EXPANSIVITY IMPLIES EXISTENCE OF HÖLDER CONTINUOUS LYAPUNOV FUNCTION

LUKASZ STRUSKI* AND JACEK TABOR
Faculty of Mathematics and Computer Science
Jagiellonian University
Lojasiewicza 6, 30-348 Kraków, Poland

(Communicated by Christian Pötzsche)

ABSTRACT. The Lyapunov function is a very useful tool in the theory of dynamical systems, in particular in the study of the stability of an equilibrium point. In this paper we construct a locally Hölder continuous Lyapunov function for a uniformly expansive set for a map $f$ in a metric space $X$. In the construction a basic role is played by the functions defining the stable and unstable cone-fields. As a tool we also use the approximately quasiconvex functions.

1. Introduction. Lyapunov function, introduced by A. M. Lyapunov [13] in 1892, can be applied to study the stability of various differential equations and dynamical systems. Intuitively, a Lyapunov function can be thought of as a generalized “energy” function for system which decreases along trajectories. Although there is no one general method for constructing Lyapunov functions, in many specific cases the theoretical or numerical construction of Lyapunov functions is known [1, 5, 8, 13, 14]. The principal advantage of a Lyapunov function is that in the study the exact analytical solution is not required. There are also other applications of Lyapunov functions, for example in [17] McCluskey shows an approach for determining terms in the system which include delay without changing the global stability.

In [10, 11] J. Lewowicz proposed to use Lyapunov functions of two variables to study structural stability and similar concepts, such as topological stability and persistence. The method has been applied in particular to study hyperbolic diffeomorphisms on manifolds. In this paper we investigate a generalization of Lyapunov function for a partial function of a metric space. We show that our Lyapunov function is Hölder continuous. Moreover, we give explicit estimations for the Hölder constants, which can be essential if one would like to use them in computer assisted dynamics [16, 25].

By a partial map $f$ from $X$ to $Y$ (denoted by $f : X \to Y$) we understand a function whose domain is a subset of $X$.

Definition 1.1. Let $f : X \to X$ be a partial function on a metric space $X$ and let $\Lambda \subset X$ be a compact invariant set for $f$ such that its some neighborhood is contained in the domain of $f$. We say that $V : X \times X \to \mathbb{R}$ is a Lyapunov function

2010 Mathematics Subject Classification. Primary: 37D20.
Key words and phrases. Lyapunov function, expansive map, Hölder continuous, cone-fields.
* Corresponding author: Lukasz Struski.

1Partial functions play a basic role in strict numerical representations of classical maps [2].
for \( f \) on \( \Lambda \) if \( \text{dom}(V) \) is an open subset of \( X \times X \), \((x,x) \in \text{dom}(V)\) for every \( x \in \Lambda \) and
\[
V(f(x), f(x)) = V(x, x) \quad \text{for} \quad x \in \Lambda,
\]
\[
V(f(x), f(y)) > V(x, y) \quad \text{for} \quad (x, y) \in \text{dom}(V), x \neq y.
\]

By using cone-fields \([8, 19]\) and approximately quasiconvex functions \([23]\) we are going to construct Lyapunov functions for uniformly expansive systems. The presented results are a metric equivalent of the implication \( i \Rightarrow ii \) in the following classical theorem.

**Theorem A** (\([24, \text{Theorem 3.2}]\)). Let \( f \) be a homeomorphism of a compact manifold \( M \). The following conditions are equivalent:

- \( i \) \( f \) is expansive;
- \( ii \) there exists a Lyapunov function for \( f \).

Let us note that the above result explains a main difference between the classical notion of Lyapunov function (as proposed in \([13]\)) and Lewowicz’s approach (see \([10, 11]\)) to which we refer in our work (see Definition \([1.1]\)). In the original (“local”) sense, Lyapunov function means a positive real-valued function, decreasing along the trajectories, whose existence in some neighborhood of a given equilibrium point of a dynamical system implies its (asymptotic) stability. In turn, in Definition \([1.1]\) we deal with a function (acting on pairs of points) that increases the distance between the consecutive iterations of points on two different trajectories, which can be considered, in fact, as some kind of a “global” property strongly related to expansivity. Let us remind that for expansive system, all different trajectories that start sufficiently close, must at some time be at a distance greater than a uniformly established positive constant. Moreover, since we work here with increasing distances between two trajectories, one can find a relationship of this approach to the Lyapunov framework for the expansion (as a direct opposite to the contraction) analysis (see, e.g., \([2, 4, 12]\)), concerning it as a tool that is well adapted for studying metric analogues of the respective results (such as, e.g., Theorem 1 in \([4]\)) demanding differential structure.

The proof of Theorem A for diffeomorphisms \( f \) can be found in \([10]\) (see Section 4 and Lemma 3.3 of that paper). Additional arguments required for the case of a homeomorphism are discussed in \([11, \text{Section 1}]\), see also \([24]\), where J. Tolosa characterized the expansivity on metric spaces with the use of Lyapunov functions basing on the results of J. Lewowicz.

Our general aim is to adapt the above and similar properties of classical notions from smooth dynamical systems such as hyperbolicity, expansivity and shadowing (also called the pseudo-orbit tracing property) to the general metric case. Note that we have recently observed an increased interest in the study of analysis of dynamical systems in metric spaces \([6, 9, 22]\). The notion of dynamical system in metric spaces can be useful in improving the understanding even of classical problems generated by differential equations, for example in differential equations with discontinuous right-hand sides or mutational equations \([3, 8]\). In these and similar cases the solutions in a natural way lead to the evolution of sets, fuzzy sets or even measures, in the spaces of which we do not have any obvious natural \( C^1 \) structure. Another motivation to study general dynamical systems in metric spaces is based on computer assisted proofs in dynamics, where the evolution of

\[^{2}\text{We do not know if the metric equivalent of the implication } ii \Rightarrow i \text{ holds, see Problem} 3]
points is typically replaced, due to necessity of using rigorous interval methods, by the evolution of their set-valued computer representations (typically given by parallelograms or sets of cubes) [16, 18].

In [21], with the use of modified cone-fields, the generalized hyperbolicity on metric spaces was defined. It occurs that similarly as in the classical case, the generalized hyperbolicity guarantees expansivity. In [15] Mazur shows a relationship between hyperbolic and cone-hyperbolic structures (which we introduced in [21]) in metric spaces. In [20] we used the same tool to describe uniform expansivity. We showed that a function $f$ is uniformly expansive if and only if there exists a generalized cone-field such that $f$ is cone-hyperbolic. We also constructed a cone-field $c_s$, $c_u$ (see Theorem B) for a uniformly expansive $f$. Moreover, our generalizations are well adapted for rigorous verification with the use of computers. In this article we show that our generalization of expansivity has similar consequences as its classical equivalent.

