Hodge operators and groups of isometries of diagonalizable symmetric bilinear forms in characteristic two

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Abstract
We study groups of isometries on non-alternating symmetric bilinear forms on vector spaces of characteristic two, and actions of these groups on exterior powers of the space, viewed as modules over algebras generated by Hodge operators.

Introduction
In [8], we have introduced Hodge operators using diagonalizable \( \sigma \)-hermitian forms on vector spaces over a field \( F \) of arbitrary characteristic. If the dimension of that space is an even number \( n = 2\ell \) then these operators help to understand exceptional homomorphisms between groups of semi-similitudes; these homomorphisms can be interpreted as representations of the group of semi-similitudes of the given form on the \( \ell \)-th exterior power of the space, where the latter is turned into a module over an algebra \( K_\ell \) generated by the Hodge operator (see [8, Sect. 2]). That algebra turns out to be a composition algebra if \( \sigma \neq \text{id} \) or if \( \text{char} \, F \neq 2 \), but it will be inseparable if \( \sigma = \text{id} \) and \( \text{char} \, F = 2 \): in that case, we obtain \( K_\ell \cong F[X]/(X^2 - \delta) \), and the group of \( F \)-linear automorphisms is trivial.

In the case where \( \sigma = \text{id} \) and the characteristic is two, the forms in question are not the ones that lead to classical groups: we then use a bilinear form that is symmetric

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but not alternating (see [1,2] below); the definitions of classical groups (i.e., symplectic, unitary, or orthogonal groups) in characteristic two employ non-degenerate forms that are alternating, $\sigma$-hermitian with $\sigma \neq \text{id}$, or quadratic forms.

Therefore, the inseparable case is treated in a cursory way only in [8]. However, it leads to phenomena that appear to be interesting, if only as a marked contrast to the results in [8]. We treat this case in a more detailed manner in the present notes, with a focus on $\ell = 2$ (and $n = 4$) because interesting phenomena are already apparent in this dimension (and the Klein quadric provides some extra geometric intuition).

1 The Hodge operator in characteristic two

Fundamental definitions and results about Hodge operators have been worked out in some detail in [8]. We repeat the fundamental facts (with simplifications due to the concentration on a special case) and refer to [8] for details and proofs.

1.1 Notation. Let $F$ be any field of characteristic 2, let $V$ be a vector space of dimension $n$ over $F$, and let $h: V \times V \to F$ be a non-degenerate symmetric bilinear form. Moreover, assume that there exists an orthogonal basis $v_1, \ldots, v_n$ with respect to $h$ (see 1.2 below).

As $\dim V$ is assumed to be finite, our assumption that $h$ be not degenerate is equivalent to the fact that (by a slight abuse of notation) we may view $h$ as a $\sigma$-linear isomorphism onto the dual space $V^\vee$, $h: V \to V^\vee: v \mapsto h(v, -)$ see e.g. [7, Ch. I, §2]. Consider now the exterior algebra $\bigwedge V$, cf. [6, VI 9]. We note that $\bigwedge$ is a functor on vector spaces and (semi)linear maps, cp. [8, 1.6]. Moreover, there is a natural isomorphism $(\bigwedge V)^\vee \cong \bigwedge (V^\vee)$, so we may write unambiguously $\bigwedge V^\vee$. Explicitly, we have $\langle f_1 \wedge \cdots \wedge f_\ell, w_1 \wedge \cdots \wedge w_\ell \rangle = \det(\langle f_i, w_k \rangle)$ for $f_i \in V^\vee$ and $w_k \in V$, see [9, I.5.6] or [1, §8, Thme. 1, p. 102]. Applying the functor $\bigwedge$ to $h: V \to V^\vee$, we obtain $\bigwedge h: \bigwedge V \to \bigwedge V^\vee$; we interpret this as a bilinear form $\bigwedge h$ on the exterior algebra $\bigwedge V$. Using the explicit formula above, we find

$$\bigwedge h(v_1 \wedge \cdots \wedge v_\ell, w_1 \wedge \cdots \wedge w_\ell) = \bigwedge^\ell h(v_1 \wedge \cdots \wedge v_\ell, w_1 \wedge \cdots \wedge w_\ell) = \det(h(v_i, w_j)).$$

In particular, the form $\bigwedge^\ell h$ is symmetric because transposition does not change the determinant.

\footnote{The treatment in [1] is quite different from that in later editions [3].}
1.2 Lemma. Let \( h: V \times V \rightarrow F \) be a non-zero symmetric bilinear form on a vector space \( V \) of finite dimension over a field \( F \) with \( \text{char } F = 2 \). Then there exists an orthonormal basis for \( V \) with respect to \( h \) if, and only if, the form \( h \) is not alternating (i.e., if there exists \( v \in V \) with \( h(v,v) \neq 0 \)).

Proof. If there exists an orthonormal basis then the Gram matrix with respect to that basis is diagonal, and will be zero if the form is alternating. Conversely, assume that there exists \( v \in V \) with \( h(v,v) \neq 0 \). If \( V^\perp \neq \{0\} \), we choose any basis for \( V^\perp \), and any vector space complement \( W \) to \( V^\perp \) with \( v \in W \). The restriction of \( h \) to \( W \) is not degenerate, and not alternating. It suffices to show that there is an orthogonal basis for \( W \). Assume that \( w_1, \ldots, w_k \) are pairwise orthogonal vectors in \( W \) with \( h(w_i, w_j) \neq 0 \). Then these vectors are linearly independent, the restriction of \( h \) to their span \( W_k \) is not degenerate, and \( W_k^\perp \cap W \) is a complement to \( W_k \) in \( W \).

We proceed by induction on \( \dim W^\perp \cap W \): If the restriction of \( h \) to \( W^\perp \cap W \) is not alternating, we apply the induction hypothesis. It remains to study the case where there exist \( x, y \in W^\perp \cap W \) with \( h(x,y) = 1 \) and \( h(x,x) = h(y,y) = 0 \). Put \( w_{k+1} := w_k + h(w_k, w_k)y, w_{k+2} := w_k + h(w_k, w_k)x + h(w_k, w_k)y, \) and \( w_k := w_k + x \). Straightforward computation yields \( h(w_{k+1}, w_{k+1}) = h(w_{k+2}, w_{k+2}) = h(w_k, w_k) \neq 0 \), and that the vectors \( w_k, w_{k+1}, w_{k+2} \in \{w_1, \ldots, w_{k-1}\}^\perp \) are pairwise orthogonal.

