Semidualities from products of trees

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Let $K$ be a global function field of characteristic $p$, and let $\Gamma$ be a finite-index subgroup of an arithmetic group defined with respect to $K$ and such that any torsion element of $\Gamma$ is a $p$–torsion element. We define semiduality groups, and we show that $\Gamma$ is a $\mathbb{Z}[1/p]$–semiduality group if $\Gamma$ acts as a lattice on a product of trees. We also give other examples of semiduality groups, including lamplighter groups, Diestel–Leader groups, and countable sums of finite groups.

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1 Introduction

1.1 Arithmetic groups

Let $K$ be a global field (number or function field), and let $S$ be a nonempty set of finitely many inequivalent valuations of $K$ including each archimedean valuation. The ring $\mathcal{O}_S \subseteq K$ will denote the corresponding ring of $S$–integers. For any $v \in S$, we let $K_v$ be the completion of $K$ with respect to $v$, so that $K_v$ is a locally compact field.

We let $G$ be a noncommutative, absolutely almost simple algebraic $K$–group, so that $G(\mathcal{O}_S)$ is a lattice, included diagonally, in the product of simple Lie groups $\prod_{v \in S} G(K_v)$. For each $v \in S$, we let $X_v$ be the symmetric space or euclidean building (depending on whether $K_v$ is an archimedean or nonarchimedean field) associated with $G(K_v)$, and we let $X_S = \prod_{v \in S} X_v$, so that $G(\mathcal{O}_S)$ acts on $X_S$ as a lattice.

We let

$$k(G, S) = \sum_{v \in S} \text{rank}_{K_v} G.$$

If $G$ is $K$–anisotropic — that is, if $G(\mathcal{O}_S)$ acts cocompactly on $X_S$ — then there is a finite-index subgroup of $G(\mathcal{O}_S)$ that is a duality group, and if $K_v$ is an archimedean field — that is, if $X_v$ is a symmetric space — for all $v \in S$, then there is a finite-index subgroup of $G(\mathcal{O}_S)$ that is a Poincaré duality group.
Borel and Serre [5; 6] showed that $G(\mathcal{O}_S)$ is also a virtual duality group when $G$ is $K$–isotropic, as long as $K$ is a number field. In particular, Borel and Serre construct a bordification of $X_S$, which we denote by $\hat{X}_S$, on which $G(K)$ acts and $G(\mathcal{O}_S)$ acts properly and cocompactly, and such that the compactly supported cohomology groups $H_c^*(\hat{X}_S; \mathbb{Z})$ are nontrivial in some single dimension, $\ell(G, S)$. The result is that any finite-index torsion-free subgroup of $G(\mathcal{O}_S)$ is a duality group of dimension $\ell(G, S)$ with dualizing module $H_c^*(\hat{G}, S)(\hat{X}_S; \mathbb{Z})$.

The purpose of this paper is to suggest a possible analogue of Borel–Serre for arithmetic groups $G(\mathcal{O}_S)$ when $K$ is a global function field.

### 1.2 Function field case

Throughout the remainder of this paper, $K$ denotes a global function field of characteristic $p$, and we suppose that $G$ is $K$–isotropic — that is, that $G(\mathcal{O}_S)$ does not act cocompactly on $X_S$.

Any finite-index subgroup of $G(\mathcal{O}_S)$ contains torsion, so it cannot be a duality group, as duality groups have finite cohomological dimension. However, there are finite-index subgroups of $G(\mathcal{O}_S)$ whose only torsion elements are $p$–elements and whose cohomological dimension over $\mathbb{Z}[1/p]$ is bounded above by $k(G, S)$. We let $\Gamma$ denote such a subgroup.

The group $\Gamma$ still has an obstruction to being a $\mathbb{Z}[1/p]$–duality group. Indeed, it is not of type $FP_k(G, S)$ over $\mathbb{Z}[1/p]$; see Kropholler [15], Bux and Wortman [11], Gandini [13], and Bux, Köhl and Witzel [10]. However, $\Gamma$ is of type $FP_{k(G, S)}$, and we conjecture that the discrepancy between types $FP_{k(G, S)}$ and $FP_{k(G, S)}$ is the only, and in some ways a minor, obstruction to $\Gamma$ being a $\mathbb{Z}[1/p]$–duality group. Before making this precise, we’ll need a definition.

For a commutative ring $R$, we say that a group $\Lambda$ is an $R$–semiduality group of dimension $d$ if

(i) $\text{cd}_R \Lambda \leq d$,

(ii) $\Lambda$ is of type $FP_d$ over $R$,

(iii) $H^k(\Lambda; R\Lambda) = 0$ if $k \neq d$, and

(iv) $H^d(\Lambda; R\Lambda)$ is a flat $R$–module.

In this definition, $H^d(\Lambda; R\Lambda)$ is called the dualizing module, and if the ring $R$ and the group $\Lambda$ are understood, then we’ll often denote the dualizing module simply as $D$. 

*Geometry & Topology, Volume 22 (2018)*
In Section 2 of this paper we’ll show the following consequence of a group being a semiduality group.

**Proposition 1** If $\Lambda$ is an $R$–semiduality group of dimension $d$, then for any $0 \leq n \leq d$ and any left $R\Lambda$–module $M$, there are natural homomorphisms of $R$–modules

$$\varphi_n^M : H_n(\Lambda; D \otimes_R M) \to H^{d-n}(\Lambda; M).$$

The $\varphi_n^M$ are compatible with the connecting homomorphisms in the long exact homology and cohomology sequences associated to a short exact sequence of coefficient modules, and if $\cdots \to Q_1 \to Q_0 \to M \to 0$ is a projective resolution of $M$ by left $R\Lambda$–modules, then $\varphi_n^M$ is injective if $Q_{n+1}$ and $Q_n$ are finitely generated, and surjective if $Q_n$ and $Q_{n-1}$ are finitely generated. By convention, $Q_{-1}$ is always finitely generated.

With the definition of semiduality and its immediate consequences listed above, we propose the following:

**Conjecture 2** Let $\mathcal{O}_S$ be a ring of $S$–integers in a global function field $K$ of characteristic $p$, and let $G$ be a noncommutative, absolutely almost simple algebraic $K$–group. If $\Gamma$ is a finite-index subgroup of $G(\mathcal{O}_S)$ such that any torsion element of $\Gamma$ is a $p$–element, then $\Gamma$ is a $\mathbb{Z}[1/p]$–semiduality group of dimension $k(G, S)$, and the dualizing module admits an action by $G(K)$.

Note that Bux, Köhl and Witzel [10] show that any $\Gamma$ as in Conjecture 2 is of type $FP_{k(G, S)-1}$ over $\mathbb{Z}[1/p]$, and it is well known that $\cd \mathbb{Z}[1/p] \Gamma \leq k(G, S)$ since the dimension of $X_S$ equals $k(G, S)$ (see Lemma 43 below). Therefore, proving Conjecture 2 would amount to proving that $H^m(\Gamma; \mathbb{Z}[1/p] \Gamma) = 0$ if $m < k(G, S)$, and that $D = H^{k(G, S)}(\Gamma; \mathbb{Z}[1/p] \Gamma)$ is flat as a $\mathbb{Z}[1/p]$–module, the latter condition being equivalent to $D$ being torsion-free since $\mathbb{Z}[1/p]$ is a principal ideal domain.

Furthermore, we conjecture that $D$ contains, and is an inverse limit of quotients of, $H_c^{k(G, S)}(X_S; \mathbb{Z}[1/p])$; i.e we can view $D$ as an augmentation of $H_c^{k(G, S)}(X_S; \mathbb{Z}[1/p])$. Thus, whereas Borel–Serre exhibits duality groups whose dualizing modules are cohomology groups of augmentations of the spaces on which arithmetic groups act, we conjecture that over function fields, arithmetic groups are semiduality groups whose dualizing modules are augmentations of cohomology groups of spaces on which the arithmetic groups act.

As an illustration, let $L$ be a field whose characteristic is not equal to $p$. Recall that $\cd L \Gamma \leq k(G, S)$. By Bux, Köhl and Witzel [10], $L$ is of type $FP_{k(G, S)-1}$ as
a \mathbb{Z}[1/p] \Gamma\text{-module. Therefore, if Conjecture 2 is true, then } H^1(\Gamma; L) \text{ is a quotient of } H_{k(G, S)-1}(\Gamma; D), \text{ and if } 2 \leq n \leq k(G, S) \text{ then}

H_{k(G, S)-n}(\Gamma; D) \cong H^n(\Gamma; L).

Note that the only dimension of } H^*(\Gamma; L) \text{ which semiduality would not be able to help determine is dimension 0, but we know } H^0(\Gamma; L) = L.

1.3 Main result

What we prove in this paper is a first case of Conjecture 2. Namely:

**Theorem 3** Conjecture 2 is true if \( \text{rank}_{K_v} G = 1 \) for all \( v \in S \). In particular, if \( P \) is a proper \( K\text{-parabolic subgroup of } G \), then there is an exact sequence

\[
0 \to H^k_c(G, S; \mathbb{Z}[1/p]) \to D \to \bigoplus_{z \in (G/P)(K)} M_z \to 0
\]

of \( \mathbb{Z}[1/p] G(K)\text{-modules, where } M_z \text{ is an uncountable } \mathbb{Z}[1/p]\text{-module for each } z \in (G/P)(K), M_z \cong M_w \text{ as } \mathbb{Z}[1/p]\text{-modules for any } z, w \in (G/P)(K), \text{ and } g(M_z) = M_{gz} \text{ for all } g \in G(K) \text{ and } z \in (G/P)(K).

For example, \( \text{SL}_2(\mathbb{F}_p[t]) \) is a semiduality group of dimension 1, \( \text{SL}_2(\mathbb{F}_p[t, t^{-1}]) \) is a semiduality group of dimension 2, and \( \text{SL}_2(\mathcal{O}_S) \) is a semiduality group of dimension \( |S| \) whose dualizing module incorporates the action of \( \text{SL}_2(K) \) on \( \mathbb{P}^1(K) \).

Our proof of Theorem 3 is geometric. That is, we will use strongly that, under the hypotheses of Theorem 3, \( X_S \) is a product of trees.

1.4 Solvable groups

Let \( B_2 \) be the group of upper triangular matrices of determinant 1. Thus, \( B_2(\mathbb{F}_p[t]) \) is commensurable to \( \mathbb{F}_p[t] \), and \( B_2(\mathbb{F}_p[t, t^{-1}]) \) is commensurable to the lamplighter group \( \mathbb{F}_p \wr \mathbb{Z} \). This paper will also show:

**Theorem 4** \( B_2(\mathcal{O}_S) \) is virtually a \( \mathbb{Z}[1/p]\text{-semiduality group of dimension } |S| \).

Thus, as the discrete group \( \text{Solv} \) is known to be a Poincaré duality group, and as the solvable Baumslag–Solitar groups are known to be duality groups, the lamplighter groups with prime-order cyclic base are semiduality groups. Notice that the three groups from the previous sentence are commensurable respectively to \( B_2(\mathbb{Z}[[\sqrt{2}]]), B_2(\mathbb{Z}[1/p]), \text{ and } B_2(\mathbb{F}_p[t, t^{-1}]) \).
We also show that certain generalizations of $B_2(F_p[t])$ and $B_2(F_p[t, t^{-1}])$ are semiduality groups, namely countable sums of finite groups and Diestel–Leader groups, respectively.

1.5 Outline of paper

In Section 2 we’ll prove Proposition 1. In Section 3 we’ll show how the cohomology of a discrete group with group-ring coefficients can be, in some cases, interpreted from the topology of a contractible space on which it acts properly, and perhaps noncocompactly. In Section 4 we’ll detail how the groups $G(O_S)$ from Theorem 3 act cocompactly on the complement of a pairwise disjoint collection of horoballs in a product of trees, and in Section 5 we’ll show that such a complement has trivial compactly supported cohomology in dimension $d - 1$, where $d$ is the number of factors in the product. Section 6 shows that $\lim}_1$ of the compactly supported cohomology of a nested sequence of regular horospheres in a product of trees is torsion-free in dimension $d - 1$, and the final section of this paper, Section 7, will combine the ingredients collected in earlier sections to prove that certain groups are semiduality groups and includes a proof of Theorem 3.

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2 Homological algebra

This section is divided in two parts. First, we’ll prove Proposition 1 from the introduction. Second, we’ll prove an equivalent characterization of semiduality groups in the form of Proposition 11. While Proposition 11 is not directly applied in this paper to other results, it seems to be of independent interest. Both the statement and proof of Proposition 11 were provided to us by a referee, to whom we are grateful.

In this section we let $R$ be a commutative ring.
2.1 Proof of Proposition 1

We’ll prove Proposition 1 in four steps. First, we’ll define \( \varphi_n^M \). Second, we’ll show how the injectivity and surjectivity of \( \varphi_n^M \) can be deduced from the finiteness properties of \( M \). Third, we’ll demonstrate the required naturality properties of \( \varphi_n^M \). Last, and not until the final sentence of Section 2.1, we’ll invoke the assumption from Proposition 1 that \( H^d(\Gamma; R\Gamma) \) is a flat \( R \)-module.

To define \( \varphi_n^M \), let \( \Gamma \) be a group of type \( FP_{d-1} \) over \( R \) with \( \text{cd}_R \Gamma \leq d \). Then there is a projective resolution of the trivial left \( R\Gamma \)-module \( R \) by left \( R\Gamma \)-modules

\[
0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to R \to 0,
\]

where \( P_i \) is finitely generated if \( i \leq d - 1 \).

We let \( P_i^* = \text{Hom}_{R\Gamma}(P_i, R\Gamma) \) and \( D = H^d(\Gamma; R\Gamma) \). Assuming that \( H^k(\Gamma; R\Gamma) = 0 \) if \( k \neq d \), we have the exact sequence of right \( R\Gamma \)-modules

\[
0 \to P_0^* \to P_1^* \to \cdots \to P_{d-1}^* \to P_d^* \to D \to 0.
\]

Let \( \cdots \to A_1 \to A_0 \to D \to 0 \) be a projective resolution of \( D \) by right \( R\Gamma \)-modules. Let \( h: A_{\bullet} \to P_{d-\bullet}^* \) be a chain map over the identity map on \( D \). Thus, for any left \( R\Gamma \)-module \( M \), there are chain maps

\[
A_{\bullet} \otimes_{R\Gamma} M \to P_{d-\bullet}^* \otimes_{R\Gamma} M \to \text{Hom}_{R\Gamma}(P_{d-\bullet}, M)
\]

that induce natural homomorphisms

\[
\text{Tor}_n^{R\Gamma}(D, M) \xrightarrow{k_n^M} H_n(P_{d-\bullet}^* \otimes_{R\Gamma} M) \xrightarrow{v_n^M} H^{d-n}(\Gamma; M).
\]

We define \( \varphi_n^M = v_n^M \circ k_n^M \). To deduce when \( \varphi_n^M \) is injective or surjective, we’ll deduce when those properties are satisfied by \( k_n^M \) and \( v_n^M \) separately, beginning with \( k_n^M \). But before that, we’ll need the following:

**Lemma 5** Let \( P \) be a projective, left \( R\Gamma \)-module, and \( P^* \) its dual. If \( \cdots \to Q_1 \to Q_0 \to M \to 0 \) is a projective resolution of a left \( R\Gamma \)-module \( M \), and if \( Q_{n+1} \) and \( Q_n \) are finitely generated, then \( \text{Tor}_n^{R\Gamma}(P^*, M) = 0 \).

**Proof** The following diagram commutes:

\[
\begin{array}{c}
P^* \otimes_{R\Gamma} Q_{n+1} \longrightarrow P^* \otimes_{R\Gamma} Q_n \longrightarrow P^* \otimes_{R\Gamma} Q_{n-1} \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{Hom}_{R\Gamma}(P, Q_{n+1}) \longrightarrow \text{Hom}_{R\Gamma}(P, Q_n) \longrightarrow \text{Hom}_{R\Gamma}(P, Q_{n-1})
\end{array}
\]

*Geometry & Topology, Volume 22 (2018)*
Since \( Q_{n+1} \) and \( Q_n \) are finitely generated projective, they are finitely presented, so the two vertical maps on the left are isomorphisms. It follows that the homology of the top row injects into the homology of the bottom row. That is, \( \text{Tor}^R_n(P^*, M) \) injects into 0, since \( P \) is projective, and thus \( \text{Hom}(P, -) \) is exact.

