Research Article

On Solutions of Fractional-Order Gas Dynamics Equation by Effective Techniques

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In this work, the novel iterative transformation technique and homotopy perturbation transformation technique are used to calculate the fractional-order gas dynamics equation. In this technique, the novel iteration method and homotopy perturbation method are combined with the Elzaki transformation. The current methods are implemented with four examples to show the efficacy and validation of the techniques. The approximate solutions obtained by the given techniques show that the methods are accurate and easy to apply to other linear and nonlinear problems.

1. Introduction

Fractional calculus (FC) has been there since classical calculus, but it has recently gained much attention due to its reaction to the requirements as mentioned above. The framework of Liouville and Riemann is used to analyze FC using differential and integral operators. Following that, it was widely used to investigate a variety of phenomena. Many academics, however, pointed out some limitations in engaging this operator, in particular, the physical meaning of the initial condition and the nonzero derivative of a constant. Caputo then presented a unique and new fractional operator that incorporated all of the abovementioned constraints. The Caputo operator is used to study most of the models studied and analyzed under the FC framework. For many ideas of FC, senior academics propose many pioneering directions, and they are the ones who provide the groundwork for the concept [1–5]. The theory and core ideas of FC have been applied to a variety of real-world problems, including biomathematics, financial models, chaos theory, optics, and other fields [6–12].

Gas dynamic equations are mathematical representations defined as the physical laws of energy conservation, mass conservation, momentum conservation, etc. Nonlinear fractional-order gas dynamics equations are applied in shock fronts, unusual factions, and connection discontinuities. Gas dynamics is a study in the field of fluid dynamics that studies gas motion and its effect on physical constructions based on the concept of fluid mechanics and fluid dynamics. The science emerges from research of gas flows, mostly around or within human minds, several instances for these research involve, and not restricted to, choked flows in nozzles and pipes, gas fuel streams in a rocket engine, aerodynamic heating on atmospheric reentry cars, and shock waves around aircraft [13, 14].

Consider the nonlinear fractional-order gas dynamics equation:

$$\frac{\partial^{\delta} v}{\partial \eta^{\delta}} + v \frac{\partial v}{\partial \eta} - \nu (1 - \nu) = 0, \quad \eta \in \mathbb{R}, \ 0 < \delta \leq 1. \quad (1)$$

The initial condition is \( \Phi(\mathfrak{3}, 0) = g(\mathfrak{3}) \), where \( \delta \) is a parameter that describes the fractional-order derivatives.
When \( \varphi = 1 \), (1) improves the equation of classical gas dynamics.

Since certain physical processes, both in engineering and applied sciences, can be successfully explained by the creation of models with the aid of fractional calculus theory. The response of the fractional-order equations eventually converges to the equations of the integer order, attracting particular interest nowadays. Due to a broad variety of applications for mathematical modeling of real-world problems, fractional differentiations are very efficient, e.g., traffic flow models, earthquake modeling, regulation, diffusion model, and relaxation processes [15–17]. In the past, traffic flow models, earthquake modeling, regulation, differential equations have been used for mathematical modeling of real-world problems, fractional differentiations are very efficient, e.g., applications for mathematical modeling of real-world models, and relaxation processes [15–17]. In the past, traffic flow models, earthquake modeling, regulation, differential equations have been used for mathematical modeling of real-world problems, fractional differentiations are very efficient, e.g., traffic flow models, earthquake modeling, regulation, diffusion model, and relaxation processes [15–17].

The goal of this study is to show how, applying the novel iterative technique and the homotopy perturbation technique, the Elzaki transform can be used to obtain approximate solutions for linear and nonlinear fractional-order differential equations. The homotopy perturbation technique was developed by Chinese mathematician J.H. In 1998, he played an important role [28]. This approach is equitable, efficient, and effective, as it eliminates an unconditioned matrix, infinite series, and complicated integrals. This technique does not necessitate the use of any unique problem parameter. Tarig Elzaki, in 2010, develops a new transformation known as the Elzaki transform (E.T). E.T. is a new transform of Laplace and Sumudu transformations [29–32]. Many other researchers use HPTM to solve various equations, such as heat-like models [33], Navier–Stokes models [34], hyperbolic and Fisher’s equations [35], and gas dynamic problem [36]. Jafari and Daftardar-Gejji presented a new iterative approach for solving nonlinear equations in 2006 [37]. Jafari et al. first apply the iterative technique and Laplace transformation and combine it. They developed an iterative Laplace transformation method, which is a modified straightforward method [38] to solve the FPDE system [39, 40].

