Apéry extensions

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Abstract

The Apéry numbers of Fano varieties are asymptotic invariants of their quantum differential equations. In this paper, we initiate a program to exhibit these invariants as (mirror to) limiting extension classes of higher cycles on the associated Landau–Ginzburg (LG) models — and thus, in particular, as periods. We also construct an Apéry motive, whose mixed Hodge structure is shown, as an application of the decomposition theorem, to contain the limiting extension classes in question. Using a new technical result on the inhomogeneous Picard–Fuchs equations satisfied by higher normal functions, we illustrate this proposal with detailed calculations for LG-models mirror to several Fano threefolds. By describing the “elementary” Apéry numbers in terms of regulators of higher cycles (i.e., algebraic $K$-theory/motivic cohomology classes), we obtain a satisfying explanation of their arithmetic properties. Indeed, in each case, the LG-models are modular families of $K3$ surfaces, and the distinction between multiples of $\zeta(2)$ and $\zeta(3)$ (or $(2\pi i)^3$) translates ultimately into one between algebraic $K_1$ and $K_3$ of the family.

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1 INTRODUCTION

Over the years since Apéry’s discovery of a proof of irrationality of $\zeta(3)$ in 1979, his recursions

\[ n^3u_n - (34n^3 - 51n^2 + 27n - 5)u_{n-1} + (n - 1)^3u_{n-2} = 0, \]  

(A3)
and
\[ n^2 u_n - (11n^2 - 11n + 3)u_{n-1} - (n-1)^2 u_{n-2} = 0, \quad (A2) \]

originally viewed as interesting individuals rather than members of an important species — to follow Siegel’s distinction — have gradually come to be seen as being the two most basic instances of Fuchsian Differential Equations (DEs) arising from the class of extremal Calabi–Yau pencils which appears in various subjects in algebraic and arithmetic geometry. This makes them natural models for case studies of this class. In each of the two cases, a unique normalized integral solution \( \{a_n\} \) exists, namely, \( a_n^{(A3)} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \), and \( a_n^{(A2)} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \). By considering the solutions \( \{b_n\} \) with the initial conditions \( b_0 = 0, b_1 = 1 \), and the rate of convergence of the ratios \( b_n/a_n \) to the limits
\[ \lim_{n \to \infty} \frac{b_n^{(A3)}}{a_n^{(A3)}} = \frac{1}{6} \zeta(3) \quad \text{and} \quad \lim_{n \to \infty} \frac{b_n^{(A2)}}{a_n^{(A2)}} = \frac{1}{5} \zeta(2), \]

Apéry was able to deduce the irrationality of \( \zeta(3) \) and improve the known order of irrationality of \( \zeta(2) \). Now, to make a story, we put \( a_n^{(B3)} = \binom{2n}{n} a_n^{(A2)} \). Clearly, the \( \{a_n^{(B3)}\} \) satisfy
\[ n^3 u_n - (22n^3 - 33n^2 + 17n - 3)u_{n-1} + 16(n - 3/2)(n - 1/2)u_{n-2} = 0, \quad (B3) \]

and the limit of the ratio \( b_n^{(B3)}/a_n^{(B3)} \) is \( \frac{1}{10} \zeta(2) \). Passing to the generating series \( \Phi(t) = \sum c_n t^n \) with \( c_n = a_n \) or \( b_n \) and translating recursions into differential equations, we have
\[ (D - 1) L^{(A3)} \Phi^{(A3)}(t) = 0 \quad \text{resp.} \quad (D - 1) L^{(B3)} \Phi^{(B3)}(t) = 0 \]

with
\[ L^{(A3)} = D^3 - t(2D + 1)(17D^2 + 17D + 5) + t^2(D + 1)^3 \]

and
\[ L^{(B3)} = D^3 - 2t(2D + 1)(11D^2 + 11D + 3) - 4t^2(D + 1)(2D + 3)(2D + 1). \]

The differential operators \( L^{(A3)} \) and \( L^{(B3)} \) have quite a lot in common:

- quite naively, both are linear third-order differential operators of degree 2 in \( t \);
- more specifically, both are operators of D3 type \([21]\), which implies that the \( i \)th coefficient in \( t \) is odd as a polynomial in \( D + i/2 \);
- each is a Picard–Fuchs operator that controls the variation of polarized Hodge structure arising in the relative \( H^2 \) in a Shioda–Inose pencil of K3 surfaces;
- both are extremal, in the sense that the parabolic cohomology groups \( H^1(P^1, -) \) of the associated local systems vanish;
- both are modular, in the sense that in each case, the solution \( \sum_{n=0}^{\infty} a_n t^n \) can be identified with the expansion of a certain weight 2 level \( N \) Eisenstein series with respect to a Hauptmodul on \( X_0^+ (N = 6 \text{ and } 5, \text{ respectively}) \);
- each is the regularized quantum differential operator of a Fano complete intersection in a \( G/P \) (namely, a sevenfold hyperplane section of \( OG(5,10) \) for \( (A3) \) and an intersection of two hyperplanes and a quadric in \( G(2,5) \) for \( (B3) \)). It is this interpretation of \( (A3) \) and \( (B3) \) as
Picard–Fuchs operators in the Landau–Ginzburg (LG) models of such “Mukai threefolds” that we are concerned with in the present paper.

Yet, in spite of all these similarities, the Apéry limit $b_n/a_n$ is proportional to $\zeta(3)$ in the first case resp. to $\zeta(2)$ in the second. Two natural questions arise:

(1) On the A side of mirror symmetry, does the Apéry limit reflect the geometry of a Fano? Inspecting the five cases of Mukai threefolds $V_{2n}$, $n = 5, \ldots, 9$, one notices that the cases with $\lim_{n \to \infty} b_n/a_n$ of weight 3 match the rational Fanos $V_{12}$, $V_{16}$, and $V_{18}$, whereas those of weight 2 correspond to the irrational $V_{10}$ and $V_{14}$.

(2) On the B side, what properties of the LG pencil control the weight of the Apéry limit?

A conceptual answer to both questions, unconditional for (2) but homological mirror symmetry (HMS) contingent for (1), is that it is ultimately the structure of the “south pole” ($t = \infty$) fiber of the LG pencil that is responsible for the weight of the Apéry limit. Indeed, by HMS for Fanos, the structure of a LG model in the neighborhood of the south pole is expected to be related to the residual category [37], that is, the semiorthogonal complement to the subcategory of $D^b_{\text{coh}}(F)$ generated by an incomplete exceptional collection. For a rational threefold, the properties of the residual category are closer, in a sense, to those of the $D^b_{\text{coh}}$ of a bona fide algebraic variety than in the irrational cases. One notices a similarity here with how the intermediate Jacobian of a threefold can have properties inconsistent with being the full motive of a curve. Nevertheless, one can unconditionally relate the structure of the south-pole fiber to the Lefschetz decomposition of the cohomology of the $\text{Fano unsections}$ — ambient varieties in which our Fano sits as a hyperplane section or a complete intersection. If the first nontrivial primitive class occurs in $H^4$, rather than in $H^6$ or higher, the Apéry constant is going to have weight 2, in agreement with the gamma conjecture.

The present paper offers mainly a Hodge-theoretic answer to (2). In a nutshell, the structure of the south-pole fiber, or more precisely, the type of the south-pole limiting mixed Hodge structure (LMHS), dictates the Hodge type of the Hodge module whose underlying differential operator is $(D - 1)L$: the $\zeta(3)$ (and $L(\zeta-3, 3)$) cases correspond to extensions encoded by a class in relative $H^3_{\mathcal{A}^m}(K3, \mathbb{Q}(3))$, whereas the $\zeta(2)$ cases correspond to extensions associated with $H^3_{\mathcal{A}^m}(K3, \mathbb{Q}(2))$. In $K$-theoretic terms, the distinction is between the $K_3$ and $K_1$ groups of relative K3 surfaces, respectively. A more general, heuristic view that suggests itself is this: just as the Fano $F$ knows the quantum cohomology of its unsection, the LG-model of $F$ remembers the asymptotics of periods of the LG-model of its unsection, expressed as solutions to inhomogeneous Picard-Fuchs (PF) equations $L(-) = g(t)$ that underlie generalized normal functions.

Arithmetic mirror symmetry
Despite a smattering of examples in recent years [14, 15, 28, 40], the role of algebraic cycles and their invariants in mirror symmetry remains something of a mystery. The above discussion suggests a new link in the context of Fano/LG-model duality, whose formulation (backed up by nontrivial evidence) is a principal goal of this paper.

One of the features of local mirror symmetry uncovered in [7, 15] was the entrance of mixed Hodge structures, whose extension classes are described on the B-model side by regulators on algebraic $K$-theory. These same regulator classes, called higher normal functions when they occur in families, are at the heart of the second author’s interpretation [30] of Apéry’s irrationality proofs for $\zeta(2)$ and $\zeta(3)$. It was in an effort to “recombine” this with the first author’s enumerative, A-model interpretation [20] of Apéry’s recurrence (see also [17]), that the animating slogan of this paper suggested itself.
Arithmetic mirror symmetry problem: For each Fano n-fold $F$ admitting a toric degeneration, show that its Apéry numbers arise as

- limits of (classical and higher) normal functions produced by cycles on a one-parameter family of Calabi-Yau (CY) $(n-1)$-folds defined over $\mathcal{Q}$, together with
- extension classes in the monodromy-invariant part of a limiting MHS of the family.

A more detailed statement of this problem may be found in §5.2.

While computations by G. da Silva [11] appeared to support this line of thinking for the rational Fano threefolds in [20], there initially seemed to be little hope for the Apéry numbers $\frac{1}{10}\zeta(2), \frac{1}{7}\zeta(2)$ of the nonrational Fanos $V_{10}, V_{14}$, with neither the “deresonation off the motivic setting” nor the “quantum Satake” argument on their regular side offering an explanation of these numbers as periods. Moreover, the model of [30], in its limitation to $K_{\text{alg}}$ of CY $(n-1)$-folds, could only produce rational multiples of $(2\pi i)^3$ or $\zeta(3)$ if $n = 3$. However, a new paradigm began to emerge around 2 years ago, allowing a much greater variety of cycles to enter. Our main result is thus the following affirmative solution to the Arithmetic Mirror Symmetry Problem (more precisely, to Problem 5.3).

**Theorem 1.1.** The Apéry numbers of the five Mukai Fano threefolds† are limits of higher normal functions arising from motivic cohomology classes on associated LG models.

The theorem is proved in §§5.3–5.5 (modulo a detail deferred to §6), using a new result on inhomogeneous Picard–Fuchs equations satisfied by higher normal functions (Theorem 5.1). In §6, we propose a theory of “Apéry extensions” on the B-model side, which encompasses these examples, and highlight some implications of an affirmative solution to Problem 5.3. In §§2–4, we place our story in context, recalling the mixed Hodge theory of GKZ systems and local mirror symmetry, quantum $\mathcal{D}$-modules and Apéry constants of Fanos, and higher normal functions on LG models. In the rest of this Introduction, we would like to convey the idea of what an Apéry extension is and why it is important.

Let $X \rightarrow \mathbb{P}^1$ be a family of compact CY $(n-1)$-folds with smooth total space and fibers‡ $X_t = \pi^{-1}(t)$, smooth off $\Sigma = \{0, t_1, \ldots, t_c, \infty\}$. Laurent polynomials $\phi(x) \in \mathcal{Q}[x^{\pm1}, \ldots, x_{n}^{\pm1}]$ with reflexive Newton polytope $\Delta$ are a key source for such families, with $\mathcal{X}$ obtained by blowing up $\mathbb{P}_\Delta$ along $\{\phi = 0\} \cap (\mathbb{P}_\Delta \setminus \mathbb{G}^n_m)$, and $\pi$ extending $1/\phi$. In particular, the mirror LG-model of a Fano $F$ degenerating to $\mathbb{P}_\Delta$ (like those in [17, 20]) arises in this way.

The cohomologies of the fibration $X_U \rightarrow U' := \mathbb{P}^1 \setminus \Sigma$ produce variation of Hodge structures (VHSs) $H^\ell = H^f \oplus H^v$ with “fixed” and “variable” parts. At each $\sigma \in \Sigma$, we have the LMHS functor $\psi_\sigma$ and monodromies $T_\sigma$. All our families will have maximal unipotent monodromy at the “north pole” $\sigma = 0$;§ for simplicity, here we also assume¶ $\text{rk}(T_\sigma - I) = 1$ if $\sigma \neq 0, \infty$, and $\ker(H^n(X_{\sigma}) \rightarrow \psi_\sigma H^n) = \{0\}(\forall \sigma)$. Then, we may write $A_{\phi}^\dagger := H^n(X \setminus X_0, X_\infty)$ as an extension

$$0 \rightarrow (\psi_{\infty} H_{U_{\infty}}^{n-1})^{T_{\infty}} \rightarrow A_{\phi}^\dagger \rightarrow \text{IH}^1 \left( \mathbb{P}^1 \setminus \{0\}, H_{U_{\infty}}^{n-1} \right) \rightarrow 0$$

(1.1)

† These are, by definition, the rank-one Fano threefolds arising as complete intersections in the Grassmannians of simple Lie groups other than projective spaces [20]; they are $V_{10}, V_{12}, V_{14}, V_{16},$ and $V_{18}$.

‡ Written $\tilde{X}_t$ in the body of the paper.

§ “North” refers to the infinity point of the LG potential; since we work primarily in a neighborhood of this point, however, it is $t = 0$ for us.

¶ In the Introduction, but not in the body of the paper.
of MHS. Now the Apéry numbers of F record limits of ratios of solutions to its quantum difference equation (Definition 3.5); and a first approximation to the Problem is to find them in the extension class of (1.1).

Unfortunately, extension classes of MHS do not produce well-defined numbers. For instance, we have Ext₁_{MHS}(ℚ(−a), ℚ(0)) ∼= ℂ/ℚ(a), which (say) would make \( \frac{1}{10} \zeta(2) \) trivial in ℂ/ℚ(2). This is where writing them as limits of admissible normal functions \(^\dagger\) enters: if (1.1) arises as \( \lim_{t \to 0} \nu(t) \) for some \( \nu \in \text{ANF}(H^{n-1}_u(r)) \), then \( \nu \) has a unique lift \( \tilde{\nu} \) on the disk \( |t| < |t_{k+1}| \) to a single-valued section of \( H^{n-1}_u \). Pairing this with a suitable section \( \omega \in \Gamma(p^1, F^{n-1} H^{n-1}_u) \) yields a truncated higher normal function (THNF) \( V(t) = \langle \tilde{\nu}, \omega \rangle \) whose first \( k \) Taylor coefficients in \( t \) are well-defined complex numbers refining the information in \( \mathcal{A}^\dagger \). So, higher normal functions get us from extension data to constants, and this is why it is better to state the Problem in terms of their limits. But which HNF to choose? Here are two candidates.

Consider the VMHS \( \mathcal{A}^\sigma_\phi := H^n(X_\sigma, X_i) \) over \( U (\sigma = 0 \text{ or } \infty) \). As an extension, it reads

\[
0 \to H^{n-1}_u \to \mathcal{A}^\sigma_\phi \to IH^1(p^1 \setminus \{\sigma\}, H^{n-1}_u) \to 0, \tag{1.2}
\]

in which the IH term is a constant VMHS. Taking first \( \sigma = 0 \), \( \mathcal{A}^\dagger_\phi = (\psi_\infty A^0_\phi)^\infty \) is recovered as the “south pole” limit of \( \mathcal{A}^0_\phi \). If \( H^{n-1}_u \) is extremal (cf. §6) with Hodge numbers all 1, then \( IH^1(p^1 \setminus \{0\}, H^{n-1}_u) \cong \mathbb{Q}(-n) \), and (1.2) gives a normal function in ANF(H^{n-1}_u(n)). According to the Beilinson–Hodge conjecture (cf. Conjecture 4.8 below), this should come from a “\( K_n \)” cycle (in CH^n(X_\sigma, X_0, n)), recovering the paradigm of [30].

The alternate (\( \sigma = \infty \)) perspective is to view \( \mathcal{A}^\dagger_\phi = [(\psi_0 A^\infty_\phi)^0 \vee (-n)] \) as a “north pole” limit. This can make a huge difference, because \( A^\infty_\phi \) and \( A^0_\phi \) are not dual in general (although the invariant parts of their limits are). Indeed, for any morphism \( \mathbb{Q}(-a) \hookrightarrow IH^1(p^1 \setminus \{\infty\}, H^{n-1}_u) \), the \( \mu^n \)-pullback

\[
0 \to H^{n-1}_u \to \mu^n A^\infty_\phi \to \mathbb{Q}(-a) \to 0 \tag{1.3}
\]

of (1.2) belongs to ANF(H^{n-1}_u(a)), and (by Beilinson–Hodge again) should arise from a “\( K_{2a-n} \)” cycle (in CH^n(X_\infty, X_\sigma, 2a - n)). It is the extensions of VMHS (1.3) that we call Apéry extensions (Definition 6.4). In the families of K3s mirror to \( V_{10} \) and \( V_{14} \), we have \( a = 2 \), and the corresponding normal functions do indeed arise from torically natural \( K_1 \)-cycles whose THNFs have the north pole limits \( \frac{1}{10} \zeta(2), \frac{1}{7} \zeta(2) \). This change in perspective came as a revelation since, for these and similar cases, the south-pole approach is not computationally viable (Remark 6.6).

2 | GENERIC LAURENT POLYNOMIALS

2.1 | GKZ system

Fix a vector \( a \in \mathbb{C}^{N+1} \) and a convex polytope \( \Delta \subset \mathbb{R}^{n+1} \) containing the origin, with vertices in \( \mathbb{Z}^{N+1} \). The corresponding toric variety \( \mathbb{P}_\Delta \) compactifies \( G_m^{N+1} \) (with coordinate \( x \)). Let \( M \subset \mathbb{Z}^{N+1} \) denote the monoid generated by \( \mathcal{M} := \Delta \cap (\mathbb{Z}^{N+1} \setminus \{0\}) \) and \( \mathcal{L} \subset \mathbb{Z}^{|\mathcal{M}|} \) the lattice of relations;

\(^\dagger\) Admissible normal functions are reviewed in §4.2 below.
we assume for simplicity that $M^{gp} = \mathbb{Z}^{N+1}$ and $M = \mathbb{Z}^{N+1} \cap \text{Cone}_\Delta$. The coefficients $\lambda$ of the generic Laurent polynomial $f(x) = \sum_{m \in \mathbb{Z}} \lambda_m x^m$ parametrize the affine parameter space on which we define the \textit{GKZ system} of partial differential operators:

\[
\begin{align*}
Z_i &= \sum_{m \in \mathbb{Z}} m_i \delta_{\lambda_m} + a_i \quad (i = 0, \ldots, N) \\
\Box_{\ell} &= \prod_{\ell \cdot m > 0} \ell \cdot m \delta_{\lambda_m} - \prod_{\ell \cdot m < 0} \ell \cdot m \delta_{\lambda_m} \quad (\ell \in \mathbb{Z})
\end{align*}
\] (2.1)

**Proposition 2.1.** For each relative cycle $\mathcal{C}$ on $(\mathbb{P}_\Delta \setminus \{f = 0\}, \mathbb{D}_\Delta \setminus \{f = 0\})$, the function

\[
\mathcal{P}_{\mathcal{C}}(\lambda) = \int_{\mathcal{C}} x^a e^{f(x)} d\log(x)
\] (2.2)

is a (local) solution of (2.1).

