General relativity and Weyl geometry

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Abstract
We show that the general theory of relativity can be formulated in the language of Weyl geometry. We develop the concept of Weyl frames and point out that the new mathematical formalism may lead to different pictures of the same gravitational phenomena. We show that in an arbitrary Weyl frame general relativity, which takes the form of a scalar–tensor gravitational theory, is invariant with respect to Weyl transformations. A key point in the development of the formalism is to build an action that is manifestly invariant with respect to Weyl transformations. When this action is expressed in terms of Riemannian geometry we find that the theory has some similarities with Brans–Dicke gravitational theory. In this scenario, the gravitational field is not described by the metric tensor only, but by a combination of both the metric and a geometrical scalar field. We illustrate this point by examining how distinct geometrical and physical pictures of the same phenomena may arise in different frames. To give an example, we discuss the gravitational spectral shift as viewed in a general Weyl frame. We further explore the analogy of general relativity with scalar–tensor theories and show how a known Brans–Dicke vacuum solution may appear as a solution of general relativity theory when reinterpreted in a particular Weyl frame. Finally, we show that the so-called WIST gravity theories are mathematically equivalent to Brans–Dicke theory when viewed in a particular frame.

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1. Introduction
We would like to start by raising two questions of a very general character. The first question is: What kind of invariance should the basic laws of physics possess? It is perhaps pertinent here to quote the following words by Dirac: ‘It appears as one of the fundamental principles of nature that the equations expressing the basic laws of physics should be invariant under the widest possible group of transformations’ [1]. The second question, which seems to be of a rather epistemological character, is: To what extent is Riemannian geometry the only possible
geometrical setting for general relativity (GR)? The purpose of this work is to address, at least partially, these two questions.

It is a very well-known fact that the principle of general covariance has played a major role in leading Einstein to the formulation of the theory of GR [2]. The idea underlying this principle is that coordinate systems are merely mathematical constructions to conveniently describe physical phenomena, and hence should not be an essential part of the fundamental laws of physics. In a more precise mathematical language, what is being required is that the equations of physics be expressed in terms of intrinsic geometrical objects, such as scalars, tensors or spinors, defined in the spacetime manifold. This mathematical requirement is sufficient to guarantee the invariance of the form of the physical laws (or covariance of the equations) under arbitrary coordinate transformations. In field theories, one way of constructing covariant equations is to start with an action in which the Lagrangian density is a scalar function of the fields. In the case of GR, as we know, the covariance of the Einstein equations is a direct consequence of the invariance of the Einstein–Hilbert action with respect to spacetime diffeomorphisms.

A rather different kind of invariance that has been considered in some branches of physics is invariance under conformal transformations. These represent changes in the units of length and time that differ from point to point in the spacetime manifold. These should not be confounded with coordinate transformations [3]. On the other hand, it has been observed that conformal invariance is a feature of Maxwell, Klein–Gordon and Dirac massless fields. This quest for this kind of invariance has also led to conformal gravity theories [4]. Conformal transformations were first introduced in physics by H Weyl in his attempt to formulate a unified theory of gravitation and electromagnetism [5]. However, in order to introduce new degrees of freedom to account for the electromagnetic field Weyl had to assume that the spacetime manifold is not Riemannian. This extension consists of introducing an extra geometrical entity in the spacetime manifold, a 1-form field $\sigma$, in terms of which the Riemannian compatibility condition between the metric $g$ and the connection $\nabla$ is redefined. Then, a group of transformations, which involves both $g$ and $\sigma$, is defined by requiring that under these transformations the new compatibility condition remains invariant. The Weyl 1-form $\sigma$ acts, in fact, as a compensating field or a ‘gauge’ field to ensure the invariance of the compatibility condition. In a certain sense, this new invariance group, which we shall call the group of Weyl transformations, includes the conformal transformations as subgroup.

It turns out that Einstein’s theory of gravity in its original formulation is not invariant, neither under conformal transformations nor under Weyl transformations. One reason for this is that the geometrical language of Einstein’s theory is completely based on Riemannian geometry. Indeed, for a long time GR has been inextricably associated with the geometry of Riemann. Further developments, however, have led to the discovery of different geometrical structures, which we might generically call ‘non-Riemannian’ geometries, Weyl geometry being one of the first examples. Many of these developments were closely related to attempts at unifying gravity and electromagnetism [6]. While the newborn non-Riemannian geometries were invariably associated with new gravity theories, one question that naturally arises is: To what extent is Riemannian geometry the only possible geometrical setting for the formulation of GR? One of our aims in this paper is to show that, surprisingly enough, one can formulate GR using the language of a non-Riemannian geometry, namely the one known as Weyl integrable geometry. In this formulation, GR appears as a theory in which the gravitational field is described simultaneously by two geometrical fields: the metric tensor and the Weyl scalar field, with the latter being an essential part of the geometry, manifesting its presence in almost all geometrical phenomena, such as curvature, geodesic motion, and so on. As we shall see, in this new geometrical setting GR exhibits a new kind of invariance, namely the
invariance under Weyl transformations. It should be mentioned that attempts to construct a conformal-invariant formulation of general relativity can be found in the literature [7]. One of these approaches, in which a massless scalar field (dilaton) plays a role similar to that of the Weyl field, leads to interesting theoretical results on the description of recent supernova data [8].

The outline of this paper is as follows. We begin by presenting the basic mathematical facts of Weyl geometry and the concept of Weyl frames. In section 3, we show how to recast GR in the language of Weyl integrable geometry. In this formulation, we shall see that the theory is manifestly invariant under the group of Weyl transformations. We proceed, in section 4, to obtain the field equations and interpret the new form of the theory as a kind of scalar–tensor theory of gravity. In sections 5 and 6, we explore the similarities of the formalism with Brans–Dicke theory of gravity. Then, in section 7, we briefly illustrate how different pictures of the same phenomena may arise in distinct frames. In section 8, we show that the so-called WIST gravity theories are mathematically equivalent to Brans–Dicke theory when viewed in a particular frame, the Riemann frame. We conclude with some remarks in section 9.

2. Weyl geometry

The geometry conceived by Weyl is a simple generalization of Riemannian geometry. Instead of postulating that the covariant derivative of the metric tensor $g$ is zero, we assume the more general condition [5]

$$\nabla_a g_{\mu\nu} = \sigma_a g_{\mu\nu}, \quad (1)$$

where $\sigma_a$ denotes the components with respect to a local coordinate basis $\{\frac{\partial}{\partial x^\alpha}\}$ of a 1-form field $\sigma$ defined on the manifold $M$. This represents a generalization of the Riemannian condition of compatibility between the connection $\nabla$ and $g$, namely the requirement that the length of a vector remains unaltered by parallel transport [9]. If $\sigma$ vanishes, then (1) reduces to the familiar Riemannian metricity condition. It is interesting to note that the Weyl condition (1) remains unchanged when we perform the following simultaneous transformations in $g$ and $\phi$:

$$\bar{g} = e^f g, \quad (2)$$

$$\bar{\sigma} = \sigma + df, \quad (3)$$

where $f$ is a scalar function defined on $M$. If $\sigma = d\phi$, where $\phi$ is a scalar field, then we have what is called a Weyl integrable manifold. The set $(M, g, \phi)$ consisting of a differentiable manifold $M$ endowed with a metric $g$ and a Weyl scalar field $\phi$ will be referred to as a Weyl frame. In the particular case of a Weyl integrable manifold (3) becomes

$$\bar{\phi} = \phi + f. \quad (4)$$

It turns out that if the Weyl connection $\nabla$ is assumed to be torsionless, then by virtue of condition (1) it gets completely determined by $g$ and $\sigma$. Indeed, a straightforward calculation shows that the components of the affine connection with respect to an arbitrary vector basis are completely given by

$$\Gamma^\alpha_{\mu\nu} = \{^a_{\mu\nu}\} + \frac{1}{2} g^{\alpha\beta} [g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}], \quad (5)$$

where $\{^a_{\mu\nu}\} = \frac{1}{2} g^{\alpha\beta} [g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}]$ represents the Christoffel symbols, i.e. the components of the Lévi-Civita connection. An important fact that deserves to be mentioned

