On sequentially $h$-complete groups

Gábor Lukács
Department of Mathematics & Statistics, York University
4700 Keele Street, Toronto, Ontario, M3J 1P3, Canada
Current address: FB 3 - Mathematik und Informatik, Universität Bremen
Bibliothekstrasse 1, 28359 Bremen, Germany
lukacs@mathstat.yorku.ca

Abstract. A topological group $G$ is sequentially $h$-complete if all the continuous homomorphic images of $G$ are sequentially complete. In this paper we give necessary and sufficient conditions on a complete group for being compact, using the language of sequential $h$-completeness. In the process of obtaining such conditions, we establish a structure theorem for $\omega$-precompact sequentially $h$-complete groups. As a consequence we obtain a reduction theorem for the problem of $c$-compactness.

All topological groups in this paper are assumed to be Hausdorff.

A topological group $G$ is sequentially $h$-complete if all the continuous homomorphic images of $G$ are sequentially complete (i.e., every Cauchy-sequence converges). $G$ is called precompact if for any neighborhood $U$ of the identity element there exists a finite subset $F$ of $G$ such that $G = UF$.

In [6, Theorem 3.6] Dikranjan and Tkachenko proved that nilpotent sequentially $h$-complete groups are precompact (also see [4]). Thus, if a group is nilpotent, sequentially $h$-complete and complete, then it is compact.

Inspired by this result, the aim of this paper is to give necessary and sufficient conditions on a complete group for being compact, using the language of sequential $h$-completeness. This aim is carried out in Theorem 6.

For an infinite cardinal $\tau$, a topological group $G$ is $\tau$-precompact if for any neighborhood $U$ of the identity element there exists $F \subset G$ such that $G = UF$ and $|F| \leq \tau$. In order to prove Theorem 6 we will first establish a strengthened version of the Guram’s Embedding Theorem for $\omega$-precompact sequentially $h$-completely groups (Theorem 5).

A topological group $G$ is $c$-compact if for any topological group $H$ the projection $\pi_H : G \times H \to H$ maps closed subgroups of $G \times H$ onto closed subgroups of $H$ (see [12], [5] and [2], as well as [3]). The problem of whether every $c$-compact

2000 Mathematics Subject Classification. Primary 22A05, 22C05; Secondary 54D30.

I gratefully acknowledge the financial support received from York University that enabled me to do this research.

©2000 American Mathematical Society
A topological group is compact has been an open question for more than ten years. As a consequence of Theorem 6, we obtain that the problem of $c$-compactness can be reduced to the second-countable case (Theorem 9).

The following Theorem is a slight generalization of Theorem 3.2 from [8]:

**Theorem 1** Let $G$ be an $\omega$-precompact sequentially $h$-complete topological group. Then every continuous homomorphism $f : G \to H$ onto a group $H$ of countable pseudocharacter is open.

In order to prove Theorem 1, we need the following three facts, two of which are due to Guran:

**Fact A (Guran’s Embedding Theorem)** A topological group is $\tau$-precompact if and only if it is topologically isomorphic to a subgroup of a direct product of topological groups of weight $\leq \tau$. (Theorem 4.1.3 in [14].)

**Fact B (Banach’s Open Map Theorem)** Any continuous homomorphism from a separable complete metrizable group onto a Baire group is open. (Corollary V.4 in [10].)

**Fact C** Let $G$ be an $\omega$-precompact topological group of countable pseudocharacter. Then $G$ admits a coarser second countable group topology. (Corollary 4 in [9].)

The proof below is just a slight modification of the proof of Theorem 3.2 from [8] mentioned above:

**Proof** First, suppose that $H$ is metrizable. Let $U$ be a neighborhood of $e$ in $G$. Since, by Fact A, $G$ embeds into a product of separable metrizable group, we may assume that $U = g^{-1}(V)$ for some continuous homomorphism $g : G \to M$ onto a separable metrizable group $M$ and some neighborhood $V$ of $e$ in $M$. Let $h = (f, g) : G \to H \times M$, and put $L = h(G)$. Let $p : L \to H$ and $q : L \to M$ be the restrictions of the canonical projections $H \times M \to H$ and $H \times M \to M$. Clearly, one has $f = ph$ and $g = qh$. Since $g$ is continuous, $W = q^{-1}(V)$ is open. We have $h(U) = h(g^{-1}(V)) = h(h^{-1}q^{-1}(V)) = W$ and $f(U) = ph(U) = p(W)$. Since $G$ is sequentially $h$-complete, the groups $L$ and $H$, being homomorphic images of $G$, are sequentially complete. Since $H$ and $L$ are metrizable, this means that they are simply complete. They are also separable, because they are metrizable and $\omega$-precompact. Thus, by Fact B, $p : L \to H$ is open, and hence $f(U) = p(W)$ is open in $H$.

