Born’s rule from statistical mechanics of classical fields: from hitting times to quantum probabilities

Andrei Khrennikov
International Center for Mathematical Modelling in Physics and Cognitive Sciences
Linnaeus University, Växjö, S-35195, Sweden
e-mail: Andrei.Khrennikov@lnu.se

May 5, 2014

Abstract
We show that quantum probabilities can be derived from statistical mechanics of classical fields. We consider Brownian motion in the space of fields and show that such a random field interacting with threshold type detectors produces clicks at random moments of time. And the corresponding probability distribution can be approximately described by the formalism of quantum mechanics. Hence, probabilities in quantum mechanics and classical statistical mechanics differ not so much as it is typically claimed. The temporal structure of the “prequantum random field” (which is the $L_2$-valued Wiener process) plays the crucial role. Moments of detector’s clicks are mathematically described as hitting times which are actively used in classical theory of stochastic processes. Born’s rule appears as an approximate rule. In principle, the difference between the “precise detection probability rule” derived in this paper and the conventional Born’s rule can be tested experimentally. In our model the presence of the random gain in detectors plays a crucial role. We also stress the role of the detection threshold. It is not merely a technicality, but the fundamental element of the model.

keywords: classical statistical mechanics, infinite-dimensional state space, random fields. quantum probability of detection, derivation of Born’s rule, threshold detectors, asymptotics of error function, distribution of hitting times, Brownian motion in the space of fields

1 Introduction

In this paper we present a novel application of statistical mechanics with the infinite-dimensional state space, the space of classical fields. We study the
problem of interaction of a random field, e.g., the electromagnetic field, with a detector of the threshold type. And we found that, for the random field, the Brownian motion in the space of fields (the $L^2$-valued Wiener process), statistics of clicks produced by such a detector can be approximately described by the Born’s rule. In quantum mechanics (QM) the latter was postulated [1]. Whether it is possible to derive this rule from some natural physical principles is still the subject of intensive debates. Thus we show that classical statistical mechanics of fields (theory of random fields [6]–[10]) provides a possibility to derive the fundamental law of QM connecting theory with experimental statistics.

We state again that the basic quantum rule is derived as an approximate rule. (Opposite to the conventional quantum mechanical viewpoint; by the latter this rule is a “precise rule”. The small parameter of the model (determining the magnitude of deviation of the Born’s rule from the “precise rule” for the threshold type detectors interacting with the $L^2$-valued Wiener process) is the quantity

$$\epsilon \equiv \frac{E_{\text{pulse}}}{E_d} << 1,$$

where $E_d$ is the detection threshold (having the physical dimension of energy) and $E_{\text{pulse}}$ is the average energy of pulses emitted (randomly) by a source. Thus the theory is about random fields of low energy interacting with detectors with sufficiently high threshold. We derived the “precise formula” for the detection probability for the threshold type detectors. The basic idea behind this derivation is embedding of the problem of detection moments for the threshold detection into well developed theory of hitting times [12]. (The latter is widely used in e.g. statistical radio-physics [13]..) This theory gives us the probability distribution of detection times. The final (precise) formula, see (38), is quite complicated analytically. At the same time it is very simple from the viewpoint of numerical computation (summation of a series with quickly decreasing terms). In principle, this formula can be tested experimentally.

This paper can be considered as development of prequantum classical statistical field theory (PCSFT) [14]–[23]. PCSFT provided representation of quantum averages and probabilities as averages and correlations with respect to ensembles of classical fields; including entangled systems (cf. Hofer [4], de la Pena and Ceto [24], Casado et al. [25], Boyer [26], Cole [27], Roychoudhuri [28], Adenier [29]). So, QM and classical statistical mechanics became essentially closer than it was commonly believed. All quantum probabilistic quantities are
represented in the classical probabilistic framework, cf. Manko et al. [53–58], Hess et al. [39, 41], De Raedt et al. [40], Garola et al. [42]. PCSFT reproduced these quantities as averages of intensities of continuous random fields. However, quantum experimental statistics is based on discrete events, clicks of detectors. To come closer to experiment, PCSFT should be completed by measurement theory describing the transition from continuous random intensities to discrete random events of detection.

The first model of the threshold detection (TSD) of continuous random fields serving PCSFT was proposed in [43]. This model is mathematically complicated. The “prequantum random fields” reproducing quantum statistics are very singular (of the white noise type). The state space of this model is the space of Schwartz tempered distributions. In [44], it was observed that the same result can be achieved by using essentially simpler model, namely, Brownian motion (Wiener process). However, in [44] from the very beginning a rough asymptotics of the detection probabilities was used; hence, directly the Born’s rule (an approximate formula for detection probabilities) was derived. In the present paper we use deeper results from theory of classical random fields (for hitting times) and obtain the “precise rule” for threshold detection which can be compared with the Born’s rule, the approximate rule. Moreover, in [44] only internal degrees of freedom such as e.g. polarization were considered. In the present paper by starting with internal degrees of freedom we proceed to the really physical case of random fields on physical space. (The latter model is classical statistical mechanics with infinite-dimensional state space.)

