Chapter 1

The principle of superposition and the Fourier series

1.1 The principle of superposition

1.1.1 Periodic motions

There are numerous periodic motions—such as oscillations and waves—observed in nature. Equations of motion describe periodic motions in the time domain. The periodic motions can also be analyzed in the frequency domain in order to acquire frequency distributions. There is a mathematical method for finding their frequency components and their amplitudes, or spectra from periodic patterns. It is called Fourier transform (FT) spectral analysis. Fourier analysis, originating from a thermal conduction problem solved by Joseph Fourier, is a powerful mathematical tool that can be also applied to various fields, including magnetic resonance and FT–IR spectroscopies, electronic circuits, telecommunication signals, and digital signal processing. The FT is also one of the fundamental theories of wave optics and quantum physics.

1.1.2 The superposition principle

Let us consider an oscillating string fixed at both ends to explain the important principle of periodic motion. A continuous train of sinusoidal waves is traveling back and forth to produce standing waves under an appropriate condition of tension in the string and its length. Figures 1.1–1.3 below are snapshots of these standing waves. Each of the standing waves corresponds to a normal mode of motion. Because the string is fixed at both ends, both ends must be nodes with motion. With the fixed boundary condition, adjacent nodes are one-half wavelength apart and the length of the string may be any integer number of one-half wavelengths. The frequency that gives the longest wavelength is called the first harmonic or the fundamental mode, and the others are integral multiples of the fundamental, called the higher harmonics.

When a string is initially struck for a short time, the subsequent oscillating string will be described by a linear combination of normal modes. In other words, there are
Figure 1.1. First harmonic (fundamental) mode.

Figure 1.2. Second harmonic mode.

Figure 1.3. Third harmonic mode.
multiple motions of different frequencies on an arbitrarily oscillating string, and the
displacement of the arbitrary point of the string is the algebraic sum of the wave
displacements of propagation and reflection. This is called the principle of super-
position. While it is a simple statement, it is an essential principle in periodic motion.
Let us see how the superposition principle appears in mathematics.

1.2 Wave equations

The normal modes are the solutions of the differential equation for the wave motion
with a particular boundary condition

$$\frac{\partial^2 u(x, t)}{\partial t^2} = v^2 \frac{\partial^2 u(x, t)}{\partial x^2} \quad (1.1)$$

where \(v\) is wave speed. By separating the variables \(u(x, t) = U(x)\Gamma(t)\), the equation for
the \(x\)-coordinate, \(U(x)\), becomes the Helmholtz equation and the one for time \(\Gamma(t)\) is
an equation of harmonic oscillation

$$\frac{d^2 U(x)}{dx^2} + \lambda U(x) = 0 \quad \text{and} \quad \frac{d^2 \Gamma(t)}{dt^2} + \lambda v^2 \Gamma(t) = 0. \quad (1.2)$$

Solutions that satisfy the boundary condition of fixed ends, \(U(0) = U(L) = 0\), are
\(\sin(2\pi mx/L)\), where \(\lambda = (2\pi m/L)^2\) and \(m = 1, 2, 3, \ldots\). By the superposition principle,
a general solution with the same boundary condition is given by

$$U(x) = \sum_{m=0}^{\infty} a_m \sin mx. \quad (1.3)$$

We can take the same variable separation approaches for obtaining solutions to
the thermal equation and the Schrödinger equation:

thermal conduction equation: \(\frac{\partial u(x, t)}{\partial t} = \kappa \frac{\partial^2 u(x, t)}{\partial x^2} \quad (1.4)\)

and

Schrödinger equation: \(i\hbar \frac{\partial u(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 u(x, t)}{\partial x^2} \quad (1.5)\)

The spatial part, \(U(x)\), of the above equations also becomes the Helmholtz equation
(1.2). Taking periodic boundary conditions for \(U(x)\) and its derivative, \(U'(x): U(0) = U(2\pi)\) and \(U'(0) = U'(2\pi)\), particular solutions are given by \(U(x) = a_m \cos(mx) + b_m \sin(mx)\) where \(\sqrt{\lambda} = m = 0, 1, 2, \ldots\). Therefore, a general solution with the periodic
boundary condition can be expressed by the linear combination of particular
solutions including a constant term, which is called the Fourier series:

$$U(x) = \frac{a_0}{2} + \sum_{m=0}^{\infty} (a_m \cos mx + b_m \sin mx). \quad (1.6)$$
1.3 Fourier series

The previous section described wave phenomena using linear differential equations where their solutions are given by sinusoidal periodic functions. Since an arbitrary periodic motion can be expressed by the superposition of sinusoidal harmonic modes, we can express an arbitrary periodic function \( f(t) \), where the variable \( t \) is time or a spatial coordinate, as a series of sinusoidal functions, constituting the Fourier series of \( f(t) \). Because the sinusoidal functions are well-known and easy to apply, the Fourier series is useful for analyzing wave phenomena.

