Convergence of metric two-level measure spaces

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Abstract

We extend the notion of metric measure spaces to so-called metric two-level measure spaces (m2m spaces): An m2m space \((X, r, \nu)\) is a Polish metric space \((X, r)\) equipped with a two-level measure \(\nu \in \mathcal{M}_f(\mathcal{M}_f(X))\), i.e. a finite measure on the set of finite measures on \(X\). We introduce a topology on the set of (equivalence classes of) m2m spaces induced by certain test functions (i.e. the initial topology with respect to these test functions) and show that this topology is Polish by providing a complete metric.

The framework introduced in this article is motivated by possible applications in biology. It is well suited for modeling the random evolution of the genealogy of a population in a hierarchical system with two levels, for example, host-parasite systems or populations which are divided into colonies. As an example we apply our theory to construct a random m2m space modeling the genealogy of a nested Kingman coalescent.

Keywords: metric two-level measure spaces, metric measure spaces, two-level measures, nested Kingman coalescent measure tree, two-level Gromov-weak topology

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1 Introduction

A metric measure space, hereinafter abbreviated as mm space, is a triple $(X, r, \mu)$ where $(X, r)$ is a Polish metric space (i.e. a complete and separable metric space) and $\mu$ is a finite Borel measure on $X$. Metric measure spaces are central objects in probability theory. Every random variable on a Polish metric space $(X, r)$ can be identified with its probability distribution $\mu$ on $X$. Hence, the random variable is represented by the metric measure space $(X, r, \mu)$. Therefore, metric measure spaces occur almost everywhere in probability theory, although most of the time only implicitly. On the other hand, notions of convergence of mm spaces and metrics on the set of (equivalence classes of) mm spaces have been of interest in geometric analysis ([Gro99, Stu06, LV09]), problems of optimal transport ([Vil09, in particular chapter 27]) and mathematical biology ([GPW09]). The introduction of these notions of convergence has allowed for the study of mm space-valued stochastic processes. This is of particular interest in mathematical biology, where such processes are used to study the evolution of genealogical (or phylogenetic) trees (cf. [GPW13, DGP12, Glö12, KW, Guf]). Typically, the metric space $(X, r)$ incorporates the genealogical tree and $\mu$ is a uniform sampling measure on the leaves of the tree.

In this article we extend the theory of metric measure spaces, in particular the definitions and results from [GPW09]. In [GPW09] the authors study metric probability measure spaces, i.e. metric measure spaces $(X, r, \mu)$ where $\mu$ is a probability measure. They introduce the Gromov-weak topology on the set $\mathbb{M}_1$ of equivalence classes of metric probability measure spaces. This topology is defined as an initial topology with respect
to a certain set of test functions on $\mathbb{M}_1$. Roughly speaking, convergence in the Gromov-weak topology is equivalent to weak convergence of the distributions of sampled finite subspaces. The authors also introduce the Gromov-Prokhorov metric and prove that it induces the Gromov-weak topology. In particular, they show that $\mathbb{M}_1$ equipped with the Gromov-weak topology is Polish and thus a suitable state space for stochastic processes.

The results in [GPW09] have been generalized to metric measure spaces with finite measures ([Glö12]) and extended to marked metric measures spaces (see [DGP11] for probability measures and [KW] for finite measures). We also seek to extend the results in [GPW09] by replacing the measure $\mu$ on the metric space $(X,r)$ with a two-level measure $\nu \in \mathcal{M}_f(\mathcal{M}_f(X))$, i.e. with a finite measure on the set of finite measures on $X$. This extension is motivated by the study of two-level branching systems in biology, e.g. host-parasite systems, where individuals of the first level are grouped together to form the second level and both levels are subject to branching or resampling mechanisms.

Let us give a few examples of models of such two-level systems that can be found in the mathematical literature:

(1) Dawson, Hochberg and Wu develop a two-level branching process in [DHW90, Wa91]. They consider particles which move in $\mathbb{R}^d$ and are subject to a birth-and-death process. Moreover, the particles are grouped into so-called superparticles, which are subject to another birth-and-death process. The state of this process is given by a two-level measure $\nu \in \mathcal{M}(\mathcal{M}(\mathbb{R}^d))$, i.e. a Borel measure on the set of Borel measures on $\mathbb{R}^d$. The authors also consider the small mass, high density limit of the discrete process. This leads to a two-level diffusion process.

(2) The authors in [BT11] provide a continuous-time two-level branching model for parasites in cells. The parasites live and reproduce (i.e. branch) inside of cells which are subject to cell division. At division of a cell the parasites inside are distributed randomly between the two daughter cells. The model originates from the discrete processes in [Kim97, Ban08] and can be seen as a diffusion limit of these processes.

(3) Another example of a two-level process is given in [MR13]. The authors model the evolution of a population together with different kinds of cells that proliferate inside of the individuals of the population. The individuals follow a birth-and-death mechanism (including mutation and selection) and the cells inside the individuals follow another birth-and-death mechanism.

(4) Dawson studies two-level resampling models in [Daw17]. He considers a random process that models a population which is divided into colonies. The type space $X$ of the individuals is finite and the state of the process is given by a two-level probability measure $\nu \in \mathcal{M}_1(\mathcal{M}_1(X))$, i.e. a Borel probability measure on the set of Borel probability measures on $X$. The individuals are subject to mutation, selection, resampling and migration mechanisms while at the same time the colonies are also subject to selection and resampling mechanisms. The author considers the finite case in which the number of colonies and the number of individuals per colony
are fixed as well as the limit case when both of these numbers go to infinity. The limit is a two-level diffusion process called the two-level Fleming-Viot process.

We want to extend the theory of metric measure spaces in such a way that the new framework is suitable for modeling the aforementioned examples. The hierarchical two-level structure is of particular interest for us. That is why we study triples \((X, r, \nu)\) where \((X, r)\) is a Polish metric space and \(\nu \in \mathcal{M}_f(\mathcal{M}_f(X))\). We call such a triple a \textit{metric two-level measure space} (abbreviated as \textit{m2m space}).

Let us give an example how we intend to use an m2m space \((X, r, \nu)\) to model a host-parasite system: The metric space \((X, r)\) represents the set of parasites \(X\) together with its genealogical (or phylogenetic) tree, which is encoded in the metric \(r\). The two-level measure \(\nu\) encodes the distribution of individuals among the hosts. For example, a single host with two parasites might be represented by a measure \(\delta_\delta + \delta_\delta\), whereas two hosts with one parasite each might be represented by a measure \(\delta_\delta + \delta_\delta\). One can easily imagine more complicated situations with more hosts and parasites. The theory developed in this article also allows to model two-level diffusions. They appear as small mass, high density limits of atomic two-level measures like above (cf. examples (1) and (4)).

In the context of our theory we are only interested in the structure of the (genealogical) trees and not in their labels. To get rid of labels, we define a notion of equivalence for m2m spaces. The focus of this equivalence is on the structure of the measure \(\nu\) and its “effective support in \(X\)”. By effective support in \(X\) we mean the smallest closed subset \(C \subset X\) with \(\text{supp}\, \mu \subset C\) for \(\nu\)-almost every \(\mu \in \mathcal{M}_f(X)\). This set is equal to the support of the first moment measure \(M_\nu(\cdot) := \int \mu(\cdot) \, d\nu(\mu)\). Roughly speaking, we identify two m2m spaces \((X, r, \nu)\) and \((Y, d, \lambda)\) if \(\lambda\) can be mapped into \(\nu\) with a function that is isometric on the effective supports of the measures. To be precise, \((X, r, \nu)\) and \((Y, d, \lambda)\) are said to be equivalent if there is a function \(f : X \to Y\) which is isometric on \(\text{supp}\, M_\nu\) (the effective support in \(X\) of \(\nu\)) and measure preserving in the sense that \(\nu = f_* \lambda\). Here, \(f_* \lambda\) denotes the two-level push-forward of \(\lambda\). It is the push-forward of the push-forward operator of \(f\). That is, \(f_* \lambda = \lambda \circ f_*^{-1}\), where \(f_*\) is the function from \(\mathcal{M}_f(X)\) to \(\mathcal{M}_f(Y)\) that maps a finite Borel measure \(\mu\) to its push-forward \(f_* \mu = \mu \circ f^{-1}\).

This notion of equivalence allows us to consider the set \(\mathcal{M}^{(2)}\) of all equivalence classes of m2m spaces. We define a set \(\mathcal{T}\) of bounded test functions on \(\mathcal{M}^{(2)}\) that is separating points in \(\mathcal{M}^{(2)}\). The set \(\mathcal{T}\) consist of three types of test functions \(\Phi : \mathcal{M}^{(2)} \to \mathbb{R}\), which serve different purposes for determining an m2m space \((X, r, \nu)\). The first kind is of the form

\[
\Phi((X, r, \nu)) = \chi(m(\nu)) \quad \text{(TF1)}
\]

where \(\chi \in C_b(\mathbb{R}_+^+)\) with \(\chi(0) = 0\) and where \(m(\nu)\) denotes the total mass of a measure \(\nu\). Test functions of this form determine the mass \(m(\nu)\) of the evaluated m2m space. The second kind is of the form

\[
\Phi((X, r, \nu)) = \chi(m(\nu)) \int \psi(m(\nu)) \, d\nu^{\otimes m}(\mu), \quad \text{(TF2)}
\]

where \(m \in \mathbb{N}\) and \(\psi \in C_0(\mathbb{R}_+^m)\) with \(\psi(a) = 0\) whenever any of the components of the vector \(a \in \mathbb{R}_+^m\) is 0 and where \(\nu^{\otimes m}(\mu)\) denotes the normalization of \(\nu\). The test functions
of the form (TF2) determine the normalized mass distribution $m_\nu \in \mathcal{M}_1(\mathbb{R}_+)$ of the evaluated $m_2m$ space. The third kind of test function is of the form

$$\Phi((X,r,\nu)) = \chi(m(\nu)) \int \psi(\mu) \int \varphi \circ R(x) d\pi^{\otimes n}(x) d\nu^{\otimes m}(\mu),$$

where $n = (n_1, \ldots, n_m) \in \mathbb{N}^m$, $\varphi \in C_b(\mathbb{R}_+^{|m|} \times |m|)$ and $\pi^{\otimes n} := \otimes_{i=1}^m \mu_i^{\otimes n_i}$. Test functions of this form determine the space $(X,r)$ (more precisely, the support $\text{supp} \ M_\nu$ equipped with the restriction of $r$) and the structure of $\nu$.

The test functions in $\mathcal{T}$ are used to induce a Hausdorff topology on $\mathcal{M}^{(2)}$. The two-level Gromov-weak topology $\tau_{2Gw}$ is defined as the initial topology with respect to $\mathcal{T}$, i.e. the coarsest topology on $\mathcal{M}^{(2)}$ such that all functions in $\mathcal{T}$ are continuous. $\mathcal{M}^{(2)}$ equipped with the two-level Gromov-weak topology is in fact a Polish space. To show this, we introduce the two-level Gromov-Prokhorov metric $d_{2GP}$, which is complete and metrizes the two-level Gromov-weak topology. Heuristically, to compute the two-level Gromov-Prokhorov distance between two $m_2m$ spaces $(X,r,\nu)$ and $(Y,d,\lambda)$ we embed $X$ and $Y$ isometrically into some common Polish metric space $Z$ and compute the Prokhorov distance between the two-level push-forwards of $\nu$ and $\lambda$. The two-level Gromov-Prokhorov distance is defined as the infimum of this value over all such embeddings.

It turns out that the functions from $\mathcal{T}$ are convergence determining for $\mathcal{M}_1(\mathcal{M}^{(2)})$. Thus, they are a suitable domain for generators of Markov processes on $\mathcal{M}^{(2)}$. In particular, this will allow us to create $m_2m$ space-valued analogs of the two-level examples given above in future research articles.

At the end of our article we apply our framework to an example. We define a two-level coalescent process called the nested Kingman coalescent, which is a coalescent model for individuals of different species. We equip the genealogical tree stemming from this coalescent with a two-level measure that contains the two-level structure of the coalescent. The result is a random $m_2m$ space called the nested Kingman coalescent measure tree.

Finally, let us summarize the two main obstacles which arise when we extend the theory of $mm$ spaces to $m_2m$ spaces:

1. In the one-level case the set of $mm$ spaces can be embedded isomorphically into $\mathcal{M}_f(\mathbb{R}^{n \times N})$ using distance matrix distributions (cf. [GPW09]). It follows directly that the Gromov-weak topology is metrizable and thus working with sequences is sufficient. Such an embedding is not possible anymore for $m_2m$ spaces. Therefore, it is not a priori clear whether the two-level Gromov weak topology is first countable and our proofs (for continuity, compactness, etc.) must not rely on sequences. Instead, we will work with nets. Nets are a generalization of sequences and most of the theorems for metric spaces using sequences (e.g. continuity of functions, closedness of sets, compactness of sets) hold true for general topological spaces when sequences are replaced by nets. After proving that $\mathcal{M}^{(2)}$ equipped with the two-level Gromov-weak topology $\tau_{2Gw}$ is in fact Polish, we then go back using ordinary sequences.

2. For characterizing compact sets of $m_2m$ spaces it is necessary to work with finite first moment measures. Unfortunately, the first moment measure $M_\nu(\cdot) =$
\[ \int \mu(\cdot) \, d\nu(\mu) \] of a two-level measure \( \nu \in \mathcal{M}_f(\mathcal{M}_f(X)) \) may be infinite. We can overcome this problem by approximating \( \nu \) sufficiently close by a two-level measure with finite moment measures. This is done by restricting \( \nu \) to \( \mathcal{M}_{\leq K}(X) = \{ \mu \in \mathcal{M}_f(X) \mid \mu(X) \leq K \} \) using a smooth density function \( f_K \). Then, \( f_K \cdot \nu \) is an element of \( \mathcal{M}_f(\mathcal{M}_{\leq K}(X)) \) and has finite moment measures. Moreover, \( f_K \cdot \nu \) converges weakly to \( \nu \) for \( K \to \infty \).

Outline: The rest of this article is organized as follows: We start with some preliminaries about finite measures, two-level measures and nets in section 2. In the subsequent section we introduce the notion of metric two-level measure spaces (m2m spaces) and the two-level Gromov-weak topology \( \tau_{2Gw} \) on the set \( \mathbb{M}^{(2)} \) of (equivalence classes of) m2m spaces. In section 4 we define the two-level Gromov-Prokhorov metric \( d_{2GP} \) on \( \mathbb{M}^{(2)} \) and show that \( (\mathbb{M}^{(2)}, d_{2GP}) \) is a Polish metric space (i.e. separable and complete). Sections 5, 6 and 7 are devoted to compact sets in \( \mathbb{M}^{(2)} \). In section 5 we define the distance distribution and the modulus of mass distribution for finite measures. In section 6 we introduce an approximation for two-level measures. The approximating two-level measures always have finite moment measures. This enables us to characterize compactness in \( \mathbb{M}^{(2)} \) in section 7. There, we give some equivalent conditions for compactness in terms of the distance distribution and the modulus of mass distribution. With these compactness conditions we are able to prove in section 8 that the topology induced by the metric \( d_{2GP} \) coincides with the two-level Gromov-weak topology \( \tau_{2Gw} \). In section 9 we investigate tightness for probability measures on \( \mathbb{M}^{(2)} \). The results about tightness are used in section 10, in which we construct a random m2m space called the nested Kingman coalescent measure tree.

2 Preliminaries

In this section we first summarize some basic properties about the weak topology and the Prokhorov metric for finite Borel measures. Then we introduce the first moment measure and the two-level push-forward. Both are essential ingredients for the definitions of m2m spaces. Finally, in the last subsection, we introduce nets in topological spaces.

Throughout the preliminaries and the rest of the article \((X, r)\) will always be a non-empty Polish metric space, unless otherwise mentioned. A Polish metric space is a complete and separable metric space, while a Polish space is a topological space which is separable and metrizable with a complete metric.

2.1 Finite measures and the Prokhorov metric

By \( \mathcal{M}_f(X) \) we denote the set of all finite Borel measures on \( X \), equipped with the weak topology. The weak topology on \( \mathcal{M}_f(X) \) is the initial topology with respect to all functions \( \mu \mapsto \int f \, d\mu \) with \( f \in \mathcal{C}_b(X) \), where \( \mathcal{C}_b(X) \) denotes the set of bounded and continuous functions from \( X \) to \( \mathbb{R} \). Recall that the initial topology on a set \( A \) with respect to a set \( \mathcal{F} \) of functions on \( A \) is defined as the coarsest topology on \( A \) such
that the functions in \( \mathcal{F} \) are continuous. By definition, a sequence \((\mu_n)\) of finite Borel measures on \( X \) converges weakly to a finite Borel measure \( \mu \) if and only if

\[
\int f \, d\mu_n \to \int f \, d\mu
\]

for every test function \( f \in C_b(X) \).

It is well known that the set \( \mathcal{M}_f(X) \) equipped with the weak topology is a Polish space and that the Prokhorov metric \( d_P \) is a complete metric for this topology (cf. for example [Pro56]). The Prokhorov distance \( d_P(\mu,\eta) \) between two finite measures \( \mu,\eta \in \mathcal{M}_f(X) \) is defined as the infimum over all \( \varepsilon > 0 \) such that

\[
\mu(A) \leq \eta(B(A,\varepsilon)) + \varepsilon \quad \text{and} \quad \eta(A) \leq \mu(B(A,\varepsilon)) + \varepsilon
\]

for all closed sets \( A \subset X \), where \( B(A,\varepsilon) = \bigcup_{a \in A} B(a,\varepsilon) \) and \( B(a,\varepsilon) \) is the open ball of radius \( \varepsilon \) around \( a \). To emphasize that we are using the Prokhorov metric for measures on a specific metric space \((X,r)\), we sometimes write \( d_P^X \) or \( d_P^{(X,r)} \) instead of \( d_P \).

For \( \mu \in \mathcal{M}_f(X) \) we define the mass of \( \mu \) by

\[
m(\mu) := \mu(X)
\]

and the normalization of \( \mu \) by

\[
\mu := \begin{cases} \frac{\mu}{m(\mu)} & \mu \neq o \\ o & \mu = o. \end{cases}
\]

Here, \( o \) denotes the null measure, which is 0 on all sets. It is easy to see that the function \( \mu \mapsto m(\mu) \) is continuous on \( \mathcal{M}_f(X) \). For a vector \( \mu = (\mu_1,\mu_2,\ldots,\mu_m) \in \mathcal{M}_f(X)^m \) we define \( m(\mu) = (m(\mu_1),\ldots,m(\mu_m)) \) and \( \mathbf{m} = (m_1,\ldots,m_m) \). Furthermore, for every \( K \geq 0 \) we define the sets

\[
\mathcal{M}_{\leq K}(X) := \{ \mu \in \mathcal{M}_f(X) \mid m(\mu) \leq K \}
\]

and

\[
\mathcal{M}_K(X) := \{ \mu \in \mathcal{M}_f(X) \mid m(\mu) = K \}.
\]

In particular, \( \mathcal{M}_1(X) \) denotes the set of probability measures on \( X \).

Recall that a set \( \mathcal{F} \subset \mathcal{M}_f(X) \) is called tight if for every \( \varepsilon > 0 \) there is a compact set \( C \subset X \) such that \( \mu(C) < \varepsilon \) for every \( \mu \in \mathcal{F} \). We say that a single measure \( \mu \in \mathcal{M}_f(X) \) is tight if the set \{\mu\} is tight. Finite measures on Polish spaces are always tight. It is well known that for probability measures tightness is equivalent to relative compactness. However, for finite measures we also need to ensure that the masses of the measures are bounded. This is part of the original theorem from Prokhorov in [Pro56, Theorem 1.12].