Check:. Applying $Z_i$ to $\mathcal{P}_{\mathcal{C}}$ gives

\[
\int_{\mathcal{C}} \sum_{m \in \mathbb{Z}} m_i \delta_{\lambda_m} x^a e^{f} d\log(x) = \int_{\mathcal{C}} d\left[\sum_{m \in \mathbb{Z}} m_i \delta_{\lambda_m} x^a e^{f} d\log(x)\right] = 0,
\]

while applying $\Box_{\ell}$ yields

\[
\int_{\mathcal{C}} x^a (\sum_{m : \ell \cdot m > 0} \ell \cdot m \delta_{\lambda_m} - \sum_{m : \ell \cdot m < 0} \ell \cdot m \delta_{\lambda_m}) e^f d\log(x) = 0.
\]

The solutions are (analytically) local because the cycles $\mathcal{C}$, and hence, their period integrals $\mathcal{P}$ have monodromy about divisors in $\mathbb{A}|_{\mathcal{C}}$.

Since the corresponding $D = \mathbb{C}[\lambda, \partial_{\lambda}]$-module

\[
\tau^\Delta_{\text{GKZ}} = D / D\langle \{Z_i\}, \{\Box_{\ell}\} \rangle
\] (2.3)

is holonomic [1, Theorem 3.9], the (local) solutions module

\[
\text{Hom}_D(\tau, \mathcal{O}_{\Delta}) \simeq \text{Hom}_C(\mathbb{C}_{\lambda} \otimes \mathbb{C}[\lambda]) \tau, \mathbb{C}
\] (2.4)

at a point $\lambda^0 \in \mathbb{C}|_{\mathcal{C}}$ is finite-dimensional. We shall think of (2.3) and (2.4) as “cohomology” and “homology,” respectively, motivated by the parametrization of solutions by relative cycles; this will be made more precise in §2.3.

### 2.2 Periods and residues

Given $m \in \mathbb{Z}^{N+1}$, write $\deg(m) =: k$ for the minimal $k \in \mathbb{Z}$ such that $k \Delta \ni m$. The ring $R = \mathbb{C}[\Delta][x^m]$, its Jacobian ideal $J_f = (\partial_{x_i} f)^N_{i=0}$, and the Jacobian ring $R/J_f$ are thereby graded by degree. Moreover, sending $p(x) \mapsto p(x) x^a e^{f(x)} d\log(x)$ induces a grading on $\tau^\Delta_{\text{GKZ}}$ and a graded
isomorphism

\[
\text{gr}(R/J_f) \xrightarrow{\cong} \text{gr}(\tau_{\Delta}^{\text{GKZ}}).
\] (2.5)

Specializing \( \lambda \) to a very general point \( \lambda^0 \) (and hence \( R \) to \( R^0 = \mathbb{C}[x^{y^0}] \)), the graded pieces have dimensions

\[
\text{dim}_\mathbb{C}(R^0/J_f)(k) = \sum_{j=0}^{N+1} (-1)^j \binom{N+1}{j} \text{dim}(R^0_{(k-j)})
\] (2.6)

(where \( \text{dim}(R^0_{(k-j)}) \) counts the points of degree \( k - j \) in \( M \)) with sum over \( k \)

\[
\text{dim}_\mathbb{C}(R^0/J_f) = (N + 1)! \text{vol}(\Delta).
\] (2.7)

See [1, (5.3) and Corollary 5.11].

Irregular case in mirror symmetry:

A polytope \( \Delta \subset \mathbb{R}^n \) with integer vertices is reflexive if and only if its polar polytope \( \Delta^c \) also has integer vertices; this implies that both have 0 as unique interior integer point. Fixing such a \( \Delta \), take \( \Delta := \Delta \) and \( \alpha = 0 \). (Note that \( N = n - 1 \).) Then we have a graded isomorphism of A- and B-model D-modules

\[
\text{QH}^*(\mathbb{P}_{\Delta^c}) \cong \text{gr}_{\text{GKZ}} \tau_{\Delta}
\] (2.8)

with the grading by \( \text{deg}_2 \) on the left-hand side (see [27]).

**Example 2.2.** Let \( \Delta \) be the triangle in the figure. Choose \( \lambda \) so that the cycle \( \mathbb{T}^2 \cong S^1 \times S^1 \) given by \( |x_1| = |x_2| = 1 \) avoids the zero locus of \( f = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_1^{-1} x_2^{-1} \), so that \( \mathcal{C} = \mathbb{T}^2 \) is a relative cycle. By (2.6)–(2.7), the rank of \( \tau \) is 3, with three graded pieces each of rank 1. Computing the period in (2.2) now gives

\[
\frac{1}{(2\pi i)^2} \mathcal{P} = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} e^f \text{dlog}(x) = \frac{1}{(2\pi i)^2} \sum_{n \geq 0} \frac{1}{n!} \int_{\mathbb{T}^2} f^n \text{dlog}(x) = \sum_{m \geq 0} \frac{(\lambda_1 \lambda_2 \lambda_3)^m}{(m!)^3},
\]

which is an irregular/exponential period. In particular, we see that \( \tau_{\Delta}^{\text{GKZ}} \) does not underlie a classical VHS or VMHS.
Regular case in mirror symmetry

With $\Delta$ as above, take $\Delta \subset \mathbb{R}^{1+n}$ to be the convex hull of the origin and $\{1\} \times \Delta$ (so that now $N = n$), and put $\varrho := (1,0)$. We denote the resulting GKZ system by $\hat{\tau}_GKZ^{\Delta}$. It has the same rank as $\tau_GKZ^{\Delta}$ since $\text{vol}(\Delta) = \frac{1}{(n+1)!} \text{vol}(\Delta)$. Rather than being isomorphic, the two are related (roughly) by Fourier–Laplace transform; and (as will be explained in §2.3) we have an isomorphism of $D$-modules

$$QH^{*+2}(K_{\Delta}^\bullet) \cong \text{gr}_{\hat{\tau}_GKZ^{\Delta}}$$

where $K_{\Delta}^\bullet$ is the total space of the canonical line bundle on $\mathbb{P}_\Delta^\bullet$.

Now let $\phi_\Delta(x) = \sum_{m \in \Delta \cap \mathbb{Z}^n} \lambda^m x^m$ be a general Laurent polynomial on $\Delta$ and $\Gamma$ a relative $n$-cycle in $(\mathbb{P}_\Delta \setminus \{\phi = 0\}, \mathbb{D}_\Delta \setminus \{\phi = 0\})$. With $f = x_0 \phi(x)$ and $\mathcal{C} = \mathbb{R}^n \times \Gamma$, the periods in (2.2) take the form

$$\mathcal{P} = \int_{\mathcal{C}} x_0 e^{f} \frac{dx_0}{x_0} \wedge \text{dlog}(x) = \int_{\Gamma} \left( \int_{-\infty}^{0} e^{x_0 f} dx_0 \right) \text{dlog}(x) = \int_{\Gamma} \frac{\text{dlog}(x)}{\phi(x)} = 2\pi i \int_{\gamma} \text{Res}_{\phi=0} \left( \frac{\text{dlog}(x)}{\phi(x)} \right)$$

if $\Gamma = \text{Tube}(\gamma)$ for $\gamma \subset \{\phi = 0\}$. In particular, these are (regular) periods of a variation of mixed Hodge structure.

**Example 2.3.** With $\Delta$ as in Example 2.2, $\Delta$ is the tetrahedron in the figure. Taking $\Gamma = \mathbb{T}^2$ in (2.10) and writing $t = \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_0}$, Cauchy residue gives for $|t| < \frac{1}{27}$

$$\frac{1}{(2\pi i)^2} \mathcal{P} = \int_{\mathbb{T}^2} \frac{\text{dlog}(x) / (2\pi i)^2}{\lambda_0 + (\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_1^{-1} x_2^{-1})} = \frac{1}{\lambda_0} \sum_{m \geq 0} \frac{(3m)!}{(m!)^3} t^m.$$

Since $\text{rk}(\hat{\tau}_GKZ^{\Delta}) = 3$, one expects three distinct periods related to the geometry of the family of elliptic curves $E_t = \{\phi_\Delta(x) = 0\}$, which has a type $I_9$ singular fiber at $t = 0$.

Fix $\lambda_0 = 1$. If $\{\alpha, \beta\}$ is a symplectic basis for $H_1(E_t)$, with $\alpha$ vanishing at $t = 0$, we can take $\Gamma$ to be $\text{Tube}(\alpha) \simeq \mathbb{T}^2$ ($\mathcal{P}$ holomorphic in $t$), $\text{Tube}(\beta)$ ($\mathcal{P} \sim \frac{9}{2\pi i} \log^2(t)$), or $\sigma := \mathbb{R}_- \times \mathbb{R}_-$ ($\mathcal{P} \sim \frac{9}{2(2\pi i)^2} \log^2(t)$). Note that only the first two are “periods of $E_t$.”

### 2.3 Mixed Hodge theory of GKZ

Continuing with the “regular case” above, and recalling that $D_\Delta := \mathbb{P}_\Delta \setminus G_m^n$, we set $X_\Delta := \{\phi_\Delta(x) = 0\}$ and $\partial X_\Delta := X_\Delta \cap D_\Delta$. By Proposition 2.1, we know that at least some solutions of $\hat{\tau}_GKZ^{\Delta}$ are parametrized by the choice of $\Gamma \in H_n(\mathbb{P}_\Delta \setminus X_\Delta, D_\Delta \setminus \partial X_\Delta)$, which (as a best-case scenario) suggests the following.
Theorem 2.4 [25]. We have a canonical isomorphism
\[
\hat{\tau}_\Delta^{GKZ} \cong H^n(\mathbb{P}_\Delta \setminus X_\Delta, D_\Delta \setminus \partial X_\Delta), \tag{2.11}
\]
in which the $D$-module structure on the RHS is defined by the Gauss–Manin connection.

The connection to mirror symmetry is amplified by the following.

Theorem 2.5 (Conjectured by [29], proved by [23] ($n = 3$) and [42]).
\[
\dim \text{Gr}^{n-k} F H^n(\mathbb{P}_\Delta \setminus X_\Delta, D_\Delta \setminus \partial X_\Delta) = \dim \mathcal{H}^k, k(\mathbb{P}_\Delta^\circ) = \dim \mathcal{H}^{k+1}, k+1 c(K \mathbb{P}_\Delta^\circ).
\]

This refines (2.11) into a graded isomorphism strongly reminiscent of Griffiths's residue theory [18], with $\text{gr}_k$ (resp. multiplication by $x_0 x^m$ as a map from $\text{gr}_k \rightarrow \text{gr}_{k+1}$) on the left matching $\text{Gr}^{n-k}_F$ (resp. $\nabla_{\partial_m} : \text{gr}^{n-k} \rightarrow \text{gr}^{n-k-1}$) on the right.

However, the RHS of (2.11) is a (variation of) mixed Hodge structure, with a nontrivial weight filtration. While intersection theory on the A-model $K \mathbb{P}_\Delta^\circ$ allows us to compute a basis of solutions to GKZ via mirror symmetry (Theorem 2.6 below), it is unclear how to see the weight filtration directly in these terms. To elaborate, we pose two questions.

1) How might one isolate the highest weight part $\text{Gr}^{W}_{n+1}$ of (2.11) (i.e., $H^{n-1}(X_\Delta)$) within the setting of GKZ solutions?

Under mirror symmetry, we have the correspondences:

- $m \in \mathcal{M}$ $\longleftrightarrow$ divisors $[D_m] \in H^2(\mathbb{P}_\Delta^\circ);$  
- relations $\ell \in \mathcal{L} \longleftrightarrow$ curves $[C_\ell] \in H_2(\mathbb{P}_\Delta^\circ);$ and  
- Mori cone $L_{\geq 0} \subset \mathcal{L} \longleftrightarrow$ effective curve classes.

We assume that $L_{\geq 0}$ is simplicial with basis $\{ L^{(i)} \}$, and put $t_i := \lambda_i^{\tau(i)}$ (e.g., $t = \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_0}$ above), $\tau_i := \frac{\log(t_i)}{2\pi i}$. The isomorphism class of $X_\Delta (=: X_\Delta)$ depends only on $t$.

Theorem 2.6 [24]. The (C-linear combinations of) periods $\mathcal{P}$ of $\hat{\tau}_\Delta^{GKZ}$ are the (C-linear combinations of) coefficients of cohomology classes in
\[
\mathcal{B}_\Delta := \sum_{\ell \in L_{\geq 0}} \prod_{m : \ell = m} \prod_{m : \ell' = 0} \prod_{m : \ell'' = 0} \prod_{m \neq n} \prod_{m \neq m} \prod_{m \neq m} (D_m - 1) \cdots (D_0 + \ell') \lambda_{\ell', \ell} D^{\ell+d} \in H^n(\mathbb{P}_\Delta^\circ) \otimes \mathbb{C}[[t]][\tau].
\]

Conjecture 2.7 (Hyperplane conjecture [24, 39]). The periods of (\nabla-flat sections of) $H^{n-1}(X_\Delta)$ are the (C-linear combinations of) coefficients of cohomology classes in $\mathcal{B}_\Delta \cup [X^\circ]$, where $X^\circ \subset \mathbb{P}_\Delta^\circ$ is an anticanonical hypersurface.

Example 2.8. With $\Delta$ as in Examples 2.2–2.3, we have $\mathbb{P}_{\Delta^\circ} = \mathbb{P}^2$, $[X^\circ] = 3[H]$ (for $H$ a hyperplane in $\mathbb{P}^2$), and
\[
\mathcal{B}_\Delta = [1](\text{holo. period}) + [H](\text{log period}) + [H]^2(\text{log}^2 \text{ period}).
\]
In this case,

\[ \mathcal{B}_\Delta \cup [X^\circ] = [H](\text{holo. period}) + [H]^2(\text{log period}), \]

and so, the hyperplane conjecture correctly asserts that the holomorphic and log periods are the actual periods of \( H^1(E_i) \).

(2) Can we compute the remaining GKZ periods, especially those which yield extension classes of \( G_{n+1}^W \) by other weight-graded pieces?

Here “compute” means using the A-model. We know at present of no (even conjectural) intrinsic A-model description of the full weight filtration. An extrinsic one, which we shall now sketch, was obtained in [7] by presenting \( \mathbb{P}_\Delta \) as the large-fiber-volume limit of compact elliptically fibered Calabi–Yau \((n + 1)\)-folds. (Though [7] treats the case \( n = 2 \), this works in general.) To obtain these families of higher dimensional CYs, let \( \mathcal{O} \subset \mathbb{R}^2 \) be the convex hull of \( \{(-1, 1), (-1, -1), (2, -1)\} \), and \( \Delta \subset \mathbb{R}^{n+2} \) be the convex hull of \( \Delta \times (-1, -1) \) and \( 0 \times \mathcal{O} \). There are torically induced morphisms \( \mathbb{P}_\Delta \rightarrow \mathbb{P}_\Delta \) and \( \mathbb{P}_\Delta \rightarrow \mathbb{P}_\Delta \) that restrict to elliptic fibrations on anticanonical (CY-)hypersurfaces \( \mathbb{X}, \mathbb{X}^\circ \).

In particular, write \( \mathbb{X}_{l,s} \) for the closure of the zero locus of \( \Phi(x, u, v) := a + bu^2v^{-1} + cu^{-1}v^{-1} + \phi_j(x)u^{-1}v^{-1} \), where \( s := \frac{\lambda_0b^2c^3}{a^6} \). Instead of the large complex-structure limit \( (t \rightarrow 0 \quad \text{and} \quad s \rightarrow 0) \), we take only \( s \rightarrow 0 \). This has the effect of degenerating the generic fiber of \( \mathbb{X}_{l,s} \rightarrow \mathbb{P}_\Delta \) and decompactifying that of \( \mathbb{X}^\circ \rightarrow \mathbb{P}_\Delta \), resulting in the diagram

\[
\begin{array}{c}
\mathbb{P}_\Delta^* \\
\uparrow_i \\
\mathbb{X}^\circ \subset \mathbb{P}_\Delta^* \\
\mathbb{P}_\Delta \\
\uparrow_i \\
\mathbb{X}_{l,s} \\
\end{array}
\]

with solid arrows labeled by generic fiber type. The singular CY \( \mathbb{X}_{l,0} \) has the Hori–Vafa model

\[ Y_L := \mathbb{X}_{l,0} \cap (G_m^n \times \mathbb{A}^3) = \{\phi_j(x) + uv = 0\} \]

a smooth noncompact CY \((n + 1)\)-fold, as a Zariski open subset, and one has the

**Theorem 2.9** [7, 14]. There are isomorphisms of \( \mathbb{Q} \)-VMHS

\[ H^{n+1}(\mathbb{X}_{l,0}) \cong H_{n+1}(Y_L)(-n - 1) \cong H^n(\mathbb{P}_\Delta \setminus X_L, \mathcal{D}_\Delta \setminus \partial X_L). \]

Now by Theorem 2.4, the RHS of (2.13) identifies with \( \mathbb{Q}_\Delta^{\text{GKZ}} \). On the other hand, Iritani’s results [26] on \( \hat{\Gamma} \)-integral structure allow us to explicitly compute the LHS of the isomorphism

\[ \text{QH}^{\text{even}}(\mathbb{X}^\circ) \cong H^{n+1}(\mathbb{X}_{l,s}) \]

of A- and B-model Z-VHS. Taking LMHS on both sides as \( s \rightarrow 0 \)

\[ \psi_H \text{QH}^{\text{even}}(\mathbb{X}^\circ) \cong \psi_H H^{n+1}(\mathbb{X}_{l,s}), \]
the (unipotent) monodromy invariants

\[ QH^*_c((+2)(K_{\mathbb{P}_{\Delta^o}}) \cong H^{n+1}(\hat{X}_{s,0}) \tag{2.16} \]

must agree as \( \mathbb{Q} \)-VMHS.

**Example 2.10.** For \( \mathbb{P}_{\Delta^o} = \mathbb{P}^2 \), the Hodge–Deligne diagrams\(^\dagger\) for (2.14)–(2.16) are

Here, \( n = 2 \) and \( h^{2,1}(\hat{X}_{s,0}^o) = 2 \), while each of the \( Gr^k \) \((k = 0, 1, 2)\) in Theorem 2.5 (visible in the right-most diagram) has rank 1.