Now we are prepared to address the injectivity and surjectivity of the homomorphisms \( \kappa_n^M : \text{Tor}^R_n(D, M) \to H_n(P_{d-*}^* \otimes R \Gamma M) \).

**Lemma 6** Let \( \cdots \to Q_1 \to Q_0 \to M \to 0 \) be a projective resolution of a left \( R \Gamma \)-module \( M \). Then \( \kappa_0^M \) is an isomorphism, \( \kappa_1^M \) is surjective, \( \kappa_n^M \) is injective if \( Q_{n+1} \) and \( Q_n \) are finitely generated, and \( \kappa_n^M \) is surjective if \( Q_n \) and \( Q_{n-1} \) are finitely generated.

**Proof** Recall that \( A_\bullet \to D \) is a projective resolution of \( D \) with a chain map \( h : A_\bullet \to P_{d-*}^* \) over the identity on \( D \), so that \( h \) is a quasi-isomorphism. We let \( P_{d-1}^* \) be the truncated complex of \( P_{d-*}^* \), so that the bottom row of this diagram of chain maps is exact:

\[
0 \to P_d^* \to P_{d-*}^* \to P_{d-*1}^* \to 0
\]

Because each \( Q_n \) is projective, the following diagram of bicomplexes has an exact row and \( (h \otimes \text{id})_* \) is a quasi-isomorphism:

\[
A_\bullet \otimes_R \Gamma Q_\bullet
\]

\[
0 \to P_d^* \otimes_R \Gamma Q_\bullet \to P_{d-*}^* \otimes_R \Gamma Q_\bullet \to P_{d-*1}^* \otimes_R \Gamma Q_\bullet \to 0
\]

Passing to homology, we have the following diagram whose row is exact and whose vertical map, \( (h \otimes \text{id})_* \), is an isomorphism:

\[
\text{Tor}^R_n(D, M)
\]

\[
\text{Tor}^R_n(P_d^*, M) \to H_n(P_{d-*}^* \otimes_R \Gamma Q_\bullet) \xrightarrow{\iota_*} H_n(P_{d-*1}^* \otimes_R \Gamma Q_\bullet) \xrightarrow{\partial} \text{Tor}^{R-1}_n(P_d^*, M)
\]

Notice that the domain of \( \partial \) is \( H_n(P_{d-*1}^* \otimes_R \Gamma M) \) since \( P_{d-*1}^* \) is a deleted projective resolution.

For \( n \geq 2 \), we have \( H_n(P_{d-*1}^* \otimes_R \Gamma M) = H_n(P_{d-*}^* \otimes_R \Gamma M) \) and \( \kappa_n^M = \iota_* \circ (h \otimes \text{id})_* \). Thus, when \( n \geq 2 \), the lemma follows from Lemma 5.
For $n = 1$, we have that $\text{Tor}^R_{n-1}(P^*_d, M) = P^*_d \otimes_R M$ and thus the image of $\iota_*$, which is the kernel of $\partial$, is $H_1(P^*_d \otimes_R M)$.

For $n = 0$, notice that $\kappa^M_0$ is the identity map on $D \otimes_R M$. \hfill \Box

For $v^M_n$: $H_n(P^*_d \otimes_R M) \to H^{d-n}(\Gamma; M)$, we have the following:

**Lemma 7** Let $M$ be a left $R\Gamma$–module. Then $v^M_1$ is injective and $v^M_n$ is bijective if $n \geq 2$. If $M$ is finitely presented, then $v^M_0$ and $v^M_1$ are bijective. If $M$ is finitely generated, then $v^M_0$ is surjective.

**Proof** If $n \geq 2$, then we see that the vertical maps in the following commutative diagram are isomorphisms since $P_{d-i}$ is finitely generated and projective if $i \geq 1$:

$$
\begin{array}{cccc}
P^*_d \otimes_R M & \longrightarrow & P^*_d \otimes_R M & \longrightarrow & P^*_d \otimes_R M \\
\downarrow & & \downarrow & & \downarrow \\
\text{Hom}_{R\Gamma}(P_{d-n}, M) & \longrightarrow & \text{Hom}_{R\Gamma}(P_{d-n}, M) & \longrightarrow & \text{Hom}_{R\Gamma}(P_{d-n}, M)
\end{array}
$$

Therefore, $v^M_n$ is an isomorphism if $n \geq 2$.

For $n = 1$, observe the following commutative diagram:

$$
\begin{array}{cccc}
P^*_d \otimes_R M & \longrightarrow & P^*_d \otimes_R M & \longrightarrow & P^*_d \otimes_R M \\
\downarrow & & \downarrow & & \downarrow \\
\text{Hom}_{R\Gamma}(P_{d-2}, M) & \longrightarrow & \text{Hom}_{R\Gamma}(P_{d-2}, M) & \longrightarrow & \text{Hom}_{R\Gamma}(P_{d-2}, M)
\end{array}
$$

Since $P_{d-2}$ and $P_{d-1}$ are finitely generated and projective, the two vertical maps on the left are isomorphisms, so $v^M_1$ is injective. If, in addition, $M$ is finitely presented, then the vertical map on the right is bijective, so $v^M_1$ is bijective.

As for $v^M_0$, the following sequence is exact by the definition of $H^d(\Gamma; M)$:

$$
\text{Hom}_{R\Gamma}(P_{d-1}, M) \to \text{Hom}_{R\Gamma}(P_{d}, M) \to H^d(\Gamma; M) \to 0
$$

Using the above for $M$ and $M = R\Gamma$, and using that the tensor product is right exact, we see that the rows of the following commutative diagram are exact:

$$
\begin{array}{cccc}
P^*_d \otimes_R M & \longrightarrow & P^*_d \otimes_R M & \longrightarrow & D \otimes_R M & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
\text{Hom}_{R\Gamma}(P_{d-1}, M) & \longrightarrow & \text{Hom}_{R\Gamma}(P_{d}, M) & \longrightarrow & H^d(\Gamma; M) & \longrightarrow 0
\end{array}
$$

The vertical map on the left is an isomorphism since $P_{d-1}$ is finitely generated and projective. Thus, $v^M_0$ is surjective (resp. bijective) if the second vertical map from
the left is, by the five lemma. Now note that the second vertical map on the left is surjective if $M$ is finitely generated, and bijective if $M$ is finitely presented.

We now have:

**Proposition 8** Suppose $\Gamma$ is a group of type $FP_{d-1}$ over $R$, that $\text{cd}_R \Gamma \leq d$, and that $H^k(\Gamma; R\Gamma) = 0$ if $k \neq d$. Let $D = H^d(\Gamma; R\Gamma)$.

If $\cdots \to Q_1 \to Q_0 \to M \to 0$ is a projective resolution of a left $R\Gamma$–module $M$, then

$$\varphi^M_n : \text{Tor}^R_{n}(D, M) \to H^{d-n}(\Gamma; M)$$

is injective if $Q_{n+1}$ and $Q_n$ are finitely generated, and surjective if $Q_n$ and $Q_{n-1}$ are finitely generated.

**Proof** Combine Lemmas 6 and 7.

Having established conditions for the injectivity and surjectivity of the $\varphi^M_n$, we turn to naturality properties of these homomorphisms, beginning with:

**Lemma 9** For a left $R\Gamma$–module $N$ and an exact sequence of left $R\Gamma$–modules $0 \to M \to M' \to M'' \to 0$, the composition

$$\text{Tor}^R_1(N^*, M'') \to N^* \otimes_{R\Gamma} M \to \text{Hom}_{R\Gamma}(N, M)$$

is 0, where $N^* = \text{Hom}_{R\Gamma}(N, R\Gamma)$.

**Proof** The statement follows from the commutative diagram with exact rows

$\begin{array}{ccccccccc}
\text{Tor}^R_1(N^*, M'') & \longrightarrow & N^* \otimes_{R\Gamma} M & \longrightarrow & N^* \otimes_{R\Gamma} M' \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Hom}_{R\Gamma}(N, M) & \longrightarrow & \text{Hom}_{R\Gamma}(N, M')
\end{array}$

We will use the previous lemma to construct a diagram of chain complexes in our proof of the following:

**Proposition 10** Suppose $\Gamma$ is a group of type $FP_{d-1}$ over $R$, that $\text{cd}_R \Gamma \leq d$, and that $H^k(\Gamma; R\Gamma) = 0$ if $k \neq d$. Let $D = H^d(\Gamma; R\Gamma)$.

For any left $R\Gamma$–module $M$, the $R$–module homomorphisms

$$\varphi^M_n : \text{Tor}^R_n(D, M) \to H^{d-n}(\Gamma; M)$$

are natural, and they are compatible with the connecting homomorphisms in the long exact homology and cohomology sequences associated to a short exact sequence of coefficient modules.
Proof  From their definition, the $\varphi_n^M$ are natural, since homology is a functor.

In order to show that the $\varphi_n^M$ are compatible with connecting homomorphisms, we let $0 \to M \to M' \to M'' \to 0$ be a short exact sequence of left $R\Gamma$–modules, and we let $J$ be the image of $\text{Tor}_1^{R\Gamma} (P^*_d, M'')$. Let $\varphi_n^M : \text{Tor}_n^{R\Gamma} (D, M) \to H_n (\varphi_n^M)$ be the connecting homomorphism, so we have an exact sequence

$$0 \to (P^*_d \otimes_{R\Gamma} M)/J \to P^*_d \otimes_{R\Gamma} M' \to P^*_d \otimes_{R\Gamma} M'' \to 0.$$ 

Furthermore, if $i \geq 1$, then $P_{d-i}$ is finitely generated, so $P^*_{d-i}$ is projective, and thus, if we let $(P^*_{d-i} \otimes_{R\Gamma} M)_J = P^*_{d-i} \otimes_{R\Gamma} M$ for $i \geq 1$ and $(P^*_{d} \otimes_{R\Gamma} M)_J = (P^*_{d} \otimes_{R\Gamma} M)/J$, then we have a short exact sequence of chain complexes

$$0 \to (P^*_{d-i} \otimes_{R\Gamma} M)_J \to P^*_{d-i} \otimes_{R\Gamma} M' \to P^*_{d-i} \otimes_{R\Gamma} M'' \to 0.$$ 

By the previous lemma, $J$ is contained in the kernel of $P^*_{d} \otimes_{R\Gamma} M \to \text{Hom}_{R\Gamma} (P^*_{d}, M)$, so the composite

$$A_0 \otimes_{R\Gamma} M \to P^*_{d} \otimes_{R\Gamma} M \to \text{Hom}_{R\Gamma} (P^*_{d}, M)$$ 

factors through $(P^*_{d-i} \otimes_{R\Gamma} M)/J$. Thus, we have a commutative diagram of chain complexes with exact columns:

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
A_\bullet \otimes_{R\Gamma} M & \to & (P^*_{d-i} \otimes_{R\Gamma} M)_J & \to & \text{Hom}_{R\Gamma} (P^*_{d-i}, M) \\
\downarrow & & \downarrow & & \downarrow \\
A_\bullet \otimes_{R\Gamma} M' & \to & P^*_{d-i} \otimes_{R\Gamma} M' & \to & \text{Hom}_{R\Gamma} (P^*_{d-i}, M') \\
\downarrow & & \downarrow & & \downarrow \\
A_\bullet \otimes_{R\Gamma} M'' & \to & P^*_{d-i} \otimes_{R\Gamma} M'' & \to & \text{Hom}_{R\Gamma} (P^*_{d-i}, M'') \\
\downarrow & & \downarrow & & \downarrow \\
0 & \to & 0 & \to & 0 \\
\end{array}
\]

Therefore, for all $n$, we have the following commutative diagram on homology, whose vertical maps are connecting homomorphisms:

\[
\begin{array}{ccc}
\text{Tor}_{n-1}^{R\Gamma} (D, M) & \to & H_{n-1} ((P^*_{d-i} \otimes_{R\Gamma} M)_J) \\
\downarrow & & \downarrow \\
\text{Tor}_n^{R\Gamma} (D, M) & \to & H_n ((P^*_d \otimes_{R\Gamma} M)/J) \\
\end{array}
\]

\[
\begin{array}{ccc}
H_n (P^*_d \otimes_{R\Gamma} M') & \to & H_{d-n} (\Gamma; M') \\
\downarrow & & \downarrow \\
H_n (P^*_d \otimes_{R\Gamma} M'') & \to & H_{d-n+1} (\Gamma; M) \\
\end{array}
\]

\text{Geometry & Topology, Volume 22 (2018)}
The top row of the above diagram is $\varphi_n^M$. The bottom row is $\varphi_n^M$ as can be seen by noting that the commutative diagram of chain maps

$$
\begin{array}{ccc}
P_{d-\bullet}^* \otimes_{R\Gamma} M & \rightarrow & A_{\bullet} \otimes_{R\Gamma} M \\
\downarrow & & \downarrow \\
(P_{d-\bullet}^* \otimes_{R\Gamma} M)_J & \rightarrow & \text{Hom}_{R\Gamma}(P_{d-\bullet}, M)
\end{array}
$$

yields the commutative diagram on homology

$$
\begin{array}{ccc}
H_{n-1}(P_{d-\bullet}^* \otimes_{R\Gamma} M) & \rightarrow & \text{Tor}^{R\Gamma}_{n-1}(D, M) \\
\downarrow & & \downarrow \\
H_{n-1}((P_{d-\bullet}^* \otimes_{R\Gamma} M)_J) & \rightarrow & H^{d-n+1}(\Gamma; M)
\end{array}
$$

and noticing that the top row of the above diagram is $\varphi_n^M$ while the bottom row coincides with the bottom row of the preceding commutative rectangle.

Last, if we assume $D$ is flat as an $R$–module, so that $\text{Tor}^{R\Gamma}_{n}(D, M) \cong H_{n}(\Gamma; D \otimes_{R} M)$, then Proposition 1 is Propositions 8 and 10.

2.2 Alternative characterization of semiduality

In what remains of this section, we’ll prove the following result, given to us by a referee.

**Proposition 11** Let $\Gamma$ be a group, and suppose $\text{cd}_R \Gamma = d$. The following are equivalent:

(i) $\Gamma$ is of type $FP_{d-1}$ over $R$, and $H^k(\Gamma; R\Gamma) = 0$ if $k \neq d$.

(ii) There is a right $R\Gamma$–module $C$ such that the natural map $C \rightarrow C^{**}$ is an injection, and such that if $0 \leq n \leq d - 2$ then there are natural isomorphisms

$$
\xi_n: \text{Tor}^{R\Gamma}_{d-1-n}(C, M) \rightarrow H^n(\Gamma; M)
$$

and

$$
\psi_n: H_n(\Gamma; N) \rightarrow \text{Ext}^{d-1-n}_{R\Gamma}(C, N)
$$

for left $R\Gamma$–modules $M$ and right $R\Gamma$–modules $N$, and such that there is an exact sequence

$$
0 \rightarrow H^{d-1}(\Gamma; M) \rightarrow C \otimes_{R\Gamma} M \rightarrow \text{Hom}_{R\Gamma}(C^*, M).
$$
(iii) All of the conditions in (ii) hold. In addition, \( C^{**}/C \cong H^d(\Gamma; R\Gamma) \), the exact sequence from (ii) extends to an exact sequence

\[
0 \to H^{d-1}(\Gamma; M) \to C \otimes_{R\Gamma} M \to \text{Hom}_{R\Gamma}(C^*, M) \to H^d(\Gamma, M) \to 0,
\]

and there is an exact sequence

\[
0 \to H^d(\Gamma; N) \to N \otimes_{R\Gamma} C^* \to \text{Hom}_{R\Gamma}(C, N) \to H_{d-1}(\Gamma; N) \to 0.
\]

**Proof** Since (iii) is a strictly larger collection of conditions than (ii), we have that (iii) implies (ii), so in this proof we’ll show that (i) implies (iii) and that (ii) implies (i). We begin with (i) implies (iii).

