A Spinor Approach to Penrose Inequality

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Abstract

Consider an asymptotically Euclidean initial data set with a smooth marginally trapped surface (possibly a union of future and past multi-connected components) as inner boundary. By a further development of the spinorial framework underlying the positive energy theorem, a refined Witten identity is worked out and in the maximal slicing case, a close connection of the identity with a conformal invariant of Yamabe type is revealed. A Kato-Yau inequality for the Sen-Witten operator is also proven from a conformal geometry perspective. Guided by the Hamiltonian picture underlying the spinorial framework, a Penrose type inequality is then proven to the effect that given the dominant energy condition, the ADM energy-momentum is, up to a non-zero constant less than unity, bounded by the areal radius of the marginally trapped surface. To establish the Penrose inequality in full generality, it is then sufficient to show that the norm of the Sen-Witten spinor, subject to the APS boundary condition imposed on a suitably defined outermost marginally trapped surface, is bounded below by that attained in the Schwarzschild metric.

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I. INTRODUCTION

When the Penrose inequality is regarded as a strengthened form of the positive energy theorem for black holes, it is natural to ask whether the spinorial proof of the positive energy theorem, first initiated by Witten [20], may be suitably generalised to tackle the Penrose inequality, particularly in the outstanding case when the initial data set is not time symmetric (see [3, 9] in the time symmetric case).

Given the three manifold of an initial data set, underlying the spinorial approach to the positive energy theorem is the physical picture that a non-zero spinor field together with its dual (defined in terms of the timelike unit normal of the three manifold in spacetime) generate a Newman-Penrose tetrad, from which an orthonormal moving frame is further defined and plays the role of canonical variables in describing the Hamiltonian dynamics of a gravitational field [1, 15]. In this sense, the Sen-Witten equation may be regarded as a gauge condition to select a moving frame on a three manifold (see also [16]) to parametrise the Hamiltonian.

To explore a spinorial approach to the Penrose inequality, so far two obstacles have been encountered. The first one is the need to further develop Witten’s spinorial technique by taking the fourth root of it, in a sense to be made precise in what follows. Another obstacle is the incompatibility of the APS boundary condition imposed on a spinor field with the marginally trapped boundary condition imposed on the inner boundary. The flagpole of the Sen-Witten spinor field subject to the APS boundary condition in general will not align with one of the two null normals of the marginally trapped surface under consideration. We shall seek to address these two issues in the present work and it turns out a better understanding of the Hamiltonian picture underlying the spinorial approach enables us to find a way to go forward. A Penrose type inequality involving the ADM energy-momentum for a generic asymptotically Euclidean initial data set then emerges naturally for the first time. The obstacle to a complete proof of the Penrose inequality is also identified.

For a good description of the Penrose inequality, see [17]. A review of the the Penrose inequality may be found in [14]. To tackle the Penrose inequality using spinors was also considered in [8] and further generalised in [12], with however only the ADM mass considered. As we shall see in what follows, the line of argument presented here is Hamiltonian in essence and in many ways distinct from the previous spinor approach.
The outline of the article may be given as follows. After certain preliminaries in Section 2, in Section 3 we shall seek to further develop the spinorial framework used in the proof of the positive energy theorem and a new refined Witten identity is worked out. Certain geometric structures underneath the refined identity will also be discussed. By twisting the Sen-Witten spinor field in a sense to be described, a new shift vector for the Hamiltonian is defined in Section 4 and its obstruction to the positivity of the refined Witten identity is addressed. A Penrose type inequality for the ADM energy momentum for a generic asymptotically Euclidean initial data set is then presented for the first time. The rest of the paper then serves to fill in the details of the proof of the main theorem presented in Section 4, including the derivation of a refined Witten identity, regularisation of zero points of the Sen-Witten spinor field and the proof of existence and uniqueness of the Sen-Witten spinor field, given the APS boundary condition at the inner boundary and appropriate falloff near spatial infinity.

II. PRELIMINARIES AND NOTATIONS

Some background materials relevant to the present work will be briefly described in this section. The notations for two spinors will follow that in [18] unless otherwise stated.

Let \((M, g_{ab})\) be a smooth, connected four dimensional spacetime manifold with Lorentzian metric signature \((+,-,-,-)\). Suppose \(N\) is an orientable, complete Riemannian three manifold identically embedded in \(M\) so that when restricted to \(N\),

\[
g_{ab} = \tau_a \tau_b - h_{ab},
\]

where \(h_{ab}\) is a smooth Riemannian metric of \(N\) and \(\tau^a\) is the unit timelike normal of \(N\) in \(M\). \(N\) is assumed to be asymptotically Euclidean in the standard sense that in the complement of some compact set in \(N\),

\[
h_{ab} = \eta_{ab} + O(1/r),
\]

\(\eta_{ab}\) is an Euclidean metric and

\[
\partial h_{ab} = O(1/r^2), \quad \partial^2 h_{ab} = O(1/r^3),
\]

where \(r\) is the standard radial parameter defined in terms of the Cartesian coordinates near infinity. When \(N\) is considered as a spacelike hypersurface identically embedded in \(M\), the
second fundamental form of $N$ in $M$ is given by $K_{ab} = h^i_a h^m_b \nabla_i \tau_m$ and in the asymptotic regime,

$$K_{ab} = O(1/r^2), \quad \partial K_{ab} = O(1/r^3).$$

As a codimension one submanifold of $(M, g_{ab})$, the geometry of $(N, h_{ab}, K_{ab})$ is also subject to the Hamiltonian and momentum constraint equations given respectively by

$$R = 2\mu + |K_{ab}|^2 - K^2, \quad (1)$$
$$j_b = D^a (K_{ab} - Kh_{ab}), \quad (2)$$

where $R$ is the scalar curvature of $(N, h_{ab})$, $K = h^{ab}K_{ab}$, $\mu$ and $j_a$ are respectively the density and current of local matter as measured by an observer at rest with respect to $N$. The four vector $(\mu, j^a)$ is required to satisfy the dominant energy condition $\mu \geq j^a j_a$ throughout the present work.

Denote by $\partial N$ the inner boundary of $N$. $\partial N$ is assumed to consist of connected components $S_i$, $i = 0, 1, \cdots, n$ with each $S_i$ a smooth spherical two surface. Let $\gamma_{ab}$ and $p$ be respectively the two metric and the mean curvature of $S_i$ defined with respect to the outward pointing normal. Then

$$\gamma^{ab} K_{ab} \pm p = 0 \quad (3)$$

characterise $S_i$ as a future (+) and past (−) marginally trapped surface.

Denote by $\tau^{AA'}$ the timelike unit normal of $N$ in spinorial indices. Let $\nabla_{AA'}$ be the spin connection lifted from the metric connection of $(M, g_{ab})$, the projection of $\nabla_{AA'}$ on $N$ may be given as $[19]$ \[D_{AB} := \sqrt{2} \tau_{(B}^A \nabla_{A')}.\] (4)

Denote by $D_{AB}$ the spin connection of $(N, h_{ab})$, it may be defined in terms of $D_{AB}$ as

$$D_{AB} \lambda_C = D_{AB} \lambda_C - \frac{1}{\sqrt{2}} K_{ABCD} \lambda^D, \quad (5)$$

where $K_{ABCD} = 2 \tau_{B}^{A'} \tau_{D}^{C'} K_{A'A'CC'}$ and $K_{A'A'CC'}$ is the second fundamental form of $N$ in spinorial indices.

We shall adopt the following Sen-Witten equation as the gauge condition to specify a spin frame in $(N, h_{ab}, K_{ab})$ given by

$$D_A^C \lambda_C = 0. \quad (6)$$
Away from the zero points of $\lambda^A$, a non-trivial dual of $\lambda^A$ may be defined in terms of $\tau_{AA'}$ as

$$\lambda^{\dagger A} = \sqrt{2} \tau^{AA'} \lambda_{A'}.$$  

We further subject $\lambda^A$ to the asymptotic boundary conditions that, near infinity,

$$\lambda^A = \lambda^A_0 + O(1/r)$$

where $\lambda^A_0$ is a covariantly constant spinor defined with respect to the flat connection of $\eta_{ab}$. At the inner boundary $S$, let $\nabla_{AC}^I$ be the spin connection pertained to the two metric of $S$. $\lambda^A$ is said to satisfy the APS (spectral) boundary condition at $S[2]$ (see also [8]) in that

$$\lambda_A = \sum_{n=0}^{\infty} a_n \lambda_{nA}, \quad a_n \in \mathbb{C}. \quad (7)$$

$\lambda_{nA}$ are eigenspinors given by

$$\nabla^A_{AC} \lambda_{nC} = \frac{1}{\sqrt{2}} \mu_n \lambda_{nA}, \quad \mu_n > 0 \quad \text{for} \quad n = 0, 1, 2 \ldots$$

and $\{\lambda^A_n\}_{n=0,1,2\ldots}$ constitute an orthonormal basis defined by the natural $l^2$ scalar product. $| \ |^2$ denotes the hermitian norm of a spinor field defined with respect to $\tau^{AA'}$.