1.3.1 Fourier theorem

**Time periodicity**

A periodic function \( f(t) \) of period \( T (-T/2 < t \leq T/2) \) can be expressed by a Fourier series [1]:

\[
 f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \tag{1.7}
\]

where \( \omega_0 = 2\pi/T \) is the angular frequency of the fundamental mode, and the Fourier coefficients \( a_m \) and \( b_m \) are given by

\[
 a_m = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(m\omega_0 t) dt \quad \text{and} \quad b_m = \frac{2}{T} \int_{-T/2}^{+T/2} f(t) \sin(m\omega_0 t) dt. \tag{1.8}
\]

The Fourier coefficients are calculated using the orthogonal property of sinusoidal functions:

\[
 \langle \cos(m\omega_0 t) | \cos(n\omega_0 t) \rangle = \int_{-T/2}^{T/2} \cos(m\omega_0 t) \cos(n\omega_0 t) dt = 0 \quad (m \neq n)
\]

\[
 \langle \sin(m\omega_0 t) | \sin(n\omega_0 t) \rangle = \int_{-T/2}^{T/2} \sin(m\omega_0 t) \sin(n\omega_0 t) dt = 0 \quad (m \neq n)
\]

and

\[
 \langle \cos(m\omega_0 t) | \sin(n\omega_0 t) \rangle = \int_{-T/2}^{T/2} \cos(m\omega_0 t) \sin(n\omega_0 t) dt = 0 \quad \text{(including } m = n).\]

Depending on the property of the original periodic function \( f(t) \), a Fourier series may have only sine-terms or cosine-terms. If the function \( f(t) \) is an even function in the interval \([-T/2, +T/2] \), the sine-terms must be excluded, and the Fourier series has only cosine terms:

\[
 f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t). \tag{1.9}
\]
Similarly, if the periodic function $f(t)$ is an odd function in the interval $[-T/2, +T/2]$, the Fourier series has only sine terms:

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t).$$  \hfill (1.10)

We can also obtain a Fourier series in a complex exponential form. By applying Euler’s formula, $e^{i\theta} = \cos \theta + i \sin \theta$, we can obtain the Fourier series of complex variables

$$f(t) = \sum_{n=-\infty}^{n=+\infty} c_n e^{in\omega_0 t} \quad \hfill (1.11)$$

where the complex Fourier coefficient is given by

$$c_n = \frac{1}{T} \int_{-T/2}^{+T/2} f(t)e^{-in\omega_0 t} \, dt. \quad \hfill (1.12)$$

The complex Fourier coefficient represents the magnitude of the frequency component and the phase.

**Spatial periodicity**

Instead of a periodic function in time, a Fourier series can be applied to a periodic function in space. The spatial periodicity can be observed through electromagnetic waves including optical interferences and diffractions, and wave packets of a particle.

An arbitrary periodic function $f(x)$ in the interval $[-\pi, +\pi]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$  \hfill (1.13)

where the coefficients $\{a_n; \ n = 0, 1, 2, 3, \ldots\}$ and $\{b_n; \ n = 1, 2, 3, \ldots\}$ are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx) \, dx \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx) \, dx. \quad \hfill (1.14)$$

The Fourier series for the spatial interval $[-\pi, +\pi]$ can be changed to $[-L, +L]$ by using another variable $z = (L/\pi)x$

$$f(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}z\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}z\right)$$  \hfill (1.15)

where

$$a_m = \frac{1}{L} \int_{-L}^{L} f(z)\cos\left(\frac{m\pi}{L}z\right)dz \quad \text{and} \quad b_m = \frac{1}{L} \int_{-L}^{L} f(z)\sin\left(\frac{m\pi}{L}z\right)dz. \quad \hfill (1.16)$$
1.3.2 Examples of the Fourier series

**Example 1: the Fourier series of a square wave.**

The first example of the Fourier series is a square wave train:

\[ f(x) = -1 \text{ if } -\pi \leq x < 0; \quad \text{and} \quad +1 \text{ if } 0 \leq x < \pi. \]  

(1.17)

The graph of this square wave train for \( x > 0 \) is shown in figure 1.4.

