Improved LCAs for constructing spanners

Rubi Arviv∗   Reut Levi†

Abstract

In this paper we study the problem of constructing spanners in a local manner, specifically in the Local Computation Model proposed by Rubinfeld et al. (ICS 2011).

We provide an LCA for constructing $(2r - 1)$-spanners with $\tilde{O}(n^{1+1/r})$ edges and probe complexity of $\tilde{O}(n^{1-1/r})$ for $r \in \{2, 3\}$, where $n$ denotes the number of vertices in the input graph. Up to polylogarithmic factors, in both cases the stretch factor is optimal (for the respective number of edges). In addition, our probe complexity for $r = 2$, i.e., for constructing 3-spanner is optimal up to polylogarithmic factors. Our result improves over the probe complexity of Parter et al. (ITCS 2019) that is $\tilde{O}(n^{1-1/2r})$ for $r \in \{2, 3\}$.

For general $k \geq 1$, we provide an LCA for constructing $O(k^2)$-spanners with $\tilde{O}(n^{1+1/k})$ edges on expectation whose probe complexity is $O(n^{2/3}\Delta^2)$. This improves over the probe complexity of Parter et al. that is $O(n^{2/3}\Delta^3)$.

∗Efi Arazi School of Computer Science, The Interdisciplinary Center, Israel. Email: rubi.arv@gmail.com.
†Efi Arazi School of Computer Science, The Interdisciplinary Center, Israel. Email: reut.levi1@idc.ac.il.
1 Introduction

A spanner is a sparse structure which is a subgraph of the input graph and preserves, up to a predetermined multiplicative factor, the pairwise distance of vertices. Formally, a $k$-spanner, where $k \geq 1$, of a graph $G = (V, E)$, is a graph $G' = (V, E')$ such that $E' \subseteq E$ in which for any pair of vertices the distance in $G'$ is at most $k$ times longer than the distance in $G$. $k$ is referred to as the stretch of the spanner.

Spanners have numerous applications in a wide variety of fields such as communication networks [3, 22, 23], biology [4] and robotics [9, 14]. Consequently, the problem of constructing spanners has been studied extensively in several models, such as the distributed model [5, 10, 11, 12, 13, 15, 24], streaming algorithms [1, 16] and dynamic algorithms [8, 7].

Recently, Parter et al. [21] considered this problem in the model of local computation algorithms, as defined by Rubinfeld et al. [25] (see also Alon et al. [2] and survey in [18]). The formulation of the problem in this model is as defined next.

Definition 1.1 ([21]). An LCA $A$ for graph spanners is a (randomized) algorithm with the following properties. $A$ has access to the adjacency list oracle $O^G$ of the input graph $G$, a tape of random bits, and local read-write computation memory. When given an input (query) edge $(u, v) \in E$, $A$ accesses $O^G$ by making probes, then returns YES if $(u, v)$ is in the spanner $H$, or returns NO otherwise. This answer must only depend on the query $(u, v)$, the graph $G$, and the random bits. For a fixed tape of random bits, the answers given by $A$ to all possible edge queries, must be consistent with one particular sparse spanner.

1.1 Our Results

We provide LCAs that with high probability constructs the following spanners.

1. A $3$-spanner with $\tilde{O}(n^{1+1/2})$ edges. The probe and time complexity of the algorithm is $\tilde{O}(n^{1/2})$ which is optimal up to polylogarithmic factors. The size-stretch trade-off is optimal as well (up to polylogarithmic factors). This improves over the algorithm of Parter et al. [21] whose probe and time complexity is $\tilde{O}(n^{3/4})$.

2. A $5$-spanner with $\tilde{O}(n^{1+1/3})$ edges (the size-stretch trade-off is optimal up to polylogarithmic factors). The probe and time complexity of the algorithm is $\tilde{O}(n^{2/3})$. This improves over the algorithm of Parter et al. [21] whose probe and time complexity is $\tilde{O}(n^{5/6})$.

3. An $O(k^2)$-spanner with $\tilde{O}(n^{1+1/k})$ edges on expectation. The probe and time complexity of the algorithm is $\tilde{O}(n^{2/3} \Delta^2)$ where $\Delta$ denotes the maximum degree of the input graph. This improves over the algorithm of Parter et al. [21] whose probe and time complexity is $\tilde{O}(n^{2/3} \Delta^4)$.

1.2 Our algorithms and techniques

We next describe our algorithms in high-level. Our LCAs for constructing 3-spanners and 5-spanners share similarities with the LCAs in [21]. The main novelty of our algorithms is in selecting several sets of centers, each designed to cluster different type of vertices as described in Subsection 1.3 and 1.4.

Our algorithm for constructing $O(k^2)$-spanners, which is described in high-level is Subsection 1.5 closely follows the construction in [17] as well as combining ideas from [21]. The main novelty in this algorithm is in the way we partition the Voronoi cells, which are formed with respected to
randomly selected centers, into clusters of smaller size. In addition, we make other adjustments in order to save an additional factor of $\Delta$ in the probe and time complexity.

We remark that in our algorithms we did not try to save on randomness. By applying techniques as in [21] it is possible to show that random bits with bounded independence suffices for our needs (and hence we may use less truly random bits). We leave the details to the full version of this paper.

1.3 Algorithm for constructing 3-spanners

We begin with describing our algorithm for constructing 3-spanners from a global point of view. The local implementation of this global algorithm is relatively straight-forward.

The high level idea is as follows. We consider a partition of the vertices into heavy and light according to their degree. All the edges incident to light vertices are added to the spanner. We now focus on the heavy vertices. As a first step, a random subset of vertices is selected. We refer to these vertices as centers. With high probability, every heavy vertex has a center in its neighborhood. Assuming this event occurs, each heavy vertex joins a cluster of at least one of the centers in its neighborhood. A cluster is composed from a center and a subset of its neighbors. On query $\{u, v\}$, where both $u$ and $v$ are heavy, we consider two cases.

1. $u$ and $v$ belong to the same cluster. In this case we add the edge $\{u, v\}$ to the spanner only in case $u$ is the center of $v$ or vice versa.

2. Otherwise, $u$ and $v$ belong to different clusters. Assume without loss of generality that the degree of $u$ is not greater than the degree of $v$. We divide the edges incident to $u$ into fixed size buckets and add the edge $\{u, v\}$ only if it has minimum rank amongst the edges in its bucket that are incident to both clusters (i.e. the cluster of $u$ and the cluster of $v$).

In order to make the above high level description concrete we need to set some parameters and describe how the centers are selected and how each vertex finds its center. We begin by defining vertices with degree larger than $\sqrt{n}$ as heavy. Thus by adding all the edges incident to light vertices we add at most $O(n^{3/2})$ edges.

The selection of the centers proceeds as follows. We define $t = \Theta(\log \sqrt{n})$ sets of centers, which are picked uniformly at random, $S_1, \ldots, S_t$ such that the size of $S_1$ is $\Theta(\sqrt{n})$ and the size of $S_{i+1}$ is roughly half of the size of $S_i$. Thus, overall the number of centers is $O(\sqrt{n})$.

We next describe how each heavy vertex finds its center. We partition the heavy vertices into $t$ sets, $V_1, \ldots, V_t$ according to their degree. The set $V_1$ contains all the vertices with degree in $[\sqrt{n} + 1, 2\sqrt{n}]$ and in general for every $i \in [t]$, the set $V_i$ contains all the vertices with degree in $[2^{i-1}\sqrt{n} + 1, 2^i\sqrt{n}]$. The centers for vertices in the set $V_i$ are taken from the set $S_i$. With high constant probability, for every $i \in [t]$, each vertex $v \in V_i$ has at least one vertex from $S_i$ in its neighborhood and at most $O(\log n)$. Thus, with high probability, each heavy vertex belongs to at least one cluster and at most $O(\log n)$ clusters.

Given a heavy vertex $v \in V_i$, the centers of $v$ are found by going over all the vertices in $S_i$, $u$, and checking if $\{u, v\}$ is an edge in the graph. Since the total number of centers is $O(\sqrt{n})$, the probe and time complexity of finding the center of a given vertex is $O(\sqrt{n})$.

It remains to set the size of the buckets. Let $\{u, v\} \in E$ be such that $v \in V_i$ and $\deg(u) \leq \deg(v)$. Since $v \in V_i$ it follows that $\deg(v) \leq 2^i\sqrt{n}$. Since $|S_i| \leq c\sqrt{n}\log n/2^i$ for some constant $c$, by setting the size of the buckets to be $\sqrt{n}$ we obtain that the total number of edges between heavy vertices that belong to different clusters is $O(n^{3/2})$, as desired.

