Brief paper

New high order sufficient conditions for configuration tracking

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In this paper, we propose new conditions guaranteeing that the trajectories of a mechanical control system can track any curve on the configuration manifold. We focus on systems that can be represented as forced affine connection control systems and we generalize the sufficient conditions for tracking known in the literature. The new results are proved by a combination of averaging procedures by highly oscillating controls with the notion of kinematic reduction.

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1. Introduction

New geometric techniques are used to generalize tracking conditions known in the literature (Barbero-Liñán & Sigalotti, 2010; Bullo & Lewis, 2005; Chambrion & Sigalotti, 2008). The tracking problem plays a key role in the performance of robots and mechanical systems such as submarines and hovercrafts in order to avoid obstacles, stay nearby a preplanned trajectory, etc. Mechanical control systems are control-affine systems on the tangent bundle of the configuration manifold $Q$. In order to simplify the motion planning tasks for these control systems, a useful tool has been introduced in the geometric control literature, namely, the notion of kinematic reduction. Such a procedure consists in identifying a control-linear system on $Q$ whose trajectories mimic those of the mechanical system. This approach has been useful to describe controllability, planning properties (Bullo & Lewis, 2005) and optimality (Barbero-Liñán & Munoz-Lecanda, 2010) of mechanical systems. However, as described in Bullo and Lewis (2005), kinematic reduction is not always possible, some conditions related to the symmetric closure of the control vector fields of both systems under study must be satisfied. In our previous work (Barbero-Liñán & Sigalotti, 2010) we proposed two extensions of the first-order sufficient conditions for tracking proposed in Bullo and Lewis (2005), each of them based on the construction of a family of ‘compatible’ vector fields, one family being infinite and the other one of infinite cardinality. Related constructions to generate admissible directions for tracking have been proposed in Bressan and Wang (2009); Martínez and Cortés (2003) (see also Agrachev & Sarychev, 2005, 2006). Our goal here is to obtain more general sufficient conditions for tracking, combining our previous results with the notion of kinematic reduction. More precisely, our aim is to identify conditions under which it is possible to associate with a mechanical system a kinematic reduction whose controlled vector fields are compatible with tracking in the sense of Barbero-Liñán and Sigalotti (2010). Trackability of the mechanical system will then follow from controllability of the kinematic reduction. The proposed approach applies directly to families of compatible vectors fields of finite cardinality (see Theorem 14). The infinite cardinality case requires some intermediate technical result. In particular, we are lead to establish a relationship between families of vector fields defined pointwise and sets of sections of the tangent bundle, in analogy to the classical Malgrange theorem (Malgrange, 1967). Based on such a pointwise characterization of infinite families of compatible vector fields, we obtain new sufficient conditions for tracking extending the results in Barbero-Liñán and Sigalotti (2010) (see Theorem 15). The newly obtained conditions are used to complete
the analysis of the control properties of an underwater vehicle initiated in Chambrión and Sigalotti (2008), proving its trackability even in the most symmetric case (see Section 4.4 for details).

2. Notation and preliminaries

Denote by $\mathbb{N}$ the set of positive natural numbers and fix $n \in \mathbb{N}$. From now on, $Q$ is a $n$-dimensional smooth manifold and $x(Q)$ denotes the set of smooth vector fields on $Q$. All vector fields are considered smooth as functions on $Q$, unless otherwise stated. Let $t_0: TQ \to Q$ be the canonical projection. We denote by $I$ a compact interval of the type $[0, \tau]$, $\tau > 0$.

2.1. Affine connection control systems

**Definition 1.** An affine connection is a mapping $\nabla: x(Q) \times x(Q) \to x(Q)$

$$(X, Y) \mapsto \nabla(X, Y) = \nabla_X Y,$$

satisfying the following properties: (1) $\nabla$ is $\mathbb{R}$-linear in $X$ and in $Y$; (2) $\nabla_{\nabla_X Y} f = \nabla_{\nabla_X Y} f$ for every $f \in \mathcal{C}^\infty(Q)$; (3) $\nabla_X f + \nabla_Y f = \nabla_{X + Y} f$, for every $f \in \mathcal{C}^\infty(Q)$. (Here $\nabla f$ denotes the derivative of $f$ in the direction $X$.)

The mapping $\nabla_X Y$ is called the covariant derivative of $Y$ with respect to $X$.

