On the Baer-Lovász-Tutte construction of groups from graphs: isomorphism types and homomorphism notions

Xiaoyu He * Youming Qiao †

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Abstract

Let $p$ be an odd prime. From a simple undirected graph $G$, through the classical procedures of Baer (Trans. Am. Math. Soc., 1938), Tutte (J. Lond. Math. Soc., 1947) and Lovász (B. Braz. Math. Soc., 1989), there is a $p$-group $P_G$ of class 2 and exponent $p$ that is naturally associated with $G$. Our first result is to show that this construction of groups from graphs respects isomorphism types. That is, given two graphs $G$ and $H$, $G$ and $H$ are isomorphic as graphs if and only if $P_G$ and $P_H$ are isomorphic as groups. Our second contribution is a new homomorphism notion for graphs. Based on this notion, a category of graphs can be defined, and the Baer-Lovász-Tutte construction naturally leads to a functor from this category of graphs to the category of groups.

1 Introduction

1.1 The results

In this note we study some basic questions regarding the following construction of finite groups from simple undirected graphs, following the classical works of Baer [Bac38], Tutte [Tut47], and Lovász [Lov79].

Notations. To introduce this construction, we set up some notations first. For $n \in \mathbb{N}$, let $[n] = \{1, \ldots, n\}$. The set of size-2 subsets of $[n]$ is denoted as $\binom{[n]}{2}$. The natural total order of $[n]$ induces the lexicographic order on $[n] \times [n]$. In this introduction, to ease the exposition, we shall only consider graphs with vertex sets being $[n]$. Therefore, a simple undirected graph with the vertex set $[n]$ is a subset of $\binom{[n]}{2}$.

For a field $\mathbb{F}$, $\mathbb{F}^n$ is the linear space consisting of length-$n$ column vectors over $\mathbb{F}$. We use $M(\ell \times n, \mathbb{F})$ to denote the linear space of $\ell \times n$ matrices over $\mathbb{F}$, and set $M(n, \mathbb{F}) := M(n \times n, \mathbb{F})$. A matrix $A \in M(n, \mathbb{F})$ is alternating, if for any $v \in \mathbb{F}^n$, $v^t Av = 0$. For $\{i, j\} \in \binom{[n]}{2}$, $i < j$, an elementary alternating matrix $A_{i,j} \in M(n, \mathbb{F})$ is the $n \times n$ matrix with the $(i, j)$th entry being 1, the $(j, i)$th entry being $-1$, and the rest entries being 0. The linear space of $n \times n$ alternating matrices over $\mathbb{F}$ is denoted by $\Lambda(n, \mathbb{F})$. The general linear group of degree $n$ over $\mathbb{F}$ is denoted by $GL(n, \mathbb{F})$.

The Baer-Lovász-Tutte construction. We now introduce the Baer-Lovász-Tutte construction of groups from graphs with vertex sets being $[n]$. See Section 3 for a description of the construction that works for any finite set without specifying some total order.

*Institute of Computing Technology, Chinese Academy of Sciences, China, and University of Chinese Academy of Sciences, China. hexiaoyu18s@ict.ac.cn
†Centre for Quantum Software and Information, University of Technology Sydney. jimmyqiao86@gmail.com
1. In the first step, we construct an alternating bilinear map from a graph, following ideas traced back to Tutte [Tut47] and Lovász [Lov79]. This step works for any field $\mathbb{F}$. Let $G \subseteq \binom{[n]}{2}$ be a graph of size $m$. Suppose $G = \{\{i_1, j_1\}, \ldots, \{i_m, j_m\}\}$, where for $k \in [m]$, $i_k < j_k$ and for $1 \leq k < k' \leq m$, $(i_k, j_k)$ is less than $(i_{k'}, j_{k'})$ in the lexicographic order. For $k \in [m]$, let $A_k$ be the $n \times n$ elementary alternating matrix $A_{i_k,j_k}$ over $\mathbb{F}$, and set $A_G = (A_1, \ldots, A_m) \in \Lambda(n, \mathbb{F})^m$. Then $A_G$ defines an alternating bilinear map $\phi_G : \mathbb{F}^n \times \mathbb{F}^n \rightarrow \mathbb{F}^m$, by $\phi_G(v, u) = (v^tA_1u, \ldots, v^tA_mu)^t$.

2. In the second step, we fix an odd prime $p$, take $\phi_G : \mathbb{F}_p^n \times \mathbb{F}_p^n \rightarrow \mathbb{F}_p^m$ from the last step over $\mathbb{F}_p$, and apply Baer’s construction [Bae38] to $\phi_G$ to obtain $P_G$ which is a $p$-group of class 2 and exponent $p$. Baer’s construction actually works for any alternating bilinear map $\phi$ over $\mathbb{F}_p$. Specifically, let $\phi$ be an alternating bilinear map $\phi : \mathbb{F}_p^n \times \mathbb{F}_p^n \rightarrow \mathbb{F}_p^m$. A $p$-group of class 2 and exponent $p$, denoted as $P_\phi$, can be constructed from $\phi$ as follows. The set of group elements of $P_\phi$ is $\mathbb{F}_p^n \times \mathbb{F}_p^n$. The group operation $\circ$ of $P_\phi$ is

$$(v_1, u_1) \circ (v_2, u_2) := (v_1 + v_2, u_1 + u_2 + \frac{1}{2} \cdot \phi(v_1, v_2)).$$

We shall review the literature related to this construction in Section 1.2. In this note we study two natural questions regarding isomorphisms and homomorphisms in this construction.

A result about isomorphisms. Given two graphs $G, H \subseteq \binom{[n]}{2}$, the Baer-Lovász-Tutte construction produce two $p$-groups of class 2 and exponent $p$, $P_G$ and $P_H$. Clearly, if $G$ and $H$ are isomorphic, then $P_G$ and $P_H$ are isomorphic. Interestingly, the converse also holds.

