Inverse Lomax-Rayleigh distribution with application

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A R T I C L E   I N F O

Keywords:
Inverse Lomax G
Moments
Probability distribution
T-X approach
Inverse Lomax Rayleigh

A B S T R A C T

In this paper, an extension of Rayleigh distribution called Inverse Lomax Rayleigh (ILR) is proposed by using the Inverse Lomax generator of [12]. Properties of ILR were derived. This includes the complete and incomplete moments, entropy, distribution of order statistics, and quantile function. A simulation study was presented to explore the properties of the estimates. This shows that they are unbiased, consistent, and efficient. An application to fatigue data shows the flexibility of ILR distribution, as it outperforms all the comparators with minimum values of all the measures.

1. Background

Using probability distributions to represent real-life situations is one of the most important tasks of a statistician. Modeling and interpreting lifetime data is essential in many practical situations, such as medical, actuarial science, engineering, and finance. In recent decades, this has prompted academics to focus on developing families of probability distributions.

Some of the recent families of distributions in the literature include Kumaraswamy Poisson G by [11], Zubair G by [3], Beta Poisson G by [15], Extended Exp G by [8], Inverse Lomax G by [13], Burr X Exponential G by [24], Odd Log-Logistic Lindley G by [7], Weibull Marshall-Olkin Lindley G by [2], Kumaraswamy-Odd Rayleigh-G by [14], Inverse Lomax Exponentiated G by [12], as well as Topp-Leone Odd Frechet G by [5], among others. Extension of probability distributions is a regular practice in the theory of statistics. Different strategies are proposed to generalize probability distributions in the literature. This is necessary so as the addition of parameter(s) will expand the adaptability of the models to catch the multifaceted nature of the data. Several generalized (or G) classes of distributions are available in the literature, but our main focus in this paper is to extend the Rayleigh distribution with the Inverse Lomax G family.

1.1. The Rayleigh distribution

The cumulative distribution function cdf and probability density function pdf of Rayleigh distribution are given by

\[ G(x; \sigma) = 1 - e^{-\frac{x^2}{2\sigma^2}}, \quad x \in \mathbb{R}, \]  

and

\[ g(x; \sigma) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, \quad x \in \mathbb{R}, \]  

where \( \sigma > 0 \).

The Rayleigh distribution has several applications, including life testing experiments, communication theory, technology, reliability analysis, applied statistics, medical testing, and clinical studies. With regard to this significance and the desire to give this distribution greater versatility, several researchers have proposed extensions to the Rayleigh distribution. This includes Odd Lindley-Rayleigh distribution by [16], Lomax-Rayleigh distribution by [4], Rayleigh-Rayleigh distribution by [6], an extension of Rayleigh distribution by [10], Weibull Rayleigh by [21], Transmuted Rayleigh by [19], New generalized Rayleigh distribution by [25], Generalized Rayleigh distribution by [17], among others.

2. Inverse Lomax family of distributions

Falgore and Doguwa [13], introduced the Inverse Lomax family of distribution by adopting T-X methodology by [9]. The cdf and pdf of the family have the following form

\[ F(x; \alpha, \lambda, \Delta) = \left( 1 + \frac{\alpha g(x; \Delta)}{G(x; \Delta)} \right)^{-\lambda}, \quad x \in \mathbb{R}, \]  

and

\[ f(x; \alpha, \lambda, \Delta) = \frac{\alpha \lambda g(x; \Delta)}{G(x; \Delta)^{\lambda+1}} \left( 1 + \frac{\alpha g(x; \Delta)}{G(x; \Delta)} \right)^{-\lambda-1}, \quad x \in \mathbb{R}, \]  

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https://doi.org/10.1016/j.heliyon.2021.e08383
Received 26 April 2021; Received in revised form 4 September 2021; Accepted 9 November 2021

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where \(a, \lambda > 0\), \(g(x; \Delta)\) is the pdf of the baseline distribution and \(G(x; \Delta)\) is the cdf of the baseline distribution, and \(\Delta = 1 - G(x; \Delta)\), and \(\Delta\) is a vector of parameter(s).

