A Lower Bound on Expected Communication Cost of Quantum State Redistribution

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Accessibility
A lower bound on expected communication cost of quantum state redistribution

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Abstract

We show a lower bound on expected communication cost of interactive entanglement assisted quantum state redistribution protocols and a slightly better lower bound for its special case, quantum state transfer. Our bound implies that the expected communication cost of interactive protocols is not significantly better than worst case communication cost, in terms of scaling of error. Furthermore, the bound is independent of the number of rounds. This is in contrast with the classical case, where protocols with expected communication cost significantly better than worst case communication cost are known.

1 Introduction

A fundamental task in quantum information theory is that of quantum state redistribution (various quantities appearing in this section have been described in Section 2):

Quantum state-redistribution: A pure state $\Psi_{RBCA}$ is shared between Alice (A,C), Bob(B) and Referee(R). For a given $\varepsilon > 0$, which we shall henceforth identify as ‘error’, Alice needs to transfer the system $C$ to Bob, such that the final state $\Psi'_{RBC_{0}A}$ (where register $C_{0} \equiv C$ is with Bob), satisfies $P(\Psi'_{RBC_{0}A}, \Psi_{RBC_{0}A}) \leq \varepsilon$. Here, $P(\ldots)$ is the purified distance.

This task has been well studied in literature in asymptotic setting ([DY08, Opp08, YBW08, YD09]) giving an operational interpretation to the quantum conditional mutual information. Recent results have obtained one-shot versions of this task ([DHO14, BCT14, AJD14]), with application to bounded-round entanglement assisted quantum communication complexity ([Tou14]).

The following upper bound has been obtained in [Tou14], developing upon the work in [BCT14], on worst case communication cost of quantum state redistribution, with error $\varepsilon$:

$$\frac{50 \cdot I(R : C|B)_{\Psi_{RABC}}}{2\varepsilon^2} + \frac{100}{\varepsilon^2} + 15.$$ 

An important application of this bound is a direct sum theorem for communication cost of bounded-round entanglement assisted quantum communication complexity, which is the main result of [Tou14].
Theorem 1.1 (Touchette [Tou14], Theorem 3). Let $C$ be the quantum communication complexity of the best entanglement assisted protocol for computing a relation $f$ with error $\rho$ on inputs drawn from a distribution $\mu$. Then any $r$ round protocol computing $f^\otimes n$ on the distribution $\mu^\otimes n$ with error $\rho - \varepsilon$ must involve at least $\Omega(n((\frac{1}{\varepsilon})^2 \cdot C - r))$ quantum communication.

Direct sum results for single-round entanglement assisted quantum communication complexity had earlier been obtained in [JRS05, JRS08, AJM+14].

A special case of quantum state redistribution is quantum state merging, in which the register $A$ is absent. It was introduced in [HOW07] as a quantum counterpart to the classical Slepian-Wolf protocol [SW73]. A one-shot quantum state merging was introduced by Berta [Ber09]. A one-shot version of classical Slepian-Wolf protocol was obtained by Braverman and Rao [BR11], in the form of the following task:

Alice is given a probability distribution $P$, Bob is given a probability distribution $Q$. Bob must output a distribution $P'$, with the property that $\|P - P'\|_1 \leq \varepsilon$.

They exhibited an interactive communication protocol achieving this task with expected communication cost

$$D(P\|Q) + O(\sqrt{D(P\|Q)}) + 2\log\left(\frac{1}{\varepsilon}\right).$$

Considering expected communication cost, instead of worst case communication cost, allowed them to obtain the following direct sum result for bounded round classical communication complexity:

Theorem 1.2 (Braverman and Rao [BR11], Corollary 2.5). Let $C$ be the communication complexity of the best protocol for computing a relation $f$ with error $\rho$ on inputs drawn from a distribution $\mu$. Then any $r$ round protocol computing $f^\otimes n$ on the distribution $\mu^\otimes n$ with error $\rho - \varepsilon$ must involve at least $\Omega(n(C - r \cdot \log(\frac{1}{\varepsilon}) - O(\sqrt{C \cdot r})))$ communication.

This result has better dependence on number of rounds $r$ in comparison to theorem 1.1. Thus, in order to obtain a stronger direct sum result for bounded-round quantum communication complexity, a possible approach would be to bound the expected communication cost of quantum state redistribution by $\approx I(R : C | B)_{\Psi_{RABC}} + O(\log(\frac{1}{\varepsilon}))$.

A special case of quantum state merging is quantum state transfer, in which register $B$ is trivial. Asymptotic version of quantum state transfer is the Schumacher compression [Sch95]. In the corresponding classical setting, when $\Psi_{RA}$ is a classical probability distribution, Alice can send register $A$ to Bob with expected communication cost $S(\Psi_A) + O(1)$, using a one-way protocol based on Huffman coding [CT91]. In fact, one can make the error arbitrarily small, at the cost of arbitrarily large worst case communication.

Our results

In this work, we show that expected communication cost for entanglement assisted quantum protocols (which we formally define in section 3) is not significantly better than the worst case communication cost. Our main theorem is the following.

Theorem 1.3. Fix a $p < 1$ and an $\varepsilon \in [0, (\frac{1}{\varepsilon})^{1-p}]$. There exists a pure state $\Psi_{RBCA}$ (that depends on $\varepsilon$) such that, any interactive entanglement assisted communication protocol for its quantum state redistribution with error $\varepsilon$ requires expected communication cost at least $I(R : C | B)_{\Psi} \cdot (\frac{1}{\varepsilon})^p$. 

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For quantum state transfer, we obtain a similar result with slightly better constants.

**Theorem 1.4.** Fix a $p < 1$ and any $\varepsilon \in [0, (\frac{1}{2})^{\frac{15}{1-p}}]$. There exists a pure state $\Psi_{RC}$ (that depends on $\varepsilon$) such that, any interactive entanglement assisted communication protocol for its quantum state transfer with error $\varepsilon$ requires expected communication cost at least $S(\Psi_R) \cdot (\frac{1}{2})^p$.

Notice that theorem 1.4 does imply theorem 1.3 as quantum state transfer is a special case of quantum state redistribution. But the quantum state $\Psi_{RBC\ A}$ that we consider in theorem 1.3 has all registers $R, A, B, C$ non-trivial and correlated with each other. Thus, a quantum state redistribution of $\Psi_{RBC\ A}$ cannot be reduced to the sub-cases of quantum state merging or quantum state transfer by any local operation, giving robustness to the bound.

**Our technique and organization**

We discuss our technique for the case of quantum state transfer. For some $\beta > 1$, we choose the pure state $\Psi_{RC}$ in such a way that its smallest eigenvalue is $\frac{1}{d\beta}$ and entropy of $\Psi_R$ is at most $2 \log(d) \beta$ ($d$ being dimension of register $R$). Let $\omega_{RC}$ be a maximally entangled state defined as $|\omega\rangle_{RC} = \Psi_{RC}^{\frac{1}{\sqrt{d}}} |\Psi\rangle_{RC}$. For any interactive protocol $P$ for quantum state transfer of $\Psi_{RC}$ with error $\varepsilon$ and expected communication cost $C$, we obtain an expression that serves as a transcript of the protocol, encoding the unitaries applies by Alice and Bob and the probabilities of measurement outcomes (Corollary 3.5). This expression is obtained by employing a technique of convex-split, introduced in [AJD14] for one-way quantum state redistribution protocols. Then, crucially relying on the fact that $\Psi_{RC}$ is a pure state, we construct a new interactive protocol $P'$ which achieves quantum state transfer of the state $\omega_{RC}$ with error $\sqrt{\beta \varepsilon} + \sqrt{\mu}$ (for any $\mu < 1$) and worst case quantum communication cost at most $\frac{C}{\mu}$. Suitably choosing the parameters $\varepsilon, \beta$ and $\mu$ and using known lower bound on worst case communication cost for state transfer of $\omega_{RC}$, we obtain the desired result. Same technique also extends to quantum state redistribution. Details appear in section 4.

In section 2 we present some notions and facts that are needed for our proofs. In section 3 we give a description of interactive protocols for quantum state redistribution and obtain the aforementioned expression that serves as a transcript of a given protocol. Section 4 is devoted to refinement of this expression and proof of main theorem. We present some discussion related to our approach and conclude in Section 5.

**2 Preliminaries**

In this section we present some notations, definitions, facts and lemmas that we will use in our proofs.

**Information theory**

For a natural number $n$, let $[n]$ represent the set $\{1, 2, \ldots, n\}$. For a set $S$, let $|S|$ be the size of $S$. A tuple is a finite collection of positive integers, such as $(i_1, i_2, \ldots, i_r)$ for some finite $r$. We let $\log$ represent logarithm to the base 2 and $\ln$ represent logarithm to the base $e$. The $\ell_1$ norm of an operator $X$ is $\|X\|_1 \overset{\text{def}}{=} \text{Tr} \sqrt{X^\dagger X}$ and $\ell_2$ norm is $\|X\|_2 \overset{\text{def}}{=} \sqrt{\text{Tr} XX^\dagger}$. A quantum state (or just a
A quantum sub-normalized state is a positive semidefinite matrix with trace less than or equal to 1. Let $|ψ⟩$ be a unit vector. We use $ψ$ to represent the state and also the density matrix $|ψ⟩⟨ψ|$, associated with $|ψ⟩$.

