Characterizing subgroup perfect codes by 2-subgroups

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Abstract
A perfect code in a graph $\Gamma$ is a subset $C$ of $V(\Gamma)$ such that no two vertices in $C$ are adjacent and every vertex in $V(\Gamma) \setminus C$ is adjacent to exactly one vertex in $C$. Let $G$ be a finite group and $C$ a subset of $G$. Then $C$ is said to be a perfect code of $G$ if there exists a Cayley graph of $G$ admitting $C$ as a perfect code. It is proved that a subgroup $H$ of $G$ is a perfect code of $G$ if and only if a Sylow 2-subgroup of $H$ is a perfect code of $G$. This result provides a way to simplify the study of subgroup perfect codes of general groups to the study of subgroup perfect codes of 2-groups. As an application, a criterion for determining subgroup perfect codes of projective special linear groups $\text{PSL}(2, q)$ is given.

Keywords Cayley graph · Perfect code · Subgroup perfect code · Projective special linear group

Mathematics Subject Classification 05C25 · 05C69 · 94B99

1 Introduction

All groups considered in the paper are finite groups with identity element denoted by 1, and all graphs considered are finite, undirected and simple. For a graph $\Gamma$, we use $V(\Gamma)$ and $E(\Gamma)$ to denote its vertex set and edge set respectively. The distance in $\Gamma$ between two vertices is the length of a shortest path between the two vertices or $\infty$ if there is no path in $\Gamma$ joining them. Let $r$ be a positive integer. A subset $C$ of $V(\Gamma)$ is called [1, 15] a perfect $r$-error-correcting code (or perfect $r$-code for short) in $\Gamma$ if every vertex of $\Gamma$ is at distance no more than $r$ to exactly one vertex in $C$. A perfect 1-code is usually called a perfect code. Equivalently, a subset $C$ of $V(\Gamma)$ is a perfect code in $\Gamma$ if $C$ is an independent set of $\Gamma$ and every vertex in $V(\Gamma) \setminus C$ is adjacent to exactly one vertex in $C$. A perfect code in a graph is also called an efficient dominating set [5] or independent perfect dominating set [18] of the graph.
The notion of perfect $r$-codes in graphs was firstly introduced by Biggs [1] as a generalization of the notions of perfect $r$-codes under the Hamming metric and Lee metric. Recall that in coding theory the Hamming distance between words of length $n$ over an alphabet of size $m \geq 2$ is precisely the graph distance in the Hamming graph $H(n, m)$ [22]. Therefore perfect $r$-codes in $H(n, m)$ are exactly those in the classical setting under the Hamming metric. Similarly, the Lee distance [12] between words of length $n$ over an alphabet of size $m \geq 3$ is precisely the graph distance in the Cartesian product $L(n, m)$ of $n$ copies of the cycle of length $m$ and therefore perfect $r$-codes in $L(n, m)$ are exactly those in the classical setting under the Lee metric.

The problem of the existence of perfect codes under a given metric is of great significance in coding theory. A complete classification of the parameters for which perfect codes over Galois fields exist have been completed in the early 1970s [29, 30, 36]. In 1970, Golomb and Welch [10] conjectured that for any $n > 2, r > 1$, and $q \geq 2r + 1$ there is no $q$-ary perfect $r$-codes of length $n$ under the Lee metric. Although this conjecture has been studied extensively for more than half a century, it is still wide open [12, 19]. The celebrated Lloyd’ Theorem [20] is one of the most powerful tools available for proving nonexistence results for perfect codes. In [1], Biggs generalized the Lloyd’s Theorem to perfect codes in distance-transitive graphs and showed that the natural setting for the problem of perfect codes is the class of distance-transitive graphs. Since the seminal works of Biggs [1] and Delsarte [6], an extensive body of research has been devoted to perfect codes in distance-transitive graphs and, in general, in distance-regular graphs and association schemes [2]. In [4], Chihara proved that many infinite families of classical distance-regular graphs have no non-trivial perfect codes, including the Grassmann graphs and the bilinear forms graphs of which the non-existence of perfect codes was reverified by a new proof in [23]. Doob graphs are an important family of distance-regular Cayley graphs. In [25], all possible parameters of subgroup perfect codes in Doob graphs were described. In [16], a necessary and sufficient condition for a Doob graph to admit perfect codes was given. The reader is referred to [11, 24, 28] for more results on perfect codes in distance-regular graphs.