It should be noted here that our research does not fit into the context of classical investigations involving various kinds of Lyapunov functions. Indeed, in most papers, including the mentioned Lyapunov [13] and Lewowicz [10, 11] original works, they are, in fact, treated as tools to prove some interesting properties of dynamical systems, whereas in our case such a possibility depends on an affirmative answer to the question posed as Problem [1]. Nevertheless, as stated above, the aim of this article is to compare the considered generalization of the concept of expansivity to its classical analogue, from a point of view of their common consequences.

This paper is organized as follows. In Section 2 we introduce basic notations and definitions concerning generalization of cone-fields to metric spaces. We recall definitions of cone-fields, cone-hyperbolicity and uniformly expansivity [20, 21]. We apply a theorem from [20, 21] (see Theorem [B]) to obtain the main result of the paper, namely the construction of a Lyapunov function for a uniformly expansive system [see Section 5]. In Sections 3 and 4 we show, by using a quasiconvex function [23], that the partial functions $c_u$, $c_s$ needed to construct this Lyapunov function are Hölder continuous.

Let’s start by recalling the definitions of a bilipschitz function. Given $L \geq 1$ and $f: X \rightarrow Y$ we call $f$ $L$-bilipschitz if

$$L^{-1}d(x, y) \leq d(f(x), f(y)) \leq Ld(x, y) \quad \text{for} \quad x, y \in \text{dom} f.$$ 

Main Result ([see Theorem 5.1]). Let $\varepsilon > 0$, $(X, d)$ be a metric space and let $f: X \rightarrow X$ be an $L$-bilipschitz map. Assume that $\Lambda \subset X$ is an invariant set for $f$ such that $f$ is uniformly expansive on $\Lambda$. Let $c_u$ and $c_s$ be defined by

$$c_s(x, v) := \inf \left\{ d\left(f^k(x), f^k(v)\right) \mid k \in \mathbb{Z}, k < 0 : f^l(v) \in B(f^l(x), \varepsilon) \text{ for } l \in [k, 0] \cap \mathbb{Z} \right\},$$

$$c_u(x, v) := \inf \left\{ d\left(f^k(x), f^k(v)\right) \mid k \in \mathbb{Z}, k \geq 0 : f^l(v) \in B(f^l(x), \varepsilon) \text{ for } l \in [0, k] \cap \mathbb{Z} \right\}.$$ 

Then $V(x, v) := c_u(x, v) - c_s(x, v)$ is a Lyapunov function for $f$ on the set $\Lambda$.

2. Cone-hyperbolic maps and uniform expansivity. In this section we introduce basic notations and definitions concerning generalization of cone-fields to metric spaces. For the convenience of the reader we recall notation from [20, 21].

By $\text{dom}(f)$ we denote the domain of a partial map $f: X \rightarrow Y$, and by $\text{im}(f)$ we denote its image. For a given $f: X \rightarrow X$ we say that a sequence $x: I \rightarrow X$ defined
on a subinterval of $I$ of $Z$ is an orbit of $f$ if
\[ x_n \in \text{dom}(f) \quad \text{and} \quad x_{n+1} = f(x_n) \quad \text{for} \quad n \in I \] such that $n + 1 \in I$.

For $\delta > 0$ and a set $A \subset X$ we define the $\delta$-neighbourhood of $A$ as
\[ A_\delta := \bigcup_{x \in A} B(x, \delta). \]

For an injective map $f: X \to X$ we call $A \subset \text{dom}(f)$ an invariant set for $f$ if $f(A) = A$. Observe that for an invariant set $A$ for $f$ we have $f(x) \in A$ and $f^{-1}(x) \in A$ for every $x \in A$.

**Definition 2.1** ([21] Definition 3.1). Let $(X, d)$ be a metric space, $\delta > 0$ and $\Lambda \subset X$ be nonempty. We say that a pair of partial functions $c_s, c_u: X \times X \to \mathbb{R}^+$ forms a $\delta$-cone-field on $\Lambda$ if
\[ \{x\} \times B(x, \delta) \subset U := \text{dom}(c_s) \cap \text{dom}(c_u) \quad \text{for} \quad x \in \Lambda. \]

We put $c(x, v) := \max\{c_s(x, v), c_u(x, v)\}$. If there exists $K > 0$ such that:
\[ \frac{1}{K} d(x, v) \leq c(x, v) \leq K \cdot d(x, v) \quad \text{for} \quad (x, v) \in U, \quad (1) \]
then we call it ($K, \delta$)-cone-field on $\Lambda$ or uniform $\delta$-cone-field on $\Lambda$.

For a more detailed explanation and motivation we refer the reader [21], Section 3. For $\delta > 0$ and each point $x \in \Lambda$ we introduce unstable and stable cones by the formulae
\[
\begin{align*}
C^u_s(\delta) & := \{v \in B(x, \delta) : c_s(x, v) \leq c_u(x, v)\}, \\
C^s_u(\delta) & := \{v \in B(x, \delta) : c_s(x, v) \geq c_u(x, v)\}.
\end{align*}
\]

We consider a partial map $f: X \to Y$ between metric spaces $X$ and $Y$ and $\Lambda \subset \text{dom}(f)$. Assume that $X$ is equipped with a uniform $\delta$-cone-field on $\Lambda$ and $Y$ is equipped with a uniform $\delta$-cone-field on a subset $Z$ of $Y$ such that $f(\Lambda) \subset Z$.

For $\delta > 0$ and every $x \in \text{dom}(f)$ we put
\[ B_f(x, \delta) := \{v \in B(x, \delta) \cap \text{dom}(f) : f(v) \in B(f(x), \delta)\}. \]

We define $u_x(f; \delta)$ and $s_x(f; \delta)$, the expansion and the contraction rates of $f$, respectively (for more information see [19] [21]).

**Definition 2.2** ([21] Definition 3.2]). Let $\delta > 0$ and $x \in \text{dom}(f)$ be given. We define
\[
\begin{align*}
u_x(f; \delta) & := \sup\{R \in [0, \infty] : c(f(x), f(v)) \geq Rc(x, v), v \in B_f(x, \delta); v \in C^u_s(\delta)\}, \\
s_x(f; \delta) & := \inf\{R \in [0, \infty] : c(f(x), f(v)) \leq Rc(x, v), v \in B_f(x, \delta); f(v) \in C^s_u(\delta)\}.
\end{align*}
\]
Let $u_\Lambda(f; \delta) := \inf_{x \in \Lambda} \{u_x(f; \delta)\}$ and $s_\Lambda(f; \delta) := \sup_{x \in \Lambda} \{s_x(f; \delta)\}$.

