ELLiptic Curves with Square-free $\Delta$

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Abstract: Under the Riemann Hypothesis for Dirichlet L-functions, we improve on the error term in a smoothed version of an estimate for the density of elliptic curves with square-free $\Delta = D/16$, where $D$ is the discriminant, by T.D. Browning and the author [1]. To achieve this improvement, we elaborate on our methods for counting weighted solutions of inhomogeneous cubic congruences to power-ful moduli. The novelty lies in going a step further in the explicit evaluation of complete exponential sums and saving a factor by averaging over the moduli.

1. Main result

Let $E$ be an elliptic curve over $\mathbb{Q}$, given in Weierstrass form

$$E = E_{A,B} : \quad y^2 = x^3 + Ax + B$$

for $A, B \in \mathbb{Z}$ with discriminant $-16(4A^3 + 27B^2) = -16\Delta_{A,B}$, say. Throughout the sequel, we shall implicitly assume that $A$ and $B$ are such that $\Delta_{A,B} \neq 0$, which is required for $E_{A,B}$ to be elliptic.

It is a very interesting question whether $\Delta_{A,B}$ is prime infinitely often. This is presently unknown. However, it is possible to estimate the density of elliptic curves with square-free $\Delta_{A,B}$.

The exponential height of $E_{A,B}$ is defined as

$$H(E_{A,B}) = \max \left\{|A|^{1/4}, |B|^{1/6}\right\}.$$  

In [1, Theorem 3], we proved the following result, building on our work about inhomogeneous cubic congruences.

**Theorem 1.1.** For $q \in \mathbb{N}$ let $\sigma(q) := \sharp\{\alpha, \beta \mod q : \Delta_{\alpha,\beta} \equiv 0 \mod q\}$. Let $\mu(n)$ be the Möbius function, with the convention that $\mu(0) = 0$ and $\mu(-n) = \mu(n)$. Then for any $\varepsilon > 0$, we have

$$\sum_{(A,B) \in \mathbb{Z}^2} \mu^2(\Delta_{A,B}) = 4X^{10} \prod_p \left(1 - \frac{\sigma(p^2)}{p^4}\right) + O\left(X^{7+\varepsilon}\right). \quad (1)$$

Here we present a smoothed version of this result with improved error term, where we assume the Riemann Hypothesis for Dirichlet L-functions.

**Theorem 1.2.** Let $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^+$ be a Schwartz class function and $\hat{\Gamma}$ its Fourier transform. Suppose that $\hat{\Gamma}(0) = 1$. Assume that the Riemann Hypothesis holds for all Dirichlet L-functions. Then for any $\varepsilon > 0$, we have

$$\sum_{(A,B) \in \mathbb{Z}^2} \Gamma\left(\frac{A}{X^4}\right) \Gamma\left(\frac{B}{X^6}\right) \mu^2(\Delta_{A,B}) = X^{10} \cdot \frac{1}{3} \cdot \prod_{p>3} \left(1 - \frac{2p-1}{p^3}\right) + O\left(X^{7-5/27+\varepsilon}\right). \quad (2)$$

Thereby, we have also computed the Euler product in Theorem 1.1 explicitly. We note that the factor 4 in (1), which comes from counting positive and non-positive $A$’s and $B$’s, is not present in (2) due to the fact that we work with smooth weights satisfying $\hat{\Gamma}(0) = 1$.

Our method elaborates on that in [1]. Rather than using our bounds for the number of solutions of inhomogeneous cubic congruences obtained in [1] directly, here we redo our treatment of them, specified for congruences of the form $4A^3 + 27B^2 \equiv 0 \mod k^2$, and go a step further in the evaluation of exponential sums. Let us briefly describe how we proceed.

2000 Mathematics Subject Classification. 11L07, 11D45.
First, we detect the square-freeness of $\Delta_{A,B}$ using the M"obius function, leading to congruences of the form $4A^3 + 27B^2 \equiv 0 \mod k^2$. To count their solutions, we apply the Poisson summation formula twice. This leads to complete cubic exponential sums to square moduli which we evaluate explicitly. Roughly, we are left to terms of the form

$$e\left(-\frac{m^2n^3}{k^2}\right),$$

where $m$, $n$ are variables. Now we flip the Kloosterman fraction by means of the identity

$$e\left(-\frac{m^2n^3}{k^2}\right) = e\left(-\frac{n^3}{m^2k^2}\right) e\left(-\frac{m^2}{n^2k^2}\right).$$

The second exponential term on the right-hand side is slowly oscillating if considered as a function of $n$. We sum up over $n$ and use Poisson summation again, but now for modulus $m^2$. This leads to cubic exponential integrals and again to complete cubic exponential sums. So far, our method agrees with that in \[1\]. The novelty comes in the next steps. We evaluate the said complete cubic exponential sums for modulus $m^2$ explicitly, in contrast to our work in \[1\], where we just bounded them. Moreover, we work with an asymptotic estimate for the said cubic exponential integrals instead of upper bounds. Then we average over the moduli, which essentially leads to linear exponential sums with M"obius function of the form

$$\sum_{k \leq x} \mu(k) \cdot e(\omega k),$$

where $\omega$ is a real number. It is this averaging which gives us an extra saving.

In our work, we shall follow the usual convention that $\varepsilon$ can change from line to line if no confusions are anticipated.

**Acknowledgement.** The author wishes to thank the Tata Institute of Fundamental Research in Mumbai (India) for its warm hospitality, excellent working conditions and financial support by an ISF-UGC grant.

2. **Reduction to inhomogeneous cubic congruences**

We start as in \[1\]. Using

$$\mu^2(n) = \sum_{k \mid n} \mu(k),$$

we write

$$\sum_{(A,B) \in \mathbb{Z}^2} \Gamma\left(\frac{A}{X^4}\right) \Gamma\left(\frac{B}{X^6}\right) \mu^2(\Delta_{A,B}) = \sum_{k=1}^{\infty} \mu(k) S(X;k^2) =: S(X),$$

where

$$S(X;k^2) := \sum_{4A^3 + 27B^2 \equiv 0 \mod k^2} \Gamma\left(\frac{A}{X^4}\right) \Gamma\left(\frac{B}{X^6}\right).$$

We split the sum over $k$ into two parts, according to the size of $k$. Let $1 \leq \xi \leq X^{100}$ be a parameter, to be fixed later. We define

$$S(X) = S_1(X) + S_2(X),$$

where

$$S_1(X) := \sum_{k \leq \xi} \mu(k) S(X;k^2)$$

and

$$S_2(X) := \sum_{k > \xi} \mu(k) S(X;k^2).$$
3. Estimation of $S_2(X)$

Since $\Gamma$ has rapid decay, we have

$$S_2(X) \ll \sum_{k > \xi} \sum_{|A| \leq X^{1+\varepsilon}, |B| \leq X^{6+\varepsilon}} 1 + O(1).$$

It follows that

$$S_2(X) \ll \sum_{|A| \leq X^{4+\varepsilon}} \sum_{0 < |m| \leq X^{12+\varepsilon} - 2} \sum_{|B| \leq X^{6+\varepsilon}} \sum_{k \in \mathbb{N}} \frac{1}{4A^2 + 2AB^2 + 1}.$$

By a classical argument of Estermann [3], we have

$$\sum_{|B| \leq X^{6+\varepsilon}} \sum_{k \in \mathbb{N}} 1 \ll O \left( (Am)^{\varepsilon} \right)$$

and hence

$$S_2(X) \ll X^{16+\varepsilon} \xi^{-2}.$$

4. Application of Poisson summation I

We now transform the term $S_1(X)$. First, we rewrite $S_1(X; k^2)$, where we assume that $k$ is square-free. As in [1], we write $k = k_2 k_3 k'$, where $k_2 = (k, 2)$ and $k_3 = (k, 3)$ and $k'$ is coprime to 6. It readily follows that $k_2 | B$ and $k_3 | A$ in the summand. Making the change of variables $A = k_3 A'$ and $B = k_2 B'$ we deduce that

$$S_1(X) = \sum_{k_2 | 2 k_3} \mu(k_2) \mu(k_3) \sum_{k' \leq \xi/(k_2 k_3)} \mu(k') \sum_{(A', B') \in \mathbb{Z}^2} \Gamma \left( \frac{k_3 A'}{X^4} \right) \Gamma \left( \frac{k_2 B'}{X_6} \right),$$

where $a = 3/k_3$ and $b = 2/k_2$. In particular it follows that $(ab, k') = 1$ and $a, b \leq 3$. We will need to account for possible common factors of $A'B'$ and $k'$. Drawing out the greatest common divisor of $B'$ and $k'$ we write $B' = hx$ and $k' = hl$, with $(x, l) = 1$. It easily follows from the square-freeness of $k$ that $h | A'$ and we can write $A' = hy$ with $(h, x, l) = 1$. Hence,

$$S_1(X) = \sum_{k_2 | 2 k_3} \sum_{3 \leq h \leq \xi/(k_2 k_3)} \mu(k_2) \mu(k_3) \mu(h) \sum_{l \leq \xi/(h k_2 k_3)} \mu(l) \times$$

$$\sum_{(x, y) \in \mathbb{Z}^2} \Gamma \left( \frac{k_3 h x}{X^6} \right) \Gamma \left( \frac{k_2 h y}{X^2} \right),$$

with $(ab, l) = 1$. Now applying the Poisson summation formula to both the sums over $x$ and $y$, we deduce that

$$S_1(X) = X^{10} \sum_{k_2 | 2 k_3} \sum_{3 \leq h \leq \xi/(k_2 k_3)} \mu(k_2) \mu(k_3) \mu(h) \sum_{l \leq \xi/(k_2 k_3 h)} \frac{\mu(l)}{l^2} \times$$

$$\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \hat{\varepsilon} \left( \frac{X^6 m}{k_2 h l^2} \right) \hat{\varepsilon} \left( \frac{X^4 n}{k_3 h l^2} \right) \mathcal{E}(m, n; l^2),$$

where

$$\mathcal{E}(m, n; l^2) = \sum_{c, d \equiv 0 \mod l^2} e \left( \frac{cm + dn}{l^2} \right).$$
As seen in [1, section 5], we can write the above complete exponential sum in two variables, \( E(m, n; l^2) \), as a complete exponential sum over a single variable, namely
\[
E(m, n; l^2) = E(a^3b^4h^2m, a^3b^2hn; l^2),
\]
where we define
\[
E(c, d; q) := \sum_{x=1}^{q} e \left( \frac{cx^3 - dx^2}{q} \right).
\]

