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Solutions of a disease model with fractional white noise

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\section*{Abstract}

We consider an epidemic disease system by an additive fractional white noise to show that epidemic diseases may be more competently modeled in the fractional-stochastic settings than the ones modeled by deterministic differential equations. We generate a new SIRS model and perturb it to the fractional-stochastic systems. We study chaotic behavior at disease-free and endemic steady-state points on these systems. We also numerically solve the fractional-stochastic systems by an trapezoidal rule and an Euler type numerical method. We also associate the SIRS model with fractional Brownian motion by Wick product and determine numerical and explicit solutions of the resulting system. There is no SIRS-type model which considers fractional epidemic disease models with fractional white noise or Wick product settings which makes the paper totally a new contribution to the related science.

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1. Introduction

Fractional-stochastic calculus consist of fractional-order derivatives, integral operators or fractional Brownian motion and a noise term representing uncertainty or randomness in modeling. These differential equations have found an outstanding role in efficiently modeling of many different phenomena in science, engineering, and economics.

Epidemic diseases including Coronavirus, Brucellosis, Chickenpox, Dengue, Ebola, Mumps, Influenza, Measles, Plague, SARS, Tetanus, Tuberculosis, Zika, West Nile Virus were studied in terms of some efficient mathematical models in the related scientific literature. These systems were generated by taking into account some facts such as duration of disease, availability and resistance against vaccination, immune systems of individuals in the population and so on. There are mathematical models employing deterministic [4–7], stochastic [1–3], fractional-order [8–15] system of differential equations. Almost each of these models were generated by compartmental models considering each compartment as individuals of susceptible (denoted by S), infected (I), exposed (E), and recovered (R) ones.

In the present research work, new SIRS models both in deterministic and fractional-stochastic settings are proposed. We are concerned with endemic equilibrium and disease-free fixed points and numerical solutions of these models with different type of numerical techniques. We also consider additive fractional white noise representation of the SIRS model and obtain numerical and exact solutions of these models. In the modeling of epidemic diseases via compartmental type mathematical models, there exists not any study considering fractional white noise, Wick product and fractional-order operators all together. Our driving motivation in this paper is to investigate the applicability of these significant and powerful techniques to the modeling of epidemic diseases. Our contributions in this paper may be listed as: (i) Creating a new SIRS model. (ii) Considering this system in terms of stochastic equations and fractional white noise. (iii) Studying chaos at disease-free and endemic steady state points. (iv) Solving perturbed systems (perturbed from deterministic to fractional-stochastic) both numerically and explicitly (exactly). We provided Figures illustrating the behavior of compartments in different environments. From this listed contributions, we can say that the present paper is totally a new contribution to mathematical biologists studying compartment models by fractional and stochastic differential equations. We believe that researchers in this area will employ the present paper and extend it to different models.

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2. SIRS system

We are concerned with a SIRS system described as:
\[
\begin{align*}
\frac{dS(t)}{dt} & = \rho M - \frac{v S(t) I(t)}{M} + k R(t) S(t) - \xi S(t), \\
\frac{dI(t)}{dt} & = \frac{v S(t) I(t)}{M} - d I(t) - \xi I(t), \\
\frac{dR(t)}{dt} & = dI(t) R(t) - k R(t) - \xi R(t),
\end{align*}
\]
(1)
in which \(S(t), I(t), R(t)\), stands for individuals of susceptible, infected and recovered, respectively, \(R(t) + I(t) + S(t) = M\). Represents population of all individuals, \(\rho\) and \(\xi\) are the birth and death rates, respectively, (we assume that they are equal to each other.) \(v\) is the probability of transmission rate between compartments \(S(t)\) and \(I(t)\), \(d\) is rate of recovery from the disease, \(k\) is the rate of being susceptible after recovered from disease. In our SIRS model, we assume that the healed people are not infected and can not spread the disease any further among other individuals in the population.