As in Section 2.1, there is a projective resolution of the trivial left \( R\Gamma \)–module \( R \)

\[
0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to R \to 0,
\]

where \( P_i \) is finitely generated if \( i \leq d-1 \), and we let \( C \) be the image of \( P_{d-1}^* \to P_d^*. \)

The map \( \xi_n \) is what we had previously named \( \psi_n^M \) in the proof of Proposition 10. The proof of the existence of the \( \psi_n \) is similar. Indeed, since the \( P_i \) are finitely generated projective if \( i \leq d-1 \), the vertical maps of the commutative diagram below are isomorphisms:

\[
\begin{array}{c}
N \otimes_{R\Gamma} P_{d-1}^* \to \cdots \to N \otimes_{R\Gamma} P_0 \to 0 \\
\text{Hom}_{R\Gamma}(P_{d-1}^*, N) \to \cdots \to \text{Hom}_{R\Gamma}(P_0^*, N) \to 0
\end{array}
\]

The isomorphisms \( \psi_n \) are the isomorphisms from the homologies of the two rows.

Now note that because \( \text{Hom}_{R\Gamma}(\mathbf{R}, R\Gamma) \) is left exact, the bottom row is exact in the following diagram:

\[
0 \to P_d \to P_{d-1} \to P_{d-2} \to 0 \\
0 \to C^* \to P_{d-1}^{**} \to P_{d-2}^{**}
\]

The top row is exact by stipulation, and since \( P_{d-1} \) and \( P_{d-2} \) are finitely generated projective, the natural vertical maps are isomorphisms. It follows that \( C^* \cong P_d \).

Applying the left-exact \( \text{Hom}_{R\Gamma}(\mathbf{R}, N) \) to \( P_{d-2}^* \to P_{d-1}^* \to C \to 0 \) yields the exact sequence

\[
0 \to \text{Hom}_{R\Gamma}(C, N) \to \text{Hom}_{R\Gamma}(P_{d-1}^*, N) \to \text{Hom}_{R\Gamma}(P_{d-2}^*, N).
\]
The isomorphisms $N \otimes_{R\Gamma} P_i \cong \text{Hom}_{R\Gamma}(P_i^*, N)$ when $i = d-1, d-2$ imply that $\text{Hom}_{R\Gamma}(C, N)$ is the kernel of $N \otimes_{R\Gamma} P_{d-1} \to N \otimes_{R\Gamma} P_{d-2}$. Therefore, we have exact sequences

$$N \otimes_{R\Gamma} P_d \to \text{Hom}_{R\Gamma}(C, N) \to H_{d-1}(\Gamma; N) \to 0$$

and

$$0 \to H_d(\Gamma; N) \to N \otimes_{R\Gamma} P_d \to \text{Hom}_{R\Gamma}(C, N).$$

Combining these two sequences and replacing $P_d$ with $C^*$ yields the exact sequence

$$0 \to H_d(\Gamma; N) \to N \otimes_{R\Gamma} C^* \to \text{Hom}_{R\Gamma}(C, N) \to H_{d-1}(\Gamma; N) \to 0.$$

Similarly, we apply the right-exact $\otimes_{R\Gamma} M$ to $P_{d-2}^* \to P_{d-1}^* \to C \to 0$ for the exact sequence $P_{d-2}^* \otimes_{R\Gamma} M \to P_{d-1}^* \otimes_{R\Gamma} M \to C \otimes_{R\Gamma} M \to 0$. When $i = d-1, d-2$, we have $P_i^* \otimes_{R\Gamma} M \cong \text{Hom}_{R\Gamma}(P_i, M)$. Hence, $C \otimes_{R\Gamma} M$ is the cokernel of $\text{Hom}_{R\Gamma}(P_{d-2}, M) \to \text{Hom}_{R\Gamma}(P_{d-1}, M)$, so we have the exact sequences

$$0 \to H^{d-1}(\Gamma; M) \to C \otimes_{R\Gamma} M \to \text{Hom}_{R\Gamma}(P_d, M)$$

and

$$C \otimes_{R\Gamma} M \to \text{Hom}_{R\Gamma}(P_d, M) \to H^d(\Gamma, M) \to 0.$$

Therefore, the sequence below is exact:

$$0 \to H^{d-1}(\Gamma; M) \to C \otimes_{R\Gamma} M \to \text{Hom}_{R\Gamma}(C^*, M) \to H^d(\Gamma, M) \to 0.$$

The final condition of (iii) to be checked is that $0 \to C \to C^{**} \to H^d(\Gamma, R\Gamma) \to 0$ is exact. For this, let $M = R\Gamma$ in the sequence immediately preceding this paragraph. Our proof that (i) implies (iii) is complete.

To show (ii) implies (i), note that the isomorphisms $\psi_n: H_n(\Gamma; N) \to \text{Ext}_{R\Gamma}^{d-n+1}(C, N)$ for $0 \leq n \leq d - 2$ and any right $R\Gamma$–module $N$ show that $H_n(\Gamma; -)$ commutes with direct products when $0 \leq n \leq d - 2$. In particular, $H_n(\Gamma; \prod R\Gamma) \cong \prod H_n(\Gamma; R\Gamma)$ for $0 \leq n \leq d - 2$. That is, for $n = 0$ we have $R \otimes_{R\Gamma} (\prod R\Gamma) \cong \prod R$ and for $1 \leq n \leq d - 2$ we have $\text{Tor}_n^{R\Gamma}(R; \prod R\Gamma) = 0$, so that $R\Gamma$ is of type $FP_{d-1}$ by Lemma 1.1 and Proposition 1.2 of [4].

For $0 \leq n \leq d - 2$, the existence of isomorphisms $\xi_n: \text{Tor}_{d-1-n}^{R\Gamma}(C, R\Gamma) \to H^n(\Gamma; R\Gamma)$ shows that $H^n(\Gamma; R\Gamma) = 0$ since $R\Gamma$ is free. That $H^{d-1}(\Gamma; R\Gamma) = 0$ follows by recalling that $C$ injects into $C^{**}$ and letting $M = R\Gamma$ in the exact sequence

$$0 \to H^{d-1}(\Gamma; M) \to C \otimes_{R\Gamma} M \to \text{Hom}_{R\Gamma}(C^*, M).$$
3 Translation from topology

3.1 Cohomology compactly supported over each compact subcomplex

If a group $\Gamma$ has a finite Eilenberg–Mac Lane complex $X = K(\Gamma, 1)$ with universal cover $\tilde{X}$, then for any ring $R$ there is an isomorphism $H^*(\Gamma; R\Gamma) = H^*_c(\tilde{X}; R)$. In this section we provide an alternative topological characterization of $H^*(\Gamma, R\Gamma)$ in the case that $X$ is not finite. Our proof uses standard techniques, which we include for completeness.

Suppose that $X$ is a locally finite cell complex with an action by a group $\Gamma$, and let $\pi: X \to \Gamma \backslash X$ denote the quotient map. Let $C^*(X; R)$ denote the cellular cochain complex of $X$ with coefficients in a ring $R$. Define a subcomplex $C^k_{cc}(X; R) \subseteq C^k(X; R)$ to contain cochains $\phi \in C^k(X; R)$ such that for every $k$–cell $\sigma \in \Gamma \backslash X$ we have $\phi(\tilde{\sigma}) = 0$ for all but finitely many $\tilde{\sigma} \in \pi^{-1}(\sigma)$. Then $d(C^k_{cc}(X; R)) \subseteq C_{cc}^{k+1}(X; R)$, and we let $H^*_{cc}(X; R)$ be the cohomology of this complex. We suppress the dependence on the action of $\Gamma$ from the notation.

**Proposition 12** Suppose $X$ is a locally finite, acyclic cell complex and $\Gamma$ is a group acting on $X$ with cell stabilizers that are finite and preserve orientation. Then

$$H^*(\Gamma; R\Gamma) \cong H^*_{cc}(X; R).$$

**Proof** Recall that the equivariant cohomology of the pair $(X, \Gamma)$ with coefficients in $R\Gamma$ is defined as

$$H^*_w(X; R\Gamma) = H^*(\Gamma; C^*(X; R\Gamma)).$$

There is an isomorphism (compare with the analogous homological result in Brown [7, VII.7.3, page 173])

$$H^*(\Gamma; R\Gamma) \cong H^*_w(X; R\Gamma).$$

There is a spectral sequence (again, compare with Brown [7, page 169])

$$E_1^{pq} = H^q(\Gamma; C^p(X; R\Gamma)) \Rightarrow H^{p+q}_\Gamma(X; R\Gamma).$$

We will show $H^q(\Gamma; C^p(X; R\Gamma)) = 0$ for all $q > 0$. Let $X_p$ denote the set of $p$–cells in $X$ and let $\Sigma_p$ be a set of representatives for $\Gamma \backslash X_p$. Letting $\Gamma_\sigma$ denote the stabilizer of $\sigma \in \Sigma_p$, there is a decomposition

$$C^p(X; R\Gamma) = \text{Hom}(C_p(X), R\Gamma) \cong \prod_{\sigma \in X_p} R\Gamma \cong \prod_{\sigma \in \Sigma_p} \text{Coind}^\Gamma_{\Gamma_\sigma}(R\Gamma).$$
Therefore, there is a decomposition of cohomology:

\[ H^q(\Gamma; C^p(X; R\Gamma)) \cong H^q(\Gamma; \prod_{\sigma \in \Sigma_p} \text{Coind}_{R\Gamma}^{\Gamma} (R\Gamma)) \cong \prod_{\sigma \in \Sigma_p} H^q(\Gamma; \text{Coind}_{R\Gamma}^{\Gamma} (R\Gamma)). \]

Applying Shapiro’s lemma yields

\[ H^q(\Gamma; C^p(X; R\Gamma)) \cong \prod_{\sigma \in \Sigma_p} H^q(\Gamma_{\sigma}; R\Gamma). \]

Since \( \Gamma_{\sigma} \) is finite, there is an isomorphism of \( R\Gamma_{\sigma} \)-modules \( R\Gamma \cong \text{Coind}_{\{1\}}^{\Gamma_{\sigma}} (\bigoplus_{\rho \in \Sigma_p} R) \). Therefore, another use of Shapiro’s lemma shows that \( H^q(\Gamma_{\sigma}; R\Gamma) = 0 \) for \( q > 0 \).

Recall that \( H^0(\Gamma_{\sigma}; R\Gamma) \cong (R\Gamma)^{\Gamma_{\sigma}} \).

It follows from the above that \( H^*(\Gamma; R\Gamma) \) is the cohomology of the cochain complex

\[ \prod_{\sigma \in \Sigma_0} (R\Gamma)^{\Gamma_{\sigma}} \to \prod_{\sigma \in \Sigma_1} (R\Gamma)^{\Gamma_{\sigma}} \to \prod_{\sigma \in \Sigma_2} (R\Gamma)^{\Gamma_{\sigma}} \to \cdots. \]

We will show this chain complex is isomorphic to the chain complex \( \{ \mathcal{C}_{cc}^k(X, R) \} \).

First we compute the coboundary maps of (1). The isomorphism

\[ C^p(X; R\Gamma) \to \prod_{\sigma \in \Sigma_p} (R\Gamma)^{\Gamma_{\sigma}} \]

sends a map \( \phi: C_p(X; R) \to R\Gamma \) to the function \( \xi: \Sigma_p \to R\Gamma \) defined by \( \xi(\sigma) = \phi([\sigma]) \) for any \( p \)-cell \( \sigma \in \Sigma_p \). From \( \Gamma \)-equivariance of \( \phi \) computation shows that the coboundary operator on the complex (1) is given by

\[ d\xi(\sigma) = \sum_i r_i \gamma_i \xi(\sigma_i) \]

if \( \partial[\sigma] = \sum n_i \gamma_i [\sigma_i] \) for ring elements \( r_i \in R \), group elements \( \gamma_i \in \Gamma \) and simplices \( \sigma_i \in \Sigma_p \).

Define an isomorphism

\[ \Theta: \prod_{\sigma \in \Sigma_p} (R\Gamma)^{\Gamma_{\sigma}} \to \mathcal{C}_{cc}^p(X, R) \]

as follows: given \( \phi \in \prod_{\sigma \in \Sigma_p} (R\Gamma)^{\Gamma_{\sigma}} \), for any \( p \)-simplex \( \rho \) in \( X \) choose \( \sigma \in \Sigma_p \) and \( \gamma \in \Gamma \) such that \( \rho = \gamma \sigma \) and set

\[ \Theta\phi(\rho) = [\phi(\sigma)]_{\gamma^{-1}}. \]
Here \([x]_{y^{-1}}\) denotes the coefficient of \([y^{-1}]\) in the formal sum \(x \in R \Gamma\). Note \(\sigma\) is uniquely specified by \(\rho\) and \(y\) is unique up to right multiplication by elements of \(\Gamma_\sigma\). Because each \(\phi(\sigma)\) is \(\Gamma_\sigma\)-invariant, \(\Theta\) does not depend on choice of \(y\). Moreover, any two \(p\)-cells in \(X\) that belong to the same \(\Gamma\) orbit will correspond to the same cell \(\sigma\) in the above equation. Since there are only finitely many terms in the formal sum \(\phi(\sigma)\), the map \(\Theta \phi\) is finitely supported above each cell in \(X\). Therefore, \(\Theta\) determines a well-defined homomorphism of \(\Gamma\)-modules.

It is clear that \(\Theta\) is injective. To see that \(\Theta\) is surjective, define an inverse \(\Theta^{-1}\) by setting \([\Theta^{-1} \xi(\sigma)]_y = \xi(y^{-1} \sigma)\). It remains only to see that \(\Theta\) is compatible with the coboundary maps. Suppose \(\rho\) is a \((p+1)\)-cell in \(X\) and \(\partial[\rho] = \sum r_i [\delta_i]\) for ring elements \(r_i \in R\) and \(p\)-cells \(\delta_i\). For each \(i\), write \(\delta_i = \gamma_i \sigma_i\) for \(\gamma_i \in \Gamma\) and \(\sigma_i \in \Sigma_p\). Then

\[
d[\Theta \phi](\rho) = \Theta \phi(\partial \rho) \\
= \sum_i r_i [\Theta \phi](\delta_i) \\
= \sum_i r_i [\phi(\sigma_i)]_{y^{-1}}.
\]

On the other hand, note that if \(\rho = \gamma \sigma\) for \(\gamma \in \Gamma\) and \(\sigma \in \Sigma_{p+1}\), then

\[
\partial[\sigma] = \sum_i r_i (\gamma^{-1} \gamma_i)[\sigma_i].
\]

Therefore,

\[
\Theta(d \phi)(\rho) = [(d \phi)(\sigma)]_{y^{-1}} \\
= [\phi(\partial \sigma)]_{y^{-1}} \\
= \left[ \sum_i r_i (\gamma^{-1} \gamma_i) \phi(\sigma_i) \right]_{y^{-1}} \\
= \sum_i r_i [(\gamma^{-1} \gamma_i) \phi(\sigma_i)]_{y^{-1}} \\
= \sum_i r_i [\phi(\sigma_i)]_{y^{-1}}.
\]

Thus \(\Theta\) commutes with the coboundary operators and hence is an isomorphism of chain complexes. This completes the proof. \(\square\)

**Lemma 13** Let \(X\) and \(\Gamma\) be as in Proposition 12. If \(G\) is a locally compact group acting cellularly on \(X\) and \(\Gamma \leq G\) then \(\text{Comm}_G(\Gamma)\) acts on \(H^*_{ce}(X;R)\).
Proof Given \( \phi \in C^k_{cc}(X) \) and \( g \in \text{Comm}_G(\Gamma) \), define \((g\phi)(\sigma) = \phi(g^{-1}\sigma)\). The condition that \( g\phi \in C^k_{cc}(X) \) is equivalent to the condition that \( \text{supp}(g\phi) \cap \Gamma K \) is compact for any compact set \( K \subseteq X \), which is equivalent to \( \text{supp}(\phi) \cap g^{-1}\Gamma K \) being compact for any compact \( K \). Fix a compact \( K \subseteq X \). Because \( g \in \text{Comm}_G(\Gamma) \), there is some compact \( K' \subseteq X \) such that \( g\Gamma K \subseteq \Gamma K' \). Therefore, \( \text{supp}(\phi) \cap g^{-1}\Gamma K \subseteq \text{supp}(\phi) \cap \Gamma K' \). The latter is compact because \( \phi \in C^k_{cc}(X) \), so \( g\phi \in C^k_{cc}(X) \). This action commutes with coboundary maps, so induces an action on cohomology.