5. Explicit evaluation of exponential sums I

Recall that \((abh, l) = 1 = (l, 6)\) and \(l\) is square-free. The exponential sum in the last section can be evaluated explicitly as follows.
\[
E(a^3b^4h^2m, a^3b^2hn; l^2)
\]
\[
= \sum_{(x, l)=1}^{l} \sum_{y=1}^{l} e \left( \frac{a^3b^4h^2m(x + ly)^3 - a^3b^2hn(x + ly)^2}{l^2} \right)
\]
\[
= \sum_{x=1}^{l} e \left( \frac{a^3b^4h^2mx^3 - a^3b^2hnx^2}{l^2} \right) \sum_{y=1}^{l} e \left( \frac{3a^3b^4h^2mx^2 - 2a^3b^2hnx}{l^2} \right)
\]
\[
= l \sum_{x=1}^{l} \sum_{(x, l)=1} \sum_{y=1}^{l} e \left( \frac{a^3b^4h^2mx^3 - a^3b^2hnx^2}{l^2} \right)
\]
\[
= l \sum_{x=1}^{l} \sum_{(x, l)=1} \sum_{y=1}^{l} e \left( \frac{a^3b^4h^2mx^3 - a^3b^2hnx^2}{l^2} \right).
\]

Let \(d = (m, l)\), \(m_1 = m/d\) and \(l_1 = l/d\). We note that \((m_1, l_1) = 1 = (l_1, d)\) by square-freeness of \(l\). The linear congruence in the last line of (10) is solvable if and only if \(d|n\), in which case we set \(n_1 = n/d\). Now we use the multiplicity of the exponential sums \(E(c, d; q)\). By [1, Lemma 6], we have
\[
E(a^3b^4h^2m, a^3b^2hn; l^2) = E(a^3b^4h^2m, a^3b^2hn; l^2)E(a^3b^4h^2m, a^3b^2hn; l^2),
\]
where \(\overline{t_1}\) is a multiplicative inverse of \(l_1\) modulo \(d\), and \(\overline{d}\) is a multiplicative inverse of \(d\) modulo \(l^2\). We further deduce that
\[
E(a^3b^4h^2m, a^3b^2hn; l^2) = dE(a^3b^4h^2m_1\overline{t_1}, a^3b^2hn_1\overline{t_1}; d)E(a^3b^4h^2m_1\overline{d}, a^3b^2hn_1\overline{d}; l^2_1).
\]
If \(m_1 = 0\), then it follows that \(m = 0, d = l\) and \(l_1 = 1\) and hence
\[
E(a^3b^4h^2m, a^3b^2hn; l^2) = dE(0, a^3b^2hn; d) = d\varphi(e)E(0, a^3b^2hn; d_2),
\]
where \(e = (n_1, d), n_2 = n_1/e, d_2 = d/e\), and \(\varphi(e)\) is the Euler totient function. The last exponential sum is a quadratic Gauss sum with coprimality constraint to square-free modulus \(d_2\). Its precise value is
\[
E(0, a^3b^2hn_2; d_2) = f(a^3b^2hn_2; d_2),
\]
where
\[
f(c; q) := \prod_{p|q} \left( \epsilon_p \left( \frac{-cq/p}{p} \right) \sqrt{p} - 1 \right)
\]
with \(\left( \frac{a}{p} \right)\) being the Legendre symbol and
\[
\epsilon_p := \begin{cases} 
  1 & \text{if } p \equiv 1 \mod 4, \\
  i & \text{if } p \equiv -1 \mod 4.
\end{cases}
\]
Altogether, if \( m_1 = 0 \), then
\[
E(a^3b^4h^2m_1a^3b^2hn_1; l_1^2) = d_1(l_1) \cdot f(a^3b^2hn_2; d_2). 
\]

Let us now assume that \( m_1 \neq 0 \). We leave the first exponential sum on the right-hand side of (11) as it is and transform the second one as in (10), leading to
\[
E(a^3b^4h^2m_1a^3b^2hn_1; l_1^2) = l_1 \sum_{x \equiv X \mod l_1} e \left( \frac{a^3b^4h^2m_1a^3b^2hn_1 - a^3b^2hn_1}{l_1} \right).
\]

Using Hensel's Lemma, we can uniquely lift the solution \( x \) to the congruence modulo \( l_1 \) in the last line to a solution of the same congruence modulo \( l_1^2 \). Then plugging this \( x \) into the exponential term, and recalling that \( a = 3/k_3 \) and \( b = 2/k_2 \), we obtain
\[
E(a^3b^4h^2m_1a^3b^2hn_1; l_1^2) = l_1 \cdot e \left( \frac{-k_2^2n_1^2}{k_3dhm_1^2l_1^2} \right) \cdot e \left( \frac{1}{l_1} \cdot \frac{k_2^2n_1^2}{k_3dhm_1^2l_1^2} \right).
\]

Flipping the Kloosterman fractions now gives
\[
E(a^3b^4h^2m_1a^3b^2hn_1; l_1^2) = l_1 \cdot e \left( \frac{-k_2^2n_1^2}{k_3dhm_1^2l_1^2} \right) \cdot e \left( \frac{1}{l_1} \cdot \frac{k_2^2n_1^2}{k_3dhm_1^2l_1^2} \right).
\]

Combining this with (11), we obtain
\[
E(a^3b^4h^2m_1a^3b^2hn_1; l_1^2) = l_1 \cdot e \left( \frac{-k_2^2n_1^2}{k_3dhm_1^2l_1^2} \right) \cdot e \left( \frac{1}{l_1} \cdot \frac{k_2^2n_1^2}{k_3dhm_1^2l_1^2} \right). 
\]

(14)

Plugging (13) and (14) into (8), we get
\[
S_1(X) = M(X) + E(X),
\]

where
\[
M(X) = X \sum_{k_2|2} \sum_{l_1 \leq k_2} \sum_{\substack{x,l_1 \leq x/l_2 \leq k_2 \\ (x,l_1) = 1}} \frac{\mu(k_2)}{k_2} \frac{\mu(k_3)}{k_3} \frac{\mu(h)}{k_2} \sum_{n_2 \in \mathbb{Z}} e^{\frac{2\pi i}{(k_2k_3h^2l_1^2)} (X^4n_2)} \cdot f(a^3b^2hn_2; d_2),
\]

(16)

and
\[
E(X) = X \sum_{k_2|2} \sum_{l_1 \leq k_2} \sum_{\substack{x,l_1 \leq x/l_2 \leq k_2 \\ (x,l_1) = 1}} \frac{\mu(k_2)}{k_2} \frac{\mu(k_3)}{k_3} \frac{\mu(h)}{k_2} \frac{\mu(d)}{d^3} \sum_{n_1 \in \mathbb{Z}} e^{\frac{2\pi i}{(k_2k_3h^2l_1^2)} (X^4n_1l_1)} \cdot \hat{f} \left( \frac{X^4n_1}{k_3hd_2l_1^2} \right) \cdot e \left( \frac{-k_2^2n_1^2}{k_3dhm_1^2l_1^2} \right)
\]

(17)

The term \( M(X) \) will be the main term, and \( E(X) \) an error term whose treatment will be the key part of this paper and carried out from section 9 onwards. First we shall deal with \( M(X) \).
6. Evaluation of the Main Term

We split $M(X)$ into two parts $M_0(X)$ and $E_0(X)$, $M_0(X)$ being the contribution of $n_2 = 0$, and $E_0(X)$ being the contribution of $n_2 \neq 0$. Hence,

$$M(X) = M_0(X) + E_0(X),$$

where

$$M_0(X) = X^{10} \sum_{k_2 | (2k_3 | 3)} \sum_{h \leq \xi/(k_2k_3)} \sum_{(h,6)=1} \sum_{e \leq \xi/(k_2k_3h)} \mu(k_2) \frac{\mu(k_3)}{k_3} \frac{\mu(h)}{h^2} \frac{\mu(e) \varphi(e)}{e^3}$$

and

$$E_0(X) = X^{10} \sum_{k_2 | (2k_3 | 3)} \sum_{h \leq \xi/(k_2k_3)} \sum_{(h,6)=1} \sum_{e \leq \xi/(k_2k_3h)} \sum_{(d_2,6h)=1} \sum_{(e,6hd_2)=1} \frac{\mu(k_2)}{k_2} \frac{\mu(k_3)}{k_3} \frac{\mu(h)}{h^2} \frac{\mu(e) \varphi(e)}{e^3} \cdot \sum_{n_2 \leq 2} \sum_{n_2 \neq 0} \frac{\mu(d_2)}{d_2^2} \frac{\mu(e) \varphi(e)}{e^3} \cdot \sum_{n_2 \leq 2} \sum_{n_2 \neq 0} f \left( \frac{X^4 n_2}{k_3 \mu d_2} \right) f \left( a^3 h^2 h n_2; d_2 \right).$$

The term $M_0(X)$ will yield the main contribution and can be easily evaluated as follows. Completing the inner-most sum over $e$ on the right-hand of (19) to a series, and writing this series as an Euler product, we obtain

$$M_0(X) = X^{10} \sum_{k_2 | (2k_3 | 3)} \sum_{h \leq \xi/(k_2k_3)} \sum_{(h,6)=1} \sum_{e \leq \xi/(k_2k_3h)} \mu(k_2) \frac{\mu(k_3)}{k_3} \frac{\mu(h)}{h^2} \left( \sum_{e=1}^{\infty} \frac{\mu(e) \varphi(e)}{e^3} + O \left( \frac{h}{\xi} \right) \right) \times \prod_{p \mid h} \left( 1 - \frac{p - 1}{p^3} \right)^{-1} \times \prod_{p > 3} \left( 1 - \frac{p - 1}{p^3} \right) + O \left( X^{10} \frac{X}{\xi} \right).$$

Now completing the sum over $h$ to a series, and writing this series as an Euler product, we further deduce that

$$M_0(X) = X^{10} \sum_{k_2 | (2k_3 | 3)} \sum_{h \leq \xi/(k_2k_3)} \sum_{(h,6)=1} \sum_{e \leq \xi/(k_2k_3h)} \mu(k_2) \frac{\mu(k_3)}{k_3} \frac{\mu(h)}{h^2} \left( \sum_{e=1}^{\infty} \frac{\mu(h)}{h^2} \prod_{p \mid h} \left( 1 - \frac{p - 1}{p^3} \right)^{-1} + O \left( \frac{1}{\xi} \right) \right) \times \prod_{p > 3} \left( 1 - \frac{p - 1}{p^3} \right) + O \left( X^{10} \frac{X}{\xi} \right) \times \prod_{p > 3} \left( 1 - \frac{p - 1}{p^3} \right) \prod_{p > 3} \left( 1 - \frac{p - 1}{p^3 - p + 1} \right) \prod_{p > 3} \left( 1 - \frac{p - 1}{p^3} \right) + O \left( X^{10} \frac{X}{\xi} \right)$$

$$= X^{10} \sum_{k_2 | (2k_3 | 3)} \sum_{k_2 | (2k_3 | 3)} \sum_{k_3 \mid (2k_3 | 3)} \sum_{p > 3} \left( 1 - \frac{2p - 1}{p^3} \right) + O \left( X^{10} \frac{X}{\xi} \right).$$
Lemma 8.1. We use the fact that \( \hat{\Gamma} \) has rapid decay. The inner-most double sum in (23) is easily estimated.

