Rotating topological black holes

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Abstract

A class of metrics solving Einstein’s equations with negative cosmological constant and representing rotating, topological black holes is presented. All such solutions are in the Petrov type-\(D\) class, and can be obtained from the most general metric known in this class by acting with suitably chosen discrete groups of isometries. First, by analytical continuation of the Kerr-de Sitter metric, a solution describing uncharged, rotating black holes whose event horizon is a Riemann surface of arbitrary genus \(g > 1\), is obtained. Then a solution representing a rotating, uncharged toroidal black hole is also presented. The higher genus black holes appear to be quite exotic objects, they lack axial symmetry, a well defined and conserved angular momentum, and have an intricate causal structure. The toroidal black holes appear to be simpler, they have rotational symmetry and the amount of rotation they can have is bounded by some power of the mass.

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1 Introduction

In the past months there has been an increasing interest in black holes whose event horizons have a nontrivial topology [1–3]. The solutions can be obtained with the least expensive modification of general relativity, the introduction of a negative cosmological constant. This is sufficient to avoid few classic theorems forbidding non-spherical black holes [4–6], and comes as a happy surprise. Charged versions of these black holes were presented in [2], they can form by gravitational collapse [7, 8] of certain matter configurations, and all together form a well behaved sequence of thermodynamically well behaved objects, obeying the well known entropy-area law [3, 9].

Up to now no rotating generalization of higher genus solutions has been known. Holst and Peldan recently showed that such solutions cannot be quotient spaces of the universal covering of (3+1)-dimensional anti-de Sitter space, at least if the fundamental group of the resulting manifold has only one generator [10]. Therefore, if we are looking for a rotating generalization of the topological black holes, we have to consider spacetimes with non constant curvature.

On the other hand, a charged rotating toroidal solution with a black hole interpretation has been presented by Lemos and Zanchin [11], following previous work on cylindrically symmetric solutions of Einstein’s equations [12–18].

In this paper a rotating generalization of higher genus black holes together with another toroidal rotating solution will be presented. We do not present uniqueness results, and apart from noticing that there are more than one non isometric torii generating black holes, we content ourselves with a discussion of some of the relevant properties they have.

We begin in Sec.(II) with the spacetime metric for the genus $g > 1$ solution and give a proof that it solves Einstein’s field equations with negative $\Lambda$-term.

In Sec.(III) we determine the black hole interpretation of the metric, and we consider in which sense mass and angular momentum are defined and conserved. We do not give a detailed description of the rather intricate causal structure (this we shall present elsewhere), nor we discuss whether the black holes can result from gravitational collapse.

In Sec.(IV) we give the rotating toroidal black hole’s metric, together with a short account of its main features.

In this paper we shall use the curvature conventions of Hawking-Ellis’ book [19] and employ Planck’s dimensionless units.

2 Spacetime metric for $g > 1$ rotating black holes

We begin by recalling the uncharged topological black holes discussed in [4–8]. The metric appropriate for genus $g > 1$ reads

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2(d\theta^2 + \sinh^2 \theta d\phi^2)$$

with the lapse function $V(r)$ given by

$$V(r) = -1 - \Lambda r^2 - \frac{2\eta}{r}$$

where $\eta$ is the mass parameter and $\Lambda = -3\ell^{-2}$ the cosmological constant. One notices that the $(\theta, \phi)$ sector of the metric describes the two-dimensional non-compact space with constant, negative curvature. As is well known, this is the universal covering space for all Riemannian surfaces with genus $g > 1$. Therefore, in order to get a compact event horizon, suitable identifications in the $(\theta, \phi)$ sector have to be carried out, corresponding to the choice of some discrete group of isometries acting on hyperbolic 2-space properly discontinuously. After this has been done, the metric (2.1) will describe higher genus black holes. The $g = 1$ case, with toroidal event
horizon, is given by the metric
\[ ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2d\sigma^2 \] (2.3)
with the lapse function \( V(r) \) given by
\[ V(r) = -\frac{\Lambda r^2}{3} - \frac{2\eta}{r} \] (2.4)
and \( d\sigma^2 \) the line element of a flat torus. Its conformal structure is completely determined by a complex parameter in the upper half complex plane, \( \tau \), which is known as the Teichmüller parameter. A representative for the flat torus metric can then be written in the form
\[ d\sigma^2 = |\tau|^2 dx^2 + dy^2 + 2 \text{Re} \tau dx dy \] (2.5)
It is quite trivial to show that all such solutions have indeed a black hole interpretation, with various horizons located at roots of the algebraic equation \( V(r) = 0 \), provided \( \eta \) be larger than some critical value depending on the genus. It can also be shown that for all genus, a ground state can be defined relative to which the ADM mass is a positive, concave function \([3]\) of the black hole’s temperature as defined by its surface gravity \([20]\), and that the entropy obeys the area law \([3, 9]\).