**Theorem 1.1.** Let $G, H, P_G$, and $P_H$ be as above. Then $G$ and $H$ are isomorphic as graphs if and only if $P_G$ and $P_H$ are isomorphic as groups.

To prove Theorem 1.1, we review the notion of isomorphisms for alternating bilinear maps. Let $\phi, \psi : \mathbb{F}^n \times \mathbb{F}^n \rightarrow \mathbb{F}^m$ be two alternating bilinear maps. Then $\phi$ and $\psi$ are isomorphic\(^1\), if there exist $C \in \text{GL}(n, \mathbb{F})$ and $D \in \text{GL}(m, \mathbb{F})$, such that for any $u, v \in \mathbb{F}^n$, we have $D(\phi(u, v)) = \psi(C(u), C(v))$. Let $\phi, \psi, P_\phi$ and $P_\psi$ be from Step 2. It is well-known that $\phi$ and $\psi$ are isomorphic if and only if $P_\phi$ and $P_\psi$ are isomorphic (see e.g. [Wil09]).

Therefore, the crux of Theorem 1.1 lies in the construction of alternating bilinear maps from graphs. Given two graphs $G, H \subseteq \binom{[n]}{2}$, construct $\phi_G$ and $\phi_H$ as in Step 1. It is obvious that if $G$ and $H$ are isomorphic, then $\phi_G$ and $\phi_H$ are isomorphic. The converse direction is the elusive one, and showing its correctness is the main technical result in this note.

**Proposition 1.2.** Let $G, H \subseteq \binom{[n]}{2}$ be two graphs, and let $\phi_G, \phi_H : \mathbb{F}^n \times \mathbb{F}^n \rightarrow \mathbb{F}^m$ be two alternating bilinear maps constructed from $G$ and $H$ in Step 1. Then $G$ and $H$ are isomorphic if and only if $\phi_G$ and $\phi_H$ are isomorphic.

A notion of graph homomorphisms. As pointed out by J. B. Wilson in [Wil09, Sec. 3.3], Baer’s construction naturally leads to a functor from the category of alternating bilinear maps over $\mathbb{F}_p$ to the category of $p$-groups of class 2 and exponent $p$. To define a category of alternating bilinear maps, let us review the notion of homomorphisms for alternating bilinear maps. Let $\phi : \mathbb{F}_p^n \times \mathbb{F}_p^n \rightarrow \mathbb{F}_p^m$ and $\psi : \mathbb{F}_p^{n'} \times \mathbb{F}_p^{n'} \rightarrow \mathbb{F}_p^{m'}$ be two alternating bilinear maps. Then a homomorphism from $\phi$ to $\psi$ is $(C, D) \in \text{M}(n \times n, \mathbb{F}) \times \text{M}(m \times m, \mathbb{F})$, such that for any $u, v \in \mathbb{F}_p^n$, we have $D(\phi(u, v)) = \psi(C(u), C(v))$.

\(^1\)This isomorphism notion of alternating bilinear maps is sometimes called pseudo-isometry in the literature [Wil09].
Our next goal is to establish a category of graphs which relates to the category of groups via the category of alternating bilinear maps. This leads us to define the following notion of graph homomorphisms.

**Definition 1.3.** Let $G \subseteq \binom{\mathbb{n}}{2}$ and $H \subseteq \binom{\mathbb{n}'}{2}$ be two graphs. An injective partial function $f : [n] \to [n']$ is a pullback homomorphism from $G$ to $H$, if $\forall\{i,j\} \in \binom{\mathbb{n}}{2}$, $\{f(i), f(j)\} \in H \Rightarrow \{i, j\} \in G$.

Note that $\{f(i), f(j)\} \in H$ implies that $f(i)$ and $f(j)$ are both defined in $f$.

The above definition has the following graph-theoretic interpretation. For a partial function $f : X \to Y$, we call $X$ the domain of $f$ and $Y$ the codomain of $f$. We use $\text{im}(f) := f(X) \subseteq Y$ to denote the image of $f$, and $D(f) = \{x \in X : f(x) \text{ is defined}\}$ to denote the domain of definition of $f$. Then a homomorphism $f$ from $G$ to $H$ just says that an isomorphic copy of the induced subgraph $H[\text{im}(f)]$ is contained in $G[D(f)]$ as a subgraph. From this, it is easy to verify that a composition of two homomorphisms is again a homomorphism; see Observation 3.1.

This pullback homomorphism notion differs from the well-established notion of homomorphisms of graphs (see e.g. [HIN04]), which defines a homomorphism from $G$ to $H$ as a (total) function $g : [n] \to [n']$ satisfying $\{i, j\} \in G \Rightarrow \{f(i), f(j)\} \in H$. We shall call such homomorphisms pushforward homomorphisms.

The alert reader may note that, to naturally arrive at pullback homomorphisms, we should view $\{i', j'\} \in \binom{\mathbb{n}'}{2}$ as some function-like object. Indeed, one can identify $S = \{i', j'\}$ with its indicator function $I_S : [n'] \to \{0, 1\}$, by $I_S(k') = 1$ if and only if $k' = i'$ or $k' = j'$. Then $f : [n] \to [n']$ naturally pulls $I_S$ back to $I_S \circ f : [n] \to \{0, 1\}$. So alternatively, we can also define a partial injective function $f : [n] \to [n']$ to be a pullback homomorphism if it sends $\{I_S : S \in H, S \subseteq \text{im}(f)\}$ to a subset of $\{I_T : T \subseteq G\}$. This provides one explanation for imposing the injectivity condition, which ensures that if $S \subseteq \text{im}(f)$, then $I_S \circ f = I_T$ for some $T \subseteq \binom{\mathbb{n}}{2}$.