So \( w_1, \ldots, w_{k-1}, w_k, w_{k+1}, w_{k+2} \) is an orthogonal basis for \( W + Fx + Fy \). Applying the induction hypothesis to \( (W + Fx + Fy)^\perp \cap W \) finishes the proof. \( \square \)

1.3 The Pfaffian form. Recall that \( \dim \Lambda^n V = 1 \) if \( \dim V = n \). We fix an isomorphism \( b: \Lambda^n V \rightarrow F \). For each positive integer \( \ell \leq n \) the map \( b \) induces an isomorphism \( \text{Pf}: \Lambda^{n-\ell} V \rightarrow \Lambda^\ell V \) given by

\[
\text{Pf}(v_1 \wedge \cdots \wedge v_{n-\ell})(w_1 \wedge \cdots \wedge w_\ell) = b(v_1 \wedge \cdots \wedge v_{n-\ell} \wedge w_1 \wedge \cdots \wedge w_\ell).
\]

This is the Pfaffian form, see [6] VI 10 Problems 23–28, VIII 12 Problem 42]. As \( \text{char } F = 2 \), the resulting bilinear map \( \text{Pf} \) on \( \Lambda V \) is symmetric,

\[
\text{Pf}(v_1 \wedge \cdots \wedge v_{n-\ell}, w_1 \wedge \cdots \wedge w_\ell) = \text{Pf}(w_1 \wedge \cdots \wedge w_\ell, v_1 \wedge \cdots \wedge v_{n-\ell}).
\]

If \( n = 2\ell \) then \( \text{Pf}(v_1 \wedge \cdots \wedge v_{\ell}, v_1 \wedge \cdots \wedge v_{\ell}) = 0 \) holds for \( v_1 \wedge \cdots \wedge v_{\ell} \).

1.4 Remark. For \( n = 4 \) and \( \ell = 2 \) we are dealing with the space \( \Lambda^2 F^4 \) that carries the Klein quadric. The \textit{quadratic form} \( Pq \) defining the Klein quadric is also referred to as a Pfaffian form (cf. [5] and [12] where this form is denoted by \( q \)), and \( \text{Pf} \) is the polar form of that quadratic form. Under the present assumption \( \text{char } F = 2 \), the polar form \( \text{Pf} \) carries less information than the quadratic form \( Pq \). Note that \( \text{Pf} \) is alternating because it is the polar form of a quadratic form in characteristic two.
If one interprets the elements of $\wedge^2 F^4$ as alternating matrices then there exists a scalar $s \in F^\times$ such that $Pq(X)^2 = s \det X$ holds for each $X \in \wedge^2 F^4$, cf. \cite{2} § 5 no. 2, Prop. 2, p. 84; the scalar $s$ reflects the choice of basis underlying that interpretation. See \cite{13} 12.14] for an interpretation of $Pq$ in terms of the exterior algebra.

1.5 The Hodge operator. We now consider the composite

$$J := Pf^{-1} \circ h : \wedge^\ell V \xrightarrow{\wedge^h} \wedge^{\ell'} V' \xrightarrow{Pf^{-1}} \wedge^{n-\ell'} V.$$ 

This semilinear isomorphism is the Hodge operator. It depends, of course, on $h$ and on $b$ but not on the choice of basis.

1.6 Explicit computation. Suppose that $v_1, \ldots, v_n$ is an orthogonal basis of $V$. For $\wedge^\ell V$ we use the basis vectors $v_{i_1} \wedge \cdots \wedge v_{i_\ell}$ with ascending $i_1 < \cdots < i_\ell \leq n$. Then $\wedge^\ell h(v_1 \wedge \cdots \wedge v_{i_\ell}, -)$ is a linear form on $\wedge^\ell V$ which annihilates each one of those basis vectors, except for $v_1 \wedge \cdots \wedge v_{i_\ell}$; in fact

$$\wedge^\ell h(v_1 \wedge \cdots \wedge v_{i_\ell}, v_1 \wedge \cdots \wedge v_{i_\ell}) = h(v_1, v_1) \cdots h(v_{i_\ell}, v_{i_\ell}).$$

In other words: $\wedge^\ell h$ is again diagonalizable. It then also follows that $\wedge^\ell h$ is not degenerate. The linear form $Pf(v_{i_\ell+1} \wedge \cdots \wedge v_n)$ annihilates the same collection of basis vectors, and

$$Pf(v_{i_\ell+1} \wedge \cdots \wedge v_n, v_1 \wedge \cdots \wedge v_{i_\ell}) = b(v_1 \wedge \cdots \wedge v_n).$$

Therefore

$$J(v_1 \wedge \cdots \wedge v_{i_\ell}) = v_{i_\ell+1} \wedge \cdots \wedge v_n \frac{h(v_1, v_1) \cdots h(v_{i_\ell}, v_{i_\ell})}{b(v_1 \wedge \cdots \wedge v_n)}.$$

Note that this last formula is correct only if $v_1, \ldots, v_n$ is an orthogonal basis, and cannot be used if $v_1 \wedge \cdots \wedge v_{i_\ell}$ corresponds to a subspace $U$ of $V$ such that $h|_{U \times U}$ is degenerate.

1.7 The square of the Hodge operator. Let $H$ be the Gram matrix of $h$ with respect to the orthogonal basis $v_1, \ldots, v_n$. The square of $J$ is a linear automorphism of $\wedge^\ell V$, and we find that

$$J^2 = \delta_\ell \id$$

where

$$\delta_\ell := \frac{\det(H)}{b(v_1 \wedge \cdots \wedge v_n)}.$$

Recall that $\det(H)$ depends on the choice of basis; the invariant would be the square class $\text{disc}(h) \in F^\times / F^\times$ of $\det(h(v_i, v_j))$. However, the whole expression depends only on $h$ and $b$. Replacing the isomorphism $b : \wedge^n V \to F$ changes $J$ by a factor and $J^2$ by the square of that factor. In particular, the isomorphism type of the algebra $K_\ell$ introduced in \cite{1.9} below does not depend on the choice of $b$. 

4
1.8 Lemma. For all $x, y \in \bigwedge^\ell V$ we have

\begin{align*}
\text{a.} & \quad \text{Pf}(J(x), y) = \bigwedge^\ell h(x, y), \\
\text{b.} & \quad \text{Pf}(J(x), J(y)) = \delta_\ell \text{Pf}(y, x), \\
\text{c.} & \quad \bigwedge^\ell h(J(x), y) = \delta_\ell \bigwedge^\ell h(x, y), \\
\text{d.} & \quad \bigwedge^\ell h(J(x), J(y)) = \delta_\ell \bigwedge^\ell h(x, y).
\end{align*}

From now on, assume $n = 2\ell$. Then $J$ is an $F$-linear endomorphism of $\bigwedge^\ell V$. We are going to use $J$ to give $\bigwedge^\ell V$ the structure of a right module over an associative algebra of dimension 2 over $F$.

