SOLUTION OF SRS ON THE FINITE INTERVAL

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The equations of transient stimulated Raman scattering on the finite interval are solved by
the spectral transform method on the semi-line. As the problem has a free end, the pump and
Stokes output at finite distance can be constructed as the solution of a linear Cauchy-Green integral
equation.

I. INTRODUCTION

Since the first observation in 1983 of the creation of a Raman spike in the pump depletion zone by Drühl, Wenzel
and Carlsten [1], there has been a large amount of studies of stimulated Raman scattering (SRS) of long laser pulses in
gas. The reason is that the Raman spike has been shown to be the macroscopic manifestation of large fluctuations of
the phase of the initial Stokes wave. Raman spike generation were then predicted [2], have been given a coherent-mode
description [3], and experiments have been preformed where SRS grows spontaneously on initial fluctuations [4]. The
Raman spike hence appears as a means to study the quantum properties of the Stokes wave initiation, which gives
informations on the phase of the electromagnetic vacuum. This comes in addition to the process of Stokes growth
which amplifies the quantum fluctuations of the medium. A general discussion of the quantum coherence properties
of SRS is given in [5].

As the SRS equations possess a Lax pair [6], there has been many att emps to modelize the Raman spike as a soliton
(see e.g. [7] [8]). But it has been proved recently that actually the Raman spike occurs in the spectral transform
theory as a manifestation of the continuous spectrum (hence it is not a soliton) when for a short period of time the
reflection coefficient becomes close to zero [10]. These results were obtained by solving the initial-boundary value
problem for the SRS equations on the infinite line, which, from a physical point of view, is inconsistent with a finite
dehasing time of the medium oscillators. However the results obtained are strikingly close to the experimental data
[11]. Here we construct a solution of SRS on the finite interval by using the spectral transform on the semi-line, first
proposed in the context of nonlinear polarization dynamics [12].

The interaction of light with a material medium, in the case when a laser pump pulse (frequency \( \omega_L \), envelope
\( A_L \)) interacts with the optical phonons (eigenfrequency \( \omega_V \), envelope \( Q \)) to give rise to a down-shifted laser pulse
(Stokes emission, frequency \( \omega_S \), envelope \( A_S \)) according to the selection rules (\( \Delta \omega \) is the the detuning from the Raman
resonance, \( \Delta k \) is the phase mismatch)

\[
\omega_S = \omega_L - \omega_V - \Delta \omega, \quad k_S = k_L - k_V - \Delta k
\]

\( I-1 \)
can be modeled by the following slowly varying envelope approximation [13]

\[
\begin{align*}
(\frac{\partial}{\partial Z} + \eta \frac{\partial}{\partial T})A_L &= i \frac{N_0}{4 \eta c} \sqrt{\omega_S \omega_L} Q A_S \exp[i(\Delta k Z - \Delta \omega T)], \\
(\frac{\partial}{\partial Z} + \eta \frac{\partial}{\partial T})A_S &= i \frac{N_0}{4 \eta c} \sqrt{\omega_S \omega_L} Q^* A_L \exp[-i(\Delta k Z - \Delta \omega T)], \\
\frac{\partial}{\partial T}Q + \frac{1}{T_2}Q &= i \frac{\epsilon_0 \alpha_0}{4 m \omega_V} \sqrt{\frac{\omega_S}{\omega_L}} A_L A_S^* \exp[-i(\Delta k Z - \Delta \omega T)].
\end{align*}
\]

\( I-2 \)
The electromagnetic wave field \( E(Z, T) \) and the material excitation \( \hat{X}(Z, T) \) are obtained as

\[
E(Z, T) = \frac{1}{2} A_L \exp[i(k_L Z - \omega_L T)] + \frac{1}{2} \sqrt{\frac{\omega_S}{\omega_L}} A_S \exp[i(k_S Z - \omega_S T)] + c.c.
\]

\( I-3 \)
\[ \tilde{X}(Z, T) = \frac{1}{2} Q \exp[i(kVZ - \omegaVT)] + c.c. \]  (1-4)

Hereabove \( \alpha'_0 \) is the differential polarizability at equilibrium, \( c/\eta \) is the light velocity in the medium, \( N \) is the density of oscillators of mass \( m \), \( T_2 \) is the relaxation time of the medium.

For a medium initially in the ground state

\[ Q(Z, 0) = 0, \]  (1-5)

and for an arbitrary set of input pump and Stokes envelopes profiles (for any value of the mismatch \( \Delta k \))

\[ A_L(0, T) = I_L(T - \frac{\eta}{c}Z), \quad A_S(0, T) = I_S(T - \frac{\eta}{c}Z), \]  (1-6)

we give here the output values of both light waves profiles \( A_L(L, T) \) and \( A_S(L, T) \) in terms of the solution of a Cauchy-Green linear integral equation.

The solution, constructed by the inverse spectral transform theory (IST), is actually exact for infinite relaxation times, and we have proposed an approximate solution for finite \( T_2 \) which matches the experiments with high accuracy [11]. The method applies also for a non-vanishing initial state of the medium but the solution in this case is in general not explicit (this is relevant in physical situations where the quantum fluctuations of the initial population density of the two-level medium is taken into account).