Throughout the present work, contraction of tensorial and spinorial indices are always defined with respect to $h_{ab}$ and the symplectic form $\epsilon_{AB}$ respectively unless otherwise stated.

**III. DEVELOPMENT OF THE SPINORIAL FRAMEWORK**

Let us begin by looking at the simple example of a constant time slice of the Schwarzschild metric, whose metric is given by

$$ds^2 = \left(1 + \frac{M}{2r}\right)^4 (dr^2 + r^2 d\Omega^2).$$

Calculations on this simple example suggest that the conventional spinorial approach will not yield an optimal Penrose inequality. Instead, we need to further develop the Witten identity by taking its fourth root in the following sense.

Define

$$u^4 = \lambda_A \lambda^{\dagger A}. \quad (8)$$
Provisionally we assume \( u > 0 \) (i.e. \( \lambda^A \) is non-zero everywhere in \( N \)) and seek to relax this later. The example of the Schwarzschild metric leads us to adopt the following definition of a two surface functional.

\[
M(S) = \frac{1}{2\pi} \int_S D_a u \, dS^a,
\]

where \( S \) is a spherical two surface embedded in \( N \). For a round sphere of radius \( r \geq \frac{M}{2} \) in a constant time slice of the Schwarzschild metric, (9) always yields \( M \). So at least in this simple example, the definition in (9) resembles the Hawking mass in that it yields the irreducible mass for a black hole at the outermost marginally trapped surface and the ADM mass at infinity.

A couple of remarks (caveats) of the definition are in order here. In the simple case of Euclidean \( \mathbb{R}^3 \) with a non-round sphere chosen as the inner boundary, the mass functional yields negative value and goes to zero at infinity from below zero. This turns out to be a blessing in disguise and is related to a more general Minkowski inequality in Euclidean \( \mathbb{R}^3 \). This problem will be taken up elsewhere. In the present context, we shall take a pragmatic stand and look on the definition as a useful handle to linking up the ADM energy at spatial infinity and a spinorial analog at a marginally trapped surface. Further, at points where \( \lambda^A \) is zero, pointwise the gradient term \( D_a u \) becomes singular. We will address this problem later on.

Given \( u \) defined in (8), the next natural step to take is to work out a Witten type identity for it. Written in terms of \( u \), the Hamiltonian part of the conventional Witten identity may be given as

\[
\triangle u^4 = \frac{4u^3}{2} \frac{\lambda^A}{u^4} + 4u^2 D_a u D^a u
\]

\[
= \left( \mu + \frac{1}{2} |K_{ab}|^2 \right) u^4 + 2 |D_{AB} \lambda_C|^2 - \sqrt{2} \lambda^A \lambda^{AB} D_{AB} K
\]

(10)

By our provisional hypothesis, \( u > 0 \), we may normalise \( \lambda_A, \lambda_B^A \) and define a spin frame \((\omega_A, \iota_A)\) by

\[
\lambda_A = u^2 \omega_A, \quad \lambda_A = u^2 \iota_A.
\]

(11)

(10) may then be written as

\[
4u^3 \triangle u + 4u^2 D_a u D^a u
\]

\[
= \left( \mu + \frac{1}{2} |K_{ab}|^2 \right) u^4 + 2 u^4 |D_{AB} \omega_C|^2 - \sqrt{2} u^4 \omega^A \iota_B D_{AB} K.
\]

(12)
To elaborate further, we shall exploit the conformal rescaling symmetries of the Sen-Witten equation. Define
\[ \hat{\tau}_a = u^2 \tau_a, \quad \hat{h}_{ab} = u^4 h_{ab}. \]  
(13)

In the simple case of the Schwarzschild metric, conformal flatness means that \( \hat{h}_{ab} \) is just the Euclidean metric. Denote by \( \hat{D}_{AB} \) the conformally rescaled Sen-Witten connection defined in terms of \( \hat{\tau}_a \) and \( \hat{h}_{ab} \) given above. Conformal rescaling symmetry of (6) means that we also have
\[ \hat{\epsilon}^{BC} \hat{\Phi}_{AB} \hat{\lambda}_C = 0 \]  
(14)

with
\[ \hat{\lambda}_C = u^{-1} \lambda_C = u \omega_C, \quad \hat{\lambda}_C^\dagger = u^{-1} \lambda_C^\dagger = u \iota_C \]  
(15)

according to (11) and \( \hat{\epsilon}^{AM} = u^{-2} \epsilon^{AM} \) is the conformally rescaled symplectic form. It may further be checked that \( \hat{\epsilon}^{AB} \hat{\lambda}_A \hat{\lambda}^*_B = 1 \) and therefore \( (\hat{\lambda}_C, \hat{\lambda}_C^\dagger) \) generate a spin frame under \( \hat{\epsilon}^{AB} \). Using the Sen-Witten equation and after some very tedious spinor calculus, we work out the following spinor identity
\[ |D_{AB} o_C|^2 = u^4 |\hat{D}_{AB} \hat{\lambda}_C|^2 + 2 |D_a \ln u|^2 - K \nu^a D_a \ln u \]  
(16)

where \( \hat{D}_{AB} \) is the conformally rescaled spin connection of \( D_{AB} \) and \( \nu_a = \sqrt{2} \omega_{(A \iota B)} \). Details of the derivation of (16) will be presented later on. Let us check that in the maximal slicing case when \( K = 0 \), we may infer from (16) the following Kato-Yau inequality for a harmonic spinor field expressed as
\[ |D_{AB} \lambda_C|^2 \geq \frac{3}{2} |D_a |\lambda||^2 \]  
(17)

where \( |\lambda| = u^2 \) (cf [6] and references therein). This may be regarded as a consistency check on the validity of the spinor identity in (16) and at the same time gives a new proof of the Kato-Yau inequality for harmonic spinor field from a conformal geometry perspective.

Given (16), (12) may be further expressed as
\[ \Delta u = \frac{1}{4} (\mu + \frac{1}{2} |K_{ab}|^2) u + \frac{1}{2} u^5 |\hat{D}_{AB} \hat{\lambda}_C|^2 \\ + \frac{1}{4} u \nu^a D_a K - \frac{1}{2} K \nu^a D_a u. \]  
(18)

With the momentum constraint further taken into account and the shift vector chosen to
be $N_a = u \nu_a$, it follows from (12) and (16) that

$$\triangle u - \frac{1}{4} D^a (K_{ab} N^b)$$

$$= \frac{1}{4} (\mu - j^a \nu_a) u + \frac{1}{2} u^5 \hat{D}_{AB} \hat{\lambda}_C |^2$$

$$+ \frac{1}{4} (\frac{1}{2} |K_{ab}|^2 - K_{ab} D^a \nu^b) u + \frac{1}{4} K_{ab} \nu^a D^b u - \frac{1}{2} K^a D_a u$$

$$= \frac{1}{4} (\mu - j^a \nu_a) u + \frac{1}{2} u^5 \hat{D}_{AB} \hat{\lambda}_C |^2$$

$$+ \frac{1}{4} (\frac{1}{2} |K_{ab}|^2 - K_{ab} D^a \nu^b) u - \frac{1}{2} K^a D_a u$$

(19)

where $|\hat{D}_{AB} \hat{\lambda}_C|$ is defined in terms of the conformally rescaled symplectic form $\hat{\epsilon}_{AB}$. In terms of the definition of $K_{ab}$ and some simple spinor calculus, it may be worked out that the spurious term $K_{ab} \nu^a D^b u$ vanishes in the first equality in (19). Subject to the conformal rescaling given in (13), we have

$$\hat{K}_{ab} = u^2 K_{ab}, \quad \hat{D}_{(a} \hat{\nu}_{b)} = u^2 D_{(a} \nu_{b)} - 2 h_{ab} \nu^a D_a u$$

(20)

where $\hat{\nu}_a = u^2 \nu_a$. From (20), it may be deduced that

$$\frac{1}{2} |K_{ab}|^2 - K_{ab} D^a \nu^b = (\frac{1}{2} |\hat{K}_{ab}|^2 - \hat{K}_{ab} \hat{D}^a \hat{\nu}^b) u^4$$

(21)

where contraction of indices on the right hand side of (21) is defined in terms of $\hat{h}_{ab}$. With (21) input into (19), we then find

$$\triangle u - \frac{1}{4} D^a (K_{ab} N^b)$$

$$= \frac{1}{4} (\mu - j^a \nu_a) u + \frac{1}{2} u^5 \hat{D}_{AB} \hat{\lambda}_C |^2$$

(22)

which may be regarded as a refinement of the conventional Witten identity, with the fourth root of the spinor norm $u$ in place of the spinor norm $\varphi$ in the identity.