We now determine at least one class of rotating generalizations of the above solutions starting with the higher genus case, namely when \( g > 1 \). The toroidal rotating black hole will be described last. The metric (2.1) looks very similar to the Schwarzschild-de Sitter metric \([21–23]\)
\[ ds^2 = -(1 - \frac{\Lambda r^2}{3} - \frac{2\eta}{r})dt^2 + (1 - \frac{\Lambda r^2}{3} - \frac{2\eta}{r})^{-1}dr^2 \\
+ r^2(d\theta^2 + \sin^2 \theta d\phi^2) \] (2.6)
(Here \( \Lambda > 0 \)).
For the latter, it is well known that a generalization to the rotating case exists, namely the Kerr-de Sitter spacetime \([21–23]\), which describes rotating black holes in an asymptotically de Sitter-space. Its metric reads in Boyer-Lindquist-type coordinates
\[ ds^2 = \rho^2(\Delta_r^{-1}dr^2 + \Delta_\theta^{-1}d\theta^2) \\
+ \rho^{-2}\Xi^{-2}\Delta_\theta[adt - (r^2 + a^2)d\phi]^2 \sin^2 \theta \\
- \rho^{-2}\Xi^{-2}\Delta_r[dt - a \sin^2 \theta d\phi]^2, \] (2.7)
where
\[ \rho^2 = r^2 + a^2 \cos^2 \theta \]
\[ \Delta_r = (r^2 + a^2)(1 - \frac{\Lambda r^2}{3}) - 2\eta r \]
\[ \Delta_\theta = 1 + \frac{\Lambda a^2}{3} \cos^2 \theta \]
\[ \Xi = 1 + \frac{\Lambda a^2}{3} \] (2.8)
and \( a \) is the rotational parameter.
Now we note that (2.1) can be obtained from (2.6) by the analytical continuation
\[ t \rightarrow it, \quad r \rightarrow ir, \quad \theta \rightarrow i\theta, \quad \phi \rightarrow \phi, \]
\[ \eta \rightarrow -i\eta, \] (2.9)
thereby changing also the sign of $\Lambda$ (this may be interpreted as an analytical continuation, too). Therefore we are led to apply the analytical continuation (2.9) also to Kerr-de Sitter spacetime (2.7), additionally replacing $a$ by $ia$. This leads to the metric

$$ds^2 = \rho^2 (\Delta_r^{-1} dr^2 + \Delta_\theta^{-1} d\theta^2) + \rho^{-2} \Xi^{-2} \Delta_\theta (adt - (r^2 + a^2) d\phi)^2 \sin^2 \theta - \rho^{-2} \Xi^{-2} \Delta_r [dt + a \sin^2 \theta d\phi]^2,$$

(2.10)

where now

$$\rho^2 = r^2 + a^2 \cosh^2 \theta$$

$$\Delta_r = (r^2 + a^2)(-1 - \frac{\Lambda r^2}{3}) - 2\eta r$$

$$\Delta_\theta = 1 - \frac{\Lambda a^2}{3} \cosh^2 \theta$$

$$\Xi = 1 - \frac{\Lambda a^2}{3}$$

(2.11)

and $\Lambda < 0$.

One observes that (2.10) describes a spacetime which reduces, in the limit $a = 0$, to the static topological black holes (2.1). For our further purpose it is convenient to write (2.10) in the form

$$ds^2 = \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 - \frac{\rho^2 \Delta_\theta \Delta_r}{\Xi^2 \Sigma^2} dt^2 + \frac{\Sigma^2 \sinh^2 \theta}{\Xi^2 \rho^2} [d\phi - \omega dt]^2$$

(2.12)

where we introduced

$$\Sigma^2 = (r^2 + a^2)^2 \Delta_\theta - a^2 \sinh^2 \theta \Delta_r$$

(2.13)

and the angular velocity

$$\omega = \frac{a[(r^2 + a^2) \Delta_\theta + \Delta_r]}{\Sigma^2}.$$

(2.14)

Our job is not yet finished however, as we have to compactify the $(\theta, \phi)$-sector into a Riemann surface while preserving the differentiability of the metric. The metric induced on each surface at fixed $r$ is foliated by surfaces at fixed $t$ and we would like these to be Riemann surfaces with $g > 1$. The metric on such surfaces is

$$d\sigma^2 = \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Sigma^2 \sinh^2 \theta}{\Xi^2 \rho^2} d\phi^2$$

(2.15)

If $\Sigma^2 > 0$, which is the case, for every $r \in \mathbb{R}$, if $-a\ell^{-2}(a^2 + \ell^2) < \eta < a\ell^{-2}(a^2 + \ell^2)$, this is actually a well defined Euclidean metric on the covering space of Riemann surfaces, the hyperbolic 2-space $H^2$. Outside the $\eta$ interval written above, the metric may become singular or may change the signature. Moreover, it is possible to show that this metric induces a global metric on any Riemann surface. First, notice that (2.15) is not invariant under isometries of $H^2$ and, a priori, it does not induce a well defined metric on the Riemann surface, unless $(\theta, \phi)$ are properly interpreted. Indeed, even for the natural metric on $H^2$ with constant, negative curvature $\theta$ is not a well defined coordinate along the boundary of the 4-gon which is identified to form the closed surface. The reason is that boundary points are, as a rule, conjugate points for geodesics reaching the boundary and $\theta$, which is the geodesic distance from a selected point, fails to be differentiable in a neighborhood of a boundary point. Therefore, we cover the Riemann surface obtained by
quotienting $H^2$ with a discrete group of isometries acting properly discontinuously on $H^2$, with a finite number (the surface is compact) of convex normal neighborhoods of the $H^2$ metric, $\{U_\alpha\}$, and declare that $\theta_\alpha(p)$, $p \in U_\alpha$, is the hyperbolic distance of $p$ from the center of $U_\alpha$. The square of the distance is infinitely differentiable on $U_\alpha$ and on $U_\alpha \cap U_\beta$, $\theta_\alpha$ is a differentiable function of $\theta_\beta$. Finally, we define the metric on each $U_\alpha$ by Eq. (2.15). As the components of this metric tensor are functions of the square of the hyperbolic geodesic distance, the metric is well defined and smooth in each convex neighborhood and thus everywhere on the Riemann surface. Notice that in the same sense the standard metric on $H^2$ with constant, negative curvature can be defined on the Riemann surface (i.e. we have not changed the differential structure to define the metric, we only used it).

