Reducing almost Lagrangian structures and almost CR geometries to partially integrable structures

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Abstract

This paper demonstrates a method for analysing almost CR geometries \((H, J)\), by uniquely defining a partially integrable structure \((H, K)\) from the same data. Thus two almost CR geometries \((H, J)\) and \((H', J')\) are equivalent if and only if they generate isomorphic induced partially integrable CR geometries \((H, K)\) and \((H', K')\), and the set of CR morphisms between these spaces contains an element that maps \(J\) to \(J'\). Similar results hold for almost Lagrangian structures.

1 Introduction

CR geometry is a particularly rich mathematical seam, spawning elegant results and successful applications all over the place with joyful abandon. Though partially integrable CR manifolds of hyper-surface type can be successfully analysed by constructing a unique normal Cartan connection (see [ˇCap02] and [ˇCS00]), the same is not true for general almost CR structures of the same type, where no such constructions exist.

However, this paper details a way of locally matching each ‘generic’ almost CR structure \((H, J)\), in a purely algebraic fashion, to a uniquely defined partially integrable CR structure \((H, K)\). The data of \((H, J)\) can thus be fully captured by the normal Cartan connection for \((H, K)\) and by \(J\). In particular, two almost CR structures are equivalent if and only if they generate isomorphic induced partially integrable CR structures \((H, K)\) and \((H', K')\), and the set of CR morphisms between these spaces contains an element that maps \(J\) to \(J'\).

Similar results hold for Lagrangian geometries, which are another real form of these CR geometries.

The construction of \(K\) from \(J\) requires certain assumptions on the eigenvalues of an automorphism \(A\) of \(H\), specifically that they all be non-real. This is the ‘generic’ condition needed for the almost CR structure – note that the condition is vacuous in the definite signature case, where \(A\) has only imaginary eigenvalues. When the condition does fail, the structure on the manifold can best be described as a mixed structure, intertwining Lagrangian and CR structures.

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2 The CR case

An almost CR structure of hypersurface type is given by:

- a manifold \(M\) of dimension \(1 + 2n\),
• a distribution \( H \subset TM \) of rank \( 2n \) generating a “contact” form \( \omega \in \Gamma(\bigwedge^2 H^* \otimes (TM/H)) \), which may be degenerate,

• an almost-complex structure \( J \) on \( H \), such that \( \nu \) is non-degenerate, where \( \nu \) is defined as

\[
\nu(X, Y) = \frac{1}{2}(\omega(X, Y) + \omega(JX, JY))
\]

for any sections \( X \) and \( Y \) of \( H \).

Note that \( \nu \) is \( J \)-hermitian, in that \( \nu(JX, JY) = \nu(X, Y) \) for all \( X, Y \in \Gamma(H) \). It defines a metric \( g \) given by

\[
g(X, Y) = \nu(X, JY).
\]

The signature of an almost CR structure is the signature of \( g \). If \( g \) is positive definite, then the non-degeneracy of \( \nu \) implies the non-degeneracy of \( \omega \); this is not the case for other signatures.

**Definition 2.1.** A partially integrable CR manifold is one defined as above with \( \nu = \omega \).

We may construct an automorphism \( A \) of \( H \) as \( A = g^{-1} \omega \). Then the core theorem of this paper is:

**Theorem 2.2.** If \( \omega \) and \( \nu \) are both non-degenerate, and none of the eigenvalues of \( A \) is real, then there exists an almost-complex structure \( K \) on \( H \), uniquely defined by the data \((H, J)\), such that \((H, K)\) defines a partially integrable CR manifold.

In the definite signature case, all the eigenvalues of \( A \) must be pure imaginary, so the restriction on \( A \) is not needed. It is an open condition that will be satisfied by “most” distributions, and certainly those obtained by small deformations of partially integrable CR structures: for in that case \( A = J \), with eigenvalues \( \pm i \).

Since partially integrable CR structures are defined by a unique normal Cartan connection ([Cap02] and [CS00]), this has the immediate corollary that:

**Corollary 2.3.** Two almost CR structures \((H, J)\) and \((H', J')\) are isomorphic if and only if their generated partially integrable CR structures \((H, K)\) and \((H', K')\) define isomorphic normal Cartan connections, and if there exists a CR morphism between them that pulls back \( J \) to \( J' \).

The methods used to prove Theorem 2.2 involve constructing \( K \) linearly from \( J \) at each point of \( M \). The procedure is easily seen to be continuous, generating a continuous \( K \). Now pick any point \( x \) in \( M \), and let \( V = H_x \); by an abuse of notation, we will drop the indexes and refer to \( \omega, \nu, g, A \) and \( J \) as \( \omega, \nu, g, A \) and \( J \).

