COULOMB POTENTIAL OF A POINT MASS IN THETA NONCOMMUTATIVE GEOMETRY

A. LEWIS LICHT

ABSTRACT. We investigate the form of the Coulomb potential of a point charge in a noncommutative geometry, using a state of minimal dispersion. We find the deviation of the potential at large distances from the point, distinguishing between coordinate distance and measured distance. Defining the “effective” value of an operator as its expectation value in a minimum dispersion state centered at a point, we find the effective potential to be finite at the origin, the effective charge density to be Gaussian and the effective total electrostatic energy to be finite. However, the true total electrostatic energy operator is shown to still be infinite.

1. INTRODUCTION

We consider the form of noncommutative geometry where the operators that measure position, the “physical” position operators, satisfy

\[ [x^i, x^j] = \theta^{ij} \]

Where \( \theta^{ij} \) is a constant antisymmetric tensor, and where the indexes \( i, j = 1, 2, 3 \). The components \( \theta^{0,i} \) are assumed to vanish. The physical momentum operators are assumed to have the conventional commutation relations \[1\]

\[ [p^i, p^j] = 0 \\
[x^i, p^j] = i\delta^{ij} \]

We investigate for such a geometry the problem of measuring distance from a point. We also investigate the operational definition of the value at a point of such quantities as the Coulomb potential, the charge distribution and the electric field.

Let \( |\Psi_0\rangle \) denote a point-like state at the origin. This could be a wave packet spherically symmetric about the origin in commutative theory, or as spherically symmetric a state as possible in NC theory. The spherical symmetry implies

\[ \langle \Psi_0 | x | \Psi_0 \rangle = 0 \]

We translate the state \( |\Psi_0\rangle \) to a point a distance \( a \) from the origin, obtaining the state

\[ |\Psi_0(a)\rangle = \exp(-ia \cdot p) |\Psi_0\rangle \]

The vector \( a \) is a coordinate displacement. Its length \( a \) is not actually the displacement distance that would be measured. The measured distance is

\[ \langle r \rangle = \langle \Psi_0(a) | \sqrt{x \cdot x} | \Psi_0(a) \rangle \]

\[ = \langle \Psi_0 | \sqrt{x \cdot x + 2x \cdot a + a^2} | \Psi_0 \rangle \]

\[ = a + \langle \Psi_0 \left| x \cdot \hat{n} + \frac{1}{2a} \left( x^2 - (x \cdot \hat{n})^2 \right) \right| \Psi_0 \rangle + O \left( \frac{1}{a^2} \right) \]
Where \( \hat{n} \) is the unit vector along \( \mathbf{a} \). Using Eq. (3), this becomes

\[
\langle r \rangle = a + \frac{1}{2a} \langle \Psi_0 \left| \left( \mathbf{x}^2 - (\mathbf{x} \cdot \hat{n})^2 \right) \right| \Psi_0 \rangle + O \left( \frac{1}{a^2} \right)
\]

The Coulomb potential that would be measured at the origin due to a point particle at the coordinate position \( \mathbf{a} \) is

\[
\langle \frac{1}{r} \rangle = \langle \Psi_0 (\mathbf{a}) | \frac{1}{\sqrt{\mathbf{x}^2}} | \Psi_0 (\mathbf{a}) \rangle = \langle \Psi_0 | \frac{1}{\sqrt{\mathbf{x}^2 + 2\mathbf{x} \cdot \mathbf{a} + a^2}} | \Psi_0 \rangle = \frac{1}{a} - \langle \Psi_0 | \left( \frac{1}{2a^3} (\mathbf{x}^2 - 3(\mathbf{x} \cdot \hat{n})^2) \right) | \Psi_0 \rangle + O \left( \frac{1}{a^4} \right)
\]

This is however expressed in terms of coordinate distance \( \mathbf{a} \). Eq. (6) may be used to express it in terms of the observed distance \( \langle r \rangle \)

\[
\langle \frac{1}{r} \rangle = \frac{1}{\langle r \rangle} + \frac{1}{\langle r \rangle^3} \left( \langle \mathbf{x}^2 \rangle - 3 \langle (\mathbf{x} \cdot \hat{n})^2 \rangle \right) + O \left( \frac{1}{\langle r \rangle^4} \right)
\]

