The relativistic mechanics in a nonholonomic setting: a unified approach to particles with non-zero mass and massless particles

Olga Rossi¹,²,⁴ and Jana Musilová³

¹ Department of Mathematics, Faculty of Science, The University of Ostrava, 30 Dubna 22, 701 03 Ostrava, Czech Republic
² Department of Mathematics and Statistics, La Trobe University, Melbourne, Victoria 3086, Australia
³ Faculty of Science, Institute of Theoretical Physics and Astrophysics, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic

E-mail: olga.rossi@osu.cz and janam@physics.muni.cz

Received 23 February 2012, in final form 2 May 2012
Published 29 May 2012
Online at stacks.iop.org/JPhysA/45/255202

Abstract
A new approach to relativistic mechanics is proposed, suitable for describing in a unified way the dynamics of different kinds of relativistic particles. Mathematically, it is an application of a geometric theory of nonholonomic systems on fibred manifolds. We propose a setting based on a naturally constructed Lagrangian and a constraint on the 4-velocity of a particle that allows a unified approach to particles with any (positive/negative/zero) square of mass. The corresponding equations of motion are obtained and discussed. In particular, new forces are found (different from the usual Lorentz force type term), arising due to the nonholonomic constraint. A possible meaning and relation with forces previously proposed by Dicke is discussed. In particular, equations of motion of massless particles and of tachyons are studied and the corresponding dynamics are investigated.

PACS numbers: 03.30.+p, 02.40.Vh, 02.40.Yy, 45.20.Jj

(Some figures may appear in colour only in the online journal)

1. Introduction
In this paper we propose a mathematical model leading to a generalized setting of relativistic dynamics, covering in a unified way massive particles with positive and negative squares of mass, and massless particles.

⁴ Formerly Krupková.
As is well known, within the framework of the special relativity theory, a point particle of a constant rest mass \( m_0 > 0 \) moving in an electromagnetic field \((\vec{A}, V)\) on \( \mathbb{R}^4 \) is usually described by the action integral
\[
\int_a^b L \, dt
\]
with the Lagrangian
\[
L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{c} \vec{A} \vec{v} - eV,
\]
providing motion equations
\[
\frac{d}{dt} \left( \frac{m_0 \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = \frac{e}{c} (\vec{v} \times \text{rot} \vec{A}) - \frac{e}{c} \frac{dA}{dt} - e \text{grad} V,
\]
with the Lorentz force, \( \vec{F}_L \), on the right-hand side. More generally, within the calculus of variations, it is known (and rather surprising) that equations
\[
\frac{d}{dt} \left( \frac{m_0 \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = \vec{F}
\]
are variational ‘as they stand’ if and only if the force on the right-hand side is a Lorentz-type force, i.e. \( \vec{F} = \vec{F}_L \), for some functions \( \vec{A}, V \) on \( \mathbb{R}^4 \) ([24], see also [15]). Mathematically, this result means that Lorentz-type force interactions are the only admissible interactions for a massive special-relativistic particle, compatible with a variational principle for curves in \( \mathbb{R}^3 \), parametrized by time \( t \in \mathbb{R} \). From a different point of view, a distinguished role of Lorentz force type interaction has also been supported by recent studies in [27]. On the other hand, there appeared hypotheses that different kinds of interactions should also be possible: a significant prediction of such a non-Lorentz-type force is due to Dicke [7]: however, a formula as well as exact arguments are missing. Hence, a question arises on a proper general mathematical setting, providing equations of motion for particles in the Minkowski spacetime, and clarifying admissible forces.

A problem of this kind can be tackled by tools of the modern calculus of variations. The above variational principle is apparently too restrictive: one should better consider a variational principle for one-dimensional submanifolds in \( \mathbb{R}^4 \). However, there is another interesting possibility, within an even more general framework, proposed and studied in [21]: one can utilize a variational principle for curves in the Minkowski spacetime \((\mathbb{R}^4, g)\) together with the relativistic constraint on the 4-velocity: from the mathematical point of view, this is a nonholonomic constraint. The constraint defines a genuine evolution space, \( \mathcal{Q} \), which is a submanifold in \( \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^4 \) of codimension 1. Then dynamics proceed in \( \mathcal{Q} \) and are governed by the so-called reduced Chetaev equations [25, 16] that become the desired equations of motion for the problem in question. In the above-mentioned paper [21], we applied the procedure directly to ‘conventional’ massive particles, moving in electromagnetic and scalar fields, and obtained a generalized formula for the force \( \vec{F} \) covering also ‘Dicke-type’ interactions.

In this paper we show that ideas of this kind can be applied to study all kinds of point particles admitted by the special relativity theory. We propose a unified mathematical setting suitable not only for ‘conventional’ particles (with a positive square of mass), but also for tachyons (particles with a negative square of mass), and particles with zero mass, moving in electromagnetic and scalar fields. The idea of [21] to tackle this problem as a variational problem with a nonholonomic constraint remains; however, the Lagrangian and the constraint are appropriately modified: the Lagrange function we propose is \emph{universal} for all kinds of particles.

\footnote{Of course, in this general setting, the meaning of functions \( \vec{A}, V \) need not be just a vector and scalar potential of an electromagnetic field. It includes also e.g. forces of inertia.}
particles (not containing the particle’s mass); however, particles are distinguished by the constraint (mass appears in the constraint condition). In the next two sections of this paper, we explain our approach in general; in the rest, we study in detail each kind of particle separately. As the main result, we obtain the corresponding equations of motion and formulae for admissible forces. We discuss the meaning of the constraint condition, and of the new forces (different from the usual Lorentz force type term), arising due to the nonholonomic constraint, and find among them a force complying with Dicke’s predictions. In particular, we pay attention to the equations of motion of massless particles and to the equations of motion of tachyons, and investigate the corresponding dynamics. In this context one should mention at least two striking properties of massless particles: first, the dynamics of these particles are found to be singular, not obeying Newton’s determinism principle, i.e. these particles belong to mechanical systems with internal constraints studied by Dirac. The last section is devoted to a summary of results of this paper, where other interesting properties of massive and massless particles, obtained within our approach, are recalled and further discussed.

2. Mathematical background

From the point of view of mathematics, the main idea is to join the variational approach, represented by the calculus of variations on fibred manifolds (see [12, 13] or [15]) with nonholonomic geometric mechanics in jet bundles as developed by one of the present authors in [16, 18, 19] (see [20] for a review); for alternative geometric approaches, we refer e.g. to [2, 4, 6, 9, 11, 23, 26, 28] and others. We emphasize that our present model is based on the former approach to nonholonomic mechanics, where in contrast to other more common settings, a nonholonomic system is considered to ‘live’ directly on the constraint manifold that serves as a genuine evolution space for the system. Other remarkable differences that make our setting of nonholonomic mechanics appropriate to apply in the study of relativistic particle dynamics are that one can deal easily and naturally with constraints nonlinear in velocities, and, last but not least, one has at hand a variational principle for nonholonomically constrained systems [19].

Throughout this paper we shall use the Einstein summation convention. Summation over Greek (respectively, Latin) indices proceeds from 1 to 4 (respectively, 1 to 3).

First, let us recall a necessary mathematical background, adapted to our situation. Consider the manifold $\mathbb{R}^4$ with canonical coordinates $(q^\sigma)$, $1 \leq \sigma \leq 4$, endowed with the Minkowski metric field

$$g = dq^4 \otimes dq^4 - \delta_{ij} dq^i \otimes dq^j.$$  \hfill (2.1)

Curves in $\mathbb{R}^4$ will be represented by their graphs, i.e. sections of the fibred manifold $\pi : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}$. If $c : s \to c(s)$ is a curve in $\mathbb{R}^4$ parametrized by a real parameter $s \in \mathbb{R}$, then the corresponding section is the mapping $\gamma : \mathbb{R} \ni s \to \gamma(s) = (s, c(s)) = (s, q^\sigma(s)) \in \mathbb{R} \times \mathbb{R}^4$. We shall need the manifold

$$J^1 \pi = \mathbb{R} \times T^1 \mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^4$$ \hfill (2.2)

(the first jet prolongation of $\pi$), called evolution space. We shall consider it with fibred coordinates $(s, q^\sigma, \dot{q}^\sigma)$; note that the ‘dot’ coordinates are defined by

$$\dot{q}^\sigma \circ J^1 \gamma = \frac{d}{ds}(q^\sigma \circ \gamma),$$ \hfill (2.3)

for every section $\gamma$ of $\pi$.

On $J^1 \pi$ we have the so-called contact structure, locally generated by basic contact 1-forms

$$\omega^\sigma = dq^\sigma - \dot{q}^\sigma ds, \quad 1 \leq \sigma \leq 4.$$ \hfill (2.4)
When considering sections of the evolution space \(J^1 \pi\), we have to distinguish general sections from the so-called \(holonomic\) sections, that is, sections of the form \(J^1 \gamma\) where \(\gamma\) is a section of \(\pi\). In coordinates, if \(\gamma(s) = (s, c(s)) = (s, q^i(s))\), then \(J^1 \gamma(s) = (s, c(s), dc/ds) = (s, q^i(s), \dot{q}^i(s))\).

In what follows, we shall often consider vector fields and 1-forms, defined on the Minkowski spacetime \(\mathbb{R}^4\) (i.e. on the typical fibre of the fibration \(\pi\)), and call them \(contravariant\) and \(covariant 4\)-vector fields, respectively. In fibred coordinates, a contravariant, respectively, covariant vector field reads
\[
\dot{u} = \dot{u}^i \frac{\partial}{\partial q^i}, \quad \text{respectively,} \quad \phi = \phi_i dq^i,
\]
where the components \(\dot{u}^i\) and \(\phi_i\), \(1 \leq i \leq 4\), are functions on \(\mathbb{R}^4\), i.e. depend upon the coordinates \((q^i)\). Due to the structure of the evolution space \(J^1 \pi\) (being a product of \(\mathbb{R}\) and the tangent space \(T\mathbb{R}^4 = \mathbb{R}^4 \times \mathbb{R}^4\)), contravariant 4-vector fields are sections of the bundle \(T\mathbb{R}^4 \to \mathbb{R}^4\), i.e. it holds \(q^i \circ \dot{u} = q^i\) and
\[
\dot{q}^i \circ \dot{u} = \ddot{u}^i (q^i, \dot{q}^i, q^j, \dot{q}^j), \quad 1 \leq i \leq 4.
\]

Covariant 4-vector fields are sections of the cotangent bundle \(T^*\mathbb{R}^4 \to \mathbb{R}^4\).

A \textit{Lagrangian} is defined to be a differential 1-form on the evolution space \(J^1 \pi\), horizontal with respect to the projection onto the base \(\mathbb{R}\). In fibred coordinates, a Lagrangian is expressed by \(\Lambda = L(s, q^i, \dot{q}^i) \, ds\), and \textit{action} (over an interval \([a, b] \subset \mathbb{R}\)) is a function
\[
S : \Gamma(\pi) \ni \gamma \to \int_a^b (L \circ J^1 \gamma) \, ds \in \mathbb{R},
\]
where \(\Gamma(\pi)\) denotes the set of sections of \(\pi\) with domains of definition containing the interval \([a, b]\).

It is important to note the geometric meaning of the Euler–Lagrange equations of a Lagrangian defined on \(J^1 \pi\). Extremals of \(\Lambda\) are \textit{holonomic integral sections of a distribution} on the evolution space \(J^1 \pi\) (by a distribution, we mean a sub-bundle of the tangent bundle to the manifold \(J^1 \pi\)). This distribution, called \textit{dynamical distribution}, arises as the \textit{characteristic distribution} of the closed 2-form
\[
\alpha = A_\sigma \omega^\sigma \wedge ds + B_{\sigma \nu} \omega^\sigma \wedge dq^\nu + F,
\]
where
\[
A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\partial^2 L}{\partial q^\sigma \partial \dot{q}^\nu} \dot{q}^\nu, \quad B_{\sigma \nu} = - \frac{\partial^2 L}{\partial q^\sigma \partial q^\nu},
\]
and \(F\) is a unique 2-contact 2-form such that \(d\alpha = 0\). In this notation, Euler–Lagrange equations of \(\Lambda\) read
\[
A_\sigma + B_{\sigma \nu} \dot{q}^\nu = 0, \quad 1 \leq \sigma \leq 4,
\]
or, in intrinsic form, \(J^1 \gamma \circ i_\xi \alpha = 0\) for every vector field \(\xi\) on \(J^1 \pi\). The rank of the dynamical distribution may be greater than 1. It is equal to 1 if and only if the matrix \((B_{\sigma \nu})\) is regular, i.e. if the Lagrangian is regular; in this case, trajectories in the evolution space are integral curves of a \textit{single vector field}. For more details on the structure of solutions of Euler–Lagrange equations and corresponding integration methods for both regular and non-regular Lagrangians, we refer to [15] and [22].

