Return probability and $k$-step measures

Nicholas Dronen
Department of Computer Science
University of Colorado at Boulder
dronen@colorado.edu
Pearson Knowledge Technologies
Boulder, Colorado, USA

Qin Lv
Department of Computer Science
University of Colorado at Boulder
qin.lv@colorado.edu

January 23, 2013

Abstract

The notion of return probability – explored most famously by George Pólya on $d$-dimensional lattices – has potential as a measure for the analysis of networks. We present an efficient method for finding return probability distributions for connected undirected graphs. We argue that return probability has the same discriminatory power as existing $k$-step measures – in particular, beta centrality (with negative $\beta$), the graph-theoretical power index (GPI), and subgraph centrality. We compare the running time of our algorithm to beta centrality and subgraph centrality and find that it is significantly faster. When return probability is used to measure the same phenomena as beta centrality, it runs in linear time – $O(n + m)$, where $n$ and $m$ are the number of nodes and edges, respectively – which takes much less time than either the matrix inversion or the sequence of matrix multiplications required for calculating the exact or approximate forms of beta centrality, respectively. We call this form of return probability the Pólya power index (PPI). Computing subgraph centrality requires an expensive eigendecomposition of the adjacency matrix; return probability runs in half the time of the eigendecomposition on a 2000-node network. These performance improvements are important because computationally efficient measures are necessary in order to analyze large networks.
1 Introduction

The probability that a random walk on a graph returns to the node where it began – the probability of returning to the origin or simply return probability – is a fairly well-known notion in the literature of random walks. Research in this area originally concentrated on return probability on infinite regular graphs. In his seminal work [1], Pólya proved that a random walk on an infinite 1- or 2-dimensional lattice returns to the origin with probability $p = 1$, but when $d > 2$, $p < 1$. Methods for determining the value of $p$ for 3-dimensional lattices were subsequently discovered [2] [3] [4] [5]. Pólya’s theorem has also been applied to electrical networks by Doyle and Snell [6]. Return probability continues to be explored in contemporary research, although the venue has shifted from graphs of fixed degree to random graphs [7] [8] [9] and to spectral methods [10].

There is a class of measures which compute some value for a node $i$ based on paths up to length $n$ originating at $i$. Some representative members of this class are degree centrality, beta centrality [11], the graph-theoretical power index (GPI) (see [12] for an overview), and subgraph centrality [13]. These measures have been called “$n$-path centralities” [14]. This term is problematic. First, a measure is only a centrality when it satisfies certain requirements, such as those proposed in [15]. Beta centrality with negative $\beta$ and the GPI, however, are not centralities. “Path” is infelicitous, too, because each measure pays attention to different entities. Beta centrality is based on walks, the GPI counts disjoint paths, and subgraph centrality is derived from closed walks. We propose to refer to them instead as “$k$-step measures.”

Return probability is a $k$-step measure as well, and it has a few virtues that distinguish it from the others: (1) being a probability, it is always in the range $[0, 1]$ and requires no normalization, so the return probability of two nodes can always be meaningfully compared, even when the nodes are in different networks; (2) it permits precise control of the length of walks over which it is computed; and (3) it can be computed very efficiently.

The notation used in this article is mostly conventional. We only consider graphs $G = (V, E)$ that are simple, connected, and undirected. Let $n = |V|$ be the number of vertices, and $m = |E|$ be the number of edges. The length of some sequence of adjacent vertices – e.g., a path or a walk – is denoted by $k$. Let $A = A(G)$ be the adjacency matrix of $G$, where $a_{ij} = 1$ if there is an edge between $i$ and $j$ and $a_{ij} = 0$ otherwise. Let $P = P(G)$ be the transition probability matrix of $G$, where $p_{ij} = 1/\deg(i)$ if $i$ and $j$ are adjacent, and $p_{ij} = 0$ otherwise. We denote the probability of an event $X$ by $P(X)$. We occasionally diverge from convention. In subsection 2.1 we use $A^{(k)}$ – with parens that distinguish it from the usual $A^k$ – to indicate a kind of $k$-th power of $A$ that is essential for computing return probability. And in subsection 3.1 we abuse the $\gg$ and $\ll$ symbols to compress our visual comparison of the power-related $k$-step measures.

Beta centrality and the GPI are measures of exclusionary power.\footnote{Alter-based centrality [16] is notably similar to return probability with $k = 2$. In its case, it has been applied to social networks by Kosinski et al. [17].} They
identify the relative power of nodes in a network by their ability to exclude their neighbors from some valuable interaction. Beta centrality of node $i$ is the $i$-th component of the vector:

$$C(\beta) = \sum_{k=0}^{\infty} \beta^{k-1} A^k 1$$

According to Bonacich [11], the “sign of $\beta$ corresponds exactly to the distinction . . . between positive and negative exchange systems” and its magnitude “affects the degree to which distant ties are taken into account”. In this article, we are only interested in beta centrality with negative values of $\beta$. The GPI is defined as:

