ON THE DIMENSION OF NON-ABELIAN TENSOR SQUARE OF LIE SUPERALGEBRAS

RUDRA NARAYAN PADHAN, IBRAHEM YAKZAN HASAN, AND SUSHREE SANGEETA PRADHAN

Abstract. In this paper, we determine upper bound for the non-abelian tensor product of finite dimensional Lie superalgebra. More precisely, if \( L \) is a non-abelian nilpotent Lie superalgebra of dimension \((k \mid l)\) and its derived subalgebra has dimension \((r \mid s)\), then \( \dim(L \otimes L) \leq (k + l - (r + s))(k + l - 1) + 2 \). We discuss the conditions when the equality holds for \( r = 1, s = 0 \) explicitly.

1. Introduction

In 1991, Rocco [15] proved that if \( G \) is a finite \( p \)-group of order \( P^n \) with derived subgroup of order \( p^m \), then \( G \otimes G \leq p^{n(n-m)} \). Further this bound was improved by Niroomand [9], i.e., \( G \otimes G \leq p^{n(n-m)} \). Ellis [1, 2] developed the theory of tensor product for Lie algebra. If \( L \) and \( K \) are two finite dimensional nilpotent Lie algebras, then the upper bound and lower bound of the dimension of the non-abelian tensor product \( L \otimes K \) have been studied by salemkar et. al [13] and the results are generalization of Rocco’s results for finite \( p \)-group. Recently, an improved upper bound on the dimension of \( L \otimes L \) has been discussed in [10], explicitly, if \( L \) is a non-abelian nilpotent Lie algebra of dimension \( n \) and its derived subalgebra has dimension \( m \), then \( \dim(L \otimes L) \leq (n - m)(n - 1) + 2 \).

Lie superalgebras have applications in many areas of Mathematics and Theoretical Physics as they can be used to describe supersymmetry. Kac [5] gives a comprehensive description of mathematical theory of Lie superalgebras, and establishes the classification of all finite dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero. In the last few years, the theory of Lie superalgebras has evolved remarkably, obtaining many results in representation theory and classification. Most of the results are extension of well known facts of Lie algebras [16, 6, 3]. Recently, Garcia-Martínez [3] introduced the notions of non-abelian tensor product of Lie superalgebras and exterior product of Lie superalgebras over a commutative ring. In this paper we determine an upper bound on the dimension of non-abelian tensor product of nilpotent Lie superalgebra.

2. Preliminaries and Auxiliary Results

Let \( \mathbb{Z}_2 = \{0, 1\} \) be a field. A \( \mathbb{Z}_2 \)-graded vector space \( V \) is simply a direct sum of vector spaces \( V_0 \) and \( V_1 \), i.e., \( V = V_0 \oplus V_1 \). It is also referred as a superspace. We consider all vector superspaces and superalgebras are over field \( \mathbb{F} \) (characteristic of \( \mathbb{F} \neq 2, 3 \)). Elements in \( V_0 \) (resp. \( V_1 \)) are called even (resp. odd) elements. Non-zero elements of \( V_0 \cup V_1 \) are called homogeneous elements. For a homogeneous element \( v \in V_0 \), with \( \sigma \in \mathbb{Z}_2 \) we set \( |v| = \sigma \) as the degree of \( v \). A subsuperspace (or, subspace) \( U \) of \( V \) is a \( \mathbb{Z}_2 \)-graded vector subspace where \( U = (V_0 \cap U) \oplus (V_1 \cap U) \). We adopt the convention that whenever the degree function appears in a formula, the corresponding elements are supposed to be homogeneous.

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A Lie superalgebra (see [5][6]) is a superspace \( L = L_0 \oplus L_1 \) with a bilinear mapping \([\cdot, \cdot] : L \times L \to L\) satisfying the following identities:

1. \([L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}\), for \(\alpha, \beta \in \mathbb{Z}_2\) (\(\mathbb{Z}_2\)-grading),
2. \([x, y] = -(\cdot 1)^{|x||y|}[y, x] \) (graded skew-symmetry),
3. \((\cdot 1)^{|x||y|}|y, z) + (\cdot 1)^{|y||z|}|y, [z, x]) + (\cdot 1)^{|z||x|}[z, [x, y]) = 0\) (graded Jacobi identity),

for all \(x, y, z \in L\). Clearly \(L_0\) is a Lie algebra, and \(L_1\) is a \(L_0\)-module. If \(L_1 = 0\), then \(L\) is just Lie algebra, but in general a Lie superalgebra is not a Lie algebra. A Lie superalgebra \(L\), is called abelian if \([x, y] = 0\) for all \(x, y \in L\). Lie superalgebras without even part, i.e., \(L_0 = 0\), are abelian. A subsuperalgebra (or subalgebra) of \(L\) is a \(\mathbb{Z}_2\)-graded vector subspace which is closed under bracket operation. The graded subalgebra \([L, L]\), of \(L\) is known as the derived subalgebra of \(L\). A \(\mathbb{Z}_2\)-graded subspace \(I\) is a graded ideal of \(L\) if \([I, L] \subseteq I\). The ideal

\[Z(L) = \{z \in L : [z, x] = 0 \text{ for all } x \in L\}\]

is a graded ideal and it is called the center of \(L\). A homomorphism between superspaces \(f : V \to W\) of degree \(|f| \in \mathbb{Z}_2\), is a linear map satisfying \(f(V_\alpha) \subseteq W_{\alpha+|f|}\) for \(\alpha \in \mathbb{Z}_2\). In particular, if \(|f| = 0\), then the homomorphism \(f\) is called homogeneous linear map of even degree. A Lie superalgebra homomorphism \(f : L \to M\) is a homogeneous linear map of even degree such that \(f([x, y]) = [f(x), f(y)]\) holds for all \(x, y \in L\). If \(I\) is an ideal of \(L\), the quotient Lie superalgebra \(L/I\) inherits a canonical Lie superalgebra structure such that the natural projection map becomes a homomorphism. The notions of epimorphisms, isomorphisms and automorphisms have the obvious meaning.

Throughout this article, for superdimension of Lie superalgebra \(L\) we simply write \(\dim L = (m \mid n)\), where \(\dim L_0 = m\) and \(\dim L_1 = n\). Also \(A(m \mid n)\) denotes an abelian Lie superalgebra where \(\dim A = (m \mid n)\). A Lie superalgebra \(L\) is said to be Heisenberg Lie superalgebra if \(Z(L) = L_1'\) and \(\dim Z(L) = 1\). According to the homogeneous generator of \(Z(L)\), Heisenberg Lie superalgebras can further split into even or odd Heisenberg Lie superalgebras [10]. By Heisenberg Lie superalgebra we mean special Heisenberg Lie superalgebra in this article, for more details on Heisenberg Lie superalgebras and its multiplier see [7] [8] [13] [12] [11] [4] [17]. Now we list some useful results from [7], for further use.

**Theorem 2.1.** [7] See Theorem 4.2, 4.3] Every Heisenberg Lie superalgebra with even center has dimension \((2m + 1 \mid n)\) and is isomorphic to \(H(m, n) = H_\Pi \oplus H_\Upsilon\), where

\[H_\Pi = \langle x_1, \ldots, x_{2m}, z \mid [x_i, x_{m+i}] = z, \ i = 1, \ldots, m > \]

and

\[H_\Upsilon = \langle y_1, \ldots, y_n \mid [y_j, y_j] = z, \ j = 1, \ldots, n > . \]

Further,

\[\dim M(H(m, n)) = \begin{cases} (2m^2 - m + n(n + 1)/2 - 1 \mid 2mn) & \text{if } m + n \geq 2 \\ (0 \mid 0) & \text{if } m = 0, n = 1 \\ (2 \mid 0) & \text{if } m = 1, n = 0. \end{cases} \]

The following is the established result for the multiplier and cover of Heisenberg Lie superalgebra of odd center.

**Theorem 2.2.** [8] See Theorem 2.8] Every Heisenberg Lie superalgebra, with odd center has dimension \((m \mid m + 1)\), is isomorphic to \(H_m = H_\Pi \oplus H_\Upsilon\), where

\[H_m = \langle x_1, \ldots, x_m, y_1, \ldots, y_m, z \mid [x_j, y_j] = z, j = 1, \ldots, m > . \]
Further,

\[ \dim \mathcal{M}(H_m) = \begin{cases} 
(m^2 - 1) & \text{if } m \geq 2 \\
1 & \text{if } m = 1.
\end{cases} \]