Defining new variables
\[
a := \frac{v}{M}, \quad e := k + \xi, \quad c := d + \xi, \quad N := \rho M,
\]
re-expression of the system (1) is:
\[
\begin{align*}
\frac{dS(t)}{dt} & = N - dI(t) S(t) + k R(t) S(t) - \xi S(t), \\
\frac{dI(t)}{dt} & = a S(t) I(t) - c I(t), \\
\frac{dR(t)}{dt} & = dI(t) R(t) - e R(t).
\end{align*}
\]
(2)

Fractional-order calculus \([16-18, 48, 51]\) (or differential equations) generated by fractional-order derivative and integral operators found a significant place in applied and computational mathematics in recent years. They were employed in many different scientific work at mathematics, physics, economics and engineering. Fractional-order operators considers historical effects and have non-local computational ability which makes these operators more powerful and desirable in modeling applications. There are many different fractional derivative or integral operators such as Grünwald-Letnikov, Riemann-Liouville, Caputo, Atangana-Baleanu and so on. Let \(g : (0, \infty) \to \mathbb{R}\) be a function. Fractional-order integral operator of order \(\beta\) for \(g\) is defined by
\[
J^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \sigma)^{\beta - 1} g(\sigma) d\sigma,
\]
in which \(\Gamma(\cdot)\) denotes Gamma function.

For \((m \in \mathbb{N})\), Caputo-type derivative operator of order \(\beta\), \(m - 1 < \beta < m\) of \(g(t)\) is given by
\[
D^\beta g(t) = J^{m - \beta} \frac{d^m}{dt^m} g(t).
\]

Next, we express derivatives on left-hand-side of system (2) by Caputo-type fractional-order time derivatives. Therefore, we write the fractional-order SIRS model as
\[
\begin{align*}
D^\alpha_S S(t) & = N - dI(t) S(t) + k R(t) S(t) - \xi S(t), \\
D^\alpha_I I(t) & = a S(t) I(t) - c I(t), \\
D^\alpha_R R(t) & = dI(t) R(t) - e R(t),
\end{align*}
\]
(3)

Disease-free equilibrium (or fixed-point or steady-state point) is determined by letting rightmost side of system (3) equal to 0, when \(I(t) = 0\). Hence, disease-free equilibrium is given by
\[
E_0 = (S(t), I(t), R(t)) = \left( \frac{N}{\xi}, 0, 0 \right).
\]

**Theorem 2.1.** Consider \(D^\alpha y(t) = g(t, y(t))\), \(y(0) = y_0\). \(0 < \alpha \leq 1\), where \(D^\alpha\) is Caputo derivative of \(g(t, y(t)) : \mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R}^m\) is a vector field. The disease free steady state point \(E_0 = 0\) is locally asymptotically stable if \(|\arg(\lambda_i)| > \frac{\pi}{\alpha}\), for \(i = 1, \ldots, 4\) where \(\lambda_i\) is the eigenvalue of Jacobian \(g(t, y(t))\) at \(E_0\).

Jacobian in \(E_0\), is
\[
J = \begin{bmatrix}
-\frac{\xi}{\xi} & -\frac{\alpha N}{\xi} & \frac{\alpha N}{\xi} \\
0 & -\frac{\xi}{\xi} & c \\
0 & 0 & -e
\end{bmatrix}.
\]
From equation \(\det \left[ \begin{bmatrix} -\xi & 0 \\ 0 & -\frac{\xi}{\xi} & c \\ 0 & 0 & -e \end{bmatrix} \right] = 0\), we obtain the eigenvalues as
\[
\lambda_1 = -\xi, \quad \lambda_2 = -\frac{\alpha N}{\xi} - c, \quad \lambda_3 = -e.
\]
It is clear that \(\lambda_1 = -\xi < 0\), and \(\lambda_3 = -e < 0\), since \(\xi > 0\) and \(e > 0\). \(|\arg(\lambda_1)| = |\arg(\lambda_3)| = \pi > \frac{\pi\alpha}{2}\), since \(0 < \alpha < 1\).

However, if \(\lambda_2 = -\frac{\alpha N}{\xi} - c < 0\), the system becomes unstable, since \(|\arg(\lambda_2)| = 0 < \frac{\pi\alpha}{2}\), for all \(0 < \alpha < 1\).

Hence, system is stable locally asymptotically only if \(\lambda_2 < 0\), that is possible when \(c > \frac{\alpha N}{\xi}\).

