Analytic Mechanics of Locally Conservative Physical Systems

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The analysis of the dynamics of a material point perfectly constrained to a submanifold of the three-dimensional euclidean space and subjected to a locally conservative force’s field, namely a force’s field corresponding to a closed but not necessarily exact differential form on such a submanifold, requires a generalization of the Lagrangian and the Hamiltonian formalism that is here developed.

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References
I. INTRODUCTION

In every elementary book of basic physics one finds the definition of a conservative force’s field as a force’s field \( \vec{f} \) such that there exists a smooth function \( V \) (called energy potential for \( \vec{f} \)) such that:

\[
\vec{f} = -\nabla V \tag{1.1}
\]

Often one finds therein the statement according to which the conservativity of a force’s field is equivalent to the condition:

\[
\nabla \wedge \vec{f} = 0 \tag{1.2}
\]

Following the strategy of converting the language of vector calculus in the language of differential forms summarized in the section \( \text{A} \) the equation 1.1 may be stated as the condition that the 1-form \( \vec{f} \flat \) is exact while the equation 1.2 may be stated as the condition that the 1-form \( \vec{f} \flat \) is closed.

So one realizes that the fact that in the ordinary three dimensional euclidean space \( \mathbb{E}^3 := (\mathbb{R}^3, \delta = dx \otimes dx + dy \otimes dy + dz \otimes dz) \) the equation 1.1 and the equation 1.2 are indeed equivalent is a consequence of the topological triviality of \( \mathbb{R}^3 \): specifically of the fact that the 1\(^{th}\) de-Rham cohomology group of \( \mathbb{R}^3 \) is trivial (and hence every closed form is exact, i.e. it can be globally integrated).

Considering the dynamics of a material point perfectly constrained to move on a topologically non-trivial submanifold of the euclidean space \( \mathbb{E}^3 \) (specifically a submanifold with non-trivial 1\(^{th}\) de-Rham cohomology group) one realizes that the condition 1.2 is a necessary but not sufficient condition for the conservativity of the involved force’s field \( \vec{f} \).

Since a force’s field satisfying the condition of equation 1.2 but violating the condition 1.1 corresponds to a 1-form \( \vec{f} \flat \) that is closed but not exact and hence, according to Poincaré Lemma \([1]\), may be integrated locally but not globally, it is natural to call such a force’s field (and the corresponding physical system too) \textit{locally conservative}.

Now nonconservative force’s fields performs a very short appearance in almost all the manuals of Classical Analytical Mechanics (see for instance \([2], [3]\)) whose attention rapidly converges on conservative systems for which the development of the lagrangian and hamiltonian formalism is reserved.

One could, at this point, argue that, from a physical viewpoint, the reason for that is simple and it is incisively expressed by Richard Feynman in the section 14.4 of \([4]\):

"We have spent a considerable time discussing conservative forces; what about nonconservative forces? We shall take a deeper view of this than usual, and state that there are no nonconservative forces! As a matter of fact, all the fundamental forces in nature appear to be conservative. This is not a consequence of Newton’s Law. In fact, so far as Newton himself knew, the forces could be nonconservative, as friction apparently is. When we say friction \textit{apparently} is, we are taking a modern view, in which it has been discovered that all the deep forces, the forces between the particles at the most fundamental level, are conservative.”

As to Analytic Mechanics (defined as the mathematical discipline dedicated to develop more advanced techniques through which Classical Newtonian Mechanics is formalized), anyway, nonconservative forces have to be taken into account.

They are indeed an experimental evidence of the Classical Newtonian Physics ruling our ordinary life (being an excellent approximation of Quantum Mechanics for macroscopic bodies as well as an excellent approximation of Special Relativity for velocities very much smaller than the velocity of light) and may be defined operatively trough dynamometers.

It should be superfluous to remind that as to dissipative physical systems (in which a portion of mechanical energy is converted into heat), the Conservation of Energy, lost in terms of the time-invariance of a suitable hamiltonian, is anyway guaranteed by the First Principle of Thermodynamics.

In \([5], [6], [7], [8], [9], [10]\) the definition of a locally-hamiltonian vector field \( X \in \Gamma(TM) \) over a symplectic manifold \((M, \omega)\) as a vector field such that the one-form \( i_X \omega \) is closed (but not necessarily exact) is presented.

No systematic analysis about how to develop the hamiltonian formalism for locally-hamiltonian vector fields is, anyway, developed therein.

Clearly, from a mathematical viewpoint, locally conservative physical systems are indeed a particular case of locally hamiltonian vector fields.

From a physical point of view, anyway, considering locally conservative physical systems in such a general framework doesn’t allow to focalize the attention to the basic physical entity: the underlying force’s field.
For this reason we have chosen to restrict our analysis to locally-conservative physical systems, whose analysis requires an extension of the lagrangian and hamiltonian formalism presented in this paper \(^1\).

The issue discussed in this paper is clearly related to the extension of Morse Theory to multivalued functions exposed in the first appendix of [12].

We won’t follow this conceptual path, observing that no systematic foundation of Mechanics based on some Principle of Minimal Action for multivalued action functionals is therein presented.

We will analyze, in particular, the paradigmatic example of the locally conservative force’s field
\[
\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy
\]
on the punctured plane \(\mathbb{R}^2 - \{0\}\).

\(^1\) As to the following quotation by Vladimir Arnold [11]:

"There are two principal ways to formulate mathematical assertions (problems, conjectures, theorems, · · ·): Russian and French. The Russian way is to choose the most simple and specific case (so that nobody could simplify the formulation preserving the main point). The French way is to generalize the statement as far as nobody could generalize it further".

the french approach to this paper would involve sheaf cohomology, Grothendieck topologies and much more; the russian approach would restrict the paper to the example II.1, example II.2, example II.3, example III.1 and example III.2. As an italian jew I have chosen an intermediate approach.
II. SET OF LOCAL ENERGY POTENTIALS FOR LOCALLY CONSERVATIVE FORCE’S FIELDS

Let us suppose to have a physical system consisting of a material point of mass $m \in (0, +\infty)$ perfectly constrained to a submanifold $^2 M$ of the 3-dimensional euclidean space $E^3 := (\mathbb{R}^3, \delta = dx \otimes dx + dy \otimes dy + dz \otimes dz)$ subjected to the force’s field $f \in \Omega^1(M)^3$, where, following the terminology and the notation of \cite{1}, $\Omega^r(M)$ is the set of the $r$-forms over $M$, $Z^r(M)$ is the $r$th cocycle group of $M$, $B^r(M)$ is the $r$th coboundary group of $M$ and $H^r(M) := \frac{Z^r(M)}{B^r(M)}$ is the $r$th de Rham cohomology group of $M$ $^4$.

Let us recall that:

**Definition II.1**

$f$ is conservative:

$$f \in B^1(M)$$

\[ (2.5) \]

We will say that:

**Definition II.2**

$f$ is locally conservative:

$$f \in Z^1(M)$$

\[ (2.6) \]

We will refer to the material point of mass $m \in (0, +\infty)$ perfectly constrained to move on $M$ under the influence of a locally conservative force’s field as to a locally conservative physical system.

**Remark II.1**

If $H^1(M) = \{\mathbb{I}\}$ a locally conservative force’s field is also conservative.

We will assume from this time forward that this is not the case and we will restrict the analysis to the situation in which $f$ is locally conservative but it is not conservative.

Hence $[f] \in H^1(M)$ is a non-trivial cohomology class.

**Remark II.2**

\[ ^2 \] To lighten the terminology we will denote from this time forward simply with the term manifold an arcwise connected differentiable manifold.

\[ ^3 \] where we have implicitly followed the strategy of converting the language of vector calculus in the language of differential forms summarized in the section $^B$ by assuming from this time forward that the Flat operation $\flat$ is understood.

\[ ^4 \] In general, given a group $G$, one defines the $n$th cohomology group of $M$ with respect to the group $G$ as $^E$:

$$H^n(M; G) := [M, K(G,n)]_0$$

\[ (2.1) \]

where $[X,Y]_0$ denotes the set of based homotopy classes of $Y^X$ and where $K(G,n)$ denotes the Eilenberg-Mac Lane spaces defined, up to homotopic equivalence, by the condition:

$$\pi_m(K(G,n)) = \begin{cases} G, & \text{if } m = n; \\ \{\mathbb{I}\}, & \text{otherwise}. \end{cases}$$

\[ (2.2) \]

Such a notion may be further generalized defining, in a suitable way, the groups of cohomology $H^n(M, S)$ of $M$ with respect to a sheaf (see appendix $^B$). Following the terminology of $^E$ we denote the $n$th group of De Rham cohomology of $M$ as:

$$H^n(M) = H^n(M, \mathbb{R}) = H^n(M, S_{constant})$$

\[ (2.3) \]