3. Non-abelian tensor products

Here we recall some of the known notation and results from [3]. Let \( P \) and \( M \) be two Lie superalgebras, then by an action of \( P \) on \( M \) we mean a \( \mathbb{K} \)-bilinear map of even degree \( P \times M \rightarrow M \),

\[ (p, m) \mapsto p \cdot m, \]

such that

1. \( [p, p'] \cdot m = p(p' \cdot m) - (-1)^{|p||p'|} p' (p \cdot m) \),
2. \( p \cdot [m, m'] = [p \cdot m, m'] + (-1)^{|p||m|}[m, p \cdot m'] \),

for all \( p, p' \in P \) and \( m, m' \in M \). For any Lie superalgebra \( M \), the Lie multiplication induces an action on itself via \( m \cdot m' = [m, m'] \). The action of \( P \) on \( M \) is called trivial if \( p \cdot m = 0 \) for all \( p \in P \) and \( m \in M \).

Given two Lie superalgebras \( M \) and \( P \) with action of \( P \) on \( M \), we define the semidirect product \( M \rtimes P \) with underlying supermodule \( M \oplus P \) endowed with the bracket given by \( ([m, p], (m', p')) = ([m, m'] + p \cdot m' - (-1)^{|m||p'|}[p', [m, m]], [p, p']) \). A crossed module of Lie superalgebras is a homomorphism of Lie superalgebras \( \partial : M \rightarrow P \) with an action of \( P \) on \( M \) satisfying

1. \( \partial(p \cdot m) = [p, \partial(m)] \),
2. \( \partial(m \cdot m') = [m, \partial(m')] \), for all \( p \in P \) and \( m, m' \in M \).

A bilinear function \( f : M \times N \rightarrow T \) is called Lie superpairing if the following relations are satisfied:

1. \( f([m, m'], n) = f(m, m'n) - (-1)^{|m||m'|} f(m', m n) \),
2. \( f(m, [n, n']) = (-1)^{|m||[n,n']} f(m'n, n) - (-1)^{|m||n|} f(m, n'n) \),
3. \( f(m \cdot m', n') = (-1)^{|m||[m', n']} f(m, n') \),

for every \( m, m' \in M_{\overline{0}} \cup M_{\overline{1}} \) and \( n, n' \in N_{\overline{0}} \cup N_{\overline{1}} \).

Let \( M \) and \( N \) be two Lie superalgebras with actions on each other. Let \( X_{M,N} \) be the \( \mathbb{Z}_2 \)-graded set of all symbols \( m \otimes n \), where \( m \in M_{\overline{0}} \cup M_{\overline{1}}, n \in N_{\overline{0}} \cup N_{\overline{1}} \) and the \( \mathbb{Z}_2 \)-gradation is given by \( |m \otimes n| = |m| + |n| \). The non-abelian tensor product of \( M \) and \( N \), denoted by \( M \otimes N \), as the Lie superalgebra generated by \( X_{M,N} \) and subject to the relations:

1. \( \lambda(m \otimes n) = \lambda m \otimes n = m \otimes \lambda n \),
2. \( (m + m') \otimes n = m \otimes n + m' \otimes n \), where \( m, m' \) have the same degree,
3. \( m \otimes (n + n') = m \otimes n + m \otimes n' \), where \( n, n' \) have the same degree,
4. \( [m, m'] \otimes n = (m \otimes m' - (-1)^{|m||m'|}m' \otimes m)n \),
5. \( m \otimes [n, n'] = (-1)^{|n||[n,n']} (m' \otimes n) - (-1)^{|m||n|} (m \otimes n') \),
6. \( [m \otimes n, m' \otimes n'] = (-1)^{|m||[m,n']|} (m \otimes m' \otimes n') \),

for every \( \lambda \in \mathbb{K}, m, m' \in M_{\overline{0}} \cup M_{\overline{1}} \) and \( n, n' \in N_{\overline{0}} \cup N_{\overline{1}} \). The tensor product \( M \otimes N \) has \( \mathbb{Z}_2 \)-grading given by \( (M \otimes N)_{\alpha} = \oplus_{\beta+\gamma=\alpha}(M_{\beta} \otimes N_{\gamma}) \) for \( \alpha, \beta, \gamma \in \mathbb{Z}_2 \). If \( M = M_{\overline{0}} \) and \( N = N_{\overline{1}} \) then \( M \otimes N \) is the non-abelian tensor product of Lie algebras introduced and studied in [2].