The upshot is that we recover the isomorphism \( QH^*_c((+2)(K_{\mathbb{P}_{\Delta^o}}) \cong \hat{\tau}_{\Delta} \) claimed in (2.9), while promoting it to an isomorphism of \( \mathbb{Q} \)-VMHS. Moreover, we obtain the promised A-model description of the weight filtration on \( \hat{\tau}_{\Delta, \text{GKZ}} \) as the monodromy weight filtration \( M_* = W(N)[-n-1] \), on LHS(2.16) \( \subset \) LHS(2.15). We may therefore use [26] to compute \( W_{\ast}, \hat{\tau}_{\Delta, \text{GKZ}} \), and the associated “mixed-\( \mathbb{Q} \)-periods,” in terms of the intersection theory of \( \mathbb{P}_{\Delta^o} \) and Gromov–Witten theory of \( \hat{X}^o \), restricted to classes of curves whose volume remains finite in the \( s \to 0 \) limit. This boils down to intersection theory and local GW theory of \( \mathbb{P}_{\Delta^o} \). (The reader who wants to see this worked out in detail in some \( n = 2 \) cases may consult [7].)

So far, we have said nothing about the extensions of MHS in (2.11) which these mixed periods are supposed to help us compute. (For instance, the right-hand term of Example 2.10 can be viewed as the dual of the extension associated to the regulator of a family of \( K_2^\text{alg} \) classes on the family \( E_t \) of elliptic curves.) The analysis of these VMHS undertaken in §§3–4 works in a “more general” setting that allows us to drop the genericity assumption on \( \phi \).

### 3 | SPECIAL LAURENT POLYNOMIALS

#### 3.1 | Landau–Ginzburg models

Instead of starting with a reflexive polytope and letting \( \phi \) vary over the corresponding parameter space minus discriminant locus, we begin by fixing a Laurent polynomial \( \phi \in \mathbb{C}[x_{1}^{\pm1}, \ldots, x_{n}^{\pm1}] \). We assume that its Newton polytope \( \Delta \) (the convex hull of those \( \{m\} \) for which \( x^m \) has nonzero coefficient) is reflexive; and fixing a maximal projective triangulation \( \text{tr}(\Delta^c) \), we also assume that the associated toric \( n \)-fold \( P_{\Delta} := P_{\Sigma(\text{tr}(\Delta^c))} \) is smooth.\(^\ddagger\) Write \( D_{\Delta} := P_{\Delta} \setminus G_n^\text{in} \) as before, \( X_t \subset P_{\Delta} \) for the Zariski closure of \( \{1 = t\phi(x)\} \), and \( Z := D_{\Delta} \cap X_0 \) for the base locus of the resulting pencil.

As in [16, Theorem 4], we may fix a sequence of blowups of \( P_{\Delta} \) typically along successive proper transforms of components of \( Z \) with composition \( \beta : X \to P_{\Delta} \), such that:

- \( X \) is smooth;

\(^\dagger\) The number of dots in the \((p, q)\) spot represents \( h^{p,q} \) of the given MHS.

\(^\ddagger\) We will write \( P_{\Delta}^\text{alg} := P_{\Sigma(\Delta^c)} \) for the singular toric \( n \)-fold (of which \( P_{\Delta} \) is a blowup).
\[ \frac{1}{\phi(x)} \] extends to a holomorphic map \( \pi : \mathcal{X} \to \mathbb{P}^1 \);

- \( \mathcal{X} \setminus \pi^{-1}(0) \) contains \( G^m_n \) as a Zariski open subset; and

- \( \text{dlog}(x) := \frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_n}{x_n} \) extends to a holomorphic \( n \)-form on \( \mathcal{X} \setminus \pi^{-1}(0) \).

We shall assume that \( \beta \) may be chosen in such a way that this extended form is nowhere vanishing, so that the \( \tilde{X}_t := \pi^{-1}(t) \) are Calabi–Yau for \( t \) not in the discriminant locus \( \Sigma \). This is weaker than assuming \( \phi \) “generic,” and implies that \( \beta_t := \beta|_\tilde{X}_t : \tilde{X}_t \to X_t \) is a crepant resolution for \( t \not\in \Sigma \) (and also, that \( \mathcal{X} \setminus \tilde{X}_0 \) is a log Calabi–Yau variety). Despite the notation, \( \tilde{X}_t \) is not smooth for \( t \in \Sigma \).

**Definition 3.1.**

(a) The compact *LG-model* associated to \( \phi \) is the family \( \pi : \mathcal{X} \to \mathbb{P}^1_t \) of CY \((n - 1)\)-folds \( \tilde{X}_t \) just constructed. We may view its total space \( \mathcal{X} \) as a smooth compactification of the pencil \( \{1 = t\phi(x)\} \subset C^m_n \times (\mathbb{P}^1_t \setminus \{0\}) \), and \( \tilde{X}_0 \) as a blowup of \( D_\Delta \).

(b) The (noncompact) *LG-model* associated to \( \phi \) is the restriction \( \mathcal{X} \setminus \tilde{X}_0 \to \mathbb{P}^1_t \setminus \{0\} \) of \( \pi \).

**Example 3.2.** Here are some Laurent polynomials in two variables (for \( n = 3 \), see §§5.3–5.5), together with the Kodaira types of the singular fibers of \( \pi \) (first and last at \( t = 0 \), resp., \( \infty \)):

| \( i \) | \( \Delta^{(i)} \) | \( \phi^{(i)} \) | Singular fibers |
|-------|-----------------|----------------|----------------|
| 1     |                 | \( x + y + \frac{1}{xy} \) | I_9, I_1, I_1, I_1(, I_0) |
| 2     |                 | \( 16x + y - 3xy \) | I_8, I_1, I_1, II |
| 3     |                 | \( \frac{(1-x)(1-y)(1-x-y)}{xy} \) | I_5, I_1, I_1, I_5 |
| 4     |                 | \( \frac{x^2 + y}{y} \) | I_6, I_1, ..., I_1, I_1 |
|       |                 | \( \frac{1}{3x^2 + 3y} \) | 5 times |
| 5     | \( \frac{1}{xy} F_3 \) | \( F_3 \) a general cubic | I_3, I_1, ..., I_1(, I_0) |
|       |                 | \( \frac{(1+x+y)^3}{xy} \) | 9 times |
| 6     |                 |                   | I_3, I_1, IV* |
For instance, the last two share the same \( \mathbb{P}_\Delta \cong \mathbb{P}^2 \), but have different \( \mathcal{X}'s \): obtained by blowing up at nine distinct points (for the general cubic), versus blowing up three times at each of three points.

### 3.2 Variation of Hodge structure

On a neighborhood of \( t = 0 \), consider the family of vanishing \((n - 1)\)-cycles \( \gamma_t \) on \( \tilde{X}_t \) whose image under \( \text{Tube} : H_{n-1}(\tilde{X}_t, \mathbb{Z}) \to H_n(\mathcal{X} \setminus \tilde{X}_t, \mathbb{Z}) \) is \( [\beta^{-1}(\mathbb{T}^n)] \), where \( \mathbb{T}^n := \cap_{i=1}^n \{ |x_i| = 1 \} \). The family of holomorphic forms

\[
\omega_t := \frac{1}{(2\pi i)^{n-1}} \text{Res}_{\tilde{X}_t} \left( \frac{d\log(x)}{1 - t\phi(x)} \right) \in \Omega^{n-1}(\mathcal{X})
\]

(3.1)

then has the holomorphic period

\[
A(t) := \int_{\gamma_t} \omega_t = \frac{1}{(2\pi i)^n} \oint_{\mathbb{T}^n} \frac{d\log(x)}{1 - t\phi(x)} = \sum_{k \geq 0} a_k t^k,
\]

(3.2)

where \( a_k = [\phi^k]_0 \) are the constant terms in powers of \( \phi \).

Writing \( \pi_U : \tilde{X}_U \to U \) for the restriction of \( \pi \) over \( U' := \mathbb{P}^1 \setminus \Sigma \), the local system \( \mathcal{H}^{n-1} := R^{n-1}(\pi_U)_* \mathbb{Q} \) has maximal unipotent monodromy\(^\dagger\) at \( t = 0 \). It underlies a (polarized) VHS with sheaf of holomorphic sections \( \mathcal{H}^{n-1} \cong \mathcal{H}_U^{n-1} \otimes \mathcal{O}_U \) and Gauss–Manin connection \( \nabla \). In fact, we will work with the sublocal-system \( \mathcal{H}_U^{n-1} \) orthogonal to the fixed part \( \mathcal{H}_f^{n-1} = H^0(U, \mathcal{H}^{n-1}) \). The corresponding sub-VHS \( \mathcal{H}_U^{n-1} \subseteq \mathcal{H}^{n-1} \) contains the Hodge line \( \mathcal{H}^{n-1,0} = (\pi_U)_* \Omega^{n-1}_{\tilde{X}_U} \). On the level of \( d\pi_U \)-closed-form representatives, the polarization \( \langle , \rangle : \mathcal{H}_U^{n-1} \times \mathcal{H}_U^{n-1} \to \mathcal{O} \) is simply given by \( \langle \omega, \eta \rangle = \int_{\tilde{X}_t} \omega_t \wedge \eta_t \).

For simplicity, we shall henceforth assume that \( \mathcal{H}_U^{n-1} \) is irreducible, not just as a \( \mathbb{Q} \)-VHS but as a \( \mathbb{C} \)-VHS.\(^\ddagger\) Let \( L \in \mathbb{C}[t, \delta_t] \) be the differential operator with \( (\mathcal{H}_U^{n-1}, \nabla) \cong D/\sqrt[\delta_t]{L} \), of degree \( d \) and order \( r = \text{rk}(\mathcal{H}_U^{n-1}) \), normalized so that the coefficient of \( \delta_t^r \) is \( 1 \) at \( t = 0 \).

A putative mirror to the LG-model is given by the following folklore.

**Conjecture 3.3.** There is a weak Fano \( n \)-fold \((X^\circ, \omega)\), determined by the triple \((\mathbb{P}_\Delta, \mathcal{D}_\Delta, Z)\) and admitting a toric degeneration to \( \mathbb{P}^n_{\Delta^\circ} \), from which one may recover \( \mathcal{H}_U^{n-1} \). (In particular, for generic \( \phi \), we have \( X^\circ = \mathbb{P}_{\Delta^\circ} \).)

Conversely, it is hoped that by studying “special” Laurent polynomials and classifying the associated local systems, one obtains a classification of weak Fano varieties admitting a toric degeneration. While Conjecture 3.3 is vague as stated, it will be refined below: a mechanism for recovering \( \mathcal{H}_U^{n-1} \) (in some cases) is given in Conjecture 3.3 bis; while the Hodge-theoretic sense in which \( X^\circ \) is mirror to \( \mathcal{X} \setminus \mathcal{X}_0 \to \mathbb{A}^1 \) is the subject of Conjecture 4.2.

\(^\dagger\) In this paper, a unipotent monodromy operator \( T = e^N \) is maximally unipotent if \( N^{n-1} \neq 0 \).

\(^\ddagger\) When \( n = 3 \), for instance, this assumption rules out (finite) monodromy in \( H^2_{\text{alg}} \); for \( n = 2 \), it is vacuous.
3.3 Quantum \( D \)-module

For simplicity, we assume in this subsection that the Picard rank \( \rho(X^\circ) = 1 \) (and \( X^\circ \) is Fano). In the standard way [19], one uses genus-zero Gromov–Witten theory to construct a quantum product ‘\( \star \)’ on \( H^\ast(X^\circ) \otimes \mathbb{C}[s^{\pm 1}] \). This is endowed with a \( D \)-module structure by letting \( \delta_s \) act via 

\[
1 \otimes \delta_s - (K_{X^\circ} \star) \otimes 1.
\]

Now let \( H^{n-1}_v \) be as in Conjecture 3.3, and \( L \) be the corresponding Picard–Fuchs equation, with Fourier–Laplace transform \( \hat{L} \). (Here we recall that the Fourier-Laplace (FL) transform and its inverse are given on functions/solutions \( f \) by 

\[
\hat{f}(s) := \frac{1}{2\pi i} \int f(t)e^{st} \frac{dt}{t} \quad \text{and} \quad \tilde{F}(t) := \frac{1}{t} \int_0^{\infty} F(s)e^{-st} ds, \tag{3.3}
\]

and on operators by replacing \( \delta_t \leftrightarrow -s \) and \( t \leftrightarrow \delta_s \).) Then we have the following amplification of that conjecture when \( \omega = -K_{X^\circ} \):

Conjecture 3.3 (bis.). As \( D \)-modules, \( H^\ast(X^\circ) \otimes \mathbb{C}[t^{\pm 1}] \cong \mathbb{R}^\ast \otimes \mathbb{Z}^\ast_L \).

That is, by applying inverse FL transform to \( \hat{L} \), we should obtain \( \mathbb{R}^n \).

Remark 3.4. Conjecture 3.3 (bis) still makes sense without the assumption that \( \rho(X^\circ) = 1 \), by pulling back the quantum \( D \)-module via the anticanonical cocharacter of the Néron–Severi torus. It has been checked for an enormous number of Fano varieties of dimension 2 and 3 [9]. A more recent general result affirms it for Fano varieties admitting a toric degeneration with terminal singularities (and preserving \( H_2 \)), in which case the nonzero integer points of \( \Delta \) are vertices and \( \phi = \sum_{m \in \Delta \cap \mathbb{Z}^n} \chi_m^\ast \) [2, Corollary 4.7].

Assuming Conjecture 3.3 (bis) holds for \( X^\circ \), we write (\( \hat{L} = \sum_{i,j} \beta_{ij} t^i \delta_t^j \)) for the (irregular) differential operator killing the generator \( 1 \otimes 1 \) of the quantum \( D \)-module, and convert this into an (irregular) quantum recursion

\[
\hat{R} : \sum_{i,j} \beta_{ij} (k-i)^j \hat{u}_{k-i} = 0 \quad (\forall k) \tag{3.4}
\]

by applying \( \hat{L} \) to a power series \( \sum_k \hat{u}_k s^k \). We consider a basis of solutions \( \{\hat{u}_i\}_{i=0}^{d-1} \), defined over the same field as \( L \) (typically \( \mathbb{Q} \)), with \( \hat{u}_i = 0 \) for \( k < i \) and \( \hat{u}_i = 1 \). Regularizing via the inverse FL transform, \( \hat{L}, \hat{R} \) become \( L, R \), with solutions \( u_k = k! \hat{u}_k \); in particular, we have \( u_k^{(i)} = 0 \) for \( k < i \) and \( u_k^{(i)} = 1 \).

We shall take the basis to be chosen so that \( u_k^{(0)} = a_k \) is as in (3.2), and impose one more assumption: that the \( \{a_k\} \) are nonzero. There are various ways to further normalize \( u^{(1)}, ..., u^{(d-2)} \). For instance, there are \( r_0 := rk((\psi_0 H^{n-1}_v)^{\ast} t_0) < d \) independent holomorphic solutions to \( \hat{L}(\cdot) = 0 \) at the origin, which we take to be given by the generating series of \( u^{(0)}, ..., u^{(r_0-1)} \). The remaining

---

\( \dagger \) If \( f(t) = \sum_k c_k t^k \) is a power series, this gives \( \hat{f}(s) = \sum_k \frac{c_k}{k!} s^k \).

\( \ddagger \) If \( (\psi_0 H^{n-1}_v)^{\ast} t_0 \) is Hodge–Tate, with \( r_0 \) distinct graded pieces \( \{\mathbb{Q}(-p_i)\}_{i=0}^{r_0-1} \) (with \( p_0 = 0 \)), one can take \( \sum_k u_k^{(i)} k (i = 0, ..., r_0 - 1) \) to be the \( \mathbb{C} \)-periods of \( \omega \), against local sections \( \varphi^{(i)} \) of \( H^{n-1}_v \otimes \mathbb{C} \) passing through \( \mathbb{C}(-p_i) \) at \( t = 0 \).
$d - r_0$ generating series will then be solutions to inhomogeneous equations $L(\cdot) = g(t) \in \mathbb{C}[t]$.\footnote{Writing $L = \sum_{r=0}^{d} t^{r} P_r(\delta_{i})$, it is reasonable to expect that $P_0(T) = \prod_{i=0}^{r_0-1} (T - i)^{r_0 - 2 P_1}$, and then we may assume that $\sum_{k \geq 1} u_k^{(1)} t^k (i = r_0, \ldots, d - 1)$ solves $L(\cdot) = P_0(i) t^i$.} In particular, it will be important in §5 that when $d = 2$ and $r = n$ ($\Rightarrow P_0(T) = T^n$), $\sum_{k \geq 1} u_k^{(1)} t^k$ solves $L(\cdot) = t$.

Slightly generalizing the definition in [20], we propose the following.

**Definition 3.5.** The Apéry constants of $X^o$ are the limits

$$\alpha(i)^{(i)}_{X^o} := \lim_{k \to \infty} \frac{u_k^{(i)}}{u_0^{(0)}} = \lim_{k \to \infty} \frac{u_k^{(i)}}{u_0^{(0)}},$$

(3.5)

for $1 \leq i \leq d - 1$. (When $d = 2$, we simply write $\alpha_{X^o}$.)

**Remark 3.6.** The closely related definition in [17] (of an Apéry class $A(X^o) \in H^*_{\text{prim}}(X^o)$, with the constants appearing as its coefficients) only considers the first $r_0 - 1$ Apéry constants, corresponding to solutions of the homogeneous equation. (As was first realized by Galkin and Iritani in the case of Grassmannians, these should correspond to the restriction of the regularized gamma class of $X^o$ to the Lefschetz coprimitive part of cohomology.)

However, the focus in [17] is on large-dimensional examples for which $r_0 = d$; taking hyperplane sections preserves $d$ as well as the $\alpha(i)^{(i)}$ (in our sense), even as $r_0$ decreases. Since four of the five 3-dimensional examples we consider in §5 are indeed obtained as multisects of homogeneous Fano varieties with ($\dim H^*_{\text{prim}}(X^o) = r_0 = 2 = d$), the “inhomogeneous” Apéry constants for the threefolds in [20] are connected to the constants in [17] in this way (albeit with a slightly different normalization).

**Remark 3.7.** Adding a constant $c$ to $\phi$ conjugates $\hat{L}$ by $e^{cs}$, which does not affect the Apéry constants. We may thus choose the constant term to make $\phi = 0$ (i.e., $X^\infty$) singular.

By “specializing” Laurent polynomials, we hope not only to classify Fanos but also to arrive at a B-model, Hodge-theoretic interpretation of their Apéry numbers. But there is a new twist. Consider the simplest case, where $d = 2$ and $r_0 = 1$, and write $u_0^{(0)} = a_k, u_1^{(1)} = : b_k = 0, 1, \ldots$, and $\alpha_{X^o} = \lim_{k \to \infty} \frac{b_k}{a_k}$. While $A(t) = \sum_{k \geq 0} a_k t^k$ is just the holomorphic period, the $\{b_k\}$ and hence $\alpha_{X^o}$ are not visible from $H^{n-1}_u$ alone. It is for this reason that we turn to variations of MHS in the next section.

