Phase Space Reduction of Star Products on Cotangent Bundles

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Abstract
In this paper we construct star products on Marsden-Weinstein reduced spaces in case both the original phase space and the reduced phase space are (symplectomorphic to) cotangent bundles. Under the assumption that the original cotangent bundle $T^*Q$ carries a symplectic structure of form $\omega_{B_0} = \omega_0 + \pi^*B_0$ with $B_0$ a closed two-form on $Q$, is equipped by the cotangent lift of a proper and free Lie group action on $Q$ and by an invariant star product that admits a $G$-equivariant quantum momentum map, we show that the reduced phase space inherits from $T^*Q$ a star product. Moreover, we provide a concrete description of the resulting star product in terms of the initial star product on $T^*Q$ and prove that our reduction scheme is independent of the characteristic class of the initial star product. Unlike other existing reduction schemes we are thus able to reduce not only strongly invariant star products. Furthermore in this article, we establish a relation between the characteristic class of the original star product and the characteristic class of the reduced star product and provide a classification up to $G$-equivalence of those star products on $(T^*Q, \omega_{B_0})$, which are invariant with respect to a lifted Lie group action. Finally, we investigate the question under which circumstances ‘quantization commutes with reduction’ and show that in our examples non-trivial restrictions arise.

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1 Introduction

Already in the first and fundamental article on deformation quantization by Bayen et al. [2], the problem how to construct a star product on a reduced phase space out of a known star product on the initial phase space has been considered. In particular, the example of the cotangent bundle $T^*S^{n-1}$ of the $n-1$-sphere has been discussed, and for reduction the Weyl-Moyal product on $T^*(\mathbb{R}^n \setminus \{0\})$ has been used. Even after general existence proofs for deformation quantizations on symplectic and Poisson manifolds have meanwhile appeared, it remains an interesting question which relations one can establish between star products on the original phase space and those on the reduced phase space. In physics terms, this corresponds to the question, whether extrinsic and intrinsic quantization are equivalent, which in some sense would mean that ‘quantization commutes with reduction’ (cf. [15] and [11] for a discussion of these topics in the framework of geometric quantization and the conventional Hilbert space approach to quantum mechanics). Moreover, the question of existence of star products on symplectic stratified spaces is still unsolved, and it appears to be very promising to attack this problem first for singular reduced phase spaces, which have been studied in detail by Sjamaar and Lerman [26].

Considering particular examples, there are various explicit constructions for phase spaces with additional structure, for instance $\mathbb{C}P^n$ [4, 27] or more general complex Grassmann manifolds [25] for which a deformation quantization analogue of phase space reduction can be constructed. But these examples all seem to be well-tailored to special situations and do not apply to arbitrary symplectic phase spaces.

In [14], Fedosov introduced a reduction scheme for arbitrary symplectic manifolds $(M, \omega)$ with a compact, free and symplectic Lie group action and a regular value of the momentum map. Fedosov starts with a certain Fedosov star product obtained from a certain $G$-invariant torsion free symplectic connection. Adapting the original star product appropriately to the fibering structure
of the principal $G$-bundle $M_0 \to M_{\text{red}} = M_0/G$, where $M_0$ denotes the inverse image of $0 \in \mathfrak{g}^*$ with respect to the classical momentum map, he showed that one can always achieve that the resulting star product on the reduced phase space $(M_{\text{red}}, \omega_{\text{red}})$ is equivalent to a canonical Fedosov star product on it. Thus, Fedosov is able to prove a ‘reduction commutes with quantization’ theorem within his particular situation. Another very general approach to reduction in the framework of deformation quantization is the BRST-method as presented by Bordemann et al. in [5]. Using a quantum BRST complex, the authors of this work are able to produce quite an explicit formula for the reduced star product under the following three assumptions:

1.) the symmetry group acts properly and freely,

2.) $0$ is a regular value of the momentum map,

3.) the initial star product on the phase space with symmetry is strongly invariant (cf. Section 2.2 for a definition).

By results obtained in [18] and [23], which reveal some obstructions in the characteristic class for a star product to be strongly invariant, the last of these assumptions imposes a restriction on the possible characteristic class of the original star product. Note that implicitly, the same restriction appears in the Fedosov reduction scheme.

The scope of the present paper is to develop a reduction scheme for star products on cotangent bundles with respect to a symplectic form which is the sum of the canonical symplectic form and the pull-back of a closed two-form on the base manifold which can be interpreted as a magnetic field. Additionally, we assume that the reduced phase space is again a cotangent bundle or, more precisely, symplectomorphic to a cotangent bundle via a non-canonical diffeomorphism. It is known that this latter assumption holds true (cf. [15, 20, 22]) if the action is the cotangent lift of a proper and free Lie group action on the base manifold and if the momentum value for which the reduced space is considered is an invariant element of the dual of the Lie algebra. Our construction is adapted to the particular geometry of a cotangent bundle and we have to restrict our reduction scheme to a certain class of star products namely those for which the space of formal functions polynomial in the momenta form a $\star$-subalgebra. Fortunately, this class of star products is rich enough to obtain star products of arbitrary characteristic class, hence we actually provide a reduction scheme which does not depend on the characteristic class of the initial star product.

Our paper is organized as follows: In Section 2 we recall from [6, 7, 8] the construction of various star products on cotangent bundles which have the common property that the formal functions polynomial in the momenta form a subalgebra. We also collect some notions of invariance with respect to Lie group actions in deformation quantization and recall the definition of the deformation quantization analogue of a $G$-equivariant classical momentum map. The resulting quantum momentum maps will play a fundamental role in our framework of phase space reduction. In Section 3 we then address the reduction of a certain class of star products on $(T^* Q, \omega_{B_0})$ which includes the examples considered in Section 2. Under the assumptions imposed on the initial data we establish a relation between the Poisson algebra of functions on $T^*(Q/G)$ polynomial in the momenta and the Poisson algebra of horizontal invariant functions on $T^* Q$ which are polynomial in the momenta as well. Moreover, we succeed to construct a deformation of this classical correspondence which enables us to define an associative product on the polynomial functions on $T^*(Q/G)$ which is induced by a star product on $(T^* Q, \omega_{B_0})$. It turns out that this product can be uniquely extended to the whole space $C^\infty(T^*(Q/G))[\nu]$. We thus obtain a star product with respect to some symplectic form on the reduced phase space which differs from the canonical symplectic structure by an additional magnetic field term and which depends on $B_0$, the chosen classical momentum map, the curvature of the chosen connection, and the momentum value used for the phase space reduction.
Also in this section we investigate the behaviour of our reduction scheme with respect to natural operations on star product algebras like isomorphisms, automorphisms, and derivations and we give conditions on which these transfer to the reduced star products. In Section 4 we relax our assumptions somewhat to arbitrary Lie group actions on $Q$ and return to consider the examples of Section 2. We derive first necessary and sufficient conditions on the geometric data which guarantee that these products admit a $G$-equivariant quantum Hamiltonian and thus a $G$-equivariant quantum momentum map in the sense of Xu [28]. In these particular cases it turns out that the quantum momentum maps are polynomials in the momenta, a result which is important for our reduction scheme to work. In addition, if there is a $G$-equivariant quantum Hamiltonian and thus a $G$-equivariant quantum momentum map in the sense of Xu [28]. In these particular cases it turns out that the quantum momentum maps are polynomials in the momenta, a result which is important for our reduction scheme to work.

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### 2 Preliminaries and Notation

In this section we first recall some notation and several canonical constructions of star products on cotangent bundles (see [6] for further details). Then we collect various notions of invariance in deformation quantization with respect to Lie group actions and recall the definition of the quantum analogue of a $G$-equivariant classical momentum map (cf. [1, 3, 28]). In our framework, this notion will turn out to be fundamental for the formulation of phase space reduction.

#### 2.1 Constructions of Star Products on Cotangent Bundles

Throughout this article, $Q$ will always denote a smooth $n$-dimensional manifold. Recall that the cotangent bundle $\pi : T^*Q \to Q$ is equipped with the canonical symplectic form $\omega_0 = -d\theta_0$, where $\theta_0$ denotes the canonical one-form. The zero section of $T^*Q$ is denoted by $i : Q \to T^*Q$ by means of which we consider $Q$ as embedded into $T^*Q$. Local coordinates on $Q$ will be denoted by $x^1, \ldots, x^n$, the induced coordinates on $T^*Q$ by $q^1, \ldots, q^n, p_1, \ldots, p_n$.

Given $k$ one-forms $\beta_1, \ldots, \beta_k \in \Gamma^\infty(T^*Q)$ we define a fiberwise acting differential operator $F(\beta_1 \vee \ldots \vee \beta_k) : \mathcal{C}^\infty(T^*Q) \to \mathcal{C}^\infty(T^*Q)$ of order $k$ by

$$
(F(\beta_1 \vee \ldots \vee \beta_k) f)(\zeta) := \left. \frac{\partial^k}{\partial t_1 \cdots \partial t_k} \right|_{t_1=\ldots=t_k=0} f(\zeta + t_1 \beta_1(x) + \ldots + t_k \beta_k(x)), \quad \zeta \in T^*_x Q. \quad (2.1)
$$

Clearly, $F$ extends to an injective algebra morphism from $\Gamma^\infty(\bigvee T^*Q)$ into the algebra of differential operators on $\mathcal{C}^\infty(T^*Q)$.

To every contravariant symmetric tensor field $T \in \Gamma^\infty(\bigvee^k T^*Q)$ one can assign a smooth function $\mathcal{P}(T) \in \mathcal{C}^\infty(T^*Q)$ by $(\mathcal{P}(T))(\zeta) := \frac{1}{k!} i^s(\zeta) \ldots i^s(\zeta) T(x)$. Note that $\mathcal{P}(T)$ is a homogeneous
polynomial of degree \( k \) in the momenta. Let us denote the space of these fiberwise polynomial functions by \( \mathcal{P}^k(Q) \). Obviously, \( \mathcal{P} \) then extends to an isomorphism between the \( \mathbb{Z} \)-graded commutative algebras \( \Gamma^\infty(\sqrt{T}Q) \) and \( \mathcal{P}(Q) = \bigoplus_{k=0}^\infty \mathcal{P}^k(Q) \).

Fixing a torsion free connection \( \nabla \) on the base manifold \( Q \) one can assign to every formal function \( f \in C^\infty(T^*Q)[[\nu]] \) a formal series with values in the differential operators on \( C^\infty(Q) \) by

\[
\varrho_0(f)\chi := i^*F(\exp(-\nu D)\chi) f = \sum_{l=0}^\infty \frac{(-\nu)^l}{l!} i^* \left( \frac{\partial^l f}{\partial p_{j_1} \cdots \partial p_{j_l}} \right) \partial_s(\partial_{x_{j_1}}) \cdots \partial_s(\partial_{x_{j_l}})^1 l! D' \chi. \tag{2.2}
\]

Here, \( D \) denotes the operator of symmetric covariant derivation which in local coordinates is given by \( D = dx^i \vee \nabla_{\partial_i} \).

It has been shown in \[8\] that the restriction of \( \varrho_0 \) to \( \mathcal{P}(Q)[[\nu]] \) is injective and that the image of \( \mathcal{P}(Q)[[\nu]] \) under \( \varrho_0 \) is closed with respect to composition of differential operators. Hence, one can define an associative product on \( \mathcal{P}(Q)[[\nu]] \) by

\[
F \star_0 F' := \varrho_0^{-1}(\varrho_0(F) \varrho_0(F')) , \quad F, F' \in \mathcal{P}(Q)[[\nu]]. \tag{2.3}
\]

Moreover, it has been shown that \( \star_0 \) can be described by bidifferential operators. Hence, one can uniquely extend \( \star_0 \) to a product on \( C^\infty(T^*Q)[[\nu]] \) yielding a star product on \( (T^*Q, \omega_0) \) which also will be denoted by \( \star_0 \) and which will be called the standard ordered star product (corresponding to \( \nabla \)). By definition of \( \star_0 \) it is obvious that \( \varrho_0 \) defines a representation of \( (C^\infty(T^*Q)[[\nu]], \star_0) \) on \( C^\infty(Q)[[\nu]] \).

Now we want to define further star products depending on a so-called order parameter \( \kappa \in [0, 1] \). To this end consider a smooth positive density \( \nu \) on \( Q \). Then \( \nu \) and the connection \( \nabla \) define a one-form \( \alpha \in \Gamma^\infty(TQ) \) by

\[
\nabla_X \nu = \alpha(X) \nu, \quad X \in \Gamma^\infty(TQ). \tag{2.4}
\]

It is immediate to check that this one-form satisfies

\[
\alpha = -\text{tr}(R), \tag{2.5}
\]

where \( R \) denotes the curvature tensor of \( \nabla \) and \( \text{tr}(R) \) the trace of the curvature endomorphism.

Let us denote by \( \text{hor}_\nabla \) the horizontal lift (with respect to \( \nabla \)) of vector fields on \( Q \) to vector fields on \( T^*Q \) (cf. also Definition A.2). Using local coordinates we then define a differential operator \( \Delta \) on \( C^\infty(T^*Q) \) by

\[
\Delta := \Delta_0 + F(\alpha) := F(dx^i) L_{\text{hor}_\nabla(\partial_{x_i})} + F(\alpha) = \frac{\partial^2}{\partial q^i \partial p_k} + \pi^2(\Gamma^l_{jk}) \frac{\partial^2}{\partial p_j \partial p_k} + \pi^2(\Gamma^l_{jk} + \alpha_k) \frac{\partial}{\partial p_k}. \tag{2.6}
\]

Here, \( \Gamma^l_{jk} \) denote the Christoffel symbols of \( \nabla \) and \( \alpha_k \) the components of \( \alpha \) in the chosen local chart. After some immediate computation it turns out that \( \Delta \) is independent of the choice of local coordinates. In \[3\] we also considered the formal series \( N_\kappa \) of differential operators given by

\[
N_\kappa := \exp(-\kappa \nu \Delta). \tag{2.7}
\]

Then this operator induces the \( \kappa \)-ordered star product

\[
f \star_\kappa f' := N_\kappa^{-1}((N_\kappa f) \star_0 (N_\kappa f')) \tag{2.8}
\]

which is obtained from \( \star_0 \) by the equivalence transformation \( N_\kappa^{-1} \). In the cases \( \kappa = 1 \) and \( \kappa = 1/2 \) the corresponding star products are called star product of anti-standard ordered type and Weyl
ordered star product. For a further discussion of these star products we again refer the reader to \cite{S}, where one can particularly find a motivation for the above definitions by means of the applicability of the GNS construction to $\star_{1/2}$. For later use let us recall the following factorization property of $N_\kappa$ from \cite[Lemma 3.6]{S}:

$$N_\kappa = \exp(-\kappa \nu \Delta_0) \exp\left( -F \left( \frac{\exp(\kappa \nu D) - \exp((\kappa - 1)\nu D)}{\nu D} \right) \right).$$ (2.9)

At this point let us mention two further important formulas which explicitly determine the $\star_\kappa$-left- and the $\star_\kappa$-right-multiplication with a formal function pulled-back from $Q$ (cf. \cite[Prop. 3.2]{S}):

$$\pi^* \chi \star_\kappa f = F(\exp(\kappa \nu D)\chi) f, \quad \tag{2.10}$$

$$f \star_\kappa \pi^* \chi = F(\exp((\kappa - 1)\nu D)\chi) f, \quad \tag{2.11}$$

where $\chi \in C^\infty(Q)[[\nu]]$ and $f \in C^\infty(T^*Q)[[\nu]]$. Let us also note that by $\varrho_\kappa(f) := \varrho_0(N_\kappa f)$ one obtains a representation of $(C^\infty(T^*Q)[[\nu]], \star_\kappa)$ on $C^\infty(Q)[[\nu]]$. Due to the properties of $\varrho_0$ and $N_\kappa$ the restriction of $\varrho_\kappa$ to $P(Q)[[\nu]]$ is injective as well.

Up to now we have only considered so-called homogeneous star products on $(T^*Q, \omega_0)$, i.e. star products for which $H = \mathcal{L}_{\xi_0} + \nu \theta_b$ is a derivation. Here $\xi_0 \in \Gamma^\infty(T(T^*Q))$ denotes the canonical Liouville vector field defined by $i_{\xi_0} \omega_0 = -\theta_0$. From this property and the fact that the bidifferential operator describing $\star_{1/2}$ at order 2 in the formal parameter is symmetric it is easy to deduce (cf. \cite[Thm. 4.6]{S}) that the characteristic class of the star products $\star_\kappa$ is given by $c(\star_\kappa) = c(\star_{1/2}) = [0]$.

Now we want to recall a construction which yields star products with characteristic class $\frac{1}{2}[\pi^* B]$, where $B \in Z^2_{\text{diff}}(Q)[[\nu]]$ is an arbitrary formal series of closed two-forms on $Q$. The thus obtained star products comprise deformation quantizations of the symplectic manifold $(T^*Q, \omega_{B_0} = \omega_0 + \pi^* B_0)$, where $B_0$ which is assumed to be real – denotes the term of zeroth order in the formal parameter of $B$.

