Weakly 1-completeness of holomorphic fiber bundles over compact Kähler manifolds

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Abstract
Diederich and Ohsawa (Publ. Res. Inst. Math. Sci. 21 (1985), no. 4, 819–833) proved that every disc bundle over a compact Kähler manifold is weakly 1-complete. In this paper, under certain conditions, we generalize this result to the case of fiber bundles over compact Kähler manifolds whose fibers are bounded symmetric domains. In particular, if the representation related to the fiber bundle is reductive, then it has a plurisubharmonic exhaustion function. If the bundle is obtained by the diagonal action on the product of bounded symmetric domains, we are able to show that it is hyperconvex.

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1 | INTRODUCTION

For a complex manifold we say that it is weakly 1-complete (or pseudoconvex) if it admits a $C^\infty$ smooth plurisubharmonic (psh, for short) exhaustion function. Two extreme examples of this concept are compact complex manifolds and Stein manifolds. On the one hand every constant function is a $C^\infty$ smooth psh exhaustion function on a compact complex manifold and on the other hand one of the equivalent conditions to be a Stein manifold is the existence of a $C^\infty$ smooth strictly psh exhaustion function [11]. Weakly 1-complete complex manifolds sit somewhere between them. For the properties of weakly 1-complete manifolds, see [18, 20, 21, 23] and the references therein. Recently Mongodi–Slodkowski–Tomassini classified weakly 1-complete complex surfaces for which such an exhaustion function can be chosen to be real analytic [17].
In this paper we investigate the weakly 1-completeness of fiber bundles whose fibers are bounded symmetric domains of higher rank. Let $M$ be a compact Kähler manifold and $\pi_1(M)$ be its fundamental group. Let $\Omega$ be a bounded symmetric domain and $E \to M$ a holomorphic fiber bundle over $M$ with fiber $\Omega$. One can express $E$ as $M \times_{\rho} \Omega$, where $\rho$ is a homomorphism from $\pi_1(M)$ to the set of automorphisms of $\Omega$, denoted by $\text{Aut}(\Omega)$, and $\pi_1(M)$ acts on the universal cover $\tilde{M}$ of $M$ as the deck transformation. We will stick to the notation introduced above throughout the whole paper. The main result is the following.

**Theorem 1.1.** Let $E = M \times_{\rho} \Omega$ be a holomorphic fiber bundle over a compact Kähler manifold $M$ with a bounded symmetric domain fiber $\Omega$ where $\rho : \pi_1(M) \to \text{Aut}(\Omega)$ is a homomorphism. Suppose that $\rho$ is reductive in $\text{Aut}(\Omega)$. Then $E$ is weakly 1-complete.

When $\rho$ is non-reductive, under some conditions, the fiber bundle is weakly 1-complete (see Theorem 5.7 and Theorem 5.10). In particular any $\mathbb{B}^2$ fiber bundle over a compact Kähler manifold is weakly 1-complete where $\mathbb{B}^2 = \{z \in \mathbb{C}^2 : |z| < 1\}$ is the two dimensional unit ball.

One of the most celebrated theorems in several complex variables is a theorem of Oka–Grauert which says that every locally pseudoconvex domain in a Stein manifold is Stein. In this theorem the Steinness of the ambient space is crucial: in the 1960’s Grauert found an example which is locally pseudoconvex but not Stein. Later Diederich–Fornæss constructed a locally trivial holomorphic disc bundle over a Hopf manifold which is locally pseudoconvex inside a $\mathbb{P}^1$-bundle but it does not admit a psh exhaustion function [6]. More results in this direction were obtained in [5, 22]. Interestingly, Diederich–Ohsawa proved that the non-Kählerness of the base manifold is necessary. More precisely they showed the following result.

**Theorem 1.2** (Diederich–Ohsawa [8]). Every holomorphic disc bundle over a compact Kähler manifold is weakly 1-complete.

The proof of this result consists of two cases corresponding to whether or not there exist harmonic sections. At first they consider the function $\phi(z, w) := 1 - \frac{|w-z|}{|wz^{-1}|}$ on the bidisc $\Delta^2$ which is invariant with respect to the diagonal action $(z, w) \mapsto (\gamma z, \gamma w)$ for all $\gamma \in \text{Aut}(\Delta)$. If there exists a harmonic section $h$, then the function $\varphi(z, w) := -\log \phi(h(z), w)$ defines a psh exhaustion function. If there is no harmonic section, they look at the $\mathbb{P}^1$-bundle $M \times_{\rho} \mathbb{P}^1$ which contains the disc bundle $M \times_{\rho} \Delta$ as an open subset. By Eells–Sampson [9] and Hamilton [12], there exists a flat section $s$ to the ambient bundle $M \times_{\rho} \mathbb{P}^1$. One can deduce that the complement of $s(M)$ on $M \times_{\rho} \mathbb{P}^1$ has the structure of a locally trivial holomorphic $\mathbb{C}$-bundle with respect to the group $\{z \mapsto az + b : |a| = 1, a, b \in \mathbb{C}\}$. Here they used the fact that $s(M)$ is contained in a real hypersurface in $M \times_{\rho} \mathbb{P}^1$ and as a consequence the normal bundle of $s(M)$ in $M \times_{\rho} \mathbb{P}^1$ is a topologically trivial line bundle. Knowing this last fact, they could construct a psh exhaustion function.

The study of bounded symmetric domain fiber bundles reduces to two cases according to whether $\rho$ is reductive. Note that for a real reductive Lie group $G$ with its Lie algebra $\mathfrak{g}$, a representation $\rho : \pi_1(M) \to G$ is said to be reductive if $\text{Ad} \circ \rho : \pi_1(M) \to \text{Aut}(\mathfrak{g})$ is completely reducible. If $G$ is algebraic, this definition is equivalent to assuming that the Zariski closure of $\rho(\pi_1(M))$ in $G$ is reductive.

At first we consider the real-valued function on $\Omega \times \Omega$ defined by

$$
\psi_{\Omega}(z, w) := \frac{K_{\Omega}(z, z)K_{\Omega}(w, w)}{|K_{\Omega}(z, w)|^2},
$$
which is invariant under the diagonal action of Aut(M). Here $K_\Omega$ denotes the Bergman kernel of $\Omega$. When $\Omega$ is the unit disc, $\psi_\Omega$ coincides with $1/\phi$. In Section 3, we show that, when $\Omega$ is a bounded symmetric domain, $\log \psi_\Omega$ is psh on $\Omega \times \Omega$. This is achieved via a straightforward calculation which uses an explicit formula for the Bergman kernel of $\Omega$. We then apply the following theorem of Corlette.

**Theorem 1.3** (Corlette [4], cf. [31, Theorem 4.7]). Let $M$ be a compact Riemannian manifold, $G$ be a semisimple algebraic group, $K \subset G$ be its maximal compact subgroup, and $\rho: \pi_1(M) \to G$ be a representation. Then a $\rho$-equivariant harmonic map $f: \tilde{M} \to G/K$ exists if and only if $\rho$ is reductive.

When $\rho$ is reductive, we can show that the function $\log \psi_\Omega(h(z), w)$ is a psh exhaustion function where $h$ is the harmonic section obtained in Theorem 1.3. It is worth pointing out that the fact that $h$ is pluriharmonic, which was proved by Siu [28] and Sampson [27] (see also [31]) is critically used in the proof.

If $\rho$ is non-reductive, there exists no harmonic section, which implies that there exists a harmonic section from $M$ to $M \times_B B$, where $B$ is a boundary component of $\Omega$ in its compact dual $\hat{\Omega}$ by Hamilton [12]. Now exploiting a generalized Cayley transformation of Korányi–Wolf [13], we are able to obtain some results in special situations (Theorem 5.7, Theorem 5.10).

Let us recall that a complex manifold is hyperconvex when it admits a bounded psh exhaustion function. Here we say that for a complex manifold $X$, $\mu: X \to (-\infty, 0)$ is a bounded exhaustion function if $\{p \in X : \mu(p) < c\}$ is relatively compact in $X$ for all negative $c \in \mathbb{R}$. It is important to stress that any hyperconvex complex manifold is weakly 1-complete but the converse is not true. For a counterexample, see [24].

Let $\Gamma \subset \text{Aut}(\Omega)$ be a cocompact discrete subgroup of $\text{Aut}(\Omega)$. Then $\Gamma \setminus \Omega$ is a compact Kähler manifold with the metric induced from the Bergman metric on $\Omega$. Consider the diagonal action of $\Gamma$ on $\Omega \times \Omega$, that is, $\gamma(z, w) = (\gamma z, \gamma w)$. Then the quotient complex manifold of $\Omega \times \Omega$ by the diagonal action, $\Omega \times \Omega / \Gamma$, is a $\Omega$-fiber bundle over $\Gamma \setminus \Omega$. In this special case we can show that $\Omega \times \Omega / \Gamma$ is hyperconvex (Theorem 6.1). If $\Omega$ is the unit ball $\mathbb{B}^n$, the fiber bundle $\mathbb{B}^n \times \mathbb{B}^n / \Gamma$ is a domain in $\mathbb{C}P^n \times \mathbb{B}^n / \Gamma$ whose real analytic boundary has Diederich–Fornæss index equal to $1/2$ (Corollary 6.2). For $n = 1$, this was proved by Adachi–Brinkschulte [2] and Fu–Shaw [10]. We exploit their theorems to compute the Diederich–Fornæss index.

## 2 PRELIMINARIES

In this section, we recall some facts about Lie group/algebra structures and Cayley transformations of bounded symmetric domains. For more details, see [13, 33, 34].

Let $X$ be an irreducible Hermitian symmetric space of non-compact type. Let $G$ be the identity component of the isometry group of $X$ with respect to the Bergman metric of $X$ and $K \subset G$ the isotropy subgroup at $o \in X$. Then $X$ is biholomorphic to $G/K$. Denote by $\mathfrak{g}$ and by $\mathfrak{f}$ the Lie algebras of $G$ and $K$, respectively. Let $\mathfrak{m} = \mathfrak{f} + \mathfrak{m}$ be the Cartan decomposition. Let $\mathfrak{g}_C = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathfrak{f}_C = \mathfrak{f} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathfrak{m}_C = \mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C}$, and $G_C$ be the complex Lie group corresponding to $\mathfrak{g}_C$. Let $\mathfrak{g}_c = \mathfrak{f} + \sqrt{-1} \mathfrak{m}$ be a Lie algebra of compact type and $G_c$ the corresponding connected Lie group of $\mathfrak{g}_c$. Then $\hat{X} := G_c / K$ is the compact dual of $X$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{f}$. Note that $\mathfrak{h}_C = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_C$. Let $\Psi$ denote the set of roots of $\mathfrak{g}_C$ with respect to $\mathfrak{h}_C$ and let $\mathfrak{g}_\alpha$ denote the root space with respect to a root $\alpha \in \Psi$. Let $\Psi_f, \Psi_m$ denote the set of compact, non-compact
roots of \( g^\mathbb{C} \) with respect to the Cartan decomposition \( g^\mathbb{C} = \mathfrak{f}^\mathbb{C} + m^\mathbb{C} \), respectively, and choose an order of \( \Psi \) such that the set of positive non-compact roots \( \Psi_+^\mathbb{C} \) satisfies that \( m^+ := \sum_{\alpha \in \Psi_+^\mathbb{C}} g^\alpha = T_0^{1,0} X \). Here \( T_0^{1,0} X \) denotes the holomorphic tangent bundle of \( X \). Denote \( m^- := \sum_{\alpha \in \Psi_-^\mathbb{C}} g^\alpha \). Let \( M^+ \) and \( M^- \) be the corresponding analytic subgroups in \( G^\mathbb{C} \). Note that \( m^+ \) and \( m^- \) are abelian subalgebras of \( g^\mathbb{C} \). The center \( z \) of \( \mathfrak{n} \) contains an element \( Z \) such that \( \text{Ad} Z E = \pm i E \) for \( E \in m^\pm \).

The center \( \mathfrak{g} \) contains an element \( Z \) such that \( \text{Ad} Z E = \pm i E \) for \( E \in \mathfrak{m} \). The center \( \mathfrak{g} \) contains an element \( Z \) such that \( \text{Ad} Z E = \pm i E \) for \( E \in \mathfrak{m} \).

Let \( J := \text{Ad} Z \) be a complex structure on \( \mathfrak{m} \). A basis of \( \mathfrak{m} \) is given by the elements \( X_\alpha := E_\alpha + E_{-\alpha} \) and \( Y_\alpha := -i(E_\alpha - E_{-\alpha}) \) where \( \alpha \) is non-compact positive. For such \( \alpha \), we have relations \( J X_\alpha = Y_\alpha \) and \( [X_\alpha, Y_\alpha] = 2iH_\alpha \). Define \( X^c_\alpha := iX_\alpha \) and \( Y^c_\alpha := iY_\alpha \). Those define a basis of \( i \mathfrak{m} \).

For \( K^\mathbb{C} \) denoting the analytic subgroup corresponding to \( \mathfrak{n}^\mathbb{C} \), \( K^\mathbb{C} \cdot M^+ \) is a semidirect product. \( \hat{X} \) is identified with \( G^\mathbb{C} / K^\mathbb{C} \cdot M^+ \) by the identity map of \( G \) into \( G^\mathbb{C} \). For \( o = e K \in \hat{X} \), the orbit \( G \cdot o \) is the image of the holomorphic embedding \( g K \mapsto g(o) \) of \( X \) into \( \hat{X} \) (Borel embedding). The map \( \zeta : \mathfrak{m}^- \to \hat{X} \) defined by

\[
\zeta(E) := \exp(E)(o)
\]

is a holomorphic homeomorphism onto a dense open subset of \( X \) and \( \zeta \) is \( \text{Ad}_K \)-equivariant. Then \( \Omega := \zeta^{-1}(G(o)) \) is a bounded symmetric domain in \( \mathfrak{m}^- \): this is the Harish–Chandra realization of \( X \).

For \( \alpha, \beta \in \Psi \), one says that \( \alpha \) and \( \beta \) are strongly orthogonal if and only if \( \alpha \pm \beta \not\in \Psi \). Let \( \Pi := \{ \alpha_1, \ldots, \alpha_r \} \) denote a maximal set of strongly orthogonal positive non-compact roots of \( g^\mathbb{C} \). Then \( X \) is of rank \( r \). For every \( \alpha \in \Pi \) let

\[
c_\alpha := \exp \left( \frac{\pi}{4} X_\alpha \right) \in G_c, \quad c := \prod_{\alpha \in \Pi} c_\alpha.
\]

c is called the Cayley transformation of \( X \). For \( \Lambda \subset \Pi \) the partial Cayley transformation is defined by

\[
c_\Lambda := \prod_{\alpha \in \Lambda} c_\alpha.
\]

Denote by \( g^\mathbb{C}_\Lambda \) the derived algebra of \( \mathfrak{n} + \sum_{\alpha \in \Pi \setminus \Lambda} g^\alpha \), where \( \perp \) is the orthogonality with respect to the inner product induced by the Killing form of \( g^\mathbb{C} \). Let \( G^\mathbb{C}_\Lambda \) denote the Lie subgroup of \( G^\mathbb{C} \) corresponding to \( g^\mathbb{C}_\Lambda \) and \( G^\mathbb{C}_\Lambda, G^\mathbb{C}, g^\mathbb{C}_\Lambda, g^\mathbb{C}, g^\mathbb{C}_\Lambda, g \) denote \( G \cap G^\mathbb{C}_\Lambda, G^\mathbb{C} \cap G^\mathbb{C}_\Lambda, g \cap g^\mathbb{C}_\Lambda, g \cap g^\mathbb{C} \), respectively. Let \( X^\mathbb{C}_\Lambda := G^\mathbb{C}_\Lambda \cdot o \subset \hat{X} \) and \( X_{\Lambda,0} := G^\mathbb{C}_\Lambda \cdot o \subset X \).

Let \( \partial X \) be the topological boundary of \( X \) in \( \hat{X} \) and \( \Delta := \{ z \in \mathbb{C} : |z| < 1 \} \) the unit disc. A holomorphic map \( g : \Delta \to \hat{X} \) such that \( g(\Delta) \subset \partial X \) is called a holomorphic arc in \( \partial X \). A finite sequence \( \{ g_1, \ldots, g_s \} \) of holomorphic arcs in \( \partial X \) is called a chain of holomorphic arcs in \( \partial X \) if \( f_j(\Delta) \cap f_{j+1}(\Delta) \neq \emptyset \) for any \( j = 1, \ldots, s-1 \). One can give an equivalence class on \( \partial X \) such that for \( z_1, z_2 \in \partial X, z_1 \sim z_2 \) if and only if there is a chain of holomorphic arcs \( \{ g_1, \ldots, g_s \} \) in \( \partial X \) with \( z_1 \in g_1(\Delta) \) and \( z_2 \in g_s(\Delta) \). The equivalence classes are called the boundary components of \( \partial X \) in \( \hat{X} \).

**Theorem 2.1** (Wolf [34]). The \( G \)-orbits on \( \partial X \) in \( \hat{X} \) are the sets

\[
G(c_{\Lambda \ominus \Lambda} o) = \bigcup_{k \in K} kc_{\Lambda \ominus \Lambda} X_{\Lambda,0} \quad \text{with} \quad \Lambda \subset \Pi,
\]
where $c_{\Pi - \Lambda}$ is the partial Cayley transformation with respect to $\Pi - \Lambda$. Furthermore the boundary components of $X$ in $\hat{X}$ are the sets $kc_{\Pi - \Lambda}X_{\Lambda,0}$ with $k \in K$ and $\Lambda \not\subseteq \Pi$. These are Hermitian symmetric spaces of non-compact type of rank $|\Lambda|$.

