Convergence analysis of a symplectic semi-discretization for stochastic NLS equation with quadratic potential

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Abstract. In this paper, we investigate the convergence in probability of a stochastic symplectic scheme for stochastic nonlinear Schrödinger equation with quadratic potential and an additive noise. Theoretical analysis shows that our symplectic semi-discretization is of order one in probability under appropriate regularity conditions for the initial value and noise. Numerical experiments are given to simulate the long time behavior of the discrete average charge and energy as well as the influence of the external potential and noise, and to test the convergence order.

1. Introduction

In this paper, we consider the following stochastic nonlinear Schrödinger equation with quadratic potential and additive noise

\[ iu_t + (\Delta u + \theta |x|^2 u + \lambda |u|^{2\sigma} u)dt = dW, \quad (t, x) \in (0, T] \times \mathcal{D}, \]

\[ u(t, 0) = u(t, 1) = 0, \quad t \in (0, T], \]

\[ u(0, x) = u_0(x), \quad x \in \mathcal{D}, \]

where \( T \in (0, +\infty), \mathcal{D} = (0, 1), \theta \in \mathbb{R}, \lambda \neq 0, 0 < \sigma < 2, \) and \( W \) denotes a Wiener process expressing the random perturbations ([14]). This equation models Bose–Einstein condensations under a magnetic trap when \( \theta < 0, \) where the quadratic potential \( |x|^2 \) describes the magnetic field whose role is to confine the movements of particles ([4]). [11] and [15] establish the well-posedness and blow up of the solution.
for (1.1). The authors in [15] indicate that the additive noise rather than the potential dominates the dynamical behaviors of the solutions.

It is known that numerical approximations have become an important tool to investigate the behaviors of the solutions. In order to guarantee the reliability and effectiveness of numerical solutions for longtime simulations, we expect numerical methods to preserve the intrinsic properties of the original systems as much as possible. For Hamiltonian systems, the symplectic schemes are shown to be superior to conventional ones especially in long time computation, attributed to their preservation of the qualitative property and the symplectic structure of the underlying continuous systems. The main goal of this work is to analyze the convergence rate of the symplectic scheme for (1.1). We propose a mid-point method in temporal direction of (1.1) in order to preserve the properties of the original problems as much as possible and to effectively simulate the influence of the external potential and noise on the long time behavior of the solution. It is shown that the mid-point semi-discretization is a symplectic scheme which preserves the symplectic structure of (1.1). The interested readers are referred to [2] and references therein for the numerical simulation of the deterministic Schrödinger equation with potentials. We also refer to [5] for the convergence analysis of the mid-point method applied to the stochastic Schrödinger equation with Lipschitz coefficients, to [6] for the mean-square convergence of a symplectic local discontinuous Galerkin method to stochastic linear Schrödinger equation with a potential and multiplicative noise and to [1] for the strong convergence rate of stochastic exponential method to the stochastic linear Schrödinger equations with a multiplicative potential.

Furthermore, we give the convergence order of the proposed scheme under non-Lipschitz condition. To this end, the higher regularity of the solution is needed due to the effect of semigroup. We get the stability of solution of (1.1) in $H^4$ by means of the estimates for the high moments of charge and energy. Because the nonlinear term of (1.1) is not global Lipschitz, it is difficult to analyze the convergence order of the symplectic scheme. Here we use the truncated technique to get the truncated equation whose nonlinear term is global Lipschitz. Then we prove that the convergence order is one in probability for the symplectic scheme under appropriate hypothesis on initial value and noise. In addition, we simulate the long time behavior of the discrete average charge and energy under the influence of the external potential and noise using a stochastic multi-symplectic scheme, owing to the multi-symplecticity of (1.1). Here we cite [12] and [13] as a partial list of the publications on the multi-symplectic scheme for the deterministic
and stochastic Schrödinger equations. Numerical experiments present that the noise dominates the dynamics of the solution stronger than external potential.

The rest of the paper is organized as follows. Some properties of the solution, including the evolution law of charge, the uniform boundedness of energy and solution, are given in Section 2. In Section 3, we first show that (1.1) owns the stochastic symplectic structure, then we construct a stochastic symplectic scheme and prove that its temporal order of convergence is one in probability. In Section 4, we perform numerical experiments to test the convergence order in Section 3, and to simulate the long time behavior of the discrete average charge and energy under the influence of the external potential and noise. In the remainder of the article, C is a generic constant whose value may vary in different occurrences, \( C(\cdot) \) denotes the constant depending on some parameters.

2. Stochastic NLS equation with quadratic potential

In order to state precisely Eq. (1.1), we consider the probability space \((\Omega, \mathcal{F}, P)\) endowed with a normal filtration \(\{\mathcal{F}_t\}_{t \geq 0}\). Let \(\beta_k = \beta^1_k + i \beta^2_k\) with \(\beta^1_k\) and \(\beta^2_k\) being its real and imaginary parts, respectively. We assume that \(\{\beta^i_k\}_{k \in \mathbb{N}, i=1,2}\) is a family of real-valued independent identified Brownian motions. Let \(\{e_k\}_{k \in \mathbb{N}}\) be an orthonormal basis of some Hilbert space \(U\). We consider the complex valued Wiener process

\[
W = \sum_{k \in \mathbb{N}} \beta_k(t, \omega) \phi e_k(x), \quad t \in [0,T], \quad x \in D, \quad \omega \in \Omega,
\]

where \(\phi \in L^2(U, H)\) the space of Hilbert–Schmidt operators from \(U\) to another Hilbert space \(H\). The corresponding norm is then given by

\[
\|\phi\|^2_{L^2(U,H)} = tr(\phi^* \phi) = \sum_{k \in \mathbb{N}} \|\phi e_k\|^2_H.
\]

In addition, \(L^2(D)\) denotes Hilbert space with inner product \(\Re \int_D f(x) \overline{g(x)} dx\) for \(f, g \in L^2(D)\). \(L^2(U, H^s)\) is denoted by \(L^2_s\), where \(H^s\) is Sobolev space consisting of functions \(f\) such that \(\nabla^k f\) exist and are square integrable for all \(k \in \{0, 1, 2 \cdots s\}\), \(s\) is positive integer. Throughout the paper, we assume that \(\phi \in L^2_s\) for a certain parameter \(s\), and Sobolev space \(H^s = \{v \in H^s; \Delta^j v = 0\ \text{on}\ \partial D, \ \text{for}\ j \leq s/2\}\) will be used.

Now, we recall the mild solution of Eq. (1.1) from [8].
An $H^1_0$-valued $\{F_t\}_{0 \leq t \leq T}$-adapted process $\{u(t); t \in [0, T]\}$, is called a mild solution of (1.1) if for every $t \in [0, T]$ holds $P$-a.s.

$$u(t) = S(t)u_0 + i\theta \int_0^t S(t-r)|x|^2 u(r)dr + i\lambda \int_0^t S(t-r)|u(r)|^{2\sigma} u(r)dr - i\int_0^t S(t-r)dW(r),$$

where $S(t) = e^{it\Delta}$ denotes the semigroup of solution operator of the deterministic linear differential equation

$$i du + \Delta u dt = 0 \text{ in } D, \ u = 0 \text{ on } \partial D \times (0, T), \ u(0) = u_0 \text{ in } D.$$

If the noise term is eliminated in (1.1), then it is the deterministic nonlinear Schrödinger equation with the quadratic potential

$$i du + (\Delta u + \theta|x|^2 u + \lambda|u|^{2\sigma} u)dt = 0,$$

it possesses the charge conservation law

$$M(u(t)) = \int_D |u(t,x)|^2 dx = \int_D |u(t_0, x)|^2 dx = M(u_0),$$

and energy conservation law

$$H(u(t)) = \frac{1}{2} \int_D |\nabla u(t,x)|^2 dx - \frac{\theta}{2} \int_D |xu(t,x)|^2 dx - \frac{\lambda}{2\sigma + 2} \int_D |u(t,x)|^{2\sigma + 2} dx = H(u_0).$$

But these conservation laws of Eq. (1.1) are invalid. Now, we state its charge evolution law and energy evolution laws, respectively.