**Proposition 2.1 (Prokhorov’s Theorem)**

*Let \( X \) be a Polish space and \( \mathcal{F} \subset \mathcal{M}_f(X) \). \( \mathcal{F} \) is relatively compact in the weak topology if and only if \( \mathcal{F} \) is tight and the set \( \{m(\mu) \mid \mu \in \mathcal{F} \} \) is bounded in \( \mathbb{R} \).*

Observe that \( \{m(\mu) \mid \mu \in \mathcal{F} \} \) is bounded if and only if \( \mathcal{F} \) is bounded in the Prokhorov metric since

\[
|m(\mu) - m(\eta)| \leq d_P(\mu,\eta) \leq \max(m(\mu),m(\eta))
\]

for all \( \mu,\eta \in \mathcal{M}_f(X) \).
2.2 The first moment measure $M_{\nu}$ and its support

In this article we deal with two-level measures of the form $\nu \in \mathcal{M}_f(\mathcal{M}_f(X))$. They are closely related to random measures, which are represented by measures in $\mathcal{M}_1(\mathcal{M}_f(X))$. An important tool in the analysis of two-level measures is the first moment measure, also called the intensity measure. The first moment measure of $\nu$ is the Borel measure on $X$ defined by

$$M_{\nu}(\cdot) := \int \mu(\cdot) \, d\nu(\mu).$$

Note that the first moment measure may be an infinite measure.

**Lemma 2.2**

Let $X$ be a Polish space and $\nu \in \mathcal{M}_f(\mathcal{M}_f(X))$. The first moment measure $M_{\nu}$ is a supporting measure of $\nu$ in the sense that

$$\int f \, dM_{\nu} = 0 \iff \int f \, d\mu = 0 \text{ for } \nu\text{-almost every } \mu \in \mathcal{M}_f(X)$$

for any non-negative measurable function $f : X \to \mathbb{R}$.

**Proof:** If $0 < \int f \, dM_{\nu} = \int \int f \, d\mu \, d\nu(\mu)$, then the set of all $\mu \in \mathcal{M}_f(X)$ with $\int f \, d\mu > 0$ cannot have $\nu$-measure zero. On the other hand, if $0 = \int f \, dM_{\nu}$, then the set of all $\mu \in \mathcal{M}_f(X)$ with $\int f \, d\mu > \frac{1}{n}$ must have $\nu$-measure zero for every $n \in \mathbb{N}$. Consequently, $\int f \, d\mu = 0$ for $\nu$-almost every $\mu \in \mathcal{M}_f(X)$. □

Recall that the support $\text{supp}\, \mu$ of a Borel measure $\mu$ is defined as the largest closed subset $A$ of $X$ with $\mu(\complement A) = 0$. Equivalently, it is the set of all $x \in X$ with $\mu(B(x, \varepsilon)) > 0$ for every $\varepsilon > 0$.

**Corollary 2.3**

Let $X$ be a Polish space and $\nu \in \mathcal{M}_f(\mathcal{M}_f(X))$. Then $\text{supp}\, \mu \subset \text{supp}\, M_{\nu}$ for $\nu$-almost every $\mu \in \mathcal{M}_f(X)$.

**Proof:** Use Lemma 2.2 with $f := 1_{(\complement \text{supp}\, M_{\nu})}$. □

The previous corollary shows that the two-level measure $\nu$ is effectively a finite measure on $\mathcal{M}_f(\text{supp}\, M_{\nu})$, i.e., we can restrict $X$ to $\text{supp}\, M_{\nu}$ without losing information about $\nu$ (cf. Definition 3.1 and Remark 3.2 for a precise statement).

2.3 Push-forward operators

Let $(X, r)$ and $(Y, d)$ be Polish metric spaces and $g$ be a Borel measurable function from $X$ to $Y$. As usual, $g_*\mu$ denotes the push-forward measure $\mu \circ g^{-1}$ for a finite Borel measure $\mu \in \mathcal{M}_f(X)$. We regard $g_*$ as an operator

$$g_* : \mathcal{M}_f(X) \to \mathcal{M}_f(Y), \quad \mu \mapsto g_* \mu = \mu \circ g^{-1}$$

(E1)
and call \( g_* \) the (**one-level** push-forward operator of \( g \). This enables us to define the **two-level push-forward operator** \( g_{**} \) of \( g \) by

\[
g_{**} : \mathcal{M}_f(\mathcal{M}_f(X)) \to \mathcal{M}_f(\mathcal{M}_f(Y))
\]

\[
\nu \mapsto g_{**} \nu := \nu \circ (g_*)^{-1}.
\]

(E2)

In this article the function \( g \) will usually be an isometry between \( X \) and \( Y \). Then, the structure of the push-forward measure \( g_* \mu \) is the same as of the original measure \( \mu \in \mathcal{M}_f(X) \). The same is true for the two-level push-forward measure \( g_{**} \nu \) with \( \nu \in \mathcal{M}_f(\mathcal{M}_f(X)) \).

Let \( \varphi : Y \to \mathbb{R} \) be measurable and let \( \mu \in \mathcal{M}_f(X) \) and \( \nu \in \mathcal{M}_f(\mathcal{M}_f(X)) \). The following transformation formulas hold true for the push-forward measures \( g_* \mu \) and \( g_{**} \nu \) (assuming that the integrals exist):

\[
\int \varphi \, d(g_* \mu) = \int \varphi \circ g \, d\mu \quad \text{(E3)}
\]

and

\[
\int_{\mathcal{M}_f(Y)} \int_Y \varphi \, d\mu \, d(g_{**} \nu)(\mu) = \int_{\mathcal{M}_f(X)} \int_Y \varphi \, d(g_* \mu) \, d\nu(\mu)
\]

\[
= \int_{\mathcal{M}_f(X)} \int_X \varphi \circ g \, d\mu \, d\nu(\mu). \quad \text{(E4)}
\]

The following lemma summarizes some useful properties of the one-level and two-level push-forward operator.

**Lemma 2.4 (Properties of push-forward operators)**

Let \( (X,d_X),(Y,d_Y) \) be Polish metric spaces and \( h,g,g_1,g_2,... \) be measurable functions from \( X \) to \( Y \). Then we have:

(1) If \( g \) is continuous, then \( g_* \) and \( g_{**} \) defined as in (E1) and (E2), respectively, are continuous.

(2) If \( g_n \) converges pointwise to \( g \), then \( g_{n*} \) and \( g_{n**} \) converge pointwise to \( g_* \) and \( g_{**} \), respectively. That is, \( g_{n*} \mu \) converges weakly to \( g_* \mu \) and \( g_{n**} \nu \) converges weakly to \( g_{**} \nu \) for every \( \mu \in \mathcal{M}_f(X) \) and \( \nu \in \mathcal{M}_f(\mathcal{M}_f(X)) \).

(3) Let \( \mu \in \mathcal{M}_f(X) \) and \( \varepsilon > 0 \). Define \( M_\varepsilon := \{ x \in X \mid d_Y(g(x), h(x)) < \varepsilon \} \) and \( \delta := \mu(\mathbb{C} M_\varepsilon) \), then we have

\[
d_P(g_* \mu, h_* \mu) \leq \max(\varepsilon, \delta).
\]

**Proof:** The proofs of (1) and (2) are straightforward with the transformation formulas (E3) and (E4). To show (3), let \( A \subset X \) be a closed set and let \( m := \max(\varepsilon, \delta) \). Then,

\[
g_* \mu(A) = \mu(g^{-1}(A))
\]

\[
= \mu(g^{-1}(A) \cap M_\varepsilon) + \mu(g^{-1}(A) \cap \mathbb{C} M_\varepsilon)
\]

\[
\leq \mu(h^{-1}(B(A, \varepsilon))) + \delta
\]

\[
\leq h_* \mu(B(A, m)) + m
\]
and in the same way we can show that

\[ h_*\mu(A) \leq g_*\mu(B(A,m)) + m. \]

This holds for every closed set \( A \subset X \) and thus

\[ d_P(g_*\mu, h_*\mu) \leq m = \max(\varepsilon, \delta). \]

\[ \square \]

### 2.4 Nets in topological spaces

This subsection is a short introduction to nets. A more comprehensive survey can be found in [Kel55]. Nets are a generalization of sequences and the reader not familiar with this topic may safely skip this part and think of sequences whenever we use nets.

A non-empty set \( A \) with a partial order \( \preceq \) is called **directed** if every pair \( \alpha_1, \alpha_2 \in A \) has a common successor \( \alpha \in A \) (i.e. \( \alpha_1 \preceq \alpha \) and \( \alpha_2 \preceq \alpha \)). A map \( x \) from a directed set \( (A, \preceq) \) to a topological space \( (X, \tau) \) is called a **net in** \( X \) similar to sequences we will denote this map by \( (x_\alpha)_{\alpha \in A} \) or \( (x_\alpha)_{\alpha} \). Observe that \( (N, \leq) \) is a directed set and that a sequence \( (x_n)_{n \in \mathbb{N}} \) is a net with index set \( \mathbb{N} \).

We say that the net \( (x_\alpha)_{\alpha} \) is **eventually** in a set \( A \subset X \) if there is an \( \alpha_0 \in A \) such that \( x_\alpha \in A \) for all \( \alpha \succeq \alpha_0 \). We say that \( (x_\alpha)_{\alpha} \) is **frequently** in \( A \) if every \( \alpha_0 \in A \) has a successor \( \alpha \succeq \alpha_0 \) with \( x_\alpha \in A \). Likewise, we say that a net eventually (resp. frequently) has a certain property if it eventually (resp. frequently) takes values in the set of elements of \( X \) with this property.

Let \( z \) be an element of \( X \). A net \( (x_\alpha)_{\alpha} \) is said to converge to \( z \) if for every neighborhood \( N \subset X \) of \( z \) there is an \( \alpha_0 \in A \) such that \( x_\alpha \in N \) for all \( \alpha \succeq \alpha_0 \) (i.e. the net is eventually in \( N \)). We denote this convergence by \( x_\alpha \to z \).

**Lemma 2.5**

Let \( X \) and \( Y \) be topological spaces and \( f : X \to Y \). The function \( f \) is continuous if and only if for every convergent net \( x_\alpha \to z \) in \( X \) we have \( f(x_\alpha) \to f(z) \).

Let \( (B, \preceq_B) \) be another directed set and \( (y_\beta)_{\beta} \) be another net in \( X \). We say that \( (y_\beta)_{\beta} \) is a **subnet** of \( (x_\alpha)_{\alpha} \) if there is a function \( T \) from \( B \) to \( A \) with \( y = x \circ T \) (i.e. \( y_\beta = x_{T(\beta)} \) for every \( \beta \)) and if for every \( \alpha_0 \in A \) there is a \( \beta_0 \in B \) such that \( T(\beta) \succeq \alpha_0 \) for every \( \beta \succeq_B \beta_0 \).

Moreover, we call a point \( z \in X \) a **cluster point** of \( (x_\alpha)_{\alpha} \) if for every neighborhood \( N \subset X \) of \( z \) and every \( \alpha_0 \in A \) there is an \( \alpha \succeq \alpha_0 \) with \( x_\alpha \in N \) (i.e. the net is frequently in \( N \)). It can be shown that \( z \) is a cluster point of \( (x_\alpha)_{\alpha} \) if and only if there is a subnet converging to \( z \).

We call a net **compact** if every subnet has a convergent subnet. The following lemma is based on [Top74, Lemma 2.3].

**Lemma 2.6**

Let \( C \) be a subset of a regular topological space \( X \). The following are equivalent:

1. \( C \) is relatively compact.
(2) Every net in $C$ has a converging subnet.
(3) Every net in $C$ has a cluster point.
(4) Every net in $C$ is a compact net.

We define the limit superior of a real-valued net $(x_\alpha)_\alpha$ by

$$\limsup x_\alpha = \lim_{\alpha} \sup_{\alpha' \geq \alpha} x_{\alpha'},$$

i.e. it is the limit of the supremum of the tails of the net. The limit superior is the largest cluster point of the net or $\infty$ if there is no largest cluster point.

Let $(\mu_\alpha)_\alpha$ be a net of finite Borel measures on $X$. We call $(\mu_\alpha)_\alpha$ tight if for every $\varepsilon > 0$ there is a compact set $C \subset X$ such that $\limsup_{\alpha} \mu_\alpha(C^c) < \varepsilon$ (i.e. $\mu_\alpha(C^c) < \varepsilon$ eventually). If $X$ is a Polish space, convergent nets and compact nets are tight.

3 M2m spaces and the two-level Gromov-weak topology

In this section we give the basic definitions of metric two-level measure spaces, equivalence of m2m spaces and the set $\mathbb{M}^{(2)}$ of equivalence classes of m2m spaces. Moreover, we define a set $\mathcal{T}$ of test functions on $\mathbb{M}^{(2)}$ and show that $\mathcal{T}$ separates points in $\mathbb{M}^{(2)}$, i.e. that an m2m space $X \in \mathbb{M}^{(2)}$ is determined by the values $\{ \Phi(X) \mid \Phi \in \mathcal{T} \}$. Then we define the two-level Gromov-weak topology on $\mathbb{M}^{(2)}$ as the initial topology induced by $\mathcal{T}$.

Definition 3.1 (Metric two-level measure spaces and equivalences)

1. A triple $(X, r, \nu)$ is called a metric two-level measure space (m2m space) if $X \subset \mathbb{R}^N$ is non-empty, $(X, r)$ is a Polish metric space and $\nu \in \mathcal{M}_f(\mathcal{M}_f(X))$.

2. Two m2m spaces $(X, r, \nu)$ and $(Y, d, \lambda)$ are called equivalent if there exists a measurable function $f : X \to Y$ such that $\lambda = f_* \nu$ and $f$ is isometric on the set $\text{supp} \, M_\nu$ (but not necessarily on the whole space $X$). The equivalence between both spaces is denoted by $(X, r, \nu) \cong (Y, d, \lambda)$ or by $(X, r, \nu) \cong_f (Y, d, \lambda)$ if we want to emphasize that $f$ is the measure-preserving isometry.

3. By $\mathbb{M}^{(2)}$ we denote the set of all equivalence classes of m2m spaces. In the following we will not distinguish between an m2m space and its equivalence class. Generic elements of $\mathbb{M}^{(2)}$ will be denoted by $X = (X, r, \nu)$, $X_n = (X, r_n, \nu_n)$ or $Y = (Y, d, \lambda)$.

Remarks 3.2

1. Note that every Polish metric space is homeomorphic to a closed subset of $\mathbb{R}^N$ (equipped with the product topology) by [Eng89, Corollary 4.3.25]. Therefore, the condition $X \subset \mathbb{R}^N$ is not a restriction and every Polish metric space with a two-level measure can be seen as an m2m space, even if $X$ is not a subset of $\mathbb{R}^N$.

2. Let $(X, r, \nu)$ be an m2m space with $S := \text{supp} \, M_\nu \neq \emptyset$. The support of $\nu$ is a subset of $\{ \mu \in \mathcal{M}_f(X) \mid \text{supp} \mu \subset S \}$ by Corollary 2.3. Thus, $(X, r, \nu)$ is equivalent
to \((S,r',\nu')\), where \(r'\) is the restriction of \(r\) to \(S \times S\) and \(\nu'\) is the restriction of \(\nu\) to \(\mathcal{M}_f(S)\). This holds for every \(\mathbb{N}^2\) space and to simplify our proofs we will often assume (without loss of generality) that \(X = \text{supp} M\).

Before we start to define the test functions which shall induce the two-level Gromov–weak topology, we need to introduce some notation: Let \(\mu = (\mu_1, \ldots, \mu_m) \in \mathcal{M}_f(X)^m\) and \(n = (n_1, \ldots, n_m) \in \mathbb{N}^m\). We define

\[
\mu^{\otimes n} := \bigotimes_{i=1}^m \mu_i^{\otimes n_i}
\]

and

\[
\mu^\otimes n := \bigotimes_{i=1}^m \mu_i^{\otimes n_i}.
\]

Moreover, for \(\mu = (\mu_1, \mu_2, \ldots) \in \mathcal{M}_f(X)^\mathbb{N}\) we define

\[
\mu^\otimes n := \bigotimes_{i=1}^\infty \mu_i^{\otimes n_i}.
\]

This notation will shorten our test functions and will be particularly convenient in the upcoming proofs. By a slight abuse of notation, we sometimes write \((i,j) \in n\), where we regard the vector \(n\) as the set \(\{(i,j) \mid i \in \{1, \ldots, m\}, j \in \{1, \ldots, n_i\}\}\). Moreover, for a Polish metric space \((X, r)\), \(m \in \mathbb{N}\) and \(n = (n_1, \ldots, n_m) \in \mathbb{N}^m\) we define the following distance operators

\[
R_m^{(X,r)} : X^m \to \mathbb{R}^{m^2}, \\
R_n^{(X,r)} : X^n \to \mathbb{R}^{n^2}, \\
R_m^{(X,r)} : X \times X^m \to \mathbb{R}^{m^2}, \\
R_n^{(X,r)} : X \times X^n \to \mathbb{R}^{n^2}.
\]

For convenience we often suppress the super- and subscript in the distance operators above and simply write \(R\) instead of \(R_m^{(X,r)}\), \(R_n^{(X,r)}\) and \(R_m^{(X,r)}\). The space and dimension should always be clear from the context.

**Definition 3.3 (Test functions)**

Define \(\mathcal{T}\) as the set of all test functions \(\Phi : \mathbb{M}^{(2)} \to \mathbb{R}\) that are of one of the following forms:

\[
\Phi((X, r, \nu)) = \chi(m(\nu)), \quad (\text{TF1})
\]

\[
\Phi((X, r, \nu)) = \chi(m(\nu)) \int \psi(m(\mu)) d\nu^{\otimes m}(\mu), \quad (\text{TF2})
\]

\[
\Phi((X, r, \nu)) = \chi(m(\nu)) \int \psi(m(\mu)) \int \varphi \circ R(x) d\nu^{\otimes n}(x) d\nu^{\otimes m}(\mu), \quad (\text{TF3})
\]

where \(m \in \mathbb{N}\), \(n = (n_1, \ldots, n_m) \in \mathbb{N}^m\), \(\chi \in C_b(\mathbb{R}_+), \psi \in C_b(\mathbb{R}_+^n), \varphi \in C_b(\mathbb{R}_+^{n^2})\) with \(\chi(0) = 0\) and \(\psi(\alpha) = 0\) whenever any of the components of the vector \(\alpha \in \mathbb{R}_+^n\) is \(0\).
The test functions are created in such a way that they are bounded on \( M^{(2)} \). This is the reason why we need to decompose the measures \( \nu \) and \( \mu \) into the masses \( m(\nu), m(\mu) \) and the normalized measures \( \overline{\nu} \) and \( \overline{\mu} \). Later in this section we will define a topology on \( M^{(2)} \) such that the test functions are continuous. Thus, they are a suitable domain for generators of stochastic processes on \( M^{(2)} \).

In Theorem 3.8 we will prove that \( \mathcal{T} \) is separating in \( M^{(2)} \). The three types of test functions serve different purposes for determining an \( m2m \) space \((X, r, \nu)\). Test functions of the form (TF1) simply determine the mass \( m(\nu) \), whereas test functions of the form (TF2) determine the normalized mass distribution \( m, \overline{\nu} \). The space \((X, r)\) (more precisely, the support \( \text{supp} M_\nu \) equipped with the restriction of \( r \)) and the structure of \( \overline{\nu} \) are determined by the test functions of type (TF3).

Note that the set \( \mathcal{T} \) is closed under multiplication, but not under addition. Moreover, the functions in \( \mathcal{T} \) are well-defined, as we can see in the next lemma.

**Lemma 3.4**

Every \( \Phi \in \mathcal{T} \) is well-defined. That is, we have \( \Phi(\mathcal{X}) = \Phi(\mathcal{Y}) \) for equivalent \( m2m \) spaces \( \mathcal{X}, \mathcal{Y} \in M^{(2)} \).

