An identity for vertically aligned entries
in Pascal’s triangle

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Abstract: The classic way to write down Pascal’s triangle leads to entries in alternating rows being vertically aligned. In this paper, we prove a linear relation on vertically aligned entries in Pascal’s triangle. Furthermore, we give an application of this relation to morphisms between hyperelliptic curves.

Keywords: Pascal’s triangle, Binomial coefficients, Hyperelliptic curves.

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1 Introduction

We consider entries in row \( n \) of Pascal’s triangle, where \( n \) is any nonnegative integer. It is well known that the \( i \)-th entry in this row can be computed as the binomial coefficient \( \binom{n}{i} \), where \( 0 \leq i \leq n \).

The entries in alternating rows of Pascal’s triangle are vertically aligned. For example, in Figure 1 below we have circled the entries that are vertically aligned with and above the third entry in Row 11.

In Figure 2 we have circled the entries that are vertically aligned with and above the sixth entry in Row 12. Note that these values are the central binomial coefficients \( \binom{2n}{n} \) and are closely related to the ubiquitous Catalan numbers \( C_n = \frac{1}{n+1} \binom{2n}{n} \) (see, for example, [6, 7]).
We can describe these entries in the following way. Starting with entry $\binom{n}{r}$ in row $n$, the elements that are vertically aligned and above are all of the form

$$\binom{n-2k}{r-k},$$

where $1 \leq k \leq r$ and $k \leq \left\lfloor \frac{n}{2} \right\rfloor$. For example, the elements that are vertically aligned with and above $\binom{11}{3}$ are $\binom{9}{2}, \binom{7}{1}, \binom{5}{0}$.
1.1 Interesting observations

Observe that
\[
\binom{11}{3} - 11 \binom{9}{2} + 44 \binom{7}{1} - 77 \binom{5}{0} = 0.
\]

When \( n = 12 \) and \( i = 6 \), we have
\[
\binom{12}{6} - 12 \binom{10}{5} + 54 \binom{8}{4} - 112 \binom{6}{3} + 105 \binom{4}{2} - 36 \binom{2}{1} + 2 \binom{0}{0} = 0.
\]

The following theorem generalizes these two observations.

2 General formula

**Theorem 2.1.** Let \( n \) be a nonnegative integer and \( 0 < r < n \). Then
\[
\sum_{k=0}^{r} (-1)^k \frac{n}{n-k} \binom{n-k}{k} \binom{n-2k}{r-k} = 0.
\]

**Remark 1.** If \( r > \lceil n/2 \rceil \), as is the case when our elements are to right of the vertical line through the middle of Pascal’s Triangle, there will be some values of \( k \) for which \( n - 2k < r - k \). But recall that \( \binom{m}{i} = 0 \) whenever \( 0 \leq m < i \) (see, for example, [2, Section 1.9]). Thus, terms for which \( 0 \leq n - 2k < r - k \) do not contribute to the sum in Theorem 2.1.

If \( n - 2k < 0 \), then \( \binom{n-2k}{r-k} \) is no longer 0. However in this case, we have \( n - k < k \) and, therefore, \( \binom{n-k}{k} = 0 \) instead. Hence, all terms for which \( r > \lceil n/2 \rceil \) do not contribute to the sum in Theorem 2.1.

**Proof of Theorem 2.1.** The following proof starts with an identity attributed to E.H. Lockwood. For any \( n \geq 1 \),
\[
x^n + y^n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k}. \tag{1}
\]
(see, for example, [2, Section 9.8]).

We separate the \( k = 0 \) term from the summation to get
\[
x^n + y^n = (x+y)^n + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k}. \tag{2}
\]

The Binomial Theorem tells us that
\[
(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r = x^n + y^n + \sum_{r=1}^{n-1} \binom{n}{r} x^{n-r} y^r. \tag{3}
\]

Substituting this expression for \( (x+y)^n \) into equation (2) yields
\[
x^n + y^n = x^n + y^n + \sum_{r=1}^{n-1} \binom{n}{r} x^{n-r} y^r + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k}.
\]
Hence,
\[
\sum_{r=1}^{n-1} \binom{n}{r} x^{n-r} y^r + \sum_{k=1}^{[n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = 0. \tag{4}
\]

Thus, when combining the two sums, the coefficient of each \(x^{n-r} y^r\) term must equal 0. We expand the second summand in order to identify all terms of the form \(x^{n-r} y^r\). The Binomial Theorem tells us that, for each \(k\),
\[
(x + y)^{n-2k} = \sum_{j=0}^{n-2k} \binom{n-2k}{j} x^{n-2k-j} y^j.
\]

