Evolving Surfaces and Evolving Implicit Differential
Equations Via Contact Geometry and Singularities

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Abstract. We present the list of unavoidable local phenomena (transitions) occurring on
the configuration of the flecnodal and parabolic curves of evolving smooth surfaces in \( \mathbb{R}^3 \)
(or \( \mathbb{R}P^3 \)). We also present the list of transitions occurring on the curve of inflections of
the solutions of evolving implicit differential equations (IDE). Our results are based on the
properties of the contours of surfaces (in a contact 3-space) for projections all whose fibres
are Legendrian.

Keywords: Surface, flecnodal curve, contact geometry, implicit differential equations.

MSC2010: 14B05, 32S25, 58K35, 58K60, 53A20, 53A15, 53A05, 53D99, 70G45.

Introduction and Main Results

We investigate two subjects: the geometry of solutions of implicit differential equations (IDE) in one variable and the local transitions of some robust properties of surfaces in 3-space. The main result on IDEs is a classification of generic 1-parameter local transitions of the discriminant (see \( \S 0.2 \)) and the curve of inflections (formed by the inflection points of the solution curves).

The main result on smooth surfaces is a classification of the local transitions of the flecnodal and parabolic curves (defined in \( \S 0.1 \)) occurring in generic 1-parameter families of smooth surfaces in \( \mathbb{R}^3 \) (or \( \mathbb{R}P^3 \)).

Our study of surfaces in 3-space is done in terms of IDEs, and in both subjects, the investigation is done in the setting of contact geometry and the Legendre singularity theory initiated by V. I. Arnold.

We assume that all manifolds and maps are smooth, which means “continuously differentiable the necessary number of times”, for example \( C^\infty \).

0.1 Transitions in Evolving Surfaces

Along the paper, a smooth evolving surface \( S_\epsilon \) means a 1-parameter family of smooth surfaces \( S \times \mathbb{R} \rightarrow \mathbb{R}P^3 \) (or \( \mathbb{R}^3 \)).

A generic smooth surface in \( \mathbb{R}^3 \) can have an open hyperbolic domain \( H \) at which the Gaussian curvature \( K \) is negative, an open elliptic domain \( E \) where \( K \) is positive and a parabolic curve \( P \) where \( K = 0 \). The flecnodal curve \( F \) is formed by the inflections of the asymptotic curves (points at which the first two derivatives of the curve are collinear). If the surface depends on one parameter (say, the time), the configuration formed by the flecnodal and parabolic curves may change.
In the middle of the 1980’s, while working in computer vision problems, D. Mumford communicated to V. Arnold the following

**Mumford’s problem:** Find all local transitions of the flecnodal and parabolic curves occurring in generic 1-parameter families of surfaces.

It turns out that there are four types of points of generic surfaces which are involved in such transitions. We have to mention them:

We distinguish two branches of the flecnodal curve, called left and right, according to the orientation of the framing (see §§4.2 for the definition).

A hyperbonode is a point of transverse intersection of the left and right branches of the flecnodal curve, and an ellipnode is a real intersection point of complex conjugate flecnodal curves. A biflecnode, noted $b$ in Fig.17, is an isolated point of the flecnodal curve at which an asymptotic curve has a bi-inflection (the first 3 derivatives are collinear - see §4.1, p. 25).

A godron is a parabolic point at which the (unique) asymptotic direction is tangent to the parabolic curve. The following theorems have already been well known:

**Theorem** [28, 21, 27, 23]. At a godron of a generic smooth surface the flecnodal curve is tangent to the parabolic curve.

**Theorem 0** [31]. Godrons locally separate the left and right branches of the flecnodal curve.

In Fig.1 we represent the elliptic domain in white, the hyperbolic domain in grey (red for online version), the right branch $F_r$ of the flecnodal curve in black, the left branch $F_l$ in white and godrons as white points on $P$.

Fig. 1. Transitions of the tangential singularities in generic evolving smooth surfaces. The sign $+$ or $-$ is the index (defined in p. 27) of the godrons taking part in the bifurcation.

**List 1.** Our first main result solves Mumford’s problem. It is the list, depicted in Fig.1 of local transitions of the flecnodal and parabolic curves (and the
above four special points) that can occur on a generic evolving smooth surface.

For example, the 4th A\textsubscript{3} transition in Fig. 1 means that (see Fig. 2):

*The birth of a hyperbolic disc generates the birth of a flecnodal curve inscribed in the bounding parabolic curve and having the shape of the figure eight. That transition occurs at an ellipnode which becomes hyperbonode after the birth.*

![Fig. 2](image)

*Fig. 2. The unavoidability of the “eight” of F at the birth of a hyperbolic disc.*

In §5, Theorem 5.1 corresponds to A\textsubscript{4} and flec-godron transitions; Theorem 5.2 to A\textsubscript{3} transitions; Theorem 5.3 to lips, bec-à-bec and swallowtail transitions, Theorem 5.4 to creation/annihilation of hyperbonodes (or ellipnodes) and flec-hyperbonode transitions; Theorem 5.5 to D\textsubscript{4} transitions.

**Fact.** Fig. 1 implies that to have an A\textsubscript{3} transition of the flecnodal curve, it is necessary to have either an ellipnode or a hyperbonode at which the transition takes place: that ellipnode is replaced by a hyperbonode (or the opposite).

(In Fig. 2 there was an ellipnode before the “birth of the 8”.)

In some transitions of Fig. 1, only the left flecnodal curve F\textsubscript{ℓ} (in white) is represented; however, the right flecnodal curve F\textsubscript{r} (in black) may have the same transitions. The branches F\textsubscript{ℓ} and F\textsubscript{r} play symmetric roles and are interchangeable in all transitions of Fig. 1.

### 0.1.1 Flecnodal curve on propagating wave fronts

The *tangential map* of a smooth surface \(S, \tau_S : S \rightarrow (\mathbb{R}P^3)^\vee\), associates to each point of \(S\) its tangent plane at that point. The *dual surface* \(S^\vee\) of \(S\) is the image of \(\tau_S\). Under \(\tau_S\) the parabolic curve of \(S\) corresponds to the cuspidal edge of \(S^\vee\) and a godron to a swallowtail point (c.f. [28]).

The natural approach to the singularities of the tangential map is via Arnold’s theory of Legendre singularities [5]. The image of a Legendre map is called its *front* (see §2). The tangential map of \(S\) is a Legendre map; so if \(S\) is in general position the only local singularities of its dual \(S^\vee\) are those of generic fronts: cuspidal edges and swallowtails. Thus the transitions of the parabolic curve may be obtained from the list of transitions of wave fronts in 3-space (appeared first in [1], later in [2, 5]).

Since the tangent planes to \(S\) along the flecnodal curve form the flecnodal curve of its dual \(S^\vee\), the projective dual to a hyperbonode (ellipnode) is a hyperbonode (resp. ellipnode) [30]. So from the A\textsubscript{3}-transitions of Fig. 1 (Th. 5.2) we get the transitions of the flecnodal curve on fronts (see Fig. 3):

**Proposition 0.1.** *The A\textsubscript{3} transitions occurring in a generic propagating wave front \(S_e\) take place at an ellipnode or at a hyperbonode which is replaced, respectively, by a hyperbonode or an ellipnode (Fig. 3). In front-time 3-space \(\{S_e \times \{\varepsilon\}\}\), the flecnodal curves of the fronts form a folded Whitney umbrella whose cuspidal edge is swiped by two swallowtail points, which collapse and disappear at the folded pinch-point (Fig. 4).*
0.2 Transitions in Evolving Implicit Differential Equations

Given a smooth function \( F : J^1(\mathbb{R}, \mathbb{R}) \to \mathbb{R} \), consider the implicit differential equation (IDE)

\[
F(x, y, p) = 0, \quad \text{where} \ p = \frac{dx}{dy}.
\]  

(0.1)

In the space of 1-jets, IDE (0.1) determines a surface \( \mathcal{V}^F \). The standard contact structure of \( J^1(\mathbb{R}, \mathbb{R}) \) endows \( \mathcal{V}^F \) with a foliation by Legendre curves, called characteristic curves of \( \mathcal{V}^F \) (see §2.1 for the definitions).

The projection of \( \mathcal{V}^F \) on \( \mathbb{R}^2 \) in the direction of the \( p \)-axis

\[
\pi_{\mathcal{V}^F} : \mathcal{V}^F \to \mathbb{R}^2, \quad \pi(x, y, p) = (x, y),
\]  

(0.2)

sends the foliation of \( \mathcal{V}^F \) to the family of solution curves of IDE (0.1). The discriminant curve \( D \) of IDE (0.1) is the set of critical values of \( \pi_{\mathcal{V}^F} \).

**Inflection.** An inflection (bi-inflection) of a plane curve is a point at which the curve has at least 3-point (resp. 4-point) contact with its tangent line.

The curve of inflections \( I \) of IDE (0.1), which consists of the inflections of the solution curves, has two branches \( I_L \) and \( I_R \), called left and right: they arise from different parts of \( \mathcal{V}^F \) (see §2.1, §2.3 for the definitions). A hyperbolic node is a point of transverse intersection of the left and right curves of inflections. A bi-inflection of IDE (0.1) is just a bi-inflection of a solution curve (left or right).

If \( \mathcal{V}^F \) is in general position, its characteristic foliation may have isolated singularities (saddle/node/focus), called characteristic points. The projection of such a point in the \( (x, y) \)-plane is called folded singularity of the IDE (13).
We restrict ourselves to \textit{binary IDE} (BIDE): such an IDE defines two (real or complex) directions, counting multiplicities, at each point in the plane
\[ F(x,y,p) = a(x,y) + 2b(x,y)p + c(x,y)p^2 = 0. \] (0.3)

\textbf{Remark.} Equation (0.3) forbids the “vertical” direction \((p = \infty)\). But we can choose any given direction as the \(x\)-axis \((p = 0)\). In the multilocal case, we can choose the \(y\)-axis to be different from the second direction.

Now we can formulate a BIDE version of Mumford’s problem:

\textbf{Mumford’s problem 2:} find all local transitions of the curve of inflections, the discriminant and the special points (hyperbolic nodes, bi-inflections, folded singularities) occurring in generic 1-parameter families of BIDE.

\textbf{List 2.} Our second main result solves Mumford’s problem 2. It is the list, depicted in Fig. 5, of local transitions of the curve of inflections, the discriminant (and the above special points) that can occur on a generic evolving BIDE. In Fig. 5 the (real) domain of definition of the BIDE is represented in grey (red for online version), the right branch \(I_r\) in black, the left branch \(I_l\) in white, and folded singularities as white points on the discriminant curve.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{Transitions of the curve of inflections, discriminant and special points in generic evolving BIDE. The signs + or − are the indices of the folded singularities (defined in p. 12).}
\end{figure}

\textbf{Elliptic Nodes.} In the domain where a BIDE determines a field of pairs of complex conjugate directions, the inflections of their solution curves form a pair of complex conjugate curves which intersect at real points called \textit{elliptic nodes}. To underline that a creation/annihilation of two elliptic nodes could have no meaning for some evolving IDEs, we include it in a dotted box.

In §1 Theorem 1.1 corresponds to \(UW\), flec-godron and \(\gamma v\) transitions; Theorem 1.2 to \(a, b, c, d, e, f\) transitions; Theorem 1.3 to \(\text{lips, bec-\text{a-bec}}\) and \(\text{swallowtail transitions, Theorem 1.4 to creation/annihilation of hyperbolic (or elliptic) nodes and flec-hyperbolic node transitions.}\n
\[5\]
0.3 Equivalence of transitions and models of transitions

A smooth 1-parameter family of plane curves $C_\varepsilon$ determines a surface $\bigcup_{\varepsilon \in \mathbb{R}} C_\varepsilon$ in plane-time 3-space, naturally foliated by its isochronal sections $C_\varepsilon$.

**Equivalence of Transitions.** Two 1-parameter families of curves $C_\varepsilon$, $C_\mu$ having a local transition at $q \in \mathbb{R}^2$ for $\varepsilon = \mu = 0$ are said to be equivalent if there is a local diffeomorphism (homeomorphism) from $\mathbb{R}^2 \times \mathbb{R}$ to itself, near $(q, 0)$, sending diffeomorphically isochronal sections to isochronal sections.

In this paper, most transitions are equivalent to the “isochronal” sections $\varepsilon = \text{const}$. of the surfaces, or pairs of surfaces (given by equations) of Table 1.

| equation(s) | name of the transition model |
|------------|----------------------------|
| $x^2 \pm y^2 - \varepsilon = 0$ | + Morse centre-type ; − Morse saddle-type |
| $x^2 \pm y^2 = \varepsilon^2$ | + elliptic cone section ; − hyperbolic cone section |
| $y^2 = \varepsilon x^2 + \alpha x^4$ | $\alpha > 0$ elliptic $A_3$; $\alpha < 0$ hyperbolic $A_3$ |
| $(y - x^2 + \varepsilon)(y + x^2 - \varepsilon) = 0$ | creation/annihilation of two crossings |
| $(y + \varepsilon)(x^2 + \varepsilon^2) = 0$ | $D_2^\pm$ transitions |
| $y = 0$; $x^2 - y(y - \varepsilon)^2 = 0$ | $\gamma_v$-transition |
| $y = 0$; $y - (x^2 - \varepsilon)^2 = 0$ | UW-transition |

Table 1: Seven surfaces whose sections $\varepsilon = \text{const}$. provide most transitions of this paper.

A centre or saddle transition is respectively called *lips* or *bec-à-bec* if it involves only the curve of inflections (or only the flecnodal curve).

We have moreover two transition types in which the involved curves (which are tangent or have a crossing) undergo no structural change, but one special point (biflecnode), which moves along one of them, pass through their common point at the transition moment. Any two such transitions with tangency (or with crossing) are equivalent.

0.4 Flecnodal Curve of Surfaces via Asymptotic IDEs

The connection between IDEs and the flecnodal curve of a surface is as follows. Given a Monge form $z = f(x, y)$ of a surface $S$ in $\mathbb{R}^3$, the asymptotic IDE

$$f_{xx}(x, y) + 2 f_{xy}(x, y)p + f_{yy}(x, y)p^2 = 0,$$

provides the flecnodal and parabolic curves. Namely, the projection of $S$ to the $(x, y)$-plane, in the $z$-direction, sends the asymptotic curves of $S$ to the solution curves of (0.4), sends the flecnodal curve to the curve of inflections (Prop. 6.2) and sends the parabolic curve to the discriminant.

The class of asymptotic IDEs is very thin in the space of all BIDEs. For this reason the local transitions for these two classes are discussed separately. They correspond respectively to List 1 and List 2 above.

0.5 Curve of Inflections and Legendrian Fibrations

A key element, necessary to investigate the curve of inflections of IDE (0.1) and its transitions, is the following fact, explained and proved in §2.3 (Fig. 10).
Consider the apparent contour $\hat{I}$ of the surface $V^F \subset J^1(\mathbb{R}, \mathbb{R})$ by the Legendrian (dual) projection $\pi^\vee: (x, y, p) \mapsto (X, Y) = (p, px - y)$. (I.e. $\hat{I}$ is the critical set of the map $\pi^\vee|_{V^F}: V^F \to (\mathbb{R}^2)^\vee$.)

**Fact.** Projection (0.2) sends $\hat{I}$ to the curve of inflections of IDE (0.1)- Fig.10

0.6 Some relevant references and the origin of this paper

At the end of the 19th Century, the flecnodal and parabolic curves of complex algebraic surfaces were investigated by Cayley, Zeuthen and Salmon ([28]). In the same years, Korteweg investigated the transitions of the parabolic and conodal curves of evolving real surfaces - in thermodynamics, he studied the evolution of the graph of the energy (as a function of the volume and the entropy) [21, 22]; so Korteweg was a founder of catastrophe theory. These Korteweg works were forgotten (or unknown) by the scientific community. (In 2004 E. Ghys and D. Serre let me know the book [24], which describes Korteweg’s studies.)

In the 1980’s, R. Thom ([20]) and V. Arnold’s school (cf. [23, 27]) revived the subject from the viewpoint of singularity theory; while others (cf. D Mumford) applied it to computer vision. Some later contributions are, for example, [25, 31, 26, 32]. The classification of jets of functions of [29] (and [18]) is complementary to our work; but to apply it to geometry of surfaces, one requires additional considerations (on the higher order terms and the moduli).

Concerning the local normal forms for IDEs, we have to mention [11, 12, 13, 14, 10, 15], and specially [13] related to godrons and [6] related to conic singularities of the surface $V^F$.

Among some pictures of computer experiments by A. Ortiz, the “8” in the transition of Fig. 2 drew V. Arnold’s attention, who suggested to prove (or disprove) the unavoidability of the “8”. Arnold considered it as a fundamental fact of geometry of surfaces (Paris, 1999). This unavoidability problem (solved by Theorem 5.2), the preprint of Panov’s paper [25] and Mumford’s problem were the motivations for this research. Many results of this paper come from Ch. 7 of my Ph. D. Thesis ([30]) and were exposed in the Singularity Theory Semester at Newton Institute (2000) and in a mini-course (Int. Conf. of Real and Complex Singularities Sao Carlos, 2002), where the preprint Surface Evolution, Implicit Differential Equations and Pairs of Legendrian Fibrations was distributed. This is a completely revamped version: shorter, clearer and includes several new results.