**Lemma 2.1.** Eq. (1.1) has the following global charge evolution law a.s.

$$M(u(t)) = M(u_0) - 2\Re \sum_{k \in \mathbb{N}} \int_0^t \int_D u(\phi e_k) dx d\beta_k(s) + \|\phi\|^2_{L^2_2}, \quad t \geq 0.$$  

Moreover, for any $m \in \mathbb{N}$, there exists a constant $C_m$.

$$E \sup_{t \in [0, T]} M^m(u(t)) \leq C_m (E(M^m(u_0)) + 1).$$

**Proof.** The proof is based on the application of Itô formula to functional $M(u) = \int_D |u(s,x)|^2 dx$. Since $M(u)$ is Fréchet derivable, the derivatives of $M(u)$ along directions $\varphi$ and $(\varphi, \psi)$ are as follows:

$$DM(u)(\varphi) = 2\Re \int_D u\varphi dx, \quad D^2 M(u)(\varphi, \psi) = 2\Re \int_D \varphi\psi dx.$$
From Itô formula, we have

\[ M(u) = M(u_0) + \int_0^t D M(u(s)) du + \frac{1}{2} \int_0^t \text{tr}[D^2 M(u)(-i\phi)(-i\phi^*)] ds \]

\[ = M(u_0) + 2 \int_0^t \Re \int_D u(-i) dx dW(s) + \int_0^t \text{tr}(\phi^*) ds \]

\[ = M(u_0) - 23 \int_0^t \int_D u dx dW(s) + t ||\phi||^2_{L_2^0} \]

To prove (2.5), we apply Itô formula to \( M^m(u) \),

\[ M^m(u) = M^m(u_0) - 2m \sum_{k \in \mathbb{N}} \int_0^t M^{m-1}(u) \int_D u \text{e}_k dx d\beta_k(s) + m ||\phi||^2_{L_2} \int_0^t M^{m-1}(u) ds \]

\[ + 2m(m-1) \int_0^t M^{m-2}(u) \Re \sum_{k \in \mathbb{N}} \left( \int_D u \text{e}_k dx \right)^2 ds, \quad t \geq 0. \]

Taking the supremum and using a martingale inequality yields

\[ \mathbb{E} \sup_{t \in [0,T]} M^m(u(t)) \leq \mathbb{E} M^m(u_0) + 6m \mathbb{E} \left[ \left( \int_0^t M^{2m-2}(u) ||\phi^* u||^2_{L_2^0} \right)^{\frac{1}{2}} \right] \]

\[ + 2m(m-1) \mathbb{E} \left[ \int_0^t M^{m-2}(u) ||\phi^* u||^2_{L_2^0} ds \right] + m ||\phi||^2_{L_2} \mathbb{E} \left( \int_0^t M^{m-1}(u) ds \right) \]

\[ \leq \mathbb{E} M^m(u_0) + 6mT ||\phi||^2_{L_2^0} \mathbb{E} \left[ \sup_{t \in [0,T]} M^{m-\frac{1}{2}}(u(t)) \right] \]

\[ + m(2m-1)T ||\phi||^2_{L_2^0} \mathbb{E} \left[ \sup_{t \in [0,T]} M^{m-1}(u(t)) \right]. \]

The lemma is proved using Hölder and Young’s inequalities in the second term of the right hand side and an induction argument. \( \square \)

Taking expectation in both sides of (2.4), we have that

\[ (2.6) \quad \mathbb{E} M(u) = \mathbb{E} M(u_0) + t ||\phi||^2_{L_2^0}, \quad t \geq 0. \]

The formula (2.6) indicates that the average charge is linear growth with respect to time \( t \). Next, we present the energy evolution law of (1.1), which can be proved by Itô formula too.
LEMMA 2.2. Eq. (1.1) has the following global energy evolution law a.s.

\[ H(u(t)) = H(u_0) + \mathcal{G} \int_{t_0}^t (\Delta u + \theta |x|^2 u + \lambda |u|^{2\sigma} u) dW(s) ds + \frac{1}{2} \int_{t_0}^t \sum_{k \in \mathbb{N}} \| \nabla \phi_k \|^2_{L^2} \]

\[ - \frac{\theta}{2} \sum_{k \in \mathbb{N}} \int_{D} |x|^2 |\phi_k|^2 dx - \frac{\lambda}{2} \sum_{k \in \mathbb{N}} \int_{t_0}^t \int_{D} |u|^{2\sigma} |\phi_k|^2 dx ds \]

\[ (2.7) \]

\[ - \sigma \lambda \sum_{k \in \mathbb{N}} \int_{0}^t \int_{D} |u|^{2\sigma - 2} (\Im(\bar{u}(\phi_k)))^2 dx ds, \quad t \geq 0. \]

From the definition of energy in (2.3) and Gagliardo–Nirenberg’s inequality, we can conclude the following result.

LEMMA 2.3. Assume that \( 0 < \sigma < 2 \). There exist constants \( C(\theta) \) and \( C(\sigma, \lambda) \) such that

(i) if \( \lambda < 0 \), then \( \| \nabla u \|^2_{L^2} \leq 2H(u) + C(\theta) \| u \|^2_{L^2} \),

(ii) if \( \lambda > 0 \), then \( \| \nabla u \|^2_{L^2} \leq 4H(u) + C(\sigma, \lambda) \| u \|^{\frac{4 + 2\sigma}{2}}_{L^2} + C(\theta) \| u \|^2_{L^2} \).

Using this lemma, we get the uniform boundedness of \( H(u(t)) \).

LEMMA 2.4. Let \( p \geq 1 \), \( \phi \in L^1_t \), \( u_0 \in H^1_0 \). There exists a constant \( C \equiv C(p, T, \theta, \sigma, \lambda) \) such that

(i) \( \sup_{0 \leq t \leq T} \mathbb{E}(H(u(t)))^p \leq C \),

(ii) \( \mathbb{E} \sup_{0 \leq t \leq T} (H(u(t)))^p \leq C \).

PROOF. We first consider the case of \( p = 1 \). If \( \lambda > 0 \), applying the expectation to (2.7), we have

\[ \mathbb{E}H(u(t)) \leq \mathbb{E}H(u_0) + \frac{1}{2} t \mathbb{E} \sum_{k \in \mathbb{N}} \| \nabla \phi_k \|^2_{L^2} + \frac{|\theta|}{2} t \mathbb{E} \sum_{k \in \mathbb{N}} \int_{D} |x|^2 |\phi_k|^2 dx \]

\[ \leq \mathbb{E}H(u_0) + tC(|\theta|) \| \phi \|^2_{L^1_t}, \]

the assertion (i) holds. If \( \lambda < 0 \),

\[ \mathbb{E}H(u(t)) \leq \mathbb{E}H(u_0) + \frac{1}{2} t \mathbb{E} \sum_{k \in \mathbb{N}} \| \nabla \phi_k \|^2_{L^2} + \frac{|\theta|}{2} t \mathbb{E} \sum_{k \in \mathbb{N}} \int_{D} |x|^2 |\phi_k|^2 dx \]

\[ + |\lambda| \mathbb{E} \sum_{k \in \mathbb{N}} \int_{0}^{t} \int_{D} \left( \frac{1}{2} |u|^{2\sigma} |\phi_k|^2 + \sigma |u|^{2\sigma - 2} (\Im(\Im(\bar{u}(\phi_k))))^2 \right) dx ds \]

\[ \leq \mathbb{E}H(u_0) + tC(|\theta|) \| \phi \|^2_{L^1_t} + |\lambda| \frac{1 + 2\sigma}{2} \mathbb{E} \sum_{k \in \mathbb{N}} \int_{0}^{t} \int_{D} |u|^{2\sigma} |\phi_k|^2 dx ds. \]
Since Hölder inequality, Young’s inequality and Gagliardo–Nirenberg’s inequality, we have

\[
\sum_{k \in \mathbb{N}} \int_D |u|^{2\sigma} |\phi e_k|^2 \, dx \leq \| u \|_{L^{2\sigma+2}}^{2\sigma} \| \phi \|_{L^2(U,L^{2\sigma+2})}^2 \leq \frac{\sigma}{\sigma + 1} \| u \|_{L^{2\sigma+2}}^{2\sigma+2} + \frac{1}{\sigma + 1} \| \phi \|_{L^2(U,L^{2\sigma+2})}^{2\sigma+2}
\]

\[
\leq \frac{\sigma}{2} \| \nabla u \|_{L^2}^2 + C(\sigma) \| u \|_{L^{2\sigma+2}}^{\frac{4+2\sigma}{2\sigma}} + \frac{1}{\sigma + 1} \| \phi \|_{L^2(U,L^{2\sigma+2})}^{2\sigma+2}
\]

\[
\leq \sigma H(u) + C(\theta) \| u \|_{L^2}^2 + C \| u \|_{L^{2\sigma+2}}^{\frac{4+2\sigma}{2\sigma}} + \frac{1}{\sigma + 1} \| \phi \|_{L^2}^{2\sigma+2},
\]

the last inequality follows from Lemma \ref{lem:2.3} and \( H^1 \) is embedded into \( L^{2\sigma+2} \). Therefore,

\[
\mathbf{E} H(u(t)) \leq Ct + C \mathbf{E} \int_0^t H(u(s)) \, ds.
\]

Then we use Gronwall’s inequality to get the assertion (i).