For any operator \( A \), the expectation value in the translated state

\[
\langle \Psi_0 (\mathbf{a}) | A | \Psi_0 (\mathbf{a}) \rangle = \langle \Psi_0 | \exp (+i\mathbf{a} \cdot \mathbf{p}) A \exp (-i\mathbf{a} \cdot \mathbf{p}) | \Psi_0 \rangle = \langle \Psi_0 | A (\mathbf{a}) | \Psi_0 \rangle
\]

can also be interpreted as measuring the operator \( A \) at the “coordinate point” \( \mathbf{a} \) in the compact state \( \Psi_0 \). We interpret this as the “effective” value of the operator \( A \) at \( \mathbf{a} \) in the state \( \Psi_0 \).

We examine measured distance and Coulomb potential for a commutative geometry in Section (2) and for \( \theta \) NC geometry in Section (3). In Section (4) it is shown that in \( \theta \) NC geometry the effective Coulomb potential, as seen from a compact state, is finite at the origin. The effective charge density is derived in Section (5), and the effective elective field in Section (6). The total “effective” electrostatic energy is calculated and found to be finite. However, in Section (7) we consider the true electric field operator and show that the integral over all space of the corresponding electrostatic energy density can be defined in \( \theta \) NC geometry and is infinite.

2. Commutative Geometry

We consider here an ordinary free particle in a Gaussian wave packet of coordinate dispersion \( \sigma \). The wave function in \( x \) space is

\[
\Psi (\mathbf{x}) = \frac{1}{(2\pi \sigma)^{3/4}} \exp \left( -\frac{\mathbf{x}^2}{4\sigma} \right)
\]

For this particle,

\[
\langle x^2 \rangle = 3\sigma,
\]

\[
\langle (\mathbf{x} \cdot \hat{n})^2 \rangle = \sigma
\]
Then the measured distance is
\[ \langle r \rangle = a + \sigma_a + O \left( \frac{1}{a^2} \right) \] (12)
and the Coulomb potential is
\[ \left\langle \frac{1}{r} \right\rangle = \frac{1}{\langle r \rangle} + \frac{\sigma}{\langle r \rangle^3} + O \left( \frac{1}{\langle r \rangle^4} \right) \] (13)

Of course, for this system \( \sigma \) can be made arbitrarily small, and the Coulomb potential arbitrarily close to \( \frac{1}{r} \).

3. Theta Noncommutativity

The operators
\[ \xi^i = x^i + \frac{\theta_i^k}{2} p^k \] (14)
have the nice property that
\[ \left[ \xi^i, \xi^j \right] = 0 \]
\[ \left[ \xi^i, p^j \right] = i \delta_{ij} \] (15)
The \( \xi^i \) can be considered as hypothetical coordinate operators, where the \( x^i \) are the operators that correspond to actual physical measurements of position.

The anti-symmetry of the \( \theta^{ij} \) matrix allows us to introduce a vector \( \theta \) such that
\[ \theta^{ij} = \epsilon^{ijk} \theta^k \] (16)
\[ \theta^k = \frac{1}{2} \epsilon^{ijk} \theta_{ij} \]
Then
\[ x = \xi + \frac{\lambda}{2} \theta \times p \] (17)
and the distance squared operator becomes
\[ x^2 = \xi^2 - \theta \cdot \xi \times p + \frac{\theta^2}{4} \left( p^2 - (\theta \cdot p)^2 \right) \] (18)

We find this operator’s spectrum by introducing creation and annihilation operators:
\[ u_{\alpha} = \frac{1}{\sqrt{\theta}} \xi_{\alpha} + i \frac{\sqrt{\theta}}{2} p_{\alpha} \]
\[ \zeta = \xi_3 \] (19)
where \( \alpha = 1, 2 \). Then
\[ x^2 = \zeta^2 + i \theta \left( u_1^\dagger u_2 - u_2^\dagger u_1 \right) + \theta \left( u_1^\dagger u_1 + u_2^\dagger u_2 + 1 \right) \] (20)
With the substitution

\[
\begin{pmatrix}
u_1 \\ u_2
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}1 & 1 \\ i & -i\end{pmatrix} \begin{pmatrix}a \\ b\end{pmatrix}
\]