3.2 Computing \( H^*_{cc}(X) \)

Let \( X \) and \( \Gamma \) be as in Proposition 12. Suppose there are subcomplexes \( X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X \) such that each \( X_k \) is closed and \( \Gamma \)–invariant, each quotient \( \Gamma \backslash X_k \) is compact, and \( X = \bigcup_i X_i \). The compactly supported cellular cochain complexes \( C^*_c(X_n; R) \) form a codirected system under the restriction maps \( r_{i,j} : X_i \to X_j \) for \( i < j \).

The chain complex \( C^k_{cc}(X; R) \) is the inverse limit of the system of chain complexes \( C^k_c(X_n; R) \) and each restriction map \( r_{i,j} \) is surjective on the chain level. It follows (see for example the “Variant” following Theorem 3.5.8 of Weibel [17, page 84]) that for any \( k \) there is a short exact sequence of cohomology

\[
0 \to \lim_{\leftarrow} H^{k-1}_c(X_n; R) \to H^k_{cc}(X; R) \to \lim_{\to} H^k_c(X_n; R) \to 0.
\]

Recall that \( \lim_{\leftarrow} H^{k-1}_c(X_n; R) \) is the cokernel of the map

\[
\Delta : \prod_{n=0}^{\infty} H^{k-1}_c(X_n; R) \to \prod_{n=0}^{\infty} H^{k-1}_c(X_n; R)
\]

defined by

\[
\Delta(x_0, x_1, x_2, \ldots) = (x_0 - r_{0,1}(x_1), x_1 - r_{1,2}(x_2), \ldots).
\]

As a straightforward application of the short exact sequence (2) we have:

**Proposition 14** Suppose \( X \) is a locally finite cell complex with an action of a group \( \Gamma \).

Suppose \( X \) is the union of an increasing sequence of \( \Gamma \)–invariant subcomplexes \( X_n \) each with cocompact \( \Gamma \) action. If there is some integer \( d \) such that \( H^*_c(X_n; R) \) is concentrated in dimension \( d \) for all \( n \) then \( H^*_c(X; R) \) is concentrated in dimensions \( d \) and \( d + 1 \). In particular, if \( X \) is \( d \)–dimensional then

\[
H^k_{cc}(X; R) = \begin{cases} 
\lim_{\to} H^d_c(X_n; R) & \text{if } k = d, \\
0 & \text{if } k \neq d.
\end{cases}
\]
4 Statement of reduction theory

In this section we’ll review the necessary results needed from reduction theory for our proof of Theorem 3. The results in this section are not new, and can be derived from Behr [2] and Harder [14], although there are some minor differences between our treatment of reduction theory here and other versions already existing in the literature. A point of difference in the proof of our formulation of these results compared with formulations in other papers is that we’ll use the reduction theory from Bestvina, Eskin and Wortman [3] as an input, which has the advantage, though not directly applied in this paper, of being equally applicable to arithmetic groups defined with respect to a number field. See also Bux and Wortman [12] and Bux, Köhl and Witzel [10].

4.1 Algebraic form of reduction theory

In this section and the next we assume that $K$ is a global field with a ring of $S$–integers $\mathcal{O}_S \leq K$ and that $G$ is a noncommutative, absolutely almost simple, $K$–isotropic, $K$–group with rank$_{K_v}(G) = 1$ for all $v \in S$.

Let $P$ be a proper $K$–parabolic subgroup of $G$. Let $A$ be a maximal $K$–split torus in $P$.

From the root system for $(G, A)$, we denote the simple root for the positive roots with respect to $P$ by $\alpha_0$.

We let $Z_G(A)$ be the centralizer of $A$ in $G$, so that $Z_G(A) = MA$ where $M$ is a reductive $K$–group with $K$–anisotropic center. We let $U$ be the unipotent radical of $P$, so that $P = UMA$. The Levi subgroup $MA$ normalizes the unipotent radical $U$, and elements of $A$ commute with those of $M$.

We denote the product over $S$ of local points of a $K$–group by “unbolding”, so that, for example,

$$G = \prod_{v \in S} G(K_v).$$

We let $\mathcal{P}$ be the set of proper $K$–parabolic subgroups of $G$. If $Q \in \mathcal{P}$, then $Q$ is conjugate in $G(K)$ to $P$. We let

$$\Lambda_Q = \{gf \in G(\mathcal{O}_S)F : (gf)P(gf)^{-1} = Q\},$$

where $F \subseteq G(K)$ is a finite set of coset representatives for $G(\mathcal{O}_S)\backslash G(K)/P(K)$.

Note that if $\gamma_1 f_1, \gamma_2 f_2 \in \Lambda_Q$, then $f_1 = f_2$. 
Given any $a = (a_v)_{v \in S} \in A$, we let

$$|\alpha_0(a)| = \prod_{v \in S} |\alpha_0(a_v)|_v,$$

where $| \cdot |_v$ is the $v$–adic norm on $K_v$.

Given any $t > 0$, we let

$$A^+(t) = \{ a \in A : |\alpha_0(a)| \geq t \},$$

and for $t > 0$, we let

$$R_Q(t) = \Lambda_Q UMA^+(t).$$

The following is a special case of Proposition 9 from Bestvina, Eskin and Wortman [3].

**Proposition 15** There exists a bounded set $B_0 \subseteq G$, and given any $N_0 \geq 0$, there exists $t_0 > 1$ and a second bounded set $B_1 \subseteq G$ such that

(i) $G = \bigcup_{Q \in \mathcal{P}} R_Q(1)B_0$;

(ii) if $Q, Q' \in \mathcal{P}$ and $Q \neq Q'$, then the distance between $R_Q(t_0)B_0$ and $R_{Q'}(t_0)B_0$ is at least $N_0$;

(iii) $G(\mathcal{O}_S) \cap R_Q(t_0)B_0 = \emptyset$; and

(iv) $G - (\bigcup_{Q \in \mathcal{P}} R_Q(2t_0)B_0)$ is contained in $G(\mathcal{O}_S)B_1$.

### 4.2 Geometric form of reduction theory

We will now reformulate Proposition 15 into a more explicit geometric statement in the form of Proposition 19 below.

For $v \in S$, we let $X_v$ be the euclidean building for $G(K_v)$, so that $X_v$ is a tree. We let $X_S = \prod_{v \in S} X_v$.

Let $\Sigma_v \subseteq X_v$ be the geodesic that $A(K_v)$ acts on by translations. We let $\Sigma_S = \prod_{v \in S} \Sigma_v$, so that $\Sigma_S$ is isometric to the euclidean space $\mathbb{R}^{\lfloor S \rfloor}$.

We define a linear functional $\hat{\alpha}_0 : \Sigma_S \to \mathbb{R}$ by associating a basepoint $e \in \Sigma_S$ with the origin as follows:

$$\hat{\alpha}_0(\alpha e) = \log_p |\alpha_0(a)| \quad \text{for } a \in A.$$

The action of $A$ on $e$ factors through $\mathbb{Z}^{\lfloor S \rfloor}$, where $\hat{\alpha}_0$ is linear, so $\hat{\alpha}_0$ extends to a functional on all of $\Sigma_S$. Furthermore, $\hat{\alpha}_0$ is nonzero since there is some $a$ with $|\alpha_0(a)| \neq 1$. 

*Geometry & Topology, Volume 22 (2018)*
For any \( r \in \mathbb{R} \), we let \( \Sigma_{S,r} \subseteq \Sigma_S \) be
\[
\Sigma_{S,r} = \{ x \in \Sigma_S : \hat{\alpha}_0(x) = r \}.
\]
Thus \( \Sigma_{S,r} \) is a hyperplane in \( \Sigma_S \) that is a finite Hausdorff distance from \( A(\Omega_S) e \subseteq \Sigma_{S,0} \).

Note that \( \Sigma_{S,r} \) is not singular if \( |S| > 1 \). That is to say, the projection of \( \Sigma_{S,r} \) to each \( \Sigma_v \) is surjective if \( |S| > 1 \). Indeed, to verify this claim observe that if \( v \in S \), then \( A(\Omega_S) / e \) has dense projection to \( A(K_v) \), and thus acts cocompactly on \( \Sigma_{S,v} \).

Now consider the geodesics \( \Sigma_v \) as unit-speed parametrizations \( \Sigma_v : \mathbb{R} \to X_v \) with \( \Sigma_v(\infty) = P \). From our description of \( \hat{\alpha}_0 : \Sigma_S \to \mathbb{R} \), we see that there are positive real numbers \( \lambda_v \) such that if \( \rho_S : \mathbb{R} \to X_S \) is given by \( \rho_S(t) = (\Sigma_v(\lambda_v t))_{v \in S} \) and if \( \beta_{\rho_S} : X_S \to \mathbb{R} \) is the Busemann function for \( \rho_S \), then \( \beta_{\rho_S} \) restricted to \( \Sigma_S \) is exactly \( \hat{\alpha}_0 \). Recall that \( \beta_{\rho_S} \) is the Busemann function for \( \rho_S \) if
\[
\beta_{\rho_S}(x) = \lim_{t \to \infty} (t - d(x, \rho_S(t))) \quad \text{for all} \quad x \in X_S.
\]
where \( d \) is the distance function on \( X_S \).

Let
\[
\Sigma_{S,r}^+ = \{ x \in \Sigma_S : \hat{\alpha}_0(x) \geq r \}
= \{ x \in \Sigma_S : \beta_{\rho_S}(x) \geq r \},
\]
so that \( \Sigma_{S,r}^+ \) is a half space in \( \Sigma_S \) whose boundary equals \( \Sigma_{S,r} \).

We let
\[
B_{P,S,r} = \{ x \in X_S : \beta_{\rho_S}(x) \geq r \}
\]
and
\[
Y_{P,S,r} = \{ x \in X_S : \beta_{\rho_S}(x) = r \}.
\]

**Lemma 16** \( B_{P,S,r} = UM \Sigma_{S,r}^+ \) and \( Y_{P,S,r} = UM \Sigma_{S,r} \).

**Proof** \( M \) is contained in both \( P \) and the parabolic group opposite to \( P \) with respect to \( A \). Also note that \( M(K_v) \) is compact for all \( v \in S \). It follows that \( M(K_v) \) fixes \( \Sigma_v \) pointwise, and thus that \( M \) fixes \( \Sigma_S \) pointwise. Therefore, \( UM \Sigma_{S,r}^+ = U \Sigma_{S,r}^+ \).

Elements of \( U(K_v) \) fix unbounded positive rays in \( \Sigma_v \), thus elements of \( U \) fix pointwise a subray of \( \rho_S \), thus \( \beta_{\rho_S} \) is invariant under multiplication by \( U \). Therefore, \( UB_{P,S,r} = B_{P,S,r} \), so \( UM \Sigma_{S,r}^+ \subseteq B_{P,S,r} \) follows from \( \Sigma_{S,r}^+ \subseteq B_{P,S,r} \).
To see that $B_{P,S,r} \subseteq U \Sigma_{S,r}^+$, let $x \in B_{P,S,r}$. Since $X_v = U(K_v) \Sigma_v$, we see that $x = u(x_v)_{v \in S}$ for some $u \in U$ and $x_v \in \Sigma_v$. Thus, $x \in U\Sigma_{S,r}^+$, again, since $\beta_{P,S}$ is invariant under multiplication by $U$.

That $Y_{P,S,r} = UM \Sigma_{S,r}$ follows similarly.

Given $t \in \mathbb{R}$, let $r_t \in \mathbb{R}$ be the supremum of all $r \in \mathbb{R}$ such that $\Sigma_{S,r_t}^+$ contains $A^+(t)e$. Notice that there is some $C > 0$, independent of $t$, such that the Hausdorff distance between $A^+(t)e$ and $\Sigma_{S,r_t}^+$ is bounded by $C$. Notice also that $t \mapsto r_t$ is an increasing function.

**Lemma 17** The Hausdorff distance between $UMA^+(t)e$ and $B_{P,S,r_t}$ is bounded independent of $t$.

**Proof** Because the Hausdorff distance between $A^+(t)e$ and $\Sigma_{S,r_t}^+$ is bounded, the Hausdorff distance between $UMA^+(t)e$ and $UM \Sigma_{S,r_t}^+ = B_{P,S,r}$ is bounded.

**Lemma 18** Let $Q \in \mathcal{P}$. If $\gamma f \in G(O_S)$ and $f \in F$ are such that $\gamma f \in \Lambda Q$, then for any $r$, we have $Q(O_S) \gamma f B_{P,S,r} = \gamma f B_{P,S,r}$.

**Proof** Note that $\gamma f B_{P,S,r}$ is given by the Busemann function for $\gamma f_{P,S}$ since $B_{P,S,r}$ is given by the Busemann function for $\rho_S$.

Since $\gamma f \in \Lambda Q$, the positive end of each $\gamma f \Sigma_v$ limits to $Q$. Thus, if $g \in Q(K_v)$, then $\gamma f \Sigma_v$ and $g \gamma f \Sigma_v$ intersect in a positive ray. Hence, if $g \in Q$, then $g \gamma f_{P,S}$ is a finite Hausdorff distance from $\gamma f_{P,S}$ and $g \gamma f B_{P,S,r} = \gamma f B_{P,S,r}$ for some $r_g \in \mathbb{R}$.

By replacing $g$ with its inverse, we may assume that $r_g \geq r$.

We may assume that the set $B_0$ from Proposition 15 is a sufficiently large neighborhood of $1 \in G$, independent of $g$, so that, in particular, there is a set $B' \subseteq B_0$ containing the point stabilizer of 1 and such that $B'B' \subseteq B_0$, and by the previous lemma, such that $UMA^+(t)B'e$ contains every vertex of $B_{P,S,r_t}$.

Let $t_0$ be as in Proposition 15. If $r_g \neq r$, then $g^n e \in \gamma f B_{P,S,r_{t_0}}$ for sufficiently large $n$. Hence, $g^n e \in \gamma f UMA^+(t_0)B'e$. Thus, $g^n \in \gamma f UMA^+(t_0)B'B' \subseteq R_Q(t_0)B_0$. We conclude, by Proposition 15 part (iii), that $g \notin G(O_S)$.

If $Q \in \mathcal{P}$, we define

$$B_{Q,S,r} = \gamma f B_{P,S,r}$$

for any $\gamma f \in \Lambda Q$. This is well defined by the previous lemma, and we also see that $Q(O_S)B_{Q,S,r} = B_{Q,S,r}$ and that the Hausdorff distance between $R_Q(t)B_0 e$...
and $B_{Q,S,r_i}$ is bounded independent of $t$ or $Q$. Using this and that the orbit map $G \to Ge \subseteq X_S$ is proper, we deduce from Proposition 15 the following:

**Proposition 19** There exists some $r_0 \in \mathbb{R}$, and given any $N \geq 0$, there is some $r_1 > r_0$ such that

1. $\bigcup_{Q \in P} B_{Q,S,r_0} = X_S$;
2. if $Q, Q' \in P$ and $Q \neq Q'$, then the distance between $B_{Q,S,r_1}$ and $B_{Q',S,r_1}$ is at least $N$;
3. $G(O_S)e \cap B_{Q,S,r_1} = \emptyset$; and
4. $X_S - \left( \bigcup_{Q \in P} B_{Q,S,r_1} \right)$ is a finite Hausdorff distance from $G(O_S)e$.

For any $r \in \mathbb{R}$, we let $X_{S,r}$ be the closure in $X_S$ of $X_S - \left( \bigcup_{Q \in P} B_{Q,S,r} \right)$.

**Lemma 20** For $r \gg 0$, $G(O_S)$ acts properly and cocompactly on $X_{S,r}$.

**Proof** Let $\gamma \in G(O_S)$. Then $\gamma B_{Q,S,r} = B_{\gamma Q^{-1},S,r}$, and therefore $G(O_S)$ acts on $\bigcup_{Q \in P} B_{Q,S,r}$ and thus on $X_{S,r}$.

Since $G(O_S)$ acts properly on $X_S$, it acts properly on $X_{S,r}$.

That $G(O_S)$ acts cocompactly on $X_{S,r}$ follows from Proposition 19(iv).

