Massless Poincaré modules
and gauge invariant equations

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Abstract

Starting with an indecomposable Poincaré module $\mathcal{M}_0$ induced from a given irreducible Lorentz module we construct a free Poincaré invariant gauge theory defined on the Minkowski space. The space of its gauge inequivalent solutions coincides with (in general, is closely related to) the starting point module $\mathcal{M}_0$. We show that for a class of indecomposable Poincaré modules the resulting theory is a Lagrangian gauge theory of the mixed-symmetry higher spin fields. The procedure is based on constructing the parent formulation of the theory. The Labastida formulation and the unfolded description of the mixed-symmetry fields are reproduced through the appropriate reductions of the parent formulation. As an independent check we show that in the momentum representation the solutions form a unitary irreducible Poincaré module determined by the respective module of the Wigner little group.
1 Introduction

Several approaches to massless mixed symmetry fields on the Minkowski space are known up to now. Although the existence of the respective irreducible modules in \( d > 5 \) was clear since the famous Wigner classification \([1]\) finding the covariant and gauge invariant field equations realizing such modules on local fields was not completely obvious. Such equations have been proposed much later by Labastida \([2]\) along with the candidate Lagrangian \([3]\). The rigorous proof that the Labastida fields indeed describe respective representations of the Wigner little group for general spins was missing till recently \([4]\) (see also \([5]\)). The unfolded form of the equations of motion and the respective local Lagrangian have been proposed in \([6, 7]\). Massless two-row fields on Minkowski space have been also analyzed within the BRST approach in \([8]\). Note also a recently proposed alternative formulation \([9]\) that treats higher spin fields by relaxing any algebraic constraints.\(^1\)

The above approaches have brought to light a number of useful algebraic structures and field-theoretical methods. However, these formulations lead to either quite involved set of covariant fields and associated algebraic constraints or hidden structure of the gauge invariance. Moreover, the interrelation between different approaches and their dynamical equivalence remains unclear beyond the case of two-row fields.

These problems are mainly due to the lack of unifying algebraic structures underlying the formulations. This calls for the proper algebraic and dynamical framework that allows one to treat the theory in model-independent terms and to use the powerful machinery of the representation theory combined with an effective technique to handle the involved gauge symmetry and the constraints present in the models.

In this paper we take a rather abstract point of view and describe a class of massless Poincaré modules in terms of Howe dual pair of Lie algebras: the Lorentz algebra \( o(1, d – 1) \) and symplectic algebra \( sp(2n) \) represented on the suitable polynomials (\( n – 1 \) corresponds to the number of rows in the Young tableau of the respective covariant field). It turns out that the massless Poincaré modules that can be realized on local fields naturally arise as quotient spaces rather than just subspaces of polynomials. This is crucial because in the field theory this quotient construction is realized through the gauge invariance.

Another important ingredient is the BRST (cohomology) technique that allows one to translate the pure algebraic definition of the Poincaré module into the genuine local gauge field theory. This can be seen as a far going generalization of the following procedure known in the literature (see, e.g., \([13]\) for the discussion in the related context): given an \( \mathfrak{g} \)-module \( \mathcal{M}_0 \) and a manifold \( X \) equipped with a flat \( \mathfrak{g} \)-connection one considers \( \mathcal{M}_0 \)-valued field subjected to the covariant constancy condition understood as an equation of

\(^1\)As far as particular cases of mixed-symmetry fields are concerned there are various successful approaches available in the literature \([10, 11, 12]\).
motion. The space of solutions to this equation coincides with (in general, is closely related to) \( \mathcal{M}_0 \) and the system is explicitly invariant under \( \mathfrak{f} \). However, this construction does not directly lead to gauge invariant equations. In particular, this makes the equations of motion in general non-Lagrangian. Moreover, studying possible interactions becomes complicated because nonlinear deformations are usually formulated in terms of gauge potentials.

The procedure proposed in this paper allows one to find a complete set of gauge fields needed for the gauge theory description of the given Poincaré module \( \mathcal{M}_0 \). More precisely we consider the indecomposable Poincaré modules induced from a given irreducible Lorentz module determined by spins \( s_{n-1} \geq s_{n-2} \geq \ldots \geq s_1 \). The idea is to realize the Poincaré module \( \mathcal{M}_0 \) as the ghost-number-zero cohomology of the appropriate BRST operator \( Q \). Using the BRST extension \([15, 16, 17]\) of the unfolded formalism \([18, 19, 20, 21, 22, 6]\) allows us to immediately construct the local gauge field theory by replacing the covariant constancy condition with its BRST extension using the generalized covariant derivative

\[
\hat{\Omega} = \nabla + Q.
\]

This derivative is naturally interpreted as a BRST operator of a first-quantized constrained system so that the constructed field theory is a free field theory associated to this quantum constrained system. In the case \( n = 2 \) (totally symmetric fields) this formulation was identified in \([15]\).

It has to be stressed that using the BRST technique brings in the ghost grading that selects physical fields (those at ghost number zero) among all the fields entering the BRST extended formulation. It turns out, that besides the \( \mathcal{M}_0 \)-valued fields one finds other ghost number zero fields that are necessarily differential forms of nonvanishing degree. These fields are automatically gauge fields, with the gauge transformations and the reducibility relations determined by the BRST operator \( \hat{\Omega} \).

Using the method developed in \([15, 16]\) allows us to extend the formulation based on \( \hat{\Omega} \) to an equivalent formulation where some of the algebraic constraints are implemented implicitly by the appropriately extended BRST operator. This determines a proper counterpart of the parent theory from \([15]\) that serves to obtain various other formulations through the equivalent reductions (elimination of generalized auxiliary fields). In particular, we show that the parent theory reduces to the well-known Labastida theory \([2, 3]\) and the recently constructed unfolded formulation \([6]\). As a byproduct this gives a proof that these two formulations are locally equivalent at the level of equations of motion, i.e. the equivalence of the metric-like and the frame-like local formulations.

Making use of the parent theory allows one to find another particular reduced theory that admits a standard Lagrangian of the form \( \langle \Psi, \Omega \Psi \rangle \). This has the same structure as the analogous Lagrangian for Fronsdal HS fields proposed in \([23, 24, 25]\). Just BRST operator \( \Omega \) entering the action is known in the literature as an appropriate truncation of the

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\(^2\)From this perspective our approach can be viewed as somewhat similar to the method of covariantized light-cone developed in \([14]\).
open bosonic string BRST operator in the tensionless limit \([26, 27]\). What we prove here is that this Lagrangian indeed describes an irreducible mixed-symmetry field provided the appropriate set of algebraic constraints are imposed on \(\Psi\).

As an independent check we show that the space of gauge inequivalent solutions of the model in the momentum representation with \(p \neq 0\) indeed coincides with the irreducible unitary module induced from the respective module of the Wigner little group with the same spins \(s_{n-1} \geq s_{n-2} \geq \ldots \geq s_1\). This shows that the constructed theory indeed describes the unitary dynamics of the right number of physical degrees of freedom.

The approach developed in the paper can be applied far beyond the context of the Poincaré invariant equations. In particular, it can be extended to cover the linear equations in the \(AdS_d\) space, where the parent formulation is known \([16]\) for the case of Fronsdal fields, i.e. \(n = 2\). There also remains to see how the massive fields can be described in this way. More precisely, how the dimensional reduction can be implemented in these terms. More ambitious perspective has to do with describing the gauge field realization of \(F\)-modules (for \(F\) sufficiently general) on the homogeneous spaces \(F/G\).

The paper is organized as follows: the main construction is presented in detail in Section 2. There we also introduce most of the technical tools needed throughout the paper. In Section 3 we show how the field content and the equations of motion of the Labastida theory can be obtained by an appropriate reduction of the parent formulation and discuss its relation to the tensionless limit of string theory. Section 4 is devoted to the analysis of various Poincaré modules appearing in the different formulations. This involves explicit reduction to the unfolded form and establishing a relationship with the modules of the Wigner approach. Conclusions and perspectives are discussed in Section 5.

2 BRST operator for mixed-symmetry fields on Minkowski space

2.1 Howe dual realization of the Poincaré algebra

Let us start with Minkowski space \(ISO(1, d-1)/SO(1, d-1)\) whose algebra of infinitesimal isometries is the Poincaré algebra \(iso(1, d-1)\). We denote the basis elements of the Poincaré algebra as \(P_a\) and \(M_{ab}\) (translations and Lorentz transformations). Suppose we are interested in the representations induced from the finite-dimensional irreducible representations of the Lorentz subalgebra \(so(1, d-1)\). It is useful to discuss first the subspaces irreducible under the Lorentz subalgebra that can be nicely described using the following oscillator realization.

Let us introduce bosonic variables \(a^a_I\) and \(\bar{a}^b_J\), \(a, b = 0, \ldots, d-1, \ I, J = 0, \ldots, n-1\).
satisfying the canonical commutation relations

\[ [\bar{a}_a^I, a_J^b] = \delta_J^I \delta_a^b. \]  

(2.1)

It is assumed that \( \bar{a}_a^I \) acts as \( \frac{\partial}{\partial a_I^a} \) on the space \( \mathcal{P}_n^d(a) \) of polynomials in \( a_I^a \)

\[
\phi(a) = \sum_{m_I} \phi_{a_1 \ldots a_{m_0}; \ldots; e_1 \ldots e_{m_{n-1}}} a_0^{a_1} \cdots a_0^{e_0} \cdots a_0^{e_{m_0}} \cdots a_{n-1}^{a_{m_1}} \cdots a_{n-1}^{e_{m_{n-1}}} ,
\]

(2.2)

where \( m_I \equiv (m_0, \ldots, m_{n-1}) \) are arbitrary non-negative integers. Introducing the Minkowski metric \( \eta_{ab} \) one can represent the Lorentz algebra on \( \mathcal{P}_n^d(a) \) as

\[
M_{ab} = a_{Ia} \bar{a}_b^I - a_{Ib} \bar{a}_a^I .
\]

(2.3)

Here and in what follows indices \( a, b \) are raised and lowered using the Minkowski metric. It follows that the expansion coefficients in (2.2) transform as Lorentz tensors. The space of all polynomials decomposes into the finite-dimensional irreducible modules of the Lorentz algebra. In order to describe all the finite-dimensional modules with integer spins in a given dimension \( d \) one needs to take \( n = [\frac{d}{2}] \).

It is useful to study the structure of \( \mathcal{P}_n^d(a) \) as the module over the orthogonal algebra \( so(1, d-1) \) using the Howe duality [28, 29]. The Howe dual algebra to the \( so(1, d-1) \) algebra is \( sp(2n) \) algebra with the basis elements given by [28, 29]

\[
T_{IJ} = a_I^a a_J^a , \quad T_I^J = \frac{1}{2} (a_I^a a_J^a + a_J^a a_I^a) , \quad T^{IJ} = \bar{a}_a^I \bar{a}^J_a .
\]

(2.4)

Their non-zero commutation relations read

\[
[T_I^J, T_K^L] = \delta_K^J T_I^L - \delta_I^J T_K^L , \quad [T_I^J, T_K^L] = \delta_K^J T_I^L + \delta_I^J T_K^L + \delta^J_L T_I^K - \delta^J_K T_I^L ,
\]

\[
[T_K^L, T_I^J] = \delta_J^L T_K^I + \delta_I^L T_K^J , \quad [T^{IJ}, T_K^L] = \delta_K^J T^{IL} + \delta^J_K T^{IL} .
\]

The diagonal elements \( T_I^I \) form a basis in the Cartan subalgebra while \( T^{IJ} \) and \( T_I^J, I > J \) are the basis elements of the upper-triangular subalgebra. Let us note that \( gl(n) \) algebra is realized by the generators \( T_I^J \) as a subalgebra of \( sp(2n) \) while its \( sl(n) \) subalgebra is generated by \( T_I^J \) with \( I \neq J \).

The finite-dimensional irreducible representations of the Lorentz algebra in the space of polynomials in \( a_I^a \) are singled out by the highest weight conditions of the dual \( sp(2n) \), \( i.e. \) annihilated by the upper triangular subalgebra of \( sp(2n) \) along with the weight conditions with respect to the Cartan subalgebra. In addition, to describe all integer spin finite-dimensional Lorentz irreps one needs to take \( n \leq \nu \), where \( \nu = [\frac{d}{2}] \) is a rank of the Lorentz algebra \( so(1, d-1) \). More precisely, let \( s_I \) be integer numbers such that \( s_I \geq s_J \) for \( I > J \). We assume that the following weight conditions are imposed

\[
T_I^I \phi = (s_I + \frac{d}{2}) \phi .
\]

(2.5)
Imposing then the tracelessness and Young symmetry conditions

\[ T^{IJ} \phi = 0, \quad T^I_J \phi = 0 \quad I > J, \tag{2.6} \]

one gets a finite-dimensional irreducible representation of the Lorentz algebra described by Young tableau of the symmetry type \((s_{n-1}, s_{n-2}, \ldots, s_0)\)

\[
\begin{array}{cccccccc}
& & & & & & & s_{n-1} \\
& & & & & & s_{n-2} \\
& & & & & s_1 \\
& & & & s_0 \\
\end{array}
\tag{2.7}
\]

Let us now briefly recall the formal structure of the polynomials in \(a^d_f\) as a module over the Howe dual \(so(1, d-1)\) and \(sp(2n)\) algebras. More detailed discussion can be found in the Appendix A where we also collect some useful statements needed in the main text. \(\mathcal{P}^d_n(a)\) considered as a \(so(1, d-1)\) and \(sp(2n)\) bimodule can be lifted to the respective complex module of the complexified algebras. The structure of the irreducible components is unchanged under the complexification. This allows us to use the results known in the literature. Since \(so(1, d-1)\) and \(sp(2n)\) algebras obviously commute, the space of polynomials \(\mathcal{P}^d_n(a)\) is a \(so(d) - sp(2n)\) bimodule. For \(n \leq \left[ \frac{d}{2} \right]\) bimodule \(\mathcal{P}^d_n(a)\) has the following structure \([30]\)

\[ \mathcal{P}^d_n = \bigoplus_{\sigma \in \Lambda} (V_\sigma \otimes U_{\theta(\sigma)}) , \tag{2.8} \]

where \(V_\sigma\) and \(U_{\theta(\sigma)}\) are respectively irreducible \(so(d)\) and \(sp(2n)\) modules with highest weights \(\sigma\) and \(\theta(\sigma)\), where \(\theta\) is some mapping (for more details see Appendix A). While \(V_\sigma\) is finite-dimensional \(U_{\theta(\sigma)}\) is the generalized Verma module induced from the finite-dimensional irreducible \(sl(n)\) module (more precisely, from the module of the corresponding parabolic subalgebra in \(sp(2n)\)). In particular, this implies that \(U_{\theta(\sigma)}\) is freely generated by generators \(T_{IJ}\) from the respective \(sl(n)\)-module\(^3\).

