 - c_0} > 0 \), \( \frac{c_1}{1 - c_0} > 0 \) and the signal \( \tilde{z}_x(1, t) \) is determined by system \( \text{(5.12)} \). Then for any initial state \((\epsilon(\cdot,0), \epsilon_t(\cdot, 0), Y(\cdot,0)) \in H \times H^1(0,1)\) with the compatibility condition \(Y(0,0) = -c_0\epsilon(0,0)\), there exists a unique solution to \((5.13)\) such that \((\epsilon, \epsilon_t, Y) \in C(0, \infty; H \times H^1(0,1))\) satisfying

\[
\lim_{t \to \infty} ||(\epsilon(\cdot, t), \epsilon_t(\cdot, t), Y(\cdot, t))||_{H \times H^1(0,1)} = 0.
\]  

**Proof.** Introduce a new variable \( \tilde{\epsilon}(x, t) = \epsilon(x, t) + Y(x, t) \). Then \((\tilde{\epsilon}(x, t), Y(x, t))\) satisfies

\[
\begin{align*}
\tilde{\epsilon}_t(x, t) &= \tilde{\epsilon}_{xx}(x, t), \\
\tilde{\epsilon}_x(0, t) &= \tilde{c}_1\tilde{\epsilon}(0, t) + \frac{c_0 - q}{1 - c_0}\tilde{\epsilon}_x(0, t), \\
\tilde{\epsilon}_x(1, t) &= \beta_x(1, t), \\
Y_t(x, t) &= -Y_x(x, t), \\
Y(0, t) &= -\tilde{c}_0\tilde{\epsilon}(0, t),
\end{align*}
\]  

with the initial state \( \tilde{\epsilon}(x, 0) = \epsilon(x, 0) + Y(x, 0), \tilde{\epsilon}_t(x, 0) = \epsilon_t(x, 0) - Y_x(x, 0), \epsilon(x, 0) = Y(x, 0) \). Comparing \((5.15)\) with \((3.6)\), we see that \((5.15)\) is exactly the same as the system \((3.6)\) by replacing \( \tilde{c}_0 \) with \( \frac{c_0 - q}{1 - c_0} \) for “\( \tilde{\epsilon} \)-part”. Thus, according to the proof of Lemma 3.1, we can conclude that \((5.13)\) admits a unique solution satisfying \((5.14)\). 

By Lemma 5.2 \((5.11)\) is indeed an observer of \((5.1)\). To find a stabilizing control law for system \((5.1)\), we introduce the following auxiliary system (here \( t \geq 0 \) and \( x \in [0, 1] \)):

\[
\begin{align*}
Z_t(x, t) &= -Z_x(x, t), \\
Z(0, t) &= -c_2\tilde{\omega}(0, t), \quad Z(x, 0) = Z_0(x).
\end{align*}
\]  

Now we introduce the new variable \( \tilde{\omega}(x, t) = \hat{\omega}(x, t) + Z(x, t) \). Then \((\tilde{\omega}, Z)\) satisfies

\[
\begin{align*}
\tilde{\omega}_t(x, t) &= \tilde{\omega}_{xx}(x, t), \\
\tilde{\omega}_x(0, t) &= \frac{c_2 - q}{1 - c_2}\tilde{\omega}_t(0, t) + c_1\epsilon(0, t), \\
\tilde{\omega}_x(1, t) &= u(t) + z_x(1, t) - Y_x(1, t) + Z_x(1, t), \\
Z_t(x, t) &= -Z_x(x, t), \quad Z(0, t) = -\frac{c_2}{1 - c_2}\tilde{\omega}(0, t).
\end{align*}
\]  

We see that the exponential stability of system \((5.16)\) is equivalent to the exponential stability of \((5.11)\). We propose the following observer-based feedback controller:

\[
u(t) = -c_3\tilde{\omega}(1, t) - c_3Z(1, t) - z_x(1, t) + Y_x(1, t) - Z_x(1, t).
\]  

21
The closed-loop system formed by (5.16) with the controller (5.17) becomes
\[
\begin{aligned}
\dot{\bar{w}}_t(x, t) &= \bar{w}_{xx}(x, t), \\
\bar{w}_x(0, t) &= \frac{c_2 - q}{1 - c_2} \bar{w}_t(0, t) + c_1 \varepsilon(0, t), \quad \bar{w}_x(1, t) = -c_3 \bar{w}(1, t), \\
Z_t(x, t) &= -Z_x(x, t), \quad Z(0, t) = -\frac{c_2}{1 - c_2} \bar{w}(0, t).
\end{aligned}
\]
(5.18)

