ON FOUR STATE HARD CORE MODELS ON THE CAYLEY TREE

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ABSTRACT. We consider a nearest-neighbor four state hard-core (HC) model on the homogeneous Cayley tree of order $k$. The Hamiltonian of the model is considered on a set of “admissible” configurations. Admissibility is specified through a graph with four vertices. We first exhibit conditions (on the graph and on the parameters) under which the model has a unique Gibbs measure. Next we turn on some specific cases. Namely, first we study, in case of particular graph (diamond), translation-invariant and periodic Gibbs measures. We provide in both cases the equations of the transition lines separating uniqueness from non-uniqueness regimes. Finally the same is done for “fertile” graphs, the so-called stick, gun, and key (here only translation invariant states are taken into account).

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1. Introduction and definitions

Hard core constraints arise in fields as diverse as combinatorics, statistical mechanics and telecommunications. In particular, hard core models arise in the study of random independent sets of graphs \cite{3}, \cite{4}, the study of gas molecules on a lattice \cite{1}, in the analysis of multi-casting in telecommunication networks(see e.g. \cite{7}, \cite{8}, \cite{11}).

We refer the reader to the nice article by Brightwell and Winkler \cite{2} on the subject, and to \cite{3} focusing on Hard core models on the Bethe lattice (Cayley tree).

Let $\Gamma^k = (V, L)$ be the uniform Cayley tree, where each vertex has $k + 1$ neighbors with $V$ being the set of vertices and $L$ the set of bonds.

We denote by $\Phi = \{0, 1, 2, 3\}$ the values of the spins $\sigma(x)$ sitting on vertices may assume. A configuration on the Cayley tree is a collection $\sigma = \{\sigma(x), x \in V\} \in \Phi^V$.

Consider a given subset $\mathcal{G}$ of pairs $(i, j) \in \Phi \times \Phi$.

A configuration will be called admissible with respect to $\mathcal{G}$ if $(\sigma(x), \sigma(y)) \in \mathcal{G}$, for any pair of nearest neighbors $x$ and $y$. For a given set $\mathcal{G}$ we denote by $\Omega$, the set of admissible configurations, and by $\Omega_A$ its restriction to a subset $A$ of $V$. 
The Hamiltonian of the model is defined through a matrix

\[ P = \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} \end{pmatrix}, \]

where \( P_{i,j} > 0 \), if \((i, j) \in G\); \( P_{i,j} = 0 \) if \((i, j) \notin G\) and \( \sum_{j \in \Phi} P_{i,j} = 1 \).

Namely, given \( G \) and \( P \), we define the HC Hamiltonian by

\[ H(\sigma) = \begin{cases} \sum_{(x,y)} \log P_{\sigma(x),\sigma(y)}, & \text{if } \sigma \in \Omega, \\ +\infty, & \text{if } \sigma \notin \Omega. \end{cases} \] (1.1)

Note that \( G \) may be viewed as a directed graph and that the correspondence is one-to-one (see Fig. 1).

![Fig. 1. The directed graph associated with \( G = \{(0, 0), (0, 2), (1, 0), (1, 2), (2, 1), (2, 3), (3, 1), (3, 3)\} \) (one put arrow when only one direction is selected).](image)

In what follows each vertex of the graph under consideration is assumed to have positive indegree and outdegree. The outdegree (respectly indegree) of a vertex is the number of ingoing (respectly outgoing) edges. The indegree will be denoted \( \text{deg}^{-}(v) \) and the outdegree by \( \text{deg}^{+}(v) \).

The remaining of the paper is organized as follows. Gibbs measures of the model with the corresponding system of recursive equations are presented in Section 2. In Section 3 we provide conditions under which the model has unique Gibbs measure. Section 4 is devoted to the diamond graph. The results for fertile graphs are given in Section 5.

2. Gibbs measures and recursive equations

On the Cayley tree, there is a natural distance to be denoted \( d(x, y) \), being the number of nearest neighbors pairs of the minimal path between the vertices \( x \) and \( y \) (by path one means a collection of nearest neighbors pairs, two consecutive pairs sharing at least a given vertex).

For a fixed \( x^0 \in V \), the root, we let

\[ V_n = \{ x \in V \mid d(x, x^0) \leq n \} \]
be the ball of radius \( n \) and
\[ W_n = \{ x \in V \mid d(x^0, x) = n \} \]
be the sphere of radius \( n \) with center at \( x^0 \). We will write \( x < y \) if the path from \( x^0 \) to \( y \) goes through \( x \).

