Fourier Law and Non-Isothermal Boundary in the Boltzmann Theory
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Steady Boltzmann Equation

\[ \nu \cdot \nabla_x F = Q(F, F) \]

- \( F(x, \nu) \): density distribution of rarefied gas
- 3 D velocity space \( \nu \in \mathbb{R}^3 \)
- \( \Omega \): bounded, connected domain in \( \mathbb{R}^d \) for \( d = 1, 2, 3 \)
- nonlinear Boltzmann operator \( Q(F_1, F_2) \):
  - quadratic, bilinear
  - non-local in \( \nu \in \mathbb{R}^3 \)
  - hard potential \( 0 \leq \gamma \leq 1 \) with angular cut-off
  - collision invariant: \( \int_{\mathbb{R}^3} \{1, \nu, |\nu|^2\} Q(F, F)(\nu) d\nu = 0 \)
- Knudsen number \( \sim 1 \) regime
Non-Isothermal Boundary and Diffusive BC

Wall temperature

\[ \theta(x) = \theta_0 + \delta\theta(x) \quad \text{on} \quad x \in \partial \Omega \]

Diffusive boundary condition on \( x \in \partial \Omega, \ n(x) \cdot v < 0 \)

\[ F(x, v) = \mu^\theta(x, v) \int_{n(x) \cdot u > 0} F(x, u) \{ n(x) \cdot u \} \, du \]

Wall Maxwellian

\[ \mu^\theta(x, v) = \frac{1}{2\pi \theta(x)^2} \exp \left[ -\frac{|v|^2}{2\theta(x)} \right] \]

with \( \int_{n(x) \cdot v > 0} \mu^\theta(x, v) \{ n(x) \cdot v \} \, dv = 0 \)
Purpose of This Work

- Analyze the thermal conduction phenomena in the kinetic regime (Knudsen number $\sim 1$)
Purpose of This Work

- Analyze the thermal conduction phenomena in the kinetic regime (Knudsen number $\sim 1$) when the wall temperature does not oscillate too much!

  \[ \theta(x) - \theta_0 \ll 1, \quad |\vartheta(x)| \leq 1 \text{ and } \delta \ll 1 \]

  \[ F_s \sim \mu \quad \text{Regime} \]
Natural Questions and Previous Works

- Existence, Uniqueness, Non-Negativity for Steady Solution
  - S.-H.Yu : existence and stability, $\Omega$ is slab ($\text{length} \ll 1$), ARMA 2009
  - L.Arkeryd, A.Nouri : Ann. Fac. Sci. Toulouse, Math. 2000 : Existence in $L^1$–space, $\Omega$ is slab

- Regularity (Continuity and Singularity)
  - Y.Guo : for IBVP, $\Omega$ convex, continuity away from $\gamma_0$ : ARMA 2010
  - C.K : for IBVP, $\Omega$ non-convex, singularity formation and propagation : CMP 2011
Natural Questions and Previous Works

- **Dynamical Stability**
  - L. Desvillettes, C. Villani: polynomial decay in $H^k$ for some BCs: Invent. Math. 2005
  - C. Villani: polynomial decay in $H^k$, diffusive BC, $\theta \equiv \theta_0$: Mem. AMS 2009
  - Y. Guo: $\theta \equiv \theta_0$, $e^{-\lambda t}$ decay in $L^\infty$ to $\mu$: ARMA 2010
  - S.-H. Yu: $e^{-\lambda t}$ decay in $L^\infty$ to the steady solution: ARMA 2009

- **Hydrodynamic Limit**
  - R. Esposito, Lebowitz, R. Marra: CMP 1994, J. Stat. Phys. 1995
Theorem: Existence, Uniqueness and Non-Negativity

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$. For all $M > 0$,
Theorem : Existence, Uniqueness and Non-Negativity

