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Numerical Exploration via Least Squares Estimation on Three Dimensional MHD Yield Exhibiting Nanofluid Model with Porous Stretching Boundaries

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Abstract: The numerical study of a three-dimensional magneto-hydrodynamic (MHD) Casson nanofluid with porous and stretchy boundaries is the focus of this paper. Radiation impacts are also supposed. A feasible similarity variable may convert a verbalized set of nonlinear “partial” differential equations (PDEs) into a system of nonlinear “ordinary” differential equations (ODEs). To investigate the solutions of the resulting dimensionless model, the least-square method is suggested and extended. Maple code is created for the expanded technique of determining model behaviour. Several simulations were run, and graphs were used to provide a thorough explanation of the important parameters on velocities, skin friction, local Nusselt number, and temperature. The comparison study attests that the suggested method is well-matched, trustworthy, and accurate for investigating the governing model’s answers. This method may be expanded to solve additional physical issues with complicated geometry.

Keywords: nanofluid; least squares method; casson fluid; magnetic field; boundary value problem; numerical solution

1. Introduction

A nano-fluid is a liquid combination of nano-particles with sizes ranging from 1 nm to 100 nm. The dispersion of nano-particles plays an important role in enhancing the thermal conductivity of the nano-fluid. Nanoparticles have many uses in heat transfer processes. Nuclear reactors, microchannel, and computer operating systems are some of the uses of nano-fluid in heat transmission. Some of the research on nanoparticles in heat transmission is covered here. Esfahani et al. [1] investigated the formulation of a novel issue for the thermal conductivity of a silica/water-ethylene glycol (40%–60%) nano-fluid under various temperature conditions. Das et al. [2] and Usman et al. [3] explore radiation nanofluids with various effects (oscillating porous plate, Hall effects, thermal radiation). Anoop et al. [4] investigate the effects of nanoparticle size on convective heat transport in nanofluids. Sheikholeslami [5] also investigates the effects of forces (Lorentz type) on nano-fluid flow in a porous cylinder using the Darcy model.
Many fluids have this characteristic of radiative energy in the arrangement of electromagnetic waves during movement. Heat transfer mechanism with significant radiation effects can be calculated with the help of Stefan Boltzmann’s law. Several scientists have examined the heat transfer processes in the occurrence of radiation. Ali et al. [6] show the influence of radiations on heat transfer in an axisymmetric penetrable contracting sheet. Bhattacharyya et al. [7] considered the essential effects of pull/blowing on a continuous stagnation-point stream and the behaviour of heat transfer. Nadeem et al. [8] investigated the effects of radiation on a Jeffery type liquid across an expanding surface. Ishaq [9] examines a micropolar liquid with radiation effects.

The flow of electrically conducting fluid becomes the cause to change the magnetic flux, and due to these changes, the current is produced. Such types of fluids are said to be MHD flow and Ohm’s law is very useful for this type of situations. Many investigators have investigated such types of flows. Usman et al. [10] examined the thermal radiation and slip influences of MHD 3D flow over a channel and formulated the solution with the help of wavelets scheme. Aoyagi et al. [11] studied MHD flow in a multi-layered channel. Finally, they suggested that complex three-dimensional inlet flow. Borrelli et al. [12] studied the buoyancy effects on three-dimensional geometry. Sher et al. [13] investigated the effects of radiation on the MHD flow stagnation point of a nano-fluid over an expanding sheet with convective boundary conditions. Motsa et al. [14] mathematically concentrate on Hall, and particle slip flows on MHD micropolar liquid stream within sight of synthetic response and warm diffusivity. Sohail et al. [15] reported the involvement of different emerging parameters on field exhibiting model with mass, thermal transport over an extending surface. The developed problem in their research is tackled via the optimal Homotopy method. The solution is drafted graphically against numerous involved parameters. Validation of the results has been checked via computing the stresses and heat transfer rate. Some recent studies can be seen in refs. [16–31].

Assessment of literature study specified in the earlier paragraphs shows a lot of research gap available in the nano-fluid research, especially for the flow via a canal or movement of liquids between the oscillating or stretching sheets/plate. The numerical study of the problem with a suitable algorithm covering physical aspects of geometry of the problem is always a big problem. It needs to be addressed more broadly. Therefore, the current study is decently dedicated to the study of numerical analysis of three dimensional MHD Casson nano-fluid with porous and stretching boundaries via the least square method. The solution (numerical) of the modelled physical problem (provided in the earlier section of this paper) is assessed using a numerical scheme named the Least square method. This suggested numerical system is improved with the help of appropriate polynomial [24–29].