**Organisation.** Part I (On IDE): In §1 we present our results on evolving IDE (Ths. 1.0 to 1.4). In §2, we explain the ideas of elementary contact geometry and singularity theory on which most theorems of the paper are based; there, we study IDE as surfaces in a contact 3-manifold and describe the curve of inflections (of the solutions of an IDE) in terms of pairs of Legendre fibrations. In §3, we state and prove the theorems of §1 in the setting of contact geometry and singularity theory (Ths. 3.0 to 3.4).

Part II (On surfaces in 3-space): In §4 we recall the classification of points of a surface (by the contact with its tangent lines) and some robust properties. In §5, we state Theorems 5.1 to 5.5 on transitions of the flecnodal curve and give further results. In §6, we explain the connection between surfaces and our contact geometry study of IDE. In §7, we prove the theorems of §5.

The reader interested only in the transitions of the flecnodal and parabolic curves on evolving surfaces in 3-space can directly go to §4 and §5 (in Part II) and then read §2, §3 and §7.

**Acknowledgements.** I thank D. Panov, M. Kazarian, P. Pushkar, F. Aicardi and T. Ohmoto for useful comments, to D. Meyer for valuable remarks to the first version (2001), to UMI2001 CNRS Solomon Lefschetz UNAM México where I wrote the last version and to the two referees whose numerous remarks help a lot to improve the presentation.

To the memory of V.I. Arnold who, in life, pressured me to publish this work.
Part I

Transitions of Evolving BIDE

1 Transitions of the curve of inflections of BIDE

Given a BIDE $F(x, y, p) = 0$, the critical set of the map $\pi_{|V^F} : V^F \to \mathbb{R}^2$ (given by $F = F_p = 0$) is called the criminant of that BIDE, and noted $\hat{D}$.

**Theorem 1.1.** (i) If a folded singular point $(x, y)$ of a BIDE $F = 0$ verifies the following genericity conditions at its characteristic point $(x, y, p)$:

a) the criminant $\hat{D}$ of $V^F$ is not tangent to the $\pi^\vee$-contour $\hat{I}$ of $V^F$;

b) the $\pi^\vee$-fibre is not tangent to the $\pi^\vee$-contour $\hat{I}$ of $V^F$;

c) the $\pi$-fibre is not tangent to the $\pi^\vee$-contour $\hat{I}$ of $V^F$;

then the curve of inflections $I$ is quadratically tangent to the discriminant $D$.

(ii) That point locally separates $I$ into its left and right branches (Fig. 6).

![Fig. 6. A folded singular point separates the left and right branches of $I$.](image)

**NOTE.** In terms of $F$, the genericity conditions $a, b, c$ are given by

\[
\begin{vmatrix}
F_y & F_{xp} & F_{pp} \\
F_{xx} + pF_{xy} & F_{xp} + pF_{yp} + F_y & \neq 0 \quad \text{b)} \quad (\partial_x + p\partial_y)^2F \neq 0;
\end{vmatrix}
\]

c) $F_{xp} + pF_{yp} + F_y \neq 0$.

Item (i) of Theorem 1.1 appeared first in [9]; however, the condition $c$, $F_{xp} + pF_{yp} + F_y \neq 0$, which prevents the curve of inflections from having a cusp, was absent, and the left-right statement of (ii) was not considered.

**Theorem 1.1.** Let $V^{F_\varepsilon} \subset J^1(\mathbb{R}, \mathbb{R})$ be a generic 1-parameter family of smooth surfaces (of IDEs). If the surface $V = V^{F_0}$ has a characteristic point $Q$ that breaks one of the conditions $a, b$ or $c$ of Theorem 1.1, then the curve of inflections $V^{F_\varepsilon}$ undergoes the following respective transitions:

a) $UW$: $(\varepsilon = 0)$ at the folded singularity $q = \pi(Q)$ the discriminant $D = \pi(\hat{D})$ has 4-point contact with the curve of inflections $I = \pi(\hat{I})$. It disappears $(\varepsilon < 0)$ or splits into two folded singular points of opposite indices $(\varepsilon > 0)$.

b) flic-folded singularity: a negative folded singular point overlaps with a bifurcnode which passes from one branch of the curve of inflections to the other.

At $\varepsilon = 0$ the curve of inflections $I = \pi(\hat{I})$ has an inflection at $q = \pi(Q)$.

c) $\gamma V$: the union of the curves of inflections of $V^{F_\varepsilon}$, with $\varepsilon \in (-1, 1)$, is a surface of the “plane-time” $\mathbb{R}^2 \times (-1, 1)$ locally diffeomorphic to the Whitney umbrella. For $\varepsilon = 0$ the curve $\pi(\hat{I})$ has a semi-cubic cusp at the folded singular point $q = \pi(Q)$. For any sufficiently small $|\varepsilon| \neq 0$ the folded singular point is generic (as in Th. 1.1).

At a critical point $Q$ of a function $F : J^1(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ the second degree terms of the Taylor series of $F$ determine a quadratic cone $K \subset T_Q J^1(\mathbb{R}, \mathbb{R})$. 

\[8\]
Theorem 1.2. Let \( \{ F(\varepsilon x, y, p) = 0, \varepsilon \in \mathbb{R} \} \) be a generic one parameter family of IDEs. If \( F = F_0 \) has a Morse critical point in general position with respect to \( \pi \) and to \( \pi^\vee \) (see §2.5), and the \( \pi \)- and \( \pi^\vee \)-fibres are not conjugate diameters of the cone \( K \) defined by \( F \) (see §2.5), then both curves \( D \) and \( I \) undergo a Morse transition, giving one of the transitions \( a \) to \( f \) in Fig.5, according to the relative position of \( K \) with the \( \pi \)- and \( \pi^\vee \)-fibres (Fig.12). Moreover, two folded singularities of equal indices are born or die.

Remark. Supposing our critical point is the origin, the conditions that guarantee it is in general position are (see example 2.6):

\[
(F^2_{xp} - F_{xx}F_{pp})(\bar{0}) \neq 0, \quad F_{pp}(\bar{0}) \neq 0 \quad \text{and} \quad F_{xx}(\bar{0}) \neq 0.
\]

The \( \pi \)- and \( \pi^\vee \)-fibres are not conjugate diameters of \( K \) iff \( F_{xp}(\bar{0}) \neq 0 \) (see Lemma 6.4 and Prop. 6.1).

The Morse transition types of \( I_\varepsilon \) and of \( D_\varepsilon \) are those of their respective degree two approximations (formed by conics). We get these types from the signs of the discriminants of their respective quadratic forms:

\[
-F_{xx}F_{xp}^2 \begin{vmatrix} F_{xx} & F_{xy} & F_{xp} \\ F_{xy} & F_{yy} & F_{yp} \\ F_{xp} & F_{yp} & F_{pp} \end{vmatrix} \quad \text{and} \quad -F_{pp} \begin{vmatrix} F_{xx} & F_{xy} & F_{xp} \\ F_{xy} & F_{yy} & F_{yp} \\ F_{xp} & F_{yp} & F_{pp} \end{vmatrix} \bigg|_{(0)}.
\]

Thus the transitions at \( \varepsilon = 0 \) of \( I_\varepsilon \) and \( D_\varepsilon \) are both of center-type or both of saddle-type iff \( F_{xx}F_{pp} > 0 \), and are of different types iff \( F_{xx}F_{pp} < 0 \).

According to [33], the fold and the Whitney pleat are the only stable singularities of maps from a surface to the plane - types 1 and 2 in Fig[7]. On movement of the surface, in a generic 1-parameter family, there are only three other singularities which can appear at isolated moments - they reduce locally to the projection, along the \( x \)-axis, of the surfaces \( z = f(x, y) \) of types 3, 4 or 5 ([4]), whose normal forms and contours are shown in Fig.7.

![Fig. 7. Generic singularities of projections in one-parameter families of surfaces.](image-url)

Theorem 1.3. If the map \( \pi^\vee_{|\mathcal{V}^F} : \mathcal{V}^F \to (\mathbb{R}^2)^\vee \), has a singularity of type 3, 4 or 5 of Fig.7, then when \( \varepsilon \) passes through 0, the curve of inflections of the solutions of the BIDE \( F_\varepsilon = 0 \) undergoes, respectively, the lips-, bec-à-bec- or swallowtail-transition in Fig.5 (Compare with Fig.8 and Figs.10 and 11).
The transitions of the curve of inflections correspond to the transitions of projections of surfaces in generic one-parameter families.

**Theorem 1.4.** Consider a generic 1-parameter family of BIDE $F_\varepsilon = 0$.

(i) If for $F_0$, $I_\ell$ and $I_r$ are quadratically tangent at $q$, then at $\varepsilon = 0$ we get a creation/annihilation transition of two hyperbolic nodes (c/a-h-n in Fig. 5).

(ii) If $q$ is a hyperbolic node for $F = F_0$ and $q$ is also a left bi-inflection, then as $\varepsilon$ passes through $0$ a left bi-inflection that moves along $I_\ell$ crosses $I_r$ at the hyperbolic node $q$ (flec-h-n in Fig. 5). (Left and right may be interchanged.)

(iii) If for $F_0$ two complex conjugate branches of $I$ are tangent at $q$, then at $\varepsilon = 0$ we get a creation/annihilation transition of two elliptic nodes.

Theorems 1.0 to 1.4 are particular cases of the contact geometry Theorems 3.0 to 3.4, which are stated and proved in Section 3.

### 1.1 Transitions of IDE in a Riemannian Plane

Suppose a smooth Riemannian metric is given in the $xy$-plane $\mathbb{R}^2$.

A geodesic inflection (resp. geodesic bi-inflection) of a smoothly immersed curve in the plane is a point at which the curve has 3-point contact (resp. 4-point contact) with its tangent geodesic at that point.

The theory developed in the next section implies the

**Claim.** Theorem 1.0 holds for the curve of geodesic inflections. The transitions of the curve of geodesic inflections and of the discriminant in generic 1-parameter families of IDEs are those described in Th. 1.1 to Th. 1.4 (Fig. 5).

### 2 IDEs as Surfaces in a Contact 3-Space

We give all necessary tools to understand the geometry of the curve of inflections of an IDE, showing the Legendrian nature of inflections and providing an “abstract” (but geometric) definition of inflection in terms of contact geometry.

**Contact Structure.** A contact structure on a smooth manifold is a maximally non-integrable field of hyperplanes - called contact hyperplanes.

Contact structures may exist only on odd-dimensional manifolds $M^{2n+1}$.

A field of hyperplanes is locally defined as the field of kernels of a differential 1-form, say $\alpha$. The *maximal nonintegrability condition* is $\alpha \wedge (d\alpha)^n \neq 0$.

**Example 2.1.** The 3-space $J^1(\mathbb{R}, \mathbb{R})$ has a contact structure whose contact planes are the kernels of the contact form $\alpha = pdx - dy$: at $Q \in J^1(\mathbb{R}, \mathbb{R})$, the contact plane is $\Pi_Q = \ker(pdx - dy)_Q \subset T_QJ^1(\mathbb{R}, \mathbb{R})$. 
2.1 Characteristics: Geometric Solutions of IDE

The geometry of IDE is part of the geometry of surfaces in contact 3-spaces.

Its description needs the most important objects of contact geometry:

Legendre submanifolds. A Legendre submanifold of a contact manifold is an integral submanifold \( L \) of the field of contact hyperplanes of the highest possible dimension (dimension \( n \) in a \( 2n + 1 \)-dimensional contact manifold). If the contact structure is locally given by the zeros of a contact form \( \alpha \), the restriction of \( \alpha \) to \( L \) vanishes: \( \alpha|_L = 0 \).

Example 2.2. The 1-graph of a smooth function \( f : \mathbb{R} \to \mathbb{R} \), which is the immersion in \( J^1(\mathbb{R}, \mathbb{R}) \) given by \( j^1 f : x \mapsto (x, f(x), f'(x)) \), is a Legendre curve of \( J^1(\mathbb{R}, \mathbb{R}) \). Indeed, \( \alpha(x, f(x), f'(x))(1, f'(x), f''(x)) = f'(x) \cdot 1 - f'(x) = 0 \).

Legendrian Fibration. A fibration \( M^{2n+1} \to B^{n+1} \) of a contact manifold \( M^{2n+1} \) is said to be Legendrian if all its fibres are Legendre submanifolds.

Example 2.3. The ‘forgetting derivatives map’ \( \pi : J^1(\mathbb{R}, \mathbb{R}) \to J^0(\mathbb{R}, \mathbb{R}) \) given by \( (x, y, p) \mapsto (x, y) \) is called standard Legendrian fibration. Indeed, it is Legendrian because the restriction of the contact form \( \alpha = pdx - dy \) to the vertical lines vanishes: \( \alpha_{(x_0, y_0, p)}(0, 0, \dot{p}) = p \cdot 0 - 0 = 0 \).

Legendre map. A Legendre map is the projection of a Legendre submanifold of the space of a Legendrian fibration to its base along its Legendre fibres.

Front. The image of a Legendre map \( L \subset M \to B \) is called the front of \( L \).

So given two Legendrian fibrations of \( M \), \( \rho_1 : M \to B_1 \) and \( \rho_2 : M \to B_2 \), every Legendre submanifold \( L \subset M \) has two fronts: the front \( \rho_1(L) \) in \( B_1 \) and the front \( \rho_2(L) \) in \( B_2 \).

Example 2.4. Under the standard fibration \( \pi : J^1(\mathbb{R}, \mathbb{R}) \to J^0(\mathbb{R}, \mathbb{R}) \), the front of the 1-graph \( j^1 f \) of a function \( f \) is the graph of \( f \) (Examples 2.2, 2.3).

On any smooth surface \( V \) of a contact 3-manifold \( M \) the contact structure defines an intrinsic field of tangent lines:

Characteristics of a surface. At almost every point \( Q \) of a smooth surface \( V \subset M \) in general position, the tangent plane of \( V \) is different from the contact plane. These planes intersect on a line tangent to \( V \) at \( Q \). This defines the field of characteristic tangent lines on \( V \). Its integral curves, called the characteristics of \( V \), are (by construction) Legendre curves of \( M^3 \).

Characteristic Fronts. Given a Legendrian fibration \( \rho : M^3 \to B \), a curve of \( B \) is called characteristic front of the surface \( V \subset M \) if it is the front of a characteristic curve of \( V \) by \( \rho : M^3 \to B \).

Example. The (geometric) solutions of a given IDE \( F(x, y, p) = 0 \) are the characteristic fronts of its surface \( V^F \subset J^1(\mathbb{R}, \mathbb{R}) \) by the standard Legendrian fibration \( \pi : (x, y, p) \to (x, y) \).

That is, the solutions are the fronts of the characteristic curves of \( V^F \).
Characteristic Point. A characteristic point of a surface \( V \subset M^3 \) is a point at which the tangent plane of \( V \) coincides with the contact plane.

Index of a Characteristic Point. For a surface \( V \subset M^3 \) in general position the set of characteristic points is discrete. At these points the field of characteristic lines is singular, having index \(+1\) or \(-1\) (Fig. 9).

Folded Singular Point. A folded singular point of a surface \( V \subset M^3 \) is the image by a Legendrian fibration \( \rho : M \to B \) of a characteristic point of \( V \).

For example, a folded singular point of the IDE \( F(x, y, p) = 0 \) is the image by \( \pi : (x, y, p) \mapsto (x, y) \) of a characteristic point of the surface \( V^F \) (see §0.2).

Near a folded singular point with smooth discriminant, the IDE is reduced to the normal form \( y = (p + kx)^2 \) by a diffeomorphism on the \( xy \)-plane [13]. Depending on the ‘modulus’ \( k \in \mathbb{R} \), the characteristic point can be a saddle, a focus or a node. Their respective folded singular points, depicted in Fig. 9, are called folded focus, folded node and folded saddle.

Index of a Folded Singularity. The index of a folded singular point is the index of its characteristic point for the characteristic field on \( V \) (Fig. 9).

Left and Right Characteristic Fronts. Let \( V \subset M^3 \) be an oriented surface and \( \rho : M^3 \to B \) a Legendre fibration with \( B \) oriented. The set of fold singularities of the projection \( \rho|_V : V \to B \) separates \( V \) into two parts \( V_\ell \) and \( V_r \), called left and right parts of \( V \), on which \( \rho|_V : V \to B \) respectively reverses or preserves the orientation. A characteristic of the left part \( V_\ell \) (resp. \( V_r \)) is said to be left (resp. right), and its projection by \( \rho \) is called a left (resp. right) characteristic front.

Left and Right Solutions of IDE. The same way, orienting the surface \( V^F \) and the \((x, y)\)-plane, the critical set of the map \( \pi|_{V^F} : (x, y, p) \mapsto (x, y) \), called criminant of the IDE (and given by \( F = F_p = 0 \)), separates \( V^F \) into its left and right parts. A solution curve of the IDE \( F = 0 \) is left (right) if it is the projection of a characteristic of the left part \( V^F_\ell \) (resp. \( V^F_r \)).

2.2 Duality as Geometry of Pairs of Legendrian Fibrations

Example 2.5. The standard dual fibration \( \pi^\vee : J^1(\mathbb{R}, \mathbb{R}) \to (\mathbb{R}^2)^\vee \) (or Legendre transform) given by \( (x, y, p) \mapsto (X, Y) = (p, px - y) \) is Legendrian. Indeed, the fibre over the point \((X, Y) = (a, b)\) is the line of equation \( ax - y = b \) (on the horizontal plane \( p = a \)) whose parametrisation \( x \mapsto (ax - b, a) \) satisfies \( \alpha(x, ax - b, a)(\dot{x}, a\dot{x}, 0) = a\dot{x} - a\dot{x} = 0 \). Observe that the front of this \( \pi^\vee \)-fibre by the map \( \pi \) of Example 2.3 is the line \( y = ax - b \) of the \( xy \)-plane.
Conversely, the front by $\pi^\vee$ of the $\pi$-fibre $(x_0, y_0, p)$ over $(x_0, y_0)$ is the line $Y = x_0X - y_0$ on the $XY$-plane $(\mathbb{R}^2)^\vee$, parametrised by $p \mapsto (p, px_0 - y_0)$.

