Integral Representation for $L$-functions for $\text{GSp}_4 \times \text{GL}_2$

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Abstract. Let $\pi$ be a cuspidal, automorphic representation of $\text{GSp}_4$ attached to a Siegel modular form of degree 2. We refine the method of Furusawa \cite{Furusawa2001} to obtain an integral representation for the degree-8 $L$-function $L(s, \pi \times \tau)$, where $\tau$ runs through certain cuspidal, automorphic representations of $\text{GL}_2$. Our calculations include the case of square-free level for the $p$-adic components of $\tau$, and a wide class of archimedean types including Maaß forms. As an application we obtain a special value result for $L(s, \pi \times \tau)$.

1 Introduction

Let $\pi = \otimes \pi_\nu$ and $\tau = \otimes \tau_\nu$ be irreducible, cuspidal, automorphic representations of $\text{GSp}_4(\mathcal{A})$ and $\text{GL}_2(\mathcal{A})$, respectively. Here, $\mathcal{A}$ is the ring of adeles of a number field $F$. We want to investigate the degree eight twisted $L$-functions $L(s, \pi \times \tau)$ of $\pi$ and $\tau$, which are important for a number of reasons. For example, when $\pi$ and $\tau$ are obtained from holomorphic modular forms, then Deligne \cite{Deligne1974} has conjectured that a finite set of special values of $L(s, \pi \times \tau)$ are algebraic up to certain period integrals. Another very important application is the conjectured Langlands functorial transfer of $\pi$ to an automorphic representation of $\text{GL}_4(\mathcal{A})$. One approach to obtaining the transfer to $\text{GL}_4(\mathcal{A})$ is to use the converse theorem due to Cogdell and Piatetski-Shapiro \cite{CogdellPiatetskiShapiro1993}, which requires precise information about the $L$-functions $L(s, \pi \times \tau)$.

In the special case that $\pi$ is generic, Asgari and Shahidi \cite{AsgariShahidi2007} have been successful in obtaining the above transfer using the converse theorem. They analyze the twisted $L$-functions using the Langlands-Shahidi method. In this method, one has to consider a larger group in which $\text{GSp}_4$ is embedded and then use the representation $\pi$ to construct an Eisenstein series on the larger group. Then the $L$-functions are obtained in the constant and non-constant terms of the Eisenstein series. Unfortunately, this method only works when $\pi$ is generic. It is known that if $\pi$ is obtained from a holomorphic Siegel modular form then it is not generic.

Another method to understand $L$-functions is via integral representations. For this method one constructs an integral that is Eulerian, i.e., one that can be written as an infinite product of local integrals, $Z(s) = \prod \nu Z_\nu(s)$. Then the local integrals are computed to obtain the local $L$-functions. In many of the constructions, the local calculations are done only when all the local data is unramified. This gives information about the partial $L$-functions, which already leads to remarkable applications. The calculations for the ramified data are unfortunately often very involved and not available in the literature. (For more on integral representations of $L$-functions, see \cite{KudlaRallisShahidi1985, KudlaRallisShahidi1987, KudlaRallisShahidi1990}.)

In the $\text{GSp}_4 \times \text{GL}_2$ case, Novodvorsky, Piatetski-Shapiro and Soudry (see \cite{Novodvorsky2000, PiatetskiShapiroSoudry2000, Soudry2000}) were the first ones to construct integral representations for $L(s, \pi \times \tau)$. Their constructions were for the special case when $\pi$ is either generic or has a special Bessel model. Examples of Siegel modular forms which do not have a special Bessel model have been constructed by Schulze-Pillot \cite{SchulzePillot1998}. The first construction of an integral representation for $L(s, \pi \times \tau)$ with no restriction on the Bessel model of $\pi$ is the work of Furusawa \cite{Furusawa2001}. In this remarkable paper, Furusawa embeds $\text{GSp}_4$ in a unitary group $\text{GU}(2,2)$ and constructs an Eisenstein series on $\text{GU}(2,2)$ using the $\text{GL}_2$ representation $\tau$. He then integrates the Eisenstein series against a vector in $\pi$. He shows that this integral is Eulerian and, when the local data is unramified, he computes the local integral to obtain the local $L$-function $L(s, \pi_\nu \times \tau_\nu)$ up to a normalizing factor. He also calculates the archimedean integral for the case that both $\pi$ and $\tau$ are holomorphic of the same weight. Thus, Furusawa obtains an integral representation for the completed $L$-function $L(s, \pi \times \tau)$ in the case when $\pi$ and $\tau$ are obtained from holomorphic modular forms of full level and same weight. He uses this to obtain a special value result, which fits into the context of Deligne’s conjectures, and to prove meromorphic continuation and functional equation.

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1 Introduction
for the $L$-function. The main limitation of [21] is that, if we fix a Siegel modular form, then the results allow us to obtain information on a very small family of twists only, namely those coming from elliptic modular forms of full level and the same weight as the Siegel modular form, which is a finite dimensional vector space.

For the applications that we discussed above, we need twists of $\pi$ by all representations $\tau$ of $\GL_2$, i.e., twists by all $\GL_2$ modular forms, holomorphic or non-holomorphic, of arbitrary weight and level. For this purpose, one needs to compute the non-archimedean local integral obtained in [9] when the local representation $\nu$ is ramified. Also, one needs to extend Furusawa’s archimedean calculation to include more general archimedean representations.

In this paper, we will compute the local non-archimedean integral from [9] in a mildly ramified case, namely when $\tau_\nu$ is an unramified twist of the Steinberg representation. We will also compute the archimedean integral for a larger family of archimedean representations $\tau_\infty$.

Before we state the results of this paper, let us recall the integral representation of [9] in some more detail. Let $L$ be a quadratic extension of the number field $F$, and let $\GU(2, 2)$ be the unitary group defined using the field $L$. Let $P$ be the standard maximal parabolic subgroup of the unitary group $\GU(2, 2)$ with a non-abelian radical. Given an irreducible, admissible representation $\tau$ of $\GL_2(\A)$ and suitable characters $\chi_0$ and $\chi$ of $\A^*_1$, one considers an induced representation $I(s, \chi, \chi_0, \tau)$ from $P$ to $\GU(2, 2)$, where $s$ is a complex parameter. Let $f(g, s)$ be an analytic family in $I(s, \chi, \chi_0, \tau)$. Define an Eisenstein series on $\GU(2, 2)$ by the formula

$$E(g, s) = E(g, s; f) = \sum_{\gamma \in P(F) \backslash \GU(2, 2)(F)} f(\gamma g, s), \quad g \in \GU(2, 2)(\A).$$

For an automorphic form $\phi$ in the space of $\pi$, consider the integral

$$Z(s) = Z(s, f, \phi) = \int_{Z(L) \backslash \GSp_4(F)} E(h, s; f)\bar{\phi}(h) dh. \quad (1)$$

In [9], Furusawa has shown that these integrals have the following two important properties.

i) There is a “basic identity”

$$Z(s) = \int_{R(\A) \backslash \GSp_4(\A)} W_f(\eta h, s)B_\phi(h) dh, \quad (2)$$

where $R \subset \GSp(4)$ is a Bessel subgroup of the Siegel parabolic subgroup, $\eta$ is a certain fixed element, $B_\phi$ corresponds to $\phi$ in the Bessel model for $\pi$, and $W_f$ is a function on $\GU(2, 2)$ obtained from the Whittaker model of $\tau$ and depending on the section $f$ used to define the Eisenstein series.

ii) $Z(s)$ is Eulerian, i.e.,

$$Z(s) = \prod_\nu Z_\nu(s) = \prod_\nu \int_{R(F_\nu) \backslash \GSp_4(F_\nu)} W_\nu(\eta h, s)B_\nu(h) dh. \quad (3)$$

In Theorem 3.9.1 below we show that if $\tau_\nu$ is mildly ramified then the local integral can be computed to give $L(3s + \frac{1}{2}, \hat{\pi}_\nu \times \tilde{\tau}_\nu)$ up to a normalizing factor.

**Theorem 1.** Let $F_\nu$ be a non-archimedean local field with characteristic zero. Let $\pi_\nu$ be an unramified, irreducible, admissible representation of $\GSp_4(F_\nu)$. Let $\tau_\nu$ be an unramified twist of the Steinberg representation. Then we can make a choice of vectors $W_\nu$ and $B_\nu$ such that the local integral in [9] is given by

$$Z_\nu(s) = \frac{q(q - 1)}{(q + 1)(q^4 - 1)} \left(1 - \left(\frac{L_\nu}{p}\right) q^{-1}\right) \frac{L(3s + \frac{1}{2}, \hat{\pi}_\nu \times \tilde{\tau}_\nu)}{L(3s + 1, \tau_\nu \times \AT(\Lambda_\nu) \times (\chi_0|_{F_\nu^S})}.\quad (4)$$

Here, $q$ is the cardinality of the residue class field of $F_\nu$, $\Lambda_\nu$ is the Bessel character on $L_\nu^\times$ used to define the Bessel model $B_\nu$, and $\AT(\Lambda_\nu)$ is the representation of $\GL_2(F_\nu)$ obtained from $\Lambda_\nu$ by automorphic induction.
We point out that the ramified calculation is not a trivial generalization of the unramified calculation in [9]. There are two main steps. First is the choice of the vector $W_\nu$ – for several obvious choices $Z_\nu(s)$ evaluates to zero. This choice depends crucially on the underlying number theory. Secondly, the actual computation of the local integral is complicated and depends heavily on the structure theory of the groups involved. We will explain this in detail in Sect. 3.

In Theorem 4.4.1 we compute the local archimedean integral in the following cases:

i) $\pi_\infty$ is the holomorphic discrete series representation of $GSp_4(\mathbb{R})$ with trivial central character and Harish-Chandra parameter $(l - 1, l - 2)$.

ii) $\tau_\infty$ is either a principal series representation of $GL_2(\mathbb{R})$ whose $K$-types have the same parity as $l$ or is a holomorphic discrete series representation of $GL_2(\mathbb{R})$ with lowest weight $l_2$ satisfying $l_2 \leq l$ and $l_2 \equiv l \pmod{2}$.

This extends the calculations in [11], where $\tau_\infty$ is only allowed to be a holomorphic discrete series representation with lowest weight $l$.

Putting together the local computations we get the following global result in Theorem 5.3.1.

**Theorem 2.** Let $\Phi$ be a cuspidal Siegel eigenform of weight $l$ with respect to $Sp_4(\mathbb{Z})$. Let $N$ be a square-free, positive integer. Let $f$ be a cuspidal Maass eigenform of weight $l_1 \in \mathbb{Z}$ with respect to $\Gamma_0(N)$. If (the adelic function corresponding to) $f$ lies in a holomorphic discrete series representation with lowest weight $l_2$, then assume that $l_2 \leq l$. Let $\pi_\Phi$ and $\tau_f$ be the corresponding cuspidal automorphic representations of $GSp_4(\mathbb{A}_\mathbb{Q})$ and $GL_2(\mathbb{A}_\mathbb{Q})$, respectively. Then a choice of local vectors can be made such that the global integral $Z(s)$ defined in (1) is given by

$$Z(s) = \frac{L(3s + \frac{1}{2}, \pi_\Phi \times \tau_f)}{\zeta(6s + 1)L(3s + 1, \tau_f \times AZ(A))},$$

(4)

where $\kappa_\infty$ and $\kappa_N$ are obtained from the local computations.

Note that the above theorem still gives information on the twisted $L$-functions for a smaller family of representations $\tau$ than is required for the application of the converse theorem mentioned earlier. For a general representation $\tau$ the local calculations are conceptually the same but the calculations are much more complicated. This is work in progress and will be a subject of a future paper.

Using [11], we get the following special value result in Theorem 5.4.1.

**Theorem 3.** Let $\Phi$ be a cuspidal Siegel eigenform of weight $l$ with respect to $Sp_4(\mathbb{Z})$. Let $N$ be a square-free, positive integer. Let $\Psi$ be a holomorphic, cuspidal Hecke eigenform of weight $l$ with respect to $\Gamma_0(N)$. Then

$$\frac{L(\frac{1}{2}, \pi_\Phi \times \tau_\Psi)}{\pi^{3l - 8}(\Phi, \Psi)_2(\Psi, \Psi)_1} \in \mathcal{C}.$$
2 General setup

In this section, we give the basic definitions and set up the data required to compute the local integrals. Let \( F \) be a non-archimedean local field of characteristic zero, or \( F = \mathbb{R} \). We fix three elements \( a, b, c \in F \) such that \( d := b^2 - 4ac \neq 0 \). Let

\[
L = \begin{cases} 
F(\sqrt{d}) & \text{if } d \notin F^{\times 2}, \\
F \oplus F & \text{if } d \in F^{\times 2}.
\end{cases}
\]

In case \( L = F \oplus F \), we consider \( F \) diagonally embedded. If \( L \) is a field, we denote by \( \overline{x} \) the Galois conjugate of \( x \in L \) over \( F \). If \( L = F \oplus F \), let \( (x,y) = (y,x) \). In any case we let \( N(x) = x\overline{x} \) and \( \text{tr}(x) = x + \overline{x} \).

2.1 The unitary group

We define the symplectic and unitary similitude groups by

\[
\begin{align*}
H(F) &= \text{GSp}_4(F) := \{ g \in \text{GL}_4(F) : t^gJg = \mu(g)J, \mu(g) \in F^{\times} \}, \\
G(F) &= \text{GU}(2,2;L) := \{ g \in \text{GL}_4(L) : t^\overline{g}Jg = \mu(g)J, \mu(g) \in F^{\times} \},
\end{align*}
\]

where \( J = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \). Note that \( H(F) = G(F) \cap \text{GL}_4(F) \). As a minimal parabolic subgroup we choose the subgroup of all matrices that become upper triangular after switching the last two rows and last two columns. Let \( P \) be the standard maximal parabolic subgroup of \( G(F) \) with a non-abelian unipotent radical. Let \( P = MN \) be the Levi decomposition of \( P \). We have \( M = M^{(1)}M^{(2)} \), where

\[
M^{(1)}(F) = \{ \begin{bmatrix} \zeta & 1 \\ 1 & \overline{\zeta} \end{bmatrix} : \zeta \in L^{\times} \},
\]

\[
M^{(2)}(F) = \{ \begin{bmatrix} 1 & \alpha & \beta \\ \mu & \gamma & \delta \end{bmatrix} \in G(F) \}.
\]

\[
N(F) = \{ \begin{bmatrix} 1 & z \\ 1 & \overline{z} \end{bmatrix} \begin{bmatrix} 1 & w \\ \overline{y} & 1 \end{bmatrix} : w \in F, y, z \in L \}.
\]

For a matrix in \( M^{(2)}(F) \) as the one above, the unitary conditions are equivalent to \( \mu = \overline{\mu} \) (i.e., \( \mu \in F^{\times} \)), \( \mu = \overline{\alpha}\delta - \beta\overline{\gamma}, \overline{\alpha}\gamma = \gamma\alpha \) and \( \overline{\delta}\beta = \beta\delta \). In addition, we have \( \overline{\alpha}\beta = \beta\alpha, \delta\gamma = \gamma\delta, \overline{\alpha}\delta = \delta\alpha, \overline{\gamma}\beta = \beta\gamma \). Hence the following holds.

**2.1.1 Lemma.** Let

\[
\begin{bmatrix}
1 & \alpha & \beta \\
\gamma & \mu & \delta
\end{bmatrix}
\]

be an element of \( M^{(2)}(F) \), as above. Then the quotient of any two entries of the matrix \( \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \), if defined, lies in \( F \). Hence, if \( \lambda \) is any invertible entry of \( \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \), then

\[
\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \lambda \begin{bmatrix} \alpha/\lambda & \beta/\lambda \\ \gamma/\lambda & \delta/\lambda \end{bmatrix}.
\]

\[\varepsilon_{\text{GL}_2(F)}\]
Consequently, the map
\[
L^\times \times GL_2(F) \longrightarrow M^{(2)}(F),
\]
\[
(\lambda, \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}) \longmapsto \begin{bmatrix} 1 \\ \lambda \alpha \\ \lambda \gamma \\ N(\lambda)(\alpha \delta - \beta \gamma) \end{bmatrix},
\]
is surjective with kernel \(\{(\lambda, \lambda^{-1}) : \lambda \in F^\times\}\).

The modular factor of the parabolic \(P\) is given by
\[
\delta_P(\begin{bmatrix} \zeta \\ 1 \\ \zeta^{-1} \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ \alpha \\ \mu \\ \gamma \\ \delta \end{bmatrix} = |N(\zeta)\mu|^{1/2}, \quad (\mu = \bar{\alpha}\delta - \beta\bar{\gamma}),
\]
where \(|\cdot|\) is the normalized absolute value on \(F\).

### 2.2 The Bessel subgroup

Recall that we fixed three elements \(a, b, c \in F\) such that \(d = b^2 - 4ac \neq 0\). Let
\[
S = \begin{bmatrix} a & b \\ \frac{b}{2} & c \end{bmatrix}, \quad \xi = \begin{bmatrix} b \\ \frac{b}{2} \end{bmatrix}.
\]

Then \(F(\xi) = F + F\xi\) is a two-dimensional \(F\)-algebra isomorphic to \(L\). If \(L = F(\sqrt{d})\) is a field, then an isomorphism is given by \(x + y\xi \rightarrow x + y\sqrt{d}\). If \(L = F \oplus F\), then an isomorphism is given by \(x + y\xi \rightarrow (x + y\sqrt{d}, x - y\sqrt{d})\). The determinant map on \(F(\xi)\) corresponds to the norm map on \(L\). Let
\[
T(F) = \{g \in GL_2(F) : {}^t g S g = \det(g) S\}.
\]

One can check that \(T(F) = F(\xi)^\times\). Note that \(T(F) \cong L^\times\) via the isomorphism \(F(\xi) \cong L\). We consider \(T(F)\) a subgroup of \(H(F) = GSp_4(F)\) via
\[
T(F) \ni g \longmapsto \begin{bmatrix} g \\ \det(g) {}^t g^{-1} \end{bmatrix} \in H(F).
\]

Let
\[
U(F) = \{ \begin{bmatrix} 1_2 & X \\ 1_2 \end{bmatrix} \in GSp_4(F) : {}^t X = X \}
\]
and \(R(F) = T(F) U(F)\). We call \(R(F)\) the Bessel subgroup of \(GSp_4(F)\) (with respect to the given data \(a, b, c\)). Let \(\psi\) be any non-trivial character \(F \rightarrow \mathbb{C}^\times\). Let \(\theta : U(F) \rightarrow \mathbb{C}^\times\) be the character given by
\[
\theta(\begin{bmatrix} 1 & X \\ 1 & \end{bmatrix}) = \psi(\text{tr}(S X)).
\]

Explicitly,
\[
\theta(\begin{bmatrix} 1 & x & y \\ 1 & y & z \\ 1 & \end{bmatrix}) = \psi(ax + by + cz).
\]
We have \(\theta(t^{-1} u t) = \theta(u)\) for all \(u \in U(F)\) and \(t \in T(F)\). Hence, if \(\Lambda\) is any character of \(T(F)\), then the map \(tu \mapsto \Lambda(t) \theta(u)\) defines a character of \(R(F)\). We denote this character by \(\Lambda \otimes \theta\).
2.3 Parabolic induction from $P(F)$ to $G(F)$

Let $(\tau, V_\tau)$ be an irreducible, admissible representation of $GL_2(F)$, and let $\chi_0$ be a character of $L^\times$ such that $\chi_0|_{F^\times}$ coincides with $\omega_{\tau}$, the central character of $\tau$. Then the representation $(\lambda, g) \mapsto \chi_0(\lambda)\tau(g)$ of $L^\times \times GL_2(F)$ factors through $\{ (\lambda, \lambda^{-1}) : \lambda \in F^\times \}$, and consequently, by Lemma 2.1.1, defines a representation of $M^{(2)}(F)$ on the same space $V_\tau$. Let us denote this representation by $\chi_0 \times \tau$. Every irreducible, admissible representation of $M^{(2)}(F)$ is of this form. If $V_\tau$ is a space of functions on $GL_2(F)$ on which $GL_2(F)$ acts by right translation, then $\chi_0 \times \tau$ can be realized as a space of functions on $M^{(2)}(F)$ on which $M^{(2)}(F)$ acts by right translation. This is accomplished by extending every $W \in V_\tau$ to a function on $M^{(2)}(F)$ via

$$W(\lambda g) = \chi_0(\lambda)W(g), \quad \lambda \in L^\times, \ g \in GL_2(F).$$

If $V_\tau$ is the Whittaker model of $\tau$ with respect to the character $\psi$, then the extended functions $W$ satisfy the transformation property

$$W(\begin{bmatrix} 1 & 1 & x \\ 1 & 1 & 1 \end{bmatrix} g) = \psi(x)W(g), \quad x \in F, \ g \in M^{(2)}(F).$$

If $s$ is a complex parameter, $\chi$ is any character of $L^\times$, and $\chi_0 \times \tau$ is a representation of $M^{(2)}(F)$ as above, we denote by $I(s, \chi, \chi_0, \tau)$ the representation of $G(F)$ obtained by parabolic induction from the representation of $P(F) = M(F)N(F)$ given on the Levi part by

$$\begin{bmatrix} \zeta & 1 & \lambda & \lambda \beta \\ 1 & \zeta^{-1} & N(\lambda)(\alpha \delta - \beta \gamma) & \lambda \delta \\ \lambda \gamma & \lambda \delta & \lambda \beta & \lambda \beta \\ 1 & 1 & \lambda \beta \gamma & \lambda \beta \delta \end{bmatrix} \mapsto |N(\zeta \lambda^{-1})(\alpha \delta - \beta \gamma)^{-1}|^{3s} \chi(\zeta)\chi_0(\lambda)\tau(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}).$$

Explicitly, the space of $I(s, \chi, \chi_0, \tau)$ consists of functions $f : G(F) \rightarrow V_\tau$ with the transformation property

$$f(\begin{bmatrix} \zeta & 1 & \lambda & \lambda \beta \\ 1 & \zeta^{-1} & N(\lambda)(\alpha \delta - \beta \gamma) & \lambda \delta \\ \lambda \gamma & \lambda \delta & \lambda \beta & \lambda \beta \\ 1 & 1 & \lambda \beta \gamma & \lambda \beta \delta \end{bmatrix} g) = |N(\zeta \lambda^{-1})(\alpha \delta - \beta \gamma)^{-1}|^{3s+\frac{3}{2}} \chi(\zeta)\chi_0(\lambda)\tau(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix})f(g).$$

Now assume that $V_\tau$ is the Whittaker model of $\tau$ with respect to the character $\psi$ of $F$. If we associate to each $f$ as above the function on $G(F)$ given by $W_\#^f(g) = f(g)(1)$, then we obtain another model of $I(s, \chi, \chi_0, \tau)$ consisting of functions $W_\# : G(F) \rightarrow \mathbb{C}$. These functions satisfy

$$W_\#(\begin{bmatrix} \zeta & 1 & \lambda & \lambda \beta \\ 1 & \zeta^{-1} & N(\lambda) & \lambda \beta \\ \lambda \gamma & \lambda \delta & \lambda \beta & \lambda \beta \\ 1 & 1 & \lambda \beta \gamma & \lambda \beta \delta \end{bmatrix} g) = |N(\zeta \lambda^{-1})(\alpha \delta - \beta \gamma)^{-1}|^{3s+\frac{3}{2}} \chi(\zeta)\chi_0(\lambda)W_\#(g), \quad \zeta, \lambda \in L^\times,$n

and

$$W_\#(\begin{bmatrix} 1 & z & w & y \\ 1 & 1 & x & 1 \\ 1 & -x & \frac{w}{y} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} g) = \psi(x)W_\#(g), \quad w, x \in F, \ y, z \in L.$$  

The following lemma gives a transformation property of $W_\#$ under the action of the elements of the Bessel subgroup $R(F)$. 