Actions of Lie superalgebras \( M \) and \( N \) on each other are said to be compatible if

1. \( (^m n)p' = (-1)^{|m||p'|}[m, n] \cdot p' \),
2. \( (^m n)p = (-1)^{|m||n|} m \cdot n \cdot p \),

for all \( m, m' \in M_{\overline{0}} \cup M_{\overline{1}} \) and \( n, n' \in N_{\overline{0}} \cup N_{\overline{1}} \). For instance if \( M, N \) are two graded ideals of a Lie superalgebra then the actions induced by the bracket are compatible.
We will denote the $K$-module tensor product of $M$ and $N$ as $M \otimes_{\text{mod}} N$. The following result comes immediately from Proposition 3.8.

**Proposition 3.1.** If $N \subseteq L^2 \cap Z(G)$, then the sequence $N \otimes L \rightarrow L \otimes L \rightarrow L/N \otimes L/N \rightarrow 0$ is exact.

**Proposition 3.2.** Proposition 3.5] If $M$ and $N$ act trivially on each other, then there is an isomorphism

$$M \otimes N \cong M^{ab} \otimes_{\text{mod}} N^{ab},$$

where $M^{ab} = M/[M, M]$ and $N^{ab} = N/[N, N]$.

**Proposition 3.3.** Proposition 3.6] Suppose the Lie superalgebras $L$ and $M$ are acting trivially on each other, then

$$(L \oplus M) \otimes (L \oplus M) \cong (L \otimes L) \oplus (L \otimes M) \oplus (M \otimes L) \oplus (M \otimes M).$$

Consider a Lie superalgebra $M = M_0 \oplus M_1$ and the identity map on $M$, i.e., the map $id : M \rightarrow M$. The map $id$ is a crossed module, and hence the exterior square $M \wedge M$ is obtained from $M \otimes M$ by imposing the additional relations:

(1) $m \otimes m' + (-1)^{|m||m'|}(m' \otimes m) = 0$

(2) $m_{\bar{m}} \otimes m_{\bar{m}'} = 0$

with $m, m' \in M_{\bar{m}} \cup M_{\bar{m}'}$, and $m' \wedge m$ is the image of $m' \otimes m$.

**Lemma 3.4.** Lemma 6.1] Let $M \sqcap M$ be the submodule of $M \otimes M$ generated by elements

(1) $m \otimes m' + (-1)^{|m||m'|}(m' \otimes m)$,

(2) $m_{\bar{m}} \otimes m_{\bar{m}'}$.

with $m, m' \in M_{\bar{m}} \cup M_{\bar{m}'}$, $m_0 \in M_{\bar{m}}$. Then $M \sqcap M$ is a central graded ideal of $M \otimes M$.

**Definition 3.5.** The exterior product of $M$ and $M$ is denoted as $M \wedge M$ and is defined as the quotient Lie superalgebra

$$M \wedge M = \frac{M \otimes M}{M \sqcap M}.$$ 

For any $m \otimes m' \in M \otimes M$ we denote the coset $m \otimes m' + M \sqcap M$ by $m \wedge m'$.

The universal quadratic functor of Lie superalgebra was introduced in [14]. The quadratic functor helps to establish the relations between the Lie exterior product and the Lie tensor product of Lie superalgebras. Here we recall the definition of universal quadratic functor of Lie superalgebra and some results from [14] which will be useful in the next section.

**Definition 3.6.** Let $L$ be a supermodule. We define the supermodule $\Gamma(M)$ as the direct sum

$$\Gamma(M) = R^{M_0} \oplus (M \otimes M) = \{ \gamma(m_0) + m' \otimes m'' | m_0 \in M_0, m', m'' \in M \},$$

and subject to the homogeneous relations

(3.1) $\gamma(\lambda m_0) = \lambda^2 \gamma(m_0)$

(3.2) $\gamma(m_0 + m_0') = \gamma(m_0) - \gamma(m_0') = m_0 \otimes m_0'$

(3.3) $m \otimes m' = (-1)^{|m||m'|}m' \otimes m$

(3.4) $m_1 \otimes m_1 = 0$

where $\lambda \in K$, $m_0, m_0' \in M_0$, $m_1 \in M_1$ and $m, m' \in M$ with the induced grading.
Proposition 3.7. For any Lie superalgebra $M$, there exists an exact sequence

\begin{equation}
\Gamma(M^{ab}) \xrightarrow{\psi} M \otimes M \xrightarrow{\pi} M \wedge M \rightarrow 0.
\end{equation}