Furthermore, to explore the physical consequences of the modelled problem, graphical study and discussion on parametric research are also presented in the coming section of the paper. This study is very influential regarding mathematical modelling and numerical algorithm. The efficiency of the designed scheme is discussed in detail in the further section and also justified concerning geometry. Therefore, the reader can study and further extend this problem and numerical method for other problems.

The current study is divided into different sections. In the first section, we have elaborated on the literature assessment of our presented work. The second section is dedicated to the mathematical modelling under the assumptions according to the geometry of the problem. The formulated problem is further non-dimensionalise to study the different aspects of the problem using different parameters. The efficient and powerful numerical algorithm is explained in Section 3. A comprehensive analysis of the along graphical representation and comparative study is presented in Section 4. Finally, we concluded our study in Section 5 with specific and important key points and further justified the numerical results.
2. Mathematical and Geometrical Analysis

Consider heat transfer of fluid containing with Copper (Cu) and Silver (Ag) nanoparticles over a three-dimensional channel in the presence of internal heat generation/absorption. \( H \) is the distance between two walls; an upper wall of the channel at \( z = H \) is uniformly injected and lower boarder of the channel at \( z = 0 \) is stretched in \( x-\) and \( y- \) direction with same rates. A steady flow of the fluid is induced due to stretching the lower wall of the channel with same velocities in opposite direction with same rates as shown in Figure 1. The applied electric field is to be considered insignificant. Also, Joule heating, viscous dissipation, and thermal radiations are considered. Conservation laws \([29,30]\) take the following structure.

Figure 1. Flow of fluid particles.

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (1)
\]

\[
\frac{u}{\rho_{nf}} \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z} = \frac{1}{\rho_{nf}} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \frac{M^2 \varphi_z}{(1 + \beta_e \beta_e)^2 + \beta_e^2} \left( \beta_e \frac{v}{1 + \beta_e \beta_e} \right), \quad (2)
\]

\[
\frac{u}{\rho_{nf}} \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z} = \frac{1}{\rho_{nf}} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) - \frac{M^2 \varphi_z}{(1 + \beta_e \beta_e)^2 + \beta_e^2} \left( \beta_e \frac{u}{1 + \beta_e \beta_e} \right), \quad (3)
\]

\[
\frac{u}{\rho_{nf}} \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z} = \frac{1}{\rho_{nf}} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right), \quad (4)
\]

\[
\frac{u}{\rho_{nf}} \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \left( \frac{K_{nf}}{\rho C_p} \right)_{nf} + \frac{16 \sigma^* T_0^3}{\rho C_p} \frac{\partial^2 T}{\partial x^2} + \left( \frac{K_{nf}}{\rho C_p} \right)_{nf} + \frac{16 \sigma^* T_0^3}{\rho C_p} \frac{\partial^2 T}{\partial y^2} + \left( \frac{K_{nf}}{\rho C_p} \right)_{nf} + \frac{16 \sigma^* T_0^3}{\rho C_p} \frac{\partial^2 T}{\partial z^2} + \frac{\sigma_{nf} \beta_e^2}{\rho C_p} \left( u^2 + v^2 \right) \quad (5)
\]

In above Equations (1)–(5), \((P)\) the pressure, \(\beta_e = \omega_e \tau_e\) the ion slip parameter, \(\beta_i = \omega_i \tau_i\) the Hall parameter, \((T)\) the temperature, \((T_w)\) the temperature far away from the surface, \((Q)\) denotes heat generation/absorption, \((u, v \text{ and } w)\) are the velocity components in \(x, y,\) and \(z-\)directions, respectively, \(\beta_0\) the constant magnetic field. The wall of channel at \(z = 0\) is stretched with different velocities \(a > 0, b > 0\) and the wall at \(z = H\) is at rest and porous; \((\rho_{nf}, \nu_{nf}, k_{nf}, C_p)_{nf}, \rho_{nf} \text{ and } \sigma_{nf}\) are the dynamic viscosity, kinematic viscosity, thermal conductivity, specific heat, density and electrical
conductivity for the nanofluid. We assumed the models for thermos-physical properties of nano-sized particles, base fluid and nanofluid.