Then the correspondence point $\leftrightarrow$ line in projective duality is provided by the Legendrian fibrations $\pi : J^1(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^2$ and $\pi^\vee : J^1(\mathbb{R}, \mathbb{R}) \to (\mathbb{R}^2)^\vee$. Namely, the respective fronts by $\pi$ and $\pi^\vee$ of a $\pi$-fibre are a point of $\mathbb{R}^2$ and a line of $(\mathbb{R}^2)^\vee$, while those of a $\pi^\vee$-fibre are a line of $\mathbb{R}^2$ and a point of $(\mathbb{R}^2)^\vee$.

**Remark.** The front of the 1-graph $J^1f$ of a function $f$ by the dual fibration $\pi^\vee : J^1(\mathbb{R}, \mathbb{R}) \to (\mathbb{R}^2)^\vee$ is the graph of the so-called Legendre transform of $f$.

**Legendre Duality and Pairs of Legendrian Fibrations.** Projective duality is generalised (at least locally) by replacing the contact space $J^1(\mathbb{R}, \mathbb{R})$ and its standard Legendrian fibrations $\pi, \pi^\vee$ with any contact 3-manifold $M$ and a pair of transverse Legendrian fibrations $\rho : M \to B$, $\rho^\vee : M \to B^\vee$ (i.e., such that their fibres are nowhere tangent). This was discussed in [3].

In the sequel $M$ denotes a contact 3-manifold, $\rho : M \to B$ and $\rho^\vee : M \to B^\vee$ a pair of transverse Legendrian fibrations and $V \subset M$ is a smooth surface.

For the standard fibrations of $J^1(\mathbb{R}, \mathbb{R})$ we shall keep the notations $\pi, \pi^\vee$.

**Generalised Lines.** We say that the front by $\rho : M \to B$ of a $\rho^\vee$-fibre is a line of $B$ and the front by $\rho^\vee : M \to B^\vee$ of a $\rho$-fibre is a line of $B^\vee$.

**Generalised Inflections.** An inflection (bi-inflection) of a smoothly immersed curve in $B$ is a point at which the curve has 3-point contact (resp. 4-point contact) with a line of $B$.

In this generalised duality, which is governed by the pair of Legendrian fibrations, the usual correspondence “inflection $\leftrightarrow$ cusp” holds.

### 2.3 Inflections, Legendrian Fibrations and Contours

**Lemma 2.1.** The 1-graphs of two smooth functions, $f, g : \mathbb{R} \to \mathbb{R}$, have 2-point contact (resp. 3-point contact) at a point $(x_0, y_0, p_0) \in J^1(\mathbb{R}, \mathbb{R})$ iff their graphs have 3-point contact (resp. 4-point contact) at $(x_0, y_0)$.

**Proof.** The graphs of $f$ and $g$ have 3-point contact at $(x_0, y_0)$ if and only if $f(x_0) = y_0 = g(x_0)$, $f'(x_0) = g'(x_0)$ and $f''(x_0) = g''(x_0)$. These three equalities imply and are implied by the following two equalities defining the tangency of the 1-graphs of $f$ and $g : (x_0, f(x_0), f'(x_0)) = (x_0, g(x_0), g'(x_0))$ and $(1, f'(x_0), f''(x_0)) = (1, g'(x_0), g''(x_0))$. The proof for the case of 4-point contact of the graphs of $f$ and $g$ is similar.

**Lemma 2.2.** Consider a contact 3-manifold $M$ and a Legendrian fibration $\rho : M \to B$. Two Legendre curves in $M$ which are not tangent to the $\rho$-fibre at a point $Q$ have 2-point contact (3-point contact) at $Q$ iff their fronts by $\rho$ in $B$ have 3-point contact (resp. 4-point contact) at $\rho(Q)$.

**Proof.** Since all Legendrian fibrations of fixed dimension are locally (in the bundle space) contact diffeomorphic (see [3]), our Legendrian curves in $M$ are locally contact-equivalent to the 1-graphs in $J^1(\mathbb{R}, \mathbb{R})$ of two smooth functions, and their fronts by $\pi$ in $B$ are diffeomorphic equivalent to the pair of graphs of that functions in $\mathbb{R}^2$. Now, Lemma 2.2 follows from Lemma 2.1. 

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Example (Inflection ↔ cusp). Let \( f : \mathbb{R} \to \mathbb{R} \) be a smooth function. By Lemma 2.2 it’s graph has a usual inflection at \( x_0 \) iff its 1-graph is tangent to the \( \pi^\vee \)-fibre of \( J^1(\mathbb{R}, \mathbb{R}) \) at \((x_0, f(x_0), f'(x_0))\), i.e. \( f''(x_0) = 0 \). This implies that the dual curve on the \( XY \)-plane \( \gamma^\vee : x \mapsto (f'(x), f'(x)x - f(x)) \) has a cusp at \( x_0 \). Namely, \((\gamma^\vee)'(x_0) = (f''(x_0), x_0 f''(x_0)) = (0, 0)\).

Example. Check that the inflection of the graph of \( f(x) = x^3 \) corresponds to the cusp of its dual curve \( \gamma^\vee : x \mapsto (X(x), Y(x)) = (3x^2, 2x^3) \).

Left and Right Curve of Inflections. The left (right) curve of inflections \( I_L \) (resp. \( I_R \)) of an IDE consists of the inflections of its left (resp. right) solution curves.

Contour and Discriminant. The set of critical points of the restriction of \( \rho \) to \( \mathcal{V} \), \( \rho |_\mathcal{V} : \mathcal{V} \to B \), noted \( \hat{D} \), is called \( \rho \)-contour of \( \mathcal{V} \). The \( \rho \)-discriminant of \( \mathcal{V} \) is the set \( D = \rho(\hat{D}) \) of critical values of \( \rho |_\mathcal{V} \).

Example. Given an IDE \( F(x, y, p) = 0 \) and the standard Legendrian fibration \( \pi : J^1(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^2 \), the \( \pi \)-contour of \( \mathcal{V}^\mathcal{F} \) (or criminant of the IDE \( F = 0 \)) is the subset \( \hat{D} \) of \( \mathcal{V}^\mathcal{F} \) at which \( F_p = 0 \). If the surface \( \mathcal{V}^\mathcal{F} \) is in general position its \( \pi \)-contour \( \hat{D} \) is a smooth curve (possibly empty).

We only consider IDEs whose contour by \( \pi \) satisfies at every point \( F_{pp} \neq 0 \) (i.e. the \( \pi \)-fibre has exactly 2-point contact with \( \mathcal{V}^\mathcal{F} \)).

The \( \pi \)-discriminant of \( \mathcal{V}^\mathcal{F} \) is the usual discriminant \( D \) of the IDE \( F = 0 \).

Remark. Given a smooth surface \( \mathcal{V} \) of a contact 3-manifold \( M \), for every Legendrian fibration \( \rho : M \to B \), all characteristic points of \( \mathcal{V} \) belong to its \( \rho \)-contour \( \hat{D} \).

Proposition 2.1. A Legendre curve \( L \) is tangent to a characteristic of \( \mathcal{V} \) at a (non characteristic) point of \( \mathcal{V} \) iff \( L \) is tangent to \( \mathcal{V} \) at that point.

Proof. A Legendre curve \( L \) is tangent to \( \mathcal{V} \) at a point iff at that point the tangent line to \( L \) lies in the tangent plane to \( \mathcal{V} \); but, since \( L \) is Legendrian, its tangent line lies also in the contact plane. So this tangent line is a characteristic line of \( \mathcal{V} \).}

Inflection-Contour Theorem. The curve of inflections of the characteristic fronts of \( \mathcal{V} \) is the image under \( \rho \) of the \( \rho^\vee \)-contour \( \tilde{I} \) of \( \mathcal{V} \) (Fig. 10).

Proof. The front \( \rho(\gamma) \) of a characteristic \( \gamma \) of \( \mathcal{V} \) has an inflection at a point \( q = \rho(Q) \) iff it has at least 3-point contact at \( q \) with a line \( \rho(\ell) \) of \( B \) (\( \ell \) being a \( \rho^\vee \)-fibre). This holds, by Lemma 2.2, iff the characteristic \( \gamma \) has at least 2-point contact with the \( \rho^\vee \)-fibre \( \ell \) at \( Q \). This means that \( Q \) belongs to the \( \rho^\vee \)-contour \( \tilde{I} \) of \( \mathcal{V} \) (by Proposition 2.1 applied to \( L = \ell \)).

Therefore, the map \( \rho \) send the fold singularities of the map \( \rho^\vee |_\mathcal{V} : \mathcal{V} \to B^\vee \) to the inflections of the characteristic fronts of \( \mathcal{V} \) – Fig. 10.

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2.4 Bi-inflections as Whitney Pleat Singularities

It is well known that a generic map between surfaces has fold singularities along a (possibly empty) curve on the source, whose closure is smooth (type 1 in Fig. 7). Besides fold singularities, such a map may only have the so-called Whitney pleat singularity at isolated points (type 2 in Fig. 7):

**Whitney pleat.** The Whitney pleat, also called the Whitney cusp, is a stable singularity and it is the singularity, at zero, of the map \((x, y) \mapsto (z, w)\):

\[
\begin{align*}
  z &= x^3 + xy, \\
  w &= y.
\end{align*}
\]

The fold and the Whitney pleat reduce locally to the projection of the surfaces \(z = x^2\) and \(z = x^3 + xy\) (in the \((x, y, z)\)-space), respectively, to the \((y, z)\)-plane along the \(x\)-axis.

**Remark.** Let \(Q\) be a point of a characteristic \(\gamma\) of \(V\). If the front \(\rho(\gamma)\) has a bi-inflection at \(q = \rho(Q)\), then the surface \(V\) has at least 3-point contact with the \(\rho^\vee\)-fibre at \(Q\) (Lemma 2.2). Conversely, if a \(\rho^\vee\)-fibre has 3-point contact with \(V\) at \(Q\) then that \(\rho^\vee\)-fibre has 3-point contact with the characteristic \(\gamma\). That is, the front \(\rho(\gamma)\) has a bi-inflection at \(q = \rho(Q)\).

So for \(V \subset M\) in general position, the Whitney pleats of \(\rho_{\vert V}^\vee : V \rightarrow B^\vee\) correspond to the bi-inflections of the characteristic fronts of \(V\) by \(\rho\) (this holds for any Legendrian fibration \(\rho : M \rightarrow B\) transverse to \(\rho^\vee\)). Therefore, we obtain the

**Bi-inflection-Pleat Theorem.** Let \(\gamma\) be a characteristic of a smooth surface \(V \subset M\) in general position. The map \(\rho_{\vert V}^\vee : V \rightarrow B^\vee\) has a Whitney pleat singularity at \(Q \in \gamma\) iff the front \(\rho(\gamma)\) has a bi-inflection at \(b = \rho(Q)\).

**Example.** If \(V^F \subset J^1(\mathbb{R}, \mathbb{R})\) is the surface of a generic IDE \(F(x, y, p) = 0\), the dual map \(\pi_{\vert V}^\vee : V^F \rightarrow (\mathbb{R}^2)^\vee\) has a Whitney pleat at a point \(Q \in V^F\) iff a solution through \(b = \pi(Q)\) of the IDE \(F = 0\) has a usual bi-inflection.

**Whitney Pleat Remark.** For a generic map between two 2-dimensional manifolds Whitney Theorem implies: 1. The critical set is a smooth curve; 2. At each point of this curve the kernel of the derivative is 1-dimensional; 3. This kernel is transverse to the critical set at its generic points, being tangent to this curve at some isolated points.

An explicit calculation (using the above normal form) shows that the kernel of the derivative is tangent to the critical set only at the Whitney pleats.
This implies the following proposition.

**Proposition 2.2.** Let $V \subset M$ be a generic smooth surface. A $\rho^{\vee}$-fibre is tangent to the $\rho^{\vee}$-contour of $V$ at a point $Q$ iff the front by $\rho$ of the characteristic curve of $V$ through $Q$ has a bi-inflection at $q = \rho(Q)$.

**Corollary 2.1** (bi-inflection characterisation). A point $b \in B$ is a bi-inflection of a given characteristic front of $V$ iff the curve of inflections (of the neighbouring characteristic fronts of $V$) is tangent, at $b$, to that given front.

**Proof.** Since the kernel of the map $\rho^{\vee} : V \to B^{\vee}$ at a point (of the $\rho^{\vee}$-contour $\hat{I}$) is the tangent space of the $\rho^{\vee}$-fibre through that point, the statement follows from Prop. 2.1, Inflection-contour Theorem and Prop. 2.2. \qed

### 2.5 Quadratic Cones and Legendrian Fibrations

The next definitions and results are needed to prove Theorems 1.2, 3.2 and 5.2.

**Cone in General Position.** A non degenerate quadratic cone in the vector space $\mathbb{R}^3$ is said to be in general position with respect to two given vectorial lines if the cone is not tangent to the plane determined by these lines and none of these two lines is a cone generatrix.

**Six Generic Positions.** There are six generic relative positions of a quadratic cone (possibly imaginary) with respect to two given vectorial lines $\ell_1$, $\ell_2$ and to the plane plane $\Pi_{\ell_1, \ell_2}$ determined by those lines (Fig 12):

- (a) the interior of the cone contains $\ell_1$ but not $\ell_2$;
- (b) the plane $\Pi_{\ell_1, \ell_2}$ intersects the cone only at the origin;
- (c) the interior of the cone contains $\ell_2$ but not $\ell_1$;
- (d) the cone is imaginary, but its vertex being real;
- (e) the interior intersects the plane $\Pi_{\ell_1, \ell_2}$ but does not contain $\ell_1$ nor $\ell_2$;
- (f) both $\ell_1$ and $\ell_2$ lie inside the cone.

**Singular Point in General Position.** A Morse conic singular point of a surface $V \subset M^3$ is in general position with respect to two transverse Legendrian fibrations if the quadratic cone determined by that surface is in general position with respect to the tangent lines to the two fibres at that point.
Remark 1. Let $\pi$ be the standard Legendre fibration of $J^1(\mathbb{R}, \mathbb{R})$ and $\pi^\vee$ its standard dual fibration. If $\ell_1$ and $\ell_2$ are the tangents to the $\pi$- and $\pi^\vee$-fibres of $J^1(\mathbb{R}, \mathbb{R})$ at a Morse conic point $Q$ of $\mathcal{V}^F$, then the above six relative positions correspond to the transitions $a$ to $f$ in Fig. 5 (Ths.1.2 and 3.2).

Example 2.6. If $F(x, y, p) = 0$ is an IDE with a Morse singularity at the point $Q = (0, 0, 0)$, noted $\bar{0}$, then the equation of the quadratic cone $K$ defined by $F$ at $\bar{0}$ is

$$F_{xx}(\bar{0})x^2 + F_{yy}(\bar{0})y^2 + F_{pp}(\bar{0})p^2 + 2F_{xy}(\bar{0})xy + 2F_{xp}(\bar{0})xp + 2F_{yp}(\bar{0})yp = 0.$$ 

At $\bar{0}$ the $\pi$-fibre is the $p$-axis and the $\pi^\vee$-fibre is the $x$-axis. Therefore, $\bar{0}$ is a critical point of $F$ in general position with respect to $\pi$ and $\pi^\vee$, iff

$$(F^2_{xp} - F_{xx}F_{pp})(\bar{0}) \neq 0, \quad F_{pp}(\bar{0}) \neq 0 \quad \text{and} \quad F_{xx}(\bar{0}) \neq 0.$$ 

In classical geometry one proves that, given a straight line $\ell$ through the vertex of a non-degenerate (possibly imaginary) quadratic cone $K \subset \mathbb{R}^3$ of equation $P_2(x, y, z) = \varepsilon$, for any $\varepsilon \in \mathbb{R}$ (including $K$), have the same diametral plane conjugate to $\ell$:

**Conjugate Diametral Plane.** The diametral plane of $K$ conjugate to $\ell$ is defined as follows: a plane $\Pi$ containing $\ell$ intersects $K$ along two lines $\ell_2$, $\ell_3$ (possibly imaginary). Take the line $\ell_4 = \ell_4(\Pi)$ that together with $\ell$ separate harmonically the pair $\ell_2$, $\ell_3$ (i.e., the cross ratio $(\ell_2, \ell_3; \ell, \ell_4)$ equals $-1$). The union of the fourth harmonic lines $\ell_4(\Pi)$, taken over all planes containing $\ell$ is a plane. It is called diametral plane of $K$ conjugate to $\ell$.

**Conjugate Diameters.** Two lines through the vertex of a non-degenerate quadratic cone $K$ are said to be conjugate diameters of $K$, if one of them is contained in the conjugate diametral plane of the other.

**Harmonicity.** We conclude that two lines $\ell$, $\ell'$ are conjugate diameters of $K$ iff their cross-ratio with the lines $\ell_1$, $\ell_2$ (real or complex) on which $K$ intersects the plane $\Pi_{4\ell'}$, determined by $\ell$, $\ell'$, equals $-1$, that is $(\ell, \ell', \ell_1, \ell_2) = -1$.