The closed-loop of observer (5.11) corresponding to controller (5.17) becomes
\[
\begin{aligned}
\dot{\bar{w}}_t(x, t) &= \bar{w}_{xx}(x, t), \\
\bar{w}_x(0, t) &= -q \bar{w}_t(0, t) + c_1 [\bar{w}(0, t) - w(0, t)], \\
\bar{w}_x(1, t) &= -c_3 \bar{w}(1, t) - c_0 Z(1, t) - Z_x(1, t), \\
Y_t(x, t) &= -Y_x(x, t), \quad Y(0, t) = -c_0 [\bar{w}(0, t) - w(0, t)], \\
Z_t(x, t) &= -Z_x(x, t), \quad Z(0, t) = -\frac{c_2}{1 - c_2} \bar{w}(0, t).
\end{aligned}
\]
(5.19)

To show the exponential stability of system (5.16) under the feedback (5.17), we consider the overall system (5.13), (5.10) and (5.18) described by
\[
\begin{aligned}
\varepsilon_t(x, t) &= \varepsilon_{xx}(x, t), \\
\varepsilon_x(0, t) &= -q \varepsilon_t(0, t) + c_1 \varepsilon(0, t), \quad \varepsilon_x(1, t) = \beta_x(1, t) - Y_x(1, t), \\
Y_t(x, t) &= -Y_x(x, t), \quad Y(0, t) = -c_0 \varepsilon(0, t), \\
\beta_t(x, t) &= \beta_{xx}(x, t), \\
\beta_x(0, t) &= \frac{c_1}{1 - c_0} \beta(0, t) + \frac{c_0 - q}{1 - c_0} \beta_t(0, t), \quad \beta(1, t) = 0, \\
\bar{w}_t(x, t) &= \bar{w}_{xx}(x, t), \\
\bar{w}_x(0, t) &= \frac{c_2 - q}{1 - c_2} \bar{w}_t(0, t) + c_1 \varepsilon(0, t), \quad \bar{w}_x(1, t) = -c_3 \bar{w}(1, t), \\
Z_t(x, t) &= -Z_x(x, t), \quad Z(0, t) = -\frac{c_2}{1 - c_2} \bar{w}(0, t),
\end{aligned}
\]
(5.20)
in the state space \( \mathcal{X} = \mathbb{H} \times H^1(0, 1) \times H^1_R(0, 1) \times L^2(0, 1) \times \mathbb{H} \times H^1(0, 1) \).

**Theorem 5.1.** Suppose that \( \frac{c_1}{1 - c_0} > 0, \frac{c_0 - q}{1 - c_0} > 0, \frac{c_2 - q}{1 - c_2} > 0 \) and \( c_3 > 0 \). For any initial state \((\bar{\varepsilon}, \bar{\varepsilon}_1, \bar{Y}, \bar{Y}_0, \bar{\beta}, \bar{\beta}_t, \bar{w}, \bar{w}_t, \bar{Z}) \in \mathcal{X}\), with the compatibility conditions \( Y_0(0) = -c_0 \bar{\varepsilon}_0(0), \)
\( Z_0(0) = -\frac{c_2}{1 - c_2} \bar{w}_0(0) \), the system (5.20) admits a unique solution \( (\varepsilon, \varepsilon_t, \bar{Y}, \bar{Y}_0, \bar{\beta}, \bar{\beta}_t, \bar{w}, \bar{w}_t, \bar{Z}) \in C(0, \infty; \mathcal{X}) \) and there exist two constants \( M, \mu > 0 \) such that
\[
\|(\varepsilon(\cdot, t), \varepsilon_t(\cdot, t), \bar{Y}(\cdot, t), \bar{\beta}(\cdot, t), \bar{\beta}_t(\cdot, t), \bar{w}(\cdot, t), \bar{w}_t(\cdot, t), \bar{Z}(\cdot, t))\|_{\mathcal{X}} \leq M e^{-\mu t}.
\]
(5.21)

