Introduction. – In recent years, there has been an increasing interest in quantum technologies. To investigate rigorously which properties of quantum mechanics are responsible for potential operational advantages, quantum resource theories were developed, see for example [1–10]. These resource theories originate from constraints that are imposed in addition to the laws of quantum mechanics, motivated either by physical or by practical considerations. From the constraints follow the free states and the free operations, which are the ones that can be prepared and executed without violation of the constraints. These two main ingredients allow for the formulation of a rigorous theoretical framework in which to analyze quantitatively the amount of the resource present in quantum states and its usefulness in operational tasks [11–13]. In addition, there exist quantum operations that can be considered resources as well, because they are not free. Therefore, a complementary question to ask is how valuable these operations are [14]. This question is often approached by the evaluation of quantities such as the resource generation capacity, i.e. the maximal increase of the resource in an input state under application of the operation, or the resource cost, i.e. the minimal amount of resources needed to simulate a non-free operation by means of free operations [15–22]. Since there exists an infinity of measures quantifying the amount of resources in states [23–25], there exist for example also an infinitude of different resource generation capacities. In addition, as we will see later, the value of operations is not necessarily connected to the creation of resources in states. Hence the situation merits a broader approach and this is why we are examining a broader framework that we exemplify at the hand of the coherence of operations. More concretely, we will build a formal resource theory on the level of operations. Based on physical considerations, we define some quantum operations to be free and others to be costly. To quantify the amount of resources present in operations rigorously, we need to define free super-operations that act on the operations and lead to the defining properties of resource measures for operations. The natural choice is to define sequential and parallel concatenations with free operations as free [26–28].

A fundamental ingredient to the departure of quantum mechanics from classical physics is the omnipresence of the superposition principle [29, 30]. This has lead to the development of rigorous resource theories of coherence [3, 9, 13, 31], which allow to investigate the role of coherence in quantum technological applications [32–34]. These theories are also formulated on the level of states and mainly focused on the inability to create coherence. Although there exist concepts in which the free operations cannot make use of coherence [31, 35–40], this is problematic in a resource theory on the level of states because we should not call a state a resource if it does not allow for an operational advantage in some task (see also [41]). Assume for example that the outputs of the free operations are independent of the coherence of the states they are applied to. Then coherence is not detectable and hence not a resource within this framework. However, if the resource theory is formulated on the level of operations, this conceptual problem can be resolved. As long as the non-free operations are distinguishable from the free ones at no cost, they can allow for an operational advantage.

But why should we study the ability to detect coherence? To exploit coherences or more generally quantum superpositions [30, 42] in technologies, it is both necessary to have access to operations that can create coherence and operations that can detect it in the sense that the presence of coherence makes a difference in the measurement statistics [43]. If we cannot detect or equivalently use coherence, there cannot be an operational advantage in its presence. Since the ability to create coherence has already been investigated in considerable detail (see [13] for a recent review), we focus on the question how well coherence can be detected by a given operation. To do this, and as already mentioned, we build a resource theory on the level of operations. The framework can be easily extended to operations that cannot create coherence and operations that can neither detect nor create it. We comment on results in this direction in the Supplemental Material (SM). In a forthcoming work, our theoretical results will be used in the

Quantifying the Coherence of Operations

Thomas Theurer,1, ∗ Dario Egloff,1, ∗ Lijian Zhang,2, 3 and Martin B. Plenio1

1 Institute of Theoretical Physics and IQST, Universität Ulm, Albert-Einstein-Allee 11, D-89069 Ulm, Germany
2 National Laboratory of Solid State Microstructures and College of Engineering and Applied Sciences, Nanjing University, Nanjing 210093, China
3 Collaborative Innovation Center of Advanced Microstructures, Nanjing University, Nanjing 210093, China

To describe certain facets of non-classicality, it is enlightening to quantify properties of operations instead of states. This is the case if one wants to quantify how well an operation detects non-classicality, which is a necessary prerequisite to use it in quantum technologies. To do this rigorously, we build resource theories on the level of operations, exploiting the concept of resource destroying maps. We discuss the two basic ingredients of these resource theories, the free operations and the free super-operations, which are sequential and parallel concatenations with free operations. This leads to defining properties of functionals that are well suited to quantify the resources of operations. We introduce these concepts at the example of coherence. In particular, we present two measures quantifying the ability of an operation to detect, i.e. to use, coherence and provide programs to evaluate them. One of them quantifies how well the map can be simulated by a stochastic process on the populations.
analysis of an experiment [44].

We first introduce the basic framework of our resource theory defining the free operations and super-operations. This leads in a natural way to defining properties of functionals that are well suited to quantify the ability of an operation to detect coherence. Then we present two functionals that satisfy these properties, one based on the diamond norm that can be calculated efficiently and the other one with a clear operational interpretation which is based on the induced trace norm. We give examples for the value of operations according to these measures and conclude with an outlook on open questions.

**Basic framework.** – Since coherence is a basis dependent concept, we fix for all systems $A$ an orthonormal basis $|i\rangle^A$ which we call incoherent. This basis is singled out by the physics of an actual system or the computational basis in a quantum algorithm. From now on, coherences and populations will be seen with respect to the incoherent basis. The incoherent basis of a system composed of two subsystems $A$ and $B$ is given by the product basis of their incoherent bases. If it is clear from the context, we will omit the superscripts indicating the systems from here on. All states $\rho$ that are a statistical mixture of the incoherent basis states, i.e.

$$\rho = \sum_i p_i |i\rangle\langle i|$$

are called incoherent. In the following, we make frequent use of the total dephasing operation $\Delta$

$$\Delta(\rho) = \sum_i |i\rangle\langle i| \rho |i\rangle\langle i|$$

which is a resource destroying map [45] in coherence theory, i.e. its output is always incoherent. The total dephasing operation on a composed system is the tensor product of the total dephasing operations on the subsystems. If we concatenate operations, we will always implicitly assume that they match, i.e. the output dimension of the first operation equals the input dimension of the second operation. In addition, we will not write the concatenation operator $\circ$ if not necessary.

To construct a resource theory that allows us to answer the question how well a quantum operation can detect coherence, we need to define the free operations and super-operations. Let us begin with the free operations. First we notice that a quantum operation free if it cannot turn a free POVM into a non-free one by applying the operation prior to the measurement. This is exactly the case if it cannot transform coherences into populations [35].

**Definition 1.** A quantum operation $\Phi_{d-inc}$ is called detection-incoherent iff

$$\Delta \Phi_{d-inc} = \Delta \Phi_{d-inc} \Delta.$$  

The set of detection-incoherent operations is denoted by $\mathcal{D}_I$.

Note that this condition has been called nonactivating in [45]. With our convention for treating subselection, this includes Def. 1 for POVMs. As we mentioned in the introduction, it is both important to create and to detect coherence, therefore one can define creation-incoherent operations, i.e. operations which cannot create coherence. In coherence theory, these operations are called MIO (for maximally incoherent operations) [3, 46] or nongenerating in a general context in [45]. Operations that can neither create nor detect coherence are called DIO (dephasing-covariant incoherent operations) [3, 46] or commutating [45].

**Definition 2.** A POVM is free iff

$$\text{tr} P_n \Delta \rho = \text{tr} P_n \rho \quad \forall \rho, n.$$  

As one expects, all free POVMs are of the following form:

**Proposition 2.** A POVM is free iff

$$P_n = \sum_i P^n_i |i\rangle\langle i| \quad \forall n.$$  

Next we define general free operations, where we need to address subselection (by measurement results) in a consistent manner. Since the ability to do subselection depends on the actual experimental implementation, we adopt the point of view that this is a resource in itself. In general, we can have a quantum instrument $I$ which allows us to do subselection according to a variable $x$, i.e. we obtain with probability $p_x = \text{tr}(\mathcal{E}_x(\rho))$ an output $\rho_x = \mathcal{E}_x(\rho)/p_x$. From the definition of the free POVMs follows that we can store the outcome $x$ in the incoherent basis of an ancillary system, which we write as

$$\mathcal{I}(\rho) = \sum_x \mathcal{E}_x(\rho) \otimes |x\rangle\langle x|,$$

and implement the subselection later using a free POVM. In the special case of a POVM $P$, we can represent it by

$$\tilde{P}(\rho) = \sum_n \text{tr}(P_n \rho) |n\rangle\langle n|.$$  

Treating subselection in this way, we can reduce our analysis to trace preserving operations.

With subselection included into our framework, we call a quantum operation free if it cannot turn a free POVM into a non-free one by applying the operation prior to the measurement. This is exactly the case if it cannot transform coherences into populations [35].

**Definition 3.** A quantum operation $\Phi_{d-inc}$ from system $A$ to $B$ is called creation-incoherent, if it cannot create coherence in system $B$ if none were present in system $A$,

$$\Phi_{c-inc} \Delta = \Delta \Phi_{c-inc} \Delta.$$  

A quantum operation $\Phi_{dc-inc}$ is called detection-creation-incoherent if it can neither detect nor create coherence,

$$\Delta \Phi_{dc-inc} = \Phi_{dc-inc} \Delta.$$
Our contribution in this work is that we show how to quantify the abilities to create and detect coherence in a rigorous manner. Note that formally, the three definitions of free operations lead to different resource theories. In the following, we will use “free operation” if it is unimportant which specific choice we are considering. This allows us to introduce the second ingredient to our resource theories, the free super-operations, in a unified manner. A super-operation is free if it is a sequential and/or parallel concatenation with free operations.

**Definition 5.** For free operations $\Phi$, elemental free super-operations are given by

\[
\begin{align*}
\mathcal{E}_{1,\Phi} [\Theta] &= \Phi \circ \Theta, & \mathcal{E}_{2,\Phi} [\Theta] &= \Theta \circ \Phi, \\
\mathcal{E}_{3,\Phi} [\Theta] &= \Theta \otimes \Phi, & \mathcal{E}_{4,\Phi} [\Theta] &= \Phi \otimes \Theta.
\end{align*}
\]  

(10)

A super-operation $F$ is free iff it can be written as a sequence of free elemental super-operations,

\[
F = \mathcal{E}_{i_1,\Phi_1} \cdots \mathcal{E}_{i_n,\Phi_n} \mathcal{E}_{i_1,\Phi_2} \mathcal{E}_{i_2,\Phi_2} \mathcal{E}_{i_1,\Phi_1}.
\]  

(11)

This definition comes from a quantum computational setting: a free super-operation is an embedding into a free network. A minimal requirement on the free super-operations is that they transform free operations into free operations, otherwise it would be possible to create resources for free. This requirement can be checked directly, see the SM. It is also straightforward to show that every free super-operation can be composed using only three elemental operations (see the SM and [26, 27]). Whilst we focus on the ability to detect coherence in the main text, we present a few results for the other two classes of free operations in the SM (see also [35, 46]). As mentioned in the introduction, the case of coherence treated here is an example of our general setup: If one exchanges the resource destroying map in Eq. 2, one can move on to Defs. 3, 4, and 5. It is also possible to define free operations without the usage of resource destroying maps and to use Def. 5 for free super-operations [27].

