Hamiltonian Analysis of Mixed Derivative Hořava-Lifshitz Gravity

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Abstract

This short note is devoted to the canonical analysis of the Hořava-Lifshitz gravity with mixed derivative terms that was proposed in arXiv:1604.04215. We determine the algebra of constraints and we show that there is one additional scalar degree of freedom with respect to the non-projectable Hořava-Lifshitz gravity.

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1 Introduction

General Relativity (GR) is one of the most beautiful physical theories that is in perfect agreement with the current experimental tests. On the other hand it is well known that this theory is in conflict with the quantum mechanics since it is not perturbatively renormalizable and hence it breaks down at high energies. In order to solve this problem P. Horava proposed very original formulation of theory of gravity [1] which is now known as Horava-Lifshitz (HL) gravity. This theory has an improved behavior at high energies due to the presence of the higher order spatial derivatives in the action which implies that the theory is not invariant under full diffeomorphism but it is invariant under so called foliation preserving diffeomorphism (DiffF)

\[ t' = f(t) \quad , \quad x'^i = x^i(x, t) \]  

(1)

This property offers the possibility that the space and time coordinates have different scaling at high energies

\[ t' = k^{-z}t \quad , \quad x'^i = k^{-1}x^i \]  

(2)

where \( k \) is a constant. Consequence of this fact is that in 3 + 1 dimensions the theory contains terms with 2 time derivatives and at least 2 spatial derivatives since the minimal amount of the scaling anisotropy that is needed for the power-counting renormalizability of this theory is \( z = 3 \). Then collecting all terms that are invariant under DiffF symmetry leads to the general action [2, 3]

\[ S = \frac{M^2_p}{2} \int dt d^3 x N \sqrt{g} K_{ij} G^{ijkl} K_{kl} - S_V \]  

(3)

where

\[ K_{ij} = \frac{1}{2N} (\partial_t g_{ij} - D_i N_j - D_j N_i) \]  

(4)

and where we introduced generalized De Witt metric \( G^{ijkl} \) defined as [4]

\[ G^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - \lambda g^{ij} g^{kl} \]  

(5)

where \( \lambda \) is an arbitrary real constant. Finally note that \( D_i \) is the covariant derivative defined with the help of the metric \( g_{ij} \). The action \( S_V \) is the potential term action in the form

\[ S_V = \frac{M^2_p}{2} \int dt d^3 x N \sqrt{g} V = \frac{M^2_p}{2} \int dt d^3 x N \sqrt{g} \left( \mathcal{L}_1 + \frac{1}{M_p^2} \mathcal{L}_2 + \frac{1}{M_4^2} \mathcal{L}_3 \right) \]  

(6)

where \( \mathcal{L}_n \) contain all terms that are invariant under foliation preserving diffeomorphism and where \( \mathcal{L}_n \) contain 2n derivatives of the ADM variables \( (N, g_{ij}) \). In the UV when \( k \gg M_* \) the dominant contributions come from the higher derivative terms that lead to the modified dispersion relation \( m^2 \propto k^6 \) that implies that this theory is power counting renormalizable. In the opposite regime \( k \ll M_* \) the dispersion relation is relativistic and it can be shown that the theory have regions in the parameter space where it is in agreement with observation.
Despite these attractive properties there is a serious problem considering the Lorentz violation operators in the matter sector. In fact, while the direct bounds on Lorentz violations in the gravity sector are weak, the bounds on the Lorentz violating operators in the matter sector are very stringent. For that reason it is very important to prevent Lorentz violations leaking from the gravity sector to the matter sector.

