Finite Time Stabilization of Nonautonomous First Order Hyperbolic Systems

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Abstract

We address nonautonomous initial boundary value problems for decoupled linear first-order one-dimensional hyperbolic systems, investigating the phenomenon of finite time stabilization. We establish sufficient and necessary conditions ensuring that solutions stabilize to zero in a finite time for any initial $L^2$-data. In the nonautonomous case we give a combinatorial criterion stating that the robust stabilization occurs if and only if the matrix of reflection boundary coefficients corresponds to a directed acyclic graph. An equivalent robust algebraic criterion is that the adjacency matrix of this graph is nilpotent. In the autonomous case we also provide a spectral stabilization criterion, which is nonrobust with respect to perturbations of the coefficients of the hyperbolic system.

Key words: Nonautonomous first-order hyperbolic systems, Reflection boundary conditions, Finite time stabilization, Stabilization criteria, Robustness

Mathematics Subject Classification: 35B40, 93D20, 93D40, 35L04, 37L15

1 Problem setting and main results

The paper concerns the finite time stabilization property in the semistrip $\Pi = \{(x, t) : 0 \leq x \leq 1, 0 \leq t < \infty\}$ of solutions to the $n \times n$-decoupled first-order system

\[
\frac{\partial}{\partial t} u_j + a_j(x, t) \frac{\partial}{\partial x} u_j + b_j(x, t) u_j = 0, \quad 0 < x < 1, \ t > 0, \ j \leq n,
\]

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endowed with the reflection boundary conditions

\[
\begin{align*}
    u_j(0, t) &= \sum_{k=1}^{m} p_{jk} u_k(1, t) + \sum_{k=m+1}^{n} p_{jk} u_k(0, t), \quad t \geq 0, \quad 1 \leq j \leq m, \\
    u_j(1, t) &= \sum_{k=1}^{m} p_{jk} u_k(1, t) + \sum_{k=m+1}^{n} p_{jk} u_k(0, t), \quad t \geq 0, \quad m < j \leq n,
\end{align*}
\]

and the initial conditions

\[
u_j(x, 0) = \varphi_j(x), \quad 0 \leq x \leq 1, \quad j \leq n.
\]

Here \( n \geq 2 \) and \( 0 \leq m \leq n \) are fixed integers. The unknown function \( u = (u_1, \ldots, u_n) \) and the initial function \( \varphi = (\varphi_1, \ldots, \varphi_n) \) are vectors of real-valued functions. The coefficients \( a_j \) and \( b_j \) are real-valued functions and the \( n \times n \)-matrix \( P = (p_{jk}) \) of the reflection boundary coefficients has real entries. The functions \( a_j \) are supposed to satisfy the following conditions:

\[
\begin{align*}
    \inf \{ a_j(x, t) : (x, t) \in \Pi, 1 \leq j \leq m \} &\geq a, \\
    \sup \{ a_j(x, t) : (x, t) \in \Pi, m + 1 \leq j \leq n \} &\leq -a
\end{align*}
\]

for some \( a > 0 \).

The purpose of the paper is to identify a class of boundary conditions of the type (1.2) ensuring that all solutions to the problem (1.1)–(1.3) stabilize to zero in a finite time not depending on the initial data. To this end, we establish several stabilization criteria in terms of the reflection boundary coefficients and the coefficients of the hyperbolic system (irrespective of the initial data). A robust combinatorial criterion will be expressed in terms of a directed graph \( G_P \) associated with the matrix \( P \). Robust algebraic criteria will be given in terms of the adjacency matrix of \( G_P \) or in terms of the matrix \( P \) itself. Moreover, we generalize these results to the case of nonautonomous boundary conditions. For autonomous problems we also give a nonrobust criterion in terms of spectral properties of the infinitesimal generator of the semigroup generated by the autonomous problem (1.1)–(1.3).

We have chosen to work in the \( L^2 \)-setting, where the existence of \( L^2 \)-generalized solutions is proved in [33]. This gives us the advantage that the stabilization criteria established in this paper apply as well to solutions of better regularity. It should be noted that they also remain to be true for solutions of worth regularity. In particular, for the strongly singular delta-wave solutions the stabilization phenomenon can easily be shown to follow from the smoothing property proved in [32, Theorem 4.5].

For a Banach space \( X \), the \( n \)-th Cartesian power \( X^n \) is considered to be a Banach space normed by

\[
\|u\|_{X^n} = \max_{j \leq n} \|u_j\|_X,
\]
where \( u = (u_1, \ldots, u_n) \) with each \( u_j \in X \). By \( C_0^\infty([0,1]) \) we denote a subspace of the vector space \( C^\infty([0,1]) \) of functions with support within \((0,1)\).

Suppose that the coefficients of (1.1) fulfill the following regularity assumptions:

\[
\text{The functions } a_j, b_j \text{ belong to } C^1(\Pi) \text{ and are bounded in } \Pi \text{ together with their first order derivatives.} (1.5)
\]

It should be noted that the boundedness assumption on \( a_j \) and \( b_j \) and the \( C^1 \)-assumption on \( b_j \) are not crucial for the results of the present paper. The former can be dropped without loss of generality, while the latter can be weakened to \( b_j \in C^{0,1}_x,0_t(\Pi) \) or \( b_j \in C^{1,0}_x,1_t(\Pi) \) accordingly to the solution concept given by Definition 1.1, or even to \( b_j \in C(\Pi) \) if one would use the solution concept as in [7, Definition A.1]. The assumptions (1.5) are imposed to simplify the presentation (in particular, they are supposed in the relevant result in [33] that we cite as Theorem 1.2 below).

As it follows from [31, Theorem 3.1], for any continuously differentiable initial function \( \varphi \) satisfying the zero-order and the first-order compatibility conditions between (1.2) and (1.3) (in particular, for \( \varphi \in C_0^\infty([0,1])^n \)), the problem (1.1)–(1.3) has a unique classical \( C^1 \)-solution in \( \Pi \). We now introduce a notion of the \( L^2 \)-generalized solution, which is analogous to that introduced in [49, §29] for initial-boundary value problems for an equation of the hyperbolic type.

**Definition 1.1** [33, Definition 4.3] Let \( \varphi \in L^2(0,1)^n \). A function \( u \in C([0,\infty),L^2(0,1))^n \) is called an \( L^2 \)-generalized solution to the problem (1.1)–(1.3) if for any sequence \( \varphi^l \in C_0^\infty([0,1])^n \) with \( \varphi^l \rightarrow \varphi \) in \( L^2(0,1)^n \) the sequence of classical \( C^1 \)-solutions \( u^l(x,t) \) to the problem (1.1)–(1.3) with \( \varphi \) replaced by \( \varphi^l \) fulfills the convergence condition

\[
\|u(\cdot,t) - u^l(\cdot,t)\|_{L^2(0,1)^n} \rightarrow_{l \to \infty} 0,
\]

uniformly in \( t \) varying in the range \( 0 \leq t \leq T \), for each \( T > 0 \).

The following theorem is obtained via extension by continuity. Generally speaking, the proof method is based on the classical result [28, Theorem 2 in Section V.8.2] stating that, for given normed spaces \( X \) and \( Y \) such that \( Y \) is complete, any linear continuous operator \( Q_0 \) from \( \Omega \subset X \) to \( Y \) admits a unique linear continuous extension \( Q \) to the closure \( \overline{\Omega} \) of \( \Omega \), and \( \|Q\| = \|Q_0\| \).

**Theorem 1.2** [33, Theorem 2.3] Suppose that the conditions (1.4) and (1.5) are fulfilled. Then, given \( \varphi \in L^2(0,1)^n \), there exists a unique \( L^2 \)-generalized solution \( u \) to the problem (1.1)–(1.3).

**Definition 1.3** [33, Definition 4.3] Problem (1.1)–(1.3) is said to be Finite Time Stabilizable (FTS) if there exists a positive real \( T_e \) such that for every \( \varphi \in L^2(0,1)^n \) the \( L^2 \)-generalized
solution to (1.1)–(1.3) is a constant zero function for all $t > T_e$. The value of $T_e$ is called the finite time stabilization. The infimum value of all $T_e$ with the above property is called the optimal stabilization time and is denoted by $T_{opt}$.

**Definition 1.4** The problem (1.1)–(1.3) is robust FTS if it is FTS for any $a_j$ and $b_j$ satisfying (1.4) and (1.5).

First we provide a spectral FTS criterion for the autonomous version of the problem (1.1)–(1.3), when $a_j(x, t) \equiv a_j(x)$ and $b_j(x, t) \equiv b_j(x)$. Introduce diagonal matrices $A(x) = \text{diag}(a_1, \ldots, a_n)$ and $B(x) = \text{diag}(b_1, \ldots, b_n)$ and write down the problem (1.1)–(1.3) as the abstract Cauchy problem in $L^2(0, 1)^n$ in the following form:

$$\frac{d}{dt}u(t) = Au(t), \quad u(0) = \varphi \in L^2(0, 1)^n,$$

(1.6)

where the operator $A : D(A) \subset L^2(0, 1)^n \mapsto L^2(0, 1)^n$ is defined by

$$(Av)(x) = -A(x)\frac{dv}{dx} - B(x)v$$

(1.7)

and

$$D(A) = \{v \in L^2(0, 1)^n : \partial_x v \in L^2(0, 1)^n, \ v_{out} = Pv_{in}\}.$$  

(1.8)

Here

$$v_{out} = (v_1(0), \ldots, v_m(0), v_{m+1}(1), \ldots, v_n(1)),$$

$$v_{in} = (v_1(1), \ldots, v_m(1), v_{m+1}(0), \ldots, v_n(0)).$$

**Theorem 1.5** The autonomous problem (1.1)–(1.3) is FTS if and only if the spectrum of the operator $A$ is empty.

It should be emphasized that the criterion stated in Theorem 1.5 is not robust (see Subsection 2.3.2), which is disadvantageous from the viewpoint of applications. To provide robust criteria, with the matrix $P$ we associate the following directed graph $G_P$:

- $\{1, \ldots, n\}$ is the vertex set of $G_P$,
- two vertices $j$ and $k$ are connected by the arrow $(j, k)$ in $G_P$ if and only if $p_{jk} \neq 0$.