## 4 | HIGHER NORMAL FUNCTIONS

### 4.1 | Variation of mixed Hodge structure

Fix a Laurent polynomial $\phi$ subject to the assumptions in §3.1, and write $X^s_t = X_t \cap G^n_m$ for the level sets of $\frac{1}{\phi}$.
**Proposition 4.1.** Suppose that $\phi$ is “generic” in the sense that it is $\Delta$-regular. Then as MHSs, we have

$$H^n(X \setminus X_0, X_t) \cong H^n(\mathbb{G}_m^n, X^*_t) \quad \text{and} \quad H^n(X \setminus X_t, X_0) \cong H^n(\mathbb{P}_\Delta \setminus X_t, D_\Delta \setminus Z)$$

for $t \neq 0$.

**Proof.** With the additional genericity assumption, the construction of $\mathcal{X}$ in §3.1 proceeds by blowing up $\mathbb{P}_\Delta$ once along each component of $Z$ (or rather, their successive strict transforms). An easy local computation shows that the restriction of $\pi$ to the exceptional divisor $\mathcal{E} \subset \mathcal{X}$ of $\beta$ is then locally constant over $\mathbb{P}^1 \setminus \{0\}$. In particular, writing $\mathcal{E}_t := \mathcal{E} \cap X_t$, $E_0$ is a deformation retract of $\mathcal{E} \setminus E_t$ for any $t \neq 0$.

Since $\mathbb{G}_m^n = X \setminus (X_0 \cup \mathcal{E})$ and $X^*_t = X_t \setminus E_t$, we have

$$H^n(\mathbb{G}_m^n, X^*_t) \cong H^n(X \setminus (X_0 \cup \mathcal{E}), X_t \setminus \mathcal{E}_t) \cong H_n(X \setminus X_t, X_0 \cup (\mathcal{E} \setminus \mathcal{E}_t))(−n)$$

and $H^n(X \setminus X_0, X_t) \cong H_n(X \setminus X_t, X_0)(−n)$, which fit together in the long-exact sequence

$$\to H_n(\mathcal{E} \setminus \mathcal{E}_t, \mathcal{E}_0)(−n) \to H^n(X \setminus X_0, X_t) \to H^n(\mathbb{G}_m^n, X^*_t) \to H_{n−1}(\mathcal{E} \setminus \mathcal{E}_t, \mathcal{E}_0)(−n) \to,$$

whose end terms are zero by the deformation retract property. The other isomorphism follows by Lefschetz duality.

The isomorphisms in Proposition 4.1 typically fail without the genericity assumption on $\phi$; that is, the restriction morphism $H^n(X \setminus X_0, X_t) \to H^n(\mathbb{G}_m^n, X^*_t)$ is not an isomorphism, and the left-hand object better reflects the topology of the LG-model. So, from here on, we shall focus primarily on the VMHS

$$\mathcal{V}_{\phi,t} := H^n(X \setminus X_0, X_t)$$

over $\mathcal{U}^* := \mathbb{P}^1 \setminus \Sigma$, and its dual

$$\mathcal{V}^*_{\phi,t}(−n) \cong H^n(X \setminus X_t, X_0).$$

Of course, if $\phi$ is generic, by Proposition 4.1 and Theorem 2.4, the right-hand term of (4.3) is nothing but a restriction of $\tilde{\tau}_\Delta^{\text{GKZ}}$. This suggests the following generalization of Theorem 2.5.

**Conjecture 4.2** [29]. For $t \notin \Sigma$, we have for each $k$

$$\mathrm{rk}(\mathrm{Gr}^{n−k}_F \mathcal{V}^*_{\phi}(−n)) = \dim(H^k(X^\circ)).$$

**Remark 4.3.**

(i) The form in which we state this conjecture combines [29, Conjecture 3.7] with [23, Theorem 3.1].

(ii) By Serre duality, (4.4) would imply that $\mathrm{rk}(\mathrm{Gr}^{n−k}_F \mathcal{V}_{\phi}) = \mathrm{rk}(\mathrm{Gr}^k_F \mathcal{V}_{\phi})$ for all of our LG-models, which has been proved by Harder [22, Corollary 2.2.7].

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†That is, the intersections of $X_i$ ($t \neq 0$) with each of the torus orbits in $D_\Delta$ are smooth and reduced; this is equivalent to the meaning of genericity in §2.

‡For example, $H^2(\mathbb{G}_m^n, X^*_t)$ does not distinguish between cases $i = 1$ and $i = 6$ in Example 3.2/4.19, and for the nongeneric case $i = 6$ does not agree with $\mathcal{V}_{\phi,t}$. 
(iii) There is a plausible amplification of (4.4) to an equality of $\mathbb{C}$-MHSs with nilpotent endomorphism: on the left-hand (B-model) side, the LMHS $(\psi_t H^n(\mathcal{X} \setminus \tilde{X}_0, \tilde{X}_t), F^\ast_{\lfloor t \rfloor, M, N_0})$ at $t = 0$; on the right-hand (A-model) side, $\bigoplus_k H^{k,k}(X^\circ)$ endowed with the “downward” Hodge filtration $F^p = \bigoplus_{k \leq n-p} H^{k,k}(X^\circ)$, together with $N = \cup [-K X^\circ]$ and its weight filtration $W(N)[-n]$, (centered about $n$).

In order to relate limits of extension classes in (4.2)–(4.3) to Apéry constants of $X^\circ$, we shall need to kill off intermediate extensions that would otherwise “obstruct” these classes. This will be accomplished by placing a “$K$-theoretic” constraint on the Laurent polynomial.

**Definition 4.4.** We say $\phi$ is tempered if the coordinate symbol $\{x_1, \ldots, x_n\} \in H^n_M(\mathbf{G}_m^n, \mathbb{Q}(n))$ lifts to a class in $H^n_M(\mathcal{X} \setminus \tilde{X}_0, \mathbb{Q}(n))$.

Henceforth, we shall mainly be concerned with the case where $\phi$ is tempered. When $n = 2$, this is just the condition that the edge polynomials of $\phi$ be cyclotomic [41]; some methods for checking temperedness for $n = 3, 4$ are given in [14, §3]. Up to scale, tempered reflexive Laurent polynomials are defined over $\bar{\mathbb{Q}}$ [41, Proposition 4.16] and are thereby rigid.

4.2 Admissible and geometric normal functions

A reference for the material that follows is [36, §2.11–12].

**Definition 4.5.** A higher normal function on $\mathcal{U}$ is (equivalently)

(i) a VMHS of the form $0 \to \mathcal{H} \to \mathcal{V} \to \mathbb{Q}(0) \to 0$, or

(ii) a holomorphic, horizontal section $\nu$ of $J(\mathcal{H}) := \mathcal{H}/(F^0 \mathcal{H} + \mathbb{H})$,

where $\mathcal{H}$ is a polarizable VHS of pure weight $-r < -1$.

Here horizontal means that, for each local holomorphic lift $\tilde{\nu}$ to $\mathcal{H}$, we have $\nabla \tilde{\nu} \in F^{-1} \mathcal{H}$.

For instance, given (i), we may locally lift $1 \in \mathbb{Q}(0)$ to $\nu_F \in F^0 \mathcal{V}$ and $\nu_Q \in \mathcal{V}$ (the local system underlying $\mathcal{V}$), and then locally define $\tilde{\nu}$ (hence $\nu$ as in (ii)) by $\tilde{\nu}(\varphi) = \nu_Q - \nu_F$.

Let $\mathcal{V}_e$ denote Deligne’s canonical extension of $\mathcal{V}$ to $\mathbb{P}^1$. Fixing disks $D_\sigma \subset \mathbb{P}^1$ at each $\sigma \in \Sigma(= \mathbb{P}^1 \setminus U)$, with coordinate $t_\sigma$, we write $T_\sigma = e^{N_\sigma} T_{\sigma}^{ss}$ for the monodromy of $\mathcal{V}$ on $D_\sigma^* = D_\sigma \setminus \{\sigma\}$, and $M_\sigma^*$ for the monodromy-weight filtration of the LMHS $\psi_\sigma \mathcal{H}$. Suppose now that there exist “lifts of 1”:

• $\nu_F' \in \Gamma(D_\sigma, \mathcal{V}_e) \rightarrow \text{holomorphic, single-valued, with } \nu_F'|_{D_\sigma^*} \in F^0 \mathcal{V}$

• $\nu_Q' \in \Gamma(D_\sigma^{\text{un}}, \mathcal{V}) T_{\sigma}^{ss} \rightarrow \text{flat, multivalued, with } N_\sigma \nu_Q' \in M_{-2}^{\sigma} \psi_\sigma \mathcal{H}$.

Then we may confer on $\mathcal{V}_e|_{t_\sigma}$ the status of a MHS $\psi_\sigma \mathcal{V}$ as follows:

• the weight filtration $M_\sigma^*$ extends that on $\psi_\sigma \mathcal{H}$, adding $\nu_Q'$ to $M_0^\sigma$;

• the Hodge filtration $F^*_{\sigma}$ extends that on $\psi_\sigma \mathcal{H}$, adding $\nu_F'(\sigma)$ to $F^0_{\sigma}$;

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1 Here, $M_\ast$ is the monodromy weight filtration of $N_0$ relative to $W_\ast$, the weight filtration on the VMHS $H^n(\mathcal{X} \setminus \tilde{X}_0, \tilde{X}_t)$; if this extended form of the conjecture holds, then $M_\ast$ is just $W(N_0)[-n]$. Moreover, if $X^\circ$ is Fano (not just weak Fano), then it implies that this LMHS is Hodge–Tate.
• the \( \mathbb{Q} \)-structure \( (\psi_\sigma V)_\mathbb{Q} \) is easiest to describe after a base change (to kill off \( T^{ss}_\sigma \)), as the specialization of \( \exp(-\frac{\log(t_\sigma)}{2\pi i} N_\sigma)\mathbb{Q} \subset V_\sigma \) at \( \sigma \).

**Definition 4.6.** The HNF \( \nu \) is admissible, written \( \nu \in \text{ANF}(H) \), if this \( \psi_\sigma V \) (equivalently, \( \nu^\sigma_\mathbb{Q} \) and \( \nu^\sigma_F \)) exists at each \( \sigma \in \Sigma \). If, in addition, we may choose \( \nu^\sigma_\mathbb{Q} \) so that \( N_\sigma \nu^\sigma_\mathbb{Q} = 0 \), then the limit \( \lim_\sigma \nu \in J((\psi_\sigma H)^{T_\sigma}) \) is defined; otherwise, \( \nu \) is singular at \( \sigma \).

**Remark 4.7.** Writing \( H = H_f \oplus H_v \) for the decomposition into fixed and variable parts (with \( H_f = H_f \otimes \mathcal{O}_{U'} \)), we claim that

\[
0 \to J(H_f) \to \text{ANF}(H) \to H g(H^1(U', H)) \to 0
\]

is exact. Indeed, since \( \text{ANF}(H) \cong \text{Ext}^1_{\text{AVMHS}(U')}(\mathbb{Q}(0), H) \), this follows at once from the spectral sequence

\[
R\text{Hom}_{\text{MHS}}(\mathbb{Q}(0), -) \circ R\Gamma_{U'} \Rightarrow R\text{Hom}_{\text{AVMHS}(U')}(\mathbb{Q}(0), -)
\]

and triviality of \( \text{Ext}^{i>1}_{\text{MHS}} \) by using the identifications \( H^0(H) = H_f, J(H_f) \cong \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H_f) \), and \( H g(H^1(H)) \cong \text{Hom}_{\text{MHS}}(\mathbb{Q}(0), H^1(H)) \).

We say that \( \nu \in \text{ANF}(H^{2p-r}(p)) \) is of geometric origin when it arises from a motivic cohomology class in

\[
H^{2p-r+1}(\mathcal{X}'_{U'}, \mathbb{Q}(p)) \cong Gr^P_r K^\text{alg}_{r-1}(\mathcal{X}'_{U'})_\mathbb{Q} \cong \text{CH}^p(\mathcal{X}'_{U'}, r-1).
\]

The most convenient representatives are found in the right-hand term, the higher Chow groups of Bloch [4, 5], which (in their cubical formulation) are defined as the \((r-1)\)st homology of a complex \( (\mathbb{Z}^p(X, \bullet), \partial) \) of codim.-p cycles on \( X \times \square \), where \( \square := \mathbb{P}^1 \setminus \{1\} \).

\[\text{Given a cycle } Z \text{ in } (4.6), \text{its restrictions } Z_t \in \text{CH}^p(X_t, r-1) \text{ have (for each } t \in U' \text{) Abel-Jacobi/regulator invariants}
\]

\[
\text{AJ}(Z_t) =: \nu_Z(t) \in J(H^{2p-r}(X_t)(p)),
\]

see Remark 4.9 below. By [8, Theorem 7.3], these glue together into an admissible normal function, so that \( Z \mapsto \nu_Z \) defines a map

\[
\text{AJ}_\phi : \text{CH}^p(\mathcal{X}'_{U'}, r-1) \to \text{ANF}(H^{2p-r}(p)).
\]

Composing with projection to \( \text{ANF}(H^{2p-r}_{0}(p)) \cong H g(H^1(U', H'_{2p-r}(p))) \) (cf. Remark 4.7) defines \( \text{AJ}_\phi^v \) and \( \nu_Z^v \), for which we have the following special case of the Beilinson–Hodge conjecture.

**Conjecture 4.8 (BHC).** For \( \mathcal{X}'_{U'} \) defined over \( \overline{\mathbb{Q}} \), \( \text{AJ}_\phi^v \) is surjective. That is, admissible and geometric HNFs with values in \( H^{2p-r}_{0}(p) \) are the same thing.

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\[1\] \( \lim_\sigma \nu \) is given by \( \nu(\lim_\sigma \nu) = \nu^\sigma_\mathbb{Q} - \nu^\sigma_F(0) \), as in the passage from (i) to (ii) above.

\[2\] Elements of \( Z^p(X, n) \) must meet all faces \( X \times \square^m \) (defined by setting \( \square \)-coordinates to 0 or \( \infty \)) properly, and \( \delta \) is given by an alternating sum of intersections with codim.-1 faces. See [14, §1] for a brief introduction to higher cycles and their Hodge-theoretic invariants.
The equivalence in Definition 4.5, as well as the notion of admissibility, persist with $H$ merely a VMHS; we shall loosely refer to such VMHSs (and the corresponding sections of $J(H)$) in this more general setting as *mixed HNFs*.

**Remark 4.9.** The Abel–Jacobi invariant $\text{AJ}(\mathcal{Z}_t)$ may be computed as the class of a closed $(2p - r)$-current $(2\pi i)^p \Pi_t + \Xi + (2\pi i)^{p-r+1}(\mathcal{Z}_t), R_{r-1}$ on $\check{X}_t$, where $R_{r-1}$ is a standard $(r-2)$-current on $\square^{-1}$, $\Gamma$ is a chain bounding on $(\mathcal{Z}_t)_*(R_{r-1}^{-1})$, and $\Xi \in F^p D^{2p-r}(\check{X}_t)$. (Here $\mathcal{Z}_t$ is thought of as a correspondence on $\check{X}_t \times \square^{-1}$, so that $(\mathcal{Z}_t)_*$ sends $(r-2)$-currents on $\square^{-1}$ to $(2p - r)$-currents on $\check{X}_t$.) One defines these *regulator currents* inductively by

$$R_{\phi}(x_1, ..., x_r) := \log(x_1) \frac{d\gamma_1}{\gamma_1} \wedge ... \wedge \frac{d\gamma_r}{\gamma_r} - 2\pi i \Gamma_{x_1} \cdot R_{r-1}(x_2, ..., x_r),$$

where $\Gamma_{x_1} := x_1^{-1}(R_{<0})$; they satisfy

$$d[R_{\phi}] = d\log(x) - (2\pi i)^p \delta_{\Gamma} + \sum_{i=1}^r (-1)^i R_{r-1}(x_1, ..., \hat{x}_i, ..., x_r) \delta_{(x_i)}. $$

These formulas were first described in [34, §5.3ff], with later amendments in [33, Lemma 8.14] and [38, §7]; the entire story is recounted succinctly in [35, §§3–4] (which is where one should start). Also see [14] ((0.17)ff in Introduction, followed by §1) for a down-to-earth summary with an eye to applications like those considered here.

### 4.3 $\mathcal{V}_{\phi}$ as a (mixed) higher normal function

First, we set (dually)

$$
\begin{aligned}
\mathcal{V}^n_{\phi,t} &:= \ker\{H^n(\Gamma^n_m, X^n_t) \to H^n(\Gamma^n_m)\} \\
\mathcal{V}^{n-1}_{\phi,t} &:= \coker\{H^{n-1}(\Gamma^n_m, X^n_t) \to H^{n-1}(\Gamma^n_m)\}
\end{aligned}
$$

and write $\mathcal{V}^n_{\phi,t} := H^n(\Gamma^n_m, X^n_t)$. Since $X^n_t$ is affine and $H^n(\Gamma^n_m) \cong \mathbb{Q}(-n)$, the relative cohomology sequence of the pair $(\Gamma^n_m, X^n_t)$ yields an exact sequence of MHS

$$0 \to \mathcal{V}^{n-1}_{\phi,t}(X^n_t) \to \mathcal{V}^n_{\phi,t} \xrightarrow{\Xi} \mathbb{Q}(-n) \to 0. \quad (4.10)$$

The kernel and cokernel of the restriction map

$$\theta : \mathcal{V}^n_{\phi,t} \to \mathcal{V}^n_{\phi,t}$$

may be read off from the end terms of (4.1). Since we are not assuming that $\phi$ is $\Delta$-regular, these may no longer be zero, but we claim that they are “mild” in the sense of having weights between $n$ and $2n - 2$. More precisely:

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\[ This last current $\Xi$ is irrelevant for computations, since \( J(H^{2p-r}(\check{X}_t)(p)) = \frac{\langle F^{p-1} H^{2n-p-1+r}(\check{X}_t, \mathbb{Q}(p)) \rangle}{\langle H^{2n-p-1+r}(\check{X}_t, \mathbb{Q}(p)) \rangle} \), and $\Xi$ dies by type against representative forms for $F^{p-1} H^{2n-p-1+r}(\check{X}_t, \mathbb{Q})$. (If $p \geq n$ or $r \geq p$, $\Xi$ is just 0.) \]
Lemma 4.10. The map \( \vartheta \) induces isomorphisms on \( \text{Gr}^W_{2n}(\cong \mathbb{Q}(-n)) \), \( \text{Gr}^W_{2n-1}(\cong \{0\}) \), and \( \text{Gr}^W_{n-1} \) (namely, \( H^{n-1}_v \cong W_{n-1}H^{n-1}(X^*_t) = W_{n-1}H^{n-1}(X^*_t) \)).