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A quantum map $E$ is a completely positive and trace preserving (CPTP) linear map: $E: {A} → {B}$. An isometry $V: {A} → {B}$ is such that $V^†V = I_A$ and $VV^† = I_B$. The set of all unitary operations on register $A$ is denoted by $U(A)$. We define

$$\rho_B \overset{\text{def}}{=} \text{Tr}_A(\rho_{AB}) \overset{\text{def}}{=} \sum_i (|i⟩ ⊗ I_B)\rho_{AB}(|i⟩ ⊗ I_B),$$

where $\{|i⟩\}_i$ is an orthonormal basis for the Hilbert space $A$ and $I_B$ is the identity matrix in space $B$. The state $\rho_B$ is referred to as the marginal state of $\rho_{AB}$ in register $B$. Unless otherwise stated, a missing register from subscript in a state will represent partial trace over that register.

A quantum map $E: A → B$ is a completely positive and trace preserving (CPTP) linear map (mapping states from $D(A)$ to states in $D(B)$). A completely positive and trace non-increasing linear map $\tilde{E}: A → B$ maps quantum states to sub-normalised states. The identity operator in Hilbert space $H_A$ (and associated register $A$) is denoted by $I_A$. A unitary operator $U_A: H_A → H_A$ is such that $U_A^†U_A = U_A U_A^† = I_A$. An isometry $V: H_A → H_B$ is such that $V^†V = I_A$ and $VV^† = I_B$. The set of all unitary operations on register $A$ is denoted by $U(A)$.

**Definition 2.1.** We shall consider the following information theoretic quantities. Let $\varepsilon ≥ 0$.

1. **generalized fidelity** For $\rho, \sigma ∈ D(\leq A)$,

$$F(\rho, \sigma) \overset{\text{def}}{=} \|\sqrt{\rho} \sqrt{\sigma}\|_1 + \sqrt{(1 - \text{Tr}(\rho))(1 - \text{Tr}(\sigma))}.$$

2. **purified distance** For $\rho, \sigma ∈ D(\leq A)$,

$$P(\rho, \sigma) = \sqrt{1 - F^2(\rho, \sigma)}.$$

3. **$\varepsilon$-ball** For $\rho_A ∈ D(A)$,

$$\mathcal{B}_\varepsilon(\rho) \overset{\text{def}}{=} \{\rho'_A ∈ D(A) | P(\rho_A, \rho'_A) ≤ \varepsilon\}.$$

4. **entropy** For $\rho_A ∈ D(A)$,

$$H(\rho_A) \overset{\text{def}}{=} -\text{Tr}(\rho_A \log \rho_A).$$

5. **relative entropy** For $\rho_A, \sigma_A ∈ D(A)$,

$$D(\rho_A∥\sigma_A) \overset{\text{def}}{=} \text{Tr}(\rho_A \log \rho_A) - \text{Tr}(\rho_A \log \sigma_A).$$


6. **max-relative entropy** For $\rho_A, \sigma_A \in \mathcal{D}(A)$,
\[
D_{\text{max}}(\rho_A\|\sigma_A) \overset{\text{def}}{=} \inf\{\lambda \in \mathbb{R} : 2^\lambda \sigma_A \geq \rho_A\}.
\]

7. **mutual information** For $\rho_{AB} \in \mathcal{D}(AB)$,
\[
I(A : B)_{\rho} \overset{\text{def}}{=} D(\rho_{AB}\|\rho_A \otimes \rho_B) = H(A)_{\rho} + H(B)_{\rho} - H(AB)_{\rho}.
\]

8. **conditional mutual information** For $\rho_{ABC} \in \mathcal{D}(ABC)$,
\[
I(A : B|C)_{\rho} \overset{\text{def}}{=} I(A : BC)_{\rho} - I(A : C)_{\rho} = I(B : AC)_{\rho} - I(B : C)_{\rho}.
\]

9. **max-information** For $\rho_{AB} \in \mathcal{D}(AB)$,
\[
I_{\text{max}}(A : B)_{\rho} \overset{\text{def}}{=} \inf_{\sigma_B \in \mathcal{D}(B)}D_{\text{max}}(\rho_{AB}\|\rho_A \otimes \sigma_B).
\]

10. **smooth max-information** For $\rho_{AB} \in \mathcal{D}(AB)$,
\[
I_{\epsilon}^\text{max}(A : B)_{\rho} \overset{\text{def}}{=} \inf_{\rho' \in \mathcal{B}(\rho)}I_{\text{max}}(A : B)_{\rho'}.
\]

11. **conditional min-entropy** For $\rho_{AB} \in \mathcal{D}(AB)$,
\[
H_{\text{min}}(A|B)_{\rho} \overset{\text{def}}{=} -\inf_{\sigma_B \in \mathcal{D}(B)}D_{\text{max}}(\rho_{AB}\|I_A \otimes \sigma_B).
\]

12. **conditional max-entropy** For $\rho_{AB} \in \mathcal{D}(AB)$,
\[
H_{\text{max}}(A|B)_{\rho_{AB}} \overset{\text{def}}{=} -H_{\text{min}}(A|R)_{\rho_{AR}},
\]
where $\rho_{ABR}$ is a purification of $\rho_{AB}$ for some system $R$.

13. **smooth conditional min-entropy** For $\rho_{AB} \in \mathcal{D}(AB)$,
\[
H_{\epsilon}^\text{min}(A|B)_{\rho} \overset{\text{def}}{=} \sup_{\rho' \in \mathcal{B}(\rho)}H_{\text{min}}(A|B)_{\rho'}.
\]

14. **smooth conditional max-entropy** For $\rho_{AB} \in \mathcal{D}(AB)$,
\[
H_{\epsilon}^\text{max}(A|B)_{\rho} \overset{\text{def}}{=} \inf_{\rho' \in \mathcal{B}(\rho)}H_{\text{max}}(A|B)_{\rho'}.
\]

We will use the following facts.

**Fact 2.2** (Triangle inequality for purified distance, [Tom12]). For states $\rho_1^A, \rho_2^A, \rho_3^A \in \mathcal{D}(A)$,
\[
P(\rho_1^A, \rho_3^A) \leq P(\rho_1^A, \rho_2^A) + P(\rho_2^A, \rho_3^A).
\]
Thus, $P(\rho,\sigma) \leq \sqrt{\|\rho - \sigma\|_1}$. 

**Fact 2.4** (Uhlmann’s theorem). [Uhl76] Let $\rho_A,\sigma_A \in \mathcal{D}(A)$. Let $|\rho\rangle_{AB}$ be a purification of $\rho_A$ and $|\sigma\rangle_{AC}$ be a purification of $\sigma_A$. There exists an isometry $V : \mathcal{H}_C \to \mathcal{H}_B$ such that,

$$F(|\theta\rangle\langle\theta|_{AB}, |\rho\rangle\langle\rho|_{AB}) = F(\rho_A,\sigma_A),$$

where $|\theta\rangle_{AB} = (I_A \otimes V)|\sigma\rangle_{AC}$.

**Fact 2.5** (Monotonicity of quantum operations). [Lin75, BCF+96, Tom12, Theorem 3.4] For states $\rho, \sigma$, and quantum operation $\mathcal{E}(\cdot)$,

$$\|\mathcal{E}(\rho) - \mathcal{E}(\sigma)\|_1 \leq \|\rho - \sigma\|_1, P(\rho,\sigma) \leq P(\mathcal{E}(\rho),\mathcal{E}(\sigma))$$

and $F(\rho,\sigma) \leq F(\mathcal{E}(\rho),\mathcal{E}(\sigma))$.

In particular, for a trace non-increasing completely positive map $\tilde{\mathcal{E}}(\cdot)$,

$$P(\rho,\sigma) \leq P(\tilde{\mathcal{E}}(\rho),\tilde{\mathcal{E}}(\sigma)).$$

**Fact 2.6** (Join concavity of fidelity). [Wat11, Proposition 4.7] Given quantum states $\rho_1,\rho_2 \ldots \rho_k, \sigma_1,\sigma_2 \ldots \sigma_k \in \mathcal{D}(A)$ and positive numbers $p_1,p_2 \ldots p_k$ such that $\sum_i p_i = 1$. Then

$$F\left(\sum_i p_i \rho_i, \sum_i p_i \sigma_i\right) \geq \sum_i p_i F(\rho_i,\sigma_i).$$

**Fact 2.7.** Let $\rho, \sigma \in \mathcal{D}(A)$ be quantum states. Let $\alpha < 1$ be a positive real number. If $P(\alpha \rho,\alpha \sigma) \leq \varepsilon$, then

$$P(\rho,\sigma) \leq \varepsilon \sqrt{\frac{\alpha}{2}}.$$ 

*Proof.* $P(\alpha \rho,\alpha \sigma) \leq \varepsilon$ implies $F(\alpha \rho,\alpha \sigma) \geq \sqrt{1 - \varepsilon^2} \geq 1 - \varepsilon^2$. But, $F(\alpha \rho,\alpha \sigma) = \alpha \|\sqrt{\rho} \sqrt{\sigma}\|_1 + (1 - \alpha)$. Thus,

$$F(\rho,\sigma) = \|\sqrt{\rho} \sqrt{\sigma}\|_1 \geq 1 - \frac{\varepsilon^2}{\alpha}.$$ 

Thus, $P(\rho,\sigma) \leq \sqrt{1 - (1 - \frac{\varepsilon^2}{\alpha})^2} \leq \sqrt{\frac{2\varepsilon^2}{\alpha}}$. 

**Fact 2.8** (Fannes inequality). [Fan73] Given quantum states $\rho_1,\rho_2 \in \mathcal{D}(A)$, such that $|A| = d$ and $P(\rho_1,\rho_2) = \varepsilon \leq \frac{1}{2d},$

$$|S(\rho_1) - S(\rho_2)| \leq \varepsilon \log(d) + 1.$$ 

**Fact 2.9** (Subadditivity of entropy). [AL70] For a quantum state $\rho_{AB} \in \mathcal{D}(AB)$, $|S(\rho_A) - S(\rho_B)| \leq S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$.