Let us recall some notions from graph theory (see, e.g., [24]). A graph is directed if its vertices are connected by edges having directions from one vertex to the other. Formally, a directed graph $G$ on a vertex set $V$ is determined by its edge set $E \subseteq V^2$, where $(i, j) \in E$ is a directed edge (arrow) from vertex $i$ to vertex $j$. Let $l \geq 1$. A cycle of length $l$ in $G$ is a sequence of pairwise distinct vertices $(k_1, \ldots, k_l)$ such that $(k_s, k_{s+1}) \in E$ for all $s < l$ and $(k_l, k_1) \in E$. Two cycles $(k_1, \ldots, k_l)$ and $(k'_1, \ldots, k'_l)$ are considered to be equal if they
can be obtained from each other by a cyclic shift or, equivalently, if the sets of their arrows
\{ (k_1, k_2), \ldots, (k_i, k_1) \} and \{ (k'_1, k'_2), \ldots, (k'_l, k'_1) \} are equal. An acyclic directed graph is a
directed graph having no cycles. For a directed graph on the vertex set \{ 1, \ldots, n \}, the
adjacency matrix is the \( n \times n \)-matrix \( W = (w_{jk}) \) such that \( w_{jk} \) is one when there is an arrow
from vertex \( j \) to vertex \( k \) and zero otherwise.

The criteria in Theorems 1.6, 1.8, 1.10, and 1.11 below are stated for the nonautonomous
system (1.1). We begin with a combinatorial criterion.

**Theorem 1.6** The problem (1.1)–(1.3) is robust FTS if and only if the directed graph \( G_P \)
is acyclic.

The following well-known result [43] yields that the combinatorial criterion is efficiently
recognizable. At the same time, it provides an algebraic criterion of finite time stabilizability
of our problem.

**Proposition 1.7** Let \( G \) be a directed graph with adjacency \( n \times n \)-matrix \( W \). Then
\( G \) is acyclic if and only if \( W \) is nilpotent, with \( W^n = 0 \).

In the following theorem we collect a number of robust algebraic criteria.

**Theorem 1.8** Let \( P = (p_{jk}) \) be an \( n \times n \)-matrix with real entries and \( W \) be the matrix with
entries \( w_{jk} = \text{sgn} |p_{jk}| \). Then the following statements are equivalent:

(i) the problem (1.1)–(1.3) is robust finite time stabilizable;
(ii) the products \( w_{i_1i_2}w_{i_2i_3}\cdots w_{i_ni_{n+1}} \) equal zero for all tuples \( (i_1, \ldots, i_{n+1}) \in \{ 1, \ldots, n \}^{n+1} \);
(iii) all principal minors of the matrix \( W \) equal zero;
(iv) the matrix \( W \) is nilpotent, with \( W^n = 0 \).

**Corollary 1.9** Let \( P = (p_{jk}) \) be an \( n \times n \)-matrix with real entries and \( W \) be the matrix
with entries \( w_{jk} = \text{sgn} |p_{jk}| \). Assume that \( W \) is nilpotent and let \( k_0 \) be the minimum value of
\( k \leq n \) such that \( W^k = 0 \).

(i) Let
\[
a_0 = \inf \{|a_j(x,t)| : (x,t) \in \Pi, j \leq n \}.
\]

Then the optimal stabilization time admits an upper bound
\[
T_{\text{opt}} \leq \frac{k_0}{a_0}.
\]

(ii) Assume that \( a_j(x,t) \equiv a_j(x) \) do not depend on \( t \) for all \( j \leq n \). Let \( I \) be the set of all
tuples \( (i_1, \ldots, i_{k_0}) \in \{ 1, \ldots, n \}^{k_0} \) such that \( w_{i_1i_2}w_{i_2i_3}\cdots w_{i_{k_0-1}i_{k_0}} \neq 0 \). Then
\[
T_{\text{opt}} = \max \left\{ \int_0^1 \frac{dx}{|a_j(x)|} : j \leq n \right\} \text{ if } k_0 = 1,
\]
\[
T_{\text{opt}} = T^* \text{ if } k_0 = 2, 3,
\]
\[
T_{\text{opt}} \leq T^* \text{ if } k_0 > 3,
\]
where
\[ T^* = \max \left\{ \sum_{k=1}^{k_0} \int_0^1 \frac{dx}{|a_{i_k}(x)|} : (i_1, \ldots, i_{k_0}) \in I \right\}. \]

Theorem 1.8 can be recast as follows.

**Theorem 1.10** Let \( P = (p_{jk}) \) be an \( n \times n \)-matrix with real entries and \( P_{\text{abs}} = (|p_{jk}|) \). Then the following statements are equivalent:

1. the problem (1.1)–(1.3) is robust finite time stabilizable;
2. the products \( p_{i_1i_2}p_{i_2i_3} \cdots p_{i_{n+1}} \) are equal to zero for all tuples \( (i_1, \ldots, i_{n+1}) \in \{1, \ldots, n\}^{n+1} \);
3. all principal minors of the matrix \( P \) equal zero;
4. the matrix \( P_{\text{abs}} \) is nilpotent, with \( P_{\text{abs}}^n = 0 \).

Our results can be extended to the case of nonautonomous boundary conditions as follows.

**Theorem 1.11** Let \( W = (w_{jk}) \) be a constant zero-one \( n \times n \)-matrix. For every \( q_{jk} \in C^1(\mathbb{R}^+) \), where \( j, k \leq n \), the problem (1.1)–(1.3) with \( p_{jk} = q_{jk}(t)w_{jk} \) is robust finite time stabilizable if and only if the matrix \( W \) fulfills one of the conditions (ii)–(iv) of Theorem 1.8 is satisfied.

The paper is organized as follows. Sections 2.1–2.2 motivate our research and describe potential applications. Extentions to evolution families and applications to nonlinear problems are discussed in Section 2.3. There we also give examples showing the non-robustness of the spectral criterion. In Section 3 we prove our main results in Theorems 1.5, 1.6, and 1.8 stating, respectively, the spectral, combinatorial, and algebraic stabilization criteria. This section also contains the proofs of Corollary 1.9 about the optimal stabilization time and Theorem 1.11 addressing nonautonomous boundary conditions.

### 2 Motivation and comments

#### 2.1 Motivation and related work

The FTS notion is motivated by the physical question whether solutions to an asymptotically stable system reach an equilibrium point (see, e.g., [3]). Last years systems with the FTS property attract more and more attention, first of all due to applications. In particular, they are well suitable to design controllers and, therefore, are intensively studied in control and system engineering [2, 5, 21, 39, 47, 48]. Starting with the work of D. Russell [42] in control theory for linear autonomous hyperbolic systems, much research is devoted to finding boundary controls transferring the system from an arbitrary initial state to the zero state, see also [9, 17, 27]. In the present paper, instead of finding boundary controls, we provide classes of first-order hyperbolic systems ensuring the above property of the solution operator.
\( \varphi \rightarrow u(\cdot, T) \), namely the property that \( u(\cdot, T) \equiv 0 \) for all initial functions \( \varphi \). The distinct feature of our systems is that the evolution processes they describe are irreversible in time. In the literature much attention is also paid to finding or estimating the optimal stabilization time \( T_{\text{opt}} \), being of special interest for engineers. In [8], the number \( T_{\text{opt}} \) is explicitly computed for autonomous one-side control systems, where \( a_j(x, t) \equiv a_j(x) \) and \( b_j(x, t) \equiv b_j(x) \). This is also one of the themes in our paper, cf. Corollary 1.9.

The concept of FTS plays an important role also in the research on the adaptive output-feedback stabilization [25, 29] and inverse problems [46].

Another motivating area is photoacoustic imaging [13, 45]. Even basic photoacoustic tomography models demonstrate mathematical properties which are crucial for reconstruction of photoacoustic wave fields and that are closely related to FTS systems.

N. Êltysheva [18], who is the second author of the present paper, identifies a class of autonomous linear first-order hyperbolic systems with FTS property; see also [22]. This approach is based on spectral analysis.

In the present paper, we give a comprehensive FTS analysis of initial-boundary value problems for a class of nonautonomous hyperbolic systems. While nonautonomous case is less studied than the autonomous one, nonautonomous phenomena occur in many physical situations [15, 19, 25, 38].

We pay a special attention to the robustness issue, which is important in the application areas mentioned above. It is absolutely clear that solving PDE systems numerically requires a certain robustness. Moreover, concrete applications usually involve measurement data known only approximately. In the control theory, the quality of control algorithms is estimated, in particular, by robustness with respect to perturbations. The robust FTS property (or, more generally, the robust stability property [2]) is important in designing robust finite time controllers for dynamical systems under modeling uncertainty.

We remark that the robustness concept for quasilinear problems is a more complicated issue (see, e.g., [8]), and this is a topic of our ongoing work. Another interesting direction is investigation of robustness not only with respect to coefficients of the hyperbolic system but also with respect to boundary coefficients.

### 2.2 Related stability concepts

Stability properties of a dynamical system are crucial for adequate description of physical phenomena. The FTS is an important instance of more general concept of asymptotic stability. The last concept is suggested by Lyapunov in 1892 [36] and is used to describe the behavior of systems within infinite time intervals. The FTS concept is used to deal with the systems operating over finite time intervals. More precisely, the FTS systems form a subclass of asymptotically stable systems characterized by superstability property which is studied in [1, 14, 41] in the autonomous case and in [33] in the nonautonomous case.
Consider an abstract evolution equation
\[ \frac{d}{dt} x(t) = B(t)x(t), \quad x(t) \in X, \] (2.1)
on a Banach space \( X \), where \( B(t) : X \to X \) for each \( t \geq 0 \) is a linear operator.