$$GPI_i(e) = \sum_{k=1}^{g^2} (-1)^{(k-1)} m_{ik}$$

where $g$ is the diameter of the network, $m_{ik}$ is the number of non-intersecting paths of length $k$ originating at node $i$, and $e$ is the number of exchange opportunities that node $i$ has in any round. Finally, subgraph centrality [13] – a measure of the number of subgraphs in which a node participates – is defined as:

$$C_s(i) = \sum_{k=0}^{\infty} (A^k)_{ii} k!$$

Later we discuss in detail other connections among beta centrality, the GPI, subgraph centrality, and return probability. For now it suffices to note a few characteristics shared just by beta centrality and subgraph centrality, and to situate return probability in relation to them. First, beta centrality and subgraph centrality are formally expressed as involving increasing powers of the adjacency matrix $A$. Return probability is expressed in a similar way (although in practice we use a stochastic matrix), but each power of $A$ must be modified before the subsequent power can be computed. Second, beta centrality and subgraph centrality are expressed as infinite sums. Since cumulative return probability converges to 1 as the walk length $k \to \infty$, the return probability for each node is the same in the limit, which is not informative. Instead we find a distribution of return probabilities over walks of length 1, . . . , $k$. Finally, beta and subgraph centrality are scalar, assigning a single real value to a node. In contrast, a distribution of return probabilities is a sequence of real values. To reduce a distribution of return probabilities for a node to a scalar value, we take either the return probability or the cumulative return probability at some chosen $k$. This allows us to compare return probability to other measures.

negative mode, it can, like return probability, be used in lieu of beta centrality with negative $\beta$.

2 The GPI has undergone many changes since its inception. This equation for the GPI appeared early in the tussle of theories competing in the exchange network literature in the ’80s and ’90s. Improved methods, results of which we use later in this paper, focus on the probability of a node being excluded in a round of exchanges.
The purpose of this article is to show that return probability is equivalent to these three $k$-step measures, and that it can be computed more efficiently, much more so in some cases. The rest of this article is organized as follows. In Section 2 we propose and validate a method for finding return probabilities. It is based on a particular kind of walk – the self-absorbing walk – which we use to model the probability of returning to the origin for the first time. If return probability is a useful measure, what does it measure? We devote Section 3 to that question, showing that with $k = 2$, return probability is strongly related to existing power measures, implying that return probability is at least an approximation of exclusionary power. We call this measure the Pólya power index (PPI). We also show that return probability with $k > 2$ is equivalent to subgraph centrality. Finally, we show that return probability is significantly more efficient to compute than beta centrality and subgraph centrality. Section 4 contains further discussion. Section 5 concludes.

2 Computing Return Probability

2.1 Algorithm

Consider a random walk on a graph $G = (V, E)$. Choose some node $i \in V$ as the origin and begin to walk. If we return to $i$, the walk terminates, and we start a new walk. To emphasize that in these walks $i$ becomes a terminating point only after the walk leaves $i$, we call this a self-absorbing walk. With such walks, returning to the origin at step $k$ is mutually exclusive with returning to the origin before step $k$. Thus the probability of returning to the origin in a $k$-step walk is related to the following two probabilities:

1. The probability of returning to $i$ at step $k$.
2. The probability of not returning to $i$ at any step $< k$.

For the first probability, let $\text{Next}_{i,k}$ be the event of returning to $i$ at step $k$ on a self-absorbing walk originated from $i$. To compute $P(\text{Next}_{i,k})$, we must know the states we can potentially be in on a walk of length $k − 1$. From there we must count the number of next steps that are possible from that set of states, taking care to distinguish those that return to $i$ from those that do not. Define $\text{Steps}_{i,k}$ as the number of possible next steps from the set of possible states after a walk of length $k − 1$ and define $\text{ReturnSteps}_{i,k}$ as the number of possible next steps that return to $i$ from the same set of possible states. Then $P(\text{Next}_{i,k})$ is:

$$P(\text{Next}_{i,k}) = \begin{cases} \frac{\text{ReturnSteps}_{i,k}}{\text{Steps}_{i,k}} & \text{if } \text{Steps}_{i,k} > 0 \\ 0 & \text{if } \text{Steps}_{i,k} = 0 \end{cases} \quad (1)$$

The second of these two probabilities is the complement of the probability of returning in any step $< k$. Let $R_{i,k}$ denote the event of returning to the origin
\( i \) at step \( k \). Then the probability of not returning to the origin in \( k - 1 \) steps is:

\[
P(R_i, 1 \wedge \ldots \wedge R_i, k - 1) = P((R_i, 1 \vee \ldots \vee R_i, k - 1)) = 1 - (P(R_i, 1) \vee \ldots \vee P(R_i, k - 1)) = 1 - (P(R_i, 1) + \ldots + P(R_i, k - 1)) = 1 - \sum_{x=1}^{k-1} P(R_i, x)
\]

(2)

Combining (1) and (2) yields our equation for the return probability for any node \( i \in V \) and any length \( k \):

\[
P(R_i, k) = \left[ 1 - \sum_{x=1}^{k-1} P(R_i, x) \right] P(\text{Next}_{i, k})
\]