**Proof.** We see that the “\((\varepsilon, Y, \beta)-part\)” of (5.20) is independent of the “\((\bar{w}, Z)-part\)” of (5.20). We first consider the “\((\varepsilon, Y, \beta)-part\)” of (5.20). Denote \( \eta(x, t) = \varepsilon(x, t) + Y(x, t) - \bar{Y}(x, t) \).
Next, we claim that the solution of

\[
\begin{aligned}
\eta_t(x, t) &= \eta_{xx}(x, t), \\
\eta_x(0, t) &= \frac{c_1}{1 - c_0} \eta(0, t) + \frac{c_0 - q}{1 - c_0} \eta_t(0, t), \quad \eta_x(1, t) = 0, \\
Y_t(x, t) &= -Y_x(x, t), \quad Y(0, t) = -\frac{c_0}{1 - c_0} [\eta(0, t) + \beta(0, t)], \\
\beta_{xt}(x, t) &= \beta_{xx}(x, t), \\
\beta_x(0, t) &= \frac{c_1}{1 - c_0} \beta(0, t) + \frac{c_0 - q}{1 - c_0} \beta_t(0, t), \quad \beta(1, t) = 0.
\end{aligned}
\]  

(5.22)

Comparing (5.22) with (4.3) and noting that \(\frac{c_1}{1 - c_0} > 0, \frac{c_0 - q}{1 - c_0} > 0\), the system (5.22) is exactly the same as the system (4.3) after replacing \(\frac{c_0}{1 - c_0} > 0\) with \(\frac{c_0 - q}{1 - c_0} > 0\). Thus, by Theorem 4.1, we can conclude that (5.22) admits a unique solution and there exist two constants \(M_1, \mu_1 > 0\) such that

\[
\| (\varepsilon(\cdot, t), \varepsilon_t(\cdot, t), \beta(\cdot, t), \beta_t(\cdot, t), Y(\cdot, t)) \|_{H \times H^1(0, 1) \times L^2(0, 1) \times H^1(0, 1)} \leq M_1 e^{-\mu_1 t}.
\]  

(5.23)

Now, we consider the “(\(\tilde{w}, Z\))-part” of (5.20) which reads as

\[
\begin{aligned}
\tilde{w}_{tt}(x, t) &= \tilde{w}_{xx}(x, t), \\
\tilde{w}_x(0, t) &= \frac{c_2 - q}{1 - c_2} \tilde{w}_t(0, t) + c_1 \varepsilon(0, t), \quad \tilde{w}_x(1, t) = -c_3 \tilde{w}(1, t), \\
Z_t(x, t) &= -Z_x(x, t), \quad Z(0, t) = -\frac{c_2}{1 - c_2} \tilde{w}(0, t),
\end{aligned}
\]  

(5.24)

By Sobolev embedding theorem and (5.23), we have

\[
|\varepsilon(0, t)| \leq \|\varepsilon(0, t)\|_{H^1(0, 1)} \leq \| (\varepsilon(\cdot, t), \varepsilon_t(\cdot, t)) \|_H \leq M_1 e^{-\mu_1 t}.
\]  

(5.25)

We can write “\(\tilde{w}\)-part” of (5.21) as

\[
\frac{d}{dt} (\tilde{w}(\cdot, t), \tilde{w}_t(\cdot, t)) = A_0(\tilde{w}(\cdot, t), \tilde{w}_t(\cdot, t)) + B_0 \varepsilon(0, t),
\]

where the operators \(A_0\) and \(B_0\) are given by

\[
A_0(\phi, \psi) = (\psi, \phi'') \quad \forall (\phi, \psi) \in \mathcal{D}(A_0),
\]

\[
\mathcal{D}(A_0) = \left\{ (\phi, \psi) \in H^2(0, 1) \times H^1(0, 1) \mid \phi'(0) = \frac{c_2 - q}{1 - c_2} \psi(0), \, \phi'(1) = -c_3 \phi(1) \right\},
\]

(5.26)

and \(B_0 = c_1(0, -\delta_0)\). It is well known (\cite{16} Proposition 2) that \(A_0\) generates an exponential stable operator semigroup \(e^{A_0 t}\) on \(\mathbb{H}\) and \(B_0\) is admissible for \(e^{A_0 t}\). It follows from (5.25) and Lemma 2.1 that there exist two constant \(M_2, \mu_2 > 0\) such that

\[
\| (\tilde{w}(\cdot, t), \tilde{w}_t(\cdot, t)) \| \leq M_2 e^{-\mu_2 t}.
\]  

(5.27)

Next, we claim that the solution of “\(Z\)-part” of (5.24) is exponentially stable. Set

\[
Z(x, t) = \begin{cases} 
-\frac{c_2}{1 - c_2} \tilde{w}(0, t - x), & t \geq x, \\
Z_0(x - t), & x > t.
\end{cases}
\]  

