Research Article

A New Extension of Serrin’s Lower Semicontinuity Theorem

Xiaohong Hu \(^1\) and Shiqing Zhang \(^2\)

\(^1\) School of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Chongqing 400065, China
\(^2\) Mathematical College, Sichuan University, Chengdu 610064, China

Correspondence should be addressed to Xiaohong Hu; huxh@cqupt.edu.cn

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We present a new extension of Serrin’s lower semicontinuity theorem. We prove that the variational functional
\[
\int_\Omega f(x, u, u') \, dx
\]
defined on \(W^{1,1}_{\text{loc}}(\Omega)\) is lower semicontinuous with respect to the strong convergence in \(L^1_{\text{loc}}\), under the assumption that the integrand \(f(x, s, \xi)\) has the locally absolute continuity about the variable \(x\).

1. Introduction and Main Results

The aim of this paper is to give some new sufficient conditions for lower semicontinuity with respect to the strong convergence in \(L^1_{\text{loc}}\) for integral functionals
\[
F(u, \Omega) = \int_\Omega f(x, u(x), Du(x)) \, dx,
\]
where \(\Omega\) is an open set of \(\mathbb{R}^n\), \(u \in W^{1,1}_{\text{loc}}(\Omega)\), defined on \(W^{1,1}_{\text{loc}}(\Omega) = \{ u : u \in L^1(K), Du \in L^1(K), \text{for all } K \subset \subset \Omega \} [1]\), \(Du\) denotes the generalized gradient of \(u\), and the integrand \(f(x, s, \xi) : \Omega \times R \times R^n \to [0, \infty)\) satisfies the following condition:

(H1) \(f\) is continuous in \(\Omega \times R \times R^n\), and \(f(x, s, \xi)\) is convex in \(\xi \in R^n\) for any fixed \((x, s) \in \Omega \times R\).

The integral functional \(F\) is called lower semicontinuous in \(W^{1,1}_{\text{loc}}(\Omega)\) with respect to the strong convergence in \(L^1_{\text{loc}}\), if, for every \(u_m, u \in W^{1,1}_{\text{loc}}(\Omega)\), such that \(u_m \to u\) in \(L^1_{\text{loc}}\), then
\[
\liminf_{m \to +\infty} F(u_m, \Omega) \geq F(u, \Omega).
\]

It is well known that condition (H1) is not sufficient for lower semicontinuity of the integral \(F\) in (1) (see book [2]). In addition to (H1), Serrin [3] proposed some sufficient conditions for lower semicontinuity of the integral \(F\). One of the most known conclusions is the following one.

Theorem 1 (see [3]). In addition to (H1), \(f\) satisfies one of the following conditions:

(a) \(f(x, s, \xi) \to +\infty\) when \(|\xi| \to +\infty\), for all \((x, s) \in \Omega \times R\),

(b) \(f(x, s, \xi)\) is strictly convex in \(\xi \in R^n\) for all \((x, s) \in \Omega \times R\),

(c) the derivatives \(f_x(x, s, \xi), f_s(x, s, \xi), \text{ and } f_{\xi x}(x, s, \xi)\) exist and are continuous for all \((x, s, \xi) \in \Omega \times R \times R^n\).

then \(F(u, \Omega)\) is lower semicontinuous in \(W^{1,1}_{\text{loc}}(\Omega)\) with respect to the strong convergence in \(L^1_{\text{loc}}\).

Conditions (a), (b), and (c) quoted above are clearly independent, in the sense that we can find a continuous function \(f\) satisfying just one of them but none of the others. Many scholars have weakened the conditions of integrand \(f\) and generalized Theorem 1, such as Ambrosio et al. [4], Cicco and Leoni [5], Fonseca and Leoni [6, 7]. In particular Gori et al. [8, 9] proved the following theorems.

Theorem 2 (see [8, 9]). Let one assume that \(f\) satisfies (H1) and that, for every compact set \(K \subset \Omega \times R \times R^n\), there exists a constant \(L = L(K)\) such that
\[
|f_x(x_1, s, \xi) - f_x(x_2, s, \xi)| \leq L|x_1 - x_2|, \\
\forall (x_1, s, \xi), (x_2, s, \xi) \in K,
\]

(3)
and, for every compact set $K_1 \subset \Omega \times R$, there exists a constant $L_1 = L_1(K_1)$ such that

$$
\left| f(x,s,\xi) \right| \leq L_1, \quad \forall (x,s) \in K_1, \forall \xi \in R^n,
$$

$$
\left| f(x,s,\xi_1) - f(x,s,\xi_2) \right| \leq L_1 |\xi_1 - \xi_2|,
$$

(4) \forall (x,s) \in K_1, \forall \xi_1, \xi_2 \in R^n.

Then $F(u,\Omega)$ is lower semicontinuous in $W^{1,1}_{loc}(\Omega)$ with respect to the strong convergence in $L^1_{loc}(\Omega)$.