\[
q = -\frac{16\sigma T^3}{3k^*} \nabla T,
\]

(6)

In Equation (6), \((\sigma^*)\) the Stefan-Boltzmann constant, and \((k^*)\) is used to present the mean absorption coefficient. The relations regarding nano-liquid [29] are illuminated as

\[
\rho_{nf} = (1 - \phi)\rho_f + \phi \rho_s, \quad \mu_{nf} = \frac{\mu_f + \phi \mu_s}{(1 - \phi)^{\frac{1}{2}}} \sigma = \frac{\sigma_s}{\sigma_f}, \quad \sigma_{nf} = \left(1 + \frac{3(\sigma - 1)\phi}{\sigma + 2 - (\sigma - 1)\phi}\right), v_{nf} = \frac{\mu_{nf}}{\rho_{nf}}K_{nf}
\]

(7)

Boundary conditions are

\[
s = \{u = a(x + y), \quad v = a(x + y), \quad w = 0, T = T_w, \quad at \quad z = 0, \quad u = 0, v = 0, w = -v_0, \quad T = T_0, \quad at \quad z = H.\}
\]

(8)

Similarity transformations suggested as follow for Eqs. (2)-(8) to acquire the non-dimensional form

\[
u = a(x + y)F'(\eta), \quad v = a(x + y)G'(\eta), \quad w = -aH(F(\eta) + G(\eta)), \quad \eta = \frac{z}{H}, \quad \theta = \frac{T - T_\infty}{T_w - T_\infty},
\]

By inserting the aforementioned nonlinear variables into the subsequent system of PDEs, we arrive at the following

\[
P^{iv} - \phi_1Re[(F' - G')F'' - (F + G)G''] + \frac{M^2\phi_2}{\left((1 + \beta_\psi)^2 + \beta_\psi^2\right)}[\beta_\psi F' - (1 + \beta_\psi)F'] = 0,
\]

(9)

\[
G^{iv} - \phi_1Re[(F' - G')G'' - (F + G)G''] - \frac{M^2\phi_2}{\left((1 + \beta_\psi)^2 + \beta_\psi^2\right)}[\beta_\psi F' + (1 + \beta_\psi)F'] = 0,
\]

(10)

\[
\left(1 + \frac{4}{3N_r}\right)\theta'' + \frac{K_f}{K_{nf}}\phi_3 Pr \theta'(F + G) + \frac{K_f}{K_{nf}}\frac{\mu_f}{\mu_s}\left(1 + \beta_\psi\right)\frac{1}{(1 + \beta_\psi)^2 + \beta_\psi^2}\phi_3 Pr E\frac{M^2}{2}(F'^2 + G'^2) - \frac{K_f}{K_{nf}}Y Pr \theta = 0,
\]

(11)

Boundary conditions for the transformed problem are

\[
(F'(0) = 1, G'(0) = 1, F(0) + G(0) = 0, \theta(0) = 1, \quad F'(1) = 0, G'(1) = 0, F(1) + G(1) = \lambda, \theta(1) = 0.
\]

(12)

In above \(\phi_1's\) \((i = 1, 2, 3)\) are defined as

\[
\phi_1 = (1 - \phi)^{\frac{5}{2}}(1 - \phi - \frac{\rho_s}{\rho_f}), \quad \phi_2 = (1 - \phi)^{\frac{5}{2}}(1 - \phi) - \frac{3(\sigma - 1)\phi}{\sigma + 2 - (\sigma - 1)\phi})\phi_3 = 1 - \phi + \phi\frac{(\rho_c)\phi}{(\rho_f)\phi}.
\]