As was already stressed, our approach is based on embedding of the detection problem into theory of random hitting times. As a consequence, we shall use the mathematical apparatus of classical probability and mathematical statistics. The error function of the Gaussian distribution and complementary error function (its asymptotic expansion) are our main mathematical tools, see, e.g., [45] for introduction and [46] for applications.

2 Threshold detectors

The detection procedure used in optical experiments is based on two steps threshold passing. The first threshold is so to say the “hardware threshold”. This is a part of the solid state physical structure of the detector, we call it the fundamental threshold. The second threshold is in some sense the “software threshold” – the discrimination threshold. This threshold is set by experiments, in the case of photo-multipliers and tungsten-based superconducting transition-edge sensors (W-TESs), or directly in the process of fabrication, in the case of silicon-avalanche-photodiodes [47]. In this paper we are basically interested in the discrimination threshold. However, we start with a brief discussion on fundamental thresholds. Hence, everywhere besides section 2.1 the “detection threshold” is just the discrimination threshold.
2.1 Fundamental thresholds of detectors

There is a difference in threshold’s structure between photo-multipliers, as the one used in [50], [51], [52], and more recent experiments (e.g., [53]–[57]) based on silicon-avalanche-photodiodes. However, detectors of both types are based on threshold processes. The total threshold is combined of a few different thresholds corresponding to different stages of the detection process. First, we describe fundamental thresholds corresponding to photo-multipliers and silicon-avalanche-photodiodes. In photo-multipliers the fundamental threshold is given by the work function, i.e., $E_{\text{fund}}$ coincides with the work function. In silicon-avalanche-photodiodes the fundamental threshold is given by the band gap, i.e., $E_{\text{fund}}$ coincides with the band gap.

2.2 Discrimination threshold

A lot of noise is involved in the process of detection, e.g. [47]–[49]. An important source of noise is the multiplication process. For example, in photo-multipliers, once an electron has been extracted from the metal, it is accelerated in vacuum by an electric field until its kinetic energy is enough to extract other bound electrons (secondary emission) when striking the surface of another metal surface (dynode), which will in turn be used accelerated onto other dynodes to free more and more electrons, until that flow of electrons becomes measurable as anodic current. The form and energy of output spikes vary significantly from one liberated energy carrier to another. Typically it is assumed that the gain is given by

$$G = \alpha \xi^N,$$

(2)

where $\xi$ is the multiplication factor of a single dynode, $\alpha$ is the fraction of photoelectrons collected by the multiplier structure, and $N$ is the number of stages in the photomultiplier. In the most simple model, $\xi$ can be assumed to follow a Poisson distribution about the average yield for each dynode, so that the gain is a compound Poisson process over $N$ identical stages. However, experimental measurements of the single photoelectron pulse height spectra from photo-multipliers exhibit a distribution with larger relative variance than predicted by the Poisson model (and in some case with a decreasing exponential distribution instead of a peaked one), and there is thus no universal description of multiplication statistics [47].

Besides of noise produced by detectors, noise responsible for so called dark counts plays an important role. This noise of the random background is inescapable. In average spikes corresponding to signal detections differ in the amplitude from noise generated spikes. This fact provides a possibility to filter noise generated spikes by using a discrimination threshold, denote the later by the symbol $E_d$. The selection of this threshold is a delicate procedure. By selecting too low $E_d$ experimenter would count too many noise generated spikes, in particular, dark counts. By selecting it too high experimenter would discard too many spikes generated by the signal. In both cases quantum statistics would be essentially disturbed.
During the last few years, a few leading groups in quantum optics started to use actively with Tungsten-based Superconducting Transition-Edge Sensors (W-TESs) – the ultra-sensitive microcalorimeters (in particular, in attempts to perform the EPR-Bell experiment with highly efficient detectors). Surprisingly usage of such modern detectors is also impossible without setting a proper discrimination threshold, see, e.g., [48], [49], for reviews. I guess that (since noise is everywhere) it is impossible to elaborate a photon detection procedure which is not of the threshold type.

3 Threshold detection scheme

3.1 Detection moment as hitting time

We consider a threshold type detector with the threshold $E_d$. It interacts with a random field $\phi(s, \omega)$, where $s$ is time and $\omega$ is a chance parameter describing randomness. For a moment, we consider the $\mathbb{C}$-valued random field (complex stochastic process). In section 4.1 we shall consider random fields valued in finite-dimensional complex space $H$. This correspond to detection of internal degrees of freedom such as e.g. polarization. We stress that the real physical situation corresponds to random fields with infinite-dimensional state space, e.g., $H = L_2(\mathbb{R}^3)$, the space of complex valued fields $\phi : \mathbb{R}^3 \to \mathbb{C}$ (or $\to \mathbb{C}^k$), see section ??

