An independence test for functional variables based on kernel normalized cross-covariance operator

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Abstract. We propose an independence test for random variables valued into metric spaces by using a test statistic obtained from appropriately centering and rescaling the squared Hilbert-Schmidt norm of the usual empirical estimator of normalized cross-covariance operator. We then get asymptotic normality of this statistic under independence hypothesis, so leading to a new test for independence of functional random variables. A simulation study that allows to compare the proposed test to existing ones is provided.

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1 Introduction

Testing the independence between two random variables has always been an important concern in statistics, and many methods have been developed to do this in the case of real or multidimensional random variables. In the case where the concerned variables are functional, one finds within Functional Data Analysis (FDA) which is a branch of statistics allowing to set up methods for processing data being digitized points of curves, some methods making it possible to test the independence or the lack of relationship between the concerned variables. This is how non-correlation tests between functional variables were proposed in [1] and [13] whereas independence tests were introduced in [6], [14] and [15]. More specifically, Górecki et al. [6] proposed
the use of the Hilbert-Schmidt Independence Criterion (HSIC), introduced by Gretton et al. [7], for testing independence of multivariate functional data, Lai et al. [14] introduced the angle covariance to characterize independence and proposed a permutation based test whereas Manfoumbi Djonguet and Nkiet [15] proposed a modification of a HSIC naive estimator that yields asymptotic normality under null hypothesis, so allowing a simple test. HSIC is one of the dependence measures proposed within the framework of kernel-based methods, that is methods based on kernel embeddings of probability measures in a reproducing kernel Hilbert space (RKHS). These are powerful methods allowing to work with high-dimensional and structured data (e.g., [11]), including functional data. Other kernel-based dependence measures have been introduced in various works, notably in [4] and [9]. Particularly, Fukumizu et al. [4] introduced a dependence measure based on the so-called normalized cross-covariance operator (NOCCO) defined in [3]. More specifically, they used the squared Hilbert-Schmidt norm of the empirical estimator of NOCCO for measuring independence and introduced a permutation test of independence based on the measure thus proposed. To the best of our knowledge, there do not exist other studies of this dependence measure which is however very interesting since, as it is proved in [4], it characterizes independence when the related kernels are characteristic ones. As a permutation test has the disadvantage of requiring high computational costs, it is of interest to perform rather the test by using the asymptotic distribution of a test statistic based on the aforementioned measure under independence hypothesis. This is the approach we take in this paper. More specifically, we introduce a test statistic obtained from appropriately centering and rescaling the squared Hilbert-Schmidt norm of the empirical estimator of NOCCO, and we get asymptotic normality for this statistic under independence hypothesis. This allows to propose a new test for independence of random variables valued into metric spaces, including functional random variables. The rest of the paper is organized as follows. Section 2 is devoted to the introduction of basic notions that are used in the paper. In Section 3, the NOCCO-based dependence measure introduced in [4] is recalled, then we defined a test statistic based on it for which we obtain an asymptotic normality under the independence hypothesis and specified assumptions. We then show how this test statistic can be computed in practice, especially when functional data are available. A simulation study on functional data that allows to compare the proposed test to those of Gretton et al. [7] and Fukumizu et al. [4] is given in Section 4. All the proofs are postponed in Section 5.
2 Preliminary notions

This section is devoted to recalling the notion of reproducing kernel hilbert space (RKHS) and defining some elements related to it that are useful in this paper. For more details on RKHS and its use in probability and statistics, one may refer to [2]. Letting \((X, B_X)\) be a measurable space, where \((X, d)\) is a metric space and \(B_X\) is the corresponding Borel \(\sigma\)-field, we consider a Hilbert space \(H_X\) of functions from \(X\) to \(\mathbb{R}\), endowed with an inner product \(<\cdot, \cdot>_X\). This space is said to be a RKHS if there exists a kernel, that is a symmetric positive semi-definite function \(K: X^2 \to \mathbb{R}\), such that for any \(f \in H_X\) and any \(x \in X\), one has \(K(x, \cdot) \in H_X\) and \(f(x) = <f, K(x, \cdot)>_X\). The feature map \(\Phi: x \in X \mapsto K(x, \cdot) \in H_X\) characterizes \(K\) since \(K(x, y) = <\Phi(x), \Phi(y)>_X\) for any \((x, y) \in X^2\). Let \(X\) be a random variable taking values in \(X\) and with probability distribution \(P_X\). If \(\mathbb{E}(\|\Phi(X)\|_{H_X}) < +\infty\), the mean element \(m_X\) associated with \(X\) is defined for all functions \(f \in H_X\) as the unique element in \(H_X\) satisfying
\[<m_X, f>_X = \mathbb{E}(f(X)) = \int_X f(x) \, dP_X(x).\]
Furthermore, if \(\mathbb{E}(\|\Phi(X)\|_{H_X}^2) < +\infty\), we can define the covariance operator associated to \(X\) as the unique operator \(\Sigma_X\) from \(H_X\) to itself such that, for any pair \((f, g) \in H_X^2\), one has
\[<f, \Sigma_X g>_X = \text{Cov}(f(X), g(X)) = \mathbb{E}(f(X)g(X)) - \mathbb{E}(f(X))\mathbb{E}(g(X)).\]
It is very important to note that if \(\|K\|_{\infty} < +\infty\), where
\[\|K\|_{\infty} := \sup_{(x, y) \in X^2} K(x, y),\]
then \(m_X\) and \(\Sigma_X\) exist. They can also be expressed as
\[m_X = \mathbb{E}(K(X, \cdot)) \quad \text{and} \quad \Sigma_X = \mathbb{E}(K(X, \cdot) \otimes K(X, \cdot)) - m_X \otimes m_X,\]
where \(\otimes\) is the tensor product such that, for any pair of vectors \((x, y)\), the product \(x \otimes y\) is the linear map defined by \((x \otimes y)(h) = <x, h \otimes y\) for all \(h\). Now, let \(Y\) be another random variable with values in a measurable space \((\mathcal{Y}, \mathcal{B}_Y)\), where \(\mathcal{Y}\) is a metric space and \(\mathcal{B}_Y\) is the corresponding Borel \(\sigma\)-field. We consider a RKHS \(H_Y\) of functions from \(\mathcal{Y}\) to \(\mathbb{R}\), endowed with
an inner product $<\cdot,\cdot>_{\mathcal{H}_Y}$, and associated with a kernel $L : \mathcal{Y}^2 \to \mathbb{R}$. If $\mathbb{E}(\|\Psi(X)\|^2_{\mathcal{H}_Y}) < +\infty$, where $\Psi = L(Y, \cdot)$ and $\|\cdot\|_{\mathcal{H}_Y}$ is the norm associated with $<\cdot,\cdot>_{\mathcal{H}_Y}$, we can define the mean element $m_Y$ and the covariance operator $\Sigma_Y$ as indicated above. We can also define the cross-covariance operator of $X$ and $Y$, denoted by $\Sigma_{XY}$, as the unique linear operator from $\mathcal{H}_Y$ to $\mathcal{H}_X$ satisfying

$$\langle f, \Sigma_{XY} g \rangle_{\mathcal{H}_X} = \text{cov}(f(X), g(Y))$$

for all $(f, g) \in \mathcal{H}_X \times \mathcal{H}_Y$. It can also be expressed as

$$\Sigma_{XY} = \mathbb{E}\left( (L(Y, \cdot) - m_Y) \otimes (K(X, \cdot) - m_X) \right) = \mathbb{E}(L(Y, \cdot) \otimes K(X, \cdot)) - m_Y \otimes m_X.$$

It is well known (see, e.g., [3, 4]) that $\Sigma_{XY}$ is an Hilbert Schmidt operator and that there exists a unique bounded operator $V_{XY} : \mathcal{H}_Y \to \mathcal{H}_X$, called the normalized cross-covariance operator (NOCCO), satisfying

$$\Sigma_{XY} = \Sigma_{X}^{1/2} V_{XY} \Sigma_{Y}^{1/2}. \quad (1)$$

From now on, we assume that $V_{XY}$ is a Hilbert-Schmidt operator. Note that, Fukumizu et al. [3] discussed conditions that ensure this later property and they show in Theorem 3 that a sufficient condition is the fact that the mean square contingency is finite, what is very natural when two different random variables are considered. As we will see later, $V_{XY}$ measures the dependence between $X$ and $Y$ and can, therefore, be the basis of a procedure for testing for independence.

### 3 NOCCO and independence testing

In this section, we consider a dependence measure based on NOCCO that permits to consider a test of independence, that is the test of the hypothesis $\mathcal{H}_0 : X \perp \perp Y$, where $\perp \perp$ denotes stochastic independence, against the alternative hypothesis $\mathcal{H}_1$ meaning that $X$ and $Y$ are not independent. Then, a consistent estimator of this measure is proposed as test statistic, and we get its asymptotic normality under $\mathcal{H}_0$. 

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3.1 A NOCCO-based dependence measure

One of the most famous RKHS-based dependence measures is the Hilbert-Schmidt independence criterion (HSIC) introduced in [7]. It is equal to $\|\Sigma_{XY}\|_{HS}^2$, where $\| \cdot \|_{HS}$ denotes the Hilbert-Schmidt norm, and is known to fully characterize independence property. Indeed, when $K$ and $L$ are characteristic kernels $\mathcal{H}_0$ is equivalent to $\Sigma_{XY} = 0$. That is why estimate of HSIC was used for testing for independence ([7, 8]). Unfortunately, asymptotic normality under $\mathcal{H}_0$ could not be obtained for the used test statistic but rather convergence in distribution to an infinite sum of chi-squared distributions, what is unusable for performing the test and requires the use of methods such as permutation methods which have high computation costs. Later, Fukumizu et al. [4] proposed to use instead the dependence measure $N(X, Y) = \|V_{XY}\|_{HS}^2$; it also characterizes independence since from (1) it is clear that, when $K$ and $L$ are characteristic kernels, $\mathcal{H}_0$ is equivalent to $N(X, Y) = 0$. In order to achieve an independence test, we have to consider a consistent estimator of $N(X, Y)$.

