Energy dissipation admissibility condition for conservation law systems admitting singular solutions

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Abstract. The main goal of the paper is to define and use a condition sufficient to choose a unique solution to conservation law systems with a singular measure in initial data. Different approximations can lead to solutions with different distributional limits. The new notion called backward energy condition is then to single out a proper approximation of the distributional initial data. The definition is based on the maximal energy dissipation defined in Dafermos (J Differ Equ 14:202–212, 1973). Suppose that a conservation law system admits a supplementary law in space–time divergent form where the time component is a (strictly or not) convex function. It could be an energy density or a mathematical entropy in gas dynamic models, for example. One of the admissibility conditions is that a proper weak solution should maximally dissipate the energy or the mathematical entropy. We show that it is consistent with other admissibility conditions in the case of Riemann problems for systems of isentropic gas dynamics with non-positive pressure in the first part of the paper. Singular solutions to these systems are described by shadow waves, nets of piecewise constant approximations with respect to the time variable. In the second part, we define and apply the backward energy condition for those systems when the initial data contains a delta measure approximated by piecewise constant functions.

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1. Introduction

A conservation law system

$$\partial_t U + \partial_x (F(U)) = 0, \ U : \Omega \rightarrow \mathbb{R}^n$$
is called physical if there exists a pair of functions \((\eta, Q)\), \(\eta\) being convex, satisfying the additional conservation law

\[
\partial_t \eta(U) + \partial_x (Q(U)) = 0,
\]

for all classical smooth solutions \(U\). The function \(\eta\) may be the energy density or the Lax (also called mathematical) entropy, for example. In this paper, we are dealing with the isentropic systems of gas dynamics and the function \(\eta\) denotes the physical energy density.

The entropy admissibility condition for conservation laws based on some well-known physical systems is introduced in [5]: The admissible weak solution to a physical system of conservation laws is the one that produces a maximal decrease of the quantity \(\int \eta dt\). It is also called the principle of maximal energy dissipation in the literature. We will call it the energy admissibility condition.

One can look in [8] for analysis of energy in compressible and incompressible isentropic Euler equations. The above admissibility condition is not as useful as the usual ones in some cases. For example, it cannot be used for the Euler system of compressible gas when \(\gamma < 5/3\), see [9]. Also, the authors in [1,11] found some examples when the use of that condition singles out physically incorrect solutions. One can also see the results of the energy dissipation condition from a standpoint of relations between self-similar and oscillating solutions constructed by the method of De Lellis and Székelyhidi for \(n\)-dimensional isentropic Euler system in [4,7].

In this paper, we will check if it is possible to use the energy admissibility condition to single out relevant solutions of conservation law systems with shadow wave solutions introduced in [12]. They are used when the standard elementary waves do not suffice for solving some Riemann problem. The delta function part annihilates a Rankine–Hugoniot deficit in the equations and the major concern is how to avoid an artificial deficit. The usual methods used in the literature are the overcompressibility (all characteristics run into a shadow wave front) or the entropy condition (using convex entropy–entropy flux pair). We will compare the energy admissibility condition with those two. The paper has two main parts.

In the first part we deal with three systems describing an isentropic flow of gas with non-positive pressure. For each of them, shadow wave solution appear for some Riemann data and we apply the energy admissibility condition to them:

In the first two systems, modeling the pressureless and Chaplygin gas, the overcompressibility and the entropy conditions suffice to single out physically meaningful solutions. That is a simple consequence of the fact that the energy is also a mathematical entropy.

In the third system, the generalized model of Chaplygin gas, overcompressibility and the entropy condition were not enough to single out a unique solution (see [16]). On the other hand, the energy condition successfully singles out a proper solution, a combination of two shock waves instead of a single shadow wave.
In the second part of the paper, we will answer the following question: How to choose an approximation of the initial data containing a combination of piecewise constant and delta function (called the delta Riemann data in the sequel) and get a physically reasonable unique solution? The principal problem here is that two different approximations of the same measure initial data give two different results in distributional sense (approximate solutions not having the same distributional limit). There is also a practical reason for using that initial data: A procedure of solving a problem with a piecewise constant approximation of smooth initial data involves shadow wave interactions. The interaction problem then reduces to a special case of the above distributional initial data in a moment of interaction.

The idea is to define so-called backward energy condition. We postulate that a proper choice of the initial data approximation should produce a solution with minimal energy dissipation in sufficiently small time interval. Note that the energy cannot rise and that a classical smooth solutions have zero energy dissipation. So, the ideal situation is to choose the approximation such that a corresponding solution is smooth. When it is not possible, the postulate means that we choose the approximation that produces a weak solution “closest” to a classical one.

Let us note that there were incomplete attempts to solve pressureless gas dynamics and related delta initial data problems by several authors, but no one raised the question about the uniqueness of a solution and a meaning of the distributional initial data in nonlinear systems (see (3.2)) up to our knowledge. With the backward energy condition, we can single out a global solution unique in the distributional sense for pressureless gas dynamics and for the Chaplygin gas. For the generalized Chaplygin model, we are also able to single out a proper initial data, but a distributional limit of the solution will be known after one calculate all possible wave interactions. That is a separate problem left for a further research.

2. Admissible shadow wave solutions

Let

\[ U_t + F(U)_x = 0 \]  

be a given conservation law system. A weak solution \( U \) is entropy admissible if

\[ \partial_t \eta(U) + \partial_x Q(U) \leq 0 \]

holds in distributional sense for each convex entropy pair \((\eta,Q)\). However, it is not always effective and there is also a question concerning its physical background for some systems. The energy admissibility condition states that the admissible solution is the one that dissipates the energy at the highest possible rate. In the recent years, a lot of systems having solutions (weak in some sense) with the delta function or its generalization are found. Let us
briefly recall the definition and some properties of shadow waves that will be used in such cases.

2.1. Shadow waves

Definition 2.1. ([12]) A shadow wave is a piecewise constant function with respect to time $t$ given by

$$U^\varepsilon(x,t) = \begin{cases} U_l, & x < c(t) - a_\varepsilon(t) \\ U_{l,\varepsilon}(t), & c(t) - a_\varepsilon(t) < x < c(t) \\ U_{r,\varepsilon}(t), & c(t) < x < c(t) + b_\varepsilon(t) \\ U_r, & x > c(t) + b_\varepsilon(t), \end{cases}$$

(2.2)

where $a_\varepsilon, b_\varepsilon \in C^1(\mathbb{R})$ and $a_\varepsilon(0) = x_{l,\varepsilon} \sim \varepsilon, b_\varepsilon(0) = x_{r,\varepsilon} \sim \varepsilon$. The states $U_{l,\varepsilon}(t)$ and $U_{r,\varepsilon}(t)$ are called intermediate states and they can have unbounded components of order $\varepsilon^{-\alpha}, \alpha \geq 0$ for each $t$. The curve $x = c(t)$ is the front of the wave, while $c'(t)$ is its speed. The external shadow wave lines are given by $x = c(t) - a_\varepsilon(t)$ and $x = c(t) + b_\varepsilon(t)$. A limit $\xi(t) := \lim_{\varepsilon \to 0}(a_\varepsilon(t)U_{l,\varepsilon}(t) + b_\varepsilon(t)U_{r,\varepsilon}(t))$ is the strength of the shadow wave.

We say that the shadow wave solve (2.1) if the expression on the left-hand side obtained by inserting (2.2) tends to 0 as $\varepsilon \to 0$. A solution of that kind is called approximate.

A distributional limit of $\lim_{\varepsilon \to 0} U^\varepsilon(x,0)$ equals

$$U|_{t=0} = \begin{cases} U_l, & x < 0 \\ U_r, & x > 0 + \gamma \delta_{(0,0)}, \end{cases} \text{ where } \gamma = \xi(0) = \lim_{\varepsilon \to 0} x_{l,\varepsilon}U_{l,\varepsilon}(0) + x_{r,\varepsilon}U_{r,\varepsilon}(0).$$

(2.3)

For a fixed $\varepsilon$, all calculations are made by using the Rankine-Hugoniot conditions for shocks. That enables us to use Lax entropy functionals contrary to some other solution concepts dealing with measure valued functions. Thus, for $f$ being a $C^1$-function we have

$$\partial_t f(U_\varepsilon) \approx \left( \frac{d}{dt}(a_\varepsilon(t)f(U_{l,\varepsilon}(t)) + b_\varepsilon(t)f(U_{r,\varepsilon}(t))) - c'(t)(f(U_1) - f(U_0)) \right) \delta_{x=ct}$$

$$- \lim_{\varepsilon \to 0} c'(t)(a_\varepsilon(t)f(U_{l,\varepsilon}(t)) + b_\varepsilon(t)f(U_{r,\varepsilon}(t)))' \delta'_{x=ct},$$

$$\partial_x f(U_\varepsilon) \approx (f(U_1) - f(U_0)) \delta_{x=ct} + \lim_{\varepsilon \to 0} \left( (a_\varepsilon(t)f(U_{l,\varepsilon}(t)) + (b_\varepsilon(t)f(U_{r,\varepsilon}(t)))' \delta'_{x=ct} \right)$$

(2.4)

Here and afterwards, the sign $\approx$ denotes a distributional limit as $\varepsilon \to 0$.

The simplest shadow wave is obtained as a solution to Riemann problem (2.1, 2.3) with $\gamma = 0$. Its central shadow wave line is given by $x = ct$ and one component in the intermediate states is constant and equals the speed $c$. The form (2.2) is a more general one suitable for shadow wave interactions and systems with initial data problems containing the delta measure. Let us note that a general form of an approximate solution to system (2.1) containing a single shadow wave is of the form (2.2) with $U_l$ and $U_r$ being classical weak solutions to (2.1).
Finding a shadow wave solution to system (2.1), i.e. functions $c(t)$ and $\xi(t)$, reduces to solving the following ODEs system

$$
\begin{align*}
  c'(t)(U_r - U_l) - (F(U_r) - F(U_l)) & \approx \partial_t (a_\varepsilon(t) U_{l,\varepsilon}(t) + b_\varepsilon(t) U_{r,\varepsilon}(t)), \\
  c'(t) (a_\varepsilon(t) U_{l,\varepsilon}(t) + b_\varepsilon(t) U_{r,\varepsilon}(t)) & \approx a_\varepsilon(t) F(U_{l,\varepsilon}(t)) + b_\varepsilon(t) F(U_{r,\varepsilon}(t)).
\end{align*}
$$

That procedure also determines type of singularity, since shadow waves approximate some well known classes of singular solutions, delta and singular shocks particularly: A shadow wave represents a delta shock wave when all but a single component of the intermediate states $U_{\{l,r\},\varepsilon}$ are bounded and that component growth order is $O(\varepsilon^{-1})$ as $\varepsilon \to 0$. If there are some other unbounded components of growth order $o(\varepsilon^{-1})$, the shadow wave represents a singular shock wave. Let us note that if the standard form of shadow wave fail to fulfill the necessary requirements for a system or initial data, then one can try with some more compound forms. One can find some examples in [12].

Up to our knowledge, all Riemann problems with delta or singular shock solutions defined in some other way (measure valued, smooth approximations, weak asymptotic solutions, . . .) from the literature have a corresponding shadow wave solution with the same distributional limit. It has recently been shown that a large class of initial data problems for the pressureless gas dynamics has a shadow wave solution that tends to a well known duality solution (see [17], Theorem 7.2).

### 2.2. Energy admissibility

Our first goal is to check if the energy admissibility condition can be applied to such solutions.

The total energy of a solution $U$ in the interval $[\overline{L}, L]$ at time $t > 0$ is given by

$$
H_{[-L,L]}(U(\cdot, t)) := \int_{-L}^{L} \eta(U(x, t)) dx.
$$

A value $L > 0$ is taken to be large enough to avoid a discussion about boundary conditions at least for some time $t < T$ significantly greater than zero and to avoid the case of total energy being infinite in finite time. For example, if the solution is given in the form (2.2), we have $U(x, t) = U_l$, $x < -L$ and $U(x, t) = U_r$, $x > L$ for $t < T$ and the curve $x = c(t)$ does not intersect the boundary of $[-L, L] \times [0, T]$. To simplify the notation, we write $H_{[-L,L]}(t)$ instead of $H_{[-L,L]}(U(\cdot, t))$.

