Characteristic polynomial and eigenvalues of anti-adjacency matrix of directed unicyclic corona graph

N Hasyyati¹, K A Sugeng¹, and S Aminah*¹

¹Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Indonesia, 16424-Depok, Indonesia

E-mail: aminah@sci.ui.ac.id

Abstract. A directed graph can be represented by several matrix representations, such as the anti-adjacency matrix. This paper discusses the general form of characteristic polynomial and eigenvalues of the anti-adjacency matrix of directed unicyclic corona graph. The characteristic polynomial of the anti-adjacency matrix can be found by counting the sum of the determinant of the anti-adjacency matrix of the directed cyclic induced subgraphs and the directed acyclic induced subgraphs from the graph. The eigenvalues of the anti-adjacency matrix can be real or complex numbers. We prove that the coefficient of the characteristic polynomial and the eigenvalues of the anti-adjacency matrix of directed unicyclic corona graph can be expressed in the function form that depends on the number of subgraphs contained in the directed unicyclic corona graphs.

1. Introduction
A graph structure can be analyzed through anti-adjacency matrix [1]. The anti-adjacency matrix is a $n \times n$ matrix, so this matrix has characteristic polynomial and its eigenvalues [2].

Wildan (2015) discussed the coefficients of the characteristic polynomial of the anti-adjacency of the directed cyclic graphs in general up to the fifth coefficients [3]. Prayitno (2019) discussed properties of adjacency in-degree Laplacian and out-degree Laplacian matrices of directed cyclic sun graph [6]. However, there are no studies about the characteristic polynomial and the eigenvalues of the anti-adjacency matrix of directed unicyclic corona graph. Hence in this paper, we will find the characteristic polynomial and its eigenvalues. The coefficients of a characteristic polynomial can be obtained by calculating the number of cyclic induced subgraphs and acyclic induced subgraphs in the directed unicyclic corona graph [3].

2. Preliminaries
This section consists of definitions, theorems, and lemmas that are required to prove the main result.

2.1. Definitions

Definition 1 [4] A directed graph $G$ is a finite nonempty set of objects called vertices together with a set $E$ of ordered pairs of distinct vertices. The element of $E$ is called directed edge or edge.
Definition 2 [1] The adjacency matrix of $G$, denoted by $A(\vec{G})$, is the $n \times n$ matrix defined as follows. The rows and the columns of $A(\vec{G})$ are indexed by $V(\vec{G})$. If $i \neq j$ then the $(i, j)$-entry of $A(\vec{G})$ is 0 if there is an edge from vertex $i$ to vertex $j$, and the $(i, j)$-entry is 1 for other cases. The $(i, i)$-entry of $A(\vec{G})$ is 0 for $i = 1, 2, \ldots, n$. For simplicity, we denote $A(\vec{G})$ by $A$.

Definition 3 [1] The matrix $B = J - A$ is called the anti-adjacency matrix of $\vec{G}$, where $J$ is the $n \times n$ matrix where all the entries are 1 and $A$ is the adjacency matrix of $\vec{G}$.

Definition 4 [5] Corona Graph $C_n \circ \overrightarrow{K_r}$ consists of a cycle $C_n$ with $n$ vertices and each vertex in the cycle $C_n$ adjacent to $r$ pendant vertices.

In this paper, the directed unicyclic corona graph is obtained by adding the direction of each arc contained in $\overrightarrow{C_n}$, and the directed edges of each pendant always lead to the vertices in the cycle graph $\overrightarrow{C_n}$, such as illustration as the following Figure 1.

Figure 1. Directed unicyclic corona graph $C_n \circ \overrightarrow{K_r}$.

The directed unicyclic corona graph $C_n \circ \overrightarrow{K_r}$ has an anti-adjacency $B \left( C_n \circ \overrightarrow{K_r} \right) = (b_{i,j})$, where

$$b_{i,j} = \begin{cases} 
0, & i = n; \\
1, & i = j + kn; \\
1, & j = 1 or \quad j = i + 1 or \quad j = 1, 2, \ldots, n; \quad k = 2, 3, \ldots, r
\end{cases}$$

Figure 2 shows the anti-adjacency matrix of the directed unicyclic corona graph $C_n \circ \overrightarrow{K_r}$.
The following theorems give the known results for anti-adjacency matrix properties.

