Low-lying mass spectra of excited doubly heavy baryons: Regge relation and mass scaling

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Abstract

In picture of heavy-diquark-light-quark endowed with heavy-pair binding, we explore the low-lying excited baryons $\Xi_{QQ'}$ and $\Omega_{QQ'}$ containing two heavy quarks($QQ' = cc, bb, bc$) by applying relations of Regge trajectory and mass scaling to excitations of light quarks and heavy diquarks, respectively. Two Regge relations, one linear and the other nonlinear, are constructed based on semi-classical analysis of massive QCD string so that they contain the short-distance correction due to heavy-pair interaction when heavy diquark excited internally. Evaluating the short-distance corrections via the binding energy of the pairs estimated by that of the heavy-anti-heavy pairs in the observed doubly heavy mesons, we compute effective masses of heavy diquarks in ground state, 2S- and 1P- waves and thereby predict low-lying mass spectra of the doubly-heavy baryons up to the 2S- and 1P- waves of the light quarks and heavy diquarks. The results indicate that the mass level spacings due to heavy diquark excitations are generally narrower in contrast with that due to the light quark excitations.

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I. INTRODUCTION

Being systems analogous to hydrogen-like atoms with notable level splitting, by which QED interaction is tested in minute detail, the doubly heavy(DH) baryon provides a unique opportunity to probe the fundamental theory of the strong interaction, quantum chromodynamics(QCD) and models inspired by it[1, 2]. It is expected that the excited states of DH baryons can form a set of nontrivial levels with various mass splitting and confront the QCD interaction with the experiments in a straightforward way.

In 2017, the LHCb Collaboration at CERN discovered the doubly charmed baryon $\Xi^{++}$ in the $\Lambda^{+}K^{-}\pi^{+}\pi^{+}$ mass spectrum[3] and reconfirms it in the decay channel of $\Xi^{+}\pi^{+}$[4], with the measured mass $3621.55 \pm 0.23 \pm 0.30$ MeV[5]. Before the LHCb observation, a $\Xi^{+}_{cc}$ candidate has been observed by the SELEX Collaboration at Fermilab[6, 7], which gives a mass about 100 MeV lighter than $3621.55$ MeV and has not been confirmed by other experiments. Further, the later observation seems to contradict with isospin symmetry of the strong interaction, by which the $\Xi^{+}_{cc}$ ground state should be close in mass to the $\Xi^{++}_{cc}$ ground state[8].

The observation of the doubly charmed baryon $\Xi^{++}_{cc}$ sets up, for the first time, a strength scale of interaction between two heavy quarks and is of help to understand strong interaction among heavy quarks[2, 9]. For instance, the mass of the $\Xi^{++}_{cc}$ helps to “calibrate” the binding energy between a pair of heavy quark governed by short-distance interaction and to answer if a DH tetraquark is stable against strongly decaying into two mesons[10]. For more examples, see Refs. [11–15] and the references therein. Very recent observation of the doubly charmed tetraquark[16] is timely to examine the assumed interquark($QQ$) interaction in hadrons and invite conversely systematic study of the doubly heavy baryons. Until now, features(masses, spin-parity and decay widths) of the $\Xi^{++}_{cc}$ and other DH baryons in their excited states remain to be explored.

In this work, we apply Regge relations for heavy hadrons and mass scaling among heavy hadrons(mesons and baryons) combinedly to explore excited doubly heavy baryons $\Xi_{QQ'}$ and $\Omega_{QQ'}$ ($QQ' = cc, bb, bc$) with the help of the observed data of the doubly heavy mesons, the $\Xi^{++}_{cc}$ as well as the commonly-used ground-state masses of the DH baryons compatible with the measured data. We construct a linear Regge relation for DH baryons, which has been applied successfully to singly heavy hadrons in the low-lying states, and extend it to
nonlinear relation of the DH baryons with excited heavy diquark where a term of heavy-pair binding due to the short-distance interquark interaction in pair is incorporated. Evaluating the short-distance corrections via the binding energy of the pairs extracted from that of the heavy-anti-heavy pairs in the observed doubly-heavy mesons, we compute effective masses of the excited heavy diquarks and predict low-lying mass spectra of the DH baryons, with comparison with other calculations.

This paper is organized as follows. In Sect. II, we apply linear Regge relation and a power law of mass scaling to estimate the spin-independent(mean) masses of the excited DH baryons \( \Xi_{QQ'} \) and \( \Omega_{QQ'} \) in 1p-waves in heavy-diquark-light-quark picture. In Sect. III, we formulated spin-dependent mass splitting of the DH baryons with the ground(1S-wave) diquark in the scheme of the \( jj \) coupling. In Sect. IV, the 1p wave masses of the DH baryons with S-wave diquark are predicted via the relation of mass scaling. In Sect. V, we extend the Regge relation to compute the 2S wave masses of the DH baryons with S-wave diquark. In Sect. VI, we improve the linear Regge relation to the DH baryons with diquark excited, and compute the mean mass shifts and the effective diquark masses due to 2S and 1P diquark excitation. In Sect. VII, the baryons mass splittings are computed for the DH baryons with diquark in 2S and 1P waves. We end with summary and discussions Sect. VIII.

II. SPIN-INDEPENDENT MASS WITH S-WAVE DIQUARK

We use heavy-diquark-quark picture for the DH baryons \( qQQ' (q = u, d \text{ and } s) \) with heavy quarks \( QQ' = cc, bb \) or \( bc \), in which the heavy pair \( QQ' \) can be scalar(spin zero) diquark denoted by \([QQ']\) or axial-vector(spin-1) diquark denoted by \(\{QQ'\}\). The charm diquark \( cc \) and the bottom diquark \( bb \) can only form axial-vector diquark, \(\{cc\}\) or \(\{bb\}\), while the bottom-charm diquark \( bc \) can form both, \(\{bc\}\) or \([bc]\). We use the notations \(N_D L_D n l\) to label the quantum numbers of the DH baryons, with the value of principal quantum number\((N_D)\) of diquark, its orbital momentum\((L_D)\) denoted by a capital letter and the principal quantum number\((n)\) for the excitations of light quark and its orbital momentum\((l)\) by a lowercase letter.

To begin with, we first consider the DH baryons with \( QQ' \) in the internal S-wave. The mass of a DH baryon \( qQQ' (Q, Q' = c, b) \) consists of the sum of two parts: \( M = \bar{M} + \Delta M \), where \( \bar{M} \) is the spin-independent mass and \( \Delta M \) is the mass splitting due to the spin-
dependent interaction. As the diquark \( QQ' \) is heavy, compared to the light quark \( q \), the heavy-light limit applies to the DH baryons \( qQQ' \).

By analogy with Ref. \[18\], one can derive a linear Regge relation from the QCD string model for the DH baryons with S-wave diquark(Appendix A) by viewing the baryon to be a \( q - QQ' \) system of a massive QCD string with diquark \( QQ' \) at one end and \( q \) at the other. Denoting the mass of a DH baryon with orbital angular momentum \( l \) of light quark by \( \bar{M}_l \), the Regge relation takes form \[18\](Appendix A)

\[
(\bar{M}_l - M_{QQ'})^2 = \pi a l + \left[ m_q + M_{QQ'} - \frac{m_{bareQQ' \prime}^2}{M_{QQ'}} \right]^2, \tag{1}
\]

where \( M_{QQ'} \) is the effective mass of the S-wave heavy diquark, \( m_q \) the effective mass of the light quark \( q \), \( a \) stands for the tension of the QCD string connecting the light quark \( q \) to \( QQ' \). Here, \( m_{bareQQ'} \) is the bare mass of \( QQ' \), given approximately by sum of the bare masses of each quark: \( m_{bareQQ'} = m_{bareQ} + m_{bareQQ'} \). Numerically, one has

\[
m_{barecc} = 2.55 \text{ GeV}, m_{barebb} = 8.36 \text{GeV}, m_{barebc} = 5.455 \text{GeV}. \tag{2}
\]

We set the values of \( m_q \) in Eq. \( \text{(1)} \) to be that in Table I, which was previously evaluated in Ref. \[18\] based on a combined Regge relation matching of the available mass(experimental) data of the excited singly heavy baryons and mesons.

**TABLE I:** The effective masses (in GeV) of the quark masses and the string tensions \( a \) (in GeV\(^2\)) of the \( D-D_s \) and \( B-B_s \) mesons in this work.

| Parameters | \( M_c \) | \( M_b \) | \( m_n \) | \( m_s \) | \( a(c\bar{n}) \) | \( a(b\bar{n}) \) | \( a(c\bar{s}) \) | \( a(b\bar{s}) \) |
|-----------|-----------|-----------|-----------|-----------|----------------|----------------|----------------|-----------|
| Input     | 1.44      | 4.48      | 0.23      | 0.328     | 0.223          | 0.275          | 0.249          | 0.313      |

To determine \( M_{QQ'} \) in Eq. \( \text{(1)} \), we apply Eq. \( \text{(1)} \) to the ground state\((n = 0 = l)\) to find \[18\]

\[
\bar{M}(1S1s) = M_{QQ'} + m_q + \frac{k_{QQ'}^2}{M_{QQ'}}, \tag{3}
\]

\[
k_{QQ'} \equiv M_{QQ'}v_{QQ'} = M_{QQ'} \left( 1 - \frac{m_{bareQQ'}^2}{M_{QQ'}^2} \right)^{1/2}, \tag{4}
\]
which agrees with the mass formula predicted by heavy quark symmetry\cite{19}. As the experimental data are still lacking except for the doubly charmed baryon $\Xi_{cc}^{++}$, we adopt the masses(Table II) of DH baryons computed by Ref. \cite{1}, which successfully predicts the ground-state mass $M(\Xi_{cc}, 1/2^+) = 3620$ MeV, close to the measured data $3621.55$ MeV for the doubly charmed baryon $\Xi_{cc}^{++}$\cite{3, 5}.

| State       | $J^P$ | Baryons | Content | Mass  | $\bar{M}_{l=0}$ |
|-------------|-------|---------|---------|-------|-----------------|
| $1^2S_{1/2}$ | $\frac{1}{2}^+$ | $\Xi_{cc}$ | $n\{cc\}$ | 3.620 | 3.6913          |
| $1^4S_{1/2}$ | $\frac{3}{2}^+$ | $\Xi_{cc}^{*}$ | $n\{cc\}$ | 3.727 |                   |
| $1^2S_{1/2}$ | $\frac{1}{2}^+$ | $\Xi_{bb}$ | $n\{bb\}$ | 10.202 | 10.2253         |
| $1^4S_{1/2}$ | $\frac{3}{2}^+$ | $\Xi_{bb}^{*}$ | $n\{bb\}$ | 10.237 |                   |
| $1^2S_{1/2}$ | $\frac{1}{2}^+$ | $\Xi_{bc}$ | $n\{bc\}$ | 6.933  | 6.9643          |
| $1^4S_{1/2}$ | $\frac{3}{2}^+$ | $\Xi_{bc}^{*}$ | $n\{bc\}$ | 6.980  |                   |
| $1^2S_{1/2}$ | $\frac{1}{2}^+$ | $\Xi_{bc}'$ | $n\{bc\}$ | 6.963  | 6.963           |

With the mass parameters $m_{q=n}$ ($n = u/d$) in Table I and $\bar{M}(1S1s)$ in Table II, one can solve Eq. \cite{3} to extract the heavy-diquark mass, with the results

$$M_{cc} = 2865.5 \text{ MeV}, \quad M_{bb} = 8916.7 \text{ MeV},$$

$$M_{bc} = 5891.8 \text{ MeV}, \quad M_{bc} = 5892.3 \text{ MeV},$$

where the subscript $cc(bb)$ stands for the vector diquark $\{cc\}(\{bb\})$ for short. The mean mass of the diquark $bc$ is $\bar{M}_{bc} = 5892.3$ MeV.

In Refs. \cite{18, 21}, the singly heavy baryon families (the $\Lambda_c/\Sigma_c$, the $\Xi_c/\Xi_c'$, the $\Lambda_b/\Sigma_b$ and the $\Xi_b/\Xi_b'$) were examined in the quark-diquark picture that the Regge slope $\pi a$ for them is the same approximately for identical flavor constituents within one family, independent of the light-diquark spin ($= 0$ or $1$). For two heavy baryons $Qqq'$ containing the identical (light) flavor $qq'$, the string tension $a$ is found to relies crucially on the heavy quark mass.
$M_Q (Q = c \text{ or } b)$, but has little dependence on the diquark spin of $qq'$. In the case of the DH baryon $qQQ'$ with given flavor $q$ one expects that the tension $a$ depends dominantly on the flavor combinations $QQ'$, that is, on the diquark mass $M_{QQ'}$ explicitly, in the heavy quark limit.

Let us consider first the nonstrange DH baryons $\Xi_{QQ'} = nQQ'$ and scale its tension from the $D/B (= Q\bar{n})$ mesons which share similar structure consisting of one heavy building block (diquark $QQ'$ or $Q$) and one light constituent with identical flavor $n$ (or $\bar{n}$) and color structure $(\bar{3}_c \otimes 3_c$ or $1_c)$. We assume, for simplicity, a power-law of the mass scaling between the string tensions of the hadrons $D/B$ and $\Xi_{QQ'}$:

$$a_D = \left( \frac{M_c}{M_b} \right)^P ,$$  \hspace{1cm} \text{(7)}

$$a_{\Xi_{QQ'}} = \left( \frac{M_c}{M_{QQ'}} \right)^P .$$  \hspace{1cm} \text{(8)}

Here, the values of the parameters $(a_D = a_{c\bar{n}}, a_B = a_{b\bar{n}}, M_c$ and $M_b$) are previously evaluated in Ref [18], as listed in Table I.

Putting the tensions and heavy-quark masses in Table I into Eq. (7) gives $P = 0.185$ and thereby predicts, by Eq. (8)

$$a_{\Xi_{cc}} = 0.2532 \text{ GeV}^2 ,$$  \hspace{1cm} \text{(9)}

$$a_{\Xi_{bb}} = 0.3123 \text{ GeV}^2 ,$$  \hspace{1cm} \text{(10)}

combining with Eqs. (5) and (6). Given the tensions in Eqs. (9) and Eq. (10), and the diquark masses in Eqs. (5), (6) and the light quark mass in Table I, one can use Eq. (1) to obtain the spin-averaged masses of the baryon system $\Xi_{QQ'}$ in p-wave($l = 1$). The results are

$$\begin{aligned}
\bar{M} (\Xi_{cc}, 1p) = 4.0813 \text{ GeV}, \quad \bar{M} (\Xi_{bb}, 1p) = 10.5577 \text{ GeV}, \\
\bar{M} (\Xi'_{bc}, 1p) = 7.3258 \text{ GeV}, \quad \bar{M} (\Xi_{bc}, 1p) = 7.3269 \text{ GeV},
\end{aligned}$$  \hspace{1cm} \text{(11)}

where $\Xi'_{bc} = n[bc]$ and $\Xi_{bc} = n\{bc\}$ stand for the bottom-charmed baryons with diquark spin 0 and 1, respectively.