**Definition 2.1** The system (2.1) is called **exponentially stable** if there exist positive reals \( \gamma \) and \( M \) such that every solution \( x(t) \) satisfies the estimate
\[ \|x(t)\| \leq Me^{-\gamma t}\|x(0)\|, \quad t \geq 0. \] (2.2)

The exponential stability for hyperbolic systems with linear and nonlinear boundary conditions has been intensively investigated in the literature by different methods, in particular, by the characteristic method [20, 33, 44], by the Lyapunov’s function approach [5, 6, 16], by the delay-equation approach [4, 8], and the backstepping approach [9, 11, 12, 26].

A stronger stability property is stated in the next definition.

**Definition 2.2** The system (2.1) is called **superstable** if the estimate (2.2) holds for every \( \gamma > 0 \) and some \( M = M(\gamma) \).

Roughly speaking, all solutions to superstable systems decay faster than any exponential as \( t \to \infty \).

It turns out that the concepts of superstability and FTS are the same in the autonomous case we consider. Note that this is not true, in general (see [14]). The following fact is proved in Section 3.1.3.

**Theorem 2.3** System (1.6) is superstable if and only if it is FTS.

We conclude this subsection with a remark about finite-dimensional spaces. Let \( B(t) = B : X \to X \) be an autonomous linear operator on a Banach space \( X \). Then the FTS property makes sense only if \( X \) is infinite dimensional, since otherwise the operator \( B \) has a non-empty point spectrum. In the nonlinear autonomous finite dimensional case, the FTS is investigated in the recent paper [35], see also references therein.

### 2.3 Further comments

#### 2.3.1 About extensions to evolution families

We formulate our results for the initial-boundary value problem (1.1)–(1.3) where the initial time, say \( \tau \), is fixed (to be zero). The established FTS criteria do not depend on the initial function \( \varphi \). From the dynamical point of view, it is interesting to know whether these criteria
do not also depend on the initial time $\tau$. The question can naturally be answered in terms of the evolution families generated by the problem (1.1), (1.2) with the initial conditions

$$u_j(x, \tau) = \varphi_j(x), \quad 0 \leq x \leq 1, \, j \leq n,$$

where $\tau \geq 0$ is arbitrary fixed. Existence of the evolution family on the space $L^2(0,1)^n$ in the case of autonomous boundary conditions is proved in [33, Theorem 2.3]. Note that the last result can be extended to nonautonomous boundary conditions with bounded coefficients.

Recall that (see [40, Definition 5.3]) a two-parameter family $\{U(t, \tau)\}_{t \geq \tau \geq 0}$ of linear bounded operators on a Banach space $X$ is called an evolution family if, first, $U(\tau, \tau) = I$ and $U(t, s)U(s, \tau) = U(t, \tau)$ for all $t \geq s \geq \tau$ and, second, the map $(t, \tau) \in \mathbb{R}^2 \rightarrow U(t, \tau) \in \mathcal{L}(X)$ is strongly continuous for all $t \geq \tau$. We say that an evolution family $\{U(t, \tau)\}_{t \geq \tau \geq 0}$ is generated by the problem (1.1), (1.2), (2.3) on the space $X = L^2(0,1)^n$ if, for each initial function $\varphi \in X$, the function $U(t, \tau)\varphi$ is the $L^2$-generalized solution at time $t$.

Let us introduce a dynamical notion of FTS generalizing [7, Definition 1.1].

**Definition 2.4** An evolution family $\{U(t, \tau)\}_{t \geq \tau \geq 0}$ on a Banach space $X$ is called FTS if there exist positive reals $T$ and $K$ such that $U(\tau+T, \tau) = 0$ for all $\tau \geq 0$ and $\|U(t, \tau)\|_{\mathcal{L}(X)} \leq K$ for all $t \geq \tau \geq 0$ with $t - \tau \leq 1$. The infimum value of all $T$ with the above property is called the optimal stabilization time.

Compared to Definition 1.3, the dynamical notion of FTS additionally requires uniform stabilization with respect to $\tau$ and uniform boundedness of the evolution family.

Note that the criteria given by Theorems 1.5, 1.6, and 1.8 do not depend on the initial time. Therefore, the uniform stabilization property in Definition 2.4 is satisfied automatically for the evolution family $U(t, \tau)$ generated by the problem (1.1), (1.2), (2.3) on the space $L^2(0,1)^n$. The uniform boundedness property is true whenever the coefficients $a_j$ and $b_j$ as well as the nonautonomous (if any) boundary coefficients $p_{jk}$ are bounded. This easily follows from the uniform stabilization property and the following exponential bound obtained in [33, Section 4.2]:

$$\|U(t, \tau)\|_{\mathcal{L}(L^2(0,1)^n)} \leq K_0 e^{C_0(t-\tau)} \text{ for all } t \geq \tau,$$

where constants $K_0 \geq 1$ and $C_0 > 0$ do not depend on $\tau$ and $t$. Consequently, if all the coefficients in (1.1) and (1.2) are bounded, then the criteria for FTS in the sense of Definition 1.3 remain to be true in the sense of Definition 2.4.

On the other hand, if the coefficients are not bounded, then the evolution family $\{U(t, \tau)\}_{t \geq \tau \geq 0}$ is not necessarily FTS. This follows from the following simple example (a similar example for one hyperbolic equation with an integral boundary condition is constructed in [7, Remark 1.2]):

$$\partial_t u_1 - \partial_x u_1 = 0, \quad \partial_t u_2 + \partial_x u_2 = 0,$$

$$u_1(1, t) = 0, \quad u_2(0, t) = p(t)u_1(0, t),$$

$$u_1(x, \tau) = \varphi_1(x), \quad u_2(x, \tau) = \varphi_2(x),$$
where $p(t)$ is a smooth function such that $p(t) = 0$ if $t \in [2k, 2k + 1]$ and $p(t) = t$ if $t \in [2k + \frac{5}{4}, 2k + \frac{7}{4}]$ for all $k \in \mathbb{N}$. Using the method of characteristics, one can easily show that the optimal stabilization time equals two. Moreover, for $\varphi = (\varphi_1, \varphi_2) = (1, 1)$ we have

$$
[U(2k + 3/2, 2k + 1)\varphi]_1 = \begin{cases} 1 & \text{if } 0 \leq x \leq 1/2 \\ 0 & \text{if } 1/2 < x \leq 1, \end{cases}
$$

$$
[U(2k + 3/2, 2k + 1)\varphi]_2 = \begin{cases} p(2k + 3/2 - x) & \text{if } 0 \leq x \leq 1/2 \\ 1 & \text{if } 1/2 < x \leq 1. \end{cases}
$$

Hence,

$$
\left\| U \left( \frac{2k + 3}{2}, 2k + 1 \right) \right\|_{L^2(0,1)^2} \geq \frac{1}{\| \varphi \|_{L^2(0,1)^2}} \left\| U \left( \frac{2k + 3}{2}, 2k + 1 \right) \varphi \right\|_{L^2(0,1)^2}
$$

$$
\geq \frac{1}{\sqrt{2}} \left( \int_0^{1/2} \left| p \left( \frac{2k + 3}{2} - x \right) \right|^2 dx \right)^{1/2} \geq \frac{1}{2\sqrt{2}} \left( 2k + \frac{5}{4} \right).
$$

It follows that, for any given $K > 0$, there exist $\tau \geq 0$ and $t \geq \tau$ such that $\|U(t, \tau)\|_{L^2(0,1)^2} \geq K$, as claimed.

### 2.3.2 About the nonrobustness of the spectral criterion

Consider the $2 \times 2$-system

$$
\partial_t u_1 + \partial_x u_1 = 0, \quad \partial_t u_2 - \partial_x u_2 = 0,
$$

subjected to the boundary conditions

$$
u_1(0, t) = u_1(1, t) - u_2(0, t), \quad u_2(1, t) = u_1(1, t) - u_2(0, t)$$

and the initial conditions

$$
u_1(x, 0) = \varphi_1(x), \quad u_2(x, 0) = \varphi_2(x).$$

Since

$$
\det(I_2 - \text{diag}(e^{-\lambda}, e^{-\lambda})P) = 1 - e^{-2\lambda} + e^{-2\lambda},
$$

the characteristic equation (see (3.9) below) for the operator $A$ generated by (2.4)–(2.5) and defined by (1.7)–(1.8) reads $1 = 0$. This means that $A$ has empty spectrum and, by Theorem 1.5, the problem (2.4), (2.5), (2.6) is FTS.
Next, consider the perturbed problem, with $\varepsilon$-perturbations in the leading part of the differential system. Specifically, we consider the system

$$\partial_t u_1 + (1 + \varepsilon)\partial_x u_1 = 0, \quad \partial_t u_2 - \partial_x u_2 = 0,$$

(2.7)
endowed with the boundary conditions (2.5) and the initial conditions (2.6). In this case
\[\det(I_2 - \text{diag}(e^{\frac{\lambda}{1 + \varepsilon}}, e^{-\lambda})P) = 1 + e^{-\lambda} - e^{\frac{\lambda}{1 + \varepsilon}},\]
and the characteristic equation is
\[1 + e^{-\lambda} - e^{\frac{\lambda}{1 + \varepsilon}} = 0.\]
Consequently, the operator $A$ generated by the perturbed problem has infinitely many eigenvalues and, therefore, the problem (2.7), (2.5), (2.6) is not FTS.