**Theorem 1** [1] Let $\tilde{G}$ be a directed acyclic graph with vertices $V(\tilde{G}) = \{v_1, v_2, ..., v_n\}$. Let $B$ be the anti-adjacency matrix of $\tilde{G}$. Then $\det B = 1$ if $\tilde{G}$ has a Hamiltonian path, and $\det B = 0$, otherwise.

**Theorem 2** [6] Let $\tilde{G}$ be a directed acyclic graph with vertices $V(\tilde{G}) = \{v_1, v_2, ..., v_n\}$ and $B$ be the anti-adjacency matrix of $\tilde{G}$. If there are two vertices whose direction of its arc direction point to the same vertex, then $\det(B) = 0$.

**Theorem 3** [7] Let $B$ be the anti-adjacency matrix of the directed cycle graph $C_n$, then $\det(B) = n - 1$.

**Theorem 4** [3] Let $P(B(\tilde{G})) = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_{n-1}\lambda + b_n$ be a characteristic polynomial of an anti-adjacency matrix $B(\tilde{G})$ of the directed graph $\tilde{G}$, then

$$b_i = (-1)^i \left( \sum_{j=1}^{w_i} |B((U)_{acyclic})_i^{(j_1)}| + \sum_{j=2}^{w_2} |B((U)_{cyclic})_i^{(j_2)}| \right),$$

where $i = 1, 2, ..., n$.

$|B((U)_{acyclic})_i^{(j_1)}|$ is the determinant of anti-adjacency matrix of directed acyclic induced subgraph

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{antiamdacency.png}
\caption{Anti-adjacency matrix of directed unicyclic corona graph $C_n \circ \overline{K_r}$.}
\end{figure}
with $i$ vertices and $j_1 = 1, 2, \ldots, w_1$; where $w_1$ is the number of directed induced subgraph $(U)_{acyclic}$ with $i$ vertices of the directed acyclic graph $\bar{G}$ and $\left|B((U)_{cyclic})_{ij_2}\right|$ is the determinant of anti-adjacency matrix of directed cyclic induced subgraph with $i$ vertices and $j_2 = 1, 2, \ldots, w_2$, where $w_2$ is the number of directed induced subgraph $(U)_{cyclic}$ with $i$ vertices of directed cyclic graph $G$.

**Theorem 5** [3] Let $P(B(\bar{G})) = \lambda^n + b_1 \lambda^{n-1} + b_2 \lambda^{n-2} + \cdots + b_{n-1} \lambda + b_n$ be a characteristic polynomial of an anti-adjacency matrix $B(\bar{G})$ from a directed graph with $n$ vertices and $m$ edges, then $b_1 = -n$ and $b_2 = m$.

Lemma 1 up to Lemma 3 are related to the value of determinant of directed acyclic induced subgraphs of $C_n \circ \overrightarrow{K_r}$. A unicyclic directed corona graph $C_n \circ \overrightarrow{K_r}$ has only one cyclic induced subgraph. Meanwhile, Figure 3 to Figure 7 show the five types of acyclic induced subgraphs of $C_n \circ \overrightarrow{K_r}$.

**Figure 3.** Type 1 of the acyclic induced subgraphs of $C_n \circ \overrightarrow{K_r}$.

Let $B_1$ be an anti-adjacency matrix of directed acyclic subgraph type 1. This type is a directed acyclic induced subgraph which contain a Hamiltonian path. Hence, according to **Theorem 1** $\det(B_1) = 1$.

**Lemma 1** Let $B$ be a directed unicyclic corona graph, then the number of induced acyclic subgraph with one vertex as in type 1 is $(r + 1)n$.

**Proof.** Induced directed subgraphs type 1 each of the subgraphs has an initial vertex that the entry degree equal to zero, that is $v_1$, or $v_2$, or ..., or $v_{(r+1)n}$. Hence, the number of subgraphs with an acyclic direction of type 1 direction equal to the number of vertices of a unicyclic corona graph, which is $(r + 1)n$.

**Figure 4.** Type 2 of the acyclic induced subgraphs of $C_n \circ \overrightarrow{K_r}$.

Let $B_2$ be an anti-adjacency matrix of directed acyclic subgraph type 2. This type is a directed acyclic induced subgraph which contain a Hamiltonian path. Hence, according to **Theorem 1** $\det(B_2) = 1$.

**Lemma 2** Let $B$ be a directed unicyclic corona graph, then the number of induced acyclic subgraph with $i$ vertices, $i = 2, 3, \ldots, n - 1$ which contains the Hamiltonian path as in type 2 is $n$ paths.
Proof. Induced directed subgraphs type 2 each of the subgraphs has an initial vertex that the entry degree equal to zero, that is $v_1$, or $v_2$, or ..., or $v_n$. Hence, the number of subgraphs with an acyclic induced type 2 is the same as the number of vertices on a cycle graph with a cyclic direction, which is $n$ paths.