The energy of the field is given by $E(s, \omega) = |\phi(s, \omega)|^2$ (hence, the random field has the physical dimension $\sim \sqrt{\text{energy}}$). (In section 5 we shall also consider spatial variables. In this case the field has the physical dimension of the energy density, i.e., energy per volume.) A threshold detector clicks at the first moment of time $\tau(\omega)$ when signal’s energy $E$ multiplied by the gain $g$, see section 2.2, exceeds the threshold:

$$gE(\tau(\omega), \omega) \geq E_d.$$  \hspace{1cm} (3)

In the mathematical model the detection moment is defined as the first hitting time:

$$\tau(\omega) = \inf \{ s \geq 0 : gE(s, \omega) \geq E_d \}.$$ \hspace{1cm} (4)

3.2 PDF of the detection moment

We proceed under the following basic assumption. After arriving to a threshold type detector a classical signal (random field) behaves inside this detector as the (complex) Brownian motion, i.e., the $\phi(s, \omega)$ is simply the Wiener process, the Gaussian process having zero average at any moment of time

$$E\phi(s, \omega) = 0.$$ \hspace{1cm} (5)

and the covariance function

$$E\phi(s_1, \omega)\overline{\phi(s_2, \omega)} = \min(s_1, s_2)\sigma^2;$$ \hspace{1cm} (6)
in particular, we can find average of its energy

\[ E\mathcal{E}(s, \omega) = \sigma^2 s. \tag{7} \]

From this equation, we see that the coefficient \( \sigma^2 \) has the physical dimension of power. We are interested in the probability distribution of the moments of the \( \mathcal{E}_d \)-threshold detection for the energy of the Brownian motion. Since moments of detection are defined formally as hitting times, we can apply theory of hitting times \[12], [8]. Consider

\[ \tau_a(\omega) = \inf\{s \geq 0 : \mathcal{E}(s, \omega) \geq a^2\} = \inf\{s \geq 0 : |\phi(s, \omega)| \geq a\}. \tag{8} \]

Its probability distribution function (PDF) is given by the complicated expression, see, e.g., Shyryaev [8]:

\[ P(\tau_a \leq \Delta t) = 4 \sum_{k=0}^{\infty} (-1)^k \left[ 1 - \Phi\left( \frac{a(1 + 2k)}{\sqrt{\sigma^2 \Delta t}} \right) \right], \tag{9} \]

where \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \) is the PDF of the standard Gaussian distribution. To show close relation to classical statistics, we also consider the error function \[45\]

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-u^2} du \]

and the complementary error function \[45\]

\[ \text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-u^2} du. \]

Now we select \( a = \sqrt{\frac{E_d}{g}} \) and set \( \tau \equiv \tau_a \). We obtain:

\[ P(\tau(\omega) \leq \Delta t) = 2 \sum_{k=0}^{\infty} (-1)^k \text{erfc}\left( (1 + 2k) \sqrt{\frac{E_d}{2\sigma^2 \Delta t g}} \right). \tag{10} \]

### 3.3 Asymptotics for the detection moment

In coming considerations, \( \Delta t \) is the \textit{average duration of the interaction of a signal with a threshold detector}. The experimental scheme can be described in the following way. There is a source of random pulses of e.g. classical electromagnetic field. (Such pulses can be identified with wave packets used in the quantum formalism.) Each pulse propagates in space (we shall discuss its dynamics) and finally arrives to a detector. The aforementioned temporal parameter \( \Delta t \) is the (average) duration of interaction of an input pulse with a detector. The quantity

\[ \overline{\tau}_{\text{pulse}} = \sigma^2 \Delta t \tag{11} \]
is the average energy of emitted pulses. We shall proceed under the basic assumption that this energy is essentially less than the threshold, i.e.,

$$\epsilon \equiv \frac{\overline{E}_{\text{pulse}}}{E_d} << 1.$$  \hspace{1cm} (12)

This is a realistic assumption, since the discrimination threshold is set for the amplified signals from the detector and the gain producing this amplification is very large. Under this assumption the probability distribution \(P(\tau(\omega) \leq \Delta t)\) can be approximated by the first term in the series (1):

$$P(\tau(\omega) \leq \Delta t) \approx 2 \text{erfc}(1/\sqrt{2\epsilon g}).$$  \hspace{1cm} (13)

In further consideration we shall use the asymptotic behavior of the complementary error function [15] for \(x \to \infty\):

$$\text{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \left[ 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2x^2)^n} \right] = \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2x^2)^n}.\hspace{1cm} (14)$$