To show the general case, suppose that the topology $T$ of the group $H$ is of countable pseudocharacter; then $H$ admits a coarser second countable topology $T'$ (by Fact C). Put $\iota : (H, T) \to (H, T')$ to be the identity map. Since $T'$ is metrizable, the continuous homomorphism $\iota \circ \varphi : G \to (H, T')$ is open by what we have already proved, and thus $\varphi$ is also open.

**Corollary 2** Let $G$ be an $\omega$-precompact sequentially $h$-complete group. The following statements are equivalent:

(i) $G$ is second countable;
(ii) $G$ contains a countable network;
(iii) $G$ has a countable pseudocharacter;
(iv) $G$ is metrizable.
Proof The implications (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) are obvious, and so is the equivalence of (i) and (iv).

(iii) \( \Rightarrow \) (iv): If \((G, T)\) is of countable pseudocharacter, it admits a coarser second countable topology \(T'\) (by Fact C); put \(\iota : G \to G_1\) to be the identity map. By Theorem 1, \(\iota\) is open, and thus \(T\) is second countable, as desired. \(\square\)

A topological group \(G\) is minimal if every continuous isomorphism \(\varphi : G \to H\) is a homeomorphism, or equivalently, if the topology of \(G\) is a coarsest (Hausdorff) group topology on \(G\). The group \(G\) is totally minimal if every continuous surjective homomorphism \(f : G \to H\) is open; in other words, \(G\) is totally minimal if every quotient of \(G\) is minimal.

**Corollary 3** Every \(\omega\)-precompact sequentially \(h\)-complete topological group of countable pseudocharacter is totally minimal and metrizable.

**Proof** Let \(G\) be an \(\omega\)-precompact sequentially \(h\)-complete group of countable pseudocharacter. By Corollary 2, \(G\) is metrizable and contains a countable network. Sequential \(h\)-completeness and the property of having a countable network are preserved under continuous homomorphic images, so for every continuous homomorphism \(\varphi : G \to H\) onto a topological group \(H\), the group \(H\) is sequentially \(h\)-complete and contains a countable network; in particular, \(H\) is \(\omega\)-precompact. Thus, by Corollary 2, \(H\) is of countable pseudocharacter. Therefore, by Theorem 1, \(\varphi\) is open. \(\square\)

Corollary 3 generalizes [8, 3.3] to the sequentially \(h\)-complete topological groups.

**Corollary 4** Every sequentially \(h\)-complete topological group with a countable network is totally minimal and metrizable. \(\square\)

A topological group is \(h\)-complete if all its continuous homomorphic images are complete (see [7]).

Using Theorem 1, we obtain the following strengthening of the Guran’s Embedding Theorem for sequentially \(h\)-complete groups:

**Theorem 5** Every \(\omega\)-precompact sequentially \(h\)-complete group \(G\) densely embeds into the (projective) limit of its metrizable quotients. In particular, if \(G\) is \(h\)-complete, then it is equal to the limit of its metrizable quotients.

**Proof** By Fact A, since \(G\) is \(\omega\)-precompact, it embeds into \(\Sigma = \prod_{\alpha \in I} \Sigma_\alpha\), a product of second countable groups. Denote by \(\pi_\alpha : G \to \Sigma_\alpha\) the restriction of the canonical projections to \(G\); without loss of generality, we may assume that the \(\pi_\alpha\) are onto. Since \(G\) is also sequentially \(h\)-complete, the \(\pi_\alpha\) are open (by Theorem 1). Thus, one has \(G/N_\alpha \cong \Sigma_\alpha\), where \(N_\alpha = \ker \pi_\alpha\). We may also assume that all metrizable quotients of \(G\) appear in the product constituting \(\Sigma\), because by adding factors we do not ruin the embedding.

