Presenting a new method for the solution of nonlinear problems

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Abstract

We present a method for the resolution of (oscillatory) nonlinear problems. It is based on the
application of the Linear Delta Expansion to the Lindstedt-Poincaré method. By applying it to
the Duffing equation, we show that our method substantially improves the approximation given
by the simple Lindstedt-Poincaré method.

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I. INTRODUCTION

In this letter we present a method for the resolution of oscillatory nonlinear problems. This new method is obtained by applying the Linear Delta Expansion (LDE) to the well-known Lindstedt-Poincaré (LP) method. We explicitly solve the Duffing Equation (Anharmonic Oscillator Equation) and find that our approximation works much better over a wider range of parameters than does the simple LP method. We also compare our results to an approximation performed using the method of the perturbative δ expansion, and find again that our approximation works with better accuracy and convergence for a wider range of parameters.

In Section II we briefly review the LP and LDE approximation methods, then in Section III we show how by applying the LDE to the LP method, one can solve the Duffing Equation. Our results are presented in Section IV and we finally present our conclusions and directions for future work in Section V.

II. APPROXIMATION SCHEMES

A. The Lindstedt-Poincaré method

In this section we introduce the Lindstedt-Poincaré distorted time (LP) method. We consider a nonlinear ODE of the form

\[ \ddot{x}(t) + \omega^2 x(t) = \varepsilon f(x(t)), \tag{2.1} \]

which describes a conservative system, oscillating with an unknown period \( T \). The nonlinear term \( \varepsilon f(x(t)) \) is treated as a perturbation. Unfortunately, when the ordinary perturbation is applied to eq. (2.1), by writing the solution as a series in \( \varepsilon \), the appearance of secular terms spoils the expansion and any predictive power is lost for sufficiently large time scales.

In order to avoid the appearance of secular terms, we switch to a scaled time \( \tau = 2\pi t/T \equiv \Omega t \), where \( T \) is the (unknown) period of the oscillations. The ODE now reads:

\[ \Omega^2 \frac{d^2 x}{d\tau^2}(\tau) + \omega^2 x(\tau) = \varepsilon f(x(\tau)). \tag{2.2} \]

We notice that the dependence upon \( \varepsilon \) in this equation enters both in the solution \( x(\tau) \)
and in the frequency $\Omega$. By assuming $\varepsilon$ to be a small parameter we write

$$\Omega^2 = \sum_{n=0}^{\infty} \varepsilon^n \alpha_n ; \quad x(\tau) = \sum_{n=0}^{\infty} \varepsilon^n x_n(\tau)$$

and expand the r.h.s of eq. (2.2) as

$$f(x) = f \left( \sum_{n=0}^{\infty} \varepsilon^n x_n(\tau) \right) \approx f(x_0) + \varepsilon x_1 f'(x_0) + \varepsilon^2 \left[ x_2 f'(x_0) + \frac{x_2^2}{2} f''(x_0) \right]$$

$$+ \varepsilon^3 \left[ x_3 f'(x_0) + x_2 x_1 f''(x_0) + \frac{x_3^2}{6} f'''(x_0) \right] + O(\varepsilon^4).$$

By using these expansions inside eq. (2.2) we obtain a system of linear inhomogeneous differential equations, each corresponding to a different order in $\varepsilon$. Let us consider the first few terms. To order $\varepsilon^0$ we obtain the equation

$$\alpha_0 \frac{d^2 x_0}{d\tau^2} + \omega^2 x_0(\tau) = 0,$$

(2.3)

describing a harmonic oscillator of frequency $\Omega = \sqrt{\alpha_0} = \omega$. To order $\varepsilon$ we obtain the equation

$$\alpha_0 \frac{d^2 x_1}{d\tau^2} + \omega^2 x_1(\tau) = s_1(\tau),$$

(2.4)

where the r.h.s. is given by

$$s_1(\tau) \equiv -\alpha_1 \frac{d^2 x_0}{d\tau^2} + f(x_0).$$

(2.5)

We remark the oscillatory behavior of the driving term $s_1(\tau)$, because of its dependence upon the order-0 solution, $x_0(\tau)$. As a result $s_1(\tau)$ will contain the fundamental frequency, corresponding to a period of $2\pi$ in the scaled time, and multiples of this frequency, appearing through the term $f(x_0(\tau))$. The presence of a driving term with the fundamental frequency leads to a resonant behavior of $x_1(\tau)$ and to the unfortunate occurrence of secular terms, which spoils our expansion. However, we can deal with this problem by fixing the coefficient $\alpha_1$ to cancel the resonant term in the r.h.s. of eq. (2.4). The iteration of this procedure to a given order $n$ allows to determine the coefficients $\alpha_0, \ldots, \alpha_n$ and therefore the frequency $\Omega = \sqrt{\alpha_0 + \alpha_1 + \ldots + \alpha_n}$.