Suppose that there is only one shadow wave in an approximate solution,

$$
U^\varepsilon(x, t) = \begin{cases} 
  U_0(x, t), & x < c(t) - \frac{\varepsilon}{2}t - x_\varepsilon \\
  U_{0,\varepsilon}(t), & c(t) - \frac{\varepsilon}{2}t - x_\varepsilon < x < c(t) \\
  U_1(x, t), & x = c(t) + \frac{\varepsilon}{2}t + x_\varepsilon \\
  U_{1,\varepsilon}(t), & c(t) < x < c(t) + \frac{\varepsilon}{2}t + x_\varepsilon \\
  U_2(x, t), & x > c(t) + \frac{\varepsilon}{2}t + x_\varepsilon
\end{cases}
$$

(2.5)

passing through an interval $[T_1, T_2] \times [-L, L]$. The function $U_i(x, t)$, $i = 1, 2$ are smooth solutions to the above system. The curve $x = c(t)$ is the front and
\[ \xi(t) = \lim_{\varepsilon \to 0} (\varepsilon^2 + x_\varepsilon)(U_{0,\varepsilon}(t) + U_{1,\varepsilon}(t)) \] is the strength of the shadow wave. Then

\[ H_{[-L,L]}(t) = \lim_{\varepsilon \to 0} \int_{-L}^L \eta(U^\varepsilon(x,t))dx \]

\[ = \int_{-L}^{c(t)} \eta(U_0(x,t))dx + \lim_{\varepsilon \to 0} \left( \frac{\varepsilon}{2} t + x_\varepsilon \right) \left( \eta(U_{0,\varepsilon}(t)) + \eta(U_{1,\varepsilon}(t)) \right) \]

\[ + \int_{c(t)}^L \eta(U_1(x,t))dx \]

for \( t \in [T_1, T_2] \). The energy production of (2.5) at a time \( t \) is

\[ \frac{d}{dt} H_{[-L,L]}(t) = \lim_{\varepsilon \to 0} \left( \left( c'(t) - \frac{\varepsilon}{2} \right) \eta(U_0(c(t)) - \frac{\varepsilon}{2} t, t) \right) \]

\[ - \int_{-L}^{c(t)-\varepsilon t/2-x_\varepsilon} Q(U_0(x,t))dx + \frac{\varepsilon}{2} \eta(U_{0,\varepsilon}(t)) + \frac{\varepsilon}{2} \eta(U_{1,\varepsilon}(t)) \]

\[ + \int_{c(t)-\varepsilon t/2-x_\varepsilon}^{c(t)} \eta(U_{0,\varepsilon}(t))dx + \int_{c(t)}^{c(t)+\varepsilon t/2+x_\varepsilon} \eta(U_{1,\varepsilon}(t))dx \]

\[ - \left( c'(t) + \frac{\varepsilon}{2} \right) \eta(U_1(c(t)) + \frac{\varepsilon}{2} t + x_\varepsilon, t) \]

\[ - \int_{c(t)+\varepsilon t/2+x_\varepsilon}^L Q(U_1(x,t))dx \]

where we have used that \( U_\varepsilon \) depends only on \( t \), the above limit exists and

\[ \eta_t = -Q_x \] for smooth solutions. Finally,

\[ \frac{d}{dt} H_{[-L,L]}(t) = -c'(t) \left( \eta(U_1(c(t), t)) - \eta(U_0(c(t), t)) \right) + Q(U_1(c(t), t)) \]

\[ - Q(U_0(c(t), t)) + Q(U_0(-L, t)) - Q(U_1(L, t)) \]

\[ + \lim_{\varepsilon \to 0} \left( \left( \frac{\varepsilon}{2} t + x_\varepsilon \right) \frac{d}{dt} \left( \eta(U_{0,\varepsilon}(t)) + \eta(U_{1,\varepsilon}(t)) \right) \right) \]

\[ + \frac{\varepsilon}{2} \left( \eta(U_{0,\varepsilon}(t)) + \eta(U_{1,\varepsilon}(t)) \right) \]

Denote by

\[ D(t) := -c'(t) \left( \eta(U_1(c(t), t)) - \eta(U_0(c(t), t)) \right) + Q(U_1(c(t), t)) - Q(U_0(c(t), t)) \]

\[ + \lim_{\varepsilon \to 0} \left( \left( \frac{\varepsilon}{2} t + x_\varepsilon \right) \left( \eta(U_{0,\varepsilon}(t)) + \eta(U_{1,\varepsilon}(t)) \right) \right) \]

\[ = \frac{d}{dt} H_{[-L,L]}(t) - \left( Q(U_0(-L, t)) - Q(U_1(L, t)) \right) \]

the local energy production of shadow wave (2.5) at time \( t \). The energy production for a shock wave

\[ U(x,t) = \begin{cases} 
U_0(x,t), & x < c(t) \\
U_1(x,t), & x > c(t) 
\end{cases} \]
$$\frac{d}{dt} H_{[-L,L]}(t) = -c'(t)(\eta(U_1(c(t), t)) - \eta(U_0(c(t), t)))$$
$$+ (Q(U_1(c(t), t)) - Q(U_0(c(t), t))) + Q(U_0(-L,t)) - Q(U_1(L,t))$$
$$= D(t) + Q(U_0(-L,t)) - Q(U_1(L,t)).$$

It equals $\frac{d}{dt} H_{[-L,L]}(t) = Q(U_0(-L,t)) - Q(U_1(L,t))$ for a rarefaction wave (see [6]), i.e. the local energy production $D(t)$ equals zero due to a continuity of a rarefaction wave.

**Lemma 2.1.** ([12]) Denote by $(\eta, Q)$ a convex entropy pair for system \((2.1)\). Shadow wave solution \((2.5)\) satisfies the entropy inequality $\partial_t \bar{\eta} + \partial_x \bar{Q} \leq 0$ in the sense of distributions if

$$D(t) = -c'(t)[\eta] + [Q] + \lim_{\varepsilon \to 0} \frac{d}{dt} \left( \frac{\varepsilon}{2} (\eta(U_{0,\varepsilon}(t)) + \eta(U_{1,\varepsilon}(t))) \right) \leq 0$$
$$\lim_{\varepsilon \to 0} \left( \frac{\varepsilon}{2} (\eta(U_{0,\varepsilon}(t)) + \eta(U_{1,\varepsilon}(t))) - (Q(U_{0,\varepsilon}(t)) + Q(U_{1,\varepsilon}(t))) \right) = 0,$$

\((2.6)\)

where $[\eta] := \eta(U_1(c(t), t)) - \eta(U_0(c(t), t))$ and $[Q] := Q(U_1(c(t), t)) - Q(U_0(c(t), t))$.

**Remark 2.1.** For gas dynamics systems endowed with entropy pair $(\eta, Q)$, $Q = u \eta$, the second condition in \((2.6)\) is automatically satisfied for shadow waves. That follows from the law of mass conservation.

**Lemma 2.2.** Let $F(U_0) \neq F(U_1)$. Suppose that shadow wave \((2.5)\) satisfies condition \((2.6)\) associated with the convex entropy pair $(\eta, Q)$ for system \((2.1)\). Then, it satisfies that condition associated with entropy pair

$$\bar{\eta}(U) = \eta(U) + \sum_{j=1}^{n} a_j U^j, \quad \bar{Q}(U) = Q(U) + \sum_{j=1}^{n} a_j f^j(U) + \bar{c},$$

\((2.7)\)

$U = (u^1, \ldots, u^n)$, $F(U) = (f^1(U), \ldots, f^n(U))$, for every $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $\bar{c} \in \mathbb{R}$. The pair $(\bar{\eta}, \bar{Q})$ is convex entropy pair for the system \((2.1)\) and $D(t)$ is invariant under the transformation \((2.7)\).

**Proof.** One can easily prove that $D \bar{Q} = D \bar{\eta} DF$ and $D^2 \bar{\eta} = D^2 \bar{\eta}$. To prove

$$\langle \partial_t \bar{\eta}(U^\varepsilon) + \partial_x \bar{Q}(U^\varepsilon), \varphi \rangle \approx \langle \partial_t \eta(U^\varepsilon) + \partial_x Q(U^\varepsilon), \varphi \rangle, \quad \varepsilon \to 0,$$

for any test function $\varphi \in C_0^\infty$, we use the results from [12]. We have

$$\Lambda_1(t) := -c'(t)[U] + [F(U)] + \frac{d}{dt} \left( \frac{\varepsilon}{2} (\eta(U_{0,\varepsilon}(t) + U_{1,\varepsilon}(t))) \right) = O(\varepsilon)$$
$$\Lambda_2(t) := \left( \frac{\varepsilon}{2} (\eta(U_{0,\varepsilon}(t) + U_{1,\varepsilon}(t)) - (F(U_{0,\varepsilon}(t)) + F(U_{1,\varepsilon}(t)))) \right) = O(\varepsilon)$$
since (2.5) is approximate solution to (2.1). Using the procedure from the proof of Lemma 10.1 in [12] we get

\[
\langle \partial_t \bar{\eta}(U^\varepsilon) + \partial_x Q(U^\varepsilon), \varphi \rangle = \langle \partial_t \eta(U^\varepsilon) + \partial_x Q(U^\varepsilon), \varphi \rangle \\
+ \sum_{j=1}^n a_j \left( \int_0^\infty \Lambda_j^1(t) \varphi(c(t), t) \, dt + \int_0^\infty \Lambda_j^2(t) \partial_x \varphi(c(t), t) \, dt \right) + O(\varepsilon),
\]

where \( \Lambda_k(t) = (\Lambda_k^1(t), \ldots, \Lambda_k^n(t)) \), \( k = 1, 2 \).

The lemma holds for classical weak solutions, too, as proved in [5]. Its consequence is that one can choose constants \( a_i, i = 1, \ldots, n \) and \( \bar{c} \) in such a way that

\[
Q(U_0(-L, t)) = 0, \quad Q(U_1(L, t)) = 0.
\]

Then, there is no energy flow on the boundaries and the total energy decreases

\[
\frac{d}{dt} \bar{H}_{[-L,L]}(t) \leq 0.
\]

(\( \bar{H}_{[-L,L]}(t) \) denotes the total energy corresponding to \( \bar{\eta} \).) So, for an energy pair \((\eta, Q)\), we can take affine transformation (2.7) such that (2.8) holds and the total energy decreases. Thus, the following definition is invariant under a change of energy flux at \( x = -L \) and \( x = L \) for shadow wave solutions also.

**Definition 2.2.** ([5]) A weak solution \( U(x, t) \) to the system (2.1) satisfies the energy admissibility condition on \( \mathbb{R} \times (0, T] \) if there is no solution \( \bar{U}(x, t) \) such that for some \( \tau \in [0, T) \) we have \( U(x, t) = \bar{U}(x, t), \ t < \tau \) and \( \frac{d}{dt} \bar{H}_{[-L,L]}(\bar{U}(\cdot, \tau)) < \frac{d}{dt} \bar{H}_{[-L,L]}(U(\cdot, \tau)) \). The derivatives are assumed to be forward in time here.

### 2.3. Pressureless gas dynamics system

Take an isentropic, inviscid and compressible flow of a gas in the absence of internal and external forces. Its dynamics is described by mass and linear momentum conservation laws

\[
\partial_t \rho + \partial_x (\rho u) = 0 \\
\partial_t (\rho u) + \partial_x (\rho u^2) = 0.
\]

Here \( \rho \geq 0 \) and \( u \) denote density and velocity of a fluid, respectively. The model can be obtained from the compressible Euler equations of gas dynamics by letting the pressure tend to zero. It is used to describe a formation of large structures in the Universe and behavior of particles that stick under collision ([2,18,19]).

System (2.9) is weakly hyperbolic with both characteristic fields being linearly degenerate and characteristic speeds equal to velocity, \( \lambda_1(\rho, u) = \lambda_2(\rho, u) = u \).

The solution to system (2.9) with the Riemann initial data

\[
(\rho, u)(x, 0) = \begin{cases} 
(\rho_0, u_0), & x < 0 \\
(\rho_1, u_1), & x > 0
\end{cases}
\]

(2.10)
consists of two contact discontinuities connected by the vacuum state

\[
U(x, t) = \begin{cases} 
(r_0, u_0), & x < u_0 t \\
(0, \frac{x}{t}), & u_0 t < x < u_1 t \\
r_1, u_1), & x > u_1 t
\end{cases}
\]

if \( u_0 \leq u_1 \). This solution will be called CD wave combination. Otherwise, there exists a singular solution given in the form of a delta shock approximated by a shadow wave. It is given by

\[
U^\varepsilon(x, t) = \begin{cases} 
(r_0, u_0), & x < c - \varepsilon t \\
(r_\varepsilon, u_\varepsilon), & c - \varepsilon t < x < c + \varepsilon t \\
r_1, u_1), & x > c + \varepsilon t,
\end{cases}
\]

where \( r_\varepsilon = \sqrt{r_0 r_1 |u_0 - u_1|^{-1}} \) and \( c = u_\varepsilon = \frac{\sqrt{r_0 u_0} + \sqrt{r_1 u_1}}{\sqrt{r_0} + \sqrt{r_1}} \) denotes speed of shadow wave. It is overcompressive, \( u_1 \leq u_\varepsilon \leq u_0 \) and its strength equals \( \xi(t) = \lim_{\varepsilon \to 0} \varepsilon r_\varepsilon = \sqrt{r_0 r_1 |u_0 - u_1| \varepsilon} \) (only for \( r \)-variable, the velocity is bounded).

The energy density is \( \eta = \frac{1}{2} r u^2 \) with the flux \( Q = \frac{1}{2} r u^3 \).

