Quantum optics in a non-inertial reference frame: the Rabi splitting in a rotating ring cavity

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We study quantum optics with the atoms coupled to the quantized electromagnetic (EM) field in a non-inertial reference frame by making use of quantum field theory in curved spacetime. We rigorously establish the microscopic model for a two-level atom interacting with the quantized EM field in a rotating ring cavity by deriving a Jaynes-Cummings (JC) type Hamiltonian. Due to the two fold degeneracy of the ring cavity modes, the Rabi splitting exhibits three rather than two resonant frequency peaks. We find that the heights of the two side peaks show a sensitive linear dependence on the rotating velocity. This high sensitivity can be utilized to detect the angular velocity of the whole system.

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Introduction — The interference of of two light beams in a rotating ring can be utilized to measure the rotating velocity. This is well known as the Sagnac effect, which bases some optical gyroscope schemes [1–6]. For the Sagnac effect in the medium with linear dispersion, there have been a lot of sophisticated studies based on classical optics. If we want to make the optical gyroscopes to an extremely high precision, it is necessary to consider the quantum fluctuations in this rotating optical system. We notice that a rigorous quantum theory about the microscopic model about the interaction between the atoms and the quantized EM field in a rotating reference frame is still not well established, but it is obviously essential for the study of quantum and nonlinear effects in a rotating optical system.

In this letter, we ascribe the effects of rotation to the “curved” spacetime metric according to the generic principles in relativity. Starting from the classical Lagrangians of the EM field and a charged particle based on the principle of least action, we obtain the covariant motion equations in the rotating non-inertial reference frame (NIRF). The variation is carried out in the rotating reference frame, and all the non-inertial physical effects rooted in the rotation are included in the “curved” spacetime metric. Then we obtain the quantized Hamiltonian for quantum optics in NIRF through the canonical quantization.

We derive a microscopic model of an atom interacting with the quantized EM field in a rotating reference frame, which then gives the Jaynes-Cummings (JC) model for a rotating ring cavity coupled with a two-level atom. For a ring cavity in the inertial rest frame, the clockwise (CW) and counter-clockwise (CCW) propagating optical modes are always exactly degenerated. Thus, in the JC model of a ring cavity, the atom couples with the two degenerate modes simultaneously. Due to the existence of two optical modes, the Rabi splitting of this system exhibits three rather than two resonant frequency peaks. More importantly, we find that the rotation of the ring would induce a detuning between the original degenerated modes in the non-inertial frame, and the heights of the two side peaks would change with the rotating speed due to this rotation induced detuning. At lower speed, the heights of the side peaks depend linearly on the rotating velocity. This sensitivity can be utilized to detect the rotation of the whole system, which can be regarded as a quantum Sagnac effect.

EM field in the rotating reference frame — We consider the EM field rotating around the z-axis in the CCW direction with angular speed $\Omega$. Physics laws must have the same covariant mathematical form in all reference frames (including the non-inertial rotating frame that we are studying). Thus, in the rotating reference frame, the Lagrangian density of the EM field is

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}. \quad (1)$$

Here $\mu_0$ is the magnetic constant, $F_{\mu\nu} := \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F^{\mu\nu} := g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$. In the above definitions, $\partial_\mu := \partial / \partial x^\mu$, where $x^\mu := (ct, x, y, z)$ is the 4-dimensional coordinate, and $A_\mu := (\phi/c, A_i) \ (i = 1, 2, 3)$ is the electromagnetic 4-potential. $\nabla_\mu$ is the covariant deriva-

![Figure 1: (Color online) Schematic setup. A two-level atom is fixed in the rotating ring, and external fibers are used for driving and probing. Two photon detectors (PD) are used to measure the photon current.](image1)

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The general solution of the above equation is
\[ F_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial A_\mu} \left[ \sqrt{-g} F^{\mu\nu} \right] = \nabla_\mu F^{\mu\nu} = 0, \]
where \( g := \det g^{\mu\nu} \). Here \( \mathcal{L} := \sqrt{-g} \mathcal{L} \) is a functional of \( A_\mu \) and \( \partial_\mu A_\nu \). This is the covariant form of the Maxwell equation, which applies to all general reference frames (both inertial and non-inertial ones) \[7\]. Substituting the spacetime metric \( g^{\mu\nu} \) into the above covariant equation, and taking the Coulomb gauge \( \nabla \cdot A = 0 \), \( A_0 = 0 \), we obtain the d’Alembert equation \( \nabla^\mu \nabla_\mu A_1 = 0 \) as
\[ \left[ -\partial_\mu^2 + \nabla^2 + 2\psi_R \cdot \nabla \partial_0 - (\tilde{\psi}_R \cdot \nabla)^2 \right] A_1 = 0. \]
Here \( \partial_0 := c^{-1} \partial / \partial t \), \( \tilde{\psi}_R = \psi_R / c \), and \( \psi_R := \Omega \times r \) is the linear velocity.

Next we reduce the problem into a quasi-1D ring configuration. We assume that \( \mathbf{R} \) is homogenous in the transverse direction, and we have \( \nabla \simeq \hat{e}_s \partial_s \) \[9\], where \( \hat{e}_s \) is the direction along the ring. Then we have
\[ \left[ -\partial_\mu^2 + 2\psi_R \partial_s \partial_0 + (1 - \nabla^2) \partial_s^2 \right] A_1 = 0. \]
Here \( \psi_R = v_R / c \) and \( v_R := \| \mathbf{v}_R \| = \Omega R \) is the linear speed.

