An improved upper bound for the number of distinct eigenvalues of a matrix after perturbation

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Abstract

An upper bound for the number of distinct eigenvalues of a perturbed matrix has been recently established by P. E. Farrell [1, Theorem 1.3]. The estimate is the central result in Farrell’s work and can be applied to estimate the number of Krylov iterations required for solving a perturbed linear system. In this paper, we present an improved upper bound for the number of distinct eigenvalues of a matrix after perturbation. Furthermore, some results based on the improved estimate are presented.

Keywords: Distinct eigenvalues; Perturbation; Defectivity; Derogatory index

1. Introduction

The spectrum of a matrix after perturbation has been investigated by many authors. However, most work is devoted to discussing some special cases, especially the case of symmetric rank-one perturbations; see, for instance, [2-5]. Recently, P. E. Farrell [1] presented an upper bound for the number of distinct eigenvalues of arbitrary matrices perturbed by updates of arbitrary rank. Let $\mathbb{C}^{n\times n}$, $\Lambda(\cdot)$, $\text{rank}(\cdot)$, and $|\cdot|$ be the set of all $n \times n$ complex matrices, the set of all distinct eigenvalues of a square matrix, the rank of a matrix, and the cardinality of a set, respectively. Assume that $A, B \in \mathbb{C}^{n\times n}$, and let $C = A + B$, it is proved by Farrell [1, Theorem 1.3] that

$$|\Lambda(C)| \leq (\text{rank}(B) + 1)|\Lambda(A)| + d(A),$$  \hspace{1cm} (1.1)

where $d(\cdot)$ denotes the defectivity of a matrix (see Definition 2.2 below). The result can be used to estimate the number of Krylov iterations for solving a linear system.

It follows from the definitions of $\Lambda(\cdot)$ and $|\cdot|$ that $|\Lambda(M)| \leq n$ for all $M \in \mathbb{C}^{n\times n}$. Given $A \in \mathbb{C}^{n\times n}$, we can observe that the estimate (1.1) is mainly of interest in the situation that $\text{rank}(B)$ is small, that is to say, $B$ is a low-rank perturbation. More specifically, if $\text{rank}(B) \leq \frac{n-d(A)}{|\Lambda(A)|} - 1$,
then \((\text{rank}(B) + 1)|\Lambda(A)| + d(A) \leq n\) is an applicable upper bound. On the other hand, if \(\text{rank}(B) > \frac{n - d(A)}{|\Lambda(A)|} - 1\), then \((\text{rank}(B) + 1)|\Lambda(A)| + d(A) > n\) is a trivial upper bound.

Nevertheless, the estimate (1.1) is not sharp in certain cases. We now give a specific example to illustrate the defect of the upper bound in (1.1). We choose a matrix \(A\) as follows:

\[
A = \begin{pmatrix}
\lambda_0 & 1 & 0 & \ldots & 0 \\
0 & \lambda_0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_0 & 1 \\
0 & 0 & \ldots & 0 & \lambda_0
\end{pmatrix},
\]

which is a \(n \times n\) Jordan block. Thus, \(|\Lambda(A)| = 1\) and \(d(A) = n - 1\). Let \(n \times n\) matrix \(B_r\) with rank \(r\) \((1 \leq r \leq n - 1)\) be defined by

\[
B_r = \begin{pmatrix} T_r & O_{r \times (n-r-1)} \\ O_{(n-r) \times (r+1)} & O_{(n-r) \times (n-r-1)} \end{pmatrix}, \quad T_r = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & r & -1 \end{pmatrix}.
\]

Hence, the upper bound in the estimate (1.1) is \((\text{rank}(B_r) + 1)|\Lambda(A)| + d(A) = n + r > n\). In this case, the upper bound in (1.1) is always invalid (i.e., the upper bound is strictly greater than order \(n\)) for all \(1 \leq r \leq n - 1\).