## 5 Cohomology of the complement of disjoint horoballs

In this section, we’ll examine the cohomology of subspaces of $X_S$ that include spaces of the form $X_{S,r}$, but are slightly more general in that we will allow ourselves to set the height of each horoball individually, rather than use a single parameter to define the height of all horoballs simultaneously. Precisely, for any tuple $(r_Q)_{Q \in P} \in (\mathbb{R} \cup \{\infty\})^P$, we let $X_{S,(r_Q)}$ be the closure of $X_S - \left( \bigcup_{Q \in P} B_{Q,S,r_Q} \right)$ in $X_S$, where $B_{Q,S,\infty}$ is taken to be the empty set.

We shall call a tuple $(r_Q)_{Q \in P} \in (\mathbb{R} \cup \{\infty\})^P$ *sufficiently large* if the resulting sets $B_{Q,S,r_Q}$ are pairwise disjoint, and if their pairwise distance is bounded below by a constant that is sufficiently large. It’s known that if $(r_Q)_{Q \in P}$ is sufficiently large then $X_{S,(r_Q)}$ is $(|S| - 2)$-connected but not $(|S| - 1)$-connected (see Stuhler [16], Bux and Wortman [12], and Bux, Köhl and Witzel [10]), but these topological properties are not directly relevant to this paper. What we require in this paper, and what we will prove
in this section, is that $H^k_c(X_{S,(r_Q)}) = 0$ if $k \leq |S| - 1$ and $(r_Q)_{Q \in \mathcal{P}}$ is sufficiently large. (See Proposition 34 below.) We will begin an inductive proof of this claim by observing that the claim is true when $|S| = 1$.

**Lemma 21** If $|S| = 1$ and $(r_Q)_{Q \in \mathcal{P}}$ is sufficiently large, then the group $H^0_c(X_{S,(r_Q)})$ is trivial, where the coefficients are in a ring $R$.

**Proof** In this case, $X_S$ is a tree, and we want to show that the components of $X_{S,(r_Q)}$ are unbounded. Indeed, choose an edge $e_0 \in X_{S,(r_Q)}$. Because $(r_Q)_{Q}$ is sufficiently large, there is an adjacent edge $e_1 \in X_{S,(r_Q)}$, and we can continue in this fashion to create an path of infinite length in $X_{S,(r_Q)}$ that begins with $e_0$.

Our proof of Proposition 34 will include an investigation of spaces that are quite similar to the space $X_{S,(r_Q)}$. Precisely, for any $(r_Q)_{Q \in \mathcal{P}}$, let $W_{S,(r_Q)}$ be the subcomplex of $X_S$ consisting of all cells of $X_S$ that are contained in $X_{S,(r_Q)}$. To see that there isn’t much difference between $X_{S,(r_Q)}$ and $W_{S,(r_Q)}$ we have:

**Lemma 22** If the tuple $(r_Q)_{Q \in \mathcal{P}}$ is sufficiently large, then there is a proper homotopy equivalence between $W_{S,(r_Q)}$ and $X_{S,(r_Q)}$.

**Proof** The proof is an observation through Morse theory. Suppose that $\mathcal{C} \subseteq X_S$ is a chamber that intersects $X_{S,(r_Q)}$ nontrivially, but is not contained in $X_{S,(r_Q)}$, and thus is not contained in $W_{S,(r_Q)}$.

Since $(r_Q)_{Q \in \mathcal{P}}$ is sufficiently large, $\mathcal{C}$ intersects $B_{Q,S,r_Q}$ for a unique $Q$. Recall that $B_{Q,S,r_Q}$ is defined as the inverse image of a positive ray with respect to the Busemann function $\beta_{yfPS} : X_S \to \mathbb{R}$ associated to the geodesic $\gamma fPS \subseteq X_S$ where $\gamma f \in \Lambda Q$.

Let $(x_v)_{v \in S}$ be the maximum point of $\mathcal{C}$ with respect to $\beta_{yfPS}$. Let $L_v$ be the descending link of $x_v$ in the tree $X_v$ with respect to $\beta_{yfS_v} : X_v \to \mathbb{R}$. We let $C_v$ be the cone on $L_v$ taken at $x_v$ in the tree $X_v$.

For $T \subseteq S$, we let $K_T = \prod_{v \in T} C_v \times \prod_{v \notin T} L_v$.

Now we are assuming that $(x_v)_{v \in S} \notin X_{S,(r_Q)}$, and note that $K_S - (x_v)_{v \in S}$ deformation retracts onto $\bigcup_{v \in S} K_{S-v}$ in such a way that the homotopy is nonincreasing with respect to $\beta_{yfPS}$. Note further that the maximum points in any $K_{S-v_0}$ with respect to $\beta_{yfPS}$ are points of the form $(y_v)_{v \in S}$ where $y_v = x_v$ if $v \neq v_0$, and if these points are not in $X_{S,(r_Q)}$, then we can further retract $K_{S-v_0}$ minus these maximums onto $\bigcup_{v \in S-v_0} K_{S-v_0}$. We continue in this fashion until all of $K_S$ has been retracted onto some union of $K_T$ with $K_T \subseteq X_{S,(r_Q)}$. 

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*Geometry & Topology, Volume 22 (2018)*
In particular, the previous two lemmas show:

**Lemma 23** If $|S| = 1$, and if $(r_Q)_{Q \in P}$ is sufficiently large, then $H^0_c(W_{S,(r_Q)}) = 0$.

This lemma will serve as the base step for our inductive proof that $H^k_c(W_{S,(r_Q)}) = 0$ if $k \leq |S| - 1$ and $(r_Q)_{Q \in P}$ is sufficiently large, which implies that $H^k_c(X_{S,(r_Q)}) = 0$ if $k \leq |S| - 1$ and $(r_Q)_{Q \in P}$ is sufficiently large.

### 5.1 Proper products

Now we will focus on the case when $|S| \geq 2$. We choose some $w \in S$ and let $\pi_w: X_S \to X_w$ be the projection.

Note that by definition of $W_{S,(r_Q)}$, if $e$ is an edge in $X_w$, and if $e^\circ$ is the interior of $e$, then $\pi_w|_{W_{S,(r_Q)}}: W_{S,(r_Q)} \to X_w$ has $\pi_w^{-1}(e^\circ) = e^\circ \times Z_e$ for some complex $Z_e \subseteq X_{S-w}$. Our inductive proof in the remainder of this section is aided by observing that the fibers $\pi_w$ restricted to one of these “$W$ spaces” is another “$W$ space”.

**Lemma 24** For any edge $e \subseteq X_w$, we have $Z_e = W_{S-w,(s^e_Q)}$ for some tuple $(s^e_Q)_{Q \in P}$. Furthermore, by choosing $(r_Q)_{Q \in P}$ sufficiently large we may assume that $(s^e_Q)_{Q \in P}$ is sufficiently large for each edge $e \subseteq X_w$.

**Proof** Let $x_w \in X_w$ be the endpoint of $e$ that maximizes $\beta_{\gamma f} \Sigma_w$ for $\gamma f \in \Lambda_Q$. Then a cell $\tilde{S} \subseteq X_{S-w}$ is contained in $Z_e$ exactly if

$$\beta_{\gamma f \rho_S}(e \times \tilde{S}) \leq r_Q,$$

which is equivalent to

$$\beta_{\gamma f \rho_S}(x_w \times \tilde{S}) \leq r_Q$$

and thus to $\beta_{\gamma f \rho_{S-w}}(\tilde{S}) \leq s^e_Q$ for some $s^e_Q$ depending on $\beta_{\gamma f} \Sigma_w(x_w)$, and thus on $e$. □

**Lemma 25** Let $\gamma f \in \Lambda_Q$. If $e_1, e_2 \in X_w$ are edges and if the maximum of $\beta_{\gamma f} \Sigma_w(e_1)$ is greater than or equal to the maximum of $\beta_{\gamma f} \Sigma_w(e_2)$, then $s^e_{Q1} \leq s^e_{Q2}$. If $\beta_{\gamma f} \Sigma_w(e_1) = \beta_{\gamma f} \Sigma_w(e_2)$, then $s^e_{Q1} = s^e_{Q2}$.

**Proof** Let $x_w \subseteq X_w$ be a geodesic that limits to $Q$, and suppose that $e_1 \subseteq x_w$.

First assume that $e_2 \subseteq x_w$. Then since $\beta_{\gamma f} \Sigma_w(e_2) \leq \beta_{\gamma f} \Sigma_w(e_1)$ we see that $s^e_{Q2} \geq s^e_{Q1}$ as desired.
If \( e_2 \) is not contained in \( \chi_w \), then there is some \( u \in U(K_w) \) such that \( u\chi_w \) does contain \( e_2 \). The result follows from the above as \( \beta_{\gamma f \rho S - w} \) and \( \beta_{\gamma f \Sigma w} \) are invariant by translations of \( U(K_w) \).

Given a vertex \( y \in X_w \), we let \( E_y \) be the set of edges in \( X_w \) that contain \( y \). Then the previous lemma produces:

**Lemma 26** For any vertex \( y \in X_w \), and any parabolic \( Q \in \mathcal{P} \), either \( \{ s^e_Q \}_{e \in E_y} \) contains a single value, or else \( \{ s^e_Q \}_{e \in E_y} \) contains exactly two values, and the minimum value is realized by a unique edge in \( E_y \).

**Proof** For \( \gamma f \in \Lambda_Q \), observe that there is a unique edge containing \( y \) that maximizes the Busemann function \( \beta_{\gamma f \Sigma w} \), and that the remaining edges minimize \( \beta_{\gamma f \Sigma w} \).

In what follows, we’ll denote the unique edge in \( E_y \) from the proof of the previous lemma as \( e.y; Q/ \). Thus if \( e; \leq e.y; Q/ \) if \( e; \subseteq e.y; Q/ \), and \( s^e_Q \neq e(y, Q) \).

We will need one more related observation about the fibers of \( \pi_w \) in the form of the following:

**Lemma 27** If there is a vertex \( y \in X_w \) and a cell \( \mathcal{F} \subseteq X_{S - w} \) such that \( y \times \mathcal{F} \subseteq W_{S, (r_Q)} \), then \( e \times \mathcal{F} \subseteq W_{S, (r_Q)} \) for each \( e \in E_y - e(y, Q) \).

**Proof** Let \( \gamma f \in \Lambda_Q \). Since \( y \times \mathcal{F} \subseteq W_{S, (r_Q)} \), the values of \( \beta_{\gamma f \rho S}(y \times \mathcal{F}) \) are bounded above by \( r_Q \). Since \( y \) maximizes the values of \( e \) under \( \beta_{\gamma f \Sigma w} \), the values of \( \beta_{\gamma f \rho S}(e \times \mathcal{F}) \) are bounded above by \( r_Q \) as well. That is, \( e \times \mathcal{F} \subseteq W_{S, (r_Q)} \).

### 5.2 Cover by fibers

Having collected some information about the fibers of \( \pi_w \mid_{W_{S, (r_Q)}} \), we will now use a collection of fibers to create a cover for \( W_{S, (r_Q)} \).

For any edge \( e \subseteq X_w \), let \( F_e = e \times W_{S - w, (s^e_Q)} \), where \( W_{S - w, (s^e_Q)} \) is as in Lemma 24.

**Lemma 28** The collection \( \{ F_e \} \) taken over all edges \( e \subseteq X_w \) is a cover for \( W_{S, (r_Q)} \).

**Proof** Suppose \( \sigma \times \mathcal{F} \) is a cell in \( W_{S, (r_Q)} \), where \( \sigma \) is a cell in an edge \( e \subseteq X_w \) and \( \mathcal{F} \) is a cell in \( X_{S - w} \).

If \( \sigma = e \), then \( \sigma \times \mathcal{F} \subseteq F_e \) by Lemma 24. If \( \sigma \) is a vertex of \( e \), say \( y \), then by Lemma 27, there is some \( e' \) such that \( y \times \mathcal{F} \subseteq e' \times \mathcal{F} \subseteq F_e' \).
For any vertex \( y \in X_w \), let \( F_y = \bigcup_{e \in E_y} F_e \). Note that there is a proper homotopy equivalence between \( F_y \) and

\[
\bigcup_{y \in e} W_{S-w,(s^e_Q)} = W_{S-w,(\max_{e \in E_y} \{s^e_Q\})}
\]

given by retracting the star of \( y \) in \( X_w \) to the point \( y \).

Further, if \( e \in E_y \), then the inclusion \( F_e \to F_y \), after proper homotopy equivalence, is the inclusion \( W_{S-w,(s^e_Q)} \to W_{S-w,(\max_{e \in E_y} \{s^e_Q\})} \). In particular, if \( e \in E_y \), then we can, and we shall, identify the map induced by inclusion

\[
\rho_y,e: H_c^{[S]-1}(F_y) \to H_c^{[S]-1}(F_e)
\]

with the map

\[
\rho_y,e: H_c^{[S]-1}(W_{S-w,(\max_{e \in E_y} \{s^e_Q\})}) \to H_c^{[S]-1}(W_{S-w,(s^e_Q)}).
\]

### 5.3 Maps between the cohomology of the fibers

For an edge \( e \subseteq X_w \), and a parabolic group \( R \in \mathcal{P} \), we let \( S_{e,R} \subseteq W_{S-w,(s^e_Q)} \) be the complex comprised of all cells \( F \subseteq W_{S-w,(s^e_Q)} \) such that there is a cell \( G \subseteq X_{S-w} \) containing \( F \) with \( \beta_{\gamma f \rho(S-w)}(G) \not\subseteq s^e_R \) where \( \gamma f \in \Lambda_R \). Thus we may informally think of the boundary of \( W_{S-w,(s^e_Q)} \) as \( \bigsqcup_{Q \in \mathcal{P}} S_{e,Q} \).

Let \( y \in X_w \) be a vertex, \( e \in E_y \), and \( R \in \mathcal{P} \). We define \( J_{y,e,R} \) to be the union of cells \( F \subseteq W_{S-w,(\max_{e \in E_y} \{s^e_Q\})} \) such that the maximum value of \( \beta_{\gamma f \rho(S-w)}(F) \) is greater than \( s^e_R \) for \( \gamma f \in \Lambda_R \). Notice that if \( J_{y,e,R} \not= \emptyset \), then \( s^e_R < s^e_R \) for some \( e \in E_y \), which, by Lemma 26, implies that \( e = e(y,R) \).

**Lemma 29**  If \( y \in X_w \), \( e \in E_y \), and \( R \in \mathcal{P} \), then \( H_c^{[S]-1}(J_{y,e,R}) = 0 \).

**Proof**  We may assume that \( J_{y,e,R} \not= \emptyset \), so that \( s^e_R < s^e_R \) for \( e \in E_y \). Then \( J_{y,e,R} \) is the complex of cells \( F \) such that the maximum value of \( \beta_{\gamma f \rho(S-w)}(F) \) is greater than \( s^e_R \) but bounded above by \( s^e_R \).

Let \( F \) be a cell as in the above paragraph of dimension \( |S| - 1 \) and assume that \( \beta_{\gamma f \rho(S-w)}(F) \) attains the minimal value for all such \( F \). Then we can retract \( F \) into \( \partial F \) along the direction of the geodesic \( \gamma f \rho(S-w) \). Repeat this process until \( J_{y,e,R} \) is retracted onto a complex of dimension \( |S|-2 \).  

*Geometry & Topology, Volume 22 (2018)*
We let $K_{y,e} \subseteq \mathcal{P}$ be the set of all $R \in \mathcal{P}$ such that $J_{y,e,R} \neq \emptyset$. If $e, \epsilon \in E_y$, and if $J_{y,e,R}$ and $J_{y,\epsilon,R}$ are each nonempty, then $e = e(y, R) = \epsilon$. Therefore, if $e$ and $\epsilon$ are distinct, we have $K_{y,e} \cap K_{y,\epsilon} = \emptyset$, so that if we let $K_y = \bigcup_{e \in E_y} K_{y,e} \subseteq \mathcal{P}$, then

$$K_y = \bigcup_{e \in E_y} K_{y,e}.$$

**Lemma 30** Given a vertex $y \in X_w$ and $e \in E_y$,

$$W_{S-w,(\max_{e \in E_y} \{s^e_Q\})} = W_{S-w,(s^e_Q)} \cup \left( \bigcup_{R \in K_{y,e}} J_{y,e(y,R),R} \right).$$

Furthermore,

$$W_{S-w,(s^e_Q)} \cap \left( \bigcup_{R \in K_{y,e}} J_{y,e(y,R),R} \right) = \bigcup_{R \in K_{y,e}} S_{e(y,R),R}.$$

**Proof** By definition, for all $R \in \mathcal{P}$, we have that $J_{y,e,R}$ and $W_{S-w,(s^e_Q)}$ are contained in $W_{S-w,(\max_{e \in E_y} \{s^e_Q\})}$.

