ON SOME PROPERTIES OF LINEAR MAPPING INDUCED 
BY LINEAR DESCRIPTOR DIFFERENTIAL EQUATION

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Abstract. In this paper we introduce linear mapping $D$ from $W^F_2 \subset L_2^n$ into $L_{m2} \times \mathbb{R}_m$, induced by linear differential equation $\frac{d}{dt}F(t)x(t) - C(t)x(t) = f(t), Fx(t_0) = f_0$. We prove that $D$ is closed dense defined mapping for any $m \times n$-matrix $F$. Also adjoint mapping $D^*$ is constructed and it’s domain $W^{F'}_2$ is described.

Some kind of so-called ”integration by parts” formula for vectors from $W^F_2, W^{F'}_2$ is suggested. We obtain a necessary and sufficient condition for existence of generalized solution of equation $Dx(\cdot) = (f(\cdot), f_0)$. Also we find a sufficient criterion for closeness of the $\mathcal{R}(D)$ in $L_{m2} \times \mathbb{R}_m$ which is formulated in terms of transparent conditions for blocks of matrix $C(t)$. Some examples are supplied to illustrate obtained results.

Introduction

System of linear differential equations in the form of

$$F(t)\dot{x}(t) + C(t)x(t) + B(t)f(t) = 0$$

is called singular or descriptor one. American mathematicians Campbell and Petzold [1] introduced a notion of central canonical form for stationary system (1). Namely, if $\det(\lambda F + C) \neq 0$ for any real $\lambda$ then we can transform (1) into independent differential and algebraic equations (for sufficiently smooth $f(\cdot)$)

$$\dot{x}_1(t) = Ax_1(t) + Kf(t), \quad x_2(t) = -Df(t) - \sum_{i=1}^{m-1} N_i Dv^{(i)}(t), \quad (x_1, x_2) = Q^{-1}x(t)$$

In case of non-constant coefficients in (1) russian mathematicians Bojarintsev and Chistjakov suggested a notion of left regularization operator

$$\Lambda_{s,r}[\frac{d}{dt}(F(t)x(t)) + C(t)x(t)] = \frac{d}{dt}x(t) + \Lambda_{s,r}[C(t)]x(t),$$
where $\Lambda_{*,r} = \sum_{j=0}^{r} L_j(t)\left(\frac{d}{dt}\right)^j$. In case of constant coefficients existence of central canonical form is equal to left regularization operator existence. In general case conditions of existance of left regularization operator depends on properties of "prolonged"-system [2].

Italian mathematician Favini [3] studied existence and fundamental solution representation of system (1) in general case where $F(t), C(t)$ supposed to be bounded linear mappings from Banach space $X$ into Banach space $Y$. Their results are based on hypothesis that interval $T := [a, b] \times [c_1, +\infty)$ consists of regular points of resolvent $(\lambda F(t) + C(t))^{-1}$ which is bounded on $T$.

**Problem statement**

Papers mentioned above ( and lots of works devoted to singular systems listed in surveys [4, 5] ) are based on the hypothesis that some canonical form of (1) exists i.e. system (1) could be transformed into implicit form. This implies we can use powerfull tool – theorems about unite points for the second type Volterra operators.

In this paper we use general approach of closed mappings theory combined with regularization methods applied to mapping induced by linear descriptor system

\[
\frac{d}{dt} Fx(t) - C(t)x(t) = f(t), \\
Fx(a) = f_0,
\]

Hence we can study some general properties of (2) – existence of solutions, continuous dependence of the solution on the right hand side of operator equation – without assuming the structure of given system to be canonical. From the other hand operator approach makes it possible to investigate noncasual systems [4].