Now, we consider a quite useful and important number; namely, the basic reproduction number described in Kermack and McKendrick [40] among many others. It is a dimensionless number and typically denoted with \(R_0\). If \(R_0 < 0\), then the disease disappears and if \(R_0 > 0\), the disease is disseminated amongst susceptible individuals. Finally, if \(R_0 = 0\), this implies that an endemic disease exists in the population with a constant rate and infected hosts convey the disease to the susceptible individuals.

Hence, we conclude from the above computations that \(R_0\), for (3) is
\[
R_0 := \frac{c \xi}{\alpha N}.
\]
Because \(R_0 > 0\), the disease is disseminated amongst susceptible individuals. In particular, if any one of the coefficients \(\xi\) or \(c\) is equal to 0, then \(R_0 = 0\), this implies that an endemic disease exists in the population with a constant rate and infected hosts convey the disease to the susceptible individuals. Next, we consider the endemic steady-state point which is obtained when \(I(t) \neq 0\). Hence, the endemic equilibrium point is
\[
E_1 = (S(t), I(t), R(t)) = \left( \frac{c}{\alpha d}, \frac{\xi c + akc + ac\xi - Na}{kcd}, \frac{ks(t)}{as(t) - c} \right).
\]
(4)

which is determined by letting the rightmost part (3) equal to 0.

\[
J(S(t), I(t), R(t)) = \begin{bmatrix}
-aI(t) + kR(t) - \xi & -aS(t) & kS(t) \\
0 & aS(t) - c & 0 \\
dR(t) & dR(t) & dR(t) - e
\end{bmatrix}.
\]

Therefore,
\[
J(E_1) = \begin{bmatrix}
A^* & -c & \frac{kc}{\xi} \\
0 & 0 & 0 \\
B^* & 0 & 0
\end{bmatrix},
\]
where
\[
A^* = -a\xi + \xi c + akc + ac\xi - Na - \xi cd \\
B^* = \frac{\xi c + akc + ac\xi - Na}{kc}
\]

It is possible to calculate the eigenvalues of this matrix for special parameter values and then study on stability, see e.g. [26], of the system.
3. Non-negative solutions

Notice that
\[(S(t), I(t), R(t)) \in R^3_i = \{(a_1, a_2, a_3) : a_i \geq 0 \text{ for every } i = 1, 2, 3\}.
\]

**Generalized mean value theorem:** (Odbat and Shawagfeh [20]) Suppose \(h(t)\) and \(\frac{D^\alpha_t}{\Gamma(\alpha)} h(t)\) are continuous functions on \([a, b]\). Then, it is possible to write that

\[h(t) = h(a) + \frac{1}{\Gamma(\alpha)} \frac{D^\alpha_t}{\Gamma(\alpha)} h(\xi) (t - a)^\alpha, \quad 0 \leq \xi \leq t \text{ for every } t \in [a, b].\]

Now, let us notice that if \(h(t) \in C([0, b])\) and \(\frac{D^\alpha_t}{\Gamma(\alpha)} h(t) \in C([0, b])\) for \(0 < \alpha \leq 1\). The theorem implies that if \(\frac{D^\alpha_t}{\Gamma(\alpha)} h(t) \geq 0\) for every \(y \in (0, b)\), \(h(t)\) is non-decreasing. Furthermore, if \(\frac{D^\alpha_t}{\Gamma(\alpha)} h(t) \leq 0\) for every \(y \in (0, b)\), \(h(t)\) is non-increasing for every \(t \in [0, b]\).

**Theorem 3.1.** There exists a unique solution of (3) belonging to \(R^3_i\).

**Proof.**
\[
\begin{align*}
D^\alpha_t S(t)_{|t=a} & = M \geq 0, \\
D^\alpha_t I(t)_{|t=a} & = 0, \\
D^\alpha_t R(t)_{|t=a} & = 0.
\end{align*}
\]

After showing existence of non-negative solutions of the system, we present numerical solutions of (3). ☐

4. Computational solutions of SIRS model

Computational or numerical solutions of models in fractional calculus have been an active research area for scientists in recent years [21–24]. We employ a numerical solution method; namely, trapezoidal rule, introduced and applied in, e.g., [19], to obtain numerical solutions of the system (3). Now, let us state the system (3) as

\[
D^\alpha_t \chi(t) = g(\chi(t)), \quad \chi^{(j)}(t_0) = \chi_0^{(j)}, \quad j = 0, 1, \ldots, k - 1.
\]

where \(g \in C^3[0, T]\). Consider \(\Delta : = \frac{T}{m}\). The discretization of the functions at the nodal points are used in the sense of

\[
\{(t_i, \chi(t_i))\}_{i=0}^m = \{(t_i, \chi_i)\}_{i=0}^m, \quad t_i = i\Delta, \quad i = 0, 1, 2, \ldots, m.
\]

