CONTROL OF FUSION BY ABELIAN SUBGROUPS OF THE HYPERFOCAL SUBGROUP

ELLEN HENKE AND JUN LIAO

Abstract. We prove that an isomorphism between saturated fusion systems over the same finite $p$-group is detected on the elementary abelian subgroups of the hyperfocal subgroup if $p$ is odd, and on the abelian subgroups of the hyperfocal subgroup of exponent at most 4 if $p = 2$. For odd $p$, this has implications for mod $p$ group cohomology.

1. Introduction

In 1971, Quillen [18] published two articles relating properties of the mod $p$ cohomology ring of a group $G$ to the elementary abelian $p$-subgroups of $G$. The results hold for any prime $p$ and any group $G$ which is a compact Lie group (e.g. a finite group). Quillen studied in particular varieties of mod $p$ cohomology rings and proved a stratification theorem stating that the variety of the mod $p$ cohomology ring of $G$ can be broken up into pieces corresponding to the $G$-conjugacy classes of elementary abelian $p$-subgroups of $G$. Therefore, it is of interest to study conjugacy relations between elementary abelian subgroups.

From now on we assume that $G$ is finite and $H$ is a subgroup of $G$ of index prime to $p$. For any two subgroups $A$ and $B$ of $G$, we write $\text{Hom}_G(A, B)$ for the set of group homomorphisms from $A$ to $B$ that are obtained via conjugation by an element of $G$. As a consequence of Quillen’s stratification theorem, $H$ controls fusion of elementary abelian subgroups in $G$, if the inclusion map from $H$ to $G$ induces an isomorphism between the varieties of the mod $p$ cohomology rings of $H$ and $G$. Here we say that the subgroup $H$ controls fusion of elementary abelian subgroups in $G$ if $\text{Hom}_H(A, B) = \text{Hom}_G(A, B)$ for all elementary abelian subgroups $A$ and $B$ of $H$. Similarly we say that $H$ controls $p$-fusion in $G$ if $\text{Hom}_H(A, B) = \text{Hom}_G(A, B)$ for all $p$-subgroups $A$ and $B$ of $H$. By the Cartan–Eilenberg stable elements formula [9, XII.10.1], the inclusion map from $H$ to $G$ induces an isomorphism in mod $p$ group cohomology if $H$ controls fusion in $G$. Together with Quillen’s fundamental results, this motivates the study of connections between control of fusion of elementary abelian subgroups and control of $p$-fusion.

If $H = S$ is a Sylow $p$-subgroup of $G$ and $p$ is odd, Quillen [17] proved as a first illustration of his theory that $G$ is nilpotent if the inclusion map from $S$ to $G$ induces an isomorphism between the corresponding varieties. We recall that, by a classical theorem of Frobenius, $G$ is nilpotent if and only if $S$ controls fusion in $G$. So Quillen showed that $S$ controls fusion in $G$ if $S$ controls fusion of elementary abelian subgroups. Variations of this theorem were proved in [12, 7, 10, 13, 8, 2], but all maintaining the hypothesis that $H = S$ is a Sylow $p$-subgroup. Only relatively recently, Benson, Grodal and the first author of this paper...
proved a result that holds more generally for any subgroup $H$ of index prime to $p$; see \[5\]. More precisely, it is shown that $H$ controls fusion in $G$ (and thus the inclusion map from $H$ to $G$ induces an isomorphism in mod $p$ group cohomology), if the inclusion map induces an isomorphism between the corresponding varieties, i.e. if $H$ controls fusion of elementary abelian subgroups of $G$. This is obtained as a consequence of a theorem that is stated and proved for saturated fusion systems; see \[5\], Theorem B]. In this short note, we point out that actually a slightly stronger version of this theorem holds. We refer the reader to \[1\], Part I for an introduction to fusion systems.

**Theorem A** (Small exponent abelian subgroups of the hyperfocal subgroup control fusion). Let $\mathcal{G} \subseteq \mathcal{F}$ be two saturated fusion systems over the same finite $p$-group $S$. Suppose that $\text{Hom}_G(A, B) = \text{Hom}_F(A, B)$ for all $A, B \leq \mathfrak{hnp}(\mathcal{F})$ with $A, B$ elementary abelian if $p$ is odd, and abelian of exponent at most 4 if $p = 2$. Then $\mathcal{G} = \mathcal{F}$.