(13)

where \((Re)\) the Reynolds number, \((M)\) Hartmann number, \((A)\) the dimensionless injection parameter, \((V_0)\) the constant injection velocity at upper wall of the channel, \((Pr)\) the Prandtl number, \((Ec)\) the “Eckert number”, \((Nr)\) the “radiation parameter”, and \((\gamma)\) the heat source/sink parameter. These dimensionless parameters appeared in Equations (9)-(13) and obey the following mathematical relation

\[
Nr = \frac{k_{nf}k^*}{4\sigma T_\infty^7}, \quad Pr = \frac{\mu_f(c_p)_f}{k_f}, \quad Y = \frac{Q}{\mu_f(c_p)_f}, \quad M^2 = \frac{\sigma_f\beta_\psi^2H^2}{\rho_f a}, \quad \frac{Re}{Re} = \frac{AH^2}{v_f}, \quad \frac{\lambda}{AH}, \frac{E}{E} = \frac{a(x + y)}{(c_p)_f}.
\]

Dimensionless form of the physical quantities i.e., “skin friction coefficients” \((C_{fx}, C_{gy})\) and the “Nusselt number” \((Nu)\) are given as

\[
C_{fx}Re^{1/2} = \frac{1}{(1 - \phi)^{\frac{2}{5}}} F''(0), \quad C_{gy}Re^{1/2} = \frac{1}{(1 - \phi)^{\frac{2}{5}}} G''(0), \quad Re^{-1/2}Nu = -\frac{K_{nf}}{K_f} \theta'(0).
\]

(14)
3. Methodology and Solution Procedure

In this part, the least-squares method is used to arrive at a numerical solution for the non-dimensional problem (9)–(12). This has the following six steps.

**Step 1:** First assume Equations (9)–(11):

\[
F^{iv} - \phi_1 Re[(F' - G')F'' - (F + G)F'''] + \frac{M^2\phi_2}{(1 + \beta \beta_e)^2 + \beta_e^2} [\beta_e G' - (1 + \beta \beta_e)F'] = 0,
\]

\[
G^{iv} - \phi_1 Re[(F' - G')G'' - (F + G)G'''] - \frac{M^2\phi_2}{(1 + \beta \beta_e)^2 + \beta_e^2} [\beta_e F' + (1 + \beta \beta_e)G'] = 0,
\]

\[
\frac{K_{nf}}{K_f} \left(1 + \frac{4}{3N_R}\right) \theta'' + \phi_3 Pr \theta'(F + G) + \frac{\mu_f/\mu_{nf}}{(1 + \beta \beta_e)^2 + \beta_e^2} \phi_2 Pr E_M^2 (F'^2 + G'^2) - Y Pr \theta = 0,
\]

**Step 2:** To get approximate system (15–17) solutions, the least-squares method proposes the following trial solutions:

\[
\tilde{F}(\eta) = \tilde{p}_0 + \tilde{p}_1 \eta + \tilde{p}_2 \eta^2 + \cdots + \tilde{p}_N \eta^N = \sum_{k=0}^{N} \tilde{p}_k \eta^k,
\]

\[
\tilde{G}(\eta) = \tilde{p}_0 + \tilde{p}_1 \eta + \tilde{p}_2 \eta^2 + \cdots + \tilde{p}_N \eta^N = \sum_{k=0}^{N} \tilde{p}_k \eta^k,
\]

\[
\tilde{\theta}(\eta) = \tilde{p}_0 + \tilde{p}_1 \eta + \tilde{p}_2 \eta^2 + \cdots + \tilde{p}_N \eta^N = \sum_{k=0}^{N} \tilde{p}_k \eta^k,
\]

**Step 3:** Trial solutions (18–20) must fulfil the boundary criteria in this technique (12).

**Step 4:** Above system of ODEs (15–17) reduces to a set of nonlinear algebraic equations using reduces trial solutions (21–23)

**Step 5:** To solve problem (9)–(12), we must construct a system of algebraic equations that must match the number of unknowns. The least-square approach suggests the following concept to create the system of equations

\[
E_F = \int R_F(\eta) \omega_k^F d\eta = 0, k = 0,1,2,\ldots,N,
\]

\[
E_G = \int R_G(\eta) \omega_k^G d\eta = 0, k = 0,1,2,\ldots,N,
\]
\[ E_\theta = \int R_\theta(\eta) \omega_k^\theta d\eta = 0, k = 0, 1, 2, \ldots, N. \]

In the above expressions (\(\omega_k^F, \omega_k^G, \) and \(\omega_k^\theta\)) indicate the “weight functions” of \(F(\eta), G(\eta),\) and \(\theta(\eta)\) respectively. Least square method suggests the following relations to find the weight functions

\[ \omega_k^F = \frac{\partial R_x}{\partial p_n}, \omega_k^G = \frac{\partial R_y}{\partial p_n}, \omega_k^\theta = \frac{\partial R_\theta}{\partial p_n}. \]

**Step 6:** Finally, we get “p’s” after computing the system of equation attained in preceding step. Injecting all “p’s” into trial solutions (21)–(23) returns the solution of (9)–(12).