These rates are a modification of the classical definition from [19] but, unlike Newhouse, we do not assume that the function $f$ is invertible.

---

3 We say that $I$ is a subinterval of $Z$ if $[k, l] \cap Z \subset I$ for any $k, l \in I$. 

Definition 2.3 ([20, Definition 4]). We say that $f$ is $\delta$-cone-hyperbolic on $\Lambda$ if

$$s_\Lambda(f; \delta) < 1 < u_\Lambda(f; \delta).$$

Definition 2.4 ([20, Definition 1]). Let $N \in \mathbb{N}$, $\varepsilon > 0$ and $\alpha \in (0, 1)$ be given. We say that $f: X \to X$ is $(N, \varepsilon, \alpha)$-uniformly expansive on a set $\Lambda \subset X$ if for any two orbits $x: \{-N, \ldots, N\} \to \Lambda$, $\nu: \{-N, \ldots, N\} \to X$ we have

$$d_{\sup}(x, \nu) \leq \varepsilon \implies d(x_0, \nu_0) \leq \alpha d_{\sup}(x, \nu),$$

where

$$d_{\sup}(x, \nu) := \sup_{-N \leq n \leq N} d(x_n, \nu_n).$$

In [20] we showed that uniform expansivity implies classical expansivity. Moreover, we give a simple example which demonstrates that the opposite implication does not hold (see [20, Ex. 2]).

Theorem B ([20, Proposition 2]). Let $\varepsilon > 0, L > 1, \beta \in (0, 1)$ and let $(X, d)$ be a metric space. Let $\Lambda \subset X$ be given and $f: X \to X$ be an $L$-bilipschitz map such that $\Lambda_{\varepsilon} \subset \text{dom}(f) \cap \text{im}(f)$. Assume that $\Lambda$ is an invariant set for $f$ and that $f$ is $(1, \varepsilon, \beta)$-uniformly expansive on $\Lambda$. Then

$$c_{\varepsilon}(x, v) := \inf \{ d(f^k(x), f^k(v)) \mid k \in \mathbb{Z}, k < 0 : f^l(v) \in B(f^l(x), \varepsilon) \text{ for } l \in [k, 0] \cap \mathbb{Z} \},$$

$$c_{\varepsilon}(x, v) := \inf \{ d(f^k(x), f^k(v)) \mid k \in \mathbb{Z}, k \geq 0 : f^l(v) \in B(f^l(x), \varepsilon) \text{ for } l \in [0, k] \cap \mathbb{Z} \},$$

define an $(L, \varepsilon/L)$ cone-field on $\Lambda$. Moreover, $f$ is cone-hyperbolic on $\Lambda$ and

$$s_\Lambda(f; \varepsilon/L) \leq \beta < \frac{1}{\beta} \leq u_\Lambda(f; \varepsilon/L).$$

This theorem plays a crucial role in the proof that for a uniformly expansive function $f$ there exists a Lyapunov function, see Section 5.

3. Approximately quasiconvex functions. Our aim in this section is to introduce an abstract class $\Phi(K, L, \varepsilon)$ of functions connected to approximately quasiconvex functions [23]. The importance of this class follows from the fact that the distance between trajectories $k \to d(f^k(x), f^k(y))$ for each pair of points $x, y$ belongs to this class. Consequently, by studying the properties of approximately quasiconvex functions we obtain more readable proofs (thanks to the simplified notation) of some basic properties of expansive systems.

For the rest of this section, let $\varepsilon > 0$ and $L > K > 1$. We put $[m, n]_{\mathbb{Z}} := [m, n] \cap \mathbb{Z}$ and $\overline{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$. 

Definition 3.1. Let $\Phi(K, L, \varepsilon)$ be a class of real nonnegative sequences $\alpha$ which satisfy the following properties:

i) $\text{dom}(\alpha) = [0, n]_{\mathbb{Z}}$ for a certain $n \in \overline{\mathbb{N}}, n \geq 1;$

ii) $\alpha(0) \in [0, \varepsilon], \forall k \in \text{dom}(\alpha) : \alpha(k) \leq \varepsilon \Rightarrow k + 1 \in \text{dom}(\alpha);$ 

iii) $\alpha(k) \leq \frac{1}{K} \max\{\alpha(k - 1), \alpha(k + 1)\}$ for $k : k - 1, k + 1 \in \text{dom}(\alpha);$ 

iv) $\frac{1}{L} \alpha(k) \leq \alpha(k + 1) \leq L \alpha(k)$ for $k : k, k + 1 \in \text{dom}(\alpha).$
For $\varepsilon > 0$ we put $\Delta_{\varepsilon}(\Lambda) := \{(x, v) : x \in \Lambda, v \in B(x, \varepsilon)\}$. For $f : \Lambda_\varepsilon \to X$ we define
\[
\text{dom}(\alpha(x,v)) : \quad = \left\{0, n + 1\right\} : n = \sup \left\{k \in \mathbb{N} : f^l(x), f^l(v) \in \text{dom}(f) \text{ for } l \in [0, k]\right\},
\]
and
\[
\alpha(x,v) := \{(d(f^k(x), f^k(v))) | k \in \text{dom}(\alpha(x,v))\}.
\]

In what follows we will usually omit $(x, v)$ at $\alpha(x,v)$. By $d((x, v); (x', v'))$ we denote the standard metric on the Cartesian product $X \times X$:
\[
d((x, v); (x', v')) = d(x, x') + d(v, v').
\]

Now we will show that the sequences $d(f^k(x), f^k(v))$ for $k \in \text{dom}(\alpha)$ belong to the class defined above.