6
2.3.1 Lemma. Let \((\tau, V_{\tau})\) be a generic, irreducible, admissible representation of \(GL_2(F)\). We assume that \(V_{\tau}\) is the Whittaker model of \(\tau\) with respect to the non-trivial character \(\psi^{-c}(x) = \psi(-cx)\) of \(F\). Let \(\chi\) and \(\chi_0\) be characters of \(L^\times\) such that \(\chi_0|_{F^\times} = \omega_{\tau}\). Let \(W^\#(\cdot, s) : G(F) \to \mathbb{C}\) be a function in the above model of the induced representation \(I(s, \chi, \chi_0, \tau)\), where \(s\) is a complex parameter. Let \(\theta\) be the character of \(U(F)\) defined in (11). Let \(\Lambda\) be the character of \(L^\times \cong T(F)\) given by

\[
\Lambda(\zeta) = \chi(\zeta^{-1})\chi_0(\zeta)^{-1}.
\]  

(18)

Let

\[
\eta = \begin{bmatrix}
1 & 0 \\
\alpha & 1 \\
1 & -\tilde{\alpha} \\
0 & 1
\end{bmatrix},
\]

where \(\alpha := \begin{cases} \frac{b + \sqrt{d}}{2c} & \text{if } L \text{ is a field,} \\ \frac{b - \sqrt{d}}{2c} & \text{if } L = F \oplus F. \end{cases}\)

(19)

Then

\[
W^\#(\eta tuh, s) = \Lambda(t)^{-1}\theta(u)^{-1}W^\#(\eta h, s)
\]

(20)

for \(t \in T(F), u \in U(F)\) and \(h \in G(F)\).

Proof. If \(L\) is a field, then the proof is word for word the same as on p. 197/198 of [9]. The case \(L = F \oplus F\) requires the only modification that the element \(\zeta = x + \frac{t}{2}\sqrt{d}\) is to be replaced by \(\zeta = x + \frac{t}{2}(\sqrt{d}, -\sqrt{d})\). \(\blacksquare\)

2.4 The local integral

Let \((\pi, V_{\pi})\) be an irreducible, admissible representation of \(H(F) = GSp_4(F)\). Let the Bessel subgroup \(R(F)\) be as defined in Section 22; it depends on the given data \(a, b, c \in F\). We assume that \(V_{\pi}\) is a Bessel model for \(\pi\) with respect to the character \(\Lambda \otimes \theta\) of \(R(F)\). Hence, \(V_{\pi}\) consists of functions \(B : H(F) \to \mathbb{C}\) satisfying the Bessel transformation property

\[
B(tuh) = \Lambda(t)\theta(u)B(h) \quad \text{for } t \in T(F), u \in U(F), h \in H(F).
\]

Let \((\tau, V_{\tau})\) be a generic, irreducible, admissible representation of \(GL_2(F)\) such that \(V_{\tau}\) is the \(\psi^{-c}\)-Whittaker model of \(\tau\) (we assume \(c \neq 0\)). Let \(\chi_0\) be a character of \(L^\times\) such that \(\chi_0|_{F^\times} = \omega_{\tau}\). Let \(\chi\) be the character of \(L^\times\) for which (18) holds. Let \(W^\#(\cdot, s)\) be an element of \(I(s, \chi, \chi_0, \tau)\) for which the restriction of \(W^\#(\cdot, s)\) to the standard maximal compact subgroup of \(G(F)\) (see below for more details) is independent of \(s\), i.e., \(W^\#(\cdot, s)\) is a “section” of the family of induced representations \(I(s, \chi, \chi_0, \tau)\). By Lemma 2.3.1 it is meaningful to consider the integral

\[
Z(s) = \int_{R(F) \setminus H(F)} W^\#(\eta h, s)B(h) \, dh.
\]

(21)

In the following we shall compute these integrals for certain choices of \(W^\#\) and \(B\). We shall only consider \(GSp_4(F)\) representations \(\pi\) that are relevant for the global application to Siegel modular forms we have in mind. In the real case we shall assume that \(\pi\) is a holomorphic discrete series representation and that \(B\) corresponds to the lowest weight vector. In the \(p\)-adic case we shall assume that \(\pi\) is an unramified representation and that \(B\) corresponds to the spherical vector.

The generic \(GL_2(F)\) representation \(\tau\), however, will be only mildly restricted in the real case, and, in the \(p\)-adic case, will be a Steinberg representation twisted by an unramified character. In the real case, the function \(W^\#\) will be constructed from a certain vector of the “correct” weight in \(V_{\tau}\). In the \(p\)-adic case, the function \(W^\#\) will be constructed from the local newform in \(V_{\tau}\).

In each case our calculations will show that the integral (21) converges absolutely for \(\text{Re}(s)\) large enough and has meromorphic continuation to all of \(\mathbb{C}\). Our choice of \(W^\#\) will be such that \(Z(s)\) is closely related to the local \(L\)-factor \(L(s, \pi \times \tau)\).
Note that the integral (21) has been calculated in [9] for \( \pi \) and \( \tau \) both holomorphic discrete series representations with related lowest weights in the real case and \( \pi \) and \( \tau \) both unramified representations in the \( p \)-adic case.

### 3 Local non-archimedean ramified theory

In this section, we evaluate (21) in the non-archimedean setting. The key steps are the choices of the vector \( W^\# \) and the actual computation of the integral \( Z(s) \).

#### 3.1 Setup

Let \( F \) be a non-archimedean local field of characteristic zero. Let \( o, p, \varpi, q \) be the ring of integers, prime ideal, uniformizer and cardinality of the residue class field \( o/p \), respectively. Recall that we fix three elements \( a, b, c \in F \) such that \( d := b^2 - 4ac \neq 0 \). Let \( L \) be as in [9]. We shall make the following assumptions:

(A1) \( a, b \in o \) and \( c \in o^\times \).

(A2) If \( d \not\in F^\times \), then \( d \) is the generator of the discriminant of \( L/F \). If \( d \in F^\times \), then \( d \in o^\times \).

**Remark:** In [9] p. 198, Furusawa makes a stronger assumption on \( a, b, c \), namely, \( \left[ \frac{a \ b/2}{b/2 \ c} \right] \in M_2(o) \).

However, it is necessary to make the weaker assumption \( a, b, c \in o \) for the global integral calculation (4.5) in [9] p. 210 to be valid for \( D \equiv 3 \) (mod 4). (This is because the matrix \( S(-D) \) on p. 208 is not in \( M_2(o^\times) \) for \( D \equiv 3 \) (mod 4).) One can check that the non-archimedean unramified calculation in [9] is valid with the weaker assumption \( a, b, c \in o \). Hence, the global result of [9] is still valid but the assumptions (A1) and (A2) above are the correct ones.

We set the Legendre symbol as follows,

\[
\left( \frac{L}{p} \right) := \begin{cases} 
-1, & \text{if } d \not\in F^\times, \ d \not\in p \\
0, & \text{if } d \not\in F^\times, \ d \in p \\
1, & \text{if } d \in F^\times 
\end{cases}
\]

(22)

If \( L \) is a field, then let \( o_L \) be its ring of integers. If \( L = F \oplus F \), then let \( o_L = o \oplus o \). Note that \( x \in o_L \) if and only if \( N(x), \text{tr}(x) \in o \). If \( L \) is a field then we have \( x \in o_L^\times \) if and only if \( N(x) \in o^\times \). If \( L \) is not a field then \( x \in o_L, N(x) \in o^\times \) implies that \( x \in o_L^\times = o^\times \oplus o^\times \). Let \( \varpi_L \) be the uniformizer of \( o_L \) if \( L \) is a field and set \( \varpi_L = (\varpi, 1) \) if \( L \) is not a field. Note that, if \( \left( \frac{L}{p} \right) \neq -1 \), then \( N(\varpi_L) \in \varpi o^\times \). Let

\[
\xi_o := \begin{cases} 
\frac{-b + \sqrt{d}}{2}, & \text{if } L \text{ is a field,} \\
\frac{-b + \sqrt{d}}{2}, \frac{-b - \sqrt{d}}{2}, & \text{if } L = F \oplus F.
\end{cases}
\]

(23)

and

\[
\alpha := \begin{cases} 
\frac{b + \sqrt{d}}{2c}, & \text{if } L \text{ is a field,} \\
\frac{b + \sqrt{d}}{2c}, \frac{b - \sqrt{d}}{2c}, & \text{if } L = F \oplus F.
\end{cases}
\]

(24)

We fix the following ideal in \( o_L \),

\[
P := p o_L = \begin{cases} 
p_L & \text{if } \left( \frac{L}{p} \right) = -1, \\
p_L^2 & \text{if } \left( \frac{L}{p} \right) = 0, \\
p \oplus p & \text{if } \left( \frac{L}{p} \right) = 1.
\end{cases}
\]

(25)
Here, \( p_L \) is the maximal ideal of \( \mathfrak{o}_L \) when \( L \) is a field extension. Note that \( \mathfrak{p} \) is prime only if \( \left( \frac{L}{p} \right) = -1 \). We have \( \mathfrak{p}^n \cap \mathfrak{o} = p^n \) for all \( n \geq 0 \). We now state a number-theoretic lemma which will be crucial in Section 3.7.

### 3.1.1 Lemma

Let notations be as above.

1. The elements 1 and \( \xi_0 \) constitute an integral basis of \( L/F \) (i.e., a basis of the free \( \mathfrak{o}_L \)-module \( \mathfrak{o}_L \)). The elements 1 and \( \alpha \) also constitute an integral basis of \( L/F \).

2. There exists no \( x \in \mathfrak{o} \) such that \( \alpha + x \in \mathfrak{P} \).

**Proof.**

1. Since \( c \in \mathfrak{o}^\times \) and \( b \in \mathfrak{o} \), the second assertion of i) follows from the first one. To prove the first assertion, first note that \( \xi_0 \) satisfies \( \xi_0^2 + \xi_0 b + ac = 0 \), and therefore belongs to \( \mathfrak{o}_L \). Since the claim is easily verified if \( L = F \oplus F \), we will assume that \( L \) is a field. Let \( A, B \in \mathfrak{F} \) be such that 1 and \( \xi_1 := A + B \sqrt{d} \) is an integral basis of \( L/F \). Then

\[
\det(\begin{bmatrix} 1 & \xi_1 \\ 1 & \xi_1 \end{bmatrix})^2 = 4B^2d
\]

generates the discriminant of \( L/F \). Since \( d \) also generates the discriminant by assumption \((A2)\), it follows that \( 2B \in \mathfrak{o}_F^\times \). Dividing \( \xi_1 \) by this unit, we may assume \( \xi_1 = A + \frac{1}{2} \sqrt{d} \) for some \( A \in \mathfrak{F} \). Now let us represent \( \xi_0 \) in this integral basis,

\[
\xi_0 = x + y \xi_1, \quad x, y \in \mathfrak{F},
\]

i.e.,

\[
\frac{-b + \sqrt{d}}{2} = x + y(A + \frac{1}{2} \sqrt{d}).
\]

Comparing coefficients, we get \( y = 1 \) and \( A = -\frac{b}{2} - x \). We may modify \( \xi_1 \) by adding the integral element \( x \) and still obtain an integral basis. But \( \xi_1 + x = \xi_0 \), and the assertion follows.

2. Let \( X \subset \mathfrak{o}_L/\mathfrak{P} \) be the image of the injection \( \mathfrak{o}/p \to \mathfrak{o}_L/\mathfrak{P} \).

Note that the field on the left hand side has \( q \) elements, and the ring on the right hand side has \( q^2 \) elements, for any value of \( \left( \frac{L}{p} \right) \). Our claim is equivalent to the statement that \( \bar{\alpha} \), the image of \( \alpha \) in \( \mathfrak{o}_L/\mathfrak{P} \), does not lie in the subring \( X \) of \( \mathfrak{o}_L/\mathfrak{P} \). Assume that \( \bar{\alpha} \in X \). By i), any element \( z \in \mathfrak{o}_L \) can be (uniquely) written as

\[
z = x \alpha + y, \quad x, y \in \mathfrak{o}.
\]

Applying the projection to \( \mathfrak{o}_L/\mathfrak{P} \), it follows that \( \bar{z} = \bar{x} \bar{\alpha} + \bar{y} \in X \). This is a contradiction, since \( \bar{z} \) runs through all elements of \( \mathfrak{o}_L/\mathfrak{P} \), but \( X \) is a proper subset.

Note that, via the identification \( T(F) = L^\times \) described in Sect. 2.2, the element \( \xi_0 \) corresponds to the matrix

\[
\begin{bmatrix} 0 & c \\ -a & -b \end{bmatrix}.
\]

Therefore, by Lemma 3.1.1 i),

\[
\mathfrak{o}_L = \mathfrak{o} \oplus \mathfrak{o} \xi_0 = \left\{ \begin{bmatrix} x & yc \\ -ya & x - yb \end{bmatrix} : x, y \in \mathfrak{o} \right\}.
\]

Since \( c \) is assumed to be a unit, it follows that

\[
\mathfrak{o}_L = T(F) \cap M_2(\mathfrak{o}) \quad \text{and} \quad \mathfrak{o}_L^\times = T(F) \cap \text{GL}_2(\mathfrak{o}).
\]
3.2 The spherical Bessel function

Let \((\pi, V_\pi)\) be an unramified, irreducible, admissible representation of \(\text{GSp}_4(F)\). Then \(\pi\) can be realized as the unramified constituent of an induced representation of the form \(\chi_1 \times \chi_2 \rtimes \sigma\), where \(\chi_1, \chi_2\) and \(\sigma\) are unramified characters of \(F^\times\); here, we used the notation of \([24]\) for parabolic induction. Let

\[
\gamma^{(1)} = \chi_1 \chi_2 \sigma, \quad \gamma^{(2)} = \chi_1 \sigma, \quad \gamma^{(3)} = \sigma, \quad \gamma^{(4)} = \chi_2 \sigma.
\]

Then \(\gamma^{(1)} \gamma^{(3)} = \gamma^{(2)} \gamma^{(4)}\) is the central character of \(\pi\). The numbers \(\gamma^{(1)}(\varpi), \ldots, \gamma^{(4)}(\varpi)\) are the Satake parameters of \(\pi\). The degree-4 \(L\)-factor of \(\pi\) is given by \(\prod_{i=1}^{4}(1 - \gamma^{(i)}(\varpi)q^{-s})^{-1}\).

Let \(\Lambda\) be any character of \(T\) unramified characters of \(G\). Then \(\Lambda\) is right \(K\)-invariant, it follows that \(\Lambda|_{\pi_\infty} = 1\). For \(l, m \in \mathbb{Z}\) let

\[
h(l, m) = \begin{bmatrix} q^{2m+l} & \varpi^{m+l} \\ \varpi^m & 1 \end{bmatrix}.
\]

Then, as in (3.4.2) of \([3]\),

\[
H(F) = \bigcup_{l \in \mathbb{Z}} \bigcup_{m \geq 0} R(F)h(l, m)K^H, \quad K^H = \text{GSp}_4(\mathfrak{o}).
\]

The double cosets on the right hand side are pairwise disjoint. Since \(B\) transforms on the left under \(R(F)\) by the character \(\Lambda \otimes \theta\) and is right \(K^H\)-invariant, it follows that \(B\) is determined by the values \(B(h(l, m))\).

In \([28]\), Proposition 2-4, we have \(B(1) \neq 0\). It follows from \(B(1) \neq 0\) and \((27)\) that necessarily \(A_{5\infty} = 1\). For \(l, m \in \mathbb{Z}\) let

\[
B(h(l, 0))y^l = \frac{1 - A_5y - A_2A_4y^2}{Q(y)}, \quad (30)
\]

where

\[
Q(y) = \prod_{i=1}^{4}(1 - \gamma^{(i)}(\varpi F)q^{-3/2}y), \quad (31)
\]

and where \(A_2, A_4, A_5\) are given in the following table. Set \(H(y) = 1 - A_5y - A_2A_4y^2\).

3.3 The local compact subgroup

We define congruence subgroups of \(\text{GL}_2(F)\), as follows. For \(n = 0\) let \(K^{(1)}(\mathfrak{p}^0) = \text{GL}_2(\mathfrak{o})\). For \(n > 0\) let

\[
K^{(1)}(\mathfrak{p}^n) = \text{GL}_2(F) \cap \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^n 1 + \mathfrak{p}^n \end{bmatrix}.
\]
The following result is well known (see [3], [S]).

3.3.1 Theorem. Let \((τ, V)\) be a generic, irreducible, admissible representation of \(GL_2(F)\). Then the spaces

\[
V(n) = \{ v ∈ V : τ(g)v = v \text{ for all } g ∈ K^{(1)}(p^n) \}
\]

are non-zero for \(n\) large enough. If \(n\) is minimal with \(V(n) ≠ 0\), then \(\dim(V(n)) = 1\).

If \(n\) is minimal such that \(V(n) ≠ 0\), then \(p^n\) is called the conductor of \(τ\). In this section we shall define a family \(K#(\mathfrak{P}^n)\), \(n ≥ 0\), of compact-open subgroups of \(G(F)\), the relevance of which is as follows. Recall that our goal is to evaluate integrals of the form

\[
\int_{H(F) \backslash H(F)} W#(ηh, s)B(h) dh,
\]

where \(W#(\cdot, s)\) is a section in a family of induced representations \(I(s, χ, χ_0, τ)\). The choice of the function \(W#(\cdot, s)\) is crucial for our purposes. We will define it in such a way that \(W#(\cdot, s)\) is supported on \(M(F)N(F)K#(\mathfrak{P}^n)\), where \(p^n\) is the conductor of the \(GL_2(F)\) representation \(τ\).

Recall that \(\mathfrak{P} = p\mathfrak{o}_L\). For \(n = 0\) we let \(K#(\mathfrak{P}^0) = G(F) ∩ GL_4(\mathfrak{o}_L)\) be the standard maximal compact subgroup of \(G(F)\). For \(n > 0\) we define

\[
K#(\mathfrak{P}^n) := \{ g ∈ G(F) : \mu(g) = 1 \} \cap \begin{bmatrix}
1 + \mathfrak{P}^n & \mathfrak{P}^n & \mathfrak{P}^n & \mathfrak{P}^n \\
\mathfrak{P}^n & 1 + \mathfrak{P}^n & \mathfrak{P}^n & \mathfrak{P}^n \\
\mathfrak{P}^n & \mathfrak{P}^n & 1 + \mathfrak{P}^n & \mathfrak{P}^n \\
\mathfrak{P}^n & \mathfrak{P}^n & \mathfrak{P}^n & 1 + \mathfrak{P}^n 
\end{bmatrix}
\]

and

\[
K#(\mathfrak{P}^n) = K#(\mathfrak{P}^0) × \{ \text{diag}(1, μ, μ, 1) : μ ∈ \mathfrak{o}^× \}.
\]

The GL_2 congruence subgroup \(K^{(1)}(p^n)\) defined above can be embedded into \(K#(\mathfrak{P}^n)\) in the following way,

\[
\begin{bmatrix}
α & β \\
γ & δ
\end{bmatrix} \mapsto \begin{bmatrix}
1 & \frac{α}{μ} & \frac{β}{δ} \\
\frac{γ}{μ} & 1 & \frac{δ}{μ}
\end{bmatrix} \begin{bmatrix}
1 & μ \\
μ & 1
\end{bmatrix}, \quad \text{where } μ = αδ - βγ.
\]

Important for us will be the intersection

\[
K#(p^n) := H(F) ∩ K#(\mathfrak{P}^n)
\]

\[
= \begin{bmatrix}
1 + p^n & p^n & 0 & 0 \\
p^n & 1 + p^n & 0 & 0 \\
p^n & p^n & 1 + p^n & p^n \\
p^n & p^n & p^n & 1 + p^n
\end{bmatrix} \begin{bmatrix}
1 & μ \\
μ & 1
\end{bmatrix}, \quad μ ∈ \mathfrak{o}^×.
\]

Note that \(H(F) ∩ K#(\mathfrak{P}^n) = K^H ∩ K#(\mathfrak{P}^n)\), where \(K^H = GSp_4(\mathfrak{o})\) is the maximal compact subgroup of \(H(F)\). It follows from Lemma 2.13 that the map

\[
\mathfrak{o}^×_L × GL_2(\mathfrak{o}) \longrightarrow M^{(2)}(F) ∩ GL_4(\mathfrak{o}_L),
\]

\[
(λ, \begin{bmatrix}
α & β \\
γ & δ
\end{bmatrix}) \mapsto \begin{bmatrix}
1 & λα & λβ \\
λα & N(c)(αδ - βγ) & λβ \\
λγ & λβ & λδ
\end{bmatrix},
\]

is surjective with kernel \(\{ (λ, λ^{-1}) : λ ∈ \mathfrak{o}_L^× \}\). For \(n > 0\) this map induces a surjection

\[
(1 + \mathfrak{P}^n) × K^{(1)}(p^n) \longrightarrow M^{(2)}(F) ∩ K#(\mathfrak{P}^n)
\]

with kernel \(\{ (λ, λ^{-1}) : λ ∈ 1 + \mathfrak{P}^n \}\).
3.4 The function \( W^\# \)

We shall now define the specific function \( W^\#(\cdot, s) \) for which we shall evaluate the integral \( 33 \). Let \((\tau, V_\tau)\) be a generic, irreducible, admissible representation of \( GL_2(F) \). We assume that \( V_\tau \) is the Whittaker model of \( \tau \) with respect to the character of \( F \) given by \( \psi^{-\tau}(x) = \psi(-cx) \). Let \( p^n \) be the conductor of \( \tau \). Let \( W^{(0)}(n) \in V(n) \) be the local newform, i.e., the essentially unique non-zero \( K^{(1)}(p^n) \) invariant vector in \( V_\tau \). We can make it unique by requiring that \( W^{(0)}(1) = 1 \).