**4. Results and Discussion**

To perceive the physical features of the modelled problem, numerical solution is assessed with the help of least squares method and further parametric graphs of dimensionless velocity and temperature are plotted. Two types of nanofluid (\(Cu, Ag\)) are used and blood is used as base fluid. Figures 2–10 are plotted the velocity behaviour (\(\alpha\) and \(y - \) components) and temperature with increasing values of parameters which is deliberated in the previous section. In order to evaluate the consequences on the boundary of the “velocity” and “temperature” of the problem, Figures 11–13 are premeditated.

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**Figure 2.** Behaviour of (a) \(F'(\eta)\) and (b) \(G'(\eta)\) as varying \(M\) when \(\phi = 0.1, \beta_1 = 1.5, \beta_c = 0.5, \lambda = 2, Re = 5.\)
Figure 3. Behaviour of (a) $F' (\eta)$ and (b) $G' (\eta)$ as varying $\beta_i$ when $\phi = 0.1, Re = 5, \beta_e = 0.5, \lambda = 2, M = 5$.

Figure 4. Behaviour of (a) $F' (\eta)$ and (b) $G' (\eta)$ as varying $\beta_e$ when $\phi = 0.1, Re = 5, \beta_i = 1.5, \lambda = 2, M = 5$. 
Figure 5. Behaviour of (a) $F'(\eta)$ and (b) $G'(\eta)$ as varying $\lambda$ when $\phi = 0.1$, $Re = 5$, $\beta_l = 1.5$, $\beta_e = 2$, $M = 5$.

Figure 6. Behaviour of (a) $F'(\eta)$ and (b) $G'(\eta)$ as varying $\phi$ when $\lambda = 2$, $Re = 5$, $\beta_l = 1.5$, $\beta_e = 2$, $M = 5$. 
Figure 7. Behaviour of $\theta(\eta)$ as varying $\phi$ when $M = 5$, $Ec = 2$, $Nr = 0.5$, $Pr = 3.9$, $Y = 0.8$, $\beta_i = \beta_c = 0.7$.

Figure 8. Behaviour of $\theta(\eta)$ against $Y$ when $M = 5$, $Ec = 2$, $Nr = 0.5$, $\phi = 0.1$, $Pr = 3.9$, $\beta_i = \beta_c = 0.7$. 

Figure 9. Behaviour of $\theta(\eta)$ against $Pr$ when $M = 5, Ec = 2, Nr = 0.5, \phi = 0.1, \gamma = 0.8, \beta_i = \beta_e = 0.7$.

Figure 10. Behaviour of $\theta(\eta)$ against $Ec$ when $M = 5, Nr = 0.5, \phi = 0.1, Pr = 3.9, \gamma = 0.8, \beta_i = \beta_e = 0.7$. 
Figure 11. Behaviour of $\text{Re}^{1/2}C_{fx}$ as varying (a) $M$ and $\text{Re}$ (b) $\phi$ and $\lambda$ (c) $\beta_i$ and $\beta_e$.

Figure 12. Behaviour of $\text{Re}^{1/2}C_{gx}$ as varying (a) $M$ and $\text{Re}$ (b) $\phi$ and $\lambda$ (c) $\beta_i$ and $\beta_e$. 
Figure 13. Behaviour of \(Re^{-\frac{1}{2}}Nu\) as varying (a) \(\beta_t\) and \(\beta_e\) (b) Pr and M (c) \(N_r\) and \(Y\) (d) \(E_c\) and \(\phi\).