The same procedure applies to the strange DH baryons, the $\Omega_{QQ'} = sQQ'$ and the associated $D_s/B_s (= c\bar{s}/b\bar{s})$ mesons, for which the mass scaling, corresponding to Eqs. (7)
and (8), has the same form

$$\frac{a_{Ds}}{a_{Bs}} = \left( \frac{M_c}{M_b} \right)^{P_{12}}$$

and

$$\frac{a_{Ds}}{a_{\Omega_{QQ'}}} = \left( \frac{M_c}{M_{QQ'}} \right)^{P_{13}}.$$  

Putting the tensions for the strange heavy mesons and the heavy-quark masses in Table I to Eq. (12) gives $P = 0.202$. One can use this $P$ value and Eq. (13) and the heavy-diquark masses in Eqs. (5) and (6) to predict

$$a_{\Omega_{cc}} = 0.2860 \text{ GeV}^2, a_{\Omega_{bb}} = 0.3596 \text{ GeV}^2,$$

$$a_{\Omega_{bc}} = 0.3308 \text{ GeV}^2 = a_{\Omega_{bc}^*},$$

where $\Omega_{bc}$ and $\Omega_{bc}^*$ stand for the DH $\Omega$ baryons with $bc$-diquark spin 0 and 1, respectively.

Given the tensions (including $a_{\Xi_{QQ'}} = a_{\Xi_{QQ'}}$) in Eqs. (14) and (15), and the heavy-diquark masses in Eqs. (5) and (6) as well as the strange quark mass $m_s = 0.328$ GeV in Table I, one can use Eq. (11) to find the mean (spin-averaged) masses of the baryon system $\Omega_{QQ'}$ in p-waves ($l = 1$). The results are

$$\begin{align*}
\bar{M} (\Omega_{cc}, 1p) &= 4.1895 \text{ GeV}, \quad \bar{M} (\Omega_{bb}, 1p) = 10.6795 \text{ GeV}, \\
\bar{M} (\Omega_{bc}', 1p) &= 7.4430 \text{ GeV}, \quad \bar{M} (\Omega_{bc}, 1p) = 7.4441 \text{ GeV},
\end{align*}$$

where $\Omega_{bc}' = s[bc]$ stands for the strange baryons $\Omega_{bc}$ with scalar diquark $[bc]$ and $\Omega_{bc} = s\{bc\}$ for the $\Omega_{bc}$ with axial vector diquark $\{bc\}$.

## III. SPIN-DEPENDENT MASS IN $jj$ COUPLING

In heavy diquark quark picture, two heavy quarks in a $\Xi_{QQ'}$ or $\Omega_{QQ'}$ system form a S-wave color anti-triplet ($\mathbf{3}_c$) diquark $(QQ')$, having spin zero ($S_{QQ'} = 0$) when $QQ' = bc$, or spin one ($S_{QQ'} = 1$) when $QQ' = cc, bc$ and $bb$. When $Q = Q'$ the diquark $QQ$ must have spin one due to full antisymmetry under exchange of two quarks. The spin $S_{QQ'}$ can couple with the spin $S_q = 1/2$ of the light quark $q (= u, d$ and $s$) to form a total spin $S_{tot} = 1 \pm 1/2 = 1/2, 3/2$ if $S_{QQ'} = 1$ or $S_{tot} = 1/2$ if $S_{QQ'} = 0$.

We assume the heavy diquark $QQ'$ in DH baryon $qQQ'$ to be in internal ground (S-wave) state and consider relative p-wave excitations of the light quark $q$ with respect to $QQ'$.
(l = 1). Here, the relative orbital angular momentum l corresponds to the angular excitation of the \( \lambda \)-mode in the quark model. In the scheme of LS coupling, coupling \( S_{\text{tot}} = 1/2 \) to \( l = 1 \) gives the states with the total angular momentums \( J = 1/2, 3/2 \), while coupling \( S_{\text{tot}} = 3/2 \) to \( l = 1 \) leads to the states with \( J = 1/2, 3/2 \) and 5/2. Normally, one uses the LS basis \( 2^{S_{\text{tot}}+1}P_J = \{ 2^{1/2}P_{1/2}, 2^{3/2}P_{3/2}, 4^{1/2}P_{1/2}, 4^{3/2}P_{3/2}, 4^{5/2} \} \) to label these multiplets in P-wave, which correspond to five allowed odd-parity states with \( J = 1/2, 1/2', 3/2, 3/2' \) and 5/2.

For mass splitting \( \Delta M = \langle H^{SD} \rangle \) due to spin interaction between heavy diquark \( QQ' \) with spin \( S_{QQ'} \) and light quark \( q \) with spin \( S_q \), we consider the spin-dependent Hamiltonian \( H^{SD} \) [43]

\[
H^{SD} = a_1 \mathbf{l} \cdot \mathbf{S}_q + a_2 \mathbf{l} \cdot \mathbf{S}_{QQ'} + b S_{12} + c \mathbf{S}_q \cdot \mathbf{S}_{QQ'},
\]

\[
S_{12} = 3 \mathbf{S}_q \cdot \mathbf{i} \mathbf{S}_{QQ'} \cdot \mathbf{r} - \mathbf{S}_q \cdot \mathbf{S}_{QQ'},
\]

where the first two terms are spin-orbit forces, the third is a tensor force, and the last describes hyperfine splitting. In the LS basis, the two \( J = 1/2 \) states and the two \( J = 3/2 \) states are unmixed unless \( a_1 = a_2 \). Otherwise they are the respective eigenstates of a 2 \( \times \) 2 matrices \( \mathcal{M}_J \) representing \( H^{SD} \) for \( J = 1/2 \) and 3/2. In the basis \( [2^1P_{1/2}, 4^1P_{3/2}] \), they can be given by the matrix [24, 26] (see Appendix B also)

\[
\mathcal{M}_{J=1/2} = \begin{bmatrix}
\frac{1}{3} (a_1 - 4a_2) & \frac{\sqrt{3}}{3} (a_1 - a_2) \\
\frac{\sqrt{3}}{3} (a_1 - a_2) & -\frac{5}{3} (a_2 + \frac{1}{2}a_1)
\end{bmatrix} + b \begin{bmatrix}
0 & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & -1
\end{bmatrix} + c \begin{bmatrix}
-1 & 0 \\
0 & \frac{1}{2}
\end{bmatrix},
\]

in the \( J = 1/2 \) subspace,

\[
\mathcal{M}_{J=3/2} = \begin{bmatrix}
-\frac{2}{3} (\frac{3}{4}a_1 - a_2) & \frac{\sqrt{5}}{3} (a_1 - a_2) \\
\frac{\sqrt{5}}{3} (a_1 - a_2) & -\frac{2}{3} (\frac{1}{2}a_1 + a_2)
\end{bmatrix} + b \begin{bmatrix}
0 & -\frac{\sqrt{5}}{10} \\
-\frac{\sqrt{5}}{10} & 4/5
\end{bmatrix} + c \begin{bmatrix}
-1 & 0 \\
0 & \frac{1}{2}
\end{bmatrix},
\]

(18, 19)
in the $J = 3/2$ subspace, and by

$$\mathcal{M}_{J=5/2} = \frac{1}{2} a_1 + a_2 - \frac{b}{5} + c. \tag{20}$$

for the $J = 5/2$. One can verify that the spin-weight sum of these matrixes over $J = 1/2, 3/2$ and $5/2$ is zero:

$$\sum_J (2J + 1) \mathcal{M}_J = 0, \tag{21}$$

as it should be for the spin interaction $H^{SD}$.

As diquark mass $M_{QQ'}$ is much larger than the light quark mass, all terms except for the first ($= a_1 l \cdot S_q$) in Eq. (17) behave as $1/M_{QQ'}$ and are then suppressed comparing with the first($a_1$-) term. Due to heavy quark spin symmetry($S_{QQ'}$ conserved), the total angular momentum of the light quark $j = l + S_q = J - S_{QQ'}$ is conserved and forms an set of the conserved operators $\{J, j\}$, together with the total angular momentum $J$ of the DH hadrons. We use then the $jj$ coupling scheme to label the spin multiplets of the DH baryons, denoted by the basis $|J, j\rangle$ (Appendix B), in which the diquark $QQ'$ is infinitely heavy and $l \cdot S_q$ becomes diagonal. As such, the formula for mass splitting $\Delta M$ can be obtained by diagonalizing $l \cdot S_q$ and treating other interactions in Eq. (17) proportional to $a_2, b$ and $c$ perturbatively.

The eigenvalues (two diagonal elements) of $l \cdot S_q$ can be obtained to be

$$\langle l \cdot S_q \rangle = \frac{1}{2} [j(j + 1) - l(l + 1) - S_q (S_q + 1)] = -1(j = 1/2) \text{ or } 1/2(j = 3/2). \tag{22}$$

Computation of the diagonal mass splitting $\Delta M(J, j) = \langle J, j \mid H^{SD} \mid J, j \rangle$ in the $jj$ coupling includes the three steps: Firstly, one solves the eigenfunctions (the LS bases) $|S_{QQ'}3, S_q3, l3\rangle$ of the $l \cdot S_q$ for its eigenvalues $j = 1/2$ and $3/2$ in the LS coupling; Secondly, one transforms the obtained LS bases into $|J, j\rangle$ in the $jj$ coupling (Appendix B); Finally, one evaluates the diagonal elements of the spin-interaction in Eq. (17) in the new bases $|J, j\rangle$. The results for $\Delta M(J, j)$ are listed in Table III.

Given Table III, one can use the lowest perturbation theory to find the mass splitting $\Delta M(J, j)$ of the DH baryons in p-wave,

$$\Delta M(1/2, 1/2) = -a_1 - \frac{4}{3} a_2 - \frac{4}{3} b + \frac{1}{3} c. \tag{23}$$
TABLE III: The matrix elements of the mass splitting operators in the p-wave heavy doubly
baryons states in the $jj$ coupling.

| $(J,j)$       | $\langle 1 \cdot S_{QQ'} \rangle$ | $\langle S_{12} \rangle$ | $\langle S_q \cdot S_{QQ'} \rangle$ |
|--------------|-----------------------------------|--------------------------|-----------------------------------|
| $(1/2,1/2)$  | $-4/3$                            | $-4/3$                   | $1/3$                             |
| $(1/2,3/2)$  | $-5/3$                            | $1/3$                    | $-5/6$                            |
| $(3/2,1/2)$  | $2/3$                             | $2/3$                    | $-5/6$                            |
| $(3/2,3/2)$  | $-2/3$                            | $2/15$                   | $-1/3$                            |
| $(5/2,3/2)$  | $1$                               | $-1/5$                   | $1/2$                             |

\[
\Delta M(1/2,3/2) = \frac{1}{2}a_1 - \frac{5}{3}a_2 + \frac{1}{3}b - \frac{5}{6}c,
\]

(24)

\[
\Delta M(3/2,1/2) = -a_1 + \frac{2}{3}a_2 + \frac{2}{3}b - \frac{1}{6}c,
\]

(25)

\[
\Delta M(3/2,3/2) = \frac{1}{2}a_1 - \frac{2}{3}a_2 + \frac{2}{15}b - \frac{1}{3}c,
\]

(26)

\[
\Delta M(5/2,3/2) = \frac{1}{2}a_1 + a_2 - \frac{1}{5}b + \frac{1}{2}c,
\]

(27)

which express the DH baryon mass splitting in terms of four parameters $(a_1, a_2, b, c)$. The
mass formula for the $1S1p$ states of DH baryons can then be given by $M(J,j) = \bar{M}(1S1p) + 
\Delta M(J,j)$, with $\bar{M}(1S1p)$ the spin-independent masses given in Eqs. (11) and (16) in section
II.

IV. MASSES OF DH BARYON WITH S-WAVE DIQUARK

To evaluate spin coupling parameters $(a_1, a_2, b, c)$ in Eq. (23) through Eq. (27), we utilize
the relations of mass scaling[17, 18], which apply successfully to the singly-heavy hadrons
associated with the singly-heavy mesons $D_s$ and singly-heavy baryons. For this, we list the
matched values of the spin couplings for the $D_s$ in Refs. [17, 18] and that of the singly heavy
baryons (the $\Sigma_Q$, the $\Xi'_Q$ and the $\Omega_Q$, with $Q = c, b$) in Refs. [21] in Table IV.

Between heavy baryons $Qqq'$ and heavy mesons we choose the $D_s = c \bar{s}$ typically, as in
Refs. [17, 18], the mass scaling relation for the spin couplings is given in Ref. [18] (see Eqs.
TABLE IV: Spin-coupling Parameters (in MeV) of singly heavy meson $D_s$ and the singly heavy baryons $\Sigma_Q$, $\Xi'_Q$ and $\Omega_Q(Q = c, b)$.

| Hadrons | $a_1$  | $a_2$  | $b$    |
|---------|--------|--------|--------|
| $D_s$   | 89.36  | 40.7   | 65.6   |
| $\Sigma_c$ | 39.96(39.34) | 21.75(26.82) | 20.70(20.05) |
| $\Sigma_b$ | 12.99(12.65) | 6.42(8.62) | 6.45(6.45) |
| $\Xi'_c$ | 32.89(33.62) | 20.16(25.35) | 16.50(17.93) |
| $\Xi'_b$ | 9.37(10.81) | 6.29(8.15) | 5.76(5.76) |
| $\Omega_c$ | 26.96(29.59) | 25.76(24.11) | 13.51(16.31) |
| $\Omega_b$ | 8.98(9.51) | 4.11(7.75) | 7.61(5.24) |

(22-24)), and can be expressed as

\[
I_{qq'} \cdot S_{qq'} : a_1 [Qqq'] = \frac{M_c}{M_Q} \cdot \frac{m_s}{m_{qq'}} \cdot a_1 (D_s),
\]

(28)

\[
I_{qq'} \cdot S_Q : a_2 [Qqq'] = \frac{M_c}{M_Q} \cdot \frac{1}{1 + m_{qq'}/M_c} \cdot a_2 (D_s),
\]

(29)

\[
S_{12} : b [Qqq'] = \frac{M_c}{M_Q} \cdot \frac{1}{1 + m_{qq'}/m_s} \cdot b (D_s),
\]

(30)

where $I_{qq'}$ and $S_{qq'}$ denote the orbital angular momentum of light diquark $qq'$ relative to the heavy quark $Q$ and the spin of the light diquark. $m_{qq'}$ is the mass of the diquark $qq'$ and $M_Q$ is heavy-quark mass in the singly heavy hadrons, and the factor of the mass ratio $M_c/M_Q$ incorporates to account for the scaling $\sim 1/M_Q$. In Eq. (30) for the tensor coupling $b$, the extra recoil factor $1/(1 + m_{qq'}/m_s)$ is used to take into account the correction due to the comparable heaviness between the diquark $qq'$ and the strange quark. Note that a similar (recoil here) factor, $1/(1 + m_{qq'}/M_c)$, entering Eq. (29), has been confirmed for the charmed and bottom baryons in P-wave and D-wave [18]. For instance, Eq. (29) in Ref. [18] (Eq. (60)) well reproduces the measured masses 6146.2 MeV and 6152.5 MeV of the $\Lambda_b(6146)$ and the $\Lambda_b(6152)$ observed by LHCb [22], respectively. Similar verifications were demonstrated in
Ref. [21] for the excited $\Omega_c$ discovered by LHCb [23], for which the $ss$-diquark is comparable with the charm quark in heaviness.