### 2.2 Poincaré modules

Remarkably the set of oscillators (2.1) allows one to realize the Poincaré algebra as well. To this end we relax some of the conditions (2.5) and (2.6) in order to describe some infinite-dimensional (indecomposable) representations of the Poincaré algebra. First of all besides the \(sp(2n)\) algebra relations there are no additional relations between elements of the form \(T_{I_1J_1}T_{I_2J_2}\cdots T_{I_kJ_k}\phi\) with \(\phi\) in \(sl(n)\)-module. For instance, as a linear space \(U_{\theta(\sigma)}\) is isomorphic to polynomials in \(T_{IJ}\) with coefficients in the \(sl(n)\)-module.
all we choose the Poincaré generators $P_a$ to act as “translations” for the $I$-th oscillators. Without loss of generality we take $I = 0$ so that

$$P_a = \frac{\partial}{\partial a_0^a}. \tag{2.9}$$

In the sequel we use the following notations $a_0 \equiv y, a_I \equiv a_i, I > 0$ with $i = 1, \ldots, n - 1$. Furthermore, it is convenient to introduce special notations for some $sp(2n)$ generators

$$S_i^\dagger \equiv T_i^0 = a_i^a \frac{\partial}{\partial y^a}, \quad \bar{S}^i_i \equiv T_0^i = y^a \frac{\partial}{\partial a_i^a}, \tag{2.10}$$
$$N_i^j \equiv T_i^j = a_i^a \frac{\partial}{\partial a_j^a} i \neq j, \quad N_i \equiv T_i^i - \frac{\ell}{2} = a_i^a \frac{\partial}{\partial a_i^a}. $$

Operators $P_a$ obviously commute with all the irreducibility conditions but $T_0^0 \phi = (s_0 + \frac{\ell}{2})\phi$. By relaxing this condition one gets a representation (in fact indecomposable) of the Poincaré algebra. This representation is finite-dimensional as the conditions $S_i^\dagger \phi = 0$ imply that for a homogeneous element a homogeneity degree in $y$ is lower than that in $a_i^a$. In other words, the operator $P_a$ acts on the last row of the corresponding Young tableau by shortening its length and the whole carrier space consists of Lorentz irreps described by Young tableaux (2.7) with $0 \leq s_0 \leq s_1$. To summarize, the resulting Poincaré module is singled out by the conditions

$$T^{IJ} \phi = 0, \quad N_i \phi = s_i \phi, \quad N_i^j \phi = 0 \quad i > j, \tag{2.11} \quad S_i^\dagger \phi = 0. \tag{2.12}$$

Although the Poincaré module just constructed plays an important role in the subsequent analysis it is not the one we are interested in now. This is because a representation realized on local fields is necessarily infinite-dimensional. In order to arrive at an infinite-dimensional module let us consider a subspace $\mathcal{M}_0$ singled out by a slight modification of (2.5) and (2.6). Namely, in addition to relaxing $T_0^0 \phi = (s_0 + \frac{\ell}{2})\phi$ we also invert the Young conditions involving $y^a \equiv a_0^a$ so that the full set of the conditions reads explicitly as

$$T^{IJ} \phi = 0, \quad N_i \phi = s_i \phi, \quad N_i^j \phi = 0 \quad i > j, \tag{2.13} \quad \bar{S}^i_i \phi = 0. \tag{2.14}$$

For the corresponding Young tableau this implies a rearranging the rows by moving the last row to the top as expressed by the last condition. The resulting module $\mathcal{M}_0$ is de-
scribed by an infinite collection of Young tableaux

\[
\begin{array}{c|c|c|c}
\hline
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\hline
s_0 & s_1 & s_2 & \ldots & s_n-1
\end{array}
\]

(2.15)

with running $s_0$ bounded from below, $s_0 \geq s_{n-1}$. Although the condition in (2.14) does not commute with $P_a$ one can consistently define the action of $P_a$ on the subspace using the appropriate projector. The quadratic Casimir operator of the Poincaré algebra $C_2 = P^2 = T^{00}$ is automatically zero on module $M_0$ because of (2.13) so that $M_0$ is the massless Poincaré module. In the unfolded description of Fronsdal fields on $AdS_d$ the respective counterpart of $M_0$ is often referred to as Weyl module.

The origin of the difference between the Poincaré module determined by (2.11),(2.12) and $M_0$ is that they are described by the highest weight conditions with respect to the two different choices of the upper triangular subalgebra of $sl(n) \subset sp(2n)$. They are generated by $(N_i^j, i > j, S^i_j)$ and $(N_i^j, i > j, \bar{S}^i_j)$, respectively. Moreover, the irreducibility conditions for these Poincaré modules contain the subalgebra formed by $N_i^j, i > j$. In fact there are other choices for the upper triangular subalgebra containing $N_i^j, i > j$ that play the essential role in the subsequent analysis.

### 2.3 BRST realization

It turns out that subspace $M_0$ defined by (2.13) and (2.14) can be represented in an explicitly Poincaré invariant way. The idea is to identify it as an appropriate quotient of a Poincaré invariant subspace with respect to a Poincaré invariant equivalence relation. Indeed, as we have noted $M_0$ is defined by the highest weight conditions (for an appropriate choice of weight ordering) of the $sl(n)$ algebra generated by $N_i^j, i > j$ along with $\bar{S}^{ti}_j$ and $S^{ti}_j$. By decomposing the entire space into the finite-dimensional irreducible $sl(n)$ components one finds that in each component the only element satisfying $N_i^j \phi = 0$ $i > j$ and not in the image of any of $S^{ti}_j$ is the highest weight vector $\bar{S}^{ti}_j \phi = 0$. Because generators $S^{ti}_j$ obviously commute with $P_a$ we arrive at the following Poincaré invariant equivalence relation

\[
\phi(y, a) \sim \phi(y, a) + S^{ti}_j \phi^{i}(y, a),
\]

(2.16)

and hence the representatives can be identified with those satisfying (2.14).

It appears useful to implement this construction in the BRST terms. To this end we
introduce fermionic ghost variables $c^i$, $b_i$, $i = 1, \ldots, n - 1$ satisfying

$$[c^i, b_j] = \delta^i_j, \quad \text{gh}(c^i) = 1, \quad \text{gh}(b_i) = -1,$$

where gh(·) denotes the ghost degree. These variables are represented on functions of $b^i$ as $c^i \phi = \frac{\partial}{\partial b_i} \phi$. We consider the following BRST operator

$$Q = c^i S^i_j.$$

(2.18)

Because the constraints form the Abelian algebra $[S^i_j, S^j_i] = 0$ the terms cubic in ghosts are absent in the BRST operator.

The space (2.13) can be identified then with the ghost-number-zero cohomology of $Q$ evaluated in the space of elements satisfying

$$\tilde{N}_i \phi \equiv (N_i + b_i c^i) \phi = s_i \phi, \quad \tilde{N}_i^j \phi \equiv (N_i^j + b_i c^j) \phi = 0 \quad i > j, \quad T^{IJ} \phi = 0.$$  

(2.19)

Here operators $\tilde{N}_i$ and $\tilde{N}_i^j$ are the BRST invariant extensions of the respective operators in (2.13), i.e. $[Q, \tilde{N}_i] = [Q, \tilde{N}_i^j] = 0$. As for the trace operators $T^{IJ}$ they are imposed directly because all the remaining conditions and the BRST operator $Q$ preserve the subspace singled out by $T^{IJ} \phi = 0$. It is easy to check that conditions (2.19) are consistent and $Q$ acts in subspace (2.19). Moreover, in the zeroth ghost degree (i.e. $b_i$-independent elements) conditions (2.19) explicitly coincides with the conditions (2.13).

A useful way to see that the construction is consistent is to observe that all the constraints (2.19) along with the constraint $S^i_j$ entering the BRST operator form the upper-triangular subalgebra of $sp(2n)$ completed by the weight conditions from the diagonal (Cartan) subalgebra. An alternative way to implement the construction is to impose all these constraints by the appropriate BRST operator and require in addition the cohomology representatives to be independent of all the ghost variables but $b_i$. In fact a similar representation is going to be useful in Sections 2.6 and 4.1.

Because translations $P_a$ and Lorentz generators $M_{ab}$ obviously commute with $Q$ and conditions (2.19) the Poincaré algebra acts in the cohomology. At the same time, the zero-ghost-number cohomology is given by the ghost-independent elements quotient over the image of $S^i_j$ leading to equivalence relation (2.16). To see that representatives of these equivalence classes can be chosen to satisfy $\tilde{S}^{ti} \phi = 0$ we note that for a ghost-independent element conditions $\tilde{N}_i^j \phi = 0 \quad i > j$ reduce to $N_i^j \phi = 0 \quad i > j$. This shows that the zero-ghost-number $Q$-cohomology indeed coincides with module $\mathcal{N}_0$.

Remarkably, $Q$-cohomology in other ghost degrees is in general nonempty. It is represented by the highest weight vectors for other choices of the upper triangular subalgebra of $sl(n)$ containing $N_i^j \quad i > j$. A detailed discussion will be given in Section 4.1.

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5Here and in what follows the commutator denotes the graded commutator, $[f, g] = fg - (-)^{|f||g|}gf$, where $|f|$ is the Grassmann parity of $f$.  

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10
2.4 Poincaré module of the solutions to PDE on Minkowski space

We now address a question of how a Poincaré module can be realized on the space of solutions of a system of differential equations on Minkowski space. This can be achieved using the construction known in the literature (see e.g. [13] for the discussion in the related context). The construction can be formulated in rather general terms. Namely, let \( M_0 \) be an \( F \)-module (\( F \) being a Lie group, not necessarily the Poincaré group). Let also \( X = F/G \) with \( G \subset F \) be a symmetric space so that there is a canonical principle \( F \)-bundle over \( X \). One then constructs the associated vector bundle with the fiber being \( M_0 \).

There is a flat \( f \)-connection (originating from the canonical \( f \)-valued form on \( F \); here \( f \) is a Lie algebra of \( F \)) on the principle \( F \)-bundle over \( X \), which determines a flat connection \( \alpha \) in the associated vector bundle.

Using the \( f \)-connection \( \alpha \) one can represent \( M_0 \) as the space of covariantly constant sections of the associated vector bundle, i.e. sections satisfying

\[
\nabla \Phi = 0, \quad \nabla = dx^a \left( \frac{\partial}{\partial x^a} + \alpha_a \right),
\]

where \( x^a \) are local coordinates on \( X \). Indeed, the space of solutions to this equation in the appropriate functional space is isomorphic to (in general, closely related to) the fiber at a given point, i.e. \( F \)-module \( M_0 \).

Let us discuss how the Lie algebra \( f \) of \( F \) acts on solutions. To this end let \( L_A \) be a basis in \( f \) and by a slight abuse of notations we also denote by \( L_A \) the action of \( L_A \) in \( M_0 \).

As usual in the field theory it is useful to define the action on fields such that the base space \( X \) is not affected. Let in a given point \( p \in X \) the algebra acts on the field according to \( \lambda|_p \Phi|_p = (\lambda^A|_p) L_A \Phi|_p \), where \( \lambda^A|_p \) are components of \( \lambda \). This action can be uniquely extended on \( X \) to the action of the form \( \lambda^A(x)L_A\Phi(x) \) by requiring

\[
[\nabla, \lambda^A(x)L_A] = d\lambda(x) + [\alpha^A L_A, \lambda^B(x)L_B] = 0,
\]

i.e. the action on the field is determined by a covariantly constant section \( \lambda^A(x) \) of the associated vector bundle with the fiber being \( f \). This guaranties that (2.20) is indeed \( F \)-invariant.

This construction is easily specialized to the case where \( F \) is a Poincaré group, \( G \) its Lorentz subgroup, \( X \) Minkowski space, and \( M_0 \) the Poincaré module considered above, and \( P_a, M_{ab} \) are the Poincaré generators in \( M_0 \). In the Cartesian coordinates \( x^a \) on \( X \) the connection form \( \alpha \) can be chosen to be \( \alpha = -dx^a P_a \) so that (2.20) takes the form

\[
dx^a \left( \frac{\partial}{\partial x^a} - P_a \right) \Phi = 0.
\]

The action of the Poincaré generators on fields (sections) can be obtained from (2.21). Namely, the translations and Lorentz rotations act respectively as

\[
\hat{P}_a \Phi = P_a \Phi, \quad \hat{M}_{ab} \Phi = M_{ab} \Phi + x_a P_b \Phi - x_b P_a \Phi .
\]
Modified generators $\hat{P}, \hat{M}$ satisfy the same algebra. Recall that $P_a$ denotes an appropriate projection of $\frac{\partial}{\partial y^a}$ on module $M_0$.

The above construction can be illustrated in the case $n = 1$ where variables $a_i^a$ are not present\(^6\) so that the Poincaré generators in $M_0$ do not require projectors and are given by $P_a = \frac{\partial}{\partial y^a}$ and $M_{ab} = y_a \frac{\partial}{\partial y^b} - y_b \frac{\partial}{\partial y^a}$. Their $x$-dependent realizations (i.e. action on $M_0$-valued sections) read as

$$\hat{P}_a \Phi = \frac{\partial}{\partial y^a} \Phi, \quad \hat{M}_{ab} \Phi = (x_a + y_a) \frac{\partial}{\partial y^b} \Phi - (x_b + y_b) \frac{\partial}{\partial y^a} \Phi. \quad (2.24)$$

In this simple example the covariant constancy condition (2.22) just says that $\Phi(x, y) = \Phi(x + y, 0) = \Phi(0, x + y)$.

The Poincaré invariant equations (2.22) are not completely satisfactory from various viewpoints. First of all, there is no gauge symmetry. More precisely, as we are going to see $M_0$-valued fields can be identified with gauge-invariant HS curvatures. What is more important, equations (2.22) are not likely to be Lagrangian even if one adds/eliminates auxiliary fields (recall that already Maxwell equations are Lagrangian only if one introduces potentials and hence the gauge symmetry). In addition, the Poincaré algebra is in general realized by the operators involving projectors in contrast to the realization on polynomials or their Poincaré invariant subspaces.

Before replacing (2.22) with a genuine gauge theory let us also note that strictly speaking, as solutions to the equation (2.22) one only gets polynomials in $x^a$ because in the fiber we have not allowed for elements non-polynomial in $y^a \equiv a_i^0$. The way out is to consider a somehow maximal fiber\(^7\) that is the space of elements that are formal power series in $y^a$ and polynomials in the remaining oscillators. In this way one can describe solutions from, e.g., $C^\infty(X)$. Note, however, that in this setting the space of solutions is not isomorphic to the fiber because there can be nonconvergent power series that cannot be extended to a smooth covariantly constant sections. In what follows we assume formal power series in $y^a$ variables.

### 2.5 Intermediate formulation

In order to be able to obtain genuine gauge symmetries in this framework we are going to replace the Poincaré module $M_0$ with a graded Poincaré module $M$ containing $M_0$ at zeroth degree and then consider a gauge theory associated to this graded space in a similar way as non-gauge theory (2.22) is associated to $M_0$. In fact, we already have all the requisites for this generalization. Indeed, the cohomology of $Q$ evaluated in (2.19)\(^6\) this case corresponds to the Klein–Gordon field. The respective equations of motions in the form (2.22) were thoroughly studied in [20].

\(^{\text{7}}\)This choice is natural from the first-quantized point of view (see [31]).
is a Poincaré module graded by the ghost degree such that $M_0$ is its degree zero subspace. Moreover, it is well known how the construction (2.20) can be generalized once the module is described in terms of the BRST operator. This generalization is known as a BRST extended unfolded formulation. It has been proposed in [15, 16] (see also [17]) in constructing the so-called parent formulations of the linear gauge theories.

The construction of the BRST extended unfolded formulation proceeds as follows. Replacing $dx^a$ with the Grassmann odd ghost variables $\theta^a$, $\text{gh}(\theta^a) = 1$ one extends the BRST operator $Q$ to

$$\hat{\Omega} = \nabla + Q, \quad \nabla = \theta^a(\frac{\partial}{\partial x^a} - \frac{\partial}{\partial y^a}),$$

and takes as a representation space functions in $x^a$ with values in the tensor product $\hat{H}$ of the representation space for $Q$ (i.e., the space of formal series in $y^a$ and polynomials in $a_i^b, b_i$ satisfying (2.19)) and the Grassmann algebra generated by $\theta^a$. Although the theory (2.25) is explicitly written in Cartesian coordinates on $X$ and the adapted local frame it can easily be rewritten in terms of arbitrary coordinates $x^\mu$ and arbitrary local frame using a more general flat covariant derivative $\nabla = \theta^\mu((\frac{\partial}{\partial x^\mu} - e^a_{\mu}(\frac{\partial}{\partial y^a} - \omega^b_{\mu a}(y^a(\frac{\partial}{\partial y^b} + a^a_{\mu}(\partial/\partial \varphi^a)))$. Here $e^a_{\mu}$ and $\omega^b_{\mu a}$ are coefficients of the flat Poincaré connection $\alpha$ and are to be identified with the vielbein and the Lorentz connection on $M_0$.

Given a BRST operator $\hat{\Omega}$ represented on $\hat{H}$-valued functions in $x^a$ the associated gauge field theory is determined by the BRST differential $s$ defined through $s \Psi = \hat{\Omega} \Psi$, where $\Psi$ is the respective string field. More precisely, if a representation space is a space of functions with values in a graded space $\hat{H}$ with basis $e_A$ then the string field is the following object (see, e.g., [15, 32])

$$\Psi(x) = \psi^A(x)e_A, \quad \text{gh}(\psi^A) = -\text{gh}(e_A), \quad |\psi^A| = |e_A|,$$

where $\psi^A$ are fields (including ghosts, antifields, etc) of the associated free field theory determined by $s$. Note that $\text{gh}(\Psi) = 0$. The relation $s \Psi = \hat{\Omega} \Psi$ indeed defines the action of $s$ on fields $\psi^A$. This action extends to space-time derivatives $\partial_{\mu_1} \ldots \partial_{\mu_k} \psi^A$ through $[s, \frac{\partial}{\partial x^\mu}] = 0$ and hence to local functions (functions of fields and their derivatives).