In this paper we shall consider Lagrangians where the Lagrange function \(L\) does not depend explicitly upon the parameter \(s\), as relevant from the physical point of view. In this case we have
\[
(L \circ J^1 \gamma)(s) = L \left( c(s), \frac{dc}{ds} \right) = L(q^i(s), \dot{q}^i(s)) = (L \circ \dot{u} \circ c)(s) = (L \circ \dot{u})(c(s));
\]

1. J. Phys. A: Math. Theor. 45 (2012) 255202
2. O Rossi and J Musilová
hence, we can consider a Lagrange function defined on $\mathbb{R} \times T\mathbb{R}^4$, of the form $L = L(\dot{q})$. Moreover, as follows from the Noether theorem, the independence of $L$ upon $s$ means that the function

$$H = -L + \frac{\partial L}{\partial \dot{q}^\sigma} \dot{q}^\sigma$$

(the Hamiltonian) is constant along extremals of $L$.

A nonholonomic constraint is a submanifold $Q$ of $J^1\pi$, fibred over $\mathbb{R} \times \mathbb{R}^4$. Whenever convenient, we denote by $\iota$ the canonical embedding of $Q$ into $J^1\pi$. We shall consider a nonholonomic constraint of codimension 1. Such a constraint is given by one first-order differential equation

$$f(s, q^\sigma, \dot{q}^\sigma) = 0,$$

such that

$$\text{rank}(\partial f / \partial \dot{q}^\sigma) = 1.$$  \hspace{1cm} (2.14)

Due to the rank condition, we may assume that the constraint is defined by equation ‘in normal form’

$$\dot{q}^4 = h(s, q^\sigma, \dot{q}^\sigma),$$

where $1 \leq \sigma \leq 4$, and $1 \leq l \leq 3$ (i.e. that $f \equiv \dot{q}^4 - h$). The given fibred coordinates induce on the submanifold $Q$ adapted coordinates $(s, q^1, q^2, q^3, q^4, \dot{q}^1, \dot{q}^2, \dot{q}^3, \dot{q}^4)$.

A constraint in the evolution space $J^1\pi$ gives rise to a constraint structure on $J^1\pi$, generated by the 1-form

$$\varphi = f ds + (\partial f / \partial \dot{q}^\sigma) \omega^\sigma.$$  \hspace{1cm} (2.16)

In [16, 23] it was found that every constraint is naturally endowed with a distribution (sub-bundle of the tangent bundle to $Q$) called canonical distribution. In our case it is annihilated by the 1-form

$$\tilde{\varphi} = \iota^* \varphi = \frac{\partial h}{\partial \dot{q}^l} \omega^l + (\dot{q}^4 - h) ds$$

(2.17)

on the manifold $Q$. As shown in [16] for the first-order and in [17] for higher order constraints, and discussed in detail in [19], the canonical distribution represents admissible directions in the evolution space (and in this way gives an explicit solution to the question of extension of the principle of virtual displacements, or D’Alembert’s principle to general velocity dependent, and to higher order constraints). Namely in the components of vector fields belonging to the canonical distribution, relations between velocity displacements (respectively, higher order displacements) and configuration displacements appear in an explicit way. For the interested reader, we refer also to [1, 8, 29] for a discussion on extension of D’Alembert’s principle to constraints nonlinear in velocities and to acceleration-dependent constraints, based on different arguments and techniques as in [16, 17]. Namely in [1] and [8], an affirmative answer to the extension problem for D’Alembert’s principle is presented via independent methods, thus giving justification to the heuristic Chetaev formula for the constraint force.

There are two models for describing a constrained system, both of which will be useful for our further considerations. The first one is more traditional, describing the constrained system as a deformation of the original unconstrained system due to a constraint force, naturally generated by the constraint. The deformed system is thus a new mechanical system defined on $J^1\pi$. It is represented by the 2-form $\alpha_C = \alpha - \varphi$, where $\alpha$ represents the original mechanical system and

$$\Phi = \lambda \varphi \wedge ds = \lambda \frac{\partial f}{\partial \dot{q}^\sigma} \omega^\sigma \wedge ds$$

(2.18)
is the constraint force, called \textit{Chetaev force}. Constrained trajectories are then integral curves of the characteristic distribution of the 2-form $\alpha_C$, passing in the manifold $Q$. Equations for these curves, called \textit{Chetaev equations} \cite{5}, depend upon one Lagrange multiplier $\lambda$ (to be determined), and read

$$\frac{\partial L}{\partial q^\sigma} - \frac{d}{dx} \frac{\partial L}{\partial \dot{q}^\sigma} = \lambda \frac{\partial f}{\partial q^\sigma}, \quad 1 \leq \sigma \leq 4. \quad (2.19)$$

The second model is more geometrical and explores the canonical distribution. A nonholonomic mechanical system is now represented as an object defined on the constraint $Q$. Hence, the manifold $Q$ has the meaning of the evolution space for the constrained system. Concretely, the constrained system is represented by the class of 2-forms $\tilde{\alpha} = \iota^* \alpha + \tilde{\varphi}$ on $Q$, where $\tilde{\varphi}$ represents 2-forms in the ideal generated by the constraint form $\tilde{\varphi}$ (annihilating the canonical distribution). Now, admissible trajectories are integral sections of the canonical distribution, and equations of motion, called \textit{Chetaev reduced equations}, are equations for admissible trajectories that are integral curves of the characteristic distribution of $\tilde{\alpha}$. Keeping the notation used so far, they take the form

$$\tilde{A}_{il} + \tilde{B}_{ils} \ddot{q}^s = 0, \quad 1 \leq l \leq 3, \quad (2.20)$$

where

$$\tilde{A}_{il} = \left( A_{il} + A_{4l} \frac{\partial h}{\partial q^l} + \left( B_{4l} + B_{il} \frac{\partial h}{\partial q^l} \right) \left( \frac{\partial h}{\partial s} + \frac{\partial h}{\partial q^\sigma} \dot{q}^\sigma \right) \right) \circ t, \quad (2.21)$$

$$\tilde{B}_{ils} = \left( B_{ils} + 2B_{il} \frac{\partial h}{\partial q^s} + B_{4l} \frac{\partial h}{\partial q^s} \frac{\partial h}{\partial q^\sigma} \right) \circ t. \quad (2.22)$$

It is important to stress that if the constraint is non-integrable (and this will be our case), the nonholonomic equations of motion cannot be derived from the ‘constrained Lagrangian’ $\tilde{L} = L \circ t = L(s, q^\sigma, \dot{q}^\sigma, h)$.}

### 3. A geometric setting for SRT particle dynamics

In this paper we propose a model for particle dynamics on the Minkowski space, suitable for a unified treatment of all kinds of point particles (‘classical’—with a positive square of mass, tachyons—with a negative square of mass and particles with zero rest mass).

This model treats a particle as a Lagrangian system on a fibred manifold $\pi : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}$ (where $\mathbb{R}^4$ is endowed with the Minkowski metric), with a nonholonomic constraint in the evolution space $J^1 \pi = \mathbb{R} \times T \mathbb{R}^4$, and is based on the following two axioms:

1. \textit{The Lagrange function is universal for all particles, and is polynomial in the ‘4-velocity’}

$$L(\dot{u}) = -\frac{1}{2} g(\dot{u}, \dot{u}) + \phi(\dot{u}) - \psi = -\frac{1}{2} \delta_{\sigma \nu} \dot{u}^\sigma \dot{u}^\nu + \phi_\sigma \dot{u}^\sigma - \psi$$

$$= -\frac{1}{2} \left( \dot{q}^4 \right)^2 - \sum_{p=1}^{3} \left( \dot{q}^p \right)^2 + \dot{q}^4 \phi_\sigma - \psi, \quad (3.1)$$

where $\phi_\sigma$, $1 \leq \sigma \leq 4$, and $\psi$ are functions on the spacetime $\mathbb{R}^4$.

Here ‘universality’ means that the Lagrangian does not contain the particle mass. Note that coefficients in the Lagrangian have the physical meaning of admissible fields: behind the (2, 0)-tensor field $g$, the Lagrangian contains a covariant 4-vector field $\phi$ and a scalar field $\psi$. 

---

6
(2) A particle is determined by a nonholonomic constraint condition on the ‘4-velocity’:
\[ g(\dot{u}, \dot{u}) = Mc^2, \]  
(3.2)
where \( M = M(q^1, q^2, q^3, \dot{q}^4) \) is a function on the spacetime \( \mathbb{R}^4 \). Later, it will turn out that \( M \) has the meaning of the square of mass. The value of the function \( M \) may vary from point to point; at a point in \( \mathbb{R}^4 \) it may be positive, negative or zero.

Note that if we assume \( M \) be continuous, then the following property holds: if \( M(x) \neq 0 \) at a point \( x \in \mathbb{R}^4 \), then there is an open neighbourhood \( U \) around \( x \) such that \( M \neq 0 \) on \( U \). This means that if \( M \) is positive (negative) at a point, it is positive (negative) in a certain neighbourhood of that point.

Since \( g(\dot{u}, \dot{u}) = ((\dot{q}^4)^2 - \sum_{p=1}^{3}(\dot{q}^p)^2)c^2 \) \( \circ \dot{u} \), condition (3.2) defines a nonholonomic constraint
\[ Q : (\dot{q}^4)^2 - \sum_{p=1}^{3}(\dot{q}^p)^2 = Mc^2 \]  
(3.3)
in \( J^1 \pi = \mathbb{R} \times T\mathbb{R}^4 \) if the rank condition (2.14) is satisfied, i.e. if
\[ \text{rank}(-\dot{q}^1, -\dot{q}^2, -\dot{q}^3, \dot{q}^4) = 1. \]  
(3.4)
Hence we have to exclude from \( \mathbb{R} \times T\mathbb{R}^4 \) the points where \( \dot{u} = 0 \) (the zero section in \( T\mathbb{R}^4 \)). However, we need more, namely the constraint \( Q \) be expressible in the normal form
\[ \dot{q}^4 = h(s, q^1, q^2, q^3). \]  
(3.5)
This is the case if
\[ \dot{q}^4 \neq 0. \]  
(3.6)
Excluding the hyperplane \( \mathcal{H}_0 \) of points in \( J^1 \pi \) where \( \dot{q}^4 = 0 \), we obtain a disconnected open submanifold of the evolution space \( J^1 \pi \), consisting from two connected components, \( J^1 \pi_+ \) and \( J^1 \pi_- \), of points where \( \dot{q}^4 > 0 \) and \( \dot{q}^4 < 0 \), respectively. On the components, we have the induced global fibred coordinates \((s, q^1, q^2, q^3)\), and the constraint condition (3.2) determines a constraint submanifold \( Q \subset J^1 \pi \setminus \mathcal{H}_0 = Q_+ \cup Q_- \), defined by the following equations in the normal form:
\[ Q_+ \subset J^1 \pi_+ : \dot{q}^4 = \sqrt{Mc^2 + \sum_{p=1}^{3}(\dot{q}^p)^2}, \quad Q_- \subset J^1 \pi_- : \dot{q}^4 = -\sqrt{Mc^2 + \sum_{p=1}^{3}(\dot{q}^p)^2}. \]  
(3.7)
In what follows, we shall choose for the evolution space of the constrained system the manifold \( Q_+ \).

