Multifractality beyond the Parabolic Approximation: Deviations from the Log-normal Distribution at Criticality in Quantum Hall Systems

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Based on differences of generalized Rényi entropies nontrivial constraints on the shape of the distribution function of broadly distributed observables are derived introducing a new parameter in order to quantify the deviation from lognormality. As a test example the properties of the two–measure random Cantor set are calculated exactly and finally using the results of numerical simulations the distribution of the eigenvector components calculated in the critical region of the lowest Landau–band is analyzed.

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Numerical simulations of disordered critical systems always face the problem of dealing with system sizes smaller then the relevant length scale therefore highly complex fluctuations appear. Consequently expectation values of observables do not represent the full picture for the description of the dynamics and physics involved, instead the whole distribution function of these observables should be studied. This problem has become the central issue of theoretical research over the past decade [1–4].

Multifractal analysis [5] has now become a standard tool to handle the aforementioned fluctuations arising near criticality. Especially the disorder induced localization-delocalization (LD) transition is an important and widely studied example of such critical systems [6].

The general method of multifractal analysis starts with the calculation of the measure $Q(\ell)$ of the observable over boxes of linear size $\ell$ in a system of length $L$ divided into $M$ pieces of these boxes. In the thermodynamic limit of $\ell/L \to 0$ the qth moment of this distribution scales with a nontrivial exponent $D_q$ that is, in general, strongly $q$–dependent. These moments are often called as generalized inverse participation numbers (IPN). The quantity $(q - 1)D_q$, also known as the mass exponent $\tau_q$, is a monotonous, nonlinear function of $q$.

Another way of describing these fluctuations is using the $f(\alpha)$ function, the Legendre–transform of $\tau_q$. The $f(\alpha)$ is a smooth, single–humped, concave function. It is the distribution of the exponents $\alpha_q = d\tau/d\alpha$ that characterize the $\mu_q(\ell) \sim (\ell/L)^{\alpha_q}$ scaling of the box–observables where $\mu_q(\ell)$ is the total measure of the qth power of the observables $Q$ over the boxes with linear length $\ell$. The $f(\alpha_q)$ gives the scaling of the number of boxes with measure $(\ell/L)^{\alpha_q}$ in the interval $(\alpha, \alpha + d\alpha)$

$$\rho(t, \alpha) = g(t, \alpha) e^{t f(\alpha)}$$  \hspace{1cm} (1)

where we have introduced the variable $t = -\ln(\ell/L)$. The above distribution describes the multifractal in the sense that $\ln \rho(t, \alpha)/t \to f(\alpha)$ if $t \to \infty$. The $f(\alpha)$ function obeys certain general rules: $f'(\alpha_0) = 0$, $f'(\alpha_1) = 1$, for some specific values of $q$

$$f'(\alpha_0) = 0, \quad f'(\alpha_1) = 1, \quad f'(\alpha_0) = D_0$$ \hspace{1cm} (2a)
$$f'(\alpha_1) = f'(\alpha_1) = \alpha_1 = D_1$$ \hspace{1cm} (2b)

where $D_0$ is the Hausdorff–dimension of the support of the observable $Q$. Furthermore the support of the $f(\alpha)$ spectrum is a finite interval $[\alpha_-, \alpha_+]$, and at the two limiting values the derivative of the $f(\alpha)$ is infinite. The importance of the study of the function $f(\alpha)$ is enhanced due to its connection to the properties of the local distribution function of the Q variables as [3]

$$\Pi(t, \ln Q) \sim \exp[t(D_0 - f(\alpha))], \quad \alpha = \ln Q/t$$ \hspace{1cm} (3)

The shape of the $f(\alpha)$ curve is close to a parabola [6] which means that the distribution $\Pi(t, \ln Q)$ is a Gaussian and the $\Pi(t, Q)$ is lognormal. Therefore any deviation from the parabolic approximation (PA) shows up as a deviation of $\Pi(t, Q)$ from lognormality. Numerical experiments show that there is a substantial deviation of the distribution function (3) from the simple Gaussian especially for large values of $\ln Q$, hence it is even asymmetrical [3] (see Fig. 3).