3.2 Test statistic and asymptotic normality

Let $\{(X_i, Y_i)\}_{1 \leq i \leq n}$ be a i.i.d. sample of $(X, Y)$; the empirical counterparts of $m_X$ and $m_Y$ are respectively given by:

$$K_n = \frac{1}{n} \sum_{i=1}^{n} K(X_i, \cdot) \quad \text{and} \quad L_n = \frac{1}{n} \sum_{i=1}^{n} L(Y_i, \cdot).$$

We then consider the following estimators of $\Sigma_X$, $\Sigma_Y$ and $\Sigma_{XY}$ defined as

$$\hat{\Sigma}_X = \frac{1}{n} \sum_{i=1}^{n} \left( K(X_i, \cdot) - K_n \right) \otimes \left( K(X_i, \cdot) - K_n \right) = \frac{1}{n} \sum_{i=1}^{n} K(X_i, \cdot) \otimes K(X_i, \cdot) - K_n \otimes K_n,$$

$$\hat{\Sigma}_Y = \frac{1}{n} \sum_{i=1}^{n} \left( L(Y_i, \cdot) - L_n \right) \otimes \left( L(Y_i, \cdot) - L_n \right) = \frac{1}{n} \sum_{i=1}^{n} L(Y_i, \cdot) \otimes L(Y_i, \cdot) - L_n \otimes L_n,$$

and

$$\hat{\Sigma}_{XY} = \frac{1}{n} \sum_{i=1}^{n} \left( L(Y_i, \cdot) - L_n \right) \otimes \left( K(X_i, \cdot) - K_n \right) = \frac{1}{n} \sum_{i=1}^{n} L(Y_i, \cdot) \otimes K(X_i, \cdot) - L_n \otimes K_n.$$
From them, we want to define an estimator of $V_{XY}$. A natural way to do this would be to invert $\hat{\Sigma}^{1/2}X + \gamma_n I_X$ and $\hat{\Sigma}^{1/2}Y + \gamma_n I_Y$ but these operators are compact ones and, therefore, do not have inverses. So, as in [4], we use a regularization approach: we consider a sequence $\{\gamma_n\}_{n \geq 1}$ of strictly positive numbers such that $\lim_{n \to +\infty} (\gamma_n) = 0$, and we estimate $V_{XY}$ by the random operator

$$\hat{V}_{XY} = (\hat{\Sigma}_X + \gamma_n I_X)^{-1/2} \hat{\Sigma}_{XY} (\hat{\Sigma}_Y + \gamma_n I_Y)^{-1/2},$$

where $I_X$ (resp. $I_Y$) denotes the identity operator of $X$ (resp. $Y$). Then, we take as estimator of $N(X, Y)$ the statistic

$$\hat{N}_n(X, Y) = \|\hat{V}_{XY}\|_{HS}^2.$$ 

Theorem 5 in [4] establishes consistency of this estimator when $\gamma_n^{-3} n^{-1} \to 0$ as $n \to +\infty$. So, $\hat{N}_n(X, Y)$ can be used as test statistic for the problem of testing for $\mathcal{H}_0$ against $\mathcal{H}_1$. It is, therefore, necessary to get asymptotic distribution under $\mathcal{H}_0$ of this statistic. In fact, we will obtain asymptotic normality after recentering and rescaling it. As in [12], for a given compact operator $A$ with decreasing eigenvalues $\lambda_p(A)$, $p \geq 1$, we define the quantity $d_r(A, \gamma)$ for all $r \in \{1, 2\}$ and $\gamma \in ]0, +\infty[$ as

$$d_r(A, \gamma) = \left\{ \sum_{p=1}^{+\infty} (\lambda_p(A) + \gamma)^{-r} \lambda_p(A)^r \right\}^{1/r}.$$

Then, for $r \in \{1, 2\}$, we put

$$\hat{D}_{r,n} = d_r(\hat{\Sigma}_X, \gamma_n) d_r(\hat{\Sigma}_Y, \gamma_n)$$

and we consider the centered and rescaled version of $\hat{N}_n(X, Y)$ given by

$$\hat{T}_n = \frac{n\hat{N}_n(X, Y) - \hat{D}_{1,n}}{\sqrt{2 \hat{D}_{2,n}}}.$$ 

In order to get the asymptotic distribution of $\hat{T}_n$, we make the following assumptions:

($\mathcal{A}_1$): the kernels $K$ and $L$ are universal and satisfy $\|K\|_{\infty} < +\infty$ and $\|L\|_{\infty} < +\infty$.
(A₂) : the sequence of decreasing eigenvalues \( \{ \lambda_p \}_{p \geq 1} \) of \( \Sigma_X \) is such that 
\[ \sum_{p=1}^{+\infty} \lambda_p^{1/2} < +\infty; \]

(A₃) : the sequence of decreasing eigenvalues \( \{ \mu_q \}_{q \geq 1} \) of \( \Sigma_Y \) is such that 
\[ \sum_{q=1}^{+\infty} \mu_q^{1/2} < +\infty; \]

(A₄) : the operators \( \Sigma_X \) and \( \Sigma_Y \) have infinitely many strictly positive eigenvalues;

(A₅) : the sequence \( \{ \gamma_n \}_{n \geq 1} \) satisfies: 
\[ \lim_{n \to +\infty} \left( \gamma_n^{-6} n^{-1} \right) = 0. \]

Then, we have:

**Theorem 1** Assume (A₁) to (A₅). Then, under \( \mathcal{H}_0 \), \( \hat{T}_n \) converges in distribution, as \( n \to +\infty \), to \( \mathcal{N}(0, 1) \).

The resulting test for independence is performed as follows: for a given significance level \( \alpha \in [0, 1] \), one has to reject \( \mathcal{H}_0 \) if 
\[ |\hat{T}_n| > F^{-1}(1 - \alpha/2), \]
where \( F \) is the cumulative distribution function of the standard normal distribution.

### 3.3 Computational aspects

For computing \( \hat{T}_n \) in practice, the kernel trick (see [16]) can be used. It is known from [4] that \( \hat{N}_n(X, Y) \) can be written as

\[
\hat{N}_n(X, Y) = \text{Tr} \left( G_X(G_X + n\gamma_n I_n)^{-1}G_Y(G_Y + n\gamma_n I_n)^{-1} \right),
\]

where \( G_X \) and \( G_Y \) are the centered \( n \times n \) Gram matrices with terms equal to

\[
(G_X)_{ij} = \left\langle K(X_i, \cdot) - \overline{K}_n, K(X_j, \cdot) - \overline{K}_n \right\rangle_{\mathcal{H}_X} \quad \text{and} \quad (G_Y)_{ij} = \left\langle L(Y_i, \cdot) - \overline{L}_n, L(Y_j, \cdot) - \overline{L}_n \right\rangle_{\mathcal{H}_Y},
\]

and \( I_n \) is the \( n \times n \) identity matrix. Clearly, using the reproducing properties of \( K \) and \( L \), the preceding terms can be written as

\[
(G_X)_{ij} = k_{ij} - \frac{1}{n} (k_{i*} + k_{j*}) + \frac{k_{*2}}{n^2} \quad \text{and} \quad (G_Y)_{ij} = \ell_{ij} - \frac{1}{n} (\ell_{i*} + \ell_{j*}) + \frac{\ell_{*2}}{n^2},
\]

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where
\[ k_{ij} = K(X_i, X_j), \quad k_\ast = \sum_{j=1}^{n} k_{ij}, \quad k_{ij} = \sum_{i=1}^{n} k_{ij}, \quad k_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij}, \]

and
\[ \ell_{ij} = L(Y_i, Y_j), \quad \ell_\ast = \sum_{j=1}^{n} \ell_{ij}, \quad \ell_{ij} = \sum_{i=1}^{n} \ell_{ij}, \quad \ell_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} \ell_{ij}. \]

Now let us look at the computation of \( \hat{D}_{1,n} \) and \( \hat{D}_{2,n} \). From the definition of \( \hat{\Sigma}_X \) it is easily seen that any eigenvector of this operator associated to a nonzero eigenvalue is a linear combination of \( u_1, \ldots, u_n \), where \( u_i = K(X_i, \cdot) - \mathcal{T}_n \). So \( f = \sum_{j=1}^{n} \alpha_j u_j \) is an eigenvector of \( \hat{\Sigma}_X \) associated to the positive eigenvalue \( \lambda \) if and only if
\[
\frac{1}{n} \sum_{j=1}^{n} \alpha_j \sum_{i=1}^{n} \langle u_i, u_j \rangle \mathcal{H}_X u_i = \sum_{j=1}^{n} \lambda \alpha_j u_j,
\]
what is equivalent to
\[
\sum_{j=1}^{n} \left( \frac{1}{n} \sum_{i=1}^{n} \alpha_i (G_X)_{ij} \right) u_j = \sum_{j=1}^{n} \lambda \alpha_j u_j.
\]
This later equality is equivalent to
\[
\lambda \alpha_j = \frac{1}{n} \sum_{i=1}^{n} \alpha_i (G_X)_{ij}, \quad \forall \ 1 \leq j \leq n,
\]
that is \( \frac{1}{n} G_X \alpha = \lambda \alpha \). So the nonzero eigenvalues of \( \hat{\Sigma}_X \) are the same as those of \( \frac{1}{n} G_X \). Similarly, the nonzero eigenvalues of \( \hat{\Sigma}_X + \gamma_n I_X \) are the same as those of \( \frac{1}{n} (G_X + n \gamma_n I_n) \). We then deduce that
\[
d_1(\hat{\Sigma}_X, \gamma_n) = \text{Tr}(\hat{\Sigma}_X (\hat{\Sigma}_X + \gamma_n I_X)^{-1}) = \text{Tr} \left( G_X (G_X + n \gamma_n I_n)^{-1} \right).
\]
From similar reasoning we also get
\[
d_1(\hat{\Sigma}_Y, \gamma_n) = \text{Tr} \left( G_Y (G_Y + n \gamma_n I_n)^{-1} \right), \quad d_2(\hat{\Sigma}_X, \gamma_n) = \text{Tr} \left( G_X^2 (G_X + n \gamma_n I_n)^{-2} \right)
\]
and
\[ d_2(\tilde{\Sigma}_Y, \gamma_n) = \text{Tr}\left( G_Y^2 (G_Y + n \gamma_n I_n)^{-2} \right). \]

Hence
\[ \tilde{D}_{1,n} = \text{Tr}\left( G_X (G_X + n \gamma_n I_n)^{-1} \right) \text{Tr}\left( G_Y (G_Y + n \gamma_n I_n)^{-1} \right) \] (3)

and
\[ \tilde{D}_{2,n} = \left\{ \text{Tr}\left( G_X^2 (G_X + n \gamma_n I_n)^{-2} \right) \text{Tr}\left( G_Y^2 (G_Y + n \gamma_n I_n)^{-2} \right) \right\}^{1/2}. \] (4)

Therefore \( \hat{T}_n \) is computed in practice by using (2), (3) and (4). The crucial point is to compute \( K(X_i, X_j) \) and \( L(Y_i, Y_j) \) for all \((i, j) \in \{1, \cdots, n\}^2\).

**Remark 1** In the case of functional data corresponding, for instance, to the case where the \( X_i \)s and the \( Y_i \)s are random functions belonging in \( L^2([0,1]) \) and observed on points \( t_1, \cdots, t_r \) and \( s_1, \cdots, s_q \), respectively, of fine grids in \([0,1]\) such that \( t_1 = s_1 = 0 \) and \( t_r = s_q = 1 \), the preceding terms can be computed or approximated easily, depending on the used kernels. For example, if the gaussian kernel is used for \( K \) and \( L \), one has

\[ K(X_i, X_j) = \exp \left( -\omega^2 \|X_i - X_j\|_X^2 \right) = \exp \left( -\omega^2 \int_0^1 (X_i(t) - X_j(t))^2 \, dt \right), \]

and

\[ L(Y_i, Y_j) = \exp \left( -\nu^2 \|Y_i - Y_j\|_Y^2 \right) = \exp \left( -\nu^2 \int_0^1 (Y_i(t) - Y_j(t))^2 \, dt \right), \]

where \( \omega > 0 \) and \( \nu > 0 \). These terms can be approximated by using trapezoidal rule, so leading to:

\[ K(X_i, X_j) \simeq \exp \left( -\omega^2 \sum_{m=1}^{r-1} \frac{t_{m+1} - t_m}{2} \left( (X_i(t_m) - X_j(t_m))^2 + (X_i(t_{m+1}) - X_j(t_{m+1}))^2 \right) \right) \] (5)

and

\[ L(Y_i, Y_j) \simeq \exp \left( -\nu^2 \sum_{m=1}^{q-1} \frac{s_{m+1} - s_m}{2} \left( (Y_i(s_m) - Y_j(s_m))^2 + (Y_i(s_{m+1}) - Y_j(s_{m+1}))^2 \right) \right). \] (6)