Now a short calculation yields

\[
E^+(X) \ll \xi^{1/2} X^{1/2+\varepsilon} + \xi^{2-\kappa} X^\varepsilon.
\]
We are left with estimating
\[
E^-(X) = \sum_{k_2|2k_3|3} \sum_{h \leq \xi/k_2k_3} \sum_{(h,d)=1} \sum_{d \leq \xi/k_2k_3} \mu(k_2) \mu(k_3) \frac{\mu(h)}{k_2} \frac{\mu(d)}{d^3} \times
\sum_{l_1 \leq \xi/(k_2kd)} \frac{\mu(l_1)}{l_1} \cdot \sum_{m_1 \in \mathbb{Z} \setminus \{0\}} \frac{\hat{\Gamma} \left( \frac{X^6m_1}{k_3hdl_1^2} \right)}{m_1} \frac{\hat{\Gamma} \left( \frac{X^4n_1}{k_3hdn_1^2} \right)}{n_1} \sum_{\nu_1 \in \mathbb{Z}} \left( \frac{X^{12}m_1}{k_3hdn_1^2l_1^4} \right) \times
\]
\[
e^{- \frac{k_2^2n_1^3}{k_3^2hdm_1^2l_1^4}} \cdot e^{- \frac{k_2^2z^3}{k_3^2hdm_1^2l_1^4}}
\]
will be interpreted as a slowly oscillating weight function of \(n_1\). We therefore define \(\Psi : \mathbb{R} \to \mathbb{C}\) as
\[
\Psi \left( \frac{X^4z}{k_3hdl_1^2} \right) := \hat{\Gamma} \left( \frac{X^4z}{k_3hdl_1^2} \right) \cdot e^{- \frac{k_2^2z^3}{k_3^2hdm_1^2l_1^4}}
\]
i.e.
\[
\Psi(z) := \hat{\Gamma}(z) e^{- \frac{k_3^2h^2d^2l_1^4}{X^{12}m_1^2} \cdot z^3}
\]
Now breaking up the summation over \(n_1\) into residue classes modulo \(k_3^2hdm_1^2\) and using the Poisson summation formula again, we get
\[
\sum_{n_1 \in \mathbb{Z}} \Psi \left( \frac{X^4n_1}{k_3hdl_1^2} \right) E(a^3b^4h^2m_1d1^2, a^3b^2hn_1l_1^2; d) \cdot e^{- \frac{k_2^2u^3}{k_3^2hdm_1^2}}
= \frac{l_1^2}{X^{4k_3^2m_1^2}} \sum_{u \in \mathbb{Z}} \Psi \left( \frac{l_1^2 u}{X^{4k_3^2m_1^2}} \right) \frac{k_3^2hdm_1^2}{\sum_{v=1}^{k_3^2hdm_1^2} E(a^3b^4h^2m_1d1^2, a^3b^2hn_1l_1^2; d) \cdot e^{- \frac{k_2^2v^3+uv}{k_3^2hdm_1^2}}}
\]
We write \(m_1 = m^*\tilde{m}\), where \(\text{rad}(m^*)\divides (6hd)\) and \((\tilde{m}, 6hd) = 1\), where \(\text{rad}(n)\) is the largest square-free divisor of the natural number \(n\). Further, we write \(q := k_3^2hd(m^*)^2\) and note that \((\tilde{m}, q) = 1\). Hence we can write the inner-most sum on the right-hand side of (27) as
\[
\sum_{v=1}^{k_3^2hdm_1^2} E(a^3b^4h^2m_1d1^2, a^3b^2hn_1l_1^2; d) \cdot e^{- \frac{k_2^2v^3+uv}{k_3^2hdm_1^2}}
= \sum_{x=1}^{q} \sum_{y=1}^{\tilde{m}^2} E(a^3b^4h^2m^*\tilde{m}l_1^2, a^3b^2h(xm^2+yg)l_1^2; d) \cdot e^{- \frac{k_2^2(xm^2+gy)l_1^2+uv}{q\tilde{m}^2}}
\]
\[
= \sum_{x=1}^{q} E(a^3b^4h^2m^*\tilde{m}l_1^2, a^3b^2hx\tilde{m}l_1^2; d) \cdot e^{- \frac{k_2^2x^3+ux}{q}} \sum_{y=1}^{\tilde{m}^2} e^{- \frac{k_2^2y^3+uy}{\tilde{m}^2}}
\]
Furthermore, we write \( q = \tilde{q}d \) and

\[
\begin{aligned}
&= \sum_{x=1}^{\tilde{q}} E(a^3b^4h^2m^a \tilde{m}^2, a^3b^2hx \tilde{m}^2; d) \cdot e \left( \frac{t_1^2k_2^2\tilde{m}^4x^3 + u}{q} \right) \\
&= \sum_{y=1}^{\tilde{q}} \sum_{z=1}^{\tilde{q}} E(a^3b^4h^2m^a \tilde{m}^2, a^3b^2h(dy + z) \tilde{m}^2; d) \cdot e \left( \frac{t_1^2k_2^2\tilde{m}^4(dy + z)^3 + u(dy + z)}{q} \right) \\
&= \sum_{z=1}^{\tilde{q}} e \left( \frac{t_1^2k_2^2\tilde{m}^4((dy + z)^3 - z^3) + udy}{\tilde{q}} \right) \\
&= \sum_{z=1}^{\tilde{q}} e \left( \frac{t_1^2k_2^2\tilde{m}^4(d^2y^3 + 3dy^2z + 3yz^2) + uy}{\tilde{q}} \right)
\end{aligned}
\]

(29)

Combining (25), (27), (28) and (29), we obtain

\[
E^*(X) = X^6 \sum_{k_2|k_3| |h| \leq \xi/(k_2k_3h)} \sum_{d \leq \xi/(k_2k_3h)} \frac{\mu(k_2)}{k_2} \cdot \frac{\mu(k_3)}{k_3} \cdot \frac{\mu(h)}{h^2} \cdot \frac{\mu(d)}{d^2} \cdot \sum_{l_1 \leq \xi/(k_2k_3hd)} \frac{\mu(l_1)}{l_1} \times
\]

(30)

\[
\sum_{m \in \mathbb{Z} \setminus \{0\}} \sum_{\tilde{m} \in \mathbb{Z} \setminus \{0\}} \frac{1}{\tilde{m}} \frac{1}{\tilde{m}} \cdot \sum_{u \in \mathbb{Z}} \psi \left( \frac{l_1^2u}{X^4k_2^2|m^a\tilde{m}|^2} \right) \\
F \left( t_1^2k_2^2\tilde{m}^2q^2, 0; \tilde{m}^2 \right) \sum_{d=1}^{\tilde{q}} E(a^3b^4h^2m^a \tilde{m}^2, a^3b^2h \tilde{m}^2; d) \cdot e \left( \frac{t_1^2k_2^2\tilde{m}^4z^3 + uz}{\tilde{q}} \right) \\
F \left( t_1^2k_2^2\tilde{m}^4d^2, 3t_1^2k_2^2\tilde{m}^4dz, 3t_1^2k_2^2\tilde{m}^4z^2 + u; \tilde{q} \right),
\]

where

\[
F(c_3, c_2, c_1; r) := \sum_{x=1}^{\tilde{q}} e \left( \frac{c_3x^3 + c_2x^2 + c_1x}{r} \right).
\]

(31)

10. ESTIMATION OF EXPONENTIAL SUMS

We shall need bounds for the two exponential sums appearing in the inner-most sum in (30).

By Lemma 8.1, we have

\[
E(a^3b^4h^2m^a \tilde{m}^2, a^3b^2h \tilde{m}^2; d) \ll d^{1/2 + \varepsilon} \left( a^3b^2h \tilde{m}^2t_1^2, d \right)^{1/2} = d^{1/2 + \varepsilon(z, d)^{1/2}}.
\]

(32)

To bound the second exponential sum, we recall [1] Lemma 5 which is due to Loxton and Schmidt [2].

**Lemma 10.1.** Let \( Q \in \mathbb{N} \) and \( f \in \mathbb{Z}[X] \). Suppose that \( f' \) has degree \( n \), precisely \( m \) distinct roots and factorization

\[
f'(X) = A(X - \zeta_1)^{n_1}(X - \zeta_2)^{n_2} \cdots (X - \zeta_m)^{n_m}.
\]

Define the semi-discriminant of \( f' \) to be

\[
\Delta = \Delta(f') := A^{2n - 2} \prod_{i \neq j} (\zeta_i - \zeta_j)^{n_i n_j}.
\]
and the exponent of $f'$ to be
\[
\eta = \eta(f') := \max\{\eta_1, ..., \eta_m\}.
\]
Then
\[
\sum_{x=1}^{Q} e \left( \frac{f(x)}{Q} \right) \leq Q^{-1/(2\eta)}(\Delta, Q)^{1/(2\eta)} \omega(Q),
\]
where $\omega(Q)$ is the number of distinct prime factors of $Q$.

We further need the following simple observation.