Remark 3.1. Recall that since the ‘universal’ Lagrangian (3.1) does not depend upon the parameter \( s \), the corresponding ‘universal’ Hamiltonian (2.12) is a constant of the (unconstrained) motion. Computing \( H \) we obtain
\[ H = -L + \frac{\partial L}{\partial \dot{q}^4} \dot{q}^4 = -\frac{1}{2} \left( (\dot{q}^4)^2 - \sum_{p=1}^{3}(\dot{q}^p)^2 \right) + \psi. \]  
(3.8)
Restricting \( H \) to the constraint submanifold, we obtain a function which is no longer ‘universal’ but carries information about the mass of the particle (and, naturally, cannot be expected to be a constant of motion):
\[ H|_Q = \psi - \frac{1}{2}Mc^2. \]  
(3.9)
The ‘universal’ Lagrangian (3.1) gives rise to four Euler–Lagrange equations
\[
\frac{\partial L}{\partial q^a} - \frac{d}{ds} \frac{\partial L}{\partial \dot{q}^a} = 0
\]
(3.10)
for graphs of curves \(c(s) = (q^a(s))\) in the spacetime \(\mathbb{R}^4\), having the following form:
\[
\begin{align*}
\bar{\dot{q}}^1 + \dot{q}^1 \left( \frac{\partial \phi_e}{\partial q^1} - \frac{\partial \phi_f}{\partial q^e} \right) - \frac{\partial \psi}{\partial q^1} = 0, \\
\bar{\dot{q}}^4 + \dot{q}^4 \left( \frac{\partial \phi_e}{\partial q^4} - \frac{\partial \phi_f}{\partial q^e} \right) - \frac{\partial \psi}{\partial q^4} = 0.
\end{align*}
\]
(3.11)

Equations we are looking for concern motions proceeding in the evolution space \(Q_+\).

Substituting
\[
A_1 = \dot{q}^p \left( \frac{\partial \phi_e}{\partial q^1} - \frac{\partial \phi_f}{\partial q^e} \right) - \frac{\partial \psi}{\partial q^1}, \quad B_{ij} = -\delta_{ij}, \quad 1 \leq i, j \leq 3,
\]
\[
A_4 = \dot{q}^p \left( \frac{\partial \phi_e}{\partial q^4} - \frac{\partial \phi_f}{\partial q^e} \right) - \frac{\partial \psi}{\partial q^4}, \quad B_{34} = B_{44} = 0, \quad B_{34} = 1,
\]
and
\[
h = \sqrt{\mathcal{M}c^2 + \sum_{p=1}^3 (\dot{q}^p)^2},
\]
(3.13)
into (2.21) and (2.22), we obtain
\[
\begin{align*}
\tilde{A}_i &= \dot{q}^i \left( \frac{\partial \phi_e}{\partial q^i} - \frac{\partial \phi_f}{\partial q^e} \right) + \left( \frac{\partial \phi_e}{\partial q^1} - \frac{\partial \phi_f}{\partial q^e} \right) \frac{\dot{q}^j}{\sqrt{\mathcal{M}c^2 + \sum_{p=1}^3 (\dot{q}^p)^2}} \\
+ &\sqrt{\mathcal{M}c^2 + \sum_{p=1}^3 (\dot{q}^p)^2} \left( \frac{\partial \phi_e}{\partial q^1} - \frac{\partial \phi_f}{\partial q^e} \right) \frac{\dot{q}^i}{\sqrt{\mathcal{M}c^2 + \sum_{p=1}^3 (\dot{q}^p)^2}}.
\end{align*}
\]
\[
\tilde{B}_{ij} = -\delta_{ij} + \frac{\dot{q}^i \dot{q}^j}{\mathcal{M}c^2 + \sum_{p=1}^3 (\dot{q}^p)^2}.
\]
(3.14)
The desired equations are three equations for sections of the fibred manifold \(Q_+\) over \(\mathbb{R}\), i.e. for curves \(c(s) = (q^a(s))\) passing in the spacetime \(\mathbb{R}^4\) and satisfying the constraint condition (3.7), as follows:
\[
\tilde{B}_{ij} \dot{q}^j = -\tilde{A}_i, \quad 1 \leq i \leq 3.
\]
(3.15)
Now, we have to distinguish two cases:

3.1. \(\mathcal{M} = 0\) at a point \(x \in Q_+\)

If \(\mathcal{M}(x) = 0\), then the matrix \((\tilde{B}_{ij})\) is singular at \(x\); hence, motion equations (3.15) cannot be put into the normal form; explicitly they read
\[
\begin{align*}
\left( \delta_{ij} - \frac{\dot{q}^i \dot{q}^j}{\sum_{p=1}^3 (\dot{q}^p)^2} \right) \ddot{q}^i &= \dot{q}^i \left( \frac{\partial \phi_e}{\partial q^i} - \frac{\partial \phi_f}{\partial q^e} \right) + \frac{\dot{q}^j \dot{q}^i}{\sqrt{\sum_{p=1}^3 (\dot{q}^p)^2}} \left( \frac{\partial \phi_e}{\partial q^1} - \frac{\partial \phi_f}{\partial q^e} \right) \\
+ &\sqrt{\sum_{p=1}^3 (\dot{q}^p)^2} \left( \frac{\partial \phi_e}{\partial q^1} - \frac{\partial \phi_f}{\partial q^e} \right) \frac{\dot{q}^i}{\sqrt{\sum_{p=1}^3 (\dot{q}^p)^2}}.
\end{align*}
\]
(3.16)
where \(1 \leq l \leq 3\). Extremals of the constrained problem are curves \(\gamma(s) = (s, q^l(s))\) satisfying the above motion equations and the equation of the constraint \(Q\).

\[
\dot{q}^4 = \sqrt{\sum_{p=1}^{3} \dot{q}^p q^p}.
\] (3.17)

The dynamics proceed in the evolution space \(Q\). Recall that on \(Q\), we have \(
\dot{q}^4 \neq 0
\).

We shall deal with these equations in more detail later when we study massless particles.

### 3.2. \(M \neq 0\) at a point \(x \in Q\)

If \(M(x) \neq 0\), then the matrix \((\tilde{B}_{ij})\) is regular at \(x\), and, due to continuity, is regular in a neighbourhood \(U\) around \(x\). Hence, on \(U\), equations (3.15) can be put into a normal form

\[
\dot{q}^j = F^j.
\] (3.18)

The inverse matrix \(\tilde{B}^{-1} = (\tilde{B}^i)\) takes the form

\[
\tilde{B}^{-1} = -\frac{1}{M\dot{c}} \begin{pmatrix}
\dot{q}^1 \dot{q}^3 - (\dot{q}^1)^2 & \dot{q}^1 \dot{q}^3 + \dot{q}^1 \dot{q}^3 & \dot{q}^1 \dot{q}^3 \\
\dot{q}^3 \dot{q}^3 & \dot{q}^3 \dot{q}^3 + (\dot{q}^3)^2 & \dot{q}^3 \dot{q}^3 \\
\dot{q}^3 \dot{q}^3 & \dot{q}^3 \dot{q}^3 & \dot{q}^3 \dot{q}^3 + (\dot{q}^3)^2
\end{pmatrix};
\] (3.19)

hence, we obtain

\[
F^j = -A^j = -\tilde{B}^{ij} \tilde{\Lambda}^i = \frac{1}{M\dot{c}} (M\dot{c} \delta^j^i + q^j \dot{q}^i) \tilde{\Lambda}^i
\]

\[
= \dot{q}^j \left( \frac{\partial \phi_1}{\partial q^j} - \frac{\partial \phi_1}{\partial q^j} \right) + \sqrt{M\dot{c}^2 + \sum_{p=1}^{3} (\dot{q}^p q^p)} \left( \frac{\partial \phi_4}{\partial q^j} - \frac{\partial \phi_4}{\partial q^j} \right)
\]

\[
- \frac{\partial \psi}{\partial q^j} \frac{\dot{q}^i}{M\dot{c}} \left( \dot{q}^j \frac{\partial \psi}{\partial q^j} + \sqrt{M\dot{c}^2 + \sum_{p=1}^{3} (\dot{q}^p q^p)} \frac{\partial \psi}{\partial q^j} \right).
\] (3.20)

**Proposition 3.2** (Equations of motion: \(M \neq 0\), four-dimensional observer). Let \(g\) be the Minkowski metric, \(\phi\) a covariant vector field and \(\psi\) a function on \(\mathbb{R}^4\). Extremals of a Lagrangian system defined by the Lagrangian

\[
L = -\frac{1}{2} g(\dot{u}, \dot{u}) + \phi(\dot{u}) - \psi = -\frac{1}{2} \left( (\dot{q}^4)^2 - \sum_{p=1}^{3} (\dot{q}^p q^p) \right) + q^4 \phi_a - \psi
\] (3.21)
on \(\mathbb{R} \times T\mathbb{R}^4\), and subject to the constraint

\[
g(\dot{u}, \dot{u}) = M\dot{c}^2, \quad \dot{u}^2 > 0, \quad M \neq 0 \text{ at each point}
\] (3.22)

are curves \(\gamma(s) = (s, q^a(s))\), satisfying the following system of mixed second- and first-order differential equations:

\[
\dot{q}^j = F^j \quad 1 \leq j \leq 3,
\] (3.23)

with \(F^j\) given by (3.20) (equations of motion) and

\[
\dot{q}^4 = \sqrt{M\dot{c}^2 + \sum_{p=1}^{3} (\dot{q}^p q^p)}
\] (equation of the constraint).

The dynamics proceed in the evolution space \(Q \subset \mathbb{R} \times T\mathbb{R}^4\) defined by the above constraint equation.
Choosing appropriate coordinates, we can express the above equations in a form adapted to a three-dimensional observer.

Consider on $J^1\pi \backslash H_0 \subset \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^4$ new coordinates $(s, q^j, t, v^i, \dot{q}^i)$, defined by the transformation rule

$$
\dot{q}^i = \frac{1}{c} v^i \dot{q}^i, \quad 1 \leq i \leq 3, \quad \dot{q}^4 = ct \quad \text{(time coordinate).}
$$

(3.25)

Note that $(s, q^j, t, v^i, \dot{q}^i)$ are global coordinates, however, are no longer fibred coordinates for the original fibration $\pi$. The meaning of the new coordinates is the following: $(t, q^j, v^i)$ are coordinates on $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$, adapted to the fibration $\mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}$ of the manifold $\mathbb{R}^4$, the fibre of our fibred manifold $\pi : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}$; note that $(q^j, q^2, q^3)$ are Cartesian coordinates on $\mathbb{R}^3$. In vector notations $\vec{r} = (q^1, q^2, q^3)$, and $\vec{v} = (v^1, v^2, v^3)$ is the usual velocity.

In these coordinates the constraint $\mathcal{Q}$ is given by equation

$$
(1 - \frac{v^2}{c^2}) (\dot{q}^4)^2 = \mathcal{M} c^2.
$$

(3.26)

Since we assume $\mathcal{M} \neq 0$ at each point, we obtain the evolution space $\mathcal{Q}_+$ expressed by equation

$$
\dot{q}^4 = \frac{\sqrt{\mathcal{M} c^2}}{1 - \frac{v^2}{c^2}}.
$$

(3.27)

where $v = \sqrt{\sum_{p=1}^{3} (v^p)^2} = \sqrt{\sum_{p=1}^{3} (dq^p/dt)^2}$ is the usual three-dimensional speed. We can also see that along every constrained path

$$
\frac{d\rho}{ds} = \frac{1}{c} \dot{q}^4 = \sqrt{\frac{\mathcal{M}}{1 - \frac{v^2}{c^2}}}.
$$

(3.28)

(We shall see later that equation (3.27) for $d\rho/ds$ has the meaning of the mass equation, or, if multiplied by $c^2$, of the energy equation.)