Let \( t : x \in V \mapsto t_x = (t_{i,x}, i \in \Phi) \in \mathbb{R}^4_+ \) be a vector-valued function on \( V \). Given \( n = 1, 2, \ldots \), consider the probability distribution \( \mu^{(n)} \) on \( \Omega_{V_n} \) defined by
\[ \mu^{(n)}(\sigma_n) = \frac{1}{Z_n} \prod_{(x,y) \subset V_n} P_{\sigma(x),\sigma(y)} \prod_{x \in W_n} t_{\sigma(x),x}. \tag{2.1} \]
where the first product runs over pairs of nearest neighbors of \( V_n \) and \( Z_n \) is the corresponding partition function.

We say that the probability distributions \( \mu^{(n)} \) are compatible if \( \forall n \geq 1 \) and \( \sigma_{n-1} \in \Omega_{V_{n-1}} \):
\[ \sum_{\omega_n \in \Omega_{W_n}} \mu^{(n)}(\sigma_{n-1}, \omega_n) = \mu^{(n-1)}(\sigma_{n-1}). \tag{2.2} \]

Such measures are usually called splitting Gibbs measures (see e.g. \cite{6,12,15}).

**Proposition 1.** The probability distributions \( \mu^{(n)}, n = 1, 2, \ldots, \) in (2.1) are compatible iff for any \( x \in V \) the following system of equations holds:
\[ z_{i,x} = \prod_{y \in S(x)} \frac{P_{i,0} + P_{i,1}z_{1,y} + P_{i,2}z_{2,y} + P_{i,3}z_{3,y}}{P_{0,0} + P_{0,1}z_{1,y} + P_{0,2}z_{2,y} + P_{0,3}z_{3,y}}. \tag{2.3} \]
where \( S(x) \) are the direct successors of \( x \) (the \( k \) nearest neighbors s.t. \( x < y \)) and \( z_{i,x} = t_{i,x}/t_{0,x}, i = 1, 2, 3 \).

**Proof.** It consists to check condition (2.2) for the measures (2.1), see e.g. the proof of Theorem 1 in \cite{14}. \( \square \)

3. Conditions of uniqueness

3.1. **Condition on the graph** \( \mathcal{G} \). As mentioned in the introduction, every vertex of the graph is assumed to have positive indegree and outdegree (to avoid trivial situations).

**Remark 1.** Notice that if \( \text{deg}^+(v) = 1 \) for all \( v = 0, 1, 2, 3 \), the corresponding hard-core model has unique Gibbs measure.

Indeed observe that in such a situation the matrix \( \mathbf{P} \) consists only 0 and 1, and the Hamiltonian (1.1) reads
\[ H(\sigma) = \begin{cases} 0, \quad & \text{if } \sigma \in \Omega, \\ +\infty, \quad & \text{if } \sigma \notin \Omega. \end{cases} \tag{3.1} \]
In addition, the set \( \Omega \) of admissible configurations is finite, so that there exists a unique Gibbs measure \( \mu(\sigma) = \frac{1}{|\Omega|}, \sigma \in \Omega \).
3.2. Condition on the matrix \( P \). Denote \( h_{i,x} = \ln z_{i,x}, i = 1, 2, 3 \). Then the equation (2.3) can be written as

\[
h_{i,x} = \sum_{y \in S(x)} \frac{P_{i,0} + P_{i,1} \exp(h_{1,y}) + P_{i,2} \exp(h_{2,y}) + P_{i,3} \exp(h_{3,y})}{P_{0,0} + P_{0,1} \exp(h_{1,y}) + P_{0,2} \exp(h_{2,y}) + P_{0,3} \exp(h_{3,y})}.
\]

(3.2)

Note that \( h_{i,x} \equiv 0, i = 1, 2, 3, x \in V \) is a solution of (3.2). Let us give a condition on \( P \) for which it will be the unique one.

We assume

\[
P_{0,1} P_{0,2} P_{0,3} > 0.
\]

(3.3)

Lemma 1. If condition (3.3) is satisfied and \( h_{x} = (\ln z_{1,x}, \ln z_{2,x}, \ln z_{3,x}) \) is a solution of (3.2) then

\[
z_{i}^{-} \leq h_{i,x} \leq z_{i}^{+},
\]

for any \( i = 1, 2, 3, \ x \in V \). Here \( (z_{1}^{-}, z_{1}^{+}, z_{2}^{-}, z_{2}^{+}, z_{3}^{-}, z_{3}^{+}) \) is a solution of

\[
\begin{align*}
z_{i}^{-} &= \min_{(x,y,z) \in D} f_{k}^{i}(x,y,z), \\
z_{i}^{+} &= \max_{(x,y,z) \in D} f_{k}^{i}(x,y,z),
\end{align*}
\]

(3.4)

where \( D = [z_{1}^{-}, z_{1}^{+}] \times [z_{2}^{-}, z_{2}^{+}] \times [z_{3}^{-}, z_{3}^{+}] \) and

\[
f_{i}(x,y,z) = \frac{P_{i,0} + P_{i,1} x + P_{i,2} y + P_{i,3} z}{P_{0,0} + P_{0,1} x + P_{0,2} y + P_{0,3} z}.
\]