We shall examine the notion of pullback homomorphisms more closely in Section 3. For now we just provide some brief remarks. First, with pullback homomorphisms and some appropriate measure to take care of graphs with vertex sets not necessarily $[n]$, we can define a category of graphs, and the Baer-Lovász-Tutte construction naturally leads to a functor from this category of graphs to the category of groups. Second, several classical algorithmic problems for graphs can be formulated naturally using pullback homomorphisms. These will be discussed in more details in Section 3. We leave a more thorough study into this notion to future works.

### 1.2 Related works

**Works related to the Baer-Lovász-Tutte construction.** Let us explain the contexts of some works related to this construction.

Tutte used the following linear algebraic construction in his study of perfect matchings on graphs [Tut47]. That is, given $G \subseteq \binom{\mathbb{n}}{2}$, Tutte constructed an alternating matrix $A = (a_{i,j})$, such that for $1 \leq i < j \leq n$, $a_{i,j} = x_{i,j}$ if $\{i, j\} \in G$, and $a_{i,j} = 0$ otherwise, where $x_{i,j}$’s are independent variables. (Then $a_{i,j}$ for $i = j$ and $i > j$ are set by the alternating condition.) This alternating matrix can be easily interpreted as a linear space of alternating matrices, spanned by elementary alternating matrices $A_{i,j}$, $\{i, j\} \in G$. Lovász explored Tutte’s construction from the perspective of randomised algorithms [Lov79], and studied linear spaces of matrices in other topics in combinatorics [Lov89].

Baer studied central extensions of abelian groups by abelian groups, and presented the construction of $p$-groups of class 2 and exponent $p$ from alternating bilinear maps in [Bae38]. Given a $p$-groups of class 2 and exponent $p$, one can take the commutator bracket to obtain an alternating
bilinear map. Taking commutators and Baer’s construction form the backbone of a pair of fun-
cctors between alternating bilinear maps over \( \mathbb{F}_p \) and \( p \)-groups of class 2 and exponent \( p \), as shown in [Wil09].

**Alternating matrix spaces and the related works.** In Step 1, we defined \( A_G \in \Lambda(n, \mathbb{F})^m \) from a graph \( G \). Following Lovász [Lov79], set \( A_G \) to be the linear span of matrices in \( A_G \), which is a subspace of \( \Lambda(n, \mathbb{F}) \). The alternating matrix space \( A_G \) is closely related to the alternating bilinear map \( \phi_G : \mathbb{F}^n \times \mathbb{F}^n \rightarrow \mathbb{F}^m \), and this perspective is easier to work with when making connections to graphs.

For example, Tutte and Lovász noted that the matching number of \( G \) is equal to one half of the maximum rank over matrices in \( A_G \). Recently in [BCG+20], it is shown that the independence number of \( G \) is equal to the maximum dimension over the totally isotropic spaces of \( A_G \). They also showed that the chromatic number of \( G \) is equal to the minimum \( c \) such that there exists a direct sum decomposition of \( \mathbb{F}^n \) into \( c \) non-trivial totally isotropic spaces for \( A_G \). In [LQ19], the vertex and edge connectivities of a graph \( G \) are shown to be equal to certain parameters related to orthogonal decompositions of \( A_G \).

Given two alternating matrix spaces \( A, B \leq \Lambda(n, \mathbb{F}) \), \( A \) and \( B \) are isomorphic, if there exists \( T \in \text{GL}(n, \mathbb{F}) \), such that \( A = T^tBT := \{ T^tBT : B \in B \} \). Theorem 1.1 then sets up another connection between graphs and alternating matrix spaces in the context of isomorphisms. In [LQ17], alternating matrix space isomorphism problem was studied as a linear algebraic analogue of graph isomorphism problem. The algorithm in [LQ17] was recently improved in [BGL+19].

**An implication: simplifying a reduction from graph isomorphism to group isomor-
phism.** To test whether two graphs are isomorphic is a celebrated algorithmic problem in computer science. Babai’s recent breakthrough showed that the graph isomorphism problem can be solved in quasipolynomial time [Bab16]. Babai proposed the group isomorphism problem as one of the next targets to study for isomorphism testing problems.

The relations between graph isomorphism problem and group isomorphism problem are as follows. On the one hand, when groups are given by Cayley tables, group isomorphism reduces to graph isomorphism [KST93]. On the other hand, graph isomorphism reduces to group isomorphism, when groups are given by generators as permutations [Luk93] or matrices over finite fields [GQ19]. The reduction in [GQ19] actually constructs \( p \)-groups of class 2 and exponent \( p \) from graphs.

The reduction in [GQ19] relies a reduction from graph isomorphism to alternating bilinear map isomorphism. However, there some gadget is needed to restrict from invertible matrices to monomial matrices. Theorem 1.1 then implies that that gadget is actually not needed, considerably simplifying the reduction.

**Structure of this note.** In the following, we shall first prove Proposition 1.2 in Section 2. We then discuss Definition 1.3 in Section 3.

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2This is straightforward to see if the underlying field \( \mathbb{F} \) is large enough. If \( \mathbb{F} \) is small, it follows e.g. as a consequence of the linear matroid parity theorem; cf. the discussion after [Lov89, Theorem 4].

3A subspace \( U \leq \mathbb{F}^n \) is a totally isotropic space of \( A \leq \Lambda(n, \mathbb{F}) \), if for any \( u, u' \in U \), and any \( A \in A \), \( u'Au' = 0 \).