Let $G$ be a group and $S$ a subset of $G$ satisfying $S^{-1} := \{x^{-1} \mid x \in S\} = S$ and $1 \notin S$. The Cayley graph $\text{Cay}(G, S)$ of $G$ with connection set $S$ is defined as the graph with vertex set $G$ such that two elements $x, y$ of $G$ are adjacent if and only if $yx^{-1} \in S$. Note that $H(n, m)$ and $L(n, m)$ are Cayley graphs of the additive group $\mathbb{Z}_m^n$ with connection sets $S_H$ and $S_L$, respectively, where $S_H$ consists of all elements of $\mathbb{Z}_m^n$ with precisely one nonzero coordinate, and $S_L$ consists of all elements of $\mathbb{Z}_m^n$ such that exactly one coordinate is $\pm 1 \pmod{m}$ and all other coordinates are zero.

Perfect codes in Cayley graphs have been extensively studied in the literature, see [13, Section 1] for a brief survey and [7, 9, 27, 35] for a few recent papers. In particular, perfect codes in Cayley graphs which are subgroups of the underlying groups are especially interesting since they are generalizations of perfect linear codes [31] in the classical setting. In [13], Huang et al introduced the following concepts: A subset $C$ of a group $G$ is called a perfect code of $G$ if it is a perfect code of some Cayley graph of $G$; if further $C$ is a subgroup of $G$, then $C$ is called a subgroup perfect code of $G$.

In [13], some interesting results on normal subgroups of a group to be perfect codes were obtained. Subsequently, these results are extended to general subgroups of a group [3, 21, 32]. Let $G$ be a group and $H$ a subgroup of $G$. In [32, 33], the author and Zhou proved that $H$ is a perfect code of $G$ if there exists a Sylow 2-subgroup of $H$ which is a perfect code of $G$. They also proved that for a metabelian group $G$, a normal subgroup $H$ of $G$ is a perfect code of $G$ if and only if a Sylow 2-subgroup of $H$ is a perfect code of $G$. This result was recently
generalized to all groups by Khaefi et al [14]. In this paper, we show that the restriction on the normality of \( H \) can also be removed. Actually, we prove the following result.

**Theorem 1.1** Let \( G \) be a finite group and \( H \) a subgroup of \( G \). Then the following statements are equivalent:

(i) every Sylow 2-subgroup of \( H \) is a perfect code of \( G \);
(ii) \( H \) has a Sylow 2-subgroup which is a perfect code of \( G \);
(iii) \( H \) is a perfect code of \( G \).

In order to reduce the problem of determining subgroup perfect codes of general groups to that of 2-groups, we prove a result by using Theorem 1.1 as follows.

**Theorem 1.2** Let \( G \) be a finite group and \( H \) a subgroup of \( G \). Let \( Q \) be a Sylow 2-subgroup of \( H \) and \( P \) a Sylow 2-subgroup of \( N_G(Q) \). Then \( H \) is a perfect code of \( G \) if and only if \( Q \) is a perfect code of \( P \).

There are four sections in this paper. After this introduction section, in Sect. 2 we fix the notations for the paper and list some known results for later use. In Sect. 3, we prove Theorems 1.1 and 1.2, and we also deduce a few other results base on Theorem 1.2. As an application of our main results, in Sect. 4 we give a criterion for determining subgroup perfect codes of the 2-dimensional projective special linear groups \( PSL(2, q) \) where \( q \) is a prime power.

