Resilience and fluctuations shape primal and dual communities in spatial networks

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Abstract

Both human-made and natural supply networks are built to operate reliably in changing conditions full of external stimuli. Many of these spatial networks exhibit community structures. Here, we show the existence of a second class of communities. These dual communities are based on an exceptionally strong mutual connectivity and can be found for example in leaf venation networks. We demonstrate that traditional and dual communities emerge naturally as two different phases of optimised network structures that are shaped by fluctuations and that they suppress failure spreading, which underlines their importance in understanding the shape of real-world supply networks.
Community structures are a fundamental trait of complex networks and have found numerous applications in systems from social networks \[1\] to biological networks \[2, 3\] and critical infrastructures \[4\]. Typically, communities are defined by a strong connectivity within each community compared to a relatively weak connectivity between them \[1, 5, 6\]. Intuitively, community structures play an important role for failure spreading, i.e. small perturbations stay within the community, which is both predicted by theory \[7, 8\] as well as observed in experiments \[9\].

Many human-made and biological networks are spatially embedded and planar \[10–13\]. For planar graphs, it is possible to define a dual graph. Every node of the dual graph corresponds to a facet of the original graph’s planar drawing (see Fig. 1a). If two facets share an edge, their corresponding dual nodes are connected by a dual edge. Dual graphs turn out to be useful not only when analysing resistor networks, power grids or natural gas networks \[14–17\], but also to study fixed points in coupled oscillator systems \[18, 19\].

Remarkably, the analysis of dual graphs allows to reveal patterns in the network structure that are hidden in the primal graph, such as dual communities. We show that not only a relatively weak connectivity, but also a relatively strong connectivity between parts of the network may be used to define community structures. This is due to the fact that strong connections in the primal graph translate into weak connections in the dual graph.

Consider a weighted, simple graph \( G(V, E, W) \) with vertex set \( V \) containing \( N = |V| \) vertices and edge set \( E \) of \( M = |E| \) edges. Spectral bisection is a commonly used method to determine the community structure of a graph that dates back to Fiedler \[20\] and makes use of the graph Laplacian \( L \). This an \( N \times N \) matrix defined as \[21\]

\[
L_{ij} = \begin{cases} 
-w_{ij} & \text{if } i \text{ is connected to } j, \\
\sum_{(i,k) \in E} w_{ik} & \text{if } i = j, \\
0 & \text{otherwise}.
\end{cases}
\] (1)

Here, \( w_{ij} > 0 \) is the weight of an edge \((i, j)\). Let us label the graph’s edges as \( \ell = 1, 2, \ldots, M \) and fix an orientation for each edge. Based on this orientation, we can define the graph’s weighted incidence matrix \( I \in \mathbb{R}^{M \times N} \) as

\[
I_{\ell n} = \begin{cases} 
\sqrt{w_{\ell}} & \text{if line } \ell \text{ starts at node } n, \\
-\sqrt{w_{\ell}} & \text{if line } \ell \text{ ends at node } n, \\
0 & \text{otherwise}.
\end{cases}
\] (2)
Figure 1. Communities and hierarchies in spatial networks. (a) A planar graph with edges characterised by either large (dark green), small (light green) or intermediate edge weights, and its dual graph. To construct the dual, each face is transformed into a dual node and a dual edge is added if two faces share an edge, where an edge with large weight is transformed into a weak link in the dual (see Eq. (4)). (d) Spectral clustering by means of the Fiedler vector $v_2$ reveals the community structure in both the graph (left) and its dual (right). (g) Based on repeated spectral clustering, the graphs are further decomposed into a hierarchy of smaller subunits which is different in the graph (left) and its (dual). (b,e,h) If we perform the same analysis on the venation network of a leaf of *Acer platanoides*, the resulting hierarchies in the original graph do not provide useful information ((h),left). The hierarchies in the dual graph, however, correspond to the functional components ((e),(h),right). (c-i) For the Continental European power grid ((i),left) and its dual (right) both primal and dual hierarchies provide different, but equally useful information about the network structure.

The Laplacian can be rewritten in terms of the incidence matrix as $L = I^T I \in \mathbb{R}^{N \times N}$, where $^\top$ denotes the matrix transpose.

