NONLINEAR WAVE EQUATION WITH BOTH STRONGLY AND WEAKLY DAMPED TERMS: SUPERCRITICAL INITIAL ENERGY FINITE TIME BLOW UP

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Abstract. By introducing a new increasing auxiliary function and employing the adapted concavity method, this paper presents a finite time blow up result of the solution for the initial boundary value problem of a class of nonlinear wave equations with both strongly and weakly damped terms at supercritical initial energy level.

1. Introduction. In this paper, we investigate the finite time blow up of the solution for the following nonlinear wave equation with strong linear damping term

\[ u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = |u|^{p-2} u, \quad (x,t) \in \Omega \times (0,\infty), \]

\[ u(x,t) = 0, \quad x \in \partial \Omega, \quad t \geq 0, \]

\[ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \]

where \( \Omega \subset \mathbb{R}^n (n \geq 1) \) is a bounded domain with a smooth boundary \( \partial \Omega \), \( \omega \geq 0 \), \( \mu \geq 0 \) and \( p \) satisfies

\[ 2 < p < \infty, \quad \text{if } n = 1,2; \quad 2 < p \leq \begin{cases} \frac{2n}{n-2}, & \text{for } \omega > 0 \text{ if } n \geq 3, \\ \frac{2n}{n-2}, & \text{for } \omega = 0 \end{cases} \]

The main aim of the present paper is to solve an open problem pointed out in [4], hence we just make a quick introduction to start this paper. We refer the reader to [4] and the references therein for the background. The comprehensive studies for

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the local and global well-posedness of the nonlinear wave equation with linear weak damping term and strong damping term were conducted in [4] for three different initial energy levels, i.e., the subcritical initial energy level $E(0) < d$ (for $\omega > 0$), the critical initial energy level $E(0) = d$ (for $\omega > 0$) and the supercritical initial energy level $E(0) > d$ (for $\omega = 0$), here and later $d$ is the depth of potential well for problem (1)-(3) and positive. The ideas in [4] especially for the high energy case have been widely employed to deal with many different classes of wave equations in the recent years, see [1]-[3] and [5]-[22] as a part of these works. The most attractive result in [4] is about the finite time blow up of the solution at the supercritical initial energy level $E(0) > d$, which as far as we know is the first blow up result of solutions for problem (1)-(3) at the supercritical initial energy level $E(0) > d$. Gazzola and Squassina in [4] proved the high energy blow up for the case without the strong damping term, i.e., $\omega = 0$ (Theorem 3.11 in [4]), and they pointed out that the high energy blowup for the case with strong damping term, i.e., $\omega > 0$, is still an open problem. The aim of the present paper is to solve this open problem. We try to find out when $E(0) > d$ (the supercritical initial energy), what kind of initial data can lead to the finite time blow up of solution for nonlinear wave equation with strong linear damping term ($\omega > 0$). Further, we like to point out that the method of the present paper establishing the supercritical initial energy finite time blow up can be also applied to the classical nonlinear wave equation involving the initial boundary value problem, that is problem (1)-(3) whatever the strong damping coefficient $\omega$ and the weak damping coefficient $\mu$ exist or vanish. In order to make it clear, we summarize the known results and open problems in the following table.

| Subcritical initial energy $E(0) < d$ | Global existence | Asymptotic behavior | Blow up |
|--------------------------------------|-----------------|-------------------|--------|
| Critical initial energy $E(0) = d$   | Reference [4]   | Reference [4]     | Reference [4] |
| Supercritical initial energy $E(0) > d$ | Open problem   | Open problem      | $\omega > 0, \mu > 0$ |

$\omega = 0, \mu > 0$  $\omega = 0, \mu = 0$  $\omega = 0, \mu > 0$  Reference [4]  Present paper

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Indeed the auxiliary function $F(t) = \|u(t)\|_2^2$ introduced in [4] (Lemma 8.2) is the key to prove the invariance of $\mathcal{N}_-$, further to prove the finite time blow up of the solution for the supercritical initial energy level ($E(0) > d$). But this auxiliary function $F(t) = \|u(t)\|_2^2$ does not work for problem (1)-(3) with the strong damping term, i.e., $\omega > 0$, and there is no any rule or hint to guide us to find and define a new $F(t)$, which is believed the main reason for that such problem remained open until now. So in the present paper we do not invent any new strategies or new structures to prove the finite time blow up of the solution, and we only introduce a new auxiliary function, i.e., $F(t) := \omega\|\nabla u(t)\|^2 + \mu\|u(t)\|^2 + 2(u, u_t)$ with some little tricks like introducing the parameter $\lambda$ for the estimates to overcome the difficulties arising in the discussion on the problem with the strong damping term, i.e., $\omega > 0$, which is the main idea of the present paper.

