By extending Ashtekar and Romano’s definition of spacelike infinity to the timelike direction, a new definition of asymptotic flatness at timelike infinity for an isolated system with a source is proposed. The treatment provides unit spacelike 3-hyperboloid timelike infinity and avoids the introduction of the troublesome differentiability conditions which were necessary in the previous works on asymptotically flat spacetimes at timelike infinity. Asymptotic flatness is characterized by the fall-off rate of the energy-momentum tensor at timelike infinity, which makes it easier to understand physically what spacetimes are investigated. The notion of the order of the asymptotic flatness is naturally introduced from the rate. The definition gives a systematized picture of hierarchy in the asymptotic structure, which was not clear in the previous works. It is found that if the energy-momentum tensor falls off at a rate faster than $\sim t^{-2}$, the spacetime is asymptotically flat and asymptotically stationary in the sense that the Lie derivative of the metric with respect to $\partial_t$ falls off at the rate $\sim t^{-2}$. It also admits an asymptotic symmetry group similar to the Poincaré group. If the energy-momentum tensor falls off at a rate faster than $\sim t^{-3}$, the four-momentum of a spacetime may be defined. On the other hand, angular momentum is defined only for spacetimes in which the energy-momentum tensor falls off at a rate faster than $\sim t^{-4}$.

I. INTRODUCTION

If gravitational collapse takes place in an asymptotically flat spacetime, we naively expect that the energy of fields and particles in the exterior region would be either radiated away to infinity or fall into the black hole and that the spacetime would approach a vacuum spacetime. Because the lost energy cannot be restored, the dynamical processes would eventually die away and the spacetime would settle down to a stationary state. Thus, we expect that the final state of gravitational collapse be described by a Kerr black hole, since the uniqueness theorem states that the Kerr spacetime is the unique asymptotically flat, stationary, vacuum solution [2]. This is the reason why the Kerr solution is used to analyze the black hole candidates observed in the universe. In this paper, to confirm the naive picture above, we shall closely examine the asymptotic structure of the very late time regime of a spacetime, i.e., timelike infinity, and see that an “asymptotic” vacuum state actually induces an “asymptotic” stationary state. This approach may be more reasonable than the assumption of the exact stationarity in the sense that exact vacuum or exact stationary state would not be achieved in a real gravitational collapse.

A solution of the Einstein equation is normally given in terms of the metric components with respect to some specific coordinates. However, it is not easy from the component expression to grasp what physical system the solution actually represents. One way to overcome the difficulty is to introduce a certain physical index that characterizes the system. For example, if the energy of the system at a given time is defined, we can picture the evolution of the system to some extent from the change in its value. Moreover, if the multipole moments of a spacetime are defined, we may obtain a precise Newtonian analog by constructing a Newtonian potential having the same moments. The analog could serve as a powerful guide for the physical interpretation of the solution. Because the gravitational field itself acts as its own source, we expect multipole moments can be defined only in spacetimes with gravitational fields satisfying a suitable fall-off condition. It is shown that stationary spacetimes are sufficiently asymptotically flat for
the multipole moments to be well defined \[8\]. However, there are no arguments on more general spacetimes. In this paper, as a first step towards defining multipole moments of general asymptotically flat spacetimes, we explore timelike infinity of a spacetime to seek what fall-off condition the gravitational field must satisfy so that four-momentum or angular-momentum may be defined.

There are some previous works on timelike infinity. A definition of timelike infinity with everywhere-smooth unphysical metric is proposed by Cutler in \[8\]. The high differentiability condition results in demanding that the spacetime have vanishing final Bondi mass at timelike infinity. This means the definition applies only for spacetimes with no bounded sources such as black holes and stars. On the other hand, Porill’s definition of the asymptotic flatness at timelike infinity is applicable to spacetimes with bounded sources \[8\]. However, as a result of the “tiewing” of the late time regime with the null regime, the treatment compresses the late time regime that has a finite volume, down to a point. As a result, we cannot demand smoothness or even differentiability of the unphysical metric at the point. Instead, we need to formulate a notion of direction-dependent “differentiability” and demand that the metric possess this awkward property. The treatment makes it difficult to understand what kinds of spacetimes are investigated.

Recently, Ashtekar and Romano have formulated an elegant treatment of spacelike infinity in a coordinate-independent manner \[1\]. In their work, the corresponding “awkward” differentiability at spacelike infinity is avoided by regarding the late time regime of a spacetime not as a point but as a 3-manifold. This is achieved by abandoning the idea of “tiewing” the late time regime and the null regime together. As a result, there is no necessity to use a conformally completed unphysical metric for the description of the gravitational field. Instead, a normal vector field and an intrinsic metric on the timelike slices foliating the unphysical spacetime are defined, and they provide the 3-manifold timelike infinity.

In the present paper, we modify the Ashtekar and Romano’s treatment of spacelike infinity to the timelike direction and investigate the asymptotic structure of spacetimes around timelike infinity. Our investigation is more or less a direct application of their work but differs in a number of important points, which enable us to discuss 1) the asymptotic vacuum state and the asymptotic stationary state of a spacetime to explore the final product of gravitational collapse, and 2) the order of asymptotic flatness to explore how flat the spacetime must be in order to define physical indices. First, we take a different view on how the fall-off condition on the energy-momentum tensor should be imposed. We impose conditions on the physical components of the physical energy-momentum tensor, while they impose conditions on the unphysical components. This change gives a picture of hierarchy in the asymptotic structure at timelike infinity and helps us to understand how the structure is systematized, which is a point that was obscure in the previous treatments. Second, a notion of asymptotic Killing fields is introduced. The formulation of the order of asymptotic symmetries is possible only by the introduction of such a notion. This is also crucial in examining and specifying how fast a spacetime becomes stationary. Also, special attention is paid to a bounded source that persists even in the very late time regime of a spacetime, in contrast with the spacelike direction.

The rest of the paper is organized as follows. In sec. \[II\] we construct timelike infinity with certain desirable features for Minkowski spacetime. Following the completion of Minkowski spacetime, we propose a definition of asymptotic flatness at future timelike infinity for a general spacetime with a source in sec. \[III\]. In sec. \[IV\], we examine the fundamental asymptotic structure of spacetimes that possess the defined asymptotic flatness. In sec. \[V\] we propose an asymptotic symmetry group which preserves the studied structure. A notion of asymptotic Killing fields is introduced in sec. \[VI\] and its relation to the fall-off of the energy-momentum tensor is clarified. In sec. \[VII\] we examine the asymptotic behavior of the gravitational fields. In sec. \[VIII\] we define conserved quantities associated with the asymptotic symmetry. The appendices give some examples of asymptotically flat spacetimes with timelike infinity defined in this paper and the flat Friedmann spacetime which is not asymptotically flat as a counter example. Throughout the paper, we follow the notation of Wald \[11\].

**II. COMPLETION OF THE MINKOWSKI SPACETIME**

In this section, we complete Minkowski spacetime \(\hat{\mathcal{M}}, \hat{\eta}_{ab}\) in such a way that future timelike infinity is represented not as a point but as a 3-manifold. The construction motivates the general definition of asymptotic flatness in the next section. Hereafter, we only consider future timelike infinity and thus omit the word future. All the discussion presented here is also applicable to past timelike infinity with minor changes.

In static spherical coordinates \((t, r, \theta, \phi)\), the metric of Minkowski spacetime takes the form

\[
\hat{\eta}_{ab} = -(dt)_a(dt)_b + (dr)_a(dr)_b + r^2(d\sigma)_{ab}
\]  

(2.1)

where \((d\sigma)_{ab} := (d\theta)_a(d\theta)_b + \sin^2 \theta(d\phi)_a(d\phi)_b\) is the unit 2-sphere metric. Introduce the standard hyperbolic coordinates \((\rho, \chi, \theta, \phi)\) in the interior of the future light cone at the origin: \(t = \rho \cosh \chi\) and \(r = \rho \sinh \chi\), where \(\chi \in [0, \infty)\) and \(\rho \in (0, \infty)\). In this chart, timelike infinity is specified by the limit surface \(\rho \to \infty\) and the metric takes the form
\[ \hat{\eta}_{ab} = -(d\Omega)_a(d\Omega)_b + \rho^2 h_{ab} \]  

(2.2)

where \( h_{ab} := (d\chi)_a(d\chi)_b + \sin^2 \chi (d\sigma)_{ab} \) is the unit spacelike 3-hyperboloid metric. If a new coordinate \( \Omega := \rho^{-1} \) is introduced and \( \mathcal{M} \) is extended to include the points on which \( \Omega = 0 \), timelike infinity is specified by the surface \( \Omega = 0 \) and the metric is found to be singular there:

\[ \hat{\eta}_{ab} = -\Omega^{-4}(d\Omega)_a(d\Omega)_b + \Omega^{-2} h_{ab}. \]  

(2.3)

To complete the spacetime conformally, we take a conformal factor of \( \Omega^4 \) and obtain a non-singular unphysical metric:

\[ \eta_{ab} := \Omega^4 \hat{\eta}_{ab} = -(d\Omega)_a(d\Omega)_b + \Omega^2 h_{ab}. \]  

(2.4)

With respect to the metric, timelike infinity \( \Omega = 0 \) is a point with no volume. This is because the components of the metric normal to and those tangent to the \( \Omega \)-const. surfaces are rescaled by the same power of \( \Omega \) although they diverge with different powers. In other words, if we rescale them according to the power of their divergence, we obtain timelike infinity that is not compressed to a point. Thus, define

\[ n^a := \Omega^{-4} \hat{\eta}^{ab}(d\Omega)_b = -(\partial \Omega)^a \]  

(2.5)

to the direction normal to the \( \Omega \)-const. surfaces and

\[ q_{ab} := \Omega^2 \left( \hat{\eta}_{ab} + \Omega^{-4} F^{-1}(d\Omega)_a(d\Omega)_b \right) = h_{ab}, \quad \text{where} \quad F := -\mathcal{L}_n \Omega = 1 \]  

(2.6)

to the direction tangent to the \( \Omega \)-const. surfaces. The information on the gravitational fields is completely given by the pair \((n^a, q_{ab})\), and we do not need the conformally rescaled unphysical metric \( \eta_{ab} \). With respect to this pair, timelike infinity is a 3-surface whose intrinsic metric is \( h_{ab} \) and whose normal is \( \partial \Omega \). That is, timelike infinity is a unit spacelike 3-hyperboloid.

Note here that the completion presented here does not conserve the causal structure of the physical spacetime, in contrast with the conformal completion. Hence, it is inappropriate, for example, to discuss the relation of timelike infinity with null infinity using the method. However, the method provides an adequate basis to investigate asymptotic structure of a spacetime at timelike infinity.

III. DEFINITIONS

In this section, we introduce a definition of asymptotic flatness at timelike infinity to order \( n \) and define some tensor fields that are useful for the coming discussion.

We have demonstrated the completion of Minkowski spacetime in the previous section. It is important to note that the way which the completion is accomplished crucially depends on the structure of Minkowski spacetime at timelike infinity. This motivates us to define an asymptotic flat spacetime as a spacetime which can be completed “asymptotically” in a similar fashion to Minkowski spacetime. More precisely, a spacetime is said to be asymptotically flat if the spacetime can be extended to enclose the points of infinity and if it is possible to construct the pair \((n^a, q_{ab})\) that has a smooth limit to the infinity.

