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On the Almeida-Thouless instability in short-range Ising spin-glasses

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We use high temperature series expansions to study the $\pm J$ Ising spin-glass in a magnetic field in $d$-dimensional hypercubic lattices for $d = 5, 6, 7$ and $8$, and in the infinite-range Sherrington-Kirkpatrick (SK) [2] model in which case there is no lattice structure. The solution of the mean-field, infinite-range, Sherrington-Kirkpatrick (SK) [2] model in the RSB phase below the AT line of the mean-field, infinite-range, Sherrington-Kirkpatrick set of “replica symmetry breaking” (RSB). The solution of the mean-field, infinite-range, Sherrington-Kirkpatrick (SK) [2] model in which case there is no lattice structure. The solution of the mean-field, infinite-range, Sherrington-Kirkpatrick (SK) [2] model in the RSB phase below the AT line of the mean-field, infinite-range, Sherrington-Kirkpatrick set of “replica symmetry breaking” (RSB). The solution of the mean-field, infinite-range, Sherrington-Kirkpatrick (SK) [2] model in the RSB phase below the AT line of the mean-field, infinite-range, Sherrington-Kirkpatrick set of “replica symmetry breaking” (RSB). The solution of the mean-field, infinite-range, Sherrington-Kirkpatrick (SK) [2] model in which case there is no lattice structure.

One of the most striking predictions of the mean field theory of spin glasses [1–3] is the existence of a line of transitions in the magnetic-field temperature plane first found by de Almeida and Thouless (AT) [4]. This transition is surprising since it occurs without the breaking of any “obvious” symmetry, and instead marks the onset of “replica symmetry breaking” (RSB). The solution of the mean-field, infinite-range, Sherrington-Kirkpatrick (SK) [2] model in which case there is no lattice structure.

We consider the Hamiltonian

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j - h \sum_{i=1}^{N} S_i,$$

where the $S_i$ are Ising spins which take values $\pm 1$, and the interactions $J_{ij}$ are quenched random variables with a bimodal distribution, i.e. $J_{ij} = \pm J$ with equal probability. The $N$ spins either lie on a hypercubic lattice, in which case the interactions are between nearest-neighbors and have $J = 1$, or correspond to the Sherrington-Kirkpatrick (SK) [2] model in which case there is no lattice structure, every spin interacts with every other spin, and $J = 1/\sqrt{N}$. We choose a bimodal distribution because the series can be worked out much more efficiently for this case than for a general distribution [15].

The AT line is characterized by the divergence of the...
spin glass susceptibility \( \chi_{SG} \) where

\[
\chi_{SG} = \frac{1}{N} \sum_{i,j=1}^{N} (\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle)^2 \text{av},
\]

where \([\cdots]_{\text{av}}\) denotes an average over the disorder. For a fixed value of \( h \) we expand the susceptibility for the hypercubic lattice in powers of

\[
w = \tanh^2(J/T).
\]

The coefficient of \( w^n \) turns out to be a polynomial of order \( 2n + 2 \) in

\[
u = \tanh^2(h/T),
\]

so

\[
\chi_{SG}(w, u) = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{2n+2} a_{n,m} u^m \right) w^n.
\]

We evaluate all the coefficients \( a_{n,m} \) up to order \( n = 10 \) for a hypercubic lattice in arbitrary dimension \( d \) [16]. The series for the SK model is obtained by setting \( J = 1/2d \) and taking \( d \to \infty \) limit, in which case the high temperature expansion variable becomes \( x = (1/T)^2 \) rather than \( w \). A ten term series is only of moderate length, but, compared with zero field, determining the series in \( d = 5 \) and higher becomes large, and, for a given order \( n \), considerable cancellations occur between the coefficients with different values of \( m \). For example, for the SK model with \( u = 0.2 \) the largest individual contribution to the coefficient of \( w^{10} \) is about \( 10^7 \) times greater than the final answer [16].