TABLE V: Effective masses (in GeV) of light diquarks in the singly heavy baryons shown. Data are from the Ref. [18] and Ref. [21].

| Hadrons | $\Sigma_c$ | $\Sigma_b$ | $\Xi'_c$ | $\Xi'_b$ | $\Omega_c$ | $\Omega_b$ |
|---------|------------|------------|----------|----------|------------|------------|
| $m_{qq}$ | 0.745     | 0.745      | 0.872    | 0.872    | 0.991      | 0.991      |

Putting the parameters of the quarks in Tables I and the light diquarks in Table V to Eqs. (28)-(30), one can estimate the parameters $a_1$, $a_2$ and $b$ for the heavy baryons $\Sigma_Q$, $\Xi_Q'$ and $\Omega_c$. We list the results in Table IV marked with parentheses adjacent to the values(data without parentheses) of the three parameters that matched in Ref. [21] to the measured data for comparison. Evidently, the mismatch shown is small: $\Delta a_1 \leq 2.63$ MeV, $\Delta a_2 \leq 5.19$ MeV and $\Delta b \leq 2.80$ MeV, and agreement between the estimates via two scalings is notable.

In the case of the DH baryons with S-wave diquark, whose quark structure of the $q - QQ'$ resembles heavy meson $D_s$ with quark structure $s - \bar{c}$, one expects that similar relations of mass scaling apply as well. Replacing the heavy-quark mass $M_Q$ in Eqs. (28)-(30) by the diquark mass $M_{QQ'}$ and the light diquark mass $m_{qq'}$ by the quark mass $m_q$, one obtains

$$a_1[QQ'q] = \frac{M_c}{M_{QQ'}} \cdot \frac{m_s}{m_q} \cdot a_1(D_s),$$

$$a_2[QQ'q] = \frac{M_c}{M_{QQ'}} \cdot \frac{1}{1 + m_q/M_c} \cdot a_2(D_s),$$

$$b[QQ'q] = \frac{M_c}{M_{QQ'}} \cdot \frac{1}{1 + m_q/m_s} \cdot b(D_s).$$

Since the hyperfine parameter $c \propto 1/(M_{QQ'}m_q)$ also scales as hadron wavefunction near the origin, it should be small in p-wave, we assume it to obey a relation of mass scaling relative to the singly charmed baryon $\Omega_c = css$ as below:

$$c[QQ'q] = \left( \frac{M_c}{M_{QQ'}} \right) \left( \frac{m_{ss}}{m_q} \right) c(\Omega_c).$$

For the excited baryons $\Omega_c$, there exists five states discovered by LHCb [23], with the masses 3000.4 MeV, 3050.2 MeV, 3065.6 MeV, 3095.2 MeV and 3119.2 MeV. Assigning these excited
baryons to be in p-wave, one can extract $c(\Omega_c)$ by matching the $jj$ mass formula to the five measured masses, with the result $^{21, 24}$

$$c(\Omega_c) = 4.07 \text{ MeV}. \quad (35)$$

The spin-averaged mass of the five excited baryons $\Omega_c$ is

$$\bar{M}(\Omega_c)_{1P} = \frac{1}{18} [(2J + 1)M(\Omega_c; J)] = 3079.94 \text{ MeV}, \quad (36)$$

where the five observed masses $M(\Omega_c; J)$, from the lightest to the heaviest, are assigned to have the increasing quantum number $J^{24}$. Further, given the observed ground-state masses, $M(\Omega_c, 1/2^+) = 2695.2 \pm 1.7 \text{ MeV}$ and $M(\Omega_c, 3/2^+) = 2765.9 \pm 2.0 \text{ MeV}^{25}$, one finds its spin-averaged mass in the s-wave to be

$$\bar{M}(\Omega_c)_{1S} = \frac{1}{6} \left(2M(1/2^+) + 4M(3/2^+)\right) = 2742.3 \pm 1.9 \text{ MeV}. \quad (37)$$

Putting the values in Eq. (37) and Eq. (36) into the Regge relation for the $\Omega_c = c(ss)$:

$$(\bar{M} - M_c)^2 = \pi a l + (m_{ss} + M_c - 1.275^2/M_c)^2,$$

where $M_c = 1.44 \text{ GeV}$ and $m_{ss}$ is the $ss$-diquark mass, one can solve the trajectory slope $\pi a$ and $m_{ss}$, with the result $^{21}$

$$a = 0.316 \text{ GeV}^2, \; m_{ss} = 991 \text{ MeV}. \quad (38)$$

Now, one can employ Eqs. (31)-(34) to estimate the four parameters of spin couplings of the DH baryons using the parameters in Table I and the diquark masses in Eqs.(5), (6) and that in Eq. (38). The results are listed collectively in Table VI. As seen in Table, the parameter $a_2$ are smaller(three times roughly) but comparable to $a_1$. The magnitudes of $b$ is as large as $a_2$ roughly while $c$ is the smallest. This agrees qualitatively with the parameter hierarchy of the spin-couplings $^{18, 21}$ implied by heavy quark symmetry.

For the bottom charmed baryons $\Xi_{bc}$ and $\Omega_{bc}$ with zero diquark spin $S_{bc} = 0$, the spin-interaction Hamiltonian is simple,

$$H^{SD} = a_1 \mathbf{l} \cdot \mathbf{S_q}, \quad (39)$$

which yields, by Eq. (22), the mass splitting ($J = 1/2, 3/2$),

$$\Delta M(\Xi_{bc}, \Omega_{bc}) = a_1(\Xi_{bc}, \Omega_{bc}) \begin{bmatrix} -1 & 0 \\ 0 & 1/2 \end{bmatrix}. \quad (40)$$
TABLE VI: Spin-coupling parameters (in MeV) in the spin interaction (17) of the DH baryons $\Xi_{QQ'}$ and $\Omega_{QQ'} (Q, Q' = c, b)$.

| DH baryons | $a_1$ | $a_2$ | $b$ | $c$ |
|------------|-------|-------|-----|-----|
| $\Xi_{cc}$ | 64.05 | 17.64 | 19.38 | 8.81 |
| $\Xi_{bb}$ | 20.58 | 5.67  | 6.23 | 2.83 |
| $\Xi_{bc}$ | 31.14 | 8.58  | 9.42 | 4.29 |
| $\Omega_{cc}$ | 44.91 | 16.66 | 16.48 | 6.18 |
| $\Omega_{bb}$ | 14.43 | 5.35  | 5.30 | 1.99 |
| $\Omega_{bc}$ | 21.84 | 8.10  | 8.01 | 3.01 |

With $a_1(D_s) = 89.4$ MeV\(^{[17]}\), one can employ the relations of mass scaling (31) relative to the $D_s$ to give

$$a_1(\Xi_{bc}) = \left( \frac{1.440 \text{ GeV}}{5.8918 \text{ GeV}} \right) \left( \frac{0.328 \text{ GeV}}{0.230 \text{ GeV}} \right) a_1(D_s) = 31.14 \text{ MeV},$$

$$a_1(\Omega_{bc}) = \left( \frac{1.440 \text{ GeV}}{5.8918 \text{ GeV}} \right) a_1(D_s) = 21.84 \text{ MeV},$$

(41)

where $M_{[bc]} = 5.8918 \text{ GeV}$ and other mass parameters are from Table I. Putting above parameters into Eq. (40), and adding $\bar{M}_L(\Xi_{bc},\Omega_{bc})$ in Eqs. (11) and (16), one obtains the P-wave masses $\bar{M}_L(\Xi_{bc},\Omega_{bc}) + \Delta M(\Xi_{bc},\Omega_{bc})$ for the baryons $\Xi_{bc}$ and $\Omega_{bc}$ (with spin-0 diquark $bc$ and $J^P = 1/2^-$ and $3/2^-$) to be

$$M(\Xi_{bc},1/2^-) = 7294.7 \text{ MeV}, \quad M(\Xi_{bc},3/2^-) = 7341.4 \text{ MeV},$$

$$M(\Omega_{bc},1/2^-) = 7421.2 \text{ MeV}, \quad M(\Omega_{bc},3/2^-) = 7453.9 \text{ MeV}.$$  

(42) (43)

We list the masses of the DH baryons in their 1S1p-wave in Tables VIII to XIV, with comparision with other predictions. They are also shown in FIGs 1-8 for the respective states of the DH baryons. Our computation suggests that the 1S1p states of the DH baryons have increasing masses with baryon spin $J$ rising from the lowest $1/2^-$ to highest $5/2^-$. 

14
TABLE VII: Mean (spin-averaged) masses (in GeV) of the excited DH baryons $\Xi_{QQ'}$ and $\Omega_{QQ'}$ predicted by Eq. (44).

| Baryons $\bar{M}(1D)$ $\bar{M}(1F)$ $\bar{M}(2S)$ $\bar{M}(2P)$ $\bar{M}(2D)$ $\bar{M}(2F)$ |
|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $\Xi_{cc}$ 4.373 4.617 4.255 4.516 4.742 4.943 |
| $\Xi_{bb}$ 10.835 11.076 10.722 10.976 11.202 11.408 |
| $\Xi_{bc}$ 7.615 7.861 7.498 7.759 7.989 8.195 |
| $\Omega_{cc}$ 4.494 4.749 4.370 4.644 4.881 5.093 |
| $\Omega_{bb}$ 10.975 11.234 10.854 11.126 11.369 11.589 |
| $\Omega_{bc}$ 7.749 8.010 7.625 7.902 8.146 8.366 |

FIG. 1: Mass spectrum of $\Xi_{cc}$ baryons (in MeV). The horizontal dashed line shows the $\Lambda_cD$ threshold.

V. DH BARYON MASSES IN 1S AND 2S WAVES

To extend the analysis of the orbitally excited baryons to radially excited states, one needs a Regge relation for the radial excitations. This type of Regge relation is proposed in
TABLE VIII: Mass spectrum of $\Xi_{cc}$ baryons (in MeV).

| State | Our | [1] | [31] | [32] | [33] | [34] | [35] | [36] | [38] |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $(NLn_l) J^P$ |     |     |     |     |     |     |     |     |     |
| $(1S1p)1/2^-$ | 3970.3 | 4053 | 3998 | 3927 | – | 3947 | 3861 | 4081 | 3910 |
| $(1S1p)1/2'^-$ | 4082.5 | 4136 | 3985 | 4052 | – | 4135 | 3871 | 4073 | 4074 |
| $(1S1p)3/2^-$ | 4039.9 | 4101 | 4014 | 4039 | – | 3994 | 3842 | 4077 | 3921 |
| $(1S1p)3/2'^-$ | 4100.6 | 4196 | 4025 | 4034 | – | 4137 | 3851 | 4079 | 4078 |
| $(1S1p)5/2^-$ | 4130.9 | 4155 | 4050 | 4047 | – | 4163 | 3888 | 4089 | 4092 |
| $(1S2s)1/2^+$ | 4216.3 | – | 4172 | – | 4183 | 4159 | 3920 | 4311 | – |
| $(1S2s)3/2^+$ | 4275.1 | – | 4193 | – | 4282 | 4131 | 3983 | 4368 | – |
| $(2S1s)1/2^+$ | 3926.6 | 3910 | 4004 | 3812 | – | 4079 | – | 4030 | 4029 |
| $(2S1s)3/2^+$ | 4022.5 | 4027 | 4036 | 3944 | – | 4114 | – | 4078 | 4042 |
| $(1P1s)1/2^-$ | 3956.0 | 3838 | 3873 | 3702 | – | 4149 | – | 3911 | – |
| $(1P1s)3/2^-$ | 4051.1 | 3959 | 3916 | 3834 | – | 4159 | – | 3917 | – |

Ref. [28] previously in consideration that the trajectory slope ratio between the radial and angular excitations is $\pi : 2$ for the heavy mesons $\bar{q}Q$ in the heavy quark limit. Extending this ratio to the heavy-diquark-quark picture of the DH system $q - QQ'$, one can have a unified Regge relation for the radially and angular excitations (view diquark $QQ'$ as one end and light quark $q$ at the other of a QCD string). The result follows via replacing heavy quark $Q$ there by the diquark $QQ'$, the light antiquark $\bar{q}$ by $q$, and $\pi al$ in Eq. (1) by $\pi a (l + \frac{\pi}{2} n)$,
TABLE IX: Mass spectrum of Ξ_{bb} baryons (in MeV).