We now construct the dual graph $G^\ast$. The concept of dual graphs is based on the duality between a graph’s cut space and its cycle space [22]. This duality may be expressed more formally by making use of the fact that the fundamental cycles of a graph form the kernel
of the weighted incidence matrix. This relationship can be written compactly introducing the weighted edge-cycle incidence matrix $C \in \mathbb{R}^{M \times N^*}$ as

$$I^T C = 0,$$  \hspace{1cm} (3)

where $N^* = M - N + 1$ is the number of independent cycles in the graph [22]. Thus, the elements of $C$ are given by

$$C_{\ell c} = \begin{cases} 
  1/\sqrt{w_\ell} & \text{if edge } \ell \text{ is part of cycle } c, \\
  -1/\sqrt{w_\ell} & \text{if reversed edge } \ell \text{ is part of cycle } c, \\
  0 & \text{otherwise.}
\end{cases}$$

Finally, we can then define the Laplacian matrix of the dual graph as

$$L^* = C^T C \in \mathbb{R}^{N^* \times N^*}.$$

Thus, edges in the dual graph are weighted with the inverse of edge weights in the primal graph.

Dual communities can be extracted by means of any community detection algorithm applied to the dual graph, see e.g. Ref. [23]. We focus on spectral graph bisection to unveil the graph’s community structure because it is immediately obtained from the graph Laplacian. Spectral graph bisection methods rely on the fact that the community structure is encoded in the second smallest eigenvalue of the graph Laplacian $\lambda_2 \geq 0$, known as the algebraic connectivity or Fiedler value, which vanishes if the graph consists of two disconnected components and increases with increasing connectivity between them. The graph’s nodes are then assigned to one of two communities based on the corresponding eigenvector $v_2^*$, the (dual) Fiedler vector [5]: two dual vertices $j^*$ and $i^*$ are in the same community if they share the same sign of the dual Fiedler vector, $\text{sign}((v_2^*)_i - m) = \text{sign}((v_2^*)_j - m)$, where $m \in \mathbb{R}$ is a threshold parameter. Here, we choose $m = 0$. As we demonstrate in Fig. 1b,e, dual communities appear naturally in real-world networks such as the vascular networks of leaves. Instead of weakly connected components, the two communities are separated by a strong vein with large edge weights [24].

Dual communities reveal hierarchical organisation of supply networks. The spectral clustering method for community detection can be applied to both the primal and the dual graph, revealing different structural information about the network (Fig. 1d). Furthermore,
Figure 2. Primal and dual communities emerge naturally in optimal supply networks. (a-c) We consider a triangular supply network with two fluctuating sources located at the leftmost and the rightmost node and Gaussian sinks attached to all other nodes. We impose additional weak (a) and strong (b) fluctuations with variance $\sigma_D^2$ on the sources and optimise the edge weights to minimise the average dissipation. We observe a transition from primal to dual communities measured by the Fiedler vectors (colour code) of (a) the primal or (b) dual graph and (c) scaling of the corresponding primal ($\lambda_2$, circles) and dual ($\lambda_2^*$, crosses) Fiedler value. (d-f) Similarly, the European power grid experiences a transition from (d) primal to (e) dual communities when transmission line capacities are optimised for different carbon-dioxide (CO$_2$) emission reduction levels. (f) Again, the result is confirmed by the scaling of primal and dual Fiedler values for increasing emission reduction which corresponds to an increasing share of fluctuating renewables (lower axis). The grids were obtained using the high-resolution European energy system model 'PyPSA-EUR' [25] to minimise the cost for transmission expansion [26]. Cost parameter for (a,b) is $\gamma = 0.9$.

we can use this approach to extract a network’s hierarchical organisation as follows. Starting from the initial network, we compute the Fiedler vector, identify the communities and then split the network into two parts at the resulting boundary by removing all edges between the communities. Then we iterate the procedure starting from the subgraphs obtained in the previous step. Repeated application of this procedure reveals different boundaries and thus different hierarchies in the primal graph and its dual (Fig. 1g).

Consider the venation network of a leaf as shown in Fig. 1b and provided by the authors
of Ref. [27]. Such a supply network consists of two clearly visible parts separated by a central vein that can be identified as dual communities, whereas the same structural pattern is not visible in the primal community structure (Fig. 1e). Remarkably, we can use the hierarchical decomposition to reveal that this central organisational pattern repeats in a hierarchical order: dual communities are split by secondary veins in a repeated manner (Fig. 1h) while the same decomposition in the primal graph does not provide useful information. Thus, leaf venation networks clearly display a dual community structure.