**2 Preliminaries**

We at first fix some notations and terminologies. For a finite set \( S \), we use \( |S| \) to denote the number of elements contained in \( S \). Let \( G \) be a group. For a subgroup \( H \) of \( G \), we use \( |G : H| \) and \( N_G(H) \) to denote the index \( |G|/|H| \) of \( H \) in \( G \) and the normalizer \( \{ g \in G \mid g^{-1}Hg = H \} \) of \( H \) in \( G \) respectively. For any subset \( X \) of \( G \), we use \( \langle X \rangle \) to denote the subgroup of \( G \) generated by \( X \). Let \( p \) be a prime divisor of \( |G| \). The Cauchy’s Theorem (see [17, 3.2.1]) states that \( G \) contains an element of order \( p \). In particular, if \( G \) is of even order, then \( G \) contains an element of order 2, called an involution of \( G \). A subgroup \( P \) of \( G \) of order a power of \( p \) is called a \( p \)-subgroup of \( G \). If further \( |G : P| \) is not divisible by \( p \), then \( P \) is called a Sylow \( p \)-subgroup of \( G \). By the famous Sylow’s Theorem (see [17, 3.2.3]), there exists a Sylow \( p \)-subgroup of \( G \), every \( p \)-subgroup of \( G \) is contained in a Sylow \( p \)-subgroup of \( G \) and all Sylow \( p \)-subgroups of \( G \) are conjugate in \( G \).

Now we list some known results for later use. The first two lemmas seems some what trivial and can be found in [32].

**Lemma 2.1** [32] Let \( G \) be a group and \( H \) a subgroup of \( G \). Then \( H \) is a perfect code of \( G \) if and only if it is a perfect code of any subgroup of \( G \) which contains \( H \).

**Lemma 2.2** [32] Let \( G \) be a group and \( H \) a subgroup of \( G \). If \( H \) is a perfect code of \( G \), then for any \( g \in G \), \( g^{-1}Hg \) is a perfect code of \( G \). More specifically, if \( H \) is a perfect code in \( \text{Cay}(G, S) \) for some connection set \( S \) of \( G \), then \( g^{-1}Hg \) is a perfect code in \( \text{Cay}(G, g^{-1}Sg) \).

The following lemma is one of the main results in [13].

**Lemma 2.3** [13] Let \( G \) be a group and \( H \) a normal subgroup of \( G \). Then \( H \) is a perfect code of \( G \) if and only if for all \( x \in G \), \( x^2 \in H \) implies \((xh)^2 = 1\) for some \( h \in H \).
The next three lemmas are from [33].

**Lemma 2.4** [33] Let $G$ be a group and $H$ a subgroup of $G$. Then $H$ is not a perfect code of $G$ if and only if there exists a double coset $D = HxH$ with $D = D^{-1}$ having an odd number of left cosets of $H$ in $G$ and containing no involution. In particular, if $H$ is not a perfect code of $G$, then there exists a 2-element $x \in G \setminus H$ such that $x^2 \in H$, $|H : H \cap xHx^{-1}|$ is odd, and $HxH$ contains no involution.

**Lemma 2.5** [33] Let $G$ be a group and $Q$ a subgroup of $G$. Suppose that either $Q$ is a 2-group or at least one of $|Q|$ and $|G : Q|$ is odd. Then $Q$ is a perfect code of $G$ if and only if $Q$ is a perfect code of $N_G(Q)$.

**Lemma 2.6** [33] Let $G$ be a group and $Q$ a subgroup of $G$. Suppose that either $Q$ is a 2-group or at least one of $|Q|$ and $|G : Q|$ is odd. Then $Q$ is a perfect code of $G$ if and only if for any $x \in N_G(Q)$, $x^2 \in Q$ implies $(xb)^2 = 1$ for some $b \in Q$.

Note that some corrigenda of the results in [32] were published in [33] and the statements of following two results from [32] listed below are unchanged and correct.

**Lemma 2.7** [32] Let $G$ be a group and $H$ a subgroup of $G$. If either the order of $H$ is odd or the index of $H$ in $G$ is odd, then $H$ is a perfect code of $G$.

**Lemma 2.8** [32] Let $G$ be a group and $H$ a subgroup of $G$. If there exists a Sylow 2-subgroup of $H$ which is a perfect code of $G$, then $H$ is a perfect code of $G$.

### 3 Main results

In this section, we present the proofs of Theorems 1.1 and 1.2. We also deduce a few corollaries of Theorem 1.2.

Let us prove Theorem 1.1 first.

**Proof of Theorem 1.1** It is obvious that (ii) is true if (i) is true. Lemma 2.8 implies (ii)⇒(iii). In what follows we deduce (iii)⇒(i).