**Definition 2.2.** If $|\Lambda| = |\Pi| - a$, then we will call $kc_{\Pi - \Lambda}X_{\Lambda,0}$ an $a$-component.

Remark that for any 1-component $kc_{\Pi - \Lambda}X_{\Lambda,0}$, there exists a unit disc $\Delta$ so that $\Delta \times kX_{\Lambda,0}$ can be embedded totally geodesically in $X$ and $kc_{\Pi - \Lambda}X_{\Lambda,0} \cong \{e^{i\theta}\} \times kX_{\Lambda,0}$ for some $e^{i\theta} \in \partial \Delta$ (see [16, Proposition 1.7]).

Let $B^\Lambda$ be the set of all elements of $G$ which preserves $c_{\Pi - \Lambda}X_{\Lambda,0}$ and $b^\Lambda$ its Lie algebra. By Korányi–Wolf, $B^\Lambda$ and $b^\Lambda$ has the following structure. We will use the notation in [13].

**Theorem 2.3** (Korányi–Wolf [13]). $B^\Lambda$ is a maximal parabolic subgroup of $G$, and is the normalizer of $Ad_{c_{\Pi - \Lambda}}^{-1} N^{\Lambda, -}$ in $G$. The identity component of $B^\Lambda$ is given by

$$
\left\{ G_\Lambda \cdot L_2^\Lambda \cdot Ad_{c_{\Pi - \Lambda}}^{-1} K_{\Pi - \Lambda,1}^\Lambda \right\} \cdot Ad_{c_{\Pi - \Lambda}}^{-1} N^{\Lambda, -}
$$

and this is the Chevalley decomposition into reductive and unipotent parts.

Note that we have

$$m^- = m_{\Pi - \Lambda,1}^- + m_{2, -}^\Lambda + m_{\Lambda}^-$$

and for $E \in m^-$, we set $E = E_1 + E_2 + E_3$ with $E_1 \in m_{\Pi - \Lambda,1}^-$, $E_2 \in m_{2, -}^\Lambda$, and $E_3 \in m_{\Lambda}^-$. Let $\nu$ be a complex antilinear map of $m^\pm$ onto $m^\mp$ preserving this direct decomposition. For all $W \in \Omega \cap m_\Lambda^-$, define the linear transformation $\mu(W) : m_{2, -}^\Lambda \rightarrow m_{2, -}^\Lambda$ by $\mu(W)U := Ad_W \tau_{\Pi - \Lambda} \nu(U)$ with $\tau_{\Pi - \Lambda} = Ad_{2} \tau_{\Pi - \Lambda}$. For all $V \in m_{\Lambda}^-$, define the linear function $f_V : m_-^\Lambda \rightarrow m_{2, -}^\Lambda$ by $f_V(W) := (I + \mu(W))V$. Finally, for all $W \in \Omega \cap m_\Lambda^-$, define the vector-valued bilinear form $F_W : m_{2, -}^\Lambda \times m_{\Pi - \Lambda,1}^- \rightarrow m_{\Pi - \Lambda,1}^-$ by $F_W(U, V) = -\frac{i}{2} [U, \tau_{\Pi - \Lambda} (\nu(I + \mu(W)))^{-1} V]$.

**Theorem 2.4** (Korányi–Wolf [13]).

1. $L_2^\Lambda \cdot Ad_{c_{\Pi - \Lambda}}^{-1} K_{\Pi - \Lambda,1}^\Lambda \cdot Ad_{c_{\Pi - \Lambda}}^{-1} N^{\Lambda, -}$ acts on $B^\Lambda$ trivially.

2. $N^{\Lambda, -}$ acts on $m^-$ by

$$g(E) = E + U + f_V(E_3) + 2iF_{E_3}(E_2, f_V(E_3)) + iF_{E_3}(f_V(E_3), f_V(E_3))$$

where $g = \exp(U + (I - \tau_{\Pi - \Lambda})V)$, $U \in n_{\Pi - \Lambda}^\Lambda$ and $V \in m_{2, -}^\Lambda$.

3. $K^{\Lambda, -}$ acts on $m^-$ by the adjoint representation and it preserves $m_{\Pi - \Lambda,1}^-$, $m_{2, -}^\Lambda$, and $m_{\Lambda}^-$.  

4. On $m_{\Pi - \Lambda,1}^-$, $K^{\Lambda, -}$ is real, $G_\Lambda$ and $L_2^\Lambda$ are trivial. These actions are $\xi$-equivariant; in particular, $K^{\Lambda, -} \cdot N^{\Lambda, -}$ preserves $\xi(m^-)$.

5. For all $k \in K^{\Gamma, -}$, $W \in \Omega_\Gamma$ and $U, V \in m_{2, -}^\Gamma$, we have

$$Ad_k F_W(U, V) = F_{Ad_k W}(Ad_k U, Ad_k V).$$
(6) \( \Omega \) can be realized as a homogeneous Siegel domain of the third type given by
\[
c_{\Pi-\Lambda} \Omega = \left\{ E \in m^- : \text{Im} E_1 - \text{Re} F_{E_3}(E_2, E_2) \in c^\Lambda, E_3 \in X_{\Lambda,0} = \Omega \cap m^-_\Lambda \right\}.
\]

Lemma 2.5. \( [[m_2^+, m_2^-], m_2^+] = 0 \).

Proof. By [13, Lemma 6.3], one sees that \( \text{Ad}_{-1} \mathfrak{r}_1^{\Lambda+} \) is an eigenspace with an eigenvalue \(-1\) of \( \text{ad}_{-Y}^{\Pi-\Lambda} \) where \( \mathfrak{r}_1^{\Lambda+} = q_2^{\Lambda+} + m_2^- \) with \( q_2^{\Lambda+} \subset \mathfrak{k}^\mathbb{C} \). Moreover \( m_2^+ \) belongs to the 0-eigenspace of \( \text{ad}_{-Y}^{\Pi-\Lambda} \). Hence
\[
[m_2^+, m_2^-] \subset (q_2^{\Lambda+} + m_2^-) \cap \mathfrak{k}^\mathbb{C} = q_2^{\Lambda+}.
\]
Note that \( [m_2^+, q_2^{\Lambda+}] \) belongs to the positive root space. Since
\[
[m_2^+, q_2^{\Lambda+}] \subset \mathfrak{r}_2^- \cap m^\mathbb{C} = m_2^-,
\]
\( [[m_2^+, m_2^-], m_2^+] \) vanishes. \( \square \)

3 PLURIHARMONICITY OF INVARIANT FUNCTIONS

In this section, we denote by \( M_{p,q}^\mathbb{C} \) the set of \( p \times q \) complex matrices. Denote by \( SM_{n,n}^\mathbb{C} \) (respectively, \( ASM_{n,n}^\mathbb{C} \)) the set of symmetric (respectively, antisymmetric) \( n \times n \) complex matrices. Irreducible bounded symmetric domains consist of the following four classical type and two exceptional type domains:

(I) \( \Omega^I_{p,q} = \{ Z \in M_{p,q}^\mathbb{C} : I_p - ZZ^* > 0 \} \),
(II) \( \Omega^I_{n,n} = \{ Z \in M_{n,n}^\mathbb{C} : I_n - ZZ^* > 0, Z^t = -Z \} \),
(III) \( \Omega^I_{n,n} = \{ Z \in M_{n,n}^\mathbb{C} : I_n - ZZ^* > 0, Z^t = Z \} \),
(IV) \( \Omega^I_{n,n} = \{ Z = (z_1, ..., z_n) \in \mathbb{C}^n : ZZ^* < 1, 0 < 1 - 2ZZ^* + |ZZ^t|^2 \} \),
(V) \( \Omega^I_{1,2} = \{ z \in M_{1,2}^\mathbb{C} : 1 - (z|z) + (z^#|z^#) > 0, 2 - (z|z) > 0 \} \), and
(VI) \( \Omega^I_{2,7} = \{ z \in H_3(\mathbb{O}_\mathbb{C}) : 1 - (z|z) + (z^#|z^#) - |\det z|^2 > 0, 3 - 2(z|z) + (z^#|z^#) > 0, 3 - (z|z) > 0 \} \).

Here \( \mathbb{O}_\mathbb{C} \) is the complex 8-dimensional algebra of complex octonions. For \( a = (a_0, a_1, ..., a_7) \in \mathbb{O}_\mathbb{C} \) with \( a_i \in \mathbb{C} \), let \( a \mapsto \bar{a} := (a_0, -a_1, ..., -a_7) \) denote the Cayley conjugation and \( a \mapsto \overline{a} := (\overline{a}_0, \overline{a}_1, ..., \overline{a}_7) \) the complex conjugation. The Hermitian scalar product is given by
\[
(a|b) := \bar{a}b + b\bar{a}.
\]
For the basis $e_0, \ldots, e_7$ of $\mathbb{O}_C$ so that $a = \sum_{i=0}^{7} a_i e_i$, we obtain

\[
(e_j | e_k) = e_j \bar{e}_k + e_k \bar{e}_j = \begin{cases} 
-e_j e_k - e_k e_j = 0 & \text{if } k \neq j \geq 1 \\
-e_0 e_k + e_k e_0 = -e_0 & \text{if } j = 0, k \geq 1 \\
e_0 e_0 + e_0 e_0 = 2e_0 & \text{if } j = k = 0 \\
2e_0 & \text{if } j = k \geq 1 \\
-2e_j e_j = 2e_0 & \text{if } j = k \geq 1 \\
2\delta_{j,k} e_0 & \text{if } j = k \geq 1 
\end{cases}
\]

(3.1)

Let $H_3(\mathbb{O}_C)$ be the complex vector space of $3 \times 3$ matrices with entries in $\mathbb{O}_C$ which are Hermitian with respect to the Cayley conjugation in $\mathbb{O}_C$. Explicitly $A \in H_3(\mathbb{O}_C)$ can be expressed as

\[
A = \begin{pmatrix} 
\alpha_1 & a_3 & \bar{a}_2 \\
\bar{a}_3 & \alpha_2 & a_1 \\
a_2 & \bar{a}_1 & \alpha_3 
\end{pmatrix}
\]

with $a_i \in \mathbb{O}_C$ and $\alpha_i \in \mathbb{C}$ for all $i = 1, 2, 3$. (3.2)

For $A \in H_3(\mathbb{O}_C)$ as in (3.2), let $A^\# \in H_3(\mathbb{O}_C)$ be the adjoint matrix of $A$ given by

\[
A^\# := \begin{pmatrix} 
\alpha_2 \alpha_3 - a_1 \bar{a}_1 & a_2 \bar{a}_1 - \alpha_3 a_3 & \bar{a}_1 a_3 - \alpha_2 a_2 \\
\bar{a}_2 a_1 - \alpha_2 a_3 & \alpha_3 a_1 - a_2 a_2 & \bar{a}_2 a_3 - a_1 a_1 \\
\bar{a}_1 a_3 - \alpha_2 a_2 & \bar{a}_3 a_2 - \alpha_1 a_1 & \alpha_1 a_2 - a_3 a_3 
\end{pmatrix}
\]

The Hermitian scalar product on $H_3(\mathbb{O}_C)$ is given by

\[
(A | B) := \sum_{i=1}^{3} \alpha_i \bar{\beta}_i + \sum_{i=1}^{3} (a_i | b_i).
\]

Explicitly, we have

\[
(A | A) = \sum_{i=1}^{3} |\alpha_i|^2 + 2 \sum_{i=1}^{3} (|a_{i0}|^2 + \ldots + |a_{i7}|^2),
\]

\[
(A^\# | A^\#) = |\alpha_2 \alpha_3 - a_1 \bar{a}_1|^2 + |\alpha_3 a_1 - a_2 \bar{a}_2|^2 + |\alpha_1 a_2 - a_3 \bar{a}_3|^2
+ (\bar{a}_3 a_2 - \alpha_1 a_1) |a_3 \bar{a}_2 - \alpha_1 a_1| + (\bar{a}_2 a_3 - \alpha_2 a_2) |a_2 \bar{a}_3 - \alpha_2 a_2|
+ (\bar{a}_1 a_3 - \alpha_3 a_3) |a_1 \bar{a}_3 - \alpha_3 a_3|,
\]

\[
\det A = \alpha_1 \alpha_2 \alpha_3 - \sum_{i=1}^{3} \alpha_i a_i \bar{a}_i + a_1 (a_2 a_3) + (\bar{a}_3 \bar{a}_2) \bar{a}_1.
\]

with $a_i = (a_{i0}, \ldots, a_{i7}) \in \mathbb{O}_C$ for $i = 1, 2, 3$. Let $M^\mathbb{O}_C_{1,2}$ denote the set of $1 \times 2$ complex octonion matrices. For $z = (z_1, z_2) \in M^\mathbb{O}_C_{1,2}$, we identify $z$ with

\[
\begin{pmatrix} 
0 & z_2 \\
\bar{z}_2 & 0 \\
z_1 & 0
\end{pmatrix} \in H_3(\mathbb{O}_C)
\]

and apply the same notation $\#$, $(\cdot, \cdot)$ and so on. We refer the reader to [26] for more details.
Lemma 3.1. Let $\Omega$ be an irreducible bounded symmetric domain and $N_\Omega$ its generic norm. Then the Bergman kernel $K(z, z)$ of $\Omega$ is of the form $c_1 N_\Omega(z)^{c_2}$ for some constant $c_1, c_2 > 0$.

The generic norms $S^I_{p,q}, S^H_n, S^{III}_n, S^{IV}_n, S^V$ and $c_2$ in Lemma 3.1 of the corresponding domains are given by

(i) $S^I_{p,q}(Z, \bar{Z}) = \det(I - ZZ^*)$ for $Z \in M_{p,q}^C$, $c_2 = p + q$,

(ii) $S^H_n(Z, \bar{Z}) = s^I_n(Z)$ for $Z \in ASM_{n,n}^C$, $c_2 = 2(n - 1)$,

(iii) $S^{III}_n(Z, \bar{Z}) = \det(I_n - ZZ^*)$ for $Z \in SM_{n,n}^C$, $c_2 = n + 1$,

(iv) $S^{IV}_n(Z, \bar{Z}) = 1 - 2ZZ^* + |ZZ'|^2$ for $Z \in C^n$, $c_2 = n$,

(v) $S^V(Z, \bar{Z}) = 1 - (Z|Z) + (Z^#|Z^#)$ for $Z \in M_{1,2}^C$, $c_2 = 12$,

(vi) $S^{VI}(Z, \bar{Z}) = 1 - (Z|Z) + (Z^#|Z^#) - |detZ|^2$ for $Z \in H_3(O_C)$, $c_2 = 18$,

with $\det(I - ZZ^*) = s^I_n(Z)^2$ for some polynomial $s^I_n(Z)$ and $Z \in ASM_{n,n}^C$.

For a bounded symmetric domain $\Omega$, possibly reducible, let $K_\Omega : \Omega \times \Omega \to C$ denote its Bergman kernel. Define a function $\psi_\Omega : \Omega \times \Omega \to \mathbb{R}$ by

$$
\psi_\Omega(z, w) := \frac{K_\Omega(z, z)K_\Omega(w, w)}{|K_\Omega(z, w)|^2}.
$$

Since $\psi_\Omega(\gamma(z), \gamma(w)) = \psi_\Omega(z, w)$ for any $\gamma \in Aut(\Omega)$ and $\psi_\Omega(z, w) > 1$ whenever $z \neq w$, $\log \psi_\Omega$ is a well-defined function on $\Omega \times \Omega$ which is invariant under the diagonal action of $Aut(\Omega)$ and positive off the diagonal.

Lemma 3.2. For any bounded symmetric domain $\Omega$, $\log \psi_\Omega$ is a $C^\infty$ psh function which is invariant under the diagonal action of $Aut(\Omega)$.

Proof. Since $\Omega$ is a finite product on irreducible bounded symmetric domains, we may assume that $\Omega$ is irreducible. Since $\psi_\Omega$ is invariant under the diagonal action of $Aut(\Omega)$, we only need to prove that $\partial \bar{\partial} \log \psi_\Omega(z, w)$ is positive semi-definite at $(0, w_0)$, when $w_0$ belongs to the maximal polydisc of $\Omega$.