From (16) together with the definition of the Sen-Witten operator in (4), a Kato-Yau inequality for the Sen-Witten operator may also be worked out for the first time to be

$$|\mathcal{D}_{AB \lambda_C}|^2 \geq \frac{3}{2} |D_a |\lambda| |^2.$$  

(23)

Further, in the maximal slicing case, the Hamiltonian part of the refined Witten identity in (22) gives

$$\triangle u = \frac{1}{8} R u + \frac{1}{2} u^5 |\hat{D}_{AB} \hat{\lambda}_C|^2.$$  

(24)
(24) resembles a conformal Laplacian if we formally identify the scalar curvature defined by the metric connection of $\hat{h}_{ab}$ as $\hat{R} = -4|\hat{D}_{AB}\hat{\lambda}_C|^2$. This formal identification actually gains weight if we work out the Witten identity for $\hat{\lambda}_A$.

The resemblance of (24) to a conformal Laplacian leads us to consider the following conformal invariant appearing naturally in the Yamabe problem. For a real valued function $f$ in $N$, consider the following functional

$$\int_N |D_{a}f|^2 + \frac{1}{8} R f^2 + \frac{1}{4}\left(\int_{S_\infty} f^2 p - \int_S f^2 p\right),$$

(25)

where $|D_{a}f|^2 = h^{ab}D_{a}fD_{b}f$, $p$ is the mean curvature of the boundary $S_\infty \cup S$ with the normal of the boundary outward pointing. $S_\infty$ is a coordinate sphere near spatial infinity while $S$ is the inner boundary. Instead of the standard choice of compactly supported test functions, we allow $f$ to behave asymptotically as $f = f_0 + O(1/r)$ for some constant $f_0$.

As the choice of test functions in the functional (25) is no longer restricted to be compactly supported and allowed to be asymptotically constant, we may choose $u^2$ as a test function and the functional in (25) becomes

$$\int_N |D_{a}u|^2 + \frac{1}{8} R u^4 + \frac{1}{4}\left(\int_{S_\infty} u^4 p - \int_S u^4 p\right)$$

$$= \int_{\hat{N}} |\hat{D}_{a}\hat{u}|^2 + \frac{1}{8} \hat{R} u^2 + \frac{1}{4}\left(\int_{\hat{S}_\infty} u^2 \hat{p} - \int_S u^2 \hat{p}\right),$$

(26)

with

$$u^4 \hat{p} = u^2 p + 2\nu^a D_{a}u^2, \quad \hat{R} = -4|\hat{D}_{AB}\hat{\lambda}_C|^2.$$

By rearranging terms in (26), we find

$$8\pi M - \int_S D_{a}u^4 dS^a = \int_N \frac{1}{2} R u^4 + 2|D_{AB}\lambda_C|^2$$

(27)

where $M$ is the ADM mass and we recover the conventional Witten identity in integral form. When the test function is chosen to be $u^{3/2}$, we have

$$2\pi M - \int_S D_{a}u dS^a = \int_N \frac{1}{8} R u + \frac{1}{2} u^5 |\hat{D}_{AB}\hat{\lambda}_C|^2$$

(28)

and this is just the Hamiltonian part of the refined Witten identity given in (24) in integral form when $K = 0$. In the maximal slicing case, both the Witten identity and its refined version in integral form are merely a rearrangement of the terms in the conformal invariant displayed in (25).
IV. TWISTED SEN-WITTEN SPINOR FIELD.

Unlike in the case of positive energy theorem, the refined Witten identity in (22) cannot be applied in a straightforward manner to generate a Penrose type inequality. Calculations of some simple examples suggest that, subject to the APS boundary condition on $\lambda^A$, the flagpole of $\lambda^A$ in general will not align with the null normals of $S$. This mismatch becomes a problem when we try to realise the marginally trapped boundary condition in terms of $\lambda^A$.

To overcome this obstacle, bear in mind that the choice of lapse and shift for a Hamiltonian is by no means unique. Consideration of the time symmetric case suggests that the fourth root of the spinor norm $u$ defined by the Sen-Witten equation remains a good choice for the lapse function. However, from a physical standpoint, a shift vector is not necessarily dictated by the flagpole of $\lambda^A$ as in the proof of the positive energy theorem. What we will do is to twist $\lambda^A$ near $S$ by the standard cut and paste technique in such a way to force the flagpole of the twisted Sen-Witten spinor to align with one of the null normals of $S$. Yet at the same time, the Sen-Witten equation satisfied by $\lambda^A$ is not disturbed.

To proceed, compactness of the inner boundary $S$ enables us to infer the existence of some sufficiently small $\delta > 0$ (to be kept fixed hereafter) such that near $S$ there exists a smooth one parameter family of two spheres $S_x$ with $x \in [0, \delta]$. Let $N_\epsilon = \cup S_x, x \in [0, \epsilon], \epsilon < \delta$ and denote by $(\tilde{\sigma}^A, \tilde{\iota}^A)$ a spin frame with the two null normals of $S$ as flagpoles. Parallel transport of $(\tilde{\sigma}^A, \tilde{\iota}^A)$ along the affinely parametrized geodesics orthogonal to $S$ generates in $N_\epsilon$ two linearly independent spinor fields again denoted by $(\tilde{\sigma}^A, \tilde{\iota}^A)$.

Introduce a cutoff function $\eta: N \to R$ such that

$$0 \leq \eta \leq 1, \quad |D_a \eta| \leq 1$$

in $N_\epsilon$ and zero elsewhere in $N$. Define a twisted spinor field $\alpha_A$ in $N$ as

$$\alpha_A = u^{\frac{1}{2}}(\eta \tilde{\sigma}_A + (1 - \eta) \sigma_A)$$

so that at $S$ the flagpole of $\alpha_A$ aligns with the null normal of $S$ defined by $\tilde{\sigma}_A$ and in $N/N_\epsilon$, up to a scaling factor $\alpha_A$ agrees with the Sen-Witten spinor field. In terms of $\alpha_A$, a shift vector of the Hamiltonian may then be defined as

$$n_a = \sqrt{2} \alpha_{(A} \alpha_{B)}.$$
It may be checked, using (29) and (30) that,

$$\alpha_a \alpha^a \leq 1$$  \hspace{1cm} (32)

and therefore the four vector \((u, n_a)\) is non-spacelike, as required by the non-spacelike Hamiltonian evolution of the initial data set \((N, h_{ab}, K_{ab})\).

When the shift vector is no longer dictated by the flagpole of the Sen-Witten spinor field \(\lambda^A\), for an arbitrary shift vector \(n^a\), the refined Witten identity in (22) may be written in a more general form as

$$\nabla u - \frac{1}{4} D^a (K_{ab} n^b)$$

$$= \frac{1}{4} (\mu u - j^a n_a) + \frac{1}{2} u \hat{\lambda}^C |D_{AC}|^2 + \frac{1}{4} \left( \frac{1}{2} |K_{ab}|^2 u - K^{ab} D_a n_b \right)$$

$$+ \frac{1}{4} u (n^a - \nu^a) D_a K. \hspace{1cm} (33)$$

Given the lapse and shift specified respectively by \(u\) and \(n_a\),

$$K_{ab} = -\frac{1}{2u} (\dot{h}_{ab} - D_a n_b - D_b n_a) \hspace{1cm} (34)$$

where \(\dot{h}_{ab}\) denotes the Lie derivative of \(h_{ab}\) with respect to the timelike vector field generating the Hamiltonian evolution of \(N\), it follows from (34) that

$$\frac{1}{2} |K_{ab}|^2 u - K^{ab} D_a n_b = \frac{1}{8u} |\dot{h}_{ab}|^2 - \frac{1}{2u} |D_{(a} n_{b)}|^2. \hspace{1cm} (35)$$

Putting (35) back into (33), we have

$$\nabla u - \frac{1}{4} D^a (K_{ab} n^b)$$

$$= \frac{1}{4} (\mu u - j^a n_a) + \frac{1}{2} u \hat{\lambda}^C |D_{AC}|^2 + \frac{1}{4} \left( \frac{1}{8u} |\dot{h}_{ab}|^2 - \frac{1}{2u} |D_{(a} n_{b)}|^2 \right)$$

$$+ \frac{1}{4} u (\nu^a - n^a) D_a K. \hspace{1cm} (36)$$

By construction, the vector \((u, n_a)\) is non-spacelike and in view of the dominant energy condition, we may see that the obstruction to positivity comes from the terms \(|D_{(a} n_{b)}|^2\) and \((\nu^a - n^a) D_a K\) in the above expression.