In (2) we set $F = \{F_{ij}\}_{1}^{m,n}$ – some rectangular matrix, $t \mapsto C(t)$ – continuous matrix-valued function, $f(\cdot)$ is some element of squared summable vector-functions space $L^2_{\infty} := L^2([t_0, T], \mathbb{R}^m)$, $T < +\infty, f_0 \in \mathbb{R}^m$. If $F$ is non-degenerate square matrix, then it’s easy to see that (2) has unique totally continuous solution $x(\cdot)$ and it satisfies Volterra integral equation

\[
Fx(t) = f_0 + \int_{a}^{t} C(s)x(s) + f(s)ds
\]
In general case of rectangular matrix $F$ we define solution of singular initial value problem (2) as follows. Set

$$Fx(t) = (Fx_1(t), ..., Fx_m(t)), Fx_i(t) = \sum_{1}^{n} F_{ij}x_j(t)$$

and let $\mathbb{W}_2^F$ be a set of all $x(\cdot)$ from $\mathbb{L}_2^n$ satisfying

$F x(\cdot)$ is totally continuous and its derivative lies in $\mathbb{L}_m^n$.

It’s easy to see that $\mathbb{W}_2^F$ is linear total subset of $\mathbb{L}_2^n$. For each $x(\cdot) \in \mathbb{W}_2^F$ we set

$$Dx(t) = (\frac{d}{dt} Fx(t) - C(t)x(t), Fx(a))$$

Now we say that $x(\cdot) \in \mathbb{W}_2^F$ is a solution of (2) if it lies in the solutions domain of operator equation

$$Dx(\cdot) = (f(\cdot), f_0) \quad (3)$$

The goal of this paper is investigation of some properties of $D$ namely closure-ness of $D$ and conditions for normal solvability of $D$. In terms of descriptor systems it can be rewrited as follows: solvability conditions for (2), conditions for continuous dependence of solution (2) on initial condition $f_0$ and perturbation $f(\cdot)$, approximation of (2) solution by sequence of functions.

**Closureness of $D$ and it’s adjoint mapping.**

Now we can introduce

**Theorem 1.** If $x(\cdot) \in \mathbb{W}_2^F$, $z(\cdot) \in \mathbb{W}_2^{F'}$, then

$$\int_{a}^{c} \left( \frac{d}{dt} Fx(t), z(t) \right) + \left( \frac{d}{dt} F'z(t), x(t) \right) dt = \left( Fx(c), F'^+ F' z(c) \right) - \left( Fx(a), F'^+ F' z(a) \right) \quad (4)$$

Moreover, $D$ is linear closed dense defined mapping and its adjoint $D'$ is given by

$$D'(z(\cdot), z_0) := -\frac{d}{dt} (F'z(\cdot) - C'z(\cdot)),$$

$$\mathcal{D}(D') = \{(z(\cdot), F'^+ F' z(a) + d) : z(\cdot) \in \mathbb{W}_2^{F'}, F'z(c) = 0, F'd = 0\},$$

$\mathbb{W}_2^{F'}$ is defined in the same way as $\mathbb{W}_2^F$ with respect to $F'$. 

Remark 1. We must stress that linear mapping
\[
  x(\cdot) \mapsto F \frac{d}{dt} x(\cdot), x(\cdot) \in W_2^n
\]
is not closed in general case. Really, let’s consider case \( n = 2, t_0 = 0, T = 1 \) and set
\[
  F = \begin{pmatrix}
  1 & 0 \\
  0 & 0
  \end{pmatrix}
\] (6)
If we denote by \( t \mapsto k(t) \) Cantor’s ”dust” function then \( v(\cdot) := (0, k(\cdot)) \not\in W_2^n \).
Let’s set
\[
  B_n(t) := \sum_0^n k\left(\frac{i}{n}\right) \binom{n}{i} t^i (1 - t)^{n - i}
\]
v\(_n(\cdot) := (0, B_n(\cdot)) \in W_2^n \) and
\[
  F \frac{d}{dt} v_n(\cdot) \to 0, v_n(\cdot) \to v(\cdot)
\]
so \( v(\cdot) \in W_2^n \) if \( x(\cdot) \mapsto F \frac{d}{dt} x(\cdot) \) is closed.
On the other hand \( v(\cdot) \in W_2^n \) \( F \frac{d}{dt} v(\cdot) = (0, 0) \). This remark implies

In general case \( W_2^n \) is not Hilbert space with respect to \( \frac{d}{dt} F \) graph-norm.