It is useful to decompose the string field according to the ghost number of fields $\psi^A$ so that $\Psi = \sum_k \Psi^{(k)}$ with $\Psi^{(k)} = \psi^{A_k}e_{A_k}$, $\text{gh}(\psi^{A_k}) = k$. The fields entering $\Psi^{(0)}$ are identified as physical fields. Gauge parameters are associated with the fields entering $\Psi^{(1)}$. The reducibility gauge parameters are then associated to fields entering $\Psi^{(2)}$ and so on. The equations of motion and gauge symmetries are then

$$\hat{\Omega} \Psi^{(0)} = 0, \quad \delta \Psi^{(0)} = \hat{\Omega} \Psi^{(1)}, \quad \delta \Psi^{(1)} = \hat{\Omega} \Psi^{(2)}, \quad \ldots,$$

where in the definition of the gauge transformations and the reducibility relations one needs to replace ghost fields with the respective gauge and reducibility parameters.
It can be useful to identify \( \Psi^{(0)} \) with a general ghost-number-zero element of the space of \( \hat{\mathcal{H}} \)-valued functions. In the same way, the gauge parameters of order \( l \) are identified with \( \hat{\mathcal{H}} \)-valued functions of ghost number \(-l+1\). For instance, in these terms the gauge transformation law takes the usual form \( \delta \Psi^{(0)} = \hat{\Omega} \xi^{(-1)} \), where \( \xi^{(-1)} \) with \( gh(\xi^{(-1)}) = -1 \) is the gauge parameter.

Let us now explicitly find the field content, equations of motion and gauge symmetries of the theory determined by \( \hat{\Omega} \) and \( \hat{\mathcal{H}} \). To this end, let us introduce the component fields entering the ghost-number-zero component of the string field

\[
\Psi^{(0)} = \psi_0 + \psi_1 + \ldots + \psi_{n-1}, \quad \psi_p = \psi^{i_1 \ldots i_p} (x; y, a) b_{i_1} \ldots b_{i_p} \theta^{a_1} \ldots \theta^{a_p}.
\]

(2.28)

Fields \( \psi_p \) are naturally identified as differential \( p \)-forms on \( X \) taking values in the space of polynomials in \( y^a, a_i \) and ghosts \( b_i \) subjected to the conditions (2.19). The equations of motion take the form

\[
\begin{align*}
\nabla \psi_0 + S_i \frac{\partial}{\partial b_i} \psi_1 &= 0, \\
\nabla \psi_1 + S_i \frac{\partial}{\partial b_i} \psi_2 &= 0, \\
\ldots \\
\nabla \psi_{n-1} &= 0.
\end{align*}
\]

(2.29)

The gauge parameters corresponds to the ghost-number-one fields entering \( \Psi^{(1)} \) and can be represented as

\[
\xi^{(-1)} = \xi_1 + \xi_2 + \ldots + \xi_{n-1}, \quad \xi_p = \xi^{i_1 \ldots i_p} (x; a, y) b_{i_1} \ldots b_{i_p} \theta^{a_1} \ldots \theta^{a_{p-1}}.
\]

(2.30)

For instance, gauge parameter \( \xi_1 = \xi^i b_i \) and is a 0-form. The gauge transformations have the form

\[
\begin{align*}
\delta_{\xi} \psi_0 &= S_i \frac{\partial}{\partial b_i} \xi_1, \\
\delta_{\xi} \psi_1 &= \nabla \xi_1 + S_i \frac{\partial}{\partial b_i} \xi_2, \\
\delta_{\xi} \psi_2 &= \nabla \xi_2 + S_i \frac{\partial}{\partial b_i} \xi_3, \\
\ldots \\
\delta_{\xi} \psi_{n-1} &= \nabla \xi_{n-1}.
\end{align*}
\]

(2.31)

In the same fashion, one can also write down the reducibility parameters of order \( l \) that are associated to the fields of ghost number \( l+1 \) and the respective reducibility relations determined by \( \hat{\Omega} \). Note that in general there are fields of ghost number up to \( n-1 \) so that there are reducibility relations of order up to \( n-2 \). The reducibility parameters of order \( l \) are \( p \)-forms with \( p \leq n-3 \).

---

\(^8\)In the case where physical fermionic fields are present this requires some care because coefficients \( \phi^A(x) \) in the expansion of a general element \( \phi = \phi^A e_A \) of the representation space are always bosonic while the fields entering \( \Psi = \psi^A e_A \) have Grassmann parity \( |\psi^A| = |e_A| \).
The formulation determined by
\[
\hat{\Omega} \Psi^{(0)} = 0, \quad \delta \xi \Psi^{(0)} = \hat{\Omega} \xi^{(-1)}, \quad \ldots, \quad \hat{\Omega} = \nabla + Q
\]  
(2.32)
is a natural generalization of the so-called intermediate form of the Fronsdal HS fields found in [15] (see also [16] for the case of \(AdS_d\) space) to the case of the mixed-symmetry fields. Although this formulation appears here on the first place we keep the term “intermediate” because as we are going to see it is an intermediate formulation between the so-called parent formulation and the unfolded one. In particular, for \(n = 2\) BRST operator \(\hat{\Omega}\) and hence the equations of motion and the gauge symmetries explicitly coincide with that identified in [15]. We claim that (2.32) defines the gaugetheory of mixed-symmetry HS fields on the Minkowski space. Namely, we show that by eliminating the generalized auxiliary fields this theory can be taken to the explicitly Lagrangian form, leading to Labastida equations of motion [2, 3]. In addition, eliminating a different collection of the generalized auxiliary fields one arrives at the unfolded form of the theory (this was recently constructed from scratch in [6]). Let us note that for the off-shell version of the Fronsdal theory the intermediate form naturally arises as a linearization of the nonlinear off-shell system in both Minkowski [33] and \(AdS_d\) space [17].

Using the gauge symmetry (2.31) one can always achieve \(\vec{S} \hat{\xi} \psi_0 = 0\), i.e. that \(\psi_0\) takes values in \(\mathcal{M}_0\) (cf. discussion before formula (2.16)). Moreover, in such a gauge the first equation reduces to \(\nabla \psi_0 = 0\), where \(\nabla\) is the Poincaré covariant derivative acting in \(\mathcal{M}_0\). In this way one shows that in the sector of 0-forms the gauge system determined by \(\hat{\Omega}\) is indeed equivalent to (2.22). In this sense, it can be understood as a gauge extension of the gauge invariant formulation (2.22). The difference is similar (up to the auxiliary fields and extra gauge symmetries) to the difference between Maxwell equations in terms of the curvature \(d^*F = 0\), \(dF = 0\) and the gauge description \(d^*F = 0\), \(F = dA\) in terms of the potential.

As we will see in Section 4.1 replacing (2.22) with (2.32) amounts, in particular, to replacing \(\mathcal{M}_0\) with the collection of the Poincaré modules \(\mathcal{M}_p\), \(0 \leq p \leq n - 1\) appearing in the ghost-number-zero \(Q\)-cohomology in the space (2.19) tensored with the Grassmann algebra in new ghost variables \(\theta^a\). It is important to stress that already this step, in general, leads to the additional fields in the theory. Moreover, more careful analysis shows that reducing to the \(Q\)-cohomology modifies the Poincaré covariant derivative entering equations of motion. Namely, it acts in the direct sum of modules in different degrees tensored with Grassmann algebra in \(\theta^a\) such that the Poincaré modules at different degrees are glued together. Note, that the naive generalization of (2.22) (see Section 4.2 for more details) would lead just to a collection of independent equations for fields at different degrees. This phenomena is well-known in the unfolded description of the Fronsdal fields: the complete unfolded system contains two type of fields – HS connections and HS curvatures and the respective equations of motion are not independent but related by
the so-called central-on-mass-shell theorem (see, e.g., [21]). As it was shown in [15] this system can still be written in terms of just one module at the price of introducing a fiber BRST operator such that the two sets of fields appear in cohomology in different degrees while the central-on-mass-shell theorem is automatically built in. From this perspective the present construction extends the one of [15] to the case of mixed-symmetry HS fields.

Let us also discuss the Poincaré invariance of the equations (2.32). Because the Poincaré module for (2.32) is just the space of all polynomials subjected to the Poincaré invariant conditions (2.19) the Poincaré generators have the usual form (without projectors, in contrast to (2.22)), i.e. the Poincaré symmetry acts on the fields according to

\[
P_a \Phi = P_a \Phi, \quad M_{ab} \Phi = M_{ab} \Phi + \left( x_a \frac{\partial}{\partial x_b} - x_b \frac{\partial}{\partial x_a} \right) \Phi + \left( \theta_a \frac{\partial}{\partial \theta^b} - \theta_b \frac{\partial}{\partial \theta^a} \right) \Phi.
\]

(2.33)

Note that the transformation of \( x^a \) implies that \( dx^a \) also transform as Lorentz vectors, which determines the transformation of \( \theta^a \).

The realizations (2.33) and (2.23) differ by \( \hat{\Omega} \)-exact term

\[
\hat{M}_{ab} - \hat{M}_{ab} = [\hat{\Omega}, x_a \frac{\partial}{\partial \theta^b} - x_b \frac{\partial}{\partial \theta^a}].
\]

(2.34)

This, in particular, implies that \([\hat{\Omega}, \hat{M}_{ab}] = 0\), because that is true for \( \hat{M}_{ab} \). Moreover, it also implies that these two representations are equivalent. Indeed, global symmetries of the theory are ghost-number-zero operators commuting with \( \hat{\Omega} \) while those in the image of the adjoint action are trivial symmetries (on-shell equivalent to the gauge symmetries). It follows that inequivalent symmetries are operator \( \hat{\Omega} \)-cohomology at zeroth ghost degree. In particular, \( \hat{M}_{ab} \) and \( M_{ab} \) are different representatives of the same cohomology class.

Note that one can also take as

\[
\bar{P}_a \Phi = P_a \Phi \quad \text{by adding} \quad [\hat{\Omega}, \frac{\partial}{\partial \theta^a}].
\]

2.6 Standard Lagrangian BRST first-quantized formulation

Given a theory of the form (2.32) determined by \( \hat{\Omega} \) one can easily eliminate variables \( y^a \) and \( \theta^a \) in order to end up with the standard first-quantized BRST description (see [15] for more details). However, this is only possible if no constraints involving \( \frac{\partial}{\partial y^a} \) are imposed on \( \Psi \) because such constraints become differential in \( x^a \) once \( y^a \) are eliminated. At the case at hand \( \Psi \) takes values in the space of elements annihilated, in particular, by

\[
\square_y \equiv T^{00} = \frac{\partial}{\partial y_a} \frac{\partial}{\partial y^a}, \quad S^i \equiv T^{i0} = \frac{\partial}{\partial \partial_{ai}} \frac{\partial}{\partial y^a}.
\]

(2.35)
These are constraints from (2.19) that involve $\frac{\partial}{\partial y^a}$. The way out is to impose these constraints through the BRST procedure.

To this end, one introduces additional Grassmann odd ghost variables $c_0$, $\bar{b}^0$, $c_i$, $\bar{b}^i$ satisfying

$$[\bar{b}^0, c_0] = 1, \quad [\bar{b}^i, c_j] = \delta^i_j, \quad \text{gh}(c_i) = \text{gh}(c_0) = 1, \quad \text{gh}(\bar{b}^i) = \text{gh}(\bar{b}^0) = -1,$$

and represented on polynomials in $c_0$, $c_i$ so that $\bar{b}^i \phi = \frac{\partial}{\partial c^i} \phi$ and $\bar{b}^0 \phi = \frac{\partial}{\partial c_0} \phi$. The extended BRST operator is given by

$$\Omega^{\text{parent}} = \theta^a \left( \frac{\partial}{\partial x^a} - \frac{\partial}{\partial y^a} \right) + c_0 \Box_y + c_i S^i + S^i_0 \frac{\partial}{\partial b_i} - c_i \frac{\partial}{\partial b_i} \frac{\partial}{\partial c_0}. \tag{2.37}$$

As a representation space one takes $\mathcal{H}^{\text{parent}}$-valued functions in $x^a$, where $\mathcal{H}^{\text{parent}}$ is a tensor product of Grassmann algebra in $\theta^a$ with polynomials in $y$, $a_i$ and ghosts $c_0$, $c_i$, $b_i$ subjected to the appropriate modification of the remaining constraints $T^{ij}$, $\tilde{N}_i - s_i$. More precisely, these constraints are modified by the ghost contributions needed to maintain their BRST invariance with respect to the extended BRST operator (2.37) and are given explicitly by

$$T^{ij} = \eta^{mn}_{ij} \left( \frac{\partial}{\partial a^m} \frac{\partial}{\partial a^n} + \frac{\partial}{\partial c_i} \frac{\partial}{\partial b_j} + \frac{\partial}{\partial c_j} \frac{\partial}{\partial b_i} \right), \quad \tilde{N}_i = a_i \frac{\partial}{\partial a^i} + c_i \frac{\partial}{\partial c_j} + b_j \frac{\partial}{\partial b_i}, \tag{2.38}$$

Note that they indeed do not involve $\frac{\partial}{\partial y^a}$. Let us also note that these constraints are BRST extensions of the generators of the upper triangular subalgebra and the Cartan elements of $sp(2n-2) \subset sp(2n)$ algebra that is a Howe dual to the Lorentz algebra acting on the space of variables $a_i^0$.

**Proposition 2.1.** Parent system $(\Omega^{\text{parent}}, \mathcal{H}^{\text{parent}})$ and intermediate system $(\tilde{\Omega}, \tilde{\mathcal{H}})$ are equivalent.

**Proof.** To prove the proposition one introduces a grading defined by a homogeneity degree in $c_0$ and $c_i$. Then using the method of the homological reduction described in Appendix B one finds that $\Omega^{\text{parent}} = \Omega^{\text{parent}}_{-1} + \Omega^{\text{parent}}_0$, where $\Omega^{\text{parent}}_{-1} = c_0 \Box_y + c_i S^i$ and the theory can be reduced to the cohomology of $\Omega^{\text{parent}}_{-1}$. We now need to invoke the homological result demonstrated in Appendix A. Namely, the crucial fact is the cohomology of the operator $\Delta_{I,J} = C_{I,J} T^{IJ}$ (no summation over $I$, $J$), where $C_{I,J}$ are some ghost variables, in the space of all polynomials is given by $C_{I,J}$-independent elements annihilated by $T^{IJ}$. In particular, the cohomology of $\Omega^{\text{parent}}_{-1} = C_{00} T^{00} + C_{0i} T^{0i}$ is concentrated in the zeroth degree and can be identified with $c_0$, $c_i$-independent elements annihilated by $T^{00}$ and $T^{0i}$. It follows that the cohomology subspace is singled out by constraints (2.19), because for $c_0$, $c_i$-independent elements constraints (2.38) along with $T^{00}$ and $T^{0i}$
explicitly coincide with (2.19). The reduced BRST operator coincides then with \( \Omega^\text{parent}_0 \) restricted to the cohomology, and hence coincides with \( \hat{\Omega} \).

Because the constraints (2.38) determining the representation space for the parent system are algebraic (do not involve \( \frac{\partial}{\partial y^a} \)) the elimination of \( y^a, \theta^a \) variables is now straightforward and amounts to dropping the first term (the one with \( \theta^a \)) and replacing \( \frac{\partial}{\partial y^a} \rightarrow \frac{\partial}{\partial x^a} \) in the remaining terms. This can be seen by reducing the theory to the cohomology of \( \theta^a \frac{\partial}{\partial y^a} \) and evaluating the reduced BRST operator (see [15] for more details and the proof). In this way one arrives at the theory determined by the following BRST operator

\[
\Omega = c_0 \Box + c_i S^i + S^i_\dagger \frac{\partial}{\partial b_i} - c_i \frac{\partial}{\partial b_i} \frac{\partial}{\partial c_0},
\]

where the constraints are given by

\[
\Box = \eta^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b}, \quad S^i_\dagger = a^a_i \frac{\partial}{\partial x^a}, \quad S^i = \frac{\partial}{\partial a^a_i} \frac{\partial}{\partial x^a},
\]

and satisfy the following algebra:

\[
[\Box, S_i] = [\Box, S^i_\dagger] = 0, \quad [S_i, S^i_\dagger] = \delta_{ij} \Box, \quad [S_i, S_j] = [S^i_\dagger, S^j_\dagger] = 0.
\]

The representation space is now given by \( \mathcal{H} \)-valued functions in \( x^a \), where \( \mathcal{H} \) is the space of polynomials in \( a_i \) and ghosts \( c_0, c_i, b_i \) satisfying

\[
T^{ij} \phi = 0, \quad N^j_i \phi = 0 \quad i > j, \quad N_i \phi = s_i \phi,
\]

where the BRST invariant constraints \( T^{ij}, N^j_i \) and \( N_i \) are defined in (2.38). In this way we have arrived at the following

**Proposition 2.2.** System \((\Omega, \mathcal{H})\) is equivalent to \((\Omega^\text{parent}, \mathcal{H}^\text{parent})\) and hence to intermediate system \((\hat{\Omega}, \hat{\mathcal{H}})\).

