MULTIDIMENSIONAL COSMOLOGY WITH m-COMPONENT PERFECT FLUID

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MULTIDIMENSIONAL COSMOLOGY WITH $m$-COMPONENT PERFECT FLUID [1]

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ABSTRACT

A cosmological model describing the evolution of $n$ Einstein spaces ($n > 1$) with $m$-component perfect-fluid matter is considered. When all spaces are Ricci-flat and for any $\alpha$-th component the pressures in all spaces are proportional to the density: $p_i^{(\alpha)} = (1 - h_i^{(\alpha)}) \rho^{(\alpha)}$, $h_i^{(\alpha)} = \text{const}$, the Einstein and Wheeler-DeWitt equations are integrated in the cases: i) $m = 1$, for all $h_i^{(\alpha)}$; ii) $m > 1$, for some special sets of $h_i^{(\alpha)}$. For $m = 1$ the quantum wormhole solutions are also obtained.

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1 Introduction. The model

Multidimensional cosmology (see, for example [1-21] and references therein) is a very interesting object of investigations both from physical and mathematical points of view.

Last decade the interest in multidimensional cosmology was stimulated mainly by Kaluza-Klein and superstring paradigmas [22,23]. The ”realistic” multidimensional cosmological models appeared mainly in a context of some unifications theories. Certainly, it is quite natural to believe that the Entire Universe is multidimensional and we live in a some sort of a (3+1)-dimensional layer, that is Our Universe. Of course, at first stage we should try to understand the structure of our 3-dimensional crude (dense) matter and the formation of Our Universe. But it seems to be very likely that at some stage of our development it will be just impossible to describe our (3+1)-dimensional layer (Our Universe) out of touch with other (multidimensional) layers and domains.

A large variety of multidimensional cosmological models is described by pseudo-Euclidean Toda-like systems [19] (see formula (1.10) below). These systems are not well studied yet. We note, that the Euclidean Toda-like systems are more or less well studied [24-28] (at least for certain sets of parameters, associated with finite-dimensional Lie algebras or affine Lie algebras). There is also a criterion of integrability by quadrature (algebraic integrability) for these (Euclidean) systems established by Adler and van Moerbeke [28]. Nevertheless, there are some indications that cosmological models may contain rather rich mathematical structures. For example, a self-dual reduction of the Bianchi-IX cosmology [29] lead us to the Halphen system of ordinary differential equations [30]. This system may be integrated in terms of modular forms [31] and is connected with a certain integrable reduction of the self-dual Yang-Mills equation [32] (with the infinite-dimensional group $SDiffSU(2)$). Another example is connected with the Kaluza-Klein dyon solution from [33]. The field equations for a spherically-symmetric Kaluza-Klein dyon in 5-dimensions were reduced in [33] to an

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open (Euclidean) Toda lattice with three points. Certainly, this problem may be formulated in terms of an appropriate cosmological model described by a pseudo-Euclidean Toda-like Lagrangian. So, we lead to an interesting nontrivial example of an integrable cosmological model.

In this paper we consider a cosmological model describing the evolution of \( n \) Einstein spaces in the presence of \( m \)-component perfect-fluid matter. The metric of the model

\[
g = -\exp[2\gamma(t)] dt \otimes dt + \sum_{i=1}^{n} \exp[2x^i(t)] g^{(i)},
\]

is defined on the manifold

\[
M = R \times M_1 \times \ldots \times M_n,
\]

where the manifold \( M_i \) with the metric \( g^{(i)} \) is an Einstein space of dimension \( N_i \), i.e.

\[
R_{m_i n_i}[g^{(i)}] = \lambda^i g_{m_i n_i}^{(i)},
\]

\( i = 1, \ldots, n; \ n \geq 2 \). The energy-momentum tensor is adopted in the following form

\[
T^M_N = \sum_{\alpha=1}^{m} T^M_N^{(\alpha)},
\]

\[
(T^M_N^{(\alpha)}) = \text{diag}(-\rho^{(\alpha)}(t), \rho^{(1)}(t)\delta_{k_1}^{m_1}, \ldots, \rho^{(n)}(t)\delta_{k_n}^{m_n}).
\]

\( \alpha = 1, \ldots, m \), with the conservation law constraints imposed:

\[
\nabla_M T^M_N^{(\alpha)} = 0
\]

\( \alpha = 1, \ldots, m - 1 \). The Einstein equations

\[
R^M_N - \frac{1}{2} \delta^M_N R = \kappa^2 T^M_N
\]

(\( \kappa^2 \) is gravitational constant) imply \( \nabla_M T^M_N = 0 \) and consequently \( \nabla_M T^M_N^{(m)} = 0 \).

We suppose that for any \( \alpha \)-th component of matter the pressures in all spaces are proportional to the density

\[
p^{(\alpha)}_i(t) = (1 - h^{(\alpha)}_i(x(t))) \rho^{(\alpha)}(t),
\]

where

\[
h^{(\alpha)}_i(x) = \frac{1}{N_i} \frac{\partial}{\partial x^i} \Phi^{(\alpha)}(x),
\]

\( i = 1, \ldots, n \), where \( \Phi^{(\alpha)}(x) \) is a smooth function on \( R^n, \ \alpha = 1, \ldots, m \).