**Observation 3.2.** Let $\varepsilon > 0$ and $L > K > 1$ be given. Let $f : \Lambda_\varepsilon \to X$ be an $L$-biLipschitz map. Assume that $\Lambda$ is an invariant set for $f$ and that $f$ is $(1, \varepsilon, \frac{1}{K})$-uniformly expansive on $\Lambda$. Then
\[
\alpha(x,v) \in \Phi(K, L, \varepsilon) \quad \text{for } (x, v) \in \Delta_{\varepsilon}(\Lambda).
\]

**Proof.** Let $(x, v) \in \Delta_{\varepsilon}(\Lambda)$. Property [iv] in Definition 3.1 is obvious and [iii] follows from uniform expansiveness. Moreover, we have $\{-1, 0, 1\} \subset \text{dom} \alpha(x,v)$. For the proof of [iii] and [iv] we can assume that $x \neq v$, because for $x = v$ these properties hold trivially. Since $f$ is $L$-biLipschitz, we know that property [iv] holds and [iii] follows from $(1, \varepsilon, \frac{1}{K})$ uniform expansiveness of $f$. \qed

**Observation 3.3.** Let $(x, v), (x', v') \in \Delta_{\varepsilon}(\Lambda)$ and $f : \Lambda_\varepsilon \to X$ be Lipschitz continuous with a Lipschitz constant $L$. Then
\[
|\alpha(x,v)(k) - \alpha(x',v')(k)| \leq L^k \cdot d((x, v); (x', v')) \quad \text{for } k \in \text{dom} \alpha \cap \text{dom} \alpha'.
\]

**Proof.** Taking any $k \in \text{dom} \alpha \cap \text{dom} \alpha'$, we have
\[
|\alpha(x,v)(k) - \alpha(x',v')(k)| = |d(f^k(x), f^k(v)) - d(f^k(x'), f^k(v'))| \\
\leq d(f^k(x), f^k(x')) + d(f^k(v), f^k(v')) \\
\leq L^k \cdot d((x, v); (x', v')) \\
= L^k \cdot d((x, v); (x', v')),
\]
where $(\ast)$ follows from the well-known inequality
\[
|d(a, b) - d(a', b')| \leq d(a, a') + d(b, b').
\]

Observe that if we assume that $f : \Lambda_\varepsilon \to X$ is $L$-biLipschitz, then we get the same inequality as above for $f^{-1}$. Now we recall the notion of $\varepsilon$-quasiconvexity.

**Definition 3.4** (Def 5). Let $I$ be a subinterval of $\mathbb{Z}$ and let $\alpha : I \to \mathbb{R}$ be a given sequence. We say that $\alpha$ is $\varepsilon$-quasiconvex if
\[
\alpha(n) \leq \max\{\alpha(n-1), \alpha(n+1)\} - \varepsilon \quad \text{for } n - 1, n, n + 1 \in I.
\]

For each $\varepsilon$-quasiconvex sequence there must be a point before which the sequence decreases and after which it grows with speed at least $\varepsilon$. 


Observation 3.5. Let $\alpha : I \to \mathbb{R}$ be a $\varepsilon$-quasiconvex sequence. Then there exists $n_0 \in \mathbb{Z} \cup \{-\infty, +\infty\}$ such that
\[
\alpha(n+1) \leq \alpha(n) - \varepsilon, \text{ for } n < n_0 \text{ such that } n, n+1 \in I,
\] (3)
and
\[
\alpha(n+1) \geq \alpha(n) + \varepsilon, \text{ for } n \geq n_0 + 1 \text{ such that } n, n+1 \in I.
\] (4)

Proof. Let $n_0 := \sup\{n \in I : \alpha(n) + \varepsilon \leq \alpha(n-1)\}$. Condition (3) automatically holds. If $n_0 \geq \sup I - 1$ or $n_0 = -\infty$ then (4) is also trivially satisfied. Suppose that $-\infty < n_0 < \sup I - 1$. Then $\alpha(n_0 + 1) > \alpha(n_0) - \varepsilon$. By quasiconvexity we obtain
\[
\alpha(n_0 + 1) + \varepsilon \leq \max\{\alpha(n_0), \alpha(n_0 + 2)\} = \alpha(n_0 + 2).
\]
Proceeding in an analogous manner, we obtain (4). \square

Remark 1. From property (iii) of Definition 3.1 and Observation 3.5 we know that there exists $k_0 \in \mathbb{Z} \cup \{-\infty, +\infty\}$ such that:
\[
\alpha(k+1) \leq \frac{1}{K} \cdot \alpha(k) \text{ for } k < k_0,
\] (5)
and
\[
\alpha(k+1) \geq K \cdot \alpha(k) \text{ for } k \geq k_0 + 1.
\] (6)

Observe that we can express $c_u$, which was defined in Theorem B, by the following formula:
\[
c_u(x,v) = \inf\{\alpha_{(x,v)} \in [0,\varepsilon] \text{ for } (x,v) \in \Delta_{\varepsilon}(\Lambda).
\] (7)
Having obtained basic properties of the space $\Phi(K,L,\varepsilon)$, we can move on to the next section.

4. Proof of the Hölder continuity. Our aim in this section is to show that the functions $c_u$, $c_s$ constructed in Theorem B are Hölder continuous. This property will play a crucial role in the proof of our main result (see Section 5).

In this part we show that the function $c_u$ defined in (7) is locally Hölder continuous. Let
\[
\text{dom}_\varepsilon \alpha := \{k \in \text{dom } \alpha : \alpha(k) < \varepsilon\},
\]
and
\[
\alpha_{\geq l} := \{\alpha(k) : k \in \text{dom } \alpha, k \geq l\}.
\]

Remark 1 guarantees that it suffices to consider only four cases:
\begin{itemize}
  \item I, which is described in Figure 1(a)
  \item II, which is described in Figure 1(b)
  \item III, which is described in Figure 1(c)
  \item IV, which is described in Figure 1(d)
\end{itemize}
(a) Both sequences are increasing on $[0, n]_{\mathbb{Z}}$.

(b) Both sequences are decreasing on $[0, n]_{\mathbb{Z}}$.

(c) For some $l \in [0, n]_{\mathbb{Z}}$ both sequences are increasing on $[l, n]_{\mathbb{Z}}$.

(d) One sequence is increasing on $[l, n]_{\mathbb{Z}}$ for some $l \in [0, n]_{\mathbb{Z}}$, and the other sequence is decreasing on $[0, n]_{\mathbb{Z}}$.

Figure 1. Sequences $\alpha, \alpha' \in \Phi(K, L, \varepsilon)$ restricted to the set $\text{dom}_\varepsilon(\alpha) \cap \text{dom}_\varepsilon(\alpha') = [0, n]_{\mathbb{Z}}, n \in \mathbb{N}$.

All possible behaviours in the set $\text{dom}_\varepsilon(\alpha) \cap \text{dom}_\varepsilon(\alpha')$ of sequences $\alpha, \alpha' \in \Phi(K, L, \varepsilon)$ that we must consider are shown in Figure 1. We will consider them separately. The following lemma concerns the situation described in Figure 1(a).