1.9 The algebra $K_\ell$. Take $\delta_\ell = \frac{\text{det}(H)}{b(v_1 \wedge \cdots \wedge v_n)^2}$ as in (1.7) and let $K_\ell$ denote the $F$-algebra consisting of all matrices of the form

$$x = \begin{pmatrix} x_0 & \delta_\ell x_1 \\ x_1 & x_0 \end{pmatrix} \in F^{2 \times 2}.$$ 

We identify $x_0 \in F$ with the diagonal matrix $\begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix}$ and put $j_\ell := \begin{pmatrix} 0 & \delta_\ell \\ 1 & 0 \end{pmatrix}$. Thus

$$\begin{pmatrix} x_0 & \delta_\ell x_1 \\ x_1 & x_0 \end{pmatrix} = x_0 + j_\ell x_1.$$ 

Note that $j_\ell^2 = \delta_\ell$.

1.10 Definition. For $v \in \bigwedge^\ell V$ we put $v j_\ell := J(v)$. In this way, the space $\bigwedge^\ell V$ becomes an $O(V, h)$-$K_\ell$-bimodule, i.e., it becomes a right module over $K_\ell$ and $O(V, h)$ acts $K_\ell$-linearly from the left. Choosing an orthogonal basis $v_1, \ldots, v_n$ for $V$ with a fixed ordering, we obtain a basis $B$ for $\bigwedge^\ell V$ consisting of all $v_{j_1} \wedge \cdots \wedge v_{j_\ell}$ where $(j_1, \ldots, j_\ell)$ is an increasing sequence of length $\ell$ in $\{1, \ldots, n\}$. The sequences with $j_1 = 1$ form a subset $B_1$ of $B$, and $J$ maps each element of $B_1$ to one of $B \setminus B_1$. Moreover, the set $B_1$ forms a basis for the $K_\ell$-module $\bigwedge^\ell V$, showing that the latter is a free module.

1.11 The bilinear form on the module. We define $g : \bigwedge^\ell V \times \bigwedge^\ell V \to K_\ell$ by

$$g(u, v) := \bigwedge^\ell h(u, v) + \bigwedge^\ell h(u, v j_\ell) j_\ell^{-1} = \bigwedge^\ell h(u, v) + j_\ell (-1)^\ell \text{Pf}(u, v);$$

see (1.8) for the description on the right hand side. This expression is $K_\ell$-bilinear.

Note that $K_\ell$ is not a field, in general: we need the more general concept of bilinear forms over rings.
1.12 Proposition. The form $g$ is diagonalizable.

For a general proof (and a general formula) see [8, 2.7]. Actually, if $K_\ell$ is a field (of characteristic two, and $\sigma = \text{id}$ by our assumptions) it suffices to note that $g$ is not alternating, see [1.2]. For the applications in Section 2 below, we give a special statement explicitly:

1.13 Example. Let $v_1, v_2, v_3, v_4$ be an orthogonal basis for $V$, with respect to $h$. Then $w_2 := v_1 \wedge v_2, w_3 := v_1 \wedge v_3, w_4 := v_1 \wedge v_4$ form an orthogonal basis for the free $K_\ell$-module $\Lambda^2 V$, with respect to $g$. Explicitly, we have $g(w_2, w_2) = h(v_1, v_1)h(v_2, v_2)$, $g(w_3, w_3) = h(v_1, v_1)h(v_2, v_2)$, and $g(w_4, w_4) = h(v_1, v_1)h(v_4, v_4)$.

From the definition of $g$ it is clear that $O(V, h)$ preserves $g$ and that $\Gamma O(V, h)$ acts by semi-similitudes of $g$, see [5, 1.8]. Thus we have, for $\dim(V) = n = 2\ell$, constructed a homomorphism $\eta_\ell: \Gamma O(V, h) \to \Gamma O(\Lambda^2 V, g)$.

1.14 Lemma. The kernel of $\eta_\ell$ is trivial.

Proof. That kernel consists of all scalar multiples $s \text{id}$ of the identity where $s^2 = 1$. As $\text{char } F = 2$, this yields $s = 1$.

We will call $K_\ell$ split whenever it contains divisors of zero. This extends the established terminology for composition algebras. Recall that $K_\ell$ is split precisely if $\delta_\ell$ is a square: $\delta_\ell = s^2$ for some $s \in F^\times$ (and this happens precisely if $h$ has discriminant 1).

In that case, we may assume $s = 1$ without loss of generality. In fact, if we replace our isomorphism $b: \Lambda^2 V \to F$ by $sb$ then the Hodge operator $J$ is replaced by $J s^{-1}: X \mapsto J(X)s^{-1}$, and we have $(J s^{-1})^2 = \text{id}$ while the algebra $K_\ell$ remains the same. If $s = 1$ then $z := 1 + j_\ell \in K_\ell$ satisfies $z^2 = 2z = 0$, and is nilpotent. Thus $K_\ell \cong F[X]/(X^2)$ is a local ring if $K_\ell$ is split.

1.15 Lemma. Let $W := \Lambda^2 V$, and assume that $K_\ell$ is split.

a. The maximal ideal in $K_\ell$ is generated by a nilpotent element $z$. The submodule $Wz$ and the quotient module $W/Wz$ are isomorphic via $\rho_z: w + Wz \mapsto wz$.

b. The restriction of the form $g$ to the subspace $Wz$ is trivial.

c. The $K_\ell$-submodule $Wz$ is invariant under $\eta(\Gamma O(V, h))$. Thus we obtain a homomorphism $\eta_\ell^0: \Gamma O(V, h) \to \Gamma \Lambda (Wz)$.

d. The group induced by $\ker \eta_\ell^0$ on $W$ is an elementary abelian 2-group, acting trivially on $W/Wz$.

Proof. The first three assertions are taken from [8, 3.6]. For the last assertion, we note that elements of $\ker \eta_\ell^0$ act trivially on $W/Wz$ because $\rho_z$ is a module homomorphism. So $\ker \eta_\ell^0$ is isomorphic to a subgroup of $\text{Hom}_F(W/Wz, Wz)$.

\[ \square \]
The precise structure of $O(V,h)$ and $\ker \eta^h_0$ depends on the defect of the form $h$.

In Section 2 we will repeatedly need the following. Recall that a local ring is a ring in which the set of non-invertible elements is closed under addition (we allow the case where that set consists of 0 alone).

1.16 Lemma. Let $K$ be a commutative local ring with $1 + 1 = 0$ in $K$, and let $SL_2(K) := \{ (a b c d) \mid a,b,c,d \in K, ad - bc = 1 \}$. We write $L_x := (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$, $U_x := (\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix})$,

\[
\hat{L}_x = \begin{pmatrix} 1 + x & 0 & x \\ 0 & 1 & 0 \\ x & 0 & 1 + x \end{pmatrix}, \quad \hat{U}_x = \begin{pmatrix} 1 + x & x & 0 \\ x & 1 + x & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad T := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.
\]

a. The set $\{ L_x \mid x \in K \} \cup \{ U_x \mid x \in K \}$ generates $SL_2(K)$.

b. The set $\{ \hat{L}_x \mid x \in K \} \cup \{ \hat{U}_x \mid x \in K \}$ generates a group $\hat{\Sigma}$ isomorphic to $SL_2(K)$.