We choose any character \( \chi_0 \) of \( L^\times \) such that
\[
\chi_0|_{F^\times} = \omega_{\tau} \quad \text{and} \quad \chi_0|_{1+\mathfrak{p}^n} = 1.
\]
This can be accomplished by extending \( \omega_{\tau}|_{\mathfrak{o}_L^\times} \) to \( \mathfrak{o}_L^\times \) using the injection \( \mathfrak{o}_L^\times/(1+p^n) \hookrightarrow \mathfrak{o}_L^\times/(1+\mathfrak{p}^n) \), and defining \( \chi_0 \) suitably on prime elements. We extend \( W^{(0)}(\cdot) \) to a function on \( M^{(2)}(F) \) via
\[
W^{(0)}(ag) = \chi_0(a)W^{(0)}(g), \quad a \in L^\times, \ g \in GL_2(F)
\] (see \( 39 \)). It follows from \( 39 \) that
\[
W^{(0)}(g\kappa) = W^{(0)}(g), \quad \text{for } g \in M^{(2)}(F) \text{ and } \kappa \in M^{(2)}(F) \cap K^\#(\mathfrak{p}^n).
\]
As in Sect. 3.2 let \((\pi, V_\pi)\) be an unramified, irreducible, admissible representation of \( GSp_4(F) \), where \( V_\pi \) is the Bessel model for \( \pi \) with respect to the character \( \Lambda \otimes \theta \) of \( R(F) = T(F)U(F) \). As was pointed out in Sect. 3.2 the character \( \Lambda \) is necessarily unramified. Let \( \chi \) be the character of \( L^\times \) given by
\[
\chi(\zeta) = \Lambda(\zeta)^{-1}\chi_0(\zeta)^{-1},
\]
so that \( 18 \) holds.

Given a complex number \( s \), there exists a unique function \( W^\#(\cdot, s) : G(F) \to \mathbb{C} \) with the following properties.

i) If \( g \notin M(F)N(F)K^\#(\mathfrak{p}^n) \), then \( W^\#(g, s) = 0 \).

ii) If \( g = mnk \) with \( m \in M(F) \), \( n \in N(F) \), \( k \in K^\#(\mathfrak{p}^n) \), then \( W^\#(g, s) = W^\#(m, s) \).

iii) For \( \zeta \in L^\times \) and \( \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in M^{(2)}(F) \),
\[
W^\# \left( \begin{bmatrix} \zeta & 1 \\ \bar{\zeta}^{-1} & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & \beta \\ \alpha & \gamma \end{bmatrix}, s \right) = |N(\zeta) \cdot \mu^{-1}|^{3(s+1/2)} \chi(\zeta) W^{(0)} \left( \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right).
\]
Here \( \mu = \bar{\alpha} \delta - \beta \gamma \).

To verify that such a function exists, use \( 42 \) and
\[
(M(F)N(F)) \cap K^\#(\mathfrak{p}^n) = (M(F) \cap K^\#(\mathfrak{p}^n)) (N(F) \cap K^\#(\mathfrak{p}^n)).
\]
Also, one has to use the fact that \( \chi|_{1+\mathfrak{p}^n} = 1 \). Note that \( W^\#(\cdot, s) \) is an element of the induced representation \( I(s, \chi, \chi_0, \tau) \) discussed in Section 2.3 In particular, Lemma 2.3.1 applies. Note that if \( n = 0 \), i.e., if \( \tau \) is unramified, then \( W^\#(\cdot, s) \) coincides with the function \( W_\psi(\cdot, s) \) defined on p. 200 of \( 9 \).
3.5 Basic local integral computation

Let $W^\#(\cdot, s)$ be the element of $I(s, \chi, \chi_0, \tau)$ defined in the previous section. Let $B$ be the spherical vector in the $\Lambda \otimes \theta$ Bessel model of the unramified representation $\pi$ of $\mathrm{GSp}_4(F)$, as in Sect. 3.2. We shall compute the integral

$$Z(s) = \int_{R(F) \backslash H(F)} W^\#(\eta h, s) B(h) \, dh.$$  \hfill (45)

By Lemma 2.3.1, the integral (45) is well-defined. By (29) and the fact that $B(h(l, m)) = 0$ for $l < 0$ ([9, Lemma 3.4.4]), we have

$$Z(s) = \sum_{l,m \geq 0} \int_{R(F) \backslash R(F) h(l,m) K^H} W^\#(\eta h, s) B(h) \, dh$$

$$= \sum_{l,m \geq 0} \int_{h(l,m)^{-1} R(F) h(l,m) \cap K^H \backslash K^H} W^\#(\eta h(l, m) h, s) B(h(l, m) h) \, dh$$

$$= \sum_{l,m \geq 0} B(h(l, m)) \int_{h(l,m)^{-1} R(F) h(l,m) \cap K^H \backslash K^H} W^\#(\eta h(l, m) h, s) \, dh. \hfill (46)$$

The function $W^\#$ is only invariant under $K^\#(\mathfrak{p}^n)$. Since our integral (46) is over elements of $H(F)$, all that is relevant is that $W^\#$ is invariant under the group $K^\#(\mathfrak{p}^n)$ defined in (37). Let us abbreviate $K_{l,m} := h(l, m)^{-1} R(F) h(l, m) \cap K^H$. Suppose we had a system of representatives $\{s_i\}$ for the double coset space $K_{l,m} \backslash K^H / K^\#(\mathfrak{p}^n)$ (it will depend on $l$ and $m$, of course). Then, from (46),

$$Z(s) = \sum_{l,m \geq 0} \sum_i B(h(l, m)) \int_{K_{l,m} \backslash K_{l,m} s_i K^\#(\mathfrak{p}^n)} W^\#(\eta h(l, m) s_i, s) \, dh$$

$$= \sum_{l,m \geq 0} \sum_i B(h(l, m)) W^\#(\eta h(l, m) s_i) \int_{K_{l,m} \backslash K_{l,m} s_i K^\#(\mathfrak{p}^n)} dh. \hfill (47)$$

In practice it will be difficult to obtain the system $\{s_i\}$. However, we can save some work by exploiting the fact that $W^\#$ is supported on the small subset $M(F)N(F)K^\#(\mathfrak{p}^n)$ of $G(F)$. Hence, we shall proceed as follows.

**Step 1:** First we determine a preliminary decomposition

$$K^H = \bigcup_j K_{l,m} s'_{j} K^\#(\mathfrak{p}^n), \hfill (48)$$

which is not necessarily disjoint. We may assume that the $s'_{j}$ are taken from the system of representatives for $K^H / K^\#(\mathfrak{p}^n)$ to be determined in the next section (but some of these will be absorbed in $K_{l,m}$, so that we get an initial reduction).

**Step 2:** Then we consider the values $W^\#(\eta h(l, m) s_i', s)$. If $\eta h(l, m) s_i' \notin M(F)N(F)K^\#(\mathfrak{p}^n)$, then $s_i'$ makes no contribution to the integral (46). Therefore, all that is relevant is the subset $\{s_i''\} \subset \{s_i'\}$ of representatives for which $\eta h(l, m) s_i'' \in M(F)N(F)K^\#(\mathfrak{p}^n)$. Hence we consider the set

$$S := \bigcup_j K_{l,m} s''_{j} K^\#(\mathfrak{p}^n).$$

**Step 3:** Now, from this much smaller set of representatives $\{s''_j\}$ we determine a subset $\{s'''_j\}$ such that this union becomes disjoint:

$$S = \bigcup_j K_{l,m} s'''_{j} K^\#(\mathfrak{p}^n).$$
From this point on we will assume that the conductor \( n \) of the given \( \text{GL}(2) \) representation \( \tau \) satisfies \( n = 1 \).

Finally, we have to compute the volumes, evaluate \( W^\# \), and carry out the summations with the help of Sugano’s formula (30).

### 3.6 The cosets \( K^\#(p^0)/K^\#(p) \)

From this point on we will assume that the conductor \( p^0 \) of the given \( \text{GL}(2) \) representation \( \tau \) satisfies \( n = 1 \).

We need to determine representatives for the coset space

\[
K^\#(p^0)/K^\#(p), \quad \text{where } K^\#(p^0) = K^H = GSp_4(\mathfrak{o}).
\]

Note that this coset space is isomorphic to \( K^\#_1(p^0)/K^\#_1(p) \), where \( K^\#_1(p) = K^\#(p) \cap \{ g \in H(F) : \mu(g) = 1 \} \).

Let

\[
s_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.
\]

It follows from the Bruhat decomposition for \( \text{Sp}_4(\mathfrak{o}/p) \) that

\[
K^\#(p^0) = \bigsqcup_{a_1, a_2 \in \mathfrak{o}^\times/(1+p)} K^\#(p) / \text{span}(a_1, a_2, a_1^{-1}, a_2^{-1})
\]

\[
\begin{bmatrix} a_1 & a_2 \\ a_1^{-1} & a_2^{-1} \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
K^\#(p^0) = \bigsqcup_{a_1, a_2 \in \mathfrak{o}^\times/(1+p)} \text{span}(a_1, a_2, a_1^{-1}, a_2^{-1}) \cdot \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}
\]
3.7 Double coset decomposition

Recall that we are interested in the double cosets \( K_{l,m} \backslash K^H / K^#(p) \), where \( K_{l,m} = h(l, m)^{-1} R(F) h(l, m) \cap K^H \).

3.7.1 Step 1: Preliminary decomposition

Observe that \( K_{l,m} \) contains all elements

\[
\begin{bmatrix}
    u \\
    u \\
    u \\
    u
\end{bmatrix}, \quad u \in o^\times, \quad \text{and} \quad \begin{bmatrix}
    1 & 0 & 0 \\
    1 & 0 & 0 \\
    1 & 0 & 0 \\
    1 & 1 & 1
\end{bmatrix},
\]

and that \( K^#(p) \) contains all elements of the form diag(1, \( \mu, 1 \), \( \mu \in o^\times \)). From (52) – (59) we therefore obtain the following preliminary decomposition, which is not disjoint.

\[
K^H = \bigcup_{u \in o^\times / (1+p)} K_{l,m} \begin{bmatrix} 1 & u & 1 \\ u & 1 & u^{-1} \end{bmatrix} K^#(p)
\] (60)

\[
\bigcup_{u \in o^\times / (1+p)} K_{l,m} \begin{bmatrix} 1 & u & 1 \\ u & 1 & u^{-1} \end{bmatrix} \begin{bmatrix} 1 & x & y \\ w & w & w \\ 1 & 1 & 1 \end{bmatrix} s_1 K^#(p)
\] (61)

\[
\bigcup_{u \in o^\times / (1+p)} K_{l,m} \begin{bmatrix} 1 & u & 1 \\ u & 1 & u^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\] (62)

\[
\bigcup_{u \in o^\times / (1+p)} K_{l,m} \begin{bmatrix} 1 & u & 1 \\ u & 1 & u^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ w & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} s_1 s_2 K^#(p)
\] (63)

\[
\bigcup_{u \in o^\times / (1+p)} K_{l,m} \begin{bmatrix} 1 & u & 1 \\ u & 1 & u^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} s_2 s_1 K^#(p)
\] (64)

\[
\bigcup_{u \in o^\times / (1+p)} K_{l,m} \begin{bmatrix} 1 & u & 1 \\ u & 1 & u^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ w & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} s_1 s_2 s_1 K^#(p)
\] (65)

\[
\bigcup_{u \in o^\times / (1+p)} K_{l,m} \begin{bmatrix} 1 & u & 1 \\ u & 1 & u^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} s_2 s_1 K^#(p)
\] (66)
Recall
\[ \eta = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \\ 1 & -\bar{\alpha} \\ 0 & 1 \end{bmatrix}, \] where \( \alpha := \begin{cases} \frac{b + \sqrt{d}}{2c} & \text{if } L \text{ is a field,} \\ \left(\frac{b + \sqrt{d}}{2c}, \frac{b - \sqrt{d}}{2c}\right) & \text{if } L = F \oplus F. \end{cases} \]

We have assumed that \( c \in \mathfrak{o}^\times \), so that \( \alpha \in \mathfrak{o}_L \). We have \( \eta h(l, m) = h(l, m) \eta_m \) where for \( m \geq 0 \) we define
\[ \eta_m = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \\ \bar{\alpha} & \bar{\alpha} \end{bmatrix}. \] (68)

Fix \( l, m \geq 0 \), and let \( r \) run through the representatives for \( K_{l,m} \setminus K^H / K^\#(\mathfrak{p}) \) from (60) – (67). We want to find out for which \( r \) is \( \eta h(l, m) r \in M(F)N(F)K^\#(\mathfrak{p}) \), since this set is the support of \( W^\# \). Since \( h(l, m) \in M(F) \), this is equivalent to \( \eta_m r \in M(F)N(F)K^\#(\mathfrak{p}) \). Hence, this condition depends only on \( m \geq 0 \) and not on the integer \( l \). Recall that
\[ K^\#(\mathfrak{p}) = \begin{bmatrix} 1 + \mathfrak{p} & \mathfrak{p} & \mathfrak{o}_L & \mathfrak{o}_L \\ \mathfrak{o}_L & 1 + \mathfrak{p} & \mathfrak{o}_L & \mathfrak{o}_L \\ \mathfrak{p} & \mathfrak{p} & 1 + \mathfrak{p} & \mathfrak{o}_L \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & 1 + \mathfrak{p} \end{bmatrix} \begin{bmatrix} 1 & \mu \\ \mu & 1 \end{bmatrix}, \quad \mu \in \mathfrak{o}^\times. \]

i) Let \( r = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \\ 1 & -\bar{\alpha} \\ 0 & 1 \end{bmatrix} \) with \( u \in \mathfrak{o}^\times / (1 + \mathfrak{p}) \). Then \( \eta_m r \in M(F)N(F)K^\#(\mathfrak{p}) \) for all \( u, m \). More precisely,
\[ \eta_m r = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \\ 1 & -\bar{\alpha} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \\ 1 & -\bar{\alpha} \\ 0 & 1 \end{bmatrix} \in M(F)K^\#(\mathfrak{p}). \] (69)

ii) Let \( r = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \\ 1 & -\bar{\alpha} \\ 0 & 1 \end{bmatrix} \) with \( u \in \mathfrak{o}^\times / (1 + \mathfrak{p}) \) and \( w \in \mathfrak{o} / \mathfrak{p} \). If \( \beta = \bar{\alpha} + uw \in \mathfrak{o}_L^\times \), then
\[ \eta_m r = \begin{bmatrix} -\beta^{-1} u \\ \beta \\ -\bar{\beta} u^{-1} \\ \bar{\beta} u^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\beta u^{-1} \\ -\beta u^{-1} & 1 \\ \beta u^{-1} & 1 \end{bmatrix} \times \begin{bmatrix} 1 & u \beta^{-1} \\ 1 & 1 - u \beta^{-1} \end{bmatrix} \in M(F)N(F)K^\#(\mathfrak{p}). \] (70)
But if $\beta \notin \mathfrak{o}_L^*$, then $\eta_{m,r} \notin M(F)N(F)K^\#(\mathfrak{p})$ since the $(3,3)$-coefficient of any matrix product of the form $\tilde{n}^{-1}\tilde{m}^{-1}\eta_{m,r}$, $\tilde{m} \in M(F)$, $\tilde{n} \in N(F)$, is always in $\beta\mathfrak{o}_L^*$.

iii) Let $r = \begin{bmatrix} 1 & u \\ u & 1 \\ u^{-1} \end{bmatrix}$ with $u \in \mathfrak{o}^\times/(1+p)$. Then $\eta_{m,r} \notin M(F)N(F)K^\#(\mathfrak{p})$, since the $(3,3)$-coefficient of any matrix product of the form $\tilde{n}^{-1}\tilde{m}^{-1}\eta_{m,r}$, $\tilde{m} \in M(F)$, $\tilde{n} \in N(F)$, is always zero.

iv) Let $r = \begin{bmatrix} 1 & u \\ u & 1 \\ u^{-1} \end{bmatrix} \begin{bmatrix} 1 & w \\ w & 1 \\ -w & 1 \end{bmatrix}$ with $u \in \mathfrak{o}^\times/(1+p)$ and $w \in \mathfrak{o}/p$. Then $\eta_{m,r} \notin M(F)N(F)K^\#(\mathfrak{p})$, since the $(3,3)$-coefficient of any matrix product of the form $\tilde{n}^{-1}\tilde{m}^{-1}\eta_{m,r}$, $\tilde{m} \in M(F)$, $\tilde{n} \in N(F)$, is always zero.

v) Let $r = \begin{bmatrix} 1 & u \\ u & 1 \\ u^{-1} \end{bmatrix} s_2 s_1$ with $u \in \mathfrak{o}^\times/(1+p)$. Then $\eta_{m,r} \notin M(F)N(F)K^\#(\mathfrak{p})$, since the $(3,3)$-coefficient of any matrix product of the form $\tilde{n}^{-1}\tilde{m}^{-1}\eta_{m,r}$, $\tilde{m} \in M(F)$, $\tilde{n} \in N(F)$, is $\varpi^m\tilde{a}u$ times the $(3,2)$-coefficient.

vi) Let $r = \begin{bmatrix} 1 & u \\ u & 1 \\ u^{-1} \end{bmatrix} \begin{bmatrix} 1 & w \\ w & 1 \\ -w & 1 \end{bmatrix}$ with $u \in \mathfrak{o}^\times/(1+p)$ and $w \in \mathfrak{o}/p$. If $\beta = \varpi^m\alpha+uw \in \mathfrak{p}$, then

$$\eta_{m,r} = \begin{bmatrix} 1 & u \\ -1/u & 1 \\ \beta u^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \beta u^{-1} & 1 \end{bmatrix} \in M(F)K^\#(\mathfrak{p}).$$

(71)

But if $\beta \notin \mathfrak{p}$, then $\eta_{m,r} \notin M(F)N(F)K^\#(\mathfrak{p})$ since the $(3,2)$-coefficient of any matrix product of the form $\tilde{n}^{-1}\tilde{m}^{-1}\eta_{m,r}$, $\tilde{m} \in M(F)$, $\tilde{n} \in N(F)$, is always in $\beta\mathfrak{o}_L^*$.

vii) Let $r = \begin{bmatrix} 1 & u \\ u & 1 \\ u^{-1} \end{bmatrix} s_2 s_1 s_2$ with $u \in \mathfrak{o}^\times/(1+p)$. Then $\eta_{m,r} \notin M(F)N(F)K^\#(\mathfrak{p})$, since the $(3,3)$-coefficient of any matrix product of the form $\tilde{n}^{-1}\tilde{m}^{-1}\eta_{m,r}$, $\tilde{m} \in M(F)$, $\tilde{n} \in N(F)$, is always zero.

viii) Let $r = \begin{bmatrix} 1 & u \\ u & 1 \\ u^{-1} \end{bmatrix} \begin{bmatrix} 1 & w \\ w & 1 \\ -w & 1 \end{bmatrix}$ with $u \in \mathfrak{o}^\times/(1+p)$ and $w \in \mathfrak{o}/p$. Then $\eta_{m,r} \notin M(F)N(F)K^\#(\mathfrak{p})$, since the $(3,3)$-coefficient of any matrix product of the form $\tilde{n}^{-1}\tilde{m}^{-1}\eta_{m,r}$, $\tilde{m} \in M(F)$, $\tilde{n} \in N(F)$, is always zero.

Hence, for every $l \geq 0$ and $m \geq 0$, the double cosets that contribute to the computation of the integral (40) are

$$\bigcup_{u \in \mathfrak{o}^\times/(1+p)} K_{l,m} \begin{bmatrix} 1 & u \\ u & 1 \\ u^{-1} \end{bmatrix} K^\#(\mathfrak{p})$$

(72)
We will now investigate possible overlaps between the double cosets given in (75) and (76) (for $3.7.3$ Step 

The case $m = m > 0$. For

By ii) of Lemma 3.1.1, the condition $\varphi^m \alpha + uw \in \mathcal{P}$ cannot be satisfied if $m = 0$. Hence, for $m = 0$ the double cosets that contribute to the computation of the integral (46) are

$$S = \bigcup_{u \in \mathfrak{o}^\times/(1+p)} K_{l,0} \begin{bmatrix} 1 & u \\ 1 & u^{-1} \end{bmatrix} K^\#(p)$$

(75) 

For $m > 0$ the condition $\varphi^m \alpha + uw \in \mathfrak{o}^\times$ (resp. $\varphi^m \alpha + uw \in \mathfrak{p}$) is satisfied if and only if $w \in \mathfrak{o}^\times$ (resp. $w \in \mathfrak{p}$). Hence, for $m > 0$ the double cosets that contribute to the computation of the integral (46) are

$$S = \bigcup_{u \in \mathfrak{o}^\times/(1+p)} K_{l,m} \begin{bmatrix} 1 & u \\ 1 & u^{-1} \end{bmatrix} K^\#(p)$$

(77) 

$$S = \bigcup_{u \in \mathfrak{o}^\times/(1+p)} K_{l,m} \begin{bmatrix} 1 & u \\ 1 & u^{-1} \end{bmatrix} s_1 K^\#(p)$$

(78) 

$$S = \bigcup_{u \in \mathfrak{o}^\times/(1+p)} K_{l,m} \begin{bmatrix} 1 & u \\ 1 & u^{-1} \end{bmatrix} s_1 s_2 s_1 K^\#(p).$$

(79) 

3.7.3 Step 3: Disjointness of double cosets

We will now investigate possible overlaps between the double cosets given in (75) and (76) (for $m = 0$) resp.

(77), (78) and (79) (for $m > 0$). Recall that $K_{l,m} = h(l,m)^{-1}R(F)h(l,m) \cap H$.

The case $m = 0$

We will now assume $m = 0$ and find all equivalences between the double cosets in (75) and (76). Let a double coset from (76) be given. Set $v = a + b(uw) + c(uw)^2$. We claim that the condition $\alpha + uw \in \mathfrak{o}^\times_L$ in (76) forces $v \in \mathfrak{o}^\times$. First observe that we have the following identity,

$$a + b(uw) + c(uw)^2 = -c(\alpha + uw)(\alpha - (uw + bc^{-1})).$$

Hence, if $v \in \mathfrak{p}$, then it would follow that $\alpha - (uw + bc^{-1}) \in \mathfrak{p} \mathfrak{o} L = \mathfrak{P}$. By Lemma 3.1.1 ii), this is impossible.

It follows that indeed $v \in \mathfrak{o}^\times$. Now let $y = -u/v$ and $x = -(u/v)(cw + b/2)$. Let $g = \begin{bmatrix} x + yb/2 & yc \\ -ya & x - yb/2 \end{bmatrix}$. 