As we are going to see system \((\Omega, \mathcal{H})\) yields a Lagrangian description of mixed-symmetry fields on the Minkowski space. In particular, we explicitly show that the appropriate reduction of this theory gives the theory proposed by Labastida in [2, 3]. In addition, we also show in Section 4.1.3 that the parent theory (2.37) (as well as the equivalent formulation (2.32)) contains the unfolded form of this model proposed in [6]. As an independent check, in Section 4.3 we observe that system \((\Omega, \mathcal{H})\) indeed describes the irreducible massless unitary representation of the Wigner little group [1] determined by the spins \( s_{n-1}, \ldots, s_1 \).

There is an alternative motivation for implementing the constraints \( S^i \) and \( \Box \) (2.35) through the BRST operator. Namely, it turns out that the BRST operator (2.39) is symmetric with respect to the standard inner product of the form

\[
\langle \phi, \psi \rangle = \int d^d x \int d c_0 \langle \phi, \psi \rangle_0,
\]
where \( \langle , \rangle_0 \) is the standard Fock inner product in the space of polynomials in \( c_i, b^i, a^a \) identified with the Fock space generated by \( c_i, b^i, a^a \) from the vacuum state \( |0\rangle \) defined by \( \bar{a}|0\rangle = \bar{c}|0\rangle = \bar{b}|0\rangle = 0 \). In our notations the conjugation rules have the following form
\[
(a^a_i)^\dagger = \bar{a}^a_i, \quad (b_i)^\dagger = -\bar{b}^i, \quad (c_i)^\dagger = \bar{c}^i, \quad (2.44)
\]
where the space-time and the internal indices are raised and lowered with the Minkowski metric \( \eta_{ab} \) and the standard Euclidean metric \( \delta_{ij} \) on the internal space, respectively. This can be seen as equipping the Grassmann odd superspace of ghosts with the super-Euclidean metric \( \epsilon_{\alpha\beta} \delta_{ij} \). Indeed, introducing the collective notation \( \chi^\alpha_i \) for \( \chi^1_i = c_i \) and \( \chi^2_i = b_i \) one finds \( \langle \chi^\alpha_i, \chi^\beta_j \rangle_0 = \epsilon_{\alpha\beta} \delta_{ij} \) (here, \( \epsilon_{\alpha\beta} \) is a 2d Levi-Civita symbol defined such that \( \epsilon_{12} = \epsilon^{12} = 1 \)). One can check that this is indeed consistent with the commutation relation\(^9\) and uniquely determines the inner product for which \( \dagger \) is the hermitian conjugation.

The inner product just defined carries ghost degree \(-1\), i.e. \( \langle \phi, \psi \rangle = 0 \) for any \( \phi, \psi \) such that \( \text{gh}(\phi) + \text{gh}(\psi) \neq 1 \).

Given a nondegenerate inner product of ghost number \(-1\) and a symmetric nilpotent BRST operator one immediately constructs the action\(^10\)
\[
S = \frac{1}{2} \langle \Psi^{(0)}, \Omega \Psi^{(0)} \rangle \quad (2.45)
\]
that determines the equations of motion \( \Omega \Psi^{(0)} = 0 \). The Batalin–Vilkovisky master action of the theory can be constructed in the form \( S_{BV} = \frac{1}{2} \langle \Psi, \Omega \Psi \rangle \) so that the fields entering the nonzero components of the string field \( \Psi \) are naturally identified with antifields and ghost fields of the BV formalism.

To make a contact to the literature and to highlight the quantum mechanical interpretation of the system let us sketch the formulation of the constrained system whose BRST operator is given by \( \Omega \). The variables are as follows. The space-time variables \( x^m, p_n \) satisfy canonical commutation relations and conjugation rules
\[
[p_m, x^n] = -\delta_m^n, \quad (x^m)^\dagger = x^m, \quad p^\dagger_n = -p_n. \quad (2.46)
\]
Internal (spin) variables (oscillators) satisfy
\[
[\bar{a}^m_i, a^n_j] = \delta_{ij} \eta^{mn}, \quad (a^m_i)^\dagger = \bar{a}^m_i, \quad i, j = 1, 2, \ldots, n - 1. \quad (2.47)
\]

The constraints are:\(^1\)
\[
\Box, \quad \mathcal{S}_i^\dagger, \quad \mathcal{S}_i, \quad (2.48)
\]
\[
T^{ij}, \quad N_i^j \quad i > j, \quad N_i - s_i. \quad (2.49)
\]

\(^9\)We use the following convention for the conjugation in the presence of fermions: \( (ab)^\dagger = (-1)^{[a][b]} b^\dagger a^\dagger \) and \( \langle \phi, a\psi \rangle = (-1)^{[a][\phi]} \langle a^\dagger \phi, \psi \rangle \), where \( [a] \) denotes the Grassmann parity of \( a \).

\(^10\)The choice of the overall sign corresponds to “almost-positive” signature of \( \eta_{ab} \).

\(^1\)Related constraints were discussed in \(^\dagger\) in the context of constant curvature spaces.
All together these constraints form an upper-triangular subalgebra of $sp(2n)$ along with $n - 1$ Cartan elements (weight conditions for spin oscillators). The second line form the subalgebra of $sp(2n - 2) \subset sp(2n)$ that do not affect the space-time variables $x^a$. The first line contains those constraints that do involve space time derivatives. These constraints also form a subalgebra (2.41).

The first-class constraint quantum system is defined by implementing the first line through the BRST operator using the ghost variables $\bar{b}_0, c_0, \bar{b}_i, c_i$ and $\bar{c}_i, b_i$ while imposing the BRST invariant extensions of the constraints from the second line directly in the representation space. The BRST operator for this system is indeed $\Omega$ given by (2.39). The respective representation $\mathcal{H}$ space is singled out by the conditions (2.42) that are BRST invariant extensions of the constraints (2.49).

It is important to stress that one can as well consider the theory determined by the action (2.45) where $\Psi^{(0)}$ is not required to satisfy constraints (2.49). Such theory describes a direct sum of irreducible fields, where any field enters the theory with (in general infinite) multiplicity. With the constraints (2.49) relaxed, action (2.45) is known in the literature and can be identified as an appropriate truncation of the open bosonic string field theory Lagrangian in the tensionless limit [26, 27].

3 Relation to the Labastida approach

In this section we analyze the dynamical content of the theory determined by the BRST operator $\Omega$ (2.39). We explicitly prove that the equations of motion for mixed-symmetry fields generated by the BRST operator $\Omega$ coincide with those originally obtained by Labastida [2, 3].

Let us sketch the main features of the Labastida equations formulated for an individual spin field with $n - 1$ rows. The kinetic operator $L$ has the form

$$L = \Box - D_i D^i + \frac{1}{2} D_i D_j T^{ij},$$

(3.1)

where $D_i = \partial^a_i \frac{\partial}{\partial x^a}$ and $D^i = \partial^a_i \frac{\partial}{\partial x^a}$, $i = 1, ..., n - 1$ and the trace annihilation $T^{ij}$ operator is defined by (2.4). The operator $L$ obviously commutes with Young symmetrizers $N_{ij}^k$ (2.10), which implies that the Young symmetry properties of the fields are shared by the operator $L$. Fields $\varphi$ satisfy the following trace constraint

$$T^{(ij} T^{kl)} \varphi = 0$$

(3.2)

that singles out double-traceless fields. The Labastida equations of motion $L \varphi = 0$ are invariant under the gauge transformations

$$\delta \varphi = D_i \Lambda^i,$$

(3.3)

[12] In the case of $n = 3$, i.e., for two-row covariant fields, this $sp(4)$ algebra has been identified in [8].
provided that parameters satisfy
\[ T^{(ij} \Lambda^{k)} = 0. \tag{3.4} \]
Let us stress that in general a parameter \( \Lambda^i \) for a given index \( i \) does not satisfy Young symmetry conditions. Instead, parameters for different \( i \) and \( j \) are related to each other by the appropriate Young symmetrizations. Such relations can be easily read off from (3.3) by imposing Young symmetry condition on the left-hand-side. It turns out that representing the space of gauge parameters by tensors \( \Lambda^i \) not satisfying Young symmetry conditions allows one to write down the gauge transformation law in a simple form.\(^\text{[13]}\)

### 3.1 Polynomials in ghosts and associated algebras

Before considering a dynamics described by BRST operator \( \Omega \) let us discuss the Fock subspace generated by ghost variables \( c_i, b_i \) from the more algebraic point of view. Introducing a collective notation \( \chi^\alpha_i = (c_i, b_i), \alpha = 1, 2 \) for ghost variables it is convenient to consider \( \chi^\alpha_i \) as coordinates on the tensor product of two superspaces with bases \( e^i \) and \( e_\alpha \). This tensor product is equipped with the metric \( \epsilon^{\alpha\beta} \) that makes a Euclidian superspace. Recall that it is this metric that induces the inner product on the Fock space of ghost variables (see Section 2.6). Note that this metric factorizes into the super-Euclidian metric \( \epsilon^{\alpha\beta} \) and the supersymplectic metric \( \delta^{ij} \). Using the \( \epsilon^{\alpha\beta} \) factor allows one to introduce oscillator realizations of \( sp(2) \) and \( sp(2n-2) \) algebras. These algebras provide convenient tools for the analysis of the dynamical content of the theory.

Remarkably, ghost variables introduce into the game one more Howe dual pair which is complementary to the previous one considered in Section 2.1. These new dual algebras are \( gl(n-1) \) and \( gl(2) \) and their generators are given by\(^\text{[14]}\)
\[
Y^i_j = \frac{1}{2} \left( \chi^\alpha_i \frac{\partial}{\partial \chi^\beta_j} - \frac{\partial}{\partial \chi^\beta_j} \chi^\alpha_i \right), \quad Z^\alpha_{\beta} = \frac{1}{2} \left( \chi^\alpha_i \frac{\partial}{\partial \chi^\beta_i} - \frac{\partial}{\partial \chi^\beta_i} \chi^\alpha_i \right). \tag{3.5}
\]

Using the \( \epsilon^{\alpha\beta} \) factor the algebra \( gl(n-1) \) can be extended by the following generators
\[
Y_{ij} = \epsilon_{\alpha\beta} \chi^\alpha_i \chi^\beta_j, \quad Y^{ij} = \epsilon^{\alpha\beta} \frac{\partial}{\partial \chi^\alpha_i} \frac{\partial}{\partial \chi^\beta_j}, \tag{3.6}
\]
so that similarly to (2.4) generators \( Y_{ij}, Y^i_j, \) and \( Y^{ij} \) form \( sp(2n-2) \) algebra. In particular, BRST extended algebraic operators (2.38) can be represented as
\[
T^{ij} = T^{ij} + Y^{ij}, \quad N^{ij}_i = N^i_j + Y^i_j + \delta^i_j. \tag{3.7}
\]
\(^\text{[13]}\)This property of the Labastida approach was originally observed in [8] within the BRST formulation of two-row field dynamics in Minkowski space.
\(^\text{[14]}\)This construction also enjoys a supersymmetric extension. To this end we note that \( \chi^\alpha_i \) transforms both as \( gl(n-1) \) and \( gl(2) \) vectors and hence we can build supercharges
\[
Q^a_a = \alpha^a_i \chi^\alpha_i, \quad Q^a_\alpha = \frac{\partial}{\partial a^\alpha} \frac{\partial}{\partial \chi^\alpha_i}, \quad \{Q^a_a, Q^b_\beta\} = \delta^a_b L^a_b - \delta^b_a Z^\alpha_{\beta},
\]
where \( L^a_b = \frac{1}{2} \left( a^a_i \frac{\partial}{\partial a^\alpha} \right) \) are \( gl(d) \) generators. The resulting superalgebra is \( gl(d|2) \).
Analogously one introduces the algebra \( sp(2) \cong sl(2) \) generated by
\[
Z_{\alpha\beta} = \epsilon_{\alpha\gamma} Z^{\gamma}_{\beta} + \epsilon_{\beta\gamma} Z^{\gamma}_{\alpha}.
\]
(3.8)
The standard basis of \( sp(2) \) algebra reads
\[
Z_+ \equiv Z^{12} = c_i \frac{\partial}{\partial b_i}, \quad Z_- \equiv Z^{21} = b_i \frac{\partial}{\partial c_i},
\]
(3.9)
\[
Z_0 \equiv Z^{11} - Z^{22} = c_i \frac{\partial}{\partial c_i} - b_i \frac{\partial}{\partial b_i}.
\]

### 3.2 The ghost-number-zero fields

It is convenient to represent string field \( \Psi \equiv \Psi(a, b, c|x) \) as follows
\[
\Psi = \Phi_1 + c_0 \Phi_2,
\]
(3.10)
For the ghost-number-zero component \( \Psi^{(0)} \) fields \( \Psi^{(0)}_1 \equiv \Phi \) and \( \Psi^{(0)}_2 \equiv C \) are the following decompositions with respect to the ghost variables:
\[
\Phi = \sum_{k=0}^{n-1} c_{i_1} \cdots c_{i_k} b_{j_1} \cdots b_{j_k} \Phi^{i_1 \cdots i_k|j_1 \cdots j_k},
\]
(3.11)
\[
C = \sum_{k=0}^{n-2} c_{i_1} \cdots c_{i_k} b_{j_1} \cdots b_{j_{k+1}} C^{i_1 \cdots i_k|j_1 \cdots j_{k+1}}.
\]
The expansion coefficients \( \Phi^{i_1 \cdots i_k|j_1 \cdots j_k}(a|x) \) and \( C^{i_1 \cdots i_k|j_1 \cdots j_{k+1}}(a|x) \) are \( gl(n-1) \) tensors antisymmetric in each group of indices, and the slash \( | \) implies that no symmetry properties between two groups of indices are assumed. In the sequel we use the notation \( \varphi \) for the \( k = 0 \) component of \( \Phi \). Note that component fields \( \Phi^{i_1 \cdots i_k|j_1 \cdots j_k} \) and \( C^{i_1 \cdots i_k|j_1 \cdots j_{k+1}} \) were considered in \([26]\). These can be seen as a generalization of the so-called triplet originally discussed in \([23]\) in the context of totally symmetric fields.

Our aim now is to find a minimal set of fields that covariantly describes an individual spin field. This is achieved in two steps. As a first step we eliminate all the generalized auxiliary fields entering the formulation determined by \( \Omega \). This is achieved using the general method of \([15]\). As a second step we subject the string field \( \Psi^{(0)} \) to the remaining irreducibility conditions, namely, the Young symmetrizer, the trace conditions, and the weight conditions.

#### 3.3 \( \Omega_{-1} \) cohomology

Let us decompose the BRST operator with respect to the homogeneity degree in \( c_0 \) as
\[
\Omega = \Omega_{-1} + \Omega_0 + \Omega_1,
\]
(3.12)
with
\[ \Omega_{-1} = -c_i \frac{\partial}{\partial b_i} \frac{\partial}{\partial c_0}, \quad \Omega_0 = c_i S^i + S^i \frac{\partial}{\partial b_i}, \quad \Omega_1 = c_0 \Box. \] (3.13)

The lowest degree component \( \Omega_{-1} \) is purely algebraic so that all the fields that are not in the cohomology of \( \Omega_{-1} \) are generalized auxiliary fields (see \[15\] for details).

In order to analyze the cohomology of \( \Omega_{-1} \) let us note that it can be represented in the form
\[ \Omega_{-1} = -Z_+ \frac{\partial}{\partial c_0}, \] (3.14)
where \( Z_+ \) is a generator of \( sl(2) \) algebra realized on ghost fields \((3.9)\).