Where \(a\) and \(b\) are annihilation operators, Eq. (20) becomes

\[
x^2 = \zeta^2 + \theta (2b^i b + 1)
\]

The angular momentum operator \(\theta \cdot \xi \times \mathbf{p}\) corresponding to rotations about the 3-axis is now

\[
\theta \cdot \xi \times \mathbf{p} = a^\dagger a - b^\dagger b
\]

The general basis state can be written as

\[
|\Psi_{nm}(\zeta)\rangle = \psi(\zeta) |nm\rangle
\]

where \(\psi(\zeta)\) is an arbitrary wavefunction, and \(|nm\rangle\) is a harmonic oscillator basis state, with

\[
|nm\rangle = \frac{a^{\dagger n} b^{\dagger m}}{\sqrt{n!m!}} |00\rangle
\]

To get the most compact state, with maximum symmetry, so that Eq. (3) is satisfied, we take \(\psi(\zeta)\) to be a gaussian with dispersion \(\sigma\), and \(n = m = 0\). Then our standard state has the wavefunction

\[
\langle \xi_\perp, \zeta | \Psi_{00}\rangle = \frac{1}{(2\pi)^{\frac{3}{2}} \sigma^\frac{3}{2} \left(\frac{\theta}{4\pi}\right)^\frac{3}{2}} \exp \left[ -\frac{\xi_\perp^2}{\theta} - \frac{\zeta^2}{4\sigma} \right]
\]

Then with \(\mathbf{\hat{n}} = (\alpha, \phi)\), we get

\[
\langle \mathbf{x}^2 \rangle = \sigma + \theta
\]

and

\[
\langle (\mathbf{x} \cdot \mathbf{\hat{n}})^2 \rangle = \sigma \cos^2 \alpha + \frac{\theta}{2} \sin^2 \alpha
\]

For this system, the deviation from the \(1/r\) Coulomb potential depends on the direction along which the potential is measured. In terms of the coordinate distance \(a\), this is

\[
\langle \frac{1}{r} \rangle = \frac{1}{a} + \frac{1}{a^3} \left( \sigma \cos^2 \alpha - 1 + \frac{\theta}{2} (3 \sin^2 \alpha - 2) \right) + O \left( \frac{1}{a^4} \right)
\]
4. Effective Potential at Origin

We distinguish the vector operator that measures position, \( x \), from the auxiliary vector operator \( \xi = (\xi_\perp, \zeta) \). Eigenvalues of the latter we will consider as a kind of “coordinate” position, analogous to the coordinate position of general relativity. Expectation values of the operator \( x \) give the “physical” position.

It is of interest to ask what an operator, in particular the Coulomb potential, looks like in terms of coordinate position. A qualitative treatment of this problem has been given by Colatto et al. \[2\] To determine this, we consider the states

\[
| \Psi (\sigma, \sigma_\perp, a) \rangle = \int d^3 \xi |\xi \rangle \frac{1}{(2\pi)^{\frac{3}{2}} \sigma x} \exp \left[ -\frac{(\zeta - a)^2}{4\sigma} - \frac{(\xi_\perp - a_\perp)^2}{4\sigma_\perp} \right]
\]

These states give us a kind of window at the point \( a \), of widths \( \sqrt{\sigma}, \sqrt{\sigma_\perp} \) in the \( z, r \) directions, to determine the “effective” value of any operator \( A \) at that point:

\[
\langle A (a) \rangle = \langle \Psi (\sigma, \sigma_\perp, a) | A | \Psi (\sigma, \sigma_\perp, a) \rangle = \langle a | A | a \rangle
\]

To look at the Coulomb potential we use the Fourier transform expression

\[
\frac{1}{r} = \frac{1}{2\pi^2} \int d^3q \frac{1}{q^2} \exp [iq \cdot x]
\]