1 Throughout this paper our convention is that Greek indices take values from 0 to $n - 1$, where $n$ is the dimension of $M$.
is the invariance of the affine connection coefficients $\Gamma^a_{\mu\nu}$ under the Weyl transformations (2) and (3). If $\sigma = \mathcal{d}\phi$, then (5) becomes

$$\Gamma^a_{\mu\nu} = \left\{ \Gamma^a_{\mu\nu} \right\} - \frac{1}{2} g^{\alpha\beta} \left[ g_{\beta\mu} \phi_{,\nu} + g_{\beta\nu} \phi_{,\mu} - g_{\mu\nu} \phi_{,\beta} \right].$$  (6)

A clear geometrical insight into the properties of Weyl parallel transport is given by the following proposition. Let $M$ be a differentiable manifold with an affine connection $\nabla$, a metric $g$ and a Weyl field of 1-forms $\sigma$. If $\nabla$ is compatible with $g$ in the Weyl sense, i.e. if (1) holds, then for any smooth curve $C = C(\lambda)$ and any pair of two parallel vector fields $V$ and $U$ along $C$, we have

$$\frac{d}{d\lambda} g(V, U) = \sigma \left( \frac{d}{d\lambda} \right) g(V, U),$$  (7)

where $\frac{d}{d\lambda}$ denotes the vector tangent to $C$ and $\sigma \left( \frac{d}{d\lambda} \right)$ indicates the application of the 1-form $\sigma$ on $\frac{d}{d\lambda}$. (In a coordinate basis, putting $\frac{d}{d\lambda} = \frac{dx^\gamma}{d\lambda} \frac{\partial}{\partial x^\gamma}$, $V = V^\partial \frac{\partial}{\partial x^\gamma}$, $U = U^\mu \frac{\partial}{\partial x^\mu}$, $\sigma = \sigma^\gamma (d\gamma)$, the above equation reads $\frac{d}{d\lambda} \left( g_{\alpha\beta} V^\gamma U^\beta \right) = \sigma^\gamma \frac{dx^\gamma}{d\lambda} g_{\alpha\beta} V^\gamma U^\beta$.)

If we integrate equation (7) along the curve $C$ from a point $P_0 = C(\lambda_0)$ to an arbitrary point $P = C(\lambda)$, then we obtain

$$g(V(\lambda), U(\lambda)) = g(V(\lambda_0), U(\lambda_0)) e^{\int_{\lambda_0}^{\lambda} \sigma \left( \frac{d}{d\lambda} \right) d\lambda}.$$  (8)

If we put $U = V$ and denote by $L(\lambda)$ the length of the vector $V(\lambda)$ at $P = C(\lambda)$, then it is easy to see that in a local coordinate system $\{x^\alpha\}$ equation (7) reduces to

$$\frac{dL}{d\lambda} = \frac{\sigma^\alpha}{2} \frac{dx^\alpha}{d\lambda} L.$$  

Consider the set of all closed curves $C : [a, b] \in R \rightarrow M$, i.e. with $C(a) = C(b)$. Then, we have the following equation:

$$g(V(b), U(b)) = g(V(a), U(a)) e^{\int a \rightarrow b \sigma \left( \frac{d}{d\lambda} \right) d\lambda}.$$  

It follows from Stokes’ theorem that if $\sigma$ is an exact form, that is, if there exists a scalar function $\phi$, such that $\sigma = \mathcal{d}\phi$, then

$$\oint_C \sigma \left( \frac{d}{d\lambda} \right) d\lambda = 0$$

for any loop. In this case the integral $e^{\int a \rightarrow b \sigma \left( \frac{d}{d\lambda} \right) d\lambda}$ does not depend on the path and (8) may be rewritten in the form

$$e^{-\phi(x(\lambda_0))} g(V(\lambda), U(\lambda)) = e^{-\phi(x(\lambda_0))} g(V(\lambda_0), U(\lambda_0)).$$  (9)

This equation means that we have an isometry between the tangent spaces of the manifold at the points $P_0 = C(\lambda_0)$ and $P = C(\lambda)$ in the ‘effective’ metric $\tilde{g} = e^{-\phi}g$.

Let us have a closer look at the correspondence between the Riemannian and Weyl integrable geometries suggested by equation (9). The first point to note is that, because $\sigma = \mathcal{d}\phi$ for some scalar field $\phi$, then if we define an ‘effective’ metric $\tilde{g} = e^{-\phi}g$, then the Weyl condition of compatibility (or, as it is sometimes called, the non-metricity condition), expressed by equation (1) or (7), is formally equivalent to the Riemannian condition imposed on $\tilde{g}$, namely

$$\nabla_\mu \tilde{g}_{\alpha\nu} = 0.$$  

It may be easily verified that (6) follows directly from $\nabla_\mu \tilde{g}_{\alpha\nu} = 0$. This simple fact has interesting and useful consequences, and later will serve as a guidance in the formulation of GR in terms of Weyl integrable geometry. One consequence is that since $\tilde{g} = e^{-\phi}g$ is
invariant under the Weyl transformations (2) and (4) any geometrical quantity constructed
with and solely with \( \hat{g} \) is invariant. Clearly, these will also be invariant under the Weyl
transformations (2) and (4). Thus, in addition to the connection coefficients \( \hat{\Gamma}^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda} \),
other geometrical objects such as the components of the curvature tensor \( \hat{R}^\alpha_{\rho\sigma\nu} = R^\alpha_{\rho\sigma\nu} = \Gamma^\alpha_{\rho\sigma\nu} - \Gamma^\alpha_{\rho\nu,\sigma} + \Gamma^\alpha_{\sigma\nu,\rho} - \Gamma^\alpha_{\sigma\rho,\nu} \), the components of the Ricci tensor \( \hat{R}_{\mu\nu} = R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} \)
and the scalar curvature \( \hat{R} = \hat{R}^\alpha_{\alpha} = \hat{R}_{\alpha\alpha} = \hat{R} \) are evidently invariant.
Moreover, in a Weyl integrable manifold it would be more natural to require this kind of
invariance to hold also in the definition of length, so we would redefine the arc length of a
curve \( x^a = x^a(\lambda) \) between \( x^a(a) \) and \( x^a(b) \) as

\[
\Delta s = \int_a^b \left( \hat{g}_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{\frac{1}{2}} d\lambda = \int_a^b e^{-\frac{\phi}{2}} \left( g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{\frac{1}{2}} d\lambda. \tag{10}
\]

The second point concerns the interplay between covariant and contravariant vectors in a Weyl
integrable manifold. Let us examine how the isomorphism that exists between vectors and
1-forms is modified when the manifold is endowed with an additional geometric field \( \phi \). This
question seems to be relevant because, as we know, it is this duality that underlies the usual
operations of raising and lowering indices of vectors and tensors. In a Weyl integrable manifold
these operations make sense only if they fulfil the requirement of Weyl invariance. Thus, let us
now briefly recall how we show, in the Riemannian context, that the tangent space \( T_p(M) \) and
the cotangent space \( T^*_p(M) \) and the tangent space \( T^*_p(M) \) at a point \( p \in M \) are isomorphic [10]. The key point is to define
the mapping \( \hat{V} : T_p(M) \to \mathbb{R} \) with \( \hat{V}(U) = g(U, V) \) for any \( U \in T_p(M) \). It is not difficult to
see that \( \hat{V} \) is a 1-form and that to any 1-form \( \sigma \in T^*_p(M) \) there corresponds a unique vector
\( V \in T_p(M) \) such that \( \sigma(U) = g(U, V) \). Now, assuming that \( \{e_\mu\} \) and \( \{e^a\} \) constitute dual
bases for \( T_p(M) \) and \( T^*_p(M) \), respectively, and putting \( V = V^\mu e_\mu \), \( \sigma = \sigma_a e^a \), we then have \( \sigma_\mu = \sigma(e_\mu) = g(e_\mu, V) = V^\nu g(e_\nu, e_\mu) \). In view of the fact that \( \sigma \) and \( V \) are isomorphic it is
natural ‘to lower’ the index \( V^\mu \) by defining \( V_\mu \equiv \sigma_\mu = g_{\mu\nu} V^\nu \), with \( g_{\mu\nu} \equiv g(e_\mu, e_\nu) \). Of course,
this procedure is not invariant under Weyl transformations since the effective metric \( \hat{g} = e^{-\phi} g \)
does not enter in any of the above operations. To remedy this situation, it suffices to redefine the
above algebra by replacing the Riemannian scalar product \( g : T_p(M) \times T_p(M) \to \mathbb{R} \) by a new
scalar product given by the bilinear form \( \hat{g} : T_p(M) \times T^*_p(M) \to \mathbb{R} \) with \( \hat{g}(U, V) = e^{-\phi} g(U, V) \). In this way the operations of raising and lowering indices when carried out with \( \hat{g} \) are clearly
invariant under (2) and (4).