2. Basic Definitions

2.1. Definition. The fractional-order Riemann operator \( D^\varphi \) of order \( \varphi \) is defined as [29]

\[
D^\varphi \nu(\zeta) = \begin{cases} 
\frac{d^\ell}{d\zeta^\ell} \nu(\zeta), \\
\frac{1}{\Delta(\ell - \varphi)} \frac{d}{d\zeta} \int_0^\zeta \frac{\nu(\zeta)}{(\zeta - \psi)^{\varphi-\ell+1}} d\psi, & \ell - 1 < \varphi < \ell,
\end{cases}
\]

where \( \ell \in Z^+, \varphi \in R^+, \) and

\[
D^{-\varphi} \nu(\zeta) = \frac{1}{\varphi(\varphi)} \int_0^\zeta (\zeta - \psi)^{\varphi-1} \nu(\psi) d\psi, \quad 0 < \varphi \leq 1.
\]

2.2. Definition. The Riemann fractional-order integral operator \( J^\varphi \) is presented by [29]

\[
J^\varphi \nu(\zeta) = \frac{1}{\varphi(\varphi)} \int_0^\zeta (\zeta - \psi)^{\varphi-1} \nu(\psi) d\psi, \quad \zeta > 0, \varphi > 0.
\]

The basic properties of the operator are presented as

\[
J^\varphi \zeta^\ell = \frac{\Gamma(\ell + 1)}{\Gamma(\ell + \varphi + 1)} \zeta^{\ell - \varphi},
\]

\[
D^\varphi \zeta^\ell = \frac{\Gamma(\ell + 1)}{\Gamma(\ell - \varphi + 1)} \zeta^{\ell - \varphi}.
\]

2.3. Definition. The Caputo fractional operator \( D^\varphi \) of \( \varphi \) is defined as [29]

\[
CD^\varphi \nu(\zeta) = \begin{cases} 
\frac{1}{\varphi(\varphi)} \int_0^\zeta \frac{\nu(\psi)}{(\zeta - \psi)^{\varphi-\ell+1}} d\psi, & \ell - 1 < \varphi < \ell,
\end{cases}
\]

2.4. Definition. The Elzaki transformation Caputo fractional-order operator is defined as

\[
E[D_\ell^\varphi g(\zeta)] = s^{-\varphi} \left[ D_\ell^\varphi g(\zeta) \right] - \sum_{k=0}^{\ell-1} s^{\ell - \varphi + k} g^{(k)}(0),
\]

where \( \ell - 1 < \varphi < \ell. \)
3. New Iterative Transformation Technique

We consider

\[ D_\eta^\ell v(\mathfrak{I}, \eta) + M v(\mathfrak{I}, \eta) + N v(\mathfrak{I}, \eta) = h(\mathfrak{I}, \eta), \quad \ell \in \mathbb{N}, \; \ell - 1 < \varphi \leq \ell, \quad (8) \]

where \( M \) and \( N \) are linear and nonlinear terms. We consider the initial condition as

\[ \nu(\mathfrak{I}, 0) = \gamma_{\ell}(\mathfrak{I}). \quad (9) \]

Using the Elzaki transformation of (8), we obtain

\[ E[D_\eta^\ell v(\mathfrak{I}, \eta)] + E[M v(\mathfrak{I}, \eta) + N v(\mathfrak{I}, \eta)] = E[h(\mathfrak{I}, \eta)]. \quad (10) \]

Implement the Elzaki differentiation property:

\[ E[v(\mathfrak{I}, \eta)] = \sum_{\ell=0}^{m} s^{2-\varphi \ell} v^{(\ell)}(\mathfrak{I}, 0) + s^\varphi E[h(\mathfrak{I}, \eta)] - s^\varphi E[M v(\mathfrak{I}, \eta) + N v(\mathfrak{I}, \eta)]. \quad (11) \]

Using the inverse Elzaki transformation (11),

\[ v(\mathfrak{I}, \eta) = E^{-1} \left[ \left\{ \sum_{\ell=0}^{m} s^{2-\varphi \ell} v^{(\ell)}(\mathfrak{I}, 0) + s^\varphi E[h(\mathfrak{I}, \eta)] \right\} - E^{-1} [s^\varphi E[M v(\mathfrak{I}, \eta) + N v(\mathfrak{I}, \eta)]] \right]. \quad (12) \]

