Polynomials: a New Tool for Length Reduction in Binary Discrete Convolutions

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Abstract

Efficient handling of sparse data is a key challenge in Computer Science. Binary convolutions, such as polynomial multiplication or the Walsh Transform are a useful tool in many applications and are efficiently solved.

In the last decade, several problems required efficient solution of sparse binary convolutions. Both randomized and deterministic algorithms were developed for efficiently computing the sparse polynomial multiplication. The key operation in all these algorithms was length reduction. The sparse data is mapped into small vectors that preserve the convolution result. The reduction method used to-date was the modulo function since it preserves location (of the ”1” bits) up to cyclic shift.

To date there is no known efficient algorithm for computing the sparse Walsh Transform. Since the modulo function does not preserve the Walsh transform a new method for length reduction is needed. In this paper we present such a new method - polynomials. This method enables the development of an efficient algorithm for computing the binary sparse Walsh Transform. To our knowledge, this is the first such algorithm. We also show that this method allows a faster deterministic computation of the sparse polynomial multiplication than currently known in the literature.

1 Introduction

Handling sparse data is one of the grails of Computer Science, and is a challenge in many of its sub-fields, from Machine Learning and Data Bases to Combinatorics and Statistics. Even in one dimension sparse data needs to be handled, and it is a tool for solving a number of problems. An example is the point-set matching problem, where two sets of points $T, P \in \mathbb{N}^d$ consisting of $n, m$ points, respectively, are given. The goal is to determine if there is a rigid transformation...
under which all the points in $P$ are covered with points in $T$. Among the important application domains to which this problem contributes are model based object recognition [16, 7], pharmacophore identification [14], searching in music archives, [20], and more.

The point-set matching problem has been studied in the literature in many variation, not the least of which in the algorithms literature (see e.g. [18, 13, 8, 2, 3, 1, 10]). The template of many of these algorithms is as follows: First do a Dimension Reduction where the inputs $T, P$ are linearized into raw vectors $T', P'$ of size polynomial in the number of non-zero values. reduced the $d$-Dimensional point set matching problem into a 1-Dimensional point set matching problem, and then solve the one dimensional problem. This problem is also known as the Sparse Convolution Problem.

We define it formally:

**Definition 1** Let $T : \{0, ..., N - 1\} \to \{0, 1\}$ and $P : \{0, ..., M - 1\} \to \{0, 1\}$ be binary functions.

**INPUT:** Binary vectors $T$ and $P$ of length $N$ and $M$, respectively.

**OUTPUT:** All indices $i$ where, for all $j$, $j = 0, ..., M - 1$, either $T[i + j] = P[j]$ or $T[i + j] = 1$ and $P[j] = 0$. We call such an index a match.

**Intuition:** If we consider a 1 as depicting a point on the line and a 0 as having no point there, then the meaning of the problem is to find all locations where every point of $P$ matches some point of $T$.

**Example 1**

$T = 000010010110001100101011100000100$, and $P = 1000101$. There is a match in location 15 because the situation is:

$T = ... 1 1 0 0 1 0 1 0 ...$

$P = ... 1 0 0 0 1 0 1$

There are matches also at locations 19 and 21.

The above problem can be trivially solved in time $O(NM)$. It can be solved in time $O(N \log M)$, in a computational model with word size $O(\log M)$, using the Fast Fourier Transform (FFT) [9].

Polynomial multiplication is a special case of a general convolution. A general convolution uses two initial functions, $v_1$ and $v_2$, to produce a third function $w$. We consider a subclass of discrete convolutions which we call discrete dot-product convolutions.