Equation (5) gives the high temperature series expansion for arbitrary values of the ratio \( h/T \). Fixing this ratio corresponds to expanding \( \chi_{SG} \) along a diagonal line in the \( h-T \) plane ending at \( h = T = 0 \), which must intersect an AT line if one exists.

Zero-field Spin-Glass Transition: It is known from earlier studies [17–20], that a 10-term series in \( w \) does not give consistent indication of a critical temperature in \( d = 3 \) and is also problematic in \( d = 4 \) giving rather large and inconsistent values of critical exponents. Hence, we confine our analysis to \( d = 5 \) and higher.

For the SK model, the zero-field spin-glass series is a simple geometrical series, \( x = 1/T^2 \), which sums to \( 1/(1-x) \) showing that the exponent \( \gamma \) equals unity. For \( d \geq 6 \), the model’s critical behavior is governed by a Gaussian fixed point with \( \gamma = 1 \) [21, 22]. We use Padé approximants directly on the series to estimate \( w_c \), the critical value of \( w \). This fixes \( \gamma \) to unity, and produces estimates for \( w_c \) shown in Table I. Note that uncertainties in series analysis are just confidence limits [23]. These \( w_c \)-values are consistent with those from large dimensionality expansions [24, 25], and agree to within around a percent with those from a more sophisticated analysis taking corrections to scaling into account [19, 20]. It is well known that estimates of critical points in series-analysis are correlated with estimates of critical exponents. Hence, by fixing the critical exponent to unity, we avoid some of the subtleties and get a fairly reasonable estimate of the critical point with a moderate length series.

| \( d \) | \( w_c \) | \( \gamma \) | \( \Delta \) |
|-----|-----|-----|-----|
| 8   | \( 0.0695 \pm 0.0002 \) | 1.20 \pm 0.1 |
| 7   | \( 0.0816 \pm 0.0004 \) | 1.20 \pm 0.2 |
| 6   | \( 0.0996 \pm 0.0008 \) | 1.20 \pm 0.5 |
| 5   | \( 0.1388 \pm 0.0009 \) | 1.9 \pm 0.1 \( \pm 0.4 \) |

In \( d = 5 \), we use standard \( d \)-log Padé approximants and differential approximants [26, 27] to analyze the series. The critical point estimate \( w_c = 0.1388 \pm 0.0009 \) is consistent with previous studies [17, 19]. Using biased approximants with critical point fixed at the central estimate \( w_c = 0.1388 \), we obtain \( \gamma = 1.9 \pm 0.1 \), again in agreement with previous studies.

Scaling dimension of the ordering field: Field theory predicts [21, 22, 28] that the scaling dimension of the ordering field \( h^2 \), or equivalently \( u \), should be \( \Delta = 2 \) at the Gaussian fixed point. In other words for \( d \geq 6 \), the variable \( u \) should scale, near \( T_c \), with the reduced temperature \( t \equiv (T - T_c)/T_c \), in the combination \( u/t^2 \).

To study this through series expansions we consider two single variable series in \( w \) defined as

\[
K_1(w) = \left. \frac{\partial u \chi_{SG}(w, u)}{\partial u \chi_{SG}(w, 0)} \right|_{u=0},
\]

and

\[
K_2(w) = \left. \frac{\partial^2 \chi_{SG}(w, u)}{\partial u \chi_{SG}(w, u)} \right|_{u=0}.
\]

Both quantities \( K_1(w) \) and \( K_2(w) \) should diverge at the critical temperature as \( 1/t^2 \) with \( \Delta = 2 \) for \( d \geq 6 \). Note that we consider the limit \( t \to 0 \) for which \( t \equiv (T - T_c)/T_c \propto (w_c - w)/w_c \).

For the SK model, these quantities sum up to

\[
K_1^{SK} = -2/(1-x)^2
\]

and

\[
K_2^{SK} = 6 - 2x - (x+7)/(1-x)^2
\]

respectively, with \( x = 1/T^2 \), clearly showing that \( \Delta = 2 \). In fact, from an asymptotic analysis of our graphical
method, one can show that the m-th derivative of $\chi_{SG}^{SK}$ with respect to $u$, evaluated at $u = 0$ diverges as $1/t^{1+2m}$, confirming that $\Delta = 2$ is true to all perturbative orders in $u$.