Then, we reach

\[ v(\mathfrak{I}, \eta) = \sum_{\ell=0}^{\infty} v_\ell(\mathfrak{I}, \eta), \quad (13) \]

\[ N \left( \sum_{\ell=0}^{\infty} v_\ell(\mathfrak{I}, \eta) \right) = \sum_{\ell=0}^{\infty} N \left[ v_\ell(\mathfrak{I}, \eta) \right]. \quad (14) \]

Replacing equations (12), (13), and (15) in (12) yields

\[ \sum_{\ell=0}^{\infty} v_\ell(\mathfrak{I}, \eta) = E^{-1} \left[ s^\varphi \left( \sum_{\ell=0}^{m} s^{2-\varphi \ell} v^{(\ell)}(\mathfrak{I}, 0) + E[h(\mathfrak{I}, \eta)] \right) \right] - E^{-1} \left[ s^\varphi \left[ M \left( \sum_{\ell=0}^{m} v_\ell(\mathfrak{I}, \eta) \right) - N \left( \sum_{\ell=0}^{m} v_\ell(\mathfrak{I}, \eta) \right) \right] \right]. \quad (16) \]

We consider the nonlinear term \( N \) by

\[ N \left( \sum_{\ell=0}^{\infty} v_\ell(\mathfrak{I}, \eta) \right) = v_0(\mathfrak{I}, \eta) + N \left( \sum_{\ell=0}^{m} v_\ell(\mathfrak{I}, \eta) \right) - M \left( \sum_{\ell=0}^{m} v_\ell(\mathfrak{I}, \eta) \right). \quad (15) \]

We describe the iterative method:

\[ v_0(\mathfrak{I}, \eta) = E^{-1} \left[ s^\varphi \left( \sum_{\ell=0}^{\ell} s^{2-\varphi \ell} v^{(\ell)}(\mathfrak{I}, 0) + s^\varphi E[g(\mathfrak{I}, \eta)] \right) \right], \]

\[ v_1(\mathfrak{I}, \eta) = -E^{-1} s^\varphi E \left[ M v_0(\mathfrak{I}, \eta) + N v_0(\mathfrak{I}, \eta) \right], \]

\[ v_{\ell+1}(\mathfrak{I}, \eta) = -E^{-1} \left[ s^\varphi E \left[ -M \left( \sum_{\ell=0}^{\ell} v_\ell(\mathfrak{I}, \eta) \right) - N \left( \sum_{\ell=0}^{\ell} v_\ell(\mathfrak{I}, \eta) \right) \right] \right], \quad \ell \geq 1. \quad (17) \]
Finally, we can write as

\[ v(F, \eta) \equiv \sum_{m=1}^{\infty} v_m(F, \eta) = v_0(F, \eta) + v_1(F, \eta) + v_2(F, \eta) + \cdots + v_\ell(F, \eta), \quad m = 1, 2, \ldots \]  

(18)

**4. Homotopy Perturbation Transform Method**

In this section, we give the general solution of FPDEs via the homotopy perturbation method:

\[ vD_\eta^\varphi v(F, \eta) + Mv(F, \eta) + Nv(F, \eta) = h(F, \eta), \quad \eta > 0, \ 0 < \varphi \leq 1, \]
\[ v(F, 0) = g(F), \quad \varphi \in \mathbb{R}. \]  

(19)

Applying Elzaki transformation of (16),

\[ E\left[D_\eta^\varphi v(F, \eta) + Mv(F, \eta) + Nv(F, \eta)\right] = E[h(F, \eta)], \quad \eta > 0, \ 0 < \varphi \leq 1, \]
\[ v(F, \eta) = s^\varphi g(F) + s^\varphi E[h(F, \eta)] - s^\varphi E[Mv(F, \eta) + Nv(F, \eta)]. \]  

(20)

Now, we use Elzaki inverse transformation, and we obtain

\[ v(F, \eta) = F(x, \eta) - E^{-1}[s^\varphi E[Mv(F, \eta) + Nv(F, \eta)]], \]  

(21)

where

\[ F(F, \eta) = E^{-1}[s^2 g(F) + s^\varphi E[h(F, \eta)]] = g(\varphi) + E^{-1}[s^\varphi E[h(F, \eta)]]. \]  

(22)

Now, the parameter \( p \) shows the producer of perturbation:

\[ v(F, \eta) = \sum_{\ell=0}^{\infty} p^\ell v_\ell(F, \eta). \]  

(23)