4A direct sum decomposition \( \mathbb{F}^n = U \oplus V \) is orthogonal with respect to \( A \leq \Lambda(n, \mathbb{F}) \), if for any \( u \in U, v \in V \), and \( A \in A \), \( u'Au = 0 \) and \( v'Av = 0 \). This definition is probably more natural from the alternating bilinear map viewpoint, but the two parameters in [LQ19] are arguably easier to motivate from the alternating matrix space perspective.
2 Proof of Proposition 1.2

Further notations. The symmetric group on a set $V$ is denoted by $\text{Sym}(V)$. For $n \in \mathbb{N}$, we let $S_n = \text{Sym}([n])$. For $B \in M(\ell \times n, \mathbb{F})$, we use $B(i,j)$ to denote the $(i,j)$th entry of $B$. We use $e_i$ to denote the $i$th standard basis vector of $\mathbb{F}^n$. For a vector $w \in \mathbb{F}^n$, we use $w[i]$ to denote the $i$th entry of $w$. For a graph $H \subseteq \binom{[n]}{2}$, $\overline{H} = \binom{[n]}{2} \setminus H$ is the complement graph of $H$.

A reformulation of the problem. Let $G, H \subseteq \binom{[n]}{2}$ be two graphs of size $m$. Recall that by Step 1, from $G$ and $H$ we construct $\mathcal{A}_G$ and $\mathcal{A}_H$ in $\Lambda(n, \mathbb{F})^m$. Let $\mathcal{A}_G$ and $\mathcal{A}_H$ be subspaces of $\Lambda(n, \mathbb{F})$ spanned by $\mathcal{A}_G$ and $\mathcal{A}_H$, respectively. Recall that $\mathcal{A}_G$ and $\mathcal{A}_H$ are isomorphic, if there exists $T \in \text{GL}(n, \mathbb{F})$, such that $\mathcal{A}_G = T^t \mathcal{A}_H T$. Clearly, $\mathcal{A}_G$ and $\mathcal{A}_H$ are isomorphic if and only if $\phi_G$ and $\phi_H$ are isomorphic. So we focus on $\mathcal{A}_G$ and $\mathcal{A}_H$ in the following.

We then need to show that if $\mathcal{A}_G$ and $\mathcal{A}_H$ are isomorphic, then $G$ and $H$ are isomorphic. Suppose that $T \in \text{GL}(n, \mathbb{F})$ satisfies $\mathcal{A}_H = T^t \mathcal{A}_G T$. Let us examine what restrictions we have on $T$. Let the $i$th row of $T$ be $w_i^t$, where $w_i \in \mathbb{F}^n$. Suppose $\{i,j\} \in G$. Then $T^t(e_i e_j^t - e_j e_i^t)T = w_i w_j^t - w_j w_i^t$, which belongs to $\mathcal{A}_H$. Note that a matrix $B \in \mathcal{A}_H$ if and only if $\forall \{k,l\} \notin H$, $B(k,l) = 0$. Therefore, for any $\{i,j\} \in G$ and $\{k,l\} \notin H$, $(w_i w_j^t - w_j w_i^t)(k,l) = w_i[k] w_j[l] - w_j[k] w_i[l] = 0$. This leads to the following observation.

Observation 2.1. Let $\mathcal{A}_G$ and $\mathcal{A}_H$ be alternating matrix spaces in $\Lambda(n, \mathbb{F})$ constructed from two graphs $G$ and $H$, respectively. Then $\mathcal{A}_G$ and $\mathcal{A}_H$ are isomorphic, if and only if there exists $T \in \text{GL}(n, \mathbb{F})$, such that for any $\{i,j\} \in G$ and $\{k,l\} \notin H$, the $2 \times 2$ submatrix of $T$ with row indices $\{i,j\}$ and column indices $\{k,l\}$ has the zero determinant.

We then show that the existence of such $T$ would imply that the graphs $G$ and $H$ are isomorphic.

Proposition 2.2. Let $G, H \subseteq \binom{[n]}{2}$ be two graphs. Suppose there exists invertible matrix $T \in \text{GL}(n, \mathbb{F})$, such that for any $\{i,j\} \in G$ and $\{k,l\} \notin H$, $\det \begin{bmatrix} T(i,k) & T(i,l) \\ T(j,k) & T(j,l) \end{bmatrix} = 0$. Then $G$ and $H$ are isomorphic.

By Observation 2.1, Proposition 2.2 implies Proposition 1.2. We then focus on proving Proposition 2.2 in the following.

2.1 Proof of Proposition 2.2

Given $\sigma \in S_n$ and $M \in M(n, \mathbb{F})$, let $f(M, \sigma) := \text{sgn}(\sigma) \prod_{i=1}^{n} M(i, \sigma(i))$. For $S \subseteq [n]$, $\sigma(S) := \{\sigma(x) : x \in S\}$. Given $A \subseteq [n]$ and $B \subseteq [n]$, let $M(A,B)$ be the submatrix of $M$ with row indices in $A$ and column indices in $B$.

To prepare for the proof, we need the following definition. For $M, N \in M(n, \mathbb{F})$, define $M \preceq N$ if $M(i,j) = 0$ or $M(i,j) = N(i,j)$ for all $(i,j)$. In other words, $M \preceq N$ if the non-zero entries of $M$ are equal to the corresponding ones in $N$. We also use $M \prec N$ to denote that $M \neq N$ and $M \preceq N$.

Definition 2.3. Let $G, H \subseteq \binom{[n]}{2}$ be two graphs. Let $T \in \text{GL}(n, \mathbb{F})$ such that for any $\{i,j\} \in G$ and $\{k,l\} \notin H$, $\det \begin{bmatrix} T(i,k) & T(i,l) \\ T(j,k) & T(j,l) \end{bmatrix} = 0$. We define a subset of matrices, $K(G, H, T) \subseteq M(n, \mathbb{F})$, such that $M \in K(G, H, T)$, if there exists a permutation $\sigma \in S_n$, and a disjoint partition of $[n] = \bigcup_{k=1}^{r} S_k$ such that:

1. $r < n$;
2. every $S_k$ is connected in $G$;
3. every $\sigma(S_k)$ is connected in $\overline{H}$;
4. $M \preceq T$;
5. $M_{ij} \neq 0$ if and only if there exists $k \in [r]$, such that $i \in S_k$ and $j \in \sigma(S_k)$;
6. $\forall k \in [r]$, the rank of the submatrix $M(S_k, \sigma(S_k))$ is 1. By (4), this implies that $T(S_k, \sigma(S_k))$ is of rank 1.