The calculation of $f(\alpha)$ and $D_q$ functions in numerical experiments suffers from several difficulties. First this analysis is suitable only if $\ell < L < \xi$ is satisfied, where $\xi$ is the localization length and still the box–length should be larger that any microscopic length scale of the system. Second, higher moments of the distribution, $|q| > 3$ are usually calculated with low precision and the PA is usually satisfactory. The PA is fixed by the values of $\alpha_0$ and $D_0$. In spite of the simplicity of the PA it breaks down at values of $q$ where the monotonicity of the $\tau_q$ is lost.
The calculation of the quantities $S_2$ and $m_2$ involve integration over the measure of the distribution in question. In our case, we have to perform integration of Eq. (6a) over the allowed range of the $\alpha$ values $[\alpha_{-\infty}, \alpha_{+\infty}]$

$$P_2^{-1}(\ell) \sim \int_{\alpha_{\min}}^{\alpha_{\max}} \ell^{2\alpha} \rho(\ell, \alpha) d\alpha$$  \hspace{1cm} (5a)

$$H(\ell) \sim -\ln \ell \int_{\alpha_{\min}}^{\alpha_{\max}} \alpha \ell^\alpha \rho(\ell, \alpha) d\alpha$$  \hspace{1cm} (5b)

Furthermore, being interested only in the small $\ell$, i.e. large $t$ limit, we can apply an asymptotic method due originally to Laplace [16] and obtain an expansion of $t$ of the form [17]

$$S_2(t) = (D_1 - D_2)t + \sum_{k=0}^{\infty} s_{2,k} t^{-k}$$  \hspace{1cm} (6a)

$$m_2(t) = (D - D_2)t + \sum_{k=0}^{\infty} m_{2,k} t^{-k}$$  \hspace{1cm} (6b)

From Eqs. (6) it is clear that their ratio tends to a constant as $t \rightarrow \infty$

$$\gamma_2 = \lim_{t \rightarrow \infty} \frac{S_2(t)}{m_2(t)} = \frac{D_1 - D_2}{D_0 - D_2}.$$  \hspace{1cm} (7)

This is the key result of our Letter. The $D_1$ and $D_2$ values in Eqs. (6) are related to the $f(\alpha)$ function, since [22] $\alpha_1 = f(\alpha_1) = D_1$ and $D_2 = 2\alpha_2 - f(\alpha_2)$. On the other hand one can calculate the quantities $S_2/m_2$ directly using the observable $Q$ at any length scale. The latter value imposes a further condition on the $f(\alpha)$ through the dimensions $D_1$ and $D_2$ using relation Eq. (8): $f(\alpha_2) = 2\alpha_2 - (\alpha_1 - \gamma_2 D_0)/(1 - \gamma_2)$.

Furthermore, the denominator of Eq. (7) is the quantity that has been recently connected to the anomalous diffusion exponent by Refs. [8] and [13].

As an illustration first let us consider the two-measure Cantor set representing a binary process [3]. In this construction one divides the $[0,1]$ interval into two equal parts receiving different, $p$ and $1 - p$ measures. While iterating this procedure for each subhalf of the interval we get a multifractal in $D_0 = 1$. This is a strictly selfsimilar multifractal if we keep the $0 \leq p \leq 1/2$ value fixed, because for the $n$th stage $S_2^{(n)} = n S_2^{(1)}$ and $m_2^{(n)} = n m_2^{(1)}$ where $S_2^{(1)}$ and $m_2^{(1)}$ are the values for the initial configuration. Hence all the $s_{2,n}$ and the $m_{2,n}$ in Eq. (6) are zero, and the $\gamma_2$ quantity is independent of $n$ and is a function of $p$ [17].

Next we consider a generalization of this Cantor set by choosing at every stage a different value for $p$, then we obtain a non-selfsimilar multifractal the random binary Cantor set (RBCS). It is straightforward to derive
the \( f(\alpha) \) function once the distribution function of the \( p_n \) sequence is further specified. Taking a uniform distribution over the interval \( [1/2 - \delta, 1/2 + \delta] \) we have found that \( \gamma_2(\delta) \) is a smooth function of \( 0 \leq \delta \leq 1/2 \). Using this simple mathematical construction we will show that the PA can be easily improved with the application of parameter \( \gamma_2 \) Eq. (7). As an example we took \( \delta = 1/3 \). The PA in Fig. 2 is denoted as a dotted line while the exact relation is represented by the solid symbols. We can see that adding a fourth order term \((\alpha - \alpha_0)^4\) to the PA fulfilling both Eqs. (2) and (6) in \( f(\alpha) \) gives the solid line that is clearly a much better approximation than the simple PA.