The nonlinear term can be defined as

\[ Nv(F, \eta) = \sum_{\ell=0}^{\infty} p^\ell H_\ell(v_\ell), \]  

(24)

where \( H_\ell \) are He’s polynomial in terms of \( v_0, v_1, v_2, \ldots, v_\ell \) and can be calculated as

\[ H_\ell(v_0, v_1, \ldots, v_\ell) = \frac{1}{\ell(n+1)} D_\eta^\ell \left[ N\left( \sum_{\ell=0}^{\infty} p^\ell v_\ell \right) \right]_{p>0}. \]  

(25)

Substituting (24) and (25) in (21), we achieve as

\[ \sum_{\ell=0}^{\infty} p^\ell v_\ell(F, \eta) = F(F, \eta) - p \times E^{-1}\left\{ s^\varphi E\left\{ M \sum_{\ell=0}^{\infty} p^\ell v_\ell(F, \eta) + \sum_{\ell=0}^{\infty} p^\ell H_\ell(v_\ell) \right\} \right\}. \]  

(26)

Comparison of coefficients \( p \) on both sides, we obtain
\( p^0: v_0(\mathfrak{F}, \eta) = F(\mathfrak{F}, \eta), \)
\( p^1: v_1(\mathfrak{F}, \eta) = E^{-1}[s^\varphi E(Mv_0(\mathfrak{F}, \eta) + H_0(\nu))], \)
\( p^2: v_2(\mathfrak{F}, \eta) = E^{-1}[s^\varphi E(Mv_1(\mathfrak{F}, \eta) + H_1(\nu))], \)
\[ \vdots \]
\( p^\ell: v_\ell(\mathfrak{F}, \eta) = E^{-1}[s^\varphi E(Mv_{\ell-1}(\mathfrak{F}, \eta) + H_{\ell-1}(\nu))], \quad \ell > 0, \; \ell \in \mathbb{N}. \]

(27)

The \( v_\ell(\mathfrak{F}, \eta) \) components can be calculated easily which is a fast convergence series. We can obtain \( p \longrightarrow 1 \):
\[
v(\mathfrak{F}, \eta) = \lim_{p \to 1} \sum_{\ell=1}^{M} v_\ell(\mathfrak{F}, \eta). \tag{28}\]

4.1. Example. Consider the fractional-order gas dynamics equation:
\[
\frac{\partial^\varphi v}{\partial \eta^\varphi} + v \frac{\partial v}{\partial \eta} - v(1 - v) = 0, \quad 0 < \varphi \leq 1, \tag{29}\]

with initial condition,
\[
v(\mathfrak{F}, 0) = e^{-\varphi}. \tag{30}\]

First on both sides apply Elzaki transformation in (29), we have
\[
E[v(\mathfrak{F}, \eta)] = s^\varphi (e^{-\varphi}) - s^\varphi E\left[ v \frac{\partial v}{\partial \eta} - v(1 - v) \right]. \tag{31}\]

Using inverse Elzaki transform on the above equation,
\[
v(\mathfrak{F}, \eta) = e^{-\varphi} - E^{-1}\left[ s^\varphi E\left[ v \frac{\partial v}{\partial \eta} - v(1 - v) \right] \right]. \tag{32}\]

We use the NITM:
\[
v_0(\mathfrak{F}, \eta) = e^{-\varphi}, \]
\[
v_1(\mathfrak{F}, \eta) = -E^{-1}\left[ s^\varphi E\left[ v_0 \frac{\partial v_0}{\partial \eta} - v_0(1 - v_0) \right] \right] = e^{-\varphi} \eta^\varphi \Gamma(\varphi + 1), \]
\[
v_2(\mathfrak{F}, \eta) = -E^{-1}\left[ s^\varphi E\left[ v_1 \frac{\partial v_1}{\partial \eta} - v_1(1 - v_1) \right] \right] = e^{-\varphi} \eta^{2\varphi} \Gamma(2\varphi + 1), \]
\[
v_3(\mathfrak{F}, \eta) = -E^{-1}\left[ s^\varphi E\left[ v_2 \frac{\partial v_2}{\partial \eta} - v_2(1 - v_2) \right] \right] = e^{-\varphi} \eta^{3\varphi} \Gamma(3\varphi + 1), \]
\[
\vdots \]
\[ v_{\ell+1}(\mathfrak{F}, \eta) = -E^{-1}\left[ s^\varphi E\left[ v_\ell \frac{\partial v_\ell}{\partial \eta} - v_\ell(1 - v_\ell) \right] \right] = e^{-\varphi} \eta^{\ell\varphi} \Gamma(l\varphi + 1). \tag{33}\]