Really, in that case \( F \frac{d}{dt} \) would be close on \( W_2^n \), because
\[
  \frac{d}{dt} F x(\cdot) = F \frac{d}{dt} x(\cdot), \forall x(\cdot) \in W_2^n
\]

Normal solvability of \( \mathcal{D} \).

For applications of linear differential equations the range of operator is very important to be closed because it implies continuous dependence of solution on initial conditions and perturbations. Next theorem introduces criterion of (2) pseudosolution existence.

\[\text{2} W_2^n \] is a set of totally continuous function in \( L_2^n \).
Theorem 2. Boundary value problem
\[
\frac{d}{dt} F x(t) = C(t) x(t) + z(t) + f(t),
\]
\[
\frac{d}{dt} F' z(t) = -C'(t) z(t) + \varepsilon^2 x(t), F' z(c) = 0,
\]
\[
F x(a) - F' x(a) + d = f_0, F' d = 0
\]
has unique solution \((x(\cdot, \varepsilon), z(\cdot, \varepsilon), d(\varepsilon))\) for any \(\varepsilon > 0\).

For given \((f(\cdot), f_0) \in \mathbb{L}_2^m \times \mathbb{R}^m\) descriptor system
\[
\frac{d}{dt} F x(t) = C(t) x(t) + f(t), F x(t_0) = f_0
\]
has the pseudosolution\(^3\) \(\hat{x}(\cdot)\) iff
\[
\|x(\cdot, \varepsilon)\|_2 \leq C \text{ while } \varepsilon \to 0
\]

Theorem 3. Let
\[
F = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}, C(t) \equiv \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, f_0 = \begin{pmatrix} f_0^1 \\ f_0^2 \end{pmatrix}
\]
where \(E_r\) is identity \(r \times r\) matrix, \(C_i\) are any matrices of appropriate dimensions. If\(^4\)
\[
\sup_{1 > \varepsilon > -1} \|Q(\varepsilon) C_2\|_{\text{mod}} < +\infty, Q(\varepsilon) := (\varepsilon^2 E + C_4 C_4)^{-1},
\]
then range of \(D\) is closed linear manifold.

Let’s illustrate above theorems by examples.

Example 1. If we set
\[
F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, C(t) \equiv \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}
\]
then \(\mathcal{N}(D) = \{0\}\), hence \(cl \mathcal{R}(D^*) = \mathbb{L}_2^n\). On the other hand
\[
\mathcal{R}(D^*) = \{(\cdot z_1 - z_1 - z_2), z_1 \in \mathbb{W}_2^1([t_0, T]), z_1(t_0) = 0, z_2 \in \mathbb{L}_2([t_0, T])\}
\]

\(^3\)We set \(\hat{x}(\cdot) \in \mathbb{W}_2^F\) is the pseudosolution of \(D x(\cdot) = (f(\cdot), f_0)\) if \(\|D \hat{x}(\cdot) - (f(\cdot), f_0)\|_2^2 = \min_{x(\cdot)} \|D x(\cdot) - (f(\cdot), f_0)\|_2^2\).

\(^4\)we set \(\|F\|_{\text{mod}} := \sum_{i,j} |F_{ij}|\) for any rectangular matrix \(F\).
Note that sufficient condition of theorem 3 does not hold in this case because $C_2 Q(\varepsilon) = -\varepsilon^{-2}$.