Proof. Clearly, \( \text{Gr}^W_{2n} \cong \mathbb{Q}(-n) \) and \( \text{Gr}^W_{2n-1} \cong \{0\} \). Since \( \text{dlog}(x) \) extends to \( \Omega^n(\mathcal{X} \setminus \mathcal{X}_0) \), the composition \( \Xi \circ \vartheta : \mathcal{V}_{\phi,t} \to H^n(\mathbb{G}_m^n) \) is surjective; and by Proposition 4.16 and Remark 4.17 below, \( \mathcal{V}_{\phi,t} := \ker(\Xi \circ \vartheta) \) has weights in \([n-1, 2n-2]\). Notice that we can write (4.2) as an extension

\[
0 \to \mathcal{V}_{\phi,t} \to \mathcal{V}_{\phi,t} \overset{\Xi \circ \vartheta}{\to} \mathbb{Q}(-n) \to 0 \quad (4.11)
\]

in analogy to (4.10).

For \( \text{Gr}^W_{n-1} \), we have \( W_{n-1} \mathcal{V}_{\phi,t} = W_{n-1} \overline{H}^{n-1}(X^*_t) = W_{n-1}H^{n-1}(X^*_t) \) by (4.9) and (4.10), and \( W_{n-1}V_{\phi,t} = \text{coker}[H^{n-1}(\mathcal{X} \setminus \mathcal{X}_0) \to H^{n-1}(X^*_t)] = H^{n-1}_v \) by the global invariant cycle theorem. Clearly, \( \vartheta \) restricts to a map \( H^{n-1}_v \to \overline{H}^{n-1}(X^*_t) \), which surjects onto the \( W_{n-1} \) part by standard mixed Hodge theory, and is injective by the assumed irreducibility of \( H^{n-1}_v \). \( \square \)

Let \( \Theta : \mathcal{V}_{\phi,t} \to \overline{H}^{n-1}(X^*_t) \) be the map induced by \( \vartheta \), so that (4.11) \( \mapsto (4.10) \) in

\[
\Theta_* : \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(-n), \mathcal{V}_{\phi,t}) \to \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(-n), \overline{H}^{n-1}(X^*_t)) \quad (4.12)
\]

That is, we may view \( \mathcal{V}_{\phi} \) and \( \mathcal{V}_{\phi}^* \) as mixed HNFs related by \( \Theta_* \), with Hodge–Deligne diagrams both of the form on the left below:

**Proposition 4.11.** \( \mathcal{V}_{\phi}^* \) is the (mixed) geometric HNF associated to the coordinate symbol \( \{x\} = \{x_1, \ldots, x_n\} \in \text{CH}^n(\mathbb{G}_m^n, n) \). Moreover, if \( \phi \) is tempered, then \( \mathcal{V}_{\phi} \) and \( \mathcal{V}_{\phi}^* \) have a common (pure) geometric sub-HNF with Hodge–Deligne diagram of the form shown on the right, associated to the lift of \( \{x\} \) to \( \text{CH}^n(\mathcal{X} \setminus \mathcal{X}_0, n) \) (cf. Definition 4.4).

Proof. Each \( \gamma \in H_{n-1}(X^*_t, \mathbb{Q}) \) may be written as \( \partial \mu \) for a \( n \)-chain \( \mu \) on \( \mathbb{G}_m^n \). The extension class \( \nu_{\phi}(t) \) of (4.10) in

\[
J(\overline{H}^{n-1}(X^*_t)(n)) \cong \text{Hom}(H_{n-1}(X^*_t, \mathbb{Q}), \mathbb{C}/\mathbb{Q}(n))
\]

is then computed on \( \gamma \) (using Stokes’s theorem) by

\[
\langle \nu_{\phi}(t), \gamma \rangle = \int_{\mu} \text{dlog}(x) \equiv \int_{\mu} d[R_n] = \int_{\gamma} R_n|_{X^*_t} = \langle \text{AJ}([x]|_{X^*_t}), \gamma \rangle.
\]
Therefore, $\nu_\phi$ is the geometric (mixed) HNF associated to the coordinate symbol $\{x\} = \{x_1, \ldots, x_n\} \in \text{CH}^n(G_m^n, n)$ (i.e., the graph of this $n$-tuple, viewed as a cycle in $G_m^n \times \square^n$).

Now setting $H^n(\mathcal{X} \setminus \mathcal{X}_0) : = \ker[H^n(\mathcal{X} \setminus \mathcal{X}_0) \to H^n(\mathcal{X}_t)]$, we can write $V_{\phi,t}$ as an extension in a manner different from (4.11), as

$$0 \to H_{\nu,t}^{n-1} \to V_{\phi,t} \to H^n(\mathcal{X} \setminus \mathcal{X}_0)^o \to 0. \quad (4.13)$$

If $\phi$ is tempered, then $\{x\}$ lifts to $\xi \in \text{CH}^n(\mathcal{X} \setminus X_0, n)$, inducing a section of the MHS-morphism $\overline{H^n(\mathcal{X} \setminus \mathcal{X}_0)^o \to H^n(G_m^n)} \cong \mathbb{Q}(-n)$ under which (4.13) pulls back to an extension of the form

$$0 \to H_{\nu,t}^{n-1} \to V_{\xi,t} \to \mathbb{Q}(-n) \to 0. \quad (4.14)$$

The resulting morphism of sequences from (4.14) to (4.11) is a partial splitting as shown in the picture above. To see what it means geometrically, observe that the fiberwise restrictions $\xi_t \in \text{CH}^n(\mathcal{X}_t, n)$ of $\xi$ compute $\nu_\phi(t)$ via the composition

$$\text{CH}^n(\mathcal{X}_t, n) \xrightarrow{\text{AJ}} J(H^{n-1}(\mathcal{X}_t)(n)) \to J(W_{n-1}H^{n-1}(X_t^s)) \hookrightarrow J(\overline{H^{n-1}(X_t^s)}(n)). \quad (4.15)$$

So, the “partial extensions” of $\mathbb{Q}(-n)$ by $\overline{H^{n-1}(X_t^s)/W_{n-1}}$ in (4.10) also split ($\forall t$). The upshot is that the extension classes of both (4.10) and (4.11) reduce to (the image of) the class of (4.14), geometrically described by $\text{AJ}(\xi_t)$ in (4.15).

Having exhibited $V^*_{\phi}$ as the regulator extension, we turn to its dual

$$0 \to \mathbb{Q}(0) \to (V^*_{\phi,t})^\vee(-n) \to H_{n-1}(X_t^s)(-n) \to 0, \quad (4.16)$$

which identifies with the localization sequence

$$0 \to H^n(\mathbb{P}_\Delta, D_\Delta) \to H^n(\mathbb{P}_\Delta \setminus X_t, D_\Delta \setminus Z) \xrightarrow{\text{Res}} H^{n-1}(X_t, Z) \to H^{n+1}(\mathbb{P}_\Delta, D_\Delta)(-1) \to 0. \quad (4.18)$$

Writing $\Omega_t := \frac{d\log(\xi)}{1-\phi(\xi)} \in \Omega^n(\mathbb{P}_\Delta \setminus X_t)$ (so that $(2\pi i)^{-n+1}\omega_i = \text{Res}(\Omega_t)$), we obtain periods of the extension by lifting $(2\pi i)^{-n+1}[\omega_i] \in F^n\{\ker(i_t)(-1)\}$ to $[\Omega_t] \in F^nH^n(\mathbb{P}_\Delta \setminus X_t, D_\Delta \setminus Z; \mathbb{C})$ and pairing with the lift of $1^\vee \in \mathbb{Q}(0)^\vee$ to $T_{\omega} := \cap_{i=1}^n T_{x_i} \in H_n(\mathbb{P}_\Delta \setminus X_t, D_\Delta \setminus Z; \mathbb{Q})$. This yields

$$\int_{(-1)^{n-1}T^\omega} \Omega_t = \int_{\mathbb{P}_\Delta} \frac{d[R_{\omega}]}{(2\pi i)^n} \land \Omega_t = \int_{\mathbb{P}_\Delta} R_{\omega} \land \frac{d[\Omega_t]}{(2\pi i)^n} \equiv \langle \tilde{\nu}_\phi(t), [\omega_t] \rangle, \quad (4.17)$$

where the last equality only holds if $T_{\omega} \cap X_t^s = \emptyset$, and only modulo relative periods $(2\pi i)^n \int_{\eta} \omega_t$ (with $\eta \in H_{n-1}(X_t, Z; \mathbb{Q})$).

**Remark 4.12.** The left-hand term of (4.17) is a special case of the GKZ integral (2.10), but with nongeneral $\phi$. This type of integral also appears in Feynman integral computations [6, 7].

When $\phi$ is tempered (so that $\tilde{\nu}_\phi(t) \in H^{n-1}(X_t, \mathbb{C})$), and certain technical assumptions hold (cf. [6, §4.2]), the last equality of (4.17) holds modulo usual periods of $\omega_t$. The VMHS picture, for
\((V^\gamma_\phi)^\gamma(-n)\) as well as \(V^\gamma_\phi(-n)\), is, of course, dual to that above:

\[\text{tempered case}\]

It will be convenient to enshrine the right-hand term of (4.17) in the following.

**Definition 4.13.** The THNF associated to a tempered \(\phi\) is (any branch of) the multivalued function \(V_\phi(t) := \langle \varpi_\phi(t), [\omega_1] \rangle\).

Later, we shall choose a branch of \(V_\phi\); but independent of this choice, it follows from [13] that the THNF satisfies an inhomogeneous Picard–Fuchs equation

\[LV_\phi(t) = g_\phi(t), \tag{4.18}\]

where \(g_\phi \in \bar{Q}(t)\) and \(L\) (from §3.2) depend only on \(\phi\).

**Remark 4.14.** Suppose \(\text{rk}(H^{n-1}_{\nu_\xi}) = n\), write \(L = \sum_{i=0}^{n} q_{n-i}(t) \delta_i\) (with \(q_0(0) = 1\)), and let \(\mathcal{Y}(t) := \langle (2\pi i)^{n-1} \omega_1, V^{n-1}_{\xi_0} \rangle\) denote the Yukawa coupling. Taking \(\gamma_t^{\nu} \in (H^{n-1}_{\nu_\xi})_T\) a local generator with \(\langle \gamma_t^{\nu}, \gamma_t^{\nu} \rangle = 1\), define \(D_\phi \in \bar{Q}^n\) by \(N^{n-1}_{\xi_0} \gamma_t^{\nu} = : D_\phi \gamma_t^{\nu}\). By [14, Corollary 4.5], we have \(q_\phi(t) = q_0(t) \mathcal{Y}(t)\). Moreover, if the \(\{\bar{X}_{\sigma} \}_{\sigma \in \Sigma \setminus \{0, \infty\}}\) have only nodal singularities, then by [31, Proposition 7.1], \(\mathcal{Y}(t) = \frac{D_\phi}{q_0(t)}\).

### 4.4 \(V_\phi\) at infinity

While \(\varpi_\phi\) is singular at 0, we can compute its limit at \(t = \infty\). First, we shall isolate a part of the extension that splits off whether or not \(\phi\) is tempered (which we do not assume here).

**Definition 4.15.** For \(\sigma \in \Sigma\), the (pure weight \(\ell\)) phantom cohomology

\[\text{Ph}_\sigma^\ell := \ker(H^\ell(X_\sigma) \to \psi_\sigma H^\ell) = \text{im}(H_{2n-\ell}(X_\sigma)(-n) \to H^\ell(X_\sigma))\]  

measures the cycles that vanish on the nearby fiber. For any subset \(\Sigma' \subseteq \Sigma\) (e.g., \(\Sigma' := \Sigma \setminus \{0, \infty\}\), put \(\text{Ph}^\ell_{\Sigma'} := \bigoplus_{\sigma \in \Sigma'} \text{Ph}_\sigma^\ell\).

(We shall also write \(X_\Sigma\) resp. \(\bar{X}_\Sigma\) for \(\pi^{-1}(S)\) when \(S\) is open resp. finite.)

\(^\dagger\) In view of the proof of Lemma 4.10 and the polarization, we have \(\text{Gr}^{W}_{n-1}H^{n-1}(X_1, Z) \cong H^{n-1}_{W,1}\).
Proposition 4.16. In AVMHS($U$), we have $\mathcal{V}_\phi = A^\dagger_\phi \oplus \text{Ph}_{\Sigma\setminus\{0\}}^n$, where $\text{Ph}_{\Sigma\setminus\{0\}}^n$ is constant of weight $n$, and $A^\dagger_\phi$ is an extension of $\text{IH}^1(\mathbb{P}^1 \setminus \{0\}, \mathcal{H}_u^{n-1})$ (also constant, but mixed) by $\mathcal{H}_u^{n-1}$. Viewing $A^\dagger_\phi$ instead as an extension of $\mathbb{Q}(n)$ recovers $\nu_\phi$.

Proof. By the Decomposition Theorem (cf. [32, (5.9)]), for any proper algebraic subset $\mathcal{S}_t \subset \mathbb{P}^1$, we have

$$H^\ell(\mathcal{S}_t) \cong H^\ell \oplus \text{IH}^1(\mathcal{S}, \mathcal{H}^{\ell-1}) \oplus \text{Ph}_{\Lambda\cap\mathcal{S}}^\ell$$

(4.20)
as MHS. (See [32, Proposition 5.5(i)] for the fact that $\text{Ph}_{\Lambda\cap\mathcal{S}}^\ell$ is pure of weight $\ell$, and also of level $\leq \ell - 2$.) The long exact sequence associated to $(\mathcal{S}_t, X_t)$ (for $t \in U$) therefore exhibits $H^n(\mathcal{S}_t)$ as an extension of $H^n(\mathcal{S}) \oplus \text{Ph}_{\Lambda\cap\mathcal{S}}^n$ by $\mathcal{H}_u^{n-1}$. But as a sub-MHS of $H^n(\mathcal{S}_t)$, $\text{Ph}_{\Lambda\cap\mathcal{S}}^n$ is the image of $H_n(\mathcal{S}_t)(-n) \cong H^n(\mathcal{S}_t) \cap \text{Ph}_{\Lambda\cap\mathcal{S}}^n$ under the Gysin map, which obviously factors through $H^n(\mathcal{S}_t)$, splitting that part of the extension. Specializing to $\mathcal{S}_t = \mathbb{P}^1 \setminus \{0\}$, we have $\text{IH}^1(\mathcal{S}, \mathcal{H}_u^{n-1}) \cong H^1(\mathcal{S}) \cap H^{n-1} = \{0\}$, and so, $\text{IH}^1(\mathcal{S}, \mathcal{H}_u^{n-1}) = \text{IH}^1(\mathcal{S}, \mathcal{H}_u^{n-1})$. Finally, since $A^\dagger_\phi$ already gives the class of $\mathcal{V}_\phi$ in LHS (4.12), applying $\Theta^*$ yields $\nu_\phi$. □

Remark 4.17. An initial observation about $A^\dagger_\phi = \mathcal{V}_\phi / \text{Ph}_{\Sigma\setminus\{0\}}^n$ is that it elucidates what temperedness achieves. Comparing with the discussion around Lemma 4.10, we see that it is built out of the three parts:

$$W_{2n-2}^A \frac{A^\dagger_\phi}{W_{2n-2}} \cong \text{Gr}_{2n}^W \text{IH}^1(\mathbb{P}^1 \setminus \{0\}, \mathcal{H}_u^{n-1}) \cong \mathbb{Q}(n),$$

(4.21)

$$W_{2n-2}^A \frac{A^\dagger_\phi}{W_{n-1}} \cong W_{2n-2} \text{IH}^1(\mathbb{P}^1 \setminus \{0\}, \mathcal{H}_u^{n-1}) \left( \cong H^{n-1}_n(\mathcal{X}_t^*) / W_{n-1} \text{ if } \phi \text{ is } \Delta\text{-regular} \right),$$

(4.22)

$$\text{Gr}_{n-1}^W A^\dagger_\phi \cong \mathcal{H}_u^{n-1}.$$  

(4.23)

Temperedness splits the extension of (4.21) by (4.22), which is a constant extension since it appears inside $\text{IH}^1(\mathbb{P}^1 \setminus \{0\}, \mathcal{H}_u^{n-1})$.

Note that if $\sigma \in \Sigma \cap S$, the same computation (together with $^\dagger$ Clemens–Schmid) exhibits $H^n(\mathcal{S}_t^*)$ as the direct sum of $\text{Ph}_{\Lambda\cap S \setminus \{\sigma\}}^n$ with an extension of $\text{IH}^1(\mathcal{S}, \mathcal{H}_u^{n-1})$ by $(\psi_\sigma \mathcal{H}_u^{n-1})^T_\sigma$. When $S = \mathbb{P}^1 \setminus \{0\}$, this yields the following.

Corollary 4.18. The MHS $A^\dagger_\phi := H^n(\mathcal{X} \setminus \mathcal{X}_0, \mathcal{X}_\infty) / \text{Ph}_{\Sigma^*}$ is isomorphic to $(\psi_\infty A^\dagger_\phi)^T_\infty$, and hence computes $\lim_\infty \nu_\phi$.

Dually, we may define

$$A_\phi := H^n(\mathcal{X} \setminus \mathcal{X}_\infty, \mathcal{X}_0) / \text{Ph}_{\Sigma^*} \cong (A^\dagger_\phi)^*(-n),$$

(4.24)

which is itself obtained as the limit at 0 (more precisely, $(\psi_0 A^\dagger_\phi)^T_0$) of

$$A^\dagger_{\phi, t} := H^n(\mathcal{X} \setminus \mathcal{X}_\infty, \mathcal{X}_t) / \text{Ph}_{\Sigma \setminus \{\infty\}}.$$

(4.25)

$^\dagger$ Here, we need not assume unipotent monodromies; see [32].
**Example 4.19.** For the six $n = 2$, Laurent polynomials in Example 3.2, the (weak) Fano varieties\(^1\) and VMHS Hodge–Deligne diagrams are:

| $i$ | tempered? | $X^\circ$ | $\mathcal{V}_\phi$ | $\mathcal{A}_\phi$ | $\mathcal{A}_\phi$ | $\mathcal{A}_\phi$ |
|-----|-----------|------------|----------------|----------------|----------------|----------------|
| 1   | Y         | $\mathbb{P}^2$ | ![Diagram](image1) | ![Diagram](image2) | ![Diagram](image3) | ![Diagram](image4) |
| 2   | N         | $\mathbb{F}_1$ | ![Diagram](image5) | ![Diagram](image6) | ![Diagram](image7) | ![Diagram](image8) |
| 3   | Y         | $\text{dP}_5$ | ![Diagram](image9) | ![Diagram](image10) | ![Diagram](image11) | ![Diagram](image12) |
| 4   | Y         | $\mathbb{P}_{[1:2:3]}$ | ![Diagram](image13) | ![Diagram](image14) | ![Diagram](image15) | ![Diagram](image16) |
| 5   | N         | $(\mathbb{P}^2)^\circ$ | ![Diagram](image17) | ![Diagram](image18) | ![Diagram](image19) | ![Diagram](image20) |
| 6   | Y         | $\text{dP}_3$ | ![Diagram](image21) | ![Diagram](image22) | ![Diagram](image23) | ![Diagram](image24) |

The red arrows in cases (2) and (5) denote nontorsion extensions of (4.21) by (4.22), reflecting the nontemperedness. (These extensions record, in $\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(-2), \mathbb{Q}(-1)) \cong \mathbb{C}/\mathbb{Q}(-1)$, the logarithms of the toric boundary coordinates of the base locus $X_i \cap D_.$) In the other cases, the limit $A_\phi$.