Now, equations of motion (3.23) can be transformed by eliminating the parameter $s$ as follows: first we have for $j = 1, 2, 3$

$$
\ddot{q}^j = \frac{d}{ds} (\dot{q}^j) = \frac{1}{c} \frac{d}{ds} (v^i \dot{q}^i) = \frac{1}{c} \frac{d}{ds} (v^i \dot{q}^i) = \frac{\mathcal{M}}{1 - \frac{v^2}{c^2}} \frac{d}{dr} \left( v^j \sqrt{\frac{\mathcal{M}}{1 - \frac{v^2}{c^2}}} \right).
$$

(3.29)

Next, anticipating the standard notation in physics, we denote

$$
\phi = \phi_0, \quad dq = \phi_0 dq + \phi_0 c dr = \frac{e}{c} \vec{A} dr - eV dr, \quad \text{or} \quad \phi = e\tilde{\phi} = e \left( \frac{1}{c} \vec{A} dr - V dr \right),
$$

(3.30)

where $e$ is the (electric) charge of the particle. Then

$$
(\phi_0)_{l=1,2,3} = e(\tilde{\phi})_{l=1,2,3} = \frac{e}{c} \vec{A}, \quad \dot{\phi}_4 = e\ddot{\phi}_4 = -\frac{e}{c} V,
$$

(3.31)

$$
\vec{F} = e \left[ \frac{\mathcal{M}}{1 - \frac{v^2}{c^2}} \left( \frac{1}{c} \vec{v} \times \vec{A} - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \text{grad} V \right) \right] - \frac{\vec{v}}{c^2 - v^2} \frac{d\psi}{dr} - \text{grad} \psi.
$$

(3.32)

With the above we can write equations of motion (3.23) in the following form:

$$
\frac{d}{dr} \left( \frac{\mathcal{M}}{1 - \frac{v^2}{c^2}} \right) = \epsilon \left( \frac{1}{c} \vec{v} \times \vec{A} - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \text{grad} V \right) - \sqrt{1 - \frac{v^2}{c^2}} \frac{\text{grad} \psi + \frac{\vec{v}}{c^2 - v^2} \frac{d\psi}{dr}}{\mathcal{M}}.
$$

(3.33)
In what follows, we shall denote by $\vec{F}_L$ the Lorentz force,

$$\vec{F}_L = e \left( \frac{1}{c} \vec{v} \times \text{rot}\vec{A} - \frac{1}{c} \frac{d\vec{A}}{dt} - \text{grad}V \right).$$

Below we shall be interested in the physical meaning of the remaining forces on the right-hand side of the equations of motion.

### 3.3. The constraint force

Chetaev equations of motion contain the constraint force $\Phi$. It is defined on $J^1 \pi = \mathbb{R} \times T\mathbb{R}^4$ and depends upon one Lagrange multiplier $\lambda$ (to be determined). For the constraint

$$f \equiv \dot{q}^i - \sqrt{\mathcal{M}c^2 + \sum_{\nu=1}^{3}(\ddot{q}^\nu)^2},$$

it reads

$$\Phi = -\sum_{l=1}^{3} \frac{\lambda \dot{q}^l}{\sqrt{\mathcal{M}c^2 + \sum_{\nu=1}^{3}(\ddot{q}^\nu)^2}} dq^l \wedge ds + \lambda dq^4 \wedge ds. \tag{3.36}$$

Chetaev equations take the form

$$-\dot{q}^l + \dot{q}^\nu \left( \frac{\partial \phi_\sigma}{\partial q^l} - \frac{\partial \phi_\nu}{\partial q^l} \right) - \frac{\partial \psi}{\partial q^l} = \frac{-\lambda \dot{q}^l}{\sqrt{\mathcal{M}c^2 + \sum_{\nu=1}^{3}(\ddot{q}^\nu)^2}}.$$ \tag{3.37}

Multiplying the $\nu$th of the above four equations by $\dot{q}^\nu$ and summing over $\nu = 1, 2, 3, 4$, we obtain

$$\left( \dot{q}^4 q^4 - \sum_{l=1}^{3} \dot{q}^l q^l \right) + \dot{q}^\nu \dot{q}^\nu \left( \frac{\partial \phi_\sigma}{\partial q^l} - \frac{\partial \phi_\nu}{\partial q^l} \right) - \dot{q}^\nu \frac{\partial \psi}{\partial q^l} = \lambda - \frac{\mathcal{M}c^2}{\sqrt{\mathcal{M}c^2 + \sum_{\nu=1}^{3}(\ddot{q}^\nu)^2}}. \tag{3.38}$$

Solutions $c(s) = (q^\nu(s))$ of Chetaev equations satisfy the equation of the constraint, and consequently, also its derivative,

$$\dot{q}^4 = \frac{1}{2} \frac{d}{ds} \mathcal{M}c^2 + \sum_{l=1}^{3} \ddot{q}^l q^l, \quad \text{and then} \quad \dot{q}^4 q^4 = \frac{1}{2} \frac{d}{ds} \mathcal{M}c^2 + \sum_{l=1}^{3} \dot{q}^l q^l. \tag{3.39}$$

Hence, the first term in brackets on the left-hand side of equation (3.38) is $\frac{d}{ds} \left( \frac{1}{2} \mathcal{M}c^2 \right)$, the second term is identically zero because of skewsymmetry of the expression in brackets and the third term equals $-\dot{\psi}/ds$. In this way, along solutions of Chetaev equations passing in the evolution space $Q_+$, we have

$$\frac{d}{ds} \left( \frac{1}{2} \mathcal{M}c^2 - \psi \right) = \lambda \frac{\mathcal{M}c^2}{\sqrt{\mathcal{M}c^2 + \sum_{\nu=1}^{3}(\ddot{q}^\nu)^2}}. \tag{3.40}$$

If $\mathcal{M} \neq 0$, we can compute the multiplier $\lambda$:

$$\lambda = \frac{\sqrt{\mathcal{M}c^2 + \sum_{\nu=1}^{3}(\ddot{q}^\nu)^2}}{\mathcal{M}c^2} \frac{d}{ds} \left( \frac{1}{2} \mathcal{M}c^2 - \psi \right). \tag{3.41}$$
Writing $\Phi = \hat{\Phi} \, dq \wedge ds$ and substituting the obtained expression for $\lambda$ into formula (3.36) for $\Phi$, we obtain components of the constraint force along solutions of the constraint equations as follows:

$$
\begin{align*}
\hat{\Phi}_1 &= -\frac{\dot{q}^i}{M c^2} \frac{d}{ds} \left( \frac{1}{2} M c^2 - \psi \right) = \frac{\dot{q}^i}{M c^2} \frac{d}{ds} \left( \psi - \frac{1}{2} M c^2 \right), \\
\hat{\Phi}_4 &= \sqrt{M c^2 + \sum_{\rho=1}^{3} (\dot{q}^\rho)^2} \frac{d}{ds} \left( \frac{1}{2} M c^2 - \psi \right),
\end{align*}
$$

(3.42)
in coordinates $(s, q^\sigma, \dot{q}^\sigma)$, and

$$
\begin{align*}
\Phi_1 &= \hat{\Phi}_1 = \frac{\dot{v}^i}{c^2 - v^2} \frac{d}{dr} \left( \psi - \frac{1}{2} M c^2 \right), \\
\Phi_4 &= c^2 \Phi_4 = -\frac{c^2}{c^2 - v^2} \frac{d}{dr} \left( \psi - \frac{1}{2} M c^2 \right)
\end{align*}
$$

(3.43)
in coordinates $(s, q^i, t, \dot{v}^i, \dot{q}^i)$.

For the sake of simplicity, we introduce the following vector notation that takes into account the formula for the constraint force and the relation between equations (3.23) and (3.33) (note that they are connected via a multiplier)

$$
\vec{F}_C = -\frac{1}{c^2} \frac{d}{dr} \left( \psi - \frac{1}{2} M c^2 \right),
$$

(3.44)
and call $\vec{F}_C$ the induced constraint force. Since the solutions have the structure of a foliation of $Q_+$, $\vec{F}_C$ is defined on the evolution space $Q_+$, and vanishes if $\vec{v} = 0$ or if

$$
\psi - \frac{1}{2} M c^2 = H|_{Q_+}
$$

(3.45)
happens to be a constant of the constrained motion.

The constraint force appears in the motion equations:

**Proposition 3.3 (Equations of motion; $\mathcal{M} \neq 0$, three-dimensional observer).** Let $\phi$ (3.30) be a 1-form (covariant 4-vector field), and $\psi$ a function on the Minkowski spacetime $\mathbb{R}^4$. Consider on $(\mathbb{R} \times T\mathbb{R}^4) \setminus \mathcal{H}_0$ adapted coordinates $(s, q^i, t, \dot{v}^i, \dot{q}^i)$ defined by (3.25). Extremals of a Lagrangian system defined by the Lagrangian $\mathcal{L} = L ds$, 

$$
L = -\frac{1}{2} \left( 1 - \frac{v^2}{c^2} \right) (\dot{q}^i)^2 + \frac{1}{c} \left( \frac{\dot{c}^i}{c} \dot{v}^i - eV \right) \dot{q}^i - \psi,
$$

(3.46)
and subject to the constraint (3.27) are curves $c(s) = (t(s), q^i(t(s)))$ in $\mathbb{R}^4$, satisfying the following system of differential equations:

$$
\frac{d}{dr} \left( \sqrt{\frac{\mathcal{M}}{1 - \frac{v^2}{c^2}}} \right) = \vec{F}_L + \vec{F}_C = \sqrt{\frac{1 - \frac{v^2}{c^2}}{\mathcal{M}}} \text{grad} \psi - \frac{1}{2} \frac{\vec{v}}{\sqrt{\mathcal{M}(1 - \frac{v^2}{c^2})}} \frac{d\mathcal{M}}{dr}
$$

(3.47)
(equations of motion), where $\vec{F}_L$ is the Lorentz force and $\vec{F}_C$ is the induced constraint force, and

$$
\frac{dr}{ds} = \sqrt{\frac{\mathcal{M}}{1 - \frac{v^2}{c^2}}}
$$

(3.48)
(equation of the constraint). The dynamics proceed in the evolution space $Q_+ \subset \mathbb{R} \times T\mathbb{R}^4$ defined by (3.27).
The above proposition gives a general form of motion equations for particles of non-zero mass. As we have seen above, the condition \( M \neq 0 \) at a point means that \( M \) is either locally positive or locally negative. In the rest of this paper, we shall be interested in the case \( M = \text{const} \). We shall discuss separately the cases \( M > 0, M < 0, \) and \( M = 0 \).

### 4. Particles with a positive square of mass

Let us suppose \( M > 0 \) and constant on \( \mathbb{R}^4 \). We set

\[
M = m_0^2. \tag{4.1}
\]

As above, we consider either fibred coordinates \((s, q^\sigma, \dot{q}^\sigma)\), \(1 \leq \sigma \leq 4\), or adapted coordinates \((s, q^i, t, v^i, q^i)\), \(1 \leq i \leq 3\), on \((\mathbb{R} \times T \mathbb{R}^4) \setminus \mathcal{H}_0\).

#### 4.1. The evolution space

For the evolution space \( Q_+ \), we then have from (3.7)

\[
\dot{q}^4 = \frac{m_0 c}{\sqrt{1 - v^2 / c^2}}, \tag{4.2}
\]

where \( m_0 > 0 \) and \( v < c \); hence, the speed of a relativistic particle with \( M > 0 \) is lower than the light speed. We can see that the constant \( m_0 \) has the meaning of the rest mass of a particle. We also denote, as usual, mass and (kinetic) energy by

\[
m = \frac{m_0}{\sqrt{1 - v^2 / c^2}}, \quad E = mc^2. \tag{4.3}
\]

We can ‘visualize’ the constraint \( Q \) and the evolution space \( Q_+ \) in the ‘velocity space’ as follows: for illustration we suppress one dimension by considering \( q^3 = 0 \) and denote \((\dot{q}^1, \dot{q}^2, \dot{q}^3) = c(x, y, w)\). Then \( Q \) is given by

\[
\frac{w^2}{m_0^2} - \frac{x^2}{m_0^2} - \frac{y^2}{m_0^2} = 1 \tag{4.4}
\]

(see figure 1).