Let us finally conclude this section with a few historical comments on Weyl gravitational
theory. Weyl developed an entirely new geometrical framework to formulate his theory, the
main goal of which was to unify gravity and electromagnetism. As is well known, although
admirably ingenious, Weyl’s gravitational theory turned out to be unacceptable as a physical
theory, as it was immediately realized by Einstein, who raised objections to the theory [9, 11].
Einstein’s argument was that in a non-integrable Weyl geometry the existence of sharp spectral
lines in the presence of an electromagnetic field would not be possible since atomic clocks
would depend on their past history [9]. However, the variant of Weyl geometry known as Weyl
integrable geometry does not suffer from the drawback pointed out by Einstein. Indeed, it is the
integral \( I(a, b) = \int_a^b \sigma \left( \frac{d\lambda}{\sqrt{\hat{g}}} \right) d\lambda \) that is responsible for the difference between the readings of
two identical atomic clocks following different paths. Because in Weyl integrable geometry
\( I(a, b) \) is not path dependent, the theory has attracted the attention of many cosmologists in
recent years as a viable geometrical framework for gravity theories [12, 13].
3. General relativity and a new kind of invariance

We have seen in the previous section that the Weyl compatibility condition (1) is preserved when we go from a frame $(M, g, \phi)$ to another frame $(\bar{M}, \bar{g}, \bar{\phi})$ through the transformations (2) and (4). This has the consequence that the components $\Gamma^\nu_{\mu\beta}$ of the affine connection are invariant under Weyl transformations, which, in turn, implies the invariance of the affine geodesics. Now, as is well known, geodesics play a fundamental role in GR as well as in any metric theory of gravity. Indeed, an elegant aspect of the geometrization of the gravitational field lies in the geodesic postulate, i.e. the statement that light rays and particles moving under the influence of gravity alone follow spacetime geodesics. Therefore a great deal of information about the motion of particles in a given spacetime is promptly available once one knows its geodesics. The fact that geodesics are invariant under (2) and (4) and that Riemannian geometry is a particular case of Weyl geometry (when $\sigma$ vanishes, or $\phi$ is constant) seems to suggest that it should be possible to express GR in a more general geometrical setting, namely one in which the form of the field equations is also invariant under Weyl transformations.

In this section, we shall show that this is indeed possible, and we shall proceed through the following steps. First, we shall assume that the spacetime manifold which represents the arena of physical phenomena may be described by a Weyl integrable geometry, which means that gravity will be described by two geometric entities: a metric and a scalar field. The second step is to set up an action $S$ invariant under Weyl transformations. We shall require that $S$ be chosen such that there exists a unique frame in which it reduces to the Einstein–Hilbert action. The third step consists of extending Einstein’s geodesic postulate to arbitrary frames, such that in the Riemann frame it should describe the motion of test particles and light exactly in the same way as predicted by GR. Finally, the fourth step is to define proper time in an arbitrary frame. This definition should be invariant under Weyl transformations and coincide with the definition of GR’s proper time in the Riemann frame. It turns out then that the simplest action that can be built under these conditions is

$$S = \int d^4x \sqrt{-g} e^{-\phi} \{ R + 2\Lambda e^{-\phi} + \kappa e^{-\phi} L_m \},$$

where $R$ denotes the scalar curvature defined in terms of the Weyl connection, $\Lambda$ is the cosmological constant, $L_m$ stands for the Lagrangian of the matter fields and $\kappa$ is the Einstein constant. In n-dimensions we would have

$$S_n = \int d^n x \sqrt{-g} e^{(1-\gamma)\phi} \{ R + 2\Lambda e^{-\phi} + \kappa e^{-\phi} L_m \}. \quad (12)$$

In order to see that the above action is, in fact, invariant with respect to Weyl transformations, we just need to recall that under (2) and (4) we have $\bar{g}^{\mu\nu} = e^{-\gamma} g^{\mu\nu}$, $\sqrt{-\bar{g}} = e^{\gamma/2} \sqrt{-g}$, $\bar{R} = R + 2\Lambda g + \kappa L_m$, $\bar{\nabla} = \nabla + \gamma \partial$. It will be assumed that $L_m$ depends on $\phi, g_{\mu\nu}$ and the matter fields, here generically denoted by $\xi$, its form being obtained from the special theory of relativity through the prescription $\eta_{\mu\nu} \to e^{-\phi} \eta_{\mu\nu}$, and $\partial_\mu \to \nabla_\mu$, where $\nabla_\mu$ denotes the covariant derivative with respect to the Weyl affine connection. If we designate the Lagrangian of the matter fields in special relativity by $L_m^\gamma = L_m^\gamma (\eta, \xi, \partial \xi)$, then the form of $L_m$ will be given by the rule $L_m(g, \phi, \xi, \nabla \xi) = L_m^\gamma (e^{-\phi} g, \tilde{\xi}, \nabla \tilde{\xi})$. As can be easily seen, these rules also ensure the invariance under Weyl transformations of part of the action that is responsible for the coupling of matter with the gravitational field, and, at the same time, reproduce the principle of minimal coupling adopted in GR when we set $\phi = 0$, that is, when we go to the Riemann frame by a Weyl transformation.

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2 Throughout this paper we shall adopt the following convention in the definition of the Riemann and Ricci tensors: $R^\rho_{\mu\nu\beta} = \Gamma^\rho_{\mu\alpha} \Gamma^\alpha_{\nu\beta} - \Gamma^\rho_{\mu\beta} \Gamma^\alpha_{\nu\alpha} - \Gamma^\rho_{\mu\alpha} \Gamma^\alpha_{\nu\beta} + \Gamma^\rho_{\mu\beta} \Gamma^\alpha_{\nu\alpha}$; $R_{\mu\nu} = R^\rho_{\rho\mu\nu}$. In this convention, we shall write the Einstein equations as $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}$, with $\kappa = \frac{8\pi G}{c^4}$. 

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We now turn our attention to the motion of test particles and light rays. Here, our task is to extend GR’s geodesic postulate in such a way to make it invariant under Weyl transformations. The extension is straightforward and may be stated as follows: if we represent parametrically a time-like curve as \( x^\mu = x^\mu(\lambda) \), then this curve will correspond to the world line of a particle free from all non-gravitational forces, passing through the events \( x^\mu(a) \) and \( x^\mu(b) \), if and only if it extremizes the functional
\[
\Delta \tau = \int_a^b e^{-\phi} \left( g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{\frac{1}{2}} d\lambda,
\]
(13)
which is obtained from the special relativistic expression of proper time by using the prescription \( \eta_{\mu\nu} \rightarrow e^{-\phi} g_{\mu\nu} \). Clearly, the right-hand side of this equation is invariant under Weyl transformations and reduces to the known expression of the proper time in GR in the Riemann frame. We take \( \Delta \tau \), as given above, as the extension to an arbitrary Weyl frame, of GR’s clock hypothesis, i.e. the assumption that \( \Delta \tau \) measures the proper time measured by a clock attached to the particle [14].