**Proof:** Let \( \mathcal{X} = (X, r, \nu) \cong \mathcal{Y} = (Y, d, \lambda) \). That is, \( f: X \to Y \) is isometric on \( \text{supp} M_\nu \) and \( \lambda = f_* \nu \). For \( n \in \mathbb{N} \) we define \( f^{*n}: X^n \to Y^n \) by \( f^{*n}(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n)) \). Because of the isometric properties of \( f \), we have \( R^{(Y,d)}(f^{*n}(x)) = R^{(X,r)}(x) \) for \( x \in (\text{supp} M_\nu)^n \). Therefore, we conclude that for every \( \Phi \) as in (TF3)

\[
\Phi((Y, d, \lambda)) = \Phi((Y, d, f_* \nu)) = \chi(m(f_* \nu)) \int \psi(m(\mu)) \int \varphi \circ R^{(Y,d)}(x) d\overline{\mu}^{\otimes n}(x) d(f_* \nu)^{\otimes m}(\mu) = \chi(m(\nu)) \int \psi(m(f^{*n} \mu)) \int \varphi \circ R^{(Y,d)}(x) d(f^{*n} \overline{\mu})^{\otimes n}(x) d\overline{\mu}^{\otimes m}(\mu) = \chi(m(\nu)) \int \psi(m(\mu)) \int \varphi \circ R^{(Y,d)}(f^{*n}(x)) d\overline{\mu}^{\otimes n}(x) d\overline{\mu}^{\otimes m}(\mu) = \Phi((X, r, \nu)).
\]

Equality for test functions as in (TF1) and (TF2) follows in the same way. \( \square \)

Before we are able to prove that \( \mathcal{T} \) is separating in \( M^{(2)} \), we need some preparatory propositions. Let \( X \) be a Polish space and \( \mu \) be a Borel probability measure on \( X \). For a sequence \( x = (x_i)_{i \in \mathbb{N}} \in X^\mathbb{N} \) and \( n \in \mathbb{N} \) we define the empirical measures

\[
\Xi_n(x) := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}.
\]

The Glivenko-Cantelli Theorem (cf. for example [Dud02, Theorem 11.4.1]) states that if \( x \) is random with law \( \mu^{\otimes \mathbb{N}} \), then almost surely the weak limit \( w{-}\lim_{n \to \infty} \Xi_n(x) \) exists and is equal to \( \mu \). In other words, the measure \( \mu \) can be reconstructed from an infinite i.i.d. sample. This also holds if \( \mu \) is a random probability measure, as can be seen from the following proposition from [Daw93, Theorem 11.2.1].
Proposition 3.5 (Reconstruction of two-level probability measures)
Let $\nu \in \mathcal{M}_1(\mathcal{M}_1(X))$ be a two-level probability measure on a non-empty Polish space $X$ and $x = (x_i)_{i \in \mathbb{N}} \in X^\mathbb{N}$ be a random sequence with law $\int \mu^\otimes \mathbb{N} \, d\nu(\mu)$. Then the weak limit
$$\mu := \lim_{n \to \infty} \Xi_n(x)$$
exists almost surely and the random probability measure $\mu$ has law $\nu$.

Our goal is to reconstruct the two-level measure $\nu \in \mathcal{M}_1(\mathcal{M}_1(X))$ from an infinite sample in $X$. To this end, we need not one random measure $\mu$, but an i.i.d. sequence of random measures $(\mu_i)_i$. If we sample an infinite i.i.d. sample $(x_{ij})_{ij}$ with each measure $\mu_i$, then, from all of the sampled points we can reconstruct first the measures $\mu_i$ and then the two-level measure $\nu$. To be precise, let $x = (x_{ij})_{ij}$ be an infinite random matrix in $X$ with law $\int \mu^\otimes \mathbb{N} \, d\nu^\otimes \mathbb{N}(\mu)$. Then, almost surely for each $i \in \mathbb{N}$ the weak limit $\mu_i := \lim_{n \to \infty} \Xi_n((x_{ij})_{ij})$ exists and each random measure $\mu_i$ has law $\nu$. Therefore, $\mu = (\mu_i)_i$ is an infinite i.i.d. sample in $\mathcal{M}_1(X)$ and $\nu$ is the weak limit of $\Xi_n(\mu)$ by the Glivenko-Cantelli Theorem.

This result can easily be extended to measures $\nu \in \mathcal{M}_1(\mathcal{M}_f(X))$ by decomposing finite measures $\mu \in \mathcal{M}_f(X)$ into their total mass $m(\mu)$ and their normalized (probability) measure $\mu$, provided that $\nu(\{o\}) = 0$ (since we cannot sample points with the null measure $o$). Let us first state a generalization of Proposition 3.5. We omit the straightforward proof.

Lemma 3.6
Let $X$ be a non-empty Polish space and $\nu \in \mathcal{M}_1(\mathcal{M}_f(X))$ be a two-level measure with $\nu(\{o\}) = 0$. Let $(m, x) \in \mathbb{R}_+ \times X^\mathbb{N}$ be random with law $\int \delta_{m(\mu)} \otimes \mu^\otimes \mathbb{N} \, d\nu(\mu)$. Then the weak limit
$$\mu := m \cdot \lim_{n \to \infty} \Xi_n(x)$$
exists almost surely and the random measure $\mu$ has law $\nu$.

Thus, we can reconstruct a randomly chosen finite measure $\mu$ from its mass and a sequence of i.i.d. samples in $X$. It follows that we can reconstruct a two-level measure $\nu \in \mathcal{M}_1(\mathcal{M}_f(X))$ from a sample of masses and points in $X$:

Proposition 3.7 (Reconstruction of two-level measures)
Let $X$ be a non-empty Polish space and let $\nu \in \mathcal{M}_1(\mathcal{M}_f(X))$ be a two-level measure with $\nu(\{o\}) = 0$. Moreover, let $((m_i), (x_{ij}))_{ij} \in \mathbb{R}^\mathbb{N} \times X^{\mathbb{N} \times \mathbb{N}}$ be random with law
$$\int \otimes_{i=1}^\infty \delta_{m(i)} \otimes \mu^\otimes \mathbb{N} \, d\nu^\otimes \mathbb{N}(\mu).$$
Then almost surely we have
1. The weak limit $\mu_i := m_i \cdot \lim_{n \to \infty} \Xi_n((x_{ij})_{ij})$ exists for every $i \in \mathbb{N}$ and the random measure $\mu_i$ has law $\nu$. 

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Without loss of generality we may assume that $m = \nu$ properties (1) to (3) from Proposition 3.7. If we define a function $Y$ Together with Proposition 3.7 this implies that there are for all $m$ $\mu$. Because $\Phi(X) = \Phi(Y)$ for every test function $\Phi \in T$. We exclude the trivial case $\nu = \lambda = o$. Because $\Phi(X) = \Phi(Y)$ for every test function $\Phi \in T$. We exclude the trivial case $\nu = \lambda = o$. Because $\Phi(X) = \Phi(Y)$ for every test function $\Phi \in T$. We exclude the trivial case $\nu = \lambda = o$. Because $\Phi(X) = \Phi(Y)$ for every test function $\Phi \in T$. We exclude the trivial case $\nu = \lambda = o$. Because $\Phi(X) = \Phi(Y)$ for every test function $\Phi \in T$. We exclude the trivial case $\nu = \lambda = o$. Because $\Phi(X) = \Phi(Y)$ for every test function $\Phi \in T$. We exclude the trivial case $\nu = \lambda = o$. Because $\Phi(X) = \Phi(Y)$ for every test function $\Phi \in T$. We exclude the trivial case $\nu = \lambda = o$. Because $\Phi(X) = \Phi(Y)$ for every test function $\Phi \in T$. We exclude the trivial case $\nu = \lambda = o$. Because $\Phi(X) = \Phi(Y)$ for every test function $\Phi \in T. \boxdot$

(2) The two-level measure $\frac{1}{n} \sum_{i=1}^{n} \delta_{\mu_i}$ converges weakly to $\nu$.

(3) The sequence $(x_{ij})_i$ is dense in supp $M_\nu$ for every positive integer $j$.

Proof: (1) If we fix $i \in \mathbb{N}$, then $(m_i, (x_{ij})_j)$ has law $\int \delta_{\mu_i} \otimes R_i^\otimes n d\nu(\mu)$ and the claim follows from Lemma 3.6.

(2) $(\mu_i)_i$ is an i.i.d. sequence of random measures, each with law $\nu$, and the claim follows from the Glivenko-Cantelli Theorem.

(3) $(x_{ij})_i$ is an i.i.d. sequence of random variables in $X$, each with law $\int \pi d\nu(\mu)$. Thus, the sequence is almost surely dense in its support, which coincides with the support of $M_\nu$. □

Property (3) of the last proposition implies that we can even reconstruct $\nu$ and $(X, r)$ if we only know the masses $(m_i)_i$ of the sampled measures and the distances $r(x_{ij}, x_{kl})$ between all points $x_{ij}$ and $x_{kl}$ with $i, j, k, l \in \mathbb{N}$. We exploit this fact in the next theorem to show that the test functions in $T$ are separating points in $M^{(2)}$.

Theorem 3.8 (Test functions are separating)

$T$ is separating in $M^{(2)}$. That is, two m2m spaces $X, Y \in M^{(2)}$ are equivalent if and only if $\Phi(X) = \Phi(Y)$ for every test function $\Phi \in T$.

Proof: We have already shown in Lemma 3.4 that the value of a test function is the same for equivalent m2m spaces. To prove the other direction, let $X = (X, r, \nu)$ and $Y = (Y, d, \lambda)$ be such that $\Phi(X) = \Phi(Y)$ for every $\Phi \in T$. We exclude the trivial case $\nu = \lambda = o$. Because $\Phi(X) = \Phi(Y)$ for every test function as in (TF1) and (TF2), we conclude that $m(\nu) = m(\lambda)$ and $m_r \nu = m_r \lambda$. In particular we have $\nu \{ o \} = \lambda \{ o \}$. Without loss of generality we may assume that $\nu$ and $\lambda$ are probability measures with $\nu \{ o \} = \lambda \{ o \} = 0$. Now, from the equality for all $\Phi \in T$ it follows that

$$\int \bigotimes_{i=1}^{m} \delta_{\mu_i} \otimes R_i^\otimes n d\nu(\mu) = \int \bigotimes_{i=1}^{m} \delta_{\mu_i} \otimes R_i^\otimes n d\lambda(\mu)$$

for all $m \in \mathbb{N}$ and $n \in \mathbb{N}^m$. By taking the projective limit we get

$$\int \bigotimes_{i=1}^{\infty} \delta_{\mu_i} \otimes R_i^\otimes n d\nu(\mu) = \int \bigotimes_{i=1}^{\infty} \delta_{\mu_i} \otimes R_i^\otimes n d\lambda(\mu).$$

Together with Proposition 3.7 this implies that there are $m \in \mathbb{R}_+^N$, $x \in X^{N \times N}$ and $y \in Y^{N \times N}$ such that $(m, R(x)) = (m, R(y))$ and that both $(m, x)$ and $(m, y)$ satisfy the properties (1) to (3) from Proposition 3.7. If we define a function $f$ by $f(x_{ij}) = y_{ij}$ for all $i, j \in \mathbb{N}$, then $f$ is isometric and can be extended continuously to an isometric function $\bar{f} : \text{supp } M_\nu \to Y$. Since we can reconstruct $\nu$ from $(m, x)$ and $\lambda$ from $(m, y)$ by taking weak limits and $f_*, f_*'$ are continuous (cf. Lemma 2.4), we see that $\lambda = f_*' \nu$. □
Remark 3.9
In the preceding proof we reconstruct an m2m space \((X, r, \nu)\) with \(\nu(\{o\}) = 0\) and \(m(\nu) = 1\) from the measure
\[
\int \bigotimes_{i=1}^{\infty} \delta_{m(\mu_i)} \otimes R_s \overline{\mu} \otimes \nu \otimes N(\mu)
\]
in \(\mathcal{M}_1(\mathbb{R}_+^N \times \mathbb{R}_+^N)\). One might be tempted to think that a general m2m space \((X, r, \nu)\) is determined by the measure (E5)
\[
m(\nu) \cdot \int \bigotimes_{i=1}^{\infty} \delta_{m(\mu_i)} \otimes R_s \overline{\mu} \otimes \nu \otimes N(\mu)
\]
in \(\mathcal{M}_f(\mathbb{R}_+^N \times \mathbb{R}_+^N)\). However, the association between an m2m space and the measure (E5) is not unique: For instance, (E5) is equal to \(o\) for every m2m space \((\emptyset, 0, c\delta_o)\) with \(c \geq 0\), even though they are different m2m spaces (cf. Remark 3.2).

Since the test functions in \(\mathcal{T}\) separate points in \(\mathcal{M}^{(2)}\), we can use them to induce a Hausdorff topology on \(\mathcal{M}^{(2)}\).

Definition 3.10 (Two-level Gromov-weak topology)
The two-level Gromov-weak topology is the initial topology on \(\mathcal{M}^{(2)}\) induced by the test functions in \(\mathcal{T}\). That is, the two-level Gromov-weak topology is the coarsest topology on \(\mathcal{M}^{(2)}\) such that the functions in \(\mathcal{T}\) are continuous. We denote this topology by \(\tau_{2Gw}\).

We say that a net of m2m spaces \((X_\alpha)_{\alpha} \subset \mathcal{M}^{(2)}\) converges two-level Gromov-weakly to an m2m space \(X \in \mathcal{M}^{(2)}\) if it converges in the two-level Gromov-weak topology, that is, if \(\Phi(X_\alpha) \to \Phi(X)\) for every test function \(\Phi \in \mathcal{T}\). We denote this by \(X_\alpha \xrightarrow{2Gw} X\).

Remark 3.11 (Weak convergence implies two-level Gromov-weak convergence)
Note that weak convergence of two-level measures implies two-level Gromov-weak convergence of the associated m2m spaces. To see this, let \((X, r)\) be a fixed Polish metric space and let \(\Phi \in \mathcal{T}\). The function
\[
\mathcal{M}_f(\mathcal{M}_f(X)) \to \mathbb{R}
\]
\(\nu \mapsto \Phi((X, r, \nu))\)
is a composition of functions which all are continuous except for the normalizing functions \(\nu \mapsto \overline{\nu}\) and \(\mu \mapsto \overline{\mu}\). However, these discontinuities are smoothed by the constraints on \(\chi\) and \(\psi\) in the definition of the test functions (recall that we have \(\chi(0) = 0\) and \(\psi(a) = 0\) whenever any of the components of the vector \(a\) is 0). Therefore, the function \(\nu \mapsto \Phi((X, r, \nu))\) is continuous for every \(\Phi \in \mathcal{T}\) and \(\nu_n \xrightarrow{w} \nu\) implies that \((X, r, \nu_n)\) converges two-level Gromov-weakly to \((X, r, \nu)\).

Remark 3.12
The theory of m2m spaces can be seen as a generalization of the theory of metric measure spaces (mm spaces). Let \((X, r, \mu), (X_1, r_1, \mu_1), (X_2, r_2, \mu_2), \ldots\) be mm spaces. Then we have
(1) \((X_1, r_1, \mu_1)\) and \((X_2, r_2, \mu_2)\) are equivalent as mm spaces if and only if \((X_1, r_1, \delta \mu_1)\)
and \((X_2, r_2, \delta \mu_2)\) are equivalent as m2m spaces.

(2) \((X_n, r_n, \mu_n)\) converges Gromov-weakly to \((X, r, \mu)\) if and only if \((X_n, r_n, \delta \mu_n)\) converges two-level Gromov-weakly to \((X, r, \delta \mu)\).

It is shown in section 8 that \((\mathbb{M}^{(2)}, \tau_{2Gw})\) is a Polish space. Thus, subsets are compact if and only if they are sequentially compact and functions on \(\mathbb{M}^{(2)}\) are continuous if and only if they are sequentially continuous. But for now we cannot use the fact that \((\mathbb{M}^{(2)}, \tau_{2Gw})\) is Polish and our proofs (for compactness or continuity) must not rely on sequences.

Sometimes it is convenient to work with simpler test functions than (TF3). For this reason we give equivalent conditions for two-level Gromov-weak convergence in the following lemma.

**Lemma 3.13**

Let \((\mathcal{X}_a)_{a \in A}\) be a net of m2m spaces with \(\mathcal{X}_a = (X_a, r_a, \nu_a)\) and let \(\mathcal{X} = (X, r, \nu)\) be another m2m space. \((\mathcal{X}_a)_{a \in A}\) converges two-level Gromov-weakly to \(\mathcal{X}\) if and only if the following two conditions hold:

1. \(m \ast \nu_a\) converges weakly to \(m \ast \nu\) in \(\mathcal{M}_f(\mathbb{R}_+)\)
2. \(\tilde{\Phi}(\mathcal{X}_a)\) converges to \(\tilde{\Phi}(\mathcal{X})\) for every \(\tilde{\Phi} : \mathbb{M}^{(2)} \to \mathbb{R}\) of the form

\[
\tilde{\Phi}(\mathcal{X}, r, \nu) = \int \psi(m(\mu)) \int \varphi \circ R d\nu_\otimes^n d\nu_\otimes^m(\mu)
\]

where \(m \in \mathbb{N}, n \in \mathbb{N}^m, \varphi \in \mathcal{C}_b(\mathbb{R}_+^{\max(|m|)}),\) and \(\psi \in \mathcal{C}_b(\mathbb{R}_+^m)\) with \(\psi(a) = 0\) whenever any of the components of the vector \(a \in \mathbb{R}_+^m\) is 0.

Note that in (TF4) we use \(\nu\) instead of the normalized version \(\overline{\nu}\).

**Proof:** (1) is equivalent to convergence of all test functions of the form (TF1) and (TF2). (2) follows from the convergence of functions of the form (TF3) by choosing the function \(\chi \in \mathcal{C}_b(\mathbb{R}_+)\) such that \(\chi(x) = x^m\) for \(x \in [0, m(\nu) + 1]\). The other direction is obvious. \(\square\)

**Remark 3.14**
The previous lemma shows that the functions \((X, r, \nu) \mapsto m(\nu)\) and \((X, r, \nu) \mapsto m_\ast \nu\) are continuous on \(\mathbb{M}^{(2)}\) in the two-level Gromov-weak topology.

## 4 The two-level Gromov-Prokhorov metric

In this section we define the two-level Gromov-Prokhorov distance \(d_{2GP}\) on the space \(\mathbb{M}^{(2)}\). At the end of this section we prove that \(d_{2GP}\) is indeed a metric and that \((\mathbb{M}^{(2)}, d_{2GP})\) is a Polish metric space. The topology induced by the metric \(d_{2GP}\) will turn out to be the same as the two-level Gromov-weak topology (cf. section 8).
Definition 4.1 (Two-level Gromov-Prokhorov metric)

Let \( \mathcal{X} = (X, r, \nu) \) and \( \mathcal{Y} = (Y, d, \lambda) \) be two \( \mathcal{m}_2 \mathcal{m} \) spaces. We define the two-level Gromov-Prokhorov distance \( d_{2\text{GP}}(\mathcal{X}, \mathcal{Y}) \) between \( \mathcal{X} \) and \( \mathcal{Y} \) by

\[
d_{2\text{GP}}(\mathcal{X}, \mathcal{Y}) := \inf_{Z, \iota_X, \iota_Y} d_{\mathcal{M}_f(Z)}(\iota_X**, \nu, \iota_Y**, \lambda),
\]

(E6)

where the infimum ranges over all isometric embeddings \( \iota_X: X \to Z, \iota_Y: Y \to Z \) into a common Polish metric space \( Z \) and where \( d_{\mathcal{M}_f(Z)} \) denotes the Prokhorov metric for measures on the space \( \mathcal{M}_f(Z) \). The topology induced by this metric is called the two-level Gromov-Prokhorov topology and denoted by \( \tau_{2\text{GP}} \).