![Figure 5. Type 3 of the acyclic induced subgraphs of $C_n \circ K_r$.](image)

Let $B_3$ be an anti-adjacency matrix of directed acyclic subgraph type 3. These subgraphs have a directed acyclic induced subgraph that contain a Hamiltonian path. Thus, according to Theorem 1, $\det(B_3) = 1$.

Lemma 3 Let $B$ be a directed unicyclic corona graph, then the number of induced acyclic subgraph with $i$ vertices, $i = 2, 3, \ldots, n$ which contains the Hamiltonian path as in type 3 is $rn$ paths.

Proof. Each of the induced directed subgraphs type 3 has an initial vertex that the entry degree equal to zero, that is $v_{n+1}$, or $v_{n+2}$, or ..., or $v_{(r+1)n}$. Hence, the number of subgraphs with an acyclic induced subgraph of type 3 is $rn$ paths.

![Figure 6. Type 4 of the acyclic induced subgraphs of $C_n \circ K_r$.](image)

Induced directed subgraphs of type 4 have more than or equal two vertices from the corona graph whose arc direction points to the same vertex, based on Theorem 2 then $\det(B_4) = 0$. 

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Induced directed subgraphs of type 5 have more than or equal to two vertices whose arc direction points to the same vertex, based on Theorem 2 then det($B_5$) = 0.

3. Main Result
In Theorem 6, we show the coefficients $b_i$; $1 \leq i \leq (r + 1)n$, of the characteristic polynomial of the anti-adjacency matrix of $\overline{C_n \circ K_r}$. In Theorem 7, we show the real eigenvalues and complex eigenvalues of the anti-adjacency matrix of $\overline{C_n \circ K_r}$.

**Theorem 6** Let $B(\overline{C_n \circ K_r})$ be the anti-adjacency matrix of a directed unicyclic corona graph. Then the characteristic polynomial of the matrix $B(\overline{C_n \circ K_r})$ is:

$$P \left( B(\overline{C_n \circ K_r}) \right) = \lambda^{(r+1)n} + \sum_{i=1}^{n-1}((r + 1)n)\lambda^{((r+1)n)-i} + (-1)^n(((r + 1)n) - 1)\lambda^n. \quad (1)$$

**Proof.**
According to Theorem 4, we have

$$b_i = (-1)^i \left( \sum_{j_1=1}^{w_1} \left| B(\langle \rangle \text{acyclic})_i^{(j_1)} \right| + \sum_{j_2=1}^{w_2} \left| B(\langle \rangle \text{cyclic})_i^{(j_2)} \right| \right).$$

a) For $i = 1, 2, \ldots, n - 1$.

- Acyclic subgraphs
  
  Because $P \left( B(\overline{C_n \circ K_r}) \right) = \lambda^{(r+1)n} + b_1\lambda^{(r+1)n-1} + \cdots + b_{(r+1)n-1}\lambda + b_{(r+1)n}$ is the characteristic polynomial of the anti-adjacency matrix of $\overline{C_n \circ K_r}$. All acyclic induced subgraphs which have one node are acyclic induced subgraphs at type 1, which based on Theorem 5 will provide $b_1 = (r + 1)n$.

For $i = 2, 3, 4, \ldots, n - 1$, the corresponding induced directed acyclic subgraphs are in type 2, type 3, type 4, and type 5. According to Lemma 2 and Lemma 3, there are $n$ paths directed acyclic induced subgraphs of type 2 and $rn$ paths directed acyclic induced subgraphs of type 3. The determinant of type 2 and type 3 subgraphs is equal to one,
meanwhile, the determinant of type 4 and type 5 subgraphs is equal to 0. Hence, according to \textbf{Theorem 4}, we obtain $\sum_{j_1}^{w_1} |B(U)_{\text{acyclic}}^{(j_1)}| = rn \cdot 1 + n \cdot 1 + 0 + 0 = (r + 1)n$.

- Cyclic subgraphs
Since a cyclic-induced subgraph in $C_n \circ K_r$ has at least $n$-vertices, then we do not have a directed cyclic-induced subgraph which has $i$-vertices; for $i = 1, 2, \cdots, n - 1$. Hence, according to \textbf{Theorem 4} $\sum_{j_2}^{w_2} |B(U)_{\text{cyclic}}^{(j_2)}| = 0$.