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Then

\[ h(l, 0)^{-1} \begin{bmatrix} g \\ \det(g)^{-1} \end{bmatrix} h(l, 0) = \begin{bmatrix} 1 & u \\ -u & 1 \end{bmatrix} \begin{bmatrix} 1 & -u \\ -u & 1 \end{bmatrix} = \begin{bmatrix} 1 & u \\ -u & 1 \\ w & 1 \\ w & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{u(b+c+u)}{v} \\ \frac{u^2}{v} & \frac{u(b+c+u)}{v} \end{bmatrix} \begin{bmatrix} 1 & w(b+c+u) \\ \frac{u}{v} & \frac{u(b+c+u)}{v} \end{bmatrix}. \] (80)

Since \( v \in \sigma \), the rightmost matrix is in \( K^\#(p) \). This identity shows that all double cosets in (78) are equivalent to double cosets in (75). So far we have shown that the set \( S \) in (75) and (76) reduces to

\[ S = \bigcup_{u \in \sigma^*/(1+p)} K_{l,0} \begin{bmatrix} 1 & u \\ 1 & u^{-1} \end{bmatrix} K^\#(p). \]

We will now show that this is a disjoint union. Let

\[ h_1 = \begin{bmatrix} 1 & u_1 \\ 1 & u_1^{-1} \end{bmatrix}, \quad h_2 = \begin{bmatrix} 1 & u_2 \\ 1 & u_2^{-1} \end{bmatrix}, \]

and assume that \( K_{l,0} h_1 K^\#(p) = K_{l,0} h_2 K^\#(p) \). Then there exists \( r \in R(F) \) such that

\[ A = h_2^{-1} h(l, 0)^{-1} rh(l, 0) h_1 \in K^\#(p). \]

Dividing the \((1,1)\) coefficient of \( A \) by the \((4,4)\) coefficient, we get \( u_1/u_2 \in 1 + p \). Hence \( h_1 \) and \( h_2 \) define the same double coset if and only if \( u_1 = u_2 \). It follows that

\[ S = \bigcup_{u \in \sigma^*/(1+p)} K_{l,0} \begin{bmatrix} 1 & u \\ 1 & u^{-1} \end{bmatrix} K^\#(p), \]

as claimed.

**The case \( m > 0 \)**

We will now assume \( m > 0 \) and find all equivalences between the double cosets in (77), (78) and (79). An argument similar to the one above shows that all the double cosets in (77) are disjoint.

**Equivalence of double cosets from (77) and (78):** Let a double coset from (78) be given. Let \( x = \frac{b(m-1)}{2w^2 - u} \) and \( y = -\frac{m}{cw^2 - u} \). Let \( g = \begin{bmatrix} x + yb/2 & yc \\ -ya & x - yb/2 \end{bmatrix} \). Then

\[ h(l, m)^{-1} \begin{bmatrix} g \\ \det(g)^{-1} \end{bmatrix} h(l, m) = \begin{bmatrix} 1 & u \\ -u & 1 \end{bmatrix} \begin{bmatrix} 1 & -u \\ -u & 1 \end{bmatrix} \begin{bmatrix} 1 & w \\ -w & 1 \end{bmatrix} \begin{bmatrix} 1 + \frac{m}{cw^2 - u} \\ \frac{m}{cw^2 - u} \end{bmatrix} \begin{bmatrix} 1 + \frac{m}{cw^2 - u} \\ \frac{m}{cw^2 - u} \end{bmatrix}. \] (81)
Equivalence amongst double coset from (79):

We will show that the double cosets in (79) are disjoint.

In the notations above we have suppressed the dependence on independent of

2 compact subgroups of GL(82), (83) are of the form

We will first show that the calculation reduces to the calculation of volumes of certain compact subgroups of GL2(F). First we make some general remarks that apply to both cases. The volumes (82), (83) are of the form

\[ \int_{K_{i,m} \setminus K_{i,m}A K^#(p)} dh, \]

The rightmost matrix lies in \( K^#(p) \) since \( w \in o^\times \). Hence cosets of (78) all coincide with cosets from (77).

Equivalence of double cosets from (77) and (79): Let \( h_1 \) be a double coset representative obtained in (77) and let \( h_2 \) be a double coset representative obtained in (79). Then the double cosets are not equivalent, since, for every \( k \in K_{i,m} \), the (2,2) entry of the matrix \( A = h_2^{-1}k h_1 \) is zero.

Equivalence amongst double coset from (79): We will show that the double cosets in (79) are disjoint.

Let

\[ h_1 = \begin{bmatrix} 1 & u_1 & s_1 s_2 s_1 \end{bmatrix}, \quad h_2 = \begin{bmatrix} 1 & u_2 & s_1 s_2 s_1 \end{bmatrix}. \]

Assume that \( K_{i,m} h_1 K^#(p) = K_{i,m} h_2 K^#(p) \). Then there exists an \( r \in R(F) \) such that

\[ A = h_2^{-1} h(l,m)^{-1} rh(l,m) h_1 \in K^#(p). \]

Let \( r = gY \), with \( g \in T(F) \) and \( Y \in U(F) \). We write an element of \( T(F) \) as \( g = \begin{bmatrix} x + \frac{yb}{2} & yc \\ -ya & x - \frac{yb}{2} \end{bmatrix} \)

with \( x, y \in F \). Looking at the (2,3) coefficient of \( A \), we see that \( y \in p \) (since \( m > 0 \)). The \( (1,1) \) coefficient gives us that \( x + \frac{yb}{2} \in 1 + p \) and hence \( x - \frac{yb}{2} \in 1 + p \). Looking at the (4,4) coefficient, we get \( u_1 / u_2 \in 1 + p \), which implies \( u_1 = u_2 \). Hence, the double cosets in (79) are disjoint.

We summarize the results of this section in the following lemma.

3.7.1 Lemma. The following are the disjoint double cosets in \( \{ K_{i,m} k K^#(p) : k \in K^H, l, m \geq 0 \} \) that have a non-trivial intersection with the support of \( W^\# \):

\[ \bigcup_{l \geq 0} \bigcup_{m \geq 0} \bigcup_{u \in o^\times / (1 + p)} K_{i,m} \begin{bmatrix} 1 & u & s_1 s_2 s_1 \end{bmatrix} K^#(p) \]

3.8 Volume computations

In Lemma 3.7.1 we obtained the double coset representatives \( \{ s_j'' \} \) needed to evaluate the integral (49). In this section we will compute the corresponding volumes. More precisely, we have to compute

\[ V_1^{l,m} := \int_{K_{i,m} \setminus K_{i,m} K^#(p)} \begin{bmatrix} 1 & u & s_1 s_2 s_1 \end{bmatrix} dh \quad \text{for } l \geq 0, \ m \geq 0, \quad (82) \]

\[ V_2^{l,m} := \int_{K_{i,m} \setminus K_{i,m} K^#(p)} \begin{bmatrix} 1 & u & s_1 s_2 s_1 \end{bmatrix} dh \quad \text{for } l \geq 0, \ m \geq 0. \quad (83) \]

In the notations above we have suppressed the dependence on \( u \) since it will turn out that these volumes are independent of \( u \). We will first show that the calculation reduces to the calculation of volumes of certain compact subgroups of \( GL_2(F) \). First we make some general remarks that apply to both cases. The volumes (82), (83) are of the form

\[ \int_{K_{i,m} \setminus K_{i,m} A K^#(p)} dh, \]
with some \( A \in K^H \). Let \( \chi_1 : K_{l,m} \backslash K^H \to \mathbb{C} \) be the characteristic function of \( K_{l,m} \backslash K_{l,m} A^\#(p) \), and let \( \delta_1 : K^H \to \mathbb{C} \) be the characteristic function of \( A^\#(p) \).

### 3.8.1 Lemma

For all \( g \in K^H \) we have

\[
\int_{K_{l,m}} \delta_1(tg) \, dt = \chi_1(g) \int_{K_{l,m} \cap (A^\#(p) A^{-1})} \, dt,
\]

where \( g \) denotes the image of \( g \) in \( K_{l,m} \backslash K^H \).

**Proof.** First assume that \( g \notin K_{l,m} A^\#(p) \). Then \( tg \notin A^\#(p) \) for all \( t \in K_{l,m} \), and hence the left side is zero. The right side is also zero by definition of \( \chi_1 \). Thus the equality holds under our assumption. Now assume that \( g \in K_{l,m} A^\#(p) \). In this case \( \chi_1(g) = 1 \). Write \( g = kA\kappa \) with \( k \in K_{l,m} \) and \( \kappa \in A^\#(p) \). We have

\[
tg \in A^\#(p) \iff tkA\kappa \in A^\#(p) \iff tkA \in A^\#(p) \iff tk \in (A^\#(p) A^{-1})A^{-1} \iff t \in (A^\#(p) A^{-1})A^{-1}.
\]

Hence the left side equals

\[
\int_{K_{l,m} \cap (A^\#(p) A^{-1})} \, dt.
\]

But since \( k \in K_{l,m} \), this integral equals \( \int_{K_{l,m} \cap (A^\#(p) A^{-1})} \, dt \). This proves the lemma. \( \blacksquare \)

Integrating both sides of (84) over \( K_{l,m} \backslash K^H \), we obtain

\[
\int_{K^H} \delta_1(g) \, dg = \left( \int_{K_{l,m} \backslash K_{l,m} A^\#(p)} dh \right) \left( \int_{K_{l,m} \cap (A^\#(p) A^{-1})} \, dt \right),
\]

so that

\[
\int_{K_{l,m} \backslash K_{l,m} A^\#(p)} dh = \text{vol}(A^\#(p)) \left( \int_{K_{l,m} \cap (A^\#(p) A^{-1})} \, dt \right)^{-1}.
\]

Note that

\[
\text{vol}(A^\#(p)) = \frac{1}{(q - 1)^2(1 + 2q + 2q^2 + 2q^3 + q^4)} = \frac{1}{(q^2 - 1)(q^4 - 1)}
\]

from (52) - (59) and the fact that \( \text{vol}(K^H) = 1 \). Hence we are reduced to computing

\[
V(l, m, A) := \int_{K_{l,m} \cap (A^\#(p) A^{-1})} \, dt.
\]

In both the volumes (52), (53) we have \( A = B w \), where \( w \) is a Weyl group element and

\[
B = \begin{bmatrix}
1 & u \\
1 & u^{-1}
\end{bmatrix}.
\]

We have

\[
V(l, m, A) = \int_{(B^{-1}K_{l,m}B) \cap (wA^\#(p) w^{-1})} dt = \int_{(B^{-1}h(l, m) A^{-1}R(F) h(l, m) B) \cap (wA^\#(p) w^{-1})} dt.
\]
The relevant Weyl group elements are \( w = 1 \) and \( w = s_1 s_2 s_1 \). We have

\[
K^\#(p) = \begin{bmatrix} 1 + p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ p & 0 & 0 \\ p & 1 + p \end{bmatrix},
\]

(90)

\[
s_1 s_2 s_1 K^\#(p) s_1 s_2 s_1 = \begin{bmatrix} 1 + p & 0 & 0 \\ 0 & 1 + p & 0 \\ p & 0 & 0 \\ p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

(91)

We have to find the intersections of these compact groups with \( B^{-1} h(l, m)^{-1} R(F) h(l, m) B \). Note that, mod \( p \), the upper left block of any matrix in any of these compact subgroups, lies in \( \text{GL}_2(\mathfrak{o}) \). Let \( L_{K,w} \) be the subgroup of \( \text{GL}_2(\mathfrak{o}) \) occurring in the upper left block of \( w K^\# (p) w^{-1} \), and let \( N_{K,w} \) be the compact subgroup of \( F^3 \) occurring in the upper right block. Write a given element of \( R(F) \) as \( t n \) with \( t \in T(F) \) and \( n \in U(F) \). Then

\[
B^{-1} h(l, m)^{-1} (t n) h(l, m) B = (B^{-1} h(l, m)^{-1} t h(l, m) B) (B^{-1} h(l, m)^{-1} n h(l, m) B).
\]

A direct computation shows that this element lies in \( w K^\#(p) w^{-1} \) if and only if

the upper left block of \( B^{-1} h(l, m)^{-1} t h(l, m) B \) lies in \( L_{K,w} \)

(92)

and

the upper right block of \( B^{-1} h(l, m)^{-1} n h(l, m) B \) lies in \( N_{K,w} \).

(93)

We have \( B = \begin{bmatrix} g & t \\ g^{-1} \end{bmatrix} \) with \( g = \begin{bmatrix} 1 \\ u \end{bmatrix} \in \text{GL}_2(\mathfrak{o}) \). The conditions (92) and (93) become

\[
\begin{bmatrix} 1 \\ u^{-1} \end{bmatrix} [\begin{bmatrix} \omega^{-m} & 1 \\ 1 & 1 \end{bmatrix}] [\begin{bmatrix} 1 \\ u \end{bmatrix}] \in L_{K,w}
\]

(94)

and

\[
\begin{bmatrix} 1 \\ u^{-1} \end{bmatrix} [\begin{bmatrix} \omega^{-2m-l} & \omega^{-m-l} \\ \omega^{-m-l} & \omega^{-m} \end{bmatrix}] [\begin{bmatrix} 1 \\ u^{-1} \end{bmatrix}] \in N_{K,w}, \quad \text{where } n = \begin{bmatrix} 1 \\ X \\ 1 \end{bmatrix}.
\]

(95)

It follows that

\[
\text{vol} \left( \{ X \in F^3 : \begin{bmatrix} 1 \\ u^{-1} \end{bmatrix} [\begin{bmatrix} \omega^{-2m-l} & \omega^{-m-l} \\ \omega^{-m-l} & \omega^{-m} \end{bmatrix}] [\begin{bmatrix} 1 \\ u^{-1} \end{bmatrix}] \in N_{K,w} \} \right)
\]

\[
= \text{vol} \left( \{ X \in F^3 : X \in [\begin{bmatrix} \omega^{2m+l} & \omega^{m+l} \\ \omega^{m+l} & \omega^{m} \end{bmatrix}] [\begin{bmatrix} 1 \\ u \end{bmatrix}] N_{K,w} [\begin{bmatrix} 1 \\ u \end{bmatrix}] [\begin{bmatrix} 1 \\ \omega^{-m} \end{bmatrix}] \} \right)
\]

\[
= \text{vol} \left( \begin{bmatrix} \omega^{m+l} & \omega^{m+l} \\ \omega^{m+l} & \omega^{m} \end{bmatrix} N_{K,w} [\begin{bmatrix} 1 \\ \omega^{-m} \end{bmatrix}] \right)
\]

\[
= \text{vol} \left( \begin{bmatrix} \omega^{2m+l} & \omega^{m+l} \\ \omega^{m+l} & \omega^{m} \end{bmatrix} N_{K,w} [\begin{bmatrix} 1 \\ \omega^{-m} \end{bmatrix}] \right)
\]

\[
= q^{-3m-u} \text{vol}(N_{K,w}).
\]

The volume of \( N_{K,w} \) is 1 if \( w = 1 \) and \( q^{-2} \) if \( w = s_1 s_2 s_1 \). Let us set

\[
L_{K,w} = \begin{cases} \begin{bmatrix} 1 + p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ p & 0 & 0 \\ p & 1 + p \end{bmatrix} & \text{if } w = 1, \\
\begin{bmatrix} 1 + p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ p & 0 & 0 \\ p & 1 + p \end{bmatrix} & \text{if } w = s_1 s_2 s_1,
\end{cases}
\]

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as above. Let

\[ T_{m,w} = \{ t \in T(F) : \begin{bmatrix} 1 & -m \\ u & 1 \end{bmatrix} [\omega^{m}]^{1} [\omega^{-m}]^{1} [1]_{u} \in L_{K,w} \} \]

\[ = T(F) \cap \begin{bmatrix} \omega^{m} \\ 1 \end{bmatrix} [\omega^{-m}]^{1} [1]_{u} \begin{bmatrix} 1 \\ u \end{bmatrix} L_{K,w} [\omega^{-m}]^{1} \]

\[ = T(F) \cap \begin{bmatrix} \omega^{m} \\ 1 \end{bmatrix} L_{K,w} [\omega^{-m}]^{1}. \]

(96)

We summarize the above considerations.

**3.8.2 Lemma.** Let \( l \) and \( m \) be non-negative integers. Let \( w \in \{1,s,s_{1},s_{2}\} \), and set \( \delta = 0 \) if \( w = 1 \), and \( \delta = 2 \) if \( w = s_{1}s_{2}s_{1} \). Let \( A = \text{diag}(1,u,1,u^{-1})w \), where \( u \in \mathfrak{o}^{\times} \). Then

\[ \int_{K_{l,m} \setminus K_{l,m} \text{AwK}(p)} dh = \text{vol}(K^{\#}(p)) \text{vol}(T_{m,w})^{-1} q^{3(m+l)+\delta}. \]

Here, \( \text{vol}(K^{\#}(p)) = (q^{2} - 1)(q^{4} - 1)^{-1} \).

Thus we are reduced to computing the volumes of the groups \( T_{m,1} \) for all \( m \geq 0 \) and \( T_{m,s_{1}s_{2}s_{1}} \) for all \( m > 0 \).

**3.8.3 Lemma.** For any \( m \geq 0 \) we have

\[ \text{vol}(T_{m,1})^{-1} = (q - 1) \left( 1 - \left( \frac{L}{p} \right) q^{-1} \right) q^{m+1}. \]

**Proof.** By definition,

\[ T_{m,1} = T(F) \cap \begin{bmatrix} \omega^{m} \\ 1 \end{bmatrix} [\omega^{-m}]^{1} [1 + p^{1} p^{1}] [\omega^{-m}]^{1} \]

Since

\[ \begin{bmatrix} \mathfrak{o}^{\times} & p \\ \mathfrak{o}^{\times} & \mathfrak{o}^{\times} \end{bmatrix} = \bigcup_{u \in \mathfrak{o}^{\times}/(1+p)} \begin{bmatrix} u \\ 1 + p^{1} p^{1} \end{bmatrix} \]

and \( \mathfrak{u} \in T(F) \), we have

\[ \int_{T(F) \cap \begin{bmatrix} \omega^{m} \\ 1 \end{bmatrix} [\omega^{-m}]^{1} [\omega^{-m}]^{1} \} \sum_{u \in \mathfrak{o}^{\times}/(1+p)} ( \int_{T(F) \cap \begin{bmatrix} \omega^{m} \\ 1 \end{bmatrix} [\omega^{-m}]^{1} [\omega^{-m}]^{1} \}) dt = \sum_{u \in \mathfrak{o}^{\times}/(1+p)} ( \int_{T(F) \cap \begin{bmatrix} \omega^{m} \\ 1 \end{bmatrix} [\omega^{-m}]^{1} [\omega^{-m}]^{1} \}) dt \]

\[ = (q - 1) \left( \int_{T(F) \cap \begin{bmatrix} \omega^{m} \\ 1 \end{bmatrix} [\omega^{-m}]^{1} [\omega^{-m}]^{1} \} dt \right). \]

Therefore,

\[ ( \int_{T(F) \cap \begin{bmatrix} \omega^{m} \\ 1 \end{bmatrix} [\omega^{-m}]^{1} [\omega^{-m}]^{1} \})^{-1} = (q - 1) \left( \int_{T(F) \cap \begin{bmatrix} \omega^{m} \\ 1 \end{bmatrix} [\omega^{-m}]^{1} [\omega^{-m}]^{1} \} \right)^{-1}. \]
Note that the group \(T(F) \cap \begin{bmatrix} \varpi^m \\ 1 \end{bmatrix} \text{GL}_2(\mathfrak{o}) \begin{bmatrix} \varpi^{-m} \\ 1 \end{bmatrix}\) lies in \(\mathfrak{o}_L^\times\), since the determinants of these matrices lie in \(\mathfrak{o}^\times\) and the trace lies in \(\mathfrak{o}\). As in \([9]\), p. 202, we define a subring \(\mathfrak{o}_m\) of \(\mathfrak{o}_L\) by

\[
\mathfrak{o}_m := \mathfrak{o}_L \cap \begin{bmatrix} \varpi^m \\ 1 \end{bmatrix} M_2(\mathfrak{o}) \begin{bmatrix} \varpi^{-m} \\ 1 \end{bmatrix}
\]

In addition, we define a smaller subring

\[
\mathfrak{o}'_m := \mathfrak{o}_L \cap \begin{bmatrix} \varpi^m \\ 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o} p \\ \mathfrak{o} \end{bmatrix} \begin{bmatrix} \varpi^{-m} \\ 1 \end{bmatrix}.
\]

We normalize the measure so that \(\text{vol}(\mathfrak{o}_L^\times) = 1\). Hence, we have

\[
\left( \int_{T(F) \cap \begin{bmatrix} \varpi^m \\ 1 \end{bmatrix} \mathfrak{o}^\times \begin{bmatrix} p \\ \mathfrak{o} \end{bmatrix} \begin{bmatrix} \varpi^{-m} \\ 1 \end{bmatrix} } dt \right)^{-1} = (\mathfrak{o}_L^\times : (\mathfrak{o}_m^\times)).
\]

From \([20]\), we have the integral basis

\[
\mathfrak{o}_L = \mathfrak{o} + \mathfrak{o} \xi_0 = \left\{ \begin{bmatrix} x \\ -ya \\ x - yb \end{bmatrix} \middle| x, y \in \mathfrak{o} \right\}, \quad \xi_0 = \begin{bmatrix} 0 & c \\ -a & -b \end{bmatrix}.
\]

Such an element lies in \(\begin{bmatrix} \varpi^m \\ 1 \end{bmatrix} M_2(\mathfrak{o}) \begin{bmatrix} \varpi^{-m} \\ 1 \end{bmatrix}\) if and only if \(y \in \mathfrak{p}^m\). Similarly, such an element lies in \(\begin{bmatrix} \varpi^m \\ 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o} p \\ \mathfrak{o} \end{bmatrix} \begin{bmatrix} \varpi^{-m} \\ 1 \end{bmatrix}\) if and only if \(y \in \mathfrak{p}^{m+1}\). Therefore,

\[
\mathfrak{o}_m = \left\{ x + \varpi^m y \xi_0 : x, y \in \mathfrak{o} \right\}
\]

and

\[
\mathfrak{o}'_m = \left\{ x + \varpi^{m+1} y \xi_0 : x, y \in \mathfrak{o} \right\}.
\]

Hence \(\mathfrak{o}'_m = \mathfrak{o}_{m+1}\), so that \((\mathfrak{o}_L^\times : (\mathfrak{o}'_m^\times)) = (\mathfrak{o}_L^\times : (\mathfrak{o}_{m+1}^\times))\). By Lemma (3.5.3) of \([9]\),

\[
(\mathfrak{o}_L^\times : (\mathfrak{o}'_m^\times)) = \left( 1 - \left( \frac{L}{p} \right) q^{-1} \right) q^{m+1}.
\]

This concludes the proof.

\[3.8.4 \text{Lemma.} \] For any \(m > 0\) we have

\[
\text{vol}(T_{m,s_1 s_2 s_1})^{-1} = (q - 1) \left( 1 - \left( \frac{L}{p} \right) q^{-1} \right) q^m.
\]

\[\text{Proof.} \] By definition

\[
T_{m,s_1 s_2 s_1} = T(F) \cap \begin{bmatrix} \varpi^m \\ 1 \end{bmatrix} \begin{bmatrix} 1 + \mathfrak{o} \\ \mathfrak{o} p 1 + \mathfrak{o} \end{bmatrix} \begin{bmatrix} \varpi^{-m} \\ 1 \end{bmatrix}.
\]