| State                  | Our       | [1] | [31] | [32] | [33] | [34] | [35] | [36] | [38] |
|------------------------|-----------|-----|------|------|------|------|------|------|------|
| \[(NLn_q) J^P\]         |           |     |      |      |      |      |      |      |      |
| \[(1S1p)\]/2^-         | 10523.1   | 10632| 10525| 10541| –    | 10476| 10511| 10694| 10493|
| \[(1S1p)\]/2'^-        | 10559.0   | 10675| 10504| 10578| –    | –    | 10514| 10691| –    |
| \[(1S1p)\]/2^-         | 10545.3   | 10647| 10526| 10567| –    | 10476| 10506| 10691| 10495|
| \[(1S1p)\]/2'^-        | 10564.8   | 10694| 10528| 10581| –    | –    | 10509| 10692| –    |
| \[(1S1p)\]/5^-         | 10574.5   | 10661| 10547| 10580| –    | 10759| 10518| 10695| 10713|
| \[(1S2s)\]/2^+         | 10707.9   | 10832| 10662| –    | 10751| 10612| 10609| 10940| –    |
| \[(1S2s)\]/3^+         | 10726.8   | 10860| 10675| –    | 10770| 10593| 10617| 10972| –    |
| \[(2S1s)\]/2^+         | 10473.4   | 10441| 10464| 10373| –    | 10571| –    | 10551| 10571|
| \[(2S1s)\]/3^+         | 10506.4   | 10482| 10480| 10413| –    | 10592| –    | 10574| 10592|
| \[(1P1s)\]/2^-         | 10523.5   | 10368| 10364| 10310| –    | 10740| –    | 10470| –    |
| \[(1P1s)\]/3^-         | 10556.4   | 10408| 10387| 10343| –    | 10742| –    | 10470| –    |

Giving rise to [28]

\[
(M - M_{QQ'})^2 = \pi a \left(l + \frac{\pi}{2} n\right) \\
+ \left(m_q + M_{QQ'} - \frac{m_{bare \, QQ'}}{M_{QQ'}}\right)^2.
\]

Applying the parameters in Eqs. (2), (5), (6), (9), (10), (14), (15) and Table I to Eq. (44), one can find the mean(spin-averaged) masses of all doubly heavy baryons (Ξ_{cc}, Ξ_{bb}, Ξ_{bc}, Ω_{cc}, Ω_{bb}, Ω_{bc}) in the radially and angularly excited states. The results are collectively in Table VII.
For the s-waves of the DH baryons, the spin-interaction Hamiltonian is simply

\[ H^{SD} = c \mathbf{S}_{QQ'} \cdot \mathbf{S}_q, \]  

(45)

in which \( \mathbf{S}_{QQ'} \cdot \mathbf{S}_q \) has the eigenvalues \( \{-1, 1/2\} \) if \( \mathbf{S}_{QQ'} = 1(l = 0) \). The mass formula for the systems \( q\{QQ'\} \) becomes \((J = 1/2, 3/2)\),

\[ M(q\{QQ'\})_{nS} = \bar{M}(q\{QQ'\})_{nS} + c(q\{QQ'\}) \begin{pmatrix} -1 & 0 \\ 0 & 1/2 \end{pmatrix}. \]  

(46)

For the s-waves of the DH system \( q[QQ'] \) with zero diquark spin \( \mathbf{S}_{QQ'} = 0(l = 0) \), mass
FIG. 2: Mass spectrum of $\Xi_{bb}$ baryons (in MeV).

FIG. 3: Mass spectrum of $\Xi_{bc}$ baryons (in MeV). The horizontal dashed line shows the $\Lambda_b D$ threshold.

of the system is simply $\bar{M}(q[bc])_{nS}$ without splitting if there is no internal diquark orbital excitations.

In 2s wave ($n = 1$), we use the mass scaling relative to the $D$ meson in 2s wave (the $D_0(2550)^0$ with mass 2549 ± 19 MeV and the $D_1^+(2600)^\pm$ with mass 2627 ± 10 MeV [25]) to
find
\[ c(\Xi_{bb}(bbu)) = \left( \frac{M_c}{M_{bb}} \right) \left[ M(D(2s), 1^-) - M(D(2s), 0^-) \right] = 12.6 \text{ MeV}, \]
\[ c(\Omega_{bb}(bbs)) = \left( \frac{M_c}{M_{bb}} \right) \left( \frac{m_u}{m_s} \right) \left[ M(D(2s), 1^-) - M(D(2S), 0^-) \right] = 8.8 \text{ MeV}. \]

Further application of the scaling relations between the DH baryons,
\[ c(\Xi_{cc})_{2s} = c(\Xi_{bb}) \left( \frac{M_{bb}}{M_{cc}} \right), \quad c(\Xi_{bc})_{2s} = c(\Xi_{bb}) \left( \frac{M_{bb}}{M_{bc}} \right), \]
\[ c(\Omega_{cc})_{2s} = c(\Omega_{bb}) \left( \frac{M_{bb}}{M_{cc}} \right), \quad c(\Omega_{bc})_{2s} = c(\Omega_{bb}) \left( \frac{M_{bb}}{M_{bc}} \right), \]
leads to the following parameter \( c \) for the 2s-wave baryons \( \Xi_{cc}, \Xi_{bc}, \Omega_{cc} \) and \( \Omega_{cc} \),
\[ c(\Xi_{cc})_{2s} = 39.2 \text{ MeV}, \quad c(\Xi_{bc})_{2S} = 19.1 \text{ MeV}, \]
\[ c(\Omega_{cc})_{2s} = 27.4 \text{ MeV}, \quad c(\Omega_{bc})_{2s} = 13.3 \text{ MeV}. \]

With the mean masses of the 2s states in Table VII, one can employ Eq. (46) to compute the 2s-wave (i.e., the 1S2s-wave) masses of the DH baryons. The results are listed in Table VIII through Table XIV, with comparison with other calculations, and shown in FIGs 1-8, respectively.
It is of interest to apply the same strategy to the ground states to see if one can reproduce the masses in Table II. For the 1s wave Ξ_{cc} firstly, the \( c \) value can be scaled to that of the ground state \( D \) mesons with mass difference \( m(D^+(2010)^+) - m(D^+) = 140.6 \text{ MeV} \) via mass scaling:

\[
c(\Xi_{cc})_{1s} = c(D)_{1s} \left( \frac{M_c}{M_{cc}} \right) = (140.6 \text{ MeV}) \left( \frac{1.44}{2.8655} \right) = 70.7 \text{ MeV},
\]

For the \( \Omega_{cc} \), the relevant meson for scaling is the ground-state \( D \) mesons with mass difference \( m(D_{s}^{±}) - m(D_{s}^{±}) = 143.8 \text{ MeV} \), and the \( c \) value is then

\[
c(\Omega_{cc})_{1s} = c(D_{s})_{1s} \left( \frac{M_c}{M_{cc}} \right) = (143.8 \text{ MeV}) \left( \frac{1.44}{2.8655} \right) = 72.3 \text{ MeV}.
\]

Using \( \tilde{M}(qQQ')_{1s} = M_{QQ'} + m_q + M_{QQ'} v_{QQ'}^2 \) in Eq. \( \text{(46)} \) with \( v_{QQ'}^2 = 1 - m_{\text{bare}QQ'}/M_{QQ'}^2 \) and putting the above \( c \) values to Eq. \( \text{(46)} \) gives

\[
(1/2, 3/2)^+ : M(\Xi_{cc})_{1s} = 3691.7 + 70.66 \{-1, 1/2\} = \{3620.8, 3726.8\} \text{ MeV},
\]

\[
(1/2, 3/2)^+ : M(\Omega_{cc})_{1s} = 3789.5 + 72.3 \{-1, 1/2\} = \{3717.2, 3825.7\} \text{ MeV},
\]

where the mean mass in 1s wave \( (v_{cc}^2 = 0.208) \)

\[
\tilde{M}(q\{cc\})_{1s} = M_{cc} + m_{n,s} + M_{cc} v_{cc}^2 = 3691.5 \text{ or } 3789.5 \text{ MeV},
\]

have been used for \( q = n \) or \( s \), respectively, and the two numbers in curly braces correspond to the respective states \( J^P = 1/2^+ \) and \( 3/2^+ \).

Next, similar calculation for the 1s wave \( \Xi_{bb} \) and \( \Omega_{bb} \) gives

\[
c(\Xi_{bb})_{1s} = c(D)_{1s} \left( \frac{M_c}{M_{bb}} \right) = (140.6 \text{ MeV}) \left( \frac{1.44}{8.9167} \right) = 22.7 \text{ MeV},
\]

\[
c(\Omega_{bb})_{1s} = c(D_{s})_{1s} \left( \frac{M_c}{M_{bb}} \right) = (143.8 \text{ MeV}) \left( \frac{1.44}{8.9167} \right) = 23.2 \text{ MeV},
\]

and the ground state masses (with \( v_{bb}^2 = 0.121 \))

\[
(1/2, 3/2)^+ : M(\Xi_{bb})_{1s} = 10226 + 22.7 \{-1, 1/2\} = \{10203.3, 10237.4\} \text{ MeV},
\]

\[
(1/2, 3/2)^+ : M(\Omega_{bb})_{1s} = 10324 + 23.2 \{-1, 1/2\} = \{10301.8, 10335.6\} \text{ MeV},
\]

where \( \tilde{M}(q\{bb\})_{1s} = M_{bb} + m_q + M_{bb} v_{bb}^2 = 10226.0(10324.0) \text{ MeV} \) have been used for \( q = n \)(or \( s \)).
Finally, for the DH systems $q(bc)$, there are three ground states, $\Xi_{bc} = n\{bc\}$ with $J^P = 1/2^+$ and $3/2^+$, $\Xi'_{bc} = n[bc]$ with $J^P = 1/2^+$, and three ground states $\Omega_{bc} = s\{bc\}$ with $J^P = 1/2^+$ and $3/2^+$, $\Omega'_{bc} = s[bc]$ with $J^P = 1/2^+$. Notice that $v_{bc}^2 = 0.1428(0.1429)$ for the diquark $\{bc\}([bc])$, one can estimate the mean masses for them by Eq. (3) to be

\[
\bar{M}(n\{bc\})_{1s} = M_{\{bc\}} + m_n + M_{\{bc\}}v_{bc}^2 = 6964.3 \text{ MeV},
\]
\[
\bar{M}(s\{bc\})_{1s} = M_{\{bc\}} + m_s + M_{\{bc\}}v_{bc}^2 = 7062.3 \text{ MeV},
\]
\[
\bar{M}(n[bc])_{1s} = 6963.0 \text{ MeV}, \quad \bar{M}(s[bc])_{1s} = 7061.0 \text{ MeV},
\]
and the $c$ parameter to be

\[
c(\Xi_{bc})_{1s} = c(D)_{1s} \left( \frac{M_c}{M_{\{bc\}}} \right) = (140.6 \text{ MeV}) \left( \frac{1.44}{5.9823} \right) = 33.8 \text{ MeV},
\]
\[
c(\Omega_{bc})_{1s} = c(D_s)_{1s} \left( \frac{M_c}{M_{\{bc\}}} \right) = (143.8 \text{ MeV}) \left( \frac{1.44}{5.9823} \right) = 34.6 \text{ MeV}.
\]

The ground state masses of the baryons $\Xi_{bc}$ and $\Omega_{bc}$ are then

\[
(1/2, 3/2)^+: M(\Xi_{bc})_{1s} = 6964.3 + 33.8 \{-1, 1/2\} = \{6930.5, 6981.2\} \text{ MeV},
\]
\[
(1/2, 3/2)^+: M(\Omega_{bc})_{1s} = 7062.3 + 34.6\{-1, 1/2\} = \{7027.7, 7079.6\} \text{ MeV},
\]
\[
1/2^+: M(\Xi'_{bc})_{1s} = 6963.0 \text{ MeV}, \quad M(\Omega'_{bc})_{1s} = 7061.0 \text{ MeV},
\]
with the two numbers in curly braces corresponding to the states $J^P = 1/2^+$ and $3/2^+$, respectively. Our predictions for the most ground-state masses of the DH baryons are in consistent with that by Ref [1], especially, the mass splitting of the systems $\Xi_{QQ'}$ and $\Omega_{QQ'}$ agree well with the later.

VI. MEAN MASS SHIFTS DUE TO DIQUARK EXCITATIONS

In this section, we consider excited DH baryons $qQQ' (QQ' = cc, bb, bc)$ with heavy diquark $QQ'$ in the internal 1P and 2S waves. The diquark $cc$ or $bb$ has to be in spin singlet(asymmetry) in the internal 1P wave or spin triplet in the internal 2S wave, while $bc$ can be either in the internal 1P or in 2S waves. As treated before, one can decompose the excited baryon mass as $M = \bar{M}_{N,L} + (\Delta M)_{N,L}$, where $\bar{M}_{N,L}$ is the spin-independent mass and $(\Delta M)_{N,L}$ is mass shift(splitting) due to the spin interactions in DH baryons $qQQ'$.

First of all, we consider excited heavy mesons in 1P and 2S waves in QCD string picture. For an excited heavy quarkonia $Q\bar{Q}$, a rotating-string picture [20] in which a heavy quark
TABLE XI: Mass spectrum of $\Omega_{cc}$ baryons (in MeV).

| State               | Our       | [1] | [31] | [33] | [34] | [38] | [39] | [40] | [41] |
|---------------------|-----------|-----|------|------|------|------|------|------|------|
| $(N_dL_n_q)J^P$     |           |     |      |      |      |      |      |      |      |
| $(1S_1p)\frac{1}{2}^-$ | 4102.0   | 4208| 4087 | 4086| 4046| 3989| 3965| 4050 |      |
| $(1S_1p)\frac{1}{2}^-$ | 1841.4   | 4271| 4081 | 4199| 4135| 3998| 3987| 4145 |      |
| $(1S_1p)\frac{3}{2}^-$ | 4165.2   | 4252| 4107 | 4086| 4052| 3972| 3956| 4176 |      |
| $(1S_1p)\frac{3}{2}^-$ | 4200.6   | 4325| 4114 | 4201| 4140| 3981| 3978| 4102 |      |
| $(1S_1p)\frac{5}{2}^-$ | 4271.1   | 4330| 4134 | 4220| 4152| 3958| 3926| 4134 |      |
| $(1S_2s)\frac{1}{2}^+$ | 4343.2   | 4270| 4268| 4295| 4180| 4041| 3964 |      |      |
| $(1S_2s)\frac{3}{2}^+$ | 4384.4   | 4288| 4334| 4265| 4188| 4096| 3979 |      |      |
| $(2S_1s)\frac{1}{2}^+$ | 4097.4   | 4075| 4118 | 4227 | 4263 | 4064 | 3925 | 4064 |      |
| $(2S_1s)\frac{3}{2}^+$ | 4162.0   | 4174| 4142 | 4263 | 4263 | 3978 | 3925 | 4064 |      |
| $(1P_1s)\frac{1}{2}^-$ | 4123.6   | 4002| 3986 | 4210 | 4210 | 3812 |      | 3812 |      |
| $(1P_1s)\frac{3}{2}^-$ | 4190.9   | 4102| 4020 | 4218 | 4218 | 3949 |      | 3949 |      |

at one end of the rotating string and and an antiquark at the other with relative orbital angular momentum $L$, infers a typical Regge relation($\bar{M}^{3/2} \sim L$), akin to Eq. (1),

$$\bar{M} = 2M_Q + 3 \left( \frac{T^2L^2}{4M_Q} \right)^{1/3}. \tag{49}$$

On the other hand, a semiclassical WKB analysis of a system of heavy quarkonia $Q\bar{Q}$ (in the color singlet) in a linear confining potential $T|r|$ leads to a quantization condition for its radial excitations(labeled by $N$, Appendix C):

$$[\bar{M} - 2M_Q]^{3/2} = \frac{3\pi T}{4\sqrt{M_Q}} (N + 2c_0), \tag{50}$$

with the constant $2c_0$ the quantum defect. Comparing the radial and angular slopes (linear coefficients in $N$ and $L$, respectively, which is $\pi : \sqrt{12}$) of the trajectory in RHS of Eq. (50)
TABLE XII: Mass spectrum of $\Omega_{bb}$ baryons (in MeV).