If \( p \geq 2 \), we apply Itô formula to \( (H(u))^p \), then

\[
(H(u))^p = (H(u_0))^p + \mathbf{E} \int_0^t \int_D p(H(u))^{p-1}(\Delta u + \theta |x|^2 u + \lambda |u|^{2\sigma} u) \, dW \, dx
\]

\[
+ \frac{1}{2} \int_0^t \int_D p(p-1)(H(u))^{p-2} \sum_{k \in \mathbb{N}} \left( \mathbf{E} \int_D (\Delta u + \theta |x|^2 u + \lambda |u|^{2\sigma} u) \, d\bar{e}_k \right)^2 \, ds
\]

\[
+ \frac{1}{2} \int_0^t \int_D \left( \sum_{k \in \mathbb{N}} \| \nabla \phi e_k \|_{L^2}^2 - \theta \sum_{k \in \mathbb{N}} \int_D |x|^2 |\phi e_k|^2 \, dx \right) \, ds
\]

\[
- \lambda \sum_{k \in \mathbb{N}} \int_D |u|^{2\sigma} |\phi e_k|^2 \, dx - 2\sigma \lambda \sum_{k \in \mathbb{N}} \int_D |u|^{2\sigma-2} (\mathbf{E} \int_D (\Delta u + \theta |x|^2 u + \lambda |u|^{2\sigma} u) \, d\bar{e}_k)^2 \, dx \right) \, ds.
\]

Since the second term on the right-hand side vanishes after taking expectation, there remains to estimate the third term and the last term. For the third term, there exists a constant \( C \) such that

\[
\sum_{k \in \mathbb{N}} \left( \mathbf{E} \int_D ((\Delta u + \theta |x|^2 u + \lambda |u|^{2\sigma} u) \, d\bar{e}_k) \right)^2 \leq C \| \phi^* (\Delta u + \theta |x|^2 u + \lambda |u|^{2\sigma} u) \|_{L^2}^2.
\]

The operator \( \phi^* \) is bounded from \( H^{-1} \) into \( L^2 \). Furthermore, \( H^1 \) is embedded into \( L^{2\sigma+2} \), \( \phi^* \) is also bounded from \( L^{\frac{2\sigma+2}{\sigma+1}} \) into \( L^2 \) and Young’s...
inequality, so we obtain
\[ \|\phi^*(\Delta u + \theta |x|^2 u + \lambda |u|^{2\sigma} u)\|_{L^2}^2 \leq \|\phi\|_{L^2}^2 (\|\nabla u\|_{L^2} + |\theta| |x|^2 \|u\|_{L^2} + \lambda \|u\|_{L^{2\sigma + 2}})^2 \]
\[ \leq 3 \|\phi\|_{L^2}^2 (\|\nabla u\|_{L^2}^2 + \theta^2 |x|^4 \|u\|_{L^2}^2 + \lambda^2 \|u\|_{L^{2\sigma + 2}}^4) \]

(2.10)
\[ \leq 3 \|\phi\|_{L^2}^4 + \frac{\lambda^2}{2\sigma + 2} \|\phi\|_{L^2}^{4\sigma + 4} + \frac{1}{2} \|\nabla u\|_{L^2}^4 + \frac{\theta^4 |x|^8}{2} \|u\|_{L^2}^4 + \frac{\lambda^2 (2\sigma + 1)}{(2\sigma + 2)^2} \|u\|_{L^{2\sigma + 2}}^4. \]

From Gagliardo–Nirenberg’s inequality and \(0 < \sigma < 2\),
\[ \|u\|_{L^{2\sigma + 2}}^4 \leq C \|\nabla u\|_{L^2}^{2\sigma} \|u\|_{L^{2\sigma + 2}}^{2\sigma + 4} \leq C(\sigma) \|\nabla u\|_{L^2}^4 + \|u\|_{L^{2\sigma + 2}}^{\frac{4\sigma + 8}{2\sigma - \sigma}}. \]

Substituting (2.11) into (2.10), and combining with (2.5) and Lemma 2.3, we have
\[ \sum_{k \in \mathbb{N}} \left( \int_D (\Delta u + \theta |x|^2 u + \lambda |u|^{2\sigma} u) \phi e_k \, dx \right)^2 \]
\[ \leq C(\|\theta\|, \|\lambda\|, \sigma) \left( \|\phi\|_{L^2}^4 + \|\phi\|_{L^2}^{4\sigma + 4} + \|\nabla u\|_{L^2}^4 + \|u_0\|_{L^2}^4 \right) \]
\[ \leq C(\|\theta\|, \|\lambda\|, \sigma) \left( H(u) \right)^2 + \|\phi\|_{L^2}^4 + \|\phi\|_{L^2}^{4\sigma + 4} + \|u_0\|_{L^2}^4 \right) \]

(2.12)
\[ \leq C(\|\theta\|, \|\lambda\|, \sigma) \left( H(u) \right)^2 + \|\phi\|_{L^2}^4 + \|\phi\|_{L^2}^{4\sigma + 4} + \|u_0\|_{L^2}^4 \right) \]

For the last term in (2.8), due to Young’s inequality, Lemma 2.3 and \(H^1\) is embedded into \(L^{2\sigma + 2}\), we get
\[ \sum_{k \in \mathbb{N}} \|\nabla \phi e_k\|_{L^2}^2 - \theta \sum_{k \in \mathbb{N}} \int_D |x|^2 |\phi e_k|^2 \, dx \]
\[ - \lambda \sum_{k \in \mathbb{N}} \int_D |u|^{2\sigma} |\phi e_k|^2 \, dx - 2\sigma \lambda \sum_{k \in \mathbb{N}} \int_D |u|^{2\sigma - 2} (3(\pi(\frac{\phi}{e_k})))^2 \, dx \]
\[ \leq \|\phi\|_{L^2}^2 + C(\|\theta\|) \|\phi\|_{L^2}^2 + \|\phi\|_{L^2}^{2\sigma + 2} + \|u_0\|_{L^2}^2 \leq \|u_0\|_{L^{2\sigma + 2}}^2 \]

(2.13)
\[ \leq C(\|\theta\|, \|\lambda\|, \sigma) \left( H(u) \right)^2 + \|\phi\|_{L^2}^4 + \|\phi\|_{L^2}^{2\sigma + 2} + \|u_0\|_{L^2}^4 \right) \]

Because of (2.12), (2.13), and Hölder inequality, we deduce
\[ \mathbf{E}(H(u(t)))^p \leq Ct + C \mathbf{E} \int_0^t (H(u(s)))^p \, ds, \]
we apply Gronwall’s inequality to obtain the estimate (i).

To show the assertion (ii) for \(p = 1\), we take the supremum over \(t \in [0, T]\) in (3) before taking the expectation. The main difference is
the appearance of the supremum of a stochastic integral compared to assertion (i). This term can be estimated by a martingale inequality,

\[ E \sup_{0 \leq t \leq T} \left( 3 \int_0^T \int_D (\Delta u + \theta |x|^2 u + \lambda |u|^{2\sigma} u) d\bar{W}(s) dx \right) \]

\[ \leq 3E \left[ \left( \int_0^T \| \phi^*(\Delta u + \theta |x|^2 u + \lambda |u|^{2\sigma} u) \|_{L^2}^2 ds \right)^{\frac{1}{2}} \right] \]

\[ \leq C(\lambda, \theta, \| \phi \|_{L^2}) E \left[ \left( \int_0^T (\| \nabla u \|_{L^2}^2 + \| u \|_{L^2}^2 + \| u \|_{L^2}^{4\sigma+2}) ds \right)^{\frac{1}{2}} \right] \]

\[ \leq C(T, \lambda, \theta, \| \phi \|_{L^2}) + \frac{1}{2} E \left( \sup_{0 \leq t \leq T} H(u(t)) \right). \]

The assertion (ii) for \( p \geq 2 \) uses arguments similar to the above estimate, so we skip the details here. \( \square \)

In [8], Theorem 4.6, a uniform boundedness for the energy is used to construct a unique mild solution with continuous \( H^4(\mathbb{R}^d) \)-valued paths for stochastic nonlinear Schrödinger equation. We can follow the same strategy to construct the unique global mild solution with continuous \( H^4(\mathcal{D}) \)-valued paths using Lemma 2.4. Moreover, we can get the same result in \( H^4(\mathcal{D}) \) and have the following uniform boundedness of the solution under \( \sigma = 1 \). Similar to Theorem 2.1 of [7], the proof of the stability of solution in Sobolev space \( H^4 \) is directly obtained by analyzing the functional

\[ f(u) = \| \nabla^4 u \|_{L^2}^2 - \lambda \Re \int_D ((-\Delta)^3 u)(|u|^2 \bar{u}) dx. \]

**Lemma 2.5.** Let \( p \geq 1, \sigma = 1, u_0 \in \dot{H}^4 \) and \( \phi \in \mathcal{L}_2^4 \). There exists a constant \( C = C(p, T, u_0, \phi, \theta, \lambda, \sigma) \) such that

\[ E \sup_{0 \leq t \leq T} \| u \|_{H^4}^{2p} \leq C. \]

### 3. Stochastic symplectic scheme

As we all know, the stochastic Schrödinger equation without quadratic potential is an infinite-dimensional stochastic Hamiltonian system, which characterize the geometric invariants of the phase flow and contribute to constructing the numerical schemes for long time computation. In the following, we show that Eq. \( \{1.1\} \) possesses stochastic symplectic structure.