(3)

It is well known that an element \( a_{i,j}^k \) of the \( k \)-th power of \( A \) is the number of walks of length \( k \) from node \( i \) to node \( j \). A non-zero element \( a_{i,i}^k \) indicates the number of closed walks of length \( k \) originating at \( i \). If the diagonal of \( A \) and its powers are left undefined, the same process computes simple paths instead of walks. However, neither technique counts self-absorbing walks. The first one fails to terminate a walk once it returns to the origin, causing it to be counted more than once; and because the diagonal is all zeros in the second one, it disallows returning to the origin altogether. To count self-absorbing walks, our computation must permit a walk to return to the origin and must terminate a walk once it returns. To accomplish this, we compute \( P(R_i, k) \) by taking modified powers of the adjacency matrix \( A \). Define \( zd(A) \) as a function that sets the diagonal entries of \( A \) to 0. Then we compute the modified \( k \)-th power of \( A \) as:

\[
A^{(k)} = \begin{cases} 
A & \text{if } k = 1, \\
zd(A^{(k-1)})A & \text{otherwise}
\end{cases}
\]

(4)

where \( A \) is the original adjacency matrix. Note that we use \( A^{(k)} \) instead of \( A^k \) to distinguish our modified matrix multiplication from ordinary matrix multiplication. To understand the purpose of setting the diagonal to 0, consider that the expression \( A^k A \) extends the \( k \)-step walks of \( A^k \) with the 1-step walks of \( A \). Setting the diagonal entries of \( A^k \) to 0 causes walks that return to the origin at step \( k \) to terminate at the origin, which satisfies the definition of a self-absorbing walk. It is easy to see analogies among the terms of Equation 3 and Equation 4 – that is, between \( 1 - \sum_{x=1}^{k-1} P(R_i, x) \) and \( zd(A^{(k-1)}) \) on the left, and \( P(\text{Next}_{i, k}) \) and \( A \) on the right.
Computing $A^{(k)}$ gives us the values of $\text{ReturnSteps}_{i,k}$ and $\text{Steps}_{i,k}$. Since $a_{i,i}^{(k)}$ is $\text{ReturnSteps}_{i,k}$ and $\sum_j a_{ij}^{(k)}$ is $\text{Steps}_{i,k}$, we redefine $P(\text{Next}_{i,k})$ as follows:

$$P(\text{Next}_{i,k}) = \begin{cases} \frac{a_{i,i}^{(k)}}{\sum_j a_{ij}^{(k)}} & \text{if } \sum_j a_{ij}^{(k)} > 0 \\ 0 & \text{if } \sum_j a_{ij}^{(k)} = 0 \end{cases}$$

(5)

Thus instead of taking increasing powers of an adjacency matrix, and counting and dividing at each step, we take increasing powers of a transition probability matrix $P$. Specifically, we design the following procedure for computing the distribution of expected return probabilities for each vertex $i \in V$ and for steps $1, \ldots, k$:

1. Initialization
   (a) Initialize $x$ to 1.
   (b) Initialize $k$ to the number of steps to perform.
   (c) Initialize the transition probability matrix $P$.

2. Iteration
   (a) If $x = k$, terminate.
   (b) Compute $P^{(x)}$ with Equation 4.
   (c) Read the values from the diagonal of $P^{(x)}$; the value $P^{(x)}_{i,i}$ is the expected return probability for node $i$ at step $x$.
   (d) Increase $x$ by 1.

The complexity of this algorithm, computed naively, is $O(kn^3)$. It can be computed much more efficiently using sparse matrices. Arithmetic operations on them are proportional to $nnz$, the number of non-zero entries. However, as $k$ increases, $P^{(k)}$ becomes less sparse and the benefits of sparse matrix multiplication decrease. Let $nnz_k$ be the number of non-zero entries in the $k$-th matrix computed by our algorithm. Then with sparse matrices the time complexity of our algorithm is $O(k \times nnz_k)$. We present another optimization in subsection 3.1 when we discuss return probability as a measure of power.

The return probability for an entire network and some $k$ can be computed by averaging the return probabilities of all nodes:

$$P(R_k) = \frac{1}{n} \sum_{i \in V} P(R_{i,k})$$

(6)

This network-wide measure can be used in the same fashion as the node-specific form of the measure. It can generate a distribution of probabilities,
Table 1: High correlation between actual and expected returns in simulations with increasing number of walks on two 100-node graphs.

| Number of walks | Complete graph | Small-world network |
|-----------------|----------------|---------------------|
| 4               | 0.87           | 0.89                |
| 6               | 0.90           | 0.91                |
| 8               | 0.95           | 0.95                |
| 10              | 0.98           | 0.78                |
| 12              | 0.91           | 0.82                |
| 14              | 0.92           | 0.94                |
| 16              | 0.96           | 0.93                |
| 18              | 0.97           | 0.98                |
| 20              | 0.97           | 0.93                |
| 22              | 0.97           | 0.98                |
| 24              | 0.99           | 0.95                |
| 26              | 0.99           | 0.99                |
| 28              | 0.99           | 0.95                |
| 30              | 0.99           | 0.99                |
| 32              | 1.00           | 0.99                |
| 34              | 1.00           | 0.95                |
| 36              | 0.99           | 0.98                |
| 38              | 0.99           | 0.99                |

either step-wise or cumulative. For easy comparison to other measures, it can be reduced to a scalar value by taking the step-wise or cumulative return probability at a given $k$. As expected, Equation 6 reaches its highest value in a dyad, where it is 1, regardless of $k$. Any other network has a network return probability $< 1$.