The fact that IST can be applied to SRS on the finite interval has been first proposed by Kaup in 1983 [8]. However the evolution of the spectral data given there does not correspond to the boundary problem (1-2), and in particular, as this evolution is homogeneous, it does not allow for the growth of the Stokes seed on a medium initially at rest. In a different context, a nonhomogeneous evolution of the spectral data has been obtained in [9] where the self-induced transparency equations are solved for an arbitrary initial \( (t = -\infty) \) population density of the two-level medium. IST has been later used to solve an initial-boundary value problem on the half-line for the nonlinear Schrödinger equation (NLS) by Fokas and Its [14], but in this case the required boundary data in \( x = 0 \) for the potential itself renders nonlinear the evolution of the spectral transform.

The property of SRS of being solvable on the finite interval results simply from the nature of the equations for which the initial-boundary value problem (1-2) is well posed and does not require new constraints when it is given on the finite interval (this is not so for NLS for which the vanishing boundary values at infinity become some prescribed boundary value in \( x = 0 \)). Consequently the method applies for every other case of solvable evolutions with nonanalytic dispersion relations when precisely passing to the finite interval does not imply adding information or constraint. However we will discover that there are requirements of analyticity of the boundary data in order for the problem to be integrable.

Before going to the method of solution, it is convenient to rescale the system (1-2) into a dimensionless system, which goes with defining the new variables

\[ x = \frac{1}{L}Z, \quad t = \frac{c}{L}(T - \frac{\eta}{c}Z). \]  (1-7)

Then the dimensionless rescaled material excitation is defined as (the differential polarizability \( \alpha'_0 \) has the dimension of a surface)

\[ q(x,t) = \frac{i}{4} \frac{N\alpha'_0}{\eta c} \sqrt{\omega_L \omega_S} \quad LQ(Z, T), \]  (I-8)

while \( A_L \) ans \( A_S \) are rescaled by using the boundary conditions as

\[ a_L(k, x, t) = A_L(\Delta k, Z, T) \frac{T_m}{I_m}, \quad a_S(k, x, t) = A_S(\Delta k, Z, T) \frac{T_m}{I_m}, \]  (I-9)

which scales to 1 the incident laser pump amplitude. The dependence on the phase mismatch is represented here by the dimensionless wave number

\[ k = \frac{1}{2} L(\Delta k - \frac{\eta}{c}\Delta \omega), \]  (I-10)

which we shall refer to as the essential mismatch parameter. Finally, for

\[ g_0 = \frac{L^2 N\alpha'_0 \eta S}{16 \eta mc^2 \omega_V} I_m^2, \]  (I-11)
(dimensionless), the system \((\text{I-2})\) becomes for infinite dephasing time
\[
\partial_x a_L = q a_S e^{-i\Delta \omega t} e^{2ikx}, \quad \partial_x a_S = -q^* a_L e^{i\Delta \omega t} e^{-2ikx},
\]
\[
q_t = -g_0 a_L a^*_S e^{i\Delta \omega t} e^{-2ikx}.
\]

Due to the dependence of the field envelopes \(a_L\) and \(a_S\) on the essential mismatch \(k\), it is necessary to consider the cooperative interaction of all \(k\)-components with the medium. Then we need actually to consider the medium excitation as resulting from all \(k\) components and replace \((\text{I-13})\) with
\[
q_t = -g_0 \int dk a_L a^*_S e^{i\Delta \omega t} e^{-2ikx},
\]
where now the input \((x = 0)\) values of the pump wave \(a_L\) and the Stokes wave \(a_S\) are also function of \(k\) sharply distributed around \(k = 0\) (the resonance), that we denote by
\[
a_L(k,0,t) = J_L(k,t), \quad a_S(k,0,t) = J_S(k,t).
\]

We will demonstrate that the system \((\text{I-12})\)\((\text{I-14})\), with the initial data \(q(x,0)\) in \(L^1\) and boundary values \((\text{I-15})\), where \(J_L(k,t)\) (resp. \(J_S(k,t)\)) has an analytic continuation in \(\text{Im}(k) > 0\) (resp. \(\text{Im}(k) < 0\)) vanishing as \(|k| \to \infty\), is integrable on the semi-line \(x > 0\). Moreover the solution furnishes the output values of the pump and Stokes waves in \(Z = L\) (i.e. \(x = 1\)), which gives the solution of the SRS equations on the finite interval.