Recall from Corollary 2.3 that for any \( \mathcal{m}_2 \mathcal{m} \) space \( (X, r, \nu) \) the support of \( \nu \) is a subset of \( \{ \mu \in \mathcal{M}_f(X) \mid \operatorname{supp} \mu \subseteq \operatorname{supp} M_{\nu} \} \).

With this fact it is easy to see that the two-level Gromov-Prokhorov distance between equivalent \( \mathcal{m}_2 \mathcal{m} \) spaces is zero. Thus, the value of \( d_{2\text{GP}}(\mathcal{X}, \mathcal{Y}) \) does not depend on the chosen representatives of the equivalence classes and is well-defined. In Proposition 4.6 we prove that \( d_{2\text{GP}} \) is in fact a metric and even complete.

Finding a “good” space \( Z \) and embeddings \( \iota_X, \iota_Y \) in (E6) can be a challenging task. If \( X \) and \( Y \) have a similar structure, there might be a natural way to overlap \( X \) and \( Y \) in such a way that the Prokhorov distance in (E6) is small. However, if \( X \) and \( Y \) are very different, it might be better to choose \( Z \) as the disjoint union \( X \sqcup Y \). Then, the problem of finding an appropriate space \( Z \) and embeddings \( \iota_X, \iota_Y \) reduces to finding a metric \( r' \) on \( X \sqcup Y \) such that it extends the old metrics \( r \) and \( d \) and connects the spaces \( X \) and \( Y \) in an optimal way. In the next lemma we prove that this procedure gives the same results as in the original definition in (E6).

Lemma 4.2 (Alternative definition of \( d_{2\text{GP}} \))

For all \( \mathcal{m}_2 \mathcal{m} \) spaces \( \mathcal{X} = (X, r, \nu) \) and \( \mathcal{Y} = (Y, d, \lambda) \) we have

\[
d_{2\text{GP}}(\mathcal{X}, \mathcal{Y}) = \inf_{r'} d_{\mathcal{M}_f(X \sqcup Y, r')}(\nu, \lambda),
\]

where the infimum ranges over all metrics \( r' \) on the disjoint union \( X \sqcup Y \) which extend the metrics \( r \) and \( d \).

Note that the metric space \( (X \sqcup Y, r') \) is always complete and separable since \( r' \) extends the complete metrics \( r \) and \( d \).

In the preceding lemma we regard \( \nu \) and \( \lambda \) as elements of the same space \( \mathcal{M}_f(\mathcal{M}_f(X \sqcup Y)) \) without using the canonical embeddings \( \iota_X: X \to X \sqcup Y \) and \( \iota_Y: Y \to X \sqcup Y \). We will continue to use this abbreviated notation in the remainder of this section whenever it seems appropriate (i.e. when we are embedding spaces into their (disjoint) union). In the same manner we will regard points \( x \in X \) and \( y \in Y \) as elements of \( X \sqcup Y \) and write \( r'(x, y) \) instead of \( r'(\iota_X(x), \iota_Y(y)) \) (where \( r' \) is a metric on \( X \sqcup Y \)).
Proof: By Definition 4.1 we always have
\[ d_{2GP}(X,Y) \leq \inf_{r'} d_P^{M_f(X \cup Y, r')} (\nu, \lambda). \]

On the other hand, let \( d_{2GP}(X,Y) < \varepsilon \). Then there is a Polish metric space \((Z, r_Z)\) and isometric embeddings \( \iota_X : X \to Z, \iota_Y : Y \to Z \) such that
\[ d_P^{M_f(Z, r_Z)} (\iota_{X**\nu}, \iota_{Y**\lambda}) < \varepsilon. \]

For \( \delta > 0 \) define \( r'_\delta \) as the metric on \( X \cup Y \) which extends \( r \) and \( d \) and satisfies
\[ r'_\delta(x,y) = \delta + r_Z(\iota_X(x), \iota_Y(y)) \]
for every \( x \in X \) and \( y \in Y \). Then \( (X \cup Y, r'_\delta) \) is complete and separable and we have
\[ d_P^{M_f(X \cup Y, r'_\delta)} (\nu, \lambda) < \varepsilon + \delta. \]

Ranging over all possible \( \varepsilon \) and \( \delta \) yields the claim. \( \square \)

In Lemma 4.4 we will see that a sequence \((X_n, r_n, \nu_n)\) of m2m spaces converges to a limit m2m space \((X, r, \nu)\) with respect to \( d_{2GP} \) if and only if we can embed all the spaces isometrically in a common Polish metric space \( Z \) such that the two-level push-forward of \( \nu_n \) converges weakly in \( M_f(M_f(Z)) \) to the two-level push-forward of \( \nu \). A similar result holds for Cauchy sequences of m2m spaces as can be seen in the following Lemma.

Lemma 4.3 (Embedding of sequences of m2m spaces)
Let \((\varepsilon_n)_n\) be a sequence of positive real numbers and let \((X_n)_n\) be a sequence of m2m spaces with \( X_n = (X_n, r_n, \nu_n) \) and
\[ d_{2GP}(X_n, X_{n+1}) < \varepsilon_n \]
for every \( n \in \mathbb{N} \). Then there is a Polish metric space \((Z, r_Z)\) and isometric embeddings \( \iota_1, \iota_2, \ldots \) of \( X_1, X_2, \ldots \), respectively, into \( Z \) such that
\[ d_P^{M_f(Z, r_Z)} (\iota_{n**\nu_n}, \iota_{n+1**\nu_{n+1}}) < \varepsilon_n \]
for all \( n \in \mathbb{N} \).

Proof: We define \( Z_n := \bigsqcup_{k=1}^n X_k \) for every \( n \in \mathbb{N} \) and \( Z_\infty := \bigsqcup_{k=1}^\infty X_k \). We will inductively define metrics \( d_n \) on \( Z_n \) using Lemma 4.2. By this lemma there exists a metric \( d_2 \) on \( Z_2 = X_1 \cup X_2 \) such that \((Z_2, d_2)\) is complete and separable and
\[ d_P^{M_f(Z_2, d_2)} (\nu_1, \nu_2) < \varepsilon_1. \]

This metric extends \( r_1 \) and \( r_2 \) and therefore we have \( (Z_2, d_2, \nu_2) \cong (X_2, r_2, \nu_2) \) and
\[ d_{2GP}((Z_2, d_2, \nu_2), \mathcal{X}_3) = d_{2GP}(X_2, \mathcal{X}_3) < \varepsilon_2. \]
By using Lemma 4.2 again we find a metric $d_3$ on $Z_3 = Z_2 \sqcup X_3$ such that $(Z_3, d_3)$ is complete and separable and

$$d_{p}^{M_{f}}(Z_3, d_3)(\nu_2, \nu_3) < \varepsilon_2.$$  

This procedure can be continued ad infinitum and in this way we get a metric $d_\infty$ on $Z_\infty$ as a “limit”. The metric space $(Z_\infty, d_\infty)$ is separable but not necessarily complete. For this reason we define $(Z_\infty, d_\infty)$ as the completion of $(Z_\infty, d_\infty)$.

Lemma 4.4 (Embedding of $d_{2\text{GP}}$-convergent sequences)

Let $(\mathcal{X}_n)_n$ be a sequence of m2m spaces with $\mathcal{X}_n = (X_n, r_n, \nu_n)$ which converges to an m2m space $\mathcal{X} = (X, r, \nu)$. Then, there is a Polish metric space $(Z, r_Z)$ and isometric embeddings $\iota, \iota_1, \iota_2, \ldots$ of $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \ldots$, respectively, into $Z$ such that

$$d_{p}^{M_{f}}(Z, r_Z)(\iota_\ast \nu_n, \iota_\ast \nu) \to 0.$$  

Proof: The proof is similar to the proof of Lemma 4.3, but this time we use $Z_n := \bigsqcup_{k=1}^{n} X_k \sqcup X$ and define the metrics $d_n$ inductively by always “routing” through the space $X$.

With the preceding embedding lemma it is easy to show that $\tau_{2\text{GP}}$-convergence implies $\tau_{2\text{Gw}}$-convergence.

Lemma 4.5 ($\tau_{2\text{GP}}$ is finer than $\tau_{2\text{Gw}}$)

Every test function $\Phi \in \mathcal{T}$ is continuous with respect to the two-level Gromov-Prokhorov topology. Therefore, two-level Gromov-Prokhorov convergence implies two-level Gromov-weak convergence and $\tau_{2\text{GP}}$ is finer than $\tau_{2\text{Gw}}$.

Proof: Let $\mathcal{X} = (X, r, \nu), \mathcal{X}_1 = (X_1, r_1, \nu_1), \ldots$ be m2m spaces with $d_{2\text{GP}}(\mathcal{X}_n, \mathcal{X}) \to 0$. By Lemma 4.4 we may assume without loss of generality that all metric spaces coincide, i.e. $(Z, r_Z) = (X, r) = (X_1, r_1) = \ldots$, and that we have $d_{p}^{M_{f}}(Z, r_Z)(\nu_n, \nu) \to 0$. For every $\Phi \in \mathcal{T}$ the function $\lambda \mapsto \Phi((Z, r_Z, \lambda))$ from $M_{f}(M_{f}(\mathcal{X}))$ to $\mathbb{R}$ is continuous with respect to weak convergence (cf. Remark 3.11) and thus we get $\Phi(\mathcal{X}_n) \to \Phi(\mathcal{X})$.

Finally, we are able to show that the two-level Gromov-Prokhorov distance satisfies the axioms of a metric.

Proposition 4.6

$d_{2\text{GP}}$ is a metric and $(\mathcal{M}^{(2)}, d_{2\text{GP}})$ is a Polish metric space.

Proof: First we prove that $d_{2\text{GP}}$ is indeed a metric on $\mathcal{M}^{(2)}$. Obviously $d_{2\text{GP}}$ is symmetric and non-negative. To prove the triangle inequality, let $\mathcal{X}_i = (X_i, r_i, \nu_i) \in \mathcal{M}^{(2)}$ for $i \in \{1, 2, 3\}$. Let $\varepsilon_1, \varepsilon_2 > 0$ such that

$$d_{2\text{GP}}(\mathcal{X}_1, \mathcal{X}_2) < \varepsilon_1 \quad \text{and} \quad d_{2\text{GP}}(\mathcal{X}_2, \mathcal{X}_3) < \varepsilon_2.$$
Then we can find a metric $d_3$ on $Z_3 := X_1 \sqcup X_2 \sqcup X_3$ that extends $r_1$, $r_2$ and $r_3$ and satisfies
\[ d_3^{M_f(Z_3,d_3)}(\nu_1, \nu_2) < \varepsilon_1 \quad \text{and} \quad d_3^{M_f(Z_3,d_3)}(\nu_2, \nu_3) < \varepsilon_2. \]

Such a metric has already been constructed in the proof of Lemma 4.3. By the triangle inequality of the Prokhorov metric we get
\[ d_{2\text{GP}}(X_1, X_3) \leq d_3^{M_f(Z_3,d_3)}(\nu_1, \nu_3) < \varepsilon_1 + \varepsilon_2. \]

The desired triangle inequality for $d_{2\text{GP}}$ follows simply by taking the infimum over all possible $\varepsilon_1$ and $\varepsilon_2$. Now assume that $d_{2\text{GP}}(\mathcal{X}, \mathcal{Y}) = 0$ for two m2m spaces $\mathcal{X} = (X, r, \nu)$, $\mathcal{Y} = (Y, d, \lambda)$. To prove that both spaces are equivalent, it suffices to show that $\Phi(\mathcal{X}) = \Phi(\mathcal{Y})$ for all test functions $\Phi \in \mathcal{T}$ (cf. Theorem 3.8). But this follows immediately from Lemma 4.5 together with Lemma 4.4 (by choosing $\mathcal{X}_n = \mathcal{Y}$ for every $n$).

This shows that $d_{2\text{GP}}$ is indeed a metric. To prove that $d_{2\text{GP}}$ is a complete metric, let $\mathcal{X}_n = (X_n, r_n, \nu_n)$ be a Cauchy sequence with respect to $d_{2\text{GP}}$. By Lemma 4.3 we can embed the metric spaces $(X_n, r_n)_{n}$ into a common Polish metric space $(Z, r)$ using isometries $(\iota_n)_{n}$. The two-level push-forward measures $(\iota_n)_n$ form a Cauchy sequence in $M_f(M_f(Z))$ and thus converge weakly to some $\nu \in M_f(M_f(Z))$. It follows that $(\mathcal{X}_n)_n$ converges to the m2m space $(Z, r, \nu)$ with respect to the two-level Gromov-Prokhorov metric.

To show that $(M^{(2)}, d_{2\text{GP}})$ is separable, we define $S$ as the set of all m2m spaces $(X, r, \nu) \in M^{(2)}$ such that $|X| < \infty$, the metric $r$ takes only rational values and
\[ \nu = \sum_{i=1}^{M} a_i \delta \left( \sum_{j=1}^{N_i} b_{ij} \delta_{x_{ij}} \right) \quad \text{with} \quad x_{ij} \in X, a_i, b_{ij} \in \mathbb{Q}_+, M, N_i \in \mathbb{N}, \quad \text{(E7)} \]

That is, $(X, r, \nu)$ is a finite m2m space with only rational distances and $\nu$ is a finite atomic measure on finite atomic measures with only rational values. The set $S$ is obviously countable. To prove density, let $(X, r, \lambda)$ be an arbitrary m2m space and let $\varepsilon > 0$. Because the set of measures of the form (E7) is dense in $M_f(M_f(X))$, there is a $\nu \in M_f(M_f(X))$ of this form with $d_{2\text{GP}}(\lambda, \nu) < \frac{\varepsilon}{2}$. Note that $S := \text{supp} \lambda$ is finite, thus $(S, r, \nu)$ is a finite m2m space that is $\frac{\varepsilon}{2}$ close to $(X, r, \lambda)$. The last step is to approximate $r$ by a rational version $r'$ such that $|r(x, y) - r'(x, y)| < \frac{\varepsilon}{2}$ for all $x, y \in S$. Then $(S, r', \nu)$ is in $S$ and we have $d_{2\text{GP}}((X, r, \lambda), (S, r', \nu)) < \varepsilon$. \(\square\)

5 Distance distribution and modulus of mass distribution

In this section we define the distance distribution and the modulus of mass distribution, which have been introduced in [GPW09]. They are indicators for the complexity of a metric measure space $(X, r, \mu)$ and will be used in the characterization of compactness in section 7.
Definition 5.1 (Distance distribution and modulus of mass distribution)

Let $\mu$ be a finite Borel measure on a Polish metric space $(X, r)$.

1. The distance distribution $w(\mu) \in \mathcal{M}_f(\mathbb{R}_+)$ of $\mu$ is defined by $w(\mu) := r_\ast \mu \otimes 2$.

2. For $\delta \geq 0$ the modulus of mass distribution $V_\delta(\mu)$ of $\mu$ is the number defined by

   $$V_\delta(\mu) := \inf \{ \varepsilon > 0 | \mu(\{ x \in X | \mu(B(x, \varepsilon)) \leq \delta \}) \leq \varepsilon \}.$$

The distance distribution reflects the effective diameter of the support of $\mu$, whereas the modulus of mass distribution measures the fineness of the measure $\mu$. Heuristically, $V_\delta(\mu)$ is the amount of “thin points” of $X$, where we think of $x \in X$ as a thin point if $\mu(B(x, \varepsilon)) \leq \delta$ for given $\varepsilon, \delta > 0$.

Note that the values of the distance distribution and the modulus of mass distribution coincide for measures from equivalent mm spaces (cf. [GPW09, Remark 2.10]).

The distance distribution and the modulus of mass distribution are constructed for one-level measures $\mu \in \mathcal{M}_f(X)$. At first sight they seem to be inappropriate to measure the complexity of a two-level measure $\nu \in \mathcal{M}_f(\mathcal{M}_f(X))$. However, we can overcome this problem by “projecting” the two levels of $\nu$ to one level, i.e. by looking at the first moment measure $M_\nu(\cdot) = \int \mu(\cdot) d\nu(\mu)$. As mentioned earlier, the moment measure of a two-level measure is not necessarily finite. This is why we will approximate $\nu$ by measures from $\mathcal{M}_f(\mathcal{M}_{\leq K}(X))$ with $K \nearrow \infty$. This approximation will be introduced in the next section.

In the rest of this section we summarize some useful properties of the modulus of mass distribution.

Lemma 5.2 (Properties of the modulus of mass distribution)

Let $\mu$ be a finite Borel measure on a Polish metric space $(X, r)$.

1. The function $\delta \mapsto V_\delta(\mu)$ is non-decreasing and bounded by the total mass $m(\mu)$. Moreover, we have $\lim_{\delta \searrow 0} V_\delta(\mu) = 0$.

2. For $\varepsilon, \delta > 0$ we have $V_\delta(\mu) < \varepsilon$ if and only if $\mu(\{ x \in X | \mu(B(x, \varepsilon)) \leq \delta \}) < \varepsilon$.

3. Let $\varepsilon$ and $\delta$ be positive real numbers with $V_\delta(\mu) < \varepsilon$. Then there is a finite set $A \subset X$ with $|A| \leq \max(1, \frac{m(\mu)}{\delta})$ such that $\mu(\cup B(A, \varepsilon)) < \varepsilon$.

Proof: 1. The claim was proved for probability measures in [GPW09, Lemma 6.5]. The same proof holds true for finite measures.

2. For the “only if direction” see Lemma 6.4 in [GPW09]. To prove the other direction, observe that the function

   $$\varepsilon \mapsto \mu(B(x, \varepsilon))$$

   is left-continuous for every $x \in X$ and that we have the equality

   $$\mu(\{ x \in X | \mu(B(x, \varepsilon)) \leq \delta \}) = \int 1_{[0,\varepsilon]}(\mu(B(x, \varepsilon))) d\mu(x).$$
It follows from the dominated convergence theorem that the function
$$
\varepsilon \mapsto \mu(\left\{ x \in X \mid \mu(B(x, \varepsilon)) \leq \delta \right\})
$$
is also left-continuous for a fixed $\delta$. If $V_\delta(\mu) \geq \varepsilon$, then we have for every $\varepsilon' < \varepsilon$ that
$$
\mu(\left\{ x \in X \mid \mu(B(x, \varepsilon')) \leq \delta \right\}) \geq \varepsilon'
$$
and therefore
$$
\mu(\left\{ x \in X \mid \mu(B(x, \varepsilon')) \leq \delta \right\}) = \lim_{\varepsilon' \uparrow \varepsilon} \mu(\left\{ x \in X \mid \mu(B(x, \varepsilon')) \leq \delta \right\}) \geq \varepsilon.
$$

(3) The assertion was proved in [GPW09, Lemma 6.9], but only for probability measures. This is why we give a full proof for finite measures: In case $m(\mu) \leq \varepsilon$ there must be an $x \in X$ with $\mu(B(x, \varepsilon)) < \varepsilon$ and we are done. Otherwise we define $D := \left\{ x \in X \mid \mu(B(x, \varepsilon)) > \delta \right\}$. Because $V_\delta(\mu)$ is less than $\varepsilon$, we have $\mu(CD) < \varepsilon < m(\mu)$ and $D$ is not empty. By [BBI01, page 278] there exists an $\varepsilon$-separated discrete subset $A$ of $D$ that is maximal. That is, we have $r(x_1, x_2) \geq \varepsilon$ for $x_1, x_2 \in A$ with $x_1 \neq x_2$ and adding a further point of $D$ would destroy this property. It follows from the maximality that $D \subset B(A, \varepsilon)$ and therefore $\mu(\bigcup B(A, \varepsilon)) \leq \mu(CD) < \varepsilon$. Moreover, we see that
$$
m(\mu) \geq \mu(B(A, \varepsilon)) = \sum_{x \in A} \mu(B(x, \varepsilon)) \geq |A|\delta,
$$
which yields the claim.