2.2 Validation

To validate that our method correctly computes expected return probabilities, we conduct an experiment similar to those in [17]. In this case we release a random walker on a 100-node scale-free network and count the number of times it returns or fails to return to the origin for walks of particular lengths.

Since the walks are random, we do not know the length of any given walk in advance. Rather, we start walking from a node $i$ and if we return to $i$ in $k$ steps, we record this fact and start another walk. When we have completed some number of walks using a node $i$ as the origin, we compute the actual return rates for $i$ on walks of length $k$ by counting the number of times we returned to $i$ on a walk of length $k$ and dividing by the total number of walks. As shown in Figure 1, for two different nodes in a scale-free network, expected return probabilities computed by our algorithm match well with the actual return rates found after 1000 walks, for different $k$ values.

In general our experience is that the number of walks required in order
Figure 1: Expected and actual return probabilities for walks of increasing length from two different nodes in a 100-node scale-free network. Actual return rates were averaged over 1000 walks started at each node.
to achieve a high correlation between expected and actual returns is small, as shown in Table 1. We have also tested with other graphs of varying sizes, which exhibit the same general correspondence between expected return probabilities and actual return rates.

3 Return Probability and Other Measures

If return probability is a meaningful measure, in what sense is it so? We answer this question by exploring relationships between return probability and two measures of power – beta centrality and the graph-theoretical power index (GPI) – and subgraph centrality. We find that return probability resembles a measure of exclusionary power when $k = 2$, which we call the Pólya power index (PPI). We also find that return probability is equivalent to subgraph centrality when $k > 2$. It also has asymptotically significant running-time advantages over both beta centrality and subgraph centrality.

3.1 Power Measures

Beta centrality and the GPI originated in the competing theories of exchange networks, thus many of the experiments conducted with them are concerned primarily with acts of exchange. However, they – like return probability – may be appropriate for identifying powerful nodes in non-exchange networks as well. When we refer to these measures, including return probability, as measures of power, we mean power in the broadest sense of the term, not just limited to exchange networks. Thus, while we rely on results in the exchange network literature to illustrate the relationships among return probability, beta centrality, and the GPI, we do not think of return probability necessarily as a mechanism for generating predictions for the outcomes of network exchange experiments. We subscribe to the distinction between “power as a potential and power as an activity” [19] and claim only that return probability can identify nodes in powerful positions.

We begin with the results of an early experiment in exchange network theory [18], using them to compare return probability and beta centrality only. The networks used in this early experiment, shown in Figure 2, were discovered later to be strong-power networks [20]. We then turn to the weak-power networks of Figure 3 and use them to compare return probability, beta centrality, and the GPI.

Beta centrality is motivated in part by the fact that in [18], classical centrality measures – degree, betweenness, and closeness – failed to predict the outcomes of experiments with negatively-connected exchange networks. In an exchange network, actors exchange objects of value. An exchange network is connected positively or negatively. Imagine an exchange network consisting of three participants $A$, $B$, and $C$: $A$ is connected to $B$, $B$ is connected to $C$. If the network is positively connected, an act of exchange between $A$ and $B$ does not preclude a concurrent act of exchange between $B$ and $C$. If the network is
Figure 2: Strong-power exchange networks illustrating the Power-Dependence Theory (PDT) experiments of Cook et al. People with structurally similar positions in the network are assigned the same category (E, D, F). People are labeled with their category and their 2-step return probability. According to PDT, power is distributed in these networks according to the relation $E > D = F$. The 2-step return probabilities agree with the predictions of Cook et al. (See also Figure 1 in [18].)
negatively connected, B cannot exchange with A and C at the same time. Here we are primarily concerned with negatively-connected networks.

The GPI originated in the network exchange literature as well. It is associated with Elementary Theory, a competitor to PDT that has itself fared quite well in the experimental literature. There are several versions of the GPI. Here we rely on Markovsky’s version and results from the Social Networks special issue on exchange networks [21]. That version is known to produce contradictory results under some conditions (see footnote 2 of [12]), but to our knowledge those conditions do not apply to these particular results. Many of the revisions of the GPI that occurred after the formulation of the original GPI – including Markovsky’s – make use of a probability of a node being excluded from exchange.