So, we can conclude that

$$b_i = (-1)^i((r + 1)n + 0) = (-1)^i(r + 1)n, \text{ for } i = 1, 2, \cdots, n - 1.$$  \hfill (2)

b) For $i = n$.

- Acyclic subgraphs
The corresponding induced directed acyclic subgraphs are of type 3, type 4, and type 5. According to Lemma 3, there are $rn$ paths directed acyclic induced subgraphs of type 3. The determinant of subgraphs of type 3 is equal to 1, meanwhile the determinant of subgraphs of type 4 and type 5 is equal to 0. Hence, according to \textbf{Theorem 4} $\sum_{j_1}^{w_1} |B(U)_{\text{acyclic}}^{(j_1)}| = rn + 0 = rn$.

- Cyclic subgraphs
The graph only has one directed cycle subgraph that has $n$-vertices. Let $B$ be an anti-adjacency matrix of its subgraph. Using \textbf{Theorem 3}, we have $\det(B) = n - 1$, then according to \textbf{Theorem 4} $\sum_{j_2}^{w_2} |B(U)_{\text{cyclic}}^{(j_2)}| = 1 \cdot (n - 1) = n - 1$.

So, we can conclude that

$$b_n = rn + n - 1 = (r + 1)n - 1.$$ \hfill (3)

c) For $i = n + 1, n + 2, \cdots, (r + 1)n$.

- Acyclic subgraphs
The corresponding induced directed acyclic subgraphs are type 5 subgraphs. Then the determinant of type 5 subgraph is equal to 0. Hence, according to \textbf{Theorem 4} $\sum_{j_1}^{w_1} |B(U)_{\text{acyclic}}^{(j_1)}| = 0$.

- Cyclic subgraphs
Let $B$ be an anti-adjacency matrix of its directed cyclic-induced subgraphs, On the directed cyclic-induced subgraphs from $C_n \circ K_r$, there are more than or equal two vertices that the direction of their edges go to the same vertex. Hence, according to \textbf{Theorem 2} $\det(B) = 0$, then according to \textbf{Theorem 4} $\sum_{j_2}^{w_2} |B(U)_{\text{cyclic}}^{(j_2)}| = 0$.

So, we can conclude that

$$b_i = 0, \text{ for } i = n + 1, n + 2, \cdots, (r + 1)n.$$ \hfill (4)

Thus, the characteristic polynomial of anti-adjacency matrix of directed unicyclic corona graph is

$$P(B(C_n \circ K_r)) = \lambda^{(r+1)n} + \sum_{i=1}^{n-1} ((r + 1)n)\lambda^{((r+1)n) - i} + (-1)^n((r + 1)n) - 1)\lambda^n.$$ \hfill (5)
Theorem 7 If $P\left( B\left( C_n \cdot K_r \right) \right) = \lambda^{(r+1)n} + \sum_{i=1}^{n-1}(r+1)\lambda^{(r+1)n-i} + (-1)^n((r+1)n - 1)\lambda^n$ is a characteristic polynomial of the anti-adjacency matrix of $C_n \cdot K_r$, then the eigenvalues of its characteristic polynomial are:

- For $n = 2k$ with $2,3,4,\ldots$; $\lambda_1 = \lambda_2 = \cdots = \lambda_{2r} = 0; \lambda_{2r+1} = (r+1)2k - 1; \lambda_{2r+2} = 1$; and $\lambda_{2r+3} = \lambda_{2r+4} = \cdots = \lambda_{(r+1)2k} = e^{i\frac{m\pi}{2k+1}}$, for $m \neq kz$, $z \in \mathbb{Z}$.

- For $n = 2k + 1$ with $1,2,3,\ldots$; $\lambda_1 = \lambda_2 = \cdots = \lambda_{r(2k+1)} = 0; \lambda_{r(2k+1)+1} = (r+1)(2k + 1) - 1$; and $\lambda_{r(2k+1)+2} = \cdots = \lambda_{(r+1)(2k+1)} = e^{i\frac{(2m+1)\pi}{2k+1}}$, for $m \neq \frac{y(2k+1)-1}{2}$, $y$ is an odd number.

Proof.