Indeed, $T^{-1} \hat{L}_x T = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$, $T^{-1} \hat{U}_x T = (\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})$, and $T^{-1} \hat{\Sigma} T = \{ (\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}) \mid A \in SL_2(K) \}$.

c. We have $\hat{\Sigma} = O(K^3, f)$, where $f$ is given by $f(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3$, and $T^{-1} \hat{\Sigma} T = O(K^3, f')$, where $f'(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3$.

Proof. Assertion a is well known for the case where $K$ is a field (e.g., see [13, p. 22]). Since matrix algebra over a local ring is less popular, we provide a direct argument (actually, in a form that works for every local ring): Let $A = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in SL_2(K)$. Then $a$ and $c$ cannot both be non-invertible. If $a$ is invertible, then

\[
\begin{pmatrix} 1 & 0 \\ a^{-1}(c - 1) & 1 \end{pmatrix} \begin{pmatrix} 1 & a - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a^{-1}(1 + b - a) \\ 0 & 1 \end{pmatrix} = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})
\]

If $c$ is invertible, then

\[
(\begin{smallmatrix} 1 & c^{-1}(a - 1) \\ 0 & 1 \end{smallmatrix}) (\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}) (\begin{smallmatrix} 1 & c^{-1}(d - 1) \\ 0 & 1 \end{smallmatrix}) = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})
\]

Assertion b follows by direct computations.

For Assertion c we note first that $O(K^3, f)$ leaves invariant the kernel of the quadratic form $g$ given by $g(x) = f(x, x)$, considered as a semilinear map from $K^3$ to the module $K$ over $K^3 := \{ x^2 \mid x \in K \}$.  

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The document discusses the properties of matrices over a local ring, specifically focusing on the group $SL_2(K)$ and its representations. The proof involves direct computations and leverages the properties of local rings and matrix algebras.
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Hodge operators in characteristic two

Conjugation by $T^{-1}$ translates the matrix description for $O(K^3, f)$ from standard coordinates into coordinates with respect to the basis $u_1 := (1, 1, 1)^\top$, $u_2 := (1, 1, 0)^\top$, $u_3 := (1, 0, 1)^\top$. The Gram matrix for $f$ with respect to that basis is $T^\top T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, where $i := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. As $\ker q = K u_2 + K u_3$ is invariant under $O(K^3, f)$, we have $T^{-1} O(K^3, f) T \subseteq \left\{ \begin{pmatrix} a \\ b \\ C \end{pmatrix} \right\}$.

Evaluating the condition $\begin{pmatrix} a & b^\top \\ 0 & C^\top \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} a \\ 0 \\ C \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$,

we find $a = 1, b = 0$, and $\det C = 1$. So $T^{-1} O(K^3, f) T = T^{-1} \Sigma T$, and $O(K^3, f) = \Sigma$ follows.

2 The four-dimensional cases

We focus on the case where $\ell = 2$ and $n = 2\ell = 4$, and write $K := K_2$. Either $K$ splits and is isomorphic to the local algebra $F[X]/(X^2)$, or we have an inseparable quadratic field extension $K|F$. The quadratic form $q: V \to F: x \mapsto h(x,x)$ is a $q$-semilinear map, where $F$ is considered as a vector space over the subfield $F[\sq}$ of squares, and $q: F \to F[\sq]: s \mapsto s^2$ is the Frobenius endomorphism. Note that $O(V, h)$ is contained in $O(V, q)$. We distinguish cases according to $\dim_{F[\sq]} q(V) \in \{1, 2, 3, 4\}$; recall that $\dim_{F[\sq]} \ker q = 4 - \dim_{F[\sq]} q(V) \in \{3, 2, 1, 0\}$ is called the defect of $q$. At several places we will use the fact that the orthogonal group of an anisotropic quadratic form is trivial if the ground field has characteristic 2; cf. [4, § 16, p. 35]. Recall from [11, § 16] that the restriction $g|_{W_2 \times W_2}, Pf|_{W_2 \times W_2}$ is trivial if $z$ is nilpotent (of course, this is of interest only if $K$ splits).

We use the standard basis $e_1, e_2, e_3, e_4$ for $V = F^4$, and write $W := \wedge^2 V$.

2.1. If $\dim_{F[\sq]} q(V) = 1$ then we may (up to similitude) assume $h(v, w) = v^t w$. This form $h$ has Witt index 2, it is equivalent to the form $\tilde{h}$ given by

$$\tilde{h}(v, w) := v^t \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} w = v_1 w_4 + v_2 w_3 + v_3 w_2 + v_4 w_1 + v_4 w_4;$$
e.g., use the basis $b_1 := (1, 1, 1, 1)^\top = e_1 + e_2 + e_3 + e_4$, $b_2 := (1, 1, 0, 0)^\top = e_1 + e_2$, $b_3 := (1, 1, 0, 0)^\top = e_1 + e_3$, $b_4 := (0, 0, 0, 1)^\top = e_4$. The orthogonal group $O(V, \tilde{h})$ leaves the quadratic form $\tilde{h}(v, v) = v_4^2$ invariant. Thus it fixes the linear form with matrix $(0, 0, 0, 1)$, and using suitable block matrices we obtain

$$O(V, \tilde{h}) \leq \left\{ \begin{pmatrix} A & x \\ 0 & a \end{pmatrix} \middle| a \in F^\times, A \in \text{GL}_3(F), x \in F^3 \right\}.$$ 

Now one computes easily that

$$O(V, \tilde{h}) = \left\{ \begin{pmatrix} 1 & t^i i & c \\ 0 & B & Bt \\ 0 & 0 & 1 \end{pmatrix} \middle| B \in \text{SL}_2(F), t \in F^2, c \in F \right\}$$

where $i := (0 1 0)$.

Note that the elements with trivial $B$ form an elementary abelian subgroup $\Xi$. In fact, we have an isomorphism

$$\xi: F^2 \times F \to \Xi: (t_1, t_2, t_3) \mapsto \begin{pmatrix} 1 & t_2 & t_1 & t_3 + t_1 t_2 \\ 0 & 1 & 0 & t_1 \\ 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

The map $\xi$ is $F$-linear if we let the scalars act via the rule $(t_1, t_2, t_3) \cdot s := (t_1 s, t_2 s, t_3 s^2)$. In particular, the dimension of $F^2 \times F \cong \Xi$ becomes $2 + \dim F^2$ which will be greater than 3 whenever the field $F$ is not perfect. The group

$$\Sigma := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}_4(F) \middle| B \in \text{SL}_2(F) \right\} \cong \text{SL}_2(F)$$

of block matrices normalizes $\Xi$ and acts in the expected way: it fixes $\xi(\{0\}^2 \times F)$ pointwise and induces the usual action on $\xi(F^2 \times \{0\})$. However, the set $\xi(F^2 \times \{0\})$ is not a subgroup; there is no $\Sigma$-invariant subgroup complement to $\xi(\{0\}^2 \times F)$.