We first show that 2-step return probability does not contradict with some PDT predictions. The people in the networks in Figure 2 are labeled with a category determined by a person’s position in the network and the person’s
| Network | Measure          | Edge  |   |
|---------|-----------------|-------|---|
| L4      | Return probability | A-B   | < |
|         | Beta centrality  |       | < |
|         | GPI             |       | < |
| L5-Stem | Return probability | A-B | < |
|         | B-C            | =  |
|         | C-D            | >  |
|         | Beta centrality  |       | ⇐ |
|         | GPI             |       | > |
| Stem    | Return probability | A-B | < |
|         | B-D            | >  |
|         | Beta centrality  |       | > |
| K-Stem  | Return probability | A-B | < |
|         | B-C            | <  |
|         | B-D            | <  |
|         | C-D            | <  |
|         | D-E            | >  |
|         | Beta centrality  |       | > |
|         | GPI             |       | ≫ |
| Borg-6  | Return probability | A-B | < |
|         | B-C            | <  |
|         | C-D            | <  |
|         | C-E            | <  |
|         | D-E            | <  |
|         | E-F            | >  |
|         | Beta centrality  |       | ≪ |
|         | GPI             |       | > |

Table 2: Relative power of nodes in weak-power networks according to return probability, beta centrality, and Markovsky’s GPI.

cumulative 2-step return probability. The networks are isomorphic to four exchange networks analyzed in both [18] and [11]. For simplicity of presentation, the figures only include the more profitable solid lines from the original network; the dashed lines are excluded. Following both Cook et al. and Bonacich, we compute return probability using only the solid lines. Basing their hypotheses on PDT, Cook et al. predicted that the power distributed among the actors in these networks would reach the equilibrium $E > D = F$. Their prediction was supported by both a laboratory experiment and computer simulations. The fact that 2-step return probability matches the predicted equilibrium for all graphs exactly is shown in the labels of Figure 2. One obtains the same results using beta centrality, with one exception: in the 7-person network, the relation remains $D > E > F$, the same as a conventional centrality.

It was discovered by Markovsky et al. that there are in fact different classes of networks [20]. The networks used in the aforementioned experiment are strong-power networks – networks in which the relations of exchange are stable and the nodes in positions of relative strength dominate their exchange partners. There are also equal power networks in which no actor has an advantage. A third class – weak-power networks – are structurally somewhere between strong and equal power networks. Figure 3 contains several examples of weak-power networks. The discovery of the different classes of networks brought a deeper un-
derstanding of the nature of the networks themselves. In strong-power networks, for example, the actors are clearly divided between those with high power and those with low power; low-power actors are only connected to high-power actors. Such networks are bipartite or very close to bipartite. We revisit bipartivity in subsection 3.2.

Table 2 shows the relative power for all connected nodes in the networks in Figure 3 computed by return probability, beta centrality, and the GPI. If the power of a node A exceeds the power of neighbor B, then the value of edge AB is listed as >. Here we consider the GPI to be a touchstone. Where beta centrality or return probability disagree with the GPI, the symbol is doubled and in bold (e.g., ≪). This abuses notation somewhat but is readable enough. Note that return probability agrees with the GPI for all edges of all networks. The only disagreements are between the GPI and beta centrality. In L5-Stem, beta centrality identifies C as the most powerful node in the network, whereas the GPI and 2-step return probability have B on equal footing with C. In K-Stem, the GPI and return probability compute that B is less powerful than D, and beta centrality holds the opposite. There is a similar disagreement over the edge BC in Borg-6. Generally, it seems that in these particular weak-power networks, beta centrality has difficulty identifying the power conferred on a node i when it is connected to a node j that has no other exchange opportunities. In such a configuration, i is always guaranteed the option of trading with a relatively powerless neighbor.

Since in this case we only need to concern ourselves with \( k = 2 \), we can reformulate the original algorithm from subsection 2.1 to be even more parsimonious. The doesn’t reduce the time complexity over our original algorithm in a meaningful way, because the number of non-zero entries in a sparse matrix is already related to the number of vertices and edges. However, it provides a form of the equation that can easily be computed when the network is represented in memory as a graph not as a matrix. In honor of George Pólya, this is the Pólya power index (PPI):

\[
PPI = \mathbb{P}(R_{i,2}) = \frac{1}{\text{deg}(i)} \sum_{j:(i,j) \in E} \frac{1}{\text{deg}(j)} \tag{7}\]

When applying this computation to an entire graph, each edge appears twice in a summation, so it runs in time \( O(n+m) \). This is significantly faster than beta centrality. Exact implementations of beta centrality require a somewhat costly \( O(n^3) \) matrix inversion. Approximate implementations sum the first \( k \) terms of beta centrality’s infinite series; just like return probability, the approximate version of beta centrality can be computed with sparse matrices, which makes the matrix multiplications sub-cubic. Even then our algorithm is faster.