The general solution of the above equation is
\[ A(s,t) = \sum_{k,\lambda} \hat{e}_{k\lambda} \mathbf{Z}_k [\alpha_{k\lambda} e^{iks - i\omega_k t} + \alpha^*_{k\lambda} e^{-iks + i\omega_k t}], \]
where \( k = 2\pi n / L \), \( n \in \mathbb{Z} \), and \( L \) is the length of the ring \[14\]; \( \hat{e}_{k\lambda} \) is the polarization directions in the transverse section, and \( \mathbf{Z}_k \) is a normalization constant. The eigenvalue equation of Eq. (6) \( \omega_k^2 + 2\psi_R k \omega_k - (c^2 - v_R^2) k^2 = 0 \), gives rise to the following dispersion relation
\[ \omega_k = \pm ck - v_R k. \]

Therefore, the rotation makes the dispersion relation anisotropic, i.e., for the two modes \( k_+ = |k| \) and \( k_- = -|k| \), their frequencies \( \omega_{k_\pm} \) no longer equal.

**Quantization of the EM field** – Under the Coulomb gauge we used above, the canonical momentums are \( \mathcal{E}^0 = 0 \) and
\[ \mathcal{E}^i = \frac{\partial \mathcal{L}}{\partial (\partial_0 A_i)} = -\varepsilon_0 (\mathbf{E} - \mathbf{v}_R \times \mathbf{B}), \]
where \( \varepsilon_0 \) is the electric constant. The Hamiltonian density \( \mathcal{H} = \varepsilon_0 \mathcal{E}^0 \) is obtained as
\[ \mathcal{H} = \frac{1}{2} (\varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2) - \frac{1}{2} \varepsilon_0 (\mathbf{v}_R \times \mathbf{B})^2. \]

Restricted in the quasi-1D ring configuration, respectively they become
\[ \mathcal{E}^i = \varepsilon_0 (\partial_i A - v_R \partial_s A)_i, \]
\[ \mathcal{H} = \frac{1}{2} \varepsilon_0 (\partial_0 A)^2 + \frac{1 - \omega_0^2}{2\mu_0} (\partial_s A)^2. \]

Notice that since \( \nabla \simeq \hat{e}_s \partial_s \), the Coulomb gauge \( \nabla \cdot A = 0 \) leads to \( \mathbf{A} \cdot \hat{e}_s = 0 \). Namely, \( \mathbf{A}(s,t) \) only has two transverse directions, and so does the canonical momentum \( \mathcal{E}(s,t) \).

We apply the following canonical quantization condition
\[ [\hat{A}_\lambda(s,t), \hat{E}^\sigma(s',t)] = i\hbar \delta^\sigma_\lambda \delta(s - s') \cdot S^{-1}, \]
\[ [\hat{A}_\lambda(s,t), \hat{A}^\sigma(s',t)] = [\hat{E}^\sigma(s,t), \hat{E}^\sigma(s',t)] = 0. \]

Here \( \lambda, \sigma = 1, 2 \) mean the two transverse directions, and \( S \) is the cross-sectional area of the ring. Since we have reduced the problem into 1-dimension the above quantization condition is consistent with the Coulomb gauge condition \( \nabla \cdot A = 0 \) automatically.

Now we write down the field operator \( \hat{\mathbf{A}}(s,t) \) as
\[ \hat{\mathbf{A}}(s,t) = \sum_{k,\lambda} \hat{e}_{k\lambda} \mathbf{Z}_k [\hat{a}_{k\lambda} e^{iks - i\omega_k t} + \hat{a}^\dagger_{k\lambda} e^{-iks + i\omega_k t}]. \]

With the help of the above canonical quantization condition (12), we can prove the following bosonic commutation relations (see Appendix B),
\[ [\hat{a}_{k\lambda}, \hat{a}^\dagger_{q\sigma}] = [\hat{a}^\dagger_{k\lambda}, \hat{a}_{q\sigma}] = 0, \]
\[ [\hat{a}_{k\lambda}, \hat{a}_{q\sigma}] = \delta_{kq} \delta_{\lambda\sigma} \frac{\hbar}{2\varepsilon_0 V c |k|}. \]

Thus, the normalization constant is taken as \( \mathbf{Z}_k = \sqrt{\hbar / 2\varepsilon_0 V c |k|} \), where \( V = L \cdot S \) is the effective volume of the ring, so that \( [\hat{a}_{k\lambda}, \hat{a}^\dagger_{q\sigma}] = \delta_{kq} \delta_{\lambda\sigma} \). Then we obtain the quantized Hamiltonian of the EM field as
\[ \hat{H}_{EM} = \int dV \mathcal{H} = \sum_{k,\lambda} \hbar \omega_k (\hat{a}^\dagger_{k\lambda} \hat{a}_{k\lambda} + \frac{1}{2}). \]

This Hamiltonian has the same form as that of the EM field in the inertial frame, but the dispersion relation \( \omega_k \) is changed due to the rotation [see Eq. (8)].
JC-model in the rotating frame – Next we study the motion of a charged particle in the EM field in the rotating reference frame. To this end, we start with the invariant Lagrangian of a charged particle in the effective curved spacetime \[ L = -mc\left[-g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}\right]^2 + eA_\mu \frac{dx^\mu}{dt}, \] where \( \tau \) is the proper time. The Hamiltonian description is obtained from the following variation (see Ref. [8] or Appendix C)

\[ \delta S = \int \frac{\delta L}{\delta a^\mu} \frac{dv^\mu}{dt} - H_c dt. \]

Here \( v^\mu := dx^\mu/dt, v_\mu := dx_\mu/dt = g_{\mu\nu}v^\nu, \) \( \hat{L} := L \cdot \Gamma^{-1} \) and

\[ \Gamma := \frac{dt}{d\tau} = \frac{c}{\sqrt{\nu_0 v^0 - v^0 v^1}}. \]

As a conservative system, the Hamiltonian \( H_c \) is the functional of \( x^i(t) \) and \( P_i(t) \) and \( P_i \) in \( H_c \), the Hamiltonian of the charged particle is obtained as (see Appendix C)

\[ H_c = \frac{v^0}{g_0^2}[g_{0i}p_i - \sqrt{(g_{0i}p_i)^2 - g_{00}\delta^2} - eA_0] \]

Substituting in the metric \( g^{\mu\nu} \) [Eq. (3)], the above Hamiltonian becomes

\[ H_c = \sqrt{(P - eA)^2 + \frac{m^2c^2}{v_R^2} + eA v_R} \]

\[ \approx \frac{1}{2m}(P - eA)^2 + v_R \cdot (P - eA) + eA, \]

where \( P := (P_1, P_2, P_3) \). In the quasi-1D ring, \( A \) is always in the transverse direction and perpendicular to the linear velocity \( v_R \), thus \( v_R \cdot A = 0 \).