To proceed towards derivation of the Born’s rule, we are fine with the rough approximation

$$\text{erfc}(x) \approx \frac{e^{-x^2}}{x\sqrt{\pi}}.$$  \hspace{1cm} (15)

(So, in our approach the Born’s rule provides a rather rough approximation of detection probabilities.) Hence,

$$P(\tau(\omega) \leq \Delta t) \approx 2 \sqrt{\frac{2\epsilon g}{\pi}} e^{-1/2\epsilon g} = 2 \sqrt{\frac{2\sigma^2 \Delta tg}{\pi E_d}} \exp\left\{ - \frac{E_d}{2\sigma^2 \Delta tg} \right\}. \hspace{1cm} (16)$$

### 3.4 Asymptotics in the presence of a random gain

As was pointed in section 2.2, the gain is by itself a random variable. Thus

$$\tau(\omega) = \inf\{ s \geq 0 : g(\omega)E(s, \omega) \geq E_d \}. \hspace{1cm} (17)$$

Let us introduce the random variable \(\eta\) by setting

$$g = 1/\eta^2.$$  

Thus in the presence of a random gain the detection moment is given by \(\tau(\omega) = \inf\{ s \geq 0 : E(s, \omega) \geq \eta^2(\omega)E_d \}\). Denote the probability distribution of \(\eta\) by \(P_\eta\). We suppose that it has the density, denote the latter by the symbol \(\rho_\eta(\lambda)\). To find the probability distribution of \(\tau\), we use the formula of total probability:

$$P(\tau \leq \Delta t) = \int d\lambda \rho_\eta(\lambda) P(\tau \leq \Delta t | \eta = \lambda),$$ \hspace{1cm} (18)
where the conditional probability has been already found:

\[ P(\tau \leq \Delta t|\eta = \lambda) \approx 2 \sqrt{\frac{2\sigma^2 \Delta t}{\pi \lambda^2 E_d}} \exp\left\{-\frac{\lambda^2 E_d}{2\sigma^2 \Delta t}\right\}. \] (19)

Finally, we have

\[ P(\tau \leq \Delta t) \approx \frac{4\sigma^2 \Delta t}{E_d} \int d\lambda \rho_\eta(\lambda) \frac{\exp\left\{-\frac{\lambda^2}{2\sigma^2 \Delta t}\right\}}{\sqrt{2\pi \sigma^2 \Delta t}}. \] (20)

### 3.5 The average number of clicks

We now recall the meaning of the time interval \( \Delta t \): the average duration of the interaction of a signal with a detector. Consider now a long run of experimental trials, emissions and detections, of the duration \( T \). Then the average number of clicks can be found as

\[ N \approx \frac{T}{\Delta t} P(\tau \leq \Delta t) \approx \frac{4\sigma^2 T}{E_d} \int d\lambda f_\eta(\lambda) \frac{\exp\left\{-\frac{\lambda^2}{2\sigma^2 \Delta t}\right\}}{\sqrt{2\pi \sigma^2 \Delta t}}, \] (21)

where \( f_\eta(\lambda) = \frac{\rho(\lambda)}{1/M} \). We now remark that

\[ D_\epsilon(\lambda) = \frac{\exp\left\{-\frac{\lambda^2}{2\epsilon}\right\}}{\sqrt{2\pi \epsilon}} \]

is the \( \delta \)-sequence for \( \epsilon \to 0 \) : \( \lim_{\epsilon \to 0} D_\epsilon(\lambda) = \delta(\lambda) \). Hence,

\[ N_s \approx \frac{4\sigma^2 T \langle \delta, f_\eta \rangle}{E_d}. \] (22)

### 4 Detection probability

#### 4.1 Derivation of the quantum (Born’s) rule for the detection probability

Consider now a random signal \( \phi(s, \omega) \) valued in the \( m \)-dimensional complex HIlbert space \( H \), where \( m \) can be equal to infinity. Let \( (e_j) \) be an orthonormal basis in \( H \). The vector-valued (classical) signal \( \phi(s, \omega) \) can be expanded with respect to this basis

\[ \phi(s, \omega) = \sum_j \phi_j(s, \omega)e_j, \quad \phi_j(s, \omega) = \langle e_j, \phi(s, \omega) \rangle. \] (23)

This mathematical operation is physically realized as splitting of the signal \( \phi(s, \omega) \) into components \( \phi_j(s, \omega) \). These components can be processed through mutually disjoint channels, \( j = 1, 2, ..., m \). We now assume that there is a threshold detector in each channel, \( D_1, ..., D_m \). We also assume that all detectors have
the same threshold \( \mathcal{E}_d > 0 \) and the same probability distribution of the gain, with the density \( \rho_d(\lambda) \).