Theorem 4. Let $\Omega \subset R$ be an open set; $f(x,s,\xi): \Omega \times R \times R \rightarrow [0, +\infty)$ satisfies the following conditions:

(H1) $f(x,s,\xi)$ is continuous on $\Omega \times R \times R$ and, $f(x,s,\xi)$ is convex in $\xi \in R$ for all $(x,s) \in \Omega \times R$;

(H2) $f(x,s,\xi)$ is continuous on $\Omega \times R \times R$ and, for every compact set of $\Omega \times R \times R$, $f(x,s,\xi)$ is absolutely continuous about $x$;

(H3) for every compact set $K_1 \subset \Omega \times R$, there exists a constant $L_1 = L_1(K_1)$, such that

$$
\left| f(x,s,\xi) \right| \leq L_1, \quad \forall (x,s) \in K_1, \forall \xi \in R^n,
$$

$$
\left| f(x,s,\xi_1) - f(x,s,\xi_2) \right| \leq L_1 |\xi_1 - \xi_2|,
$$

(7) \forall (x,s) \in K_1, \forall \xi_1, \xi_2 \in R^n.

Then the functional $F(u,\Omega) = \int_{\Omega} f(x,u(x),u'(x))dx$ is lower semicontinuous in $W^{1,1}_{loc}(\Omega)$ with respect to the strong convergence in $L^1_{loc}(\Omega)$.

Theorem 5. Let $\Omega \subset R$ be an open set; $f(x,s,\xi): \Omega \times R \times R \rightarrow [0, +\infty)$ satisfies (H1) and the following condition:

(H4) for every compact set of $\Omega \times R \times R$, $f(x,s,\xi)$ is absolutely continuous about $x$.

Then the functional $F(u,\Omega)$ is lower semicontinuous in $W^{1,1}_{loc}(\Omega)$ with respect to the strong convergence in $L^1_{loc}(\Omega)$.

2. Preliminaries

In this section, we will collect some basic facts which will be used in the proofs of Theorems 4 and 5.

It is well known that a real function $f:[a,b] \rightarrow R$ is called an absolutely continuous function on $[a,b]$, if, for all $\varepsilon > 0, 3\delta > 0$, such that for any finite disjoint open interval $\{ (a_i,b_i) \}_{i=1}^n$ on $[a,b]$, when $\sum_{i=1}(b_i - a_i) < \delta$, we have

$$
\sum_{i=1}^n |f(b_i) - f(a_i)| < \varepsilon.
$$

Obviously, if $f(x)$ is Lipschitz continuous on $[a,b]$, $f(x)$ is absolutely continuous on $[a,b]$.

One of the main tools, used in the present paper, in order to prove the lower semicontinuity of the functional $F(u,\Omega)$ in (1), is an approximation result for convex functions due to De Giorgi [10].

Lemma 6 (see [10]). Let $U \subseteq R^d$ be an open set and $f: U \times R^d \rightarrow [0, +\infty)$ a continuous function with compact support on $U$, such that, for every $t \in U, f(t,\cdot)$ is convex on $R^d$. Then there exists a sequence $\{\eta_{ij}\}_{i,j=1} \subseteq C_\infty^0(R^d)$, $\eta_{ij} \geq 0, \int_{R^n} \eta_{ij} d\rho = 1$, and supp$(\eta_{ij}) \subseteq B(0,1)$, such that, if we let

$$
a_{ij}(t) = \int_{R^n} f(t,\rho) \left((n+1)\eta_{ij}(\rho) + \langle \nabla \eta_{ij}(\rho),\rho \rangle \right) d\rho,
$$

$$
b_{ij}(t) = \int_{R^n} f(t,\rho) \nabla \eta_{ij}(\rho) d\rho,
$$

one has

$$
f_j(t,\xi) = \max_{i \leq j} \{ 0, a_{ij}(t) + \langle b_{ij}(t),\xi \rangle \}, \quad j \in N,
$$

(10) satisfying the following results:

(i) for every $j \in N, f_j : U \times R^d \rightarrow [0, +\infty)$ is a continuous function with compact support on $U$ such that, for all $t \in U, f_j(t,\cdot)$ is convex on $R^d$. Moreover, for all $(t,\xi) \in U \times R^d$, $f(j,t,\xi) \leq f_{j+1}(t,\xi)$, and

$$
f(t,\xi) = \sup_{j \in N} f_j(t,\xi),
$$

(11)

(ii) for every $j \in N$, there exists a constant $M_j > 0$, such that, for all $(t,\xi) \in U \times R^d$,

$$
|f_j(t,\xi)| \leq M_j (1 + |\xi|),
$$

(12)

and, for all $t \in U$, and $\xi_1, \xi_2 \in R^d$,

$$
|f_j(t,\xi_1) - f_j(t,\xi_2)| \leq M_j |\xi_1 - \xi_2|.
$$

(13)
3. Proof of Theorem 4

We will divide four steps to complete the proof of Theorem 4.