To analyze $K_1(w)$ and $K_2(w)$ series in finite dimensions, we use d-log Padé and differential approximants. We fix the critical point at those values estimated from the zero-field susceptibility series, see Table I. A histogram of $\Delta$ values estimated from the analysis are shown in Fig. 1. It is clear that in $d \geq 6$ the exponent $\Delta$ remains equal to 2. However, in $d = 5$ it is closer to 3.

**Analysis of series in a finite field:** We fix a value of $u$ and study the series in $w$. In the SK model the spin-glass susceptibility diverges as a simple pole at the AT line. Unlike the case of zero field the series for SK model are no longer simple in a field, and indeed no truncated series can reproduce exactly the violation of scaling encapsulated in the fact that, along the AT line, $T - T_c$ scales as $h^\theta$ with $\theta = 2/3$ rather than as $h^{2/\Delta}$ as expected from scaling. In fact, for any finite length series, at sufficiently small $h$, such a non-linear relation can not follow. Hence, our focus will be on fields which are not too small to be dominated by just the leading order field terms.

We have found that the finite-field series for the SK model do not converge well close to the AT line. The series analysis works better in the variable $w = \tanh^2 h/T$. Two diagonal Padé approximants for $\chi_{SG}^{-1}$ using the variable $w$ for $u \equiv \tanh^2 (h/T) = 0.1$ and $u = 0.2$ are shown in Fig. 2 along with the exact value computed numerically. The critical point is located reasonably well at $u = 0.1$ but not at $u = 0.2$. This is found to be true for a majority of Padés, including off-diagonal ones.

For different values of $u$, we carry out a large number of Padé approximants and determine the critical point from the set of approximants which are bunched closest to each other. The estimated phase boundary is shown in Fig. 3. The exact value of AT line for the SK model, determined numerically, and its asymptotic small $h$-t limit are also shown in the figure. The series analysis is consistent with the correct $\theta = 2/3$ value for the AT line but overestimates the extent of the paramagnetic phase for larger $u$. Any significant improvement will need substantially longer series. Note that as discussed in the previous paragraph, for very small $h$, the analysis is dominated by the leading $h$-terms and shows only a small shift in the critical point. While the convergence is not excellent, it is clear that high temperature expansion with a moderate number of terms can capture the highly non-trivial Almeida-Thouless instability in the SK model.

Fisher and Sompolinsky [28] have shown that between $d = 8$ and $d = 6$ the AT-line exponent becomes $\theta = 4/(d-2)$, which goes from the SK value of $\theta = 2/3$ in $d = 8$ to $\theta = 1$ in $d = 6$. Thus usual scaling relation $\theta = 2/\Delta$ is restored in $d = 6$. For $d = 6, 7, 8$, we repeat the same analysis as for the SK model. The results for the estimated AT lines are shown in Fig. 4. The uncertainties in locating the AT line are too large to allow an unbiased fit to a power-law. However, a few points can clearly be noted: (i) In both $d = 7$ and $d = 8$, the small-field behavior differs qualitatively from that at larger fields and is very similar to the behavior seen in the SK model. (ii) For $d = 7$ and $8$ it is only after $u$ exceeds a certain value that a more consistent behavior with $\theta < 1$ emerges. As a guide, we have drawn curves with $\theta = 2/3$ and $\theta = 4/5$ in $d = 8$ and $d = 7$ respectively, as expected from the analysis of Fisher and Sompolinsky. (iii) In $d = 6$, we do not see the clear discrepancy at low-fields and the behavior is more consistent with $\theta = 1$ as expected from scaling, which is predicted to be restored [28] in $d = 6$. This suggests that the series analysis is capturing some
FIG. 3: The solid (blue) line is the expression for the AT line in the SK model while the dashed (green) line is its asymptotic small $h$ limit. The points joined by lines are the results of the Padé analysis of the 10-term series, including just the Padé approximants that are bunched together.