**Lemma 10.2.** Let $Q \in \mathbb{N}$ and $f(X) = c_nX^n + c_{n-1}X^{n-1} + ... + c_1X \in \mathbb{Z}[X]$. Suppose that $\delta|(c_n, ..., c_2, Q)$. Then
\[
\sum_{x=1}^{Q} e \left( \frac{f(x)}{Q} \right) \neq 0 \implies \delta|c_1 \text{ and } \sum_{x=1}^{Q} e \left( \frac{f(x)}{Q} \right) = \delta \sum_{x=1}^{\tilde{Q}} e \left( \frac{\tilde{f}(X)}{Q} \right),
\]
where $\tilde{f}(X) = f(X)/\delta \in \mathbb{Z}[X]$ and $\tilde{Q} = Q/\delta \in \mathbb{N}$.

**Proof.** By the conditions in Lemma 10.2 we have
\[
\sum_{x=1}^{Q} e \left( \frac{f(x)}{Q} \right) = \sum_{y=1}^{\tilde{Q}} \sum_{z=1}^{\delta} e \left( \frac{f(z\tilde{Q} + y) - c_1(z\tilde{Q} + y)}{Q} \right) e \left( \frac{c_1(z\tilde{Q} + y)}{\tilde{Q}} \right)
\]
\[
= \sum_{y=1}^{\tilde{Q}} e \left( \frac{f(y)}{\tilde{Q}} \right) \sum_{z=1}^{\delta} e \left( \frac{c_1z}{\delta} \right)
\]
\[
= \sum_{y=1}^{\tilde{Q}} e \left( \frac{\tilde{f}(y)}{\tilde{Q}} \right) \cdot \left\{ \begin{array}{ll}
\delta & \text{if } \delta|c_1 \\
0 & \text{if } \delta \nmid c_1.
\end{array} \right.
\]

Now we are ready to estimate $F \left( \overline{T}_1^2 k_2^2 \tilde{m}^4 d^2, 3 \overline{T}_1^2 k_2^2 \tilde{m}^4 dz, 3 \overline{T}_1^2 k_2^2 \tilde{m}^4 z^2 + u; \tilde{q} \right)$. We set $\delta := (d, |m^*|)$. Then from Lemma 10.2 we deduce that
\[
F \left( \overline{T}_1^2 k_2^2 \tilde{m}^4 d^2, 3 \overline{T}_1^2 k_2^2 \tilde{m}^4 dz, 3 \overline{T}_1^2 k_2^2 \tilde{m}^4 z^2 + u; \tilde{q} \right) \neq 0 \implies \delta| \left( \overline{T}_1^2 k_2^2 \tilde{m}^4 z^2 + u \right) \text{ and }
\]
\[
F \left( \overline{T}_1^2 k_2^2 \tilde{m}^4 d^2, 3 \overline{T}_1^2 k_2^2 \tilde{m}^4 dz, 3 \overline{T}_1^2 k_2^2 \tilde{m}^4 z^2 + u; \tilde{q} \right)
= \delta F \left( \overline{T}_1^2 k_2^2 \tilde{m}^4 d^2 \delta^{-1}, 3 \overline{T}_1^2 k_2^2 \tilde{m}^4 dz \delta^{-1}, (3 \overline{T}_1^2 k_2^2 \tilde{m}^4 z^2 + u) \delta^{-1}; \tilde{q} \delta^{-1} \right). \tag{33}
\]

We assume that this is the case. Then we compute that the derivative of the polynomial
\[
f(X) := \overline{T}_1^2 k_2^2 \tilde{m}^4 d^2 \delta^{-1}X^3 + 3 \overline{T}_1^2 k_2^2 \tilde{m}^4 d \delta^{-1}X^2 + \left( 3 \overline{T}_1^2 k_2^2 \tilde{m}^4 z^2 + u \right) \delta^{-1} \in \mathbb{Z}[X]
\]
has exponent
\[
\eta(f') = \begin{cases} 
2 & \text{if } u = 0 \\
1 & \text{if } u \neq 0
\end{cases}
\]
and semi-discriminant
\[
\Delta(f') = \begin{cases} 
\left( \overline{T}_1^2 k_2^2 \tilde{m}^4 d^2 \delta^{-1} \right)^2 & \text{if } u = 0 \\
12 \overline{T}_1^2 k_2^2 \tilde{m}^4 d^2 \delta^{-2} u & \text{if } u \neq 0.
\end{cases}
\]

Now from 33 and Lemma 10.1 it follows that
\[
F \left( \overline{T}_1^2 k_2^2 \tilde{m}^4 d^2, 3 \overline{T}_1^2 k_2^2 \tilde{m}^4 dz, 3 \overline{T}_1^2 k_2^2 \tilde{m}^4 z^2 + u; \tilde{q} \right) \ll \delta^{1/4} \tilde{q}^{3/4 + \varepsilon} \left( \left( \overline{T}_1^2 k_2^2 \tilde{m}^4 d^2 \delta^{-1} \right)^2, \tilde{q} \delta^{-1} \right)^{1/4}
\]
\[
\ll \delta^{3/4} \tilde{q}^{3/4 + \varepsilon} = (d, |m^*|)^{3/4} \tilde{q}^{3/4} \text{ if } u = 0 \tag{34}
\]
and

\[
F \left( \frac{1}{k^2 \tilde{m}^4 d^2}, 3 \frac{1}{k^2 \tilde{m}^4 dz_1}, 3 \frac{1}{k^2 \tilde{m}^4 z^2 + u; \tilde{q}} \right)
\leq \delta^{1/2} \tilde{q}^{1/2 + \varepsilon} \left( 12 \frac{1}{k^2 \tilde{m}^4 (d\tilde{\delta})^2 u, \tilde{q}^{-1} \right)^{1/2}
\]

where we note that \((d\tilde{\delta})^2\) and \(\tilde{q}\tilde{\delta}^{-1}\) are coprime. Putting (32) and (35) together and summing over \(z\), we see that the inner-most sum in (30) is bounded by

\[
\sum_{z=1}^{d} E(a^2 h^2 \tilde{m}^4 d^2, a^2 h \tilde{z} \tilde{m}^2 \tilde{d}; d) \cdot e \left( \frac{1}{k^2 \tilde{m}^4 z^3 + u \tilde{z}} \right) \times F \left( \frac{1}{k^2 \tilde{m}^4 d^2}, 3 \frac{1}{k^2 \tilde{m}^4 d, 3 \frac{1}{k^2 \tilde{m}^4 z^2 + u; \tilde{q}} \right)
\leq \delta^{1/2 + \varepsilon} \tilde{q}^{1/2 + \varepsilon} (d, \tilde{m}^* \frac{1}{2})^{1/2} \left( u, hd|m^*|^2 \right)^{1/2} \text{ if } u \neq 0.
\]

Similarly, we deduce

\[
\sum_{z=1}^{d} E(a^2 h^2 \tilde{m}^4 d^2, a^2 h \tilde{z} \tilde{m}^2 \tilde{d}; d) \cdot e \left( \frac{1}{k^2 \tilde{m}^4 z^3 + u \tilde{z}} \right) \times F \left( \frac{1}{k^2 \tilde{m}^4 d^2}, 3 \frac{1}{k^2 \tilde{m}^4 d, 3 \frac{1}{k^2 \tilde{m}^4 z^2 + u; \tilde{q}} \right)
\leq \delta^{1/2 + \varepsilon} \tilde{q}^{3/4 + \varepsilon} (d, \tilde{m}^* \frac{3}{2})^{1/2} \left( u, hd|m^*|^2 \right)^{1/2} \text{ if } u = 0.
\]
11. Explicit evaluation of exponential sums II

Now we turn to the key point of this paper, an explicit evaluation of the cubic exponential sum $F_1(k_2^2q^2, 0; u, \tilde{m}^2)$ appearing in (26), followed by an averaging over $l_1$. We write

$$F_1(k_2^2q^2, 0; u, \tilde{m}^2) = \sum_{x=1}^{\tilde{m}^2} e \left( \frac{l_1 k_2^2 q^2 (y \tilde{m} + x)^3 + u(y \tilde{m} + x)}{\tilde{m}^2} \right)$$

$$= \sum_{x=1}^{\tilde{m}^2} e \left( \frac{l_1 k_2^2 q^2 x^3 + ux}{\tilde{m}^2} \right) \sum_{y=1}^{\tilde{m}} e \left( \frac{3l_1 k_2^2 q^2 y^2}{\tilde{m}} + uy \right)$$

$$= \tilde{m} \sum_{x=1}^{\tilde{m}^2} e \left( \frac{l_1 k_2^2 q^2 x^3 + ux}{\tilde{m}^2} \right)$$

where the multiplicative inverses are modulo $\tilde{m}^2$.

12. Asymptotic estimation of exponential integrals

We also need to estimate asymptotically the Fourier transform $\hat{\Psi}(z)$ of the function $\Psi(z)$ defined in (30). We have

$$\hat{\Psi}(\alpha) = I(\alpha, \beta) := \int_{-\infty}^{\infty} \Psi(z) e(-\alpha z) dz = \int_{-\infty}^{\infty} \hat{\Gamma}(z) e(-\beta z^3 - \alpha z) dz,$$

where $\beta$ is of the form

$$\beta := \frac{k_2^2 h^2 d^2 s^4}{X_{12}^2 |m^*\tilde{m}|^2}$$

with $s = l_1$. We shall be interested in values of $\alpha$ of the form

$$\alpha := \frac{us^2}{X^4 k_2^2 |m^*\tilde{m}|^2}$$

In [1 subsection 3.2.], we provided estimates for $I(\alpha, \beta)$ for the special case when $\hat{\Gamma}(z)$ is a Gaussian, i.e.

$$\hat{\Gamma}(z) = e^{-\pi z^2}.$$

These results can be carried over to any Schwartz class function with minor modifications. We prove the following general estimates, similar to those in [1 subsection 3.2.], using standard estimates for exponential integrals.