**Step 1.** Let \( \{ \beta_j(x,s) \}_{j \in \mathbb{N}} \) be a sequence of smooth functions satisfying

1. there exists a compact set \( \Omega' \times H \subset \Omega \times R \), such that \( \beta_j(x,s) = 0 \), for all \( (x,s) \in (\Omega' \setminus \Omega') \times (R \setminus H) \);
2. for every \( i \in \mathbb{N} \), \( \beta_i(x,s) \leq \beta_{i+1}(x,s) \), for all \( (x,s) \in \Omega' \times H \);
3. \( \lim_{i \rightarrow +\infty} \beta_j(x,s) = 1 \), for all \( (x,s) \in \Omega' \times H \).

Let

\[
\ell_j(x,s,\xi) = \beta_j(x,s) f(x,s,\xi), \quad i = 1, 2, \ldots
\]

(14)

It is clear that, for each \( i \in \mathbb{N} \), \( \ell_j \) satisfies all the hypotheses in Theorem 4 and also vanishes if \( (x,s) \) is outside \( \Omega' \times H \); thus

\[
\lim_{i \rightarrow +\infty} \ell_j(x,s,\xi) = f(x,s,\xi), \quad \forall (x,s,\xi) \in \Omega' \times H \times R,
\]

\[
\ell_j(x,s,\xi) \leq \ell_{j+1}(x,s,\xi) \leq f(x,s,\xi),
\]

\[
\forall i \in \mathbb{N}, \quad \forall (x,s,\xi) \in \Omega' \times H \times R.
\]

(15)

By Levi’s Lemma, we have

\[
\lim_{i \rightarrow +\infty} \int_{\Omega'} \ell_j(x,s,\xi) \, dx = \int_{\Omega'} f(x,s,\xi) \, dx.
\]

(16)

Thus, without loss of generality, we can assume that there exists an open set \( \Omega' \times H \subset \subset \Omega \times R \), such that

\[
f(x,s,\xi) = 0, \quad \forall (x,s,\xi) \in (\Omega' \setminus \Omega') \times (R \setminus H) \times R.
\]

(17)

Let \( u_m, u \in W^{1,1}_{\text{loc}}(\Omega) \) such that \( u_m \rightharpoonup u \) in \( L^1_{\text{loc}}(\Omega) \). We will prove that

\[
\liminf_{m \rightarrow +\infty} F(u_m,\Omega) \geq F(u,\Omega).
\]

(18)

Without loss of generality, we can assume that

\[
\liminf_{m \rightarrow +\infty} F(u_m,\Omega) = \lim_{m \rightarrow +\infty} F(u_m,\Omega) < + \infty.
\]

(19)

By (17), we have \( F(u_m,\Omega) = F(u_m,\Omega') \) and \( F(u,\Omega) = F(u,\Omega') \); thus we will only prove the following inequality:

\[
\lim_{m \rightarrow +\infty} F(u_m,\Omega') \geq F(u,\Omega').
\]

(20)

**Step 2.** Let \( \eta_\varepsilon \in C_c^\infty(R) \) be a mollifier, and, for \( \varepsilon > 0 \), define

\[
\nu_\varepsilon(x) = \eta_\varepsilon \ast u(x)
\]

\[
= \int_\Omega \eta_\varepsilon(x-y) u(y) \, dy, \quad x \in [\Omega],
\]

(21)

where \([\Omega] \triangleq \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \varepsilon \}\). We have

\[
[u_\varepsilon(x)]' = \left[ \int_\Omega \eta_\varepsilon(x-y) u(y) \, dy \right]_x
\]

\[
= \int_\Omega [\eta_\varepsilon(x-y)]_x u(y) \, dy
\]

\[
= \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y) [u(y)]_x \, dy = [u'_\varepsilon](x),
\]

\[
x \in \Omega.
\]

In the following, we denote the derivative of \( u \) by \( u'_\varepsilon \). When \( u \in W^{1,1}_{\text{loc}}(\Omega) \), we know \( u'_\varepsilon \in L^1_{\text{loc}}(\Omega) \). By the properties of convolution, we know \( u'_\varepsilon \in C_c^\infty(\Omega) \) and

\[
u_\varepsilon \rightharpoonup u'_\varepsilon \quad \text{in} \quad L^1_{\text{loc}}(\Omega) \quad \text{as} \quad \varepsilon \rightarrow 0^+.
\]

(23)

That is, for all \( \delta > 0 \), \( \exists \varepsilon > 0 \), such that

\[
\int_{\Omega'} [u'_\varepsilon - u'](x) \, dx < \delta.
\]

(24)

Now we estimate the difference for the integrand values on different points:

\[
f(x,u_m,u'_m) - f(x,u,u'_\varepsilon)
\]

\[
= f(x,u_m,u'_m) - f(x,u_m, u'_\varepsilon)
\]

\[
+ f(x,u_m, u'_\varepsilon) - f(x,u, u'_\varepsilon)
\]

\[
+ f(x,u, u'_\varepsilon) - f(x,u,u'_\varepsilon).
\]

(25)