**Detecting coherence.** – To quantify the amount of a resource present in an operation, we follow the usual axiomatic approach of quantum resource theories [1–10]. From physical considerations, we collect a set of defining properties that every measure of the resource should obey. The first property is that the measure should be faithful, which means that it needs to be zero on the set of free operations and larger than zero on non-free operations. The second property is monotonicity under the free super-operations, i.e. the amount of resource can only decrease under the application of a free super-operation. With our convention concerning subselection, this ensures monotonicity under subselection as well [47]. The third property is convexity and can be seen as a matter of convenience. It ensure that mixing does not create resources. These properties lead to the following definition.

**Definition 6.** A functional $M$ from quantum operations to the positive real numbers is called a resource measure iff

\[
\begin{align*}
M (\Theta) &= 0 \Leftrightarrow \Theta \text{ free}, \\
M (\Theta) &\geq M (F [\Theta]) \quad \forall \Theta, \quad \forall F \text{ free}, \\
M (\Theta) &\text{ is convex.}
\end{align*}
\]  

(12)

A functional that is a measure according to the above definition is of special interest if it has a clear operational interpretation, i.e. if the number it puts on a resource is directly connected to its value in a specific application. Often resource measures are hard to evaluate, thus measures that have a closed form expression or can be calculated efficiently using numerical methods are important as well. In the following, we will give one resource measure with respect to the ability to detect coherence that can be calculated efficiently and another one with an operational interpretation. Both involve norms on quantum operations. Therefore we review some related terminology first. A norm $\| \cdot \|$ on quantum operations is called sub-multiplicative iff

\[
\| \Theta_1 \circ \Theta_2 \| \leq \| \Theta_1 \| \| \Theta_2 \| \quad \forall \Theta_1, \Theta_2
\]  

(13)

and sub-multiplicative with respect to tensor products iff

\[
\| \Theta_1 \otimes \Theta_2 \| \leq \| \Theta_1 \| \| \Theta_2 \| \quad \forall \Theta_1, \Theta_2.
\]  

(14)

Norms with the above properties can be used to define measures.

**Proposition 7.** Let $\| \cdot \|$ denote a sub-multiplicative norm on quantum operations which is sub-multiplicative with respect to tensor products. If $\| \Phi \| \leq 1$ for all $\Phi$ detection-incoherent, the functional

\[
M (\Theta) = \min_{\Phi \in D^2} \| \Delta \Theta - \Delta \Phi \|
\]  

(15)

is a measure in the detection-incoherent setting.

Choosing a particular norm in the above proposition, the so-called completely bounded trace norm or diamond norm [48], we find a measure that can be calculated efficiently. The diamond norm is based on the trace norm, which is defined for a linear operator $A$ by [49]

\[
\| A \|_1 = \text{tr} \left( \sqrt{A^\dagger A} \right).
\]  

(16)

The induced trace norm on a quantum operation (or more general a super-operator) $\Theta$ is, as the name suggests, defined by

\[
\| \Theta \|_1 = \max \{ \| \Theta (X) \|_1 : \| X \|_1 \leq 1 \}.
\]  

(17)

Finally, the completely bounded trace norm or diamond norm of a quantum channel is given by

\[
\| \Theta^{B\rightarrow A} \|_\diamond = \sup_Z \| \Theta^{B\rightarrow A} \otimes 1^Z \|_1 = \| \Theta^{B\rightarrow A} \otimes 1^A \|_1
\]  

and has multiple applications in quantum information [48–50]. With these definitions at hand, we are ready to present our first measure.
Theorem 8. The functional
\[ M_n(\Theta) = \min_{\Phi \in D^n} \|\Delta(\Theta - \Phi)\|_1 \] (18)
is a measure in the detection-incoherent setting. We call this measure the diamond-measure.

Rather surprisingly, we show in the SM that this measure can be calculated efficiently using a semidefinite program [51] which is based on [52]. A related measure is given in the following theorem.

Theorem 9. The functional
\[ \tilde{M}_n(\Theta) = \min_{\Phi \in D^n} \max_{\psi} \|\Delta(\Theta - \Phi)|\psi\rangle\langle\psi|\|_1 \] (19)
is a measure in the detection-incoherent setting. We call it the nSID-measure (non-stochasticity in detection).

As we prove in the SM, this measure has an operational interpretation in our framework. Assume you obtain a single copy of a quantum channel which is equal to \( \Theta_0 \) or \( \Theta_1 \) with probability 1/2 each. The optimal probability \( P_c(1/2, \Theta_0, \Theta_1) \) to correctly guess \( i = 0, 1 \) if one can perform only incoherent measurements is given by (see also [53])
\[ P_c(1/2, \Theta_0, \Theta_1) = \frac{1}{2} + \frac{1}{4} \max_{\psi} \|\Delta(\Theta_0 - \Theta_1)|\psi\rangle\langle\psi|\|_1. \]

Therefore, in a single shot regime, \( 1/2 + 1/4 \tilde{M}_n(\Theta) \) is the optimal probability to guess correctly if one obtained \( \Theta \) or the least distinguishable free operation, provided we can use only free measurements. Therefore, the measure determines how well the operation can be approximated by a free one, i.e., by a stochastic map from populations to populations, if we are only considering the populations of the output state. This operational interpretation is the reason for the choice of the name nSID-measure. Note that a similar interpretation holds for the diamond-measure with the only difference that, on the auxiliary system, non-free measurements are allowed as well. Therefore the diamond-measure is an upper bound on the nSID-measure.

As we mentioned, the diamond-measure can be calculated efficiently using a semidefinite program. In the following, we sketch a method which allows us to calculate the nSID-measure. From the definition of the trace norm, we sketch a method which allows us to calculate the nSID-measure efficiently using a semidefinite program. In the following, the nSID-measure.

Therefore, the diamond-measure is an upper bound on the name nSID-measure. Note that a similar interpretation in our framework. Assume you obtain a nSID-measure (non-stochasticity in detection).

This operational interpretation is the reason for the choice of the nSID-measure. This can be seen as a reason why for example the Deutsch–Jozsa algorithm [54, 55] not only starts but also finishes with Hadamard gates. It is not enough to create coherence, it also has to be detected, i.e., used, in order to exploit it.

Conclusions. – At the example of coherence theory, we showed how to construct rigorous resource theories on the level of operations using resource destroying maps [45]. These theories are based on two main ingredients, the free operations and the free super-operations. The free super-operations are sequential and parallel concatenations with free operations, i.e., the embedding into a network of free operations. Based on physical considerations, we defined properties that a measure of resource in an operation should obey, for example monotonicity under the free super-operations. We focused particularly on the question how well a quantum operation can detect coherence. This is important, since both the ability to create and to detect coherence are necessary prerequisites for operational advantages of quantum computation over classical computation. We presented two measures quantifying the ability of an operation to detect coherence. The first
can be calculated efficiently using a semidefinite program. The second, named the nSID-measure, can be evaluated in an iterative manner and has a clear operational interpretation. Its value determines how well we can distinguish the given quantum operation from the free operations in a single try. Finally, we proved that Fourier transforms and measurements in a Fourier basis maximize the nSID-measure and can therefore be considered optimal in the task of measuring coherence.

Completion of the resource theories provided here is a sizable task. It includes the question of manipulation, quantification, and exploitation of the resource operations using free super-operations. A thorough answer to these questions may lead to a better understanding of operational advantages provided by quantum devices, which in turn may lead to improved designs. Working out our approach in scenarios different from coherence theory will shed new light on other quantum properties.

We acknowledge helpful discussions with Andrea Smirne, Joachim Roskopf, Feixiang Xu, and Huichao Xu. TT, DE, and MBP acknowledge financial support by the ERC Synergy Grant BioQ (grant no 319130). LZ is grateful to the financial support from National Natural Science Foundation of China under Grants No. 11690032 and No. 11474159.