Finally, consider $\varepsilon$-perturbations in the lower-order part of the hyperbolic system, namely

$$\partial_t u_1 + \partial_x u_1 + \varepsilon u_1 = 0, \quad \partial_t u_2 - \partial_x u_2 = 0,$$

(2.8)
endowed with the conditions (2.5) and (2.6). The characteristic equation here reads
\[1 + e^{-\lambda} - e^{-(\lambda + \varepsilon)} = 0.\]
Again, the operator $A$ generated by this problem has infinitely many eigenvalues and, therefore, the problem (2.8), (2.5), (2.6) is not FTS (even under small perturbations of $a_j$ or $b_j$).

Remark that the matrix $P$ in the boundary conditions (2.5) does not satisfy the conditions of the robust FTS Theorem 1.8. Indeed,
\[P = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\]
and, hence, $W^2 \neq 0$.

2.3.3 About applications to nonlinear problems
In [33, Examples 3.4–3.6] we discuss problems from chemical kinetics and boundary control theory modeled by nonlinear hyperbolic initial boundary value problems. Their linearizations at stationary solutions include boundary conditions fulfilling the assumptions of Theorem 1.8. This makes the linearized problems to be perturbations of FTS problems, to which one can apply [33, Theorem 2.7] about stability properties of the perturbed problems. This, in its turn, allows one to prove the asymptotic stability of stationary solutions to the original nonlinear problems.
3 Stabilization criteria

3.1 Spectral criterion: Proof of Theorem 1.5

Here we consider the autonomous version of our problem in the abstract form (1.6).

3.1.1 Preliminary statements

By means of the (non-degenerate) change of variables \( u_j \mapsto w_j \) where

\[
w_j(x, t) = u_j(x, t) \exp \int_0^x \frac{b_j(\xi)}{a_j(\xi)} d\xi, \quad j \leq n,
\]

we rewrite the problem (1.6) in the following equivalent form:

\[
\begin{align*}
\partial_t w + A(x) \partial_x w &= 0, \quad (x, t) \in \Pi, \\
out(t) &= P_1 \vin(t), \quad t \geq 0,
\end{align*}
\]

where \( w = (w_1, \ldots, w_n) \) and

\[
P_1 = \text{diag}(1, \ldots, 1, e^{\beta_{m+1}}, \ldots, e^{\beta_n}) \text{diag}(e^{-\beta_1}, \ldots, e^{-\beta_m}, 1, \ldots, 1),
\]

\[
\beta_j = \int_0^1 \frac{b_j(\xi)}{a_j(\xi)} d\xi.
\]

Note that the problem (1.6) is FTS if and only if the problem (3.1), (3.2) with the initial conditions

\[
w_j(x, 0) = \varphi_j(x) \exp \int_0^x \frac{b_j(\xi)}{a_j(\xi)} d\xi, \quad x \in [0, 1], \quad j \leq n,
\]

is FTS.

Switching now to an abstract setting, we proceed with the problem

\[
\frac{d}{dt} u(t) = \mathcal{A}_0 u(t), \quad u(0) = \varphi \in L^2(0, 1)^n,
\]

where the operator \( \mathcal{A}_0 : D(\mathcal{A}_0) \subset L^2(0, 1)^n \mapsto L^2(0, 1)^n \) is defined by

\[
(\mathcal{A}_0 v)(x) = -A(x) \frac{dv}{dx},
\]

and

\[
D(\mathcal{A}_0) = \{ v \in L^2(0, 1)^n : \partial_x v \in L^2(0, 1)^n, \out = P_0 \vin \},
\]
the matrix $P_0$ being an arbitrary $n \times n$-matrix with constant entries. Note that $\sigma(A) = \sigma(A_0)$ where operator $A_0 : D(A_0) \subset L^2(0,1)^n \rightarrow L^2(0,1)^n$ is defined by the right hand sides of (3.5)–(3.6) with $P_0$ replaced by $P_1$ as in (3.3).

It is known from [30, Theorem 6.29] that the operator $A_0$ is closed and has finitely or countably many eigenvalues, each of finite multiplicity. Furthermore, $\sigma(A_0)$ has no finite limit points.

In [33, Lemma 4.2] we proved the following apriori estimate for the $L^2$-generalized solution $u$ to the problem (3.4):

$$
\|u(\cdot, t)\|_{L^2(0,1)^n} \leq K_0 e^{C_0 t} \|\varphi\|_{L^2(0,1)^n}, \quad t \geq 0,
$$

the positive constants $K_0$ and $C_0$ being independent of $t$ and $\varphi$. It follows that the spectrum of $A_0$ lies in the semistrip $\Re \lambda \leq C_0$. Denote

$$
\alpha_j(x) = -\int_0^x \frac{1}{a_j(\xi)} d\xi \quad \text{and} \quad \tau_j = |\alpha_j(1)| \quad \text{for all} \quad j \leq n. \quad (3.7)
$$

Set

$$
\Delta(\lambda) = \det \left( I_n - \text{diag} \left( e^{-\lambda \tau_1}, ..., e^{-\lambda \tau_n} \right) P_0 \right), \quad (3.8)
$$

where $I_n$ is the unit $n \times n$-matrix.

**Lemma 3.1** A complex number $\lambda$ is an eigenvalue of the operator $A_0$ if and only if $\lambda$ satisfies the characteristic equation

$$
\Delta(\lambda) = 0. \quad (3.9)
$$

**Proof.** By definition, $\lambda \in \sigma(A_0)$ if and only if there exists a nonzero function $y(x, \lambda) = (y_1(x, \lambda), \ldots, y_n(x, \lambda))$ in $D(A_0)$ fulfilling the equation $\lambda y = A_0 y$ or, the same, the equation

$$
\lambda y = -A(x) \frac{dy}{dx}. \quad (3.10)
$$

The general solution to (3.10) is given by the formula

$$
y_j(x, \lambda) = c_j e^{\lambda \alpha_j(x)}, \quad j \leq n, \quad (3.11)
$$

where $c_j$ are arbitrary reals. To determine $c = (c_1, \ldots, c_n)^T$, we use the boundary conditions $y_{out} = P_0 y_{in}$, which gives us the equation

$$
X(\lambda)c = 0, \quad (3.12)
$$

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where
\[ X(\lambda) = \text{diag} \left( \frac{1}{m}, \ldots, \frac{1}{m}, e^{\lambda_{m+1}(1)}, \ldots, e^{\lambda_{n}(1)} \right) \]
\[ -P_0 \text{diag} \left( e^{\lambda_1(1)}, \ldots, e^{\lambda_{m}(1)}, 1, \ldots, 1 \right) \]
\[ = \text{diag} \left( e^{-\lambda_1(1)}, \ldots, e^{-\lambda_{m}(1)}, e^{\lambda_{m+1}(1)}, \ldots, e^{\lambda_{n}(1)} \right) \]
\[ \times \left( I - \text{diag} \left( e^{-\lambda_{1}}, \ldots, e^{-\lambda_{n}} \right) P_0 \right) \text{diag} \left( e^{\lambda_1(1)}, \ldots, e^{\lambda_{m}(1)}, 1, \ldots, 1 \right). \]

(3.13)

It follows that
\[ \det(X(\lambda)) = \Delta(\lambda) \exp \left\{ \lambda \sum_{j=m+1}^{n} \alpha_j(1) \right\}. \]

(3.14)

Therefore, the equation (3.12) has a nonzero solution if and only if \( \lambda \) is a solution to (3.9), as desired.

\( \square \)

Remark 3.2 The function \( \Delta(\lambda) \) given by (3.8) is a Dirichlet polynomial. Specifically,
\[ \Delta(\lambda) = 1 + \sum_{k=1}^{M} E_k e^{-\lambda r_k}, \]
where \( M \geq 1 \) is an integer, \( r_1 < r_2 < \ldots < r_M \) are reals expressed in terms of \( \tau_k \), and \( E_k \) are reals expressed in terms of the entries of the matrix \( P_0 \) as well as the coefficients of the hyperbolic system.

It is known from [34, p. 266-268] that if at least one of the coefficients \( E_k \) is nonzero, then \( \Delta(\lambda) \) has a countable number of zeros, lying in a strip which is parallel to the imaginary axis. Consequently, the equation \( \Delta(\lambda) = 0 \) has no solutions in the complex plane if and only if \( \Delta(\lambda) \equiv 1 \) or, the same, if and only if \( E_k = 0 \) for all \( k \leq M \).

We conclude this subsection with a technical lemma.