By the convexity of \( f(x,s,\xi) \) with respect to \( \xi \), we have

\[
f(x,u_m,u'_m) - f(x,u_m, u'_\varepsilon)
\]

\[
\geq f(x,u_m,u'_m) \cdot (u'_m - u'_\varepsilon).
\]

\[
= f(x,u_m, u'_\varepsilon) \cdot u'_m - f(x,u,u'_\varepsilon) \cdot u'_m
\]

\[
+ f(x,u, u'_\varepsilon) \cdot (u'_\varepsilon - u'_m)
\]

\[
+ f(x,u,u'_\varepsilon) \cdot (u'_\varepsilon - u'_m).
\]

(26)

By (25) and (26), we have

\[
\int_{\Omega'} [f(x,u_m,u'_m) - f(x,u,u'_\varepsilon)] \, dx
\]

\[
\geq \int_{\Omega'} [f(x,u_m,u'_m) \cdot (u'_m - u'_\varepsilon)] \, dx
\]

\[
+ \int_{\Omega'} [f(x,u,u'_\varepsilon) \cdot (u'_\varepsilon - u'_m)] \, dx
\]

\[
+ \int_{\Omega'} [f(x,u,u'_\varepsilon) - f(x,u,u'_\varepsilon)] \, dx
\]

\[
+ \int_{\Omega'} [f(x,u,u'_\varepsilon) - f(x,u,u'_\varepsilon)] \, dx.
\]

(27)
Step 3. Now, we estimate the right side of inequality (27).

By (6) and (24), we have

$$\int_{\Omega'} [f_\xi(x,u,u'_\epsilon) \cdot (u' - u'_\epsilon)] \, dx \\ \geq -L_1 \int_{\Omega'} |u' - u'_\epsilon| \, dx \geq -L_1 \delta.$$  

(28)

Thus

$$\lim_{\epsilon \to 0} \int_{\Omega'} [f_\xi(x,u,u'_\epsilon) \cdot (u' - u'_\epsilon)] \, dx \geq 0.$$  

(29)

Since $f(x,s,\xi)$ and $f_\xi(x,s,\xi)$ are continuous functions, they are bounded functions on compact subset. By Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{m \to \infty} \int_{\Omega} [f_\xi(x,u_m,u'_\epsilon) \cdot u'_\epsilon] \, dx = 0,$$

$$\lim_{m \to \infty} \int_{\Omega} [f(x,u_m,u'_\epsilon) - f(x,u,u'_\epsilon)] \, dx = 0.$$  

(30)

Now, we will prove

$$\lim_{\epsilon \to 0} \int_{\Omega'} [f(x,u,u'_\epsilon) - f(x,u,u')] \, dx \geq 0.$$  

(31)

By Lemma 6, there exists a sequence of nonnegative continuous functions $f_j(x,s,\xi)$ ($j \in N$), such that $f_j(x,s,\xi)$ is convex on $\xi$, and, for all $(x,s,\xi) \in \Omega' \times H \times R$,

$$f_j(x,s,\xi) \leq f_{j+1}(x,s,\xi),$$

$$f(x,s,\xi) = \sup_{j \in N} f_j(x,s,\xi),$$

(32)

$$|f_j(x,s,\xi_1) - f_j(x,s,\xi_2)| \leq M_j |\xi_1 - \xi_2|.$$  

By Levi’s Lemma, we obtain

$$\lim_{j \to \infty} \int_{\Omega} f_j(x,u,u'_\epsilon) \, dx = \int_{\Omega} f(x,u,u'_\epsilon) \, dx,$$

$$\lim_{j \to \infty} \int_{\Omega} f_j(x,u,u') \, dx = \int_{\Omega} f(x,u,u') \, dx.$$  

(33)

In order to prove (31), we only need to prove

$$\lim_{\epsilon \to 0} \int_{\Omega'} [f_j(x,u,u'_\epsilon) - f_j(x,u,u')] \, dx \geq 0,$$

(34)

$$\forall j \in N.$$  

By (33), we have

$$\int_{\Omega} [f_j(x,u,u'_\epsilon) - f_j(x,u,u')] \, dx$$

$$\geq -M_j \int_{\Omega} |u'_\epsilon - u'| \, dx \geq -M_j \delta.$$  

(35)

Thus (31) holds.

Step 4. Now, we need to prove

$$\lim_{m \to +\infty} \int_{\Omega'} [f_\xi(x,u_m,u'_\epsilon) \cdot u'_\epsilon - f_\xi(x,u,u'_\epsilon) \cdot u'] \, dx = 0.$$  

(36)

Let

$$g(x,s) \equiv f_\xi(x,s,u'_\epsilon), \quad x \in \Omega',$$

$$G_m(x) \equiv \int_{u(x)}^{u_m(x)} g(x,s) \, ds, \quad x \in \Omega'.$$  

(37)

(38)