Proof. We rewrite (3.2) as

\[
z_{i,x} = \prod_{j=1}^{k} f_{i}(z_{1,x,j}, z_{2,x,j}, z_{3,x,j}),
\]

where \( x_{j} \) are the direct successors of \( x \). The condition (3.3) guarantees that the functions \( f_{i} \) are bounded. It is not difficult to see that

\[
z_{i,1}^{-} < z_{i,x} < z_{i,1}^{+}, \ i = 1, 2, 3,
\]

where

\[
\begin{align*}
z_{i,1}^{+} &= \max_{x,y,z>0} f_{i}^{k}(x,y,z), \\
z_{i,1}^{-} &= \min_{x,y,z>0} f_{i}^{k}(x,y,z).
\end{align*}
\]

(3.5)

Consider now the functions \( f_{i}(x,y,z) \) on the sets \( D_{1} = [z_{1,1}^{-}, z_{1,1}^{+}] \times [z_{2,1}^{-}, z_{2,1}^{+}] \times [z_{3,1}^{-}, z_{3,1}^{+}] \). A second step of the procedure leads to

\[
z_{i,1}^{-} < z_{i,2}^{-} < z_{i,x} < z_{i,1}^{+}, \ i = 1, 2, 3,
\]

and by iteration we get the following inequalities

\[
z_{i,n}^{-} < z_{i,x} < z_{i,n}^{+},
\]

where the \( z_{i,n}^{\pm}, n = 1, 2, ..., \) satisfy
\[
z_{i,n+1}^{-} = \min_{(x,y,z) \in D_n} f_i^k(x, y, z), \\
z_{i,n+1}^{+} = \max_{(x,y,z) \in D_n} f_i^k(x, y, z),
\]
with \( z_{i,1}^{\pm} \) defined in (3.5) and

\[
D_n = [z_{1,n}^{-}, z_{1,n}^{+}] \times [z_{2,n}^{-}, z_{2,n}^{+}] \times [z_{3,n}^{-}, z_{3,n}^{+}].
\]

It is easy to see that this construction leads to bounded increasing (resp. decreasing) sequences \( z_{i,n}^{-} \) (resp. \( z_{i,n}^{+} \)). As a consequence, we get the existence of

\[
\lim_{n \to \infty} z_{i,n}^{\pm} = z_i^{\pm}.
\]

This completes the proof. \(\square\)

Consider the function \( h = (h_1, h_2, h_3) \to F(h) = (F_1(h), F_2(h), F_3(h)) \), defined by

\[
F_i(h) = \ln \frac{P_{i,0} + P_{i,1} \exp(h_1) + P_{i,2} \exp(h_2) + P_{i,3} \exp(h_3)}{P_{0,0} + P_{0,1} \exp(h_1) + P_{0,2} \exp(h_2) + P_{0,3} \exp(h_3)}.
\]

We denote \( \|h\| = \max\{|h_1|, |h_2|, |h_3|\} \) and put

\[
\theta_{ij} = \max_{(x,y,z) \in \mathcal{D}} \left| \frac{\partial F_i(x, y, z)}{\partial h_j} \right|, \quad i, j = 1, 2, 3.
\]

Condition (3.3) implies that \( \theta_{ij} < 1 \) for any \( i, j = 1, 2, 3 \). Let us denote

\[
\theta = \max_{i,j} \theta_{ij}.
\]

**Lemma 2.** For any \( h, l \in \mathcal{D} \) one has

a) \( \|F(h) - F(l)\| \leq 3\theta \|h - l\| \),

b) \( \|F(h)\| \leq 3\theta \|h\| \).

**Proof.** a) We have

\[
\|F(h) - F(l)\| = \max_{i=1,2,3} \{|F_i(h) - F_i(l)|\} \leq 3\theta \|h - l\|.
\]

b) follows from a) taking into account that \( F(0,0,0) = 0 \) by letting \( l = (0,0,0) \). \(\square\)

**Theorem 1.** Under condition (3.3) and

\[
3k\theta < 1
\]

the system of equations (3.2) has a unique solution \( h_{1,x} = h_{2,x} = h_{3,x} = 0 \). Consequently there exists a unique splitting Gibbs measure.
Proof. Using (3.2) and Lemma 2 we have
\[ \|h_x\| \leq \sum_{y \in S(x)} \|F(h_y)\| \leq k \max_{y \in S(x)} \|F(h_y)\| = (3k\theta)\|h_\tilde{y}\|. \]
Iterating this inequality leads to
\[ \|h_x\| \leq (3k\theta)^n\|h_u\|, \tag{3.8} \]
where \(u\) is such that \(d(x,u) = n\). Since \(\|h_u\| \leq C = \max\{\ln z_1^+, \ln z_2^+, \ln z_3^+\}\), we get \(h_x \equiv 0\). This completes the proof. □