It is not difficult to verify that the extremization condition of the functional (13) leads to the equations
\[
\frac{d^2 x^\mu}{d\lambda^2} + \left( \{ a^\mu_{\alpha\beta} \} - \frac{1}{2} g^{\alpha\beta} (g_{\alpha\gamma} \phi,_{\beta\gamma} + g_{\beta\gamma} \phi,_{\alpha\gamma} - g_{\alpha\beta} \phi,_{\gamma}) \right) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0,
\]
(14)
where \( \{ a^\mu_{\alpha\beta} \} \) denotes the Christoffel symbols calculated with \( g_{\mu\nu} \). Let us recall that in the derivation of the above equations the parameter \( \lambda \) has been chosen such that
\[
e^{-\phi} g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = K = \text{const.}
\]
(15)
along the curve, which, up to an affine transformation, permits the identification of \( \lambda \) with the proper time \( \tau \). It turns out that these equations are exactly those that yield the affine geodesics in a Weyl integrable spacetime, since they can be rewritten as
\[
\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0,
\]
(16)
where \( \Gamma^\mu_{\alpha\beta} = \{ a^\mu_{\alpha\beta} \} - \frac{1}{2} g^{\mu\gamma} (g_{\alpha\gamma} \phi,_{\beta\gamma} + g_{\beta\gamma} \phi,_{\alpha\gamma} - g_{\alpha\beta} \phi,_{\gamma}) \), according to (6), may be identified with the components of the Weyl connection. Therefore, the extension of the geodesic postulate by requiring that the functional (13) be an extremum is equivalent to postulating that the particle motion must follow affine geodesics defined by the Weyl connection \( \Gamma^\mu_{\alpha\beta} \). It will be noted that, as a consequence of the Weyl compatibility condition (1) between the connection and the metric, (15) holds automatically along any affine geodesic determined by (16). Because both the connection components \( \Gamma^\mu_{\alpha\beta} \) and the proper time \( \tau \) are invariant when we switch from one Weyl frame to the other, equations (16) are invariant under Weyl transformations.

As we know, the geodesic postulate not only makes a statement about the motion of particles, but also regulates the propagation of light rays in spacetime. Because the path of light rays are null curves, one cannot use the proper time as a parameter to describe them. In fact, light rays are supposed to follow null affine geodesics, which cannot be defined in terms of the functional (13), but, instead, they must be characterized by their behavior with respect to parallel transport. We shall extend this postulate by simply assuming that light rays follow Weyl null affine geodesics.

It is well known that null geodesics are preserved under conformal transformations, although one needs to reparametrize the curve in the new gauge. In the case of Weyl transformations, null geodesics are also invariant with no need of reparametrization, since, again, the connection components \( \Gamma^\mu_{\alpha\beta} \) do not change under (2) and (4), while condition
(15) is obviously not altered. As a consequence, the causal structure of spacetime remains unchanged in all Weyl frames. This seems to complete our programme of formulating GR in a geometrical setting that exhibits a new kind of invariance, namely that with respect to Weyl transformations.

4. General relativity as a scalar–tensor theory

In the present formalism it is interesting to rewrite the action (12) in Riemannian terms. This is done by expressing the Weyl scalar curvature $R$ in terms of the Riemannian scalar curvature $\tilde{R}$ and the scalar field $\phi$, which gives

$$R = \tilde{R} - (n - 1)\Box \phi + \frac{(n - 1)(n - 2)}{4} g^{\mu \nu} \phi_{,\mu} \phi_{,\nu},$$

where $\Box \phi$ denotes the Laplace–Beltrami operator. It is easily shown that, by inserting $R$ as given by (17) into equation (12) and using Gauss’ theorem to neglect divergence terms in the integral, one obtains

$$S_n = \int d^n x \sqrt{-g} e^{(1 - \frac{1}{2})\phi} \left\{ \tilde{R} + \omega g^{\mu \nu} \phi_{,\mu} \phi_{,\nu} + 2\Lambda e^{-\phi} + \kappa e^{-\phi} L_m \right\},$$

where $\omega = \frac{(n - 1)(2 - n)}{4}$. For $n = 4$ we have $\omega = -\frac{3}{2}$ and the action becomes

$$S = \int d^4 x \sqrt{-g} e^{-\phi} \left\{ \tilde{R} - \frac{3}{2} g^{\mu \nu} \phi_{,\mu} \phi_{,\nu} + 2\Lambda e^{-\phi} + \kappa e^{-\phi} L_m \right\}.$$  

In the next section, it will be convenient to change the scalar field variable $\phi$ by defining $\Phi = e^{-\phi}$. In terms of the new field $\Phi$, the action (19) takes the form

$$S = \int d^4 x \sqrt{-g} \left\{ \Phi \tilde{R} - \frac{3}{2\Phi} g^{\mu \nu} \Phi_{,\mu} \Phi_{,\nu} + 2\Lambda \Phi^2 + \kappa \Phi^2 L_m \right\}.$$  

If we take variations of $S$, as given by (19), with respect to $g_{\mu \nu}$ and $\phi$, these being considered as independent fields, we shall obtain, respectively,

$$\tilde{G}_{\mu \nu} - g_{\mu \nu} \Box \phi - \frac{1}{2} (g_{\mu \nu} \phi_{,\nu} + \frac{1}{2} g_{\mu \nu} \phi_{,\alpha} \phi_{,\alpha}^a) = e^{-\phi} \Lambda g_{\mu \nu} - \kappa T_{\mu \nu},$$

$$\tilde{R} - 3\Box \phi + \frac{1}{2} \phi_{,a} \phi_{,a}^a = \kappa T - 4e^{-\phi} \Lambda,$$

where $\tilde{G}_{\mu \nu}$ and $\tilde{R}$ denote the Einstein tensor and the curvature scalar, both calculated with the Riemannian connection, and $T = g^{\mu \nu} T_{\mu \nu}$. It should be noted that (22) is just the trace of (21), and so the above equations are not independent. This is consistent with the fact that we have complete freedom in the choice of the Weyl frame. It also means that $\phi$ may be viewed as an arbitrary gauge function and not as a dynamical field.

It is straightforward to verify that in terms of the variable $\Phi = e^{-\phi}$, equations (21) and (22) read

$$\tilde{G}_{\mu \nu} = -\kappa T_{\mu \nu} + \Lambda \Phi g_{\mu \nu} + \frac{3 \Phi^2}{2} \left( \Phi_{,\mu} \Phi_{,\nu} - \frac{1}{2} g_{\mu \nu} \Phi_{,a} \Phi_{,a}^a \right) - \frac{1}{\Phi} (\Phi_{,\mu} \Phi_{,\nu} - g_{\mu \nu} \Box \Phi),$$

$$\tilde{R} + 3 \Box \Phi - \frac{3}{2\Phi^2} \Phi_{,a} \Phi_{,a}^a = \kappa T - 4 \Phi \Lambda.$$  

Some considerations should be made on the form taken by the energy–momentum tensor $T_{\mu \nu}$, which appears on the right-hand side of equations (21) and (23). Here, as well as in the previous development of the formalism that leads to the formulation of GR in a Weyl integrable