**DEFINITION:** A physical spacetime \((\hat{\mathcal{M}}, \hat{g}_{ab})\) is said to possess an asymptote at future timelike infinity \( \mathcal{I}^+ \) to order \( n \) (AFTI-\( n \)) for a non-negative integer \( n \), if there exists a manifold \( \mathcal{M} \) with boundary \( \mathcal{H} \), a smooth function \( \Omega \) defined on \( \mathcal{M} \), and an imbedding \( \Psi \) of an open subset \( \hat{\mathcal{F}} \) in \( \hat{\mathcal{M}} \) to \( \mathcal{M} - \mathcal{H} \) satisfying the following conditions:

1. \( \mathcal{I}^+ := \partial \mathcal{F} \cap (\mathcal{M} - \Psi(\hat{\mathcal{M}})) \) is not empty and \( \mathcal{I}^+ + \mathcal{I}^- = I^+ (\mathcal{F}) \) where \( \mathcal{F} := \Psi(\hat{\mathcal{F}}) \);
2. \( \Omega \equiv 0 \) and \( \nabla \Omega \hat{\Omega} \neq 0 \), where \( \hat{\Omega} \) denotes the equality evaluated on \( \mathcal{I}^+ \);
3. \( n^a := \Omega^{-4} \Psi^{-1} \hat{g}^{ab} \nabla_b \Omega \) and \( q_{ab} := \Omega^2 (\Psi^{-1} \hat{g}_{ab} + \Omega^{-4} F^{-1} \nabla_a \Omega \nabla_b \Omega) \) admit smooth limits to \( \mathcal{I}^+ \) with \( q_{ab} \) having signature \((+++) \) on \( \mathcal{I}^+ \), where \( F := -\mathcal{L}_n \Omega \); and
4. \( \lim_{\tau \to \infty} \Omega^{-2} \nabla_a \Omega \hat{T}_{\mu \nu} = 0 \) where \( \hat{T}_{\mu \nu} := \Psi^{-1} \hat{T}_{\mu \nu} \) and \( \{\hat{\eta}_{a}^{\sigma}\}_{\mu = 0,1,2,3} \) is a tetrad of \( (\mathcal{M}, \hat{g}_{ab}) \).

Let us now discuss the meaning of the conditions. First of all, note that the asymptotic flatness is defined for an open subset \( \hat{\mathcal{F}} \) of the spacetime manifold. This is because isolated bodies may be present at the late time regime of a spacetime and thus the whole spacetime is not expected to be asymptotically flat. If there are isolated bodies at the late time regime, the region excluding the bodies should be chosen as \( \hat{\mathcal{F}} \). The former part of the first condition ensures that the infinity \( \mathcal{I}^+ \) consists of points that are included by extending the physical manifold. Such \( \mathcal{I}^+ \) resides not in
the region $\Psi(M)$, where the physical manifold is mapped, but on the boundary of the region of interest, $\partial F$. The latter part assures that $i^+$ exists in the future of the physical spacetime so that $i^+$ represents future timelike infinity. The second condition assures that $i^+$ is infinitely far away. The third demands that the asymptotic spacetime can be completed asymptotically in a similar fashion to the case of Minkowski spacetime and that the infinity $i^+$ is timelike. This means that $\Omega$ asymptotically plays a role of $t^{-1}$ in Minkowski spacetime. The fourth condition specifies how the physical components of the physical energy-momentum tensor should decay. This condition is different from the corresponding one in the definition given by Ashtekar and Romenko [1]. If we literally translated their prescription for spacelike infinity to our timelike infinity, we would demand that the energy-momentum tensor fall off faster than $\Omega^3$, $\lim_{\Omega^{-1}\Psi^i T_{ab}} = 0$, and consider such terms as $q_{a}^{\mu}q_{b}^{\nu}\Omega^{-3}\Psi^i R_{mn}$ vanish on $i^+$. Although it is not explicitly stated in [1], this means that the energy-momentum tensor is evaluated by the components with respect to a unphysical coordinate basis, $(\epsilon_i)^a(\epsilon_i)^b\Psi^i T_{ab}$ where $\{(\epsilon_i)^a\}_{n=0,1,2,3}$ is a tetrad of the unphysical spacetime. Here, we take an alternative view. We evaluate the physical energy-momentum tensor by its components with respect to a physical coordinate basis, $\tilde{T}_{\mu\nu} = \Psi^i(\epsilon_i)^a(\epsilon_i)^b T_{ab}$. In this view, it is found that the richness of asymptotic structures that a spacetime possesses depends on how fast the energy-momentum tensor falls off in the spacetime. Thus, we need to specify the rate of the fall-off when defining asymptotic flatness. To evaluate $\tilde{T}_{\mu\nu}$ hereafter, we use a tetrad consisting of a unit normal vector to $\Omega$-const. surfaces, $\tilde{n}^a$, and the triad on $\Omega$-const. surfaces, $\{(\epsilon_i)^a\}_{i=1,2,3}$. They satisfy

$$\tilde{g}^{ab} = -\tilde{n}^a \tilde{n}^b + (\epsilon_i)^a (\epsilon_i)^b$$

$$\Psi^i \tilde{n}^a = \sqrt{-\Psi^i \tilde{g}^{ab} \tilde{n}^b} = \Omega^2 F^{-1/2} n^a,$$

$$\Psi^i (\epsilon_i)^a = \Omega (\epsilon_i)^a$$

where $\{(\epsilon_i)^a\}_{i=1,2,3}$ is the smooth triad of $q_{ab}$. Eq. (3.3) will be obtained below. Since all the equations appearing in the following discussion are those on $M$, unless it may cause ambiguity, we omit hereafter $\Psi^i$ in front of the tensors defined on $M$ for brevity.

It is meaningful to note here that in a scalar field $\phi$ in the Schwarzschild spacetime decays asymptotically at the rate $\sim t^{-2}$ or faster, according to the Price’s theorem [3]. Thus, the Schwarzschild spacetime with such a scalar field is asymptotically flat to order 1.

Now consider the Schwarzschild spacetime. We can take the region $r > 2m$ for $F$. Then, all the conditions above are satisfied and $i^+$ is represented by a unit spacelike 3-hyperboloid with a point removed from it. (See appendix A for the details.) The removed point corresponds to the future of the region $r \leq 2m$, or the region excluded from $M$. Recalling that a unit spacelike 3-hyperboloid has a topology of $S^2 \times \Omega$, we see that the topology of $i^+$ is now $\Omega^2 \times (\Omega^+ \{p\})$ where $\Omega^+$ denotes $[0, \infty)$. This result may be regarded as a common feature of the spacetimes with sources and motivates the following definition.

DEFINITION: A spacetime $(M, \tilde{g}_{ab})$ is said to be asymptotically flat at future timelike infinity with a source to order $n$ (AFFTIS-$n$) if $(M, \tilde{g}_{ab})$ possesses an asymptote at timelike infinity to order $n$ and $i^+$ has topology of $S^2 \times (\Omega^+ \{p\})$.

For the convenience of the following discussion, we define some useful tensors and show the relations among them. From the definition, the physical metric $g_{ab}$ and its inverse $\tilde{g}^{ab}$ are given by

$$\tilde{g}_{ab} = -\Omega^{-4} F^{-1} \nabla_a \Omega \nabla_b \Omega + \Omega^{-2} q_{ab}$$

$$\tilde{g}^{ab} = -\Omega^{4} F^{-1} n^a n^b + \Omega^2 q^{ab}$$

respectively with functions and tensors that are smooth in $F$ and on $i^+$. From eqs. (3.3) and (3.1), we obtain eq. (3.3). A smooth projection operator on the $\Omega$-const. surfaces can be defined from smooth $q^{ab}$ and $q_{ab}$:

$$q^a_{\ b} := q^{ac} q_{cb}$$

$$= \delta^a_{\ b} + F^{-1} n^a \nabla_b \Omega. \quad (3.6)$$

The operator is the same as the one defined from $q^{ab}$ and $\tilde{q}_{ab}$. Thus, derivative operators $D_a$ and $\tilde{D}_a$ that are associated with $q_{ab}$ and $\tilde{q}_{ab}$ respectively do not differ. Hence,

$$\tilde{R}_{abc}^d = \tilde{R}_{abc}^d \quad \text{and} \quad 3 \tilde{R}_{abc} = 3 \tilde{R}_{abc}$$

where $(\tilde{D}_a \tilde{D}_b - \tilde{D}_b \tilde{D}_a) \omega_c = 3 \tilde{R}_{abc} \omega_d, 3 \tilde{R}_{abc} := 3 \tilde{R}_{abc}$, $(D_a D_b - D_b D_a) \omega_c = 3 \tilde{R}_{abc} \omega_d$ and $3 \tilde{R}_{abc} := 3 \tilde{R}_{abc}$.

On the other hand, the physical extrinsic curvature of the $\Omega$-const. surfaces does not admit a smooth limit to $i^+$:
\[ K_{ab} := \frac{1}{2} \mathcal{L}_\xi q_{ab} \]
\[ = \Omega^{-1} F^{1/2} q_{ab} + \frac{1}{2} F^{-1/2} \mathcal{L}_n q_{ab}. \]  

(3.8)

Thus, define a smooth tensor
\[ K_{ab} := \Omega \hat{K}_{ab} \]
\[ = F^{1/2} q_{ab} + \frac{1}{2} \Omega F^{-1/2} \mathcal{L}_n q_{ab}. \]  

(3.9)

Hereafter, we raise and lower the indices of tensors without hats on \( \Omega \)-const. surfaces by the smooth metric \( q_{ab} \) and the inverse \( q^{ab} \). For example, \( K^{ab} := q^{ac} K_{cb} \).

