On Lie point symmetry of classical Wess-Zumino-Witten Model

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Abstract

We perform the group analysis of Witten’s equations of motion for a particle moving in the presence of a magnetic monopole, and also when constrained to move on the surface of a sphere, which is the classical example of Wess-Zumino-Witten model. We also consider variations of this model. Our analysis gives the generators of the corresponding Lie point symmetries. The Lie symmetry corresponding to Kepler’s third law is obtained in two related examples.

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1 Introduction

The Wess-Zumino model\cite{1} has the distinction of being applicable via the current algebra conditions to explain the process $K^+K^-\rightarrow\pi^+\pi^0\pi^-$ as well as in setting up the string theory actions through the generalizations made by Witten\cite{2}. To motivate before arriving at the generalized form, Witten had considered an example of the equation of motion of a particle in three dimensions constrained to move on the surface of a sphere in the presence of a magnetic monopole. The equations of motion of such a system in the presence of a magnetic monopole can not be obtained from the usual Lagrangian formulation. Hence, to look for the continuous symmetries associated with such a system one has to analyse directly the equations of motion. This procedure is more fundamental in some sense, from the point of view of symmetries, as in certain cases many different Lagrangians may give rise to the same equations of motion. The group analysis of the equations of motion gives all the Lie point group symmetry generators. In the cases where a Lagrangian formulation is possible, a subset of these generators acting on the Lagrangian giving zero are the Noether symmetries\cite{3}. However besides these there may be other generators obtained through group analysis which have direct physical significance, but not explicitly available from the consideration of Noether symmetries alone. The reproduction of Kepler’s third law in the planetary motion problem is such an example. This is obtained without solving the equations of motion. The extension of this idea to the notion of Lie dynamical symmetries contains similarly a subclass known as Cartan symmetries. The Runge-Lenz vector can be obtained from such considerations. These symmetries are, further, related to the Lie- Bäcklund symmetries.

The application of these types of analysis to nonlocal cases have been widely studied through Bäcklund transformations and related techniques in the context of integrable systems containing infinite number of conservation laws\cite{4}. The Thirring model, which corresponds to coupled partial differential equations has been analysed by Morris\cite{5}. The differential geometric forms developed earlier\cite{6} are used in this analysis to obtain the prolongation structure. Some other applications of these ideas to important problems from physics is comprehensibly covered in\cite{7}.

In this paper we restrict ourselves to finding the Lie point symmetries of the coupled set of differential equations representing:

(i) a charged particle moving in three dimensions in the presence of a magnetic field
proportional to the coordinate vector,

(ii) as in (i) and the constraint that the particle moves on the surface of a sphere, the situation being equivalent to the motion of the particle in the presence of a magnetic monopole stationed at the centre of the sphere.

(iii) a particle in a particular type of velocity dependent potential.

Section II provides the outline of the method for group analysis of the equations of motion. In section III we use this to find the generators of the point symmetries for the above examples. Section IV is devoted to physical interpretations and conclusions.

It should be noted that in the context of the symmetries of Wess-Zumino-Witten models, the ones corresponding to non Lie symmetries plays by far the most important role and these have been fruitfully exploited [8].

## 2 Group analysis

Typically we are interested in the coupled nonlinear set of equations representing the equations of motion of a particle in three dimensions. These are of the form

\[ \ddot{q}^a = \beta \omega^a(q^i, \dot{q}^i, t) \] (2.1)

where a dot is a derivative with respect to time, \( a, i = 1, 2, \) and \( 3, \) and \( \beta \) is a constant involving mass, coupling constant etc. The expressions for the function \( \omega \) are given explicitly for each example.