From the fact that the size of the buckets is $\sqrt{n}$ it follows that the total probe and time complexity of our algorithms is $O(\sqrt{n})$. From the fact that the diameter of every cluster is 2 we
obtain that for every edge \( \{u, v\} \) which we remove from the graph, there exists a path of length at most 3 between \( u \) and \( v \). Hence, the stretch factor of our spanner is 3, as desired.

### 1.4 Algorithm for constructing 5-spanners

We extend the ideas form the previous section to obtain our algorithm for constructing 5-spanners as follows. We partition the vertices in the graph into three sets: heavy, medium, and light. The set of light vertices is defined to be the set of all vertices with degree at most \( n^{1/3} \) and the set of heavy vertices is defined to be the set of all vertices of degree at least \( n^{2/3} \). The set of the medium vertices is defined to be all vertices which are not light nor heavy.

As before, we add to the spanner all the edges incident to light vertices and cluster all the heavy vertices into cluster of diameter 2. The difference is that now when we partition the heavy vertices into sets according to their degree the first set consists of all vertices with degree in \([n^{2/3} + 1, 2n^{2/3}]\).

We partition the set of medium vertices into two sets according to the following random process. Each medium vertex, \( v \) samples uniformly at random \( \Theta(\log n) \) of its neighbors. If one of the vertices in the sample is heavy then \( v \) joins the cluster of the heavy vertex in the sample that has minimum rank. Otherwise we say that \( v \) is bad. This forms clusters of diameter 4.

In a similar manner to the process described above we define another a new collection of sets of centers for the bad vertices such that the total number of such centers is \( \tilde{O}(n^{2/3}) \) and each bad vertex belongs to at least one cluster and at most \( O(\log n) \) clusters. The new centers are selected (randomly) only from the set of vertices which are not heavy. We call the corresponding clusters light-clusters since they contain at most \( n^{2/3} \) vertices and have diameter 2. Since the total number of light-clusters is \( \tilde{O}(n^{2/3}) \) we can afford to take an edge between every pair of adjacent light-clusters. Moreover, we partition each light-cluster into buckets of size \( n^{1/3} \) and take an edge between every pair of adjacent buckets. Since each bad vertex belongs to \( O(\log n) \) light-clusters, the total number of pairs of buckets is \( \tilde{O}(n^{4/3}) \). The time and probe complexity of finding all the edges incident to two buckets is \( O(n^{2/3}) \), as desired.

To analyse the stretch factor we partition the edges we remove into three types. The first type of edges are edges between vertices in the same cluster. The second type of edges are edges between a vertex \( v \) and a cluster \( C \) which is not light, in which case there at least one edge in the spanner which is incident to both \( v \) and \( C \). The third type of edges are edges which are incident to a pair of light-clusters, in which case there exists at least one edge in the spanner which is incident to each pair of such clusters. Thus, overall the stretch factor is 5, as claimed.

### 1.5 Algorithm for constructing \( O(k^2) \)-spanners

The high-level idea of the algorithm, which we describe from a global point of view, is as follows. The vertices of the graph are first partitioned into \( \tilde{O}(n^{2/3}) \) Voronoi cells which are formed with respect to a randomly selected set of \( \tilde{O}(n^{2/3}) \) centers. Each Voronoi cell is then partitioned into clusters of size \( \tilde{O}(n^{1/3}) \). In addition each Voronoi cell is marked with probability \( 1/n^{1/3} \) which respectively also marks all the clusters of the cell. Each non-marked cluster connects to all the adjacent marked clusters using a single edge. This forms clusters-of-clusters around marked clusters. In addition each non-marked cluster \( A \) is engaged with a single marked cluster which is adjacent to it according to some predetermined rule. The idea is that instead of connecting every pair of adjacent clusters \( A \) and \( B \), which we can not afford, our goal is to connect \( A \) with the cluster-of-clusters of \( C \) where \( C \) is the marked cluster that \( B \) is engaged with. The only problem with this approach is that we can not afford to reconstruct the cluster-of-clusters of \( C \). Instead we are able to find the identity of all the Voronoi cells which are adjacent to \( C \) and try to connect \( A \) with at least one of these cells. We
show that this is indeed the case although $A$ may not be connected directly to anyone of these cells. By applying an inductive argument we show that the number of hops we make, where traversing from one Voronoi cell to another is considered as one hop, to form the connection between $A$ and $B$ is $O(k)$. Since the diameter of each Voronoi cell is $O(k)$ we obtain a stretch factor of $O(k^2)$.

One of the challenges in reducing the dependence on $\Delta$ in the complexity of our algorithm is in being able to partition the Voronoi cells into clusters of size $O(n^{1/3})$ while having only $O(n^{2/3})$ clusters. Our technique refines the approach taken in [17] in which the BFS tree of each Voronoi cell is broken into clusters that consists of singletons, which correspond to vertices for which the respective subtree contains too many vertices, and the remaining small subtrees. As in [21] our idea is to join some of these small subtrees into one cluster, which saves a factor of $\Delta$ in the total number of clusters. However, as opposed to [21] our technique allows us to do so without increasing the dependence on $\Delta$ in the probe complexity.

1.6 Related work

As mentioned above, the work which is the most closely related to our work is by Parter et al. [21]. In addition to the upper bounds mentioned in Section 1.1 they also provide an LCA for constructing 5-spanners with $O(n^{1+1/k})$ edges and probe complexity $O(n^{1-1/(2k)})$ for the special case in which the minimum degree is known to be at least $n^{1/2-1/(2k)}$.

On the negative side, they provide a lower bound of $\Omega(\min\{\sqrt{n}, n^2/m\})$ probes for the simpler task of constructing a spanning graph with $o(m)$ edges, where $m$ denotes the number of edges in the input graph.

Our work also builds on the upper bound in [17], designed originally for bounded degree graphs, which provide a spanner with $(1 + \epsilon)n$ edges on expectation, where $\epsilon$ is a parameter, stretch factor $O(\log^2 n \cdot \text{poly}(\Delta/\epsilon))$ and probe complexity of $O(\text{poly}(\Delta/\epsilon) \cdot n^{2/3})$. The work in [17] is a follow-up of [19, 20] which initiated the study of LCAs for constructing ultra-sparse (namely, with $(1 + \epsilon)n$ edges) spanning subgraphs.

2 Preliminaries

A local algorithm has access to the adjacency list oracle $O^G$ which provides answers to the following probes (in a single step):

- **Degree probe:** Given $v \in V$, returns the degree of $v$, denoted by $\deg(v)$.
- **Neighbour probe:** Given $v \in V$ and an index $i$, returns the $i$-th neighbor of $v$ if $i \leq \deg(v)$. Otherwise, $\perp$ is returned.
- **Adjacency probe:** Given an ordered pair $(u, v)$ where $u \in V$ and $v \in V$, if $v$ is a neighbor of $u$ then $i$ is returned where $v$ is the $i$-th neighbor of $u$. Otherwise, $\perp$ is returned.

The number of vertices in the graph is $n$ and we assume that each vertex $v$ has an id, $id(v)$, where there is a full order over the ids.

Let $G = (V, E)$ be a graph. We denote the distance between two vertices $u$ and $v$ in $G$ by $d(u, v)$ and the set of neighbours of $v$ in $G$ by $N_G(v)$. We denote by $N_G(v)[i]$ the $i$-th neighbour of $v$ in $G$. For vertex $v \in V$ and an integer $k$, let $\Gamma_k(v, G)$ denote the set of vertices at distance at most $k$. When the graph $G$ is clear from the context, we shall use the shorthand $d(u, v)$, $N_G(v)$ and $\Gamma_k(v)$ for $d_G(u, v)$, $N(v)$ and $\Gamma_k(v, G)$, respectively. We define a ranking $r$ of the edges as follows:
If $r(u, v) < r(u', v')$ if and only if $\min\{id(u), id(v)\} < \min\{id(u'), id(v')\}$ or $\min\{id(u), id(v)\} = \min\{id(u'), id(v')\}$ and $\max\{id(u), id(v)\} < \max\{id(u'), id(v')\}$.

We shall use the following definitions in our algorithms for constructing 3-spanners and 5-spanners.

**Definition 2.1.** We say that a vertex $v \in V$ is in class $i$ w.r.t. $\Delta$ if $\deg(v) \in [2^{i-1}\Delta + 1, 2^i\Delta]$.