Type I. $\Omega^I_{p,q}$: Since $K^I_{\Omega_{p,q}}(z, w) = c_1 \det(I - z\bar{w})^{c_2}$ for some constants $c_1, c_2 > 0$, we have

$$
\partial \bar{\partial} \log \psi_{\Omega^I_{p,q}}(z, w)
= \partial \bar{\partial} \left( \log K_{\Omega^I_{p,q}}(z, z) + \log K_{\Omega^I_{p,q}}(w, w) - \log K_{\Omega^I_{p,q}}(z, w) - \log K_{\Omega^I_{p,q}}(w, z) \right)
= -c_2 \partial \bar{\partial} \left( \log det(I - z\bar{z}^t) + \log det(I - w\bar{w}^t) - \log det(I - z\bar{w}^t) - \log det(I - w\bar{z}^t) \right)
= c_2 \left( \begin{array}{cc}
-\partial_z \bar{\partial}_z \log det(I - z\bar{z}^t) & \partial_z \bar{\partial}_w \log det(I - z\bar{w}^t) \\
\partial_w \bar{\partial}_z \log det(I - w\bar{z}^t) & -\partial_w \bar{\partial}_w \log det(I - w\bar{w}^t)
\end{array} \right)
= c_2 \left( \begin{array}{cc}
I & -I \\
-I & -\partial_w \bar{\partial}_w \log det(I - w\bar{w}^t)
\end{array} \right) \text{ at } (0, w_0).
$$

(3.4)
If $-\partial_w \bar{\partial}_w \log \det(I - w \bar{w}^t) \geq I$, for $X, Y \in \mathbb{C}^{p,q}$, we have
\[
(X, Y) \begin{pmatrix} I & -I \\ -I & -\partial_w \bar{\partial}_w \log \det(I - w \bar{w}^t) \end{pmatrix} \begin{pmatrix} \bar{X}^t \\ \bar{Y}^t \end{pmatrix} = X\bar{X}^t - Y\bar{X}^t - X\bar{Y}^t - \partial_w \bar{\partial}_w \log \det(I - w \bar{w}^t)(Y, \bar{Y}^t) \geq (X - Y)(X - Y)^t
\]
and hence $\partial \bar{\partial} \log \psi_{\Omega_{p,q}}$ is plurisubharmonic. Note that
\[
-\frac{\partial^2 \log \det(I - w \bar{w}^t)}{\partial w_{ij} \partial \bar{w}_{kl}} = -\frac{1}{\det(I - w \bar{w}^t)} \frac{\partial^2 \det(I - w \bar{w}^t)}{\partial w_{ij} \partial \bar{w}_{kl}} + \frac{1}{\det(I - w \bar{w}^t)^2} \frac{\partial \det(I - w \bar{w}^t)}{\partial w_{ij}} \frac{\partial \det(I - w \bar{w}^t)}{\partial \bar{w}_{kl}}. \tag{3.5}
\]

Let $w = (w_{ij}) \in M_{p,q}^{\mathbb{C}}$ with $p \leq q$. Denote $w = (w_{ij}^1, \ldots, w_{ij}^p)$ with $w_j = (w_{j1}, \ldots, w_{jq})$. Let us consider $\partial \bar{\partial} \det(I - w \bar{w}^t)$ at
\[
w_0 = \text{diag}(w_{11}, \ldots, w_{pp}) = \begin{pmatrix} w_{11} & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & w_{pp} & 0 \end{pmatrix}.
\]
Since we have
\[
I - w \bar{w}^t = \begin{pmatrix} 1 - w_1 \bar{w}_1 & -w_1 \bar{w}_2^t & \ldots & -w_1 \bar{w}_p^t \\ -w_2 \bar{w}_1^t & 1 - w_2 \bar{w}_2^t & \ldots & -w_2 \bar{w}_p^t \\ \vdots & \vdots & \ddots & \vdots \\ -w_p \bar{w}_1^t & -w_p \bar{w}_2^t & \ldots & 1 - w_p \bar{w}_p^t \end{pmatrix},
\]
by the derivative formula of the determinant, one obtains
\[
\frac{\partial \det(I - w \bar{w}^t)}{\partial \bar{w}_{ij}} = -\delta_{ij} \frac{\bar{w}_{ii} \prod_{s=1}^{p} (1 - |w_{s0}|^2)}{1 - |w_{ii}|^2}, \tag{3.6}
\]
and
\[
\frac{\partial^2 \det(I - w \bar{w}^t)}{\partial w_{ij} \partial \bar{w}_{kl}} = \det \begin{pmatrix} 1 - w_1 \bar{w}_1 & -w_1 \bar{w}_2^t & \ldots & -w_1 \bar{w}_p^t \\ \vdots & \ddots & \ddots & \vdots \\ -\frac{\partial w_1}{\partial w_{ij}} \bar{w}_1 & \frac{\partial w_1}{\partial w_{ij}} \bar{w}_2^t & \ldots & \frac{\partial w_1}{\partial w_{ij}} \bar{w}_p^t \\ -w_p \bar{w}_1^t & -w_p \bar{w}_2^t & \ldots & 1 - w_p \bar{w}_p^t \end{pmatrix}. \tag{i^{th}}
\]
Since we have
\[
\frac{\partial w_i}{\partial w_{ij}} = \delta_{ij}, \quad \frac{\partial w_i}{\partial w_{ij}} = \delta_{il} \quad \text{and} \quad \frac{\partial w_i^t}{\partial w_{kl}} = w_{il}
\]
for \(\sigma = 1, \ldots, p\), one obtains
\[
\frac{\partial^2 \det(I - w^t w)}{\partial w_{ij} \partial w_{kl}} = \det\begin{pmatrix}
1 - w_1^t w_1 & -w_1^t w_2 & \cdots & -w_1^t w_p \\
\vdots & \ddots & \ddots & \vdots \\
-w_i^t w_1 & -w_i^t w_2 & \cdots & -w_i^t w_p \\
-w_p^t w_1 & -w_p^t w_2 & \cdots & 1 - w_p^t w_p
\end{pmatrix}
\]
\(l^{th}\). (3.7)

If \(i < k\), the \(k\)th row of the matrix in (3.7) is \((0, \ldots, 0, -w_{kl}, 0, \ldots, 0)\) when \(w = \text{diag}(w_{11}, \ldots, w_{pp})\) where \(-w_{kl}\) is located in the \(k\)th component. Hence if \(k \neq l\), then (3.7) should be zero. Through the same computation one obtains
\[
\frac{\partial^2 \det(I - w^t w)}{\partial w_{ij} \partial w_{kl}} = 0 \quad \text{for} \quad i \neq k \quad \text{and} \quad k \neq l.
\]
Hence by (3.5), (3.6), and (3.7)
\[
\frac{\partial^2 \log \det(I - w^t w)}{\partial w_{ij} \partial w_{kl}} = 0 \quad \text{for} \quad i \neq k \quad \text{and} \quad k \neq l.
\]
(3.8)

Suppose that \(i = k = l \neq j\). Then \(i\)th row of (3.7) is zero at \(w_0 = \text{diag}(w_{11}, \ldots, w_{pp})\). Hence by (3.5), (3.6), and (3.7)
\[
\frac{\partial^2 \log \det(I - w^t w)}{\partial w_{ij} \partial w_{ii}} = 0 \quad \text{for} \quad i \neq j.
\]
(3.9)

Suppose that \(i < k\) and \(k = l\). Then at \(w_0 = \text{diag}(w_{11}, \ldots, w_{pp})\) Equation (3.7) is equal to
\[
\frac{\partial^2 \det(I - w^t w)}{\partial w_{ij} \partial w_{kk}} = \det\begin{pmatrix}
M_1 & 0 & 0 & 0 & 0 \\
0 & -w_{ij} & 0 & -\delta_{kj} & 0 \\
0 & 0 & M_2 & 0 & 0 \\
0 & 0 & 0 & -w_{kk} & 0 \\
0 & 0 & 0 & 0 & M_3
\end{pmatrix},
\]
(3.10)
where \(M_1 = \text{diag}(1 - |w_{11}|^2, \ldots, 1 - |w_{i-1,i-1}|^2), \ M_2 = \text{diag}(1 - |w_{i+1,i+1}|^2, \ldots, 1 - |w_{k-1,k-1}|^2), \) and \(M_3 = \text{diag}(1 - |w_{k+1,k+1}|^2, \ldots, 1 - |w_{pp}|^2)\). Hence by (3.10)
\[
\frac{\partial^2 \log \det(I - w^t w)}{\partial w_{ij} \partial w_{kk}} = 0 \quad \text{for} \quad i \neq k \quad \text{and} \quad i \neq j.
\]
(3.11)
By a similar computation one also obtains

$$\frac{\partial^2 \det(I - w\bar{w}')}{\partial w_i \partial \bar{w}_{kk}} = \frac{\bar{w}_{ii} w_{kk} \Pi_{\sigma=1}^p (1 - |w_{\sigma\sigma}|^2)}{(1 - |w_{ii}|^2)(1 - |w_{kk}|^2)}$$ for \( i \neq k \)

and

$$\frac{\partial^2 \det (I - w\bar{w}')}{\partial w_{ii} \partial \bar{w}_{ii}} = -\frac{\Pi_{\sigma=1}^p (1 - |w_{\sigma\sigma}|^2)}{1 - |w_{ii}|^2}.$$ 

Hence we obtain

$$\frac{\partial^2 \log \det (I - w\bar{w}')}{\partial w_{ii} \partial \bar{w}_{kk}} = \frac{\bar{w}_{ii} w_{kk}}{(1 - |w_{ii}|^2)(1 - |w_{kk}|^2)} - \frac{\bar{w}_{ii}}{(1 - |w_{ii}|^2)} \frac{w_{kk}}{(1 - |w_{kk}|^2)} = 0 \quad \text{for} \quad i \neq k \quad (3.12)$$

and

$$\frac{\partial^2 \log \det(I - w\bar{w}')}{\partial w_{ii} \partial \bar{w}_{ii}} = -\frac{1}{1 - |w_{ii}|^2} - \frac{\bar{w}_{ii}}{1 - |w_{ii}|^2} \frac{w_{ii}}{1 - |w_{ii}|^2} = -\frac{1}{(1 - |w_{ii}|^2)^2}. \quad (3.13)$$

Suppose that \( i = k \) and \( j = l \leq p \) but \( i \neq j \). Then

$$\frac{\partial^2 \det(I - w\bar{w}')}{\partial w_{ij} \partial \bar{w}_{ij}} = \det \begin{pmatrix} N_1 & N_2 & 0 \\ \bar{N}_2 & -1 & N_3 \\ 0 & \bar{N}_3 & N_4 \end{pmatrix} = -\frac{\Pi_{\sigma=1}^p (1 - |w_{\sigma\sigma}|^2)}{(1 - |w_{ii}|^2)(1 - |w_{jj}|^2)},$$

where \( N_1 = \text{diag}(1 - |w_{11}|^2, \ldots, 1 - |w_{i-1,i-1}|^2) \), \( N_2 = (-w_{1j}, \ldots, -w_{i-1,j})^t \), \( N_3 = (-\bar{w}_{i+1,j}, \ldots, -\bar{w}_{p,j}) \), and \( N_4 = \text{diag}(1 - |w_{i+1,i+1}|^2, \ldots, 1 - |w_{pp}|^2) \). Hence

$$\frac{\partial^2 \log \det(I - w\bar{w}')}{\partial w_{ij} \partial \bar{w}_{ij}} = -\frac{1}{(1 - |w_{ii}|^2)(1 - |w_{jj}|^2)} \quad \text{for} \quad i \neq j \leq p. \quad (3.14)$$

Suppose that \( i = k \) and \( j = l \geq p + 1 \) but \( i \neq j \). Then

$$\frac{\partial^2 \log \det(I - w\bar{w}')}{\partial w_{ij} \partial \bar{w}_{ij}} = -\frac{1}{1 - |w_{ii}|^2} \quad \text{for} \quad i \leq p < j \leq q. \quad (3.15)$$

Suppose that \( i = k \), \( j \neq l \) but \( j \neq i, l \neq i \). If \( j < i \) and \( l < i \), we have

$$\frac{\partial^2 \log \det(I - w\bar{w}')}{\partial w_{ij} \partial \bar{w}_{il}} = \frac{1}{\det(I - w\bar{w}')} \det \begin{pmatrix} L_1 & L_2 & 0 \\ L_3 & 0 & 0 \\ 0 & 0 & L_4 \end{pmatrix} = 0 \quad (3.16)$$

with \( N_1 = \text{diag}(1 - |w_{11}|^2, \ldots, 1 - |w_{i-1,i-1}|^2) \), \( L_2 = (0, \ldots, 0, -w_{ii}, 0, \ldots, 0)^t \) where \( -w_{ii} \) is located in the \( i \)th component, \( L_3 = (0, \ldots, 0, -w_{jj}, 0, \ldots, 0) \) where \( -w_{jj} \) is located in the \( j \)th component and \( L_4 = \text{diag}(1 - |w_{i+1,i+1}|^2, \ldots, 1 - |w_{pp}|^2) \). This holds since \( i \)th row and \( j \)th row are dependent.
If $i < j \neq l$ or $l < i < j \leq p$ or $j < i < l \leq p$, by a similar computation, we obtain
\[
\frac{\partial^2 \log \det(I - w\overline{w})}{\partial w_{ij} \partial \overline{w}_{il}} = 0.
\] (3.17)

As a result, by (3.8), (3.9), (3.11), (3.12), (3.13), (3.14), (3.15), and (3.17) for a tangent vector $X = \sum_{i,j} X_{ij} \frac{\partial}{\partial w_{ij}}$ at $w_0$, we have
\[
- \sum_{1 \leq i, k \leq p} X_{ij} \frac{\partial^2 \log \det(I - w\overline{w})}{\partial w_{ij} \partial \overline{w}_{kl}} X_{kl} = \sum_{1 \leq i, j \leq p} \frac{|X_{ij}|^2}{1 - |w_{ii}|^2 (1 - |w_{jj}|^2)} + \sum_{1 \leq i \leq p, p+1 \leq j \leq q} \frac{|X_{ij}|^2}{1 - |w_{ii}|^2} \geq \sum_{1 \leq i \leq p} |X_{ij}|^2,
\] (3.18)
and hence $\log \psi_\Omega$ is psh.

**Type II and III:** Since $\psi_{\Omega_n^{II}}, \psi_{\Omega_n^{III}}$ are the restriction of $\psi_{\Omega_n^{I}}, \frac{1}{2} \psi_{\Omega_n^{I}}$ on $\Omega_n^{II}, \Omega_n^{III}$, respectively, $\log \psi_{\Omega_n^{II}}, \log \psi_{\Omega_n^{III}}$ are psh.

**Type IV:** Since $K_{\Omega_n^{IV}}(z, \overline{z}) = c_1 (1 - 2z\overline{z} + |z\overline{w}|^2)^{-c_2}$ for some constant $c_1, c_2 > 0$, we have
\[
\partial \overline{\partial} \log \psi_{\Omega_n^{IV}}(z, w) = c_2 \begin{pmatrix} 2I & -2I \\ -2I & -\partial_w \overline{\partial}_w \log \left(1 - 2w\overline{w} + |w\overline{w}|^2\right) \end{pmatrix}
\] (3.19)
at $(0, w)$. A maximal polydisc in $\Omega_n^{IV}$ is given by
\[
\Delta^2 := \{(w_1, w_2, 0, \ldots, 0) : w_1 = \lambda(\zeta_1 + \zeta_2), w_2 = i\lambda(\zeta_1 - \zeta_2), |\zeta_1| < 1, |\zeta_2| < 1\},
\]
with $\lambda^2 = \frac{i}{4}$. For type IV domain, we denote $\sqrt{-1}$ by $i$. Note that on $\Delta^2$,
\[
ww' = 4\lambda^2 \zeta_1 \zeta_2 = i\zeta_1 \zeta_2, \quad w\overline{w}' = \frac{1}{2} (|\zeta_1|^2 + |\zeta_2|^2),
\]
\[
1 - 2w\overline{w}' + |w\overline{w}'|^2 = (1 - |\zeta_1|^2) (1 - |\zeta_2|^2),
\]
\[
4w_1 w_2 = - (\zeta_1^2 - \zeta_2^2),
\]
\[
4w_1 \overline{w}_2 = 4\lambda(\zeta_1 + \zeta_2)\overline{\lambda}(\zeta_1 - \zeta_2) = -i(\zeta_1 + \zeta_2)(\overline{\zeta_1} - \overline{\zeta_2}).
\] (3.20)

Since $\frac{\partial^2}{\partial w_j \partial \overline{w}_k} (1 - 2w\overline{w}' + |w\overline{w}'|^2) = -2\delta_{jk} + 4w_j \overline{w}_k$, on $\Delta^2$, we obtain
\[
\partial \overline{\partial} \left(1 - 2w\overline{w}' + |w\overline{w}'|^2\right) = \begin{pmatrix} -2 + 4|w_1|^2 & 4w_1 \overline{w}_2 \\ 4w_2 \overline{w}_1 & -2 + 4|w_2|^2 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ \vdots \\ 0 \end{pmatrix}.
\]
Since \( \frac{\partial}{\partial \omega_j} (1 - 2w \bar{w} + |w w'|^2) = -2(\bar{w}_j - w_j(\bar{w} w')) \) on \( \Delta^2, \partial S^{IV}_n \bar{S}^{IV}_n \) is
\[
\begin{pmatrix}
4|\bar{w}_1 - w_1(\bar{w} w')|^2 & 4(\bar{w}_1 - w_1(\bar{w} w'))(w_2 - \bar{w}_2(w w')) & 0 \\
4(\bar{w}_2 - w_2(\bar{w} w'))(w_1 - \bar{w}_1(w w')) & 4|\bar{w}_2 - w_2(\bar{w} w')|^2 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Hence by a straightforward calculation we obtain
\[
- S^{IV}_n \frac{\partial^2 S^{IV}_n}{\partial \omega_1 \partial \omega_2} + \frac{\partial S^{IV}_n}{\partial \omega_1} \frac{\partial S^{IV}_n}{\partial \omega_2} = i (|\xi_1|^2 - |\xi_2|^2)(2 - |\xi_1|^2 - |\xi_2|^2),
\]
\[
- S^{IV}_n \frac{\partial^2 S^{IV}_n}{\partial \omega_1 \partial \omega_1} + \frac{\partial S^{IV}_n}{\partial \omega_1} \frac{\partial S^{IV}_n}{\partial \omega_1} = 2(1 - |\xi_1|^2)(1 - |\xi_2|^2) + (|\xi_1|^2 - |\xi_2|^2)^2
\]
and
\[
- S^{IV}_n \frac{\partial^2 S^{IV}_n}{\partial \omega_2 \partial \omega_2} + \frac{\partial S^{IV}_n}{\partial \omega_2} \frac{\partial S^{IV}_n}{\partial \omega_2} = 2(1 - |\xi_1|^2)(1 - |\xi_2|^2) + (|\xi_1|^2 - |\xi_2|^2)^2.
\]

To show that (3.19) is semi-positive definite, we only need to prove that \(-\partial \bar{\partial} \log S^{IV}_n - 2I\) is positive semi-definite, that is,
\[
\begin{pmatrix}
M & 0 \\
0 & \frac{2}{S^{IV}_n} I
\end{pmatrix} - 2I
\]
is semi-positive definite where \(M\) is given by
\[
\begin{pmatrix}
\frac{1}{(S^{IV}_n)^2} \left(2(1 - |\xi_1|^2)(1 - |\xi_2|^2) + (|\xi_1|^2 - |\xi_2|^2)^2\right) & \frac{i}{(S^{IV}_n)^2} \left(|\xi_1|^2 - |\xi_2|^2\right)(2 - |\xi_1|^2 - |\xi_2|^2) \\
\frac{-i}{(S^{IV}_n)^2} \left(|\xi_1|^2 - |\xi_2|^2\right)(2 - |\xi_1|^2 - |\xi_2|^2) & \frac{1}{(S^{IV}_n)^2} \left(2(1 - |\xi_1|^2)(1 - |\xi_2|^2) + (|\xi_1|^2 - |\xi_2|^2)^2\right)
\end{pmatrix}.
\]

**Lemma 3.3.** \(M - \frac{2}{S^{IV}_n} I \geq 0.\)

**Proof.** We only need to check
\[
\begin{pmatrix}
2(1 - |\xi_1|^2)(1 - |\xi_2|^2) + (|\xi_1|^2 - |\xi_2|^2)^2 - 2S^{IV}_n & i(|\xi_1|^2 - |\xi_2|^2)(2 - |\xi_1|^2 - |\xi_2|^2) \\
-i(|\xi_1|^2 - |\xi_2|^2)(2 - |\xi_1|^2 - |\xi_2|^2) & 2(1 - |\xi_1|^2)(1 - |\xi_2|^2) + (|\xi_1|^2 - |\xi_2|^2)^2 - 2S^{IV}_n
\end{pmatrix}
\]
(3.22)
is positive semi-definite. Since (3.22) is equal to
\[
\begin{pmatrix}
(|\zeta_1|^2 - |\zeta_2|^2)^2 & i(|\zeta_1|^2 - |\zeta_2|^2)(2 - |\zeta_1|^2 - |\zeta_2|^2) \\
-i((|\zeta_1|^2 - |\zeta_2|^2)(2 - |\zeta_1|^2 - |\zeta_2|^2)) & (|\zeta_1|^2 - |\zeta_2|^2)^2
\end{pmatrix}
\]
and the determinant of it is
\[
(|\zeta_1|^2 - |\zeta_2|^2)^4 - (|\zeta_1|^2 - |\zeta_2|^2)(2 - |\zeta_1|^2 - |\zeta_2|^2)^2 = 4(|\zeta_1|^2 - |\zeta_2|^2)^2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2) \geq 0,
\]
we obtain the lemma. □

Since \( \frac{2}{s_n} I_{n-2} - 2I_{n-2} \) is positive definite and \( M - 2I \succ M - \frac{2}{s_n} I > 0 \), as a result, \( \bar{\partial}\bar{\partial} \log \psi_{\Omega_{n}^V}(z, w) \) is positive semi-definite.