If $\mathcal{F} \subseteq W_{S-w,(\max_{e \in E_y} \{s^e_Q\})}$ is a cell, and if $\mathcal{F}$ is not contained in $W_{S-w,(s^e_Q)}$, then the maximum value of $\beta_{\gamma f,\rho_{(S-w)}}(\mathcal{F})$ is greater than $s^e_R$ for some $R \in \mathcal{P}$ and $\gamma f \in \Lambda_R$, so that $\mathcal{F} \subseteq J_{y,e,R}$, which is to say that

$$W_{S-w,(\max_{e \in E_y} \{s^e_Q\})} \subseteq W_{S-w,(s^e_Q)} \cup \bigcup_{R \in \mathcal{P}} J_{y,e,R},$$

so we have equality. Furthermore, by the definition of $K_{y,e}$, and since $(s^e_Q)_{Q \in \mathcal{P}}$ is sufficiently large, we have

$$W_{S-w,(\max_{e \in E_y} \{s^e_Q\})} = W_{S-w,(s^e_Q)} \cup \bigcup_{R \in K_{y,e}} J_{y,e,R}.$$

Now suppose that there is a cell $\mathcal{F}$ contained in both $W_{S-w,(s^e_Q)}$ and $J_{y,e,R}$ for some $R \in K_{y,e}$. The latter inclusion implies that there is some $G \subseteq X_{S-w}$ such that the maximum value of $\beta_{\gamma f,\rho_{(S-w)}}(\mathcal{G})$ is greater than $s^e_R$ for $\gamma f \in \Lambda_R$. That is, $\mathcal{F} \subseteq S_{e,R}$.

To show the other inclusion, let $\mathcal{F} \subseteq W_{S-w,(s^e_Q)}$ be such that there is a cell $G \subseteq X_{S-w}$ containing $\mathcal{F}$ with $\beta_{\gamma f,\rho_{(S-w)}}(\mathcal{G}) \leq s^e_R$ for some $R \in K_{y,e}$ where $\gamma f \in \Lambda_R$. Then $\mathcal{F} \subseteq J_{y,e,R}$.

We also have the following lemma whose proof is similar.
Lemma 31  Given a vertex $y \in X_w$, 
\[ W_{S-w}(\max_{e \in E_y} \{ s'_Q \}) = W_{S-w}(s'^y_Q) \cup \left( \bigsqcup_{R \in K_y} \mathcal{J}_{y,e(y,R),R} \right). \]
Furthermore, 
\[ W_{S-w}(s'^y_Q) \cap \left( \bigsqcup_{R \in K_y} \mathcal{J}_{y,e(y,R),R} \right) = \bigsqcup_{R \in K_y} \mathcal{S}_{e(y,R),R}. \]

The Mayer–Vietoris sequence for the pair in Lemma 30 yields the coboundary homomorphism 
\[ \delta_{y,e} : \bigoplus_{R \in K_y,e} H^{|S|-2}_c(S_{e(y,R),R}) \to H^{|S|}_c(W_{S-w}(\max_{e \in E_y} \{ s'_Q \})). \]
Similarly, Lemma 31 yields 
\[ \delta_y : \bigoplus_{R \in K_y} H^{|S|-2}_c(S_{e(y,R),R}) \to H^{|S|}_c(W_{S-w}(\max_{e \in E_y} \{ s'_Q \})), \]
so that $\delta_y = \bigoplus_{e \in E_y} \delta_{y,e}$. 

Lemma 32  Suppose that $H^{|S|-2}_c(W_{S-w}(r_Q)) = 0$ for any sufficiently large sequence $(r_Q)_{Q \in \mathbb{P}}$. If $\sum_{R \in K_y} v_R \in \bigoplus_{R \in K_y} H^{|S|-2}_c(S_{e(y,R),R})$ and $\delta_y(\sum v_R) = 0$, then $\delta_y(v_R) = 0$ for all $R \in K_y$. 

Proof  Since $H^{|S|-2}_c(W_{S-w}(\max_{e \in E_y} s'_Q))$ and $H^{|S|-2}_c(W_{S-w}(s'^y_Q))$ are trivial by assumption, we have the following portion of the Mayer–Vietoris sequence for the pair from Lemma 31: 
\[ 0 \to \bigoplus_{R \in K_y} H^{|S|-2}_c(J_{y,e(y,R),R}) \to \bigoplus_{R \in K_y} H^{|S|-2}_c(S_{e(y,R),R}) \to H^{|S|}_c(W_{S-w}(\max_{e \in E_y} \{ s'_Q \})). \]
where $\delta_y$ is the rightmost map on the line above. Therefore, if $\delta_y(\sum v_R) = 0$, then $\sum v_R \in \bigoplus_{R \in K_y} H^{|S|-2}_c(J_{y,e(y,R),R})$, and in particular, for each $R$ we have $v_R \in H^{|S|-2}_c(J_{y,e(y,R),R})$, so that $\delta_y(v_R) = 0$ for each $R$. \hfill \square 

Lemma 33  Suppose that $H^{|S|-2}_c(W_{S-w}(r_Q)) = 0$ for any sufficiently large sequence $(r_Q)_{Q \in \mathbb{P}}$. Let $y \in X_w$ be a vertex, and suppose that $x \in H^{|S|-1}_c(F_y)$ is nonzero. Then there is at most one $e \in E_y$ such that $\rho_{y,e}(x) = 0$. 

Geometry & Topology, Volume 22 (2018)
We are now ready to prove the main result of this section.

where the map on the left is \( \delta_{y,e} \) and the map on the right is \( \rho_{y,e} \). Therefore, \( \rho_{y,e}(x) = 0 \) implies \( x = \delta_{y,e}(\sum v_R) = \delta_y(\sum v_R) \) for some \( \sum v_R \).

Now if \( \rho_{y,e}(x) = 0 \), then similarly, \( x = \delta_y(\sum w_R) \) for some \( \sum w_R \). Therefore, \( \delta_y(\sum (v_R - w_R)) = x - x = 0 \), which implies that \( \delta_y(v_R - w_R) = 0 \) for each \( R \) by the previous lemma.

Now fix some \( R \) with \( \delta_{y,e}(v_R) \neq 0 \). Then

\[ \delta_y(w_R) = \delta_y(v_R) = \delta_{y,e}(v_R) \neq 0, \]

from which we deduce that \( R \) is contained in \( K_{y,e} \) and \( K_{y,e} \). Thus, \( e = \epsilon \). \( \square \)

We are now ready to prove the main result of this section.

**Proposition 34** If \( (r_Q)_{Q \in \mathcal{P}} \) is sufficiently large, then \( H_c^k(W_{S,(r_Q)}) = 0 \) if \( k \neq |S| \) and thus \( H_c^k(X_{S,(r_Q)}) = 0 \) if \( k \neq |S| \).

**Proof** If \( |S| = 1 \), then we have proved this proposition in Lemma 23, so we assume the proposition is true for \( S - w \) and prove it is true for \( S \).

By Lemma 28, and since for any vertex \( y \in X_w \) we have \( F_y = \bigcup_{e \in E_y} F_e \), we see that the collection \( \{F_y\} \) taken over all vertices \( y \in X_w \) is a cover for \( W_{S,(r_Q)} \). Note also that if \( y \) and \( z \) are the endpoints of an edge \( e \subseteq X_w \), then \( F_e = F_y \cap F_z \). Thus, the nerve of \( \{F_y\} \) can be identified with \( X_w \), and there’s an associated spectral sequence with \( E_2^{pq} = H_c^p(X_w, \{H_c^q(F_*)\}) \) that converges to \( H_c^{p+q}(W_{S,(r_Q)}) \). (See eg [7, VII.4] for the analogous homology sequence. The derivation of the sequence we use here is a straightforward adaptation of that one.)

Since \( F_x = e \times W_{S-w,(s_Q^e)} \), our induction hypothesis implies \( H_c^q(F_x) = 0 \) for \( q \neq |S| - 1 \) and, together with \( X_w \) being 1–dimensional, that implies \( H_c^p(X_w, \{H_c^q(F_*)\}) = 0 \) if \( q \neq |S| - 1 \) or if \( p \geq 2 \). Thus, we will have \( H_c^k(W_{S,(r_Q)}) = 0 \) for \( k \neq |S| \) if \( H_c^0(X_w, \{H_c^{|S|-1}(F_*)\}) = 0 \), so we will verify that the kernel of the map

\[ d: \bigoplus_{y \in X_w^{(0)}} H_c^{|S|-1}(F_y) \to \bigoplus_{e \in X_w^{(1)}} H_c^{|S|-1}(F_e) \]

is trivial.
To do this, suppose \( \sum g_y \in \bigoplus H^{|S|-1}_c(F_y) \) is nonzero. Choose some vertex \( y \in X_w \) with \( g_y \neq 0 \) and such that \( y \) is contained in an edge \( e \) and in the component of \( X_w - e^\circ \) all of whose vertices \( y' \neq y \) have \( g_{y'} = 0 \).

By Lemma 33, there is an edge \( e \in E_y - e \) such that \( \rho_{y,e}(g_y) \neq 0 \). Therefore, the \( H^{|S|-1}_c(F_e) \) component of \( d(\sum g_y) \) is nonzero, and thus \( d(\sum g_y) \neq 0 \).

We have seen that \( H^p_c(X_w,\{H^q(F_*)\}) = 0 \) if \( (p,q) \neq (1,|S|-1) \).

If the sequence \((r_Q)_{Q \in \mathcal{P}}\) from Proposition 34 is constant, then we have the immediate:

**Corollary 35** If \( r \gg 0 \) and \( k \neq |S| \), then \( H^k_c(X_{S,r}) = 0 \).

The proof of Proposition 34 given above applies to horosphere complements in products of trees that are more general than those arising from arithmetic groups. In particular, suppose \( d \in \mathbb{N} \) and that \( T_i \), for \( 1 \leq i \leq d \), is a locally finite tree with no vertices of valence 1. Choose geodesics \( \Sigma_i: \mathbb{R} \to T_i \) parametrized such that the integer values of \( \Sigma_i \) are exactly the vertices in \( T_i \) in the image of \( \Sigma_i \). For any collection of positive numbers \( \lambda_i \), let \( \beta: \prod_{i=1}^d T_i \to \mathbb{R} \) be the Busemann function for \( \rho: \mathbb{R} \to \prod_{i=1}^d T_i \) given by \( \rho(t) = (\Sigma_i(\lambda_i t))_{i=1}^n \). Let \( Z = \beta^{-1}((-\infty, r]) \) for any given \( r \in \mathbb{R} \).

**Corollary 36** If \( k \neq d \), then \( H^k_c(Z) = 0 \).

**Proof** Apply Proposition 34 to a sequence \((r_Q)_{Q \in \mathcal{P}}\) that has exactly one finite value, and the result is the statement of this corollary. The only exception is that Proposition 34 applies to trees whose valences are dictated by an arithmetic group, and it applies to Busemann functions for rays whose slopes (the \( \lambda_i \)) are determined by an arithmetic group. But neither of these explicit data are used in the proof of Proposition 34.

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6 Topology of horospheres

Let \( X = \prod_{i=1}^d T_i \) where each \( T_i \) is a locally finite tree with no vertices of valence 1. Suppose each edge length in \( T_i \) equals 1. For each tree \( T_i \), choose a geodesic \( \Sigma_i \subseteq T_i \) and label its vertices \( x_{i,n} \) for \( n \in \mathbb{Z} \). This induces a height function \( h_i \) on the vertices of \( T_i \) where \( h_i(x_{i,0}) = 0 \) and \( h_i(v) = n - d(v, x_{i,n}) \) if the closest vertex of \( \Sigma_i \) is \( x_{i,n} \). Extend each \( h_i \) linearly over edges to produce a height function \( h_i \) defined on all of \( T_i \). For \( 1 \leq i \leq d \), we choose \( \lambda_i > 0 \) and we define a Busemann function \( \beta: X \to \mathbb{R} \) by \( \beta(x_1,\ldots,x_d) = \sum_i \lambda_i h_i(x_i) \).
Say that a vertex \( v \in T_i \) is below a vertex \( w \in T_i \) if there is a path \( \gamma \) from \( v \) to \( w \) such that \( h_i \circ \gamma \) is strictly increasing. In this case we say \( w \) is above \( v \). Note that for any \( x_i \in T_i \) and \( t > 0 \) there is a unique point \( y_i \in T_i \) above \( x_i \) such that \( h_i(y_i) = h_i(x_i) + t \). Using this notation, the assignment \( x_i \leftrightarrow y_i \) defines a flow \( \phi_{i,t} : T_i \to T_i \). These then define a flow on \( X \) by

\[
\phi_t(x_1, \ldots, x_d) = (\phi_{1,t}/(\lambda_1 \sqrt{d})(x_1), \ldots, \phi_{d,t}/(\lambda_d \sqrt{d})(x_d)).
\]

Note that \( \beta(\phi_t(x)) = \beta(x) + t \).

For \( r \in \mathbb{R} \), we define

\[
Y_r = \beta^{-1}(r), \quad X_r = \beta^{-1}(-\infty, r] \quad \text{and} \quad B_r = \beta^{-1}[r, \infty).
\]

The space \( X \) naturally has the structure of a cube complex. Subdivide this structure to give \( X \) the structure of a cell complex such that \( Y_r \) and \( X_r \) are subcomplexes. In particular, for each \((d-1)\)-cell \( e \) of \( Y_r \) there is a unique \( d \)-cell \( \hat{e} \) of \( X \) lying above \( e \) such that \( e^\circ \cap \hat{e} \neq \emptyset \).

In this section, all cohomology groups will be understood to have coefficients in some ring \( R \).

### 6.1 Horoball cohomology

**Lemma 37** \( H^k_c(B_r) = 0 \) for all \( r \) and \( k \).

**Proof** For any number \( m > n \) let

\[
C(m) = \{ x = (x_1, \ldots, x_d) \in X : \beta(x) \geq r \text{ and } x_i \text{ lies below } x_{i,m} \}
\]

and let

\[
\partial^\dagger C(m) = \{ x \in C(m) : h_i(x) = m \text{ for some } i \}.
\]

Note that \( C(m) \) deformation retracts onto \( \partial^\dagger C(m) \). Thus \( H^k_c(C(m), \partial^\dagger C(m)) = 0 \).

Note that the sets \( C(m) \) form an exhaustion of \( B_r \) by compact sets. Let \( c(m) \) be the closed subset of \( C(m) \) consisting of points whose distance from \( \partial^\dagger C(m) \) is at least \( \varepsilon \), for some small \( \varepsilon > 0 \). The compact sets \( c(m) \) also form an exhaustion of \( B_r \), so it suffices to show \( H^k_c(B_r, B_r - c(m)) = 0 \) for all \( m \). By excision we have

\[
H^k_c(B_r, B_r - c(m)) \cong H^k_c(C(m), C(m) - c(m)).
\]

Because \( C(m) - c(m) \) deformation retracts onto \( \partial^\dagger C(m) \), we have

\[
H^k_c(C(m), C(m) - c(m)) \cong H^k_c(C(m), \partial^\dagger C(m)) = 0.
\]

\[\Box\]
6.2 Horosphere cohomology

For \( n \in \mathbb{N} \), the collection \( \{H^d_c(Y_n)\}_{n \in \mathbb{N}} \) forms a directed system under the maps \((\phi_1)^*: H^d_c(Y_{n+1}) \to H^d_c(Y_n)\), where \( \phi_1 \) is the time-1 flow on \( X \). The goal of this section is to show \( \lim_n H^d_c(Y_n) \) is trivial and \( \lim_n H^d_c(Y_n) \) is torsion-free.