Then, \( \hat{T}_n \) is to be computed by using (5) and (6).
4 Simulations

In this section, we investigate the finite sample performances of the proposed test, that we denote by PT, and compare it to two other kernel-based independence tests: the permutation test using HSIC introduced in [7], denoted here by HSIC, and the classical NOCCO-based permutation test of [4] that we denote by pNOCCO. We computed empirical sizes and powers through Monte Carlo simulations. We considered the case where $X = Y = L^2([0,1])$ and generated the data according to the two following models:

**Model 1:** $X(t) = t + \sqrt{2} \sum_{k=1}^{2} \alpha_{k,1} \sin(k\pi t) + \alpha_{k,2} \cos(k\pi t) + \varepsilon_1(t)$ and $Y(t) = t + \sqrt{2} \sum_{k=1}^{2} \beta_{k,1} \sin(k\pi t) + \beta_{k,2} \cos(k\pi t) + \varepsilon_2(t)$, where $\varepsilon_1(t)$ and $\varepsilon_2(t)$ are independent and independently sampled from the normal distribution $N(0, 0.25)$, the Fourier coefficients are independently sampled from the standard normal distribution with the condition $\alpha_{2,2} = \beta_{2,2}$;

**Model 2:** $X(t) = \sqrt{2} \sum_{k=1}^{30} \xi_k \cos(k\pi t)$ and $Y(t) = \sqrt{2} \sum_{k=1}^{30} \nu_k \cos(k\pi t)$, where the $\xi_k$s are independent and distributed as the Cauchy distribution $C(0, 0.5)$ and, for a given $m \in \{0, \ldots, 30\}$, $\nu_k = f(\xi_k)$ for $k = 1, \ldots, m$ and the $\nu_k$s with $k = m + 1, \ldots, 50$ are sampled independently from the standard normal distribution. Here, $f$ is a given function that establishes dependence between $X$ and $Y$.

In model 1, $X$ and $Y$ are dependent due to a shared coefficient. In model 2, $X$ and $Y$ are independent when $m = 0$, and they are dependent for $m \in \{1, \ldots, 30\}$, the dependence level increasing as $m$ increases. Empirical sizes and powers were computed on the basis of 500 independent replicates. For each of them, we generated a sample of size $n = 20, 30, 50, 60, 100$ of the above processes in discretized versions on equispaced values $t_1, \ldots, t_{101}$ in $[0,1]$, where $t_j = (j - 1)/100$, $j = 1, \ldots, 101$. For performing our method, we used the gaussian kernels $K(x, y) = L(x, y) = \exp\left(-\omega^2 \int_0^1 (x(t) - y(t))^2 dt\right)$ with bandwith $\omega^2$ equal to the heuristic median computed from the data; it is the most popular bandwidth choice in simulations and while it has no guarantee of optimality, it remains a safe choice in most of the cases (see [5]). The terms $K(X_i, X_j)$ and $L(Y_i, Y_j)$ were computed by approximating integrals involved in these kernels by using the trapezoidal rule as indicated in (5) and (6). The HSIC and pNOCCO methods were used with 100 permutations. For PT and pNOCCO methods we used $\gamma_n = n^{-1/7}$ for the regularization.
sequence. The nominal significance level is taken as $\alpha = 0.05$ for all tests. Table 1 and Table 2 report the obtained results. Table 1 shows a superiority of PT over the two other methods. Indeed, the obtained values for the power are higher for this method than for the two others for each sample size. In Table 2, the obtained values for $m = 0$ are closer to the nominal size for PT, and this method outperforms HSIC and pNOCCO for each values of $m$ greater than 0. For all the three methods the power increases as $m$ increases, but it is seen that the power of PT increases faster to 1. In the particular case of $f(x) = \cos(x)$, all the three methods do not sufficiently detect the dependence since they give low power for all values of $m$.

| $n$ | PT    | HSIC  | pNOCCO |
|-----|-------|-------|--------|
| 20  | 0.504 | 0.328 | 0.354  |
| 30  | 0.842 | 0.586 | 0.638  |
| 40  | 0.964 | 0.784 | 0.823  |
| 50  | 0.992 | 0.908 | 0.942  |
| 60  | 1.00  | 0.966 | 0.980  |

Figure 1: Empirical powers over 500 replications for Model 1 with nominal significance level $\alpha = 0.05$.  

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| Relationship   | Method | $m = 0$ | $m = 1$ | $m = 3$ | $m = 5$ | $m = 10$ |
|---------------|--------|---------|---------|---------|---------|----------|
| $f(x) = x^3$  | PT     | 0.030   | 0.184   | 0.558   | 0.858   | 1.000    |
|               | HSIC   | 0.010   | 0.028   | 0.174   | 0.430   | 0.934    |
|               | pNOCCO | 0.008   | 0.036   | 0.192   | 0.450   | 0.942    |
| $f(x) = x^2$  | PT     | 0.023   | 0.170   | 0.530   | 0.844   | 0.996    |
|               | HSIC   | 0.010   | 0.032   | 0.164   | 0.432   | 0.926    |
|               | pNOCCO | 0.008   | 0.030   | 0.204   | 0.452   | 0.941    |
| $f(x) = x^2 \sin(x)$ | PT     | 0.030   | 0.198   | 0.542   | 0.836   | 0.996    |
|               | HSIC   | 0.010   | 0.048   | 0.212   | 0.456   | 0.932    |
|               | pNOCCO | 0.008   | 0.048   | 0.218   | 0.462   | 0.936    |
| $f(x) = \sin(x)$ | PT     | 0.030   | 0.036   | 0.016   | 0.026   | 0.028    |
|               | HSIC   | 0.010   | 0.012   | 0.014   | 0.018   | 0.010    |
|               | pNOCCO | 0.008   | 0.014   | 0.012   | 0.016   | 0.018    |

Figure 2: Empirical sizes and powers over 500 replications for Model 2, with sample size $n = 100$ and significance level $\alpha = 0.05$.

5 Proofs

5.1 Preliminary lemmas

First, putting

$$\tilde{\Sigma}_{XY} = \frac{1}{n} \sum_{i=1}^{n} (L(Y_i, )-m_Y)\otimes(K(X_i, )-m_X), \quad \tilde{V}_{XY} = (\Sigma_X+\gamma_n I)^{-1/2}\tilde{\Sigma}_{XY}(\Sigma_Y+\gamma_n I)^{-1/2}$$

and

$$\tilde{N}_n(X,Y) = \|\tilde{V}_{XY}\|_{HS}^2,$$

we have:
**Lemma 1** Assume ($\mathcal{A}_1$) to ($\mathcal{A}_5$). Then, under $\mathcal{H}_0$, we have:

$$
\left| \widehat{N}_n(X, Y) - \tilde{N}_n(X, Y) \right| = O_P(\gamma^{-3}n^{-3/2}), \quad \tilde{N}_n(X, Y) = O_P(\gamma^{-2}n^{-1})
$$

and

$$
\tilde{N}_n(X, Y) = O_P(\gamma^{-2}n^{-1}).
$$

**Proof.** We have

$$
\widehat{N}_n(X, Y) = \left\langle \left( \widehat{\Sigma}_X + \gamma n I_X \right)^{-1} \widehat{\Sigma}_{XY}, \widehat{\Sigma}_{XY} \left( \widehat{\Sigma}_Y + \gamma n I_Y \right)^{-1} \right\rangle_{HS}
$$

(7)

and, similarly,

$$
\tilde{N}_n(X, Y) = \left\langle (\Sigma_X + \gamma n I_X)^{-1} \widetilde{\Sigma}_{XY}, \widetilde{\Sigma}_{XY} \left( \Sigma_Y + \gamma n I_Y \right)^{-1} \right\rangle_{HS},
$$

(8)

so that

$$
\widehat{N}_n(X, Y) - \tilde{N}_n(X, Y)
$$

$$
= \left\langle \left( \left( \widehat{\Sigma}_X + \gamma n I_X \right)^{-1} - (\Sigma_X + \gamma n I_X)^{-1} \right) \widehat{\Sigma}_{XY}, \widehat{\Sigma}_{XY} \left( \widehat{\Sigma}_Y + \gamma n I_Y \right)^{-1} \right\rangle_{HS}
$$

$$
+ \left\langle (\Sigma_X + \gamma n I_X)^{-1} \widetilde{\Sigma}_{XY}, \widetilde{\Sigma}_{XY} \left( (\widehat{\Sigma}_Y + \gamma n I_Y)^{-1} - (\Sigma_Y + \gamma n I_Y)^{-1} \right) \right\rangle_{HS}
$$

$$
:= A_1 + A_2.
$$

Using the formula $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$ we get

$$
\left( \left( \widehat{\Sigma}_X + \gamma n I_X \right)^{-1} - (\Sigma_X + \gamma n I_X)^{-1} \right) \widehat{\Sigma}_{XY} = \left( \left( \Sigma_X + \gamma n I_X \right)^{-1} \right) \left( \Sigma_X - \widehat{\Sigma}_X \right) \left( \Sigma_X + \gamma n I_X \right)^{-1} \widehat{\Sigma}_{XY}.
$$

Then, from the Cauchy-Schwartz inequality and $\| (\Sigma + \gamma n I)^{-1} \| \leq \gamma^{-1}$ for all compact operator $\Sigma$, we get

$$
|A_1| \leq \gamma^{-3} \| \Sigma_X - \widehat{\Sigma}_X \|_{HS} \| \widehat{\Sigma}_{XY} \|_{HS}^2 \tag{9}
$$

and, similarly,

$$
|A_2| \leq \gamma^{-3} \| \Sigma_Y - \widehat{\Sigma}_Y \|_{HS} \| \widehat{\Sigma}_{XY} \|_{HS}^2. \tag{10}
$$

Lemma 5 in [3] ensures that $\| \widehat{\Sigma}_{XY} - \Sigma_{XY} \|_{HS} = O_P(n^{-1/2}), \| \Sigma_Y - \widehat{\Sigma}_Y \|_{HS} = O_P(n^{-1/2})$ and $\| \Sigma_X - \widehat{\Sigma}_X \|_{HS} = O_P(n^{-1/2})$. Under $\mathcal{H}_0$ one has $\Sigma_{XY} = 0$, hence $\| \widehat{\Sigma}_{XY} \|_{HS} = O_P(n^{-1/2})$. We the deduce from (9) and (10) that...
|A_1| = O_P(\gamma_n^{-3}n^{-3/2}), and |A_2| = O_P(\gamma_n^{-3}n^{-3/2}). Consequently, |\tilde{N}_n(X,Y) - \hat{N}_n(X,Y)| = O_P(\gamma_n^{-3}n^{-3/2}). Now, from (11) and (12) we get by the Cauchy-Schwartz inequality |\tilde{N}_n(X,Y)| \leq \gamma_n^{-2}\|\Sigma_{XY}\|_{HS}^2 and |\tilde{N}_n(X,Y)| \leq \gamma_n^{-2}\|\Sigma_{XY}\|_{HS}^2 ensuring that \tilde{N}_n(X,Y) = O_P(\gamma_n^{-2}n^{-1}) and \hat{N}_n(X,Y) = O_P(\gamma_n^{-2}n^{-1}).