**Lemma 4.1.** Let $\alpha, \alpha' \in \Phi(K, L, \varepsilon)$ and $n \in \mathbb{N}$. Assume that $n \in \text{dom}_\varepsilon(\alpha) \cap \text{dom}_\varepsilon(\alpha')$ and $\alpha, \alpha'$ are increasing on $[0, n]_{\mathbb{Z}}$. Then

$$|\inf(\alpha) - \inf(\alpha')| = |\alpha(0) - \alpha'(0)|.$$

**Proof.** It is easy to see that

$$\inf(\alpha) = \alpha(0) \quad \text{and} \quad \inf(\alpha') = \alpha'(0).$$

Hence $|\inf(\alpha) - \inf(\alpha')| = |\alpha(0) - \alpha'(0)|$. \qed

Next, we consider the situation described in Figure 1(b).

**Lemma 4.2.** Let $\alpha, \alpha' \in \Phi(K, L, \varepsilon)$ and $n \in \mathbb{N}$. Assume that $n \in \text{dom}_\varepsilon(\alpha) \cap \text{dom}_\varepsilon(\alpha')$ and $\alpha, \alpha'$ are decreasing on $[0, n]_{\mathbb{Z}}$. Then

$$|\inf(\alpha) - \inf(\alpha')| \leq \frac{\varepsilon}{K^n}.$$

**Proof.** Clearly, we have

$$\inf(\alpha) \in [0, \alpha(n)], \quad \inf(\alpha') \in [0, \alpha'(n)]$$
and, by Remark 3.1 and Definition 3.1 (ii) we get
\[
\alpha(N) \leq \frac{\varepsilon}{K^n}, \quad \alpha'(n) \leq \frac{\varepsilon}{K^n}.
\]
Hence \( \inf(\alpha), \inf(\alpha') \in [0, \frac{\varepsilon}{K^n}] \), which implies
\[
|\inf(\alpha) - \inf(\alpha')| \leq \frac{\varepsilon}{K^n}.
\]

Now we move on to the case described in Figure 1(c). Note that in this case Property (iii) from Definition 3.1, which determines the speed of the increase or decrease of sequences, does not imply the desired result. Instead, we use the relationship between the elements of the sequence.

**Lemma 4.3.** Let \( \alpha, \alpha' \in \Phi(K, L, \varepsilon) \) and \( n \in \mathbb{N} \). Assume that there exists \( l \in \text{dom}_x(\alpha) \cap \text{dom}_x(\alpha') = [0, n] \) such that \( \alpha, \alpha' \) are increasing on \([l, n] \) and
\[
|\alpha(k) - \alpha'(k)| \leq L^k \cdot |\alpha(0) - \alpha'(0)| \quad \text{for} \quad k \in [0, n] \mathbb{Z}.
\] (8)

Then
\[
|\inf(\alpha) - \inf(\alpha')| \leq L^l \cdot |\alpha(0) - \alpha'(0)|.
\]

**Proof.** Clearly, we have
\[
|\inf(\alpha) - \inf(\alpha')| = \left| \inf_k \{\alpha(k) - \alpha'(k)\} \right| \leq \sup_{k \in [0, l]} |\alpha(k) - \alpha'(k)| \leq \sup_{k \in [0, l]} L^k \cdot |\alpha(0) - \alpha'(0)| \leq L^l \cdot |\alpha(0) - \alpha'(0)|.
\]

Finally, we consider the situation illustrated by Figure 1(d).

**Lemma 4.4.** Let \( \alpha, \alpha' \in \Phi(K, L, \varepsilon) \). Let \( n \) be such that \([0, n] = \text{dom}_x(\alpha) \cap \text{dom}_x(\alpha') \). Assume that \( \alpha' \) is decreasing on \([0, n] \), \( \alpha \) is increasing on \([l, n] \) for some \( l \in [0, n] \), and
\[
|\alpha(k) - \alpha'(k)| \leq L^k \cdot |\alpha(0) - \alpha'(0)| \quad \text{for} \quad k \in [0, n] \mathbb{Z}.
\] (9)

Then
\[
|\inf(\alpha) - \inf(\alpha')| \leq L^n \cdot |\alpha(0) - \alpha'(0)| + \frac{\varepsilon}{K^n}.
\]

**Proof.** It is clear that
\[
\inf(\alpha) \leq \alpha(n) \quad \text{and} \quad \inf(\alpha') \leq \alpha'(n),
\]
so we have
\[
|\inf(\alpha) - \inf(\alpha')| \leq \max\{\alpha(n), \alpha'(n)\}.
\]
We will consider two cases: \( \max\{\alpha(n), \alpha'(n)\} = \alpha(n) \) and \( \max\{\alpha(n), \alpha'(n)\} = \alpha'(n) \). Suppose that \( \max\{\alpha(n), \alpha'(n)\} = \alpha'(n) \). We use the fact that \( \alpha' \) is decreasing on \([0, n] \mathbb{Z} \) and Definition 3.1 (ii). We have
\[
\alpha'(n) \leq \frac{\alpha'(0)}{K^n} \leq \frac{\varepsilon}{K^n}.
\]

Suppose that \( \max\{\alpha(n), \alpha'(n)\} = \alpha(n) \). Since \( \alpha' \) is decreasing on \([0, n] \mathbb{Z} \), it follows from (9) that we get
\[
\alpha(n) = |\alpha(n) \pm \alpha'(n)| \leq |\alpha(n) - \alpha'(n)| + \alpha'(n) \leq L^n \cdot |\alpha(0) - \alpha'(0)| + \frac{\varepsilon}{K^n}.
\]
Finally, we obtain
\[ |\inf(\alpha) - \inf(\alpha')| \leq L^n \cdot |\alpha(0) - \alpha'(0)| + \frac{\varepsilon}{K^n}. \]

For the rest of this section let \( \delta := |\alpha(0) - \alpha'(0)|. \) Let \( \hat{n} \) stand for the minimal number of iterations of the sequence \( \alpha \) such that \( [0, \hat{n}] \subset \text{dom}_L(\alpha) \). We will now show that in Case IV (Figure 1(d)) and under some additional assumptions \( \hat{n} \) can be estimated from below.