B. Linear delta expansion

The linear delta expansion (LDE) is a powerful technique which has been originally introduced to deal with problems of strong coupling Quantum Field Theory, for which the
naive perturbative approach is not useful. Since then this method has been applied to a wide class of problems [4, 5, 6, 7, 8, 9]. In its original formulation a lagrangian density $\mathcal{L}$, which is not exactly solvable, is interpolated with a solvable lagrangian $\mathcal{L}_0(\mu)$, depending upon one (or more) parameters $\mu$:

$$\mathcal{L}_\delta = \mathcal{L}_0(\mu) + \delta (\mathcal{L} - \mathcal{L}_0(\mu)) . \quad (2.6)$$

For $\delta = 0$ one obtains $\mathcal{L}_0(\mu)$, whereas for $\delta = 1$ one recovers the full lagrangian $\mathcal{L}_\delta$. The term $\delta (\mathcal{L} - \mathcal{L}_0)$ is treated as a perturbation and $\delta$ is used to keep track of the perturbative order. Eventually $\delta$ is set to be 1.

We notice that the interpolation of the full lagrangian with the solvable one, $\mathcal{L}_0(\mu)$, brings an artificial dependence upon the arbitrary parameter $\mu$. Such dependence, which would vanish if all perturbative orders were calculated, can be milder to a finite perturbative order, by requiring some physical observable $\mathcal{O}$ to be locally insensitive to $\mu$, i.e:

$$\frac{\partial \mathcal{O}(\mu)}{\partial \mu} = 0.$$

This condition is known as Principle of Minimal Sensitivity (PMS) and is normally seen to improve the convergence to the exact solution.

III. ANHARMONIC OSCILLATOR

Following the techniques described in the previous Section, we can now apply the LDE to the LP for the solution of the Duffing Equation (the Anharmonic oscillator equation):

$$\frac{d^2x}{dt^2}(t) + \omega^2 \cdot x(t) = -\mu x^3(t) . \quad (3.1)$$

This equation describes a conservative system, where the total energy is given by

$$E = \frac{\dot{x}^2}{2} + \left[\frac{\omega^2 x^2}{2} + \mu \frac{x^4}{4}\right]. \quad (3.2)$$

The period of the oscillation can be calculated in terms of an elliptic integral

$$T_{exact} = 2 \int_{-A}^{A} dx \frac{1}{\sqrt{2(E - V(x))}} , \quad (3.3)$$

where $A$ is the amplitude of the oscillations.
Following the procedure explained in the previous Section, we write Eq. (3.1) as
\[ \Omega^2 \frac{d^2 x}{d\tau^2}(\tau) + \left(\omega^2 + \lambda^2\right) x(\tau) = \delta \left[-\mu x^3(\tau) + \lambda^2 x(\tau)\right] , \] (3.4)
where an arbitrary parameter \( \lambda \) with dimension of frequency has been introduced. Clearly for \( \delta = 1 \), Eq. (3.4) reduces to Eq. (3.1). We repeat the procedures previously explained and find a hierarchy of linear inhomogeneous differential equations to be solved sequentially.

To zeroth order we obtain the equation
\[ \alpha_0 \frac{d^2 x_0}{d\tau^2} + (\omega^2 + \lambda^2) x_0(\tau) = 0 , \] (3.5)
with solution
\[ x_0(\tau) = A \cos \tau . \] (3.6)
The zeroth order frequency is then given by
\[ \alpha_0 = \omega^2 + \lambda^2 . \] (3.7)

We proceed to compute the first order and find that
\[ \alpha_0 \frac{d^2 x_1}{d\tau^2} + (\omega^2 + \lambda^2) x_1(\tau) = S_1(\tau) , \] (3.8)
where
\[ S_1(\tau) = A \cos \tau \left[\alpha_1 + \lambda^2 - \frac{3A^2\mu}{4}\right] - \frac{A^3\mu}{4} \cos 3\tau . \] (3.9)
Now \( \alpha_1 \) is fixed by eliminating the term proportional to \( \cos \tau \):
\[ \alpha_1 = \frac{3A^2\mu}{4} - \lambda^2 . \] (3.10)