Let $\mu : \mathbb{R}^3 \to \mathbb{R}^2$ be a fibration of parallel lines and write $\ell$ for the $\mu$-fibre through the origin. The basic properties of conjugated diameters imply:
Proposition 2.3. Let $K \subset \mathbb{R}^3$ be a non-degenerate quadratic cone of equation $P_2(x,y,z) = 0$. If the $\mu$-fibre $\ell$ does not lie on $K$, then the union of the $\mu$-contours of the quadrics $P_2(x,y,z) = \varepsilon$, with $\varepsilon \in \mathbb{R}$, is the diametral plane conjugate to $\ell$ of any of these quadrics (including $K$).

In fact, the $\mu$-contour of a quadric $P_2(x,y,z) = \varepsilon$ is the intersection of that quadric with the diametral plane of $K$ conjugate to $\ell$.

Proposition 2.4. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a function whose level surface $F = 0$ has a conic Morse singularity at $\bar{0}$. If the $\mu$-fibre $\ell$ at $\bar{0}$ does not lie on the quadratic cone $K$ defined by $F$, then the surface formed by the $\mu$-contours of the level surfaces $F = \varepsilon$ is smooth at $\bar{0}$, its tangent plane at $\bar{0}$ is the diametral plane of $K$ conjugate to $\ell$ and this plane does not contain $\ell$.

Proof. By Proposition 2.3 near $\bar{0}$, the $\mu$-contours of the surfaces $F = \varepsilon$ form a surface which is a deformation (by higher order terms) of the diametral plane of $K$ conjugate to $\ell$. The tangent plane of this surface at $\bar{0}$ is the diametral plane of $K$ conjugate to $\ell$ because it depends only on the quadratic terms of $F$. The plane does not contain $\ell$ because a diametral plane of $K$ conjugate to $\ell$ contains $\ell$ iff that plane is tangent to $K$ along $\ell$.

3 Local Transitions of the Curve of Inflections

We separate the local transitions of the curve of inflections and discriminant of BIDEs in Fig. 5 (and those of the flecnodal and parabolic curves of surfaces in Fig. 4) into classes of distinct natures:

I. Transitions involving characteristic points: UW, flec-folded singularity and $\gamma_v$ for BIDE ($A_4$ and flec-godron for surfaces in 3-space);

II. Transitions involving Morse conic points of $V^F$: a to f for BIDE ($A_3$ for surfaces in 3-space);

III. Transitions involving Whitney pleats (lips, bec-à-bec, swallowtail);

IV. Transitions involving multi-singularities: creation/annihilation of hyperbolic (elliptic) nodes, flec hyperbolic node for BIDE (creation/annihilation of hyperbonodes (or ellipnodes) and flec-hyperbonode for surfaces in 3-space);

For surfaces in 3-space there is an additional type: $D_4$ transitions.

Along this section we consider a contact 3-manifold $M$ and we take two fixed transverse Legendre fibrations $\rho : M \to B$ and $\rho^\vee : M \to B^\vee$.

3.1 Transitions Involving Characteristic Points

Theorem 3.0. Let $Q$ be a characteristic point of a smooth surface $V \subset M$ such that the $\rho$-fibre at $Q$ has exactly 2-point contact with $V$ and

a) the $\rho$-contour $\hat{D}$ of $V$ is not tangent to the $\rho^\vee$-contour $\hat{I}$ of $V$;

b) the $\rho^\vee$-fibre is not tangent to the $\rho^\vee$-contour $\hat{I}$ of $V$;

c) the $\rho$-fibre is not tangent to the $\rho^\vee$-contour $\hat{I}$ of $V$.

Then the discriminant $D = \rho(\hat{D})$ and the curve of inflections $I = \rho(\hat{I})$ are quadratically tangent at $q = \rho(Q)$. This point $q = \rho(Q)$ locally separates $I$ into its left and right branches $I_\ell$, $I_r$ (Fig. 13).
Proof. The derivative of $\rho : V \rightarrow B$ at $Q$ sends tangent lines of $\hat{D}$ and $\hat{I}$ to the image line of the tangent plane of $V$ at $Q$. So $I$ and $D$ are tangent.

The condition of 2-point contact of the $\rho$-fibre at $Q$ with $V$ implies that there are local smooth coordinates $(x,y,z)$ in $M$ (near $Q$) and $(x,y)$ in $B$ (near $q = \rho(Q)$) such that $V$ is given by the equation $y = z^2$; the $\rho$-contour $\hat{D}$ is the $x$-axis and the map $\rho$ is given by $(x,y,z) \mapsto (x,y)$. 

In the sequel, the notation “h.o.t.($x$)” means “higher order terms in $x$”.

By condition (b), the contour $\hat{I}$ may be parametrised by the $x$-coordinate

$$x \mapsto (x, (\alpha x + \beta x^2 + \text{h.o.t.}(x))^2, \alpha x + \beta x^2 + \text{h.o.t.}(x)). \tag{3.1}$$

Its image $I = \rho(\hat{I})$ is the curve $(x, \alpha^2 x^2 + 2\alpha \beta x^3 + \beta^2 x^4 + \text{h.o.t.}(x))$. Condition (a) implies $\alpha \neq 0$, proving the quadratic tangency with $D$ (the $x$-axis).

Since $\hat{D}$ and $\hat{I}$ intersect each other transversely at $Q$, the contour $\hat{D}$ locally separates $\hat{I}$ at $Q$ [Fig. 13]. That is, $\hat{I}$ has one branch on the left component of $V$ and the other branch on the right one. This implies that $q = \rho(Q)$ locally separates the left and right branches of the curve of inflections.

In generic 1-parameter families of IDE, one of the conditions a), b) or c) can fail at isolated parameter values at which occur the following respective transitions:

**Theorem 3.1.** Let $V_\varepsilon \subset M$ be a generic 1-parameter family of smooth surfaces. If the surface $V = V_0$ has a characteristic point in which one of the conditions a, b or c of Th. 3.0 is broken, then the curve of inflections of $V_\varepsilon$ undergoes the following respective transitions at $\varepsilon = 0$ (Fig. 5):

a) **UW:** ($\varepsilon = 0$) the point $q = \rho(Q)$ is a folded singular point at which the discriminant $D = \rho(\hat{D})$ has 4-point contact with $I = \rho(\hat{I})$. It disappears ($\varepsilon < 0$) or splits into two folded singular points of opposite indices ($\varepsilon > 0$).

b) **flec-folded singularity:** a negative folded singularity overlaps (at $\varepsilon = 0$) with a biflecnodel which passes from one branch of the curve of inflections to the other (as $\varepsilon$ changes sign). At $\varepsilon = 0$ the curve of inflections $\rho(\hat{I})$ has itself an inflection at $q = \rho(Q)$.

c) **$\gamma_v$:** for $\varepsilon \in (-1, 1)$ the surface of the 3-space $B \times (-1, 1)$ formed by the curves of inflections of $V_\varepsilon$ is locally diffeomorphic to the Whitney umbrella. For $\varepsilon = 0$ the curve $\rho(\hat{I})$ has a semi-cubic cusp at the folded singular point $q = \rho(Q)$. For any sufficiently small $|\varepsilon|$ the folded singular point is generic.
Proof. Case a. The curve $\hat{I}$ can be parametrised by (3.1), with $\alpha = 0$ and $\beta \neq 0$. So its image $I = \rho(\hat{I})$ is the curve $(x, \beta^2 x^4 + \h.o.t.(x))$, that is, $I$ has 4-point contact with $D = \rho(\hat{D})$ (the $x$-axis).

A small generic deformation of the function $\beta x^2 + \h.o.t.(x)$ is given (up to a ‘translation in the $x$-coordinate’) by $\varepsilon + \beta x^2 + \h.o.t.(x)$, with $\beta \neq 0$. The corresponding deformation of the contour $\hat{I}$, $x \mapsto (x, (\beta x^2 + \h.o.t.(x))^2, \beta x^2 + \h.o.t.(x))$, is given by $x \mapsto (x, (\varepsilon + \beta x^2 + \h.o.t.(x))^2, \varepsilon + \beta x^2 + \h.o.t.(x))$. So the family of curves of inflections is equivalent to the standard $UW$ family of curves $x \mapsto (x, (\varepsilon + \beta x^2)^2)$.

Suppose $\beta < 0$, i.e. $\beta < 0$ (the case $\beta > 0$ is equivalent). For small $\varepsilon < 0$, the function $\varepsilon + \beta x^2 + \h.o.t.(x)$ has no zeros near the origin. Thus the curves $\hat{D}_\varepsilon$ and $\hat{I}_\varepsilon$ have no point of intersection (see Fig. 14 left).

For small $\varepsilon > 0$ the function $\varepsilon + \beta x^2 + \h.o.t.(x)$ has two zeros near the origin. Thus the curves $\hat{D}_\varepsilon$ and $\hat{I}_\varepsilon$ intersect at two neighbouring folded singular points (see Fig. 14 right).

Case b. By Whitney Pleat Remark and Proposition 2.2 (p.110), condition (b) implies that the map $\rho^\nu : V \to B^\nu$ has a Whitney pleat at $Q$. The $\rho$-contour $\hat{D}_\varepsilon$ of $V_\varepsilon$ locally separates $V_0$ into its left and right components and, hence, it also separates $\hat{I}_\varepsilon$ into its left and right branches. Since the characteristic points and the Whitney pleats are both stable, also the surfaces $V_{\varepsilon \neq 0}$ have a characteristic point and have a Whitney pleat. For a generic family $V_\varepsilon$ the Whitney pleat crosses transversely the $\rho$-contour, passing from one branch of $\hat{I}_\varepsilon$ to the other at $\varepsilon = 0$. Theorem 3.1b follows from this fact.

Case c. By condition (c), the $\rho^\nu$-contour $\hat{I}$ can be parametrised by the $z$-coordinate as $(\beta z^2 + \gamma_0 z^3 + \h.o.t.(z), z^2)$. Its image under $\rho$ is the curve

$$(\beta z^2 + \gamma_0 z^3 + \h.o.t.(z), z^2),$$

which is sent by the affine transformation $(x, y) \mapsto (x - \beta y, y)$ to the semi-cubic cusp $(\gamma_0 z^3 + \h.o.t.(z), z^2)$, $\gamma_0 \neq 0$ (the condition $\gamma_0 = 0$ is of codimension 2 and would give rise to a ramphoid cusp $(\delta z^4 + \h.o.t.(z), z^3)$, $\delta \neq 0$).

A deformation of $V = V_0$ inside a generic 1-parameter family $V_\varepsilon \subset M$ induces a deformation of the curve of inflections $z \mapsto (\gamma_0 z^3 + \h.o.t.(z), z^2)$, which is given (up to a ‘translation in the first coordinate’ and an affine transformation $(x, y) \mapsto (x - \beta y, y)$) by the parametrisation $z \mapsto (\varepsilon z + \gamma z^3 + \h.o.t.(z), z^2)$, where $\gamma = \gamma(\varepsilon)$ and $\gamma(0) = \gamma_0$.

As the parameter $\varepsilon$ changes, the tangent line to the $\rho^\nu$-contour of $V_\varepsilon$ at the characteristic point passes through the ‘vertical’ position (at $\varepsilon = 0$) with non zero angular velocity.
The surface in plane-time \((x, y, \varepsilon)\) formed by the union of the curves of this family is the image of the map \((z, \varepsilon) \mapsto (\varepsilon z + \gamma z^3 + \text{h.o.t.}(\varepsilon, z), z^2, \varepsilon)\). This map is a perturbation with higher order terms of the (Whitney) map \((z, \varepsilon) \mapsto (\varepsilon z + \gamma_0 z^3, z^2, \varepsilon)\), which is stable (Appendix B). For fixed \(\varepsilon < 0\), the parametrised curve \(z \mapsto (\varepsilon z + \gamma_0 z^3, z^2)\) has a double point which is the image of the values \(z = \pm \sqrt{-\varepsilon/\gamma_0}\), like in Fig. 15.

3.2 Transitions Involving Quadratic Conic Points

**Theorem 3.2.** Let \(\{V^F \subset M\}_{\varepsilon \in \mathbb{R}}\) be a generic 1-parameter family of surfaces given by functions \(F^\varepsilon: M \to \mathbb{R}\). If \(F = F_0\) has a Morse conic point \(Q\) in general position with respect to \(\rho\) and \(\rho^\vee\), and the tangents to the two fibres at \(Q\) are not conjugate diameters of the cone \(K\) defined by \(V\), then both curves \(I\) and \(D\) undergo a Morse transition - we get one of the transitions \(a\) to \(f\) in Fig. 5, in accordance with the position of the cone \(K\), \(a\) to \(f\) in Fig. 12. Moreover, two folded singularities of equal indices are born or die (Fig. 5).

In Fig. 16, we show a deformation of the cone (in the cases \(a\) and \(d\)) and the projections to the \(xy\)-plane of its contours by \(\pi\) and by \(\pi^\vee\). See also Fig. 5.

**Proof of Theorem 3.2**

Consider a Legendrian fibration \(\mu: M \to B\) of our contact 3-manifold \(M\).

**Lemma 3.1.** Let \(F: M \to \mathbb{R}\) be a function whose level surface \(F = 0\) has a Morse conic singular point \(Q\). If the tangent line \(\ell\) of the \(\mu\)-fibre at \(Q\) does not lie on the quadratic cone \(K \subset T_Q M\) defined by \(F\), then

(i) at \(Q\), the surface swiped by the \(\mu\)-contours of the level surfaces \(F = \varepsilon\) (that we note \(\Sigma^F_\mu\)) is smooth, its tangent plane is the diametral plane of \(K\) conjugate to \(\ell\) and this plane does not contain \(\ell\);
(ii) near $Q$, the foliation of the surface $\Sigma^\mu_F$, which is naturally defined by the $\mu$-contours of the surfaces $F = \varepsilon$, is diffeomorphic to the foliation of the plane by the level curves of a function near a Morse critical point.

**Proof.** (i) It follows from Proposition 2.4 use a local diffeomorphism sending $Q$ to $0 \in \mathbb{R}^3$ and rectifying the fibres of $\mu$ to parallel lines - its differential preserves the conjugacy relations.

(ii) By item (i), we can take local coordinates $x, y, z$ near $Q$ such that the surface $\Sigma^\mu_F$ is locally given by the equation $z = 0$ and the surface $\nu^F_\varepsilon$ has equation $\varepsilon = ax^2 + by^2 + cz^2 + dxy + exz + gyz + \psi_{\geq 3}(x, y, z)$, where the quadratic cone $K$, of equation $ax^2 + by^2 + cz^2 + dxy + exz + gyz = 0$, is non degenerate and not tangent to the surface $z = 0$.

So, in these coordinates, the foliation of the surface $\Sigma^\mu_F$ is given by the level curves $\varepsilon = ax^2 + by^2 + dxy + \psi_{\geq 3}(x, y, 0)$. Lemma 3.1 is proved.

Let us come back to our fixed Legendrian fibrations $\rho : M \rightarrow B$ and $\rho^\vee : M \rightarrow B^\vee$. The following lemma is the simplest case of Theorem 3.2.

**Lemma 3.2.** Let $F : M \rightarrow \mathbb{R}$ be a function whose level surface $F = 0$ has a Morse conic point $Q$ in general position with respect to $\rho$ and $\rho^\vee$. If the tangents to the $\rho$ and $\rho^\vee$-fibres at $Q$ are not conjugate diameters of the cone $K$, then both curves $D_\varepsilon, I_\varepsilon$ undergo a Morse transition at $\varepsilon = 0$: We get one of the transitions $a \rightarrow f$ in Fig. 5 according to the position of $K$ in Fig. 12.

**Proof.** We apply Lemma 3.1 for our fixed fibrations $\mu = \rho$ and $\mu = \rho^\vee$. 

To prove Theorem 3.2, we need the following notion:

**Flec-Surface.** The flec-surface of a function $F : M \rightarrow \mathbb{R}$ is the union of the $\rho^\vee$-contours $\tilde{I}_\varepsilon$ of the level surfaces $F = \varepsilon$. More generally, the flec-surface $\mathcal{F}(\mathcal{V}_\varepsilon)$ of a $1$-parameter family of surfaces $\mathcal{V}_\varepsilon \subset M$ is the surface swiped by the $\rho^\vee$-contours $\tilde{I}_\varepsilon$ of the surfaces $\mathcal{V}_\varepsilon$.

**Remark.** Thus the $\rho^\vee$-contour $\tilde{I}_\varepsilon$ of the surface $\mathcal{V}_\varepsilon$ is the intersection of $\mathcal{V}_\varepsilon$ with the flec-surface of the function $F_\varepsilon$. The $\rho^\vee$-contours $\tilde{I}_\varepsilon, \varepsilon \in \mathbb{R}$, define a (singular) foliation on the flec-surface $\mathcal{F}(\mathcal{V}_\varepsilon)$ (by its definition).

**Proof of Theorem 3.2** Write $K$ for the quadratic cone of $T_Q M$ defined by the surface $\mathcal{V}_0, \mathcal{F}(\mathcal{V}_\varepsilon)$ for the flec-surface of the family $\mathcal{V}_\varepsilon$ and $\ell$ for the tangent line of the $\rho^\vee$-fibre at $Q$. The flec-surface $\mathcal{F}(\mathcal{V}_\varepsilon)$ is smooth at $Q$ and very close to the diametral plane conjugate to $\ell$ of $K$ (being a smooth deformation of this plane, see Lemma 3.1).