For $A \in \Gamma^\infty(T^*Q)[[\nu]]$ with real $A_0$ and all $\kappa \in [0,1]$ consider the operator $A_\kappa : C^\infty(T^*Q)[[\nu]] \to C^\infty(T^*Q)[[\nu]]$ defined by

$$A_\kappa := t_{-A_0} \exp \left( -F \left( \frac{\exp(\kappa \nu D) - \exp((\kappa - 1)\nu D)}{\nu D} A - A_0 \right) \right), \quad \tag{2.12}$$

where $t_{-A_0}$ denotes the fiberwise translation by $-A_0$ that is $t_{-A_0}(\xi_x) = \xi_x - A_0(x)$. The important property of $A_\kappa$ is that it defines an automorphism of $\star_\kappa$ if and only if $dA = 0$ (cf. \cite[Thm. 3.4]{S}). To define star products reflecting the presence of a magnetic field $B$ on $Q$ consider a good open cover $\{O_j\}_{j \in I}$ of $Q$ together with local formal potentials $A_j \in \Gamma^\infty(T^*O_j)[[\nu]]$ of $B$. This means $B|_{O_j} = dA_j$, where in addition $A_0$ is chosen to be real. Let us denote by $A^B_\kappa$ the operator determined by Eq. (2.12) with $A^j$ used instead of $A$. Then one defines an associative product $\star^B_\kappa$ on $C^\infty(T^*O_j)[[\nu]]$ by

$$f \star^B_\kappa f' := A^B_\kappa \left( ((A^B_\kappa)^{-1} f) \star_\kappa ((A^B_\kappa)^{-1} f') \right), \quad f, f' \in C^\infty(T^*O_j)[[\nu]]. \quad \tag{2.13}$$

Now one makes the crucial observation that the operator $(A^B_\kappa)^{-1}A^B_\kappa$ corresponds via \cite[2.12]{S} to the closed formal one-form $A_j|_{O_j \cap O_k} - A^B_k|_{O_j \cap O_k}$. Hence, this operator is an automorphism of $(C^\infty(T^*(O_j \cap O_k))[\nu], \star_\kappa)$ and one can define a star product $\star^B_\kappa$ on $C^\infty(T^*Q)[[\nu]]$ by setting

$$f \star^B_\kappa f' |_{T^*O_j} = f |_{T^*O_j} \star^B_\kappa f' |_{T^*O_j}, \quad f, f' \in C^\infty(T^*Q)[[\nu]]. \quad \tag{2.14}$$

These star products do not depend on the particular choice of the covering nor on the choice of the potentials but only on $B$ (and of course on $\star_\kappa$). By a straightforward computation one checks
that \( \star^B_\kappa \) is a star product on \((T^*Q, \omega_{B_0})\) for all \( \kappa \in [0,1] \). Moreover, in [S Thm. 4.6] it has been shown that the characteristic class of \( \star^B_\kappa \) is given by \( c(\star^B_\kappa) = \frac{1}{\nu} [\pi^* B] \). Thus there exists for every equivalence class of star products on \((T^*Q, \omega_{B_0})\) a representative \( \star^B_\kappa \).

2.2 Notions of Invariance in Deformation Quantization

Let \( G \) be a Lie group and denote by \( g = \text{Lie}(G) \) its Lie algebra. In addition, assume \( \varphi : G \times M \to M \) to be a (left) action of \( G \) on a manifold \( M \) equipped with a symplectic form \( \omega \), and denote for every \( g \in G \) by \( \varphi_g : M \to M \) the map \( m \mapsto \varphi(g, m) \). Then a star product \( \star \) on \((M, \omega)\) is called \( \varphi \)-invariant if it is a quantum Hamiltonian and additionally satisfies

\[
\varphi_g^* f \equiv \varphi_g^* \star f \quad \text{for all } f \in C^\infty(M)[[\nu]], \ g \in G.
\]

In other words this means that \( r \) defined by \( r(g) f := \varphi_g^{-1} \star f \) defines a left action on \((C^\infty(M) [[\nu]], \star)\) by automorphisms. By anti-symmetrization of Eq. (2.15) with respect to \( f \) and \( f' \) one checks that the action \( \varphi \) necessarily has to be symplectic.

Given a \( G \)-invariant star product \( \star \) on a symplectic manifold \((M, \omega)\), differentiation of \( r \) yields an action \( \rho \) of the Lie algebra \( g = \text{Lie}(G) \) on \( C^\infty(M) [[\nu]] \) by derivations of \( \star \). Explicitly,

\[
\rho(\xi) f = -L_{\xi_M} f \quad \text{for all } f \in C^\infty(M) [[\nu]], \ \xi \in g,
\]

where \( \xi_M \) denotes the fundamental vector field associated to \( \xi \). Clearly, in the \( G \)-invariant case, every fundamental vector field \( \xi_M \) is symplectic.

Now let \( \star \) be a \( G \)-invariant star product and denote by \( C^1(g, C^\infty(M)) \) the space of linear forms on \( g \) with values in \( C^\infty(M) \). Then an element \( J = J_0 + J_+ \in C^1(g, C^\infty(M))[[\nu]] \) with real-valued \( J_0 \in C^1(g, C^\infty(M)) \) and \( J_+ \in \nu C^1(g, C^\infty(M))[[\nu]] \) is called a quantum Hamiltonian for \( r \), if

\[
\rho(\xi) = -L_{\xi_M} = \frac{1}{\nu} \text{ad}_r(J(\xi)) \quad \text{for all } \xi \in g.
\]

In other words this means that the Lie derivative with respect to the generating vector fields is a quasi-inner (or essentially inner) derivation of \( \star \). \( J \) is called a \( G \)-equivariant quantum Hamiltonian, if it is a quantum Hamiltonian and additionally satisfies

\[
\varphi_g^* J(\xi) = J(\text{Ad}(g^{-1})\xi) \quad \text{for all } \xi \in g, \ g \in G.
\]

A quantum Hamiltonian \( J \) is called a quantum momentum map if in addition

\[
\frac{1}{\nu} (J(\xi) \star J(\eta) - J(\eta) \star J(\xi)) = J([\xi, \eta]) \quad \text{for all } \xi, \eta \in g.
\]

Clearly, differentiation of (2.18) with respect to \( g \) at \( e \) shows that a \( G \)-equivariant quantum Hamiltonian always defines a quantum momentum map. Note that the converse generally does not hold true. In the sequel we will refer to a \( G \)-equivariant quantum Hamiltonian as a \( G \)-equivariant quantum momentum map.

The zeroth order parts of (2.17) and (2.18) mean that \( J_0 \) is a \( G \)-equivariant classical momentum map for \( r \), i.e. \( \xi_M \) is a Hamiltonian vector field with Hamiltonian function \( J_0(\xi) \) and the smooth mapping \( J_0 : M \to g^* \) defined by \( \langle J_0(m), \xi \rangle = J_0(\xi)(m) \) is \( \text{Ad}^* \)-equivariant. Note that we follow the convention to denote by \( \text{Ad}^*(g) = (\text{Ad}(g^{-1}))^* \) the coadjoint action of \( g \).

Now recall that a \( G \)-invariant star product is called strongly \( G \)-invariant, if \( J = J_0 \) defines a quantum Hamiltonian, where \( J_0 \) is a \( G \)-equivariant classical momentum map.
Finally, let us briefly introduce the notions of isomorphisms and equivalence transformations in the $G$-invariant framework. A star product $\ast$ on $(M, \omega)$ is called $G$-isomorphic to the star product $\ast'$ on $(M, \omega')$, if one can find an isomorphism from $(C^\infty(M)[[\nu]], \ast)$ to $(C^\infty(M)[[\nu]], \ast')$ which commutes with $r(g)$ for every $g \in G$. If in addition $\omega = \omega'$ and there exists an equivalence transformation from $(C^\infty(M)[[\nu]], \ast)$ to $(C^\infty(M)[[\nu]], \ast')$ which commutes with $r(g)$ for every $g \in G$, one calls $\ast$ a star product $G$-equivalent to $\ast'$. From these definitions it is obvious how to define the notions of $G$-automorphisms and $G$-self equivalences. As an immediate consequence note that for a $G$-isomorphism $T$ from $(C^\infty(M)[[\nu]], \ast)$ to $(C^\infty(M)[[\nu]], \ast')$ and $J$ a $G$-equivariant quantum momentum map for $\ast$ the transformed map $J'(\xi) := T J(\xi)$ is a $G$-equivariant quantum momentum map for $\ast'$. But one has to observe that the notion of strong $G$-invariance is not preserved under $G$-isomorphisms, in general, that means that for a strongly $G$-invariant $\ast$, a $G$-isomorphic $\ast'$ need not be strongly $G$-invariant.

3 Reduction of Star Products on Cotangent Bundles

In this section, we present a general procedure for the phase space reduction of a certain class of star products on cotangent bundles. As we show in Section 3.1 this reduction method applies in particular to the star products $\ast_k$ and $\ast^G$ constructed in the previous section. Since we will be concerned with different cotangent bundles $T^*Q$, $T^*\overline{Q}$ at the same instance, we adopt the convention to use $\overline{\ }$ to indicate objects related to the cotangent bundle $\overline{\pi} : T^*\overline{Q} \to \overline{Q}$.

3.1 Classical Phase Space Reduction of $(T^*Q, \omega_0 + \pi^*B_0)$: Geometric and Algebraic Properties

Given a Lie group $G$ we denote by $\{e_i\}_{1 \leq i \leq \dim(G)}$ a basis of its Lie algebra $\mathfrak{g}$ and by $\{\xi^i\}_{1 \leq i \leq \dim(G)}$ the corresponding dual basis of $\mathfrak{g}^*$. Assume that $\phi : G \times Q \to Q$ is a left action on the base manifold $Q$. This action gives rise to a $G$-action $\Phi : G \times T^*Q \to T^*Q$ on $T^*Q$, the cotangent lift of $\phi$. Explicitly, it is defined by $\Phi(g, \xi) := \Phi_g(\xi)$ and satisfies $\pi \circ \Phi_g = \phi_g \circ \pi$. We then have $\Phi_g^*\theta_0 = \theta_0$ and consequently $\Phi_g^*\omega_0 = \omega_0$. Let us denote the fundamental vector fields of $\phi$ by $\xi_Q \in \Gamma^\infty(TQ)$ and those of $\Phi$ by $\xi_{\Phi,Q} \in \Gamma^\infty(T(T^*Q))$. Clearly, these vector fields are $\pi$-related, i.e. $T \pi \circ \xi_{\Phi,Q} = \xi_Q \circ \pi$ for all $\xi \in \mathfrak{g}$. Recall that a $G$-equivariant classical momentum map for the canonical symplectic form $\omega_0$ is given by $J_0(\xi) = \theta_0(\xi_{\Phi,Q}) = P(\xi_Q)$, where $P(\xi_Q) \in P^1(Q)$ denotes the function linear in the momenta corresponding to the vector field $\xi_Q$. In canonical coordinates this reads $J_0(\xi) = p_i \pi^*\xi_Q^i$.

For the symplectic form $\omega_{B_0} = \omega_0 + \pi^*B_0$ with $B_0 \in Z^2(G)$ the following well-known result gives first conditions which have to be satisfied in order to be able to construct the reduced phase space.

Lemma 3.1 Let $\Phi$ act on $(T^*Q, \omega_{B_0})$ as above. Then the following holds true:

i.) $\Phi$ is a symplectic action with respect to $\omega_{B_0}$ if and only if $B_0$ is $G$-invariant.

ii.) If $B_0$ is $G$-invariant, then there is a $G$-equivariant classical momentum map for $\Phi$ if and only if there is a real-valued element $j_0 \in C^1(\mathfrak{g}, C^\infty(Q))$ such that

$$d j_0(\xi) = i_{\xi_Q} B_0 \quad \text{and} \quad \phi_g^* j_0(\xi) = j_0(\Ad(g^{-1})\xi) \quad \text{for all } g \in G, \xi \in \mathfrak{g}. \quad (3.1)$$

In this case $J_0(\xi) = P(\xi_Q) + \pi^* j_0(\xi)$ defines a $G$-equivariant classical momentum map which is unique up to elements of the space $\mathfrak{g}^G$ of invariants.
iii.) If the relations (3.1) are satisfied, one has in particular

$$j_0([\xi, \eta]) = B_0(\xi_Q, \eta_Q) \quad \text{for all } \xi, \eta \in \mathfrak{g}. \quad (3.2)$$

PROOF: The proof of i.) is obvious, so let us show ii.). To this end check first that $J_0$ defines a classical Hamiltonian for $\Phi$ if and only if $d(j_0(\xi) - P(\xi_Q)) = \pi^*i_{\xi_Q}B_0$ for all $\xi \in \mathfrak{g}$. After application of $i^*$ one notes that this is equivalent to the existence of $J_0$ such that the first condition in Eq. (3.1) is fulfilled. Now observe that the canonical momentum map $J^0$ for the case $B_0 = 0$, which is given by $J^0(\xi) = P(\xi_Q)$, is $G$-equivariant. Therefore, $G$-equivariance of $J_0$ is equivalent to the $G$-equivariance of $j_0$. The statement about the ambiguity of $J_0$ is a general fact which holds true for arbitrary Hamiltonian $G$-spaces. Assertion iii.) is obtained by differentiating $\phi^*_j J_0(\xi) = j_0(\text{Ad}(g^{-1})\xi)$ with respect to $g$ and using the first condition in Eq. (3.1). $\Box$

From now on we will assume that the above conditions for the existence of a $G$-equivariant classical momentum map are satisfied and that the action of $G$ on $Q$ is proper and free. This implies in particular that the orbit space of the $G$-action is smooth and even that $p : Q \to \overline{Q} = Q/G$ is a left principal $G$-bundle. Now fix an element $\mu_0 \in \mathfrak{g}^*G$ and recall that it gives rise to the reduced phase space $\tilde{J}_0^{-1}(\mu_0)$, where $\tilde{J}_0 : T^*Q \to \mathfrak{g}^*$ is defined by $\langle \tilde{J}_0(\xi_Q), \xi \rangle = J_0(\xi(\xi_Q))$. Since $\mu_0$ is a regular value of $J_0$, classical Marsden–Weinstein reduction applies and the reduced phase space naturally carries the structure of a symplectic manifold. The induced symplectic form on the reduced space will be denoted by $\omega_{\mu_0}$. It is uniquely characterized by the relation

$$\pi^*_{\mu_0}\omega_{\mu_0} = i_{\mu_0}\omega_{B_0}, \quad (3.3)$$

where $i_{\mu_0} : \tilde{J}_0^{-1}(\mu_0) \to T^*Q$ denotes the inclusion and $\pi_{\mu_0}$ the projection of $\tilde{J}_0^{-1}(\mu_0)$ onto the orbit space.

In case $B_0 = 0$, $J_0(\xi) = P(\xi_Q)$ it is well known that the reduced phase space is symplectomorphic to $T^*Q$ equipped with a symplectic structure of the form $\omega_0 + \pi^*b_0$, where $b_0$ is a closed two-form on $Q$ which vanishes for $\mu_0 = 0$. Note that the construction of an appropriate symplectomorphism is not canonical unless $\mu_0 = 0$ (cf. [15] [20] [22]).

An analogous result holds in the case of non-vanishing $B_0$. Since neither a precise statement of this nor a proof seems to have been appeared in the literature we briefly present it here. To this end we need some tools from the theory of (left) principal $G$-bundles. Let $\gamma$ be a connection one-form on the principal bundle $p : Q \to \overline{Q}$. Note that $\gamma$ transforms according to the rule $\phi^*_j \gamma = \text{Ad}(g)\gamma$ since $G$ acts from the left on $Q$. Moreover, let $\lambda := d\gamma - \frac{1}{2}[\gamma, \gamma]$, denote the corresponding curvature form; observe the minus sign in front of the bracket which is due to the fact that we work with a left principal $G$-bundle. Recall that $\lambda$ is a $\mathfrak{g}$-valued horizontal two-form on $Q$. Finally, we associate to every smooth map $j : Q \to \mathfrak{g}^*$ a one-form $\Gamma_j \in \Gamma^\infty(T^*Q)$ by

$$\Gamma_j := \langle j, \gamma \rangle. \quad (3.4)$$

Then $\phi^*_j \Gamma_j = \Gamma_{\text{Ad}^*(g^{-1})\phi^*_j}$ for all $g \in G$. Thus, if $j$ is $G$-equivariant, then $\Gamma_j$ is $G$-invariant as well. Now we are prepared to formulate our result.

**Theorem 3.2 (cf. [15] [20] [22])** With notations and assumptions from above, the reduced phase space $\tilde{J}_0^{-1}(\mu_0)/G$ with symplectic form $\omega_{\mu_0}$ induced by $\omega_{B_0}$ is symplectomorphic to $(T^*Q, \omega_{\mu_0}) = (T^*(Q/G), \omega_0 + \pi^*b_0)$, where $b_0$ is the uniquely determined closed two-form on $Q$ which satisfies $p^*b_0 = B_0 + d\Gamma_j - \mu_0$:

$$(T^*Q)_{\mu_0}, \omega_{\mu_0} = (\tilde{J}_0^{-1}(\mu_0)/G, \omega_{\mu_0}) \cong (T^*Q, \omega_{\mu_0}). \quad (3.5)$$
PROOF: Let us consider the one-form $\Gamma_{j_0-\mu_0}$, where the smooth mapping $\tilde{j}_0 : Q \to \mathfrak{g}^*$ is defined by $\langle \tilde{j}_0(x), \xi \rangle = j_0(\xi)(x)$. This one-form induces a fiber translation $tr_{\tilde{j}_0-\mu_0}$ given by $tr_{\tilde{j}_0-\mu_0}(\xi_x) = \xi_x + \Gamma_{j_0-\mu_0}(x)$. Due to the properties of the connection one-form $tr_{\tilde{j}_0-\mu_0}$ maps $\tilde{J}_0^{-1}(\mu_0)$ to $(T^*Q)^0 := \{ \xi_x \in T^*Q \mid \xi_x(\xi_Q(x)) = 0 \text{ for all } \xi \in \mathfrak{g} \}$. Next observe that $tr_{\tilde{j}_0-\mu_0}$ commutes with every $\Phi_g$ by the $G$-invariance of $\mu_0$ and the $G$-equivariance of $J_0$. Therefore, $tr_{\tilde{j}_0-\mu_0}$ passes to the quotient and defines a diffeomorphism $\Psi_{\mu_0}$ from $\tilde{J}_0^{-1}(\mu_0)/G$ to $(T^*Q)^0/G \cong T^*(Q/G)$. In order to determine the symplectic form that is carried over to $T^*\overline{Q} = T^*(Q/G)$ via $(\Psi_{\mu_0}^{-1})^*$, one first has to compute $t^*\Gamma_{j_0-\mu_0} \omega_{B_0} = \omega_0 + \pi^*(B_0 + d\Gamma_{j_0-\mu_0})$. Now an easy computation using the relations between $i_{\pi^{0*}} \omega_{B_0} = \pi^* \omega_{\mu_0}$ and the commutative diagram

\[
\begin{array}{cccc}
\tilde{J}_0^{-1}(\mu_0) & \xrightarrow{tr_{\tilde{j}_0-\mu_0}} & (T^*Q)^0 & \xrightarrow{\pi^{0*}} & Q \\
\downarrow{\pi_{\mu_0}} & & \downarrow{\pi^0} & & \downarrow{p} \\
\tilde{J}_0^{-1}(\mu_0)/G & \xrightarrow{\Psi_{\mu_0}} & (T^*Q)^0/G \cong T^*\overline{Q} & \xrightarrow{\pi} & \overline{Q},
\end{array}
\]

where $i_{\mu_0}, i^0$ are the inclusions into $T^*Q$ and $\pi_{\mu_0}, \pi^0$ the projections onto the respective orbit spaces.