We consider an atom with the nucleus fixed at a certain position inside the ring cavity. Around the nucleus, the electron is trapped by the central force potential \( \varphi \sim \frac{1}{r} \). In the expansion of the above equation, \( H_{\text{atom}} := \frac{P^2}{2m} + e\varphi + v_R \cdot P \) is exactly the Hamiltonian of a hydrogen-like atom plus a perturbation term \( v_R \cdot P \), which comes from the non-inertial effect of rotation, and the term \( P \cdot A \) describes the coupling between the atom and the EM field.

Treating \( \hat{P} \) and \( \hat{x} \) as operators, we obtain the energy levels of the hydrogen-like atom. Then we focus on two energy levels \( \{e\} \) and \( \{g\} \) with a dipole transition. Neglecting the \( A^2 \) term, with the help of Eq. (13), the total Hamiltonian \( \hat{H} = \hat{H}_c + \hat{H}_{\text{EM}} \) reads,

\[ \hat{H} = \hbar\Omega \hat{\sigma}^z + \xi \hat{\sigma}^y + \sigma^y \sum_{k,\lambda} (\eta_{k\lambda} \hat{a}_{k\lambda} + \eta_{k\lambda}^* \hat{a}_{k\lambda}^\dagger) + \sum_{k,\lambda} h_{\omega_k} \hat{a}_{k\lambda}^\dagger \hat{a}_{k\lambda}, \]

where \( \hat{\sigma}^z := |e\rangle\langle e| - |g\rangle\langle g|, \hat{\sigma}^y := |e\rangle\langle g| + |g\rangle\langle e| \) and \( \hat{\sigma}^y = i[\hat{\sigma}^z, \hat{\sigma}^y] \). The coefficients \( \xi \) and \( \eta_{k\lambda} \) are explicitly given as

\[ \xi = \frac{i(e | v_R \cdot P | g) = -\frac{\hbar \nu}{m} \cdot (\vec{p} \cdot \hat{\epsilon}_k),} \]

\[ \eta_{k\lambda} = \frac{-ie}{m} \sum \langle \vec{z}_k e^{ik\vec{r}}(\hat{P} \cdot \hat{\epsilon}_k) | g \rangle \approx \frac{\Omega(\vec{p} \cdot \hat{\epsilon}_k)}{2\pi a | \vec{v} c |} \frac{\hbar}{4} e^{ik_0}, \]

where \( \vec{P} := e | \vec{r} | g \) is the dipole moment [10]. Here the dipole approximation is applied, and \( k_0 \) is the position of the atom. We choose \( s_0 = 0 \) to cancel the phase factors. Comparing with the inertial case \( v_R = 0 \), we see that \( \Omega \) and \( \eta_{k\lambda} \) is unchanged, and the contribution of rotation appears in the correction term \( \xi \hat{\sigma}^y \) and the dispersion relation \( \omega_k \).

We set the two polarization directions to be parallel and vertical to the projection of \( \vec{p} \) in the transversal section respectively. Then the vertical optical modes are decoupled with the atom [see Eq. (21)]. When the ring is not too long, the frequencies of different eigen modes of the EM field are well separated from each other, so we only consider the modes nearly resonant with the atom. But we should notice that, different from the Fabry-Perot type, the \( k_+ \) and \( k_- \) modes are always nearly degenerate in ring cavity (unless \( k = 0 \)). When \( v_R = 0 \), they are exactly degenerated \( \omega_{k_+} = \omega_{k_-} \). That means, the \( k_+ \) and \( k_- \) modes must be considered together. Therefore, by omitting the double creation and annihilation terms, we establish the JC-model of two-level atom in a rotating ring cavity with the JC Hamiltonian

\[ \hat{H} = \frac{\hbar \Omega}{2} \hat{\sigma}^z + \xi \hat{\sigma}^y + \hbar \omega_+ \hat{a}_+^\dagger \hat{a}_+ + \hbar \omega_- \hat{a}_-^\dagger \hat{a}_- \]

\[ + g \hat{\sigma}^+(\hat{a}_+ + \hat{a}_-) + g^* \hat{\sigma}^-(\hat{a}_+^\dagger + \hat{a}_-^\dagger), \]

where the coupling strength \( g \) is

\[ g = i\Omega(\vec{p} \cdot \hat{\epsilon}_k) \frac{\hbar}{2\pi a | \vec{v} c |} \frac{1}{k}. \]

Here \( \hat{a}_\pm \) is the annihilation operator for the modes \( k_\pm \), and their frequencies are \( \omega_{\pm} := (c \pm v_R k) := \omega_0 \pm \Delta \), where we define \( \omega_0 := c | k | \) as the rest frequency of the ring cavity and \( \Delta := -v_R k \) as the rotation detuning. For simplicity, hereafter we absorb the phase of \( g \) into the operators to make \( g = g^* > 0 \) and set \( \hbar = 1 \).