In Sec. 2 the Einstein equations (1.7) for the model are reduced to the equations of motion for some Lagrange system with the energy constraint \( E = 0 \) imposed. When \( m = 1 \) and all spaces are Ricci-flat (\( \lambda^i = 0 \) in (1.3), \( i = 1, \ldots, n \)) such reduction was performed previously in [9].
In Sec. 3 we consider the Einstein equations, when all spaces are Ricci-flat and \( h_i^{(\alpha)} = \text{const}, \; i = 1, \ldots, n, \; \alpha = 1, \ldots, m \). In this case we deal with pseudo-Euclidean Toda-like system with the Lagrangian

\[
L_A = \frac{1}{2} G_{ij} \dot{x}^i \dot{x}^j - \sum_{\alpha=1}^{m} \kappa^2 A^{(\alpha)} \exp(u_i^{(\alpha)} x^i),
\]  

(1.10)

where \( \text{sign}(G_{ij}) = (-, +, \ldots, +) \) \cite{14,15}, \( u_i^{(\alpha)} = N_i h_i^{(\alpha)} \) and \( A^{(\alpha)} = \text{const} \; i = 1, \ldots, n, \; \alpha = 1, \ldots, m \). The Einstein equations are integrated in the following cases: 1) \( m = 1 \); 2) \( n = 2, \; m \geq 2, \; A^{(\alpha)} \neq 0, \; u^{(\alpha)} - u^{(1)} = b^{(\alpha)} u, \; \alpha = 1, \ldots, m \), where \( u^2 = G^{ij} u_i u_j = 0, \; u \neq 0 \); 3) \( u^{(\alpha)} = b^{(\alpha)} u, \; u^2 < 0, \; A^{(\alpha)} > 0, \; \alpha = 1, \ldots, m \).

In Sec. 4 the Wheeler-DeWitt (WDW) equation for the model is considered. When all spaces are Ricci-flat the WDW equation is integrated in all listed above cases. For \( m = 1 \) the solution of the WDW equation, satisfying so-called quantum wormhole boundary conditions \cite{34}, are obtained.

2 The equations of motion

The non-zero components of the Ricci-tensor for the metric (1.1) are following

\[
R_{00} = - \sum_{i=1}^{n} N_i [\ddot{x}^i - \dot{\gamma} \dot{x}^i + (\dot{x}^i)^2],
\]

(2.1)

\[
R_{m n_i} = g_{m n_i}^{(i)} [\dot{x}^i + \exp(2x^i - 2\gamma)(\ddot{x}^i + \dot{x}^i (\sum_{i=1}^{n} N_i \dot{x}^i - \dot{\gamma}))],
\]

(2.2)

\( i = 1, \ldots, n \).

We put

\[
\gamma = \gamma_0 \equiv \sum_{i=1}^{n} N_i x^i
\]

(2.3)

in (1.1) (the harmonic time is used). Then it follows from (2.1) and (2.2) that the Einstein equations (1.7) for the metric (1.1) with \( \gamma \) from (2.3) and the energy-momentum tensor from (1.4), (1.5) are equivalent to the following set of equations

\[
\frac{1}{2} G_{ij} \dot{x}^i \dot{x}^j + V_c + \kappa^2 \sum_{\alpha=1}^{m} \rho^{(\alpha)} \exp(2\gamma_0) = 0,
\]

(2.4)

\[
\lambda^i + \ddot{x}^i \exp(2x^i - 2\gamma_0) = \kappa^2 \exp(2x^i) \sum_{\alpha=1}^{m} [p_i^{(\alpha)} + (D - 2)^{-1} (\rho^{(\alpha)} - \sum_{j=1}^{n} N_j p_j^{(\alpha)})],
\]

(2.5)

\( i = 1, \ldots, n \). Here

\[
G_{ij} = N_i \delta_{ij} - N_i N_j
\]

(2.6)

are the components of the minisuperspace metric,

\[
V_c = -\frac{1}{2} \sum_{i=1}^{n} \lambda^i N_i \exp(-2x^i + 2\gamma_0)
\]

(2.7)
is the potential and $D \equiv \dim M = 1 + \sum_{i=1}^{n} N_{i}$.

The conservation law constraint (1.6) for $\alpha \in \{1, \ldots, m\}$ reads

$$\dot{\rho}^{(\alpha)} + \sum_{i=1}^{n} N_{i} \dot{x}^{i} (\rho^{(\alpha)} + p_{i}^{(\alpha)}) = 0. \quad (2.8)$$

We impose the conditions of state in the form (1.8), (1.9). Then eq. (2.8) gives

$$\rho^{(\alpha)}(t) = A^{(\alpha)} \exp[-2N_{i}x^{i}(t) + \Phi^{(\alpha)}(x(t))], \quad (2.9)$$

where $A^{(\alpha)} = \text{const}$ and eqs. (2.4), (2.5) may be written in the following manner

$$\frac{1}{2}G_{ij} \dot{x}^{i} \dot{x}^{j} + V + \kappa^{2} \sum_{\alpha=1}^{m} A^{(\alpha)} \exp \Phi^{(\alpha)} = 0, \quad (2.10)$$

$$\lambda^{i} + \ddot{x}^{i} \exp(2x^{i} - 2\gamma_{0}) = -\kappa^{2} \sum_{\alpha=1}^{m} u_{i}^{(\alpha)} A^{(\alpha)} \exp(2x^{i} - 2\gamma_{0} + \Phi^{(\alpha)}), \quad (2.11)$$

$i = 1, \ldots, n$. In (2.11) we denote

$$u_{i}^{(\alpha)} \equiv N_{i}h_{i}^{(\alpha)} = \partial_{i} \Phi^{(\alpha)}, \quad u_{i}^{(\alpha)} = G^{ij} u_{j}^{(\alpha)}, \quad (2.12)$$

where [15]

$$G^{ij} = \frac{\delta_{ij}}{N_{i}} + \frac{1}{2-D} \quad (2.13)$$

are the components of the matrix inverse to the matrix $(G_{ij})$ (2.6).