**Definition 2.** A forced affine connection control system (FACCS) is a control mechanical system given by $\Sigma = (Q, \nabla, Y, \theta)$ where: $Q$ is a smooth $n$-dimensional manifold called the configuration manifold, $Y: \mathbb{R} \times TQ \to TQ$ is smooth, affine with respect to the velocities, and such that $(t_0 \circ Y)(t, \cdot) = \gamma(t)$ for every $t$, $\theta$ is a finite set $\{I_1, \ldots, I_m\}$ of control vector fields on $Q$, and a trajectory $\gamma: I \subset \mathbb{R} \to Q$ is admissible for $\Sigma$ if $\dot{\gamma}: I \to TQ$ is absolutely continuous and there exists a measurable and bounded control $u: I \to \mathbb{R}^k$ such that the dynamical equations of the control system $\Sigma$

$$\nabla_{\dot{\gamma}}(t) = Y(t, \dot{\gamma}(t)) + \sum_{a=1}^k u_a(t) Y_a(\gamma(t)), \quad (1)$$

are fulfilled (for almost every $t \in I$).

The vector field $Y$ includes all the non-controlled external forces; e.g., the potential and the non-potential forces. The assumption that $Y$ is affine with respect to the velocities means that, for every $q \in Q$ and $t \in \mathbb{R}$, the map $T_t Q \ni v \mapsto Y(t, v) \in T_t Q$ is affine.

3. Tracking problem

We consider here the problem arising when one tries to follow a particular trajectory on the configuration manifold, called reference or target trajectory, which is in general not a solution of the FACCS considered. A trajectory is successfully tracked if there exist solutions to the FACCS that approximate it arbitrarily well. Consider in what follows any distance $d: Q \times Q \to \mathbb{R}$ on $Q$ whose corresponding metric topology coincides with the topology on $Q$.

**Definition 3.** A curve $\gamma: I \to Q$ of class $\mathcal{C}^1$ is trackable for the FACCS $\Sigma$ if, for every strictly positive tolerance $\epsilon$, there exist a control $u^* \in L^2(I, \mathbb{R}^k)$ and a solution $\xi^*: I \to \Sigma$ corresponding to $u^*$ such that $\xi^*(0) = \gamma(0)$ and $d(\gamma(t), \xi^*(t)) < \epsilon$ for every $t \in I$. The trajectory is said to be strongly trackable for $\Sigma$ if, in addition to the above requirements, for every $\epsilon > 0$ the approximating trajectory $\xi^*$ may be found also satisfying $\xi^*(0) = \gamma(0)$.

**Remark 4.** Since any $\mathcal{C}^1$ curve can be uniformly approximated, with arbitrary precision, by a smooth curve having the same tangent vector at its initial point, then $\Sigma$ satisfies the CTP (respectively, the SCTP) if and only if every curve on $Q$ of class $\mathcal{C}^\infty$ is trackable (respectively, strongly trackable) for $\Sigma$.

3.1. Tracking results for control-linear systems

A control-linear system (also called driftless kinematic system) on $Q$ is a pair $(Q, \theta)$ where $\theta$ is a finite subset $\{x_1, \ldots, x_m\}$ of $x(Q)$, identified with the control system

$$\dot{\gamma}(t) = \sum_{a=1}^m u_a(t) X_a(\gamma(t)), \quad \gamma(t) \in Q,$$

where $u_1, \ldots, u_m$ are $\mathcal{C}^\infty$ real-valued functions.

**Proposition 5** (See Liu, 1997; Sussmann & Liu, 1991). Let $X_1, \ldots, X_m$ be smooth vector fields on $Q$ and take $\kappa \in \mathbb{N}$. Let $(X_1, \ldots, X_m)$ be the set of all Lie brackets of the vector fields $X_1, \ldots, X_m$ of length less than or equal to $\kappa$. Assume that $\gamma: I \to Q$ is a $\mathcal{C}^\infty$ curve such that $\dot{\gamma}(t) = \sum_{a=1}^m u_a(t) X_a(\gamma(t))$, with $u: I \to \mathbb{R}^m$ smooth. Then, for every $\epsilon > 0$ there exists a solution $\gamma_{\epsilon}$ of the control-linear system $(Q, \{X_1, \ldots, X_m\})$ with smooth control $u_{\epsilon}: I \to \mathbb{R}^m$ and initial condition $\gamma_{\epsilon}(0) = \gamma(0)$ such that $d(\gamma_{\epsilon}(t), \gamma(t)) < \epsilon$ for every $t \in I$.