Rabi splitting in the rotating frame – We consider the case that the dipole moment is parallel to the transverse direction, thus \( \xi = 0 \) and we can choose the polarization direction to satisfy \( \vec{p} \cdot \hat{\epsilon}_k = | \vec{p} | \). In this case, the excitation number \( \hat{N} := \hat{\sigma}^z + \hat{a}_+^\dagger \hat{a}_+ + \hat{a}_-^\dagger \hat{a}_- \) is conserved, i.e., \( [\hat{N}, \hat{H}] = 0 \). The ground state is \( | G \rangle = | g, 0, 0 \rangle \) and the eigen energy is \( E_G = -\Omega/2 \). But generally we cannot give an analytical solution for the whole energy spectrum. When \( \omega_0 = \Omega \), we can obtain the eigen energy of the first three excited levels [Fig. 2(d)], and they are

\[ E_0 = \frac{\Omega}{2}, \quad E_\pm = \frac{\Omega}{2} \pm \Delta g. \]
where $\Delta_g := \sqrt{\Delta^2 + 2g^2}$ (see Appendix D or Ref. [11]).

Therefore, if we use an external input to drive the ring cavity weakly, we predict to see Rabi splitting with three resonant peaks [12]. We consider a probing setup as demonstrated in Fig. 1. An external driving laser is input to drive the $k_+$-mode of the ring cavity. The photons in the ring cavity can leak into the output fiber, and then be probed by photon detectors.

We use the following master equation to describe the system,

$$\dot{\rho} = i[\rho, \hat{H}_d(t)] + \sum_{\alpha = +, -} \frac{\gamma}{2} \{2\hat{a}_\alpha \rho \hat{a}^\dagger_\alpha - \{\rho, \hat{a}^\dagger_\alpha \hat{a}_\alpha \}\},$$  

(25)

where $\hat{H}_d(t) = \mathcal{E}(e^{i\omega_d t} \hat{a}_+ + e^{-i\omega_d t} \hat{a}^\dagger_-)$ is the driving term. Here we consider the case $\Omega = \omega_0$. When the driving strength $\mathcal{E}$ is weak, the cavity modes will not be excited to states with large photon numbers, and we obtain (see Appendix D)

$$\pi_+ \simeq 4\mathcal{E}^2 \frac{M}{F}, \quad \pi_- \simeq 16\mathcal{E}^2 g^4 \frac{F}{M},$$

(26)

where

$$M := 4[g^2 - \tilde{\Omega}(\tilde{\Omega} - \Delta)]^2 + 2g^2 \tilde{\Omega}^2,$$

(27)

$$F := \gamma^4 \tilde{\Omega}_d^2 + 2g^2 [2g^4 + \tilde{\Omega}_d^2 - g^2 \tilde{\Omega}_d^2 + \Delta^2 \tilde{\Omega}_d^2]$$

$$+ 16\tilde{\Omega}_d^2 [2g^2 - \tilde{\Omega}_d^2 + \Delta^2]_d^2,$$

and $\tilde{\Omega} := \Omega - \omega_d$.

When $\gamma \to 0$, the denominator $F(\tilde{\Omega})$ has three minima around $\tilde{\Omega} = 0$ and $\tilde{\Omega} = \pm \sqrt{\Delta^2 + 2g^2}$, which give rise to three peaks in $\pi_\pm(\tilde{\Omega})$ corresponding to the first three excited energy Eq. (24). We can also explicitly see that $F(\tilde{\Omega})$ is symmetric for $\pm \tilde{\Omega}$, i.e., $F(\tilde{\Omega}) = F(-\tilde{\Omega})$, but $M(\tilde{\Omega})$ is not. Therefore, $\pi_- (\tilde{\Omega})$ is symmetric for $\pm \tilde{\Omega}$, but $\pi_+ (\tilde{\Omega})$ is not. We plot the steady photon number $\pi_{\pm}(\tilde{\Omega})$ of the cavity modes in Fig. 2. When there is no rotation, $\nu_R = 0$, the steady photon number of the two modes $\pi_{\pm}(\tilde{\Omega})$ are both symmetric with respect to the driving detuning $\pm \tilde{\Omega}$ [Fig. 2(a)]. It is worth noticing that when there is a small rotating velocity, the heights of the side peaks of $\pi_{\pm}(\tilde{\Omega})$, which correspond to the mode $\hat{a}_+$ being driven, change sensitively with rotating velocity [Fig. 2(b)].

Since the positions of the side peaks are $\tilde{\Omega} \simeq \pm \sqrt{\Delta^2 + 2g^2}$, we plot the height of $\pi_{\pm}(\Omega = \sqrt{2g})$ with respect to the rotation detuning $\Delta = -\nu_R k$ around $\Delta \simeq 0$ [Fig. 2(c)]. When $\Delta$ is quite small (in this example, we have $|\Delta/\tilde{\Omega}| = |\nu_R/c| < 10^{-5}$), $\pi_{\pm}(\Delta)$ depend linearly on $\Delta = -\nu_R k$ around $\Delta = 0$. From Eqs. (26, 27), we obtain the slope of $\pi_{\pm}(\Omega, \tilde{\Omega} = \sqrt{2g})$ around $\Delta = 0$ as

$$\frac{\partial \pi_{\pm}(\Omega, \Delta)}{\partial \Delta} \bigg|_{\Omega = \sqrt{2g}} \simeq \frac{64\sqrt{2g}\mathcal{E}^2}{\gamma^2 (\gamma^2 + 8g^2)}.$$

(28)

From these results, we see that the sensitivity of this height change with respect to the rotating velocity can be controlled by the coupling strength $g$ and the decay rate $\gamma$. A ring cavity with high quality promises a sensitive measurement.