With respect to the $K$-basis $b_1 \wedge b_2$, $b_2 \wedge b_4$, $b_3 \wedge b_4$, the Gram matrix for $g$ is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. The elements of $O(W, g)$ are thus described (with respect to the same basis) by the block matrices $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$, with $A \in \text{SL}_2(K)$, see \[\text{[14]}\].

\[\text{[14]}\] This phenomenon also plays its role in the study of duality of symplectic quadrangles, cf. \[\text{[10]}\].
We fix the isomorphism \( b: \wedge^4 V \to F \) in such a way that \( f(e_{\pi(1)} \wedge e_{\pi(2)}) = e_{\pi(3)} \wedge e_{\pi(4)} \) for each permutation \( \pi \) of \( \{1, 2, 3, 4\} \); recall that the standard basis \( e_1, e_2, e_3, e_4 \) is an orthonormal basis with respect to the form \( h \). In particular, we now find \( \delta = 1 \), the algebra \( K_\ell \) splits, and \( z := 1 + j \) is nilpotent. Using the basis \( b_1, \ldots, b_4 \) from above we observe that

\[
\begin{align*}
Y_1 &:= (b_1 \wedge b_4)z = (e_1 \wedge e_4 + e_2 \wedge e_4 + e_3 \wedge e_4)z = b_1 \wedge b_4 + b_2 \wedge b_3, \\
Y_2 &:= (b_2 \wedge b_4)z = (e_1 \wedge e_4 + e_2 \wedge e_4)z = b_1 \wedge b_2, \\
Y_3 &:= (b_3 \wedge b_4)z = (e_1 \wedge e_4 + e_3 \wedge e_4)z = b_1 \wedge b_3
\end{align*}
\]

form a basis for \( Wz \).

Evaluating \( \xi(t_1, t_2, t_3) \in \Xi \) at \( Y_1, Y_2, \) and \( Y_3 \), we see that \( \Xi \) acts trivially on \( Wz \) (cp. [11.14]). For \( B \in \text{SL}_2(F) \) we find that \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) maps \( Y_1a_1 + Y_2a_2 + Y_3a_3 \) to \( Y_1a_1 + Y_2a'_2 + Y_3a'_3 \) with \( (a_2, a'_3)^T = B(a_2, a_3)^T \). In other words, the image of that element of \( \Xi \) under \( \eta \) is described by the block matrix \( \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \).

This action of \( \text{SL}_2(F) \) is an action by isometries of the \( F \)-bilinear form \( f' \) on \( Wz \) defined by \( f'(Y_1a_1 + Y_2a_2 + Y_3a_3, Y_1x_1 + Y_2x_2 + Y_3x_3) := a_1x_1 + a_2x_2 + a_3x_3 \); see [11.16]. Note that \( \Xi \cong \text{SL}_2(F) \) induces the full group \( O(Wz, f') \). However, the form \( f' \) is not \( g^0 \) because \( g^0 \equiv 0 \), see [11.15].

The range \( q(V) \) of the quadratic form \( q \) is just \( F^\bullet \). So every similitude has an element of \( F^\bullet \) as multiplier, and belongs to \( F^\bullet O(V, q) \).

2.2. If \( \dim_F q(V) = 2 \), we may assume \( h(x, y) = x_1c_1y_1 + x_2c_2y_2 + x_3c_3y_3 + x_4c_4y_4 \), where \( c_3 \) and \( c_4 \) lie in \( c_1F^\bullet + c_2F^\bullet \), and \( c_1, c_2 \) are linearly independent over \( F^\bullet \).

If \( c_3 \in F^\bullet c_1 \) we may assume \( c_3 = c_1 \). If \( c_3 \notin F^\bullet c_1 \) then there exist \( s, t \in F \) with \( c_2 = s^2c_1 + t^2c_3 \). Then \( T := \begin{pmatrix} tc_3 \\ sc_1 \\ 1 \end{pmatrix} \) is invertible, and \( T^T \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} T = \begin{pmatrix} c_1^2 & 0 \\ 0 & c_2^2 \end{pmatrix} \) holds with \( c_1^2 := (tc_3)^2c_1 + (sc_1)^2c_3 \). Thus \( f_1 := tc_3e_1 + sc_1e_2 \), \( f_2 := e_2 \), \( f_3 := sc_1 + te_3 \), \( f_4 := e_4 \) is an orthogonal basis, and the Gram matrix for \( h \) with respect to that basis has diagonal entries \( c_1^2, c_2^2, c_2, c_4 \). Repeating the argument, we obtain that either there exists a diagonal Gram matrix with three identical diagonal entries (if \( c_4 \notin F^\bullet c_1^2 \)), or there exists a diagonal Gram matrix with two pairs of identical diagonal entries.

Up to similitude, we may thus assume that the Gram matrix (with respect to the standard basis) is one of

\[
H_1 := \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad H_2 := \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

respectively, with \( m \in F \setminus F^\bullet \).
a. If the form \( h \) is described by \( H_1 \) then its discriminant \( m \) is not a square, and \( K \) is not split; in fact, we have \( K = F(j) \cong F[X]/(X^2 - m) \), with \( j^2 = m \notin F \). With respect to the \( K \)-basis \( e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4 \), the Gram matrix for \( g \) is \( \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 \end{pmatrix} \). From 1.16c we know that \( O(W, g) \cong SL_2(K) \).

In order to understand the group \( O(V, h) \), we first study the quadratic form given by \( q(x) = h(x, x) = mx_1^2 + x_2^2 + x_3^2 + x_4^2 = mx_1^2 + (x_2 + x_3 + x_4)^2 \). As \( m \) is not a square in \( F \), the kernel of \( q \) is the hyperplane \( \{(0, x_2, x_3, x_4)^T \mid x_2 + x_3 + x_4 = 0 \} \). This hyperplane is invariant under \( O(V, h) \); it will be convenient to use the basis \( b_1 := e_1, b_2 := e_2 + e_3 + e_4, b_3 := e_2 + e_3, b_4 := e_2 + e_4 \). With respect to that basis, the Gram matrix of \( h \) is the block matrix \( \begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix} \), where \( N := \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} \) is a Gram matrix for the norm form of \( K[F] \), and \( i := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). In coordinates with respect to that basis, the isometry group of \( h \) consists of the block matrices of the form \( \begin{pmatrix} E & F \\ 0 & A \end{pmatrix} \) with the \( 2 \times 2 \) identity matrix \( E \), and \( A \in SL_2(F) \).