One possibility how to resolve this problem was suggested in [5], where the Lorentz violating gravity sector couples to the Standard model through power-suppressed operators. However it turns out that this generic mechanism is not entirely successful in case of HL gravity due to the fact that non-dynamical vector gravitons are not modified with respect to GR which leads to the quadratic divergences that should be fine-tuned away. It was proposed in [5] to include a single term $D^i K_{jk} D^j K^i_{jk}$ to the action. The presence of this term modifies the vector graviton sector at linear order while leaves the tensor and scalar dispersion relation qualitatively unchanged. This proposal was further studied in [6] where the contributions of all terms of the form $(D^i K_{jk})^2$ were analyzed. It was shown there that all dispersion relations in the UV now become of the type $\omega^2 \propto k^4$ that are not enough for the standard power-counting renormalizability of HL gravity. However then it was argued in [6] that in the presence of the mixed derivative terms the power-counting relation is modified and these dispersion relations provide sufficient momentum suppressions in the amplitudes. These mixed derivative extensions were further analyzed in [7] where the new class of Lifshitz-like scalar theories that are power-counting renormalizable and unitary were introduced. In [8] this construction was extended to the case of HL gravity where the most general mixed derivative form of HL was proposed. However a careful perturbative analysis performed there shows that the presence of the new mixed derivative terms has a dramatic impact on the theory since they generate new degree of freedom. This is very interesting fact that certainly deserves to be analyzed further. Natural mechanism how to identify the number of degrees of freedom is to perform Hamiltonian analysis of this theory and this is exactly the goal of this paper. It turns out that this analysis is rather straightforward when we introduce appropriate auxiliary fields in order to replace mixed derivative terms with ordinary ones. The presence of these auxiliary fields then imply new second class constraints that can be solved for them at least in principle. In other words terms like $(D^i K_{jk})^2$ do not generate new dynamical degree of freedom. In fact, we show that this scalar degree of freedom is related to the existence of the time derivative of the lapse $N$.

The structure of this note is as follows. In the next section (2) we introduce mixed derivative HL gravity following [8]. Then in section (3) we find Hamiltonian formulation of this theory and determine number of degrees of freedom. In Summary (4) we outline our results.
2 Mixed Derivative HL Gravity

In order to construct mixed derivative HL gravity new additional part to the action was introduced in

\[ S_\kappa = \frac{M_p^2}{2M_2^2} \int dt d^3x \sqrt{g} \mathcal{L}_\kappa \]

that contains all Diff\(_F\) invariant operators that involve two spatial and two time derivatives. Generally the number of independent operators is of order \(10^2\). However we focus on terms that contribute to the quadratic action. Then we consider following contend of HL gravity

\[ \mathcal{L}_1 = 2\alpha a_i a^i + \beta R, \]
\[ \mathcal{L}_2 = \alpha_1 R D_i a^i + \alpha_2 D_i a_j D^j a^i + \beta R_{ij} R^{ij} + \beta_2 R^2, \]
\[ \mathcal{L}_3 = \alpha_3 D_i D^i R D_j a^j + \alpha_4 D^k D_k a_i D_j D^j a^i + \beta_3 D_i R_{jk} D^i R^{jk} + \beta_4 D_i R D^i R \]

while the mixed derivative terms have the form

\[ \mathcal{L}_\kappa = D_i K_{jk} D_j K_{mn} M^{ijklmn} + 2(\sigma_1 \mathcal{A}_i \mathcal{A}^i + \sigma_2 \mathcal{A}_i D^i K + \sigma_3 \mathcal{A}_i D_j K^{ij}), \]

where

\[ M^{ijklmn} = \gamma_1 g^{ij} g^{lm} g^{kn} + \gamma_2 g^{il} g^{jm} g^{kn} + \gamma_3 g^{ij} g^{jk} g^{mn} + \gamma_4 g^{ij} g^{kl} g^{mn}, \]

and where

\[ \mathcal{A}_i = \frac{1}{2N} (\partial_i a_i - N_j D_j a_i - a_j D_i N^j) \]

is Diff\(_F\) covariant combination that contains the time derivative of \(N\). There is also Diff\(_F\) covariant combination that contains the time derivative of the 3–curvature

\[ r_{ij} = \frac{1}{2N} (\dot{R}_{ij} - N_k D_k R_{ij} - R_{ik} D_j N^k - R_{jk} D_i N^k) \]

so that terms like \(K^{ij} r_{ij}\) and \(K r\) are Diff\(_F\) scalars with the right number of derivatives.