The endomorphism space \( V \otimes V^* \) decomposes under the triple \((\nu, g, J)\) as

\[
V \otimes V^* = \mathfrak{su}(g, J) \oplus (\bigwedge^1 V) \oplus (\bigwedge^2 V) \oplus (\bigwedge^3 V) \oplus (\mathbb{R} \cdot J) \oplus (\mathbb{R} \cdot \text{Id}).
\]

Relevant subalgebras of this are the complex algebra \( \mathfrak{sl}(V, J) = \mathfrak{su}(g, J) \oplus (\bigwedge^1 V) \), the symplectic algebra \( \mathfrak{sp}(\nu) = \mathfrak{su}(g, J) \oplus (\bigwedge^2 V) \oplus (\mathbb{R} \cdot J) \), the orthogonal algebra \( \mathfrak{so}(g) = \mathfrak{su}(g, J) \oplus (\bigwedge^2 V) \oplus (\mathbb{R} \cdot J) \) and the conformal unitary algebra \( \mathfrak{su}(g, J) \oplus (\mathbb{R} \cdot J) \oplus (\mathbb{R} \cdot \text{Id}) \). Following the tradition of parabolic geometry ([CG00], [CG02]), this last algebra will be designated \( \mathfrak{g}_0 \) and the corresponding group \( G_0 \). It is the largest group that preserves a given CR structure. Upon constructing \( K \) from \( J \), we will see that the definition is unique up to \( G_0 \) action, hence that it defines a unique CR structure.

Note that \( \mathfrak{so}(g) \cap \mathfrak{gl}(V, J) = \mathfrak{g}_0 \), hence that \( SO(g) \cap GL(V, J) = G_0 \). The process for constructing \( K \) flows from:
Proposition 2.4. For any given collection of non-degenerate $\omega$, $J$, $\nu$ and $g$, defined as above with $A$ having only non-real eigenvalues, there exists an element $e$ in $SO(g)$ such that $\omega$ is hermitian for the almost-complex structure $K = e^{-1}Je$. This $e$ is defined up to the left action of $G_0$; since $G_0$ preserves $J$, this defines $K$ uniquely.

Constructing this $e$ is basic linear algebra. We will be operating in the complexified space $V_C = V \otimes \mathbb{C}$, keeping track of the subspace $V = V \otimes 1$ via complex conjugation.

Definition 2.5. The space $D^k_\alpha$ is defined to be the $k$-th generalised eigenspace for $A$ with eigenvalue $\alpha$, i.e. the kernel of the linear map $(A - \alpha Id)^k$. The Jordan normal form decomposition of $A$ demonstrates that

$$V_C = \oplus_j D_{\alpha_j},$$

where $\alpha_j$ are the eigenvalues of $A$ and $D_{\alpha_j} = D^k_{\alpha_k}$ for some $k$ where $D^k_{\alpha_k} = D^1_{\alpha_k}$.

Definition 2.6. The set $S$ is defined to be the set of eigenvalues of $A$; since $A$ is non-degenerate, $0 \notin S$, and by assumption, $S \cap \mathbb{R} = \emptyset$.

Lemma 2.7. The space $D_{\alpha}$ is $g$-orthogonal to all eigenspaces $D_{\beta}$, except when $\beta = -\alpha$. Moreover, $g$ gives a non-degenerate pairing between $D_{\alpha}$ and $D_{-\alpha}$.

Proof of Lemma. Let $u \in D^1_\alpha$, $v \in D^1_\beta$. Then since $g(Au,v) = -g(Av,u)$, we must have $\alpha g(u,v) = -\beta g(u,v)$. Hence either $g(u,v) = 0$, or $\alpha = -\beta$. Assume for the moment that $\alpha \neq -\beta$; hence $D^1_\alpha \perp D^1_\beta$.

Reasoning by induction, assume that $D^j_\alpha \perp g D^k_\beta$, and let $u \in D^j_\alpha$, $v \in D^{k+1}_\beta$. Then

$$\alpha g(u,v) = g(Au,v) = -g(Av,u) = -g(u,\beta v + v^k) = -g(u,\beta v).$$

Hence $D^j_\alpha \perp g D^{k+1}_\beta$. Since we may induct both $j$ and $k$, and since the generalised eigenspaces stabilise after finitely many steps, we must have $D_{\alpha} \perp D_{\beta}$.

Consequently, the pairing under $g$ gives $D_{\alpha} \supset D^*_{-\alpha}$ and $D_{-\alpha} \supset D^*_{\alpha}$. Dimensional considerations imply that $g$ pairs $D_{\alpha}$ and $D_{-\alpha}$ in a non-degenerate fashion. $\blacksquare$

Since $A$ is real, it commutes with complex conjugation, implying that $\overline{S} = S$ and that $\overline{D_{\alpha}} = D_{-\alpha}$.