\(^3\)To compute beta centrality, we use UCINET 6 for Windows [22] with \( \beta = -0.2 \). For GPI, we use the values of GPI3 and \( p \) whenever GPI3 = 1 in Markovsky’s Table 1 [21].
Figure 4: A regular graph in which degree, closeness, betweenness, and eigenvector centrality are the same for all nodes, but return probability and subgraph centrality vary according to each node's position in the graph. Vertices are labeled with the node number, its return probability for $k = 5$, and its subgraph centrality. Return probability identifies the same sets of nodes ($\{1, 2, 8\}$, $\{3, 5, 7\}$, $\{4, 6\}$) as subgraph centrality.
3.2 Subgraph Centrality

Return probability can also be used to identify interesting nodes when $k > 2$. Both [13] and [23] have noted that in some regular graphs, eigenvector centrality [24] is equivalent to degree centrality. Both subgraph centrality and return probability are able to distinguish nodes from one another in such graphs. The label of each node in Figure 4 shows cumulative return probability for $k = 5$ and subgraph centrality. Both measures identify the same groupings of nodes and thus have the same discriminatory power. This is no surprise, because both measures are expressed as diagonals of powers of some matrix representation of a graph. Additionalluy, both measures count trivial closed walks (i.e. a closed walk made from a path starting at node $i$ and the return path to $i$ along the same edges). However, return probability counts only \textit{unique} trivial and non-trivial closed walks – which it accomplishes by way of self-absorbing walks – whereas subgraph centrality counts \textit{all} trivial and non-trivial closed walks.

For small motifs, return probability runs more quickly than subgraph cen-
trality regardless of the size of the network, albeit at the cost of greater memory consumption due to sparse matrices not being multiplied in place. This can be seen in Figure 5, which shows elapsed time for 10-step return probability and subgraph centrality in small-world networks of varying size. The benchmarking program is written in the Python programming language and makes use of the SciPy library for scientific computing. The eigendecomposition function is scipy.linalg.eigh, which in turn uses the robust LAPACK and BLAS linear algebra libraries. We ran the program on a computer with a 1.8 Ghz AMD Athlon 2200 processor and 768 MB RAM. Given that matrix multiplication is highly parallel, a distributed MapReduce-style system is an appropriate solution to the problem of computing return probability for networks too large to fit into the memory of a single computer.

In a complete graph, both the number of closed walks of length \( k \) and the factorial of \( k \) grow quickly. Ultimately, for sufficiently large \( k \), \( k! \) reduces \( A^{k}_{ii}/k! \) to 0. However, the rate of growth of \( A^{k}_{ii} \) also increases rapidly with the order of \( A \). In Figure 6, the value of \( k \) at which \( A^{k}_{ii}/k! \) reaches its maximum increases as the number of nodes increases. Thus, the lengths of the closed walks counted by subgraph centrality are sensitive to the size of the network. By contrast, return probability allows one to select exactly the lengths of the walks considered by the measure.

Both subgraph centrality and return probability can be used to quantify bipartivity degree, a measure of how close a network is to being bipartite. Since a bipartite network contains no odd cycles, the number of even cycles divided by the number of cycles is 1. When computed for an entire graph, subgraph centrality is expressed as:

\[
SC = \frac{1}{n} \sum_{i=1}^{n} SC_i = \frac{1}{n} \sum_{i=1}^{n} e^\lambda_i
\]

Which leads to the following equation for a network’s subgraph centrality:

\[
\beta(G) = \frac{SC_{\text{even}}}{SC} = \frac{\sum_{j=1}^{n} \cosh \lambda_j}{\sum_{j=1}^{n} e^{\lambda_j}}
\]

The same can be computed in terms of a network’s cumulative return probability for walks up to length \( k \):

\[
\beta(G_k) = \frac{\mathbb{P}(R_{k\text{even}})}{\mathbb{P}(R_k)} = \frac{\sum_{j:(j \equiv 0 \mod 2)}^{k} \mathbb{P}(R_j)}{\sum_{j=1}^{k} \mathbb{P}(R_j)}
\]

Or of a node \( i \)’s cumulative return probability:

\[
\beta(G_{i,k}) = \frac{\mathbb{P}(R_{i,k\text{even}})}{\mathbb{P}(R_{i,k})} = \frac{\sum_{j:(j \equiv 0 \mod 2)}^{k} \mathbb{P}(R_{i,j})}{\sum_{j=1}^{k} \mathbb{P}(R_{i,j})}
\]

Bipartivity degree is typically a value in the range \([0.5, 1]\). For a bipartite network – such as the strong-power networks in Figure 2 – \( \beta(G) \) is 1. When
Figure 6: The number of closed walks of length $k$ counted by subgraph centrality for complete graphs of different size.
Figure 7: The network bipartivity degree of non-bipartite weak-power networks for walks of increasing length.
Figure 8: The node bipartivity degree of nodes in a network that consists of a complete bipartite network joined by one edge with a complete graph. Note in (b) that a node’s bipartivity degree remains 1 longer depending on its proximity to the border node.
even and odd closed walks contribute equally $\beta(G)$ is 0.5; the bipartivity degree of a complete graph approaches 0.5 as both $k$ and the size of the graph grow. Figure 7 shows that the bipartivity degree of the non-bipartite weak-power networks differs even as $k$ increases.