**Definition 2** Let $v_1 = v_1[0], ..., v_1[N - 1]; v_2 = v_2[0], ..., v_2[N - 1]$ be vectors in $\mathbb{R}^N$. The dot product of $v_1$ and $v_2$, denoted as $v_1 \cdot v_2$, is:

$$v_1 \cdot v_2 = \sum_{i=0}^{N-1} v_1[i]v_2[i].$$

A discrete dot product convolution, operates $O(N)$ bijections on the indices of $v_1$ and computes the dot product. Formally:

**Definition 3** Let $v_1 = v_1, v_2 \in \mathbb{R}^N, c \in \mathbb{N}$. Let $B = \{\beta_0, ..., \beta_{cN-1}\}$ be bijections from $\{0, ..., N - 1\}$ to $\{0, ..., N - 1\}$. Then the discrete dot-product $B$ convolution of $v_1$ and $v_2$, denoted as $v_1 \otimes_B v_2$, is the vector of length $cN$ defined as:

$$v_1 \otimes_B v_2[j] = \sum_{i=0}^{N-1} v_1[\beta(i)]v_2[i], \quad j = 0, ..., cN - 1.$$
The polynomial multiplication we have seen above, as well as the Discrete Walsh Transform (used e.g. in [4] for computing matching with flipped bits), are special cases of discrete dot-product convolutions. Formally:

**Definition 4**  
1. The Discrete Walsh Transform (DWT) of $v_1, v_2 \in \mathbb{R}^N$ is the discrete dot-product $B$ convolution of $v_1$ and $v_2$ where $B = \{\beta_0, ..., \beta_{N-1}\}$ and $\beta_j : \{0, ..., N-1\} \rightarrow \{0, ..., N-1\} \rightarrow \{0, ..., N-1\}$ is defined as: $\beta_j(i) = i \oplus j$. $i$ and $j$ are the binary number representations of $i$ and $j$ (i.e. binary strings of length $\log N$) and $\oplus$ is the exclusive or operation.

2. The polynomial multiplication of $v_1 \in \mathbb{R}^N, v_2 \in \mathbb{R}^M$ is the discrete dot-product $B$ convolution of $v_1$ and $v_2$ where $B = \{\beta_0, ..., \beta_{N-1}\}$ and $\beta_j : \{0, ..., M-1\} \rightarrow \{j, ..., j + M - 1\}$ is defined as: $\beta_j(i) = i + j$. In this definition the dot product is of vectors in $\mathbb{R}^M$, i.e.

$$v_1 \otimes v_2[j] = \sum_{i=0}^{M-1} v_1[i + j]v_2[i], \quad j = 0, ..., N - M + 1.$$  

A special feature both these convolutions have is that they can be solved in time $O(N \log N)$ via a divide-and-conquer approach. The algorithm for the polynomial multiplication uses the FFT and the algorithm for the Discrete Walsh Transform is the Fast Walsh Transform (FWT). Of course, similar to the DFT case, one may also consider a binary version of the DWT. In both these cases, the $O(N \log N)$ divide-and-conquer algorithm is a wonderful solution if the input vectors have many points. Suppose, however, that our arrays are sparse. In the sparse case, many values of $T$ and $P$ are 0. Thus, they do not contribute to the convolution value.

**Convention:** Throughout this paper, a capital letter (e.g. $N$) is used to denote the size of the vector, which is equivalent to the largest index of a non-zero value, and a small letter (e.g. $n$) is used to denote the number of non-zero values. It is assumed that the vectors are not given explicitly, rather they are given as a set of $(index, value)$ pairs, for all the non-zero values.

In our convention, the number of non-zero values of $T(P)$ is $n(m)$. Clearly, we can compute the convolution either in time $O(N \log M)$ or in time $O(nm)$. The challenge was (see e.g. [15]) whether the convolution can be computed in time $o(nm)$.

To our knowledge, the only efficient algorithms for computing sparse discrete binary convolutions, are for the FFT. The state-of-the art in computing such sparse convolutions is to use a locality preserving function to reduce the length of the sparse vectors, and then to use the fast convolution algorithm. In Section 2 we summarize the details of current length reduction methods. The locality preserving function is the modulo function.

**Paper’s Contribution:** The main contribution of this paper is introducing a novel tool for length reduction – polynomials. We show a number of advantages to using polynomials:

1. The new polynomials technique leads to an elegant algorithm for the sparse DWT. To our knowledge, no such algorithm is known to-date.