We obtain the solution
\[ x_1(\tau) = -\frac{A^3 \mu}{32(\omega^2 + \lambda^2)} \cos \tau + \frac{A^3 \mu}{32(\omega^2 + \lambda^2)} \cos 3\tau , \]
and the frequency
\[ \Omega^2 = \alpha_0 + \alpha_1 = \omega^2 + \frac{3A^2\mu}{4} , \] (3.11)
which is observed to be independent of \( \lambda \).
The next order gives:

\[
\alpha_0 \frac{d^2 x_2}{d\tau^2} + (\omega^2 + \lambda^2) x_2(\tau) = S_2(\tau),
\]  

(3.12)

where now

\[
S_2(\tau) = \frac{A (3 A^4 \mu^2 + 128 \alpha_2 (\omega^2 + \lambda^2))}{128 (\omega^2 + \lambda^2)} \cos \tau \\
+ \frac{A^3 \mu (3 A^2 \mu - 4 \lambda^2)}{16 (\omega^2 + \lambda^2)} \cos 3\tau \\
- \frac{3 A^5 \mu^2}{128 (\omega^2 + \lambda^2)} \cos 5\tau.
\]  

(3.13)

As before \(\alpha_2\) is fixed by eliminating the term proportional to \(\cos \tau\):

\[
\alpha_2 = -\frac{3 A^4 \mu^2}{128 (\omega^2 + \lambda^2)}.
\]  

(3.14)

We obtain the solution

\[
x_2(\tau) = \frac{A^3 \mu (23 A^2 \mu - 32 \lambda^2)}{1024 (\omega^2 + \lambda^2)^2} \cos \tau + \frac{A^3 \mu (-3 A^2 \mu + 4 \lambda^2)}{128 (\omega^2 + \lambda^2)^2} \cos 3\tau \\
+ \frac{A^5 \mu^2}{1024 (\omega^2 + \lambda^2)^2} \cos 5\tau
\]  

(3.15)

and the frequency

\[
\Omega^2 = \alpha_0 + \alpha_1 + \alpha_2 = \omega^2 + \frac{3 A^2 \mu}{4} - \frac{3 A^4 \mu^2}{128 (\omega^2 + \lambda^2)}.
\]  

(3.16)

Note that at this order the frequency now depends on the arbitrary parameter \(\lambda\). However, due to the explicit dependence, by applying the PMS, we would obtain the same solution as in the simple LP method. In order to get a different solution, we must go to the next order in the expansion.

Finally, following the same procedure, we obtain the expression for the third order:

\[
\alpha_0 \frac{d^2 x_3}{d\tau^2} + (\omega^2 + \lambda^2) x_3(\tau) = S_3(\tau),
\]  

(3.17)

where

\[
s_3(\tau) = \left[ A \alpha_3 - \frac{3 A^5 \mu^2 (3 A^2 \mu - 4 \lambda^2)}{512 (\omega^2 + \lambda^2)^2} \right] \cos \tau \\
- \frac{(A^3 \mu (297 A^4 \mu^2 - 768 A^2 \mu \lambda^2 + 512 \lambda^4))}{2048 (\omega^2 + \lambda^2)^2} \cos 3\tau \\
+ \frac{3 A^5 \mu^2 (3 A^2 \mu - 4 \lambda^2)}{256 (\lambda^2 + \omega^2)^2} \cos 5\tau - \frac{3 A^7 \mu^3}{2048 (\lambda^2 + \omega^2)^2} \cos 7\tau.
\]  

(3.18)
By eliminating the term proportional to \( \cos \tau \) we determine \( \alpha_3 \) to be

\[
\alpha_3 = \frac{3 A^4 \mu^2 (3 A^2 \mu - 4 \lambda^2)}{512 (\lambda^2 + \omega^2)^2},
\]

and the solution

\[
x_3(\tau) = -\frac{A^3 \mu}{32768} \frac{547 A^4 \mu^2 - 1472 A^2 \mu \lambda^2 + 1024 \lambda^4}{(\lambda^2 + \omega^2)^3} \cos \tau \\
+ \frac{A^3 \mu}{16384} \frac{297 A^4 \mu^2 - 768 A^2 \mu \lambda^2 + 512 \lambda^4}{(\lambda^2 + \omega^2)^3} \cos 3\tau \\
+ \frac{A^5 \mu^2}{2048} \frac{(-3 A^2 \mu + 4 \lambda^2)}{(\lambda^2 + \omega^2)^3} \cos 5\tau + \frac{A^7 \mu^3}{32768} \frac{1}{(\lambda^2 + \omega^2)^3} \cos 7\tau.
\]