FIG. 4: Estimates of the AT line in $d = 6$, $d = 7$ and $d = 8$ obtained from Padé analyses of the series. The formulae for the lines are given in the legend and are discussed in the text.

FIG. 5: Approximants showing divergence of the spin-glass susceptibility in $d = 5$, and the range of their exponent $\gamma$.

of the key features of the finite-field behavior in short-range spin-glasses and its changes with dimensionality, and that the Almeida-Thouless instability does exist for $d \geq 6$.

In $d = 5$, the system is no longer governed by a Gaussian fixed point. However, the increased $\Delta$ value shown in Table I suggests that if there is an AT line it should have a power $\theta = 2/\Delta$ which is again close to $2/3$. For this case, we analyze our series by d-log Padé approximants and differential approximants. At very small $u$ values, we see a singularity that is very similar to what is seen in the SK model and in higher $d$. There is only a small shift in the critical point. But, once $u$ exceeds a certain value most approximants do not show a consistent divergence. As seen in Fig. 5, only a handful of approximants show any divergence at all. These predict a reduced exponent $0.7 < \gamma < 1.3$. This could imply that the series are too short to see the non-trivial critical behavior in $d < 6$ or it could mean that there is no Almeida-Thouless instability below $d = 6$. We especially note that one difference in our analysis of the susceptibility in a field in $d = 5$ versus higher-$d$ is that in higher dimensions we biased the critical exponents to have mean-field values. The absence of such a bias contributes to the uncertainty in the $d = 5$ analysis and may be partly responsible for the lack of a more definitive answer in $d = 5$.

Conclusions: In conclusion, we have studied the problem of short-range Ising spin-glasses in a field by high temperature series expansion methods. We have presented evidence for violation of scaling along the AT line in high dimension and its restoration as $d \to 6$, as first shown by Fisher and Sompolinsky [28]. Within the convergence of our analysis, we have presented evidence for the existence of the AT line of instabilities for $d \geq 6$. In $d = 5$, the critical exponents $\gamma$ and $\Delta$ are significantly larger than the mean-field values but no consistent evidence for the AT line is found. Thus, our results are consistent with 6 being the lower critical dimension for the AT line. However it is also possible that an AT line does occur for $d < 6$ but the series are too short to see it.

Finally we compare our results with other work. The early renormalization group calculation of Bray and Roberts [14] did not find a stable, perturbative fixed point corresponding to an AT line below the upper critical dimension of 6. While this result is consistent with
there being no AT line below $d = 6$ it is also possible that a non-perturbative fixed point is present for this range of dimension. Other studies used Monte Carlo (MC) simulations on one-dimensional long-range interactions models [9, 10, 12]. The results were analysed using finite-size scaling (FSS) theory. While all analyses found evidence for AT line for interactions corresponding to $d$ above 6, different conclusions were reached in lower dimensions [9, 10, 12].

Our approach is complimentary to MC in that we study short-range models directly on d-dimensional hypercubic lattices and that the series represent equilibrium property of the infinite system. Thus, the combined MC and series evidence provides a strong case for an AT line in short-range models at least in high enough dimensions. It would be challenging, but worthwhile, to try to extend the series approach to include higher order terms.