We note that

\[
\exp [iq \cdot x] = \exp \left[ iq \cdot \left( \xi + \frac{1}{2} \theta \times p \right) \right] = \exp [iq \cdot \xi] \exp \left[ iq \cdot \frac{1}{2} \theta \times p \right]
\]

since

\[
[q \cdot \xi, q \cdot \theta \times p] = iq \cdot \theta \times q = 0
\]

also

\[
\exp \left[ iq \cdot \frac{1}{2} \theta \times p \right] |\xi \rangle = | \xi + \frac{1}{2} \theta \times q \rangle
\]

Then

\[
\langle a | \frac{1}{r} | a \rangle = \frac{1}{2\pi^2 (2\pi)^{\frac{3}{2}}} \int d^3\xi' d^3\xi d^3q \frac{1}{q^2} \exp \left[ iq \cdot \xi' - \frac{(\zeta' - a)^2}{4\sigma} - \frac{(\xi_\perp' - a_\perp)^2}{4\sigma_\perp} \right] \times \langle \xi' | \xi + \frac{1}{2} \theta \times q \rangle
\]

which becomes

\[
\langle a | \frac{1}{r} | a \rangle = \frac{1}{2\pi^2} \int d^3q \frac{1}{q^2} \exp \left[ iq \cdot a - \frac{\sigma (q^3)}{2} - \frac{\sigma_\perp q_\perp^2}{2} \right]
\]
where

\( \sigma_p = \sigma_\perp + \frac{\theta^2}{16\sigma_\perp} \)

At very large \(|a|\), we can write this to first order in the dispersions,

\[
\langle a | \frac{1}{r} | a \rangle = \left[ 1 + \frac{\sigma}{2} \left( \frac{\partial}{\partial a^3} \right)^2 + \frac{\sigma_p}{2} \nabla_\perp^2 \right] \frac{1}{a}
\]

\( = \frac{1}{a} + \frac{1}{a^3} \left[ \sigma \left( 3 \cos^2 \alpha - 1 \right) + \sigma_p \left( 3 \sin^2 \alpha - 2 \right) \right] \)

The dispersion of Eq. (38) is minimized by \( \sigma_\perp = \frac{\theta}{4} \), which makes \( \sigma_p = \frac{\theta}{2} \) and Eq. (39) becomes identical with Eq. (29).

Unlike the Coulomb potential in commuting geometry, the effective potential in \( \theta \) NC is finite at \( a = 0 \), where we have

\[
\langle 0 | \frac{1}{r} | 0 \rangle = \frac{1}{2\pi^2} \int \frac{d^3q}{q^2} \exp \left[ -\frac{\sigma (q^3)^2}{2} - \frac{\sigma_p q_\perp^2}{2} \right]
\]

\( = \frac{1}{2\pi} \int_{-1}^{+1} dc \frac{1}{\sqrt{\sigma_p - (\sigma_p - \sigma)c^2}} \)

If \( \sigma > \sigma_p \), this becomes

\[
\langle 0 | \frac{1}{r} | 0 \rangle = \frac{1}{\pi \sqrt{\sigma - \sigma_p}} \sinh^{-1} \left( \frac{\sqrt{\sigma - \sigma_p}}{\sigma_p} \right)
\]

At \( \sigma = \sigma_p \) this becomes \( \frac{1}{\pi \sqrt{\sigma_p}} \), and for \( \sigma < \sigma_p \) we get

\[
\langle 0 | \frac{1}{r} | 0 \rangle = \frac{1}{\pi \sqrt{\sigma_p - \sigma}} \arcsin \left( \frac{\sqrt{\sigma_p - \sigma}}{\sigma_p} \right)
\]

which is finite at \( \sigma = 0 \) and equal to \( \frac{1}{2\sqrt{\sigma_p}} \).