Our goal is to perform the Hamiltonian analysis of this theory in order to confirm that it contains new scalar degree of freedom. Before we proceed to this analysis we will argue that we can consistently ignore terms containing \(r_{ij} K^{ij}\) and \(r K\). To do this we use the formula for the variation of the Ricci tensor

\[ \delta R_{ij} = \frac{1}{2} g^{kl} [D_k D_j \delta g_{il} + D_k D_i \delta g_{jl} - D_i D_j \delta g_{kl} - D_k D_l \delta g_{ij}] \]
and hence
\[
\dot{R}_{ij} = \frac{1}{2} g^{kl} (D_k D_j \dot{g}_{il} + D_k D_i \dot{g}_{jl} - D_i D_j \dot{g}_{kl} - D_k D_l \dot{g}_{ij}) .
\] (14)

Now we use the fact that
\[
\dot{g}_{ij} = 2 N K_{ij} + D_i N_j + D_j N_i
\] (15)
so that \( r_{ij} \) has the form
\[
\begin{align*}
\dot{r}_{ij} &= \frac{1}{2N} g^{kl} (D_k D_j (2 NK_{il} + D_i N_l + D_l N_i) + D_k D_i (2 NK_{jl} + D_j N_l + D_l N_j) \\
&\quad - D_i D_j (2 NK_{kl} + D_k N_l + D_l N_k) - D_k D_l (2 NK_{ij} + D_i N_j + D_j N_i) \\
&\quad - N^k D_k R_{ij} - R_{ik} D_j N^k - R_{jk} D_i N^k) .
\end{align*}
\] (16)

Then certainly terms like \( K^{ij} r_{ij} \) and \( K r \) add additional terms in the action that contain \( a_i \) and its covariant derivative together with covariant derivative of \( K_{ij} \). As we will argue in the next section it is natural to replace \( K_{ij} \) with auxiliary field whenever covariant derivative acts on \( K_{ij} \). Then we see that these terms contribute as additional potential terms for these auxiliary fields and vector \( a_i \) and hence do not have an impact on the canonical structure of the theory. Then in order to simplify resulting analysis we will not include terms like \( K^{ij} r_{ij} \) and \( K r \) into the action.

3 Canonical Analysis

The specific property of the mixed derivative form of HL gravity is that it contains the covariant derivative of \( K_{ij} \) so that when we proceed to the Hamiltonian formalism we would get relation between conjugate momenta \( \pi_{ij} \) and differential operator of the second order in the spatial derivatives acting on \( K_{ij} \). Then in order to invert this relation we should introduce some kernel of this differential operator and we would find that \( K_{ij} \) is given as an integral over the second argument of this kernel multiplied by some functions of \( \pi_{ij} \). However then we would find that the Hamiltonian is non-local functional of conjugate variables and hence it is very difficult to show that this Hamiltonian has correct Poisson brackets with the generators of the spatial diffeomorphism. For that reason it is more natural to introduce auxiliary fields \( A_{ij} \) instead \( K_{ij} \) in the expressions that contain covariant derivative of \( K_{ij} \). Explicitly, we consider the action in the form
\[
\begin{align*}
S &= \frac{M_p^2}{2} \int dt d^3 x N \sqrt{g} (K_{ij} \mathcal{G}^{ijkl} K_{kl} - V) \\
&\quad + \frac{M_p^2}{2M^2} \int dt d^3 x N \sqrt{g} (D_i A_{jk} D_l A_{mn} M^{ijklmn} \\
&\quad + 2(\sigma_1 A_i A^i + \sigma_2 A_i D^i A + \sigma_3 A_i D_j A^{ij}) + B^{ij} (A_{ij} - K_{ij})) .
\end{align*}
\] (17)
However we still see that there is the second problem with the variable $A_i$ that contains the time derivative of $a_i = \frac{\partial N}{N}$. In order to simplify given expression let us begin with the following observations

\begin{equation}
\partial_t a_i = \partial_i \left( \frac{\partial_i N}{N} \right), \quad D_j a_j = D_j a_i
\end{equation}

so that $A_i$ can be written as

\begin{equation}
A_i = \frac{1}{2N} \partial_i \left( \frac{\partial_i N}{N} - \frac{N j}{N} \partial_j N \right).
\end{equation}

