Let $T$ be a countable abelian group and let $(X,T)$ be a minimal flow; i.e. $X$ is a compact Hausdorff space and $T$ acts on it as a group of homeomorphisms in such a way that for each $x \in X$ its $T$-orbit, $Tx = \{tx : t \in T\}$, is dense in $X$. Following [6] and [9] we introduce the following notations (generalizing from the case $T = \mathbb{Z}$ to the case of a general $T$ action). For an integer $d \geq 1$ let $X^d = X^{2d}$. We index the coordinates of an element $x \in X^d$ by subsets $\epsilon \subseteq \{1, \ldots, d\}$. Thus $x = (x_\epsilon : \epsilon \subseteq \{1, \ldots, d\})$, where for each $\epsilon$, $x_\epsilon \in X_\epsilon = X$. E.g. for $d = 2$ we have $x = (x_{\emptyset}, x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}})$.

Let $\pi_* : X^d \to X^{2d-1}$ denote the projection onto the last $2^d - 1$-coordinates; i.e., the map which forgets the $\emptyset$-coordinate. Let $X_*^d = \pi_*(X^d) = X^{2^d-1} = \prod \{X_\epsilon : \epsilon \neq \emptyset\}$ and for $x \in X^d$ let $x_* = \pi_*(x) \in X_*^d$ denote its projection; i.e. $x_*$ is obtained by omitting the $\emptyset$-coordinate of $x$. For each $\epsilon \subseteq \{1, \ldots, d\}$ we denote by $\pi_\epsilon$ the projection map from $X^d$ onto $X_\epsilon = X$. For a point $x \in X$ we let $x^d \in X^d$ and $x_*^d \in X_*^d$ be the diagonal points all of whose coordinates are $x$. $\Delta^d = \{x^d : x \in X\}$ is the diagonal of $X^d$ and $\Delta_*^d = \{x_*^d : x \in X\}$ the diagonal of $X_*^d$. Another convenient representation of $X^d$ is as a product space $X^d = X^{d-1} \times X^{d-1}$ (with $X^{[0]} = X$). When using this decomposition we write $x = (x', x'')$. More explicitly, for for $\epsilon \subseteq \{1, \ldots, d-1\}$ let $\epsilon d = \epsilon \cup \{d\}$, and define the identification $X^{d-1} \times X^{d-1} \to X^d$ by $(x', x'') \mapsto x$ with $x_\epsilon = x'_\epsilon$ and $x_{\epsilon d} = x''_\epsilon$. We will refer to $x'$ and $x''$ as the first and second $2^{d-1}$ coordinates, respectively.

We next define two group actions on $X^d$, the face group action $F_d$ and the total group action $G_d$. These actions are representations of $T^d = T \times T \times \cdots \times T$ ($d$ times) and $T^{d+1}$, respectively, as subgroups of Homeo $(X^d)$. For the $F_d$ action, $F_d \times X^d \to X^d$,

\[ (t_1, \ldots, t_d), (x_\epsilon : \epsilon \subseteq \{1, \ldots, d\}) \mapsto (t_\epsilon x_\epsilon : \epsilon \subseteq \{1, \ldots, d\}), \]

Let $\mathbb{R}P^d$ IS AN EQUIVALENCE RELATION AN ENVELOPING SEMIGROUP PROOF

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Abstract. We present a purely enveloping semigroup proof of a theorem of Shao and Ye which asserts that for an abelian group $T$, a minimal flow $(X,T)$ and any integer $d \geq 1$, the regional proximal relation of order $d$ is an equivalence relation.

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where \( t_{\epsilon}x_{\epsilon} = t_{n_{1}} \cdots t_{n_{k}}x_{\epsilon} \), if \( \epsilon = \{n_{1}, \ldots, n_{k}\} \) and \( t_{\emptyset}x_{\emptyset} = x_{\emptyset} \). We can then represent the homeomorphism \( \tau \in \mathcal{F}_d \) which corresponds to \((t_{1}, \ldots, t_{d}) \in T^{d}\) as

\[
\tau = \tau_{(t_{1}, \ldots, t_{d})}^{[d]} = (t_{\epsilon} : \epsilon \subset \{1, \ldots, d\}).
\]

We will also consider the restriction of the \( \mathcal{F}_d \) action to \( X^{\prime [d]}_d \) which is defined by omitting the \( \emptyset \) coordinate. Note that under the action of \( \mathcal{F}_d \) on \( X^{[d]} \) the \( \emptyset \)-coordinate is left fixed.