(where $S_{constant}$ is the sheaf of constant functions over $M$). As to homology groups let us observe that, under mild topological conditions (see the section 6.1.1 of $^E$ and the section 1.6 of $^{12}$) that we will assume from this time forward, $H_n(M, \mathbb{Z})$ and $H_n(M, \mathbb{R})$ are isomorphic; hence we will simply talk about the $n$th homology group of $M$:

$$H_n(M) = H_n(M, \mathbb{R})$$

\[ (2.4) \]
A physical system whose mathematical structure is similar (but different) to the one discussed in this paper consists of a material point of mass $m \in (0, +\infty)$ and electric charge $e \in \mathbb{R} - \{0\}$ perfectly constrained to move on a submanifold $M$ of the 3-dimensional euclidean space $\mathbb{E}^3 := (\mathbb{R}^3, \delta := dx \otimes dx + dy \otimes dy + dz \otimes dz)$ under the influence of the Lorentz’s force $f = ev \wedge B$ (where $v = v_x dx + v_y dy + v_z dz \in \Omega^1(M)$ is the velocity of the material point while $B = B_x dy \wedge dz - B_y dx \wedge dz + B_z dx \wedge dy \in \Omega^2(M)$ is the magnetic field) induced by a magnetic field $B \in B^2(M)$ (and hence $[B] \in H^2(M; \mathbb{R})$ is a not-trivial cohomology class) extensively studied in the literature (see for instance [14, 7]).

Example II.1

Let us suppose that $M = \mathbb{R}^2 - \{0\}$ and let us introduce the following force’s field:

$$f := f_x dx + f_y dy \in \Omega^1(M)$$  \hspace{1cm} (2.6)

$$f_x := \frac{-y}{x^2 + y^2}$$  \hspace{1cm} (2.7)

$$f_y := \frac{x}{x^2 + y^2}$$  \hspace{1cm} (2.8)

represented in the figure[11]

Since:

$$\frac{\partial f_x}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial f_y}{\partial x}$$  \hspace{1cm} (2.9)

it follows that:

$$df = (\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}) dx \wedge dy = 0$$  \hspace{1cm} (2.10)

and hence $f \in Z^1(M)$. Let us now introduce the map $\theta : M - \{(0, y), y \in (-\infty, 0) \cup (0, +\infty)\} \rightarrow \mathbb{R}$:

$$\theta(x, y) := \arctan(\frac{y}{x})$$  \hspace{1cm} (2.11)

One may easily verify that:

$$d\theta := \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy = f_{|M - \{(0, y), y \in (-\infty, 0) \cup (0, +\infty)\}}$$  \hspace{1cm} (2.12)

where $f_{|M - \{(0, y), y \in (-\infty, 0) \cup (0, +\infty)\}} \in \Omega^1(M - \{(0, y), y \in (-\infty, 0) \cup (0, +\infty)\})$ is the restriction of $f$ to $M - \{(0, y), y \in (-\infty, 0) \cup (0, +\infty)\}$. It follows that:
1. 
\[ f_{\mathcal{M} - \{(0,y), y \in (-\infty,0) \cup (0,+)\}} \in B^1(\mathcal{M} - \{(0,y), y \in (-\infty,0) \cup (0,+)\}) \]

(2.13)

2. 
\[ f \notin B^1(\mathcal{M}) \]

(2.14)

**Remark II.3**

The closed but not exact 1-form of the example II.1 has been extensively studied in a different physical context as the vector-potential 1-form on the punctured plane \( x, y \in \mathbb{R}^2 - \{0\} \) generated by a magnetic flux line along the z-axis.

The more famous physical appearance of the 1-form 
\[ \frac{x^2}{x^2+y^2} dx + \frac{y^2}{x^2+y^2} dy \]

is in the context of the hydrodynamical analogue of such a system (where hence such a form represents an Eulerian velocity’s field) as a vortex.

Following once more the terminology and the notation of [1] let \( C_r(\mathcal{M}) \) be the \( r^{th} \) chain group of \( \mathcal{M} \), let \( Z_r(\mathcal{M}) \) be the \( r^{th} \) cycle group of \( \mathcal{M} \), let \( B_r(\mathcal{M}) \) be the \( r^{th} \) boundary group of \( \mathcal{M} \), and let \( H_r(\mathcal{M}) := \frac{Z_r(\mathcal{M})}{B_r(\mathcal{M})} \) be the \( r^{th} \) homology group of \( \mathcal{M} \).

Let us recall that given a field force \( f \in \Omega^1(\mathcal{M}) \) and a 1-cycle \( c \in C_1(\mathcal{M}) \):

**Definition II.3**

work made by \( f \) along \( c \):
\[ W(c, f) := \int_c f \]

(2.15)

Given a conservative field force \( f \in B^1(\mathcal{M}) \):

**Definition II.4**

potential energy of \( f \):
\[ V \in \Omega^0(\mathcal{M}) : f = -dV \]

(2.16)

**Remark II.4**

Let us remark that \( f \) is invariant under the following action of \( \mathbb{R} \) (seen as an abelian group) over \( \Omega^0(\mathcal{M}) \):
\[ a \in \mathbb{R} : V \mapsto V + a \]

(2.17)

in the following sense: if \( V \) is an energy potential of \( f \) then \( V + a \) is also an energy potential of \( f \) for every \( a \in \mathbb{R} \).

Applying Stokes’ Theorem the work made by a conservative force’s field with energy potential \( V \) along the 1-cycle \( c \in C_1(\mathcal{M}) \) may be written as:
\[ W(c, f) = -W(c, dV) = -W(\partial c, V) \]

(2.18)

So the work made by a conservative force’s field with energy potential \( V \) along a 1-cycle \( c \) depends only by the values taken by \( V \) on the boundary of \( c \).

Let us now recall the Hurewicz’s isomorphism:
\[ H_1(\mathcal{M}) = \frac{\pi_1(\mathcal{M})}{[\pi_1(\mathcal{M}), \pi_1(\mathcal{M})]} \]

(2.19)

where we have denoted by \( [G, G] \) the commutator subgroup of an arbitrary group \( G \) defined as:
\[ [G, G] := \{ x \cdot y \cdot x^{-1} \cdot y^{-1} : x, y \in G \} \]

(2.20)

It follows that if \( \pi_1(\mathcal{M}) \) is abelian then \( H_1(\mathcal{M}) = \pi_1(\mathcal{M}) \).
Remark II.5

Let us remark, by the way, that equation 2.19 implies that a locally conservative but not conservative force field may exist only on a multiply connected manifold.

It is curious, with this regard, that though the formalism of Quantum Mechanics on multiply connected configuration spaces, pioneered at the end of the sixties and the beginning of the seventies by Larry Schulman and Cecile Morette De Witt, is nowadays commonly founded in the literature (see for instance the 23th chapter of [16], the 7th chapter of [17] and the 8th chapter of [18] as to its implementation, at different levels of mathematical rigor, in the path-integration’s formulation, as well as the 8th chapter of [19], the 3th chapter of [20] and the section 6.8 of [21] for its formulation in the operatorial formulation) 5, the role of the multiple-connectivity of the configuration space in Classical Mechanics is, at least up to our knowledge, largely unexplored.

For instance, up to our knowledge, no systematic comparison of the left generalized rigid body of a Lie group G (defined, according to the 2th appendix of [2], as the dynamical system with lagrangian $L: TG \mapsto \mathbb{R}$:

$$L(q, \dot{q}) := \frac{1}{2} |\dot{q}|^2_g$$

(2.21)

where $g$ is the left-invariant riemannian metric on $G$) and the left generalized rigid body of $\tilde{G}$ (where $\tilde{G}$ is the universal covering group of G) exists in the literature, though its importance, for instance, in the cases $G := SO(n)$, $\tilde{G} := Spin(n)$.

The same can be said as to the comparison between the right generalized rigid body of a Lie group G (defined as the dynamical system with lagrangian $L: TG \mapsto \mathbb{R}$ given by the equation 2.21 where $g$ is the right-invariant riemannian metric of G) and the right generalized rigid body of $\tilde{G}$ though its importance, for instance, in Fluid Dynamics, where G is the group of the diffeomorphisms of the manifold on which the fluid moves, and in the related Physics of solitons (considering for example that the Korteweg de Vries equation may be interpreted as the motion’s equation of the right generalized rigid body of the Virasoro group $Diff(S^1)$ [6]).

Let us assume that $\pi_1(M)$ is abelian.

Given a path $\alpha: [0, 1] \mapsto M$ one has then that:

$$W(\alpha, f) = V(\alpha(1)) - V(\alpha(0))$$

(2.22)

It follows that the work made by a conservative force’s field along a loop is equal to zero.

The work made by a locally conservative force’s field along a loop may, contrary, be different from zero.