**Definition 2.2.** We say that an index $i \in \mathbb{N}$ is in bucket $j \in \mathbb{N}$ w.r.t. $\Delta$ if $i \in [(j-1)\cdot \Delta + 1, j\cdot \Delta]$.

## 3 LCA for constructing 3-spanners

In this section we prove the following theorem.

**Theorem 3.1.** There exists an LCA that given access to an $n$-vertex simple undirected graph $G$, constructs a 3-spanner of $G$ with $\tilde{O}(n^{1+1/2})$ edges whose probe complexity and time complexity are $\tilde{O}(n^{1/2})$.

The selection of the centers proceeds as follows. We define $t \overset{\text{def}}{=} \log \sqrt{n}$ sets of centers $S_1, \ldots, S_t$. For every $i \in [t]$, we pick u.a.r. $x_i$ vertices to be in $S_i$ where $x_1 = \sqrt{n}\log n$ and $x_{i+1} = x_i/2$ for every $i \in [t-1]$.

The next claim states that With high probability every heavy vertex has at least one center and $O(\log n)$ centers in its neighborhood.

**Claim 3.2.** With high probability, for every $i \in [t]$ and every vertex $v \in V$ that is in class $i$ w.r.t. $\sqrt{n}$ it holds that $N(v) \cap S_i \neq \emptyset$ and that $|N(v) \cap S_i| = O(\log n)$.

### Algorithm 1 LCA for constructing 3-spanners

**Input:** Access to an undirected graph $G = (V, E)$ and a query $\{u, v\} \in E$ where we assume w.l.o.g. that $\deg(u) \geq \deg(v)$.

**Output:** Returns whether $\{u, v\}$ belongs to the spanner or not.

1. If $\deg(v) \leq n^{1/2}$ then return YES (recall that $\deg(u) \geq \deg(v)$).
2. Otherwise, let $c$ denote the class of $u$ w.r.t. $\sqrt{n}$ (see Definition 2.1).
3. If $v \in S_c$ then return YES.
4. Otherwise, let $C \overset{\text{def}}{=} S_c \cap N(u)$. If $C = \emptyset$ then return YES.
5. Let $i$ denote the index of $u$ in $N(v)$ and let $b$ denote the bucket of $i$ w.r.t. $\sqrt{n}$ (see Definition 2.2).
6. For each $x \in C$:
   
   (a) Go over every $j < i$ such that $j$ is in bucket $b$ and return YES if for every such $j$, $N(v)[j]$ does not belong to the cluster of $x$.
7. Return NO.
Claim 3.3. With high probability, the stretch factor of the spanner constructed by Algorithm 1 is 3.

Proof. Let \( \{u, v\} \) be an edge in \( E \) such that \( \deg(u) \geq \deg(v) \). We will show that there exists a path of length at most 3 between \( u \) and \( v \) in the spanner constructed by Algorithm 1, denoted by \( G' = (V, E') \). If \( \deg(v) \leq \sqrt{n} \) then \( \{u, v\} \in E' \) and we are done. Otherwise, if there exists a cluster \( C \) such that \( u \) and \( v \) are both belong to \( C \) then in \( G' \) they are both connected by an edge to the center of \( C \). Thus there exists a path of length at most 2 between \( u \) and \( v \) in \( G' \). Otherwise, let \( C' \) be a cluster for which \( u \) belongs to. We claim that \( v \) is adjacent to \( C' \) in \( G' \). This follows by induction on the index of \( u \) in \( N(v) \) and Sub-Step 6a.

Claim 3.4. The probe and time complexity of Algorithm 1 is \( O(\sqrt{n} \log n) \).

Steps 1-2 can be implemented by accessing the random coins. To implement Step 4 we need to go over all the vertices in \( S_c \) (we may assume w.l.o.g. that we generate all the centers in advance) as there are only \( O(\sqrt{n} \log n) \) centers) and check whether they are in \( N(v) \) (by making a single adjacency probe). Thus the probe (and time) complexity of this step is \( O(\sqrt{n} \log n) \). Step 5 can be implemented by a single adjacency probe. The total number of vertices we check in Sub-Step 6a is bounded by the size of \( C \) times the size of a bucket which is \( \sqrt{n} \). For each vertex we check we make a single neighbor and then we check whether it belongs to the cluster of a specific center. The latter can be implemented by making a single degree probe and a single adjacency probe. By Claim 3.2 the size of \( C \) is bounded by \( O(\log n) \), thus the probe (and time) complexity of Step 6 is \( O(\sqrt{n} \log n) \). The claim follows.

Claim 3.5. With high probability, the number of edges of the spanner constructed by Algorithm 1 is \( O(n^{1+1/2}) \).

Proof. The number of edges added to \( E' \) due to Step 1 is at most \( n^{3/2} \). By the bound on the number of centers, the number of edges added to \( E' \) due to Step 2 is \( O(n^{3/2} \log n) \). To analyse the number of edges added to \( E' \) due to Step 3 consider an edge \( \{u, v\} \) such that \( \deg(u) \geq \deg(v) \), \( \deg(v) \geq \sqrt{n} \) and \( v \notin S_c \), where \( c \) denotes the class of \( u \) w.r.t. \( \sqrt{n} \). Since \( u \) is in class \( c \) it follows that \( \deg(u) \leq 2^{c} \sqrt{n} \). Since \( \deg(v) \leq \deg(u) \) it follows that \( N(v) \) has at most \( 2^{c} \) buckets. By Sub-Step 6a for any cluster \( C' \), the number of edges in \( E' \) that are incident to \( v \) and a vertex from \( C \) is at most \( 2^{c} \) (since we add to \( E' \) at most a single edge for each bucket of \( N(v) \)). Since the number of clusters of class \( c \) is \( O(\sqrt{n} \log n/2^{c}) \), the total number of clusters of class greater or equal to \( c \) is \( O(\sqrt{n} \log n/2^{c}) \) as well. Therefore, the total number of edges that are incident to \( v \) and added to \( E' \) due to Step 4 is \( O(\sqrt{n} \log n) \). We conclude that the \( |E'| = O(n^{2/3} \log n) \), as desired.

4 LCA for constructing 5-spanners

In this section we prove the following theorem.

Theorem 4.1. There exists an LCA that given access to an \( n \)-vertex simple undirected graph \( G \), constructs a 5-spanner of \( G \) with \( \tilde{O}(n^{1+1/3}) \) edges whose probe complexity and time complexity are \( \tilde{O}(n^{2/3}) \).

Our algorithm for constructing 5-spanners also proceeds by forming clusters around centers and connecting the different clusters. For the sake of presentation we first describe our local algorithm.
from a global point of view (see algorithm 2). In Section 4.1 we describe the local implementation of this algorithm.

As in the algorithm for constructing 3-spanners, the clusters are formed around randomly selected centers only that now we have two types of clusters (and centers), heavy-clusters and light-clusters that will be described in the sequel.

The selection of the first type of centers. The selection of the first type of centers proceeds as follows. We define \( a \overset{\text{def}}{=} \log n^{1/3} \) sets of centers \( S_1^1, \ldots, S_a^1 \). For every \( i \in [a] \), we pick u.a.r. \( y_i \) vertices to be in \( S_i^1 \) where \( y_1 = n^{1/3} \log n \) and \( y_{i+1} = y_i/2 \) for every \( i \in [a-1] \). The clusters which are formed around the first type of centers are the heavy-clusters. The formation of the heavy-clusters is described in Step 2 of Algorithm 2.

The selection of the second type of centers. The selection of the second type of centers proceeds as follows. We define \( b \overset{\text{def}}{=} \log n^{2/3} \) sets of centers \( S_1^2, \ldots, S_b^2 \). For every \( i \in [b] \), we pick u.a.r. \( x_i \) vertices to be in \( S_i^2 \) where \( x_1 = n^{2/3} \log n \) and \( x_{i+1} = x_i/2 \) for every \( i \in [b-1] \). The clusters which are formed around the second type of centers are the light-clusters. The formation of the light-clusters is described in Step 3 of Algorithm 2.

The way we connect the different clusters is described in Steps 4 and 5.

In the next couple of claims we prove that with high probability every vertex \( v \) such that \( \deg(v) > n^{1/3} \) joins at least one cluster and at most \( O(\log n) \) heavy-clusters.