Secondly, for \( r \in \{1,2\} \), we consider

\[
D_{r,n} = d_r(\Sigma_X,\gamma_n)d_r(\Sigma_Y,\gamma_n) = \left( \sum_{p=1}^{+\infty} \frac{\lambda_p}{(\lambda_p + \gamma_n)^r} \right) \left( \sum_{p=1}^{+\infty} \frac{\mu_p}{(\mu_p + \mu_n)^r} \right),
\]

where \( \{\lambda_p\}_{p \geq 1} \) (resp. \( \{\mu_q\}_{q \geq 1} \)) is the sequence of decreasing eigenvalues of \( \Sigma_X \) (resp. \( \Sigma_Y \)), and we put:

\[
\tilde{T}_n = \frac{n\tilde{N}_n(X,Y) - D_{1,n}}{\sqrt{2D_{2,n}}}.
\]

Then, we have:

**Lemma 2** Assume \( (\mathcal{A}_1) \) to \( (\mathcal{A}_5) \). Then, under \( \mathcal{H}_0 \), we have \( \tilde{T}_n = \tilde{T}_n + o_p(1) \).

**Proof.** We have

\[
\sqrt{2}(\tilde{T}_n - \tilde{T}_n) = \left( \frac{n\tilde{N}_n(X,Y)}{D_{2,n}} - \frac{n\tilde{N}_n(X,Y)}{D_{2,n}} \right) - \left( \frac{\tilde{D}_{1,n}}{D_{2,n}} - \frac{D_{1,n}}{D_{2,n}} \right) := B - C.
\]

First,

\[
|B| \leq \left| \frac{1}{D_{2,n}} - \frac{1}{\tilde{D}_{2,n}} \right| n\tilde{N}_n(X,Y) + \frac{1}{D_{2,n}} \left| \tilde{N}_n(X,Y) - \hat{N}_n(X,Y) \right|
\]

\[
\leq \left| \frac{D_{2,n} - \tilde{D}_{2,n}}{D_{2,n}\tilde{D}_{2,n}} \right| n\tilde{N}_n(X,Y) - n\tilde{N}_n(X,Y)
\]

\[
+ \left| \frac{D_{2,n} - \tilde{D}_{2,n}}{D_{2,n}\tilde{D}_{2,n}} \right| n\tilde{N}_n(X,Y) + \frac{1}{D_{2,n}} \left| n\tilde{N}_n(X,Y) - n\tilde{N}_n(X,Y) \right|
\]

\[
:= B_1 + B_2 + B_3.
\]

Clearly,

\[
\frac{D_{2,n} - \tilde{D}_{2,n}}{D_{2,n}\tilde{D}_{2,n}} = \frac{d_2(\Sigma_X,\gamma_n) - d_2(\hat{\Sigma}_X,\gamma_n)}{d_2(\Sigma_X,\gamma_n)d_2(\hat{\Sigma}_X,\gamma_n)d_2(\hat{\Sigma}_Y,\gamma_n)} + \frac{d_2(\Sigma_Y,\gamma_n) - d_2(\hat{\Sigma}_Y,\gamma_n)}{d_2(\Sigma_X,\gamma_n)d_2(\Sigma_Y,\gamma_n)d_2(\hat{\Sigma}_Y,\gamma_n)}
\]

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and, therefore,

$$|B_1| \leq \frac{|d_2(\Sigma X, \gamma_n) - d_2(\hat{\Sigma} X, \gamma_n)|}{d_2(\Sigma X, \gamma_n)d_2(\hat{\Sigma} X, \gamma_n)d_2(\Sigma Y, \gamma_n)} \times |n \bar{N}_n(X, Y) - n \bar{N}_n(X, Y)|$$

$$+ \frac{|d_2(\Sigma Y, \gamma_n) - d_2(\hat{\Sigma} Y, \gamma_n)|}{d_2(\Sigma X, \gamma_n)d_2(\Sigma Y, \gamma_n)d_2(\hat{\Sigma} Y, \gamma_n)} \times |n \bar{N}_n(X, Y) - n \bar{N}_n(X, Y)|,$$

$$|B_2| \leq \frac{|d_2(\Sigma X, \gamma_n) - d_2(\hat{\Sigma} X, \gamma_n)|}{d_2(\Sigma X, \gamma_n)d_2(\hat{\Sigma} X, \gamma_n)d_2(\Sigma Y, \gamma_n)} \times n \bar{N}_n(X, Y)$$

$$+ \frac{|d_2(\Sigma Y, \gamma_n) - d_2(\hat{\Sigma} Y, \gamma_n)|}{d_2(\Sigma X, \gamma_n)d_2(\Sigma Y, \gamma_n)d_2(\hat{\Sigma} Y, \gamma_n)} \times n \bar{N}_n(X, Y).$$

Proposition 12 in [12] and Assumption ($\mathcal{A}_2$) ensure that

$$\|\hat{\Sigma} X - \Sigma X\|c_1 := \sum_{p=1}^{+\infty} \|(\hat{\Sigma} X - \Sigma X)e_p\|_h x = O_P(n^{-1/2});$$

applying now Lemma 23 in [12] with $S = \Sigma X$ and $\Delta = \hat{\Sigma} X - \Sigma X$ we get

$$|d_2(\Sigma X, \gamma_n) - d_2(\hat{\Sigma} X, \gamma_n)| \leq \frac{\gamma_n^{-1}||\hat{\Sigma} X - \Sigma X||c_1}{1 + \gamma_n^{-1}||\hat{\Sigma} X - \Sigma X||c_1} = O_P(\gamma_n^{-1}n^{-1/2}) \quad (13)$$

and, similarly, $|d_2(\Sigma Y, \gamma_n) - d_2(\hat{\Sigma} Y, \gamma_n)| = O_P(\gamma_n^{-1}n^{-1/2})$, $|d_1(\Sigma X, \gamma_n) - d_1(\hat{\Sigma} X, \gamma_n)| = O_P(\gamma_n^{-1}n^{-1/2})$ and $|d_1(\Sigma Y, \gamma_n) - d_1(\hat{\Sigma} Y, \gamma_n)| = O_P(\gamma_n^{-1}n^{-1/2})$. Plugging these equalities in (13) and using Lemma 1 and Lemma 18 in [12], we deduce from (12) that $B = o_P(1)$. On the other hand,

$$C = \frac{d_1(\hat{\Sigma} X, \gamma_n)d_1(\hat{\Sigma} Y, \gamma_n)}{d_2(\hat{\Sigma} X, \gamma_n)d_2(\hat{\Sigma} Y, \gamma_n)} - \frac{d_1(\Sigma X, \gamma_n)d_1(\Sigma Y, \gamma_n)}{d_2(\Sigma X, \gamma_n)d_2(\Sigma Y, \gamma_n)}$$

$$= \frac{(d_1(\hat{\Sigma} X, \gamma_n) - d_1(\Sigma X, \gamma_n))d_1(\hat{\Sigma} Y, \gamma_n) + d_1(\Sigma X, \gamma_n)(d_1(\hat{\Sigma} Y, \gamma_n) - d_1(\Sigma Y, \gamma_n))}{d_2(\Sigma X, \gamma_n)d_2(\Sigma Y, \gamma_n)}$$

$$+ \frac{d_1(\Sigma X, \gamma_n)d_1(\Sigma Y, \gamma_n)(d_2(\hat{\Sigma} X, \gamma_n) - d_2(\Sigma X, \gamma_n))}{d_2(\Sigma X, \gamma_n)d_2(\Sigma Y, \gamma_n)}$$

$$+ \frac{d_1(\Sigma X, \gamma_n)d_1(\Sigma Y, \gamma_n)(d_2(\Sigma Y, \gamma_n) - d_2(\hat{\Sigma} Y, \gamma_n))}{d_2(\Sigma X, \gamma_n)d_2(\Sigma Y, \gamma_n)}.$$
so that
\[
|C| \leq \frac{O_P(\gamma_n^{-1}n^{-1/2})d_1(\tilde{\Sigma}_Y, \gamma_n)}{d_2(\Sigma_X, \gamma_n)d_2(\Sigma_Y, \gamma_n)} + \frac{O_P(\gamma_n^{-1}n^{-1/2})d_1(\tilde{\Sigma}_X, \gamma_n)}{d_2(\Sigma_X, \gamma_n)d_2(\Sigma_Y, \gamma_n)}
\]
\[
+ \frac{O_P(\gamma_n^{-1}n^{-1/2})d_1(\Sigma_X, \gamma_n)d_1(\Sigma_Y, \gamma_n)}{d_2(\Sigma_X, \gamma_n)d_2(\Sigma_Y, \gamma_n)}
\]
\[
+ \frac{O_P(\gamma_n^{-1}n^{-1/2})d_1(\Sigma_X, \gamma_n)d_1(\Sigma_Y, \gamma_n)}{d_2(\Sigma_X, \gamma_n)d_2(\Sigma_Y, \gamma_n)}
\]
\[
= \frac{O_P(\gamma_n^{-2}n^{-1/2})d_1(\tilde{\Sigma}_Y, \gamma_n)}{d_2(\Sigma_X, \gamma_n)d_2(\Sigma_Y, \gamma_n)} + \frac{O_P(\gamma_n^{-2}n^{-1/2})d_1(\tilde{\Sigma}_X, \gamma_n)}{d_2(\Sigma_X, \gamma_n)d_2(\Sigma_Y, \gamma_n)}
\]
\[
+ \frac{O_P(\gamma_n^{-3}n^{-1/2})d_1(\Sigma_X, \gamma_n)d_1(\Sigma_Y, \gamma_n)}{d_2(\Sigma_X, \gamma_n)d_2(\Sigma_Y, \gamma_n)}
\]
\[
+ \frac{O_P(\gamma_n^{-3}n^{-1/2})d_1(\Sigma_X, \gamma_n)d_1(\Sigma_Y, \gamma_n)}{d_2(\Sigma_X, \gamma_n)d_2(\Sigma_Y, \gamma_n)}
\]

Lemmas 18 and 19 in [12] and Assumption (\(\mathcal{A}_0\)) allow to conclude that \(C = o_P(1)\). Then from (11) we deduce that, under \(\mathcal{H}_0\), \(\tilde{T}_n = T_n + o_p(1)\).

Next, we can write \(\tilde{\Sigma}_{XY}\) as
\[
\tilde{\Sigma}_{XY} = \frac{1}{n} \sum_{i=1}^{n} \left( L(Y_i, \cdot) - m_Y \right) \otimes \left( K(X_i, \cdot) - m_X \right) - \left( \mathcal{T}_n - m_Y \right) \otimes \left( \mathcal{K}_n - m_X \right),
\]
where
\[
\mathcal{T}_n = \frac{1}{n} \sum_{i=1}^{n} L(Y_i, \cdot) \quad \text{and} \quad \mathcal{K}_n = \frac{1}{n} \sum_{i=1}^{n} K(X_i, \cdot).
\]

Let \(\{e_p\}_{p \geq 1}\) (resp. \(\{f_q\}_{q \geq 1}\)) be an orthonomal basis of \(\mathcal{H}_X\) (resp. \(\mathcal{H}_Y\)) consisting of eigenvectors of \(\Sigma_X\) (resp. \(\Sigma_Y\)) and being such that \(e_p\) (resp. \(f_q\)) is eigenvector associated with the eigenvalue \(\lambda_p\) (resp. \(\mu_q\)). We have the the eigen-decompositions
\[
(\Sigma_X + \gamma_n \mathbb{I}_X)^{-1/2} = \sum_{p=1}^{+\infty} (\lambda_p + \gamma_n)^{-1/2} e_p \otimes e_p, \quad (\Sigma_Y + \mathbb{I}_Y)^{-1/2} = \sum_{q=1}^{+\infty} (\mu_q + \gamma_n)^{-1/2} f_q \otimes f_q
\]

lead to \(\tilde{V}_{XY} = \tilde{V}_{XY}^{(1)} - \tilde{V}_{XY}^{(2)}\), where
\[
\tilde{V}_{XY}^{(1)} = \frac{1}{n} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} \sum_{i=1}^{n} (\lambda_p + \gamma_n)^{-1/2} (\mu_q + \gamma_n)^{-1/2} (e_p \otimes e_p) \left( (L(Y_i, \cdot) - m_Y) \otimes (K(X_i, \cdot) - m_X) \right) (f_q \otimes f_q)
\]

\[
\tilde{V}_{XY}^{(2)} = \frac{1}{n} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} \sum_{i=1}^{n} (\lambda_p + \gamma_n)^{-1/2} (\mu_q + \gamma_n)^{-1/2} (e_p \otimes e_p) \left( (L(Y_i, \cdot) - m_Y) \otimes (K(X_i, \cdot) - m_X) \right) (f_q \otimes f_q)
\]
and
\[ \tilde{V}^{(2)}_{XY} = \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1/2} (\mu_q + \gamma_n)^{-1/2} (e_p \otimes e_p) ((L_n - m_Y) \otimes (K_n - m_X)) (f_q \otimes f_q). \]

The following lemma gives properties of the last two terms that will be useful later for proving the main theorem of the paper.