Example II.2

In the framework of the example II.1 let us observe first of all that the fundamental group of the punctured plane is abelian:

$$\pi_1(M) = \mathbb{Z}$$

(2.23)

and hence:

$$H_1(M) = \pi_1(M) = \mathbb{Z}$$

(2.24)

Given a loop $c: [0, 1] \mapsto M$ it may be easily verified that:

$$W(f, c) = 2\pi n_{\text{winding}}(c, \bar{0})$$

(2.25)

where in general $n_{\text{winding}}(c, q)$ is the winding number of the loop c with respect to the point q.

Let us now consider a locally conservative field force $f \in Z^1(M)$.

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5 in presence of a multiply connected configuration space a suppletive topological superselection rule exists, the involved superselection charge, taking values on $\text{Hom}(H_1(\text{configuration space}), U(1))$, appearing in physically very different contexts going from the $\theta$-angle of Yang-Mills quantum field theories (see for instance the 10th chapter of [22] or the section 23.6 of [23]) to the magnetic flux of the solenoid involved in the Aharonov-Bohm effect and to the fractional statistic of a quantum system of identical particles living on the plane (see for instance [22] and [13])
Let us recall that by Poincaré Lemma it follows that for every contractible open set $U$ there exists a $V_U \in \Omega^0(U)$ such that:

$$f_U = -dV_U \quad (2.26)$$

where $f_U \in B^1(U)$ is the restriction of $f$ to the open set $U$.

This allows to introduce the following:

**Definition II.5**

*set of local energy potentials for $f$:*

a set $\{V_i\}_{i=1}^n$ such that:

1. 

$$n \in \mathbb{N}_+ \quad (2.27)$$

2. 

$$V_i \in \Omega^0(U_i) \quad i = 1, \cdots, n \quad (2.28)$$

3. \( \{U_i\}_{i=1}^n \) is a covering of $M$ such that:

$$U_i \text{ is contractible} \quad \forall i \in \{1, \cdots, n\} \quad (2.29)$$

4. 

$$f_{|U_i} = -dV_i \quad \forall i \in \{1, \cdots, n\} \quad (2.30)$$

**Remark II.6**

Given a locally conservative force’s field $f$ and a set $\{V_i\}_{i=1}^n$ of local energy potentials for $f$, let us observe that definition II.5 implies that if $U_i \cap U_j \neq \emptyset$, then the map $c_{ij} : U_i \cap U_j \to \mathbb{R}$ defined by:

$$c_{ij}(q) := V_i(q) - V_j(q) \quad (2.31)$$

is constant.

Let us observe furthermore that:

1. 

$$c_{ii} = 0 \quad (2.32)$$

2. 

$$c_{ij} = -c_{ji} \quad (2.33)$$

3. if $U_i \cap U_j \cap U_k \neq \emptyset$ then:

$$c_{ij} + c_{jk} = c_{ik} \quad (2.34)$$

**Remark II.7**
Let us remark that \( f \) is invariant under the following action of \( \mathbb{R}^n \) (seen as an abelian group):

\[
(a_1, \ldots, a_n) \in \mathbb{R}^n : V_i \mapsto V_i + a_i \quad \forall i \in \{1, \ldots, n\}
\]  

(2.35)

in the following sense: if \( \{V_i\}_{i=1}^n \) is a set of local energy potentials of the locally conservative force’s field \( f \) then \( \{V_i + a_i\}_{i=1}^n \) is also a set of local energy potentials for \( f \).

Clearly:

\[
(a_1, \ldots, a_n) \in \mathbb{R}^n : c_{ij} \mapsto c_{ij} + a_i - a_j \quad \forall i, j \in \{1, \ldots, n\}
\]  

(2.36)

Example II.3

In the framework of the example II.1 and of the example II.2 let us observe first of all that:

\[
\lim_{x \to 0^+} \theta(x, y) = +\frac{\pi}{2} \quad \forall y \in (0, +\infty)
\]

(2.37)

\[
\lim_{x \to 0^-} \theta(x, y) = -\frac{\pi}{2} \quad \forall y \in (0, +\infty)
\]

(2.38)

\[
\lim_{x \to 0^+} \theta(x, y) = -\frac{\pi}{2} \quad \forall y \in (-\infty, 0)
\]

(2.39)

\[
\lim_{x \to 0^-} \theta(x, y) = +\frac{\pi}{2} \quad \forall y \in (-\infty, 0)
\]

(2.40)

and hence in particular:

\[
\not\exists \lim_{x \to 0} \theta(x, y) \quad \forall y \in (-\infty, 0) \cup (0, +\infty)
\]

(2.41)

so that it doesn’t exist a continuous extension of the map \( \theta \) to the whole \( M \).

Let us consider the following covering \( \{U_i\}_{i=1}^4 \) of \( M \):

\[
U_1 := \{(x, y) \in \mathbb{R}^2 : x \geq 0 \land y \geq 0\} - \{\vec{0}\}
\]

(2.42)

\[
U_2 := \{(x, y) \in \mathbb{R}^2 : x \leq 0 \land y \geq 0\} - \{\vec{0}\}
\]

(2.43)

\[
U_3 := \{(x, y) \in \mathbb{R}^2 : x \leq 0 \land y \leq 0\} - \{\vec{0}\}
\]

(2.44)

\[
U_4 := \{(x, y) \in \mathbb{R}^2 : x \geq 0 \land y \leq 0\} - \{\vec{0}\}
\]

(2.45)

Clearly:

\( U_i \) is contractible \( \forall i \in \{1, \ldots, 4\} \)

(2.46)

Let us the introduce the following maps:

1. \( V_1 \in \Omega^0(U_1) \):

\[
V_1(x, y) := \begin{cases} -\theta(x, y), & \text{if } x > 0; \\ +\frac{\pi}{2}, & \text{if } x = 0. \end{cases}
\]

(2.47)

2. \( V_2 \in \Omega^0(U_2) \):

\[
V_2(x, y) := \begin{cases} -\theta(x, y), & \text{if } x < 0; \\ -\frac{\pi}{2}, & \text{if } x = 0. \end{cases}
\]

(2.48)
3. $V_3 \in \Omega^0(U_3)$:

$$V_3(x, y) := \begin{cases} -\theta(x, y), & \text{if } x < 0; \\ +\frac{\pi}{2}, & \text{if } x = 0. \end{cases}$$

(2.49)

4. $V_4 \in \Omega^0(U_4)$:

$$V_4(x, y) := \begin{cases} -\theta(x, y), & \text{if } x > 0; \\ -\frac{\pi}{2}, & \text{if } x = 0. \end{cases}$$

(2.50)

By construction we have that \( \{V_i\}_{i=1}^4 \) is a set of local energy potentials for \( f \).

Clearly:

$$c_{12} = +\pi$$

(2.51)

$$c_{23} = -\pi$$

(2.52)

$$c_{34} = +\pi$$

(2.53)

$$c_{41} = -\pi$$

(2.54)

while, since \( U_1 \cap U_3 = U_2 \cap U_4 = \emptyset \), \( c_{13} \) and \( c_{24} \) are undefined.
III. THE GENERALIZATION OF THE LAGRANGIAN AND THE HAMILTONIAN FORMALISM REQUIRED TO ANALYZE LOCALLY CONSERVATIVE PHYSICAL SYSTEMS

Given the locally conservative (but not conservative) physical system (that we will denote as $PS_f$) consisting of a material point of mass $m \in (0, +\infty)$ perfectly constrained to a differentiable submanifold $M$ of the 3-dimensional euclidean space $\mathbb{E}^3 := (\mathbb{R}^3, \delta = dx \otimes dx + dy \otimes dy + dz \otimes dz)$ subjected to the locally conservative (but not conservative) force’s field $f$ we will show how its analysis in the framework of Analytical Mechanics requires the introduction of a suitable generalization of both the Lagrangian and the Hamiltonian formalism.

Let us observe first of all that (assuming that $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is an inertial frame) the dynamics of $PS_f$ is ruled by the Newton’s Law:

$$m \ddot{\vec{r}} = \vec{f} + \vec{f}_{\text{constraint}}$$ (3.1)

where $\vec{f}_{\text{constraint}}$ is the force’s field constraining the material point to $M$.