If one replaces $\mathfrak{hnp}(\mathcal{F})$ by $S$, then the above theorem coincides with \[5\], Theorem B]. We recall that the hyperfocal subgroup $\mathfrak{hnp}(\mathcal{F})$ is the subgroup of $S$ generated by all elements of the form $x^{-1}\varphi(x)$ where $x \in Q$ and $\varphi \in \text{O}^p(\text{Aut}_2(Q))$ for some subgroup $Q$ of $S$. If $\mathcal{F} = \mathcal{F}_S(G)$ is the fusion system of a finite group $G$ with Sylow $p$-subgroup $S$, then Puig’s hyperfocal subgroup theorem \[16\], §1.1] states that $\mathfrak{hnp}(\mathcal{F}) = \text{O}^p(G) \cap S$. In the situation of Theorem A, Quillen’s example $Q_8 \leq Q_8 : C_3$ shows that it is indeed not enough to consider only elementary abelian subgroups for $p = 2$.

A fusion system $\mathcal{F}$ on $S$ is called nilpotent if $\mathcal{F} = \mathcal{F}_S(S)$. Restricting attention to subgroups of the hyperfocal subgroup is motivated by a theorem of the second author of this paper together with Zhang, which characterizes $p$-nilpotency of a saturated fusion system $\mathcal{F}$ by the fusion on certain subgroups of the hyperfocal subgroup of $\mathcal{F}$; see \[14\]. Another motivation comes from work of Ballester-Bolinches, Ezquerro, Su and Wang \[2\] showing that, in certain special cases, fusion is detected on the subgroups of the focal subgroup of $\mathcal{F}$ which are cyclic of order $p$ or 4. We show here that in Theorem A and C of \[2\], the focal subgroup can actually be replaced by the hyperfocal subgroup. More precisely, we prove the following theorem which gives in particular a new characterization of nilpotent fusion systems:

**Theorem B.** Let $\mathcal{F}$ be a saturated fusion system over a finite $p$-group $S$, and let $\mathcal{G} = N_{\mathcal{F}}(S)$ or $\mathcal{G} = \mathcal{F}_S(S)$. Suppose that $\text{Hom}_G(A, B) = \text{Hom}_F(A, B)$ for all $A, B \leq \mathfrak{hnp}(\mathcal{F})$ which are cyclic subgroups of order $p$ or 4. Then $\mathcal{G} = \mathcal{F}$.

We remark that, in general, it is not the case that the subgroup $H$ controls fusion in $G$ if it controls fusion on cyclic subgroups of order $p$ for odd $p$, or on subgroups of order at most 4 for $p = 2$. This is not even the case if $G$ has a normal Sylow $p$-subgroup as the following example shows: Let $n$ be an integer such that $n \geq 2$ and $p$ does not divide $n$. Let $S$ be the field of order $p^n$, so that $S$ under addition forms in particular an elementary abelian group of order $p^n$. Note that every non-zero element of $S$ induces a group automorphism of $S$ via multiplication. Let $D$ be the group of all these automorphisms. Then $D$ is a subgroup of $\text{Aut}(S)$ of order $p^n - 1$ acting freely and transitively on the non-trivial elements of $S$. Let $\sigma$ be the Frobenius automorphism of the field $S$. Then $\sigma$ has order $n$ and is also a group automorphism of $S$. Moreover, $\sigma$ normalizes $D$, as conjugation by $\sigma$ takes every element of $D$ to its $p$th power. Hence, $\hat{D} = D \times \langle \sigma \rangle$ is a group of order $(p^n - 1)n$. Since $p$ does not divide $n$, it follows that $S$ is a normal Sylow $p$-subgroup of $G := S \rtimes \hat{D}$. Moreover, $H := S \rtimes D$
is a subgroup of $G$ of index prime to $p$. Note also that $S = [S, D] = \mathfrak{up}(\mathcal{F}_S(G))$. Let $\mathcal{V}$ be the set of subgroups of $S$ of order $p$. Then $\mathcal{V}$ has $\frac{p^n-1}{p-1}$ elements. As $D$ acts freely and transitively on the non-trivial elements of $S$, it follows that $D$ acts also transitively on $\mathcal{V}$, and that $C_D(A) = 1$ for all $A \in \mathcal{V}$. Thus $|\text{Aut}_D(A)| = |N_D(A)| = \frac{|D|}{|C_D(A)|} = p - 1$ for every $A \in \mathcal{V}$. As any two elements of $\mathcal{V}$ are conjugate under $D$, it follows $|\text{Hom}_D(A, B)| = p - 1$ for all $A, B \in \mathcal{V}$. Thus, $\text{Hom}_D(A, B) = \text{Hom}_D(A, B)$ is the set $\text{Inj}(A, B)$ of injective group homomorphism from $A$ to $B$. As $\text{Hom}_D(A, B) \subseteq \text{Hom}_G(A, B) \subseteq \text{Inj}(A, B)$, this implies $\text{Hom}_D(A, B) = \text{Hom}_G(A, B)$ for all $A, B \in \mathcal{V}$. So $H$ controls fusion in $G$ of the cyclic subgroups of order $p$ (and thus for $p = 2$ also of the cyclic subgroups of order at most 4). However, as $D \neq \hat{D}$, the subgroup $H$ does not control fusion in $G$.