From the above proposition we deduce the following result. (Similar arguments can be found in Jakubczyk (2002).)

**Corollary 6.** If the Lie algebra $\text{Lie}(X_1, \ldots, X_m)$ generated by $X_1, \ldots, X_m$ has constant rank on $Q$, then for every smooth curve $\gamma: I \to Q$ such that $\dot{\gamma}(t) \in \text{Lie}_{\inf} (X_1, \ldots, X_m)$ for every $t \in I$ and for every $\epsilon > 0$ there exists a solution $\gamma_{\epsilon}$ of the control-linear system $(Q, \{X_1, \ldots, X_m\})$ with smooth control $u_{\epsilon}: I \to \mathbb{R}^m$ and initial condition $\gamma_{\epsilon}(0) = \gamma(0)$ such that $d(\gamma_{\epsilon}(t), \gamma(t)) < \epsilon$ for every $t \in I$.

**Proof.** The proof works by covering the compact set $\gamma(I)$ by finitely many open sets $\Omega_1, \ldots, \Omega_K$ of $Q$ such that for every $j = 1, \ldots, K$ there exists on $\Omega_j$ a basis of the distribution $\text{Lie}(X_1, \ldots, X_m)$ made of Lie brackets of $X_1, \ldots, X_m$. Let $\kappa$ be the maximum of the length of the brackets used to construct such bases and let $(X_1, \ldots, X_m)$ be the set of all Lie brackets of the vector fields $X_1, \ldots, X_m$ of length less than or equal to $\kappa$. Then $\gamma(\epsilon) = \sum_{a=1}^m u_{\epsilon}(t) X_a(\gamma(t))$, with $u: I \to \mathbb{R}^m$ smooth, where smoothness follows from the fact that $\text{Lie}(X_1, \ldots, X_m)$ has constant rank. We then conclude by Proposition 5. $\blacksquare$

3.2. Previous strong configuration tracking results

Conditions guaranteeing the SCTP have been obtained in Barbero-Liñán and Sigalotti (2010), generalizing previous results presented in Bullo and Lewis (2005) (in particular Theorem 12.26) and in Chambrión and Sigalotti (2008). We recall them here below in a version adapted to what follows. The main difference of these statements from the ones of Theorem 4.4 and Corollary 4.7 in Barbero-Liñán and Sigalotti (2010) is that here we focus on the strong configuration trackability of a given trajectory, instead of looking at the SCTP. The proof is however the same, since the proof proposed in Barbero-Liñán and Sigalotti (2010) is based on an argument where the target trajectory is also fixed.

We need to introduce the symmetric product in $x(Q)$ defined by $(X: Y) = \nabla_X Y + \nabla_Y X$ for every $X, Y \in x(Q)$. For subsets $A, B$ of $x(Q)$, $A - B = \{X - Y \mid X \in A, Y \in B\}$, $\text{cl}(A) = A \cap (-A)$ denotes the convex hull of $A$, and $\text{cl}(A)$ denotes the closure of $A$ in $x(Q)$ with respect to the topology of the uniform convergence on compact sets.
Proposition 7. Let $\Sigma = (Q, \nabla, Y, \mathscr{H})$ be a FACCS. Let $X_0 = \text{span}_{\epsilon \in (0, \epsilon_0)}(q)$ and, for $l \in \mathbb{N}$,

$$X_l = X_{l-1} - \co\{[Z; Z] \mid Z \in L(X_{l-1})\}.$$  \hspace{1cm} (2)

Fix a smooth reference trajectory $\gamma_{\text{ref}} : I \to Q$ of class $C^\infty$. Assume that there exist $I, N \subseteq \mathbb{N}, \lambda_1, \ldots, \lambda_N \subseteq C^\infty(I, [0, +\infty)), \Lambda = \{x \in \mathbb{R} : x < 0\}$ such that $\int_{\gamma_{\text{ref}}(t)} Y(t, \gamma_{\text{ref}}(t)) = \sum_{n=1}^{N} \lambda_n(t) Z_n(t)$ for all $t \in I$, then $\gamma_{\text{ref}}$ is strongly trackable.