For example, if we consider a minimal cascade \((X, f)\), taking \( T = \mathbb{Z} = \{f^{n} : n \in \mathbb{Z}\}, d = 3 \) and \( \tau = \tau_{(2,5,11)}^{[3]} \in \mathcal{F}_3 \cong \mathbb{Z}^{3}, \) we have:

\[
\tau(x) = (x_{\emptyset}, f^{2}x_{\{1\}}, f^{5}x_{\{2\}}, f^{2+5}x_{\{1,2\}}, f^{11}x_{\{3\}}, f^{2+11}x_{\{1,3\}}, f^{2+5+11}x_{\{1,2,3\}}),
\]

and

\[
\tau(x_{\epsilon}) = (f^{2}x_{\{1\}}, f^{5}x_{\{2\}}, f^{2+5}x_{\{1,2\}}, f^{11}x_{\{3\}}, f^{2+11}x_{\{1,3\}}, f^{2+5+11}x_{\{1,2,3\}}).
\]

Note that the fact that the \( \mathcal{F}_d \) action is well defined depends on the commutativity of the group \( T \).

The action of \( T^{d+1} \) on \( X^{[d]} \), denoted by \( \mathcal{S}_d \), is the action generated by the face group action \( \mathcal{F}_d \) and the diagonal \( \emptyset \)-action of \( T, T \times X^{[d]} \rightarrow X^{[d]} \), defined by

\[
(t, x) \mapsto \theta_{t}^{[d]}x = (tx_{\epsilon} : \epsilon \subset \{1, \ldots, d\}).
\]

Thus for the \( \mathcal{S}_d \) action \( \mathcal{S}_d \times X^{[d]} \rightarrow X^{[d]} \),

\[
\left((t_{1}, \ldots, t_{d}, t_{d+1}), (x_{\epsilon} : \epsilon \subset \{1, \ldots, d\})\right) \mapsto (t_{d+1}t_{\epsilon}x_{\epsilon} : \epsilon \subset \{1, \ldots, d\}),
\]

where \( t_{\epsilon}x_{\epsilon} = t_{n_{1}} \cdots t_{n_{k}}x_{\epsilon} \), if \( \epsilon = \{n_{1}, \ldots, n_{k}\} \) and \( t_{\emptyset}x_{\emptyset} = x_{\emptyset} \). In other words, the \( \mathcal{S}_d \) action on \( X^{[d]} \) is given by the representation:

\[
T^{d+1} \rightarrow \text{Homeo} (X^{[d]}), \quad (t_{1}, \ldots, t_{d}, t_{d+1}) \mapsto \theta_{t_{d+1}}^{[d]} \tau_{(t_{1}, \ldots, t_{d})}^{[d]}.
\]

Notice that

\[
\tau_{(t_{1}, \ldots, t_{d})}^{[d]}(x_{\epsilon}, x_{\prime}) = (\tau_{(t_{1}, \ldots, t_{d-1})}^{[d-1]}x_{\epsilon}, \theta_{t_{d}}^{[d-1]} \tau_{(t_{1}, \ldots, t_{d-1})}^{[d-1]}x_{\prime}).
\]

In their paper [9] Shao and Ye prove that \( RP^{[d]} \), the generalized regionally proximal relation of order \( d \), is always an equivalence relation for a minimal cascade \((X, T)\). Their proof is based on the detailed analysis of the \( \mathcal{S}_d \) action provided by Host Kra and Mass in [6], where the authors treated the distal case. The main tool used by Shao and Ye is a theorem which asserts that, for each \( x \in X \), the action of the face group \( \mathcal{F}_d \) on the orbit closure \( \text{cls } \mathcal{F}_d x_\epsilon^{[d]} \) is minimal. Their proof of this latter theorem was based on the general structure theory of minimal flows due to Ellis-Glasner-Shapiro [3], McMahon [8] and Veech [10]. Now, it turns out that there is a direct enveloping semigroup proof of this theorem which is very similar to the proof by Ellis and Glasner given in [7, page 46]. The possibility of applying the Ellis-Glasner proof as a shortcut to Shao and Ye’s proof was also discovered by Ethan Akin. In the next section I present this short proof, established for a general commutative group. For the interested reader I will, in a subsequent section, reproduce the beautiful proof of Shao and Ye of the fact that for each \( d \geq 1 \), \( RP^{[d]} \) is an equivalence relation.

