Computing scalar products via a two-terminal quantum transmission line

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Abstract

The scalar product of two vectors with \( K \) real components can be computed using two quantum channels, that is, information transmission lines in the form of spin-1/2 XX chains. Each channel has its own \( K \)-qubit sender and both channels share a single two-qubit receiver. The \( K \) elements of each vector are encoded in the pure single-excitation initial states of the senders. After time evolution, a bi-linear combination of these elements appears in the only matrix element of the second-order coherence matrix of the receiver state. An appropriate local unitary transformation of the extended receiver turns this combination into a renormalized version of the scalar product of the original vectors. The squared absolute value of this scaled scalar product is the intensity of the second-order coherence which consequently can be measured, for instance, employing multiple-quantum NMR. The unitary transformation generating the scalar product of two-element vectors is presented as an example.

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I. INTRODUCTION

The realization of classical computational algorithms on the basis of quantum operations has become a promising direction in the development of quantum information processing. A well known algorithm of this type is the HHL (Harrow-Hassidim-Lloyd) algorithm for solving systems of linear algebraic equations \[1\], which was realized for solving the simplest systems of two equations on the basis of photonic systems \[2\] and superconducting quantum gates \[3\]. Some other applications of this algorithm can be found in \[4, 5\].

Among the powerful and effective quantum algorithms included in most of the contemporary quantum counterparts of the classical algorithms of computational algebra (both linear and nonlinear) are the quantum Fourier transform (QFT) \[6\], Hamiltonian simulation \[7, 8\] and phase estimation \[6, 9, 10\] (based on QFT). These subroutines are used in the above mentioned HHL-algorithm and in the algorithms for solving systems of nonlinear equations \[11\], executing various elementary matrix operations (including addition, multiplication and tensor product) with a method based on the Trotter product formula \[12\], performing the effective measurement of the desired observable \[13\] (which is applicable to various problems of linear algebra), and solving linear differential equations \[14\]. In all these quantum algorithms, the phase estimation as an essential subroutine allows to increase the accuracy of calculations using an additional quantum subsystem whose dimensionality increases with an increase in the desired accuracy.

We consider a quantum counterpart of a particular algebraic problem widely applicable in different fields of physics and mathematics, namely, the scalar product of arbitrary vectors with real entries (the way to generalize it for complex vectors is also discussed). A probabilistic protocol of scalar product estimation was proposed in \[15\] and then resumed in the Appendix B of \[12\]. In that protocol, both vectors to be multiplied are encoded into the same system, they are multiplied by the states, respectively, \(|0\rangle\) and \(|1\rangle\) of an additional qubit (ancilla). Then, after applying the Hadamard operator (or phase rotation) to the ancilla, one obtains the real (imaginary) part of the scalar product as the probability of measuring \(|0\rangle\) at the ancilla. The precision of the result depends on the number of repetitions of the protocol.

In contrast, our protocol is a single-shot operation. The vectors to be multiplied are the initial states of two different senders. Therefore, these states can be output states of
other quantum algorithms. Senders can be remote one from another, so that time evolution is required to transfer sender’s states to the receiver. Applying the proper unitary transformation to the extended receiver we obtain the scalar product of the input vectors as a particular element (second-order coherence matrix) of the two-qubit receiver’s density matrix. Being localized in the particular matrix element, the scalar product can be used as input data for other quantum operations. Thus, the quantum schemes proposed in our paper can be blocks in a more general device, and the protocol of scalar product can be used as a subroutine in other algorithms.

The hardware used in our algorithm is a two-channel communication line. Each channel includes the $K$-qubit sender, the transmission line and one qubit of the receiver. Thus, the receiver consists of two qubits independently of the dimensionality of the vectors to be multiplied. However, the number of qubits $K$ in each sender equals the dimensionality of these vectors. The components of the vectors to be multiplied are the probability amplitudes of the senders’ initial states, therefore the norm of these vectors is bounded by one. These amplitudes are real in the case of real vector components. The transmission lines are necessary to connect the remote senders with the receiver and can consist of different numbers of qubits.

After evolution, the elements of the vectors $\mathbf{v}(i), i = 1, 2$, encoded into the initial states of the senders appear in a bi-linear combination in the "corner" element (the element $\rho_{14}$ of the 4-dimensional receiver’s density matrix which is also the only element of the 2nd-order MQ coherence matrix). Then, using the unitary transformation of the extended receiver, we eliminate all those terms in that bilinear combination which do not contribute to the desired scalar product. In the end, the corner element of the density matrix is equal to the scalar product of the original vectors multiplied by a scalar factor. We show that this scalar factor equals $s\sqrt{(1 - |\mathbf{v}(1)|^2)(1 - |\mathbf{v}(2)|^2)}$ where $s$ takes its maximal value $s = \frac{1}{\sqrt{K}}$ in the shortest communication line of $2K$ qubits (there are no transmission lines in this case) and decreases with an increase in the length of the channels.