In Figure 2a, the effects of \(x\) — component of velocity with increasing values of Hartman number is presented. Figure 2a is strategized to clarify the performance of velocity, which is diminishing with the growing standards of Hartman number. This parameter seems in the modelled problem to be due to the magnetic field implementation. Increases in the strength of magnetic field oppose the flow velocity of a liquid; thus, velocity is decreasing between \(0.4 \leq \eta \geq 1\). Hartmann number is treated has a retardation role regarding particles. Physically, Lorentz force appeared as negative force in dimensionless momentum equation. Direction of flow and applied magnetic field is noticed as opposite. Hence, flow in directions of horizontal and vertical is slow down against higher values of Hartmann number. Further, opposite behaviour can be seen for \(0 \leq \eta > 0.4\). Similar performance of \(y\) — component of velocity for increasing values of Hartman number can be seen in Figure 2b. With the help of a literature study, it is concluded that “Ion” slip parameter \((\beta_t)\) is the “product” between ion cyclotron frequency and ion collision time. The attitude of velocity components with varied values of \((\beta_t)\) are demonstrated in Figure 3a. It is clear from Figure 3a that \(x\) — components \((F'(\eta))\) of velocities upsurges when the “ion” slip parameter is enlarged because the “magnetic” field has reverse attitude due to
ion slip. On the other hand, opposite performance of \( y \) \( – \) component of velocity is offered in Figure 3b. The impact of Hall parameter \( (\beta_i) \) on \( f'(\eta) \) and \( g'(\eta) \) are offered in Figure 4a,b. It should be mentioned that \( x \) \( – \) and \( y \) \( – \) components, \( F'(\eta) \) and \( G'(\eta) \), of velocities have grown \((0.4 \leq \eta \geq 1)\) trend when Hall parameter \( (\beta_i) \) is more significant than before. Increasing values of slip parameter depreciate the Magnetic field and Lorentz force. It is revealed that Hall parameter is appeared due to presence of generalized Ohm’s law in current analysis. Generalized Ohm’s law predicts that direct propositional relation occurs among Lorentz force and Hall number. So, an enhancement in Lorentz force reveals that inclination in Hall number. Higher Lorentz force behaves like a frictional force against a flow of particles. Frictional force becomes the reason for slowdown in flow. On the other hand, the reverse attitude of velocity can be perceived for \( 0 \leq \eta > 0.4 \). In the next Figure 5, both velocity components present the increasing attitude with the growing values of the injection parameter \( (\lambda) \). The increasing attitude of velocity \( (F'(\eta), G'(\eta)) \) for \( 0.4 \leq \eta \geq 1 \) with growing values of solid volume fraction \( (\phi) \) can be observed in Figure 6a,b. Figure 7 depicts the behaviour of dimensionless temperature with varying values of the radiation parameter and explains how it behaves. The temperature of the system is rising as a result of radiation processes, which are transferring energy to the fluid’s particle particles. As a result, the temperature of the system rises in direct proportion to the increase in radiation parameters. Decreasing temperature behaviour for increasing parameter values can be seen in Figure 8, and the reverse attitude of the temperature can be seen for the Prandtl number (see Figure 8). Figure 10 is demonstrated to investigate the properties of “Eckert” number \( (Ec) \) on the dimensionless temperature \( \theta(\eta) \). This graph 10 illustrates that as the Eckert number \( (Ec) \) becomes more important than previously, the “temperature” increases. Since Eckert number \( (Ec) \) is the coefficient of viscous dissipation, an increase in Eckert number correlates to an increase in heat dissipated owing to resistive force. Eckert number is formulated due to viscous dissipation in energy equation (dimensionless equation). Rate of work done (viscous dissipation) is increased versus higher values of Eckert number. This means that temperature into fluid particles is enhanced when Eckert number is increased.

Figure 11a–c are strategized to clarify the performance of \( x \) \( – \) component of velocity on the boundaries of the problem. It is obvious that when the parameters are increased, the system’s velocity increases. \( (Re, \lambda, \beta_i) \). Furthermore, similar behaviour can be observed for \( y \) \( – \)component of velocity with various values of \( (Re) \) and \( (\lambda) \) (see Figure 12a,b). On the other hand, \( (\beta_i) \) has the opposite performance for \( x \) and \( y \) \( – \)velocity components. In order to make an observation about the temperature at the problem’s borders, the Nusselt number’s graphical behaviour is shown in Figure 13a–d. It is detected that temperature reductions with the growth of different parameters except \( (\beta_i) \), i.e., the temperature are growing with the increasing values of \( (\beta_i) \).