Let us note that all the results of this work extend to the case where \( T \) is a commutative separable topological group which acts continuously and minimally on
a compact metric space \( X \). In fact, both minimality and the regionally proximal
relations are the same for the group and for a countable dense subgroup. Moreover,
most of the results (like Theorem 1.1, parts (1) - (3), as well as Theorem 2.5) hold for
actions of \( T \) on a general compact space (not necessarily metrizable). I wish to thank
Ethan Akin whose suggestions led to improvements of a first draft of this work.

1. The minimality of the face action on \( Q_x^d \)

Let \((X,T)\) be a minimal flow with \( T \) abelian. Let

\[
Q^d = \text{cls} \left\{ gx^d : x \in X, \ g \in S_d \right\} = \overline{S_d \Delta^d} = \overline{F_d \Delta^d},
\]

\[
Q_x^d = Q^d \cap (\{x\} \times X^{d-1}), \quad \text{and} \quad Q_x^d = \pi_*(Q_x^d).
\]

For each \( x \in X \) let \( Y_x^d = \overline{F_d(x^d)} \subset Q_x^d \) be the orbit closure of \( x^d \) under \( F_d \). Finally,
let \( Y_x^d = \pi_*(Y_x^d) \).

1.1. Theorem (Shao and Ye). 1. The flow \((Q^d,S_d)\) is minimal.

2. For each \( x \in X \), the flows \((Y_x^d,F_d)\), and hence also \((Y_x^d,F_d)\), are minimal.

3. For each \( x \in X \) the set \( Y_x^d \) is the unique minimal subflow of the \( F_d \)-flow
\((Q_x^d,F_d)\). Hence also \( Y_x^d \) is the unique minimal subflow of the \( F_d \)-flow \((Q_x^d,F_d)\).

4. \(^*\) For a dense \( G_\delta \) subset \( X_0 \subset X \) we have \( Y_x^d = Q_x^d \).

Proof. 1. Let us denote \( N := Q^d \) and \( \mathcal{J} := S_d \). Let \( E = E(N,\mathcal{J}) \) be the enveloping
semigroup of \((N,\mathcal{J})\). Let \( \pi_\epsilon : N \to X_\epsilon = X \) be the projection of \( N \) on the \( \epsilon \) coordinate,
where \( \epsilon \subset \{1,...,d\} \). We consider the action of the group \( \mathcal{J} \) on the \( \epsilon \) coordinate via the
projection \( \pi_\epsilon \), that is, for \( \epsilon \subset \{1,...,d\}, (t_1,...,t_d,t_{d+1}) \in T^{d+1} \)
and \( x \in X_\epsilon = X \),

\[
\mathcal{J} \times X_\epsilon \to X_\epsilon, \quad (\theta_{i+1}^{[d]} \tau_{[d]}^{(t_1,...,t_d)}, x) \mapsto t_d t_{d+1} x.
\]

With respect to this action of \( \mathcal{J} \) on \( X_\epsilon = X \) the map \( \pi_\epsilon : (N,\mathcal{J}) \to (X_\epsilon,\mathcal{J}) \)
\( X \) is a flow homomorphism. Let \( \pi_\epsilon^* : E(N,\mathcal{J}) \to E(X_\epsilon,\mathcal{J}) \) be the corresponding homomorphism
of enveloping semigroups. Notice that for the action of \( \mathcal{J} \) on \( X_\epsilon \), \( E(X_\epsilon,\mathcal{J}) = E(X,T) \)
as subsets of \( X^X \) (as \( t_{d+1}, t_{d+1} \in T \)).

Let now \( u \in E(X,T) \) be any minimal idempotent. Then \( \widetilde{u} = (u,u,...,u) \in E(N,\mathcal{J}) \).
Choose \( v \) a minimal idempotent in the closed left ideal \( E(N,\mathcal{J})\tilde{u} \). Then \( \nu \tilde{u} = v \). We
want to show that \( \nu \tilde{u} = \nu \tilde{u} \). Set, for \( \epsilon \subset \{1,...,d\}, \nu_\epsilon = \pi_\epsilon^* v \). Note that, as an
element of \( E(N,\mathcal{J}) \) is determined by its projections, it suffices to show that for each \( \epsilon, \ \nu \epsilon = u \).
Since \( \nu_\epsilon = u \), the map \( \pi_*^\epsilon \) is a semigroup homomorphism, we have that \( \nu_\epsilon u = v \) as \( \nu \tilde{u} = v \).
In particular we deduce that \( \nu_\epsilon \) is an idempotent belonging to
the minimal left ideal \( E(X_\epsilon,T) = E(X,T) \) which contains \( u \). This implies (see [7,
Exercise 1.23.2.(b)]) that