II. THE SPECTRAL PROBLEM ON THE SEMI-LINE

Basic definitions. We briefly give the basic notions on the Zakharov-Shabat spectral problem on the semi-line for the \(2 \times 2\) matrix \(\nu(k,x,t)\) in the potentials \(q(x,t)\) and \(r(x,t)\)
\[
\nu_x + ik[\sigma_3, \nu] = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \nu, \quad x \geq 0, \quad t \geq 0.
\]

The solution \(\nu\) can be verified to obey
\[
\frac{\partial}{\partial x} \det\{\nu\} = 0.
\]

Two fundamental solutions, say \(\nu^\pm\), are defined by
\[
\begin{pmatrix} \nu^+_{11} \\ \nu^+_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \int_0^x dx' q_{21}^+ e^{2ik(x-x')} \right)
\]
\[
\begin{pmatrix} \nu^+_{12} \\ \nu^+_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \left( \begin{pmatrix} f_x^\infty dx' q_{11}^+ e^{-2ik(x-x')} \end{pmatrix} \right)
\]
\[
\begin{pmatrix} \nu^-_{11} \\ \nu^-_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \int_0^x dx' q_{11}^- e^{2ik(x-x')} \right)
\]
\[
\begin{pmatrix} \nu^-_{12} \\ \nu^-_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \left( \begin{pmatrix} f_x^\infty dx' q_{22}^- e^{-2ik(x-x')} \end{pmatrix} \right)
\]
The determinants of $e^{\nu x}$ for the Riemann-Hilbert relations (II-11), the method consists in comparing the integral equations for $k$ equations (II-3)-(II-6) are analytic (entire functions of $k$). Indeed it obeys

$$\nu(k,x,t) = \begin{cases} \nu^+(k,x,t), & \text{Im}(k) > 0 \\ \nu^-(k,x,t), & \text{Im}(k) < 0 \end{cases},$$

(II-7)

which is obtained from the integral equations by integration by part. The scattering coefficients (functions of $k$ and parametrized by $t$) are defined for $k$ real, as

$$\rho^+ = -\int_0^\infty dx' q\nu_{22}^+ e^{2ikx'}, \quad \rho^- = -\int_0^\infty dx' r\nu_{11}^- e^{-2ikx'},$$

(II-9)

Following standard methods (see e.g. [13], especially the appendix), one can prove from the above integral equations the solution $\nu$ obeys on the real axis, following the Riemann-Hilbert problem

$$\nu_1^+ - \nu_1^- = -e^{2ikx}\rho^- \nu_2^-,$$

$$\nu_2^+ - \nu_2^- = e^{-2ikx}\rho^+ \nu_1^-.$$  

(II-11)

The method consists simply in comparing e.g. the integral equations for $(\nu_1^+ - \nu_1^-)e^{-2ikx}$ to the one for $\rho^- \nu_2^-$. Bound states. As the integrals run on the finite support $[0,x]$, the solutions $\nu_1^+$ and $\nu_2^-$ of the Volterra integral equations (II-3) are analytic (entire functions of $k$) in the complex plane. The property (II-6) is used to compute the determinants of $(\nu_1^+\nu_2^-)$ and $(\nu_1^-\nu_2^+)$ both at $x = 0$ and $x = \infty$, which gives

$$\nu_{11}^+(k,\infty,t) = \frac{1}{\tau^+(k,t)}, \quad \nu_{22}^+(k,\infty,t) = \frac{1}{\tau^-(k,t)}.$$  

(II-12)

Hence the quantity $1/\tau^+$ ($1/\tau^-$) is an entire functions of $k$ and can have a number $N^+$ ($N^-$) of zeroes $k_n^+$ ($k_n^-$) which, for bounded $r$ and $q$ are simple an of finite number.

Consequently, the solutions $\nu_1^+$ and $\nu_2^-$ of the Fredholm integral equations (II-5) and (II-4), which can be written also

$$\begin{pmatrix} \nu_{11}^+ \\ \nu_{22}^+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tau^- - \begin{pmatrix} \int_x^\infty dx' q\nu_{22}^+ e^{2ik(x-x')} \\ \int_x^\infty dx' r\nu_{11}^- e^{2ik(x-x')} \end{pmatrix},$$

(II-13)

$$\begin{pmatrix} \nu_{11}^- \\ \nu_{22}^- \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tau^+ - \begin{pmatrix} \int_x^\infty dx' q\nu_{22}^- e^{-2ik(x-x')} \\ \int_x^\infty dx' r\nu_{11}^+ e^{-2ik(x-x')} \end{pmatrix},$$

(II-14)

have respectively the $N^-$ and $N^+$ simple poles $k_n^-$ and $k_n^+$ of $\tau^-$ and $\tau^+$. Then the integral equations for the residues allow to obtain

$$\text{Res}_{k_n^-} \nu_1^- = C_n^- \nu_2^-(k_n^-) \exp[2ik_n^-x], \quad C_n^- = \text{Res}_{k_n^-} \rho^-,$$

(II-15)

$$\text{Res}_{k_n^+} \nu_2^+ = C_n^+ \nu_1^+(k_n^+) \exp[-2ik_n^+x], \quad C_n^+ = \text{Res}_{k_n^+} \rho^+.$$  

(II-16)

As for the Riemann-Hilbert relations (II-11), the method consists in comparing the integral equations for $\exp[-2ik_n^-x]\text{Res} \nu_1^-$ and for $C_n^- \nu_2^-(k_n^-)$.
Boundary behaviors. An important relation for the following is the so-called unitarity relation
\[ \tau^+ \tau^- = 1 - \rho^+ \rho^-, \tag{II-17} \]
obtained by computing \( \nu_{\pm 2}(k, \infty, t) \) by using (II-11) and by comparing the result with (II-12).