II. SCHEME OF COMMUNICATION LINE AND ITS EVOLUTION.

The general setup of the two-channel spin chain proposed for implementing the protocol of the scalar product of two $K$-element vectors initially remote from each other is illustrated
FIG. 1: General scheme of the spin-1/2 two-channel communication line for performing the scalar product of remote $K$-element vectors encoded in the pure states of the $K$-qubit senders $S_1$ and $S_2$. The result of the scalar product appears in the element $\rho_{14}^{(R)}$ of the 2-qubit receiver's density matrix. The extended receiver $ER$ consists of $(K_1 + K_2)$ qubits in Fig. 1. It consists of two $K$-qubit senders $S_1$ and $S_2$, which are used to encode the elements of the vectors as is shown below, the two-qubit receiver $R$ for registration of the result, the $(K_1 + K_2)$-qubit extended receiver $ER$ (the values of $K_i$, $i = 1, 2$, will be determined below) for applying the required unitary transformations, and two transmission lines $TL_i$, $i = 1, 2$, connecting the senders with the receiver. We emphasize that the lengths $L_1$ and $L_2$ of the first and second channels might be different (due to the different lengths of $TL_1$ and $TL_2$), and the full length of the communication line is $N = L_1 + L_2$. The receiver consists of two qubits, which are the end-nodes of the channels. The extended receiver $ER$ encompasses $K_1$ and $K_2$ qubits from, respectively, the first and second channel.

In the simplest case of the $2K$-spin chain ($N = 2K$) the scheme reduces to the one shown in Fig. 2. There is no $TL_i$, $i = 1, 2$, in this scheme. In addition, $R$ overlaps with $S_1$ and $S_2$, and the extended receiver $ER$ is identical to the complete spin system.

A. Evolution of the communication line and final state of the receiver.

Let the evolution of the spin chain be governed by the $XX$ Hamiltonian

$$
H = \sum_{i=1}^{N-1} D_i(I_{i,x}I_{i+1,x} + I_{i,y}I_{i+1,y}),
$$

$$
[H, I_z] = 0.
$$

The $I_{i,\alpha}(i = 1, ..., N; \alpha = x, y, z)$ are spin-1/2 operators with eigenvalues $\pm 1/2$ and $I_z$ is the total $z$ component, $I_z = \sum_{i=1}^{N} I_{i,z}$. We also need a unitary transformation $U^{(ER)}(\varphi)$ of the
FIG. 2: The simplest 2K-qubit spin-1/2 system allowing to perform the scalar product of K-element vectors encoded in the pure states of the K-qubit senders $S_1$ and $S_2$. The result of the scalar product appears in the element $\rho^{(R)}_{14}$ of the 2-qubit receiver’s density matrix.

extended receiver, where $\varphi$ denotes a list of free parameters of this transformation. The conservation law

$$[U^{(ER)}, I_z^{(ER)}] = 0$$

(3)

(where $I_z^{(ER)}$ is the total $z$ spin component of the extended receiver) together with (2) prevents the mixing of matrix elements from the coherence matrices of different orders [16].

After initialization, we first allow the spin chain to evolve till some optimal time instant $t_0$ (which will be determined below for a particular example, see Sec. IV A). Then, at time $t_0$, we apply the unitary transformation to the extended receiver. Thus, the complete unitary transformation of the communication line reads

$$W = \left( E_{S_1,TL_1'} \otimes U^{(ER)}(\varphi) \otimes E_{TL_2',S_2} \right) V(t),$$

(4)

$$V(t) = e^{-iHt},$$

(5)

where $TL_i'$, $i = 1, 2$, are the transmission lines $TL_i$ without the nodes of the extended receiver and $E_{S_i,TL_i'}$ is the identity operator in the space of the system $S_i \cup TL_i'$, $i = 1, 2$.

In the case shown in Fig. 2, the united senders, $S_1 \cup S_2$, the extended receiver, and the complete system coincide. Hence, it is not necessary to wait for the time evolution to transfer the quantum information from the senders to the extended receiver. Therefore, we can disregard time evolution and apply the unitary transformation $U^{(ER)}$ immediately (at $t = 0$), so that the operator $W$ reads

$$W = U^{(ER)}(\varphi).$$

(6)
The receiver density matrix reads
\[ \rho^{(R)} = \text{Tr}_{/R} (\rho), \] (7)
where the trace is calculated over all the nodes except those of the receiver. In the following section, the “senders only” scheme from Fig.2 will be discussed in more detail.