It is not difficult to verify that equations (2.11) are equivalent to the Lagrange equations for the Lagrangian

$$L = \frac{1}{2}G_{ij} \dot{x}^{i} \dot{x}^{j} - V \quad (2.14)$$

where

$$V = V(x) = V_{c}(x) + \sum_{\alpha=1}^{m} \kappa^{2} A^{(\alpha)} \exp[\Phi^{(\alpha)}(x)]. \quad (2.15)$$

Eq. (2.10) is the zero-energy constraint

$$E = \frac{1}{2}G_{ij} \dot{x}^{i} \dot{x}^{j} + V = 0. \quad (2.16)$$

Remark 1. In terms of 1-forms $u^{(\alpha)} = u_{i}^{(\alpha)} dx^{i}$, the relations (1.9) read: $u^{(\alpha)} = d\Phi^{(\alpha)}$, $\alpha = 1, \ldots, m$. In this case

$$du^{(\alpha)} = 0, \quad (2.17)$$

$\alpha = 1, \ldots, m$. The set of eqs. (2.17) (on $\mathbb{R}^{n}$) is equivalent to (1.9). An open problem is to generalize the considered here formalism for the following cases: a) $du^{(\alpha)} \neq 0$ for some $\alpha \in \{1, \ldots, m\}$; b) $du^{(\alpha)} = 0$ for all $\alpha = 1, \ldots, m$, but $u^{(\alpha)}$ are defined on an open submanifold $\Omega \subset \mathbb{R}^{n}$ with the non-trivial cohomology group $H^{1}(\Omega, \mathbb{R}) \neq 0$. 

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Using eqs. (2.1) and (2.2), it is not difficult to verify that the Einstein equations (1.7) for the metric (1.1) and the energy-momentum tensor from (1.4), (1.5), (1.8), (1.9) are equivalent to the Lagrange equations for the following degenerate Lagrangian (see also [15])

\[
L = \frac{1}{2} \exp(-\gamma + \gamma_0(x))G_{ij}\dot{x}^i\dot{x}^j - \exp(\gamma - \gamma_0(x))V(x) \tag{2.18}
\]

\((L = L(\gamma, x, \dot{x})).\) Fixing the gauge

\[
\gamma = \gamma_0(x) - 2f(x), \tag{2.19}
\]

where \(f = f(x)\) is a smooth function on \(\mathbb{R}^n,\) we get the Lagrangian

\[
L_f = \frac{1}{2} \exp(2f(x))G_{ij}\dot{x}^i\dot{x}^j - \exp(-2f(x))V(x). \tag{2.20}
\]

For \(f = 0\) we have the harmonic-time gauge (2.3). The set of Lagrange equations for the Lagrangian (2.18) (or equivalently the set of the Einstein equations) with \(\gamma\) from (2.19) is equivalent to the set of Lagrange equations for the Lagrangian (2.20) with the energy constraint imposed

\[
E_f = \frac{1}{2} \exp(2f(x))G_{ij}\dot{x}^i\dot{x}^j + \exp(-2f(x))V(x) = 0. \tag{2.21}
\]

Remark 2. We remind that the action of the relativistic particle of mass \(m,\) moving in the pseudo-Euclidean background space with the metric \(\hat{G}_{ij}(x)\) has the following form

\[
S = \int d\tau [\hat{G}_{ij}(x(\tau))\dot{x}^i\dot{x}^j - \frac{m^2}{2}e(\tau)], \tag{2.22}
\]

where \(e = e(\tau)\) is 1-bein. Comparing (2.18) and (2.22), we find that for \(V(x) > 0\) the cosmological model (2.18) is equivalent to the model of relativistic particle with the mass \(m = 1,\) moving in the conformally-flat (pseudo-Euclidean) space with the metric \(\hat{G}_{ij}(x) = 2V(x)G_{ij}\) (see also [37]). In this case \(e = 2V(x)\exp(\gamma - \gamma_0(x))\). For \(V(x) < 0\) we have a tachyon. The problem may be also reformulated in terms geodesic-flow problem for conformally-flat metric (this follows from (2.22) or from a more general scheme [43]).

## 3 Classical solutions

Now, we consider the following case: \(\lambda^i = 0\) (all spaces are Ricci-flat), \(u_i^{(\alpha)} = N_i h_i^{(\alpha)} = \text{const},\) \(i = 1, \ldots, n.\) Then \(V_c = 0\) and we put \(\Phi^{(\alpha)} = u_i^{(\alpha)}x^i\) in (2.15). In this case the Lagrangian (2.14) has the form (1.10).