**Lemma 3** Assume \((\mathcal{A}_1)\) to \((\mathcal{A}_5)\). Then, under \(\mathcal{H}_0\), we have:

(i) \[ \mathbb{E}(n \| \tilde{V}^{(1)}_{XY} \|_{HS}^2) = D_{1,n}; \]

(ii) \[ \mathbb{E}(n \| \tilde{V}^{(2)}_{XY} \|_{HS}^2) = n^{-1} D_{1,n}; \]

(iii) \[ n \| \tilde{V}^{(2)}_{XY} \|_{HS}^2 = o_P(1); \]

(iv) \[ n \langle \tilde{V}^{(1)}_{XY}, \tilde{V}^{(2)}_{XY} \rangle_{HS} = o_P(1). \]

**Proof.**

(i). Using twice the formula \((a \otimes b)(c \otimes d) = \langle a, d \rangle c \otimes b\) and the reproducing properties of \(K\) and \(L\), we get

\[ \tilde{V}^{(1)}_{XY} = \frac{1}{n} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1/2} (\mu_q + \gamma_n)^{-1/2} \left( \sum_{i=1}^{n} (e_p(X_i) - \mathbb{E}(e_p(X_i))) (f_q(Y_i) - \mathbb{E}(f_q(Y_i))) \right) f_q \otimes e_p; \]

so that

\[ \| \tilde{V}^{(1)}_{XY} \|_{HS}^2 = \frac{1}{n^2} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1} (\mu_q + \gamma_n)^{-1} \left\{ \sum_{i=1}^{n} \alpha_{p,i,q} \right\}^2 = \frac{1}{n^2} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1} (\mu_q + \gamma_n)^{-1} \left( \sum_{i=1}^{n} \alpha_{p,i,q}^2 + 2 \sum_{i=2}^{n} \alpha_{p,i,q} \alpha_{p,i-1,q} \right), \]  

where

\[ \alpha_{p,i,q} = (e_p(X_i) - \mathbb{E}(e_p(X_i))) (f_q(Y_i) - \mathbb{E}(f_q(Y_i))) \]

(14)

(15)
and

\[ M_{p,j,q} = \sum_{l=1}^{j} \alpha_{p,l,q}. \]  \hfill (16)

Since \( \alpha_{p,i,q} \) and \( \alpha_{p,j,q} \) are independent for \( i \neq j \) and, under \( \mathcal{H}_0 \), \( \mathbb{E}(\alpha_{p,i,q}) = 0 \), we get

\[
\mathbb{E}\left(n\|\tilde{V}_{XY}^{(1)}\|_{\text{HS}}^2\right) = \frac{1}{n} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1}(\mu_q + \gamma_n)^{-1} \\
\left( \sum_{i=1}^{n} \mathbb{E}\left(\alpha_{p,i,q}^2\right) + 2 \sum_{i=2}^{n} \mathbb{E}\left(\alpha_{p,i,q}M_{p,i-1,q}\right) \right)
\]

\[
= \frac{1}{n} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1}(\mu_q + \gamma_n)^{-1} \\
\left( \sum_{i=1}^{n} \mathbb{E}\left(\alpha_{p,i,q}^2\right) + 2 \sum_{i=2}^{n} \mathbb{E}\left(\alpha_{p,i,q}\right) \mathbb{E}\left(M_{p,i-1,q}\right) \right)
\]

\[
= \frac{1}{n} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1}(\mu_q + \gamma_n)^{-1} \left( \sum_{i=1}^{n} \mathbb{E}\left(\alpha_{p,i,q}^2\right) \right)
\]

\[
= \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1}(\mu_q + \gamma_n)^{-1} \text{var}(e_p(X_1))\text{var}(f_q(Y_1))\]

\[
= \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1}(\mu_q + \gamma_n)^{-1} \langle e_p, \Sigma_X e_p \rangle_{\mathcal{H}_X} \langle f_q, \Sigma_Y f_q \rangle_{\mathcal{H}_Y}\]

\[
= \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1}(\mu_q + \gamma_n)^{-1} \lambda_p \mu_q\]

\[
= \left\{ \sum_{p=1}^{+\infty} \frac{\lambda_p}{\lambda_p + \gamma_n} \right\} \left\{ \sum_{q=1}^{+\infty} \frac{\mu_q}{\mu_q + \gamma_n} \right\}\]

\[
= d_1(\Sigma_X, \gamma_n)d_1(\Sigma_Y, \gamma_n) = D_{1,n}.\]  \hfill (17)

(ii). Using again the formula \((a \otimes b)(c \otimes d) = \langle a, d \rangle c \otimes b\) and the reproducing
properties of $K$ and $L$, we get

$$
\tilde{V}_{XY}^{(2)} = \frac{1}{n^2} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1/2} (\mu_q + \gamma_n)^{-1/2} \left( \sum_{i=1}^{n} e_p(X_i) - \mathbb{E}(e_p(X_i)) \right) \left( \sum_{i=1}^{n} f_q(Y_i) - \mathbb{E}(f_q(Y_i)) \right) f_q \otimes e_p,
$$

so that

$$
\|\tilde{V}_{XY}^{(2)}\|_{HS}^2 = \frac{1}{n^4} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1} (\mu_q + \gamma_n)^{-1} \left( \sum_{i=1}^{n} e_p(X_i) - \mathbb{E}(e_p(X_i)) \right)^2 \left( \sum_{i=1}^{n} f_q(Y_i) - \mathbb{E}(f_q(Y_i)) \right)^2
$$

and, under $\mathcal{H}_0$,

$$
\mathbb{E}\left( n \|\tilde{V}_{XY}^{(2)}\|_{HS}^2 \right) = \frac{1}{n^3} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1} (\mu_q + \gamma_n)^{-1} \mathbb{E}\left( \left( \sum_{i=1}^{n} e_p(X_i) - \mathbb{E}(e_p(X_i)) \right)^2 \right) \mathbb{E}\left( \left( \sum_{i=1}^{n} f_q(Y_i) - \mathbb{E}(f_q(Y_i)) \right)^2 \right)
$$

$$
= \frac{1}{n^3} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1} (\mu_q + \gamma_n)^{-1} \left( \sum_{i=1}^{n} \text{var}\left( e_p(X_i) \right) \right) \left( \sum_{j=1}^{n} \text{var}\left( f_q(Y_j) \right) \right)
$$

$$
= \frac{1}{n} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1} (\mu_q + \gamma_n)^{-1} \lambda_p \mu_q
$$

$$
= \frac{1}{n} D_{1,n}.
$$

(iii). Lemma 19 in [12] and Assumptions ($\mathcal{A}_2$) and ($\mathcal{A}_3$) imply that

$$
\gamma_n D_{1,n} \leq 4 \left( \sum_{p \geq 1} \lambda_p^{1/2} \right) \left( \sum_{q \geq 1} \mu_q^{1/2} \right).
$$

Then, using Assumption ($\mathcal{A}_5$), we obtain the inequality

$$
\mathbb{E}\left( n \|\tilde{V}_{XY}^{(2)}\|_{HS}^2 \right) = n^{-1} D_{1,n} \leq 4 \gamma_n^{-1} n^{-1} \left( \sum_{p \geq 1} \lambda_p^{1/2} \right) \left( \sum_{q \geq 1} \mu_q^{1/2} \right) = o(1) \quad (18)
$$
from which, and Markov inequality, we deduce that $n \| \tilde{V}_X(2)_{\text{HS}} \| = o_P(1)$ since $\lim_{n \to +\infty} (\gamma_n^{-1} n^{-1}) = 0$.

(iv). Using the Cauchy-Schwartz and Hölder inequalities, the previous properties (i) and (ii), and (18), we obtain the inequality

$$E \left( n \left\| \langle \tilde{V}_X(1), \tilde{V}_X(2) \rangle_{\text{HS}} \right\| \right) \leq n \left( E \left\| \tilde{V}_X(1) \right\|^2_{\text{HS}} \right)^{1/2} \left( E \left\| \tilde{V}_X(2) \right\|^2_{\text{HS}} \right)^{1/2} = n^{-1/2} D_{1,n}$$

$$\leq 4 \gamma_n^{-1} n^{-1/2} \left( \sum_{p \geq 1} \lambda_p^{1/2} \right) \left( \sum_{q \geq 1} \mu_q^{1/2} \right)$$

from which, and Markov inequality, we deduce that $\langle \tilde{V}_X(1), \tilde{V}_X(2) \rangle_{\text{HS}} = o_P(1)$ since $\lim_{n \to +\infty} (\gamma_n^{-1} n^{-1/2}) = 0$.