By triangle inequality and (7), we have

$$|f_\xi(y_i,s,u'_\epsilon(y_i)) - f_\xi(x_i,s,u'_\epsilon(x_i))|$$

$$\leq |f_\xi(y_i,s,u'_\epsilon(y_i)) - f_\xi(x_i,s,u'_\epsilon(y_i))|$$

$$+ |f_\xi(x_i,s,u'_\epsilon(y_i)) - f_\xi(x_i,s,u'_\epsilon(x_i))|$$

(39)

$$+ L_1 |u'_\epsilon(y_i) - u'_\epsilon(x_i)|.$$  

By (39), condition (H2) and $u'_\epsilon \in C^0_0(\Omega)$, we know that $g(x,s)$ is a locally absolute continuous function about $x$. So $g(x,s)$ is almost everywhere differentiable; that is, $\partial g/\partial x$ exists almost everywhere. Taking derivatives in both sides of (38), we obtain

$$G'_m(x) = g(x,u_m) \cdot u'_m - g(x,u) \cdot u'$$

$$+ \int_{u(x)}^{u_m(x)} \frac{\partial g}{\partial x} \, ds, \quad a.e. \ x \in \Omega'.$$  

(40)

Because $G_m(x)$ vanishes outside $\Omega'$, we obtain

$$\int_{\Omega'} G'_m(x) \, dx = 0.$$  

(41)

By (40), we have

$$\left|\int_{\Omega'} [f_\xi(x,u_m,u'_\epsilon) \cdot u'_m - f_\xi(x,u,u'_\epsilon) \cdot u'] \, dx\right|$$

$$= \left|\int_{\Omega} [g(x,u_m) \cdot u'_m - g(x,u) \cdot u'] \, dx\right|$$

$$= -\int_{\Omega'} \frac{\partial g}{\partial x} \, ds \, dx \leq \int_{D_m} \frac{\partial g}{\partial x} \, dx \, ds,$$

where

$$D_m = \{(x,s) \in \Omega' \times H \mid \min |u_m(x), u(x)|$$

$$\leq s(x) \leq \max |u_m(x), u(x)|\}.$$  

(43)

We note

$$|D_m| = \int_{\Omega'} \int_{u(x)}^{u_m(x)} ds \, dx$$

$$\leq \int_{\Omega'} |u_m - u| \, dx \to 0 \quad (m \to +\infty).$$  

(44)
By Fubini’s Theorem, we have
\[
\int_{\Omega \times R} \frac{\partial g}{\partial x} \, dx \, ds = \int_{H} ds \int_{\Omega} \frac{\partial g}{\partial x} \, dx.
\]
(45)
Since \( g(x, s) \) is absolutely continuous about \( x \), \( \frac{\partial g}{\partial x} \) is integrable and absolutely integrable with respect to \( x \); that is,
\[
\int_{\Omega} \left| \frac{\partial g}{\partial x} \right| \, dx < +\infty.
\]
(46)
By (42), we obtain
\[
\int_{\Omega \times R} \left| \frac{\partial g}{\partial x} \right| \, dx \, ds < +\infty.
\]
(47)
Because of the absolute continuity of integral, we have
\[
\lim_{m \to +\infty} \int_{D_{m}} \left| \frac{\partial g}{\partial x} \right| \, dx = 0.
\]
(48)
By (42), we obtain
\[
\lim_{m \to +\infty} \int_{\Omega} \left[ f \left( x, u_{m}, u'_{m} \right) \cdot u'_{m} - f \left( x, u, u' \right) \right] \, dx = 0.
\]
(49)
Thus we just proved (36). By (29)–(31) and (36), we have
\[
\lim_{m \to +\infty} \int_{\Omega} \left[ f \left( x, u_{m}, u'_{m} \right) - f \left( x, u, u' \right) \right] \, dx \geq 0.
\]
(50)
Thus we deduce that the functional \( F(u, \Omega) \) is lower semicontinuous in \( W_{\text{loc}}^{1,1}(\Omega) \) with respect to the strong convergence in \( L_{\text{loc}}^{1}(\Omega) \). We complete the proof.

4. Proof of Theorem 5

In order to prove Theorem 5, we will verify all the conditions in Theorem 4 under the assumptions in Theorem 5. Now we will divide three steps to complete the proof of Theorem 5.