Remark 3. The curvature induced term \(V_c\) (2.7) may be generated in the framework of the model with the Ricci-flat spaces \(M_i\) by the addition of \(n\) new components of the perfect fluid with \(u_i^{(k)} = 2N_i - 2\delta_i^k\) and \(\kappa^2 A^{(k)} = -\lambda^k N_k/2, i, k = 1, \ldots, n.\) The introduction of the cosmological constant \(\Lambda\) into the model is equivalent to the addition of a new component with \(u_i^{(n+1)} = 2N_i\) and \(\kappa^2 A^{(n+1)} = \Lambda.\)
3.1 One-component matter

We consider the case \( m = 1, A^{(1)} = A \neq 0 \). We denote \( h_i^{(1)} = h_i, u_i^{(1)} = u_i = N_i h_i \).

We remind [14, 15] that the minisuperspace metric

\[
G = G_{ij} dx^i \otimes dx^j \tag{3.1}
\]

has pseudo-Euclidean signature \((-,+,...,+\)), i.e. there exist a linear transformation

\[
z^a = V^a_i x^i, \tag{3.2}
\]

diagonalizing the minisuperspace metric (3.1)

\[
G = \eta_{ab} \, dz^a \otimes dz^b = -dz^0 \otimes dz^0 + \sum_{i=1}^{n-1} dz^i \otimes dz^i, \tag{3.3}
\]

where

\[
(\eta_{ab}) = (\eta^{ab}) \equiv \text{diag}(-1,+1,...,+1), \tag{3.4}
\]

\( a,b = 0,...,n-1 \).

Proposition 1. For any \( u = (u_i) \in \mathbb{R}^n, u \neq 0 \), there exists a (nondegenerate) \( n \times n \) matrix \((V^a_i)\) such that

\[
\eta_{ab} V^a_i V^b_j = G_{ij} \tag{3.5}
\]

and a) \( V^0_i = u_i/\sqrt{-u^2} \), for \( u^2 < 0 \); b) \( V^1_i = u_i/\sqrt{u^2} \), for \( u^2 > 0 \); c) \( V^0_i + V^1_i = u_i \), for \( u^2 = 0 \);

Here and below \( (u = (u_i) = (N_i h_i)) \)

\[
u^2 \equiv u_i u^i = G^{ij} u_i u_j = \sum_{i=1}^{n} N_i (h_i)^2 + \frac{1}{2 - D} (\sum_{i=1}^{n} N_i h_i)^2. \tag{3.6}
\]

(We note that in notations of [14] \( u^2 = \hat{\Delta}(h)/(2 - D) \).)

This proposition follows from the fact that \( < u, v > \equiv G^{ij} u_i v_j \) is bilinear symmetric 2-form of signature \((-,+,...,+\)) and the following quite obvious

Proposition 2. Let \( v \in E = \mathbb{R}^n, n \geq 2 \), and \( < u, v > : E \times E \rightarrow \mathbb{R} \) is a bilinear symmetric 2-form of signature \((-,+,...,+\)). Then there exists a basis \( v^0, ... , v^{n-1} \) in \( E \), such that \( < v^a, v^b > = \eta^{ab} \) and a) \( v = v^0 \), b) \( v = v^1 \), c) \( v = v^0 + v^1 \), in the cases: a) \( u^2 \equiv < v, v > = -1 \), b) \( u^2 = 1 \), c) \( u^2 = 0 \) respectively.

Let \( u \neq 0 \). In \( z = (z^a)\)-coordinates (3.2) with the matrix \((V^a_i)\) from the Proposition 1 the Lagrangian (2.14) has the following form

\[
L_A = \frac{1}{2} \eta_{ab} \dot{z}^a \dot{z}^b - V_A = -\frac{1}{2} (\dot{z}^0)^2 + \sum_{i=1}^{n-1} \frac{1}{2} (\dot{z}^i)^2 - V_A, \tag{3.7}
\]

where

\[
V_A = \kappa^2 A \exp(2qz^0), \quad u^2 < 0, \tag{3.8}
\]

\[
= \kappa^2 A \exp(2qz^1), \quad u^2 > 0, \tag{3.9}
\]

\[
= \kappa^2 A \exp(z^0 + z^1), \quad u^2 = 0. \tag{3.10}
\]
is the potential (2.15). Here we denote

$$2q \equiv \sqrt{|u^2|}. \tag{3.11}$$

The Lagrange equations for the Lagrangian (3.7)

$$\ddot{z}^a = -\eta^{ab}\partial_b V_A \tag{3.12}$$

with the energy constraint (2.16)

$$E_A = \frac{1}{2}\eta_{ab}\dot{z}^a\dot{z}^b + V_A = 0, \tag{3.13}$$

can be easily solved. We present the solutions.

a) For $u^2 < 0$

$$z^i = p^i t + q^i, \quad i = 1, \ldots, n - 1, \tag{3.14}$$

$$2qz^0 = y(t), \tag{3.15}$$

where $p^i$ and $q^i$ are constants and

$$y(t) = \begin{cases} \ln[C/D \sinh^2(\frac{1}{2}\sqrt{C}(t - t_0))], & C \neq 0, D > 0, \\ \ln[4/D(t - t_0)^2], & C = 0, D > 0, \\ \ln[-C/D \cosh^2(\frac{1}{2}\sqrt{C}(t - t_0))], & C > 0, D < 0, \end{cases} \tag{3.16-3.18}$$