**Lemma 12.1.** Suppose that $\alpha$ and $\beta > 0$ are real numbers. Set

$$\delta(\alpha) = \begin{cases} 1 & \text{if } \alpha < 0, \\ 0 & \text{if } \alpha \geq 0. \end{cases}$$
and define the function \( I(\alpha, \beta) \) as in (39). Let \( \Delta > 1 \), \( C > 0 \) and suppose that \( \beta \geq \Delta^{-1} \). Then we have the estimates
\[
I(\alpha, \beta) = O(1) \quad \text{if} \quad \alpha = 0, \tag{43}
I(\alpha, \beta) = G(\alpha, \beta) + O\left(\Delta \log(2 + \beta) \cdot |\alpha|^{-1}\right) \quad \text{if} \quad 0 < |\alpha| \leq \Delta^2 \beta, \tag{44}
I(\alpha, \beta) = O\left(\Delta^{-C}\right) \quad \text{if} \quad \Delta^2 \beta < \alpha. \tag{45}
\]

**Proof.** The estimate (43) follows by bounding the integral in question trivially by
\[
I(\alpha, \beta) \ll \int_{-\infty}^{\infty} |\hat{G}(z)| \, dz = O(1), \quad (46)
\]
and (44) follows by iterated integration by parts, saving a factor of size \( \gg \Delta \) in each step. If \( \alpha > 0 \), then \( G(\alpha, \beta) = 0 \) in (44), and the estimate in (44) follows from [4, Lemma 8.10] for \( k = 1 \) using integration by parts upon noting that
\[
\left| \frac{d}{dz}(-\beta z^3 - \alpha z) \right| = 3\beta z^2 + \alpha \geq \alpha.
\]
If \( \alpha < 0 \), then we are in the stationary phase case with stationary points
\[
x_0 := \pm \frac{|\alpha|^{1/2}}{(3 \beta)^{1/2}}.
\]
Set
\[
a := \frac{|\alpha|^{1/2}}{2 \beta^{1/2}} \quad \text{and} \quad b := \frac{|\alpha|^{1/2}}{\beta^{1/2}}.
\]
We first deal with the partial integral over the interval \([a, b]\). Employing [4, Corollary 8.15] together with [4, Lemma 8.10] for \( k = 2 \), and using integration by parts, we asymptotically evaluate this integral as
\[
\int_{b}^{a} \hat{G}(z) e\left(-\beta z^3 - \alpha z\right) \, dz = \hat{G} \left(-\frac{|\alpha|^{1/2}}{(3 \beta)^{1/2}}\right) \cdot \frac{1}{2^{1/2}(3 |\alpha| \beta)^{1/4}} \cdot e \left(-\frac{2 |\alpha|^{3/2}}{3^{1/2} \beta^{3/2}}\right) \tag{47}
\]
where we recall that \( |\alpha| \leq \Delta^2 \beta \). Similarly, we find
\[
\int_{a}^{-b} \hat{G}(z) e\left(-\beta z^3 - \alpha z\right) \, dz = \hat{G} \left(-\frac{|\alpha|^{1/2}}{(3 \beta)^{1/2}}\right) \cdot \frac{1}{2^{1/2}(3 |\alpha| \beta)^{1/4}} \cdot e \left(-\frac{1}{8} + \frac{2 |\alpha|^{3/2}}{3^{1/2} \beta^{3/2}}\right) \tag{48}
\]
Using (47) and (48), and estimating the remaining integrals over \((\infty, -b), (-a, a)\) and \((b, \infty)\) again using [4, Lemma 8.10] for \( k = 1 \), we obtain (41).

Since we shall apply partial summation over \( l_1 \), we shall also need the following asymptotic evaluation for
\[
\frac{\partial}{\partial s} I(\alpha, \beta) = I_1(\alpha, \beta) := 2\pi i \int_{-\infty}^{\infty} \hat{G}(z) e\left(-\frac{4 \beta}{s} \cdot z^3 - \frac{2 \alpha}{s} \cdot z\right) \cdot e\left(-\beta z^3 - \alpha z\right) \, dz, \tag{49}
\]
with $\alpha$ and $\beta$ being defined as in (40) and (41). The following result can be proved in a similar way as Lemma 12.1, where

$$\hat{\Gamma}(z) \cdot \left( -\frac{4\beta}{s} \cdot z^3 - \frac{2\alpha}{s} \cdot z \right)$$

now takes the rule of $\hat{\Gamma}(z)$.

**Lemma 12.2.** Suppose that $\alpha$ and $\beta > 0$ are real numbers. Set

$$\delta(\alpha) = \begin{cases} 1 & \text{if } \alpha < 0, \\ 0 & \text{if } \alpha \geq 0 \end{cases}$$

and

$$G_1(\alpha, \beta) := \delta(\alpha) \cdot \left( -\hat{\Gamma} \left( \frac{|\alpha|^{1/2}}{(3\beta)^{1/2}} \right) \cdot \frac{2^{1/2} |\alpha|^{5/4}}{3^{3/4} \beta^{3/4} s} \cdot e \left( \frac{1}{8} - \frac{2|\alpha|^{3/2}}{3^{3/2} \beta^{1/2}} \right) \right) + \hat{\Gamma} \left( -\frac{|\alpha|^{1/2}}{(3\beta)^{1/2}} \right) \cdot \frac{2^{1/2} |\alpha|^{5/4}}{3^{3/4} \beta^{3/4} s} \cdot e \left( \frac{1}{8} + \frac{2|\alpha|^{3/2}}{3^{3/2} \beta^{1/2}} \right)$$

and define the function $I_1(\alpha, \beta)$ as in (39). Let $\Delta > 1$, $C > 0$ and suppose that $\beta \geq \Delta^{-1}$. Then we have the estimates

$$I_1(\alpha, \beta) = G_1(\alpha, \beta) + O \left( \Delta \log(2 + \beta) \cdot s^{-1} \right) \text{ if } 0 < |\alpha| \leq \Delta^2 \beta,$$

$$I_1(\alpha, \beta) = O \left( \Delta^{-C} \right) \text{ if } \Delta^2 \beta < \alpha.$$ We note that for $\alpha$ and $\beta$ as in (40) and (41), we have

$$G(\alpha, \beta) = \delta(u) \times$$

$$\hat{\Gamma} \left( \frac{X^4 |u|^{1/2}}{3^{1/2} k_2 k_3 \h \h ds} \right) \cdot \frac{X^4 k_3^{1/2} |m^* \tilde{m}|}{12^{1/2} (k_2 \h \h) \frac{1}{1/2} |u|^{1/4} s^{1/2}} \cdot e \left( \frac{1}{8} - \frac{2|u|^{3/2}}{3^{3/2} k_2 \h \h \h |m^* \tilde{m}|^2} \right) +$$

$$\hat{\Gamma} \left( -\frac{X^4 |u|^{1/2}}{3^{1/2} k_2 k_3 \h \h ds} \right) \cdot \frac{2^{1/2} X^4 k_3^{1/2} |m^* \tilde{m}|}{12^{1/2} (k_2 \h \h) \frac{1}{1/2} |u|^{1/4} s^{1/2}} \cdot e \left( \frac{1}{8} + \frac{2|u|^{3/2}}{3^{3/2} k_2 \h \h \h |m^* \tilde{m}|^2} \right)$$

and

$$G_1(\alpha, \beta) = \frac{2^{1/2} X^4 |u|^{1/2}}{3^{1/2} (k_2 \h \h) \frac{1}{1/2} |u|^{1/4} s^{1/2}} \cdot e \left( \frac{1}{8} - \frac{2|u|^{3/2}}{3^{3/2} k_2 \h \h \h |m^* \tilde{m}|^2} \right) +$$

$$\frac{2^{1/2} X^4 |u|^{1/2}}{3^{1/2} (k_2 \h \h) \frac{1}{1/2} |u|^{1/4} s^{1/2}} \cdot e \left( \frac{1}{8} + \frac{2|u|^{3/2}}{3^{3/2} k_2 \h \h \h |m^* \tilde{m}|^2} \right).$$

Further, we observe that

$$\frac{\partial}{\partial s} G(\alpha, \beta) = G_1(\alpha, \beta) + O \left( \frac{X^4 |m^* \tilde{m}|}{(k_2 \h \h) \frac{1}{1/2} |u|^{1/4} s^{1/2}} + \frac{X^8 |u|^{1/4} |m^* \tilde{m}|}{(k_2 \h \h) \frac{1}{3/2} s^{7/2}} \right).$$

Now we suppose that

$$s \geq \left( \frac{|m^* \tilde{m}|}{k_2 \h \h} \right)^{1/2} X^{3-\varepsilon}$$

so that $\beta$, as specified in (40), satisfies $\beta \geq X^{-\varepsilon}$. Further, we set

$$K := \frac{(k_2 k_3 \h \h \h)^2}{X^{8-2\varepsilon}}$$

and define

$$\Omega(\alpha, \beta) = I(\alpha, \beta) - G(\alpha, \beta).$$
Then, for \( \alpha \) and \( \beta \) as in \((40)\) and \((41)\), it follows that

\[
\Omega(\alpha, \beta) = \begin{cases} O(1) & \text{if } \alpha = \beta = 0, \\
O\left(X^{4+2\varepsilon}|m^* \tilde{m}|^2|u|^{-1}s^{-2}\right) & \text{if } 0 < |u| \leq K, \\
O\left(X^{-C}\right) & \text{if } |u| > K,
\end{cases}
\]

\( (56) \)

\[
\frac{\partial}{\partial s} \Omega(\alpha, \beta) = \begin{cases} O\left(X^{2\varepsilon}(s^{-1} + X^4|m^* \tilde{m}|((hd)^{-1/2}|u|^{-1/4}s^{-5/2})\right) & \text{if } 0 < |u| \leq K, \\
O_{\Gamma,C}\left(X^{-C}\right) & \text{if } |u| > K.
\end{cases}
\]

\( (57) \)

We also note that

\[
G(\alpha, \beta) = O\left(X^{-C}\right) \text{ if } |u| > K
\]

\( (58) \)

by rapid decay of \( \hat{\Gamma} \). The last three estimates and equations are the results on exponential integrals we shall work with in the following.