Here (7) rewrites as
\[
\begin{align*}
\dot{x}_1(t) &= x_1(t) + (1 + \varepsilon^{-2}) z_1(t) + f_1(t), \\
\dot{z}_1(t) &= -z_1(t) + (1 + \varepsilon^2) x_1(t) + f_2(t), \\
x_1(t_0) - z_1(t_0) &= f_{01}, z_1(T) = 0, x_2(t) = -\varepsilon^2 z_1(t, \varepsilon)
\end{align*}
\]
so if we denote by $k(\cdot)$ solution of
\[
\dot{k}(t) = 2k(t) + (1 + \varepsilon^{-2}) - (1 + \varepsilon^2) k^2(t) := U(t, k), k(t_0) = 1
\]
then it’s easy to see that
\[
k^- < k(t, \varepsilon) < k^+, 0 < \varepsilon < \varepsilon_0, t > t_0, \tag{11}
\]
where
\[
k^- := \frac{\varepsilon^2 - \sqrt{\varepsilon^2 + 3\varepsilon^4 + \varepsilon^6}}{\varepsilon^2 + \varepsilon^4}, \quad k^+ := \frac{\varepsilon^2 + \sqrt{\varepsilon^2 + 3\varepsilon^4 + \varepsilon^6}}{\varepsilon^2 + \varepsilon^4}
\]
So equality $U(t, k) = (1 + \varepsilon^2)(k - k^-)(k^+ - k)$ implies $k(t, \varepsilon) > 0, t \geq t_0, 0 < \varepsilon < \varepsilon_0$, therefore $k(t, \varepsilon) \geq k(t_0, \varepsilon) > 0$ for $t \geq t_0, 0 < \varepsilon < \varepsilon_0$.

Let’s denote by $q(\cdot, \varepsilon)$ solution of
\[
q_{tt}(t) - 2q_t(t) + (1 + \varepsilon^{-2})(1 + \varepsilon^2) q(t) = 0, q_t(t_0) = 1 + \varepsilon^2, q(t_0) = 1
\]
It’s clear that
\[
q(t, \varepsilon) = e^{\int_0^t (1 + \varepsilon^2) k(s, \varepsilon) ds} > 0 \Rightarrow q(t, \varepsilon) \geq 0, t \geq t_0, 0 < \varepsilon < \varepsilon_0 \tag{\!*}
\]
If we set
\[
\varphi(t, \varepsilon) = \frac{e^{t-t_0}}{q(t, \varepsilon)} \left\{ f_1(t) + \int_{t_0}^{t} \frac{q(\tau, \varepsilon)}{e^{\tau-t_0}} f_1(\tau) - \frac{q(\tau, \varepsilon) f_2(\tau)}{e^{\tau-t_0}(1 + \varepsilon^2)} d\tau \right\}
\]
and
\[
z(t, \varepsilon) = -\frac{q(t, \varepsilon)}{e^t} \int_t^T \frac{e^s}{q(s, \varepsilon)} (f_2(s) + (1 + \varepsilon^2) \varphi(s, \varepsilon)) ds
\]
then it’s obvious that $x_1(t, \varepsilon) = k(t, \varepsilon) z(t, \varepsilon) + \varphi(t, \varepsilon), x_2(t, \varepsilon) = -\varepsilon^{-2} z(t, \varepsilon)$.

Let’s set $f_1(t) \equiv 0, f_2(t) = -e^{t-t_0}, f_1^0 = 1$. Then
\[
\varphi(t, \varepsilon) = \frac{\varepsilon^2 e^{t-t_0}}{(1 + \varepsilon^2) q(t, \varepsilon)} + \frac{e^{t-t_0}}{1 + \varepsilon^2} z(t, \varepsilon) = \frac{-\varepsilon^2 q(t, \varepsilon)}{e^{t+t_0}} \int_t^T \frac{e^{2s}}{q^2(s, \varepsilon)} ds
\]
and \( x(t) = \left( \frac{x_1(t)}{x_2(t)} \right), x_1(t) = -f_2(t), x_2(t) \equiv 0 \) is unique solution of \( D x(\cdot) = (f(\cdot), f_0). \)

We'll show that \( x_1(\cdot, \varepsilon) \to x_1, x_2(\cdot, \varepsilon) \to 0 \) in \( \mathbb{L}_2^2 \). (*) implies

\[
\frac{e^{t-t_0}}{1+\varepsilon^2} < \frac{e^{t-t_0}}{1+\varepsilon^2} + \frac{\varepsilon^2 e^{t-t_0}}{(1+\varepsilon^2)q(t_0, \varepsilon)}, z(t, \varepsilon) \leq -\varepsilon^2 \frac{q(t, \varepsilon)}{e^{t+t_0} q^2(t, \varepsilon)} \int_t^T e^{2s} ds,
\]