As in 1.16b we generate the group \( SL_2(F) \) by the matrices \( L_x \) and \( U_x \), with \( x \in F \). Transforming \( \begin{pmatrix} \xi & 0 \\ 0 & L_x \end{pmatrix} \) and \( \begin{pmatrix} \xi & 0 \\ 0 & U_x \end{pmatrix} \) back into the description with respect to standard coordinates, we obtain that \( O(V, h) \) is generated by the matrices \( \tilde{A}_x := \tilde{T}^{-1} \begin{pmatrix} \xi & 0 \\ 0 & A_x \end{pmatrix} \tilde{T} \) and \( \tilde{B}_x := \tilde{T}^{-1} \begin{pmatrix} \xi & 0 \\ 0 & B_x \end{pmatrix} \tilde{T} \), where \( \tilde{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( \tilde{A}_x = \begin{pmatrix} 1 & 0 \\ 0 & L_x \end{pmatrix} \), and \( \tilde{B}_x = \begin{pmatrix} 1 & 0 \\ 0 & U_x \end{pmatrix} \), with \( x \in F \).

With respect to the \( K \)-basis \( e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4 \) for \( W \), we then find that the action of these elements on \( W \) is described by the matrices \( \tilde{L}_x \) and \( \tilde{U}_x \), respectively, with \( x \in F \). The same matrices, but with \( x \in K \) instead of \( x \in F \), generate \( O(W, g) \cong SL_2(K) \); see 1.16c. Therefore, the image of \( O(V, h) \) under \( \eta \) equals \( O(W, g) \cap GL_3(F) \).

For each similitude \( \varphi \in GO(V, h) \), the multiplier \( r_\varphi \) lies in the range \( q(V) = F^d + F^d, m \) because that range contains 1. We note that \( q(V) \) forms the quadratic extension field \( F^d(m) \cong F(\sqrt{m}) \) of \( F^d \). Every element \( a^2 + b^2m \in F^d(m) \setminus \{0\} \) is the multiplier of some similitude of the form \( h \); in coordinates with respect to the basis, \( b_1, b_2, b_3, b_4 \), the block matrix \( \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \) with \( A := \begin{pmatrix} a & b \\ bm & a \end{pmatrix} \in GL_2(F) \) describes a similitude with multiplier \( a^2 + b^2m \).

b. Now consider the case where \( h \) is described by \( H_2 \). Then the discriminant is a square, and \( K \) is split; we normalize such that \( j^2 = 1 \) and \( K \cong F[X]/(X^2) \). The kernel of the quadratic form given by \( q(x) = h(x, x) \) is spanned by \( d_1 := e_1 + e_2 \) and \( d_2 := e_3 + e_4 \). We use \( d_3 := e_1 \) and \( d_4 := e_4 \) to extend this to a basis for \( V \). With respect to that basis, the Gram matrix for \( h \) is the block matrix \( \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix} \); again, \( N = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} \) (but this is no longer a Gram matrix for the norm of \( K[F] \)). In these coordinates, the isometry group of \( h \) consists of the block matrices of the form \( \begin{pmatrix} E & F \\ 0 & B \end{pmatrix} \) with the \( 2 \times 2 \) identity matrix \( E \) and \( B \in F^{2 \times 2} \) such that \( NB \) is a symmetric matrix. Transforming this description into standard coordinates, we obtain \( O(V, h) = \left\{ \begin{pmatrix} E + aM & bM \\ mbM & E + cM \end{pmatrix} \mid a, b, c \in F \right\} \), where \( M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \).
This shows that $O(V,h) \cong F^2$ is an elementary abelian 2-group, and we can determine the action on $W$ using $z = 1 + j$: with respect to the $F$-basis

$w_1z := (v_1 \wedge v_4)z = v_1 \wedge v_4 + v_2 \wedge v_3,$

$w_2z := (v_1 \wedge v_3)z = v_1 \wedge v_3 + v_2 \wedge v_4,$

$w_3z := (v_1 \wedge v_2)z = v_1 \wedge v_2 + v_3 \wedge v_4,$

for $W$ obtained from the $K$-basis $w_1 := v_1 \wedge v_4, w_2 := v_1 \wedge v_3, w_3 := v_1 \wedge v_2$ for $W$, the action of $\left(\begin{array}{ccc} E + aM & bM \\ mbM & E + cM \end{array}\right) \in O(V,h)$ on $W$ is described by the matrix

$\left(\begin{array}{ccc} 1 + a + c & a + c & (m + 1)b \\ a + c & 1 + a + c & (m + 1)b \\ 0 & 0 & 1 \end{array}\right).$

This shows that the action of $O(V,h)$ on $W$ is not faithful, the kernel is

$\left\{\left(\begin{array}{ccc} E + aM & 0 \\ 0 & E + aM \end{array}\right) \mid a \in F\right\}.$

We remark that, in coordinates with respect to the basis $w_1, w_2, w_3$, the group $O(W,g)$ is described as $\{\hat{U}_x \mid x \in K\} \cong (K,+)$, cp. [16].

As in case [23] we have $q(V) = F^b(m)$, and $F^b(m) \setminus \{0\}$ is the set of all multipliers of similitudes of $h$; in coordinates with respect to the basis $d_1,d_2,d_3,d_4$, the block matrix $\left(\begin{array}{cc} A & 0 \\ 0 & A \end{array}\right)$ with $A := \left(\begin{array}{cc} a & b \\ bm & a \end{array}\right) \in GL_2(F)$ describes a similitude with multiplier $a^2 + b^2m$.

2.3. If $\dim_F q(V) = 3$ then there is $u_1 \in V \setminus \{0\}$ with $h(u_1,u_1) = 0$. As $h$ is not degenerate, there exists $u_2 \in V$ with $h(u_1,u_2) = 1$. We write $s := h(u_2,u_2)$. If $s = 0$ then dim ker $q > 1$, contradicting our assumption that the range of $q$ has dimension 3. Up to similarity (and rescaling $u_1$), we may thus assume $s = 1$.

The restriction of $h$ to $Fu_1 + Fu_2$ is not degenerate, so $\{u_1,u_2\}^\perp$ forms a vector space complement for that subspace in $V$. If the restriction of $h$ to that complement were isotropic then dim ker $q$ would be greater than 1.

So the restriction of $h$ to $\{u_1,u_2\}^\perp$ is anisotropic, and diagonalizable by [12]. Choosing an orthonormal basis $u_3,u_4$ for $\{u_1,u_2\}^\perp$, we obtain that the Gram matrix for $h$ with respect to the basis $u_1,u_2,u_3,u_4$ is

$H := \left(\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 \end{array}\right);
the discriminant of \( h \) is represented by \( \delta := c_3 c_4 \). Now \( \delta \notin F^{q} \); otherwise, the vector \( u_3 c_4 + u_4 \sqrt{c_3 c_4} \) would be isotropic, and \( \dim \ker q \) would be greater than 1. So \( K \cong F(\sqrt{\delta}) \) is not split. We note that \( c_3 \notin K^{q} \), in fact \( c_3 = (r + t \sqrt{\delta})^2 \) with \( r, t \in F \) would imply \( 0 = 1 + c_3 (r/c_3)^2 + c_4^2 \), contradicting the fact that \( \dim \ker q = 1 \).