The numerical values for nano-particles which are utilized during the numerical simulation are explained in the Table 1. In order prove the efficiency and compatibility of the numerical algorithm with the defined problem we have presented the residual in Table 2. There are also included comparative study can be seen in Tables 3 and 4.

**Table 1.** Thermophysical properties (TP) of base fluids and nano-particles [1,31].

| TP         | \( k \text{(W/mK)} \) | \( \rho \text{(kg/m}^3\) | \( c_p \text{(J/kgK)} \) | \( \Sigma \) |
|------------|------------------------|------------------------|------------------------|---------|
| Water      | 0.613                  | 997.1                  | 4179                   | 1.19    |
| Ag         | 429                    | 10,500                 | 235                    | \( 6.63 \times 10^4 \) |
| Cu         | 400                    | 8933                   | 385                    | \( 5.96 \times 10^4 \) |
Table 2. Analysis of the square residual error of least square methods with various degree of approximation.

| $M$ | $\epsilon_F$ | $\epsilon_E$ | $\epsilon_G$ |
|-----|---------------|---------------|---------------|
| 5   | $1.1337 \times 10^{-1}$ | $2.3403 \times 10^{-6}$ | $2.3140 \times 10^{-6}$ |
| 8   | $8.8110 \times 10^{-19}$ | $1.5896 \times 10^{-17}$ | $3.7655 \times 10^{-13}$ |
| 12  | $1.0671 \times 10^{-28}$ | $1.2267 \times 10^{-26}$ | $2.7063 \times 10^{-22}$ |
| 16  | $8.0748 \times 10^{-38}$ | $1.4473 \times 10^{-36}$ | $7.1356 \times 10^{-29}$ |
| 21  | $2.4514 \times 10^{-49}$ | $2.7840 \times 10^{-48}$ | $3.2630 \times 10^{-38}$ |

Table 3. Comparison of the obtained solutions for $F''$ and $G''$ with the existing results [30] when $N_R = \infty, M = \phi = Ec = 0$ and $Pr = 0.5$.

| $n$ | $\lambda$ | Obtained $F''$ [30] | obtained $G''$ [30] |
|-----|-----------|---------------------|---------------------|
| 1   | 0.5       | 1.22391             | 0.61245             |
| 1   |           | 1.41540             | 0.61237             |
| 3   | 0.5       | 1.98814             | 0.99357             |
| 1   | 0.5       | 2.29704             | 2.29719             |

Table 4. Comparison of the obtained solutions for $-\theta'(0)$ with the existing results [30] when $N_R = \infty, M = \phi = Ec = 0$ and $Pr = 0.5$.

| $n$ | $Pr$ | $\lambda$ | $-\theta'(0)$ [30] |
|-----|------|-----------|--------------------|
| 1   | 0.7  | 0.5       | 0.97210            |
| 1   |      | 1.12227   | 1.12241            |
| 1   | 0.5  | 1.22465   | 1.22475            |
| 1   |      | 1.42426   | 1.42421            |
| 7   | 0.5  | 3.76267   | 3.76272            |
| 7   |      | 4.34474   | 4.34482            |
| 3   | 0.7  | 0.5       | 1.58257            |
| 1   |      | 1.82738   | 1.82744            |
| 1   | 0.5  | 1.98937   | 1.98942            |
| 1   |      | 2.29721   | 2.29719            |

5. Conclusions

Three-dimensional MHD Casson nano-fluid with porous and stretching boundaries have been investigated. The prevailing system of nonlinear PDEs converted into simplified ODEs utilizing apposite similarity variables. Furthermore, for the purpose of making physical observations of the suggested model, a numerical scheme (Least-square method) is employed, and the mean of graphical figures explains numerical solution with parametric effects. Hence, key results with physical justifications are specified below:

- Flow and thermal energy into particles is inclined for $Ag$ rather than flow and thermal energy into particles for case of $Cu$;
- Reduction into flow analysis is noticed versus change in Hartmann and volume fraction numbers;
- Thermal energy boosts for variation in Eckert number while flow is declined when Hall number is boosted;
- Rate of heat flux is decayed versus change in Prandtl, ion slip, Eckert and thermal radiation numbers whereas rate of heat flux is boosted against higher values of Reynolds and ion slip numbers;
- Error analysis is computed within help of least square approach.

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