In this paper, we give an improved upper bound for the number of distinct eigenvalues of a matrix after perturbation. Under the same assumptions, we establish that

\[
|\Lambda(C)| \leq (\text{rank}(B) + 1)|\Lambda(A)| + d(A) - d(C).
\]  

Applying (1.2) to the above example, we can derive that the improved upper bound of \(|\Lambda(C_r)|\) (here \(C_r = A + B_r\)) is \((\text{rank}(B_r) + 1)|\Lambda(A)| + d(A) - d(C_r) = (n + r) - (n - 1 - r) = 2r + 1\), which is an applicable upper bound, especially in low-rank perturbations.

We give another numerical example to explain our estimate in (1.2). Let

\[
A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\]
Thus, the perturbed matrix $C$ and its Jordan canonical form $J$ (regardless of the permutations of its diagonal Jordan blocks) are

$$
C = \begin{pmatrix}
2 & 2 & 1 & 1 & 1 \\
1 & 2 & 2 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

and

$$
J = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \frac{5-\sqrt{13}}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{5+\sqrt{13}}{2}
\end{pmatrix},
$$

respectively. Since, $|\Lambda(A)| = 1$, $d(A) = 2$, $\text{rank}(B) = 1$, and $d(C) = 1$, then the upper bounds in (1.1) and (1.2) are

$$(\text{rank}(B) + 1)|\Lambda(A)| + d(A) = 4 \quad \text{and} \quad (\text{rank}(B) + 1)|\Lambda(A)| + d(A) - d(C) = 3,$$

respectively. Notice that $|\Lambda(C)| = 3$, which attains the upper bound in (1.2).

2. Preliminaries

In this section, we introduce some basic notations and concepts which are frequently used in the subsequent content. Let $\lambda \in \Lambda(M)$. The multiplicity of $\lambda$ as a zero of the characteristic polynomial $f_M(\cdot)$ is called its \textit{algebraic multiplicity}. The dimension of the eigenspace of $M$ corresponding to $\lambda$ is called its \textit{geometric multiplicity}. Let $m_a(M, \lambda)$ and $m_g(M, \lambda)$ denote the algebraic and geometric multiplicity of $\lambda$ as an eigenvalue of $M$, respectively.

Actually, the geometric multiplicity of an eigenvalue $\lambda \in \Lambda(M)$ coincides with the number of Jordan blocks corresponding to $\lambda$ in the Jordan canonical form of $M$. Evidently, $m_g(M, \lambda) \geq 1$ for all $\lambda \in \Lambda(M)$. If $m_g(M, \lambda) \equiv 1$ for all $\lambda \in \Lambda(M)$, then $M$ is said to be \textit{nonderogatory}; otherwise, $M$ is called a \textit{derogatory} matrix. Recall that the geometric multiplicity of an eigenvalue is not greater than its algebraic multiplicity. If $m_g(M, \lambda) < m_a(M, \lambda)$ for some $\lambda \in \Lambda(M)$, then $M$ is called a \textit{defective} matrix. If $m_g(M, \lambda) = m_a(M, \lambda)$ for all $\lambda \in \Lambda(M)$, then $M$ is said to be \textit{nondefective}. Thus, a matrix $M$ is diagonalizable if and only if $M$ is nondefective; see, for example, [6, p. 58].

We next introduce the definitions of the \textit{defectivity} of an eigenvalue and a matrix; see [1, Definitions 1.1 and 1.2]. In addition, we define the \textit{derogatory index} of a matrix in Definition 2.3 below.

\textbf{Definition 2.1.} The \textit{defectivity of an eigenvalue} $\lambda \in \Lambda(M)$ is denoted by $d(M, \lambda)$, which is the
difference between its algebraic and geometric multiplicities, i.e.,
\[ d(M, \lambda) := m_a(M, \lambda) - m_g(M, \lambda). \]  
(2.1)

**Definition 2.2.** The *defectivity of a matrix* \( M \) is denoted by \( d(M) \), which is the sum of the defectivities of its eigenvalues, i.e.,
\[ d(M) := \sum_{\lambda \in \Lambda(M)} (m_a(M, \lambda) - m_g(M, \lambda)) = \sum_{\lambda \in \Lambda(M)} d(M, \lambda). \]  
(2.2)

Because of \( m_a(M, \lambda) \geq m_g(M, \lambda) \geq 1 \) for all \( \lambda \in \Lambda(M) \), we obtain \( d(M, \lambda) \geq 0 \) and \( d(M) \geq 0 \). Indeed, the defectivity of a matrix can be considered as a quantitative measure of its nondiagonalizability since \( M \) is diagonalizable if and only if \( d(M) = 0 \). Moreover, it is clear that the defectivity of a matrix is equal to the number of off-diagonal ones in its Jordan canonical form; see [1, Remark 1].