III. SCALAR PRODUCT VIA THE “SENDERS ONLY” SCHEME

First, we consider the scalar product of \( K \)-element vectors using the scheme in Fig.2 where \( N = 2K \) so that there are no transmission lines and even the receiver nodes are contained in the senders. We start with the pure initial states of the senders \( S_1 \) and \( S_2 \)
\[ |\psi_i \rangle = a^{(i)}_0 |0 \rangle + \sum_{n=1}^{K} a^{(i)}_{K-n+1} |n \rangle, \quad \sum_{n=0}^{K} |a^{(i)}_n|^2 = 1, \quad i = 1, 2, \quad a^{(i)}_0 \neq 0. \] (8)
Here \( |n \rangle \) means the one-excitation state with the \( n \)-th spin of \( S_1 \) or \( S_2 \) excited. The initial state of the complete system reads
\[ \rho(0) = \rho^{(S_1)}(0) \otimes \rho^{(S_2)}(0), \quad \rho^{(S_i)} = |\psi_i \rangle \langle \psi_i |, \quad i = 1, 2. \] (9)
This initial state has no more than two excitations, therefore the system evolves in the zero-, one- and two excitation state subspaces due to the conservation laws (2) and (3). In this section, we will use capital Latin multi-indices, where the subscripts 1 and 2 are related, respectively, to \( S_1 \) and \( S_2 \) (for instance, \( I_1, I_2 \)), the multi-index with the subscript \( R \) is related to the receiver \( (I_R) \), and primed multi-indices are related to the appropriate sender without the receiver’s nodes (for instance, \( I'_1, I'_2 \)).

We write the receiver density matrix (7) in components, explicitly performing the matrix multiplications and also the trace over all degrees of freedom except those of the receiver:
\[ \rho^{(R)}_{\text{tr}_{/R}} = \sum_{N'_1, N'_2, I'_1, I'_2, J'_1, J'_2} W_{N'_1 N'_2 I'_1 J'_1; I'_2 J'_2} \rho^{(S_1)}_{I'_1 J'_1} \rho^{(S_2)}_{I'_2 J'_2} \] (10)
Since the receiver is a 2-qubit system, its coherence matrix of order +2 has only a single element, connecting the two-qubit states with multi-indices \( N_R = \{0, 0\} \) and \( M_R = \{1, 1\} \), which we now construct. Due to (3) the matrix \( W \) is block-diagonal with respect to the number of excitations. Since \( M_R = \{1, 1\} \), the multi-indices \( N'_1 \) and \( N'_2 \) on \( W^+ \) can only contain zero entries, which we denote by \( 0'_1 \) and \( 0'_2 \), respectively. Because of the one-excitation
initial states of the senders, $J_1$ and $J_2$ must contain exactly one entry equal to unity, which we denote by $|J_1| = |J_2| = 1$. Similarly, since $N_R = \{0, 0\}$, and $N'_1 = 0'_1$ and $N'_2 = 0'_2$, the matrix $W$ operates in the zero-excitation subspace and hence has only one element $W_{0'_1 0'_2; 0'_1 0'_2} = 1$, where again $0_1$ and $0_2$ denote the zero-excitation states of the senders $S_1$ and $S_2$, respectively. The desired element thus reads

$$\rho^{(R)}_{00;11} = \sum_{|J_1| = 1} \rho^{(S_1)}_{0_1; J_1} \rho^{(S_2)}_{0_2; J_2} W_{J_1 J_2; 0'_1 110'_2}. \quad (11)$$

This equation involves only one column of $W^+$, indexed by $(0'_1 110'_2)$. The other columns must fulfill the unitarity condition $WW^+ = E_{2K}$ (the $2K \times 2K$ unit matrix) and they are arbitrary otherwise.