We claim that

\[
T_{m,s_1 s_2 s_1} = T(F) \cap \begin{bmatrix} \varpi^m \\ 1 \end{bmatrix} \begin{bmatrix} 1 + \mathfrak{o} \\ \mathfrak{o} p \end{bmatrix} \begin{bmatrix} \varpi^{-m} \\ 1 \end{bmatrix}.
\]

Indeed, assume that

\[
\begin{bmatrix} x + yb/2 \\ -ya \\ x - yb/2 \end{bmatrix} \in \begin{bmatrix} \varpi^m \\ 1 \end{bmatrix} \begin{bmatrix} 1 + \mathfrak{o} \\ \mathfrak{o} p \end{bmatrix} \begin{bmatrix} \varpi^{-m} \\ 1 \end{bmatrix}.
\]
Then in particular \( yc \in p^m \subset p \), since \( m > 0 \). Since \( c \) is a unit, we get \( y \in p \). Thus, \( x + yb/2 \in 1 + p \) implies \( x - yb/2 \in 1 + p \), as claimed. Since

\[
\begin{bmatrix}
o^x o \\
p^e o^e
\end{bmatrix} = \bigcup_{u \in o^x/(1+p)} \begin{bmatrix} u \\ 1+p o \\
p^e o^e
\end{bmatrix}
\]

and \( \begin{bmatrix} u \\ u \end{bmatrix} \in T(F) \), we have

\[
\int_{T(F) \cap [\varpi^{-m} 1]} \begin{bmatrix} o^x o \\
p^e o^e
\end{bmatrix} \left( \int_{T(F) \cap [\varpi^{-m} 1]} \begin{bmatrix} u \\ 1+p o \\
p^e o^e
\end{bmatrix} \right) dt
\]

\[
= \sum_{u \in o^x/(1+p)} \left( \int_{T(F) \cap [\varpi^{-m} 1]} \begin{bmatrix} u \\ 1+p o \end{bmatrix} \right) \left( \int_{T(F) \cap [\varpi^{-m} 1]} \begin{bmatrix} o^x o \\
p^e o^e
\end{bmatrix} \right) dt
\]

\[
= (q-1) \int_{T(F) \cap [\varpi^{-m} 1]} \begin{bmatrix} 1+p o \end{bmatrix} \left( \int_{T(F) \cap [\varpi^{-m} 1]} \begin{bmatrix} o^x o \\
p^e o^e
\end{bmatrix} \right) dt.
\]

Therefore,

\[
\left( \int_{T(F) \cap [\varpi^{-m} 1]} \begin{bmatrix} o^x o \\
p^e o^e
\end{bmatrix} \right)^{-1} = (q-1) \left( \int_{T(F) \cap [\varpi^{-m} 1]} \begin{bmatrix} o^x o \\
p^e o^e
\end{bmatrix} \right)^{-1}.
\]

Let \( o_m \) be the subring of \( o_L \) as defined in the proof of Lemma 3.8.3. In addition, we define another subring

\[
o_m'' := o_L \cap \begin{bmatrix} o \\
1 \\
o \\
0 \\
0 \\
\end{bmatrix} \begin{bmatrix} o^x o \\
p^e o^e
\end{bmatrix} \begin{bmatrix} o \\
1 \\
0 \\
0 \\
0 \\
\end{bmatrix}.
\]

Since \( \text{vol}(o_L^x) = 1 \), we have

\[
\left( \int_{T(F) \cap [\varpi^{-m} 1]} \begin{bmatrix} o^x o \\
p^e o^e
\end{bmatrix} \right)^{-1} = (o_L^x : (o_m'')^x).
\]

As above we have the integral basis \( o_L = o + o\xi_0 = \left\{ \begin{bmatrix} x \\
yc \\
x \\
y \\
\end{bmatrix} : x, y \in o \right\} \). Such an element lies in \([\varpi^{-m} 1] M_2(o) \begin{bmatrix} o \\
1 \\
o \\
0 \\
0 \\
\end{bmatrix}\) if and only if \( y \in p^m \). Similarly, such an element lies in \([\varpi^{-m} 1] \begin{bmatrix} o \\
1 \\
o \\
0 \\
0 \\
\end{bmatrix}\) if and only if \( y \in p^m \). Therefore,

\[
o_m = \left\{ x + \varpi^m y \xi_0 : x, y \in o \right\}
\]

and

\[
o_m'' = \left\{ x + \varpi^m y \xi_0 : x, y \in o \right\},
\]

so that actually \( o_m = o_m'' \). Hence \( (o_L^x : (o_m'')^x) = (o_L^x : (o_m)^x) \). By Lemma (3.5.3) of [9],

\[
(o_L^x : (o_m)^x) = 1 - \left( \frac{L}{p} \right)^{-1} q^m.
\]

This concludes the proof.

Let us summarize the volume computations.
3.8.5 Lemma. Let $V_{1}^{l,m}$ and $V_{2}^{l,m}$ be the volumes defined in (58), (59).

i) For any $l, m \geq 0$ we have

$$V_{1}^{l,m} = \frac{1}{(q+1)(q^l-1)} \left[ 1 - \left( \frac{L}{p} \right) q^{-1} \right] q^{4m + 3l + 1}.$$ 

ii) For any $l \geq 0$ and $m > 0$ we have

$$V_{2}^{l,m} = \frac{1}{(q+1)(q^l-1)} \left[ 1 - \left( \frac{L}{p} \right) q^{-1} \right] q^{4m + 3l + 2}.$$ 

Proof. This follows from Lemmas 3.8.2, 3.8.3 and 3.8.4.

3.9 Main local theorem

In Sections 3.3 and 3.5, we have defined the functions $W^\#$ and the integral $Z$ for any ramified representation $\tau$ of $GL_2(F)$. We have computed the relevant double cosets and their corresponding volumes under the assumption that $\tau$ has conductor $p$. We will now assume that $\tau = \Omega St_{GL(2)}$, where $\Omega$ is an unramified character of $F^\times$, and $St_{GL(2)}$ is the Steinberg representation of $GL(2, F)$. Then $\tau$ has conductor $p$, and the central character of $\tau$ is $\omega_\tau = \Omega^2$. We work in the $\psi^{-c}$ Whittaker model for $\tau$. In this model, the newform $W^{(0)}$ has the properties

$$W^{(0)} \begin{pmatrix} a & \omega \cr 1 & 1 \end{pmatrix} = \begin{cases} |a| \Omega(a) & \text{if } a \in \mathfrak{o}, \\
0 & \text{otherwise}, \end{cases}$$

and

$$W^{(0)}(g \begin{pmatrix} \varpi & 1 \\
0 & 1 \end{pmatrix}) = -\Omega(\varpi)W^{(0)}(g) \quad \text{for all } g \in GL_2(F).$$

We refer to [25] for details. Using Lemma 3.8.1, we have

$$Z(s) = \sum_{l \geq 0} \sum_{m > 0} B(h(l,m)) \sum_{u \in \mathfrak{o}^\times/(1+p)} \left( W^\#(\eta h(l,m)) \begin{bmatrix} 1 & u \\
1 & u^{-1} \end{bmatrix}, s \right) V_{1}^{l,m}$$

$$+ W^\#(\eta h(l,m)) \begin{bmatrix} 1 & u \\
1 & u^{-1} \end{bmatrix} s_1 s_2 s_1, s \right) V_{2}^{l,m}$$

$$+ \sum_{l \geq 0} B(h(l,0)) \sum_{u \in \mathfrak{o}^\times/(1+p)} W^\#(\eta h(l,0)) \begin{bmatrix} 1 & u \\
1 & u^{-1} \end{bmatrix}, s \right) V_{1}^{l,0}. \quad (99)$$

Recall formula (44) for the function $W^\#(\cdot, s)$. Note that for $\zeta \in F^\times$ we have $\chi(\zeta) = \omega_\tau(\zeta)^{-1} \omega_\tau(\zeta)^{-1} \Omega(\zeta)^{-1}$. Using (69) and (71), we write the argument of $W^\#$ as an element of $M(F)N(F)K^\#(\mathfrak{q})$. Then, from (44),

$$Z(s) = \sum_{l \geq 0} \sum_{m > 0} B(h(l,m)) \sum_{u \in \mathfrak{o}^\times/(1+p)} \begin{bmatrix} \varpi^{m+l} \zeta^{(s+\frac{1}{2})} \omega_\tau(\varpi^{m+l})^{-1} \Omega(\varpi^{m+l})^{-2} \Omega^2(\varpi^{m}) \\
1 \end{bmatrix} V_{1}^{l,m}$$

$$\times \left( W^{(0)}(\begin{pmatrix} \varpi^l & 1 \\
0 & 1 \end{pmatrix}) V_{1}^{l,m} + W^{(0)}(\begin{pmatrix} \varpi^l \\
0 & 1 \end{pmatrix}) V_{2}^{l,m} \right)$$

$$+ \sum_{l \geq 0} B(h(l,0)) \sum_{u \in \mathfrak{o}^\times/(1+p)} \begin{bmatrix} \varpi^l \zeta^{(s+\frac{1}{2})} \omega_\tau(\varpi^l)^{-1} \Omega(\varpi^l)^{-2} W^{(0)}(\begin{pmatrix} \varpi^l \\
0 & 1 \end{pmatrix}) V_{1}^{l,0}. \quad (100)$$

26
Here, we have used the fact that $\Omega^2$ is the central character of $\tau$ and $W(\Omega)$ is right invariant under $\begin{bmatrix} o & x \\ p & o \end{bmatrix}$.

It follows from (97) and (98) that $W(\Omega)\left(\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right) = -\Omega(\varpi^l)|\varpi|^{l+1}$ for all $l \geq 0$. Hence, we get

$$Z(s) = (q-1) \sum_{l \geq 0} \sum_{m>0} B(h(l,m))|\varpi|^{2m+l}|\varpi|^{3(s+\frac{1}{2})}\omega_{\pi}(\varpi)^{-2}$$

$$\times \left(|\varpi|^l \Omega(\varpi) \nu_1^{l,m} - |\varpi|^{l+1} \Omega(\varpi) \nu_1^{l,m} \right)$$

$$+ (q-1) \sum_{l \geq 0} B(h(l,0))|\varpi|^{3(s+\frac{1}{2})} \omega_{\pi}(\varpi)^{-2} |\varpi|^l \nu_1^{l,0}$$

$$= (q-1) \sum_{l \geq 0} B(h(l,m))|\varpi|^{3(s+\frac{1}{2})} (\omega_{\pi}(\varpi))^{-2}$$

$$\times (|\varpi|^l \nu_1^{l,m} - q^{-1} |\varpi|^{l,0}).$$

(101)

By Lemma (5.5.3 (i) and ii) the first sum is zero. Hence

$$Z(s) = (q-1) \sum_{l \geq 0} B(h(l,0))q^{-(3s+5/2)} (\omega_{\pi}(\varpi))^{-1} \nu_1^{l,0}$$

$$= \frac{q-1}{(q+1)(q^4-1)} \sum_{l \geq 0} B(h(l,0))q^{-(3s+5/2)} (\omega_{\pi}(\varpi))^{-1}$$

$$\times \left(1 - \left(\frac{L}{p}\right) q^{-1}\right) q^{3l+1}$$

$$= \frac{q-1}{(q+1)(q^4-1)} \sum_{l \geq 0} B(h(l,0))q^{-(3s+1/2)} (\omega_{\pi}(\varpi))^{-1}.$$  

(102)

Let $\pi = \chi_1 \times \chi_2 \times \sigma$ be an unramified principal series representation of $GSp_4(F)$; in case $\chi_1 \times \chi_2 \times \sigma$ is not irreducible, take its unramified constituent. Recall the characters $\gamma^{(1)}, \ldots, \gamma^{(4)}$ defined in Sect. 3.2. Let $\nu$ be the absolute value in $F$ normalized by $\nu(\varpi) = q^{-1}$. Set

$$L(s, \tilde{\pi} \times \tilde{\tau}) = \prod_{i=1}^{4} \left(1 - \left((\gamma^{(i)})^{-1} \Omega^{-1/2} \nu^{1/2} \nu_F q^{-s}\right)^{-1}\right).$$  

(103)

Then $L(s, \tilde{\pi} \times \tilde{\tau})$ is the standard $L$-factor attached to the representation $\tilde{\pi} \times \tilde{\tau}$ of $GSp_4(F) \times GL_2(F)$ by the local Langlands correspondence. Here, $\tilde{\pi}$ (resp. $\tilde{\tau}$) denotes the contragredient representation of $\pi$ (resp. $\tau$). Denote by $A(T)(s)$ the irreducible, admissible representation of $GL_2(F)$ obtained by automorphic induction from the character $\Lambda$ of $L^\times$. Set

$$L(s, \tau \times A(T)(s) \times \chi|_{F^\times}) = \begin{cases} (1 - \chi(\varpi)^{-1} \nu^{-2s})^{-1}, & \text{if } (\frac{L}{p}) = -1, \\
(1 - \Lambda(\varpi_L)(\chi\Omega)(\varpi)^{1/2} \nu^{-3s-1})^{-1}, & \text{if } (\frac{L}{p}) = 0, \\
(1 - \Lambda(\varpi_L)(\chi\Omega)(\varpi)^{1/2} \nu^{-3s-1})^{-1}, & \text{if } (\frac{L}{p}) = 1. \\
\end{cases}$$

(104)

Then $L(s, \tau \times A(T)(s) \times \chi|_{F^\times})$ is the standard $L$-factor attached to the representation $\tau \times A(T)(s) \times \chi|_{F^\times}$ of $GL_2(F) \times GL_2(F) \times GL_1(F)$ by the local Langlands correspondence. We now state the main theorem of the local non-archimedean theory.

3.9.1 Theorem. Let $\pi$ be an unramified, irreducible, admissible representation of $GSp_4(F)$ (not necessarily with trivial central character), and let $\tau = \Omega St_{GL(2)}$ with an unramified character $\Omega$ of $F^\times$. Let $Z(s)$ be the integral (39), where $W^\#$ is the function defined in Sect. 3.4 and $B$ is the spherical Bessel function defined in Sect. 3.2. Then

$$Z(s) = \frac{q(q-1)}{(q+1)(q^4-1)} \left(1 - \left(\frac{L}{p}\right) q^{-1}\right) \frac{L(3s + \frac{1}{2}, \tilde{\pi} \times \tilde{\tau})}{L(3s + 1, \tau \times A(T)(s) \times \chi|_{F^\times})}. $$

(105)
Proof. By \(31\) and \(32\),

\[
Z(s) = \frac{q(q - 1)}{(q + 1)(q^2 - 1)} \left( 1 \left( \frac{L}{p} \right) q^{-1} \right) H(q^{-3s+1/2}(\omega \pi)(\omega F)^{-1}) \frac{Q(q^{-3s+1/2}(\omega \pi)(\omega F)^{-1})}{Q(q^{-3s+1/2}(\omega \pi)(\omega F)^{-1})}.
\]

By \(31\),

\[
Q(q^{-3s+1/2}(\omega \pi)(\omega F)^{-1})) = \prod_{i=1}^{4} (1 - \gamma^{(i)}(\omega F)q^{-3s-\frac{1}{2}}(\omega \pi)(\omega F)^{-1})
\]

= \[4 \prod_{i=1}^{4} (1 - (\gamma^{(i)}(\omega \pi)q^{-1} \nu^{1/2})(\omega F)q^{-3s-1/2})\]

= \[4 \prod_{i=1}^{4} (1 - ((\gamma^{(i)}q^{-1} \nu^{1/2})(\omega F)q^{-3s-1/2})\]

To compute the numerator of \(106\), we distinguish cases. If \((\frac{L}{p}) = -1\), then \(H(y) = 1 - q^{-4} \Lambda(\omega F)y^2\), and hence

\[
H(q^{-3s+1/2}(\omega \pi)(\omega F)^{-1}) = 1 - q^{-4} \Lambda(\omega F)(q^{-3s+1/2}(\omega \pi)(\omega F)^{-1})^2
\]

= \[1 - (\Lambda \omega^{-2} \Omega^{-2})(\omega F)q^{-6s-3}\]

= \[1 - (\omega^{-1} \omega^{-1})(\omega F)q^{-6s-3}\]

= \[1 - \chi(\omega F)q^{-1}q^{-6s-2}\]

\[= L(3s + 1, \tau \times \mathcal{A} \mathcal{T} \Lambda \times \chi | F^\times)^{-1}.\]

If \((\frac{L}{p}) = 0\), then \(H(y) = 1 - q^{-2} \Lambda(\omega F)y\), and hence

\[
H(q^{-3s+1/2}(\omega \pi)(\omega F)^{-1}) = 1 - q^{-2} \Lambda(\omega F)q^{-3s+1/2}(\omega \pi)(\omega F)^{-1}
\]

= \[1 - \Lambda(\omega F)(\omega \pi)(\omega F)^{-1}q^{-3s-3/2}\]

= \[1 - \Lambda(\omega F)(\omega \pi)(\omega F)^{-1}q^{-1}q^{-3s-1}\]

\[= L(3s + 1, \tau \times \mathcal{A} \mathcal{T} \Lambda \times \chi | F^\times)^{-1}.\]

If \((\frac{L}{p}) = 1\), then \(H(y) = (1 - q^{-2} \Lambda(\omega F)y)(1 - q^{-2} \Lambda(\omega F \omega^{-1})y)\), and hence

\[
H(q^{-3s+1/2}(\omega \pi)(\omega F)^{-1}) = (1 - q^{-2} \Lambda(\omega F)(\omega \pi)(\omega F)^{-1})
\]

\[
(1 - q^{-2} \Lambda(\omega F \omega_{\omega}^{-1})(\omega F)^{-1}q^{-3s+1/2}(\omega \pi)(\omega F)^{-1})
\]

= \[1 - \Lambda(\omega F)(\omega \pi)(\omega F)^{-1}(\omega \pi)(\omega F)^{-1}q^{-3s-3/2}\]

= \[1 - \Lambda(\omega F)(\omega \pi)(\omega F)^{-1}(\omega \pi)(\omega F)^{-1}q^{-3s-3/2}\]

= \[1 - \Lambda(\omega F)(\omega \pi)(\omega F)^{-1}(\omega \pi)(\omega F)^{-1}q^{-3s-3/2}\]

\[= L(3s + 1, \tau \times \mathcal{A} \mathcal{T} \Lambda \times \chi | F^\times)^{-1}.\]

Hence \(H(q^{-3s+1/2}(\omega \pi)(\omega F)^{-1}) = L(3s + 1, \tau \times \mathcal{A} \mathcal{T} \Lambda \times \chi | F^\times)^{-1}\) in all cases. This concludes the proof of the theorem.

\[ \square \]

4 Local archimedean theory

In this section we compute the local archimedean integral. As in Sect. 3 the key step is the choice of vectors \(B\) and \(W^\#\).
Consider the symmetric domains \( \mathbb{H}_2 := \{ Z \in M_2(\mathbb{C}) : \ i(Z - Z)^* \text{ is positive definite} \} \) and \( \mathfrak{h}_2 := \{ Z \in \mathbb{H}_2 : t(Z - Z) = 0 \} \). The group \( G^+(\mathbb{R}) := \{ g \in G(\mathbb{R}) : \mu_2(g) > 0 \} \) acts on \( \mathbb{H}_2 \) via \((g, Z) \mapsto g(Z)\), where

\[
g(Z) = (AZ + B)(CZ + D)^{-1}, \quad g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G^+(\mathbb{R}), \ Z \in \mathbb{H}_2.
\]

Under this action, \( \mathfrak{h}_2 \) is stable by \( H^+(\mathbb{R}) = \text{GSp}^+_2(\mathbb{R}) \). The group \( K_\infty = \{ g \in G^+(\mathbb{R}) : \mu_2(g) = 1, g(I) = I \} \) is a maximal compact subgroup of \( G^+(\mathbb{R}) \). Here, \( I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{H}_2 \). Explicitly,

\[
K_\infty = \{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} : A, B \in \text{GL}(2, \mathbb{C}), \ tAB = tBA, \ tAA + tBB = 1 \}.
\]

By the Iwasawa decomposition

\[
G(\mathbb{R}) = M^{(1)}(\mathbb{R})M^{(2)}(\mathbb{R})N(\mathbb{R})K_\infty,
\]

where \( M^{(1)}(\mathbb{R}), M^{(2)}(\mathbb{R}) \) and \( N(\mathbb{R}) \) are as defined in [19], [27], [6]. A calculation shows that

\[
M^{(1)}(\mathbb{R})M^{(2)}(\mathbb{R})N(\mathbb{R}) \cap K_\infty
= \{ \begin{bmatrix} \zeta & \alpha \\ -\beta & \zeta \end{bmatrix} : \zeta, \alpha, \beta \in \mathbb{C}, |\zeta| = 1, |\alpha|^2 + |\beta|^2 = 1, \alpha \beta = \beta \alpha \}.
\]

Note also that

\[
M^{(2)}(\mathbb{R}) \cap K_\infty = \left\{ \begin{bmatrix} 1 & \alpha \\ -\beta & 1 \end{bmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1, \alpha \beta = \beta \alpha \right\},
\]

and that there is an isomorphism

\[
(S^1 \times \text{SO}(2))/\{(\lambda, \begin{bmatrix} \lambda & \lambda \\ \lambda & \lambda \end{bmatrix}) : \lambda = \pm 1 \} \xrightarrow{\sim} M^{(2)}(\mathbb{R}) \cap K_\infty,
\]

\[
(\lambda, \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}) \mapsto \begin{bmatrix} 1 & \lambda \alpha \\ -\lambda \beta & 1 \end{bmatrix}.
\]

For \( g \in G^+(\mathbb{R}) \) and \( Z \in \mathbb{H}_2 \), let \( J(g, Z) = CZ + D \) be the automorphy factor. Then, for any integer \( l \), the map

\[
k \mapsto \det(J(k, I))^l
\]

defines a character \( K_\infty \to \mathbb{C}^\times \). If \( k \in M^{(2)}(\mathbb{R}) \cap K_\infty \) is written in the form \( \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \), then \( \det(J(k, I))^l = \lambda^l e^{-il\theta} \), where \( \alpha = \cos(\theta) \), \( \beta = \sin(\theta) \). Let \( K^H_\infty = K_\infty \cap H^+(\mathbb{R}) \). Then \( K^H_\infty \) is a maximal compact subgroup, explicitly given by

\[
K^H_\infty = \{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} : tAB = tBA, \ tAA + tBB = 1 \}.
\]

Sending \( \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \) to \( A - iB \) gives an isomorphism \( K^H_\infty \cong \text{U}(2) \). Recall that we have chosen \( a, b, c \in \mathbb{R} \) such that \( d = b^2 - 4ac \neq 0 \). In the archimedean case we shall assume that \( d < 0 \) and let \( D = -d \). Then \( \mathbb{R}(\sqrt{-D}) = \mathbb{C} \). As in Sect. 2.2 we have

\[
T(\mathbb{R}) = \left\{ \begin{bmatrix} x + yb/2 & yc \\ -ya & x - yb/2 \end{bmatrix} : x, y \in \mathbb{R}, x^2 + y^2D/4 > 0 \right\}.
\]
Let
\[ T^1_\infty = T(\mathbb{R}) \cap SL(2, \mathbb{R}) = \left\{ \begin{bmatrix} x + yb/2 & yc \\ -ya & x - yb/2 \end{bmatrix} : x, y \in \mathbb{R}, x^2 + y^2D/4 = 1 \right\}. \tag{113} \]
We have \( T(\mathbb{R}) \cong \mathbb{C}^\times \) via \( \begin{bmatrix} x + yb/2 \\ -ya \\ x - yb/2 \end{bmatrix} \mapsto x + y\sqrt{-D}/2. \) Under this isomorphism \( T^1_\infty \) corresponds to the unit circle. We have
\[ T(\mathbb{R}) = T^1_\infty \cdot \left\{ \begin{bmatrix} \zeta \\ \zeta \end{bmatrix} : \zeta > 0 \right\}. \tag{114} \]
As in [9], p. 211, let \( t_0 \in GL_2(\mathbb{R})^+ \) be such that \( T^1_\infty = t_0SO(2)t_0^{-1}. \) We will make a specific choice of \( t_0 \) when we choose the matrix \( S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \) below. By the Cartan decomposition,
\[ GL_2^+(\mathbb{R}) = SO(2) \cdot \left\{ \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} : \zeta_1, \zeta_2 > 0, \zeta_1 \geq \zeta_2 \right\} \cdot SO(2). \tag{115} \]
Therefore,
\[ GL_2^+(\mathbb{R}) = t_0SO(2) \cdot \left\{ \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} : \zeta_1, \zeta_2 > 0, \zeta_1 \geq \zeta_2 \right\} \cdot SO(2) \]
\[ = T^1_\infty t_0 \cdot \left\{ \begin{bmatrix} \sqrt{\zeta_1 \zeta_2} \\ \sqrt{\zeta_1} \zeta_2 \end{bmatrix} \begin{bmatrix} \sqrt{\zeta_1} \zeta_2 \\ \sqrt{\zeta_2} \zeta_1 \end{bmatrix} : \zeta_1, \zeta_2 > 0, \zeta_1 \geq \zeta_2 \right\} \cdot SO(2) \]
\[ = T(\mathbb{R})t_0 \cdot \left\{ \begin{bmatrix} \zeta \\ \zeta^{-1} \end{bmatrix} : \zeta \geq 1 \right\} \cdot SO(2). \tag{116} \]

Using this, it is not hard to see that
\[ H(\mathbb{R}) = R(\mathbb{R}) \cdot \left\{ \begin{bmatrix} \lambda_0 \zeta \\ \zeta^{-1} \end{bmatrix} \right\}^{t_0^{-1}} : \lambda \in \mathbb{R}^\times, \zeta \geq 1 \right\} \cdot K^H_\infty. \tag{117} \]

Here, \( R(\mathbb{R}) = T(\mathbb{R})U(\mathbb{R}) \) is the Bessel subgroup defined in Sect. 2.2. One can check that all the double cosets in (117) are disjoint.