Suppose that $H$ is a perfect code of $G$. Let $Q$ be an arbitrary Sylow 2-subgroup of $H$. It suffices to show that $Q$ is a perfect code of $G$. If $|N_G(Q) : Q|$ is odd, then Lemma 2.7 implies that $Q$ is a perfect code of $N_G(Q)$ and it follows from Lemma 2.5 that $Q$ is a perfect code of $G$. In what follows, we assume that $|N_G(Q) : Q|$ is even. Consider an arbitrary element $x \in N_G(Q) \setminus Q$ with $x^2 \in Q$. Set $P = \langle Q, x \rangle$. Then $Q$ is a normal subgroup of $P$ of index 2. In particular, $P = Q \cup xQ$. Since $Q$ is a Sylow 2-subgroup of $H$, we get $x \notin H$ and $x^2 \in H$. Therefore $(HxH)^{-1} = Hx^{-1}H = HxH$. Furthermore, $|H : H \cap xHx^{-1}|$ is odd as $Q = xQx^{-1} \subseteq H \cap xHx^{-1}$. Since

$$h_1xH = h_2xH \iff x^{-1}h_2^{-1}h_1x \in H$$

$$\iff h_2^{-1}h_1 \in xHx^{-1}$$

$$\iff h_2^{-1}h_1 \in H \cap xHx^{-1}$$

$$\iff h_1(H \cap xHx^{-1}) = h_2(H \cap xHx^{-1})$$

for each pair of elements $h_1, h_2 \in H$, it follows that $HxH$ is a union of an odd number of left cosets of $H$ in $G$. Since $H$ is a perfect code of $G$, it follows from Lemma 2.4 that
$H \times H$ contains an involution, say $h_1 h_2$. Let $h = h_2 h_1$. Since $xh = h_1^{-1}(h_1 h_2)h_1$, $xh$ is an involution in $xH$. Set $L = \langle Q, x, h \rangle$. Since $xh x^{-1} = (xh)^2 h^{-1} x^{-2} = h^{-1} x^{-2} \in \langle Q, h \rangle$ and $x Q x^{-1} = Q$, we conclude that $(Q, h)$ is a normal subgroup of $L$. Since $x \notin \langle Q, h \rangle$ and $x^2 \in \langle Q, h \rangle$, we obtain that $(Q, h)$ is of index 2 in $L$. Therefore $|L|/|P| = |\langle Q, h \rangle|/|Q|$. Since $Q$ is a Sylow 2-subgroup of $H$ and $(Q, h)$ is a subgroup of $H$ containing $Q$, we have that $|\langle Q, h \rangle|/|Q|$ is odd. Therefore $|L|/|P|$ is odd and it follows that $P$ is a Sylow 2-subgroup of $L$. Since $xh$ is an involution contained in $L$, there exists a Sylow 2-subgroup $R$ of $L$ containing $xh$. By the Sylow’s Theorem, the Sylow 2-subgroups of $L$ are conjugate in $L$. Therefore there exists an element $b \in L$ such that $b R b^{-1} = P$. It follows that $b x b h b^{-1} \in P$. Since $x \notin H$ and $b, h \in H$, we get $b x h b^{-1} \notin H$. Thus $b x b h b^{-1} \notin P$. It follows that $b x h b^{-1} \in x Q$ as $P = Q \cup x Q$ and $b x h b^{-1} \in P$. Since $xh$ is an involution, we have that $b x h b^{-1}$ is an involution. Now we have proved that $xQ$ contains an involution. By Lemma 2.6, $Q$ is a perfect code of $G$.

Now we prove Theorem 1.2.

**Proof of Theorem 1.2 ⇒)** Suppose that $H$ is a perfect code of $G$. By Theorem 1.1, we have that $Q$ is a perfect code of $G$ and it follows from Lemma 2.1 that $Q$ is a perfect code of $P$.