Define $S_+$ as the set of elements $\alpha$ in $S$ such that the argument of $\alpha$ is in $(0,\pi/2]$. Then $S = S_+ \cup \overline{S_+} \cup -S_+ \cup -\overline{S_+}$. For any $\alpha$ in $S_+$, define the space

$$C_{\alpha} = D_{\alpha} + D_{\pi} + D_{-\alpha} + D_{-\pi}.$$ 

These $C_{\alpha}$ are mutually orthogonal, non-degenerate under $g$ and closed under complex conjugation. This means that $C_{\alpha} = C_{\alpha}^R \otimes \mathbb{C}$, where $C_{\alpha}^R$ is a real subspace of $V$. If $\alpha$ is not pure imaginary, then $C_{\alpha}^R$ must be of split signature $(2p,2p)$, since $D_{\alpha} + D_{-\pi}$ must be the complexification of an even dimensional isotropic space, pairing non-degenerately with $D_{-\alpha} + D_{-\pi}$.

If $\alpha$ is pure imaginary, let $v + \overline{v}$ be an orthonormal element of $C_{\alpha}^R$, for $v \in D_{\alpha}$. Then

$$||iv - i\overline{v}||^2 = -2g(iv,i\overline{v}) + g(iv,i\overline{v}) + g(v,\overline{v}) = 2g(v,\overline{v}) = ||v + \overline{v}||^2,$$

since $v$ and $\overline{v}$ are isotropic. Hence $C_{\alpha}^R$ is of signature $(2p,2q)$.

Let $L_{\pm}$ be the $\pm 1$ eigenspace of $J$. These two spaces must be isotropic with $L_+ \oplus L_- = V$, by the properties of $J$. Note that $\overline{L_+} = L_-$. By the signature results for $C_{\alpha}^R$, we have the following lemma:
Lemma 2.8. There exists a decomposition of \( L_+ \) as
\[
L_+ = \oplus_{\alpha \in S} P_\alpha,
\]
such that the spaces \( Q_\alpha = P_\alpha \oplus \overline{P_\alpha} \) are mutually orthogonal, non-degenerate under \( g \), closed under complex conjugation, and of same dimension and signatures as \( C_\alpha \).

Proof of Lemma. Pick an orthonormal basis in \( V \) for the hermitian metric \( g + iv \), group the basis elements together to generate subspaces of the correct signature, and complexify into subspaces of \( V_C \).

Since these spaces are all preserved by \( J \), they give the required splitting of \( L_+ \).

We now choose a map \( e \) on \( V_C \), defined in the following way: for \( \alpha \) not purely imaginary, let \( P_\alpha = P_\alpha^1 \oplus P_\alpha^2 \), where \( P_\alpha^1 \oplus P_\alpha^2 \) is isotropic. Then map \( D_\alpha \) into \( P_\alpha^1 \) in any fashion, map \( D_\alpha \) into \( P_\alpha^1 \) by conjugation, \( D_{-\alpha} \) into \( P_\alpha^2 \) by duality under \( g \), and \( D_{-\alpha} \) into \( P_\alpha^2 \) by duality then conjugation (or conjugation then duality then duality)

For \( \alpha \) purely imaginary, pick an orthonormal basis \( \nu_j \oplus \overline{\nu_j} \) of \( C_\alpha^R \) with \( \nu_j \in D_\alpha \), an orthonormal basis \( \nu_j \oplus \overline{\nu_j} \) of \( Q_\alpha^R \) for \( u \in L_+ \), and map one basis to the other, sending \( D_\alpha \) into \( P_\alpha \).

By construction, \( e \) preserves the metric \( g \) and complex conjugation; thus it is an element of \( SO(g) \). Let \( e' \) be another element of \( SO(g) \) that maps \( D_\alpha \) to \( L_+ \); whenever \( \alpha \) has positive imaginary part; then \( e' = fe \), where \( f \) is an element of:
\[
(GL(L_+ \oplus GL(L_-)) \cap SO(g).
\]

But this intersection is \( G_0 \), as the real part of \( (GL(L_+ \oplus GL(L_-)) \cap SO(g) \) is just \( GL(V, J) \). Thus \( e \) is unique up to left \( G_0 \) action. By construction, the endomorphism \( e Ae^{-1} \) must have eigenspaces that are subspaces of \( L_\pm \), and hence commute with \( J \). Since \( e \) is orthogonal, this implies that \( e^{-1} \cdot \omega \) is \( J \)-hermitian, where
\[
(\omega^{-1} \cdot \omega)(X, Y) = \omega(\omega^{-1}X, \omega^{-1}Y).
\]

Lemma 2.9. The form \( \omega \) is hermitian under the complex structure \( K = e^{-1}Je \), i.e.
\[
\omega(KX, KY) = \omega(X, Y),
\]
and \( K \) is invariantly defined independently of the choice of \( e \).