### 4.2 The Bessel function

Recall that we have chosen three elements \( a, b, c \in \mathbb{R} \) such that \( d = b^2 - 4ac \neq 0. \) We will now make the stronger assumption that \( S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \in M_2(\mathbb{R}) \) is a positive definite matrix. Set \( D = 4ac - b^2 > 0, \) as above. Given a positive integer \( l \geq 2, \) consider the function \( B : H(\mathbb{R}) \to \mathbb{C} \) defined by
\[ B(h) := \left\{ \begin{array}{ll} \mu_2(h)^l \det(J(h, I))^{-l} e^{-2\pi i \text{tr}(Sk^H)} & \text{if } h \in H^+(\mathbb{R}), \\
0 & \text{if } h \notin H^+(\mathbb{R}), \end{array} \right. \tag{118} \]
where \( I = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}. \) Note that the function \( B \) only depends on the choice of \( S \) and \( l. \) Recall the character \( \theta \) of \( U(\mathbb{R}) \) defined in (12). It depends on the choice of additive character \( \psi, \) and throughout we choose \( \psi(x) = e^{-2\pi ix}. \) Then the function \( B \) satisfies
\[ B(tuh) = \theta(u)B(h) \quad \text{for } h \in H(\mathbb{R}), t \in T(\mathbb{R}), u \in U(\mathbb{R}), \tag{119} \]
and
\[ B(hk) = \det(J(k, I))^l B(h) \quad \text{for } h \in H(\mathbb{R}), k \in K^H_\infty. \tag{120} \]
Property (119) means that $B$ satisfies the Bessel transformation property with the character $\Lambda \otimes \theta$ of $R(\mathbb{R})$, where $\Lambda$ is trivial. In fact, by the considerations in [28] 1-3, $B$ is the lowest weight vector in a holomorphic discrete series representation of $\text{PGSp}(4, \mathbb{R})$ corresponding to Siegel modular forms of degree 2 and weight $l$. By (119) and (120), the function $B$ is determined by its values on a set of representatives for $R(\mathbb{R}) \setminus H(\mathbb{R})/K^H$. Such a set is given in (117).

### 4.3 The function $W^#$

Let $(\tau, V_\tau)$ be a generic, irreducible, admissible representation of $\text{GL}_2(\mathbb{R})$ with central character $\omega_\tau$. We assume that $V_\tau = W(\tau, \psi(-\cdot))$ is the Whittaker model of $\tau$ with respect to the non-trivial additive character $x \mapsto \psi(-cx)$. Note that $S$ positive definite implies $c > 0$. Let $W^{(0)} \in V_\tau$ have weight $l_1$. Then $W^{(0)}$ has the following properties.

i) $$W^{(0)}(gr(\theta)) = e^{it_1 \theta} W^{(0)}(g) \quad \text{for } g \in \text{GL}_2(\mathbb{R}), \ r(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \in \text{SO}(2).$$

ii) $$W^{(0)}(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}) = \psi(-cx) W^{(0)}(g) \quad \text{for } g \in \text{GL}_2(\mathbb{R}), \ x \in \mathbb{R}.$$ 

Let $\chi_0$ be the character of $\mathbb{C}^\times$ with the properties

$$\chi_0|_{\mathbb{R}^\times} = \omega_\tau, \quad \chi_0(\zeta) = \zeta^{-l_1} \quad \text{for } \zeta \in \mathbb{C}^\times, |\zeta| = 1. \quad (121)$$

Such a character exists since $\omega_\tau(-1) = (-1)^{l_1}$. We extend $W^{(0)}$ to a function on $M^{(2)}(\mathbb{R})$ via

$$W^{(0)}(\zeta g) = \chi_0(\zeta) W^{(0)}(g), \quad \zeta \in \mathbb{C}^\times, \ g \in \text{GL}_2(\mathbb{R}) \quad (122)$$

(see Lemma 2.1.1). Then it is easy to check that

$$W^{(0)}(gk) = \det(J(k, I))^{-l_1} W^{(0)}(g) \quad \text{for } g \in M^{(2)}(\mathbb{R}) \text{ and } k \in M^{(2)}(\mathbb{R}) \cap K_\infty. \quad (123)$$

We will need values of $W^{(0)}$ evaluated at $\begin{bmatrix} t \\ 1 \end{bmatrix}$ for $t \neq 0$. For this we look at the Lie algebra $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{R})$ and consider the elements

$$R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

In the universal enveloping algebra $U(\mathfrak{g})$ let

$$\Delta = \frac{1}{4}(H^2 + 2RL + 2LR). \quad (124)$$

Then $\Delta$ lies in the center of $U(\mathfrak{g})$ and acts on $V_\tau$ by a scalar, which we write in the form $-(\frac{1}{4} + (\frac{r}{2})^2)$ with $r \in \mathbb{C}$. In particular,

$$\Delta W^{(0)} = -\left(\frac{1}{4} + (\frac{r}{2})^2\right) W^{(0)}. \quad (125)$$

If one restricts the function $W^{(0)}$ to $\begin{bmatrix} t^{1/2} \\ t^{-1/2} \end{bmatrix}$, $t > 0$, then (125) reduces to the differential equation satisfied by the classical Whittaker functions. Hence, there exist constants $a^+, a^- \in \mathbb{C}$ such that

$$W^{(0)}(\begin{bmatrix} t \\ 0 \end{bmatrix}) = \begin{cases} a^+ \omega_\tau((4\pi ct)^{1/2}) W^{(4\pi ct)}_{-\frac{1}{4}, \frac{1}{4}} & \text{if } t > 0, \\
\quad a^- \omega_\tau((-4\pi ct)^{1/2}) W^{(-4\pi ct)}_{-\frac{1}{4}, \frac{1}{4}} & \text{if } t < 0. \end{cases} \quad (126)$$
Here, $W_{\pm \frac{1}{2} \frac{1}{2}}$ denotes a classical Whittaker function; see [9] p. 244, [17]. Let $\chi$ be the character of $\mathbb{C}^\times$ given by

$$\chi(\zeta) = \chi_0(\zeta)^{-1}. \quad (127)$$

We interpret $\chi$ as a character of $M^{(1)}(\mathbb{R})$; see [3]. Given a complex number $s$, we define a function $W^\#(\cdot, s) : G(\mathbb{R}) \to \mathbb{C}$ as follows. Given $g \in G(\mathbb{R})$, write $g = m_1 m_2 n k$ according to (107) Then set

$$W^\#(g, s) = \delta^{s+1/2}(m_1 m_2) \det(J(k, I))^{-l_1} \chi(m_1) W^0(m_2). \quad (128)$$

Property (123) shows that this is well-defined. Explicitly, for $\zeta \in \mathbb{C}^\times$ and $[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}] \in M^{(2)}(\mathbb{R})$,

$$W^\#(\begin{bmatrix} \zeta & 1 \\ 1 & \bar{\zeta}^{-1} \end{bmatrix} \begin{bmatrix} 1 & \beta \\ \alpha & \gamma \end{bmatrix}, s) = ||\zeta|^2 \cdot \mu^{-1}||^{3(s+1/2)} \chi(\zeta) W^0(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}). \quad (129)$$

Here, $\mu = \bar{\alpha} \delta - \beta \bar{\gamma}$. It is clear that $W^\#(\cdot, s)$ satisfies

$$W^\#(gk, s) = \det(J(k, I))^{-l_1} W^\#(g, s) \quad \text{for } g \in G(\mathbb{R}), \ k \in K_\infty. \quad (130)$$

By Lemma [23.1] we have

$$W^\#(\eta tuh, s) = \theta(u)^{-1} W^\#(\eta h, s) \quad (131)$$

for $t \in T(\mathbb{R})$, $u \in U(\mathbb{R})$, $h \in G(\mathbb{R})$ and

$$\eta = \begin{bmatrix} 1 & 1 \\ \alpha & -\bar{\alpha} \\ 1 & 1 \end{bmatrix}, \quad \alpha = \frac{b + \sqrt{d}}{2c}, \ d = b^2 - 4ac. \quad (132)$$

### 4.4 The local archimedean integral

Let $B$ and $W^\#$ be as defined in Sections [4.2] and [4.3]. By (119) and (131), it makes sense to consider the integral

$$Z_\infty(s) = \int_{R(\mathbb{R}) \setminus H(\mathbb{R})} W^\#(\eta h, s) B(h) dh. \quad (132)$$

Our goal in the following is to evaluate this integral. It follows from (120) and (130) that this integral is zero if $l_1 \neq l$. We shall therefore assume that $l_1 = l$. Then the function $W^\#(\eta h, s) B(h)$ is right invariant under $K_\infty^H$. From the disjoint double coset decomposition (117) and the fact that $W^\#(\eta h, s) B(h)$ is right invariant under $K_\infty^H$ we obtain

$$Z_\infty(s) = \pi \int_{R^\times} \int_{1}^\infty W^\#(\eta \begin{bmatrix} \lambda t_0 & \zeta \\ \zeta^{-1} & t_0^{-1} \end{bmatrix}, s) B(\begin{bmatrix} \lambda t_0 & \zeta \\ \zeta^{-1} & t_0^{-1} \end{bmatrix}) (\zeta - \zeta^{-3}) \lambda^{-4} d\zeta \ d\lambda; \quad (133)$$

see [9] (4.6) for the relevant integration formulas. The above calculations are valid for any choice of $a, b, c$ as long as $S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$ is positive definite. To compute (133), we will fix $D = 4ac - b^2$ and make
special choices for \( a, b, c \). First assume that \( D \equiv 0 \pmod{4} \). In this case let \( S(-D) := \begin{bmatrix} D/2 & 0 \\ 0 & 1 \end{bmatrix} \). Then

\[
\eta = \begin{bmatrix} \sqrt{-D}/2 & 1 \\ 1 & \sqrt{-D}/2 \end{bmatrix}, \text{ and we can choose } t_0 = \begin{bmatrix} 2^{1/2}D^{-1/4} \\ 2^{-1/2}D^{1/4} \end{bmatrix}.
\]

From (118) we have

\[
B\left( \begin{bmatrix} \lambda t_0 \left[ \begin{array}{c} \zeta \\ \zeta^{-1} \end{array} \right] \\ t_0^{-1} \left[ \begin{array}{c} \zeta^{-1} \\ \zeta \end{array} \right] \end{bmatrix} \right) = \begin{cases} \lambda t e^{-2\pi\lambda D^{1/2} z^2 z^{-2}} & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda < 0. \end{cases}
\]

(134)

Next we rewrite the argument of \( W^\# \) as an element of \( MNK_\infty \),

\[
\eta \begin{bmatrix} \lambda t_0 \left[ \begin{array}{c} \zeta \\ \zeta^{-1} \end{array} \right] \\ t_0^{-1} \left[ \begin{array}{c} \zeta^{-1} \\ \zeta \end{array} \right] \end{bmatrix} = \begin{cases} \lambda D^{1/4} \left( \frac{\zeta^2 + \zeta^{-2}}{2} \right)^{-1/2} \\ D^{1/4} \left( \frac{\zeta^2 + \zeta^{-2}}{2} \right)^{1/2} \end{cases}
\]

\[
\begin{bmatrix} 1 & -i\zeta^2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -i\zeta^2 & 1 \\ 0 & 0 \end{bmatrix},
\]

where \( k_0 \in SU(2) = \{ g \in SL_2(\mathbb{C}) : g^{-1}gg = I_2 \} \). Hence, using (129) and (130), we get

\[
W^\# \left( \eta \begin{bmatrix} \lambda t_0 \left[ \begin{array}{c} \zeta \\ \zeta^{-1} \end{array} \right] \\ t_0^{-1} \left[ \begin{array}{c} \zeta^{-1} \\ \zeta \end{array} \right] \end{bmatrix}, s \right)
\]

\[
= \begin{cases} \lambda D^{1/4} \left( \frac{\zeta^2 + \zeta^{-2}}{2} \right)^{-1} \omega_{\tau}(\lambda)^{-1} W(0) \left( \begin{array}{c} \lambda D^{1/2} \left( \frac{\zeta^2 + \zeta^{-2}}{2} \right)^{1/2} \\ 0 \\ 1 \end{array} \right) \end{cases}.
\]

(135)

Let \( q \in \mathbb{C} \) be such that \( \omega_{\tau}(y) = y^q \) for \( y > 0 \). It follows from (126), (134) and (135) that

\[
Z_{\infty}(s) = a^+ \pi D^{-\frac{1}{4} + \frac{1}{2}} \frac{1}{\Phi(4\pi)^2} \int_0^\infty \lambda^{3s+\frac{1}{2}+\frac{1}{2}} \left( \frac{\zeta^2 + \zeta^{-2}}{2} \right)^{-\frac{3s}{2}+\frac{1}{2}} W_{\frac{1}{2}}(\Phi(4\pi\lambda D^{1/2} \zeta^2 + \zeta^{-2}))
\]

\[
e^{-2\pi\lambda D^{1/2} \zeta^2 + \zeta^{-2}} (\zeta - \zeta^{-3}) \lambda^{-4} d\zeta d\lambda.
\]

(136)

Substituting \( u = (\zeta^2 + \zeta^{-2})/2 \) we get

\[
Z_{\infty}(s) = a^+ \pi D^{-\frac{1}{2} + \frac{1}{2}} \frac{1}{\Phi(4\pi)^2} \int_0^\infty \lambda^{3s+\frac{1}{2}+\frac{1}{2}} u^{-3s+\frac{1}{2}+\frac{1}{2}} W_{\frac{1}{2}}(\Phi(4\pi\lambda D^{1/2}u)) e^{-2\pi\lambda D^{1/2}u} \frac{d\lambda}{\lambda} du.
\]

We will first compute the integral with respect to \( \lambda \). For a fixed \( u \) substitute \( x = 4\pi\lambda D^{1/2}u \) to get

\[
Z_{\infty}(s) = a^+ \pi D^{-3s+\frac{1}{2}+\frac{1}{2}} \frac{1}{\Phi(4\pi)^2} \int_0^\infty u^{-3s+\frac{1}{2}+\frac{1}{2}} \int_0^\infty W_{\frac{1}{2}}(x) e^{-\frac{\phi}{\Phi}x^{3s-\frac{1}{2}+\frac{1}{2}}} dx \frac{d\lambda}{\lambda} du.
\]

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Using the integral formula for the Whittaker function from [17, p. 316], we get

\[
Z_\infty(s) = a^+ \pi D^{-3s-\frac{1}{2}+\frac{3}{2}(4\pi)^{-3s+\frac{3}{2}+l+q}} \frac{\Gamma(3s+l-1+\frac{ir}{2}-\frac{3}{2})\Gamma(3s+l-1-\frac{ir}{2}-\frac{3}{2})}{\Gamma(3s+\frac{1}{2}-\frac{r}{2}-\frac{3}{2})} \int_1^{\infty} u^{-6s-l+q} du \\
= a^+ \pi D^{-3s-\frac{1}{2}+\frac{3}{2}(4\pi)^{-3s+\frac{3}{2}+l+q}} \frac{\Gamma(3s+l-1+\frac{ir}{2}-\frac{3}{2})\Gamma(3s+l-1-\frac{ir}{2}-\frac{3}{2})}{6s+l-q-1} \\
= \frac{a^+}{2} \pi D^{-3s-\frac{1}{2}+\frac{3}{2}(4\pi)^{-3s+\frac{3}{2}+l+q}} \frac{\Gamma(3s+l-1+\frac{ir}{2}-\frac{3}{2})\Gamma(3s+l-1-\frac{ir}{2}-\frac{3}{2})}{\Gamma(3s+\frac{1}{2}-\frac{r}{2}-\frac{3}{2})}. \quad (137)
\]

Here, for the calculation of the \( u \)-integral, we have assumed that \( \operatorname{Re}(6s+l-q) > 0 \). — Now assume that \( D \equiv 3 \mod 4 \). In this case we choose

\[
S(-D) = \begin{bmatrix} \frac{1+D}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{D}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}.
\]

Let \( T(\mathbb{R}), R(\mathbb{R}), \eta, B \) be the objects defined with this \( \begin{bmatrix} a & b/2 \\ c & d \end{bmatrix} = \begin{bmatrix} 1+D & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \), and let \( \tilde{T}(\mathbb{R}), \tilde{R}(\mathbb{R}), \tilde{\eta}, \tilde{B} \) be the corresponding objects defined with \( \begin{bmatrix} \tilde{a} & \tilde{b}/2 \\ \tilde{b}/2 & \tilde{c} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 1 \end{bmatrix} \). Let

\[
h_0 = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}.
\]

Then

\[
T^1(\mathbb{R}) = h_0 \tilde{T}^1(\mathbb{R}) h_0^{-1}, \quad T(\mathbb{R}) = h_0 \tilde{T}(\mathbb{R}) h_0^{-1}, \quad R(\mathbb{R}) = h_0 \tilde{R}(\mathbb{R}) h_0^{-1}.
\]

Furthermore, \( \eta = \tilde{\eta} h_0^{-1} \). The integral (132) becomes

\[
Z_\infty(s) = \int_{R(\mathbb{R}) \setminus H(\mathbb{R})} W^\#(\eta h, s) B(h) \, dh
\]

\[
= \int_{h_0 \tilde{R}(\mathbb{R}) h_0^{-1} \setminus H(\mathbb{R})} W^\#(\tilde{\eta} h_0^{-1} h, s) B(h_0 h_0^{-1} h) \, dh
\]

\[
= \int_{h_0 \tilde{R}(\mathbb{R}) h_0^{-1} \setminus H(\mathbb{R})} W^\#(\tilde{\eta} h_0^{-1} h h_0, s) B(h_0 h_0^{-1} h h_0) \, dh
\]

\[
= \int_{\tilde{R}(\mathbb{R}) \setminus H(\mathbb{R})} W^\#(\tilde{\eta} h, s) \tilde{B}(h) \, dh.
\]

This integral can be computed just like the one in the case \( D \equiv 0 \mod 4 \), and we get the exactly same answer as in (137).

4.4.1 Theorem. Let \( l \) and \( D \) be positive integers such that \( D \equiv 0, 3 \mod 4 \). Let \( S(-D) = \begin{bmatrix} D/4 \\ 1 \end{bmatrix} \) if \( D \equiv 0 \mod 4 \) and \( S(-D) = \begin{bmatrix} (1+D)/4 & 1/2 \\ 1/2 & 1 \end{bmatrix} \) if \( D \equiv 3 \mod 4 \). Let \( B : \operatorname{GSp}_4(\mathbb{R}) \to \mathbb{C} \) be the function
defined in (138), and let \( W^\#(\cdot, s) \) be the function defined in (128). Then, for \( \text{Re}(6s + l - q) > 0 \),

\[
Z_\infty(s) := \int_{R(\mathbb{R}) \setminus H(\mathbb{R})} W^\#(\eta h, s) B(h) \, dh
\]

\[
= \frac{a^+}{2} pi D^{-3s-\frac{l}{2}+\frac{q}{2}} (4\pi)^{-3s+\frac{3}{2}-l+q} \frac{\Gamma(3s + l - 1 + \frac{q}{2} - \frac{q}{4}) \Gamma(3s + l - 1 - \frac{q}{2} - \frac{q}{4})}{\Gamma(3s + \frac{l+1-\alpha l}{2})}.
\] (138)

Here, \( q \in \mathbb{C} \) is related to the central character of \( \tau \) via \( \omega_r(y) = y^q \) for \( y > 0 \). The number \( r \in \mathbb{C} \) is such that (125) holds.