Note that for any $M \in K(G, H, T)$, we have $\det(M) = 0$. This is because

$$\det(M) = \pm \prod_{k=1}^{r} \det(M(S_k, \sigma(S_k))),$$

and since $r < n$, there exists $k \in [r]$ such that $|S_k| \geq 2$ with $\text{rk}(M(S_k, \sigma(S_k))) = 1$.

We then state the following two lemmas, whose proofs will be postponed to the end of this subsection.

**Lemma 2.4.** Given a matrix $T \in M(n, \mathbb{F})$, if there exists a set of matrices $\{T_1, T_2, \ldots, T_s\} \subseteq M(n, \mathbb{F})$ such that

1. $T_i \preceq T$ for all $i$,
2. for any $\sigma \in S_n$, if $f(T, \sigma) \neq 0$ then there exists $i \in [s]$ such that $f(T_i, \sigma) = f(T, \sigma)$,
3. for any $\sigma \in S_n$ and $i \neq j$, $f(T_i, \sigma)f(T_j, \sigma) = 0$,

then $\det(T) = \sum_{i=1}^{s} \det(T_i)$.

We defined a set of matrices $K(G, H, T)$ in Definition 2.3. Note that $K(G, H, T)$ is a finite set, so there are maximal elements in it with respect to the $\preceq$ order. We now show that the set of all maximal matrices in $K(G, H, T)$ satisfies the conditions of Lemma 2.4.

**Lemma 2.5.** If $G$ is not isomorphic to $H$, then the set of all maximal matrices in $K(G, H, T)$ satisfies the conditions of Lemma 2.4.

Given these two lemmas, we can prove Proposition 2.2 by way of contradiction. That is, suppose $H$ is not isomorphic to $G$, but there exists $T \in \text{GL}(n, \mathbb{F})$, such that for any $(i, j) \in G$ and $(k, l) \notin H$, $\det \begin{bmatrix} T(i, k) & T(i, l) \\ T(j, k) & T(j, l) \end{bmatrix} = 0$. Let $\{T_1, T_2, \ldots, T_m\}$ be the set of all maximal matrices in $K(G, H, T)$, then $\det(T) = \sum_{i=1}^{m} \det(T_i)$ by Lemma 2.4 and Lemma 2.5. Note that the determinant of any matrix in $K(G, H, T)$ is 0, so $\det(T) = 0$. This contradicts with the fact that $T$ is invertible.

We now prove the two key lemmas.

**Proof of Lemma 2.4.** The statement can be verified as follows.

$$\sum_{i=1}^{s} \det(T_i) = \sum_{i=1}^{s} \sum_{\sigma \in S_n} f(T_i, \sigma)$$

$$= \sum_{\sigma \in S_n} \sum_{i=1}^{s} f(T_i, \sigma)$$

$$= \sum_{\sigma \in S_n: \exists i, f(T_i, \sigma) \neq 0} \sum_{i=1}^{s} f(T_i, \sigma).$$

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Then by condition 3, for any \( \sigma \in S_n \) with \( \exists i \in [s], f(T_i, \sigma) \neq 0 \), there is actually a unique such \( i \). We denote it by \( i_\sigma \). Let \( P = \{ \sigma \in S_n : \exists i \in [s], f(T_i, \sigma) \neq 0 \} \). Let \( Q = \{ \sigma \in S_n : f(T, \sigma) \neq 0 \} \). By condition 2, we have \( Q \subseteq P \). By condition 1, we have \( P \subseteq Q \). So we can continue extending the equations from the above as follows.

\[
\sum_{i=1}^{s} \det(T_i) = \sum_{\sigma \in P} f(T_i, \sigma) = \sum_{\sigma \in Q} f(T, \sigma) = \sum_{\sigma \in S_n} f(T, \sigma) = \det(T).
\]

**Proof of Lemma 2.5.** Let \( \{T_1, T_2, \ldots, T_m\} \) be the set of all maximal matrices in \( K(G, H, T) \). Let us verify the conditions in Lemma 2.4 one by one.

- **Condition 1:** \( T_i \preceq T \) for all \( i \).

  This follows directly from condition 4 in the definition of \( K(G, H, T) \).

- **Condition 2:** For any \( \sigma \in S_n \), if \( f(T, \sigma) \neq 0 \) then there exists \( i \) that \( f(T_i, \sigma) = f(T, \sigma) \).

  Since \( G \) is not isomorphic to \( H \), for any permutation \( \sigma \in S_n \) with \( f(T, \sigma) \neq 0 \), there exists \( \{i, j\} \in \binom{[n]}{2} \) such that \( \{i, j\} \in G \) and \( \{\sigma(i), \sigma(j)\} \notin H \). This implies that

  \[
  \det \begin{bmatrix} T(i, \sigma(i)) & T(i, \sigma(j)) \\ T(j, \sigma(i)) & T(j, \sigma(j)) \end{bmatrix} = 0,
  \]

  that is, \( T(i, \sigma(j))T(j, \sigma(i)) = T(i, \sigma(i))T(j, \sigma(j)) \neq 0 \). Note that \( T(i, \sigma(i))T(j, \sigma(j)) \neq 0 \), as \( f(T, \sigma) \neq 0 \). Construct a matrix \( T' \in M(n, \mathbb{F}) \), such that \( T' \) coincide with \( T \) in these \( n + 2 \) entries

  \[
  \{(1, \sigma(1)), \ldots, (n, \sigma(n)), (i, \sigma(j)), (j, \sigma(i))\},
  \]

  and then set other entries of \( T' \) to be 0. Then \( T' \in K(G, H, T) \), and \( f(T', \sigma) = f(T, \sigma) \). It follows that there exists \( T_i \) such that \( T_i \) is maximal in \( K(G, H, T) \), \( T_i \preceq T' \), and it satisfies that \( f(T_i, \sigma) = f(T, \sigma) \).