5. The Charge Density

The charge density, expressed as

\[
\rho = -\frac{1}{4\pi} \nabla^2 \frac{1}{r}
\]

\( = \frac{1}{4\pi} \left[ \mathbf{p} \cdot \left[ \mathbf{p}, \frac{1}{r} \right] \right] \)

has expectation value in the coordinate states given by

\[
\rho(a) = \langle a | \rho | a \rangle
\]

\( = -\frac{1}{4\pi} \nabla^2 \langle a | \frac{1}{r} | a \rangle \)
Using Eq. (37) this becomes a Gaussian distribution:

\[ \rho(a) = \frac{1}{(2\pi\sigma^2)^{3/2}} \exp \left( -\frac{(a^3)^2}{2\sigma^2} - \frac{a^2}{2\sigma} \right) \]

6. The “Effective” Electric Field

With an effective potential at the point \( a \) defined as

\[ \phi(a) = \langle a | \frac{1}{r} | a \rangle \]

we can define an effective field at \( a \) as

\[ E(a) = -\nabla_a \phi(a) \]

The “effective” energy contained in this field may be of some interest. Defining it as

\[ U_E = \frac{1}{8\pi} \int d^3a E^2(a) \]

we find

\[ U_E = \frac{1}{8\pi} \int d^3a (\nabla \phi)^2 \]

\[ = -\frac{1}{8\pi} \int d^3a \phi \nabla^2 \phi \]

\[ = \frac{1}{2} \int d^3a \phi \rho \]

From Eqs. (37) and (45) we get, for \( \sigma < \sigma_p \),

\[ U_E = \frac{1}{2\sqrt{2\pi}\sqrt{\sigma_p - \sigma}} \arcsin \left( \frac{\sqrt{\sigma_p - \sigma}}{\sigma_p} \right) \]

At the extreme limit, when \( \sigma = 0 \) and \( \sigma_p = \frac{\theta}{2} \), this becomes \( U_E = \frac{1}{4\sqrt{\theta}} \).

7. The True Electric Field

Although the integral over all space of the square of the expectation value of the electric field operator is finite, this is not true of the operator \( E^2 \) itself. Let \( A(x) \) be any operator function of \( x \). It can be written as

\[ A(x) = \int \frac{d^3q}{(2\pi)^3} \hat{A}(q) \exp \left[ iq \cdot \left( \xi + \frac{1}{2} \theta \times p \right) \right] \]

where

\[ \hat{A}(q) = \int d^3a A(x + a) \exp (-iq \cdot a) \exp (-iq \cdot \xi) \]
in particular, the operator’s space integral can be found from

$$\tilde{A}(0) = \int d^3a \, A(x + a)$$

The operator $E^2$ can be written as

$$E^2(x) = \frac{1}{4\pi^4} \int \frac{d^3q' d^3q}{q'^2 (q + q')^2} q' \cdot (q + q') \exp \left[-\frac{i}{2} q' \cdot \theta \times p\right] \exp (iq \cdot x) \exp \left[\frac{i}{2} q' \cdot \theta \times p\right]$$

from which it follows that

$$\int d^3a \, E^2(x + a) = \frac{1}{4\pi^4} \int \frac{d^3q}{q^2}$$

and is infinite.

8. Conclusions

In $\theta$ NC geometry the physical operators corresponding to the measurement of position, the $x^i$, are non-commuting, but can be expressed in terms of the momentum operators $p^i$, and commuting “coordinate” operators $\xi^i$. States of minimum dispersion in $x^i$ can be expressed as Gaussian eigenfunctions of the $\xi^i$. These states are the closest one can get in $\theta$ NC to a point particle.

The antisymmetric tensor $\theta^{ij}$ defines a vector $\theta^k$. The physical distance to a point charge, for fixed coordinate distance, depends on the direction relative to $\theta$ along which the distance is measured. The departures of the measured Coulomb potential from $1/r$ are also direction dependent.

In commutative geometry, the expectation value of a field operator such as the potential, the charge distribution, or the electric field, in a Gaussian state located at a point, becomes equal to the value of the field at that point when the state’s dispersion goes to zero. One might therefore consider in $\theta$ NC the expectation of the value of the field in a minimum dispersion state centered at some point to be the “effective” equivalent of the value of the field at that point. We find the effective value of the potential at zero position to be finite. Also the integral over all space of the “effective” electrostatic energy density of a point charge is finite, as discussed by Colatto et al. \[2\] However, the “effective” value involves just one matrix element. We show that the integral over all space of the energy density operator is still infinite, just as in commutative geometry.

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References

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