Let \(M - |P|\) be the Minkowski norm of the ADM energy-momentum four vector at spatial infinity. By integrating (36) over a region of \(N = N_e \cup N/N_e\) bounded by the inner boundary
By the Cauchy-Schwarz inequality, 

\[
2\pi(M - |P|) \geq \int_{N_c \cup N/N_c} \left[ \frac{1}{4} (\mu u - j^a n_a) + \frac{1}{2} u^5 |\hat{D}_{AB}\hat{\lambda}_C|^2 
+ \frac{1}{4} \left( \frac{1}{8u} |\hat{h}_{ab}|^2 - \frac{1}{2u} |D_{(a}n_{b)}|^2 \right) + \frac{1}{4} u(\nu^a - n^a)D_a K. \right]
+ \frac{1}{4} \int_S -\sqrt{2} u^{-3}(\lambda^{\dagger A}\nabla_A^C\lambda_C + \lambda_A\nabla^{AC}\lambda_C^\dagger)
-(Ku - K_{ab} n^b \nu^a + pu). \tag{37}
\]

From (32), we see that \( n^a = u \tilde{\nu}^a \) at \( S \) where \( \tilde{\nu}^a \) is the outward pointing normal of \( S \). It then follows from the marginally trapped condition given in (3) that the curvature term in the inner boundary integral in (37) vanishes. Further, by (30) and (31), in \( N/N_c \), (36) is equal to the refined Witten identity displayed in (22). As a result, (37) may further be elaborated to become

\[
2\pi(M - |P|) 
\geq \int_{N_c} \left[ \frac{1}{4} (\mu u - j^a n_a) + \frac{1}{2} u^5 |\hat{D}_{AB}\hat{\lambda}_C|^2 
+ \frac{1}{4} \left( \frac{1}{8u} |\hat{h}_{ab}|^2 - \frac{1}{2u} |D_{(a}n_{b)}|^2 \right) + \frac{1}{4} u(\nu^a - n^a)D_a K. \right]
+ \frac{1}{4} \int_{N/N_c} \left( \frac{1}{4} (\mu - j^a \nu_a)u + \frac{1}{2} u^5 |\hat{D}_{AB}\hat{\lambda}_C|^2 
+ \frac{1}{4} \int_S -\sqrt{2} u^{-3}(\lambda^{\dagger A}\nabla_A^C\lambda_C + \lambda_A\nabla^{AC}\lambda_C^\dagger) \right) \tag{38}
\]

Within \( N_c \), from (30), we have

\[
n_a = u(1 - \eta)^2 \nu_a + \sqrt{2} u \left[ \eta^2 \tilde{\alpha}(C \tilde{i}_D) + \eta(1 - \eta)(\tilde{\alpha}(C \tilde{i}_D) + o(C \tilde{i}_D)) \right]. \tag{39}
\]

By the Cauchy-Schwarz inequality,

\[
|D_{(a}n_{b)}|^2 
\leq \frac{1}{4} \left| \alpha_b D_a u + \alpha_a D_b u \right|^2 + u^2 (1 - \eta)^2 |D_{(a}\nu_{b)}|^2 
+ 2u^2 |\eta^2 D_{AB}\tilde{\alpha}(C \tilde{i}_D) + \eta(1 - \eta) [D_{AB}\tilde{\alpha}(C \tilde{i}_D) + D_{AB}o(C \tilde{i}_D)] 
+ 2\tilde{\alpha}(C \tilde{i}_D) \eta D_{AB}\eta - 2o(C \tilde{i}_D)(1 - \eta)D_{AB}\eta 
+ \left[ \tilde{\alpha}(C \tilde{i}_D) + o(C \tilde{i}_D) \right][(1 - \eta)|D_{AB}\eta - \eta D_{AB}\eta|^2] \right|^2 
\leq u^2 |D_{(a}\nu_{b)}|^2 + C_1 \tag{40}
\]
for some constant $C_1$ determined by $\sup_{N_{\varepsilon}}(u, |D_a u|, |D_{AB} \phi_C|, |D_{AB} \bar{\phi}_C|)$ and we have used (32) in arriving at the final inequality. In a similar way,

\[
\left| (u\nu^a - n^a) D_a K \right| < \sqrt{2} u |\eta^2 \bar{o}^A \tau^B + (2\eta - \eta^2) o^A \tau^B + \eta (1 - \eta)(\bar{o}^A \tau^B + o^A \tau^B) | |D_{AB} K| < C_2
\]

(41)

for some constant $C_2$ determined by $\sup_{N_{\varepsilon}}(u, |D_a K|)$.

By construction, $N_{\varepsilon}$ is generated by a one parameter family of spheres $S_x, x \in [0, \varepsilon)$ and denote by $A_x$ the area of $S_x$, we have from (40) and (41) and the foliated structure of $N_{\varepsilon}$ that, for $\varepsilon < \delta$,

\[
\int_{N_{\varepsilon}} \frac{1}{2u} |D(a n_b)|^2 + (u\nu^a - n^a) D_a K \\
\leq \left( \int_{N_{\varepsilon}} \frac{u}{2} |D(a \nu_b)|^2 \right) + (C_1 + C_2) \int_0^\varepsilon A_x \, dx \\
< \left( \int_{N_{\varepsilon}} \frac{u}{2} |D(a \nu_b)|^2 \right) + (C_1 + C_2) \varepsilon \left( \sup_{x \in [0, \varepsilon]} A_x \right) \\
< \left( \int_{N_{\varepsilon}} \frac{u}{2} |D(a \nu_b)|^2 \right) + C \varepsilon
\]

(42)

where $C = (C_1 + C_2) \sup_{x \in [0, \delta]} A_x$.

In view of (12), the integral over $N_{\varepsilon}$ in (38) may further be expressed as

\[
\int_{N_{\varepsilon}} \left[ \frac{1}{4} (\mu u - j^a n_a) + \frac{1}{2} u^5 |\hat{D}_{AB} \lambda_C|^2 + \frac{1}{4} \left( \frac{1}{8u} |\hat{h}_{ab}|^2 - \frac{1}{2u} |D(a n_b)|^2 \right) \right. \\
+ \left. \frac{1}{4} u(\nu^a - n^a) D_a K \right] \\
> \int_{N_{\varepsilon}} \left[ \frac{1}{4} (\mu u - j^a n_a) + \frac{1}{2} u^5 |\hat{D}_{AB} \lambda_C|^2 + \frac{1}{4} \left( \frac{1}{8u} |\hat{h}_{ab}|^2 - \frac{u}{2} |D(a \nu_b)|^2 \right) \right] - C \varepsilon \\
= \int_{N_{\varepsilon}} \left[ \frac{1}{4} (\mu u - j^a n_a) + \frac{1}{2} u^5 |\hat{D}_{AB} \lambda_C|^2 - C \varepsilon \right]
\]

(43)

where the last equality follows from the definition of the Sen-Witten operator together with (35) with $u\nu_a$ in place of $n_a$ in it. Putting (43) back into (38), we then find

\[
2\pi (M - |P|) \\
\geq \int_{N} \frac{1}{4} (\mu u - j^a n_a) + \frac{1}{2} u^5 |\hat{D}_{AB} \lambda_C|^2 - C \varepsilon \\
+ \frac{1}{4} \int_{S} -\sqrt{2} u^{-3} (\lambda^A \nabla_A \lambda_C + \lambda_A \nabla^A \lambda_C^\dagger).
\]

(44)
The term \( C_\epsilon \) is an additional term to an otherwise manifestly positive volume integral in (44) that generates by the twisting of \( \lambda_A \). This additional term may be suppressed to be sufficiently small provided the annular region \( N_\epsilon \) is chosen to be sufficiently small by shrinking \( \epsilon \). The arbitrariness of \( \epsilon \) then means that the positivity of the integral over \( N \) is not disturbed. With all these considerations, we may then infer from (44) that

\[
2\pi(M - |P|) \geq \frac{1}{4} \int_S -\sqrt{2} f^{-4} (\lambda^{\dagger A} \nabla_A C \lambda_C + \lambda_A \nabla^A C \lambda_C^{\dagger})
\]

where for notational convenience later on, we have written

\[
f^4 = u^3.
\]

Likewise, in the past trapped case when \( N_a \) is chosen to be inward pointing and given by \( N_a = -u \nu_a \), we deduce in a similar way the validity of (45).

V. EVALUATION OF THE INNER BOUNDARY TERM

In our next step, we shall evaluate the inner boundary term worked out in (45). The presence of \( f^{-4} \) in the integrand of (45) means that the calculation will not be entirely straightforward. We will have to appeal to the APS boundary condition satisfied by \( \lambda^A \) in a less obvious way and the arguments are more intricate than originally anticipated.