Remark 2. To check the condition of Theorem 1 one needs a solution of the system (3.4). But the analysis of solutions of (3.4) is rather tricky. However our construction gives a convenient way to check the condition. Namely, one can check the condition \(3k\theta^{(m)} < 1\) where \(\theta^{(m)} = \max_{i,j} \{\theta_{i,j}^{(m)}\}\). Here \(\theta_{i,j}^{(m)}\) is defined by (3.7) with \(z_i^+\) replaced by \(z_i^{+m}\). By construction \(\theta_{i,j}^{(m)} \geq \theta_{i,j}\) and \(\lim_{m \to \infty} \theta_{i,j}^{(m)} = \theta_{i,j}\). Thus \(\theta^{(m)}\) gives an approximation for \(\theta\).

4. The diamond graph

Consider the graph
\[ G_{\text{diamond}} = \{(0,0), (0,2), (1,0), (1,2), (2,1), (2,3), (3,1), (3,3)\}. \tag{4.1} \]
It is the graph shown in Fig. 1. It may be seen as compatibility rules on edges for a two state model.

Consider then the matrix
\[ P = \begin{pmatrix} P_{0,0} = \alpha & P_{0,1} = 0 & P_{0,2} = 1 - \alpha & P_{0,3} = 0 \\ P_{1,0} = \beta & P_{1,1} = 0 & P_{1,2} = 1 - \beta & P_{1,3} = 0 \\ P_{2,0} = 0 & P_{2,1} = 1 - \beta & P_{2,2} = 0 & P_{2,3} = \beta \\ P_{3,0} = 0 & P_{3,1} = 1 - \alpha & P_{3,2} = 0 & P_{3,3} = \alpha \end{pmatrix}, \tag{4.2} \]
where \(\alpha, \beta \in (0,1)\). This is a simplified version of the diamond HC model with obvious symmetries between the parameters.

The corresponding set of recursive equations (2.3) reads
\[ f_x = \prod_{y \in S(x)} \frac{\beta + (1 - \beta)g_y}{\alpha + (1 - \alpha)g_y}, \quad g_x = \prod_{y \in S(x)} \frac{\beta h_y + (1 - \beta)f_y}{\alpha + (1 - \alpha)g_y}, \quad h_x = \prod_{y \in S(x)} \frac{\alpha h_y + (1 - \alpha)f_y}{\alpha + (1 - \alpha)g_y}, \tag{4.3} \]
where \(f_x = z_{1,x}', \quad g_x = z_{2,x}', \quad h_x = z_{3,x}'.\)

4.1. Translation invariant measures.
In this subsection we look for solutions of the form \( f_x = f, \ g_x = g, \ h_x = h \), for all \( x \in V \).

In this situation we get from (4.3): \[
\begin{align*}
f &= \left( \frac{\beta + (1 - \beta)g}{\alpha + (1 - \alpha)g} \right)^k, \\
g &= \left( \frac{\beta h + (1 - \beta)f}{\alpha + (1 - \alpha)g} \right)^k, \\
h &= \left( \frac{\alpha h + (1 - \alpha)f}{\alpha + (1 - \alpha)g} \right)^k.
\end{align*}
\] (4.4)

Denoting \( u = f^{1/k}, \ v = g^{1/k} \) and \( w = h^{1/k} \) we obtain
\[
\begin{align*}
u &= \beta + (1 - \beta)v^k, \\
v &= \beta w^k + (1 - \beta)u^k, \\
w &= \frac{\alpha w^k + (1 - \alpha)u^k}{\alpha + (1 - \alpha)v^k}.
\end{align*}
\] (4.5)

The expression of \( w \) as a function of \( v \) reads
\[
w = \left\{ \beta \left[ v(\alpha + (1 - \alpha)v^k) - (1 - \beta) \left( \frac{\beta + (1 - \beta)v^k}{\alpha + (1 - \alpha)v^k} \right)^k \right] \right\}^{1/k}.
\]

Then we get
\[
v = \eta(v) = \frac{1}{\alpha + (1 - \alpha)v^k} \left[ (1 - \beta) \left( \frac{\beta + (1 - \beta)v^k}{\alpha + (1 - \alpha)v^k} \right)^k + \beta^{1-k} \left( \frac{\alpha v + (\beta - \alpha) (\beta + (1 - \beta)v^k)^k}{\alpha + (1 - \alpha)v^k} \right)^{k} \right].
\] (4.6)

**Lemma 3.** The function \( \eta \) has the following properties:
1. \( \eta \) is a bounded function and \( \eta(0) > 0, \ \eta(+\infty) < +\infty \).
2. \( \eta(1) = 1, \ \eta'(1) = k \left[ 2\alpha - (1 + k(\beta - \alpha))^2 + k(\beta^2 - \alpha^2) \right] \).