#### 4.2. 4-momentum

Since \( \mathbb{R}^4 \) is equipped with the Minkowski metric field \( g \), which is a regular symmetric \((2, 0)\)-tensor field on \( \mathbb{R}^4 \), every contravariant 4-vector field \( \hat{\mathbf{u}} \) on \( \mathbb{R}^4 \) is canonically associated with a covariant 4-vector field (a 1-form) \( g(\hat{\mathbf{u}}, \cdot) \) on \( \mathbb{R}^4 \). We put

\[
p = g(\hat{\mathbf{u}}, \cdot) \tag{4.5}
\]

and call \( p \) the 4-momentum associated with \( \hat{\mathbf{u}} \). In fibred coordinates, where the section \( \hat{\mathbf{u}} \) has components \( \hat{u}^\sigma = \dot{q}^\sigma \circ \hat{\mathbf{u}}, 1 \leq \sigma \leq 4 \), we have \( p = \hat{p}_\sigma dq^\sigma \), where

\[
\hat{p}_\sigma = g_{\sigma\nu} \hat{u}^\nu = g_{\sigma\nu} (\dot{q}^\nu \circ \hat{\mathbf{u}}), \quad \text{i.e.} \quad (\hat{p}_\sigma)_{1 \leq \sigma \leq 4} = (-\dot{q}^1, -\dot{q}^2, -\dot{q}^3, \dot{q}^4) \circ \hat{\mathbf{u}}. \tag{4.6}
\]

For simplicity of notation, we shall denote components of \( p \) associated with \( \hat{\mathbf{u}} \) simply by

\[
(p_\sigma)_{1 \leq \sigma \leq 4} = (-\dot{q}^1, -\dot{q}^2, -\dot{q}^3, \dot{q}^4), \tag{4.7}
\]

where \( \dot{q}^\prime \)'s are components of \( \hat{\mathbf{u}} \).

In adapted coordinates \((s, q^i, t, v^i, q^i)\) we similarly have \( p = p_i dq^i + p_4 dt, \)

\[
p_1 = -\frac{v^i \dot{q}^i}{c}, \quad p_4 = c \dot{q}^4. \tag{4.8}
\]
Note that in the latter case, the corresponding ‘contravariant components’ are
\[ p^1 = \frac{\nu_1 q^4}{c}, \quad p^4 = cq^4. \]  \hspace{1cm} (4.9)
We denote \( \hat{p} = (p^l)_{1 \leq l \leq 3} \), and write
\[ (p^l)_{1 \leq l \leq 4} = \left( -\frac{\nu q^4}{c}, \ c q^4 \right), \quad (p^\sigma)_{1 \leq \sigma \leq 4} = \left( \frac{\nu q^4}{c}, \ c q^4 \right). \]  \hspace{1cm} (4.10)
With the help of the 4-momentum, the equation of the constraint \( Q \), i.e. \( g(\hat{u}, \hat{u}) = m_0^2 c^2 \), reads
\[ \hat{p}_\sigma \hat{p}^\sigma = m_0^2 c^2, \quad \text{or} \quad p_1 p^1 + \frac{1}{c^2} p_4 p^4 = m_0^2 c^2. \]  \hspace{1cm} (4.11)
As expected, the ‘universal’ 4-momentum becomes a specific particle characteristic on the particle’s evolution space \( Q^+ \) (see (4.2)).
Components of the 4-momentum then take the form
\[ (\hat{p}_\sigma)_{1 \leq \sigma \leq 4} = (-mv, \ E/c), \quad (\hat{p}^\sigma)_{1 \leq \sigma \leq 4} = (mv, \ E/c). \]  \hspace{1cm} (4.12)
Finally, with the help of (4.2), which now takes the form \( q^4 = mc = E/c \), we can compute the energy; we obtain the familiar formula
\[ E = c \sqrt{m_0^2 c^2 - p_1 p^1} = c \sqrt{m_0^2 c^2 + p^2}, \]  \hspace{1cm} (4.13)
where \( p^2 = \sum_{l=1}^{3} (p^l)^2 \) denotes the usual square of length of the 3-vector \( \hat{p} \).

4.3. Equations of motion

Rewriting motion equations for the particular case \( \mathcal{M} = m_0^2 = \text{const} \), we obtain

**Proposition 4.1** (Equations of motion: usual particles, four-dimensional observer). Let \( g \) be the Minkowski metric, \( \phi \) a covariant vector field and \( \psi \) a function on \( \mathbb{R}^4 \). Extremals of a Lagrangian system defined by the Lagrangian \( L = L_0 \),
\[ L = -\frac{1}{2} g(\hat{u}, \hat{u}) + \phi(\hat{u}) - \psi = -\frac{1}{2} \left( (\hat{q}^4)^2 - \sum_{l=1}^{3} (\hat{q}^p)^2 \right) + \hat{q}^4 \phi_0 - \psi, \]  \hspace{1cm} (4.14)
on $\mathbb{R} \times T\mathbb{R}^4$, and subject to the constraint

$$\dot{q}^4 = \sqrt{m_0^2 c^2 + \sum_{l=1}^{3} (\dot{q}^l)^2},$$

(4.15)

are curves $\gamma(s) = (s, q^a(s))$, satisfying the system of mixed second- and first-order differential equations as follows: (3.23) where $\mathcal{M} = m_0^2$ (equations of motion), and $cq^4 = E$ (equation of the constraint, energy equation).

**Proposition 4.2** (Equations of motion: usual particles, three-dimensional observer). Let $\phi$ (3.30) be a 1-form (covariant 4-vector field), and $\psi$ a function on the Minkowski spacetime $\mathbb{R}^4$.

Consider on $(\mathbb{R} \times T\mathbb{R}^4) \setminus H_0$ adapted coordinates $(s, q^l, t, v^l, \dot{q}^4)$ defined by (3.25). Extremals of a Lagrangian system defined by the Lagrangian $L = L ds$,

$$L = -\frac{1}{2} \left(1 - \frac{v^2}{c^2}\right) (\dot{q}^4)^2 + e \left(\frac{1}{c} \vec{A} \vec{v} - V\right) \dot{q}^4 / c - \psi,$$

(4.16)

and subject to the constraint (4.2) are curves $c(s) = (t(s), q^l(t(s)))$ in $\mathbb{R}^4$, satisfying the following system of differential equations:

$$\frac{d}{dt} \left(\frac{m_0 \vec{v}}{\sqrt{1 - v^2 / c^2}}\right) = \vec{F}_L + \vec{F}_C - \frac{1}{m_0} \sqrt{1 - v^2 / c^2} \text{ grad } \psi, \quad \vec{F}_C = -\frac{1}{m_0 c^2} \frac{\vec{v}}{\sqrt{1 - v^2 / c^2}} \frac{d\psi}{dt},$$

(4.17)

(equations of motion), together with the following equation of the constraint (mass equation, resp. energy equation):

$$\frac{dr}{ds} = m, \quad \text{resp.} \quad c^2 \frac{dt}{ds} = E.$$  

(4.18)

### 4.4. Variational principle for time-dependent curves

Taking into account equations of motion (4.17), we can ask if they are equations for extremals of an (unconstrained) variational principle for curves $t \to (q^l(t), q^4(t))$ in $\mathbb{R}^3$. This means, we ask about a Lagrangian 1-form $L(t, q^l, v^l) dt$ on $\mathbb{R} \times \mathbb{R}^3$ such that equations (4.17) would identify with Euler–Lagrange equations

$$\frac{\partial L}{\partial q^l} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^l} = 0.$$  

(4.19)

The problem can be tackled directly by applying the Helmholtz conditions (necessary and sufficient conditions for a system of differential equations be variational ‘as it stands’) [10]. However, the answer is known: as shown in [24], equations with the left-hand side as in (4.17) are variational if the force on the right-hand side is a Lorentz-type force, i.e. if we are in the well-known situation of a particle in an electromagnetic field. Hence, we can conclude:

**Proposition 4.3.** If the scalar field $\psi$ is a constant, then equations (4.17) are variational as equations for extremals of a Lagrangian $L(t, q^l, v^l) dt$ on $\mathbb{R} \times \mathbb{R}^3$, such that

$$L = -m_0 c^2 \sqrt{1 - v^2 / c^2} + e \frac{\vec{A} \vec{v}}{c} - eV.$$  

(4.20)

Note an interesting connection between the vanishing of the induced constraint force and the existence of a Lagrangian, namely $\vec{F}_C \equiv 0$ implies that $d\psi/dt = 0$, i.e. $\psi = \text{const.}$
Remark 4.4. Within the theory of nonholonomic systems, it is known that motion equations of a Lagrangian system subject to constraints need not be Euler–Lagrange equations of the constrained Lagrangian. This is the case also in our situation. Indeed, the unconstrained Lagrangian 1-form
\[ \Lambda = L ds = - \left( \frac{1}{2} \left( 1 - \frac{v^2}{c^2} \right) (q^4)^2 - \frac{e}{c} \lambda - eV \right) \frac{q^4}{c} + \psi \]  
(4.21)
gives rise to the constrained Lagrangian
\[ \Lambda_C = L_C ds = - \left( \frac{1}{2} m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} - \frac{e}{c} \lambda - eV + \frac{\psi}{m} \right) \frac{dt}{t} \]
(4.22)
defined on the constraint \( Q^+ \). The relevant part of the Lagrangian \( \Lambda_C \) for considering time-parametrized curves is its horizontal part with respect to the projection onto the time axis, i.e. the 1-form \( L dt \), with
\[ L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{c} \lambda - eV - \frac{\psi}{m}. \]  
(4.23)
However, Euler–Lagrange equations of \( L \) do not coincide with equations (4.17).

It should be also noted that a naïve idea to add a scalar potential \( \psi \) to the conventional Lagrangian (4.20), indeed does not provide equations of motion (4.17).

4.5. Dicke force

Equations of motion in proposition 4.2 can be equivalently expressed in a way that admits comparison of the obtained forces with a hypothesis due to Dicke on non-Lorentz-type interactions [7].

Denote \( \exp \mu = e^\mu \) and put
\[ \mu = \frac{\psi}{m_0 c^2}, \quad \tilde{m}_0 = m_0 e^\mu. \]  
(4.24)
Let us express motion equations (4.17) in terms of the new scalar potential \( \mu \) and ‘mass’ \( \tilde{m}_0 \). We obtain
\[ \frac{d}{dr} \left( \frac{\tilde{m}_0 e^{-\mu} \dot{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) + \frac{m_0 \dot{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{d\mu}{dr} = \mathcal{F}_L - m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} \operatorname{grad} \mu, \]
(4.25)
and
\[ \frac{de^{-\mu}}{dr} = -e^{-\mu} \frac{d\mu}{dr}; \]
(4.26)
hence,
\[ e^{-\mu} \frac{d}{dr} \left( \frac{\tilde{m}_0 \dot{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = \mathcal{F}_L - m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} \operatorname{grad} \mu. \]
(4.27)
Multiplying this equation by \( e^\mu \), we have
\[ \frac{d}{dr} \left( \frac{\tilde{m}_0 \dot{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = e^\mu \mathcal{F}_L - m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} \operatorname{grad} \mu. \]  
(4.28)
However,\[ \nabla \tilde{m}_0 = m_0 \nabla e^\mu = m_0 e^\mu \nabla \mu = \tilde{m}_0 \nabla \mu , \tag{4.29} \]
and we finally obtain
\[
\frac{d}{dr} \left( \frac{\tilde{m}_0 \vec{v}}{\sqrt{1 - v^2/c^2}} \right) = e^\mu \mathcal{F}_L - c^2 \sqrt{1 - v^2/c^2} \nabla \tilde{m}_0. \tag{4.30} \]

**Proposition 4.5.** With the help of the mass function \( \tilde{m}_0 \), depending upon a scalar potential \( \psi \) on \( \mathbb{R}^4 \), and defined by
\[
\tilde{m}_0 = m_0 \exp \left( \frac{\psi}{m_0^2 c^2} \right), \tag{4.31} \]
equations of motion in proposition 4.2 take the following equivalent form:

- **in the absence of the electromagnetic field:**
\[
\frac{d}{dr} \left( \frac{\tilde{m}_0 \vec{v}}{\sqrt{1 - v^2/c^2}} \right) = -c^2 \sqrt{1 - v^2/c^2} \nabla \tilde{m}_0, \tag{4.32} \]

- **in the presence of the electromagnetic field:**
\[
\frac{d}{dr} \left( \frac{\tilde{m}_0 \vec{v}}{\sqrt{1 - v^2/c^2}} \right) = \frac{\tilde{m}_0}{m_0} \mathcal{F}_L - c^2 \sqrt{1 - v^2/c^2} \nabla \tilde{m}_0. \tag{4.33} \]

Hence, the particle moves like it has a non-constant rest mass and is subject to a force depending upon the particle speed and the gradient of the mass. The influence of an electromagnetic field on the particle’s dynamics depends not only upon the particle charge, but also upon \( \exp \left( \psi / (m_0^2 c^2) \right) = \tilde{m}_0/m_0 \).