**Remark III.1**

Let us remark that we define a perfect constraint as one satisfying the following Generalized D’Alambert - Lagrange’s Principle:

$$W(f_{\text{constraint}}, c) = 0 \forall c \in C^1(M)$$ (3.2)

that, if $\pi_1(M)$ is abelian, (owing to equation 2.19) reduces to the (Ordinary) D’Alambert - Lagrange’s Principle:

$$W(f_{\text{constraint}}, \alpha) = 0 \forall \text{path } \alpha \text{ on } M$$ (3.3)

Let us now introduce the following:

**Definition III.1**

*set of local lagrangians of $PS_f$:*

a set $\{L_i\}_{i=1}^n$ such that $L_i : T^*U_i \mapsto \mathbb{R}$ is defined as:

$$L_i(q, \dot{q}) := \frac{m}{2} |\dot{q}|^2 - V_i(q)$$ (3.4)

where $\{V_i\}_{i=1}^n$ is a set of local energy potentials for $f$ and where $g := i^*\delta$ is the riemannian metric over $M$ induced by the inclusion’s embedding $i : M \mapsto \mathbb{R}^3$.

**Definition III.2**

*set of local hamiltonians of $PS_f$:*

a set $\{H_i\}_{i=1}^n$ such that $H_i : T^*U_i \mapsto \mathbb{R}$ is defined as:

$$H_i(p, q) := \frac{|p|^2}{2m} + V_i(q)$$ (3.5)

where $\{V_i\}_{i=1}^n$ is a set of local energy potentials for $f$ and where $g := i^*\delta$ is the riemannian metric over $M$ induced by the inclusion’s embedding $i : M \mapsto \mathbb{R}^3$.

**Remark III.2**

Let us remark that each local lagrangian $L_i$ of a set of local lagrangians $\{L_i\}_{i=1}^n$ for $PS_f$ is regular.

Each local hamiltonian $H_i$ of the set of local hamiltonians $\{H_i\}_{i=1}^n$ for $PS_f$ constructed with the same set of local energy potentials of $\{L_i\}_{i=1}^n$ can then be simply obtained from $L_i$ through Legendre’s transform.

Given $i \in \{1, \cdots, n\}$:

**Definition III.3**

dynamical system $DS_i$:
the dynamical system having lagrangian $L_i$ (and hence having hamiltonian $H_i$)

Remark III.3

To each dynamical system $DS_i$ one can apply the theorem of the section 21 of the 4th chapter of [2] stating its equivalence with the (Ordinary) D’Alambert Principle as well as its equivalence with the limit $N \to +\infty$ of a system of energy potential $V_i + NV_{\text{constraint}}$ where:

$$V_{\text{constraint}}(\vec{x}, M) := \text{distance}_\delta^2(\vec{x}, M)$$  \hspace{1cm} (3.6)

($\text{distance}_\delta(\vec{x}, M)$ being of course the distance, with respect of the euclidean metric $\delta$, of the point $\vec{x}$ from $M$).

Remark III.4

The definition [11,2] allows to appreciate how locally conservative physical systems can be seen as particular cases of locally hamiltonian vector fields [5, 6].

Given a symplectic manifold $(Q, \omega)$ and a vector field $X \in \Gamma(TQ)$ let us recall that:

Definition III.4

$X$ is hamiltonian:

$$i_X \omega \in B^1(Q)$$  \hspace{1cm} (3.7)

Definition III.5

$X$ is locally-hamiltonian:

$$i_X \omega \in Z^1(Q)$$  \hspace{1cm} (3.8)

An hamiltonian vector field is obviously also a locally hamiltonian vector field.

Given an hamiltonian vector field $X$, a function $H \in \Omega^0(Q)$ such that $i_X \omega = dH$ is called an hamiltonian of $X$.

If $H^1(Q) \neq \{I\}$ a locally hamiltonian vector field $X \in \Gamma(TQ)$ is not in general an hamiltonian vector field.

Let us introduce the following:

Definition III.6

Lie group of the symplectomorphisms of $(Q, \omega)$:

$$\text{Simp}(Q, \omega) := \{ \psi \in \text{Diff}(Q) : \psi^* \omega = 0 \}$$  \hspace{1cm} (3.9)

If $Q$ is closed (i.e. compact and without boundary) then the set of the locally-hamiltonian vector fields is the Lie algebra of the Lie group $\text{Simp}(Q, \omega)$.

A general analysis concerning the flows of locally hamiltonian vector fields is, up to our knowledge, still lacking.

For the reasons explained in section [3] we will not pursue such a general approach, limiting our attention to locally conservative physical systems that, as will now show, are nothing but a particular case.

Poincaré Lemma assures us that given a covering $\{U_i\}_{i=1}^n$ of $Q$ such that:

$$U_i \text{ is contractible} \hspace{0.5cm} \forall i \in \{1, \cdots, n\}$$  \hspace{1cm} (3.10)

one has that:

$$i_X \omega \in B^1(U_i)$$  \hspace{1cm} (3.11)

and hence there exists a set $\{H_i\}_{i=1}^n$ such that:

---

We advise the reader that in [2] locally hamiltonian vector fields are called symplectic vector fields.
1. 
\[ H_i \in \Omega^0(U_i) \quad \forall i \in \{1, \ldots, n\} \quad (3.12) \]

2. 
\[ i_X \omega = dH_i \quad \forall i \in \{1, \ldots, n\} \quad (3.13) \]

It is natural to call the set \( \{H_i\}_{i=1}^n \) a set of local hamiltonians for \( X \).

Let us now consider the particular case in which the symplectic manifold \( (Q, \omega) \) is of the form \( (T^*M, \omega_{can}) \) for a suitable differential submanifold \( M \) of the three dimensional euclidean space \( \mathbb{E}^3 :=(\mathbb{R}^3, \delta = dx \otimes dx + dy \otimes dy + dz \otimes dz) \) (where \( \omega_{can} \) is the canonical symplectic form of the cotangent bundle \( T^*M \)).

Given a locally conservative force’s field \( f \in Z^1(M) \) one can find a suitable locally hamiltonian vector field \( X_f \in \Gamma(T^*M) \) whose corresponding set of local hamiltonians is a set of local hamiltonians for \( PS_f \) in the sense of the definition III.2.

A local lagrangian \( L_i \) or hamiltonian \( H_i \) can be used to derive, through respectively the Euler-Lagrange’s equation or the Hamilton’s equations, the motion \( q(t) \) associated to an initial condition \( q(0) \in U_i \) and such that:

\[ q(t) \in U_i \quad \forall t \in (0, +\infty) \quad (3.14) \]

**Example III.1**

In the framework of the example [III.1] of the example [III.2] and of the example [III.3] let us observe first of all that obviously, according to Newton’s Law (assuming that \( \{0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \) is an inertial frame), the dynamics of \( PS_f \) is ruled by the differential equations:

\[ m\ddot{x} = -\frac{y}{x^2 + y^2} \quad (3.15) \]

\[ m\ddot{y} = \frac{x}{x^2 + y^2} \quad (3.16) \]

\[ m\ddot{z} = f_{\text{constraint}} \quad (3.17) \]

A set of local lagrangians for \( PS_f \) is given by:

\[ L_i(x, y, \dot{x}, \dot{y}) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V_i(x, y) \quad i \in \{1, \ldots, 4\} \quad (3.18) \]

while a set of local hamiltonians for \( PS_f \) is given by:

\[ H_i(x, y, p_x, p_y) = \frac{p_x^2 + p_y^2}{2m} + V_i(x, y) \quad i \in \{1, \ldots, 4\} \quad (3.19) \]

where:

\[ H_i(x, y, p_x, p_y) = p_x \dot{x} + p_y \dot{y} - L_i(x, y, \dot{x}, \dot{y}) \quad i \in \{1, \ldots, 4\} \quad (3.20) \]

\[ p_x = \frac{\partial L_i}{\partial \dot{x}} = m\dot{x} \quad i \in \{1, \ldots, 4\} \quad (3.21) \]

\[ p_y = \frac{\partial L_i}{\partial \dot{y}} = m\dot{y} \quad i \in \{1, \ldots, 4\} \quad (3.22) \]

Let us now concentrate our attention to \( U_1 \).
Passing to polar coordinates \( \delta = dr \otimes dr + r^2d\theta \otimes d\theta \) and observing that:

\[
V_1(r, \theta) = -\theta \quad \forall r \in (0, +\infty), \forall \theta \in [0, \frac{\pi}{2}]
\]  

(3.23)

we have that:

\[
L_1(r, \theta, \dot{r}, \dot{\theta}) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \theta \quad \forall r \in (0, +\infty), \forall \theta \in [0, \frac{\pi}{2}], \dot{r} \in \mathbb{R}, \dot{\theta} \in \mathbb{R}
\]  

(3.24)

\[
H_1(r, \theta, p_r, p_\theta) = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \theta \quad \forall r \in (0, +\infty), \forall \theta \in [0, \frac{\pi}{2}], p_r \in \mathbb{R}, p_\theta \in \mathbb{R}
\]  

(3.25)

where:

\[
H_1(r, \theta, p_r, p_\theta) = p_r\dot{r} + p_\theta\dot{\theta} - L_1(r, \theta, \dot{r}, \dot{\theta})
\]  