It is easy to see that all these solutions satisfy the energy admissibility condition: A contact discontinuity does not change energy and a non-overcompressive shadow wave increase the energy: The situation is clear when \( u_0 > u_1 \) since there is only a shadow wave solution (which satisfy both overcompressibility and Lax entropy condition). Similarly, there is only a contact discontinuity solution when \( u_0 = u_1 \). Let \( u_1 > u_0 \). Then, there is a CD wave combination solution \( U \), and a shadow wave \( \tilde{U}^\varepsilon \) given by (2.11) with the same \( \xi \) and \( c = u_\varepsilon \) as above. Using the formulas (2.4) we get

\[
\partial_t \eta(\tilde{U}^\varepsilon) + \partial_x Q(\tilde{U}^\varepsilon) \approx (\xi c^2 + r_1 u_1^2 (u_1 - c) + r_0 u_0^2 (c - u_0)) \delta_{x=ct} > 0,
\]

in the sense of distributions since \( \xi > 0 \), \( u_0 < c < u_1 \). (Details may be found in [17].) For \( U \), consisting of contact discontinuities and the vacuum state, we have

\[
\partial_t \eta(U) + \partial_x Q(U) = 0.
\]

Then,

\[
0 < \int_{-L}^L (\partial_t \eta(\tilde{U}^\varepsilon) + \partial_x Q(\tilde{U}^\varepsilon)) \, dx = \partial_t \int_{-L}^L \eta(\tilde{U}^\varepsilon) \, dx + \underbrace{Q(\tilde{U}^\varepsilon(L, t)) - Q(\tilde{U}^\varepsilon(-L, t))}_{=Q(U_1) - Q(U_0)} = Q(U_1) - Q(U_0)
\]

\[
0 = \int_{-L}^L (\partial_t \eta(U) + \partial_x Q(U)) \, dx = \partial_t \int_{-L}^L \eta(U) \, dx + \underbrace{Q(U(L, t)) - Q(U(-L, t))}_{=Q(U_1) - Q(U_0)} = Q(U_1) - Q(U_0).
\]

That proves that \( U \) satisfy the energy admissibility condition.
2.4. Chaplygin gas

The gas dynamics model with negative pressure

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0 \\
\partial_t (\rho u) + \partial_x \left( \rho u^2 - \frac{1}{\rho} \right) &= 0
\end{align*}
\]

(2.12)

is introduced to describe the aerodynamics force acting on a wing of an airplane in [3]. In recent years, it is found that it can be used in cosmology as a model of the dark energy in the Universe, see [10]. Here \( u \) and \( \rho \) denote the velocity and the density of a fluid, respectively. The Riemann problem has a global solution consisting of elementary waves and shadow waves, as shown in [13]. That paper also contains wave interaction analysis that will be used in the second part.

System (2.12) is strictly hyperbolic having two distinct eigenvalues

\[
\lambda_1(\rho, u) = u - \frac{1}{\rho} < \lambda_2(\rho, u) = u + \frac{1}{\rho}
\]

with both fields being linearly degenerate. If \( \lambda_1(\rho_0, u_0) < \lambda_2(\rho_1, u_1) \), there exists classical solution to the Riemann problem (2.12, 2.10). It consists of two contact discontinuities connected by the constant intermediate state \( \left( \rho_m, u_m \right) = \left( \frac{\lambda_2(\rho_1, u_1)}{\lambda_1(\rho_1, u_1)} - \frac{\lambda_1(\rho_0, u_0)}{\lambda_2(\rho_0, u_0)} \right)^2 \).

Otherwise, the solution is overcompressive shadow wave propagating with the constant speed

\[
s = \frac{[\rho u]}{[\rho]} + \frac{1}{[\rho]} \sqrt{[\rho u]^2 - [\rho][\rho u^2 - \rho^{-1}]} = \frac{[\rho u]}{[\rho]} + \frac{\kappa}{[\rho]},
\]

(2.13)

where \([\cdot]\) is a jump across a discontinuity, and \( \kappa \) is the Rankine–Hugoniot deficit, \( \kappa := s[\rho] - [\rho u] \). It is worthwhile to mention that the term under the square root in (2.13) can be written as

\[
\rho_0 \rho_1 \left( \lambda_1(\rho_0, u_0) - \lambda_1(\rho_1, u_1) \right) \left( \lambda_2(\rho_0, u_0) - \lambda_2(\rho_1, u_1) \right).
\]

So, the shadow wave solution exists only if \( \lambda_1(\rho_0, u_0) \geq \lambda_2(\rho_1, u_1) \), i.e. if there is no classical weak solution. Also, the overcompressibility and the entropy condition for any entropy pair are equivalent. The most interesting choice is \( \eta \) being the physical energy density \( \eta = \rho u^2 + \frac{1}{\rho} \) with corresponding flux \( Q = \rho u^3 - \frac{u}{\rho} \) because this choice leads directly to the fact that the energy admissibility condition is satisfied, too. For more details about existence of a shadow wave solution and the equivalency of two admissibility conditions see [13]. A solution to Riemann problem for Chaplygin gas system is weakly unique (the distributional limit is unique).

2.5. Generalized Chaplygin gas model

In this section, we will see a real use of the energy admissibility condition for unbounded solutions. It suffices to single out a proper (physically relevant) solution contrary to other admissibility criteria.
The generalized Chaplygin gas model consists of the mass and momentum conservation laws

\[ \begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2 - \frac{1}{\rho^\alpha}) &= 0,
\end{align*} \tag{2.14} \]

where \(0 < \alpha < 1\). That is a strictly hyperbolic system with eigenvalues

\[ \lambda_1(\rho, u) = u - \sqrt{\alpha \rho^{-\frac{1+\alpha}{2}}} < \lambda_2(\rho, u) = u + \sqrt{\alpha \rho^{-\frac{1+\alpha}{2}}}. \]

Suppose that the left-hand side \((\rho_0, u_0)\) in the Riemann data (2.10) is fixed. Both characteristic fields are genuinely nonlinear and elementary wave curves are given by the following relations:

\[ \begin{align*}
R_1: u &= u_0 + \frac{2\sqrt{\alpha}}{1+\alpha} \left( \rho^{-\frac{1+\alpha}{2}} - \rho_0^{-\frac{1+\alpha}{2}} \right), \quad \rho < \rho_0 \\
R_2: u &= u_0 - \frac{2\sqrt{\alpha}}{1+\alpha} \left( \rho^{-\frac{1+\alpha}{2}} - \rho_0^{-\frac{1+\alpha}{2}} \right), \quad \rho > \rho_0 \\
S_1: u &= u_0 - \sqrt{\frac{\rho - \rho_0}{\rho_0} \left( \rho_0^{-\alpha} - \rho^{-\alpha} \right)}, \quad \rho > \rho_0 \\
S_2: u &= u_0 - \sqrt{\frac{\rho - \rho_0}{\rho_0} \left( \rho_0^{-\alpha} - \rho^{-\alpha} \right)}, \quad \rho < \rho_0
\end{align*} \]

A classical weak solution to the Riemann problem (2.14, 2.10) being a combination of the elementary waves exists if the right initial state \((\rho_1, u_1)\) is above the curve

\[ \Gamma_{ss}(\rho_0, u_0): u = u_0 - \rho_0^{-\frac{1+\alpha}{2}} - \rho^{-\frac{1+\alpha}{2}}. \]

For \((\rho_1, u_1)\) below and at the curve \(\Gamma_{ss}(\rho_0, u_0)\) a unique solution is the shadow wave as proved in [16].

**Lemma 2.3.** There exists a shadow wave solution (2.5) with \(U = (\rho, u)\) to (2.14, 2.10) if

\[ \rho_0 \rho_1 (u_0 - u_1)^2 > (\rho_0 - \rho_1)(\rho_1^{-\alpha} - \rho_0^{-\alpha}). \]

Its speed is given by

\[ c = \begin{cases} 
\frac{u_1 \rho_1 - u_0 \rho_0 + \kappa_1}{\rho_1 - \rho_0}, & \text{if } \rho_0 \neq \rho_1 \\
\frac{1}{2}(u_0 + u_1), & \text{if } \rho_0 = \rho_1,
\end{cases} \]

where \(\kappa_1 = \lim_{\varepsilon \to 0} \varepsilon \rho_\varepsilon = \sqrt{\rho_0 \rho_1 (u_0 - u_1)^2 - (\rho_0 - \rho_1)(\rho_1^{-\alpha} - \rho_0^{-\alpha})}\) denotes Rankine–Hugoniot deficit and \(\delta\) is supported by the line \(x = c(t) = ct\).

In most of the cases from the literature, it suffices to use overcompressibility condition to exclude non-wanted singular solutions containing the delta function. However, that is not the case here: As it has been proved in [16],
shadow wave solutions are overcompressive if \((\rho_1, u_1)\) is below the curve

\[
\Gamma_{oc}(\rho_0, u_0) : u = \begin{cases} 
0 & \text{if } \rho_0 \leq \rho, \\
0 - \frac{1}{\rho \rho_0} \left( \sqrt{\alpha \rho_0} \rho^{1-\alpha} + \frac{1}{2} \rho_0^2 \rho_0^{\alpha} - (1 - \alpha) \rho_0^{-\alpha} \right), & \text{if } \rho_0 > \rho.
\end{cases}
\]

However, the curve \(\Gamma_{oc}\) is located above the curve \(\Gamma_{ss}\) which means that for \((\rho_1, u_1)\) located in the area between two curves both classical two shock and overcompressive shadow wave solution exist. The first shock propagates with speed \(c_1 = u_0 - A(\rho_0, \rho_m)\) and joins \((\rho_0, u_0)\) on the left to \((\rho_m, u_m)\) on the right, while the second one propagates with speed \(c_2 = u_1 + A(\rho_1, \rho_m)\) = \(u_1 + A(\rho_1, \rho_m)\) and joins \((\rho_m, u_m)\) to \((\rho_1, u_1)\).

Here,

\[
A(\rho_i, \rho_j) := \sqrt{\frac{\rho_j}{\rho_i} \rho_i^{\alpha} - \rho_j^{\alpha}},
\]

and intermediate state \((\rho_m, u_m)\) is determined by the following relations

\[
u_m = u_0 - \sqrt{\frac{\rho_m - \rho_0}{\rho_0 \rho_m} (\rho_0^{-\alpha} - \rho_m^{-\alpha})}, \quad \rho_m > \rho_0
\]

\[
u_m = u_1 + \sqrt{\frac{\rho_m - \rho_1}{\rho_1 \rho_m} (\rho_1^{-\alpha} - \rho_m^{-\alpha})}, \quad \rho_m > \rho_1.
\]

Thus, one has to find an additional admissibility criterion which will exclude the overcompressive shadow wave solution in the area above \(\Gamma_{ss}\) curve. The system (2.14) possesses infinitely many convex entropies, so one can use entropy condition given in [12]. The shadow wave solution (2.11) with a constant speed \(c\) is admissible if

\[
D := -c[\eta] + [Q] + \lim_{\varepsilon \to 0} \varepsilon \eta(\rho_\varepsilon, u_\varepsilon) \leq 0
\]

\[
\lim_{\varepsilon \to 0} (-c\varepsilon \eta(\rho_\varepsilon, u_\varepsilon) + \varepsilon Q(\rho_\varepsilon, u_\varepsilon)) = 0
\]

for each convex entropy pair \((\eta, Q)\). The idea was implemented in [16], but without a complete success probably due to the lack of precise approximations of the modified Bessel functions of the second kind. However, all numerical simulations have shown that shadow wave solution is not admissible above \(\Gamma_{ss}\). Here, we are about to show that energy admissibility condition successfully excludes the unwanted shadow wave solution in that area using the following energy–energy flux pair:

\[
\eta(\rho, u) = \frac{1}{2} \rho u^2 + \frac{1}{1 + \alpha} \rho^{-\alpha} \quad \text{and} \quad Q(\rho, u) = \frac{1}{2} \rho u^3 - \frac{\alpha}{1 + \alpha} \rho^{-\alpha} u.
\]

### 2.5.1. The energy dissipation.

The energy admissibility condition will exclude the unwanted shadow wave solution above \(\Gamma_{ss}\) if its energy production is greater than energy production of classical \(S_1 + S_2\) solution. The local energy production of the two shock combination connecting states \((\rho_0, u_0)\) and
$(\rho_1, u_1)$ is constant and given by
\[
D^{cl} := -c_1(\eta(\rho_m, u_m) - \eta(\rho_0, u_0)) + Q(\rho_m, u_m) - Q(\rho_0, u_0) - c_2(\eta(\rho_1, u_1) - \eta(\rho_m, u_m)) + Q(\rho_1, u_1) - Q(\rho_m, u_m),
\]
while
\[
D^{sdw} := -c[\eta] + [Q] + \lim_{\varepsilon \to 0} \varepsilon \eta(\rho_\varepsilon, u_\varepsilon) = -c[\eta] + [Q] + \frac{1}{2} c^2 \kappa_1
\]
is the local energy production for the shadow wave. The following two relations
\[
\begin{align*}
0_0 - u_1 &= \frac{\rho_m - \rho_0}{\rho_m} A(\rho_0, \rho_m) + \frac{\rho_m - \rho_1}{\rho_m} A(\rho_1, \rho_m), \\
\kappa_1 &= \sqrt{\sqrt{\rho_m - \rho_0}(\rho_m - \rho_1)(c_2 - c_1)}
\end{align*}
\]
are consequences of (2.16). Suppose that $\rho_1 \neq \rho_0$. Then, (2.19) implies
\[
\begin{align*}
c - c_1 &= \frac{\rho_1}{\rho_1 - \rho_0} (u_1 - u_0) + \frac{\kappa_1}{\rho_1 - \rho_0} + A(\rho_0, \rho_m) \\
&= \frac{\rho_m - \rho_1}{\rho_1 - \rho_0} \left( -1 + \sqrt{\frac{\rho_m - \rho_0}{\rho_m - \rho_1}} \right) (c_2 - c_1) > 0 \\
c - c_2 &= \frac{\rho_0}{\rho_1 - \rho_0} (u_1 - u_0) + \frac{\kappa_1}{\rho_1 - \rho_0} - A(\rho_1, \rho_m) \\
&= \frac{\rho_m - \rho_0}{\rho_1 - \rho_0} \left( -1 + \sqrt{\frac{\rho_m - \rho_1}{\rho_m - \rho_0}} \right) (c_2 - c_1) < 0.
\end{align*}
\]
We have used that $\frac{\rho_m - \rho_0}{\rho_m - \rho_0} > 1$ if $\rho_1 > \rho_0$. The inequalities in (2.20) have the opposite signs if $\rho_1 < \rho_0$. That proves that the speed of shadow wave is between shock speeds, $c_1 < c < c_2$. If $\rho_0 = \rho_1$, then $c - c_1 = c_2 - c = \frac{\rho_0}{\rho_m} A(\rho_0, \rho_m)$.