Let \(\iota\) be the embedding of \(G\) into the product of its metrizable quotients, and put \(L = \lim_{\alpha \in I} G/N_\alpha\). The image of \(\iota\) is obviously contained in \(L\). (We note that if \(G/N_\alpha\) and \(G/N_\beta\) are metrizable quotients, then \(G/N_\alpha N_\beta\) and \(G/N_\alpha \cap N_\beta\) are also metrizable quotients; the latter one is metrizable, because the continuous
homomorphism $G \to G/N_{\alpha} \times G/N_{\beta}$ is open onto its image, as the codomain is metrizable.) In order to show density, let $x = (x_{\alpha}N_{\alpha})_{\alpha \in I} \in L$ and let

$$U = U_{\alpha_1} \times \cdots \times U_{\alpha_k} \times \prod_{\alpha \neq \alpha_i} G/N_{\alpha}$$

be a neighborhood of $x$. By the consideration above, the quotient $G/\bigcap_{i=1}^{k} N_{\alpha_i}$ is metrizable, so $\bigcap_{i=1}^{k} N_{\alpha_i} = N_{\gamma}$ for some $\gamma \in I$. Thus, $\pi_{\alpha_i}(x_{\gamma}) = x_{\alpha_i}N_{\alpha_i}$, therefore $\iota(G)$ intersects $U$, and hence $\iota(G)$ is dense in $L$. \hfill \square

A topological group $G$ is \textit{maximally almost-periodic} (or briefly, MAP) if it admits a continuous monomorphism $m : G \to K$ into a compact group $K$, or equivalently, if the finite-dimensional unitary representations of $G$ separate points.

Theorem 6 Let $G$ be a complete topological group. Then the following assertions are equivalent:

(i) the closed normal subgroups of closed separable subgroups of $G$ are $h$-complete and MAP;

(ii) every closed separable subgroup $H$ of $G$ is sequentially $h$-complete, and its metrizable quotients $H/N$ are MAP;

(iii) $G$ is compact.

The following easy consequence of a result by Dikranjan and Tkáčenko plays a very important role in proving Theorem 6:

Fact D $G$ is precompact if and only if every closed separable subgroup of $G$ is precompact. (Theorem 3.5 in [6].)

Proof (i) $\Rightarrow$ (ii): If $H/N$ is a quotient described in (ii), then it is clearly $h$-complete, as the homomorphic image of $H$, which is assumed to be $h$-complete in (i). Let $m : H \to K$ be a continuous injective homomorphism into a compact group $K$. Since $H$ is $h$-complete, $m(H)$ is closed in $K$, so we may assume that $m$ is onto. The subgroup $m(N)$ is normal in $K$, because $m$ is bijective, and it is closed because $N$ is $h$-complete. Therefore, $\bar{m} : H/N \to K/m(N)$ is a continuous injective homomorphism, showing that $H/N$ is MAP.

(ii) $\Rightarrow$ (iii): Since $G$ is complete, in order to show that $G$ is compact, we show that it is also precompact. By Fact D it suffices to show that every closed separable subgroup of $G$ is precompact.

Let $H$ be a closed separable subgroup of $G$. The group $H$ is $\omega$-precompact (because it is separable), and by (ii) $H$ is sequentially $h$-complete. Applying Theorem 3 $H$ densely embeds into the product $P$ of its metrizable quotient. In order to show that $H$ is precompact, we show that each factor of the product $P$ is precompact.

Let $Q = H/N$ be a metrizable quotient of $H$; since $H$ is sequentially $h$-complete, so is $Q$. The group $Q$ is second countable, because (being the continuous image of $H$) it is separable. Thus, by Corollary 5 $Q$ is totally minimal, because it is sequentially $h$-complete. According to (ii), $Q$ is MAP, and together with minimality this implies that $Q$ is precompact. \hfill \square
Remark The implication (i) ⇒ (iii) can also be proved directly, without applying Theorem 5 by using Fact 1 and 3.4.

Since subgroups and continuous homomorphic images of c-compact groups are c-compact again, and are in particular h-complete, we obtain:

**Corollary 7** If $G$ is c-compact and MAP, then $G$ is compact.

**Proof** If $G$ is MAP, then in particular all its subgroups are so, and all its closed subgroups are c-compact, thus h-complete. Hence Theorem applies. □

A group is minimally almost periodic (or briefly, m.a.p.) if it has no non-trivial finite-dimensional unitary representations.

**Corollary 8** Every c-compact group $G$ has a maximal compact quotient $G/M$, where $M$ is a closed characteristic m.a.p. subgroup of $G$.