3. Inverse Lomax-Rayleigh distribution

By considering Rayleigh as baseline distribution, we have the cdf and pdf of the Inverse Lomax-Rayleigh distribution that follows from equations (3) and (4) given by

\[
F_{I,LR}(x; a, \lambda, \sigma) = \left[1 + a \left( \frac{e^{\frac{x^2}{\sigma^2}}}{1 - e^{\frac{x^2}{\sigma^2}}} \right) \right]^{-1}, \quad x \in \mathbb{R} 
\]

(5)

and

\[
f_{I,LR}(x; a, \lambda, \sigma) = \frac{a \lambda x e^{\frac{x^2}{\sigma^2}}}{\sigma^2(1 - e^{\frac{x^2}{\sigma^2}})^2} \left[1 + a \left( \frac{e^{\frac{x^2}{\sigma^2}}}{1 - e^{\frac{x^2}{\sigma^2}}} \right) \right]^{-1}, \quad x \in \mathbb{R} 
\]

(6)

where \(a, \sigma > 0\) are scale parameters & \(\lambda > 0\) is a shape parameter. The survival, hazard, and reverse hazard functions can be represented as

\[
S_{I,LR}(x; a, \lambda, \sigma) = \frac{a \lambda x e^{\frac{x^2}{\sigma^2}}}{\sigma^2(1 - e^{\frac{x^2}{\sigma^2}})^2} \left[1 + a \left( \frac{e^{\frac{x^2}{\sigma^2}}}{1 - e^{\frac{x^2}{\sigma^2}}} \right) \right]^{-1} 
\]

(7)

\[
h_{I,LR}(x; a, \lambda, \sigma) = \frac{a \lambda x e^{\frac{x^2}{\sigma^2}}}{\sigma^2(1 - e^{\frac{x^2}{\sigma^2}})^2} \left[1 + a \left( \frac{e^{\frac{x^2}{\sigma^2}}}{1 - e^{\frac{x^2}{\sigma^2}}} \right) \right]^{-1} 
\]

(8)

\[
r_{I,LR}(x; a, \lambda, \sigma) = \frac{a \lambda x e^{\frac{x^2}{\sigma^2}}}{\sigma^2(1 - e^{\frac{x^2}{\sigma^2}})^2} \left[1 + a \left( \frac{e^{\frac{x^2}{\sigma^2}}}{1 - e^{\frac{x^2}{\sigma^2}}} \right) \right]^{-1} 
\]

(9)

Validity Test The pdf in equation (6) is valid. For a pdf to be valid, the \(\int_{-\infty}^{\infty} f(x) dx \) must be 1. In this case,

\[
\int_{-\infty}^{\infty} f(x; a, \lambda, \sigma) dx = \int_{0}^{\infty} \frac{a \lambda x e^{\frac{x^2}{\sigma^2}}}{\sigma^2(1 - e^{\frac{x^2}{\sigma^2}})^2} \left[1 + a \left( \frac{e^{\frac{x^2}{\sigma^2}}}{1 - e^{\frac{x^2}{\sigma^2}}} \right) \right]^{-1} dx
\]

(10)

which is the same as

\[
a d \int_{0}^{\infty} \frac{x e^{\frac{x^2}{\sigma^2}}}{(1 - e^{\frac{x^2}{\sigma^2}})^2} \left[1 + a \left( \frac{e^{\frac{x^2}{\sigma^2}}}{1 - e^{\frac{x^2}{\sigma^2}}} \right) \right]^{-1} dx,
\]

substituting \(y = a \left( \frac{e^{\frac{x^2}{\sigma^2}}}{1 - e^{\frac{x^2}{\sigma^2}}} \right)\),

we arrived at

\[
\lambda \int_{0}^{\infty} (1 + y)^{-1-a} dy = 1 
\]

4. Quantile function

The quantile function of ILD can be derived by inverting equation (5) as follows

Let

\[
U = \left[1 + a \left( \frac{e^{\frac{z^2}{\sigma^2}}}{1 - e^{\frac{z^2}{\sigma^2}}} \right) \right]^{-1}
\]