**Theorem 2.1.** The solution to the Riemann problem (2.14, 2.10) is a combination of classical elementary waves if
\[
u_1 > 0_0 - \rho_0^{\frac{1+\alpha}{1-\alpha}} - \rho_1^{\frac{1+\alpha}{1-\alpha}},
\]
or shadow wave (2.5) otherwise. Both solutions satisfy the energy admissibility condition.

**Proof.** If $(\rho_1, u_1)$ lies in the area where only one solution exists, there is nothing to prove. Suppose that there exist two solutions for $(\rho_1, u_1)$ above the curve $\Gamma_{ss}(\rho_0, u_0)$, a classical two shock solution and a shadow wave. To prove the theorem it thus suffices to show that $D^{sdw} - D^{cl} > 0$ in that area. We have
\[
D^{sdw} - D^{cl} = \eta(\rho_m, u_m)(c_1 - c_2) + \eta(\rho_0, u_0)(c_1 - c_1) + \eta(\rho_1, u_1)(c_2 - c) + \frac{1}{2} c^2 \kappa_1.
\]
Suppose that $\rho_0 \neq \rho_1$. Using (2.20), we get
\[
\frac{D^{sdw} - D^{cl}}{c_2 - c_1} = -\frac{\rho_m - \rho_1}{\rho_1 - \rho_0} \left( -1 + \sqrt{\frac{\rho_m - \rho_0}{\rho_m - \rho_1}} \right) \eta(\rho_0, u_0) + \frac{\rho_m - \rho_0}{\rho_1 - \rho_0} \left( -1 + \sqrt{\frac{\rho_m - \rho_1}{\rho_m - \rho_0}} \right) \eta(\rho_1, u_1) + \frac{1}{2} c^2 \sqrt{(\rho_m - \rho_0)(\rho_m - \rho_1)}
\]
\[
= \frac{1}{1 + \alpha} J_1 + \frac{1}{2} J_2,
\]
where
\[
J_1 := -\frac{\rho_m - \rho_1}{\rho_1 - \rho_0} \left( -1 + \sqrt{\frac{\rho_m - \rho_0}{\rho_m - \rho_1}} \right) \eta(\rho_0, u_0),
\]
\[
J_2 := \frac{\rho_m - \rho_0}{\rho_1 - \rho_0} \left( -1 + \sqrt{\frac{\rho_m - \rho_1}{\rho_m - \rho_0}} \right) \eta(\rho_1, u_1) + \frac{1}{2} c^2 \sqrt{(\rho_m - \rho_0)(\rho_m - \rho_1)}.
\]
where

\[ I_1 = \rho_1^{-\alpha} - \rho_0^{-\alpha} - \frac{\rho_m - \rho_1}{\rho_1 - \rho_0} \left( -1 + \sqrt{\frac{\rho_m - \rho_0}{\rho_m - \rho_1}} \right) (\rho_1^{-\alpha} - \rho_0^{-\alpha}), \]

\[ I_2 = -\rho_m u_m^2 + \frac{\rho_m - \rho_1}{\rho_1 - \rho_0} \left( -1 + \sqrt{\frac{\rho_m - \rho_0}{\rho_m - \rho_1}} \right) \rho_0 u_0^2 + \frac{\rho_m - \rho_0}{\rho_1 - \rho_0} \left( 1 - \sqrt{\frac{\rho_m - \rho_1}{\rho_m - \rho_0}} \right) \rho_1 u_1^2 + c^2 (\rho_m - \rho_0)(\rho_m - \rho_1). \]

Let us prove that \( I_1 > 0. \)

\[ I_1 = (\rho_1^{-\alpha} - \rho_0^{-\alpha}) \frac{\rho_m - \rho_0}{\rho_1 - \rho_0} - (\rho_0^{-\alpha} - \rho_m^{-\alpha}) \frac{\rho_m - \rho_1}{\rho_1 - \rho_0} + (\rho_0^{-\alpha} - \rho_1^{-\alpha}) \frac{\rho_m - \rho_0}{\rho_1 - \rho_0} \sqrt{\frac{\rho_m - \rho_1}{\rho_m - \rho_0}} = \frac{\rho_m - \rho_0}{\rho_1 - \rho_0} (\rho_m - \rho_1) \left( \frac{\rho_1^{-\alpha} - \rho_m^{-\alpha}}{\rho_m - \rho_1} \left( -1 + \sqrt{\frac{\rho_m - \rho_1}{\rho_m - \rho_0}} \right) \right) + \frac{\rho_0^{-\alpha} - \rho_m^{-\alpha}}{\rho_m - \rho_0} \left( -1 + \sqrt{\frac{\rho_m - \rho_0}{\rho_m - \rho_1}} \right). \]

One can easily see that

\[ a_1 := 1 - \sqrt{\frac{\rho_m - \rho_1}{\rho_m - \rho_0}} > 0 \quad \text{and} \quad a_2 := -1 + \sqrt{\frac{\rho_m - \rho_0}{\rho_m - \rho_1}} > 0 \]

if \( \rho_1 > \rho_0. \) Both \( a_1 \) and \( a_2 \) are negative if \( \rho_1 < \rho_0 \) and \( I_1 > 0 \) because \( \rho_m > \rho_0 \) and \( \rho_m > \rho_1. \)

Eliminating \( u_0 \) and \( u_m \) from \( I_2, \) we get

\[ I_2 = \frac{\rho_m - \rho_0}{(\rho_1 - \rho_0)^2} \frac{\rho_m - \rho_1}{\rho_m} \left( \frac{1}{2} \left( \rho_1 a_2 - \rho_0 a_1 \right) \left( \sqrt{\rho_0^{-\alpha} - \rho_m^{-\alpha}} - \sqrt{\rho_1^{-\alpha} - \rho_m^{-\alpha}} \right)^2 \right) + \sqrt{\left( \rho_0^{-\alpha} - \rho_m^{-\alpha} \right) \left( \rho_1^{-\alpha} - \rho_m^{-\alpha} \right)} \left( \rho_1 a_2 - \rho_0 a_1 \right) - \sqrt{\rho_0 \rho_1} \left( \frac{\sqrt{\rho_m - \rho_1} - \sqrt{\rho_m - \rho_0}}{\sqrt{\rho_m - \rho_1} (\rho_m - \rho_0)} \right)^2). \]

Then

\[ \rho_1 a_2 - \rho_0 a_1 = \left( 1 - \frac{\sqrt{\rho_m - \rho_0}}{\sqrt{\rho_m - \rho_1}} \right) \left( \rho_0 \sqrt{\frac{\rho_m - \rho_1}{\rho_m - \rho_0} - \rho_1} \right) > 0. \]
The above inequality follows from the fact that $a > 1$ for $\rho_1 > \rho_0$. That implies $(1 - a) < 0$ and $\frac{\rho_0}{a} - \rho_1 < 0$. The same holds for $\rho_1 < \rho_0$. Finally,

\[
(\rho_1 a_2 - \rho_0 a_1) - \sqrt{\rho_0 \rho_1} \left( \frac{\sqrt{\rho_m - \rho_1} - \sqrt{\rho_m - \rho_0}}{(\rho_m - \rho_1)(\rho_m - \rho_0)} \right)^2
\]

\[
= - \left( \sqrt{\rho_0} - \sqrt{\rho_1} \right)^2 + \left( \sqrt{\rho_0} - \sqrt{\rho_1} \right) \left( \sqrt{\rho_0 a_1} - \sqrt{\rho_1 a} \right) > 0
\]

since

\[
\frac{\sqrt{\rho_0 a_1} - \sqrt{\rho_1 a}}{\sqrt{\rho_0} - \sqrt{\rho_1}} > 1.
\]

Thus, $I_2 > 0$ and $D^{sdw} - D^{cl} > 0$.

The case $\rho_0 = \rho_1$ is simpler,

\[
D^{sdw} - D^{cl} = (c_2 - c_1)(\rho_0^{\alpha} - \rho_m^{\alpha}) \left( \frac{1}{2} \frac{\rho_m - \rho_0}{\rho_m} + \frac{1}{1 + \alpha} \right) > 0.
\]

Remark 2.2. Note that this result coincides with the one from [14] obtained using the so-called vanishing pressure method.

3. Backward energy condition

It is known that in the case of hyperbolic systems singularities naturally develop even if initial data are smooth. Now, suppose that we have singular initial data. The idea behind the new admissibility criteria is to single out a maximally regular weak solution with such data. We believe that such a solution is the best approximation of a real-world process. Let us give a simple illustration of the idea. Consider the inviscid Burgers’ equation with arbitrary initial data and suppose that a solution to such problem is a step function at fixed time $t = T$, meaning that it is given in the form of shock wave at least for $t > T$. In general, that happens even if the initial data is smooth. The value $T$ may be the point where a classical smooth solution if it exists, breaks down, i.e. where its gradient explodes. Starting from that moment, we have to use another more robust solution definition and apply new methods mostly based on some kind of approximation that will give a complete solution to the initial data problem. If the initial data is discontinuous, then the use of such methods is inevitable. In order to avoid the possibility of choosing some other weak over classical smooth solution as long as such solution exists, the backward energy condition has to be defined in such a way that a smooth solution is favored. The philosophy behind that idea is that smooth solutions are the most natural ones. By using that kind of reasoning, we will choose the approximation of the initial that gives a unique weak solution (distributional limit) with a minimal energy dissipation.

The main example will be the system of gas dynamics

\[
\partial_t \rho + \partial_x (\rho u) = 0
\]

\[
\partial_t (\rho u) + \partial_x (\rho u^2 + p(\rho)) = 0,
\]

(3.1)
with pressure functions \( p(\rho) \leq 0 \) satisfying \( p'(\rho) \geq 0 \) and \( pp''(\rho) + 2p'(\rho) \geq 0 \). Due to the lack of a positive pressure we expect to have solutions with mass being infinite. Thus, take the measure (or distributional) initial data

\[
(r, u)(x, 0) = \begin{cases} 
(r_0, u_0), & x < 0 \\
(r_1, u_1), & x > 0 + (\xi_\delta, 0)\delta,
\end{cases} \tag{3.2}
\]

where \( r_0, r_1 > 0, \xi_\delta > 0 \).

Like in the first part, we analyze three different types of pressure:

- \( p(\rho) \equiv 0 \) (pressureless gas dynamics),
- \( p(\rho) = -\rho^{-1} \) (Chaplygin model) and
- \( p(\rho) = -\rho^{-\alpha}, \alpha \in (0, 1) \) (generalized Chaplygin model).

In all cases, there exists a unique solution to Riemann problem being the shadow wave or a combination of elementary waves.

We will use wave tracking approach from [17] to solve initial data problem (3.1, 3.2). The initial data are approximated by the piecewise constant function

\[
U_\mu(x, 0) := (r_\mu, u_\mu)(x, 0) = \begin{cases} 
(r_0, u_0), & x < -\frac{\mu}{2} \\
(r_\delta, \frac{\xi_\delta}{\mu}, u_\delta), & -\frac{\mu}{2} < x < \frac{\mu}{2} \\
(r_1, u_1), & x > \frac{\mu}{2}
\end{cases} \tag{3.3}
\]

depending on the parameter \( \mu \gg \varepsilon \). The parameter \( 1 \gg \varepsilon > 0 \) is used in a construction of shadow waves (2.5) used for solving (3.1, 3.3) together with the other elementary waves. The parameter \( \mu \) tends to zero but significantly slower than \( \varepsilon \). A value \( u_\delta \) is artificially added into (3.3) and it does not have any influence on a distributional value of the initial data. A way to single out a proper solution is to find a value of \( u_\delta \) that eliminates all unphysical solutions. A distributional physically meaningful solution to the problem (3.1, 3.2) is then obtained by letting \( \mu \to 0 \) (that makes \( \varepsilon \to 0 \) even faster). We are looking for initial data that gives a minimal energy dissipation (or equivalently, a maximal local energy production) of a unique solution corresponding to initial data (3.3).

**Definition 3.1.** (Backward energy condition) Denote by \( U_{\mu}^{u_\delta} \) the admissible solution to (3.1, 3.3) and by \( \frac{d}{dt}H_{\mu}^{u_\delta} \) its energy dissipation. Take the value \( \tilde{u}_\delta \) such that the corresponding solution \( \tilde{U}_{\mu} \) has a minimal energy dissipation \( \frac{d}{dt}\tilde{H}_{\mu}(\cdot, t) \) for \( t \) and \( \mu \) small enough, i.e.

\[
\frac{d}{dt}\tilde{H}_{\mu}(\cdot, 0+) \leq \inf_{u_\delta \in \mathbb{R}} \frac{d}{dt}H_{\mu}^{u_\delta}(\cdot, 0+) \text{ for } \mu \text{ being small enough}. \tag{3.4}
\]

Then a solution \( U \) to (3.1, 3.2) that satisfies the backward energy condition is given by \( U(x, t) = \lim_{\mu \to 0} \tilde{U}_{\mu}(x, t) \). If there is more than one solution that satisfies the above condition, we choose \( u_\delta \) such that approximated initial data (3.3) has a minimal initial energy.