Finally, considering

$$E_n = \frac{1}{n} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1} (\mu_q + \gamma_n)^{-1} \sum_{i=1}^{n} \left( \alpha_{p,i,q}^2 - E \left( \alpha_{p,i,q}^2 \right) \right), \quad (19)$$

where $\alpha_{p,i,q}$ is defined in (15), we have:

**Lemma 4** Assume ($\mathcal{A}_1$) to ($\mathcal{A}_5$). Then, under $\mathcal{H}_0$, we have $E_n = o_P(1)$.

**Proof.** Since $E_n$ is centered it is enough to show that $\lim_{n \to +\infty} \text{var}(E_n) = 0$, then Markov inequality allows to conclude. We have

$$\text{var}(E_n) = \frac{1}{n^2} \sum_{i=1}^{n} \text{var} \left[ \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1} (\mu_q + \gamma_n)^{-1} \left\{ \alpha_{p,i,q}^2 - E \left( \alpha_{p,i,q}^2 \right) \right\} \right]$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} v_{n,i},$$

where

$$v_{n,i} = \sum_{p_1,q_1=1}^{+\infty} \sum_{p_2,q_2=1}^{+\infty} (\mu_{q_1} + \gamma_n)^{-1} (\lambda_{p_1} + \gamma_n)^{-1} (\mu_{q_2} + \gamma_n)^{-1} (\lambda_{p_2} + \gamma_n)^{-1} \text{cov}(\alpha_{p_1,i,q_1}^2, \alpha_{p_2,i,q_2}^2). \quad (20)$$
Further,
\[
\left| \text{cov}(\alpha_{p_1, i, q_1}^2, \alpha_{p_2, i, q_2}^2) \right| \leq \mathbb{E} \left( \alpha_{p_1, i, q_1}^2 \alpha_{p_2, i, q_2}^2 \right) + \mathbb{E} \left( \alpha_{p_1, i, q_1}^2 \mathbb{E} \left( \alpha_{p_2, i, q_2}^2 \right) \right) \\
\leq 2 \mathbb{E}^{1/2} (\alpha_{p_1, i, q_1}^4) \mathbb{E}^{1/2} (\alpha_{p_2, i, q_2}^4)
\]
(21)

and
\[
\alpha_{p, i, q}^2 = \left( e_p(X_i) - \mathbb{E} (e_p(X_i)) \right)^2 \left( f_q(Y_i) - \mathbb{E} (f_q(Y_i)) \right)^2 \\
= \left( e_p^2(X_i) + \mathbb{E}^2 (e_p(X_i)) - 2e_p(X_i) \mathbb{E} (e_p(X_i)) \right) \\
\left( f_q^2(Y_i) + \mathbb{E}^2 (f_q(Y_i)) - 2f_q(Y_i) \mathbb{E} (f_q(Y_i)) \right) \\
\leq \left( e_p^2(X_i) + \mathbb{E}^2 (|e_p(X_i)|) + 2 |e_p(X_i)| \mathbb{E} (|e_p(X_i)|) \right) \\
\left( f_q^2(Y_i) + \mathbb{E}^2 (|f_q(Y_i)|) + 2 |f_q(Y_i)| \mathbb{E} (|f_q(Y_i)|) \right)
\]
(22)

Since
\[
|e_p(X_i)| = |\langle e_p, K(X_i, \cdot) \rangle_{\mathcal{H}_x}| \leq \|K(X_i, \cdot)\|_{\mathcal{H}_x} = K(X_i, X_i)^{1/2} \leq \|K\|^{1/2}
\]
(23)

and, similarly,
\[
|f_q(Y_i)| \leq \|L\|^{1/2},
\]
(24)

it follows from (22) that \(\alpha_{p, i, q}^2 \leq 16 \|K\| \|L\| \). Then, from (21) and the equality
\[
\mathbb{E} \left( \alpha_{p, i, q}^2 \right) = \text{var} (e_p(X_i)) \text{var} (f_q(Y_i)) = \langle e_p, \Sigma_X e_p \rangle_{\mathcal{H}_X} \langle f_q, \Sigma_Y f_q \rangle_{\mathcal{H}_Y} = \lambda_p \mu_q
\]
(25)

we get
\[
\left| \text{cov}(\alpha_{p_1, i, q_1}^2, \alpha_{p_2, i, q_2}^2) \right| \leq 32 \|K\| \|L\| \lambda_p^{1/2} \lambda_q^{1/2} \mu_p^{1/2} \mu_q^{1/2}
\]
(26)

Using this later inequality together with the inequality \((a + \gamma_n)^{-1} \leq \gamma_n^{-1}\) for all positive number \(a\), we deduce from (20) that
\[
u_{n, i} \leq 32 \|K\| \|L\| \gamma_n \left( \sum_{p=1}^{+\infty} \lambda_p^{1/2} \right)^2 \left( \sum_{q=1}^{+\infty} \mu_q^{1/2} \right)^2.
\]
Thus
\[ \text{var}(E_n) \leq 32 \|K\|_{\infty} \|L\|_{\infty} \gamma_n^{-4} n^{-1} \left( \sum_{p=1}^{+\infty} \lambda_p^{1/2} \right)^2 \left( \sum_{q=1}^{+\infty} \mu_q^{1/2} \right)^2, \]

Assumptions (\(A_2\)), (\(A_3\)) and (\(A_5\)) allow to conclude that \(E_n = o_p(1)\).

### 5.2 Proof of Theorem 1

Lemma 2 and an application of Slutsky’s theorem show that it is enough to prove that
\[ \tilde{T}_n \xrightarrow{d} N(0, 1) \text{ as } n \to +\infty. \tag{27} \]

From (27) we have
\[
\tilde{T}_n = \frac{n\|\tilde{V}_{XY}^{(1)} - \tilde{V}_{XY}^{(2)}\|_{\text{HS}}^2 - D_{1,n}}{\sqrt{2D_{2,n}}} - \frac{n\|\tilde{V}_{XY}^{(1)}\|_{\text{HS}}^2 - D_{1,n}}{\sqrt{2D_{2,n}}} + n\|\tilde{V}_{XY}^{(2)}\|_{\text{HS}}^2 - 2n \langle \tilde{V}_{XY}^{(1)}, \tilde{V}_{XY}^{(2)} \rangle_{\text{HS}}.
\]

Furthermore, using Lemma 3(i), (14) and the third line in (17), we obtain
\[
n\|\tilde{V}_{XY}^{(1)}\|_{\text{HS}}^2 - D_{1,n} = n\|\tilde{V}_{XY}^{(1)}\|_{\text{HS}}^2 - \mathbb{E}\left(n\|\tilde{V}_{XY}^{(1)}\|_{\text{HS}}^2\right) = E_n + 2F_n
\]

where \(E_n\) is given in (19) and
\[
F_n = \frac{1}{n} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1}(\mu_q + \gamma_n)^{-1} \left\{ \sum_{i=2}^{n} \alpha_{p,i,q} M_{p,i-1,q} \right\},
\]

the random variables \(\alpha_{p,i,q}\) and \(M_{p,i-1,q}\) being defined in (15) and (16). Therefore, we can write
\[
\tilde{T}_n = \sqrt{2} \frac{C_n}{D_{2,n}} + \frac{B_n}{\sqrt{2D_{2,n}}} + \frac{n\|\tilde{V}_{XY}^{(2)}\|_{\text{HS}}^2 - 2n \langle \tilde{V}_{XY}^{(1)}, \tilde{V}_{XY}^{(2)} \rangle_{\text{HS}}}{\sqrt{2D_{2,n}}}
\]

and since, from Lemma 18 in [12], we have \(\lim_{n \to +\infty} (D_{2,n}) = +\infty\), we deduce from Lemma 3 and Lemma 4 that
\[
\tilde{T}_n = \sqrt{2}D_{2,n}^{-1}F_n + o_P(1)
\]

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So, in order to get (27) it suffices to prove that

\[ D_{2,n}^{-1}F_n \xrightarrow{p} N(0, 1/2) \quad \text{as} \quad n \to +\infty. \quad (28) \]

We will show it by using the central limit theorem for triangular arrays of martingale differences. For all \((p, q)\) we set \(\alpha_{p,0,q} = 0\), \(\mathcal{F}_{n,0} = \{\emptyset, \Omega\}\) and for \(i \geq 1\)

\[ \mathcal{F}_{n,i} = \sigma(\alpha_{p,l,q}, 1 \leq l \leq i, 1 \leq p \leq n, 1 \leq q \leq n). \]

Defining \(\epsilon_{n,1} = 0\) and for all \(i \geq 2\),

\[ \epsilon_{n,i} = D_{2,n}^{-1}n^{-1} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1}(\mu_q + \gamma_n)^{-1} \alpha_{p,i,q}M_{p,i-1,q}, \]

we have \(D_{2,n}^{-1}F_n = \sum_{i=1}^{n} \epsilon_{n,i}\) and \(\epsilon_{n,i}\) is a martingale increment since \(E(\epsilon_{n,i} | \mathcal{F}_{n,i-1}) = 0\). Then, from Theorem 3.2 and Corollary 3.1 in [10], we will obtain (28) if we show that

\[ S_n^2 := \sum_{i=1}^{n} E\left( \epsilon_{n,i}^2 | \mathcal{F}_{n,i-1} \right) \xrightarrow{p} \frac{1}{2} \quad \text{as} \quad n \to +\infty \quad (29) \]

and

\[ E\left( \max_{1 \leq i \leq n} \epsilon_{n,i}^2 \right) = o(1). \quad (30) \]

Proof of (29): We have

\[ S_n^2 = D_{2,n}^{-2}n^{-2} \sum_{i=1}^{n} E\left( \left( \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1}(\mu_q + \gamma_n)^{-1} \alpha_{p,i,q}M_{p,i-1,q} \right)^2 | \mathcal{F}_{n,i-1} \right) \]

\[ = G_n + H_n, \]

where

\[ G_n = D_{2,n}^{-2}n^{-2} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\mu_q + \gamma_n)^{-2}(\lambda_p + \gamma_n)^{-2} \sum_{i=1}^{n} M_{p,i-1,q}^2 E\left( \alpha_{p,i,q}^2 \right) \]

and

\[ H_n = D_{2,n}^{-2}n^{-2} \sum_{p_1,q_1=1}^{+\infty} \sum_{p_2,q_2=1}^{+\infty} \sum_{i=1}^{n} \sum_{(p_1,q_1) \neq (p_2,q_2)} (\mu_{q_1} + \gamma_n)^{-1}(\lambda_{p_1} + \gamma_n)^{-1} \]

\[ \quad \left( \mu_{q_2} + \gamma_n \right)^{-1}(\lambda_{p_2} + \gamma_n)^{-1} M_{p_1,i-1,q_1}M_{p_2,i-1,q_2} E\left( \alpha_{p_1,i,q_1}\alpha_{p_2,i,q_2} \right). \]

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First, notice that under $\mathcal{H}_0$ one has:

\[
\mathbb{E}\left(\alpha_{p_1,i,q_1}\alpha_{p_2,i,q_2}\right) = \mathbb{E}\left(\left(e_{p_1}(X_i) - \mathbb{E}(e_{p_1}(X_i))\right)\left(e_{p_2}(X_i) - \mathbb{E}(e_{p_2}(X_i))\right)\right)
\]

\[
= \mathbb{E}\left(\left(f_{q_1}(Y_i) - \mathbb{E}(f_{q_1}(Y_i))\right)\left(f_{q_2}(Y_i) - \mathbb{E}(f_{q_2}(Y_i))\right)\right)
\]

\[
= \mathbb{E}\left(\langle e_{p_1}, K(X_i, \cdot) - m_X \rangle_{\mathcal{H}_X} \langle e_{p_2}, K(X_i, \cdot) - m_X \rangle_{\mathcal{H}_X}\right)
\]

\[
= \langle e_{p_1}, \Sigma_X e_{p_2} \rangle_{\mathcal{H}_X} \langle f_{q_1}, \Sigma_Y f_{q_2} \rangle_{\mathcal{H}_Y}
\]

\[
= \lambda_{p_1,q_1} \delta_{p_1,p_2} \delta_{q_1,q_2},
\]

where $\delta$ is the Kronecker symbol. Thus, $H_n = 0$ and it remains to prove that $G_n$ converges in probability to $1/2$ as $n \to +\infty$. For doing that, we will show that $\lim_{n \to +\infty} (\mathbb{E}(G_n)) = 1/2$ and $G_n - \mathbb{E}(G_n) = o_p(1)$. Using the independence of the family $(\alpha_{p,i,q})_{1 \leq i \leq n}$ and the fact that these random variables are centered, we can write

\[
\mathbb{E}(G_n) = D_{2,n}^{-2} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\mu_q + \gamma_n)^{-2} (\lambda_p + \gamma_n)^{-2} \sum_{i=1}^{n} \mathbb{E}(M_{p,i-1,q}^2) \mathbb{E}(\alpha_{p,i,q}^2)
\]

\[
= D_{2,n}^{-2} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\mu_q + \gamma_n)^{-2} (\lambda_p + \gamma_n)^{-2} \sum_{i=1}^{n} \text{var} \left(\sum_{\ell=0}^{i-1} \alpha_{p,\ell,q}\right) \mathbb{E}(\alpha_{p,i,q}^2)
\]

\[
= D_{2,n}^{-2} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\mu_q + \gamma_n)^{-2} (\lambda_p + \gamma_n)^{-2} \sum_{i=1}^{n} \sum_{\ell=0}^{i-1} \mathbb{E}(\alpha_{p,\ell,q}^2) \mathbb{E}(\alpha_{p,i,q}^2).
\]

using the formula

\[
\left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2 + 2 \sum_{i=1}^{n} \sum_{\ell=0}^{i-1} a_i a_\ell
\]

\[
(32)
\]