IV. THE ZERO-TH ORDER ASYMPTOTIC STRUCTURE

In this section, we examine the asymptotic structure of an AFTI-0 spacetime using the Einstein equation. Consider the momentum constraint equation expressed in terms of smooth tensors and \( \hat{T}^\mu_\nu \):
\[ \Omega^{-2} (\hat{e}_i)^a \hat{e}_b \hat{T}_{ab} = [D_c K^c_a - D_a K] (e_i)^a. \]  

(4.1)

In AFTI-0 spacetimes, the left-hand side vanishes on \( \hat{\mathcal{I}}^+ \) because of the fall-off condition on the energy-momentum tensor. Therefore, we get
\[ D_c K^c_a - D_a K = 0. \]  

(4.2)

Substituting eq. (3.9) into this equation, we find
\[ F = \text{const}. \]  

(4.3)

By virtue of the conformal freedom of the function \( F \), the above constant can be set to unity without any loss of generality. To see this, complete the spacetime with a function \( \Omega' := \alpha \Omega \) instead of \( \Omega \), where \( \alpha \) is a smooth function that is constant on \( \hat{\mathcal{I}}^+ \). Then, the function \( F' := -\mathcal{L}_n \Omega' \) in this completion is given by \( F' = \alpha^{-2} F \). Thus, there is always a function \( \Omega' \) that induces \( F' = 1 \) for any given function \( \Omega \). We call a completion with such a function a \textit{unit hyperboloid completion} for the reason that will be clarified below. Because there is no loss of generality, hereafter we consider only unit hyperboloid completions. In the completion,
\[ F = 1 + \sum_{\ell=1}^{\infty} (\ell) F \cdot \Omega^\ell, \]  

(4.4)

where \( (\ell) F \) are defined by
\[ (\ell) F := \lim_{\rightarrow \hat{\mathcal{I}}^+} \Omega^{-1} (F - 1) \text{ and } (n) F := \lim_{\rightarrow \hat{\mathcal{I}}^+} \Omega^{-n} (F - 1 - \sum_{\ell=1}^{n-1} (\ell) F) \text{ for } n \geq 2; \]  

(4.5)

and
\[ K_{ab} = q_{ab} + \frac{1}{2} \Omega (1) F q_{ab} + \mathcal{L}_n q_{ab} + O(\Omega^2). \]  

(4.6)

Now consider the space-space components of the Ricci tensor:
\[ \Omega^{-2} (\hat{e}_i)^a \hat{e}_b \hat{R}_{ab} = [\hat{\mathcal{R}}_{ab} + KK_{ab} - F^{1/2} K_{ab} - 2K_{ac} K^c_b \]
\[ + \Omega F^{-1/2} \mathcal{L}_n K_{ab} + F^{-1/2} D_a D_b F^{1/2} ] (\hat{e}_i)^a (e_i)^b \]  

(4.7)

In AFTI-0 spacetimes, the left-hand side vanishes on \( \hat{\mathcal{I}}^+ \) by virtue of the fall-off condition on the energy-momentum tensor. Substituting eqs. (4.4) and (4.6) into eq. (4.7), we obtain
\[ \hat{\mathcal{R}}_{ab} = -2q_{ab}. \]  

(4.8)
Here, recall that $i^+$ is a sub-manifold of $\mathcal{M}$ and thus there is an identity imbedding $\Pi$ of $i^+$ in $\mathcal{M}$. Using the map $\Pi$, a tensor field $A_{ab}$ defined on $\mathcal{M}$ induces a tensor field $\Pi^* A_{ab}$ on $i^+$. (Hereafter, we denote the tensor fields on the manifold $i^+$ in boldface.) It is important to note here that if a tensor field is tangential to $i^+$, or $q_a^m q_b^n A_{mn} := A_{ab}$, the induced tensor field contains the same information as the original. Hence, the investigation of an equation on $i^+$ with tensor fields tangential to $i^+$ can be done by considering the induced equation on $i^+$. Thus, let us consider the following equation induced by eq.(4.8) on $i^+$.

\begin{equation}
\tilde{R}_{ab} := -2q_{ab}.
\end{equation}

The equation tells us that the unphysical 3-spacetime $(i^+, q_{ab})$ has negative constant curvature:

\begin{equation}
\tilde{R}_{abed} := -2q_{a[d]q_{d]b}}.
\end{equation}

In other words, timelike infinity $(i^+, q_{ab})$ is isometric to a unit spacelike hyperboloid in Minkowski spacetime and $q_{ab} := h_{ab}$. Note here that it is isometric to a unit hyperboloid because the unit hyperboloid completion induces $F \equiv 1$. This is where the name comes from.

The fall-off condition on the time-time component of the energy-momentum tensor gives nothing new.

Now that we have explored everything that can be derived from the fall-off condition which AFTI-0 spacetimes satisfy, the asymptotic structure of AFTI-0 spacetime can be summarized as follows.

**DEFINITION:**

The function $F$ which is unity on $i^+$, i.e., $F \equiv 1$ and the manifold $i^+$ with normalized negative-constant-curvature, i.e., $q_{ab} := h_{ab}$ are called the zero-th order asymptotic structure of an AFTI-0 spacetime.

The zero-th order asymptotic structure corresponds to the universal structure of $\mathcal{M}$ spacetimes in \textsuperscript{[1]}. However, they differ in that the former does not contain the equivalence class of $\Omega$ constructed by regarding two completions $\Omega$ and $\Omega'$ equivalent if and only if they satisfy

\begin{equation}
\lim_{\longrightarrow i^+} \frac{\Omega' - \Omega}{\Omega^2} = 0
\end{equation}

while the latter does. The reason is that the equivalence class does not arise on its own from the definition of an AFTI-0 spacetime. This observation is also supported by the discussion of asymptotic symmetries in section \textsuperscript{V}.

Recalling that the difference between an AFTI-0 spacetime and an AFFTIS-0 spacetime is the topology of $i^+$, we define the asymptotic structure of an AFFTIS-0 spacetime as follows.

**DEFINITION:**

The function $F$ which is unity on $i^+$, i.e., $F \equiv 1$ and the manifold $i^+$ with normalized negative-constant-curvature, i.e., $q_{ab} := h_{ab}$ and topology of $S^2 \times (\mathbb{R}^+ \setminus \{0\})$ are called the zero-th order asymptotic structure of an AFFTIS-0 spacetime.

Let us consider the form of the physical metric $\hat{g}_{ab}$ in an AFTI-0 spacetime to understand the zero-th order asymptotic structure more. From the fact that an AFTI-0 spacetime possesses the zero-th order asymptotic structure, we see that the expansions of the function $F$ and the tensor field $q_{ab}$ are given by

\[ F = 1 + O(\Omega) \]
\[ q_{ab} = h_{ab} + f(d\Omega)a(d\Omega)b + f^1(d\Omega)(a(e_1)b) + O(\Omega) \]

in an AFTI-0 spacetime where $f$ and $f^1$ are smooth functions. With the expansion above, the definitions of $F$ and $q_{ab}$ give

\begin{equation}
\hat{g}_{ab} = -\Omega^{-4}[1 + O(\Omega)](d\Omega)a(d\Omega)b + \Omega^{-2}[h_{ab} + f(d\Omega)a(d\Omega)b + f^1(d\Omega)(a(e_1)b) + O(\Omega)].
\end{equation}

Introducing a conformal time $\eta := \ln \Omega$, we obtain

\begin{equation}
\hat{g}_{ab} = (\tilde{g}_{ab} + t \tilde{g}_{ab} + \cdots)
\end{equation}

where

\begin{equation}
(\tilde{g}_{ab} := (e^{-\eta})^2[-(d\eta)a(d\eta)b + h_{ab}]
\end{equation}

and $e^{2\eta}, (\tilde{g}_{ab}, e^{2\eta}, (\tilde{g}_{ab}, \cdots$ are tensor fields that do not depend on $\Omega$. If we transform the coordinates, it can be seen that $\tilde{g}_{ab}$ is a metric of Minkowski spacetime. Note that $\tilde{g}_{ab}$ is arbitrary in an AFTI-0 spacetime. That is, eq. \textsuperscript{(4.13)} tells us that an AFTI-0 spacetime is asymptotically Minkowskian but how it approaches an asymptotically Minkowski spacetime is not specified in any way.
V. ASYMPTOTIC SYMMETRIES

In this section, we define asymptotic symmetries by considering the transformations that preserve the zero-th order asymptotic structure of an AFTI-0 and an AFFTIS-0 spacetime and analyze their structures. Basically, this section is a review of the corresponding work of [1] but differs in the following points. First, the definition of the asymptotic symmetries is different due to the difference in the asymptotic structure they are to preserve. Second, one of the subgroups of the asymptotic symmetry group is examined in detail for the explicit evaluation of four-momentum in the sec. VIII. Third, the structure of the asymptotic symmetries differ due to the presence of a bounded source at the late time regime in an AFFTIS spacetime.

In Minkowski spacetime, the symmetry group, i.e., the Poincaré group is a group of diffeomorphisms \( \{ \phi \} \) that preserve the structure of the spacetime \( \hat{\eta}_{ab} \): \( \hat{\eta}_{ab} \phi^a \hat{\eta} = \eta_{ab} \). Because the structure that is universal to AFTI-0 or AFFTIS-0 spacetimes is the zero-th order asymptotic structure, we define the asymptotic symmetries of such spacetimes to be transformations that preserve the asymptotic structures.

First we examine the infinitesimal asymptotic symmetries. As they are local transformations, the structure they preserve is that of an AFTI-0 spacetime.

**DEFINITION:** The infinitesimal Ti symmetry group \( \mathcal{L}_G \) is a group of equivalence classes of a \( C^\infty \) vector field \( \xi^a \) that satisfies the following conditions:

1. \( \xi^a \) induces a smooth function \( \alpha \) given by
   \[
   \alpha := \lim_{\rightarrow i^+} \frac{\mathcal{L}_{\xi} \Omega}{\Omega^2};
   \]  
   (5.1)

   and

2. \( \xi^a \) induces a Killing vector of \( (i^+, h_{ab}) \),
   \[
   \mathcal{L}_{\xi} h_{ab} = 0.
   \]  
   (5.2)

Two vector fields \( \xi_1^a \) and \( \xi_2^a \) are regarded as equivalent if and only if \( (\alpha_1, \xi_1^a) \equiv (\alpha_2, \xi_2^a) \).

(Ti stands for timelike infinity and rhymes with Scri(\( \mathbb{I}^\pm \)) as Ashtekar’s Spi(\( i^0 \)) does.)

The first condition demands that \( \Psi_\xi^*(\Omega) \) differs from \( \Omega \) by terms of \( O(\Omega^2) \) so that \( \Psi_\xi^*(\Omega) \) also induces a unit hyperboloid completion and thus \( \xi^a \) preserves the zero-th order asymptotic structure \( F^{\xi \infty} = 1 \), where \( \Psi \) denotes an infinitesimal map generated by \( \xi \). The second condition ensures that \( \xi^a \) preserves the zero-th order asymptotic structure \( q_{ab} = h_{ab} \). The equivalence class is defined because the vector fields that belong to the same class cannot be distinguished as a generator of symmetries.

Let us investigate the structure of the infinitesimal Ti group. Because its element is a \( C^\infty \) vector field, \( \mathcal{L}_G \) has the structure of a Lie algebra. The Lie bracket on the pairs is given by

\[
[(\alpha, \xi^a), (\beta, \eta^a)] = (\mathcal{L}_{\xi}\beta - \mathcal{L}_{\eta}\alpha, [\xi, \eta]^a).
\]  
(5.3)

The introduction of the following subgroup of \( \mathcal{L}_G \) helps the investigation of the structure on \( \mathcal{L}_G \).

**DEFINITION:** The infinitesimal Ti supertranslation group \( \mathcal{L}_S \) is a subgroup of \( \mathcal{L}_G \) whose element \( (\alpha, \xi) \) satisfies

\[
\xi^a = 0.
\]  
(5.4)

Because the subgroup contains the following infinitesimal Ti translation group, we call it infinitesimal Ti supertranslation group.

**DEFINITION:** The infinitesimal Ti translation group \( \mathcal{L}_T \) is a subgroup of \( \mathcal{L}_S \) whose element \( (\alpha, 0) \) satisfies

\[
D_a D_b \alpha - a h_{ab} = 0
\]  
(5.5)

where \( D_a \) is the derivative operator associated with the intrinsic metric \( h_{ab} \) on \( i^+ \).

(The group does not depend on the function \( \Omega \) used in the completion of a spacetime because \( \xi^a \) vanishes on \( i^+ \) and thus \( \alpha \) does not depend on \( \Omega \).)