We are now going to find \( \Omega_{-1} \) cohomology in the subspace of elements satisfying \( T^{ij} \phi = 0, \quad N^i \phi = 0 \quad i > j \) and \( N_i \phi = (s_i + \frac{c}{2}) \phi \). To this end it is useful to identify first the cohomology in the entire representation space and then impose the conditions. This is legitimate because of the following argument: the conditions we are dealing with are the highest weight conditions for the \( sp(2n-2) \) algebra formed by \( T^{ij}, N^i, T_{ij} \), where \( T_{ij} \) is a BRST invariant extension of \( T_{ij} \). Decomposing the entire space into the direct sum of irreducible highest weight \( sp(2n-2) \)-modules one finds that any element can be represented as a sum of elements of the form \( \phi = \phi_0 + T_A \phi^A \), where \( \phi_0 \) satisfy the highest weight conditions and \( T_A \) is a collective notation for all the generators from the lower-triangular subalgebra \((i.e. \quad T_{ij}, N^i \quad j > i)\). Because \( sp(2n-2) \) commutes with \( \Omega_{-1} \) one concludes that \( \Omega_{-1} \) does not map elements of the form \( T_A \phi^A \) to elements satisfying highest weight conditions. This implies that the coboundary condition is not affected by restricting to the subspace so that \( \Omega_{-1} \) cohomology in the subspace coincides with the restriction to the subspace of the \( \Omega_{-1} \) cohomology in the entire space.

Using the representation \((3.14)\) the searched-for cohomology \( H(\Omega_{-1}) \) in the entire space can be readily found (see also Section \(4.1.1\)). Indeed \( \text{Im} \Omega_{-1} \) is given by \( c_0 \)-independent elements that are in the image of \( Z_+ \). It follows that one can represent the \( c_0 \)-independent cohomology by elements annihilated by \( Z_- \). Let us consider then \( \text{Ker} \Omega_{-1} \) for \( c_0 \)-dependent elements (for \( c_0 \)-independent the cocycle condition is satisfied trivially). It follows that elements annihilated by \( Z_+ \) satisfy the cocycle condition and are in cohomology. Decomposing a general element \( \phi \) into \( c_0 \)-(in)dependent elements according to \( \phi = \phi_1 + c_0 \phi_2 \) we can formulate cohomological conditions as \( Z_- \phi_1 = 0 \) and \( Z_+ \phi_2 = 0 \). Eigenvalues of the generator \( Z_0 \) are integer numbers, \( Z_0 \phi_1 = m \phi_1 \) and \( Z_0 \phi_2 = (m-1) \phi_2 \), where \( m = gh(\phi) \) is a ghost number. For instance, for \( m = 0 \) we obtain \( \phi_2 = 0 \) and \( Z_+ \phi_1 = 0 \) and, hence, \( c_0 \)-independent component is \( sl(2) \) invariant. Let us also note that the choice of representatives is consistent with the conditions \((2.42)\) because both \( Z_+ \) and \( Z_- \) commute with \((2.42)\). This determines the structure of the physical fields entering \( \Psi^{(0)} \).

For positive values of the ghost number \( +m, \quad 0 \leq m \leq n-1 \) the cohomology \( H^m \) is given by elements \( \phi_1 = 0 \) and \( Z_+ \phi_2 = 0 \). This implies that the string field \( \Psi^{(-m)} \) takes
the following form

$$\Psi^{(-m)} = \Psi^{(-m)}_2 = \sum_{k=0}^{n-m-2} c_{i_1} \cdots c_{i_{k+m}} b_{j_1} \cdots b_{j_{k+1}} \Psi_2^{i_1 \cdots i_{k+m}, j_1 \cdots j_{k+1}},$$

(3.15)

where all components $\Psi_2^{i_1 \cdots i_{k+m}, j_1 \cdots j_{k+1}}$ are $gl(n-1)$ Young tableaux with columns of heights $k + m$ and $k + 1$. Elements $\Psi_2^{(m)}$ of the cohomology are irrelevant in the present analysis and correspond to the antifields of the Batalin-Vilkovisky formulation of the theory.

For non-positive values of the ghost number $-m$, $0 \leq m \leq n - 1$ the cohomology $H^{-m}$ is given by elements with $\phi_2 = 0$ and $Z_\phi_1 = 0$, so that the string field takes the form

$$\Psi_1^{(m)} = \sum_{k=0}^{n-m-1} c_{i_1} \cdots c_{i_k} b_{j_1} \cdots b_{j_{k+m}} \Psi_1^{j_1 \cdots j_{k+m}, i_1 \cdots i_k},$$

(3.16)

where all components $\Psi_1^{j_1 \cdots j_{k+m}, i_1 \cdots i_k}$ are $gl(n-1)$ Young tableaux with columns of heights $k + m$ and $k$. The expansion coefficients of $\Psi_1^{(m)}$ are identified with dynamical fields (at $m = 0$) and ghost field of $(m - 1)$-th level of reducibility (for $m \neq 0$) associated to the respective gauge parameters. Recall that in addition one needs to impose the conditions (2.42) in order to describe cohomology in the subspace.

More detailed discussion of the gauge symmetries of the theory will be given in Section 3.5.

### 3.4 BRST extended algebraic conditions

Let us now analyze algebraic irreducibility conditions (2.42) imposed on the representatives of $\Omega_{-1}$ cohomology. Representing the string field as $\Psi = \Psi_1 + c_0 \Psi_2$ the BRST extended trace constraint (2.38) takes the form

$$(T^{ml} + Y^{ml})(\Psi_1 + c_0 \Psi_2) = 0.$$  

(3.17)

Obviously, it does not mix up traces of $\Psi_1$ and $\Psi_2$ and hence they can be analyzed separately. In both sectors constraint (3.17) relates $k + 1$-th component of the cohomology to the trace of $k$-th one. Applying $T^{ps}$ and $Y^{ps}$ to the left-hand-side of the above expression yields the relation $T^{ml}T^{ps} \Psi_{1,2} = Y^{ml}Y^{ps} \Psi_{1,2}$. Observing then that a symmetrized combination $Y^{(ml)Y^{ps}}$ is identically zero one obtains that $\Phi$ satisfies the double trace constraint $T^{(ml)T^{ps}} \Psi_{1,2} = 0$.

For the dynamical fields associated to the lowest component of the cohomology $H^0$ we recover the familiar Labastida constraint (3.2)

$$T^{(ps)T^{ml}} \varphi = 0$$

(3.18)
and find that all other components $\Phi^{j_1 \ldots j_k, i_1 \ldots i_k}$ for $k > 0$ are expressed in terms of the traces of $\varphi$. For instance, for lowest values of $k$ the corresponding expressions read off from $T^{m l} \Phi + Y^{m l} \Phi = 0$ are

$$T^{m l} \varphi - \left( \Phi^{|m|} + \Phi^{|l|} \right) = 0 ,$$

(3.19)

$$T^{p s} T^{m l} \varphi + 4 \left( \Phi^{|p| m} + \Phi^{|l| s} + \Phi^{|s m| p} + \Phi^{|p m| s} \right) = 0 .$$

(3.20)

Identifying the right-hand-sides with appropriate symmetrizations of $\Phi^{m, l}$ and $\Phi^{s l, p m}$ respectively, we obtain formulas that express these components through the field $\varphi$.

The cohomology $H^{-1}$ that corresponds to gauge parameters of the zeroth level can be analyzed along the same lines. In particular, for $k = 0$ component $\Lambda^i \equiv \Psi^i_1$ of $H^{-1}$ one obtains the relation

$$T^{(m n) \Lambda^1} = 0$$

(3.21)

which is Labastida constraint (3.4) for the gauge parameters. Quite analogously to the dynamical fields, higher order components of $H^{-1}$ are expressed via traces of the gauge parameter $\Lambda^i$.

To analyze Young symmetry types of the fields we impose the BRST extended algebraic conditions

$$(N^i_j + Y^i_j)(\Psi_1 + c_0 \Psi_2) = 0 \quad i > j ,$$

(3.22)

and

$$(N^i_i + Y^i_i - s_i - 1)(\Psi_1 + c_0 \Psi_2) = 0 ,$$

(3.23)

where $s_i$ are integer spins and $Y^i_i \equiv Y^{i i}$ for a fixed $i$. In particular, for the field $\varphi$ we obtain $N^i_j \varphi = 0 \quad i > j$, i.e., it is described by Young tableau of the type $(s_{n-1}, s_{n-2}, \ldots, s_1)$ and the corresponding gauge parameter $\Lambda^i$ has one less $i$-th oscillator $a^i_a$. Let us note that BRST extended conditions (3.23) do not in general lead to Young symmetries of $\Lambda^i$. Instead there appears a set of recurrent relations between $\Lambda^i$ generated by Young symmetrizers $N^i_j \quad i > j$. Their form can be easily read off from (3.22).

Finally, let us note that the representative of the cohomology $H^{-(n-1)}$ has the following form $\Psi^{(n-1)}_1 = b_1 \cdots b_{n-1} \Psi^{1 \ldots n-1}_1$, i.e. corresponds to the maximally antisymmetric tensor. As discussed above it corresponds to the gauge parameter of the maximal depth of reducibility $n - 1$. The conditions (3.22), (3.23) applied to $\Psi^{(n-1)}_1$ reduce to $N^i_j \Psi^{(n-1)}_1 = 0 \quad i > j$ and $N^i_i \Psi^{(n-1)}_1 = (s_i - 1) \Psi^{(n-1)}_1$. In terms of Lorentz irreps it corresponds to Young tableau with one leftmost column cut off compared to the tableau associated with the dynamical field $\varphi$.

### 3.5 Gauge transformations and field equations

In order to describe the theory reduced to $\Omega_{-1}$-cohomology $H$ one is to compute the reduced operator $\tilde{\Omega}$ acting in $H$. $\tilde{\Omega}$ determines the equations of motion, gauge symmetries,
and reducibility relations of the reduced theory and can be found using the standard co-

homological technique (see [15] for an exposition in the similar terms). We now take a
different route and obtain the explicit form of the reduced equation of motion and gauge
symmetries by explicitly eliminating the generalized auxiliary fields associated to the
contractible pairs for Ω_{−1}.

The gauge transformations

\[ \delta \Psi^{(0)} = \Omega \xi^{(-1)} , \quad \text{gh}(\xi^{(-1)}) = -1 . \]  

involve the gauge parameters of the form \( \xi^{(-1)} = \Lambda + c_0 \Upsilon \), where \( \text{gh}(\Lambda) = -1 \) and \( \text{gh}(\Upsilon) = -2 \), and

\[ \Lambda = \sum_{k=0}^{n-2} c_{i_1} \cdots c_{i_k} b_{j_1} \cdots b_{j_k+1} A^{i_{i_1}\cdots i_{i_k}j_{j_1}\cdots j_{j_k+1}} , \]

\[ \Upsilon = \sum_{k=0}^{n-3} c_{i_1} \cdots c_{i_k} b_{j_1} \cdots b_{j_k+2} \Upsilon^{i_{i_1}\cdots i_{i_k}j_{j_1}\cdots j_{j_k+2}} . \]  

(3.25)

The gauge symmetry is reducible and there exists the set of level-\((l-1)\) \((1 \leq l \leq n-1)\) gauge parameters and gauge transformations of the form

\[ \delta \xi^{(-l)} = \Omega \xi^{(-l-1)} , \quad \text{gh}(\xi^{(-l)}) = -l . \]  

(3.26)

Recall that gauge parameters are also subjected to the BRST extended irreducibility conditions \((2.42)\).

For fields \( \Phi \) and \( C \) the gauge transformations take the form

\[ \delta \Phi = Z_{+} \Upsilon + \Omega_{0} \Lambda , \]

\[ \delta C = \Box \Lambda - \Omega_{0} \Upsilon . \]  

(3.27)

We observe that the transformation for fields \( \Phi \) contains an algebraic term \( Z_{+} \Upsilon \). It
means precisely that the part of components of fields \( \Phi \) are Stueckelberg-like and can be
gauged away by imposing the proper gauge condition. Using the cohomological analysis of Section \[3.3\] we conclude that the remaining components of fields \( \Phi \) are described
by rectangular \( gl(n-1) \) Young tableaux. The consideration of gauge symmetries on
\((m-1)\)-th level goes the same way via identification of Stueckelberg-like contributions
to the transformation law and shows that the reducibility parameters of the reduced theory
indeed corresponds to \( H^{-m} \) \((3.16)\).

Noting that \( \Omega_{0} \) acts by a linear combination of \( S^i \) and \( S_i^{\dagger} \) we obtain for \( k = 0 \) compo-

tent \( \varphi \) the following transformation:

\[ \delta \varphi = S_i^{\dagger} \Lambda^i . \]  

(3.28)
We see that identification \( S_i^\dagger \equiv D_i \) yields the Labastida gauge law (3.3).

The equations of motion that follow from the action (2.45) have the form

\[
\Omega \Psi^{(0)} = 0 ,
\]

(3.29)

and are invariant with respect to the gauge transformation (3.24). In terms of the components \( \Psi^{(0)} = \Phi + c_0 C \) equations take the form

\[
\Box \Phi - \Omega_0 C = 0 ,
\]

(3.30)

\[
\Omega_0 \Phi + Z_+ C = 0 .
\]

(3.31)

We observe that all fields \( C \) enter the second field equation algebraically and hence can be fully eliminated by expressing in terms of the first derivatives of fields \( \Phi \). Indeed, similarly to the gauge transformation law analysis the corresponding term in the field equations is expressed as \( Z_+ C \). Noting that fields \( C \) are not in the kernel of \( Z_+ \) we conclude that all of them can be expressed through the appropriate combinations of \( \Omega_0 \Phi \).

To analyze the field equations for the component \( \varphi \) we start with \( k = 0 \) and obtain

\[
\Box \varphi - S^\dagger_m C^m = 0 ,
\]

(3.32)

and

\[
S^n \varphi - S^\dagger_m \Phi^{n|m} - C^m = 0 .
\]

(3.33)

By solving the second equation for the auxiliary field \( C^m \) and substituting the result in the first equation we obtain \( \Box \varphi - S^\dagger_m S^m \varphi + S^\dagger_m S^\dagger_n \Phi^{m|n} = 0 \). Taking into account trace relation (3.19) we finally get the Labastida field equation

\[
\left( \Box - S^\dagger_m S^m + \frac{1}{2} S^\dagger_m S^\dagger_n T^{mn} \right) \varphi = 0 .
\]

(3.34)

4 \hspace{1em} (Generalized) Poincaré modules

4.1 \hspace{1em} \( Q \)-cohomology and the unfolded formulation

According to the general strategy [15] given a parent form of the theory, the unfolded formulation can be obtained reducing to the cohomology of the fiber part of the BRST operator (2.37). This is equivalent to reducing the theory (2.32) to the cohomology of \( Q \). Eliminating the generalized auxiliary fields associated to the contractible pairs for \( Q \) the theory reduces to that determined by the reduced BRST operator of the form [15]

\[
\Omega_{unf} = d - \bar{\sigma} ,
\]

(4.1)
where $d$ is de Rham differential $\theta^a \frac{\partial}{\partial x^a}$ and $\tilde{\sigma}$ is the reduction of $\sigma = \theta^a \frac{\partial}{\partial y^a}$ to $Q$-cohomology. In this way one describes the theory in terms of the fields taking values in $Q$-cohomology only.

In order to explicitly describe the unfolded form of the theory one needs to know $Q$-cohomology. In the vanishing ghost number it has been already computed in Section 2.3. In order to compute $Q$-cohomology at all the remaining ghost numbers, i.e. $-(n-1), -(n-2), \ldots, 0$, we need some additional algebraic tools.