* These two authors contributed equally

[1] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. 78, 2275 (1997).
[2] M. Horodecki, P. Horodecki, and J. Oppenheim, Phys. Rev. A 67, 062304 (2003).
[3] J. Åberg, arXiv:quant-ph/0612146 (2006).
[4] G. Gour and R. W. Spekkens, New J. Phys. 10, 033023 (2008).
[5] M. Horodecki and J. Oppenheim, Int. J. Mod. Phys. B 27, 1345019 (2013).
[6] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Phys. Rev. Lett. 111, 250404 (2013).
[7] V. Veitch, S. A. H. Mousavian, D. Gottesman, and J. Emerson, New J. Phys. 16, 013009 (2014).
[8] A. Grudka, K. Horodecki, M. Horodecki, P. Horodecki, R. Horodecki, P. Joshi, W. Kłobus, and A. Wójcik, Phys. Rev. Lett. 112, 120401 (2014).
[9] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. 113, 140401 (2014).
[10] L. Del Rio, L. Kraemer, and R. Renner, arXiv:1511.08818 (2015).
[11] M. B. Plenio and S. Virmani, Quant. Inf. Comp. 7, 1 (2007).
[12] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
[13] A. Streltsov, G. Adesso, and M. B. Plenio, Rev. Mod. Phys. 89, 041003 (2017).
[14] M. A. Nielsen, C. M. Dawson, J. L. Dodd, A. Gilchrist, D. Mortimer, T. J. Osborne, M. J. Bremner, A. W. Harrow, and A. Hines, Phys. Rev. A 67, 052301 (2003).
[15] J. Eisert, K. Jacobs, P. Papadopoulos, and M. B. Plenio, Phys. Rev. A 62, 052317 (2000).
[16] D. Collins, N. Linden, and S. Popescu, Phys. Rev. A 64, 032302 (2001).
[17] C. H. Bennett, A. W. Harrow, D. W. Leung, and J. A. Smolin, IEEE Trans. Inf. Theory 49, 1895 (2003).
[18] A. Mani and V. Karimipour, Phys. Rev. A 92, 032331 (2015).
[19] Z. Xi, M. Hu, Y. Li, and H. Fan, arXiv:1510.06473 (2015).
[20] M. García-Díaz, D. Egloff, and M. B. Plenio, Quant. Inf. Comp. 16, 1282 (2016).
[21] K. Bu, A. Kumar, L. Zhang, and J. Wu, Phys. Lett. A 381, 1670 (2017).
[22] K. Ben Dana, M. García-Díaz, M. Mejatty, and A. Winter, Phys. Rev. A 95, 062327 (2017).
[23] G. Vidal, J. Mod. Opt. 47, 355 (2000).
[24] S. Du, Z. Bai, and X. Qi, Quant. Inf. Comp. 15, 1307 (2015).
[25] S. Du, Z. Bai, and X. Qi, Quant. Inf. Comp. 17, 503 (2017).
[26] G. Chiribella, G. M. D’Ariano, and P. Perinotti, Europhys. Lett. 83, 30004 (2008).
[27] B. Coecke, T. Fritz, and R. W. Spekkens, Inf. Comput. 250, 59 (2016).
[28] Q. Zhuang, P. W. Shor, and J. H. Shapiro, arXiv:1803.07580 (2018).
[29] E. Schrödinger, Naturwissenschaften 23, 807 (1935).
[30] N. Killoran, F. E. S. Steinhoff, and M. B. Plenio, Phys. Rev. Lett. 116, 080402 (2016).
[31] A. Winter and D. Yang, Phys. Rev. Lett. 116, 120404 (2016).
[32] M. Hillery, Phys. Rev. A 93, 012111 (2016).
[33] C. Napoli, T. R. Bromley, M. Cianciaruso, M. Piani, N. Johnston, and G. Adesso, Phys. Rev. Lett. 116, 150502 (2016).
[34] J. M. Matera, D. Egloff, N. Killoran, and M. B. Plenio, Quantum Sci. Technol. 1, 01LT01 (2016).
[35] S. Meznaric, S. R. Clark, and A. Datta, Phys. Rev. Lett. 110, 070502 (2013).
[36] E. Chitambar and G. Gour, Phys. Rev. Lett. 117, 030401 (2016).
[37] B. Yadin, J. Ma, D. Girolami, M. Gu, and V. Vedral, Phys. Rev. X 6, 041028 (2016).
[38] I. Marvian and R. W. Spekkens, Phys. Rev. A 94, 052324 (2016).
[39] E. Chitambar and G. Gour, Phys. Rev. A 94, 052336 (2016).
[40] E. Chitambar and G. Gour, Phys. Rev. A 95, 019902 (2017).
[41] D. Egloff, J. M. Matera, T. Theurer, and M. B. Plenio, arXiv:1802.02021 (2018).
[42] T. Theurer, N. Killoran, D. Egloff, and M. B. Plenio, Phys. Rev. Lett. 119, 230401 (2017).
[43] A. Smirne, D. Egloff, M. García Díaz, M. B. Plenio, and S. F. Huelga, arXiv:1709.05267 (2017).
[44] In preparation.
[45] Z.-W. Liu, X. Hu, and S. Lloyd, Phys. Rev. Lett. 118, 060502 (2017).
[46] M. García Díaz, K. Fang, X. Wang, M. Rosati, M. Skotiniotis, J. Calsamiglia, and A. Winter, arXiv:1805.04045 (2018).
[47] X.-D. Yu, D.-J. Zhang, G. F. Xu, and D. M. Tong, Phys. Rev. A 94, 060302 (2016).
[48] A. Y. Kitaev, Russ. Math. Surv. 52, 1191 (1997).
[49] J. Watrous, *Theory of quantum information* (Cambridge University Press, 2018).
[50] A. Gilchrist, N. K. Langford, and M. A. Nielsen, Phys. Rev. A 71, 062310 (2005).
[51] S. Boyd and L. Vandenberghe, *Convex optimization* (Cambridge University Press, 2004).
[52] J. Watrous, *Theory Comput.* 5, 217 (2009).
[53] W. Matthews, S. Wehner, and A. Winter, Commun. Math. Phys. 291, 813 (2009).
[54] D. Deutsch, Proc. R. Soc. A 400, 97 (1985).
[55] D. Deutsch and R. Jozsa, Proc. R. Soc. A 439, 553 (1992).
[56] M.-D. Choi, Linear Algebra Appl. 10, 285 (1975).
[57] M. Jiang, S. Luo, and S. Fu, Phys. Rev. A 87, 022310 (2013).
Simplifying results

Here we present some results that will simplify the proofs of the results in the main text. For completeness, we first proof the following lemma, which is basically clear by definition.

**Lemma 11.** If both \( \Phi \) and \( \Theta \) are free, \( \Phi \circ \Theta \) and \( \Phi \otimes \Theta \) are free as well.

**Proof.** First we note that

\[
\Phi_{\text{c-inc}} \Theta_{\text{c-inc}} \Delta = \Phi_{\text{c-inc}} \Delta \Theta_{\text{c-inc}} \Delta = \Delta \Phi_{\text{c-inc}} \Delta \Theta_{\text{c-inc}} \Delta = \Delta \Phi_{\text{c-inc}} \Theta_{\text{c-inc}} \Delta,
\]

\[
\Delta \Phi_{\text{d-inc}} \Theta_{\text{d-inc}} \Delta = \Delta \Phi_{\text{d-inc}} \Delta \Theta_{\text{d-inc}} \Delta = \Delta \Phi_{\text{d-inc}} \Delta \Theta_{\text{d-inc}} \Delta = \Delta \Phi_{\text{d-inc}} \Theta_{\text{d-inc}} \Delta,
\]

\[
(\Phi_{\text{c-inc}} \otimes \Theta_{\text{c-inc}}) \Delta = (\Phi_{\text{c-inc}} \Delta) \otimes (\Theta_{\text{c-inc}} \Delta) = (\Delta \Phi_{\text{c-inc}} \Delta) \otimes (\Delta \Theta_{\text{c-inc}} \Delta) = \Delta (\Phi_{\text{c-inc}} \otimes \Theta_{\text{c-inc}}) \Delta,
\]

\[
\Delta (\Phi_{\text{d-inc}} \otimes \Theta_{\text{d-inc}}) = (\Delta \Phi_{\text{d-inc}} \Delta) \otimes (\Delta \Theta_{\text{d-inc}} \Delta) = \Delta (\Phi_{\text{d-inc}} \otimes \Theta_{\text{d-inc}}) \Delta.
\]

This shows that our free super-operations indeed preserve the set of free operations. In addition, it allows us to give the following simplified characterization of the free super-operations (see also [26] and lemma 3.11 in [27]).

**Proposition 12.** A super-operation \( \mathcal{F} \) is free if and only if it can be written as

\[
\mathcal{F} [\Theta] = \mathcal{F}_{\Phi_2, \Phi_1} [\Theta] := \Phi_2 \circ (\Theta \otimes 1) \circ \Phi_1
\]

where \( \Phi_1 \) and \( \Phi_2 \) are free.

**Proof.** First we note that

\[
\mathcal{F}_{\Phi_2, \Phi_1} [\Theta] = \mathcal{E}_{1, \Phi_2} \circ \mathcal{E}_{2, \Phi_1} \circ \mathcal{E}_{3, 1} [\Theta]
\]

is always free by definition. Defining the (in all three frameworks) free operation

\[
\Phi_S (\rho \otimes \sigma) = \sigma \otimes \rho,
\]

we can write all elemental free operations in this form,

\[
\mathcal{E}_{1, \Phi} [\Theta] := \Phi \circ \Theta = \Phi \circ (\Theta \otimes 1) \circ 1 = \mathcal{F}_{\Phi, 1} [\Theta],
\]

\[
\mathcal{E}_{2, \Phi} [\Theta] := \Theta \circ \Phi = 1 \circ (\Theta \otimes 1) \circ \Phi = \mathcal{F}_{1, \Phi} [\Theta],
\]

\[
\mathcal{E}_{3, \Phi} [\Theta] := \Theta \otimes \Phi = (\Theta \otimes 1) \circ (1 \otimes \Phi) = 1 \circ (\Theta \otimes 1) \circ (1 \otimes \Phi) = \mathcal{F}_{1, \otimes \Phi} [\Theta],
\]

\[
\mathcal{E}_{4, \Phi} [\Theta] := \Phi \otimes \Theta = \Phi_S \circ (\Theta \otimes \Phi) \circ \Phi_S = \Phi_S \circ (\Theta \otimes 1) \circ (1 \otimes \Phi) \circ \Phi_S = \mathcal{F}_{\Phi_S, (1 \otimes \Phi) \circ \Phi_S} [\Theta].
\]

Therefore

\[
\mathcal{F}_{\Phi_4, \Phi_3} \circ \mathcal{F}_{\Phi_2, \Phi_1} [\Theta] := \Phi_4 \circ ((\Phi_2 \circ (\Theta \otimes 1) \circ \Phi_1) \otimes 1) \circ \Phi_3
\]

\[
= \Phi_4 \circ ((\Phi_2 \otimes 1) \circ ((\Theta \otimes 1) \circ \Phi_1) \otimes 1) \circ \Phi_3
\]

\[
= \Phi_4 \circ (\Phi_2 \otimes 1) \circ ((\Theta \otimes 1) \otimes 1) \circ (\Phi_1 \otimes 1) \circ \Phi_3
\]

\[
= \Phi_4 \circ (\Phi_2 \otimes 1) \circ (\Theta \otimes 1) \circ (\Phi_1 \otimes 1) \circ \Phi_3
\]

\[
= \mathcal{F}_{\Phi_4 \circ (\Phi_2 \otimes 1) \circ (\Phi_1 \otimes 1) \circ \Phi_3
\]

finishes the proof.

As shown in the following proposition, this simplifies the defining properties of a measure.

**Proposition 13.** \( M \) is a measure of resource of operations iff

\[
M (\Theta) = 0 \iff \Theta \text{ free},
\]

\[
M (\Theta) \geq M (\mathcal{E}_{1, \Phi} [\Theta]) \forall \Theta, \forall \Phi \text{ free},
\]

\[
M (\Theta) \geq M (\mathcal{E}_{2, \Phi} [\Theta]) \forall \Theta, \forall \Phi \text{ free},
\]

\[
M (\Theta) \geq M (\mathcal{E}_{3, 1} [\Theta]) \forall \Theta,
\]

\( M (\Theta) \) is convex.

\[
(30)
\]
Proof. Assume $M$ is a measure. Then, by definitions, the conditions hold. Now assume the conditions hold. Using Prop. 12, we can write
\[
M(\mathcal{F}[\Theta]) = M(\mathcal{E}_{1,\Phi_2} \circ \mathcal{E}_{2,\Phi_1} \circ \mathcal{E}_{3,1} [\Theta]) \leq M(\mathcal{E}_{2,\Phi_1} \circ \mathcal{E}_{3,1} [\Theta]) \leq M(\mathcal{E}_{3,1} [\Theta]) \leq M(\Theta)
\] (31)
\[
\square
\]

Next we have the following lemma, which we will use frequently.

**Lemma 14.** The eigenvectors of $\mathbb{I}^A \otimes \Delta^B(\rho^{A,B})$ are separable and of the form $|\phi_{a|b}\rangle^A \otimes |b\rangle^B$.