2. Our technique can be used without preprocessing to achieve a Las-Vegas algorithm for the sparse FFT which runs in time $O(n \log^2 n)$. This matches the expected running time of the Las-Vegas algorithm presented by Cole and Hariharan in [8]. However, our algorithm has the added advantage of guaranteeing a worst case time of $O(n^2 \log n)$ whereas the algorithm in [8] has no bound on its worst case running time.
3. Allowing $O(n^2)$ preprocessing time on the text, our technique can then *deterministically* solve the sparse FFT for incoming patterns in time $O(n \log^2 n)$. This improves the best known deterministic solution to the problem ([5]), whose time was $O(n \log^3 n)$, with the same preprocessing time.

2  **Summary of Length Reduction Methods**

Recently [11, 12], Hassanjeh et al. considered algorithms for approximating the sparse Fourier transform. We are interested in exactly computing sparse convolutions (the Walsh transform, as well as the fast Fourier transform-based polynomial multiplication). In this section we summarize Cole and Hariharan’s *length reduction* method, which exactly computes sparse polynomial multiplication.

Suppose we can map all the non-zero values into a smaller vector, say of size $O(n \log m)$. Suppose also that this mapping is alignment preserving in the sense that applying the same transformation on $P$ will guarantee that the alignments are preserved. Then we can simply map the the vectors $T$ and $P$ into the smaller vectors and then use FFT for the convolutions on the smaller vectors, achieving time $O(n \log^2 m)$. We then map the results back to the original vectors.

The problem is that to-date there is no known mapping with that alignment preserving property. Cole and Hariharan [8] suggested using the modulo function as the alignment preserving mapping, in a randomized setting that answers the problem with high probability. The reason their algorithm is not deterministic is the following: In their length reduction phase, several indices of non-zero values in the original vector may be mapped into the same index in the reduced size vector. If the index of only one non-zero value is mapped into an index in the reduced size vector, then this index is denoted as *singleton* and the non-zero value is said to appear as a *singleton*. If more then one non-zero value is mapped into the same index in the reduced size vector, then this index is denoted as *multiple*. The multiple case is problematic since we can not be sure of the right alignment. The proposed solution of Cole and Hariharan was to create a set of $\log n$ pairs of vectors using $\log n$ hash function rather then a single pair of vectors. They showed that in $O(\log n)$ attempts, the probability that some index will always be in a multiple situation is small.

A different, deterministic, solution was shown in [5]. The idea was to find $\log n$ hash functions that reduce the size of the vectors to $O(n \log n)$. The algorithm guaranteed that each non-zero value appears with no collisions in *at least* one of the vectors, thus eliminating the possibility of an error.

The ultimate goal of the length reduction is as follows: Given two vectors $T, P$ whose sizes are $N, M$, with $n, m$ non-zero elements respectively (where $n > m$), obtain two vectors $T', P'$ of size $O(n)$ such that all the non-zero elements in $T$ and in $P$ will appear as singletons in $T'$ and in $P'$ respectively while maintaining the distance property.

The *distance property* which need to be maintained is defined as follows: If $P'[f(0)]$ is aligned with $T'[f(i)]$, then $P'[f(j)]$ will be aligned with $T'[f(i+j)]$.

This goal was not reached yet, rather a set of $O(\log n)$ vectors of size $O(n \log n)$ were obtained in [5], where each non-zero in the text appears at least once as a singleton in the set of vectors. This length reduction gave an $O(n \log^3 n)$ algorithm for convolution in sparse data with a preprocessing time of $O(n^2)$.

The general outline of this idea could, conceivably, work for the sparse FWT case. However, the modulo function can not serve as a locality preserving mapping for the Walsh transform, since
the convolution bijection is an exclusive or rather than a shift. Thus to-date there is no known efficient algorithm for the sparse DWT.