Here $t_0$ is an arbitrary constant, $D = -2u^2\kappa^2 A$, $C = -u^2(p^0)^2$ and $(\vec{p})^2 = \sum_{i=1}^{n-1}(p^i)^2$.

b) For $u^2 > 0$ we have

$$z^i = p^i t + q^i, \quad i = 0, 2, \ldots, n - 1, \tag{3.19}$$

$$2qz^1 = y(t), \tag{3.20}$$

with $(\vec{p})^2 = (p^0)^2 - \sum_{i=2}^{n-1}(p^i)^2$ in (3.15)-(3.18).

c) $u^2 = 0, u \neq 0$. In this case

$$z^i = p^i t + q^i, \quad i = 2, \ldots, n - 1, \tag{3.21}$$

$$z^+ = z^0 + z^1 = p^+ t + q^+, \tag{3.22}$$

$$z^- = z^0 - z^1 = p^- t + q^- + \kappa^2 A\zeta(t), \tag{3.23}$$

where for $p^+ \neq 0$

$$z(t) = 2(p^+)^{-2}\exp(p^+ t + q^+), \quad p^+ p^- = (\vec{p})^2 \tag{3.24}$$

($p^- = 0$ for $n = 2$) and for $p^+ = 0$

$$z(t) = t^2 \exp q^+, \quad (\vec{p})^2 + 2\kappa^2 A \exp q^+ = 0. \tag{3.25}$$

Here $(\vec{p})^2 = \sum_{i=2}^{n-1}(p^i)^2$.  

For $u = 0$ we have

$$z^a = p^a t + q^a, \quad a = 0, \ldots, n - 1, \quad (3.26)$$

$$\frac{1}{2} \eta_{ab} p^a p^b + \kappa^2 A = 0. \quad (3.27)$$

**Kasner-like parametrization.** Here we consider the case $u^2 < 0, A \neq 0$. For $C = -u^2(\bar{p})^2 > 0$ we reparametrize the time variable

$$\tau = \frac{T}{\sqrt{\varepsilon}} \ln \frac{\exp(\sqrt{C}(t - t_0)) + \sqrt{\varepsilon}}{\exp(\sqrt{C}(t - t_0)) - \sqrt{\varepsilon}}. \quad (3.28)$$

where

$$\varepsilon \equiv A/|A| = \pm 1, \quad T \equiv (2/\kappa^2|A||u^2|)^{1/2}. \quad (3.29)$$

We introduce new (Kasner-like) parameters

$$\alpha^i \equiv -2V^i_s p^s / \sqrt{-u^2(\bar{p})^2}, \quad (3.30)$$

where $(V^i_s) = (V^a_i)^{-1}$ and the summation parameter $s$ runs: $s = 1, \ldots, n - 1$. Then, due to relations (3.2), (3.5), (3.14)-(3.16), (3.18) and Proposition 1 we get the following expression for the metric (1.1) \[40\]

$$g = -(\prod_{i=1}^n (a_i(\tau))^{2N_i - u_i}) d\tau \otimes d\tau + \sum_{i=1}^n a_i^2(\tau) g^{(i)}, \quad (3.31)$$

where

$$a_i(\tau) = A_i^{\left[\frac{\sinh(\tau \sqrt{\varepsilon}/T)}{\sqrt{\varepsilon}}\right]^{2u_i/u^2} \left[\tanh(\tau \sqrt{\varepsilon}/2T)\right]^{\alpha^i}}, \quad (3.32)$$

$i = 1, \ldots, n; A_i > 0$ are constants and the parameters $\alpha^i$ satisfy the relations

$$u_i \alpha^i = 0, \quad (3.33)$$

$$G_{ij} \alpha^i \alpha^j = -4/u^2 \quad (3.34)$$

(see Proposition 1 and (3.30)). For the density (2.15) we have

$$\rho(\tau) = A \prod_{i=1}^n (a_i(\tau))^{u_i - 2N_i}. \quad (3.35)$$

We note, that $(\bar{p})^2 = 2\kappa^2|A| \prod_{i=1}^n A_i^{u_i}$

For $A > 0$ we have an exceptional solution (3.31), (3.33), (3.34) with the scale factors

$$a_i(\tau) = \bar{A}_i \exp(\pm 2u^i \tau / u^2 T), \quad (3.36)$$

$\bar{A}_i > 0, i = 1, \ldots, n$. This solution correspond to $C = 0$ case (3.17).

Remark 4. In [19] the Einstein equations (2.10), (2.11) were solved for $A^{(\alpha)} = 0, \alpha = 1, \ldots, m, \lambda^1 \neq 0, \lambda^i = 0, i > 1$. The solutions [19] may be also obtained from the formulas (3.31)-(3.34). We note that the spherically-symmetric analogue of the solution
was considered in [36] (the case $d = 2$ was considered previously in [35]). There exists an interesting special case of the solutions [35, 36]. It is the $n$-time generalization of the Schwarzschild solution
\[ g = -[(1 - \frac{L}{R})A]_{ab}dt^a \otimes dt^b + (1 - \frac{L}{R})^{-spA}dR \otimes dR + (1 - \frac{L}{R})^{1-spA}R^2d\Omega^2, \]
where $L \neq 0$ and $A = (A_{ab})$ is symmetric $n \times n$ matrix, satisfying the relation $sp(A^2) + (spA)^2 = 2$.

We consider this solution in a separate publication.