In experiments, the steady photon number $\pi_{\pm}$ can be measured by the average output photon current. In a probe setup as shown in Fig. 1, we have $\hat{a}_{\pm, OUT} = \hat{a}_{\pm, IN} + \sqrt{\gamma} \hat{a}_\pm$, and the average output photon current is $\langle \hat{a}_{\pm, OUT}^\dagger \hat{a}_{\pm, OUT} \rangle = \gamma \pi_{\pm}$ [13]. This photon current directly characterizes the steady photon numbers of the cavity modes, and can be measured by the photon detectors.

Summary – In conclusion, we have generally studied the quantum optics for the interaction between light and atom in a non-inertial reference frame by the approach of quantum field theory in curved spacetime. For the two-level atom interacting with the quantized EM field in a rotating ring cavity, our approach built a microscopic model described by a two-mode JC Hamiltonian. Based on this generalized JC model, our study predicts that that the heights of the side peaks in the Rabi splitting show a sensitive linear dependence on the rotating velocity at low speed. Therefore, this model can not only be utilized for hybrid optical gyroscope design, but also provide the future development of quantum gyroscope with a solid physical base to consider the effect of quantum fluctuation.

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Appendix A: spacetime metric

Considering the system rotating along the z-axial in the counter-clockwise direction, we have [2, 9]

\[ t = T, \]
\[ x = X \cos \Omega t + Y \sin \Omega t, \]
\[ y = -X \sin \Omega t + Y \cos \Omega t, \]
\[ z = Z. \]

Here, \( X = (cT, X, Y, Z) \) is the coordinates in the inertial lab frame, while \( x = (ct, x, y, z) \) is in the co-rotating frame. We obtain the general coordinate transformation matrix as

\[ [A^\alpha_\beta] = \left[ \begin{array}{cccc} 1 & \Omega y/c \cos \Omega t & -\sin \Omega t \\ -\Omega x/c & -\sin \Omega t & \cos \Omega t \\ 0 & 0 & 1 \end{array} \right] . \]  

The spacetime metric \( g^{\mu \nu} \) in the co-rotating frame, as a covariant tensor, can be calculated by

\[ g^{\mu \nu} = \left[ \begin{array}{cccc} \frac{\partial x^\mu}{\partial X^\alpha} \cdot \eta^{\alpha \beta} \frac{\partial x^\nu}{\partial X^\beta} \end{array} \right] = \left[ \begin{array}{cccc} -1 & -\frac{\Omega y}{c} & \frac{\Omega x}{c} \\ -\frac{\Omega x}{c} & 1 - \frac{\Omega^2 y^2}{c^2} & \frac{\Omega y x}{c^2} \\ \frac{\Omega x}{c} & \frac{\Omega y x}{c^2} & 1 - \frac{\Omega^2 x^2}{c^2} \end{array} \right] , \]  

where \( \eta^{\mu \nu} = \text{diag}\{-1, 1, 1, 1\} \) is the metric in the inertial lab frame. And we also have

\[ g_{\mu \nu} = [g^{\mu \nu}]^{-1} = \left[ \begin{array}{cccc} -1 + \frac{\Omega^2 (x^2 + y^2)}{c^2} & \frac{\Omega y}{c} & \frac{\Omega x}{c} \\ -\frac{\Omega y}{c} & 1 & \frac{\Omega y x}{c^2} \\ -\frac{\Omega x}{c} & \frac{\Omega y x}{c^2} & 1 \end{array} \right] . \]  

Appendix B: Canonical quantization of the rotating EM field

In the canonical quantization of the rotating EM field, it is crucial to find a proper orthogonal relation of the field operator \( \hat{A}(s, t) \). This relation can be obtained with the help of a density-flow relation derived from the equation of motion. From the equation of motion \( \partial_0 \hat{A}^* \hat{D} \hat{A} = \partial_x [\hat{A}^* \hat{J} \hat{A}] \),

\[ -\partial_0^2 + 2\Omega R \partial_x \partial_0 - (\Omega^2 R^2 + \Omega_R^2) \partial_0^2 \hat{A} = 0, \]  

we obtain the following density-flow relation

\[ \partial_0 [A^* \partial_0 A] = \partial_x [A^* \hat{J} A], \]  

where

\[ A^* \partial_0 A := (A^* \cdot \partial_0 A - A \cdot \partial_0 A^*), \]
\[ A^* \hat{J} A := (1 - \hat{A}^2) (A^* \cdot \partial_s A - A \cdot \partial_s A^*). \]

Integrating Eq. (B2) over the whole space, we obtain

\[ \partial_0 \left( \int dS : \int_0^L ds [A^* \partial_0 A] \right) = 0, \]

where \( \int dS \) is the integral over the cross section and gives a constant area \( S \). Then we know that this integral is a conserved constant independent of time (but still not determined yet).

Then we can find some orthogonal relations. For the eigen solution

\[ A_{k\lambda}(s, t) := \hat{e}_{k\lambda} \hat{a}_{k\lambda}(s, t) := \hat{e}_{k\lambda} \hat{Z}_k e^{iks - i\omega_b t}, \]

we have the following orthogonal relation

\[ \int_0^L ds [A_{k\mu}^* \partial_0 A_{k\lambda}] = \int_0^L ds [A_{\alpha \sigma}^* \partial_0 A_{k\lambda}] = \delta_{k\lambda} \delta_{\alpha \sigma} , \]

\[ \int_0^L ds [A_{q\mu}^* \partial_0 A_{k\lambda}] = \int_0^L ds [A_{q\sigma}^* \partial_0 A_{k\lambda}] = 0. \]  

Notice that no matter \( k > 0 \) and \( k < 0 \), we both have \( \omega_k + v_R k = c|k| \).