- **Condition 3:** For any \( \sigma \in S_n \) and \( i \neq j \), \( f(T_i, \sigma)f(T_j, \sigma) = 0 \).

  For the sake of contradiction, suppose there exist \( \sigma \in S_n \), and \( i \neq j \), such that \( f(T_i, \sigma) \neq 0 \) and \( f(T_j, \sigma) \neq 0 \). Since \( T_i \) and \( T_j \) satisfy Definition 2.3, there are partitions of \( [n] \) associated with them, and we denote the partition of \( [n] \) associated with \( T_i \) (resp. \( T_j \)) by \( U^{(i)} \) (resp. \( U^{(j)} \)). In the following we show how to construct \( P \in K(G, H, T) \) such that \( T_i \preceq P \), arriving at the desired contradiction.

  Let \( U \) be a set of subsets of \( [n] \), obtained by taking the union of those subsets in \( U^{(i)} \) and \( U^{(j)} \). Recall that for any \( S \in U \), \( S \) is connected in \( G \), \( \sigma(S) \) is connected in \( \overline{H} \), all elements in submatrix \( T(S, \sigma(S)) \) are nonzero, and the rank of \( T(S, \sigma(S)) \) is 1.

  We now implement the following procedure, which transforms \( U \) into a partition of \( [n] \).

  - As long as there exit \( S_1, S_2 \in U \) such that \( S_1 \cap S_2 \neq \emptyset \), we perform one of the following:
    - If \( S_1 \subseteq S_2 \), delete \( S_1 \) from \( U \).
by conditions (e) and (f), and since \( v \) which satisfy the following conditions:

**Proof.** The connectivities of \( v \) and the rank of \( T(v, \sigma(S)) \) is 1. When \( S_1 \subseteq S_2 \) or \( S_1 \supseteq S_2 \), this is clear. We shall prove that this also holds in the last case in Claim 2.6.

Therefore, after performing the above procedure, \( U \) becomes a partition of \( \{1, 2, \ldots, n\} \), and for any \( S \in U \), all elements in the submatrix \( T(S, \sigma(S)) \) are nonzero, and the rank of \( T(S, \sigma(S)) \) is 1. It follows that there exists \( P \in K(G, H, T) \) corresponding to the partition \( U \) and \( \sigma \in S_n \), by Definition 2.3. It is clear that \( T_i \prec P \), giving us the desired contradiction. □

**Claim 2.6.** Suppose we have two graphs \( G, H \subseteq \binom{[n]}{2} \), \( T \in \text{GL}(n, \mathbb{F}) \), \( \sigma \in S_n \), and \( S_1, S_2 \subseteq [n] \), which satisfy the following conditions:

(a) \( \forall (i, j) \in G \) and \( \forall (k, l) \in \overline{H} \), det \[
\begin{bmatrix}
T(i, k) & T(i, l) \\
T(j, k) & T(j, l)
\end{bmatrix} = 0,
\]

(b) \( S_1 \cap S_2 \neq \emptyset \),

(c) \( S_1 \) and \( S_2 \) are both connected in \( G \),

(d) \( \sigma(S_1) \) and \( \sigma(S_2) \) are both connected in \( \overline{H} \),

(e) elements in the submatrix \( T(S_1, \sigma(S_1)) \) are all nonzero, and \( \text{rk}(T(S_1, \sigma(S_1))) = 1 \),

(f) elements in the submatrix \( T(S_2, \sigma(S_2)) \) are all nonzero, and \( \text{rk}(T(S_2, \sigma(S_2))) = 1 \).

Then \( S_1 \cup S_2 \) is connected in \( G \), \( \sigma(S_1 \cup S_2) \) is connected in \( \overline{H} \), all elements in the submatrix \( T(S_1 \cup S_2, \sigma(S_1 \cup S_2)) \) are nonzero, and \( \text{rk}(T(S_1 \cup S_2, \sigma(S_1 \cup S_2))) = 1 \).

**Proof.** The connectivities of \( S_1 \cup S_2 \) in \( G \) and \( \sigma(S_1 \cup S_2) \) in \( \overline{H} \) are evident by (c) and (d). We then focus on the last two properties relating \( T(S_1 \cup S_2, \sigma(S_1 \cup S_2)) \) in the statement.

As \( S_1 \cap S_2 \neq \emptyset \) by (b), there exists some \( u \in S_1 \cap S_2 \). Fix one such \( u \in S_1 \cap S_2 \). Our goal is to prove that for any \( v \in S_1 \cup S_2 \), and \( w \in \sigma(S_1 \cup S_2) \), \( T(v, w) = \lambda_v T(u, w) \) holds, for some \( \lambda_v \neq 0 \in \mathbb{F} \) that only depends on \( v \). We shall achieve this by applying double inductions on the distance between \( v \) and \( u \) (the outer induction), and on the distance between \( w \) and \( \sigma(u) \) (the inner induction). By conditions (e) and (f), and since \( u \in S_1 \cap S_2 \), we have \( T(v, \sigma(u)) \neq 0 \) for any \( v \in S_1 \cup S_2 \). So for any \( v \in S_1 \cup S_2 \), we can set

\[
\lambda_v = T(v, \sigma(u))/T(u, \sigma(u)).
\]

Let \( d_G(u, v) \) denote the distance from \( u \) to \( v \) in graph \( G \).