The organization of the paper is as follows: In Section 2, we introduce some preliminaries. In Section 3 we give the proof of the finite time blow up result of the solution with supercritical initial energy.
2. Preliminaries. In order to be a continuous study of [4], we shall use the same notions introduced in [4]. We denote by $\|u\|_p$ the $L^p(\Omega)$ norm ($2 \leq p < \infty$), by $\|u\|$ the $L^2(\Omega)$ norm and by $\langle \cdot, \cdot \rangle$ the $L^2$ inner product. And also the notion $\langle \cdot, \cdot \rangle$ is used in this paper to denote the duality paring between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

As did in [4], for problem (1)-(3) we also introduce the energy functional
\[
E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 - \frac{1}{p} \|u\|_p^p,
\] (5)
the Nehari functional
\[
I(u) = \|\nabla u\|^2 - \|u\|_p^p
\] (6)
and an unstable set (the outer region of potential well)
\[
N_\omega = \{u \in H^1_0(\Omega) | I(u) < 0\}.
\] (7)

Subsequently we state the following definition of solution and the local existence theorem for problem (1)-(3) in [4].

Definition 2.1 ([4]). By solution of problem (1)-(3) over $[0, T_0]$ we mean a function $u \in C^0 ([0, T_0), H^1_0(\Omega)) \cap C^1 ([0, T_0), L^2(\Omega)) \cap C^2 ([0, T_0), H^{-1}(\Omega))$ with $u_t \in L^2 ([0, T_0), H^1_0(\Omega))$ such that $u(0) = u_0$, $u_t(0) = u_1$ and
\[
\langle u_{tt}(t), v \rangle + \int_\Omega \nabla u(t) \nabla vdx + \omega \int_\Omega \nabla u_t(t) \nabla vdx + \mu \int_\Omega u_t(t)vdx = \int_\Omega |u|^{p-2}uvdx
\]
for all $v \in H^1_0(\Omega)$ and a.e. $t \in [0, T_0]$.

Theorem 2.2 ([4]). Suppose that $u_0(x) \in H^1_0(\Omega)$ and $u_1(x) \in L^2(\Omega)$. Then problem (1)-(3) admits a unique local solution $u(x, t)$ defined on a maximum time interval $[0, T_0)$. Moreover if
\[
\sup_{t \in [0, T_0)} \|u(x, t)\| < +\infty,
\]
then $T_0 = +\infty$.

In the end we show the energy identity for problem (1)-(3).

Lemma 2.3. The total energy functional $E(t)$ of the solution $u(t)$ to problem (1)-(3) satisfies
\[
E(t) + \omega \int_0^t \|\nabla u_\tau\|^2 d\tau + \mu \int_0^t \|u_\tau\|^2 d\tau = E(0).
\] (8)

Proof. Multiplying Equation (1) by $u_t$ and integrating the obtained results over $\Omega$, we obtain (8). \qed

3. Finite time blow up at supercritical initial energy level. We first prove the following lemma to obtain that the unstable set $N_\omega$ is invariant under the flow of problem (1)-(3) at the supercritical initial energy level $E(0) > d$, where $d$ is the depth of potential well for problem (1)-(3) and positive.

Lemma 3.1. Let $u_0(x) \in H^1_0(\Omega)$ and $u_1(x) \in L^2(\Omega)$. Suppose
\[
(u_0, u_1) > \frac{p(\mu + 1 + C\delta)}{(p-2)c} E(0) > \frac{p(\mu + 1 + C\delta)}{(p-2)c} d,
\] (9)
hereafter $\delta := \max\{1, \omega\}$ and $C$ is the best constant of Poincaré inequality
\[
\|\nabla u\|^2 \geq C\|u\|^2.
\] (10)
Then
\[ \{ t \mapsto \omega \| \nabla u(t) \|^2 + \mu \| u(t) \|^2 + 2(u, u_t) \} \]
is positive and strictly increasing provided that \( u(t) \in \mathcal{N}_- \).