3 We found that, in [16], a similar action, in the case of vacuum, was obtained by using an argument based on the Palatini approach.
manifold, we use the effective metric \( \hat{g} = e^{-\phi} g \) as a guide to ensure Weyl invariance. In this way, it is natural to define the energy–momentum tensor \( T_{\mu\nu}(\phi, g, \xi, \nabla \xi) \) of the matter field \( \xi \), in an arbitrary Weyl frame \((M, g, \phi)\), by the formula

\[
\delta \int d^4x \sqrt{-\hat{g}} L_m(g, \phi, \xi, \nabla \xi) = \int d^4x \sqrt{-g} e^{-2\phi} T_{\mu\nu}(\phi, g, \xi, \nabla \xi) \delta(e^\phi g^{\mu\nu}), \tag{25}
\]

where the variation on the left-hand side must be carried out simultaneously with respect to both \( g_{\mu\nu} \) and \( \phi \). In order to see that the above definition makes sense, first recall that \( L_m(g, \phi, \xi, \nabla \xi) \) is given by the prescription \( \eta_{\mu\nu} \rightarrow e^{-\phi} \hat{g}_{\mu\nu} \) and \( \partial_\mu \rightarrow \nabla_\mu \), where \( \nabla_\mu \) denotes the covariant derivative with respect to the Weyl affine connection. Let us recall here that \( L_m(g, \phi, \xi, \nabla \xi) \equiv L^\mu_\nu(g, \phi, \xi, \nabla \xi) \), where \( L^\mu_\nu \) denotes the Lagrangian of the field \( \xi \) in flat Minkowski spacetime. Secondly, it should be clear that the left-hand side of equation (25) can always be put in the same form of the right-hand side of the same equation. This can easily be seen from the fact that \( \delta L_m = \frac{\partial L_m}{\partial \phi} \delta e^\phi + \frac{\partial L_m}{\partial \phi} \delta \phi = \frac{\partial L_m}{\partial (e^\phi g^{\mu\nu})} \delta (e^\phi g^{\mu\nu}) \) and that \( \delta (\sqrt{-g} e^{-2\phi}) = -\frac{1}{2} \sqrt{-\hat{g}} e^{-3\phi} \hat{g}_{\mu\nu} \delta (e^\phi g^{\mu\nu}) \). Finally, it is clear that the definition of \( T_{\mu\nu}(\phi, g, \xi, \nabla \xi) \) given by (25) is invariant under the Weyl transformations (2) and (4).

We would like to conclude this section with a brief comment on the form that the equation that expresses the energy–momentum conservation law takes in an arbitrary Weyl frame. We start with the Einstein equations written in the Riemann frame \((M, \hat{g}, 0)\):

\[
G_{\mu\nu}(\hat{g}, 0) = -\kappa T_{\mu\nu}(\hat{g}, 0). \tag{26}
\]

Because \( G_{\mu\nu}(\hat{g}, 0) \) is divergenceless with respect to the metric connection \( \{^\alpha_{\mu\nu}\}_{\hat{g}} = \frac{1}{2} \hat{g}^{\alpha\beta} \{^\beta_{\mu\nu}\}_{\hat{g}} \) it follows from (26) that

\[
\nabla^\alpha T_{\mu}^\alpha = \nabla^\alpha (\hat{g}^{\mu\nu} T_{\nu}) = 0, \tag{27}
\]

where the symbol \( \nabla^\alpha \) denotes the covariant derivative defined by \( \{^\alpha_{\mu\nu}\}_{\hat{g}} \). If we now go to an arbitrary Weyl frame \((M, g = e^\phi \hat{g}, \phi)\), then a straightforward calculation shows that (27) takes the form

\[
\nabla^\alpha T_{\mu}^\alpha = T_{\mu}^\alpha \phi_{,\alpha} - \frac{1}{2} T \phi_{,\mu}, \tag{28}
\]

where \( T = \hat{g}^{\mu\beta} T_{\mu\beta} \hat{g} \) and \( \nabla^\alpha \) stands for the covariant derivative defined by the metric connection calculated with \( g \).

At first sight, due to the presence of non-vanishing terms on the right-hand side of (28) one may be led to think, erroneously, that in the Weyl frame we have an apparent violation of the energy–momentum conservation law. Nonetheless, we must remember that if one is not working in the Riemann frame, then the Weyl scalar field \( \phi \) is an essential part of the geometry and necessarily should appear in any equation describing the behavior of matter in spacetime. This explains the presence of \( \phi \) coupled with \( T_{\mu\nu} \) in (28). Note that if \( \phi = \text{const} \) we recover the familiar general-relativistic energy–momentum conservation equation. Finally, it is not difficult to verify that the above equation is invariant under the Weyl transformations (2) and (4).

5. Similarities with Brans–Dicke theory

We shall now take a look at some similarities between the Brans–Dicke theory of gravity and GR, when the latter is expressed in the formalism we have developed in the previous section. Possible relationships between Weyl integrable geometry and Jordan–Brans–Dicke theories have already been pointed out in the literature, although it should be clear that the transformation between the so-called Jordan and Einstein frames is not identical to a Weyl transformation [15].
Let us start by recalling that the field equations of Brans–Dicke theory of gravity may be written in the form [17]

\[ \widetilde{G}_{\mu\nu} = -\frac{\kappa^*}{\Phi} T_{\mu\nu} - \frac{\omega}{\Phi^2} \left( \Phi_{,\mu} \Phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \Phi_{,\alpha} \Phi^{,\alpha} \right) - \frac{1}{\Phi} (\Phi_{,\mu;\nu} - g_{\mu\nu} \Box \Phi), \] (29)

\[ \widetilde{R} - 2\omega \frac{\Box \Phi}{\Phi} + \frac{\omega}{\Phi^2} \Phi_{,\alpha} \Phi^{,\alpha} = 0, \] (30)

where \( \kappa^* = \frac{8\pi G}{c^4} \), and we are keeping the notation of the previous section, in which \( \widetilde{G}_{\mu\nu} \) and \( \widetilde{R} \) denote the Einstein tensor and the curvature scalar, respectively, calculated with respect to the metric \( g_{\mu\nu} \).

By combining (29) and (30) we can easily derive the equation

\[ \Box \Phi = \frac{\kappa^* T}{2\omega + 3}, \] (31)

which is the most common form of the scalar field equation usually found in the literature. Equation (31), however, is not defined for \( \omega = -\frac{3}{2} \), so for this value of \( \omega \) one has to use (30) instead, which then, becomes

\[ \widetilde{R} + \frac{3}{2\Phi^2} \Phi_{,\alpha} \Phi^{,\alpha} = 0. \] (32)

On the other hand, equation (29) for \( \omega = -\frac{3}{2} \) reads

\[ \widetilde{G}_{\mu\nu} = -\frac{\kappa^*}{\Phi} T_{\mu\nu} + \frac{3}{2\Phi^2} \left( \Phi_{,\mu} \Phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \Phi_{,\alpha} \Phi^{,\alpha} \right) - \frac{1}{\Phi} (\Phi_{,\mu;\nu} - g_{\mu\nu} \Box \Phi). \] (33)

Now, if we take the trace of (33) with respect to \( g_{\mu\nu} \) we obtain

\[ \widetilde{R} + \frac{3}{2\Phi^2} \Phi_{,\alpha} \Phi^{,\alpha} = \kappa^* T. \] (34)

Of course (32) and (34) are not compatible, unless \( T = 0 \), which, then, implies that when \( \omega = -\frac{3}{2} \) the Brans–Dicke field equations (29) and (30) cease to be independent, and the system of differential equations for \( g_{\mu\nu} \) and \( \Phi \) becomes undertermined. As a consequence, one may freely choose an arbitrary \( \Phi \) and work out a solution for \( g_{\mu\nu} \) from (33).