Suppose now that \( \phi(s, \omega) \) is the \( H \)-valued Brownian motion (the Wiener process in \( H \)). This process is determined by the covariance operator \( B : H \to H \). Any covariance operator is Hermitian, positive, and the trace-class and vice versa. The complex Wiener process is characterized by the Hermitian covariance operator. (We remark that complex-valued random signals are widely used in e.g. radio-physics.) We have, for \( y \in H \),

\[
E(y, \phi(s, \omega)) = 0,
\]

and, for \( y_j \in H, j = 1, 2, \)

\[
E\langle y_1, \phi(s_1, \omega) \rangle \langle \phi(s_2, \omega), y_2 \rangle = \min(s_1, s_2) \langle By_1, y_2 \rangle.
\]

The latter is the covariance function of the stochastic process; in the operator form: \( B(s_1, s_2) = \min(s_1, s_2) B \). We note that the dispersion of the \( H \)-valued Wiener process (at the instant of time \( s \)) is given by

\[
\Sigma_s^2 \equiv E\|\phi(s, \omega)\|^2 = s \text{Tr} B.
\] (24)

This is the average energy of this random signal at the instant of time \( s \). Hence, the quantity

\[
\Sigma^2 = \frac{\Sigma_s^2}{s} = \text{Tr} B
\] (25)

has the physical dimension of power; this is the average power of the signal of the Brownian motion type (it does not depend on time).

We also remark that by normalization of the covariance function for the fixed \( s \) by the dispersion we obtain the operator,

\[
\rho = B/\text{Tr} B,
\] (26)

which formally has all properties of the density operator used in quantum theory to represent quantum states. Its matrix elements have the form

\[
\rho_{ij} = b_{ij}/\Sigma^2.
\] (27)

These are dimensionless quantities. The relation (26) plays a fundamental role in our approach [14–23]: each classical random process generates a quantum state (in general mixed) which is given by the normalized covariance operator of the process. One can proceed the other way around as well: each density operator determines a class of classical random processes [58].

We now consider components of the random signal \( \phi(s, \omega) \) and their correlations:

\[
E\phi_i(s_1, \omega)\overline{\phi_i(s_2, \omega)} = \min(s_1, s_2) \langle Be_i, e_j \rangle = \min(s_1, s_2) b_{ij}.
\] (28)
In particular,
\[ \sigma_j^2(s) \equiv E\mathcal{E}_j(s, \omega) \equiv E|\phi_j(s, \omega)|^2 = s b_{jj}. \]  
(29)
This is the average energy of the \( j \)th component at the instant of time \( s \). We also consider its average power:
\[ \sigma_j^2 = b_{jj}. \]  
(30)
We remark that the average power of the total random signal is equal to the sum of powers of its components:
\[ \Sigma^2 = \sum_j \sigma_j^2. \]  
(31)
Consider now a run of experiment of the duration \( T \). The average number of clicks for the \( j \)th detector can be approximately expressed as
\[ N_j \equiv N_{\sigma_j} \approx \frac{\sigma_j^2 T \langle \delta_f \eta \rangle}{\mathcal{E}_d}. \]  
(32)
This asymptotics is valid under the assumption (11). Here
\[ \mathcal{E}_{\text{pulse}} = \Sigma^2 \Delta t \]
is the average energy of emitted pulses. (We state again that we consider the process of detection in the absence of losses, cf. footnote 3.)
The total number of clicks is (again approximately) given by
\[ N = \sum_j N_j \approx \frac{4\Sigma^2 T \langle \delta_f \eta \rangle}{\mathcal{E}_d}. \]  
(33)
Hence, for the detector \( D_j \), the probability of detection can be expressed as
\[ P_j = \frac{N_j}{N} \approx \frac{\sigma_j^2}{\Sigma^2} = \rho_{jj}. \]  
(34)
This is, in fact, the Born’s rule for the quantum state \( \rho \) and the projection operator \( \hat{C}_j = |e_j\rangle\langle e_j| \) on the vector \( e_j \) : Hence, for the detector \( D_j \), the probability of detection can be expressed as
\[ P_j = \text{Tr}\rho\hat{C}_j. \]  
(35)

4.2 Ercf-rule for detection probability
We state again that the expression (22) was based on an approximation of the PDF of the detection moment for the detection threshold \( \mathcal{E}_d \gg \mathcal{E}_{\text{pulse}} \).
By using the precise formula (10) and by taking into account the presence of the random gain we obtain
\[ N_j = \frac{T}{\Delta t} \int d\lambda \rho_\eta(\lambda) P(\tau_j \leq \Delta t|\eta = \lambda) \]
\[ N = \frac{2T}{\Delta t} \sum_{k=0}^{\infty} (-1)^k \int d\lambda \rho_\eta(\lambda) \sum_j \text{erfc}\left( (1 + 2k) \sqrt{\frac{\lambda^2 \Delta t}{2 \sigma_j^2}} \right). \]  

(36)