(5.28)
Then a direct computation shows that $Z(x, t)$ solves “$Z$-part” of (5.24). Thus, to show the exponentially stability of “$Z$-part” of (5.23), it suffices to prove that there exist two constants $M_3, \mu_3 > 0$ such that
\[
\int_0^1 \tilde{w}_t^2(0, t - x) dx \leq M_3 e^{-\mu_3 t}.
\] (5.29)

Indeed, (5.29) can be proved by defining $\rho(t) = 2\int_0^1 (x - 1) \tilde{w}_t(x, t) \tilde{w}_x(x, t) dx$. Since the proof of (5.29) is very similar to the proof of (1.19), we omit the details. Combining (5.23), (5.27), (5.28) and the exponential stability of $Z(\cdot, t)$ on $H^1(0, 1)$, we get (5.21).

### 5.3 Well-posedness and exponential stability of the closed-loop system

We go back to the closed-loop system of (5.1) under the feedback (5.17):

\[
\begin{cases}
    w_{tt}(x, t) = w_{xx}(x, t), \\
    w_x(0, t) = -qw_t(0, t), \\
    w_x(1, t) = -c_3 \tilde{w}(1, t) - c_3 Z(1, t) - Z_x(1, t) + Y_x(1, t) \\
    -Z_x(1, t) + f(w(\cdot, t), w_t(\cdot, t)) + d(t), \\
    v_{tt}(x, t) = v_{xx}(x, t), \\
    v_x(0, t) = -qv_t(0, t) + c_1 [v(0, t) - w(0, t)], \\
    v_x(1, t) = -c_3 \tilde{w}(1, t) - c_3 Z(1, t) - Z_x(1, t) + Y_x(1, t) \\
    -Z_x(1, t) - W_x(1, t), \\
    z_{tt}(x, t) = z_{xx}(x, t), \\
    z_x(0, t) = \frac{c_1}{1 - c_0} z(0, t) + \frac{c_0 - q}{1 - c_0} z_x(0, t), \\
    z(1, t) = -v(1, t) - W(1, t) + w(1, t), \\
    \tilde{w}_{tt}(x, t) = \tilde{w}_{xx}(x, t), \\
    \tilde{w}_x(0, t) = -qw_t(0, t) + c_1 [\tilde{w}(0, t) - w(0, t)], \\
    \tilde{w}_x(1, t) = -c_3 \tilde{w}(1, t) - c_3 Z(1, t) - Z_x(1, t), \\
    W_t(x, t) = -W_x(x, t), \\
    W(0, t) = -c_0 [v(0, t) - w(0, t)], \\
    Y_t(x, t) = -Y_x(x, t), \\
    Y(0, t) = -c_0 [\tilde{w}(0, t) - w(0, t)], \\
    Z_t(x, t) = -Z_x(x, t), \\
    Z(0, t) = -c_2 \tilde{w}(0, t).
\end{cases}
\] (5.30)

We consider the system (5.30)-(5.31) in the state space $\mathcal{H} = \mathbb{H}^3 \times H^1(0, 1) \times \mathbb{H} \times [H^1(0, 1)]^2$.

**Theorem 5.2.** Suppose that $\frac{c_1}{1 - c_0} > 0$, $\frac{c_0 - q}{1 - c_0} > 0$, $\frac{c_3 - q}{1 - c_3} > 0$ and $c_3 > 0$. Suppose that $f : \mathbb{H} \to \mathbb{R}$ is continuous, and $d \in L^\infty[0, \infty)$ or $d \in L^2[0, \infty)$. For any initial state $(w_0, w_1, v_0, v_1, z_0, z_1, W_0, \tilde{w}_0, \tilde{w}_1, Y_0, Z_0) \in \mathcal{H}$ with the compatibility conditions

\[-z_0(1) - v_0(1) - W_0(1) + w_0(1) = 0, \quad Z_0(0) + c_2 \tilde{w}_0(0) = 0, \quad W_0(0) + c_0 [v_0(0) - w_0(0)] = 0, \quad Y_0(0) + c_0 [\tilde{w}_0(0) - w_0(0)] = 0,\]