**Type V:** Since \( \Omega_{16}^V \subset \Omega_{27}^VI \) and \( S^V \) is the restriction of \( S^VI \) on \( \Omega_{16}^V \), we only need to show the plurisubharmonicity of \( \log \psi_{\Omega_{27}^V} \) on \( \Omega_{27}^VI \).

**Type VI:** We will show the pluriharmonicity of \( \log \psi_{\Omega_{27}^V} \) on \( \Omega_{27}^VI \) at \( (0, w_0) \) where \( w_0 \in \Delta^3 \subset \Omega_{27}^VI \). We put
\[
z = \begin{pmatrix}
\beta_1 & b_3 & \tilde{b}_2 \\
\tilde{b}_3 & \beta_2 & b_1 \\
b_2 & \tilde{b}_1 & \beta_3
\end{pmatrix}, \quad w = \begin{pmatrix}
\alpha_1 & a_3 & \tilde{a}_2 \\
\tilde{a}_3 & \alpha_2 & a_1 \\
a_2 & \tilde{a}_1 & \alpha_3
\end{pmatrix},
\]
and \( w_0 = \text{diag}(\alpha_1, \alpha_2, \alpha_3) \). Define
\[
\hat{\Gamma} := \begin{pmatrix}
I_3 & 0 \\
0 & 2I_{24}
\end{pmatrix},
\]
where \( I_n \) denotes the \( n \times n \) identity matrix for any \( n \in \mathbb{N} \). Since
\[
S^VI(z, \bar{z}) = 1 - (z|z) + (z|^z^\#) - |\det z|^2
\]  
(3.23)
and \( (z|^z^\#), \ |\det z|^2 \) are homogeneous polynomials of degree 4, 6, respectively, in \( \beta_i, \ \tilde{\beta}_i, \ b_{ij}, \ \tilde{b}_{ij} \) with \( i = 1, 2, 3, \ j = 0, 1, \ldots, 7 \), second derivatives of \( (z|^z^\#) - |\det z|^2 \) vanish at \( z = 0 \), and it implies
\[
\bar{\partial}\bar{\partial} \log \psi_{\Omega_{27}^V}(z, w)|_{z=0} = c_2 \begin{pmatrix}
\hat{\Gamma} & -\hat{\Gamma} \\
-\hat{\Gamma} & -\partial_w \partial_w \log S^VI(w, \bar{w})
\end{pmatrix}
\]  
(3.24)
at \( (0, w_0) \). Since
\[
w^\# = \begin{pmatrix}
\alpha_2 \alpha_3 - a_1 \tilde{a}_1 & a_1 a_2 - \alpha_3 \tilde{a}_3 & a_3 a_1 - \alpha_2 \tilde{a}_2 \\
a_1 a_2 - \alpha_3 \tilde{a}_3 & \alpha_3 \alpha_3 - a_2 \tilde{a}_2 & a_2 a_3 - \alpha_1 \tilde{a}_1 \\
a_3 a_1 - \alpha_2 \tilde{a}_2 & a_2 a_3 - \alpha_1 \tilde{a}_1 & \alpha_1 \alpha_2 - a_3 \tilde{a}_3
\end{pmatrix},
\]
WEAKLY 1-COMPLETENESS OF HOLOMORPHIC FIBER BUNDLES OVER COMPACT KÄHLER MANIFOLDS

We have

\[
\frac{\partial w^#}{\partial a_{1j}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha_1 e_j \\ 0 & -\alpha_1 e_j & 0 \end{pmatrix}, \quad \frac{\partial w^#}{\partial a_{2j}} = \begin{pmatrix} 0 & 0 & -\alpha_2 e_j \\ 0 & 0 & 0 \\ -\alpha_2 e_j & 0 & 0 \end{pmatrix}, \quad \frac{\partial w^#}{\partial a_{3j}} = \begin{pmatrix} 0 & -\alpha_3 e_j & 0 \\ 0 & -\alpha_3 e_j & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

(3.25)

Moreover we have

\[
\frac{\partial w^#}{\partial \alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_3 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}, \quad \frac{\partial w^#}{\partial \alpha_2} = \begin{pmatrix} \alpha_3 & 0 & 0 \\ 0 & 0 & \alpha_1 \\ 0 & \alpha_1 & 0 \end{pmatrix}, \quad \frac{\partial w^#}{\partial \alpha_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha_3 \\ 0 & \alpha_3 & 0 \end{pmatrix}.
\]

(3.26)

Moreover we have

\[
\frac{\partial \det w}{\partial a_{ij}} = 0
\]

(3.27)

at \( w = w_0 \). By (3.23), (3.25), and (3.27) one has

\[
\frac{\partial S^{VI}(w, \bar{w})}{\partial a_{ij}} = \frac{\partial}{\partial a_{ij}} \left( 1 - (w \cdot w) + (w^# \cdot w^#) - |\det w|^2 \right)
\]

\[= - \left( \frac{\partial w}{\partial a_{ij}} \right| w + \left( \frac{\partial w^#}{\partial a_{ij}} \right| w^# \right) - \frac{\partial \det w}{\partial a_{ij}} \det w = 0,
\]

(3.28)

and by (3.23), (3.25), (3.26), and (3.27) one has

\[
\frac{\partial^2 S^{VI}(w, \bar{w})}{\partial \alpha_j \partial \bar{a}_{kl}} = - \left( \frac{\partial w}{\partial \alpha_j} \right| \frac{\partial w}{\partial \bar{a}_{kl}} + \left( \frac{\partial w^#}{\partial \alpha_j} \right| \frac{\partial w^#}{\partial \bar{a}_{kl}} \right) - \frac{\partial \det w}{\partial \alpha_j} \left( \frac{\partial \det w}{\partial \bar{a}_{kl}} \right) = 0.
\]

(3.29)

Moreover by (3.1), (3.23), (3.27), and (3.25)

\[
- \frac{\partial^2 S^{VI}(w, \bar{w})}{\partial a_{ij} \partial \bar{a}_{kl}} = \left( \frac{\partial w}{\partial a_{ij}} \right| \frac{\partial w}{\partial \bar{a}_{kl}} - \left( \frac{\partial w^#}{\partial a_{ij}} \right| \frac{\partial w^#}{\partial \bar{a}_{kl}} \right)
\]

\[= \delta_{ik}(e_j | e_l) - \delta_{ik} \alpha_i \bar{\alpha}_k (e_j | e_l) = 2 \delta_{ik} \delta_{ji} e_0 - 2 \delta_{ik} \delta_{ji} \alpha_i \bar{\alpha}_k e_0,
\]

(3.30)

and since we have \( S^{VI}(w, \bar{w}) = (1 - |\alpha_1|^2)(1 - |\alpha_2|^2)(1 - |\alpha_3|^2) \) at \( (0, w_0) \), we obtain

\[
\frac{\partial^2 \log S^{VI}(w, \bar{w})}{\partial \alpha_i \partial \bar{\alpha}_k} = \frac{-\delta_{ik}}{(1 - |\alpha_i|^2)^2}.
\]

(3.31)

For a vector \( (Y, Z) \) on \( T_{w_0} \Omega^{VI}_{27} \) with \( Y = \sum Y_j \frac{\partial}{\partial \alpha_j} \) and \( Z = \sum Z_{kl} \frac{\partial}{\partial \bar{a}_{kl}} \), by (3.28), (3.29), (3.30), and (3.31) we have
\[
\frac{1}{c_2} \partial_w \bar{\partial}_w \log \psi_{\Omega_{27}}((Y, Z), (\bar{Y}, \bar{Z})) = - \sum Z_{ij} \frac{\partial^2 \log S^{27}(w, \bar{w})}{\partial \alpha_i \partial \alpha_k} Z_{kl} - \sum Y_i \frac{\partial^2 \log S^{27}(w, \bar{w})}{\partial \alpha_i \partial \alpha_k} \bar{Y}_k
\]
\[
= \frac{2}{\Pi^3 (1 - |\alpha_i|^2)} \sum |Y_i|^2 + \sum \frac{2 \sum_{ij} (1 - |\alpha_i|^2) |Z_{ij}|^2}{(1 - |\alpha_i|^2)^2}.
\]

By Equation (3.24), \( \partial \bar{\partial} \log \psi_{\Omega_{27}} \) is positive semi-definite. \( \square \)

4 WHEN \( \rho \) IS REDUCTIVE

Let \( M_1 \) and \( M_2 \) be Riemannian manifolds with Riemannian metrics \( d s^2_{M_1} = \sum g_{\alpha \beta} dx^\alpha dx^\beta \), \( d s^2_{M_2} = \sum h_{ij} dy^i dy^j \), respectively. Let \( f : M_1 \to M_2 \) be a map. The energy \( E(f) \) of \( f \) is defined by

\[
\frac{1}{2} \int_{M_1} \text{trace}_{d s^2_{M_1}} (f^* d s^2_{M_2}) = \frac{1}{2} \int_{M_1} \sum g^{\alpha \beta} h_{ij} \circ f \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta}.
\]

The Euler–Lagrange equation for the energy functional \( E \) is \( \Delta f := \text{trace} \nabla d f = 0 \), and this can be expressed in a local coordinate by

\[
\Delta_{M_1} f^i + \sum \Gamma^i_{jk} \circ f \frac{\partial f^j}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} g^{\alpha \beta} = 0
\]

for all \( i \), where \( \Delta_{M_1} \) is the Laplace–Beltrami operator of \( M_1 \) and \( \Gamma^i_{jk} \) is the Christoffel symbol of \( M_2 \). The map \( f \) is said to be harmonic if \( f \) satisfies the Euler–Lagrange equation for the energy functional.

If \( M_1 \) and \( M_2 \) are Kähler, the map \( f : M_1 \to M_2 \) is said to be pluriharmonic if \( \nabla_{1,0} \partial f \equiv 0 \). In local coordinates, pluriharmonic map \( f \) satisfies

\[
\frac{\partial^2 f^i}{\partial \zeta^\alpha \partial \bar{\zeta}^\beta} + \sum \Gamma^i_{jk} \circ f \frac{\partial f^j}{\partial \zeta^\alpha} \frac{\partial f^k}{\partial \bar{\zeta}^\beta} = 0
\]

for all \( i \). We remark that if \( f \) is pluriharmonic, then \( f \) is also harmonic and the pluriharmonicity is preserved by the isometries.

Proof of Theorem 1.1. By Theorem 1.3, there exists a \( \rho \)-equivariant harmonic map from \( \tilde{M} \) to \( \Omega \) and hence there exists a harmonic section of \( E \). By Siu [28] and Sampson [27], the given harmonic section \( s \) is pluriharmonic. For a local coordinate \( U \) on \( M \) such that \( \pi^{-1}(U) \cong U \times \Omega \), we may express \( s|_U(\xi) = (\xi, h(\xi)) \) for some \( h : U \to \Omega \). Define a map \( \psi \) by

\[
\psi(\xi, w) = \log \psi_{\Omega}(h(\xi), w).
\]

Since \( \log \psi_{\Omega} \) is invariant with respect to the diagonal action of \( \text{Aut}(\Omega) \) and the automorphisms are isometries with respect to the Bergman metric, we may assume that \( h(\xi) = 0 \) and \( w \) is contained in a maximal totally geodesic polydisc of \( \Omega \). Consider the pluriharmonic map \( (\xi, w) \mapsto (h(\xi), w) \) from \( U \times \Omega \to \Omega \times \Omega \), with respect to the product metrics \( g_{U} \otimes g_{\text{std}} \) and \( g_{\Omega} \otimes g_{\text{std}} \) with the standard Euclidean metric \( g_{\text{std}} \) of the ambient Euclidean space of \( \Omega \), the Bergman metric \( g_{\Omega} \)

\[
\psi(\xi, w) = \log \psi_{\Omega}(h(\xi), w).
\]
of \( \Omega \), and the Kähler metric \( g \) of \( M \). Since \( \Gamma^k_{ij}(0) = 0 \) where \( \Gamma^k_{ij} \) is the Christoffel symbol of \( g_{\Omega} \), pluriharmonicity of \( h \) implies

\[
\frac{\partial^2 h^k}{\partial \xi_i \partial \xi_j}(\xi) = 0 \quad \text{for any } i, j, k. \tag{4.1}
\]

By the chain rule and the equality

\[
\frac{\partial^2}{\partial z_k \partial \overline{w}_j} \log \psi_{\Omega}(z, w) = \frac{\partial^2}{\partial z_k \partial \overline{w}_j} \log K(z, z)K(w, w) = 0,
\]

we have

\[
\frac{\partial^2}{\partial \xi_i \partial \overline{w}_j} \log \psi_{\Omega}(h(\xi), w) = \sum_k \frac{\partial^2 \log \psi_{\Omega}(0, w)}{\partial z_k \partial \overline{w}_j} \frac{\partial h^k}{\partial \xi_i}(\xi) + \sum_k \frac{\partial^2 \log \psi_{\Omega}(0, w)}{\partial \overline{z}_k \partial \overline{w}_j} \frac{\partial \overline{h}^k}{\partial \xi_i}(\xi), \tag{4.2}
\]

\[
\frac{\partial^2}{\partial w_i \partial \overline{w}_j} \log \psi_{\Omega}(h(\xi), w) = \frac{\partial^2 \log \psi_{\Omega}}{\partial w_i \partial \overline{w}_j}(0, w), \tag{4.3}
\]

and by (4.1)

\[
\frac{\partial^2}{\partial \xi_i \partial \overline{\xi}_j} \log \psi_{\Omega}(h(\xi), w) = \sum_{k,l} \left( \frac{\partial^2 \log \psi_{\Omega}}{\partial z_k \partial z_l} \frac{\partial h^k}{\partial \xi_i} \frac{\partial h^l}{\partial \overline{\xi}_j} + \frac{\partial^2 \log \psi_{\Omega}}{\partial \overline{z}_k \partial \overline{z}_l} \frac{\partial \overline{h}^k}{\partial \xi_i} \frac{\partial \overline{h}^l}{\partial \overline{\xi}_j} + \frac{\partial^2 \log \psi_{\Omega}}{\partial z_k \partial \overline{z}_l} \frac{\partial h^k}{\partial \xi_i} \frac{\partial \overline{h}^l}{\partial \overline{\xi}_j} + \frac{\partial^2 \log \psi_{\Omega}}{\partial \overline{z}_k \partial z_l} \frac{\partial \overline{h}^k}{\partial \xi_i} \frac{\partial h^l}{\partial \overline{\xi}_j} \right). \tag{4.4}
\]

**Type I, \( \Omega_{p,q} \):** At \((0, w_0)\) with \( w_0 = \text{diag}(w_{11}, \ldots, w_{pp}), |w_{jj}| < 1 \) for all \( j \), one obtains

\[
\frac{\partial \det(I - z\overline{w})}{\partial z_{i,j}} = -\overline{w}_{ij},
\]

\[
\frac{\partial^2 \det(I - z\overline{w})}{\partial z_{i,j} \partial z_{k,l}} = \det \begin{pmatrix} \overline{w}_{ij} & \overline{w}_{kj} \\ \overline{w}_{il} & \overline{w}_{kl} \end{pmatrix},
\]

and hence

\[
\frac{\partial^2}{\partial z_{i,j} \partial z_{k,l}} \log \psi_{\Omega_{p,q}^i}(z, w) = -c_2 \frac{\partial^2}{\partial z_{i,j} \partial z_{k,l}} \log \frac{\det(I - z\overline{w}) \det(I - w\overline{z})}{\det(I - z\overline{w}) \det(I - w\overline{z})} \tag{4.5}
\]

\[
= c_2 \frac{\partial^2}{\partial z_{i,j} \partial z_{k,l}} \log \det(I - z\overline{w}) = -c_2 \overline{w}_{il} \overline{w}_{kj}.
\]
Moreover we have
\[
\frac{\partial^2}{\partial z_{ij} \partial \bar{w}_{kl}} \log \det(I - z \bar{w}) = -\delta_{ik} \delta_{jl}. \tag{4.6}
\]