The series solution form is given as
\[
v(\mathfrak{F}, \eta) = v_0(\mathfrak{F}, \eta) + v_1(\mathfrak{F}, \eta) + v_2(\mathfrak{F}, \eta) + v_3(\mathfrak{F}, \eta) \]
\[ + \cdots + v_\ell(\mathfrak{F}, \eta). \tag{34}\]

The approximate solution is achieved as
\[
v(\mathfrak{F}, \eta) = e^{-\varphi} \sum_{\ell=0}^{\infty} p^\ell H_\ell(v) \tag{35}\]
\[
\sum_{\ell=0}^{\infty} p^\ell v_\ell(\mathfrak{F}, \eta) = e^{-\varphi} + p\left\{ E^{-1}\left[ s^\varphi E\left( \sum_{\ell=0}^{\infty} p^\ell H_\ell(v) \right) \right] \right\}. \tag{36}\]

Then, we have
\[
p^0: v_0(\mathfrak{F}, \eta) = e^{-\varphi}, \]
\[
p^1: v_1(\mathfrak{F}, \eta) = \left[ E^{-1}\left[ s^\varphi E(H_0(\nu)) \right] \right] = e^{-\varphi} \eta^\varphi \Gamma(\varphi + 1), \]
\[
p^2: v_2(\mathfrak{F}, \eta) = \left[ E^{-1}\left[ s^\varphi E(H_1(\nu)) \right] \right] = e^{-\varphi} \eta^{2\varphi} \Gamma(2\varphi + 1), \tag{37}\]
\[
p^3: v_3(\mathfrak{F}, \eta) = \left[ E^{-1}\left[ s^\varphi E(H_2(\nu)) \right] \right] = e^{-\varphi} \eta^{3\varphi} \Gamma(3\varphi + 1), \]
\[
\vdots \]
\[
p^\ell: v_\ell(\mathfrak{F}, \eta) = \left[ E^{-1}\left[ s^\varphi E(H_{\ell-1}(\nu)) \right] \right] = e^{-\varphi} \eta^{\ell\varphi} \Gamma(l\varphi + 1). \tag{38}\]

Then, the series form solution of HPTM is presented:
\[
v(\mathfrak{F}, \eta) = \sum_{n=0}^{\infty} p^n v_n(\mathfrak{F}, \eta). \tag{39}\]

The approximate solution of Example 1 is given by
\[
v(\mathfrak{F}, \eta) = e^{-\varphi} + e^{-\varphi} \eta^\varphi \Gamma(\varphi + 1) + e^{-\varphi} \eta^{2\varphi} \Gamma(2\varphi + 1) + e^{-\varphi} \eta^{3\varphi} \Gamma(3\varphi + 1) \]
\[ + \cdots + e^{-\varphi} \eta^{n\varphi} \Gamma(n\varphi + 1). \]

The exact result of (29):
\[ v(\mathfrak{F}, \eta) = e^{-\alpha \eta}. \]  

(40)

In Figure 1, the actual and analytical solutions are proved at \( \varphi = 1 \) of Example 4.1. In Figure 2, the three-dimensional figure for numerous fractional orders are described which demonstrates that the modified decomposition technique and new iterative transform technique approximated obtained results are in close contact with the analytical and the exact results. In Figure 3, the analytical solution graph of fractional order \( \varphi = 0.4 \) of problem 3.1. This comparative shows a strong connection among the homotopy perturbation transform method and actual solutions. Consequently, the homotopy perturbation transform method and new iterative transformation technique are accurate innovative techniques which need less calculation time and is very simple and more flexible as compared to other methods.

4.2. Example. We take into consideration

\[ v_0(\mathfrak{F}, \eta) = b^{-\alpha}, \]

\[ v_1(\mathfrak{F}, \eta) = -E^{-1}\left[s^\varphi E\left[v_0 \frac{\partial v_0}{\partial \eta} - v_0 (1 - v_0) \log b\right]\right] = b^{-\alpha} \frac{\log b \eta^\varphi}{\Gamma(\varphi + 1)}, \]

\[ v_2(\mathfrak{F}, \eta) = -E^{-1}\left[s^\varphi E\left[v_1 \frac{\partial v_1}{\partial \eta} - v_1 (1 - v_1) \log b\right]\right] = b^{-\alpha} \frac{(\log b)^2 \eta^{2\varphi}}{\Gamma(2\varphi + 1)}, \]