**Proof.** This results from tedious but straightforward computations. \( \square \)

**Theorem 2.** If \( |\eta'(1)| > 1 \), there exist at least three translation-invariant Gibbs measures.

**Proof.** By Lemma 3, \( v = 1 \) is a solution of (4.6). When \( |\eta'(1)| > 1, \ v = 1 \) is unstable. So there exists a small neighborhood \( 1 - \varepsilon, 1 + \varepsilon \) of \( v = 1 \) such that for \( v \in (1 - \varepsilon, 1) \) \( \eta(v) < v \), and for \( v \in (1, 1 + \varepsilon) \) \( \eta(v) > v \). Since \( \eta(0) > 0 \), there exists a solution \( v^* \) between 0 and 1. Similarly since \( \eta(+\infty) < +\infty \) there is another solution \( v^{**} \) between 1 and \(+\infty\). Thus, there exist at least three solutions. This completes the proof. \( \square \)

The corresponding phase diagram is shown in Fig. 2.
4.1.2. In this subsection we look for solutions of the form $f_x = g_x$, $h_x = 1$. Note that $f_x = g_x$, $h_x = 1$ satisfies the system of equations (4.3) for any function $f_x$ which satisfies the following equation

$$f_x = \prod_{y \in S(x)} \frac{\beta + (1 - \beta)f_y}{\alpha + (1 - \alpha)f_y}.$$  

(4.7)

**Remark 3.** Recall that the functional equation of the Ising model on the Cayley tree is given by:

$$f_x = \prod_{y \in S(x)} \frac{1 + \theta f_y}{\theta + f_y},$$  

(4.8)

where $\theta = e^{2J/T}$, $J$ denoting the strength of the interaction and $T$ the temperature. The equations (4.7) thus coincide with (4.8) when

$$\alpha = \frac{\theta}{\theta + 1}, \quad \beta = \frac{1}{\theta + 1}.$$  

(4.9)

Consequently all known results for Ising model can be reformulated for these particular values.

To give a non-uniqueness condition for the solutions of the equation (4.7), we will use the following
Lemma 4. Let
\[ z = \left( \frac{\beta + (1 - \beta)z}{\alpha + (1 - \alpha)z} \right)^k, \quad z > 0. \] (4.10)

Then,
1) If \((\alpha, \beta) \in \left\{ (x, y) \in [0, 1]^2 : x \leq \frac{y(k+1)^2}{4ky+(k-1)^2} \right\}\) the equation (4.10) has a unique solution \(z = 1\).
2) If \((\alpha, \beta) \in \left\{ (x, y) \in [0, 1]^2 : x > \frac{y(k+1)^2}{4ky+(k-1)^2} \right\}\) the equation (4.10) has three solutions.

Proof. Denoting \(x = \frac{1-\beta}{\beta} z\), \(A = \frac{\beta(1-\alpha)^k}{(1-\beta)^k}\) and \(B = \frac{\alpha(1-\beta)}{\beta(1-\alpha)}\) we get
\[ Ax = \left( \frac{1 + x}{B + x} \right)^k. \] (4.11)
This equation is studied in [12], Proposition 10.7. The equation (4.11) with \(x \geq 0, k \geq 1, A, B > 0\) has a unique solution if either \(k = 1\) or \(B \leq \left( \frac{k+1}{k-1} \right)^2\). If \(k > 1\) and \(B > \left( \frac{k+1}{k-1} \right)^2\) then there exist \(\nu_1(B, k), \nu_2(B, k)\), with \(0 < \nu_1(B, k) < \nu_2(B, k)\), such that the equation has three solutions if \(\nu_1(B, k) < A < \nu_2(B, k)\) and has two if either \(A = \nu_1(B, k)\) or \(A = \nu_2(B, k)\). In fact:
\[ \nu_i(B, k) = \frac{1}{x_i} \left( \frac{1 + x_i}{B + x_i} \right)^k, \] (4.12)
where \(x_1, x_2\) are the solutions of
\[ x^2 + [2 - (B - 1)(k - 1)]x + B = 0. \]
The critical line is obtained by inserting the critical value \(B = \left( \frac{k+1}{k-1} \right)^2\) (equivalent to \(\alpha = \frac{\beta(k+1)^2}{4k\beta+(k-1)^2}\)) in (4.12).

Lemma 5. The solutions \(f_x\) of (4.7) satisfy
\[ z^- \leq f_x \leq z^+, \]
where \(z^- \leq 1 \leq z^+ \) solve the equation (4.10).