To do so, we partition the vertices with degree greater than \( n^{1/3} \) into 3 sets. The first set, denoted by \( H \), is the set of vertices, \( v \), such that \( \deg(v) \geq n^{2/3} \).

The second set is the set of vertices, \( v \), such that \( n^{1/3} < \deg(v) < n^{2/3} \) for which at least half of the vertices in \( N(v) \) have degree at least \( n^{2/3} \). We denote this set by \( M_1 \).

\( M_2 \) consists of the remaining vertices. Namely, \( M_2 \) is the set of vertices, \( v \), such that \( n^{1/3} < \deg(v) < n^{2/3} \) and for which less than half of the vertices in \( N(v) \) have degree at least \( n^{2/3} \).

The implication of the next claim is that w.h.p. every vertex in \( H \) joins at least one heavy-cluster and at most \( O(\log n) \) heavy-clusters.

**Claim 4.2.** With high probability, for every \( v \in H \) it holds that \( N(v) \cap S_1^1 \neq \emptyset \) and that \( |N(v) \cap S_1^1| = O(\log n) \) where \( c \in [a] \) is the class of \( v \) w.r.t. \( n^{2/3} \).

The implication of the next claim (when combined with Claim 4.2) is that w.h.p. every vertex in \( M_1 \) joins, via a representative, at least one heavy-cluster and at most \( O(\log n) \) heavy-clusters.

**Claim 4.3.** With high probability, for every \( v \in M_1 \) it holds that \( v \) has a representative.

**Proof.** Let \( v \in M_1 \). Consider Step 2 of Algorithm 2. Since at least half of the neighbors of \( v \) have degree at least \( n^{2/3} \), it follows that w.h.p. \( R_v \neq \emptyset \) and so \( v \) has a representative.

The implication of the next claim is that w.h.p. every vertex in \( M_2 \) that does not have a representative joins at least one light-cluster and at most \( O(\log n) \) light-clusters.

**Claim 4.4.** With high probability, for every \( v \in M_2 \) it holds that \( N(v) \cap S_1^2 \neq \emptyset \) and that \( |N(v) \cap S_1^2| = O(\log n) \) where \( c \in [b] \) is the class of \( v \) w.r.t. \( n^{1/3} \).

The following corollary follows directly from Claims 4.2, 4.3, and 4.4.

**Corollary 4.4.1.** With high probability every vertex \( v \) such that \( \deg(v) > n^{1/3} \) joins at least one cluster and at most \( O(\log n) \) clusters.
Algorithm 2 Global algorithm for constructing 5-spanners

Input: A graph $G = (V, E)$.

Output: Constructs a 5-spanner of $G$, $G' = (V, E')$.

1. For every $v$ such that $\text{deg}(v) \leq n^{1/3}$ add to $E'$ all the edges that are incident to $v$.

2. Forming heavy-clusters:
   (a) For each vertex $v$ such that $\text{deg}(v) \geq n^{2/3}$ we define the centers of $v$ to be $N(v) \cap S_c^1$ where $c$ is the class of $v$ w.r.t. $n^{2/3}$ (see Definition 2.1). For every center $s$ of $v$, $v$ joins the cluster of $s$ by adding the edge $\{s, v\}$ to $E'$.
   (b) Each vertex $v$ such that $n^{1/3} < \text{deg}(v) < n^{2/3}$ sample u.a.r. $y \overset{\text{def}}{=} \Theta(\log n)$ of its neighbors. Let $R_v$ denote this set. The representative of $v$ is defined to be the vertex, $r$, of minimum id in $R_v$ such that $\text{deg}(r) \geq n^{2/3}$ (if such vertex exists). If $v$ has a representative, $r$, then the edge $\{v, r\}$ is added to $E'$ (and hence $v$ joins all the clusters of $r$).

3. Forming light-clusters:
   (a) For each vertex $v$ such that $n^{1/3} < \text{deg}(v) < n^{2/3}$ for which $v$ does not have a representative we define the centers of $v$ to be $N(v) \cap S_c^2$ where $c$ is the class of $v$ w.r.t. $n^{1/3}$ (see Definition 2.1). For every center $s$ of $v$, $v$ joins the cluster of $s$ by adding the edge $\{s, v\}$ to $E'$.

4. Connecting vertices to adjacent heavy-clusters:
   (a) Let $\{u, v\}$ be such that $\text{deg}(u) \geq \text{deg}(v)$ and $u$ belongs to a heavy-cluster. For each cluster $C$ that $u$ belongs to, do:
      i. Partition the interval $[\text{deg}(v)]$ into sequential intervals, which we refer to as buckets, of size $n^{2/3}$: $b_1, \ldots, b_s$ (where only $b_s$ may have size which is smaller than $n^{2/3}$).
      ii. For each $i \in [s]$, go over every $j \in b_i$ in increasing order and check if $N(v)[j]$ belongs to $C$. If such $j$ is found, add $\{v, N(v)[j]\}$ to $E'$ and move to the next bucket.

5. Connecting adjacent light-clusters:
   (a) Let $\{u, v\}$ be such that both $u$ and $v$ belong to different light-clusters. For each light clusters $C_u$ and $C_v$ that $u$ and $v$ belong to, respectively, do:
      i. Let $s_u$ and $s_v$ denote the centers of $C_u$ and $C_v$, respectively. Let $c_u$ and $c_v$ denote the classes of $u$ and $v$ w.r.t. $n^{1/3}$, respectively.
      ii. Partition the vertices in $N(s_u)$ that belong to the cluster $C_u$ (namely, the neighbors of $s_u$ that are in class $c_u$ w.r.t. $n^{1/3}$) into subsets of size $n^{1/3}$ greedily by their index in $N(s_u)$, $S_u^{t_1}, \ldots, S_u^{t_t}$ (all the subsets are of size $n^{1/3}$ except from perhaps $S_u^{t_t}$).
      iii. Repeat Step 5(a)(ii) for the vertices in $N(s_v)$ that belong to $C_v$ and let $S_v^{t_1}, \ldots, S_v^{t_r}$ denote the resulting subsets.
      iv. For each $t \in [t]$ and $r \in [r]$, add the edge of minimum rank in $E(S_u^{t_t}, S_v^{t_r})$ to $E'$ (if such edge exists).
Claim 4.5. With high probability, $|E'| = \tilde{O}(n^{1+1/3})$.

Proof. The number of edges added to $E'$ due to Step 1 is at most $n^{1+1/3}$. By Claims 4.2 and 4.3 the number of edges add to $E'$ due to Steps 2a and 3a is $\tilde{O}(n)$. Since each vertex has at most one representative the number of edges added to $E'$ due to Step 2b is at most $n$.

Consider $\{u, v\}$ such that $\deg(u) \geq \deg(v)$ and $u$ belongs to a heavy-cluster $C$. According to Step 4(a)i we connect $v$ to $C$ by adding to $E'$ at most $\lceil\deg(v)/n^{2/3}\rceil$ edges (at most one edge for each bucket of $N(v)$).

If $u$ has degree at most $n^{2/3}$ then $\lceil\deg(v)/n^{2/3}\rceil$ and so the total number of edges that are incident to $v$ and added to $E'$ due to Step 4(a)i is bounded by the total number of centers of the first type which is $\tilde{O}(n^{1/3})$.

Otherwise, let $c$ denote the class of $u$ w.r.t. $n^{2/3}$, then by definition $\deg(u) \leq 2^c \cdot n^{2/3}$. Therefore $\deg(v) \leq 2^c \cdot n^{2/3}$. The number of centers in $S^i_k$ is $n^{1/3} \log n/2^c$ and so the total number of centers in $\bigcup_{0 \leq i < a} S^i_k$ is $O(n^{1/3} \log n/2^c)$. Observe that the number of edges which are incident to $v$ and added to $E'$ due to Step 4(a)i is at most $\lceil\deg(v)/n^{2/3}\rceil$ times the number of centers in $\bigcup_{c \leq i \leq a} S^i_k$. Thus the total number of edges that are incident to $v$ and added to $E'$ due to Step 4(a)i is $\tilde{O}(n^{1/3})$ in this case as well. Therefore, the total number of edges which are added to $E'$ in Step 4(a)i is $\tilde{O}(n^{1+1/3})$.

By Claim 4.4 it follows that the total number of subsets partitioning the light clusters is $\tilde{O}(n^{2/3})$ as the size of each subset is $n^{1/3}$ except for at most $\tilde{O}(n^{2/3})$ subsets, and since each vertex may belong to $O(\log n)$ different clusters. Since in Step 5(a)iv we add at most a single edge between a pair of subsets the total number of edges added to $E'$ due to this step is $\tilde{O}(n^{1+1/3})$. This concludes the proof of the claim.