Step 1. Similar to the first step of the proof in Theorem 4, without loss of generality, we assume that the integrand \( f(x, s, \xi) \) vanishes outside a compact subset of \( \Omega' \times H \subset \subset \Omega \times R \). Thus we assume that there exists an open set \( \Omega' \times H \subset \subset \Omega \times R \), such that
\[
f(x, s, \xi) \equiv 0, \quad \forall (x, s, \xi) \in \left( \Omega \setminus \Omega' \right) \times (R \setminus H) \times R.
\]
(51)
Let \( u_{m}, u \in W_{\text{loc}}^{1,1}(\Omega) \), such that \( u_{m} \to u \) in \( L_{\text{loc}}^{1}(\Omega) \); we need to prove
\[
\lim_{m \to +\infty} F \left( u_{m}, \Omega' \right) \geq F \left( u, \Omega' \right).
\]
(52)
By Lemma 6, there exists a function sequence \( \{f_{j}(x, s, \xi)\}_{j \in N} \), such that, for all \( j \in N \), \( f_{j} \) is a continuous function on \( \Omega' \times H \subset \subset \Omega \times R \), for all \( (x, s) \in \Omega' \times H \), \( f_{j}(x, s, \cdot) \) is convex on \( R \), and, for all \( (x, s, \xi) \in \Omega' \times H \times R \),
\[
f_{j}(x, s, \xi) \leq f_{j+\frac{1}{2}}(x, s, \xi),
\]
(53)
\[
f(x, s, \xi) = \sup_{j \in N} f_{j}(x, s, \xi),
\]
(54)
\[
\left| f_{j}(x, s, \xi) - f_{j}(x, s, \xi) \right| \leq M_{j} \left| \xi_{1} - \xi_{2} \right|, \quad (x, s) \in \Omega' \times H, \quad \xi_{1}, \xi_{2} \in R.
\]
(55)
Let \( \eta_{k} \in C_{c}^{\infty}(R) \) \((0 < \epsilon < 1) \) be a mollifier, and define the \( f_{j, \epsilon} = f_{j} \ast \eta_{k} \); that is,
\[
f_{j, \epsilon}(x, s, \xi) = \int_{R} f_{j}(x, s, \xi - \epsilon) \eta_{k}(\epsilon) \, d\epsilon.
\]
(56)
By (55), we have
\[
\left| f_{j, \epsilon}(x, s, \xi) - f_{j}(x, s, \xi) \right| \leq \int_{R} \left| f_{j}(x, s, \xi - \epsilon) - f_{j}(x, s, \xi) \right| \eta_{k}(\epsilon) \, d\epsilon
\]
\[
\leq \int_{\sup \eta_{k}} M_{j} \left| \xi \right| \eta_{k}(\epsilon) \, d\epsilon \leq M_{j} \cdot \epsilon.
\]
(57)
Choose \( \epsilon = \epsilon_{j} = 1/jM_{j} \to 0 \). By (57), we have
\[
\left| f_{j, \epsilon}(x, s, \xi) - f_{j}(x, s, \xi) \right| \leq M_{j} \epsilon_{j} = \frac{1}{j}.
\]
(58)
So
\[
f_{j}(x, s, \xi) - \frac{2}{j} \leq f_{j, \epsilon}(x, s, \xi) - \frac{1}{j}
\]
\[
\leq f_{j}(x, s, \xi) \leq f_{j}(x, s, \xi).
\]
(59)
By (53), (54), and Levi’s Lemma, we have
\[
\lim_{j \to +\infty} \int_{\Omega'} f_{j}(x, u(x), u'(x)) \, dx = \int_{\Omega'} f(x, u(x), u'(x)) \, dx.
\]
(60)
Let
\[
F_{j}(u, \Omega') = \int_{\Omega'} \left[ f_{j, \epsilon}(x, u(x), u'(x)) - \frac{1}{j} \right] \, dx.
\]
(61)
By (59)–(61), we have
\[
\lim_{j \to +\infty} F_{j}(u, \Omega') = F(u, \Omega')
\]
(62)
Obviously,
\[
F_{j}(u, \Omega') \leq F(u, \Omega'), \quad \forall j \in N.
\]
(63)
Thus
\[ \sup_{j \in \mathbb{N}} F_j(u, \Omega') = F(u, \Omega'). \]  
(64)

Therefore \( F(u, \Omega') \), being the supremum of the family of functionals \( \{ F_j(u, \Omega') \}_{j \in \mathbb{N}} \), will be lower semicontinuous if every \( \{ F_j(u, \Omega') \} \) is lower semicontinuous.

**Step 2.** In order to prove that, for all \( j \in \mathbb{N}, F_j(u, \Omega') \) is lower semicontinuous in \( W^{1,1}_{\text{loc}}(\Omega) \) with respect to the strong convergence in \( L^1_{\text{loc}}(\Omega) \), we will prove that, for all \( j \in \mathbb{N} \), the integrand \( f_{j, \varepsilon}(x, u(x), u'(x)) \) satisfies all conditions of Theorem 4. Obviously, for all \( j \in \mathbb{N}, f_{j, \varepsilon} \) satisfies condition (H1).

For all \( (x, s) \in \Omega' \times H \) and \( \xi_1, \xi_2 \in R \), by (55), we have
\[
\left| f_{j, \varepsilon}(x, s, \xi_1) - f_{j, \varepsilon}(x, s, \xi_2) \right| \\
\leq \int_{R} \left| f_{j}(x, s, \xi_1 - z) - f_{j}(x, s, \xi_2 - z) \right| \cdot \eta_{\varepsilon}(z) \, dz \\
\leq \int_{\text{supp} \eta_{\varepsilon}} M_j \left| \xi_1 - \xi_2 \right| \eta_{\varepsilon}(z) \, dz \leq M_j \left| \xi_1 - \xi_2 \right|. 
\]  
(65)

Thus
\[ \frac{\partial f_{j, \varepsilon}}{\partial \xi} (x, s, \xi) \leq M_j. \]  
(66)

So \( f_{j, \varepsilon} \) satisfies (6) in condition (H3) of Theorem 4.