In particular, one can set \( \Phi = \Phi_0 = \text{const} \), in which case (33) becomes formally identical to the Einstein equations constant with the gravitational constant \( G \) replaced by \( \frac{1}{\Phi_0} \). At this point, it is interesting to note that one obtains the same result by means of the conformal transformation \( g_{\mu\nu} = e^{\Phi_0} g_{\mu\nu} \), since the conformally transformed Einstein tensor \( \widetilde{G}_{\mu\nu} \) is given by

\[ \widetilde{G}_{\mu\nu} = \frac{3}{2\Phi^2} \left( \Phi_{,\mu} \Phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \Phi_{,\alpha} \Phi^{,\alpha} \right) + \frac{1}{\Phi} (\Phi_{,\mu;\nu} - g_{\mu\nu} \Box \Phi). \] (It is curious that one could use this property to generate an infinite class of Brans–Dicke theory for \( w = -\frac{3}{2} \) from known solutions of the Einstein equations.) This known mathematical fact is often interpreted in the literature as representing a conformal equivalence between Brans–Dicke gravity for \( w = -3/2 \) and GR [19, 20]. It will be noted, however, that, in spite of the amazing similarity of the field equations, we are far from having a complete analogy between the two theories. Indeed, when we turn to the motion of test particles, we immediately realize that in the Brans–Dicke theory it is postulated that these particles must follow Riemannian geodesics, whereas in the case of GR formulated in a Weyl frame (or in the case of conformal relativity) these must follow geodesics that are not Riemannian. In the following section, we shall illustrate this point with a simple example taken from a known vacuum solution of Brans–Dicke theory, namely the O’Hanlon–Tupper vacuum solution [21].
6. Brans–Dicke vacuum solutions for \( w = -3/2 \)

In the case of vacuum and vanishing cosmological constant, equations (23) and (24) reduce to

\[
\bar{G}_{\mu\nu} = \frac{3}{2\Phi^2} \left( \Phi_{,\mu} \Phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \Phi_{,\alpha} \Phi^{,\alpha} \right) - \frac{1}{\Phi} (\Phi_{;\mu;\nu} - g_{\mu\nu} \Box \Phi) \quad \text{and} \quad (35)
\]

\[
\bar{R} + \frac{3}{\Phi^2} \Box \Phi - \frac{3}{2\Phi^2} g_{\alpha\beta} \Phi^{;\alpha;\beta} = 0,
\]

respectively. As we have just mentioned, in this situation the equations of GR in an arbitrary Weyl frame ((23) and (24)) are identical to those of Brans–Dicke theory ((29) and (30)) for \( \omega = -\frac{3}{2} \), provided that we identify the Weyl scalar field with the Brans–Dicke scalar field. At this point, suppose we want to see how a solution of the above equations, regarded as a relativistic solution, i.e. Minkowski spacetime [23, 24]. For \( w = -\frac{3}{2} \), this solution has a big bang singularity as \( t \to 0 \). When \( \omega \to \infty \) it has the limit \( A(t) = A_0 t^p, \Phi = \Phi_0 t^q \), with \( p = \frac{1}{\omega + 1} (\omega + 1 \pm \sqrt{(2\omega + 3)/3}) \), and \( q = \frac{1}{\omega + 1} (1 \mp \sqrt{(2\omega + 3)/3}) \), with \( A_0 \) and \( \Phi_0 \) being integration constants [21]. For \( w > -\frac{3}{2} \), this solution does not go over the corresponding general relativistic solution, i.e. Minkowski spacetime [23, 24]. For \( w = -\frac{3}{2} \), we have \( A(t) = A_0 t^0 \) and \( \Phi = \Phi_0 t^{-2} \). This represents a model in which the so-called Dirac’s hypothesis does not hold, since the Newtonian gravitational ‘constant’, interpreted in Brans–Dicke theory as the inverse of the scalar field \( G \propto 1/\Phi \), decreases as the universe expands [25].

In order to interpret the O’Hanlon–Tupper model in the light of a general relativistic picture, we start by putting (36a) in the conformally flat form

\[
dx^2 = \bar{g}^{\mu\nu}(d\tau^2 - dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2),
\]

where we have made the coordinate transformation \( t = e^{\phi t} \) and defined \( \Psi(t) = 2(\tau + \ln A_0) \). In terms of the new coordinate, the Brans–Dicke scalar field is given by \( \Phi = \Phi_0 A_0^2 e^{-\Psi(t)} \).

Regarding both \( g_{\mu\nu} \) given by (37) and \( \Phi \) as describing the gravitational field in the Weyl frame \((M, g, \Phi)\), we now want to know how they will appear in a Riemann frame \((\bar{M}, \bar{g}, \bar{\Phi})\), that is, in a frame, where \( \bar{\Phi} \) is constant and, hence, the geometry is Riemannian. Recalling that the general form of the Weyl transformations (2) and (4) in terms of the variables \( \Phi = e^{-\phi} \) and \( \bar{\Phi} = e^{-\phi} \) is given by

\[
\bar{g}_{\mu\nu} = e^{\phi} g_{\mu\nu},
\]

\[
\bar{\Phi} = e^{-\phi} \Phi,
\]

it is clear that the natural choice of \( f \) that will turn \( \Phi \) into a constant is \( f = -\Psi(t) \). We thus are led to the Riemann frame \((M, \bar{g} = \eta, \bar{\Phi} = \Phi_0 A_0^2)\), where \( \eta \) denotes the Minkowski metric. Therefore, we conclude that the O’Hanlon–Tupper cosmological model, when regarded

\footnote{The O’Hanlon–Tupper solution for \( \omega = -\frac{3}{2} \) is identical to the cosmological model found by Singh and Shridhar for a radiation-filled Robertson–Walker universe [26].}
formally as a general relativistic solution in the Weyl frame \((M, g, \Phi)\), is equivalent to Minkowski spacetime, whose geodesics consist of straight lines satisfying the equations
\[
\frac{d^2 x^\mu}{d\tau^2} = 0. \tag{40}
\]
From the fact the affine geodesics are invariant under the Weyl transformations (2) and (4), and since in the Riemann frame \((\tilde{M}, \tilde{g} = \eta, \tilde{\Phi} = \Phi_0 A_0^2)\) the Weyl affine geodesics coincide with the metric geodesics, it is evident that in the Weyl frame \((M, g, \Phi)\) the affine geodesics will also be given by (40).

As we have already pointed out, the formal equivalence exhibited above between Brans–Dicke vacuum solutions for \(w = -\frac{1}{2}\) and general relativistic vacuum solutions expressed in a Weyl geometric setting is not complete. The reason is that we have not taken into account an aspect that is fundamental to any metric theory of gravity: How do we determine the motion of test particles and light? Indeed, as we have mentioned earlier, in the case of GR, the geodesic equations that govern the motion of test particles and light in an arbitrary Weyl frame are constructed with the affine connection coefficients, which explicitly involves the Weyl scalar field, and are invariant under Weyl transformations. Of course, we have a different situation in the case of Brans–Dicke theory, where, even in the presence of the scalar field, the geodesics are defined by the Lévi-Civitá connection. Therefore, in the O’Hanlon–Tupper model the geodesic motion of particles and light will not be given by (40). A short calculation shows that the Brans–Dicke geodesic equations are
\[
\frac{d^2 x^\mu}{d\tau^2} + \frac{d\Psi}{dr} \frac{dx^\mu}{d\tau} + \frac{e^{-\Psi}}{2} \Psi_{\mu}^\nu = 0.
\]
To conclude this section, we would like to show how the formal equivalence discussed above can be used to generate a whole class of vacuum solutions of Brans–Dicke field equations for \(\omega = -\frac{1}{2}\), which includes the O’Hanlon–Tupper model as a particular case. To do this, let us suppose that we want to obtain a solution of the field equations (35) corresponding to a homogeneous and isotropic spacetime. As we know, the most general form of the metric of such spacetime may be written as
\[
d s^2 = dt^2 - \frac{A(t)^2}{1 + \frac{k}{4}} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2), \tag{41}
\]
where \(k = 0, \pm 1\) represents the curvature of the spatial sections. We now regard (35) as the Einstein field equations in the Weyl frame \((M, g, \Phi)\), so that we can go to the Riemann frame \((\tilde{M}, \tilde{g}, \tilde{\Phi}) = 1\) through the Weyl transformations (38) and (39) by choosing \(f = \ln \Phi\). In the Riemann frame, the line element corresponding to \(\tilde{g}\) will be
\[
d \tilde{s}^2 = \Phi(t) \, dt^2 - \frac{\Phi(t)A(t)^2}{1 + \frac{\tilde{k}}{4}} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2). \tag{42}
\]
Defining a new time coordinate \(\tilde{t}\) by \(\Phi(t)^{1/2} \, dt = d\tilde{t}\) and putting \(\Phi(t)A(t)\tilde{t} = \tilde{A} \, \tilde{t}\), \(\tilde{t}\), \(41\) takes the form
\[
d \tilde{s}^2 = d\tilde{t}^2 - \frac{\tilde{A}(\tilde{t})^2}{1 + \frac{\tilde{k}}{4}} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2). \tag{43}
\]
Now, in the Riemann frame \(\tilde{g}_{\mu\nu}\) simply becomes
\[
\tilde{g}_{\mu\nu} = 0,
\]
with \(\tilde{g}_{\mu\nu}\) calculated with the metric \(\tilde{g}\). It may be readily verified that this yields only one independent equation, namely
\[
\left(\frac{d\tilde{A}}{d\tilde{t}}\right)^2 = -k\tilde{c}^2. \tag{44}
\]
An obvious conclusion that can be drawn from the above equation is that there are no solutions for \( k = 1 \) (this has been pointed out in [20]). If we take \( k = 0 \), then \( \lambda(t) = B \), where \( B \) is an arbitrary constant. Thus, from the definition of \( \lambda(t) \), we have \( \Phi(t)A(t)^2 = B \). This means that we have an infinite number of Brans–Dicke vacuum solutions for \( \omega = -\frac{3}{2} \), O’Hanlon–Tupper model merely corresponding to the particular choice \( A(t) \sim t \).