(3.26)

\[
p_r = \frac{\partial L_1}{\partial \dot{r}} = mr
\]  

(3.27)

\[
p_\theta = \frac{\partial L_1}{\partial \dot{\theta}} = mr^2\dot{\theta}
\]  

(3.28)

Hamilton’s equations are:

\[
p_r = -\frac{\partial H_1}{\partial r} = \frac{p_r^2}{mr^3}
\]  

(3.29)

\[
p_\theta = -\frac{\partial H_1}{\partial \theta} = 1
\]  

(3.30)

\[
\dot{r} = \frac{\partial H_1}{\partial p_r} = \frac{p_r}{m}
\]  

(3.31)

\[
\dot{\theta} = \frac{\partial H_1}{\partial p_\theta} = \frac{p_\theta}{mr^2}
\]  

(3.32)

and hence:

\[
p_\theta(t) = p_\theta(0) + t
\]  

(3.33)

\[
\dot{p}_r(t) = \frac{(p_\theta(0) + t)^2}{mr^3}
\]  

(3.34)

Given a set \( \{V_i\}_{i=1}^n \) of local energy potentials for \( f \) let us observe that if \( U_i \cap U_j \neq \emptyset \) then the map \( t_{ij} : U_i \cap U_j \rightarrow \mathbb{R} \):

\[
t_{ij}(q) := \exp(c_{ij}(q))
\]  

(3.35)

is constant.

Furthermore equation \( 2.32 \), equation \( 2.33 \) and equation \( 2.34 \) imply that:

1. \( t_{ii}(q) = 1 \)  

(3.36)

2. \( t_{ij}(q) = t_{ji}(q)^{-1} \)  

(3.37)
3. if \( U_i \cap U_j \cap U_k \neq \emptyset \) then:

\[
t_{ij}(q) \cdot t_{jk}(q) = t_{ik}(q)
\]  

(3.38)

Let us now observe that equations 3.36, 3.37 and 3.38 are nothing but the consistency conditions satisfied by the transition functions of a fibre bundle.

We can consequentially follow the strategy indicated in the section 9.2.2 of [1] to construct a principal bundle \( P(M, \mathbb{R}) \) given \( M, \{U_i\}_{i=1}^n \), the transition functions \( \{t_{ij}(q)\} \) (that in our case are constant) and the structure group \( \mathbb{R} \):

let us start introducing the set:

\[
X := \bigcup_{i=1}^n U_i \times \mathbb{R}
\]  

(3.39)

Given \( (q_1, a_1) \in U_i \times \mathbb{R} \) and \( (q_2, a_2) \in U_j \times \mathbb{R} \) and introduced the following equivalence relation:

\[
(q_1, a_1) \sim (q_2, a_2) := q_1 = q_2 \land a_2 = t_{ij}(q)a_1
\]  

(3.40)

the total space \( P \) is simply defined as the quotient set:

\[
P := \frac{X}{\sim}
\]  

(3.41)

Denoted an element of \( P \) as \([q, a] \) the projection \( \pi : P \mapsto M \) is defined as:

\[
\pi([q, a]) := q
\]  

(3.42)

The local trivialization \( \phi_i : U_i \times \mathbb{R} \mapsto \pi^{-1}(U_i) \) is defined as:

\[
\phi_i : (q, a) \mapsto [(q, a)]
\]  

(3.43)

A set of local energy potentials of the locally conservative force’s field \( f \) may be interpreted as a set of local sections of \( P(M, \mathbb{R}) \) by considering a set of maps \( \{\hat{V}_i\}_{i=1}^n \) such that:

1. \( \hat{V}_i \in \Gamma(U_i, P) \ \forall i \in \{1, \cdots , n\} \)

(3.44)

2. \( \hat{V}_i(q) := \phi_i(V_i(q), 0) = [(V_i(q), 0)] \)

(3.45)

From the principal bundle \( P(M, \mathbb{R}) \) one can then naturally derive:

1. the principal bundle \( P_L(TM, \mathbb{R}) \) defined as:

\[
P_L := \frac{\bigcup_{i=1}^n TU_i \times \mathbb{R}}{\sim_L}
\]  

(3.46)

where:

\[
((q_1, v_1), a_1) \sim_L ((q_2, v_2), a_2) := (q_1, a_1) \sim (q_2, a_2) \ \forall v_1 \in T_{q_1}U_i, \forall v_2 \in T_{q_2}U_j, \forall (q_1, a_1) \in U_i \times \mathbb{R}, \forall (q_2, a_2) \in U_j \times \mathbb{R}
\]  

(3.47)

2. the principal bundle \( P_H(T^*M, \mathbb{R}) \) defined as:

\[
P_H := \frac{\bigcup_{i=1}^n T^*U_i \times \mathbb{R}}{\sim_H}
\]  

(3.48)

where:

\[
((q_1, p_1), a_1) \sim_H ((q_2, p_2), a_2) := (q_1, a_1) \sim (q_2, a_2) \ \forall p_1 \in T^*_{q_1}U_i, \forall p_2 \in T^*_{q_2}U_j, \forall (q_1, a_1) \in U_i \times \mathbb{R}, \forall (q_2, a_2) \in U_j \times \mathbb{R}
\]  

(3.49)
The set of local lagrangians \( \{ L_i \}_{i=1}^n \) for the locally conservative physical system \( PS_f \) may be interpreted as a set of local sections of the bundle \( P_L(TM, \mathbb{R}) \) by considering the set of maps \( \{ \tilde{L}_i \}_{i=1}^n \) such that:

1. 
   \[ \tilde{L}_i \in \Gamma(TU_i, P_L) \quad \forall i \in \{1, \cdots, n\} \]  
   \( (3.50) \)

2. 
   \[ \tilde{L}_i(q, \dot{q}) := \frac{m}{2} |\dot{q}|^2 - \bar{V}_i(q) \]  
   \( (3.51) \)

In an analogous way the set of local hamiltonians \( \{ H_i \}_{i=1}^n \) for the locally conservative physical system \( PS_f \) may be interpreted as a set of local sections of the bundle \( P_H(T^*M, \mathbb{R}) \) by considering the set of maps \( \{ \tilde{H}_i \}_{i=1}^n \) such that:

1. 
   \[ \tilde{H}_i \in \Gamma(T^*U_i, P_H) \quad \forall i \in \{1, \cdots, n\} \]  
   \( (3.52) \)

2. 
   \[ \tilde{H}_i(p, \dot{q}) := \frac{|p|^2}{2m} + \bar{V}_i(q) \]  
   \( (3.53) \)

**Remark III.5**

If the locally conservative force’s field \( f \) is conservative then the principal bundles \( P(M, \mathbb{R}) \), \( P_L(TM, \mathbb{R}) \) and \( P_H(T^*M, \mathbb{R}) \) are trivial and hence they admit global sections that are, respectively, a globally defined energy potential, a globally defined lagrangian and a globally defined hamiltonian.

If, contrary, the locally conservative force’s field \( f \) is not conservative then the principal bundles \( P(M, \mathbb{R}) \), \( P_L(TM, \mathbb{R}) \) and \( P_H(T^*M, \mathbb{R}) \) are not trivial and hence they don’t admit global sections.

Anyway the same existence of these bundles is sufficient to guarantee that all the local descriptions of \( PS_f \) define in a consistent way a global dynamics.

**Remark III.6**

Let us remark that in a locally conservative but not conservative physical system the mechanical energy is conserved locally but not globally.

As a whole such a system is dissipative, converting a portion of work into heat \(^7\).

The physical source of such a dissipation is, in the final analysis, the topological non-triviality of the involved manifold.

**Example III.2**

In the framework of the example II.1 of the example II.2 of the example II.3 and of the example III.1 let us construct the principal bundle \( P(\mathbb{R}^2 - \{\vec{0}\}, \mathbb{R}) \).