The group \( O(V, h) \) leaves invariant the quadratic form \( q \), and also the subspace \( \ker q \).

Using this fact facilitates to see that the elements of \( O(V, h) \) are described (with respect to the basis \( u_1, u_2, u_3, u_4 \)) by matrices of the form \( \left( \begin{smallmatrix} U & 0 \\ 0 & E \end{smallmatrix} \right) \) with \( x \in F, E = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \), and \( U \) as in \( 1.16 \). So \( O(V, h) \cong (F, +) \).

In coordinates with respect to the basis \( v_1 := u_1 + u_2, v_2 := u_2, v_3 := u_3, v_4 := u_4 \), we obtain that the group \( O(V, h) \) consists of the block matrices \( U_\lambda := \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right) \), with \( x \in F \).

With respect to the \( K \)-basis \( w_1 := v_1 \land v_2, w_2 := (v_2 \land v_3) \sqrt{\delta^{-1}}, w_3 := v_1 \land v_3 \), the Gram matrix for the bilinear form \( g \) is \( \left( \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{smallmatrix} \right) \). The form \( g \) is isotropic (as prophesied by \( 8.2.9 \)); in fact, the vector \( w_1 + w_2 \) is isotropic with respect to \( g \).

In order to determine \( O(W, g) \), we consider the quadratic form \( q_g \) obtained by evaluating \( g \) on the diagonal. That quadratic form has one-dimensional kernel because \( c_3 \notin K^{q} \). So \( K(w_1 + w_2) \) is that kernel, and invariant under the group \( O(W, g) \). It turns out that the elements of \( O(W, g) \) are described (with respect to the basis \( w_1, w_2, w_3 \)) by matrices of the form \( U_\lambda \) as in \( 1.16 \) with \( x \in K \). This yields \( O(W, g) \cong (K, +) \).

For each similitude \( h \), the multiplier lies in \( q(V) = F^{q} + F^{q} c_3 + F^{q} c_4 \). If \( m \in q(V) \setminus F^{q} \) were a multiplier of some similitude then \( q(V) \) would be a vector space over the field \( F^{q}(m) \). This is impossible, and we obtain \( \text{GO}(V, h) = F^{x} O(V, h) \).

***

2.4. If \( \dim_{q} q(V) = 4 \) then \( q \) is anisotropic, and both \( O(V, q) \) and \( O(V, h) \) are trivial. Depending on the field \( F \), it is possible that the discriminant is trivial, or non-trivial. We may assume that \( q(V) \) contains 1, and pick \( v_1 \in V \) with \( q(v_1) = 1 \).

If every similitude has a multiplier in \( F^{q} \) then \( \text{GO}(V, h) = F^{x} O(V, h) = F^{x} \text{id} \).

Now assume that there exists a similitude \( \lambda \) with multiplier \( r \in q(V) \setminus F^{q} \). Then \( r^{-1} \lambda^2 \) lies in the (trivial) group \( O(V, h) \), and we find \( \lambda^2 = r \text{id} \). The vectors \( v_1 \) and \( v_2 := \lambda(v_1) \) are linearly independent because \( q(v_2) = r \notin F^{q} = q(Fv_1) \), and \( \lambda(v_2) = rv_1 \) yields that \( Fv_1 + Fv_2 \) is invariant under \( \lambda \). Pick \( v_3 \in \{v_1, v_2\}^\perp \setminus \{0\} \). Then \( v_3 \) and \( v_4 := \lambda(v_3) \) span \( \{v_1, v_2\}^\perp \), and \( v_1, v_2, v_3, v_4 \) is a basis for \( V \). With respect to that basis, the Gram matrix for \( h \) is

\[
\begin{pmatrix}
1 & s & 0 & 0 \\
0 & r & 0 & 0 \\
0 & 0 & c & ct \\
0 & 0 & ct & cr
\end{pmatrix}
\]
where $s := h(v_1, v_2)$, $c := h(v_3, v_2) t := h(v_3, v_4) / c$. The discriminant of $h$ is represented by $(r + s^2)(r + t^2)$; it is trivial if, and only if, we have $s = t$.

In coordinates with respect to the basis $u_1 := v_1$, $u_2 := sv_1 + v_2$, $u_3 := v_3$, $u_4 := tv_3 + v_4$, the bilinear form $h$ is described by the diagonal matrix $D$ with entries $1$, $a$, $c$, $cb$, where $a := r + s^2$ and $b := (r + t^2)$. By our present assumption, these entries are linearly independent over $F^a$. So $\sqrt{c} \notin (F(\sqrt{r} + s)) = (F(\sqrt{r}))(\sqrt{c}) = F^a(r)$, and $E := F(\sqrt{r}, \sqrt{c})$ is an extension of degree $4$ over $F$, with the additional property $E^c \subseteq F$.

For $k \in F$, we write $M_k := (\begin{smallmatrix} 1 & 0 \\ 0 & k \end{smallmatrix})$, and abbreviate $i := M_1$. Then $i M_k i = M_k^r$, and $M_k^2 = kE$, where $E$ denotes the $2 \times 2$ identity matrix.

The set $L_k := \{ xE + yM_k \mid x, y \in F \}$ forms a subalgebra of $F^{2 \times 2}$. We note that $L_k \cong F[X] / (X^2 - k)$. In particular, the algebras $L_a$ and $L_k$ are fields. We abbreviate $\rho := \sqrt{a + b} = s + t \in F$, and obtain that $\psi : L_a \to L_k : xE + yM_a \mapsto (x + \rho y)E + yM_b$ is an isomorphism. So the set $L$ of all block matrices $\left( \begin{array}{cc} A & 0 \\ 0 & \psi(A) \end{array} \right)$ is a subfield of $End(V)$.

Straightforward computation yields $M_k^r(i M_k)i M_k = k I M_k$. From this it follows easily that every non-zero element of $L$ is a similitude of the bilinear form with Gram matrix $D = \left( \begin{smallmatrix} M_a & 0 \\ 0 & cE \end{smallmatrix} \right)$. So $R_a := F^a(a) \setminus \{0\}$ is contained in the set $R$ of multipliers of similitudes of the form with Gram matrix $D$. On the other hand, the set $R$ is contained in $q(V) \setminus \{0\} = (F^a + F^a a + F^a c + F^a cb) \setminus \{0\} = (F^a(a) + F^a(a)c) \setminus \{0\}$, and it is a union of sets of the form $R_a(x + yc)$, with $x, y \in F^a$ and $(x, y) \neq (0, 0)$. For the determination of $R$, it thus suffices to determine $R \cap \{1 + yc \mid y \in F^a \}$.