3 The New Length Reduction Technique for the FWT

We employ a length reduction algorithm that applies to any convolution. The algorithm has four stages:

1. **Reduction**: A locality preserving function that reduces the range of indices.
2. **Convolution**: Perform the convolution on the shorter strings.
3. **Verification**: Verify that the results are consistent, i.e. that all multiplied elements indeed needed to be multiplied.
4. **Solution Expansion**: Map the solution of the shorter strings, to the original longer strings.

Below we explain the intuition for the reduction function chosen for the DWT, and then give the details of how polynomials perform the required length reduction.

Recall that in the case of FWT both the text and pattern have the same length $N$. For ease of exposition we assume that $N = 2^L$. Assume that the text has $n$ non-zero values, and the pattern has $m$ non-zeros values. Recall also that we denote the exclusive or operation by $\oplus$. As in the sparse FFT case, we want to check the value of the FWT for all $i$ for which $\sum_{j=0}^{n-1} T[i \oplus j]P[j]$ has $m$ non-zero summands. As we mentioned above, the algorithm is the same as in the case of the DFT. The problem is that here the ‘modulo’ function is no help at all as a length reduction function.

3.1 Intuition

Split the text to two strings of size $2^{L-1} = \frac{N}{2}$ and merge them with the “or” operator. Due to the fact that $N > n$, with high probability there will be no collisions, and thus all cases will be singletons. We do the same thing to the pattern and then run the FWT on the smaller strings. The multiples are handled similarly to the FFT case, i.e. ignore at this stage and in another reduction they will be unlikely to collide. We also need a verification stage to make sure that we have not multiplied values from different substrings.

This argument works well if the inputs are generated under a uniform random distribution. For supporting all inputs we need some randomization tool. The simple way to do it is as follow: Let $T_1$ be the first half of $T$ and $T_2$ the second. Choose a random bit string $\text{mask}$ of length $L - 1$, set its most significant bit to be 1, and calculate $T_2^{\text{mask}}[i] = T_2[i \oplus \text{mask}]$. Now do the ‘or’ operation between $T_1$ and $T_2^{\text{mask}}$. In effect, we have randomly permuted $T_2$. The following lemma, which immediately follows from the commutativity of the exclusive-or operation, guarantees that this random permutation preserves the locality for the Walsh transform.

**Lemma 1** Pattern index $i$ matches text index $k$ for location $j$ of the Walsh transform iff pattern index $i \oplus \text{mask}$ matches text index $k \oplus \text{mask}$. 

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Example 2  Consider the following text and pattern:

| address | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---------|-----|-----|-----|-----|-----|-----|-----|-----|
| T       | 0   | 1   | 0   | 0   | 0   | 1   | 0   |     |
| P       | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   |

The Walsh transform is: 02000020 (since at locations 001 = 1 and 110 = 6 the 1’s in the text and pattern match).

Consider now mask = 101. (The most significant bit (MSB) is always 1 at the mask because it is a mask on \( T_2 \), which is the second half of the text). The smaller text and pattern are as below:

| address | 000 | 001 | 010 | 011 |
|---------|-----|-----|-----|-----|
| \( T' \) | 0   | 1   | 0   | 1   |
| \( P' \) | 1   | 0   | 1   | 0   |

The Walsh transform is 0202.

The above example lacks the verification and solution expansion stages. We need to verify that the results are consistent. Consistency means that one of two cases happen:

1. Every text value with an unchanged address is multiplied by a pattern value with an unchanged address, and every text value with a masked address is multiplied by a pattern value with a masked address.

2. Every text item with an unchanged address is multiplied by a pattern item with a masked address and every text item with a masked address is multiplied by a pattern item with an unchanged address.

In addition it is necessary to be able to identify which of the non-zero results belong to their location (01 in the example) and which should be xor-ed with the mask (11 in the example).

**Verification Idea:** In the reduction stage, construct two strings of length \( 2^{L-1} \) where for each non-zero value write \( s \) if it is static, i.e., was not moved by the mask, and \( m \) if it was moved by the mask. We are interested in products where all values are either \( s \cdot s \), or \( m \cdot m \), and in products where all values are \( m \cdot s \). This can be calculated by a constant number of appropriate binary DWTs.