Denote by \( p \) and \( q \) the real and imaginary parts of \( u \), respectively. Let \( \hat{W} = \sum_{k \in \mathbb{N}} \beta_k(t, \omega)e_k(x) \) be a cylindrical Wiener process with \( \hat{W}_1 \)
and $\hat{W}_2$ being its real and imaginary parts. Then the Wiener process $W = \phi \hat{W}_1 + i\phi \hat{W}_2 =: W_1 + iW_2$. Then Eq. (1.1) is equivalent to

$$d_t p + \Delta q dt + \theta |x|^2 q dt + \lambda |p^2 + q^2|^\sigma q dt = dW_2,$$

$$d_t q - \Delta p dt - \theta |x|^2 p dt - \lambda |p^2 + q^2|^\sigma p dt = -dW_1$$

with initial datum $(p(0), q(0)) = (p_0, q_0)$. Set

$$H_1 = -\frac{1}{2} \int_D |\nabla u|^2 dx + \frac{\theta}{2} \int_D |xu|^2 dx + \frac{\lambda}{2\sigma + 2} \int_D |u|^{2\sigma + 2} dx.$$

Then the equation (3.1) can be rewritten as

$$d_t p = -\frac{\delta H_1}{\delta q} dt + dW_2,$$

$$d_t q = \frac{\delta H_1}{\delta p} dt - dW_1,$$

where $\frac{\delta H_1}{\delta q}$ and $\frac{\delta H_1}{\delta p}$ denote the variational derivative of $H_1$ with respect to $p$ and $q$, respectively. In fact, (3.2) is an infinite-dimensional stochastic Hamiltonian system. Using the same procedure as [5], one can derive that (3.2) possesses the symplectic structure

$$\omega(t) := \int_{x_0}^{x_1} dp \wedge dq dx.$$

**Theorem 3.1.** The phase flow of Eq. (1.1) preserves the symplectic structure (3.3).

In order to preserve the stochastic symplectic structure, we consider the mid-point scheme of the temporal discretization for (1.1),

$$i \frac{u^{n+1} - u^n}{\tau} + \Delta u^{n+1/2} + \theta |x|^2 u^{n+1/2} + \lambda |u^{n+1/2}|^{2\sigma} u^{n+1} = \frac{\Delta_n W}{\tau}, \quad n = 0, 1, \cdots, N - 1,$$

where $\tau = \frac{T}{N}$ is time step size, $\Delta_n W = W(t_{n+1}) - W(t_n)$, $u^{n+1/2} = \frac{1}{2}(u^n + u^{n+1})$. Denote

$$\bar{\omega}^n := \int_{x_0}^{x_1} dp^n \wedge dq^n dx,$$

we have the following result.

**Theorem 3.2.** The scheme (3.4) possesses the discrete symplectic structure, i.e.

$$\bar{\omega}^{n+1} = \bar{\omega}^n.$$
PROOF. For convenience, denote \( \Phi(p, q) = \frac{1}{2\sigma^2}(p^2 + q^2)^{\sigma + 1} \), then (3.4) can be rewritten as
\[
p^{n+1} = p^n - \Delta q^{n+\frac{1}{2}} - \theta |x|^2 q^{n+\frac{1}{2}} - \lambda \frac{\partial \Phi}{\partial q}(p^{n+\frac{1}{2}}, q^{n+\frac{1}{2}}) + \Delta_n W_2,
\]
\[
q^{n+1} = q^n + \Delta p^{n+\frac{1}{2}} + \theta |x|^2 p^{n+\frac{1}{2}} + \lambda \frac{\partial \Phi}{\partial p}(p^{n+\frac{1}{2}}, q^{n+\frac{1}{2}}) - \Delta_n W_1.
\]
Differentiating the above equation on the phase space, we obtain
\[
dp^{n+1} = dp^n - \Delta dq^{n+\frac{1}{2}} - \theta |x|^2 dq^{n+\frac{1}{2}} - \lambda \frac{\partial^2 \Phi}{\partial q \partial p} dp^{n+\frac{1}{2}} - \lambda \frac{\partial^2 \Phi}{\partial q^2} dq^{n+\frac{1}{2}} ,
\]
\[
dq^{n+1} = dq^n + \Delta dp^{n+\frac{1}{2}} + \theta |x|^2 dp^{n+\frac{1}{2}} + \lambda \frac{\partial^2 \Phi}{\partial p \partial q} dq^{n+\frac{1}{2}} + \lambda \frac{\partial^2 \Phi}{\partial p^2} dp^{n+\frac{1}{2}} .
\]
Then
\[
dp^{n+1} \wedge dq^{n+1}
= dp^n \wedge dq^n + \left( - \Delta dq^{n+\frac{1}{2}} - \theta |x|^2 dq^{n+\frac{1}{2}} - \lambda \frac{\partial^2 \Phi}{\partial q \partial p} dp^{n+\frac{1}{2}} - \lambda \frac{\partial^2 \Phi}{\partial q^2} dq^{n+\frac{1}{2}} \right) \wedge dq^n
+ dp^n \wedge \left( \Delta dp^{n+\frac{1}{2}} + \theta |x|^2 dp^{n+\frac{1}{2}} + \lambda \frac{\partial^2 \Phi}{\partial p \partial q} dq^{n+\frac{1}{2}} + \lambda \frac{\partial^2 \Phi}{\partial p^2} dp^{n+\frac{1}{2}} \right)
+ \left( - \Delta dq^{n+\frac{1}{2}} - \theta |x|^2 dq^{n+\frac{1}{2}} - \lambda \frac{\partial^2 \Phi}{\partial q \partial p} dp^{n+\frac{1}{2}} - \lambda \frac{\partial^2 \Phi}{\partial q^2} dq^{n+\frac{1}{2}} \right) \\
\wedge \left( \Delta dp^{n+\frac{1}{2}} + \theta |x|^2 dp^{n+\frac{1}{2}} + \lambda \frac{\partial^2 \Phi}{\partial p \partial q} dq^{n+\frac{1}{2}} + \lambda \frac{\partial^2 \Phi}{\partial p^2} dp^{n+\frac{1}{2}} \right) .
\]
Substituting
\[
dp^n = dp^{n+\frac{1}{2}} + \Delta dq^{n+\frac{1}{2}} + \theta |x|^2 dq^{n+\frac{1}{2}} + \lambda \frac{\partial^2 \Phi}{\partial q \partial p} dp^{n+\frac{1}{2}} + \lambda \frac{\partial^2 \Phi}{\partial q^2} dq^{n+\frac{1}{2}} ,
\]
\[
dq^n = dq^{n+\frac{1}{2}} - \Delta dp^{n+\frac{1}{2}} - \theta |x|^2 dp^{n+\frac{1}{2}} - \lambda \frac{\partial^2 \Phi}{\partial p \partial q} dq^{n+\frac{1}{2}} - \lambda \frac{\partial^2 \Phi}{\partial p^2} dp^{n+\frac{1}{2}} ,
\]
into the above equality, we have
\[
\int_{x_0}^{x_1} dp^{n+1} \wedge dq^{n+1} dx = \int_{x_0}^{x_1} dp^n \wedge dq^n dx, \text{ a.s.}
\]

\[
□
\]

In fact, we can prove that the scheme (3.4) exists a numerical solution.