So consider $y \in F^a(a) = F^a(b)$. Then $y = y_1^2 + y_2^2 b$ with $y_1, y_2 \in F$, and $1 + yc = 1 + y_1^2 c + y_2^2 cb$. If $1 + yc \in R$ then there exists a similitude $\lambda$ with $q(\lambda(u_1)) = 1 + yc$. As the map $q$ is injective, we infer $\lambda(u_1) = u_1 + u_3 y_1 + u_4 y_2$. From $q(\lambda(u_2)) = (1 + yc)a = a + (y_1 \rho + y_2 b)^2 c + (y_1 + y_2 \rho)^2 cb$ we infer $\lambda(u_2) = u_2 + u_3 (y_1 \rho + y_2 b) + u_4 (y_1 + y_2 \rho)$. As $u_1$ and $u_2$ are orthogonal, their images under $\lambda$ are orthogonal, as well. Evaluating the bilinear form, we obtain the condition $0 = y_1 (y_1 \rho + y_2 b)c + y_2 (y_1 + y_2 \rho)cb = \rho c(y_1^2 + y_2^2 b)$.

The non-split case. If $\rho \neq 0$ (i.e., if the discriminant of $h$ is not trivial) then $y = 0$ follows from the fact that $b \notin F^a$. So we have $R_a = R$ in that case, and have found all similitudes: with respect to the orthogonal basis $u_1, u_2, u_3, u_4$, the similitudes are those endomorphisms with a matrix in $L \setminus \{0\}$. We note that in this case, the algebra $K$ is not split; in fact, we have $K \cong F(\sqrt{\det D}) = F(\sqrt{ab}) = F(\sqrt{a^2 + b^2}) = F(\sqrt{a})$, and $R = K^a$.

The split case. Now assume that the discriminant is trivial, i.e., $a = b$ (and $\rho = 0$). We use the block matrices $A := \left( \begin{array}{cc} M_a & 0 \\ 0 & M_a \end{array} \right)$ and $C := \left( \begin{array}{cc} 0 & cE \\ E & 0 \end{array} \right)$, with $M_a := \left( \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right)$ as above. Straightforward computation yields $A^l D A = a D$, $A^r D = D A$, $C^l D C = c D$, and
$C^t D = DC$. From this it follows easily that every non-zero element of the $F$-subalgebra $F(A, C)$ generated by $A$ and $C$ in $\text{End}(V)$ is a similitude of the bilinear form with Gram matrix $D$. Now $F(A, C)$ is a field isomorphic to $F(\sqrt{a}, \sqrt{c}) \cong E$, and the multiplicative group of that field acts (by similitudes) sharply transitively on $V \setminus \{0\}$.

In this case, the discriminant is trivial, and $K$ splits. For each similitude in $F(A, C)^\times$, the action on $W_z$ is rather easy to understand: First we choose $b: \Lambda^4 V \to F$ in such a way that $b(u_1 \wedge u_2 \wedge u_3 \wedge u_4) = \det D = ac$. Then $J^2 = \text{id}$; explicitly, we note $J(u_3 \wedge u_4) = u_1 \wedge u_2 c$, $J(u_2 \wedge u_4) = u_1 \wedge u_3 a$, $J(u_2 \wedge u_3) = u_1 \wedge u_4$. Applying the nilpotent element $z = 1 + j \in K$, we obtain the basis $w_1 := (u_3 \wedge u_4)z = u_3 \wedge u_4 + u_1 \wedge u_2 c$, $w_2 := (u_2 \wedge u_4)z = u_2 \wedge u_4 + u_1 \wedge u_3 a$, $w_3 := (u_2 \wedge u_3)z = u_2 \wedge u_3 + u_1 \wedge u_4$ for $W_z$.

In coordinates with respect to the basis $u_1, u_2, u_3, u_4$, the alternating forms given by those elements of $\Lambda^4 V$ are described by the block matrices $W_1 := \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$, $W_2 := \begin{pmatrix} 0 & M_1 \\ M_2 & 0 \end{pmatrix}$, and $W_3 := \begin{pmatrix} 0 & M_1 \\ M_2 & 0 \end{pmatrix}$, respectively. Using $A, C$ as above, we consider $X := x_1 E + x_2 A + x_3 C + x_4 AC \in F(A, C)$. A straightforward calculation yields that this endomorphism of $V$ induces multiplication by the scalar $x_1^4 + x_2^3 a + x_3^2 c + x_4^2 ac$ on $W_z$; if $X \neq 0$ then this scalar is the multiplier of the similitude described by the matrix $Y$. Note also that $X^2 = (x_1^2 + x_2^2 a + x_3^2 c + x_4^2 ac)\text{id}$.

2.5 Theorem. Assume $\text{char } F = 2$, $\sigma = \text{id}$ and $\ell = 2$. Then one of the following holds.

a. If $q$ has defect 3, then $K$ splits and $O(V; h) \cong (\text{SL}_2(F) \times F^2) \times F$. The normal subgroup $\Xi \cong F^2 \times F$ is the kernel of the action on $W_z$. See 2.1.

We obtain $\text{SL}_2(F) \cong \eta(O(V; h)) = O(W; g) \cap \text{GL}_F(W) < O(W; g) \cong \text{SL}_2(K) \cong \text{SL}_2(F) \times \text{SL}_2(F)$; note that $K \cong F \times F$ is not a field. Every multiplier is a square, the group of similitudes is $\text{GO}(V; h) = F^\times \circ O(V; h)$.

b. If $q$ has defect 2 and $K$ is not split, then $O(V; h) \cong \text{SL}_2(F)$. See 2.2.

We have $\text{SL}_2(F) \cong \eta(O(V; h)) = O(W; g) \cap \text{GL}_F(W) < O(W; g) \cong \text{SL}_2(K)$. The multipliers of similitudes of $h$ form the multiplicative group of an inseparable quadratic extension field of $F$.

c. If $q$ has defect 2 and $K$ is split, then $O(V; h) \cong F^3$ is abelian (and its action on $W_z$ is neither trivial nor faithful). See 2.2.

We then have $O(V; h) \cong (F^3, +)$, and $O(W; g) \cong (K, +)$. Again, the multipliers of similitudes of $h$ form the multiplicative group of an inseparable quadratic extension field.

d. If $q$ has defect 1, then $O(V, h) \cong F$ is abelian, and $K$ is not split. See 2.3.

We have $O(V; h) \cong (F, +)$, and $\eta(O(V; h)) = O(W; g) \cap \text{GL}_F(W) < O(W; g) \cong (K, +)$. Every multiplier is a square, the group of similitudes is $\text{GO}(V; h) = F^\times \circ O(V; h)$.

e. If $q$ has defect 0, then $q$ is anisotropic and both $O(V, q)$ and $O(V, h)$ are trivial.

In any case, the range of the quadratic form $q$ is an inseparable extension field $E$ of degree 4 over $F$.\]
If the discriminant is not a square then the multipliers of similitudes form the multiplicative group of an inseparable quadratic field extension.

If the discriminant is a square then the multipliers of similitudes form the multiplicative group of an inseparable quadratic field extension. In that case, every element of $E$ occurs as a multiplier, and $GO(V, h)$ is sharply transitive on $V \setminus \{0\}$.

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