Example 3  In Example 2 we have

| address | 000 | 001 | 010 | 011 |
|---------|-----|-----|-----|-----|
| \( T' \) | 0   | \( s \) | 0   | \( m \) |
| \( P' \) | \( s \) | 0   | \( m \) | 0   |

Locations 00 and 10 have value 0. In location 01 we have a product of \( m \cdot m \) and \( s \cdot s \), and in location 11 we have products of \( m \cdot s \). Thus the reduction gives a consistent result and, furthermore, it means that the result in location 01 remains in that location, and the result in location 11 belongs to the index xor-ed with the mask, i.e. \( 11 \oplus 101 = 110 = 6 \).

The idea above reduces the length by a half, from \( N \) to \( \frac{N}{2} \). It is easy to see that this process can be recursively repeated. However, the analysis of the probability of collision is then a bit tedious. Our polynomial framework for length reduction enables this analysis in a clear and elegant manner.
3.2 Polynomials over a Finite Field

We will consider our indices as elements in the finite field $F_{2^L}$. The motivation for this is that $x + y = x \oplus y$ in $F_{2^L}$. Now in order to reduce the length of the string we will write each element in $F_{2^L}$ as a polynomial in $F_{2^L}[X]$ of degree $d = \frac{L}{\ell} - 1$. In order to do that, let us look at the binary presentation of a non-zero element index $i = a_{L-1}a_{L-2}...a_0$. The index binary presentation is divided into set of $\ell$ bits starting from the LSB. By this division we obtain a set of $\frac{L}{\ell}$ numbers in $F_{2^\ell}$. These numbers will be used as the coefficients of the polynomial representing this index, thus this polynomial belongs to $F_{2^\ell}[X]$ and its degree is $d = \frac{L}{\ell} - 1$.

Example 4 Consider the non-zero element index $i = 17$. let us assume that $\ell = 2$ by breaking the binary presentation of $i$ into blocks of 2 bits we obtain $i = 17 = 10001 = (01) \times X^2 + (00) \times X + (01) \times 1 = X^2 + 1$.

Note that under $F_{2^\ell}[X]$, addition is calculated as XOR, and multiplication is calculated as polynomial multiplication modulo some irreducible polynomial. Below is an example of the multiplication table of $F_{2^2}$.

Example 5 Multiplication table of $F_{2^2}$.

|   | 0 | 1 | a | b |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | a | b |
| a | 0 | a | b | 1 |
| b | 0 | b | 1 | a |

Note that $a$ and $b$ can be thought as representing 10 and 11 respectively.

After we obtain the polynomials representing the indices of the non-zeros elements, we can obtain different reduced size vector by applying different values to $X$ in the polynomials.

To prove the correctness of this length reduction for FWT, we have to prove that the following lemma still holds:

**Lemma 2** For any assignment of $X$, if $P[0]$ is aligned with the base polynomial representing $T[i]$, then $P[j]$ will be aligned with one of the polynomials representing $T[i \oplus j]$.

**Proof:**

The proof is simple, and it is based on the building method. The polynomial representing the index of $P[0]$ is $P_0[X] = 0$ and it is aligned with $P_i[X]$, which is the polynomial representing the index $T[i]$, thus the polynomial representing $P[j]$ is $P_j[X]$, and it is aligned with the polynomial $P_i[x] + P_j[x]$. Recall that the coefficients of the polynomials $P_i[x]$ and $P_j[x]$ are the bits representing $i$ and $j$, and the addition under $F_{2^\ell}$ is XOR, thus we get that $P_i[x] + P_j[x]$ is the polynomial representing the index $i \oplus j$.