First, we have to prove that the tangent plane of $\mathcal{F}(\mathcal{V}_\varepsilon)$ at $Q$ is still the diametral plane of $K$ conjugate to $\ell$. The flec-surface $\mathcal{F}(\mathcal{V}_\varepsilon)$, being the union of the $\rho^\vee$-contours $\tilde{I}_\varepsilon$ of the surfaces $\mathcal{V}_\varepsilon$ contains the curve $\tilde{I}_0$ (singular at $Q$) which is the intersection of the surface $\mathcal{V}_0$ (having a conic Morse singularity at $Q$) with the (smooth) flec-surface of the function $F := F_0$ (see Lemma 3.1).

So the curve $\tilde{I}_0$ has a Morse double point at $Q$. If the branches of $\tilde{I}_0$ are real, then their tangent lines determine the tangent plane of the surface $\mathcal{F}(\mathcal{V}_\varepsilon)$, which is the tangent plane of the flec-surface of $F_0$ at $Q$ and is therefore the diametral plane of $K$ conjugate to $\ell$. If the branches of $\tilde{I}_0$ are complex conjugate, then (locally) $\tilde{I}_0$ is just the point $Q$ but the tangent plane of the
flec-surface $\mathcal{F}_\ell(V_\varepsilon)$ at $Q$ is still the tangent plane of the flec-surface of the function $F_0$, that is, the diametral plane of $K$ conjugate to $\ell$.

The fact that the tangent lines of the fibres of $\rho$ and $\rho^\vee$ at $Q$ are not conjugate diameters of $K$ iff the tangent plane to $\mathcal{F}_\ell(V_\varepsilon)$ at $Q$ is not vertical (i.e., it does not contain the $\rho$-direction) is stated below in Proposition 6.1 for $M = J^1(\mathbb{R}, \mathbb{R})$, $\rho = \pi$ and $\rho^\vee = \pi^\vee$. This fact holds for arbitrary $M$ because all Legendre fibrations are locally contactomorphic. Thus, the projection $\rho|_{\mathcal{F}_\ell(V_\varepsilon)}: \mathcal{F}_\ell(V_\varepsilon) \rightarrow B$ of the flec-surface $\mathcal{F}_\ell(V_\varepsilon)$ at $Q$ is a local diffeomorphism.

The local transition of the curve of inflections is thus described by the singular foliation of $\mathcal{F}_\ell(V_\varepsilon)$ in the neighbourhood of $Q$. Since the curve $\hat{I}_0$ has a Morse singularity at $Q$, the curve of inflections undergoes a Morse transition. The change (by 2 or $-2$) of the Euler characteristic of the surface $V_\varepsilon$ at the transition, implies that the number of folded singular points changes locally by 2 and also provides the indices of those folded singular points. □

### 3.3 Transitions Involving Whitney Pleats

Let $V_\varepsilon$ be a generic 1-parameter family of smooth surfaces of $M$, with $V = V_0$.

**Theorem 3.3.** If the map $\rho^\vee|_V : V \rightarrow B^\vee$, has a singularity of type 3, 4 or 5 of Fig. 7, then when the parameter $\varepsilon$ passes through 0, the curve of inflections of the $\rho$-solutions of $V_\varepsilon$ undergoes a lips-, bec-à-bec- or swallowtail-transition, respectively, described in Figs. 5 and 1.

**Proof.** Theorem 3.3 follows from the known fact (c.f. [5]) that under a generic 1-parameter deformation of the singularities of types 3, 4 and 5 (in Fig. 7), their respective contours undergo a transition in which two Whitney pleats are born or die (see Fig. 8).

A generic 1-parameter deformation of the surfaces $z = x^3 + xy^2$ and $z = x^3 - xy^2$ whose projections along the $x$-axis have the singularities of the types 3 and 4, is given by the families $z = x^3 + xy^2 + \varepsilon x$ and $z = x^3 - xy^2 + \varepsilon x$, respectively. For each $\varepsilon$, the intersection of the plane $y = y_0 = \text{const}$. with the curve of singular points of the projection of the surface along the $x$-axis consists of the points of the graph of the function $f|_{y=y_0}(x) = x^3 \pm xy_0^2 + \varepsilon x$ at which the tangent line is horizontal. Thus, the projection of the curve of singular points to the $(x, y)$-plane consists of the points at which the partial derivative with respect to $x$ vanishes, $f_x(x, y; \varepsilon) = 0$. One gets the family of conics $3x^2 \pm y^2 + \varepsilon = 0$ which, at $\varepsilon = 0$, has a Morse transition (lips or bec-à-bec in Fig. 8). The born or dying bi-inflections in these transitions correspond to the born or dying Whitney pleats of the $\rho^\vee$-contour of $V$.

Similarly, for the type 5 singularity, we have the surfaces $z = x^4 + xy + \varepsilon x^2$ and we obtain the curve $4x^3 + y + 2\varepsilon x = 0$. For $\varepsilon > 0$ this curve has two points whose tangent is parallel to the $x$-axis (corresponding to two Whitney pleats), at $\varepsilon = 0$ these points collapse and for $\varepsilon < 0$ they disappear. □

### 3.4 Transitions Involving Multisingularities

The bec-à-bec transition of the curve of inflections of the fronts of $V$ comes from a Morse transition at an unstable double point of the $\rho^\vee$-contour of $V$. 

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The stable self-intersection points of the curve of inflections are multisingularities: if a generic surface $V$ has two parts $V_\ell$ and $V_r$, their $\rho^\vee$-contours $\hat{I}_\ell$ and $\hat{I}_r$ are smooth curves whose images $I_\ell = \rho(\hat{I}_\ell)$ and $I_r = \rho(\hat{I}_r)$ are smooth curves of $B$ which may have points of transverse intersection (hyperbolic nodes), which are stable multisingularities. For a BIDE they are also hyperbolic nodes and for a smooth surface they are hyperbonodes (Fig. 17).

**Theorem 3.4.** Consider a generic 1-parameter family of surfaces $V_\epsilon \subset M$.

(i) If, for $V_0$, the curves $I_\ell, I_r$ are quadratically tangent at $q \in B$, then we get a creation/annihilation transition of two hyperbolic nodes (c/a-h·n in Fig. 5).

(ii) If $q \in B$ is a hyperbolic node for $V_0$ and $q$ is also a left bi-inflection, then as $\epsilon$ passes through 0 a left bi-inflection moves along $I_\ell$, crossing $I_r$ at the hyperbolic node $q$ (flec-hn in Fig. 5). (Left and right may be interchanged.)

**Proof.** We have two parts $V_\ell, V_r$ (of $V$) whose $\rho^\vee$-contours $\hat{I}_\ell, \hat{I}_r$ project under $\rho$ to $I_\ell$ and $I_r$. Item (i) follows from the following fact: a generic deformation of $V_\ell$ and $V_r$ will not keep the tangency of their corresponding curves of inflections $I_\ell$ and $I_r$, giving rise to a creation/annihilation transition.

For the proof of Item (ii), first, we can fix $V_1$ in such a way that the projection $\rho^\vee|_V : V \to C$ has a Whitney pleat at a point of $V_1$, that is, there is a bi-inflection point lying in $I_1 = \rho(\hat{I}_1)$. Next, one obtains the transition 14 of Fig. 5 by moving $V_2$ in a generic way: the curve $I_2 = \rho(\hat{I}_2)$ passes trough the bi-inflection point in $I_1$. The stability of the transition (ii) follows from the stability of the Whitney pleat singularity. □

**Part II**

**Transitions on Evolving Surfaces**

**4 Basic Properties of Smooth Surfaces in 3-Space**

**4.1 Classification of Points by the Contact with Lines**

The points of a generic smooth surface in a real 3-space (projective, affine or Euclidean) are classified by the contact of the surface with its tangent lines.

![Diagram of tangential singularities](image)

**Fig. 17.** The 8 tangential singularities of a generic smooth surface.

A generic smooth surface $S$ has three (possibly empty) parts:

- **(E)** an open domain of elliptic points: no real tangent line exceeds 2-point
contact with $S$; (H) an open domain of hyperbolic points: there are two such lines, called asymptotic lines (they determine two asymptotic directions); and (P) a smooth curve of parabolic points: a unique (double) asymptotic line.

The parabolic curve separates the elliptic and hyperbolic domains of $S$.

In the closure of the hyperbolic domain there is (F) a smooth immersed flecnodal curve formed by the points at which an asymptotic tangent line exceeds 3-point contact with $S$.

There are also four types of isolated points: (g) a godron is a parabolic point at which the (unique) asymptotic line is tangent to the parabolic curve; (h) a hyperbonode is a point of transverse self-intersection of the flecnodal curve; (b) a biflecnode is a point of the flecnodal curve at which one asymptotic tangent has 5-point contact with $S$; (e) an ellipnode is a real point in the elliptic domain of the simplest self-intersection of the complex conjugate flecnodal curves associated to the complex conjugate asymptotic lines.

**Inflections of Plane and Space Curves.** For a smooth curve in 3-space or in the plane an inflection (resp. bi-inflection) is a point at which the first two (resp. first three) derivatives are colinear. Equivalently, the curve has 3-point (resp. 4-point) contact with its tangent line at that point. If a Euclidean structure is admitted, the curvature satisfies $k = 0$ (resp. $k = k' = 0$).

A generic smooth curve of $\mathbb{RP}^3$ (or $\mathbb{R}^3$) has no inflection. However, the asymptotic curves of a generic smooth surface may have inflections; being ‘stable’ under small perturbations of the surface. Moreover, if an asymptotic curve has an inflection, then the neighbouring asymptotic curves have also a neighbouring inflection. In §6.4 (Proposition 6.3), we prove that

The curve formed by the inflections of the asymptotic curves of a smooth surface coincides with the flecnodal curve of that surface.

**Hyperbonodes and ellipnodes.** If the surface is locally presented in Monge form

$$z = f_2(x, y) + f_3(x, y) + \ldots,$$

where $f_j$ is a homogeneous polynomial of degree $j$, the point is a hyperbonode or an ellipnode iff the quadratic form $f_2$ divides the cubic form $f_3$, that is, the cubic form is zero modulo the quadratic form [19].

**Example.** A smooth surface in $\mathbb{RP}^3$ is a one-sheet hyperboloid (ellipsoid) iff all its points are hyperbonodes (resp. ellipnodes).

**Remark.** Seven of the above tangential singularities were well known in the 19th Century [25], but their normal forms up to the 5-jet, under projective transformations, were given in [23, 27] and ellipnodes were found by Panov [25]. The numbers of ellipnodes, hyperbonodes and godrons on a surface are related to the Euler characteristic of that surface by formulas which determine the possible coexistences of these points on the surface [19]. A projective invariant and an index for hyperbonodes were introduced in [32].

**4.2 Some Relevant Properties of Smooth Surfaces**

An asymptotic curve is an integral curve of a field of asymptotic directions.

**Left-Right.** Fix an orientation in $\mathbb{RP}^3$ (or $\mathbb{R}^3$). A regular smooth curve is said to be left (right) at the points where its first three derivatives form a
negative (resp. positive) frame. **At a hyperbolic point of a generic smooth surface one asymptotic curve is left and the other is right** (cf. [31]): one of them twists like a left screw and the other like a right screw. Their respective asymptotic tangent lines are called **left and right asymptotic tangents**.

**Remark.** At an inflection of an asymptotic curve the above frame is not defined. But, since the osculating plane to an asymptotic curve of a generic surface \( S \) at a non-inflation \( q \) is the tangent plane to \( S \) at \( q \); for an arbitrary surface \( S \) we say that an asymptotic curve \( \gamma \) is left (right) at \( q \) if the tangent plane to \( S \) along \( \gamma \) twists negatively (resp. positively) at \( q \); if \( T := \gamma' \) and \( N \in T_pS \) is a vector orthogonal to \( T \), the frame \((T, N, N')\) is negative (resp. positive) at \( q \). This enables, for example, to distinguish the two families of straight lines that foliate a one-sheet hyperboloid (its asymptotic curves).

**Left and Right Flecnodal Curve.** The **left (right) flecnodal curve** \( F^\ell \) (resp. \( F^r \)) of a surface \( S \) consists of the points of the flecnodal curve of \( S \) at which the over-osculating tangent line is a left (resp. right) asymptotic line.

**Remark.** An observer ignoring this distinction will be unable to understand why the points of transverse self intersection of \( F \) are sometimes stable and sometimes unstable, and why the transitions of \( F \) in Fig. 18 are both stable.

![Fig. 18. Two stable transitions of the flecnodal curve.](image)

Presenting \( S \) in Monge form and using the description of the asymptotic curves in terms of IDEs (§6.1), Corollary 2.1 provides the

**Biflecnode’s Characterisation.** A point of a smooth surface \( S \) in \( \mathbb{R}P^3 \) (or \( \mathbb{R}^3 \)) is a left biflecnode of \( S \) iff at that point the left flecnodal curve is tangent to the left asymptotic direction (the same holds for the right biflecnodes).

**Godron-Swallowtail.** A godron is also defined in terms of singularities of the contact with the tangent plane – materialised as singularities of the dual surface. An ordinary godron corresponds to a swallowtail point of the dual surface (an \( A_3 \) Legendre singularity - cf. [3]). A double godron is unstable and corresponds to a transition where two swallowtails are born or die (an \( A_4 \) Legendre singularity). A godron is simple if it corresponds to a swallowtail of the dual surface. All godrons of a surface in general position are simple.

**Index of a Godron.** A simple godron is said to be positive or of index +1 (resp. negative or of index −1) if, at the neighbouring parabolic points, the half-asymptotic lines directed to the hyperbolic domain point towards (resp. away from) the godron (Fig. 19).

![Fig. 19. The index of a godron.](image)
Five characterisations and several geometric properties of positive and negative godrons and swallowtails are described in [31].

**Natural Orientation of the Parabolic Curve.** At each elliptic point, the surface locally lies in one of the two half-spaces determined by the tangent plane. This half-space, called *positive*, determines a *natural co-orientation* on each connected component of the elliptic domain. By continuity we get a positive half-space on the smooth part of the parabolic curve [31].

Let $q$ be a parabolic point of a generic smooth surface $S$. Take an affine coordinate system $x,y,z$ at $q$ such that the $xy$-plane is tangent to $S$ and the $x$-axis is tangent to the parabolic curve at $q$. Direct the positive $z$-axis to the positive half-space at $q$ and the positive $y$-axis to the hyperbolic domain. Finally, direct the positive $x$-axis in such way that any basis $(e_x, e_y, e_z)$ of $x,y,z$ forms a positive frame for the chosen orientation of $\mathbb{R}^3$ (or of $\mathbb{RP}^3$).

The vector $e_x$ determines a *natural orientation* of the parabolic curve.

Of course, the natural orientation fits with the local behaviour of the parabolic curve at the $A_3$ and $D_4$ transitions (Fig. 20).

**Fig. 20.** Natural orientation of the parabolic curve.

5 Transitions of flecnodal and parabolic curves

5.1 The Five Main Theorems on Evolving Surfaces

Theorems 5.1 to 5.5 describe the local transitions of the flecnodal and parabolic curves and tangential singularities occurring in generic evolving surfaces.

**Monge form.** We present the germs of surfaces in $\mathbb{R}^3$ at the origin in Monge form $z = f(x,y)$ with $f(0,0) = 0$ and $df(0,0) = 0$, expressing the partial derivatives of $f$ with numerical subscripts:

$$f_{ij}(x,y) := \frac{\partial^{i+j}f}{\partial x^i \partial y^j}(x,y) \quad \text{and} \quad f_{ij} := f_{ij}(0,0).$$

Suppose $S_\varepsilon$ is a generic 1-parameter family of smooth surfaces in $\mathbb{RP}^3$ (or $\mathbb{R}^3$) depending smoothly on the parameter $\varepsilon$. It will be given in Monge form $z = f^\varepsilon(x,y)$. Below, all conditions for $f = f^0$ are given at $(x,y) = (0,0)$.

**Theorem 5.1.** Suppose $S_0$ has a godron $g$ and $P$ is smooth at $g$.

a) If $P$ and $F$ have 4-point contact at $g$, then we get a bigodron transition ($A_4$, equivalent to UW): two godrons of opposite index are born or die.

b) If $S_0$ has 5-point contact with the asymptotic tangent at $g$, then $g$ has index $-1$ and we get a flec-godron transition: a biflenode, which passes from one local branch of the flecnodal curve to the other, overlaps with $g$ at $\varepsilon = 0$. 27
NOTE 1. To get a godron we need \( f_{20} = f_{11} = f_{30} = 0 \) with \( f_{02}f_{21}f_{40} \neq 0 \).

\( a \) the equality \( 3f_{21}^2 - f_{02}f_{40} = 0 \) guarantees a double godron and the condition
\( 9f_{21}f_{31} - 4f_{12}f_{40} - f_{02}f_{50} \neq 0 \) guarantees that it is not triple.

\( b \) conditions \( f_{40} = 0 \) and \( f_{50} \neq 0 \) guarantee the godron is also a biflecnode.

Example. For an ordinary godron Platonova’s normal form of the 4-jet of the surface is
\[ f(x, y) = y^2/2 - x^2y + \rho x^4/2 \] with \( \rho \neq 1, 0. \)

If \( \rho = 1 \) (i.e. \( 3f_{21}^2 - f_{02}f_{40} = 0 \)) this normal form provides an infinitely degenerate godron: \( f(x, y) = \frac{1}{2}(y - x^2)^2. \) To get just a bigodron it suffices to
add a multiple of \( x^3y, \) by our second condition. The bigodron and flec-godron transitions, in generic 1-parameter families, were studied in [31].