It is important to note here that there are four independent functions that satisfy eq. (5.5):

\[
\alpha_t \equiv \cosh \chi, \quad \alpha_x \equiv \sinh \chi \sin \theta \cos \phi, \quad \alpha_y \equiv \sinh \chi \sin \theta \sin \phi \quad \text{and} \quad \alpha_z \equiv \sinh \chi \cos \theta
\]  
(5.6)
where \((\chi, \theta, \phi)\) is the standard hyperbolic coordinate on \(i^+\). It can be shown that if \(i^+\) is timelike infinity of \(n\)-dimensional spacetime and thus \(h_{ab}\) in eq.(5.3) is the \((n - 1)\)-hyperboloid metric, there are \(n\) independent functions satisfying the equation. The number corresponds to the number of possible “directions” of translation in the spacetime. When Minkowski spacetime is completed, the translational Killing fields correspond to the elements of \(\mathcal{L}_L\). It will be shown that the group is closely related with the asymptotic translation Killing fields introduced in sec.VII and the four-momentum of the asymptotic translation introduced in sec.VII.

The structure of \(\mathcal{L}_G\) can be examined by noting that \(\mathcal{L}_S\) is a Lie ideal of \(\mathcal{L}_G\) . Hence, the quotient \(\mathcal{L}_G/\mathcal{L}_S\) has the structure of a Lie algebra consisting of cosets \(\{\alpha, \xi^a\}\) where \(\{\alpha\}\) is a set of all the functions smooth on \(i^+\). Thus, each element of the quotient is characterized by \(\xi^a\). In other words, \(\mathcal{L}_G/\mathcal{L}_S\) is isomorphic to the group of the Killing fields of \((i^+, h_{ab})\). Recalling that such Killing fields is in the Lie algebra of the Lorentz group \(\mathcal{L}_L\), we obtain

\[
\mathcal{L}_G/\mathcal{L}_S \cong \mathcal{L}_L \quad \text{and thus} \quad \mathcal{L}_G \cong \mathcal{L}_S \circ \mathcal{L}_L
\]  

(5.7)

where \(\circ\) denotes the semi-direct sum here.

Next, let us consider the asymptotic symmetry group. Following the symmetry group of Minkowski spacetime, we define the asymptotic symmetry group as a group of diffeomorphisms. Because \(\text{AFTI-0}\) spacetimes possess only the zero-th order asymptotic structure in common, the diffeomorphisms can be only defined on \(i^+\). However, \(i^+\) of an \(\text{AFTI-0}\) spacetime is not specified enough to consider diffeomorphisms generated on itself. Hence, we consider \(i^+\) of an \(\text{AFTI-0}\) spacetimes whose topology is restricted and demand that \(i^+\) be geodesically complete so that diffeomorphisms could be generated on it.

**DEFINITION:** The Ti group \(\mathcal{G}\) is a group of diffeomorphisms generated by elements of \(\mathcal{L}_G\) on \(i^+\) that is geodesically complete except for the geodesics through a point and that has topology of \(S^2 \times (\mathbb{R}^+ - \{p\})\).

Recall that a 3-manifold with negative-constant curvature with topology of \(S^2 \times (\mathbb{R}^+ - \{p\})\) is a spacelike 3-hyperboloid with a removed point. Hence, it is only when the removed point is chosen as the origin of the Lorentz transformations that the elements of \(\mathcal{L}_L\) do generate diffeomorphisms. Otherwise, the integral curves of the elements of \(\mathcal{L}_L\) would be incomplete due to the point. Hence, the Ti group \(\mathcal{G}\) is a semi-direct product of the infinite dimensional additive group of smooth functions on the unit spacelike hyperboloid with the Lorentz group around the removed point, \(\mathcal{L}_{RP}\):

\[
\mathcal{G} \cong S \circ \mathcal{L}_{RP}.
\]  

(5.8)

The above additive group is *infinite* dimensional in contrast with *four* dimensional additive subgroup of the Poincaré group’s, the translation group. This is because, as for transformations of the zero-th order asymptotic structures, translations are the changes of the function \(\Omega\) used in completions and the changes are possible in “infinitely different ways”. It is interesting that this structure is very similar to that of the BMS group defined at null infinity which is a semi-direct product of the infinite dimensional additive group of *conformally weighted functions on the 2-sphere* and the Lorentz group \([7, 8]\).

However, it is important to note here that the additive group \(S\) in eq.(5.8) consists of all the possible finite supertranslation. This is against our intuition that there are no spacelike translations when there exists a source, or a removed point on \(i^+\), that breaks the symmetries. The reason traces back to the fact that the Ti group is defined only to preserve the structure of \(i^+\) but the structure of \(i^+\) is preserved as long as the function \(\alpha\) is smooth, regardless of the detailed nature of \(\alpha\), which characterizes supertranslations. In the next section, we find that a translation whose associated function is \(\alpha\) transforms the points in the vicinity of \(i^+\) in \(\mathcal{F}\) by the generator \(\Omega D_{a} \alpha\). Hence, to take the actions of the supertranslations into the symmetry group more appropriately, we need to define the group \(\mathcal{G}'\) to consist of diffeomorphisms on the vicinity of \(i^+\). Now, the generators of the group must be vanishing in the vicinity of the removed point on \(i^+\), and thus \(\mathcal{G}'\) is a semi-direct product of the infinite dimensional additive group \(\mathcal{S}_{RP}\) of smooth functions whose gradient vanishes in the vicinity of the removed point and the Lorentz group around the removed point:

\[
\mathcal{G}' \cong \mathcal{S}_{RP} \circ \mathcal{L}_{RP}.
\]  

(5.9)

This modified Ti group \(\mathcal{G}'\) does not contain “spacelike supertranslation” because, as we shall see in the next section, the gradients of the functions \(\alpha\) associated with spacelike translations do not vanish in the vicinity of the removed point.
VI. ASYMPTOTIC KILLING FIELDS

In this section, we introduce the notion of asymptotic Killing fields. The introduction helps us to understand that the asymptotic symmetries are closely related with the hierarchy found in the asymptotic structure and gives a basis for discussing how fast a spacetime becomes stationary.

If a vector field $\hat{\xi}^a$ is a Killing vector field, the Lie derivative of the metric with respect to $\hat{\xi}$ vanishes. This fact motivates us to call the vector field $\hat{\xi}^a$ an asymptotic Killing field if $\mathcal{L}_{\hat{\xi}^a}$ asymptotically vanishes to the limit $\Omega \to 0$.

**DEFINITION:** A spacetime $(\hat{M}, \hat{g}_{ab})$ is said to admit an asymptotic Killing field $\hat{\xi}^a$ to order $n$ if

$$
\lim_{\Omega \to n} \Omega^{-\nu} \mathcal{L}_{\hat{\xi}^a} \hat{g}^{\mu \nu} = 0
$$

(6.1)

where $\mathcal{L}_{\hat{\xi}^a} \hat{g}^{\mu \nu} := \Psi^*((\hat{e}_a)^b \hat{g}_{ab})$.

The local asymptotic symmetries defined above are related to the infinitesimal Ti group defined in the previous section as follows.

**PROPOSITION 1:** In an AFTI-0 spacetime, a vector field $\hat{\xi}^a$ belongs to the infinitesimal Ti group if and only if $\hat{\xi}^a$ is an asymptotic Killing field to order 0.

**Proof of If:** The components of the Lie derivative of the metric with respect to $\hat{\xi}^a$ are given by

$$
\hat{n}^a \hat{n}^b \mathcal{L}_{\hat{\xi}^a} \hat{g}^{ab} = 4\Omega^{-1} \mathcal{L}_{\hat{\xi}^a} \hat{g}^{ab} - 2\mathcal{F}^{(1)} F + O(\Omega^2)
$$

(6.2)

in AFTI-0 spacetimes where eq.(4.4) is used in the evaluation. Next, from the space-space component, we find

$$
\hat{q}_m^a \hat{q}_n^b \mathcal{L}_{\hat{\xi}^a} \hat{g}^{ab} = 0
$$

(6.3)

must be satisfied so that all the components of $\mathcal{L}_{\hat{\xi}^a} \hat{g}^{ab}$ are of $O(\Omega^1)$. The equation above induces

$$
\mathcal{L}_{\hat{\xi}^a} \hat{h}_{ab} = 0.
$$

(6.4)

on $\hat{\xi}^a$. The equation tells us that $\hat{\xi}^a$ induces a Killing field of $(\hat{\iota}^+, \hat{h}_{ab})$.

To summarize, an asymptotic Killing field to order 0 generates a smooth function $\alpha$ on $\hat{\iota}^+$ and induces a Killing field of $(\hat{\iota}^+, \hat{h}_{ab})$. The vector field belongs to the infinitesimal Ti group. □

**Proof of Only If:** If $\hat{\xi}^a$ belongs to the infinitesimal Ti group, $\alpha := \mathcal{L}_{\hat{\xi}^a} \hat{g}^{ab}$ is a smooth function from the definition. Thus, in AFTI spacetimes eqs. (6.3) are satisfied. Further, $\hat{\xi}^a$ induces a Killing field of $(\hat{\iota}^+, \hat{h}_{ab})$. Hence, the first term in the space-space component vanishes resulting in $(\mathcal{L}_{\hat{\xi}^a} \hat{g}^{ab})_{\hat{\iota}^+} = O(\Omega^1)$. That is, $\hat{\xi}^a$ is an asymptotic Killing field to order 0. □

Eqs.(6.3) show that it was appropriate to define the zero-th order asymptotic structure such that it does not contain the equivalence class of $\Omega$ defined by eq.(4.4). As we have seen above, if the components of $\mathcal{L}_{\hat{\xi}^a} \hat{g}^{ab}$ are of order $O(\Omega)$, then the vector field $\hat{\xi}^a$ preserves the zero-th order asymptotic structure. However, the vector field $\hat{\xi}^a$ preserves the
the equivalence class of \( \Omega \), only when \( D_m \alpha - q_m \mathcal{L}_\xi n^a \equiv 0 \) were satisfied in addition, or the time-space components of \( \mathcal{L}_\xi \hat{g}_{ab} \) were of \( O(\Omega^2) \), as it can be seen from eqs. (1). (For the details on the preservation of the class, see [1].) The fact that the time-space components of \( \mathcal{L}_\xi \hat{g}_{ab} \) must be \( O(\Omega^2) \) instead of \( O(\Omega^1) \) to preserve the class, indicates that the equivalence class of \( \Omega \) is of an order higher asymptotic structure.

Next, we investigate the asymptotic translational Killing fields. This is to see how fast an AFTI-0 spacetime becomes stationary.

**Proposition 2:** In an AFTI-0 spacetime,

\[
\hat{\xi}_t^a = \Omega [\sinh \chi (e_1) a] + \Omega^2 F^{-1/2} \cosh \chi n^a
\]

\[
\hat{\xi}_x^a = \Omega [\cosh \chi \sin \theta \cos \phi (e_1) a] + \cos \theta \cos \phi (e_2) a - \sin \phi (e_3) a] + \Omega^2 F^{-1/2} \sinh \chi n^a
\]

\[
\hat{\xi}_y^a = \Omega [\cosh \chi \sin \phi (e_1) a] + \cos \theta \cos \phi (e_2) a + \sin \phi (e_3) a] + \Omega^2 F^{-1/2} \sinh \chi n^a
\]

\[
\hat{\xi}_z^a = \Omega [\cosh \chi \cos \theta (e_1) a - \sin \theta (e_2) a] + \Omega^2 F^{-1/2} \sinh \chi n^a,
\]

(6.6)

are admitted as asymptotic Killing fields to order one, where \( e_1 = \partial_\chi, e_2 = (\sinh^{-1} \chi) \partial_\theta \) and \( e_3 = (\sinh^{-1} \chi \sin^{-1} \theta) \partial_\psi \).