Now consider the space $C^\infty(T^*Q)^G$ of $G$-invariant smooth functions on $T^*Q$ and the space

\[
I_{\mu_0,*} := \left\{ f \in C^\infty(T^*Q)^G \mid f = \sum_{i=1}^{\dim(G)} h^i(J_0(e_i) - \langle \mu_0, e_i \rangle) \right\} \text{ with } h^i \in C^\infty(T^*Q).
\]

Since $I_{\mu_0,*}$ is a Poisson ideal in $C^\infty(T^*Q)^G$, the pointwise product and the Poisson bracket $\{ , \}_{B_0}$ corresponding to $\omega_{B_0}$ induce the structure of a Poisson algebra on the quotient $C^\infty(T^*Q)^G/I_{\mu_0,*}$ by

\[
\{ f, f' \}_{\mu_0,*} := \{ f, f' \}_{B_0,*}, \quad \{ [f, f'], [f', f''] \}_{\mu_0,*} := \{ [f, f'] \}_{B_0,*}.
\]

The thus obtained Poisson algebra is known to be isomorphic to $(C^\infty(T^*\overline{Q}), \{ , \}_{B_0})$, where $\{ , \}_{B_0}$ denotes the Poisson bracket corresponding to the symplectic form $\omega_{B_0}$. Unfortunately, there is no canonical construction of such an isomorphism, but restricting to functions polynomial in the momenta, we are able to find a natural isomorphism between the Poisson subalgebras $\mathcal{P}(Q)^G/I_{\mu_0,*} \subseteq C^\infty(T^*Q)^G/I_{\mu_0,*}$ and $\mathcal{P}(\overline{Q}) \subseteq C^\infty(T^*\overline{Q})$ which depends only on the choice of the connection $\gamma$. Hereby we have used the abbreviations $\mathcal{P}(Q)^G := \mathcal{P}(Q) \cap C^\infty(T^*Q)^G$ and $I_{\mu_0,*} := \mathcal{P}(Q) \cap I_{\mu_0,*}$.

Let us now provide the details. Due to the choice of the connection $\gamma$ the tangent bundle of $Q$ can be written as the direct sum of the horizontal bundle $HQ$ and the (trivial) vertical bundle $VQ$. Clearly, this decomposition induces a decomposition of the symmetric powers of $TQ$. Hence the space of sections $\Gamma^\infty(VTQ)$ can be written as

\[
\Gamma^\infty(VTQ) = \bigoplus_{k=0}^{\infty} \Gamma^\infty(H^kQ) \oplus \bigoplus_{k=1}^{\infty} \bigoplus_{r=1}^{k} \Gamma^\infty(H^{k-r}Q \vee V^rQ).
\]
Obviously, \( \Gamma^\infty(\mathbb{V}^l H \mathbb{Q} \vee \mathbb{V}^r V \mathbb{Q}) \) is bigraded by the horizontal degree \( l \) and the vertical degree \( r \). We will refer to the spaces \( \bigoplus_{k=0}^\infty \Gamma^\infty(\mathbb{V}^k H \mathbb{Q}) \) and \( \bigoplus_{k=1}^\infty \bigoplus_{r=1}^k \Gamma^\infty(\mathbb{V}^{k-r} H \mathbb{Q} \vee \mathbb{V}^r V \mathbb{Q}) \) as the space of totally horizontal sections and partially vertical sections in \( \mathbb{V} T \mathbb{Q} \), respectively. Moreover, we denote by \( H \) the projection onto the totally horizontal sections and by \( P^\mathbb{Q} \) the projection onto the partially vertical sections. Since \( \{ e_{iQ} \}_i \) is a set of basis sections of the vertical bundle, there exist for every \( T \in PV(\Gamma^\infty(\mathbb{V} T \mathbb{Q})) \) uniquely determined tensor fields \( R^i(T) \in \Gamma^\infty(\mathbb{V} T \mathbb{Q}) \) such that \( T = \sum_{i=1}^{\dim(G)} R^i(T) \vee e_{iQ} \). Hence \( \Gamma^r : PV(\Gamma^\infty(\mathbb{V} T \mathbb{Q})) \rightarrow \Gamma^\infty(\mathbb{V} T \mathbb{Q}) \) is a well-defined mapping that extends to all of \( \Gamma^\infty(\mathbb{V} T \mathbb{Q}) \) by setting \( R^i(T) := 0 \) for \( T \in H(\Gamma^\infty(\mathbb{V} T \mathbb{Q})) \). Using the isomorphism \( P : \Gamma^\infty(\mathbb{V} T \mathbb{Q}) \rightarrow \mathcal{P}(\mathbb{Q}) \) we then get the following decomposition of \( \mathcal{P}(\mathbb{Q}) \) into the spaces of so-called totally horizontal and partially vertical polynomial functions:

\[
\mathcal{P}(\mathbb{Q}) = h(\mathcal{P}(\mathbb{Q})) \oplus pv(\mathcal{P}(\mathbb{Q})).
\]

Hereby, we have used \( h = P \circ H \circ P^{-1} \) and \( pv = P \circ PV \circ P^{-1} \). Under the isomorphism \( P \) the mapping \( R^i \) transforms to \( r^i : \mathcal{P}(\mathbb{Q}) \rightarrow \mathcal{P}(\mathbb{Q}) \) which explicitly is given by

\[
r^i(F) = \begin{cases} \frac{1}{r} F(\Gamma^i_0) F & \text{if } F \text{ is vertical of degree } r \geq 1, \\ 0 & \text{if } F \text{ is vertical of degree } 0. 
\end{cases}
\]  

Thus, every \( F \in \mathcal{P}(\mathbb{Q}) \) can be written as

\[
F = h(F) + \sum_{i=1}^{\dim(G)} r^i(F) P(e_{iQ}).
\]

Now consider again the \( G \)-invariant one-form \( \Gamma_0^*_{j_0-\mu_0} \) on \( T^* \mathbb{Q} \) defined in the proof of Theorem iii.) The space \( \mathcal{P}(\mathbb{Q}) \) decomposes into the direct sum

\[
\mathcal{P}(\mathbb{Q}) = h(\mathcal{P}(\mathbb{Q})) \oplus \left\{ F \in \mathcal{P}(\mathbb{Q}) \mid F = \sum_{i=1}^{\dim(G)} H^i(J_0(e_i) - \langle \mu_0, e_i \rangle) \text{ with } H^i \in \mathcal{P}(\mathbb{Q}) \right\}
\]

and this decomposition is \( G \)-invariant. Moreover, the projections onto the respective subspaces are given by \( h_{\mu_0} := h \circ t^*_{\Gamma_0-\mu_0} \) and \( pv_{\mu_0} := t^*_{\Gamma_0-\mu_0} \circ pv \circ t^*_{\Gamma_0-\mu_0} \).

i.) According to i.) the space of \( G \)-invariant polynomial functions \( \mathcal{P}(\mathbb{Q})^G \) decomposes into the direct sum

\[
\mathcal{P}(\mathbb{Q})^G = h(\mathcal{P}(\mathbb{Q})^G) \oplus P^{\mu_0}.
\]

iii.) The space \( h(\mathcal{P}(\mathbb{Q})^G) \) of totally horizontal \( G \)-invariant polynomial functions becomes a Poisson algebra with the usual pointwise product of functions and the Poisson bracket \( \{ \cdot, \cdot \}_{j_0-\mu_0} \) defined by

\[
\{ F, F' \}_{j_0-\mu_0} := h_{\mu_0} \left( \{ F, F' \}_{B_0} \right) = h(t^*_{\Gamma_0-\mu_0} \{ F, F' \}_{B_0}), \quad F, F' \in h(\mathcal{P}(\mathbb{Q})^G).
\]
iv.) As a Poisson algebra, \( (h(\mathcal{P}(Q)^G), \{ , \}_J) \) is isomorphic to \( \mathcal{P}(\overline{Q}) \) with the Poisson bracket induced by \( \omega_{b_0} = \overline{\omega}_0 + \overline{\pi} \omega_{b_0} \), where \( b_0 \) denotes the uniquely determined closed two-form on \( \overline{Q} \) such that \( p^*b_0 = B_0 + d\Gamma_{j_0-\mu_0} \). If \( h^*: \Gamma^\infty(\sqrt{T\overline{Q}}) \to \Gamma^\infty(\sqrt{TQ}) \) denotes the horizontal lift (which is obtained by extension from \( \Gamma^\infty(T\overline{Q}) \) to \( \Gamma^\infty(TQ) \) as homomorphism with respect to \( \sqrt{\cdot} \), particularly \( \chi^h = p^*\chi \) for \( \chi \in C^\infty(\overline{Q}) \), an explicit Poisson algebra isomorphism is given by

\[
l: \mathcal{P}(\overline{Q}) \ni \mathcal{P}(t) \mapsto \mathcal{P}(\mathcal{P}(Q))^G, \quad t \in \Gamma^\infty(\sqrt{T\overline{Q}}). \tag{3.14}
\]

v.) Finally, \( (h(\mathcal{P}(Q)^G), \{ , \}_J) \) is isomorphic to \( \mathcal{P}(Q)^G/I_{\mu_0} \), with Poisson algebra structure defined by Eq. \((3.13)\). An isomorphism is given by

\[
h(\mathcal{P}(Q)^G) \ni F \mapsto [F]_{\mu_0} \in \mathcal{P}(Q)^G/I_{\mu_0}. \tag{3.15}
\]

PROOF: Statements i.) and ii.) are obvious from the above considerations. Since the kernel of \( h_{i_0-\mu_0}|_{\mathcal{P}(Q)^G} \) is a Poisson ideal in \( \mathcal{P}(Q)^G \), it is straightforward to verify that \( \{ , \}_J \) defines a Poisson bracket on \( h(\mathcal{P}(Q)^G) \). For the proof of iv.), first note that \( l: \mathcal{P}(\overline{Q}) \to h(\mathcal{P}(Q)^G) \) is an isomorphism of commutative algebras. Thus it remains to show that the Poisson bracket obtained on \( \mathcal{P}(\overline{Q}) \) by pull-back by \( l \) of the Poisson bracket \( \{ , \}_J \) on \( h(\mathcal{P}(Q)^G) \) coincides with the Poisson bracket induced by \( \omega_{b_0} \). To check this, it is enough to compute the induced bracket of \( \mathcal{P}(t) \) and \( \mathcal{P}(s) \) for \( t, s \in \Gamma^\infty(T\overline{Q}) \). By a straightforward computation one obtains

\[
\{ l(\mathcal{P}(t)), l(\mathcal{P}(s)) \}_J = -\mathcal{P}(\{ [t, s]^h \} - \pi^*(\{ (B_0 + d\Gamma_{j_0-\mu_0})(t^h, s^h) \})).
\]

Since \( Tp^{th} = t \circ p \) and analogously for \( s \), we get \( (B_0 + d\Gamma_{j_0-\mu_0})(t^h, s^h) = p^*(b_0(t, s)) \) by definition of \( b_0 \). Then, using \( l(\mathcal{P}(\chi)) = \pi^*p^*\chi \) for \( \chi \in C^\infty(\overline{Q}) \), we find

\[
l^{-1}(\{ l(\mathcal{P}(t)), l(\mathcal{P}(s)) \}_J) = -\mathcal{P}(\{ [t, s] \} - \pi^*(b_0(t, s))).
\]

But this proves iv.) since this last expression coincides with the Poisson bracket of \( \mathcal{P}(t) \) and \( \mathcal{P}(s) \) with respect to the symplectic form \( \omega_{b_0} = \overline{\omega}_0 + \overline{\pi} \omega_{b_0} \). For the proof of v.), one again has to check the compatibility of the Poisson brackets, but this is straightforward observing that the kernel of \( h_{j_0-\mu_0} \) restricted to \( \mathcal{P}(Q)^G \) is the ideal \( I_{\mu_0} \).

\[\Box\]

3.2 Reduction of a Certain Class of Star Products on Cotangent Bundles

In view of the reduction of the classical structures considered in the preceding section, it should be possible to analogously determine the reduction of certain star products on \( (T^*Q, \omega_{B_0}) \) in order to obtain star products on \( (T^*\overline{Q}, \omega_{B_0}) \). To this end recall first our general assumptions that the \( G \)-action on \( Q \) is proper and free and that \( \mu_0 \in \mathfrak{g}^* \), which is a regular value of \( J_0 \), is chosen to be invariant with respect to the coadjoint action of \( G \). For the quantum reduction we have to assume in addition that we are given a star product \( \star \) on \( (T^*Q, \omega_{B_0}) \), a \( G \)-equivariant quantum momentum map \( J \) and a deformation \( \mu \) of the classical momentum value \( \mu_0 \) such that the following properties hold true:

- \( \star \) is \( G \)-invariant, i.e. invariant with respect to the lifted action \( \Phi \) of \( G \) on \( T^*Q \).
- \( J = J_0 + J_+ \in C^1(\mathfrak{g}, C^\infty(T^*Q))[[\mu]] \) is a \( G \)-equivariant quantum momentum map for \( \star \), where \( J_0 \) denotes a \( G \)-equivariant classical momentum map of the form \( J_0(\xi) = P(\xi_Q) + \pi^*J_0(\xi) \) as in Lemma 3.1.
• $\mathcal{P}(Q)[[\nu]]$ is a $\ast$-subalgebra and $J(\xi) \in \mathcal{P}(Q)[[\nu]]$ for all $\xi \in \mathfrak{g}$.

• The quantum momentum value has the form $\mu = \mu_0 + \mu_+$ with $\mu_+ \in \nu \mathfrak{g}^*[[\nu]]$ and $\mathfrak{g}^*$ the complexification of $\mathfrak{g}$. Moreover, $\mu$ is invariant with respect to the coadjoint action of $G$.

The third of the above assumptions will enable us to compute the reduced star product by means of polynomial functions, only. Note at this point that later in Corollary 4.14 ii.) we will prove that the assumption $J(\xi) \in \mathcal{P}(Q)[[\nu]]$ for all $\xi \in \mathfrak{g}$ is actually no additional assumption but a consequence of the fact that $\mathcal{P}(Q)[[\nu]]$ is a $\ast$-subalgebra.

As for the algebraic part of the classical reduction consider $C^\infty(T^*Q)^G[[\nu]]$ and the subspace

$$I_{\mu,*} := \left\{ f \in C^\infty(T^*Q)^G[[\nu]] \left| f = \sum_{i=1}^{\dim(G)} h^i \ast (J(e_i) - \langle \mu, e_i \rangle) \text{ with } h^i \in C^\infty(T^*Q)[[\nu]] \right. \right\}.$$ 

Then $I_{\mu,*}$ is a two-sided ideal of $C^\infty(T^*Q)^G[[\nu]]$, since $J$ is a quantum Hamiltonian, and the quotient space $C^\infty(T^*Q)^G[[\nu]]/I_{\mu,*}$ becomes an associative algebra by

$$[f]_{\mu,*} \ast_{\text{red}} [f']_{\mu,*} := [f \ast f']_{\mu,*}. \quad (3.16)$$

In order to interpret the associative product $\ast_{\text{red}}$ as a star product on the reduced phase space, one has to find a $\mathbb{C}[[\nu]]$-module isomorphism between $C^\infty(T^*Q)^G[[\nu]]/I_{\mu,*}$ and $(C^\infty(T^*Q)^G/I_{\mu_0,*})[[\nu]]$. Since the latter space is isomorphic to $C^\infty(T^*\mathcal{O})[[\nu]]$ one then defines an associative product on $C^\infty(T^*\mathcal{O})[[\nu]]$ by declaring the isomorphism in question to be a homomorphism of associative algebras. If the thus constructed product is a star product indeed, then we have obtained the reduced star product we are looking for. At this point one has to mention that there is no canonical construction of such an isomorphism for the space of all smooth functions, but like in the classical case one can find a natural isomorphism between the subspaces obtained by restriction to polynomial functions.

Let us now provide the details for the construction of this latter isomorphism. To this end observe first that by the third assumption above the product $\ast_{\text{red}}$ can be restricted to $\mathcal{P}(Q)^G[[\nu]]/I_{\mu,*}$ to $C^\infty(T^*\mathcal{O})[[\nu]]$ where $I_{\mu,*} := \mathcal{P}(Q)[[\nu]] \cap I_{\mu,*}$. We now claim that there is a naturally constructed $\mathbb{C}[[\nu]]$-module isomorphism between $\mathcal{P}(Q)^G[[\nu]]/I_{\mu,*}$ and $(\mathcal{P}(Q)^G/I_{\mu_0,*})[[\nu]]$ where the latter space is isomorphic to $\mathcal{P}(\mathcal{O})[[\nu]]$ by Proposition 3.3 iv.) and v.).