**Definition 2.3.** The *derogatory index* of a matrix \( M \) is denoted by \( I(M) \), which is defined by
\[ I(M) := \sum_{\lambda \in \Lambda(M)} (m_g(M, \lambda) - 1). \]  
(2.3)

Let \( M \in \mathbb{C}^{n \times n} \). Notice that since \( \sum_{\lambda \in \Lambda(M)} m_a(M, \lambda) = n \), from (2.2), we have \( d(M) = n - \sum_{\lambda \in \Lambda(M)} m_g(M, \lambda) \). Thus, we obtain \( |\Lambda(M)| + d(M) \leq n \) and \( I(M) = \sum_{\lambda \in \Lambda(M)} (m_g(M, \lambda) - 1) = n - d(M) - |\Lambda(M)| \). Therefore, the quantity \( I(M) \) satisfies \( 0 \leq I(M) \leq n - 1 \) for all \( M \in \mathbb{C}^{n \times n} \).

We can find that \( I(M) = n - 1 \) if and only if \( M \) is a scalar matrix (i.e., \( M = cI \) for some \( c \in \mathbb{C} \)), and \( I(M) = 0 \) if and only if \( M \) is nonderogatory.

We finally mention the relation between the degree of minimal polynomial and the number of distinct eigenvalues of a matrix. It is well known that a matrix \( M \) can be diagonalized if and only if every zero of its minimal polynomial \( q_M(t) \) has multiplicity one; see, for instance, [6, Corollary 3.3.10]. If \( M \) is diagonalizable, then \( |\Lambda(M)| \) is equal to the degree of \( q_M(t) \) since all distinct eigenvalues of \( M \) are the roots of \( q_M(t) = 0 \); otherwise, \( |\Lambda(M)| \) is strictly less than the degree of \( q_M(t) \).

### 3. Main result

We now give the main result of this paper, which provides an improved upper bound for the number of distinct eigenvalues of a perturbed matrix.
Theorem 3.1. Assume that $A, B \in \mathbb{C}^{n \times n}$ and let $C = A + B$, then

$$|\Lambda(C)| \leq (\text{rank}(B) + 1)|\Lambda(A)| + d(A) - d(C).$$

Proof. We first define $S_1 := \Lambda(C) \cap \Lambda(A)$ and $S_2 := \Lambda(C) \setminus \Lambda(A) = \Lambda(C) \cap (\Lambda(A))^c$, where $(\cdot)^c$ denotes the complement of a set. Then we have

$$|\Lambda(C)| = |\Lambda(C) \cap \Lambda(A)| + |\Lambda(C) \setminus \Lambda(A)| = |S_1| + |S_2|. \tag{3.1}$$

We next take into account the case that $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. Notice that an upper bound for $\sum_{\lambda \in S_2} m_a(C, \lambda)$ is established in the proof of Farrell’s work [1, Theorem 1.3] and it is crucial for Farrell’s proof. Alternatively, we intend to seek an appropriate upper bound for the quantity $\sum_{\lambda \in S_2} m_g(C, \lambda)$. For any $M \in \mathbb{C}^{n \times n}$, we have $\sum_{\lambda \in \Lambda(M)} m_a(M, \lambda) = n$, and it follows that

$$\sum_{\lambda \in S_1} m_a(C, \lambda) + \sum_{\lambda \in S_2} m_a(C, \lambda) = \sum_{\lambda \in \Lambda(A)} m_a(C, \lambda) + \sum_{\lambda \in S_2} m_a(C, \lambda) = n, \tag{3.2}$$

with the convention that $m_a(M, \lambda) = 0$ (hence, $m_g(M, \lambda) = 0$) if and only if $\lambda \notin \Lambda(M)$.