Consequently the boundary behaviors of \( \nu \) can be written
\[ x = 0 : \quad \nu^+ = \begin{pmatrix} 1/	au^+ & \rho^+ \\ 0 & 1 \end{pmatrix}, \quad \nu^- = \begin{pmatrix} 1 & 0 \\ \rho^- & 1 \end{pmatrix}, \tag{II-18} \]
\[ x = \infty : \quad \nu^+ = \begin{pmatrix} 1 \rho^+ e^{-2ikx} & 0 \\ -e^{2ikx}/\tau^- & 1/\tau^+ \end{pmatrix}, \quad \nu^- = \begin{pmatrix} \tau^- & 0 \\ e^{-2ikx}/\rho^-/\tau^- & 1/\tau^- \end{pmatrix}. \tag{II-19} \]

The \( \bar{\partial} \) problem. The analytic properties of \( \nu \) can be summarized in the following formula
\[ \frac{\partial \nu}{\partial k} = \nu R \exp[2i\lambda_3 x], \tag{II-20} \]
with the spectral transform
\[ R = \frac{i}{2} \left( \begin{array}{cc} 0 & \rho^+ \delta^+ \\ -\rho^- \delta^- & 0 \end{array} \right) - 2i\pi \left( \sum C_n^+ \delta(k - k_n^+) - \sum C_n^- \delta(k - k_n^-) \right), \tag{II-21} \]
where the distributions \( \delta^\pm \) and \( \delta(k - k_n) \) are defined as
\[ \int \int d\lambda \wedge d\bar{\lambda} f(\lambda) \delta^\pm(\lambda) = -2i \int_{-\infty}^{+\infty} d\lambda_R f(\lambda_R \pm i0), \]
\[ \int \int d\lambda \wedge d\bar{\lambda} f(\lambda) \delta(\lambda - k_n) = f(k_n) \tag{II-22} \]
with the notation \( \lambda = \lambda_R + i\lambda_I \).

### III. INVERSE SPECTRAL PROBLEM

The Cauchy-Green integral equation. The inverse problem is solved by integrating the \( \bar{\partial} \)-equation (II-21) with the boundary (II-8). We prove here the following theorem: the solution \( f(k, x, t) \) of the Cauchy-Green integral equation
\[ f(k, x, t) = 1 + \frac{1}{2i\pi} \int \frac{d\lambda \wedge d\bar{\lambda}}{\lambda - k} f(\lambda, x, t) R(\lambda, t) \exp[2i\lambda_3 x]. \tag{III-1} \]
coincides with the solution \( \nu(k, x, t) \) of (II-3)-(II-6) if the reflection coefficient possess a meromorphic continuation with simple poles \( k_n^\pm \) and residues \( C_n^\pm \), namely
\[ \pm \text{Im}(k) > 0 \quad \Rightarrow \quad \frac{\partial}{\partial k} \rho^\pm(k) = 2i\pi \sum C_n^\pm \delta(k - k_n^\pm). \tag{III-2} \]
Such analytical property of the spectral data occur when the potentials are on compact support. Therefore we shall assume here that the initial datum \( q(x, 0) \) is indeed on compact support and we will have to demonstrate that the time evolution conserves this analytical property. Actually the physically interesting case is when \( q(x, 0) \) vanish identically. This implies no bound states which will be assumed in the following (everything can be extended to the case with bound states as they appear as the poles \( k_n^\pm \) of \( \rho^+ \) in \( \text{Im}(k) > 0 \), and those \( (k_n^-) \) of \( \rho^- \) in \( \text{Im}(k) < 0 \).
Proof of the Theorem. The first step is to verify from (II-20) that the function $f$ is solution of the differential problem (II-1). This is easily done by following the method of [15], in short: prove the relation

$$\frac{\partial}{\partial k}[(f_x - ikf_\sigma)i f^{-1}] = 0$$

(III-3)

and integrate it. Then the solution $f$ of (III-1) solves the spectral problem (II-1) with

$$\begin{pmatrix} 0 & \nu \rho \\ r & 0 \end{pmatrix} = i[\sigma, f^{(1)}]$$

(III-4)

where $f^{(1)}$ is the coefficient of $1/k$ in the Laurent expansion of $f(k)$.