To proceed further we declare that $A_i$ is an independent variable when we introduce auxiliary field $Y^i$ and add following term to the action $Y^i \left( A_i - \frac{1}{2N} \partial_i \left( \frac{\partial_i N}{N} - \frac{N j}{N} \partial_j N \right) \right)$. Then with the help of the integration by parts we can rewrite the action into the form

\begin{equation}
S = \frac{M_p^2}{2} \int dtd^3x N \sqrt{g} \left( K_{ij} G^{ijkl} K_{kl} - V \right) + \frac{M_p^2}{2M_*^2} \int dtd^3x N \sqrt{g} \left( D_i A_{jk} D_t A_{mn} M^{ijklmn} \right) + 2(\sigma_1 A_i A^i + \sigma_2 A_i D^i A + \sigma_3 A_i D_j A^{ij}) + B^{ij} (A_{ij} - K_{ij})
+ \int dtd^3x \left( NY^i A_i + \frac{1}{2} \partial Y^i \left( \frac{\partial_i N}{N} - \frac{N j}{N} \partial_j N \right) \right)
\end{equation}

which is suitable for the Hamiltonian analysis. Explicitly from (20) we obtain

\begin{align*}
\pi^{ij} &= \frac{\delta L}{\delta \partial_i g_{ij}} = \frac{M_p^2}{2} \sqrt{g} G^{ijkl} K_{kl} - \frac{M_p^2}{2M_*^2} \sqrt{g} B^{ij}, \quad \pi^i = \frac{\delta L}{\delta \partial_i N_i} \approx 0, \\
p_{ij} &= \frac{\delta S}{\delta \partial_i B^{ij}} \approx 0, \quad q^{ij} = \frac{\delta S}{\delta \partial_i A_{ij}} \approx 0, \\
p^i &= \frac{\delta L}{\delta \partial_i A_i} \approx 0, \quad p_i = \frac{\delta L}{\delta \partial_i Y^i} \approx 0, \quad \pi_N = \frac{1}{2N} \partial_i Y^i,
\end{align*}

where the last relation implies following primary constraint

\begin{equation}
\Phi_N = \pi_N N - \frac{1}{2} \partial_i Y^i \approx 0.
\end{equation}

\footnote{Since we are interested in the local properties of this theory and the number of physical degrees of freedom we can ignore boundary terms.}
Then the Hamiltonian has the form
\[ H = \int d^3x (\pi_N \partial_t N + \pi^{ij} \partial_i g_{ij} - \mathcal{L}) = \int d^3x \left( N\mathcal{H}_0 + N^i \left( -2g_{ik}D_j\pi^{jk} + \frac{1}{2N} \partial_i N \partial_j Y^j \right) \right), \tag{23} \]

where
\[ \mathcal{H}_0 = \frac{2}{M_p^2 \sqrt{g}} \left( \pi^{ij} + \frac{M_p^2}{2M_*^2} \sqrt{g}B^{ij} \right) g_{ijkl}(\pi^{kl} + \frac{M_p^2}{2M_*^2} \sqrt{g}B^{kl}) + \frac{M_p^2}{2} \sqrt{g} V - \frac{M_p^2}{2M_*^2} \sqrt{g} (D_i A_{jk} D_l A_{mn} M^{ijklmn} \right) + 2(\sigma_1 A_i A^i + \sigma_2 A_i D^i A + \sigma_3 A_i D_j A^{ij}) + B^{ij} A_{ij} - Y^i A_i. \tag{24} \]

Observe that with the help of the constraint \( \Phi_N \) we can rewrite an expression in the second bracket into the form
\[ -2g_{ik}D_j\pi^{jk} + \frac{1}{2N} \partial_i N \partial_j Y^j = -2g_{ik}D_j\pi^{jk} + \pi_N \partial_i N - N^i \partial_i N \Phi_N. \tag{25} \]