In this framework, the analysis of the probability of collision is easy. Two indices $i = p_i(X)$ and $j = p_j(X)$ can collide only on $d = \frac{L}{\ell} - 1$ different evaluations, as any collision implies that the value assigned to $X$ in the polynomials is a root of the difference polynomial representing those two indices. Since the degree of this polynomial is bounded by $d$, then there can be no more then $d$ different roots to this polynomial. We conclude:
Lemma 3  The probability of a collision when choosing a random evaluation $r$ is no greater than $\frac{d^2}{2^\ell}$.

It is easy to see that appropriate binary Walsh transforms allow us to detect singletons and delete them from the next rounds. Appropriate binary Walsh transforms also easily provide the expansion.

4  The New Length Reduction Technique for the FFT

4.1  Sparse Vector of Polynomial Length

The proposed technique deals with the case that $N$ is polynomial in $n$, thus the indices are bounded by $n^c$. In the case where, $N$ is exponential in $n$, the reduction to a polynomial case can be used.

**Main Idea:** Derive a set of unique polynomials from each non-zero index in $T$, and one polynomial for each non-zero in $P$. Each assignment for the polynomials in $\mathbb{F}_q$, where $q$ is a prime number of size $\Theta(n)$ will give a different mapping of the non-zeros in $T$ and in $P$ to vectors of size $q$. The convolution will be performed between the vectors obtained from $T$ and $P$ under the same assignments.

The first step of the algorithm is to choose a prime number of size $\Theta(n)$, and create a polynomial for each non-zero index in $T$. The created polynomial of index $i$ will be denoted as the base polynomial of $T[i]$. The creation of the polynomial is done by representing the index as a number in base $\frac{(q-1)}{2}$. Each digit is interpreted as a coefficient of the polynomial.

**Example 6** Let $q = 13$. Consider index 95 in base 10. This equals 235 in base $\frac{(13-1)}{2} = 6$. Each digit is a coefficient, thus the polynomial representing index 95 is $2X^2 + 3X + 5$.

Since the indices in $T$ are bounded by $n^c$, and $q$ is $\Theta(n)$, then the degree of the polynomials which created in this step is bounded by $c$. In the next step, from each polynomial we create $2^c$ polynomials. This is done by giving two choices for each coefficient of the polynomial: (1) Leave it as is. (2) Add $\frac{q-1}{2}$ to the coefficient and decrease by 1 the coefficient of the higher degree. We do this for all the coefficients of the polynomial except for the coefficient of the highest degree.

**Example 7** Suppose we have a non-zero index 95, using $q = 13$ we get the base polynomial $2X^2 + 3X + 5$. After the second step we will obtain 4 polynomials: $2X^2 + 3X + 5$, $2X^2 + 2X + 11$, $X^2 + 9X + 5$, $X^2 + 8X + 11$.

The first polynomial is the base polynomial. The second polynomial was obtained by adding 6 to the first coefficient and decreasing the second coefficient by one. The 3rd and the 4th polynomials were created by adding 6 to the second coefficient of the first and second polynomials respectively, and decreasing the third coefficient by one.

The duplication of the polynomials was made to meet the distance preserving requirement of the length reduction. The following lemma formalizes it.

**Lemma 4** For any assignment of $X$, if $P[0]$ is aligned with the base polynomial representing $T[i]$, then $P[j]$ will be aligned with one of the polynomials representing $T[i+j]$.
Proof: Let \( q \) be the chosen prime number. Index 0 in \( P \) is represented by the polynomial 0, and index \( j \) in \( P \) is represented by the polynomial \( A = a_cX^c + a_{c-1}X^{c-1} + \ldots + a_0 \). Index \( i \) in \( T \) is represented by a polynomial of the form \( B = b_cX^c + b_{c-1}X^{c-1} + \ldots + b_0 \), and index \( i + j \) in \( T \) is represented by a polynomial \( D = d_cX^c + d_{c-1}X^{c-1} + \ldots + d_0 \). Note that the coefficients \( a_i \) and \( b_i \) are smaller than \( \frac{(q-1)}{2} \).