**Proof.** Define
\[
\rho^{A,B} = \sum_{i,j,k,l} \rho_{ijkl} |i,j\rangle\langle k,l|.
\] (32)

If we do a projective measurement $\{\langle i | i \rangle\}$ on system $B$ with outcome $b$, the post-measurement state of system $A$ is given by
\[
\rho_{i|b} = \sum_{i,k} \rho_{i|k} \frac{p_k}{p_b} |k\rangle\langle k| = \sum_a q_{a|b} |\phi_{a|b}\rangle \langle \phi_{a|b}|
\] (33)

where the right side is its eigendecomposition. We define the orthonormal set of states
\[
|\psi_{a,b}\rangle = |\phi_{a|b}\rangle^A \otimes |b\rangle^B.
\] (34)

Then
\[
\mathbb{I}^A \otimes \Delta^B(\rho^{A,B}) |\psi_{a,b}\rangle = \sum_{i,j,k} \rho_{i|j} |i,j\rangle \langle k| \langle \phi_{a|b}, b\rangle
\]
\[
= \sum_{i,j} q_{i|j} |\phi_{i|j}\rangle \langle \phi_{i|j}| \langle \phi_{a|b}, b\rangle
\]
\[
=q_{a|b} p_b |\psi_{a,b}\rangle.
\] (35)

Thus all eigenvectors of $\mathbb{I}^A \otimes \Delta^B(\rho^{A,B})$ are of the form $|\phi_{a|b}\rangle^A \otimes |b\rangle^B$. \qed

In general, we can check directly if a quantum operation is free: Every quantum operation $\Phi$ is linear and thus completely determined by the coefficients $\Phi_{b,d}^{a,c}$ defined through
\[
\Phi(\langle i | j \rangle) = \sum_{k,l} \Phi_{k,l}^{i,j} |k\rangle\langle l|.
\] (36)

**Proposition 15.** Let us represent a linear map $\Phi(\rho)$ by the coefficients $\Phi_{a,c}^{b,d}$ as above. Then $\Phi$ is completely positive iff
\[
\Phi_{a,c}^{b,d} = \sum_n K_{a,n} K_{n,c}^* K_{a,b}^*.
\] (37)

Under this condition, $\Phi$ is a detection-incoherent quantum operation iff
\[
\Phi_{a,a}^{b,d} = p(a|b) \delta_{b,d} \quad \forall a,b,d.
\] (38)

$\Phi$ is a creation-incoherent quantum operation iff
\[
\Phi_{b,c}^{a,a} = p(b|a) \delta_{b,c} \land \sum_a \Phi_{a,a}^{b,d} = \delta_{b,d} \quad \forall a,b,d.
\] (39)

$\Phi$ is a detection-creation-incoherent operation iff
\[
\Phi_{a,a}^{b,d} = p(a|b) \delta_{b,d} \quad \forall a,b,d,
\]
\[
\Phi_{b,c}^{a,a} = p(b|a) \delta_{b,c} \quad \forall a,b,d.
\] (40)
Proof. \( \Phi \) being completely positive is equivalent to \( \Phi (\rho) = \sum_{n} K_{n} \rho K_{n}^{\dagger} \) which we can write as
\[
\Phi(\rho) = \sum_{n,i,j,a,b,c,d} K_{n} a,b |a \rangle \langle b | j \rangle |K_{n,a,b,c,d}^{*} |d \rangle \langle c | = \sum_{n,a,b,c,d} K_{n,a,b,c,d}^{*} \rho_{b,d} |a \rangle \langle c |
\] (41)
from which follows
\[
\Phi_{a,c}^{b,d} = \sum_{n} K_{n,a,b,c,d}^{*}.
\] (42)
Now we come to the first part of the proposition. We have
\[
\Delta \Phi \rho = \sum_{a,b,d} \Phi_{b,d}^{b,a} \rho_{b,d} |a \rangle \langle a |, \quad \Delta \Phi \Delta \rho = \sum_{a,b} \Phi_{b,b}^{a,a} \rho_{b,b} |a \rangle \langle a |.
\] (43)
Thus \( \Phi \) being detection-incoherent is equivalent to
\[
\sum_{b} \Phi_{b,d}^{b,s} \rho_{b,d} = \sum_{b} \Phi_{b,b}^{s,s} \rho_{b,b} \quad \forall s, \rho,
\] (44)
which is exactly the case if this condition holds for all pure \( \rho = |\psi \rangle \langle \psi | \) with
\[
|\psi \rangle = \sum_{j} \sqrt{q_{j}} e^{i \gamma_{j}} |j \rangle,
\] (45)
meaning \( \Phi \) is detection-incoherent iff
\[
\sum_{b \neq d} \Phi_{s,b}^{b,d} \sqrt{q_b \sqrt{q_d}} e^{i(\gamma_b - \gamma_d)} = 0 \quad \forall s, \{q_i\} \text{prob. distr., } \gamma_i \in \mathbb{R}.
\] (46)
From this follows the necessary condition
\[
0 = \frac{1}{2} \left( \Phi_{s,b}^{b,d} e^{i(\gamma_b - \gamma_d)} + \Phi_{s,b}^{d,b} e^{i(\gamma_d - \gamma_b)} \right)
= \Re \left( \sum_{n} K_{n,a,b}^{s,s} K_{n,b,d}^{*} e^{i(\gamma_b - \gamma_d)} \right) \quad \forall s, b \neq d, \gamma_b, \gamma_d \in \mathbb{R}
\] (47)
and thus
\[
\sum_{n} K_{n,a,b}^{s,s} K_{n,b,d}^{*} = \Phi_{s,b}^{b,d} = 0 \quad \forall s, b \neq d.
\] (49)
In addition, since we work with trace preserving operations, we have
\[
p(a|b) = \langle a | \sum_{n} K_{n} |b \rangle \langle b | K_{n}^{\dagger} |a \rangle = \sum_{n} K_{n,a,b}^{s} K_{n,b,d}^{*} = \Phi_{a,a}^{b,b}
\] (50)
and therefore necessarily
\[
\sum_{n} K_{n,a,b}^{s,s} K_{n,b,d}^{*} = \Phi_{s,s}^{b,d} = p(a|b) \delta_{b,d} \quad \forall a, b, d.
\] (51)
This is already sufficient for trace preservation, since
\[
1 = \sum_{n} K_{n}^{\dagger} K_{n} \iff \sum_{n,a,b,c,d} K_{n,a,b}^{*} K_{n,b,d}^{*} = \sum_{a} \Phi_{a,a}^{b,d} = \delta_{b,d}.
\] (52)
To show that condition (38) is also sufficient, we can plug it into condition (46)

\[ \sum_{n,b \neq d} K_{n,b}^* K_{n,d} \sqrt{q_b} \sqrt{q_d} e^{i(\gamma_b - \gamma_d)} = \sum_{b \neq d} p(s|b) \delta_{b,d} \sqrt{q_b} \sqrt{q_d} e^{i(\gamma_b - \gamma_d)} = 0 \quad \forall s, \{q_i\} \text{prob. distr., } \gamma_i \in \mathbb{R}, \]  

and see that it is satisfied.

Now we come to the second part. Assume \( \Phi \) is creation-incoherent. From

\[ \Phi (|i\rangle\langle i|) = \sum_{k,l} \Phi_{k,l}^i |k\rangle\langle l| \]  

and the condition \( \Delta \Phi \Delta = \Phi \Delta \), we find

\[ \Phi_{k,l}^i = \delta_{k,l} \]  

This time, we need to enforce trace preservation separately, again by the condition

\[ \sum_a \Phi_{b,a}^i = 0 \]  

 Sufficiency is straightforward, the third statement a trivial consequence of the others.

Proofs of the results in the main text

Here we give the proofs of our results in the main text, which we restate for readability.

**Proposition (2).** A POVM is free iff

\[ P_n = \sum_i P_i^n |i\rangle\langle i| \quad \forall n. \]  

**Proof.** Let us use the notation

\[ P_n = \sum_{i,j} P_{i,j}^n |i\rangle\langle j| \]  

and

\[ |\psi\rangle = \frac{1}{\sqrt{2}} (|a\rangle + e^{i\phi}|b\rangle) \]  

with \( a \neq b \) and \( \phi \in \mathbb{R} \). Now assume that the POVM is free. From Def. 1 follows for \( \tilde{\rho} = |\psi\rangle\langle \psi| \)

\[ \text{tr} P_n \Delta \tilde{\rho} = \text{tr} P_n \tilde{\rho} \]

\[ \iff P_{a,a}^n + P_{b,b}^n = P_{a,a}^n + P_{b,b}^n + P_{a,b}^n e^{-i\phi} + P_{b,a}^n e^{i\phi} \]

\[ \iff 0 = 2 \text{Re} (P_{a,b}^n e^{-i\phi}) \]

such that

\[ P_n = \sum_i P_i^n |i\rangle\langle i| \quad \forall n \]

is a necessary condition. It is obviously also sufficient.
Proposition (7). Let $\| \cdot \|$ denote a sub-multiplicative norm on quantum operations which is sub-multiplicative with respect to tensor products. If $\| \Phi \| \leq 1$ for all $\Phi$ detection-incoherent, the functional

$$M(\Theta) = \min_{\Phi \in DI} \| \Delta \Theta - \Delta \Phi \|$$

(63)

is a measure in the detection-incoherent setting.