Also, given two graded ideals $I$ and $J$ of $M$, the following sequence is exact:

\begin{equation}
\Gamma\left(\frac{I \cap J}{I, J}\right) \xrightarrow{\psi} I \otimes J \xrightarrow{\pi} I \wedge J \rightarrow 0.
\end{equation}

Proposition 3.8. [14] Proposition 7.2.4] Let $M$ and $N$ be two supermodules. Then

\begin{equation}
\Gamma(M \oplus N) \cong \Gamma(M) \oplus \Gamma(N) \oplus (M \otimes_{\text{mod}} N).
\end{equation}

Definition 3.9. A quadratic map between two supermodules $M$ and $N$ is a map $\varphi : M \rightarrow N$ satisfying:

1. $\varphi(\lambda m) = \lambda^2 \varphi(m)$.
2. The associated symmetric function $b_\varphi : M \times M \rightarrow N$ defined by $b_\varphi(m, m') = \varphi(m + m') - \varphi(m) - \varphi(m')$ is bilinear.

Proposition 3.10. [14] Proposition 7.2.2] The pair $\gamma = \gamma_M = (\gamma_0, b_\gamma)$, with $\gamma_0 : M_0 \rightarrow \Gamma(M_0)$ defined by $\gamma_0(x_0) = \gamma(x_0)$ and $b_\gamma : M \times M \rightarrow \Gamma(M)$ defined by $b_\gamma(x, y) = x \otimes y$ is a quadratic map.

Proposition 3.11. [14] Proposition 7.2.6] Let $M$ be a free supermodule, and let $\{\vec{x}_i\}_{i \in I_0} \cup \{\bar{x}_i\}_{i \in I_1}$ be an ordered basis of $M$ composed by homogeneous elements and such that the elements $\{\bar{x}_i\}_{i \in I_0}$ of $M_0$ are less or equal than those of $M_1$, $\{\vec{x}_i\}_{i \in I_1}$. Then $\Gamma(M)$ is free with basis $\{\gamma_0(x_i)\}_{i \in I_0} \cup \{b_\gamma(x_i, x_j)\}_{i, j \in I_0 \cup I_1, i < j}$.

Proposition 3.12. [14] Proposition 7.2.9] Let $M$ be a Lie superalgebra such that $M^{ab}$ is a free supermodule. Then the homomorphism $\psi$ in the sequence 3.5 is injective.

Corollary 3.13. Let $M$ be a Lie superalgebra such that $M^{ab}$ is a free supermodule. Then the following sequence is exact

\begin{equation}
0 \rightarrow \Gamma(M^{ab}) \xrightarrow{\psi} M \otimes M \xrightarrow{\pi} M \wedge M \rightarrow 0.
\end{equation}

4. Main results

In this section, we show that any $(k \mid l)$-dimensional non-abelian nilpotent Lie superalgebra $L$ with derived supersubalgebra of dimension $(r \mid s)$ satisfies $\dim(L \otimes L) \leq (k+l-(r+s))(k+l-1)+2$. In particular, for $r = 1, s = 0$, we determine the structure of $L$ when the equality holds.

Lemma 4.1. For any two abelian Lie superalgebras $M$ and $N$ there is an isomorphism

\begin{equation}
(M \oplus N) \square (M \oplus N) \cong (M \square M) \oplus (N \square N) \oplus (M \otimes N).
\end{equation}

Proof. The proof follows from Proposition 3.8 and Lemma 4.2.

Lemma 4.2. Let $M$ be a finite dimension Lie superalgebra. Then

\begin{equation}
\Gamma(M/M^2) \cong M/M^2 \square M/M^2.
\end{equation}

Proof. From the exact Sequence 3.5 we have $\text{Im} \psi = \ker \pi = M \square M$. Now we shall prove that

\begin{equation}
\Gamma(A(m \mid n)) \cong A(m \mid n) \square A(m \mid n)
\end{equation}