These set of equations can be analysed by means of one parameter groups by infinitesimal transformations. We demand the equation to be invariant under infinitesimal changes of the explicit variable \( t, \) as well as simultaneous infinitesimal changes of the dependent functions \( q^a \) in the following way,

\[ t \to t_1 = t + \epsilon \xi(t, q^1, q^2, q^3) + O(\epsilon^2), \]

\[ q^a \to q_1^a = q^a + \epsilon \eta^a(t, q) + O(\epsilon^2). \] (2.2)

Under \( t \to t_1 \) and \( q^a \to q_1^a, \) the equation changes to,

\[ \ddot{q}_1^a = \beta \omega_1^a(t_1, q_1^i, \dot{q}_1^i) \] (2.3)
To illustrate the procedure consider the simple case in one space dimension. We express the above equation in terms of $t$ and $q$ by using the transformation (2.2). Then the invariance condition implies that an expression containing various partial derivatives of $\xi$ and $\eta$ is obtained which equates to zero. For example, we get

$$\frac{dq_1}{dt} = \frac{dq}{dt} + \epsilon (\frac{\partial \eta}{\partial q} - \frac{\partial \xi}{\partial t}) \frac{dq}{dt} + \epsilon^2 (\frac{\partial q}{\partial q} \frac{\partial\eta}{\partial t})^2 + O(\epsilon^2).$$

(2.4)

and now relate the left hand side with $\frac{dq}{dt}$ by using binomial theorem for the denominator to obtain

$$\frac{dq_1}{dt} = \frac{dq}{dt} + \epsilon (\frac{\partial \eta}{\partial q} - \frac{\partial \xi}{\partial t}) \frac{dq}{dt} - \frac{\partial \xi}{\partial q} (\frac{\partial q}{\partial t})^2 + O(\epsilon^2).$$

(2.5)

A similar procedure is followed to express $\frac{d^2 q_1}{dt^2}$ likewise. By substituting equations (2.2) - (2.5) for a given explicit expression for $\omega_1$ and remembering that $\frac{d^2 q}{dt^2} - \omega$ is zero, we obtain the desired partial differential equation whose solution would determine $\xi(t, q)$ and $\eta(t, q)$. In our case, of course, we have to find $\xi(t, q_1, q_2, q_3)$ and $\eta(t, q_a)$'s.

To relate these to the generators of the infinitesimal transformations we write

$$t_1(t, q^i; \epsilon) = t + \epsilon \xi(t, q^i) + \cdots = t + \epsilon X t + \cdots$$

(2.6)

$$q^a_1(t, q^i; \epsilon) = q^a + \epsilon \eta(t, q^i) + \cdots = q^a + \epsilon X q^a + \cdots$$

(2.7)

where the functions $\xi$ and $\eta^a$ are defined by

$$\xi(t, q^i) = \frac{\partial t_1}{\partial \epsilon} \bigg|_{\epsilon=0},$$

(2.8)

$$\eta^a(t, q^i) = \frac{\partial q^a_1}{\partial \epsilon} \bigg|_{\epsilon=0}$$

(2.9)

and the operator $X$ is given by

$$X = \xi(t, q^i) \frac{\partial}{\partial t} + \eta^a(t, q^i) \frac{\partial}{\partial q^a}.$$  

(2.10)

Following Stephani [3], we will consider the equation having the symmetry generated by $X$ and its extension

$$\dot{X} = \xi \frac{\partial}{\partial t} + \eta^a \frac{\partial}{\partial q^a} + \dot{\eta}^a \frac{\partial}{\partial q^a}.$$ 

(2.11)

and the symmetry condition determines $\dot{\eta}^a$. The symmetry condition is given by

$$\xi \omega^a, t + \eta^b \omega^a, b + (\dot{\eta}^b, t + \dot{q}^b \eta^a, c - \dot{q}^b \xi, c) \frac{\partial \omega^a}{\partial q^b}$$

$$+ 2 \omega^a (\xi, t + \dot{q}^b \xi, b) + \omega^b (q^a \xi, b - \eta^a, b) + \dot{q}^a q^b \dot{q}^c \xi, bc$$

$$+ 2 q^a q^c \xi, tc - q^a q^b \eta^a, bc + q^a \xi, ut - 2 q^a \eta^a, tb - \eta^a, tt = 0$$

(2.12)
where \( f_t = \frac{\partial f}{\partial t} \) and \( f_c = \frac{\partial f}{\partial q^c} \). By herding together coefficients of the terms with cubic, quartic, and linear in \( \dot{q}^a \), and the ones independent of \( \dot{q}^a \) separately, and equating each of these to zero we obtain an over determined set of partial differential equations and solve for \( \xi \) and \( \eta^a \).