Clearly, if \( P[i][0] \) is aligned with \( T[i] \), then for any assignment of \( X \), \( P[j] \) will be aligned with the polynomial \( A + B = (a_c + b_c)X^c + (a_{c-1} + b_{c-1})X^{c-1} + \ldots + (a_0 + b_0) \). Now consider the first coefficient of \( D \). Since \( a_0 \) and \( b_0 \) are smaller then \( \frac{(q-1)}{2} \), then there are only two cases: (1) \( (a_0 + b_0) < \frac{(q-1)}{2} \), thus \( d_0 = a_0 + b_0 \). (2) \( (a_0 + b_0) \geq \frac{(q-1)}{2} \), thus \( d_0 = a_0 + b_0 - \frac{(q-1)}{2} \) which is covered by the polynomial where \( \frac{(q-1)}{2} \) was added to the first coefficient.

In the later case, one was added to the second coefficient, thus we decrease the next coefficient whenever we add \( \frac{(q-1)}{2} \) to the current coefficient. The same cases exist also in all the coefficients, but a polynomial was created for each possible case (\( 2^c \) cases), thus one of the created polynomials will be equal to the polynomial \( A + B \).

Note that all the \( 2^c \times n \) created polynomials are unique, and in \( \mathbb{F}_q \). Assigning a value to the polynomials in \( \mathbb{F}_q \) will give a vector of size \( q \).

Lemma 5 Any two polynomials can be mapped to the same location in at most \( c \) assignments.

Proof: The distance between any two polynomials gives a polynomial, where the degree of the difference polynomial is bounded by \( c \). Since both polynomials give the same index under the selected assignment, then the assigned value is a root of the difference polynomial. The degree of this polynomial is bounded by \( c \), thus it can have at most \( c \) different roots in \( \mathbb{F}_q \).

Since any polynomial can be mapped into the same location with at most \( 2^c \times n \) other polynomials, and with each of them at most \( c \) times, due to Lemma 5 then we get the following Corollary:

Corollary 1 Any polynomial can appear as a multiple in not more then \( c \times 2^c \times n \) vectors.

The last step of the length reduction algorithm is to find a set of \( O(\log n) \) assignments which will ensure that each polynomial will appear as a singleton at least once.

The selection of the \( O(\log n) \) assignments is done as follows: Construct table \( A \) with \( 2^c \times n \) columns and \( c \times 2^{c+1} \times n \) rows. Row \( i \) correspond to an assigned value \( a_i \) and the corresponding reduced length vector \( T'_i \). A column corresponds to a polynomial \( P_j \). The value of \( A_{ij} \) is set to 1 if polynomial \( j \) appears as a singleton in vector \( T'_i \). Due to Corollary 1 the number of zeros in each column can not exceed \( c \times 2^c \times n \). Thus, in each column there are 1’s in at least half of the rows, which means that the table is at least half full. Since the table is at least half full there exists a row in which there is one in at least half of the columns. The assignment value which generated this row is chosen, and all the columns where there was a 1 in the selected row are deleted from the table.

Recursively another assignment value is chosen and the table size is halved again, until all the columns are deleted. Since at each step at least half of the columns are deleted, the number of prime number chosen can not exceed \( \log(2^c \times n) = c \log n \).

Time: Creating vector \( T'_i \) (row \( i \)) takes \( O(n) \) time. Since we start with a full matrix of \( O(n) \) rows then the initialization takes \( O(n^2) \) time. Choosing the \( O(\log n) \) assignment values is done
recursively. The recurrence is:

\[ t(n^2) = n^2 + t\left(\frac{n^2}{2}\right) \]

The closed form of this recurrence is \( O(n^2) \).

Note that if we choose the assignments randomly in running time we can skip the preprocessing step. In the expected case we will succeed after \( O(\log n) \) assignments. But even in the worst case \( O(n) \) assignments will guarantee success.