Now, we will prove that \( f_{j, \varepsilon} \) satisfies (7) in condition (H3) of Theorem 4. By \( \text{supp} \eta_{\varepsilon} \subseteq B(0, \varepsilon_j) \), we have
\[
\frac{\partial f_{j, \varepsilon}}{\partial \xi} (x, s, \xi) = \int_{R} \frac{\partial f_{j}}{\partial \xi} (x, s, \xi - z) \cdot \eta_{\varepsilon}(z) \, dz \\
= - \int_{R} \frac{\partial f_{j}}{\partial z} (x, s, \xi - z) \cdot \eta_{\varepsilon}(z) \, dz \\
= \int_{R} f_{j}(x, s, \xi - z) \frac{\partial \eta_{\varepsilon}(z)}{\partial z} \, dz. 
\]  
(67)

By (55) and (67), we have
\[
\left| \frac{\partial f_{j, \varepsilon}}{\partial \xi} (x, s, \xi_1) - \frac{\partial f_{j, \varepsilon}}{\partial \xi} (x, s, \xi_2) \right| \\
\leq \int_{R} \left| f_{j}(x, s, \xi_1 - z) - f_{j}(x, s, \xi_2 - z) \right| \cdot \left| \frac{\partial \eta_{\varepsilon}(z)}{\partial z} \right| \, dz \\
\leq M_j \left| \xi_1 - \xi_2 \right| \int_{R} \left| \frac{\partial \eta_{\varepsilon}(z)}{\partial z} \right| \, dz = L_j M_j \left| \xi_1 - \xi_2 \right|, 
\]  
(68)

where
\[ L_j = \int_{R} \left| \frac{\partial \eta_{\varepsilon}(z)}{\partial z} \right| \, dz. \]  
(69)

is a constant depending on \( \varepsilon_j \). Thus \( f_{j, \varepsilon} \) satisfies (7). So we proved that \( f_{j, \varepsilon} \) satisfies condition (H3).

**Step 3.** Next we will prove that \( f_{j, \varepsilon} \) satisfies condition (H2).

By condition (H4), for every compact subset \( \Omega' \times H \times K, f(x, s, \xi) \) is absolutely continuous about \( x \), that is, for all \( \varepsilon_0 > 0, \exists \delta > 0 \) such that for any finite disjoint open interval \( \{y_i, y_j\}_{i=1}^{n} \in \Omega' \), when \( \sum_{i=1}^{n} (y_j - y_i) < \delta \), we have
\[
\sum_{i=1}^{n} \left| f(y_j, s, \xi) - f(x_j, s, \xi) \right| < \varepsilon_0. 
\]  
(70)

By Step 1, \( \{ f_{j}(x, s, \xi) \}_{j \in \mathbb{N}} \) satisfies (53)-(55) and the following property:
\[
f_{j}(x, s, \xi) = \max_{1 \leq q \leq j} \left\{ a_q(x, s) + b_q(x, s) \xi \right\}, \quad j \in \mathbb{N}, \]  
(71)

where
\[
a_q(x, s) = \int_{R} f(x, s, \rho) \left[ 2\eta_q(\rho) + \rho \frac{\partial \eta_q(\rho)}{\partial \rho} \right] d\rho, \]  
(72)

\[
b_q(x, s) = - \int_{R} f(x, s, \rho) \frac{\partial \eta_q(\rho)}{\partial \rho} d\rho, \]  
(73)

And, for all \( (x, s, \xi) \in \Omega' \times H \times R, \eta_q \in C^0_c(R) (q \in \mathbb{N}) \) are mollifiers satisfying \( \eta_q \geq 0, \int_{R} \eta_q(\rho) d\rho = 1 \), and \( \text{supp} \eta_q \subseteq B(0, 1) \), for all \( j \in \mathbb{N} \). By (71), without of loss generality, we assume that there exists \( l \in \{1, \ldots, j\} \), such that
\[
f_{j}(x, s, \xi) = a_l(x, s) + b_l(x, s) \cdot \xi, \]  
(74)

where \( a_l, b_l \) are given by (72). By (70), we obtain
\[
\sum_{i=1}^{n} \left| a_l(y_j, s) - a_l(x_j, s) \right| \leq \int_{R} \left| f(y_j, s, \rho) - f(x_j, s, \rho) \right| d\rho \leq \varepsilon_0 \int_{B(0,1)} \left[ 2\eta_l(\rho) + \rho \frac{\partial \eta_l(\rho)}{\partial \rho} \right] d\rho \\
\leq \frac{1}{2} A_1 \varepsilon_0, 
\]  
(75)
is a constant. Similar to the above proof, we have

$$
\sum_{i=1}^{n} [b_i (y_i, s) - b_i (x_i, s)] \\
\leq \int_{R} \sum_{i=1}^{n} [f (y_i, s, \rho) - f (x_i, s, \rho)] \cdot \left| \frac{\partial \eta_i (\rho)}{\partial \rho} \right| d\rho \\
\leq \varepsilon_0 \int_{B(0,1)} \left| \frac{\partial \eta_i (\rho)}{\partial \rho} \right| d\rho \leq A_1 \cdot \varepsilon_0.
$$