Proof. Since $\| \cdot \|$ is a norm, $M(\Theta)$ is faithful. Remember that $\tilde{\Phi} \Phi \in DI$ if both $\tilde{\Phi}, \Phi \in DI$. For $\tilde{\Phi} \in DI$, we therefore have

$$M(\Theta \tilde{\Phi}) = \min_{\Phi \in DI} \| \Delta \Theta \tilde{\Phi} - \Delta \Phi \|
\leq \min_{\Phi \in DI} \| \Delta \Theta \tilde{\Phi} - \Delta \Phi \|
\leq \min_{\Phi \in DI} \| \Delta \Theta - \Delta \Phi \| \| \tilde{\Phi} \|
= M(\Theta)$$

(64)

and

$$M(\tilde{\Phi} \Theta) = \min_{\Phi \in DI} \| \Delta \tilde{\Phi} \Theta - \Delta \Phi \|
= \min_{\Phi \in DI} \| \Delta \tilde{\Phi} \Delta \Theta - \Delta \Phi \|
\leq \min_{\Phi \in DI} \| \Delta \tilde{\Phi} \Delta \Theta - \Delta \Phi \Phi \|
= \min_{\Phi \in DI} \| \Delta \tilde{\Phi} \Delta \Theta - \Delta \tilde{\Phi} \Delta \Phi \|
\leq \min_{\Phi \in DI} \| \Delta \tilde{\Phi} \| \| \Delta \Theta - \Delta \Phi \|
\leq \min_{\Phi \in DI} \| \Delta \Theta - \Delta \Phi \|
= M(\Theta).$$

(65)

Since the norm is sub-multiplicative with respect to tensor products, we have

$$M(\Theta \otimes 1) = \min_{\Phi \in DI} \| \Delta (\Theta \otimes 1) - \Delta \Phi \|
\leq \min_{\Phi = \Phi_1 \otimes 1 \in DI} \| (\Delta \Theta) \otimes \Delta - (\Delta \Phi_1) \otimes \Delta \|
= \min_{\Phi_1 \in DI} \| (\Delta \Theta - \Delta \Phi_1) \otimes \Delta \|
\leq \min_{\Phi_1 \in DI} \| \Delta \Theta - \Delta \Phi_1 \| \| \Delta \|
\leq M(\Theta).$$

(66)

Convexity is a consequence of absolute homogeneity and the triangle inequality. Choose $\Phi_1, \Phi_2$ such that

$$M(\Theta) = \| \Delta \Theta - \Delta \Phi_1 \|,$n
$$M(\Psi) = \| \Delta \Psi - \Delta \Phi_2 \|. $$

(67)

Then, for $0 \leq t \leq 1$, we find

$$M(t \Theta + (1-t) \Psi) = \min_{\Phi \in DI} \| \Delta (t \Theta + (1-t) \Psi) - \Delta \Phi \|
\leq \| \Delta (t \Theta + (1-t) \Psi) - \Delta (t \Phi_1 + (1-t) \Phi_2) \|
= \| (t \Delta \Theta - t \Delta \Phi_1) + (1-t) (\Delta \Psi - \Delta \Phi_2) \|
\leq t \| \Delta \Theta - \Delta \Phi_1 \| + (1-t) \| \Delta \Psi - \Delta \Phi_2 \|
= t M(\Theta) + (1-t) M(\Psi).$$

(68)
Theorem (8). The functional
\[ M_0(\Theta) = \min_{\Phi \in DI} \| \Delta \Theta - \Delta \Phi \|_0 \] (69)
is a measure in the detection-incoherent setting. We call this measure the diamond-measure.

Proof. The diamond norm of a quantum operation is equal to one, the diamond norm is sub-multiplicative and sub-multiplicative with respect to tensor products [49]. \qed

Theorem (9). The functional
\[ \tilde{M}_0(\Theta) = \min_{\Phi \in DI} \max_{|\psi\rangle} \| \Delta (\Theta - \Phi) |\psi\rangle\langle\psi| \|_1 \] (70)
is a measure in the detection-incoherent setting. We call it the nSID-measure (non-stochasticity in detection).

Proof. Since the induced trace norm is not sub-multiplicative with respect to tensor products, we cannot use Prop. 7. To begin the proof, we notice that by convexity,
\[ \max_{|\psi\rangle} \| \Delta (\Theta - \Phi) |\psi\rangle\langle\psi| \|_1 = \max_{\sigma} \| \Delta (\Theta - \Phi) \sigma \|_1 \] (71)
For \( \tilde{\Phi} \in DI \), we have
\[ \tilde{M}_0(\tilde{\Phi}) = \min_{\Phi \in DI} \max_{|\psi\rangle} \| \Delta (\tilde{\Phi} - \Phi) |\psi\rangle\langle\psi| \|_1 \leq \max_{|\psi\rangle} \| \Delta (\tilde{\Phi} - \Phi) |\psi\rangle\langle\psi| \|_1 = 0 \] (72)
and for \( \Delta \Theta \neq \Delta \Theta \Delta \), there exists for all \( \Phi \in DI \) a \( |\psi\rangle \) such that we have \( (\Delta \Theta - \Delta \Phi \Delta) |\psi\rangle |\phi\rangle \neq 0 \). Since \( \| \cdot \|_1 \) is a norm, this proves faithfulness.

From this follows, again for \( \tilde{\Phi} \in DI \),
\[ \tilde{M}_0(\Theta \tilde{\Phi}) = \min_{\Phi \in DI} \max_{\sigma} \| \Delta (\Theta \tilde{\Phi} - \Phi) \sigma \|_1 \leq \min_{\Phi \in DI} \max_{\sigma} \| \Delta (\Theta \tilde{\Phi} - \Phi \Phi) \sigma \|_1 = \min_{\Phi \in DI} \max_{\rho = \Phi \sigma} \| \Delta (\Theta - \Phi) \rho \|_1 \leq \min_{\Phi \in DI} \max_{\sigma} \| \Delta (\Theta - \Phi) \sigma \|_1 = \tilde{M}_0(\Theta), \] (73)
where we used in the second line that \( \tilde{\Phi} \Phi \in DI \) if \( \Phi, \tilde{\Phi} \in DI \). Using that the trace norm is contractive under CPTP maps, we find
\[ \tilde{M}_0(\Phi \Theta) = \min_{\Phi \in DI} \max_{\sigma} \| \Delta (\Phi \Theta - \Phi) \sigma \|_1 \leq \min_{\Phi \in DI} \max_{\sigma} \| \Delta (\Phi \Theta - \Phi \Phi) \sigma \|_1 = \min_{\Phi \in DI} \max_{\rho = \Phi \sigma} \| \Delta (\Theta - \Phi) \rho \|_1 \leq \min_{\Phi \in DI} \max_{\sigma} \| \Delta (\Theta - \Phi) \sigma \|_1 = \tilde{M}_0(\Theta). \] (74)
With the help of Lem. 14 follows

\[
\hat{M}_0(\Theta \otimes 1) = \min_{\Phi \in DI} \max_{\sigma} \left\| \Delta \left( \Theta \otimes 1 - \hat{\Phi} \right) \sigma \right\|_1
\]

\[
\leq \min_{\Phi \in DI} \max_{\sigma} \left\| \Delta \left( \Theta \otimes 1 - \Phi \otimes 1 \right) \sigma \right\|_1
\]

\[
= \min_{\Phi \in DI} \max_{\sigma} \left\| \left( \Delta \Theta \otimes 1 - \Delta \Phi \otimes 1 \right) \left( 1 \otimes \Delta \right) \sigma \right\|_1
\]

\[
= \min_{\Phi \in DI \rho = (1 \otimes \Delta)\sigma} \max_{\rho \|_1} \left\| \left( \Delta \Theta \otimes 1 - \Delta \Phi \otimes 1 \right) \rho \right\|_1
\]

\[
\leq \min_{\Phi \in DI} \sum_{a,b} q_a \left( \Phi(a,b) \right) \max_{\|,i} \left\| \left( \Delta \Theta \otimes 1 - \Delta \Phi \otimes 1 \right) \left| \phi, i \right> \left< \phi, i \right| \right\|_1
\]

\[
= \min_{\Phi \in DI} \max_{\|,i} \left\| \left( \Delta \Theta - \Delta \Phi \right) \left| \phi \right> \left< i \right| \right\|_1
\]

\[
= \min_{\Phi \in DI} \max_{\|,i} \left\| \left( \Delta \Theta - \Delta \Phi \right) \left| \phi \right> \left< i \right| \right\|_1
\]

\[
= \hat{M}_0(\Theta).
\]

Convexity is a consequence of absolute homogeneity and the triangle inequality. Choose \( \Phi_1, \Phi_2, \sigma_1, \sigma_2 \) such that

\[
\hat{M}_0(\Theta) = \left\| \left( \Delta \Theta - \Delta \Phi_1 \right) \sigma_1 \right\|_1
\]

\[
\hat{M}_0(\Psi) = \left\| \left( \Delta \Psi - \Delta \Phi_2 \right) \sigma_2 \right\|_1.
\]

Then, for \( 0 \leq t \leq 1 \), we find

\[
\hat{M}_0(t\Theta + (1-t)\Psi) = \min_{\Phi \in DI} \max_{\sigma} \left\| \left[ \Delta(t\Theta + (1-t)\Psi) - \Delta \Phi \right] \sigma \right\|_1
\]

\[
\leq \max_{\sigma} \left\| \left[ \Delta(t\Theta + (1-t)\Psi) - \Delta(t\Phi_1 + (1-t)\Phi_2) \right] \sigma \right\|_1
\]

\[
= \max_{\sigma} \left\| t(\Delta \Theta - \Delta \Phi_1)\sigma + (1-t)(\Delta \Psi - \Delta \Phi_2)\sigma \right\|_1
\]

\[
\leq \max_{\sigma} \left[ t \left\| \left( \Delta \Theta - \Delta \Phi_1 \right) \sigma \right\|_1 + (1-t) \left\| \left( \Delta \Psi - \Delta \Phi_2 \right) \sigma \right\|_1 \right]
\]

\[
\leq \max_{\sigma} t \left\| \left( \Delta \Theta - \Delta \Phi_1 \right) \sigma \right\|_1 + \max_{\sigma} (1-t) \left\| \left( \Delta \Psi - \Delta \Phi_2 \right) \sigma \right\|_1
\]

\[
= tM(\Theta) + (1-t)M(\Psi).
\]

Proposition (10). The maximum value of \( \hat{M}_0(\Theta) \) for \( \Theta \) a quantum channel with input of dimension \( n \) and output of dimension \( m \) is given by

\[
\frac{2(N_0 - 1)}{N_0},
\]

where \( N_0 = \min\{n, m\} \). It is both saturated by a Fourier transform in a subspace of dimension \( N_0 \) and by a measurement in the Fourier basis, encoding the outcomes in the incoherent basis.