Claim 4.6. With high probability, the stretch factor of the spanner constructed by Algorithm 2 is 5.

Proof. Let $\{u, v\}$ be an edge which is not included in $E'$. By Step 1 of the algorithm it follows that the degree of both $u$ and $v$ is greater than $n^{1/3}$. By Corollary 4.4.1 w.h.p. all vertices with degree greater than $n^{1/3}$ join at least one cluster. In the rest of the proof we condition on the event that both $u$ and $v$ join at least one cluster.

Assume w.l.o.g. that $\deg(u) \geq \deg(v)$. If both $u$ and $v$ belong to the same cluster (either heavy or light) then there exists a path of length at most 4 in $G'$ between $u$ and $v$ as the diameter of each cluster is at most 4.

If $u$ belongs to a heavy cluster, $C$, then by Step 4(a)i of the algorithm it follows that there exists at least one edge in $E'$ which is incident to $v$ and a vertex in $C$. Since the diameter of $C$ is at most 4 it follows that there exists a path in $G'$ from $v$ to $u$.

Otherwise, both $u$ and $v$ belong to different light clusters $C_u$ and $C_v$. By Step 5(a)iv there exists at least one edge in $E'$ which is incident to a vertex in $C_u$ and a vertex in $C_v$. Since the diameter of a light cluster is at most 2 we obtain that there exists a path in $G'$ from $u$ to $v$ of length at most 5. This concludes the proof of the claim.

4.1 The local implementation

In this section we prove the following claim. In the proof of the claim we also describe the local implementation of Algorithm 2.

Claim 4.7. The probe and time complexity of the local implementation of Algorithm 2 is $\tilde{O}(n^{2/3} \log n)$. 

9
First case: \( \deg(u) \geq n^{2/3} \). In this case we find the centers of \( u \) by going over all the centers, \( s \), in \( S_c^1 \) where \( c \) is the class of \( u \) w.r.t. \( n^{2/3} \) and preforming the adjacency probe \( \langle u, s \rangle \). If \( v \) belongs to the set of centers of \( u \) then we return YES. Overall, since the number of centers of of the first type is \( \tilde{O}(n^{1/3}) \), finding the centers of \( u \) requires \( \tilde{O}(n^{1/3}) \) probes and time.

We then find the bucket, \( b \), of \( u \) in \( N(v) \) w.r.t. \( n^{2/3} \) (see Definition 2.2) by preforming the adjacency probe \( \langle v, u \rangle \). Let \( i \) denote the index of \( u \) in \( N(v) \). For each center of \( u, s \) and for each \( j \in b \) such that \( j < i \), we check if \( N(v)[j] \) belongs to the cluster of \( s \). In order to do so we first probe the degree of \( y = N(v)[j] \). If \( \deg(y) \geq n^{2/3} \) then \( v \) is in the cluster of \( s \) if and only if it is a neighbour of \( s \) and is in class \( c \) w.r.t. \( n^{2/3} \) where \( c \) is such that \( s \) belongs to \( S_c^1 \). If \( \deg(y) < n^{2/3} \) then we first find the representative of \( y \) and if it has a representative we check if it belongs to the cluster of \( s \). Since we have to check this for at most \( n^{2/3} \) vertices and for \( O(\log n) \) centers, overall the probe and time complexity of preforming this task is \( \tilde{O}(n^{2/3}) \).

Second case: \( n^{1/3} < \deg(u) < n^{2/3} \) and either \( u \) or \( v \) have a representative. In this case we proceed as in the previous case only that we preform all the checks with respect to the centers of the representative of \( u \) (and/or the representative of \( v \)). Since finding the representative of a vertex requires \( O(\log n) \) probes and time the probe and time complexity in this case is \( \tilde{O}(n^{2/3}) \) as well.

Third case: \( n^{1/3} < \deg(u) < n^{2/3} \) and both \( u \) and \( v \) do not have representatives. This corresponds to the case in which both \( u \) and \( v \) belong to light clusters. In order to find the centers of \( u \) we simply go over all vertices, \( y \), in \( N(u) \) and check if \( y \) is in \( S_c^2 \) where \( c \) denotes the class of \( u \) w.r.t. \( n^{1/3} \). We repeat the same process for \( v \). Since checking if a vertex belongs to \( S_c^2 \) can be done in \( O(\log n) \) time (we can generate all the centers in advance and store them in a binary search tree) this task requires \( \tilde{O}(n^{2/3}) \) probes and time.

Finally, for each pair of centers \( s_u \) and \( s_v \) of \( u \) and \( v \), respectively, we go over all the neighbours of \( s_u \) and \( s_v \) and determine for each one, according to its degree, whether it belongs to the cluster of \( s_u \) and \( s_v \), respectively. We then find the subsets that \( u \) and \( v \) belong to as defined in Steps 5(a)ii and 5(a)iii and return YES if and only of \( \{u, v\} \) is the edge of minimum rank that connects these subsets.

The above three cases cover all possible scenarios which implies that the time (and probe) complexity of the local implementation of Algorithm 2 is \( \tilde{O}(n^{2/3}) \) as claimed. \( \square \)

5 LCA for constructing \( O(k^2) \)-spanners

5.1 The algorithm that works under a promise

We begin by describing a global algorithm for constructing an \( O(k^2) \)-spanner which works under the following promise on the input graph \( G = (V, E) \). Let \( L \overset{\text{def}}{=} cn^{1/3} \log n \), where \( c \) is a constant that will be determined later. For every \( v \in V \), let \( i_v \overset{\text{def}}{=} \min_{r} \{|\Gamma_r(v)| \geq L\} \). We are promised that \( \max_{v \in V} \{i_v\} \leq k \). In words, we assume that the \( k \)-hop neighborhood of every vertex in \( G \) contains at least \( L \) vertices.
In addition, we assume without loss of generality that \( k = O(\log n) \) as already for \( k = \log n \) our construction yields a spanner with \( \tilde{O}(n) \) edges on expectation.

Our algorithm builds on the partition of \( V \) which is described next.

### 5.2 The Underlying Partition

**Centers.** Pick a set \( S \subset |V| \) of \( r \overset{\text{def}}{=} \Theta(n^{2/3} \log n) \) vertices u.a.r. We shall refer to the vertices in \( S \) as centers. For each vertex \( v \in V \), its center, denoted by \( c(v) \), is the center which is closest to \( v \) amongst all centers (break ties between centers according to the id of the center).

**Voronoi cells.** The Voronoi cell of a vertex \( v \), denoted by \( \text{Vor}(v) \), is the set of all vertices \( u \) for which \( c(u) = c(v) \). Additionally, we assign to each cell a random rank, so that there is a uniformly random total order on the cells; note carefully that the rank of a cell thus differs from the rank of its center (which is given by its identifier, which is not assigned randomly). We remark that we can determine the rank of the cell from the shared randomness and the cell’s identifier, for which we simply use the identifier of its center.

#### 5.2.1 Clusters

The Voronoi cells are partition into clusters which are classified into a couple of categories as described next.

**Singleton Clusters.** For each Voronoi cell, consider the BFS tree spanning it, which is rooted at the respective center. For every \( v \in V \), let \( p(v) \) denote the parent of \( v \) in this BFS tree. If \( v \) is a center then \( p(v) = v \). For every \( v \in V \setminus S \), let \( T(v) \) denote the subtree of \( v \) in the above-mentioned BFS tree when we remove the edge \( \{v, p(v)\} \); for \( v \in S \), \( T(v) \) is simply the entire tree. Now consider a Voronoi cell. If the cell contains at most \( L \) vertices, then the cluster of all the vertices in the Voronoi cell is the cell itself. Otherwise, there are two cases. If \( T(v) \) contains more than \( L \) vertices, then we say that \( v \) is heavy and define the cluster of \( v \) to be the singleton \( \{v\} \). Otherwise, we say that \( v \) is light and its cluster is defined as follows.

**Non-singleton clusters.** Observe that if \( v \) is light then it has a unique ancestor \( u \) (including \( v \)) such that \( u \) is not heavy and \( p(u) \) is heavy. We define the cluster of \( v \) to consist of \( T(u) \) and possibly additional subtrees, \( T(u') \), where \( u' \) is a also a child of \( p(u) \) (in \( T(p(u)) \)), as described next.