Let $\lambda$ be an arbitrary eigenvalue of $A$. If $\lambda \in S_1$, by $\text{rank}(M_1 + M_2) \leq \text{rank}(M_1) + \text{rank}(M_2)$ for all $M_1, M_2 \in \mathbb{C}^{n \times n}$, we obtain

$$m_g(C, \lambda) = n - \text{rank}(\lambda I - C) \geq n - \text{rank}(\lambda I - A) - \text{rank}(B) = m_g(A, \lambda) - \text{rank}(B).$$

If $\lambda \in \Lambda(A) \setminus S_1$, then $\lambda I - C$ is nonsingular, which implies that $n - \text{rank}(\lambda I - C) = 0$, and we have $m_g(C, \lambda) = 0$ and $m_g(A, \lambda) - \text{rank}(B) \leq 0$. Thus, we obtain

$$\sum_{\lambda \in \Lambda(A)} m_a(C, \lambda) = \sum_{\lambda \in S_1} m_a(C, \lambda) \tag{3.3a}$$

$$= \sum_{\lambda \in S_1} (m_g(C, \lambda) + d(C, \lambda)) \tag{3.3b}$$

$$\geq \sum_{\lambda \in S_1} (m_g(A, \lambda) - \text{rank}(B) + d(C, \lambda))$$

$$+ \sum_{\lambda \in \Lambda(A) \setminus S_1} (m_g(A, \lambda) - \text{rank}(B)) \tag{3.3c}$$

$$= \sum_{\lambda \in \Lambda(A)} (m_a(A, \lambda) - \text{rank}(B)) + \sum_{\lambda \in S_1} d(C, \lambda) \tag{3.3d}$$

$$= \sum_{\lambda \in \Lambda(A)} (m_a(A, \lambda) - \text{rank}(B) - d(A, \lambda)) + \sum_{\lambda \in S_1} d(C, \lambda) \tag{3.3e}$$

$$= n - |\Lambda(A)| \cdot \text{rank}(B) - d(A) + \sum_{\lambda \in S_1} d(C, \lambda), \tag{3.3f}$$
where we have used the definitions (2.1) and (2.2). It is worth mentioning that the key difference with the proof of Farrell [1] is the steps (3.3b) and (3.3c). By relation (3.2), we can obtain

\[ \sum_{\lambda \in S_2} m_g(C, \lambda) + \sum_{\lambda \in S_2} d(C, \lambda) = \sum_{\lambda \in S_2} m_a(C, \lambda) \]

\[ \leq n - \left( n - |\Lambda(A)| \cdot \text{rank}(B) - d(A) + \sum_{\lambda \in S_1} d(C, \lambda) \right) \]

\[ = |\Lambda(A)| \cdot \text{rank}(B) + d(A) - \sum_{\lambda \in S_1} d(C, \lambda). \]

Then

\[ \sum_{\lambda \in S_2} m_g(C, \lambda) \leq |\Lambda(A)| \cdot \text{rank}(B) + d(A) - \left( \sum_{\lambda \in S_1} d(C, \lambda) + \sum_{\lambda \in S_2} d(C, \lambda) \right), \]

in other words,

\[ \sum_{\lambda \in S_2} m_g(C, \lambda) \leq |\Lambda(A)| \cdot \text{rank}(B) + d(A) - d(C). \] (3.5)

Since \(|S_1| \leq |\Lambda(A)|\) and \(|S_2| \leq \sum_{\lambda \in S_2} m_g(C, \lambda)\), from (3.1) and (3.5), respectively, we deduce that

\[ |\Lambda(C)| = |S_1| + |S_2| \leq (\text{rank}(B) + 1)|\Lambda(A)| + d(A) - d(C). \] (3.6)

We finally consider two special cases: either \(S_1\) or \(S_2\) is empty. If \(S_1 = \emptyset\), i.e., \(\Lambda(C) \cap \Lambda(A) = \emptyset\), then \(|S_1| = 0\) and \(S_2 = \Lambda(C)\). Repeating the derivation in (3.4) yields

\[ |\Lambda(A)| \cdot \text{rank}(B) + d(A) \geq n. \]