### 3.2 Two spaces with $m$-component matter

Here we consider the following case: $n = 2, m \geq 2, A^{(\alpha)} \neq 0, u^{(\alpha)}(\alpha) = \alpha, \ldots, m$, where $u^2 = 0, u \neq 0$ and $b^{(\alpha)}$ are constants.

In $z$-coordinates (3.2), where the matrix $(V_i^a)$ satisfies the Proposition 1 (see the case c) $u^2 = 0$) we have

\[ z^+ = z^0 + z^1 = (V^0_i + V^1_i)x^i = u_i x^i, \]
\[ \Phi^{(1)} = u^{(1)}_i x^i = \alpha_+ z^+ + \alpha_- z^-, \]
where $2\alpha_+ = - < u^1, u^* >, 2\alpha_- = - < u^1, u >$, and $u^* = (u^*_i)$ is defined by the relation: $u_i^* x^i = z^-$ (or equivalently $< u^*, u^* > = 0, < u^*, u > = -2$).

Due to (3.37)-(3.39) the potential in (1.10) is factorized
\[ V = V_+(z^+)V_-(z^-), \]
where
\[ V_+(z^+) = \exp(\alpha_+ z^+) (\kappa^2 A^{(1)} + \sum_{k=2}^{m} \kappa^2 A^{(k)} \exp(\alpha^*(z^+), \]
\[ V_-(z^-) = \exp(\alpha_- z^-). \]

Let $A^{(\alpha)} > 0, \alpha = 1, \ldots, m$. We consider the $f$-gauge (2.19) with
\[ F = e^{2f} = V. \]

In this gauge the Lagrangian (2.20) reads
\[ L_f = -\frac{1}{2} V_+(z^+) \dot{z}^+ V_-(z^-) \dot{z}^- - 1. \]

In the variables
\[ w^\pm = w^\pm(z^\pm) = \int_{z_0}^{z^\pm} dxV_{\pm}(x) \]
the Lagrangian (3.44) has rather simple form
\[ L_f = -\frac{1}{2} w^+ \dot{w}^- - 1. \]
The equations of motion for (3.46) give
\[ w^\pm(t) = p^\pm t + q^\pm. \] (3.47)

The parameters \(p^\pm\) satisfy the energy constraint
\[ 2E_f = -p^+p^- + 2 = 0. \] (3.48)

Remark 5. It is interesting to note that the so-called \(D\)-dimensional Schwarzschild-deSitter solution [44,45] may be obtained from the considered here cosmological solution with \(n = m = 2\) and \(N_1 = 1, N_2 = D - 2\).

3.3. \(n\) spaces with \(m\) component matter

Now we consider the simplest case of the multicomponent matter. We put in (1.10) \(n \geq 2, A^{(\alpha)} > 0, u^{(\alpha)} = b^{(\alpha)}u, u^2 < 0\), where \(b^{(\alpha)}\) are constants, \(\alpha = 1, \ldots, m\).

In \(z\)-coordinates (3.2), corresponding to the case a) from the Proposition 1, the Lagrangian (1.10) has the form (3.7) with the potential
\[ V_A = V_A(z^0) = \sum_{i=1}^m \kappa^2 A^{(\alpha)} \exp(2q b^{(\alpha)} z^0), \] (3.49)

where \(q\) is defined in (3.11) \((A = (A^{(\alpha)})\). The solutions of the equations (3.12) and (3.13) are expressed by the formula (3.14) and the following relation
\[ \int_{z_0}^{z_0} dx [2E + 2V_A(x)]^{-1/2} = \pm(t - t_0), \] (3.50)

where \(2E = \sum_{i=1}^{n-1} (p^i)^2\), and \(c_0, t_0\) are constants.

4 Quantum solutions

The WDW equation for the model in harmonic time gauge (2.3) reads as follows:
\[ (-\frac{1}{2\mu}G^{ij} \partial_i \partial_j + \mu V)\Psi = 0, \] (4.1)

where \(\Psi = \Psi(x)\) is "the wave function of the Universe", \(V\) is the potential (2.15), \(\partial_i = \partial/\partial x^i\) and \(G^{ij}\) are defined in (2.13). The relation (4.1) is a result of a trivial quantization of the zero energy constraint (2.16), written in the form \(\mu E = 0\). Here \(\mu\) is a fundamental quantum parameter of the theory.

In \(f\)-gauge (2.19) the WDW equation should be written in the conformally covariant form [15, 37] (such form of the WDW equation was discussed earlier by Misner [38])
\[ (-\frac{1}{2\mu} \Delta[e^{2f} \hat{G}] + \frac{a_n}{\mu} R[e^{2f} \hat{G}] + e^{-2f} \mu V)\Psi^f = 0, \] (4.2)

where \(\Delta[\hat{G}]\) and \(R[\hat{G}]\) are the Laplace-Beltrami operator and the scalar curvature of \(\hat{G}\) respectively, \(a_n = (n - 2)/8(n - 1)\) and
\[ \Psi^f = \exp[(2 - n)f/2] \Psi. \] (4.3)
Without loss of generality we put $\mu = 1$ below.