there exists a unique solution to [5.30]-[5.31] such that
\((w, w_1, v, v_1, z, z_1, W, \tilde{w}_0, \tilde{w}_1, Y, Z) \in C(0, \infty; \mathcal{X})\) satisfies
\[
\|(w(\cdot, t), w_t(\cdot, t), \tilde{w}(\cdot, t), \tilde{w}_t(\cdot, t), Y(\cdot, t), Z(\cdot, t))\|_{\mathcal{H}^2 \times \mathcal{H}^1(0,1)}^2 \leq M e^{-\mu t}, \quad t \geq 0, \tag{5.32}
\]
with some \(M, \mu > 0\), and
\[
\sup_{t \geq 0} \|(v(\cdot, t), v_t(\cdot, t), z(\cdot, t), z_t(\cdot, t), W(\cdot, t))\|_{\mathcal{H}^2 \times \mathcal{H}^1(0,1)} < \infty. \tag{5.33}
\]
If we assume further that \(f(0,0) = 0\) and \(d \in L^2[0, \infty)\), then
\[
\lim_{t \to \infty} \|(v(\cdot, t), v_t(\cdot, t), z(\cdot, t), z_t(\cdot, t), W(\cdot, t))\|_{\mathcal{H}^2 \times \mathcal{H}^1(0,1)} = 0. \tag{5.34}
\]
If we assume that \(f \equiv 0\) and \(d \equiv 0\), then there exist two constants \(M', \mu' > 0\) such that
\[
\|(v(\cdot, t), v_t(\cdot, t), z(\cdot, t), z_t(\cdot, t), W(\cdot, t))\|_{\mathcal{H}^2 \times \mathcal{H}^1(0,1)} \leq M' e^{-\mu' t}, \quad t \geq 0. \tag{5.35}
\]

**Proof.** Using the variables \(\varepsilon(x, t), \beta(x, t)\) and \(\tilde{\varepsilon}(x, t)\) given by [5.12], [5.9] and [5.3], respectively, and the invertible transformation \(\tilde{w}(x, t) = \tilde{w}(x, t) + Z(x, t)\), we can write a system equivalent to [5.30]-[5.31] as follows:
\[
\begin{align*}
\varepsilon_t(x, t) &= \varepsilon_{xx}(x, t), \\
\varepsilon_x(0, t) &= -q\varepsilon_t(0, t) + c_1 \varepsilon(0, t), \quad \varepsilon_x(1, t) = \beta_x(1, t) - Y_x(1, t), \\
Y_t(x, t) &= -Y_x(x, t), \quad Y(0, t) = -c_0 \varepsilon(0, t), \\
\beta_t(x, t) &= \beta_{xx}(x, t), \\
\beta_x(0, t) &= \frac{c_1 - q}{1 - c_0} \beta(0, t) + \frac{c_0}{1 - c_0} \beta_t(0, t), \quad \beta(1, t) = 0,
\end{align*}
\]
\[
\begin{align*}
\tilde{w}_t(x, t) &= \tilde{w}_{xx}(x, t), \\
\tilde{w}_x(0, t) &= \frac{c_2 - q}{1 - c_2} \tilde{w}_t(0, t) + c_1 \varepsilon(0, t), \quad \tilde{w}_x(1, t) = -c_3 \tilde{w}(1, t), \\
Z_t(x, t) &= -Z_x(x, t), \quad Z(0, t) = -\frac{c_2}{1 - c_2} \tilde{w}(0, t), \\
\tilde{v}_t(x, t) &= \tilde{v}_{xx}(x, t), \\
\tilde{v}_x(0, t) &= -q \tilde{v}_t(0, t) + c_1 \tilde{v}(0, t), \\
\tilde{v}_x(1, t) &= -f(w(\cdot, t), v_t(\cdot, t)) - d(t) - W(x, t), \\
W_t(x, t) &= -W_x(x, t), \quad W(0, t) = -c_0 \tilde{v}(0, t).
\end{align*}
\]