The \( \gamma \nu \)-transition of Fig[3] which depicts item c of Theorems [31] and [11] is absent from Fig.[1]. This absence is explained by the following result:

No-Cusp Theorem. The flecnodal curve of a smooth surface in \( \mathbb{RP}^3 \) (or in \( \mathbb{R}^3 \)) has never a cusp at a godron at which the parabolic curve is smooth.

This fact imposes topological restrictions on the possible configurations of the flecnodal curve of a smooth surface. For example,

Theorem [31]. Inside a hyperbolic disc bounded by a closed parabolic curve there is an odd number of hyperbonodes (at least one).

Theorem 5.2. If \( S_0 \) has a point \( q \) at which \( P \) has a Morse singularity with a unique asymptotic direction in general position (transverse to the branches of \( P \)), then the flecnodal curve has an \( A_3 \) transition at \( q \). We have four cases:

\( a \) \( q \) is an isolated parabolic point inside a hyperbolic region. An elliptic island is born or disappears.

\( b \) and \( c \) \( q \) is a crossing of two branches of \( P \) at which the asymptotic line is pointing, respectively, to the hyperbolic sectors and to the elliptic sectors. Two locally disjoint hyperbolic (elliptic) regions merge, while an elliptic (hyperbolic) region is separated into two locally disjoint regions.

\( d \) \( q \) is an isolated parabolic point inside an elliptic region. A hyperbolic island is born or disappears.

In the four cases, two godrons are born or die, and in “surface-time” 3-space, \( \{S_\varepsilon \times \{\varepsilon\}\} \), the flecnodal curves of the surfaces \( S_\varepsilon \) form a Whitney umbrella whose self-intersection line consists of hyperbonodes and whose handle consists of ellipnodes. The section \( \varepsilon = 0 \) is tangent to the umbrella at its pinch-point, giving an \( A_3 \) curve singularity (see Appendix [B] and Fig.[21]).

NOTE 2. To take the unique asymptotic tangent as x-axis imposes \( f_{20} = 0. \)
From eq. of \( P \), \( \Delta(x, y) := (f_{20}f_{02} - f_{11}^2)(x, y) = 0 \), at \((0, 0)\) we get \( f_{11} = 0 \) and \( f_{02} \neq 0 \). The condition that \((0, 0)\) is a critical point of \( \Delta(x, y) \),
\[ f_{20}f_{12} + f_{30}f_{02} - 2f_{11}f_{21} = 0 \quad \text{and} \quad f_{20}f_{03} + f_{21}f_{02} - 2f_{11}f_{12} = 0, \]
implies \( f_{30} = f_{21} = 0. \) It is of Morse type if \( f_{40}(f_{22}f_{02} - 2f_{12}^2) - f_{02}f_{31}^2 \neq 0. \)

Finally, the x-axis is transverse to both branches of \( P \) if \( f_{40} \neq 0. \)

Theorem 5.3. \( a \)\ If \( S_0 \) has a point \( q \) at which \( F_\ell \) has a Morse singularity and the left asymptotic line is transverse to both branches of \( F_\ell \) (if they are real), then \( q \) is a degenerate left biflecnode. We get a lips transition if \( F_\ell \) is
an isolated point, and a bec-à-bec transition if \( F_t \) has a local crossing (Fig. 11). In both cases two biflecnodes are born or die. (The same holds for \( F_r \).)

b) If \( q \in S_0 \) is a left triflecnode (6-point contact with the asymptotic tangent) and \( F_t \) is smooth at \( q \), then we get a “swallowtail” transition - two left biflecnodes are born or die, collapsing like the maximum and minimum values of the functions \( f_{\varepsilon}(x) = x^3 + \varepsilon x \), at \( \varepsilon = 0 \) - Fig. 11. (The same holds for \( F_r \)).

**Note 3.** The asymptotic lines are coordinate axes iff \( f_{20} = f_{02} = 0, f_{11} \neq 0 \).

a) The condition \( f_{30} \neq 0 \) guarantees a simple biflecnode.

Write \( A := f_{50} f_{11}, B := f_{41} f_{11} - 2 f_{31} f_{21}, C := 3 f_{31} f_{12} - 2 f_{22} f_{21} + f_{32} f_{11} \). The equality \( 3 f_{31}^2 - 2 f_{31} f_{11} = 0 \) implies the flecnodal curve of \( S = S_0 \) has a singularity at the origin, which is of Morse type if \( AC - B^2 \neq 0 \).

b) The condition \( 3 f_{31}^2 - 2 f_{31} f_{11} \neq 0 \) guarantees the flecnodal curve is smooth at the origin; and the conditions \( f_{30} = f_{40} = f_{50} = 0 \) with \( f_{60} \neq 0 \) guarantee the origin is a triflecnode but not a quadriflecnode.

**Example.** By Theorem 5.3 a), the flecnodal curve of the surface given by

\[
    f(x, y) = x y + x^5 + y^3 + \alpha x^2 y + \alpha^2 x^3 y + \gamma x^3 y^2
\]

has a Morse singularity at the origin if \( \alpha \neq 0 \) and \( \gamma \neq \frac{2}{3} \alpha^6 \) because \( AC - B^2 = 2 \cdot 6! \gamma - 24^2 \alpha^6 \). But if \( \gamma = \frac{2}{3} \alpha^6 \), the flecnodal curve has a semi cubic cusp.

**Theorem 5.4.** a) If \( F_t \) and \( F_r \) are tangent at \( q \in S_0 \), then at \( \varepsilon = 0 \) we get a creation/annihilation transition of two hyperbonodes.

b) If \( q \in S_0 \) is a hyperbolic point at which the asymptotic tangents have 4- and 5-point contact with \( S_0 \), then we get a “flec-hyperbonode” transition: a left biflecnode moves along \( F_t \) and, at \( \varepsilon = 0 \), crosses \( F_r \) overlapping with a hyperbonode. (\( F_t \) and \( F_r \) may be interchanged.)

c) If two complex conjugate branches of \( F \) are tangent at \( q \in S_0 \), then at \( \varepsilon = 0 \) we get a creation/annihilation transition of two ellipnodes.

**Note 4.** If we take the asymptotic lines as coordinate axes, the condition \( f_{30} = f_{03} = 0 \) guarantees the origin is a hyperbonode. Then

a) the condition \( 4 f_{40} f_{04} f_{11} - (3 f_{21}^2 - 2 f_{31} f_{11})(3 f_{32}^2 - 2 f_{13} f_{11}) = 0 \) guarantees the tangency of \( F_t \) and \( F_r \).

b) the condition \( (3 f_{31}^2 - 2 f_{31} f_{11})(3 f_{12}^2 - 2 f_{13} f_{11}) \neq 0 \) guarantees the transversality of \( F_t \) and \( F_r \), and \( f_{40} = 0, f_{50} \neq 0 \) guarantee our point is a biflecnode which is not triflecnode.

c) At an ellipnode we can choose affine coordinates such that \( f_{20} = f_{02} \) and \( f_{30} = f_{03} = f_{21} = f_{12} = 0 \). In such coordinate system, the equality \( (f_{40} - 3 f_{22})(f_{04} - 3 f_{22}) - (f_{31} - 3 f_{31})(f_{13} - 3 f_{31}) = 0 \) guarantees the tangency of two complex conjugate branches of the flecnodal curve at \( q = (0, 0) \).
Example. The respective normal forms for hyperbonodes of Landis-Platonova [23, 27] and Tabachnikov-Ovsienko [24] are the following:

\[ f(x, y) = xy \pm x^4 + \alpha x^3 y + \beta xy^3 + y^4, \]

\[ f(x, y) = xy \pm Ix^4 + x^3 y + xy^3 + Jy^4. \]

Both satisfy \( f_{21} = f_{12} = 0 \). So, to get the tangency of \( F_t \) et \( F_r \) we need to have \( f_{10}, f_{04} - f_{31}, f_{13} = 0 \). We respectively get \( \alpha \beta = \pm 16 \) and \( \pm 16IJ = 1 \).

Flat Umbilics. A point \( q \) at which the quadratic part of \( f \) is zero and has nondegenerate cubic form is called nondegenerate flat umbilic. The projective dual surface at \( q \) has a \( D_4 \) front singularity. All tangent lines at \( q \) are asymptotic, but either one or three of those lines have 4-point contact with \( S \) at \( q \) because the cubic form vanishes along either one line \((D_4^+)\) or three lines \((D_4^-)\). Thus we have either one or three branches of \( F \) through \( q \) (Fig. 22).

Clearly, a vector \( \vec{u} = (u_1, u_2) \) belongs to the kernel of the cubic form of \( f \) iff \( \partial^3_\alpha f = 0 \), where \( \partial_\alpha := u_1 \partial_x + u_2 \partial_y \). Thus in the \( D_4^- \) case there are three vectors \( \vec{u}, \vec{v}, \vec{w} \) satisfying \( \partial^3_\alpha f = \partial^3_\beta f = \partial^3_\gamma f = 0 \).

Theorem 5.5. Suppose \( S_0 \) has a non degenerate flat umbilic at \( q \) and no asymptotic line has 5-point contact with \( S_0 \). Then

a) If \( S_0 \) has three asymptotic tangents with 4-point contact, then \( S_0 \) has locally three transverse flecnodal curves and we get a \( D_4^- \) transition: an elliptic disc with tree negative godrons on \( P \) and three neighbouring hyperbonodes collapse and reappear (the reappeared disc has opposite natural co-orientation.)

b) If \( S_0 \) has one asymptotic tangent with 4-point contact, then we get a \( D_4^+ \) transition: (say \( \varepsilon < 0 \)) two elliptic regions of opposite co-orientation approach each other; \((\varepsilon = 0)\) one branch of \( F \) and two branches of \( P \) intersect transversely at \( q \); \((\varepsilon > 0)\) the elliptic regions do not merge, but repulse each other.

In the process, an ellipnode passes from one elliptic domain to the other and a negative godron passes from one parabolic curve to the other.

Note 5. a) Let \( \vec{u}, \vec{v}, \vec{w} \) be the vectors satisfying \( \partial^3_\alpha f = \partial^3_\beta f = \partial^3_\gamma f = 0 \). The condition \( \partial^5_\alpha f \partial^3_\alpha f \partial^4_\alpha f \neq 0 \) ensures each over-osculating asymptotic line has exactly 4-point contact with \( S_0 \): this bans the local presence of biflecnodes.

The analogue genericity condition in b) is \( \partial^3_\alpha \neq 0 \).

Degenerate flat umbilics. A flat umbilic \( D_4^+ \) or \( D_4^- \) satisfying \( \partial^3_\alpha f = 0 \) (for some vector \( \vec{u} \)) is not stable in generic 1-parameter families of surfaces but is stable in 2-parameter ones if \( \partial^3_\alpha f \neq 0 \) - for small values of the parameter the surfaces have a biflecnode on the flecnodal curve corresponding to the \( \vec{u} \)-direction. Similarly, a flat umbilic \( D_4^- \) with \( \partial^3_\alpha f = \partial^3_\beta f = 0 \) is not stable in generic 2-parameter families of surfaces but is stable in 3-parameter ones - two flecnodal curves of the surfaces have one biflecnode.

Example. The \( D_4^- \) flat umbilic point of the surface given in Monge form by

\[ f(x, y) = x^3 - xy^2 + \alpha x^3 y + \beta y^4 + \psi(x, y) \quad (5.2) \]

is not stable in generic 1-parameter families of smooth surfaces if \( \partial^3_\alpha f = 0 \), \((\partial_x + \partial_y)^4 f = 0 \) or \((\partial_x - \partial_y)^4 f = 0 \), that is, if \( \beta = 0, \beta = -\alpha \) or \( \beta = \alpha \).

So the normal form \( (5.2) \) in Table 2 of [29] should be accompanied by the restrictions \( \beta \neq 0, \beta \neq -\alpha \) and \( \beta \neq \alpha \). Similarly, one has to impose \( \beta \neq 0 \) to the \( D_4^+ \) normal form of [29].
5.2 Further Results and Comments

**Extension to Surfaces in other Spaces.** Since the tangential singularities depend only on the contact of the surface with its tangent lines, all our results hold for surfaces in the 3-sphere $S^3$; in Lobachevsky 3-space $\Lambda^3$; in de Sitter world, etc. Namely, $S^3$ is the double covering of $\mathbb{RP}^3$; the lines of the Klein model of $\Lambda^3$ are lines of the affine space $\mathbb{R}^3$; etc.

5.2.1 First Degenerate “Conic” Points of the surface $a^f = 0$

Consider a surface $z = f(x,y)$ such that the function $a^f$ has a critical point at $\bar{0}$ with $a^f(\bar{0}) = 0$. Assume that $f_{40} \neq 0$ and $f_{02} = 1$, and write

$$\Delta = 3f_{40}[2f_{12}(f_{12}f_{33}f_{40} + 3f_{12}f_{40}f_{31} - f_{12}f_{13}) + f_{31}(f_{31}f_{41} - f_{40}f_{32})] + f_{31}f_{50} + f_{30}f_{51}.$$

**Theorem 5.2.1** The parabolic curve $P$ and the flecnodal curve $F$ of the surface $z = f(x,y)$ have the singularities shown in Table 2 at $(0,0)$:

| $P$ | $F$ | picture | conditions | codim |
|-----|-----|---------|------------|-------|
| $A_1$ | $A_3$ | ![Picture](image) | $f_{40}(f_{22} - 2f_{12}^2) - f_{31}^2 \neq 0.$ | 3     |
| $A_2$ | $A_6$ | ![Picture](image) | $f_{40}(f_{22} - 2f_{12}^2) - f_{31}^2 = 0, \quad \Delta \neq 0.$ | 4     |
| $A_3$ | $A_9$ | ![Picture](image) | $f_{40}(f_{22} - 2f_{12}^2) - f_{31}^2 = 0, \quad \Delta = 0.$ | 5     |

Table 2: A Morse conic singularity and the simplest non Morse conic singularities.

**On the proof.** The first case is part of Theorem 5.2. In the second case, we make a suitable linear change of coordinates so that the relevant terms of the Newton diagram of the Hessian polynomial are $H_f(x,y) = x^2 + \Delta y^3 + \ldots$, giving an $A_2$ singularity for the parabolic curve - the $A_6$ singularity of the flecnodal curve is obtained by a calculation and using the Newton diagram of the polynomial $f^f(x,y)$. The condition $\Delta = 0$, of third case, clearly provides an $A_3$ singularity for the parabolic curve, and the singularity $A_9$ of the flecnodal curve is obtained similarly as we did in the second case.  

**Remark.** In table 2 we made one picture for each case, but the 1st and 3rd cases have four real realisations and the 2nd case has two real realisations.

**Example.** The surface whose local Monge form is

$$f(x,y) = y^2/2 \pm x^4 + \alpha x^3 y + (2\beta^2 \pm \frac{3}{8}\alpha^2)x^2 y^2 + \beta xy^2,$$

satisfies $f_{40}(f_{22}f_{02} - 2f_{12}^2) - f_{02}f_{31}^2 = 0$ (see Note 2). So if $\alpha\beta(\alpha^2 \pm 8\beta^2) \neq 0$ (i.e. $\Delta \neq 0$) the parabolic and flecnodal curves of the surface have, respectively, the singularities $A_2$ (a cusp) and $A_6$ at $0$. If $\beta = 0$ we get the normal form $f(x,y) = y^2/2 \pm x^4 + \alpha x^3 y \pm \frac{3}{8}\alpha^2 x^2 y^2$ of [29] whose parabolic and flecnodal curves have the respective singularities $A_3$ and $A_9$ because $\Delta = 0$. 

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5.2.2 Comparing the transitions of Figures 1 and 5

On one hand, the curve of inflections of a BIDE \( F(x, y, p) = 0 \) is a particular case of the curve of (generalised) inflections of the characteristic fronts of a surface \( \mathcal{V} \subset M \) in a contact 3-manifold \( M \). On the other hand, the flecnodal curve of a surface \( z = f(x, y) \) leads to study the curve of inflections of the asymptotic IDE \( a^f = 0 \), which is a very special class of BIDE (of infinite codimension). So, we have three families of objects summarised in Table 3.

| Surface in \( \mathbb{RP}^3 (\mathbb{R}^3) \) | IDE \( F(x, y, p) = 0 \) and \( \mathcal{V}' \) | \( \mathcal{V} \subset M, \rho, \rho' \) |
|-------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| parabolic curve                          | discriminant                                   | \( \rho \)-discriminant                       |
| asymptotic curve                         | solution of the IDE \( F = 0 \)               | characteristic front                           |
| flecnodal curve                          | curve of inflections                           | curve of inflections                           |
| godron                                    | folded singular point                          | \( \rho \)-folded singular point               |
| hyperbonode                               | hyperbolic node                                | hyperbolic node                               |
| ellipnode                                 | (elliptic node)                                | (elliptic node)                               |
| biflecnode                                | bi-inflection                                  | bi-inflection                                 |

Table 3: Asymptotic curves | Solutions of IDEs | Characteristic fronts.

\( A_3 \) vs (a to d) transitions. The \( A_3 \) transitions in Fig. 1 are analogues to the transitions a to d in Fig. 5; in both, the surface of the BIDE has a Morse singularity. But in cases a to d, the curve of inflections has a Morse transition, while in the \( A_3 \) cases the flecnodal curve undergoes an \( A_3 \) transition, as a family of sections of a Whitney umbrella one of which is tangent to the umbrella at its pinch point (see Appendix B and Fig. 21).