Clearly, the projections $h_{\gamma_0 - \mu_0}$, $pv_{\gamma_0 - \mu_0}$ and the maps $r_{\gamma_0 - \mu_0}^i := t_{\gamma_0 - \mu_0}^i \circ r_i \circ t_{\gamma_0 - \nu_0}^i$ extend by $\mathbb{C}[[\nu]]$-linearity from $\mathcal{P}(Q)$ to $\mathcal{P}(Q)[[\nu]]$. Hence, by Proposition 3.3 i.), every $F \in \mathcal{P}(Q)[[\nu]]$ decomposes uniquely into the sum of the form

$$F = h_{\gamma_0 - \mu_0}(F) + pv_{\gamma_0 - \mu_0}(F) = h_{\gamma_0 - \mu_0}(F) + \sum_{i=1}^{\dim(G)} r_{\gamma_0 - \mu_0}^i(F)(J_0(e_i) - \langle \mu_0, e_i \rangle).$$

After defining $\Delta_{\mu,*} : \mathcal{P}(Q)[[\nu]] \rightarrow \mathcal{P}(Q)[[\nu]]$ by the equation

$$\Delta_{\mu,*}F := \frac{1}{\nu} \sum_{i=1}^{\dim(G)} \left( r_{\gamma_0 - \mu_0}^i(F)(J_0(e_i) - \langle \mu_0, e_i \rangle) - r_{\gamma_0 - \mu_0}^i(F) \ast (J(e_i) - \langle \mu, e_i \rangle) \right), \quad (3.17)$$

the above decomposition can be rewritten as

$$F = h_{\gamma_0 - \mu_0}(F) + \sum_{i=1}^{\dim(G)} r_{\gamma_0 - \mu_0}^i(F) \ast (J(e_i) - \langle \mu, e_i \rangle) + \nu \Delta_{\mu,*}F.$$
Now repeat this ad infinitum and decompose at every step the remaining term not of the form 
\[ h_{j_0 - \mu_0}(F') + \sum_{i=1}^{\dim(G)} r_{j_0 - \mu_0}^i (F') \star (J(e_i) - \langle \mu, e_i \rangle) \]. This procedure finally yields

\[ F = h_{j_0 - \mu_0} \left( \frac{\id}{\id - \nu \Delta_{\mu,\ast}} F \right) + \sum_{i=1}^{\dim(G)} r_{j_0 - \mu_0}^i \left( \frac{\id}{\id - \nu \Delta_{\mu,\ast}} F \right) \star (J(e_i) - \langle \mu, e_i \rangle). \tag{3.18} \]

Like in the classical case we obtain:

**Lemma 3.4** i.) The space \( \mathcal{P}(Q)[[\nu]] \) decomposes into the direct sum

\[ \mathcal{P}(Q)[[\nu]] = h(\mathcal{P}(Q))[[\nu]] \oplus \left\{ F \in \mathcal{P}(Q)[[\nu]] \mid F = \sum_{i=1}^{\dim(G)} H_i \star (J(e_i) - \langle \mu, e_i \rangle) \text{ with } H_i \in \mathcal{P}(Q)[[\nu]] \right\} \quad (3.19) \]

and this decomposition is \( G \)-invariant. Moreover, the projections onto the respective subspaces are given by

\[ h_{j_0 - \mu_0} \circ \frac{\id}{\id - \nu \Delta_{\mu,\ast}} \quad \text{and} \quad \sum_{i=1}^{\dim(G)} r_{j_0 - \mu_0}^i \left( \frac{\id}{\id - \nu \Delta_{\mu,\ast}} F \right) \star (J(e_i) - \langle \mu, e_i \rangle), \quad F \in \mathcal{P}(Q)[[\nu]]. \]

ii.) According to i.) the space of formal series of \( G \)-invariant polynomial functions \( \mathcal{P}(Q)^G[[\nu]] \) decomposes into the direct sum

\[ \mathcal{P}(Q)^G[[\nu]] = h(\mathcal{P}(Q)^G)[[\nu]] \oplus I_{\mu,\ast}^G. \tag{3.20} \]

**Proof:** The fact that every element of \( \mathcal{P}(Q)[[\nu]] \) can be decomposed as stated in i.) is obvious from the above considerations. To see that the sum in \[ (3.19) \] is direct, one just has to observe that the lowest order in \( \nu \) of \( \sum_{i=1}^{\dim(G)} H_i \star (J(e_i) - \langle \mu, e_i \rangle) \) is given by \( \sum_{i=1}^{\dim(G)} H_0^i (J_0(e_i) - \langle \mu_0, e_i \rangle) \) (where we have written \( H_i = \sum_{k=0}^{\infty} \nu^k H^k_i \)). But then assuming that \( \sum_{i=1}^{\dim(G)} H_i \star (J(e_i) - \langle \mu, e_i \rangle) \) is horizontal, the fact that the sum in Eq. \[ (3.11) \] is direct implies that \( H_0^i = 0 \) for all \( 1 \leq i \leq \dim(G) \). Repeating this argument order by order, we finally get \( H_i = 0 \) for all \( 1 \leq i \leq \dim(G) \) proving that the decomposition is direct. Using the invariance properties of \( \mu, J \) and \( \star \), it is easy to check that \( \Phi^g \) maps elements of the form \( \sum_{i=1}^{\dim(G)} H^i \star (J(e_i) - \langle \mu, e_i \rangle) \) to elements of the same form. Thus, the above decomposition turns out to be \( G \)-invariant, since obviously \( \Phi^g \) maps totally horizontal polynomials to totally horizontal polynomials. ii.) is a direct consequence of i.) and the definition of \( I_{\mu,\ast}^G \).

After these preparations we obtain one of the main results of this section.

**Theorem 3.5** i.) The space \( h(\mathcal{P}(Q)^G)[[\nu]] \) of formal series of totally horizontal \( G \)-invariant polynomial functions becomes an associative algebra by means of the product \( \bullet^{J,\mu} \) defined by

\[ F \bullet^{J,\mu} F' := h_{j_0 - \mu_0} \left( \frac{\id}{\id - \nu \Delta_{\mu,\ast}} (F \star F') \right), \quad F, F' \in h(\mathcal{P}(Q)^G)[[\nu]]. \tag{3.21} \]

ii.) The pull back of \( \bullet^{J,\mu} \) to \( \mathcal{P}(\overline{G})[[\nu]] \) via the isomorphism \( l : \mathcal{P}(\overline{G})[[\nu]] \rightarrow h(\mathcal{P}(Q)^G)[[\nu]] \) defined in Eq. \[ (3.12) \] gives rise to a star product \( \star^{J,\mu} \) on \( \mathcal{P}(\overline{G})[[\nu]] \), where the underlying Poisson bracket is induced by the symplectic form \( \omega_{b_0} = \overline{\omega}_0 + \pi^g b_0 \) given in Proposition \[ iv.) \].
iii.) (h(\mathcal{P}(Q)^G)[[\nu]], \cdot^{J,\mu}) is isomorphic to \mathcal{P}(Q)^G[[\nu]]/I_{\mu,*}^P, with the associative algebra structure defined in Eq. (3.21). An isomorphism is given by

\[ h(\mathcal{P}(Q)^G)[[\nu]] \ni F \mapsto [F]_{\mu,*} \in \mathcal{P}(Q)^G[[\nu]]/I_{\mu,*}^P. \]  

(3.22)

iv.) The star product \( \star^{J,\mu} \) on \( \overline{\mathcal{P}(Q)}[[\nu]] \) can be described by bidifferential operators, hence can be uniquely extended to a star product on \( C^\infty(T^*\overline{Q})[[\nu]] \), which will also be denoted by \( \star^{J,\mu} \).

PROOF: Using the fact that the kernel of the projection \( h_{j_0-\mu_0} \circ \text{id} \) restricted to \( \mathcal{P}(Q)^G[[\nu]] \) is a two-sided ideal in \( \mathcal{P}(Q)^G[[\nu]] \), it is straightforward to see that the composition \( \cdot^{J,\mu} \) defined in (3.21) on \( h(\mathcal{P}(Q)^G)[[\nu]] \) is associative. For the proof of ii.), first note that \( F \cdot^{J,\mu} 1 = h_{j_0-\mu_0}(\text{id} \cdot^{\nu \Delta_{\mu,*}} F) = F = 1 \cdot^{J,\mu} F \) for all \( F \in h(\mathcal{P}(Q)^G)[[\nu]] \) since \( \Delta_{\mu,*} F = 0 \) and \( t^*_{\Gamma_{j_0-\mu_0}} F = F \) for totally horizontal polynomial functions, implying that \( f \star^{J,\mu} 1 = f = 1 \star^{J,\mu} f \) for \( f \in \mathcal{P}(\overline{Q})[[\nu]] \). Now, an immediate computation yields

\[ F \cdot^{J,\mu} F' - F' \cdot^{J,\mu} F = \nu h_{j_0-\mu_0}(\{F, F'\}_R) + O(\nu^2), \quad \text{for all} \ F, F' \in h(\mathcal{P}(Q)^G), \]

but according to statement iv.) of Proposition 3.3, this implies that for \( f, f' \in \mathcal{P}(\overline{Q}) \), the lowest order in \( f \star^{J,\mu} f' - f' \star^{J,\mu} f \) is given by the Poisson bracket corresponding to \( \omega_{\mu_0} \). Assertion iii.) is obvious from Lemma 3.3 ii.) and the definition of the associative product on \( h(\mathcal{P}(Q)^G)[[\nu]] \) resp. \( \mathcal{P}(Q)^G[[\nu]]/I_{\mu,*}^P. \) Since \( \star \) is a differential star product and \( \Delta_{\mu,*} \) a differential operator, it is obvious that the product \( \cdot^{J,\mu} \) on \( h(\mathcal{P}(Q)^G)[[\nu]] \) can be described by bidifferential operators. But this also implies that the corresponding star product on \( \mathcal{P}(\overline{Q})[[\nu]] \) is given by bidifferential operators. Since bidifferential operators are completely determined by their values on polynomial functions, \( \star^{J,\mu} \) extends to a star product on \( C^\infty(T^*\overline{Q})[[\nu]] \) in a unique way.

\[ \Box \]

Remark 3.6 Clearly, pulling back the star product \( \star^{J,\mu} \) on \( C^\infty(T^*\overline{Q})[[\nu]] \) to \( C^\infty((T^*\mu_0)^+)[[\nu]] \) via the symplectomorphism \( \Psi_{\mu_0} \) constructed in the proof of Theorem 3.22 one obtains a star product \( \star^{J,\mu}_{\Psi_{\mu_0}} := \Psi_{\mu_0} \circ \star^{J,\mu} \) on the symplectic manifold \( (J_0^{-1}(\mu_0)/G, \omega_{\mu_0}) \). All results we derive in the sequel about the star products \( \star^{J,\mu} \) will then transfer almost literaly to the star products \( \star^{J,\mu}_{\Psi_{\mu_0}} \).

In the remaining section we prove several properties of the reduction scheme introduced above. This will partially clarify the dependence of \( \star^{J,\mu} \) on the chosen \( \mu \in \mathfrak{g}^G + \nu \mathfrak{g}_c^G[[\nu]] \). Moreover, we thus explain the relation between automorphisms resp. derivations of \( \star \) and those of \( \star^{J,\mu} \). In particular, we show that our reduction is natural with respect to \( G \)-isomorphisms of star products which preserve \( \mathcal{P}(Q)[[\nu]] \) and satisfy an appropriate compatibility relation for the momentum values.

Proposition 3.7 i.) Let \( \cdot^{J,\mu} \) and \( \cdot^{J',\mu'} \) denote the products on \( h(\mathcal{P}(Q)^G)[[\nu]] \) obtained by reduction of \( (\star, J, \mu) \) and \( (\star', J', \mu') \). Moreover, let \( T \) be an isomorphism (equivalence transformation) from \( \star \) to \( \star' \) which preserves \( \mathcal{P}(Q)[[\nu]] \) and which commutes with every \( \Phi_g^* \). Define \( T : h(\mathcal{P}(Q)^G)[[\nu]] \to h(\mathcal{P}(Q)^G)[[\nu]] \) by

\[ TF := h_{j_0-\mu_0} \left( \frac{\text{id}}{\text{id} - \nu \Delta_{\mu,*}', T} \right), \quad F \in h(\mathcal{P}(Q)^G)[[\nu]]. \]  

(3.23)

Then \( T \) is an isomorphism (equivalence transformation) from \( \cdot^{J,\mu} \) to \( \cdot^{J',\mu'} \), if \( \mu' = \mu + \delta \mu \), where \( \delta \mu \in \mathfrak{g}^G + \nu \mathfrak{g}_c^G[[\nu]] \) is given by \( (\delta \mu, \xi) = J'(\xi) - TJ(\xi) \). The inverse is given by

\[ T^{-1} F := h_{j_0-\mu_0} \left( \frac{\text{id}}{\text{id} - \nu \Delta_{\mu,*}} T^{-1} \right), \quad F \in h(\mathcal{P}(Q)^G)[[\nu]]. \]  

(3.24)
If $J'(\xi) = TJ(\xi)$, then $T$ is an isomorphism (equivalence transformation) from $\cdot^{J,\mu}$ to $\cdot^{'J,\mu}$.

ii.) Let $\cdot^{J,\mu}$ and $\cdot^{J,\mu'}$ denote the products on $h(\mathcal{P}(Q)^G)[[\nu]]$ obtained by reduction of $(\ast, J, \mu)$ and $(\ast, J, \mu')$. Assume $\Phi$ to be an automorphism of $\ast$ (starting with id) which preserves $\mathcal{P}(Q)[[\nu]]$ and commutes with every $\Phi^*_g$. Define $\Phi : h(\mathcal{P}(Q)^G)[[\nu]] \to h(\mathcal{P}(Q)^G)[[\nu]]$ by

$$AF := h_{j_0-\mu_0} \left( \frac{id}{id - \nu \Delta_{\mu',*}} A F \right), \quad F \in h(\mathcal{P}(Q)^G)[[\nu]]. \quad (3.25)$$

Then $\Phi$ is an isomorphism (equivalence transformation) from $\cdot^{J,\mu}$ to $\cdot^{J,\mu'}$, if $\mu' = \mu + \delta \mu$, where $\delta \mu \in g^* + \nu g^*_e[[\nu]]$ is given by $(\delta \mu, \xi) = J(\xi) - A J(\xi)$. The inverse is given by

$$A^{-1} F = h_{j_0-\mu_0} \left( \frac{id}{id - \nu \Delta_{\mu',*}} A^{-1} F \right), \quad F \in h(\mathcal{P}(Q)^G)[[\nu]]. \quad (3.26)$$

If $J(\xi) = AJ(\xi)$, then $\Phi$ is an automorphism of $\cdot^{J,\mu}$ (starting with id).

iii.) Let $D$ denote a $\mathbb{C}[[\nu]]$-linear derivation of $\ast$ which preserves $\mathcal{P}(Q)[[\nu]]$ and commutes with every $\Phi^*_g$. Define $D : h(\mathcal{P}(Q)^G)[[\nu]] \to h(\mathcal{P}(Q)^G)[[\nu]]$ by

$$DF := h_{j_0-\mu_0} \left( \frac{id}{id - \nu \Delta_{\mu',*}} D F \right), \quad F \in h(\mathcal{P}(Q)^G)[[\nu]]. \quad (3.27)$$

Then $D$ is a $\mathbb{C}[[\nu]]$-linear derivation of $\cdot^{J,\mu}$, if $D J(\xi) = 0$ for all $\xi \in g$.

iv.) Every isomorphism resp. equivalence transformation, automorphism (starting with id) or derivation constructed according to i.), ii.) or iii.) transfers to an isomorphism resp. equivalence transformation, automorphism (starting with id), or derivation on $\mathcal{P}(Q)[[\nu]]$ with the reduced star product as algebra structure. Moreover, the induced map extends in a unique way to an isomorphism resp. equivalence transformation, automorphism (starting with id), or derivation on $C^\infty(T^*\mathcal{Q})[[\nu]]$ with the reduced star product as algebra structure.

v.) All the above constructions can be localized in the sense that starting from a local isomorphism resp. equivalence transformation, automorphism (starting with id) or derivation on $C^\infty(T^*U)[[\nu]]$ resp. $\mathcal{P}(U)[[\nu]]$, where $U \subseteq Q$ is an open subset of $Q$, one obtains a local isomorphism resp. equivalence transformation, automorphisms (starting with id) or derivation on $\mathcal{P}(U)[[\nu]]$ resp. $C^\infty(T^*\mathcal{U})[[\nu]]$, where $\mathcal{U} = p(U) \subseteq \mathcal{Q}$.

**Proof:** For the proof of the first three statements it suffices to show i.) since ii.) is just a special case of i.) and the proof of iii.) is only a slight adaption of the argument which proves i.). Since $T$ commutes with $\Phi^*_g$ it also commutes with $\mathcal{L}_{\xi^*}$ for every $\xi \in g$. By definition of a quantum Hamiltonian this implies that $\frac{1}{\hbar} \mathcal{L}_{\xi^*}(J'(\xi) - TJ(\xi)) = 0$. But this entails that for every such isomorphism $T$ one has $J'(\xi) - TJ(\xi) \in \mathbb{C}[[\nu]]$, hence $(\delta \mu, \xi) = J'(\xi) - TJ(\xi)$ defines an element $\delta \mu \in g^* + \nu g^*_e[[\nu]]$. Using the $G$-equivariance of the quantum momentum maps and the $G$-invariance of $T$ it is evident that $\delta \mu$ is $G$-invariant. Moreover, it is straightforward to see that the condition $\mu' = \mu + \delta \mu$ implies that $T$ maps elements of $\mathcal{I}_{\mu,*}^p$ to elements of $\mathcal{I}_{\mu',*}^p$. Using the definition of the products $\cdot^{J,\mu}$ and $\cdot^{J',\mu'}$ on $h(\mathcal{P}(Q)^G)[[\nu]]$ an easy computation then shows that $T$ as defined in Eq. satisfies the stated properties and that its inverse is given by Eq. Statement iv.) is obvious by observing that the induced mappings on $\mathcal{P}(Q)[[\nu]]$ can be described as formal series of differential operators possibly composed with the pull-back of a diffeomorphism of $T^*\mathcal{Q}$ preserving
\[ \mathcal{P}(Q) \]. This latter situation occurs in case \( \mathcal{T} \) and \( \mathcal{A} \) do not start with id. Statement v.) is evident from the fact that we are only concerned with local operators. \( \square \)

Note that, evidently, the above construction of mappings on the reduced star product algebras is unfortunately not rich enough to yield all these mappings but only describes under which circumstances such mappings on the original star product on \( \mathcal{C}^\infty(T^*Q)[[\nu]] \) descend to such mappings on them.

