Research Article

Laplace Decomposition Method to Study Solitary Wave Solutions of Coupled Nonlinear Partial Differential Equation

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1. Introduction

Systems of partial differential equations have attracted much attention in a variety of applied sciences because of their wide applicability. These systems were formally derived to describe wave propagation to model the shallow water waves [1–5] and to examine some chemical reaction-diffusion model of Brusselator [4–6]. Some of the commonly used methods to solve these equations are the method of characteristics, the Riemann invariants, and Adomian decomposition method [6].

In this work, we used Laplace decomposition method introduced by Khuri [7, 8] which is further used by Yusufoglu to solve Duffing equation [9] and Elgasery for Falkner-Skan equations [10]. This technique, modified by Hussain and Khan [11], illustrates how the Laplace transform may be used to approximate the solutions of the nonlinear partial differential equations by extending the decomposition method [12, 13].

2. Laplace Decomposition Method (LDM)

In this section, we outline the main steps of the method. We consider the nonlinear partial differential equations in an operator form:

\[
\begin{align*}
L_t u + R_1(u, v) + N_1(u, v) &= f_1, \\
L_t v + R_2(u, v) + N_2(u, v) &= f_2.
\end{align*}
\]

With initial data

\[
\begin{align*}
u(x, 0) &= g_1(x), \\
v(x, 0) &= g_2(x),
\end{align*}
\]

where \(L_t = \partial / \partial t\) is a first-order partial differential operator, \(R_1, R_2, N_1, N_2\) are linear and nonlinear operators, respectively, and \(f_1, f_2\) are the source terms. By applying the Laplace transform to both sides of (1) and using initial conditions (2), we have

\[
\begin{align*}
L(L_t u) + L(R_1(u, v)) + L(N_1(u, v)) &= L(f_1), \\
L(L_t v) + L(R_2(u, v)) + L(N_2(u, v)) &= L(f_2).
\end{align*}
\]

By the differentiation property of Laplace transform, we get

\[
L(u) = \frac{g_1(x)}{p} + \frac{L(f_1)}{p} - \frac{1}{p} [L(R_1(u, v)) + L(N_1(u, v))],
\]
The following recursive relations:

\[ L(v) = \frac{g_2(x)}{p} + \frac{L(f_1)}{p} - \frac{1}{p} [L(R_2(u, v)) + L(N_2(u, v))] \]

(4)

where “\( p \)” is Laplace domain function. Solutions \( u(x, t) \) and \( v(x, t) \) in LDM are defined as

\[ u(x, t) = \sum_{n=0}^{\infty} u_n, \quad v(x, t) = \sum_{n=0}^{\infty} v_n. \]

(5)

The nonlinear terms \( N_1, N_2 \), represented by infinite series,

\[ N_1(x, t) = \sum_{n=0}^{\infty} A_n, \quad N_2(x, t) = \sum_{n=0}^{\infty} B_n \]

(6)

are the Adomian polynomials [14], generated for all forms of nonlinearity and they are determined by the following relations:

\[ A_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} \left( N_1 \sum_{i=0}^{\lambda} \lambda^i u_i \right) \right]_{\lambda=0}, \]

(7)

\[ B_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} \left( N_2 \sum_{i=0}^{\lambda} \lambda^i v_i \right) \right]_{\lambda=0}. \]

Substituting (5) and (6) into (4), gives

\[ L \left( \sum_{n=0}^{\infty} u_n \right) = \frac{g_1(x)}{p} + \frac{L(f_1)}{p} - \frac{1}{p} \left[ L \left( R_1 \left( \sum_{n=0}^{\infty} u_n, \sum_{n=0}^{\infty} v_n \right) \right) \right]_t + L \left( \sum_{n=0}^{\infty} A_n \right), \]

(8)

\[ L \left( \sum_{n=0}^{\infty} v_n \right) = \frac{g_2(x)}{p} + \frac{L(f_2)}{p} - \frac{1}{p} \left[ L \left( R_2 \left( \sum_{n=0}^{\infty} u_n, \sum_{n=0}^{\infty} v_n \right) \right) \right]_t + L \left( \sum_{n=0}^{\infty} B_n \right). \]

Applying the linearity of the Laplace transform, we deduce the following recursive relations:

\[ L(u_0) = \frac{g_1(x)}{p} + \frac{L(f_1)}{p}, \]

\[ L(v_0) = \frac{g_2(x)}{p} + \frac{L(f_2)}{p}, \]

(9)

\[ L(u_1) = -\frac{1}{p} L \left( [R_1(u_0, v_0)] \right) - \frac{1}{p} L[A_0], \]

\[ L(v_1) = -\frac{1}{p} L \left( [R_2(u_0, v_0)] \right) - \frac{1}{p} L[B_0]. \]

In general, for \( k \geq 1 \), the recursive relations are given by

\[ L(u_{k+1}) = -\frac{1}{p} L \left( [R_1(u_k, v_k)] \right) - \frac{1}{p} L[A_k], \]

\[ L(v_{k+1}) = -\frac{1}{p} L \left( [R_2(u_k, v_k)] \right) - \frac{1}{p} L[B_k]. \]

(10)

Applying the inverse Laplace transform, we can evaluate \( u_k \) and \( v_k \) \( (k \geq 0) \). In some cases the exact solution in the closed form can be obtained.

3. Application

At the classical level, a set of coupled nonlinear wave equations describes the interaction between high-frequency and low-frequency waves [15], and the calculation of exact and numerical solutions of the equations, in particular, travelling wave solutions, plays an important role in wave-wave interaction and soliton theory [1, 16].