With the help of the above orthogonal relation, for the field operator

\[ \hat{A}(s, t) = \sum_{k, \lambda} \hat{e}_{k\lambda} \hat{Z}_k [\hat{a}_{k\lambda} e^{iks - i\omega_k t} + \hat{a}_{k\lambda}^\dagger e^{-iks + i\omega_k t}], \]
we have
\[
\int_0^L ds [A_{k\lambda}^* \mathcal{D} \hat{A}] = -2i(k\mathcal{Z}_k)^2 |L|k |\partial_{k\lambda}^{|},
\]
\[
\int_0^L ds [A_{k\lambda}^* \mathcal{D} \hat{A}] = 2i(k\mathcal{Z}_k)^2 |L|k |\partial_{k\lambda}^{|}.
\] (B6)

Notice that indeed the above terms in the integrals can be written as
\[
\begin{align*}
A_{k\lambda}^* \mathcal{D} \hat{A} &= a_{k\lambda}^* \cdot (\partial_0 - v_0^R \partial_\lambda) \hat{A} - \hat{A} \cdot (\partial_0 - v_0^R \partial_\lambda) a_{k\lambda}^* \\
&= \frac{a_{k\lambda}^* \hat{\epsilon}^\lambda}{\zeta_0} - i |k| a_{k\lambda}^* \hat{\lambda},
\end{align*}
A_{k\lambda} \mathcal{D} \hat{A} = a_{k\lambda} \cdot (\partial_0 - v_0^R \partial_\lambda) \hat{A} - \hat{A} \cdot (\partial_0 - v_0^R \partial_\lambda) a_{k\lambda} \\
&= \frac{a_{k\lambda} \hat{\epsilon}^\lambda}{\zeta_0} + i |k| a_{k\lambda} \hat{\lambda}.
\]

Then, together with the canonical quantization conditions \([\hat{A}_\lambda(s, t), \hat{\epsilon}^\sigma(s', t)] = i\hbar \delta^\sigma_\sigma(s - s') \cdot \hat{A}^{-1}\) and \([\hat{A}_\lambda(s, t), \hat{A}_\lambda(s', t)] = [\hat{\epsilon}^\lambda(s, t), \hat{\epsilon}^\sigma(s', t)] = 0\), we can calculate \([\hat{a}_{k\lambda}, \hat{a}_{k\sigma}]\) from Eq. (B6) as follows
\[
\begin{align*}
\langle 2| & (\mathcal{Z}_k)^2 |L|k |\hat{a}_{k\lambda}, 2|\mathcal{Z}_q)^2 |L|q |\hat{a}_{q\sigma} \\
= \int ds & ds' [A_{k\lambda}^* (s, t) \mathcal{D} \hat{A}(s, t), A_{q\sigma} (s', t) \mathcal{D} \hat{A}(s', t)] \\
= \int ds & ds' [\frac{a_{k\lambda} \hat{\epsilon}^\lambda(s)}{\zeta_0} - i |k| a_{k\lambda} \hat{\lambda}(s), \frac{a_{q\sigma} \hat{\epsilon}^\sigma(s')}{\zeta_0} + i |q| a_{q\sigma} \hat{\lambda}(s')].
\end{align*}
\]
\[
= \frac{2\hbar |L|k |\langle \mathcal{Z}_k |^2 \cdot \hat{\epsilon}^\lambda \delta_{kq}}{\zeta_0}.
\]

Therefore, we choose the normalization constant \(\mathcal{Z}_k\) to be
\[
\mathcal{Z}_k = \sqrt{\frac{\hbar}{2 \varepsilon_0 V (\omega_k + \nu_R k)}} = \sqrt{\frac{\hbar}{2 \varepsilon_0 V c |k|}}
\] (B7)
where \(V := L \cdot S\) is the volume of the quasi-1D ring, so that we obtain the bosonic commutation relation \([\hat{a}_{k\lambda}, \hat{a}_{q\sigma}] = \delta^\sigma_\sigma \delta_{kq}\). With the same method, we can also check \([\hat{a}_{k\lambda}, \hat{a}_{q\sigma}] = [\hat{a}_{k\lambda}, \hat{a}_{q\sigma}^\dagger] = 0\).

After we have got the normalization constant \(\mathcal{Z}_k\), it is straightforward to verify that the Hamiltonian of the rotating EM field is
\[
\hat{H}_EM = \int dV \mathcal{H}.
\]
\[
= \int dV \left[ \frac{1}{2} \varepsilon_0 (\partial_0 \hat{A})^2 + \frac{1}{2 \mu_0} (\partial_\lambda \hat{A})^2 \right] \\
= \sum_{k, \lambda} \frac{1}{2} \omega_k (\hat{a}_{k\lambda}^\dagger \hat{a}_{k\lambda} + \hat{a}_{k\lambda} \hat{a}_{k\lambda}^\dagger).
\]

Appendix C: Hamiltonian description of a charged particle in the co-rotating EM field

The Hamiltonian equation can be derived from the following variation,
\[
\delta \int Ld\tau = \delta \int \hat{L} dt = \delta \int (P_i \frac{dx^i}{dt} - \hat{H}) dt = 0,
\] (C1)
where \(H[x_i(t), P^i(t)]\) is the functional of independent variables \(x^i(t)\) and \(P_i(t)\). The above variation gives
\[
0 = \delta \int \left( P_i \frac{dx^i}{dt} - H[x^i, P_i] \right) dt
\]
\[
= P_i \delta x^i \bigg|_A^B + \delta \int \left( \frac{dx^i}{dt} \frac{\partial H}{\partial P_i} \delta P_i + \frac{dP_i}{dt} \frac{\partial H}{\partial x^i} \delta x^i \right) dt.
\]