13. Rearranging summations and partial estimation of \( E^-(X) \)

Now we pull in the sum over \( l_1 \) in \((39)\), getting

\[
E^-(X) = X^6 \sum_{k_1^2 | k_3^3} \sum_{l_1 \leq \xi/(k_2 k_3 d)} \sum_{h d \leq \xi \nu^2} \sum_{m^* \in \mathbb{Z} \setminus \{0\}} \sum_{\tilde{m} \in \mathbb{Z} \setminus \{0\}} \sum_{l_1 \leq \xi/(k_3 h k_3 d)} \sum_{(l_1, h d m^* \tilde{m}) = 1} \frac{1}{(m^*)^2} \cdot \frac{1}{m^*} \cdot \sum_{l_2 \in \mathbb{Z}} \mu(k_2) k_2 \cdot \mu(k_3) k_3 \cdot \mu(h) h^2 \cdot \mu(d) d^2 \times
\]

\[
\frac{\mu(l_1)}{l_1} \cdot \hat{\Gamma} \left( \frac{X^{6} m^* \tilde{m}}{k_2 h d l_1^2} \right) \cdot \hat{\Psi} \left( \frac{l_2^3 u}{X^{4} k_3^2 |m^* \tilde{m}|^2} \right) \times
\]

\( (59) \)

\[
E(a^3 b^4 h^2 m^* \tilde{m} \tilde{t}_1^{-2}, a^3 b^2 h \tilde{m} \tilde{t}_1^{-2} d; \epsilon \left( \frac{l_1 k_2 \tilde{m}^4 \tilde{m}^3 + u \tilde{m}}{q} \right) \times
\]

\[
F \left( \frac{l_1^2 k_2 \tilde{m}^4 d^2; \tilde{t}_1^2 k_2 \tilde{m}^4 d z, \tilde{t}_1^2 k_2 \tilde{m}^4 z^2 + u; \tilde{q} \right) \cdot F(l_2^2 k_2^2 q^2, 0, u; \tilde{m}^2).
\]

Next, upon recalling \((39)\) and \((55)\), we split the function \( \hat{\Psi} \) (with \( s = l_1 \)) into

\[
\hat{\Psi}(\alpha) = \Omega(\alpha, \beta) + G(\alpha, \beta)
\]

and accordingly \( E^-(X) \) into

\[
E^-(X) = E_\Omega(X) + E_G(X).
\]

\( (60) \)

Further, we cut summations, at the cost of errors of size \( O(1) \), taking into account that \( \hat{\Gamma} \), \( \Omega \) and \( G \) have rapid decay, which, in the case of \( \Omega \), follows from the last case in \((39)\), and in the case of \( G \) follows from \((58)\). We set

\[
M := \frac{\xi^2}{hdX^{6-\varepsilon}}, \quad U := \frac{\xi^2}{X^8-2\varepsilon}, \quad L := \max \left\{ \left( \frac{|m^* \tilde{m}|}{k_2 h d} \right)^{1/2} X^{3-\varepsilon}, \frac{|u|^{1/2} X^{4-\varepsilon}}{k_2 k_3 h d} \right\}, \quad \tilde{L} := \frac{\xi}{k_2 k_3 h d}.
\]
Then

\[ E_f(X) \ll 1 + X^6 \sum_{k_2 \mid k_3} \sum_{k_2, k_3 \mid 3 \ h \leq \xi} \sum_{(h, d) = 1} \sum_{d \leq \xi/(k_2 k_3 h)} \frac{\mu(k_2)}{k_2} \cdot \frac{\mu(k_3)}{k_3} \cdot \frac{\mu(h)}{h^2} \cdot \frac{\mu(d)}{d^3} \times \]

\[
\sum_{m \in \mathbb{Z} \setminus \{0\}} \sum_{\tilde{m} \in \mathbb{Z} \setminus \{0\}} \frac{1}{(m \cdot \tilde{m})^2} \cdot \sum_{|u| \leq U} \sum_{j=1}^{q_1} \frac{\mu(l_1)}{l_1} \cdot \tilde{\Gamma} \left( \frac{X^6 m \cdot \tilde{m}}{k_2 h d l_1^2} \right) \cdot f \left( \frac{l_1^2 u}{X^4 k_2^3 |m \cdot \tilde{m}|^2} \right) \times \]

\[
E(a^b b^2 h^2 m^2 m_1^{-2}, a^3 b^2 h z m_1^{-2} \cdot d) \cdot e \left( \frac{\tilde{\Gamma} l_1^{-2} k_2^2 \tilde{m}^2 z^2 + u z}{q} \right) \times \]

\[
F \left( \tilde{\Gamma} z^2 k_2^2 \tilde{m}^2 d^2, 3 \tilde{\Gamma} z^2 k_2^2 \tilde{m}^2 d z, 3 \tilde{\Gamma} z^2 k_2^2 \tilde{m}^2 z^2 + u z, \tilde{q} \right) \cdot F \left( \tilde{\Gamma} z^2 k_2^2 q^2, 0, \tilde{m}^2 \right), \]

for \( f = \Omega, G \). Breaking the summation over \( l_1 \) into residue classes modulo

\[ q_1 = [d, \tilde{q}] = \frac{d q}{(d, m^2)} \]

(recall that \( q = \tilde{q} d \) and \( q = k_3^3 h d (m^2) \)), we get

\[ E_f(X) = 1 + X^6 \sum_{k_2 \mid k_3} \sum_{k_2, k_3 \mid 3 \ h \leq \xi/(k_2 k_3 h)} \sum_{(h, d) = 1} \sum_{d \leq \xi/(k_2 k_3 h)} \frac{\mu(k_2)}{k_2} \cdot \frac{\mu(k_3)}{k_3} \cdot \frac{\mu(h)}{h^2} \cdot \frac{\mu(d)}{d^3} \times \]

\[
\sum_{m \in \mathbb{Z} \setminus \{0\}} \sum_{\tilde{m} \in \mathbb{Z} \setminus \{0\}} \frac{1}{(m \cdot \tilde{m})^2} \cdot \sum_{|u| \leq U} \sum_{j=1}^{q_1} \frac{\mu(l_1)}{l_1} \cdot \tilde{\Gamma} \left( \frac{X^6 m \cdot \tilde{m}}{k_2 h d l_1^2} \right) \cdot f \left( \frac{l_1^2 u}{X^4 k_2^3 |m \cdot \tilde{m}|^2} \right) \times \]

\[
F \left( \tilde{\Gamma} z^2 k_2^2 \tilde{m}^2 d^2, 3 \tilde{\Gamma} z^2 k_2^2 \tilde{m}^2 d z, 3 \tilde{\Gamma} z^2 k_2^2 \tilde{m}^2 z^2 + u z, \tilde{q} \right) \times \]

\[
\sum_{l_1 \leq L} \sum_{(l_1, d \tilde{m}) = 1} \sum_{l_1 \equiv j \mod q_1} \mu(l_1) = \Omega(1) \text{ and } \Omega(0) = O(1), \text{ which latter is the bound in the first case in } (56), \text{ and estimating the sum over } l_1 \text{ trivially using } |\mu(l_1)| \leq 1, \text{ we get}
\]

\[ R_{\Omega}(X) \ll X^{6+\varepsilon} \sum_{h d \leq \xi^6} \sum_{\tilde{m} \in \mathbb{Z} \setminus \{0\}} \sum_{\tilde{m} \in \mathbb{Z} \setminus \{0\}} \sum_{|u| \leq U} \sum_{j=1}^{q_1} \frac{1}{(h d)^{3/4}} \cdot \sum_{|m \cdot \tilde{m}| \leq M} 1. \]
The inner-most sum can be evaluated explicitly. If \( s(\bar{m}) \) is the largest square dividing \( \bar{m} \), then

\[
\sum_{x_1=1}^{[\bar{m}]} \frac{1}{x^2 \equiv 0 \mod |\bar{m}|} = \sqrt{s(\bar{m})}.
\]

By a short calculation, it follows that \( R(x) \ll X^{6+\varepsilon} \) and hence, we deduce from (50), (53) and (61) that

\[
E_f(X) \ll X^{6+\varepsilon} + X^{6+\varepsilon} \sum_{k_2|2, k_3|3} \sum_{h d \leq \xi} \left( \frac{1}{(hd)^{3/2}} \cdot \sum_{m^* \in \mathbb{Z}/(0)} \sum_{\tilde{m}^* \in \mathbb{Z}/(0)} \frac{(d, |m^*|)^{1/2}}{|m^* \tilde{m}|} \cdot \sum_{m^* \tilde{m}^* \in \mathbb{Z}/(0)} \frac{(d, |m^*|)^{1/2}}{|m^* \tilde{m}|} \cdot \sum_{|m^* \tilde{m}| \leq M} \sum_{0 \leq |u| \leq U} \sum_{j \equiv |u| \mod q_1} |Z_f(j)|
\]

for \( f = \Omega, G \), where

\[
Z_f(j) := \sum_{L \leq l_1 \leq L} \frac{\mu(l_1)}{l_1} \cdot \tilde{\Gamma} \left( \frac{X^6 m^* \tilde{m}}{k_2 h d l_1^2} \right) \cdot f \left( \frac{p_1^2 u}{X^4 k^2_2 m^* \tilde{m}^2} \right) \cdot e \left( \gamma l_1 \right),
\]

with

\[
\gamma := \frac{(x^2 + u x_1) k_2 q}{m^2}.
\]

14. Partial summation over \( l_1 \)

In this section, we transform the inner-most sum in (64)

\[
\sum_{j \equiv |u| \mod q_1} |Z_f(j)|
\]

by applying partial summation over \( l_1 \) to \( Z_f(j) \). We shall assume that the variables \( k_2, k_3, h, d, m^*, \tilde{m}, u, x_1 \) satisfy the summation conditions in (50). In particular, (51) holds, and we are in the case \( 0 < |u| \leq K \) in (56) and (57).