Let \( \Omega \subset \mathbb{R}^d, \ d = 1, 2, 3. \)
For all \( M > 0, \) there exists \( \delta_0 > 0 \) such that for \( 0 < \delta < \delta_0 \) in

\[
|\theta(x) - \theta_0| \leq \delta, \quad \text{on} \quad x \in \partial \Omega,
\]
then there exists a non-negative solution \( F_s = M\mu + \sqrt{\mu}f_s \geq 0 \) with

\[
\int\int_{\Omega \times \mathbb{R}^3} f_s \sqrt{\mu} = 0 \quad \text{to the problem}
\]

\[
v \cdot \nabla_x F_s = Q(F_s, F_s), \quad F_s|_{\gamma^-} = \mu^\theta \int_{\gamma^+} F_s d\gamma,
\]
Theorem : Existence, Uniqueness and Non-Negativity

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$.
For all $M > 0$, there exists $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ in

$$|\theta(x) - \theta_0| \leq \delta, \quad \text{on } x \in \partial \Omega,$$

then there exists a non-negative solution $F_s = M\mu + \sqrt{\mu}f_s \geq 0$ with $\int\int_{\Omega \times \mathbb{R}^3} f_s \sqrt{\mu} = 0$ to the problem

$$v \cdot \nabla_x F_s = Q(F_s, F_s), \quad F_s|_{\gamma_-} = \mu^\theta \int_{\gamma_+} F_s d\gamma,$$

such that, for all $0 \leq \zeta < \frac{1}{4+2\delta}$, $\beta > 4$,

$$||\langle v \rangle^\beta e^{\zeta|v|^2} f_s||_{\infty} + |\langle v \rangle^\beta e^{\zeta|v|^2} f_s|_{\infty} \lesssim \delta.$$

If $M\mu + \sqrt{\mu}g_s$ is an another solution with $\int\int_{\Omega \times \mathbb{R}^3} g_s \sqrt{\mu} = 0$ such that, for $\beta > 4$

$$||\langle v \rangle^\beta g_s||_{\infty} + |\langle v \rangle^\beta g_s|_{\infty} \ll 1,$$

then $f_s \equiv g_s$. 
If $\theta(x)$ is continuous on $\partial \Omega$ then $F_s$ is continuous away from $\mathcal{D}$. In particular, if $\Omega$ is convex then $\mathcal{D} = \gamma_0$. On the other hand, if $\Omega$ is not convex then we can construct a continuous function $\theta(x)$ on $\partial \Omega$ such that the corresponding solution $F_s$ is not continuous.
Theorem: Dynamical Stability

Let $0 \leq \zeta < \frac{1}{4+2\delta}$, $\beta > 4$. There exists $\varepsilon_0 > 0$, depends on $\delta_0$, and $\lambda > 0$ such that if

$$||\langle v \rangle^\beta e^{\zeta |v|^2} [f(0) - f_s]||_{\infty} \leq \varepsilon_0$$

then there exists a unique non-negative dynamic solution $F(t) = \mu + f_s \sqrt{\mu} + f(t) \sqrt{\mu} \geq 0$ to the dynamical problem

$$\partial_t F + v \cdot \nabla x F = Q(F, F), \quad F(x, v) = \mu^\theta \int_{n(x) \cdot v > 0} F \ n \cdot v$$

for $x \in \partial \Omega$ and $n(x) \cdot v < 0$ such that

$$||\langle v \rangle^\beta e^{\zeta |v|^2} [f(t) - f_s]||_{\infty} \lesssim e^{-\lambda t} ||\langle v \rangle^\beta e^{\zeta |v|^2} [f(0) - f_s]||_{\infty}$$
Why $\delta -$ Expansion?

Fourier Law: a relation between the temperature and the heat flux

$$q_s = -\kappa(\theta_s) \partial_x \theta_s$$

for suitable positive smooth function $\kappa$.