**Proposition 3.1.** Let \( \phi \in \mathcal{L}_2^4 \), and \( u_0 \) be \( F_0 \)-measurable with values in \( H_0^1 \), then for sufficiently small \( \tau \), there exists an \( H_0^1(D) \)-valued \( \{ F_t \}_{0 \leq n \leq N} \)-adapted solution \( \{ u^n; 0 \leq n \leq N \} \) of \( \text{(3.4)} \).
Proof. Fix a family \( \{ \xi^n \}_{0 \leq n \leq N-1} \) of deterministic functions in \( H^1(D) \), we also fix \( \tilde{u}^n \in H^1_0(D) \), the existence of solution \( \tilde{u}^{n+1} \in H^1_0(D) \) of

\[
\frac{\tilde{u}^{n+1} - \tilde{u}^n}{\tau} + \Delta \tilde{u}^{n+\frac{1}{2}} + \theta |x|^{2} \tilde{u}^{n+\frac{1}{2}} + \lambda |\tilde{u}^{n+\frac{1}{2}}|^2 \tilde{u}^{n+\frac{1}{2}} = \xi^n
\]  
(3.5)

follows from a standard Galerkin method and Brouwer theorem (see [9]). Assuming that \( \tilde{u}^{n+1} \in H^1_0(D) \) is a solution of (3.5), multiplying (3.5) by the complex conjugate \( \tilde{u}^{n+\frac{1}{2}} \) of \( \tilde{u}^{n+\frac{1}{2}} \), integrating over \( D \) and taking the imaginary part of the resulting identity, we have

\[
\|\tilde{u}^{n+1}\|_2^2 = \|\tilde{u}^n\|_2^2 + 2\tau \Im \int_D \xi^n \bar{\tilde{u}}^{n+\frac{1}{2}} dx \\
\leq \|\tilde{u}^n\|_2^2 + \tau \left( \|\xi^n\|_2^2 + \frac{1}{2} \|\tilde{u}^n\|_2^2 + \frac{1}{2} \|\tilde{u}^{n+1}\|_2^2 \right).
\]

Therefore,

\[
\|\tilde{u}^{n+1}\|_2^2 \leq \frac{2 + \tau}{2 - \tau} \|\tilde{u}^n\|_2^2 + \frac{2\tau}{2 - \tau} \|\xi^n\|_2^2.
\]  
(3.6)

Using the same method, we multiply (3.5) by \( -\Delta \tilde{u}^{n+\frac{1}{2}} - \theta |x|^{2} \tilde{u}^{n+\frac{1}{2}} - \lambda |\tilde{u}^{n+\frac{1}{2}}|^2 \tilde{u}^{n+\frac{1}{2}} \), integrate over \( D \) and take the imaginary part of the
resulting identity. Therefore, using Hölder inequality and Young’s inequality, we obtain
\[
H(\tilde{u}^{n+1}) - H(\tilde{u}^n) = \tau \int_D \xi^n \left( -\Delta \tilde{u}^{n+\frac{1}{2}} - \theta |x|^{2\tilde{u}^{n+\frac{1}{2}}} - \lambda |\tilde{u}^{n+\frac{1}{2}}|^{2\sigma} \tilde{u}^{n+\frac{1}{2}} \right) dx
\]
\[
\leq \tau \left( \frac{1}{2} \| \nabla \xi^n \|_{L^2}^2 + \frac{1}{4} \| \nabla \tilde{u}^n \|_{L^2}^2 + \frac{1}{4} \| \nabla \tilde{u}^{n+1} \|_{L^2}^2 + C(\theta)(\| \xi^n \|_{L^2}^2 + \| \tilde{u}^n \|_{L^2}^2 + \| \tilde{u}^{n+1} \|_{L^2}^2) \right)
\]
\[
+ \frac{1}{2} \left( \frac{1}{4} \| \nabla \xi^n \|_{L^2}^2 + \frac{1}{4} \| \nabla \tilde{u}^n \|_{L^2}^2 + \frac{1}{4} \| \nabla \tilde{u}^{n+1} \|_{L^2}^2 + C(\theta)(\| \xi^n \|_{L^2}^2 + \| \tilde{u}^n \|_{L^2}^2 + \| \tilde{u}^{n+1} \|_{L^2}^2) \right)
\]
\[
+ \frac{1}{2} \left( \frac{1}{4} \| \nabla \xi^n \|_{L^2}^2 + \frac{1}{4} \| \nabla \tilde{u}^n \|_{L^2}^2 + \frac{1}{4} \| \nabla \tilde{u}^{n+1} \|_{L^2}^2 + C(\theta)(\| \xi^n \|_{L^2}^2 + \| \tilde{u}^n \|_{L^2}^2 + \| \tilde{u}^{n+1} \|_{L^2}^2) \right)
\]
\[
+ \frac{1}{2} \left( \frac{1}{4} \| \nabla \xi^n \|_{L^2}^2 + \frac{1}{4} \| \nabla \tilde{u}^n \|_{L^2}^2 + \frac{1}{4} \| \nabla \tilde{u}^{n+1} \|_{L^2}^2 + C(\theta)(\| \xi^n \|_{L^2}^2 + \| \tilde{u}^n \|_{L^2}^2 + \| \tilde{u}^{n+1} \|_{L^2}^2) \right)
\]
\[
+ C(\theta, \lambda)(\| \tilde{u}^{n+1} \|_{L^2}^{2 + \frac{4\sigma}{2 - \sigma}} + C(\theta, \lambda)(\| \tilde{u}^{n+1} \|_{L^2}^{2 + \frac{4\sigma}{2 - \sigma}})
\]
\[
\leq \frac{1}{4} \| \nabla \tilde{u}^{n+1} \|_{L^2}^2 + C(\tau, \theta, |\lambda|, |\theta|, \| \xi^n \|_{H^1}, \| \tilde{u}^n \|_{H^1}, \| \tilde{u}^{n+1} \|_{L^2})
\]

where the second inequality follows from Gagliardo-Nirenberg’s inequality and the last inequality follows from the fact that \( \tau \) is sufficiently small. From (3.6) and Lemma 2.3, we have
\[
\| \tilde{u}^{n+1} \|_{H^1}^2 \leq C(\tau, \sigma, |\lambda|, |\theta|, \| \xi^n \|_{H^1}, \| \tilde{u}^n \|_{H^1}, \| \tilde{u}^{n+1} \|_{L^2})
\]

Define a map
\[
\Lambda : H^1_0 \times H^1 \ni (\tilde{u}, \xi) \rightarrow \Lambda(\tilde{u}, \xi) \in \mathcal{P}(H^1_0),
\]
where \( \mathcal{P}(H^1_0) \) is the set of subsets of \( H^1_0(D) \), \( \Lambda(\tilde{u}, \xi) \) is the set of solutions \( \tilde{u}^{n+1} \) of (3.5). From the closedness of the graph of \( \Lambda \) and a selector theorem, there exists a universal and Borel measurable map \( \kappa : H^1_0 \times H^1 \rightarrow H^1_0 \) such that \( \kappa(u, \xi) \in \Lambda(u, \xi) \) for \( (u, \xi) \in H^1_0 \times H^1 \). Assume that \( u^n \in H^1_0 \) is \( \mathcal{F}_{\tau_n} \)-measurable random variable, then \( u^{n+1} = \kappa(u^n, \Delta_n W) \) is \( H^1_0 \)-valued solution of (3.4).

Now we investigate the convergence rate of the scheme (3.4) under \( \sigma = 1 \). To deal with the power law of the nonlinear term, we introduce a cut-off function \( \mu \in C^\infty(D) \) such that \( \text{supp} \mu \in [0, 2] \) and \( \mu = 1 \) on [0, 1]
Write $\mu_{R}(u) = \mu(\frac{|u|}{R})$, then we consider the truncated equation

$$i\mu_{R}(u) + (\Delta u_{R} + \theta |x|^{2}u_{R} + \lambda \mu_{R}(u_{R})|u_{R}|^{2}u_{R})dt = dW.$$  

Denote $f(u_{R}) = \mu_{R}(u_{R})(|u_{R}|^{2}u_{R})$, the mild form of the corresponding mid-point scheme of (3.7) is

$$u_{n+1,R} = S_{\tau}u_{n,R} + i\tau \theta \sqrt{T_{\tau}}u_{n,R} + i\tau \lambda T_{\tau}f(u_{n,R}) - i\sqrt{T_{\tau}}\chi_{n+1},$$

where

$$S_{\tau} = \left(1 - \frac{i\tau}{2}\Delta\right)^{-1} \left(1 + \frac{i\tau}{2}\Delta\right)^{-1}, \quad T_{\tau} = \left(1 - \frac{i\tau}{2}\Delta\right)^{-1}, \quad \chi_{n+1} = \frac{W(t_{n+1}) - W(t_{n})}{\sqrt{T_{\tau}}}.$$  

We have the following estimates to operators $S_{\tau}$ and $T_{\tau}$ (see [10]) with $\alpha \in [0, 1]$, which are are useful in the convergence analysis below:

$$\|S_{\tau}\|_{L(L^{2})} \leq 1, \quad \|T_{\tau}\|_{L(L^{2})} \leq 1, \quad \|S_{\tau} - I\|_{L(H^{2+\alpha},H^{\alpha})} \leq K\tau,$$

$$\|S(t_{n}) - S_{n}\|_{L(H^{2+\alpha},H^{\alpha})} \leq K\tau \|S(-s) - S_{-s}T_{\tau}\|_{L(H^{2+\alpha},L^{2})} \leq K\tau^{\alpha}.$$  

The scheme (3.8) is well-defined and $\{u_{n,R}\}_{0 \leq n \leq N}$ are uniformly bounded provided that the nonlinear term $f(\cdot)$ is global Lipschitz. This Lipschitz continuity can be guaranteed by Proposition 2.2 in [3] because $H^{1}$ is an algebra.