\( \{\xi_{\mu\nu}\} \) are four independent asymptotic translational Killing fields in the sense that, with respect to the physical metric \( \hat{g}_{ab} \), their norms are constant and they are orthogonal to each other:

\[
\hat{g}_{ab} (\hat{\xi}_{\mu\nu})^a (\hat{\xi}_{\mu\nu})^b = \mp \delta_{\mu\nu}
\]

where \( - \) for \( \mu = 0 \) and \( + \) for \( \mu = 1, 2, 3 \).

**Proof:** The space components of the above asymptotic translational Killing vector fields are of \( O(\Omega) \) and the time components are of \( O(\Omega^2) \). Thus, the expansion of the vector field \( \hat{\xi}^a \) is given by

\[
\hat{\xi}^a = \sum_{\ell=1}^{\infty} \Omega^{\ell} (\xi^a)_{\ell} + \sum_{\ell=2}^{\infty} \Omega^{\ell} (\xi^a)_{\ell} (\partial_\Omega)^a
\]

(6.7)

where \( \{^{(\ell)} \xi^a\} \) is a set of smooth functions independent of \( \Omega \). Hence,

\[
\alpha := \mathcal{L}_\xi \Omega / \Omega^2 = \sum_{\ell=2}^{\infty} (^{(\ell)} \xi^a) \Omega^{\ell-2}
\]

(6.8)

is a smooth function. Thus, the components of \( \mathcal{L}_\xi \hat{g}_{ab} \) are given by

\[
\hat{n}^a \hat{n}^b \mathcal{L}_\xi \hat{g}_{ab} = 0 + O(\Omega^2)
\]

\[
(\hat{e}_1)^a \hat{e}_1^b \mathcal{L}_\xi \hat{g}_{ab} = \left[ \Omega (D_m \alpha - (^{(1)} \xi^a)_{m}) + O(\Omega^2) \right] (e_1)^a
\]

\[
(\hat{e}_1)^a \hat{e}_1^b \mathcal{L}_\xi \hat{g}_{ab} = \left[ \Omega (q_m a q_n b \mathcal{L}_\xi \xi_1 (e_1) a q_m b + 2 \alpha q_{mn}) + O(\Omega^2) \right] (e_1)^m (e_1)^n
\]

(6.9)

From the equations above, we see that \( (\mathcal{L}_\xi \hat{g})_{\mu\nu} = O(\Omega^2) \) if the space components of the \( O(\Omega^1) \) terms of the vector field \( \hat{\xi}^a \) satisfy

\[
D_m \alpha - (^{(1)} \xi^a)_{m} = 0 \quad \text{and} \quad q_m a q_n b \mathcal{L}_\xi \xi_1 (e_1) a q_m b - \alpha q_{mn} = \theta.
\]

(6.10)

These equations are tangential to \( \mathcal{i}^+ \) and thus induce

\[
^{(1)} \xi^a = D^a \alpha \quad \text{and} \quad \mathcal{L}_\xi h_{ab} - 2 \alpha h_{ab} = \theta
\]

(6.11)

on \( \mathcal{i}^+ \) without any loss of information where \( ^{(1)} \xi^a \) and \( \alpha \) denote the vector field and the function induced by \( ^{(1)} \xi^a (e_1) \), and \( \lim_{\tau \to \mathcal{i}^+} \alpha := \alpha_{i^+} \) on \( \mathcal{i}^+ \) respectively. Thus, the conditions can be restated as

\[
D_a D_b \alpha - \alpha h_{ab} = 0
\]

(6.12)

\[
^{(1)} \xi^a = D^a \alpha
\]

(6.13)
In other words, \((L\bar{\xi})_{\mu\nu} = O(\Omega^2)\) is satisfied if the leading term of the time component of \(\bar{\xi}, \alpha = (\bar{\xi}^a, \bar{\xi}^b, \bar{\xi}^c, \bar{\xi}^d)\), satisfies eq. (6.12) and if those of the space components \((\bar{\xi})^a\) are given by eq. (6.13). Note here that eq. (6.12) is the same as eq. (5.3) used in defining the infinitesimal Ti translation and that \(\{\alpha_{\mu\nu}\}\) in eq. (6.6) are the four independent smooth functions that satisfy the equation. Then, from eq. (6.13), we see that the corresponding independent space components are

\[
\begin{align*}
(\bar{\xi})^a_{\mu} & \equiv \sinh \chi (e_{\mu})^a \\
(\bar{\xi})^b_{\mu} & \equiv \cosh \chi \sin \theta \cos \phi (e_{\mu})^b + \cos \theta \cos \phi (e_{\mu})^a - \sin \phi (e_{\mu})^b \\
(\bar{\xi})^c_{\mu} & \equiv \cosh \chi \sin \theta \sin \phi (e_{\mu})^c + \cos \theta \sin \phi (e_{\mu})^a + \cos \phi (e_{\mu})^c \\
(\bar{\xi})^d_{\mu} & \equiv \cosh \chi \cos \theta (e_{\mu})^d - \sin \theta (e_{\mu})^a.
\end{align*}
\] (6.14)

The space components of the vector fields in eqs. (6.6) do take this form and thus are asymptotic Killing fields to order one in \(\Omega\). □

As a corollary of proposition 2, we obtain the relation between the infinitesimal Ti translational group \(L_T\) and the above asymptotic translational Killing fields.

**Corollary:** The asymptotic translational Killing fields to order one, eqs. (6.6), belong to the infinitesimal Ti translational group \(L_T\).

**Proof:** Admitted as asymptotic Killing fields to order one, the asymptotic translational Killing fields belong to the infinitesimal Ti group according to proposition 1. In addition, the fields vanish on \(\bar{t}^+\) and generate smooth functions satisfying eq. (5.3). □

Let us consider the finite translations in AFFTIS-0 spacetimes. From eqs. (6.7) and (6.13), we see that the generators of translations vanish on \(\bar{t}^+\) but transform the points in vicinity of \(\bar{t}^+\) in the direction of \(\Omega D^a \alpha\), or \((\bar{\xi})^a\). Although the generator \((\bar{\xi})^a\) of a timelike translation vanishes in the vicinity of the removed point \(\chi = 0\) on \(\bar{t}^+\), the generators \((\bar{\xi})^a, (\bar{\xi})^b, (\bar{\xi})^c\) and \((\bar{\xi})^d\) of spacelike translations do not. Hence, there exist only timelike translation symmetry \(\bar{\xi}\) and no spacelike translation symmetries in AFFTIS-0 spacetimes. In other words, the existence of source in an AFFTIS-0 spacetime breaks the spacelike translation symmetries but leaves the timelike symmetries.

From the above discussion, it can be said that if the energy-momentum tensor of a spacetime falls off at a rate faster than \(\Omega^2\), the spacetime becomes asymptotically stationary at the rate of \(\Omega^2\).

We conclude the section with a remark on the relation of the asymptotic symmetries with the hierarchy in asymptotic structure. We have shown that asymptotic translational Killing fields are admitted to order 1 in AFFTIS-0 spacetimes. However, we can see that the fields are not admitted to order 2 in general. For the vector fields to be admitted to order 2, the coefficients of the \(O(\Omega^2)\) terms in eqs. (6.6) need to vanish. These terms contain the higher order asymptotic structure, such as \((\bar{\xi}) F\). For example, the \(O(\Omega^2)\) term of the time-time component is \(\Omega^2[\bar{D}ab \alpha + (\bar{\xi})^a - 2\nu a \nabla a]\). These higher order asymptotic quantities are arbitrary in AFFTIS-0 spacetimes. Hence, in general, the \(O(\Omega^2)\) terms in eqs. (6.6) do not vanish and thus asymptotic translational Killing fields to order two are not admitted in general AFFTIS-0 spacetimes. The asymptotic translational Killing fields to order 2 are the asymptotic symmetries that belong to the zero-th order asymptotic structure.

VII. GRAVITATIONAL FIELDS

In this section, we investigate the gravitational fields and find hierarchy in their asymptotic behaviors.

Because the Ricci tensor of \(g_{ab}\) is reserved for the definition of AFFTIS-\(n\) spacetimes, we examine the Weyl tensor \(\bar{C}_{abcd}\) to investigate the gravitational fields. For convenience of the discussion, decompose the Weyl tensor into the electric part \(\bar{E}_{ab} := \bar{C}_{ambn} \bar{n}^m \bar{n}^n\) and the magnetic part \(\bar{B}_{ab} := \bar{C}_{ambn} \bar{n}^m \bar{n}^n\) where \(\bar{C}_{ambn}\) denotes the dual of the 2-form \(\bar{C}_{ambn}\). In terms of quantities associated with the \(\Omega\)-const. surfaces and the physical components of the energy-momentum tensor, \(\bar{E}_{ab}\) and \(\bar{B}_{ab}\) are given by

\[
\begin{align*}
\bar{E}_{ab} & = K_{ar} \bar{K}^r_b - L^r_a \bar{K}_{ab} + \bar{D}_a \bar{a}_b + \bar{a}_a \bar{a}_b + \frac{1}{2} (\bar{q}_a \bar{q}_b - \bar{q}_{ab} \bar{n}^r \bar{n}^s) \bar{L}_{rs} \\
\bar{B}_{ab} & = \bar{e}_r \bar{s} \bar{D}^r \bar{K}_{bs} + \frac{1}{2} \bar{L}_{ab} \bar{n}^r \bar{L}_{rs}
\end{align*}
\] (7.1)

where \(\bar{L}_{ab} := \bar{R}_{ab} - \frac{1}{6} \bar{R} \bar{g}_{ab}\) and \(\bar{a}_a := \bar{q}_a \bar{n}^r \nabla_a \bar{n}^r\).
In AFTI-0 spacetimes, we find that $\hat{E}_{ab}$ and $\hat{B}_{ab}$ vanish on $\hat{i}^+$:

$$\hat{E}_{ab} = \sum_{\ell=1}^{\infty} \Omega^{\ell} E_{ab} \quad \text{and} \quad \hat{B}_{ab} = \sum_{\ell=1}^{\infty} \Omega^{\ell} B_{ab}$$  \hspace{1cm} (7.2)

where $^{(n)}E_{ab}$ and $^{(n)}B_{ab}$ are defined by

$$^{(1)}E_{ab} := \lim_{\hat{r} \to \hat{0}^+} \Omega^{-1} \hat{E}_{ab}, \quad ^{(n)}E_{ab} := \lim_{\hat{r} \to \hat{0}^+} \Omega^{-n} (\hat{E}_{ab} - \sum_{\ell=1}^{n-1} \Omega^{\ell} E_{ab}) \quad \text{for} \ n \geq 2,$$ \hspace{1cm} (7.3)

$$^{(1)}B_{ab} := \lim_{\hat{r} \to \hat{0}^+} \Omega^{-1} \hat{B}_{ab} \quad \text{and} \quad ^{(n)}B_{ab} := \lim_{\hat{r} \to \hat{0}^+} \Omega^{-n} (\hat{B}_{ab} - \sum_{\ell=1}^{n-1} \Omega^{\ell} B_{ab}) \quad \text{for} \ n \geq 2.$$ \hspace{1cm} (7.4)