We conclude by stating a version of Theorem A in terms of varieties of cohomology rings. We continue to assume that $G$ is a finite group and we fix moreover an algebraically closed field $\Omega$ of prime characteristic $p$. We either set $k = \Omega$ or $k = \mathbb{F}_p$. Moreover, set $H^*(G) := H^*(G, k)$ and define the variety $V_G$ to be the variety $\text{Hom}_k(H^*(G), \Omega)$ of $k$-algebra homomorphisms from $H^*(G)$ to $\Omega$; see Remark 3.2 for alternative definitions of $V_G$. Then every $k$-algebra homomorphism $\alpha : H^*(G) \to H^*(H)$ induces a map of varieties $\alpha^* : V_H \to V_G$ by sending any homomorphism $\beta \in V_H = \text{Hom}_k(H^*(H), \Omega)$ to $\beta \circ \alpha \in V_G = \text{Hom}_k(H^*(G), \Omega)$. For an arbitrary subgroup $H$ of $G$, we write $\text{res}_{G, H} : H^*(G) \to H^*(H)$ for the map induced by the inclusion map $H \to G$, and hence $\text{res}_{G, H}^* : V_H \to V_G$ for the corresponding map of varieties.

If $H$ is a subgroup of $G$ containing a Sylow $p$-subgroup $S$ of $G$, then we have the inclusion maps $S \cap O^p(G) \hookrightarrow H \hookrightarrow G$ which induce the following maps of varieties:

$$V_{S \cap O^p(G)} \xrightarrow{\text{res}_{H, S \cap O^p(G)}^*} V_H \xrightarrow{\text{res}_{G, H}^*} V_G$$

So in particular, we can consider the restriction of the map $\text{res}_{G, H}^* : V_H \to V_G$ to the subvariety $\text{res}_{H, S \cap O^p(G)}^* V_{S \cap O^p(G)}$ of $V_H$. If $p$ is an odd prime and $H$ is a subgroup of $G$ of index prime to $p$, then the results in [5] say basically that $H$ controls fusion in $G$ if $\text{res}_{G, H}^* : V_H \to V_G$ is an isomorphism of varieties. Theorem A implies a slightly stronger statement which is stated in the next theorem. Notice that a subgroup $H$ of $G$ has index prime to $p$ if and only if $H$ contains a Sylow $p$-subgroup of $G$.

**Theorem C.** Let $G$ be a finite group, let $p$ be an odd prime, and let $H$ be a subgroup of $G$ containing a Sylow $p$-subgroup $S$ of $G$. Suppose the restriction of the map $\text{res}_{G, H}^*$ to $\text{res}_{H, S \cap O^p(G)}^* V_{S \cap O^p(G)}$ is injective. Then $H$ controls fusion in $G$ and the restriction map $\text{res}_{G, H} : H^*(G) \to H^*(H)$ is an isomorphism.

Note that Theorem C says in particular that the map $\text{res}_{G, H}^* : V_H \to V_G$ is an isomorphism of varieties if its restriction to $\text{res}_{H, S \cap O^p(G)}^* V_{S \cap O^p(G)}$ is injective. One sees easily that the converse of Theorem C holds as well: If $\text{res}_{G, H} : H^*(G) \to H^*(H)$ is an isomorphism then $\text{res}_{G, H}^* : V_H \to V_G$ is an isomorphism. In particular, the restriction of $\text{res}_{G, H}^*$ to $\text{res}_{H, S \cap O^p(G)}^* V_{S \cap O^p(G)}$ is injective.

We remark also that a theorem analogous to Theorem C can be proved for saturated fusion systems rather than for groups. For more details, we refer the reader to Remark 3.3.

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The proof of Theorem \( \mathbf{A} \) is very similar to the proof of Theorem \( \mathbf{B} \) in \[5\]. We need the following variation of \[5\] Theorem 2.1.

**Theorem 2.1.** Let \( P \) be a finite \( p \)-group and let \( G \) be a subgroup of \( \text{Aut}(P) \) containing the group \( \text{Inn}(P) \) of inner automorphism. Then there exists a \( G \)-invariant subgroup \( D \) of \( [P,O^p(G)] \), of exponent \( p \) if \( p \) is odd and exponent at most 4 if \( p = 2 \), such that \( [D,P] \leq Z(D) \), and such that every non-trivial \( p' \)-automorphism in \( G \) restricts to a non-trivial \( p' \)-automorphism of \( D \). Furthermore, for any such \( D \) and any maximal (with respect to inclusion) abelian subgroup \( A \) of \( D \) it follows that \( A \leq P \) and \( C_G(A) \) is a \( p \)-group.