Approximate solution of \(u(t)\) depending on (\(\alpha, \Delta\)) is given by

\[
\begin{align*}
N(\alpha, \Delta) = & \sum_{i=0}^{m-1} \frac{(t - t_0)}{i!} \chi^{(i)}(t_0) + \frac{\Delta^\alpha g(c)}{\Gamma(\alpha + 2)} \\
& + \sum_{i=1}^{m-1} \frac{\Delta^{i+1} g(t_i)}{\Gamma(\alpha + 2)} b_{m-i},
\end{align*}
\]

in which

\[
a_m = \frac{\Delta^\alpha g(0)}{\Gamma(\alpha + 2)} \left[(-1 + m)^{\alpha+1} - m^\alpha (-1 + m - \alpha)\right],
\]

\[
b_{m-i} = (-1 + m - i)^{\alpha+1} - 2(m - i)^{\alpha+1} + (m + 1 - i)^{\alpha+1},
\]

for \(m = 1, 2, 3, \ldots\).

\(N(\alpha, \Delta) - E_\alpha(\alpha, \Delta, \Delta),\)

is approximate solution of (5) with

\[
|E_\alpha(\alpha, \Delta, \Delta)| \leq C(\alpha) ||g||_{\infty} \Delta^{2+\alpha} = O(\Delta^2).
\]

**Simulations**

The following parameters are considering: \(N = 150, a = 10/(58)(365)\) per year, \(k = 3/(84)(365)\) per year, \(\epsilon = 1/22\) per year, \(c = 1/23\) per year, \(d = 1/3\) per year and \(e = 0.24\) per year, with \(S(0) = 30, I(0) = 1.0\) and \(R(0) = 1.0\). All values were assumed. The dynamics of the new fractional SIRS model given by Eq. (3) for several values of \(\alpha \in (0, 1]\), arbitrarily chosen, are plotted in Fig. 1a–c.

5. Fractional-stochastic SIRS model

The mathematical models describing epidemic diseases are generated by deterministic, stochastic or fractional-order system of ordinary differential equations. Each of these models explains the advantages of employed differential equation in their system. Fractional-order models indicates that these types of models takes into account past effects of the system, stochastic models [27–29,52] emphasis on probabilistic transmissions of the diseases among the individuals in the population and so on. To the best of our knowledge, there exists not any mathematical model for a epidemic disease which considers both fractional-order operators and white noise together. Now, we perturb the fractional-order system (3) into a new system (mostly known as fractional-stochastic system) by an additive white noise to the rightmost side of each equation in the system. Our major aim in the section is to illustrate the use of both fractional-order operators and white noise together and test the effectiveness of resulting new model. It is possible to generate some other fractional-order stochastic SIRS models simply by a new type of fractional order operator or the changing the way of adding the noise to the system. We consider the perturbed fractional-order stochastic SIRS model:

\[
\begin{align*}
D^\alpha_t S(t) & = N - aS(t)I(t) + kR(t)S(t) - \xi S(t) - cI S(t)I(t) dW_1(t), \\
D^\alpha_t I(t) & = aS(t)I(t) - cI(t) - cS(t)I(t) dW_2(t), \\
D^\alpha_t R(t) & = dI(t)R(t) - eR(t) - cR(t)I(t) dW_3(t),
\end{align*}
\]

where \(dW_i\) is a Wiener process for each \(i = 1, 2, 3\). Disease-free point, obtained when \(I(t) = 0\), is the same point with the deterministic system (3). Stability of the system (6) may be investigated by Lyapunov stability method [25] which is applied to the system (6):

\[
\begin{align*}
S(t)D^\alpha_t S(t) + I(t)D^\alpha_t I(t) + R(t)D^\alpha_t R(t) & = S^2(t) + S(t)I(t) + S(t)R(t) - a^2S^2(t)I(t) + kR(t)S^2(t) - \xi S^2(t) + cI(S(t)I(t) dW_1(t) + aS(t)^2I(t) - c^2I(t)^2 + cS(t)^2I(t^2) dW_2(t) + dI(t)^2) - e^2R(t) \\
& + cI(t)R^2(t) dW_3(t) = S^2(t)(1 - a\Delta - \xi - cI(t) dW_1(t) + I(t) + R(t)) + I(t) \Delta I(t)(aS(t) - c - cS(t) dW_2(t)) \\
& + R^2(t)(dI(t) - e - cI(t) dW_3(t)) \leq 0.
\end{align*}
\]