### 4.1.1 $sl(2)$ cohomology

Let $a^A, y^A$ be two sets of variables which we allow to be bosonic or fermionic of the same Grassmann parity, $|a^A| = |y^A|$. On the space of polynomials in $a, y$ we define the $sl(2)$ algebra

$$
J = a^A \frac{\partial}{\partial y^A}, \quad \bar{J} = y^A \frac{\partial}{\partial a^A}, \quad h = [J, \bar{J}] = a^A \frac{\partial}{\partial a^A} - y^A \frac{\partial}{\partial y^A} .
$$

Extending the space by the ghost variable $b$ with gh$(b) = -1$ one considers the following operators

$$
q = J \frac{\partial}{\partial b}, \quad \bar{q} = \bar{J} b .
$$

Both are obviously nilpotent and act on the space of polynomials $\phi = \phi_1 + b\phi_2$. The operator $q$ has the same structure as $Q$ we are interested in while $\bar{q}$ is a kind of anti-BRST operator associated to $q$. The cohomology of both $q$ and $\bar{q}$ can easily be computed using the $sl(2)$ representation theory. Namely, the representatives for both $q$ and $\bar{q}$ can be taken in the form with $\bar{J}\phi_1 = 0$ and $J\phi_2 = 0$. We see that the $q$ and $\bar{q}$ cohomology are not only isomorphic but are represented by the same elements. Moreover, one observes that the cohomology representatives chosen in this way can be singled out by

$$
q\phi = \bar{q}\phi = 0 .
$$

The same is of course true if instead of the space of polynomials and the algebra \[4.2\] one takes an arbitrary representation space of the $sl(2)$ algebra formed by $J, \bar{J}, h = [J, \bar{J}]$. The only requirement is that the entire representation space is decomposable into the direct sum of finite-dimensional irreducible $sl(2)$-modules. In particular, if among variables $a^A, y^A$ there is a fermionic pair $\alpha, \gamma$ then the statement is also true if instead of $\alpha \frac{\partial}{\partial \gamma}$ and $\gamma \frac{\partial}{\partial \alpha}$ terms in $J, \bar{J}$ one takes $\alpha \gamma$ and $\frac{\partial}{\partial \gamma} \frac{\partial}{\partial \alpha}$, respectively. This is because $\alpha \gamma \frac{\partial}{\partial \gamma} \frac{\partial}{\partial \alpha}$ and $[\alpha \gamma, \frac{\partial}{\partial \gamma} \frac{\partial}{\partial \alpha}] = \alpha \frac{\partial}{\partial \alpha} + \gamma \frac{\partial}{\partial \gamma} - 1$ also form $sl(2)$.

### 4.1.2 $Q$-cohomology

We now turn to the computation of the $Q$-cohomology in the subspace \[2.19\]. As we have seen the representatives of $Q$-cohomology at zeroth ghost degree can be chosen to
be annihilated by the upper-triangular subalgebra

\[ U_0 = \left\{ N_{ij} \mid i > j, \bar{S}^{ji} \right\}, \quad (4.5) \]

of the \( sl(n) \) algebra generated by \( N_{ij}, i \neq j, S^i_1, \bar{S}^{ji} \).

In fact there are other choices for the upper-triangular subalgebra containing \( N_{ij}, i > j \). More precisely, there are \( n \) subalgebras

\[ U_p = \left\{ N_{ij} \mid i > j, \bar{S}^{ji}, i = 1, \ldots, n-p-1, S^j_j, j = n-p, \ldots, n-1 \right\}, \quad (4.6) \]

which are upper-triangular and contain \( N_{ij}, i > j \). For \( p = 0 \) this indeed gives (4.5).

Each \( U_p \) define a Poincaré module \( M_p \) (one can consistently define the Poincaré module structure in the same way as for \( M_0 \)). The conditions read explicitly as

\[ T^{I J} \phi = 0, \quad N_{i j} \phi = 0, \quad i > j, \quad (4.7) \]

\[ \bar{S}^{ji} \phi = 0, \quad N_i \phi = s_i \phi, \quad i = 1, \ldots, n-p-1, \]

\[ S^j_j \phi = 0, \quad N_j \phi = (s_j - 1) \phi, \quad j = n-p, \ldots, n-1, \quad (4.8) \]

and can be represented by Young tableaux of the form

\[
\begin{array}{ccccccc}
\hline
\hline
s_2 & s_1 & & & & & s_{n-1} - 1 \\
\hline
\hline
& s_0 & & & & & s_{n-2} - 1 \\
\hline
. & . & . & & & & \\
\hline
\end{array}
\]

(4.9)

with the weight \( s_0 \) such that

\[ s_{n-p-1} \leq s_0 \leq s_{n-p} - 1. \quad (4.10) \]

From the above inequality it follows that if \( s_i = s_{i-1} \) then module \( M_{n-i} \) is empty. Note that whatever weights \( s_i \) are module \( M_0 \) is always nonempty. Let us also note that \( M_{n-1} \) coincides with the module defined by (2.11), (2.12) if one shifts weights \( s_i \to s_i - 1 \).

We have the following

**Proposition 4.1.** The cohomology of \( Q \) in subspace (2.19) at ghost degree \( p \) can be identified with the subspace \( M_p \). In particular, \( M_p \) is naturally a Poincaré module for any \( p \).

The second statement immediately follows from \( [Q, M_{ab}] = [Q, P_a] = 0 \). The proof of the Proposition is given in Appendix C. In what follows we explicitly demonstrate the computation of \( Q \)-cohomology for the first nontrivial case \( n = 3 \).
The cohomology of the BRST operator \( Q = S_1^+ \frac{\partial}{\partial b_1} + S_2^+ \frac{\partial}{\partial b_2} \) in the subspace (2.19) can be identified with the cohomology of

\[
\hat{Q}_0 = \chi N_2^1 + S_1^+ \frac{\partial}{\partial b_1} + S_2^+ \frac{\partial}{\partial b_2} + \chi b_2 \frac{\partial}{\partial b_1} \equiv \chi N_2^1 + S_2^+ \frac{\partial}{\partial b_2} + (S_1^+ + \chi b_2) \frac{\partial}{\partial b_1}
\]  

(4.11)

evaluated in the space of elements satisfying \( T^{IJ} \phi = 0 \) along with the weight conditions and represented by \( \chi \)-independent elements. Here we have introduced ghost variable \( \chi \) associated to the constraint \( N_2^1 \) that generates the cubic ghost term in \( \hat{Q}_0 \). Indeed, for a \( \chi \)-independent element the cocycle condition implies \( N_2^1 \phi = 0 \).

Along with \( \hat{Q}_0 \) let us consider another nilpotent operator

\[
\hat{Q}_1 = \chi N_2^1 + \bar{S}^{11} b_1 + S_2^+ \frac{\partial}{\partial b_2} + \frac{\partial}{\partial b_2} \frac{\partial}{\partial \chi} b_1 = \chi N_2^1 + S_2^+ \frac{\partial}{\partial b_2} + (\bar{S}^{11} + \frac{\partial}{\partial b_2} \frac{\partial}{\partial \chi}) b_1 .
\]  

(4.12)

This can be seen as a BRST operator implementing the conditions from upper-triangular subalgebra \( U_1 \) if one flips the ghost number assignment for the variable \( b_1 \). The difference between \( \hat{Q}_0 \) and \( \hat{Q}_1 \) is in

\[
q_1 = (S_1^+ + \chi b_2) \frac{\partial}{\partial b_1}
\]  

(4.13)

replaced by

\[
\bar{q}_1 = (\bar{S}^{11} + \frac{\partial}{\partial b_2} \frac{\partial}{\partial \chi}) b_1 .
\]  

(4.14)

As suggested by the notations these two operators are indeed particular cases of \( q \) and \( \bar{q} \) discussed above for \( a^A = \{a^a_1, \chi\} \) and \( y^A = \{y^a_2, b_2\} \).

In fact, cohomology of \( \hat{Q}_0 \) and \( \hat{Q}_1 \) are identical. To see this let us reduce both cohomological problems to the cohomology of \( q_1 \) and \( \bar{q}_1 \), respectively (this can be achieved decomposing the operators in the homogeneity degree in \( b_1 \)). Choosing as representatives the subspace \( q_1 \phi = \bar{q}_1 \phi = 0 \) one observes that \( \hat{Q}_0 \) and \( \hat{Q}_1 \) act in this subspace. Moreover, in this subspace they simply coincide. This proves that cohomology of \( \hat{Q}_0 \) and \( \hat{Q}_1 \) are isomorphic and the representatives can be taken the same.

In exactly the same way one proves that the cohomology of \( \hat{Q}_1 \) is identical to the cohomology of \( \hat{Q}_2 \) given by

\[
\hat{Q}_2 = \chi N_2^1 + \bar{S}^{12} b_1 + (\bar{S}^{12} + \chi \frac{\partial}{\partial b_1}) b_2,
\]  

(4.15)

which can be considered a BRST operator implementing the conditions from \( U_2 \) if one in addition changes the ghost number assignment for \( b_2 \). Analogously, the difference between \( \hat{Q}_1 \) and \( \hat{Q}_2 \) is in

\[
q_2 = (S_2^+ - b_1 \frac{\partial}{\partial \chi}) \frac{\partial}{\partial b_2}
\]  

(4.16)

replaced by

\[
\bar{q}_2 = (\bar{S}^{12} - \chi \frac{\partial}{\partial b_1}) b_2 .
\]  

(4.17)

Once again, above operators are particular cases of \( q \) and \( \bar{q} \) with \( a^A = \{a^a_2, -b_1\} \) and \( y^A = \{y^a, \chi\} \).
Operators $\hat{Q}_0, \hat{Q}_1, \hat{Q}_2$ allow us to immediately compute the cohomology on the representation space of elements $\phi = \phi^0 + b_1\phi^1 + b_2\phi^2 + b_1b_2\phi^{12}$. In particular, for elements whose representatives have the form $b_1\phi^1 + b_2\phi^2$ the cocycle conditions with respect to $\hat{Q}_0$ and $\hat{Q}_1$ imply $\phi_1 = 0$ and
\[
N_2^1\phi^2 = 0, \quad \tilde{S}^{t1}\phi^2 = S_2^t\phi^2 = 0, \quad (4.18)
\]
i.e. the conditions for $M_1$. For elements of the form $b_1b_2\phi^{12}$ the cocycle condition with respect to $\hat{Q}_0$ gives
\[
N_2^1\phi^{12} = 0, \quad \tilde{S}^{t1}\phi^{12} = \tilde{S}^{t2}\phi^{12} = 0, \quad (4.19)
\]
i.e. the conditions for $M_2$. Finally, for ghost-independent $\phi^0$ the cocycle conditions with respect to $\hat{Q}_2$ give the conditions identified in Section $2.3$, i.e.
\[
N_2^1\phi^0 = 0, \quad \tilde{S}^{t1}\phi^0 = \tilde{S}^{t2}\phi^0 = 0. \quad (4.20)
\]

The above consideration results in the observation that any cohomology class has (in fact, a unique) representative satisfying
\[
\hat{Q}_0\phi = \hat{Q}_1\phi = \hat{Q}_2\phi = 0. \quad (4.21)
\]

Let us now recall that in addition $\phi$ satisfies the tracelessness condition $T^{IJ}\phi = 0$, and Young symmetry and the weight conditions $N_i^j\phi = 0$, $i > j$, $\tilde{N_i}\phi = s_i\phi$. Below we describe all the solutions to these conditions.

- Module $M_0$ singled out by $N_2\phi^0 = s_2\phi^0$, $N_1\phi^0 = s_1\phi^0$ is described by the following Young tableaux

\[
M_0 : \begin{array}{cccccc}
& & & & & \\
& & & & & \\
s_2 & & & & & \\
s_1 & & & & & \\
& & & & & \\
& & & & & \\
s_0 & & & & & \\
\end{array}, \quad s_0 \geq s_2 . \quad (4.22)
\]

- Module $M_1$ singled out by $N_2\phi^2 = (s_2 - 1)\phi^2$, $N_1\phi^2 = s_1\phi^2$ is described by the following Young tableaux

\[
M_1 : \begin{array}{cccccc}
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
s_0 & & & & & \\
s_1 & & & & & \\
& & & & & \\
\end{array}, \quad s_1 \leq s_0 \leq s_2 - 1 . \quad (4.23)
\]
Module $M_2$ singled out by $N_2\phi^{12} = (s_2 - 1)\phi^{12}$, $N_1\phi^{12} = (s_1 - 1)\phi^{12}$ is described by the following Young tableaux

$$M_2 : \begin{array}{cccc}
& & s_2 - 1 & \\
& s_1 - 1 & & \\
& s_0 & & \\
\end{array}, \quad 0 \leq s_0 \leq s_1 - 1 .
$$

(4.24)

In the case of coinciding weights $s_1 = s_2 = s$ one gets $N_1\phi^2 = s\phi^2$ and $N_2\phi^2 = (s - 1)\phi^2$. But these contradict $N_2\phi^2 = 0$ because it implies that the number of $a_2$ is greater or equal than that of $a_1$. It follows that module $M_1$ is empty and the remaining modules are described by

$$M_0 : \begin{array}{cccc}
& & s_0 & \\
& s & & \\
& s & & \\
\end{array}, \quad s_0 \geq s .
$$

(4.25)

$$M_2 : \begin{array}{cccc}
& & s - 1 & \\
& s - 1 & & \\
& s_0 & & \\
\end{array}, \quad 0 \leq s_0 \leq s - 1 .
$$

(4.26)

The discussed above $M_0$, $M_1$, and $M_2$ can be recognized as modules appearing within the unfolded formulation for two-row fields [6]. More precisely, $p$-form fields with $p = 0, 1, 2$ of the unfolded approach take values in $M_p$. The case $s_1 = s_2$ was also considered in [35, 50]. A detailed discussion of the relationship with the unfolded formulation for any $n$ is given in the next Section.

4.1.3 Unfolded formulation

Any cohomology class of the form $b_{n-1}b_{n-2} \ldots b_{n-p}\phi_p$ gives rise to the ghost-number-zero element of the form $\theta^{a_1} \ldots \theta^{a_p} b_{n-1}b_{n-2} \ldots b_{n-p} \phi_{a_1 \ldots a_p}$. These in turn give rise to the physical fields that are $p$-forms on Minkowski space. One then finds that the space of physical fields of the theory (2.32) reduced to $Q$-cohomology is given by differential forms of degrees $0, 1, \ldots, n - 1$ taking values in respectively the cohomology spaces at ghost number $0, -1, \ldots, -n + 1$ described by Lorentz Young tableaux (4.9), (4.10). If $s_l = s_{l-1}$ then the $(n - l)$-form is missing so that remaining fields correspond to the rectangular blocks of the Young tableaux with rows of the length $s_l$. One then concludes that the spectrum of unfolded fields coincide with that proposed in [6].

In order to identify the unfolded equations and gauge symmetries one is to find a reduced BRST operator $\tilde{\Omega}$ acting in the $Q$-cohomology. More precisely, the reduced
operator have the form $\tilde{\Omega} = d - \bar{\sigma}$ where $\bar{\sigma}$ is the differential $\sigma = \theta^a \frac{\partial}{\partial y^a}$ reduced to the $Q$-cohomology. We also save notation $d$ for the restriction of $d = \theta^m \frac{\partial}{\partial \theta^m}$ to the $Q$-cohomology. In order to compute $\bar{\sigma}$ we follow the procedure of [15]. To this end we introduce minus the target-space ghost number as an additional grading. Then the entire representation space $H$, i.e. the space (2.19) tensored with the Grassmann algebra in $\theta^a$ is decomposed into the direct sum $H = \mathcal{E} \oplus \mathcal{G} \oplus \mathcal{F}$, where $\mathcal{E}$ is the subspace of representatives of the $Q$-cohomology, $\mathcal{G} = \text{Im} \ Q$, and $\mathcal{F}$ the complementary subspace. $Q$ determines the invertible map from $\mathcal{F}$ to $\mathcal{G}$. Let also $\rho : \mathcal{G} \to \mathcal{F}$ be the inverse to $Q$, i.e. $Q \rho g = g$ for any $g \in \mathcal{G}$. It follows that operator $\rho$ can be chosen to have a degree $+1$ with respect to ghosts $b_i$ and variables $y^a$, and a degree $-1$ with respect to variables $a_i^a$. We also assume that $\rho$ is extended to $\text{Ker} \ Q = \mathcal{E} \oplus \mathcal{G}$ such that $\rho e = 0$ for any $e \in \mathcal{E}$. Given such $\rho$ the expression for $\bar{\sigma} : \mathcal{E} \to \mathcal{E}$ reads as [15]

$$\bar{\sigma} = \Pi_\mathcal{E} (\sigma - (\sigma \rho) \sigma + (\sigma \rho) (\sigma \rho) \sigma - \ldots) , \quad (4.27)$$

where $\Pi_\mathcal{E}$ denotes the projector to $\mathcal{E}$. If $\mathcal{E}$ contains cohomology classes with ghost numbers from 0 to $n - 1$ then in general only first $n$ terms can be non-vanishing in this series. Also in $d$-dimensions the $(d + 1)$-st term necessarily vanish but this does not play a role because $n \leq \lfloor \frac{d}{2} \rfloor$.