Lemma 3.3 Let \( r \in C^1([0,1]) \) and let \( q(x, \xi) \) be a continuously differentiable function in \( x \in [0,1] \) and two times continuously differentiable in \( \xi \in [0,1] \). Suppose that \( \partial_{\xi} q(x, \xi) \neq 0 \) and there exists \( q_0 \geq 0 \) such that
\[ q(x, \xi) \geq -q_0 \]
for all \( x, \xi \in [0,1] \). Then for all \( t > q_0 \) and \( \gamma > 0 \) it holds
\[ \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \int_{0}^{1} e^{\lambda q(x, \xi)} r(\xi) d\xi d\lambda \equiv 0. \]

(3.16)
Proof. Denote the left hand side of (3.16) by $I(x, t)$. Integrating by parts, we get

$$I(x, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{\lambda} \left( e^{\lambda q(x, \xi)} r(\xi) \right)_{\xi=0}^{\xi=1} - \int_{0}^{1} e^{\lambda q(x, \xi)} \partial_{\xi} \left( \frac{r(\xi)}{\partial_{\xi} q(x, \xi)} \right) d\xi \, d\lambda. \quad (3.17)$$

It is known that for all $\gamma > 0$

$$\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda(t-t_0)}}{\lambda} d\lambda = \begin{cases} 1 & \text{if } t > t_0 \\ 0 & \text{if } t < t_0. \end{cases}$$

Introduce a function

$$g(t, x, \xi) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda(t+q(x, \xi))}}{\lambda} d\lambda = \begin{cases} 1 & \text{if } t + q(x, \xi) > 0 \\ 0 & \text{if } t + q(x, \xi) < 0. \end{cases}$$

Suppose that $t \geq q_0 + \delta$ for some $\delta > 0$. By (3.15), we have $g(t, x, \xi) \equiv 1$ for all $x, \xi \in [0, 1]$. By the Dirichlet criterion of the uniform convergence of improper integrals, the integral

$$\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda(t+q(x, \xi))}}{\lambda} \partial_{\xi} \left( \frac{r(\xi)}{\partial_{\xi} q(x, \xi)} \right) d\lambda$$

converges uniformly in $\xi \in [0, 1]$ and, moreover, equals $\partial_{\xi} \left( \frac{r(\xi)}{\partial_{\xi} q(x, \xi)} \right)$. Changing the order of integration in the right hand side of (3.17), we get

$$I(x, t) = \left. \frac{r(\xi)}{\partial_{\xi} q(x, \xi)} \right|_{\xi=0}^{\xi=1} - \int_{0}^{1} \partial_{\xi} \left( \frac{r(\xi)}{\partial_{\xi} q(x, \xi)} \right) d\xi \equiv 0,$$ 

as desired. \hfill \Box

### 3.1.2 Proof of Theorem 1.5

As it follows from Section 3.1.1, it suffices to prove that the problem (3.4) is FTS if and only if $\sigma(A_0) = \emptyset$.

**Necessity.** Let the problem (3.4) be FTS. If $\lambda \in \sigma(A_0)$, then, on the account of (3.11), the function $[u(t)](x) = u(x, t) = e^{\lambda t} y(x, \lambda)$ is a solution to the problem (3.4) with $\varphi(x) = y(x, \lambda)$, that is not equal to zero for all $t \geq 0$. This contradicts to the assumption that (3.4) is FTS. Therefore, $\sigma(A_0) = \emptyset$.

**Sufficiency.** Suppose that $\sigma(A_0) = \emptyset$. Due to Definition 1.1, it suffices to prove that there is $T > 0$ such that, given $\varphi \in C_0^\infty([0, 1])^n$, the corresponding continuously differentiable solution to the problem (3.4) equals zero for $t > T$. 

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Fix an arbitrary $\varphi \in C_\infty^0([0,1])^n$. Let $u(t)$ be the continuously differentiable solution to the problem (3.4) in $\Pi$. We use the following apriori estimate (see, e.g., [37, Estimate (6)]):
\[
\|u(t)\|_{C^1([0,1])^n} \leq K_1 e^{C_1 t}\|\varphi\|_{C^1([0,1])^n}, \quad t \geq 0,
\]
with constants $K_1$ and $C_1$ not depending on $t$ and $\varphi$. This estimate allows us to apply to (3.4) the Laplace transform in $t$, with the parameter $\lambda$. In the new unknown
\[
\tilde{u}(x, \lambda) = \int_0^\infty e^{-\lambda t} u(x, t) \, dt, \quad t > 0, \quad \text{Re} \lambda > C_1,
\]
the problem (3.4) reads
\[
\lambda \tilde{u} - A_0 \tilde{u} = \varphi(x). \tag{3.18}
\]
Now we solve (3.18) and show that the solution
\[
u(x, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda} \tilde{u}(x, \lambda) \, d\lambda, \quad t > 0, \quad \gamma > C_1, \tag{3.19}
\]
to the problem (3.4) becomes zero for $t > T$ for some $T \geq 0$ to be specified below.

A general solution to the equation (3.18) is given by the formula
\[
\tilde{u}(x, \lambda) = z(x, \lambda) + y(x, \lambda), \tag{3.20}
\]
where for a partial solution $z = (z_1, \ldots, z_n)$ to the nonhomogeneous equation (3.18) we have the representation
\[
z_j(x, \lambda) = \int_0^x \exp \left\{ -\lambda \int_\xi^x \frac{d\tau}{a_j(\tau)} \right\} \varphi_j(\xi) a_j(\xi) \, d\xi, \quad j \leq n, \tag{3.21}
\]
while $y(x, \lambda)$ is a general solution to the homogeneous equation (3.10) and therefore is defined by the formula (3.11). We now determine $c$ in (3.11). For that we use the fact that the function $\tilde{u}(x, \lambda)$ fulfills the boundary conditions $\tilde{u}_{out} = P_0 \tilde{u}_{in}$ and get
\[
X(\lambda)c = b, \tag{3.22}
\]
where
\[
b = P_0(z_1(1, \lambda), \ldots, z_m(1, \lambda), 0, \ldots, 0)^T - (0, \ldots, 0, z_{m+1}(1, \lambda), \ldots, z_n(1, \lambda))^T.
\]
Since $\sigma(A_0) = \emptyset$, the function $X(\lambda)$ is invertible for all complex numbers $\lambda$. From Lemma 3.1 and Remark 3.2 it follows that $\Delta(\lambda) \equiv 1$. Therefore, the representation of $\det(X(\lambda))$ given by (3.14) reads
\[
\det(X(\lambda)) = \exp \left\{ \lambda \sum_{j=m+1}^n \alpha_j(1) \right\}.
\]
Set $X^{-1}(\lambda) = \{\kappa_{ij}(\lambda)\}_{i,j=1}^n$ and find $\kappa_{ij}$. On the account of the representation (3.13) for the matrix $X(\lambda)$ we conclude that $\kappa_{ij}(\lambda)$ are entire functions of $\lambda$ of the type $\kappa_{ij}(\lambda) = \sum_{k=1}^{m_{ij}} \gamma_k e^{\lambda \mu_k}$, where the reals $\mu_k$ are determined by $\alpha_j(1)$, $j \leq n$, while the reals $\gamma_k$ are determined by the elements of the matrix $P_0$. Moreover, there exists a positive real $\mu^*$ such that for all $k$

$$-\mu^* \leq \mu_k \leq \mu^*. \quad (3.23)$$

Taking into account the equation (3.22) which reads $c = X^{-1}(\lambda)b$, for all $j \leq n$ we obtain the following formula:

$$c_j = \sum_{k=1}^m \kappa_{jk}(\lambda) \sum_{i=1}^m p^0_{ki} z_i(1, \lambda) + \sum_{k=m+1}^n \kappa_{jk}(\lambda) \left( \sum_{i=1}^m p^0_{ki} z_i(1, \lambda) - z_k(1, \lambda) \right),$$

where $p^0_{jk}$ are the entries of the matrix $P_0$. Therefore, for every $j \leq n$ the function $y_j(x, \lambda)$ is a linear combination of entire functions in $\lambda$, of the following type:

$$\int_0^1 \exp \left\{ \lambda \left( \alpha_j(x) - \int_0^1 \frac{d\tau}{a_j(\tau)} + \mu_k \right) \right\} \frac{\varphi_l(\xi)}{a_l(\xi)} d\xi, \quad (3.24)$$

where $l \in \{1, \ldots, n\}$ and $\mu_k$ satisfy (3.23).

Next we apply Lemma 3.3 to the functions $y(x, \lambda)$ and $z(x, \lambda)$ in (3.20). On the account of (3.21) and (3.24), the function $u(x, t)$ given by (3.19) equals zero for $t > T$, where

$$T = \max_{j,l \leq n} \max_{\mu_k, x, \xi \in [0,1]} \left( \int_0^1 \frac{d\tau}{a_j(\tau)} - \alpha_j(x) - \mu_k, \int_0^x \frac{d\tau}{a_j(\tau)} \right).$$

Since $\sigma(A_0) = \emptyset$, the problem (3.4) is FTS. Theorem 1.5 is therefore proved.

**Corollary 3.4** If the problem (1.1)–(1.3) is robust FTS, then all principal minors of the matrix $P$ are equal to zero.

**Proof.** Due to Definition 1.4, it suffices to prove the desired statement for a partial case of the problem (1.1)–(1.3), namely for (3.4) (autonomous version of (1.1)–(1.3) with $b_j \equiv 0$). Assume that (3.4) is robust FTS and prove that all principal minors of the matrix $P_0$ equal zero.

Let $(P_0)_{j_1j_2\ldots j_l}$ be the principal minor of the matrix $P_0$, i.e. the determinant of the restriction of $P_0$ to the rows and columns with indices $j_1, j_2, \ldots, j_l$. By Theorem 1.5, the problem (3.4) is FTS iff the characteristic equation $\Delta(\lambda) = 0$ has no solutions. The function $\Delta(\lambda)$ can be expressed as follows:

$$\Delta(\lambda) = \det \left( I_n - \text{diag}(e^{-\lambda \tau_1}, \ldots, e^{-\lambda \tau_n})P_0 \right) =$$

$$1 + \sum_{l=1}^n (-1)^l \sum_{1 \leq j_1 < j_2 < \cdots < j_l \leq n} e^{-\lambda(\tau_{j_1} + \tau_{j_2} + \cdots + \tau_{j_l})} (P_0)_{j_1j_2\ldots j_l}, \quad (3.25)$$

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Indeed, recall the following well-known formula for the characteristic polynomial of the matrix $P_0$:

$$
\det(\lambda I_n - P_0) = \lambda^n - \lambda^{n-1} \sum_{j=1}^{n} (P_0)_j + \sum_{k=2}^{n} (-1)^k \lambda^{n-k} \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} (P_0)_{j_1 j_2 \cdots j_k}. \quad (3.26)
$$

To derive (3.25), we apply the same argument as for deriving (3.26) and, therefore, get

$$
\Delta(\lambda) = \det(I_n - \text{diag}(e^{-\lambda \tau_1}, \ldots, e^{-\lambda \tau_n}) P_0)
= e^{-\lambda \sum_{i=1}^{n} \tau_i} \det(\text{diag}(e^{\lambda \tau_1}, \ldots, e^{\lambda \tau_n}) I_n - P_0)
= 1 - \sum_{j=1}^{n} e^{-\lambda\tau_j} (P_0)_j + \sum_{k=2}^{n} (-1)^k \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} e^{-\lambda(\tau_{j_1} + \tau_{j_2} + \cdots + \tau_{j_k})} (P_0)_{j_1 j_2 \cdots j_k}.
$$

The formula (3.25) now easily follows.