(11)

\[
U^{-\frac{1}{a}} - 1 = \left( \frac{e^{\frac{2}{\sigma^2}}}{1 - e^{\frac{2}{\sigma^2}}} \right)
\]

\[
e^{-\frac{2}{\sigma^2}} - 1 \quad e^{-\frac{2}{\sigma^2}} - 1 + e^{-\frac{2}{\sigma^2}} = ae^{-\frac{2}{\sigma^2}}
\]

\[
e^{-\frac{2}{\sigma^2}} - 1 = U^{-\frac{1}{a}} - 1
\]

by taking log of both sides and some simplifications, we have the quantile function as

\[
x = \left[ -2\sigma^2 \log \left( \frac{1 - U^{-\frac{1}{a}}} {1 - a - U^{-\frac{1}{a}}} \right) \right]^\frac{1}{2}
\]

(12)

5. Mixture form

The pdf in equation (6) can be re-written in closed form. This form can be used in deriving basic properties such as moments, entropies, and distribution of order statistics.

\[
f(x; a, \lambda, \sigma) = \frac{a \lambda x e^{\frac{x^2}{\sigma^2}}}{\sigma^2(1 - e^{\frac{x^2}{\sigma^2}})^2} \left[1 + a \left( \frac{e^{\frac{x^2}{\sigma^2}}}{1 - e^{\frac{x^2}{\sigma^2}}} \right) \right]^{-1}
\]

(13)

\[
\Rightarrow \quad \left[1 + a \left( \frac{e^{\frac{x^2}{\sigma^2}}}{1 - e^{\frac{x^2}{\sigma^2}}} \right) \right]^{-1} = \sum_{i=1}^{n} \frac{(-1)^i \Gamma(\lambda + 1 + i)}{i \Gamma(\lambda + 1)} \left( \frac{e^{\frac{x^2}{\sigma^2}}}{1 - e^{\frac{x^2}{\sigma^2}}} \right)^{i}
\]

(14)

\[
\Rightarrow \quad (1 - e^{-\frac{2}{\sigma^2}})^{-1} = \sum_{i=1}^{n} \frac{\Gamma(2 + i + j)}{j \Gamma(2 + i)} \left( e^{-\frac{2}{\sigma^2}} \right)^{i}
\]

(15)
6. Moment and moment generating function

6.1. Moment

The moments of the ILR distribution can be given in terms of the mixture representations discussed in section 5.

\[
f(x; \alpha, \lambda, \sigma) = \frac{a\lambda}{\sigma^2[\Gamma(\lambda + 1)]} \sum_{ij} \xi_{ij} x \left( \frac{1}{2}\sigma^2 \right)^{1/2} x^{1/2} e^{-x/2}.
\]

Substituting \( y = \frac{x^2}{2\sigma^2} (1 + i + j) \), we arrived at

\[
I_{\gamma} = \int_{-\infty}^{\infty} f(x) dx = \frac{a\lambda}{\sigma^2[\Gamma(\lambda + 1)]} \sum_{ij} \xi_{ij} \left( \frac{1}{2}\sigma^2 \right)^{1/2} x^{1/2} e^{-x/2}.
\]

6.2. Moment generating function

The moment generating function of ILR distribution can be given in terms of the moment as shown below

\[
M(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \sum_{t} r e^{\gamma t} P_{\gamma}
\]

7. Renyi entropy of ILR distribution

The entropy considered here is the Renyi entropy by [23]. The Renyi entropy for the ILR random variable is given by

\[
I_{\gamma}(c) = \frac{1}{\gamma-1} \log \int_{0}^{\infty} f(x) dx, \quad c \geq 0 \quad \& \quad c \neq 1
\]

\[
f(x; \alpha, \lambda, \sigma) = \frac{a\lambda}{\sigma^2[\Gamma(\lambda + 1)]} \sum_{ij} \xi_{ij} x \left( \frac{1}{2}\sigma^2 \right)^{1/2} x^{1/2} e^{-x/2}.
\]