Lemma 5.3 ($V_\delta$ is upper semi-continuous)

Let $\mathcal{M}$ denote the set of metric measure spaces equipped with the Gromov-weak topology and let $\delta > 0$ be fixed. The function
$$
\mathcal{M} \to \mathbb{R}_+
$$
$$(X, r, \mu) \mapsto V_\delta(\mu)
$$
is upper semi-continuous. That is, if a net (or a sequence) $((X_\alpha, r_\alpha, \mu_\alpha))_\alpha$ of metric measure spaces converges Gromov weakly to $\mathcal{M}$, then
$$
\limsup_\alpha V_\delta(\mu_\alpha) \leq V_\delta(\mu).
$$

Proof: The proof for sequences of metric probability measure spaces can be found in [GPW09, Proposition 6.6]. It remains valid even if we replace metric probability measure spaces by metric measure spaces with finite measures.

The following technical lemma is a preparation for the example in section 10.

Lemma 5.4

Let $\mathcal{M}$ denote the set of metric measure spaces equipped with the Gromov-weak topology and let $\varepsilon, \delta > 0$ be fixed. The function
$$
\mathcal{M} \to \mathbb{R}_+
$$
$$(X, r, \mu) \mapsto \mu(\left\{ x \in X \mid \mu(B(x, \varepsilon)) < \delta \right\})
$$

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is lower semi-continuous. That is, if a net (or a sequence) \((X_\alpha, r_\alpha, \mu_\alpha)\) of metric measure spaces converges Gromov weakly to \((X, r, \mu)\), then
\[
\liminf_\alpha \mu_\alpha(\{ x \in X \mid \mu_\alpha(B(x, \varepsilon)) < \delta \}) \geq \mu(\{ x \in X \mid \mu(B(x, \varepsilon)) < \delta \}).
\]

Proof: The proof of [GPW09, Proposition 6.6 (iv)] actually shows that
\[
M \rightarrow \mathbb{R}_+
\]
\[(X, r, \mu) \mapsto \mu(\{ x \in X \mid \mu(B(x, \varepsilon)) \leq \delta \})
\]
is upper semi-continuous by using the Portmanteau Theorem for closed sets. The proof of our claim is similar, but uses the Portmanteau Theorem for open sets instead. □

6 Approximation of m2m spaces

As mentioned before, we need to approximate two-level measures \( \nu \in \mathcal{M}_f(\mathcal{M}_f(X)) \) by measures \( \nu_K \) from \( \mathcal{M}_f(\mathcal{M}_{\leq K}(X)) \) to ensure that the first moment measure is finite. The simplest choice for \( \nu_K \) would be the restriction of \( \nu \) to \( \mathcal{M}_{\leq K}(X) \), i.e. \( \nu_K(\cdot) = \nu(\cdot \cap \mathcal{M}_{\leq K}(X)) \). However, this rough cut-off may lead to discontinuities if \( \nu(\mathcal{M}_K(X)) \) is greater than 0. Therefore it is better to “cut off” \( \nu \) with a continuous density \( f_K \) as defined below.

Let \( \{ g_K \in C_b(\mathbb{R}_+) \mid K > 0 \} \) be a set of functions having the following properties:

1. \( 0 \leq g_K \leq 1 \) for every \( K \),
2. \( g_K(x) = 0 \) for \( x \geq K \),
3. \( g_K \rightarrow 1 \) for \( K \rightarrow \infty \) uniformly on every bounded interval.

For example we may define
\[
g_K(x) := \begin{cases} 
1 & 0 \leq x \leq \frac{K}{2} \\
2 - \frac{2x}{K} & \frac{K}{2} < x \leq K \\
0 & K < x.
\end{cases}
\]

Moreover, let \( f_K := g_K \circ m \). For a two-level measure \( \nu \in \mathcal{M}_f(\mathcal{M}_f(X)) \) we denote by \( f_K \cdot \nu \) the measure that has density \( f_K \) with respect to \( \nu \). That is, it is the unique measure which satisfies
\[
f_K \cdot \nu(A) = \int_A f_K(\mu) d\nu(\mu) = \int_A g_K(m(\mu)) d\nu(\mu)
\]
for every measurable \( A \subset \mathcal{M}_f(X) \). The measures \( \{ f_K \cdot \nu \mid K > 0 \} \) will serve as approximations of \( \nu \). Clearly we have \( f_K \cdot \nu \xrightarrow{w} \nu \) for \( K \rightarrow \infty \). Moreover, if \( \nu_n \xrightarrow{w} \nu \), we have \( f_K \cdot \nu_n \xrightarrow{w} f_K \cdot \nu \) for every \( K > 0 \). Observe that \( f_K \cdot \nu \in \mathcal{M}_f(\mathcal{M}_{\leq K}(X)) \), thus the first moment measure is finite.
Lemma 6.1 (Properties of the approximation $f_K \cdot \nu$)

The following statements hold true in the two-level Gromov-Prokhorov and in the two-level Gromov-weak topology:

1. The function $(X, r, \nu) \mapsto (X, r, f_K \cdot \nu)$ is continuous for every $K > 0$.

2. For every $K > 0$ the function $(X, r, \nu) \mapsto (X, r, M f_K \cdot \nu)$ is continuous from $\mathcal{M}^{(2)}$ to $\mathcal{M}$, where $\mathcal{M}$ denotes the set of metric measure spaces equipped with the Gromov-weak topology.

3. The function $(X, r, \nu) \mapsto w(M f_K \cdot \nu)$ is continuous for every $K > 0$.

4. $(X, r, f_K \cdot \nu) \to (X, r, \nu)$ for $K \to \infty$ and for every m2m space $(X, r, \nu) \in \mathcal{M}^{(2)}$.

Proof: (1) with $\tau_{2GP}$: Fix $K > 0$ and let the net $((X_\alpha, r_\alpha, \nu_\alpha))_\alpha$ converge to $(X, r, \nu)$ in the two-level Gromov-Prokhorov topology. We use Lemma 3.13 to show that $(X_\alpha, r_\alpha, f_K \cdot \nu_\alpha)$ converges to $(X, r, f_K \cdot \nu)$. Since $m_\ast \nu_\alpha \xrightarrow{w} m_\ast \nu$ and $g_K$ is continuous and bounded, we have

$$m(f_K \cdot \nu_\alpha) = \int g_K(m) \, dm_\ast \nu_\alpha(m) \to \int g_K(m) \, dm_\ast \nu(m) = m(f_K \cdot \nu).$$

Moreover, for every $h \in C_b(\mathbb{R}_+)$

$$\int h \, dm_\ast (f_K \cdot \nu_\alpha) = \int h(m(\mu)) g_K(m(\mu)) \, d\nu_\alpha(\mu) = \int h(m) g_K(m) \, dm_\ast \nu_\alpha(m)$$

and this converges to

$$\int h(m) g_K(m) \, dm_\ast \nu(m) = \int h \, dm_\ast (f_K \cdot \nu).$$

It follows that $m_\ast (f_K \cdot \nu_\alpha)$ converges weakly to $m_\ast (f_K \cdot \nu)$. Now let $\Phi$ be as in (TF4). Then

$$\tilde{\Phi}((X_\alpha, r_\alpha, f_K \cdot \nu_\alpha)) = \int \psi(m(\mu)) \int \varphi \circ R \, d\mu^{\otimes n} \, d(f_K \cdot \nu_\alpha)^{\otimes m}(\mu)$$

$$= \int \psi(m(\mu)) \prod_{i=1}^m g_K(m(\mu_i)) \int \varphi \circ R \, d\mu^{\otimes n} \, d\nu_\alpha^{\otimes m}(\mu)$$

and this converges to

$$\int \psi(m(\mu)) \prod_{i=1}^m g_K(m(\mu_i)) \int \varphi \circ R \, d\mu^{\otimes n} \, d\nu^{\otimes m}(\mu) = \tilde{\Phi}((X, r, f_K \cdot \nu)).$$

Therefore, all three conditions of Lemma 3.13 are satisfied and $(X_\alpha, r_\alpha, f_K \cdot \nu_\alpha)$ converges to $(X, r, f_K \cdot \nu)$.

(1) with $\tau_{2GP}$: If we have a converging sequence of m2m spaces $(X_n, r_n, \nu_n) \to (X, r, \nu)$, we can embed all the metric spaces isometrically into a common Polish metric space
is a test function as in (TF4). Since thus weakly, we also have convergence of all test functions of this form (by Lemma 3.13) and weak convergence implies convergence of the corresponding m2m spaces in the two-level Gromov-Prokhorov metric.

(2): Recall from Lemma 4.5 that the two-level Gromov-weak topology \( \tau_{2Gw} \) is coarser than the two-level Gromov-Prokhorov topology \( \tau_{2GP} \). Thus, it suffices to show continuity only with respect to \( \tau_{2Gw} \). Let \( K > 0 \) and let the net \( ((X_\alpha, r_\alpha, \nu_\alpha))_\alpha \) converge to \( (X, r, \nu) \) in the two-level Gromov-weak topology. By assertion (1) \( (X_\alpha, r_\alpha, f_K \cdot \nu_\alpha) \) converges two-level Gromov-weakly to \( (X, r, f_K \cdot \nu) \). We want to show that the mm spaces \( \left((X_\alpha, r_\alpha, M_{f_K \cdot \nu_\alpha})\right)_\alpha \) converge Gromov-weakly to \( (X, r, M_{f_K \cdot \nu}) \). By the definition of the Gromov-weak topology (cf. [GPW09]) we need to show that \( \hat{\Phi}((X_\alpha, r_\alpha, M_{f_K \cdot \nu_\alpha})) \) converges to \( \hat{\Phi}((X, r, M_{f_K \cdot \nu})) \) for every test function \( \Phi: M \rightarrow \mathbb{R}_+ \) of the form

\[
\hat{\Phi}((X, r, \mu)) = \int \varphi \circ R \, d\mu^{\otimes m}
\]

with \( m \in \mathbb{N} \) and \( \varphi \in C_b(\mathbb{R}^{m \times m}) \). Fix such a \( \hat{\Phi} \) and let \( n := (1, \ldots, 1) \in \mathbb{N}^m \). Then

\[
\hat{\Phi}((X_\alpha, r_\alpha, M_{f_K \cdot \nu_\alpha})) = \int \varphi \circ R \, d(M_{f_K \cdot \nu_\alpha})^{\otimes m} = \int \int \varphi \circ R \, d\mu^{\otimes m} \, d(f_K \cdot \nu_\alpha)^{\otimes m}(\mu).
\]

If we choose a function \( \psi \in C_b(\mathbb{R}^m) \) such that \( \psi(x_1, \ldots, x_m) = \prod_{i=1}^m x_i \) on \([0, K]^m\), the right hand side of the last equation can be written as

\[
\hat{\Phi}((X_\alpha, r_\alpha, f_K \cdot \nu_\alpha)) := \int \psi(\mu) \int \varphi \circ R \, d\mu^{\otimes n} \, d(f_K \cdot \nu_\alpha)^{\otimes m}(\mu).
\]

\( \hat{\Phi} \) is a test function as in (TF4). Since \( (X_\alpha, r_\alpha, f_K \cdot \nu_\alpha) \) converges two-level Gromov-weakly, we also have convergence of all test functions of this form (by Lemma 3.13) and thus

\[
\hat{\Phi}((X_\alpha, r_\alpha, M_{f_K \cdot \nu_\alpha})) = \hat{\Phi}((X_\alpha, r_\alpha, f_K \cdot \nu_\alpha)) \rightarrow \hat{\Phi}((X, r, f_K \cdot \nu)) = \hat{\Phi}((X, r, M_{f_K \cdot \nu})).
\]

(3): The claim follows immediately with assertion (2) and the fact that the function

\[
M \rightarrow M_1(\mathbb{R}_+)
\]

\( (X, r, \mu) \rightarrow \varphi(\mu) \)

is continuous (cf. [GPW09, Proposition 6.6]).

(4): \( f_K \cdot \nu \) converges weakly to \( \nu \) for \( K \rightarrow \infty \). Therefore, \( (X, r, f_K \cdot \nu) \) converges to \( (X, r, \nu) \) in the two-level Gromov-Prokhorov metric. Since two-level Gromov-Prokhorov convergence implies two-level Gromov weak convergence (cf. Lemma 4.5), assertion (4) is true for both topologies. \( \square \)

By combining the preceding lemma with Lemma 5.3 we get the following corollary.
Corollary 6.2
For all $\delta, K > 0$ the function

$$M^{(2)} \rightarrow \mathbb{R}_+$$

$$ (X, r, \nu) \mapsto V_\delta(M_{fK} \cdot \nu) $$

is upper semi-continuous in the two-level Gromov-weak topology (and in the two-level Gromov-Prokhorov topology). That is, if a net (or a sequence) $((X_\alpha, r_\alpha, \nu_\alpha))_\alpha$ of m2m spaces converges Gromov weakly to $(X, r, \nu)$, then

$$\limsup_\alpha V_\delta(M_{fK} \cdot \nu_\alpha) \leq V_\delta(M_{fK} \cdot \nu).$$

In particular this implies $\limsup_\alpha V_\delta(M_{fK} \cdot \nu_\alpha) \rightarrow 0$ for $\delta \searrow 0$.

7 Compactness

In this section we examine compactness in $(M^{(2)}, d_{2GP})$. The main result of this section is Theorem 7.2, in which we give several equivalent criteria for relative compactness. In Theorem 7.3 we characterize compact nets. It might seem odd to look at nets in a metric space. However, we need to show in the proof of Theorem 8.1 that every $\tau_{2Gw}$-convergent net is also $\tau_{2GP}$-convergent. Thus it is useful for us to have a better understanding about $\tau_{2GP}$-compact nets.

But first we introduce a sequence of compact subsets of $M^{(2)}$ whose union is dense. For every $N \in \mathbb{N}$ we define $A_N \subset M^{(2)}$ as the set of all m2m spaces $(X, r, \nu)$ such that

- $\text{supp} M_\nu$ consists of at most $N$ points,
- the diameter of $\text{supp} M_\nu$ is at most $N$,
- $\nu \in M_{\leq N}(M_{\leq N}(X))$.

Observe that the union $\bigcup_{N \in \mathbb{N}} A_N$ is dense in $(M^{(2)}, d_{2GP})$ since it contains the dense set $S$ from the proof of Theorem 4.6.

Lemma 7.1

$A_N$ is compact in the two-level Gromov-Prokhorov topology for every $N \in \mathbb{N}$.

Proof: Let $((X_n, r_n, \nu_n))_n$ be a sequence in $A_N$. Without loss of generality we assume $X_n = \text{supp} M_{\nu_n}$. The finite metric spaces $((X_n, r_n))_n$ are determined by the number of points and the mutual distances between the points. All of these are bounded by $N$. By [BBI01, Theorem 7.4.15] $((X_n, r_n))_n$ is relatively compact in the Gromov-Hausdorff topology and there is a subsequence which converges to some compact metric space $(X, r)$. For the sake of convenience we denote this subsequence again by $((X_n, r_n))_n$. We have $|X| \leq N$ and $\text{diam} X \leq N$. By [GPW09, Lemma A.1] there is a compact metric space $(Z, r_Z)$ and isometric embeddings $\iota, \iota_1, \iota_2, \ldots$ of $X, X_1, X_2, \ldots$ into $Z$ such that

$$d_H^Z(\iota_n(X_n), \iota(X)) \rightarrow 0.$$
Here $d_H^*$ denotes the Hausdorff metric on $Z$. Because $Z$ is compact, both $\mathcal{M}_{\leq N}(Z)$ and $\mathcal{M}_{\leq N}(\mathcal{M}_{\leq N}(Z))$ are compact too. Therefore, $((\nu_n))_n$ has a subsequence which converges weakly to some measure $\nu \in \mathcal{M}_{\leq N}(\mathcal{M}_{\leq N}(Z))$. Then, the same subsequence of $((X_n,r_n,\nu_n))_n$ converges to $(Z,r_Z,\nu)$ and $(Z,r_Z,\nu) \cong (X,r,\nu) \in \mathcal{A}_N$. \hfill \Box

**Theorem 7.2 (Characterization of compact sets)**

Let $\Gamma \subset \mathcal{M}^{(2)}$ be a set of $m^2m$ spaces. The following are equivalent:

1. $\Gamma$ is relatively compact in the two-level Gromov-Prokhorov topology.
2. For every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d_{2GP}(X,\mathcal{A}_N) < \varepsilon$ for every $X \in \Gamma$.
3. $\{m_n \cdot \nu \mid (X,r,\nu) \in \Gamma\}$ is relatively compact in $\mathcal{M}_f(\mathbb{R}_+)$ and for every $K > 0$ we have
   - $\sup_{(X,r,\nu) \in \Gamma} V_\delta(M_{fK} \cdot \nu) \to 0$ for $\delta \searrow 0$,
   - $\{w(M_{fK} \cdot \nu) \mid (X,r,\nu) \in \Gamma\}$ is relatively compact in $\mathcal{M}_f(\mathbb{R}_+)$.
4. $\{m_n \cdot \nu \mid (X,r,\nu) \in \Gamma\}$ is relatively compact in $\mathcal{M}_f(\mathbb{R}_+)$ and for every $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that for every $X = (X,r,\nu) \in \Gamma$ there exists a measurable subset $X_{\varepsilon} \subset X$ with
   - $\nu(\{\mu \in \mathcal{M}_f(X) \mid \mu(\mathcal{C}_{X_{\varepsilon}}) \leq \varepsilon\}) < \varepsilon$,
   - $X_{\varepsilon}$ can be covered by at most $N_\varepsilon$ balls of radius $\varepsilon$,
   - the diameter of $X_{\varepsilon}$ is at most $N_\varepsilon$.
5. $\{m_n \cdot \nu \mid (X,r,\nu) \in \Gamma\}$ is relatively compact in $\mathcal{M}_f(\mathbb{R}_+)$ and for every $\varepsilon > 0$ and $X = (X,r,\nu)$ there is a compact subset $C_{X,\varepsilon} \subset X$ such that
   - $\nu(\{\mu \in \mathcal{M}_f(X) \mid \mu(\mathcal{C}_{X_{\varepsilon}}) \leq \varepsilon\}) < \varepsilon$,
   - $C_{\varepsilon} := \{C_{X,\varepsilon} \mid X \in \Gamma\}$ is relatively compact in the Gromov-Hausdorff topology.

Note that in assertion (3) is suffices to have the property only for a diverging sequence $(K_n)_n \not\nearrow \infty$.

**Proof (of Theorem 7.2):** The equivalence of assertions (1) to (4) follows easily from Theorem 7.3 and the fact that a set is relatively compact if and only if every net in it is compact (cf. Lemma 2.6). In the remainder of this proof we show that assertion (4) and (5) are equivalent.

Note that a set $C$ of compact metric spaces is relatively compact in the Gromov-Hausdorff-topology if and only if the diameter of the elements of $C$ is uniformly bounded and for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that every element of $C$ can be covered by at most $N$ balls of radius $\varepsilon$. This fact can be found in [BBI01, Section 7.4.2]. Therefore, assertion (5) readily implies assertion (4).