**Proposition 1.** Let \( \alpha, \alpha' \in \Phi(K, L, \varepsilon) \) and \( \delta \leq \varepsilon \cdot \min \left\{ \frac{K-1}{KL}, \frac{1}{KL}, 2^{-\log_K KL} \right\} \). Assume that \( \alpha' \) is decreasing on \([0, n]_Z \) and
\[ |\alpha(k) - \alpha'(k)| \leq L^k \cdot \delta \quad \text{for} \quad k \in [0, n]_Z \cap \text{dom}(\alpha). \]

Then
\[ \alpha(k) \in [0, \varepsilon] \quad \text{for} \quad k \leq \log_{KL} \left( \frac{\varepsilon}{\delta} \right). \]

**Proof.** The case of \( k = 0 \) is trivial (see Definition 3.1 (ii)). Assume that \( k \geq 1 \). If \( \alpha \) is decreasing on \([0, n]_Z \), the claim obviously holds. Let us thus suppose that \( \alpha \) is not decreasing.

It follows easily that \( \alpha(k) \in [0, \varepsilon] \) for \( k \in \text{dom}(\alpha') \cap [0, n]_Z \), since we have
\[
\alpha(k) = |\alpha(k) - \alpha'(k) + \alpha'(k)| \\
\leq |\alpha(k) - \alpha'(k)| + \alpha'(k) \\
\leq L^k \cdot \delta + \frac{\varepsilon}{K^L} \quad \text{for} \quad k \in [0, n]_Z.
\]

Consider the following function
\[
F: \left[ 1, \log_{KL} \left( \frac{\varepsilon}{\delta} \right) \right] \ni x \mapsto L^x \cdot \delta + \frac{\varepsilon}{K^x} \in [0, \infty).
\]
Since it is continuous and convex, it suffices to check whether the value of \( F \) does not exceed at the endpoints of its domain, i.e.: \( F(1) \leq \varepsilon, F \left( \log_{KL} \left( \frac{\varepsilon}{\delta} \right) \right) \leq \varepsilon \). Observe that for \( \delta \leq \frac{\varepsilon}{K^L} \) we have \( \log_{KL} \left( \frac{\varepsilon}{\delta} \right) \geq 1 \).

Since \( \delta \leq \frac{\varepsilon}{K^L} \cdot \varepsilon \), we (easily) get
\[
F(1) = L \delta + \frac{\varepsilon}{K} \leq \varepsilon.
\]

We will now show that \( F(\hat{x}) \leq \varepsilon \) for \( \hat{x} := \log_{KL} \left( \frac{\varepsilon}{\delta} \right) \). We have
\[
(K^{\hat{x}})^{\log_K KL} = (KL)^{\hat{x}} = (KL)^{\log_{KL} \left( \frac{\varepsilon}{\delta} \right)} = \frac{\varepsilon}{\delta}, \quad \text{which implies} \quad K^{\hat{x}} = \left( \frac{\varepsilon}{\delta} \right)^{\log_K KL}.
\]
Moreover
\[
L^{\log_{KL} \left( \frac{\varepsilon}{\delta} \right) \cdot \delta} = L^{\log_{KL} \left( \frac{\varepsilon}{\delta} \right) \cdot \log_{KL} L} = \left( \frac{\varepsilon}{\delta} \right)^{\log_K KL \cdot \delta} = \frac{\varepsilon}{K^{\hat{x}}},
\]
As we also have \( \delta \leq 2^{-\log_K KL \varepsilon} \), we finally conclude that
\[
F(\hat{x}) = L^x \cdot \delta + \frac{\varepsilon}{K^x} + \frac{2 \varepsilon}{K^x} + \frac{\delta}{\varepsilon} \cdot \left( \frac{\varepsilon}{\delta} \right)^{\log_K KL} = 2 \varepsilon \cdot \log_{KL} L \cdot \delta^{\log_{KL} K} \leq \varepsilon.
\]

The following theorem shows that for a certain set the difference of the infima of sequences \( \alpha, \alpha' \in \Phi(K, L, \varepsilon) \) satisfies the Hölder condition. This property allows us to show that the function \( c_u \) defined by (2) is locally Hölder continuous.
Theorem 4.5. Let \( \alpha, \alpha' \in \Phi(K,L,\varepsilon) \) and \( \delta \leq \varepsilon \cdot \min \left\{ \frac{K-1}{KL}, \frac{1}{KL}, 2^{-\log KL} \right\} \) be such that \( |\alpha_k - \alpha'_k| \leq L^k \cdot \delta \) for \( k \in \text{dom}_\varepsilon(\alpha) \cap \text{dom}_\varepsilon(\alpha') \). Then

\[
|\inf(\alpha) - \inf(\alpha')| \leq C \cdot \delta^\gamma,
\]

where \( C = 2K \varepsilon^{\log KL} L, \gamma = \log KL K \).

Proof. Similarly as before, we have to consider four cases, which are described in Figure 1. First we discuss Case IV, which is illustrated in Figure 1(d). By Proposition 1 we can take 

\[
n = \left\lfloor \log KL \left(\frac{\varepsilon}{\delta}\right) \right\rfloor \geq 1.
\]

Let \( \tilde{n} = \left\lfloor \log KL \left(\frac{\varepsilon}{\delta}\right) \right\rfloor \). From Lemma 4.4 we know that

\[
|\inf(\alpha) - \inf(\alpha')| \leq L^n \cdot \delta + \varepsilon K^n.
\]

We have

\[
\tilde{n} > \log KL \left(\frac{\varepsilon}{\delta}\right) - 1,
\]

\[
(K^{\tilde{n}})^{\log KL} = (KL)^{\tilde{n}} > (KL)^{\log KL \left(\frac{\varepsilon}{\delta}\right) - 1} = \frac{\varepsilon}{\delta KL},
\]

\[
K^{\tilde{n}} > \left(\frac{\varepsilon}{\delta KL}\right)^{\log KL K},
\]

\[
1 - K^{\tilde{n}} < \left(\frac{\delta KL}{\varepsilon}\right)^{\log KL K}.
\]

Finally, we obtain

\[
|\inf(\alpha) - \inf(\alpha')| \leq L^{\tilde{n}} \cdot \delta + \frac{\varepsilon}{K^{\tilde{n}}} \leq 2\varepsilon \left(\frac{\delta KL}{\varepsilon}\right)^{\log KL K} = 2K \varepsilon^{\log KL} L \cdot \delta^{\log KL} K.
\]

Case I (Figure 1(a)) is obvious, because \( \tilde{n} \geq 1 \) and from Lemma 4.1 we have

\[
|\inf(\alpha) - \inf(\alpha')| = \delta \leq L^{\tilde{n}} \cdot \delta + \frac{\varepsilon}{K^{\tilde{n}}} \leq 2K \varepsilon^{\log KL} L \cdot \delta^{\log KL} K.
\]