The second step consists in proving

$$f^+_l(k, x, t) = \nu^+_l(k, x, t), \quad f^-_l(k, x, t) = \nu^-_l(k, x, t),$$

(III-5)

which is achieved just by comparing the values of these vectors in $x = 0$. The functions $f^+_l$ and $f^-_l$ solve the following coupled vectorial system (forget for a while the $(x, t)$-dependence) on the real axis $\text{Im}(k) = 0$:

$$f^+_l(k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2i\pi} \int \frac{d\lambda}{\lambda - (k + i0)} \rho^-(\lambda)f^-_2(\lambda) e^{2i\lambda x},$$

(III-6)

$$f^-_l(k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2i\pi} \int \frac{d\lambda}{\lambda - (k - i0)} \rho^+(\lambda)f^+_1(\lambda) e^{-2i\lambda x}. $$

(III-7)

Now, by closing the contour of integration in $\text{Im}(\lambda) < 0$ for $f^+_l$ and $\text{Im}(\lambda) > 0$ for $f^-_l$, the Cauchy theorem for the analyticity requirement (III-2) leads to

$$\forall x < 0, \text{Im}(k) = 0 : f^+_l(k, x, t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f^-_l(k, x, t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

(III-8)

Consequently, the two matrices $(f^+_l, f^-_l)$ and $(\nu^+_l, \nu^-_l)$ solve the same first order differential problem and have the same values in $x = 0$, so (III-5) is proved.

The third step results in proving

$$f^+_l(k, x, t) = \nu^+_l(k, x, t), \quad f^-_l(k, x, t) = \nu^-_l(k, x, t),$$

(III-9)

which is performed by evaluating the functions

$$f^+_l = \frac{1}{2i\pi} \int \frac{d\lambda}{\lambda - (k + i0)} \rho^+ f^+_1 e^{-2i\lambda x},$$

$$f^-_l = -\frac{1}{2i\pi} \int \frac{d\lambda}{\lambda - (k - i0)} \rho^- f^-_2 e^{2i\lambda x}$$

(III-10)

as $x \to \infty$. A useful formula is

$$\lim_{x \to \pm \infty} Pf \int \frac{d\lambda}{\lambda - k} e^{i(\lambda - k)x} f(\lambda) = \pm i\pi f(k),$$

(III-11)

where $Pf$ denotes the Cauchy principal value integral, and the Sokhotski-Plemelj formula

$$\int \frac{d\lambda}{\lambda - (k \pm i0)} f(\lambda) = \pm i\pi f(k) + Pf \int \frac{d\lambda}{\lambda - k} f(\lambda).$$

(III-12)

Since it has been already proved that $f^+_l = \nu^+_l$ and $f^-_l = \nu^-_l$, the functions $f^+_l$ and $f^-_l$ have from (II-4) and (II-5) bounded behaviors as $x \to \infty$. Consequently

$$\lim_{x \to \pm \infty} f^+_l = 0, \quad \lim_{x \to \pm \infty} f^-_l = 0.$$

(III-13)
The remaining functions $f_{11}$ and $f_{22}^\pm$ are obtained by using the differential form of (III-1), namely

$$f_{11}^+ - f_{11}^- = -e^{2ikx} \rho^- f_{12}^- , \quad f_{22}^+ - f_{22}^- = e^{-2ikx} \rho^+ f_{21}^+.$$  (III-14)

With (III-8), this gives

$$\forall x < 0 \, , \, \text{Im}(k) = 0 \, : \, f_{11}^-(k, x, t) = 1, \quad f_{22}^+(k, x, t) = 1.$$  (III-15)

Finally, the two matrices $(f_{11}^-, f_{22}^+)$ and $(\nu_{11}^-, \nu_{22}^+)$ solve the same first order differential problem and have the same values in $x = 0$ for the diagonal elements and in $x = \infty$ for the off-diagonal, hence they are equal. The theorem is proved and an immediate consequence is that the coefficients $\rho^\pm$ are effectively given by (II-9).

**Reduction.** It is convenient for physical applications to assume the following reduction

$$r = -\bar{q}$$  (III-16)

for which

$$(\nu_{11}^+)^* = \nu_{22}^-, \quad (\nu_{12}^+)^* = -\nu_{21}^-,$$  (III-17)

$$\rho^+ = -(\rho^-)^*, \quad \tau^+ = (\tau^-)^*, \quad k_n^+ = \bar{k}_n, \quad C_n^+ = \bar{C}_n.$$  (III-18)

The overbar denotes the complex conjugation and the “star”

$$f^*(k) = \overline{f(k)}.$$  (III-19)

As a consequence the boundary values of $\nu$ become

$$x = 0 \, : \, \nu^+ = \begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix}, \quad \nu^- = \begin{pmatrix} 1 & -\rho^* \\ 0 & 1 \end{pmatrix},$$  (III-20)

$$x = \infty \, : \, \nu^+ = \begin{pmatrix} \tau & 0 \\ e^{2ikx} \rho^*/\tau^* & 0 \end{pmatrix}, \quad \nu^- = \begin{pmatrix} \tau^* & -e^{-2ikx} \rho/\tau \\ 0 & 1/\tau^* \end{pmatrix},$$  (III-21)

where we use the notation $\rho = \rho^+$ and $\tau = \tau^+$, for real $k$.