**Proof** Let $G$ be a c-compact group, and let $M = n(G)$, the von Neumann radical of $G$ (the intersection of the kernels of all the finite-dimensional unitary representations of $G$). By its definition, $G/M$ is the maximal MAP quotient of $G$, and according to Corollary 4 this is the same as the maximal compact quotient of $G$, because each quotient of $G$ is c-compact.

The subgroup $M$ is also c-compact, so by what we have proved so far, it has a maximal compact quotient $M/L$, where $L$ is characteristic in $M$, thus $L$ is normal in $G$. By the Third Isomorphism Theorem, $G/M \cong (G/L)/(M/L)$, and since both $M/L$ and $G/M$ are compact, by the three space property of compactness in topological groups, the quotient $G/L$ is compact too. Thus, $M \subseteq L$, and therefore $M = L$. Hence, $M$ is minimally almost periodic. □

We conclude with a reduction theorem. If $G$ is c-compact, then so is every closed subgroup of $G$. Thus, each closed subgroup $H$ and separable metrizable quotient $H/N$ mentioned in (ii) of Theorem 5 is also c-compact. Therefore, the equivalence of (ii) and (iii) in Theorem 5 yields:

**Theorem 9** The following statements are equivalent:
(i) every c-compact group is compact;
(ii) every second countable c-compact group is MAP (and thus compact);
(iii) every second countable c-compact m.a.p. group is trivial. □

It is not known whether (ii) is true, but by Corollary 4 every second-countable c-compact group is totally minimal, a fact that may help in proving or giving a counterexample to (ii).

**Acknowledgements**

I am deeply indebted to Prof. Walter Tholen, my PhD thesis supervisor, for giving me incredible academic and moral help.

I am thankful to Prof. Mikhail Tkachenko for sending me his papers 14 and 15 by e-mail.

I am grateful for the constructive comments of the anonymous referee that led to an improved presentation of the results in this paper.

Last but not least, special thanks to my students, without whom it would not be worth it.
References

[1] A. V. Arhangel’skii. Cardinal invariants of topological groups. Embeddings and condensations. *Dokl. Akad. Nauk SSSR*, 247(4):779–782, 1979.

[2] M. M. Clementino, E. Giuli, and W. Tholen. Topology in a category: compactness. *Portugal. Math.*, 53(4):397–433, 1996.

[3] M. M. Clementino and W. Tholen. Tychonoff’s theorem in a category. *Proc. Amer. Math. Soc.*, 124(11):3311–3314, 1996.

[4] D. N. Dikranjan. Countably compact groups satisfying the open mapping theorem. *Topology Appl.*, 98(1-3):81–129, 1999. II Iberoamerican Conference on Topology and its Applications (Morelia, 1997).

[5] D. N. Dikranjan and E. Giuli. Compactness, minimality and closedness with respect to a closure operator. In *Categorical topology and its relation to analysis, algebra and combinatorics (Prague, 1988)*, pages 284–296. World Sci. Publishing, Teaneck, NJ, 1989.

[6] D. N. Dikranjan and M. G. Tkačenko. Sequential completeness of quotient groups. *Bull. Austral. Math. Soc.*, 61(1):129–150, 2000.

[7] D. N. Dikranjan and A. Tonolo. On characterization of linear compactness. *Riv. Mat. Pura Appl.*, 17:95–106, 1995.

[8] D. N. Dikranjan and V. V. Uspenski. Categorically compact topological groups. *J. Pure Appl. Algebra*, 126(1-3):149–168, 1998.

[9] I. Guran. On topological groups close to being Lindelöf. *Soviet Math. Dokl.*, 23:173–175, 1981.

[10] T. Husain. *Introduction to topological groups*. W. B. Saunders Co., Philadelphia, Pa., 1966.

[11] G. Lukács. On locally compact c-compact groups. Preprint, 2003.

[12] E. G. Manes. Compact Hausdorff objects. *General Topology and Appl.*, 4:341–360, 1974.

[13] L. S. Pontryagin. *Selected works. Vol. 2*. Gordon & Breach Science Publishers, New York, third edition, 1986. Topological groups, Edited and with a preface by R. V. Gamkrelidze, Translated from the Russian and with a preface by Arlen Brown, With additional material translated by P. S. V. Naidu.

[14] M. G. Tkačenko. Topological groups for topologists. I. *Bol. Soc. Mat. Mexicana (3)*, 5(2):237–280, 1999.

[15] M. G. Tkačenko. Topological groups for topologists. II. *Bol. Soc. Mat. Mexicana (3)*, 6(1):1–41, 2000.