**Fact 2.10** (Concavity of entropy). [Wat11, Theorem 10.9] For quantum states $\rho_1,\rho_2 \ldots \rho_n$, and positive real numbers $\lambda_1,\lambda_2 \ldots \lambda_n$ satisfying $\sum_i \lambda_i = 1$,

$$S\left(\sum_i \lambda_i \rho_i\right) \geq \sum_i \lambda_i S(\rho_i).$$
**Fact 2.11.** For a quantum state $\rho_{ABC}$, it holds that

$$I(A : C)_\rho \leq 2S(\rho_C),$$

$$I(A : C|B)_\rho \leq I(AB : C)_\rho \leq 2S(\rho_C).$$

**Proof.** From Fact 2.9, $I(A : C)_\rho = S(\rho_A) + S(\rho_C) - S(\rho_{AC}) \leq 2S(\rho_C)$. \hfill \Box

**Fact 2.12.** For a bipartite quantum state $\rho_{AB}$, $I_{\max}(A : B)_\rho \geq -H_{\min}(A|B)_\rho$.

**Proof.** Let $\sigma_B$ be the state achieved in infimum in the definition of $I_{\max}(A : B)_\rho$. Let $\lambda \overset{\text{def}}{=} I_{\max}(A : B)_\rho$. Consider,

$$\rho_{AB} \leq 2^\lambda \rho_A \otimes \sigma_B \leq 2^\lambda I_A \otimes \sigma_B.$$

Thus, we have

$$-H_{\min}(A|B)_\rho = \inf_{\sigma_B \in D(B)} D_{\max}(\rho_{AB}|| I_A \otimes \sigma_B') \leq D_{\max}(\rho_{AB}|| I_A \otimes \sigma_B) \leq \lambda = I_{\max}(A : B)_\rho.$$

This gives,

$$\inf_{\rho'_{AB} \in B_{\epsilon}(\rho_{AB})} \rho_{AB} - H_{\min}(A|B)_{\rho'} \leq I_{\max}(A : B)_\rho.$$

\hfill \Box

**Fact 2.13.** For a classical-quantum state $\rho_{AB}$ of the form $\rho_{AB} = \sum_j p(j) |j\rangle \langle j| \otimes \sigma_j^B$, it holds that

$I_{\max}(A : B)_\rho \leq \log(|B|)$.

**Proof.** By definition, $I_{\max}(A : B)_\rho \leq D_{\max}\left(\rho_{AB}|| I_A \otimes \frac{I_B}{|B|}\right)$. Also,

$$\rho_{AB} = \sum_j p(j) |j\rangle \langle j| \otimes \sigma_j^B \leq |B| \sum_j p(j) |j\rangle \langle j| \otimes \frac{I_B}{|B|} = |B| \rho_A \otimes \frac{I_B}{|B|}.$$

Thus, the fact follows. \hfill \Box

**Fact 2.14.** For a classical-quantum state $\rho_{ABC} = \sum_j p(j) |j\rangle \langle j| \otimes \rho^j_{BC}$, it holds that $I(AB : C)_\rho \geq \sum_j p(j) I(B : C)_{\rho^j}$

**Proof.** Consider,

$$I(AB : C)_\rho = S(\rho_{AB}) + S(\rho_C) - S(\rho_{ABC})$$

$$= S(\sum_j p(j) |j\rangle \langle j| \otimes \rho^j_B) + S(\sum_j p(j) \rho^j_C) - S(\sum_j p(j) |j\rangle \langle j| \otimes \rho^j_BC)$$

$$= \sum_j p(j) S(\rho^j_B) + S(\sum_j p(j) \rho^j_C) - \sum_j p(j) S(\rho^j_BC)$$

$$\geq \sum_j p(j) S(\rho^j_B) + \sum_j p(j) S(\rho^j_C) - \sum_j p(j) S(\rho^j_BC) \quad \text{(Fact 2.10)}$$

$$= \sum_j p(j) I(B : C)_{\rho^j}$$

\hfill \Box
Lemma 2.15. Fix a $\beta \geq 1$ and an integer $d > 1$. There exists a probability distribution $\mu = \{e_1, e_2, \ldots, e_d\}$, with $e_1 \geq e_2 \geq \cdots \geq e_d$, such that $e_d = \frac{1}{d^2}$ and entropy $S(\mu) \leq 2 \log(d)$. 

Proof. Set $e_2 = e_3 = \cdots = e_d = \frac{1}{d^2}$. Then $e_1 = 1 - \frac{d - 1}{d\beta}$. Using $x \log(\frac{x}{e}) \leq \frac{\log(e)}{e} < 1$ for all $x > 0$, we can upper bound the entropy of the distribution as

$$
\sum_i e_i \log \frac{1}{e_i} = (1 - \frac{d - 1}{d\beta}) \log(\frac{1}{1 - \frac{d - 1}{d\beta}}) + \frac{d - 1}{d\beta} \log(d) < 2 + \frac{\log(d)}{\beta} \leq 2 \log(d).
$$

$\square$

3 Interactive protocol for quantum state redistribution

In this section, we describe general structure of an interactive protocol for quantum state redistribution and its expected communication cost.

Let quantum state $|\Psi\rangle_{RBCA}$ be shared between Alice ($A$, $C$), Bob ($B$) and Referee ($R$). Alice and Bob have access to shared entanglement $\theta_{E_AE_B}$ in registers $E_A$ (with Alice) and $E_B$ (with Bob). Using quantum teleportation, we can assume without loss of generality that Alice and Bob communicate classical messages, which involves performing a projective measurement on registers they respectively hold, and sending the outcome of measurement to other party. This allows for the notion of expected communication cost.

A $r$-round interactive protocol $\mathcal{P}$ (where $r$ is an odd number) with error $\varepsilon$ and expected communication cost $C$ is as follows.

**Input:** A quantum state $|\Psi\rangle_{RBCA}$, error parameter $\varepsilon < 1$.

**Shared entanglement:** $|\theta\rangle_{E_AE_B}$.

- Alice performs a projective measurement $\mathcal{M} = \{M_{AC}^1, M_{AC}^2, \ldots\}$. Probability of outcome $i_1$ is $p_{i_1} \equiv \text{Tr}(M_{AC}^{i_1} |\Psi_{CA} \otimes \theta_{E_A}\rangle \langle \theta_{E_A}|)$. Let $\phi_{RBACE_AE_B}^{i_{1}}$ be the global normalized quantum state, conditioned on this outcome. She sends message $i_1$ to Bob.

- Upon receiving the message $i_1$ from Alice, Bob performs a projective measurement

$$
\mathcal{M}^{i_1} = \{M_{BE_B}^{i_1}, M_{BE_B}^{2i_1}, \ldots\}.
$$

Probability of outcome $i_2$ is $p_{i_2|i_1} \equiv \text{Tr}(M_{BE_B}^{i_2} \phi_{RBACE_AE_B}^{i_1}\rangle \langle \phi_{RBACE_AE_B}^{i_1}|)$. Let $\phi_{RBACE_AE_B}^{i_{2},i_{1}}$ be the global normalized quantum state conditioned on this outcome $i_2$ and previous outcome $i_1$. Bob sends message $i_2$ to Alice.

- Consider any odd round $1 < k \leq r$. Let the measurement outcomes in previous rounds be $i_1, i_2, \ldots, i_{k-1}$ and global normalized state be $\phi_{RBACE_AE_B}^{i_{k-1},i_{k-2},\ldots,i_{1}}$. Alice performs the projective measurement $\mathcal{M}^{i_{k-1},i_{k-2},\ldots,i_{2},i_{1}} = \{M_{AC}^{1,i_{k-1},i_{k-2},\ldots,i_{2},i_{1}}, M_{AC}^{2,i_{k-1},i_{k-2},\ldots,i_{2},i_{1}}, \ldots\}$ and obtains outcome $i_k$ with probability $p_{i_k|i_{k-1},i_{k-2},\ldots,i_{2},i_{1}} \equiv \text{Tr}(M_{AC}^{i_{k},i_{k-1},i_{k-2},\ldots,i_{2},i_{1}} |\phi_{RBACE_AE_B}^{i_{k-1},i_{k-2},\ldots,i_{1}}\rangle \langle \phi_{RBACE_AE_B}^{i_{k-1},i_{k-2},\ldots,i_{1}}|)$. Let the global normalized state after outcome $i_k$ be $\phi_{RBACE_AE_B}^{i_{k},i_{k-1},i_{k-2},\ldots,i_{1}}$. Alice sends the outcome $i_k$ to Bob.


The expected communication cost is as follows.

**Fact 3.1.** Expected communication cost of $F$ is $\mathbf{E}_F$ of a probability distribution $\{p_1, p_2 \ldots p_m\}$ is $\sum_{i=1}^{m} p_i \log(i)$.

Proof. The expected communication cost is the expected length of the messages over all probability outcomes. It can be evaluated as

$$\sum_{i_1} p_{i_1} \log(i_1) + \sum_{i_1,i_2} p_{i_1}p_{i_2|i_1} \log(i_2) + \ldots \sum_{i_1,i_2 \ldots i_r} p_{i_1,i_2 \ldots i_r}p_{i_r|i_1,i_2 \ldots i_r} \log(i_r)$$

$$= \sum_{i_1,i_2 \ldots i_r} p_{i_1,i_2 \ldots i_r} (\log(i_1) + \log(i_2) + \ldots \log(i_r)).$$

This allows us to define

**Definition 3.2. Communication weight** of a probability distribution $\{p_1, p_2 \ldots p_m\}$ is $\sum_{i=1}^{m} p_i \log(i)$. 


The following lemma is a coherent representation of above protocol.

**Lemma 3.3.** For every $k \leq r$, let $O_k$ represent the set of all tuples $(i_1, i_2 \ldots i_k)$ which satisfy: \( \{i_1, i_2 \ldots i_k\} \) is a sequence of measurement outcomes that occurs with non-zero probability upto $k$-th round of $P$.