Finally, we remark that (2.10) is a limit case of the metric of Plebanski and Demianski [25], which is the most general known Petrov type-D solution of the source-free Einstein-Maxwell equations with cosmological constant. In the case of zero electric and magnetic charge it reads

$$ds^2 = \frac{1}{(1-pq)^2} \left\{ \frac{p^2 + q^2}{p} dp^2 + \frac{\mathcal{P}}{p^2 + q^2} (d\tau + q^2 d\sigma)^2 + \frac{p^2 + q^2}{\mathcal{L}} d\xi^2 - \frac{\mathcal{L}}{p^2 + q^2} (d\tau - p^2 d\sigma)^2 \right\},$$

(2.16)

where the structure functions are given by

$$\mathcal{P} = \left(-\frac{\Lambda}{6} + \gamma\right) + 2np - \epsilon p^2 + 2\eta p^3 + \left(-\frac{\Lambda}{6} - \gamma\right) p^4$$

$$\mathcal{L} = \left(-\frac{\Lambda}{6} + \gamma\right) - 2\eta q + \epsilon q^2 - 2\eta q^3 + \left(-\frac{\Lambda}{6} - \gamma\right) q^4.$$  

(2.17)

$\Lambda$ is the cosmological constant, $\eta$ and $n$ are the mass and nut parameters, respectively, and $\epsilon$ and $\gamma$ are further real parameters. (For details c.f. [25]). Rescaling the coordinates and the constants according to

$$p \rightarrow L^{-1} p, \quad q \rightarrow L^{-1} q, \quad \tau \rightarrow L \tau, \quad \sigma \rightarrow L^3 \sigma,$$

$$\eta \rightarrow L^{-3} \eta, \quad \epsilon \rightarrow L^{-2} \epsilon, \quad n \rightarrow L^{-3} n, \quad \gamma \rightarrow L^{-4} \gamma + \frac{\Lambda}{6}, \quad \Lambda \rightarrow \Lambda$$

(2.18)

and taking the limit as $L \rightarrow \infty$, one obtains

$$ds^2 = \frac{p^2 + q^2}{\mathcal{P}} dp^2 + \frac{\mathcal{P}}{p^2 + q^2} (d\tau + q^2 d\sigma)^2 + \frac{p^2 + q^2}{\mathcal{L}} d\xi^2 - \frac{\mathcal{L}}{p^2 + q^2} (d\tau - p^2 d\sigma)^2,$$

(2.19)

where now

$$\mathcal{P} = \gamma + 2np - \epsilon p^2 - \frac{\Lambda}{3} p^4$$

$$\mathcal{L} = \gamma - 2\eta q + \epsilon q^2 - \frac{\Lambda}{3} q^4.$$  

(2.20)

Setting now

$$q = r, \quad p = a \cosh \theta, \quad \tau = \frac{t - a \phi}{a \Xi}, \quad \sigma = -\frac{\phi}{a \Xi},$$

$$\epsilon = -1 - \frac{\Lambda a^2}{3}, \quad \gamma = -a^2, \quad n = 0,$$

(2.21)

one gets our solution (2.10). As we said, the metric (2.10) results to be a limit case of the more general solution (2.16) of Einstein’s equation. This formally shows that (2.10) should solve
Einstein’s field equations with cosmological constant, i.e. the analytical continuation (2.9) of the Kerr-de Sitter metric should yield again a solution. Anyhow, one could doubt of the procedure as it involves an infinite limit of some parameters in the initial solution of Einstein’s equations. Therefore, let us sketch a short independent proof of the fact that (2.10) still satisfies Einstein’s equations.

Generally speaking, all functions which appear in the left hand side of Einstein’s equations containing the cosmological constant are polynomial in metric tensor components, components of the inverse metric tensor and derivatives of metric tensor components. Considering all these functions as independent variables, the l.h.s of Einstein’s equations defines analytic functions in these variables. Let us consider Kerr-de Sitter spacetime defined above. Then the metric, its inverse and its derivatives define locally analytic functions of the (generally complex) variables \( t, r, \theta, \phi, \eta, a, \Lambda \). We conclude that the l.h.s of Einstein’s equations defines analytic functions of \( (t, r, \theta, \phi, \eta, a, \Lambda) \) in open connected domains away from singularities corresponding to zeroes of \( \Delta_r \) and the determinant of the analytically continued metric \( (g = -\Xi^{-4}(r^2 + a^2 \cos^2 \theta)^2) \). Moreover, we know that, for real values of \( (t, r, \theta, \phi, \eta, a, \Lambda) \), \( \Lambda > 0 \), these functions vanish because Kerr-de Sitter metric is a solution of Einstein’s equations. Hence, due to the theorem of uniqueness of the analytical continuation of a function of several complex variables, they must vanish concerning all (generally complex) remaining values of \( (t, r, \theta, \phi, \eta, a, \Lambda) \), provided they belong to the same domain of analyticity of the previously considered real values. In particular, we can pick out the set of values determining the metric (2.10) as final values. Notice that these values belong to the same analyticity domain of the values determining Kerr-de Sitter parameters to parameters appearing in the metric (2.10), and skipping all singularities.