Let us denote
\[
\mathcal{F}_D = -c^2 \sqrt{1 - \frac{v^2}{c^2}} \nabla \tilde{m}_0 \tag{4.34} \]
and call \( \mathcal{F}_D \) the Dicke force.

**Remark 4.6.** The obtained formula for the force \( \mathcal{F}_D \) complies with a prediction of a relativistic non-Lorentz-type interaction due to Dicke [7]. Although in his paper a formula or, at least, exact arguments are missing, Dicke conjectured the existence of a force, originating from the mass distribution of the universe, and acting on particles with non-zero (positive) rest mass. By his hypothesis, the particle should move like it has a nonconstant mass depending upon a scalar field, and subject to a force depending upon the particle’s nonconstant mass, and upon its speed proportionally to the factor \(-\sqrt{1 - v^2/c^2}\). Remarkably, this conjecture was influenced by Brans andDicke’s modification of general relativity, published earlier, attempting to adapt the general relativity theory to the Mach principle (see [3]).

5. Particles with a negative square of mass—tachyons

Let us suppose \( \mathcal{M} < 0 \) and constant on \( \mathbb{R}^4 \). We set
\[
\mathcal{M} = -m_0^2, \tag{5.1} \]
where \( m_0 > 0 \). Particles of this kind are *particles with a negative square of mass, called tachyons*.

Again we consider either fibred coordinates \((s, q^\sigma, q^4)\), \(1 \leq \sigma \leq 4\), or adapted coordinates \((s, q^\tau, t, v^l, q^4)\), \(1 \leq l \leq 3\), on \((\mathbb{R} \times T\mathbb{R}^4) \setminus \mathcal{H}_0\).
5.1. The evolution space

The evolution space $Q_+$ is the submanifold

$$
π^4 = \frac{m_0 c}{\sqrt{v^2 c^2 - 1}}. \quad (5.2)
$$

For tachyons one can introduce instantaneous mass and (kinetic) energy by the following formulæ:

$$
m = \frac{m_0 \sqrt{v^2 c^2 - 1}}{\sqrt{v^2 c^2 - 1}} \quad \text{where} \quad v > c \quad \text{and} \quad E = mc^2. \quad (5.3)
$$

By the above, tachyons can be regarded as particles characterized by a real positive mass $m_0$ moving with a speed $v > c$.

Note that if the speed of a tachyon is increasing, its mass $m$ is decreasing ($v \to \infty$ means that $m \to 0$ and $E \to 0$, and conversely, $v \to c$ means that $m \to \infty$ and $E \to \infty$), and that

$$
m = m_0 \quad \text{if} \quad v = c\sqrt{2}. \quad (5.4)
$$

This formula demonstrates a different meaning of the positive constant $m_0$ characterizing a ‘normal particle’ and a tachyon: while for a ‘normal particle’ $m_0$ has the meaning of the mass at rest, for a tachyon this is the mass it would possess when moving with speed $c\sqrt{2}$.

Similarly to the case $\mathcal{M} > 0$, we can ‘visualize’ the constraint $Q$ and the evolution space $Q_+$ in the ‘velocity space’. Suppressing the $\dot{q}^1$ dimension and denoting again $(\dot{q}^1, \dot{q}^2, \dot{q}^4) = c(x, y, w)$, the manifold $Q$ is given by the equation

$$
\frac{w^2}{m_0^2} - \frac{x^2}{m_0^2} - \frac{y^2}{m_0^2} = -1 \quad (5.5)
$$

(see figure 2).

5.2. 4-momentum and energy

We can introduce the 4-momentum in a full analogy with the $\mathcal{M} > 0$ case. Starting from the ‘universal momentum’

$$
p = g(\dot{u}, \cdot), \quad (5.6)
$$
in coordinates \( p = \hat{p}_a \, dq^a = p_l \, dq_l + p_4 \, dt \), we obtain formulae for the components (see the previous section)

\[
(\hat{p}_a)_{1 \leq a \leq 4} = (-\dot{q}^1, -\dot{q}^2, -\dot{q}^3, \dot{q}^4),
\]

\[
(\hat{p}_a)_{1 \leq a \leq 4} = \left(-\frac{\ddot{u}q^4}{c}, c\dot{q}^4\right), \quad (\hat{p}_a^c)_{1 \leq a \leq 4} = \left(\frac{\ddot{u}q^4}{c}, c\dot{q}^4\right).
\]  

(5.7)

A concrete particle attains its momentum on the constraint \( Q \) given by the equation

\[
g(\hat{u}, \hat{v}) = -m_0^2c^2, \text{ i.e.}
\]

\[
\hat{p}_a \hat{p}^a = -m_0^2c^2, \quad \text{or} \quad p_l \hat{p}^l + \frac{1}{c} \lambda_0 p_4 = -m_0^2c^2.
\]  

(5.8)

Hence, on the evolution space \( Q_+ \), components of a tachyon’s 4-momentum take the form

\[
(\hat{p}_a)_{1 \leq a \leq 4} = (-mv_4, E/c), \quad (\hat{p}_a^c)_{1 \leq a \leq 4} = (mv_4, E/c).
\]  

(5.9)

Finally, with the help of (5.8), we can compute the energy:

\[
\mathcal{E} = c\sqrt{-p_4 \hat{p}^4 - m_0^2c^2} = c\sqrt{p^2 - m_0^2c^2},
\]  

(5.10)

where, as in the previous sections, \( p^2 = \vec{p} \vec{p} \) is the usual square of length of the 3-vector \( \vec{p} \).

5.3. Equations of motion

Rewriting motion equations (3.23) for \( M = -m_0^2 = \text{const} \), we obtain

**Proposition 5.1** (Equations of motion: tachyons, four-dimensional observer). Let \( g \) be the Minkowski metric, \( \phi \) a covariant vector field and \( \psi \) a function on \( \mathbb{R}^4 \). Extremals of a Lagrangian system defined by the Lagrangian \( \Lambda = L \delta s \),

\[
L = -\frac{1}{2} g(\hat{u}, \hat{v}) + \phi(\hat{u}) - \psi = -\frac{1}{2} \left( (\dot{q}^4)^2 - \sum_{p=1}^{3} (\dot{q}^p)^2 \right) + \dot{q}^4 \phi_0 - \psi,
\]  

(5.11)

on \( \mathbb{R} \times \mathbb{T} \mathbb{R}^4 \), and subject to the constraint (5.2) are curves \( \gamma(s) = (s, q^a(s)) \), satisfying the following system of mixed second- and first-order differential equations: (3.23) where \( M = -m_0^2 \) (equations of motion), and \( c\dot{q}^a = \mathcal{E} \) (equation of the constraint, energy equation).

**Proposition 5.2** (Equations of motion: tachyons, three-dimensional observer). Let \( \phi \) (3.30) be a 1-form (covariant 4-vector field), and \( \psi \) a function on the Minkowski spacetime \( \mathbb{R}^4 \). Consider on \( (\mathbb{R} \times \mathbb{T} \mathbb{R}^4) \backslash \mathcal{H}_0 \) adapted coordinates \( (s, q^l, t^l, \dot{q}^l) \) defined by (3.25). Extremals of a Lagrangian system defined by the Lagrangian \( \Lambda = L \delta s \),

\[
L = -\frac{1}{2} \left( 1 - \frac{v^2}{c^2} \right) (\dot{q}^4)^2 + c \left( \frac{1}{c} \vec{A} \vec{v} - V \right) \frac{\dot{v}^4}{c} - \psi,
\]  

(5.12)

and subject to the constraint

\[
\dot{q}^4 = \frac{m_0c}{\sqrt{v^2/c^2 - 1}}.
\]  

(5.13)

are curves \( c(s) = (t(s), \dot{q}^l(t(s))) \) in \( \mathbb{R}^4 \), satisfying the following system of differential equations:

\[
\frac{d}{ds} \left( \frac{m_0 \vec{v}}{\sqrt{v^2/c^2 - 1}} \right) = \tilde{F}_L + \tilde{F}_C - \left. \frac{1}{m_0} \frac{\sqrt{v^2/c^2 - 1}}{c} \right| \text{grad} \psi, \quad \text{where} \quad \tilde{F}_C = -\frac{1}{m_0c^2} \frac{\vec{v}}{\sqrt{v^2/c^2 - 1}} \frac{d\psi}{ds}
\]  

(5.14)
(equations of motion), together with the following equation of the constraint (mass equation, resp. energy equation):
\[
\frac{dr}{ds} = m, \quad \text{resp.} \quad c^2 \frac{dr}{ds} = \mathcal{E}.
\] (5.15)

**Proposition 5.3.** If \( \psi = \text{const} \), equations of motion (5.14) are variational as equations for extremals of a Lagrangian
\[
\mathcal{L} = m_0 c^2 \sqrt{\frac{v^2}{c^2} - 1} + e \frac{eA}{c} \mathbf{v} - eV.
\] (5.16)

Finally, let us investigate a Dicke-type influence on tachyons. Denote \( \mu = \psi m_0^2 c^2 \), \( \tilde{m}_0 = m_0 e^\mu \). (5.17)

An analogous procedure as in the case of ‘normal particles’ directly leads to the following result:

**Proposition 5.4.** With the help of the mass function \( \tilde{m}_0 \), depending upon a scalar potential \( \psi \) on \( \mathbb{R}^4 \), and defined by
\[
\tilde{m}_0 = m_0 \exp \left( \frac{\psi}{m_0^2 c^2} \right),
\] (5.18)
equations of motion for tachyons take the following equivalent form:

- in the absence of an electromagnetic field:
\[
\frac{d}{dt} \left( \tilde{m}_0 \mathbf{v} \sqrt{\frac{\mathbf{v}^2}{c^2} - 1} \right) = -c^2 \sqrt{\frac{\mathbf{v}^2}{c^2} - 1} \text{grad} \tilde{m}_0,
\] (5.19)

- in the presence of an electromagnetic field:
\[
\frac{d}{dt} \left( \frac{\tilde{m}_0 \mathbf{v}}{\sqrt{\frac{\mathbf{v}^2}{c^2} - 1}} \right) = \frac{\tilde{m}_0}{m_0} \mathbf{F}_L - c^2 \sqrt{\frac{\mathbf{v}^2}{c^2} - 1} \text{grad} \tilde{m}_0.
\] (5.20)

We can see that in the case of particles with a negative square of mass, the Dicke force takes the form
\[
\mathbf{F}_D = -c^2 \sqrt{\frac{\mathbf{v}^2}{c^2} - 1} \text{grad} \tilde{m}_0.
\] (5.21)

Thus, the results for tachyons are formally the same as for particles with a positive square of mass.

6. Particles with zero mass

It remains to study the case \( \mathcal{M} = 0 \), i.e. particles with zero mass. Let us again consider on \((\mathbb{R} \times T\mathbb{R}^4)\setminus \mathcal{H}_0\) adapted coordinates \((s, q^l, t, v^l, \dot{q}^l)\), \(1 \leq l \leq 3\).
6.1. The evolution space

The constraint \( Q \subset (\mathbb{R} \times T\mathbb{R}^4) \setminus \mathcal{H}_0 \) is now given by the equation

\[
(q^4)^2 - \sum_{p=1}^{3} (\dot{q}^p)^2 = 0, \quad \text{resp.} \quad (q^4)^2 \left( 1 - \frac{v^2}{c^2} \right) = 0.
\]  

(6.1)

Since \( \dot{q}^4 \neq 0 \), it holds \( v = c \), i.e. particles of this kind move with the light speed.