Hence, the system (6) is stable to 0, when

\[
1 - a\Delta - \xi - cI(t) dW_1(t) + I(t) + R(t) < 0,
\]
\[
aS(t) - c - cS(t) dW_2(t) < 0,
\]
\[
dI(t) - e - cI(t) dW_3(t) < 0,
\]

6. Computational solutions

We present an approximate solution of (6). By employing an Euler type numerical method, discretized equations are expressed as

\[
\begin{align*}
S(n) & = \left[N(n) - a(n-1)S(n-1) + kS(n-1)R(n-1) - \xi S(n-1) - cI(n-1)\sqrt{\Delta t} \mu_0 + \frac{c^2}{2} S(n-1)I(n-1)(\mu_0^2 - 1)\Delta t \right]^{\alpha^n} \sum_{m=1}^{n} \tau(m)S(n-m) \\
I(n) & = \left[aS(n-1)I(n-1) - cI(n-1) + cS(n-1)I(n-1)\sqrt{\Delta t} \mu_0 + \frac{c^2}{2} S(n-1)I(n-1)(\mu_0^2 - 1)\Delta t \right]^{\alpha^n} \sum_{m=1}^{n} \tau(m)I(n-m) \\
R(n) & = \left[dI(n-1)R(n-1) - eR(n-1) - cR(n-1)\sqrt{\Delta t} \eta_0 + \frac{c^2}{2} R(n-1)I(n-1)(\eta_0^2 - 1)\Delta t \right]^{\alpha^n} \sum_{m=1}^{n} \tau(m)R(n-m)
\end{align*}
\]
where $\xi_n$, $\eta_n$, $\eta_n$ are Gaussian random variables $N(0, 1)$. The element, $\int \tau(m)A(n - m)$. $A \in \{S, I, R\}$, on the rightmost side of the equations represent the historical (or memory effects) of the mapping at each time step. $\tau$ is defined as

$$\tau(0) = 1, \ \tau(m) = \left[1 - \frac{1 + \alpha}{m}\right] \tau(m - 1), \ 1 \leq m,$$

with the initial conditions $S(0) = S_0$, $I(0) = I_0$, $R(0) = R_0$.

**Simulations**

Following parameters are considering: $N = 150$, $a = 10/(58)(365)$ per year, $k = 3/(84)(365)$ per year, $c = 1/22$ per year, $d = 1/3$ per year and $e = 0.24$ pear year, with $S(0) = 30$, $I(0) = 1.0$ and $R(0) = 1.0$, all values were assumed. The dynamics of the new fractional-order stochastic SIRS model given by Eq. (5) for several values of $\alpha \in (0, 1]$, arbitrarily chosen, are plotted in Fig. 2a-c.

**7. SIRS model with fractional Brownian motion**

Most of the SIR, SIS, SEIR, SEIRS type of epidemic disease models in the mathematical biology are generated by systems of ordinary differential equations. It is possible to assume that these models involve uncertainty, noise effects, undetermined outside forces and randomness in almost each time of the duration of the disease. These uncertainties may be strength of immune system of individuals, availability and resistance against vaccination, transmission rates of disease among individuals, and so on. The mathematical models in epidemiology is produced by means of either deterministic or stochastic systems of differential equations. We consider the deterministic fractional SIRS system (2) by an additive fractional white noise [30–33] which is stated via Wick product. By using the equality:

$$\frac{dB^H(t)}{dt} = W^H(t),$$

we present a numerical solution of the Wick product added SIRS model with an Euler type numerical method. In the history of research so far, there exists not any mathematical model considering Wick product interpretation of any epidemic disease. From this point of view, the present study is the first research work associating Wick product with a SIRS model.