⇐ Suppose that $Q$ is a perfect code of $P$. Consider an arbitrary element $x \in N_G(Q)$ with $x^2 \in Q$. Then $\langle Q, x \rangle$ is a 2-subgroup of $N_G(Q)$. Since $P$ is a Sylow 2-subgroup of $N_G(Q)$, we conclude that $\langle Q, x \rangle$ is contained in $b^{-1} P b$ for some $b \in N_G(Q)$. Thus $b x h b^{-1} \in P$ and $(b x h b^{-1})^2 \in Q$. Since $Q$ is normal in $P$ and a perfect code of $P$, by Lemma 2.3 we obtain $(b x h b^{-1})^2 c^2 = 1$ for some $c \in Q$. It follows that $b^{-1} c b \in Q$ and $(b^{-1} c b)^2 = 1$. By Lemma 2.6, we have that $Q$ is a perfect code of $G$. Recall that $Q$ is a Sylow 2-subgroup of $H$. By Theorem 1.1, $H$ is a perfect code of $G$.

In what follows, we give three corollaries of Theorem 1.2. These corollaries involve special groups $G$ or special subgroups $H$.

**Corollary 3.1** Let $G$ be a group and $H$ a subgroup of $G$. Let $Q$ be a Sylow 2-subgroup of $H$ and $P$ a Sylow 2-subgroup of $G$ containing $Q$. If $P \cap N_G(Q)$ is of odd index in $N_G(Q)$, then $H$ is a perfect code of $G$ if and only if $Q$ is a perfect code of $P$.

**Proof** Note that $N_P(Q) = P \cap N_G(Q)$. Since $P \cap N_G(Q)$ is of odd index in $N_G(Q)$, we have that $N_P(Q)$ is a Sylow 2-subgroup of $N_G(Q)$. By Lemma 2.5, $Q$ is a perfect code of $P$ if and only if $Q$ is a perfect code of $N_P(Q)$. Together with Theorem 1.2, we conclude that $H$ is a perfect code of $G$ if and only if $Q$ is a perfect code of $P$.

**Corollary 3.2** Let $G$ be a group having a normal Sylow 2-subgroup $P$. Let $H$ be a subgroup of $G$ and $Q$ a Sylow 2-subgroup of $H$. Then $H$ is a perfect code of $G$ if and only if $Q$ is a perfect code of $P$.

**Proof** Since the Sylow 2-subgroup $P$ is normal in $G$, we have that $P$ is the unique Sylow 2-subgroup of $G$. Therefore every Sylow 2-subgroup of $N_G(Q)$ is contained in $P$. This implies that $Q$ is a subgroup of $P$ and $N_P(Q)$ is the unique Sylow 2-subgroup of $N_G(Q)$. Since $N_P(Q) = P \cap N_G(Q)$, it follows that $P \cap N_G(Q)$ is of odd index in $N_G(Q)$. By Corollary 3.1, $H$ is a perfect code of $G$ if and only if $Q$ is a perfect code of $P$.

**Corollary 3.3** Let $G$ be a group having an abelian Sylow 2-subgroup and $H$ a subgroup of $G$. Let $Q$ be a Sylow 2-subgroup of $H$ and $P$ a Sylow 2-subgroup of $G$ containing $Q$. Then $H$ is a perfect code of $G$ if and only if $Q$ is a perfect code of $P$.
Proof By the Sylow’s Theorem, all Sylow 2-subgroups of \( G \) are conjugate in \( G \). Since \( G \) has an abelian Sylow 2-subgroup and \( P \) is a Sylow 2-subgroup of \( G \), we have that \( P \) is abelian and therefore \( Q \) is normal in \( P \). Thus \( P \) is a Sylow 2-subgroup of \( N_G(Q) \). By Theorem 1.2, we obtain that \( H \) is a perfect code of \( G \) if and only if \( Q \) is a perfect code of \( P \). \( \square \)

As a contrast to Corollary 3.3, we introduce the following proposition of which the statement is an equivalent expression of a theorem [21, Theorem 1.1] of Ma et al. Note that a 2-group has no element of order 4 if and only if it is elementary abelian.

Proposition 3.4 Let \( G \) be a group. Then every subgroup of \( G \) is a perfect code if and only if 
\( G \) has an elementary abelian Sylow 2-subgroup.