3 Some properties of \( g > 1 \) rotating black holes

We shall briefly discuss now the black hole interpretation of the proposed solutions, starting with the case \( g > 1 \), and some of their physical properties.

3.1 Curvature

Let us begin by looking at the curvature of the spacetime metric (2.10). The only nonvanishing complex tetrad component of the Weyl tensor is given by

\[
\Psi_2 = -\frac{2\eta}{(r + ia \cosh \theta)^2}
\]  

(The \( \Psi_i \), \( i = 0, \ldots, 4 \), are the standard complex tetrad components describing the conformal curvature. For details c.f. [23], [26]). For \( \eta = 0 \) the Weyl tensor vanishes and, since \( R_{ij} = \Lambda g_{ij} \), our manifold is a space of constant curvature, \( k = -\ell^{-2} \), i.e. a quotient space of the universal covering of anti-de Sitter space. This situation is comparable to that of the Kerr metric, which, for vanishing mass parameter, is simply the Minkowsky metric written in oblate spheroidal coordinates.

One further observes that \( \Psi_2 \) is always nonsingular, in particular the curvature singularity in Kerr-de Sitter space at \( \rho^2 = 0 \), i.e. \( r = 0, \theta = \pi/2 \) vanishes after the analytical continuation, as \( r^2 + a^2 \cosh^2 \theta \) is always positive. Hence the manifold may be extended to values \( r < 0 \). This is similar to the black hole of BTZ [27], where no curvature singularity occurs (see also [28] for an exhaustive determination of \((2 + 1)\)-black holes and their topology).

\footnote{A more detailed description including Penrose’s diagrams and a discussion of the conserved charges will be contained in a forthcoming paper.}
3.2 Singularity structure and horizons

The metric (2.10) becomes singular at $\Delta_r = 0$. With $\Lambda = -3\ell^{-2}$ this equation reads

$$(r^2 + a^2) \left( \frac{r^2}{\ell^2} - 1 \right) - 2\eta r = 0 \tag{3.2}$$

There are several cases. In all cases, before the leftmost zeroes and after the rightmost zero, $\Delta_r$ is positive. Here the various cases:

i) If $\ell^2 < a^2(7 + 4\sqrt{3})$ and $\eta \in \mathbb{R}$ there is only one positive solution $r_+$ of (3.2) and only one negative solution $r_-$. For $r > r_+$ and $r < r_-$, $\Delta_r$ is positive and $\partial_r$ is spacelike. For $r_- < r < r_+$, $\partial_r$ becomes timelike. $r_-$ and $r_+$ are first order zeroes.

ii) If $\ell^2 = a^2(7 + 4\sqrt{3})$ and

a) $\eta \neq \pm \eta_0$, $\eta_0 = 4\ell/3(26\sqrt{3} - 45)^{1/2} > 0$, then the solutions behave as in the case (i);
b) $\eta = -\eta_0$, then there is a first order root $r_-$ for $r < 0$ and a third order root $r_+ = [((\ell^2 - a^2)/6)^{1/2}$ for $r > 0$. $\Delta_r$ changes its sign in those roots;
c) $\eta = \eta_0$, then there is a first order root $r_+$ for $r > 0$ and a third order root $r_- = -[((\ell^2 - a^2)/6)^{1/2}$ for $r < 0$. $\Delta_r$ changes its sign in those roots.

iii) If $\ell^2 > a^2(7 + 4\sqrt{3})$ and $\eta \leq 0$ we have again several subcases. Let

$$R_{\pm} = \sqrt{\frac{1}{6}(\ell^2 - a^2) \pm \frac{1}{2}\sqrt{(\ell^2 - a^2)^2 - 12\ell^2a^2}}$$

$$\eta_{\pm} = -\frac{2R_{\pm}}{3\ell^2} \left[ (\ell^2 - a^2) \pm \frac{1}{2}\sqrt{(\ell^2 - a^2)^2 - 12\ell^2a^2} \right] \tag{3.3}$$

(Note that $\eta_- < \eta_+ < 0$).

a) for $0 \geq \eta > \eta_+$, $\Delta_r$ behaves as in (i);
b) for $\eta = \eta_+$, $\Delta_r$ has two positive zeroes $r_+ = R_+$ and $r_{++} > r_+$ and a negative zero $r_-$. At $r = r_+$ the graph of $\Delta_r$ versus $r$ is tangent to the $r$-axis and $\Delta_r$ does not change sign ($r_+$ is a second order root), whereas at $r = r_{++}$ and $r = r_-$, $\Delta_r$, $r$ increasing, changes sign from negative to positive values and from positive to negative values respectively. These are first order zeroes;
c) for $\eta_- < \eta < \eta_+$, $\Delta_r$ has three positive zeroes and one negative zeroes where $\Delta_r$ changes sign. These zeroes are first order;
d) in the case $\eta = \eta_-$ one obtains again two positive roots $r_+$ and $r_{++} = R_+ > r_+$, and a negative root $r_-$. At $r_+$ and $r_-$, which are first order roots, $\Delta_r$ changes sign from $-r$ to $+1$ and from $+r$ to $-1$ respectively, whereas at $r_{++}$, which is a second order root, $\Delta_r$ doesn’t change sign;
e) for $\eta < \eta_-$ we get again the same behaviour as in (i).

iv) If $\ell^2 > a^2(7 + 4\sqrt{3})$ and $\eta > 0$ the discussion of the roots is symmetric to that for the case (iii) considering the symmetry of (3.2) under the combined inversion $r \rightarrow -r$, $\eta \rightarrow -\eta$. In general we have one positive first order zero and up to three negative zeroes.