- For $n = 2k$ with $k = 2,3,4,\ldots$, then equation (1) can be written as follows.

$$P\left( B\left( C_n \cdot K_r \right) \right) = \lambda^{(r+1)2k} + \sum_{i=1}^{2k-1}(-1)^i((r+1)2k)\lambda^{(r+1)2k-i} + (-1)^{2k}((r+1)2k - 1)\lambda^{2rk} = 0$$

$$\Leftrightarrow P\left( B\left( C_n \cdot K_r \right) \right) = \lambda^{2rk}\left( \lambda^{2k} - ((r+1)2k)\lambda^{2k-1} + ((r+1)2k)\lambda^{2k-2} - \cdots - ((r+1)2k)\lambda + ((r+1)2k - 1) \right) = 0. \quad (5)$$

We use the Horner method to factor equation (5). Then we have the following:

$$\begin{array}{cccccc}
(r+1)2k-1 & -r+1 & 2k & -(r+1)2k & r+1 & 2k \\
& 1 & -1 & 1 & -1 & \vdots & -1 \\
1 & 0 & 1 & \vdots & 1 & \vdots & 1 \\
& & 1 & & & & \\
1 & & 1 & & & & \\
1 & & 1 & & & &
\end{array}$$

Hence, $P\left( B\left( C_n \cdot K_r \right) \right) = \lambda^{2rk}\left[ \lambda - ((r+1)2k - 1) \right][\lambda - 1][1 + \sum_{i=1}^{k-1}(\lambda^2)^i] = 0 \quad (6)$

From equation (6), we can see the polynomial $[1 + \sum_{i=1}^{k-1}(\lambda^2)^i]$ is a geometric series. So, with the geometric series formula, we obtain $1 + \sum_{i=1}^{k-1}(\lambda^2)^i = \frac{\lambda^{2k} - 1}{\lambda^2 - 1}$.

$$\Rightarrow P\left( B\left( C_n \cdot K_r \right) \right) = \lambda^{2rk}\left[ \lambda - ((r+1)2k - 1) \right]\left[ \frac{\lambda^{2k} - 1}{\lambda^2 - 1} \right] = 0. \quad (7)$$

From equation (7) we will have:

- $\lambda_1 = \lambda_2 = \cdots = \lambda_{2r} = 0, \lambda_{2r+1} = (r+1)2k - 1, \text{and} \lambda_{2r+2} = 1.$

- $\lambda_{2r+3} = \lambda_{2r+4} = \cdots = \lambda_{(r+1)2k} = e^{i\frac{m\pi}{2k+1}}$, for $m \neq kz$ with $z \in \mathbb{Z}$. 

Hence, from equation (9), we have $\lambda_{2r+3} = \lambda_{2r+4} = \cdots = \lambda_{(r+1)2k} = e^{i\frac{m\pi}{2k+1}}$, for $m \neq kz$ with $z \in \mathbb{Z}$. 


• For \( n = 2k + 1 \) with \( k = 1,2,3, \ldots \) then equation (1) can be written as follows.

\[
P \left( B \left( C_n \cdot K_r \right) \right) = \lambda^{(r+1)(2k+1)} + \sum_{i=1}^{2k} (-1)^i ((r + 1)(2k + 1)\lambda^{(r+1)(2k+1)-i} + (-1)^{2k+1}((r + 1)(2k + 1) - 1)\lambda^{(2k+1)} = 0.
\]

\[
\Rightarrow P \left( B \left( C_n \cdot K_r \right) \right) = \lambda^{(2k+1)}(\lambda^{(2k+1)} - (r + 1)(2k + 1)\lambda^{(2k+1)-1} + (r + 1)(2k + 1)\lambda^{(2k+1)-2} - \cdots + (r + 1)(2k + 1)\lambda^{(2k+1)-2k} + ((r + 1)(2k + 1) - 1)) = 0.
\]

We use the Horner method to factor equation (10). Then we have the following:

\[
\begin{array}{cccccc}
(r + 1)(2k + 1) - 1 & 1 & - (r + 1)(2k + 1) & (r + 1)(2k + 1) - 1 & \cdots & (r + 1)(2k + 1) - 1 \\
(r + 1)(2k + 1) - 1 & 1 & - (r + 1)(2k + 1) & (r + 1)(2k + 1) - 1 & \cdots & (r + 1)(2k + 1) - 1 \\
... & ... & ... & ... & ... & ... \\
1 & 1 & 1 & 1 & \cdots & 0
\end{array}
\]

Hence, \( P \left( B \left( C_n \cdot K_r \right) \right) = \lambda^{(2k+1)}[\lambda - ((r + 1)(2k + 1) - 1)] \left[ 1 + \sum_{i=1}^{2k} (-\lambda)^i \right] = 0 \) \hspace{1cm} (11)

From equation (11), we can see the polynomial \( 1 + \sum_{i=1}^{2k} (-\lambda)^i = 0 \) is a geometric series. So, with the geometric series formula, we will have \( 1 + \sum_{i=1}^{2k} (-\lambda)^i = \frac{-\lambda^{-1}}{1 - \lambda} \).