By (4.4) and (4.5)
\[
\frac{1}{c^2} \frac{\partial^2}{\partial \xi_\alpha \partial \xi_\beta} \log \psi_{\Omega_{p,q}}(h(\xi), w) = \sum_{i,k=1}^p \left( -\bar{w}_{i\alpha} \frac{\partial h_{i\beta}^k}{\partial \xi_\alpha} \frac{\partial h_{i\beta}^k}{\partial \xi_\beta} - w_{i\alpha} \frac{\partial h_{i\beta}^k}{\partial \xi_\alpha} \frac{\partial h_{i\beta}^k}{\partial \xi_\beta} \right) + \sum_{i=1}^p \sum_{j=1}^q \left( \frac{\partial h_{ij}^j}{\partial \xi_\alpha} \frac{\partial h_{ij}^j}{\partial \xi_\beta} + \frac{\partial h_{ij}^j}{\partial \xi_\alpha} \frac{\partial h_{ij}^j}{\partial \xi_\beta} \right). \tag{4.7}
\]

As a consequence of (4.2), (4.3), (4.7), and (4.6) for \( Z = \sum_{\alpha} X_\alpha \frac{\partial}{\partial \xi_\alpha} + \sum_{i,j} Y_{ij} \frac{\partial}{\partial z_{ij}} \) with \( X = \sum_{\alpha} X_\alpha \frac{\partial}{\partial \xi_\alpha} \), we obtain
\[
\frac{1}{c^2} \frac{\partial^2}{\partial \xi_\alpha} \log \psi_{\Omega_{p,q}}(h(\xi), w)(Z, \bar{Z}) \\
= \sum_{i=1}^p \sum_{j=1}^q \left( |X h_{ij}^j|^2 + |X h_{ij}^j|^2 - 2 \text{Re}\left( (X h_{ij}^j)Y_{ij} \right) \right) \\
- \sum_{i,j=1}^p 2 \text{Re}\left( w_{ii} w_{jj} (X h_{ij}^j) (\bar{X} h_{ij}^j) \right) + \partial_w \partial_{\bar{w}} \log \psi_{\Omega_{p,q}}(0, w)(Y, \bar{Y}) \\
\geq \sum_{i=1}^p \sum_{j=1}^q \left( |X h_{ij}^j|^2 + |X h_{ij}^j|^2 - 2 \text{Re}\left( (X h_{ij}^j)Y_{ij} \right) + \frac{|Y_{ij}|^2}{1 - |w_{ii}|^2} \right) - \sum_{i,j=1}^p 2 \text{Re}\left( w_{ii} w_{jj} (X h_{ij}^j) (\bar{X} h_{ij}^j) \right) \\
= \sum_{i=1}^p \sum_{j=1}^q \left( |w_{ii} (X h_{ij}^j) - \bar{w}_{jj} (X h_{ij}^j)|^2 + (1 - |w_{ii}|^2) |X h_{ij}^j|^2 \right) + \sum_{i=1}^p \sum_{j=p+1}^q \left( |X h_{ij}^j|^2 + |w_{ii}|^2 |X h_{ij}^j|^2 \right) \\
+ \sum_{i=1}^p \sum_{j=1}^q \left| \sqrt{1 - |w_{ii}|^2} |X h_{ij}^j| - \frac{Y_{ij}}{\sqrt{1 - |w_{ii}|^2}} \right|^2 \geq 0
\]
by (3.18). Hence \( \partial \partial \log \psi_{\Omega_{p,q}}(h(\xi), w) \) is positive semi-definite.

**Type II, III:** Since \( \psi_{\Omega_{ij}}^{II} \) and \( \psi_{\Omega_{ij}}^{III} \) are the restriction of \( \psi_{\Omega_{ii,n}}^{II} \) and \( \frac{1}{2} \psi_{\Omega_{ii,n}}^{II} \) on \( \Omega_{ii}^{II} \) and \( \Omega_{ii}^{III} \), respectively, \( \partial \partial \log \psi_{\Omega_{ii,n}}^{II}(h(\xi), w), \partial \partial \log \psi_{\Omega_{ii,n}}^{III}(h(\xi), w) \) are positive semi-definite.

**Type IV:** At \((0, w)\), we have
\[
\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \psi_{\Omega_n^{IV}}(z, w) = c^2 \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log(1 - 2z w^* + (zz^*)(ww^*)) = 2 \delta_{ij}(ww^*) - 4 \bar{w}_i \bar{w}_j.
\]

Therefore we have
\[
\frac{\partial^2 \log \psi_{\Omega_n^{IV}}}{\partial z_1 \partial \bar{z}_1} = -2c^2(\bar{w}_1 - \bar{w}_2), \quad \frac{\partial^2 \log \psi_{\Omega_n^{IV}}}{\partial z_2 \partial \bar{z}_2} = 2c^2(\bar{w}_1 - \bar{w}_2), \quad \frac{\partial^2 \log \psi_{\Omega_n^{IV}}}{\partial z_1 \partial \bar{z}_2} = -4c^2 \bar{w}_1 \bar{w}_2,
\]
and hence by (4.4)

\[
\frac{1}{2c_2} \frac{\partial^2}{\partial \xi^2} \log \psi_{\Omega IV} (h(\xi), w) = \sum_j \left( \frac{\partial \bar{h}^j}{\partial \xi} \frac{\partial h^j}{\partial \bar{\xi}} + \frac{\partial h^j}{\partial \xi} \frac{\partial \bar{h}^j}{\partial \bar{\xi}} \right) + 2 \text{Re} \left( - (w_1^2 - w_2^2) \frac{\partial h^1}{\partial \xi} \frac{\partial h^1}{\partial \bar{\xi}} + (w_1^2 - w_2^2) \frac{\partial h^2}{\partial \xi} \frac{\partial h^2}{\partial \bar{\xi}} - 2 w_1 w_2 \frac{\partial h^1}{\partial \xi} \frac{\partial h^2}{\partial \bar{\xi}} - 2 w_1 w_2 \frac{\partial h^2}{\partial \xi} \frac{\partial h^1}{\partial \bar{\xi}} \right).
\]

This implies for \( X = \sum_{\alpha=1}^n X_{\alpha} \frac{\partial}{\partial \xi^\alpha} \)

\[
\frac{1}{2c_2} \sum_{\alpha, \beta} X_{\alpha} \frac{\partial^2}{\partial \xi^\alpha \partial \bar{\xi}^\beta} \log \psi_{\Omega IV} (h(\xi), w) X_{\beta} \\
= n \sum_{j=1}^n \left( |X_{\beta}^j|^2 + |X h^j|^2 \right) + 2 \text{Re} \left( - (w_1^2 - w_2^2)(X h^1)(\bar{X} h^1) + (w_1^2 - w_2^2)(X h^2)(\bar{X} h^2) - 2 w_1 w_2 (X h^1)(\bar{X} h^2) - 2 w_1 w_2 (X h^2)(\bar{X} h^1) \right).
\]

For \( \zeta_1, \zeta_2 \) satisfying \( w_1 = \lambda (\zeta_1 + \zeta_2) \) and \( w_2 = i \lambda (\zeta_1 - \zeta_2) \) and for \( X = \sum X_{\alpha} \frac{\partial}{\partial \xi^\alpha} \), we have

\[
\frac{1}{2c_2} \frac{\partial^2}{\partial \bar{\xi}^2} \log \psi_{\Omega IV} (h(\xi), w)(X, \bar{X}) \\
= n \sum_{j=1}^n \left( |X_{\beta}^j|^2 + |X h^j|^2 \right) + 2 \text{Re} \left( \frac{i}{2} (\zeta_1^2 + \zeta_2^2)(X h^1)(\bar{X} h^1) - \frac{i}{2} (\zeta_1^2 + \zeta_2^2)(X h^2)(\bar{X} h^2) \\
+ \frac{1}{2} (\zeta_1^2 - \zeta_2^2)(X h^1)(\bar{X} h^2) + \frac{1}{2} (\zeta_1^2 - \zeta_2^2)(X h^2)(\bar{X} h^1) \right) \\
= n \sum_{j=3}^n \left( |X_{\beta}^j|^2 + |X h^j|^2 \right) + \frac{1}{2} \left( |X h^1 - i X h^2|^2 + |\bar{X} h^1 - i \bar{X} h^2|^2 + |X h^1 + i X h^2|^2 + |\bar{X} h^1 + i \bar{X} h^2|^2 \right) \\
+ 2 \text{Re} \left( \frac{\zeta_1^2}{2} i(X h^1 - i X h^2)(\bar{X} h^1 - i \bar{X} h^2) + \frac{\zeta_2^2}{2} i(X h^1 + i X h^2)(\bar{X} h^1 + i \bar{X} h^2) \right) \\
= \frac{1}{2} \left[ (1 - |\zeta_1|^2)(|X h^1 - i X h^2|^2 + |\bar{X} h^1 - i \bar{X} h^2|^2) + (1 - |\zeta_2|^2)(|X h^1 + i X h^2|^2 + |\bar{X} h^1 + i \bar{X} h^2|^2) \right] \\
+ \frac{|\zeta_1|^2}{2} \left( |X h^1 - i X h^2|^2 + |\bar{X} h^1 - i \bar{X} h^2|^2 \right) + \frac{|\zeta_2|^2}{2} \left( |X h^1 + i X h^2|^2 + |\bar{X} h^1 + i \bar{X} h^2|^2 \right) \\
+ \sum_{j=3}^n \left( |X_{\beta}^j|^2 + |X h^j|^2 \right). \tag{4.8}
\]
As a consequence of (4.2), (4.3), (4.8), and (3.21), for $Z = \sum \alpha \frac{\partial}{\partial \alpha} + \sum Y_k \frac{\partial}{\partial w_k}$, one obtains

$$\frac{1}{2c^2} \frac{\partial}{\partial \log \Omega_{\mathcal{L}}(\xi, \omega)}(Z, Z) = \frac{1}{2} \left[ (1 - |\xi_1|^2) \left( |Xh^1 - iXh^2|^2 + |\bar{X}h^1 - i\bar{X}h^2|^2 \right) + (1 - |\xi_2|^2) \left( |Xh^1 + iXh^2|^2 + |\bar{X}h^1 + i\bar{X}h^2|^2 \right) \right]$$

$$+ \frac{|\xi_1|^2}{2} \left( |Xh^1 - iXh^2|^2 + |\bar{X}h^1 + i\bar{X}h^2|^2 \right) + \frac{|\xi_2|^2}{2} \left( |Xh^1 + iXh^2|^2 + |\bar{X}h^1 - i\bar{X}h^2|^2 \right)$$

$$- 2 \sum_{j=1}^n \text{Re}(Xh^j)\bar{Y}_j + \frac{M_{11}}{2} |Y_1|^2 + \frac{M_{22}}{2} |Y_2|^2 + \frac{M_{12}}{2} Y_1 \bar{Y}_2 + \frac{M_{21}}{2} Y_2 \bar{Y}_1$$

$$+ \sum_{j=3}^n \left( |X\bar{h}_j|^2 + |Xh_j|^2 \right) - (1 - |\xi_1|^2)(1 - |\xi_2|^2) |Xh_j|^2$$

$$+ \sum_{j=3}^n \left( |X\bar{h}_j|^2 + |Xh_j|^2 - (1 - |\xi_1|^2)(1 - |\xi_2|^2) |Xh_j|^2 \right),$$

(4.9)

where $M_{11} = M_{22} = \frac{2(1 - |\xi_1|^2)(1 - |\xi_2|^2) + (|\xi_1|^2 - |\xi_2|^2)^2}{(1 - |\xi_1|^2)(1 - |\xi_2|^2)^2}$ and $M_{12} = -M_{21} = i \frac{(2 - |\xi_1|^2 - |\xi_2|^2)(|\xi_1|^2 - |\xi_2|^2)}{(1 - |\xi_1|^2)^2(1 - |\xi_2|^2)^2}$.

Since we have

$$|Xh^1|^2 + |Xh^2|^2 = \frac{1}{2} \left( |Xh^1 - iXh^2|^2 + |Xh^1 + iXh^2|^2 \right)$$

and

$$- 2 \sum_{j=1}^2 \text{Re}(Xh^j)\bar{Y}_j + \frac{M_{11}}{2} |Y_1|^2 + \frac{M_{22}}{2} |Y_2|^2 + \frac{M_{12}}{2} Y_1 \bar{Y}_2 + \frac{M_{21}}{2} Y_2 \bar{Y}_1$$

$$\geq 2 \sum_{j=1}^2 \left( -2 \text{Re}(Xh^j)\bar{Y}_j + \frac{|Y_j|^2}{S_{n}^{IV}} \right) = 2 \sum_{j=1}^2 \left( \left| \sqrt{S_{n}^{IV}} Xh^j - \frac{Y_j}{S_{n}^{IV}} \right|^2 - S_{n}^{IV} |Xh_j|^2 \right),$$

Equation (4.9) is greater than or equal to 0.
**Type V, VI:** We need to prove the theorem only for type VI domain. We put

\[
  z = \begin{pmatrix}
  \beta_1 & b_3 & \tilde{b}_2 \\
  b_3 & \beta_2 & b_1 \\
  \tilde{b}_2 & b_1 & \beta_3
  \end{pmatrix}, \quad
  w = \begin{pmatrix}
  \alpha_1 & a_3 & \tilde{a}_2 \\
  \tilde{a}_3 & \alpha_2 & a_1 \\
  a_2 & \tilde{a}_1 & \alpha_3
  \end{pmatrix},
\]

and \( w_0 = \text{diag}(\alpha_1, \alpha_2, \alpha_3) \in \Delta^3 \subset \Omega_{27}^{VI} \). Since we have \( \frac{\partial^2 \det z}{\partial \beta_i \partial b_{kl}} = 0 \) at \( z = 0 \) and

\[
  \frac{\partial \Delta^{VI}(z, \bar{w})}{\partial b_{kl}} = 0
\]

at \( (0, w_0) \), by (3.23) one has

\[
  \frac{\partial^2}{\partial \beta_i \partial b_{kl}} \log \psi_{\Omega_{27}^{VI}}(z, w) = c_2 \frac{\partial^2}{\partial \beta_i \partial b_{kl}} \log S^{VI}(z, \bar{w}) = c_2 \left( \frac{\partial^2 z^\#}{\partial \beta_i \partial b_{kl}} \left| w^\# \right. \right).
\]

Since \( w^\# \) is a diagonal matrix and \( \frac{\partial^2 z^\#}{\partial \beta_i \partial b_{kl}} \) is off-diagonal at \( (0, w_0) \), those scalar product vanishes and hence we have

\[
  \frac{\partial^2}{\partial \beta_i \partial b_{kl}} \log \psi_{\Omega_{27}^{VI}}(z, w) = 0
\]

at \( (0, w_0) \). By considering diagonal elements in \( \frac{\partial^2 z^\#}{\partial \beta_i \partial \beta_j} \), at \( (0, w_0) \) we obtain

\[
  \frac{\partial^2 S^{VI}(z, \bar{w})}{\partial \beta_i \partial \beta_k} = \left( \frac{\partial^2 z^\#}{\partial \beta_i \partial \beta_k} \left| w^\# \right. \right) = \begin{cases} 
\overline{\alpha}_i \overline{\alpha}_k & \text{if } i \neq k, \\
0 & \text{if } i = k,
\end{cases}
\]

\[
  \frac{\partial S^{VI}(z, \bar{w})}{\partial \beta_i} = -\left( \frac{\partial z}{\partial \beta_i} \left| w_0 \right. \right) = -\overline{\alpha}_i,
\]

and therefore

\[
  \frac{1}{c_2} \frac{\partial^2}{\partial \beta_i \partial \beta_k} \log \psi_{\Omega_{27}^{VI}}(z, w) = \begin{cases} 
0 & \text{if } i \neq k, \\
-\overline{\alpha}_i^2 & \text{if } i = k.
\end{cases}
\]