Consider the following operator

\[
L_A^C = -\nabla_A^C - \mu_0 \epsilon_A^C.
\]

In order to obtain a lower bound of the inner boundary term in (45) in terms of the areal radius of the marginally trapped surface, it is sufficient to prove that

\[
\int_S dS \ f^{-4} (\lambda^{\dagger A} L_A^C \lambda_C + \lambda_A L^A C \lambda_C^{\dagger}) \geq 0.
\]

To begin with, it is not difficult to see that, when restricted to the Hilbert space spanned by the eigenvectors of \( \{\mu_n\} \), \( L_A^C \) becomes a positive operator and therefore admits a unique square root operator \( T_A^C \) so that

\[
L_A^C = T_A^B T_B^C.
\]
The inner boundary integral in (48) may then be further expressed as

\[
\int_S f - 4 (\lambda^\dagger_A L_A^C \lambda_C + \lambda_A L^{AC} \lambda^\dagger_C) = \int_S dS f - 4 (\lambda^\dagger_A T_A^M T_M^N \lambda_N + \lambda_A T^{AM} T_M^N \lambda^\dagger_N).
\]

(49)

The formal analogy between \(L_A^C\) and \(T_A^C\) plus a large amount of calculations in terms of \(T_A^C\) raise the question whether it is feasible to develop the calculus of \(T_A^C\) similar to that of \(L_A^C\). It turns out that this expectation is not far off the mark and, perhaps in a way not entirely expected, we need some holomorphic functional calculus to realise it.

For a spherical two surface, the inverse operator \(L_A^{-1}^C\) exists. It is bounded and again positive. It admits a square root operator \(T_A^{-1}^B\) so that

\[
L_A^{-1}^C = T_A^{-1}^B T_B^{-1}^C.
\]

By the Cauchy integral formula for the analytic function of a bounded operator, \(T_A^{-1}^C\) admits an integral representation

\[
T_A^{-1}^C = \frac{1}{2\pi i} \int_G dz z^{-\frac{1}{2}} R_A^C,
\]

(50)

where

\[
R_A^C := (z \epsilon^A_C - L_A^C)^{-1}
\]

is the resolvent operator of \(L_A^C\) defined in the standard way and \(G\) is a contour closed at \(\infty\) that encloses the eigenvalues of \(L_A^{-1}^C\) along the positive real axis. To be concrete, choose the contour \(G = \bigcup_{n=0}^{\infty} \gamma_n\) so that, for each \(n\), \(\gamma_n\) is a small circle centered at \(\mu_n\) defined by

\[
\gamma_n = \{ z \in \mathbb{C} | z = -\mu_n + \mu_0 + \epsilon e^{i\theta} \text{ for some sufficiently small } \epsilon \in \mathbb{R} \text{ and } \theta \in [0, 2\pi) \}.
\]

From (50), we then have (10), Chapter 5, Section 10

\[
T_A^C = \frac{1}{2\pi i} \int_G dz z^{-\frac{1}{2}} R_A^M L_M^C.
\]

(51)

It then follows from the definition of \(R_A^C\) that it commutes with \(L_A^C\) which, in terms of the index notation, may be written as

\[
L_A^M R_M^C = R_A^M L_M^C.
\]

(52)
Given (51) and (52), we may rewrite (49) as

\[
\int_S f^{-4} (\lambda^A L_A^C \lambda_C + \lambda_A L^{AC} \lambda_C^+) \\
= \frac{1}{2\pi i} \int dS \quad d\Gamma \quad dw \quad z^{-\frac{i}{2}} w^{-\frac{i}{2}} f^{-4} \left[ \lambda^A R_A^M L_M^B L_B^C R_C^D \lambda_D \right. \\
+ \left. \lambda_A R_A^M L_M^B L_B^C R_C^D \lambda_D^+ \right].
\] (53)

This suggests to us to define

\[
\omega_A = \frac{1}{2\pi i} \int dS \quad z^{-\frac{i}{2}} R_A^M \lambda_M.
\] (54)

In terms of spectral representation of the resolvent operator \( R_A^C \) given as

\[
R_A^C = \sum_{n=0}^{\infty} \frac{1}{z - \mu_n} \lambda_n A \lambda_n^+, \n\]

it may be checked that

\[
\omega_A^+ = \frac{1}{2\pi i} \int dS \quad z^{-\frac{i}{2}} R_A^M \lambda_M^+.
\] (55)

(53) may then be written in a more compact form as

\[
\int_S f^{-4} (\lambda^A L_A^C \lambda_C + \lambda_A L^{AC} \lambda_C^+) \\
= \int_S dS \quad f^{-4} (\omega^A L_A^M L_M^N \omega_N + \omega_A L^{AM} L_M^N \omega_N^+) \] (55)

From this point on, we may evaluate the integrand in (55) in terms of standard spinor calculus. From the definition of \( L_A^M \) in (47), we have

\[
f^{-4} \omega^A L_A^M L_M^N \omega_N \\
= f^{-4} \omega^A (-\nabla_A^M - \mu_0 \epsilon_A^M)(-\nabla_M^N - \mu_0 \epsilon_M^N) \omega_N \\
= f^{-4} \left[ \omega^A \nabla_A^M \nabla_M^N \omega_N + 2 \mu_0 \omega^A \nabla_A^M \omega_M + \mu_0^2 \omega^A \omega_A \right] \\
= f^{-4} \left[ \nabla_A^M (\omega^A \nabla_M^N \omega_N) - (\nabla_C^N \omega_C^+) (\nabla_N^M \omega_M) \right. \\
+ \left. 2 \mu_0 \omega^A \nabla_A^M \omega_M + \mu_0^2 \omega_A \omega_A \right].
\] (56)

Likewise, the term \( f^{-4} \omega_A L^{AM} L_M^N \omega_N^+ \) in (55) may be calculated in a similar manner and together with (56) we have

\[
f^{-4} (\omega^A L_A^M L_M^N \omega_N + \omega_A L^{AM} L_M^N \omega_N^+) \\
= f^{-4} \left[ \nabla_A^M (\omega^A \nabla_M^N \omega_N) + \nabla^{AM} (\omega_A \nabla_M^N \omega_N^+) \right. \\
- 2 (\nabla_C^N \omega_C^+) (\nabla_N^M \omega_M) \\
+ \left. 2 \mu_0 (\omega^A \nabla_A^M \omega_M + \omega_A \nabla^{AM} \omega_M^+) + 2 \mu_0^2 \omega_A \omega_A^+ \right].
\] (57)
We shall now evaluate (57) term by term. Define

\[ \beta^A = f^{-2}\omega^A, \quad \beta^1A = f^{-2}\omega^1A. \]

Consider first

\[
\begin{align*}
-4 \mathcal{X}_C^N(\omega^1C \mathcal{X}_N^M \omega_M) &= \mathcal{X}_C^N(f^2\beta^1C \mathcal{X}_N^M f^2) \\
&= \mathcal{X}_C^N[f^2\beta^1C (\omega_M \mathcal{X}_N^M f^2 + f^2 \mathcal{X}_N^M \omega_M)] \\
&= -\frac{1}{2} \mathcal{X}_C^N[(\beta^1C \beta^M \mathcal{X}_{MN} f^4) + f^{-3} \mathcal{X}_C^N(f^{-4}\beta^1C \mathcal{X}_N^M \beta_M)] \\
&= -\frac{1}{2} \mathcal{X}_C^N(\beta^1C \beta^M \mathcal{X}_{MN} f^4) + \mathcal{X}_C^N(\beta^1C \mathcal{X}_{NM} \beta_M) \\
&= 4(\beta^1C \mathcal{X}_C^N \ln f)(\mathcal{X}_N^M \beta_M).
\end{align*}
\] (58)

Likewise, the term \( f^{-4} \mathcal{X}^{AM}(\omega_A \mathcal{X}_M^N \omega_1^N) \) may be calculated in a similar way and we find

\[
\begin{align*}
-4 \mathcal{X}^{AN}(\omega_A \mathcal{X}_M^N \omega_1^N) &= -\frac{1}{2} \mathcal{X}_C^N[(\beta^1C \beta^M \mathcal{X}_{MN} f^4) + f^{-3} \mathcal{X}_C^N(\beta^1C \mathcal{X}_{NM} \beta_M)] \\
&= 4(\beta^1C \mathcal{X}_C^N \ln f)(\mathcal{X}_N^M \beta_M).
\end{align*}
\] (59)

Adding up (58) and (59), we have

\[
\begin{align*}
&\quad \begin{align*}
&= f^{-4} (\mathcal{X}_A^M(\omega_A^1 \mathcal{X}_M^N \omega_N) + \mathcal{X}^{AM}(\omega_A \mathcal{X}_M^N \omega_1^N)] \\
&= \mathcal{X}_C^N(\beta^1C \mathcal{X}_N^M \beta_M) + \mathcal{X}^{AM}(\beta_A \mathcal{X}_M^N \beta_1^N) \\
&= 4(\beta^1C \mathcal{X}_C^N \ln f)(\mathcal{X}_N^M \beta_M) + 4(\beta_A \mathcal{X}^{AM} \ln f)(\mathcal{X}_M^N \beta_1^N).\tag{60}
\end{align*}
\end{align*}
\]