The frequency to order \( \delta^3 \) is now obtained to be

\[
\Omega^2 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = \omega^2 + \frac{3 A^2 \mu}{4} - \frac{3 A^4 \mu^2}{128 (\omega^2 + \lambda^2)} + \frac{3 A^4 \mu^2 (3 A^2 \mu - 4 \lambda^2)}{512 (\lambda^2 + \omega^2)^2}.
\]

This time, the frequency depends upon the arbitrary parameter \( \lambda \) in a nontrivial way and we can apply the PMS in order to fix the value of \( \lambda \). We do this by imposing that \( \frac{d\Omega^2}{d\lambda} = 0 \), which leads to the following result:

\[
\lambda = A \sqrt{\frac{3 \mu}{2}}.
\]

Notice that since \( \lambda \) depends linearly upon \( A \) the formula for \( \Omega^2 \) obtained in this case *does not simply correspond to an expansion in \( A \).* As a matter of fact we find that the frequency corresponding to this value of \( \lambda \) is

\[
\Omega^2 = \frac{64 A^4 \mu^2 + 192 A^2 \mu \omega^2 + 128 \omega^4}{96 A^2 \mu + 128 \omega^2}.
\]

Notice that the Duffing equation (3.1) is left invariant under the simultaneous rescaling of the anharmonic coupling \( \mu \) and of the amplitude, i.e. \( \mu \to \mu' \) and \( A \to A' = A \sqrt{\mu/\mu'} \). This invariance is manifest in the equation (3.22), which is function of \( A^2 \mu \), which is invariant under this rescaling.

**IV. RESULTS**

We now present the results following from this analysis. In Fig. 1 we compare the exact frequency, calculated with Eq. (3.3) with the frequency obtained with our method (LPLDE), equation (3.22), and with the LP method, equation (3.20) taking \( \lambda = 0 \), both to third order.
in perturbation theory. We take $\omega = \mu = 1$ and vary the amplitude of the oscillations. We observe that our method yields an excellent approximation to the exact result even for large amplitudes, where the simple LP approximation fails.

In Fig. 2 we compare the period obtained with our method to the exact period of Eq. (3.3) and to the one obtained with the formulae of [3], which are obtained by applying the nonlinear delta expansion. Our method provides an excellent approximation to the exact period over a wide range of the parameter $\mu$, which controls the nonlinearity. The plots are obtained assuming $\omega = 1$ and the boundary conditions $x(0) = 1$ and $\dot{x}(0) = 0$. The formulae of [3] behave badly in the region $\mu < 0$, which corresponds to a potential well of finite depth centered around $x = 0$, and yield a precision comparable to the one achieved with our method for $\mu > 0$. Corresponding to the value $\mu = 0$ the oscillator is in a position of (unstable) equilibrium and the exact period diverges. Notice that for large values of $\mu$ all the methods seem to give a good approximation to the exact solution, including the LP method (to first order), which (to third order) was behaving poorly in the case previously studied. Unfortunately the equations of [3] are not suitable to be analyzed as in Fig. 1, and thus we cannot fully test the efficiency of this method.

In Fig. 3 we plot the relative error corresponding to the different approximations for $\mu > 0$. Our method to third order in perturbation theory yields an error typically smaller than the errors of the other methods and with a magnitude of about 0.1%.

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V. CONCLUSIONS

We have presented a method for the solution of (oscillatory) nonlinear problems. It is based on the application of the Linear Delta Expansion to the Lindstedt-Poincaré method. We applied it to the Duffing Equation and find that the new method converges faster and with greater accuracy than the simple LP method. Also, by comparing it with methods based on the perturbative $\delta$ expansion, we show that our solution not only converges faster and more accurately, but it also works for a much wider range of parameters. We are currently applying our method to a wider class of nonlinear problems [10], and we are also
interested in considering its possible extension to quantum systems.

VI. ACKNOWLEDGMENTS

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FIG. 1: Squared frequency of the anharmonic oscillator as a function of the amplitude (arbitrary units). $\omega = \mu = 1$.

FIG. 2: Period of the anharmonic oscillator.
FIG. 3: Error corresponding to the different approaches for the case studied in Fig. 3