Since \( \det z \) is a homogeneous polynomial of degree 3 in \( \beta_i, b_{kl} \) with \( i, k = 1, 2, 3, j = 0, 1, \ldots, 7 \), we have

\[
  \frac{\partial^2 \det z}{\partial b_{ij} \partial b_{kl}} = 0
\]

at \( z = 0 \) and hence by (4.10)

\[
  \frac{1}{c_2} \frac{\partial^2}{\partial b_{ij} \partial b_{kl}} \log \psi_{\Omega_{27}^{VI}}(z, w) = \frac{\partial^2}{\partial b_{ij} \partial b_{kl}} \log S^{VI}(z, \bar{w}) = \left( \frac{\partial^2 z^\#}{\partial b_{ij} \partial b_{kl}} \left| w^\# \right. \right).
\]
Since $w^\#$ is diagonal at $(0, w_0)$, it is enough to consider the diagonal terms of $\tfrac{\partial^2 w^\#}{\partial b_{ij} \partial b_{kl}}$. If $i \neq k$, then
\[
(\tfrac{\partial^2 w^\#}{\partial b_{ij} \partial b_{kl}} | w^\#) = 0.
\]
If $i = k$, then
\[
(\tfrac{\partial^2 w^\#}{\partial b_{ij} \partial b_{kl}} | w^\#) = -\left(\begin{array}{ccc}
\alpha_2 \alpha_3 & 0 & 0 \\
0 & \alpha_1 \alpha_3 & 0 \\
0 & 0 & \alpha_1 \alpha_2
\end{array}\right).
\]
Since
\[
\frac{\partial^2 b_m}{\partial b_{ij} \partial b_{il}} = \frac{\partial b_m}{\partial b_{il}} \frac{\partial b_m}{\partial b_{ij}} = \delta_{im}(e_i e_j + e_j e_i) = \delta_{im}(e_j e_i) = 2 \delta_{im} \delta_{ji} e_0,
\]
one has
\[
\frac{1}{c_2} \tfrac{\partial^2}{\partial b_{ij} \partial b_{kl}} \log \psi_{\Omega V_{27}}^\#(z, w) = -2 \delta_{ji} \delta_{ik} \left(\frac{\alpha_1 \alpha_2 \alpha_3}{\alpha_i}\right) e_0.
\]
\[
+ \frac{2 \sum_{i,j}(1 - |\alpha_i|^2)|Z_{ij}|^2}{\prod_{i=1}^{3}(1 - |\alpha_i|^2)} + \sum \frac{|Y_i|^2}{(1 - |\alpha_i|^2)^2} - 2Re \sum X_{\eta} \overline{Y}_i - 4Re \sum X_{h_{ij}} \overline{Z}_{ij}
\]
\[
= \sum |\overline{\alpha_i} X_{\eta_i} - \alpha_i X_{\overline{\eta_i}}|^2 + \sum (1 - |\alpha_i|^2)|X_{\eta_i}|^2 + \sum (1 - |\alpha_i|^2)|X_{\overline{\eta_i}}|^2
+ 2 \sum_j \left( |\overline{\alpha_2} X_{h_{1j}} - \alpha_3 X_{\overline{h}_{1j}}|^2 + |\overline{\alpha_3} X_{h_{2j}} - \alpha_1 X_{\overline{h}_{2j}}|^2 + |\overline{\alpha_1} X_{h_{3j}} - \alpha_2 X_{\overline{h}_{3j}}|^2 \right)
\]
\[
+ 2 \sum_j \left( (1 - |\alpha_2|^2)|X_{h_{1j}}|^2 + (1 - |\alpha_3|^2)|X_{\overline{h}_{1j}}|^2 \right)
+ 2 \sum_j \left( (1 - |\alpha_3|^2)|X_{h_{2j}}|^2 + (1 - |\alpha_1|^2)|X_{\overline{h}_{2j}}|^2 \right)
+ 2 \sum_j \left( (1 - |\alpha_1|^2)|X_{h_{3j}}|^2 + (1 - |\alpha_2|^2)|X_{\overline{h}_{3j}}|^2 \right)
\]
\[
\geq \sum |\overline{\alpha_i} X_{\eta_i} - \alpha_i X_{\overline{\eta_i}}|^2 + \sum (1 - |\alpha_i|^2)|X_{\eta_i}|^2 + \sum_i \left| (1 - |\alpha_i|^2)X_{\eta_i} - \frac{Y_i}{1 - |\alpha_i|^2} \right|^2
+ 2 \sum_j \left( |\overline{\alpha_2} X_{h_{1j}} - \alpha_3 X_{\overline{h}_{1j}}|^2 + |\overline{\alpha_3} X_{h_{2j}} - \alpha_1 X_{\overline{h}_{2j}}|^2 + |\overline{\alpha_1} X_{h_{3j}} - \alpha_2 X_{\overline{h}_{3j}}|^2 \right)
\]
\[
+ 2 \sum_j \left( (1 - |\alpha_2|^2)|X_{h_{1j}}|^2 + (1 - |\alpha_3|^2)|X_{\overline{h}_{1j}}|^2 \right)
+ 2 \sum_j \left( (1 - |\alpha_3|^2)|X_{h_{2j}}|^2 + (1 - |\alpha_1|^2)|X_{\overline{h}_{2j}}|^2 \right)
+ 2 \sum_j \left( (1 - |\alpha_1|^2)|X_{h_{3j}}|^2 + (1 - |\alpha_2|^2)|X_{\overline{h}_{3j}}|^2 \right)
\]
\[
+ 2 \sum_j \left( \frac{Z_{1j}}{\sqrt{(1 - |\alpha_2|^2)(1 - |\alpha_3|^2)}} - \sqrt{(1 - |\alpha_2|^2)(1 - |\alpha_3|^2)}X_{h_{1j}} \right)^2
+ 2 \sum_j \left( \frac{Z_{2j}}{\sqrt{(1 - |\alpha_1|^2)(1 - |\alpha_3|^2)}} - \sqrt{(1 - |\alpha_1|^2)(1 - |\alpha_3|^2)}X_{h_{2j}} \right)^2
+ 2 \sum_j \left( \frac{Z_{3j}}{\sqrt{(1 - |\alpha_1|^2)(1 - |\alpha_2|^2)}} - \sqrt{(1 - |\alpha_1|^2)(1 - |\alpha_2|^2)}X_{h_{3j}} \right)^2 \geq 0.
\]

As a result, \( \psi \) is psh. Since \( \log \psi_{\Omega} \) is invariant under the diagonal action of \( \text{Aut}(\Omega) \), \( \psi \) is well defined on \( M \). By the construction of \( \psi_{\Omega} \), it is an exhaustion function. \( \square \)

5 \hspace{1cm} WHEN \( \rho \) IS NON-REDUCTIVE

Consider the heat equation according to Eells–Sampson given in [9]:

\[
\frac{d}{dt} s(p, t) = \Delta s(p, t),
\]
\[
s(p, 0) = s_0(p),
\]

(5.1)
for a map \( s : M \times [0, \tau) \to E \) with \( \tau > 0 \) where \( s_0 : M \to E \) is a continuous section of \( E \). Note that since \( \Omega \) is contractible, there exists such \( s_0 \) (see [29], for example). Denote \( g_{\Omega} \) the Bergman metric on \( \Omega \), \( g_M \) the Kähler metric on \( M \), and \( g_E \) the induced metric from \( g_{\Omega} \), \( g_M \) on \( E \). Let \( d_{\Omega} \) and \( d_E \) denote the distances induced from \( g_{\Omega} \) and \( g_E \), respectively.

**Lemma 5.1** (Hamilton [12], Diederich–Ohsawa [8]). *The family \( \{ s_t = s(\cdot, t) \} \) is well defined for any \( t \in \mathbb{R}^+ \) and \( s_t \) is also a section for any \( t \). Moreover the family \( \{ s_t = s(\cdot, t) : t \in \mathbb{R}^+ \} \) is uniformly equicontinuous on \( M \) with respect to \( g_E \) and \( g_M \).*

Let \( \hat{\Omega} \) denote the compact dual of \( \Omega \). We can choose \( 0 < t_1 < t_2 < \cdots \) with \( \lim_{k \to \infty} t_k = \infty \) so that

\[
 s_\infty : M \to \tilde{M} \times \hat{\Omega}, \quad s_\infty(p) := \lim_{k \to \infty} s_{t_k}(p)
\]

exists (cf. [8, 12]).

**Lemma 5.2.** Let \( \{ p_j \}_{j=1}^{\infty}, \{ q_j \}_{j=1}^{\infty} \) be sequences in \( \Omega \) and \( p, q \) be points on \( \partial \Omega \) such that \( p_j \to p, q_j \to q \) as \( j \to \infty \). If \( \liminf_{j \to \infty} d_\Omega(p_j, q_j) < \infty \), then \( p \) and \( q \) belong to the same boundary component.

**Proof.** Since the Bergman distance and the Kobayashi distance are equivalent on \( \Omega \), the condition \( \liminf_{j \to \infty} d_\Omega(p_j, q_j) < \infty \) implies \( \liminf_{j \to \infty} d^K_\Omega(p_j, q_j) < \infty \) with the Kobayashi distance \( d^K_\Omega \) on \( \Omega \). By [35, Proposition 3.5], for a complex line \( L \) containing \( p \) and \( q \), the interior of \( \overline{\Omega} \cap L \) in \( L \) contains \( p \) and \( q \). Since the interior of \( \overline{\Omega} \cap L \) is contained in the boundary component of \( \Omega \), we obtain the lemma. \( \square \)

**Proposition 5.3.** If \( \rho \) is non-reductive, then \( \rho(\pi_1(M)) \) is contained in a maximal real parabolic subgroup in \( \text{Aut}(\Omega) \).

**Proof.** Since \( \rho \) is non-reductive, by Theorem 1.3 there exists no harmonic section from \( M \) to \( M \times \rho \Omega \). This implies that there exists a family \( \{ s_t = s(\cdot, t) : t \in [0, \infty) \} \) satisfying (5.1) which is uniformly equicontinuous on \( M \) with respect to \( g_M \) and \( g_E \).

Let \( p, q \in M \). Choose \( t_1 < t_2 < \cdots \) with \( \lim_{k \to \infty} t_k = \infty \) so that \( s_\infty := \lim_{k \to \infty} s_{t_k} \) is harmonic. For simplicity we will denote \( s_{t_k} \) by \( s_k \). Then we have \( \lim_{k \to \infty} d_E(s_k(p), s_k(q)) < \infty \) and \( \lim_{k \to \infty} s_k(p), \lim_{k \to \infty} s_k(q) \in M \times \rho \partial \Omega \). Denote

\[
 s_\infty(p) = [\tilde{p}, z], \quad s_\infty(q) = [\tilde{q}, w], \quad s_t(p) = [\tilde{p}, z_t] \quad \text{and} \quad s_t(q) = [\tilde{q}, w_t],
\]

where \( z, w \in \partial \Omega, z_t, w_t \in \Omega \) and \( \tilde{p}, \tilde{q} \in \tilde{M} \). Since we have

\[
 \infty > \lim_{t \to \infty} d_E(s_t(p), s_t(q)) = \lim_{t \to \infty} d_E([\tilde{p}, z_t], [\tilde{q}, w_t])
\]

\[
 \geq \lim_{t \to \infty} \min_{\gamma \in \pi_1(M)} (d_M(\gamma \tilde{p}, \gamma \tilde{q}) + d_\Omega(\rho(\gamma)z_t, \rho(\gamma)w_t)),
\]

there exists \( \gamma \in \pi_1(M) \) such that \( d_\Omega(\rho(\gamma)z, \rho(\gamma)w) < \infty \). Hence \( z \) and \( w \) should be contained in a boundary component of \( \partial \Omega \), say \( B \), by Lemma 5.2.
Let $\gamma \in \pi_1(M)$. Since we have $s_\infty(p) = [\tilde{p}, z] = [\gamma \tilde{p}, \rho(\gamma)z]$ for any $p \in M$, $z$ and $\rho(\gamma)z$ belong to $B$. Since automorphisms of $\Omega$ permute boundary components and $z$, $\rho(\gamma)z$ both belong to $B$, $\rho(\gamma)$ is a normalizer of $B$ which is a maximal real parabolic subgroup of $\text{Aut}(\Omega)$.

**Remark 5.4.** Let $\Omega = \Omega_1 \times \cdots \times \Omega_k$ be a bounded symmetric domain with irreducible factors $\Omega_i$, $i = 1, \ldots, k$. If $\rho$ is non-reductive, then $\rho(\pi_1(M))$ is contained in $P_1 \times \cdots \times P_k$ where $P_i$ is a maximal parabolic subgroup in $\text{Aut}(\Omega_i)$ or $\text{Aut}(\Omega_i)$ itself for each $i$.

**Lemma 5.5.** Suppose that $\rho$ is non-reductive. Then a limit map $s_\infty$ of the family $\{s_t = s(\cdot, t) : t \in [0, \infty)\}$ is a harmonic map with respect to the induced metric from the Kähler metric on $M$ and the Bergman metric of the boundary component where $s_t$ converges to.

**Proof.** Since the Bergman metric on the boundary component is the limit of $g_{\Omega}$ restricted to the corresponding characteristic symmetric subspaces of it, we obtain the lemma.

**Lemma 5.6.** For $\Lambda \subset \Pi$ with $|\Lambda| = 1$, consider a totally geodesic subspace $\Delta \times X_{\Pi - \Lambda, 0} \subset \Omega$. Let $\sigma : \Omega \to m_{\Lambda, 1}^{-}$ be the projection. If $\sigma(p)$ tends to $\partial \Delta$, then $p$ tends to $\partial \Delta \times X_{\Pi - \Lambda, 0}$.

**Proof.** Let $m^{-} = \mathbb{C}\alpha \oplus H_\alpha \oplus N_\alpha$ be the decomposition with respect to the tangent unit vector $\alpha \in T_0\Delta$ such that the bisectional curvature of $\Omega$ with respect to the Bergman metric in directions $\alpha$ and $\xi \in H_\alpha$ (respectively, $\xi \in N_\alpha$) equals to $1/2$ (respectively, 0) where $\xi$ is a root vector. Remark that $\mathbb{C}\alpha \cap \Omega \cong \Delta$ and $N_\alpha \cap \Omega \cong X_{\Pi - \Lambda, 0}$. (For more detail, see [14].) Therefore we only need to show that $H_\alpha$-component vanishes as the modulus of $\mathbb{C}\alpha$-component tends to 1. For each root vector of unit norm $\xi \in H_\alpha$, either

1. $(\mathbb{C}\alpha + \mathbb{C}\xi) \cap \Omega \cong \mathbb{B}^2$ is totally geodesic in $\Omega$, or
2. there exists $\eta \in N_\alpha$ such that $(\mathbb{C}\alpha + \mathbb{C}\xi + \mathbb{C}\eta) \cap \Omega \cong \Omega_{x\xi}^\prime$ is totally geodesic in $\Omega$ by [16, Lemma 3.6]. If $(\mathbb{C}\alpha + \mathbb{C}\xi) \cap \Omega \cong \mathbb{B}^2$, then $\mathbb{C}\xi$-component of $p$ tends to zero since the modulus of $\mathbb{C}\alpha$-component tends to 1. Now let us consider the case (2). By (3.20) for each $w = (w_1, w_2, w_3)$ with $|\zeta_1| = 1$ we have

$$0 \leq 1 - 2w^i + |w^i|^2 = (1 - |\zeta_1|^2)(1 - |\zeta_2|^2) - 2|w_3|^2 + |i\zeta_1\zeta_2 + w_3^2|^2 - |i\zeta_1\zeta_2|^2$$

$$= -2|w_3|^2 + |w_3|^4 + 2\text{Re}(i\zeta_1\zeta_2\overline{w_3}) \leq -2|w_3|^2 + |w_3|^4 + 2|\zeta_2\overline{w_3}|^2.$$ 

Therefore we obtain

$$2 \leq |w_3|^2 + 2|\zeta_2|^2 \quad (5.2)$$

and

$$1 \geq w^i\overline{w} = \frac{1}{2} + \frac{1}{2}|\zeta_2|^2 + |w_3|^2, \quad (5.3)$$

since $w_1^2 + w_2^2 = i\zeta_1\zeta_2$ for $w_1 = \lambda(\zeta_1 + \zeta_2)$, $w_2 = i\lambda(\zeta_1 - \zeta_2)$ with $\lambda^2 = i/4$. By (5.3) and (5.2), we have

$$2 \leq |w_3|^2 + 2|\zeta_2|^2 \leq |w_3|^2 + 2(1 - 2|w_3|^2),$$

and we induce $w_3 = 0$. This completes the proof.
For a non-reductive representation $\rho: \pi_1(M) \to \text{Aut}(\Omega)$, let $B$ be the boundary component to where a subsequence of a family of solutions of the heat equation (5.1) converges. Denote by $N(B) := \{ g \in \text{Aut}(\Omega) : gB = B \}$ the set of normalizers of $B$ in $\text{Aut}(\Omega)$ and by $c_B$ the Cayley transformation with respect to $B$. Define $\rho_c : \pi_1(M) \to \text{Ad}_{c_B} N(B) \subset G^C$ by

$$\rho_c(\gamma) := c_B \circ \gamma = c_B(c_B^{-1} \circ \gamma) : c_B(\Omega) \to c_B(\Omega).$$

Then $M \times_{\rho_c} c_B(\Omega)$ is holomorphically equivalent to $M \times_{\rho} \Omega$ as a fiber bundle over $M$ and $c_B(\Omega)$ has the form (6) in Theorem 2.4.

**Theorem 5.7.** Let $E = M \times_{\rho} \Omega$ be a holomorphic fiber bundle over a compact Kähler manifold $M$ with an irreducible bounded symmetric domain fiber $\Omega$ where $\rho: \pi_1(M) \to \text{Aut}(\Omega)$ is a non-reductive representation. Suppose that there exists a family of solutions of (5.1) which has a subsequence that converges to a 1-component. Then there exist a plurisubharmonic function $\psi$ on $E$ and a totally geodesic subspace $\Delta \times \Delta^\perp \subset \Omega$ which is invariant under $\rho(\pi_1(M))$ such that

$$\lim_{[p,z] \to \partial E \setminus M \times_{\rho} (\partial \Delta \times \Delta^\perp)} \psi([p,z]) = \infty$$

where $\{e^{i\theta}\} \times \Delta^\perp$ is a maximal boundary component of $\Omega$ for each $\theta \in \mathbb{R}$.

**Proof of Theorem 5.7.** By the assumption and Lemma 5.5 there exist a maximal boundary component $B$ and a $\rho$-equivariant harmonic map from $\tilde{M}$ to $B$ with respect to the Kähler metric induced from $M$ and the Bergman metric on $B$. Moreover $\rho(\pi_1(M)) \subset N(B)$ and any element in $\rho(\pi_1(M))$ has the decomposition of the form (2.1). Hence there exists a psf exhaustion function, say $\psi_B$, on $M \times_{\rho} B \cong M \times_{\rho_c} X_{\Lambda,0}$ by Theorem 1.1.

From what follows we will denote $K_\Omega(z,z)$ by $K_\Omega(z)$ for a domain $\Omega$ for simplicity.