\[
u \epsilon = u,
\]

and it follows that indeed \( \nu \tilde{u} = \nu \). Thus, \( \tilde{u} \) is an element of the minimal left ideal \( E(N,\mathcal{J})\tilde{v} \) which contains \( v \),
and therefore \( \tilde{u} \) is a minimal idempotent of \( E(N,\mathcal{J}) \).

Now let \( x \in X \) and let \( u \) be a minimal idempotent in \( E(X,T) \) with \( u x = x \) (since \( (X,T) \) is minimal there always exists such an idempotent). By the above argument, \( \tilde{u} \)

\(^*\)This seems to be a new observation.
is also a minimal idempotent of \((N, T)\) which implies that \(N = Q^{[d]}\), the orbit closure of \(x^{[d]} = \tilde{u}x^{[d]}\), is \(T\) minimal (see [7, Exercise 1.26.2]).

2. Given \(x \in X\) we now let \(N := Q^{[d]}_{x_n}\) and \(T := T_{d}\). The proof of the minimality of the flow \((Q^{[d]}_{x_n}, T_{d})\) is almost verbatim the same, except that here the claim that for \(u\) a minimal idempotent in \(E(X, T)\), the map \(\tilde{u} = (u, u, \ldots, u)\) \((2^d - 1\) times) is in \(E(Q^{[d]}_{x_n}, T_{d})\), is not that evident. However, as \(u\) is an idempotent this fact follows from the following lemma (with \(p_1 = \cdots = p_d = u\)).

1.2. Lemma. Let \(p_1, \ldots, p_d \in E(X, T)\) and for \(\epsilon = \{n_1, \ldots, n_k\} \subset \{1, \ldots, d\}\), with \(n_1 < \cdots < n_k\), let \(q_\epsilon = p_{n_k} \cdots p_{n_1}\). Then the map \((q_\epsilon : \epsilon \subset \{1, \ldots, d\}, \epsilon \neq \emptyset)\) is an element of \(E(Q^{[d]}_{x_n}, T_{d})\).

Proof. By induction on \(d\), using the identity (1), or more specifically

\[
\tau^{[d]}_{(e, \ldots, e, t_d)}(x', x'') = (x', \theta^{[d-1]}_d x''),
\]

and the fact that right multiplication in \(E(X, T)\) is continuous. \(\square\)

3. We first reproduce the ingenious “useful lemma” from [9].

1.3. Lemma. If \((x^{[d-1]}, w) \in Q^{[d]}\) for some \(x \in X\) and \(w \in X^{[d-1]}\) and \((x^{[d-1]}, w)\) is an \(T_{d}\)-minimal point, then \((x^{[d-1]}, w) \in Y^{[d]}_x\).

Proof. Since \((x^{[d-1]}, w) \in Q^{[d]}\) it follows that \((x^{[d-1]}, w)\) is in the \(S_{d}\)-orbit closure of \(x^{[d]}\), i.e. there is a sequence \(\{(t_{1k}, \ldots, t_{dk}, t_{d+1k})\}_k \subset T^{d+1}\) such that

\[
\theta^{[d]}_{t_{d+1k}} \tau^{[d]}_{(t_{1k}, \ldots, t_{dk})}(x^{[d]}) \to (x^{[d-1]}, w).
\]

Now

\[
\theta^{[d]}_{t_{d+1k}} \tau^{[d]}_{(t_{1k}, \ldots, t_{dk})}(x^{[d]}) = \theta^{[d]}_{t_{d+1k}} (\text{id}^{[d-1]} \times \theta^{[d-1]}_{t_{dk}}) \tau^{[d-1]}_{(t_{1k}, \ldots, t_{d-1k})}(x^{[d-1]}),
\]

and letting \(a_k := \theta^{[d-1]}_{t_{d+1k}} \tau^{[d-1]}_{(t_{1k}, \ldots, t_{d-1k})}(x^{[d-1]}),\) we have:

\[
(\text{id}^{[d-1]} \times \theta^{[d-1]}_{t_{dk}})(a_k, a_k) \to (x^{[d-1]}, w).
\]