4.2 Sparse Vector of Exponential Length

In this case, as proposed in [5], each of the vectors \( T \) and \( P \) is reduced into a single vector of size \( O(n^4) \), where all the non-zeros appear as singletons. The reduction is performed using the modulus function with a prime number \( q \) of size \( O(n^4) \). It was already proven there that there are at most \( n^3 \) prime number of size \( O(n^4) \), which generate at least one multiple. Thus, by testing \( n^3 + 1 \) prime numbers we ensure that at least one of them produces a vector with no multiples.

In order to find such a prime number, we find \( n^3 + 1 \) prime numbers of size \( O(n^4) \). Then we multiply all the prime numbers to receive a large number \( Q \). In addition we have at most \( n^2 \) different distances between any two non-zeros. We multiply all of them to receive a large number \( D \). The next step is to find the greatest common divisor (GCD) between \( Q \) and \( D \). Since there is at least one prime number in \( Q \) which does not divide \( D \), then \( GCD(Q,D) \) is less then \( Q \). Dividing \( Q \) by the \( GCD(Q,D) \) will give \( R \) which is the multiplication of all the prime numbers that create only singletons. The last step is to find at least one of them. This is done using a binary search on the prime numbers. We take the multiplication of half of the prime numbers \( Q' \), and find the \( GCD(Q',R) \). If \( GCD(Q',R) > 1 \) we continue with this set of prime numbers and multiply half of them iteratively. Otherwise, we continue with the other half of the prime numbers. After \( O(\log n) \) iterations we will find one prime number which will generate only singletons.

The algorithm appears in detail below.

**Algorithm – \( N \) is exponential in \( n \)**

1. Find \( n^3 + 1 \) prime numbers of size \( O(n^4) \).
2. Multiply all the prime numbers to obtain \( Q \).
3. Multiply all the difference between any two non-zero indices to obtain \( D \).
4. Set \( R = \frac{Q}{GCD(Q,D)} \).
5. Let \( S \) be the set of all prime numbers.
6. While the size of \( S \) is larger then 1 do:
   (a) Let \( S' \) be a set of the first half of prime numbers in \( S \).
   (b) Set \( Q' \) to be the multiplication of all the prime numbers in \( S' \).
   (c) If \( GCD(Q',R) > 1 \) then set \( S = S' \), otherwise set \( S = S/S' \).

**Correctness:** Immediately follows from the discussion.
Time: Step 1 is performed in time $O(n^3 \text{polylog}(n))$ using the primality testing described in [6]. Step 2 is done by building a binary tree of products where each node contain the product of the two number in the lower level. This tree has $O(\log n)$ levels. In the leaves there are $n^3$ prime numbers with $\log n$ bits, so the total number of bits in each level is $O(n^3 \log n)$. A product of two numbers can be computed in time $O(b \log b \log \log b)$ [17], where $b$ is the number of bits. Thus each level can be computed in time $O(n^3 \text{polylog}(n))$ and the total time for step 2 is $O(n^3 \text{polylog}(n))$. Step 3 is performed in the same way, but this time in the leaves there are $n^2$ numbers with $n$ bits, thus each level has $n^3$ bits and the time for this step is $O(n^3 \log n)$. In step 4 we calculate the GCD of two numbers with $O(n^3 \log n)$ bits. This can be done in time $O(n^3 \text{polylog}(n))$ using [19]. The calculation for step 6(b) was already performed in step 2, and step 6(c) can be calculated in time $O(n^3 \text{polylog}(n))$, thus the time of step 6 is $O(n^3 \text{polylog}(n))$. Following this discussion the total time of this algorithm is $O(n^3 \text{polylog}(n))$.

5 Conclusion and Open Problems

A new tool for Length Reduction and Sparse Convolution was introduced in this paper - encoding the indices as polynomials. This enabled the first algorithm for efficient calculation of sparse binary discrete Walsh transform. It also provided better and faster algorithms for several well known problems, such as randomized efficient sparse FFT and deterministic efficient sparse FFT computation.

An important problem remains: Can the Length Reduction and Sparse DFT problems be solved deterministically without the need of the preprocessing step or, alternately, can the preprocessing time be reduced from quadratic?

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