Hence,
\[
(xy)^k (x + y)^{n-2k} = \sum_{j=0}^{n-2k} \binom{n-2k}{j} x^{n-2k-j} y^j.
\]  

The values of \(j\) that yield \(x^{n-r} y^r\) terms are \(j = r - k\). Note that we must have \(k \leq r\), since otherwise \(j \leq 0\). Thus, the coefficient of \(x^{n-r} y^r\) in equation (5) is
\[
\frac{1}{r-k} \sum_{k=1}^{r} \binom{n-2k}{n-r-k}.
\]

Hence, the sum of the coefficients of the \(x^{n-r} y^r\) terms in equation (4) is
\[
\sum_{k=0}^{r} (-1)^k \frac{n}{n-k} \binom{n-k}{k} \binom{n-2k}{r-k} = 0,
\]
where the \(k = 0\) term is \(\binom{n}{n-r}\), which comes from the first summation in equation (4).

**Remark 2.** The expressions \(\frac{n}{n-k} \binom{n-k}{k}\) that appear in Theorem 2.1 are referred to as the Triangle of coefficients of Lucas (or Cardan) polynomials, denoted \(T(n, k)\), in the On-Line Encyclopedia of Integer Sequences [5]. We also recall that the \(n\)th Lucas number, \(L_n\), is given by
\[
L_n = \sum_{k=0}^{[n/2]} \frac{n}{n-k} \binom{n-k}{k}
\]
(see, for example, [2]).

### 3 Application to hyperelliptic curves

We now give an application of the identity in Theorem 2.1. Work on this application in [1, Section 5.1] is what led the author to discover the identity in Theorem 2.1.

Let \(K\) be a field with \(\text{char}(K) \neq 2\). A hyperelliptic curve is a compact Riemann surface defined by a nonsingular equation of the form \(y^2 = f(x)\), where \(f(x) \in K[x]\). The degree of the polynomial \(f(x)\) is either \(2g + 2\) or \(2g + 1\), where \(g\) is the genus of the curve. A defining property of hyperelliptic curves is that they have degree two maps with \(2g + 2\) branch points onto the projective line \(\mathbb{P}^1\) (see, for example, [3, Chapter 3], [4, Chapter 2]).
In the section we will start with genus $g$ hyperelliptic curves $C$ of the form $y^2 = x^{2g+1} + x$. The map

$$\phi(x, y) = \left( \frac{x^2 + 1}{x}, \frac{y}{x^a} \right),$$

where $a = \frac{g+1}{2}$, is a nonconstant morphism from $C$ to some curve $C'$. Note that the curve $C'$ will also be hyperelliptic. We initially define $C'$ to be of the form

$$y^2 = c_d x^d + \cdots + c_{d-i} x^{d-i} + \cdots + c_0$$

and we will apply the transformation of variables given by $\phi$ to determine the coefficients $c_j$. Applying the transformation yields

$$\left( \frac{y}{x^a} \right)^2 = c_d \left( \frac{x^2 + 1}{x} \right)^d + \cdots + c_{d-i} \left( \frac{x^2 + 1}{x} \right)^{d-i} + \cdots + c_0$$

Note that the degree of the expression in $x$ will be $g + 1 - d + 2d = g + 1 + d$. In order for $\phi$ to be a morphism from $C$ to $C'$, this last equation should, in fact, be the equation for the curve $C$. Hence, we need $c_d = 1$ and $g + 1 + d = 2g + 1$, which implies $d = g$. Consequently,

$$y^2 = x(x^2 + 1)^g + \cdots + c_g x^{1+g}(x^2 + 1)^g + \cdots + c_0 x^{g+1}. \quad (6)$$

In order to determine the coefficients $c_j$, we need to expand the right-hand side of the equation and match coefficients with those of $C$. We now work through two examples to better understand what the coefficients of $C'$ will be.

**Example 3.1.** Let $g = 5$, so that $C$ is the hyperelliptic curve $y^2 = x^{11} + x$. From our above work we know that the degree of $C'$ will be 5. Letting

$$A_1 = x(x^2 + 1)^5$$

$$= x^{11} + 5x^9 + 10x^7 + 10x^5 + 5x^3 + x,$$

$$A_2 = x^3(x^2 + 1)^3$$

$$= x^9 + 3x^7 + 3x^5 + x^3,$$

$$A_3 = x^5(x^2 + 1)^1$$

$$= x^7 + x^5,$$

we see that $A_1 - 5A_2 + 5A_3 = x^{11} + x$. Hence, $\phi$ is a morphism from $C$ to $y^2 = x^5 - 5x^3 + 5x$.