Let $F_s$ be the solution to the steady Boltzmann equation

$$\theta_s(x) = \frac{1}{3 \rho_s} \int_{\mathbb{R}^3} |v - u_s|^2 F_s(x, v) dv$$
$$u_s(x) = \frac{1}{\rho_s} \int_{\mathbb{R}^3} v F_s(x, v) dv$$
$$\rho_s(x) = \int_{\mathbb{R}^3} F_s(x, v) dv$$
$$q_s(x) = \frac{1}{2} \int_{\mathbb{R}^3} (v - u_s(x)) |v - u(x)|^2 F_s(x, v) dv.$$

Purpose: See the first order characterization of $F_s$
What is $\delta$—Expansion? : $\mu_\delta$—Expansion

Wall Temperature

$$\theta(x) = \theta_0 + \delta \vartheta(x), \quad |\vartheta(x)| \leq 1, \quad x \in \partial \Omega.$$  

Wall Maxwellian

$$\mu_\delta(x, \nu) = \frac{1}{2\pi[\theta_0 + \delta \vartheta(x)]^2} \exp \left( -\frac{|\nu|^2}{2[\theta_0 + \delta \vartheta(x)]} \right)$$

Taylor Expansion in $\delta$ ($\mu_\delta$ is analytic in $\delta$)

$$\mu_\delta = \mu + \delta \mu_1 + \delta^2 \mu_2 + \cdots + \delta^m \mu_m + \cdots$$
What is $\delta$—Expansion? : $f_s \sim \delta f_1 + \delta^2 f_2 + \cdots$

Formal Expansion:

$$F_s = \mu + \sqrt{\mu} \{\delta f_1 + \delta^2 f_2 + \cdots\}$$
$$f_s = \delta f_1 + \delta^2 f_2 + \cdots$$

Plug in

$$v \cdot \nabla_x F_s = Q(F_s, F_s)$$

with Diffusive Boundary Condition to get the linear equation for $f_i$ (comparing the coefficients of power of $\delta$)

Once we solve $f_i$, define the Remainder $f_m^\delta$ such that

$$f_s = \delta f_1 + \delta^2 f_2 + \cdots + \delta^m f_m^\delta$$
Theorem: $\delta$–Expansion

$\delta$–Expansion is valid!

There exist $f_1, f_2, \cdots, f_{m-1}$ and for all $i = 1, 2, \cdots m - 1$

$$||\langle v \rangle^{\beta} e^{\zeta|v|^2} f_i||_{\infty} \lesssim 1$$

for all $0 \leq \zeta < \frac{1}{4}, \beta > 4$

and the remainder $f_m^{\delta}$ exits and

$$||\langle v \rangle^{\beta} e^{\zeta|v|^2} f_m^{\delta}||_{\infty} \lesssim 1$$

for all $0 \leq \zeta < \frac{1}{4+2\delta}, \beta > 4$
Let $\Omega = [0, 1]$. If the Fourier Law holds for $F_s = \mu + \sqrt{\mu} f_s$, 

$$F_s = \mu + \delta f_1 \sqrt{\mu} + O(\delta^2) \sqrt{\mu}$$

$$\theta_s = \theta_0 + \delta \theta_1 + O(\delta^2)$$

then 

$$\theta_1(x)$$ is a linear function on $[0, 1]$
From an available numeric simulation (Ohwada, Aoki, Sone, 1989) \( \theta_1 \) is not linear!

\[ \Downarrow \]

Fourier Law is not valid at the kinetic regime!
Linearized Boltzmann operator

\[ Lf = -\frac{1}{\sqrt{\mu}} [Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}f, \mu)] = \nu(v)f - Kf \]

- semi-positive: \[ \langle Lg, g \rangle \gtrsim \|\{ I - P \}g \|_V^2 \]
- kernel = ‘hydrodynamic part’

\[ Pg \equiv \left\{ a_g(t, x) + \nu \cdot b_g(t, x) + \frac{|v|^2 - 3}{2} c_g(t, x) \right\} \sqrt{\mu} \]