Let us observe first of all that:

1. 
   \[ U_1 \cap U_2 = \{(0, y) \mid y \in (0, +\infty)\} \]  
   \( (3.54) \)

2. 
   \[ t_{12}(q) = \exp(+\pi) \quad \forall q \in U_1 \cap U_2 \]  
   \( (3.55) \)

\(^7\) Of course, as we have already remarked, the Conservation of Mechanical Energy, lost in terms of the time-invariance of a globally defined hamiltonian, is replaced by the more general Conservation of Energy (in all its forms) guaranteed by the First Principle of Thermodynamics.
2.  
\[ U_2 \cap U_3 = \{(x, 0) \mid x \in (-\infty, 0)\} \]  
\[ t_{23}(q) = \exp(-\pi) \quad \forall q \in U_2 \cap U_3 \]  
\[ \tag{3.56} \]  

3.  
\[ U_3 \cap U_4 = \{(0, y) \mid y \in (-\infty, 0)\} \]  
\[ t_{34}(q) = \exp(+\pi) \quad \forall q \in U_3 \cap U_4 \]  
\[ \tag{3.57} \]  

4.  
\[ U_4 \cap U_1 = \{(x, 0) \mid x \in (0, +\infty)\} \]  
\[ t_{41}(q) = \exp(-\pi) \quad \forall q \in U_4 \cap U_1 \]  
\[ \tag{3.58} \]  

Introduced the set:  
\[ X := \bigcup_{i=1}^{4} U_i \times \mathbb{R} \]  
\[ \tag{3.62} \]  

the equivalence relation involved in the definition of  
\[ P := \frac{X}{\sim} \]  
is the following:  

1. given \((q_1, a_1) \in U_1 \times \mathbb{R}\) and \((q_2, a_2) \in U_2 \times \mathbb{R}\):  
\[ (q_1, a_1) \sim (q_2, a_2) \Leftrightarrow q_1 = q_2 \land a_2 = \exp(+\pi)a_1 \]  
\[ \tag{3.63} \]  

2. given \((q_1, a_1) \in U_2 \times \mathbb{R}\) and \((q_2, a_2) \in U_3 \times \mathbb{R}\):  
\[ (q_1, a_1) \sim (q_2, a_2) \Leftrightarrow q_1 = q_2 \land a_2 = \exp(-\pi)a_1 \]  
\[ \tag{3.64} \]  

3. given \((q_1, a_1) \in U_3 \times \mathbb{R}\) and \((q_2, a_2) \in U_4 \times \mathbb{R}\):  
\[ (q_1, a_1) \sim (q_2, a_2) \Leftrightarrow q_1 = q_2 \land a_2 = \exp(+\pi)a_1 \]  
\[ \tag{3.65} \]  

4. given \((q_1, a_1) \in U_4 \times \mathbb{R}\) and \((q_2, a_2) \in U_1 \times \mathbb{R}\):  
\[ (q_1, a_1) \sim (q_2, a_2) \Leftrightarrow q_1 = q_2 \land a_2 = \exp(-\pi)a_1 \]  
\[ \tag{3.66} \]  

The set of local sections \(\tilde{V}_i \in \Gamma(U_i, P)\)\(\bigcup_{i=1}^{4}\) is:  

1.  
\[ \tilde{V}_1(x, y) = \left[ (-\arctan \left( \frac{y}{x} \right), 0) \right] \quad \forall (x, y) \in U_1 : (x, y) \notin U_1 \cap U_2 \]  
\[ \tag{3.67} \]  

\[ \tilde{V}_1(x, y) = \left[ (+\frac{\pi}{2}, 0) \right] \quad \forall (x, y) \in U_1 \cap U_2 \]  
\[ \tag{3.68} \]  

2.  
\[ \tilde{V}_1(x, y) = \left[ (-\arctan \left( \frac{y}{x} \right), 0) \right] \quad \forall (x, y) \in U_2 : (x, y) \notin U_2 \cap U_3 \]  
\[ \tag{3.69} \]  

\[ \tilde{V}_1(x, y) = \left[ (-\frac{\pi}{2}, 0) \right] \quad \forall (x, y) \in U_2 \cap U_3 \]  
\[ \tag{3.70} \]
\[ \tilde{V}_3(x, y) = \left[ (-\arctan \frac{y}{x}, 0) \right] \\forall (x, y) \in U_3 : (x, y) \notin U_3 \cap U_4 \] (3.71)

\[ \tilde{V}_3(x, y) = \left[ (\frac{\pi}{2}, 0) \right] \\forall (x, y) \in U_3 \cap U_4 \] (3.72)

\[ \tilde{V}_4(x, y) = \left[ (-\arctan \frac{y}{x}, 0) \right] \\forall (x, y) \in U_4 : (x, y) \notin U_4 \cap U_1 \] (3.73)

\[ \tilde{V}_4(x, y) = \left[ (-\frac{\pi}{2}, 0) \right] \\forall (x, y) \in U_4 \cap U_1 \] (3.74)

from which the construction of \( \{ \tilde{L}_i \in \Gamma(TU_i, P_L) \}_{i=1}^4 \) and of \( \{ \tilde{H}_i \in \Gamma(T^*U_i, P_H) \}_{i=1}^4 \) may be immediately derived.

Let us now introduce an alternative approach to the formulation of the Analytic Mechanics of \( PS_f \) essentially consisting in the lifting to the universal covering space \( \tilde{M} \).

Given a set \( \{ V_i \}_{i=1}^n \) of local energy potentials for the locally conservative force's field \( f \) let us introduce the following:

**Definition III.7**

*lift of \( \{ V_i \}_{i=1}^n \) to the universal covering space:*

the set of functions \( \{ \tilde{V}_i \}_{i=1}^n \) such that:

1. \( \tilde{V}_i \in \Gamma(U_i, \tilde{M}) \) where \( \Gamma(U_i, \tilde{M}) \) is the set of the local sections defined on \( U_i \) of the principal bundle \( \tilde{M}(M, \pi_1(M)) \)

2. \( \tilde{V}_i(\tilde{q}) := V_i(\pi(\tilde{q})) \quad \forall i \in \{1, \ldots, n\} \) (3.75)

Given the set \( \{ L_i \}_{i=1}^n \) of local lagrangians associated to the set \( \{ V_i \}_{i=1}^n \) of local energy potentials let us introduce the following:

**Definition III.8**

*lift of \( \{ L_i \}_{i=1}^n \) to the universal covering space:*

the set of functions \( \{ \tilde{L}_i \}_{i=1}^n \) such that:

1. \( \tilde{L}_i \in \Gamma(U_i, T\tilde{M}) \) where \( \Gamma(U_i, \tilde{M}) \) is the set of the local sections defined on \( U_i \) of the tangent bundle of \( \tilde{M} \)

2. \( \tilde{L}_i(\tilde{q}, \dot{\tilde{q}}) := \frac{m}{2} |\dot{\tilde{q}}|_g^2 - \tilde{V}_i(\tilde{q}) \) (3.76)

where \( \tilde{q} \) is the lift of the riemannian metric \( g \) to \( \tilde{M} \).

In an analogous way, given the set \( \{ H_i \}_{i=1}^n \) of local hamiltonians associated to the set \( \{ V_i \}_{i=1}^n \) of local energy potentials \( \{ V_i \}_{i=1}^n \):

**Definition III.9**

*lift of \( \{ H_i \}_{i=1}^n \) to the universal covering space:*

the set of functions \( \{ \tilde{H}_i \}_{i=1}^n \) such that:

1. \( \tilde{H}_i \in \Gamma(U_i, T^*\tilde{M}) \) where \( \Gamma(U_i, \tilde{M}) \) is the set of the local sections defined on \( U_i \) of the cotangent bundle of \( \tilde{M} \)

2. \( \tilde{H}_i(\tilde{q}, \tilde{p}) := \frac{|\tilde{p}|_g^2}{2m} + \tilde{V}_i(\tilde{q}) \) (3.77)
Example III.3

In the framework of the example II.1 of the example III.1 of the example III.2 and of example III.3, let us endow $\mathbb{R}^2$ with the following binary inner operations:

\[(x_1,y_1) + (x_2,y_1) := (x_1 + x_2, y_1 + y_2)\] (3.78)

\[(x_1,y_1) \cdot (x_2,y_1) := (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)\] (3.79)

and let us make the usual identification $(\mathbb{R}^2,+,\cdot) \equiv \mathbb{C}$.

Let us then observe that:

\[f^\sharp(z) = \frac{i}{z}\] (3.80)

(analytic on $\mathbb{C} - \{0\}$ with a pole in $z = 0$) and that given a loop $c : [0,1] \mapsto M$ and remembering that $Res(\frac{1}{z},0) = 1$ one has that:

\[\int_c f(z)dz = -2\pi n_{\text{winding}}(c,0) = -W(f,c)\] (3.81)

Since $M = \mathbb{C}^\times := \mathbb{C} - \{0\}$ is the punctured complex plane it follows that (see the example 2 of the section 2.9 of [25]):

\[\tilde{M} = \mathbb{C}\] (3.82)

the universal covering map $\pi : \mathbb{C} \mapsto \mathbb{C}^\times$ being the exponential:

\[\pi(z) = \exp(z)\] (3.83)

Obviously:

\[\pi_1(\mathbb{C}^\times) = \mathbb{Z}\] (3.84)

Considered the following action of $\pi_1(\mathbb{C}^\times)$ on $\mathbb{C}$:

\[n : z \mapsto z + 2\pi ni\] (3.85)

one has that:

\[\mathbb{C}^\times = \frac{\mathbb{C}}{\mathbb{Z}}\] (3.86)

Remark III.7

Let us recall that, in general, given a discrete group $G$ acting on a manifold $M$ the quotient space $\overline{M} = \mathbb{C}$ is not a manifold, but it is only an orbifold (according to the definition given by William Thurston about which we demand to the appendix E of [26]).