To prove the other direction, let $\varepsilon > 0$. For $n \in \mathbb{N}$ let $\varepsilon_n := \frac{\varepsilon}{2} \cdot \left(\frac{1}{2}\right)^n$ and let $N_{\varepsilon_n}$ and $X_{\varepsilon_n}$ be as in assertion (4). Without loss of generality we may assume that every
The set \( K_{X,\varepsilon_n} \) is closed. For every \( X = (X,r,\nu) \in \Gamma \) and every \( n \in \mathbb{N} \) there is a compact set \( K_{X,\varepsilon_n} \subset M_f(X) \) with \( \nu(K_{X,\varepsilon_n}) < \varepsilon_n \). Because \( K_{X,\varepsilon_n} \) is tight, we have
\[
K_{X,\varepsilon_n} \subset \{ \mu \in M_f(X) \mid \mu(C_{X,\varepsilon_n}) < \varepsilon_n \}
\]
for some compact set \( C_{X,\varepsilon_n} \subset X \) and thus
\[
\nu(C \{ \mu \in M_f(X) \mid \mu(C_{X,\varepsilon_n}) \leq \varepsilon \}) 
\leq \nu \left( C \bigcap \{ \mu \in M_f(X) \mid \mu(C_{X,\varepsilon_n}) \leq \varepsilon_n, \nu(C_{X,\varepsilon_n}) \leq \varepsilon_n \} \right)
\leq \sum_{n \in \mathbb{N}} \left( \nu(C \{ \mu \in M_f(X) \mid \mu(C_{X,\varepsilon_n}) \leq \varepsilon_n \}) 
+ \nu(C \{ \mu \in M_f(X) \mid \mu(C_{X,\varepsilon_n}) \leq \varepsilon_n \}) \right)
< 2 \sum_{n \in \mathbb{N}} \varepsilon_n = \varepsilon.
\]
The set \( C = \{ C_{X,\varepsilon} \mid X \in \Gamma \} \) is relatively compact in the Gromov-Hausdorff-topology because the diameter of each \( C_{X,\varepsilon} \) is bounded by \( N_{\varepsilon_1} \) and for every \( \delta > 0 \) there is a natural number \( N \) such that each \( C_{X,\varepsilon} \) can be covered by no more than \( N \) balls of radius \( \delta \) (cf. the remark at the beginning of this proof). Thus, assertion (5) is fulfilled and the proof is complete. \( \square \)

**Theorem 7.3 (Characterization of compact nets)**

Let \((\mathcal{A}, \preceq)\) be a directed set and let \((X_\alpha)_{\alpha \in \mathcal{A}}\) be a net in \( \mathbb{M}^{(2)} \) with \( X_\alpha = (X_\alpha, r_\alpha, \nu_\alpha) \). The following are equivalent:

1. \((X_\alpha)_{\alpha} \) is a compact net with respect to the two-level Gromov-Prokhorov topology.
2. For every \( \varepsilon > 0 \) there is an \( N \in \mathbb{N} \) such that \( d_{2GP}(X_\alpha, K_N) < \varepsilon \) eventually.
3. \((m_\varepsilon \nu_\alpha)_{\alpha} \) is a compact net in \( M_f(\mathbb{R}^+) \) and for every \( K > 0 \) we have:
   - For every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( V_\delta(M_f K_{\varepsilon_0}) < \varepsilon \) eventually,
   - \((w(M_f K_{\varepsilon_0}))_{\alpha} \) is a compact net in \( M_f(\mathbb{R}^+) \).
4. \((m_\varepsilon \nu_\alpha)_{\alpha} \) is a compact net and for every \( \varepsilon > 0 \) there is an \( N_\varepsilon \in \mathbb{N} \) such that for every \( \alpha \) there is a set \( X_{\alpha,\varepsilon} \subset X_\alpha \) such that we eventually have
   - \( \nu_\alpha(C \{ \mu \in M_f(X_\alpha) \mid \mu(C X_{\alpha,\varepsilon}) \leq \varepsilon \}) < \varepsilon \),
   - \( X_{\alpha,\varepsilon} \) can be covered by at most \( N_\varepsilon \) balls of radius \( \varepsilon \),
   - the diameter of \( X_{\alpha,\varepsilon} \) is at most \( N_\varepsilon \).
Proof: \((1) \Rightarrow (2)\): We prove this assertion by contradiction and assume \((2)\) does not hold. That is, there is an \(\varepsilon > 0\) such that for every \(N \in \mathbb{N}\) the distance \(d_{2GP}(X_\alpha, A_N)\) is frequently greater than or equal to \(\varepsilon\). Therefore, there is a subnet \((X_{T(\beta)})_{\beta}\) of \((X_\alpha)_\alpha\) (with a directed set \(B\) and a function \(T\) from \(B\) to \(A\)) such that \(d_{2GP}(X_{T(\beta)}, A_N)\) is eventually at most \(\varepsilon\) for every positive integer \(N\).

Since \((X_{T(\beta)})_{\beta}\) is compact, it has a convergent subnet. Let \(X\) be the limit of this subnet. Observe that the set \(\bigcup_{N \in \mathbb{N}} A_N\) is dense in \(M(2)\) since it contains the dense set \(S\) from the proof of Theorem 4.6. Thus, there is a natural number \(N_0\) with \(d_{2GP}(X, A_{N_0}) < \varepsilon\). Consequently, the subnet converging to \(X\) is eventually \(\varepsilon\)-close to \(A_{N_0}\). But this contradicts the construction of the net \((X_{T(\beta)})_{\beta}\).

\((2) \Rightarrow (1)\): Observe that assertion \((2)\) also holds for any subnet of \((X_\alpha)_\alpha\). Thus, it is enough to show that any net with \((2)\) has a convergent subnet. Inductively we can construct subnets \((X_{T_{n}(\beta)})_{\beta \in B_n}\) for each \(n \in \mathbb{N}\) such that \((X_{T_{n}(\beta)})_{\beta \in B_n}\) is a subnet of \((X_{T_{n-1}(\beta)})_{\beta \in B_{n-1}}\) and eventually contained in a ball of radius \(1/n\). By a diagonal argument we can construct a subnet that is a Cauchy net. That is, for every \(\varepsilon > 0\) the subnet is eventually contained in a ball of radius \(\varepsilon\). Since \((M(2), d_{2GP})\) is complete, this subnet net is convergent.

\((1) \Rightarrow (3)\): Note that both functions \((X, r, \nu) \mapsto m_\nu\) and \((X, r, \nu) \mapsto w(M_{f_{K(\nu)}})\) are continuous by Lemma 6.1 and that the continuous image of a compact net is again compact. Thus, \((m_\nu)_{\alpha}\) and \((w(M_{f_{K(\nu)}}))_{\alpha}\) are both compact.

To prove the last property, assume that it does not hold, i.e. there are \(K, \varepsilon > 0\) such that \(V_\delta(M_{f_{K(\nu)}}) \geq \varepsilon\) frequently for every \(\delta > 0\). Inductively we can construct subnets \((X_{T_{n}(\beta)})_{\beta \in B_n}\) for each \(n \in \mathbb{N}\) such that \((X_{T_{n}(\beta)})_{\beta \in B_n}\) is a subnet of \((X_{T_{n-1}(\beta)})_{\beta \in B_{n-1}}\) and \(V_\delta(M_{f_{K(\nu)}}) \geq \varepsilon\) eventually. By a diagonal argument we can construct a subnet \((X_{T(\beta)})_{\beta \in B}\) such that \(V_\delta(M_{f_{K(\nu)}}) \geq \varepsilon\) eventually for every \(\delta > 0\). By Corollary 6.2 this subnet cannot have a convergent subnet in contradiction to \((1)\).

\((3) \Rightarrow (4)\): Let \(1 > \varepsilon > 0\). The net \((m_\nu)_{\alpha}\) is compact and thus tight. Therefore, there is a \(K\) such that eventually \(m_\nu - m_{f_{K(\nu)}} < \varepsilon/3\). Then, assertion \((4)\) is a consequence of the following two claims, which we will prove later.

Claim 1: There is a positive integer \(N_1\) and for every \(\alpha \in A\) there is a bounded set \(C_\alpha \subset X_\alpha\) with \(\text{diam} C_\alpha \leq N_1\) such that eventually

\[
f_K \cdot \nu_\alpha \left( \mathcal{C} \{ \mu \in M_f(X_\alpha) \mid \mu(C C_\alpha) \leq \frac{\varepsilon}{3} \} \right) \leq \frac{\varepsilon}{3}.
\]

Claim 2: There is a positive integer \(N_2\) and for every \(\alpha \in A\) there is a finite set \(A_\alpha \subset X_\alpha\) with \(|A_\alpha| \leq N_2\) such that eventually

\[
f_K \cdot \nu_\alpha \left( \mathcal{C} \{ \mu \in M_f(X_\alpha) \mid \mu(C B(A_\alpha, \varepsilon)) < \frac{\varepsilon}{3} \} \right) \leq \frac{\varepsilon}{5}.
\]

With these two claims we can define the subsets \(X_{\alpha,\varepsilon} = C_\alpha \cap B(A_\alpha, \varepsilon)\) and \(N_\varepsilon = \max(N_1, N_2)\). Then eventually the set \(X_{\alpha,\varepsilon}\) has diameter of at most \(N_\varepsilon\) and can be
covered by at most $N_\varepsilon$ balls of radius $\varepsilon$. Moreover, eventually we have
\[
\nu_a \left( \{ \mu \in \mathcal{M}_f(X_a) \mid \mu(\mathcal{C}_{X_a,\varepsilon}) \leq \varepsilon \} \right) \\
< \left( m(\nu_a) - m(f_k \cdot \nu_a) \right) + f_k \cdot \nu_a \left( \{ \mu \in \mathcal{M}_f(X_a) \mid \mu(\mathcal{C}_{X_a,\varepsilon}) \leq \frac{\varepsilon}{3} \} \right) \\
\leq \varepsilon + f_k \cdot \nu_a \left( \{ \mu \in \mathcal{M}_f(X_a) \mid \mu(\mathcal{C} \cup \mathcal{C}_{X_a,\varepsilon}) \leq \frac{\varepsilon}{3} \} \right) \\
+ f_k \cdot \nu_a \left( \{ \mu \in \mathcal{M}_f(X_a) \mid \mu(\mathcal{C}B(A_\alpha,\varepsilon)) \leq \frac{\varepsilon}{3} \} \right) \\
< \varepsilon,
\]
and the assertion is proved.

**Proof of Claim 1:** The net $(w(M_{f_k \cdot \nu_a}))_\alpha$ is compact and thus tight. Therefore, there is an $a > 0$ such that eventually
\[
w(M_{f_k \cdot \nu_a})([a, \infty)) < \frac{1}{2} \left( \frac{\varepsilon}{3} \right)^4.
\]
Let $N_1$ be a positive integer with $2a \leq N_1$. We will construct bounded sets $C_\alpha \subset X_a$ with $\text{diam} C_\alpha < 2a \leq N_1$ such that eventually (E8) holds. If
\[
f_k \cdot \nu_a \left( \{ \mu \in \mathcal{M}_f(X_a) \mid m(\mu) \leq \frac{\varepsilon}{3} \} \right) \leq \varepsilon,
\]
then this is satisfied for $C_\alpha := B(x, a)$ for any $x \in X_a$. In the case where
\[
f_k \cdot \nu_a \left( \{ \mu \in \mathcal{M}_f(X_a) \mid m(\mu) \leq \frac{\varepsilon}{3} \} \right) > \frac{\varepsilon}{3}
\]
we define $C_\alpha := \{ x \in X_a \mid M_{f_k \cdot \nu_a}(\mathcal{C}B(x, a)) < \frac{1}{2} \left( \frac{\varepsilon}{3} \right)^2 \}$. Then, $\text{diam} C_\alpha < 2a$ follows by the following contradiction: If there are $x, y \in C_\alpha$ with $r_\alpha(x, y) \geq 2a$ we have
\[
\left( \frac{\varepsilon}{2} \right)^2 > M_{f_k \cdot \nu_a}(\mathcal{C}B(x, a)) + M_{f_k \cdot \nu_a}(\mathcal{C}B(y, a)) \\
\geq M_{f_k \cdot \nu_a}(\mathcal{C}B(x, a) \cap B(y, a)) \\
= M_{f_k \cdot \nu_a}(X_a) = \int \mu(X_a) d(f_k \cdot \nu_a)(\mu) \\
\geq \frac{\varepsilon}{3} f_k \cdot \nu_a \left( \{ \mu \in \mathcal{M}_f(X_a) \mid m(\mu) > \frac{\varepsilon}{3} \} \right).
\]
This contradicts (E10), so the diameter of $C_\alpha$ is less than $2a$.

Furthermore, we eventually have
\[
\frac{1}{2} \left( \frac{\varepsilon}{3} \right)^4 > w(M_{f_k \cdot \nu_a})([a, \infty)) \\
= (M_{f_k \cdot \nu_a})^\otimes \left( \{(x,y) \in X_a^2 \mid y \notin B(x, a)\} \right) \\
\geq (M_{f_k \cdot \nu_a})^\otimes \left( \{(x,y) \in X_a^2 \mid x \notin C_\alpha, y \notin B(x, a)\} \right) \\
\geq \frac{1}{2} \left( \frac{\varepsilon}{3} \right)^2 M_{f_k \cdot \nu_a}(\mathcal{C}C_\alpha).
\]
In the last inequality we used the fact that
\[
M_{f_K \cdot \nu_0}(\mathcal{C} B(x, \varepsilon)) \geq \frac{1}{2} \left( \frac{\varepsilon}{2} \right)^2
\]
for \( x \notin C_\alpha \) by the very definition of \( C_\alpha \). We conclude that eventually we have
\[
\left( \frac{\varepsilon}{2} \right)^2 > M_{f_K \cdot \nu_0}(\mathcal{C} C_\alpha) = \int \mu(\mathcal{C} C_\alpha) \, \text{d}(f_K \cdot \nu_0)(\mu)
\geq \frac{\varepsilon}{3} \left( f_K \cdot \nu_0 \right) \left( \{ \mu \in M_f(X_\alpha) \mid \mu(\mathcal{C} C_\alpha) > \frac{\varepsilon}{3} \} \right)
\]
and this yields (E8).

**Proof of Claim 2:** Because \((m \cdot \nu_0)_\alpha\) is a compact net, \( m(\nu_0) \) is eventually bounded by some positive real number \( m \). By assumption there is a \( \delta > 0 \) such that eventually \( V_\delta(M_{f_K \cdot \nu_0}) < \varepsilon^2 \). Set \( N_2 := \max(1, \lfloor \frac{KM}{\delta} \rfloor) \). By Lemma 5.2 we can find for every \( \alpha \in A \) a finite set \( A_\alpha \subset X_\alpha \) with \(|A_\alpha| \leq N_2\) such that eventually
\[
\frac{\varepsilon^2}{9} > M_{f_K \cdot \nu_0}(\mathcal{C} B(A_\alpha, \frac{\varepsilon^2}{9}))
\geq M_{f_K \cdot \nu_0}(\mathcal{C} B(A_\alpha, \varepsilon))
= \int \mu(\mathcal{C} B(A_\alpha, \varepsilon)) \, \text{d}(f_K \cdot \nu_0)(\mu)
\geq \frac{\varepsilon}{3} \left( f_K \cdot \nu_0 \right) \left( \{ \mu \in M_f(X_\alpha) \mid \mu(\mathcal{C} B(A_\alpha, \varepsilon)) \geq \frac{\varepsilon}{3} \} \right).
\]
This leads to the desired inequality (E9).

(4) \( \Rightarrow \) (2): Let \( \varepsilon > 0 \) be arbitrary and let \( N_\varepsilon \) and \( X_{\alpha, \varepsilon} \) be as in assertion (4). \( X_{\alpha, \varepsilon} \) can eventually be covered by \( N_\varepsilon = N_\varepsilon \) balls \( B(x_1^{(\alpha)}, \varepsilon), \ldots, B(x_{N_\varepsilon}^{(\alpha)}, \varepsilon) \). Define a function \( F_\alpha : X_\alpha \rightarrow X_\alpha \) by
\[
F_\alpha(x) = \begin{cases} x_1^{(\alpha)}, & \text{if } x \in B(x_1^{(\alpha)}, \varepsilon) \text{ or } x \notin \bigcup_{j=1}^{N_\varepsilon} B(x_j^{(\alpha)}, \varepsilon) \\ x_j^{(\alpha)}, & \text{if } x \in B(x_j^{(\alpha)}, \varepsilon) \setminus \bigcup_{j=1}^{N_\varepsilon} B(x_j^{(\alpha)}, \varepsilon) \text{ for } i \in \{ 2, \ldots, N_\varepsilon \}. \end{cases}
\]
By assertion (3) of Lemma 2.4 we have \( d_P(\mu, F_\alpha \ast \mu) \leq \varepsilon \) for every \( \mu \in M_f(X_\alpha) \) with \( \mu(\mathcal{C} X_{\alpha, \varepsilon}) \leq \varepsilon \). Thus, eventually we have \( d_P(\nu_0, F_\alpha \ast \nu_0) \leq \varepsilon \).

Because \((m \cdot \nu_0)_\alpha\) is compact, \( m(\nu_0) \) is eventually bounded from above by some positive number \( m \) and \((m \cdot \nu_0)_\alpha\) is tight. Therefore, there is a \( K > 0 \) such that eventually \( d_P(\nu_0, F_\alpha \ast \nu_0) < \varepsilon \), where \( \nu_0' \) is the restriction of \( \nu_0 \) to \( M_{\leq K}(X_\alpha) \). By the triangle inequality we eventually have \( d_P(\nu_0, F_\alpha \ast \nu_0') \leq 2 \varepsilon \). Moreover, \((X_\alpha, r_\alpha, F_\alpha \ast \nu_0')\) is eventually in \( A_N \) for \( N := \max(m, K, N_\varepsilon) \). This proves the claim since \( \varepsilon \) was arbitrary. \( \square \)

8 Equivalence of both topologies

We are finally able to prove the fact that the two-level Gromov-weak topology and the two-level Gromov-Prokhorov topology on \( \mathbb{M}^{(2)} \) coincide. Our proof relies mostly on the compactness criteria of Theorem 7.3.
Theorem 8.1
The two-level Gromov-weak topology and the two-level Gromov-Prokhorov topology coincide.

Proof: We will show that the identity function $\text{id}: (M^{(2)}, \tau_{2Gw}) \to (M^{(2)}, \tau_{2GP})$ and its inverse $\text{id}^{-1}$ are both continuous. Because $\tau_{2GP}$ stems from a metric, $\text{id}^{-1}$ is continuous if and only if it is sequentially continuous and the latter was proved in 4.5. However, we do not know yet if $\tau_{2Gw}$ is metrizable (nor if it is first countable). Therefore, it is not enough to show that $\text{id}$ is sequentially continuous. Instead we have to work with nets and show that each $\tau_{2Gw}$-convergent net $X_\alpha = (X_\alpha, r_\alpha, \nu_\alpha)$ $\xrightarrow{2Gw} X = (X, r, \nu)$ also converges in the two-level Gromov-Prokhorov topology. We will do so by proving that $(X_\alpha)_\alpha$ is a compact net with respect to $\tau_{2GP}$. Because $T$ is separating (cf. Theorem 3.8), this implies that every subnet has a converging subnet with the same limit $X$. Hence, $(X_\alpha)_\alpha$ itself converges to $X$ in the $d_{2GP}$-metric.

To prove compactness, we use property (3) of Theorem 7.3. First of all, the functions $(X, r, \nu) \mapsto m_\ast \nu$ and $(X, r, \nu) \mapsto w(M_{fK} \cdot \nu)$ are continuous with respect to $\tau_{2Gw}$ by Remark 3.14 and Lemma 6.1. Hence, both $(m_\ast \nu_\alpha)_\alpha$ and $(w(M_{fK} \cdot \nu_\alpha))_\alpha$ are compact. Furthermore, by Corollary 6.2 for every $\varepsilon > 0$ there is a $\delta > 0$ such that eventually $V_\delta(M_{fK} \cdot \nu_\alpha) < \varepsilon$. Thus, the assumptions of Theorem 7.3 are satisfied and $(X_\alpha)_\alpha$ is a compact net in the two-level Gromov weak topology. □

Note that now every statement we made about one of the two topologies (e.g. embedding theorems, compactness criteria, etc.) also holds true for the other topology.