Since

\[ d(C) = \sum_{\lambda \in \Lambda(C)} d(C, \lambda) = \sum_{\lambda \in \Lambda(C)} (m_a(C, \lambda) - m_g(C, \lambda)) = n - \sum_{\lambda \in \Lambda(C)} m_g(C, \lambda) \]

and \(m_g(C, \lambda) \geq 1\) for all \(\lambda \in \Lambda(C)\), we conclude that \(|\Lambda(C)| + d(C) \leq n\) is always valid. Hence,

\[ |\Lambda(C)| + d(C) \leq n < (\text{rank}(B) + 1)|\Lambda(A)| + d(A). \] (3.7)

If \(S_2 = \emptyset\), i.e., \(\Lambda(C) \subseteq \Lambda(A)\), then \(|\Lambda(C)| \leq |\Lambda(A)|\), \(S_1 = \Lambda(C)\), and \(|S_2| = 0\). We repeat the derivation in (3.3) and obtain

\[ n = \sum_{\lambda \in \Lambda(A)} m_a(C, \lambda) \geq \sum_{\lambda \in \Lambda(A)} (m_a(A, \lambda) - \text{rank}(B) - d(A, \lambda)) + \sum_{\lambda \in \Lambda(C)} d(C, \lambda) \]

\[ = n - |\Lambda(A)| \cdot \text{rank}(B) - d(A) + d(C), \]
then
\[ d(C) \leq |\Lambda(A)| \cdot \text{rank}(B) + d(A). \]

Using \(|\Lambda(C)| \leq |\Lambda(A)|\) yields
\[ |\Lambda(C)| + d(C) \leq (\text{rank}(B) + 1)|\Lambda(A)| + d(A). \tag{3.8} \]

Consequently, from (3.6)-(3.8), we conclude that \(|\Lambda(C)| \leq (\text{rank}(B) + 1)|\Lambda(A)| + d(A) - d(C)\) always hold. This completes the proof. \(\square\)

**Remark 3.2.** If the perturbed matrix \(C\) is diagonalizable (i.e., \(d(C) = 0\)), by Theorem 3.1, we obtain
\[ |\Lambda(C)| \leq (\text{rank}(B) + 1)|\Lambda(A)| + d(A), \]
which is exactly the inequality (1.1); otherwise, we have \(d(C) \geq 1\), then Theorem 3.1 gives
\[ |\Lambda(C)| \leq (\text{rank}(B) + 1)|\Lambda(A)| + d(A) - 1, \]
which is a smaller upper bound than the estimate (1.1).

4. Applications

In view of the improved upper bound for the number of distinct eigenvalues of a matrix after perturbation, we can establish some interesting results.

The following Corollary 4.1 provides a lower bound for the derogatory index of a matrix after perturbation.

**Corollary 4.1.** Let \(A, B \in \mathbb{C}^{n \times n}\) and \(C = A + B\), the derogatory index of the perturbed matrix \(C\) satisfies
\[ \mathcal{I}(C) \geq \mathcal{I}(A) - \text{rank}(B) \cdot |\Lambda(A)|. \]

**Proof.** We note that the definition (2.3) implies \(\mathcal{I}(M) = n - (|\Lambda(M)| + d(M))\) for all \(M \in \mathbb{C}^{n \times n}\). The statement follows immediately from Theorem 3.1. \(\square\)

The next corollary plays an important role in the estimate for the number of Krylov iterations after a rank one update.

**Corollary 4.2.** Suppose that \(A \in \mathbb{C}^{n \times n}\) is diagonalizable, \(\text{rank}(B) = 1\), and let \(C = A + B\). If \(C\) is also diagonalizable, then \(|\Lambda(C)| \leq 2|\Lambda(A)|\). If \(C\) is not diagonalizable, then \(|\Lambda(C)| \leq 2|\Lambda(A)| - 1\).
Proof. On the basis of Theorem 3.1, we have $|\Lambda(C)| \leq 2|\Lambda(A)| + d(A) - d(C) = 2|\Lambda(A)| - d(C)$ since $A$ is diagonalizable. If $C$ is diagonalizable, then $d(C) = 0$ which yields $|\Lambda(C)| \leq 2|\Lambda(A)|$. If $C$ cannot be diagonalized, then $d(C) \geq 1$ and we obtain $|\Lambda(C)| \leq 2|\Lambda(A)| - 1$. □