In order to connect to the pure initial states we rewrite the element as follows:

$$\rho^{(R)}_{00;11} = \sum_{j=1}^{K} \sum_{i=1}^{K} s_{ij} \rho^{(S_1)}_{0_1; J_1^{(i)}} \rho^{(S_2)}_{0_2; J_2^{(j)}}, \quad (12)$$

$$s_{ij} = W^+_{J_1^{(i)} J_2^{(j)}; 0'_1 110'_2}. \quad (13)$$

The coefficients $s_{ij}$ are related to the elements (which are still free) of the unitary transformation $W \equiv U^{(ER)}$ and $J_k^{(i)}$ is the multi-index $J_k$ with all zeros except the $i$th entry which equals 1. We set certain coefficients $s_{ij}$ equal to zero, keeping in mind that the matrix $W$ has to be unitary:

$$s_{ij} = \delta_{ij} s, \quad i, j = 1, \ldots, K. \quad (14)$$

Then (12) reduces to

$$\rho^{(R)}_{00;11} = s \sum_{i=1}^{K} \rho^{(S_1)}_{0_1; J_1^{(i)}} \rho^{(S_2)}_{0_2; J_2^{(i)}} = S \sum_{i=1}^{K} \left( a^{(1)}_i a^{(2)}_i \right)^*, \quad (15)$$

$$S = a^{(1)}_0 a^{(2)}_0 s, \quad (16)$$

since the elements of the initial senders’ density matrices are given by

$$\rho^{(S_k)}_{0_k; J_k^{(i)}} = a^{(k)}_0 (a^{(k)}_i)^*. \quad (17)$$

If the $a^{(j)}_i$ are all real, we can introduce the vectors

$$\mathbf{v}^{(j)} = \begin{pmatrix} a^{(j)}_1 \\ \vdots \\ a^{(j)}_K \end{pmatrix}, \quad j = 1, 2, \quad (18)$$
then the rhs of eq.(15) is proportional to the scalar product of these vectors:

\[ \rho^{(R)}_{00;11} = S \mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)}, \quad S = s \sqrt{(1 - |\mathbf{v}^{(1)}|^2)(1 - |\mathbf{v}^{(2)}|^2)}. \]  

(19)

We remark that the norm of the vectors \( \mathbf{v}^{(j)}, j = 1, 2 \), defined in (18) is less than one (as mentioned in the Introduction) and \( a_0^{(j)}, j = 1, 2 \), can not equal zero (otherwise \( S = 0 \)), as noted in Eq.(8). It is simple to show that in this case

\[ s = 1/\sqrt{K}. \]  

(20)

In fact, Eq.(14) means that there are \( K \) nonzero elements in the selected column of \( W^+ \) and each of them equals \( s \). Then Eq.(20) follows from the normalization condition for this column. This is the maximal value of the scale coefficient \( s \) in our protocol, \( s \) decreases with an increase in the length of the channels as will be seen in the example of Sec.IV A.

The final step of the protocol is the measurement of the intensity \( I_2 \) of the second-order coherence, given by the second-order coherence matrices \( \rho^{(R; \pm 2)} \) of the receiver:

\[ I_2 = \text{Tr} \rho^{(R; 2)} \rho^{(R; -2)} = (\rho^{(R)}_{00;11})^2 = S^2 (\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)})^2. \]  

(21)

Thus the measured quantity \( I_2 \) is proportional to the square of the scalar product of the vectors \( \mathbf{v}^{(1)} \) and \( \mathbf{v}^{(2)} \).

A. Generalization to complex vectors \( \mathbf{v}^{(i)} \)

The reason requiring the reality of \( \mathbf{v}^{(i)} \) is in the structure of the matrix element \( \rho^{(R)}_{00;11} \) given in (15). For the scalar product of complex vectors \( \mathbf{v}^{(i)} \) this formula should be replaced by

\[ \rho^{(R)}_{00;11} = (a_0^{(1)})^* a_0^{(2)} s \sum_{i=1}^{K} a_i^{(1)} (a_i^{(2)})^*, \]  

(22)

which can be achieved by a simple unitary transformation of the initial state of the first sender. This transformation must transfer the amplitude \( a_0^{(1)} \) in the state \( |\psi_1\rangle \) (8) to a state with two excited spins, say, the 1st and 2nd, \( |12\rangle \):

\[ U : \quad a_0^{(1)} |0\rangle + \sum_{n=1}^{K} a_n^{(1)} |K - n + 1\rangle \rightarrow \sum_{n=1}^{K} a_n^{(1)} |K - n + 1\rangle + a_0^{(1)} |12\rangle. \]  

(23)
In terms of the MQ-coherence matrices, this transformation means transferring the elements of the $-1$st-order coherence matrix to the 1st-order coherence matrix, while the elements of the 1st-order coherence matrix of the state $|\psi_1\rangle$ become zeros. There is no principal difficulty in constructing the appropriate unitary transformation of the extended receiver in this case, but the spin dynamics involved in the time evolution then must be extended to the three-excitation subspace. We will not study the scalar product of complex vectors in more detail.