There exist registers $M_1, M_2 \ldots M_r$ and isometries

\[
\{U_{i_1 k-1, i_k k-2 \ldots i_2, i_1} \colon ACE_A \rightarrow ACE_A M_k \mid k > 1, k \text{ odd }, (i_1, i_2 \ldots i_k-1) \in O_{k-1}\},
\]

\[
\{U_{i_1 k-1, i_k k-2 \ldots i_2, i_1} \colon BE_B \rightarrow BE_B M_k \mid k \text{ even }, (i_1, i_2 \ldots i_k-1) \in O_{k-1}\}
\]

and $U : ACE_A \rightarrow ACE_A M_1$, such that

\[
|\Psi\rangle_{RBC A} |\theta\rangle_{E_B E_B} = U^\dagger \sum_{i_1, i_2 \ldots i_r} \sqrt{p_{i_1, i_2 \ldots i_r}} U_{i_1, i_2 \ldots i_r}^\dagger U_{i_1, i_2 \ldots i_r}^\dagger \left| i_{i_1}, i_{i_2 \ldots i_r-1} \right\rangle_{RBC A C_0 T_B E_A} |i_r\rangle_{M_r} \ldots |i_1\rangle_{M_1}.
\]

**Proof.** Fix an odd $k > 1$. Let the messages prior to $k$-th round be $(i_1, i_2 \ldots i_{k-1})$. As defined in protocol $P$, global quantum state before $k$-th round is $\phi_{RBC A E_B E_B}^{i_{k-1}, i_k, i_{k-2}, \ldots i_1}$. Alice performs the measurement

\[
\{M_{ACE_A}^{i_{k-1}, i_k, i_{k-2}, \ldots i_1}, M_{AXE_A}^{i_{k-1}, i_k, i_{k-2}, \ldots i_1} \ldots \}.
\]

This leads to a **convex-split** (introduced in [AJD14]):

\[
\phi_{RBE_B}^{i_{k-1}, i_k, i_{k-2}, \ldots i_1} = \sum_{i_k} \text{Tr}_{ACE_A} \left(M_{ACE_A}^{i_{k-1}, i_k, i_{k-2}, \ldots i_1} \phi_{RBC A E_B E_B}^{i_{k-1}, i_k, i_{k-2}, \ldots i_1}\right)
\]

\[
= \sum_{i_k} p_{i_k} |i_{k-1}, i_k, i_{k-2}, \ldots i_1\rangle \text{Tr}_{ACE_A} \left(M_{ACE_A}^{i_{k-1}, i_k, i_{k-2}, \ldots i_1} \phi_{RBC A E_B E_B}^{i_{k-1}, i_k, i_{k-2}, \ldots i_1} M_{ACE_A}^{i_{k-1}, i_k, i_{k-2}, \ldots i_1}\right)
\]

\[
= \sum_{i_k} p_{i_k} |i_{k-1}, i_k, i_{k-2}, i_1\rangle \phi_{RBE_B}^{i_{k-1}, i_k, i_{k-2}, i_1} \tag{1}
\]

A purification of $\phi_{RBE_B}^{i_{k-1}, i_k, i_{k-2}, \ldots i_1}$ on registers $RBC A E_B E_A$ is $\phi_{RBC A E_B E_A}^{i_{k-1}, i_k, i_{k-2}, i_1}$. Introduce a register $M_k$ (of sufficiently large dimension) and consider the following purification of

\[
\sum_{i_k} p_{i_k} |i_{k-1}, i_k, i_{k-2}, i_1\rangle \phi_{RBE_B}^{i_{k-1}, i_k, i_{k-2}, i_1}
\]

on register $RBC A E_B E_A M_k$ :

\[
\sum_{i_k} \sqrt{p_{i_k}} |i_{k-1}, i_k, i_{k-2}, i_1\rangle \phi_{RBE_B}^{i_{k-1}, i_k, i_{k-2}, i_1} \mid RBC A E_B E_A |i_k\rangle M_k.
\]

By Uhlmann’s theorem, there exists an isometry $U_{i_{k-1}, i_k, i_{k-2}, i_1} : ACE_A \rightarrow ACE_A M_k$ such that

\[
U_{i_{k-1}, i_k, i_{k-2}, i_1} |\phi_{RBE_B}^{i_{k-1}, i_k, i_{k-2}, i_1} \rangle_{RBC A E_B E_A} = \sum_{i_k} \sqrt{p_{i_k}} |i_{k-1}, i_k, i_{k-2}, i_1\rangle \phi_{RBE_B}^{i_{k-1}, i_k, i_{k-2}, i_1} \mid RBC A E_B E_A |i_k\rangle M_k \tag{2}
\]
For \( k = 1 \), introduce register \( M_1 \) of sufficiently large dimension. Similar argument implies that there exists an isometry \( U : ACE_A \rightarrow ACE_A M_1 \) such that

\[
U |\Psi\rangle_{RBACE_B E_A} = \sum_{i_1} \sqrt{p_{i_1}} |\phi^{i_1}\rangle_{RBACE_B E_A} |i_1\rangle_{M_1}
\]

(3)

For \( k \) even, introduce a register \( M_k \) of sufficiently large dimension. Again by similar argument, there exists an isometry \( U_{i_{k-1},i_{k-2}...i_1} : BE_B \rightarrow BE_B M_k \) such that

\[
U_{i_{k-1},i_{k-2}...i_1} |\phi^{i_{k-1},i_{k-2}...i_1}\rangle_{RBACE_B E_A} = \sum_{i_k} \sqrt{p_{i_k}} |\phi^{i_{k-1},i_{k-2}...i_1}\rangle_{RBACE_B E_A} |i_k\rangle_{M_k}
\]

(4)

Now, we recursively use equations 2, 3 and 4. Consider,

\[
|\Psi\rangle_{RBACA} |\theta\rangle_{E_A E_B} = U^\dagger \sum_{i_1} \sqrt{p_{i_1}} |\phi^{i_1}\rangle_{RBACE_B E_A} |i_1\rangle_{M_1}
\]

\[
= U^\dagger \sum_{i_1} \sqrt{p_{i_1}} U_k^\dagger \sum_{i_2} \sqrt{p_{i_2}} |\phi^{i_2,i_1}\rangle_{RBACE_B E_A} |i_2\rangle_{M_2} |i_1\rangle_{M_1}
\]

\[
= U^\dagger \sum_{i_1,i_2} \sqrt{p_{i_1,i_2}} U_{i_{k-1},i_{k-2}...i_1} |\phi^{i_2,i_1}\rangle_{RBACE_B E_A} |i_2\rangle_{M_2} |i_1\rangle_{M_1}
\]

\[
= U^\dagger \sum_{i_1,i_2...i_r} \sqrt{p_{i_1,i_2...i_r}} U_{i_{k-1},i_{k-2}...i_1} U_{i_{k-2},i_{k-3}...i_2} |\phi^{i_r,i_{r-1}...i_1}\rangle_{RBACE_B T_B E_A} |i_r\rangle_{M_r} ... |i_1\rangle_{M_1}
\]

Last equality follows by recursion. This completes the proof.

\[\square\]

**Definition 3.4.** We introduce the following useful definitions.

- Let \( k > 1 \) be odd. Isometry \( U_k : ACE_A M_1 M_2 ... M_{k-1} \rightarrow ACE_A M_1 M_2 ... M_{k-1} M_k \),

\[
U_k \overset{\text{def}}{=} \sum_{i_1,i_2...i_{k-1}} |i_1\rangle_{M_1} \otimes |i_2\rangle_{M_2} \otimes ... |i_{k-1}\rangle_{M_{k-1}} \otimes U_{i_{k-1},i_{k-2}...i_1}.
\]

- For \( k \) even, Isometry \( U_k : BE_B M_1 M_2 ... M_{k-1} \rightarrow BE_B M_1 M_2 ... M_{k-1} M_k \),

\[
U_k \overset{\text{def}}{=} \sum_{i_1,i_2...i_{k-1}} |i_1\rangle_{M_1} \otimes |i_2\rangle_{M_2} \otimes ... |i_{k-1}\rangle_{M_{k-1}} \otimes U_{i_{k-1},i_{k-2}...i_1}.
\]

- Unitary \( U_{r+1}^a : ACE_A M_1 M_2 ... M_r \rightarrow ACE_A M_1 M_2 ... M_r \),

\[
U_{r+1}^a \overset{\text{def}}{=} \sum_{i_1,i_2...i_r} |i_1\rangle_{M_1} \otimes |i_2\rangle_{M_2} \otimes ... |i_r\rangle_{M_r} \otimes U_{i_r,i_{r-1}...i_1}.
\]

- Unitary \( U_{r+1}^b : BE_B M_1 M_2 ... M_r \rightarrow BC_0 T_B M_1 M_2 ... M_r \),

\[
U_{r+1}^b \overset{\text{def}}{=} \sum_{i_1,i_2...i_r} |i_1\rangle_{M_1} \otimes |i_2\rangle_{M_2} \otimes ... |i_r\rangle_{M_r} \otimes U_{i_r,i_{r-1}...i_1}.
\]
• Unitary $U_{r+1} : ACE_ABE_BM_1M_2\ldots M_r \rightarrow ACE_ABC_0T_BM_1M_2\ldots M_r,$

$$U_{r+1} \equiv \sum_{i_1,i_2,\ldots i_r} |i_1\rangle\langle i_1|_M_1 \otimes |i_2\rangle\langle i_2|_M_2 \otimes \ldots |i_r\rangle\langle i_r|_M_r \otimes U_{i_r,i_{r-1}\ldots i_1}.$$ 

This leads to a more convenient representation of lemma 3.3.