All zeroes of $\Delta_r$ in the examined cases are merely coordinate singularities, similar to the Schwarzschild case. They represent horizons, as the normals to the constant $t$ and constant $r$ surfaces become null when $r$ is a root of $\Delta_r = 0$. The couple of outermost horizons (e.g. $r_H = r_{++}$ and $r_H = r_-$ in case (iii,d)) are also event horizons as the Killing trajectories in the exterior static domains never intersect the surfaces $r = r_H$. The future parts of these event horizons are the boundary of the causal past of all time-like inextendible geodesics contained in the respective static regions which reach the future time-like infinity.

However, the resulting causal structure is rather intricate. We notice the complete absence of metric singularities at $r = 0$. This allows one to consider the coordinate $r$ in the complete range $(-\infty, +\infty)$ as we did above.

We remark that there is an extreme case which for $a \rightarrow 0$ gives the naked singularity discussed in [3,4], but for $a > 0$ still represents a black hole. Hence the non rotating naked singularity is
unstable, as it turns into a black hole by an infinitesimal addition of angular momentum. This seems to lend some support to the cosmic censorship conjecture.

In all cases discussed above, the outermost zeroes are event horizons. One gets for the Gaussian curvature of the event horizons

\[
K = -\frac{1}{\rho_H^6} \left[ (\rho_H^2 - 4a^2 \cosh^2 \theta) \left( (r_H^2 + a^2) \Delta \theta + \frac{\rho_H^2 a^2}{l^2} \sinh^2 \theta \right) + \frac{4a^2 \rho_H^4}{l^2} \cosh^2 \theta \right],
\]

where the index \( H \) indicates that the corresponding quantities are to be evaluated on the event horizon \( r_H \). It can be shown that, at least for \( \eta > 0 \), \( K \) is everywhere negative on positive outermost horizons, hence the integration of \( K \) over a compact domain yields via the Gauss-Bonnet theorem a direct check that the event horizon is in fact a Riemann surface \( S_g \) of genus \( g > 1 \). The same result is obtained for \( \eta < 0 \) and negative outermost horizons. In other cases \( K \) could be locally positive.

Note that the horizon has been warped due to the rotation. It is no more a surface of constant curvature as in the nonrotating case. Actually, each surface with fixed \( r \) outside the horizon takes on this topology with genus \( g > 1 \). Therefore, the topology of the external manifold is \( \mathbb{R}^2 \times S_g \).

### 3.3 Angular velocity and surface gravity

At least for \(-a\ell^{-2}(a^2 + \ell^2) < \eta < a\ell^{-2}(a^2 + \ell^2)\), the event horizon (as well as any \( r = \text{constant} \) surface) rotates relative to the stationary frame at infinity with angular velocity \( \Omega_H = \omega(r_H, \theta) \), where \( \omega \) is given by Eq. (2.14), which yields

\[
\Omega_H = \frac{a}{r_H^2 + a^2}
\]

(3.5)

Notice that \( \omega(r, \theta) \) is just given by \( d\phi/dt \) along time-like trajectories with fixed values for \( r \) and \( \theta \), \( t \) being proportional to the proper time \( \tau \) according to \( t = (\Xi^2 \Sigma^2 / \rho^2 \Delta \Delta r) \tau \).

There also exists a dragging effect at infinity, as \( \omega \) is non vanishing there, its value being \( \Omega_\infty = a/(a^2 + \ell^2) \).

The surface gravity \( \kappa \) is another important property of the event horizon. It is normally defined in terms of the null generator of the horizon, \( l^a = \partial_t + \Omega_H \partial_\phi \), using

\[
l^c \nabla_c l^a = \kappa l^a
\]

(3.6)

However, although in the present case \( \partial_t \) still is a global Killing field, the vector \( \partial_\phi \) is only a local Killing field, because of the identifications used to build up \( S_g \). This agrees with the known result that Riemann surfaces with \( g > 1 \) admit no global Killing fields, nor even global conformal Killing fields. Nevertheless, the surface gravity can still be defined as the acceleration per unit coordinate time which is necessary to hold in place a co-rotating particle (i.e. one at some fixed \( r \) and \( \theta \)) near the event horizon. Such a particle will move on an integral curve of the vector \( u = N^{-1}(\partial_t + \omega \partial_\phi) \), which is timelike everywhere in the exterior domain bounded by the event horizon. The function \( N \) normalizing the four-velocity is the lapse function of the foliation determined by the Killing coordinate time \( t \), and is

\[
N^2 = \frac{\rho_H^2 \Delta \Delta r}{\Xi^2 \Sigma^2}
\]

(3.7)

By computing the four-acceleration, one obtains in this way

\[
\kappa = \frac{1}{2(a^2 + \ell^2)(r_H^2 + a^2)} \left[ 3r_H^3 + (a^2 - \ell^2)r_H + \frac{a^2 \ell^2}{r_H} \right]
\]

(3.8)
Remarkably, this is constant over the event horizon even in the absence of a true rotational symmetry. In view of this last fact, the meaning of the surface gravity as the quantum temperature of the black hole remains a little bit obscure. The fact is that, although one can define a conserved mass by using the time translation symmetry of the metric, one cannot define a strictly conserved angular momentum, but only a conserved angular momentum with respect to a special choice of the observers at infinity. Hence the status of the first law for such black holes certainly needs further clarifications. As we will see, the situation will be rather different for toroidal black holes, which behave quite similarly to the Kerr solution. This also suggests that higher genus rotating black holes may be a kind of stable soliton solutions in anti-de Sitter gravity.