Now we want to consider star products which possess certain additional properties and will show under which preconditions these properties transfer to the reduced star products. Let us first recall some notions of special star products (cf. e.g. [8, 18]):

i.) A star product \( \ast_s \) resp. \( \ast_{as} \) on \( T^*Q \) is said to be of standard ordered resp. anti-standard ordered type, if for all \( f \in \mathcal{C}^\infty(T^*Q)[[\nu]] \) and \( \chi \in \mathcal{C}^\infty(Q)[[\nu]] \)
\[
\pi^* \chi \ast_s f = \pi^* f \chi \quad \text{resp.} \quad f \ast_{as} \pi^* \chi = f \pi^* \chi.
\]

iii.) A star product is called of Vey type or a natural star product, if the bidifferential operator \( \ast_\nu \) of complex conjugation \( C \), where we set \( C = -\nu \mathcal{B}_\mathcal{Q} \), is Hermitian and the star product of Weyl type. In contrast to the above lemma, where the properties of the original star product transfer to the reduced star product without any further conditions, the property of being Hermitian and the \( \nu \)-parity are not stable with respect to reduction, in general, unless certain additional conditions on the quantum momentum map \( J \) and the momentum value \( \mu \) are satisfied.
Lemma 3.9  
i.) Let $\star$ be a Hermitian star product on $(T^*Q, \omega_{B_0})$. If the relation
\begin{equation}
C \left( J_+ (\xi) - \langle \mu_+, \xi \rangle - \frac{\nu}{2} \text{tr}(\text{ad}(\xi)) \right) = J_+ (\xi) - \langle \mu_+, \xi \rangle - \frac{\nu}{2} \text{tr}(\text{ad}(\xi)) \tag{3.29}
\end{equation}
is satisfied for all $\xi \in \mathfrak{g}$, then the reduced star product $\star^{J,\mu}$ is Hermitian as well.

ii.) Let $\star$ be a star product on $(T^*Q, \omega_{B_0})$ which has the $\nu$-parity property. If the equation
\begin{equation}
P \left( J_+ (\xi) - \langle \mu_+, \xi \rangle - \frac{\nu}{2} \text{tr}(\text{ad}(\xi)) \right) = J_+ (\xi) - \langle \mu_+, \xi \rangle - \frac{\nu}{2} \text{tr}(\text{ad}(\xi)) \tag{3.30}
\end{equation}
holds true for all $\xi \in \mathfrak{g}$, then the reduced star product $\star^{J,\mu}$ also has the $\nu$-parity property.

iii.) Let $\star$ be a star product of Weyl type on $(T^*Q, \omega_{B_0})$. If Eqs. (3.29) and (3.30) are satisfied, then the reduced star product $\star^{J,\mu}$ is of Weyl type, too.

Proof: To prove i.) we observe first that $CJ$ defines a $G$-equivariant quantum Hamiltonian since $\star$ is assumed to be Hermitian. Suppose that Eq. (3.29) holds true. Then it is easy to verify that $C(\nu \Delta_\mu, F) = \nu \Delta_\mu, CF$ for all $F \in \mathcal{P}(Q)^G[[\nu]]$. Since $h_{J_0} - \nu \partial_\nu$ and $l$ obviously commute with $C$, this implies that $\star^{J,\mu}$ is Hermitian. The proof of ii.) is completely analogous to the one of i.) replacing $C$ by $P$ and iii.) follows by combination of i.) and ii.).

Now we consider homogeneous star products on $(T^*Q, \omega_0)$, i.e. star products for which $H = \mathcal{L}_{\xi_0} + \nu \partial_\nu$ is a derivation. Observe that for every $k \in [0,1]$ both the star product $\star_k$ and the star product $\star^E_k$, $B = \nu B_1$ constructed in Section 2.1 are of this kind. Therefore, the following results directly apply to some of the examples we will discuss in more detail in Sections 4.1 and 6.1.

Lemma 3.10  
i.) Let $\star$ denote a homogeneous star product on $(T^*Q, \omega_0)$ and recall that the $G$-equivariant quantum momentum map $J$ has the form $J(\xi) = P(\xi_0) + \langle \mu_0, \xi \rangle + J_+ (\xi)$ with $\mu_0 \in \mathfrak{g}^{*G}$. Then the reduced star product $\star^{J,\mu}$ on $(T^*Q, \omega_0)$ turns out to be homogeneous as well, if $J$ and the momentum value $\mu$ satisfy
\begin{equation}
\mathcal{H} J(\xi) - J(\xi) = \nu \partial_\nu (\mu, \xi) - \langle \mu, \xi \rangle \quad \text{for all } \xi \in \mathfrak{g}. \tag{3.31}
\end{equation}

ii.) Under the preconditions of i.), there exists another $G$-equivariant quantum momentum map $J'$ of particular form $J'(\xi) = P(\xi_0) + \nu J_1 (\xi) = P(\xi_0) + \nu \pi^* j_1 (\xi)$ with $j_1 \in C^1(\mathfrak{g}, C^\infty(Q))$ and another momentum value $\mu' = \nu \mu_1$ such that $\star^{J',\mu'}$ coincides with $\star^{J,\mu}$.

Proof: First observe that Eq. (3.31) implies $\hat{\mu}_0 = \mu_0$, since $\mathcal{L}_{\xi_0} P(\eta_0) = P(\eta_0)$. Therefore $\star^{J,\mu}$ is a star product with respect to the canonical symplectic form on $T^*Q$. Using (3.31) it is easy to show that the mapping $\mathcal{H}$, which evidently commutes with every $\Phi^{\nu}_g$, preserves $I_{\mu,\star}^P$. This implies that the mapping $\mathcal{H} : h(\mathcal{P}(Q)^G)[[\nu]] \to h(\mathcal{P}(Q)^G)[[\nu]]$, which is defined by
\begin{equation}
\mathcal{H} F := h \left( \frac{id}{id - \nu \Delta_\mu, \star} \mathcal{H} F \right), \quad F \in h(\mathcal{P}(Q)^G)[[\nu]],
\end{equation}
is a derivation with respect to the product $\bullet^{J,\mu}$. Since according to Eq. (3.39) the map $r^i$, $1 \leq i \leq \text{dim} (G)$ is homogeneous of degree $-1$ with respect to $\xi_0$, an easy computation shows that $\mathcal{H}$ commutes with $\nu \Delta_\mu, \star$. Thus $\mathcal{H} F = \nu \partial_\nu F + h(\mathcal{L}_{\xi_0} F)$ holds true. From this observation it is evident that the composition $l^{-1} \circ \mathcal{H} \circ l$, which is a derivation of the star product $\star^{J,\mu}$, equals $\mathcal{H} = \nu \partial_\nu + \mathcal{L}_{\xi_0}$, where $\xi_0$ denotes the canonical Liouville vector field on $(T^*Q, \omega_0)$. This proves i.) For the proof of ii.) one has to analyse Eq. (3.31) by $J(\xi) \in \mathcal{P}(Q)[[\nu]]$ and since every eigenvalue of $\mathcal{L}_{\xi_0}$ has to be a non-negative integer, one observes that $J$ has the form $J(\xi) = P(\xi_0) + \langle \mu_0, \xi \rangle + \nu \pi^* j_1 (\xi) + \sum_{r=2}^\infty \nu^r \langle \mu_r, \xi \rangle$. Using $\mu_0 = \mu_0$ we then obtain the relation $J(\xi) - \langle \mu, \xi \rangle = P(\xi_0) + \nu \pi^* j_1 (\xi) - \nu (\mu_1, \xi)$. This implies that $\star^{J,\mu}$ equals $\star^{J',\mu'}$ which proves ii.).
Remark 3.11 Proposition ii.) of the preceding lemma shows in particular that two reduced star products \(*^{\mu,\nu}\) and \(*^{\mu',\nu'}\) coincide, if \(J(\xi) - \langle \mu, \xi \rangle = J'(\xi) - \langle \mu', \xi \rangle\). This follows from the fact that only the difference of the quantum momentum map and the momentum value enter the construction of the reduced star products. In view of this observation it is equivalent to either fix a quantum momentum map and vary the momentum values or vary the quantum momentum map and choose the momentum value to be zero. In the general considerations of the following sections we will nevertheless treat \(J\) and \(\mu\) as independent parameters of the construction, but in concrete examples it will turn out to be convenient to restrict the consideration to one fixed quantum momentum map and to vary the momentum values arbitrarily.

4 Invariant Star Products on \(T^*Q\) and Quantum Moment Maps

In this section we prepare the grounds for the phase space reduction of the star products \(*_\kappa\) and \(*^B\) defined in Section 2.1. To this end we will provide conditions on the data entering the construction of \(*_\kappa\) and \(*^B\) which guarantee that the obtained star products are \(G\)-invariant. Furthermore, we provide conditions which are necessary and sufficient for the existence of a \(G\)-equivariant quantum momentum map. Note that all results derived in this section hold for the lifted action of an arbitrary Lie group action on the base manifold \(Q\) and that the assumption made earlier for phase space reduction, namely that the action is proper and free, is not needed here. Only assuming that there exists a \(G\)-invariant torsion free connection on \(Q\) we will show in particular that every \(G\)-invariant star product on \((T^*Q, \omega_{B_0})\) is \(G\)-equivalent to some star product \(*^B\). This will actually turn out to be one of the key results for the computation of the characteristic class of the reduced star products in Section 5. In the course of these investigations, we also obtain a complete classification up to \(G\)-equivalence of star products on \((T^*Q, \omega_{B_0})\) which are invariant with respect to lifted group actions. This turns out to be a slight refinement of the classification results in [3] for our special geometric situation.

4.1 Invariance of \(*_\kappa\) and \(*^B\) and their Quantum Moment Maps

In order to derive necessary and sufficient conditions for the \(G\)-invariance of the star products \(*_\kappa\), we prove the following:

**Proposition 4.1** Let \(\phi\) denote a diffeomorphism of \(Q\), let \(\Phi = T^*(\phi^{-1})\) be the lift to the cotangent bundle and assume that the connection \(\nabla\) is invariant with respect to \(\phi\), i.e. that \(\phi^*\nabla_X Y = \nabla_{\phi^*X} \phi^*Y\) holds true for all \(X, Y \in \Gamma^\infty(TQ)\). Then there exists for every \(\kappa \in [0, 1]\) a uniquely determined formal series of differential operators \(S_{\kappa,\phi}\) on \(C^\infty(T^*Q)\) which starts with \(\text{id}\) and commutes with \(\mathcal{H} = L_{\xi_0} + \nu \partial_\nu\) such that

\[
\phi^* \rho_\kappa (f) (\phi^{-1})^* = \rho_\kappa (S_{\kappa,\phi} \Phi^* f) \quad \text{for all} \; f \in C^\infty(T^*Q)[[\nu]].
\]

In addition \(S_{\kappa,\phi}\) is explicitly given by

\[
S_{\kappa,\phi} = \exp \left( -F \left( \frac{\exp(\nu D) - \text{id}}{D} (\phi^* \alpha - \alpha) \right) \right).
\]

Moreover, \(S_0,\phi\) and \(S_1,\phi\) are automorphisms of \(*_0\) and \(*_1\), respectively. Furthermore, \(S_{\kappa,\phi}\) is an automorphism of \(*_\kappa\) for \(\kappa \neq 0, 1\) if and only if \(D(\phi^* \alpha - \alpha) = 0\).

**Proof:** A straightforward computation using the \(\phi\)-invariance of \(\nabla\) and that \(F\) satisfies Eq. (A.8) yields \(\phi^* \rho_\kappa (f) (\phi^{-1})^* = \rho_\kappa \left( N^\phi_{\kappa} \Phi^* N_\kappa f \right)\). Next observe that \(\Phi^* \Delta_0 f = \Delta_0 \Phi^* f\) by Eqs. (A.7). Using
the factorization property of $N_\kappa$ given in Eq. (2.9), this proves that the formal series of differential operators $S_{\kappa,\phi}$ given in (1.2) satisfies (4.1). The facts that $S_{\kappa,\phi}$ starts with id and that it commutes with $\mathcal{H}$ are obvious from its explicit form. For the proof of the uniqueness assume that $S'_{\kappa,\phi}$ is a second formal series of differential operators having the same properties as $S_{\kappa,\phi}$. Restricting to the space of functions polynomial in the momenta we obtain $g_\kappa (S'_{\kappa,\phi} F) = g_\kappa (S_{\kappa,\phi} F)$ for all $F \in \mathcal{P}(Q)[[\nu]]$. Now observe that the homogeneity of $S'_{\kappa,\phi}$ and $S_{\kappa,\phi}$ implies that they map elements $F \in \mathcal{P}(Q)[[\nu]]$ to elements of $\mathcal{P}(Q)[[\nu]]$. Since the restriction of $g_\kappa$ to $\mathcal{P}(Q)[[\nu]]$ is injective one thus concludes $S'_\kappa F = S\kappa F$. But this implies $S'_{\kappa,\phi} = S_{\kappa,\phi}$ since differential operators on $C^\infty (T^*Q)$ are completely determined by their values on $\mathcal{P}(Q)$, proving the uniqueness of $S_{\kappa,\phi}$. In case $\kappa = 0$, $S_{0,\phi}$ is an automorphism of $\star_0$ since it coincides with id. For $\kappa = 1$ and $A = \nu (\phi^* \alpha - \alpha)$ consider the operator $A_1$ defined by Eq. (2.12). Then $A_1$ coincides with $S_{1,\phi}$ and is an automorphism of $\star_1$ if and only if $d(\phi^* \alpha - \alpha) = 0$. Now recall that $d\alpha = -\text{tr} (R)$ and that $\phi^* R = R$ due to the $\phi$-invariance of $\nabla$. Then $\phi^* \alpha - \alpha$ is obviously closed, hence $S_{1,\phi}$ is an automorphism of $\star_1$. Finally, let us consider the case $\kappa \neq 0, 1$. Assuming that $D(\phi^* \alpha - \alpha) = 0$, the map $S_{\kappa,\phi} = \exp (-\kappa \nu F (\phi^* \alpha - \alpha))$ coincides with $A_\kappa$ for $A = \kappa \nu (\phi^* \alpha - \alpha)$, hence it is an automorphism of $\star_\kappa$ due to the closedness of $\phi^* \alpha - \alpha$. Conversely, let us assume that $S_{\kappa,\phi}$ is an automorphism of $\star_\kappa$. Then $S_{\kappa,\phi} (A_\kappa) = 1$ is again an automorphism of $\star_\kappa$, where the latter is taken for $A = \kappa \nu (\phi^* \alpha - \alpha)$. Now a straightforward expansion of this automorphism yields $S_{\kappa,\phi} (A_\kappa)^{-1} = \exp \left( \frac{\kappa (\kappa - 1)}{2} \nu^2 F (\phi^* \alpha - \alpha) + O(\nu^3) \right)$. But since every star product automorphism starting with id is of the form $\exp(\nu D)$, where $D$ is a derivation of the star product, and since the lowest order of a derivation is given by a symplectic vector field, one concludes that \( \frac{\kappa (\kappa - 1)}{2} \nu^2 F (\phi^* \alpha - \alpha) \) must be zero since otherwise it can never define a symplectic vector field. Together with the injectivity of $F$ and the precondition $\kappa \neq 0, 1$ we thus obtain $D(\phi^* \alpha - \alpha) = 0$ and the proposition is proved. □

Using the explicit formulas for the $\star_\kappa$-left- and $\star_\kappa$-right-multiplication with functions $\pi^* \chi$, $\chi \in C^\infty (Q)$, we also get the following.

**Lemma 4.2** Let $\phi$ denote a diffeomorphism of $Q$ and let $\star_\kappa$ be invariant with respect to $\Phi = T^* (\phi^{-1})$, i.e. $\Phi^* (f \star_\kappa f') = \Phi^* f \star_\kappa \Phi^* f'$ for all $f, f' \in C^\infty (T^*Q)[[\nu]]$. Then the connection $\nabla$ is invariant with respect to $\phi$.