For the third term on the right hand side of (57),

\[
\begin{align*}
&= 2 f^{-4} (\mathcal{X}_C^N(\omega^1C) \mathcal{X}_N^M \omega_M) \\
&= 2 f^{-4} (\mathcal{X}_C^N f^2 \beta^1C) \mathcal{X}_N^M f^2 \beta_M \\
&= 2 f^{-4} [(\beta^1C \mathcal{X}_C^N f^2 + f^2 \mathcal{X}_C^N \beta^1C)] [\beta_M \mathcal{X}_N^M f^2 + f^2 \mathcal{X}_N^M \beta_M] \\
&= 2(\mathcal{X}_C^N \beta^1C) (\mathcal{X}_N^M \beta_M) + 8(\beta^1C \mathcal{X}_C^N \ln f)(\beta_M \mathcal{X}_N^M \ln f) \\
&= 4(\beta_M \mathcal{X}_N^M \ln f)(\mathcal{X}_C^N \beta^1C) + 4(\beta^1C \mathcal{X}_C^N \ln f)(\mathcal{X}_N^M \beta_M).\tag{61}
\end{align*}
\]
Next consider the term

\[ 2\mu_0 f^{-4} (\omega^A \nabla_A \omega_M + \omega_A \nabla^{AM} \omega^M) \]

\[ = 2\mu_0 f^{-4} (f^2 \beta^A \nabla_A \beta_M + f^2 \beta_A \nabla^{AM} \beta^M) \]

\[ = 2\mu_0 (\beta^A \nabla_A \beta_M + \beta_A \nabla^{AM} \beta_M) \]

\[ + 2\beta_M \beta^A \nabla_A \beta_M \ln f + 2\beta_M \beta^A \nabla^{AM} \beta_M \ln f \]

\[ = 2\mu_0 (\beta^A \nabla_A \beta_M + \beta_A \nabla^{AM} \beta_M) \]

\[ - 2\beta_M \beta^A \nabla_A \beta_M \ln f + 2\beta_M \beta^A \nabla^{AM} \beta_M \ln f \]

\[ = 2\mu_0 (\beta^A \nabla_A \beta_M + \beta_A \nabla^{AM} \beta_M). \quad (62) \]

Substituting (60), (61) and (62) back into (57), we see that many terms not manifestly positive in (60) and (61) mutually cancel each other and we finally have

\[ f^{-4} (\omega^A L^M_A L^N_M \omega_N + \omega_A L^{AM} L^N_M \omega^N_M) \]

\[ = |\nabla^A \beta_M|^2 + 2\mu_0 (\beta^A \nabla_A \beta_M + \beta_A \nabla^{AM} \beta_M) + 2\mu_0 |\beta|^2 \]

\[ + \nabla^C \nabla^M \beta_M + \nabla^A \nabla^M \nabla^N \beta^N_M \]

\[ = (\nabla^A \beta_M - \mu_0 \epsilon^A_M) \beta_M \nabla^C \nabla^M \beta_M + \nabla^A \nabla^M \nabla^N \beta^N_M \]

\[ = |L^N_M \omega_N|^2 + \nabla^C \nabla^M (\beta^C \nabla^N_M \beta_M) \]

\[ + \nabla^A \nabla^M (\beta_A \nabla^N_M \beta^N_M) \]

\[ = |L^N_M \omega_N|^2 + \nabla^C \nabla^M (\beta^C \nabla^N_M \beta_M) \]

\[ = 0 \]

\[ \text{according to the definition of } L^M_N \text{ given in (47).} \]

When integrating (63) over \( S \), the divergence terms in (63) vanish and we get from (55) and (63) that

\[ \int_S f^{-4} (\omega^A L^C_A \lambda_C + \lambda_A L^{AC} \lambda^C) \]

\[ \int_S f^{-4} (\omega^A L_A^M L_M^N \omega_N + \omega_A L^{AM} L_M^N \omega^N_M) \]

\[ = \int_S |L^N_M \omega_N|^2 > 0 \]

\[ \text{(64)} \]
as desired. From (47) and (64), we may infer

\[- \int_S f^{-4} (\lambda^{\dagger A} \nabla_A \lambda_C + \lambda_A \nabla^{AC} \lambda^\dagger_C) \geq 2 \mu_0 \int_S f^{-4} \lambda_A \lambda^{\dagger A} \]

\[= 2 \mu_0 \int_S u \]

according to the definition stated in (8) and (46). (45) then becomes

\[M - |P| \geq cr \] (65)

where we have used \( \mu_0 \geq \frac{1}{r} \) for a spherical surface [3] and

\[c = \inf_S u. \] (66)

So far we have been assuming that \( u \) is strictly positive. This hypothesis may be relaxed by a suitable regularization (or cutoff) of the zero points of \( \lambda_A \) and details will be presented in Section 9. As the final step, we shall estimate the upper bound of the constant \( c \) to complete the proof.

Suppose for some \( x \in \partial N \), \( \lambda_A = 0 \). The APS boundary condition in (7) then implies that \( \lambda_A \) vanishes everywhere in \( \partial N \) and \( \partial N \) is a set of zero points of infinite order. Subject to (6), we have the following elliptic system

\[D^2 \lambda_A = \frac{1}{2} (\mu \epsilon_A^L - j_A^L) \lambda_L, \quad A = 0, 1. \] (67)

where \( D^2 = -D_{AB} D^{AB} \) is a generalised Laplacian. It may be checked that \( |D^2 \lambda_A| \leq C|\lambda_A| \) for some constant \( C \). For a sufficiently small coordinate ball \( B \) centered at \( x \), standard reflection across \( \partial N \cap B \) enables us to extend (67) from \( B \cap R_+^3 \) to the entire \( B \) as an elliptic system with Lipshitz coefficients. Unique continuation at the point \( x \) then implies \( \lambda_A = 0 \) everywhere in \( N \) (see [11], Theorem 1.8) and this contradicts the asymptotic boundary condition satisfied by \( \lambda_A \) near infinity. We may then infer \( c > 0 \) and the inequality in (65) is not vacuous.

To estimate the upper bound of \( c \), we revert to the spinor norm \( \varphi = u^4 \) and we have

\[D^2 \varphi = D^a (D_a \varphi - K_{ab} \nu^b \varphi) \]

\[= (\mu - j_a \nu^a) \varphi + 2 |D_{AB} \lambda_C|^2 \] (68)
where \( \nu^a = \sqrt{2} o(A_{L'B}) \). The dominant energy condition implies that \( D^2 \varphi > 0 \). Further, (68) is an elliptic PDE of divergence form. The maximum principle applies and the maximum of \( \varphi \) will occur either at inner boundary \( S \) or at infinity. Suppose on the contrary that the maximum of \( \varphi \) occurs at some \( x \in S \). It follows that \( \frac{\partial \varphi}{\partial \nu} < 0 \) at \( x \). Continuity implies there exists a neighbourhood \( U \subset S \) centered at \( x \) such that \( \frac{\partial \varphi}{\partial \nu} < 0 \) in \( U \). Fix a cutoff function \( \eta > 0 \) in \( U \), by twisting \( \lambda_A \) in an appropriate way as before, the integral form of (68) together with the Sen-Witten equation in (6) and the marginally trapped condition on \( S \) give

\[
\int_S \eta^4 \frac{\partial \varphi}{\partial \nu} = - \int_S \sqrt{2} \eta^4 (\lambda^{\dagger A} \nabla_A C \lambda_C + \lambda_A \nabla^{AC} \lambda^\dagger C) \quad (69)
\]

Given the APS boundary condition, \(-\nabla_M N\) is a positive operator and admits a unique square root operator. With \( \eta^4 \) and \(-\nabla_M N\) in place of \( u^{-3} \) and \( L_M N \) respectively in (55) and by repeating the arguments leading to (64), we have \( \int_S \eta^4 \frac{\partial \varphi}{\partial \nu} > 0 \) and this contradicts our initial hypothesis that \( \frac{\partial \varphi}{\partial \nu} < 0 \) at \( U \). Therefore, the maximum of \( \varphi \) will occur at the asymptotic regime and we necessarily have \( c < 1 \). As a result, we have \( 1 > c > 0 \) in (65).

With the trivial generalisation to the case of multi-connected horizon, we are then finally in a position to state the following theorem.

**Theorem.**

Let \((N, h_{ab}, K_{ab})\) be an asymptotically Euclidean initial data set with inner boundary \( \partial N = \bigcup_{i=0}^{n-1} S_i \), where \( S_i, i = 0, \ldots n-1 \) are disjoint, smooth future or past marginally trapped surfaces with spherical topology and areal radius \( r_i \). Subject to the dominant energy condition, we have

\[
M - |P| \geq c \sum_{i=1}^{n} r_i, \quad 0 < c < 1.
\]

**VI. DERIVATION OF THE REFINED WITTEN IDENTITY**

We will now go back to fill in certain details in the steps leading to the proof of the theorem just stated. In this section, we shall first provide more details on the derivation of the spinor identity stated in (16).