Thus

$$
\sum_{i=1}^{n} [f_i (y_i, s, \xi) - f_i (x_i, s, \xi)] \\
\leq \sum_{i=1}^{n} [a_i (y_i, s) - a_i (x_i, s)] \\
+ \sum_{i=1}^{n} [b_i (y_i, s) - b_i (x_i, s)] \cdot |\xi| \\
\leq (2 + A_1) \varepsilon_0 + A_1 \varepsilon_0 K_1 = (2 + A_1 + A_1 K_1) \varepsilon_0 \equiv \sigma.
$$

Since $\xi$ belongs to a compact set, then $K_1 = \sup_{|\xi|} = +\infty$. Choose $\varepsilon_0$ sufficient small, so that $\sigma$ is small enough. Thus $f_i (x, s, \xi)$ is absolutely continuous about $x$ for all $(x, s, \xi) \in A$, which is a compact subset of $\Omega \times R \times R$. By (56) and (77), we have

$$
\sum_{i=1}^{n} \left| f_{i, \varepsilon_i} (y_i, s, \xi) - f_{i, \varepsilon_i} (x_i, s, \xi) \right| \\
\leq \int_{R} \sum_{i=1}^{n} \left| f_j (y_i, s, \xi - z) - f_j (x_i, s, \xi - z) \right| \cdot \eta_i (z) dz \\
\leq \sigma \int_{B(0,\varepsilon_i)} \eta_i (z) dz = \sigma.
$$

(78)

By (67) and (78), we obtain

$$
\sum_{i=1}^{n} \left| \frac{\partial f_{i, \varepsilon_i}}{\partial \xi} (y_i, s, \xi) - \frac{\partial f_{i, \varepsilon_i}}{\partial \xi} (x_i, s, \xi) \right| \\
\leq \int_{R} \sum_{i=1}^{n} \left| f_j (y_i, s, \xi - z) - f_j (x_i, s, \xi - z) \right| \cdot \left| \frac{\partial \eta_i (z)}{\partial z} \right| dz \\
\leq \sigma \int_{R} \left| \frac{\partial \eta_i (z)}{\partial z} \right| dz = L_1 \sigma.
$$

(79)

where $L_1$ are constants depending on $\varepsilon_i$ and given by (69) (for all $j \in N$). By (79), for every compact subset on $\Omega \times R \times R$, $\partial f_{j, \varepsilon_i} / \partial \xi$ is absolutely continuous about $x$. Thus $f_{j, \varepsilon_i}$ satisfies condition (H2).

Now, we have proved $f_{j, \varepsilon_i}$ satisfies all conditions in Theorem 4, so $F_j (u, \Omega')$ is lower semicontinuous in $W_{loc}^{1,1} (\Omega)$ with respect to the strong convergence in $L^1_{loc} (\Omega)$. Thus $F(u, \Omega)$ has the same lower semicontinuity. This completes the proof of Theorem 5.

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References

[1] R. A. Adams and J. F. Fournier, Sobolev Space, Academic Press, New York, NY, USA, 2nd edition, 2003.
[2] C. Y. Pauc, La Méthode Métrique en Calcul des Variations, Hermann, Paris, France, 1941.
[3] J. Serrin, "On the definition and properties of certain variational integrals," Transactions of the American Mathematical Society, vol. 101, pp. 139–167, 1961.
[4] L. Ambrosio, N. Fusco, and D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Monographs, Oxford University Press, New York, NY, USA, 2000.
[5] V. de Cicco and G. Leoni, "A chain rule in $L^1 (div; \Omega)$ and its applications to lower semicontinuity," Calculus of Variations and Partial Differential Equations, vol. 19, no. 1, pp. 23–51, 2004.
[6] G. Fonseca and G. Leoni, "Some remarks on lower semicontinuity," Indiana University Mathematics Journal, vol. 49, no. 2, pp. 617-655, 2000.
[7] G. Fonseca and G. Leoni, "On lower semicontinuity and relaxation," Proceedings of the Royal Society of Edinburgh A, vol. 131, no. 3, pp. 519–565, 2001.
[8] M. Gori and P. Marcellini, "An extension of the Serrin’s lower semicontinuity theorem," Journal of Convex Analysis, vol. 9, no. 2, pp. 475–502, 2002.
[9] M. Gori, F. Maggi, and P. Marcellini, "On some sharp conditions for lower semicontinuity in $L^1$," Differential and Integral Equations, vol. 16, no. 1, pp. 51–76, 2003.
[10] E. de Giorgi, Teoremi di semicontinuit'a nel calcolo delle variazioni, Istituto Nazionale di Alta Matematica, Rome, Italy, 1968.