Fractional Brownian motion (fBm) is a process introduced by Kolmogorov [41], and employed by Mandelbrot [42]. fBm is a Gaussian process denoted typically by $B^H(t)$ depending on the Hurst index, $H$, located in $0 < H < 1$. fBm has zero mean and covariance function defined as

$$E\left[B^H(t)B^H(u)\right] = \frac{1}{2}\left[u^{2H} + t^{2H} - |t - u|^{2H}\right]. \ t,u \in [0, \infty),$$

when $H = \frac{1}{2}$; $B^{1/2}(t)$ is a standard Wiener process or Brownian motion. Some interesting properties of fBm including self-similarity, being centered Gaussian process, having stationary increments, homogeneity in time, being symmetric and so on may be seen in, e.g. [43–46,50].

Another quite useful and important tool in fBm and its applications is a product known as Wick product. It is a paired (binary) type operation, typically denoted with a symbol $\diamond$, and it is employed in famous Wiener-Ito-Chaos decomposition of function in $L^2$–measurable space. Wick product is a significant phenomena in stochastic and fractional stochastic calculus, mostly applied in the solutions of nonlinear differential equations in financial mathematics and fluid mechanics. A detailed treatment to Wick prod-
uct and its applications can be seen in Kaligotla et al. [34], Duncan et al. [35], Levajkovic et al. [36], Parczewski [37], Biagini et al. [38], Lebovits et al. [39], Kim et al. [49]. In this paper, we employ the Wick product in the fractional Brownian motion settings to represent fractional white noise. Addressing an fractional Wick-ito integral, following rule holds:

$$\int_{R}Y(t,w)dB^{H}(t) = \int_{R}Y(t)\circ W^{H}(t)dt = \int MY(tdB^{H}(t)),$$

in which the operator $M$ is defined as

$$Mf(x) = C_{H}\int_{R}\frac{f(t)}{|t-x|^{1/2-\pi}}dt,$$

where,

$$C_{H} = [2\Gamma(H-1/2)\cos(1/2)\pi (H-1/2)]^{-1}[\Gamma(2H+1)\sin\pi(H)]^{1/2},$$

and

$$H_{n,\beta}(w) = \int_{R}\eta_{n}(t)dB(t).$$

in which

$$\eta_{n}(x) = \pi^{-1/4}((n-1)!)^{-1/2}h_{n-1}(\sqrt{2}x)e^{-x^{2}/2}, \quad n = 1,2,\ldots$$

The $h_{n}(x)$ defined in this equation is given as

$$h_{n}(x) = (-1)^{n}x^{n/2}\frac{d^{n}}{dx^{n}}(e^{x^{2}/2}), \quad n = 0,1,2,\ldots$$

Fractional white noise $W^{H}(t)$ also has a series representation given by

$$W^{H}(t) = \sum_{k=1}^{\infty}M\eta_{k}(t)H_{0,\beta}(w),$$

where the operator $M$ is defined by Eq. (10).

Now, by adding fractional white noise to the terms in the right-hand side of the system of equations (2), we get

$$\frac{dS(t)}{dt} = N - a(t)S(t) + kR(t)S(t) - \xi S(t)\circ W^{H}(t),$$

$$\frac{dI(t)}{dt} = al(t)S(t) - cl(t)\circ W^{H}(t),$$

$$\frac{dR(t)}{dt} = dl(t)R(t) - eR(t)\circ W^{H}(t),$$

with some suitable supplementary conditions $S(0) = S_{0}$, $I(0) = I_{0}$, $R(0) = R_{0}$. This system may be restated as

$$\frac{dS(t)}{dt} = (N - al(t)S(t) + kR(t)S(t))dt - \xi S(t)\circ W^{H}(t)dt,$$

$$\frac{dI(t)}{dt} = al(t)S(t)dt - cl(t)\circ W^{H}(t)dt,$$

$$\frac{dR(t)}{dt} = dl(t)R(t)dt - eR(t)\circ W^{H}(t)dt.$$

Writing in the integral form:

$$S(t) = S_{0} + \int_{0}^{t}(N - aS(s)I(s) + kR(s)S(s))ds - \xi \int_{0}^{t}S(s)\circ W^{H}(s)ds,$$