\[ v_3(\mathfrak{F}, \eta) = -E^{-1}\left[s^\varphi E\left[v_2 \frac{\partial v_2}{\partial \eta} - v_2 (1 - v_2) \log b\right]\right] = b^{-\alpha} \frac{(\log b)^3 \eta^{3\varphi}}{\Gamma(3\varphi + 1)}, \]

\[ \vdots \]

\[ v_{n+1}(\mathfrak{F}, \eta) = -E^{-1}\left[s^\varphi E\left[v_n \frac{\partial v_n}{\partial \eta} - v_n (1 - v_n) \log b\right]\right] = b^{-\alpha} \frac{(\log b)^n \eta^{n\varphi}}{\Gamma(n\varphi + 1)}. \]

(45)

The series solution form is presented by

\[ v(\mathfrak{F}, \eta) = v_0(\mathfrak{F}, \eta) + v_1(\mathfrak{F}, \eta) + v_2(\mathfrak{F}, \eta) \]

\[ + v_3(\mathfrak{F}, \eta) + \cdots + v_n(\mathfrak{F}, \eta). \]

The approximate solution is achieved as

\[ v(\mathfrak{F}, \eta) = b^{-\alpha} + b^{-\alpha} \frac{\log b \eta^\varphi}{\Gamma(\varphi + 1)} + b^{-\alpha} \frac{(\log b)^2 \eta^{2\varphi}}{\Gamma(2\varphi + 1)} + b^{-\alpha} \frac{(\log b)^3 \eta^{3\varphi}}{\Gamma(3\varphi + 1)} + \cdots + b^{-\alpha} \frac{(\log b)^n \eta^{n\varphi}}{\Gamma(n\varphi + 1)}, \]

\[ v(\mathfrak{F}, \eta) = b^{-\alpha} \sum_{n=0}^{\infty} \frac{(\log b)^n \eta^{n\varphi}}{\Gamma(n\varphi + 1)} = b^{-\alpha} E_\varphi(\log b \eta^\varphi). \]

(46)
Now, we apply the HPTM; we obtain

\[
\sum_{\ell=0}^{\infty} p^\ell \nu_\ell(\mathfrak{F}, \eta) = (b^{-\mathfrak{F}}) + p \left\{ E^{-1} \left( s^p E \left[ \sum_{\ell=0}^{\infty} p^\ell H_\ell(\nu) \right] \right) \right\},
\]

where the polynomial signifying the nonlinear expressions is \( H_\ell(\nu) \). For instance, the components of He’s polynomials are obtained through the recursive correlation

\[
H_\ell(\nu) = \nu_\ell(\partial \nu_\ell/\partial \eta) - \nu_\ell(1 - \nu_\ell) \log b, \ \forall \ell \in \mathbb{N}.\]

Now, both sides as the equivalent power coefficient of \( p \) are compared; the following calculation is obtain by
\[ p^0: \upsilon_0(\mathfrak{F}, \eta) = b^{-\alpha}, \]
\[ p^1: \upsilon_1(\mathfrak{F}, \eta) = [E^{-1}s^\alpha E(H_0(\upsilon)))] = b^{-\alpha} \frac{\log b \eta^\alpha}{\Gamma(\alpha + 1)}, \]
\[ p^2: \upsilon_2(\mathfrak{F}, \eta) = [E^{-1}s^\alpha E(H_1(\upsilon)))] = b^{-\alpha} \frac{(\log b)^2 \eta^{2\alpha}}{\Gamma(2\alpha + 1)}, \]
\[ p^3: \upsilon_3(\mathfrak{F}, \eta) = [E^{-1}s^\alpha E(H_2(\upsilon)))] = b^{-\alpha} \frac{(\log b)^3 \eta^{3\alpha}}{\Gamma(3\alpha + 1)}, \]
\[ \vdots \]
\[ p^n: \upsilon_n(\mathfrak{F}, \eta) = [E^{-1}s^\alpha E(H_{n-1}(\upsilon)))] = b^{-\alpha} \frac{(\log b)^n \eta^{n\alpha}}{\Gamma(n\alpha + 1)}. \]

Thus, we obtain
\[ \upsilon(\mathfrak{F}, \eta) = \sum_{i=0}^{\infty} p^i \upsilon_i(\mathfrak{F}, \eta). \]