**B. Example: scalar product of two-element vectors, $K = 2$**

We illustrate our protocol for the simplest case of two-dimensional vectors. The unitary transformation in this case is block-diagonal

$$W^+ = \text{diag}(1, W_1, W_2), \quad (24)$$

$$W_2 = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\
0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 & 0 \\
0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 & 0 \\
0 & 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad (25)$$

where the two excitation block $W_2$ refers to the basis of the extended receiver states

$$|0011\rangle, |0101\rangle, |0110\rangle, |1001\rangle, |1010\rangle, |1100\rangle. \quad (26)$$

In (24), $W_1$ is the $4 \times 4$ block in the one-excitation subspace, which is not important in our case. Eq.\,(20) yields $s = \frac{1}{\sqrt{2}}$. We note that, according to eq.\,(11), only the third column of $W_2$ in (25) is important for the scalar product. The other columns only serve to fulfill the unitarity condition for $W_2$.

From the point of view of quantum information processing, it is important to note that the block $W_2$ in the form (25) can be written in terms of standard quantum gates, namely the one-qubit rotations and the two-qubit CNOT gate. We denote by $C_{ij}$ the CNOT entangling the $i$th and $j$th spins with the control qubit $i$ written in the basis $|0\rangle$, $|j\rangle$, $|i\rangle$, $|1\rangle$. 

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$|ji\rangle$:

$$C_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (27)

Let $R_{yi}(\beta)$ be the $y$-rotation of the $i$th qubit:

$$R_{yi}(\beta) = e^{i\beta I_{yi}} = \begin{pmatrix} \cos \frac{\beta}{2} & \sin \frac{\beta}{2} \\ -\sin \frac{\beta}{2} & \cos \frac{\beta}{2} \end{pmatrix}. \hspace{1cm} (28)$$

We can introduce a two-qubit operation with the qubits $i$ and $j$ which commutes with $I_{zi} + I_{zj}$:

$$E_{ij}(\beta) = C_{ij} R_{yi}(\beta) C_{ji} R_{yi}(-\beta) C_{ij}. \hspace{1cm} (29)$$

Then the unitary transformation

$$W^+ = E_{12}(0) E_{34}(\frac{\pi}{4}) E_{23}(\frac{2\pi}{2}) E_{34}(\frac{\pi}{4}) E_{12}(\frac{\pi}{4})$$

produces the block $W_2$ \textit{(25)} in the two-excitation subspace.

\section*{IV. SCALAR PRODUCT OF TWO REMOTE VECTORS, FIG.1}

In this section we consider the general situation shown in Fig.1 where the $K$-qubit senders $S_1$ and $S_2$ are connected to the 2-qubit receiver $R$ by transmission lines of, in general, different lengths. We construct the general protocol for calculating the scalar product of $K$-dimensional vectors ($K > 1$) with real elements \textit{(18)}. To this end we consider the pure states \textit{(8)} of two $K$-qubit senders each containing at most a single excitation and with real coefficients $a_k^{(i)}$. The initial state of the entire quantum system reads:

$$\rho(0) = \rho^{(S_1)}(0) \otimes \rho^{(TL_1,R,TL_2)}(0) \otimes \rho^{(S_2)}(0), \quad \rho^{(S_1)}(0) = |\psi_i\rangle \langle \psi_i|, \quad i = 1, 2,$$  \hspace{1cm} (31)

where $|\psi_i\rangle, \; i = 1, 2,$ are the sender’s pure states \textit{(8)} and $\rho^{(TL_1,R,TL_2)}(0)$ is the initial state of the remainder of the spin chain.

We show, similar to Sec.\textit{III} that the scalar product of two vectors appears in the only element of the second-order coherence matrix of the two-qubit receiver.
We consider the unitary transformation $W$ defined in (4) and define the multi-indices $I_1, \ldots, I_5$ related with, respectively, subsystems $S_1, TL_1, R, TL_2$ and $S_2$. Each multi-index $I_i$ of a $k_i$-qubit subsystem consists of a set of $k_i$ zeros and ones where the one at the $k$th position corresponds to the excited $k$th spin. Then, using the initial state (31) and time evolution included in the unitary operator $W$, we obtain, in a way analogous to (10)

$$\rho^{(R)} = \text{Tr}_{S_1, TL_1, TL_2, S_2} (\rho) \Rightarrow \rho^{(R)}_{N_3; M_3} = \sum_{N_1, N_2, N_4, N_5} W_{N_1, N_2, N_4, N_5, I_1, I_2, I_3, I_4, I_5} \rho^{(S_1)}_{I_1, I_2, I_3, I_4, I_5} \rho^{(S_2)}_{I_5, J_5} W^+_{J_1, J_2, J_3, J_4, J_5} \rho^{(TL_1, R, TL_2)}_{S_1, S_2, T_1, T_2, T_3, T_4},$$