| State | Our [1] | [31] | [32] | [33] | [34] | [38] | [39] | [40] | [41] |
|-------|---------|------|------|------|------|------|------|------|------|
| $(N_dL_{nq}l) J^P$ | | | | | | | | | |
| $(1S1p)1/2^-$ | 10651.8 | 10771 | 10605 | – | 10607 | 10616 | 10646 | 10968 | 10651 |
| $(1S1p)1/2'^-$ | 10678.2 | 10804 | 10591 | – | 10796 | 10763 | 10648 | 10976 | 10700 |
| $(1S1p)3/2^-$ | 10672.1 | 10785 | 10610 | – | 10608 | 10619 | 10641 | 10957 | 10661 |
| $(1S1p)3/2'^-$ | 10683.5 | 10802 | 10611 | – | 10797 | 10765 | 10643 | 10963 | 10720 |
| $(1S1p)5/2^-$ | 10692.3 | 10798 | 10625 | – | 10808 | 10766 | 10637 | 10956 | 10670 |
| $(1S2s)1/2^+$ | 10845.2 | 10970 | 10751 | 10830 | 10744 | 10693 | 10736 | 10969 | – |
| $(1S2s)3/2^+$ | 10858.4 | 10992 | 10763 | 10839 | 10730 | 10765 | 10643 | 10976 | 10720 |
| $(2S1s)1/2^+$ | 10633.7 | 10610 | 10566 | – | 10707 | – | – | – | 10493 |
| $(2S1s)3/2^+$ | 10656.8 | 10645 | 10579 | – | 10723 | – | – | – | 10540 |
| $(1P1s)1/2^-$ | 10683.6 | 10532 | 10464 | – | 10803 | – | – | – | 10416 |
| $(1P1s)3/2^-$ | 10706.8 | 10566 | 10482 | – | 10805 | – | – | – | 10462 |

and in that of Eq. (49), one can combine two trajectories into (Appendix C)

\[
[M_{N,L} - 2M_Q - B(Q\bar{Q})_{N,L}]^{3/2} = \frac{3\sqrt{3}T}{2\sqrt{M_Q}} \left(L + \frac{\pi N}{\sqrt{12}} + 2c_0\right),
\]  

where a term of extra energy $B(Q\bar{Q})_{N,L}$, named heavy-pair binding energy, enters to represent a corrections to the picture of Regge spectra due to the effective quark masses and short-distance interquark forces when two heavy quarks ($Q$ and $\bar{Q}$) come close each other. Such a term is ignored in the semiclassical picture of massive QCD string as well as in the WKB analysis (Appendix C). For the ground state of DH hadrons, a similar binding between two heavy quarks was considered in Ref. [2, 42].

For applications in this work, we rewrite Eq. (51) and extend it to a general form in
TABLE XIII: Mass spectrum of $\Omega_{bc}$ baryons (in MeV).

| State                  | Our  | [33] | [38] | [39] | [40] |
|------------------------|------|------|------|------|------|
| $(N_d L_n q) J^P$      |      |      |      |      |      |
| $(1S1p) 1/2^-$         | 7401.8 | −   | 7335 7386 7476 |
| $(1S1p) 1/2^-$         | 7441.7 | −   | 7346 7392 7490 |
| $(1S1p) 3/2^-$         | 7432.5 | −   | 7334 7373 7470 |
| $(1S1p) 3/2^-$         | 7449.7 | −   | 7349 7379 7486 |
| $(1S1p) 5/2^-$         | 7463.0 | −   | 7362 7363 7451 |
| $(1S2s) 1/2^+$         | 7611.5 7559 | −   | 7480 7475 |
| $(1S2s) 3/2^+$         | 7631.7 7571 | −   | 7497 7485 |
| $(2S1s) 1/2^+$         | 7373.3 | −   | −    | −    | −    |
| $(2S1s) 3/2^+$         | 7407.8 | −   | −    | −    | −    |
| $(1P1s) 1/2^-$         | 7411.6 | −   | −    | −    | −    |
| $(1P1s) 3/2^-$         | 7446.1 | −   | −    | −    | −    |

which $Q\bar{Q}'$ can be $b\bar{c}$ in flavor (by heavy quark symmetry),

$$\bar{M}(Q\bar{Q})_{N,L} = M_Q + M_{Q'} + B(Q\bar{Q'})_{N,L} + 3\left[T \left(L + \pi N/\sqrt{12} + 2c_0\right)\right]^{2/3}/[2(M_Q + M_{Q'})^{1/3}].$$  \hspace{1cm} \text{(52)}$$

Setting $N = 0 = L$ in Eq. (52) gives

$$Tc_0(Q\bar{Q}') = \frac{\sqrt{2}(M_Q + M_{Q'})}{6\sqrt{3}} \left[\bar{M}(Q\bar{Q})_{1S} - M_Q - M_{Q'} - B(Q\bar{Q'})_{1S}\right]^{3/2},$$  \hspace{1cm} \text{(53)}$$

which yields numerically,

$$Tc_0(cc) = 0.0689 \text{ GeV}^2, \quad Tc_0(bb) = 0.4363 \text{ GeV}^2, \quad Tc_0(bc) = 0.2136 \text{ GeV}^2. \hspace{1cm} \text{(54)}$$

On the phenomenological side, one can estimate the binding energy between $Q$ and $\bar{Q}'$ via the following relation,

$$B(Q\bar{Q'})_{N,L} = \bar{M}(Q\bar{Q}') - \bar{M}(Q\bar{n}) - \bar{M}(Q\bar{n}') + \bar{M}(n\bar{n}').$$  \hspace{1cm} \text{(55)}$$
TABLE XIV: Mass spectrum of $\Xi_{bc}'/\Omega_{bc}'$ baryons (in MeV).

| State | $J^P$ | $\text{Our}(\Xi_{bc}')$ | $\text{Our}(\Omega_{bc}')$ |
|-------|-------|--------------------------|--------------------------|
| $(N_dLn_ql)$ | $J^P$ | $N_{bc}$ | $\bar{N}_{bc}$ | $b\bar{c}$ | $\bar{b}c$ | $b\bar{q}$ | $\bar{b}q$ | $q\bar{q}$ |
| $(1S1p)1/2^-$ | 7294.7 | 7388 | 7421.2 | - |
| $(1S1p)3/2^-$ | 7341.4 | 7390 | 7453.9 | - |
| $(1S2s)1/2^+$ | 7497.0 | 7645 | 7624.2 | - |
| $(2S1s)1/2^+$ | 7243.3 | 7333 | 7396.3 | - |
| $(1P1s)1/2^-$ | 7281.6 | 7230 | 7434.6 | - |
| $(1P1s)1/2'$ | 7249.0 | 7199 | 7411.7 | - |
| $(1P1s)3/2^-$ | 7297.9 | 7228 | 7446.0 | - |
| $(1P1s)3/2'$ | 7232.7 | 7201 | 7400.3 | - |
| $(1P1s)5/2^-$ | 7314.2 | 7265 | 7457.4 | - |

in which $\bar{M}$ represents the mean (spin-averaged) masses of the respective mesons formed by the quark pairs $Q\bar{Q}'$, $Q\bar{n}$ and $n\bar{n}'(n = u, d)$, shown in Table XV.

TABLE XV: Means spin-average masses (MeV) of heavy quarkonia, $B_c$ mesons, the $B$ mesons, $D$ mesons and the light unflavored mesons

| Mesons | $c\bar{c}$ | $b\bar{b}$ | $b\bar{c}$ | $c\bar{q}$ | $b\bar{q}$ | $q\bar{q}$ |
|--------|-------------|-------------|-------------|-------------|-------------|-------------|
| $\bar{M}(1S)$ | 3068.65 | 9444.9 | 6320.25 | 1973.23 | 5313.40 | 619.98 |
| $\bar{M}(2S)$ | 3627.50 | 10017.3 | 6901.20 | 2627.00 | 5917.00 | 1423.75 |
| $\bar{M}(1P)$ | 3525.26 | 9899.7 | 6765.32 | 2435.72 | 5737.17 | 1245.79 |
FIG. 5: Mass spectrum of $\Omega_{cc}$ baryons (in MeV). The horizontal dashed line shows the $\Lambda_c D_s$ threshold.

FIG. 6: Mass spectrum of $\Omega_{bb}$ baryons (in MeV).
FIG. 7: Mass spectrum of Ω_{bc} baryons (in MeV). The horizontal dashed line shows the Λ_{bD_s} threshold.

In the case of QQ' = cc, Eq. (55) gives

\[ B(cc)_{1S} = \bar{M}(cc)_{1S} - 2\bar{M}(c\bar{n})_{1S} + \bar{M}(n\bar{n}')_{1S}, \]

\[ = 3068.65 - 2(1973.23) + 619.98, \]

\[ = -257.83 \text{ MeV}, \quad (56) \]

where we have used the mean masses in the ground states, \( \bar{M}(cc)_{1S} = [3M(J/\psi) + M(\eta_c)]/4 = 3068.65 \text{ MeV}, \) \( \bar{M}(c\bar{n})_{1S} = [3M(D^*) + M(D)]/4 = 1973.23 \text{ MeV}, \) \( \bar{M}(n\bar{n}')_{1S} = [3M(\rho) + 154]/4 = 619.98 \text{ MeV}. \) Here, we have used the predicted 1S-wave mass 154.0 MeV of the pion in Ref. [44], instead of the observed pion mass as it is too light to be like mass of a normal n\bar{n} meson in that contributions from light quarks must be concealed in Eq. (56).

With the 1S wave data of the respective bottomonium, the B meson and \( \bar{M}(n\bar{n}')_{1S} = 619.98 \text{ MeV in Table XV, one can estimate } B(b\bar{b}) \text{ similarly via Eq. (55).} \)

For the pair bc, only the 1S and 2S wave mesons (the B_c) are available for \( \bar{M}(bc) \) experimentally, and they give, by Eq. (55),

\[ B(bc)_{1S} = \bar{M}(bc)_{1S} - \bar{M}(b\bar{n})_{1S} - \bar{M}(\bar{c}n)_{1S} + \bar{M}(n\bar{n})_{1S}, \]

\[ = 6320.3 - 5313.4 - 1973.2 + 619.98, \]

\[ = -346.4 \text{ MeV,} \]
for the 1S wave pair $b\bar{c}$, in which the ground-state mass $\bar{M}(b\bar{c})_{1S}$ is estimated by the mean mass of the measured $M(B_c,0^-)_{1S} = 6274.5$ MeV and the predicted mass splitting $\Delta M(B_c)_{1S} = 61$ MeV between the $1^-$ and $0^-$ states by Ref. [43]:

$$\bar{M}(b\bar{c})_{1S} = \frac{1}{4}[6274.5 + 3 \times (6274.5 + \Delta M(B_c)_{1S})] = 6320.3 \text{ MeV}.$$ 

For 2S wave $b\bar{c}$, one can use the measured 2S-wave mass $M(B_c,0^-)_{2S} = 6871.2$ MeV of the $B_c$ meson and the predicted mass splitting $\Delta M(B_c)_{2S} = 40$ MeV by Ref. [43] to get

$$\bar{M}(b\bar{c})_{2S} = \frac{1}{4}[6871.2 + 3 \times (6871.2 + \Delta M(B_c)_{2S})] = 6901.2 \text{ MeV},$$

and thereby to find

$$B(b\bar{c})_{2S} = 6901.2 - 5917.0 - 2607.5 + 1423.7$$

$$= -199.6 \text{ MeV},$$

where $\bar{M}(c\bar{n})_{2S} = 2607.5 \text{ MeV}$[25], and $\bar{M}(b\bar{n})_{2S} = 5917.0 \text{ MeV}$ taken from the predicted mean mass[28] of the $B$ mesons in 2S-wave. Also, $B(c\bar{c})_{2S} = -117.3 \text{ MeV}$ and $B(c\bar{c})_{2S} = -392.98 \text{ MeV}$. Inspired by atomic spectra in a purely Coulombic potential, we assume the binding energy $B(Q\bar{Q}')$ to depend upon the reduced pair mass $\mu_{QQ'} = M_QM_{Q'}/(M_Q + M_{Q'})$
in the $P$-power form \cite{42,45},

$$B(Q\bar{Q}') - B_0 = k[\mu_{QQ'}]^P = k \left( \frac{M_Q M_{Q'}}{M_Q + M_{Q'}} \right)^P. \quad (57)$$

where $B_0$ are a constant while the parameters $k = k_{N,L}$ and $P = P_{N,L}$ depends on the radial and angular quantum numbers of the excited $Q\bar{Q}'$. In the ground $(1S1s)$ state, three values of the binding $B(c\bar{c}) = -257.83$ MeV, $B(b\bar{b}) = -561.92$ MeV and $B(b\bar{c}) = -346.40$ MeV in Table XVI correspond to the parameters $\{B_0 = 62.066$ MeV, $P_{0,0} = 0.58805$, $k_{0,0} = -388.342\}$.