**Proof.** By \[5\] Theorem 2.1, there exists a characteristic subgroup \( D_1 \) of \( P \), of exponent \( p \) if \( p \) is odd and exponent at most 4 if \( p = 2 \), such that \( [D_1,P] \leq Z(D_1) \), and such that every non-trivial \( p' \)-automorphism of \( P \) restricts to a non-trivial \( p' \)-automorphism of \( D_1 \). Set \( D := [D_1,O^p(G)] \). As \( D_1 \) is \( G \)-invariant and as \( O^p(G) \) is normal in \( G \), the subgroup \( D \) is \( G \)-invariant. In particular, as \( \text{Inn}(P) \leq G \) by assumption, we have \( [D,P] \leq D \). Using \( [D_1,P] \leq Z(D_1) \) we obtain thus \( [D,P] \leq [D_1,P] \cap D \leq Z(D_1) \cap D \leq Z(D) \). If \( \varphi \) is a \( p' \)-automorphism of \( P \) with \( \varphi|_D = \text{Id}_D \) then \( [D,\varphi] = 1 \) and \( [D_1,\varphi] \leq [D_1,O^p(G)] = D \). Thus, by \[11\] Theorem 5.3.6, we have \( [D_1,\varphi] = [D_1,\varphi,\varphi] \leq [D,\varphi] = 1 \) and \( \varphi|_{D_1} = \text{Id}_{D_1} \). Because of the way \( D_1 \) was chosen, this implies that \( \varphi = \text{Id}_P \). So we have shown that every non-trivial \( p' \)-automorphism in \( G \) restricts to a non-trivial automorphism of \( D \).

For the last part let \( A \) be a maximal subgroup of \( D \) with respect to inclusion. Then \( [A,P] \leq Z(D) \leq A \) and thus \( A \leq P \). Furthermore, if \( B \leq C_G(A) \) is a \( p' \)-subgroup, then \( A \times B \) acts on \( D \). Since \( A \) is maximal abelian, it follows \( C_D(A) = A \leq C_D(B) \). Thompson’s \( A \times B \)-lemma \[11\] Theorem 5.3.4 now says that \( [D,B] = 1 \) and so \( B = 1 \). Since \( B \) was arbitrary, it follows that \( C_G(A) \) is a \( p \)-group.

We need the following crucial lemma, which is \[5\] Main Lemma 2.4.

**Lemma 2.2.** Let \( G \subseteq F \) be two saturated fusion systems on the same finite \( p \)-group \( S \), and \( P \leq S \) an \( F \)-centric and fully \( F \)-normalized subgroup, with \( \text{Aut}_F(R) = \text{Aut}_G(R) \) for every \( P < R \leq N_S(P) \). Suppose that there exists a subgroup \( Q \leq P \) with \( \text{Hom}_F(Q,S) = \text{Hom}_G(Q,S) \). Then \( \text{Aut}_F(P) = \langle \text{Aut}_G(P), C_{\text{Aut}_F(P)}(Q) \rangle \).

**Proof of Theorem \( \mathbf{A} \).** By Alperin’s fusion theorem \[11\] Theorem I.3.6, \( F \) is generated by \( F \)-automorphisms of fully \( F \)-normalized \( F \)-centric subgroups. We want to show that \( \text{Aut}_F(P) = \text{Aut}_G(P) \) for all \( P \leq S \). By induction on \( |S:P| \), we can assume that \( \text{Aut}_F(R) = \text{Aut}_G(R) \) for all \( R \leq S \) with \( |R| > |P| \). Furthermore, by Alperin’s fusion theorem, we can choose \( P \) to be fully \( F \)-normalized and \( F \)-centric. By Theorem 2.1 we can pick an \( \text{Aut}_F(P) \)-invariant subgroup \( D \) of \( [P,O^p(\text{Aut}_F(P))] \), of exponent \( p \) if \( p \) is odd and of exponent at most 4 if \( p = 2 \), such that every non-trivial \( p' \)-automorphism \( \varphi \in \text{Aut}_F(P) \) restricts to a non-trivial automorphism of \( D \) and, for any maximal (with respect to inclusion) abelian subgroup \( A \) of \( D \), \( A \leq P \) and \( C_{\text{Aut}_F(P)}(A) \) is a \( p \)-group. As \( P \) is fully \( F \)-normalized, \( \text{Aut}_S(P) \) is a Sylow \( p \)-subgroup of \( \text{Aut}_F(P) \), and so if we replace \( A \) by a conjugate of \( A \) under \( \text{Aut}_F(P) \), we can arrange that \( C_{\text{Aut}_F(P)}(A) \leq \text{Aut}_S(P) \leq \text{Aut}_G(P) \). As \( D \) has exponent \( p \) if \( p \) is odd and exponent at most 4 if \( p = 2 \), we have by assumption \( \text{Hom}_F(A,S) = \text{Hom}_G(A,S) \). So by Lemma 2.2 applied with \( A \) in place of \( Q \), we obtain that \( \text{Aut}_F(P) = \langle \text{Aut}_G(P), C_{\text{Aut}_F(P)}(A) \rangle = \text{Aut}_G(P) \) as wanted. \( \square \)
Let $P$ be a set of representatives of the $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-essential subgroups. A version of the Alperin–Goldschmidt Theorem for fusion systems states that $\mathcal{F}$ is generated by the $\mathcal{F}$-automorphism groups of the elements of $\mathcal{P} \cup \{S\}$. Analyzing what is used in the proof above, one sees that we only need the following condition in Theorem A: For every $P \in \mathcal{P} \cup \{S\}$ and every abelian subgroup $A$ of the commutator subgroup $[P, O^p(\text{Aut}_P(P))]$ which is of exponent $p$ or 4, we have $\text{Hom}_\mathcal{F}(A, S) = \text{Hom}_G(A, S)$.