Then it is easy to see that an extended Hamiltonian has the form
\[ H_E = \int d^3x (N\mathcal{H}_0 + N^i \mathcal{H}_i + v_N \Phi_N + P^iv_i + v^i p_i + v^{ij} p_{ij} + w^{ij} q^{ij}) \tag{26} \]

where
\[ \mathcal{H}_i = -2g_{ik}D_i\pi^{kl} + \pi_N \partial_i N. \tag{27} \]

As the next step we analyze the requirement of the preservation of all primary constraints. We begin with \( \pi_i \approx 0 \)
\[ \partial_t \pi_i = \{\pi_i, H_E\} = -\mathcal{H}_i \approx 0. \tag{28} \]

However \( \mathcal{H}_i \) is not the correct form of the spatial diffeomorphism constraints since it has vanishing Poisson brackets with auxiliary fields. In order to find the correct form of the spatial diffeomorphism constraints we add appropriate linear combinations of the primary constraints to it so that we define \( \tilde{\mathcal{H}}_i \) as
\[ \tilde{\mathcal{H}}_i = \mathcal{H}_i - 2\partial_h(A_{ij} g^{jk}) + \partial_i A_{jk} q^{jk} + 2\partial_j (B^{jk} p_{ik}) + \partial_l B^{jk} p_{jk} - \partial_i p_j Y^j + \partial_j (p_i Y^i) \tag{29} \]

and its smeared form
\[ T_S(N^i) = \int d^3x N^i \tilde{\mathcal{H}}_i. \tag{30} \]
Note that the extra terms that we added to $\mathcal{H}_i$ are proportional to the primary constraints. Then we obtain the following Poisson brackets

$$\{ T_S(N^i), A_{mn} \} = -\partial_n N^i A_{im} - \partial_m N^i A_{in} - N^i \partial_i A_{mn} ,$$
$$\{ T_S(N^i), B^{mn} \} = \partial_j N^m B^{jn} + \partial_j N^m B^{jm} - N^i \partial_i B^{mn} ,$$
$$\{ T_S(N^i), a_j \} = -N^k \partial_k a_i - \partial_i N^k a_k ,$$
$$\{ T_S(N^i), Y^i \} = -\partial_k (N^k Y^i) + \partial_j N^j Y^i .$$

(31)

Note that $Y^i$ transforms as a vector density. These relations show that $\tilde{\mathcal{H}}_i$ are correct generators of spatial diffeomorphism. Further, they are preserved during the time evolution of the system due to the fact that the Hamiltonian is manifestly invariant under spatial diffeomorphism and also due to the fact that the Poisson brackets between the smeared form of these constraints is equal to

$$\{ T_S(N^i), T_S(M^j) \} = T_S(N^j \partial_j M^i - M^j \partial_j N^i) .$$

(32)

Now we consider time evolution of the constraint $P^i \approx 0$

$$\partial_t P^i = \{ P^i, H \} = N \left( \frac{M_p^2}{M_*^2} \sqrt{g} (2\sigma_1 A^i + \sigma_2 D^i A + \sigma_3 D^j A^{ij}) + Y^i \right) \equiv N \Phi^i_{II} \approx 0$$

(33)

that implies an existence of the secondary constraints $\Phi^i_{II} \approx 0$. Let us now analyze the time evolution of the constraint $\Phi_N \approx 0$

$$\partial_t \Phi_N = \{ \Phi_N, H \} = \left\{ \Phi_N(x), \int d^3y N(y) \mathcal{H}_0(y) \right\} + \frac{1}{2} \partial_i v^i = 0 .$$

(34)