Finally we will apply our analysis to the widely studied LD transition that takes place in a two dimensional disordered system subject to a perpendicular strong magnetic field: the quantum Hall system \([19]\). This LD transition is responsible to the integer quantum Hall effect. It is described by the critical index \( \nu \) of the localization length and the \( f(\alpha) \) spectrum, that is believed to be universal \([20]\), however, up to now the focus has been set on the PA only and the value of \( \alpha_0 \) fixing the position of the maximum of it. The value of \( \alpha_0 \) describes the scaling behavior of the typical local electron density. We show how the deviation from the lognormal distribution can be measured using Eq. (3) and propose an optimal modification of both the \( f(\alpha) \) and the distribution function \( \Pi(t, \ln Q) \). For that purpose we have performed calculations of two dimensional systems of linear size of 200 magnetic lengths. We have obtained 134 eigenfunctions and calculated the overall joint probability distribution function of the local amplitudes \( Q = |\psi|^2 \). In Fig. 1 we have plotted the logarithm of the histogram of \( \ln Q \) as a function of \( \ln Q/t \). On the same figure the PA is also presented. The discrepancy is clear especially for larger values of \( Q \). On the same figure we have given a possible improvement of the analytical \( f(\alpha) \) that shows a much better resemblance to the numerical data.

The improvement in Fig. 1 was achieved similarly as for the case of the RBCS. For that purpose we have calculated the \( m_2(t)/t \) and \( S_2(t)/t \) values and plotted as a function of \( 1/t \). Unfortunately the low system size is a severe limitation on our problem, since it is desired to obtain the values of \( m_2(t)/t \) and \( S_2(t)/t \) for \( t \to \infty \). However, it is not possible to go beyond \( t_{\text{max}} = \ln L \) and current computation possibilities impose a limit of \( t_{\text{max}} \approx 6.0 \). In order to obtain an acceptable value for the values \( m_2(t)/t \) and \( S_2(t)/t \) as \( 1/t \to \infty \) we have performed an \([K/L]\) order Padé approximation \([20]\) for both quantities. Besides finite size scaling this type of approximation is ready to give an estimate of the coefficients in the expansions Eq. (6). Not having data obtained for different system sizes at hand we can show that the Padé approximation indeed gives acceptable results, as well. We assumed that the \([K/L]\) order approximation should have no singularity in the given interval and has to fall to zero for \( 1/t \to \infty \) since on the scale \( \ell \approx L \) the distribution is homogeneous, thus reducing to the possibilities to \( K < L \) and \( K = 0 \). In Fig. 3 we have plotted the calculated quantities together with several Padé approximants. The limit \( 1/t \to 0 \) gives the approximate value for the parameter \( \gamma_2 \). This way we obtained 0.445 for \([0/2]\) 0.439 for \([0/3]\) and 0.436 for \([0/4]\). Therefore we set the value of \( \gamma_2 \approx 0.44 \). On the other hand, using standard multifractal analysis, our data gives \( \alpha_0 = c < \ln Q > /t = 2.24 \), \( D_1 = 1.78 \) \( D_2 = 1.61 \) that are all well in the generally accepted range \([19]\). The latter values yield \( \gamma_2 \approx 0.44 \) again.

Having \( \alpha_0 \) and \( \gamma_2 \) at hand we may extend the functional form of \( f(\alpha) \) beyond the PA in order to get a bet-
agreement with the numerical distribution function (see Eq. (3)). For the improvement of the functional form of the $f(\alpha)$ entering in Eq. (3) there should be a number of possibilities. We used the addition of the term $(\alpha-\alpha_0)^3$ and $(\alpha-\alpha_0)^5$ to the PA fixing their coefficients requiring the constraints Eqs. (2) and (7) within an error of $10^{-4}$. Therefore the $\Pi(x)$ function seems to be modified as $\Pi(x) = \Pi_{PA}(x) \exp[-t(c_3x^3 + c_5x^5)]$ where $x = \ln Q/t - \alpha_0$, $c_3 \approx -0.03$ and $c_5 \approx 0.12$.

In this Letter we have introduced the differences of Rényi entropies in order to describe the behavior of any multifractal as a function of $t = -\ln(\ell/L)$. These parameters enable to look beyond the parabolic approximation and obtain further constraints on the shape of the $f(\alpha)$ function as well as the distribution function of the multifractal quantity. In order to get insight in the $t$ dependence we have applied Laplace’s method and tried the results on exactly known deterministic and random multifractals. In the former case the self similarity property is also obtained while the latter case was a good test how the improvement from PA can be done.

Finally we have analyzed the joint distribution function of the local amplitudes of wave functions calculated in the critical regime of the Quantum Hall System. We have applied the Padé approximation in order to mimic the thermodynamic limit. The results show a possible tail of the distribution function that is irregular for smaller amplitudes.

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