Proposition 8. Let $\Sigma = (Q, \nabla, Y, \mathscr{H})$ be a FACCS. Define $X_0 = \mathscr{H}$ and, for $l \in \mathbb{N}$,

$$X_l = X_{l-1} \cup \{[Z; Z_0] \mid Z, Z_0 \in X_{l-1}\}.$$  \hspace{1cm} (3)

Assume that there exists $I \subseteq \mathbb{N}$ such that for each $i \in \{0, \ldots, l-1\}$, for each $Z \in X_{l-1}, (Z; Z) \in \text{span}_{\epsilon \in (0, \epsilon_0)}(Q)$. Let $X_l = \{Z_1, \ldots, Z_n\}$. Fix a smooth reference trajectory $\gamma_{\text{ref}} : I \to Q$ of class $C^\infty$. If there exist $\lambda_1, \ldots, \lambda_N \subseteq C^\infty(I, \mathbb{R})$ such that $\int_{\gamma_{\text{ref}}(t)} Y(t, \gamma_{\text{ref}}(t)) = \sum_{n=1}^{N} \lambda_n(t) Z_n(t)$ for all $t \in I$, then $\gamma_{\text{ref}}$ is strongly trackable.

4. A generalization of tracking conditions

Before introducing the new results about tracking, we need some technical lemmas described in Section 4.1 and to define kinematic reduction in Section 4.2. All that is necessary to prove the new theorem about trackability is defined in Section 4.3.

4.1. Pointwise and sectionwise characterization of $\mathfrak{k}$

The results in this section, in the spirit of the classical Malgrange theorem (see Malgrange, 1967), aim at characterizing the sets $\mathfrak{k}$ of sections of $\mathfrak{t}$, introduced above, in terms of iterated computations of subsets of $\mathfrak{t}$. First, we recall here Malgrange’s theorem. Let $C^m(Q, E)$ be the space of all $C^m$ functions from $Q$ to a given finite-dimensional vector space $E$.

Theorem 9 (Malgrange, 1967, Theorem 2.1.3). If $M$ is a submodule of $C^0(Q, E), M$ is the closure of $M$ in $C^0(Q, E)$ and $\overline{M}$ is the module of all functions $f \in C^0(Q, E)$ that are pointwise in $M$, then $M = \overline{M}$.

Let us associate with a family $\mathscr{H} = \{Y_1, \ldots, Y_n\} \subseteq \mathfrak{t}(Q)$, in addition to the family $\mathfrak{k}$, the family of subsets of $\mathfrak{t}$ defined pointwise, for every $q \in Q$, as

$$\mathfrak{X}_{0, q} = \text{span}_{\epsilon \in (0, \epsilon_0)}(q),$$  \hspace{1cm} (4)

where $A = B = \{u - v : u, v \in A\}$ for every $A, B \subseteq T_Q$ and $\Lambda_{1-\epsilon}$ is the convex hull in $T_Q$ of all vectors of the form $(Z; Z)$ for $Z \in \mathfrak{t}(Q)$ such that $Z(q') \in L(X_{1-\epsilon})$ for all $q' \in Q$. Recall that each $\mathfrak{k}$ is a convex cone in $\mathfrak{t}(Q)$, closed for the $C^0_{\text{loc}}$ topology (see Barbero-Liñán & Sigalotti, 2010, Proposition 4.1). By a straightforward recurrence argument, moreover, one easily gets that $\mathfrak{X}_{l, q}$ in (4) is a closed convex cone of $T_Q$. $Q$ we need a preliminary result to establish the equivalence between the two definitions.

Lemma 10. Let $\mathscr{H}$ be a $C^0_{\text{loc}}$-closed set of $\mathfrak{t}(Q)$ and assume that $\mathfrak{H}$ is closed with respect to finite linear combinations with coefficients in $C^\infty(Q, [0, \infty))$. Then $\mathfrak{H} = \{V \in \mathfrak{t}(Q) \mid V(q) \in \mathfrak{H} \forall q \in Q\}$.