7. Different pictures of the same physical phenomena

As we have seen, when we go from one frame \((M, g, \phi)\) to another frame \((\overline{M}, \overline{g}, \overline{\phi})\) through the Weyl transformations (2) and (4), the pattern of affine geodesic curves does not change. However, distinct geometrical and physical pictures may arise in different frames. This is particularly evident in the case of a conformally flat spacetime, i.e. when we have in a Riemann frame \( g = e^\eta \eta \). In such situations, one can completely gauge away the Riemannian curvature by a frame transformation, thereby going to a frame in which one is left with a geometrical scalar field in a Minkowski background [27]. This is well illustrated, for instance, when we consider the class of Robertson–Walker (RW) spacetimes \((k = 0, \pm 1)\), which are known to be conformally flat [28]. If we go to the Weyl frame \((M, \eta, \phi)\) by means of a Weyl transformation we arrive at a new cosmological scenario in which the Riemannian curvature ceases to determine the cosmic expansion and other phenomena, these effects being now attributed to the sole action of a scalar field living in flat spacetime. There are many other examples of how distinct physical interpretations of the same phenomena are possible in different frames. By way of illustration, we shall consider, in this section, how one would describe, in a general Weyl frame, an important effect predicted by GR: the so-called gravitational spectral shift.

Let us consider the gravitational field generated by a massive body, which in an arbitrary Weyl frame \((M, g, \phi)\) is described by both the metric tensor \( g_{\mu\nu} \) and the scalar field \( \phi \). For the sake of simplicity, let us restrict ourselves to the case of a static field, in which neither \( g_{\mu\nu} \) nor \( \phi \) depends on time. Let us suppose that a light wave is emitted on the body at a fixed point with spatial coordinates \((r_E, \theta_E, \varphi_E)\) and received by an observer at a fixed point \((r_R, \theta_R, \varphi_R)\). Denoting the coordinate times of emission and reception by \( t_E \) and \( t_R \), respectively, the light signal, which in the Weyl frame corresponds to a null affine geodesic, connects the event \((t_E, r_E, \theta_E, \varphi_E)\) with the event \((t_R, r_R, \theta_R, \varphi_R)\). Let \( \lambda \) be an affine parameter along this null geodesic with \( \lambda = \lambda_E \) at the event of emission and \( \lambda = \lambda_R \) at the event of reception. If we write the line element in the form of \( ds^2 = g_{00}(r, \theta, \varphi)\, dx^2 = g_{\mu\nu}(r, \theta, \varphi)\, dx^\mu\, dx^\nu \), then, since the geodesic is null, we must have

\[
g_{00}(r, \theta, \varphi)\left(\frac{dt}{d\lambda}\right)^2 = g_{\mu\nu}(r, \theta, \varphi)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda},
\]

so we can write

\[
\frac{dr}{d\lambda} = \left[\frac{g_{\mu\nu}(r, \theta, \varphi)\, dx^\mu\, dx^\nu}{g_{00}(r, \theta, \varphi)\, dx^0\, dx^0}\right]^{1/2}.
\]

On integrating between \( \lambda = \lambda_E \) and \( \lambda = \lambda_R \) we have

\[
t_R - t_E = \int \left[\frac{g_{\mu\nu}(r, \theta, \varphi)\, dx^\mu\, dx^\nu}{g_{00}(r, \theta, \varphi)\, dx^0\, dx^0}\right]^{1/2}d\lambda.
\]

Because the integral on the right-hand side of the above equation depends only on the light path through space, and since the emitter and observer are at fixed positions in space, then \( t_R - t_E \) has the same value for all signals sent. This implies that for any two signals emitted at coordinate times \( t_E^{(1)} \) and \( t_E^{(2)} \) and received at \( t_R^{(1)} \) and \( t_R^{(2)} \), we have \( t_R^{(1)} - t_E^{(1)} = t_R^{(2)} - t_E^{(2)} \).
which means that the coordinate time difference \( \Delta t_E = t_E^{(2)} - t_E^{(1)} \) at the event of emission is equal to the coordinate time difference \( \Delta t_R = t_R^{(2)} - t_R^{(1)} \) at the event of reception. On the other hand, we know from section 3 that the proper time recorded by clocks in a general Weyl frame must be calculated by using the formula

\[
\Delta \tau = \int_a^b e^{-\frac{\phi}{\Delta}} \left( \mathcal{G}^\mu_\nu \frac{dx^\mu}{dx^\nu} \right)^\frac{1}{2} \, d\lambda.
\]

Therefore, the proper time recorded by the clocks of observers situated at the body and at the point of reception will be given by

\[
\Delta \tau_E = e^{-\frac{\phi}{\Delta}} \sqrt{g_{00}(r_E, \theta_E, \psi_E)} \Delta t_E,
\]

and

\[
\Delta \tau_R = e^{-\frac{\phi}{\Delta}} \sqrt{g_{00}(r_R, \theta_R, \psi_R)} \Delta t_R,
\]

where \( \phi_E = \phi(r_E, \theta_E, \psi_E) \) and \( \phi_R = \phi(r_R, \theta_R, \psi_R) \). Since \( \Delta t_E = \Delta t_R \), we have

\[
\frac{\Delta \tau_R}{\Delta \tau_E} = \frac{e^{-\frac{\phi_R}{\Delta}} \sqrt{g_{00}(r_R, \theta_R, \psi_R)}}{e^{-\frac{\phi_E}{\Delta}} \sqrt{g_{00}(r_E, \theta_E, \psi_E)}}.
\]

Suppose now that \( n \) waves of frequency \( v_E \) are emitted in proper time \( \Delta t_E \) from an atom situated on the body. Then \( v_E = \frac{n}{\Delta \tau_E} \) is the proper frequency measured by an observer situated at the body. On the other hand, the observer situated at the fixed point \( (r_R, \theta_R, \psi_R) \) will see these \( n \) waves in a proper time \( \Delta t_R \), hence will measure a frequency \( v_R = \frac{n}{\Delta \tau_R} \). Therefore, we have

\[
\frac{v_R}{v_E} = \frac{e^{-\frac{\phi_R}{\Delta}} \sqrt{g_{00}(r_R, \theta_R, \psi_R)}}{e^{-\frac{\phi_E}{\Delta}} \sqrt{g_{00}(r_E, \theta_E, \psi_E)}}.
\]

(47)

We, thus, see that \( v_R \neq v_E \), i.e. the observed frequency differs from the frequency measured at the body, and this constitutes the spectral shift effect in a general Weyl frame.

To conclude, two points related to the above equation are worth noting. The first is that since in a Riemann frame \( \phi = 0 \) equation (47) reduces the well-known general relativistic formula for the gravitational spectral shift. The second point is that if we go to a Weyl frame where \( g_{00} \) is constant, then equation (47) becomes simply

\[
\frac{v_R}{v_E} = e^{\frac{1}{2}(\phi_R - \phi_E)}.
\]

As we see, in this frame all information concerning the gravitational field is contained in the Weyl scalar field.