**Corollary 3.5.** It holds that

$$|\Psi\rangle_{RBCA} \otimes |\theta\rangle_{E_AE_B} = U_1^\dagger U_2^\dagger \ldots U_{r+1}^\dagger \sum_{i_1,i_2,\ldots i_r} \sqrt{p_{i_1,i_2,\ldots i_r}} \left|\tau_{i_r,i_{r-1}\ldots i_1}\right\rangle_{RBCA0T_BE_A} |i_r\rangle_{M_r} \ldots |i_1\rangle_{M_1},$$

and

$$\mathcal{P}(\Psi_{RBCA}, \sum_{i_1,i_2,\ldots i_r} p_{i_1,i_2,\ldots i_r} \tau_{RBCA}) \leq \varepsilon.$$ 

**Proof.** The corollary follows immediately using Definition 3.4 and lemma 3.3. \qed

4 Lower bound on expected communication cost

In this section, we obtain a lower bound on expected communication cost of quantum state redistribution and quantum state transfer, by considering a class of states defined below.

Let register $R$ be composed of two registers $R_A, R'$, such that $R \equiv R_A R'$. Let $d_a$ be the dimension of registers $R_A$ and $A$. Let $d$ be the dimension of registers $R', C$ and $B$. Consider,

**Definition 4.1.** $|\Psi\rangle_{RBCA} \equiv \frac{1}{\sqrt{d_a}} \sum_{i=1}^{d_a} |\alpha\rangle_{R_A} |\alpha\rangle_A |\psi\rangle_{R'BC}$. Let $|\psi\rangle_{R'BC} = \sum_{j=1}^{d} \sqrt{e_j} |u_j\rangle_{R'} |v_j(a)\rangle_{B} |w_j(a)\rangle_C$ where $e_1 \geq e_2 \geq \ldots e_d > 0$, $\sum_{i=1}^{d} e_i = 1$ and $\{|u_1\rangle, \ldots |u_d\rangle\}, \{|v_1(a)\rangle, \ldots |v_d(a)\rangle\}, \{|w_1(a)\rangle, \ldots |w_d(a)\rangle\}$ form an orthonormal basis (second and third bases may depend arbitrarily on $a$) in their respective Hilbert spaces.

For quantum state transfer, we consider a pure state $\Psi_{RC}$ with Schmidt decomposition $\sum_{i=1}^{d} \sqrt{e_j} |u_j\rangle_{R} |w_j\rangle_{C}$.

Given the state $|\psi\rangle_{R'BC}$ from definition 4.1, we define a 'GHZ state' corresponding to it: $|\omega\rangle_{R'BC} \equiv \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |u_j\rangle_{R'} |v_j(a)\rangle_{B} |w_j(a)\rangle_{C}$. Using this, we define $\omega_{RBCA} \equiv \frac{1}{\sqrt{d_a}} \sum_{i=1}^{d_a} |\alpha\rangle_{R_A} |\alpha\rangle_A |\omega\rangle_{R'BC}$. Similarly, given the bipartite state $\Psi_{RC}$, we define a maximally entangled state $\omega_{RC} \equiv \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |u_j\rangle_{R} |w_j\rangle_{C}$.

Following two relations are easy to verify.

$$|\omega\rangle_{RBCA} = \frac{1}{\sqrt{d_a \cdot d}} \Psi_{R}^{-\frac{1}{2}} |\Psi\rangle_{RBCA} \text{ and } |\omega\rangle_{RC} = \frac{1}{\sqrt{d}} (\Psi_{R})^{-\frac{1}{2}} |\tilde{\Psi}\rangle_{RC} \quad (5)$$

As noted in section 3, the protocol $\mathcal{P}$ achieves quantum state redistribution of $\Psi_{RBCA}$ with error $\varepsilon$ and expected communication cost $C$. Following lemma is a refined form of corollary 3.5 and is also applicable to state $\Psi_{RBCA}$ not of the form given in definition 4.1.

**Lemma 4.2.** There exists a probability distribution $\{p'_{i_1,i_2,\ldots i_r}\}$ and pure states $\kappa_{C'E_AT_B}^{'i_r,i_{r-1}\ldots i_1}$ such that

$$\mathcal{P}(\Psi_{RBCA} \otimes \theta_{E_AE_B}, U_1^\dagger U_2^\dagger \ldots U_{r+1}^\dagger \sum_{i_1,i_2,\ldots i_r} p'_{i_1,i_2,\ldots i_r} \Psi_{RBCA0A} \otimes \kappa_{C'E_AT_B}^{'i_r,i_{r-1}\ldots i_1} |i_r\rangle_{M_r} \ldots |i_1\rangle_{M_1}) \leq 2\sqrt{\varepsilon},$$

and the communication weight of $p'_{i_1,i_2,\ldots i_r}$ is at most $\frac{C}{2\sqrt{\varepsilon}}$. 

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Proof. Let $\mathcal{B}$ be the set of tuples $(i_1, i_2, \ldots, i_r)$ for which $F^2(\Psi_{RBC_0A}, \tau_{RBC_0A}^{i_r, i_{r-1} \ldots i_1}) \leq 1 - \varepsilon$. Let $\mathcal{G}$ be remaining set of tuples. From corollary 3.5 and purity of $\Psi_{RBC_0A}$, it holds that

$$\sum_{i_1, i_2, \ldots, i_r} p_{i_1, i_2, \ldots, i_r} F^2(\Psi_{RBC_0A}, \tau_{RBC_0A}^{i_r, i_{r-1} \ldots i_1}) \geq 1 - \varepsilon^2.$$ 

Thus,

$$(1 - \varepsilon) \sum_{(i_1, i_2, \ldots, i_r) \in \mathcal{B}} p_{i_1, i_2, \ldots, i_r} + \sum_{(i_1, i_2, \ldots, i_r) \in \mathcal{G}} p_{i_1, i_2, \ldots, i_r} \geq 1 - \varepsilon^2,$$

which implies $\sum_{(i_1, i_2, \ldots, i_r) \in \mathcal{G}} p_{i_1, i_2, \ldots, i_r} \leq \varepsilon$. Thus we have $\sum_{(i_1, i_2, \ldots, i_r) \in \mathcal{G}} p_{i_1, i_2, \ldots, i_r} \geq 1 - \varepsilon$.

Define $\lambda'_{i_1, i_2, \ldots, i_r} \overset{\text{def}}{=} \frac{p_{i_1, i_2, \ldots, i_r}}{\sum_{i_1, i_2, \ldots, i_r \in \mathcal{G}} p_{i_1, i_2, \ldots, i_r}}$, if $(i_1, i_2 \ldots i_r) \in \mathcal{G}$ and $\lambda'_{i_1, i_2, \ldots, i_r} \overset{\text{def}}{=} 0$ if $(i_1, i_2 \ldots i_r) \in \mathcal{B}$.

For all $(i_1, i_2 \ldots i_r) \in \mathcal{G}$, $F^2(\Psi_{RBC_0A}, \tau_{RBC_0A}^{i_r, i_{r-1} \ldots i_1}) \geq 1 - \varepsilon$. Thus by Fact 2.2, there exists a pure state $\kappa_{CEA T_B}$ such that

$$F^2(\Psi_{RBC_0A} \otimes \kappa_{CEA T_B}, \tau_{RBCAC_0T_B E_A}) \geq 1 - \varepsilon \quad (6)$$

Consider,

$$\mathcal{P}(\sum_{i_1, i_2, \ldots, i_r} \sqrt{p_{i_1, i_2, \ldots, i_r} \tau_{RBCAC_0T_B E_A}^{i_r, i_{r-1} \ldots i_1}} |i_r\rangle_{M_r} \cdots |i_1\rangle_{M_1}, \sum_{i_1, i_2, \ldots, i_r} \sqrt{\lambda'_{i_1, i_2, \ldots, i_r} \tau_{RBCAC_0T_B E_A}^{i_r, i_{r-1} \ldots i_1}} |i_r\rangle_{M_r} \cdots |i_1\rangle_{M_1})$$

$$= \sqrt{1 - \left( \sum_{i_1, i_2, \ldots, i_r \in \mathcal{G}} p_{i_1, i_2, \ldots, i_r} \right)^2} \leq \sqrt{\varepsilon}$$

and

$$\mathcal{P}(\sum_{i_1, i_2, \ldots, i_r} \sqrt{\lambda'_{i_1, i_2, \ldots, i_r} \tau_{RBCAC_0T_B E_A}^{i_r, i_{r-1} \ldots i_1}} |i_r\rangle_{M_r} \cdots |i_1\rangle_{M_1}, \sum_{i_1, i_2, \ldots, i_r} \sqrt{p_{i_1, i_2, \ldots, i_r} \Psi_{RBC_0A} \otimes \kappa_{CEA T_B}^{i_r, i_{r-1} \ldots i_1}} |i_r\rangle_{M_r} \cdots |i_1\rangle_{M_1})$$

$$= \sqrt{1 - \left( \sum_{i_1, i_2, \ldots, i_r} \lambda'_{i_1, i_2, \ldots, i_r} \right)^2} \leq \sqrt{\varepsilon} \quad \text{(Equation 6)}$$