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and equation (25) we obtain the equality

\[ E(G_n) = 2^{-1}D_{2,n}^{-2}n^{-2} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\mu_q + \gamma_n)^{-2}(\lambda_p + \gamma_n)^{-2} \left( \left( \sum_{i=1}^{n} E(\alpha_{p,i,q}^2) \right) - \sum_{i=1}^{n} E(\alpha_{p,i,q}^2) \right) \]

\[ = 2^{-1}D_{2,n}^{-2}n^{-2} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\mu_q + \gamma_n)^{-2}(\lambda_p + \gamma_n)^{-2} \left( \sum_{i=1}^{n} \mu_q \lambda_p \right)^2 - \sum_{i=1}^{n} \mu_q^2 \lambda_p^2 \]

\[ = 2^{-1}D_{2,n}^{-2}n^{-2} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\mu_q + \gamma_n)^{-2}(\lambda_p + \gamma_n)^{-2} \mu_q^2 \lambda_p^2 (n^2 - n) \]

\[ = \frac{1}{2} - \frac{1}{2n} \]

from which we deduce that \( \lim_{n \to +\infty} (E(G_n)) = 1/2 \). For proving that \( G_n - E(G_n) = o_p(1) \) it is enough to show that \( \lim_{n \to +\infty} (\text{var}(G_n)) = 0 \). We have

\[ G_n - E(G_n) = D_{2,n}^{-2}n^{-2} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\mu_q + \gamma_n)^{-2}(\lambda_p + \gamma_n)^{-2}Q_{n,p,q} \]

where \( Q_{n,p,q} = \sum_{i=1}^{n} E(\alpha_{p,i,q}^2) \left( M_{p,i-1,q}^2 - E(M_{p,i-1,q}^2) \right) \). So that

\[ \text{var}(G_n) = D_{2,n}^{-4}n^{-4}E \left( \left( \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\mu_q + \gamma_n)^{-2}(\lambda_p + \gamma_n)^{-2}Q_{n,p,q} \right)^2 \right) = s_{1,n} + s_{2,n} \]

(33)

where

\[ s_{1,n} = D_{2,n}^{-4}n^{-4} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\mu_q + \gamma_n)^{-4}(\lambda_p + \gamma_n)^{-4}E(\alpha_{p,i,q}^2) \]

(34)

and

\[ s_{2,n} = D_{2,n}^{-4}n^{-4} \sum_{p_1,q_1=1}^{+\infty} \sum_{p_2,q_2=1}^{+\infty} \sum_{(p_2,q_2)\neq(p_1,q_1)} (\mu_{q_1} + \gamma_n)^{-2}(\lambda_{p_1} + \gamma_n)^{-2}(\mu_{q_2} + \gamma_n)^{-2}(\lambda_{p_2} + \gamma_n)^{-2} \]

\[ \times E(Q_{n,p_1,q_1}Q_{n,p_2,q_2}) \]

(35)

Putting

\[ v_{p,i,q} = M_{p,i,q}^2 - E(M_{p,i,q}^2) - (M_{p,i-1,q}^2 - E(M_{p,i-1,q}^2)) \]

\[ = \alpha_{p,i,q}^2 - E(\alpha_{p,i,q}^2) + 2\alpha_{p,i,q}M_{p,i-1,q} \]

(36)
and using the formula
\[
\sum_{i=1}^{n} a_i b_i = a_n \sum_{i=1}^{n} b_i - \sum_{i=1}^{n-1} \sum_{j=1}^{i} b_j (a_{i+1} - a_i)
\]

with \( a_i = M_{p,i-1,q}^2 - \mathbb{E} \left( M_{p,i-1,q}^2 \right) \) and \( b_i = \mathbb{E} \left( \alpha_{p,i,q}^2 \right) \) we get

\[
Q_{n,p,q} = \left( M_{p,n-1,q}^2 - \mathbb{E} \left( M_{p,n-1,q}^2 \right) \right) \sum_{i=1}^{n} \mathbb{E} \left( \alpha_{p,i,q}^2 \right) - \sum_{i=1}^{n-1} v_{p,i,q} \sum_{j=1}^{i} \mathbb{E} \left( \alpha_{p,j,q}^2 \right)
\]

\[
= \left( M_{p,n-1,q}^2 - \mathbb{E} \left( M_{p,n-1,q}^2 \right) - \sum_{i=1}^{n-1} v_{p,i,q} \right) \sum_{i=1}^{n} \mathbb{E} \left( \alpha_{p,i,q}^2 \right)
\]

\[
+ \sum_{i=1}^{n-1} v_{p,i,q} \sum_{j=1}^{n} \mathbb{E} \left( \alpha_{p,j,q}^2 \right) - \sum_{i=1}^{n-1} v_{p,i,q} \sum_{j=1}^{i} \mathbb{E} \left( \alpha_{p,j,q}^2 \right)
\]

\[
= \left( M_{p,n-1,q}^2 - \mathbb{E} \left( M_{p,n-1,q}^2 \right) - \sum_{i=1}^{n-1} v_{p,i,q} \right) \sum_{i=1}^{n} \mathbb{E} \left( \alpha_{p,i,q}^2 \right) + \sum_{i=1}^{n-1} v_{p,i,q} \sum_{j=i+1}^{n} \mathbb{E} \left( \alpha_{p,j,q}^2 \right)
\]

\[
= \sum_{i=1}^{n-1} v_{p,i,q} \sum_{j=i+1}^{n} \mathbb{E} \left( \alpha_{p,j,q}^2 \right),
\]

so that

\[
\mathbb{E} \left( Q_{n,p_1,q_1} Q_{n,p_2,q_2} \right) = \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} \left( \sum_{j=i+1}^{n} \mathbb{E} \left( \alpha_{p_1,j,q_1}^2 \right) \right) \left( \sum_{\ell=k+1}^{n} \mathbb{E} \left( \alpha_{p_2,\ell,q_2}^2 \right) \right) \mathbb{E} \left( v_{p_1,i,q_1} v_{p_2,k,q_2} \right).
\]
From (36), we get for $i \neq k$ with $i > k$ (without loss of generality):

$$
E(v_{p_1,i,q_1}v_{p_2,k,q_2}) = E \left( \left( \alpha_{p_1,i,q_1}^2 - E(\alpha_{p_1,i,q_1}) + 2\alpha_{p_1,i,q_1}M_{p_1,i-1,q_1} \right)
\times \left( \alpha_{p_2,k,q_2}^2 - E(\alpha_{p_2,k,q_2}) + 2\alpha_{p_2,k,q_2}M_{p_2,k-1,q_2} \right) \right)
$$

$$
= E \left( \left( \alpha_{p_1,i,q_1}^2 - E(\alpha_{p_1,i,q_1}) \right) \left( \alpha_{p_2,k,q_2}^2 - E(\alpha_{p_2,k,q_2}) \right) \right)
+ 2\alpha_{p_1,i,q_1} \left( \alpha_{p_2,k,q_2}^2 - E(\alpha_{p_2,k,q_2}) \right) M_{p_1,i-1,q_1}
+ 2\alpha_{p_2,k,q_2} \left( \alpha_{p_1,i,q_1}^2 - E(\alpha_{p_1,i,q_1}) \right) M_{p_2,k-1,q_2}
+ 4\alpha_{p_1,i,q_1}\alpha_{p_2,k,q_2} M_{p_1,i-1,q_1} M_{p_2,k-1,q_2}
$$

and since, from independence properties, we have

$$
E \left( \left( \alpha_{p_1,i,q_1}^2 - E(\alpha_{p_1,i,q_1}) \right) \left( \alpha_{p_2,k,q_2}^2 - E(\alpha_{p_2,k,q_2}) \right) \right) = 0,
$$

$$
E \left( \alpha_{p_1,i,q_1} \left( \alpha_{p_2,k,q_2}^2 - E(\alpha_{p_2,k,q_2}) \right) M_{p_1,i-1,q_1} \right) = 0,
$$

$$
E \left( \alpha_{p_2,k,q_2} \left( \alpha_{p_1,i,q_1}^2 - E(\alpha_{p_1,i,q_1}) \right) M_{p_2,k-1,q_2} \right) = 0,
$$

$$
\text{(37)}
$$
\[ E\left(\alpha_{p_1,i,q_1}\alpha_{p_2,k,q_2}M_{p_1,i-1,q_1}M_{p_2,k-1,q_2}\right) = E\left(\alpha_{p_1,i,q_1}\right)E\left(\alpha_{p_2,k,q_2}M_{p_1,i-1,q_1}M_{p_2,k-1,q_2}\right) = 0, \]

it follows that \( E\left(v_{p_1,i,q_1}v_{p_2,k,q_2}\right) = 0 \). Thus

\[ E\left(Q_{n,p_1,q_1}Q_{n,p_2,q_2}\right) = \sum_{i=1}^{n-1} \left( \sum_{j=i+1}^{n} E\left(\alpha_{p_1,j,q_1}\right) \right) \left( \sum_{\ell=i+1}^{n} E\left(\alpha_{p_2,\ell,q_2}\right) \right) E\left(v_{p_1,i,q_1}v_{p_2,i,q_2}\right), \]

and

\[
\left| E\left(Q_{n,p_1,q_1}Q_{n,p_2,q_2}\right) \right| \leq \left( \sum_{j=1}^{n} E\left(\alpha_{p_1,j,q_1}\right) \right) \left( \sum_{\ell=1}^{n} E\left(\alpha_{p_2,\ell,q_2}\right) \right) \sum_{i=1}^{n-1} \left| E\left(v_{p_1,i,q_1}v_{p_2,i,q_2}\right) \right| \leq n^2 \mu_{q_1} \lambda_{p_1} \lambda_{p_2} \sum_{i=1}^{n-1} \left| E\left(v_{p_1,i,q_1}v_{p_2,i,q_2}\right) \right|. \tag{38} \]