### 3 The symmetry generators

For the first example, the motion of a particle in presence of a magnetic field, the equations of motion are

\[
\ddot{q}^a = \beta \varepsilon^{abc} \dot{q}^b q^c
\]

where the right hand side, which is our \( \omega^a \), is the Lorentz force acting on a charged particle, the magnetic field for this case being proportional to \( q^c \).

As was pointed out by Witten [9], one faces trouble in attempting to derive these equations of motion by using the usual procedure of variation of a Lagrangian as no obvious term can be included in the Lagrangian whose variation would give the right hand side of equation (2.12). Hence it would be more appropriate here to consider the group analysis of the equations of motion directly to obtain all the Lie point symmetries, with the Noether symmetries being a subclass of these. However, the present analysis cannot give any of the non-Lie symmetries.

Substituting

\[
\omega^a = \varepsilon^{abc} \dot{q}^b q^c
\]

into equation (2.12) we obtain, in general, coupled partial differential equations for \( \xi \) and \( \eta \) by equating to zero the terms corresponding to various powers of \( \dot{q}^l \).

Consideration of the term with \( \dot{q}^a \dot{q}^b \dot{q}^c \) tells us

\[
\xi_{,bc} = 0.
\]

Hence we may have

\[
\xi = A_i(t)q^i + B(t) + C.
\]

The terms quadratic in \( \dot{q} \) give

\[
-\beta \dot{q}^a \dot{q}^b q^m \xi_{,c} \varepsilon^a_{bm} + 2\beta \dot{q}^a \dot{q}^b q^m \xi_{,b} \varepsilon^a_{tm} \\
+ \beta \dot{q}^a \dot{q}^b \dot{q}^c \xi_{,b} \varepsilon^c_{rs} + 2q^a \dot{q}^c \xi_{,tc} - \dot{q}^c \dot{q}^b \eta^a_{,bc} = 0
\]

(3.17)
This shows that $\xi$ has to be independent of $q^l$,

$$\xi = B(t) + C,$$  \hspace{1cm} (3.18)

and $\eta$ may have the form

$$\eta^a = D(t)q^a + E(t)\varepsilon^{la}_m q^m + F(t) + G.$$  \hspace{1cm} (3.19)

The terms linear in $\dot{q}^l$ provide

$$\beta \dot{q}^l \varepsilon^a_{\ b} \eta^b + \beta \dot{q}^s \varepsilon^a_{\ b m} \eta^c_{\ b} - \beta \dot{q}^b q^m \varepsilon^a_{\ b m} \xi_{\ t}$$
$$+ 2 \beta \dot{q}^l \varepsilon^a_{\ b m} \xi_{\ t} - \beta \dot{q}^r q^s \varepsilon^a_{\ r s} \eta^b_{\ a} - 2 \dot{q}^b \eta_a + \dot{q}^a \xi_{\ t t} = 0.$$  \hspace{1cm} (3.20)

This demands

$$B(t) = tH + C,$$  \hspace{1cm} (3.21)

and also $\eta^a$ has to be independent of $t$,

$$\eta^a = -H q^a + E(t)\varepsilon^{la}_m q^m.$$  \hspace{1cm} (3.22)

Thus we obtain five generators

$$X_a = \varepsilon^a_{\ b k} q^b \frac{\partial}{\partial q^k},$$  \hspace{1cm} (3.23)

$$X_4 = \frac{\partial}{\partial t},$$  \hspace{1cm} (3.24)

$$X_5 = t \frac{\partial}{\partial t} - q^a \frac{\partial}{\partial q^a}.$$  \hspace{1cm} (3.25)