The proof of Theorem B is essentially the same as the one of [2, Theorem A] except that we use Theorem 2.1 instead of [5, Theorem 2.1]. Essentially, Theorem B is a consequence of the following lemma:

**Lemma 2.3.** Let $\mathcal{F}$ be a saturated fusion systems over a finite $p$-group $S$. Suppose that $\text{Hom}_\mathcal{F}(A, B) \subseteq \text{Hom}_{N_\mathcal{F}(S)}(A, B)$ for all subgroups $A, B \leq \mathfrak{hnp}(\mathcal{F})$ which are cyclic of order $p$ or 4. Then $\mathcal{F} = N_\mathcal{F}(S)$.

**Proof.** Suppose that $Q$ is an $\mathcal{F}$-essential subgroup. Then by definition, $Q$ is in particular fully normalized and thus $\text{Aut}_S(Q)$ is a Sylow $p$-subgroup of $\text{Aut}_\mathcal{F}(Q)$. By Theorem 2.1 there is an $\text{Aut}_\mathcal{F}(Q)$-invariant subgroup $D \leq [Q, O^p(\text{Aut}_\mathcal{F}(Q))] \leq Q \cap \mathfrak{hnp}(\mathcal{F})$ such that every non-trivial $p'$-element of $\text{Aut}_\mathcal{F}(Q)$ restricts to a non-trivial automorphism of $D$, and $D$ is of exponent $p$ or 4. Let $Z_i(S)$ be the $i$-th center of $S$ and $D_i = D \cap Z_i(S)$. We argue now that $D_i = \text{Aut}_\mathcal{F}(Q)$-invariant: For every $x \in D_i$ and any $\varphi \in \text{Aut}_\mathcal{F}(Q)$, $\varphi|_{D_i}$ extends by hypothesis to an element of $\text{Aut}_\mathcal{F}(S)$ which clearly normalizes $Z_i(S)$. As $\varphi$ normalizes $D_i$, it follows $\varphi(x) \in Z_i(S) \cap D = D_i$. So $D_i$ is indeed $\text{Aut}_\mathcal{F}(Q)$-invariant. Thus, for some $n \in \mathbb{N}$, the series $1 = D_0 \leq D_1 \leq \cdots \leq D_n = D$ is a normal subgroup of $\text{Aut}_\mathcal{F}(Q)$. For any $p'$-element $\varphi$ of $H$, we have $\varphi|_D = \text{Id}_D$ by [11, Theorem 5.3.2], and thus $\varphi = \text{Id}_Q$ by the choice of $D$. Therefore, the stabilizer $H$ is a $p$-group and so $H \leq O_p(\text{Aut}_\mathcal{F}(Q))$. Since $\text{Aut}_S(Q)$ stabilizes the series $D_0 \leq D_1 \leq \cdots \leq D_n = D$, it follows that $\text{Aut}_S(Q) = O_p(\text{Aut}_\mathcal{F}(Q))$, which is a contradiction as every $\mathcal{F}$-essential subgroup is centric and radical. Hence there is no $\mathcal{F}$-essential subgroup. Thus, $\mathcal{F} = N_\mathcal{F}(S)$ by Alperin’s fusion theorem [11, Theorem I.3.6]. □