We now turn to another type of supply networks: power grids. Fig. 1c shows the European power transmission grid and its dual graph. Again, a hierarchical decomposition reveals different levels of hierarchies in the grid that correspond to its functional components. These components may also be interpreted geographically: the mountain ranges such as the Pyrenees or the Alps as well as the former Iron Curtain are clearly visible in the decomposition. Remarkably, both primal and dual decompositions provide useful structural information here. In particular, there is a dual community boundary at cut level three that closely corresponds to a system split in Eastern Europe that occurred on January 8th, 2021, where the European power grid was split into two parts along this boundary.

Although mathematically similar [24, 28, 29], the two types of networks we studied display different structural hierarchies and communities. Whereas leaf venation networks are evolutionarily optimised, the structure of power grids depends strongly on historical aspects and their ongoing transition to include a higher share of renewable energy sources. This transition aspect also manifests in their community structure as we will see in the next section.

**Fluctuations shape community structures in optimal flow networks.** Understanding how the structure of optimal supply networks emerges is an important part of complex networks research [30–33]. For networks where a single source supplies the entire network, it is well established that fluctuations in the supply can cause a transition from a tree-like to loopy network [29, 31]. We extend this result by studying how does the increase in fluctuations influences the optimal network structure in supply networks with multiple strongly fluctuating sources and weakly fluctuating sinks.

To interpolate between strongly fluctuating sources and weakly fluctuating ones, we use a similar setup as in Ref. [31]. We consider a linear flow network consisting of a triangular lattice with $N$ nodes of which $N_s$ are sources and with sinks whose outflows are fluctuating
iid Gaussian random variables. Additionally, we add fluctuations only to the sources of the networks that can be tuned by the additional variance of fluctuations \( \sigma_D^2 \) without affecting the statistics of the sinks. We then tune the edge weights \( w_\ell \) such that they minimize the average network dissipation \( \langle D \rangle = \text{trace} \left( P^\top L^\dagger P \right) \) while fixing the overall cost \( \sum_{\ell=1}^N w_\ell^\gamma \) to build the network. Here \( L^\dagger \) is the Moore-Penrose pseudoinverse of the Laplacian, \( P = (P_1, ..., P_N)^\top \in \mathbb{R}^N \) is the vector of sources and sinks attached to the nodes 1, ..., \( N \) of the network, and \( \gamma \in \mathbb{R} \) is a cost parameter [26].

Whereas the optimal network structure shows primal communities for weakly fluctuating sources, \( \sigma_D^2 \approx 1 \), it undergoes a transition to a dual community structure for strong fluctuations, \( \sigma_D^2 \gg 1 \) (see Fig. 2). We can capture this transition in terms of the primal and dual Fiedler values (Fig. 2c). Thus, optimal supply networks have a community structure – whether it is primal or dual depends on the degree of fluctuations.

Strikingly, an analogous transition is observed for actual power transmission grids when optimising the network structure for different levels of fluctuating renewable energy sources. We consider the European power transmission grid and optimise its network structure for different carbon dioxide (CO\(_2\)) emission reduction targets compared to the year 1990 ranging from 60% to 100% reduction using the open energy system model 'PyPSA-Eur' [25]. We then obtain the network structure by setting the edge weight to the overall transmission capacity. With increasing penetration of fluctuating renewables, we observe a decrease in the dual Fiedler value \( \lambda_2^* \) and an increase in the primal Fiedler value \( \lambda_2 \) and thus a transition from primal to dual communities. Note that the generation mix in the optimised power system changes for different emission scenarios from conventionally-dominated grids to highly-renewable grids [26].

**Dual communities determine robustness of supply networks.** To study the robustness of a linear flow network with respect to small perturbations, we make use of a sensitivity factor. We add an inflow \( \Delta P \) at a node \( e_1 \) and an outflow of the same amount at another node \( e_2 \) and study how much this will change the flow \( \Delta F_\ell \) on a link \( \ell \). Here, we focus on the case where \( e_1 \) and \( e_2 \) are the terminal nodes of an edge \( e = (e_1, e_2) \) and treat the more general case of inflows at two arbitrary nodes in the SI [26]. The sensitivity factor \( \eta_{e_1, e_2, \ell} = \frac{\Delta F_\ell}{\Delta P} \) that relates the flow changes to the inflow is given by [26, 34, 35]