Note that in coordinates \((s, q^1, t, v^1, q^4)\), the constraint \( Q_+ \) cannot be expressed in the form \( \dot{q}^4 = h(s, q^1, t, v^1) \), so that we have to represent the evolution space in the form

\[
Q_+ : \quad v = c, \quad \dot{q}^4 > 0.
\]  

(6.2)

Drawing the constraint \( Q \) and the evolution space \( Q_+ \) in the ‘velocity space’ similarly to the previous two cases, i.e. suppressing the \( \dot{q}^3 \) dimension and denoting \((\dot{q}^1, \dot{q}^2, \dot{q}^4) = c(x, y, w)\), we obtain for \( Q \)

\[
w^2 - x^2 - y^2 = 0
\]  

(6.3)

(see figure 3).

6.2. 4-momentum and energy

Now, with the help of the ‘universal 4-momentum’ \( p = g(\dot{u}, \cdot) \), the equation of the constraint \( g(\dot{u}, \dot{u}) = 0 \) appears in the form

\[
\hat{p}_\alpha \hat{p}^\alpha = 0, \quad \text{or} \quad p_1 p^1 + \frac{1}{c^2} p_4 p^4 = 0, \quad \text{i.e.} \quad p_4 p^4 - c^2 p^2 = 0,
\]  

(6.4)

where, as usual, \( p^2 = \vec{p} \cdot \vec{p} = \sum_{l=1}^{3} (p^l)^2 \).

On the evolution space \( Q_+ \) it holds \( v = c \). Introducing the unit vector \( \vec{e}_v \) in the direction of \( \vec{v} \), i.e.

\[
\vec{v} = c \vec{e}_v,
\]  

(6.5)
we obtain the components of the 4-momentum on $Q_+^+$ as follows:
\[
(p_\sigma)_{1 \leq \sigma \leq 4} = q^4(-e_\sigma, c), \quad (p^\sigma)_{1 \leq \sigma \leq 4} = q^4(e_\sigma, c).
\] (6.6)
Let us introduce energy by the same formula as in the $M \neq 0$ cases:
\[
E = c q^4.
\] (6.7)
Then the equation of the constraint $Q_+$ takes the form $E^2 - c^2 p^2 = 0$, $E > 0$; hence, the energy of a massless particle (i.e. the function $E$ on the evolution space) is
\[
E = c p.
\] (6.8)

6.3. Equations of motion

It is to be stressed that the case $M = 0$ is different from the above two, when $M \neq 0$. The reason is that the constrained equations of motion are implicit equations: singular in the sense that they cannot be put into the normal form. Recall that they describe dynamics proceeding in the evolution space $Q_+$ and read as follows:
\[
\begin{align*}
\left(\delta_{ij} - \frac{q^4 q_{\ell} q^j}{\sum_{\ell=1}^{3} (q^\rho)^2}\right) \dot{q}^j &= q^j \left(\frac{\partial \phi_1}{\partial q^j} - \frac{\partial \phi_1}{\partial q^j}\right) + \frac{q^j q^i}{\sqrt{\sum_{\ell=1}^{3} (q^\rho)^2}} \left(\frac{\partial \phi_1}{\partial q^i} - \frac{\partial \phi_2}{\partial q^i}\right) \\
&+ \frac{1}{\sqrt{\sum_{\ell=1}^{3} (q^\rho)^2}} \left(\frac{\partial \phi_4}{\partial q^j} - \frac{\partial \phi_4}{\partial q^j}\right) - \frac{\partial \psi}{\partial q^j} - \frac{\partial \psi}{\partial q^j} \frac{q^j}{\sqrt{\sum_{\ell=1}^{3} (q^\rho)^2}}.
\end{align*}
\] (6.9)
where $1 \leq l \leq 3$.

Let us express these equations in adapted coordinates $(s, q^j, t, v^j, \dot{q}^j)$. Since along the constrained paths we have
\[
\dot{q}^j = \frac{d q^j}{ds} = \frac{1}{c} \frac{d}{dt} \frac{d s}{d t} = \frac{\dot{q}^j}{c} \left(\frac{v^j}{c} \frac{d q^j}{d t} + \dot{q}^4 \frac{d v^j}{d t}\right),
\] (6.10)
we obtain after some calculations
\[
\dot{q}^j = \frac{d q^j}{d s} = \frac{1}{c} \frac{d}{dt} \frac{d s}{d t} = \frac{\dot{q}^j}{c} \left(\frac{v^j}{c} \frac{d q^j}{d t} + \dot{q}^4 \frac{d v^j}{d t}\right),
\] (6.11)
where we have used the constraint conditions $v = c$ and $\dot{q}^4 > 0$.

With notations $(\phi_1)_{1,2,3} = \xi A, \phi_4 = -\xi V,$ and the unit vector $e_v$ in the direction of $v$, the motion equations take the vector form as follows:
\[
\begin{align*}
\dot{e}_v^j &= \frac{d e_v^j}{d t} = e_v^j \left(\dot{e}_v \times \text{rot} A - \frac{1}{c} \frac{\partial A}{\partial t} - \text{grad} V + \left[\dot{e}_v \left(\frac{1}{c} \frac{\partial A}{\partial t} + \text{grad} V\right)\right]\right)
\end{align*}
\] (6.12)
Multiplying this vector equation by $e_v$, we immediately obtain
\[
c e_v \text{ grad} \psi + \frac{\partial \psi}{\partial t} = \dot{v} \text{ grad} \psi + \frac{\partial \psi}{\partial t} = \frac{d \psi}{d t} = 0.
\] (6.13)
(Note that the same result provides equation (3.40) where $M = 0$.)

---

6 As is known, for implicit equations initial conditions (in our case a point in the evolution space) generally do not determine a unique solution. Lagrangian dynamics of this kind are studied by Dirac’s theory of constrained systems, or by direct methods developed in [14] (see also [15] and [22]). The latter methods are also suitable for systems that—as in our case—are not Lagrangian (do not come from a variational principle).
Now, taking into account that every unit vector is perpendicular to its derivative, i.e. 
\( \vec{e}_\psi \frac{d\vec{e}_\psi}{dt} = 0 \), putting the derivative \( \partial\psi/\partial t \) expressed from (6.13) to (6.12), and using the relation 
\( \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{b}) \vec{c} - (\vec{a} \cdot \vec{c}) \vec{b} \), we can rewrite the vector equation (6.12) as follows:

\[
\dot{q}^2 \frac{\vec{d} \vec{e}_\psi}{dt} = \vec{J} - \frac{c}{q^2} [\text{grad } \psi - \vec{e}_\psi (\vec{e}_\psi \cdot \text{grad } \psi)] \implies \frac{\mathcal{E}}{c} \frac{\vec{d} \vec{e}_\psi}{dr} = \vec{J} + \frac{c^2}{\mathcal{E}} \vec{e}_\psi \times (\vec{e}_\psi \times \text{grad } \psi),
\]

(6.14)

where

\[
\vec{J} = e \left( \vec{e}_\psi \times \text{rot } \vec{A} - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \text{grad } V + \left[ \vec{e}_\psi \left( \frac{1}{c} \frac{\partial \vec{A}}{\partial t} + \text{grad } V \right) \right] \vec{e}_\psi \right) \quad \text{and}
\]

\[
\frac{d\psi}{dt} = c \vec{e}_\psi \cdot \text{grad } \psi + \frac{\partial \psi}{\partial t} = 0,
\]

(6.17)

together with the equation of the constraint (energy equation)

\[
\frac{c^2}{\mathcal{E}} \frac{dt}{dr} = \mathcal{E} > 0.
\]

(6.18)

The motion proceeds in such a way that the vector fields \( \vec{v} \) and \( d\vec{v}/dt, \vec{J} \) and \( \vec{v} \), and \( \text{grad } \psi \)
and \( d\psi/dt \) are always orthogonal, where \( \vec{v} = c \vec{e}_\psi \), and \( \vec{J} \) is given by (6.15).

Let us study the motion equations in more detail. Assume that \( \vec{J} = 0 \) (this can be fulfilled for a chargeless particle or for zero 4-potential, i.e. \( \vec{A} = 0 \) and \( V = 0 \)). The motion is then governed by equations

\[
\frac{\mathcal{E}}{c} \frac{\vec{d} \vec{e}_\psi}{dr} = \frac{c^2}{\mathcal{E}} \vec{e}_\psi \times (\vec{e}_\psi \times \text{grad } \psi), \quad \frac{d\psi}{dr} = c \vec{e}_\psi \cdot \text{grad } \psi + \frac{\partial \psi}{\partial t} = 0.
\]

(6.19)

(Note that \( \vec{J} \) can be chosen zero independently of \( \psi \), and the second equation in (6.19) holds for an arbitrary \( \psi \).) Multiplying the first equation by \( \text{grad } \psi \) and taking into account that \( d\psi/dt \text{ grad } \psi = 0 \) (see (6.16)), we obtain

\[ 0 = \text{grad } \psi \left[ \vec{e}_\psi \times (\vec{e}_\psi \times \text{grad } \psi) \right] = (\vec{e}_\psi \text{ grad } \psi)^2 - \text{grad }^2 \psi, \]

and thus, denoting as \( \theta \) the angle made by vectors \( \vec{e}_\psi \) and \( \text{grad } \psi \), we have

\[ |\text{grad } \psi|^2 = |\text{grad } \psi|^2 \cos^2 \theta \implies \vec{e}_\psi \parallel \text{grad } \psi, \quad \text{or} \quad \text{grad } \psi = 0. \]

(6.20)

Because of the relation just obtained between \( \vec{e}_\psi \) and \( \text{grad } \psi \), it holds \( \vec{e}_\psi \times \text{grad } \psi = 0 \) in (6.19). This means that the following motion equations should be fulfilled simultaneously:

\[ \frac{d\vec{e}_\psi}{dr} = 0, \quad \frac{d\psi}{dr} = 0. \]
The first of them gives us that \( \vec{e}_v = \vec{k} = \text{const} \), i.e. for \( \vec{F} = 0 \) the particle moves in \( \mathbb{R}^3 \) along a straight line (with the speed of light), i.e. more precisely, every straight line in \( \mathbb{R}^3 \) is an allowed trajectory, depending upon initial conditions. Together with the second equation, solutions are all straight lines passing in the surfaces \( \psi = \text{const} \). This is, however, in contradiction with condition (6.20). A release is to conclude that massless particles do not feel the scalar field \( \psi \). On the other hand, we have seen earlier (cf the Dicke force) that particles with non-zero mass do feel the field \( \psi \). Hence, we can conjecture that every particle possesses a scalar field charge depending upon the particle’s mass, such that for massless particles the scalar field charge is zero. It is natural to assume this charge being \( m_0 \) and write

\[
\psi = m_0 \bar{\psi}. \tag{6.21}
\]

Now the above results can be reformulated as follows:

**Proposition 6.1** (Equations of motion: particles with zero mass, three-dimensional observer).

Let \( \phi \) (3.30) be a 1-form (covariant 4-vector field), and \( \bar{\psi} \) a function on the Minkowski spacetime \( \mathbb{R}^4 \). Consider on \( \mathbb{R} \times T \mathbb{R}^4 \backslash H_0 \) adapted coordinates \((s, q^i, t, v^i, \dot{q}^i)\) defined by (3.25). A massless particle is described by a Lagrangian

\[
L = -\frac{1}{2} \left( 1 - \frac{v^2}{c^2} \right) (q^i)^2 + e \left( \frac{1}{c} \vec{A} \vec{v} - V \right) \frac{\dot{q}^i}{c} - a \bar{\psi}, \tag{6.22}
\]

where the charge \( a = 0 \), subject to the constraint

\[
v = c, \quad \dot{q}^i > 0. \tag{6.23}
\]

Trajectories are curves \( c(s) = (t(s), q^i(t)) \) in \( \mathbb{R}^4 \), satisfying the following system of differential equations:

\[
\frac{\dot{c}}{c} \frac{d\vec{e}_v}{dt} = e \left\{ \vec{e}_v \times \text{rot}\vec{A} - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \text{grad} V + \left[ \vec{e}_v \left( \frac{1}{c} \frac{\partial \vec{A}}{\partial t} + \text{grad} V \right) \right] \vec{e}_v \right\},
\]

\[
c^2 \frac{dt}{ds} = E > 0, \tag{6.24}
\]

where \( E \) is a function on \( \mathbb{R}^4 \), having the meaning of the kinetic energy of the particle.