Note here that, although they depend on the choice of $\Omega$, $^{(n)}E_{ab}$ and $^{(n)}B_{ab}$ are defined, regardless of whether their lower order terms vanish. This is in contrast with the treatment of [1] where the subsequent order terms are defined only when their lower order terms vanish. Their leading terms are given by

$$^{(1)}E_{ab} = \frac{1}{2} (D_a D_b - h_{ab}) (^{(1)}F + \frac{1}{2} (e_{ij} \epsilon_{ij} b_{ab} (^{(3)}L_{ij}) + \delta_{1,1} (^{(3)}L_{00}))$$ \hspace{1cm} (7.5)

$$^{(1)}B_{ab} = \frac{1}{2} \epsilon_{ra} s D_{(r} ^{(1)}K_{bs) + \frac{1}{2} \epsilon_{ab} r (^{(3)}L_{0i})$$ \hspace{1cm} (7.6)

where $^{(1)}K_{ab} := \lim_{\hat{r} \to \hat{0}^+} (K_{ab} - h_{ab})$, $^{(3)}L_{00} := \Psi^* [\hat{L}_{ab} \hat{n}^a \hat{n}^b]$, $^{(3)}L_{ij} := \Psi^* [\hat{L}_{ab} \hat{n}^a \hat{e}^b_i]$ and $^{(3)}L_{\mu \nu}$ denotes the coefficient of the $O(\Omega^n)$ term of $^{(1)}F$ and $^{(1)}K_{ab}$. Because these quantities are arbitrary in AFTI-0 spacetimes, we cannot conclude anything more than that $\hat{E}_{ab}$ and $\hat{B}_{ab}$ vanish on $\hat{i}^+$.

Let us examine the gravitational fields in AFTI-1 spacetimes. The fall-off condition on the energy-momentum tensor demands $^{(3)}L_{\mu \nu}$ vanish on $\hat{i}^+$ and we obtain

$$^{(1)}E_{ab} = \frac{1}{2} (D_a D_b - h_{ab}) (^{(1)}F) \quad \text{and} \quad ^{(1)}B_{ab} = \frac{1}{2} \epsilon_{ra} s D_{(r} ^{(1)}K_{bs).$$ \hspace{1cm} (7.7)

The equation on $^{(1)}E_{ab}$ induces

$$^{(1)}E_{ab} = \frac{1}{2} (D_a D_b - h_{ab}) (^{(1)}F)$$ \hspace{1cm} (7.8)

on $\hat{i}^+$. Using eq.(4.10), it can be shown that $^{(1)}E_{ab}$ obeys the field equation

$$D_a (^{(1)}E_{bc}) = 0.$$ \hspace{1cm} (7.9)

Contracting over $a$ and $b$ and recalling that $^{(1)}E_{ab}$ is traceless by definition, we find that $^{(1)}E_{ab}$ is divergence-free:

$$D_a (^{(1)}E^{ab}) = 0.$$ \hspace{1cm} (7.10)

It can be shown that $^{(1)}B_{ab}$ satisfies the same field equation. Contracting the Bianchi identity $\nabla_a \hat{R}_{bc} \mathring{d} e = 0$ with $\hat{n}^a \mathring{d} e : \mathring{e}_{mn} \mathring{e}_{d} a$, we obtain

$$\epsilon^{ac} \mathring{e}_{m} \mathring{D}_{a} \mathring{B}_{cn} - \hat{n}^a \mathring{\nabla} a \mathring{E}_{mn} + \epsilon^{ac} \mathring{m} \mathring{e}_{de} \mathring{K}_{ad} \mathring{E}_{ce}$$ \hspace{1cm} (7.11)

$$- 2 \epsilon^{bc} (m \mathring{B}_{n} a) \mathring{a}_b + \frac{1}{2} (\hat{n}^a \mathring{q}^c \mathring{q}_{mn} - \hat{n}^a \mathring{q}_{mn} \mathring{q}^c) \mathring{\nabla} a \mathring{L}_{ce} = 0$$

from which we find, in AFTI-$n$ spacetimes,

$$D_a (^{(1)}B_{bc}) = O(\Omega) + O(\Omega^n)$$ \hspace{1cm} (7.12)

where we have used the equations $n^r \nabla_r (\hat{e}_{ij})_a = -(\hat{e}_{ij})_a + O(\Omega)$, $\Omega \cdot n^r \nabla_r \mathring{L}_{ij} = O(\Omega^{n+3})$, $\mathring{L}_{\mu \alpha} = O(\Omega^{n+3})$ and $D_a \mathring{L}_{0i} = O(\Omega^{n+3})$ to evaluate the derivative of $\mathring{L}_{ab}$.
\[
\left( \hat{\eta}^{[a} q^{c]} e q_{mn} - \hat{\eta}^{[a} q^{c]} q^{m}_{n} \right) \hat{\nabla}_{a} L_{cc} = \Omega^{-1} \left[ q_{mn} \left( \Omega a^{a} \nabla_{a} \hat{L}_{1} - 2 \hat{L}_{1} - 2 \hat{L}_{ab} - (e_{a})^{c} D_{e} \hat{L}_{a} \right) \right.
\]
\[
\left. + (e_{1})_{m} (e_{1})_{n} \left( \Omega n^{a} \nabla_{a} \hat{L}_{1} - \hat{L}_{1} \right) + (e_{1})_{m} D_{n} \hat{L}_{a} + O(\Omega^{n+4}) \right] = O(\Omega^{n+2}).
\]

Hence, in AFT1-1 spacetimes, the equation \( D_{a}^{(1)} B_{bc} \equiv 0 \) is satisfied and induces
\[
D_{a}^{(1)} B_{bc} \equiv 0 \tag{7.13}
\]
on \( i^{+} \). Thus \( B_{ab} \) is also divergence-free:
\[
D_{a}^{(1)} B_{ab} \equiv 0. \tag{7.14}
\]

Here, it is important to note that the explicit form of the function \( F \) can be obtained. Because \( E_{ab} \) is traceless by definition, the contraction of eq.\((7.8)\) gives a differential equation for the function \( F \):
\[
D^{a} D_{a}^{(1)} F - 3 \Omega^{(1)} F \equiv 0. \tag{7.15}
\]

We have obtained the non-trivial equation because the Einstein equation is used in a non-trivial way to obtain eq.\((7.8)\). That is, the terms that are not traceless, \( \frac{1}{2} (e_{1})_{a} (e_{1})_{b} \left( \hat{L}_{1} + \delta_{i}^{(1)} L_{ab} \right) \), are discarded to obtain the equation. In contrast, we do not obtain any non-trivial equations from noting that \( B_{ab} \) is traceless because the discarded terms to obtain the corresponding equation \((7.7)\) are traceless. The general solution of eq.\((7.15)\) takes the form
\[
^{(1)} F(\chi, \theta, \phi) = \sum_{\ell \geq 0} a_{\ell m}^{(1)} F_{\ell m}(\chi, \theta, \phi) \tag{7.16}
\]
where
\[
^{(1)} F_{\ell m}(\chi, \theta, \phi) := \frac{1}{\sqrt{\sinh \chi}} P_{\ell}^{\pm \frac{1}{2}}(\cosh \chi) Y_{\ell m}(\theta, \phi). \tag{7.17}
\]

Here, \( P_{\ell}^{\mu}(z) \) denote the associated Legendre functions defined by
\[
P_{\ell}^{\mu}(z) := \frac{1}{1!(1-\mu)} \left( \frac{z+1}{z-1} \right)^{\mu/2} F \left( -\nu, \nu + 1, 1 - \mu; \frac{1-z}{2} \right) \quad \text{for} \quad z > 1 \tag{7.18}
\]
where \( F(\alpha, \beta, \gamma; z) \) are the hypergeometric functions. \( Y_{\ell m} \) stands for the spherical harmonics with their coefficients chosen as
\[
Y_{\ell m}(\theta, \phi) = \begin{cases} 
\sqrt{2\pi} P_{\ell}^{m}(\cos \theta) \cos m\phi & m \geq 0 \\
\sqrt{2\pi} P_{\ell}^{m}(\cos \theta) \sin m\phi & m < 0
\end{cases} \tag{7.19}
\]
so that their physical meanings can be conveniently read in the next section. Explicitly, we obtain
\[
^{(1)} F_{00} = 2 \left( 2 \sinh \chi + \frac{1}{\sinh \chi} \right) \tag{7.20}
\]
\[
^{(1)} F_{11} = 2 \left( 2 \cosh \chi - \frac{\cosh \chi}{\sinh^{2} \chi} \right) \sin \theta \cos \phi. \tag{7.21}
\]

When the Schwarzschild spacetimes are completed, the function \( F \) indeed consists of the above functions. See the appendix for the explicit calculations.

To define an angular momentum, it is important to see the conditions for \( B_{ab} \) to be divergence-free. This can be done using the Bianchi identity \( \nabla_{(a} R_{bc)} \, dc = 0 \) contracted with \( \epsilon^{a b c} \eta_{a} \hat{q}_{c} e \),
\[
\hat{D}_{e} B^{rs} = \frac{1}{2} \hat{L}^{t}_{s r} \hat{\nabla}_{e} L_{pq}. \tag{7.22}
\]
Keeping in mind that \((1)B_{ab}\) is divergence-free, we find that the above equation induces
\[
D_\tau (2)B^{rs} = \epsilon^{rspt} (L_{ij}q_p e_i)_r (e_j)_t.
\]  
(7.23)

Hence, \((2)B_{ab}\) is not divergence-free in general AFTI-1 spacetimes. However, in AFTI-2 spacetimes \((4)L_{ij}\) vanishes on \(i^+\) and \((2)B_{ab}\) is divergence-free:
\[
D_\tau (2)B^{rs} \equiv 0.
\]  
(7.24)

Note here that \((1)B_{ab}\) does not need to vanish on \(i^+\) for \((2)B_{ab}\) to be divergence-free. This point is unclear in \([\text{?}]\).

We conclude the section with a remark on the relation of the asymptotic behavior of gravitational fields with the hierarchy in the asymptotic structure. The gravitational fields of each order, \((n)E_{ab}\) and \((n)B_{ab}\), are constructed from the asymptotic structure of the corresponding order. Hence, if the energy-momentum tensor falls off at a low rate and the spacetime lacks the higher order asymptotic structure, the asymptotic behaviors of the higher order gravitational fields are arbitrary. In an AFTI-0 spacetime, \((0)E_{ab}\) and \((0)B_{ab}\) vanish on \(i^+\) and \((4)E_{ab}\) and \((4)B_{ab}\) take arbitrary forms for \(\ell \geq 1\). In an AFTI-1 spacetime, \((1)E_{ab}\) and \((1)B_{ab}\) induces divergence-free tensor fields on \(i^+\) that satisfy the field equations \((7.9)(7.10)(7.7)\) respectively, and \((1)E_{ab}\) takes the form given by \((7.14)(7.7)\). \((4)E_{ab}\) and \((4)B_{ab}\) take arbitrary forms for \(\ell \geq 2\). In an AFTI-2 spacetime, \((2)B_{ab}\) induce a divergence-free tensor field on \(i^+\).