\(^1\) This is meant in the sense of Conjectures 3.3 and 4.2 only, although Conjecture 3.3 bis does in fact hold (modulo constant terms) for $\phi^{(i)}$ if $i = 1, 3, 4, 6$. (The nontempered examples were included for variety.) Note that the cases $i = 4, 5$ are weak Fano (not Fano) since they are resolutions of singular toric surfaces.
contains only torsion extensions.† In all cases, we have \( \text{rk}(\text{Gr}^W_{k} \mathcal{Y}) = \dim(H^k,k(X^o)) \) in accordance with Conjecture 4.2.

More striking is the disparity in form between \( \mathcal{A}_{\phi} \) and \( \mathcal{A}_{\phi} \). While both share \( \text{Gr}^W_1 \cong \mathbb{Q} \) and \( \text{Gr}^W_2 \cong \text{IH}_1(\mathbb{P}^1, \mathbb{Q}) \), we have

\[
\frac{W_1}{W_2} \mathcal{A}_{\phi} \cong \mathbb{Q}(-2) \quad \text{versus} \quad \frac{W_1}{W_2} \mathcal{A}_{\phi} \cong \mathbb{Q}(-1).
\]

Only in cases (3) and (4) does \( \mathcal{A}_{\phi} \) yield a "\( K_2 \)-type" normal function in \( \text{ANF}(H^1(2)) \), which for (3) is due to an involution of \( \mathcal{X} \) over \( t \to -\frac{1}{t} \) [30, §5.2]. However, we may regard (in (2), (4), and (5)) extensions of \( \mathbb{Q}(-1) \subseteq \text{IH}_1(\mathbb{P}^1, \mathbb{Q}) \) by \( \mathbb{Q} \) as "\( K_0 \)-type" normal functions, whose image in \( \text{ANF}(H^1(1)) \) generate the Mordell–Weil group of \( \pi_{\mathbb{Q}} \). That their limits at 0 capture part (or all, as in (2)) of \( \text{lim}_{\infty} \nu_{\phi} \) is essentially the fact that the limits in \( X_0 \) of Abel–Jacobi of differences of sections in \( X_t \cap D_\Delta \) are given by ratios of toric coordinates on \( D_\Delta \).

As mentioned in the Introduction, \( \mathcal{A}_{\phi} \) and \( \mathcal{A}_{\phi} \) do not share the dual relationship (4.24) with their limits. Indeed, as we have just seen, \( \mathcal{A}_{\phi} \) is not even an HNF in a canonical way (unlike \( \mathcal{A}_{\phi} \)). However, for \( n \geq 3 \), it is precisely this lack of canonicity which makes \( \mathcal{A}_{\phi} \) better adapted to exhibiting \( \text{lim}_{\infty} \nu_{\phi} \) in terms of limits of truncated HNFs.

### 5 | THE PROBLEM AND SOME FANO THREEFOLD EXAMPLES

In this section, we state a precise but restricted version of the Arithmetic Mirror Symmetry Problem (see §5.2), and then solve it when \( X^o \) is one of the Mukai Fano threefolds \( V_{2N} \ (5 \leq N \leq 9) \) [20]. For each of these, [10] provides many LG-models of the form in Definition 3.1 — corresponding to the many possible toric degenerations \( \mathbb{P}_{\Delta^o} \) of \( X^o \) — satisfying Conjectures 3.3 and 3.3 bis. They are found by taking \( \phi \) to be (up to an additive constant) the Minkowski polynomial [9] for the corresponding (reflexive) \( \Delta \), which is tempered in view of [11, Prop. 2.4].

Our job is then to exhibit \( \mathcal{A}_{\phi,t} \) as a geometric HNF in the sense of (4.8), and the Apéry constant \( \alpha_{X^o} \) as the limit at \( t = 0 \) of the corresponding THNF, canonically normalized as described in §5.1. In contrast to \( \mathcal{A}_{\phi,t} \), this cannot arise from the lift of the coordinate symbol \( \{x_1, x_2, x_3\} \) to \( \text{CH}^3(\bar{X}_t, 3) \), since that HNF is singular at 0. Rather, we are looking for an extension

\[
0 \to H_v^2(p) \to \mathcal{A}_{\phi,t} \to \mathbb{Q}(0) \to 0
\]

arising from

\[
\mathcal{Z} \mapsto \nu_{\mathcal{Z}} : \text{CH}^p(\mathcal{X} \setminus \bar{X}_\infty, r - 1) \to \text{ANF}(H_v^{2p-r}(p))
\]

with \( 2p - r = 2 \), which forces \( (p, r) = (3, 3) \) (\( \mathcal{Z} \) belongs to the \( K_3^{\text{alg}} \) of the K3 fibers \( \bar{X}_t \)) or \( (2, 1) \) (\( \mathcal{Z} \) lies in \( K_1 \) of the fibers).‡ It is these cycles \( \mathcal{Z} \) which (in §§5.3–5.5) we will show how to construct in each case.

---

† In case (1) \( (X_\infty \text{ smooth}) \), this is by a computation in \( K_3(X_\infty) \); in (3) and (4) \( (X_\infty \text{ singular}) \), it is because \( K_3^{\text{ind}}(\mathbb{Q}) \) is torsion. Later, we will see how torsion extensions actually may lift to well-defined invariants in \( C \).

‡ Taking \( p > 3 \) yields \( F^{-1}H_v^2(p) = \{0\} \), making the extension class of (5.1) horizontal (by transversality) with rational monodromy (images under \( T_\sigma - I \)), hence trivial (since monodromy acts irreducibly on \( H_v^2 \)).
The inhomogeneous equation of a normal function

Given \( \nu \in ANF(H_{u}^{n-1}(p)) \), let \( \tilde{\nu} := \nu_{Q} - \nu_{F} \) be a multivalued holomorphic lift to \( H_{u}^{n-1} \). (Here \( u \) can be a higher or classical normal function, i.e., \( p \geq \frac{n}{2} \).) We may generalize Definition 4.13 and (4.18) by setting

\[
V(t) := \langle \tilde{\nu}(t), [\omega_{i}] \rangle,
\]

and \( g(t) := LV(t) \in \mathbb{C}(t) \), which is zero if and only if \( \nu \) is torsion \([13]\). (Note that since \( \langle F_{1}, \omega \rangle = 0 \) and \( L(H_{u}^{n-1}, \omega) = 0 \), \( g \) is independent of the choices of \( \nu_{Q} \) and \( \nu_{F} \).) In a special case, in which \( \nu \) is singular at 0, we have a formula for \( g(t) \) (Remark 4.14).

The next result summarizes what we can say more generally about this inhomogeneous term. It is motivated as follows. Suppose that \( \nu \) is nonsingular at 0 (Definition 4.6), so that the truncated normal function (NF) has a power-series expansion \( V(t) = \sum_{k=0}^{\infty} v_{k} t^{k} \). There follow some bounds one can make on \( g \). For its statement, we shall assume only that:

- \( \{\tilde{X}_{\sigma}\} \) is a family of CY \((n-1)\)-folds over \( \mathbb{P}^{1} \) (smooth off \( \Sigma \));
- \( \{\omega_{i}\} \) is a section of \( H_{u}^{n-1,0} \cong \mathcal{O}_{\mathbb{P}^{1}}(h) \), with divisor \( h[\infty] \);
- \( L = \sum_{j=0}^{d} t^{j} P_{j}(\delta_{i}) \in \mathbb{C}[t, \delta_{i}] \) is its PF operator, of degree \( d \); and
- \( H_{u}^{n-1} \) has maximal unipotent monodromy at 0.

This is somewhat more general than the setting of the rest of this paper, which takes \( \{\tilde{X}_{\sigma}\} \) to arise from the level sets of a Laurent polynomial; in this case, we have \( h = 1 \) (see [31, Example 4.5]), and frequently only nodal singularities on the \( \{\tilde{X}_{\sigma}\}_{\sigma \in \Sigma^{*}} \).

**Theorem 5.1.** Assume that \( \nu \) is nonsingular away from 0 and \( \infty \). Then \( g \) is a polynomial of degree \( \leq d - h \). If \( \nu \) is also nonsingular at 0, then \( t \mid g \). If \( \nu \) is also nonsingular at \( \infty \) and \( T_{\infty} \) is unipotent, then \( \deg(g) \leq d - h - 1 \).

**Proof.** Let \( u \) be a local coordinate on a disk \( D_{\sigma} \) about \( \sigma \in \Sigma \), and \( H_{e} \) resp. \( H^{e} \) the canonical resp. dual-canonical extensions of \( H_{u}^{n-1}|_{D_{\sigma}} \) to \( D_{\sigma} \). (That is, the eigenvalues of \( \nabla_{\delta_{u}} \) are in \((-1,0]\) resp. \([0,1)\).) Assuming that \( \nu \) is nonsingular at \( \sigma \), we may choose \( \nu_{Q} \) so that \( N_{\sigma}\nu_{Q} = 0 \); thus, \( \tilde{\nu} \) is \( T_{\sigma} \)-invariant, and extends to a section of \( H_{e} \). Since \( \omega \) is a section of \( H_{e} \), and \( \langle \cdot, \cdot \rangle \) extends to \( H_{e} \times H_{e} \to \mathbb{C} \), \( V = \langle \tilde{\nu}, \omega \rangle \) extends to a holomorphic function on \( D_{\sigma} \). For \( \sigma \in \Sigma^{*} \), we have \( L \in \mathbb{C}[u, \delta_{u}] \) hence \( g|_{D_{\sigma}} \) holomorphic. At \( \sigma = 0 \), maximal unipotency forces the indicial polynomial \( P_{0}(T) \) to be divisible by \( T \), so that \( L \) sends \( \mathcal{O}(D_{0}) \to t \mathcal{O}(D_{0}) \) and \( g(0) = 0 \). If \( \sigma = \infty \) and \( u = t^{-1} \), our assumption that \( \langle \omega \rangle = h[\infty] \) gives \( V|_{D_{\infty}} \in u^{h}\mathcal{O}(D_{\infty}) \); applying \( L = \sum_{j=0}^{d} u^{-j} P_{j}(\delta_{u}) \) yields \( g|_{D_{\infty}} \in u^{h-d}\mathcal{O}(D_{\infty}) \).

We can refine the result at \( \infty \), and deal with singularities at 0 and \( \infty \), by writing \( \tilde{\nu} \) and \( \omega \) locally in terms of bases of the canonical extension. With \( \sigma, u \) as above, \( H_{u}^{n-1} = H_{u}^{\text{un}} \oplus H_{u}^{\text{non}} \) decomposes into unipotent (\( T_{\sigma}^{\text{ss}} \)-invariant) and nonunipotent parts, with (multivalued) bases \( \{e_{i}\} \) and \( \{e_{i}^{*}\} \), the latter chosen so that \( T_{\sigma}^{\text{ss}} e_{i}^{*} = \zeta_{k}^{a_{j}} e_{j}^{*} \) (\( \zeta_{k} := e^{\frac{2\pi i}{k}} \)). Writing \( \epsilon(u) := \frac{\log(u)}{2\pi i} \), a basis of \( H_{e} = H_{e}^{\text{un}} \oplus H_{e}^{\text{non}} \) is given by \( e_{i} := e^{-\epsilon(u) N_{\sigma}} e_{i} \) and \( e_{i}^{*} := e^{-\epsilon(u) N_{\sigma}} u^{-\frac{a_{j}}{k}} e_{j}^{*} \), which have the property that \( \nabla_{\delta_{u}} e_{i}, \nabla_{\delta_{u}} e_{i}^{*} \in H_{e} \). Admissibility says that the Hodge lift takes the form

\[
\nu_{F}(u) = u \sum_{i} f_{i}(u) e_{i} + u \sum_{j} f_{j}^{*}(u) e_{j}^{*} + \tilde{e}_{C} \in \Gamma(D_{\sigma}, V^{c}),
\]
where \( e_C \) is a \( \mathbb{C} \)-lift of \( 1 \) to \( \mathbb{V}_C^{un} \) and \( \bar{e}_C := e^{-\ell(u)N_{\sigma}}e_C \). If \( \nu \) is nonsingular at \( \sigma \), then \( \bar{e}_C = e_C \) and \( \nabla_{\delta_u} \bar{e}_C = 0 \); if it is singular at \( \sigma \), then we may assume \( \bar{e}_C = e_C + \ell(u)\delta_1 \), so that \( \nabla_{\delta_u} \bar{e}_C \in H^{un}_e \). Write \( \text{ord}_\sigma(\omega) := o \) (this is \( h \) if \( \sigma = \infty \) and \( 0 \) if \( \sigma = 0 \)).

Replacing \( \hat{\nu} \) by \( \hat{\nu} := \bar{e}_C - \nu_F \) changes it by a \( \mathbb{C} \)-period hence does not affect \( g \). Writing \( L = \sum_{k \geq 0} q^\tau_k(u)\delta_u^k \), we have

\[
g = L(\hat{\nu}, \omega) = \sum_{k \geq 1} \sum_{j=1}^k q^\tau_k(u)\langle \nabla_j \hat{\nu}, \nabla_{k-j} \omega \rangle
\]

since \( \nabla_L \omega = 0 \). Clearly, \( \nabla_{\delta_u}^{k-j} \omega \in u^\omega \mathcal{H}_e \), while \( \nabla_{\delta_u}^{j} \hat{\nu} \in u \mathcal{H}_e = u \mathcal{H}^{un}_e \oplus \mathcal{H}^e \) resp. \( \mathcal{H}^{un}_e \oplus \mathcal{H}^e = \mathcal{H}^e \) for \( \nu \) nonsingular resp. singular at \( \sigma \). Hence, \( \langle \nabla_{\delta_u}^{j} \hat{\nu}, \nabla_{\delta_u}^{k-j} \omega \rangle \) belongs to \( u^\omega \mathcal{O}(D_\sigma) \) if \( \nu \) is nonsingular and \( T_\sigma \) is unipotent, and otherwise to \( u^\omega \mathcal{O}(D_\sigma) \). For \( \sigma = \infty \), multiplying by \( q^\tau_k(u) \) introduces \( u^{-d} \)-divider. The result follows. \( \square \)

Remark 5.2. Different choices of \( \nu_\mathbb{Q} \) yield branches of \( V \) that differ by \( \mathbb{Q}(p) \)-periods \((2\pi i)^p \int_{\varphi} \omega_1 \), \( \varphi_1 \in H_{n-1}(\bar{X}_1, \mathbb{Q}) \). If the \( \{T_\sigma - I\}_{\sigma \in \Sigma^*} \) have rank 1, and there are \( d \) of them (i.e., \( H^{n-1}_u \) has no “removable singularities”), and one \( \sigma_0 \in \Sigma^* \) has greater modulus than the others, then we say that \( \mathcal{X} \) is of normal conifold type. In this case \( V \) can be chosen uniquely by maximizing its radius of convergence; that is, there is a unique branch that is single-valued on the complement of the interval \([\sigma_0, \infty]\).

5.2 | The arithmetic mirror symmetry problem

Rather than reiterating the general but vague version from the Introduction, we give a more precise variant in a restricted setting. Assume that our Fano variety and LG-model satisfy the following:

- \( H^*_{\text{prim}}(X^\circ) \) and \( (\psi_0 H^{n-1}_u)^{T_0} \) are Hodge–Tate of rank \( r_0 \), with isomorphic associated grades (as predicted by Conjecture 4.2);
- \( H^n_{\text{prim}}(X^\circ) = \{0\} \) (if \( n \) is even), and \( \rho(X^\circ) = 1 \);
- \( \mathcal{X} \) is of normal conifold type (Remark 5.2), and satisfies Conjecture 3.3 bis;
- \( \phi \) (and thus \( \mathcal{X} \), and \( L \)) is defined over \( \hat{\mathbb{Q}} \);
- \( d = r_0 + 1 \); and
- \( P_0(d - 1) \neq 0 \).

Referring to §3.3, we write \( b_j := u_j^{(d-1)} \) and \( B(t) := \sum_{j \geq d-1} b_j t^j \), so that \( \alpha^{(d-1)}_{\mathcal{X}^\circ} = \lim_{j \to \infty} \frac{b_j}{a_j} \) and \( LB = P_0(d - 1)t^{d-1} \).

Problem 5.3. Exhibit the Apéry constants \( \{\alpha^{(i)}_{\mathcal{X}^\circ}\}_{i=1}^{d-1} \) as periods, by showing that:

(a) The first \( d - 2 \) constants \( \{\alpha^{(i)}_{\mathcal{X}^\circ}\}_{i=1}^{d-2} \) are (up to \( \hat{\mathbb{Q}}^e \)-multiples) extension classes in \( (\psi_0 H^{n-1}_u)^{T_0} \) that are torsion (i.e., powers of \( 2\pi i \)) if \( \phi \) is tempered.
(b) There is (up to scale) a unique HNF $\nu \in \text{ANF}(H^n_{H}(p)) \setminus \{0\}$ singular only at $t = \infty$, for some (unique) $p \in \left[ \frac{n+1}{2}, n \right] \cap \mathbb{Z}$. This HNF is motivic, that is, arises from some $Z \in \text{CH}^p(\mathcal{X} \setminus \tilde{X}_{\infty}, 2p - n)_{\mathbb{Q}}$; and $LV = -\mathfrak{t}^d - 1$ for some $^1 \mathfrak{t} \in \mathbb{Q}^*$.

(c) Normalize $V$ uniquely as in Remark 5.2, and set $\hat{V}(t) := \frac{P_0(d-1)}{t} V(t)$. Then $\alpha_{\mathcal{X}^e}^{(d-1)} = \hat{V}(0) + \sum_{i=1}^{d-2} \beta_i \alpha_{\mathcal{X}^e}^{(i)}$, where $\beta_i \in \mathbb{Q}(\hat{V}(0), \hat{V}'(0), \ldots, \hat{V}^{(i)}(0))$.

Remark 5.4. Stated in this way, the thrust of the Arithmetic Mirror Symmetry Problem is somewhat obscured. What it really proposes is that given a Fano $X^e$ with $H^*_{\text{prim}}$ as above, of rank 1 less than the degree of its quantum differential equation, there exists an LG-model $\mathcal{X}$ (and cycle $Z$) satisfying the remaining hypotheses together with the content of Problem 5.3 (a)–(c).

In the three subsections that follow, we solve this problem in several cases with $n = 3 = r$ and $d = 2$. The situation simplifies, since $P_0(1) = 1$ and there is only one Apéry constant $\alpha_{\mathcal{X}^e}$; moreover, Theorem 5.1 guarantees that $LV = -\mathfrak{t} t$ for some $\mathfrak{t} \in \mathbb{C}^*$ once we have a HNF of the type described. So, it remains to produce $Z$ (hence $\nu$), and show $\mathfrak{t} \in \mathbb{Q}^*$ and $\alpha_{\mathcal{X}^e} = \hat{V}(0)$ in each case; we defer the uniqueness to §6.