**Proof of Theorem B** Lemma 2.3 $\mathcal{F} = N_\mathcal{F}(S)$. So for $\mathcal{G} = N_\mathcal{F}(S)$ the assertion follows immediately. Assume now $\mathcal{G} = \mathcal{F}_S(S)$. As $\mathcal{F} = N_\mathcal{F}(S)$, it is sufficient to show that $\text{Aut}_\mathcal{F}(S) = \text{Inn}(S)$. By Theorem 2.1 there is an $\text{Aut}_\mathcal{F}(S)$-invariant subgroup $D \leq [S, O^p(\text{Aut}_\mathcal{F}(S))] \leq S \cap \mathfrak{hnp}(\mathcal{F})$ such that every non-trivial $p'$-element of $\text{Aut}_\mathcal{F}(S)$ restricts to a non-trivial automorphism of $D$, and $D$ is of exponent $p$ or 4. Let $D_i = D \cap Z_i(S)$ and $n \in \mathbb{N}$ such that $D_n = D$. By hypothesis, every element of $\text{Aut}_\mathcal{F}(S)$ acts on every element of $D$ as conjugation by an element of $S$. Hence, $\text{Aut}_\mathcal{F}(S)$ stabilizes the series $1 = D_0 \leq D_1 \leq \cdots \leq D_n = D$ and is thus a $p$-group by [11, Theorem 5.3.2]. Since $\text{Inn}(S) \in \text{Syl}_p(\text{Aut}_\mathcal{F}(S))$, it follows $\text{Aut}_\mathcal{F}(S) = \text{Inn}(S)$ as required. □

3. **Proof of Theorem C**

Throughout, assume that $G$ is a finite group and that $\Omega$ is an algebraically closed field of prime characteristic $p$. Let $H^*(G)$ and $V_G$ be as in the introduction. Recall that, for any subgroup $H$ of $G$, we write $\text{res}_{G,H}^\ast: H^*(G) \to H^*(H)$ for the map induced by the inclusion map from $H$ to $G$, and $\text{res}_{G,H}^* : V_H \to V_G$ for the corresponding map of varieties.
For the proof of Theorem 6.3 we will need some more notation: For every elementary abelian $p$-group $A$, we set

$$V^+_A := V_A \setminus \bigcup_{A' < A} \text{res}^*_{A,A'} V_{A'}.$$ 

If $A$ is an elementary abelian subgroup of $G$, set

$$V^+_{G,A} = \text{res}^*_{G,A} V_A^+.$$ 

We start with the following elementary observation:

**Remark 3.1.** Let $A \leq K \leq G$ such that $A$ is elementary abelian. Then $\text{res}^*_{G,K} V^+_{K,A} = V^+_{G,A}$.

**Proof.** As $\text{res}^*_{G,K} \circ \text{res}^*_{K,A} = \text{res}^*_{G,A}$, we have $V^+_{G,A} = \text{res}^*_{G,A} V_A^+ = \text{res}^*_{G,K} (\text{res}^*_{K,A} V_A^+) = \text{res}^*_{G,K} V^+_{K,A}$. $\Box$

**Remark 3.2.** Write $H^{ev}(G)$ for the subring of $H^*(G)$ of elements of even degree. If $k = \mathbb{F}_p$ notice that the $k$-algebra homomorphisms from $H^*(G)$ to $\Omega$ are the same as the ring homomorphisms from $H^*(G)$ to $\Omega$. So if $k = \mathbb{F}_p$ then, upon replacing $H^*(G)$ by $H^{ev}(G)$ if $p$ is odd, the variety $V_G$ corresponds to the variety $H_G(X)(\Omega)$ studied by Quillen [18] in the special case that $X$ is a point. If $k = \Omega$, it follows from Hilbert’s Nullstellensatz that $V_G$ is homeomorphic to the maximal ideal spectrum of $H^*(G)$ via the map sending every homomorphism in $V_G$ to its kernel; see Theorem 5.4.2 and the surrounding discussion in [3]. So again upon replacing $H^*(G)$ by $H^{ev}(G)$, the variety $V_G$ as defined in this paper corresponds to the variety $V_G$ as defined by Benson [3].

It is common to study the variety of $H^{ev}(G)$ rather than the variety of $H^*(G)$, because $H^{ev}(G)$ is commutative, whereas $H^*(G)$ is only graded commutative, and texts on commutative algebra are written for strictly commutative rings. As pointed out by Benson [4, p.9], the results from commutative algebra which are needed in the theory hold accordingly for graded commutative rings. Moreover, it is pointed out that any graded commutative ring $A$ is commutative modulo its nilradical, and every element of odd degree lies in the nilradical if $p$ is odd. So writing $\mathfrak{nil}$ for the nilradical of $H^*(G)$, it follows that $H^*(G)/\mathfrak{nil}$ is isomorphic to $H^{ev}(G)/(H^{ev}(G) \cap \mathfrak{nil})$. As the nilradical $\mathfrak{nil}$ is contained in the kernel of every $k$-algebra homomorphism from $H^*(G)$ to $\Omega$, the variety $\text{Hom}_k(H^*(G),\Omega)$ is canonically homeomorphic to the variety $\text{Hom}_k(H^{ev}(G),\Omega)$.