$$I(t) = I_{0} + \int_{0}^{t}aS(s)I(s)ds - \int_{0}^{t}cl(s)\circ W^{H}(s)ds,$$

$$R(t) = R_{0} + \int_{0}^{t}dl(s)R(s)ds - \int_{0}^{t}eR(s)\circ W^{H}(s)ds.$$

It is further possible that we can express system (13) as

$$S(t) = S_{0} + \int_{0}^{t}(N - aS(s)I(s) + kR(s)S(s))ds - \xi \int_{0}^{t}S(s)\circ dB^{H}(s)ds,$$

$$I(t) = I_{0} + \int_{0}^{t}aS(s)I(s)ds - \int_{0}^{t}cl(s)\circ dB^{H}(s)ds,$$

(14)
\[
R(t) = R_0 + \int_0^t dI(s) R(s) ds - \int_0^t eR(s) \circ dB^H(s) ds.
\]

Now, we solve the system (14) numerically employing an Euler's type of numerical solution technique as follows:

\[
S_n(t) = S_{n-1}(t) + S_0 - \left[ N - aS_{n-1}(t)I_{n-1}(t) + kR_{n-1}(t)S_{n-1}(t) \right] (t_n - t_{n-1})
\]

\[
-\xi S_{n-1}(t) \circ (B^H_n(t) - B^H_{n-1}(t)).
\]

\[
I_n(t) = I_{n-1}(t) + I_0 + aS_{n-1}(t)I_{n-1}(t)(t_n - t_{n-1}) + cI_{n-1}(t) \circ (B^H_n(t) - B^H_{n-1}(t)).
\]

\[
R_n(t) = R_{n-1}(t) + R_0 + dI_{n-1}(t) R_{n-1}(t)(t_n - t_{n-1}) + eR_{n-1}(t) \circ (B^H_n(t) - B^H_{n-1}(t)).
\]

(15)

Simulations

The following parameters are considering: \( N = 150, a = 10/(58)(365) \) per year, \( k = 3/(84)(365) \) per year, \( \epsilon = 1/22 \) per year, \( c = 1/23 \) per year, \( d = 1/3 \) per year and \( e = 0.24 \) per year, with \( S(0) = 30, I(0) = 1.0 \) and \( R(0) = 1.0 \), all values were assumed. The dynamics of the new SIRS model with additive fractional white noise stated in terms of the Wick product given by Eq. (15) for several values of \( H \in (0, 1] \), arbitrarily chosen, are plotted in Fig. 3a–c.

8. Explicit solutions of fractional stochastic SIR model

In this section, we present two different exact (or sometimes known as explicit) solutions of the system given by (11). In the first technique that we call it direct method, we solve the equations in the system (11) separately by considering each equation in the system alone. Now, the first stochastic equation

\[
dS(t) = \left[ N - aI(t)S(t) + kR(t)S(t) \right] dt - \Xi S(t) \circ W^H(t) dt.
\]

appearing in system of equations (11). By using the identity given for total population of all individuals, we get that

\[
dS(t) = (\rho S(t) + \rho I(t) + \rho R(t) - aS(t)I(t) + kR(t)S(t)) dt - \Xi S(t) \circ W^H(t) dt.
\]

Without loss of generality, let us ignore the term \( \rho I(t) + \rho R(t) \), since we deal with the population of susceptible individuals and by the fact that infected and recovered individuals are opposite to one another. Then, \( dS(t) \) may be written as

\[
dS(t) = [\rho S(t) - aI(t)S(t) + kR(t)S(t)] dt - \Xi S(t) \circ W^H(t) dt.
\]

\[
\frac{dS(t)}{dt} = S(t) \circ \left[ \rho - aI(t) + kR(t) - \Xi W^H(t) \right].
\]

\[
S(t) = S_0 \circ \exp \left[ \int_0^t (\rho - aI(s) + kR(s)) ds - \Xi \int_0^t dB^H(s) \right].
\]

\[
S(t) = S_0 \circ \exp \left[ \int_0^t (\rho - aI(s) + kR(s)) ds - \Xi B^H(t) \right].
\]