Remark 5.5. A general result (for $n = r$ and $d = 2$) encompassing these cases appeared in [31, Thm. 10.11], making essential use of Theorem 5.1 above. It reduces the Arithmetic Mirror Symmetry Problem to establishing the existence of a “good” LG-model and checking the Beilinson–Hodge conjecture; once the cycle is found, the equality $\alpha_{\mathcal{X}^e} = \hat{V}(0)$ is automatic. But we shall explicitly compute $\hat{V}(0)$ below in each case anyway, both to provide explicit (and instructive) solutions to the problem, and to check that $\mathfrak{t} \neq 0$ and the cycle indeed produces a nontrivial HNF. Moreover, with [31] in hand, one may regard these computations of $\hat{V}(0)$ as illustrations of a regulator calculus that is available for calculating the Apéry constant when modular and other methods (such as in [20]) are unavailable.

5.3 $K_1$ of a $K3$: The $V_{10}$ HNF

The irrational Fano threefold $V_{10} = G(2, 5) \cap \mathbb{Q} \cap P_1 \cap P_2$ (quadric and linear sections of the Plücker embedding) has a mirror LG model with discriminant locus $\Sigma = \{0, \sigma_+, \sigma_-, \infty\}$ (where $\sigma_{\pm} = \frac{-11 \pm 5\sqrt{5}}{4}$), given by the Laurent polynomial

$$\phi(x) = \frac{(1 - x_3)(1 - x_1 - x_3)(1 - x_2 - x_3)(1 - x_1 - x_2 - x_3)}{-x_1 x_2 x_3}.$$ (5.2)

Namely, compactifying $\{t = \phi(x)\}$ in $P_\Delta$ yields a family $\{X_t\}_{t \in P^1}$, whose fibers over $P^1 \setminus \Sigma$ are singular $K3$s with one $A_3$ and six $A_1$ singularities; these are resolved (to Picard-rank 19 $K3$s) by $\beta : \tilde{X}_t \to X_t$. The Newton polytope $\Delta$, together with a portion

$$X_t \cap \{x_3 = 0\} = \{x_1 = 1\} \cup \{x_2 = 1\} \cup \{x_1 + x_2 = 1\} = : C_1 \cup C_2 \cup C_3$$

$^1$ More precisely, $\mathfrak{t}$ should belong to the common field of definition of $\mathcal{X}$ and $Z$. 

of the base locus (red), are displayed in the figure. The PF operator and period sequence are given by

\[ L = δ_t^3 - 2t(2δ_t + 1)(11δ_t^2 + 11δ_t + 3) - 4t^2(δ_t + 1)(2δ_t + 3)(2δ_t + 1), \]

\[ a_k := [φ^k]_2 = \sum_{i=0}^{k} \sum_{j=0}^{k-i} \frac{k!2k!}{i!2j!2(k-i)!2(k-j)!(k-i-j)!} = 1, 6, 114, \ldots; \]

while the monodromy operators \( T_0, T_\pm, T_\infty \) have Jordan forms \( J(3), (-1) \oplus 1^2, (-1)^2 \oplus 1 \) and LMHS types

where the \( T_\sigma \)-invariant classes are circled. The Apéry constant is \( \alpha = \frac{1}{10} \zeta(2) \) [20].

To construct the cycle \( Z \in CH^2(\mathcal{X} \setminus \tilde{X}_\infty, 1) \), we shall make use of the rational curves \( \{C_i\} \). On \( X_i \), a higher Chow cycle is given by \( (C_1, g_1 := \frac{x_2}{x_2 - 1}) + (C_2, g_2 := \frac{x_1^{-1}}{x_1}) + (C_3, g_3 := \frac{x_1}{x_1 - 1}) \), since the sums of divisors cancel on \( X_i \). To lift this to a cycle \( Z_i \) on \( \tilde{X}_i \) (say, for \( t \notin \Sigma \)), one adds two more terms \( (D_1, f_1) + (D_2, f_2) \) supported on the exceptional divisors over the nodes of \( X_i \) at \( C_1 \cap C_3 \) and \( C_2 \cap C_3 \). These \( \{Z_i\} \) are the restrictions of an obvious precycle \( Z \) on \( \mathcal{X} \), whose boundary fails to vanish only on \( \tilde{X}_\infty \).

The next step is to find a family of closed 2-currents \( R_t \) on \( \tilde{X}_t \) representing the class \( \nu_Z(t) \in J(H^2(\tilde{X}_t)(1)) \), or more precisely a lift to \( H^2(\tilde{X}_t, \mathbb{C}) \) which is single-valued on \( D_t \). Writing \( \mu := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_2 \leq 1, 1 - x_2 \leq x_1 \leq 1\} \), for \( |t| \ll 1 \) let \( \Gamma_t \) denote the branch of \( \{(x_1, x_2, x_3) \in \tilde{X}_t \mid (x_1, x_2) \in \mu\} \) with \( x_3 \) small. Then we have \( R_t = (2\pi i)^2 δ_{\Gamma_t} + 2\pi i \sum_{i=1}^{3} \log(g_i)\delta_{C_i} + 2\pi i \sum_{i=1}^{2} \log(f_i)\delta_{D_i} \) (plus an irrelevant \( (2,0) \)-current), which yields the

\[\text{\footnotesize† The successive blowups along the components of the base locus occurring in the construction of } \mathcal{X} \text{ generate additional exceptional curves on } \tilde{X}_\infty \text{ that disconnect the 5-gon } D_1 \cup D_2 \cup C_1 \cup C_2 \cup C_3, \text{ and it is on these that } \delta Z \text{ is supported.}\]

\[\text{\footnotesize‡ In the context of regulator currents, } \log(-) \text{ means the single-valued branch with discontinuity along } \mathbb{R}_-.\]
\[ V(t) = \langle [R_t], [\omega_t] \rangle = (2\pi i)^2 \int_{\Gamma_t} \omega_t = \frac{1}{2\pi i} \int_{|x_3| = \epsilon} \frac{d \log(x_3)}{1-t x_3} = \sum_{k \geq 0} t^k \int_{\mu} [\phi^k] x_3 \frac{dx_1}{x_1} \frac{dx_2}{x_2} = \sum_{k \geq 0} u_k t^k. \]

(Here \([-]_{x_3}^3\) takes terms of the Laurent polynomial constant in \(x_3\).) By Theorem 5.1, it suffices to compute

\[ v_0 = \int_0^1 \int_{1-x_2}^1 \frac{dx_1}{x_1} \frac{dx_2}{x_2} = -\int_0^1 \log(1-x_2) \frac{dx_2}{x_2} = \text{Li}_2(1) = \zeta(2) \quad \text{and} \]

\[ v_1 = \int_0^1 \int_{1-x_2}^1 \left\{ x_1^{-1} (4x_2^{-2} - 6 + 2x_2) + (-6x_2^{-1} + 6 - x_2) + x_1(2x_2^{-1} - 1) \right\} \frac{dx_1}{x_1} \frac{dx_2}{x_2} = -10 + 6\zeta(2) \]

to conclude that \(LV = -10t\). Normalization therefore yields

\[ \hat{V}(t) = \frac{1}{10} \zeta(2) + (-1 + \frac{3}{5} \zeta(2))t + \cdots, \quad (5.3) \]

as desired.

### 5.4 \( K_3 \) of a \( K3 \): the \( V_{12} \) HNF

The LG mirror of the rational Fano \( V_{12} = \text{OG}(5,10) \cap P_1 \cdots \cap P_7 \) is given by

\[ \phi(x) = \frac{(1-x_1)(1-x_2)(1-x_3)(1-x_1-x_2 + x_1x_2-x_1x_2x_3)}{-x_1x_2x_3}. \quad (5.4) \]

This time the Picard-rank 19 3 × 3s \( X_t \), smooth for \( t \notin \Sigma = \{0, \sigma_+, \sigma_-, \infty\} (\sigma_\pm = (-1 \pm \sqrt{2})^4) \), resolve seven \( A_1 \) singularities on \( X_t \). The family \( \mathcal{X} \) is birational to that of [3] and underlies the proof of irrationality of \( \zeta(3) \) [30]; indeed, \( \alpha = \frac{1}{6} \zeta(3) \) [20]. Its PF operator

\[ L = \delta_1^3 - t(2\delta_1 + 1)(17\delta_1^2 + 17\delta_1 + 5) + t^2(\delta_1 + 1)^3 \]

and (Apéry) period sequence

\[ a_k := \sum_{\ell=0}^{k} \binom{k}{\ell}^2 \binom{k + \ell}{\ell}^2 = 1, 5, 73, \ldots \]
reflect a VHS with monodromies of the same types as in §5.3 except at \( t = \infty \) (where we get maximal unipotent monodromy).

Since \( \phi \) is tempered, the symbol \( \{x\} \) lifts to \( \xi \in \text{CH}^3(\mathcal{X} \setminus \tilde{X}_0,3) \). The birational map \( I : (x,t) \mapsto \left(\frac{x_3}{1-x_3}, \frac{-(1-x_1)(1-x_2)}{1-x_1-x_2+x_1x_2}, \frac{x_1}{1-x_1}, \frac{1}{t}\right) \) from \( \mathcal{X} \) to itself, viewed as a correspondence, allows us to define \( Z := I^*\xi \in \text{CH}^3(\mathcal{X} \setminus \tilde{X}_\infty,3) \). The resulting THNF

\[
V(t) = \langle \hat{v}_Z(t), \omega_t \rangle = \langle \hat{v}_Z(t^{-1}), t^{-1} \omega_{t^{-1}} \rangle = \int_{X_{t^{-1}}} R_3(x) \wedge d \left[ \frac{1}{(2\pi i)^3} \frac{\text{dlog}(x)}{t - \phi(x)} \right] \\
= \sum_{k \geq 0} t^k \left( \int_{[0,1]^3} \left( \frac{\prod_{j=1}^3 X_i^k(1-X_j)}{(1-X_3(1-X_1X_2))^{k+1}} \right) \right) =: \sum_{k \geq 0} u_k t^k
\]

has \( u_0 = 2\zeta(3) \) and \( u_1 = -12 + 10\zeta(3) \), which (again by Theorem 5.1) is enough to conclude that \( LV = -12t \). But then normalization gives

\[
\hat{V}(t) = \frac{1}{6} \zeta(3) + (-1 + \frac{5}{6}\zeta(3))t + \cdots,
\]

and, in particular, \( \hat{V}(0) = \alpha \).

### 5.5 HNFs for \( V_{14}, V_{16}, V_{18} \)

Again, the LG models are families of Picard-rank 19 K3s. The irrational case \( V_{14} = G(2,6) \cap P_1 \cap \cdots \cap P_5 \) is similar to \( V_{10} \), with Laurent polynomial

\[
\phi(x) = \frac{(1-x_1-x_2-x_3)(1-x_2-x_3)(1-x_3)^2 - x_2(1-x_1-x_2-x_3)}{-x_1x_2x_3},
\]
discriminant locus \( \Sigma = \{0, \frac{1}{27}, -1, \infty\} \), and PF operator

\[
L = \delta_i^3 - t(1 + 2\delta_i)(13\delta_i^2 + 13\delta_i + 4) - 3t^2(\delta_i + 1)(3\delta_i + 4)(3\delta_i + 2).
\]

The monodromy types are the same as for \( V_{10} \), except for \( T_\infty \), which acts on \( (\phi_\infty H^2_u)^{2,0} \) resp. \( (\phi_\infty H^2_u)^{0,2} \) by \( e^{-\frac{2\pi i}{3}} \) resp. \( e^{\frac{2\pi i}{3}} \).
The toric boundary divisor $x_1 = 0$ intersects $X_t$ in $C_1 = \{x_2 = 1 - x_3\}$ and $C_2 = \{x_2 = (1 - x_3)^2\}$, and a cycle $Z \in \text{CH}^2(\tilde{X} \setminus \tilde{X}_\infty, 1)$ is given by $(C_1, \frac{x_3}{1-x_3}) + (C_2, \frac{1-x_3}{x_3})$. Arguing as before, this yields

$$V(t) = \sum_{k \geq 0} v_k t^k = \sum_{k \geq 0} t^k \int_{\mu} [\phi^k]_0 \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3},$$

where $\mu := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_3 \leq 1, (1 - x_3)^2 \leq x_2 \leq 1 - x_3\}$. We compute $v_0 = \zeta(2)$ and

$$v_1 = \int_0^1 \int_0^{1-x_3} \left\{ \frac{2x_2 x_3^{-1} + (-x_3 + 4 - 3x_3^{-1})}{x_2^{-1} x_3^{-1} (x_3 - 1)^3} \right\} \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3} = -7 + 4\zeta(2),$$

hence that $LV = -7t$. Renormalizing this gives

$$\hat{V}(t) = \frac{1}{7\zeta(2)}(1 + \frac{4}{7}\zeta(2))t + \cdots,$$

and $\hat{V}(0) = \frac{1}{7\zeta(2)}$ indeed matches the $\alpha$ from [20].

Turning to $V_{16} = LG(3, 6) \cap P_1 \cap P_2 \cap P_3$ and $V_{18} = (G_2/P_2) \cap P_1 \cap P_2$, we use

$$\phi(x) = \frac{(1 - x_1 - x_2 - x_3)(1 - x_1)(1 - x_2)(1 - x_3)}{-x_1 x_2 x_3}$$
resp.

$$\frac{(x_1 + x_2 + x_3)(x_1 + x_2 + x_3 - x_1 x_2 - x_2 x_3 - x_1 x_3 + x_1 x_2 x_3)}{-x_1 x_2 x_3}$$

from [11] for our LG models, with $\Sigma = \{0, 12 \pm 8\sqrt{2}, \infty\}$ resp. $\{0, 9 \pm 6\sqrt{3}, \infty\}$ and

$L = \delta^3 - 4t(1 + 2\delta)t(3\delta^2 + 3\delta t + 1) + 16t^2(\delta t + 1)^3$

resp. $\delta^3 - 3t(1 + 2\delta)t(3\delta^2 + 3\delta t + 1) - 27t^2(\delta t + 1)^3$.

The monodromy/LMHS types are the same as for $V_{12}$; we write $N = 6, 8, 9$ for $V_{2N}$, and put $I(t) := \frac{1}{M_N}$ with $M_N = 1, \frac{15}{16}, -\frac{1}{27}$, respectively. In each case, there is an isomorphism $H^2_\mu \cong I^*H^2_\nu$ of $\mathbb{Q}$-VHS.† For $N = 8, 9$, this is not an integral isomorphism so is induced by correspondences $I, I^{-1} \in Z^2(\tilde{X} \times I^*X)_0$ (with $I^*(I^{-1})^* = \text{id}_{H^2_{\mu}}$) rather than a birational map; nevertheless, we may still define $Z := I^*\xi \in \text{CH}^2(\tilde{X} \setminus \tilde{X}_\infty, 3)$. Here, we normalize $I$ to pull back an integral generator $\zeta_s^\vee$ of $(H^2_{\mu})^T_{\infty}$ back to $\gamma_{I(s)} \in (H^2_{\nu})^T_0$, where $s = I(t)$.

Since the monodromy/LMHS types are the same as for $V_{12}$, we write $N = 6, 8, 9$ for $V_{2N}$, and put $I(t) := \frac{1}{M_N}$ with $M_N = 1, \frac{1}{16}, -\frac{1}{27}$, respectively. In each case, there is an isomorphism $H^2_\mu \cong I^*H^2_\nu$ of $\mathbb{Q}$-VHS.† For $N = 8, 9$, this is not an integral isomorphism so is induced by correspondences $I, I^{-1} \in Z^2(\tilde{X} \times I^*X)_0$ (with $I^*(I^{-1})^* = \text{id}_{H^2_{\mu}}$) rather than a birational map; nevertheless, we may still define $Z := I^*\xi \in \text{CH}^2(\tilde{X} \setminus \tilde{X}_\infty, 3)$. Here, we normalize $I$ to pull back an integral generator $\zeta_s^\vee$ of $(H^2_{\mu})^T_{\infty}$ back to $\gamma_{I(s)} \in (H^2_{\nu})^T_0$, where $s = I(t)$.

Since the integrals $\int_{\mathbb{R}^3} \frac{d\log s}{(\phi(x))^{k+1}}$ are quite difficult for $N = 8, 9$, we use a different strategy than in §5.4. As a section of $\phi_{\nu,e} \cong \Theta_{\nu,1}(1)$, $\omega_I$ has divisor $[\infty]$, and so, $(I^{-1})^*\omega_{I(t)} = C_N t\omega_I$ for some $C_N \in C^*$. Write $\gamma_{s}^\vee \in (H^2_{\nu})^T_\infty$ for the element dual to $\zeta_s$, so that $\lim_{s \to \infty} s\omega_s = -\frac{1}{(2\pi i)^2}\text{Res}_{\infty}^\vee(\frac{d\log s}{\phi(x)}) = r_N\gamma_{s}^\vee$ in $H_2(\tilde{X}_\infty)$ where $r_N := \frac{1}{(2\pi i)^3}\text{Res}_{p}^\vee(\frac{d\log s}{\phi(x)}) = 1, \frac{1}{2}$, resp. $\frac{1}{\sqrt{-3}}$ (for some triple-normal-crossing point $p \in X_\infty$). This yields

$$C_N = \lim_{t \to 0} \left(\gamma_{I(t)} - (I^{-1})^*\omega_{I(t)}\right) = \lim_{s \to \infty} M_N s((I^{-1})^*\gamma_{I(s)}, \omega_s) = \lim_{s \to \infty} M_N (\zeta_s^\vee, s\omega_s) = M_N r_N.$$

† This is easiest to see from the differential equation, but also follows from the fact that (for all five cases) the LG model of $V_{2N}$ realizes the canonical weight-2 rank-3 VHS over $X_0(N) + N$, which for $N$ composite has an additional Fricke involution.
Write Λ for L with t replaced by s, we have

\[ L = \frac{-1}{M_N s} \Lambda^1_s. \]

Applying this to

\[ V(t) = \langle \tilde{v}_z(t), \omega_t \rangle = \frac{1}{C_N t} \langle (I^* \tilde{v}_\varphi(s), (I^{-1})^* \omega_s) \rangle = \frac{M_N s}{C_N} \langle \tilde{v}_\varphi(s), \omega_s \rangle \]

(5.6)

yields \( LV = \frac{-1}{C_N s} \Lambda \langle \tilde{v}_\varphi(s), \omega_s \rangle = \frac{-D_N}{C_N s} = \frac{-D_N}{C_N t} \), where \( D_N = 12, 16 \), resp. 9 is the constant from Remark 4.14. Moreover, thinking of \( \zeta^\vee_\infty \) as a “membrane stretched once around \( \mathcal{X}_\infty \),” taking \( \lim_{s \to \infty} \) of (5.6) gives

\[ V(0) = -\frac{1}{r_N} \int_{\mathcal{X}_\infty} R_3(x) \wedge \frac{1}{(2\pi i)^2} \text{Res}_{\mathcal{X}_\infty} \left( \frac{\text{dlog}(x)}{\varphi(x)} \right) = \int_{\mathcal{X}_\infty^\vee} R_3(x) \vert_{\mathcal{X}_\infty^\vee} \]

(5.7)

for \( \zeta^\vee_\infty \) in suitably general position; \( ^{†} \) and \( \hat{V}(0) = \frac{r_N}{D_N} V(0) \).