Proof. Let \(\alpha > \beta\).

By using properties of the function
\[ \varphi(t) = \frac{\beta + (1 - \beta)t}{\alpha + (1 - \alpha)t} \]
it is not difficult to see that
\[ z^-_1 \equiv \left( \frac{\beta}{\alpha} \right)^k < f_x < z^+_1 \equiv \left( \frac{1 - \beta}{1 - \alpha} \right)^k \] (4.13)

Now consider the function \(\varphi\) on the interval \([z^-_1, z^+_1]\). A new iteration of the construction leads to
\[ z^-_2 < z^-_1 < f_x < z^+_2 < z^+_1. \]
The process of iterations give $z_n^- < z_{i,x} < z_n^+$, where $z_n^\pm$, $n = 1, 2, ...$ satisfy

$$z_{n+1}^- = \varphi^k(z_n^-), \quad z_{n+1}^+ = \varphi^k(z_n^+)$$

It is easy to see that $z_n^-$ (resp. $z_n^+$) are bounded increasing (resp. decreasing) sequences. This shows that the limits $\lim_{n \to \infty} z_n^\pm = z^\pm$ exist. Moreover $\varphi^k(z^\pm) = z^\pm$.

(The case $\beta > \alpha$ is similar; the case $\alpha = \beta$ is trivial).

\[\square\]

**Theorem 3.**

1) If $(\alpha, \beta) \in \{(x, y) \in [0, 1]^2 : x \leq \frac{y(k+1)^2}{4(k+1)}\}$ the equation (4.7) has a unique solution $f_x \equiv 1$.

2) If $(\alpha, \beta) \in \{(x, y) \in [0, 1]^2 : x > \frac{y(k+1)^2}{4(k+1)}\}$ the equation (4.7) has at least three solutions.

These solutions lead to translation-invariant Gibbs measures.

**Proof.** 1) It is easy to see that $f_x \equiv 1$ is a solution of (4.7). From Lemma 4 it follows that under the conditions of theorem, the equation (4.10) has unique solution $z^- = z^+ = 1$. Then from Lemma 5 one gets $f_x \equiv 1$.

2) It is a consequence of Lemma 4. In this case, equation (4.7) has at least three constant solutions $f_x = 1$, $f_x = z^+$ and $f_x = z^-$. (see Fig. 3.)

![Fig. 3](image_url)

4.2. **Periodic measures.** In this subsection, we consider periodic solutions of (4.8). We will use the group structure of the Cayley tree. It is known (see [5]) that there exists a one-to-one correspondence between the set of vertices $V$ of a Cayley tree of order
The group $G_k$, free product of $k+1$ second-order cyclic groups with generators $a_1, a_2, \ldots, a_{k+1}$.

**Definition 1.** Let $\tilde{G}$ be a normal subgroup of the group $G_k$. The set $z = \{ z_x : x \in G_k \}$ is said to be $\tilde{G}$-periodic if $z_{yx} = z_x$ for any $x \in G_k$ and $y \in \tilde{G}$.

**Definition 2.** The Gibbs measure corresponding to a $\tilde{G}$-periodic set of quantities $z$ is said to be $\tilde{G}$-periodic.

It is easy to see that a $G_k$-periodic measure is translation invariant. Denote

$$G_k^{(2)} = \{ x \in G_k : \text{the length of word } x \text{ is even} \}. $$

This set is a normal subgroup of index two [5]. Note that $G_k^{(2)}$ is either the subset of even vertices (i.e. with even distance to the root).

The following proposition characterizes the set of all periodic solutions.

**Proposition 2.** For $\alpha \neq \beta$. Let $\tilde{G}$ be a normal subgroup of finite index in $G_k$. Then each $\tilde{G}$-periodic solutions of equation (4.7) is either translation-invariant or $G_k^{(2)}$-periodic.

**Proof.** It is easy to see that for $\alpha \neq \beta$ the function $\varphi(t) = (\beta + (1 - \beta)t)/((\alpha + (1 - \alpha)t)$ is one-to-one. Using this property together with arguments similar to the ones given in the proof of Theorem 2 in [9] lead to the statement. $\square$

By Proposition 2, the description of a $\tilde{G}$-periodic solutions of (4.7) is reduced to the solutions of system (4.14) below. This system describes periodic solutions with period two, more precisely, $G_k^{(2)}$-periodic solutions. They correspond to functions

$$f_x = \begin{cases} 
  z_1, & \text{if } x \in G_k^{(2)}, \\
  z_2, & \text{if } x \in G_k \setminus G_k^{(2)}. 
\end{cases} $$