In a similar manner, the second equation

\[
dI(t) = aI(t)S(t) dt - cI(t) \circ W^H(t) dt.
\]
in the system (11) may be expressed as
\[
\frac{dI(t)}{dt} = I(t) \circ \left[ aS(t) - cW^H(t) \right].
\]
\[
I(t) = I_0 \circ \exp \left[ \int_0^t a(s)ds - c \int_0^t dB^H(s) \right].
\]
\[
I(t) = I_0 \circ \exp \left[ \int_0^t a(s)ds - cB^H(t) \right].
\]

Finally, the third equation
\[
dR(t) = dR(t)R(t)dt - eR(t) \circ W^H(t)dt,
\]
in the system (11) may be expressed as
\[
\frac{dR(t)}{dt} = R(t) \circ \left[ dI(t) - e\Psi(t) \right].
\]
\[
R(t) = R_0 \circ \exp \left[ \int_0^t I(s)ds - e \int_0^t dB^H(s) \right].
\]
\[
R(t) = R_0 \circ \exp \left[ \int_0^t I(s)ds - eB^H(t) \right].
\]

Next, we present exact (or explicit) solutions of the system (2) having multiplicative fractional white noise by another method; namely, semi-martingale method [47].

Let us write the equations of the system (2) with multiplicative fractional white noise. First equation in the system (2):
\[
\frac{dS(t)}{dt} = S(t) + I(t) + R(t) - aS(t)I(t) + kR(t)S(t) - \xi S(t) \frac{dB^H(t)}{dt},
\]
\[
\frac{dS(t)}{dt} = [I(t) + R(t) + (1 + kR(t) - aI(t))S(t)]dt - \xi S(t)dB^H(t),
\]
from which \(S(t)\) is obtained as
\[
S(t) = \Psi_1(t) \left[ S_0 + \int_0^t \left[ S(s) + I(s) \right] \left( \Psi_1^{-1} \right)(s)ds \right],\]
where,
\[
\Psi_1(t) = \exp \left[ \int_0^t [1 + kR(s) - aI(s)]ds + \xi B^H(t) \right].
\]

Now, we express the second equation in the system (2) with multiplicative fractional white noise as follows:
\[
\frac{dI(t)}{dt} = aS(t)I(t) - cI(t) \frac{dB^H(t)}{dt},
\]
\[
\frac{dI(t)}{dt} = [I(t) + (aS(t) - 1)I(t)]dt - cI(t)dB^H(t),
\]
from which \(I(t)\) is obtained as
\[
I(t) = \Psi_2(t) \left[ I_0 + \int_0^t I(s) \Psi_2^{-1}(s)ds \right],\]
in which
\[
\Psi_2(t) = \exp \left[ \int_0^t (aS(s) - 1)ds - cB^H(t) \right].
\]

Finally, the third equation in the system (2) with multiplicative fractional white noise is:
\[
\frac{dR(t)}{dt} = dR(t)I(t) - eR(t) \frac{dB^H(t)}{dt},
\]
\[
\frac{dR(t)}{dt} = [R(t) + (dI(t) - 1)R(t)]dt - eR(t)dB^H(t),
\]
from which \(R(t)\) is obtained as
\[
R(t) = \Psi_3(t) \left[ R_0 + \int_0^t R(s) \Psi_3^{-1}(s)ds \right],\]
in which
\[
\Psi_3(t) = \exp \left[ \int_0^t (dI(s) - 1)ds - eB^H(t) \right].
\]

9. Conclusions and outlook

Fractional-stochastic differential equations and fractional Brownian motion are very powerful modeling tools in science, engineering and economics. They found a very widespread applications in each of these areas. In the present research study, we pointed an original SIRS model for a modeling of a possible epidemic disease under certain assumptions. We study on chaos (stability) analysis, numerical and explicit solutions of fractional-stochastic SIRS models in fractional calculus and fractional Brownian motion settings.

As a future extension of the present research work, we plan to set up connections between SIRS types models and control optimization problems. Furthermore, because each of these epidemic disease models include parameters, we will study on problems of parameter estimation, identification and parameter sensitivity analyses of fractional-stochastic models in mathematical biology.

Declaration of Competing Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

CRediT authorship contribution statement

M.A. Akinlar: Conceptualization, Data curation. Mustafa Inc: Formal analysis. J.F. Gómez-Aguilar: Writing - original draft, Visualization. B. Bourtafa: Formal analysis, Writing - original draft.

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