We see that the “\((\varepsilon, \beta, \tilde{w}, Y, Z)\)-part” of [5.30]-[5.31] is independent of the “\((\tilde{v}, W)\)-part” of [5.36]-[5.37]. By Theorem 5.11, there exist two constants \(M_1, \mu_1 > 0\) such that the solution \((\varepsilon, \varepsilon_t, Y, \beta, \beta_t, \tilde{w}, \tilde{w}_t) \in C(0, \infty; \mathcal{X})\) satisfies
\[
\|(\varepsilon(\cdot, t), \varepsilon_t(\cdot, t), Y(\cdot, t), \beta(\cdot, t), \beta_t(\cdot, t), \tilde{w}(\cdot, t), \tilde{w}_t(\cdot, t), Z(\cdot, t), Z_t(\cdot, t))\|_{\mathcal{X}} \leq M_1 e^{-\mu_1 t}. \tag{5.38}
\]
Since \(\tilde{w}(x, t) = \tilde{w}(x, t) - Z(x, t)\) and \(\tilde{w}_t(x, t) = \tilde{w}_t(x, t) + Z_x(x, t)\), we have that
\[
\|\tilde{w}(\cdot, t), \tilde{w}_t(\cdot, t)\|_{\mathcal{H}} \leq \|\tilde{w}(\cdot, t), \tilde{w}_t(\cdot, t)\|_{\mathcal{H}} + \|\tilde{Z}(\cdot, t), \tilde{Z}_t(\cdot, t)\|_{\mathcal{H}} \leq 3M_1 e^{-\mu_1 t}. \tag{5.39}
\]
Since \( w(x, t) = \hat{w}(x, t) - \varepsilon(x, t) \), \( w_t(x, t) = \hat{w}_t(x, t) - \varepsilon_t(x, t) \), we obtain
\[
\|(w(\cdot, t), w_t(\cdot, t))\|_H \leq \|(\hat{w}(\cdot, t), \hat{w}_t(\cdot, t))\|_H + \|(\varepsilon(\cdot, t), \varepsilon_t(\cdot, t))\|_H \leq 4M_1 e^{-\mu_1 t}.
\]
(5.40)

It follows from (5.38), (5.39) and (5.40) that (5.32) holds with \( M = 6M_1 \) and \( \mu = \mu_1 \).

Now we consider the “\((\hat{v}, W)\)-part” which reads as
\[
\begin{aligned}
\hat{v}_H(x, t) &= \hat{v}_{xx}(x, t), \\
\hat{v}_x(0, t) &= -q\hat{v}_t(0, t) + c_1\hat{v}(0, t), \\
\hat{v}_x(1, t) &= -f(w(\cdot, t), w_t(\cdot, t)) - d(t) - W_x(1, t), \\
W_t(x, t) &= -W_x(x, t), \quad W(0, t) = -c_0\hat{v}(0, t).
\end{aligned}
\]
(5.41)

Since \( f : H^1(0, 1) \times L^2(0, 1) \to \mathbb{R} \) is continuous and \((w, w_t)\) is bounded (since it tends to zero), \( f(w(\cdot, t), w_t(\cdot, t)) \in L^\infty(0, \infty) \). Since \( d \in L^\infty(0, \infty) \) or \( d \in L^2(0, \infty) \), it follows from Lemma 5.1 that system (5.41) admits a unique bounded solution, i.e.,
\[
\sup_{t \geq 0} \|(\hat{v}(\cdot, t), \hat{v}_t(\cdot, t), W(\cdot, t))\|_{H^1(0, 1)} < \infty.
\]
(5.42)

Noting that \( W_t(x, t) = -W_x(x, t) \), it follows from (2.6), (2.22) and (4.21) that
\[
\|(v(\cdot, t), v_t(\cdot, t))\|_H \leq \|(\hat{v}(\cdot, t), \hat{v}_t(\cdot, t))\|_H + \|(w(\cdot, t), w_t(\cdot, t))\|_H,
\]
\[
\|(z(\cdot, t), z_t(\cdot, t))\|_H \leq \|(\beta(\cdot, t), \beta_t(\cdot, t))\|_H + \|(\hat{v}(\cdot, t), \hat{v}_t(\cdot, t))\|_H + \|(W(\cdot, t), W_x(\cdot, t))\|_H,
\]
which gives (5.33), because both right-hand sides are bounded.

Now suppose that \( f(0, 0) = 0 \) and \( d \in L^2(0, \infty) \). By (5.40) and the continuity of \( f \), we have \( \lim_{t \to \infty} |f(w, w_t)| = 0 \). By Lemma 5.1 we obtain
\[
\lim_{t \to \infty} \|(\hat{v}(\cdot, t), \hat{v}_t(\cdot, t), W(\cdot, t))\|_{H^1(0, 1)} = 0.
\]
(5.43)

By (5.38), (5.40), (5.42) and (5.43), we derive
\[
\|(v(\cdot, t), v_t(\cdot, t))\|_H \leq \|(\hat{v}(\cdot, t), \hat{v}_t(\cdot, t))\|_H + \|(w(\cdot, t), w_t(\cdot, t))\|_H \to 0, \text{ as } t \to \infty,
\]
\[
\|(z(\cdot, t), z_t(\cdot, t))\|_H \leq \|(\tilde{z}(\cdot, t), \tilde{z}_t(\cdot, t))\|_H + \|(\hat{v}(\cdot, t), \hat{v}_t(\cdot, t))\|_H + \|(W(\cdot, t), W_x(\cdot, t))\|_H,
\]
which is bounded. Next, suppose that \( f \equiv 0 \) and \( d \equiv 0 \). It follows from Lemma 5.1 that there exist two constants \( M_2, \mu_2 > 0 \) such that for all \( t \geq 0, \)
\[
\|(\hat{v}(\cdot, t), \hat{v}_t(\cdot, t), W(\cdot, t))\|_{H^1(0, 1)} \leq M_2 e^{-\mu_2 t}.
\]
(5.44)