We start with the case \( f = G \). In this case, we have, by (51) with \( s = l_1 \),

\[
Z_G(j) = Z_{G, 1}(j) + Z_{G, -1}(j),
\]

where

\[
Z_{G, \omega}(j) := \delta(u) \cdot e \left( \frac{\omega}{s} \right) \cdot \frac{X^4 k_2^2 m^* \tilde{m}}{(18k_2 h d)^{1/2}|u|^{1/4}} \cdot \sum_{L \leq l_1 \leq L} \frac{\mu(l_1)}{l_1^{1/2}} \cdot \tilde{\Gamma} \left( \frac{X^6 m^* \tilde{m}}{k_2 h d l_1^2} \right) \cdot f \left( \frac{p_1^2 u}{X^4 k_2^3 m^* \tilde{m}^2} \right) \cdot e \left( \gamma l_1 \right)
\]

with

\[
\gamma_\omega := \gamma - \omega \cdot \frac{2|u|^{1/2}}{3^{1/2} k_2 k_3 h d |m^* \tilde{m}|^2}.
\]

Next, we write

\[
S_j(s, w) := \sum_{L \leq n \leq s} \mu(n) \cdot e \left( w n \right)
\]
and remove the weight functions on the right-hand side of (65) using partial summation, leading to

\[
\begin{align*}
Z_{G, \omega}(j) &= \delta(u) \cdot e \left( \frac{\omega}{8} \right) \cdot \frac{X^4 \kappa^{1/2} |m^* \tilde{m}|}{(18k_2 hd)^{1/2} |u|^{1/4}} \times \\
& \quad \left( \frac{1}{L^{5/2}} \cdot \tilde{\Gamma} \left( \frac{X^6 m^* \tilde{m}}{k_2 hd L^2} \right) \cdot \tilde{\Gamma} \left( \omega \cdot \frac{X^4 |u|^{1/2}}{3! / k_2 k_3 hd L} \right) \cdot S_j(L, \gamma) + \\
& \quad \frac{5}{2} \int L \frac{1}{s^{3/2}} \cdot \tilde{\Gamma} \left( \frac{X^6 m^* \tilde{m}}{k_2 hd s^2} \right) \cdot \tilde{\Gamma} \left( \omega \cdot \frac{X^4 |u|^{1/2}}{3! / k_2 k_3 hd s} \right) \cdot S_j(s, \gamma \omega) \ ds + \\
& \quad 2X^6 m^* \tilde{m} \frac{1}{k_2 hd} \int L \frac{1}{s^{11/2}} \cdot \tilde{\Gamma} \left( \frac{X^6 m^* \tilde{m}}{k_2 hd s^2} \right) \cdot \tilde{\Gamma} \left( \omega \cdot \frac{X^4 |u|^{1/2}}{3! / k_2 k_3 hd s^2} \right) \cdot S_j(s, \gamma \omega) \ ds + \\
& \quad \omega X^4 |u|^{1/2} \frac{3! / k_2 k_3 hd ^2}{3! / k_2 k_3 hd ^2} \int L \frac{1}{s^{9/2}} \cdot \tilde{\Gamma} \left( \frac{X^6 m^* \tilde{m}}{k_2 hd s^2} \right) \cdot \tilde{\Gamma} \left( \omega \cdot \frac{X^4 |u|^{1/2}}{3! / k_2 k_3 hd s^2} \right) \cdot S_j(s, \gamma \omega) \ ds \right) \cdot \tilde{\Gamma}(z, \tilde{\Gamma}(z) = O(1), \text{ it follows that} \]

\[
\begin{align*}
\sum_{j=1}^{q_1} |Z_{G, \omega}(j)| &\ll \frac{X^4 |m^* \tilde{m}|}{(hd)^{1/2} |u|^{1/4} L^{5/2}} \cdot \sum_{j=1}^{q_1} |S_j(L, \gamma)| + \\
& \quad \int L \left( \frac{X^4 |m^* \tilde{m}|}{(hd)^{1/2} |u|^{1/4} s^{7/2}} + \frac{X^{10} |m^* \tilde{m}|^2}{(hd)^{3/2} |u|^{1/4} s^{11/2}} + \frac{X^8 |u|^{1/4} |m^* \tilde{m}|}{(hd)^{3/2} s^{9/2}} \right) \cdot \sum_{j=1}^{q_0} |S_j(s, \gamma | \omega) \ ds. \tag{67}
\end{align*}
\]

Now we turn to the case \( f = \Omega \). Then using (65) and partial summation, we get

\[
\begin{align*}
Z_{\Omega}(j) &= \frac{1}{L} \cdot \tilde{\Gamma} \left( \frac{X^6 m^* \tilde{m}}{k_2 hd L^2} \right) \cdot \Omega \left( \frac{u L^2}{X^4 k_2^2 |m^* \tilde{m}|^2} \cdot \frac{k_2^3 h^2 d^2 L^4}{X^4 |m^* \tilde{m}|^2} \right) \cdot S_j(L, \gamma) + \\
& \quad \int L \frac{1}{s^{3/2}} \cdot \tilde{\Gamma} \left( \frac{X^6 m^* \tilde{m}}{k_2 hd s^2} \right) \cdot \Omega \left( \frac{u s^2}{X^4 k_2^2 |m^* \tilde{m}|^2} \cdot \frac{k_2^3 h^2 d^2 s^4}{X^4 |m^* \tilde{m}|^2} \right) \cdot S_j(s, \gamma) \ ds + \\
& \quad 2X^6 m^* \tilde{m} \frac{1}{k_2 hd} \int L \frac{1}{s^{11/2}} \cdot \tilde{\Gamma} \left( \frac{X^6 m^* \tilde{m}}{k_2 hd s^2} \right) \cdot \Omega \left( \frac{u s^2}{X^4 k_2^2 |m^* \tilde{m}|^2} \cdot \frac{k_2^3 h^2 d^2 s^4}{X^4 |m^* \tilde{m}|^2} \right) \cdot S_j(s, \gamma) \ ds - \\
& \quad \int L \frac{1}{s} \cdot \frac{d}{ds} \Omega \left( \frac{u s^2}{X^4 k_2^2 |m^* \tilde{m}|^2} \cdot \frac{k_2^3 h^2 d^2 s^4}{X^4 |m^* \tilde{m}|^2} \right) \cdot S_j(s, \gamma) \ ds.
\end{align*}
\]
Hence, using $\hat{\Gamma}(z) = O(1)$ and the bounds for $\Omega$ and $\Omega'$ in the case $0 < |u| \leq K$ in (56) and (57), it follows that

$$\sum_{j=1}^{q_1} |Z_{\Omega}(j)| \ll X^{\varepsilon} \cdot \frac{X^4|m^*\tilde{m}|^2}{|u|L^3} \cdot \sum_{j=1}^{q_1} \left| S_j(\tilde{L}, \gamma) \right| + X^{\varepsilon} \times$$

$$\int_{L}^{L} \left( \frac{X^4|m^*\tilde{m}|^2}{|u|s^4} + \frac{X^{10}|m^*\tilde{m}|^3}{hd|u|s^6} + \frac{1}{s^2} + \frac{X^4|m^*\tilde{m}|}{(hd)^{1/2}|u|^{1/4}s^{7/2}} + \frac{X^8|u|^{1/4}|m^*\tilde{m}|}{(hd)^{3/2}s^{3/2}} \right) \times (68)$$

$$\sum_{j=1}^{q_1} \left| S_j(s, \gamma) \right| ds.$$

15. Averaging over $j$ and $l_1$

The next step is to estimate

$$\sum_{j=1}^{q_1} \left| S_j(s, w) \right|$$

for any $w \in \mathbb{R}$, where $S_j(s, w)$ is defined as in (60). Let $R \leq X^{100}$ be a positive integer, to be specified later. Using Dirichlet’s approximation theorem, there exist an integer $a$ and a positive integer $r$ such that $(a, r) = 1$, $r \leq R$ and $w = a/r + \beta$ with $|\beta| \leq 1/(rR)$. Using partial summation, it follows that

$$S_j(s, w)$$

$$= e(\beta s) \sum_{L \leq n \leq s} \mu(n) \cdot e \left( \frac{a}{r} \cdot n \right) = 2\pi i \beta \int_{L}^{s} e(\beta t) \cdot \left( \sum_{L \leq n \leq t} \mu(n) \cdot e \left( \frac{a}{r} \cdot n \right) \right) dt$$

$$\ll \left| \sum_{L \leq n \leq s} \mu(n) \cdot e \left( \frac{a}{r} \cdot n \right) \right| + \frac{1}{rR} \int_{L}^{s} \left| \sum_{L \leq n \leq t} \mu(n) \cdot e \left( \frac{a}{r} \cdot n \right) \right| dt.$$

Now we write $f := (a, r)$, $n_1 := n/f$ and $r_1 := r/f$. Then

$$e \left( \frac{a}{r} \cdot n \right) = e \left( \frac{a}{r_1} \cdot n_1 \right) = \frac{1}{\varphi(r_1)} \cdot \sum_{\chi_1 \mod r_1} \chi(an_1)\tau(\chi_1),$$

where

$$\tau(\chi_1) := \sum_{x=1}^{r_1} \chi_1(x) \cdot e \left( \frac{x}{r_1} \right)$$

is the Gauss sum for the Dirichlet character $\chi_1$. Using this and

$$\mu(n) = \begin{cases} \mu(n_1)\mu(f) & \text{if } (n_1, f) = 1, \\ 0 & \text{if } (n_1, f) > 1, \end{cases}$$

and detecting the coprimality condition $(n_1, 6\tilde{m}f) = 1$ using the principal character $\chi_0$ modulo $6\tilde{m}f$ and the congruence condition $n_1f \equiv j \mod q_1$ by Dirichlet characters $\chi$ modulo $q_1$ (recall
that \((j, q) = 1\), we arrive at

\[
\sum_{\substack{L < n \leq t \\
(n, 6n\beta) = 1 \\
n \equiv j \mod q_1}} \mu(n) \cdot e \left( \frac{a}{r} \cdot n \right) = \frac{1}{\varphi(q_1)} \cdot \sum_{\chi \mod q_1} \chi(j) \sum_{f | r} \mu(f) \chi(f) \cdot \frac{1}{\varphi(r_1)} \cdot \sum_{\chi_1 \mod r_1} \chi_1(a) \chi_1(f) \tau(\chi_1) \times 
\sum_{L/f < n_1 \leq t/f} \chi(n_1) \mu(n_1).
\]