Clearly it is possible to use bipartivity degree to distinguish communities that tend towards homophily from communities that tend towards heterophily. Figure 8a shows the node bipartivity degree for all nodes in a graph that consists of a complete bipartite graph and a complete graph joined by a single edge $e$. The “border” nodes are the nodes made adjacent by $e$. Not only are the nodes in the two different networks clearly distinguishable from each other, but the border nodes show a clear divergence away from the bipartivity degree of the other nodes in their cohort and towards each other. Narrowing in on the nodes in the bipartite graph, we can see in Figure 8b that the bipartivity degree of the nodes farthest from the border node remains 1 for longer than that of the nodes in the same set as the border node.

4 Discussion

For some measures, the $k$ in “$k$-step” is a parameter of the measure itself – for example, the $k$ parameter of return probability. When $k = 1$, return probability is the inverse of degree. As $k$ increases, the walks touch a larger neighborhood of nodes. Thus if one wishes to compute return probability for a neighborhood of a specific size, one simply chooses the appropriate value of $k$. The $\beta$ parameter of beta centrality also suggests the size of the neighborhood around node $i$ that is included in the computation. When $\beta$ is 0, beta centrality is akin to degree. When the absolute value of $\beta$ is small, only proximal neighbors are considered, and the neighborhood grows as $|\beta|$ increases. However, $\beta$ itself does not indicate the exact length of walks. However, a cut-off point can be clearly defined when computing the approximation of beta centrality, because then one can sum the first $k$ terms of the series and terminate. The GPI also has a parameter, $e$, but it is unrelated to the scope of the computation; the GPI is computed from one up to the diameter of the network. Subgraph centrality has no walk length parameter, but as we describe in subsection 3.2, the size of the neighborhood it considers varies, being determined only by size of the network and the factorial denominator.

Both beta centrality and subgraph centrality, being expressed as infinite sums of open or closed walks, have to cope with the problem of convergence. The root of the problem is that walks never terminate. In an undirected graph, the number of walks of length $k$ is always greater than the number of walks of length $k - 1$. They deal with this by weighing walks inversely by length (beta centrality) or by factorial of length (subgraph centrality). When computing beta centrality or subgraph centrality, the weighting is the same regardless of the structure of the network. Return probability also converges, but for different reasons, and weighting is determined by the underlying structure of the network itself. In the star graph $S_{1,n}$, 2-step return probability for the cen-
ter node is 1, and for the others it is $1/n$. Take $S_{1,4}$. Since it is bipartite, there are no odd-length paths, and the return probabilities for steps 1, . . . , 6 are 0, 0, 0.25, 0.0, 0.1875, 0.0, 0.1407. Equation 2 automatically scales step $k$ by the complement of step $k - 1$ so that it is a portion of what remains. Without Equation 2, they are 0, 0.25, 0.0, 0.25, 0.0, 0.25, so Equation 1 is 0.25 at every even-numbered step. This is obvious when you consider that if the walk has not returned to the origin, it is always at the center of the star. Equation 2 scales the return probability of step 4 down by the portion of probability already accounted for by step 2. So if the 2-step return probability for a node $i$ increases due to an edge being added to one of $i$’s neighbors, the return probability for $i$ at all subsequent steps is reduced in proportion to the increase at step 2.

Beta centrality and the GPI measure power by having odd-length paths contribute positively and even-length paths contribute negatively to the value the measure computes for a node $i$. The 1-paths contribute positively to $i$’s power, just as a node with higher degree has higher degree centrality. The 2-paths detract from $i$’s power because they provide neighbors with the opportunity to exchange with some node other than $i$. Two-step return probability functions similarly in that it increases when the degree of $i$’s neighbors decreases. The less the opportunity $i$’s neighbors have to exchange with a node other than $i$, the greater is $i$’s 2-step return probability and the greater its power.

Using “$k$-step measure” as the name for the measures discussed in this paper makes the category somewhat more general, albeit in name only. The category could be made even more general. $K$-step measures compute some value for a node $i$ by considering a sequence of increasingly larger sets of nodes by starting with a set containing only $i$, then adding the neighbors of $i$, and the neighbors of those neighbors, and so on. In mathematical morphology, dilation $\delta_d$ is an operation on a subgraph $g$ of graph $G$ which adds to $g$ the nodes of $G$ adjacent to those of $g$ [20]. A $d$-dilation is the application of $\delta_d$ times:

$$\delta_d(g) = \delta(\delta(\ldots(g)\ldots))$$

The process of constructing the sequence of sets considered by a $k$-step measure is just a series of $k$ dilations. This process is a constrained form of dilation in so far as it always starts with a subgraph of one node. A more general definition – one that permits inclusion of a greater number of measures – would have two components: (1) a generic process that constructs a sequence of node sets by repeatedly dilating an arbitrary subgraph $g$ of $G$; and (2) an unspecified computation which takes as input the sequence of node sets generated by the first component.