- **The initial condition for the outer induction.** By condition (e) and (f), for any \( w \in \sigma(S_1 \cup S_2) \) we have \( T(u, w) \neq 0 \). So when \( d_G(u, v) = 0 \), i.e. \( u = v \), it trivially holds that for any \( w \in \sigma(S_1 \cup S_2) \), \( T(u, w) = \lambda_v T(u, w) \neq 0 \) where \( \lambda_v = 1 \).

- **The outer inductive step.** Suppose \( T(v, w) = \lambda_v T(u, w) \neq 0 \) holds for any \( v \in S_1 \cup S_2 \) satisfying that \( d_G(u, v) \leq t_1 \), and any \( w \in \sigma(S_1 \cup S_2) \). Consider some vertex \( j \in S_1 \cup S_2 \) with \( d_G(u, j) = t_1 + 1 \). Then there exists some other vertex \( i \in S_1 \cup S_2 \) such that \( \{i, j\} \in E \) and \( d_G(u, i) = t_1 \).

We now prove that \( T(j, w) = \lambda_j T(u, w) \) for any \( w \in \sigma(S_1 \cup S_2) \), by induction on \( d_{\overline{H}}(\sigma(u), w) \).
• The initial condition for the inner induction. As the base case, suppose \( d_H(\sigma(u), w) = 0 \), i.e. \( w = \sigma(u) \). By conditions (e) and (f), \( T(j, \sigma(u)) \neq 0 \), so \( T(j, \sigma(u)) = \lambda_j T(u, \sigma(u)) \neq 0 \) by the definition of \( \lambda_j \) in Equation 2.

• The inner inductive step. Suppose we have \( T(j, k) = \lambda_j T(u, k) \neq 0 \) for any \( k \in \sigma(S_1 \cup S_2) \) satisfying \( d_H^1(\sigma(u), k) \leq t_2 \). Consider \( l \in \sigma(S_1 \cup S_2) \) satisfying \( d_H^1(\sigma(u), l) = t_2 + 1 \). Then there exists \( k \in \sigma(S_1 \cup S_2) \) such that \( \{k, l\} \in \overline{H} \) and \( d_H^1(\sigma(u), k) = t_2 \). We have \( \{i, j\} \in G \), \( \{k, l\} \in \overline{H} \), \( T(i, k) = \lambda_i T(u, k) \neq 0 \) (by the inductive hypothesis on \( d_G(u, i) \)), \( T(i, l) = \lambda_i T(u, l) \neq 0 \) (by the inductive hypothesis on \( d_G(u, i) \)), and \( T(j, k) = \lambda_j T(u, k) \neq 0 \) (by the inductive hypothesis on \( d_H(\sigma(u), k) \)). It follows from condition (a) that \( T(j, l) = \lambda_j T(u, l) \neq 0 \).

• Concluding the inner induction. By the above, we have that \( T(j, w) = \lambda_j T(u, w) \) for any \( w \in \sigma(S_2) \). So for any \( w \in \sigma(S_1 \cup S_2) \), \( T(j, w) = \lambda_j T(u, w) \neq 0 \) when \( d_G(u, j) = t_1 + 1 \).

• Concluding the outer induction. By the above, we have that \( T(v, w) = \lambda_v T(u, w) \neq 0 \) for any \( v \in S_1 \cup S_2 \) and \( w \in \sigma(S_1 \cup S_2) \).

\[ \square \]

3 Discussions on pullback homomorphisms

A functor based on the Baer-Lovász-Tutte construction. Let \( X \) be a finite set. A (simple and undirected) graph \( G = (X, E) \) consists of a vertex set \( X \) and an edge set \( E \subseteq \binom{X}{2} \). Pullback homomorphisms as in Definition 1.3 can be easily adapted to accommodate graphs \( G = (X, E) \) and \( H = (Y, F) \) with arbitrary vertex sets.

Let us also present a variant of the Baer-Lovász-Tutte construction for graphs on arbitrary vertex sets. Note that a little twist is needed so that this construction does not require total orders on the vertex and edge sets.

Let \( X \) be a finite set. Given a field \( \mathbb{F} \), we use \( \mathbb{F}X \) to denote the \( \mathbb{F} \)-vector space spanned by elements in \( X \). For \( v \in \mathbb{F}X \) and \( x \in X \), we use \( v[x] \) to denote the coefficient of \( x \) in \( v \).

Suppose we have a graph \( G = (X, E) \). For any \( E \subseteq \binom{X}{2} \), let \( \tilde{E} = \{(x, x') \in X \times X : \{x, x'\} \in E\} \). For \( (x, x') \in X \times X \), the elementary alternating bilinear form \( A_{(x, x')}: \mathbb{F}X \times \mathbb{F}X \to \mathbb{F} \) is defined as \( A_{(x, x')}(u, v) = u[x]v[x'] - u[x']v[x] \). Then an alternating bilinear map \( \phi_G: \mathbb{F}X \times \mathbb{F}X \to \mathbb{F} \tilde{E} \) can be defined as \( (\phi_G(u, v))(x, x') = A_{(x, x')}(u, v) \). Note that \( \dim(\text{im}(\phi_G)) = |E| \), as \( A_{(x, x')}(u, v) = -A_{(x', x)}(u, v) \) for any \( u, v \in \mathbb{F}X \). We then apply Baer’s construction to \( \phi_G: \mathbb{F}pX \times \mathbb{F}pX \to \mathbb{F} \tilde{E} \) to obtain a p-group of class 2 and exponent \( p \), \( P_G \). In particular, the set of group elements of \( P_G \) is \( \mathbb{F}pX \times \text{im}(\phi_G) \).

We record a basic property of pullback homomorphisms in the following fact.

**Fact 3.1.** Let \( G_i = (X_i, E_i), \ i \in [3], \) be three graphs. Let \( f_j : X_j \to X_{j+1}, \ j \in [2], \) be pullback homomorphisms from \( G_j \) to \( G_{j+1} \). Then \( f_2 \circ f_1 : X_1 \to X_3 \) is a pullback homomorphism from \( G_1 \) to \( G_3 \).