(33)

where $\{I\}$ is short for $(I_1, I_2, I_3, I_4, I_5)$ and $\{J\}$, $\{N\}$ analogously. Note that we have suppressed the time dependence of $W$ for simplicity. The density matrices on the rhs are to be taken at $t = 0$, while $\rho^{(R)}$ depends on time. We can reasonably assume the system to be initialized such that at $t = 0$ excitations are only present in the senders. The initial density operator of the connection between $S_1$ and $S_2$ then is

$$\rho^{(TL_1, R, TL_2)} = |0_20_30_4 \rangle \langle 0_20_30_4|,$$

(34)

and its only nonzero element is $\rho^{(TL_1, R, TL_2)}_{0_20_30_4; 0_20_30_4} = 1$, where 0 is the zero value of the multi-index, associated with the ith subsystem. Then expression (33) becomes simpler:

$$\rho^{(R)}_{N_3; M_3} = \sum_{N_1, N_2, N_4, N_5} \sum_{I_1, I_2, I_3, I_4, I_5} W_{N_1, N_2, N_4, N_5, I_1, I_2, I_3, I_4, I_5} \rho^{(S_1)}_{I_1, I_2, I_3, I_4, I_5} \rho^{(S_2)}_{I_5, J_5} W^+_{J_1, J_2, J_3, J_4, J_5} \rho^{(TL_1, R, TL_2)}_{S_1, S_2, T_1, T_2, T_3, T_4},$$

(35)

Now we consider the second-order coherence matrix which consists of a single element with $N_3 = \{0, 0\}$, $M_3 = \{1, 1\}$, and take into account that we stay in the subspace with two excitations at most, and that $W_{0_10_20_30_4; 0_10_20_30_4} = 1$. By an argument similar to that leading from (10) to (11), (35) reduces to the following form:

$$\rho^{(R)}_{0_0; 11} = \sum_{|J_1| = 1, |J_5| = 1} \rho^{(S_1)}_{0_1, J_1} \rho^{(S_2)}_{0_5, J_5} W^+_{J_1, J_2, J_3, J_4, J_5} \rho^{(TL_1, R, TL_2)}_{S_1, S_2, T_1, T_2, T_3, T_4}.$$

(36)

Using the free parameters of the unitary transformation $U^{(ER)}$ we can achieve

$$W^+_{J_1, J_2, J_3, J_4, J_5} = 0, \quad J_1 \neq J_5.$$

(37)

There are $K(K - 1)$ equations in this system. If we also satisfy the $K$ equations

$$W^+_{J_1, J_2, J_3, J_4, J_5} = s = \text{const}, \quad |J_1| = 1,$$

(38)
then (36) gets the form \((0_5 \equiv 0_1)\)

\[
\rho_{00;11}^{(R)} = s \sum_{i=1}^{K} \rho_{0i;1i}^{(S_1)} \rho_{0i;3i}^{(S_2)} = s v^{(1)} \cdot v^{(2)},
\]

(39)

where the vectors \(v^{(i)}, i = 1, 2\), are defined in (18) and (17).

In order to construct the required element \(\rho_{00;11}^{(R)}\) of the two-qubit receiver’s density matrix the unitary transformation \(W\) has to satisfy conditions (37) and (38). These conditions involve only one column of \(W^+\), namely the one with the index \((0_10_2110_40_5)\). This column corresponds to the column with the index \((0 \ldots 01 10 \ldots 0)\) in \((U^{(ER)})^+\), similar to Sec III.

The ones in the above multi-index refer to the nodes of the receiver. We denote this column \((U^{(ER)})^+\) by \(K_1 \ldots K_2\). Due to the conservation law (3), the nonzero elements of the column are those that connect to other two-excitation states of the extended receiver. The dimension of the space spanned by these states, and hence, the number of nonzero elements in the column under discussion, is \(P = \frac{1}{2} N^{(ER)}(N^{(ER)} - 1)\), where we temporarily denote the number of qubits in the extended receiver by \(N^{(ER)} = K_1 + K_2\). Due to unitarity the nonzero elements are points on the \(P\)-dimensional unit sphere and can be parametrized in terms of \(2P - 1\) angles:

\[
(U^{(ER)})^+ = \begin{pmatrix} e^{i\varphi_1} \sin \alpha_1 \sin \alpha_2 \ldots \sin \alpha_{P-1} \\ e^{i\varphi_2} \cos \alpha_1 \sin \alpha_2 \ldots \sin \alpha_{P-1} \\ e^{i\varphi_3} \cos \alpha_2 \sin \alpha_3 \ldots \sin \alpha_{P-1} \\ \vdots \\ e^{i\varphi_P} \cos \alpha_{P-1} \end{pmatrix}.
\]