Let

\[
\pi_1 : (X^{[d]}, T_{d}) \to (X^{[d-1]}, T_{d}), \quad (x', x'') \mapsto x',
\]

\[
\pi_2 : (X^{[d]}, T_{d}) \to (X^{[d-1]}, T_{d}), \quad (x', x'') \mapsto x'',
\]

be the projections to the first and last \(2^{d-1}\) coordinates respectively. For \(\pi_1\) we consider the action of the group \(T_{d}\) on \(X^{[d-1]}\) via the representation

\[
\tau^{[d-1]}_{(t_1, \ldots, t_d)} \mapsto \tau^{[d-1]}_{(t_1, \ldots, t_{d+1})},
\]

and for \(\pi_2\) the action is via the representation

\[
\tau^{[d-1]}_{(t_1, \ldots, t_d)} \mapsto \theta^{[d-1]}_{t_d} \tau^{[d-1]}_{(t_1, \ldots, t_{d+1})}.
\]
Denote the corresponding semigroup homomorphisms of enveloping semigroups by
\[ \pi_i^*: E(X^{[d]}, F_d) \rightarrow E(X^{[d-1]}, F_d), \quad i = 1, 2. \]

Notice that for these actions of $F_d$ on $X^{[d-1]}$, as subsets of $X^{[d]} X^{[d]}$,
\[ \pi_1^*(E(X^{[d]}, F_d)) = E(X^{[d-1]}, F_{d-1}) \quad \text{and} \quad \pi_2^*(E(X^{[d]}, F_d)) = E(X^{[d-1]}, S_{d-1}). \]

Thus for $p \in E(X^{[d]}, F_d)$ and $x \in X^{[d]}$, we have:
\[ px = p(x', x'') = (\pi_1^*(p)x', \pi_2^*(p)x''). \]

Now fix a minimal left ideal $L$ of $E(X^{[d]}, F_d)$. By (2) $a_k \rightarrow x^{[d-1]}$ and, since $(Q^{[d-1]}, S_{d-1})$ is minimal, there exist $p_k \in L$ such that $a_k = \pi_2^*(p_k)x^{[d-1]}$. Without loss of generality we assume that $p_k \rightarrow p \in L$. Then
\[ \pi_2^*(p_k)x^{[d-1]} = a_k \rightarrow x^{[d-1]} \quad \text{and} \quad \pi_2^*(p_k)x^{[d-1]} \rightarrow \pi_2^*(p)x^{[d-1]}.
\]

Hence
\[ \pi_2^*(p)x^{[d-1]} = x^{[d-1]}. \]

Since $L$ is a minimal left ideal and $p \in L$ there exists a minimal idempotent $v \in J(L)$ such that $vp = p$. Then
\[ \pi_2^*(v)x^{[d-1]} = \pi_2^*(v)\pi_2^*(p)x^{[d-1]} = \pi_2^*(vp)x^{[d-1]} = \pi_2^*(p)x^{[d-1]} = x^{[d-1]}. \]

Let
\[ F = G(F_{d-1}(x^{[d-1]}), x^{[d-1]}) = \{ \alpha \in vL : \pi_2^*(\alpha)x^{[d-1]} = x^{[d-1]} \} \]
be the Ellis group of the pointed flow $(F_{d-1}(x^{[d-1]}), x^{[d-1]})$. Then $F$ is a subgroup of the group $vL$. By (3), we have $p \in F$ and since $F$ is a group, we have $pFx^{[d]} = Fx^{[d]} \subset \pi_2^{-1}(x^{[d]}).$ Since $vx^{[d]} \in Fx^{[d]} = pFx^{[d]}$, there is some $x_0 \in Fx^{[d]}$ such that $vx^{[d]} = px_0$. Set $x_k = p_kx_0$, then
\[ x_k = p_kx_0 \rightarrow px_0 = vx^{[d]} = (\pi_1^*(v)x^{[d-1]}, x^{[d-1]}). \]

As $x_0 \in Fx^{[d]}$, it follows that $\pi_2(x_0) = x^{[d-1]}$, hence
\[ \pi_2(x_k) = \pi_2(p_kx_0) = \pi_2^*(p_k)\pi_2(x_0) = \pi_2^*(p_k)x^{[d-1]} = a_k = x^{[d-1]}.
\]