These together imply, using triangle inequality for purified distance (Fact 2.2),

$$\mathcal{P}(\sum_{i_1, i_2, \ldots, i_r} \sqrt{p_{i_1, i_2, \ldots, i_r} \tau_{RBCAC_0T_B E_A}^{i_r, i_{r-1} \ldots i_1}} |i_r\rangle_{M_r} \cdots |i_1\rangle_{M_1}, \sum_{i_1, i_2, \ldots, i_r} \sqrt{p_{i_1, i_2, \ldots, i_r} \Psi_{RBC_0A} \otimes \kappa_{CEA T_B}^{i_r, i_{r-1} \ldots i_1}} |i_r\rangle_{M_r} \cdots |i_1\rangle_{M_1})$$

$$\leq 2\sqrt{\varepsilon}$$

Thus, from corollary 3.5, we have

$$\mathcal{P}(\Psi_{RBCA} \otimes \theta_{E_A E_B}, \mathcal{U}^\dagger U_1^\dagger \ldots \mathcal{U}_{r+1}^\dagger \sum_{i_1, i_2, \ldots, i_r} p_{i_1, i_2, \ldots, i_r} \Psi_{RBC_0A} \otimes \kappa_{CEA T_B}^{i_r, i_{r-1} \ldots i_1} |i_r\rangle_{M_r} \cdots |i_1\rangle_{M_1}) \leq 2\sqrt{\varepsilon}.$$ 

The communication weight of $p'_{i_1, i_2, \ldots, i_r}$ is

$$\sum_{i_1, i_2, \ldots, i_r} p'_{i_1, i_2, \ldots, i_r} \log(i_1 \cdot i_2 \ldots i_r) \leq \frac{1}{1 - \varepsilon} \sum_{i_1, i_2, \ldots, i_r \in \mathcal{G}} p_{i_1, i_2, \ldots, i_r} \log(i_1 \cdot i_2 \ldots i_r)$$

$$\leq \frac{1}{1 - \varepsilon} \sum_{i_1, i_2, \ldots, i_r \in \mathcal{G}} p_{i_1, i_2, \ldots, i_r} \log(i_1 \cdot i_2 \ldots i_r) = \frac{C}{1 - \varepsilon}.$$
This completes the proof.

We now use Lemma 4.2 to prove the following for the state $\omega_{RBCA}$. Recall that $e_d$ is the smallest eigenvalue of $\psi_{R'}^a$, independent of $a$.

**Lemma 4.3.** It holds that

$$P(\omega_{RBCA} \otimes \theta_{E_A E_B}, U^\dagger U_2 \ldots U_{r+1}^\dagger \sum_{i_1, i_2, \ldots i_r} \sqrt{p'_{i_1, i_2, \ldots i_r}} \omega_{RC0} \otimes \kappa_{CEAT_B}^{i_r, \ldots i_1} |i_r\rangle_{M_r} \ldots |i_1\rangle_{M_1}) \leq \sqrt{\frac{8\varepsilon}{e_d \cdot d}}.$$  

*Communication weight of distribution $p'_{i_1, i_2, \ldots i_r}$ is $\frac{C}{1 - \varepsilon}$.***

**Proof.** Define a completely positive map $\tilde{\psi} : R \rightarrow R$ as $\tilde{\psi}(\rho) = \frac{e_d}{d} (\Psi_{R'}^{-\frac{1}{2}} \rho \Psi_{R'}^{-\frac{1}{2}})$, which is trace non-increasing since $\Psi_{R'}^{-1} \leq \frac{d}{e_d} I_R$. Using equation 5, observe that

$$\tilde{\psi}(\Psi_{RBCA}) = e_d \cdot d \cdot \omega_{RBCA}.$$  

Consider,

$$2\sqrt{\varepsilon} \geq P(\Psi_{RBCA} \otimes \theta_{E_A E_B}, U^\dagger U_2 \ldots U_{r+1}^\dagger \sum_{i_1, i_2, \ldots i_r} \sqrt{p'_{i_1, i_2, \ldots i_r}} \Psi_{RC0} \otimes \kappa_{CEAT_B}^{i_r, \ldots i_1} |i_r\rangle_{M_r} \ldots |i_1\rangle_{M_1})$$

(Lemma 4.2)

$$\geq P(\tilde{\psi}(\Psi_{RBCA}) \otimes \theta_{E_A E_B}, U^\dagger U_2 \ldots U_{r+1}^\dagger \sum_{i_1, i_2, \ldots i_r} \sqrt{p'_{i_1, i_2, \ldots i_r}} \tilde{\psi}(\Psi_{RC0}) \otimes \kappa_{CEAT_B}^{i_r, \ldots i_1} |i_r\rangle_{M_r} \ldots |i_1\rangle_{M_1})$$

(Fact 2.5)

$$= P(d \cdot e_d \cdot \omega_{RBCA} \otimes \theta_{E_A E_B}, U^\dagger U_2 \ldots U_{r+1}^\dagger \sum_{i_1, i_2, \ldots i_r} \sqrt{p'_{i_1, i_2, \ldots i_r}} \omega_{RC0} \otimes \kappa_{CEAT_B}^{i_r, \ldots i_1} |i_r\rangle_{M_r} \ldots |i_1\rangle_{M_1})$$

Using Fact 2.7, we thus obtain

$$P(\omega_{RBCA} \otimes \theta_{E_A E_B}, U^\dagger U_2 \ldots U_{r+1}^\dagger \sum_{i_1, i_2, \ldots i_r} \sqrt{p'_{i_1, i_2, \ldots i_r}} \omega_{RC0} \otimes \kappa_{CEAT_B}^{i_r, \ldots i_1} |i_r\rangle_{M_r} \ldots |i_1\rangle_{M_1}) \leq \sqrt{\frac{8\varepsilon}{d \cdot e_d}}.$$  

Furthermore, there is no change in communication weight. This completes the proof.

**Corollary 4.4.** It holds that

$$P(\omega'_{RC} \otimes \theta_{E_A E_B}, U^\dagger U_2 \ldots U_{r+1}^\dagger \sum_{i_1, i_2, \ldots i_r} \sqrt{p'_{i_1, i_2, \ldots i_r}} \omega'_{RC0} \otimes \kappa_{CEAT_B}^{i_r, \ldots i_1} |i_r\rangle_{M_r} \ldots |i_1\rangle_{M_1}) \leq \sqrt{\frac{8\varepsilon}{e_d \cdot d}}.$$  

*Communication weight of distribution $p'_{i_1, i_2, \ldots i_r}$ is $\frac{C}{1 - \varepsilon}$.***

Now we exhibit an interactive entanglement assisted communication protocol for state-redistribution of $\omega_{RBCA}$ with suitably upper bounded worst case communication cost.
Lemma 4.5. Fix an error parameter $\mu > 0$. There exists an entanglement assisted $r$-round quantum communication protocol for state redistribution of $\omega_{RBCA}$ with worst case quantum communication cost at most $\frac{2C}{\mu(1-\varepsilon)}$ and error at most $\sqrt{\frac{8\varepsilon}{ed \cdot d}} + \sqrt{\mu}$.

Proof. From lemma 4.3 we have that

$$\Pr(\omega_{RBCA} \otimes \theta_{EAEB}, U_1^T U_2^T \ldots U_r^T \sum_{i_1,i_2 \ldots i_r} \sqrt{p'_{i_1,i_2 \ldots i_r} \omega_{RBCA} \otimes \kappa_{CEA TB}^{i_r,i_{r-1} \ldots i_1} | i_r \rangle_M \ldots | i_1 \rangle_M}) \leq \sqrt{\frac{8\varepsilon}{ed \cdot d}},$$

and

$$\sum_{i_1,i_2 \ldots i_r} p'_{i_1,i_2 \ldots i_r} \log(i_1 \cdot i_2 \ldots i_r) \leq \frac{C}{1-\varepsilon}.$$

Consider the set of tuples $(i_1,i_2 \ldots i_r)$ which satisfy $i_1 \cdot i_2 \ldots i_r > \frac{C}{(1-\varepsilon)\mu}$ Let this set be $B'$ and $G'$ be the set of rest of the tuples. Then

$$\frac{C}{(1-\varepsilon)} > \sum_{i_1,i_2 \ldots i_r \in B'} p_{i_1,i_2 \ldots i_r} \log(i_1 \cdot i_2 \ldots i_r) > \frac{C}{(1-\varepsilon)\mu} \sum_{i_1,i_2 \ldots i_r \in B'} p_{i_1,i_2 \ldots i_r}.$$

This implies $\sum_{i_1,i_2 \ldots i_r \in B'} p_{i_1,i_2 \ldots i_r} < \mu$. Define a new probability distribution $q_{i_1,i_2 \ldots i_r} \overset{\text{def}}{=} \frac{p_{i_1,i_2 \ldots i_r}}{\sum_{i_1,i_2 \ldots i_r \in B'} p_{i_1,i_2 \ldots i_r}}$ for all $(i_1,i_2 \ldots i_r) \in G'$ and $q_{i_1,i_2 \ldots i_r} = 0$ for all $(i_1,i_2 \ldots i_r) \in B'$. Consider,

$$\Pr(\sum_{i_1,i_2 \ldots i_r} \sqrt{p'_{i_1,i_2 \ldots i_r} \omega_{RBCA} \otimes \kappa_{CEA TB}^{i_r,i_{r-1} \ldots i_1} | i_r \rangle_M \ldots | i_1 \rangle_M}, \sum_{i_1,i_2 \ldots i_r} \sqrt{q_{i_1,i_2 \ldots i_r} \omega_{RBCA} \otimes \kappa_{CEA TB}^{i_r,i_{r-1} \ldots i_1} | i_r \rangle_M \ldots | i_1 \rangle_M}) \leq \sqrt{\mu}.$$

Thus, triangle inequality for purified distance (Fact 2.2) implies

$$\Pr(\omega_{RBCA} \otimes \theta_{EAEB}, U_1^T U_2^T \ldots U_r^T \sum_{i_1,i_2 \ldots i_r} \sqrt{q_{i_1,i_2 \ldots i_r} \omega_{RBCA} \otimes \kappa_{CEA TB}^{i_r,i_{r-1} \ldots i_1} | i_r \rangle_M \ldots | i_1 \rangle_M}) \leq \sqrt{\frac{8\varepsilon}{ed \cdot d}} + \sqrt{\mu}.$$

Defining $\pi_{RBCAEB} \overset{\text{def}}{=} U_1^T U_2 \ldots U_r^T \sum_{i_1,i_2 \ldots i_r \in G'} \sqrt{q_{i_1,i_2 \ldots i_r} \omega_{RBCA} \otimes \kappa_{CEA TB}^{i_r,i_{r-1} \ldots i_1} | i_r \rangle_M \ldots | i_1 \rangle_M}$, we have

$$\Pr(\omega_{RBCA} \otimes \theta_{EAEB}, \omega'_{RBCAEB}) \leq \sqrt{\frac{8\varepsilon}{ed \cdot d}} + \sqrt{\mu} \quad (7)$$

Let $T$ be the set of all tuples $(i_1,i_2 \ldots i_k)$ (with $k \leq r$) that satisfy the following property: there exists a set of positive integers $\{i_{k+1},i_{k+2} \ldots i_r\}$ such that $(i_1,i_2 \ldots i_k,i_{k+1} \ldots i_r) \in G'$. Consider the following protocol $P'$. 