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[1] S. F. Edwards and P. W. Anderson, Theory of spin glasses, J. Phys. F 5, 965 (1975).
[2] D. Sherrington and S. Kirkpatrick, Solvable model of a spin glass, Phys. Rev. Lett. 35, 1792 (1975).
[3] K. Binder and A. P. Young, Spin glasses: Experimental facts, theoretical concepts and open questions, Rev. Mod. Phys. 58, 801 (1986).
[4] J. R. L. de Almeida and D. J. Thouless, Stability of the Sherrington-Kirkpatrick solution of a spin glass model, J. Phys. A 11, 983 (1978).
[5] G. Parisi, The order parameter for spin glasses: a function on the interval 0–1, J. Phys. A 13, 1101 (1980).
[6] G. Parisi, Order parameter for spin-glasses, Phys. Rev. Lett. 50, 1946 (1983).
[7] D. S. Fisher and D. A. Huse, Absence of many states in realistic spin glasses, J. Phys. A 20, L1005 (1987).
[8] D. S. Fisher and D. A. Huse, Equilibrium behavior of the spin-glass ordered phase, Phys. Rev. B 38, 386 (1988).
[9] M. Baity-Jesi et al, The three dimensional Ising spin glass in an external magnetic field: the role of the silent majority, J. Stat. Mech. p. P05014 (2014), (arXiv:1403.2622).
[10] R. A. Baños et al, Thermodynamic glass transition in a spin glass without time-reversal symmetry, Proc. Natl. Acad. Sci. USA 109, 6452 (2012), (arXiv:1202.5593).
[11] H. G. Katzgraber and A. P. Young, Probing the Almeida-Thouless line away from the mean-field model, Phys. Rev. B 72, 184416 (2005).
[12] D. Larson, H. G. Katzgraber, M. A. Moore, and A. P. Young, Spin glasses in a field: Three and four dimensions as seen from one space dimension, Phys. Rev. B 87, 024414 (2013), (arXiv:1211.7297).
[13] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, Diluted one-dimensional spin glasses with power law decaying interactions, Phys. Rev. Lett 101, 107203 (2008).
[14] A. J. Bray and S. A. Roberts, Renormalisation-group approach to the spin glass transition in finite magnetic fields, J. Phys. C 13, 5405 (1980).
[15] R. R. P. Singh and A. P. Young, (unpublished), Details of the series generation method will be published elsewhere.
[16] R. R. P. Singh and A. P. Young, The series coefficients can be found in the Supplementary Materials.
[17] R. Fisch and A. B. Harris, Series study of a spin-glass model in continuous dimensionality, Phys. Rev. Lett. 38, 785 (1977).
[18] R. R. P. Singh and S. Chakravarty, Critical behavior of an Ising spin-glass, Phys. Rev. Lett. 57, 245 (1986).
[19] L. Klein, J. Adler, A. Aharony, A. B. Harris, and Y. Meir, Series expansions for the Ising spin glass in general dimension, Phys. Rev. B 43, 11249 (1991).
[20] D. Daboul, I. Chang, and A. Aharony, Test of universality in the Ising spin glass using high temperature graph expansion, Physics of Condensed Matter 41, 231 (2004), arXiv:cond-mat/0408167.
[21] A. B. Harris, T. C. Lubensky, and J.-H. Chen, Critical properties of spin-glasses, Phys. Rev. Lett. 36, 415 (1976).
[22] J. H. Chen and T. C. Lubensky, Mean field and e-expansion study of spin glasses, Phys. Rev. B p. 2106 (1977).
[23] J. Oitmaa, C. Hamer, and W. Zheng, Series Expansion Methods for strongly interacting lattice models (Cambridge University, Cambridge, 2006).
[24] R. R. P. Singh and M. E. Fisher, Short-range Ising spin glasses in general dimensions, J. Appl. Phys. 63, 3994 (1988).
[25] M. E. Fisher and R. R. P. Singh, Critical points, large-dimensionality expansions, and the ising spin-glass, in Disorder in Physical Systems, edited by G. Grimmett and D. J. A. Welsh (Oxford University Press, Oxford, 1990).
[26] D. L. Hunter and G. A. Baker, Methods of series analysis. III. Integral approximant methods, Phys. Rev. B 19, 3808 (1979).
[27] M. E. Fisher and H. Au-Yang, Inhomogeneous differential approximants for power series, J. Phys. A 12, 1677 (1979).
[28] D. S. Fisher and H. Sompolinsky, Ordered phase of short-range Ising spin-glasses, Phys. Rev. Lett. 54, 1063 (1985).