Returning to the metric (2.10), a few algebra yields

\[ g_{tt} = \frac{a^2 \sinh^2 \theta - \Delta_r}{\rho^2 \Xi^2} \]  

(3.9)

From this expression, we realize that \( g_{tt} \) may change sign within all regions where \( \Delta_r > 0 \). Anyhow, \( g_{tt} < 0 \) for \(|r|\) sufficiently large. The surface where \( g_{tt} = 0 \) inside any region where \( \Delta_r > 0 \) is one of the boundaries of an "ergo-pseudosphere" which is defined as a region where both \( \partial_t \) and \( \partial_r \) are spacelike. This is therefore a stationary limit surface, the equation of which reads

\[ \theta = \pm \sinh^{-1} \frac{\sqrt{\Delta_r}}{a}. \]  

(3.10)

The remaining boundaries of the same ergo-pseudosphere are event horizons located at roots of \( \Delta_r = 0 \). These are general features of rotating black hole metrics. Furthermore, we notice that, similarly to the Kerr solution, the event horizon and the surrounding stationary limit surface meet at \( \theta = 0 \), where they are smoothly tangent to each other provided \( \Delta_r = 0 \) in a first-order zero.

### 3.4 Mass and angular momentum

Waiting for a more detailed analysis, we would like to give some comments about the status of the two conserved charges, mass and angular momentum, which one expects to be associated to a rotating self-gravitating system.

One approach to a general and sensible definition of conserved charges associated to a given spacetime, is the canonical ADM analysis appropriately extended to include non-asymptotically flat solutions. This led to the introduction of the more general concept of quasi-local energy \[29\] for a spatially bounded self-gravitating system, and more generally, to various other quasi-local conserved charges. By a careful handling of all the boundary terms in the Hamiltonian for general relativity, one arrives at the following expression \[29–31\] for the mass

\[ M = -\frac{1}{8\pi} \int_{S_\rho} [N(\Theta - \Theta_0) - 16\pi (P_{ab}V^a\xi^b - (P_{ab}V^a\xi^b)_{[0]})] \sqrt{\sigma} d^2x \]  

(3.11)

where quantities with a subscript 0 denote background subtractions, chosen so that \( M \) be a function of the canonical data alone \[29\]. Here \( S_\rho \) is an asymptotic Riemann surface embedded in a \( t = constant \) slice, with outward pointing normal \( \xi^a \) and extrinsic curvature \( \Theta, P_{ab} \) is the canonically conjugate momentum to the metric induced on the slice, and \((N,V^a)\) are the lapse function and the shift vector of the \( t = constant \) foliation. Similarly, when a rotational Killing symmetry exists, generated by \( K^a \), a conserved angular momentum can be defined by the formula \[29\]

\[ J = 2 \int_{S_\rho} [P_{ab}K^a\xi^b - (P_{ab}K^a\xi^b)_{[0]}] \sqrt{\sigma} d^2x \]  

(3.12)
Unlike the case of non-rotating topological black holes, where a natural choice for the background can be made, no distinctive background metric has been found in the present case. The best we are able to do is to define the mass relative to some other solution with the same topology and rotation parameter. In spite of the dragging effect at infinity and the intricate form of the metric, what we get is the very simple result

\[ M = \Xi^{-2}(\eta - \eta_0)(g - 1) \]  

and \( \eta \) can be expressed as a function of the outermost horizon location \( r_H \), by using \( \Delta_r(r_H, \eta) = 0 \). Thus \( \eta \) really is related to the Hamiltonian mass, albeit in a relative sense. As for the angular momentum we are in a different position, since there is no rotational Killing symmetry. However the vector \( K = \partial_\phi \), although it is not a Killing vector, obeys asymptotically \( \nabla_{(a}K_{b)} \approx O(r^{-3}) \) and is therefore a kind of approximate symmetry. We may try to compute \( J \) using Eq. (3.12) with \( K = \partial_\phi \). Then one finds that \( J \) is finite without the need of a subtraction and we get

\[ J = \eta aI, \text{ where the integral } \]

\[ I = \frac{3}{8\pi(g - 1)} \int_{S_g} \sinh^3 \theta \, d\theta \, d\phi \]

(3.14)
can be performed over a fundamental domain of the Riemann surface \( S_g \). This is weakly conserved in the sense that it depends on the choice of a spatial slice in the three-boundary at infinity. Due to these facts, one may doubt about the existence of a thermodynamical "first law", and the full subject of black hole thermodynamics needs here further clarifications.

One may note, among other things, that \( J = 0 \) for the locally anti-de Sitter solution corresponding to \( \eta = 0 \), in agreement with Holst’s and Peldan’s theorem [10].