(40)

The remaining columns of the unitary transformation can be constructed to satisfy the unitarity condition \(U^{(ER)}(U^{(ER)})^+ = E_{ER}\), where \(E_{ER}\) is the unit operator on the extended receiver. Thus, we have \(2P - 1\) real parameters \(\alpha_i\) and \(\varphi_i\) to satisfy the \(K^2\) complex equations (37) and (38). The condition

\[
2P - 1 = N^{(ER)}(N^{(ER)} - 1) - 1 \geq 2K^2 \Rightarrow N^{(ER)} \geq \frac{1}{2}(1 + \sqrt{5 + 8K^2})
\]

(41)

defines the minimal size of the extended receiver. In particular, the choice \(K_1 = K_2 = K\) obviously fulfills this condition for all \(K \geq 2\).
A. Example: scalar product of two-element vectors using two channels of 20 nodes

We consider the two-channel communication line governed by the $XX$-Hamiltonian \( \Pi \). Each channel consists of 20 nodes \( N = 40 \) with two pairs of coupling constants adjusted for the high probability state transfer between the end nodes of an isolated channel \[17\] (see Fig.1):

\[
D_1 = D_{N/2-1} = D_{N/2+1} = D_{N-1} = 0.55, \quad (42)
\]
\[
D_2 = D_{N/2-2} = D_{N/2+2} = D_{N-2} = 0.817.
\]

The coupling between the two channels is weak, the coupling constant being \( D_N = 0.006 \). In this case the evolution operator \( V(t) \) is essential in \( W(t) \) (see Eq.\( (41) \)) and there is an optimal time instant providing the maximal value for the parameter \( s \) in \( (39) \).

As an example, we consider two-qubit senders \( i = 1, 2 \) encoding two-element real vectors \( v^{(i)} \). This corresponds to \( K = 2 \) in \( (8) \) and in \( (18) \). We use a four-qubit extended receiver in the communication line, setting \( K_1 = K_2 = 2 \). Then \( P = 6 \), so that we have 11 real parameters in \( (40) \) to satisfy 4 complex equations \( (37), (38) \). The optimization yields that the maximal \( s = 0.6813 \) is achieved at \( t = 26.441 \) (this time instance coincides with that for the high-probability state transfer \[18\] between the end nodes of a single channel) with the parameters in \( (40) \)

\[
\alpha_1 = 3.135160, \quad \alpha_2 = 1.570857, \quad \alpha_3 = 4.712397, \quad \alpha_4 = 5.497855, \quad \alpha_5 = 1.581785, \quad (43)
\]
\[
\varphi_1 = 5.526328, \quad \varphi_2 = 0.000065, \quad \varphi_3 = 1.497402, \quad \varphi_4 = 0.999731, \quad \varphi_5 = 3.141532,
\]
\[
\varphi_6 = 1.319482.
\]

We note that the above calculated \( s \) is less than the one obtained from formula \( (20) \) in Sec.\ref{sec:III.B} In addition, the values of \( \alpha_i, i = 1, \ldots, 5 \), and \( \varphi_2, \varphi_5 \) are close to multiples of \( \pi/4 \):

\[
\alpha_1 \approx \pi, \alpha_2 \approx \frac{\pi}{2}, \alpha_3 \approx \frac{3\pi}{2}, \alpha_4 \approx \frac{7\pi}{4}, \alpha_5 \approx \frac{\pi}{2}, \varphi_2 \approx 0, \varphi_5 \approx \pi. \quad (44)
\]

This is not by chance. With these \( \alpha_i \) and \( \varphi_i \), the column \( (40) \) only by sign differs from the third column of the matrix \( W_2 \) in \( (25) \) (which is responsible for the scalar product in the "senders only" scheme). This change in sign as well as the deviation of parameters \( (43) \) from values \( (44) \) is due to the evolution and imperfection of state transfer.
Again, the constructed unitary transformation can be generated by the set of CNOTs and one-qubit rotations as follows. Let $C_{ij}$ be CNOT with control qubit $i$, $R_{yi}(\beta)$ be the $y$-rotation (28) and $R_{zi}(\beta)$ be the $z$-rotation of the $i$th spin:

$$R_{zi}(\beta) = e^{i\beta I_{zi}} = \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix}.$$  \hfill (45)