The approximate solution of Example 2 is given as
\[ \upsilon(\mathfrak{F}, \eta) = b^{-\alpha} + b^{-\alpha} \frac{\log b \eta^\alpha}{\Gamma(\alpha + 1)} + b^{-\alpha} \frac{(\log b)^2 \eta^{2\alpha}}{\Gamma(2\alpha + 1)} + b^{-\alpha} \frac{(\log b)^3 \eta^{3\alpha}}{\Gamma(3\alpha + 1)} + \cdots + b^{-\alpha} \frac{(\log b)^n \eta^{n\alpha}}{\Gamma(n\alpha + 1)}. \] (51)

The exact result of (40) is
\[ \upsilon(\mathfrak{F}, \eta) = b^{-\alpha} \sum_{m=0}^{\infty} \frac{(\log b)^m \eta^m}{\Gamma(m\alpha + 1)} = b^{-\alpha} E_\alpha(\log b \eta^\alpha). \]

In Figure 4, the actual and analytical solutions are proved at \( \phi = 1 \) of Example 4.2. In Figure 5, the three-dimensional figure for numerous fractional order is described which demonstrates that the modified decomposition technique and new iterative transform technique approximated obtained results are in close contact with the analytical and the exact results. In Figure 6, the analytical solution graph of fractional order \( \phi = 0.4 \) of problem 3.2. This comparative result shows a strong connection between the homotopy perturbation transform method and actual solutions. Consequently, the homotopy perturbation transform
method and new iterative transformation technique are accurate innovative techniques which needs less calculation
time and is very simple and more flexible as compare to other methods.

4.3. Example. We take into consideration the fractional-order nonlinear homogeneous gas dynamics equation:
\[
\frac{\partial^\varphi u}{\partial \eta^\varphi} + u \frac{\partial v}{\partial \eta} - v(1 - u) + e^{-3\eta} = 0, \quad 0 < \varphi \leq 1, 
\]  
(53)
with initial condition,
\[
u(0, 0) = 1 - e^{-3}.
\]  
(54)
Applying the Elzaki transformation in (52) yields
\[
E[u(\eta, \eta)] = s^2(1 - e^{-3}) - s^\varphi E\left[\frac{\partial u}{\partial \eta} - v(1 - u) + e^{-3\eta}\right].
\]  
(55)
Using inverse Elzaki transform on the above equation,
\[
u(\eta, \eta) = 1 - e^{-3} - E^{-1}\left[s^\varphi E\left[\frac{\partial u}{\partial \eta} - v(1 - u) + e^{-3\eta}\right]\right].
\]  
(56)
We use the NITM:

Figure 4: Simulations of the solutions of problem 3.2.

Figure 5: The fractional order of \(\varphi = 0.8\) and \(0.6\) of problem 3.2.
The series solution form is given as

$$v(\mathcal{S}, \eta) = v_0(\mathcal{S}, \eta) + v_1(\mathcal{S}, \eta) + v_2(\mathcal{S}, \eta) + v_3(\mathcal{S}, \eta) + \cdots + v_n(\mathcal{S}, \eta).$$  

The approximate solution is achieved as

$$v(\mathcal{S}, \eta) = 1 - e^{-\alpha} - e^{-\alpha} \frac{\eta^\varphi}{\Gamma(\varphi + 1)} - e^{-\alpha} \frac{\eta^{2\varphi}}{\Gamma(2\varphi + 1)} - e^{-\alpha} \frac{\eta^{3\varphi}}{\Gamma(3\varphi + 1)} - \cdots - e^{-\alpha} \frac{\eta^{m\varphi}}{\Gamma(m\varphi + 1)}.$$  

$$v(\mathcal{S}, \eta) = e^{-\alpha} \sum_{m=0}^{\infty} \frac{(\eta^\varphi)^m}{\Gamma(m\varphi + 1)} = e^{-\alpha} E_{\varphi}(\eta^\varphi).$$

**Figure 6:** The fractional order of $\varphi = 0.4$ of problem 3.2.
Now, we apply the HPTM, and we obtain

\[
\sum_{\ell=0}^{\infty} p^\ell \nu_\ell (\mathfrak{G}, \eta) = 1 - e^{-3} + p \left\{ E^{-1} \left( s^\kappa E \left[ \sum_{\ell=0}^{\infty} p^\ell H_\ell (\nu) + e^{-3+\eta} \right] \right) \right\},
\]

where the polynomial signifying the nonlinear expressions is \( H_\ell (\nu) \). For instance, the components of He’s polynomials are obtained through the recursive correlation

\[ H_\ell (\nu) = \nu_\ell (\partial \nu_\ell / \partial \eta) - \nu_\ell (1 - \nu_\ell) \log b, \quad \forall \ell \in \mathbb{N}. \]