Let $m + n = 1$, then by Corollary 3.13 we have

\begin{align*}
\Gamma(A(1 \mid 0)) & \cong A(1 \mid 0) \square A(1 \mid 0) \\
\Gamma(A(0 \mid 1)) & \cong A(0 \mid 1) \square A(0 \mid 1)
\end{align*}
Now assume that \(m+n \geq 2\). Then from Proposition 3.2 and using induction hypothesis we have
\[
\Gamma(A(m+1 \mid n)) \cong \Gamma(A(m \mid n) \oplus A(1 \mid 0))
\]
\[
\cong \Gamma(A(m \mid n)) \oplus \Gamma(A(1 \mid 0)) \oplus A(m \mid n) \otimes A(1 \mid 0)
\]
\[
\cong A(m \mid n) \Box A(m \mid n) \oplus A(1 \mid 0) \Box A(1 \mid 0) \oplus A(m \mid n) \otimes A(1 \mid 0)
\]
\[
\cong (A(m \mid n) \oplus A(1 \mid 0)) \Box (A(m \mid n) \oplus A(1 \mid 0))
\]
\[
\cong A(m+1 \mid n) \Box A(m+1 \mid n).
\]
Similarly, we can see that \(\Gamma(A(m \mid n+1)) \cong A(m \mid n+1) \Box A(m \mid n+1)\). Thus the Relation 4.1 is true for all \(m+n \geq 1\).

**Corollary 4.3.** Let \(M\) be a finite dimension Lie superalgebra, then
\[
M \Box M \cong M/M^2 \Box M/M^2 \cong \Gamma(M/M^2).
\]

**Proof.** From Lemma 4.2 and the exact Sequence 5.5, we have the following epimorphism \(M/M^2 \Box M/M^2 \rightarrow M \Box M\). Now by using Lemma 4.2 and the natural epimorphism \(M \Box M \rightarrow M/M^2 \Box M/M^2\) we get the desire result.

**Proposition 4.4.** For \(m+n \geq 2\), \(H(m,n) \otimes H(m,n) \cong H(m,n)/H^2(m,n) \otimes H(m,n)/H^2(m,n)\). Moreover, \(H(1,0) \otimes H(1,0) \cong A(6 \mid 0)\) and \(H(0,1) \otimes H(0,1) \cong A(1 \mid 0)\).

**Proof.** Suppose that \(m+n \geq 2\). From Theorem 3.11 and Lemma 4.2 we have
\[
\dim(H(m,n) \Box H(m,n)) = \dim\Gamma(H(m,n)/H^2(m,n)) = (2m^2 + m + \frac{n(n-1)}{2}) \mid 2mn,
\]
and by Theorem 2.1 we have \(\dim M(H(m,n)) = (2m^2 - m + (n+1)/2 - 1 \mid 2mn)\). Then
\[
\dim(H(m,n) \otimes H(m,n)) = \dim(H(m,n) \Box H(m,n)) + \dim M(H(m,n)) + \dim H^2(m,n)
\]
\[
= (2m^2 + m + \frac{n(n-1)}{2}) + (2m^2 - m + (n+1)/2 - 1) + (1 \mid 0)
\]
\[
= (4m^2 + n^2 \mid 4mn)
\]
\[
= 4m^2 + 4mn + n^2.
\]
On the other hand, \(H(m,n)/H^2(m,n) \otimes H(m,n)/H^2(m,n) = (2m + n)^2 = 4m^2 + 4mn + n^2\). Now the result can be obtained by the natural epimorphism \(H(m,n) \otimes H(m,n) \rightarrow H(m,n)/H^2(m,n) \otimes H(m,n)/H^2(m,n)\).

Similarly one can see that \(H(1,0) \otimes H(1,0)\), (and \(H(0,1) \otimes H(0,1)\)) is Lie superalgebra with dimension \((6 \mid 0)\), (resp. of dimension \((1 \mid 0)\)). One can easily check that \(H(1,0) \otimes H(1,0)\) and \(H(0,1) \otimes H(0,1)\) are abelian too.

Now we state the same result for a Heisenberg Lie superalgebra with odd center, as the proof is quite similar to the Proposition 4.4 so we omit it.

**Proposition 4.5.** When \(m \geq 2\), \(H_m \otimes H_m \cong H_m/H^2_m \otimes H_m/H^2_m\). Moreover, \(H_1 \otimes H_1 \cong A(2 \mid 3)\).

**Theorem 4.6.** Let \(L\) be an \((k \mid l)\)-dimensional non-abelian nilpotent Lie superalgebra with derived supersubalgebra of dimension \((r \mid s)\). Then
\[
\dim(L \otimes L) \leq (k + l - (r + s))(k + l - 1) + 2.
\]
In particular, for \(r = 1, s = 0\) the equality holds if and only if \(L \cong H(1,0) \oplus A(k-3 \mid l)\).