**Proof.** We claim that the following introduced new auxiliary function
\[ F(t) := \omega \| \nabla u(t) \|^2 + \mu \| u(t) \|^2 + 2(u, u_t) \]  
(11)
is positive and strictly increasing provided that \( u(t) \in \mathcal{N}_- \). In fact from Equation (1) it follows
\[ F'(t) = 2\omega (\nabla u, \nabla u_t) + 2\mu (u, u_t) + 2(u_{tt}(t), u) + 2\| u_t \|^2 \]
(12)
which together with \( u(t) \in \mathcal{N}_- \) shows
\[ F'(t) > 0 \text{ for } t \in [0, +\infty). \]  
(13)
Moreover from (9) it implies that
\[ F(0) = \omega \| \nabla u_0 \|^2 + \mu \| u_0 \|^2 + 2(u_0, u_1) \]
(14)
\[ > 2(u_0, u_1) \]
and
\[ \frac{2(p\mu + 1 + C\delta)}{(p-2)C} E(0) > 0. \]
Combining (13) and (14) yields \( F(t) > F(0) > 0 \), which tells us that
\[ \{ t \mapsto \mu \| u(t) \|^2 + \omega \| \nabla u(t) \|^2 + 2(u, u_t) \} \]
is positive and increasing strictly.

Next we display the sign preserving properties of the unstable set \( \mathcal{N}_- \) for problem (1)-(3) with supercritical initial energy.

**Lemma 3.2** (Invariant set \( \mathcal{N}_- \)). Let \( u_0(x) \in H^1_0(\Omega) \) and \( u_1(x) \in L^2(\Omega) \). Then the solution to problem (1)-(2) belongs to \( \mathcal{N}_- \), provided that \( u_0 \in \mathcal{N}_- \) and the initial data satisfy (9).

**Proof.** We claim \( u(t) \in \mathcal{N}_- \) for \( t \in [0, T_0) \), where \( T_0 \leq +\infty \) is the maximum existence time of the solution \( u(t) \). Arguing by contradiction, we suppose that \( t_0 \in (0, T_0) \) is the first time such that
\[ I(u(t_0)) = 0 \]  
(15)
and
\[ I(u(t)) < 0 \text{ for } t \in [0, t_0). \]
Hence from Lemma 3.1 it follows that
\[ \{ t \mapsto \omega \| \nabla u(t) \|^2 + \mu \| u(t) \|^2 + 2(u, u_t) \} \]
is positive and increasing strictly on the interval \([0, t_0)\), which together with (9) gives
\[ \omega \| \nabla u(t) \|^2 + \mu \| u(t) \|^2 + 2(u(t), u_t(t)) \]
\[ > \omega \| \nabla u_0 \|^2 + \mu \| u_0 \|^2 + 2(u_0, u_1) \]
\[ > 2(u_0, u_1) \]
\[ > \frac{2(p\mu + 1 + C\delta)}{(p-2)C} E(0), \ t \in (0, t_0). \]
Moreover, from the continuity of \(u(t)\) and \(u_t(t)\) in \(t\), we obtain
\[
\omega \|\nabla u(t_0)\|^2 + \mu \|u(t_0)\|^2 + 2(u(t_0), u_t(t_0)) \geq \frac{2p(\mu + 1 + C\delta)}{(p - 2)C} E(0). \tag{17}
\]

Recalling (8) and (5), we have
\[
E(0) = E(t) + \omega \int_0^t \|\nabla u_x\|^2 \, dx + \mu \int_0^t \|u_x\|^2 \, dx
= \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 - \frac{1}{p} \|u\|^p + \int_0^t (\omega \|\nabla u_x\|^2 + \mu \|u_x\|^2) \, dx
= \frac{1}{2} \|u_t\|^2 + \frac{p - 2}{2p} \|\nabla u\|^2 + \frac{1}{p} F(u) + \int_0^t (\omega \|\nabla u_x\|^2 + \mu \|u_x\|^2) \, dx, \tag{18}
\]
which together with (15), \(\omega \geq 0, \mu \geq 0, p > 2\), (10) and the Cauchy-Schwarz inequality shows that
\[
E(0) \geq \frac{1}{2} \|u_t(t_0)\|^2 + \frac{p - 2}{2p} \|\nabla u(t_0)\|^2
\geq \frac{(p - 2)C}{2p(\mu + 1 + C\delta)} |u_t(t_0)|^2 + \frac{(p - 2)C}{2p(\mu + 1 + C\delta)} \omega \|\nabla u(t_0)\|^2
+ \frac{(p - 2)(\mu + 1)}{2p(\mu + 1 + C\delta)} \|\nabla u(t_0)\|^2
\geq \frac{(p - 2)C}{2p(\mu + 1 + C\delta)} |u_t(t_0)|^2 + \frac{(p - 2)C}{2p(\mu + 1 + C\delta)} \omega \|\nabla u(t_0)\|^2
+ \frac{(p - 2)(\mu + 1)}{2p(\mu + 1 + C\delta)} \|u(t_0)\|^2
\geq \frac{(p - 2)C}{2p(\mu + 1 + C\delta)} (2(u(t_0), u_t(t_0)) + \mu \|u(t_0)\|^2 + \omega \|\nabla u(t_0)\|^2). \tag{19}
\]
Clearly (19) contradicts (17). So the proof is completed. \(\Box\)