**Proof:** Let us consider the case $\kappa = 0$ first. Using Eq. (2.11) and property (A.8) for $F$, a comparison of the terms of second order in the formal parameter within the equation $\Phi^* (f \star_0 \pi^* \chi) = \Phi^* f \star_0 \pi^* \phi^* \chi$ yields $F ((\phi^* D) d\chi') = (D (D \chi'))$ for all $\chi' \in C^\infty (Q)$. Since $F$ is injective, we thus conclude that $(\phi^* D) d\chi' = D d\chi'$. Evaluating this for local coordinate functions $x^i$ and using $\phi^* D = D - dx^i \wedge dx^j \wedge i_S (S_\phi (\partial_{x^i}, \partial_{x^j}))$, we thus obtain that the tensor field $S_\phi$ from Lemma A.1 has to vanish. Hence $\nabla$ has to be invariant with respect to $\phi$. For $\kappa \neq 0$ we consider the second order terms of $\Phi^* (\pi^* \chi \star_\kappa f) = \pi^* \phi^* \chi \star_\kappa \Phi^* f$. Then Eq. (2.10) yields $F (\kappa^2 (\phi^* D) d\chi') = (\kappa^2 D d\chi')$ for all $\chi' \in C^\infty (Q)$. Since $\kappa \neq 0$, we may conclude as above that this implies $\phi$-invariance of $\nabla$. □

Combining the results of Proposition 4.1 and Lemma 4.2 we get:

**Theorem 4.3**

i.) The star products $\star_0$ and $\star_1$ are $G$-invariant if and only if the connection $\nabla$ is $G$-invariant.

ii.) For $\kappa \neq 0, 1$ the star product $\star_\kappa$ is $G$-invariant if and only if the connection $\nabla$ is $G$-invariant and

$$D(\phi^*_g \alpha - \alpha) = 0 \quad \text{for all } g \in G. \quad (4.3)$$

iii.) In either case we have

$$\phi^*_g g_\kappa (f) \phi^*_g = g_\kappa (S_{\kappa,\phi^*_g} \Phi^*_g f), \quad (4.4)$$
where according to Eq. (4.2) the automorphism \( S_{\kappa,\phi_g} \) of \(*_\kappa\) is given by

\[
S_{\kappa,\phi_g} = \exp \left( -F \left( \exp(\kappa \nu D) - \text{id} \right) \left( \phi_g^* \alpha - \alpha \right) \right). \tag{4.5}
\]

More explicitly, this means \( S_{0,\phi_g} = \text{id} \), \( S_{\kappa,\phi_g} = \exp \left( -F \left( \kappa \nu \left( \phi_g^* \alpha - \alpha \right) \right) \right) \) for \( \kappa \neq 0,1 \), and

\[
S_{1,\phi_g} = \exp \left( -F \left( \exp(\nu D) - \text{id} \right) \left( \phi_g^* \alpha - \alpha \right) \right).
\]

**Proof:** Let us assume that \( \nabla \) is \( G \)-invariant. Then apply (4.1) with \( \phi = \phi_g \) and use the representation property of \( \varrho_\kappa \) to check

\[
\varrho_\kappa \left( S_{\kappa,\phi_g} \Phi^*_{\phi} f \ast f' \right) = \varrho_\kappa \left( S_{\kappa,\phi_g} \Phi^*_{\phi} f \ast S_{\kappa,\phi_g} \Phi^*_{\phi} f' \right), \quad f, f' \in C^\infty (T^*Q)[[\nu]].
\]

Restricting to \( F, F' \in \mathcal{P}(Q)[[\nu]] \), the injectivity of \( \varrho_\kappa \) and the fact that \( S_{\kappa,\phi_g} \Phi^*_{\phi} \) preserves \( \mathcal{P}(Q)[[\nu]] \)

implies that

\[
S_{\kappa,\phi_g} \Phi^*_{\phi} (F \ast F') = S_{\kappa,\phi_g} \Phi^*_{\phi} F \ast S_{\kappa,\phi_g} \Phi^*_{\phi} F' \quad \text{for all } F, F' \in \mathcal{P}(Q)[[\nu]].
\]

Since \(*_\kappa\) is described by bidifferential operators and since these are completely determined by their values on \( \mathcal{P}(Q)[[\nu]] \), this yields that \( S_{\kappa,\phi_g} \Phi^*_{\phi} \) is an automorphism of \(*_\kappa\). For \( \kappa = 0 \) resp. \( \kappa = 1 \) we already know by Proposition 4.1 and the \( G \)-invariance of \( \nabla \) that \( S_{0,\phi_g} \) resp. \( S_{1,\phi_g} \) is an automorphism of \(*_\kappa\), and so is \( \Phi^*_{\phi} \). In case \( \kappa \neq 0,1 \) the additional condition \( D(\phi_g^* \alpha - \alpha) = 0 \) is equivalent to \( S_{\kappa,\phi_g} \) being an automorphism of \(*_\kappa\), but this is equivalent to \( \Phi^*_{\phi} \) being an automorphism of \(*_\kappa\). Conversely, let us assume that \(*_\kappa\) is \( G \)-invariant. Then Lemma 4.2 implies that \( \nabla \) is also \( G \)-invariant. But now we can use the above observation to conclude that for \( \kappa \neq 0,1 \) the additional equation \( D(\phi_g^* \alpha - \alpha) = 0 \) must hold. Together this proves claims i.) and ii.) of the theorem. Assertion iii.) is obvious by the proof of i.), ii.), and Eq. (4.2). \( \square \)

Finally, we consider the star products \(*^B_\kappa\). Before we can state the generalization of Theorem 4.3 we provide some rather trivial but nevertheless crucial results:

**Lemma 4.4**

i.) For every \( \kappa \in [0,1] \) and formal series \( B \in Z^2_{3\mathbb{R}}(Q)[[\nu]] \) of closed two-forms on \( Q \) with real \( B_0 \), \(*^B_\kappa\)-left-multiplication by \( \pi^* \chi \), \( \chi \in C^\infty (Q)[[\nu]] \)

coincides with \(*_\kappa\)-left-multiplication by \( \pi^* \chi \). Analogously, \(*^B_\kappa\)-right-multiplication by \( \pi^* \chi \) coincides with \(*_\kappa\)-right-multiplication by \( \pi^* \chi \).

ii.) For every \( \kappa \in [0,1] \) a mapping \( A_\kappa \) of the form given in Eq. (2.7.13) is an automorphism of \(*_\kappa\), if and only if it is an automorphism of \(*^B_\kappa\).

**Proof:** By definition of \(*^B_\kappa\) we have \( \pi^* \chi \ast_\kappa f |_{T^*O_j} = A_\kappa^j \left( \left( (A_\kappa^j)^{-1} \pi^* \chi |_{T^*O_j} \ast_\kappa \left( (A_\kappa^j)^{-1} f |_{T^*O_j} \right) \right) \right) \).

By the explicit form of \( A_\kappa^j \) it is obvious that \( (A_\kappa^j)^{-1} \pi^* \chi = \pi^* \chi \) holds true and that \( (A_\kappa^j)^{-1} \) commutes with \( F(\beta) \) for every \( \beta \in \Gamma^\infty (\nabla T^*Q)[[\nu]] \). Using these observations together with the expression for \( \pi^* \chi \ast_\kappa f \) given in Eq. (2.10), one obtains \( \pi^* \chi \ast_\kappa f = \pi^* \chi \ast_\kappa f \). The proof for \(*^B_\kappa\)-right-multiplication is completely analogous. Assertion ii.) is obvious from the fact that mappings as given in Eq. (2.12) commute with the local isomorphisms \( A_\kappa^j \) from \(*_\kappa\) to \(*^B_\kappa\). \( \square \)

After these preparations we can state one of the main results of this section:

**Theorem 4.5**

i.) The star products \(*^B_0\) and \(*^B_1\) are \( G \)-invariant, if and only if \( \nabla \) is \( G \)-invariant and \( \phi_g^* B = B \) for all \( g \in G \).

ii.) For \( \kappa \neq 0,1 \) the star product \(*^B_\kappa\) is \( G \)-invariant, if and only if \( \nabla \) and \( B \) are \( G \)-invariant and \( D(\phi_g^* \alpha - \alpha) = 0 \) for all \( g \in G \).
PROOF: First assume that ∇ and B are G-invariant. Additionally, assume for κ ≠ 0,1 that D(φ^κ_0α − α) = 0 for all g ∈ G. Then the star products *κ are all G-invariant by Theorem 4.3. But this implies that

\[ A_k^i \Phi^*_g(A_k^i)^{-1} \]

is a local automorphism of \((C^\infty(T^*O_j)[[\nu]], \star^B_\kappa)\). Obviously, the equality \(\phi_g^*B = B\) entails the relation \(d(A^j - \phi_g^*A^j) = 0\). By Lemma 4.4 ii.) this implies that

\[ t^*_{-(A_0^i - \phi_g^*A_0^i)} \exp \left( -F \left( \frac{\exp(\kappa \nu D) - \exp((\kappa - 1)\nu D)}{\nu D}(A^j - \phi_g^*A^j) - (A_0^j - \phi_g^*A_0^j) \right) \right) \Phi_g^* \]

is a local automorphism of \((C^\infty(T^*O_j)[[\nu]], \star^B_\kappa)\). But then \(\Phi_g^*\) is also an automorphism of \(*^B_\kappa\), proving one direction of i.) and ii.). For the converse statement assume that \(*^B_\kappa\) is G-invariant. Then Lemma 4.4 i.) and Lemma 4.4 ii.) imply that ∇ is G-invariant. For the cases κ = 0 and κ = 1 this implies that \(*_0\) and \(*_1\) are invariant. Together with the above considerations this entails by Lemma 4.4 ii.) that

\[ t^*_{-(A_0^i - \phi_g^*A_0^i)} \exp \left( -F \left( \frac{\mathrm{id} - \exp(-\nu D)}{\nu D}(A^j - \phi_g^*A^j) - (A_0^j - \phi_g^*A_0^j) \right) \right) \]

and

\[ t^*_{-(A_0^i - \phi_g^*A_0^i)} \exp \left( -F \left( \frac{\exp(\nu D) - \mathrm{id}}{\nu D}(A^j - \phi_g^*A^j) - (A_0^j - \phi_g^*A_0^j) \right) \right) \]

define local automorphisms of \((C^\infty(T^*O_j)[[\nu]], \star_0)\) and \((C^\infty(T^*O_j)[[\nu]], \star_1)\), respectively. But this implies that \(A^j - \phi_g^*A^j\) is closed, hence \(B\) is G-invariant. For the case κ ≠ 0,1 we need a more detailed argument. By invariance of the connection the mapping \(S_{\kappa,\phi_g^*\Phi_g^*}(A^i)^{-1}\) is a local automorphism of \(*_\kappa^B\), hence \(A_k^i S_{\kappa,\phi_g^*\Phi_g^*}(A_k^i)^{-1}\) is a local automorphism of \(*_\kappa^B\). Since by assumption \(*_\kappa^B\) is G-invariant, this yields that

\[ t^*_{-(A_0^i - \phi_g^*A_0^i)} \exp \left( -F \left( \frac{\exp(\kappa \nu D) - \exp((\kappa - 1)\nu D)}{\nu D}(A^j - \phi_g^*A^j) - (A_0^j - \phi_g^*A_0^j) \right) \right) \times \exp \left( -F \left( \frac{\exp(\kappa \nu D) - \mathrm{id}}{\nu D}(A^j - \phi_g^*A^j) - (A_0^j - \phi_g^*A_0^j) \right) \right) \]

is a local automorphism of \(*_\kappa^B\). Considering the order zero part in the formal parameter this implies that \(t^*_{-(A_0^i - \phi_g^*A_0^i)}\) has to be a local symplectomorphism with respect to \(\omega_{B_0}\). Thus \(A_0^j - \phi_g^*A_0^j\) is closed. Factorizing the local automorphism corresponding to \(A_0^j - \phi_g^*A_0^j\) we now obtain that

\[ \exp \left( -F \left( \frac{\exp(\kappa \nu D) - \exp((\kappa - 1)\nu D)}{\nu D}(A^j - A_0^j - \phi_g^*(A^j - A_0^j)) + \frac{\exp(\kappa \nu D) - \mathrm{id}}{\nu D}(A^j - A_0^j - \phi_g^*(A^j - A_0^j)) \right) \right) \]

is a local automorphism of \(*_\kappa^B\). In order one of \(\nu\) this means that \(-F \left( A^j_1 - \phi_g^*(A^j_1 + \kappa(\phi_g^*\alpha - \alpha) \right)\) defines a symplectic vector field with respect to \(\omega_{B_0}\). Therefore \(A_0^j - \phi_g^*A_0^j + \kappa(\phi_g^*\alpha - \alpha)\) is closed. But since ∇ is invariant, \(\phi_g^*\alpha - \alpha\) is closed and so is \(A^j_1 - \phi_g^*A^j_1\). Factorizing again the local
automorphism corresponding to the closed one-form \( \nu(A_1^j - \phi_g^*A_1^j + \kappa(\phi_g^*\alpha - \alpha)) \), we end up with another local automorphism of \( \ast^B \). To lowest order in the formal parameter the exponent of this automorphism is given by \( F \left(-A_2^j + \phi_g^*A_2^j + \frac{\kappa(\kappa - 1)}{2}\mathcal{D}(\phi_g^*\alpha - \alpha)\right) \). This term again has to define a symplectic vector field. For \( \kappa \neq 0,1 \) this is only possible, if \( \mathcal{D}(\phi_g^*\alpha - \alpha) = 0 \), hence \( \mathcal{S}_{\kappa,\phi_g} \) is an automorphism of \( \ast \) and \( \ast \) is \( G \)-invariant. This means that the mapping \( \mathcal{A}_\kappa^B \Phi_g^*(A_1^j)^{-1}\Phi_{g^{-1}} \) from Eq. (4.6) is a local automorphism of \( \ast^B \). Now this local automorphism is an operator of form \( \mathcal{A}_\kappa \) as given in Eq. (2.12) which in turn is an automorphism of \( \ast^B \) (and hence of \( \ast^B \) by Lemma 4.3(ii.)), if and only if \( A^j - \phi_g^*A^j \) is closed. But this implies finally that \( \phi_g^*B - B = d(\phi_g^*A^j - A^j) = 0 \), i.e. that \( B \) is \( G \)-invariant.

After having investigated the invariance of the star products \( \ast \) and \( \ast^B \), we will next consider \( G \)-equivariant quantum momentum maps for these star products. More precisely, we now state one of the key results on phase space reduction of the star products \( \ast \). To this end we henceforth assume \( \ast \) to be \( G \)-invariant. Moreover, we assume the connection \( \nabla \) to be \( G \)-invariant and that for \( \kappa \neq 0,1 \) the relation \( \mathcal{D}(\phi_g^*\alpha - \alpha) = 0 \) holds true for all \( g \in G \).

**Proposition 4.6** For every \( \kappa \in [0,1] \), the \( G \)-invariant star product \( \ast \) on \( (T^*Q,\omega_0) \) is strongly \( G \)-invariant, i.e. the map \( J \in C^1(g,\mathcal{C}^\infty(T^*Q)) \) with \( J(\xi) := J_0(\xi) = P(\xi) \) is a \( G \)-equivariant quantum momentum map for the lifted Lie group action. In case \( J' \), where \( J'_0 \) is assumed to be real, is another \( G \)-equivariant quantum momentum map for the same action, then \( J' \) is given by \( J'(\xi) = J_0(\xi) + (\tilde{\mu},\xi) \) with \( \tilde{\mu} \in \mathfrak{g}^G + \nu\mathfrak{g}^G(\nu) \).

**Proof:** Put \( g = \exp(\xi) \) in Eq. (14.1), differentiate the resulting equation with respect to \( t \) and evaluate at \( t = 0 \). Then one obtains \( [\mathcal{L}_{\xi_0},\varrho_h(f)] = \varrho_h\left(-F\left(\frac{\exp(\nu\mathcal{D})-\text{id}}{\mathcal{D}}\mathcal{L}_{\xi_0}\alpha\right)f + \mathcal{L}_{\xi_0}\ast f\right) \). From the definition of \( \varrho_h \) one concludes immediately that \( \varrho_h(P(\xi)) = -\nu\mathcal{L}_{\xi_0} - \kappa\alpha \varrho_h(\pi^*(\text{div}(\xi) + \alpha(\xi))) \), where \( \text{div}(\xi) = \text{tr}(Y \mapsto \nabla Y \xi_0) \) denotes the covariant divergence of the vector field \( \xi_0 \). Using the representation property of \( \varrho_h \) this implies

\[
\varrho_h\left(-\frac{1}{\nu}\text{ad}_{\ast}(P(\xi) + \kappa\nu\pi^*(\text{div}(\xi) + \alpha(\xi)))f + F\left(\frac{\exp(\nu\mathcal{D})-\text{id}}{\mathcal{D}}\mathcal{L}_{\xi_0}\alpha\right)f\right) = \varrho_h\left(\mathcal{L}_{\xi_0}\ast f\right).
\]

At this point we need a little technical result.

**Sublemma 4.7** If the star product \( \ast \) is \( G \)-invariant, then the following equality holds true:

\[
\kappa \text{ad}_{\ast}(\pi^*(\text{div}(\xi) + \alpha(\xi)))f = F\left(\frac{\exp(\nu\mathcal{D})-\text{id}}{\mathcal{D}}\mathcal{L}_{\xi_0}\alpha\right)f \quad \text{for all } f \in \mathcal{C}^\infty(T^*Q).
\] (4.7)

**Proof:** For \( \kappa = 0 \) there is nothing to show, since both sides coincide with 0. For \( \kappa = 1 \) we obtain from (2.10) and (2.11) that \( \text{ad}_{\ast}(\pi^*(\text{div}(\xi) + \alpha(\xi))) = F\left(\frac{\exp(\nu\mathcal{D})-\text{id}}{\mathcal{D}}\text{d}(\text{div}(\xi) + \alpha(\xi))\right) \). Now observe that \( (\mathcal{L}_{\xi_0}\nabla)Y = \mathcal{L}_{\xi_0}\nabla_0Y - \nabla\mathcal{L}_{\xi_0}Y - \nabla Y \mathcal{L}_{\xi_0}Z = 0 \) by invariance of \( \nabla \). A straightforward computation then shows \( -\text{tr}(R)(\xi_0,Y) = d\alpha(\xi_0,Y) = d(\text{div}(\xi_0))(Y) \), but from this equation and Cartan's formula Eq. (1.7) is immediate in case \( \kappa = 1 \). For \( \kappa \neq 0,1 \) the additional relation \( D(\phi_g^*\alpha - \alpha) = 0 \) obviously implies \( D\mathcal{L}_{\xi_0}\alpha = 0 \), hence \( F\left(\frac{\exp(\nu\mathcal{D})-\text{id}}{\mathcal{D}}\mathcal{L}_{\xi_0}\alpha\right) = \kappa
\]
By the sublemma we now obtain \( g_\kappa \left( -\frac{1}{\nu} \text{ad}_{g_\kappa} (P(\xi)) f \right) = g_\kappa \left( \mathcal{L}_{\xi T^* Q} f \right) \). Recall that \( g_\kappa \) restricted to \( \mathcal{P}(Q)[[\nu]] \) is injective and observe that \( \mathcal{L}_{\xi T^* Q} \) and \( \text{ad}_{g_\kappa} (P(\xi)) \) preserve \( \mathcal{P}(Q)[[\nu]] \). Then we obtain

\[
-\frac{1}{\nu} \text{ad}_{g_\kappa} (P(\xi)) F = \mathcal{L}_{\xi T^* Q} F \quad \text{for all } F \in \mathcal{P}(Q)[[\nu]].
\]

Since \( \star_\kappa \) is described by bidifferential operators, this implies \( -\mathcal{L}_{\xi T^* Q} = \frac{1}{\nu} \text{ad}_{g_\kappa} (P(\xi)) \), hence \( J(\xi) = J_0(\xi) = \mathcal{L}(\xi) \) is a quantum Hamiltonian which is known to be \( G \)-equivariant. The claim about the ambiguity of the quantum momentum map is a general result which holds over arbitrary symplectic manifolds (cf. [23] Prop. 6.3).