**Example 3.2.** Now let $g = 6$, so that $C$ is the hyperelliptic curve $y^2 = x^{13} + x$. From our above work we know that the degree of $C'$ will be 6. Letting
\[ B_1 = x(x^2 + 1)^6 \]
\[ = x^{13} + 6x^{11} + 15x^9 + 2 - x^7 + 15x^5 + 6x^3 + x, \]
\[ B_2 = x^3(x^2 + 1)^4 \]
\[ = x^{11} + 4x^9 + 6x^7 + 4x^5 + x^3, \]
\[ B_3 = x^5(x^2 + 1)^2 \]
\[ = x^9 + 2x^7 + x^5, \]
\[ B_4 = x^7(x^2 + 1)^0 \]
\[ = x^7, \]

we see that \( B_1 - 6B_2 + 9B_3 - 2B_4 = x^{13} + x. \) Hence, \( \phi \) is a morphism from \( C \) to the curve \( y^2 = x^6 - 6x^4 + 9x^2 - 2. \)

While working on [1, Section 5.1], the author determined (by hand) the curve \( C' \) for \( g = 11, \) obtaining 1, 11, 44, 77, 55, and 11, with alternating signs (see Table 1 below). The author entered this sequence of numbers into the On-line Encyclopedia of Integer Sequences [5] search bar and found that these numbers are the Triangle of coefficients of Lucas (or Cardan) polynomials, \( T(n, k). \) The coefficients that appear in Examples 3.1 and 3.2 are also of the form \( T(n, k). \) As noted in Remark 2,

\[ T(n, k) = \binom{n}{k} \binom{n - k}{k}. \]

This leads us to the following theorems.

**Theorem 3.3.** Let \( C \) be the hyperelliptic curve \( y^2 = x^{2g+1} + x \) and let \( C' \) be the hyperelliptic curve

\[ y^2 = \sum_{k=0}^{[g/2]} (-1)^k \frac{g}{g-k} \binom{g-k}{k} x^{g-2k}. \]

Then the map

\[ \phi(x, y) = \left( \frac{x^2 + 1}{x}, \frac{y}{x^a} \right), \]

where \( a = \frac{g+1}{2}, \) is a nonconstant morphism from \( C \) to \( C'. \)

We can generalize Theorem 3.3. Let \( c \in \mathbb{Q}^* \) be constant and \( \zeta \) be a primitive \( g \)-th root of unity. In the following theorem we work over the field \( \mathbb{F} = \mathbb{Q}(\zeta, c^{1/g}). \)

**Theorem 3.4.** Let \( C \) be the hyperelliptic curve \( y^2 = x^{2g+1} + cx \) and let \( C_i \) be the hyperelliptic curve

\[ y^2 = \sum_{k=0}^{[g/2]} (-1)^k \frac{g}{g-k} \binom{g-k}{k} \zeta^{ik} c^{k/g} x^{g-2k} \]

for \( i = 0, 1. \) Then the map

\[ \phi_i(x, y) = \left( \frac{x^2 + \zeta^i c^{1/g}}{x}, \frac{y}{x^a} \right), \]

where \( a = \frac{g+1}{2}, \) is a nonconstant morphism from \( C \) to \( C_i. \)
Theorem 3.3 follows from Theorem 3.4 by letting $c = 1$ and $i = 0$. Furthermore, since
\[
\frac{g}{g-k}\left(\frac{g-k}{k}\right) = \left(\frac{g-k}{k}\right) + \left(\frac{g-k-1}{k-1}\right)
\]
(see, for example, [2, Section 9.9]), Theorem 3.4 also generalizes Lemma 5.1 in [1] because we are no longer restricting $g$ to be odd. Though the proof of Theorem 3.4 is nearly identical to the proof of Lemma 5.1 in [1], we include it here for the sake of completion.