9 Tightness and convergence determining sets

In this section we show that the set of test functions $T$ is convergence determining for $M_1(M^{(2)})$ and characterize tight subsets of $M_1(M^{(2)})$. Let us first recall the definition of convergence determining sets of functions.

Definition 9.1 (Convergence determining sets)
Let $(X, r)$ be a metric space. A set $F \subset C^b(X)$ is called convergence determining for $M_1(X)$ if for $\mu, \mu_1, \mu_2, \ldots \in M_1(X)$ weak convergence $\mu_n \xrightarrow{w} \mu$ is equivalent to

$$\int f \, d\mu_n \to \int f \, d\mu \quad \forall f \in F.$$

Theorem 9.2
The set of test functions $T$ is convergence determining for $M_1(M^{(2)})$.

Proof: The set $T$ separates points, is closed under multiplication and induces the topology of $M^{(2)}$. Therefore, it is convergence determining for $M_1(M^{(2)})$ by [HJ77, Lemma 4.1]. □
This means that a sequence \((P_n)_{n}\) in \(\mathcal{M}_1(\mathbb{M}^{(2)})\) converges to \(P \in \mathcal{M}_1(\mathbb{M}^{(2)})\) if and only if \(P_n[\Phi]\) converges to \(P[\Phi]\) for every \(\Phi \in T\). Here, \(P[\Phi]\) denotes the expectation \(\int \Phi(X)\,dP(X)\).

We now give a characterization of tight subsets of \(\mathcal{M}_1(\mathbb{M}^{(2)})\). Since tightness is defined in terms of compact sets, it is not surprising that we use Theorem 7.2 to find conditions for tightness.

**Proposition 9.3 (Characterization of tight sets)**

A set \(\mathcal{P} \subset \mathcal{M}_1(\mathbb{M}^{(2)})\) is tight if and only if for every \(\varepsilon > 0\) and \(K > 0\) there are \(\delta > 0\) and \(c > 0\) such that for every \(P \in \mathcal{P}\) we have

1. \(P(m(\nu)) \geq c < \varepsilon\),
2. \(P(m_*\nu([c, \infty])) \geq \varepsilon) < \varepsilon\),
3. \(P(V_\delta(M_{fK}\nu)) \geq \varepsilon) < \varepsilon\),
4. \(P(w(M_{fK}\nu)([c, \infty])) \geq \varepsilon) < \varepsilon\).

**Proof:** Let \(\varepsilon, K > 0\). If \(\mathcal{P}\) is tight, there is a compact set \(C \subset \mathbb{M}^{(2)}\) with \(P(\bar{C}) < \varepsilon\) for every \(P \in \mathcal{P}\). By property (3) of Theorem 7.2 we can choose \(\delta > 0\) and \(c > \sup \{m(\nu) | (X,r,\nu) \in C\}\) such that for every \((X,r,\nu) \in C\) we have

\[m_*\nu([c, \infty)) < \varepsilon,\]
\[V_\delta(M_{fK}\nu) < \varepsilon,\]
\[w(M_{fK}\nu)([c, \infty)) < \varepsilon\]

and the claim follows immediately.

To prove the other direction, let \(\varepsilon > 0\). We are going to construct a relatively compact set \(C \subset \mathbb{M}^{(2)}\) such that \(P(\bar{C}) < \varepsilon\) for every \(P \in \mathcal{P}\). First, define \(\varepsilon_n := \frac{\varepsilon}{4} \cdot 2^{-n}\) and \(K_n := n\) for every \(n \in \mathbb{N}\). By assumption there are \(c, \delta_n, c_n > 0\) such that

\[P(m(\nu)) \geq c < \frac{\varepsilon}{4},\]
\[P(m_*\nu([c_n, \infty)) \geq \varepsilon_n) < \varepsilon_n,\]
\[P(V_\delta(M_{fK_n}\nu)) \geq \varepsilon_n) < \varepsilon_n,\]
\[P(w(M_{fK_n}\nu)([c, \infty))) \geq \varepsilon_n) < \varepsilon_n\]

for every \(P \in \mathcal{P}\) and \(n \in \mathbb{N}\). Let \(C_n\) be the set of all \((X,r,\nu) \in \mathbb{M}^{(2)}\) with

\[m_*\nu([c_n, \infty)) < \varepsilon_n,\]
\[V_\delta(M_{fK_n}\nu) < \varepsilon_n,\]
\[w(M_{fK_n}\nu)([c_n, \infty)) < \varepsilon_n.\]

We have \(P(\bar{C}C_n) < 3\varepsilon_n\) for every \(P \in \mathcal{P}\). With \(C := \{(X,r,\nu) \in \mathbb{M}^{(2)} | m(\nu) < c\} \cap \bigcap_{n \in \mathbb{N}} C_n\),
we get

$$P(C) \leq \frac{\varepsilon}{4} + \sum_{n \in \mathbb{N}} 3\varepsilon_n = \varepsilon$$

for every $P \in \mathcal{P}$. Moreover, $C$ satisfies the compactness criterion given in property (3) of Theorem 7.2 and thus is relatively compact. □

If an m2m space $(X, r, \nu)$ satisfies $\nu \in \mathcal{M}_1(\mathcal{M}_1(X))$, its first moment measure $M_\nu$ is always finite and we do not need to approximate it by $M_{f_k\nu}$. Let us define the subset of all m2m spaces with this property.

**Definition 9.4**

We define the set of metric two-level probability measure spaces $\mathcal{M}^{(2)}_{1,1}$ by

$$\mathcal{M}^{(2)}_{1,1} := \{ (X, r, \nu) \in \mathcal{M}^{(2)} \mid \nu \in \mathcal{M}_1(\mathcal{M}_1(X)) \}.$$

$\mathcal{M}^{(2)}_{1,1}$ is a closed subset of $\mathcal{M}^{(2)}$ and thus a Polish metric space with the restriction of the metric $d_{\mathcal{GP}}$.

Proposition 9.3 boils down to a much simpler version if $P(\mathcal{M}^{(2)}_{1,1}) = 1$ for all $P \in \mathcal{P}$. We omit the obvious proof.

**Corollary 9.5**

A set $\mathcal{P} \subset \mathcal{M}_1(\mathcal{M}^{(2)}_{1,1})$ is tight if and only if for every $\varepsilon > 0$ there are $\delta > 0$ and $c > 0$ such that for every $P \in \mathcal{P}$ we have

1. $P(w(M_\nu)([c, \infty))) \geq \varepsilon < \varepsilon$
2. $P(V_\delta(M_\nu) \geq \varepsilon) < \varepsilon$.

The following version of the preceding corollary is particularly useful for the application in section 10.

**Corollary 9.6**

A set $\mathcal{P} \subset \mathcal{M}_1(\mathcal{M}^{(2)}_{1,1})$ is tight if the following two conditions hold:

1. There is a finite Borel measure $\mu$ on $\mathbb{R}_+$, such that

   $$P[w(M_\nu)] = \int w(M_\nu) dP((X, r, \nu)) \leq \mu$$

   for every $P \in \mathcal{P}$.

2. $\limsup_{\delta \searrow 0} P[M_\nu(\{ x \in X \mid M_\nu(B(x, \varepsilon)) < \delta \})] = 0$ for every $\varepsilon > 0$.

**Remark 9.7**

In case $\mathcal{P}$ is a sequence $(P_n)_n$, we can replace $\lim$ in property (2) by $\limsup$.
Proof (of Corollary 9.6): We show that \( \mathcal{P} \) satisfies both properties of Corollary 9.5. Let \( \varepsilon > 0 \). There exists a \( c > 0 \) such that \( \mu([c, \infty)) < \varepsilon^2 \). By Markov’s inequality we get

\[
P(w(M_\nu)([c, \infty)) \geq \varepsilon) \leq \frac{P[w(M_\nu)([c, \infty))]}{\varepsilon} \leq \frac{\mu([c, \infty))}{\varepsilon} < \varepsilon
\]

for every \( \mathcal{P} \in \mathcal{P} \). Moreover, by condition (2) there is a \( \delta > 0 \) such that

\[
P(M_\nu(\{ x \in X \mid M_\nu(B(x, \varepsilon/2)) < 2\delta \}) < \varepsilon^2
\]

for all \( \mathcal{P} \in \mathcal{P} \). With Lemma 5.2 and with Markov’s inequality we get

\[
P(V_\delta(M_\nu) \geq \varepsilon) = P(M_\nu(\{ x \in X \mid M_\nu(B(x, \varepsilon)) \leq \delta \}) \geq \varepsilon)
\leq \varepsilon^{-1}P(M_\nu(\{ x \in X \mid M_\nu(B(x, \varepsilon)) \leq \delta \})]
\leq \varepsilon^{-1}P(M_\nu(\{ x \in X \mid M_\nu(B(x, \varepsilon)) < 2\delta \})]
< \varepsilon
\]

for all \( \mathcal{P} \in \mathcal{P} \). Thus, \( \mathcal{P} \) is tight by Corollary 9.5 \( \square \)

10 Example: The nested Kingman coalescent measure tree

In this section we define the finite and infinite nested Kingman coalescent. Moreover, we introduce the nested Kingman coalescent measure tree, which is a random m2m space defined as the weak limit of finite random m2m spaces. To show convergence we will apply the tightness criteria from section 9.

10.1 The nested Kingman coalescent

Nested coalescents were introduced in [BB16,BBDLSJ] to jointly model the species and the gene coalescents of a population of multiple species. The nested Kingman coalescent is a special case of the model developed in these publications (cf. also [BBRSSJ] for further research about the nested Kingman coalescent). In this subsection we give a definition of the nested Kingman coalescent first for a finite set \( I \subset \mathbb{N}^2 \) of individuals and then for infinitely many individuals (i.e. \( I = \mathbb{N}^2 \)). We use \( \mathbb{N}^2 \) to encode individuals because we think of \( (i,j) \in \mathbb{N}^2 \) as the \( j \)-th individual of the \( i \)-th species.

For a non-empty set \( I \) let \( \mathcal{E}(I) \subset I^2 \) denote the set of equivalence relations on \( I \) equipped with the discrete topology. The equivalence classes of an equivalence relation are called blocks. We say that a pair \( (\mathcal{R}_1, \mathcal{R}_2) \in \mathcal{E}(I) \times \mathcal{E}(I) \) is nested (or that \( \mathcal{R}_2 \) is nested in \( \mathcal{R}_1 \)) if \( \mathcal{R}_1 \supset \mathcal{R}_2 \). Note that \( (\mathcal{R}_1, \mathcal{R}_2) \) is nested if and only if for every block \( \pi_2 \) of \( \mathcal{R}_2 \) there is a block \( \pi_1 \) of \( \mathcal{R}_1 \) with \( \pi_2 \subset \pi_1 \). Let

\[
\mathcal{N}(I) := \{(\mathcal{R}_1, \mathcal{R}_2) \in \mathcal{E}(I)^2 \mid \mathcal{R}_2 \subset \mathcal{R}_1 \}
\]

denote the set of nested equivalence relations equipped with the discrete topology.
Moreover, we define the following equivalence relations on $\mathbb{N}^2$, which will be the initial states of the nested Kingman coalescent:

$$G_0 := \{(x, x) \mid x \in \mathbb{N}^2\},$$
$$S_0 := \{(i, j), (i, k) \mid i, j, k \in \mathbb{N}\}.$$

$G_0$ is the equivalence relation with only singleton blocks and $S_0$ is the equivalence relation whose blocks are the different species of the population.

**Definition 10.1 (Finite nested Kingman coalescent)**

Let $I$ be a finite subset of $\mathbb{N}^2$ and $\gamma_s, \gamma_g > 0$. Let $R^{(I)} = (R^{(I)}(t))_{t \geq 0} = (R_s^{(I)}(t), R_g^{(I)}(t))_{t \geq 0}$ be a continuous-time Markov process with values in $\mathcal{N}(I)$. We call $R^{(I)}$ the finite nested Kingman coalescent on $I$ with rates $(\gamma_s, \gamma_g)$ if it has the following properties:

- The initial state is $R_s^{(I)}(0) = S_0 \cap I^2$ and $R_g^{(I)}(0) = G_0 \cap I^2$, i.e. for $x = (x_1, x_2) \in I^2$ and $y = (y_1, y_2) \in I^2$ we have
  $$(x, y) \in R_s^{(I)}(0) \iff x_1 = y_1,$$
  $$(x, y) \in R_g^{(I)}(0) \iff x = y.$$

- The species coalescent $R_s^{(I)} = (R_s^{(I)}(t))_{t \geq 0}$ behaves like a Kingman coalescent with rate $\gamma_s$, i.e. any two blocks in $R_s^{(I)}(t)$ merge at rate $\gamma_s$.

- The gene coalescent $R_g^{(I)} = (R_g^{(I)}(t))_{t \geq 0}$ behaves in the following way: any two blocks $\pi_1, \pi_2$ of $R_g^{(I)}$ such that $\pi_1 \cup \pi_2$ is contained in a single block of $R_s^{(I)}(t)$ merge at rate $\gamma_g$. Other blocks cannot merge.

The definition of the finite nested Kingman coalescent describes the behavior of a Markov process with only finitely many states. Thus, it is clear that such a process exists and is unique in distribution. Figure 1 shows a realization of a finite nested Kingman coalescent.

We now define the nested Kingman coalescent for an infinite set of individuals (that is, with $I = \mathbb{N}^2$). For the existence of this process we refer to the construction of (more general) nested coalescents in [BBDLSJ, section 5].

**Definition 10.2 (Nested Kingman coalescent)**

Let $\gamma_s, \gamma_g > 0$. The nested Kingman coalescent with rates $(\gamma_s, \gamma_g)$ is a continuous-time Markov process $R = (R(t))_{t \geq 0} = (R_s(t), R_g(t))_{t \geq 0}$ with values in $\mathcal{N}(\mathbb{N}^2)$ such that for any finite $I \subset \mathbb{N}^2$ the restriction of $R$ to $\mathcal{N}(I)$ is a finite nested Kingman coalescent on $I$ with rates $(\gamma_s, \gamma_g)$.

It follows immediately from the definition that the initial states of the species and the gene coalescent are

$$R_s(0) = S_0 \quad \text{and} \quad R_g(0) = G_0.$$
Figure 1 – A realization of a finite nested Kingman coalescent on the set $I = \{(1,1),(1,2),(2,1),(2,2),(3,1)\}$. The species coalescent on the left starts with the equivalence classes $\{(1,1),(1,2)\},\{(2,1),(2,2)\}$ and $\{(3,1)\}$ (here abbreviated by 1, 2 and 3). Its branch points are speciation events. The tree on the right side depicts the gene coalescent. Notice that merging events in the gene coalescent can only happen after the corresponding species have merged in the species coalescent.

It is well-known that the standard Kingman coalescent immediately comes down from infinity, meaning that after any positive time the coalescent almost surely has only finitely many blocks left, even if we start with infinitely many blocks. The same is true for the nested Kingman coalescent as stated in the next lemma. A proof can be found in [BBDLSJ, section 6].

**Lemma 10.3**

The nested Kingman coalescent immediately comes down from infinity. That is, if $\mathcal{R} = (\mathcal{R}_s(t),\mathcal{R}_g(t))_{t \geq 0}$ is a nested Kingman coalescent, then for every $t > 0$ both $\mathcal{R}_s(t)$ and $\mathcal{R}_g(t)$ almost surely consist of only finitely many blocks.

### 10.2 The nested Kingman coalescent measure tree

In this subsection we define a random m2m space called the nested Kingman coalescent measure tree. Roughly speaking, it is the genealogical tree of the gene coalescent of a nested Kingman coalescent equipped with a two-level measure that represents uniform sampling of species on the second level and uniform sampling of individuals in a single species on the first level.

Let $\mathcal{R} = (\mathcal{R}_s,\mathcal{R}_g)$ be a nested Kingman coalescent and $\mathbb{P}$ its law. For $x = (x_1,x_2) \in \mathbb{N}^2$ and $y = (y_1,y_2) \in \mathbb{N}^2$ we define the coalescence time of $x$ and $y$ by

$$r_g(x,y) := \inf \{ t \geq 0 \mid (x,y) \in \mathcal{R}_g(t) \}.$$

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The law of \( r_g(x,y) \) is

\[
\begin{aligned}
r_g(x,y) \sim \begin{cases} 
\text{Exp}(\gamma) \ast \text{Exp}(\gamma), & \text{if } x_1 \neq y_1 \\
\text{Exp}(\gamma), & \text{if } x_1 = y_1 \text{ and } x_2 \neq y_2 \\
\delta_0, & \text{if } x = y,
\end{cases}
\end{aligned}
\]  
(E11)

where \( \text{Exp}(\gamma) \) denotes the exponential distribution with parameter \( \gamma > 0 \) and \( \mu \ast \eta \) denotes the convolution of two distributions \( \mu \) and \( \eta \). The function \( r_g \) satisfies the triangle inequality and is thus a (random) metric on \( \mathbb{N}^2 \) (in fact it is even an ultra-metric). Let \((Z,r)\) denote the completion of the metric space \((\mathbb{N}^2,r_g)\).

**Remark 10.4 (Partial exchangeability)**

The distances between individuals of the nested Kingman coalescent are partially exchangeable in the following sense. Let \( \tilde{P} \) be the set of finite permutations \( p \) on \( \mathbb{N}^2 \) with the property that \( \pi_1 p(i,j) = \pi_1 p(i,k) \) for all \( i,j,k \in \mathbb{N} \), where \( \pi_1 \) denotes the projection to the first component of a vector (i.e. the first components of \( p(x) \) and \( p(y) \) coincide whenever the first components of \( x,y \in \mathbb{N}^2 \) coincide). Then, for every \( x_1, \ldots, x_n \in \mathbb{N}^2 \) the law of \( R(x_1, \ldots, x_n) \) is equal to the law of \( R(p(x_1), \ldots, p(x_n)) \) for every permutation \( p \in \tilde{P} \). In other words, the law of \( R(x_1, \ldots, x_n) \) is invariant under \( \tilde{P} \).

For all \( M, N \in \mathbb{N} \) we define the two-level measure \( \nu_{M,N} \in \mathcal{M}_1(\mathcal{M}_1(Z)) \) by

\[
\nu_{M,N} := \frac{1}{M} \sum_{i=1}^{M} \delta\left(\frac{i}{M} \sum_{j=1}^{N} \delta_{(0,j)}\right).
\]  
(E12)

\( \nu_{M,N} \) samples uniformly one of the first \( M \) species and then samples uniformly one of the first \( N \) individuals in that species. Let \( H_{M,N} \) be the function that maps a realization of the nested Kingman coalescent to the \( \text{m}2\text{m} \) space \((Z,r,\nu_{M,N})\) and define

\[
Q_{M,N} := H_{M,N} \ast \mathbb{P} \in \mathcal{M}_1(\mathcal{M}^{(2)}).
\]

**Theorem 10.5**

1. The sequence \((Q_{M,N})_N\) is weakly convergent for every \( M \in \mathbb{N} \). We denote its limit by \( Q_M \).

2. The sequence \((Q_M)_M\) is weakly convergent.

Therefore, the limit

\[
Q := \text{w-lim}_{M \to \infty} \text{w-lim}_{N \to \infty} Q_{M,N} = \text{w-lim}_{M \to \infty} Q_M
\]

exists. We call any random variable with values in \( \mathcal{M}^{(2)} \) and law \( Q \) a nested Kingman coalescent measure tree.