hence according to (11) and \( q(t_0, \varepsilon) = 1 \) we get

\[
\int_{t_0}^T (\varphi(t, \varepsilon) + f_2(t))^2 dt \leq \int_{t_0}^T \left( \frac{e^{t-t_0}}{1+\varepsilon^2} + \frac{\varepsilon^2 e^{t-t_0}}{(1+\varepsilon^2)} - e^{t-t_0} \right)^2 dt \to 0, \varepsilon \to 0,
\]
\[
\int_{t_0}^T k^2(t, \varepsilon) z^2(t, \varepsilon) dt \leq \int_{t_0}^T \left( -\varepsilon^2 k^2 \frac{e^{2T} - e^{2t}}{2e^{t+t_0}} \right)^2 dt \to 0, \varepsilon \to 0
\]

therefore \( x_1(\cdot, \varepsilon) \to -f_2(\cdot) \). One can show

\[
q(t, \varepsilon) = \frac{\left( e^{4} + \sqrt{e^{2} + 3\varepsilon^{4} + \varepsilon^{6}} \right) e^{\frac{1}{2} \sqrt{e^{2} + 3\varepsilon^{4} + \varepsilon^{6}} (t-t_0)}}{2\sqrt{e^{2} + 3\varepsilon^{4} + \varepsilon^{6}}},
\]

\[
\left( \sqrt{e^{2} + 3\varepsilon^{4} + \varepsilon^{6}} - \varepsilon^{4} \right) e^{\frac{1}{2} \sqrt{e^{2} + 3\varepsilon^{4} + \varepsilon^{6}} (t-t_0)}
\]

\[
2\sqrt{e^{2} + 3\varepsilon^{4} + \varepsilon^{6}}
\]

hence \( \|q(\cdot, \varepsilon)\|_2 \to +\infty \) while \( \varepsilon \to 0 \). On the other hand

\[
-\varepsilon^{-2} z(t, \varepsilon) \leq \frac{e^{2T} - e^{2t}}{2e^{t+t_0} q(t, \varepsilon)}, \]

therefore \( x_2(\cdot, \varepsilon) \to 0 \).

**Example 2.** If we set

\[
F = \begin{pmatrix} -2 & 6 \\ 2 & -6 \end{pmatrix}, C(t) \equiv \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}
\]

then (2) may be rewritten as

\[
\frac{d}{dt} (-2x_1 + 6x_2)(t) = x_1(t) - 3x_2(t) + f_1(t),
\]
\[
\frac{d}{dt} (2x_1 - 6x_2)(t) = 2x_1(t) - 6x_2(t) + f_2(t),
\]
\[
(-2x_1 + 6x_2)(t_0) = f^0_1, (2x_1 - 6x_2)(t_0) = f^0_2
\]
One can see that $F_1 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, C_0 := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, where $L = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}, R = \begin{pmatrix} 0 & \frac{1}{6} \\ -1 & -\frac{1}{2} \end{pmatrix}$, so equation (12) is equal to

$$\frac{d}{dt} F_1 R^{-1} x(t) = C_0 R^{-1} x(t) + L f(t), F_1 R^{-1} x(t_0) = L f_0$$

Note that $\det(\lambda F_1 + C_0) \equiv 0$. On the other hand theorem 3 ( $C_2 Q(e) \equiv 0$ ) implies that range of $y(\cdot) \mapsto \mathcal{D}_1 y(\cdot) = (\frac{d}{dt} F_1 y(\cdot) - C_0 y(\cdot), F_1 y(t_0))$ is closed. In this case it’s simple to verify last sentence. Really, adjoint mapping is defined by rule

$$(z(\cdot), z_0) \mapsto (-\dot{z}_1(t), -z_2(t)), z_2 \in L_2(t_0, T), z_1 \in W_2^1(t_0, T), z_1(T) = 0$$

so $\mathcal{R}(\mathcal{D}_1) = L_2(t_0, T) \times \{0\}$ implies $\text{cl} \mathcal{R}(\mathcal{D}_1) = \mathcal{R}(\mathcal{D}_1)$.

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