by replacing back, we have

\[
I_{\gamma}(c) = \frac{1}{\gamma-1} \log \int_{0}^{\infty} \left( \frac{a\lambda}{\sigma^2[\Gamma(\lambda + 1)]} \sum_{ij} \xi_{ij} x \left( \frac{1}{2}\sigma^2 \right)^{1/2} x^{1/2} e^{-x/2} \right)^c dx, \quad c \neq 1
\]

\[
I_{\gamma}(c) = \frac{1}{\gamma-1} \log \int_{0}^{\infty} \left( \frac{a\lambda}{\sigma^2[\Gamma(\lambda + 1)]} \sum_{ij} \xi_{ij} x \left( \frac{1}{2}\sigma^2 \right)^{1/2} x^{1/2} e^{-x/2} \right)^c dx,
\]

for simplicity, we use

\[
[1 - F(x)]^{n-k} = \sum_{a=0}^{\infty} \left( \frac{n-k}{a} \right) (-1)^a [F(x)]^a.
\]

8. Pdf of the kth order statistics for the ILR distribution

Order statistics are important in statistical theory especially in the theory of extreme value. The pdf of the kth order statistics of the ILR is derived here

\[
f_{k,n}(x) = \frac{n!}{k!(n-k)!} f(x) [F(x)]^{k-1} [1 - F(x)]^{n-k},
\]

where

\[
I_{\gamma}(c) = \frac{1}{\gamma-1} \log \left[ \frac{2^{\gamma-1} a^\gamma \lambda^\gamma \sigma^2}{[\Gamma(\lambda + 1)]^\gamma} \sum_{ij} \xi_{ij} \left( \frac{1}{2}\sigma^2 \right)^{1/2} \right],
\]

for simplicity, we use

\[
[1 - F(x)]^{n-k} = \sum_{a=0}^{\infty} \left( \frac{n-k}{a} \right) (-1)^a [F(x)]^a.
\]

9. Incomplete moment of X

Incomplete moments play a significant role in computing measures of statistical theory. The sth incomplete moments \( m_s(y) \) of the ILR distribution are

\[
m_s(x) = \int_{0}^{y} x^s f(x) dx
\]
Let \( x_1, x_2, x_3, \ldots, x_n \) be the observed values of \( n \) observations independently drawn from the ILR distribution with parameter vector \( \theta = (\alpha, \lambda, \sigma)^T \). Then,

\[
f(x; \theta) = \frac{a \lambda x^2 e^{-\frac{x^2}{2\sigma^2}}}{\alpha^2 (1 - e^{-\frac{x^2}{2\sigma^2}})^{\frac{3}{2}}} \left[ 1 + \alpha + \frac{e^{\frac{x^2}{2\sigma^2}}}{1 - e^{-\frac{x^2}{2\sigma^2}}} \right]^{\alpha-1}. \tag{33}
\]

The log-likelihood (II) function for \( \theta \) denoted by \( l(\theta) \) can be expressed as

\[
l(\theta) = n \log \left( \frac{a \lambda}{\alpha^2} \right) + \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \frac{x_i^2}{2\sigma^2} - \frac{3}{2} \sum_{i=1}^{n} \log \left( 1 - e^{-\frac{x_i^2}{2\sigma^2}} \right) - (\alpha + 1) \sum_{i=1}^{n} \log \left[ 1 + \alpha + \frac{e^{\frac{x_i^2}{2\sigma^2}}}{1 - e^{-\frac{x_i^2}{2\sigma^2}}} \right]. \tag{34}
\]

have taken the partial derivatives of Equation (34) with respect to \( \alpha, \lambda, \) and \( \sigma \) we derived \( U(\theta) \) i.e. the Score Vector components are as follows

\[
\frac{d\ell}{d\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \left( 1 + \alpha + \frac{e^{\frac{x_i^2}{2\sigma^2}}}{1 - e^{-\frac{x_i^2}{2\sigma^2}}} \right) \tag{35}
\]

\[
\frac{d\ell}{d\lambda} = \frac{n}{\lambda} \tag{36}
\]

\[
\frac{d\ell}{d\sigma} = -2n \sigma + \sum_{i=1}^{n} \frac{x_i^2}{\sigma^3} - \frac{2}{\sigma^3} \sum_{i=1}^{n} \left( 1 - \frac{x_i^2}{2\sigma^2} \right) \tag{37}
\]

Setting Equations (35), (36), and (37) to zero and also solving simultaneously yields the MLE (\( \hat{\theta} \)) of \( \theta \). However, these equations can not be solved analytically. Therefore, statistical software can be employed to solve the equations numerically through iterative methods.