Let us first make some general observations on the explicit structure of $\bar{\sigma}$. Let $f_i \in \mathcal{E}$, $i = 0, \ldots, n - 1$ has the form $f_i = b_{i+1} \cdots b_{n-1} \phi_i \in M_{n-i-1}$ (it is assumed that $f_{n-1} = \phi_{n-1}$ and $\phi_i$ depends also on $\theta^a$). Then the term $\Pi_\mathcal{E} (\sigma \rho)^l \sigma f_i$ with $l \geq 1$ in $\bar{\sigma} f_i$ can be nonvanishing only if $\# y = s_i = s_{i-1} = \ldots = s_{i-l+1}$, where $\# y$ equals to $s_0$ and denotes the homogeneity degree of $f_i$ in variables $y^a$. This can be easily seen by counting the number of ghosts and oscillators and then comparing with the structure of the $Q$-cohomology. Moreover, this is the only nonvanishing terms in the whole series for $\bar{\sigma} f_i$ (if $s_0 \neq s_i$ then the only nonvanishing term is $\Pi_\mathcal{E} \sigma$) provided $s_{i-1} \neq s_{i-1}$. This implies that for a given $f_i$ the only term that contributes corresponds to the rectangular block with the upper row representing $a_i^a$ of the Young tableau encoding the symmetry properties of $f_i$.

Let now $f_i \in \mathcal{E}$ be of the form above and such that in addition

$$S_i^l f_i = \tilde{S}^{ii} f_i = 0 , \quad \ldots , \quad S_i^{l-1} f_i = \tilde{S}^{ii-l} f_i = 0 . \quad (4.28)$$

This means that the Young tableaux representing $f_i$ contains the rectangular block of the height $l + 1$ (with the rows corresponding to $y, a_i, \ldots, a_{i-1}$). Let us consider $Q(\tilde{S}^{ii} b_i) \sigma f_i$. Among all the operators $S_i^l \frac{\partial}{\partial b_m}$ entering $Q$ those with $m < i$ act trivially because $f_i$ does not depend on $b_m$ with $m < i$ while those with $m > i$ commute with $\tilde{S}^{ii}$ in the subspace of elements satisfying the Young symmetry conditions $N^i_j \phi = 0$ $i > j$. Moreover $S_i^l \frac{\partial}{\partial b_m} \sigma f_i = 0$ for $m > i$ because of the cocycle condition $Q \sigma f_i = 0$. The remaining term $S_i^l \frac{\partial}{\partial a_m}$ gives $[S_i^l, \tilde{S}^{ii}] \sigma f_i - \tilde{S}^{ii} \sigma S_i^l f_i$. The second term vanishes because $[S_i^l, \sigma] = 0$
and $S_i^\dagger f_i = 0$ according to the assumption. The first term gives $(a_i \frac{\partial}{\partial a_i} - y \frac{\partial}{\partial y}) \sigma f_i$. Because for $f_i$ one has $(a_i \frac{\partial}{\partial a_i} - y \frac{\partial}{\partial y}) f_i = 0$ one gets $(a_i \frac{\partial}{\partial a_i} - y \frac{\partial}{\partial y}) \sigma f_i = \sigma f_i$ so that

$$Q(S_i^\dagger b_i) \sigma f_i = \sigma f_i.$$  (4.29)

This shows that for an element of the form $\sigma f_i$ one can consistently define $\rho$ according to $\rho \sigma f_i = \bar{S}_i^\dagger b_i \sigma f_i$. Using $\bar{S}_i^\dagger b_i \sigma f_i = \bar{\sigma}^i b_i f_i$ where $\bar{\sigma}^i = \theta^a \frac{\partial}{\partial a_i}$ it is easy to see that $f_{i-1} = S_i^\dagger b_i \sigma f_i$ again satisfies $Q \sigma f_{i-1} = 0$ and

$$S_{i-1}^\dagger f_{i-1} = S_i^\dagger f_{i-1} = 0, \quad \ldots, \quad S_{i-l}^\dagger f_i = S_i^\dagger f_i = 0,$$  (4.30)

so that the construction can be iterated defining the action of $\rho$ in all the nonvanishing terms in (4.27). For instance in the setting above one gets

$$(\sigma \rho)^l \sigma f_i = \sigma \bar{\sigma}^i b_{i-l} \bar{\sigma}^i b_{i-l+1} \ldots \bar{\sigma}^i b_i f_i.$$  (4.31)

It turns out that using the fact that all the other terms in (4.27) for a particular $f_i$ vanish one can write a closed expression for $\bar{\sigma}$

$$\bar{\sigma} = \Pi \varepsilon \left(\sigma - \sum_{i=1}^{n-1} \sigma \bar{\sigma}^i b_i + \sum_{i<j} \sigma \bar{\sigma}^i b_i \bar{\sigma}^j b_j - \ldots\right).$$  (4.32)

For example for $n = 3$ one gets

$$\bar{\sigma} = \Pi \varepsilon \left(\sigma - \sigma \bar{\sigma}^1 b_1 - \sigma \bar{\sigma}^2 b_2 + \sigma \bar{\sigma}^1 b_1 \bar{\sigma}^2 b_2\right).$$  (4.33)

Note that if $s_1 = s_2$ only the first and the last terms contribute because the cohomology class $\mathcal{M}_1$ is missing in this case.

### 4.2 Generalized Poincaré module

As we have seen the spectrum of the unfolded fields can be described as a zero-ghost-number $Q$-cohomology $\mathcal{M}$ evaluated in the space (2.19) tensored with the Grassmann algebra in $\theta^a$. This space is graded by the homogeneity in $b_i$ (this degree is known in the literature as the target space ghost number) and its zeroth degree component coincides with $\mathcal{M}_0$ while the higher degree components are $\mathcal{M}_p$ tensored with the $p$-th homogeneous subspace of the Grassmann algebra in $\theta^a$. In the gauge description of the model this space replaces the starting point module $\mathcal{M}_0$ entering the gauge invariant description (2.20). In fact it is easy to see that the unfolded equations of motion for $b_i$-independent fields indeed reproduce (2.22) for $\mathcal{M}_0$-valued 0-form. This suggests that $\mathcal{M}$ is a natural generalization (extension) of $\mathcal{M}_0$ referred to in what follows as the Generalized Poincaré module. Note that from the BRST theory viewpoint it can be natural to consider all the $Q$-cohomology (not only at zeroth ghost degree). This can be seen as a BRST extension of $\mathcal{M}$. Along
with the fields of the unfolded formulation it contains all the respective ghost fields and antifields.

Let us briefly discuss the Poincaré module structure of $M$. To identify this structure it is convenient to use the alternative realization (2.33) of the Poincaré algebra in the theory determined by (2.25). In reducing the intermediate formulation (2.32) to $Q$-cohomology the generators (2.33) are also reduced to some operators acting on $Q$-cohomology valued fields. Because Lorentz generators $\bar{M}_{ab}$ maps representatives (C.10) to themselves and therefore their reduction is given by the same formulas. To obtain a reduction of $\bar{P}_a$ one should be more careful. The form of the reduced operator can be computed using, e.g., the formulas from [37]. It turns out, however, that this can equivalently be inferred from the $\tilde{\sigma}$ through $\tilde{\sigma} = \theta^a \tilde{P}_a$. Indeed, because $\theta$-variables enter the $Q$-cohomology through the tensor factor reducing $\sigma$ to $Q$-cohomology is the same as reducing $P_a$ to $Q$-cohomology and then constructing $\tilde{\sigma} = \theta^a \tilde{P}_a$.

Inspecting the explicit form of the reduced generators one finds that they also define the Poincaré module structure on $M$. This can be seen by, e.g., identifying $M$ with constant $M$-valued fields. As it follows from the explicit form of $\tilde{\sigma}$ generator $\tilde{P}_a$ (in contrast to the Lorentz generators that are unchanged) does act between different degree components of $M$. From this point of view $M$ can be thought as modules $M_p$ tensored with the algebra of $\theta^a$ and nontrivially glued together.

Remarkably, $M$ can be equipped with two in general different Poincaré module structures. The one determined by $P_a$ (the operator $\partial^a / \partial y^a$ acting in the cohomology of $Q$), for which the generalized Poincaré module is a direct sum of Poincaré modules appearing in different degrees and another one determined by $\tilde{P}$ (the reduction of $\partial^a / \partial y^a$ to the unfolded formulation) for which the generalized Poincaré module is not a direct sum in general. The tricky point here is that in both cases one reduces $\partial^a / \partial y^a$ to the $Q$-cohomology but the form of the reduced operator depends on the total BRST operator. In the algebraic setting of Section 2.3 this total operator is $Q$ itself while for $\tilde{P}$ the total BRST operator is $\tilde{\Omega} = d - \sigma + Q$.

As a final remark note that one can also define two different theories determined by the BRST operators $d - \tilde{\sigma}$ and $d - \theta^a P_a$. While the first one is the genuine gauge theory the second one is the direct sum of the gauge-invariant theory (2.20) and a bunch of the decoupled topological theories for differential forms of nonzero degrees.

### 4.3 Wigner approach

Another Poincaré module associated to the starting point module $M_0$ (Weyl module) can be identified by considering the space of gauge inequivalent solutions of the theory in the appropriate functional space. To construct this module explicitly we use the standard
BRST first-quantized description of the theory constructed in Section 2.6. Recall that the theory is determined by the BRST operator $\Omega$ given by (2.39) and the representation space is formed by $y^a, \theta^a$-independent elements satisfying the BRST extended trace, Young symmetry, and the weight conditions (2.42).

Let us now describe the space of gauge inequivalent configurations of the theory in the space of functions where $\frac{\partial}{\partial x^a}$ act diagonally (i.e. in the momentum representation). This space of inequivalent configurations can be identified with the zero-ghost-number $\Omega$-cohomology. Because $\Omega$ commutes with $\frac{\partial}{\partial x^a}$ it is enough to compute cohomology in a momentum eigenspace where $\frac{\partial}{\partial p_i^a} \phi = p_a \phi$. Assuming $p_a \neq 0$ (and hence disregarding the so-called zero-momentum cohomology) the cohomology can be easily computed using the arguments similar to the standard light-cone gauge. We follow [37] (see also [25] for a more traditional approach) where this computation has been explicitly carried over in the similar terms for $n = 2$.

Let us introduce the light-cone components $+, -, \alpha$ of the momenta and the oscillators. Assuming $p^+ \neq 0$ consider the following degree in the representation space

$$
\text{deg}(a^+_i) = 2, \quad \text{deg}(a^-_i) = -2, \\
\text{deg}(c_i) = 1, \quad \text{deg}(b_i) = -1,
$$

(4.34)

with all the other variables carrying vanishing degree [38,39]. The BRST operator (2.39) can be expanded into the components of definite degree as $\Omega = \Omega_{-1} + \Omega_0 + \Omega_1 + \Omega_2$. The lowest degree component of $\Omega$ reads as

$$
\Omega_{-1} = p^+ (c_i \frac{\partial}{\partial a^+_i} + a^-_i \frac{\partial}{\partial b_i})
$$

(4.35)

and can be seen as a version of de Rham differential multiplied by $p^+$. Because the degree is bounded from below (in the space of polynomials in oscillators $a_i$) one can first reduce the problem to the cohomology of $\Omega_{-1}$ in the subspace singled out by the conditions (2.42) (this is consistent as (2.42) commute with $\Omega_{-1}$).

It turns out that this cohomology can be obtained by restricting the $\Omega_{-1}$-cohomology evaluated in the space of all polynomials to the subspace (2.42). This is obvious for the constraints $N_i - s_i$ because $\Omega_{-1}$ do not mix different eigenspaces. As for the remaining constraints $T^{ij}$ and $N^{ij}_i, i > j$ they can be added as the additional constraints to BRST operator $\Omega_{-1}$ with their own ghost variables $\xi_A$ so that the required cohomology can be identified with cohomology of the extended BRST operator whose representatives can be chosen $\xi_A$-independent. The extended BRST operator has the structure $\Omega' = \Omega_{-1} + \xi_A T^A + \text{ghost terms}$, where $T^A$ is a collective notation for the constraints $T^{ij}$ and $N^{ij}_i, i > j$. Observing that the constraints $T^A$ carry vanishing degree one reduces the cohomological problem for $\Omega'$ to the cohomology of $\Omega_{-1}$. In the space of all polynomials $\Omega_{-1}$-cohomology is given by a subspace $\mathcal{E}$ of $c_i, b_i, a^+_i, a^-_i$-independent elements. The
reduced BRST operator has the form \( \xi_A \tilde{T}^A \) + ghost terms, where the reduced constraints \( \tilde{T}^A \) can be shown to be just original constraints \( T^A \) restricted to \( \mathcal{E} \). For a \( \xi_A \)-independent element from \( \mathcal{E} \) the cocycle condition imply

\[
\tilde{T}^{ij} \phi = \eta^\alpha \frac{\partial}{\partial a_i^\alpha} \frac{\partial}{\partial a_j^\alpha} \phi = 0, \quad \tilde{N}^i \phi = a^\alpha_i \frac{\partial}{\partial a_j^\alpha} \phi = 0 \quad i > j, \quad (4.36)
\]

This gives an explicit description of \( \Omega_{-1} \)-cohomology in the subspace (2.42).

The reduced theory is then determined by the reduced BRST operator

\[
\tilde{\Omega} = c_0 (p^i p_i - 2 p^+ p^-), \quad (4.37)
\]

defined on the subspace (4.36). Note that \( \Omega_1 \) and \( \Omega_2 \) do not contribute because the cohomology is concentrated in zeroth degree. The zero-ghost-number \( \tilde{\Omega} \)-cohomology in the momenta eigenspace is given by an arbitrary \( c_0 \)-independent elements satisfying (4.36) multiplied by \( \delta(p^2) \). Finally, the cohomology can be identified with the “Wigner module”, i.e. functions on the mass-shell \( p^2 = 0 \) with values in the subspace (4.36). One can speculate that the procedure above establishes an explicit duality transform between the Weyl module (zero-ghost-number cohomology of \( \Omega \) in the space of polynomials in \( x^a \)) and the Wigner module (zero-ghost-number cohomology of \( \Omega \) in the space where \( \frac{\partial}{\partial x^a} \) is diagonalizable). Note that because \( \Omega_{-1} \) is symmetric with respect to the inner product the reduction is consistent with the inner product.

The reduced action can be readily obtained in the form (see [37] for details)

\[
S^{lc} = \frac{1}{2} \int d^d p \, dc_0 \langle \tilde{\Psi}^{(0)}, \tilde{\Omega} \tilde{\Psi}^{(0)} \rangle_0, \quad (4.38)
\]

where the field \( \Psi^{(0)} \) now takes values in the subspace (4.36) and \( \langle \cdot, \cdot \rangle_0 \) denotes the Fock space inner product (see (2.43)) restricted to the subspace generated by the transversal oscillators. This is indeed the standard light-cone action for the transversal degrees of freedom. As it should be there is no leftover gauge symmetry. It is easy to see that conditions (4.36) are the irreducibility conditions for the \( so(d - 2) \) which is a Lie algebra of Wigner little group. One then concludes that the transversal degrees of freedom form an irreducible representation of \( so(d - 2) \) determined by weights \( s_{n-1} \geq s_{n-2} \geq \ldots \geq s_1 \). Note also that these conditions form an upper triangular subalgebra of \( sp(2n - 2) \) that is Howe dual to \( so(d - 2) \) on the Fock space of transversal oscillators \( a_i^\alpha \). Let us stress that contrary to the computation of the cohomology in the space of polynomials in variables \( y \) there are no cohomology classes depending on \( c_i, b_i \). In particular, the states analogous to those in the gauge modules do not appear. This is also due to the assumption that \( p^+ \neq 0 \). That is why the states from the gauge modules are often called zero momentum cohomology. These states are ignored in the Wigner approach.
Because the procedure just described follows the standard steps of the Wigner description of the unitary irreps one concludes that the gauge theory determined by (2.39) along with the trace, Young symmetry, and the weight conditions indeed describe a unitary irrep of the Poincaré group in the sense of Wigner approach.

5 Conclusions and outlooks

The above study could clearly be extended in various directions. A rather natural generalization is to allow for non-vanishing cosmological constant that implies the $(A)dS_d$ background geometry. For totally symmetric fields the corresponding parent formulation was developed in [16] and its extension to the case of arbitrary symmetry type will be considered elsewhere.