Thus \( P \left( B \left( C_n \cdot K_r \right) \right) = \lambda^{(2k+1)}[\lambda - ((r + 1)(2k + 1) - 1)] \left[ \frac{-\lambda^{-1}}{1 - \lambda} \right] = 0. \) \hspace{1cm} (12)

From equation (12), we have:

\[
\begin{align*}
\lambda_1 &= \lambda_2 = \cdots = \lambda_r(2k + 1) = 0 \text{ and } \lambda_r(2k + 1) + 1 = (r + 1)(2k + 1) - 1. \\
\end{align*}
\]

\[
\lambda^{-1} \neq \frac{-\lambda^{-1}}{1 - \lambda} = 0 \Rightarrow -\lambda - 1 = 0 \iff \lambda = -1. \text{ hence } \lambda \neq -1.
\]

According to [8], to find the roots of equation (8), we can set \( \lambda = re^{i\theta} \) be the solution of the equation \( \lambda^{2k+1} = -1 \), then we have \( (re^{i\theta})^{2k+1} = -1 \iff r^{2k+1} e^{i(2k+1)\theta} = 1 \).

\( e^{i(\pi + 2m\pi)} \). We have \( r = 1 \) and \( \theta = \frac{(2m+1)\pi}{2k+1} \) for \( m \equiv \frac{y(2k+1)-1}{2} \) with \( y \) is the odd number.(14)

Hence, from equation (14) we will have \( \lambda_r(2k + 1) + 2 = \lambda_r(2k + 1) + 3 = \cdots = \lambda_r(2k + 1) = e^{i\pi \frac{(2m+1)\pi}{2k+1}} \), form \( \frac{y(2k+1)-1}{2} \) with \( y \) is the odd numbers.

\section{Conclusion}

The directed unicyclic corona graph \( C_n \cdot K_r \) has the characteristic polynomial of its anti-adjacency matrix as follows.

\[
P \left( B \left( C_n \cdot K_r \right) \right) = \lambda^{(r+1)n} + \sum_{i=1}^{n-1} ((r + 1)n)\lambda^{(r+1)n-i} + (-1)^n ((r + 1)n - 1)\lambda^n.
\]

The values of the coefficients of the characteristic polynomial, and the eigenvalues of the anti-adjacency matrix of the directed unicyclic corona graph \( C_n \cdot K_r \), can be seen in Table 1 and Table 2.

\begin{table}[h]
\centering
\caption{Coefficient of the characteristic polynomial of the directed unicyclic corona graph.}
\begin{tabular}{ccc}
\hline
Coefficient & Coefficient Value \\
\hline
\( b_1, \ldots, b_{n-1} \) & \(-1)^i((r + 1)n) \)
\hline
\( b_n \) & \(-1)^n((r + 1)n - 1) \)
\hline
\( b_{n+1}, \ldots, b_{(r+1)n} \) & 0 \\
\hline
\end{tabular}
\end{table}
Table 2. Eigenvalues of the directed unicyclic corona graph.

| $n$ | Real | Complex |
|-----|------|---------|
| $n=2k$; $k=2,3,4...$ | $\lambda_{1,2,...,2rk} = 0$. | $\lambda_{2rk+3,2rk+4,...,(r+1)2k} = e^{im\pi}$, for $m \neq kz$ with $z \in \mathbb{Z}$. |
| $n=2k+1$; $k=1,2,3,...$ | $\lambda_{1,2,...,r(2k+1)} = 0$ and $\lambda_{r(2k+1)+1} = (r+1)(2k+1) - 1$ | $\lambda_{r(2k+1)+2,r(2k+1)+3,...,(r+1)(2k+1)} = e^{im\left(\frac{2m+1}{2k+1}\right)}$, for $m \neq \frac{y(2k+1)-1}{2}$ with $y$ is the odd number. |

An open problem for the research is finding whether there is any relation between the eigenvalues and the structure of the graph.

Acknowledgment
This research is funded by Hibah PUTI Prosiding Universitas Indonesia 2020 (Number: NKB-1008/UN2.RST/HKP.05.00/2020).

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