Therefore, we have used the following relation
\[
\delta \int P_i v^i dt = \int \left[ v^i \delta P_i + P_i \frac{d}{dt} \delta x^i \right] dt
\]
\[
= \int \left[ v^i \delta P_i + P_i \frac{d}{dt} \delta x^i \right] dt
\]
\[
= P_i \delta x^i \bigg|_A^B + \int \left[ \frac{d}{dt} \delta P_i - \frac{\partial H}{\partial x^i} \delta x^i \right] dt
\]
\[
= - \Gamma m v_0 \delta x^i - e A_0 v^0
\]
\[
= - \Gamma m v_0 \delta x^i - e A_0 v^0
\]
\[
= \Gamma m v_0 \delta x^i - e A_0 v^0
\]
\[
= \Gamma m v_0 \delta x^i - e A_0 v^0
\]
\[
= \Gamma m v_0 \delta x^i - e A_0 v^0
\]

Here we denote \(v^\mu := dx^\mu /dt, v^\mu := dx^\mu /dt\). And we denote \(p_i := \Gamma m v_0\) as the mechanical momentum. Here we take \(x^\mu := (ct, x_1, x_2, x_3)\) and \(x_\mu = g_{\mu\nu} x^\nu\), thus we have \(v^0 = c\).

Further, we still need to replace \(\Gamma m v_0\) by the canonical momentum \(P_i\) in the Hamiltonian \(H\). We emphasize that here we only have got the relation between \(P_i, p_i\) and \(v_i\) from Eq. (C5), but we did not have the definition of \(P_0, p_0\), so we cannot just naively replace \(\Gamma m v_0\) as \(p_0\). To find the relation between \(\Gamma m v_0\) and \(P_i, p_i,\) we should notice the following two relations,
\[
\Gamma m v_i v^i = \Gamma^2 m^2 v_i (g^{0i} v_0 + g^{ij} v_j)
\]
\[
= g^{0i} p_i \cdot \Gamma m v_0 + g^{ij} p_j
\]
\[
\Gamma m v_i v^i = \Gamma^2 m^2 \cdot \frac{v_i v^i}{v_0 v_0 - v_i v^i}
\]
\[
= - m^2 c^2 - \Gamma m v_0 \cdot \Gamma m v_0
\]
\[
= - g^{0i} (\Gamma m v_0)^2 - g^{0i} p_i \cdot (\Gamma m v_0) - m^2 c^2
\]

Therefore, we obtain an equation about \(\Gamma m v_0\), i.e.,
\[
\Gamma m v_0 \leq 2 g^{0i} p_i \cdot (\Gamma m v_0) + g^{ij} p_j + m^2 c^2 = 0
\] (C8)
which leads to the solution \((\Gamma_{mv0})\) should be negative, so the other positive solution is invalid)

\[
\Gamma_{mv0} = g_0^0 p_i - \sqrt{g_0^0 p_i^2 - g_0^0 (g_0^0 p_j + m^2 c^2)}
\]

Therefore, we have

\[
H_e = \frac{v_0}{g_0^0} [g_0^0 p_i - \sqrt{(g_0^0 p_i)^2 - g_0^0 (g_0^0 p_j + m^2 c^2)}] - ev_0 A_0,
\]

where \(p_i = P_i - e A_i\).

**Appendix D: Steady photon number**

For the JC-model for a two-level atom in a rotating ring cavity

\[
\hat{H} = \frac{\hbar \Omega}{2} \hat{\sigma}^z + \xi \hat{\sigma}^y + \hbar \omega_+ \hat{a}^\dagger_+ \hat{a}_+ + \hbar \omega_- \hat{a}^\dagger_- \hat{a}_-
\]

\[
+ g \hat{\sigma}^+ (\hat{a}_+ + \hat{a}_-) + g^* \hat{\sigma}^- (\hat{a}_+^\dagger + \hat{a}_-^\dagger),
\]

we can directly verify that the ground state is \(|g, 0, 0\rangle\) and the eigen energy is \(E_G = -\hbar \Omega/2\).

The excitation number \(N := \langle \hat{\sigma}^z + \hat{a}_+ \hat{a}_+ + \hat{a}_- \hat{a}_- \rangle\) is always conserved, i.e., \([N, \hat{H}] = 0\). Thus, the single excitation subspace, which is spanned by \(|e, 0, 0\rangle\), \(|g, 1, 0\rangle\), \(|g, 0, 1\rangle\), is closed, i.e.,

\[
\hat{H}|e, 0, 0\rangle = \frac{\Omega}{2}|e, 0, 0\rangle + g|g, 1, 0\rangle + g|g, 0, 1\rangle
\]

\[
\hat{H}|g, 1, 0\rangle = g|e, 0, 0\rangle + (\omega_+ + \Delta)|g, 1, 0\rangle
\]

\[
\hat{H}|g, 0, 1\rangle = g|e, 0, 0\rangle + (\omega_- - \Delta)|g, 0, 1\rangle
\]

When \(\Omega = \omega_0\), the above eigen equations can be diagonalized exactly and we can obtain the eigen energy states in the single excitation subspace. The eigen energies are \(E_0 = \Omega/2\) and \(E_{\pm} = \Omega/2 \pm \Delta_g\), where \(\Delta_g := \sqrt{\Delta^2 + 2g^2}\), and the eigenstates are

\[
|E_0\rangle = \frac{1}{Z_0} \left[ \Delta|e, 0, 0\rangle - g|g, 1, 0\rangle + g|g, 0, 1\rangle \right], \tag{D3}
\]

\[
|E_{\pm}\rangle = \frac{1}{Z_{\pm}} \left[ g(|\Delta \pm \Delta_g|)|e, 0, 0\rangle + \frac{1}{2}(|\Delta \pm \Delta_g^2)|g, 1, 0\rangle

+ |g|2|^g, 0, 1\rangle \right],
\]

where \(Z_0\) and \(Z_{\pm}\) are normalization constants.