When $Q\bar{Q}'$ excited to the $1P$ wave, one can utilizes the measured mean masses of the charmonium and bottomonium shown in Table XV to give, by Eq. (55),

$$B(c\bar{c})_{1P} = -100.38\text{ MeV}, \quad B(b\bar{b})_{1P} = -328.81\text{ MeV},$$

and thereby predict (for $b\bar{c}$), by Eq. (57), $P_{0,1} = 0.773645$, $k_{0,1} = -0.209448$ and

$$B(b\bar{c})_{1P} = B_0 + \frac{B(c\bar{c})_{1P} - B_0}{[(1 + 1.44/4.48)/2]^{P_{0,1}}},$$

$$= 62.066 + \frac{-100.377 - (62.066)}{[(1 + 1.44/4.48)/2]^{0.773645}}$$

$$= -161.78\text{ MeV}. \quad (58)$$

For the heavy quark-antiquark pairs $\{c\bar{c}, b\bar{b}, b\bar{c}\}$, three values of string tension $T$ that reproduce, by Eq. (52), the mean masses in Table XV and the binding energies in Table XVI are

$$\{T(c\bar{c}), T(b\bar{b}), T(b\bar{c})\} = \{0.21891, 0.43367, 0.34278\}\text{ GeV}^2. \quad (59)$$

Next, we consider binding energy $B(QQ')$ of heavy quark pair $QQ'$ in the color antitriplet($\bar{3_c}$). Such a binding corrects the energy of the QCD string connecting $Q$ and $Q'$ in short-distance, as indicated by heavy-quarkonia spectra\cite{23}. Note that the $QQ'$ interaction strength in a color triplet is half that of $Q\bar{Q}'$ in a color singlet by color-$SU(3)$ argument, we assume, as in Ref. \cite{2}, that

$$B(QQ') = \frac{1}{2}B(Q\bar{Q}'),$$

with $Q\bar{Q}'$ in color singlet$(1_c)$. It follows that for the shift $\Delta B_{N,L} \equiv B(QQ')_{N,L} - B(QQ')_{0,0}$ of the binding energy relative to the ground states$(1S1s)$,

$$\Delta B(QQ)_{N,L} = \frac{1}{2}[B(Q\bar{Q}')_{N,L} - B(Q\bar{Q}')_{0,0}]. \quad (61)$$
Applying the binding data in Table XVI to Eq. (60) and Eq. (61) gives

\[ B(cc)_{1S} = -128.92 \text{ MeV}, \quad B(bb)_{1S} = -280.96 \text{ MeV}, \quad B(bc)_{1S} = -173.20 \text{ MeV}, \]
\[ B(cc)_{2S} = -58.65 \text{ MeV}, \quad B(bb)_{2S} = -196.49 \text{ MeV}, \quad B(bc)_{2S} = -99.78 \text{ MeV}, \]
\[ B(cc)_{1P} = -50.19 \text{ MeV}, \quad B(bb)_{1P} = -164.41 \text{ MeV}, \quad B(bc)_{1P} = -80.89 \text{ MeV}, \]  

for the binding of the 1S-wave heavy pair and

\[ \Delta B(cc)_{1P} = \frac{1}{2}(-100.38 + 257.83) = 78.73 \text{ MeV}, \]
\[ \Delta B(bb)_{1P} = \frac{1}{2}(-328.81 + 561.92) = 116.56 \text{ MeV}, \]
\[ \Delta B(bc)_{1P} = \frac{1}{2}(-161.78 + 346.40) = 92.32 \text{ MeV}, \]

\[ \Delta B(cc)_{2S} = \frac{1}{2}(-117.3 + 257.83) = 70.27 \text{ MeV}, \]
\[ \Delta B(bb)_{2S} = \frac{1}{2}(-392.98 + 561.92) = 84.47 \text{ MeV}, \]
\[ \Delta B(bc)_{2S} = \frac{1}{2}(-199.55 + 346.40) = 73.43 \text{ MeV}, \]

for shifts of the binding of the 1P and 2S waves of the pair, with the results shown collectively in Table XVI.

TABLE XVI: The binding energy \(B(QQ')\), \(B(QQ')\) and their shift \(\Delta B\) relative to the ground state. All items are in MeV.

| Binding energy | 1S   | 2S   | 1P   | \(\Delta B(2S)\) | \(\Delta B(1P)\) |
|----------------|------|------|------|------------------|------------------|
| \(B(cc)\)      | -257.83 | -117.30 | -100.38 | 140.53           | 157.45           |
| \(B(bb)\)      | -561.92 | -392.98 | -328.81 | 168.94           | 233.11           |
| \(B(bc)\)      | -346.40 | -199.55 | -161.78 | 146.85           | 184.32           |
| \(B(cc)\)      | -128.92 | -58.65 | -50.19 | 70.27            | 78.73            |
| \(B(bb)\)      | -280.96 | -196.49 | -164.41 | 84.47            | 116.55           |
| \(B(bc)\)      | -173.20 | -99.78 | -80.89 | 73.43            | 92.32            |
Finally, we consider DH baryons $qQQ'$ with excited diquark $QQ'$. In the picture of QCD string, one can view the baryon as a string system $[Q - Q]q$ consisting of a heavy subsystem of massive string $[Q - Q]$ (with each $Q$ at one of ends) being rotating or vibrating and a light subsystem of a light quark $q$ together with a string connected to it at the third end. For a 1S or 1P wave diquark in DH baryon labelled by the quantum numbers $N$ and $L$, respectively, the Regge relation similar to Eq. (52) takes the form (Appendix D),

$$\bar{M}(QQ')_{N,L} = M_Q + M_{Q'} + \Delta B(QQ')_{N,L} + 3\left[\frac{T_{QQ'} \left(\frac{L + \pi N/\sqrt{12}}{2} + T c_0\right)}{2(M_Q + M_{Q'})^{1/3}}\right]^{2/3} + c_1,$$

(65)

with $T_{QQ'}$ the tension of string within the subsystem $[Q - Q]$ and $c_0$ given by Eq. (53).

Here, $c_1$ is an additive constant, defined up to the ground state of the whole DH system.

Notice again that the interaction strength between $QQ'$ in a color $\bar{3}_c$ is half that of $Q\bar{Q}'$ in a color singlet by a color-$SU(3)$ argument, we shall assume

$$T_{QQ'}[\bar{3}_c] = \frac{1}{2} T_{Q\bar{Q}'}[1_c],$$

(66)

for heavy quark-quark pairs indicated. So, $2T_{QQ'}c_0 = T_{Q\bar{Q}'}c_0$. Accounting the excitation energy $(\Delta \bar{M}^{(s)})_{N,L} = \bar{M}^{(s)}_{N,L} - \bar{M}_{0,0}$ of the DH baryon for the energy shift of the heavy subsystem $[Q - Q]$ relative to its ground state, one can write the excitation energy of the DH baryon with excited diquark ($T_{QQ'} = T$ given in Eq. (53)) as

$$\bar{M}^{(s)}_{N,L} - \bar{M}_{0,0} = \Delta B(QQ')_{N,L} + 3\left[\frac{T(L + \pi N/\sqrt{12})/2 + T c_0}{2(M_Q + M_{Q'})^{1/3}}\right]^{2/3} + C_1,$$

(67)

where $C_1$ is a constant related to $c_1$ and determined by (setting $N = 0 = L$),

$$C_1[qQQ'] = -3\left[\frac{(T c_0)^2}{2(M_Q + M_{Q'})}\right]^{1/3},$$

with $\Delta B(QQ')_{0,0} = 0$. So, we obtain the baryon mass shift due to diquark excitations,

$$(\Delta \bar{M}^{(s)})_{N,L} = \Delta B(QQ')_{N,L} + 3\left[\frac{T(L + \pi N/\sqrt{12})/2 + T c_0}{2(M_Q + M_{Q'})^{1/3}}\right]^{2/3} - (T c_0)^{2/3},$$

(68)

with $T c_0$ given by Eq. (53).

Using the binding data in Eqs. (63) and (64), and the values of $T$ in Eq. (59), Eq. (68) gives rise to the mean-mass shifts of the DH baryons due to the 2S and 1P diquark excitations relative to their ground states (1S1s),

$$(\Delta \bar{M}^{(s)}(\Xi_{cc,bb,lc})_{2S} = \{298.87, 270.12, 280.29\} \text{ MeV},$$

(69)

$$(\Delta \bar{M}^{(s)}(\Xi_{cc,bb,lc})_{1P} = \{327.73, 320.09, 318.56\} \text{ MeV}.$$

(70)
and the same values for the $\Omega_{QQ'}(QQ' = cc, bb, bc)$. This enable us to define effective masses $E_{QQ'}$ of heavy-diquark in its excited state via the energy shift due to diquark excitations:

$$(E_{QQ'})_{N,L} = M_{QQ'} + (\Delta M^{(*)})_{N,L},$$

which gives explicitly (with $M_{cc,bb,bc} = \{2865.5, 8916.7, 5892.2\}$)

$$(E_{cc,bb,bc})_{2S} = \{3.1644, 9.1868, 6.1726\} \text{ GeV}, \quad (71)$$

$$(E_{cc,bb,bc})_{1P} = \{3.1932, 9.2368, 6.2109\} \text{ GeV}. \quad (72)$$

VII. MASS SPLITTINGS DUE TO 2S AND 1P DIQUARK EXCITATIONS

We consider the low-lying DH baryons with excited diquarks. For the DH baryons $\Xi_{QQ'}$ and $\Omega_{QQ'}$ with excited (2S or 1P wave) diquark $QQ'$, the mass splitting arises from the spin interaction between diquark with effective magnetic moments $e_{QQ'}(L + S_{QQ'})/(2M_{QQ'})$ and the light quark $q$ with spin magnetic moment of $e_{q}S_{q}/m_{q}$. Here, $e_{QQ'}$ and $e_{q}$ stand for the charges of diquark with internal orbital angular moment $L$ and light quark with spin $S_{q}$. The spin-interaction of the baryon system is

$$H^{SD} = c^{*}(L + S_{QQ'}) \cdot S_{q},$$

where the spin $S_{QQ'}$ of diquark is integral, and it has to be unit ($S_{QQ'} = 1$) in its $S$ wave and to be zero ($S_{QQ'} = 0$) in its $P$ wave when $QQ' = cc$ or $bb$. In the $bc$-diquark case, if $QQ' = [bc]$, its spin $S_{[bc]} = 0$ in its $S$ wave and $S_{[bc]} = 1$ in its $P$ wave, and if $QQ' = \{bc\}$, one has $S_{\{bc\}} = 1$ in its $S$ wave and $S_{\{bc\}} = 0$ in its $P$ wave by the symmetry of the baryon states $q[bc]$ and $q\{bc\}$. Hence, for the system $q[bc]$, $J_{d} = L \oplus S_{QQ'} = \{0\}$ in $S$ wave of diquark or $\{0, 1, 2\}$ in $P$ wave of diquark while for the system $q\{bc\}$, $J_{d} = L \oplus S_{QQ'} = \{1\}$ in the 2S and 1P wave diquark.

We consider the 2S1s and 1P1s states of the DH baryons in this section. To find the mass splitting of the baryon multiplets, we utilize the mass scaling relative to the ground state $D$ mesons (the $D^{\pm}$ with mass 1869.66 $\pm$ 0.05 MeV and the $D^{*}(2010)^{\pm}$ with mass 2010.26 $\pm$ 0.05 MeV [28]),

$$c^{*}(\Xi_{bb}(bbq))_{2S,1P} = \left(\frac{M_{c}}{(E_{bb})_{2S,1P}}\right)\left(\frac{m_{u}}{m_{q}}\right)c(D)_{1S}, \quad (74)$$

in which $c(D)_{1S} = M(D^{*\pm}) - M(D^{\pm}) = 140.6$ MeV is one of the spin-coupling parameters (that corresponds to the contact-term) for the $D$ mesons, as Ref. [18].
(1) The 2S1s states. For the DH system $q[bc]$, as $L = 0 = S_{[bc]}$, the baryon mass is simply $\bar{M}(q[bc])_{nS}$ without multiplet splitting. For the system $q\{QQ\}'$, as $L = 0 = S_{[bc]}$, the baryon mass is simply $\bar{M}(q\{bc\})_{nS}$ without multiplet splitting. For the system $q\{QQ\}'$, as $L = 0$, the spin-interaction reduces to $H^{SD}(2S) = c^* S_{QQ'} \cdot S_q$, in which $S_{QQ'} \cdot S_q$ has the eigenvalues $\{-1, 1/2\}$ for the spin of total system $J = 1/2$ or $3/2$, respectively. The mass formula for the 2S1s wave baryons $q\{QQ\}'$ becomes (we use $2S$ for $2S1s$ for short),

$$M(q\{QQ\}')_{2S} = \bar{M}_{1S1s} + (\Delta \bar{M}^{(*)})_{2S} + c^*(q\{QQ\}') \begin{bmatrix} -1 & 0 \\ 0 & 1/2 \end{bmatrix},$$

with $\bar{M}_{1S} \equiv \bar{M}_{0,0}$ the baryon ground-state masses and $\Delta \bar{M}^{(*)}$ the baryon mass shift in Eq. (68). Using Eq. (74), one gets

$$c^*(\Xi_{bb}(bbn))_{2S} = \left( \frac{M_c}{E_{bb}} \right) (140.6 \text{ MeV}) = 22.04 \text{ MeV},$$

$$c^*(\Omega_{bb}(bbs))_{2S} = \left( \frac{M_c}{E_{bb}} \right) \left( \frac{m_n}{m_s} \right) (140.6 \text{ MeV}) = 15.45 \text{ MeV},$$

where $(E_{bb})_{2S} = 9.1869 \text{ GeV}$. Further application of the scaling relations between the DH baryons

$$c^*(\Xi_{cc})_{2S} = c^*(\Xi_{bb}) \left( \frac{E_{bb}}{E_{cc}} \right), \quad c^*(\Xi_{bc})_{2S} = c^*(\Xi_{bb}) \left( \frac{E_{bb}}{E_{bc}} \right),$$

$$c^*(\Omega_{cc})_{2S} = c^*(\Omega_{bb}) \left( \frac{E_{bb}}{E_{cc}} \right), \quad c^*(\Omega_{bc})_{2S} = c^*(\Omega_{bb}) \left( \frac{E_{bb}}{E_{bc}} \right),$$

(77)

gives

$$c^*(\Xi_{cc})_{2S} = 63.98 \text{ MeV}, \quad c^*(\Xi_{bc})_{2S} = 32.80 \text{ MeV},$$

$$c^*(\Omega_{cc})_{2S} = 44.87 \text{ MeV}, \quad c^*(\Omega_{bc})_{2S} = 23.00 \text{ MeV}.$$

(78)

Given the masses in Table II and the mean masses of the ground-state baryons $\Omega_{cc}$, $\Omega_{bb}$ and $\Omega_{bc}$ in Ref. [1], $\Delta \bar{M}^{(*)}$ in Eq. (69) and the values of $c^*$ in Eqs. (76) and (77), one gets, by Eq. (75), the multiplet masses of the DH baryons $\Xi_{QQ'}$ and $\Omega_{QQ'}$ in the 2S1s state, listed in Table VIII through Table XIV, and shown in FIGs 1-8, respectively.