**Transmission coefficient.** It is useful for the following to derive the relationship between $\tau$ and $\rho$. Although this can be done in general, we still consider only the case when no bound states are present (this will be the case of interest). From the definitions (II-10), we have for large $k$

$$\tau^\pm(k) \sim 1 + \mathcal{O}(1/k).$$  (III-22)

Defining then $h(k)$ as $\ln(\tau^+)$ for $\text{Im}(k) \geq 0$, and $-\ln(\tau^-)$ for $\text{Im}(k) \leq 0$, its discontinuity on the real $k$-axis can be written by means of (II-17)

$$h^+ - h^- = \ln(1 - \rho^+ \rho^-).$$  (III-23)

As from (II-22), $h(k)$ vanishes for large $k$, the above Riemann-Hilbert problem has the following solution

$$\text{Im}(k) \neq 0 \, : \, h(k) = \frac{1}{2i\pi} \int \frac{d\lambda}{\lambda - k} \ln(1 - \rho^+ \rho^-),$$  (III-24)

and hence $\tau^\pm$ are given from $\rho^\pm$. In the reduction (II-16), this relation can be written

$$\tau = \sqrt{1 + |\rho|^2} \, e^{i\theta}, \quad \theta = -\frac{1}{2\pi} \int \frac{d\lambda}{\lambda - k} \ln(1 + |\rho|^2).$$  (III-25)
**IV. LAX PAIR AND GENERAL EVOLUTION**

*Lax pair.* The compatibility

\[ U_t - V_x + [U, V] = 0 \]  \hspace{1cm} (IV-1)

between the two following spectral problems

\[ \nu_x = \nu \text{i}k \sigma_3 + U \nu, \quad U = -\text{i}k \sigma_3 + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \]  \hspace{1cm} (IV-2)

\[ \nu_t = \nu \Omega + V \nu, \quad V = \frac{1}{2\text{i}\pi} \int \frac{d\lambda \wedge d\bar{\lambda}}{\lambda - k} \nu (M e^{2\text{i}\lambda x} - \frac{\partial \Omega}{\partial \lambda}) \nu^{-1}, \]  \hspace{1cm} (IV-3)

leads to the evolutions

\[ \frac{\partial}{\partial t} \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} = -\frac{1}{2\pi} \int \left[ \sigma_3, \nu (M e^{2\text{i}\lambda x} - \frac{\partial \Omega}{\partial \lambda}) \nu^{-1} \right], \]  \hspace{1cm} (IV-4)

\[ \frac{\partial}{\partial t} R = [R, \Omega] + M. \]  \hspace{1cm} (IV-5)

This general result is not proved here (see [13] or [14]), but we give in the next section the computation of the time evolution of \( \rho(x, t) \) (given here by (IV-5)) directly from the Lax pair, when the evolution of \( q(x, t) \) is given by (IV-10).

*Solvable evolution.* The entries \( \Omega(k, t) \) (diagonal matrix-valued function) and \( M(k, t) \) (off-diagonal matrix-valued distribution) are arbitrary, which allows to solve evolutions with nonanalytic dispersion relations \( \frac{\partial \Omega}{\partial \bar{k}} \neq 0 \) and arbitrary boundary values [16] (the problem of arbitrary boundary values has been first solved for the SIT equations in [3]). As we are interested in solving evolutions as (I-14) where the integral runs on the real \( k \)-axis, we work within the reduction (III-16) and chose (remember the definition (III-19)):

\[ M(k, t) = \begin{pmatrix} 0 & m(k, t) \delta^+ \\ -m^*(k, t) \delta^- & 0 \end{pmatrix}, \]  \hspace{1cm} (IV-6)

and the dispersion relation

\[ \Omega(k, t) = \omega(k, t) \sigma_3, \quad \omega(k, t) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda - k} \varphi(\lambda, t), \quad k \notin \mathbb{R}, \]  \hspace{1cm} (IV-7)

analytic except on the real axis where it stands the discontinuity

\[ \frac{\partial \omega(k, t)}{\partial k} = \frac{i}{2} (\omega^+(k_R, t) - \omega^-(k_R, t)) \delta(k_I) = \varphi(k_R, t)\delta(k_I). \]  \hspace{1cm} (IV-8)

Then the quantity \( \nu(\partial \Omega/\partial \bar{k})\nu^{-1} \) is ill-defined because \( \nu(x, t) \) itself is discontinuous on the real axis. To compute it we need to use the identity [13]

\[ \nu \frac{\partial \Omega}{\partial k} \nu^{-1} = \frac{\partial}{\partial k} (\nu \Omega \nu^{-1}) - \nu |Re e^{2\text{i}kx} \Omega| \nu^{-1}. \]  \hspace{1cm} (IV-9)

Within the reduction and with the above choice of \( M \) and \( \Omega \), the evolution (IV-4), becomes by means of (IV-9)

\[ q_t = \frac{2i}{\pi} \int_{-\infty}^{+\infty} dk \left[ 2\varphi \nu_1^+ \nu_1^- + m(\nu_1^+)^2 e^{-2\text{i}kx} + m^*(\nu_1^-)^2 e^{2\text{i}kx} \right]. \]  \hspace{1cm} (IV-10)