We can introduce the two-qubit operation on the qubits $i$ and $j$ which commutes with $I_{zi} + I_{zj}$:

$$E_{ij}(\alpha, \beta) = C_{ij}R_{zi}(\alpha)R_{yi}(\beta)R_{zi}(-\alpha)C_{ji}R_{zi}(\alpha)R_{yi}(-\beta)R_{zi}(-\alpha)C_{ij}. \hfill (46)$$

Then the unitary transformation $U^{ER}$ with the values (43) for the parameters in (40) can be represented, for instance, in the following form:

$$(U^{ER})^+ = E_{12}(\alpha_1, \beta_1)E_{23}(\alpha_2, \beta_2)E_{34}(\alpha_3, \beta_3)E_{43}(\alpha_4, \beta_4)E_{32}(\alpha_5, \beta_5)E_{21}(\alpha_6, \beta_6), \hfill (47)$$

where, (with the accuracy $\sim 10^{-6}$),

$$\alpha_1 = 0.056765, \beta_1 = 5.496129, \alpha_2 = 6.276980, \beta_2 = 1.577448,$$

$$\alpha_3 = 6.134857, \beta_3 = 0.320085, \alpha_4 = 6.184914, \beta_4 = 0.471440,$$

$$\alpha_5 = 6.263752, \beta_5 = 1.562174, \alpha_6 = 6.226368, \beta_6 = 0.786962.$$  \hfill (48)

It is interesting to note that the constraints (14) for the four-node chain ($K = 2$) can be also satisfied with the unitary transformation $U^{(ER)}$ having the structure (47):

$$W^+ = (U^{ER})^+ = E_{12}(0, 2\varphi)E_{23}(0, 4\varphi)E_{34}(0, -\varphi)E_{43}(0, -\varphi)E_{32}(0, 4\varphi)E_{21}(0, -2\varphi), \quad \varphi = \frac{\pi}{8}.$$  \hfill (49)

We do not present the explicit formula for the block $W_2$ in this case. It differs from (25), but the third column of this block, which controls the constraints (14), is the same.

V. CONCLUSION

Using a local unitary transformation of the so-called extended receiver we obtain a scalar product of two real vectors and place the result in the element of the second-order MQ-coherence matrix of the receiver. These vectors are initially encoded in the pure states of
two senders which are, generically, remote from the receiver. Thus, the encoded vectors evolve along the spin-1/2 channels to the receiver, therefore getting mixed. The unitary transformation at the extended receiver (which includes $K_1$ and $K_2$ qubits from, respectively, the first and second channels) is used to remove extra terms in the resulting expression for the above element, so that the remaining terms form an expression proportional to the scalar product of the original vectors. The factor $s$ in the proportionality coefficient $S$ can not exceed $1/\sqrt{K}$, which is found for the scheme without finite-length transmission lines $TL_i$, Fig.2 and decreases with an increase in the channel lengths. For the simplest example of the scalar product of two-element vectors (4-node extended receiver), we show that the unitary transformation $U^{ER}$ can be represented as a combination of CNOTs and one-qubit rotations, which is important for programming the scalar product on quantum computers.

The dimensionality of the quantum system used for implementing the scalar-multiplication protocol does not depend on the required accuracy of calculations, but only on the dimensionality of the original vectors and on the distance between the senders and the receiver (the lengths of $TL_i$, $i = 1, 2$). In addition, the result of the multiplication is transferred to a particular element of the receiver’s density matrix without performing measurements on any particular subsystem. Therefore, the protocol is completely quantum and does not involve any classical step except for the initialization of the vectors $v^{(i)}$. Hence, the obtained scalar product can be used in further quantum calculations.

The derived unitary transformation of the extended receiver depends on the parameters of the communication line, on the Hamiltonian governing the quantum evolution and on the length $K$ of the vectors to be multiplied. Once constructed, this transformation can be used for multiplying any pair of $K$-dimensional vectors.

We wish to emphasize that the proposed protocol allows to multiply vectors with real elements and is based on unitary transformations (the time evolution operator and the local unitary transformation $U^{(ER)}$ on the extended receiver) which conserve the excitation number in the spin system and therefore do not mix coherence matrices of different orders. However, the generalization to complex vectors is quite straightforward and was outlined in Sec. IIIA.

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