Proof. The inclusion $\mathfrak{H} \subseteq \{V \in \mathfrak{t}(Q) \mid V(q) \in \mathfrak{H} \forall q \in Q\}$ is trivial and we are left to prove the opposite one. If $\mathfrak{k}$ is closed with respect to finite linear combinations with coefficients in $C^\infty(Q, [0, \infty))$, then $\mathfrak{H}$ is closed with respect to finite linear combinations with coefficients in $C^\infty(Q, [0, \infty))$. Moreover, $\mathfrak{H}$ is closed with respect to finite linear combinations with coefficients in $C^\infty(Q, [0, \infty))$. Hence, applying Lemma 10 to $\mathfrak{H}$ is $L(X_{1-\epsilon})$ we deduce that $A, A' \subseteq B$ for some smooth function $f$. By induction hypothesis, $B$ is in $X_{1-\epsilon}$. Hence, $\alpha A' \subseteq B$ in $X_{1-\epsilon}$. Therefore, concluding the proof of the identity $\mathfrak{H} = \{V \in \mathfrak{t}(Q) \mid V(q) \in \mathfrak{H} \forall q \in Q\}$.

4.2. Kinematic reduction

It is known from the literature that to perform certain motion planning tasks for a mechanical control system it may be useful to exploit trajectories of a suitably adapted control-linear system. In some circumstances, indeed, it is possible to identify a control-linear system whose trajectories are admissible for the mechanical system as well, as recalled in Theorem 13. Motion planning for control-linear systems is a widely studied issue (see Chitour, Jean, & Long, 2013, and references therein) and is in general easier than for control-affine systems such as forced affine connection control systems. Before proceeding, we introduce some necessary definitions. Let $\mathfrak{X} = \{X_1, \ldots, X_n\} \subseteq \mathfrak{t}(Q)$ and consider the control-linear system $(Q, \mathfrak{X})$ (defined in Section 3.1). Let $\text{Sym}^0(\mathscr{H}) = \text{span}_{\epsilon \in (0, \epsilon_0)}(\mathscr{H})$ and $\text{Sym}^1(\mathscr{H}) = \text{Sym}^0(\mathscr{H}) + \text{span}_{\epsilon \in (0, \epsilon_0)}(W; Z)$ for $W, Z \in \mathfrak{H}$. 


Definition 12. Let $\Sigma = (Q, \mathcal{V}, 0, \emptyset')$ be a FACCs. A driftless kinematic system $\Sigma_{kin} = (Q, \mathcal{X})$ is a kinematic reduction of $\Sigma$ if for every controlled trajectory $(\gamma, u_{kin})$ of $\Sigma_{kin}$ with $u_{kin}$ smooth there exists $u$ smooth such that $(\gamma, u)$ is a controlled trajectory for $\Sigma$.

Let us recall the following result from Bullo and Lewis (2005).

Theorem 13 (Bullo & Lewis, 2005, Theorem 8.18). Let $\Sigma$ and $\Sigma_{kin}$ be as in Definition 12. Assume that $\mathcal{X}$ and $\mathcal{Y}$ generate constant-rank distributions. Then $\Sigma$ is a kinematic reduction of $\Sigma_{kin}$ if and only if $\Sym^1(\mathcal{X}) \subset \Span_{\gamma}(\mathcal{Y})$ for every $q \in Q$.

4.3. A new criterion for trackability

We generalize in this section the sufficient conditions for tracking recalled in Section 3.2. Theorems 14 and 15 are both based on the idea that if a FACCs admits a kinematic reduction whose admissible vector fields are constructed as in Propositions 7 and 8, then all trajectories of such a kinematic reduction can be tracked with arbitrarily precision by trajectories of the mechanical system. If the kinematic reduction satisfies in addition the Lie bracket generating condition, it then follows that the CTP holds true.

Theorem 14. Let $\Sigma = (Q, \mathcal{V}, \mathcal{Y}, \emptyset')$ be a FACCs. Define the families $\mathcal{X}_i$, $i \in \mathbb{N}$, of vector fields on $Q$ as in (3). Assume that there exists $l \in \mathbb{N}$ such that (i) $Y(t, p) \in \Span_{\gamma}(\mathcal{X}_j(p))$ for every $p \in TQ$, (ii) the distributions $\Span_{\mathcal{X}_i}$, $i \in \mathbb{N}$, and $\Lie(\mathcal{X}_j)$ have constant rank; (iii) for all $i \in \mathbb{N}$ and $j \in \mathbb{N}$, $(\gamma, t) \in \Span_{\mathcal{X}_i}(\mathcal{Y}) \times \Span_{\mathcal{X}_j}(\mathcal{Y})$.