To get the above result, it is necessary to use the Riemann-Hilbert relations (II-11) to express \( \nu_1^+ \) and \( \nu_1^- \) in terms of \( \nu_{11}^+ \) and \( \nu_{12}^- \). This equation, coupled to the system (IV-2), is the general solvable evolution with a nonanalytic dispersion law vanishing at large \( k \). It is used now to solve the boundary value problem (I-12) (I-14).
V. SOLUTION OF SRS

The above theory allows to solve the nonlinear evolution problem (I-12) (I-14) with the initial data of \( q(x,0) \) (which actually will be taken to vanish) and boundary value (I-13). This is done in the following way

**Basic relations.** As the 3 functions

\[
\left( \frac{a_L}{a_S \exp[2ikx-i\Delta \omega t]} \right), \quad \left( \frac{\nu_{11}^+}{\nu_{21}^+} \right), \quad \left( \frac{\nu_{12}^-}{\nu_{22}^-} \right) \exp[2ikx],
\]

(V-1)
solve the same first-order differential system, they are uniquely related by their values in \( x=0 \) and comparing (I-15) with (III-20) we obtain readily

\[
\left( \frac{a_L}{a_S \exp[2ikx-i\Delta \omega t]} \right) = J_L \left( \frac{\nu_{11}^+}{\nu_{21}^+} \right) + J_S \left( \frac{\nu_{12}^-}{\nu_{22}^-} \right) \exp[2ikx-i\Delta \omega t].
\]

(V-2)

This formula actually gives the solution to the physical problem (compute the output values from the input values) as soon as the function \( \rho^+(k,t) \) is calculated (\( \tau \) is expressed from \( \rho \) in (III-23)).

**Computation of \( \rho(k,t) \).** This computation is performed by first determining the functions \( \varphi(k,t) \) and \( m(k,t) \). This is done by equating the evolution (I-14) with (IV-10) by means of the relation (V-2) between \( (a_L,a_S) \) and \( \nu \). Using the reduction relations (III-17), we finally get that the evolution (I-14), where \( (a_L,a_S) \) is expressed in terms of \( \nu \) through (V-2), reads exactly as (IV-10) for

\[
\varphi = -\frac{\pi}{3} \gamma_0 (|J_L|^2 - |J_S|^2), \quad m = i\frac{\pi}{2} \gamma_0 J_L J_S^* e^{i\Delta \omega t}.
\]

(V-3)

The corresponding evolution (IV-3), which reads

\[
\rho^+_t = -2\omega^+ \rho^+ - 2im,
\]

(V-4)

then gives by means of (V-7)

\[
\rho^+_t(k,t) = \pi \gamma_0 J_L(k,t) J_S^*(k,t) e^{i\Delta \omega t} - \frac{i}{2} \gamma_0 \rho^+(k,t) \int \frac{d\lambda}{\lambda - (k+i0)} (|J_L|^2 - |J_S|^2),
\]

(V-5)

\[
C^+_{n,t} = C^+_n \left( -\frac{i}{2} \gamma_0 \int \frac{d\lambda}{\lambda - k_n} (|J_L|^2 - |J_S|^2) \right).
\]

(V-6)

For everything to work, we have seen in sec. 3 that the analytical requirement (III-2) is essential. It is clear on the above time evolution that this is ensured for all \( t \) if we require that \( J_L(k,t) \) be analytic in the upper half plane and \( J_S(k,t) \) in the lower one (note from (III-19) that \( J_S^*(k,t) \) is a function of \( k \) analytic in the upper half plane). In short

\[
\text{Im}(k) > 0 \Rightarrow \frac{\partial}{\partial k} J_L(k,t) = 0, \quad \text{Im}(k) < 0 \Rightarrow \frac{\partial}{\partial k} J_S(k,t) = 0.
\]

(V-7)

The other constraint is that, for all \( t \), the function \( \rho^+(k,t) \) vanish at large \( k \) in Im(\( k \)) > 0. This is guaranteed here because \( J_L(k,t) \), \( J_S^*(k,t) \) and \( \omega^+(k,t) \) do vanish at large \( k \) in Im(\( k \)) > 0. Note that it would not be true for an analytic dispersion relation (like the polynomial \( k^2 \) for NLS).

A medium initially at rest corresponds to \( q(x,0) = 0 \), and hence from (I-9) and (I-11) to

\[
\rho^+(k,0) = 0, \quad C^+_n(0) = 0.
\]

(V-8)

Consequently the evolution (V-4) ensures that no bound states are created and the equation (V-5) can be solved explicitly. Then the solution of the related integral equation (III-6) (III-7) gives \( \nu^\pm(k,x,t) \) which in turn allows to compute the light field envelopes \( a_L(k,x,t) \) and \( a_S(k,x,t) \). In particular their values in \( x = 1 \) furnishes the output in the case of the finite interval of physical length \( L \). For instance, with the solution \( \rho^+(k,t) \) of (V-5), the pump output is obtained by solving (with \( \rho^- = -\rho^+(\ast) \)):

\[
\nu^+_{11}(k,x,t) = 1 - \frac{1}{2i\pi} \int \frac{d\lambda}{\lambda - (k+i0)} \rho^-(\lambda,t) \nu^+_{12}(\lambda,x,t) e^{2i\lambda x},
\]