Proof \( \Rightarrow \) If \( G \) has a Sylow 2-subgroup \( P \) which is not an elementary abelian group, then \( P \) contains an element \( z \) of order 4. In particular, \( z \in N_G(\langle z^2 \rangle) \) and \( z \langle z^2 \rangle \) contain no involution. By Lemma 2.6, \( \langle z^2 \rangle \) is not a perfect code of \( G \). Therefore, if every subgroup of \( G \) is a perfect code, then every Sylow 2-subgroup of \( G \) is elementary abelian.

\( \Leftarrow \) If \( G \) has an elementary abelian Sylow 2-subgroup, then every Sylow 2-subgroup of \( G \) is elementary abelian as all Sylow 2-subgroups of \( G \) are conjugate. Let \( H \) be a subgroup of \( G \). Let \( Q \) be a Sylow 2-subgroup of \( H \) and \( P \) a Sylow 2-subgroup of \( N_G(Q) \). Then \( P \) is an elementary abelian group containing \( Q \) and therefore \( Q \) is a perfect code of \( P \). By Theorem 1.2, we obtain that \( H \) is a perfect code of \( G \). \( \square \)

Let \( Q \) be a Sylow 2-subgroup of \( H \) which is contained in a Sylow 2-subgroup \( P \) of \( G \). In general, \( Q \) being a perfect code of \( P \) does not guarantee that \( H \) is a perfect code of \( G \). See the following example.

Example 3.5 Let \( G = S_6 \), \( H = \langle (12)(35), (345) \rangle \), \( P = \langle (12), (35), (3456) \rangle \) and \( Q = \langle (12)(35) \rangle \). Then \( Q \) is a Sylow 2-subgroup of \( H \) and \( P \) is a Sylow 2-subgroup of \( G \) containing \( Q \). It is obvious that \((1325) \in N_G(Q), (1325)^2 \in Q \) and \((1325)Q = \{(1325), (1523)\} \). Note that \((1325)Q \) contains no involution. By Lemma 2.6, \( Q \) is not a perfect code of \( G \). It follows from Theorem 1.1 that \( H \) is not a perfect code of \( G \). However, \( Q \) is a perfect code of \( P \) as \( Q \) has a complement \( \{(12), (3456)\} \) in \( P \).

4 Subgroup perfect codes of \( \text{PSL}(2, q) \)

Throughout this section, we assume that \( q \) is a prime power. Let \( d = 1 \) if \( q \) is even and \( d = 2 \) if \( q \) is odd. Recall that \( |\text{PSL}(2, q)| = \frac{1}{2}q(q-1)(q+1) \). Note that all subgroups of \( \text{PSL}(2, q) \) were first known by Dickson in [8]. The main results of this section can be used to check whether a given subgroup of \( \text{PSL}(2, q) \) is perfect code. We will use the following result of Dickson which gives a classification of maximal subgroups of \( \text{PSL}(2, q) \).

Lemma 4.1 [8, 26] A maximal subgroup of \( \text{PSL}(2, q) \) is isomorphic to one of the following groups:

(i) the dihedral group of order \( \frac{2(q-1)}{d} \) when \( q \neq 3, 5, 7, 9, 11 \);
(ii) the dihedral group of order \( \frac{2(q+1)}{d} \) when \( q \neq 2, 7, 9 \);
(iii) a semidirect product of an elementary abelian group of order \( q \) by a cyclic group of order \( \frac{q-1}{d} \);
(iv) \( S_4 \) when \( q \) is an odd prime number and \( q \equiv \pm 1 \pmod{8} \);
(v) \( A_4 \) when \( q \) is a prime number \( > 3 \) and \( q \equiv 3, 13, 27, 37 \pmod{40} \);

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(vi) As when \( q \) is one of the following forms: \( q = 5^m \) or \( 4^m \) where \( m \) is a prime, \( q \) is a prime number congruent to \( \pm 1 \) (mod 5), or \( q \) is the square of an odd prime number which satisfies \( q \equiv -1 \) (mod 5);

(vii) \( \text{PSL}(2, r) \) when \( q = r^m \) and \( m \) is an odd prime number;

(viii) \( \text{PSL}(2, r) \) when \( q = r^2 \).

The following lemma maybe well known and it can be deduced directly from Lemma 4.1.

**Lemma 4.2** Let \( q \) be a prime power with \( q \equiv \pm 1 \) (mod 8). Then every Sylow 2-subgroup of \( \text{PSL}(2, q) \) is a dihedral group.