By (5.38), (5.40) and (5.44), we obtain that for all \( t \geq 0, \)
\[
\|(v(\cdot, t), v_t(\cdot, t))\|_H \leq \|(\hat{v}(\cdot, t), \hat{v}_t(\cdot, t))\|_H + \|(w(\cdot, t), w_t(\cdot, t))\|_H \leq M_2 e^{-\mu_2 t} + 4M_1 e^{-\mu_1 t},
\]
\[
\|(z(\cdot, t), z_t(\cdot, t))\|_H \leq \|(\beta(\cdot, t), \beta_t(\cdot, t))\|_H + \|(\hat{v}(\cdot, t), \hat{v}_t(\cdot, t))\|_H + \|(W(\cdot, t), W_x(\cdot, t))\|_H
\]
\[
\leq M_1 e^{-\mu_1 t} + 2M_2 e^{-\mu_2 t},
\]
which, combining with (5.44), implies that (5.35) holds. \( \square \)
Remark 5.1. Similarly to Remark 4.2, we point out that the output measurement signals \( w(0, t), w(1, t) \) are also almost a minimal set of measurement signals to exponentially stabilize the system \( (5.1) \). Theorem 5.2 shows that we can design disturbance estimator and state observer by using \( \{w(0, t), w(1, t)\} \) only and that the system \( (5.1) \) can be exponentially stabilized by using \( \{w(0, t), w(1, t)\} \) only. However,

(a). Each of the observation \( \{w(0, t), w(1, t)\} \) is not enough for exact observability, i.e., for any \( T > 0 \), there is no constant \( C_T \) such that

\[
\int_0^T w^2(0, s) ds \geq C_T \|w(\cdot, 0), w_t(\cdot, 0)\|_\mathcal{H},
\]

\[
\int_0^T w^2(1, s) ds \geq C_T \|w(\cdot, 0), w_t(\cdot, 0)\|_\mathcal{H}.
\]

(b). The \( y(t) = w(1, t) \) is also not enough for exponential stabilizability. Actually, let \( f(w) \equiv 0, d(t) = \mu e^{i\mu t} \) and \( \phi(x) = \sin \mu (x - 1) \), where \( \mu \) satisfies \( \cosh i\mu = q \sinh i\mu \).

Then, system \( (5.1) \) admits a solution \( (w, w_t) = (e^{i\mu t} \phi(x), i\mu e^{i\mu t} \phi(x)) \) which makes the output \( y(t) = w(1, t) \equiv 0 \).

From (a), (b), \( w(0, t) \) seems to be necessary to ensure the possibility of stabilization. We leave two open question here: (I): Can we design a state observer for system \( (5.1) \) using only \( y(t) = w(0, t) \)? (II): Is \( y(t) = w(0, t) \) only enough to make system \( (5.1) \) stabilizable?

6 Concluding remarks

We have studied the exponential stabilization problem for the one dimensional unstable or anti-stable wave equation with Neumann boundary control subject to an unknown bounded disturbance, using only two measurement signals. We have designed disturbance estimators that do not use high gain and, based on these, have proposed state observers. We have shown that the total disturbance is estimated by the disturbance estimator in the sense that the error is in \( L^2[0, \infty) \), and that the state of the original system is recovered by the proposed state observer. We have constructed a state observer based output feedback controller that guarantees that the signals in the original system are exponentially stable. This is a first output feedback controller that can exponentially stabilize a system described by PDEs with both internal uncertainty and external disturbance. This shows that exponential stability can be achieved without sliding mode control even for a very general type of disturbance. Our approach can be generalized to deal with other PDEs such as unstable/anti-stable wave equation with Dirichlet boundary control matched with the internal uncertainty and the external disturbance, again using two measurement signals. We have posed open questions in Remarks 4.2 and 5.1 concerning a stabilizing controller using only one output measurement signal.