It may be that 2-step return probability is a close approximation of structural exclusive power because it resembles the model of Markovsky’s WeakNet simulation software in which actors “(i) seek one exchange per round, (ii) seek to exchange with a randomly selected other, and (iii) keep seeking exchange in a given round until no more potential exchange partners remain” [21]. If $A$ and $B$ are connected, the probability that $A$ seeks to exchange with $B$ and $B$ seeks
to exchange with $A$ is just the product of the inverse of the degrees of $A$ and $B$. Similarly, 2-step return probability is the probability that $A$ will be chosen as an exchange partner by any randomly-selected neighbor.

The performance improvements that return probability shows over beta centrality and subgraph centrality should be understood in the context of practical analysis. If one wishes to compute subgraph centrality and one already has the eigenvalues and eigenvectors of the adjacency matrix – say, for use by some other algorithm – computing return probability would be wasteful. An analyst who needs to identify powerful nodes in a somewhat large network may be able to get an answer in a reasonable amount of time by using the approximate version of beta centrality algorithm and stopping the computation after a small number of walks. Where we imagine return probability being most useful is in the analysis of extremely large networks. When a network’s nodes number in the millions (e.g. online social networks, the World Wide Web), the computational efficiency of return probability makes it an attractive alternative to beta centrality or subgraph centrality.

5 Conclusion

We have presented an algorithm for computing return probability for networks. The measure is probabilistic, so it requires no normalization, and it permits exact control over the size $k$ of the neighborhood it forms around a node. Because it shows agreement with beta centrality and the GPI, the Pólya power index appears to be a measure of relative power in networks. It is also just as capable as subgraph centrality at classifying nodes based on features of a network that can only be identified by looking at longer-distance relationships. Further, the time complexity of return probability – $O(n + m)$ for the Pólya power index and proportional to $k \times \text{nnz}_k$ when $k > 2$ – is less than either beta centrality or subgraph centrality and lends itself to easier analysis of extremely large networks.

References

[1] G. Pólya, Math. Ann. 84 (1921) 149.

[2] G.N. Watson, The Quarterly Journal of Mathematics os-10 (1939) 266, http://qjmath.oxfordjournals.org/content/os-10/1/266.full.pdf+html.

[3] W.H. McCrea and F.J.W. Whipple, Proc. Roy. Soc. Edinburgh 60 (1940) 281.

[4] C. Domb, Mathematical Proceedings of the Cambridge Philosophical Society 50 (1954) 586, http://journals.cambridge.org/article_S0305004100029716.
[5] M.L. Glasser and I.J. Zucker, Proceedings of the National Academy of Sciences of the United States of America 74 (1977) 1800, http://www.pnas.org/content/74/5/1800.full.pdf+html.

[6] P.G. Doyle and J.L. Snell, ArXiv Mathematics e-prints (2000), arXiv:math/0001057.

[7] G. Grimmett, Probability on Graphs: Random Processes on Graphs and Lattices, 1 ed. (Cambridge University Press, 2010).

[8] O.C. Martin and P. Šulc, Phys. Rev. E 81 (2010) 031111.

[9] D. Heicklen and C. Hoffman, Electron. J. Probab 10 (2005).

[10] F. Sobieczky, Journal of Theoretical Probability 23 (2010) 639, 10.1007/s10959-010-0298-3.

[11] P. Bonacich, American Journal of Sociology 92 (1987) 1170.

[12] D. Willer and P. Emanuelson, Journal of Mathematical Sociology 32 (2008) pp. 165.

[13] E. Estrada and J.A. Rodríguez-Velázquez, Phys. Rev. E 71 (2005) 056103.

[14] S.P. Borgatti and M.G. Everett, Social Networks 28 (2006) 466.

[15] B. Ruhnau, Social Networks 22 (2000) 357.

[16] Z.P. Neal, Centpow: Stata module to calculate centrality and power indices for each node in a network, Statistical Software Components, Boston College Department of Economics, 2011.

[17] S.P. Borgatti, Social Networks 27 (2005) 55.

[18] K.S. Cook et al., The American Journal of Sociology 89 (1978) 275.

[19] D. Willer, Social Networks 14 (1992) 187.

[20] B. Markovsky et al., American Sociological Review 58 (1993) pp. 197.

[21] B. Markovsky, Social Networks 14 (1992) 267.

[22] S.P. Borgatti, M.G. Everett and L.C. Freeman, UCINET 6 For Windows: Software for Social Network Analysis, 2002.

[23] P. Bonacich, Social Networks 29 (2007) 555.

[24] P. Bonacich, Journal of Mathematical Sociology 2 (1972) 113.

[25] E. Jones et al., SciPy: Open source scientific tools for Python, 2001.

[26] E. Anderson et al., LAPACK Users’ Guide, Third ed. (Society for Industrial and Applied Mathematics, Philadelphia, PA, 1999).
[27] P. Holme et al., Phys. Rev. E 68 (2003) 056107.

[28] E. Estrada and J.A. Rodríguez-Velázquez, Phys. Rev. E 72 (2005) 046105.

[29] L.D.F. Costa et al., Advances in Physics 56 (2007) 167, arXiv:cond-mat/0505185.