In the end we present the finite time blow up result of the solution to problem (1)-(3) with supercritical initial energy. In order to show the proof of the finite time blow up result more clearly, we introduce the star inner product
\[
(u, v)_* = \omega \int_{\Omega} \nabla u \nabla v \, dx + \mu \int_{\Omega} uv \, dx
\]
and the norm \(\|u\|_*^2 = (u, u)_*\) for the Sobolev space \(H_0^1(\Omega)\).

**Theorem 3.3** (Blow up for \(E(0) > d\)). Let \(u_0(x) \in H_0^1(\Omega)\) and \(u_1(x) \in L^2(\Omega)\). Assume that \(u_0 \in N_-\) and (9) holds, then the solution \(u(t)\) to problem (1)-(3) blows up in finite time.

**Proof.** Clearly from Lemma 3.2 it follows that \(u(t) \in N_-\). Assume by contradiction that \(u(t)\) is global. Then we define
\[
B(t) := \|u\|^2 + \int_0^t \|u(x, \tau)\|^2 \, d\tau + (T_0 - t)\|u_0\|^2, \quad \forall \; T_0 > 0, \tag{20}
\]
which together with the continuity of \(u(t)\) in \(t\) implies
\[
B(t) \geq \rho > 0, \quad \forall \; t \in [0, T_0], \tag{21}
\]
where \(\rho\) is independent of the choice of \(T_0\).
Furthermore, we can derive
\[
B'(t) = 2(u, u_t) + \|u\|^2 - \|u_0\|^2
\]
\[
= 2(u, u_t) + 2 \int_0^t (u(\tau), u_\tau(\tau))_\ast d\tau, \; t \in [0, T_0] \tag{22}
\]
and
\[
B''(t) = 2 \|u_t\|^2 + 2(u_{tt}(t), u) + 2(u, u_t)_\ast
\]
\[
= 2 \|u_t\|^2 - 2I(u), \; t \in [0, T_0]. \tag{23}
\]
From (22) it implies that
\[
(B'(t))^2 \leq 4 \left( (u, u_t)^2 + 2(u, u_t) \int_0^t (u(\tau), u_\tau(\tau))_\ast d\tau \right)
\]
\[
+ 4 \left( \int_0^t (u(\tau), u_\tau(\tau))_\ast d\tau \right)^2. \tag{24}
\]
Then from the Cauchy-Schwarz inequality it follows
\[
(u, u_t)^2 \leq \|u\|^2 \|u_t\|^2,
\]
\[
\left( \int_0^t (u(\tau), u_\tau(\tau))_\ast d\tau \right)^2 \leq \int_0^t \|u(\tau)\|^2_\ast d\tau \int_0^t \|u_\tau(\tau)\|^2_\ast d\tau
\]
and
\[
2(u, u_t) \int_0^t (u(\tau), u_\tau(\tau))_\ast d\tau
\]
\[
\leq 2\|u\| \|u_t\| \left( \int_0^t \|u(\tau)\|^2_\ast d\tau \right)^{1/2} \left( \int_0^t \|u_\tau(\tau)\|^2_\ast d\tau \right)^{1/2}
\]
\[
\leq \|u\|^2 \int_0^t \|u_\tau(\tau)\|^2_\ast d\tau + \|u_t\|^2 \int_0^t \|u(\tau)\|^2_\ast d\tau.
\]
Therefore (24) becomes
\[
(B'(t))^2 \leq 4 \left( \|u\|^2 + \int_0^t \|u(\tau)\|^2_\ast d\tau \right) \left( \|u_t\|^2 + \int_0^t \|u_\tau(\tau)\|^2_\ast d\tau \right)
\]
\[
\leq 4B(t) \left( \|u_t\|^2 + \int_0^t \|u_\tau(\tau)\|^2_\ast d\tau \right), \tag{25}
\]
which together with (23) gives
\[
B''(t)B(t) - \frac{\lambda + 2}{4} (B'(t))^2
\]
\[
\geq B(t) \left( B''(t) - (\lambda + 2) \left( \|u_t\|^2 + \int_0^t \|u_\tau(\tau)\|^2_\ast d\tau \right) \right) \tag{26}