8. WIST theory viewed in the Riemann frame

In section 2, we have commented briefly on the close correspondence between the mathematical structure of Weyl integral geometry and Riemannian geometry. More precisely, we have shown that to each Weyl frame \((M, g, \phi)\) there corresponds a unique Riemann frame \((M, \tilde{g} = e^{-\phi} g, 0)\), such that geometrical objects constructed from \( g \) and \( \phi \) in the frame \((M, g, \phi)\), such as the affine connection coefficients, curvature, geodesics, etc, can be carried over to \((M, \tilde{g}, 0)\) without ambiguity, and vice versa. This fact makes us wonder how some gravity theories formulated in a Weyl integral spacetime would then appear when viewed in the Riemann frame \((M, \tilde{g} = e^{-\phi} g, 0)\). A good representative of these theories, on which we would like to focus our attention now, is a proposal known as the Weyl integrable spacetime (WIST) [12]. Let us recall the basic tenets of this theory.
The WIST approach starts by postulating the action

$$S = \int d^4x \sqrt{-g} \left\{ R + \omega \phi^{\mu} \phi_{,\mu} + e^{-2\phi} L_m \right\},$$  \hspace{1cm} (48)

where $R$ denotes the Weylian curvature, $\phi$ is the scalar Weyl field, $\omega$ is a dimensionless parameter and $L_m$ is the Lagrangian of the matter fields. It is also postulated that the form of $L_m$ is obtained from the corresponding Lagrangian in special relativity by substituting simple derivatives by covariant derivatives with respect to the Weyl connection. As regards to the above action, two comments are in order. The first is that it is not invariant under the Weyl transformations (2) and (4). The second, as we shall show now, is that when we go to the Riemann frame (\(M, \tilde{g} = e^{-\phi} g, 0\)) through the Weyl transformations \(\tilde{g}_{\mu\nu} = e^{-\phi} g_{\mu\nu}, \tilde{\phi} = \phi - \phi = 0\), then (48) becomes

$$S = \int d^4x \sqrt{-\tilde{g}} \tilde{e}^{\phi} \left\{ \tilde{R} + \omega \tilde{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + L_m \right\},$$  \hspace{1cm} (49)

where by \(\tilde{R}\) we are denoting the scalar curvature defined in terms of \(\tilde{g}_{\mu\nu}\). Changing to the field variable \(\Phi = e^\phi\), we finally obtain

$$S = \int d^4x \sqrt{-\tilde{g}} \tilde{e}^{\Phi} \left\{ \tilde{R} + \omega \tilde{g}^{\mu\nu} \Phi_{,\mu} \Phi_{,\nu} + L_m \right\},$$  \hspace{1cm} (50)

which we immediately recognize as the action of Brans–Dicke theory of gravity written in units such that \(8\pi c^4 = 1\) [17]. We thus see that in the Riemann frame the WIST action (48) is formally identical to the Brans–Dicke action (50), where $\Phi$ is no longer interpreted as a geometrical field. We believe this to be a new result, which seems to have some relevance given the relatively large literature on WIST theories. This duality also reminds us of a similar situation in which Brans–Dicke theory is interpreted in two different frames, the Jordan and Einstein frames, an issue widely discussed in the literature [29].

The mathematical analogy between WIST and Brans–Dicke theories works in both directions. Thus, one may start the action (50), which gives Brans–Dicke theory in the usual Riemannian (Jordan) frame, and then go to the Weyl frame (Einstein frame) in which the action takes the form of (48), where the scalar field $\phi$ might be interpreted as a geometrical field. The usual view—let us say, the non-geometrical view—is that we have the same Brans–Dicke theory in two different frames, the Jordan and Einstein frame. The physical interpretation of the two pictures has been widely discussed in the literature [29]. However, a characteristic feature of Brans–Dicke theory is that Newton’s gravitational constant $G$ is replaced by the inverse of the scalar field, i.e. $G = \Phi^{-1}$, an idea that goes back to Dirac [25]. Similar to the original Weyl theory, which represents an elegant way of geometrizing the electromagnetic field [5], the same can be said of the WIST theory as regards to the scalar field: we have here a geometrization of a scalar field. In view of this analogy, the passage from the Jordan frame to the Einstein frame may be interpreted as a ‘geometrization’ of $G$, the empirical physical quantity that sets the strength of the gravitational force, now promoted to the status of a field. One may perhaps feel inclined to regard this geometrical attempt to explain the origin of $G$ as being in accordance with the Machian view that local physical laws are determined by the large-scale structure (geometry) of the universe [31].

It is worth noting that a connection between Brans–Dicke theory and Weyl integrable geometry appears in a different context. In fact, this connection has been proved to exist for any scalar–tensor theory in which the scalar field is non-minimally coupled to the metric [16, 32]. Without going into the details, the argument is as follows. We start with the action (49) in the absence of matter and consider variations in the sense of the Palatini approach, i.e. treating the metric and the affine connection separately as dynamical variables. It is then not difficult to show that the variation with respect to the connection leads to equation (1), that is, the compatibility condition that defines a Weyl integrable manifold.
9. Final remarks

As we have seen, it is possible to set up a different scenario of general relativity theory in which the gravitational field is not associated with the metric tensor only, but with the combination of both the metric $g_{\mu\nu}$ and a geometrical scalar field $\phi$. In this scenario, we have a new kind of invariance and the same physical phenomena may appear in different pictures and distinct representations. This can be well illustrated when we consider, for instance, homogeneous and isotropic cosmological models. All these have a conformally flat geometry, and as a consequence, there is a frame in which the geometry of these models becomes that of flat Minkowski spacetime. In the Riemann frame, the spacetime manifold is endowed with a metric that leads to the Riemannian curvature, while in the Weyl frame spacetime is flat. In this case, all information about the gravitational field is encoded in the scalar field. Another example is given by the gravitational spectral shift, in which the Weyl scalar field plays an essential role.

The presence of a scalar field in an arbitrary Weyl frame also leads to formal analogy with Brans–Dicke theory, a fact that has already been known and mentioned in the literature [19]. Because of this O’Hanlon–Tupper, spacetime in Brans–Dicke theory with $\omega = -\frac{1}{3}$ can be regarded as the Minkowski spacetime in a Weyl frame, although the analogy is not perfect since in Brans–Dicke theory test particles follow metric geodesics rather than affine Weyl geodesics.

An important conclusion to be drawn from what has been presented in this paper is that general relativity can ‘survive’ perfectly in a non-Riemannian environment. Moreover, as far as physical observations are concerned, all Weyl frames, each one determining a specific geometry, are completely equivalent. In a certain sense, this would remind us of the view conceived by Henri Poincaré that the geometry of spacetime is perhaps a convention that can be freely chosen by the theoretician [33]. In particular, according to this view, general relativity might be rewritten in terms of an arbitrary conventional geometry [34].

Finally, we should also note that the same formalism we have used to recast general relativity in a form that is manifestly invariant under Weyl transformations may be extended in a straightforward way to the so-called $f(R)$ theories [35], where the issue of physical interpretation between the Einstein and Jordan frames may be of interest [36]. The basic idea here is to start with the action $S = \int d^4x \sqrt{-g} \left( f(R) + \kappa L_m(g, \xi) \right)$, where $\xi$ stands generically for the matter fields. We then follow the same procedure presented in section 3 and postulate that this action may be regarded as defined in a Weyl integral spacetime in a particular frame where the Weyl scalar field vanishes, that is, in the Riemann frame. The next step is almost obvious: using the fact that the combination $e^{\phi}R$ is invariant under (2) and (4) the sought-after action in an arbitrary Weyl frame will be given by $S = \int d^4x \sqrt{-ge^{-2\phi}} \left( f(e^{\phi}R) + \kappa L_m(g, \xi) \right)$, where for the definition of $L_m(g, \xi)$ in an arbitrary frame the prescriptions outlined in section 3 still apply. We leave the details of this extension for a separate publication.

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