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Input: A quantum state in registers $RBCAE_AEB$.

- Alice applies the isometry $U : ACE_A \to ACE_A M_1$ (definition 3.4). She introduces a register $M'_1 \equiv M_1$ in the state $|0\rangle_{M'_1}$ and performs the following unitary $W_1 : M_1 M'_1 \to M_1 M'_1$:

$$W_1 |i\rangle_{M_1} |0\rangle_{M'_1} = |i\rangle_{M_1} |i\rangle_{M'_1}$$

if $(i) \in T$ and $$W_1 |i\rangle_{M_1} |0\rangle_{M'_1} = |i\rangle_{M_1} |0\rangle_{M'_1}$$

if $(i) \notin T$.

She sends $M'_1$ to Bob.

- Bob introduces a register $M'_2 \equiv M_2$ in the state $|0\rangle_{M'_2}$. If he receives $|0\rangle_{M'_1}$ from Alice, he performs no operation. Else he applies the isometry $U_2 : BEM'_1 \to BEM'_1 M_2$ and then performs the following unitary $W_2 : M'_1 M_2 M'_2 \to M'_1 M_2 M'_2$:

$$W_1 |i\rangle_{M'_1} |j\rangle_{M_2} |0\rangle_{M'_2} = |i\rangle_{M'_1} |j\rangle_{M_2} |j\rangle_{M'_2}$$

if $(i, j) \in T$

and

$$W_1 |i\rangle_{M'_1} |j\rangle_{M_2} |0\rangle_{M'_2} = |i\rangle_{M'_1} |j\rangle_{M_2} |0\rangle_{M'_2}$$

if $(i, j) \notin T$.

He sends $M'_2$ to Alice.

- For every odd round $k > 1$, Alice introduces a register $M'_k \equiv M_k$ in the state $|0\rangle_{M'_k}$. If she receives $|0\rangle_{M'_{k-1}}$ from Bob, she performs no further operation. Else, she applies the isometry

$$U_k : ACE_A M_1 M'_2 M'_3 \ldots M'_{k-1} \to ACE_A M_1 M'_2 M'_3 \ldots M'_{k-1} M_k$$

and performs the following unitary $W_k : M_1 M'_2 \ldots M'_{k-1} M_k M'_k \to M_1 M'_2 \ldots M'_{k-1} M_k M'_k$:

$$W_k |i_1\rangle_{M_1} |i_2\rangle_{M'_2} \ldots |i_k\rangle_{M_k} |0\rangle_{M'_k} = |i_1\rangle_{M_1} |i_2\rangle_{M'_2} \ldots |i_k\rangle_{M_k} |i_k\rangle_{M'_k}$$

if $(i_1, i_2 \ldots i_k) \in T$

and

$$W_k |i_1\rangle_{M_1} |i_2\rangle_{M'_2} \ldots |i_k\rangle_{M_k} |0\rangle_{M'_k} = |i_1\rangle_{M_1} |i_2\rangle_{M'_2} \ldots |i_k\rangle_{M_k} |0\rangle_{M'_k}$$

if $(i_1, i_2 \ldots i_k) \notin T$.

She sends $M'_k$ to Bob.

- For every even round $k > 2$, Bob introduces a register $M'_k \equiv M_k$ in the state $|0\rangle_{M'_k}$. If he receives $|0\rangle_{M'_{k-1}}$ from Alice, he performs no further operation. Else, he applies the isometry $U_k : BEM'_1 M'_2 M'_3 \ldots M'_{k-1} \to BEM'_1 M'_2 M'_3 \ldots M'_{k-1} M_k$ and performs the following unitary $W_k : M'_1 M_2 \ldots M'_{k-1} M_k M'_k \to M'_1 M_2 \ldots M'_{k-1} M_k M'_k$:

$$W_k |i_1\rangle_{M'_1} |i_2\rangle_{M_2} \ldots |i_k\rangle_{M_k} |0\rangle_{M'_k} = |i_1\rangle_{M'_1} |i_2\rangle_{M_2} \ldots |i_k\rangle_{M_k} |i_k\rangle_{M'_k}$$

if $(i_1, i_2 \ldots i_k) \in T$
and
\[ W_k |i_1\rangle_{M'_1} |i_2\rangle_{M_2} \cdots |i_k\rangle_{M_k} |0\rangle_{M'_k} = |i_1\rangle_{M'_1} |i_2\rangle_{M_2} \cdots |i_k\rangle_{M_k} |0\rangle_{M'_k} \quad \text{if } (i_1, i_2 \ldots i_k) \notin T. \]

He sends \( M'_k \) to Alice.

- After round \( r \), if Bob receives \( |0\rangle_{M'_r} \) from Alice, he performs no further operation. Else he applies the unitary \( U_{r+1}^b : BE_M |M'_1 M'_2 M'_3 \cdots M'_r \rightarrow BC_M |M'_1 M'_2 M'_3 \cdots M'_r \). Alice applies the unitary \( U_{r+1}^a : ACE_A |M_1 M'_2 M_3 \cdots M_r \rightarrow ACE_A |M'_1 M'_2 M'_3 \cdots M'_r \). They trace out all of their registers except \( A, B, C \).

Let \( E : RBCAE_B \rightarrow RBC_0A \) be the quantum map generated by \( P' \). For any \( k \), if any of the parties receive the state \( |0\rangle_{M'_k} \), let this event be called \( \text{abort} \).

We show the following claim.

**Claim 4.6.** It holds that \( E(\pi_{RBCAE_B}) = \omega_{RBC_0A} \)

**Proof.** We argue that the protocol never aborts when acting on \( \pi_{RBCAE_B} \). Consider the first round of the protocol. Define the projector \( \Pi \defeq \sum_{i : (i \notin T)} |i\rangle_{M_1} \). From definition 3.4, it is clear that the isometry \( U^t_2U^t_3 \cdots U^t_{r+1} \) is of the form \( \sum_i |i\rangle_{M_1} \otimes V_i \) for some set of isometries \( \{V_i\} \). Thus, from the definition of \( \pi_{RBCAE_B} \) (in which the summation is only over the tuples \( (i_1, i_2 \ldots i_r) \in G' \)), it holds that
\[ \Pi U_{r+1} \pi_{RBCAE_B} = 0. \]
This implies that Bob does not receive the state \( |0\rangle_{M'_1} \) and hence he does not aborts.

Same argument applies to other rounds, which implies that the protocol never aborts. Thus, the state at the end of the protocol is
\[ \text{Tr}_{CEATB}(U_{r+1}U_r \cdots U_2 U \pi_{RBCAE_B}) = \omega_{RBC_0A}. \]

Thus, from equation 7 it holds that
\[ P(E(\omega_{RBCA} \otimes \theta_{E_B}), \omega_{RBC_0A}) \leq \sqrt{\frac{8\epsilon}{\epsilon_d \cdot d}} + \sqrt{\mu}. \]

Quantum communication cost of the protocol is at most
\[ \max_{(i_1, i_2 \ldots i_r) \in G'} (\log((i_1 + 1) \cdot (i_2 + 1) \cdots (i_r + 1))) \leq 2 \cdot \max_{(i_1, i_2 \ldots i_r) \in G'} (\log(i_1 \cdot i_2 \ldots i_r)) \leq \frac{2C}{(1 - \epsilon)\mu}. \]

This completes the proof.

Similarly, we have the corollary for quantum state transfer.
Corollary 4.7. Fix an error parameter $\mu > 0$. There exists a $r$-round communication protocol for state transfer of $\omega'_{RC}$ with worst case quantum communication cost at most $\frac{2C}{\mu(1-\varepsilon)}$ and error at most $\sqrt{\frac{8\varepsilon}{\mu d^2}} + \sqrt{\mu}$.

Next two lemmas obtain lower bound on worst case quantum communication cost of quantum state redistribution of $\omega'_{RBCA}$ and quantum state transfer of $\omega'_{RC}$.

Lemma 4.8. Let $d$, the local dimension of register $B$, be such that $d > 2^{18}$. Then worst case quantum communication cost of any interactive entanglement assisted quantum state redistribution protocol of the state $\omega'_{RBCA}$, with error $\delta < \frac{1}{6}$, is at least $\frac{1}{6} \log(d)$.