VIII. CONSERVED QUANTITIES

In this section, we define two conserved quantities, namely four-momentum of AFTI-1 spacetimes and angular momentum of AFTI-2 spacetimes, which are associated with the asymptotic symmetries that the spacetimes possess.

As Minkowski spacetime admits infinitesimal translation symmetries, it possesses a conserved four-momentum. The four-momentum is defined as a linear mapping from the four-dimensional vector space of infinitesimal translations to real numbers. Because AFTI-1 spacetimes possess infinitesimal Ti translations, we expect that four-momentum of AFTI-1 spacetimes can be also defined as a linear mapping of infinitesimal Ti translations to real numbers.

The four-momentum representing the state of the spacetime should be defined with information from the gravitational fields. However, AFTI-0 spacetimes possess no asymptotic gravitational fields, as we have seen in the previous section. This means that four-momentum cannot be defined in AFTI-0 spacetimes. Now, consider AFTI-1 spacetimes. Here, we have gravitational fields \((1)E_{ab}\) and \((1)B_{ab}\). On the other hand, infinitesimal Ti translations may be represented by \(D^\nu \alpha^\tau\), where \(\alpha^\tau\) is a smooth function generated by the elements of \(\mathcal{L}_\tau\). The vector field is a conformal Killing field of \((i^+, h_{ab})\) by definition:
\[
D_{(a} D_{b)} \alpha^\tau = 2 \alpha^\tau h_{ab}.
\]  
(8.1)

With these \((1)E_{ab}\) and \(D^n \alpha^\tau\), the four-momentum of an AFTI-1 spacetime can be defined as follows.

**DEFINITION:** Four-momentum \(P\) of an AFTI-1 spacetime with a region \(A\) on \(i^+\) that satisfies

1) \(A\) is bounded by cross sections \(S_a\) and \(S_b\) that do not intersect with each other, i.e., \(\partial A = S_a \cup S_b\) and \(S_a \cap S_b = \emptyset\),

2) \(q_{ab}\) and \(n^a\) have smooth limit to \(A \subset i^+\) and functions \(\alpha^\tau\) generated by elements of \(\mathcal{L}_\tau\) are smooth on \(A\)

is a linear mapping of \(\alpha^\tau\) to real numbers defined by
\[
P(A) := -\frac{1}{8\pi} \int_S \epsilon^{mn} h_{ab} m^{(1)} E_{mn} D^n \alpha^\tau
\]  
(8.2)

where \(S\) is a cross section of the region \(A\).

Note that, in general, AFTI-1 spacetimes do not possess a region \(A\) satisfying the above properties; \(q_{ab}\) and \(n^a\) are demanded to have smooth limits to \(i^+ \cap F\) in AFTI-1 spacetimes but the region \(i^+ \cap F\) does not need to satisfy the above properties 1) and 2). However, we need such a region on \(i^+\) to define a “conserved quantity” as we shall see below.

The four-momentum can be considered as a “conserved quantity” because it is invariant under a change of the cross section \(S\) chosen for the evaluation. This is because the 2-form integrand is closed in \(A\): the 3-form \(D_{[a} \epsilon_{b]c} m^{(1)} E_{mn} D^n \alpha^\tau\) is proportional to
where we used, in the first line, the fact that $D^n \alpha_T$ is a conformal Killing field and, in the second line, the facts that $(1)^* \tilde{E}_{ab}$ is divergence-free and traceless. In other words, the four-momentum is “conserved” in $\mathcal{A}$ because it consists of a smooth conformal Killing field and a smooth divergence-free, traceless gravitational field.

Recall that there are four independent functions of $\alpha_T$ as shown in eq.(5.6). Thus, there are four independent components of four momentum $P$ as we expect.

Now let us evaluate the four-momentum defined above and see where the information is contained. First, we evaluate the $\alpha_t$ component. For simplicity, choose the $\chi = \text{const.}$ surface in $\mathcal{A}$ as $S$. We obtain

$$P(\alpha_t) = \frac{1}{16\pi} \int_{\chi = \text{const.}} \left( \frac{\partial^2}{\partial \chi^2} (1)F - (1)F \right) \sinh \chi \sqrt{\det h} \, d\theta d\phi,$$

(8.4)

where $h := \det(h_{ij}) = \sinh^2 \chi \sin \theta$. Substituting the general solution of $(1)F$, eq.(7.16), and using the orthogonality of spherical harmonics

$$\int Y_{\ell m}(\cos \theta) d\phi = 4\sqrt{\pi} \int_0^\pi \delta_{\ell,0} \delta_{m,0},$$

(8.5)

we find that all the terms but $(1)F_{00}$ of $(1)F$ vanish and obtain

$$P(\alpha_t) = a_{00}.$$  

(8.6)

The spatial momentum can be evaluated likewise as

$$P(\alpha_x) = a_{11},$$

$$P(\alpha_y) = a_{1-1},$$

$$P(\alpha_z) = a_{10}.$$  

(8.7)

Here, we find that the $\ell = 0, 1$ modes of $(1)F$ contain the information on the four-momentum of an AFTI-0 spacetime. Since the function $(1)F$ is of the first order asymptotic structure, the coefficients of $(1)F$ and thus the four momentum $P$ is also of the first order asymptotic structure. If we complete the Kerr spacetime and the Schwarzschild spacetime, we obtain the expected answers for the four-momentum $P$: $a_{00}$ equals the mass parameter $m$. See eqs.(A3) and (A9) for the explicit calculations. Moreover, $a_{11}$ equals $-mv$ in the leading order of $v$ for the Schwarzschild spacetime boosted by a constant velocity $v$ along the $x$-axis. See eq.(A11).

Because $(1)B_{ab}$ is also divergence-free and traceless and $\xi \in \mathcal{L}_L$ is a conformal Killing field of $(\tilde{\mathcal{I}}, \tilde{h}_{ab})$, we obtain a new “conserved quantity” if we replace $(1)E_{mn}D^n \alpha_T$ in eq.(8.2) with one of $(1)B_{mn}D^n \alpha_T$, $(1)E_{mn}\xi^n$ and $(1)B_{mn}\xi^n$. However, it is found that they all give exact 2-form integrands. Hence, the “conserved quantities” identically vanish [1]. Because there are no more gravitational fields in an AFTIS-1 spacetime, four-momentum $P$ can be considered as the sole “conserved quantity” of AFTIS-1 spacetimes.

Next, consider AFTIS-2 spacetimes. There exists a gravitational field $(2)B_{ab}$ that is divergence-free and traceless in such spacetimes. Hence, we yield another “conserved quantity” by replacing $(1)E_{mn}D^n \alpha_T$ in eq.(8.2) with $(2)B_{mn}\xi^n$. This quantity can be regarded as angular momentum if a gauge condition $(1)B_{ab} = 0$ is imposed. This is to reduce the Ti group to the Poincaré group so that the angular momentum is defined around an origin in the four-dimensional affine space of the infinitesimal Ti translations [1]. Note here that it is clarified that the gauge condition is needed for the quantity to be defined around the four-dimensional origin, not for the quantity to be “conserved”. This point was left unclear in [1].

We conclude the section with a remark on how the conserved quantities are associated with the hierarchy in the asymptotic structure. To define the conserved quantities, we needed vector fields that respect the asymptotic symmetries and gravitational fields that are traceless and divergence-free. However, the asymptotically flat spacetimes that satisfy only the weakest fall-off condition on the energy-momentum tensor do not possess sufficient asymptotic structures that the spacetimes lack for the conserved quantities to be well defined. As a result, four-momentum is defined for an AFTIS-1 spacetime and angular momentum is defined for an AFTIS-2 spacetime.
IX. SUMMARY AND DISCUSSIONS

In this paper, we have proposed a new definition of asymptotic flatness at timelike infinity. Instead of rescaling the metric of a spacetime conformally as in the previous works \cite{3,4,5}, we rescaled 3-metrics and normal vector fields to the time slices with different powers to provide the information on the gravitational fields. As a result, the causal structure of the physical spacetime is not manifest in the unphysical spacetime and the timelike infinity arises not as a point but as a $3$-manifold. The 3-manifold timelike infinity enables us to characterize the asymptotically flat spacetime under investigation with a fall-off condition on the physical components of the physical energy-momentum tensor. Hence, it is possible to discuss the properties of such a spacetime with the energy-momentum tensor falling off at a certain rate where the rate is specified with the function $\Omega$ that reads asymptotically the inverse of $t$, the observer’s proper time. This makes a sharp contrast with the treatment using the conformal completion in which the spacetime is characterized with the awkward differentiability conditions on the unphysical metric and thus such an argument is not possible. Hereafter in this section, we refer to the physical components of the physical energy-momentum tensor simply by the energy-momentum tensor for brevity.

We found the following hierarchy in the asymptotic structure of an asymptotically spacetime. If the energy-momentum tensor falls off faster than the rate $\Omega^2$, the spacetime possesses an intrinsic structure called the zero-th order asymptotic structure. The structure consists of a function $F$ that is unity on timelike infinity $\mathbf{i}^+$, and the manifold $\mathbf{i}^+$ with a unit spacelike 3-hyperboloid metric $h_{\alpha\beta}$. The presence of the structure means that the spacetime is asymptotically Minkowskian. An asymptotic symmetry group, called Ti group, can be defined to preserve the structure. The group is found to be similar to the Poincaré group but differs in that the 4-dimensional translation group is replaced with an infinite dimensional additive group. Further, by introducing the notion of an asymptotic Killing field, it is found that such a spacetime becomes asymptotically stationary at the rate $\Omega^2$ and in general does not become stationary at a faster rate. Although there exists an asymptotically symmetry group, the spacetime is just asymptotically Minkowskian and thus does not possess gravitational fields with sufficient properties to have four-momentum defined. If the energy-momentum tensor falls off faster than the rate $\Omega^3$, there is enough higher order asymptotic structure for the four-momentum to be defined. The form of one of the asymptotic structure $^{(1)}F$ is determined in such spacetimes. A conserved quantity associated with angular momentum can be defined only for spacetimes with even more asymptotic structure: the energy-momentum tensor needs to fall off faster than the rate $\Omega^4$. For the quantity to be regarded as angular-momentum in the sense that the angular-momentum should be defined around the four dimensional origin, the leading order of one of the gravitational fields, $^{(1)}B_{\alpha\beta}$, must vanish.

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APPENDIX A: SOME AFFTIS SPACETIMES AND SOME THAT ARE NOT

We give the explicit construction of the completion of the Schwarzschild spacetime and show that it is an AFFTIS spacetime. Then, it is shown that the nature of the charge and the angular momentum of the Kerr-Newmann spacetime appears in the second order asymptotic structure. The function $F := - \mathcal{L}_n \Omega$ which contains the information on four-momentum is explicitly calculated for a boosted Schwarzschild spacetime. At the end, an example of spacetimes which do not satisfy the conditions of AFFTIS-$n$ spacetimes is given.