Let $x_k = (b_k, a_k) \in F_d(x^{[d]}); \text{then, by (4), } \lim b_k = \pi_1^*(v)x^{[d-1]}.$ By (2), $\theta_{t_{dk}}^{[d-1]}a_k \rightarrow w$, hence
\[ (id^{[d-1]} \times \theta_{t_{dk}}^{[d-1]})(b_k, a_k) = (b_k, \theta_{t_{dk}}^{[d-1]}a_k) \rightarrow (\pi_1^*(v)x^{[d-1]}, w). \]

Since $id^{[d-1]} \times \theta_{t_{dk}}^{[d-1]} = \tau_{(e, \ldots, e, t_{dk})}^{[d]} \in F_d$ and $(b_k, a_k) \in F_d(x^{[d]}),$ we have
\[ (\pi_1^*(v)x^{[d-1]}, w) \in \overline{F_d(x^{[d]})}. \]

Since, by assumption, $(x^{[d-1]}, w)$ is $F_d$ minimal, there is some minimal idempotent $u \in J(L)$ such that $u(x^{[d-1]}, w) = (\pi_1^*(u)x^{[d-1]}, \pi_2^*(u)w) = (x^{[d-1]}, w)$. Since $u, v \in L$ are minimal idempotents in the same minimal left ideal $L$, we have $uw = u$. Thus
\[ u(\pi_1^*(v)x^{[d-1]}, w) = (\pi_1^*(u)\pi_1^*(v)x^{[d-1]}, \pi_2^*(u)w) = (\pi_1^*(uv)x^{[d-1]}, w) = (\pi_1^*(u)x^{[d-1]}, w) = (x^{[d-1]}, w). \]

By (5), we have $(x^{[d-1]}, w) \in \overline{F_d(x^{[d]})}$ and the proof of the lemma is completed. \qed
We are now ready to complete the proof of part (3) of the theorem. We assume by induction that this assertion holds for every $1 \leq j \leq d-1$ and now, given $x \in X$, consider a minimal subflow $Y$ of the flow $(Q_x^{[d]}, \mathcal{F}_d)$. With notations as in the previous lemma, we observe that $Y_1 = \pi_1(Y)$ is a minimal subflow of the flow $(Q_x^{[d-1]}, \mathcal{F}_{d-1})$ and therefore, by the induction hypothesis $Y_1 = Y_x^{[d-1]} = \mathcal{F}_{d-1} x^{[d-1]}$. But then for some $w \in Q^{[d-1]}$ we have $(x^{[d-1]}, w) \in Y$ and, applying Lemma 1.3, we conclude that $(x^{[d-1]}, w) \in Y_x^{[d]}$. Thus $Y = Y_x^{[d]}$ and the proof is complete.

4. Let $2^X$ be the compact hyperspace consisting of the closed subsets of $X$ equipped with the (compact metric) Vietoris topology. Let $\Phi : X \to 2^X$ be the map $x \mapsto Y_x^{[d]}$. It is easy to check that this map is lower-semi-continuous (i.e. $x_1 \to x \Rightarrow \lim \inf \Phi(x_i) \supset \Phi(x)$). It follows then that the set of continuity points of $\Phi$ is a dense $G_\delta$ subset $X_0 \subset X$ (see e.g. [2]). Since the set $\mathcal{F}_d X^{[d]}$ is dense in $Q^{[d]}$, it follows that at each point of $X_0$ we must have $Y_x^{[d]} = Q_x^{[d]}$. \hfill \Box

2. $RP^{[d]}$ IS AN EQUIVALENCE RELATION

In this section we outline the Shao-Ye proof that $RP^{[d]}$ is an equivalence relation. We assume that $(X, T)$ is a minimal compact metrizable $T$-flow, where $T$ is an abelian group. We fix a compatible metric $\rho$ on $X$.

2.1. Definition. The regionally proximal relation of order $d$ is the relation $RP^{[d]} \subset X^{[d]} \times X^{[d]}$ defined by the following condition: $(x, y) \in RP^{[d]}$ iff for every $\delta > 0$ there is a pair $x', y' \in X$ and $(t_1, \ldots, t_d) \in T^d$ such that:

\[ \rho(x, x') < \delta, \quad \rho(y, y') < \delta \quad \text{and} \quad \rho^d((\tau_{(t_1, \ldots, t_d)} x'), (\tau_{(t_1, \ldots, t_d)} y')) := \sup \{ \rho(t, x', t, y') : \epsilon \subset \{1, \ldots, d\}, \epsilon \neq \emptyset \} < \delta. \]

For $d = 1$ this relation is the classical regionally proximal relation, see e.g. [1].