(V-9)
\[ \nu_{12}(k, x, t) = \frac{1}{2i\pi} \int \frac{d\lambda}{\lambda - (k - i0)} \rho^+(\lambda, t) \nu^+_{11}(\lambda, x, t) e^{-2i\lambda x}, \]  

(V-10)

and then it is given by

\[ a_L(k, 1, t) = J_L(k, t) \nu^+_{11}(k, 1, t) + J_S(k, t) \nu^-_{12}(k, 1, t) \exp[2ik - i\Delta t]. \]  

(V-11)

We note finally that, in the case of the semi-infinite line \((L \to \infty)\), from the boundary values \([III-21]\), we have the following pump output

\[ a_L(k, \infty, t) = \frac{1}{\tau^+} (J_L - \rho^+ J_S e^{-i\Delta t}), \]  

(V-12)

which has been used in \([1]\) to interpret the experiments of \([1]\).

VI. EVOLUTION OF THE SPECTRAL TRANSFORM FROM THE LAX PAIR

The auxiliary Lax operator given in \([IV-3]\) can be simplified by using \([IV-9]\) and the property that \(\nu \Omega \nu^{-1}\) vanishes as \(O(1/k)\). Then \([IV-3]\) reduces to

\[ \nu_t = X \nu, \quad X = \frac{1}{2i\pi} \int \frac{d\lambda}{\lambda - k} \nu e^{-2i\lambda x} (M + [R, \Omega]) e^{i\lambda k x} \nu^{-1}. \]  

(VI-1)

For the particular choices \([IV-6]\) and \([IV-7]\), the above equation reads

\[ \nu_t = X \nu, \quad X(k, x, t) = -\frac{1}{\pi} \int \frac{d\lambda}{\lambda - k} \chi(\lambda, x, t), \]  

(VI-2)

\[ \chi(k, x, t) = (m - i\rho^+ \omega^+) \left( \begin{array}{cc} \nu^-_{12} & i \nu^+_{11} \\ i \nu^-_{12} & \nu^+_{11} \end{array} \right) \exp[-2ikx] + \]  

\[ + (m^* - i(\rho^+)^* \omega^-) \left( \begin{array}{cc} \nu^-_{22} & i \nu^+_{21} \\ i \nu^-_{22} & \nu^+_{21} \end{array} \right) \exp[2ikx]. \]  

(VI-3)

Our purpose here is to recover now the time evolution \([V-4]\) of the reflection coefficient \(\rho(k, t)\) by means of the usual method which consists in evaluating \([VI-4]\) at one boundary, say in \(x = 0\). The boundary values of both \(X^+\) and \(X^-\) in \(x = 0\) are easily calculated from those of \(\nu^+\) and \(\nu^-\) in \([II-18]\). Then the equation \([VI-2]\) gives for \(\nu^+\) and \(\nu^-\) successively

\[ \rho^+_t = \frac{1}{\pi} \int \frac{d\lambda}{\lambda - (k + i0)} (i\omega^+ \rho^+ - m), \]  

(VI-4)

\[ 0 = \frac{1}{\pi} \int \frac{d\lambda}{\lambda - (k + i0)} (-i\omega^- (\rho^+)^* + \tilde{m}), \]  

(VI-5)

\[ 0 = \frac{1}{\pi} \int \frac{d\lambda}{\lambda - (k - i0)} (i\omega^- \rho^- - m), \]  

(VI-6)

\[ - (\rho^-)^+_t = \frac{1}{\pi} \int \frac{d\lambda}{\lambda - (k - i0)} (-i\omega^- (\rho^+)^* + \tilde{m}). \]  

(VI-7)

First, these equations are compatible if \(\omega^- = (\omega^+)^*\), which is guaranteed through \([V-7]\) by \(\varphi \in i\mathbb{R}\) (see \([V-3]\)). Then the above four equations reduce to two and we select \([VI-4]\) and \([VI-6]\) whose solution goes in two steps. 

First step: the equation \([VI-4]\) is realized if the function \(i\omega^+ \rho^+ - m\) is analytic in \(\text{Im}(k) > 0\) and vanishes as \(k \to \infty\). This is true by construction for \(\omega(k, t)\) and this is true also for \(m(k, t)\) because we have required that \(J_L(k, t)\) be analytic in the upper half plane and \(J_S(k, t)\) in the lower one \(\text{(remember that from \([III-19]\) that \(J_S^*(k, t)\) is a function...)}\)

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of $k$ analytic in the upper half plane). Finally this holds also for the reflection coefficient $\rho^+(k,t)$ by the requirement (II-2).

Second step: by subtraction of (VI-4) and (VI-6), and by use of the Sokhotski formula (III-12) we arrive precisely at the time evolution (V-4).

Finally, using the behaviors in $x = \infty$ instead of $x = 0$, furnishes the time evolution of the transmission coefficient $\tau^\pm$.

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