The crux of the calculations leading to (16) is to evaluate the term

\[
|D_{AB} o_C|^2 = - D_{AB} o_C D^{AB} \iota_C. \quad (70)
\]
Given (13), we have

\[
D_{AB\partial C} D^{AB} C
= D_{AB}(u^{-1}\dot{\lambda}_C)D^{AB}(u\dot{\lambda}^C)
= \left(-\dot{\lambda}_C u^{-2}D_{AB} u + u^{-1}D_{AB}\dot{\lambda}_C\right)
\left(\dot{\lambda}^C D^{AB} u + uD^{AB}\dot{\lambda}^C\right).
\]

(71)

It is standard to work out that, under the conformal rescaling \(h_{ab} \to \hat{h}_{ab} = u^4 h_{ab}\),

\[
\hat{D}_{AB}\dot{\lambda}_C = D_{AB}\dot{\lambda}_C - \dot{\lambda}_B D_{CA} \ln u - \dot{\lambda}_A D_{CB} \ln u
\]

(72)

\[
\hat{D}_{AB}\dot{\lambda}^C = D_{AB}\dot{\lambda}^C + \epsilon_A^C \dot{\lambda}^M D_{BM} \ln u + \epsilon_B^C \dot{\lambda}^M D_{AM} \ln u
\]

Substitute (72) into (71), we then have

\[
D_{AB\partial C} D^{AB} C
= \left(u^{-1}\hat{D}_{AB}\dot{\lambda}_C + u^{-2}\dot{\lambda}_B D_{CA} u + u^{-2}\dot{\lambda}_A D_{CB} u - u^{-2}\dot{\lambda}_C D_{AB} u\right)
\left(u\hat{D}^{AB}\dot{\lambda}^C - \epsilon^{AC} \dot{\lambda}^M D_{BM} u - \epsilon^{BC} \dot{\lambda}^M D_{AM} u + \dot{\lambda}^C D^{AB} u\right)

= \hat{D}_{AB}\dot{\lambda}_C \hat{D}^{AB}\dot{\lambda}^C
+ u^{-1}\hat{D}_{AB}\dot{\lambda}_C \left(-\epsilon^{AC} \dot{\lambda}^M D_{BM} u - \epsilon^{BC} \dot{\lambda}^M D_{AM} u + \dot{\lambda}^C D^{AB} u\right)
+ u\hat{D}^{AB}\dot{\lambda}^C \left(u^{-2}\dot{\lambda}_B D_{CA} u + u^{-2}\dot{\lambda}_A D_{CB} u - u^{-2}\dot{\lambda}_C D_{AB} u\right)
+ \left(u^{-2}\dot{\lambda}_B D_{CA} u + u^{-2}\dot{\lambda}_A D_{CB} u - u^{-2}\dot{\lambda}_C D_{AB} u\right)
\left(-\epsilon^{AC} \dot{\lambda}^M D_{BM} u - \epsilon^{BC} \dot{\lambda}^M D_{AM} u + \dot{\lambda}^C D^{AB} u\right).
\]

(73)

Subject to the Sen-Witten equation together with its conformal symmetries, after some standard calculations, we have

\[
u^{-1}\hat{D}_{AB}\dot{\lambda}_C \left(-\epsilon^{AC} \dot{\lambda}^M D_{BM} u - \epsilon^{BC} \dot{\lambda}^M D_{AM} u + \dot{\lambda}^C D^{AB} u\right)
= (D^{AB} \ln u)(\dot{\lambda}^C \hat{D}_{AB}\dot{\lambda}_C + \frac{1}{2} K \nu^a D_a \ln u)
\]

(74)

Further, using the identity

\[
D_{AB}\lambda^C = D_{A(B\lambda^C)} + D_A[B\lambda^C],
\]

\[
= D_{A(B\lambda^C)} + \frac{1}{2} \epsilon_{BCD} D_{AN} \lambda^{CN}
\]
together with again the Sen-Witten equation and its conformal symmetries, we may work out

\[ u \hat{D}^{AB} \hat{\lambda}^C \left( u^{-2} \hat{\lambda}_B D_{CA} u + u^{-2} \hat{\lambda}_A D_{CB} u - u^{-2} \hat{\lambda}_C D_{AB} u \right) \]

\[ = (D_{AB} \ln u) \left( \hat{\lambda}_C \hat{D}^{AB} \hat{\lambda}^C \right) + \frac{1}{2} K \nu^a D_a \ln u \]

(75)

Since \( \hat{\epsilon}^{AB} \hat{\lambda}_A \hat{\lambda}_B = 1 \), summing terms in (74) and (75), we have

\[ u^{-1} \hat{D}_{AB} \hat{\lambda}_C \left( -\epsilon^{AC} \hat{\lambda}^M D_B^M u - \epsilon^{BC} \hat{\lambda}^M D_A^M u + \hat{\lambda}^C D^{AB} u \right) \]

\[ + u \hat{D}^{AB} \hat{\lambda}^C \left( u^{-2} \hat{\lambda}_B D_{CA} u + u^{-2} \hat{\lambda}_A D_{CB} u - u^{-2} \hat{\lambda}_C D_{AB} u \right) \]

\[ = K \nu^a D_a \ln u \]

(76)

To evaluate in (73) the term

\[ \left( u^{-2} \hat{\lambda}_B D_{CA} u + u^{-2} \hat{\lambda}_A D_{CB} u - u^{-2} \hat{\lambda}_C D_{AB} u \right) \]

\[ \left( -\epsilon^{AC} \hat{\lambda}^M D_B^M u - \epsilon^{BC} \hat{\lambda}^M D_A^M u + \hat{\lambda}^C D^{AB} u \right) \]

(77)

further calculations enable us to infer that (77) is equal to

\[ 6u^{-2}(o^B D_{AB} u)(\iota^N D^A u) + u^2 D_a u \nu^a u. \]

(78)

From the Newman-Penrose tetrad constructed from the spin frame \((o^A, \iota^A)\), a moving three frame intrinsic to \(N\) may be defined as

\[ m^a = o^A \iota^A, \quad \bar{m}^a = o^A \iota^A, \quad \nu^a = \frac{1}{\sqrt{2}} (o^A \iota^A - \iota^A o^A). \]

In terms of \((\nu^a, m^a, \bar{m}^a)\), we have

\[ o^B D_{AB} u = -\frac{1}{\sqrt{2}} (\nu^a D_a u) o_A - (m^a D_a u) \iota_A \]

(79)

and

\[ \iota^N D^A u = -(\bar{m}^a D_a u) o^A + \frac{1}{\sqrt{2}} (\nu^a D_a u) \iota^A. \]

(80)

Using \(h_{ab} = \nu_a \nu_b + 2 m_{(a} \bar{m}_{b)}\), we may deduce from (79) and (80) that

\[ u^{-2}(o^B D_{AB} u)(\iota^N D^A u) = -\frac{1}{2} u^{-2} D_a u D^a u. \]

(81)
Therefore we finally obtain from (78) and (81) that the term in (77) is equal to $2u^{-2}D_a u D^a u$.

Putting all these together with (76) back to (73), we then have

$$\left|D_{AB} \partial_C \right|^2 = -D_{AB} \partial_C D^{AB} \partial_C$$

$$= u^4 |\hat{D}_{AB} \hat{\lambda}_C|^2 + 2 |D_a \ln u|^2 - K \nu^a D_a \ln u$$

(82)

which is the spinor identity stated in (16). Note that $|\hat{D}_{AB} \hat{\lambda}_C|^2$ is evaluated in terms of the conformally rescaled symplectic form $\hat{\epsilon}_{AB}$.

VII. REGULARISATION OF ZERO POINTS OF A SPINOR FIELD

We shall now outline a way to relax the provisional hypothesis that $\lambda^A$ is non-zero everywhere in $N$. Given the APS boundary condition, zero points of $\lambda_A$ stay away from the inner boundary $S$. Denote by $X \subset N/\partial N$ the set of zero points of finite order. The asymptotic boundary condition for $\lambda^A$ means that $X$ is a subset of some compact set in $N$. $X$ is closed then further implies that $X$ is compact.

It is also known that $X$ is contained in a countable union of smooth curves in $N\{4\}$. Compactness of $X$ implies that $X \subset \bigcup_{k=1}^n C_k$ for some natural number $n$ and $C_k : [0, 1] \to N$ for $k = 1,..n$ are smooth curves. A smooth tubular neighbourhood $T_k : [0, L_k] \times D_\epsilon \to N$ may be constructed so that $C_k \subset T_k$, $D_\epsilon$ is a geodesic disk of radius $\epsilon$ centered at a point in $C_k$. In place of $N$, we consider instead

$$N' = N/\{\text{interior of } \cup T_k\}.$$

The integral in (37) then acquires extra boundary terms

$$\sum_{k=1}^n \int_{\partial T_k} \left( \frac{\partial u}{\partial \nu} - u K_{ab} r^a \nu^b \right)$$

where $r^a$ is the normal to $\partial T_k$.