Proof. We first prove the bound given in the proposition. We need to distinguish two cases.
For \( n \leq m \):
\[
\begin{align*}
\min_{\Phi \in \mathcal{D}I} \max_{\rho} \| \Delta (\Lambda - \Phi) \rho \|_1 \\
\leq \min_{\Phi \in \mathcal{D}I} \max_{\rho} \| \Delta (\Lambda - \Lambda \Delta \Phi) \rho \|_1 \\
= \min_{\Phi \in \mathcal{D}I} \max_{\rho} \| \Delta \Lambda (I_n - \Delta \Phi) \rho \|_1 \\
\leq \min_{\Phi \in \mathcal{D}I} \max_{\rho} \| (I_n - \Delta \Phi) \rho \|_1 \\
\leq \max_{\rho} \| \rho - \frac{I_n}{n} \|_1 \\
= \left\| |1\rangle \langle 1| - \frac{I_n}{n} \right\|_1 \\
= (1 - 1/n) + (n - 1)/n \\
= \frac{2(n - 1)}{n}.
\end{align*}
\] (79)

For \( n \geq m \):
\[
\begin{align*}
\min_{\Phi \in \mathcal{D}I} \max_{\rho} \| \Delta (\Lambda - \Phi) \rho \|_1 \\
\leq \max_{\rho} \| \Delta \Lambda \rho - \frac{I_m}{m} \|_1 \\
\leq \left\| |1\rangle \langle 1| - \frac{I_m}{m} \right\|_1 \\
= (1 - 1/m) + (m - 1)/m \\
= \frac{2(m - 1)}{m}.
\end{align*}
\] (80)

That the Fourier transform (\( FT \)), or the measurement in the Fourier basis, saturate the bound follows from the fact that inputting the \( N_0 \) states that are sent to the respective orthogonal incoherent states, one gets \( \Delta \Phi \rho = \Delta \Phi \Delta N_0 \rho = \Delta \Phi \frac{I_{N_0}}{N_0} \), where we assumed without loss of generality that \( FT \) acts non-trivially on the span of the first \( N_0 \) states and \( I_{N_0} \) denotes the identity on this space (and the first equality comes from \( \Phi \in \mathcal{D}I \)). Assuming that \( \Phi \) does not act as the identity superoperator for these test-states, results in one of the respective resulting states having a bigger distance than \( \frac{2(N_0 - 1)}{N_0} \). Therefore the distance is at least given by \( \frac{2(N_0 - 1)}{N_0} \); i.e. the Fourier transform saturates the bound.

\[ \square \]

**Semidefinite program for the diamond-measure.**

For a quantum operation \( \Theta = \Theta^{B \rightarrow A} \), we define its corresponding Choi state \([56, 57]\) by
\[
J(\Theta) = \sum_{i,j} \Theta(|i\rangle\langle j|) \otimes |i\rangle\langle j|.
\] (81)

The diamond-measure can be calculated efficiently using the semidefinite program

\[
\begin{align*}
\text{Primal problem} & \quad \text{Dual problem} \\
\text{minimize:} & \quad 2 \| \text{tr}_B(Z) \|_\infty & \quad \text{maximize:} & \quad 2 (\text{tr}(J(\Delta \Theta) X) - \text{tr}(Y_2)) \\
\text{subject to:} & \quad Z \geq J(\Delta \Theta) - W, & \quad \text{subject to:} & \quad X \leq I_B \otimes \rho : \rho \geq 0, \text{tr}(\rho) = 1, \\
& \quad [I - \Delta]W = 0, & & \quad [I - \Delta]Y_1 - X + I_B \otimes Y_2 \geq 0, \\
& \quad \text{tr}_B(W) = I_A, & & \quad X \geq 0, \\
& \quad Z \geq 0, & & \quad Y_1 = Y_1^\dagger, \\
& \quad W \geq 0, & & \quad Y_2 = Y_2^\dagger.
\end{align*}
\] (82)
which is based on [52]. Strong duality holds. Note that $\text{tr}_B$ is the partial trace over the first subsystem since $J(\Delta \Theta) \in B \otimes A$ (see Eq. (81)).

**Proof.** According to [52], $\|\Delta \Theta - \Delta \Phi\|_o$ is the optimal value of

\[
\begin{align*}
\text{minimize:} & \quad 2\|\text{tr}_B(Z)\|_\infty \\
\text{subject to:} & \quad Z \geq J(\Delta \Theta - \Delta \Phi), \\
& \quad Z \geq 0.
\end{align*}
\]

(83)

Therefore, $M_o(\Theta) = \min_{\Phi \in \mathcal{D} \mathcal{I}} \|\Delta \Theta - \Delta \Phi\|_o$ is the optimal value of

\[
\begin{align*}
\text{minimize:} & \quad 2\|\text{tr}_B(Z)\|_\infty \\
\text{subject to:} & \quad Z \geq J(\Delta \Theta - \Delta \Phi), \\
& \quad \Phi \in \mathcal{D} \mathcal{I}, \\
& \quad Z \geq 0.
\end{align*}
\]

(84)

For $\Phi \in \mathcal{D} \mathcal{I}$, we find

\[
J(\Delta \Phi) = J(\Delta \Phi \Delta) = \sum_{i,k} \Phi_{k,k}^{i,i} |k\rangle_B \langle k|_B \otimes |i\rangle_A |i\rangle_A
\]

(85)

with $\Phi_{k,k}^{i,i} = p(k|i)$ according to Eq. (38). Thus (84) is equivalent to

\[
\begin{align*}
\text{minimize:} & \quad 2\|\text{tr}_B(Z)\|_\infty \\
\text{subject to:} & \quad Z \geq J(\Delta \Theta) - W, \\
& \quad [1 - \Delta]W = 0, \\
& \quad \text{tr}_B(W) = \mathbb{1}_A, \\
& \quad Z \geq 0, \\
& \quad W \geq 0.
\end{align*}
\]

(86)

which is the primal problem. This can be reformulated as

\[
\begin{align*}
\text{minimize:} & \quad a \\
\text{subject to:} & \quad a\mathbb{1}_A - 2\text{tr}_B(Z) \geq 0, \\
& \quad Z \geq J(\Delta \Theta) - W, \\
& \quad [1 - \Delta]W = 0, \\
& \quad \text{tr}_B(W) = \mathbb{1}_A, \\
& \quad Z \geq 0, \\
& \quad W \geq 0, \\
& \quad a \geq 0.
\end{align*}
\]

(87)

The corresponding Lagrangian is given by

\[
L \left( a, Z, W, \tilde{X}, X, Y_1, Y_2 \right) = a + \text{tr} \left( (2 \text{tr}_B Z - a\mathbb{1}_A) \tilde{X} \right) + \text{tr} \left( (J(\Delta \Theta) - W - Z) X \right) \\
+ \text{tr} \left( (1 - \Delta) W Y_1 \right) + \text{tr} \left( \text{tr}_B W - \mathbb{1}_A \right) Y_2
\]

(88)

and the dual function by

\[
q \left( \tilde{X}, X, Y_1, Y_2 \right) = \inf_{a, Z, W \geq 0} L \left( a, Z, W, \tilde{X}, X, Y_1, Y_2 \right) = \inf_{a, Z, W \geq 0} \text{tr} \left( J(\Delta \Theta) X \right) - \text{tr} \left( Y_2 \right) + a \left( 1 - \text{tr} \tilde{X} \right) + 2 \text{tr} \left( \text{tr}_B(Z) \tilde{X} \right) \\
- \text{tr} \left( ZX \right) + \text{tr} \left( WY_1 \right) - \text{tr} \left( \Delta |W| Y_1 \right) - \text{tr} \left( WX \right) + \text{tr} \left( \text{tr}_B(W) Y_2 \right).
\]

(89)
With
\[
\text{tr} \left( \text{tr}_B (Z) \tilde{X} \right) = \text{tr} \left( Z \left( \mathbb{1}_B \otimes \tilde{X} \right) \right) \tag{90}
\]
and
\[
\text{tr} \left( \Delta [W] Y_1 \right) = \text{tr} \left( W \Delta [Y_1] \right) \tag{91}
\]
follows
\[
q \left( \tilde{X}, X, Y_1, Y_2 \right) = \inf_{a,Z,W} \text{tr} \left( J (\Delta \Theta) X \right) - \text{tr} (Y_2) + a \left( 1 - \text{tr} \tilde{X} \right) + \text{tr} \left( Z \left( 2 \mathbb{1}_B \otimes \tilde{X} - X \right) \right) + \text{tr} \left( W \left( Y_1 - \Delta Y_1 - X + \mathbb{1}_B \otimes Y_2 \right) \right) = \begin{cases} \text{tr} \left( J (\Delta \Theta) X \right) - \text{tr} (Y_2) & \text{if } \text{tr} \tilde{X} \leq 1 \land 2 \mathbb{1}_B \otimes \tilde{X} - X \geq 0 \land Y_1 - \Delta Y_1 - X + \mathbb{1}_B \otimes Y_2 \geq 0, \\ -\infty & \text{else.} \end{cases} \tag{92}
\]
Thus the dual problem is given by
\[
\begin{align*}
\text{maximize:} & \quad \text{tr} \left( J (\Delta \Theta) X \right) - \text{tr} (Y_2) \\
\text{subject to:} & \quad \text{tr} \tilde{X} \leq 1, \\
& \quad 2 \mathbb{1}_B \otimes \tilde{X} - X \geq 0, \\
& \quad Y_1 - \Delta Y_1 - X + \mathbb{1}_B \otimes Y_2 \geq 0, \\
& \quad \tilde{X} \geq 0, \\
& \quad X \geq 0, \\
& \quad Y_1 = Y_1^+, \\
& \quad Y_2 = Y_2^+. \tag{93}
\end{align*}
\]
Assume \( \tilde{X} \geq 0 \) and \( \text{tr} \tilde{X} < 1 \). Then \( \tilde{X}^\prime := \frac{1}{\text{tr} \tilde{X}} \tilde{X} = (1 + c) \tilde{X} \) has trace one, is positive semidefinite and
\[
2 \mathbb{1}_B \otimes \tilde{X}^\prime - X = 2 \mathbb{1}_B \otimes \tilde{X} - X + 2c \mathbb{1}_B \otimes \tilde{X} \tag{94}
\]
is positive semidefinite for all \( X \) that satisfy \( 2 \tilde{X} \otimes \mathbb{1}_B - X \geq 0 \). Thus we can simplify the dual problem to
\[
\begin{align*}
\text{maximize:} & \quad \text{tr} \left( J (\Delta \Theta) X \right) - \text{tr} (Y_2) \\
\text{subject to:} & \quad X \leq 2 \mathbb{1}_B \otimes \rho : \rho \geq 0, \text{tr}(\rho) = 1, \\
& \quad Y_1 - \Delta Y_1 - X + \mathbb{1}_B \otimes Y_2 \geq 0, \\
& \quad X \geq 0, \\
& \quad Y_1 = Y_1^+, \\
& \quad Y_2 = Y_2^+. \tag{95}
\end{align*}
\]
Finally we can define \( X^\prime = \frac{1}{2} X, Y_1^\prime = \frac{1}{2} X \) and \( Y_2^\prime = \frac{1}{2} X \) to arrive at the dual problem stated.

To show that strong duality holds, we write the primal problem as
\[
\begin{align*}
\text{minimize:} & \quad 2 \| \text{tr}_B (Z) \|_\infty \\
\text{subject to:} & \quad J(\Delta \Theta) - W - Z \leq 0, \\
& \quad - Z \leq 0, \\
& \quad - W \leq 0, \\
& \quad [\mathbb{1} - \Delta] W = 0, \\
& \quad \text{tr}_B (W) = \mathbb{1}_A. \tag{96}
\end{align*}
\]
According to [51], strong duality holds if there exist $Z', W'$ such that the equality constraints are satisfied and the inequality constraints are strictly satisfied. If we choose
\[ Z' = \mathbb{1}_B \otimes \mathbb{1}_A + J(\Delta \Theta), \]
\[ W' = \frac{1}{\dim B} \mathbb{1}_B \otimes \mathbb{1}_A, \tag{97} \]
this is obviously the case.