We begin with some definitions and notations. In order to determine the cluster of \( u \) (which is also the cluster of \( v \)) consider transforming the heavy vertex \( r = p(u) \) into a binary tree which we call the auxiliary tree of \( r \), \( B_r \), as follows. \( B_r \) is rooted at \( r \) and has \( i \) complete layers where \( i \) is such that \( 2^i < \deg(r) \) and \( 2^{i+1} \geq \deg(r) \). These layers consists of auxiliary vertices, namely they do not correspond to vertices in \( G \). We then add another layer to \( B_r \) consisting of the neighbors of \( r \), sorted from left to right according to their index in \( N(r) \). Note that except from the root and the vertices at the last layer of \( B_r \), all vertices in \( B_r \) are auxiliary vertices. This complete the definition of \( B_r \).

For each vertex \( x \in B_r \) we define \( B_r(x) \) to be the subtree of \( B_r \) rooted at \( x \). We define \( S(x) \overset{\text{def}}{=} B_r(x) \cap N(r) \), namely \( S(x) \) is the set of vertices of \( N(r) \) which are in subtree of \( B_r \) rooted at \( x \). The descendants of \( x \), denoted by the set \( D(x) \), are defined to be the union of the vertices in \( T(y) \) for every \( y \in S(x) \), namely \( D(x) \overset{\text{def}}{=} \bigcup_{y \in S(x)} T(y) \). The weight of \( x \) is defined to be the number of vertices in \( D(x) \), namely, \( w(x) \overset{\text{def}}{=} |D(x)| \).
We are now ready to define the cluster of \( u \). Let \( z(u) \) be the unique ancestor of \( u \) in \( B_r \) (including \( r \)), \( z \), for which \( w(z) \leq L \) and \( w(p(z)) > L \) (where \( p(z) \) denotes the parent of \( z \) in \( B_r \)). The cluster of \( u \) (and \( v \)) is defined to be the set \( D(z) \). This completes the description of how the Voronoi cell into clusters.

**Special vertices** In order to bound the number of clusters (see Section 5.4) we shall use the following definitions.

**Definition 5.1** (Special vertex). *We say that a vertex \( u \) is special if \( |T(u)| > L \) and for every child of \( u \) in \( T(u) \), \( t \), it holds that \( |T(t)| \leq L \).*

Analogously we define special auxiliary vertex as follows.

**Definition 5.2** (Special auxiliary vertex). *We say that an auxiliary vertex \( y \) is a special auxiliary vertex if either of the following conditions hold:

1. \( y \) is a parent of a (non auxiliary) vertex \( v \) which is heavy. In this case we say that \( y \) is a type \((a)\) special vertex.
2. \( w(y) > L \) and for every child of \( y \), \( t \), it holds that \( w(t) \leq L \). In this case we say that \( y \) is a type \((b)\) special vertex.*

For a cluster \( C \), let \( c(C) \) denote the center of the vertices in \( C \) (all the vertices in the cluster have the same center). Let \( \text{Vor}(C) \) denote the Voronoi cell of the vertices in \( C \).

This describes a partition of \( V \) into Voronoi cells, and a refinement of this partition into clusters.

### 5.3 The Edge Set

Our spanner, \( G' = (V, E') \), initially contains, for each Voronoi cell, \( \text{Vor} \), the edges of the BFS tree that spans \( \text{Vor} \), i.e., the BFS tree rooted at the center of \( \text{Vor} \) spanning the subgraph induced by \( \text{Vor} \). Clearly, the spanner spans the subgraph induced on every Voronoi cell. Next, we describe which edges we add to \( E' \) in order to connect adjacent clusters of different Voronoi cells.

**Marked Clusters and Clusters-of-Clusters**

Each center is marked independently with probability \( p \overset{\text{def}}{=} 1/n^{1/3} \). If a center is marked, then we say that its Voronoi cell is marked and all the clusters in this cell are marked as well.

**Cluster-of-clusters.** For every marked cluster, \( C \), define the cluster-of-clusters of \( C \), denoted by \( C(C) \), to be the set of clusters which consists of \( C \) and all the clusters which are adjacent to \( C \). Let \( B \) be a non-marked cluster which is adjacent to at least one marked cluster. Let \( Y \) denote the set of all edges such that one endpoint is in \( B \) and the other endpoint belongs to a marked cluster. The cluster \( B \) is engaged with the marked cluster \( C \) which is adjacent to \( B \) and for which the edge of minimum rank in \( A \) has its other endpoint in \( C \).

**The Edges between Clusters**

By saying that we connect two adjacent subsets of vertices \( A \) and \( B \), we mean that we add the minimum ranked edge in \( E(A, B) \) to \( E' \). For a cluster \( A \), define its adjacent centers \( \text{Cen}(\partial A) \overset{\text{def}}{=} \{c(v) \mid u \in A \land \{u, v\} \in E \} \setminus \{c(A)\} \), i.e., the set of centers of Voronoi cells that are adjacent to \( A \). This definition explicitly excludes \( c(A) \), as there is no need to connect \( A \) to its own Voronoi cell.
We next describe how we connect the clusters. The high-level idea is to make sure that for every adjacent clusters \( A \) and \( B \) we connect \( A \) with the cluster engaging \( B \) (perhaps not directly) and vice versa. For clusters which are not adjacent to any marked cluster and hence not engaged with any cluster we make sure to keep them connected to all adjacent Voronoi cells. Formally:

1. We connect every cluster to every adjacent marked cluster.
2. Each cluster \( A \) that is not engaged with any marked cluster (i.e., no cell adjacent to \( A \) is marked) we connect to each adjacent cell.
3. Suppose cluster \( A \) is adjacent to cluster \( B \), where \( B \) is adjacent to a marked cell. Denote by \( C \) the (unique) marked cluster that \( B \) is engaged with. We connect \( A \) with \( B \) if the following conditions hold:
   - the minimum ranked edge in \( E(A, \text{Vor}(B)) \) is also in \( E(A, B) \)
   - \( c(B) \) is amongst the \( n^{1/k} \log n \) lowest ranked centers in \( Cen(\partial A) \cap Cen(\partial C) \)

5.4 Sparsity

Claim 5.3. The number of clusters, denoted by \( s \), is at most \( |S| + (4nk \log \Delta)/L \).

Proof. We first observe that, due to the promise on \( G \), it follows that for every \( v \in V \), the distance between \( v \) and \( c(v) \) is at most \( k \). Recall the terminology from Subsection 5.2.

Consider \( v \) which is heavy and therefore its cluster is the singleton \( \{ v \} \). By an inductive argument, it follows that \( v \) is an ancestor of a special vertex (see Definition 5.1). Since for every pair of special vertices \( u \) and \( w \), \( T(u) \) and \( T(w) \) are vertex disjoint, we obtain that there are at most \( n/L \) special vertices. Since for every special vertex, there are at most \( k \) ancestors, the total number of heavy vertices is bounded by \( nk/L \).

Observe that any cluster either (i) is an entire Voronoi cell (ii) is a singleton \( \{ v \} \) where \( v \) is heavy (iii) is not a singleton and contains a node \( v \) such that \( p(v) \) is heavy. The number of type (i) clusters is bounded by the number of Voronoi cells \( |S| \). We just bounded the number of clusters of type (ii) by \( nk/L \). Thus it remains to bound the number of type (iii) clusters.

Let \( A \) be a type (iii) cluster. Namely, \( A \) is not a singleton and contains a node \( v \) such that \( p(v) \) is heavy. Let \( r = p(v) \). We say that the cluster \( A \) is assigned to \( r \). Since \( r \) is a singleton, some of its children may be singletons and the rest of its children belong to a type (iii) cluster which is assigned to \( r \).

Let \( u \) be a child of \( r \) which belongs to \( A \). As described in Section 5.2.1, the cluster of \( u \) is defined to be \( D(z(u)) \) where \( z(u) \) is an auxiliary vertex and the weight of the parent of \( z(u) \) in \( B_r \), \( p \), is greater than \( L \). By an inductive argument, it follows that \( p \) is an ancestor (in \( B_r \)) of an auxiliary special vertex (see Definition 5.2). Since the depth of \( B_r \) is bounded by \( \log \Delta \) we obtain that the number of vertices in \( B_r \) which are parents (in \( B_r \)) of vertices, \( z \), such that \( D(z) \) is a cluster is at most \( \log \Delta \) times the number of auxiliary special vertices in \( B_r \). Since \( B_r \) is a binary tree it follows that the number of clusters which are assigned to \( r \) is bounded by \( 2 \log \Delta \) times the number of auxiliary special vertices in \( B_r \).