There is a Mayer–Vietoris exact sequence

\[
\cdots \to H^d_c(X_n) \oplus H^d_c(B_n) \to H^d_c(Y_n) \to H^d_c(X) \to \cdots.
\]

We know \( H^d_c(X_n) = 0 \) by Corollary 36 and \( H^d_c(B_n) = 0 \) by Lemma 37. Therefore the connecting map \( H^d_c(Y_n) \to H^d_c(X) \) is injective. In this way we consider each module \( H^d_c(Y_n) \) as a submodule of \( H^d_c(X) \). Note that \((\phi_1)^*: H^d_c(X) \to H^d_c(X)\) is the identity map. It therefore follows from naturality of the Mayer–Vietoris sequence that \((\phi_1)^*: H^d_c(Y_{n+1}) \to H^d_c(Y_n)\) is the inclusion map of subgroups of \( H^d_c(X) \).

We will prove \( \lim_n H^d_c(Y_n) = 0 \), for which we set up notation. For any vertex \( v \in T_i \) let \( g_i(v) \) be the unique vertex above \( v \) such that \( h_i(g_i(v)) = h_i(v) + 1 \). Under the identification of \( X^{(0)} \) with \( \prod_i T_i^{(0)} \), let \( g: X^{(0)} \to X^{(0)} \) be the function defined to be \( g_i \) in the \( i \)-th coordinate, so that \( g(w)(i) = g_i(w(i)) \) for any \( w \in \prod_i T_i^{(0)} \).

Given \( i \) and \( n \), let \( C_{i,n} \subseteq T_i \) be the subtree of \( T_i \) spanned by the set of vertices

\[ \{v \in T_i : v \text{ is below } x_{i,n} \text{ and } h_i(v) \geq -n\} \]

Let \( K_n = \prod_{i=1}^d C_{i,n} \). The collection \( \{K_n\}_{n=0}^\infty \) forms an exhaustion of \( X \) by compact sets, and so \( H^k_c(X) = \lim_n H^k_c(X, X \setminus K_n) \).

We compute each relative cohomology group \( H^k_c(X, X \setminus K_n) \) as the cohomology of the quotient space \( X/(X \setminus K_n) \). Let \( EC_{i,n} \) be the set of vertices \( v \in C_{i,n} \) such that \( h_i(v) = -n \). Each set \( C_{i,n} \) is homotopy equivalent relative \( EC_{i,n} \cup \{x_{i,n}\} \) to a star with \#(\( EC_{i,n} \)) leaves, and so if \( \partial K_n \) is comprised of points in \( K_n \) whose \( i \)-th coordinate for some \( i \) is contained in \( EC_{i,n} \cup \{x_{i,n}\} \), then \( K_n \) is homotopy equivalent relative \( \partial K_n \) to a cube complex with a top-dimensional cube for each vertex in \( \prod_{i=1}^d EC_{i,n} \). It follows that there is a homotopy equivalence

\[ X/(X \setminus K_n) \simeq \bigvee_{v \in \prod_{i=1}^d EC_{i,n}} S^d. \]

To simplify notation, let \( \Lambda_n \) be the set of vertices in \( \prod_{i=1}^d EC_{i,n} \). With this notation, it follows that there is an isomorphism

\[
H^k_c(X, X \setminus K_n) \cong \begin{cases} R^{\Lambda_n} & \text{if } k = d, \\ 0 & \text{otherwise}. \end{cases}
\]
Under this identification, the map $f_n: R^\Lambda_n \to R^\Lambda_{n+1}$ induced by the map of pairs $(X, X \setminus K_{n+1}) \to (X, X \setminus K_n)$ is described as follows: given a function $\alpha: \Lambda_n \to R$, define $f_n(\alpha)(w) = \alpha(g(w))$ if $g(w) \in \Lambda_n$ and $f_n(\alpha)(w) = 0$ otherwise.

Given a vertex $v \in \Lambda_n$, let $x_{i,n}, v(i)$ denote the geodesic segment between $x_{i,n}$ and $v(i)$, and let $F_v$ be the cube $\prod_{i=1}^d x_{i,n}, v(i)$. Note there is an equality of spaces $K_n = \bigcup_{v \in \Lambda_n} F_v$. Given a compactly supported cellular cochain $\phi \in Z^d_c(X)$, the assignment $[\phi] \mapsto (v \mapsto \phi(F_v))_n$ gives the isomorphism $H^d_c(X) \cong \lim_{\to n} R^\Lambda_n$.

Note that the above is a proof of this well-known result:

**Proposition 38** $H^k_c(X) = 0$ if $k \neq d$ and $H^d_c(X)$ is a free $R$–module.

See Borel and Serre [6] for a more general theorem about the compactly supported cohomology of euclidean buildings.

We now observe that $H^{*}_c(Y_r)$ is concentrated in dimension $d - 1$.

**Proposition 39** $H^k_c(Y_r) = 0$ if $k \neq d - 1$ and $H^{d-1}_c(Y_r)$ is a free $R$–module.

**Proof** There is a Mayer–Vietoris exact sequence

$$\cdots \to H^k_c(X_r) \oplus H^k_c(B_r) \to H^k_c(Y_r) \to H^{k+1}_c(X) \to \cdots.$$ 

By Corollary 36 we know $H^k_c(X_r) = 0$ for $k \leq d - 1$. By Lemma 37 we know $H^k_c(B_r) = 0$ for all $k$. And by Proposition 38 we know $H^{k+1}_c(X) = 0$ for $k \leq d - 2$. The result follows.

Using the notation that we established prior to Proposition 38, we will prove:

**Lemma 40** $\lim_{\to n} H^{d-1}_c(Y_n) = 0$.

**Proof** Take any cohomology class $[\phi] \in H^d_c(X)$. Choose $n \in \mathbb{N}$ so that the support of $\phi$ is contained in $K_n$.

Consider any vertex $v \in \Lambda_n$, and choose $m$ such that $m > \beta(K_{n+1})$. Suppose $[\phi] = [\delta \psi]$ for some $\psi \in H^{d-1}_c(Y_m)$, where $\delta$ is the chain map inducing the connecting homomorphism in the Mayer–Vietoris sequence. Choose $N > n + 1$ such that the support of $\psi$, and hence also the support of $\delta \psi$, is contained in $K_N$. Since $[\phi]$ and $[\delta \psi]$ are equal in $H^d_c(X)$, their images in $R^\Lambda_N$ are equal.

Choose any $w \in \Lambda_N$ so that $g^{N-n}(w) = v$. Then $F_v \subseteq F_w$. Since the support of $\phi$ is contained in $K_n$ and $F_v = F_w \cap K_n$, we have $\phi(F_v) = \phi(F_w)$.
For each $1 \leq i \leq d$ choose a vertex $e_i \in EC_{i,N}$ such that $g^N_{i-n-1}(e_i) \in EC_{i,n+1}$ but $g^N_{i-n} \notin EC_{i,n}$. Let $P(d)$ be the set of subsets of $\{1, \ldots, d\}$. For each $\sigma \in P(d)$ define $w_\sigma \in \Lambda_n$ by

$$w_\sigma(i) = \begin{cases} w(i) & \text{if } i \notin \sigma, \\ e_i & \text{if } i \in \sigma. \end{cases}$$

Note that $w_\emptyset = w$. If $\sigma \neq \emptyset$, then $F_{w_\sigma} \cap K_1 = \emptyset$ and so $\phi(F_{w_\sigma}) = 0$. Because $[\phi]$ and $[\delta \psi]$ have the same image in $R^{\Lambda_n}$ we see $\phi(F_{w_\sigma}) = \delta \psi(F_{w_\sigma})$ for all $\sigma \in P(d)$.

We claim the $(d-1)$–chain $\sum_{\sigma \in P(d)} (-1)^{\lvert \sigma \rvert} (F_{w_\sigma} \cap Y_m)$ is the zero chain. Indeed, for $1 \leq i \leq d$, let $u_i$ be an order-2 isometry of $T_i$ with $u_i(w(i)) = e_i$ and $u_i(x_i) = x_i$ if $h_i(x_i) \geq n + 1$. For $\sigma \in P(d)$ we let $u_\sigma$ be the product of $u_i$ with $i \in \sigma$. In particular, $u_\emptyset = 1$. Notice that $u_\sigma F_{w_\sigma} = F_{w_\sigma}$, and that $u_\sigma Y_m = Y_m$.

We let $P(d)^* = P(d) - \{1, \ldots, d\}$, and for each $\tau \in P(d)^*$, we define $R_\tau = \{(x_i) \in F_w \cap Y_m : h_i(x_i) \leq n + 1 \text{ precisely when } i \in \tau\}$.

Thus, $\bigcup_{\tau \in P(d)^*} R_\tau = F_w \cap Y_m$; if $\tau_1 \neq \tau_2$ then $R_{\tau_1}$ and $R_{\tau_2}$ do not contain a common $(d-1)$–cell, and if $\sigma \in P(d)$ then $u_\sigma R_\tau = u_\sigma \cap \tau R_\tau$.

Recall that by the binomial theorem, if $k \in \mathbb{N}$, then $\sum_{\mu \in P(k)} (-1)^{\lvert \mu \rvert} = 0$. Therefore,

$$\sum_{\sigma \in P(d)} (-1)^{\lvert \sigma \rvert} (F_{w_\sigma} \cap Y_m) = \sum_{\sigma \in P(d)} (-1)^{\lvert \sigma \rvert} (u_\sigma F_w \cap Y_m) = \sum_{\sigma \in P(d)} (-1)^{\lvert \sigma \rvert} u_\sigma (F_w \cap Y_m) = \sum_{\tau \in P(d)^*} \sum_{\sigma \in P(d)} (-1)^{\lvert \sigma \rvert} u_\sigma (F_w \cap Y_m) = \sum_{\tau \in P(d)^*} \sum_{\sigma \in P(d)} (-1)^{\lvert \sigma \rvert} u_\sigma R_\tau = \sum_{\tau \in P(d)^*} \sum_{\rho \in P(\lvert \tau \rvert)} (u_\rho) R_\tau = 0.$$ 

This establishes our claim that $\sum_{\sigma \in P(d)} (-1)^{\lvert \sigma \rvert} (F_{w_\sigma} \cap Y_m)$ is the zero chain.
Using the definition of the connecting map \( \delta \) we therefore have

\[
0 = \psi \left( \sum_{\sigma \in P(d)} (-1)^{|\sigma|} F_{w_\sigma} \cap Y_{\sigma} \right)
= \sum_{\sigma \in P(d)} (-1)^{|\sigma|} \delta \psi (F_{w_\sigma})
= \sum_{\sigma \in P(d)} (-1)^{|\sigma|} \phi (F_{w_\sigma})
= \phi (F_w)
= \phi (F_\nu).
\]

This shows that if \([\phi] \in H_c^d(X)\) and \([\phi]\) is contained in \(H_c^{d-1}(Y_m) \subseteq H_c^d(X)\) for some sufficiently large value of \(m\), then \([\phi] = 0\). That is, \(\lim n^{-1} H_c^{d-1}(Y_n) = \cap_m H_c^{d-1}(Y_m) = 0\).

Our next goal is to prove that \(\lim n^{-1} H_c^{d-1}(Y_m)\) is torsion-free.

**Lemma 41** Suppose \([\psi] \in H_c^{d-1}(Y_m)\) and suppose there are \([\phi] \in H_c^d(X)\) and \(r \in R\) such that \(r[\phi] = [\delta \psi]\). Then there is some \([\tilde{\phi}] \in H_c^{d-1}(Y_m)\) such that \(r[\tilde{\phi}] = [\psi]\).

**Proof** Let \(\hat{Y}_m\) be the subcomplex of \(X\) containing only the (subdivided) \(d\)–cells \(e \subseteq X\) such that \(\beta(e) \geq m\) and \(e \cap Y_m \neq \emptyset\). Let \(i: \hat{Y}_m \to X\) be the inclusion map. The Mayer–Vietoris connecting map induces a homomorphism \(\hat{\delta}: H_c^{d-1}(Y_m) \to H_c^d(\hat{Y}_m)\) such that the following diagram commutes:

\[
\begin{array}{ccc}
H_c^{d-1}(Y_m) & \xrightarrow{\delta} & H_c^d(\hat{Y}_m) \\
\downarrow \phi & & \downarrow \phi \\
H_c^d(X) & \xrightarrow{i^*} & H_c^d(\hat{Y}_m)
\end{array}
\]

Given any \((d-1)\)–cell \(c \subseteq Y_m\), let \(\hat{c} \subseteq \hat{Y}_m\) be the unique \(d\)–cell in \(\hat{Y}_m\) with \(c \subseteq \hat{c}\).

If \([\phi] \in H_c^d(\hat{Y}_m)\), then define \([\epsilon \phi] \in H_c^{d-1}(Y_m)\) by \(\epsilon \phi (c) = \phi (\hat{c})\). Then \(\epsilon: H_c^d(\hat{Y}_m) \to H_c^{d-1}(Y_m)\) is the inverse of \(\hat{\delta}\), so there is an isomorphism \(H_c^{d-1}(Y_m) \cong H_c^d(\hat{Y}_m)\). The lemma follows by setting \([\tilde{\phi}] = \epsilon i^*([\phi])\).

**Lemma 42** \(\lim n^{-1} H_c^{d-1}(Y_m)\) is torsion-free as an \(R\)–module.
Proof  Recall that \( \lim^1 H_{c}^{d-1}(Y_m) \) is the cokernel of the map \[
\Delta: \prod_m H_{c}^{d-1}(Y_m) \to \prod_m H_{c}^{d-1}(Y_m),
\]
where \( \Delta(([\psi_m])_m) = ([\psi_m] - [\psi_{m+1}])_m. \)

Suppose \( ([\xi_m])_m \in \prod_m H_{c}^{d-1}(Y_m) \) and that there is some regular element \( r \in R \) such that \( (r[\xi_m])_m \) is in the image of \( \Delta \). Let \( ([\psi_m])_m \in \prod_m H_{c}^{d-1}(Y_m) \) be a sequence such that \( r[\xi_m] = [\psi_m] - [\psi_{m+1}] \) for all \( m \). Note that this implies that for any \( M > m \) there is some \( \zeta_{m,M} \in H_{c}^{d-1}(Y_m) \) such that \( r[\zeta_{m,M}] = [\psi_m] - [\psi_M] \).

Fix any \( m \) and choose \( n \in \mathbb{N} \) such that \( \psi_m \) is supported on \( K_n \). Take any \( v \in \Lambda_n \). Choose any \( M > m \) such that \( M > \beta(K_n) \). Choose \( N \in \mathbb{N} \) such that \( \psi_M \) is supported on \( K_N \).

Choose any \( w \in \Lambda_N \) such that \( g^{N-n}(w) = v \). Construct vertices \( w_\sigma \in \Lambda_N \) for each \( \sigma \in P(d) \) as in the proof of Lemma 40, so that \( F_{w_\sigma} = F_v \) and \( \psi_m(F_{w_\sigma}) = 0 \) if \( \sigma \neq \emptyset \) and \( \sum_{\sigma \in P(d)} (-1)^{|\sigma|} F_{w_\sigma} \cap Y_M \) is the zero chain. Then

\[
\psi_M \left( \sum_{\sigma \in P(d)} (-1)^{|\sigma|} F_{w_\sigma} \cap Y_M \right) = 0,
\]

and so

\[
\delta \psi_M \left( \sum_{\sigma \in P(d)} (-1)^{|\sigma|} F_{w_\sigma} \right) = 0.
\]

It follows that \( r \) divides the quantity

\[
(\delta \psi_m - \delta \psi_M) \left( \sum_{\sigma \in P(d)} (-1)^{|\sigma|} F_{w_\sigma} \right) = \delta \psi_m \left( \sum_{\sigma \in P(d)} (-1)^{|\sigma|} F_{w_\sigma} \right) - \delta \psi_M \left( \sum_{\sigma \in P(d)} (-1)^{|\sigma|} F_{w_\sigma} \right)
\]

\[
= \delta \psi_m (F_{w_\emptyset})
\]

\[
= \psi_m (F_v \cap Y_M).
\]

This holds for any vertex \( v \in \Lambda_n \), so \( r \) divides the image of \( [\psi_m] \) in \( R^{\Lambda_n} \). Therefore, \( r \) divides the image of \( [\psi_m] \) in \( H_{c}^{d}(X) \). By Lemma 41, for each \( m \) there is some \( [\phi_m] \in H_{c}^{d-1}(Y_m) \) such that \( r[\phi_m] = [\psi_m] \). It follows that \( ([\xi_m])_m = \Delta([\phi_m]) \), which completes the proof. \( \square \)

Geometry & Topology, Volume 22 (2018)
7 Examples of semiduality groups

Below we provide examples of semiduality groups. In order to verify the condition on the cohomological dimension, we recall the following standard result.