### 12. Simulation

In this section, a Monte Carlo simulation analysis is performed and the findings are presented to demonstrate the performance of the estimates at different true parameter values. We set the true parameter values as \( (\alpha = 0.6, \lambda = 0.5, \sigma = 0.3) \). The numerical study is described as follows:

(i). For true parameter values i.e. \( \theta = (\lambda, \sigma, a)^T \), we simulated a random sample of size \( n \) from the ILR distribution using the quantile function defined in Equation (11).

(ii). We then Estimate the parameters of the ILR distribution from the sample using method of maximum likelihood.

(iii). We conduct \( N = 1,000 \) replications of steps (i) and (ii).

(iv). For each of the three (3) estimated parameters of the ILR, from the \( N \) replicates, we compute the mean estimate, Bias, and MSE. The statistics are given by

\[
\hat{\theta} = \frac{1}{N} \sum_{i=1}^{N} \tilde{\theta}_i, \quad Bias(\hat{\theta}) = \hat{\theta} - \theta, \quad \text{var}(\hat{\theta}) = \frac{1}{N} \sum_{i=1}^{N} (\tilde{\theta}_i - \hat{\theta})^2
\]

where the vector of estimated parameters \( \tilde{\theta}_i \) is the maximum likelihood estimate for each iteration (\( n = 30, 75, 150, 300, 500, 1,000 \)).

The simulation results are presented in Table 1. The simulation study has shown that irrespective of the parameter values chosen, the Bias and MSE of the parameter estimate decay as the sample size \( n \) increases. Thus, the larger the sample size, the more consistent are the estimates of the parameters. The estimates are good as they approach the true parameter values as the sample size increases.

### 13. Application

A demonstration of the applicability of the ILR was demonstrated using fatigue data as in [1]. The summary of the data is as follows: \( n = 76, \) minimum = 0.0251, maximum = 9.9096, mean = 1.9592, mode = 1.5, median = 1.7362, variance = 2.4774, skewness = 1.9796, as well as kurtosis = 5.1608. Table 2 summarizes the comparators with their


Table 1. The Estimate, Bias, and MSE for the parameters of ILR distribution.

| n   | Properties | $\alpha = 0.6$ | $\lambda = 0.5$ | $\sigma = 0.3$ |
|-----|------------|----------------|-----------------|----------------|
| 30  | Est.       | 0.6014         | 0.5003          | 0.1119         |
|     | Bias       | 0.0014         | 0.0003          | -0.1881        |
|     | MSE        | 0.0009         | 0.0002          | 0.0436         |
| 75  | Est.       | 0.6013         | 0.5004          | 0.1357         |
|     | Bias       | 0.0013         | 0.0004          | -0.1643        |
|     | MSE        | 0.0009         | 7.4e-05         | 0.0389         |
| 150 | Est.       | 0.6003         | 0.5006          | 0.1419         |
|     | Bias       | 0.0003         | 0.0006          | -0.1581        |
|     | MSE        | 1.7e-07        | 6.01e-07        | 0.0396         |
| 300 | Est.       | 0.6003         | 0.5006          | 0.1485         |
|     | Bias       | 0.0003         | 0.0006          | -0.1515        |
|     | MSE        | 1.72e-07       | 5.81e-07        | 0.0399         |
| 500 | Est.       | 0.6012         | 0.5005          | 0.1531         |
|     | Bias       | 0.0012         | 0.0005          | -0.1469        |
|     | MSE        | 0.0008         | 1.14e-05        | 0.0398         |
| 1000| Est.       | 0.6003         | 0.5006          | 0.1637         |
|     | Bias       | 0.0003         | 0.0006          | -0.1363        |
|     | MSE        | 1.61e-07       | 5.47e-07        | 0.0380         |