As a consequence of these equations, the motion proceeds in such a way that the vector fields \( \vec{v} \) and \( d\vec{v}/dt \), and \( \vec{F} \) and \( \vec{v} \) are always orthogonal, where \( \vec{v} = c \vec{e}_v \), and \( \vec{F} \) is the force given by the relation (6.15).

Note that from the form (6.24) of the equations of motion, it is explicitly clear that equations for massless particles are indeed singular: they do not provide a uniquely determined motion unless energy of the particle is fixed.

### 6.3. A brief summary and conclusions

We proposed a new approach to foundations of relativistic dynamics. It is based on a treatment of a particle as a Lagrangian system subject to a nonholonomic constraint, compatible with the special relativity theory. We explored a geometric setting to nonholonomic mechanics, proposed by one of the present authors [16], where the constrained system is considered as ‘living’ on the submanifold defined by the constraint: the constraint submanifold thus becomes a genuine evolution space for the constrained system.

The key idea is that relativistic particles of any mass moving in an electromagnetic field (defined by a 4-potential \( \phi \)) and a scalar field (defined by a scalar potential \( \bar{\psi} \)) can be described by
(1) the Lagrange function

\[ L = -\frac{1}{2} g_{\sigma\nu} \dot{q}^\sigma \dot{q}^\nu + e \bar{\phi} \dot{q}^\sigma - a \bar{\psi}, \]  

(6.25)

defined on \( \mathbb{R} \times T\mathbb{R}^4 \), where \( g \) is the Minkowski metric on \( \mathbb{R}^4 \), \( g = (-1, -1, -1, 1) \), and \( e, \) resp. \( a \), is the particle’s charge corresponding to the electromagnetic and scalar fields, respectively, and

(2) the nonholonomic constraint

\[ g_{\sigma\nu} \dot{q}^\sigma \dot{q}^\nu = M c^2, \]  

(6.26)
in \( \mathbb{R} \times T\mathbb{R}^4 \), where \( M \) is a function on the spacetime, representing the particle’s mass.

As a main result, we found equations of motion as they appear to a four-dimensional, and to a three-dimensional observer.

We discussed in detail the cases when \( M = \text{const} = m_0^2 \) (usual particles), \( M = \text{const} = -m_0^2 \) (tachyons) and \( M = 0 \) (massless particles).

As expected, massless particles move with the light speed, and particles with a positive square of mass move with a speed lower than the speed of light; the constant \( m_0 \) has the meaning of rest mass. On the other hand, particles with a negative square of mass (tachyons) are particles with real positive mass \( m_0 \), moving with a speed greater than the light speed. The constant \( m_0 \) is then the mass of a tachyon moving with the speed \( c\sqrt{2} \).

It turned out that the scalar field charge \( a \) is closely connected with the mass of the particle: for massless particles \( a = 0 \), and \( a \neq 0 \) in the other cases. This means that the presence of a scalar field does not affect the motion of massless particles. On the other hand, all massive particles are influenced by a scalar field. We have found that the corresponding force takes the form for ‘normal’ particles, resp. for tachyons:

\[ \vec{F}_D = -c^2 \sqrt{1 - \frac{v^2}{c^2}} \text{grad} \tilde{m}_0, \quad \text{respectively} \quad \vec{F}_D = -c^2 \sqrt{\frac{v^2}{c^2} - 1} \text{grad} \tilde{m}_0, \]  

(6.27)

where

\[ \tilde{m}_0 = m_0 \exp \left( \frac{\bar{\psi} m_0}{c^2} \right). \]  

(6.28)

and we have put \( a = m_0 \).

We called \( \vec{F}_D \) the Dicke force in honour of Dicke who predicted the existence of a force of this kind [7].

The charge \( e \) has the usual meaning; however, the corresponding force acting on massive and massless particles is different. For massive particles we obtain the usual formula for the Lorentz force (the same for ‘normal’ particles and tachyons). For massless particles the corresponding force takes the form

\[ \vec{F} = e \left\{ \vec{e}_v \times \text{rot} \vec{A} - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \text{grad} V + \left[ \vec{e}_v \left( \frac{1}{c} \frac{\partial \vec{A}}{\partial t} + \text{grad} V \right) \right] \vec{e}_v \right\}. \]  

(6.29)

It is important to stress a striking difference between dynamics of massive and massless particles. Massive particles are regular mechanical systems in the sense that the dynamics in the evolution space obey Newton’s determinism principle (motion equations can be put into the normal form, i.e. every trajectory is uniquely determined by initial conditions). On the other hand, massless particles are singular mechanical systems: the dynamics in the evolution space do not obey Newton’s determinism principle (motion equations cannot be put into the normal form, i.e. trajectories are not completely determined by initial conditions). To obtain a
concrete trajectory in the spacetime \( \mathbb{R}^4 \), one has to fix initial conditions and the energy of the particle.

We also introduced the 4-momentum as the 1-form, canonically related to the 2-tensor field \( g \). With the help of 4-momentum \( \hat{p} = (\hat{p}_\alpha) \), the evolution space (= the constraint submanifold) is given by equation

\[
\hat{p}_\alpha \hat{p}^\alpha = m^2_0 c^2,
\]

respectively

\[
\hat{p}_\alpha \hat{p}^\alpha = -m^2_0 c^2,
\]

respectively

\[
\hat{p}_\alpha \hat{p}^\alpha = 0,
\]

for ‘normal’ particles, tachyons and massless particles, respectively. On the evolution space, the above ‘universal’ 4-momentum gives rise to the 4-momentum of a corresponding particle; components of the 4-momentum were shown to be as follows:

\[
(\hat{p}_\alpha) = (-m_\mathbf{v}, E/c), \quad (\hat{p}_\sigma) = \left( -\frac{\mathbf{E} \cdot \mathbf{v}}{c^2}, \frac{\mathbf{E}}{c} \right) = \frac{\mathbf{E}}{c} (-\mathbf{v}_e, 1)
\]

for massive, respectively, massless particles, where

\[
m = \frac{m_0 c}{\sqrt{1 - v^2 c^2}}, \quad m = \frac{m_0 c}{\sqrt{v^2 c^2 - 1}},
\]

for ‘normal’ particles, respectively, for tachyons, and \( \mathcal{E} \) is the kinetic energy of the particle, defined (in all the cases) by formula

\[
\mathcal{E} = \mathcal{L}^4.
\]

Consequently, on the evolution space, energy is given by the following formulae:

\[
\mathcal{E} = c \sqrt{p^2 + m^2_0 c^2}, \quad \mathcal{E} = c \sqrt{p^2 - m^2_0 c^2}, \quad \mathcal{E} = c p,
\]

for ‘normal’ particles, tachyons and massless particles, respectively, where \( p = \sqrt{\mathbf{p} \cdot \mathbf{p}} \) is the usual length of the momentum 3-vector \( \mathbf{p} \).

Finally let us mention the variational aspects of the obtained equations of motion. In the particular case of ‘usual’ particles with a positive square of mass moving in an electromagnetic field, the ‘(3 + 1)-dimensional’ equations of motion we have obtained take the usual form. Hence, as is known, they can be obtained from a variational principle as Euler–Lagrange equations of the Lagrangian (1.1). On the other hand, rather surprisingly, in the presence of a scalar field, the obtained equations do not come from an (unconstrained) variational principle\(^7\). In particular, it might be interesting that trying a ‘natural’ extension of the Lagrangian (1.1) by adding to the Lagrangian a scalar field term does not provide the correct equations of motion (recall section 4.4).

Acknowledgments

Research was supported by grants GACR 201/09/0981 of the Czech Science Foundation. OR also wishes to acknowledge support of the IRSES project GEOMECH (no. 246981) within the 7th European Community Framework Programme.

References

[1] Bates L M and Nester J M 2011 On D’Alembert’s principle Commun. Math. 19 57–72
[2] Bloch A M (with the collaboration of Baillieul J, Crouch P and Marsden J) 2003 Nonholonomic Mechanics and Control (Berlin: Springer)
[3] Brans C andDicke R H 1961 Mach’s principle and a relativistic theory of gravitation Phys. Rev. 124 925–35

\(^7\) The reader might be interested to note that the equations of motion we have obtained in this paper are ‘variational’ in a generalized sense: one can find a nonholonomic variational principle providing these equations (see [19]).
[4] Carinena J F and Rañada M F 1993 Lagrangian systems with constraints: a geometric approach to the method of Lagrange multipliers J. Phys. A: Math. Gen. 26 1335–51
[5] Chetaev N G 1932–33 On the Gauss principle Izv. Kazan. Fiz.-Mat. Obsc. 6 323–6 (in Russian)
[6] de León M, Marrero J C and de Diego D M 1997 Non-holonomic Lagrangian systems in jet manifolds J. Phys. A: Math. Gen. 30 1167–90
[7] Dicke R 1965 The influence of the time dependent gravitation interaction on the Solar system Gravity and Relativity ed W F Hoffmann (Moscow: Mir) (in Russian)
[8] Flament M R 2011 D’Alembert–Lagrange analytical dynamics for nonholonomic systems J. Math. Phys. 52 032705
[9] Giachetta G 1992 Jet methods in nonholonomic mechanics J. Math. Phys. 33 1652–65
[10] Helmholtz H 1887 Ueber die physikalische Bedeutung des Prinzips der kleinsten Wirkung J. Reine Angew. Math. 100 137–66
[11] Koon W S and Marsden J E 1997 The Hamiltonian and Lagrangian approaches to the dynamics of nonholonomic systems Rep. Math. Phys. 40 21–62
[12] Krupka D 1973 Some geometric aspects of variational problems in fibered manifolds Folia Fac. Sci. Nat. Univ. Park. Brunensis Physica 14 (Brno, Czechoslovakia) 1–65 (arXiv:math-ph/0110005)
[13] Krupka D 2008 Global variational theory in fibered spaces Handbook of Global Analysis (Amsterdam: Elsevier) pp 755–839
[14] Krupková O 1994 A geometric setting for higher-order Dirac–Bergmann theory of constraints J. Math. Phys. 35 6557–76
[15] Krupková O 1997 The Geometry of Ordinary Variational Equations (Lecture Notes in Mathematics vol 1678) (Berlin: Springer)
[16] Krupková O 1997 Mechanical systems with non-holonomic constraints J. Math. Phys. 38 5098–126
[17] Krupková O 2000 Higher-order mechanical systems with constraints J. Math. Phys. 41 5304–24
[18] Krupková O 2002 Recent results in the geometry of constrained systems Rep. Math. Phys. 49 269–78
[19] Krupková O 2009 The nonholonomic variational principle J. Phys. A: Math. Theor. 42 185201
[20] Krupková O 2010 Geometric mechanics on nonholonomic submanifolds Commun. Math. 18 51–77
[21] Krupková O and Musilová J 2001 The relativistic particle as a mechanical system with non-holonomic constraints J. Phys. A: Math. Gen. 34 3859–75
[22] Krupková O and Prince G E 2008 Second order ordinary differential equations in jet bundles and the inverse problem of the calculus of variations Handbook of Global Analysis (Amsterdam: Elsevier) pp 841–908
[23] Massa E and Pagani E 1997 Classical mechanics of non-holonomic systems: a geometric approach Ann. Inst. H Poincaré 66 1–36
[24] Novotný J 1980 On the inverse variational problem in the classical mechanics Proc. Conf. on Differential Geometry and Its Applications (Universita Karlova, Prague, 1981) ed O Kowalski pp 189–95
[25] Sarlet W 1996 A direct geometrical construction of the dynamics of non-holonomic Lagrangian systems Extracta Math. 11 202–12
[26] Sarlet W, Cantrijn F and Saunders D J 1995 A geometrical framework for the study of non-holonomic Lagrangian systems J. Phys. A: Math. Gen. 28 3252–68
[27] Sarlet W, Prince G, Mestdag T J and Krupková O 2012 Time-dependent kinetic energy metrics for Lagrangians of electromagnetic type J. Phys. A: Math. Theor. 45 085208
[28] Saunders D J, Sarlet W and Cantrijn F 1996 A geometrical framework for the study of non-holonomic Lagrangian systems II J. Phys. A: Math. Gen. 29 4265–74
[29] Sumbatov A S 2002 Nonholonomic systems Regul. Chaotic Dyn. 7 231–8