Proof of Lemma. First note that
\[
\omega(KX, KY) = \omega(e^{-1}J(eX), e^{-1}J(eY))
\]
\[
= (e^{-1} \cdot \omega)(J(eX), J(eY))
\]
\[
= (e^{-1} \cdot \omega)(eX, eY)
\]
\[
= \omega(e^{-1}(eX), e^{-1}(eY))
\]
\[
= \omega(X, Y),
\]
since \( (e^{-1} \cdot \omega) \) is \( J \)-hermitian. Now let \( e' = fe \) be another suitable map. Then
\[
(e')^{-1}Je' = e^{-1}f^{-1}Je = e^{-1}Je = K,
\]
since \( f \in G_0 \) preserves \( J \).

3 The Lagrangian case

Almost Lagrangian structures are given by a distribution \( H \) with contact form \( \omega \), as above, and by a decomposition
\[
H = E \oplus F,
\]
into two bundles of equal size. This can be characterised by the existence of a trace-free involution $\sigma$ squaring to the identity, with $E$ as its $+1$ eigenspace and $F$ its $-1$ eigenspace. Partial integrability is given by the relation:

$$\omega(\sigma(X), \sigma(Y)) = -\omega(X, Y),$$

equivalent to the isotropy of $E$ and $F$ under $\omega$ (notice the change in sign compared with the CR case). The canonical two-form that we will need is $\nu$, defined by

$$\nu(X, Y) = \frac{1}{2} (\omega(X, Y) - \omega(\sigma(X), \sigma(Y))),$$

while the (split) metric $g$ is

$$g(X, Y) = \nu(X, \sigma(Y)).$$

If $\omega$ and $\nu$ are non-degenerate, and the automorphism $A = g^{-1} \omega$ does not have any purely imaginary eigenvalues, then the proof proceed as in the CR case (except that now $D_\alpha \oplus D_{-\alpha}$ will be mapped into $L_+$, rather than $D_\alpha \oplus D_{-\alpha}$ as was the case then; note also that $L_\pm = L_\pm$, $L_+ = E \otimes \mathbb{C}$ and $L_- = F \otimes \mathbb{C}$).

## 4 Real and imaginary eigenvalues

If $A$ in the CR case has a real eigenvalue, the above procedure does not work. Since a $g$-skew automorphism with real eigenvalues can be approximated arbitrarily closely by those with complex eigenvalues, it must still remain the case that $D_\alpha$ is even-dimensional, and that $C_\alpha = D_\alpha \oplus D_{-\alpha}$ is of split signature $(2p, 2p)$. The only obstruction to choosing $e$ as above is that $e$ cannot now be chosen to lie inside $SO(g)$, but only in its complexification $SO(g, \mathbb{C})$ (uniqueness of the definition of $e$ is preserved by assigning $D_\alpha$ to $L_+$ when $\alpha > 0$). However, a different approach can bear fruit.

Define $M_A$ to be the subset of $M$ where $A$ has real eigenvalues. It must be closed, implying that $M^c = M - M_A$ is open, hence that $M^c$ is a submanifold of $M$. On $M^c$, we have a unique choice of $K$, but this choice cannot necessarily be extended continuously across $M_A$. If $M_A$ has empty interior, and does not separate $M$ into components, then the degeneracy on $M_A$ does not matter much: the equivalence problem for $(H, J)$ can be analysed away from $M_A$, and extended to $M_A$ by continuity. Even if $M$ does get separated in to components, the equivalence problem can still be analysed on each component separately, and the resulting limits “glued together”.

If $M_A$ does have a non-empty interior, then the natural structure on it is an intertwined structure: a decomposition of $H$ into isotropic $H_1 \oplus H_2$, such that there exists a $J$ on $1$ and a $\sigma$ on $H_2$ with $\omega|_{\lambda^2 H_1}$ being $J$-hermitian, and $\omega|_{\lambda^2 H_2}$ being $\sigma$-Lagrangian. On any connected subset of $M_A$ where the rank of the generalised eigenspaces for real eigenvalues is constant, such a structure can be defined. Since that rank is upper semi continuous, and bounded above by $2n$, this allows us to partition $M_A$ into components where such structures are defined, excluding only sets of small dimension.

Of course, the converse results hold for almost Lagrangian structures with an $A$ with pure imaginary eigenvalues.

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