Since $B$ is a 1-component, we have $|\Lambda| = 1$ and hence $\dim m_{\Pi^{-\Lambda,1}} = 1$. Let $H$ denote the upper half space $\{ z \in \mathbb{C} : \text{Im } z > 0 \}$. Define a function $\delta : M \times_{\rho_c} c_B(\Omega) \to \mathbb{R}$ by

$$\delta([p,(E_1,E_2,E_3)]) := \frac{1}{2a} \log \left( \frac{K_{\rho_c}(\Omega)(E_1,E_2,E_3)}{K_H(E_1)^a K_{X,0}(E_3)^a} \right)$$

(5.4)

where $a = 1 + \frac{1}{2} \dim m_{\Lambda^{-,-}}$. First we claim that $\delta$ is a well-defined function. For a biholomorphism $f : D_1 \to D_2$ with domains $D_1, D_2$ in $\mathbb{C}^n$, we have $K_{D_1}(z,z) = |\det J(f)(z)|^2 K_{D_2}(f(z),f(z))$, where we denote by $J(f)$ the Jacobian matrix of $f$. Hence, for the claim to hold, we only need to show

$$|\det(J(g))| = |\det(J(g|m_{\Lambda}^-))| \det(J(g|m_{\Pi^{-\Lambda,1}}^-))^a$$

(5.5)

for any $g \in \rho_c(\pi_1(M))$. Since $\rho_c(\pi_1(M)) \subset \text{Ad}_{c_B} N(B)$, we may decompose $g$ by $g_1 g_2 g_3$ with $g_1 \in G_{\Lambda}$, $g_2 \in L_{\Pi^{-\Lambda,1}} K_{\Pi^{-\Lambda,1}}^*$, $g_3 \in N_{\Lambda^{-,-}}$.

Since $g_3$ acts on $c_B(\Omega)$ by the expression (2.2), the left- and the right-hand sides of Equation (5.5) are both equal to 1 and hence (5.5) holds for $g_3$.

Note that $L_{\Pi^{-\Lambda,1}} K_{\Pi^{-\Lambda,1}}^*$ acts on $m_{\Lambda}^-$ by the adjoint representation and it preserves $m_{\Pi^{-\Lambda,1}}^-$, $m_{\Lambda}^{\Lambda^{-,-}}$ and $m_{\Lambda}^-$. Since the action of $g_2$ on $m_{\Pi^{-\Lambda,1}}^-$ is real, there exists a positive constant $C$ such that
$C \text{Im} (E_1) = \text{Im} (g_2 E_1)$ for any $E_1$. This implies that the right-hand side of Equation (5.5) equals to $C^a$. On the other hand since $g_2$ acts on $m_{\Lambda}^-$ trivially, by (2.3) we have

$$C \left( \text{Im} E_1 - \text{Re} F_{E_1}(E_2, E_2) \right) = \text{Im} (g_2 E_1) - \text{Re} F_{E_1}(g_2 E_2, g_2 E_2), \quad (5.6)$$

which implies that $C \text{ Re} F_{E_1}(E_2, E_2) = \text{ Re} F_{E_1}(g_2 E_2, g_2 E_2)$. Hence the action of $g_2$ on $m_{\Lambda}^{-\Lambda}$ is equivalent to the action of $\sqrt{\mathsf{C}U}$ for some unitary transformation $U$ on $m_{\Lambda}^{-\Lambda}$. As a result $|\det(J(g))|$ equals to $C^a$.

Remark that $G_{\Lambda}$ acts on $m_{\Pi-\Lambda, 1}^-$ trivially. Let $g_1 = \exp(m_\Lambda^+ k \exp(m_\Lambda^-) \in G_{\Lambda}$ where $m_\Lambda^+ \in m_\Lambda^+$, $m_\Lambda^- \in m_\Lambda^-$, and $k \in K^C$. Since $\exp(m_\Lambda^-)$ acts on $m^-$ as a translation and $k$ belongs to the isotropy subgroup at 0, we have $\det(J(k \exp(m_\Lambda^-))) = 1$. Hence we only need to consider the case when $g = \exp(m_\Lambda^+)$. Let $E = (E_1, E_2, E_3) \in m^- = m_{\Pi-\Lambda, 1}^- + m_{\Lambda}^{-\Lambda} + m_\Lambda^-$ and $e = (e_1, e_2, e_3) \in T_E m^- \cong m_{\Pi-\Lambda, 1}^- + m_{\Lambda}^{-\Lambda} + m_\Lambda^-$. By the Hausdorff-f-Campbell formula, we have

$$\exp(m_\Lambda^+) \exp(E + te) \cdot K^C M^+ = \exp \left( E + te + [m_\Lambda^+, E + te] + \frac{1}{2}[m_\Lambda^+, [m_\Lambda^+, E + te]] + \cdots \right) K^C M^+.$$  

By Lemma 2.5 we obtain

$$E + te + [m_\Lambda^+, E + te] + \frac{1}{2}[m_\Lambda^+, [m_\Lambda^+, E + te]] + \cdots$$

$$= \text{Ad}_{\exp(m_\Lambda^+)}(E) + t \left( e_1 + e_2 + [m_\Lambda^+, e_2] + \text{Ad}_{\exp(m_\Lambda^+)}(e_3) \right),$$

and

$$[m_\Lambda^+, e_2] \in \mathfrak{k}^C.$$

As a consequence, we obtain (5.5) for $g_1$ and hence we complete the proof of the claim.

On the other hand, the relation (5.5) implies that for any $g_3 \in G_{\Lambda}$ we have

$$K_{c_{\mu}(\Omega)}(i, 0, E_3) = K_{c_{\mu}(\Omega)}(i, 0, g_3 E_3) \left| \det J(g_3 \mid m_{\Lambda}^-)(E_3) \right|^2,$$

which implies that there exists a positive constant $\nu$ such that

$$K_{c_{\mu}(\Omega)}(i, 0, E_3) = \nu K_{X_{\Lambda, 0}}(E_3)$$

for any $E_3 \in X_{\Lambda, 0}$ since $K_{X_{\Lambda, 0}}(E_3) = K_{X_{\Lambda, 0}}(g_3 E_3) \left| \det J(g_3 \mid m_{\Lambda}^-)(E_3) \right|^2$. As a consequence we have

$$K_{c_{\mu}(\Omega)}(E_1, E_2, E_3) = K_{c_{\mu}(\Omega)}(i(\text{Im} E_1 - \text{Re} F_{E_1}(E_2, E_2)), 0, E_3)$$

$$= \frac{K_{c_{\mu}(\Omega)}(i, 0, E_3)}{\left( \text{Im} E_1 - \text{Re} F_{E_1}(E_2, E_2) \right)^{2a}}$$

$$= \nu K_{X_{\Lambda, 0}}(E_3) \left( i(\text{Im} E_1 - \text{Re} F_{E_1}(E_2, E_2)) \right)^a.$$
Since
\[
\delta([p, (E_1, E_2, E_3)]) = \frac{1}{2a} \log \left( \frac{K_{\mathcal{B}(\Omega)}(E_1, E_2, E_3)}{K_{\mathcal{H}}(E_1)^a K_{X_{\Lambda,0}}(E_3)} \right)
\]
\[= \frac{1}{2a} \log \left( \frac{\nu K_{\mathcal{H}}(i(\text{Im } E_1 - \text{Re } F_{E_3}(E_2, E_2))^a)}{K_{\mathcal{H}}(E_1)^a} \right)
\]
\[= -\log \left( \frac{\text{Im } E_1 - \text{Re } F_{E_3}(E_2, E_2)}{\text{Im } E_1} \right) + \frac{1}{2a} \log \nu,
\]
one obtains that \(\delta([p, (E_1, E_2, E_3)])\) diverges to infinity whenever \(\text{Im } E_1 - \text{Re } F_{E_3}(E_2, E_2)\) tends to zero unless \(E_2 = 0\). Remark that \(\text{Re } F_{E_3}\) is a positive definite bilinear form since \(c_B(\Omega)\) is Kobayashi hyperbolic. Hence by a straightforward calculation, the last expression in (5.7) is psh.

Now define a function on \(M \times \mathbb{R}_+ c_B(\Omega)\) by
\[
\psi_{c_B(\Omega)}([p, (E_1, E_2, E_3)]) := \psi_B([p, E_3]) + \delta([p, (E_1, E_2, E_3)]).
\]
Let \(\Delta_\Lambda\) be the unit disc such that \(\Delta_\Lambda \times X_{\Lambda,0}\) is a totally geodesic subspace in \(\Omega\) and let \(\sigma : \Omega \to \Delta_\Lambda\) be a projection to \(m_{\Pi_{-\Lambda,1}}\). Note that \(\Delta_\Lambda \subset m_{\Pi_{-\Lambda,1}}\) and \(\sigma(\Omega) = \Delta_\Lambda\). For \((E_1, E_2, E_3) \in c_B(\Omega)\), if \(|E_1|\) tends to infinity or \(\text{Im } E_1\) tends to zero, then
\[
\sigma \circ c^{-1}_B(E_1, E_2, E_3) \to \partial \Delta_\Lambda,
\]
since \(c_B\) maps \(\Delta_\Lambda\) onto \(\mathbb{H} = \{ (E_1, 0, 0) \in m^- : \text{Im } E_1 > 0 \}\) biholomorphically. Hence \(c_B^{-1}(E_1, E_2, E_3)\) converges to \(\partial \Delta_\Lambda \times X_{\Lambda,0}\) by Lemma 5.6 and hence the induced function \(\psi\) from \(\psi_{c_B(\Omega)}\) to \(M \times \mathbb{R}_+ \Omega\) can be finite only when \([p, z]\) tends to \(M \times \mathbb{R}_+ (\partial \Delta_\Lambda \times X_{\Lambda,0})\) since \(\psi_B([p, E_3])\) diverges to infinity as \(E_3\) converges to \(\partial X_{\Lambda,0}\). As a result it is a desired plurisubharmonic function. □

**Remark 5.8.**

1. For each irreducible bounded symmetric domain, \(a\) is given as follows:

| \(\Omega\) | \(\Omega^I_{\mathcal{P}_\Lambda}(p \leq q)\) | \(\Omega^I_{\mathcal{H}}\) | \(\Omega^{IH}_{\mathcal{P}_\Lambda}\) | \(\Omega^{IV}_{\mathcal{P}_\Lambda}(n > 2)\) | \(\Omega^V_{\mathcal{P}_\Lambda}\) | \(\Omega^{IV}_{\mathcal{P}_\Lambda}\) |
|---|---|---|---|---|---|---|
| \(a\) | \(\frac{n+q}{2}\) | \(n-1\) | \(\frac{n+1}{2}\) | \(\frac{n}{2}\) | \(6\) | \(9\) |

2. The boundary component \(B\) in the proof of Theorem 5.7 is of the form \(\{e^{i\theta}\} \times \Delta^\perp\) for some \(\theta\).

**Corollary 5.9.** Let \(E = M \times_{\mathcal{P}} \Omega\) be a holomorphic fiber bundle over a compact Kähler manifold \(M\) with an irreducible bounded symmetric domain fiber \(\Omega\) where \(\rho : \pi_1(M) \to \text{Aut}(\Omega)\) is a non-reductive representation. Suppose that \(\rho\) is a maximal 1-parabolic representation. If there are two families of solutions of (5.1) which have subsequences that converge to 1-components \(\{e^{i\theta_1}\} \times \Delta^\perp_1\) and \(\{e^{i\theta_2}\} \times \Delta^\perp_2\), respectively, for some \(\theta_1, \theta_2\) with different \(\Delta_1\) and \(\Delta_2\), then \(E\) is weakly 1-complete.

**Proof.** Let \(B_1\) and \(B_2\) be the 1-components where the subsequences of the families of the solution of (5.1) converge. Let \(\psi_1\) and \(\psi_2\) be the psh functions which are constructed in Theorem 5.7 with
respect to $B_1$ and $B_2$, respectively. Then $\psi_1 + \psi_2$ gives a psh exhaustion on $E$ since $\delta \Delta_1 \times \Delta_1^\perp$ and $\delta \Delta_2 \times \Delta_2^\perp$ do not have intersection if $\Delta_1$ and $\Delta_2$ are different.

From what follows we consider when $\Omega$ is the unit ball $\mathbb{B}^N$. In this case the boundary component $B$ is a point on the boundary and

$$c_B(\mathbb{B}^N) = \left\{ E = (E_1, E_2) \in \mathfrak{m}^- = \mathfrak{m}_{\Pi - \Lambda, 1}^- + \mathfrak{m}_{2}^{\Lambda, -} : \text{Im} E_1 - \text{Re} F(E_2, E_2) > 0 \right\}.$$ 

Moreover $M \times_{\rho_c} \mathfrak{m}^- \cong M \times_{\rho_c} (\mathfrak{m}_{\Pi - \Lambda, 1}^- + \mathfrak{m}_{2}^{\Lambda, -})$ is an affine bundle, that is, transition maps are affine since $G_{\Lambda} = \{ \text{id} \}$ which yields $\rho_c(\pi_1(M)) \subset L_{2}^{\Lambda} K_{\Pi - \Lambda, 1}^* N^{\Lambda, -}$. Let

$$\mu : \pi_1(M) \rightarrow L_{2}^{\Lambda} K_{\Pi - \Lambda, 1}^*$$

denote the representation $\rho_c$ project to $L_{2}^{\Lambda} K_{\Pi - \Lambda, 1}^*$ by ignoring translations. Then $M \times_{\mu} \mathfrak{m}^-$ is a holomorphic vector bundle.

The holomorphic vector bundle is said to be polystable when it is a direct sum of stable vector bundles and each component has the same slope. See [32] for the definitions of stable vector bundles and their slope.

**Theorem 5.10.** Let $E = M \times_{\rho} \mathbb{B}^N$ be a holomorphic $\mathbb{B}^N$-fiber bundle over a compact Kähler manifold $M$ where $\rho : \pi_1(M) \rightarrow \text{Aut}(\Omega)$ is a non-reductive representation. Suppose that $M \times_{\mu} \mathfrak{m}^-$ is a polystable vector bundle. Then $M$ is weakly 1-complete.

**Proof.** Since all Chern classes of the flat bundle vanish, by Uhlenbeck–Yau [32] $M \times_{\mu} \mathfrak{m}^-$ has a flat Hermitian structure, that is, there exists a trivializing cover $\{ U_\alpha \}$ of $M \times_{\rho_c} \mathfrak{m}^- \rightarrow M$ with fiber coordinates $\{ t_\alpha \}$ satisfying

$$t_\alpha = a_\alpha^\beta t_\beta + b_\alpha^\beta,$$

where $a_\alpha^\beta \in U(N)$ and $b_\alpha^\beta \in \mathbb{C}^N$. Moreover there exists a pluriharmonic functions $c_\alpha : U_\alpha \rightarrow \mathbb{C}^N$ such that $b_\alpha^\beta = c_\alpha - a_\alpha^\beta c_\beta$ on $U_\alpha \cap U_\beta$ since $M$ is Kähler (see [7]). Then the function $\phi := |t_\alpha - c_\alpha|^2$ defines a psh exhaustion function on $M \times_{\rho_c} \mathfrak{m}^- \rightarrow M$. Now define

$$\psi_{c_B(\mathbb{B}^N)}([p, E]) := \phi([p, E]) - \log \text{dist}([p, E], \partial(M \times_{\rho_c} c_B(\Omega))),$$

where dist is induced from the flat Hermitian structure of $M \times_{\mu} \mathfrak{m}^-$. Then this is the desired psh exhaustion function.

**Corollary 5.11.** Any $\mathbb{B}^2$ fiber bundle over a compact Kähler manifold is weakly 1-complete.

**Proof.** By Theorem 1.1, we only need to consider when $\rho$ is non-reductive. Since every boundary component of $\mathbb{B}^2$ is an I-component and $M \times_{\mu} (\mathfrak{m}_{\Pi - \Lambda, 1}^- + \mathfrak{m}_{2}^{\Lambda, -})$ is a direct sum of flat line bundles (hence of degree zero) $M \times_{\mu} \mathfrak{m}_{\Pi - \Lambda, 1}^-$ and $M \times_{\mu} \mathfrak{m}_{2}^{\Lambda, -}$ by Lemma 5.12 and Theorem 2.4 (3), the fiber bundle is weakly 1-complete by Theorem 5.10.
Lemma 5.12. Let $\mu : \pi_1(M) \to GL(V)$ be a representation of a vector space $V$. Suppose $V = V_1 + V_2$ and $\mu$ acts invariantly on $V_1$ and $V_2$. Then $M \times_\mu V$ is isomorphic to $M \times_\mu V_1 \oplus M \times_\mu V_2$.

Proof. Since $V_1$ and $V_2$ are invariant under the action of $\mu$, the transition functions of $M \times_\mu V$ are of the form $\begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & h_{\gamma\delta} \end{pmatrix}$ which implies that $M \times_\mu V$ is isomorphic to $M \times_\mu V_1 \oplus M \times_\mu V_2$. \hfill \Box

6 | Ω-FIBER BUNDLES OVER COMPACT QUOTIENTS OF BOUNDED SYMMETRIC DOMAINS

6.1 | Hyperconvexity

Let $\Gamma \subset Aut(\Omega)$ be a cocompact discrete subgroup of Aut($\Omega$). Then $\Gamma \setminus \Omega$ be a compact Kähler manifold with respect to the Bergman metric on $\Omega$. Consider the diagonal action of $\Gamma$ on $\Omega \times \Omega$ defined by

$$\gamma(z, w) = (\gamma z, \gamma w).$$

(6.1)

We denote the quotient manifold of $\Omega \times \Omega$ with respect to the action (6.1) by $\Omega \times \Omega / \Gamma$. One can notice that $\Omega \times \Omega / \Gamma$ is an $\Omega$-fiber bundle over $\Gamma \setminus \Omega$. For a generic norm $N_{\Omega}$ given in Section 3, define

$$\delta(z, w) := \frac{N_{\Omega}(z, z)N_{\Omega}(w, w)}{|N_{\Omega}(z, w)|^2}.$$ 

Theorem 6.1. Let $\Omega$ be an irreducible bounded symmetric domain. Then $\Omega \times \Omega / \Gamma$ is hyperconvex. More precisely, $-\delta^{1/r}$ with $\frac{1}{r} \leq \frac{1}{2 \text{rank}(\Omega)}$ is a bounded psh exhaustion function.