If, as in our case, such an action is free then the quotient space is also a manifold.

Instead of introducing a lift of the local energy potentials $\{V_i\}_{i=1}^4$ to the universal covering space $\tilde{M} = \mathbb{C}$ let us observe that the angle $\text{Arg}(z)$ may be seen as the complete analytic function $\theta : \Sigma_{\log} \mapsto \mathbb{C}$:

\[\theta(z) := \frac{1}{i} \log\left(\frac{z}{|z|}\right)\] (3.87)

where $\Sigma_{\log}$ is the Riemann surface of the logarithm (see the section [3] and the figure [2]) so that it is natural to introduce the Riemann-surface lifting $\tilde{V} : \Sigma_{\log} \mapsto \mathbb{C}$:

\[\tilde{V}(z) := -\theta(z)\] (3.88)

The dynamics of $PSL_f$ should then be describable through the lagrangian $\tilde{L} : T\Sigma_{\log} \mapsto \mathbb{R}$:

\[\tilde{L}(z,\dot{z}) := \frac{1}{2} |\dot{z}|^2 - \tilde{V}(z)\] (3.89)

where $\delta$ is the riemannian metric on $\Sigma_{\log}$ induced by the euclidean metric on the plane.
APPENDIX A: PASSING FROM THE LANGUAGE OF VECTOR CALCULUS TO THE LANGUAGE OF DIFFERENTIAL FORMS

Let us consider the three-dimensional euclidean manifold \( E^3 := (\mathbb{R}^3, \delta := dx \otimes dx + dy \otimes dy + dz \otimes dz) \).

Introduced the canonical basis \( \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \) of \( \mathbb{R}^3 \) defined as:

\[
(\vec{e}_i)_j := \delta_{i,j} \quad i, j = 1, \cdots, 3
\]  

(A1)

let us adopt the following Sharp and Flat notation of [6]:

Given the 1-form \( \alpha = \alpha_x dx + \alpha_y dy + \alpha_z dz \in \Omega^1(\mathbb{R}^3) \) let us introduce the following:

**Definition A.1**

**vector field associated to** \( \alpha \):

\[
\alpha^\sharp := \alpha_x \vec{e}_1 + \alpha_y \vec{e}_2 + \alpha_z \vec{e}_3
\]  

(A2)

Given, contrary, the vector field \( \vec{v} := v_x \vec{e}_1 + v_y \vec{e}_2 + v_z \vec{e}_3 \):

**Definition A.2**

**1-form associated to** \( \vec{v} \):

\[
\vec{v}^\flat := v_x dx + v_y dy + v_z dz
\]  

(A3)

Then:

**Proposition A.1**

1. Cross Product:

\[
\vec{v} \wedge \vec{w} = \star (\vec{v}^\flat \wedge \vec{w}^\flat)^\sharp
\]  

(A4)

2. Scalar Product:

\[
(\vec{v} \cdot \vec{w})dx \wedge dy \wedge dz = \vec{v}^\flat \wedge \star (\vec{w}^\flat)
\]  

(A5)

3. Gradient:

\[
\vec{\nabla} f := \text{grad} f = (df)^\sharp
\]  

(A6)
4. Curl:
\[ \nabla \wedge \vec{v} := \text{curl}\vec{v} = [\star (d\vec{v}^\flat)]^2 \]  
(A7)

5. Divergence:
\[ \nabla \cdot \vec{v} := \text{div}\vec{v} = \star d(\star \vec{v}^\flat) \]  
(A8)

where \( \star \) is the Hodge Star operator of \( \mathbb{E}^3 \) satisfying the following equations:

\[ \star 1 = dx \wedge dy \wedge dz \]  
(A9)

\[ \star dx = dy \wedge dz \]  
(A10)

\[ \star dy = -dx \wedge dz \]  
(A11)

\[ \star dz = dx \wedge dy \]  
(A12)

\[ \star (dx \wedge dy) = dz \]  
(A13)

\[ \star (dx \wedge dz) = -dy \]  
(A14)

\[ \star (dy \wedge dz) = dx \]  
(A15)

\[ \star (dx \wedge dy \wedge dz) = 1 \]  
(A16)
APPENDIX B: THE DOUBLE MEANING OF THE LOCUTION "RIEMANN SURFACE"

The locution "Riemann surface" appears in the Theoretical Physics' literature in two distinct meanings:

1. as a bidimensional submanifold of $\mathbb{R}^3$ satisfying suitable conditions

2. as the domain of definition (a finite or countable set of copies of $\mathbb{C}$ patched together through a suitable number of "cut and paste" operations) on which a multivalued complex function of a complex variable becomes single-valued (we will refer to this approach as to the carpenter’s definition of a Riemann surface)

The deep link existing between the two meanings is something usually taken for granted though it is never clarified. Actually its full comprehension requires to give up the carpenter’s definition and to introduce more advanced mathematical concepts that we think it may be appropriate to review here (demanding to [27], [28], [29] for further details).

**Example B.1**

Let us consider the carpenter’s definition of the Riemann surface of the logarithm.

Given $z_1, z_2 \in \mathbb{C}$:

**Definition B.1**

$z_1$ is a logarithm of $z_2$:

$$\exp(z_1) = z_2$$  \hspace{1cm} (B1)

Given $r \in (0, +\infty)$ and $\theta \in [0, 2\pi)$ one may easily verify that the set of the logarithms of $r \exp(i\theta)$ is \{log($r$) + i($\theta$ + $2\pi n$) $n \in \mathbb{Z}$\}.

The carpenter’s definition of $\Sigma_{\log}$ proceeds in the following way:

1. one considers a sequence $\{\text{sheet}(n), n \in \mathbb{Z}\}$ such that:

   $$\text{sheet}(n) := \mathbb{C} - \{0\} \ \forall n \in \mathbb{Z}$$  \hspace{1cm} (B2)

2. for every $n \in \mathbb{Z}$ one defines the map $\phi_n : \text{sheet}(n) \mapsto \mathbb{C}$:

   $$\phi_n(z) := \log(|z|) + i(\text{Arg}(z) + 2\pi n)$$  \hspace{1cm} (B3)

3. one defines $\log : \cup_{n \in \mathbb{Z}} \text{sheet}(n) \mapsto \mathbb{C}$ such that:

   $$\log|_{\text{sheet}(n)} := \phi_n$$  \hspace{1cm} (B4)

4. one "cuts" each sheet($n$) along the negative real semiaxis ($-\infty, 0$) so that such an interval is replaced with two copies of it that one calls the upper border and the lower border of the cut sheet, i.e.:

   $$\text{border}(n, +) := (-\infty, 0)$$  \hspace{1cm} (B5)

   $$\text{border}(n, -) := (-\infty, 0)$$  \hspace{1cm} (B6)

   $$\text{sheet}(n) = \mathbb{C} - \{0\} \mapsto \text{sheet}(n) := (\mathbb{C} - (-\infty, 0] \cup \text{border}(n, +) \cup \text{border}(n, -) \ \forall n \in \mathbb{Z}$$  \hspace{1cm} (B7)

5. one "welds together" $\text{border}(n, +)$ and $\text{border}(n + 1, -)$ by making the identifications:

   $$\text{border}(n, +) \equiv \text{border}(n + 1, -) \ \forall n \in \mathbb{Z}$$  \hspace{1cm} (B8)

obtaining a surface $\Sigma_{\log}$ on which the logarithm results defined.

Given a topological space $(M, T)$ [13]:
Definition B.2

sheaf of germs $S$ on $(M, T)$:
The assignment to each $U \in T$ of a group $S(U)$ (called the group of sections of $S$ over $U$) such that:

1. Given $U, V \in T$ such that $U \subseteq V$ there exist a map $r^U_V : S(V) \rightarrow S(U)$ (called the restriction map from $V$ to $U$) such that:

$$r^U_U = \mathbb{I} \quad \forall U \in T$$

$$r^U_V \circ r^W_V = r^V_V \quad \forall U, V, W \in T$$ (B9)

2. Given $U \in T$ such that $U = \bigcup_i U_i : U_i \in T \forall i$:

$$s_1, s_2 \in F(U) : r^U_{U_i}(s_1) = r^U_{U_i}(s_2) \forall i \Rightarrow s_1 = s_2$$ (B11)

$$r^U_{U_i \cap U_j}(s_i) = r^U_{U_i \cap U_j}(s_j) \forall i, j \Rightarrow \exists s \in F(U) : r^U_{U_i}(s) = s_i \forall i$$ (B12)

Given a sheaf $S$ on $(M, T)$ and given $x \in M$:

Definition B.3

set of the germs of $S$ at $x$:

$$S_x := \lim_{\rightarrow x \in U} F(U)$$ (B13)

where $\lim_{\rightarrow x \in U}$ is a direct limit.