For the study of reduction of the invariant star products \( \star_\kappa \) the \( G \)-equivariant quantum momentum map given by \( J(\xi) = \mathcal{L}(\xi) \) appears to be the preferred one, since it coincides with the (canonical) \( G \)-equivariant classical momentum map. In the sequel we therefore will mostly work with this momentum map in order to compute the reduced products of \( \star_\kappa \).

Now we consider the star products \( \star_\kappa^B \) on \((T^* Q, \omega_{B_0})\) and will derive necessary and sufficient conditions for the existence of \( G \)-equivariant quantum momentum maps for \( G \)-invariant star products of form \( \star_\kappa^B \). In view of Theorem [15] we assume additionally to the conditions which guarantee \( \star_\kappa \) to be \( G \)-invariant that \( B \) is an element of \( Z_1^G(\mathcal{Q}) G[[\nu]] \).

**Proposition 4.8** Suppose that the star product \( \star_\kappa^B \) on \((T^* Q, \omega_{B_0})\) is \( G \)-invariant. Then there is a \( G \)-equivariant quantum momentum map for the \( G \)-action under consideration, if and only if there is an element \( j \in C^1(\mathfrak{g}, C^\infty(\mathcal{Q}))[\nu] \) with real-valued \( j_0 \) such that

\[
d j(\xi) = i_{\xi_Q} B \quad \text{and} \quad \phi_g^* j(\xi) = j(\text{Ad}(g^{-1}) \xi) \quad \text{for all } \xi \in \mathfrak{g}, g \in G. \tag{4.8}
\]

In this case we particularly have

\[
j([\xi, \eta]) = B(\xi_Q, \eta_Q) \quad \text{for all } \xi, \eta \in \mathfrak{g}. \tag{4.9}
\]

Moreover, the map \( J \in C^1(\mathfrak{g}, C^\infty(T^* Q))[\nu] \) given by \( J(\xi) := \mathcal{P}(\xi_Q) + \pi^* j(\xi) \) defines a \( G \)-equivariant quantum momentum map, which is unique up to elements of \( \mathfrak{g}^* G + \nu \mathfrak{g}^* G[[\nu]] \).

**PROOF:** Consider the following equation over \( T^* O_j \):

\[
\Phi_{g_0 A_j} \Phi_{g_0^{-1}} = t^* \phi_{g_0 A_j} \exp \left( -\frac{\nu}{D} \left( \exp(\kappa \nu D) - \exp((\kappa - 1) \nu D) \phi_{g_0 A_j} - \phi_{g_0 A_j} \right) \right). \]

By differentiation at the neutral element of \( G \) we obtain

\[
\mathcal{L}_{\xi T^* Q} = A_j \mathcal{L}_{\xi T^* Q} (A_j^{-1}) - F \left( \frac{\exp(\kappa \nu D) - \exp((\kappa - 1) \nu D)}{\nu D} \mathcal{L}_{\xi Q} A_j \right).
\]

Now the invariance of \( \star_\kappa^B \) implies that \( \star_\kappa \) is invariant as well, hence we conclude from Proposition [14] that \( \mathcal{L}_{\xi T^* Q} = -\frac{1}{\nu} \text{ad}_{g_\kappa} (P(\xi_Q)) \). By a direct computation using that \( A_j \mathcal{L}_{\xi T^* Q} (A_j^{-1}) = -\frac{1}{\nu} \text{ad}_{g_\kappa} (P(\xi_Q) - \pi^* (\mathcal{A}_j(\xi_Q))) \). On the other hand \( \mathcal{L}_{\xi Q} A_j = i_{\xi_Q} B + d(\mathcal{A}_j(\xi_Q)) \). By the explicit formula for \( \text{ad}_{g_\kappa} (\pi^* (\mathcal{A}_j(\xi_Q))) \) we thus obtain

\[
\mathcal{L}_{\xi T^* Q} = -\frac{1}{\nu} \left( \text{ad}_{g_\kappa} (P(\xi_Q)) + F \left( \frac{\exp(\kappa \nu D) - \exp((\kappa - 1) \nu D)}{D} i_{\xi_Q} B \right) \right).
\]
From this equation and from the explicit form of the inner derivations \( \text{ad}_g^{\kappa}(\pi^*\chi), \chi \in \mathcal{C}^\infty(Q)[[\nu]] \) it is clear that there is a quantum Hamiltonian for the considered action, if and only if there is an element \( j \in C^1(g,\mathcal{C}^\infty(Q))[[\nu]] \) such that \( dj(\xi) = i_{\xi Q} B \) for all \( \xi \in g \). Observe that the condition necessary for the solvability of this equation is satisfied, since by invariance of \( \star^B_{\kappa} \) we have that \( di_{\xi Q} B = L_{\xi Q} B = 0 \). Evidently, \( J \) with \( J(\xi) = P(\xi_Q) + \pi^* j(\xi) \) is additionally \( G \)-equivariant, if and only if the second condition in Eq. \[\text{(1.1)}\] is satisfied. Finally, differentiating this equation in \( \nu \) using the equality \( i_{\xi Q} B = dj(\xi) \), it is straightforward to check that Eq. \[\text{(1.9)}\] holds true. The claim about the ambiguity of \( J \) is well known, hence the proposition is proved. \( \square \)

**Remark 4.9** It is immediate to show that \( \star^B_{\kappa} \) admits a quantum Hamiltonian \( J \) which additionally satisfies \( \frac{1}{\nu} \text{ad}_g^{\kappa}(J(\xi)).J(\eta) = J([\xi,\eta]) \), if and only if there is an element \( j \in C^1(g,\mathcal{C}^\infty(Q))[[\nu]] \) such that \( dj(\xi) = i_{\xi Q} B \) and \( B(\xi_Q,\eta_Q) = j([\xi,\eta]) \). Moreover, these conditions determine \( j \) up to elements of \( \mathfrak{g}^* + \nu\mathfrak{g}^*[[\nu]] \) which are invariant with respect to the coadjoint action of \( g \), i.e. which vanish on \([\mathfrak{g},\mathfrak{g}]\).

**Corollary 4.10** Suppose that the star product \( \star^B_{\kappa} \) on \((T^*Q,\omega_{B_0})\) is \( G \)-invariant and that \( J_0 \) with \( J_0(\xi) = P(\xi_Q) + \pi^* j_0(\xi) \) is a \( G \)-equivariant classical momentum map. Then \( \star^B_{\kappa} \) is strongly \( G \)-invariant, if and only if \( B_+ \in \nu Z^2_{\text{ad}}(Q)^G[[\nu]] \), where \( B = B_0 + B_+ \), is horizontal, i.e. if and only if

\[
{i_{\xi Q} B_+ = 0 \quad \text{for all } \xi \in g.} \tag{4.10}
\]

### 4.2 General Invariant Star Products on \( T^*Q \): Relations to \( \star^B_{0} \) and Classification

In this section we consider star products \( \star \) on \((T^*Q,\omega_{B_0})\) which are \( G \)-invariant with respect to a lifted group action. Under the assumption that there is a \( G \)-invariant torsion free connection \( \nabla \) on \( Q \) we will in particular construct for every such \( \star \) a \( G \)-equivalent star product of form \( \star^B_{0} \). Incidentally, our results show that there is a \( G \)-equivariant quantum momentum map for an arbitrary \( G \)-invariant star product in the above sense if and only if there is a \( G \)-equivariant quantum momentum map for a certain star product \( \star^B_{0} \). But for these star products we have already derived criteria for the existence of \( G \)-equivariant quantum momentum maps. Thus we obtain necessary and sufficient conditions for the existence of \( G \)-equivariant quantum momentum maps for an arbitrary \( G \)-invariant star product. Finally, we use the \( G \)-equivalence between a \( G \)-invariant \( \star \) and an appropriate star product \( \star^B_{0} \) to give a classification of star products on \((T^*Q,\omega_{B_0})\) up to \( G \)-equivalence. Actually, our result is a slight refinement of the general classification results of [3] in the particular case of a cotangent bundle with lifted \( G \)-action.

First we need a few results which allow for a comparison of two different star products \( \star^B_{0} \) and \( \star^B_{0} \) in case \( B^{(k)} = \sum_{l=0}^{k} \nu^l B_l \) and \( B^{(k+1)} = \sum_{l=0}^{k+1} \nu^l B_l \). Note that the following lemma holds for arbitrary ordering parameters \( \kappa \in [0,1] \) but since we only need it for \( \kappa = 0 \) we restrict the proof to this particular case. The proof for general \( \kappa \) is an immediate adaption.

**Lemma 4.11** For \( k \in \mathbb{N} \) let \( B^{(k+1)} = \sum_{l=0}^{k+1} \nu^l B_l \) and \( B^{(k)} = \sum_{l=0}^{k} \nu^l B_l \) be series of closed two-forms on \( Q \). Then the describing bidifferential operators \( C^{B_{0}^{(k+1)}}_{r} \) and \( C^{B_{0}^{(k)}}_{r} \) of the corresponding star products \( \star_{0}^{B_{0}^{(k+1)}} \) and \( \star_{0}^{B_{0}^{(k)}} \) on \((T^*Q,\omega_{B_0})\) satisfy

\[
C^{B_{0}^{(k+1)}}_{r} = C^{B_{0}^{(k)}}_{r} \quad \text{for } r = 0, \ldots, k + 1 \quad \text{and} \quad C^{B_{0}^{(k+1)}}_{k+2} (f,f') - C^{B_{0}^{(k+1)}}_{k+2} (f',f) = C^{B_{0}^{(k)}}_{k+2} (f,f') - C^{B_{0}^{(k)}}_{k+2} (f',f) - (\pi^* B_{k+1}) (X^{B_0}_{f}, X^{B_0}_{f'}). \tag{4.12}
\]
where \( X_f^{B_0} \) denotes the Hamiltonian vector field of \( f \in \mathcal{C}^\infty(T^*Q) \) with respect to the symplectic form \( \omega_{B_0} \).

**Proof:** Let \( O \) be an element of a good open cover of \( Q \). Over \( O \) consider a local potential \( A^{(k+1)} = A^{(k)} + \nu^{k+1} A_{k+1} \) of \( B^{(k+1)} \), where \( A^{(k)} \) is a local potential of \( B^{(k)} \) and \( A_{k+1} \) is a local potential of \( B_{k+1} \). From the very definition of the star products corresponding to \( B^{(k+1)} \) and \( B^{(k)} \) one obtains that \( S_{k+1} := \exp \left( - F \left( \frac{\partial}{\partial y} \nu^{k+1} A_{k+1} \right) \right) \) defines a local equivalence from \((\mathcal{C}^\infty(T^*O)[\nu], \star_0^{B^{(k+1)}}) \) to \((\mathcal{C}^\infty(T^*O)[\nu], \star_0^{B^{(k)}}) \). Expanding the products \( \star_0^{B^{(k+1)}} \) and expanding the products \( \star_0^{B^{(k)}} \) and \( \star_0^{B^{(k+1)}} \) one obtains Eqs. (111) and (112) by an immediate computation.

The following results (which are essentially due to Lichnerowicz [21] Lemma 1 and 2 and Bertelson et al. [3] Prop. 2.1)] will turn out to be crucial for our further investigations.

**Lemma 4.12**

i.) Suppose that there is a \( G \)-invariant torsion free connection \( \nabla \) on \( Q \). Then every \( G \)-invariant differential \( \mathcal{C}^\infty(T^*Q) \)-Hochschild p-coboundary \( C \) \((p \geq 1) \) which vanishes on constants is the coboundary of a \( G \)-invariant differential \( p-1 \)-cochain \( c \) vanishing on constants. In case \( C(F_1, \ldots, F_p) \in \mathcal{P}(Q) \) for all \( F_1, \ldots, F_p \in \mathcal{P}(Q) \) one can additionally achieve that \( c(F_1, \ldots, F_{p-1}) \in \mathcal{P}(Q) \) for all \( F_1, \ldots, F_{p-1} \in \mathcal{P}(Q) \).

ii.) For every closed \( G \)-invariant p-form \( \Omega \) \((p \geq 1) \) on \( T^*Q \) there exists a \( G \)-invariant closed p-form \( \beta \) on \( Q \) and a \( G \)-invariant \( p-1 \)-form \( \Xi \) on \( T^*Q \) such that \( i^* \Xi = 0 \) and

\[
\Omega = d\Xi + \pi^* \beta. \tag{4.13}
\]

If \( \Omega(X_{F_1}^{B_0}, \ldots, X_{F_p}^{B_0}) \in \mathcal{P}(Q) \) for all \( F_1, \ldots, F_p \in \mathcal{P}(Q) \), then \( \Xi \) can be chosen such that \( \Xi(X_{F_1}^{B_0}, \ldots, X_{F_p}^{B_0}) \in \mathcal{P}(Q) \) for all \( F_1, \ldots, F_{p-1} \in \mathcal{P}(Q) \).

**Proof:** For the proof of i.) recall from [3] Def. 4 that every torsion free connection \( \nabla \) on \( Q \) defines a torsion free connection \( \nabla^{T^*Q} \) on \( T^*Q \) which is \( G \)-invariant if the original connection \( \nabla \) is invariant. Moreover, having chosen a torsion free connection \( \nabla^M \) on a manifold \( M \) it is well known that every p-cochain \( C \) on \( \mathcal{C}^\infty(M) \) can be uniquely written as \( C(f_1, \ldots, f_p) = C^{I_1, \ldots, I_p} \left( \left( \partial y_{I_1} \right) f_1 \right) \ldots \left( \left( \partial y_{I_p} \right) f_p \right) \), where \( I_1, \ldots, I_p \) denote multi-indices and the \( C^{I_1, \ldots, I_p} \) are components of tensor fields in \( \mathcal{C}^\infty(V \left| I_1 \right| TM \otimes \ldots \otimes V \left| I_p \right| TM) \) with respect to local coordinates \( y^1, \ldots, y^m \) of \( M \). In case \( C \) is a coboundary one can explicitly build a \( p-1 \)-cochain \( c \) such that \( \delta c = C \) \((\delta \text{Hochschild differential}) \), where the tensor fields defining \( c \) are given as combinations of those of \( C \). But this implies that in case \( C \) and the connection are invariant \( c \) is also invariant (cf. [3] Remark 2.1) proving the first part of i.). For the proof of the second part of i.) one first has to observe that the covariant derivative with respect to the above connection preserves \( \mathcal{P}(Q) \) since the Christoffel symbols of \( \nabla^{T^*Q} \) in a local bundle chart are polynomials in the momenta. Together with the assumption that \( C \) preserves \( \mathcal{P}(Q) \), which implies that the components of the corresponding tensor fields in the momenta, this entails that \( c \) also preserves \( \mathcal{P}(Q) \). For the proof of ii.) consider the closed p-form \( \beta := i^* \Omega \) on \( Q \) which is evidently \( G \)-invariant by \( G \)-invariance of \( \Omega \). Therefore the closed two-form \( \Omega - \pi^* \beta \) is \( G \)-invariant and \( i^* (\Omega - \pi^* \beta) = 0 \). Now, consider the homotopy \( H : \mathbb{R} \times T^*Q \to T^*Q, \ (t, \zeta_x) \mapsto H(t, \zeta_x) := t \zeta_x \). By means of this homotopy one can explicitly define a \( p-1 \)-form \( \Xi \) by \( \Xi(X_{1}, \ldots, X_{p-1}) := \int_0^1 (H^*(\Omega - \pi^* \beta))(\partial_t, X_1, \ldots, X_{p-1}) dt \). This \( \Xi \) satisfies \( d\Xi = \Omega - \pi^* \beta \) by the classical proof of Poincaré’s lemma. Due to the compatibility of the above homotopy with the \( \Phi_y \) the thus defined \( p-1 \)-form is \( G \)-invariant. Since \( H(t, i(x)) = i(x) \) we also have \( i^* \Xi = 0 \). From the explicit shape of \( \Xi \) it is also obvious that \( \Xi(X_{F_1}^{B_0}, \ldots, X_{F_p}^{B_0}) \in \mathcal{P}(Q) \) for all \( F_1, \ldots, F_{p-1} \in \mathcal{P}(Q) \) in case \( \Omega(X_{F_1}^{B_0}, \ldots, X_{F_p}^{B_0}) \in \mathcal{P}(Q) \) for all \( F_1, \ldots, F_p \in \mathcal{P}(Q) \). \( \square \)
Using these technical preparations we can adapt the proof of [3, Prop. 4.1] to the present situation and obtain:

**Proposition 4.13** Suppose that there is a G-invariant torsion free connection \( \nabla \) on \( Q \). Then we have:

i.) For every star product \( \star \) on \((T^*Q, \omega_{B_0})\) which is invariant with respect to the lifted action of a \( G \)-action on \( Q \) there is a formal series \( i_{\star}^t \in T_1 B_+ = \sum_{k=1}^\infty v^k B_k \in Z^G_1(Q)^G[\{v\}] \) of \( G \)-invariant closed two-forms on \( Q \) and a \( G \)-equivariance transformation \( \mathcal{T} \) from \( \star \) to the \( G \)-invariant star product \( \star_0^B \), where \( B = B_0 + B_+ \).

ii.) In case \( \mathcal{P}(Q)[[\nu]] \) is a \( \star \)-subalgebra one can find \( B_+ \) and a \( G \)-equivariance transformation \( \mathcal{T} \) from \( \star \) to \( \star_0^B \) as in i.) such that \( \mathcal{T} F \in \mathcal{P}(Q)[[\nu]] \) for all \( F \in \mathcal{P}(Q)[[\nu]] \).