We will state two special cases of formula (138). First assume that \( \tau = \chi_1 \times \chi_2 \), an irreducible principal series representation of \( \text{GL}(2, \mathbb{R}) \), where \( \chi_1 \) and \( \chi_2 \) are characters of \( \mathbb{R}^\times \). Let \( \varepsilon_i \in \{0, 1\} \) and \( s_i \in \mathbb{C} \) be such that \( \chi_i(x) = \text{sgn}(x)^{\varepsilon_i} |x|^{s_i} \), for \( i = 1, 2 \). Then \( \Delta \) acts on \( \tau \) by multiplication with \( -\frac{1}{4}(1 - (s_1 - s_2)^2) \). Comparing with (125), we get \( (s_1 - s_2)^2 = -r^2 \), so that \( ir = \pm (s_1 - s_2) \). Furthermore, \( q = s_1 + s_2 \). Therefore,

\[
Z_\infty(s) = \frac{a^+}{2} pi D^{-3s-\frac{l}{2}+\frac{q}{2}} (4\pi)^{-3s+\frac{3}{2}-l+s_1+s_2} \frac{\Gamma(3s + l - 1 - s_1) \Gamma(3s + l - 1 - s_2)}{\Gamma(3s + \frac{l+1-\alpha l}{2})}.
\] (139)

Now assume that \( l_1 \) is a positive integer, that \( q \in \mathbb{C} \), and that \( \tau = \mathcal{D}_q(l_1) \), the discrete series (or limit of discrete series) representation of \( \text{GL}(2, \mathbb{R}) \) with a lowest weight vector of weight \( l_1 \) for which the central element \( Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) acts by multiplication with \( q \). Then \( ir = \pm (l_1 - 1) \), so that, from (138),

\[
Z_\infty(s) = \frac{a^+}{2} pi D^{-3s-\frac{l}{2}+\frac{q}{2}} (4\pi)^{-3s+\frac{3}{2}-l+q} \frac{\Gamma(3s + l - 1 + \frac{l_1-1}{2} - \frac{q}{4}) \Gamma(3s + l - 1 - \frac{l_1-1}{2} - \frac{q}{4})}{\Gamma(3s + \frac{l+1-\alpha l}{2})}.
\] (140)

5 Modular Forms

Let \( \mathbb{A} \) be the ring of adeles of \( \mathbb{Q} \). In this section we will consider a cuspidal, automorphic representation \( \pi \) of \( \text{GSp}_4(\mathbb{A}) \), obtained from a Siegel cusp form, and a cuspidal, automorphic representation \( \tau \) of \( \text{GL}_2(\mathbb{A}) \), obtained from a Maass form. We want to obtain an integral formula for the \( L \)-function \( L(s, \pi \times \tau) \). We will use the local calculations from the previous two sections to achieve this.

Given a quadratic field extension \( L/\mathbb{Q} \), we define the groups \( G = GU(2, 2), H = \text{GSp}_4, P = MN \) and \( R = TU \) as in Sect. 2.1 and 2.2 but now considered as algebraic groups over \( \mathbb{Q} \).

5.1 Siegel modular forms and Bessel models

Let \( \Gamma_2 = \text{Sp}_4(\mathbb{Z}) \). For a positive integer \( l \) denote by \( S_l(\Gamma_2) \) the space of Siegel cusp forms of degree 2 and weight \( l \) with respect to \( \Gamma_2 \). If \( \Phi \in S_l(\Gamma_2) \) then \( \Phi \) satisfies

\[
\Phi(\gamma(Z)) = \det(J(\gamma, Z))^l \Phi(Z), \quad \gamma \in \Gamma_2, \ Z \in \mathfrak{g}_2.
\]

Let us assume that \( \Phi \in S_l(\Gamma_2) \) is a Hecke eigenform. It has a Fourier expansion

\[
\Phi(Z) = \sum_{S > 0} a(S, \Phi) e^{2\pi i r tr(SZ)},
\]

where \( S \) runs through all symmetric semi-integral positive definite matrices of size two. Let us make the following two assumptions about the function \( \Phi \).

**Assumption 1**: \( a(S, \Phi) \neq 0 \) for some \( S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \) such that \( b^2 - 4ac = -D < 0 \) where \( -D \) is the discriminant of the imaginary quadratic field \( \mathbb{Q}(\sqrt{-D}) \).
Assumption 2: The weight $l$ is a multiple of $w(-D)$, the number of roots of unity in $\mathbb{Q}(\sqrt{-D})$. We have

$$w(-D) = \begin{cases} 
4 & \text{if } D = 4, \\
6 & \text{if } D = 3, \\
2 & \text{otherwise.}
\end{cases}$$

Let us define a function $\phi = \phi_\Phi$ on $H(\mathfrak{A}) = \text{GSp}_4(\mathfrak{A})$ by

$$\phi(\gamma h, k_0) = \mu_2(h) \det(J(h, I))^{-l} \Phi(h, I),$$

where $\gamma \in H(\mathbb{Q})$, $h, k_0 \in \prod_{p < \infty} H(\mathbb{Z}_p)$. Here $I = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$. Note that $\phi$ is invariant under the center $Z_H(\mathfrak{A})$ of $H(\mathfrak{A})$. It can be shown (see [1, p. 186]) that the function $\phi_\Phi$ is a cuspidal automorphic form. Let $V_\Phi$ be the automorphic representation generated by $\phi_\Phi$. This representation may not be irreducible, but decomposes into a direct sum of finitely many irreducible, cuspidal, automorphic representations of $H(\mathfrak{A})$. Let $\pi_\Phi$ be one of these irreducible components, and write $\pi_\Phi$ as a restricted tensor product $\pi_\Phi \cong \otimes_p \pi_p$, where $\pi_p$ is an irreducible, admissible, unitarizable representation of $H(\mathbb{Q}_p)$. Since $\phi_\Phi$ is $H(\mathbb{Z}_p)$-invariant for all finite primes $p$, the representation $\pi_p$ has a non-zero, essentially unique $H(\mathbb{Z}_p)$-invariant vector. The same calculations as in [1] show that the equivalence class of $\pi_p$ depends only on $\Phi$ and not on the chosen global irreducible component $\pi_\Phi$.

Let $\psi = \prod_p \psi_p$ be a character of $\mathbb{Q}\backslash \mathfrak{A}$ which is unramified at every finite prime and such that $\psi_\infty(x) = e^{-2\pi i x}$ for $x \in \mathbb{R}$. Let

$$S(-D) = \begin{cases} 
\begin{bmatrix} D & 0 \\ 0 & 1 \end{bmatrix} & \text{if } D \equiv 0 \pmod{4}, \\
\begin{bmatrix} 1+D & 1/2 \\ 1 & 1 \end{bmatrix} & \text{if } D \equiv 3 \pmod{4}.
\end{cases}$$

Our quadratic extension is $L = \mathbb{Q}(\sqrt{-D})$. We have $T(\mathbb{Q}) \simeq \mathbb{Q}(\sqrt{-D})^\times$. Let $\Lambda$ be an ideal class character of $\mathbb{Q}(\sqrt{-D})$, i.e., a character of $T(\mathfrak{A})/T(\mathbb{Q})T(\mathbb{R}) \prod_{p < \infty} (T(\mathbb{Q}_p) \cap \text{GL}_2(\mathbb{Z}_p))$, to be chosen below. We define the global Bessel function of type $(\Lambda, \theta)$ associated to $\tilde{\phi}$ by

$$B_{\tilde{\phi}}(h) = \int_{Z_H(\mathfrak{A})H(\mathbb{Q}) \backslash H(\mathfrak{A})} (\Lambda \otimes \theta)(r)^{-1} \tilde{\phi}(rh) dr,$$

where $\theta(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}) = \psi(\text{tr}(S(-D)X))$ and $\tilde{\phi}(h) = \overline{\phi(h)}$. For a finite prime $p$, the function $B_p(h_p) := B_{\tilde{\phi}}(h_p)$, with $h_p \in H(\mathbb{Q}_p)$, is in the Bessel model for $\pi_p$ with respect to the character $\Lambda_p \otimes \theta_p$ of $R(\mathbb{Q}_p)$. The uniqueness of the Bessel model for $\text{GSp}_4$ (see [1]) gives us

$$B_{\tilde{\phi}}(h) = B_{\tilde{\phi}}(h_\infty) \prod_{p < \infty} B_p(h_p),$$

where $h = \otimes h_p$. From [28] (1-17), (1-19), (1-26)), we have, for $h_\infty \in H^+(\mathbb{R})$,

$$B_{\tilde{\phi}}(h_\infty) = |\mu_2(h_\infty)|^l \det(J(h_\infty, I))^{-l} e^{-2\pi i \text{tr}(S(-D)h_\infty, I)} \sum_{j=1}^{h(-D)} \Lambda(t_j)^{-1} a(S_j, \Phi),$$

and $B_{\tilde{\phi}}(h_\infty) = 0$ for $h_\infty \not\in H^+(\mathbb{R})$. Here, $h(-D)$ is the class number of $\mathbb{Q}(\sqrt{-D})$, the elements $t_j, j = 1, \ldots, h(-D)$, are representatives of the idele classes of $\mathbb{Q}(\sqrt{-D})$ and $S_j, j = 1, \ldots, h(-D)$, are the representatives of $\text{SL}_2(\mathbb{Z})$ equivalent classes of primitive semi-integral positive definite matrices of discriminant $-D$ corresponding to $t_j$. Thus, by Assumption 1, there exists a $\Lambda$ such that $B_{\tilde{\phi}}(I_4) \neq 0$. We fix such a $\Lambda$.

Note that $B_{\tilde{\phi}}(h_\infty)$ is a non-zero constant multiple of $\text{188}$. Let us abbreviate $a(\Lambda) = \sum_{j=1}^{h(-D)} \Lambda(t_j) a(S_j, \Phi)$.
5.2 Maaß forms and Eisenstein series

Let $\mathcal{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ be the complex upper half plane. Fix a square-free integer $N$. Let $
abla_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\} \subset \text{SL}_2(\mathbb{Z}) : N|c\}$. A smooth function $f : \mathcal{H} \to \mathbb{C}$ is called a Maaß cusp form of weight $l_1$ with respect to $\Gamma_0(N)$ if

i) For every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$ and $z \in \mathcal{H}$ we have

$$f \left( \frac{az + b}{cz + d} \right) = \left( \frac{cz + d}{|cz + d|} \right)^{l_1} f(z).$$

ii) $f$ is an eigenfunction of $\Delta_{l_1}$, where

$$\Delta_{l_1} = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - il_1y \frac{\partial}{\partial x},$$

iii) $f$ vanishes at the cusps of $\Gamma_0(N)$.

Denote the space of Maaß cusp forms of weight $l_1$ with respect to $\Gamma_0(N)$ by $S_{l_1}^M(N)$. A function $f \in S_{l_1}^M(N)$ has the Fourier expansion

$$f(x + iy) = \sum_{n \neq 0} a_n W_{\text{sgn}(n)} \frac{1}{4} + \frac{y}{4} (4\pi |n| y) e^{2\pi i nx},$$

where $W_{\nu,\mu}$ is a classical Whittaker function (the same function as in [120]) and $(\Delta_{l_1} + \lambda)f = 0$ with $\lambda = 1/4 + (r/2)^2$. Let $f \in S_{l_1}^M(N)$ be a Hecke eigenform.

If $ir/2 = (l_2 - 1)/2$ for some integer $l_2 > 0$, then the cuspidal, automorphic representation of $\text{GL}_2(\mathbb{A})$ constructed below is holomorphic at infinity of lowest weight $l_2$. In this case we make the additional assumptions that $l_2 \leq l$ and $l_2 \leq l_1$, where $l$ is the weight of the Siegel cusp form $\Phi$ from the previous section.

Starting from $f$, we obtain another Maaß form $f_1 \in S_{l_1}^M(N)$ by applying the raising and lowering operators. The raising operator $R_*$ maps $S_{l_1}^M(N)$ to $S_{l_1+2}^M(N)$ and the lowering operator $L_*$ maps $S_{l_1}^M(N)$ to $S_{l_1-2}^M(N)$; for more details on these operators, see [23], pp. 3925. In particular, we have

$$f_l = \begin{cases} R_{l-2} R_{l-4} \cdots R_{l_1+2} R_{l_1} f & \text{if } l_1 < l, \\
 f & \text{if } l_1 = l, \\
 L_{l_1+2} L_{l_1+4} \cdots L_{l_1-2} L_{l_1} f & \text{if } l_1 > l. 
\end{cases} \quad (147)$$

Note that, by Assumption 2 on the Siegel cusp form $\Phi$, the weight $l$ is always even. Also, $S_{l_1}^M(N)$ is empty if $l_1$ is odd. Hence, (147) makes sense. If $ir/2 = (l_2 - 1)/2$, then the assumption $l_2 \leq l$ guarantees that $f_l \neq 0$. Suppose $\{c(n)\}$ are the Fourier coefficients of $f_l$. In later calculations we will need $c(1)$. By [23, Lemma 2.5],

$$c(1) = \begin{cases} a_1 & \text{if } l_1 \leq l, \\
 \prod_{t \equiv l_1+2 \pmod{2}} \left( \frac{ir}{2} + \frac{t}{2} \right) \left( \frac{ir}{2} - \frac{t}{2} + \frac{l}{2} \right)^{-l_1} a_1 & \text{if } l_1 > l. 
\end{cases} \quad (148)$$

Define a function $\hat{f}$ on $\text{GL}_2(\mathbb{A})$ by

$$\hat{f}(\gamma_0 m k_0) = \left( \frac{\gamma i + \delta}{|\gamma i + \delta|} \right)^{-l_1} f_l \left( \frac{\alpha i + \beta}{\gamma i + \delta} \right), \quad (149)$$

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where $\gamma_0 \in \text{GL}_2(\mathbb{Q})$, $m = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \text{GL}_2^+(\mathbb{R})$, $k_0 \in \prod_{p \mid N} K^{(1)}(p) \prod_{p \nmid N} \text{GL}_2(\mathbb{Z}_p)$. Here, for $p \mid N$ we have $K^{(1)}(p) = \text{GL}_2(\mathbb{Q}_p) \cap \left[ \frac{\mathbb{Z}_p^\times}{p^t} \mathbb{Z}_p \right]$ with $p = p\mathbb{Z}_p$, as in (32). Then $\hat{f}$ satisfies

$$\hat{f}(gr(\theta)) = e^{it\theta} \hat{f}(g), \quad g \in \text{GL}_2(\mathbb{A}), \quad r(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$  \hspace{1cm} (150)

Let $(\tau_f, V_f)$ be the cuspidal, automorphic representation of $\text{GL}_2(\mathbb{A})$ generated by $\hat{f}$. By strong multiplicity one, $\tau_f$ is irreducible. Note that $\tau_f$ has trivial central character. Write $\tau_f$ as a restricted tensor product $\tau_f = \otimes'_p \tau_p$. If $p \nmid N$ is a finite prime, then $\tau_p$ is an irreducible, admissible, unramified representation of $\text{GL}_2(\mathbb{Q}_p)$. If $p|N$, then $\tau_p$ is an irreducible, admissible representation of $\text{GL}_2(\mathbb{Q}_p)$ with conductor $p = p\mathbb{Z}_p$. Since $\tau$ has trivial central character, $\tau_p$ is a twisted Steinberg representation given by $\tau_p = \Omega_p \text{St}_{\text{GL}_2(\mathbb{Q}_p)}$, where $\Omega_p$ is an unramified, quadratic character of $\mathbb{Q}_p^\times$. Let

$$W^{(0)}(g) := \int_{\mathbb{A}} \hat{f}(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g) \psi(x) dx,$$

where $\psi$ is the additive character fixed in the previous section. Then $W^{(0)}$ is in the Whittaker model of $\tau_f$ with respect to the character $\psi^{-1}$. By (150),

$$W^{(0)}(gr(\theta)) = e^{it\theta} W^{(0)}(g), \quad g \in \text{GL}_2(\mathbb{A}), \quad r(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}. \hspace{1cm} (151)$$

For any finite prime $p$, the function $W_p(g_p) := W^{(0)}(g_p)$, for $g_p \in \text{GL}_2(\mathbb{Q}_p)$, is in the Whittaker model of $\tau_p$. By the uniqueness of Whittaker models for $\text{GL}_2$, we get

$$W^{(0)}(g) = W^{(0)}(g_\infty) \prod_{p < \infty} W_p(g_p)$$

for $g = \otimes g_p$. Using the definition (149) for $\hat{f}$ we get, for $t \in \mathbb{R}^\times$,

$$W^{(0)}(\begin{bmatrix} t \\ 1 \end{bmatrix}) = \begin{cases} c(1) W_{\frac{t}{2}}(4\pi t) & \text{if } t > 0, \\
c(-1) W_{-\frac{t}{2}}(-4\pi t) & \text{if } t < 0. \end{cases} \hspace{1cm} (152)$$

We want to extend $\hat{f}$ to a function on $\text{GU}(1, 1; \mathbb{L})(\mathbb{A})$. For this, we need to construct a suitable character $\chi_0$ on $\mathbb{L}^\times \backslash \mathbb{A}_L^\times$.

### 5.2.1 Lemma. Let $S$ be a divisible group, i.e., a group with the property that $S = \{s^n : s \in S\}$ for all positive integers $n$. Let $A$ and $B$ be abelian groups, and assume that $B$ is finite. Then every exact sequence

$$1 \rightarrow S \rightarrow A \rightarrow B \rightarrow 1$$

splits.

**Proof.** Write $B$ as a product of cyclic groups $(b_i)$. Choose pre-images $a_i$ of $b_i$ in $A$. Modifying $a_i$ by suitable elements of $S$, we may assume that $a_i$ has the same order as $b_i$. Then the group generated by all $a_i$ is isomorphic to $B$. \hfill \blacksquare

### 5.2.2 Lemma. Let $L = \mathbb{Q}(\sqrt{-D})$ with $D > 0$ be an imaginary quadratic number field. Let $\mathbb{A}_L^\times$ be the group of ideles of $L$. Let $K_f$ be the subgroup given by $\prod_{v < \infty} \mathfrak{o}_L^\times$, where $v$ runs over all finite places of $L$, and $\mathfrak{o}_L$ is the ring of integers in the completion of $L$ at $v$. The archimedean component of $\mathbb{A}_L^\times$ is isomorphic to $\mathbb{R}^\times \times S^1$, where $S^1$ is the unit circle. Let $l \in \mathbb{Z}$ be a multiple of $w(-D)$, the number of roots of unity in $L$. Then there exists a character $\chi_0$ of $\mathbb{A}_L^\times$ with the properties

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i) $\chi_0$ is trivial on $\mathbb{A}_L^\times K_f L^\times$; and

ii) $\chi_0(\zeta) = \zeta^{-l}$ for all $\zeta \in S^1$.

**Proof.** First note that $\mathbb{A}_L^\times K_f L^\times = \mathbb{R}_{>0} K_f L^\times$. There is an exact sequence

$$1 \rightarrow W\backslash S^1 \rightarrow \mathbb{R}_{>0} K_f L^\times \backslash \mathbb{A}_L^\times \rightarrow \mathbb{C}^\times K_f L^\times \backslash \mathbb{A}_L^\times \rightarrow 1,$$

where $W$ is the group of roots of unity in $L$. The group on the right is the ideal class group of $L$. By Lemma 5.2.1

$$\mathbb{R}_{>0} K_f L^\times \backslash \mathbb{A}_L^\times \cong (W\backslash S^1) \times (\mathbb{C}^\times K_f L^\times \backslash \mathbb{A}_L^\times).$$

By hypothesis, the map $S^1 \ni \zeta \mapsto \zeta^l$ factors through $W\backslash S^1$. The assertion follows. $\blacksquare$

Let $\chi_0$ be a character of $\mathbb{A}_L^\times$ as in Lemma 5.2.2 (observe our Assumption 2 above). We extend $\hat{f}$ to $\text{GU}(1, 1; L)(\mathbb{A})$ by

$$\hat{f}(\zeta g) = \chi_0(\zeta) \hat{f}(g) \quad \text{for } \zeta \in \mathbb{A}_L^\times, \ g \in \text{GL}_2(\mathbb{A}).$$

(153)

Since $l$ is even, this is well-defined; see (110) and (150). Let $\chi$ be the character of $L^\times \backslash \mathbb{A}_L^\times$ given by $\chi(\zeta) = \Lambda(\zeta)^{-1} \chi_0(\zeta)^{-1}$. Let $K_G^\#(N)$ be the compact subgroup $\prod_{\nu|\nu} K^\#(\nu^\#) \prod_{\nu|\nu} K^\#(\nu^0)$ of $\text{GU}(2, 2; L)(\mathbb{A})$, where $K^\#(\nu^\#)$ is as defined in (35). For a complex variable $s$, let us define a function $f_\Lambda(\cdot, s)$ on $\text{GU}(2, 2; L)(\mathbb{A})$ by

i) $f_\Lambda(g, s) = 0$ if $g \notin M(\mathbb{A}) N(\mathbb{A}) K_\infty K_G^\#(N)$.

ii) If $m = m_1 m_2, m_i \in M^{(i)}(\mathbb{A}), n \in N(\mathbb{A}), k = k_0 k_\infty, k_0 \in K_G^\#(N), k_\infty \in K_\infty$, then

$$f_\Lambda(mnk, s) = \delta_P^{\cdot + s}(m) \chi(m_1) \hat{f}(m_2) \det(J(k_\infty, I))^{-l}.$$ (154)

Recall from (10) that $\delta_P(m_1 m_2) = |N_L/\mathbb{Q}(m_1)\mu_1(m_2)|^{-\frac{1}{2}}$.

Here, $M^{(1)}(\mathbb{A}), M^{(2)}(\mathbb{A}), N(\mathbb{A})$ are the adelic points of the algebraic groups defined by (6), (7) and (8). The groups $K^\#(\nu^\#)$ are as defined in (35), and $K_\infty$ is as defined in Sect. 4.1. In fact, $f_\Lambda$ is a section in the representation $I(s, \chi, \chi_0, \tau)$ of $\text{GU}(2, 2; L)(\mathbb{A})$ obtained by parabolic induction from $P$; see Sect. 2.3.

Let us define the Eisenstein series on $\text{GU}(2, 2; L)(\mathbb{A})$ by

$$E_\Lambda(g, s) = \sum_{\gamma \in P(\mathbb{Q}) G(\mathbb{Q})} f_\Lambda(\gamma g, s).$$ (155)

This series is absolutely convergent for $\Re(s) > 1/2$, uniformly convergent in compact subdomains and has a meromorphic continuation to the whole complex plane; see [10].

**Remark:** Note that our definition (154) differs from the formula for $f_\Lambda$ given on p. 209 of [9]. In fact, the function $f_\Lambda$ in [9] is not well-defined, since there is a non-trivial overlap between $M^{(2)}(\mathbb{R})$ and $K_\infty$. It is necessary to extend the function $\hat{f}$ to $\text{GU}(1, 1; L)(\mathbb{A})$ using the character $\chi_0$ as in [158], not the trivial character.