Let \( x \) and \( y \) be auxiliary special vertices of type (b). It follows that \( D(x) \) and \( D(y) \) are disjoint. Thus the number of auxiliary special vertices of type (b) is bounded by \( n/L \). Therefore the total number of auxiliary special vertices is bounded by the number of heavy vertices, which is at most \( nk/L \), plus \( n/L \).

We conclude that the total number of clusters is bounded by \( |S| + (4nk \log \Delta)/L \), as desired.

Claim 5.4. The expected number of edges in \( E' \) is \( O(n^{1+1/k} \cdot k^2 \log^3 n) \).

Proof. The number of edges we add due to the BFS trees of the Voronoi cells is at most \( |V| - 1 \).
The number of edges which are taken due to Condition 1 is at most $s$ times the number of marked clusters, denoted by $m$. The expectation of $m$ is exactly $s \cdot p$. Since $s = O(n^{2/3}k \log n)$ and $p = 1/n^{1/3}$ we obtain that the expected number of edges which are taken due to Condition 1 is bounded by $s^2 p = O(nk^2 \log^2 n)$.

Observe that the probability that cluster $A$ is not adjacent to a marked cell is $(1 - p)^{|\text{Cen}(\partial A)|} \leq e^{-p|\text{Cen}(\partial A)|}$. Hence, if $|\text{Cen}(\partial A)| \geq 3p^{-1} \ln n$, $A$ is w.h.p. adjacent to a marked cell. Using a union bound over all clusters, it follows that with probability at least $1 - 1/n^2$ each cluster $A$ without an adjacent marked cell satisfies that $|\text{Cen}(\partial A)| \leq 3p^{-1} \ln n$; the probability of the event that this bound is violated cannot contribute more than $|E|/n^2 < 1$ to the expectation. Therefore, the expected number of edges which are taken due to Condition 2 is bounded by $(3\ln n)/p + 1 = O(nk \log n)$.

Let $A$ be a cluster. The number of edges which are adjacent to $A$ and are taken due to Condition 3 is bounded by the total number of marked clusters times $n^{1/k} \log n$. Thus, the total number of edges which are taken due to Condition 3 is bounded by $s \cdot m \cdot n^{1/k} \log n$.

To conclude, the expected total number of edges in $E'$ is $O(n^{1+1/k} \cdot k^2 \log^3 n)$, as desired.

### 5.5 Connectivity and Stretch

**Claim 5.5.** $G'$ is connected.

**Proof.** Recall that $G'$ contains a spanning tree on every Voronoi cell, hence it suffices to show that we can connect any pair of Voronoi cells by a path between some of their vertices. Moreover, the facts that $G$ is connected and the Voronoi cells are a partition of $V$ imply that it is sufficient to prove this for any pair of adjacent Voronoi cells. Accordingly, let Vor and Vor$_1$ be two cells such that $E(\text{Vor}, \text{Vor}_1) \neq \emptyset$.

Consider clusters $A \subseteq \text{Vor}$ and $B \subseteq \text{Vor}_1$ such that the edge $e$ of minimum rank in $E(\text{Vor}, \text{Vor}_1)$ is in $E(A, B)$. If $B$ is not adjacent to a marked cell, then Condition 2 implies that $e$ is selected into $H$. Thus, we may assume that $B$ is adjacent to a marked cell Vor$'$ such that there exists a marked cluster $C \subseteq \text{Vor}'$ such that $B$ is engaged with $C$.

If the rank of Vor$_1$ is minimum in $\text{Vor}(\partial C) \cap \text{Vor}(\partial A)$, then $e$ is selected into $G'$ by Condition 3 and we are done. Otherwise, observe that Vor$_1$ is connected to Vor$'$, as the edge of minimum rank in $E(B, C)$ is selected into $G'$ by Condition 1. Therefore, it suffices to show that Vor gets connected to Vor$'$. Let Vor$_2$ be the cell of minimum rank among $\text{Vor}(\partial C) \cap \text{Vor}(\partial A)$. Let $D \subseteq \text{Vor}_2$ be the cluster satisfying that the edge $e'$ of minimum rank in $E(A, \text{Vor}_2)$ is in $E(A, D)$. Note that Vor$_2$ is connected to Vor$'$ (which we saw to be connected to Vor$_1$), as there is some cluster $D' \subseteq \text{Vor}(D)$ that is adjacent to $C$ and selects the edge of minimum rank in $E(D', C)$ by Condition 1.

Overall, we see that it is sufficient to show that Vor gets connected to Vor$_2$, where Vor$_2$ has smaller rank than Vor$_1$. We now repeat the above reasoning inductively. In step $i$, we either succeed in establishing connectivity between Vor and Vor$_i$, or we determine a cell Vor$_{i+1}$ that has smaller rank than Vor$_i$ and is connected to Vor$_i$. As any sequence of Voronoi cells of descending ranks must be finite, the induction halts after finitely many steps. Because the induction invariant is that Vor$_{i+1}$ is connected to Vor$_i$, this establishes connectivity between Vor and Vor$_1$, completing the proof.

**Claim 5.6.** Denote by $G_{\text{Vor}}$ the graph obtained from $G$ by contracting Voronoi cells and by $G'_{\text{Vor}}$ its subgraph obtained when doing the same in $G'$. If the cells’ ranks are uniformly random, w.h.p. $G_{\text{Vor}}$ is a spanner of $G_{\text{Vor}}$ of stretch $O(k)$.  

14
Proof. Recall the proof of Lemma 5.5. We established connectivity by an inductive argument, where each step increased the number of traversed Voronoi cells by two. Hence, it suffices to show that the induction halts after \( O(k) \) steps w.h.p.

To see this, observe first that \( G_{VOR} \) is independent of the ranks assigned to Voronoi cells and pick any pair of adjacent cells \( Vor, Vor_1 \), i.e., neighbors in \( G_{VOR} \). We perform the induction again, assigning ranks from high to low only as needed in each step, according to the following process. In each step, we query the rank of some cells, and given an answer of rank \( r \), the ranks of all cells of rank at least \( r \) are revealed as well. In step \( i \), we begin by querying the rank of \( Vor_i \). Consider the cluster \( D_i \subseteq Vor_1 \) adjacent to \( A \) satisfying that the edge with minimum rank in \( E(Vor_1, A) \) is also in \( E(D_i, A) \). We can assume without loss of generality that \( D_i \) is engaged with a marked cluster \( F_i \) (as otherwise \( D_i \) connects to \( A \) directly and we can terminate the process). If the rank of anyone of \( n^{1/k} \log n \) lowest ranked cells which are adjacent to both \( F_i \) and \( A \) was already revealed, then the process terminates. Otherwise, we query the rank of all the cells which are adjacent to both \( F_i \) and \( A \) whose rank is still unrevealed. We set the cell of the queried cluster that has minimum rank to be \( Vor_{i+1} \) and we continue to the next step.

We claim that, in each step \( i \), either the process terminates, or the rank of \( Vor_{i+1} \) is at most \( 1/n^{1/k} \) of the rank of \( Vor_i \), with high probability. To verify this, observe that in the beginning of step \( i \), any cell center whose rank was not revealed so far has rank which is uniformly distributed in \([r_{i-1}, r_{i}]\), where \( r_{i} \) is the rank of \( Vor_i \). Since there are at least \( n^{1/k} \log n \) such cells, we obtain that with high probability the rank of the minimum ranked cell is at most \( r_{i}/n^{1/k} \), as desired. Hence, with high probability the process terminates after \( O(k) \) steps as \( r_1 \) is bounded by the number of Voronoi cells, which itself is trivially bounded by \( n \). By the union bound over all pairs of cells \( Vor \) and \( Vor_1 \), we get the desired guarantee.

Claim 5.7. W.h.p., \( G' \) is a spanner of \( G \) of stretch \( O(k^2) \).

Proof. Due to the promise on \( G \), w.h.p. the spanning trees on Voronoi cells have depth \( O(k) \). Hence, the claim holds for any edge within a Voronoi cell. Moreover, for an edge connecting different Voronoi cells, by Lemma 5.6 w.h.p. there is a path of length \( O(k) \) in \( G'_{VOR} \) connecting the respective cells. Navigating with at most \( O(k) \) hops in each traversed cell, we obtain a suitable path of length \( O(k^2) \) in \( G' \). \( \square \)

5.6 The algorithm for general graphs

We use a combination of the algorithm in Section 5 with the algorithm by Baswana and Sen [6] which has the following guarantees.