\]
\[
\geq B(t) \left( -\lambda \|u_t\|^2 - 2I(u) - (\lambda + 2) \int_0^t \|u_\tau(\tau)\|^2_\ast d\tau \right),
\]
where \( p > \lambda > 2 \) will be decided later. Now we define
\[
\xi(t) := -\lambda \|u_t\|^2 - 2I(u) - (\lambda + 2) \int_0^t \|u_\tau(\tau)\|^2_\ast d\tau,
\]
which together with (18), (10) and the Cauchy-Schwarz inequality yields
\[
\xi(t) = (p - \lambda)\|u_t\|^2 + (p - 2)\|\nabla u\|^2 - 2pE(0) + (2p - 2 - \lambda) \int_0^t \|u_\tau(\tau)\|^2 d\tau
\]
\[
= (p - \lambda)\|u_t\|^2 + \frac{p - \lambda}{C} (\mu + 1)\|\nabla u\|^2 - 2pE(0)
\]
\[
+ \left(p - 2 - \frac{p - \lambda}{C} (\mu + 1)\right) \|\nabla u\|^2 + (2p - 2 - \lambda) \int_0^t \|u_\tau(\tau)\|^2 d\tau
\]
\[
\geq (p - \lambda)\|u_t\|^2 + (p - \lambda) (\mu + 1)\|u\|^2 - 2pE(0)
\]
\[
+ \left(p - 2 - \frac{p - \lambda}{C} (\mu + 1)\right) \frac{\omega}{\delta} \|\nabla u\|^2 + (2p - 2 - \lambda) \int_0^t \|u_\tau(\tau)\|^2 d\tau
\]
\[
\geq (p - \lambda) (2(u, u_t) + \mu\|u\|^2) + \left(p - 2 - \frac{p - \lambda}{C} (\mu + 1)\right) \frac{\omega}{\delta} \|\nabla u\|^2
\]
\[
- 2pE(0) + (2p - 2 - \lambda) \int_0^t \|u_\tau(\tau)\|^2 d\tau.
\]
At this point, we choose
\[
\lambda := p - \frac{C(p - 2)}{\mu + 1 + C\delta},
\]
which guarantees that \(\lambda \in (2, p)\), since \(p > 2\) and \(\delta = \max\{1, \omega\}\). Then, by \(\lambda < 2p - 2\), Lemma 3.2 and Lemma 3.1, a simple computation on (27) gives
\[
\xi(t) \geq \frac{C(p - 2)}{\mu + 1 + C\delta} (\omega\|\nabla u(t)\|^2 + \mu\|u(t)\|^2 + 2(u, u_t)) - 2pE(0)
\]
\[
> \frac{C(p - 2)}{\mu + 1 + C\delta} (\omega\|\nabla u_0\|^2 + \mu\|u_0\|^2 + 2(u_0, u_1)) - 2pE(0)
\]
(28)
\[
> \frac{2C(p - 2)}{\mu + 1 + C\delta} (u_0, u_1) - 2pE(0) := \eta > 0.
\]
Therefore by (26)-(28) and (21), we have
\[
B''(t)B(t) - \frac{\lambda + \frac{2}{4}}{B'(t)^2} > \eta \rho > 0, \quad t \in [0, T_0].
\]
(29)
Taking a simple calculation on \(y(t) = B(t)^{-\frac{\lambda + \frac{2}{4}}{4}}\) together with (29) yields
\[
y''(t) < \frac{\lambda - \frac{2}{4}}{\eta \rho y(t)^{\frac{\lambda + \frac{2}{4}}{4}}}, \quad t \in [0, T_0],
\]
which shows
\[
\lim_{t \to T_*} y(t) = 0,
\]
where \(T_*\) is independent of the initial choice of \(T_0\). At this point we may suppose \(T_* < T_0\). Then we have
\[
\lim_{t \to T_*} B(t) = +\infty,
\]
which is a contradiction with the assumption that the solution \(u(t)\) is global. Hence we complete the proof. \(\square\)

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