### 5.3 Global integral and L-functions

Let $\phi$ be as in [111]. Let $f_\Lambda(\cdot, s)$ and $E_\Lambda(\cdot, s)$ be as in the previous section. We shall evaluate the global integral

$$Z(s, \Lambda) = \int_{\mathbb{Z}_H(\mathbb{A}) H(\mathbb{Q}) \backslash H(\mathbb{A})} E_\Lambda(h, s) \phi(h) dh.$$ (156)
In Theorem 2.4 of [9], the following basic identity has been proved.

\[ Z(s, \Lambda) = \int_{R(\Lambda) \setminus H(\Lambda)} W_{\Lambda}(\eta h, s) B_\phi(h) \, dh, \tag{157} \]

where

\[ W_{\Lambda}(g, s) = \int_{\mathbb{Q} \setminus \mathbb{A}} f_{\Lambda} \left( \begin{array}{ccc} 1 & x & 0 \\ \alpha & 1 & 1 \\ 0 & 1 & 1 \end{array} \right) g(s) \psi(x) \, dx, \tag{158} \]

\[ \eta = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \\ 0 & 1 \end{bmatrix}, \quad \alpha = \frac{b + \sqrt{-D}}{2}, \]

and \( B_\phi \) is as defined in (143). Note that the value of \( b \) above depends on the choice of \( S(-D) \) in (142). For the choice of \( f_{\Lambda} \) in the previous section, we get

\[ W_{\Lambda}(g, s) = W_\infty(g, s) \prod_{p < \infty} W_p(g, s), \]

where \( W_p \) is the function \( W^\# \) defined in Sect. 3.4. For \( g_\infty \in G(\mathbb{R}) \), the function \( W_\infty(g, s) \) is exactly the function \( W^\# \) from (128). Note that, in this case, the values of \( a^+, a^- \) in (126) are given by \( a^+ = c(1) \) and \( a^- = c(-1) \). From the basic identity (157) we therefore have

\[ Z(s, \Lambda) = \prod_{p \leq \infty} Z_p(s), \quad Z_p(s) = \int_{R(\mathbb{Q}_p) \setminus H(\mathbb{Q}_p)} W_p(\eta h_p, s) B_p(h_p) \, dh_p. \]

Here, \( B_\infty \) is the function given in (145). If \( p \) is a finite prime such that \( p \nmid N \), then all the local data satisfies the hypothesis of Theorem 3.7 from [9], where the corresponding local integral is computed. For \( p|N \), we apply Theorem 3.9.1 and for the archimedean integral we apply Theorem 4.4.1. We obtain the following integral representation.

**5.3.1 Theorem.** Let \( \Phi \in S_l(\Gamma_2) \) be a cuspidal Siegel eigenform of degree 2 and even weight \( l \) satisfying the two assumptions from Sect. 5.1. Let \( L = \mathbb{Q}(\sqrt{-D}) \), where \( D \) is as in Assumption 1. Let \( N \) be a square-free, positive integer. Let \( f \) be a Maaß Hecke eigenform of weight \( l_1 \in \mathbb{Z} \) with respect to \( \Gamma_0(N) \). If \( f \) lies in a holomorphic discrete series with lowest weight \( l_2 \), then assume that \( l_2 \leq l \). Then the integral (157) is given by

\[ Z(s, \Lambda) = \kappa_\infty \kappa_N \frac{L(3s + \frac{1}{2}, \pi_\Phi \times \tau_f)}{\zeta(6s + 1)L(3s + 1, \tau_f \times \mathcal{A}(\Lambda))}, \tag{159} \]

where

\[ \kappa_\infty = \frac{1}{2} \lambda(\Lambda) c(1) \pi D^{-3s - \frac{1}{2}} (4\pi^{-3s + \frac{1}{2}} - 1) \Gamma(3s + l - 1 + \frac{i\pi}{2}) \Gamma(3s + l - 1 - \frac{i\pi}{2}) \Gamma(3s + \frac{l+1}{2}), \]

\[ \kappa_N = \prod_{p|N} \frac{p(p-1)}{(p+1)(p^2-1)} (1 - \left( \frac{L}{p} \right) p^{-1}) (1 - p^{-6s-1})^{-1}. \]

Here, the non-zero constant \( c(1) \) is given by (147), the non-zero constant \( \lambda(\Lambda) \) is defined at the end of Sect. (147) and

\[ \left( \frac{L}{p} \right) = \begin{cases} -1 & \text{if } p \ is \ inert \ in \ L, \\ 0 & \text{if } p \ ramifies \ in \ L, \\ 1 & \text{if } p \ splits \ in \ L. \end{cases} \]

The quantity \( \frac{L}{p} \) is as in (140).
5.4 The special value

In this section, we will apply Theorem 5.3.1 to a special case – when \( f \), from the previous section, is a holomorphic cusp form of the same weight \( l \) as the Siegel cusp form \( \Phi \) – to obtain a special \( L \)-value result. This result fits into the general conjecture of Deligne on special values of \( L \)-functions.

Let \( \Psi(z) = \sum_{\alpha \geq 0} \frac{b_\alpha e^{2\pi i n z}}{\alpha!} \) be a holomorphic cuspidal eigenform of weight \( l \) with respect to \( \Gamma_0(N) \). Here, \( l \) is the same as the weight of the Siegel modular form \( \Phi \) from Sect. 5.1 and \( N \) is a square-free, positive integer.

Let us normalize \( \Psi \) so that \( b_1 = 1 \). The function \( f_\Psi \) defined by \( f_\Psi(z) = y^{l/2} \Psi(z) \) is a Maass form in \( S_l^M(N) \). Let \( \{c(n)\} \) be its Fourier coefficients; see (149). It follows from the formula \( W_{\mu+1/2,\mu}(z) = e^{-\pi z} z^{\mu+1/2} \) for the Whittaker function that

\[
c(n) = \left\{ \begin{array}{ll}
0 & \text{if } n < 0, \\
b_n(4\pi n)^{-l/2} & \text{if } n > 0,
\end{array} \right.
\]

(160)

From (149), we have

\[
\hat{f}_\Psi(\gamma_0 m k_0) = \left( \frac{\gamma_0 + \delta}{\gamma' + \delta} \right)^{-l} f_\Psi \left( \frac{\alpha + \beta}{\gamma'} \right) = \frac{\det(m)^{l/2}}{\gamma'} \Psi \left( \frac{\alpha + \beta}{\gamma'} \right),
\]

where \( \gamma_0 \in \text{GL}_2(\mathbb{Q}) \), \( m = \left[ \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right] \in \text{GL}_2(\mathbb{R}) \), \( k_0 \in \prod_{p \mid N} \text{K}(\mathfrak{p}) \prod_{p \not\mid N} \text{GL}_2(\mathbb{Z}_p) \). Let us denote \( z_{22} \) by \( \mathbb{H}^* \) for \( Z = \left[ \begin{array}{cc} * & * \\ * & z_{22} \end{array} \right] \in \mathbb{H}_2 \). Let us set \( \bar{Z} = \frac{1}{2} (\bar{Z} - Z) \) for \( Z \in \mathbb{H}_2 \). Let \( \text{Im}(z) \) denote the imaginary part of a complex number \( z \). Let \( f_\Lambda \) be as defined in (153) and \( I = \left[ \begin{array}{c} i \\ i \end{array} \right] \in \mathbb{H}_2 \).

5.4.1 Lemma. For \( g \in G^+(\mathbb{R}) \), we have

\[
f_\Lambda(g, s) = \mu_2(g)^l \det(J(g, I))^{-l} \frac{(\det g(I))^{3s+\frac{3}{2}-\frac{1}{4}}}{\text{Im}(g(I))^s} \Psi((g(I))^s).
\]

(161)

Proof. For \( g \in G^+(\mathbb{R}) \) and \( Z \in \mathbb{H}_2 \) we have \( g(Z) = \mu_2(g)^l J(g, Z)^{-1} \bar{Z} J(g, Z) \). This implies that \( \det(g(I)) = \mu_2(g)^2 \det(J(g, I))^2 \det(I) = \mu_2(g)^2 \det(J(g, I))^2 \). It follows from (108) that we can write the element \( g \in G^+(\mathbb{R}) \) as

\[
g = \left[ \begin{array}{cc} \zeta & 1 \\ 1 & \zeta^{-1} \end{array} \right] \left[ \begin{array}{cc} 1 & \alpha \\ \mu & \beta \end{array} \right] \left[ \begin{array}{cc} 1 & x & y \\ \bar{y} & 1 & \bar{y} \end{array} \right] \left[ \begin{array}{cc} 1 & \bar{x} \\ -\bar{x} & 1 \end{array} \right] k,
\]

where \( \zeta \in \mathbb{R}^* \), \( \left[ \begin{array}{c} \alpha \\ \gamma \end{array} \right] \in \text{GL}_2^+(\mathbb{R}) \), \( x, y \in \mathbb{C} \), \( w \in \mathbb{R} \) and \( k \in K_{\mathbb{R}} \). Then we have

\[
\det(J(g, I)) = \zeta^{-1} \mu(\gamma i + \delta) \det(J(k, I)) \quad \text{and} \quad (g(I))^s = \frac{\alpha + \beta}{\gamma i + \delta}.
\]

Hence, the right hand side of (161) is equal to

\[
\mu^l \left( \zeta^{-1} \mu(\gamma i + \delta) \det(J(k, I)) \right)^{-l} \frac{\frac{\zeta^{-1} \mu(\gamma i + \delta) \det(J(k, I))}{\mu |\gamma i + \delta|^2}}{\left| \frac{\alpha + \beta}{\gamma i + \delta} \right|^{3s+\frac{3}{2}-\frac{1}{4}}} \Psi \left( \frac{\alpha + \beta}{\gamma i + \delta} \right)
\]

Using the fact that \( |\det(J(k, I))|^{-2} = \det(k(I)) = 1 \), we get the lemma.

Remark: Eq. (4.4.2) of [9] claims that, for \( g \in G^+(\mathbb{R}) \), the function \( f_\Lambda(g, s) \) satisfies a formula different from (101). In this formula, the term \( \det(\text{Im}(g(I))) \) replaces the term \( \det(g(I)) \) from (161). Note that
\[ 5.4.2 \text{ Lemma.} \quad \text{We have} \quad E_\Lambda(156) \text{ and the Eisenstein series} \]

Since the functions \( f \) Explicitly, \( \text{Re}((156)) \) only depends on \( Z = g(I) \). Hence, we can define a function \( E_\Lambda \) on \( \mathbb{H}_2 \) by the formula

\[ E_\Lambda(Z, s) = \mu_2(g)^{-1} \det(J(g, I))^{l} E_\Lambda(g, s) \]

where \( g \in G^+(\mathbb{R}) \) is such that \( g(I) = Z \). The series that defines \( E_\Lambda(Z, s) \) is absolutely convergent for \( \text{Re}(s) > 3 - 1/2 \) (see 15). Since \( l \geq 12 \), we can set \( s = 0 \) and obtain a holomorphic Eisenstein series \( E_\Lambda(Z, 0) \) on \( \mathbb{H}_2 \). For a finite place \( p \) of \( \mathbb{Q} \) recall the local congruence subgroups \( K^\#(\mathbb{Q}_p) \subset G(\mathbb{Z}_p) \) and \( K^\#(p^n) = K^\#(\mathbb{Q}_p) \cap H(\mathbb{Z}_p) \) defined in \( \mathbb{Q} \) resp. \( \mathbb{Q}_p \). We let

\[ \Gamma^\#(N) = G(\mathbb{Q}) \cap G(\mathbb{R})^+ K^\#(N), \quad K^\#(N) = \prod_{p\nmid N} K^\#(\mathbb{Q}_p) \prod_{p\mid N} K^\#(p^0), \]

and

\[ \hat{\Gamma}^\#(N) = H(\mathbb{Q}) \cap H(\mathbb{R})^+ K^\#(N), \quad \hat{K}^\#(N) = \prod_{p\nmid N} (Z_H(\mathbb{Z}_p)K^\#(p)) \prod_{p\mid N} (Z_H(\mathbb{Z}_p)K^\#(p^0)). \]

Since the functions \( f_\Lambda \) and \( E_\Lambda \) are also invariant under the center, we let

\[ \hat{\Gamma}^\#(N) = H(\mathbb{Q}) \cap H(\mathbb{R})^+ \hat{K}^\#(N), \quad \hat{K}^\#(N) = \prod_{p\nmid N} (Z_H(\mathbb{Z}_p)K^\#(p)) \prod_{p\mid N} (Z_H(\mathbb{Z}_p)K^\#(p^0)). \]

Explicitly,

\[ \hat{\Gamma}^\#(N) = \{ h = (h_{ij}) \in \text{Sp}(4, \mathbb{Z}) \cap \begin{bmatrix} Z & NZ & Z & Z \\ Z & Z & Z & Z \\ NZ & NZ & Z & Z \\ NZ & NZ & NZ & Z \end{bmatrix} : h_{11} \equiv h_{44} \text{ mod } N, h_{22} \equiv h_{33} \text{ mod } N \}. \]

Then \( E_\Lambda(Z, 0) \) is a modular form of weight \( l \) with respect to \( \Gamma^\#(N) \). Its restriction to \( h_2 \) is a modular form of weight \( l \) with respect to \( \hat{\Gamma}^\#(N) \). We see that \( E_\Lambda(Z, 0) \) has a Fourier expansion

\[ E_\Lambda(Z, 0) = \sum_{S \geq 0} b(S, E_\Lambda)e^{2\pi i \text{tr}(SZ)}, \]

where \( S \) runs through all Hermitian half-integral (i.e., \( S = \begin{bmatrix} t_1 & t_2 \\ t_2 & t_3 \end{bmatrix}, t_1, t_2 \in \mathbb{Z}, \sqrt{-d}t_2 \in \mathbb{O}_{\mathbb{Q}(\sqrt{-d})} \) positive semi-definite matrices of size 2. By \( [14], \)

\[ b(S, E_\Lambda) \in \bar{\mathbb{Q}} \quad \text{for any } S. \] (163)

Here \( \bar{\mathbb{Q}} \) denotes the algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \). The relation between the global integral \( Z(s, \Lambda) \) defined in \( [15] \) and the Eisenstein series \( E_\Lambda \) is given in the following lemma.

5.4.2 Lemma. We have

\[ Z(\frac{l}{6} - \frac{1}{2}, \Lambda) = \frac{1}{2} V_N \int_{\hat{\Gamma}^\#(N) \backslash \mathbb{H}_2} E_\Lambda(Z, 0) \tilde{\Phi}(Z)(\det(Y))^{l-3} dX dY, \]

where \( V_N = \prod_{p\mid N} (p^s - 1)(p^s - 1) \) and \( Z = X + iY \).
Proof. By definition,
\[
Z(\tfrac{t}{6} - \frac{1}{2}, \Lambda) = \int_{Z_{\tilde{H}(\mathbb{Q}) \setminus \tilde{\mathcal{H}}(\Lambda)}} E_{\Lambda}(h, \tfrac{t}{6} - \frac{1}{2}) \tilde{\phi}_\Lambda(h) \, dh.
\]
Note that the integrand is right invariant under \( K_\infty^H K^#(N) \). Since \( \text{vol}(K_\infty^H K^#(N)) = \prod_p \frac{1}{(p^{s_1}-1)(p^{s_2}-1)} = V_N \), it follows that
\[
Z(\tfrac{t}{6} - \frac{1}{2}, \Lambda) = V_N \int_{Z_{\tilde{H}(\mathbb{Q}) \setminus \tilde{\mathcal{H}}(\Lambda)}} E_{\Lambda}(h, \tfrac{t}{6} - \frac{1}{2}) \tilde{\phi}_\Lambda(h) \, dh.
\]
Note that
\[
Z_{\tilde{H}(\mathbb{Q}) \setminus \tilde{H}(\mathbb{R})} / H^\#(N) / K^\#_\infty = \tilde{\mathcal{H}}(\Lambda) \setminus h_2.
\]
The \( H(\mathbb{R})^+ \)-invariant measure on \( h_2 \) is given by \( \frac{1}{2} \det(dX)^{-3} dX dY \). From (141) and (102) we get, for \( h \in H(\mathbb{R})^+ \),
\[
E_{\Lambda}(h, \tfrac{t}{6} - \frac{1}{2}) \tilde{\phi}_\Lambda(h) = \mu_2(h)^t \det(J(h, I))^{-1} E_{\Lambda}(h(I), 0) \mu_2(h)^t \det(J(h, I))^{-1} \tilde{\Phi}(h(I)) = \det(Y)^t E_{\Lambda}(Z, 0) \tilde{\Phi}(Z),
\]
where \( Z = h(I) = X + iY \). We get the last equality because, for \( Z \in h_2 \) and \( h \in H(\mathbb{R})^+ \),
\[
\text{Im}(h(Z)) = \mu_2(h)^t J(h, Z)^{-1} \text{Im}(Z) J(h, Z)^{-1}.
\]
This completes the proof of the lemma. \( \square \)

### 5.4.3 Lemma
With notations as above, we have
\[
\frac{Z(\frac{t}{6} - \frac{1}{2}, \Lambda)}{(\Phi, \Phi)_2} \in \bar{\mathbb{Q}},
\]
where
\[
(\Phi, \Phi)_2 = \int_{\mathfrak{g}/\mathfrak{k}_2} |\Phi(Z)|^2 \det(Y)^{l-3} dX dY.
\]

Proof. Let \( \Gamma(2)(N) := \{ g \in \text{Sp}_4(\mathbb{Z}) : g \equiv 1 \pmod{N} \} \). Since \( \Gamma(2)(N) \subset \tilde{\Gamma}(N) \) we know that \( \mathcal{E}_{\Lambda}|_{h_2} \) is a holomorphic Siegel modular form of weight \( l \) with respect to \( \Gamma(2)(N) \). Let us denote the space of all holomorphic Siegel modular forms of weight \( l \) with respect to \( \Gamma(2)(N) \) by \( M_l(\Gamma(2)(N)) \) and its subspace of cusp forms by \( S_l(\Gamma(2)(N)) \). For \( \Phi_1, \Phi_2 \in M_l(\Gamma(2)(N)) \) with one of the \( \Phi_i \) a cusp form, one can define the Petersson inner product \( \langle \Phi_1, \Phi_2 \rangle_N \) in the usual way. Let \( V \) be the orthogonal complement of \( S_l(\Gamma(2)(N)) \) in \( M_l(\Gamma(2)(N)) \) with respect to the Petersson inner product. In Corollary 2.4.6 of [14], it is shown, using the Siegel operator, that \( V \) is generated by Eisenstein series. By Theorem 3.2.1 of [14], one can choose a basis \( \{ E_j \} \) such that all the Fourier coefficients of each \( E_j \) are algebraic. From [10, p. 460], we can find an orthogonal basis \( \{ \Phi_i \} \) of \( S_l(\Gamma(2)(N)) \) such that \( \Phi = \Phi_1 \) and all the Fourier coefficients of the \( \Phi_i \) are algebraic. Let us write
\[
\mathcal{E}_{\Lambda}|_{h_2} = \sum_i \alpha_i \Phi_i + \sum_j \beta_j E_j.
\]
Given a \( F \in M_l(\Gamma(2)(N)) \) and \( \sigma \in \text{Aut}(\mathbb{C}/\bar{\mathbb{Q}}) \), let \( F^\sigma \in M_l(\Gamma(2)(N)) \) be defined by applying the automorphism \( \sigma \) to the Fourier coefficients of \( F \). Applying \( \sigma \) to \( \mathcal{E}_{\Lambda}|_{h_2} \) we get
\[
\mathcal{E}_{\Lambda}|_{h_2} = \sum_i \sigma(\alpha_i) \Phi_i + \sum_j \sigma(\beta_j) E_j.
\]
This follows from the construction of the bases \( \{ \Phi_i \}, \{ E_j \} \) and the property \( \mathcal{E}_{\Lambda}|_{h_2} \). From (166) and (167) we now get
\[
\sigma \left( \langle \mathcal{E}_{\Lambda}|_{h_2}, \Phi_1 \rangle \right) = \langle \mathcal{E}_{\Lambda}|_{h_2}, \Phi_1 \rangle \quad \text{for all } \sigma \in \text{Aut}(\mathbb{C}/\bar{\mathbb{Q}}) \quad \Rightarrow \quad \langle \mathcal{E}_{\Lambda}|_{h_2}, \Phi_1 \rangle \in \mathbb{Q}.
\]
Now, using Lemma 5.4.2, we get the result.

Let \((\Psi, \Psi)_1 = (\text{SL}_2(\mathbb{Z}) : \Gamma_1(N))^{-1} \int_{\Gamma_1(N) \backslash \mathbb{H}} |\Psi(z)|^2 y^{-2} dx dy\), where \(\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a, d \equiv 1 \pmod{N} \right\}\). We have the following generalization of Theorem 4.8.3 of [9].

**5.4.4 Theorem.** Let \(\Phi\) be a cuspidal Siegel eigenform of weight \(l\) with respect to \(\Gamma_2\) satisfying the two assumptions from Section 5.1. Let \(\Psi\) be a normalized, holomorphic, cuspidal eigenform of weight \(l\) with respect to \(\Gamma_0(N)\), with \(N\) a square-free, positive integer. Then

\[
L\left(\frac{l}{2} - 1, \pi_\Phi \times \tau_\Psi\right) \pi^{l-8}(\Phi, \Phi)_2(\Psi, \Psi)_1 \in \bar{\mathbb{Q}}
\]

**Proof.** By Theorem 5.3.1, we have

\[
Z\left(\frac{l}{2} - 1, \frac{1}{2}, \Lambda\right) = C \pi^{4-2l} \frac{L\left(\frac{l}{2} - 1, \pi_\Phi \times \tau_\Psi\right)}{\zeta(l-2) L\left(\frac{l-1}{2}, \tau_\Psi \times \text{AI}(\Lambda)\right)},
\]

where

\[
C = a(\Lambda) D^{-l+2} 2^{-4l+6} (2l-5)! \prod_{p \mid N} \frac{p(p-1)}{(p+1)(p^2-1)} \left(1 - \left(\frac{\sqrt{-D}}{p}\right)p^{-1}(1-p^{-l+2})^{-1}\right) \in \bar{\mathbb{Q}};
\]

observe that \(\frac{\pi}{2} = \frac{\sqrt{l}}{2}\), and that \(c(1) = (4\pi)^{-l/2}\) by \([160]\). It is well known that \(\zeta(l-2)\pi^{2-l} \in \mathbb{Q}\). Using \([27]\), by the same argument as in the proof of Theorem 4.8.3 in [9], we get

\[
L\left(\frac{l}{2} - 1, \tau_\Psi \times \text{AI}(\Lambda)\right) \pi^{l-2}(\Psi, \Psi)_1 \in \bar{\mathbb{Q}}.
\]

Together with \([165]\), this implies the theorem.

We remark that it would be interesting to know the behavior of the quantity \(\frac{L\left(\frac{l}{2} - 1, \pi_\Phi \times \tau_\Psi\right)}{\pi^{l-8}(\Phi, \Phi)_2(\Psi, \Psi)_1}\) under the action of \(\text{Aut}(\mathbb{C})\). This subject will be considered in a future work.

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