We shall also obtain a saving by averaging over \(j \mod q\), where \((j, q) = 1\). Using the orthogonality relations for Dirichlet characters, the bound \(|\tau(\chi)| \leq \sqrt{r_1}\), and the Riemann Hypothesis for Dirichlet \(L\)-functions which implies that

\[
\sum_{L/f < n_1 \leq t/f} \chi(n_1) \mu(n_1) \ll \left( \frac{f}{t} \right)^{1/2} X^\varepsilon,
\]

we deduce that

\[
\sum_{j=1 \atop (j, q_1) = 1}^{q_1} \left| \sum_{\substack{L < n \leq t \\
(n, 6n\beta) = 1 \\
n \equiv j \mod q_1}} \mu(n) \cdot e \left( \frac{a}{r} \cdot n \right) \right|^2 = \frac{1}{\varphi(q_1)} \sum_{\chi \mod q_1} \left| \sum_{f | r} \mu(f) \chi(f) \cdot \frac{1}{\varphi(r_1)} \cdot \sum_{\chi! \mod r_1} \chi(a) \chi_1(f) \tau(\chi_1) \sum_{L/f < n_1 \leq t/f} \chi_1(n_1) \mu(n_1) \right|^2 \ll rtX^\varepsilon,
\]

and therefore, by Cauchy-Schwarz,

\[
\sum_{j=1 \atop (j, q_1) = 1}^{q_1} \left| \sum_{\substack{L < n \leq t \\
(n, 6n\beta) = 1 \\
n \equiv j \mod q_1}} \mu(n) \cdot e \left( \frac{a}{r} \cdot n \right) \right| \ll (q_1 rt)^{1/2} X^\varepsilon.
\]

Using this together with (69), we get

\[
\sum_{j=1 \atop (j, q_1) = 1}^{q_1} |S_j(s, w)| \ll q_1^{1/2} \left( r^{1/2}s^{1/2} + s^{3/2} \right) \leq q_1^{1/2} \left( R^{1/2}s^{1/2} + s^{3/2} \right),
\]

Now fixing \(R := s^{2/3}\), it follows that

\[
\sum_{j=1 \atop (j, q_1) = 1}^{q_1} |S_j(s, w)| \ll q_1^{1/2} s^{5/6} X^\varepsilon.
\]

(70)

16. PROOF OF THEOREM 1.2

To prove Theorem 1.2 it remains to bound the error term \(E^{-}(X)\). Combing (63), (67), (68) and (70), we obtain

\[
\sum_{j=1 \atop (j, q_1) = 1}^{q_1} \left( |Z_G(j)| + |Z_0(j)| \right) \ll q_1^{1/2} X^\varepsilon \left( \frac{X^4|m^*\tilde{n}|}{(hd)^{1/2}|u|^{1/2}L^{5/3}} + \frac{X^{10}|m^*\tilde{n}|^2}{(hd)^{1/2}|u|^{1/4}L^{11/3}} + 
\frac{X^8|m^*\tilde{n}|^2}{(hd)^{1/2}L^{8/3}} + \frac{X^4|m^*\tilde{n}|^2}{|u|L^{13/6}} + \frac{X^{10}|m^*\tilde{n}|^3}{hd|u|L^{25/6} + \frac{1}{L^{1/6}}} \right).
\]

(71)
Recalling

\[ q_1 = \frac{q}{(d, |m^*|)} = \frac{k_1^2 hd(m^*)^2}{(d, |m^*|)} \]

and

\[ L := \max \left\{ \left( \frac{|m^* \bar{m}|}{k_2 hd} \right)^{1/2} X^{3-\varepsilon}, \left| \frac{u}{k_2 k_3 hd} \right| \right\}, \]

we deduce that

\[ \sum_{j=1}^{q_1} (|Z_G(j)| + |Z_G(j)|) \ll X^\varepsilon \cdot \frac{|m^*|}{(d, |m^*|)^{1/2}} \cdot \sum_{i=1}^{4} u^{\alpha_i} |m^* \bar{m}|^{\beta_i} (hd)^{\gamma_i} X^{\delta_i}, \]

where

\[ (\alpha_1, \beta_1, \gamma_1, \delta_1) := \left( -\frac{1}{4}, \frac{5}{6}, -\frac{1}{6}, -1 \right), \]

\[ (\alpha_2, \beta_2, \gamma_2, \delta_2) := \left( -\frac{13}{12}, \frac{1}{3}, -\frac{8}{3}, -\frac{1}{3} \right), \]

\[ (\alpha_3, \beta_3, \gamma_3, \delta_3) := \left( -\frac{1}{12}, \frac{19}{12}, -\frac{5}{2}, -\frac{1}{2} \right), \]

\[ (\alpha_4, \beta_4, \gamma_4, \delta_4) := \left( -\frac{2}{12}, \frac{0}{3}, -\frac{2}{3}, -\frac{2}{3} \right). \]

Here we use the second term in the maximum in (72) to bound the third term and sixth term in the sum on the right-hand side of (71) and the first term in the said maximum to bound all the other terms in the said sum. Combining (60), (62) and (73), it follows that

\[ E^{-}(X) \ll X^{6+\varepsilon} + \sum_{i=1}^{4} A(\alpha_i, \beta_i, \gamma_i, \delta_i), \]

where

\[ A(\alpha, \beta, \gamma, \delta) := X^{6+\varepsilon} \sum_{(hd) \leq \tilde{c}} \sum_{m^* \in \mathbb{Z}/(d, |m^*|) \setminus \{0\}} \sum_{\bar{m} \in \mathbb{Z}/|m^* \bar{m}| \leq M} \left( \frac{1}{|m^* \bar{m}|} \right)^{\alpha_i} \sum_{u, hd|\tilde{m}|} \left( u, hd\right)^{\beta} \gamma_i X^{\delta_i}, \]

\[ \sum_{0 < |u| \leq U} \sum_{x_i \equiv \bar{m} \mod |\tilde{m}|} |u|^\alpha |m^* \bar{m}|^{\beta} (hd)^{\gamma} X^{\delta}, \]

Now to estimate \( E^{-}(X) \), we bound \( A(\alpha, \beta, \gamma, \delta) \) for general parameters. We first examine the cardinality

\[ \sum_{x_i \equiv \bar{m} \mod |\tilde{m}|} \sum_{x_1 = 1}^{\tilde{m}} 1 = \sharp \{ x \in \{1, \ldots, \tilde{m}\} : x_1 \equiv -\mathfrak{m} \mod |\tilde{m}| \}, \]

where we recall that \((|\tilde{m}|, 6) = 1\) and \(\mathfrak{m}\) is a multiplicative inverse of 3 modulo \(|\tilde{m}|\). We observe that a necessary condition for the above congruence to be solvable is that \((u, |\tilde{m}|)\) is a perfect square. In this case, let \((u, |\tilde{m}|) = u^2, v > 0\). Then \(v|x_1\), and the congruence reduces to \(x_2^2 \equiv \mathfrak{m} u_2 \mod m_2\), where \(x_2 = x_1/v, u_2 = u/v^2\) and \(m_2 = |\tilde{m}|/v^2\). Hence,

\[ \sharp \{ x \in \{1, \ldots, \tilde{m}\} : x_1 \equiv -\mathfrak{m} \mod |\tilde{m}| \} = v^2 \cdot \sharp \{ x \in \{1, \ldots, m_2\} : x_2^2 \equiv -\mathfrak{m} u_2 \mod m_2 \} \]

\[ \ll v^2 \cdot m_2^2 \ll (u, |\tilde{m}|) X^\varepsilon. \]

We further observe that

\[ \sum_{0 < |u| \leq U} \left( u, hd|\tilde{m}| \right)^{1/2} |u|^{\alpha} \ll U^{\tilde{m}} (hd |m^* \bar{m}|)^{\varepsilon} \ll U^{\tilde{m}} X^\varepsilon \ll \frac{\xi^{2\varepsilon}}{X^{\tilde{m}}}, X^\varepsilon. \]
with \( \bar{\alpha} := \max(0, 1 + \alpha) \), where we recall \( U = \xi^2 X^{2\epsilon - 8} \). Combining the above gives

\[
\sum_{0 < \vert u \vert \leq U} \sum_{x_1 = 1}^{\vert \bar{m} \vert} \left( u, hd \vert m^* \vert^{2} \bar{m} \right) \vert u \vert^{\alpha} \ll \frac{\xi^{2\bar{\alpha}} X^{8\bar{\alpha}}}{X^{8\bar{\alpha}}} \cdot X^{\epsilon}.
\] (77)

Furthermore, we have

\[
\sum_{m^* \in \mathbb{Z} \setminus \{0\}} \sum_{\bar{m} \in \mathbb{Z} \setminus \{0\}} \frac{\vert m^* \vert^{1+\bar{\beta}} \bar{m} \vert^{\bar{\beta}}}{\rad(m^*)^{6 \omega}} \ll M^{\bar{\beta} + \epsilon} \sum_{0 < \vert m^* \vert \leq M} \frac{1}{\rad(m^*)^{6 \omega}} M^{\bar{\rho}} (hd)^{\epsilon} \ll \frac{\xi^{2\bar{\beta}} (hd)^{\epsilon}}{(hd)^{\delta} X^{6\bar{\gamma}}} \cdot X^{\epsilon}
\] (78)

with \( \bar{\beta} := \max(0, \beta) \), where we recall \( M = \xi^2 / (hdX^{6-\epsilon}) \). Using (76), (77) and (78), we get

\[
A(\alpha, \beta, \gamma, \delta) \ll \xi^{2\bar{\alpha} + 2\bar{\beta}} X^{6 + \delta - 8\bar{\alpha} - 6\bar{\beta} + \epsilon} \sum_{h \leq \xi^\gamma} h^{\gamma - \bar{\beta} - 3/2} \sum_{d \leq \xi^\frac{\gamma}{h}} d^{\gamma - \bar{\beta} - 3/2}
\] (79)

with \( \tilde{\gamma} := \max\{0, \gamma - \bar{\beta} - 1/2\} \).

Now using (73), (75) and (79), we compute that

\[
E^{-}(X) \ll X^{\epsilon} \left( X^{6 + \xi^{11/6 + \kappa/6} X^{-2} + \xi^2 X^{-8/3}} \right).
\] (80)

Combining (4), (7), (15), (18), (20), (21), (22), (24) and (80), we arrive at

\[
S(X) = X^{10} \cdot \frac{1}{3} \prod_{p > 3} \left( 1 - \frac{2p - 1}{p^3} \right) + O \left( X^{\epsilon} \left( X^{6 + \frac{X^{16}}{\xi^2} + \frac{X^{10}}{\xi} + \xi^{(1-\kappa)/2} X^4 + \xi^{2-\kappa} + \frac{\xi^{11/6 + \kappa/6}}{X^2} + \frac{\xi^2}{X^{8/3}} \right) \right).
\]

Finally, taking

\[
\kappa := \frac{16}{31} \quad \text{and} \quad \xi := X^{124/27},
\]

we obtain the result in Theorem 1.2.

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