**Remark 3.1.** Note that a choice of an energy pair \( \tilde{\eta} \) and \( \tilde{Q} \) from Lemma 2.2 has a direct impact on the second criterion in Definition 3.1. The first one, (3.4), is independent of the choice.
The general algorithm goes as follows. Assume that a small parameter \( \mu > 0 \) and a value of \( u_\delta \) from (3.3) are given. Initially, an approximate solution is a solution to double Riemann problem (3.1.3.3). Those waves are uniquely determined by the relationship between \( u_0, u_\delta \) and \( u_1 \) as we have seen in the first part. The first wave or wave combination emanates from \( ( - \frac{\xi}{2} \mu, 0) \) and connects the state \( U_0 = (\rho_0, u_0) \) with \( U_\delta = \left( \frac{\xi}{\mu}, u_\delta \right) \), while the second one emanates from \( (\frac{\xi}{2}, 0) \) and connects the state \( U_\delta \) with \( U_1 = (\rho_1, u_1) \). Then, continue by following further wave interactions. The procedure resembles the method of higher order shadow waves introduced in [13].

Now, we shall apply the backward energy condition to three different systems of the form (3.1) described above.

3.1. Pressureless gas dynamics system

3.1.1. Weak solution and the backward energy condition. Recall, the energy–energy flux pair for the system is \( \eta = \frac{1}{2} \rho u^2 \) and \( Q = \eta u \). The local energy production of a shadow wave at time \( t \) is

\[
\mathcal{D}(t) = -u_s(t)[\eta] + [Q] + \lim_{\varepsilon \to 0} \frac{d}{dt} \left( 2(\xi t + x_\varepsilon) \eta(U(t)) \right) = -\frac{1}{2} \left( \rho_0(u_0 - u_s(t))^3 + \rho_1(u_s(t) - u_1)^3 \right).
\]

It is negative for a shadow wave and zero for a contact discontinuity. These facts will be used to choose a proper initial data.

Case A₁ \( (u_\delta < u_0 < u_1) \). The solution is given by the shadow wave connecting \( U_0 \) and \( U_\delta \) and the CD wave combination connecting \( U_\delta \) and \( U_1 \). The local energy production of the solution fully comes from the shadow wave,

\[
\mathcal{D} = -\frac{1}{2} \rho_0(u_0 - y_0,\delta)^3 - \frac{1}{2} \frac{\xi}{\mu} (y_0,\delta - u_\delta)^3 = -\frac{1}{2} \frac{\rho_0 \xi}{\mu + \sqrt{\rho_0}} (u_0 - u_\delta)^3 < 0
\]

for \( t \) small enough. It is clear that \( \mathcal{D} < 0 \) and increases to zero as \( u_\delta \to u_0 \).

Case A₂ \( (u_0 \leq u_\delta \leq u_1) \). In this case, the energy is conserved and we have \( \mathcal{D} = 0 \) for each \( t \geq 0 \) and each \( u_\delta \in [u_0, u_1] \). The total initial energy has a minimum at

\[
u_\delta = \begin{cases}
0, & \text{if } \text{sign}(u_0u_1) < 0 \\
\min\{u_0, u_1\}, & \text{if } \text{sign}(u_0), \text{sign}(u_1) \geq 0 \\
\max\{u_0, u_1\}, & \text{if } \text{sign}(u_0), \text{sign}(u_1) \leq 0.
\end{cases}
\]

Case A₃ \( (u_0 < u_1 < u_\delta) \). We have \( \mathcal{D} < 0 \) and \( \mathcal{D} \to 0 \) as \( u_\delta \to u_1 \) since

\[
\mathcal{D} = -\frac{1}{2} \frac{\rho_1 \xi}{\mu + \sqrt{\rho_1}} (u_0 - u_\delta)^3 < 0
\]

for \( t \) small enough (like in Case A₁).

Case B₁ \( (u_\delta < u_1 < u_0) \). Now,

\[
\mathcal{D} = -\frac{1}{2} \rho_0 \frac{\xi}{\mu + \sqrt{\rho_0}} (u_0 - u_\delta)^3 < -\frac{1}{2} \rho_0(u_0 - u_1)^3
\]

for \( t \) small enough.
Case $B_2$ ($u_1 \leq u_\delta \leq u_0$ with at least one inequality being strict). In this case
\[ D = -\frac{1}{2} \frac{\rho_0 \xi_\delta / \mu}{(\sqrt{\xi_\delta / \mu} + \sqrt{\rho_0})^2} (u_0 - u_\delta)^3 - \frac{1}{2} \frac{\rho_1 \xi_\delta / \mu}{(\sqrt{\xi_\delta / \mu} + \sqrt{\rho_1})^2} (u_\delta - u_1)^3 \]
for $t$ small enough. $D$ is a cubic function in $u_\delta$ having a maximum in
\[ y^\mu = \frac{u_0 \sqrt{r_0^\mu} + u_1 \sqrt{r_1^\mu}}{\sqrt{r_0^\mu} + \sqrt{r_1^\mu}}, \]
where $r_0^\mu = \frac{\rho_0 \xi_\delta / \mu}{(\sqrt{\xi_\delta / \mu} + \sqrt{\rho_0})^2}$ and $r_1^\mu = \frac{\rho_1 \xi_\delta / \mu}{(\sqrt{\xi_\delta / \mu} + \sqrt{\rho_1})^2}$
regardless of a relation between $\rho_0$ and $\rho_1$. Since $y^\mu \to y : = \frac{\sqrt{\rho_0 u_0} + \sqrt{\rho_1 u_1}}{\sqrt{\rho_0} + \sqrt{\rho_1}}$ as $\mu \to 0$, we have
\[ \lim_{\mu \to 0} \sup_{u_\delta \in (u_1, u_0)} D = D_{\text{max}} := -\frac{1}{2} \frac{\rho_0 \rho_1 (u_0 - u_1)^3}{(\sqrt{\rho_0} + \sqrt{\rho_1})^2}. \quad (3.6) \]
Case $B_3$ ($u_1 < u_0 < u_\delta$). Like in Case $B_1$, we have
\[ D = -\frac{1}{2} \rho_1 \frac{\xi_\delta / \mu (u_\delta - u_1)^3}{(\sqrt{\xi_\delta / \mu} + \sqrt{\rho_1})^2} < \frac{1}{2} \rho_1 (u_0 - u_1)^3 \]
for $t$ small enough.

Obviously, if $u_0 > u_1$, then
\[ \frac{\rho_0 \rho_1}{(\sqrt{\rho_0} + \sqrt{\rho_1})^2} < \min\{\rho_0, \rho_1\} \text{ and } D_{\text{max}} > -\frac{1}{2} \min\{\rho_0, \rho_1\} (u_0 - u_1)^3. \]
That is, $u_\delta = y^\mu$ is the value in (3.3) for which the corresponding solution satisfies the backward energy condition.

3.1.2. Limit of an energy admissible solution. For (2.9) we are able to easily obtain complete solution for $t > 0$ using procedures from [17]. Thus, there exists a distributional limit of the constructed approximate solution satisfying the backward energy condition as $\mu \to 0$. It will be taken to be the physically meaningful distributional solution to system (2.9) with the distributional initial data (3.2). The energy dissipation analysis described in the details below will show very interesting facts.

Lemma 3.1. (Lemma 3.1. from [17]) Let
\[ (\rho, u)(x, 0) = \begin{cases} (\rho_1, u_1), & x < 0 \\ (\rho_0, u_0) \delta, & (\rho u)(x, 0) = u_\delta \xi_\delta \delta, \end{cases} \]
be the initial data for (2.9). If $u_0 \geq u_\delta \geq u_1$, $\xi_\delta \geq 0$ and $\rho_0, \rho_1 \geq 0$, then the solution is the overcompressive shadow wave with strength $\xi(t)$ and speed $u_s(t)$ given by
\[ \xi(t) = \sqrt{\xi_\delta^2 + \rho_0 \rho_1 [u]^2 t^2 + 2\xi_\delta (u_\delta \rho - [\rho u])t}, \quad \xi(0) = \xi_\delta \]
\[ u_s(t) = \begin{cases} \left( \frac{1}{\xi(t)} [\rho u] + \rho_0 \rho_1 [u]^2 t + \xi_\delta (u_\delta \rho - [\rho u]) \right), & \text{if } \rho_0 \neq \rho_1, \\ \frac{\xi_\delta^2}{\xi_\delta^2(t)} (u_\delta - \frac{u_0 + u_1}{2}) + \frac{u_0 + u_1}{2}, & \text{if } \rho_0 = \rho_1, \end{cases} \quad (3.7) \]
The front of the resulting shadow wave is $x = c(t) = \int_0^t u_s(\tau) d\tau$. 

An interaction problem between two waves when at least one is a shadow wave can be interpreted as the initial value problem (2.9, 3.2) with the initial data translated to the interaction point \((X, T)\), where

\[
\xi_\delta = \xi_l(T) + \xi_r(T), \quad u_\delta = \frac{u_{s,l}(T)\xi_l(T) + u_{s,r}(T)\xi_r(T)}{\xi_l(T) + \xi_r(T)}.
\]

The values \(\xi_l(T)\) and \(u_{s,i}(T)\), \(i = l, r\) denote strengths and speeds of the left and right incoming wave at the interaction time \(t = T\).

For the readers convenience, we will present all possible solutions in the above cases. One can find some interesting things in their behavior.

**Case A1.** The local solution consists of a shadow wave emanating from \(x = -\mu/2\) and CD wave combination emanating from \(x = \mu/2\). The shadow wave is supported by \(x = -\frac{\mu}{2} + y_0 t\) with the speed \(y_0 := \frac{\mu\sqrt{\rho_0 + \rho_\delta\sqrt{\frac{\mu}{\mu} + \xi_\delta}}}{\sqrt{\rho_0 + \xi_\delta}}\). The speed \(y_0\) is greater than the slope \(u_\delta\) of the first contact discontinuity, so two waves will interact at the time \(t = T_1 = \frac{-\frac{\mu}{2} + u_\delta}{y_0} \sim \sqrt{\mu}\). Note that we have used that the external shadow wave line given by \(x = -\frac{\mu}{2} + y_0 t + \frac{\mu}{2} t\) first intersects the line \(x = \frac{\mu}{2} + u_\delta t\), but the term \(\frac{\mu}{2} t\) can be neglected since \(\varepsilon \ll \mu\) and \(\rho_\varepsilon(t) \sim \varepsilon^{-1}\) (Lemma 3.2. from [17]). The resulting shadow wave connects \(U_0\) and vacuum state and propagates with the strength \(\xi(t) = \sqrt{\xi_0^2 T_1^2 + 2\rho_0 \xi_0 T_1 (u_0 - y_0)(t - T_1)}\), where \(\xi_0 := \sqrt{\rho_0 \frac{\xi_\delta}{\mu}(u_0 - u_\delta)}\), and the speed \(u_s(t) = u_0 - \frac{\xi_0}{\xi(t)} (u_0 - y_0)\). Both functions are increasing, \(\xi(t)\) is non-negative and \(u_s(t) \to u_0 < u_1\) as \(t \to \infty\). Thus, the resulting shadow wave will not interact with second contact discontinuity. One can see that \(y_0 \to u_\delta\) and \(\xi_0 T_1 \to \xi_\delta\) as \(\mu \to 0\) by using the above expressions. Straightforward calculation gives that

\[
(\rho, u)(x, t) = \begin{cases} 
(\rho_0, u_0), & x < c(t) \\
(0, \frac{x}{t}), & c(t) < x < u_1 t + (\xi(t), 0)\delta(x - c(t)), \\
(\rho_1, u_1), & x > u_1 t
\end{cases}
\]

as the distributional limit of the approximate solution as \(\mu \to 0\). The functions \(\xi(t)\) and \(u_s(t)\) are given in (3.7), while \(c(t) = \int_0^t u_s(s) ds\).

**Case A2.** The solution consists of two CD wave combinations, one emanating from \(x = -\mu/2\) and the other one from \(x = \mu/2\). The approximate solution is given by

\[
U^\mu(x, t) = \begin{cases} 
(\rho_0, u_0), & x < -\frac{\mu}{2} + u_0 t \\
(0, \frac{x}{t}), & -\frac{\mu}{2} + u_0 t < x < -\frac{\mu}{2} + u_\delta t \\
(\frac{\xi_\delta}{\mu}, u_\delta), & -\frac{\mu}{2} + u_\delta t < x < \frac{\mu}{2} + u_\delta t \\
(0, \frac{x}{t}), & \frac{\mu}{2} + u_\delta t < x < \frac{\mu}{2} + u_1 t \\
(\rho_1, u_1), & x > \frac{\mu}{2} + u_1 t
\end{cases}
\]

If \(u_0 < u_\delta < u_1\), the distributional limit of (3.9) is a combination of two contact discontinuities and a delta shock supported by line \(x = u_\delta t\). If \(u_\delta\) coincides with \(u_0\) or \(u_1\), then there is one instead of two contact discontinuities and the shadow wave has a constant strength and characteristics speed. It is called the
delta contact discontinuity (see [15]). If \( u_0 = u_\delta = u_1 \), the solution is a single delta contact discontinuity.

Case A3. The local solution consists of the CD wave combination emanating from \( x = -\mu/2 \) and the shadow wave from \( x = \mu/2 \). The second contact discontinuity in the combination (of the slope \( u_\delta \)) interacts with the shadow wave having the speed \( y_1 := \frac{u_s \sqrt{\frac{\mu}{x} + u_1 \sqrt{\mu}}}{\sqrt{\frac{\mu}{x} + \sqrt{\mu}}} \). The resulting shadow wave propagates with a speed \( u_s(t) \) that increases over time and satisfies \( u_s(t) \to u_1 > u_0 \), \( t \to \infty \). That means that the left contact discontinuity will not overtake the resulting shadow wave. The distributional limit of approximate solution is a combination of contact discontinuity connecting \( U_0 \) and the vacuum state, and a weighted delta shock connecting the vacuum state to \( U_1 \), similarly to (3.8).