In this case, we have from (4.7):

$$z_1 = \left( \frac{\beta + (1 - \beta)z_2}{\alpha + (1 - \alpha)z_2} \right)^k, \quad z_2 = \left( \frac{\beta + (1 - \beta)z_1}{\alpha + (1 - \alpha)z_1} \right)^k. \quad (4.14)$$

Namely, $z_1$ and $z_2$ satisfy

$$z = g(g(z)), \quad \text{where } g(z) = \left( \frac{\beta + (1 - \beta)z}{\alpha + (1 - \alpha)z} \right)^k. \quad (4.15)$$

Note that to get periodic (non translation invariant) measure we must find solutions of (4.14) with $z_1 \neq z_2$. Obviously, such solutions are roots of the equation

$$g(g(z)) - z = 0. \quad (4.16)$$

For $k = 2$, simple but long computations show that the last equation is equivalent to the equation

$$Az^2 + Bz + C = 0, \quad (4.17)$$
where
\[ A = [\alpha(1 - \alpha) + (1 - \beta)^2]^2, \quad C = [\alpha^2 + \beta(1 - \beta)]^2, \]
\[ B = 4\alpha\beta(1 - \alpha)(1 - \beta) + 2\beta(1 - \beta)^3 + \alpha^2(1 - \beta)^2 + 2\alpha^3(1 - \alpha) - (1 - \alpha)^2\beta^2. \]
The discriminant of the equation (4.17) has the following form
\[ D = D(\alpha, \beta) = [3\alpha\beta(1 - \alpha)(1 - \beta) + \beta(1 - \beta)^3 + \alpha^3(1 - \alpha) - (1 - \alpha)^2\beta^2] \times \]
\[ [5\alpha\beta(1 - \alpha)(1 - \beta) + 3\beta(1 - \beta)^3 + 2\alpha^2(1 - \beta)^2 + 3\alpha^3(1 - \alpha) - (1 - \alpha)^2\beta^2]. \]

It is easy to see that \( D(\alpha, \beta) = D(1-\beta, 1-\alpha). \) If \( \alpha \) is small enough and \( \beta \) is large enough then \( D > 0 \) and \( B < 0 \), in this case the equation (4.17) has two positive solutions. Thus we have proved the following

**Theorem 4.** If \( D > 0 \) and \( B < 0 \) then the model corresponding to the matrix (4.2) has at least two \( G_2^{(2)} \)-periodic (non-translation-invariant) Gibbs measures (see Fig. 4).

Fig. 4. Case of periodic solutions of equations (4.7) for \( k = 2 \). Two periodic solutions exist in the shaded area, no solution in the white area.

In the next picture (Fig. 5), we collect the last three diagrams. One remarks that the transition curves of the cases \( f_x = g_x, h_x = 1 \) and \( f_x = f; g_x = g; h_x = h \) get tangent (this arises for all \( k \)).
5. Fertile graphs

In this section, we consider symmetric graphs, more precisely three types of fertile graphs, the so-called stick, gun, and key [2]:

\[
\begin{align*}
G_{\text{stick}} &= \{(0,1),(0,3),(2,3)\}, \\
G_{\text{gun}} &= \{(0,0),(0,1),(0,2),(0,3)(1,2)\}, \\
G_{\text{key}} &= \{(0,1),(0,2),(0,3),(1,2)\}.
\end{align*}
\]

There, the above graphs are undirected, meaning that if \((a,b)\) belongs to the graph, then it is also the case for \((b,a)\).

5.1. The stick graph. For this graph, shown in Fig 6, the system of equations (2.3) reads

\[
\begin{align*}
f_x &= \prod_{y \in S(x)} \frac{1}{\alpha f_y + (1 - \alpha) h_y}, \\
g_x &= \prod_{y \in S(x)} \frac{h_y}{\alpha f_y + (1 - \alpha) h_y}, \\
h_x &= \prod_{y \in S(x)} \frac{\beta + (1 - \beta) g_y}{\alpha f_y + (1 - \alpha) h_y}, \quad (5.1)
\end{align*}
\]

where \(\alpha, \beta \in (0,1)\).
Let us exhibit conditions on $\alpha$ and $\beta$ under which the system of equations (5.1) has more than one constant solutions, i.e. $f_x = f, g_x = g, h_x = h$.