Theorem 5.8 ([6]). There exists a randomized \( k \)-round distributed algorithm for computing a \((2k - 1)\)-spanner \( G' = (V, E') \) with \( O(kn^{1+1/k}) \) edges for an unweighted input graph \( G = (V, E) \). More specifically, for every \( \{u, v\} \in E' \), at the end of the \( k \)-round procedure, at least one of the endpoints \( u \) or \( v \) (but not necessarily both) has chosen to include \( \{u,v\} \) in \( E' \).

We call a vertex \( v \) remote if the \( k \)-hop neighborhood of \( v \) contain less than \( L \) vertices. We denote by \( \bar{R} \) the set of vertices which are not remote.

**First Step.** Run the algorithm from Section 5 on the subgraph induced by \( \bar{R} \), i.e., \( \{u,v\} \in E \) with \( u,v \in \bar{R} \) is added to \( E' \) if and only if the algorithm outputs the edge.
Second Step. Run the algorithm of Baswana and Sen on the subgraph $H = (V, \{\{u, v\} \in E \mid u \in R \text{ or } v \in R\})$, i.e., $\{u, v\} \in E$ with $u \in R$ or $v \in R$ is added to $E'$ if and only if the algorithm outputs the edge.

5.7 Stretch Factor

Consider any edge $e = \{u, v\} \in E \setminus E'$ we removed. If both $u$ and $v$ are in $\bar{R}$, then $e$ was removed by the Algorithm from Section 5.6, which was applied to the subgraph induced by $\bar{R}$. Applying Claim 5.6 to the connected component of $e$, we get that w.h.p. there is a path of length $O(k^2)$ from $u$ to $v$ in $G'$. If $u$ or $v$ are in $R$, by Theorem 5.8 there is a path of length $O(k)$ from $u$ to $v$ in $G'$.

Corollary 5.8.1. The above algorithm guarantees stretch $O(k^2)$ w.h.p. and satisfies that the expected number of edges in $E'$ is $O(n^{1+1/k} \cdot k^2 \log^3 n)$

5.8 The local implementation

In this section we prove the following theorem.

Theorem 5.9. There exists an LCA that given access to an $n$-vertex simple undirected graph $G$, with high probability constructs a $O(k^2)$-spanners with $\tilde{O}(n^{1+1/k})$ edges in expectation whose probe complexity and time complexity are $\tilde{O}(n^{2/3} \Delta^2)$.

Algorithm 3 LCA for constructing $O(k^2)$-spanners

Input: $\{u, v\} \in E$
Output: whether $\{u, v\}$ is in $E'$ or not.

1. If $u$ or $v$ are in $R$, simulate the algorithm of Baswana and Sen at $u$ and $v$ when running it on the connected component of $u$ and $v$ in the subgraph $H$ (see Section 5.6). Return YES if either $u$ or $v$ has chosen to include $\{u, v\}$ and NO otherwise.

2. Otherwise, $u, v \in \bar{R}$ and we proceed according to Section 5.1, where all nodes in $R$ are ignored:
   (a) If $\text{Vor}(u) = \text{Vor}(v)$, return YES if $\{u, v\}$ is in the BFS tree of $\text{Vor}(u)$ and NO otherwise.
   (b) Otherwise, let $Q$ and $W$ denote the clusters of $u$ and $v$, respectively. Return YES if at least one of the following conditions hold for $A = Q$ and $B = W$, or symmetrically, for $A = W$ and $B = Q$, and NO otherwise.
      i. $A$ is a marked cluster and $\{u, v\}$ has minimum rank amongst the edges in $E(A, B)$.
      ii. $A$ is not engaged with any marked cluster. Namely, all the clusters which are adjacent to $A$ are not marked. In this case, we take $\{u, v\}$ if it has minimum rank amongst the edges in $E(A, \text{Vor}(B))$.
      iii. There exists a marked cluster $C$ such that $B$ is engaged with $C$, and the following holds:
         • $\{u, v\}$ has minimum rank amongst the edges in $E(A, \text{Vor}(B))$.
         • The cell $\text{Vor}(B)$ is amongst the $n^{1/k} \log n$ minimum ranked cells in $\text{Cen}(\partial A) \cap \text{Cen}(\partial C)$

\footnote{The algorithm is described for connected graphs; we simply apply it to each connected component of $H$.}
Proof. The local implementation of the algorithm which is described in the previous section is listed in Algorithm 3. The correctness of the algorithm follows from the previous sections. We shall prove that its complexity is as claimed.

The local implementation for remote vertices. For Step 1 we need to determine for both $u$ and $v$ if they are remote. Recall that a vertex $u$ is remote if its $k$-hop neighborhood contains less than $L$ vertices. Therefore, we can decide for any vertex $u$ whether it is in $R$ with probe and time complexity $O(L\Delta)$. If either $u$ or $v$ are remote then we need to determine for each vertex in their $k$-hop neighborhood whether it is remote or not. If either $u$ or $v$ are remote then the $k$-hop neighborhood of each of them contain at most $L\Delta$ vertices. We obtain that the probe and time complexity of this step is $O(L^2\Delta^2)$, in total.

If $u, v \in R$, the algorithm proceeds as in Section 5.1

Finding the center and reconstructing the BFS tree. We first analyse the probe and time complexity of determining the center of a vertex. Given a vertex $v$ we preform a BFS from $v$ layer by layer and stop at the first layer in which we find a center or after exploring at least $L$ vertices. Let $i$ denote the layer in which the execution of the BFS stops. It follows that up to layer $i-1$ we explored strictly less than $L$ vertices. Thus the probe and time complexity of finding the center is $O(L\Delta)$. At the same cost we also determine the path from $c(v)$ to $v$ in the BFS tree rooted at $c(v)$ as follows. The parent of $v$ in the tree is the neighbour of $v$ that has minimum id amongst all neighbour of $v$ that are closer than $v$ to $c(v)$. Similarly, we can determine the parent of the parent of $v$ and so on until we reach $c(v)$.

Determining if a vertex is heavy In order to reconstruct the clusters we need to be able to determine if a vertex is heavy or not. Recall that a vertex $v$ is heavy if $|T(v)| > L$. We explore $T(v)$ be preforming a find center procedure on all the neighbours of $v$ and then continue recursively on all the neighbours of $v$ that belong to $\text{Vor}(v)$. Since finding the center takes $O(L\Delta)$ probes (and time) we conclude that we can determine whether $v$ is heavy or light at probe complexity $O(L^2\Delta^2)$, by partially or completely exploring $T(v)$, as we need to find the center of at most $L\Delta$ vertices.

Reconstructing the clusters Given a vertex $v$ we reconstruct it cluster by first finding its unique ancestor in the BFS tree, $u$ such that $u$ is not heavy but $r = p(u)$ is heavy. This can be done by preforming a binary search on the path from $v$ to its centers and applying the is heavy procedure. We then know that the cluster of $v$ consists of $T(u)$ and perhaps the tree rooted at other children of $p(u)$. To reconstruct the cluster of $u$ (which is also the cluster of $v$) we need to find in $B_r$ the unique ancestor of $u$, $z = z(u)$, such that $w(z) \leq L$ and $w(p(z)) > L$ (where $p(z)$ denotes the parent of $z$ in $B_r$). We can determine if the weight of an auxiliary vertex is greater than $L$ or not similarly to determining if a vertex is heavy or not at probe and time complexity $O(L^2\Delta^2)$. By preforming a binary search on the path from $u$ to $r$ in $B_r$, we can find $z$ at probe and time complexity $O(L^2\Delta^2)$. We then reconstruct $D(z)$ by exploring $T(y)$ for every $y \in S(z)$ which overall takes $O(L^2\Delta^2)$ probes and time.

Determining the cells adjacent to clusters For Step 2 we need to reconstruct the cluster of $u$ the cluster of $v$ and the clusters that $u$ and $v$ are engaged with. In addition, for each one of these clusters, $C$ we need to determine for each vertex $y$ which is adjacent to $C$ its center. Since the size of the clusters is bounded by $L$ the latter task requires $O(L^2\Delta^2)$ probes and time.
We conclude that we can perform all necessary checks to decide whether \( \{u, v\} \in E' \) or not using \( \tilde{O}(L^2 \Delta^2) \) probes and time. \( \square \)

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