Now, both sides of the equivalent power coefficient of \( p \) is compared; the following calculation is obtain by

\[ H_\ell (\nu) = \nu_\ell (\partial \nu_\ell / \partial \eta) - \nu_\ell (1 - \nu_\ell) \log b, \quad \forall \ell \in \mathbb{N}. \]
\begin{align}
 p^0: \quad & \psi_0(\mathfrak{I}, \eta) = 1 - e^{-\frac{\eta}{(\mathfrak{I} + 1)}} \\
 p^1: \quad & \psi_1(\mathfrak{I}, \eta) = \left[ E^{-1} \left\{ \frac{s \phi}{E} H_0(\psi) + e^{-\frac{3 \eta}{(\mathfrak{I} + 1)}} \right\} \right] = -e^{-\frac{\eta}{(\mathfrak{I} + 1)}} \\
 p^2: \quad & \psi_2(\mathfrak{I}, \eta) = \left[ E^{-1} \left\{ \frac{s \phi}{E} H_1(\psi) + e^{-\frac{3 \eta}{(2 \mathfrak{I} + 1)}} \right\} \right] = -e^{-\frac{\eta}{(2 \mathfrak{I} + 1)}} \\
 p^3: \quad & \psi_3(\mathfrak{I}, \eta) = \left[ E^{-1} \left\{ \frac{s \phi}{E} H_2(\psi) + e^{-\frac{3 \eta}{(3 \mathfrak{I} + 1)}} \right\} \right] = -e^{-\frac{\eta}{(3 \mathfrak{I} + 1)}} \\
 \vdots
 \end{align}

\begin{align}
 p^n: \quad & \psi_n(\mathfrak{I}, \eta) = \left[ E^{-1} \left\{ \frac{s \phi}{E} H_{n-1}(\psi) + e^{-\frac{3 \eta}{(n \mathfrak{I} + 1)}} \right\} \right] = -e^{-\frac{\eta}{(n \mathfrak{I} + 1)}} \\
 \end{align}

Then, the series-form solution of HPTM is given as

\begin{equation}
 \psi(\mathfrak{I}, \eta) = \sum_{\ell=0}^{\infty} \psi_{\ell}(\mathfrak{I}, \eta). \tag{62}
\end{equation}

The approximate solution of example in this section is given as

\begin{align}
 \psi(\mathfrak{I}, \eta) &= 1 - e^{-\frac{\eta}{(\mathfrak{I} + 1)}} - e^{-\frac{\eta}{(2 \mathfrak{I} + 1)}} - e^{-\frac{\eta}{(3 \mathfrak{I} + 1)}} - \cdots - e^{-\frac{\eta}{(n \mathfrak{I} + 1)}} \\
 \psi(\mathfrak{I}, \eta) &= e^{-\frac{\eta}{(\mathfrak{I} + 1)}} \sum_{\ell=0}^{\infty} \frac{\eta^\ell}{(\mathfrak{I} + 1)} = e^{-\frac{3 \eta}{(\mathfrak{I} + 1)}}. \tag{63}
\end{align}

The exact result of (52) is

\begin{equation}
 \psi(\mathfrak{I}, \eta) = 1 - e^{-\frac{3 \eta}{(\mathfrak{I} + 1)}}. \tag{64}
\end{equation}

In Figure 7, the actual and analytical solutions are proved at \(\mathfrak{I} = 1\) of Example 4.3. In Figure 8, the three-dimensional figure for numerous fractional order is described, which demonstrates that the modified decomposition technique and new iterative transform technique approximated obtained results are in close contact with the analytical and the exact results. In Figure 9, the analytical solution graph of fractional order \(\mathfrak{I} = 0.4\) of problem 3.4. This comparative shows a strong connection among the homotopy.
perturbation transform method and actual solutions. Consequently, the homotopy perturbation transform method and new iterative transformation technique are accurate innovative techniques which need less calculation time and is very simple and more flexible as compared to other methods.

5. Conclusion

In this paper, we analyzed the time fractional of gas dynamics equation by applying two analytical techniques. It is also used that the suggested methods’ rate of convergence is sufficient for the solution of fractional-order partial differential equations. The computations of these methods are very straightforward and simple. Therefore, these methods can be applied to fractional partial differential equations.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this study.

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