(2) The 1P1s states. For the systems $q\{cc\}$, $q\{bb\}$ and $q\{bc\}$, $S_{QQ'} = 0$ and the system spin $J = L \oplus 1/2$ takes values $J = 1/2$ and $3/2$. The spin interaction becomes $c^*L \cdot S_q$, which equals to $c^*\text{diag}[-1, 1/2]$. So, the DH baryon mass is
\[ M(q\{QQ'\})_{1P} = \bar{M}_{1S1s} + (\Delta \bar{M}^{(s)})_{1P} + c^* \begin{pmatrix} -1 & 0 \\ 0 & 1/2 \end{pmatrix}, \]
\[ J = 1/2, \]
\[ J = 3/2. \]  

(79)

For the systems \( q[b]c \), \( S_{QQ'} = 1 \), and \( J_d = L \oplus S_{QQ'} \) can take values \( J_d = 0, 1 \) or 2 and \( J = 1/2, 1/2', 3/2, 3'/2, 5/2 \). The baryon systems can be labelled by \( |J, J_d\rangle \) and the relation \( J_d \cdot S_q \equiv [J(J + 1) - J_d(J_d + 1) - 3/4]/2 \) infers

\[ M(q[b]c)_{1P} = \bar{M}_{1S1s} + (\Delta \bar{M}^{(s)})_{1P} + c^* \text{diag} \begin{pmatrix} 0, -1/2, -3/2, 1 \end{pmatrix}, \]

in the subspace of \( \{|J, J_d\rangle = \{|1/2, 0\rangle, |1/2, 1\rangle, |3/2, 1\rangle, |3/2, 2\rangle, |5/2, 2\rangle\} \).

Application of the mass scaling Eq. (74) leads to (\( E_{bb} = 9.2368 \text{ GeV} \))

\[ c^*(\Xi_{bb})_{1P} = \left( \frac{M_c}{E_{bb}} \right) (140.6 \text{ MeV}) = 21.92 \text{ MeV}, \]
\[ c^*(\Omega_{bb})_{1P} = \left( \frac{M_c}{E_{bb}} \right) \left( \frac{m_n}{m_s} \right) (140.6 \text{ MeV}) = 15.37 \text{ MeV}, \]

and

\[ c^*(\Xi_{cc})_{1P} = c^*(\Xi_{bb})_{1P} \left( \frac{9.2368}{3.1933} \right) = 63.40 \text{ MeV}, \]
\[ c^*(\Xi_{bc})_{1P} = c^*(\Xi_{bb})_{1P} \left( \frac{9.2368}{6.2108} \right) = 32.60 \text{ MeV}, \]
\[ c^*(\Omega_{cc})_{1P} = c^*(\Omega_{bb})_{1P} \left( \frac{9.2368}{3.1933} \right) = 44.46 \text{ MeV}, \]
\[ c^*(\Omega_{bc})_{1P} = c^*(\Omega_{bb})_{1P} \left( \frac{9.2368}{6.2108} \right) = 22.86 \text{ MeV}, \]

(81)

(82)

Given the masses in Table II and the mean masses of the ground-state baryons \( \Omega_{cc}, \Omega_{bb} \) and \( \Omega_{bc} \) in Ref. [1], \( \Delta \bar{M}^{(s)} \) in Eq. (70) and the \( c^* \) values in Eqs. (81) and (82), one can apply Eq. (79) and Eq. (80) to obtain the mean mass \( \bar{M}_{1S} + (\Delta \bar{M}^{(s)})_{1P} \) and the multiplet masses of the 1P1s-wave DH baryons. The results are listed in Table XIII and XIV and shown in FIGs. 1-8, respectively, for the 2S1s and 1P1s states of the DH baryons.

**VIII. SUMMARY AND REMARKS**

In this work, we apply a set of relations of mass scaling between heavy hadrons which is based on chromodynamics similarity among these hadrons to explore excited doubly heavy
baryons $\Xi_{QQ'}$ and $\Omega_{QQ'}$. The low-lying S- and P-wave masses of the excited DH baryons are predicted in the picture of heavy-diquark-light-quark by computing the mass parts of spin-independent and spin-dependent separately and compared to other calculations. For the spin-independent mass of the DH baryons, two Regge trajectories, one linear and the other nonlinear, are employed to describe the respective excitations of the light quark and diquark internally up to $2S$ and $1P$ waves, which are derived semiclassically from the model of the QCD string and matched successfully to the observed heavy baryons and mesons. In the case of diquark excitation with light quark in ground state, the short-distance correction of heavy-pair binding are extracted from the excitation energies of heavy diquarks phenomenologically and incorporated into Regge relation, by which effective masses of excited diquarks are estimated and mass scaling is constructed.

Our mass analysis suggests (see FIGs. 1-8) that the mean mass-level spacings $\Delta \bar{M}$ of heavy diquark excitations are generally narrower than that of light-quark excitations. This agrees qualitively with a harmonic-oscillator approximation of the QCD string with massive ends that the level spacing of baryons $\Delta \bar{M} \sim \omega \sim 1/m_q$ or $\Delta \bar{M} \sim 1/{\text{Mass}}(QQ)$, where $\omega$ stands for oscillator frequency related to the effective mass $m_q$ of the light quark or that of the diquark ${\text{Mass}}(QQ)$, which equals $E_{QQ}$ for excited diquark and $M_{QQ}$ for the ground state diquark. This approximation applies preferably in the low-lying states of a baryon system with smaller size. Meanwhile, it seems (see FIGs. 1-8) that the mass splitting of spin multiplets of the excited DH baryons are wider for the diquark excitations than for the excitation of light quarks, which remain to be explained in the further explorations. We hope the oncoming experiments like LHCb and Belle to examine our predictions for the excited DH resonances addressed.

By the way, the binding function Eq. (57), inspired by atomic spectra, mimic the non-relativistic spectra of a system of heavy quark and antiquark bounded by the short-distance Coulomb-like force of single gluon exchange. In a purely Coulombic potential $V(r) = -(4/3)\alpha_s/r$ the energy levels is given by $E_n = -[(4/3)\alpha_s]^2\mu/(2n^2)$ ($n = n_r + L + 1$), linear on the reduced mass $\mu$ of system. In a quarkonia or the heavy meson $B_c$, Eq. (57) holds qualitatively as $B_0$ may also depend weakly on the quantum numbers $(N,L)$ and upon the effective mass of the quarks $[30, 42, 45]$. So, the prediction Eq.(58) for the 1P-wave binding $B(b\bar{c})$ and ensuing 1P-wave mass calculations of the baryon $\Xi_{bc}$ and $\Omega_{bc}$ are of approximated (uncertain within 10 MeV roughly).
Low-energy QCD interaction is known to be involved in many aspects, especially in excited hadrons, and there exist various approaches exploring the excited doubly heavy baryons [1, 31–36, 38]. The relations of Regge trajectory are established phenomenologically and rooted in the string picture of QCD interaction. Some remarks and discussions are in order:

(i) For the DH baryons with 1S-wave diquark, the Regge trajectory stems from excitation of the light quark \( q \) which moves relativistically and is away from the heavy mass-center of diquark most of time so that the short-distance interaction between the diquark \( QQ' \) and \( q \) is ignorable, especially for excited states. The string picture without short-distance correction is appropriate in this case and the linear Regge relation Eq. (1) based on applies.

(ii) When diquark of DH baryons excited, an improved (nonlinear) Regge relation with a short-distance correction due to extra heavy-pair binding is required, especially for the low-lying excitations of diquarks for which heavy-pairs are deep in attractive Coulomb-like potential. This is supported by the energy drop (roughly half or 1/3) of the extracted binding of the 2S and 1P wave baryons relative to the ground state, as Table XVI and Eq. (55) indicated phenomenologically.

(iii) While the mass predictions for excited DH baryons in this work is by no means regorous, our estimation of the baryon spin-multiplet splitting is general in that mass scaling used is based on the known chromodynamics similarity between heavy baryons and heavy mesons. The parameterized spin-dependent interactions Eq. (17) and Eq. (73) is built generally according to Lorentz structure and tensor nature of interaction [26, 27].

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**Appendix A**

For the orbitally excitations of a DH baryon \( q(QQ') \) with S-wave diquark \( QQ' \), the classical energy and orbital angular momentum for the rotating QCD string read

\[
E = \sum_{i=QQ',q'} \left[ \frac{m_{barei}}{\sqrt{1 - (\omega r_i)^2}} + \frac{a}{\omega} \int_0^{\omega r_i} \frac{du}{\sqrt{1 - u^2}} \right], \tag{A1}
\]
\[
l = \sum_{i=QQ',q} \left[ \frac{m_i \omega r_i^2}{\sqrt{1 - (\omega r_i)^2}} + \frac{a}{\omega} \int_{0}^{\omega r_i} \frac{u^2 du}{\sqrt{1 - u^2}} \right], \tag{A2}
\]

where \(m_{\text{bare}QQ'}\) and \(m_{\text{bare}q}\) are the respective bare masses of the heavy diquarks \(QQ'\) and light quark \(q\), and \(\omega r_i = v_i\) are the velocity of the string end tied to the quark \(i = QQ', q\), \(a\) stands for the tension of the QCD string. We define the effective (dynamical) masses of the heavy diquarks and the light quark in the CM frame of the baryon by

\[
M_{QQ'} = \frac{m_{\text{bare}QQ'}}{\sqrt{1 - v_{QQ'}^2}}, \quad m_q = \frac{m_{\text{bare}q}}{\sqrt{1 - v_q^2}}, \tag{A3}
\]

Integrating Eq. (A1) and Eq. (A2) gives

\[
E = M_{QQ'} + m_q + \frac{a}{\omega} \left[ \arcsin (v_{QQ'}) + \arcsin (v_q) \right], \tag{A4}
\]

\[
l = \frac{1}{\omega} \left( M_{QQ'} v_{QQ'}^2 + m_q v_q^2 \right) + \sum_{i=QQ',q} \left[ \arcsin (v_i) - v_i \sqrt{1 - v_i^2} \right], \tag{A5}
\]

The boundary condition of string at ends linked to heavy quark gives

\[
\frac{a}{\omega} = \frac{(m_{\text{bare}QQ'}) v_{QQ'}}{1 - v_{QQ'}^2} = \frac{M_{QQ'} v_{QQ'}}{\sqrt{1 - v_{QQ'}^2}}. \tag{A6}
\]

As the diquark \(QQ'\) is very heavy and moves nonrelativistically in hadrons, \(v_{QQ'}\) is small in the limit of heavy quark \((m_{\text{bare}QQ'} \rightarrow 0)\). A series expansion of Eq. (A6) upon \(v_{QQ'}\) gives

\[
\frac{a}{\omega} \simeq M_{QQ'} v_{QQ'} + \frac{1}{2} M_{QQ'} v_{QQ'}^3 = P_{QQ'} + \frac{P_{QQ'}^3}{2 M_{QQ'}^2}, \tag{A7}
\]

From Eq. (A3) one has \(v_q = \sqrt{1 - (m_{\text{bare}q}/m_q)^2}\).

Assuming \(q\) to move relativistically \((v_q \rightarrow 1)\), or, \(m_{\text{bare}q}/m_q \ll 1\), one finds

\[
\arcsin (v_q) = \arcsin \left( \sqrt{1 - (m_{\text{bare}q}/m_q)^2} \right) \simeq \frac{\pi}{2} - \frac{m_{\text{bare}q}}{m_q}, \tag{A8}
\]

\[
\arcsin (v_{QQ'}) = v_{QQ'} + \frac{1}{6} v_{QQ'}^3 + \mathcal{O} (v_{QQ'}^5), \tag{A9}
\]

\[
v_{QQ'} \sqrt{1 - v_{QQ'}^2} = v_{QQ'} - \frac{1}{2} v_{QQ'}^3 + \mathcal{O} (v_{QQ'}^5), \tag{A10}
\]

Substitute the above relations into Eqs. (A4) and (A5) yields

\[
E \simeq M_{QQ'} + m_q + \frac{a}{\omega} \left[ \frac{\pi}{2} - \frac{m_{\text{bare}q}}{m_q} + v_{QQ'} + \frac{1}{6} v_{QQ'}^3 \right], \tag{A11}
\]

38
\[
\omega l \simeq MQQ'v_{QQ'}^2 + m_q + \frac{a}{\omega} \left[ \frac{\pi}{4} - \frac{m_{bare}}{m_q} \right] + \frac{\alpha}{3\omega} v_{QQ'}^3. \quad (A12)
\]

Using Eq. (A7) and eliminating \(\omega\), Eqs. (A11) and (A12) combine to give, when ignoring the tiny term \(m_{bare}/m_q\),

\[
(E - MQQ')^2 = \pi al + \left( m_q + \frac{P_{QQ'}^2}{MQQ'} \right)^2 - 2m_{bare}P_{QQ'}.
\]

where \(P_{QQ'} \equiv MQQ'v_{QQ'} \simeq MQQ' \left( 1 - \frac{m_{bare}^2/QQ'}/M_{QQ'}^2 \right)^{1/2} \). Taking the small bare-mass limit \(m_{bare} \to 0\), Eq. (A13) leads to Eq. (1), where \(P_{QQ'}^2/M_{QQ'} = MQQ' - m_{bare}^2/QQ' /M_{QQ'}\).

**Appendix B**

Given the eigenvalues \(j = 1/2\) and \(3/2\), one solves the bases(eigenfunctions) \(|S_{QQ'3}, S_{q3}, l_3\rangle\) of the \(l \cdot S_q\) in the LS coupling. The mass formula \(\Delta M = \langle H^{SD} \rangle\) for a DH baryon \(qQQ'\) with S-wave diquark \(QQ'\) can be obtained by diagonalizing the dominate interaction \(a_1 l \cdot S_q\) and adding the diagonal elements of other perturbative spin-interactions in Eq. (17). This can be done by evaluating the matrix elements of \(H^{SD}\) in the LS coupling and then changing the bases \(|S_{QQ'3}, S_{q3}, l_3\rangle\) to the new bases \(|J, j\rangle\) in the \(jj\) coupling to find the mass formula \(\Delta M = \langle H^{SD} \rangle\).