Infinite Degeneracy of the asymptotic IDE at \( D_4 \) transitions. If a surface in \( \mathbb{RP}^3 \) has a flat umbilic point \( q \) (a \( D_4 \)-singularity), then the surface \( \mathcal{A}_0 \subset J^1(\mathbb{R}, \mathbb{R}) \) of its asymptotic IDE contains the whole fibre of \( \pi: J^1(\mathbb{R}, \mathbb{R}) \rightarrow J^0(\mathbb{R}, \mathbb{R}) \) over \( q \) (see Fig. 22). Thus, the IDE defined by \( \mathcal{A}_0 \) is not binary and has infinite codimension in the space of IDEs. This explains the absence of \( D_4 \) transitions in Fig. 5.

Fig. 22. Evolution of the projection \( \pi: \mathcal{A}' \rightarrow \mathbb{R}^2 \) for the \( D_4^+ \) and \( D_4^- \) perestroikas.
6 From Surfaces to IDEs: Preparatory Results

6.1 Asymptotic Double and Asymptotic IDE

Contact Elements. Let $N$ be an $n$-dimensional smooth manifold. A contact element at a point of $N$ is a vectorial hyperplane (of co-dimension 1) of the tangent space of $N$ at that point (called the point of contact). The set of all contact elements of $N$ is a $(2n - 1)$-dimensional manifold (noted by $PT^*N$) endowed with a natural contact structure.

Asymptotic double. Now, consider a generic smooth surface $S$ in $\mathbb{R}P^3$. The asymptotic-double $A$ of $S$ is the (smooth) surface of $PT^*S$ consisting of the asymptotic directions along $S$: at each hyperbolic point, the asymptotic lines determine two contact elements, while at the parabolic points there is only one asymptotic contact element (with multiplicity two). See Fig. 23.

![Fig. 23. The asymptotic-double of a surface.](image)

So, under the natural projection $PT^*S \to S$ (sending each contact element to its point of contact) the asymptotic double $A \subset PT^*S$ doubly covers the hyperbolic domain with a fold singularity over the parabolic curve.

To study the flecnodal curve, we shall use the asymptotic double $A$ of $S$.

Consider a projection of $S$ to an affine plane $\mathbb{R}^2 \subset \mathbb{R}P^3$, from a point exterior to the projective plane that contains $\mathbb{R}^2$, $\pi : S \to \mathbb{R}^2 \subset \mathbb{R}P^3$, and assume it is a local diffeomorphism at every point of $S$.

On one hand, any such projection sends (bijectively) the flecnodal curve of $S$ onto the curve of $\mathbb{R}^2$ formed by the inflections of the images (by $\pi$) of the asymptotic curves of $S$ (see Proposition 6.2 in §6.4).

On the other hand, the derivative of $\pi$ sends the contact elements of $S$ to the contact elements of $\pi(S) \subset \mathbb{R}^2$ and it induces a contactomorphism $PT^*S \to PT^*(\pi(S)) \subset PT^*\mathbb{R}^2$ sending $A$ to a surface $\tilde{A} \subset PT^*\mathbb{R}^2$ which we still call the asymptotic-double of $S$ and which doubly covers (under the natural projection $PT^*\mathbb{R}^2 \to \mathbb{R}^2$) the image by $\pi$ in $\mathbb{R}^2$ of the hyperbolic domain. We note it $A'$ when $S$ is given in Monge form $z = f(x, y)$.

Consider the surface $S$ as the graph of a function $z = f(x, y)$, and take the projection $(x, y, z) \to (x, y)$, along the $z$-axis. Then, the asymptotic-double $A' \subset PT^*\mathbb{R}^2$ is given by the contact elements of the $xy$-plane for which

$$f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 = 0. \quad (\ast)$$

In order to make calculations, we take the space $J^1(\mathbb{R}, \mathbb{R})$ (with coordinates $x, y, p$) as an ‘affine’ chart of $PT^*\mathbb{R}^2$. Namely, $J^1(\mathbb{R}, \mathbb{R})$ parametrises all non vertical contact elements of the $xy$-plane: the contact element with slope $p_0 \neq \infty$ at $(x_0, y_0)$ is represented by the point $(x_0, y_0, p_0)$ in $J^1(\mathbb{R}, \mathbb{R})$. 

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The asymptotic-double $A^f$ is the surface in $J^1(\mathbb{R}, \mathbb{R})$ given by the asymptotic IDE
\[ a^f(x, y, p) := f_{xx} + 2f_{xy}p + f_{yy}p^2 = 0, \] (a)
obtained from eq. (*) by taking $p = dy/dx$. The asymptotic curves of $S$ are the images by $f$ of the solutions of the IDE (a), whose discriminant (in the $xy$-plane) corresponds to the parabolic curve of $S$.

### 6.2 Facts and Lemmas on IDEs $F(x, y, p) = 0$

Consider a smooth function $F : J^1(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, (x, y, p) \mapsto F(x, y, p)$.

**Inflection Function.** The inflection function of $F$, $I^F : J^1(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, (x, y, p) \mapsto I^F(x, y, p)$, is defined by
\[ I^F(x, y, p) := F_x(x, y, p) + pF_y(x, y, p). \]

**Flec-Surface.** The flec-surface of $F$, defined in §3.2, is the surface given by the equation $I^F(x, y, p) = 0$ in the space $J^1(\mathbb{R}, \mathbb{R})$.

**Lemma 6.1.** The contour of the surface $V^F$ by the projection $\pi^\vee$ (i.e. the $\pi^\vee$-contour) is the intersection of $V^F$ with the flec-surface of the function $F$.

**Proof.** An easy computation shows (and it is well known, [3]) that the characteristic direction ($\S$2.1) is generated by the vector field
\[(\dot{x}, \dot{y}, \dot{p}) = (-F_p, -pF_p, F_x + pF_y). \] (*)
The direction field (*) is tangent to the $\pi^\vee$-fibre at a point of $V^F$ iff this field is horizontal, i.e. iff $F_x + pF_y = 0$. Thus the $\pi^\vee$-contour of $V^F$ consists of the points of $V^F$ at which $I^F(x, y, p) = 0$.

**Corollary.** A point of the surface $V^F$ is a characteristic point iff $dF \neq 0$, $F_p = 0$ and $F_x + pF_y = 0$ at this point.

**Lemma 6.2.** The origin belongs to the flec-surface of $F$ iff $F_x(\bar{0}) = 0$.

**Proof.** $I^F(\bar{0}) = (F_x + pF_y)(\bar{0}) = F_x(\bar{0})$.

**Lemma 6.3.** The flec-surface of $F$ has vertical tangent plane at $\bar{0}$ iff
\[ F_{xp}(\bar{0}) + F_y(\bar{0}) = 0. \]

**Proof.** The tangent plane to the flec-surface is vertical at a point iff $I^F_p = 0$ at that point, i.e. iff $F_{xp} + pF_{yp} + F_y = 0$ at that point.

**Lemma 6.4.** Suppose that the origin is a Morse critical point of $F$. The flec-surface of $F$ has vertical tangent plane at the origin iff
\[ F_{xp}(\bar{0}) = 0. \]

**Proof.** The statement follows from Lemma 6.3.

**Proposition 6.1.** Let $Q \in V^F$ a Morse critical point of $F$. The flec-surface of $F$ has vertical tangent plane at $Q$ iff the tangents to the $\pi$ and $\pi^\vee$-fibres are conjugate diameters of the cone $K \subset T_QJ^1(\mathbb{R}, \mathbb{R})$ determined by $F$. 

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Proof. At $Q = \bar{0} \in J^1(\mathbb{R}, \mathbb{R})$ the contact plane $\Pi$ coincides with the $xp$-plane and the tangents to the $\pi$ and $\pi^\vee$-fibres coincide with the $x$- and $p$-axes, $\ell_x$ and $\ell_p$. So, a simple calculation shows that the cross-ratio of the $\pi$- and $\pi^\vee$-tangents, $\ell_x, \ell_p$, with the lines $\ell_1, \ell_2$, on which the contact plane $\Pi$ intersects the cone $K$ (being real or complex lines), is equal to

$$ (\ell_x, \ell_p, \ell_1, \ell_2) = \frac{-F_{xp} + \sqrt{F_{xp}^2 - F_{xx}F_{pp}}}{-F_{xp} - \sqrt{F_{xp}^2 - F_{xx}F_{pp}}}. \tag{6.1} $$

The tangents to the $\pi$ and $\pi^\vee$-fibres at $\bar{0}$ are conjugate diameters of $K$ iff the cross-ratio $(\ell_x, \ell_p, \ell_1, \ell_2)$ equals $-1$ (§2.5 (harmonicity)). This holds, by eq. (6.1), iff $F_{xp}(\bar{0}) = 0$. But, by Lemma 6.4, $F_{xp}(\bar{0}) = 0$ iff the tangent plane to the flec-surface of $F$ at $\bar{0}$ is vertical.

6.3 Lemmas for the Asymptotic IDE $a^f(x, y, p) = 0$

Lemma 6.5. The function $a^f$ satisfies the equalities

$$ a^f_{xp}(\bar{0}) + a^f_y(\bar{0}) = \frac{3}{2} a^f_{xp}(\bar{0}) = 3a^f_y(\bar{0}) = 3f_{xx}(0, 0). $$

Proof. Direct, short and easy calculation (by hand).

Lemma 6.6. The flec-surface of $a^f$ has vertical tangent plane at $\bar{0}$ iff $a^f_y(\bar{0}) = 0$.

Proof. Lemma 6.3 (applied to $F = a^f$) and Lemma 6.5 imply Lemma 6.6.

Lemma 6.7. If the function $a^f$ has a critical point then its flec-surface has vertical tangent plane at that point.

Proof. Since we are supposing that the critical point is the origin, we have $a^f_x(\bar{0}) = a^f_y(\bar{0}) = a^f_p(\bar{0}) = 0$. By Lemma 6.2 (for $F = a^f$), the point belongs to the flec-surface. Now Lemma 6.7 follows from Lemma 6.6.

Lemma 6.8. If the flec-surface of $a^f$ has vertical tangent plane at a point of the criminant then this point is critical for the function $a^f$.

Proof. Supposing our point is $\bar{0}$, we have $a^f_x(\bar{0}) = 0$ (by Lemma 6.2) and $a^f_p(\bar{0}) = 0$ (because $\bar{0}$ is in the criminant). By Lemma 6.6, $a^f_y(\bar{0}) = 0$.

6.4 Flecnodal Curve and Curves of Inflections

Proposition 6.2. Given a smooth surface $S$ in Monge form $z = f(x, y)$, the projection $(x, y, z) \mapsto (x, y)$ (along the $z$-axis) sends the flecnodal curve of $S$ onto the curve of inflections of the asymptotic IDE $a^f(x, y, p) = 0$:

$$ f_{xx} + 2f_{xy}p + f_{yp}^2 = 0. \tag{a} $$

Of course, the discriminant (in the $xy$-plane) of the IDE (a) corresponds to the parabolic curve (in the $xyz$-space) of the surface $z = f(x, y)$.
Proof of Proposition 6.2. Consider a regularly parametrised curve on 
\[ \gamma(t) = (\gamma_1(t), \gamma_2(t), f(\gamma_1(t), \gamma_2(t))), \quad \gamma(0) = (0, 0, 0), \quad \gamma'(0) \neq (0, 0, 0). \]
Its tangent line at \( \gamma(0) \) is parametrised by \( \ell(s) = \gamma(0) + s\gamma'(0) \). Consider the function \( G(x, y, z) = f(x, y) - z \). The line \( \ell \) has 4-point contact with \( S \) at the origin iff the function \( g(s) = (G \circ \ell)(s) \) has a 0 of multiplicity 4 at \( s = 0 \). We already have \( g(0) = g'(0) = 0 \) because \( \ell \) is tangent to \( S \) at the origin. (To simplify, we write \( \gamma'_1 \) and \( \gamma'_2 \) instead of \( \gamma'_1(0) \) and \( \gamma'_2(0) \); and since the function \( f \) and its partial derivatives are evaluated at \( (s\gamma'_1, s\gamma'_2) \), we omit to note this point.) The function \( g \) is given by 
\[ g(s) = s(f_x\gamma'_1 + f_y\gamma'_2) - f. \]
We have therefore 
\[ g''(s) = s(f_{xxx}(\gamma'_1)^3 + 3f_{xxy}(\gamma'_1)^2\gamma'_2 + 3f_{xyy}(\gamma'_1)(\gamma'_2)^2 + f_{yyy}(\gamma'_2)^3) + \]
\[ + f_{xx}(\gamma'_1)^2 + 2f_{xy}\gamma'_1\gamma'_2 + f_{yy}(\gamma'_2)^2. \]
So \( g''(0) = 0 \) iff the vector \( (\gamma'_1, \gamma'_2) \) satisfies the equation 
\[ (f_{xx}(\gamma'_1)^2 + 2f_{xy}\gamma'_1\gamma'_2 + f_{yy}(\gamma'_2)^2)|_{s=0} = 0. \]
\[ g'''(s) = f_{xxx}(\gamma'_1)^3 + 3f_{xxy}(\gamma'_1)^2\gamma'_2 + 3f_{xyy}(\gamma'_1)(\gamma'_2)^2 + f_{yyy}(\gamma'_2)^3 + s \cdot \text{(something)}. \]
Thus, \( g'''(0) = 0 \) iff the vector \( (\gamma'_1, \gamma'_2) \) satisfies the equation 
\[ (f_{xxx}(\gamma'_1)^3 + 3f_{xxy}(\gamma'_1)^2\gamma'_2 + 3f_{xyy}(\gamma'_1)(\gamma'_2)^2 + f_{yyy}(\gamma'_2)^3)|_{s=0} = 0. \]
But the curve of inflections of the asymptotic curves of a smooth surface coincides with the flecnodal curve of that surface. Proof. Let \( \gamma \) be a regularly parametrised left (or right) asymptotic curve of \( S \) such that \( \gamma(0) = (0, 0, 0) \). Along our asymptotic curve \( \gamma \) we have 
\[ f_{xx}(\gamma'_1)^2 + 2f_{xy}\gamma'_1\gamma'_2 + f_{yy}(\gamma'_2)^2 = 0 \] (6.2)
Deriving (6.2) we get that the following equation holds along \( \gamma \)
\[ (f_{xxx}(\gamma'_1)^3 + 3f_{xxy}(\gamma'_1)^2\gamma'_2 + 3f_{xyy}(\gamma'_1)(\gamma'_2)^2 + f_{yyy}(\gamma'_2)^3) + \]
\[ + 2(f_{xx}\gamma''_1 + f_{xy}\gamma'_1\gamma'_2 + f_{yy}\gamma'_2^2) = 0. \] (6.3)
The origin, \( \gamma(0) \), belongs to the left flecnodal curve iff the asymptotic tangent line \( \ell(s) = \gamma(0) + s\gamma'(0) \) has at least 4-point contact with \( S \) at \( \gamma(0) \). According to the proof of Proposition 6.2, this 4-point contact is attained iff 
\[ f_{xx}(\gamma'_1)^2 + 2f_{xy}\gamma'_1\gamma'_2 + f_{yy}(\gamma'_2)^2 = 0 \] and \( (6.4) \)
\[ f_{xxx}(\gamma'_1)^3 + 3f_{xxy}(\gamma'_1)^2\gamma'_2 + 3f_{xyy}(\gamma'_1)(\gamma'_2)^2 + f_{yyy}(\gamma'_2)^3 = 0 \] at the origin.
Equation (6.3) is satisfied because \( \gamma \) is an asymptotic curve. Thus, the last equation holds at the origin iff the second term of the left-hand side of (6.3) vanishes at the origin. This last condition holds iff \( \gamma''(0) \) is a multiple \( \gamma'(0) \), that is, iff \( \gamma(0) \) is an inflection of our asymptotic curve \( \gamma \). \( \square \)
7 Proofs of the Theorems on Evolving Surfaces

On the proofs of Theorem 5.1 and of No Cusp Theorem

Items a) and b) of Theorem 5.1 coincide with items a) and b) of Theorem 3.1 and of Theorem 1.1 (and were discussed in [31]). It remains to prove that, in the case of surfaces in 3-space, item c) of Theorem 3.1 never occurs:

The flecnodal curve of a smooth surface in \( \mathbb{R}P^3 \) (or in \( \mathbb{R}^3 \)) has never a cusp at a godron at which the parabolic curve is smooth. (No Cusp Theorem.)

Proof of No Cusp Theorem. Assume the surface has a godron \( g \) at the point \( z_0 = f(x_0, y_0) \) with smooth parabolic curve. Since its corresponding point \( (x_0, y_0, p_0) \in \mathcal{A}f \subset J^1(\mathbb{R}, \mathbb{R}) \) is a characteristic point of the surface \( \mathcal{A}f \), the point \( (x_0, y_0) \) is a folded singular point of the IDE \( a_f = 0 \) with smooth discriminant. Therefore \( (x_0, y_0, p_0) \) is not a critical point of the function \( a_f \).

The curve of inflections of the IDE \( a_f = 0 \) is the image under the map \( \pi: (x, y, p) \mapsto (x, y) \) of the \( \pi^\vee \)-contour of \( \mathcal{A}f \). So, the curve of inflections has a cusp at \( (x_0, y_0) \) iff the tangent line to the \( \pi^\vee \)-contour of \( \mathcal{A}f \) at \( (x_0, y_0, p_0) \) is vertical. But since the \( \pi^\vee \)-contour of \( \mathcal{A}f \) is the intersection of \( \mathcal{A}f \) with the flec-surface of \( a_f \), this intersection has vertical tangent line at a point only if the flec-surface of \( a_f \) has vertical tangent plane at that point. By Lemma 6.8, this is impossible for a non-critical point of the function \( a_f \).