\[
\eta_{e_1, e_2, \ell} = \sqrt{\frac{w_\ell}{w_e} l_\ell^\top L L^\dagger L^\top l_e},
\]
Figure 3. Primal and dual communities inhibit failure spreading. A square grid is divided into (a) two primal communities by weakening the links connecting two parts of the network or (b) into two dual communities by strengthening the links horizontally separating the two parts. The Fiedler vector (colour code) reveals the community structure. (c,d) Both primal and dual communities inhibit flow changes $|\Delta F|$ (colour coded) in the other community after the failure of a single link (red) with unit flow. (e) We interpolate between primal and dual communities in a square grid of size $21 \times 10$ by tuning the weight $w_e$ of the horizontal edges or vertical edges (see a,b). The flow ratio $R$ reveals that failure spreading to the other community is largest for $w_e = 1$. It decays for either type of community as measured by primal and dual Fiedler values $\lambda_2$ (crosses) and $\lambda_2^*$ (circles), respectively. The green line represents the median value and the shaded regions indicate the 25% and 75% quantiles.

where $l_e \in \mathbb{Z}^M$ is the indicator vector of edge $e$ which is equal to one at the positions indicated by the subscript and zero otherwise. Importantly, the sensitivity factor may also be used to simulate the failure of a link $e = (e_1, e_2)$ by choosing the inflow $\Delta P$ accordingly [26] and is well-known in the context of power system security analysis, where it is referred to as power transfer distribution factor [34].

Sensitivity to changes in the flow patterns is determined by the primal community structure of a linear flow network measured by the Fiedler value $\lambda_2$ [7, 26, 28, 36]. Remarkably,
we can find an analogous description in the dual graph \[15\]

$$\eta_{e_1,e_2,\ell} = -\sqrt{\frac{w_{\ell}}{w_e}} C(L^*)^\dagger C^\dagger e.$$ (6)

The dual Laplacian $L^*$ contributes to the sensitivity factor $\eta_{e_1,e_2,\ell}$ in the same way as the primal Laplacian. Hence, primal and dual community structures determine network flows in an equivalent manner: if a network admits a dual community structure and $\lambda_2^*$ is small, then changes in one community will only weakly affect the other one.

Consider an inflow and simultaneous outflow $\Delta P$ at two nodes $\ell_1$ and $\ell_2$, respectively, that are connected via an edge $\ell = (\ell_1, \ell_2)$. We compare the resulting flow changes in the same (S) and the other (O) community as the given edge by evaluating their ratio $R(\ell, d)$ at a given distance $d$ to the link \[28\]

$$R(\ell, d) = \frac{\langle |\Delta F_k| \rangle_{d}^{k \in O}}{\langle |\Delta F_r| \rangle_{d}^{r \in S}} = \frac{\langle |\eta_{\ell_1,\ell_2,k}| \rangle_{d}^{k \in O}}{\langle |\eta_{\ell_1,\ell_2,r}| \rangle_{d}^{r \in S}}.$$ We then average over all possible trigger links and distances to arrive at the mean flow ratio $R = \langle R(\ell, d) \rangle_{\ell,d}$. The mean flow ratio ranges from $R \approx 0$ if the other module is weakly affected, i.e. there is a strong community effect, to $R \approx 1$ if there is no noticeable effect. We note that $R$ describes flow changes after perturbations in the in- and outflows as well as flow changes as a result of the complete failure of links \[26\].

Fig 3 illustrates that both primal and dual communities suppress flow changes in the other community. The flow ratio $R$ decays for either community structure and the decay is well-captured by the Fiedler value of the primal ($\lambda_2$) and the dual ($\lambda_2^*$) graph (see SI \[26\]).

**Discussion and Conclusion.** We have introduced a new way to define and identify community structures in planar graphs that we refer to as dual communities. We demonstrated that both primal and dual community structures emerge as different phases of optimized networks – whether the one or the other is realised in a given optimal network depends on the degree of fluctuations. In addition to that, both types of communities have the ability to suppress failure spreading – they are thus optimised to limit the effect of link failures or other perturbations.

An important difference between primal and dual communities is the fact that the former are based on a weak connectivity, while dual communities require a strong connectivity. This has important consequences for supply networks such as power grids. While preventing failure spreading, dual communities will not affect the network’s ability to transport energy...
between them, or may even increase the supply capabilities. This is in stark contrast to primal communities that limit failure spreading from one community to the other one, but also supply. Thus, the construction of dual communities may also serve as a strategy against failure spreading, in line with other ideas brought forward recently [28].

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[26] See SI for further examples of dual communities in leaf venation networks, additional figures on the open energy system models and extended methods describing linear flow networks, link failures and the fluctuating sink model in detail, which includes Refs. [35, 37–50].

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