**4.1 One-component matter**

Here we find the quantum analogues of the classical solutions from 3.1, i.e. we integrate the WDW equation

$$(-\frac{1}{2} \eta^{ab} \frac{\partial}{\partial z^a} \frac{\partial}{\partial z^b} + V_A) \Psi = 0. \quad (4.4)$$

with the potential (3.8)-(3.10). We note, that the WDW equation for 1-component model with $n$ Ricci-flat spaces was considered previously in [17].

a) $u^2 < 0$. In this case the WDW equation (4.4) reads

$$[-\frac{\partial}{\partial z^0}^2 - \sum_{i=1}^{n-1} \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^i} + 2\kappa^2 A \exp(2qz^0)] \Psi = 0. \quad (4.5)$$

We are seeking solutions of (4.5) in the following form

$$\Psi(z) = \exp(i\bar{p}z^0) \Phi(z^0), \quad (4.6)$$

where $\bar{p} = (p_1, \ldots, p_{n-1})$ is a constant vector (generally from $C^{n-1}$), $\bar{z} = (z^1, \ldots, z^{n-1})$, $\bar{p} \bar{z} \equiv \sum_{i=1}^{n-1} p_i z^i$. The substitution of (4.6) into (4.5) gives

$$[-(\frac{\partial}{\partial z^0})^2 - 2\kappa^2 A \exp(2qz^0)] \Phi = 2\mathcal{E} \Phi, \quad (4.7)$$

where $2\mathcal{E} = \sum_{i=1}^{n-1} p_i^2$. Solving (4.7), we get two linearly independent solutions

$$\Phi(z^0) = B_\nu(\sqrt{-2\kappa^2 A q^{-1}} e^{qz^0}), \quad (4.8)$$

where $\nu = i\sqrt{2\mathcal{E}}/q = i|\bar{p}|/q$, and $B_\nu = I_\nu, K_\nu$ is modified Bessel function. We note, that

$$\nu = \exp qz^0 = \exp(\frac{1}{2} u_i x^i) = \prod_{i=1}^n a_i^{u_i/2} \quad (4.9)$$

is a natural scale factor for the model ($a_i = e^{x^i}$).

The general solution of eq. (4.5) has the following form

$$\Psi(z) = \sum_{B=I,K} \int d^{n-1}\bar{p} C_B(\bar{p}) \Psi_{\bar{p}}^B(z), \quad (4.10)$$

where

$$\Psi_{\bar{p}}^B(z) = e^{i\bar{p}z} B_{|\bar{p}|/q}(\sqrt{-2\kappa^2 A q^{-1}} e^{qz^0}), \quad (4.11)$$

and functions $C_B$ ($B = I,K$) belong to an appropriate class of (generalized) functions.

b) $u^2 > 0$. In this case the WDW equation (4.4) reads

$$[-\frac{\partial}{\partial z^1} \frac{\partial}{\partial z^1} + \frac{\partial}{\partial z^0} \frac{\partial}{\partial z^0} - \sum_{i=2}^{n-1} \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^i} + 2\kappa^2 A \exp(2qz^1)] \Psi = 0. \quad (4.12)$$
An analogous consideration in this case gives the general solution (4.10) with
\[ \Psi^B_\tilde{p}(z) = e^{i\tilde{p}\tilde{z}} B_{\nu(\tilde{p})} (\sqrt{2\kappa^2 A} e^{qz^0}). \] (4.13)
Here \( \tilde{p} = (p_0, p_2, \ldots, p_{n-1}) \), \( \tilde{z} = (z^0, z^2, \ldots, z^{n-1}) \), \( \nu(\tilde{p}) = i\sqrt{2\tilde{E}/q} \), and \( 2\tilde{E} = p_0^2 - \sum_{i=2}^{n-1} p_i^2 \).

c) \( u^2 = 0 \) for \( u \neq 0 \) the WDW equation reads
\[ [-4\partial_+\partial_- + \sum_{i=1}^{n-1} \left( \frac{\partial}{\partial z^i} \right)^2 - 2\kappa^2 A \exp(z^+) ]\Psi = 0, \] (4.14)
where \( z^\pm = z^0 \pm z^1 \), \( \partial_\pm = \partial/\partial z^\pm \). The substitution
\[ \Psi(z) = \exp(i\tilde{p}\tilde{z}) \Phi(z^+, z^-), \] (4.15)
with \( \tilde{p} = (p_2, \ldots, p_{n-1}) \), \( \tilde{z} = (z^2, \ldots, z^{n-1}) \) entails
\[ [4\partial_+\partial_- + 2\tilde{E} + 2\kappa^2 A \exp(z^+) ]\Phi = 0, \] (4.16)
where \( 2\tilde{E} = \sum_{i=2}^{n-1} p_i^2 \). Introducing new variables \( u^0, u^1 \), where \( u^0 \pm u^1 = u^\pm \) and
\[ u^+ = 2\tilde{E} z^+ + 2\kappa^2 A \exp(z^+), \quad u^- = z^- \] (4.17)
we get the Klein-Gordon equation for \( \Phi \) with \( m^2 = 1 \)
\[ ((\partial/\partial u^0)^2 - (\partial/\partial u^1)^2 + 1)\Phi = 0. \] (4.18)

It is quite obvious how to write the general solution of (4.22).