Table 2. Competing Models with ILR distribution.

| Models | References |
|--------|------------|
| TGR    | [23]       |
| WR     | [24]       |
| TLR    | [25]       |
| R      | [26]       |

Table 3. Maximum Likelihood Estimates (with standard errors) for the ILR distributions and other comparators.

| Models | MLEs          |
|--------|---------------|
|        | $\hat{\alpha}$ | $\hat{\lambda}$ | $\hat{\delta}$ |
| ILR    | 0.1816        | 0.7525           | 3.2501          |
| TGR    | (0.1139)      | (0.1389)         | (0.7495)        |
| WR     | 0.6342        | 0.2649           | 0.7179          |
| T2R    | 0.0765        | 0.0389           | 0.2284          |
| R      | 7.9           | 0.6336           | 0.0152          |
|        | (3.0821)      | (0.0529)         | (0.0079)        |
|        | 1.3066        | 0.0028           | 0.1825          |
|        | (0.646)       | (0.0025)         | (0.0227)        |
|        | 1.7725        |                 |                 |
|        | (0.1017)      |                 |                 |

The Maximum Likelihood estimates with the standard errors in (parentheses) for the ILR distribution are presented in Table 3. These are the estimated values of the parameters of the ILR distribution based on the Fatigue data set. The standard errors can be used to compute the confidence interval for drawing inferences. In Table 4, ILR seems to be the best with large P-Value and smaller AIC, CAIC, BIC, HQIC, -ll, and KS, respectively. This shows that the proposed ILR distribution fits the Fatigue data better than the compared distributions.

14. Concluding remarks

In this paper, a new sub-model of the Inverse Lomax G family of distributions was proposed. It is called Inverse Lomax Rayleigh (ILR) distribution. Some of the properties of ILR distribution were presented. Figs. 1, 2, and 3 show the pdfs, hazards, cdf, and survival functions, respectively. This indicates that the ILR distribution can take symmetric and asymmetric shapes depending on the values of the parameters. To test the proposed distribution, a simulation study was conducted by setting the initial values of the parameters as ($\alpha = 0.6, \lambda = 0.5, \sigma = 0.3$) for 1,000 iterations. Furthermore, the ILR distribution was fitted to a Fatigue data set alongside some other distributions in the literature. Based on the results in Table 4, the proposed ILR distribution seems to

Fig. 1. Plots of the ILR density function.

Fig. 2. Plots of the ILR cdf and sf functions.
Table 4. Measurement criteria for the ILR and other comparators.

| Models | AIC     | CAIC    | BIC     | HQIC    | LI     | F-Value | K.S     |
|--------|---------|---------|---------|---------|--------|---------|---------|
| ILR    | 249.1795| 249.1717| 256.1717| 251.9739| 121.5898| 5.43E-01| 0.0897  |
| TGR    | 251.6984| 252.0317| 258.6906| 254.4928| 122.8486| 2.27E-01| 0.1174  |
| WR     | 252.6481| 252.9815| 259.6403| 255.4426| 123.3241| 1.37E-01| 0.1306  |
| TLR    | 388.2653| 388.5986| 395.2575| 391.0597| 191.1227| 3.01E-01| 0.2175  |
| R      | 276.6394| 276.6935| 278.9702| 277.5709| 137.3197| 2.90E-03| 0.2043  |

Fig. 3. Plots of the ILR hazard function.

Fig. 4. Fitted Densities for the Fatigue data set.

be the best. Fig. 4 also indicated that the ILR distribution fitted the data best than the other comparators.

Declarations

Author contribution statement

M. I. Nazir: Conceived and designed the experiments. H. A. Abdul-salam: Performed the experiments; Wrote the paper. J. Y. Falgore: Analyzed and interpreted the data; Wrote the paper.

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Data availability statement

Data included in article/supplementary material/referenced in article.

Declaration of interests statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

Acknowledgements

The authors appreciate the efforts of the editor and the reviewers.

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