For Case II (Figure 1(b)), it follows from Lemma 4.1 that we know

\[
|\inf(\alpha) - \inf(\alpha')| \leq \frac{\varepsilon}{K^{\tilde{n}}} \quad \text{for} \quad \text{dom}_\varepsilon(\alpha) \cap \text{dom}_\varepsilon(\alpha') = [0,n]
\]

and, since \( \tilde{n} \leq n \), we obtain

\[
|\inf(\alpha) - \inf(\alpha')| \leq \frac{\varepsilon}{K^{\tilde{n}}} \leq \frac{\varepsilon}{K^{n}} \leq L^n \cdot \delta + \frac{\varepsilon}{K^{n}} \leq 2K \varepsilon^{\log KL} L \cdot \delta^{\log KL} K.
\]

Let us move on to Case III (Figure 1(c)). Figure 2 shows possible subcases.
Figure 2. All possible situations in Case III (see Figure 1(c)) depending on the position of $\tilde{n}$, where $\text{dom}(\alpha) \cap \text{dom}(\alpha') = [0, n]_Z$.

The cases shown in Figure 2(a)-2(b) have already been studied (see Lemmas 4.2 and 4.4), so we only have to consider the situation depicted in Figure 2(c). It follows from Lemma 4.3 that

$$|\inf(\alpha) - \inf(\alpha')| \leq L^l \cdot \delta$$ for some $l \in [0, \tilde{n}]_Z$,

hence

$$|\inf(\alpha) - \inf(\alpha')| \leq L^l \cdot \delta \leq L^{\tilde{n}} \cdot \delta \leq L^{\tilde{n}} \cdot \delta \leq \frac{\varepsilon}{K^{\tilde{n}}} \leq 2K^{\varepsilon \log_{KL} L} \cdot \delta \log_{KL} K.$$

As an obvious consequence of this theorem we have the following corollaries.

**Corollary 1.** Let $\varepsilon > 0, L > K > 1$ and $\Lambda \subset X$ be given and $f: X \to X$ be an $L$-bilipschitz map such that $\Lambda \subset \text{dom}(f) \cap \text{im}(f)$. Assume that $\Lambda$ is an invariant set for $f$ and that $f$ is $(1, \varepsilon, \frac{1}{K})$-uniformly expansive on $\Lambda$. Then the function $c_u$ constructed in (2) is locally Hölder continuous with coefficients $C = 2K^{\varepsilon \log_{KL} L}$, $\gamma = \log_{KL} K$.

**Proof.** The claim follows directly form Observations 3.2, 3.3 and Theorem 4.5.

The following corollary shows that we get a similar result for the function $c_s$ by using the same tools.
Corollary 2. Let the assumptions be as in Corollary \[\text{1}\]. Then the function \(c_u\) constructed in \[\text{2}\] is locally Hölder continuous with coefficients \(C = 2K^{\log K_L} KL^2 \varepsilon^{\log K_L} L\), \(\gamma = \log K_L K\).

Proof. The difference between \(c_u\) and \(c_s\) in terms of \(\alpha\) is that in the former case the function \(f\) is iterated forward and \(\alpha\) starts from \(\alpha(0) = d(x, v)\), while in the latter the function \(f\) is iterated backward and \(\alpha\) is defined as \(\alpha(k) := d(f^{-k-1}(x), f^{-k-1}(v))\) for \(k \in \text{dom}(\alpha)\).

Let us consider \(\alpha := \{d(f^{-k-1}(x), f^{-k-1}(v))\}_{k \in \text{dom}(\alpha)}\). Since the function \(f\) is \(L\)-bilipschitz, we get
\[
d \left( (f^{-1}(x), f^{-1}(v)), (f^{-1}(x'), f^{-1}(v')) \right) \leq L \cdot d((x, v), (x', v')).
\]
We put \(\delta = d((x, v), (x', v'))\) and, by Theorem \[\text{4.5}\], we have:
\[
|c_u(x, v) - c_u(x', v')| \leq 2K^{\log K_L} L \cdot (Ld)^{\log K_L} K
\]
\[
\leq 2K^{\log K_L} L^{\log K_L} K \cdot \delta^{\log K_L} K
\]
\[
\leq 2K^{\log K_L} KL^2 \varepsilon^{\log K_L} L \cdot \delta^{\log K_L} K.
\]

\[\square\]

5. Construction of Lyapunov function. As we have already mentioned, the concept of a Lyapunov function is a very useful tool in the theory of dynamical systems. Finding suitable Lyapunov functions is difficult because there is no general method to construct a Lyapunov (candidate) function \[\text{13}\].

In this section we show how to construct a Lyapunov function for \(f\) by using functions \(c_u\) and \(c_s\), which are defined in \[\text{2}\]. We recall that the definition of a generalized Lyapunov function was given in the Introduction, see Definition \[\text{1.1}\].

Theorem 5.1. Let \(\varepsilon > 0, L > 1, \beta < 1\) be given and let \((X, d)\) be a metric space. Let \(f: \Lambda_{\varepsilon} \to X\) be an \(L\)-bilipschitz map. Assume that \(\Lambda\) is an invariant set for \(f\) and \(f\) is \((1, \varepsilon, \beta)\)-uniformly expansive on \(\Lambda\). Let \(c_u\) and \(c_s\) be defined as in \[\text{2}\] and denote
\[
V(x, v) := c_u(x, v) - c_s(x, v).
\]
Then \(V\) is a locally Hölder continuous Lyapunov function and
\[
\Delta_f V(x, v) \geq \min \left( \frac{u_\Lambda(f; \delta) - 1}{L}, \frac{1 - s_\Lambda(f; \delta)}{L^2 \cdot s_\Lambda(f; \delta)} \right) \cdot d(x, v).
\]
(15)

Proof. From Theorem \[\text{13}\] we deduce that \(c_u(x, v), c_s(x, v)\) define an \((L, \varepsilon/L)\) cone-field on \(\Lambda\) such that \(f\) is cone-hyperbolic on \(\Lambda\). It follows from local Hölder continuity of \(c_u\) and \(c_s\), that \(V\) is also locally Hölder continuous. From definitions \(c_u\) and \(c_s\), we conclude that \(V(x, v) = 0\) for \(x = v\). Thus we assume that \(x \neq v\) and we will show that \(\Delta_f V(x, v) > 0\).