A convenient characterization of $RP^{[d]}$ is provided by Host-Kra-Maass in [6, Lemma 3.3]. Among its implications one has the corollary that the relation $RP^{[d]}$ is preserved under factors (Corollary 2.3 below).

2.2. Lemma. Let $(X, T)$ be a minimal flow. Let $d \geq 1$ and $x, y \in X$. Then $(x, y) \in RP^{[d]}$ if and only if there is some $a_* \in X^{[d]}$ such that $(x, a_*, y, a_*) \in Q^{[d+1]}$.

Proof. Suppose first that $(x, y) \in RP^{[d]}$. Fix an arbitrary point $z \in X$. Then, given $\delta > 0$, we first find a pair $x', y' \in X$ and $(t_1, \ldots, t_d) \in T^d$ which satisfy the requirements in Definition 2.1, and then replace $x'$ by $sz$ and $y'$ by $tsz$, with appropriate $s, t \in T$, so that $\rho(x, sz) < \delta$, $\rho(y, tsz) < \delta$ and

\[ \rho^d((\tau_{(t_1, \ldots, t_d)} (sz)^{[d]}), (\tau_{(t_1, \ldots, t_d)} (tsz)^{[d]})) < \delta. \]

Denoting $a_\delta = (sz)^{[d]}$, we have

\[ \tau^{[d+1]}_{(t_1, \ldots, t_d, t)} (sz, a_\delta, sz, a_\delta) = (\tau_{(t_1, \ldots, t_d)} (sz, a_\delta), \tau_{(t_1, \ldots, t_d)} (sz, a_\delta)) \in Q^{[d+1]} \]

Now, chose a convergent subsequence to get

\[ \lim_{\delta \to 0} \tau^{[d+1]}_{(t_1, \ldots, t_d, t)} (sz, a_\delta, sz, a_\delta) = (x, a_*, y, a_*) \in Q^{[d+1]} \]
Conversely, assume that there is some $a_+ \in X_s^{[d]}$ such that $(x, a_+, y, a_+) \in Q^{[d+1]}$. Then, there exist sequences $x_n \in X$ and $F_n \in \mathcal{F}_{d+1}$ such that

$$F_n((x_n)^{[d+1]}) \rightarrow (x, a_+, y, a_+).$$

Now $F_n$ has the form $F_n = (\tau_n^{[d]}, \theta_n^{[d]}, \tau_n^{[d]})$ with $t_n \in T$ and $\tau_n^{[d]} \in \mathcal{F}_d$, so that $x_n \rightarrow x$ and $t_n x_n \rightarrow y$, and it follows that $(x, y) \in RP^{[d]}$, as required.

\[ \square \]

It follows directly from the definition that $RP^{[d]}$ is a symmetric and $T$-invariant relation. It is also easy to see that it is closed. However, even for $d = 1$ there are easy examples which show that, in general, it need not be an equivalence relation (not being transitive). The remarkable fact that when $(X, T)$ is minimal, and $T$ is abelian, the relation $RP^{[1]}$ is an equivalence relation (and therefore coincides with the equicontinuous structure relation; i.e., the smallest closed invariant relation $S \subset X \times X$ such that the quotient flow $(X/S, T)$ is equicontinuous) is due to Ellis and Keynes [4] (see also [8]).

2.3. Corollary. If $\pi : (X, T) \rightarrow (Y, T)$ is a homomorphism of minimal $T$-flows then

$$(\pi \times \pi)(RP^{[d]}(X)) \subset RP^{[d]}(Y).$$

Equipped with Theorem 1.1 we will now show that for every $d \geq 1$ the relation $RP^{[d]}$ is an equivalence relation. First we prove two more necessary and sufficient conditions on a pair $(x, y) \in X \times X$ to belong to $RP^{[d]}$.

2.4. Proposition. Let $(X, T)$ be a minimal flow and $d \geq 1$. The following conditions are equivalent:

1. $(x, y) \in RP^{[d]}$.
2. $(x, y, y, \ldots, y) = (x, y_s^{[d+1]}) \in Q^{[d+1]}$.
3. $(x, y, y, \ldots, y) = (x, y_s^{[d+1]}) \in \mathcal{F}_{d+1}(x^{[d+1]})$.