4 The rotating toroidal black hole

We discuss now another black hole solution in anti-de Sitter gravity which represents a rotating torus hidden by an event horizon. The first solution of this kind has been discovered by Lemos and Zanchin [11] by compactifying a charged open black string. This is a solution that can be obtained from the non rotating toroidal metric by mixing time-angle variables into new ones. This is not a permissible coordinate transformation in the large as angles, unlike time, are periodic variables. This is why the solutions are globally different, as clearly showed by Stachel while investigating the gravitational analogue of the Aharonov-Bohm effect [32].

The metric we shall present cannot be obtained by forbidden coordinate mixing, but it can be obtained from the general Petrov type-D solution already presented by a simple choice of parameters. By requiring the existence of the non-rotating solution (which we know to exist) and the time inversion symmetry, \( t \to -t, \phi \to -\phi \), we get a metric of the form

\[ ds^2 = -N^2 dt^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_P} dP^2 + \frac{\Sigma^2}{\rho^2} (d\phi - \omega dt)^2 \]

(4.1)

where \( P \) is a periodic variable with some period \( T \), \( \phi \) is another angular variable with period 2\( \pi \) and

\[ \rho^2 = r^2 + a^2 P^2 \quad \Delta_P = 1 + \frac{a^2}{\ell^2} P^4 \]

(4.2)

\[ \Delta_r = a^2 - 2mr + \ell^{-2} r^4 \quad \Sigma^2 = r^4 \Delta_P - a^2 P^4 \Delta_r \]

(4.3)

Finally, the angular velocity and the lapse are given by, respectively

\[ \omega = \frac{\Delta_r P^2 + r^2 \Delta_P}{\Sigma^2} a, \quad N^2 = \frac{\rho^2 \Delta_P \Delta_r}{\Sigma^2}. \]

(4.4)
The solution is obtained as a limit case of the Plebanski-Demianski metric by setting $\varepsilon = 0$, $\gamma = a^2$ and rescaling $p = aP$ (this last to have the limit $a \to 0$).

The metric induced on the stationary three-surfaces at some constant $r$ is then

$$d\sigma^2 = \frac{a^2 P^2 + r^2}{\Delta P} dP^2 + \frac{\Sigma^2}{a^2 P^2 + r^2} d\phi^2$$

As long as $\Sigma^2 > 0$, this is a well defined metric on a cylinder, but as it stands it cannot be defined on the torus which one gets identifying some value of $P$, say $P = T/2$, with $P = -T/2$. This is because the components of the metric are even, rational functions of $P$ but have unequal derivatives at $\pm T/2$. We see again the singularities we encountered in the case of Riemann surfaces, when trying to define the metric at boundary points. Thus we need to cover $S^1 \times S^1$ with four coordinate patches, and set $P = \lambda \sin \theta$ in a neighborhood of $\theta = 0$ and $\theta = \pi$ and $P = \lambda \cos \theta$ in a neighborhood of $\theta = \pi/2$ and $\theta = 3\pi/2$, where $\lambda$ is a constant needed to match the length of the circle to the chosen value $T$. On the overlap $\cos \theta$ is a $C^\infty$ function of $\sin \theta$ and vice versa, so now the metric is well defined and smooth on a torus.

Even on the cylinder, the metric (4.1) represents a rotating cylindrical black hole not isometric to the one discussed by Lemos [11, 18] or Santos [17], which are stationary generalizations of the general static cylindrical solution founded by Linet [16]. Thus in this case we have not a unique solution, but rather a many-parameter family of stationary, locally static metrics. This was to be expected as whenever the first Betti number of a static manifold is non vanishing, there exists in general a many-parameter family of locally static, stationary solutions of Einstein’s equations, a fact which can be regarded as a gravitational analogue of the Aharonov-Bohm effect [32].

We shall study now the metric (4.1) for $m > 0$. Notice the symmetry under the combined inversion $r \to -r$, $m \to -m$. The metric coefficients are functions of $(r, P)$ and $P$ is identified independently of $\phi$. Therefore the metric has a global rotational symmetry (unlike the higher genus solutions) and is stationary. We shall consider almost only the region $r \geq 0$ which has a black hole interpretation and is the physically relevant region for black holes forming by collapse. Anyhow, the metric (4.1) admits a sensible continuation to $r < 0$.

### 4.1 Singularity and horizons

The event horizons arise from the zeroes of $\Delta_P$. In the case $m > 0$ that we are considering, all zeroes may appear in the region $r \geq 0$ only. Considering the metric (4.1), one finds that there is a critical value, $a_c$, for the rotation parameter $a$, such that for $a > a_c$ the solution is a naked singularity. For $0 \leq a < a_c$ there are two positive first order roots, $r_+$ and $r_-$ with $r_+ \geq r_-$, which coalesce at the second order root $r_+ = r_- = (m\ell^2/2)^{1/3}$ when $a = a_c$. This critical value is

$$a_c = \sqrt[3]{3(m/2)^{2/3}\ell^{1/3}}$$

The event horizon is located at the larger value $r_+$, and has a surface gravity

$$\kappa = \frac{2r_+^3 - m\ell^2}{\ell^2 r_+^2}$$

The surface gravity vanishes when $a = a_c$ and the metric describes an extreme black hole. Finally, there is a curvature singularity at $r^2 = 0$, namely at $r = P = 0$. As a point set at fixed time, this is $\{p, q\} \times S^1$, where $\{p, q\}$ are the two points on the torus at $r = 0$ which correspond to $P = 0$ (this looks like a pair of disjoint ring singularities, but we have not analysed this in detail). As we can see, the situation is quite similar to the Kerr metric, except that the Euler

\[\text{footnote 1}\]
characteristic of the horizon now vanishes. To check this, notice that the metric on the horizon is, locally
\[ d\sigma^2 = \frac{a^2 P^2 + r_+^2}{\Delta_P} P^2 dP^2 + \frac{\Delta_P r_+^4}{a^2 P^2 + r_+^2} d\phi^2 \] (4.8)