Let \(x \in \Lambda\) and \(v \in B_f(x, \varepsilon/L)\) such that \(v \in C_x^u\). We have
\[
\Delta_f V(x, y) = c_u(f(x), f(v)) - c_s(f(x), f(v)) - c_u(x, v) + c_s(x, v)
\]
\[
\geq u_\Lambda(f; \delta) \cdot c_u(x, v) - c_s(f(x), f(v)) - c_u(x, v) + c_s(x, v)
\]
\[
\geq (u_\Lambda(f; \delta) - 1) \cdot c_u(x, v) = (u_\Lambda(f; \delta) - 1) \cdot c(x, v)
\]
\[
\geq \frac{1}{L} \cdot (u_\Lambda(f; \delta) - 1) \cdot d(x, v) \quad \text{\(f\) cone-hyperbolic}
\]
\[\geq 0.\]
Let $x \in \Lambda$ and $v \in B_f(x, \varepsilon/L)$ be such that $f(v) \in C^s_f(x)$. We get

$$
\Delta_f V(x, y) = c_u(f(x), f(v)) - c_s(f(x), f(v)) - c_u(x, v) + c_s(x, v)
+ \frac{1}{sA(f; \delta)} \cdot c_s(f(x), f(v))
\geq c_u(f(x), f(v)) - c_s(f(x), f(v)) - c_u(x, v)
\geq \left( \frac{1}{sA(f; \delta)} - 1 \right) \cdot c_s(f(x), f(v))
= \left( \frac{1}{sA(f; \delta)} - 1 \right) \cdot c(f(x), f(v))
$$

by (1)

$$
f \text{ L-bilipschitz } \Rightarrow \left[ \frac{1}{L^2} \cdot \left( \frac{1}{sA(f; \delta)} - 1 \right) \cdot d(x, v) \right] \text{ cone-hyperbolic } \geq 0,
$$

which concludes the proof. $\square$

From the previous results we conclude that if $f$ is uniformly expansive, then there exists a Lyapunov function for $f$. We ask whether the opposite implication is true.

**Problem 1.** Let $\varepsilon > 0$ and $L > 1$ be given. Let $f : \Lambda_\varepsilon \to X$ be an L-bilipschitz map.

Does the existence of a Lyapunov function for $f$ satisfying $\Delta_f V(x, y) \geq C d(x, y)$ for $x, y \in X$ and certain $C > 0$ imply that $f$ is uniformly expansive on $\Lambda$?

**REFERENCES**

[1] E. Akin, *The General Topology of Dynamical Systems*, volume 1 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1993.

[2] D. Angeli, A Lyapunov approach to the incremental stability properties. *IEEE Trans. Automat. Control*, **47** (2002), 410–421.

[3] J. P. Aubin, Mutational equations in metric spaces. *Set-Valued Analysis*, **1** (1993), 3–46.

[4] F. Forni and R. Sepulchre, A differential lyapunov framework for contraction analysis. *IEEE Trans. Automat. Control*, **59** (2014), 614–628.

[5] P. Giesl and S. Hafstein, Review on computational methods for Lyapunov functions. *Discrete and Continuous Dynamical Systems-Series B*, **20** (2015), 2291–2331.

[6] J. Harjani and K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations. *Nonlinear Analysis: Theory, Methods & Applications*, **72** (2010), 1188–1197.

[7] T. Kaczynski, K. Mischaikow and M. Mrozek, *Computational Homology*, volume 157, Springer-Verlag, New York, 2004.

[8] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, volume 54. Cambridge University Press, Cambridge, 1995.

[9] V. Lakshmikantham, T. G. Bhaskar and J. V. Devi, *Theory of Set Differential Equations in Metric Spaces*, volume 54. Cambridge University Press, Cambridge, 2006.

[10] J. Lewowicz, Lyapunov functions and topological stability. *Journal of Differential Equations*, **38** (1980), 192–209.

[11] J. Lewowicz, Persistence of semi-trajectories. *Journal of Dynamics and Differential Equations*, **18** (2006), 1095–1102.

[12] W. Lohmiller and J.-J. E. Slotine, On contraction analysis for non-linear systems. *Automatica J. IFAC*, **34** (1998), 683–686.

[13] A. M. Lyapunov, The general problem of the stability of motion. *International Journal of Control*, **55** (1992), 521–790.

[14] M. Malisoff and F. Mazenc, *Constructions of Strict Lyapunov Functions* Springer Science & Business Media, 2009.
EXPANSIVITY IMPLIES EXISTENCE

[15] M. Mazur, On the relationship between hyperbolic and cone-hyperbolic structures in metric spaces. *Annales Polonici Mathematici*, 109 (2013), 29–38.

[16] M. Mazur and J. Tabor, Computational hyperbolicity. *Discrete and Continuous Dynamical Systems (DCDS-A)*, 29 (2011), 1175–1189.

[17] C. C. McCluskey, Using Lyapunov functions to construct Lyapunov functionals for delay differential equations. *SIAM Journal on Applied Dynamical Systems*, 14 (2015), 1–24.

[18] M. Mrozek, Topological dynamics: Rigorous numerics in cubical homology. In *Advances in Applied and Computational Topology: Proc. Symp. Amer. Math. Soc*, volume 70, pages 41–73. American Mathematical Society, Providence, RI, 2012.

[19] S. Newhouse, Cone-fields, domination, and hyperbolicity. *Modern dynamical systems and applications*, (2004), 419–432.

[20] L. Struski and J. Tabor, Expansivity and cone-fields in metric spaces. *Journal of Dynamics and Differential Equations*, 26 (2014), 517–527.

[21] L. Struski, J. Tabor and T. Kulaga, Cone-fields without constant orbit core dimension. *Discrete and Continuous Dynamical Systems*, 32 (2012), 3651–3664.

[22] J. Tabor, Differential equations in metric spaces. *Proceedings of Equadiff 10*, 127 (2002), 353–360.

[23] J. Tabor, Jó. Tabor and M. Złodak, On ω-strongly quasiconvex and ω-strongly quasiconcave sequences. *Aequationes mathematicae*, 82 (2011), 255–256.

[24] J. Tolosa, The method of Lyapunov functions of two variables. *Contemporary Mathematics*, 440 (2007), 243–271.

[25] D. Wilczak and P. Zgliczyński, Computer assisted proof of the existence of homoclinic tangency for the Hénon map and for the forced damped pendulum. *SIAM Journal on Applied Dynamical Systems*, 8 (2009), 1632–1663.

Received August 2016; revised May 2017.

E-mail address: lukasz.struski@im.uj.edu.pl
E-mail address: jacek.tabor@ii.uj.edu.pl