Naively we should say that this equation determines the Lagrange multipliers $v^i$ which is not correct since it would imply that one equation determines three components of $v^i$. In order to resolve this issue note that we can replace $v^i$ with arbitrary combinations of another Lagrange multipliers and hence let us presume $v^i$ in the form $v^i = \epsilon^{ijk} \partial_j \tilde{v}_k$ that obeys $\partial_i v^i = 0$. With this ansatz the equation (33) implies an existence of the secondary constraint $C$. In order to find its explicit form we use the fact that

$$\{ \pi_N(x), a_i(y) \} = \frac{1}{N} \partial_i \delta(x - y)$$

(35)

and hence

$$\left\{ \Phi_N(x), \int d^3y N(y) \mathcal{H}_0(y) \right\} - N(x) \mathcal{H}_0(x) + \frac{\partial}{\partial x^i} \left( \frac{\delta \mathcal{H}_0}{\delta a_i(x)} \right) \equiv -NC(x) ,$$

(36)

where $C = \mathcal{H}_0 - \frac{1}{N} \partial_i \left[ \frac{\partial \mathcal{H}_0}{\partial a_i} \right]$. Note that this constraint is the generalization of the constraint $C$ known from the non-projectable HL gravity \[^9,^10,^11\] to its mixed derivative generalization.
It is also important to stress that when we replace the Lagrange multipliers $v^i$ with $\tilde{v}^i$ we find that after integration by parts the original constraints $p_i \approx 0$ are replaced with equivalent ones

$$\psi_p^i = \epsilon^{ijk} \partial_j p_k \approx 0 .$$

(37)

Now the requirement of the preservation of the constraint $\psi_p^i \approx 0$ implies

$$\partial_t \psi_p^i = \left\{ \psi_p^i, H \right\} = \epsilon^{ijk} \partial_j (N A_k) - \frac{1}{2} \epsilon^{ijk} \partial_j \partial_k \pi_N = \epsilon^{ijk} \partial_j (N A_k) \equiv \Psi_{II}^i \approx 0 .$$

(38)

Note that $\psi_p^i$ and $\Psi_{II}^i$ are not independent but obey the relation

$$\partial_i \psi_p^i = 0 , \partial_i \Psi_{II}^i = 0 .$$

(39)

This is very important result which will be useful when we calculate the number of physical degrees of freedom.

Now we analyze the requirement of the preservation of the constraints $p_{ij} \approx 0$

$$\partial_t p_{ij} = \left\{ p_{ij}, H_E \right\} = \frac{4}{M_p^2} G_{ijkl} (\pi^{kl} + \frac{1}{2} \sqrt{g} B^{kl}) - N \sqrt{g} A_{ij} \equiv -N \Phi_{II}^{ij} \approx 0 ,$$

(40)

and $q^{ij} \approx 0$

$$\partial_t q^{ij} = \left\{ Q^{ij}, H_E \right\} = N \left( \frac{M_p^2}{M_*^2} \sqrt{g} a_k M^{kijl} D_l A_{mn} - \frac{M_p^2}{M_*^2} \sqrt{g} \sigma_2 a^k A_k g^{ij} \right.$$

$$- \frac{M_p^2}{2M_*^2} (a^i A^j + a^j A^i) - \sqrt{g} B^{ij} + \frac{M_p^2}{M_*^2} \sqrt{g} D_k (M^{kijl} D_l A_{mn}) - \frac{M_p^2}{M_*^2} \sqrt{g} \sigma_2 D^k (A_k) g^{ij}$$

$$- \frac{M_p^2}{2M_*^2} (D^i (A^j) + D^j (A^i)) - \sqrt{g} B^{ij} \right) \equiv N \psi_{II}^{ij} \approx 0 .$$

(41)

It is easy to see that $\Phi_{II}^{ij}, \Psi_{II}^{ij}$ are the second class constraints with $q^{ij} \approx 0 , p_{ij} \approx 0$ so that they vanish strongly and should be solved at least in principle. Further, these constraints do not depend on $N$ which is also very important.