For a zero point $x \in N/\partial N$, both $u$ and $D_a u$ vanish at $x$ and therefore

$$u = O(r^{1/2}), \quad \frac{\partial u}{\partial r} = O(r^{-1/2})$$

in $D_\epsilon$ where $r$ is the geodesic distance from $C_k$. Using the compactness of $\bigcup_{k=1}^n T_k$ and by means of further calculations, we have

$$\left| \int_{\partial T_k} \frac{\partial u}{\partial r} - \frac{1}{4} u K_{ab} r^a \nu^b \right| \leq \alpha \epsilon^{1/2}.$$

(83)
for some constant $\alpha$ independent of $\epsilon$. In view of (83), the integral form of the refined Witten identity then becomes

$$2\pi(M - |P|) = \int_{N'} \Delta u - \frac{1}{4} D^a(K_{ab} N^b) + \int_S \left( \frac{\partial u}{\partial \nu} - K_{ab} N^a b^b \right) + o(\epsilon^{1/2}).$$

(84)

By shrinking the radius of the tubes $T_k, k = 1, \cdots n$ to a sufficiently small $\epsilon$, we see that the standard positivity argument continues to hold for (84) when zero points of $\lambda_A$ are taken into consideration.

VIII. EXISTENCE AND UNIQUENESS OF SEN-WITTEN SPINOR FIELD AND THE APS BOUNDARY CONDITION.

We will complete the proof of the above stated theorem by proving the existence and uniqueness of solution to the Sen-Witten equation in (6), subject to the APS boundary condition and the asymptotic boundary condition displayed in (II). Once we realise that a suitable amount of twisting of a spinor field described in (30) will not disturb the positivity argument, the proof becomes quite standard elliptic estimates in terms of the Lax-Milgram approach. For completeness, we shall briefly sketch it here.

Denote by $N_R$ the subset in $N$ bounded by a coordinate ball $B_R$ of Euclidean radius $R$ near infinity. Fix a real valued function $\sigma$ in $N$ such that $\sigma \geq 1$ and $\sigma = 1$ in $N$, $\sigma = r$ in $N/N_{2R}$ where $r$ is the Euclidean radial distance in the asymptotic regime. Let $W^{k,p}_\sigma$ be the weighted Sobolev spaces defined in the standard way with $p = 2$ and we define the norm of $W^{k,p}_\sigma$ in terms of $\mathcal{D}_{AB}$. Denote the weighted Sobolev norm of $W^{1,2}_{-1}$ by $\|\|$. It is also sufficient to define $\|\|$ in terms of $\mathcal{D}_{AB}$ alone. Further restrict the domain of the Sen-Witten operator $\mathcal{D}_A C$ to a closed space $H_- \subset W^{1,2}_{-1}$ such that $\psi_A \in H_-$ if and only if $\psi_A \in W^{1,2}_{-1}$ and $\psi_A|_S$ satisfies the APS boundary condition given in (II) and (7).

Extend the covariantly constant spinor $\lambda_{0A}$ near infinity in an obvious way to $N$ and denote it by $\eta_A$. Fix a sequence of Euclidean radius $R_i$ near infinity indexed by natural numbers with $R_{i+1} > R_i$ for all $i$ and $\lim_{i \to \infty} R_i \to \infty$. Then consider a sequence $\{\eta_{iA}\}$ with support in $N/R_i$ such that $\lim_{i \to \infty} \eta_{iA} \to \eta_A$. Define

$$\lambda_{iA} = \psi_{iA} + \eta_{iA}.$$  (85)
For notation convenience, the \(i\) th dependence of \(\psi_A\) and \(\eta_A\) will be suppressed in what follows and \(\psi_A\) is assumed to have support in \(N_{R_i}\).

Subject to the dominant energy condition, \(\mathcal{D}_A^C\) is injective. It is then sufficient to consider the following elliptic operator

\[
\mathcal{D}_A^C \mathcal{D}_C^N \psi_N = - \mathcal{D}_A^C \mathcal{D}_C^N \eta_N
\]

with the prescribed APS boundary condition at the inner boundary and the asymptotic fall off near spatial infinity.

As in the standard Lax-Milgram approach, define a bilinear form in \(H\) as

\[
a(\alpha, \lambda) = \int_N (\mathcal{D}_C^N \alpha_N)^\dagger (\mathcal{D}_C^L \lambda_L)
\]

(87)

together with the linear functional in \(H\) defined by

\[
f(\alpha) = -\int_N (\mathcal{D}_C^N \alpha_N)^\dagger (\mathcal{D}_C^L \eta_L).
\]

(87)

Using the identity

\[
\mathcal{D}_{AB} \lambda_C = \mathcal{D}_A(B \lambda_C) + \mathcal{D}_A[\lambda_C] \\
= \mathcal{D}_A(B \lambda_C) + \frac{1}{2} \epsilon_{BC} \mathcal{D}_A \lambda_N
\]

it may be checked that

\[
|a(\lambda, \alpha)| \leq C' ||\lambda|| ||\alpha||
\]

for some constant \(C\) and the linear functional \(f\) is bounded.

To prove the coercivity of the bilinear form \(a(\alpha, \lambda)\), given \(\psi_A\) is supported in \(N_{R_i}\), in general we have

\[
4 \int_{N_{R_i}} |\mathcal{D}_A^N \psi_N|^2 \\
= \int_{N_{R_i}} \left[ (\mu |\psi|^2 - j^n a) + \frac{1}{8} |\psi|^2 |\dot{h}_{ab}|^2 \\
+ 2 |D_{AB} \psi_C|^2 - \frac{1}{2} |\psi|^2 |D(a n_b)|^2 \\
+ \int_S -\sqrt{2} (\psi^A \mathcal{F}_A^C \psi_C + \psi_A \mathcal{F}_A^{AC} \psi_C^\dagger) \\
- \int_S (K_{ab} \gamma_{ab} + p |\psi|^2) \right] (88)
\]
where $|\psi|^2 = \psi^A \psi_A$ and $n_a$ is a shift vector to be specified. As that in the previous section, consider the partition $N = N_e \cup N/N_e$ and define a twisting of $\psi_A$ in $N_e$ by

$$\tilde{\alpha}_A = \eta \tilde{\lambda}_A + (1 - \eta)\psi_A,$$

with $\tilde{\lambda}_A = |\psi|\tilde{\sigma}_A$ where the flagpole of $\tilde{\sigma}_A$ aligns with the future pointing null normal of $S$. A shift vector is chosen to be $n_a = \sqrt{2} \tilde{\alpha}_A \tilde{\alpha}_B$. Subject further to the marginally trapped boundary condition imposed on $S$, (88) then becomes

$$4 \int_{N_{R_i}} |D^N_A \psi_N|^2 = \left[ \int_{N_{R_i}} (\mu |\psi|^2 - j^a n_a) + |D_{AB} \psi_C|^2 - C \epsilon \\
+ \int_S -\sqrt{2} (\psi^A \nabla_A^C \psi_C + \psi_A \nabla^{AC} \psi_C^\dagger) \right].$$

(89)

The APS boundary condition means that the inner boundary term in (89) is positive. Together with the dominant energy condition and that $\epsilon$ is arbitrary, (89) may further be given as

$$4 \int_{N_{R_i}} |D^N_A \psi_N|^2 \geq \int_{N_{R_i}} |D_{AB} \psi_C|^2.$$  

(90)

With the index $i$ reinstated into the spinor field $\psi_A$ and from the definition of $|| \cdot ||$, we may further infer from (90) that

$$a(\psi_i, \psi_i) \geq C ||\psi_i||^2.$$

for some constant $C$ independent of $i$. With $\eta_i$ in place of $\eta$ in (86), a weak solution $\psi_{iA}$ exists for (86). It may also be checked that $\psi_{iA}$ is uniformly bounded in $H_-$, by passing to a subsequence if necessary $\psi_{iA}$ converges weakly to some $\psi_A \in H_-$. Moreover, it follows from the injectivity of the Sen-Witten operator that $\psi_A$ is necessarily unique. Elliptic regularity then implies that $\psi_A$ is a strong, smooth solution to (86) with the prescribed boundary conditions at the inner boundary and that near spatial infinity.

\section{IX. CONCLUDING REMARKS}

The contribution of the present work lies in suggesting that a spinor approach to the Penrose inequality is viable to a certain extent. The next step towards a complete proof of
the Penrose inequality is to give an appropriate geometric characterisation of an outermost trapped surface and see whether spin geometry is capable of giving a lower bound of the norm of the Sen-Witten spinor field at the outermost trapped surface in terms of that of the Schwarzschild metric. In the course of development of the spinorial framework of the positive energy theorem, we have also uncovered certain geometric structures of an initial data set underlying the spinorial framework and might worth pursuing further. From a physical standpoint, the insights we gain from the proof itself concerning the global structure and geometry of an initial data set describing gravitational collapse seem to be as valuable as the Penrose inequality itself.

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