Boltzmann equation \(\rightarrow\) macroscopic equation for \(b_f\)

\[ \Delta_x b_f = \partial_x^2 \{ I - P \} f + \cdots \]

ellipticity in \(H^k\) \(\rightarrow\) Guo:VMB(Invent.Math.2003), VPL(JAMS2011); Gressman-Strain:BE without angular cut-off(JAMS2011)
Difficulties with boundary conditions

\[ Pf \equiv \left\{ a_f(t, x) + v \cdot b_f(t, x) + \frac{|v|^2 - 3}{2} c_f(t, x) \right\} \sqrt{\mu} \]

- \( Pf \) and \( \{ I - P \} f \) do not make sense at the boundary
- no boundary condition for \( a_f, c_f \), only \( b_f \cdot n(x) = 0 \) on \( \partial \Omega \)
Mathematical Framework: $L^2 - L^\infty$ Frame

Y. Guo: Initial Boundary Value Problem of BE, ARMA 2010

- $L^2$ Posivity: We Need A New Method!
- $L^\infty$ Bound: We Need A New Method!
New $L^2$ Positivity Estimate

\[ \mathbf{v} \cdot \nabla f + Lf = g, \quad f_{\gamma} = P_{\gamma}f + r \]

with
\[ \int_{\Omega \times \mathbb{R}^3} f \sqrt{\mu} = 0 = \int_{\Omega \times \mathbb{R}^3} g \sqrt{\mu} = \int_{n \cdot \mathbf{v} < 0} r \]

\[ \Rightarrow \quad \| Pf \|_\nu \leq M \{ \| (I - P)f \|_\nu + \| (1 - P_{\gamma})f \|_{2,+} \} + \cdots \]

- weak formulation (Green’s identity) + test functions
- constructive estimate with an explicit $M$
- dimension of $\Omega = 1, 2, 3$
New $L^2$ Positivity Estimate

Weak formulation (Green’s identity)

$$\int_\gamma \psi f - \int\int_{\Omega \times \mathbb{R}^3} v \cdot \nabla \psi f = - \int\int_{\Omega \times \mathbb{R}^3} \psi L(I - P)f + \int\int_{\Omega \times \mathbb{R}^3} \psi g$$

bulk \hspace{1cm} f = \{a_f + v \cdot b_f + \frac{|v|^2 - 3}{2} c_f\} \sqrt{\mu} + (I - P)f

boundary \hspace{1cm} f_\gamma = P_\gamma f + (1 - P_\gamma)f \mathbf{1}_{\gamma^+} + r \mathbf{1}_{\gamma^-}$$
New $L^2$ Positivity Estimate

Test functions

- for $c_f$: $\psi_c = (|v|^2 - \beta_c)\sqrt{\mu}\{v \cdot \nabla_x\}(-\Delta_0)^{-1}c_f$
  with $\int_{\mathbb{R}^3}(|v|^2 - \beta_c)v_i^2\mu(v)dv = 0$

- for $b_f$:
  - $\psi^{i,j}_b = (v_i^2 - \beta_b)\sqrt{\mu}\partial_j(-\Delta_0)^{-1}(b_f)_j$ for all $i, j = 1, 2, \cdots d$
    with $\int_{\mathbb{R}^3}(v_i^2 - \beta_b)\mu(v)dv = 0$, for all $i$
  - $\phi^{i,j}_b = v_iv_j|v|^2\sqrt{\mu}\partial_j(-\Delta_0)^{-1}(b_f)_i$ for all $i \neq j$

- for $a_f$: $\psi_a = (|v|^2 - \beta_a)\{v \cdot \nabla_x\}(\nabla^2_{N})^{-1}a_f$
  with $\int_{\mathbb{R}^3}(|v|^2 - \beta_a)(\frac{|v|^2}{2} - \frac{3}{2})(v_i)^2\mu(v)dv = 0$ for all $i$
Future

Thanks!