We can find their extension from the formula

$$\dot{\eta}^a = \frac{d\eta^a}{dt} - \dot{q}^a \frac{d\xi}{dt}$$  \hspace{1cm} (3.26)

and obtain, with the extensions

$$\dot{X}_a = \varepsilon^a_{\ b k} (q^b \frac{\partial}{\partial q^k} + \dot{q}^b \frac{\partial}{\partial q^k})$$  \hspace{1cm} (3.27)

$$\dot{X}_4 = \frac{\partial}{\partial t}$$  \hspace{1cm} (3.28)

$$\dot{X}_5 = t \frac{\partial}{\partial t} - q^a \frac{\partial}{\partial q^a} - 2 \dot{q}^a \frac{\partial}{\partial q^a}.$$  \hspace{1cm} (3.29)
A comparison with the results of similar analysis for the Kepler problem \cite{3} shows that the first four generators are identical, the first three corresponding to the generators of the three dimensional rotation group and $X_4$ is the generator for time translation. However in this case the law corresponding to Kepler’s third law goes instead like

$$t_1 r_1 = tr$$ (3.30)

If the particle is further constrained to move on the surface of a sphere of unit radius, the equation of motion becomes

$$\ddot{q}^a = \beta \varepsilon^{abc} \dot{q}^b \dot{q}^c - q^c \dot{q}^k \dot{q}_k.$$ (3.31)

This is equivalent to the case of a particle moving in the presence of a magnetic monopole centered at the origin of the sphere. Witten has generalised this idea to arbitrary dimensions for field theoretic considerations.

With $\omega^a$ being equal to the right hand side of equation (3.31) the group analysis shows that there is only one trivial time translation generator for this problem,

$$X = \frac{\partial}{\partial t}.$$ (3.32)

Same is the case if we ignore the term containing $\varepsilon^{abc}$ in equation (3.31).

But for equations of motions of the form

$$\ddot{q}^a = \dot{q}^a q^k q_k$$ (3.33)

we again find five symmetry generators, the first four being the same as $\dot{X}_a$ and $\dot{X}_4$ while

$$X'_5 = 2t \frac{\partial}{\partial t} - q^a \frac{\partial}{\partial q^a},$$ (3.34)

and with its extension,

$$\dot{X}'_5 = 2t \frac{\partial}{\partial t} - q^a \frac{\partial}{\partial q^a} - 3 \dot{q}^a \frac{\partial}{\partial q^a}.$$ (3.35)

The length and time scale in this case as

$$t_1 r_1^2 = tr^2.$$ (3.36)
4 Conclusion

It is well known that if the right hand side of equation (3.13) is zero, it would admit eight symmetries, which is the maximum number for an ordinary second order differential equation. By including different $q^a$, $\dot{q}^a$ dependent terms in the equations we do explicitly see which generators survive and also can find the complete Lie algebra. We have chosen three cases motivated by problems from physics. The original motivation of including Wess-Zumino terms in the Lagrangian has been to reduce some of its symmetries\cite{9}, but as we see in three dimensions the effect is rather drastic for the point symmetries. Only the trivial time translation generator survives. However, for the other examples considered we get some interesting result in the form of Kepler's scaling law. This we get without solving the equations of motion. Besides this, if a Langrangian could be set up, those generators operating on the Lagrangian to produce zero include all the Noether symmetries\cite{10}.

There has been much current interest in the gauge theories on noncommutative spaces in connection with the quantization of D-branes\cite{10}. There the models considered are usually in two dimensional space and relates to the phenomena of quantum Hall effect as the vector potential taken is proportional to the cordinates. This gives rise to the necessary constant (strong) magnetic field. However, in our case the setting is in three space dimensions and it is the magnetic field which is proportional to the coordinate vectors.

It is also expected that related analysis may provide useful information when terms are modified in the Lagrangian, due to quantum corrections for example. By explicitly showing how many and which generators remain as symmetries we will have a better understanding of the breaking of continuous symmetries.

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