As in (37) we get

\[
E\left(v_{p_1,i,q_1}v_{p_2,i,q_2}\right) = E\left(\alpha_{p_1,i,q_1}^2 - E\left(\alpha_{p_1,i,q_1}\right)^2\right) \left( \alpha_{p_2,i,q_2}^2 - E\left(\alpha_{p_2,i,q_2}\right)^2\right) + 2\alpha_{p_1,i,q_1}\left(\alpha_{p_2,i,q_2}^2 - E\left(\alpha_{p_2,i,q_2}\right)^2\right) M_{p_1,i-1,q_1} + 2\alpha_{p_2,i,q_2}\left(\alpha_{p_1,i,q_1}^2 - E\left(\alpha_{p_1,i,q_1}\right)^2\right) M_{p_2,i-1,q_2} + 4\alpha_{p_1,i,q_1} \alpha_{p_2,i,q_2} M_{p_1,i-1,q_1} M_{p_2,i-1,q_2} \\
= \text{cov}\left(\alpha_{p_1,i,q_1}, \alpha_{p_2,i,q_2}\right) + 2E\left(\alpha_{p_1,i,q_1}\left(\alpha_{p_2,i,q_2}^2 - E\left(\alpha_{p_2,i,q_2}\right)^2\right)\right) E\left(M_{p_1,i-1,q_1}\right) + 2E\left(\alpha_{p_2,i,q_2}\left(\alpha_{p_1,i,q_1}^2 - E\left(\alpha_{p_1,i,q_1}\right)^2\right)\right) E\left(M_{p_2,i-1,q_2}\right) + 4E\left(\alpha_{p_1,i,q_1} \alpha_{p_2,i,q_2}\right) E\left(M_{p_1,i-1,q_1} M_{p_2,i-1,q_2}\right) \\
= \text{cov}\left(\alpha_{p_1,i,q_1}, \alpha_{p_2,i,q_2}\right) + 4E\left(\alpha_{p_1,i,q_1} \alpha_{p_2,i,q_2}\right) E\left(M_{p_1,i-1,q_1} M_{p_2,i-1,q_2}\right). \tag{39} \]
Moreover the independence of the variables $\alpha_{p_1,i,q_1}$ and $\alpha_{p_2,j,q_2}$ for $i \neq j$ gives

$$
\mathbb{E}
\left(
M_{p_1,i-1,q_1} \cdot M_{p_2,i-1,q_2}
\right) = \sum_{\ell=0}^{i-1} \mathbb{E}
\left(
\alpha_{p_1,\ell,q_1} \cdot \alpha_{p_2,\ell,q_2}
\right)
$$

and using again (32) and (31), we obtain

$$
\sum_{i=1}^{n-1} \left| \mathbb{E}
\left(
\alpha_{p_1,i,q_1} \cdot \alpha_{p_2,i,q_2}
\right) \mathbb{E}
\left(
M_{p_1,i-1,q_1} \cdot M_{p_2,i-1,q_2}
\right) \right| = \sum_{i=1}^{n-1} \mathbb{E}
\left(
\alpha_{p_1,i,q_1} \cdot \alpha_{p_2,i,q_2}
\right) \sum_{\ell=0}^{i-1} \mathbb{E}
\left(
\alpha_{p_1,\ell,q_1} \cdot \alpha_{p_2,\ell,q_2}
\right)

\leq \frac{1}{2} \left( \sum_{i=1}^{n-1} \left| \mathbb{E}
\left(
\alpha_{p_1,i,q_1} \cdot \alpha_{p_2,i,q_2}
\right) \right| \right)^2 - \sum_{i=1}^{n-1} \mathbb{E}^2
\left(
\alpha_{p_1,i,q_1} \cdot \alpha_{p_2,i,q_2}
\right)

\leq \frac{1}{2} \left( \sum_{i=1}^{n-1} \left| \mathbb{E}
\left(
\alpha_{p_1,i,q_1} \cdot \alpha_{p_2,i,q_2}
\right) \right| \right)^2

\leq \frac{1}{2} \mu_{q_1} \cdot \lambda_{p_1} \cdot \mu_{q_2} \cdot \lambda_{p_2} \cdot \delta_{p_1} \cdot \delta_{q_1}.
$$

(40)

Finally, (39), (26) and (40) give

$$
\sum_{j=1}^{n-1} \left| \mathbb{E}[v_{p_1,j,q_1} \cdot v_{p_2,j,q_2}] \right| \leq 32n \cdot K \cdot \|L\|_\infty \cdot \lambda_{p_1}^{1/2} \cdot \lambda_{p_2}^{1/2} \cdot \mu_{q_1}^{1/2} \cdot \mu_{q_2}^{1/2} + 2n^2 \cdot \mu_{q_1} \cdot \lambda_{p_1} \cdot \mu_{q_2} \cdot \lambda_{p_2} \cdot \delta_{p_1} \cdot \delta_{q_1}
$$

and from (38) we obtain:

$$
\left| \mathbb{E}
\left(
Q_{n,p_1,q_1} \cdot Q_{n,p_2,q_2}
\right) \right| \leq 2n^4 \left( 16n^{-1} \cdot K \cdot \|L\|_\infty \cdot \lambda_{p_1}^{3/2} \cdot \mu_{q_1}^{3/2} \cdot \lambda_{p_2}^{3/2} \cdot \mu_{q_2}^{3/2} + n^2 \cdot \mu_{q_1} \cdot \lambda_{p_1} \cdot \mu_{q_2} \cdot \lambda_{p_2} \cdot \delta_{p_1} \cdot \delta_{q_1} \right).
$$

(41)

This inequality allows to obtain bounds for the terms $s_{1,n}$ and $s_{2,n}$ given in (34) and (35). Indeed, from (31), we deduce that

$$
\mathbb{E}
\left(
Q_{n,p,q}^2
\right) \leq 2n^4 \left( 16n^{-1} \cdot K \cdot \|L\|_\infty \cdot \lambda_{p}^3 \cdot \mu_{q}^3 + \lambda_{p}^4 \cdot \mu_{q}^4 \right)
$$

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and, therefore, that
\[ s_{1,n} \leq 2D_{2,n}^{-4} \left\{ 16n^{-1} \| K \|_{\infty} \| L \|_{\infty} \sum_{p=1}^{+\infty} \frac{\lambda_p^3}{(\lambda_p + \gamma_n)^4} \sum_{q=1}^{+\infty} \frac{\mu_q^3}{(\mu_q + \gamma_n)^4} + \sum_{p=1}^{+\infty} \left( \frac{\lambda_p}{\lambda_p + \gamma_n} \right)^4 \sum_{q=1}^{+\infty} \left( \frac{\mu_q}{\mu_q + \gamma_n} \right)^4 \right\}. \]

Since
\[ \frac{\lambda_p^3}{(\lambda_p + \gamma_n)^4} \leq \gamma_n^{-1} \left( \frac{\lambda_p}{\lambda_p + \gamma_n} \right)^2, \quad \left( \frac{\lambda_p}{\lambda_p + \gamma_n} \right)^4 \leq \frac{\lambda_p}{\lambda_p + \gamma_n}, \]
and, similarly,
\[ \frac{\mu_q^3}{(\mu_q + \gamma_n)^4} \leq \gamma_n^{-1} \left( \frac{\mu_q}{\mu_q + \gamma_n} \right)^2, \quad \left( \frac{\mu_q}{\mu_q + \gamma_n} \right)^4 \leq \frac{\mu_q}{\mu_q + \gamma_n}, \]
it follows
\[ s_{1,n} \leq 2D_{2,n}^{-4} \left\{ \sum_{p=1}^{+\infty} \left( \frac{\lambda_p}{\lambda_p + \gamma_n} \right)^2 \sum_{q=1}^{+\infty} \left( \frac{\mu_q}{\mu_q + \gamma_n} \right)^2 \right\} \left\{ 16\gamma_n^{-2}n^{-1} \| K \|_{\infty} \| L \|_{\infty} + 1 \right\} \]
\[ = 2D_{2,n}^{-2} \left\{ 16\gamma_n^{-2}n^{-1} \| K \|_{\infty} \| L \|_{\infty} + 1 \right\} \]
and using Lemma 18 in [12] and Assumption (A5) we deduce that \( \lim_{n \to +\infty} (s_{1,n}) = 0. \) On the other hand, (41) implies that for \((p_2, q_2) \neq (p_1, q_1):\)
\[ |E \left( Q_{n,p_1,q_1} Q_{n,p_2,q_2} \right) | \leq 32n^3 \| K \|_{\infty} \| L \|_{\infty} \mu_{q_1}^{3/2} \mu_{p_1}^{3/2} \lambda_{p_2}^{3/2}. \]
Hence
\[ |s_{2,n}| \leq 32D_{2,n}^{-4} n^{-1} \| K \|_{\infty} \| L \|_{\infty} \left( \sum_{p=1}^{+\infty} \frac{\lambda_p^{3/2}}{(\lambda_p + \gamma_n)^2} \right)^2 \left( \sum_{q=1}^{+\infty} \frac{\mu_q^{3/2}}{(\mu_q + \gamma_n)^2} \right)^2. \]
Since
\[ \frac{\lambda_p^{3/2}}{(\lambda_p + \gamma_n)^2} = \frac{\lambda_p}{\lambda_p + \gamma_n} \frac{\lambda_p^{1/2}}{\lambda_p + \gamma_n} \leq \frac{\lambda_p^{1/2}}{\lambda_p + \gamma_n} \leq \gamma_n^{-1} \lambda_p^{1/2} \quad \text{and} \quad \frac{\mu_q^{3/2}}{(\mu_q + \gamma_n)^2} \leq \gamma_n^{-1} \mu_q^{1/2}, \]
it follows

\[ |s_{2,n}| \leq 32D_{2,n}^{-4}n^{-4}n^{-1}\|K\|_\infty\|L\|_\infty\left(\sum_{p=1}^{+\infty} \lambda_p^{1/2}\right)^2 \left(\sum_{q=1}^{+\infty} \mu_q^{1/2}\right)^2. \]

Lemma 18 in [12] and Assumptions (A_2), (A_3) and (A_5) allow to deduce that
\[ \lim_{n \to +\infty} (s_{2,n}) = 0. \]
Finally, from (33) we conclude that
\[ \lim_{n \to +\infty} (\text{var}(G_n)) = 0, \]
what ensures that
\[ G_n - E(G_n) = o_P(1). \]

Proof of (30): Using (23) et (24) we obtain
\[ |\alpha_{p,i,q}| \leq 4\|K\|_\infty\|L\|_\infty. \]
Thus
\[ \max_{1 \leq i \leq n} |\epsilon_{n,i}| \leq 4D_{2,n}^{-1}n^{-1}\|K\|_\infty^{1/2}\|L\|_\infty^{1/2}\sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1} (\mu_q + \gamma_n)^{-1} \max_{1 \leq i \leq n} |M_{p,i-1,q}|, \]
and from the Minkowski inequality:
\[ E^{1/2}\left[ \max_{1 \leq i \leq n} \epsilon_{n,i}^2 \right] \leq 4D_{2,n}^{-1}n^{-1}\|K\|_\infty^{1/2}\|L\|_\infty^{1/2}\sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} (\lambda_p + \gamma_n)^{-1} (\mu_q + \gamma_n)^{-1} E^{1/2}\left[ \max_{1 \leq i \leq n} |M_{p,i-1,q}|^2 \right]. \]

One easily verifies that \((M_{p,i,q})_{1 \leq i \leq n}\) is a \(F_{n,i}\)-martingale. Doob’s inequality gives
\[ E^{1/2}\left[ \max_{1 \leq j \leq n} |M_{p,i-1,q}|^2 \right] \leq E^{1/2}\left[ M_{i,n-1,k}^2 \right] \leq n^{1/2} \mu_q^{1/2} \lambda_p^{1/2}, \]
so that
\[ E\left[ \max_{1 \leq i \leq n} |\epsilon_{n,i}| \right] \leq E^{1/2}\left[ \max_{1 \leq j \leq n} \epsilon_{n,i}^2 \right] \leq 4D_{2,n}^{-1}n^{-1/2}\gamma_n^{-2}\|K\|_\infty^{1/2}\|L\|_\infty^{1/2}\left(\sum_{p=1}^{+\infty} \lambda_p^{1/2}\right) \left(\sum_{q=1}^{+\infty} \mu_q^{1/2}\right). \]

Assumptions (A_2), (A_3) and (A_5) allow to conclude (30).

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