This metric can be written in conformally flat form by factoring out the \( \phi \phi \)-component, which is smooth and positive. The conformal metric has \( \sigma_{\phi\phi} = 1 \) and it turns out to be flat. The actual metric is thus conformally flat and defined on a compact domain. The scalar curvature of a conformally flat manifold is a total divergence and vanishes when integrated over a closed manifold. Therefore the Euler characteristic vanishes and the horizon, which we assumed to be compact and orientable, must be a torus. Furthermore, by rescaling the metric with a constant parameter \( \lambda \), we can see that the periods scale as \( 2\pi \rightarrow 2\pi \lambda, T \rightarrow \lambda T \). Therefore it is the ratio of the periods that is conformally invariant. This ratio determines the conformal class of the torus and is the analogue of the more familiar Teichmüller parameter.

Since all surfaces at constant \( r \) take on the topology of a torus, \( T^2 \), the topology of the external region (the domain of outer communication in Carter’s language) is that of \( \mathbb{R}^2 \times T^2 \).

Finally, few comments on the presence of ergo-regions are in order. Consideration of the metric (4.1) lead us to
\[ g_{tt} = \frac{a^2 \Delta_P - \Delta_r}{\rho^2}. \] (4.9)

From this expression, we see that \( g_{tt} > 0 \) within the regions where \( \Delta_r < 0 \). Conversely, within regions where \( \Delta_r > 0 \), outside the external horizon in particular, \( g_{tt} \) may change its sign becoming positive for \( |r| \) sufficiently small and negative for large \( |r| \). In fact, as in the previously considered case, ergo-tori appear within the regions \( \Delta_r > 0 \). In particular, this happens outside the outermost event horizon. Ergo-tori, where both \( \partial_t \) and \( \partial_r \) are space-like, are bounded by event horizons and the surfaces at \( g_{tt} = 0 \), given by the implicit equation
\[ P^4 = \frac{r^4 - 2m\ell^2 r}{a^4}. \] (4.10)

Differently from the previously examined class of topological rotating black holes, surfaces at \( g_{tt} = 0 \) and horizons do not meet in the case of a toroidal rotating black hole. Indeed, surfaces at \( g_{tt} = 0 \), in the region outside the external horizon fill the interval \( [r_m, r_e] \) where \( r_+ < r_m = (2m\ell^2)^{1/3} \) and \( r_e \) is the positive root of \( r^4 - 2m\ell^2 r - \lambda^4 a^4 = 0 \). (There is another surface at \( g_{tt} = 0 \) for \( r < 0 \), filling the interval \( [r_e', 0] \) where \( r_e' \) is the remaining negative solution of the equation above.)

### 4.2 Mass, angular momentum and comments on thermodynamics

With the given choice of the periods (\( T \) for \( P \) and \( 2\pi \) for \( \phi \)), we determine the area of the event horizon to be
\[ A = 2\pi T r_+^2 \] (4.11)

and the angular velocity, \( \Omega_H = a r_-^{-2} \). The Hamiltonian mass of the given spacetime, relative to the background solution with toroidal topology but \( m = a = 0 \), can be computed by carefully handling the divergent terms appearing when the boundary of spacetime is pushed to spatial infinity. Also, the Killing observers at infinity relative to which the mass is measured, have a residual, \( P \)-dependent angular velocity
\[ \Omega_\infty = a\ell^{-2} P^2 \] (4.12)
and this also must be taken into account. All calculations done, we get a conserved mass $M$ and a conserved angular momentum $J$ according to

$$M = \frac{m T}{2\pi}, \quad J = Ma,$$

(4.13)
giving to $a$ the expected meaning. As $a$ is bounded from above by $a_c$, in order for the solution to have a black hole interpretation, we see that the angular momentum is bounded by a power $M^{5/3}$ of the mass. Moreover, by comparing two solutions with the same parameter $a$ but slightly different masses, one obtains $dM = \kappa dA/8\pi$, showing that the first law in its usual form is not obeyed. We point out, however, that the surfaces defining the stationary frame at infinity are not isotherm, the red-shifted temperature being position dependent. Hence the canonical partition function of the black hole, which requires that the temperature be fixed on the boundary, may eventually permit a first law only in integrated form.

5 Conclusion

We have presented a class of exact solutions of Einstein’s equations with negative cosmological constant, having many of the features which are characteristics of black holes. All these solutions are of Petrov-type $D$ and the horizons, when they exist, have the topology of Riemann surfaces and therefore they lack rotational symmetry for genus $g > 1$. Among the solutions there is also a toroidal black hole, still different from the Lemos-Zanchin solution. The toroidal metric has an exact rotational symmetry and a well-defined mass and angular momentum. From this perspective, it is more promising as a thermodynamical object and one may hope to find suitable generalizations of the ”four laws of black hole mechanics”. Apart from this, the solutions seem to be interesting in their own rights, they have intriguing properties and may provide further ground to test string theory ideas in black hole’s physics.

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\(^3\)Identical conclusions have been stated recently in [34], in connection with constant curvature black holes.
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