**Quantum wormholes.** In the case a) \( u^2 < 0 \) for \( A < 0 \) there exist so-called quantum wormhole solutions of the WDW equation [34]. We present here a continuous spectrum family of these solutions. The wave functions are following
\[ \hat{\Psi}_{\lambda,\tilde{n}}(z) = \exp[-q^{-1}\sqrt{-2\kappa^2 A} e^{qz^0} \cosh(\lambda - q\tilde{n}^i \tilde{n}^i)]. \] (4.19)
where \( \lambda \in R \) and \( \tilde{n} \) is unit vector: \( (\tilde{n})^2 = 1 \) \( (\tilde{n}^i \in S^{n-1}) \). These solutions are related with the solutions (4.11) (with \( B = K \)) by the formula
\[ \hat{\Psi}_{\lambda,\tilde{n}}(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dk \Psi_{k\tilde{n}}(z) e^{-ik\lambda}, \] (4.20)
(such trick was suggested in [39], see also [20,41]). The solutions (4.19) satisfy the quantum wormhole boundary conditions (in terms of parameter \( v \) (4.9): i) the wave function is exponentially damped for large space geometries \( (v \to +\infty) \); ii) the wave function is regular when the spatial geometry degenerates \( (v \to 0) \).

We also note that the the functions
\[ \Psi_{m,\tilde{n}} = H_m(x^0)H_m(x^1) \exp[-(x^0)^2 + (x^1)^2] \] (4.21)
where $H_m$ are Hermite polynomials, $m = 0, 1, \ldots$,
\[
x^i = \left(\frac{2}{q}\right)^{1/2}(-2\kappa^2 A)^{1/4} \exp(q z^0/2) f^i(\frac{1}{2} q \tilde{\eta}), \quad i = 0, 1,
\]
\[
(f^0, f^1) = (\sinh, \cosh) \text{ are also solutions of the WDW equation with the quantum wormhole boundary conditions. (They are called discrete spectrum quantum wormholes.)}
\]
We note that the special cases of the solutions (4.19), (4.21) for $u_i = 2N_i$ ($\Lambda$-term case) and $u_i = 2N_i - 2\delta^1_1$ (1-curvature case, $\lambda^1 \neq 0$) were considered in [41] and [20] respectively.

We also note that for b) $u^2 > 0$ and $A > 0$ there also exist quantum wormhole solutions. (In this case $z^0$ should be replaced by $z^1$ in (4.19), $\vec{z}$ is defined in 3.1 b) and $\vec{n}$ belongs to hypersphere.)

4.2. Two spaces with $m$-component matter

For the model from the subsection 3.2 the WDW equation (4.2) in the $f$-gauge (3.43) has the following form ($\mu = 1$)
\[
(2 \frac{\partial}{\partial V_+(z^+) \partial z^+} + 1) \Psi = 0.
\] (4.22)

Indeed, for $n = 2$ we have
\[
\Delta[e^{2f}G] = e^{-2f} \Delta[G], \quad a_2 = 0 \quad \text{and} \quad \Psi^f = \Psi \text{ (see (4.3)}.
\]
In $w$-variable $w = (w^0, w^1)$, where $w^0 \pm w^1 = w^\pm$, where $w^\pm$ are defined in (3.45), we get the Klein-Gordon equation with $m^2 = 2$
\[
[(\frac{\partial}{\partial w^0})^2 - (\frac{\partial}{\partial w^1})^2 + 2] \Psi = 0.
\] (4.23)

4.3 $n$-spaces with $m$ component matter

Here we present the solutions of the WDW equation (4.4) with the potential (3.49), i.e. quantum analogues of the classical solutions from 3.3 are considered.

Repeating all arguments from 4.1 (case a)), we get the general solution of (4.4)
\[
\Psi(z) = \sum_{* = \pm} \int d^{n-1} \vec{p} C_\ast(p) \Psi_{\ast}^\ast(p(z)),
\] (4.24)

where
\[
\Psi_{\ast}^\ast(p(z)) = \exp(i \vec{p} \vec{z}) \Phi_{\ast}^\ast(p(z)^0),
\] (4.25)

$* = \pm$, and $\Phi^* = \Phi_{\ast}^\ast(p(z)^0)$ are two linearly independent solutions of the equation
\[
[-(\frac{\partial}{\partial z^0})^2 - 2V_A(z^0)] \Phi = 2 \mathcal{E}_{\vec{p}} \Phi,
\] (4.26)

with the notations for $\mathcal{E} = \mathcal{E}_{\vec{p}}, \vec{p}, \vec{z}$ from 4.1 a).

We note, that for special values of parameters $A^{(a)}$ and $b^{(a)}$ in the potential (3.49) the equation (4.26) describes the quantum spin systems [42].

5 Concluding remarks

In this paper we investigated the multidimensional cosmological model with $n$ ($n > 1$) Ricci-flat spaces, filled by $m$-component perfect fluid. In some sense, this model may be considered as "universal" cosmological model: a lot of cosmological models may be obtained from it
under a suitable choice of parameters. This fact may be used for ”Toda-like” classification of
known exact cosmological (and spherically-symmetric) solutions of the Einstein equations.
(We note, that the Bianchi-IX cosmological model is described by the ”Toda-like” Lagrangian
(1.10) with $n = 3$ and $m = 6$.)

Here we integrated the Einstein and Wheeler-DeWitt equations for some sets of parameters. But an open problem is the problem of integrability of the considered here model (at classical and quantum levels) for arbitrary values of the parameters $m, n, N_i$ and $u_i^{(a)}$. We hope to continue the investigation of this problem in forthcoming publications.

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