We denote $u = f^{1/k}, v = g^{1/k}$ and $w = h^{1/k}$ we get from (5.1) the following

$$
u = \frac{1}{\alpha u^k + (1-\alpha)w^k}, \quad v = \frac{w^k}{\alpha u^k + (1-\alpha)w^k}, \quad w = \frac{\beta + (1-\beta)v^k}{\alpha u^k + (1-\alpha)w^k}.$$  \hspace{1cm} (5.2)

One easily finds that

$$
u = \left(v(\beta + (1-\beta)v^k)^{-k}\right)^{1/(k+1)}, \quad w = \left(v(\beta + (1-\beta)v^k)^{1/(k+1)}\right).$$

Then from the second equation of (5.2) we get

$$v = Y(v) = \frac{1}{\alpha(\beta + (1-\beta)v^k)^{-k} + (1-\alpha)}.$$  \hspace{1cm} (5.3)

It is clear that $Y$ is an increasing, bounded function and

$$Y(0) = \frac{\beta^k}{\alpha + (1-\alpha)\beta^k} > 0, \quad Y(+\infty) = \frac{1}{1-\alpha} < +\infty.$$

$$Y(1) = 1, \quad Y'(1) = k^2\alpha(1-\beta).$$

**Theorem 5.** If $k^2\alpha(1-\beta) > 1$ then there are at least three translation-invariant Gibbs measures.

**Proof.** By properties of $Y(v)$ we know that $v = 1$ is a solution of (5.3). Under $|Y'(1)| > 1$, $v = 1$ is unstable. So there is sufficiently small neighborhood of $v = 1$: $(1 - \varepsilon, 1 + \varepsilon)$ such that $Y(v) < v$, for $v \in (1 - \varepsilon, 1)$ and $Y(v) > v$, for $v \in (1, 1 + \varepsilon)$. Since $Y(0) > 0$ there is a solution $v^*$ between 0 and 1, similarly since $Y(+\infty) < +\infty$ there is an other solution $v^{**}$ between 1 and $+\infty$. Thus there are at least three solutions. This completes the proof. \hfill $\square$

5.2. **The gun graph.** For this graph (see Fig 7) the system of recursive equations (2.3) is given by

$$f_x = \prod_{y \in S(x)} \frac{\alpha + (1-\alpha)g_y}{af_y + bg_y + ch_y + d},$$

$$g_x = \prod_{y \in S(x)} \frac{\beta + (1-\beta)f_y}{af_y + bg_y + ch_y + d},$$

$$h_x = \prod_{y \in S(x)} \frac{1}{af_y + bg_y + ch_y + d}.$$  \hspace{1cm} (5.4)

where $\alpha, \beta, a, b, c, d \in (0, 1); a + b + c + d = 1.$
In this case for simplicity we assume $\alpha = \beta$. Then for constant solutions, denoting $u = (f_x)^{1/k}$, $v = (g_x)^{1/k}$ and $w = (h_x)^{1/k}$ we get from (5.4) that

$$u = \frac{\alpha + (1 - \alpha)v^k}{au^k + bv^k + cw^k + d}, \quad v = \frac{\alpha + (1 - \alpha)u^k}{au^k + bv^k + cw^k + d}, \quad w = \frac{1}{au^k + bv^k + cw^k + d}. \quad (5.5)$$

From this system we get

$$u = w(\alpha + (1 - \alpha)v^k), \quad v = w(\alpha + (1 - \alpha)u^k).$$

Consequently

$$u(\alpha + (1 - \alpha)u^k) = v(\alpha + (1 - \alpha)v^k).$$

This gives $u = v$ and then $w = u(\alpha + (1 - \alpha)u^k)^{-1}$. Hence we have

$$u = U(u) = \frac{(\alpha + (1 - \alpha)u^k)^{k+1}}{(a + b)(\alpha + (1 - \alpha)u^k)^{k} + c} u^k + d (\alpha + (1 - \alpha)u^k)^{k}. \quad (5.6)$$

The following properties of $U(u)$ are clear: $U$ is bounded and $U(0) = \frac{\alpha}{d}$, $U(+\infty) < +\infty$, $U(1) = 1$, $U'(1) = k\{kc + d - \alpha(kc + 1)\}$.

Using these properties one can prove the following

**Theorem 6.** If $k|kc + d - \alpha(kc + 1)| > 1$, there exists at least three translation-invariant Gibbs measures.

**Proof.** The proof is similar to the proof of Theorem 5 \hfill \Box

### 5.3. The key graph.

The system of recursive equations for this final fertile graph under consideration (see Fig 8), is as follows:

$$f_x = \prod_{y \in S(x)} \frac{\alpha + (1 - \alpha)g_y}{af_y + bg_y + ch_y},$$

$$g_x = \prod_{y \in S(x)} \frac{\beta + (1 - \beta)f_y}{af_y + bg_y + ch_y},$$

$$h_x = \prod_{y \in S(x)} \frac{1}{af_y + bg_y + ch_y}. \quad (5.7)$$
where $\alpha, \beta, a, b, c \in (0, 1); a + b + c = 1$.

Fig. 7. The key graph

This is a particular case of the previously analysed gun graph (obtained with $d = 0$). Hence Theorem 6 remains true with $\alpha = \beta$ and $k|kc - \alpha(\alpha c + 1)| > 1$.

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