Assume that $u_{0} \in \dot{H}^{4}$, $\phi \in L_{2}^{4}$. From the global Lipschitz continuity of the nonlinear term, we obtain that

$$\{f(u_{R}(t))\}_{t \in [0, T]} \in L^{2p}(\Omega; L^{\infty}(0, T; \dot{H}^{4}))$$  

and

$$E \sup_{0 \leq t \leq T} \|u_{R}(t)\|_{\dot{H}^{4}}^{2p} \leq C(p, T, R, \|u_{0}\|_{\dot{H}^{4}}, \|\phi\|_{L_{2}^{2}})$$

using Sobolev embedding and Gronwall’s inequality. Now, we prove the following error estimate, the key of its proof lies in the mild solution and the unitarity of the both $S(t, r)$ and $S_{\tau}$.

**PROPOSITION 3.2.** Let $\sigma = 1$, $u_{0} \in \dot{H}^{4}$ and $\phi \in L_{2}^{4}$, then for any $T \geq 0$, there exists a constant $C$ such that

$$E \max_{n=0, \ldots, N} \|u_{R}(t_{n}) - u_{n,R}\|_{H^{4}}^{2p} \leq C\tau^{2p}.$$
\textbf{Proof.} For simplicity, we omit the dependence $R$ of $u^n$ and $u$. Assume that $\tau < 2/(C(\theta) + C(L_f, |\lambda|))$, here $L_f$ denotes the Lipschitz constant of $f$. Clearly,

$$u^n = S^n u_0 + i\tau \sum_{l=1}^n S^n_{\tau} T_\tau |x|^2 u^{l-\frac{1}{2}} + i\tau \lambda \sum_{l=1}^n S^n_{\tau} T_\tau f(u^{l-\frac{1}{2}}) - i\sqrt{\tau} \sum_{l=1}^n S^n_{\tau} T_\tau \chi_l.$$

Define the mapping $g_n$ which maps $u^0, \ldots, u^{n-1}$ to $u^n$. For any sequences $\{u^n\}_{0 \leq n \leq N}$ and $\{v^n\}_{0 \leq n \leq N}$ with $u_0 = v_0$, we have

$$\|g_n(u^0, \ldots, u^{n-1}) - g_n(v^0, \ldots, v^{n-1})\|_{H^1}$$

$$\leq \tau \left\| \sum_{l=1}^n S^n_{\tau} T_\tau |x|^2 (u^{l-\frac{1}{2}} - v^{l-\frac{1}{2}}) \right\|_{H^1} + \tau \left\| \sum_{l=1}^n S^n_{\tau} T_\tau (f(u^{l-\frac{1}{2}}) - f(v^{l-\frac{1}{2}})) \right\|_{H^1}$$

$$\leq \tau C(\theta) \sum_{l=1}^{n-1} \|u^l - v^l\|_{H^1} + \frac{\tau C(\theta)}{2} \|g_n(u^0, \ldots, u^{n-1}) - g_n(v^0, \ldots, v^{n-1})\|_{H^1}$$

$$+ \tau C(L_f, |\lambda|) \sum_{l=1}^{n-1} \|u^l - v^l\|_{H^1} + \frac{\tau C(L_f, |\lambda|)}{2} \|g_n(u^0, \ldots, u^{n-1}) - g_n(v^0, \ldots, v^{n-1})\|_{H^1}.$$  

This fact provides that

$$(3.11)$$

$$\|g_n(u^0, \ldots, u^{n-1}) - g_n(v^0, \ldots, v^{n-1})\|_{H^1} \leq \frac{2\tau (C(\theta) + C(L_f, |\lambda|))}{2 - \tau (C(\theta) + C(L_f, |\lambda|))} \sum_{l=1}^{n-1} \|u^l - v^l\|_{H^1}.$$  

We know that

$$u(t_n) = S(t_n) u_0 + i\theta \int_0^{t_n} S(t_n - r) |x|^2 u(r) dr$$

$$+ i\lambda \int_0^{t_n} S(t_n - r) f(u(r)) dr - i \int_0^{t_n} S(t_n - r) dW(r)$$
and deduce

\[ u(t_n) - g_n(u^0, \ldots, u(t_{n-1})) = (S(t_n) - S^n_i)u_0 \]
\[ + \, i\theta \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} (S(t_n - r)|x|^2u(r) - S^{n-i}_{\tau}T_r|x|^2u(t_{l-\frac{1}{2}}))dr \]
\[ + \, i\lambda \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} (S(t_n - r)f(u(r)) - S^{n-i}_{\tau}T_r f(u(t_{l-1})))dr \]
\[ - \, i \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} (S(t_n - r) - S^{n-i}_{\tau}T_r) dW(r) \]
\[ =: A_n + B_n + C_n + D_n. \]

From (3.9), the terms \( A_n \) and \( D_n \) are estimated as following,

\[
(3.12) \quad \mathbf{E} \max_{n=0,\ldots,N} \|A_n + D_n\|_{H^1}^{2p} \leq C\tau^{2p}(\mathbf{E}(\|u_0\|_{H^4}^{2p}) + \|\phi\|_{C^2_1}^{2p}).
\]

For the term \( B_n \), we divide it into the following parts

\[
\begin{align*}
&i\theta \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} (S(t_n - r)|x|^2u(r) - S^{n-i}_{\tau}T_r|x|^2u(t_{l-\frac{1}{2}}))dr \\
&= i\theta \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} (S(t_n - r) - S^{n-i}_{\tau}T_r)|x|^2u(r)dr + i\theta \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} S^{n-i}_{\tau}T_r|x|^2(u(r) - u(t_{l-1}))dr \\
&+ i\theta \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} S^{n-i}_{\tau}T_r|x|^2(u(t_{l-1}) - u(t_{l-\frac{1}{2}}))dr \\
&=: B^1_n + B^2_n + B^3_n.
\end{align*}
\]
Concerning the first term $B_n^1$, we have

$$
\left\| \frac{i}{2} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} \left( S(t_n - r) - S_{r}^{n-l}T \right) |x|^2 u(r) dr \right\|_{H_1} 
\leq \left\| \frac{i}{2} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} \left( S(t_n - r) - S(t_n) S_{r}^{n-l}T \right) |x|^2 u(r) dr \right\|_{H_1} 
+ \left\| \frac{i}{2} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} \left( S(t) - S_{r}^{n-l}T \right) |x|^2 u(r) dr \right\|_{H_1} 
\leq \left\| \frac{i}{2} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} \left( S(-r) - S_{r}^{n-l}T \right) |x|^2 u(r) dr \right\|_{H_1} 
+ \left\| S(t_n) - S_{r}^{n} \right\|_{L^2(H^1)} \left\| \frac{i}{2} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} S_{r}^{n-l}T |x|^2 u(r) dr \right\|_{H_4} 
\leq CT \sup_{t \in [0,T]} \left\| u(t) \right\|_{H^4},
$$

so that

$$
E \max_{n=0,\ldots,N} \left\| B_n^1 \right\|_{H^1}^{2p} \leq C T^{2p}.
$$

In the context of mild solution of (3.7), we have

$$
u(r) = S(r - t_{l-1}) u(t_{l-1}) + \int_{t_{l-1}}^{r} S(r - s) |x|^2 u(s) ds 
+ i \frac{\lambda}{2} \int_{t_{l-1}}^{r} S(r - s) f(u(s)) ds - i \int_{t_{l-1}}^{r} S(r - s) dW(s).
$$

Therefore,

$$
B_n^2 = i \frac{\lambda}{2} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} S_{r}^{n-l}T \left\| x \right\|^2 \left( S(r - t_{l-1}) - Id \right) u(t_{l-1}) dr 
+ i \frac{\lambda}{2} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} S_{r}^{n-l}T \left\| x \right\|^2 \left( \frac{i}{2} \int_{t_{l-1}}^{r} S(r - s) |x|^2 u(s) ds \right) dr 
+ i \frac{\lambda}{2} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} S_{r}^{n-l}T \left\| x \right\|^2 \left( i \int_{t_{l-1}}^{r} S(r - s) f(u(s)) ds \right) dr 
- i \frac{\lambda}{2} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} S_{r}^{n-l}T \left\| x \right\|^2 \left( i \int_{t_{l-1}}^{r} S(r - s) dW(s) \right) dr 
=: B_n^{21} + B_n^{22} + B_n^{23} + B_n^{24}.
$$
For $\rho \in \mathbb{R}$, $S(\rho)$ is an isometry and
\[ \|S(\rho) - I\|_{L(H^4, H^2)} \leq C\rho. \]