By assumption, the problem (3.4) is FTS for any matrix $A$. Hence, due to Theorem 1.5, for any fixed $P_0$ and all positive reals $\tau_1, \ldots, \tau_n$ we have $\Delta(\lambda) \equiv 1$. On the account of (3.25), this means that

$$
\sum_{l=1}^{n} (-1)^l \sum_{1 \leq j_1 < j_2 < \cdots < j_l \leq n} e^{-\lambda(\tau_{j_1} + \tau_{j_2} + \cdots + \tau_{j_l})} (P_0)_{j_1 j_2 \cdots j_l} = 0 \quad \text{for all } \lambda \in \mathbb{C}. \quad (3.27)
$$

If $b_1$ and $b_2$ are distinct reals, then the functions $e^{-\lambda b_1}$ and $e^{-\lambda b_2}$ of a complex variable $\lambda$ are linearly independent. Appropriately choosing $\tau_l$ for each $l \leq n$, we will show that the principal minors $(P_0)_{j_1 j_2 \cdots j_l}$ are equal to zero for all $l \leq n$. Set $\tau_0 = 1/(n + 1)$. Fix arbitrary $1 \leq j_1 < j_2 < \cdots < j_l \leq n$. Put $\tau_{j_1} = \tau_{j_2} = \cdots = \tau_{j_l} = 1$ and $\tau_i = \tau_0$ for all $i \neq j_k, k \leq l$. Then one summand in (3.27) is $(P_0)_{j_1 j_2 \cdots j_l} e^{-\lambda \tau}$, while all other summands include factors of the type $e^{-\lambda \tau_r}$ with $r \neq l$. It follows that $(P_0)_{j_1 j_2 \cdots j_l} = 0$, as desired. \hfill \Box

### 3.1.3 Proof of Theorem 2.3

If the problem (1.6) is superstable, then the resolvent $R(\lambda; A)$ of the infinitesimal operator $A$ generated by (1.6) is an entire function of a complex variable $\lambda$. This follows from the resolvent formula [40]

$$
R(\lambda; A)x = \int_{0}^{\infty} e^{-\lambda t} T(t)x \, dt, \quad x \in L^2(0,1)^n,
$$

where $T(t)$ is a $C_0$-semigroup generated by the operator $A$ and satisfying the estimate $\|T(t)\| \leq M(\gamma) e^{-\gamma t}$ for any $\gamma > 0$. Consequently, $\sigma(A) = \emptyset$.

On the other side, Theorem 1.5 says that the operator $A$ given by (1.7), (1.8) has empty spectrum if and only if the problem (1.6) is finite time stabilizable.
Remark 3.5 As pointed out by an anonymous reviewer, Theorems 1.5 and 2.3 (autonomous setting) in the case $b_j \equiv 0$ (this involves no loss of generality) and $a_j > 0$ could also be derived using time-delay equations. Indeed, the integration of the system along the characteristic directions results in the following functional-delay system; see also [6, (3.16)]:

$$
\phi_j(t) = \sum_{k=1}^{n} p_{jk} \phi_k(t - \tau_k), \quad i \leq n,
$$

(3.28)

where $\phi_j(t) = u_j(0, t)$ and $\phi_j(\xi)$ for $\xi \in [-\tau_j, 0]$ are known functions of the initial conditions (1.3). By a classical result on linear time-delay systems (see, e.g., [23, Theorem 3.5] and [6, (3.17)]), system (3.28) is exponentially stable if and only if there exists $\delta > 0$ such that

$$
\Delta(\lambda) = 0 \text{ implies } \Re \lambda \leq -\delta,
$$

(3.29)

where $\Delta(\lambda)$ is given by (3.8) with the matrix $P = (p_{jk})$ in place of $P_0$; see also Lemma 3.1. From (3.29) it follows that the system (3.28) (and, therefore, the original autonomous problem (1.1)–(1.3)) is superstable if and only if

$$
\Delta(\lambda) \neq 0 \quad \text{for all } \lambda \in \mathbb{C}.
$$

(3.30)

Under the condition (3.30), the FTS property can be obtained from appropriate estimates for the exponential decay as applied in [5] and [10]. We are thankfull to the reviewer for drawing our attention to this approach.

### 3.2 Sufficient condition for robust FTS

Due to the discussion in Subsection 2.3.2, the spectral criterion is not robust in the sense of Definition 1.4. To prove robust FTS criteria stated in Theorems 1.6, 1.8, and 1.10, we first give a sufficient condition for robust FTS, namely the condition (3.33) in Lemma 3.6 below.

The regularity assumptions (1.5) imposed on the coefficients of the system (1.1) allow us to put the problem (1.1)–(1.3) into a smooth setting whenever the initial function $\varphi$ is sufficiently smooth. We use integration along characteristic curves: For given $j \leq n$, $x \in [0, 1]$, and $t \geq 0$, the $j$-th characteristic of (1.1) passing through the point $(x, t) \in \Pi$ is defined as the solution $\omega_j(\xi) = \omega_j(\xi, x, t)$ to the initial value problem

$$
\partial_\xi \omega_j(\xi, x, t) = \frac{1}{a_j(\xi, \omega_j(\xi, x, t))}, \quad \omega_j(0, x, t) = t.
$$

The characteristic curve $\tau = \omega_j(\xi, x, t)$ reaches the boundary of $\Pi$ in two points with distinct ordinates. Let $x_j(x, t)$ denote the abscissa of that point whose ordinate is smaller. Denote

$$
c_j(\xi, x, t) = \exp \int_{x}^{\xi} \left( \frac{b_j}{a_j} \right) (\eta, \omega_j(\eta, x, t)) d\eta
$$

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and define a linear operator \( R : C(\Pi)^n \to C(\mathbb{R}_+)^n \) by
\[
(Ru)_j(t) = \sum_{k=1}^m p_{jk}u_k(1, t) + \sum_{k=m+1}^n p_{jk}u_k(0, t), \quad j \leq n.
\]

It is straightforward to show that a \( C^1 \)-map \( u : \Pi \to \mathbb{R}^n \) is a classical solution to the problem (1.1)–(1.3) if and only if it satisfies the following system of functional equations:
\[
u_j(x, t) = \begin{cases} 
  c_j(x_j(x, t), x, t) (Ru)_j(\omega_j(x_j(x, t))) & \text{if } x_j(x, t) = 0 \text{ or } x_j(x, t) = 1 \\
  c_j(x_j(x, t), x, t) \varphi_j(x_j(x, t)) & \text{if } x_j(x, t) \in (0, 1).
\end{cases}
\tag{3.31}
\]

A continuous function \( u \) satisfying (3.31) in \( \Pi \) is called a continuous solution to (1.1)–(1.3).

Introduce a linear operator \( S : C(\mathbb{R}_+)^n \to C(\Pi)^n \) by
\[
(Sv)_j(x, t) = c_j(x_j(x, t), x, t)v_j(\omega_j(x_j(x, t), x, t)), \quad j \leq n.
\]

Due to (3.31), for given \( l \in \mathbb{N} \), the continuous solution to (1.1)–(1.3) satisfies the equation
\[
u(x, t) = [(SR)^lu](x, t) \quad \text{for all } t > l/a.
\tag{3.32}
\]

Hence, the stabilization properties of the problem under consideration are closely related to the powers of the linear operator \( SR : C(\Pi)^n \to C(\Pi)^n \).

Lemma 3.6 Let \( n \geq 2 \). Assume that
\[
p_{i_1i_2i_3\ldots i_{n+1}} = 0 \quad \text{for all tuples } (i_1, i_2, \ldots, i_n, i_{n+1}) \in \{1, \ldots, n\}^{n+1}.
\tag{3.33}
\]
Then the problem (1.1)–(1.3) is robust finite time stabilizable.

Proof. Since (3.33) does not depend on \( a_j \) and \( b_j \), then, on the account of (3.32), we are done if we prove that
\[
[(SR)^lu](x, t) \equiv 0 \quad \text{for all } t > n/a, \ u \in C(\Pi)^n.
\tag{3.34}
\]

Assume first that \( n = 2 \). The condition (3.34) in this case reads
\[
c_j(x_j, x, t) (RSRu)_j(\omega_j(x_j, x, t)) \equiv 0 \quad \text{for all } t > 2/a, \ u \in C(\Pi)^2, \text{ and } j \leq 2,
\]
and is satisfied whenever
\[
(RSRu)(t) \equiv 0 \quad \text{for all } t > 2/a \text{ and } u \in C(\Pi)^2.
\tag{3.35}
\]
Here

\[(RSRu)_j(t) = \sum_{k=1}^{2} p_{jk}(SRu)_k(1 - x_k, t)\]
\[= \sum_{k=1}^{2} p_{jk}c_k(x_k, 1 - x_k, t)(Ru)_k(\omega_k(x_k, 1 - x_k, t))\]
\[= \sum_{k=1}^{2} p_{jk}c_k(x_k, 1 - x_k, t) \sum_{i=1}^{2} p_{ki}u_i(1 - x_i, \omega_k(x_k, 1 - x_k, t)),\]  

(3.36)

where we used the shorthand notation \(x_j = x_j(x, t)\) for \(j \leq n\). At the same time, the condition (3.33) for \(n = 2\) reads

\[p_{jk}p_{ki} = 0 \quad \text{for all } j, k, i \leq 2.\]  

(3.37)

The lemma now follows immediately from the equations (3.35)–(3.37).