The total number of clicks in all detectors is the sum of \( N_j \):

\[ N_j = \frac{2T}{\Delta t} \sum_{k=0}^{\infty} (-1)^k \int d\lambda \rho_\eta(\lambda) \sum_j \text{erfc}\left( (1 + 2k) \sqrt{\frac{\lambda^2 \Delta t}{2 \sigma_j^2}} \right). \]  

(37)

Hence, the probability of a click in the \( j \)th detector is given by sufficiently complex formula (generalized Born’s rule):

\[ P_j = \frac{\sum_{k=0}^{\infty} (-1)^k \int d\lambda \rho_\eta(\lambda) \sum_j \text{erfc}\left( (1 + 2k) \sqrt{\frac{\lambda^2 \Delta t}{2 \sigma_j^2}} \right)}{\sum_{k=0}^{\infty} (-1)^k \int d\lambda \rho_\eta(\lambda) \sum_j \text{erfc}\left( (1 + 2k) \sqrt{\frac{\lambda^2 \Delta t}{2 \sigma_j^2}} \right)}. \]  

(38)

We can easily rewrite this formula by using the original probability distribution of the gain \( g = g(\omega) \) and not of the random variable \( \eta(\omega) \). (We state again that the latter random variable is related to the gain by the relation \( g(\omega) = 1 / \eta^2(\omega) \).) We remark that densities of the probability distributions for \( g \) and \( \eta \) (for nonnegative \( \eta \)) are connected by the rule:

\[ \rho_g(\lambda) = \frac{1}{2\lambda^3/\pi} \rho_\eta\left(\frac{1}{\sqrt{\lambda}}\right), \quad \rho_\eta(\lambda) = \frac{1}{2\lambda^3} \rho_g\left(\frac{1}{\lambda^2}\right). \]  

(39)

We have

\[ N_j = \frac{T}{\Delta t} \int d\lambda \rho_g(\lambda) P(\tau_j \leq \Delta t | g = \lambda) \]

\[ = \frac{2T}{\Delta t} \sum_{k=0}^{\infty} (-1)^k \int d\lambda \rho_g(\lambda) \text{erfc}\left( (1 + 2k) \sqrt{\frac{\lambda^2 \Delta t}{2 \sigma_j^2}} \right). \]  

(40)

The total number of clicks in all detectors is the sum of \( N_j \):

\[ N = \frac{2T}{\Delta t} \sum_{k=0}^{\infty} (-1)^k \int d\lambda \rho_g(\lambda) \sum_j \text{erfc}\left( (1 + 2k) \sqrt{\frac{\lambda^2 \Delta t}{2 \sigma_j^2}} \right). \]  

(41)

Hence, the probability of a click in the \( j \)th detector is given by sufficiently complex formula (generalized Born’s rule):

\[ P_j = \frac{\sum_{k=0}^{\infty} (-1)^k \int d\lambda \rho_g(\lambda) \text{erfc}\left( (1 + 2k) \sqrt{\frac{\lambda^2 \Delta t}{2 \sigma_j^2}} \right)}{\sum_{k=0}^{\infty} (-1)^k \int d\lambda \rho_g(\lambda) \sum_j \text{erfc}\left( (1 + 2k) \sqrt{\frac{\lambda^2 \Delta t}{2 \sigma_j^2}} \right)}. \]  

(42)

As we have seen, it is not easy to derive the canonical Born’s rule from (38). Although this expression is very complex from the analytic viewpoint, it can be easy and quickly calculated by computer. It would be interesting to compare the result which can be obtained on the basis of (38) with experimental
data. TSD predicts that the experimental statistical data should match with this generalized Born’s rule better than with the canonical Born’s rule.

We also remark that, for sufficiently large values of $E_d$, the series inside expressions (38), (42) converge very quickly. Therefore by using only the first terms of these series we obtain very good approximation. Hence the following approximate Ercf-expressions for the detection probabilities can be used

$$P_j = \frac{\int d\lambda \rho_\eta(\lambda) \text{erfc}\left(\frac{\lambda^2 E_d}{2\sigma_j^2 \Delta t}\right)}{\int d\lambda \rho_\eta(\lambda) \sum_j \text{erfc}\left(\frac{\lambda^2 E_d}{2\sigma_j^2 \Delta t}\right)}.$$  \tag{43}

or directly in the gain variable

$$P_j \approx \frac{\int d\lambda \rho_g(\lambda) \text{erfc}\left(\frac{\lambda^2 E_d}{2\sigma_j^2 \Delta t}\right)}{\int d\lambda \rho_g(\lambda) \sum_j \text{erfc}\left(\frac{\lambda^2 E_d}{2\sigma_j^2 \Delta t}\right)}.$$  \tag{44}

We remark that, although the gain variable has the straightforward physical meaning, and hence the representation (44) is better from the physical viewpoint, the $\eta$-representation is better from the mathematical viewpoint: the integrals in (43) are Gaussian. The same can be said about complete Ercf-representations of the detection probabilities, cf. (42) with (38).