In summary we have following collection of the second class constraints

$$\psi_p^i \approx 0 , \quad \Psi_{II}^i \approx 0 , \quad \mathcal{P}^i \approx 0 , \quad \Phi_{II}^i \approx 0 , \quad \Phi_{N} \approx 0 , \quad \mathcal{C} \approx 0 .$$

(42)

From $\Phi_{II}^i$ we can express $A_i$ as function of the canonical variables at least in principle. Using $\mathcal{C}$ and $\Phi_{N}$ we can eliminate $\pi_N$ and $N$. Finally note that $\psi_p^i$ and $\Psi_{II}^i$ can be solved as

$$p_i = \partial_t \pi , N A_i = \partial_t \phi$$

(43)

which implies an existence of one scalar degree of freedom $\phi$ with momenta $\pi$. This result also explains why we used the Lagrange multipliers $\tilde{v}_i$ instead of $v^i$. The goal
was to separate the second class constraints into two ones $C, \Phi_N$ that eliminate $N$ and conjugate momenta and the remaining ones that implies an existence of the scalar degree of freedom. Finally note that $\Phi^{ij}_{II}$ and $\Psi^{ij}_{II}$ can be solved for $A_{ij}$ and $B_{ij}$ at least in principle so that there are no new additional dynamical degrees of freedom. This is expected result since the spatial derivative of $K_{ij}$ cannot generate new dynamical degrees of freedom. Finally note that three first class constraints $\bar{\mathcal{H}}_i$ can be gauge fixed and we eliminate three degrees of freedom from $g_{ij}$. The remaining three degrees of freedom correspond to the massless graviton and one scalar degree of freedom. In summary, mixed derivative HL gravity contains two additional scalar degrees of freedom with respect to GR.

As the last point we proceed to the question of an existence of the global constraints. Note that the action is invariant under foliation preserving diffeomorphism and hence we expect an existence of two global first class constraints as in case of non-projectable HL gravity $[11]$. Let us introduce the first one

$$\Pi_N = \int d^3x \Phi_N = \int d^3x \pi_N N$$

that has following Poisson brackets

$$\{\Pi_N, N\} = -N, \quad \{\Pi_N, \pi_N\} = \pi_N, \quad \{\Pi_N, a_i\} = 0$$

and

$$\{\Pi_N, \Psi^{ij}_{II}\} = -\epsilon^{ijk} \partial_j (N A_k) = -\Psi^{ij}_{II} \approx 0.$$  

In other words $\Pi_N$ poisson commutes with all second class constraints that do not depend on $N$ explicitly. The situation is slightly more complicated in case of the constraint $C(x)$ we use following result

$$\{\Pi_N, C(x)\} = \{\Pi_N, \{\Phi_N(x), H_E\}\} =$$

$$\{\Phi_N(x), \{\Pi_N, H_E\}\} = -\left\{\Phi_N(x), \int d^3y N(y) \mathcal{H}_0(y)\right\} = NC(x) \approx 0.$$  

In summary we found that $\Pi_N$ has vanishing Poisson bracket with all second class constraints on the constraint surface.

Then, following $[9]$ we split $\Phi_N$ as

$$\ddot{\Phi}_N = \Phi_N - \frac{\sqrt{g_N}}{\int d^3x \sqrt{g_N}} \Pi_N ,$$

that obeys the equation

$$\int d^3x \ddot{\Phi}_N = 0.$$  

In other words we have $\infty^3 - 1$ local second class constraints $\ddot{\Phi}_N$ and one global first class constraint $\Pi_N$. Finally the requirement of the preservation of the constraint

$^3$We use notations introduced in $[12]$. It is important to stress that $\ddot{\Phi}_N$ and $\Pi_N$ consist $\infty^3$ constraints which corresponds to the number of the constraints $\Phi_N$. 

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\[ \Pi_N \text{ implies} \]
\[ \partial_t \Pi_N = \{ \Pi_N, H_E \} = - \int d^3 x N \mathcal{H}_0 \equiv - \Pi_T \approx 0 , \tag{50} \]
where we introduced new first class constraint \( \Pi_T \approx 0 \) and we used the fact that \( \mathcal{H}_0 \) depends on \( a_i \) only and hence Poisson commutes with \( \Pi_N \). Observe that using the explicit form of \( C \) we have the relation
\[ \int d^3 x N C = \int d^3 x N \mathcal{H}_0 . \tag{51} \]
Then we can introduce the constraint
\[ \tilde{C}(x) = C(x) - \frac{\sqrt{g}}{\int d^3 x \sqrt{g} N} \Pi_T \tag{52} \]
that obeys the relation
\[ \int d^3 x N \tilde{C} = 0 \tag{53} \]
so that there is \( \infty^3 - 1 \) second class constraints \( \tilde{C} \).