The proof for \(n = 3\) is similar, with analogs of (3.35), (3.36), and (3.37) being

\[(RSRSRu)(t) \equiv 0 \quad \text{for all } t > 3/a \text{ and } u \in C(\Pi)^3,\]

\[(RSRSRu)_j(t) = \sum_{k=1}^{3} p_{jk}c_k(x_k, 1 - x_k, t) \sum_{i=1}^{3} p_{ki}c_i(x_i, 1 - x_i, \omega_k(x_k, 1 - x_k, t))\]
\[\times \sum_{s=1}^{3} p_{is}u_s(1 - x_s, \omega_i(x_i, 1 - x_i, \omega_k(x_k, 1 - x_k, t))),\]  

(3.38)

and

\[p_{jk}p_{ki}p_{is} = 0 \quad \text{for all } j, k, i, s \leq 3,\]  

respectively.

It is clear that a similar argument works for any subsequent \(n\). \qed

Remark 3.7 Lemma 3.6 proves the implication \((i) \implies (ii)\) of Theorem 1.10 by showing that the condition (3.33) is sufficient for the robust FTS property. In the autonomous setting, the proof of this lemma also suggests a possibility of proving the inverse implication \((ii) \implies (i)\) using the apparatus of delay equations. Assume, for simplicity, that \(n = 3\) and \(b_j = 0\) for \(j \leq 3\). According to (3.32) and (3.38), the solution to the autonomous system restricted to \(\partial\Pi\) satisfies the following delay system:

\[\phi_j(t) = \sum_{k=1}^{3} \sum_{i=1}^{3} \sum_{s=1}^{3} p_{jk}p_{ki}p_{is}\phi_s(t - \tau_i - \tau_k - \tau_j), \quad j \leq 3, \ t > 3/a,\]  

(3.39)
where \( \phi_j(t) = u_j(x_j, t) \) and \( \tau_j \) are defined by (3.7). This system could be used to derive the condition (3.33) from the robust FTS property, completing the proof of the equivalence \((i) \iff (ii)\) of Theorem 1.10. Note, however, that this way requires a certain amount of tedious work, mainly because the right hand side of (3.39) involves all functions \( \phi_j, j \leq 3 \).

We follow a different, combinatorial approach as presented in the next subsection, which allows us to treat the general nonautonomous setting.

### 3.3 Combinatorial criterion: Proof of Theorem 1.6

Recall that, given an \( n \times n \)-matrix \( P = (p_{jk}) \), the graph \( G_P \) was defined as the directed graph with the adjacency matrix \( W = (\text{sgn}|p_{jk}|) \).

**Sufficiency.** Suppose that \( G_P \) is acyclic. Consider an arbitrary sequence \( j_1, \ldots, j_{n+1} \), where \( j_i \in \{1, \ldots, n\} \). It must contain two equal elements. Let \( j_k = j_l \) for \( k \leq l \) such that the difference \( l - k \) is minimum possible. Since the subsequence \( j_k, \ldots, j_l \) does not form a cycle in \( G_P \), there exists \( s \) such that \( k \leq s < l \) and \( (j_s, j_{s+1}) \) is not a directed edge of \( G_P \), that is \( p_{j_s,j_{s+1}} = 0 \). It follows that \( p_{j_1j_2}p_{j_2j_3}\cdots p_{j_{n}j_{n+1}} = 0 \) and, therefore, the condition (3.33) is fulfilled. By Lemma 3.6, the problem (1.1)–(1.3) is robust FTS.

**Necessity.** Assume that the problem (1.1)–(1.3) is robust FTS. By Corollary 3.4, all principal minors of the matrix \( P \) are equal to zero. The acyclicity of \( G_P \) follows from the following lemma.

**Lemma 3.8** \( G_P \) is acyclic if and only if all principal minors of the matrix \( P \) equal zero.

**Proof.** **Necessity.** Assume that \( G_P \) is acyclic. We first show that \( \det P = 0 \). Indeed,

\[
\det P = \sum_{\sigma} \text{sgn}(\sigma)p_{1\sigma(1)}p_{2\sigma(2)}\cdots p_{n\sigma(n)} = \sum_{\sigma} s(\sigma),
\]

the sum being taken over all permutations \( \sigma \) of the set \( \{1, 2, \ldots, n\} \). Consider a permutation \( \sigma \). This permutation decomposes into the product of independent cycles. Let \( (j_1, \ldots, j_l) \) be one of them. The sequence \( j_1, \ldots, j_l \) does not form a cycle in \( G_P \), which means that there exists \( s \leq l \) such that \( (j_s, j_{s+1}) \) is not a directed edge of \( G_P \), where \( \oplus \) is addition modulo \( l \), that is \( p_{j_s,j_{s+1}} = 0 \). It follows that \( p_{j_1j_2}p_{j_2j_3}\cdots p_{j_{l}j_1} = 0 \). Therefore, \( s(\sigma) = 0 \) for each \( \sigma \), implying that \( \det P = 0 \).

To finish this part of the proof, it remains to note that every principal minor of \( P \) determines the adjacency matrix of a subgraph of \( G_P \). As all subgraphs of a directed acyclic graph are acyclic, the same argument as above implies that all principal minors of \( P \) are equal to zero.

**Sufficiency.** We prove that, if \( G_P \) has a cycle, then \( P \) has a nonzero principle minor. The proof is by induction on \( n \).

The base case \( n = 1 \) is trivial: If \( G_P \) is a cycle, then \( w_{11} = 1 \) and, hence, \( p_{11} \neq 0 \).
Assume that the claim is true for all matrices of order $1, \ldots, n - 1$. Given a matrix $P$ of order $n$, consider two cases.

Case 1: $G_P$ has a cycle $j_1, \ldots, j_l$ of length $l < n$. Let $G'_P$ be a subgraph of $G_P$ induced on the vertices $j_1, \ldots, j_l$, and $P'$ be the restriction of $P$ to the rows and columns with indices $j_1, \ldots, j_l$. Since $G'$ has a cycle, the induction assumption implies that $P'$ has a nonzero principle minor, which is also a principle minor of $P$.

Case 2: $G_P$ has a cycle $C$ of length $n$ and does not have any shorter cycles. Without loss of generality we can assume that $C$ is formed by the vertices $1, 2, \ldots, n$ in this order. Note that $C$ is a unique cycle of length $n$ in $G_P$. Indeed, any other cycle would contain an arrow $(i, j)$ absent in $C$. This is, however, impossible because this arrow would form a shorter cycle along with a part of $C$.

Let us prove that $\det P \neq 0$. If a permutation $\sigma$ in the equality (3.40) is not cyclic, then $s(\sigma) = 0$ as in the necessity part. If $\sigma$ is cyclic but different from $(1, 2, \ldots, n)$, then $s(\sigma) = 0$ as well, because $s(\sigma)$ contains a factor $p_{ij} = 0$ for an arrow $(i, j)$ not in $C$. Therefore, $\det P = s(\sigma)$ for $\sigma = (1, 2, \ldots, n)$, that is, $\det P = \text{sgn}(\sigma)p_{12}p_{23}\cdots p_{nn+1} \neq 0$. The proof is therewith complete. □

**Corollary 3.9** A constant zero-one $n \times n$-matrix $W = (w_{jk})$ is the adjacency matrix of an acyclic directed graph if and only if all principal minors of $W$ are equal to zero.

### 3.4 Algebraic criterion

#### 3.4.1 Proof of Theorem 1.8

Claims $(i)$ and $(iv)$ are equivalent accordingly to Theorem 1.6 and Proposition 1.7. The equivalence of Claims $(i)$ and $(iii)$ follows from Theorem 1.6 (see also Corollary 3.9).

To finish the proof, let us show the equivalence of Claims $(i)$ and $(ii)$. Assume to the contrary that the problem (1.1)–(1.3) is robust FTS and there exists a nonzero product of the type $w_{i_1i_2}w_{i_2i_3}\cdots w_{i_ni_{n+1}}$ where $(i_1, i_2, \ldots, i_n, i_{n+1}) \in \{1, \ldots, n\}^{n+1}$. Since $W$ is an adjacency matrix of a directed graph $G_P$, then from the definition of an adjacency matrix we conclude that then there is a cycle in $G_P$. In other words, $G_P$ is not acyclic, contradicting Theorem 1.6.

#### 3.4.2 Proof of Corollary 1.9

Similarly to the proof of Lemma 3.6, if $W^{k_0} = 0$, then

$$(SR)^{k_0}u(x, t) = 0 \quad \text{for all } t > k_0/a_0 \text{ and } u \in C(\Pi)^n.$$ 

Taking this into account, Part $(i)$ readily follows from (3.32) with $l = k_0$.
Part \((u)\) for \(k_0 = 1\) straightforwardly follows from the solution formula (3.31) and the fact that \(p_{jk} = 0\) for all \(j \leq n\) and \(k \leq n\) whenever \(k_0 = 1\).

Now, let \(k_0 \geq 2\) and prove that \(T_{opt} \leq T^*\). We rewrite the problem (1.1)–(1.3) in an equivalent form. Since the coefficients \(a_j\) do not depend on \(t\), the characteristic curves are well defined for each \(t \in \mathbb{R}\). We extend the functions \(b_j(x, t)\) by setting \(b_j(x, t) \equiv 0\) for \(t < 0\). In the domain \(\Pi\), consider the problem (1.1), (1.2) with the delay boundary conditions

\[
    u_j(x_j, t) = \varphi_j(0, x_j, t), \quad t \in [-\tau_j, 0], \quad j \leq n, \tag{3.41}
\]

where \(\tau_j\) are defined by (3.7), \(x_j = 0\) for \(j \leq m\) and \(x_j = 1\) for \(m + 1 < j \leq n\), and the map \(\tau \mapsto \sigma_j(\tau, x_j, t)\) is the inverse of the characteristic \(\xi \mapsto \omega_j(\xi, x_j, t)\).