Proof. (3) $\Rightarrow$ (2) is obvious. The implication (2) $\Rightarrow$ (1) follows from Lemma 2.2. Thus it suffices to show that (1) $\Rightarrow$ (3). Let $(x, y) \in RP^{[d]}$, then by Lemma 2.2, there is some $a \in X^{[d]}$ such that $(x, a_+, y, a_+) \in Q^{[d+1]}$. Observe that $(y, a_+) \in Q^{[d]}$. By Theorem 1.1.(3), there is a sequence $\{F_k\} \subset \mathcal{F}_d$ such that $F_k(y, a_+) \rightarrow y^{[d]}$. Hence

$$(F_k \times F_k)(x, a_+, y, a_+) \rightarrow (x, y_s^{[d]}, y_s^{[d]}) = (x, y_s^{[d+1]}).$$

Since $F_k \times F_k \in \mathcal{F}_{d+1}$ and $(x, a_+, y, a_+) \in Q^{[d+1]}$, we have that $(x, y_s^{[d+1]}) \in Q^{[d+1]}$. By Theorem 1.1.(2), $y^{[d+1]}$ is $\mathcal{F}_{d+1}$-minimal. It follows that $(x, y_s^{[d+1]})$ is also $\mathcal{F}_{d+1}$-minimal. Now $(x, y_s^{[d+1]}) \in Q^{[d+1]}[x]$ is $\mathcal{F}_{d+1}$-minimal and by Theorem 1.1.(3), $\mathcal{F}_{d+1}(x^{[d+1]})$ is the unique $\mathcal{F}_{d+1}$-minimal subset in $Q^{[d+1]}[x]$. Hence we have that $(x, y_s^{[d+1]}) \in \mathcal{F}_{d+1}(x^{[d+1]})$, and the proof is completed.

As an easy consequence of Proposition 2.4 we now have the following theorem.

2.5. Theorem. Let $(X, T)$ be a minimal metric flow, where $T$ an abelian group, and $d \geq 1$. Then $RP^{[d]}$ is an equivalence relation.
Proof. It suffices to show the transitivity, i.e. if \((x, y), (y, z) \in RP^{[d]}\), then \((x, z) \in RP^{[d]}(X)\). Since \((x, y), (y, z) \in RP^{[d]}\), by Proposition 2.4 we have
\[(y, x, x, \ldots, x), (y, z, z, \ldots, z) \in F_{d+1}(y^{[d+1]})\].
By Theorem 1.1.(2) \((F_{d+1}(y^{[d+1]}), F_{d+1})\) is minimal, whence
\[(y, z, z, \ldots, z) \in F_{d+1}(y, x, x, \ldots, x)\].
Finally, as \(F_{d+1}\) acts as the identity on the \(\emptyset\)-coordinate, it follows that also
\[(x, z, z, \ldots, z) \in F_{d+1}(x^{[d+1]})\].
By Proposition 2.4 again, \((x, z) \in RP^{[d]}\). □

2.6. Remark. From Proposition 2.4 we deduce that in the definition of the regionally proximal relation of order \(d\) the point \(x'\) can be replaced by \(x\). More precisely, a pair \((x, y) \in X \times X\) is in \(RP^{[d]}\) if and only if for every \(\delta > 0\) there is a point \(y' \in X\) and \((t_1, \ldots, t_d) \in T^d\) such that:
\[\rho(y, y') < \delta\] and
\[\rho^{[d]}(r_{[t_1, \ldots, t_d]}x^{[d]}, r_{[t_1, \ldots, t_d]}y^{[d]}) := \sup \{\rho(t_\epsilon x, t_\epsilon y') : \epsilon \subset \{1, \ldots, d\}, \epsilon \neq \emptyset\} < \delta\].
Again for \(d = 1\) this is a well known result (see [10] and [8]).

Let us conclude with the following remark. It is not hard to see that the proximal relation \(P \subset X \times X\) is a subset of \(RP^{[d]}\) for each \(d \geq 1\) (see Proposition 3.1 in [6]). Thus for every \(d \geq 1\) the quotient flow \(X/PR^{[d]}\) is a minimal distal flow. Of course the main result of Host, Kra and Maass in this work [6] is the fact that, for \(T = \mathbb{Z}\), this minimal distal factor flow is the maximal factor of \((X, T)\) which is a system of order \(d - 1\); i.e., a \(T\)-flow which is an inverse limit of \((d - 1)\)-step minimal \(T\)-nilflows. In turn, the results in [6] are based on the profound analogous ergodic theoretical theorems obtained by Host and Kra in [5].

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