Corollary 4.3. Assume that $A, B \in \mathbb{C}^{n \times n}$ and let $C = A + B$. If $C$ is nonderogatory, then the number of distinct eigenvalues of $A$ satisfies

$$\frac{n - d(A)}{\text{rank}(B) + 1} \leq |\Lambda(A)| \leq n - d(A).$$

Proof. Since $C$ is nonderogatory, we have $\mathcal{I}(C) = 0$, i.e., $|\Lambda(C)| + d(C) = n$. According to Theorem 3.1, we obtain $n \leq (\text{rank}(B) + 1)|\Lambda(A)| + d(A)$, which implies that $|\Lambda(A)| \geq \frac{n - d(A)}{\text{rank}(B) + 1}$. Notice that $|\Lambda(A)| \leq n - d(A)$ is clear. □

Given any $A \in \mathbb{C}^{n \times n}$, it can be decomposed as $A = \mathcal{H}(A) + S(A)$, where $\mathcal{H}(A) = \frac{1}{2}(A + A^*)$ and $S(A) = \frac{1}{2}(A - A^*)$ are the Hermitian and skew-Hermitian part of $A$, respectively. By Theorem 3.1, we can obtain the following inequality:

Corollary 4.4. Let $A \in \mathbb{C}^{n \times n}$, we have the following estimate

$$|\Lambda(A)| \leq \min \{ (\text{rank}(\mathcal{H}(A)) + 1) |\Lambda(S(A))|, (\text{rank}(S(A)) + 1) |\Lambda(\mathcal{H}(A))| \} - d(A). \tag{4.1}$$

Proof. Notice that both $\mathcal{H}(A)$ and $S(A)$ are normal matrices, which can be unitarily diagonalized. Thus, the inequality (4.1) follows from Theorem 3.1. □

Remark 4.5. Since $\mathcal{H}(A)$ and $S(A)$ are diagonalizable, we have $|\Lambda(\mathcal{H}(A))| \leq \text{rank}(\mathcal{H}(A)) + 1$ and $|\Lambda(S(A))| \leq \text{rank}(S(A)) + 1$. From inequality (4.1), we can deduce that $|\Lambda(A)| \leq (\text{rank}(\mathcal{H}(A)) + 1) (\text{rank}(S(A)) + 1) - d(A)$.

Remark 4.6. For any $A \in \mathbb{C}^{n \times n}$ and $\alpha \in \mathbb{C}$, $A$ can also be decomposed as

$$A = (\mathcal{H}(A) + \alpha I) + (S(A) - \alpha I) = \mathcal{H}_\alpha(A) + S_\alpha(A),$$

where $\mathcal{H}_\alpha(A) := \mathcal{H}(A) + \alpha I$ and $S_\alpha(A) := S(A) - \alpha I$. Notice that $\mathcal{H}_\alpha(A)$ and $S_\alpha(A)$ are normal for all $\alpha \in \mathbb{C}$, then $\mathcal{H}_\alpha(A)$ and $S_\alpha(A)$ can be unitarily diagonalized. Since the parameter $\alpha$ is arbitrary, the estimate (4.1) can be modified as

$$|\Lambda(A)| \leq \inf_{\alpha \in \mathbb{C}} \min \{ (\text{rank}(\mathcal{H}_\alpha(A)) + 1) |\Lambda(S_\alpha(A))|, (\text{rank}(S_\alpha(A)) + 1) |\Lambda(\mathcal{H}_\alpha(A))| \} - d(A).$$

Similarly, we can obtain

$$|\Lambda(A)| \leq \inf_{\alpha \in \mathbb{C}} \{ (\text{rank}(\mathcal{H}_\alpha(A)) + 1) (\text{rank}(S_\alpha(A)) + 1) \} - d(A).$$
Moreover, it is easy to see that other decompositions of $A$ can lead to some different estimates of $|\Lambda(A)|$.

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