When we use an external laser to drive the \(\hat{a}_+\) mode, the master equation of the system is

\[
\dot{\rho} = i[\rho, \hat{H}_d(t)] + \sum_{\alpha = +, -} \frac{\gamma}{2} \{2\rho \alpha \rho \tilde{\alpha} - \{\rho, \tilde{\alpha}_\alpha \}, \tilde{\alpha}_\alpha \}, \tag{D4}
\]

where \(\hat{H}_d = E (e^{i\omega_d t} \hat{a}_+ + e^{-i\omega_d t} \hat{a}_+^\dagger)\) is the driving term. We make a unitary transformation by \(\exp[i\omega_d (\hat{a}_+^\dagger + \hat{a}_-^\dagger \hat{a}_- + \hat{a}_- \hat{a}_-^\dagger)\) and the equation becomes time independent. Then we obtain equations for observable expectations as follows

\[
0 = -i(\hat{\omega}_+ - i\frac{\gamma}{2})[\alpha_+ - ig(\hat{\sigma}^z) - iE]
\]

\[
0 = -i(\hat{\omega}_- - i\frac{\gamma}{2})[\alpha_- - ig(\hat{\sigma}^z)]
\]

\[
0 = -i(\hat{\omega}_+ - i\frac{\gamma}{2})S_+ - iE(\hat{\sigma}^z) + ig(\langle \hat{\sigma}^z \rangle) + i\frac{\gamma}{2}((\hat{\sigma}^z) + 1)
\]

\[
0 = -i(\hat{\omega}_- - i\frac{\gamma}{2})S_- + ig(\langle \hat{\sigma}^z \rangle) + i\frac{\gamma}{2}((\hat{\sigma}^z) + 1)
\]

\[
0 = \langle \tilde{\Omega}(\hat{\sigma}^z)\rangle + ig(S_+ + S_-)
\]

\[
0 = (Z_+ - Z_+) + (Z_- - Z_-)
\]

Here we denote \(Z_{\pm} := \langle \hat{\sigma}^z \hat{a}_\pm \rangle\), \(S_{\pm} := \langle \hat{\sigma}^z \hat{a}_{\pm} \rangle\), \(\alpha_{\pm} := \langle \hat{\sigma}^z \hat{a}_{\pm} \rangle\), and \(\hat{\omega}_{\pm} := \omega_{\pm} - \omega_d\); \(\hat{\Omega} := \Omega - \omega_d\). In this equation we have omitted terms of higher orders, like \(\langle \hat{\sigma}^z \hat{a}_\pm^\dagger \hat{a}_\pm \rangle\), \(\langle \hat{\sigma}^z \hat{a}_\pm \hat{a}_\pm^\dagger \rangle\), so the above linear equations become complete. This approximation requires that the driving strength is weak thus the excitation is very low.

From this set of linear equations, we obtain the solution of \(\alpha_{\pm} = \langle \hat{\sigma}^z \rangle\), and then we obtain \(\tilde{\pi}_{\pm} \simeq |\alpha_{\pm}|^2\) as follows

\[
\tilde{\pi}_+ \simeq |\alpha_+|^2 = \frac{4E^2(M^2 F^2 D + o(E^2))}{(FD + 4E^2G)^2}, \tag{D5}
\]

\[
\tilde{\pi}_- \simeq |\alpha_-|^2 = \frac{16E^2g^4 F^2 D}{(FD + 4E^2G)^2},
\]

where

\[
M := 4(g^2 - \hat{\Omega}\hat{\omega}_-)^2 + \gamma^2 \hat{\Omega}^2,
\]

\[
D := \gamma^2 + 4\Delta^2,
\]

\[
F := \gamma^4 \hat{\Omega}^2 + 4\gamma^2 (2g^2 + (\hat{\omega}_+ \hat{\Omega} - g^2)^2 + (\hat{\omega}_- \hat{\Omega} - g^2)^2)

+ 16\hat{\Omega}^2 [2g^2 - \hat{\Omega}^2 + \Delta^2]^2,
\]

\[
G := 16\Delta^2 [(2g^2 - \hat{\Omega}^2)\hat{\omega}_-^2 + 2g\hat{\omega}_- \hat{\Omega} \Delta] + \gamma^4 (2g^2 - \hat{\Omega}^2)

+ 4\gamma^2 [(2g^2 - \hat{\Omega}^2)^2 - (2g^2 - \hat{\Omega}^2)^2 + 2g^2 \Delta^2].
\]

When the driving strength \(E\) is weak, we omit terms of \(o(E^2)\) in Eq. (D5) and obtain

\[
\tilde{\pi}_+ \simeq \frac{4E^2M}{F}, \quad \tilde{\pi}_- \simeq \frac{16E^2g^4}{F}. \tag{D6}
\]

For the case that the atom is resonant with the resonant frequency, i.e., \(\Omega = \omega_0\), we have \(\hat{\omega}_{\pm} = \hat{\Omega} \pm \Delta\), and we obtain

\[
M := 4(g^2 - \hat{\Omega}(\hat{\Omega} - \Delta)^2 + \gamma^2 \hat{\Omega}^2,
\]

\[
F := \gamma^4 \hat{\Omega}^2 + 8\gamma^2 [2g^4 + \hat{\Omega}^4 - 2g^2 \hat{\Omega}^2 + \Delta^2 \hat{\Omega}^2]

+ 16\hat{\Omega}^2 [2g^2 - \hat{\Omega}^2 + \Delta^2]^2.
\]

This is what we have shown in the text.

The above master equation can be also solved numerically by setting a cutoff on the Hilbert dimension of the cavity.
modes. We find that when the driving strength $\mathcal{E}$ is weak, the above analytical results of the steady photon number show well accordance with the numerical calculations.