As we argued previously \( \Pi_N \) is the first class constraint. Further, since \( \mathcal{H}_0 \) does not depend on \( N \) but on \( a_i \) only we find that
\[ \{ \Pi_N, \Pi_T \} = 0 . \tag{54} \]
On the other hand it is clear that \( \Pi_T \) does not have vanishing Poisson brackets with all second class constraints and hence we cannot say that it is the first class constraint. In order to resolve this problem we proceed as follows. As the first step we introduce common notation for all second class constraints \( \Psi_A \) and denote their Poisson brackets as
\[ \{ \Psi_A(x), \Psi_B(y) \} = \triangle_{AB}(x, y) \tag{55} \]
with inverse
\[ \int d^3 y \triangle_{AB}(x, y) \triangle_{BC}(y, z) = \delta_A^C \delta(x, z) . \tag{56} \]
Then we define following constraint
\[ \tilde{\Pi}_T = \Pi_T - \int d^3 x d^3 y \{ \Pi_T, \Psi_A(x) \} \triangle_{AB}(x, y) \Psi_B(y) \tag{57} \]
that clearly obeys the relation
\[ \{ \tilde{\Pi}_T, \Pi_N \} = 0 , \{ \tilde{\Pi}_T, \Psi_A(x) \} = 0 \tag{58} \]
and also
\[ \{ \tilde{\Pi}_T, \tilde{\Pi}_T \} = 0 . \tag{59} \]

In summary we have found the second first class constraint \( \tilde{\Pi}_T \) that reflects the invariance of the action under foliation preserving diffeomorphism. Of course, this constraint reduces to \( \Pi_T \) when all second class constraints vanish strongly.
4 Summary

This short note was devoted to the Hamiltonian analysis of the mixed derivative extension of HL gravity. We showed that there is a new scalar mode with agreement with the perturbative analysis performed in [8]. In other words mixed derivative HL gravity contains two additional scalar modes with respect to the GR. The presence of these modes could have huge impact on the consistency of the theory and on its phenomenological applications as was discussed in [8].

Naively we should say that when we add terms with time derivative of the lapse $N$ then the presence of the new dynamical degree of freedom is obvious but the situation is not so simple. The reason is that $N$ is not ordinary scalar but it transforms under foliation preserving diffeomorphism as

$$N'(t', x') = (1 - f(t))N(t, x).$$ (60)

Than the naive time derivative $\nabla_n N = \frac{1}{N}(\partial_t N - N^i \partial_i N)$ is not invariant under foliation preserving diffeomorphism and it turns out that the only possible covariant expression that contains time derivative of $N$ is a vector $A_i$ and hence terms that are presented in $L_\kappa$ that contain mixed derivatives. As we saw in the main body of this paper these terms make the Hamiltonian analysis rather non-trivial.

The second important point was to identify two global first class constraints whose existence is predicted by the fact that the mixed derivative HL gravity is invariant under foliation preserving diffeomorphism. We found these constraints and we showed that they are the first class constraints.

We can also comment the issue of two additional degrees of freedom. In principle they could be eliminated when we introduce some Lagrange multiplier modified terms to the action as for example in [9]. These new second class constraints could eliminate these additional degrees of freedom at least in principle. However it is important to stress that the resulting theory will be very complicated. Even without these additional terms the mixed derivative HL gravity has very complicated symplectic structure due to the fact that there are second class constraints that contain differential operators. These facts together with the existence of two scalar degrees of freedom is very important problem when we consider mixed derivative HL gravity as a candidate of the renormalizable theory of gravity.

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