For each interaction terms of Eq. (17), one can evaluate its matrix elements by explicit construction of states with a given \(J_3\) as linear combinations of the baryon states \(|S_{QQ'3}, S_{q3}, l_3\rangle\) in the LS coupling where \(S_{q3} + S_{QQ'3} + l_3 = J_3\). Due to the rotation invariance of the matrix elements, it suffices to use a single \(J_3\) for each term. Then, one can use

\[
l \cdot S_i = \frac{1}{2} \left[ l_+ S_{i-} + l_- S_{i+} \right] + l_3 S_{i3}, \quad (B1)
\]

to find their elements by applying \(l \cdot S_i\) \((i = q, QQ')\) on the the third components of angular momenta. For projected states of baryon with given \(J_3\), they can be expressed, in the LS coupling, as

\[
\begin{aligned}
\left| ^2P_{J=1/2, J_3} = \frac{1}{2} \rightangle &= \sqrt{\frac{2}{3}} \left| 1, -\frac{1}{2}, 0 \rightangle - \frac{1}{3} \left| 0, \frac{1}{2}, 0 \rightangle \\
&- \sqrt{\frac{2}{3}} \left| 0, \frac{1}{2}, 1 \rightangle + \frac{2}{3} \left| -1, \frac{1}{2}, 1 \rightangle,
\end{aligned}
\]

(B2)
\[
\begin{align*}
|^{4}P_{J=1/2}, J_3 = 1/2 \rangle &= \frac{1}{\sqrt{2}} |1, 1/2, -1 \rangle - \frac{1}{3} |1, -1/2, 0 \rangle \\
&\quad - \frac{\sqrt{2}}{3} |0, 1/2, 0 \rangle + \frac{1}{3} |0, -1/2, 1 \rangle + \frac{1}{3\sqrt{2}} |-1, 1/2, 1 \rangle, \\
|^{2}P_{J=3/2}, J_3 = 3/2 \rangle &= \sqrt{\frac{2}{3}} |1, -1/2, 1 \rangle - \sqrt{\frac{1}{3}} |0, 1/2, 1 \rangle, \\
|^{4}P_{J=3/2}, J_3 = 3/2 \rangle &= \sqrt{\frac{3}{5}} |1, 1/2, 0 \rangle - \sqrt{\frac{2}{15}} |1, -1/2, 1 \rangle \\
&\quad - \frac{2}{\sqrt{15}} |0, 1/2, 1 \rangle, \\
|^{4}P_{J=5/2}, J_3 = 5/2 \rangle &= |1, 1/2, 1 \rangle.
\end{align*}
\]

and use them to compute the matrix elements of \(1 \cdot S_i \), \(S_{QQ'} \cdot S_q \) and \(S_{12}/2 \) in the basis \([^{2}P_{J}, ^{4}P_{J}]\). With some algebra, one can find

\[
\langle 1 \cdot S_{QQ'} \rangle_{J=1/2} = \begin{bmatrix} -\frac{4}{3} & -\sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & -\frac{5}{3} \end{bmatrix}, \langle 1 \cdot S_q \rangle_{J=1/2} = \begin{bmatrix} \frac{1}{3} & \sqrt{\frac{2}{3}} \\ \frac{\sqrt{2}}{3} & -\frac{5}{6} \end{bmatrix}, \\
\langle 1 \cdot S_{QQ'} \rangle_{J=3/2} = \begin{bmatrix} \frac{2}{3} & -\frac{\sqrt{5}}{3} \\ -\sqrt{\frac{5}{3}} & -\frac{2}{3} \end{bmatrix}, \langle 1 \cdot S_q \rangle_{J=3/2} = \begin{bmatrix} -\frac{1}{6} & \frac{\sqrt{5}}{3} \\ \frac{\sqrt{5}}{3} & -\frac{1}{3} \end{bmatrix}, \\
\langle S_{QQ'} \cdot S_q \rangle_{J=1/2} = \begin{bmatrix} -1 & 0 \\ 0 & 1/2 \end{bmatrix}, \langle S_{QQ'} \cdot S_q \rangle_{J=3/2} = \begin{bmatrix} -1 & 0 \\ 0 & 1/2 \end{bmatrix}, \\
\langle S_{12}/2 \rangle_{J=1/2} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -1 \end{bmatrix}, \langle S_{12}/2 \rangle_{J=3/2} = \begin{bmatrix} 0 & -\frac{\sqrt{5}}{10} \\ -\frac{\sqrt{5}}{10} & \frac{4}{5} \end{bmatrix}.
\]

In the subspace of \(J = 5/2\), one finds

\[
\langle 1 \cdot S_{QQ'} \rangle_{J=5/2} = 1, \quad \langle 1 \cdot S_q \rangle_{J=5/2} = \frac{1}{2}, \\
\langle S_{QQ'} \cdot S_q \rangle_{J=5/2} = \frac{1}{2}, \quad \langle S_{12}/2 \rangle_{J=5/2} = -\frac{1}{5}.
\]
Given the above matrices, one can solve each eigenvalue $\lambda$ of $\mathbf{1} \cdot \mathbf{S}_q$ and the corresponding eigenvectors for a given $J$, and for that $J$ one can write the baryon states $|J,j\rangle$ in the $jj$ coupling. They are linear combinations of the $LS$ bases $|2S+1P_J\rangle$ with definite coefficients specified by the respective eigenvector solved for $J$:

$$\lambda = -1: |J = \frac{1}{2}, j = \frac{1}{2}\rangle = \frac{1}{3} |1^2 P_{1/2}\rangle - \frac{2\sqrt{2}}{3} |1^4 P_{1/2}\rangle,$$

$$\lambda = +\frac{1}{2}: |J = \frac{1}{2}, j = \frac{3}{2}\rangle = \frac{2\sqrt{2}}{3} |1^2 P_{1/2}\rangle + \frac{1}{3} |1^4 P_{1/2}\rangle,$$

$$\lambda = -1: |J = \frac{3}{2}, j = \frac{1}{2}\rangle = \frac{2}{3} |1^2 P_{3/2}\rangle - \frac{\sqrt{5}}{3} |1^4 P_{3/2}\rangle,$$

$$\lambda = +\frac{1}{2}: |J = \frac{3}{2}, j = \frac{3}{2}\rangle = \frac{\sqrt{5}}{3} |1^2 P_{3/2}\rangle + \frac{2}{3} |1^4 P_{3/2}\rangle,$$

$$\lambda = +\frac{1}{2}: |J = \frac{5}{2}, j = \frac{3}{2}\rangle = |1^4 P_{5/2}\rangle,$$

This gives the required baryon states in the heavy diquark limit, by which the diagonal matrix elements of $\mathbf{1} \cdot \mathbf{S}_Q \cdot \mathbf{S}_Q^\prime$, $S_{12}/2$ and $\mathbf{S}_Q \cdot \mathbf{S}_q$ can be obtained. The detailed results are collected in Table VIII.

### Appendix C

(1) Quantization condition for heavy quarkonia $Q\bar{Q}$

Consider a heavy quarkonia $Q\bar{Q}$ (at a distance of $r$) in a linear confining potential $T|r|$ for which the system Hamiltonian is $H^{Q\bar{Q}} = 2\sqrt{p^2 + M_Q^2} + T|r|$. Here, we have ignored the short-distance interaction. In the frame of center-of-mass, the heavy quark $Q$ moves equivalently in a confining potential $T\bar{x}$, with $\bar{x} = r/2$, and then the Hamiltonian for $Q$ as a half system of the quarkonia $Q\bar{Q}$, becomes

$$H^Q = \frac{1}{2} H^{Q\bar{Q}} = \sqrt{p^2_Q + \frac{l_q^2}{\bar{x}^2} + M_Q^2} + T|x|.$$  

(C1)

Using the semiclassical WKB analysis upon Eq. (C1) for the radial excitations($l_q = 0$), one has a WKB quantization condition [46,47],

$$2 \int_{x_-}^{x_+} p_x(x)dx = 2\pi (N + c_0),$$  

(C2)
with \( p_x(x) = |\mathbf{p}_x| = \sqrt{[(E(Q) - T|x|)^2 - M_Q^2]} \), \( x_+ = (E(Q) - M_Q)/T = -x_- \) the classical “turning points”, \( N \) the radial quantum number, \( c_0 \) a constant. Here, \( E(Q) \) is the semiclassical value of \( H^Q \), and the factor 2 before the integral in Eq. (C2) arises from underlying spinor nature of a quark whose wave function returns to original value after double cycle of journey in the position space. \(^{28}\) Assuming \( Q \) to be in S wave (moving radially), integration of Eq. (C2) gives

\[
\pi(N + c_0) = \frac{E(Q)^2}{T} \left[ \sqrt{1 - B^2} + B^2 \ln \left( \frac{1 - \sqrt{1 - B^2}}{B} \right) \right], \quad \text{(C3)}
\]

with \( B \equiv M_Q/E(Q) \).

Above result is only for the half system. Transforming to the whole system of quarkonia by mapping \( E(Q) \to E/2 \) and \( N \to N/2 \), Eq. (C3) gives \( B \to \tilde{B} = 2M_Q/E \)

\[
2\pi T(N + 2c_0) = E^2 \left[ \sqrt{1 - \tilde{B}^2} + \tilde{B}^2 \ln \left( \frac{1 - \sqrt{1 - \tilde{B}^2}}{\tilde{B}} \right) \right] \\
\approx E^{1/2} \left[ \frac{4\sqrt{2}}{3} E^{3/2} \epsilon^{3/2} - \frac{7\sqrt{2}}{15} E^{3/2} \epsilon^{5/2} + \mathcal{O}(\epsilon^{7/2}) \right] \quad \text{(C4)}
\]

where \( \epsilon \equiv 1 - \tilde{B} = 1 - 2M_Q/E \), which is small in heavy quark limit and \( E = 2M_Q/(1 - \epsilon) \approx 2M_Q \). Thus, Eq. (C4) leads to, to the leading order of \( \epsilon \),

\[
3\pi T(N + 2c_0) = 4\sqrt{M_Q}(E - 2M_Q)^{3/2},
\]

which gives Eq. (50).

(2) Improved (nonlinear) Regge relation

Rewriting in a typical form of standard trajectory, \((\bar{M} - \text{const.})^{3/2} \sim \) quantum numbers, for the heavy quarkonia system, the Regge relation Eq. (49) in Ref. \(^{29}\) becomes

\[
(\bar{M} - 2M_Q)^{3/2} = \frac{3\sqrt{3}}{2\sqrt{M_Q}} TL. \quad \text{(C5)}
\]

Comparing the radial and angular slopes (linear coefficients in \( N \) and \( L \)) in RHS of Eq. (50) and Eq. (C5), which is \( \pi : \sqrt{12} \), one can combine two equations into one unified form:

\[
[\bar{M}_{N,L} - 2M_Q]^{3/2} = \frac{3\sqrt{3}T}{2\sqrt{M_Q}} \left( L + \frac{\pi N}{\sqrt{12}} + 2c_0 \right). \quad \text{(C6)}
\]

In the derivation \(^{29}\) of Eq. (49) and that of Eq. (50) shown in Appendix Eq. (C1), the linear confining interaction between \( Q \) and \( \bar{Q} \) is assumed, with the short-distance force
between them ignored. As the short-distance force is required for low-lying quarkonia system and violates the typical linear trajectory in Eq. (C6), one way out is to assume that the short-distance attractive force between $Q$ and $\bar{Q}$ in the color-singlet($1_c$) would provide an extra(negative) energy $B(Q\bar{Q})_{N,L}$ to the spectra of the $Q\bar{Q}$ system and to deduct $B(Q\bar{Q})_{N,L}$ (named the binding energy) from the hadron mass $\bar{M}_{N,L}$ in Eq. (C6) so that the LHS of Eq. (C6) becomes the remaining string energy solely: $(\bar{M}_{N,L} - 2M_Q)^{3/2} \rightarrow (\bar{M}_{N,L} - B(Q\bar{Q})_{N,L} - 2M_Q)^{3/2}$. In doing so, the arguments by the classical string picture[29] and that in Eq. (1) given above remain valid and the formula Eq. (C6) remains intact formally up to a replacement $\bar{M}_{N,L} \rightarrow \bar{M}_{N,L} - B(Q\bar{Q})_{N,L}$. This gives rise to Eq. (51).

The binding energy $B(Q\bar{Q})_{N,L}$(in the excited states $|N, L\rangle$ of system) depends on the quantum numbers$(N, L)$ of the system considered [42] and thereby violates the linearity of the Regge relations Eq. (C5) and Eq. (50). Normally, such a term is negative since when two quarks($Q$ and $\bar{Q}$) are heavy enough to stay close each other, they both experiences an attractive Coulomb-like force of single gluon exchange, as the observed heavy-quarkonia spectra[25] of the cascade type indicated.

Appendix D

In the QCD string picture, one can view a DH baryons $qQQ'$, with excited diquark $QQ'$, as a string system $|Q - Q\rangle q$, consisting of a (heavy) subsystem of massive string $|Q - Q\rangle$ (each $Q$ at one of ends) and a light subsystem of a light-quark $q$ and the string connected to it.

In the semiclassical approximation, one can assume the light subsystem of $q$ and attached string to it to be in stationary state while the heavy subsystem $|Q - Q\rangle$ is excited to the excited state(denoted by $|N, L\rangle$, say). As such, the excitation of the heavy subsystem $|Q - Q\rangle$ in color antitriplet($\bar{3}_c$) with string tension $T_{QQ}$ resembles the excitation of a heavy quarkonia $Q\bar{Q}$ in color singlet($1_c$) with string tension $T$ up to a color(strength) factor of string interaction which is taken to be half commonly. Based on this similarity, one can write the excitation energy of the heavy system, by analogy with Eq. (52),

$$\bar{M}(QQ)_{N,L} - 2M_Q = \Delta B(QQ)_{N,L} + 3\left[\frac{T_{QQ} \left((L + \pi N/\sqrt{12}) + 2c_0\right)}{[4M_Q]^{1/3}}\right]^{2/3} + c_1,$$

(D1)
in which the first term $\Delta B(QQ)_{N,L} = B(QQ)_{N,L} - B(QQ)_{0,0}$ in the RSH accounts for the
short-distance contribution due to heavy-quark binding and the second term for the excited string energy of the heavy subsystem in the $|N,L\rangle$ state, $c_0$ is given by Eq. (53). We add an addictive constant $c_1$ since $\tilde{M}(QQ)$ is defined up to the ground state of the whole DH system containing $QQ$.

Extending Eq. (D1) to the diquark case of $QQ = bc$, one gets Eq. (65) generally, by heavy quark symmetry.

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