Proof. Since $\delta$ is an invariant function on $\Omega \times \Omega$ under the diagonal action of Aut($\Omega$), $\delta$ is a well-defined function on $\Omega \times \Omega / \Gamma$. Since the proofs are similar, we will only show when $\Omega$ is type I or type IV domain.

Type I, $\Omega_{p,q}^I$; At $(0, w_0)$ with $w_0 = \text{diag}(w_{11}, \ldots, w_{pp})$ since we have

$$\frac{\partial \log \delta(z, w)}{\partial w_{ij}} = \frac{\partial}{\partial w_{ij}} \log \frac{\det(I - zz') \det(I - ww')}{\det(I - z w') \det(I - w z')} = \frac{1}{\det(I - w w')} \frac{\partial}{\partial w_{ij}} \det(I - w w') = \frac{-w_{ij} \delta_{ij}}{1 - |w_{ij}|^2}$$

and

$$\frac{\partial \log \delta(z, w)}{\partial z_{ij}} = \frac{\partial}{\partial z_{ij}} \log \frac{\det(I - zz') \det(I - ww')}{\det(I - z w') \det(I - w z')} = -\frac{\partial}{\partial z_{ij}} \det(I - z w') = w_{ij},$$
one obtains
\[ \frac{\partial \log \delta}{\partial z_{ij}} \frac{\partial \log \delta}{\partial \overline{z}_{kl}} = \frac{\overline{w}_{ij} w_{kl}}{(1 - |w_{ii}|^2)(1 - |w_{kk}|^2)}, \]
\[ \frac{\partial \log \delta}{\partial w_{ij}} \frac{\partial \log \delta}{\partial \overline{w}_{kl}} = \frac{\overline{w}_{ii} w_{kk} \delta_{ij} \delta_{kl}}{(1 - |w_{ii}|^2)(1 - |w_{kk}|^2)}. \]

Therefore for \( Z = \sum X_{ij} \frac{\partial}{\partial z_{ij}} + \sum Y_{ij} \frac{\partial}{\partial w_{ij}} \), by (3.4) and (3.18) we have
\[
\frac{r}{\delta^{1/r}} \frac{\partial}{\partial (\delta^{1/r})} (Z, Z) = -\overline{\partial} \log \delta(Z, \overline{Z}) - \frac{1}{r} \partial \log \delta(Z) \wedge \overline{\partial} \log \delta(\overline{Z})
\]
\[ \begin{align*}
= \sum_{i,j=1}^{p} \left( |X_{ij}|^2 - 2 \text{Re}(X_{ij} \overline{Y}_{ij}) + \frac{|Y_{ij}|^2}{(1 - |w_{ii}|^2)(1 - |w_{jj}|^2)} \right) \\
- \frac{1}{r} \sum_{i=1}^{p} \left| \sum_{j=1}^{p} \overline{w}_{ij} X_{ii} - \sum_{j=1}^{q} \overline{w}_{ii} Y_{ij} \right|^2 \\
+ \sum_{i=1}^{p} \sum_{k=p+1}^{q} \left( \frac{|Y_{ik}|^2}{1 - |w_{ii}|^2} + |X_{ij}|^2 - 2 \text{Re}(X_{ij} \overline{Y}_{ij}) \right).
\end{align*}
\]

Since
\[
\sum_{i=1}^{p} \left( |X_{ii}|^2 - 2 \text{Re}(X_{ii} \overline{Y}_{ii}) + \frac{|Y_{ii}|^2}{(1 - |w_{ii}|^2)^2} \right) - \frac{1}{r} \sum_{i=1}^{p} \left( \sum_{j=1}^{p} \overline{w}_{ij} X_{ii} - \sum_{j=1}^{q} \overline{w}_{ii} Y_{ij} \right)^2
\geq \sum_{i=1}^{p} \left( |X_{ii}|^2 - 2 \text{Re}(X_{ii} \overline{Y}_{ii}) + \frac{|Y_{ii}|^2}{(1 - |w_{ii}|^2)^2} \right) - \frac{2p}{r} \sum_{i=1}^{p} \left( \sum_{j=1}^{p} \overline{w}_{ij} X_{ii}^2 + \sum_{j=1}^{q} \overline{w}_{ii} Y_{ij}^2 \right)^2
\geq \sum_{i=1}^{p} \left| \sqrt{1 - |w_{ii}|^2} X_{ii} - \frac{Y_{ii}}{\sqrt{1 - |w_{ii}|^2}} \right|^2 \geq 0,
\]
we obtain the proposition for \( \Omega_{p,q} \).

**Type IV:** At \((0, w_0)\) with \(w_0 = (w_1, w_2, 0, \ldots, 0)\) where \(w_1 = \lambda(\zeta_1 + \zeta_2)\), \(w_2 = i\lambda(\zeta_1 - \zeta_2)\) as in the proof of Lemma 3.2 for type IV domain, since we have
\[
\frac{\partial \log \delta}{\partial w_j} = \frac{1}{1 - 2w_j \overline{w} + |w w'|^2} \frac{\partial}{\partial w_j} (1 - 2w_j \overline{w} + |w w'|^2) = \frac{-2\overline{w}_j + 2w_j \overline{w}}{1 - 2w_j \overline{w} + |w w'|^2}
\]
and
\[
\frac{\partial \log \delta}{\partial z_j} = -\frac{\partial}{\partial z_j} (1 - 2z \overline{w} + z z' \overline{w} w') = 2\overline{w}_j,
\]
for \( Z = \sum X_j \frac{\partial}{\partial z_j} + \sum Y_j \frac{\partial}{\partial w_j} \) one obtains

\[
r \frac{\partial \bar{\delta}(-\delta^{1/r}(Z, \bar{Z}))}{\partial^{1/r}} = -\frac{\partial \bar{\delta}}{\partial \log \delta(Z, \bar{Z})} - \frac{1}{r} \frac{\partial \log \delta(Z, \bar{Z})}{\partial \log \delta(Z, \bar{Z})}
\]

\[
= \sum_{j=1}^n 2|X_j|^2 - 4\text{Re} \sum_{j=1}^n X_j \bar{Y}_j + M_{11}|Y_1|^2 + M_{12}Y_1 \bar{Y}_2 + M_{21}Y_2 \bar{Y}_1 + M_{22}|Y_2|^2
\]

\[
- \frac{1}{r} \left| \sum_{j=1}^2 \left( 2\bar{w}_jX_j + \frac{-2\bar{w}_j + 2w_j\bar{w}|w|^2}{1 - 2w\bar{w} + |w|^2}Y_j \right) \right|^2.
\]

By substituting \( \zeta_1, \zeta_2 \), we have

\[
\sum_{j=1}^2 \left( \bar{w}_jX_j + \frac{-\bar{w}_j + w_j\bar{w}|w|^2}{1 - 2w\bar{w} + |w|^2} \right)
\]

\[
= \lambda \left( \bar{\zeta}_1(X_1 - iX_2) + \bar{\zeta}_2(X_1 + iX_2) - \frac{\bar{\zeta}_1}{S_n^V}(1 - |\zeta_2|^2)(Y_1 - iY_2) - \frac{\bar{\zeta}_2}{S_n^V}(1 - |\zeta_1|^2)(Y_1 + iY_2) \right)
\]

and

\[
M_{11}|Y_1|^2 + M_{12}Y_1 \bar{Y}_2 + M_{21}Y_2 \bar{Y}_1 + M_{22}|Y_2|^2
\]

\[
= \frac{1}{S_n^V} \left( (1 - |\zeta_1|^2)^2|Y_1 + iY_2|^2 + (1 - |\zeta_2|^2)^2|Y_1 - iY_2|^2 \right),
\]

by relations

\[
M_{11} = M_{22} = \frac{1}{S_n^V} \left( (1 - |\zeta_1|^2)^2 + (1 - |\zeta_2|^2)^2 \right),
\]

\[
M_{12} = -M_{21} = \frac{i}{S_n^V} \left( (1 - |\zeta_2|^2)^2 - (1 - |\zeta_1|^2)^2 \right),
\]

\[
M_{12}Y_1 \bar{Y}_2 + M_{21}Y_2 \bar{Y}_1 = \frac{iM_{12}}{2} \left( |Y_1 + iY_2|^2 - |Y_1 - iY_2|^2 \right).
\]

Therefore Equation (6.2) is greater than or equal to

\[
(1 - |\zeta_1|^2)|X_1 - iX_2|^2 + (1 - |\zeta_2|^2)|X_1 + iX_2|^2 - 4\text{Re}X_1 \bar{Y}_1 - 4\text{Re}X_2 \bar{Y}_2 + \frac{|Y_1 - iY_2|^2}{1 - |\zeta_1|^2} + \frac{|Y_1 + iY_2|^2}{1 - |\zeta_2|^2}
\]

\[
= \left| \sqrt{1 - |\zeta_1|^2}X_1 - iX_2 \right|^2 + \left| \sqrt{1 - |\zeta_2|^2}X_1 + iX_2 \right|^2,
\]

and hence \( -\delta^{1/r} \) is psh.
Recall the definition of the Diederich–Fornæss index: for a domain $D$ in a complex manifold of dimension $n$, $n \geq 2$ with smooth boundary, the Diederich–Fornæss index of $D$ is defined by

$$
\sup \left\{ \mu \in (0, 1) : -\delta(-\nu)^\mu > 0 \text{ on } D \right\}
$$

where the supremum is taken over all defining function $\nu$ of $D$.

**Corollary 6.2.** The Diederich–Fornæss index of $\mathbb{B}^n \times \mathbb{B}^n / \Gamma$ in $\mathbb{B}^n \times \mathbb{CP}^n / \Gamma$ is $1/2$.

**Proof.** Since $-\delta = -\frac{1}{(1-|z|^2)(1-|w|^2)}$ is invariant with respect to the action of $\Gamma$, it gives a real analytic defining function of $\mathbb{B}^n \times \mathbb{B}^n / \Gamma \subset \mathbb{B}^n \times \mathbb{CP}^n / \Gamma$. Adachi–Brinkschulte [2, Main Theorem] and Fu-Shaw [10] proved independently that for a relatively compact domain with $C^3$ boundary in a complex manifold of dimension $N$, if the Levi form of the domain has at least $k$ zero eigenvalues everywhere on the boundary with $0 \leq k \leq N - 1$, then the Diederich–Fornæss index should be less or equal to $\frac{N-k}{N}$. Since the Levi form of $\mathbb{B}^n \times \mathbb{B}^n / \Gamma$ has at least $n$ number of zero eigenvalues, the Diederich–Fornæss index should be less than or equal to $1/2$. On the other hand by Theorem 6.1 the index should be greater or equal to $1/2$, and hence, we complete the proof. □

### 6.2 $k$-Twisted bounded symmetric domains

Let us consider the diagonal action of $\rho : \Gamma \to \text{Aut}(\Omega)$ on $\Omega^k := \Omega \times \cdots \times \Omega$ given by $\gamma(z_1, \ldots, z_k) = (\gamma z_1, \ldots, \gamma z_k)$. Let $\Omega^k / \Gamma$ be the quotient of $\Omega^k$ by this diagonal action. Then it is a holomorphic $\Omega^{k-1}$-fiber bundle over $\Omega / \Gamma$. Define a function $\psi_k$ on $\Omega^k$ by

$$
\psi_k(z) := \left| \frac{\prod_{j=1}^{k} K_\Omega(z_j, z_j)}{K_\Omega(z_1, z_2)K_\Omega(z_2, z_3) \cdots K_\Omega(z_{k-1}, z_k)K_\Omega(z_k, z_1)} \right|^2.
$$

Then $\psi_2 = \psi_\Omega^2$, where $\psi_\Omega$ is given in (3.3), and one has

$$
\psi_k(z_1, \ldots, z_k) = \psi_\Omega(z_1, z_2) \cdots \psi_\Omega(z_{k-1}, z_k) \psi_\Omega(z_k, z_1).
$$

Since

$$
\delta \bar{\partial} \log \psi_k(z_1, \ldots, z_k) = \sum_{j=1}^{k-1} \delta \bar{\partial} \log \psi_\Omega(z_j, z_{j+1}) + \delta \bar{\partial} \log \psi_\Omega(z_k, z_1) \geq 0,
$$

we have the following.

**Corollary 6.3.** For any irreducible bounded symmetric domain $\Omega$ and $k \geq 2$, $\Omega^k / \Gamma$ is hyperconvex.

### 6.3 Steinness

Let $\Gamma \subset G$ be a cocompact discrete subgroup of $\text{Aut}(\Omega)$. Then $\Gamma \setminus \Omega$ is a compact Kähler manifold with respect to the metric induced from the Bergman metric on $\Omega$. Now consider the diagonal
action of $\Gamma$ on $\Omega \times \Omega$ defined by
\begin{equation}
\gamma(z, w) = \left(\gamma z, \overline{\gamma w}\right).
\end{equation}

Denote by $\Omega \times \Omega / \Gamma$ the quotient manifold of $\Omega \times \Omega$ by the action (6.3). Then $\Omega \times \Omega / \Gamma$ is an $\Omega$-fiber bundle over $\Gamma \setminus \Omega$. Now consider the function on $\Omega \times \Omega$ defined by
\begin{equation}
\tilde{\psi}_\Omega(z, w) := \psi_\Omega(z, \overline{w}) = \frac{K_\Omega(z, z)K_\Omega(w, w)}{|K_\Omega(z, \overline{w})|^2}.
\end{equation}

Since
\begin{equation}
\tilde{\psi}_\Omega(\gamma(z, w)) = \tilde{\psi}_\Omega(\gamma z, \gamma \overline{w}) = \frac{K_\Omega(\gamma z, \gamma z)K_\Omega(\gamma \overline{w}, \gamma \overline{w})}{|K_\Omega(\gamma z, \gamma \overline{w})|^2} = \tilde{\psi}_\Omega(z, w),
\end{equation}

$\tilde{\psi}_\Omega$ induces a function on $\Omega \times \Omega / \Gamma$. Hence $\tilde{\delta}(z, w) := \frac{N_\Omega(z, z)N_\Omega(w, w)}{|N_\Omega(z, \overline{w})|^2}$ also induces a function on $\Omega \times \Omega / \Gamma$ and it is an exhaustion function.

**Theorem 6.4.** Let $\Omega$ be an irreducible bounded symmetric domain. Then $\Omega \times \Omega / \Gamma$ admits a bounded strictly psh exhaustion function. More precisely, $-\tilde{\delta}^{1/r}$ with $\frac{1}{r} \leq \frac{1}{2\text{rank}(\Omega)}$ is a bounded strictly psh exhaustion function.

**Proof.** Type I, $\Omega^I_{p,q}$: Since $N_\Omega(z, \overline{w})$ is holomorphic in the functions of $z_i$ and $w_i$, we have
\begin{equation}
-\partial \bar{\partial} \log \hat{\delta} = \begin{pmatrix}
I_{pq} & 0 \\
0 & -\partial_w \hat{\delta}_w \log \det(I - w\overline{w}^t)
\end{pmatrix}
\end{equation}
at $(0, w_0)$ with $w_0 = \text{diag}(w_{11}, \ldots, w_{pp})$, and hence for a non-zero vector $Z = \sum X_{jk} \frac{\partial}{\partial z_{jk}} + \sum Y_{jk} \frac{\partial}{\partial w_{jk}}$ at $(0, w_0)$ we obtain
\begin{equation}
\frac{r}{\delta^{1/r}} \frac{\partial \tilde{\delta}(-\delta^{1/r})}{\delta^{1/r}} = -\partial \tilde{\delta} \log \hat{\delta} - \frac{1}{r} \partial \log \hat{\delta} \land \tilde{\delta} \log \hat{\delta}
\end{equation}

\begin{equation}
= \sum_{j,k=1}^p \left( |X_{jk}|^2 + \frac{|Y_{jk}|^2}{(1 - |w_{kk}|^2)(1 - |w_{jj}|^2)} \right)
\end{equation}

\begin{equation}
= -\frac{1}{r} \sum_{j=1}^p w_{jj}X_{jj} - \frac{p}{1 - |w_{jj}|^2} \left( \sum_{j=1}^p \bar{w}_{jj}Y_{jj} \right)^2 + \sum_{j=1}^p \sum_{k=p+1}^q \left( \frac{|Y_{jk}|^2}{1 - |w_{jj}|^2} + |X_{jk}|^2 \right)
\end{equation}
\[ \geq \sum_{j,k=1, j \neq k}^{p} \left( |X_{jk}|^2 + \frac{|Y_{jk}|^2}{(1 - |w_{kk}|^2)(1 - |w_{jj}|^2)} \right) \]
\[ + \sum_{j=1}^{p} \left( (1 - |w_{jj}|^2)|X_{jj}|^2 + \frac{|Y_{jj}|^2}{1 - |w_{jj}|^2} \right) \]
\[ + \sum_{j=1}^{p} \sum_{k=p+1}^{q} \left( \frac{|Y_{jk}|^2}{1 - |w_{jj}|^2} + |X_{jk}|^2 \right) > 0. \]

Therefore \( -\delta^{1/r} \) is strictly psh exhaustion function.

**Other cases:** We omit the proof since we can perform a similar computation. \( \square \)

By a proof similar to that of Corollary 6.2, we obtain the following.

**Corollary 6.5.** The Diederich–Fornaess index of \( \mathbb{B}^n \times \mathbb{B}^n/\Gamma \) in \( \mathbb{B}^n \times \mathbb{C}P^n/\Gamma \) is \( 1/2 \).

**Remark 6.6.** In case \( \Omega \) is the unit disc in \( \mathbb{C} \), Theorem 6.4 was proved by Adachi in [1].

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