Case B1. Like in Case A1, the approximate solution initially consists of the shadow wave emanating from \( x = -\mu/2 \) and the CD wave combination from \( x = \mu/2 \). The shadow wave interacts with the first contact discontinuity at \( t = T_1 \) to form new shadow wave with speed \( u_s(t) \to u_0 \), \( t \to \infty \). Since \( u_0 > u_1 \), at \( t = T_2 \) the shadow wave will overtake the second contact discontinuity whose slope is \( u_1 \). The resulting one connects \( U_0 \) to \( U_1 \). Its speed is increasing, so \( T_2 < \frac{\mu}{y_0 - u_1} \sim \sqrt{\mu} \to 0 \) as \( \mu \to 0 \). Thus, both interactions occur in the time \( t \sim \sqrt{\mu} \) and the distributional limit of approximate solution is

\[
U(x, t) = \begin{cases} U_0, & x < c(t) \\ U_1, & x > c(t) + (\xi(t), 0)\delta(x - c(t)), \end{cases}
\]  

(3.10)

where \( \xi(t) \) and \( u_s(t) \) are given by (3.7) and \( c(t) = \int_0^t u_s(s) ds \).

Case B2. The solution to (2.9, 3.3) consists of two overcompressive shadow waves interacting at time \( T = \frac{\mu}{y_0 - y_1} \sim \sqrt{\mu} \). One from \( x = -\mu/2 \) propagates with speed \( y_0 \) and strength \( \xi_0 \), while the other one propagates from \( x = \mu/2 \) with speed \( y_1 \) and strength \( \xi_1 \). The initial speed \( u_s := u_s(T + 0) \) and strength \( \xi := (T + 0) \) of the resulting overcompressive shadow wave are

\[
u_s = \frac{y_0 \xi_0 + u_0 \xi_1}{\xi_0 T + \xi_1 T}, \quad \xi = \xi_0 T + \xi_1 T.
\]

We have \( y_0 - y_1 \sim \sqrt{\frac{\mu}{\xi_0}(\sqrt{\mu}(u_0 - u_\delta) + \sqrt{\mu}(u_\delta - u_1))} \) as \( \mu \to 0 \). That fact together with \( y_0, y_1 \to u_\delta \) as \( \mu \to 0 \) implies \( \xi \to \xi_\delta \) and \( u_s \to u_\delta \) as \( \mu \to 0 \). A distributional limit as \( \mu \to 0 \) is an overcompressive delta shock (3.10) with \( u_s(t) \) and \( \xi(t) \) given by (3.7).

Case B3. Immediately after the initial time, the solution consists of the CD wave combination and the shadow wave as in Case A3. The result of interaction between the second contact discontinuity and the shadow wave is a new shadow wave that propagates with a speed \( u_s(t) \to u_1 \), \( t \to \infty \). Now, \( u_1 < u_0 \) and the first contact discontinuity will overtake the shadow wave when \( t = T_2 \sim \sqrt{\mu} \). Both interactions occur when \( t \sim \sqrt{\mu} \), The distributional limit of approximate solution is (3.10).

Let us now write down solutions satisfying the backward energy condition. If \( u_0 \leq u_1 \), the weak solution that solves (2.9, 3.2) and satisfies backward energy condition is one or two (when \( \text{sign}(u_0 u_1) < 0 \)) contact discontinuities.
combined with the delta contact discontinuity given by

\[ U(x, t) = \begin{cases} 
(\rho_0, u_0), & x < u_0 t \\
(0, \frac{\xi_0}{\sqrt{\rho_0}}), & u_0 t < x < u_\delta t \\
(0, \frac{\xi_0}{\sqrt{\rho_1}}), & u_\delta t < x < u_1 t + (\xi_\delta, 0)\delta(x - u_\delta t) \\
(\rho_1, u_1), & x > u_1 t,
\end{cases} \quad (3.11) \]

where \( u_\delta \in [u_0, u_1] \) is given in (3.5) so that the total initial energy is minimized.

If \( u_0 > u_1 \), the weak solution that satisfies the backward energy condition is given by

\[ U(x, t) = \begin{cases} 
(\rho_0, u_0), & x < y t \\
(\rho_1, u_1), & x > y t + (\xi + \xi_0, 0)\delta(x - y t),
\end{cases} \quad (3.12) \]

where \( \xi = \sqrt{\rho_0 \rho_1} (u_0 - u_1) \), \( y = \frac{u_0 \sqrt{\rho_0} + u_1 \sqrt{\rho_1}}{\sqrt{\rho_0} + \sqrt{\rho_1}} \). The speed and strength of (3.12) are obtained from (3.7) by putting \( u_\delta = y \).

The above results are summarized in the following theorem.

**Theorem 3.1.** Let \((2.9, 3.2)\) be given, where \( \rho_0, \rho_1 > 0 \) and \( \xi_\delta > 0 \). If \( u_0 > u_1 \), the admissible weak solution satisfying the backward energy condition is the delta shock wave propagating with constant speed \( y \) given by (3.12). If \( u_0 \leq u_1 \), the backward admissible solution locally conserves energy in the sense of distributions and it is given by (3.11).

**Remark 3.2.** Additionally, if the initial speed of a shadow wave (2.11) equals \( y \), its local energy production is maximal and equals \( D_{\text{max}} \) defined in (3.6) for each \( t \). That is a consequence of the fact that for a shadow wave with the initial speed \( y \) we have \( u_s(t) = y, t \geq 0 \). We say that such a wave is in its equilibrium state. Since \( u_s(t) \to y \) as \( t \to \infty \) the following holds: A (shadow) wave that starts in the non-equilibrium state (a non-positive local energy production across its front has not reached a maximum value) will adjust its speed to approach the equilibrium state. That is not the case in general, as it will be demonstrated for systems (2.12) and (2.14).

### 3.2. Chaplygin model

The backward energy condition can also be applied successfully for the Chaplygin model (2.12). The energy density is \( \eta = \rho u^2 + \frac{1}{\rho} \) with the flux \( Q = \rho u^3 - \frac{u}{\rho} \). Using (2.17) we get that the local energy production of a shadow wave connecting \((\rho_0, u_0)\) to \((\rho_1, u_1)\) equals

\[ D = -s[\eta] + [Q] + \kappa s^2, \]

where \( \kappa \) and \( s \) are defined in (2.13). The structure of approximate solution and its analysis are similar to the one for pressureless gas dynamics system.

**Theorem 3.2.** Let the problem \((2.12, 3.2)\) be given, where \( \rho_0, \rho_1 > 0 \) and \( \xi_\delta > 0 \). If \( \lambda_1(\rho_0, u_0) > \lambda_2(\rho_1, u_1) \), the admissible weak solution that meets the backward energy condition is a weighted delta shock. If \( \lambda_1(\rho_0, u_0) \leq \lambda_2(\rho_1, u_1) \), the solution satisfying the backward energy condition locally conserves energy in the
sense of distributions and

\[ u_\delta = \begin{cases} 
\lambda_2(\rho_1, u_1), & \text{if } \lambda_2(\rho_1, u_1) < 0 \\
\lambda_1(\rho_0, u_0), & \text{if } \lambda_1(\rho_0, u_0) > 0 \\
0, & \text{otherwise.}
\end{cases} \]

**Proof.** Let us first consider the case \( \lambda_1(\rho_0, u_0) \leq \lambda_2(\rho_1, u_1) \). The situation is analogous to the one obtained for \( u_0 \leq u_1 \) in the pressureless system.

If \( u_\delta \in [\lambda_1(\rho_0, u_0), \lambda_2(\rho_1, u_1)] \), the solution to the problem \((2.12, 3.3)\) is a combination of contact discontinuities, and its local energy production equals zero for each \( u_\delta \). Otherwise it would be negative due to presence of a shadow wave. The distributional limit of such solution is

\[ U(x, t) = \begin{cases} 
(\rho_0, u_0), & x < \lambda_1(\rho_0, u_0)t \\
(\rho_{m_1}, u_{m_1}), & \lambda_1(\rho_0, u_0)t < x < u_\delta t \\
(\rho_{m_2}, u_{m_2}), & u_\delta t < x < \lambda_2(\rho_1, u_1)t + (\xi_\delta, 0)\delta(x - u_\delta t), \\
(\rho_1, u_1), & x > \lambda_2(\rho_1, u_1)t 
\end{cases} \]

where

\[
(\rho_{m_1}, u_{m_1}) = \left( \frac{2}{u_\delta - u_0 + \rho_0^{-1}}, \frac{u_\delta + u_0 - \rho_0^{-1}}{2} \right),
\]

\[
(\rho_{m_2}, u_{m_2}) = \left( \frac{2}{u_1 + \rho_1^{-1} - u_\delta}, \frac{u_\delta + u_1 + \rho_1^{-1}}{2} \right).
\]

The value of \( u_\delta \) that minimizes \( u_\delta^2 \), and consequently the total initial energy is a proper choice for \( u \) component of initial data.

There are three possible types of solution when \( \lambda_1(\rho_0, u_0) > \lambda_2(\rho_1, u_1) \). Let \( u_\delta < \lambda_1(\rho_0, u_0) \). Then \( U_0 = (\rho_0, u_0) \) and \( U_\delta = (\xi_\delta, u_\delta) \) are connected by the shadow wave with strength \( \kappa = \sqrt{[\rho u^2]_0 - [\rho u^2 - \rho^{-1}]_0} \) and speed \( s = [\rho u]_0 [\rho u^2]_0 + \kappa [\rho]_0 \). We have used the notation \([\cdot]_0 = \cdot|_{u_\delta} - \cdot|_{u_0} \), see (2.13). Then

\[
s - u_\delta = (u_\delta - u_0) \left( \frac{\rho_0}{\xi_\delta / \mu - \rho_0} + \frac{\kappa}{(\xi_\delta / \mu - \rho_0)(u_\delta - u_0)} \right) =: (u_\delta - u_0)d_1^\mu \quad \text{and}
\]

\[
s - u_0 = (u_\delta - u_0) \left( \frac{\xi_\delta / \mu}{\xi_\delta / \mu - \rho_0} + \frac{\kappa}{(\xi_\delta / \mu - \rho_0)(u_\delta - u_0)} \right) =: (u_\delta - u_0)d_2^\mu.
\]

The local energy production of the shadow wave connecting \( U_0 \) to \( U_\delta \) equals

\[
D = \frac{\xi_\delta}{\mu} (u_\delta - s)^3 + \rho_0(s - u_0)^3 + \rho_0^{-1}(u_0 - s) + \frac{\mu}{\xi_\delta} (s - u_0)
\]

\[
= (u_0 - u_\delta)^3 \left( (d_1^\mu)^3 \frac{\xi_\delta}{\mu} - (d_2^\mu)^3 \rho_0 \right) + (u_0 - u_\delta) \left( \rho_0^{-1}d_2^\mu - \mu \frac{\xi_\delta}{\mu} \right).
\]

We have \( d_1^\mu = O(\sqrt{\mu}) \) and \( d_2^\mu = 1 + O(\sqrt{\mu}), \mu \to 0 \). Thus,

\[
D \approx \rho_0(u_0 - u_\delta)^3 + \frac{1}{\rho_0}(u_0 - u_\delta) \text{ for } \mu \text{ small enough.}
\]
Similarly, if \( u_\delta > \lambda_2(U_1) \), then \( U_\delta \) and \( U_1 = (\rho_1, u_1) \) are connected by a shadow wave and

\[
D \approx \rho_1(u_1 - u_\delta)^3 + \frac{1}{\rho_1}(u_\delta - u_1) \text{ for } \mu \text{ small enough.}
\]

If \( u_\delta \leq \lambda_2(U_1) \) one can see that \( D \) has its maximum at \( u_\delta = \lambda_2(U_1) \). In the case \( u_\delta \geq \lambda_1(U_0) \), \( D \) has it at \( u_\delta = \lambda_1(U_0) \) as it was the case for the pressureless gas dynamics. In those two cases solutions to (2.12, 3.3) are weighted delta shock and its local energy production is not constant with \( U \).

There are the following possibilities for a short time solution. Depending on relations between the above constants, called the switch points, the last model analyzed in the paper is system (2.14) with energy pair (2.18).

3.3. Generalized Chaplygin gas model

The last model analyzed in the paper is system (2.14) with energy pair (2.18). A type of solution changes when \( u_\delta \) takes the following values:

\[
A_1 = u_0 - \rho_0^{-\frac{1+\alpha}{2}}, \quad A_2 = u_0 + \frac{2\sqrt{\alpha}}{1+\alpha} \rho_0^{-\frac{1+\alpha}{2}},
\]

\[
B_1 = u_1 - \frac{2\sqrt{\alpha}}{1+\alpha} \rho_1^{-\frac{1+\alpha}{2}}, \quad B_2 = u_1 + \rho_1^{-\frac{1+\alpha}{2}}.
\]

Depending on relations between the above constants, called the switch points, there are the following possibilities for a short time solution.

- The state \((\rho_0, u_0)\) is connected to \((\xi\delta, u_\delta)\) by
  - \( R_1 + R_2 \) if \( u_\delta > A_2 \)
  - \( S_1 + R_2 \) if \( A_1 < u_\delta \leq A_2 \)
\begin{itemize}
  \item shadow wave if \( u_\delta \leq A_1 \)
  \item The state \( (\xi_0, u_\delta) \) is connected to \((\rho_1, u_1)\) by
    \begin{itemize}
      \item \( R_1 + R_2 \) if \( u_\delta < B_1 \)
      \item \( R_1 + S_2 \) if \( B_1 \leq u_\delta < B_2 \)
      \item shadow wave if \( u_\delta \geq B_2 \).
    \end{itemize}
\end{itemize}

Note that the local energy production for a rarefaction wave equals zero due to their continuity and it is negative for other waves. The following lemmas will describe the production in all the above cases.