Proof of Theorem 3.4. Recall Lockwood’s identity from equation (1)
\[
A^n + B^n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{k} \binom{n-k}{k} (AB)^k (A + B)^{n-2k}.
\]
Letting $n = g$, $A = x^2$, and $B = \zeta^{i/g} c$ yields
\[
x^{2g} + c = \sum_{k=0}^{\lfloor g/2 \rfloor} (-1)^k \frac{g}{g-k}\left(\frac{g-k}{k}\right) \zeta^{ik} c^{k/g} x^{2k} (x^2 + \zeta^{i/g} c)^{g-2k},
\]
since $\zeta^{ig} = 1$. We multiply both sides by $x$ to get
\[
x^{2g+1} + cx = \sum_{k=0}^{\lfloor g/2 \rfloor} (-1)^k \frac{g}{g-k}\left(\frac{g-k}{k}\right) \zeta^{ik} c^{k/g} x^{2k+1} (x^2 + \zeta^{i/g} c)^{g-2k}, \quad (7)
\]
We now demonstrate that $\phi_i$ is indeed a morphism between $C$ and $C_i$. We apply the transformation of variables to $C_i$ to get
\[
\left(\frac{y}{x^n}\right)^2 = \sum_{k=0}^{\lfloor g/2 \rfloor} (-1)^k \frac{g}{g-k}\left(\frac{g-k}{k}\right) \zeta^{ik} c^{k/g} \left(\frac{x^2 + \zeta^{i/g} c}{x}\right)^{g-2k}
\]
\[
y^2 = \sum_{k=0}^{\lfloor g/2 \rfloor} (-1)^k \frac{g}{g-k}\left(\frac{g-k}{k}\right) \zeta^{ik} c^{k/g} x^{2k+1} (x^2 + \zeta^{i/g} c)^{g-2k}
\]
\[
= x^{2g+1} + cx,
\]
where the last equality holds by equation (7). Hence, we have shown that $\phi_i$ is a morphism from $C$ to $C_i$. \qed

Table 1 below gives $C_i$ for values of $g$ up to 11 and for $c = 1$. Note that this table expands on the table that appears in [1, Section 5.1].
3.1 Higher genus observations

The following corollaries to Theorem 3.3 describe patterns for some of the above coefficients.

**Corollary 3.4.1.** For all $g$, the coefficient of $x^{g-2}$ will always be $-g\zeta^i$.

**Proof.** This coefficient corresponds to $k = 1$, which equals

$$(-1)^{g-1} \frac{g}{g-1} \left( \frac{g-1}{1} \right) \zeta^i = -g\zeta^i.$$  

**Corollary 3.4.2.** When $g$ is even, the lowest degree term will always be $(-1)^{g/2} 2\xi_{g/2}$.

**Proof.** Note that when $g$ is even, the lowest degree term corresponds to $k = g/2$, which yields $x^0$. We compute the coefficient to be

$$(-1)^{g/2} \frac{g}{g-g/2} \left( \frac{g-g/2}{g/2} \right) \xi_{g/2} = (-1)^{g/2} 2\xi_{g/2}.$$  

**Corollary 3.4.3.** When $g$ is odd, the lowest degree term will always be $(-1)^{(g-1)/2} g x \xi_{(g-1)/2}$.

**Proof.** When $g$ is odd, the lowest degree term corresponds to $k = (g-1)/2$, which yields $x^1$. We compute the coefficient to be

$$(-1)^{(g-1)/2} \frac{g}{g-(g-1)/2} \left( \frac{g-(g-1)/2}{(g-1)/2} \right) \xi_{(g-1)/2} = (-1)^{(g-1)/2} \frac{g}{g+1/2} \left( \frac{(g-1)/2 + 1}{(g-1)/2} \right) \xi_{(g-1)/2} = (-1)^{(g-1)/2} g \xi_{(g-1)/2}.$$  

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References

[1] Emory, M., Goodson, H., & Peyrot, A. (2018). Towards the Sato–Tate Groups of Trinomial Hyperelliptic Curves. ArXiv e-prints, page arXiv:1812.00242, Dec. 2018.

[2] Koshy, T. (2014). Pell and Pell–Lucas numbers with applications. Springer, New York.

[3] Miranda, R. (1995). Algebraic curves and Riemann surfaces, Volume 5 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI.

[4] Silverman, J. H. (2009). The arithmetic of elliptic curves, Volume 106 of Graduate Texts in Mathematics. Springer, Dordrecht, Second edition.

[5] Sloane, N. J. A. (2018). Sequence A034807. The On-Line Encyclopedia of Integer Sequences. Available online at: https://oeis.org/A034807.

[6] Stanley, R. P. (1999). Enumerative Combinatorics. Volume 2, Cambridge Studies in Advanced Mathematics, Volume 62. Cambridge University Press, Cambridge.

[7] Stanley, R. P. (2012). Enumerative Combinatorics. Volume 1, Cambridge Studies in Advanced Mathematics, Volume 49. Cambridge University Press, Cambridge, Second edition.