Suitable generalizations of the definition have been the starting point from which Alexander Grothendieck’s extraordinary abstraction’s skills have led to strongly remarkable results for a taste of which we demand to [30].

Though fibre-bundles are nowadays considered as part of the differential geometric tool-bag of a theoretical physicist the same cannot be said about sheaves of germs (as remarked by Chris Isham in the section 5.1.4 of [31] and still valid after almost a decade).

One of the reasons is that every sheaf of germs is the sheaf of germs of the local sections of a suitable fibre bundle.

The notion of a sheaf of germs plays, anyway, a crucial role in the rigorous definition of a Riemann surface of a complex function of a complex variable.

Definition B.4

region of $\mathbb{C}$:

$$U \subset \mathbb{C} \text{ open and connected}$$ (B14)

Definition B.5

function element:
a couple $(f, G)$ such that:

1. $G$ is a region of $\mathbb{C}$

2. $f$ is an analytic function on $G$

Given a function element $(f, G)$ and a point $a \in G$:

Definition B.6

germs of $f$ at $a$:

$$[f]_a := \{(g, D) \text{ function element} : a \in D \land \exists U \text{ neighborhood of } a : g(z) = f(z) \forall z \in U\}$$ (B15)

Given a path $\gamma : [0, 1] \rightarrow \mathbb{C}$ and a family $\{(f_t, D_t), t \in [0, 1]\}$ of function elements:
Definition B.7

\((f_1, D_1)\) is the analytic continuation of \((f_0, D_0)\) along \(\gamma\):

\[
\forall t \in [0, 1] \exists \delta > 0 : (|s - t| < \delta \Rightarrow \gamma(s) \in D_t) \land [f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}
\]  

(B16)

Proposition B.1

HP:

\[
\gamma : [0, 1] \mapsto \mathbb{C} : \gamma(0) = a \land \gamma(1) = b \text{ path}
\]

\(\{(f_t, D_t) t \in [0, 1]\}, \{(g_t, B_t) t \in [0, 1]\}\) analytic continuations of \((f_0, D_0)\) along \(\gamma\) such that \([f_0]_a = [g_0]_a\)

TH:

\[
[f_1]_b = [g_1]_b
\]

Given a path \(\gamma : [0, 1] \mapsto \mathbb{C} : \gamma(0) = a \land \gamma(1) = b\) and given \(\{(f_t, D_t) t \in [0, 1]\}, \{(g_t, B_t) t \in [0, 1]\}\) analytic continuations of \((f_0, D_0)\) along \(\gamma\) such that \([f_0]_a = [g_0]_a\) Proposition B.1 justifies the following:

Definition B.8

analytic continuation of \([f_0]_a\) along \(\gamma\):

\([f_1]_b\)

Given a function element \((f, G)\):

Definition B.9

complete analytic function obtained from \((f, G)\):

\[
\mathcal{F}[(f, G)] := \{[g]_b : \exists a \in G, \exists \gamma : [0, 1] \mapsto \mathbb{C} : \gamma(0) = a \land \gamma(1) = b \text{ path} : [g]_b \text{ is the analytic continuation of } [f]_a \text{ along } \gamma\} \quad (B17)
\]

Given \(\mathcal{F}\):

Definition B.10

\(\mathcal{F}\) is a complete analytic function:

\[
\exists (f, G) \text{ function element} : \mathcal{F} = \mathcal{F}[(f, G)]
\]  

(B18)

Given \(G \subset \mathbb{C}\) open:

Definition B.11

sheaf of germs of analytic functions on \(G\):

\[
\mathcal{S}(G) := \{(z, [f]_z) : z \in G, f \text{ is analytic at } z\}
\]  

(B19)

Remark B.1

It may be proved (see \[30\]) that the definition B.11 is a particular case of the definition B.2

Definition B.12
projection map of $S(G)$:
the map $\pi : S(G) \mapsto \mathbb{C}$:

$$\pi[(z, [f]_z)] := z$$  \hspace{1cm} (B20)

Then:

**Proposition B.2**

$$S_z(G) = \pi^{-1}(z) \ \forall z \in G$$  \hspace{1cm} (B21)

where $S_z(G)$ is the set of germs in $z$ of $S_z(G)$ defined, for an arbitrary sheaf of germs, by the definition $\text{[B.3]}$.

Given $D \subset G$ open:

**Definition B.13**

$$N(f, D) := \{(z, [f]_z) : z \in D\}$$  \hspace{1cm} (B22)

Given $(a, [f]_a) \in S(G)$:

**Definition B.14**

$$N(a, [f]_a) := \{N(g, B) : a \in B \land [g]_a = [f]_a\}$$  \hspace{1cm} (B23)

Then:

**Proposition B.3**

1. $\{N(a, [f]_a) : (a, [f]_a) \in S(G)\}$ is a neighborhood system on $S(G)$
2. the induced topology is Hausdorff
3. the projection map $\pi$ of $S_z(G)$ is continuous with respect to such a topology.

Given a complete analytic function $F$ we have now all the required ingredients to introduce the following:

**Definition B.15**

Riemann surface of $F$:

$$\Sigma_F := \{(z, [g]_z) : [g]_z \in F\}$$  \hspace{1cm} (B24)

**Remark B.2**

The definition $\text{[B.15]}$ allows to show that a complete analytic function is indeed a function in the ordinary meaning of such a term: it may be seen as the map $F : \Sigma_F \mapsto \mathbb{C}$:

$$F[(z, [f]_z)] := f(z)$$  \hspace{1cm} (B25)

Let us now introduce the second meaning of the locution "Riemann surface":

**Definition B.16**

Riemann surface:

a one-complex-dimensional connected complex analytic manifold

The link existing between the definition $\text{[B.15]}$ and the definition $\text{[B.16]}$ is given by the following:

**Theorem B.1**

Link between the two meanings of the locution "Riemann surface":

1. the Riemann surface $\Sigma_f$ of a complete analytic function $f$ (according to the definition $\text{[B.15]}$) is a Riemann surface (according to the definition $\text{[B.16]}$)
2. for every Riemann surface $\Sigma$ (according to the definition $\text{[B.16]}$) satisfying suitable regularity conditions, there exists a complete analytic function $f$ such that $\Sigma$ is the Riemann surface of $f$ (according to the definition $\text{[B.15]}$), i.e such that:

$$\Sigma = \Sigma_f$$  \hspace{1cm} (B26)
## APPENDIX C: NOTATION

| Symbol | Description |
|--------|-------------|
| ∧      | and (logical conjunction) |
| ∨      | or (logical disjunction) |
| ¬      | not (logical negation) |
| ∇ f    | gradient of the scalar field f |
| ∇ ∧ f’ | rotor of the vector field f’ |
| ∇ · f’ | divergence of the vector field f’ |
| α⃗f   | vector field associated to the 1-form α |
| β⃗f   | 1-form associated to the vector field f’ |
| πₙ(M)  | nᵗʰ homotopy group of M |
| Cₙ(M)  | nᵗʰ chain group of M |
| Zₙ(M)  | nᵗʰ cycle group of M |
| Bₙ(M)  | nᵗʰ boundary group of M |
| Hₙ(M)  | nᵗʰ homology group of M |
| Ωⁿ(M)  | set of the n-forms over M |
| α ∧ β  | exterior product of the form α with respect to the form β |
| dω     | exterior derivative of the form ω |
| iₓω    | interior product of the form ω with respect to the vector field X |
| Cⁿ(M)  | nᵗʰ cochain group of M |
| Zⁿ(M)  | nᵗʰ cocycle group of M |
| Bⁿ(M)  | nᵗʰ coboundary group of M |
| Hⁿ(M)  | nᵗʰ de Rham cohomology group of M |
| [G,G]  | commutator subgroup of the group G |
| P(M,G) | principal bundle with base space M, total space P and structure group G |
| TM     | tangent bundle of M |
| T* M   | cotangent bundle of M |
| ωᵦ     | canonical symplectic form of a cotangent bundle |
| Γ(T(r,s)M) | set of the global sections of the (r,s) tensor bundle over M |
| Γ(U,E) | set of the local sections over U of the fibre bundle E |
| M      | universal covering space of M |
| fₙ     | differential map of f |
| f*     | pullback of f |
| ⋆      | Hodge Star operator associated to a riemannian metric |
| ℂₓ     | punctured complex plane |
| Ref(f,z) | residue of the function f in z |
| S(U)   | group of sections of the sheaf S over U |
| Sₓ     | set of the germs of the sheaf S in x |
| Σ₀ f   | Riemann surface of the complete analytic function f |