**Lemma 3.2.** The local energy production of the shadow wave connecting \((\rho_0, u_0)\) and \((\xi_0, u_\delta)\), \( u_\delta \leq A_1 \) equals
\[
D = \frac{1}{2} \rho_0 (u_\delta - u_0)^3 - \frac{\alpha}{1 + \alpha} \rho_0^{-\alpha} (u_\delta - u_0) + O(\sqrt{\mu}) \quad \text{as} \quad \mu \to 0
\]
and the limit has a maximum at \( A_1 \).

For the shadow wave connecting \((\xi_0, u_\delta)\) and \((\rho_1, u_1)\), \( u_\delta \geq B_2 \) we have
\[
D = \frac{1}{2} \rho_1 (u_1 - u_\delta)^3 - \frac{\alpha}{1 + \alpha} \rho_1^{-\alpha} (u_1 - u_\delta) + O(\sqrt{\mu}) \quad \text{as} \quad \mu \to 0.
\]
The limit has a maximum at \( B_2 \).

The solution composed of two shadow waves exists if \( B_2 \leq u_\delta \leq A_1 \) and its local energy production has a maximum at
\[
u_\delta = \frac{[\rho u]}{[\rho]} + \frac{1}{[\rho]} \sqrt{\rho_0 \rho_1 [u]^2 + 2 \frac{\alpha}{3(1+\alpha)} [\rho][\rho^{-\alpha}]} =: x_0
\]
if \( x_0 \in [B_2, A_1] \). Otherwise, a maximum point will be \( B_2 \) or \( A_1 \).

**Proof.** By using the cubic function properties, we can see that \( f_1 \) has a maximum at \( u_\delta = u_0 - \sqrt{\alpha} \sqrt{\frac{2}{3(1+\alpha)} \rho_0^{\frac{1+\alpha}{2}}} > A_1 \), while \( f_2 \) has a maximum at \( u_\delta = u_1 + \sqrt{\alpha} \sqrt{\frac{2}{3(1+\alpha)} \rho_1^{\frac{1+\alpha}{2}}} < B_2 \). The sum \( f_1 + f_2 \) has a maximum at
\[
x_0 = \frac{[\rho u]}{[\rho]} + \frac{1}{[\rho]} \sqrt{\rho_0 \rho_1 [u]^2 + 2 \frac{\alpha}{3(1+\alpha)} [\rho][\rho^{-\alpha}]}.
\]
If \( x_0 \in (B_2, A_1) \), then \( u_\delta = x_0 \). Otherwise, \( u_\delta \) is exactly one of the endpoints of the interval, \( u_\delta = B_2 \) or \( u_\delta = A_1 \) depending on the given values for \((u_0, \rho_0)\) and \((u_1, \rho_1)\), since a solution composed of two shadow waves exists if \( B_2 \leq u_\delta \leq A_1 \).

**Lemma 3.3.** The local energy production of the \( S_1 + R_2 \) solution connecting \((\rho_0, u_0)\) and \((\xi_0, u_\delta)\) with an intermediate state \((\rho_m, u_m)\) is
\[
D = A(\rho_0, \rho_m) \left( \frac{1}{2} \frac{\rho_m - \rho_0}{\rho_m} (\rho_0^{-\alpha} - \rho_m^{-\alpha}) + \frac{1}{1 + \alpha} (\rho_m^{-\alpha} - \rho_0^{-\alpha}) + \frac{\alpha}{1 + \alpha} \rho_m^{-\alpha} \frac{\rho_m - \rho_0}{\rho_m} \right).
\]
The value $A(\rho_0, \rho_m)$ is defined in (2.15), while $\rho_m$ satisfies
\[
u_0 - \frac{\rho_m - \rho_0}{\rho_m} A(\rho_0, \rho_m) = u_\delta - \frac{2\sqrt{\alpha}}{1 + \alpha} \frac{\rho_m^{-\alpha}}{\rho_m} - \mathcal{O}(\mu^{\frac{1+\alpha}{2}}), \quad \rho_m > \rho_0. \quad (3.14)
\]
Also, $\mathcal{D}$ increases with respect to $u_\delta$ and $0 \leq \frac{\partial \mathcal{D}}{\partial u_\delta} \leq m_1 := \left(\frac{3}{2} - \frac{\alpha}{1+\alpha}\right) \rho_0^{-\alpha}$ for $u_\delta \in [A_1, A_2]$.

Proof. Put
\[
a := \frac{1}{2} \frac{\rho_m - \rho_0}{\rho_m} (\rho_0^{-\alpha} - \rho_m^{-\alpha}),
\]
\[
b := \frac{1}{1 + \alpha} (\rho_0^{-\alpha} - \rho_m^{-\alpha}) - \frac{\alpha}{1 + \alpha} \rho_m^{-\alpha} \rho_m - \rho_0.
\]
Then $\mathcal{D} = A(\rho_0, \rho_m)(a - b)$. The function $a$ increases with $\rho_m$ and $a = 0$ for $\rho_m = \rho_0$, i.e., $a > 0$. Also, $b$ increases with $\rho_m$, $b = 0$ if $\rho_m = \rho_1$, so $b > 0$.

Using The Chain Rule and the Implicit Function theorem we obtain
\[
\frac{\partial \mathcal{D}}{\partial u_\delta} = \frac{\partial \mathcal{D}}{\partial \rho_m} \frac{\partial \rho_m}{\partial u_\delta},
\]
where
\[
\frac{\partial \mathcal{D}}{\partial \rho_m} = -\frac{1}{2} A(\rho_0, \rho_m) (2a - (1 + \alpha)b) \frac{a + b}{(\rho_m - \rho_0)(\rho_0^{-\alpha} - \rho_m^{-\alpha})}.
\]
\[
\frac{\partial \rho_m}{\partial u_\delta} = -\frac{(\rho_0^{-\alpha} - \rho_m^{-\alpha}) (\sqrt{\alpha} \rho_m^{-\frac{\alpha}{1+\alpha}} + \rho_0 \rho_m A(\rho_0, \rho_m)) + \frac{1}{2} A(\rho_0, \rho_m)(2a - (1 + \alpha)b).}
\]
The inequality $\frac{\partial \mathcal{D}}{\partial u_\delta} \geq 0$ follows from $\rho_m \geq \rho_0$ and the fact that $\rho_m (2a - (1 + \alpha)b)$ increases with $\rho_m$ increases, so $2a - (1 + \alpha)b \geq 0$. Next, we have
\[
\frac{\partial \mathcal{D}}{\partial u_\delta} \leq \frac{\rho_m}{\rho_m - \rho_0} (a + b) \leq m_1.
\]
The above inequality holds because the values $\frac{\rho_m}{\rho_m - \rho_0} a$ and $\frac{\rho_m}{\rho_m - \rho_0} b$ are non-negative, increasing with $\rho_m$ and
\[
\lim_{\rho_m \to \infty} \frac{\rho_m}{\rho_m - \rho_0} a = \frac{1}{2} \rho_0^{-\alpha}, \quad \lim_{\rho_m \to \infty} \frac{\rho_m}{\rho_m - \rho_0} b = \frac{1}{1 + \alpha} \rho_0^{-\alpha}.
\]

The following lemma can be proved in the same way.

**Lemma 3.4.** The local energy production of the $R_1 + S_2$ solution connecting $(\xi_\mu, u_\delta)$ and $(\rho_1, u_1)$ with an intermediate state $(\rho_m, u_m)$ is
\[
\mathcal{D} = A(\rho_1, \rho_m) \left(\frac{1}{2} \frac{\rho_m - \rho_1}{\rho_m} (\rho_1^{-\alpha} - \rho_m^{-\alpha}) + \frac{1}{1 + \alpha} (\rho_m^{-\alpha} - \rho_1^{-\alpha}) + \frac{\alpha}{1 + \alpha} \rho_m^{-\alpha} \rho_m - \rho_1 \right),
\]
where $\rho_m$ satisfies
\[
u_1 + \frac{\rho_m - \rho_1}{\rho_m} A(\rho_1, \rho_m) = u_\delta + \frac{2\sqrt{\alpha}}{1 + \alpha} \rho_m^{-\frac{1+\alpha}{2}} + \mathcal{O}(\mu^{\frac{1+\alpha}{2}}), \quad \rho_m > \rho_1. \quad (3.16)
\]
Also, \( D \) decreases with respect to \( u_\delta \) and \(-m_2 := -\left(\frac{3}{2} - \frac{\alpha}{1+\alpha}\right)\bar{\rho}_1^{-\alpha} \leq \frac{\partial D}{\partial u_\delta} \leq 0 \) for \( u_\delta \in [B_1, B_2] \).

All the above lemmas have to be used in the proof of following theorem.

**Theorem 3.3.** There exists \( u_\delta \in [\min\{A_2, B_1\}, \max\{A_2, B_1\}] \) and the corresponding solution to the problem (2.14, 3.2) satisfying backward energy condition:

1. If \( A_1 < A_2 < B_1 < B_2 \), a solution in a short time interval is \( R_1 + R_2 \) followed by \( R_1 + R_2 \).
2. If \( A_1 < B_1 < A_2 < B_2 \), we have \( S_1 + R_2 \) followed by \( R_1 + S_2 \) as a solution.
3. If \( A_1 < B_1 < B_2 < A_2 \), we have \( S_1 + R_2 \) followed by \( R_1 + S_2 \) or \( S_1 + R_2 \) followed by SDW as a solution.
4. If \( B_1 < A_1 < A_2 < B_2 \), we have SDW followed by \( R_1 + S_2 \) or \( S_1 + R_2 \) followed by \( R_1 + S_2 \) as a solution.
5. If \( B_1 < A_1 < B_2 < A_2 \), we have SDW followed by \( R_1 + S_2 \) or \( S_1 + R_2 \) followed by \( R_1 + S_2 \) followed by SDW as a solution.
6. If \( B_1 < B_2 < A_1 < A_2 \), we have SDW followed by \( R_1 + S_2 \) or SDW followed by SDW or \( S_1 + R_2 \) followed by SDW as a solution.

**Remark 3.3.** To obtain the form of approximate solution for each \( t > 0 \) and a corresponding distributional limit, it is necessary to know the result of all interaction problems including at least one shadow wave. That also includes interactions between shadow and rarefaction waves. But, contrary to the pressureless and Chaplygin system, we still do not know a result of all wave interactions for the generalized Chaplygin model. That is left for future work.

**Proof.** The proof of this theorem reduces to a search for \( u_\delta \) that maximizes non-positive local energy. The choice of \( u_\delta \) will depend on the relation between the switch points \( A_1, A_2, B_1, B_2 \) from both sides is negligible with respect to \( \mu \).

First, we will prove that \( D \), defined in Lemmas 3.2–3.4 converges to a continuous function as \( \mu \to 0 \). That is, the difference in the points \( A_1, A_2, B_1, B_2 \) from both sides is negligible with respect to \( \mu \).

Looking from left to right, when \( R_1 + R_2 \) goes to \( S_1 + R_2 \) or \( R_1 + S_2 \) and vice versa, the function \( D \), given in (3.13) and (3.15) is continuous since a strength of the shock wave tends to zero as \( u_\delta \) tends to a switch point. Now, let us look at the case when a shadow wave lies on the left while \( S_1 \) lies on the right-hand side of a switch point. Let \( u_\delta \to A_1 = u_0 - \rho_0^{-\frac{1+\alpha}{\alpha}} \) from the right-hand side. \( S_1 \) connects the states \( (\rho_0, u_0) \) and \( (\rho_m, u_m) \). Substitution of \( u_\delta = A_1 \) into (3.14) and taking a limit as \( \mu \to 0 \) gives the following equation for \( \rho_m \)

\[
\frac{2\sqrt{\alpha}}{1+\alpha} \rho_m^{-\frac{1+\alpha}{2}} - \sqrt{\frac{\rho_m - \rho_0}{\rho_0}} \left( \rho_0^{-\alpha} - \rho_m^{-\alpha} \right) = -\rho_0^{-\frac{1+\alpha}{\alpha}}.
\]

\( =: f(\rho_m) \)
We have
\[ f' = -\sqrt{\alpha} \rho_m^{-\frac{3+\alpha}{2}} - \frac{\rho_0^{-\alpha} \rho_m^{-2} + \rho_0^{-1} \rho_m^{-\alpha-1} - (1+\alpha)\rho_m^{-\alpha-2}}{2\sqrt{\rho_m-\rho_0}} < 0 \quad \text{for } \rho_m > \rho_0, \]
so the function \( f \) is decreasing. Also, \( f(\rho_0) > -\rho_0^{\frac{1+\alpha}{2}} \) and \( \lim_{\rho_m \to \infty} f(\rho_m) = -\rho_0^{\frac{1+\alpha}{2}} \). That means \( \rho_m \to \infty \) as \( u_\delta \to A_1 \). Finally, \( \lim_{\rho_m \to \infty} D = f_1(A_1) \).

The similar arguments apply when \( S_2 \) is on the left-hand and a shadow wave on the right-hand side of the point \( B_2 \). One just has to use (3.15) instead od (3.13) and (3.16) instead of (3.14). Thus, the continuity of the limit of \( D \) on \( \mathbb{R} \) follows.

Now, we will explain all possible 6 cases for switch points order.