The Fourier series of the square wave train is given by:

\[
\sum_{n=1}^{\infty} \frac{1}{\pi n (2n-1)} \sin[(2n - 1)x]
\]

\[= \frac{4}{\pi} \left( \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \frac{1}{7} \sin 7x + \ldots \right).\]

Figure 1.5 is the actual Fourier series of the first 10 terms—except the multiplication factor \( \pi/4 \)—by iterative summation using EXCEL®. Its macro source program is listed in the appendix, A1.4.1. If readers are not familiar with EXCEL macro, refer to appendix A1.3.

If we calculate the sum of 100 terms, the Fourier series gets much closer to the square wave train with noticeable oscillations at the rising and falling edges (figure 1.6).

**Remark: Gibbs phenomenon**

The fine oscillations at the edges do not disappear even if the Fourier series takes many more terms. This overshoot behavior occurs at a jump discontinuity, and it is called the Gibbs phenomenon [2]. The size of the overshoot is tuned out to be proportional to the magnitude of the discontinuity. For the square wave train, the maximum peak value of the partial sum approaches approximately.
$\frac{2}{\pi} \int_{0}^{\pi} \frac{\sin \xi}{\xi} d\xi \approx 1.1789$. Its calculation offers a good mathematical exercise and readers should try it. Refer to appendix A2 for a detailed calculation.

Example 2: the Fourier series of a sawtooth wave.
The sawtooth wave is a repetition of the function $f(t) = x$ for $-\pi < x < +\pi$ and the period is $2\pi$. Figure 1.7 shows this signal for $x \geq 0$. 
The Fourier series of the above sawtooth wave is

\[ f(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) = 2 \left[ \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \cdots \right]. \]

Figure 1.7. Sawtooth wave.

Figure 1.8. Fourier transform of sawtooth wave (10 terms).
The Fourier series up to 10 terms and 100 terms are shown in figures 1.8 and 1.9, respectively. The Gibbs phenomenon is also noticeable in this case.

1.4 Orthonormal basis

The concept of orthonormal basis is the foundation of the superposition principle and the Fourier series theory. Let us consider an $n$-dimensional Euclidian vector space for an intuitive discussion of the orthonormal basis. In this space, an arbitrary vector can be expressed as a linear combination of unit vectors:

$$\vec{v} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + \cdots + v_n \vec{e}_n$$ (1.18)

where $\{v_j; j = 1, 2, \ldots, n\}$ are the components of the vector $\vec{v}$, and $\{\vec{e}_i; i = 1, 2, \ldots, n\}$ form a set of unit vectors associated with the given Cartesian coordinate frame. The component $v_j$ is given by the inner product of the vector and the unit vector

$$v_j = \langle \vec{v} | \vec{e}_j \rangle = \sum_{j=1}^{n} v_j \langle \vec{e}_j | \vec{e}_i \rangle$$ (1.19)

because the unit vectors are orthonormal: $\langle \vec{e}_j | \vec{e}_i \rangle = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker’s delta, i.e. $\delta_{ij} = 1$ if $i = j$; and 0, otherwise. Furthermore, there are no other unit vectors required to express an arbitrary vector in the $n$-dimensional space. Thus, the unit vectors $\{\vec{e}_i; i = 1, 2, \ldots, n\}$ form an orthonormal basis of the vector space.

As discussed in section 1.2, the Helmholtz differential equation has sinusoidal solutions as the normal modes in the given Cartesian coordinates. Because
sinusoidal functions are orthogonal, for the string oscillation, the normal modes can be regarded as the ‘unit vectors’ and an arbitrary string wave can be expressed in the form of the linear combination (superposition) of the normal modes, constituting the Fourier series.

References

[1] Stewart J 2016 *Fourier Series—Stewart Calculus* (Boston, MA: Cengage Learning)
[2] *Gibbs Phenomena*, MIT OpenCourseWare (https://ocw.mit.edu/courses/mathematics/18-03sc-differential-equations-fall-2011/unit-iii-fourier-series-and-laplace-transform/operations-on-fourier-series/MIT18_03SCF11_s22_7text.pdf).