**Operational interpretation of the nSID-measure**

In the following, we complete the proof of the operational interpretation of the nSID-measure given in the main text. What remains to show is the identity
\[ P_c(1/2, \Theta_0, \Theta_1) = \frac{1}{2} + \frac{1}{4} \max_{\psi} \|\Delta (\Theta_0 - \Theta_1) \mid \psi \rangle \langle \psi \|_1. \tag{98} \]

First we show the following proposition, which is a special case of results in [53]. For completeness, we give a direct proof.

**Proposition 16.** Assume you obtain a single copy of a quantum state which is with probability $\lambda$ equal to $\rho_0$ and with probability $1 - \lambda$ equal to $\rho_1$. The optimal probability $P_c(\lambda, \rho_0, \rho_1)$ to correctly guess $i = 0, 1$ when one can perform only incoherent measurements is given by
\[ P_c(\lambda, \rho_0, \rho_1) = \frac{1}{2} + \frac{1}{2} \|\Delta (\lambda \rho_0 - (1 - \lambda) \rho_1) \|_1. \]

**Proof.** The optimal strategy to correctly guess $i$ is based on the outcome of an dichotomic POVM $\{P_0, P_1 = 1 - P_0\}$ in the set of allowed measurements where we guess $i$ whenever we measured $i$. This can be seen by the following arguments: In the end, we have to make a dichotomic guess. This can only be based on the outcomes of an (not necessarily dichotomic) incoherent POVM. In principle, we could post-process the measurement outcomes in a stochastic manner to arrive at our dichotomic guess. However, this stochastic post-processing can be incorporated into the definition of a new incoherent POVM. In addition, an optimal strategy includes the usage of all information obtainable, therefore the task consists in finding an optimal $P_0$.

Let us define
\[ \rho = \lambda \rho_0 + (1 - \lambda) \rho_1, \]
\[ X = \lambda \rho_0 - (1 - \lambda) \rho_1 \tag{99} \]
and $I = \{i : X_{i,i} = \langle i | X | i \rangle > 0\}$. For a fixed and not necessarily optimal $P_0$, the probability $P_c(P_0; \lambda, \rho_0, \rho_1)$ to guess correctly is then given by
\[ P_c(P_0; \lambda, \rho_0, \rho_1) = \lambda \text{tr} (P_0 \rho_0) + (1 - \lambda) \text{tr} (P_1 \rho_1) \]
\[ = \text{tr} \left( \frac{P_0 \rho + X}{2} \right) + \text{tr} \left( \frac{P_1 \rho - X}{2} \right) \]
\[ = \text{tr} \left[ (P_0 + P_1) \frac{\rho}{2} + (P_0 - P_1) \frac{X}{2} \right] \]
\[ = \frac{1}{2} + \text{tr} \left[ (2P_0 - 1) \frac{X}{2} \right] \]
\[ = \frac{1}{2} + \text{tr} [P_0 X] - \frac{1}{2} (\lambda - (1 - \lambda)) \]
\[ = (1 - \lambda) + \sum_{i} P_i^0 X_{i,i} \tag{100} \]
and
\[ P_c(\lambda, \rho_0, \rho_1) = \max_{P_0} P_c(P_0; \lambda, \rho_0, \rho_1) \]
\[ = (1 - \lambda) + \max_{P_0} \sum_{i} P_i^0 X_{i,i} \]
\[ = (1 - \lambda) + \sum_{i \in I} X_{i,i}. \tag{101} \]
In addition,

\[
\frac{1}{2} + \frac{1}{2} \sqrt{\lambda (\rho_0 - (1 - \lambda) \rho_1)} \|_{1} = \frac{1}{2} + \frac{1}{2} \left\| \sum_{i} (\lambda \rho_{i,i}^0 - (1 - \lambda) \rho_{i,i}^0) |i\rangle\langle i| \right\|_{1}
\]

\[
= \frac{1}{2} + \frac{1}{2} \sum_{i} |\lambda \rho_{i,i}^0 - (1 - \lambda) \rho_{i,i}^0|
\]

\[
= \frac{1}{2} + \frac{1}{2} \sum_{i} |X_{i,i}|
\]

\[
= \frac{1}{2} + \frac{1}{2} \left[ \sum_{i \in \mathcal{I}} X_{i,i} - \sum_{i \notin \mathcal{I}} X_{i,i} \right]
\]

\[
= \frac{1}{2} + \frac{1}{2} \left[ 2 \sum_{i \in \mathcal{I}} X_{i,i} - \sum_{i \in \mathcal{I}} X_{i,i} \right]
\]

\[
= \frac{1}{2} + \frac{1}{2} \left[ 2 \sum_{i \in \mathcal{I}} X_{i,i} - (2 \lambda - 1) \right]
\]

\[
= (1 - \lambda) + \sum_{i \in \mathcal{I}} X_{i,i},
\]

which finishes the proof.

This allows us to obtain a slightly more general result than needed.

**Proposition 17.** Assume you obtain a single copy of a quantum channel which is with probability \( \lambda \) equal to \( \Theta_0 \) and with probability \( 1 - \lambda \) equal to \( \Theta_1 \). The optimal probability \( P_c(\lambda, \Theta_0, \Theta_1) \) to correctly guess \( i = 0, 1 \) if one can perform only incoherent measurements is given by

\[
P_c(\lambda, \Theta_0, \Theta_1) = \frac{1}{2} + \frac{1}{2} \max_{T} \| T |\psi\rangle\langle \psi| \|_1
\]

for

\[
T = \Delta [\lambda \Theta_0 - (1 - \lambda) \Theta_1].
\]

**Proof.** The optimal probability to guess correctly is given by the optimal probability to distinguish \( \sigma_0 = (\Theta_0 \otimes \mathbb{I}) \sigma \) and \( \sigma_1 = (\Theta_1 \otimes \mathbb{I}) \sigma \) for optimal \( \sigma \) (note that this includes the strategy of applying \( \Theta_i \otimes \mathcal{E} \) to \( \sigma \) for an quantum channel \( \mathcal{E} \)). Therefore we have

\[
P_c(\lambda, \Theta_0, \Theta_1) = \max_{\sigma} \frac{1}{2} + \frac{1}{2} \sqrt{\lambda (\sigma_0 - (1 - \lambda) \sigma_1)} \|_{1}.
\]
However, using Lem. 14

$$\max \| \Delta [\lambda \Theta_0 \otimes 1 - (1 - \lambda)\Theta_1 \otimes 1] \sigma \|_1 = \max \| (T \otimes \Delta) \sigma \|_1$$

$$= \max \| (T \otimes 1) (1 \otimes \Delta) \sigma \|_1$$

$$= \max_{\rho = (1 \otimes \Delta) \sigma} \| (T \otimes 1) \rho \|_1$$

$$= \max \left\{ \sum_{a,b} q_a b p_b (T \otimes 1) |\phi_a b, b \rangle \langle \phi_a b, b | \right\}_1$$

$$\leq \max \sum_{a,b} q_a b p_b \| (T \otimes 1) |\phi_a b, b \rangle \langle \phi_a b, b | \|_1$$

$$\leq \sum_{a,b} q_a b p_b \max_{|\phi\rangle, |i\rangle} \| (T \otimes 1) |\phi\rangle \langle \phi | \otimes |i\rangle \langle i | \|_1$$

$$= \max_{|\phi\rangle, |i\rangle} \| T |\phi\rangle \langle \phi | \|_1 \| i \rangle \langle i | \|_1$$

$$= \max_{|\phi\rangle} \| T |\phi\rangle \langle \phi | \|_1.$$  (106)

\[ \square \]

## Evaluating the nSID-measure

In this section, we show how the nSID-measure can be calculated. First we formulate the program for the upper bound,

$$\max \rho \text{tr} \left[ \Delta (\Theta - \Phi_n) \right] \| \rho \|,$$  (107)

as a branch and bound problem. Using the short hand notation $J_n = J(\Delta (\Theta - \Phi_n))$, we will show that the optimal value of the above optimization problem is equivalent to the optimal value of

$$\text{minimize:} \quad -2 \text{tr}[X J_n]$$

$$\text{subject to:} \quad X = \bigoplus_i \rho_i,$$

$$0 \leq \rho_i \leq \rho,$$

$$\text{tr}[\rho_i] = B(i),$$

$$\rho \geq 0,$$

$$\text{tr}(\rho) = 1,$$

$$B(i) \in \{0, 1\}.$$  (108)

Note that for fixed $B$, this is a semidefinite program. We thus just need to minimize the different programs over the possible choices of $B$.

We first show now that (107) is equivalent to

$$\text{minimize:} \quad -2 \text{tr}\left[ P \text{tr}_2[(1 \otimes \rho) J_n] \right]$$

$$\text{subject to:} \quad \rho \geq 0,$$

$$\text{tr}(\rho) = 1,$$

$$P^2 = P,$$

$$P \geq 0,$$  (109)

where the last line means that $P$ is a projector, and the minimization implies that the optimal $P_0$ for any fixed state $\rho_0$ is the projector onto the positive part of $\text{tr}_2[(1 \otimes \rho_0) J_n] = (\Delta (\Theta - \Phi_n))|\rho_0\rangle$. Now we note that

$$\text{tr}[P_0 (\Delta (\Theta - \Phi_n))|\rho_0\rangle] = -\text{tr}[(1 - P_0) (\Delta (\Theta - \Phi_n))|\rho_0\rangle],$$  (110)
since $(\Delta(\Theta - \Phi))$ is the difference of two trace preserving maps. This implies that

$$2 \text{tr}[P_0 \text{tr}_2[(1 \otimes \rho_0)J_n]] = \text{tr}[(\Delta(\Theta - \Phi))][\rho_0]],$$

(111)

for any fixed state $\rho_0$, which gives the equivalence to (107).