Proof. Let us recall the following result from Bullo and Lewis (2005).

Theorem 15. Let $\Sigma = (Q, \mathcal{V}, \mathcal{Y}, \emptyset')$ be a FACCs. Define the families $\mathcal{X}_i$, $i \in \mathbb{N}$, of vector fields on $Q$ as in (2). Assume that there exists $l \in \mathbb{N}$ such that (i) $Y(t, p) \in \Span_{\gamma}(\mathcal{X}_j(p))$ for every $p \in TQ$, (ii) for all $q \in Q$, $\Lie(\mathcal{X}_j(q)) = \Span_{\gamma}(q)$ and $\Lie(\mathcal{X}_j(q)) = \Span_{\gamma}(q)$, (iii) the distributions $\mathcal{X}_j$ and $\mathcal{Y}$ and $\Lie(\mathcal{X}_j)$ have constant rank. Fix a smooth reference trajectory $\gamma_{ref} : I \to Q$. If $\gamma_{ref}(t) \in \Lie_{\gamma_{ref}(t)}(\mathcal{X}_j)$ for every $t \in I$, then $\gamma_{ref}$ is trackable. In particular, if $\Lie_{\gamma_{ref}(t)} = T_{\gamma_{ref}}Q$ for every $q \in Q$ then the CTP holds.
and
\[
\frac{d\Pi}{dr} = \Pi \times \omega + P \times v + \left( \begin{array}{c} u_1 \\ u_2 \\ 0 \end{array} \right), \quad \frac{dP}{dr} = P \times \omega + \left( \begin{array}{c} 0 \\ 0 \\ u_3 \end{array} \right). \tag{9}
\]

The controls correspond to a linear acceleration along one of the axes of the ellipsoidal vehicle and to two angular accelerations around the other two axes. Eqs. (8)–(9) are a first-order formulation in TSE(3) of a FACCS in SE(3) corresponding to the vector fields \( Y_1 = e_1/J_1, \ Y_2 = e_2/J_2, \ Y_3 = e_3/M_3 \), where left-invariant vector fields are identified with elements of the Lie algebra se(3) = span\{e_1, \ldots, e_6\} of SE(3) (see Barbero-Liñán & Sigalotti, 2010, for details). The Lie group structure of the configuration manifold can be exploited to compute Lie brackets and symmetric products of the control vector fields (see Bullo, 1995; Bullo & Lewis, 2005). It was proven in Chambon and Sigalotti (2008) that if \( M_1 \neq M_2 \) then system (8)–(9) satisfies the SCTP. In Barbero-Liñán and Sigalotti (2010), moreover, based on Proposition 8, an explicit tracking algorithm was proposed.

We focus here on the case that was left unanswered in Barbero-Liñán and Sigalotti (2010) and Chambon and Sigalotti (2008), namely, we prove that the CTP holds when
\[
J_1 = J_2, \quad M_1 = M_2. \tag{10}
\]

Such a case cannot be studied using the most general sufficient conditions for trackability given in Barbero-Liñán and Sigalotti (2010) (recalled in Section 4.1, as illustrated here below. Under condition (10), one easily computes using the results in Bullo (1995); Bullo and Lewis (2005) that \( \langle Y_1; Y_2 \rangle = 0 \). \( \langle Y_1; Y_3 \rangle = -e_3(J_1M_3) = Y_3 \). \( \langle Y_2; Y_3 \rangle = e_3(J_2M_3) = Y_2 \). Moreover, \( (e_i; e_j) = 0 \) for \( j = 1, \ldots, 6 \). Hence, \( \forall i \geq 1 \), \( \forall i \geq 1 \), condition (iii) in the statement of Theorem 14 is satisfied for every \( i \geq 0 \). Indeed, \( \forall i \geq 1 \), \( \forall i \geq 1 \), the CTP cannot be deduced from Proposition 7. Note that \( \langle Y_1; Y_2 \rangle = e_3/J_2^2 \). Thus, \( \text{Lie}(x_i) = T_iQ \) for every \( q \in \text{SE}(3) \). By Theorem 14 the CTP is guaranteed for these control vector fields. This completes the results in Barbero-Liñán and Sigalotti (2010) and Chambon and Sigalotti (2008), allowing to conclude that system (8)–(9) satisfies the CTP for any choice of the (positive definite) diagonal inertial matrix \( M \).

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