**acknowledgements**

This paper was started during my visit to Quantum Optics and Quantum Information Center of Austrian Academy of Science (March 2012); I would like to thank Anton Zeilinger and his coworkers, especially Rupert Ursin and Sven Ramelow, for the discussions on experimental verification of PCSFT and the knowledge transfer of the details on “experimental technicalities”. This paper was finished during the school "Quantum Foundations and Open Systems", Joao Pessoa, Brazil (15-28 July, 2012) and I would like to thank Inacio de Almeida Pedrosa and Claudio Furtado for hospitality. I also thank my colleagues at the International Center for Mathematical Modeling (at LNU) Irina Basieva, Astrid Hilbert, Borje Nilsson, and Sven Nordebo for stimulating discussions. Irina Basieva performed numerical simulation on asymptotic convergence of the detection probability to the quantum probability. The presence of the graphical illustrations essentially improved the presentation of the material of the paper. This research was supported by the grant “Mathematical Modeling” of the Faculty of Natural Science and Engineering of Linnaeus University.

**References**

[1] M. Born, *Zeitschrift für Physik* 37 (1926) 863.
[2] N. P. Landsman, in *Compendium of Quantum Physics*, F. Weinert, K. Hentschel, D. Greenberger, and B. Falkenburg (eds.), Springer, Berlin, 2008.

[3] A. Khrennikov, *Contextual Approach to Quantum Formalism* Springer, Heidelberg-Berlin-New York, 2009.

[4] W. A. Hofer, *Found. Phys.* 41, 754-791 (2011).

[5] I. Ojima, Micro-Macro Duality and Space-Time Emergence. ADVANCES IN QUANTUM THEORY, AIP Conf. Proc. 1327, 197-206 (2011).

[6] E. L. O’Neill, *Introduction to statistical optics*, Dover Publ., Inc., Mineola, N.Y., 1991.

[7] W. B. Davenport, W. L. Root, *An introduction to the theory of random signals and noise*, Wiley-IEEE Press, 1987.

[8] A. N. Shiryayev, *On Martingale Methods in Problems for Boundary Crossing by Brownian Motion*. Ser. Contemporary Math. Problems 8 (Stecklov Math. Institute Press, Moscow, 2007).

[9] S. M. Ritov, *Introduction to statistical radiophysics*, Nauka, Fizmatlit, Moscow, 1966.

[10] P. M. Morse and K. U. Ingard, *Theoretical acoustics*, Mc Graw-Hill Book Company, New York, St. Louis, San Francisco, Toronto, London, Sydney, 1968.

[11] Sinha, U., Couteau, C., Medendorp, Z., Sollner, I., Laflamme, R., Sorkin, R., and Weihs, G.: Testing Born’s rule in quantum mechanics with a triple slit experiment. In: Accardi, L., Adenier, G., Fuchs, C., Jaeger, G., Khrennikov, A., Larsson, J.-A., and Stenholm, S. (eds). Foundations of Probability and Physics-5, American Institute of Physics, Ser. Conference Proceedings, vol. 1101. Melville, NY, 2009, pp. 200-207.

[12] I. I. Gikhman and A. V. Skorokhod, *The Theory of Stochastic Processes*, Springer, Berlin-Heidelberg-New York, 1979.

[13] M. Zakai and J. Ziv, “On the threshold effect in radar range estimation,” *IEEE Trans. Information Theory* IT-15, pp. 167-70, 1969.

[14] A. Khrennikov, *J. Phys. A: Math. Gen.* 38 (2005) 9051.

[15] A. Khrennikov, *Found. Phys. Letters* 18 (2006) 637.

[16] A. Khrennikov, *Physics Letters A* 357 (2006) 171.

[17] A. Khrennikov, *Found. Phys. Lett.* 19 (2006) 299.

[18] A. Khrennikov, *Nuovo Cimento B* 121 (2006) 505.

[19] A. Khrennikov, *EPL* 88 (2009) 40005.1.
[20] A. Khrennikov, *Europhysics Lett.* **90** (2010) 40004.

[21] A. Khrennikov, M. Ohya, N. Watanabe, *International J. of Quantum Information* **9** (2011) 281.

[22] A. Khrennikov, M. Ohya, N. Watanabe *J. Russian Laser Research* **31** (2010) 462.

[23] A. Khrennikov, *J. of Russian Laser Research* **31** (2010) 191.

[24] L. De la Pena and A. Cetto, *The Quantum Dice: An Introduction to Stochastic Electrodynamics,* Kluwer, Dordrecht, 1996.