**Remark 10.6 (Generalization of Theorem 10.5)**

Recall that the \( \Lambda \)-coalescent is a generalization of the Kingman coalescent that allows multiple mergers and in which the merging rates are described by a finite measure \( \Lambda \in \)
\(M_f([0,1])\). In a similar manner we can generalize the nested Kingman coalescent to a \textit{nested \((\Lambda_s,\Lambda_g)\)-coalescent}, where the species coalescent behaves like a \(\Lambda_s\)-coalescent and the gene coalescent behaves like a \(\Lambda_g\)-coalescent (inside of single species blocks). The proof of Theorem 10.5 is valid even for the nested \((\Lambda_s,\Lambda_g)\)-coalescent as long as the nested coalescent immediately comes down from infinity. The latter condition is true if and only if both \(\Lambda_s\) and \(\Lambda_g\) are such that the corresponding \(\Lambda\)-coalescents immediately come down from infinity (cf. [BBRSSJ]).

It is an open question whether these conditions can be relaxed to the more general dust free property (cf. [GPW09, Theorem 4] in which the authors construct a \(\Lambda\)-coalescent measure tree for all \(\Lambda\)-coalescents which satisfy the dust free property).

\section{Proof of Theorem 10.5}

The proofs of both statements of Theorem 10.5 use the same kind of argument. First we show that the sequence under consideration has at most one limit point. Then we show that the sequence is relatively compact, i.e. every subsequence has a convergent subsequence. Consequently, because the limit point is unique, we may conclude that the original sequence is convergent.

Two main tools when working with coalescents are exchangeability and relative frequencies of blocks. We already showed in Remark 10.4 that the nested Kingman coalescent is partially exchangeable. Let us now define the relative frequencies of blocks.

\textbf{Definition 10.7}

For all \(i,l \in N\) and \(t \in \mathbb{R}_+\) we define

\[
f_{i,l}(t) := \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{[0,t]}(r_g((i,1),(l,n))).
\]

\(f_{i,l}(t)\) is the relative frequency of the block of \((i,1)\) w.r.t. species \(l\) at time \(t\).

It is a standard fact from coalescent theory that in a Kingman coalescent the relative frequencies of blocks exist. By definition the gene coalescent restricted to a single species \(i \in N\) is a Kingman coalescent. Thus, the relative frequency \(f_{i,i}(t)\) exists for every \(t \geq 0\). Moreover, the Kingman coalescent almost surely has proper frequencies (cf. [Pit99, Theorem 8]), implying that

\[
P(f_{i,i}(t) = 0) = 0
\]

for every \(t > 0\).

For \(f_{i,l}(t)\) with \(i \neq l\) the situation is a little different since the species \(i\) and \(l\) have to merge first. Let \(\tau_s(i,l)\) denote the coalescent time of the species \(i\) and \(l\) in the species tree. Clearly, we have \(f_{i,l}(t) = 0\) for \(t < \tau_s(i,l)\). For \(t \geq \tau_s(i,l)\) the gene coalescent restricted to \(\{l,j\} | j \in N\} \cup \{(i,1)\}\) behaves like a Kingman coalescent. Thus, the relative frequency \(f_{i,l}(t)\) also exist for \(t \geq \tau_s(i,l)\).

Because the nested Kingman coalescent starts with singleton blocks at time \(t = 0\), we have \(f_{i,l}(0) = 0\) for all \(i,l \in N\). Moreover, almost surely the function \(t \mapsto f_{i,l}(t)\) is
non-decreasing (the relative frequency increases when blocks merge) and converges to 1 (eventually all blocks have merged to a single block).

In the next lemma we state an analogon of equation (E13) for the relative frequencies \( f_{i,l}(t) \) with \( i \neq l \). Though it may happen that \( f_{i,l}(t) = 0 \) for some \( l \neq i \), it may not happen for all \( l \neq i \). This follows from the partial exchangeability explained in Remark 10.4.

**Lemma 10.8**

For every \( t > 0 \) and every \( i \in \mathbb{N} \)

\[ P(f_{i,l}(t) = 0 \text{ for all } l \neq i) = 0. \]  

(E14)

**Proof:** We define for every \( n \in \mathbb{N} \) and \( t' \in \mathbb{R}_+ \) the infinite vector \( f_n(t') \in [0,1]^\mathbb{N} \) by

\[ f_n(t') := (f_{n,l}(t'))_{l \neq n} = (f_{n,1}(t'), f_{n,2}(t'), \ldots, f_{n,n-1}(t'), f_{n,n+1}(t'), \ldots). \]

We fix \( t > 0 \) and \( i \in \mathbb{N} \). Observe that equation (E14) is equivalent to \( P(f_i(t) = 0) = 0 \) and that the sequence of vectors \( (f_n(t))_{n \in \mathbb{N}} \) is exchangeable (cf. Remark 10.4).

By de Finetti’s theorem there is a random probability measure \( \Xi \) on \( [0,1]^\mathbb{N} \) such that \( \Xi \otimes \mathbb{N} \) is a regular conditional distribution of \( (f_n(t))_n \) given \( \sigma(\Xi) \) (cf. [Ald85, Theorem 3.1]).

We prove the claim by contradiction. Assume that

\[ 0 < P(f_i(t) = 0) = \int \Xi(0) dP. \]

This is true if and only if \( P(\Xi(0) > 0) > 0 \) and in this case

\[ P(f_i(t) = 0) = \begin{cases} 0 & \text{for infinitely many } n \in \mathbb{N} \\ \Xi(0) > 0 & \text{for infinitely many } n \in \mathbb{N} \mid \Xi(0) > 0 \end{cases} \cdot P(\Xi(0) > 0) \]

(E15)

However, if \( f_n(t) = 0 \) for infinitely many \( n \in \mathbb{N} \), then there is an increasing sequence of positive integers \( (n_k)_{k} \) with \( f_{n_k}(t) = 0 \). Observe that \( f_{n_k}(t) = 0 \) implies that at time \( t \) the block of \( (n_k,1) \) has not merged with a block of another species. Therefore, each \( (n_l,1) \) is in a different block and \( R_g(t) \) contains an infinite number of blocks. But by Lemma 10.3 we know that almost surely \( R_g(t) \) contains only finitely many blocks. This is a contradiction to (E15). Consequently, we must have \( P(f_i(t) = 0) = 0. \)

\[ \square \]

Before we start to prove Theorem 10.5, observe that the two-level measure \( \nu_{M,N} \) from (E12) has the first moment measure

\[ M_{\nu_{M,N}} = \frac{1}{M} \sum_{i=1}^M \frac{1}{N} \sum_{j=1}^N \delta_{(i,j)}, \]

i.e. \( M_{\nu_{M,N}} \) is a uniform distribution on the set \( \{1,\ldots,M\} \times \{1,\ldots,N\} \).

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10.3.1 Weak convergence of \((Q_{M,N})_N\) for fixed \(M\)

Let \(M \in \mathbb{N}\) be fixed.

**Uniqueness:** Recall that the test functions from \(\mathcal{T}\) are convergence determining for \(\mathcal{M}_1(\mathbb{M}^{(2)})\). Because all our \(m2m\) spaces are elements of \(\mathbb{M}^{(2)}\) (i.e., the measures on both levels have mass 1), it is enough to consider test functions \(\Phi\) of the form

\[
\Phi((X,r,\nu)) = \int \int \varphi \circ R d\mu \otimes_{n} d\nu \otimes_{m}(\mu)
\]

(E16)

with \(m \in \mathbb{N}, n = (n_1, \ldots, n_m) \in \mathbb{N}^m\) and \(\varphi \in C_b(\mathbb{R}^{|n|\times|m|})\). We will explain why for each such \(\Phi\) the limit \(\lim_{N \to \infty} Q_{M,N}[\Phi]\) exists. Thus, the sequence \((Q_{M,N})_N\) has at most one limit point.

Fix a test function \(\Phi\) as in (E16). Then, we have

\[
Q_{M,N}[\Phi] = \int \Phi dQ_{M,N} = \int \Phi((Z,r,\nu_{M,N})) dP = \int \int \int \varphi \circ R d\mu \otimes_{n} d(\nu_{M,N}) \otimes_{m}(\mu) dP.
\]

One can show that this converges to

\[
\int \frac{1}{M^m} \sum_{i_1,\ldots,i_m=1}^{M} \varphi(R((i_1,1),\ldots,(i_{1},n_1),(i_{2},n_1+1),\ldots,(i_{2},n_1+n_2),(i_{3},n_1+n_2+1),\ldots,(i_{m},|n|))) dP
\]

(E17)

for \(N \to \infty\) using the partial exchangeability of the distances under \(P\) (cf. Remark 10.4). However, writing down a formal proof for general \(m\) and \(n\) is cumbersome and we would have to introduce a lot of notation. For this reason we omit the proof. The reader may easily verify our claim for small \(m\) and \(n\) to understand what is going on here. Heuristically, \(\Phi\) corresponds to sampling \(m\) species, then sampling \(n_1,\ldots,n_m\) individuals in these species and then evaluating the (genetic) distances between the individuals. We sample with the two-level measure \(\nu_{M,N}\), which means we uniformly sample from the first \(M\) species and in each of these species we uniformly sample from the first \(N\) individuals.

Since \(M\) is finite, it is possible that some species are sampled more than once. But for \(N \to \infty\) the probability to sample a single individual more than once goes to 0.

**Relative compactness:** We use Corollary 9.6. Thus, we have to show the following:

1. there is a finite Borel measure \(\mu_0\) on \(\mathbb{R}_+\) with \(Q_{M,N}[w(M_\nu)] \leq \mu_0\) for all \(N \in \mathbb{N}\),
2. \(\lim\limsup_{\delta \searrow 0, N \to \infty} Q_{M,N}[M_\nu(\{ x \in X \mid M_\nu(\overline{B}(x,\varepsilon)) < \delta \})] = 0\) for every \(\varepsilon > 0\).

(1) Define the finite measure

\[
\mu_0 := \delta_0 + \text{Exp}(\gamma_s) + \text{Exp}(\gamma_g) \ast \text{Exp}(\gamma_s),
\]

(E18)
where $\text{Exp}(\gamma)$ denotes the exponential distribution with parameter $\gamma > 0$ and $\mu * \eta$ denotes the convolution of two distributions $\mu$ and $\eta$. The law of $r(x, y)$ is bounded from above by the finite measure $\mu_0$ for all $x, y \in \mathbb{N}^2$ (cf. (E11)). Since $w(M_{\nu, M, N}) = r * M_{\nu, M, N}$, we get the desired inequality

$$Q_{M, N}[w(M_\nu)] = \int w(M_{\nu, M, N}) \, d\mathbb{P} \leq \mu_0.$$  

(2) Let $\varepsilon > 0$. Then we have

$$Q_{M, N}[M_\nu(\{ x \in X \mid M_\nu(B(x, \varepsilon)) < \delta \})]$$

$$= \int M_{\nu, M, N}(\{ x \in Z \mid M_{\nu, M, N}(B(x, \varepsilon)) < \delta \}) \, d\mathbb{P}$$

$$= \frac{1}{M} \sum_{i=1}^{M} \frac{1}{N} \sum_{j=1}^{N} \int 1_{[0, \delta)}(M_{\nu, M, N}(B((i, j), \varepsilon))) \, d\mathbb{P}$$

$$= \mathbb{P}(M_{\nu, M, N}(B((1, 1), \varepsilon)) < \delta)$$

$$\leq \mathbb{P}(M_{\nu, M, N}(B((1, 1), \varepsilon)) \leq \delta),$$

where we used the fact that $\mathbb{P}(M_{\nu, M, N}(B((i, j), \varepsilon)) < \delta)$ is the same for all $i, j \in \mathbb{N}$. By the definition of the relative frequencies $f_{1, l}(\varepsilon)$ we have

$$M_{\nu, M, N}(B((1, 1), \varepsilon)) \rightarrow \frac{1}{M} \sum_{l=1}^{M} f_{1, l}(\varepsilon)$$

almost surely for $N \rightarrow \infty$ and Fatou’s lemma yields

$$\limsup_{N \rightarrow \infty} \mathbb{P}(M_{\nu, M, N}(B((1, 1), \varepsilon)) \leq \delta) \leq \mathbb{P}\left( \frac{1}{M} \sum_{l=1}^{M} f_{1, l}(\varepsilon) \leq \delta \right).$$

Combining (E19), (E20) and (E13) we get

$$\lim_{\delta \searrow 0} \limsup_{N \rightarrow \infty} Q_{M, N}[M_\nu(\{ x \in X \mid M_\nu(B(x, \varepsilon)) < \delta \})]$$

$$\leq \lim_{\delta \searrow 0} \mathbb{P}\left( \frac{1}{M} \sum_{l=1}^{M} f_{1, l}(\varepsilon) \leq \delta \right)$$

$$= \mathbb{P}\left( \frac{1}{M} \sum_{l=1}^{M} f_{1, l}(\varepsilon) = 0 \right)$$

$$\leq \mathbb{P}(f_{1, 1}(\varepsilon) = 0)$$

$$= 0.$$

10.3.2 Weak convergence of $(Q_M)_M$

We have no information about $Q_M$ other than that it is the weak limit of $(Q_{M, N})_N$. Thus we must derive its properties from the approximating measures $Q_{M, N}$. 

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**Uniqueness:** We show that the sequence \((Q_M)_M\) has at most one limit point by proving existence of the limit \(\lim_{M \to \infty} Q_M[\Phi]\) for each \(\Phi\) as in (E16). Because these test functions are convergence determining, it follows that the sequence \((Q_M)_M\) has at most one limit point.

Fix a test function \(\Phi\) of the form (E16). Since \(\Phi\) is continuous and bounded, we have \(\lim_{M \to \infty} Q_M[\Phi] = \lim_{M \to \infty} \lim_{N \to \infty} Q_{M,N}[\Phi]\). Using (E17) and the partial exchangeability of the distances (Remark 10.4) one can show that
\[
\lim_{M \to \infty} Q_M[\Phi] = \lim_{M \to \infty} \lim_{N \to \infty} Q_{M,N}[\Phi] = \int \varphi(R((1,1),\ldots,(1,n_1),(2,1),\ldots,(m,n_m))) \, dP.
\]

Again, we omit the cumbersome proof. Heuristically, if \(M \to \infty\), then the probability to sample a species more than once goes to 0.

**Relative compactness:** Again, we use Corollary 9.6. We will show the following:

1. \(Q_M[w(M_{\nu})] \leq \mu_0\) for every \(M \in \mathbb{N}\), where \(\mu_0\) is defined in (E18),

2. \(\lim_{\delta \searrow 0} \limsup_{M \to \infty} Q_M(M_{\nu}(\{x \in X \mid M_{\nu}(B(x, \varepsilon)) < \delta\}) = 0\) for every \(\varepsilon > 0\).

(1) Fix \(M \in \mathbb{N}\). Observe that for finite measures \(\eta, \mu \in \mathcal{M}_f(\mathbb{R}_+\) we have \(\eta \leq \mu\) if and only if \(\int f \, d\eta \leq \int f \, d\mu\) for every non-negative, bounded and continuous function \(f\). For such \(f\) the function
\[
M_{1,1}(2) : \mathbb{R}_+ \to \mathbb{R}_+ \\
(X, r, \nu) \mapsto \int f \, dw(M_{\nu})
\]
is bounded and continuous as a concatenation of bounded and continuous functions. Because \(Q_M\) is the weak limit of \((Q_{M,N})_N\) and because \(\int f \, dQ_{M,N}[w(M_{\nu})] \leq \int f \, d\mu_0\), we get
\[
\int f \, dQ_M[w(M_{\nu})] = \int \int f \, dw(M_{\nu}) \, dQ_M \\
= \lim_{N \to \infty} \int \int f \, dw(M_{\nu}) \, dQ_{M,N} \\
= \lim_{N \to \infty} \int f \, dQ_{M,N}[w(M_{\nu})] \\
\leq \int f \, d\mu_0.
\]
Therefore, we have \(Q_M[w(M_{\nu})] \leq \mu_0\) for every \(M \in \mathbb{N}\).
(2) Fix $\varepsilon > 0$. Because $Q_{M,N}$ converges weakly to $Q_M$, we have
\[ \liminf_{N \to \infty} Q_{M,N}[f] \geq Q_M[f] \]
for every bounded lower semi-continuous function $f : M^{(2)} \to \mathbb{R}$ (cf. [Bog07, Corollary 8.2.5]). By Lemma 6.1 and Lemma 5.4 the function
\[ M^{(2)}_{1,1} \to \mathbb{R}_+ \]
\[ (X,r,\nu) \mapsto M_{\nu}(\{ x \in X \mid M_{\nu}(\mathcal{B}(x,\varepsilon) < \delta) \}) \]
is lower semi-continuous (and obviously bounded by 1). Therefore, we have
\[ Q_M[M_{\nu}(\{ x \in X \mid M_{\nu}(\mathcal{B}(x,\varepsilon)) < \delta \})] \]
\[ \leq \liminf_{N \to \infty} Q_{M,N}[M_{\nu}(\{ x \in X \mid M_{\nu}(\mathcal{B}(x,\varepsilon)) < \delta \})]. \]

Using inequalities (E19) and (E20) we get
\[ Q_M[M_{\nu}(\{ x \in X \mid M_{\nu}(\mathcal{B}(x,\varepsilon)) < \delta \})] \leq \mathbb{P}\left( \frac{1}{M} \sum_{l=1}^{M} f_{1,l}(\varepsilon) \leq \delta \right) \]  \hspace{1cm} (E21)

$(f_{1,l}(\varepsilon))_{l \geq 2}$ is a sequence of exchangeable random variables. By de Finetti’s Theorem (cf. [Ald85, Theorem 3.1]) there exists a random probability measure $\Xi$ with values in $M_f([0,1])$ such that $\Xi^{\otimes N}$ is a regular conditional distribution of $(f_{1,l}(\varepsilon))_{l \geq 2}$ given $\sigma(\Xi)$. In other words, $(f_{1,l}(\varepsilon))_{l \geq 2}$ is conditionally i.i.d. given $\sigma(\Xi)$. It follows that
\[ \frac{1}{M} \sum_{l=2}^{M} f_{1,l}(\varepsilon) \xrightarrow{M \to \infty} \mathbb{E}(f_{1,2}(\varepsilon) \mid \Xi) = \int x \, d\Xi(x) \]
almost surely (cf. [Ald85, Equation 2.24]). Fatou’s lemma yields
\[ \lim \sup_{\delta \searrow 0} \mathbb{P}\left( \frac{1}{M} \sum_{l=1}^{M} f_{1,l}(\varepsilon) \leq \delta \right) \leq \lim_{\delta \searrow 0} \mathbb{P}\left( \int x \, d\Xi(x) \leq \delta \right) \]
\[ = \mathbb{P}\left( \int x \, d\Xi(x) = 0 \right) \]
\[ = \mathbb{P}(\Xi = \delta_0). \]  \hspace{1cm} (E22)

Since $\Xi^{\otimes N}$ is the conditional distribution of $(f_{1,l}(\varepsilon))_{l \geq 2}$ given $\sigma(\Xi)$, we have
\[ \mathbb{P}(f_{1,l}(\varepsilon) = 0 \text{ for all } l \geq 2 \mid \Xi = \delta_0) = 1. \]

It follows that
\[ \mathbb{P}(\Xi = \delta_0) = \mathbb{P}(\Xi = \delta_0) \cdot \mathbb{P}(f_{1,l}(\varepsilon) = 0 \text{ for all } l \geq 2 \mid \Xi = \delta_0) \]
\[ = \mathbb{P}(f_{1,l}(\varepsilon) = 0 \text{ for all } l \geq 2). \]  \hspace{1cm} (E23)

The latter probability is 0 by Lemma 10.8. By combining (E21), (E22) and (E23) we get the claim.
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