**Proof** Since \( q \equiv \pm 1 \) (mod 8), we have \( |\text{PSL}(2, q)| = \frac{1}{2}q(q - 1)(q + 1) \). Therefore 8 \mid |\text{PSL}(2, q)| and exactly one of \( \frac{1}{2}q(q + 1) \) and \( \frac{1}{2}q(q - 1) \) is odd. By Lemma 4.1, if \( q \neq 7, 9 \), then \( \text{PSL}(2, q) \) has dihedral subgroups of order \( q \pm 1 \); if \( q = 7, 9 \), then \( \text{PSL}(2, q) \) has maximal subgroups isomorphic to \( S_4 \) (note that \( \text{PGL}(2, 3) \cong S_4 \)). Thus \( \text{PSL}(2, q) \) has a dihedral Sylow 2-subgroup and it follows that every Sylow 2-subgroup of \( \text{PSL}(2, q) \) is a dihedral group. \( \square \)

In [21], it was shown that every subgroup of \( \text{PSL}(2, q) \) is a perfect code if \( q \) is even or \( q \equiv \pm 3 \) (mod 8). The following theorem gives a criterion for determining subgroup perfect codes of \( \text{PSL}(2, q) \) for the remainder case when \( q \equiv \pm 1 \) (mod 8).

**Theorem 4.3** Let \( q \) be a prime power with \( q \equiv \pm 1 \) (mod 8), \( H \) a subgroup of \( \text{PSL}(2, q) \) and \( Q \) a Sylow 2-subgroup of \( H \). Then \( H \) is a perfect code of \( \text{PSL}(2, q) \) if and only if one of the followings holds:

(i) \( Q \) is trivial;

(ii) \( Q \) is noncyclic;

(iii) \( Q \) is a cyclic 2-group of maximal order.

**Proof** Write \( G = \text{PSL}(2, q) \). We at first prove the sufficiency. If \( Q \) is trivial, then \( H \) is of odd order. By Lemma 2.7, \( H \) is a perfect code of \( G \). In what follows, we assume that \( Q \) is nontrivial. By Lemma 4.2, every Sylow 2-subgroup of \( G \) is dihedral. By the Sylow’s Theorem, \( Q \) is contained in a Sylow 2-subgroup of \( G \). Therefore, if \( Q \) is noncyclic, then it is either a dihedral group or an elementary abelian group of order 4. Let \( P \) be a Sylow 2-subgroup of \( N_G(Q) \). If \( Q \) is noncyclic, then \( P \) is a dihedral group and \( Q \) is of index 2 in \( P \). If \( Q \) is a cyclic 2-group of maximal order, then \( P \) is a Sylow 2-subgroup of \( G \) and therefore dihedral. In both cases, \( Q \) has a complement of order 2 in \( P \) and therefore is a perfect code of \( P \). By Lemma 1.2, \( H \) is a perfect code of \( G \). This completes the proof of the sufficiency.

Now we prove the necessity. If \( Q \) is a nontrivial proper subgroup of a cyclic 2-group, then there exists \( x \in G \) such that \( x \notin Q \) and \( Q = \langle x^2 \rangle \). Clearly, \( x \in N_G(Q) \) and \( xQ \) contains no involution. By Lemma 2.6, \( Q \) is not a perfect code of \( G \). It follows from Theorem 1.1 that \( H \) is not a perfect code of \( G \). Thus, if \( H \) is a perfect code of \( G \), then one of the three conditions (i), (ii) and (iii) holds. \( \square \)

It is straightforward to check that every maximal subgroup of \( G \) listed in Lemma 4.1 contains a Sylow 2-subgroup which is either noncyclic or a cyclic 2-group of maximal order. Combining the result of Ma et al [21] that every subgroup of \( \text{PSL}(2, q) \) is a perfect code if \( q \) is even or \( q \equiv \pm 3 \) (mod 8), Theorem 4.3 implies the following result.

**Corollary 4.4** Let \( q \) be a prime power. Then every maximal subgroup of \( \text{PSL}(2, q) \) is a perfect code of \( \text{PSL}(2, q) \).
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