In static spherical coordinates, the metric of Schwarzschild spacetime takes the form

$$ \tilde{g}_{\alpha\beta} = -(1 - 2m/r)(1 - 2m/r)(dt)^2 + \frac{4m}{c} \frac{4m}{\sinh \chi} (\partial_\chi)^a - \frac{1 - 2m\Omega}{1 - 2m\Omega} \frac{1 - 2m\Omega}{\sinh \chi} (\partial_\chi)^a. \quad (A1) $$

Introducing new coordinates $t = \Omega^{-1} \cosh \chi$ and $r = \Omega^{-1} \sinh \chi$, we obtain smooth $n^a$, $F$ and $q_{ab}$:

$$ n^a = - (\frac{1 + 4m\Omega \sinh \chi - 4m^2\Omega^2}{1 - 2m\Omega} (\partial_\chi)^a - \frac{4m \cosh \chi - m\Omega}{1 - 2m\Omega} \frac{4m \cosh \chi - m\Omega}{\sinh \chi} (\partial_\chi)^a - \frac{4m \cosh \chi - m\Omega}{1 - 2m\Omega} \frac{4m \cosh \chi - m\Omega}{\sinh \chi} (\partial_\chi)^a. \quad (A2) $$

$$ F = \frac{1 - 4m\Omega \sinh \chi}{1 - 2m\Omega} \frac{4m^2}{\sinh \chi} = 1 + \Omega \frac{F_{00}}{\Omega^2} + \frac{4m^2}{\Omega^2} \frac{4m^2}{\sinh \chi} + \frac{4m^2}{\sinh \chi} = \frac{4m^2}{\Omega^2} (\partial_\chi)^a, \quad (A3) $$

$$ (\partial_\chi)^a = [\Omega m \frac{F_{00}}{\Omega^2} + \frac{4m^2}{\sinh \chi} (\partial_\chi)^a - [\Omega m \frac{F_{00}}{\Omega^2} + \frac{4m^2}{\sinh \chi} (\partial_\chi)^a ] (\partial_\chi)^a. \quad (A2) $$

$$ F = \frac{1 - 4m\Omega \sinh \chi - 4m^2\Omega^2}{1 - 2m\Omega} \frac{4m^2}{\sinh \chi} = 1 + \Omega \frac{F_{00}}{\Omega^2} + \frac{4m^2}{\Omega^2} \frac{4m^2}{\sinh \chi} + \frac{4m^2}{\sinh \chi} = \frac{4m^2}{\Omega^2} (\partial_\chi)^a. \quad (A3) $$

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and

\[ q_{ab} = 16m^2 \frac{\cosh^2 \chi - 2m\Omega \cosh \chi/\tanh \chi + m^2\Omega^2/\tanh^2 \chi}{1 + 2m\Omega (2 \sinh \chi - 1/\sinh \chi) - 12m^2\Omega^2 + 8m^3\Omega^3/\sinh \chi} \text{aftis-}(d\Omega)_a(d\Omega)_b \]

\[ -4m = \cosh \chi - m\Omega/\tanh \chi \]

\[ = h_{ab} + [16m^2 \cosh^2 \chi - \Omega 64m^3 \cosh \chi \sinh \chi/(\tanh \chi) + O(\Omega^2)](d\Omega)_a(d\Omega)_b \]

Together with the fact that it is a vacuum solution of the Einstein equation, the above calculations show that the Schwarzschild spacetime is an AFIT-∞ spacetime, where we denote a spacetime that is an AFIT-n spacetime for arbitrary n by an AFIT-∞ spacetime. Now take the outside of the event horizon \( r > 2m \) in the original coordinates as \( \tilde{F} \). Because the region \( r \leq 2m \) corresponds to the point \( \chi = 0 \) on the \( \Omega = 0 \) surface, the choice provides timelike infinity \( \tilde{r}^+ \) with the point missing. Thus, topology of \( \tilde{r}^+ \) is \( S^2 \times \{ \Re \} \) and the Schwarzschild spacetime satisfies the condition of an AFIT-n spacetime to arbitrary \( n \). It is important to note here that the Oppenheimer-Snyder spacetime, which describes dust collapsing to form a Schwarzschild black hole, can be completed to possess \( \tilde{r}^+ \) with the same structure as the above completed Schwarzschild spacetime does. The spacetime has the Schwarzschild metric outside the dust and the surface of the dust collapses along \( r = (r_{ini}/2)(1 + \cos \eta) \) where \( r \) and \( r_{ini} \) are the Schwarzschild radial coordinate and the initial dust radius at time \( t = 0 \), respectively; and \( \eta \) is related to the Schwarzschild time coordinate \( t \) by

\[ t = 2m \ln \left| \frac{\sqrt{r_{ini}/2m - 1} + \tan \eta/2}{\sqrt{r_{ini}/2m - 1} - \tan \eta/2} \right| + 2m \sqrt{r_{ini}/2m - 1} (\eta + (\eta + \sin \eta)r_{ini}/4m). \] (A4)

If we choose the region \( r > r_{ini} \) as region \( \tilde{F} \), the metric of the region is that of the Schwarzschild spacetime and thus \( \tilde{F} \) satisfies all the conditions of an AFIT-n spacetime for arbitrary \( n \) and can be completed to have exactly the same \( n^a \), \( F \) and \( q_{ab} \) in \( \tilde{F} \) as those of the Schwarzschild spacetime. Since the region \( r \leq r_{ini} \) corresponds to the point \( \chi = 0 \) on \( \tilde{r}^+ \) in this completion also, the global structure of \( \tilde{r}^+ \) is also the same as that of the above completed Schwarzschild spacetime.

In static spherical coordinates, the metric of Kerr-Newmann spacetime is given by

\[ \hat{g}_{ab} = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} (dt)_a(dt)_b - 2a^2 \sin^2 \theta(r^2 + a^2 - \Delta) (dt)_a(d\phi)_b + \frac{\Sigma}{(r^2 + a^2 - \Delta a^2 \sin^2 \theta)} \sin^2 \theta(d\phi)_a(d\phi)_b \]

\[ + (dr)_a(dr)_b + \Sigma(d\theta)_a(d\theta)_b \]

where \( \Sigma = r^2 + a^2 \cos^2 \theta \) and \( \Delta = r^2 + a^2 + e^2 - 2mr \). Introducing \( \eta := \ln \Omega \), we clearly see that the nature of the charge and the angular momentum of the momentum in the spacetime appear in the second order asymptotic structure and do not alter the lower order asymptotic structure:

\[ n^a = e^{-\eta}[(0)n^a + \Omega (1)n^a + \Omega^2 (2)n^a + O(\Omega^3)] \]

\[ F = (0)F + \Omega (1)F + \Omega^2 (2)F + O(\Omega^3) \]

\[ q_{ab} = (0)q_{ab} + \Omega (1)q_{ab} + \Omega^2 (2)q_{ab} + O(\Omega^3) \] (A7)

where

\[ (0)n^a = -\partial_\eta \]

\[ (1)n^a = m (F_{00}\partial_\eta - 4m \cosh \chi \partial_\eta) \]

\[ (2)n^a = \frac{4m^2 - e^2}{\tanh \chi} - e^2 - a^2 \sin^2 \theta(\partial_\eta)^2 - 4m^2 - 2e^2 - a^2 \sin^2 \theta \tanh \chi (\partial_\chi)^2 + 2ma \frac{\cosh \chi}{\sinh \chi} (\partial_\phi)^2; \] (A8)

\[ (0)F = 1, \quad (1)F = m \quad (2)F = \frac{4m^2 - e^2}{\tanh \chi} - e^2 - a^2 \sin^2 \theta; \] (A9)
\(q_{ab} = h_{ab}, \quad q_{ab} = -2(4m \cosh \chi)(d \eta)_a (d \chi)_b + 2m F^0_0(d \chi)_a (d \chi)_b \quad \text{and} \quad q_{ab} = 16m^2 \cosh^2 \chi (d \eta)_a (d \eta)_b - 2 \frac{4m^2 - e^2 - a^2 \sin \theta}{\tanh \chi} (d \eta)_a (d \chi)_b + 4ma \sin^2 \theta \frac{(d \eta)_a (d \phi)_b}{\tanh \chi} - (d \chi)_a (d \phi)_b, \quad (A10)\)

In the boosted Schwarzschild spacetime with velocity of \(v\) along \(x\)-axis, or to be precise in the Schwarzschild spacetime that is observed by an observer with velocity of \(v\) along \(x\)-axis

\[F = 1 + \Omega \left\{ 2m(2 \sinh \chi + \frac{1}{\sin \chi}) - 2mv(2 \cosh \chi - \frac{\cosh \chi}{\sinh \chi} \sin \theta \cos \phi) + O(\Omega^2) \right\}
= 1 + \Omega \left\{ m F^0_0 - mv F_{11} + O(v^2) \right\} + O(\Omega^2). \quad (A11)\]

Let us consider the flat Friedmann-Robertson-Walker spacetime, which does not qualify as an AFTI spacetime as one expects. The metric of the spacetime filled with perfect fluid is

\[\hat{g}_{ab} = -(dt)_a (dt)_b + a^2(t) [(dr)_a (dr)_b + \nu^2 (d\sigma)_{ab}],\]

where \(a(t) = a_0(t/t_0)^{2/3(1+w)}\) if the energy density \(\rho\) and the pressure \(p\) is related by \(p = \rho \mu\). In order to get smooth \(n^a\) and \(q_{ab}\), we need to choose the new coordinates \((\bar{\Omega}, \chi)\) as

\[t = a_0 \frac{1 + 3w}{1 + w} \bar{\Omega}^{-1} (\cosh \chi)^{3+3w}, \quad r = t_0 \frac{2}{1 + w} \bar{\Omega}^{-1}(\sinh \chi).\]

They give us

\[n^a = \frac{1}{a_0^2} \left( \frac{3 + 3w}{1 + 3w} \right)^2 (\cosh \chi)^{-\frac{4}{1 + 3w}} (\partial t)^a \quad (A12)\]
\[q_{ab} = a_0^2 (\cosh \chi)^{\frac{4}{1 + 3w}} [(d \chi)_a (d \chi)_b + \sinh^2 \chi (d \sigma)_{ab}]. \quad (A13)\]

However, with this choice of \(\bar{\Omega}\), the time-time component of the energy-momentum tensor decays as \(\Omega^2\):

\[\hat{n}^a \hat{n}^b T_{ab} = 9 \rho_0 \left( \frac{t}{t_0} \right)^{-2} \cosh^2 \chi + O(\Omega) \]
\[= \Omega^2 \rho_0 \left( \frac{3t_0(1 + w)}{a_0(1 + 3w)} \right)^2 (\cosh \chi)^{-\frac{4}{1 + 3w}} + O(\Omega^2)\]

from which we see that the spacetime does not satisfy the third condition of AFTI-\(n\) spacetimes for non-negative \(n\) and thus the spacetime is not an AFTI-\(n\) spacetime. Hence, the spacetime does not need to possess the properties of an AFTI-\(n\) spacetimes: the function

\[F = \frac{1}{a_0^2} \left( \frac{3 + 3w}{1 + 3w} \right)^2 (\cosh \chi)^{-\frac{4}{1 + 3w}}, \quad (A14)\]

is not unity on \(\vec{r}^+\); and the spacetime does not admit asymptotic stationary Killing field.

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