We now use (5.7) to verify that \( \hat{V}(0) \) recovers the Apéry constants in [20]. For \( N = 6 \), the computation in [30, §5.3] (with \( \zeta^\vee_\infty = -\psi \)) gives \( \int_{\mathcal{X}_\infty^\vee} R_3(x) = 2\zeta(3) \), recovering \( \hat{V}(0) = \frac{\zeta(3)}{6} \). For \( N = 8 \), putting \( \zeta^\vee_\infty \) in general position is tricky, so we use the first expression in (5.7). As \( R_3(x) = \log(x_1) \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3} + 2\pi i \log(x_2) \frac{dx_3}{x_3} \delta_{T x_1} + (2\pi i)^2 \log(x_3) \delta_{T x_1 \cap T x_2} \) is nontrivial only on the component \( \{x_1 = 1 - x_2 - x_3\} \subset \mathcal{X}_\infty \), with only its third term surviving against the (2,0) residue form, this yields

\[ V(0) = -2 \int_{T^1-x_2-x_3 \cap T x_2} \log(x_3) \frac{dx_2 \wedge dx_3}{(1-x_2)(1-x_3)(x_2+x_3)} \]

\[ = -2 \int_1^\infty \log(x_3) \left( \left| \int_0^1 \frac{dx_2}{1-x_2} (x_2+x_3) \right| \right) dx_3 \]

\[ = 4 \int_0^1 \frac{\log^2(u)}{1-u^2} du \quad (\text{where } u = x_3^{-1}) \]

\[ = 4(\text{Li}_3(1) - \text{Li}_3(-1)) = 7\zeta(3), \]

hence \( \hat{V}(0) = \frac{7}{32}\zeta(3). \)

Finally, for \( N = 9 \), we first replace \( \{x_1\} \) (hence \( \xi \), and \( \mathcal{Z} \)) by the equivalent symbol \( \{z\} \), where \( z_1 = \frac{-x_1}{x_1+x_2}, z_2 = \frac{-x_1}{x_2}, \) and \( z_3 = \frac{x_1 x_2}{x_1+x_2} \). In these coordinates,

\[ \varphi(x(z)) = z_1^{-1} z_3^{-1} (1-z_1) (1-z_3) z_1 (1-1-z_2) z_3 (1-(1-z_2^{-1}) z_3) \]

and so, \( \mathcal{X}_\infty = \mathcal{X}_\infty^\prime \cup \mathcal{X}_\infty^\prime\prime = \{z_1 = 1\} \cup \{z_1 = \varphi(z_2, z_3) : = \frac{1-z_3}{1-(1-z_2) z_3 (1-(1-z_2^{-1}) z_3)} \} \), with \( C_\infty := \mathcal{X}_\infty^\prime \cap \mathcal{X}_\infty^\prime\prime \) described by \( z_3 = \mathcal{g}(z_2) : = \frac{1-z_3^{-1}}{(1-z_2)(1-z_3^{-1})} \). Clearly, \( R_3(z) \vert_{\mathcal{X}_\infty^\prime} = 0 \). For \( \zeta^\vee_\infty \cap \mathcal{X}_\infty^\prime\prime \), which must bound on \( C_\infty \), we may take the two-chain parametrized by \( (z_2, z_3) = \{(e^{i\theta}, \rho \mathcal{g}(e^{i\theta})) \mid \theta \in [-\frac{\pi}{3}, \frac{\pi}{3}], \rho \in [0,1]\} \), this yields

\[ ^{†} \text{Compare [30, Theorem 4.2(b) + Corollary 4.3], which this generalizes.} \]
\[ V(0) = \int_{\mathbb{C}^2 \cap X''_{\infty}} \log(\varphi(z_2, z_3)) \frac{dz_2}{z_2} \wedge \frac{dz_3}{z_3} = \int_{\mathbb{C}^2 \cap X''_{\infty}} \left( \text{Li}_2(\varphi(z_2)) - \text{Li}_2((1 - z_2)\varphi(z_2)) \right) \frac{dz_2}{z_2} \]

\[ = \int_{-\frac{\pi i}{3}}^{\frac{\pi i}{3}} (4 \log(1 - u) + \log(u)) \log(u) \frac{du}{u} = \left[ 4 \text{Li}_3(u) - 4 \text{Li}_2(u) \log(u) + \frac{1}{3} \log^3(u) \right]_{-\frac{\pi i}{3}}^{\frac{\pi i}{3}} = \frac{4\pi^3 i}{27}, \]

whereupon \( \hat{V}(0) = \frac{V(0)}{9\sqrt{-3}} = \frac{4\pi^3}{3^5 \sqrt{3}} = \frac{1}{3}L(\chi_3, 3). \)

## 6 APÉRY AND NORMAL FUNCTIONS

In this brief final section, we introduce a framework for studying the normal functions arising in connection with the Arithmetic Mirror Symmetry Problem (including the examples in §5), and propose some terminology.

**Definition 6.1.** The Apéry motive is \( \mathbb{A}_\phi := H^n(\mathcal{X} \setminus \bar{X}_\infty, \bar{X}_0)/\text{Ph}_{\Sigma^*} \) from (4.24), or (if one prefers) its underlying mixed motive.

We dig into its structure a bit: there are exact sequences of MHS

\[
\begin{array}{cccccc}
0 & \uparrow & \uparrow \\
(\phi_{\infty} H_{u}^{n-1})_{\mathcal{A}} & (-1) \\
0 & (\phi_{0} H_{u}^{n-1})_{\mathcal{T}^0} & \mathbb{A}_\phi & H^1(\mathbb{A}_{\mathcal{X}}, H_{u}^{n-1}) & 0 \\
& & & IH^1(\mathbb{P}^1, H_{u}^{n-1}) & \\
0 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
& & & IH^1(\mathbb{P}^1, H_{u}^{n-1}) & 0 & \\
\end{array}
\]

where \((\cdot)_{\mathcal{A}} = \text{coker}(T_{\infty} - I), (\cdot)_{\mathcal{T}^0} = \text{ker}(T_0 - I), \) and \( \mathbb{A}^1 \) means \( \mathbb{P}^1 \setminus \{t = \infty\} \). The parabolic cohomology \( IH^1(\mathbb{P}^1, H_{u}^{n-1}) \) is pure of weight \( n \) and rank

\[ ih^1(\mathbb{P}^1, H_{u}^{n-1}) = \sum_{\sigma \in \Sigma} \text{rk}(T_{\sigma} - I) - 2r. \]

**Definition 6.2.** \( H_{u}^{n-1} \) (or \( \phi \)) is extremal if (6.2) is zero.

Recall that if \( \phi \) is tempered, the coordinate symbol \( \{x\} \) lifts to \( \xi \in CH^n(\mathcal{X} \setminus \bar{X}_0, n) \). The cycle class of \( \text{Res}(\xi) \in CH^{n-1}(\bar{X}_0, n - 1) \) yields an embedding \( \mathbb{Q}(-n) \hookrightarrow H^{n+1}_{\bar{X}_0}(\mathcal{X}) \cong H_{n-1}(\bar{X}_0)(-n) \), or dually\(^1\) a splitting

\[ \varepsilon : (\phi_{0} H_{u}^{n-1})_{\mathcal{T}^0} \rightarrow \mathbb{Q}(0). \]

\(^1\)The first map in the portion \( H_{n+1}(\bar{X}_0)(-n) \rightarrow H^{n-1}(\bar{X}_0) \rightarrow (\phi_{0} H_{u}^{n-1})_{\mathcal{T}^0} \rightarrow 0 \) of the Clemens–Schmid sequence has pure weight \( n - 1 \), and so, the second map has a splitting \((\phi_{0} H_{u}^{n-1})_{\mathcal{T}^0} \hookrightarrow H^{n-1}(\bar{X}_0) \) that is an isomorphism in weights \( < n - 1 \). Dualizing the embedding yields \( H^{n-1}(\bar{X}_0) \rightarrow \mathbb{Q}(0) \), and (6.3) is the composition.
Suppose then that \( \phi \) is tempered and extremal, and that (for some \( p \in \mathbb{N} \)) there exists an embedding
\[
\mu : \mathbb{Q}(-p) \hookrightarrow (\psi_\infty H_u^{n-1})_{T_{\infty}}(-1).
\] (6.4)
Then from (6.1), we obtain the diagram
\[
0 \rightarrow \mathbb{Q}(0) \rightarrow \mu^* \varepsilon_* A_\phi \rightarrow \mathbb{Q}(-p) \rightarrow 0 \quad \| \mu^* \varepsilon_* A_\phi \rightarrow (\psi_\infty H_u^{n-1})_{T_{\infty}}(-1) \rightarrow 0
\]
(6.5)
with exact rows. Under \( \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(-p), \mathbb{Q}(0)) \cong \mathbb{C}/\mathbb{Q}(p) \), define
\[
\alpha_\phi(\mu) \in \mathbb{C}/\mathbb{Q}(p)
\] (6.6)
to be the image of the extension class of the top row of (6.5).

**Example 6.3.** The Laurent polynomials considered in §§5.3–5.5 are tempered and extremal, with \((\psi_\infty H_u^2)_{T_{\infty}}(-1) \cong \mathbb{Q}(-2)\) resp. \(\mathbb{Q}(-3)\) for \(V_{10}, V_{14}\) resp. \(V_{12}, V_{16}, V_{18}\). (Indeed, \(\mu\) and \(\varepsilon\) are both isomorphisms.) In view of (6.10) below, in each case, \(\alpha_\phi(\mu)\) is just \(V(0)\) viewed modulo \(\mathbb{Q}(p)\). But for \(V_{10}, V_{14}, \text{and } V_{18}\), \(V(0)\) is in \(\mathbb{Q}(p)\) and so \(\alpha_\phi(\mu)\) is trivial!

From the example, we see the importance of presenting (6.6) as a limit of an HNF, since by canonically normalizing the latter (Remark 5.2), we may then refine (6.6) to a well-defined complex number. To do this, note that the same proof as for Proposition 4.16 expresses the VMHS \(A_{\phi,t} := H^n(X \setminus \overline{X}, \overline{X})/\text{Ph}_{\Sigma \setminus \{\infty\}}\) as an extension
\[
0 \rightarrow H_u^{n-1} \rightarrow A_\phi \rightarrow \text{IH}^1(A^1, H_u^{n-1}) \rightarrow 0.
\] (6.7)
So, we arrive at this article’s eponymous

**Definition 6.4.** The pullback
\[
0 \rightarrow H_u^{n-1} \rightarrow \mu^* A_\phi \rightarrow \mathbb{Q}(-p) \rightarrow 0
\] (6.8)
of (6.7) under (6.4) is called an Apéry extension.

We may view (6.8) as a higher normal function
\[
\nu_\mu \in \text{ANF}(H_u^{n-1}(p)),
\] (6.9)
which is singular at \( t = \infty \) and only there,† and we define and normalize \( V_\mu(t) := \langle \nu_\mu, \omega_t \rangle \) as in §5.1. Since \( \mu^* A_\phi \cong (\psi_0 \mu^* A_\phi)^{T_0} \) and \(\varepsilon\) is induced by pairing with \( \lim_{t \to 0} \omega_t \), we conclude that
\[
V_\mu(0) \mapsto \alpha_\phi(\mu) \quad \text{under } \mathbb{C} \rightarrow \mathbb{C}/\mathbb{Q}(p).
\] (6.10)

† See the proof of [31, Theorem 10.8].
At least when $\phi$ is tempered and extremal, and $\IH^1(\mathbb{A}^1, H_u^{n-1})$ is split Hodge–Tate, it is these $V_\mu(0)$ which are expected to produce (up to $\bar{\mathbb{Q}}^*$) the interesting Apéry constants. (In the absence of these conditions, of course, the situation will be more complicated.)

Conversely, any $\nu \in \ANF(H_u^{n-1}(p))$ nonsingular off $\infty$ arises as in (6.8). In our fine examples, $\IH^1(\mathbb{A}^1, \mathbb{P}_{n-1}^n) \cong \nu$ has rank 1. So, this finishes off the uniqueness part of Problem 5.3 in each case, completing the proof of Theorem 1.1.

Now according to the BHC (since $\mathcal{X} \setminus \bar{\mathcal{X}}_\infty$ is defined over $\bar{\mathbb{Q}}$), there exists a higher cycle $\mathcal{Z}_\mu \in CH_p(\mathcal{X} \setminus \bar{\mathcal{X}}_\infty, 2p-n)_\mathbb{Q}$ with $\nu_\mu = \nu_{\mathcal{Z}_\mu}$. This provides a mechanism for explaining the arithmetic content of the Apéry constants. Let $K \subset \bar{\mathbb{Q}}$ be the field of definition of $\mathcal{Z}_\mu$. By [12, Corollary 5.3ff], $V_\mu(0)$ may be interpreted as the image of $\tilde{t}_X \mathcal{Z}_\mu$ under

$$H^n_{\mathcal{M}}(\mathcal{X}_0, K(p)) \cong J(H^{n-1}(\mathcal{X}_0)_\mathbb{Q})$$

where the second map comes from temperenedness of $\phi$. Since $\mathcal{X}_0 = \bigcup_i Y_i$ is an normal crossing divisor (NCD), we have a spectral sequence $E_1^{a, b} = Z^p(\mathcal{X}_0^{[a]}, 2p-n-b) \Rightarrow H^{n+p}_{\mathcal{M}}(\mathcal{X}_0, K(p))$

where $\mathcal{X}_0^{[a]} := \coprod_{|I| = a+1} (\cap_{i \in I} Y_i)$. The induced filtration $\mathcal{W}$, [12, §3] has bottom piece

$$\mathcal{W}_{-n+1} H^n_{\mathcal{M}}(\mathcal{X}_0, K(p)) \cong \coker(CH^p(\mathcal{X}_0^{[-n+2]}, 2p-1) \to CH^p(\mathcal{X}_0^{[-n+1]}, 2p-1))$$

and (6.11) restricts to the Borel regulator on this piece.

**Example 6.5.** In §§5.3–5.5, $Z_0$ belongs to $\mathcal{W}_{-2} H^3_{\mathcal{M}}(\mathcal{X}_0, K(p))$, with $K = \mathbb{Q}(\sqrt{-3})$ for $V_{18}$ and $K = \mathbb{Q}$ for the other $V_{2N}'s$. Since each $\alpha_{X_0}$ is also real by construction, and $\mathfrak{f}$ belongs to $K$, Borel's theorem (together with part (c) of Problem 5.3) forces $\alpha_{X_0}$ to be in $\mathbb{Q}(2)(V_{10}, V_{14}), \mathbb{Q}(3)(V_{12}, V_{16})$, and $\mathbb{Q}(3)(V_{18})$, respectively, before any computation is done.

**Remark 6.6.** We finally owe the reader an explanation regarding the flip in perspective from $\psi_0 \mathcal{A}_{\phi}^\dagger$ (and the limit of the coordinate-symbal normal function at $\infty$) to $\psi_0 \mathcal{A}_{\phi}$ (and the limit of Apéry normal functions at 0), specifically the “computational nonviability” claimed for the former. First of all, if we choose the lift $\bar{\nu}_\phi$ to be single-valued around $\infty$, it is a section of the dual-canonical extension ($H^{n-1}_u$); since $\omega$ is a section of $H_u^{n-1}$ with a simple zero at $\infty$, they do indeed pair to a holomorphic function $V_\phi$ on a disk $D_\infty$, but one with $V_\phi(\infty) = 0$. Replacing $\omega$ by $\hat{\omega} := t\omega$ gives $V_\phi := t V_\phi$, from which we can in principle read off the limiting extension class if we know the limits of the invariant periods of $\hat{\omega}$ at $\infty$ (which is already nontrivial); and this was essentially the method used for $V_{16}$ and $V_{18}$.

But this approach becomes problematic when $T_\infty$ is nonunipotent, intuitively because $\hat{\omega}$ then has periods that blow up at $\infty$, and we lack a suitable representative for $\hat{\omega}(\infty)$ on $\mathcal{X}_\infty$. More precisely, if we let $\rho : D_\infty \to D_\infty$ be the base change (ramified at $\infty$) that kills $T_{\infty}^{SS}$, the pullback $\rho^* \omega$ is not a section of $(\rho^* H^{n-1}_u) \circ D_\infty$, and we cannot use [12, Corollary 5.3] to compute $V_\phi(\infty)$. So, while, for (say) $V_{10}$ and $V_{14}$, one can show (abstractly, from its inhomogeneous equation) that $V_\phi(\infty)$ is a nonzero complex number, it does not seem nearly as accessible as the $V_\mu(0)$ values computed in §§5.3 and §§5.5.
Remark 6.7. In addition to its implications for the arithmetic of $\alpha_{X^o}$, Problem 5.3 appears to encode interesting algebro-geometric predictions about Fanos. To just give the idea in the simplest possible case, suppose that $F^m$ is a Fano $m$-fold, with $H^e_{\text{prim}}(F^m)$ of rank 2, concentrated in weights 0 and $2w$, with degree 2 QDE. Let $P_i$ denote general hyperplanes in some $\mathbb{P}^M$ in which $F^m$ is minimally embedded, and accept the idea that — as long as $F^{m-\ell}$ : $= F^m \cap P_1 \cap \cdots \cap P_{\ell}$ remains Fano — Problem 5.3 (a)–(c) continue to hold and the Apéry constant $\alpha$ remains unchanged (cf. Remark 3.6). At first, $\alpha = \alpha_{F^m}$ is computed by the LMHS of the LG model; but after hyperplane sections kill off the second Lefschetz string in $H^*$, $\alpha = \alpha_{F^{m-\ell}}$ is computed by the limit of a nontorsion extension of $H^{m-\ell-1}$ by $\mathbb{Q}(-w)$ (even if $\alpha$ is “torsion”). As soon as $F^{w-1} H^{m-\ell-1} = \{0\}$, however, Griffiths transversality forces such normal functions to be flat and thus torsion. So, for Problem 5.3 to be consistent, $F^{m-\ell}$ cannot be Fano for $m - \ell < w$; that is, the index $i(F^m)$ is $\leq m - w + 1$. (Recall that $i$ is defined by $-K_F = i h$.) A quick perusal of examples in this paper suggests that this is sharp: $G(2, 5)$, $OG(5, 10)$, and $LG(3, 6)$ (but not $G(2, 6)$ or $G_2/P_2$) each have rank two $H^e_{\text{prim}}$ and $d = 2$; while their respective $(m, w, i)$ are $(6, 2, 5)$, $(10, 3, 8)$, resp. $(6, 3, 4)$.

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