In particular, the Quillen Stratification Theorem as stated in [18 Theorem 10.2] and [3 Theorem 5.6.3] can be proved accordingly with our definitions:

**Theorem 3.3** (Quillen’s Stratification Theorem). Let $A$ be a set of representatives of the $G$-conjugacy classes of elementary abelian subgroups of $G$. Then $V_G$ is the disjoint union

$$V_G = \coprod_{A \in A} V^+_{G,A}.$$ 

of locally closed subvarieties $V^+_{G,A}$. Moreover, for every $A \in A$, the automorphism group $\text{Aut}_G(A)$ acts freely on $V^+_A$ and the map $\text{res}^*_{G,A}$ induces a homeomorphism $V^+_A/\text{Aut}_G(A) \to V^+_{G,A}$.

The fact that $V_G = \coprod_{A \in A} V^+_{A,G}$ for any set $A$ of representatives of the $G$-conjugacy classes of the elementary abelian subgroups of $G$, will be used in our proof in the following form:
Remark 3.4. Let $A$ and $A'$ be elementary abelian subgroups of $G$. If $A$ and $A'$ are $G$-conjugate then we have $V_{G,A}^+ = V_{G,A'}^+$, and if $A$ and $A'$ are not $G$-conjugate then $V_{G,A}^+$ and $V_{G,A'}^+$ are disjoint.

Proof of Theorem C. Assume that the restriction of the map $\text{res}^*_H: V_H \to V_G$ to the subvariety $\text{res}^*_{H,S\cap O^p(G)} V_{S\cap O^p(G)}$ of $V_H$ is injective.

Step 1: Let $A$ be an elementary abelian subgroup of $S \cap O^p(G)$. We show that the map $\text{res}^*_H$ induces a bijection from $V_{H,A}^+$ to $V_{G,A}^+$. Moreover, if $A'$ is another elementary abelian subgroup of $S \cap O^p(G)$ such that $V_{G,A}^+ = V_{G,A'}^+$, then we show $V_{H,A}^+ = V_{H,A'}^+$. To see this note that, by Remark 3.1, we have that $\text{res}^*_H \circ \text{res}^*_{H,S\cap O^p(G)} V_{S\cap O^p(G)}$ is injective. By a symmetric argument, it follows that $V_{H,A'}^+$ is contained in $\text{res}^*_{H,S\cap O^p(G)} V_{S\cap O^p(G)}$. As the actions of $\text{Aut}_H(G,A)$-orbits on $V_{A,H}^+$ are the same as the $\text{Aut}_H(G)$-orbits. As the actions of $\text{Aut}_H(G,A)$ and $\text{Aut}_H(A)$ on $V_{A,H}^+$ are free, this implies that $|\text{Aut}_H(A)| = |\text{Aut}_H(A)|$. Thus, since $\text{Aut}_H(A) \subseteq \text{Aut}_G(A)$, it follows $\text{Aut}_G(A) = \text{Aut}_H(A)$.

Step 2: Let $A$ and $A'$ be two $G$-conjugate elementary abelian subgroups of $S \cap O^p(G)$. We show that $\text{Aut}_G(A) = \text{Aut}_H(A)$. By the Quillen stratification theorem Theorem 3.3, the group $\text{Aut}_G(A)$ acts freely on $V_{A,G}^+$, and the map $\text{res}^*_G$ induces a homeomorphism $V_{A,G}^+ / \text{Aut}_G(A) \to V_{G,A}^+$. In particular, the fibres of the map $\text{res}^*_G: V_{G,A}^+ \to V_{G,A}^+$ are precisely the orbits of $\text{Aut}_G(A)$ on $V_{G,A}^+$. Similarly, applying the Quillen stratification theorem with $H$ in place of $G$, we get that $\text{Aut}_H(A)$ acts freely on $V_{A,H}^+$, and the fibres of the map $\text{res}^*_H: V_{A,H}^+ \to V_{H,A}^+$ are precisely the orbits of $\text{Aut}_H(A)$ on $V_{H,A}^+$. Note that $\text{res}^*_G = \text{res}^*_H \circ \text{res}^*_H$. As the map $\text{res}^*_G: V_{H,A}^+ \to V_{G,A}^+$ is by Step 1 a bijection, it follows that the maps $\text{res}^*_G: V_{A,G}^+ \to V_{G,A}^+$ and $\text{res}^*_H: V_{A,H}^+ \to V_{H,A}^+$ have the same fibres. So the $\text{Aut}_G(A)$-orbits on $V_{G,A}^+$ are the same as the $\text{Aut}_H(A)$-orbits. As the actions of $\text{Aut}_G(A)$ and $\text{Aut}_H(A)$ on $V_{A,G}^+$ are free, this implies that $|\text{Aut}_G(A)| = |\text{Aut}_H(A)|$. Thus, since $\text{Aut}_H(A) \subseteq \text{Aut}_G(A)$, it follows $\text{Aut}_G(A) = \text{Aut}_H(A)$.