Proof of Theorem 5.2 (On \( A_3 \)-transitions)

The conditions that guarantee the surface of equation \( F = 0 \) has a “Morse conic point in general position” (§2.5) are stated in Example 2.6. For \( F = a_f \), these conditions are equivalent to the genericity conditions of Th. 5.2, stated in Note 2 in terms of the Monge form \( z = f(x, y) \) at \( q = (0, 0, 0) \).

Proposition 7.1. If the function \( a_f \) has a Morse critical point, then the quadratic cone \( K \) determined by the equation \( a_f = 0 \) cannot have positions e and f with respect to the fibrations \( \pi \) and \( \pi^\vee \) (Fig. 12). Moreover, the tangent lines to the fibres of \( \pi \) and \( \pi^\vee \) are conjugate diameters of \( K \).

Proof. Lemma 6.5 implies that \( a_f(0) = 0 \). By Example 2.6 the intersection of the cone with the plane \( y = 0 \) consists of the two lines given by the equation \( a_{xx}(0)x^2 + a_{xp}(0)p^2 = 0 \). Therefore the positions e and f of the cone are impossible (see Fig. 12) and the cross-ratio of these lines with the x- and p-axes (the directions of the \( \pi^\vee \)- and \( \pi \)-fibres) is evidently \(-1\).

So, for surfaces, only the positions a,b,c,d of the cone \( K \) in Fig. 12 can take place.

Proof of Theorem 5.2. A generic family of smooth surfaces \( \{ z = f_\varepsilon(x, y) \} \) defines the family of asymptotic IDEs \( \{ a_f = 0 \} \). By the stability of the Morse critical points, the functions \( a_f \) have a Morse critical point near the origin; but we can suppose that that critical point is the origin (by a translation and a rotation in the \( xy \)-plane, depending on \( \varepsilon \)).
Then, by Lemma 6.7 and Lemma 6.4 the 2-jet (in the variables \(x, y, p\)) of our family \(\Phi^f\) of asymptotic IDE can be written in the form

\[
j^2\Phi^f(x, y, p; \varepsilon) = Ax^2 + 2Bxy + Cy^2 + Dp^2 + 2Gyp - \mathcal{E},
\]

where \(A = a + a_1\varepsilon + \cdots\), \(B = b + b_1\varepsilon + \cdots\), \(C = c + c_1\varepsilon + \cdots\), \(E = e + e_1\varepsilon + \cdots\), \(G = g + g_1\varepsilon + \cdots\), and \(\mathcal{E} = \varepsilon + h.o.t.(\varepsilon)\).

The first approximation in \(\varepsilon\) is given by the family if IDEs

\[
\Psi^f(x, y, p; \varepsilon) = a^f - \varepsilon = ax^2 + 2bxy + cy^2 + dp^2 + 2gyp - \varepsilon = 0.
\]

The fact that \(a^f(\bar{0}) = 0\) and that \(\bar{0}\) is a critical point of \(a^f(x, y, p)\) implies that at \((x, y) = (0, 0)\) we have

\[
f_{20} = f_{11} = f_{30} = f_{21} = 0.
\]

So the part of the Taylor expansion of \(f\) determining the cone of \(a^f = 0\) is

\[
f(x, y) = \frac{ax^4}{12} + \frac{bx^3y}{3} + \frac{cx^2y^2}{2} + \frac{gxxy^2}{2} + \frac{dy^4}{2} + \cdots . \tag{7.1}
\]

With a scaling we make \(d = 1\). The condition \(f_{40}(f_{22} - 2f_{21}^2) - f_{31}^2 \neq 0\) means \(a(c - g^2) - b^2 \neq 0\). We can make \(g = 0\) by a projective transformation (using the fact that \(xy^2\) is a multiple of the quadratic part of \(f\)). Thus the above first approximation in \(\varepsilon\) of the family of asymptotic IDEs becomes

\[
a^f(x, y, p) - \varepsilon = ax^2 + 2bxy + cy^2 + p^2 - \varepsilon = 0. \tag{7.2}
\]

Then the flec-surface is given by the equation

\[
Ia^f/2 = (ax + by) + (cy + bx)p = 0.
\]

Making the change of variables \(Y = ax + by\), \(X = bx + cy\) (which is possible only if \(\Delta = ac - b^2 \neq 0\)) we get the equalities

\[
x = (bX - cY)/\Delta, \quad y = (bY - aX)/\Delta \quad \text{and} \quad p = -Y/X.
\]

Inserting them in (7.2) and putting \(\alpha = -a/\Delta\), we get the equation

\[
Y^2 = \varepsilon X^2 + \alpha X^4 - (c/\Delta)X^2Y^2 + 2(b/\Delta)X^3Y, \tag{7.3}
\]

of a Whitney umbrella (elliptic if \(\alpha > 0\) and hyperbolic if \(\alpha < 0\)) whose isochronal section \(\varepsilon = 0\) is tangent to the umbrella at the pinch-point.

For \(g \neq 0\) we also get a Whitney umbrella, as expected, whose isochronal plane section \(\varepsilon = 0\) is diffeomorphic to two tangent parabolas given by the equation \((3g^2a^2 + \Delta a)Y^4 + 4g^3a^2XY^2 + \Delta^2X^2 = 0\), with \(\Delta \neq ag^2\).

**Stability.** The preceding calculations show that for each \(\varepsilon\) the flecnodal curve of the IDE \(a^f - \varepsilon = 0\) is the vertical projection of the intersection curve of the flec-surface of \(a^f\) with the surface \(\mathcal{A}_\varepsilon\) (of the IDE \(a^f - \varepsilon = 0\)). The flec-surface of \(a^f\) is thus foliated by curves labelled by \(\varepsilon\) whose vertical projection is the flecnodal curve of the corresponding IDE \(a^f - \varepsilon = 0\). This defines a map from the flec-surface of \(a^f\) to the plane-time 3-space \((x, y, \varepsilon)\), whose image is
the Whitney umbrella (without its handle), swiped by the flecnodal curves of this family. This map singularity is stable \[33\] (see AppendixB).

**Parabolic Curve.** The parabolic curve is provided by the Hessian curve, \((f_{20}f_{02} - f_{11}^2)(x, y) = 0\). At the transition moment it is given by the equation:

\[
(ax^2 + 2bxy + cy^2)(cx^2 + gx + 1) - (bx^2 + 2cxy + gy)^2 = 0.
\] (7.4)

Then (for \(\Delta \neq ag^2\)) the parabolic curve is diffeomorphic, near the origin, to the two transverse lines (may be imaginary) given by the equation

\[
ax^2 + 2bxy + (c - g^2)y^2 = 0.
\]

\[\square\]

### Proof of Theorem 5.3 (Lips, bec-à-bec, swallowtail)

**Proof of Theorem 5.3 (a).** For generic 1-parameter families of surfaces in \(\mathbb{RP}^3\), the corresponding pairs of surfaces \(a^f = 0\) and \(I^f = 0\) (in \(J^1(\mathbb{R}, \mathbb{R})\)) may have one point of tangency for isolated parameter values. At such a point the differentials of \(a^f\) and \(I^f\) are proportional; and at \(\bar{0}\) this proportionality is equivalent to the equality \(3f_{21}^2 - 2f_{31}f_{11} = 0\) because \(f_{30} = f_{40} = 0\).

If we consider these surfaces as graphs of functions \(p = p_a(x, y)\) and \(p = p_I(x, y)\), the projection of their intersection to the \(xy\)-plane consists of the zeroes of the difference \(p_a(x, y) - p_I(x, y)\), whose Taylor series is given by \(Ax^2 + 2Bxy + Cy^2 + \cdots\), where \(A = f_{50}f_{11}, B = f_{41}f_{11} - 2f_{31}f_{21}\) and \(C = 3f_{31}f_{12} - \frac{2}{3}f_{22}f_{21} + f_{32}f_{11}\).

The statements of Theorem 5.3 (a) follow from this fact.

**Proof of Theorem 5.3 (b).** Since \(f_{30} = f_{40} = 0\), the pair of surfaces given by the equations \(a^f(x, y, p) = 0\) and \(I^f(x, y, p) = 0\) are smooth and transverse at \(\bar{0}\) because \(3f_{21}^2 - 2f_{31}f_{11} \neq 0\) (just compute the differentials). If we consider these surfaces as graphs of functions \(p = p_a(x, y)\) and \(p = p_I(x, y)\), then the projection of their intersection to the \(xy\)-plane (that is, the flecnodal curve) is given by the zeroes of the difference \(p_a(x, y) - p_I(x, y)\). A direct computation shows that the Taylor series of this difference has the form

\[
\left(\frac{f_{31}}{3f_{21}} - \frac{f_{21}}{2f_{11}}\right)y^2 + 2\beta xy + \gamma y^2 + \frac{f_{60}}{3f_{21}}x^3 + \cdots,
\]

which implies the left (right) flecnodal curve has second order tangency with the left (resp. right) asymptotic line (for any value of \(\beta\) and \(\gamma\)). Then we have a double biflecnode, which can disappear or split into two simple biflecnodes.

This proves Theorem 5.3 (b).

### On the proof of Theorem 5.5 (On \(D_4\)-transitions)

We shall study the local transition at \(\varepsilon = 0\) of the flecnodal and parabolic curve of the surfaces \(S_\varepsilon\) given by the following 1-parameter family of functions:

\[
f_\varepsilon(x, y) = \frac{\varepsilon}{2}(x^2 \pm y^2) + \frac{1}{2}(x^2y + y^3).
\] (7.5)

(A generic family of functions would contain terms of degree \(\geq 4\), which will only break the symmetries - the terms of the form \(\alpha x^3 y\) and \(\beta y^4\) will ban
the local presence of biflecnode as we have seen at the end of §5 - see also Fig. 22)

The parabolic curves of this family of surfaces are the conics of equation

\[ \pm (3y + 2\varepsilon)^2 - 3x^2 = \pm \varepsilon^2, \]

(hyperbolas in the case \( D^+ \)) and ellipses in the case \( D^- \) which undergo respectively a hyperbolic and an elliptic cone section transition.

To describe the flecnodal curves, we use the equations \( a^{I\varepsilon} = 0 \), \( I^{I\varepsilon} = 0 \), that is

\[ (y + \varepsilon) + 2xp \pm (\varepsilon + 3y)p^2 = 0, \quad 3p(1 \pm p^2) = 0. \]

**Case \( D^+_1 \).** Insert the solutions of the equation \( 3p(1 + p^2) = 0 \) (\( p = 0 \) and \( p = \pm i \)) in the equation \( (y + \varepsilon) + 2xp + (\varepsilon + 3y)p^2 = 0 \). We respectively get the line of equation \( y = -\varepsilon \), tangent to the parabolic curve at the godron \((0, -\varepsilon)\), and the complex conjugate lines \( y = \pm ix \) whose real intersection point, the origin, is an ellipnode.

**Case \( D^-_1 \).** Insert the solutions of the equation \( 3p(1 - p^2) = 0 \) in the equation \( (y + \varepsilon) + 2xp + (\varepsilon + 3y)p^2 = 0 \). We get the lines \( y = -\varepsilon \) and \( y = \pm x \), tangent to the parabolic curve at the respective godrons \((0, -\varepsilon)\) and \((\pm \varepsilon/2, -\varepsilon/2)\). These lines intersect at the three hyperbonodes \((0, 0), (\pm \varepsilon, -\varepsilon)\).

Theorem 5.5 follows from these facts because the flecnodal and parabolic curves, and the godrons, hyperbonodes and ellipnodes are stable. The introduction of terms of higher degree (to get a generic 1-parameter family) would deform the flecnodal curve, which consists of three straight lines, into smooth curves, but would not change the configurations.

**Appendix A  Examples of families of BIDEs**

**Example.** Consider the family of implicit differential equations

\[ \{ F_\varepsilon(x, y, p) = y - x^3 - \varepsilon x - p^2 = 0 \}. \]

For a fixed value of \( \varepsilon \), the criminant is provided by the system of equations \( F_\varepsilon = 0 \) and \( (\partial F_\varepsilon / \partial p) = 0 \). So, the discriminant is given by the equation:

\[ y = x^3 + \varepsilon x. \]

By Lemma 6.1 and Inflection-contour Theorem, the curve of inflections of the equation \( F_\varepsilon = 0 \) is the projection along the \( p \)-direction of the intersection of the surfaces given by the equations \( F(x, y, p) = 0 \) and \( I^F(x, y, p) = 0 \):

\[ y = x^3 + \varepsilon x + (3x^2 + \varepsilon)^2. \]

For each \( x \), the coordinate \( y \) of the curve of inflections is not less than the coordinate \( y \) of the discriminant. For \( \varepsilon > 0 \) these curves have no intersection point. For \( \varepsilon < 0 \) they have two points of tangency, i.e. two folded singular points. For \( \varepsilon = 0 \) these two folded singular points collapse into a folded singular point with multiplicity 2.
Example (multiple folded singular point). Consider the implicit differential equation
\[ F(x, y, p) = y - f(x) - p^2 = 0, \]
where \( f : \mathbb{R} \to \mathbb{R} \) is a smooth function. Its discriminant curve and its curve of inflections are defined by the respective equations
\[ y = f(x), \quad \text{and} \quad y = f(x) + (f'(x))^2. \]
If \( f \) is a polynomial of degree \( k + 1 \) then \( f(x) + (f'(x))^2 \) has degree \( 2k \). Moreover, if \( f \) has an \( A_k \) singularity at the origin then \( f(x) + (f'(x))^2 \) has also an \( A_k \) singularity at the origin. In particular, if \( f(x) = x^{k+1} \), the multiplicity of intersection at the origin of the criminant with the curve of inflections is \( 2k \), i.e. we have a folded singular point with multiplicity \( k \).

Example (perestroikas \( \gamma_v \) and \( d \), by P. Pushkar). Take the family of BIDEs
\[ F_\varepsilon(x, y, p) = x^2 + xp + p^2 + \frac{1}{2}(y - 1)^2 - \varepsilon = 0 \quad (\text{with } \varepsilon \in \mathbb{R}). \]
whose surfaces \( V^{F_\varepsilon} \) are ellipsoids for \( \varepsilon > 0 \).
Combining the equations \( F_\varepsilon(x, y, p) = 0 \) and \( I^{F_\varepsilon}(x, y, p) = 0 \), one obtains the curve of inflections of the BIDE \( F_\varepsilon(x, y, p) = 0 \):
\[ 4x^2 + (\frac{1}{2} - \varepsilon)y^2 - y^3 - 2x^2y + x^2y^2 + \frac{1}{2}y^4 = 0. \]
The discriminant of the BIDE \( F_\varepsilon(x, y, p) = 0 \) is the ellipse of equation
\[ \frac{3}{4}x^2 + \frac{1}{2}(y - 1)^2 = \varepsilon. \]
In Fig.[24], the curve of inflections and the discriminant curve are depicted for \( \varepsilon = 1, 1/2, 2/9 \) and 0. At \( \varepsilon = 1/2 \), the curve of inflections undergoes a \( \gamma_v \)-perestroika; at \( \varepsilon = 0 \), we have the transition \( d \) of Fig.[5]

![Fig. 24. Discriminant curves and curve of inflections of P. Pushkar’s Example.](image-url)

Appendix B  The Whitney Umbrella

The standard Whitney umbrella is the germ at the origin of the “surface”, in 3-space, given by the equation \( x^2 = zy^2 \) (Fig.[24]). It intersects the planes \( z = \text{const} > 0 \) in pairs of lines \( x^2 = ay^2 \) and the planes \( y = \text{const} \) in parabolas \( z = bx^2 \). This “surface” contains the \( z \)-axis and has the form of an eccentric
umbrella, whose handle is the negative $z$-axis. A Whitney umbrella is a germ of surface diffeomorphic to the standard Whitney umbrella. For example, the equation $y^2 = zx^2 + \alpha x^4$ determines the so-called elliptic ($\alpha > 0$) or hyperbolic ($\alpha < 0$) Whitney umbrella (Fig. 25).

**Whitney Singularity.** The Whitney singularity of a map $\mathbb{R}^2 \to \mathbb{R}^3$ is the germ at zero of the map which, in suitable coordinates, is given by the formula $(u, v) \mapsto (uv, v, u^2)$. Its image, called cross-cap, is the standard Whitney umbrella without its handle. *This singularity is stable* (see Whitney [33]). The image of the map $(u, v) \mapsto (u, uv, v^2 - \alpha u^2)$ is an elliptic ($\alpha > 0$) or hyperbolic ($\alpha < 0$) cross-cap.

**Remark.** A generic transverse section of the Whitney umbrella, through its pinch-point, is a curve having a semi-cubic cusp (Fig. 15).

**Single Tangent Line.** At the pinch-point of a Whitney umbrella, there is a special tangent line: at each point of the self-intersection curve (the curve of double points) the surface has two transverse tangent planes (one for each local branch). Going along the self-intersection curve towards the pinch point, these planes tend to a single plane, and coincide with that plane at the pinch-point. In this plane there are two distinguished lines, one of them is the tangent line to the self-intersection curve and the other, called the *single tangent line*, is the line tangent to the surface (the $y$-axis in Fig. 25).

**Tangent Planes to the Umbrella.** If a map $\mathbb{R}^2 \to \mathbb{R}^3$ has the Whitney singularity at a point, then the image of its differential (at that point) is the single tangent line to the surface (the $y$-axis in Fig. 25).

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