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Similarly to the above, by (7.4), the admissibility of $\mathbb{B}_2$ is nondecreasing in $t$, hence $C_{1H} \leq C_{1T}$ for any $t \in [0, T]$. It is easy to see from (7.3) that $|\varphi(0)| \leq \|\varphi, \psi\|_H$ holds for all $(\varphi, \psi) \in \mathbb{H}$. Thus, for any $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in C(0, T; \mathbb{H})$, 

\[
\left\| \int_0^t e^{A(t-s)} \mathbb{B}_1(\varphi_1(0, s) - \varphi_2(0, s))ds \right\|_H^2 \leq C_{1T} \int_0^T \left\| \begin{bmatrix} \varphi_1(\cdot, s) \\ \psi_1(\cdot, s) \end{bmatrix} - \begin{bmatrix} \varphi_2(\cdot, s) \\ \psi_2(\cdot, s) \end{bmatrix} \right\|_H^2 ds.
\]  

From [35, Proposition 2.3] we know that $C_{1H}$ is nondecreasing in $t$, hence $C_{1H} \leq C_{1T}$ for any $t \in [0, T]$. It is easy to see from (7.3) that $|\varphi(0)| \leq \|\varphi, \psi\|_H$ holds for all $(\varphi, \psi) \in \mathbb{H}$. Thus, for any $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in C(0, T; \mathbb{H})$, 

\[
\left\| \int_0^t e^{A(t-s)} \mathbb{B}_1(\varphi_1(0, s) - \varphi_2(0, s))ds \right\|_H^2 \leq C_{1T} \int_0^T \left\| \begin{bmatrix} \varphi_1(\cdot, s) \\ \psi_1(\cdot, s) \end{bmatrix} - \begin{bmatrix} \varphi_2(\cdot, s) \\ \psi_2(\cdot, s) \end{bmatrix} \right\|_H^2 ds.
\]  

Similarly to the above, by (7.4), the admissibility of $\mathbb{B}_2$ implies that for all $t > 0$,
\[
\left\| \int_0^t e^{\mathcal{A}(t-s)} \mathbb{H} \left[ f \left[ \begin{array}{c} \varphi_1(s) \\ \psi_1(s) \end{array} \right] - f \left[ \begin{array}{c} \varphi_2(s) \\ \psi_2(s) \end{array} \right] \right] \right\|_\mathbb{H}^2 \leq C_2 T \left\| f \left[ \begin{array}{c} \varphi_1(s) \\ \psi_1(s) \end{array} \right] - f \left[ \begin{array}{c} \varphi_2(s) \\ \psi_2(s) \end{array} \right] \right\|_{L^2[0,t]}^2 \leq C_2 T L^2 \int_0^t \left\| f \left[ \begin{array}{c} \varphi_1(s) \\ \psi_1(s) \end{array} \right] - f \left[ \begin{array}{c} \varphi_2(s) \\ \psi_2(s) \end{array} \right] \right\|_{\mathbb{H}}^2 \, ds.
\]

It follows from here and (7.3), (7.5) that for any \((\varphi_1, \varphi_2), (\varphi_2, \psi_2) \in C(0,T; \mathbb{H})\),

\[
\left\| F \left[ \psi_1(t) \right] - F \left[ \psi_2(t) \right] \right\|_{\mathbb{H}}^2 
\leq (C_1 T (q+1)^2 + C_2 T L^2) \int_0^t \left\| \varphi_1(s) - \varphi_2(s) \right\|_{\mathbb{H}}^2 \, ds
\]

\[
= (C_1 T (q+1)^2 + C_2 T L^2) \int_0^t e^{2\lambda s} e^{-2\lambda s} \left\| \varphi_1(s) - \varphi_2(s) \right\|_{\mathbb{H}}^2 \, ds
\]

\[
\leq (C_1 T (q+1)^2 + C_2 T L^2) \frac{e^{2\lambda t} - 1}{2\lambda} \left\| \varphi_1(t) - \varphi_2(t) \right\|_{\mathbb{H}}^2 \quad \forall \, t \in [0,T].
\]

Choose \(\lambda > \frac{1}{2} [C_1 T (q+1)^2 + C_2 T L^2]\) in (7.2), then the above estimate implies that \(F\) is a strict contraction on \(C(0,T; \mathbb{H})\). By the contraction mapping theorem (see, for instance, [2]), (7.3) has a unique fixed point \((\phi, \psi) \in C(0,T; \mathbb{H})\), which is then a solution of (1.5) in \([0,T]\), which implies that \(\psi = \varphi_t\). Since the above reasoning works for any \(T > 0\), (1.1) admits a unique global solution. \(\square\)