Step 4: We are now in a position to complete the proof. Let $A$ and $A'$ be elementary abelian subgroups of $S \cap O^p(G)$. We want to show that $\text{Hom}_G(A,A') = \text{Hom}_H(A,A')$ and can assume without loss of generality that $A$ and $A'$ are $G$-conjugate. Then $A$ and $A'$ are $H$-conjugate by Step 1 and thus there exists $\psi \in \text{Hom}_H(A,A')$. Let $\varphi \in \text{Hom}_G(A,A')$. Note that $\varphi = \psi \circ (\psi^{-1} \circ \varphi)$. By Puig’s hyperfocal subgroup theorem [16 §1.1], we have $S \cap O^p(G) = \text{hyp}(F_S(G))$. So using Theorem A we can conclude that $F_S(G) = F_S(H)$. Thus, by the Cartan–Eilenberg stable elements formula [9 XII.10.1], the map $\text{res}^*_G: H^*(G) \to H^*(H)$ is an isomorphism.

Remark 3.5. A version of Theorem C can also be formulated and proved for abstract saturated fusion systems rather than for groups. Let $F$ be a saturated fusion system over a finite $p$-group $S$. Assume that $k$ is an algebraically closed field of characteristic $p$. The
cohomology ring $H^*(\mathcal{F}) = H^*(\mathcal{F}, k)$ of the saturated fusion system $\mathcal{F}$ is the subring of $\mathcal{F}$-stable element in $H^*(S) = H^*(S, k)$, which is the subring of $H^*(S)$ consisting of elements $\xi \in H^*(S)$ such that $\text{res}^S_\phi(\xi) = \text{res}_\phi(\xi)$ for any $\phi \in \text{Hom}_\mathcal{F}(P, S)$ and any subgroup $P \leq S$. The ring $H^*(\mathcal{F})$ is a graded commutative ring. We write $V_\mathcal{F}$ for the maximal ideal spectrum of $H^*(\mathcal{F})$, or alternatively for the variety of $k$-algebra homomorphisms from $H^*(\mathcal{F})$ to $k$.

Let $\mathcal{G}$ be a saturated fusion subsystem of $\mathcal{F}$. Note that any $\mathcal{F}$-stable element of $H^*(S)$ is in particular $\mathcal{G}$-stable, so we can consider the inclusion map $\text{res}_{\mathcal{F}, \mathcal{G}}: H^*(\mathcal{F}) \rightarrow H^*(\mathcal{G})$ which then gives us a map $\text{res}_{\mathcal{F}, \mathcal{G}}^*: V_\mathcal{G} \rightarrow V_\mathcal{F}$ of varieties. Similarly, if $Q \leq S$, we are given a $k$-algebra homomorphism $\text{res}_{\mathcal{F}, Q}: H^*(\mathcal{F}) \rightarrow H^*(Q)$ by composing the inclusion map $H^*(\mathcal{F}) \hookrightarrow H^*(S)$ with the restriction map $\text{res}_{S,Q}: H^*(S) \rightarrow H^*(Q)$. Again, this induces a map of varieties $\text{res}_{\mathcal{F}, Q}^*: V_Q \rightarrow V_\mathcal{F}$. In particular, if $A \leq S$ is elementary abelian, one can define $V_{\mathcal{F}, A}^+ = \text{res}_{\mathcal{F}, A}^* V_A^+$. In an unpublished preprint, Markus Linckelmann [15, Theorem 1] proves a version of the Quillen stratification theorem; see also Theorem 1.3 and Remark 1.1 in [19]. Using this, one can similarly prove the following version of Theorem C for fusion systems:

Let $\mathcal{G} \subseteq \mathcal{F}$ be an inclusion of saturated fusion systems over the same finite $p$-group $S$, and $p$ an odd prime. If the restriction of the map $\text{res}_{\mathcal{F}, \mathcal{G}}^*: V_\mathcal{G} \rightarrow V_\mathcal{F}$ to $\text{res}_{\mathcal{G}, \text{hyp}(\mathcal{F})}^* V_{\text{hyp}(\mathcal{F})}$ is injective, then $\mathcal{F} = \mathcal{G}$ and in particular $H^*(\mathcal{F}) = H^*(\mathcal{G})$.

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Institute of Mathematics, University of Aberdeen, Fraser Noble Building, Aberdeen AB24 3UE, U.K.
E-mail address: ellen.henke@abdn.ac.uk

School of Mathematics and Statistics, Hubei University, Wuhan, 430062, P. R. China.
Current address: Institute of Mathematics, University of Aberdeen, Fraser Noble Building, Aberdeen AB24 3UE, U.K.
E-mail address: jliao@pku.edu.cn