We consider the Schrödinger-KdV (Sch-KdV) equation as a model for the interaction of long and short nonlinear waves:

\[ i\eta_t = E_{xx} + E\eta, \]

(11)

\[ \eta_t = -6E\eta_x - \eta_{xxx} + \left([E]^2\right)_x. \]

With initial conditions

\[ E(x, 0) = 2\sqrt{2}k^2 \left( 1 - 3 \tanh^2(kx) \right)e^{jxx}, \]

\[ \eta(x, 0) = \frac{8k^2 - \alpha}{3} - 6k^2\tanh^2(kx), \]

(12)

where \( \alpha, k \) are arbitrary constant.

With Laplace decomposition method on (11) and using the differentiation property of Laplace transform, initial conditions and the inverse Laplace transforms are

\[ \sum_{n=0}^{\infty} E_n(x, t) = E(x, 0) - iL^{-1} \left[ \frac{1}{p} L \left( E_{xxx} + \sum_{n=0}^{\infty} A_n(\eta, E) \right) \right], \]

\[ \sum_{n=0}^{\infty} \eta_n(x, t) \]

\[ = \eta(x, 0) + L^{-1} \left[ \frac{1}{p} L \left( \sum_{n=0}^{\infty} B_n(E) - 6 \sum_{n=0}^{\infty} C_n(\eta, E) - \eta_{xxx} \right) \right], \]

(13)

where \( (x, t) = \sum_{n=0}^{\infty} E_n \eta_n(x, t) = \sum_{n=0}^{\infty} \eta_n \), and

\[ \sum_{n=0}^{\infty} A_n(\eta, E) = \eta E \sum_{n=0}^{\infty} B_n(E) = \left([E]^2\right)_x, \]

\[ \sum_{n=0}^{\infty} C_n(\eta, E) = E\eta_x \] are adomian polynomials that represent nonlinear terms.
Figure 1: The plots of results for solution of Sch-KdV equations with a fixed values of \( \alpha = k = 0.05 \) and for different values of time. (a) Analytical solutions for \( E(x, t) \). (b) Numerical results for \( E(x, t) \) by means of LDM. (c) Analytical solutions for \( \eta(x, t) \). (d) Numerical results for \( \eta(x, t) \) by means of LDM.

So the recursive relation is deduced as

\[
E(x, 0) = E_0 = 2\sqrt{2}k^2 \left( 1 - 3 \tanh^2(kx) \right) e^{\text{ix}x},
\]

\[
E_1(x, t) = -iL^{-1} \left[ \frac{1}{p} L \left( E_{0xx} + \sum_{n=0}^{\infty} A_n(\eta, E) \right) \right],
\]

\[
E_{n+1}(x, t) = -iL^{-1} \left[ \frac{1}{p} L \left( E_{nxx} + \sum_{n=0}^{\infty} A_n(\eta, E) \right) \right],
\]

\[
\eta(x, 0) = \eta_0 = \frac{8k^2 - \alpha}{3} - 6k^2 \tanh^2(kx),
\]

\[
\eta_1(x, t) = L^{-1} \left[ \frac{1}{p} L \left( \sum_{n=0}^{\infty} B_n(E) - 6 \sum_{n=0}^{\infty} C_n(\eta, E) - \eta_{0xxx} \right) \right],
\]

\[
\eta_{n+1}(x, t) = L^{-1} \left[ \frac{1}{p} L \left( \sum_{n=0}^{\infty} B_n(E) - 6 \sum_{n=0}^{\infty} C_n(\eta, E) - \eta_{nxxx} \right) \right],
\]

\[
\eta_{n+1}(x, t) = L^{-1} \left[ \frac{1}{p} L \left( \left( |E_0|^2 \right)_x - 6E_0\eta_0 - \eta_{0xxx} \right) \right],
\]

By this recursive relation we can find other components of the solution as

\[
E_1(x, t) = -iL^{-1} \left[ \frac{1}{p} L (E_{0xx} + \eta_0 E_0) \right],
\]

\[
E_{n+1}(x, t) = \frac{2\sqrt{2}k^2 t}{3} \times \left[ (3\alpha^2 + \alpha + 10k^2)(1 - 3 \tanh(kx)) \right.
\]

\[
\times \cos(\alpha x) + 36k \sech^2(kx)
\]

\[
\times \left( 3\alpha^2 + \alpha + 10k^2 \right)
\]

\[
\times (1 - 3 \tanh(kx)) \sin(\alpha x)
\]

\[
- 36k \sech^2(kx) \tanh(kx) \cos(\alpha x) \right],
\]

\[
\eta_{n+1}(x, t) = L^{-1} \left[ \frac{1}{p} L \left( \left( |E_0|^2 \right)_x - 6E_0\eta_0 - \eta_{0xxx} \right) \right],
\]
The other components of the decomposition series can be determined in a similar way; we can obtain the expression of $E(x, t)$ in a Taylor series, which gives the closed form solutions as:

$$E(x, t) = 2\sqrt{2k^2} \left(1 - 3 \tanh^2(k(x + 2at)) \right) \times \exp \left( \left[ \alpha x + \frac{1}{3} (3\alpha^2 + \alpha - 10k)t \right] \right),$$

$$
\eta(x, t) = \frac{8k^2 - \alpha}{3} - 6k^2 \tanh^2(k(x + 2at)).
$$

(16)

4. Numerical Description of the Solution

The Laplace decomposition method is used for finding the exact and approximate travelling-waves solutions of the Sch-KdV equation. Both the exact and approximate solutions obtained for $n = 2$ using LDM are plotted in Figure 1. It is evident that when compute more terms for the decomposition series the numerical results are getting much closer to the corresponding analytical solutions.

5. Conclusion

The Laplace decomposition method is a powerful method which has provided an efficient potential for the solution of physical applications modeled by nonlinear differential equations. The algorithm can be used without any need to complex calculations except for simple and elementary operations.

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