Fact 3.1 suggests the following category of graphs \( \text{Graph}_{\leftarrow} \), where \( \leftarrow \) denotes “pullback.” In \( \text{Graph}_{\leftarrow} \), the objects are graphs of the form \( G = (V, E) \), the morphisms are pullback homomorphisms, and the identity homomorphism on \( G = (V, E) \) is the identity map from \( V \) to \( V \). The identity axiom and the associativity axiom hold trivially.

Let \( \text{Group} \) be the category of groups. The Baer-Lovász-Tutte construction then naturally leads to the following functor from \( \text{Graph}_{\leftarrow} \) to \( \text{Group} \).

**Definition 3.2.** Let BLT be a pair of maps from objects and morphisms in \( \text{Graph}_{\leftarrow} \) to those in \( \text{Group} \) as follows.
• For $G \in \text{Graph}_c$, set $\text{BLT}(G)$ to be the $p$-group of class $2$ and exponent $p$ via the Baer-Lovász-Tutte construction.

• Given two graphs $G = (X, E)$ and $H = (Y, F)$, let $f : X \to Y$ be a pullback homomorphism. As $f$ is injective, $f$ induces an injective partial function $\tilde{f} : \tilde{E} \to \tilde{F}$, in which $(x, x') \in \tilde{E}$ is defined in $\tilde{f}$ if and only if both $x$ and $x'$ are defined in $\tilde{f}$. Furthermore, $f$ induces a linear map $\ell : \mathbb{F}_pX \to \mathbb{F}_pY$ as follows: for $u \in \mathbb{F}_pX$ and $y \in Y$, if $y \notin \text{im}(f)$, then $(\ell(u))[y] = 0$. Otherwise, $(\ell(u))[y] = u[f^{-1}(y)]$. Analogously, $\tilde{f}$ induces a linear map $\tilde{\ell} : \mathbb{F}_p\tilde{E} \to \mathbb{F}_p\tilde{F}$ as follows: for $v \in \mathbb{F}_p\tilde{E}$ and $e \in \tilde{F}$, if $e \notin \text{im}(\tilde{f})$, then $(\tilde{\ell}(v))[e] = 0$. Otherwise, $(\tilde{\ell}(v))[e] = v[\tilde{f}^{-1}(e)]$. Clearly, $\tilde{\ell}$ sends $\text{im}(\phi_G)$ to $\text{im}(\phi_H)$. We then set $\text{BLT}(f) : \text{BLT}(G) \to \text{BLT}(H)$ as sending $(u, v) \in \mathbb{F}_pX \times \text{im}(\phi_G)$ to $(\ell(u), \tilde{\ell}(v)) \in \mathbb{F}_pY \times \text{im}(\phi_H)$.

We first verify that $\text{BLT}(f)$ is a group homomorphism. To see this, by [Wil09, Sec. 3.4], it is enough to verify that $(\ell, \tilde{\ell})$ is an alternating bilinear map homomorphism from $\phi_G$ to $\phi_H$. To see this, note that for any $u, u' \in \mathbb{F}X$ and $(y, y') \in Y \times Y$, we have the following.

• Suppose both $y, y'$ are in $\text{im}(f)$. On the one hand, we have

$$((\tilde{\ell}(\phi_G(u, u')))[(y, y')] = u[f^{-1}(y)] \cdot u'[f^{-1}(y')] - u[f^{-1}(y')] \cdot u[f^{-1}(y)].$$

On the other hand,

$$\phi_H(\ell(u), \ell(u'))[(y, y')] = (\ell(u))[y] \cdot (\ell(u'))[y'] - (\ell(u))[y'] \cdot (\ell(u'))[y]$$

$$= u[f^{-1}(y)] \cdot u'[f^{-1}(y')] - u[f^{-1}(y')] \cdot u'[f^{-1}(y)].$$

• Suppose one of $y, y'$ are not in $\text{im}(f)$. Then on the one hand, $(\tilde{\ell}(\phi_G(u, u')))[(y, y')] = 0$. On the other hand, $(\phi_H(\ell(u), \ell(u')))[(y, y')]$ is also $0$, as either $(\ell(u))[y] = (\ell(u'))[y] = 0$, or $(\ell(u))[y] = (\ell(u'))[y'] = 0$.

We then verify that $\text{BLT}$ forms a functor from the category of graphs with pullback homomorphisms to the category of $p$-groups of class $2$ and exponent $p$. For this, we need to verify that for two pullback homomorphisms $f, g : X \to Y$, $\text{BLT}$ satisfies that $\text{BLT}(f \circ g) = \text{BLT}(f) \circ \text{BLT}(g)$. To see this, we only need to verify that the composition of partial injective functions is preserved in the composition of linear maps, and this is straightforward.

Algorithmic aspects of pullback homomorphisms. Let $G = (X, E)$ and $H = (Y, F)$ be two graphs. Suppose $f : X \to Y$ is a pullback homomorphism. Let $K = H[\text{im}(f)]$. We define the order (resp. size) of $f$ as the order (resp. size) of $K$. We can rephrase several classical algorithmic problem in terms of pullback homomorphisms by restricting to various classes of $G, H,$ and $f$.

• Suppose $H$ is the graph consisting of a perfect matching of large enough order. Then asking to maximize the size over all pullback homomorphisms is the maximum matching problem on $G$.

• Suppose $H$ is the complete graph of large enough order. Then asking to maximize the order over all pullback homomorphisms is the maximum clique problem on $G$.

• Suppose $G$ is the empty graph of large enough order. Then asking to maximize the order over all pullback homomorphisms is the independent set problem on $H$.

• If $f$ is surjective, then the corresponding algorithmic problem is the well-known subgraph isomorphism problem, asking whether $H$ is isomorphic to a subgraph of $G$.

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