Proof. Following lower bound on worst case quantum communication cost for interactive quantum state redistribution of the state $\omega'_{RBCA}$, with error $\delta$, has been shown ([BCT14], Section 5, Proposition 2):

$$\frac{1}{2} (I^\delta_{\max}(R : BC)_{\omega} - I_{\max}(R : B)_{\omega}).$$

Recall, from definition 4.1, that $\omega_{RBC} = \frac{1}{d_a} \sum_{a=1}^{d_a} |a\rangle|a\rangle_{R} \otimes \omega^0_{R'} BC$ is a classical-quantum state. Consider,

$$I^\delta_{\max}(R : BC)_{\omega} \geq \inf_{\rho_{RBC} \in B^{\delta}(\omega_{RBC})} I(R : BC)_{\rho}$$

$$\geq \inf_{\rho_R \in B^{\delta}(\omega_R)} S(\rho_R) + \inf_{\rho_{RBC} \in B^{\delta}(\omega_{RBC})} S(\rho'_{BC}) - \sup_{\rho_{RBC} \in B^{\delta}(\omega_{RBC})} S(\rho_{RBC})$$

$$\geq I(R : BC)_{\omega} - 3\delta \log(d) - 3 \quad (\text{Fact 2.8})$$

$$\geq \frac{1}{d_a} \sum_a I(R' : BC)_{\omega_a} - 3\delta \log(d) - 3 \quad (\text{Fact 2.13})$$

$$= 2\log(d) - 3\delta \log(d) - 3.$$

To bound $I_{\max}(R : B)_{\omega}$, notice that $\omega_{RB} = \frac{1}{d_a} \sum_{a=1}^{d_a} \sum_{j=1}^{d} |a\rangle|a\rangle_{R} \otimes |u_j\rangle |u_j\rangle_{R} \otimes |v_j(a)\rangle |v_j(a)\rangle_{B}$ is also a classical-quantum state. Using Fact 2.13 we obtain $I_{\max}(R : B)_{\omega} \leq \log(|B|) = \log(d)$.

Thus, communication cost is lower bounded by

$$\frac{1}{2} (I^\delta_{\max}(R : BC)_{\omega} - I_{\max}(R : B)_{\omega}) \geq \frac{\log(d) - 3\delta \log(d) - 3}{2} = \frac{1 - 3\delta}{2} \log(d) - 1.5 > \frac{1}{6} \log(d),$$

for $d > 2^{18}$. \hfill \qed

For quantum state transfer, we have following bound.

Lemma 4.9. Worst case quantum communication cost for state transfer of the state $\omega'_{RC}$, with error $\delta < \frac{1}{2}$, is at least $\frac{1}{2} \log(d) + \frac{1}{2} \log(1 - \delta^2)$.

Proof. The following lower bound on worst case interactive quantum communication cost of state transfer of $\omega'_{RC}$ has been shown ([BCT14], Section 5, Proposition 2):

$$\frac{1}{2} I^\delta_{\max}(R : C)_{\omega'}. $$
Consider,
\[
\begin{align*}
I^\delta_{\max} (R : C) & \geq -H^\delta_{\min} (R|C) \quad \text{(Fact 2.12)} \\
& \geq -H_{\max} (R|C) + \log(1 - \delta^2) \quad \text{(Proposition 6.3, [Tom15])} \\
& = \log(d) + \log(1 - \delta^2)
\end{align*}
\]

Now we proceed to proof of Theorem 1.3.

**Proof: Theorem 1.3** Suppose there exists a \(r\)-round communication protocol \(P\) for entanglement assisted quantum state redistribution of the pure state \(\Psi_{RBCA}\) with error \(\varepsilon\) and expected communication cost at most \(I(R : C|B)_{\Psi} \cdot (\frac{1}{\varepsilon})^p\). Then we show a contradiction for \(p < 1\).

For a \(\beta \geq 1\) to be chosen later, and \(d > 2^{18}\), we choose \(\{e_1, e_2 \ldots e_d\}\) (Definition 4.1) as constructed in lemma 2.15. Thus,
\[
I(R : C|B)_{\Psi} \leq 2S(\Psi_{C}) \leq 4 \frac{\log(d)}{\beta} \quad \text{(Fact 2.11)}.
\]

Fix an error parameter \(\mu\). From lemma 4.5, there exists a communication protocol \(P'\) for quantum state redistribution of \(\omega_{RBCA}\), with error at most \(\sqrt{\mu} + \sqrt{8\beta \varepsilon}\) and worst case quantum communication cost at most
\[
2 \cdot I(R : C|B)_{\Psi} \cdot (\frac{1}{\varepsilon})^p \leq 8 \frac{\log(d)}{\beta \mu (1 - \varepsilon)} \cdot (\frac{1}{\varepsilon})^p \leq 16 \frac{\log(d)}{\beta \mu} \cdot (\frac{1}{\varepsilon})^p.
\]

Last inequality holds since \(\varepsilon < 1/2\). Let \(\beta \mu \varepsilon^p = 128\). Then \(\sqrt{\mu} + \sqrt{8\beta \varepsilon} = \sqrt{128 + \frac{32}{\sqrt{\mu} \varepsilon^{1 - p}}}\), which is minimized at \(\mu = 32 \cdot \varepsilon^{\frac{1-p}{4}}\). This gives \(\sqrt{128 + \frac{32}{\sqrt{\mu} \varepsilon^{1 - p}}} = 8\sqrt{2} \cdot \varepsilon^{\frac{1-p}{4}}\) and \(\beta = 4/\varepsilon^{\frac{1-p}{4}} > 1\).

As in the theorem, let \(\varepsilon \in [0, (\frac{1}{70})^{\frac{1}{1-p}}]\). Thus, we have a protocol for state redistribution of \(\omega_{RBCA}\), with error at most \(8\sqrt{2} \cdot \varepsilon^{\frac{1-p}{4}} < \frac{1}{6}\) and worst case communication at most \(\frac{1}{8} \log(d)\), in contradiction with lemma 4.8.

Above argument does not hold for any \(p \geq 1\) since we need to simultaneously satisfy \(\beta \geq 1\), \(8\beta \varepsilon < 1\) and \(\mu < 1\).

On similar lines, we prove Theorem 1.4 below.

**Proof: Theorem 1.4** Suppose there exists a communication protocol for state transfer of the pure states \(\tilde{\Psi}_{RC}\) with error \(\varepsilon < \frac{1}{2}\) and expected communication cost at most \(S(\tilde{\Psi}_{R}) \cdot (\frac{1}{\varepsilon})^p\). Then we show a contradiction for \(p < 1\).

For a \(\beta \geq 1\) to be chosen later, choose \(a_i\) as constructed in lemma 2.15. Then \(S(\tilde{\Psi}_{R}) \leq 2 \frac{\log(d)}{\beta}\).

Fix an error parameter \(\mu\). From corollary 4.7, there exists a communication protocol for state transfer of \(\omega'_{RC}\), with error at most \(\sqrt{\mu} + \sqrt{8\beta \varepsilon}\) and worst case quantum communication cost at most
\[
2S(\tilde{\Psi}_{R}) \cdot (\frac{1}{\varepsilon})^p \leq 4 \frac{\log(d)}{\beta \mu (1 - \varepsilon)} \cdot (\frac{1}{\varepsilon})^p \leq 8 \frac{\log(d)}{\beta \mu} \cdot (\frac{1}{\varepsilon})^p.
\]
Let $\beta \mu^p = 16$. Then $\sqrt{\mu} + \sqrt{8\beta \varepsilon} = \sqrt{\mu} + \frac{8\sqrt{2}}{\sqrt{\mu}} \varepsilon^{\frac{1-p}{2}}$, which is minimized at $\mu = 8\sqrt{2} \varepsilon^{\frac{1-p}{2}}$. This gives $\sqrt{\mu} + \sqrt{8\beta \varepsilon} = \sqrt{32\sqrt{2}} \varepsilon^{\frac{1-p}{2}}$ and $\beta = \sqrt{2}/\varepsilon^{\frac{1-p}{2}} > 1$.

As in the theorem, let $\varepsilon \in [0, (\frac{1}{2})^{1-p}]$. Thus, we have a protocol for state transfer of $\omega'_R$, with error at most $\sqrt{32} \varepsilon^{\frac{1-p}{2}} < \frac{1}{2}$ and worst case communication at most $\frac{1}{2} \log(d)$, in contradiction with lemma 4.9.

5 Conclusion

We have shown a lower bound on expected communication cost of interactive quantum state redistribution and quantum state transfer. Main technique that we use is to construct an interactive protocol for quantum state redistribution of $\omega_{RBCA}$, using any interactive protocol for quantum state redistribution of the state $\Psi_{RBCA}$. To justify why this seems to be a necessary step, consider the sub-case of quantum state transfer. Suppose there exists a protocol for quantum state transfer of $\Psi_R$ with expected communication cost $S(\Psi_R)$ and error $\varepsilon$. We can use lemma 4.5 to obtain another protocol with error $\varepsilon + \sqrt{\mu}$ and worst case communication cost at most $S(\Psi_R)/\mu$. But this does not lead to any contradiction, since it is straightforward to exhibit a protocol for state transfer of $\Psi_R$ with error $\varepsilon + \sqrt{\mu}$ and worst case communication cost $S(\Psi_R)/(\varepsilon + \sqrt{\mu})^2 < S(\Psi_R)/\mu$.

Furthermore, our argument does not apply to classical setting. This follows from the fact that we are considering a pure state $\Psi_{RBCA}$ and this allows us to obtain lemma 4.3 without changing the probability distribution $p'_{i_1,i_2,...,i_r}$ (and hence the corresponding communication weight), when we apply the map $E$.

Some questions related to our work are as follows.

1. Can the bounds obtained in theorems 1.3 and 1.4 be improved, or shown to be tight?

2. What are some applications of theorems 1.3 and 1.4 in quantum information theory? An immediate application is that we obtain a lower bound on worst case communication cost of quantum state redistribution, since worst case communication cost is always larger than expected communication cost of a protocol.

3. Is it possible to improve the direct sum result for entanglement assisted quantum information complexity obtained in [Tou14]?

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