Note that the functions \(c_j(x_j, x, t)\) are continuous in \((x, t) \in \Pi\). In the framework of the method of characteristics, any continuous solution \(u\) to the problem (1.1), (1.2), (3.41) in \(\Pi\) is determined by the formula

\[
    u_j(x, t) = \begin{cases} 
    c_j(x_j, x, t) (Ru)_j (\omega_j(x_j, x, t)) & \text{if } \omega_j(x_j, x, t) \geq 0 \\
    c_j(x_j, x, t) \varphi_j(0, x_j, \omega_j(x_j, x, t)) & \text{if } \omega_j(x_j, x, t) < 0,
    \end{cases} \tag{3.42}
\]

where \(j \leq n\). Since \(b_j \equiv 0\) for \(t < 0\), we have the obvious equality \(c_j(x_j, x, t) = c_j(x_j(x, t), x, t)\) for all \((x, t) \in \Pi\). Hence, the functions \(u_j\) given by (3.42) do not depend on \(b_i\) for \(t < 0\) and \(i \leq n\). Now, we conclude that the formula (3.42) for continuous solutions to the problem (1.1), (1.2), (3.41) is an equivalent form of (3.31) for continuous solutions to the problem (1.1)–(1.3).

Suppose that \(k_0 = 3\) (the same argument works for any other \(k_0 \geq 2\)). Let us introduce the following notation:

\[
    y_j = 1 - x_j, \\
    t_{jki}(x) = \int_{x_j}^x \frac{dx}{a_j(x)} + \int_0^1 \frac{dx}{|a_k(x)|} + \int_0^1 \frac{dx}{|a_i(x)|}, \\
    c_{jki}(x, t) = c_j(x_j, x, t) c_k(x_k, y_k, \omega_j(x_j, x, t)) c_i(x_i, y_i, \omega_k(x_k, y_k, \omega_j(x_j, x, t))).
\]

Notice that \(c_{jki}(x, t) \neq 0\) for all \((x, t) \in \Pi\) and \(j, k, i \leq n\).

To prove that \(T_{opt} \leq T^*\), it suffices to show that for \(u\) given by (3.42) it holds \(u_j \equiv 0\) for all \(t \geq T^*\) and \(j \leq n\). Let \(t \geq T^*\) be arbitrary fixed. We have \(t - t_{jki}(x) \geq 0\) for all \((j, k, i) \in I\) and \(x \in [0, 1]\). For every \(j \leq n\) and \(x \in [0, 1]\), the value \(u_j(x, t)\) can be computed by iterating three times the first line of the formula (3.42). We, therefore, get

\[
    u_j(x, t) = [(SR)^3] u_j(x, t) = \sum_{k,i,s=1}^{n} \sum_{(j,k,i) \in I} p_{jkp_{kis}} c_{jki}(x, t) u_s(x_s, t - t_{jki}(x)), \tag{3.43}
\]

where all products \(p_{jkp_{kis}}\) equal zero by the assumption. Hence, \(u_j(\cdot, t) \equiv 0\), as desired.
Finally, let \( k_0 = 3 \) and show that \( T_{opt} = T^* \) in this case (the same argument works for \( k_0 = 2 \)). We are done if we prove that for any \( \varepsilon \in (0, \min_{j \leq n} \tau_j) \) there exists a function \( \varphi \) and an index \( j \leq n \) such that \( u_j(y_j, T^* - \varepsilon) \neq 0 \), where \( u_j \) is given by (3.42). Fix an arbitrary \( \varepsilon \in (0, \min_{j \leq n} \tau_j) \). Due to the definition of \( T^* \), there exists a triple, say \( (\alpha, \beta, \gamma) \in I \), such that \( T^* = t_{\alpha\beta\gamma}(y_\alpha) \). Since \( t_{jki}(y_j) \leq T^* \), we have \( T^* - \varepsilon - t_{jki}(y_j) > -\min_{j \leq n} \tau_j \) for all \( (j, k, i) \in I \). Moreover, \( T^* - \varepsilon - t_{\alpha\beta\gamma}(y_\alpha) < 0 \). This means that we can compute the value 
\( u_\alpha(y_\alpha, T^* - \varepsilon) \) by iterating twice the first line of the formula (3.42) and applying once the operator \( S \) of integration along characteristics. Specifically,

\[
\begin{align*}
    u_\alpha(y_\alpha, T^* - \varepsilon) &= [(SR)^2 Su]_\alpha(y_\alpha, T^* - \varepsilon) \\
    &= \sum_{k, i, j} p_{akp} c_{aki}(y_\alpha, T^* - \varepsilon) u_i(x_i, T^* - \varepsilon - t_{akl}(y_\alpha)).
\end{align*}
\]

Consider an arbitrary pair \((k, i)\) such that \((\alpha, k, i) \in I \) and \( T^* - \varepsilon - t_{akl}(y_\alpha) \geq 0 \). The value \( u_i(x_i, T^* - \varepsilon - t_{akl}(y_\alpha)) \) can be computed using the first line of (3.42). Doing so, we see that the corresponding summand in (3.44) is equal to zero. If \((\alpha, k, i) \in I \) but \( T^* - \varepsilon - t_{akl}(y_\alpha) < 0 \), which includes the pair \((k, i) = (\beta, \gamma)\), then the corresponding summand in (3.44) can be computed using the second line of (3.42). We now choose \( \varphi \) in (3.42) such that, for all triples \((\alpha, k, i)\) of this kind except \((\alpha, \beta, \gamma)\) we have \( \varphi_i(\sigma_1(0, x_i, T^* - \varepsilon - t_{akl}(y_\alpha))) = 0 \), while \( \varphi_\gamma(\sigma_\gamma(0, x_\gamma, T^* - \varepsilon - t_{\alpha\beta\gamma}(y_\alpha))) \neq 0 \). This is possible, since, by our construction, \( 0 < \sigma_\gamma(0, x_\gamma, T^* - \varepsilon - t_{\alpha\beta\gamma}(y_\alpha)) < 1 \) and there is no triple \((\alpha, k, \gamma) \in I \) such that \( k \neq \beta \). This completes the proof.

3.4.3 Proof of Theorem 1.10

The equivalence of \((i)\) and \((ii)\) follows from Theorem 1.8 and the obvious property

\[ w_{i_1i_2}w_{i_2i_3} \cdots w_{i_ni_n+1} = 0 \text{ if and only if } p_{i_1i_2}p_{i_2i_3} \cdots p_{i_ni_{n+1}} = 0. \]

Assume that the problem (1.1)–(1.3) is robust FTS. For given \( i, j \leq n \), the \( ij \)-th element \((P^n)_{ij}\) of the matrix \( P^n \) is expressed by the formula

\[
(P^n)_{ij} = \sum_{i_2 = 1}^{n} p_{i_2} \sum_{i_3 = 1}^{n} p_{i_2i_3} \cdots \sum_{i_{n+1} = 1}^{n} p_{i_nj} = \sum_{i_2, i_3, \ldots, i_n = 1}^{n} p_{i_2} p_{i_2i_3} \cdots p_{i_nj}. \quad (3.45)
\]

Due to Claim \((ii)\), all summands in the right hand side equal zero. In other words, the matrix \( P \) is nilpotent. This yields that the matrix \( P_{abs} \) of the absolute values of \( p_{ij} \), namely \( (P_{abs})_{ij} = |p_{ij}| \), is nilpotent as well. The implication \((i) \Rightarrow (iv)\) is, therefore, proved. The implication \((iv) \Rightarrow (i)\) follows from Claim \((ii)\), the formula (3.45), and the property that

\[ P_{abs}^n = 0 \text{ if and only if } |p_{i_1i_2}| |p_{i_2i_3}| \cdots |p_{i_ni_{n+1}}| = 0 \]

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for all tuples \((i_1, i_2, \ldots, i_{n}, i_{n+1})\) ∈ \(\{1, \ldots, n\}^{n+1}\).

Finally, combining Theorem 1.6 with Lemma 3.8, we conclude that \((\iota)\) and \((\iota\iota)\) are equivalent.

### 3.4.4 Proof of Theorem 1.11

We use the following statement, which is proved similarly to Lemma 3.6.

**Lemma 3.10** Let \(n \geq 2\), \(W = (w_{jk})\) be a constant zero-one \(n \times n\)-matrix, and \(q_{jk} \in C^1(\mathbb{R}_+)\), where \(j, k \leq n\). Assume that

\[
w_{i_1 i_2}w_{i_2 i_3}\cdots w_{i_n i_{n+1}} = 0 \quad \text{for all tuples } (i_1, i_2, \ldots, i_n, i_{n+1}) \in \{1, \ldots, n\}^{n+1}.
\]

Then the problem (1.1)–(1.3) with \(p_{jk} = q_{jk}(t)w_{jk}\) is robust finite time stabilizable.

Since the matrix \(W\) satisfies the condition \((\iota\iota)\) of Theorem 1.8, the sufficiency part follows. Since the problem is FTS for every \(P(t)\), let us fix \(P(t)\) such that \(P(t)\) does not depend on \(t\). The necessity part follows now from Theorem 1.10.

**Remark 3.11** The proof of Theorems 1.8, 1.10, and 1.11 use robustness in an essential way. Note that the robust criteria are formulated in terms of the boundary coefficients only. The FTS concept is technically more complicated. Indeed, even in the autonomous case, the spectral criterion shows that an FTS criterion can hardly be expected to be formulated by means of the boundary coefficients only.

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