**Case** \( A_1 < A_2 < B_1 < B_2 \). This case is the simplest one, since for \( u_\delta \notin [A_2,B_1] \) the local energy production is negative, so the solution dissipates the energy. For \( u_\delta \in [A_2,B_1] \) the solution is given as a combination of rarefaction waves \((R_1 + R_2) connecting (\rho_0, u_0) and (\xi_\mu^-, u_\delta)\) followed by \( R_1 + R_2 \) connecting \((\xi_\mu^-, u_\delta)\) and \((\rho_1, u_1)\). Being a smooth solutions, rarefaction waves conserve energy, so the optimal choice for \( u_\delta \) is value from \( (A_2,B_1) \) chosen such that the initial total energy is minimal. That reduces to finding \( u_\delta \) such that \( u_\delta^2 \) is minimal.

**Case** \( A_1 < B_1 < A_2 < B_2 \). In this case the solution is given in the form of \( R_1 + R_2 \) which follows (is being followed by) SDW or \( S_1 + R_2 \) (\( R_1 + S_2 \) or SDW) for \( u_\delta \notin (B_1,A_2) \). Using Lemmas 3.2, 3.3 and 3.4 one easily proves that the total energy production increases at least for \( u_\delta < B_1 \) and decreases at least for \( u_\delta > A_2 \). Furthermore, \( \frac{\partial D}{\partial u_\delta} u_\delta = B_1 > 0 \) and \( \frac{\partial D}{\partial u_\delta} u_\delta = A_2 < 0 \), meaning that there exists \( u_\delta \in (B_1,A_2) \) such that \( \frac{\partial D}{\partial u_\delta} = 0 \). The corresponding admissible solution is given by \( S_1 + R_2 \) connecting \((\rho_0, u_0)\) and \((\xi_\mu^- u_\delta)\) followed by \( R_1 + S_2 \) connecting \((\xi_\mu^-, u_\delta)\) and \((\rho_1, u_1)\).

**Case** \( A_1 < B_1 < B_2 < A_2 \). This case is possible only if \( \rho_0 \leq \rho_1 \). Like in the previous case \( D \) increases for \( u_\delta < B_1 \) and decreases for \( u_\delta > A_2 \). Next, \( \frac{\partial D}{\partial u_\delta} u_\delta = B_1 > 0 \) and \( \frac{\partial D}{\partial u_\delta} u_\delta = A_2 < 0 \), so there exists \( u_\delta \in (B_1,A_2) \) such that \( \frac{\partial D}{\partial u_\delta} = 0 \). If such \( u_\delta \) belongs to the interval \((B_1,B_2)\), then the admissible solution is given in the form of \( S_1 + R_2 \) followed by \( R_1 + S_2 \). Otherwise, if \( u_\delta \in (B_2,A_2) \), then the solution is \( S_1 + R_2 \) followed by SDW.

**Case** \( B_1 < A_1 < A_2 < B_2 \). This case is also possible only if \( \rho_1 \leq \rho_0 \). \( D \) is increasing for \( u_\delta < B_1 \) and decreasing for \( u_\delta > A_2 \) and it achieves maximum value for some \( u_\delta \in (B_1,A_2) \). If such \( u_\delta \) belongs to \((B_1,A_1)\) the solution is given in the form of SDW followed by \( R_1 + R_2 \), otherwise it is \( S_1 + R_2 \) followed by \( R_1 + S_2 \).

**Case** \( B_1 < A_1 < B_2 < A_2 \). The admissible solution depends on the relation between \( \rho_0 \) and \( \rho_1 \). If \( \rho_0 \leq \rho_1 \), then \( m_1 \geq m_2 \) and
\[ \frac{\partial D}{\partial u_\delta} > \frac{\partial D}{\partial u_\delta} u_\delta = A_1 > m_1 - m_2 \geq 0 \quad \text{for } u_\delta < A_1. \]
Thus, $D$ decreases for $u_δ > B_2$. Maximal value is achieved in the interval $(A_1, A_2)$ and the solution is given in the form of $S_1 + R_2$ followed by $R_1 + S_2$ or SDW. If $ρ_1 ≤ ρ_0$, we have $m_1 ≤ m_2$, so

$$\frac{∂D}{∂u_δ} < m_1 - m_2 ≤ 0 \text{ for } u_δ > B_2$$

and $D$ increases for $u_δ < B_1$. Maximal value is achieved in the interval $(B_1, B_2)$ and the solution is given in the form of SDW or $S_1 + R_2$ followed by $R_1 + S_2$ solution.

**Case $B_1 < B_2 < A_1 < A_2$.** Again, if $ρ_0 ≤ ρ_1$,

$$\frac{∂D}{∂u_δ} > m_1 - m_2 ≥ 0 \text{ for } u_δ < B_2$$

and $D$ decreases for $u_δ > A_2$. Maximal value is achieved in the interval $(B_2, A_2)$ and the solution is given in the form of SDW or $S_1 + R_2$ followed by SDW. If $ρ_1 ≤ ρ_0$,

$$\frac{∂D}{∂u_δ} < m_1 - m_2 ≤ 0 \text{ for } u_δ > A_1$$

and $D$ increases for $u_δ < B_1$, so it achieves maximal value in the interval $(B_1, A_1)$. The admissible solution is given in the form of SDW followed by $R_1 + S_2$ or SDW.

In this case, knowing the values $ρ_0, ρ_1, u_0$ and $u_1$, it is possible to determine the explicit admissible solution form.

If $x_0$ from Lemma 3.2 belongs to the interval $(B_2, A_1)$ the admissible solution is given as a combination of two shadow waves.

Let $ρ_0 ≤ ρ_1$. The relation $A_1 ≥ \bar{A} := u_1 + ρ_1^{-\frac{1}{α}}$

$$\sqrt{ρ_0^{-α}(1 - \frac{2}{3}\frac{α}{1+α}) + \frac{2}{3}\frac{α}{1+α}ρ_0^{-α}}$$

implies $B_2 ≤ x_0 ≤ A_1$ and $\frac{∂D}{∂u_δ} < 0$ for $u_δ ≥ A_1$, so the optimal solution is combination of two shadow waves with $u_δ = x_0$. If $A_1 < \bar{A}$, $\frac{∂D}{∂u_δ} > 0$ for $u_δ ∈ (B_2, A_1)$ and $x_0 > A_1$ so the admissible solution is given by $S_1 + R_2$ followed by shadow wave.

Similarly, if $ρ_0 ≥ ρ_1$ and $B_2 ≤ \bar{B} := u_0 - ρ_0^{-\frac{1}{α}}$

$$\sqrt{ρ_1^{-α}(1 - \frac{2}{3}\frac{α}{1+α}) + \frac{2}{3}\frac{α}{1+α}ρ_0^{-α}}$$

we have $x_0 ∈ (B_2, A_1)$ and the admissible solution is combination of two shadow waves. Otherwise, the maximum value for $D$ is achieved for $u_δ$ in $(B_1, B_2)$ and the optimal solution is shadow wave followed by $R_1 + S_2$.

The above results are illustrated in the table in Appendix. □

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### Appendix

1. \[
\begin{align*}
\text{SDW} - R_1 + R_2 & \quad S_1 + R_2 - R_1 + R_2 & \quad R_1 + R_2 - R_1 + S_2 & \quad R_1 + S_2 - R_1 + S_2 & \quad R_1 + R_2 - SDW \\
D \text{ increases} & & & & \\
A_1 & & & & \\
\end{align*}
\]

\[D = 0 \quad B_1 \quad D \text{ decreases} \quad B_2 \quad D \text{ decreases} \]

\[\text{Minimal initial total energy achieved for } u_0 \text{ given in (3.34)}\]

2. \[
\begin{align*}
\text{SDW} - R_1 + R_2 & \quad S_1 + R_2 - R_1 + R_2 & \quad S_1 + R_2 - R_1 + S_2 & \quad R_1 + R_2 - R_1 + S_2 & \quad R_1 + R_2 - SDW \\
D \text{ increases} & & & & \\
A_1 & & & & \\
\end{align*}
\]

\[\alpha D = 0 \text{ for some } u_0 \]

3. \[
\begin{align*}
\text{SDW} - R_1 + R_2 & \quad S_1 + R_2 - R_1 + R_2 & \quad S_1 + R_2 - R_1 + S_2 & \quad S_1 + R_2 - SDW & \quad R_1 + R_2 - SDW \\
D \text{ increases} & & & & \\
A_1 & & & & \\
\end{align*}
\]

\[\alpha D = 0 \text{ for some } u_0 \in (B_1, A_2)\]

Remark: \(A_1 < B_1 < B_2 < A_2\) only if \(p_0 \leq p_1\)

4. \[
\begin{align*}
\text{SDW} - R_1 + R_2 & \quad \text{SDW} - R_1 + S_2 & \quad S_1 + R_2 - R_1 + S_2 & \quad S_1 + R_2 - SDW & \quad R_1 + R_2 - SDW \\
D \text{ increases} & & & & \\
B_1 & & A_1 & & \\
\end{align*}
\]

\[\alpha D = 0 \text{ for some } u_0 \in (B_1, A_2)\]

Remark: \(B_1 < A_1 < A_2 < B_2\) only if \(p_1 \leq p_0\)

5. \[
\begin{align*}
\text{SDW} - R_1 + R_2 & \quad \text{SDW} - R_1 + S_2 & \quad S_1 + R_2 - R_1 + S_2 & \quad S_1 + R_2 - SDW & \quad R_1 + R_2 - SDW \\
D \text{ increases} & & & & \\
B_1 & & A_1 & & \\
\end{align*}
\]

\[D \text{ increases if } p_0 \leq p_1 \]

\[\text{If } p_0 \leq p_1, \alpha D = 0 \text{ for some } u_0 \in (A_1, A_2)\]

\[\text{If } p_1 \leq p_0, \alpha D = 0 \text{ for some } u_0 \in (B_1, B_2)\]

\[D \text{ decreases if } p_1 \leq p_0\]

6. \[
\begin{align*}
\text{SDW} - R_1 + R_2 & \quad \text{SDW} - R_1 + S_2 & \quad \text{SDW} - SDW & \quad S_1 + R_2 - SDW & \quad R_1 + R_2 - SDW \\
D \text{ increases} & & & & \\
B_1 & & B_2 & & \\
\end{align*}
\]

\[D \text{ increases if } p_0 \leq p_1 \]

\[\text{If } p_0 \leq p_1, \alpha D = 0 \text{ for some } u_0 \in (B_2, A_2)\]

\[\text{If } p_1 \leq p_0, \alpha D = 0 \text{ for some } u_0 \in (B_1, A_1)\]

\[D \text{ decreases if } p_1 \leq p_0\]
References

[1] Barbera, E.: On the principle of minimal entropy production for Navier–Stokes–Fourier fluids. Continuum Mech. Thermodyn. 11, 327–330 (1999)

[2] Brenier, Y., Grenier, E.: Sticky particles and scalar conservation laws. SIAM J. Numer. Anal. 35(6), 2317–2328 (1998)

[3] Chaplygin, S.: On gas jets. Sci. Mem. Moscow Univ. Math. Phys. 21, 1–121 (1904)

[4] Chiodaroli, E., Kreml, O.: On the energy dissipation rate of solutions to the compressible isentropic Euler system. Arch. Ration. Mech. Anal. 214, 1019–1049 (2014)

[5] Dafermos, C.: The entropy rate admissibility criterion for solutions of hyperbolic conservation laws. J. Differ. Equ. 14, 202–212 (1973)

[6] Dafermos, C.: Maximal dissipation in equations of evolution. J. Differ. Equ. 252(1), 567–587 (2012)

[7] Feireisl, E.: Maximal dissipation and well-posedness for the compressible Euler system. J. Math. Fluid Mech. 16, 447–461 (2014)

[8] Feireisl, E., Gwiazda, P., Świerczewska-Gwiazda, A., Wiedemann, E.: Regularity and energy conservation for the compressible Euler equations. Arch. Ration. Mech. Anal. 223, 1375–1395 (2017)

[9] Hsiao, L.: The entropy rate admissibility criterion in gas dynamics. J. Differ. Equ. 38, 226–238 (1980)

[10] Kamenshchik, A., Moschella, U., Pasquier, V.: An alternative to quintessence. Phys. Lett. 511, 265–268 (2001)

[11] Müller, I., Weiss, W.: Thermodynamics of irreversible processes—past and present. Eur. Phys. J. H 37, 139–236 (2012)

[12] Nedeljkov, M.: Shadow waves, entropies and interactions for delta and singular shocks. Arch. Ration. Mech. Anal. 197(2), 489–537 (2010)

[13] Nedeljkov, M.: Higher order shadow waves and delta shock blow up in the Chaplygin gas. J. Differ. Equ. 256(11), 3859–3887 (2014)

[14] Nedeljkov, M.: Admissibility of a solution to generalized Chaplygin gas. Theor. Appl. Mech. 46(1), 89–96 (2019)

[15] Nedeljkov, M., Oberguggenberger, M.: Interactions of delta shock waves in a strictly hyperbolic system of conservation laws. J. Math. Anal. Appl. 344, 1143–1157 (2008)

[16] Nedeljkov, M., Ružičić, S.: On the uniqueness of solution to generalized Chaplygin gas. Discrete Contin. Dyn. Syst. 37(8), 4439–4460 (2017)
[17] Nedeljkov, M., Ružičić, S.: Shadow wave tracking procedure and initial data problem for pressureless gas model. Acta Appl. Math. 171, 1–36 (2021)

[18] Weinan, E., Rykov, Y.G., Sinai, Y.G.: Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion dynamics. Commun. Math. Phys. 177(2), 349–380 (1996)

[19] Zeldovich, Y.B.: Gravitational instability: an approximate theory for large density perturbations. Astron. Astrophys. 5, 84–89 (1970)

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