[25] A. Casado, T. Marshall, E. Santos, *J. Opt. Soc. Am.* B **14** 494

[26] Boyer T 1980 A brief survey of stochastic electrodynamics. *Foundations of Radiation Theory and Quantum Electrodynamics* ed Barut A (New York: Plenum) p 141

[27] Cole D C, Rueda A, Danley K 2001 *Phys. Rev.* A **63** 054101

[28] Ch. Roychoudhuri, Shall we climb on the shoulders of the giants to extend the reality horizon of physics?” In *Quantum Theory: Reconsideration of Foundations-4*, G. Adenier, A. Yu. Khrennikov, P. Lahti, V. I. Manko, and Th.M. Nieuwenhuizen, eds., **962**, pp. 195-205, American Institute of Physics, Ser. Conference Proceedings, Melville, NY, 2007.

[29] G. Adenier, *J. Russian Laser Research* **29**, pp. 409-417, 2008.

[30] Manko V I 1996 *J. of Russian Laser Research* **17** 579

[31] Manko V I and Shchukin E V 2001 *J. Russian Laser Research* **22** 545

[32] Manko M A, Manko V I, and Mendes R V 2006 *J. Russian Laser Research* **27** 507

[33] De Nicola S, Fedele R, Man’ko M A, and Man’ko V I 2004 *J. of Russian Laser Research,* **25** 1071

[34] Manko M A and Manko V I 2012 Statistics of observables in the probability representation of quantum and classical system states. AIP Conf. Proc. **1424** 234

[35] Chernega V N and Manko V I 2012 System with classical and quantum subsystems in tomographic probability representation. AIP Conf. Proc. **1424** 33

[36] Manko O 2009 Tomographic Representation of quantum mechanics and statistical physics. AIP Conf. Proc. **1101** 104

[37] Manko M A 2007 Tomographic entropy and new entropic uncertainty relations. AIP Conf. Proc. **962** 132
[38] Manko V I 2007 Probability Instead of Wave Function and Bell Inequalities as Entanglement Criterion AIP Conf. Proc. 962 140

[39] K. Hess and W. Philipp EPL 57 (2002) 775.

[40] H. De Raedt, K. De Raedt , K. Michielsen, EPL 69 (2005) 861.

[41] K. Hess, K. Michielsen, H. De Raedt, EPL 87 (2009) 60007.

[42] C. Garola, S. Sozzo, EPL 86 (2009) 20009.

[43] A. Khrennikov, Annals of Physics 327 (2012) 1786-1802.

[44] A. Khrennikov, B. Nilsson and S. Nordebo, Journal of Physics: Conference Series 361 (2012) art N 012030.

[45] L. C. Andrews, Special Functions of Mathematics for Engineers, SPIE Publ. , Bellingham, 1998.

[46] B. Van Zeghbroeck, Principles of Semiconductor Devices, Denver, University of Colorado, 2011

[47] W. D. Oliver, E. Diamanti E. Waks, K. Inoue, and Y. Yamamoto, Selected Topics in Quant. Electronics 2003 9 (2003) 1502.

[48] S. V. Polyakov, A. Migdall, and Sae Woo Nam, Real-Time Data-Acquisition Platform for Pulsed Measurements In: Advances in Quantum Theory, AIP Conf. Proc. 1327, Melville, NY, 2010. pp. 505-519.

[49] M. D. Eisaman, J. Fan, A. Migdall, and S. V. Polyakov, Rev. Sci. Instrum. 82 (2011) 071101.

[50] A. Aspect, Three experimental tests of Bell inequalities by the measurement of polarization correlations between photons, PhD Thesis 2674( Orsay), 1983 (In French).

[51] P. Grangier, Etude expérimentale de propriétés non-classiques de la lumière: interférence à un seul photon. Université de Paris-Sud, Centre D’Orsay (1986)

[52] P. Grangier, G. Roger, and A. Aspect, EPL 1, 173 (1986)

[53] G.Brida, E. Cagliero, G. Falzetta, M.Genovese, M. Gramegna and C. Novero, J. Phys. B: At. Mol. Opt. Phys. 35 (2002) 4751.

[54] G. Brida, M.Genovese, M. Gramegna, E.Caglierio, J. Phys. B: At. Mol. Opt. Phys. 37 (2004) 3781.

[55] Brida, N. Antonietti, M. Gramegna, L. Krivitsky, F. Piacentini, M.L. Rastello, I. Ruo Berchera, P. Traina, E. Predazzi, International Journal of Quantum Information 5 (2007) 265.
[56] G. Brida, I. Degiovanni, M. Genovese, V. Schettini, S. Polyakov, A. Migdall, *Opt. Exp.* 16 (2008) 11750.

[57] M. Genovese, G. Brida, M. Gramegna, F. Piacentini, E. Predazzi, I. Ruoi-Berchera *Journal of Physics: Conference Series* 67 (2007) 01204.

[58] M. Ohya, and N. Watanabe, Japan J. Industrial and Appl. Math. 3 (1986), 197.