**Lemma 43** Suppose $\Lambda$ is a group acting on an acyclic cell complex $X$ with finite cell stabilizers. Suppose $R$ is a commutative ring such that $|\Lambda_\sigma|$ is invertible in $R$ for any cell stabilizer $\Lambda_\sigma$. Then $\text{cd}_R \Lambda \leq \dim(X)$.

**Proof** Suppose $M$ is an $R\Lambda$–module. For each $j$ let $\Sigma_j$ be a set of representatives of $\Lambda$–orbits of $j$–cells of $X$. There is a spectral sequence (compare to the homology version appearing in Brown [7, VII.7.7, page 173])

$$E^{j,q}_1 = \prod_{\sigma \in \Sigma_j} H^q(\Lambda_\sigma; M_\sigma) \Rightarrow H^{j+q}(\Lambda; M).$$

For any $q > 0$ the module $H^q(\Lambda_\sigma, M_\sigma)$ is annihilated by $|\Lambda_\sigma|$. But $M_\sigma$ is an $R$–module and $|\Lambda_\sigma|$ is invertible in $R$ for any cell $\sigma$, so the groups $H^q(\Lambda_\sigma; M_\sigma)$ are trivial if $q > 0$. By definition, the groups $E^{j,q}_1$ are trivial for $j > \dim(X)$. It follows that $\text{cd}_R \Lambda \leq \dim(X)$.

7.1 Rank-one arithmetic groups

In this section we prove Theorem 3. To that end, suppose $\mathcal{O}_S$ is the ring of $S$–integers in a global function field $K$ of characteristic $p$. Suppose $G$ is a noncommutative, absolutely almost simple algebraic $K$–group and $\Gamma$ is a finite-index subgroup of $G(\mathcal{O}_S)$ such that any torsion element of $\Gamma$ is a $p$–element.

**Proposition 44** $\Gamma$ is a $\mathbb{Z}[1/p]$–semiduality group of dimension $k(G, S)$.

**Proof** $\Gamma$ acts on the product of trees $X_S$ with finite $p$–group stabilizers. It follows from Lemma 43 that $\text{cd}_{\mathbb{Z}[1/p]} \Gamma \leq \dim(X_S) = k(G, S)$. It is known that $\Gamma$ is type $FP_{k(G,S)-1}$ over $\mathbb{Z}[1/p]$; see Stuhler [16], Bux and Wortman [12], and Bux, Köhl and Witzel [10]. It remains to show that $H^*(\Gamma, \mathbb{Z}[1/p] \Gamma)$ is concentrated in dimension $k(G, S)$, where it is flat as a $\mathbb{Z}[1/p]$–module.

For sufficiently large $n \in \mathbb{N}$, we have by Lemma 20 that $\Gamma$ acts properly cocompactly on each $X_{S,n}$. As $n$ tends to infinity, the spaces $X_{S,n}$ exhaust $X_S$. By Proposition 12 there is an isomorphism $H^k_{\text{cc}}(X_S) \cong H^k(\Gamma, \mathbb{Z}[1/p] \Gamma)$, where here and for the rest of the proof we take cohomology of spaces with $\mathbb{Z}[1/p]$ coefficients. We proved in Proposition 34 that $H^r_c(X_{S,n})$ is concentrated in dimension $d$. It
follows from Proposition 14 that $H^*_{cc}(X_S)$ is concentrated in dimension $d$, where $H^d_{cc}(X_S) = \lim_n H^d_c(X_{S,n})$.

It remains only to show that $\lim_n H^d_c(X_{S,n})$ is $\mathbb{Z}[1/p]$–torsion-free, since $\mathbb{Z}[1/p]$ is a principal ideal domain. The closure of the complement of $X_{S,n}$ is a disjoint union of horoballs $\bigcup_{Q \in P} B_{Q,S,n}$. Up to proper homotopy, each set $B_{Q,S,n}$ is a set of the form $B_n$ as defined in Section 6, and $X_{S,n} \cap B_n = Y_n$, where $Y_n = Y_{Q,S,n}$. There is a Mayer–Vietoris exact sequence

$$H^{d-1}_c(X_{S,n}) \oplus H^{d-1}_c \left( \bigcup_p B_n \right) \to H^{d-1}_c \left( \bigcup_p Y_n \right) \to H^d_c(X_S) \to H^d_c(X_{S,n}) \oplus H^d_c \left( \bigcup_p B_n \right) \to H^d_c \left( \bigcup_p Y_n \right).$$

Because the unions are disjoint, for each $k$ there are isomorphisms

$$H^k_c \left( \bigcup_p B_n \right) \cong \bigoplus_p H^k_c(B_n) \quad \text{and} \quad H^k_c \left( \bigcup_p Y_n \right) \cong \bigoplus_p H^k_c(Y_n).$$

We know $H^k_c(B_n) = 0$ for all $k$ by Lemma 37. We also know $H^{d-1}_c(X_{S,n}) = 0$ by Proposition 34. Clearly $H^d_c(Y_n) = 0$ since $Y_n$ is $(d-1)$–dimensional. Therefore, we have a short exact sequence

$$0 \to \bigoplus_p H^{d-1}_c(Y_n) \to H^d_c(X_S) \to H^d_c(X_{S,n}) \to 0.$$

These maps are compatible with the maps induced by inclusion $i_n: X_{S,n} \to X_{S,n+1}$, the time-1 flow $\phi_1: Y_n \to Y_{n+1}$, and the identity map $X_S \to X_S$. The above short exact sequence therefore gives rise to a short exact sequence of codirected systems of compactly support cohomology, from which there is an exact sequence

$$0 \to \lim_p \bigoplus H^{d-1}_c(Y_n) \to H^d_c(X_S) \to \lim_p H^d_c(X_{S,n}) \to \lim^1_p H^{d-1}_c(Y_n) \to 0.$$

The maps of the system $\{ \bigoplus H^{d-1}_c(Y_n) \}$ preserve the direct sum structure. We know $\lim_p H^{d-1}_c(Y_n)$ is trivial by Lemma 40 and $\lim^1_p H^{d-1}_c(Y_n)$ is torsion-free by Lemma 42. Since $H^d_c(X_S)$ is torsion-free by Proposition 38, it follows that $\lim_p H^d_c(X_{S,n})$ is torsion-free by the short exact sequence

$$0 \to H^d_c(X_S) \to \lim_p H^d_c(X_{S,n}) \to \bigoplus_p \lim^1_p H^{d-1}_c(Y_n) \to 0.$$

This proves Theorem 3, as every module in the above sequence is a $\mathbb{Z}[1/p]G(K)$–module by Lemma 13.
7.2 Solvable groups

In this section we prove groups of the form $B_2(\mathcal{O}_S)$ are semiduality groups. We then prove that some generalizations of certain groups of this form are also semiduality groups, namely lamplighter groups, Diestel–Leader groups, and countable direct sums of finite groups. All are straightforward applications of the following lemma.

**Lemma 45** Let $X$ be a product of $d$ trees with Busemann function $\beta : X \to \mathbb{R}$ as described in Section 6. Suppose a group $\Lambda$ acts on $X$ cellularly, with finite cell stabilizers, and cocompactly on subsets of the form $\beta^{-1}(I)$ for closed intervals $I$. Suppose $R$ is a principal ideal domain such that $|\Lambda_\sigma|$ is invertible for every cell stabilizer $\Lambda_\sigma$. Then $\Lambda$ is an $R$–semiduality group of dimension $d$.

**Proof** Define

$$Y_n = \beta^{-1}(\{n\}), \quad X_n = \beta^{-1}[0, n] \quad \text{and} \quad B_n = \beta^{-1}[n, \infty).$$

The space $X$ is contractible, so by Lemma 43 we know $\text{cd}_R \Lambda \leq \dim(X) = d$. Since $\Lambda$ acts cocompactly with finite stabilizers on a horosphere $Y_n$ and $\tilde{H}_k(Y_n) = 0$ for $k < n - 1$ by a result of Bux [9, 3.1], Brown’s criterion implies that $\Lambda$ is type $FP_{d-1}$; see for example Brown [8, 1.1].

The complexes $X_n$ form an exhaustion of $B_0$ by closed, $\Gamma$–invariant sets such that $\Gamma \backslash X_n$ is compact. Therefore, by the results of Section 3 there is an isomorphism $H^*(\Lambda; RA) \cong H^k_{cc}(B_0)$ and for each $k$ there is an exact sequence

$$0 \to \lim^1 H^k_{cc}(X_n) \to H^k_{cc}(B_0) \to \lim H^k_{cc}(X_n) \to 0.$$

Note that the flow $\phi_t$ provides a proper deformation retraction of $X_n$ to $Y_n$ so $H^*_c(X_n) \cong H^*_c(Y_n)$. We know $H^*_c(Y_n)$ is concentrated in dimension $d - 1$ by Proposition 39. Now Lemma 40 says $\lim H^d_{cc}(Y_n) = 0$ so $H^d_{cc}(B_0)$ is concentrated in dimension $d$. In that dimension there is an isomorphism $H^d_{cc}(B_0) \cong \lim H^d_{cc}(Y_n)$, which is torsion-free by Lemma 42 and hence flat as an $R$–module.

Suppose $\mathcal{O}_S$ is the ring of $S$–integers in a global function field $K$ of characteristic $p$. Let $B_2$ be the group of upper triangular matrices of determinant 1.

**Theorem 46** Suppose $\Gamma$ is a finite-index subgroup of $B_2(\mathcal{O}_S)$ such that the order of every finite-order element is a power of $p$. Then $\Gamma$ is a $\mathbb{Z}[1/p]$–semiduality group of dimension $|S|$.
In the notation of Section 4, we may choose \( P = B_2 \). Then applying Lemma 18 with \( D_1 \) and \( f \), we see that \( P \) acts on the horoball \( B_{P, S, n} \) for all sufficiently large \( n \in \mathbb{N} \). In fact \( P(O_S) \) is the entire stabilizer of \( B_{P, S, n} \) in \( SL_2(O_S) \) since if \( y \in SL_2(O_S) \) and \( yB_n = B_n \) then \( B_{P, S, n} = B_{yP, y^{-1}, S, n} \), which by Proposition 19(ii) means \( P = yP^{-1} \) and so \( y \in P \). It follows that the action of \( P \) on \( Y_{P, S, n} \) is proper and cocompact since the action of \( SL_2(O_S) \) is proper and cocompact on \( X_S \). In particular, cell stabilizers are finite.

Suppose \( F \) is a finite group. The lamplighter group with base group \( F \) is \( \Gamma_F = F \rtimes \mathbb{Z} = (\bigoplus_{i \in \mathbb{Z}} F) \rtimes \mathbb{Z} \), where \( \mathbb{Z} \) acts by shifting the indices of a sequence \((f_i)\).

**Theorem 47** The lamplighter group with base group \( F \) is a \( \mathbb{Z}[1/|F|] \)--semiduality group of dimension 2.

**Proof** Let \( T_1 \) and \( T_2 \) be copies of a \((|F|+1)--\)regular tree. The lamplighter group \( \Gamma_F \) acts on \( T_1 \times T_2 \) in a natural way; for a description of the action see Wortman [18, Section 4]. This action preserves a Busemann function \( \beta \) and is cocompact on any set of the form \( \beta^{-1}(I) \) for closed intervals \( I \subset \mathbb{R} \). Stabilizers of cells are finite sums of copies of \( F \). Therefore Lemma 45 applies.

There are “higher-rank” generalizations of lamplighter groups known as Diestel–Leader groups \( \Gamma_d(q) \) which act on a product of \( d \) regular trees of valence \( q+1 \). These are constructed by Bartholdi, Neuhauser and Woess [1] for any values of \( d \) and \( q \) such that \( d \leq p+1 \) for any prime \( p \) dividing \( q \); a lamplighter group with base group \( F \) is an example of \( \Gamma_2(|F|) \). The proof of Theorem 47 easily generalizes to prove:

**Theorem 48** A Diestel–Leader group \( \Gamma_d(q) \) is a \( \mathbb{Z}[1/q] \)--semiduality group of dimension \( d \).

As a final remark, consider a countable collection of finite groups \( \{F_i\}_{i \in \mathbb{N}} \) and let \( \Lambda = \bigoplus_{i \in \mathbb{N}} F_i \). (This is not necessarily solvable.) Then \( \Lambda \) is an \( R \)--semiduality group of dimension 1 for any principal ideal domain \( R \) in which \( |F_i| \) is invertible for every \( i \). (So, for example, any countable sum of finite groups is a \( \mathbb{Q} \)--semiduality group.) To see
this, let $\Lambda_n = \bigoplus_{i=0}^n F_i$. Form a graph of groups with underlying graph a simplicial ray whose $n$th vertex and proceeding edge are labeled by $\Lambda_n$, with inclusion maps from edge groups to incident vertex groups. Then $\Lambda$ is the fundamental group of this graph of groups. It acts on the Bass–Serre tree preserving a height function inherited from the base ray, and is cocompact on preimages of closed intervals. Cell stabilizers are isomorphic to some $\Lambda_n$, so Lemma 45 produces the desired result.

References

[1] L Bartholdi, M Neuhauser, W Woess, Horocyclic products of trees, J. Eur. Math. Soc. 10 (2008) 771–816 MR
[2] H Behr, Endliche Erzeugbarkeit arithmetischer Gruppen über Funktionenkörpern, Invent. Math. 7 (1969) 1–32 MR
[3] M Bestvina, A Eskin, K Wortman, Filling boundaries of coarse manifolds in semisimple and solvable arithmetic groups, J. Eur. Math. Soc. 15 (2013) 2165–2195 MR
[4] R Bieri, B Eckmann, Finiteness properties of duality groups, Comment. Math. Helv. 49 (1974) 74–83 MR
[5] A Borel, J-P Serre, Corners and arithmetic groups, Comment. Math. Helv. 48 (1973) 436–491 MR
[6] A Borel, J-P Serre, Cohomologie d’immeubles et de groupes $S$–arithmétiques, Topology 15 (1976) 211–232 MR
[7] K S Brown, Cohomology of groups, Graduate Texts in Mathematics 87, Springer (1982) MR
[8] K S Brown, Finiteness properties of groups, J. Pure Appl. Algebra 44 (1987) 45–75 MR
[9] K-U Bux, Finiteness properties of soluble arithmetic groups over global function fields, Geom. Topol. 8 (2004) 611–644 MR
[10] K-U Bux, R Köhl, S Witzel, Higher finiteness properties of reductive arithmetic groups in positive characteristic: the rank theorem, Ann. of Math. 177 (2013) 311–366 MR
[11] K-U Bux, K Wortman, Finiteness properties of arithmetic groups over function fields, Invent. Math. 167 (2007) 355–378 MR
[12] K-U Bux, K Wortman, Connectivity properties of horospheres in Euclidean buildings and applications to finiteness properties of discrete groups, Invent. Math. 185 (2011) 395–419 MR
[13] G Gandini, Bounding the homological finiteness length, Bull. Lond. Math. Soc. 44 (2012) 1209–1214 MR

Geometry & Topology, Volume 22 (2018)
[14] **G Harder**, *Minkowskische Reduktionstheorie über Funktionenkörpern*, Invent. Math. 7 (1969) 33–54 MR

[15] **P H Kropholler**, *On groups of type (FP)_∞*, J. Pure Appl. Algebra 90 (1993) 55–67 MR

[16] **U Stuhler**, *Homological properties of certain arithmetic groups in the function field case*, Invent. Math. 57 (1980) 263–281 MR

[17] **C A Weibel**, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics 38, Cambridge Univ. Press (1994) MR

[18] **K Wortman**, *A finitely presented solvable group with a small quasi-isometry group*, Michigan Math. J. 55 (2007) 3–24 MR

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