Thus,
\[ \|B_{n}^{21}\|_{H^1} \leq C T \sup_{t \in [0, T]} \|u(t)\|_{H^4}, \]

which leads to
\[ E \max_{n=0, \ldots, N} \|B_{n}^{21}\|_{H^1}^{2p} \leq C T^{2p}. \]

Because $\|B_{n}^{22}\|_{H^1} \leq C T \sup_{t \in [0, T]} \|u(t)\|_{H^4}$, we have
\[ E \max_{n=0, \ldots, N} \|B_{n}^{22}\|_{H^1}^{2p} \leq C T^{2p}. \]

Due to $\|B_{n}^{23}\|_{H^1} \leq C T \sup_{t \in [0, T]} \|f(u(t))\|_{H^4}$, we get
\[ E \max_{n=0, \ldots, N} \|B_{n}^{23}\|_{H^1}^{2p} \leq C T^{2p}. \]

Using Fubini’s theorem and martingale inequality, we have
\[ E \max_{n=0, \ldots, N} \|B_{n}^{24}\|_{H^1}^{2p} \]

\[ = E \max_{n=0, \ldots, N} \left\| -i \theta \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l}} S_{\tau}^{n-l} T_{\tau} |x|^2 \left( i \int_{s}^{t_{l}} S(r - s) dr \right) dW(s) \right\|_{L^1_{2}}^{2p} \]
\[ \leq C_{p} E \left( \sum_{l=1}^{N} \int_{t_{l-1}}^{t_{l}} \sum_{k \in \mathbb{N}} \| \theta S_{\tau}^{n-l} T_{\tau} e_{k} |x|^2 \left( \int_{s}^{t_{l}} S(r - s) \phi e_{k} dr \right) \|_{L^1_{2}}^{2} ds \right)^{p} \leq C T^{2p}. \]

For $B_{n}^{3}$,
\[ \|B_{n}^{3}\|_{H^1} = \left\| i \theta \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l}} S_{\tau}^{n-l} T_{\tau} (|x|^2 u(t_{l-1}) - |x|^2 u(t_{l-\frac{1}{2}})) dr \right\|_{H^1} \]
\[ \leq C T \|u(t_{l-1}) - u(t_{l-\frac{1}{2}})\|_{H^1}. \]

Using the definition of $u(t)$, we can estimate $B_{n}^{3}$ similarly.
It remains to estimate the term \( C_n \). It can be decomposed into a sum

\[
\begin{align*}
&i\lambda \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l}} \left( S(t_{n} - r) f(u(r)) - S^{n-l}_{\tau} T_{\tau} f(u(t_{l-1})) \right) dr \\
= &\ i\lambda \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l}} \left( S(t_{n} - r) - S^{n-l}_{\tau} T_{\tau} \right) f(u(r)) dr \\
+ &\ i\lambda \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l}} S^{n-l}_{\tau} T_{\tau} \left( f(u(r)) - f(u(t_{l-1})) \right) dr \\
+ &\ i\lambda \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l}} S^{n-l}_{\tau} T_{\tau} \left( f(u(t_{l-1})) - f(u(t_{l-1}^{\frac{1}{2}})) \right) dr \\
=: &\ C_{1}^{1} + C_{2}^{n} + C_{3}^{n}.
\end{align*}
\]

For the term \( C_{1}^{1} \), we have

\[
\left\| i\lambda \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l}} \left( S(t_{n} - r) - S^{n-l}_{\tau} T_{\tau} \right) f(u(r)) dr \right\|_{H_{1}} \\
\leq \left\| i\lambda \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l}} \left( S(t_{n} - r) - S(t_{n}) S^{n-l}_{\tau} T_{\tau} \right) f(u(r)) dr \right\|_{H_{1}} \\
+ \left\| i\lambda \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l}} \left( S(t_{n}) - S^{n}_{\tau} \right) S^{n-l}_{\tau} T_{\tau} f(u(r)) dr \right\|_{H_{1}} \\
\leq \left\| i\lambda \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l}} \left( S(-r) - S^{n-l}_{\tau} T_{\tau} \right) f(u(r)) dr \right\|_{H_{1}} \\
+ \| S(t_{n}) - S^{n}_{\tau} \|_{L^{3}(H^{3}, L^{2})} \left\| i\lambda \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l}} S^{n-l}_{\tau} T_{\tau} f(u(r)) dr \right\|_{H_{4}}.
\]

Similar to the estimates of \( B_{1}^{n} \), we have

\[
\| C_{1}^{1} \|_{H^{1}} \leq C T \tau \sup_{t \in [0, T]} \| f(u(t)) \|_{H^{4}}.
\]

Therefore,

\[
\sum_{n=0}^{N} \left\| C_{n}^{1} \right\|_{H^{1}}^{2p} \leq C \tau^{2p}.
\]
Using the definition of the mild solution and Taylor formula,

\[ C_n^2 = i\lambda \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} S_{r}^{n-l} T_{r} f'(u(t_{l-1})) \left( (S(r-t_{l-1}) - Id)u(t_{l-1}) \right) \]

\[ + i\theta \int_{t_{l-1}}^{r} (S(r-s)|x|^2u(s)ds + i\lambda \int_{t_{l-1}}^{r} S(r-s)f(u(s))ds \right) dr \]

\[ - \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} S_{r}^{n-l} T_{r} f'(u(t_{l-1})) \left( \int_{t_{l-1}}^{r} S(r-s)dW(s) \right) dr \]

\[ + \frac{i}{2} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} S_{r}^{n-l} T_{r} \int_{0}^{1} f''(\rho u(r) + (1-\rho)u(t_{l-1})) \]

\[ \cdot (u(r) - u(t_{l-1}), u(r) - u(t_{l-1})) d\rho dr. \]

Combined with the estimate of \( B_n \) and (3.10),

\[ E \max_{n=0, \ldots, N} \| C_n^2 \|_{H^1}^{2p} \leq C_T^{2p}. \]

For the term \( C_n^3 \), the estimate of \( B_n^3 \) and Lipschitz continuity of \( f \) yield that

\[ E \max_{n=0, \ldots, N} \| C_n^3 \|_{H^1}^{2p} \leq C_T^{2p}. \]

Summarize the above estimates, we obtain

\[ E \max_{n=0, \ldots, N} \| u(t_n) - g_n(u^0, \ldots u(t_{n-1})) \|_{H^1}^{2p} \leq C_T^{2p}. \]

Then, by (3.11) and Minkowski inequality, we have

\[ (E \max_{n=0, \ldots, N} \| \tilde{u}^n - u(t_n) \|_{H^1}^{2p})^{1/2p} \]

\[ \leq C \frac{2(C(\theta) + C(|\lambda|, L_f))}{2 - \tau(C(\theta) + C(|\lambda|, L_f))} \tau \sum_{l=1}^{n-1} \left( E \left( \max_{l=0, \ldots, d} \| \tilde{u}^l - u(t_l) \|_{H^1}^{2p} \right) \right)^{1/2p} + C_T. \]

The result follows from the discrete Gronwall’s inequality. \( \Box \)

**Theorem 3.3.** Let \( \sigma = 1, u_0 \in \dot{H}^4 \) and \( \phi \in \mathcal{L}_4 \), then for any \( 0 \leq n \leq N \),

\[ \lim_{C \to \infty} P \left( \max_{n=0, \ldots, N} \| u(t_n) - u^n \|_{H^1} \geq C_T \right) = 0. \]

**Proof.** Define a stopping time

\[ \tau_R = \inf_{n=0, \ldots, N} \{ t_n : \| u_{R_n} - 1 \|_{H^1} \geq R \text{ or } \| u_R^n \|_{H^1} \geq R \}, \]

Using the definition of the mild solution and Taylor formula,
and the discrete solution \( u^n = u^n_R \) if \( t_n \leq t_R \). From Proposition 3.2, we have

\[
\mathbb{E} \max_{n=0,\ldots,N} \| u_R(t_n) - u_R^n \|_{H^1}^{2p} \leq C(R, p) \tau^{2p}.
\]