Our formulation can be also generalized to describe massive fields of any symmetry type on Minkowski space. This could be done using a standard dimesional reduction $d + 1 \rightarrow d$ thereby obtaining massive field dynamics in $d$ dimensional Minkowski space. This procedure can be implemented in the BRST theory terms [40, 41] and hence is directly applicable to the present formulation.

An interesting topic is to develop supersymmetric extensions which assume an appropriate inclusion of fermionic mixed-symmetry fields. Within our approach addressing the problem seems to be straightforward and reduces to introducing spin-tensors in an appropriate fashion. This can be achieved either by considering polynomials with coefficients in spinorial modules as, e.g., in [42, 43, 44] or by introducing additional oscillators transforming as $so(1, d - 1)$ spinors [45]. Both ways are equivalent and leave intact the main ingredients of our construction.

Another possible extension has to do with describing dual formulations (see, e.g., [46, 47], and references therein) of the mixed symmetry fields. These can be expected to arise through the different realizations of the Poincaré translations. Much less trivial seems the possibility to give a realization of the same module in the space-time of different geometry and/or dimension in the spirit of [48, 49].

A natural question that can be asked using the formulation developed in the paper is whether there exist a mixed symmetry counterparts of the well known higher spin algebras. Although in the case of symmetric fields a consistent HS algebra exists only on AdS space, at the off-shell level one can identify the analogous structure also in the Minkowski space. Moreover, in the symmetric field case a natural framework [33, 17] to study this structure is provided by a version of the intermediate formulation (2.32). From this perspective, the approach developed in the paper can be a natural tool to study candidate HS algebras for mixed symmetry fields that in turn can be a first step towards constructing consistent interactions for mixed-symmetry fields.
We choose orthonormal basis

\[ \mathbf{E}_I = T_{I+1}^I \]
\[ \mathbf{H}_I = T_{I+1}^{I+1} - T_I^I \quad \text{for} \ 0 \leq I \leq n - 2 \quad \text{and} \quad \mathbf{H}_{n-1} = -T_{n-1}^{n-1} \]
\[ \mathbf{F}_I = T_I^{I+1} \]
\[ \mathbf{F}_{n-1} = -\frac{1}{4}T_{n-1}^{n-1} \]  \hspace{1cm} (A.1)

We choose orthonormal basis \( h_I = -T_I^I, 0 \leq I \leq n - 1 \) in the Cartan subalgebra. Then \( H_I = h_I - h_{I+1} \) for \( 0 \leq I \leq n - 2 \) and \( H_{n-1} = h_{n-1} \). The dual basis \( \epsilon_I \) in the space dual to the Cartan subalgebra satisfy \( \langle \epsilon_I, h_J \rangle = \delta_{IJ} \). The simple positive roots are \( \alpha_I = \epsilon_I - \epsilon_{I+1} \) for \( 0 \leq I \leq n - 2 \) and \( \alpha_{n-1} = 2\epsilon_{n-1} \). The half of the sum of the positive roots is \( \rho = n\epsilon_0 + (n - 1)\epsilon_1 + \cdots + \epsilon_{n-1} \). We note also that \( E_I, H_I, \) and \( F_I \) with \( 0 \leq I \leq n - 2 \) form a Chevalley basis of the \( sl(n) \) subalgebra in \( sp(2n) \).

We describe in details the structure of the \( so(d) - sp(2n) \) bimodule \( \mathcal{P}_n^d(a) \) in the case \( n = [d/2] \) and give several notes in the case \( n > [d/2] \). We suppose \( n = [d/2] \) and let \( \Lambda \) denote the space of vectors \( \sigma = (s_{n-1}, s_{n-2}, \cdots, s_0) \) with integer components satisfying \( s_{n-1} \geq s_{n-2} \geq \cdots \geq s_0 \geq 0 \). Thus \( \sigma \) defines a highest weight of \( so(d) \) and \( \Lambda \) is the space of dominant highest weights corresponding to tensor modules. We also define a mapping \( \theta \) from \( \Lambda \) to the space of \( sp(2n) \) highest weights

\[ \theta(\sigma) = -\sum_{I=0}^{n-1} \left( s_I + \frac{d}{2} \right) \epsilon_I. \]  \hspace{1cm} (A.2)

The highest weight \( \theta(\sigma) \) can be at the same time considered a \( sl(n) \) highest weight. By Howe duality \( so(d) \) and \( sp(2n) \) algebras mutually centralize each other in \( \mathcal{P}_n^d(a) \), (2.2). The \( so(d) - sp(2n) \) bimodule \( \mathcal{P}_n^d(a) \) has the structure [30]

\[ \mathcal{P}_n^d(a) = \bigoplus_{\sigma \in \Lambda} (V_\sigma \otimes U_{\theta(\sigma)}) , \]  \hspace{1cm} (A.3)

where \( V_\sigma \) and \( U_{\theta(\sigma)} \) are irreducible \( so(d) \) and \( sp(2n) \) modules with highest weights \( \sigma \) and \( \theta(\sigma) \) respectively. The module \( U_{\theta(\sigma)} \) is the generalized Verma module induced from the
finite dimensional irreducible $sl(n)$ module $W_{θ(σ)}$ with integer dominant $sl(n)$ highest weight $θ(σ)$. In other words, it means that $U_{θ(σ)}$ is freely generated by generators $T_{IJ}$ from $sl(n)$ module $W_{θ(σ)}$. The check that the $sp(2n)$ generalized Verma module $U_{θ(σ)}$ with the highest weight $θ(σ)$ (A.2) is simple is reduced to the standard application of the Kac–Kazhdan criterion [50] that image of $θ(σ)+ρ$ under a reflection from the Weyl group can not belong to the lattice of weights of $U_{θ(σ)}$.

In particular, the module $U_{θ(σ)}$ is cofree with respect to generators $T_{IJ}$ and therefore the cohomology of the operator $∆_{IJ} = C_{IJ}T_{IJ}$ in $U_{θ(σ)}$ are

$$H^n(∆_{IJ}) = \begin{cases} W_{θ(σ)}, & n = 0, \\ 0, & n > 0. \end{cases}$$ (A.4)

In the case $n > [d/2]$ the module $U_{θ(σ)}$ is not isomorphic to a generalized Verma module but is a quotient of a generalized Verma module. In other words there are some relations between generators $T_{IJ}$. Whenever, $n > [d/2]$ the decomposition (2.8) is the same but the mapping $θ$ is defined as follows. Let $σ = (s_{n−1}, s_{n−2}, \cdots, s_{n−[d/2]})$ with integer components satisfying $s_{n−1} ≥ s_{n−2} ≥ \cdots ≥ s_{n−[d/2]} ≥ 0$ be a dominant integer $so(d)$ highest weight. We set $s_0 = s_1 = \cdots = s_{n−[d/2]−1} = 0$. Then, the mapping $θ$ is given by (A.2). In this case the generalized Verma module induced from the finite-dimensional irreducible $sl(n)$ module $W_{θ(σ)}$ contains singular vectors and its structure for large $n−[d/2]$ is quite complicated.

### B Homological reduction

Here we reproduce the proposition on the homological reduction proved in [15][16]. Let $H$ be a vector (super)space. Consider a bundle $H = X \times H \rightarrow X$, where $X$ is a space-time manifold with local coordinates $x^a$, and denote by $Γ(H)$ the space of sections of $H$. There are two gradings, the Grassmann parity and ghost number defined on $H$ which are naturally extended to $H$-valued sections. The BRST operator $Ω : Γ(H) → Γ(H)$ is a Grassmann odd differential of finite order in $x$-derivatives with coefficients in linear operators in $H$.

We assume that an additional grading in $H$ can be introduced such that each graded component is finite-dimensional.

**Proposition B.1.** Suppose $H$ to be equipped with an additional grading besides the ghost number,

$$H = \bigoplus_{i ≥ 0} H_i, \quad \deg(H_i) = i, \quad (B.1)$$

and let the BRST operator $Ω$ have the form

$$Ω = Ω_{−1} + Ω_0 + \sum_{i ≥ 1} Ω_i, \quad \deg(Ω_i) = i, \quad (B.2)$$
with \( \Omega_i : \Gamma(\mathcal{H})_j \rightarrow \Gamma(\mathcal{H})_{i+j} \). If \( \Omega_{-1} \) is independent of \( x \) and contains no \( x \)-derivatives then the cohomology \( H(\Omega_{-1}, \Gamma(\mathcal{H})) \cong \Gamma(\mathcal{E}) \) for some vector bundle \( \mathcal{E} \subset \mathcal{H} \) and the system \( (\Omega, \Gamma(\mathcal{H})) \) can be consistently reduced to \( (\tilde{\Omega}, \Gamma(\mathcal{E})) \), where the operator \( \tilde{\Omega} \) is the differential induced by \( \Omega \) in the cohomology of \( \Omega_{-1} \).

Let us note that operator \( \Omega_{-1} \) acting on \( \mathcal{H} \) induces a triple decomposition \( \mathcal{H} = \mathcal{E} \oplus \mathcal{F} \oplus \mathcal{G} \), where \( \text{Ker} \Omega_{-1} = \mathcal{E} \oplus \mathcal{G} \), \( \mathcal{E} \cong H(\Omega_{-1}, \mathcal{H}) \), \( \mathcal{G} = \text{Im} \Omega_{-1} \), and \( \mathcal{F} \) is a complementary subbundle. Then \( \tilde{\Omega} \) is algebraically invertible and \( \tilde{\Omega} \) is given by

\[
\tilde{\Omega} = (\Omega - \tilde{\Omega} (\Omega)^{-1} \Omega) , \quad \tilde{\Omega} : \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E}) . \quad (B.3)
\]

An explicit recursive construction for \( \tilde{\Omega} \) can be found in [15]. Note also that if the cohomology of \( \Omega_{-1} \) is concentrated in one degree then \( \tilde{\Omega} = \Omega_0 \) considered as acting in \( \Gamma(\mathcal{E}) \).

In the case where equations of motion have the unfolded form \( \Omega \Phi(p) = 0 \) with \( \Omega \) being a flat covariant differential acting on differential \( p \)-forms \( \Phi(p) \), the respective \( \Omega_{-1} \) was originally identified as the \( \sigma_- \)-operator [20][21].

### C Q-cohomology for any \( n \)

The proof in the general case goes in exactly the same way as for \( n = 3 \). Namely, one constructs operators \( \hat{\Omega}_l \) associated to the upper-triangular subalgebras \( U_l \) using the same rule as before, i.e. ghosts \( b_t \) enter the respective terms either as \( S_{i}^t \frac{\partial}{\partial b_i} \) or as \( \tilde{S}^{t\dagger} b_t \). For any \( \hat{\Omega}_l \) and \( \hat{\Omega}_{l+1} \) one finds that their difference is in the term \( q_{l+1} \) replaced by \( \bar{q}_{l+1} \) that shows that the cohomology of all \( \hat{\Omega}_l \) is identical. The difference between \( \hat{\Omega}_l \) and \( \hat{\Omega}_{l+1} \) originates from the relation between the upper-triangular subalgebras \( U_l \) and \( U_{l+1} \) that can be visualized as the exchange \( S_{i}^t \rightarrow \tilde{S}^{t\dagger} \).

The representation space of operators \( \hat{\Omega}_l \) is given by

\[
\phi = \phi^{(0)} + \phi^{(1)} + \cdots \phi^{(n-1)} \equiv \sum_{k=0}^{n-1} b_{i_{1}} \cdots b_{i_{k}} \phi^{i_{1} \cdots i_{k}} , \quad (C.1)
\]

where \( \phi^{i_{1} \cdots i_{k}} \) are anti-symmetric tensors, and \( gh \phi^{(k)} = k \).

Let us start the analysis of \( Q \)-cohomology with operator \( \hat{\Omega}_0 \) that can be represented as

\[
\hat{\Omega}_0 = \sum_{i>j} \chi^{i}_{j} N_{i}^{j} + \sum_{i>j,j \neq 1} \chi^{i}_{j} b_{i} \frac{\partial}{\partial b_{j}} + \sum_{j \neq 1} S_{j}^{t} \frac{\partial}{\partial b_{j}} + (S_{1}^{t} + \chi^{2}_{1} b_{1}) \frac{\partial}{\partial b_{1}} , \quad (C.2)
\]

where \( \chi^{i}_{j} i > j \) are ghosts associated to \( N_{i}^{j} i > j \). At the minimal ghost number \(-(n-1)\) the cohomology of \( \hat{\Omega}_0 \) is obviously given by

\[
\hat{\Omega}_0 \phi = 0 , \quad (C.3)
\]
where elements $\phi = \phi^{(n-1)} \equiv b_1 \ldots b_{n-1}\phi_{n-1}$ are such that $\phi_{n-1}$ satisfies

$$N_{i}^{j}\phi_{n-1} = 0 \quad i > j, \quad S_{i}^{t}\phi_{n-1} = 0, \quad i = 1, \ldots, n - 1,$$

i.e. the highest weight vectors for the upper-triangular subalgebra $U_{n-1}$.

Then one again finds that the last term of $\hat{Q}_{0}$ can be treated as $q_{1} = \left(S_{i}^{t} + \chi_{i}^{2}b_{j}\right)\frac{\partial}{\partial b_{0}}$, and consistently replaced with the respective $\bar{q}_{1}$. The resulting operator is $\hat{Q}_{1}$. Applying the same reasoning as in the case of $n = 3$ one concludes that the cohomology of operators $\hat{Q}_{0}$ and $\hat{Q}_{1}$ are isomorphic and the representatives can be taken the same

$$\hat{Q}_{0}\phi = \hat{Q}_{1}\phi = 0.$$  \tag{C.5}

Solving these relations one gets

$$\frac{\partial}{\partial b_{1}}\phi^{(k)} = 0, \quad k = 0, 1, \ldots, n - 2,$$  \tag{C.6}

along with a set of linear combinations of Young symmetrizers applied to $\phi^{(k)}$. One observes then that in the ghost number $n - 2$ relation (C.6) means that the only non-zero component of $\phi^{(n-2)}$ is $b_{2} \ldots b_{n-1}\phi_{n-1}^{23 \ldots n-1} \equiv b_{2} \ldots b_{n-1}\phi_{n-2}$ which satisfies

$$N_{i}^{j}\phi_{n-2} = 0 \quad i > j, \quad S_{i}^{1}\phi_{n-2} = 0 \quad i = 2, 3, \ldots, n - 1, \quad \bar{S}_{1}^{1}\phi_{n-2} = 0,$$  \tag{C.7}

i.e. the highest weight vectors for the upper-triangular subalgebra $U_{n-2}$.

Repeating the procedure for $q_{2}$ etc one finds $n$ operators $\hat{Q}_{i}$ such that the cohomology representatives can be taken to satisfy

$$\hat{Q}_{m}\phi = \hat{Q}_{m+1}\phi = 0, \quad m = 0, \ldots, n - 1.$$  \tag{C.8}

At the last step of the above iterative procedure one is left with operator $\hat{Q}_{n-1}$ which defines the cohomology in the maximal ghost degree 0 through the cocycle condition

$$\hat{Q}_{n-1}\phi = 0.$$  \tag{C.9}

This immediately gives the answer for the cohomology. Namely the representative at ghost number $-p, 0 \leq p \leq n - 1$ is given by $b_{n-p}b_{n-p+1} \ldots b_{n-1}\phi_{p}$ with $\phi_{p}$ satisfying

$$N_{i}^{j}\phi_{p} = 0 \quad i > j,$$

$$S_{n-p}^{t}\phi_{p} = \ldots = S_{n-1}^{t}\phi_{p} = 0, \quad \bar{S}_{1}^{1}\phi_{p} = \ldots = \bar{S}_{n-p-1}^{1}\phi_{p} = 0,$$  \tag{C.10}

i.e. the highest weight vectors for the upper-triangular subalgebra $U_{p}$.

Summarizing the above one concludes that any cohomology class of the original BRST operator $Q$ has a representative that can be chosen to satisfy

$$\hat{Q}_{0}\phi = \hat{Q}_{1}\phi = \ldots = \hat{Q}_{n-1}\phi = 0.$$  \tag{C.11}

Recall that in addition $\phi_{p}$ satisfies $N_{i}^{j}\phi_{p} = 0 \quad i > j, \quad T_{i}^{j}\phi_{p} = 0$, and $\bar{N}_{i}(b_{n-p} \ldots b_{n-1}\phi_{p}) = s_{i}(b_{n-p} \ldots b_{n-1}\phi_{p})$.  

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