Since $(\Delta(\Theta - \Phi))][\rho_0] = \text{tr}_2[(1 \otimes \rho_0)J_n]$ is diagonal in the incoherent basis, the optimal $P_0$ for (109) is diagonal as well. Then we can restrict the optimization to these $P$. But diagonal $P$ can be rewritten as $P = \text{diag}(B)$, with $B$ a vector with components either 0 or 1. The next step is to write

$$\text{tr}[P \text{tr}_2[(1 \otimes \rho)J_n]] = \text{tr}[(P \otimes 1)(1 \otimes \rho)J_n] = \text{tr}[(P \otimes \rho)J_n] = \text{tr}[(\text{diag}(B) \otimes \rho)J_n] = \text{tr}[(\oplus_i B(i)\rho)J_n].$$

(112)

This means that (107) is equivalent to the problem

minimize: $-2 \text{tr}[XJ_n]$
subject to: $X = \bigoplus_i B(i)\rho_i,$
$\rho_i \geq 0,$
$\text{tr}(\rho_i) = 1,$
$B(i) \in \{0, 1\}$

(113)

or

minimize: $-2 \text{tr}[XJ_n]$
subject to: $X = \bigoplus_i B(i)\tilde{\rho}_i,$
$B(i)\tilde{\rho}_i = B(i)\rho_i,$
$\text{tr}[B(i)\tilde{\rho}_i] = B(i),$
$\rho_i \geq 0,$
$\text{tr}(\rho_i) = 1,$
$B(i) \in \{0, 1\}.$

(114)

Note that the constraint $\text{tr}[B(i)\tilde{\rho}_i] = B(i)$ in the above program is always satisfied if the other constraints hold. We arrive at (108) by defining $B(i)\tilde{\rho}_i = \rho_i$ and relaxing the constraint $\rho_i = B(i)\rho$ to $0 \leq \rho_i \leq \rho$. On the other hand, given the constraints of (108) are valid, if $B(i) = 0$, from $0 \leq \rho_i$ and $\text{tr}[\rho_i] = B(i) = 0$ it follows that $\rho_i = 0$ and therefore $\rho_i = 0 = B(i)\rho$. If $B(i) = 1$, $\rho - \rho_i$ is a traceless hermitian operator and by the condition $\rho_i \leq \rho$ it is also positive. Hence it is zero and $\rho_i = \rho = B(i)\rho$. This proves the equivalence of (108) to (114) and finally to (107).

Next we formulate the linear program for the lower bound. The outer problem, giving the lower bound, is given by

minimize: $\max_{\rho_i \in D(n)} \text{tr}[(\Delta(\Theta - \Phi))][\rho_i]]$
subject to: $\Phi \in D\mathcal{L}.$

(115)

First we note that $\Phi \in D\mathcal{L}$ implies that $S := J(\Delta \Phi)$ is diagonal and therefore only defined by the transition probabilities $p(k|l)$ from the populations of the input states to the ones of the output states. Secondly we can calculate $\sigma_i = \Delta \Theta[\rho_i]$ for each $\rho_i \in D(n)$. Then we see that the only quantities that matter are the diagonal elements $r_i$ of $\rho_i$, and $s_i$ of $\sigma_i$ and we get the program

minimize: $\max_i \sum_k \left| s_i(k) - \sum_l p(k|l)r_i(l) \right|$
subject to: $\sum_k p(k|l) = 1 \quad \forall l,$
$p(k|l) \geq 0 \quad \forall k, l.$

(116)

Since both $\Delta \Theta$ and $\Delta \Phi$ are trace preserving operations,

$$\text{tr}[(\Delta(\Theta - \Phi))][\rho_i]] = 2 \text{tr} (\text{Pos}((\Delta(\Theta - \Phi)))[\rho_i]))$$

(117)
where $\text{Pos}$ denotes the positive part of an operator. From this follows that (116) is equivalent to

\[
\begin{align*}
\text{minimize:} & & 2x \\
\text{subject to:} & & x \geq \sum_k T_{k,i} \quad \forall i, \\
& & T_{k,i} \geq 0 \quad \forall i, k, \\
& & T_{k,i} \geq s_i(k) - \sum_l p(k|l) r_i(l) \quad \forall i, k, \\
& & \sum_k p(k|l) = 1 \quad \forall l, \\
& & p(k|l) \geq 0 \quad \forall k, l, \\
\end{align*}
\]

(117)

For $N = |D(n)|$ and $\rho_i \in D(n)$, introducing the matrices $R \in \mathbb{R}^{d_{\text{out}} \times N}$ by $R(:, i) = \text{diag}(\rho_i)$, and $S \in \mathbb{R}^{d_{\text{out}} \times N}$ by $S(:, i) = \text{diag}(\Delta\Theta[\rho_i])$ leads to

\[
\begin{align*}
\text{minimize:} & & 2x \\
\text{subject to:} & & x \geq \sum_k T_{k,i} \quad \forall i, \\
& & T_{k,i} \geq 0 \quad \forall i, k, \\
& & T_{k,i} \geq S_{k,i} - (PR)_{k,i} \quad \forall i, k, \\
& & \sum_k P_{k,l} = 1 \quad \forall l, \\
& & P_{k,l} \geq 0 \quad \forall k, l, \\
& & T \in \mathbb{R}^{d_{\text{out}} \times N}, \\
& & P \in \mathbb{R}^{d_{\text{out}} \times d_{\text{in}}}. \\
\end{align*}
\]

(118)

Examples

In this section, we calculate the measures introduced in the main text for two quantum channels acting on qutrits. For the first example, we mix the total dephasing operation $\Delta$ from the main text, which is free, with the quantum Fourier transformation $FT$, which is most valuable according to the nSID-measure (see Prop. 10). For $0 \leq p \leq 1$, we denote the resulting map by $\Theta$,

\[
\Theta(\rho) = (1 - p)\Delta\rho + pFT\rho. 
\]

(120)

Since $\Theta$ is free for $p = 0$, both measures are zero in this case. For $p \neq 0$, the operation is non-free, leading to non-zero measures. This is shown in Fig. 1, where it is also shown that in the case of $\Theta$, the two measures are equal within numerical precision. To show that this is not always the case, we present a second example, which is given by

\[
\Lambda(\rho) = (1 - p)\Delta\rho + p \sum_{n=1}^{3} K_n \rho K_n^\dagger, 
\]

(121)

where again $0 \leq p \leq 1$ and

\[
K_1 = \frac{1}{\sqrt{4}} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad K_2 = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \frac{1}{\sqrt{4}} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}. 
\]

(122)

As shown in Fig. 1, the two measures are different for $\Lambda$ and the diamond-measure is, for given $p$, larger than the nSID-measure. In general, as can be seen directly form the definitions, the diamond-measure is an upper bound to the nSID-measure. As expected from Prop. 10, $\Theta$ is more ”valuable” than $\Lambda$ for the same $p$. 
A measure for detection-creation incoherent operations

Finally we want to give at least one example for a measure in the detection-creation-incoherent setting, which has been introduced in [35]. Therefore we denote by $S(\cdot \| \cdot)$ the (quantum) relative entropy.

**Proposition 18.** The functional $M_c$ defined as [35]

$$M_c(\Theta) = \sup_{\rho} S(\Theta \Delta \rho \| \Delta \Theta \rho)$$

is a measure in the detection-creation-incoherent setting.

**Proof.** Monotonicity under left and right composition is shown in [35] and faithfulness is given by the property of the relative entropy that $S(\rho \| \sigma) = 0$ iff $\rho = \sigma$. From Lem. 14, we know that the eigenvectors of $\mathbb{1} \otimes \Delta B(\rho^{A,B})$ are separable. For
\[ \Gamma_b(\rho) = \rho \otimes |b\rangle\langle b| \] and using joint convexity and contractivity, we can then prove
\[
M_c(\Theta \otimes I) = \sup_{\rho} S\left( (\Theta \otimes I) \Delta \rho \big\| (\Theta \otimes I) \rho \right) \\
= \sup_{\rho} S\left( ((\Theta \Delta) \otimes I)(I \otimes \Delta) \rho \big\| ((\Delta \Theta) \otimes I)(I \otimes \Delta) \rho \right) \\
= \sup_{\sigma = (I \otimes \Delta) \rho} S\left( ((\Theta \Delta) \otimes I) \sigma \big\| ((\Delta \Theta) \otimes I) \sigma \right) \\
= \sup_{q_{a|b|p|b}, |\phi_{a|b}\rangle} S\left( ((\Theta \Delta) \otimes I) \sum_{a,b} q_{a|b|p|b} |\phi_{a|b}\rangle \langle \phi_{a|b}| b \rangle \langle b | \rho \big\| ((\Delta \Theta) \otimes I) \sum_{a,b} q_{a|b|p|b} |\phi_{a|b}\rangle \langle \phi_{a|b}| b \rangle \langle b | \rho \right) \\
\leq \sup_{q_{a|b|p|b}, |\phi_{a|b}\rangle} \sum_{a,b} q_{a|b|p|b} S\left( ((\Theta \Delta) \otimes I) |\phi_{a|b}\rangle \langle \phi_{a|b}| b \rangle \langle b | \rho \big\| ((\Delta \Theta) \otimes I) |\phi_{a|b}\rangle \langle \phi_{a|b}| b \rangle \langle b | \rho \right) \\
= \sup_{|\phi\rangle} S\left( \Gamma_b \Theta |\phi\rangle \big\| \Gamma_b \Delta |\phi\rangle \right) \\
\leq \sup_{|\phi\rangle} S\left( \Theta |\phi\rangle \big\| \Delta |\phi\rangle \right) \\
= M_c(\Theta). \quad (124)
\]

In addition, we have
\[
M_c(\Theta) = \sup_{\rho} S\left( \Theta \Delta \rho \big\| \Delta \Theta \rho \right) \\
= \sup_{\rho} S\left( \text{tr}_B \left( ((\Theta \Delta) \otimes I) (\rho \otimes |1\rangle\langle 1|) \right) \big\| \text{tr}_B \left( ((\Delta \Theta) \otimes I) (\rho \otimes |1\rangle\langle 1|) \right) \right) \\
\leq \sup_{\rho} S\left( ((\Theta \Delta) \otimes I) (\rho \otimes |1\rangle\langle 1|) \big\| ((\Delta \Theta) \otimes I) (\rho \otimes |1\rangle\langle 1|) \right) \\
= \sup_{\rho} S\left( (\Theta \otimes I) \Delta (\rho \otimes |1\rangle\langle 1|) \big\| (\Delta \Theta \otimes I) (\rho \otimes |1\rangle\langle 1|) \right) \\
\leq \sup_{\rho} S\left( (\Theta \otimes I) \Delta \rho \big\| (\Delta \Theta \otimes I) \rho \right) \\
= M_c(\Theta \otimes I) \quad (125)
\]

and thus
\[
M_c(\Theta) = M_c(\Theta \otimes I). \quad (126)
\]

Convexity is given by
\[
M_c \left( \sum_i q_i \Theta_i \right) = \sup_{\rho} S\left( \sum_i q_i \Theta_i \Delta \rho \big\| \sum_i q_i \Delta \Theta_i \rho \right) \\
\leq \sup_{\rho} \sum_i q_i S\left( \Theta_i \Delta \rho \big\| \Delta \Theta_i \rho \right) \\
\leq \sum_i q_i \sup_{\rho} S\left( \Theta_i \Delta \rho \big\| \Delta \Theta_i \rho \right) \\
= \sum_i q_i M_c(\Theta_i). \quad (127)
\]