This yields that \( \max_{n=0,\ldots,N} \| u_R(t_n) - u_R^n \|_{H^1} \) converges to 0 in probability as \( \tau \to 0 \). Similar to [10], by Chebyshev inequality, we have

\[
P\left( \max_{n=0,\ldots,N} \| u(t_n) - u^n \|_{H^1} \geq C \tau \right) \\
\leq P\left( \max_{n=0,\ldots,N} \| u(t_n) \|_{H^1} \geq R \right) + P\left( \max_{n=0,\ldots,N} \| u^n \|_{H^1} \geq R \right) \\
+ P\left( \max_{n=0,\ldots,N} \| u_R(t_n) - u_R^n \|_{H^1} \geq C \tau \right) \\
\leq P\left( \max_{n=0,\ldots,N} \| u(t_n) \|_{H^1} \geq R \right) + P\left( \max_{n=0,\ldots,N} \| u^n \|_{H^1} \geq R \right) \\
E \max_{n=0,\ldots,N} \| u_R(t_n) - u_R^n \|_{H^1}^2 \\
+ \frac{C^2 \tau^2}{R^2}.
\]

Following from the uniform boundedness of \( u^n \) and \( u(t_n) \) together with Proposition 3.2, we have \( P\left( \max_{n=0,\ldots,N} \| u(t_n) - u^n \|_{H^1} \geq C \tau \right) \) converges to 0 as \( C \) and \( R \) tend to \( \infty \). \( \square \)

4. Numerical experiments

In this section, we focus on the following example

\[
\begin{align*}
 idu(t, x) + (\Delta u(t, x) + \theta x^2 u(t, x) + \lambda |u(t, x)|^2 u(t, x))dt &= \epsilon dW, \\
u(0, x) &= \sin(\pi x).
\end{align*}
\]

(4.1)

Here, \( \epsilon \) denotes the size of noise and \( \epsilon = 0 \) can be considered as the deterministic case in some sense. Next, we first present numerical experiments to verify the convergence order of the proposed stochastic symplectic scheme (3.4) on \([0, 1] \times [0, T]\). In order to investigate the influence of quadratic potential and noise, we give some numerical experiment on the solution and evolution laws of charge and energy in the sense of expectation for stochastic multi-symplectic scheme

\[
i(\delta_t^+ u_{j+\frac{1}{2}}^n + \delta_t^- u_{j-\frac{1}{2}}^n) = -2\delta_x^+ \delta_x^- u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - \theta |x_{j+\frac{1}{2}}|^2 u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - \theta |x_{j-\frac{1}{2}}|^2 u_{j-\frac{1}{2}}^{n+\frac{1}{2}} \\
-\lambda |u_{j+\frac{1}{2}}^{n+\frac{1}{2}}|^2 \sigma x_{j+\frac{1}{2}} + \lambda |u_{j-\frac{1}{2}}^{n+\frac{1}{2}}|^2 \sigma x_{j-\frac{1}{2}} + \frac{\epsilon}{\tau} (\Delta_n W_{j+\frac{1}{2}} + \Delta_n W_{j-\frac{1}{2}}).
\]

(4.2)

where

\[
\delta_x^+ u_j := \frac{u_{j+1} - u_j}{h}, \delta_t^+ u_n := \frac{u_{n+1} - u_n}{\tau}, \delta_x^+ \delta_x^- u_j := \frac{u_{j+1} + 2u_j + u_{j-1}}{h^2}.
\]
It is obtained by applying mid-point scheme to (4.1) in both temporal and spatial directions [13].

Under the homogeneous Dirichlet boundary condition, (4.2) possesses the discrete charge and energy properties deduced by similar method to [13], respectively:

\[ h \sum_j |u_{j+\frac{1}{2}}^{n+1}|^2 = h \sum_j |u_{j+\frac{1}{2}}^n|^2 + h\epsilon \Im \sum_j \left( \Delta_n W_{j+\frac{1}{2}} + \Delta_n W_{j-\frac{1}{2}} \right) \overline{u_{j+\frac{1}{2}}^{n+1}}, \]

and

\[ H^{n+1} - H^n = \frac{\lambda}{2} h \sum_j |u_{j+\frac{1}{2}}^{n+1}|^2 (|u_{j+\frac{1}{2}}^{n+1}|^2 - |u_{j+\frac{1}{2}}^n|^2) - \frac{\lambda}{2\sigma + 2h} \sum_j (|u_{j+\frac{1}{2}}^{n+1}|^4 - |u_{j+\frac{1}{2}}^n|^4) \]

\[ + h\epsilon \Re \sum_j \left( \Delta_n W_{j+\frac{1}{2}} + \Delta_n W_{j-\frac{1}{2}} \right) (\delta^+_i \overline{u_j^n}), \]

Here, the global energy of (4.2) at time \( t_n \) is defined as

\[ H^n = \frac{1}{2} h \sum_j |\delta^+_x u_j^n|^2 - \frac{\theta}{2} h \sum_j |x_{j+\frac{1}{2}} u_{j+\frac{1}{2}}^n|^2 - \frac{\lambda}{2\sigma + 2} h \sum_j |u_{j+\frac{1}{2}}^n|^{2\sigma+2}. \]

In the sequel, we choose \( \theta = 1, \lambda = 1 \), and consider the real-valued Wiener process \( W(t, x) = \sum_{k=1}^M \frac{1}{\sqrt{\kappa_k}} \epsilon_k(x) \beta_k(t) \) with the truncated number \( M = 50 \), where \( \{ \beta_k; 1 \leq k \leq M \} \) is a family of independent \( \mathbb{R} \)-valued Brownian motions, and \( \epsilon_k(x) = \sin(\pi k x) \), \( k = 1, ..., M \) denotes the orthonormal basis of \( L^2(0,1) \). Let \( I_\tau = \{ t_n; 0 \leq n \leq N \} \) be the uniform discretization of \( [0,1] \) of size \( \tau > 0 \), and apply the uniform discretization of \( [0,1] \) of size \( h = \frac{1}{256} \). The reference values are generated for the smallest mesh size \( \tau_{ref} = 2^{-14} \). In Fig.4.1, we plot the convergence curves based on the errors \( \| u_{ref} - u_\tau \|_{L^2} \) with \( \tau = 2^p \tau_{ref}, p = 1, 2, 3, 4, 5 \). We can see that the convergence order for the \( L^2 \)-error of the mid-point scheme is 2 if \( \epsilon = 0 \) and the slope of our scheme (3.4) in stochastic case is 1. This observation verifies the theoretical result in Section 3.

We now investigate the behaviors of solitary wave solution under the influence of quadratic potential and noise. In these experiments, we take \( \lambda = 1, \sigma = 1 \) and the step sizes \( h = 0.1, \tau = 0.01 \). The profiles of amplitude \( |u(t, x)| \) are presented in Fig.4.2 and Fig.4.3. In Fig.4.2, these two figures give the propagation of solitary wave when taking different size of noise \( \epsilon = 0.02, 0.2 \) and \( \theta = -1 \). We find that the waveform of solution is obviously disturbed as the scale of noise becomes larger, that is the velocity of solitary wave is influenced. Fig.4.3
Figure 1. Rates of convergence for $\epsilon = 0$ (left) and $\epsilon = \sqrt{2}$ (right).

Figure 2. The profile of numerical solution $|u(x, t)|$ for one trajectory with different noise when $\theta = -1$. The left figure is the case of $\epsilon = 0.02$, The right figure is the case of $\epsilon = 0.2$.

presents the long time behaviors of solution when we take the different kind of quadratic potential $\theta = -1, 0, 1$ with $\epsilon = 0.2$. Combining these three figures, we find that the external potential influences the velocity of solitary wave, it can neither prevent the propagation, nor destroy the solitary. Moreover, it dominates the dynamics of the solution weaker than noise.

The average charge conservation law $\mathbf{E}(M(u(t, x)))$ follows linearly grow evolution law with respect to time and the average energy conservation law $\mathbf{E}(H(u(t, x)))$ follows linear evolution law at most. These phenomena are reflected in Fig.4.4 and Fig.4.5, respectively, where the evolution of the average discrete charge and energy obey nearly linear growth over 100 trajectories. In Fig.4.4, the different external potential
have small effects on the average charge and energy. But different size noises have obvious effects on them, the evolution laws of the charge and energy more and more tend to the conservation laws when $\epsilon$ tends to 0 especially.

Figure 3. The profile of $|u(t, x)|$ when $\theta = -1$ (left), 0 (middle), 1 (right), respectively.

Figure 4. The evolution of the average discrete charge obeys linear growth if $\theta = -1, 0, 1, \lambda = 1, \sigma = 1$ (left); $\epsilon = 0.01, 0.1, 0.2, \theta = -1, \lambda = 1, \sigma = 1$ (right).

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Figure 5. The evolution of the average discrete energy obeys linear growth if $\theta = -1, 0, 1, \lambda = 1, \sigma = 1$ (left); $\epsilon = 0.01, 0.1, 0.2, \theta = -1, \lambda = 1, \sigma = 1$ (right).

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