**Proof.** First, for \(r + s = 1\), we have the following cases

1. \(r = 1, s = 0\);
(2) $r = 0$, $s = 1$.

(1) Let $r = 1$, $s = 0$, then from [8 Proposition 3.4], we have $L \cong H(m, n) \oplus A(k - 2m - 1 \mid l - n)$ for $m + n \geq 1$. Thus by Proposition 3.3,

$$\dim(H(m, n) \oplus A(k - 2m - 1 \mid l - n)) = \dim(H(m, n) \oplus A(k - 2m - 1 \mid l - n))$$
$$+ 2 \dim(H(m, n) \otimes_{\text{mod}} A(k - 2m - 1 \mid l - n)) + \dim(A(k - 2m - 1 \mid l - n) \oplus A(k - 2m - 1 \mid l - n)).$$

From Proposition 3.2,

$$(H(m, n) \otimes A(k - 2m - 1 \mid l - n)) \cong (H(m, n)/H^2(m, n) \otimes_{\text{mod}} A(k - 2m - 1 \mid l - n)).$$

Now consider the following cases:

(i) When $m = 1, n = 0$, then by using Proposition 4.4,

$$\dim(H(1, 0) \oplus A(k - 3 \mid l)) \otimes (H(1, 0) \oplus A(k - 3 \mid l)) = (6 \mid 0) + 2(2 \mid 0)(k - 3 \mid l) + (k - 3 \mid l)^2$$
$$= (k + l - 1)^2 + 2.$$

(ii) When $m = 0, n = 1$, then again from Proposition 4.4,

$$\dim(H(0, 1) \oplus A(k - 1 \mid l - 1)) \otimes (H(0, 1) \oplus A(k - 1 \mid l - 1)) = (1 \mid 0) + 2(0 \mid 1)(k - 1 \mid l - 1) + (k - 1 \mid l - 1)^2$$
$$= (k + l - 1)^2.$$

(iii) When $m + n \geq 2$, then from Proposition 4.4,

$$\dim(H(m, n) \oplus A(k - 2m - 1 \mid l - n)) \otimes (H(m, n) \oplus A(k - 2m - 1 \mid l - n))$$
$$= (4m^2 + n^2 \mid 4mn) + 2(2m \mid n)(k - 2m - 1 \mid l - n) + (k - 2m - 1 \mid l - n)^2$$
$$= (k + l - 1)^2.$$

(2) Similarly for $r = 0, s = 1$, we can find that $L \cong H_m \oplus A(k - m \mid l - m - 1)$ and $\dim(L \otimes L) \leq (k + l - 1)^2 + 2$.

Let $r + s \geq 2$ and assume that the result is true for $r + s - 1$. Then for a graded ideal $N$ contained in $Z(L) \cap L^2$ with $\dim N = (1 \mid 0)$ (or $\dim N = (0 \mid 1)$), we have by induction hypothesis and Proposition 3.1,

$$\dim(L \otimes L) \leq \dim(L/N \otimes L/N) + \dim(L \otimes N),$$

and

$$\dim(L/N \otimes L/N) \leq (k + l - 1 - (r + s - 1))(k + l - 2) + 2.$$

Now from Proposition 3.2,

$$\dim(L \otimes N) = \dim(L/L^2 \otimes_{\text{mod}} N).$$

Therefore,

$$\dim(L \otimes L) \leq (k + l - (r + s)) + (k + l - (r + s))(k + l - 2) + 2$$
$$= (k + l - (r + s))(k + l - 1) + 2.$$

$\square$
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Centre for Data Science, Institute of Technical Education and Research, Siksha ‘O’ Anusandhan (A Deemed to be University), Bhubaneswar-751030, Odisha, India
Email address: rudra.padhan6@gmail.com, rudranarayanpadhan@soa.ac.in

Centre for Applied Mathematics and Computing, Institute of Technical Education and Research, Siksha ‘O’ Anusandhan (A Deemed to be University), Bhubaneswar-751030, Odisha, India
Email address: ibrahemhasan898@gmail.com

Department of Mathematics, C. V. Raman Global University, Bhubaneswar-752054, Odisha, India
Email address: sushreesp1992@gmail.com