**Proof:** Let \( \star \) be an arbitrary \( G \)-invariant star product on \((T^*Q, \omega_{B_0})\) as above and consider the \( G \)-invariant star product \( \star_0^B \), where \( B_0 = B_0 \). In order zero of the formal parameter, the star products \( \star \) and \( \star_0^B \) trivially coincide. Since both are star products with respect to \( \omega_{B_0} \) the anti-symmetric part of \( C^B_1 \) vanishes. Therefore Lemma 4.12 implies that there is a \( G \)-invariant 1-cochain \( c_1 \) with \( C^B_1 - C^* - \delta c_1 \). Now put \( T_0 := i_{\star}^t \), which is clearly \( G \)-invariant. Define another \( G \)-invariant star product by \( \star^B := T_0 \star \), i.e. let \( T_0 (f \star f') = (T_0 (f) \star (T_0 (f')) \). Then an easy computation shows that \( C^B_1 = C^B_1 \) for \( r = 0, 1 \). By associativity of these two star products one obtains that the anti-symmetric part of \( C^B_2 - C^B_2 \) defines a \( G \)-invariant closed two-form \( \Omega_1 \) on \( T^*Q \) via

\[
C^B_2 (f, f') - C^B_2 (f', f) - C^B_2 (f, f') + C^B_2 (f') = \Omega_1 (X^{B_0}, X^{B_0})
\]

Again from Lemma 4.12 we get that \( B_1 := \omega_1 \) is a \( G \)-invariant closed two-form on \( Q \) and that there is a \( G \)-invariant one-form \( \Xi_1 \) on \( T^*Q \) such that \( \Omega_1 = d \Xi_1 + \pi^* B_1 \). Then we consider the \( G \)-invariant star product \( \star^B \) with \( B_1 = \nu B_0 + \nu B_1 \) and the \( G \)-invariant star product \( \star^B := T_0 \star^B \), where \( T_0 (f) := f + \nu \Xi_1 (X^{B_0}) \). According to Eq. (4.14) we have \( C^B_0 \) for \( r = 0, 1 \).

Now it is straightforward to check that \( C^B_1 = C^B_1 \) for \( r = 0, 1 \) and that the anti-symmetric part of \( C^B_2 - C^B_2 \) vanishes due to the definition of \( \star^B \) and Eq. (4.12). But then Lemma 4.12 i.) yields \( C^B_2 - C^B_2 \) with \( \delta c_2 \) with a \( G \)-invariant 1-cochain \( c_2 \). Putting \( \star^B := T^{(1)} \star^B \) with \( T^{(1)} := \id - \nu^2 c_2 \) we then obtain \( C^B_1 = C^B_1 \) for \( r = 0, 1, 2 \). Proceeding inductively we thus can find \( G \)-invariant operators \( T^{(l)} = \id - \nu^l c_{l+1} \) for \( l = 0, \ldots, k \), \( G \)-invariant \( \tilde{T}^{(m)} \) for \( m = 0, \ldots, k-1 \) with \( \tilde{T}^{(m)} f = f + \nu^{m+1} \Xi_{m+1} (X^{B_0}) \) and \( G \)-invariant closed two-forms \( B_1, \ldots, B_k \) on \( Q \) such that \( C^B_1 = C^B_1 \) for \( r = 0, \ldots, k + 1 \). Hereby, \( B^{(k)} = \sum_{l=0}^k \nu^l B_l \) and \( \star^{(k)} = \mathcal{T}^{(k)} \). Hence \( \mathcal{T} \) is a \( G \)-equivalance from \( \star \) to \( \star_0^B \) and \( B_+ \) is given by \( B_+ = B - B_0 \) proving i.). For the proof of ii.) one just has to observe that \( \mathcal{P}(Q)[[\nu]] \) is a \( \star \)-subalgebra and that by assumption \( \mathcal{P}(Q)[[\nu]] \) is a \( \star \)-subalgebra. Using Lemma 4.12 this implies that in every step of the above construction one can achieve that \( T^{(l)} \) and \( \tilde{T}^{(m)} \) map elements of \( \mathcal{P}(Q)[[\nu]] \) to elements of \( \mathcal{P}(Q)[[\nu]] \). To verify this check
by induction that $\mathcal{P}(Q)[[\nu]]$ is both a $\star^{(k)}$-subalgebra as well as a $\wedge^{(k)}$-subalgebra since $\mathcal{P}(Q)[[\nu]]$ is a $\mathfrak{g}^B$-subalgebra for all occurring $B(l)$.

Note that even after having fixed a $G$-invariant torsion free connection on $Q$ one cannot use the construction of $\frac{\alpha}{2}B_\pm$ in the proof of the preceding proposition to define a map from the space of $G$-invariant star products on $(T^*Q,\omega_{B_0})$ to the space of formal series of closed $G$-invariant two-forms on $Q$. This fact is caused by the freedom of choice in the equivalence transformations which in fact affects the explicit form of $\frac{\alpha}{2}B_\pm$. For instance, in the definition of $T^{(0)}$ we could have replaced $c_1$ by $c_1 + L_X$, where $X$ is a $G$-invariant vector field on $T^*Q$. An easy computation then shows that this gives rise to a modification of $B_1$ by the additional term $-\text{d}i^*(i_X\omega_{B_0})$. Later on, we will show that the above construction nevertheless induces a bijection between the $G$-equivalence classes of $G$-invariant star products and formal series of elements of the space $H^2_{\text{dR},G}(Q) = Z^2_{\text{dR}}(Q)^G / \text{d}(\Gamma^\infty(T^*Q)^G)$ of second degree cohomology classes of $G$-invariant de Rham cohomology. Moreover, this bijection will actually turn out to be independent of the chosen connection, hence is canonical (cf. [3, Thm. 4.1] for an analogous statement on general symplectic manifolds).

As an immediate corollary Proposition 4.13 implies:

**Corollary 4.14** Under the assumptions of the proposition the following holds true:

i.) There is a $G$-equivariant quantum momentum map for a $G$-invariant star product $\star$ on $(T^*Q,\omega_{B_0})$, if and only if there is a $G$-equivariant quantum momentum map for the star product $\star^B_0$, where $B = B_0 + B_\pm$ denotes a formal series of closed $G$-invariant two-forms on $Q$ as in Proposition 4.13 i.).

ii.) If $\mathcal{P}(Q)[[\nu]]$ is in addition a $\star$-subalgebra, then every $G$-equivariant quantum momentum map $J$ for $\star$ satisfies $J(\xi) \in \mathcal{P}(Q)[[\nu]]$ for all $\xi \in \mathfrak{g}$.

**Proof:** For the proof of i.) consider a $G$-equivariant quantum momentum map $J$ for $\star$ and a $G$-equivariant $T$ to the star product $\star^B_0$. Then $J^B$ with $J^B(\xi) := T^B J(\xi)$ clearly defines a $G$-equivariant quantum momentum map for $\star^B_0$. Vice versa, every $G$-equivariant quantum momentum map $J^B$ for $\star^B_0$ defines a $G$-equivariant quantum momentum map $J$ for $\star$ via $J(\xi) := T^{-1} J^B(\xi)$. For the proof of ii.) apply Proposition 4.13 to show that $T$ can be chosen to preserve $\mathcal{P}(Q)[[\nu]]$. Given a $G$-equivariant quantum momentum map $J$ for $\star$ we then get one for $\star^B_0$ by $J^B(\xi) := T^B J(\xi)$. But from Proposition 4.13 we have that $J^B$ is of form $J^B(\xi) = P(\xi_\varnothing + \pi^j(\xi) \in \mathcal{P}(Q)[[\nu]]$, where $j \in \mathcal{C}^1(\mathfrak{g},\mathcal{C}_\infty(Q))[[\nu]]$ satisfies the conditions in Eq. 4.8. This implies in particular that $J(\xi) = T^{-1} J^B(\xi) \in \mathcal{P}(Q)[[\nu]]$. Since any other $G$-equivariant quantum momentum map $J'$ for $\star$ differs from $J$ by an element of $\mathfrak{g}^G + \nu \mathfrak{g}_G^G[[\nu]]$ we obtain $J'(\xi) \in \mathcal{P}(Q)[[\nu]]$ for every $G$-equivariant $J'$ which is a quantum momentum map for $\star$.

In view of the second part of the above corollary it now becomes clear that one of the assumptions we made for our reduction scheme – namely that $J(\xi) \in \mathcal{P}(Q)[[\nu]]$ – is in fact not an additional assumption but a consequence of the assumption that $\mathcal{P}(Q)[[\nu]]$ is a $\star$-subalgebra.

For the purposes of the following section, where we will compute the characteristic class of a reduced star product the results achieved so far would completely suffice. But with a little more effort we can give a classification of the $G$-invariant star products on cotangent bundles up to $G$-equivalence, a result which is of independent interest. To this end we show in a first step the following proposition. Its proof is rather technical but yields the key tools for the main results of the last part of this section.

**Proposition 4.15** Let $\star^B_0$ resp. $\star^{B'}_0$ denote a $G$-invariant star products on $(T^*Q,\omega_{B_0})$ which is obtained from a $G$-invariant torsion free connection $\nabla$ resp. $\nabla'$ and a $G$-invariant formal series
of closed two-forms $B$ resp. $B'$ on $Q$ starting with $B_0$. Then $*^B_0$ and $*^{B'}_0$ are $G$-equivalent, if and only if $\frac{1}{\nu}B_+ = \frac{1}{\nu}(B - B_0)$ and $\frac{1}{\nu}B'_+ = \frac{1}{\nu}(B' - B_0)$ are $G$-cohomologous, i.e. if and only if there is a $G$-invariant formal series of one-forms $\beta$ on $Q$ such that $\frac{1}{\nu}B_+ = \frac{1}{\nu}B'_+ + d\beta$.

**Proof:** Let us first assume that $B_+$ and $B'_+$ are $G$-cohomologous. Then we want to prove that $*^B_0$ and $*^{B'}_0$ are $G$-equivalent. To this end we need the following result about standard ordered star products obtained from torsion free connections.

**Sublemma 4.16** Let $\nabla$ and $\nabla'$ denote two torsion free connections on $Q$ and $\varrho_0$ resp. $\varrho'_0$ the corresponding standard ordered representation of the star product $*_0$ resp. $*'_0$ on $(T^*Q, \omega_0)$. Then there is a uniquely determined formal series $S$ of differential operators on $C^\infty(T^*Q)$ such that $SF \in \mathcal{P}(Q)[[\nu]]$ for all $F \in \mathcal{P}(Q)[[\nu]]$ and

$$\varrho'_0(f) = \varrho_0(Sf) \text{ for all } f \in C^\infty(T^*Q)[[\nu]].$$

This implies that $S$ is an equivalence transformation from $(C^\infty(T^*Q)[[\nu]], *'_0)$ to $(C^\infty(T^*Q)[[\nu]], *_0)$. Moreover, $S$ satisfies $S\pi^*\chi = \pi^*\chi$ for all $\chi \in C^\infty(Q)[[\nu]]$. Consequently, one has

$$\frac{1}{\nu}S\text{ad}*_0(\pi^*\chi)S^{-1} = \frac{1}{\nu}\text{ad}*_0(\pi^*\chi).$$

**Proof:** First recall that the operators of symmetric covariant derivation $D$ and $D'$ satisfy $D' = D - dx^i \nabla x^j \partial_{x^j}$, where the symmetric tensor field $s$ is given by $\nabla' Y = \nabla Y + S(X,Y)$. Now it is easy to see that $\varrho'(F)$ lies in the image of $\varrho_0$ for all $F \in \mathcal{P}(Q)[[\nu]]$. By injectivity of the restriction of $\varrho_0$ to $\mathcal{P}(Q)[[\nu]]$ one can define a map $S : \mathcal{P}(Q)[[\nu]] \to \mathcal{P}(Q)[[\nu]]$ by $SF := \varrho_0^{-1}(\varrho'_0(F))$. Using the explicit form of the standard ordered representations it is immediate to check that this map is given by a formal series of differential operators on $\mathcal{P}(Q)$ and that this series starts with id. By definition, $S$ satisfies Eq. (4.14) on polynomial functions in the momenta and $S(F *'_0 F') = SF *_0 SF'$ for all $F, F' \in \mathcal{P}(Q)[[\nu]]$. This implies that $S$, which clearly extends uniquely to a mapping on $C^\infty(T^*Q)[[\nu]]$, is an equivalence from $*'_0$ to $*_0$ and satisfies Eq. (4.14).

Uniqueness of $S$ is again a direct consequence of $\varrho_0$ being injective when restricted to $\mathcal{P}(Q)[[\nu]]$. For the proof of the further properties of $S$ observe first that $\varrho_0$ satisfies $\varrho_0(\pi^*\chi F) \psi = \chi \varrho_0(F) \psi$ for all $\chi, \psi \in C^\infty(Q)[[\nu]]$ and that an analogous relation holds for $\varrho'_0$. Using the definition of $S$ this yields that $S(\pi^*\chi F) = \pi^*\chi SF$ for all $F \in \mathcal{P}(Q)[[\nu]]$, hence $S$ commutes with left-multiplications by formal functions pulled-back from $Q$. In particular, we obtain by setting $F = 1$ that $S\pi^*\chi = \pi^*\chi$. From this relation Eq. (4.15) is immediate, since $S$ is an equivalence from $*'_0$ to $*_0$.  

Using $G$-equivariance of $\varrho'_0$ and $\varrho_0$ the preceding sublemma entails that $\varrho'_0(f) = \varrho_0(Sf) = \varrho_0\left(\Phi^*_g S\Phi^*_g f\right)$ for all $f \in C^\infty(T^*Q)[[\nu]]$. Clearly, $\Phi^*_g S\Phi^*_g$ preserves $\mathcal{P}(Q)[[\nu]]$. But since $S$ is the uniquely determined map which satisfies Eq. (4.14) and preserves $\mathcal{P}(Q)[[\nu]]$ we conclude that $S = \Phi^*_g S\Phi^*_g$, i.e. $S$ is a $G$-equivalence from $*'_0$ to $*_0$. We claim that if $\frac{1}{\nu}B_+ = \frac{1}{\nu}B'_+ + d\beta$ with $\beta \in \Gamma^\infty(T^*Q)^G[[\nu]]$ one can use $S$ to build a $G$-equivalence from $*'_0$ to $*_0$. To this end consider $A^j = A^j_0 + A^j_+$ with local potentials $A^j_0$ of $B_0$ and $A^j_+$ of $B'_+$ over some $O_j$ which is assumed to be an element of a good open cover of $Q$. Then $A^j = A^j_0 + A^j_+ + \nu \beta$ is a local potential of $B$. Composition of $S$ with the local isomorphisms $(A^j_0)^{-1}$ and $A^j_0$ defined in Eq. (4.12) gives rise to a local isomorphism $T_j := S(A^j_0)^{-1} \left( (C^\infty(T^*O_j)[[\nu]], *'_0) \right) \to (C^\infty(T^*O_j)[[\nu]], *^B_j)$. We now have to show that these local isomorphisms actually glue together to a globally defined $G$-equivalence. To verify this recall from [8, Thm. 3.4] that $(A^j_0)^{-1} A^j_0 f = f(1)$ for every $f \in C^\infty(T^*(O_j \cap O_i))[[\nu]]$, where $f$ is given by the unique solution of the differential equation $\phi \frac{d}{d\nu} \tilde{f}(t) = \frac{1}{\nu} \text{ad}*_0(\pi^* a_x j) \tilde{f}(t)$ with $\tilde{f}(0) = f$. Here, $a_{xj} \in C^\infty(O_j \cap O_i)[[\nu]]$ satisfies $da_{xj} = A^j_0 - A^j_0$ over $O_j \cap O_i$.
Since $S$ is an equivalence from $\star_0$ to $\star_0$ and $S\pi^*\chi = \pi^*\chi$, this implies that $S(A^0_{ij})^{-1}A^0_iS^{-1}f = \tilde{f}(1)$, where $\tilde{f}$ solves $\frac{d}{dt}\tilde{f}(t) = \frac{1}{\nu}\text{ad}_{a_{ij}}(\pi^*a_{ij})\tilde{f}(t)$ with $\tilde{f}(0) = f$. But from the choice of the local potentials $A^j$ and $A^j\tilde{T}$ we obtain $da_{ij} = A^i - A^j$ over $O_j\cap O_i$. Therefore, $S(A^0_{ij})^{-1}A^0_iS^{-1}$ is equal to $(A^0_{ij})^{-1}A^0_i$, i.e. over $C^\infty(T^*(O_j\cap O_i))[[\nu]]$ we have $T_jT_i^{-1} = id$. But this entails that we can define a global isomorphism $T$ from $(C^\infty(T^*Q)[[\nu]],\star_0^B)$ to $(C^\infty(T^*Q)[[\nu]],\star_0^B)$ by $(Tf)|_{T(O_j\cap O_i)} := T_jf|_{T(O_j\cap O_i)}$. From the fact that $A^j\tilde{T}$ and $A^j\tilde{T}$ coincide at zeroth order in $\nu$ it is obvious that this isomorphism is in fact an equivalence transformation. It remains to show that $T$ is $G$-invariant. Let us fix $g \in G$. Using that $S$ is $G$-invariant we obtain $\Phi_g^*T_{g^{-1}}\Phi_g^* = A^i - A^j$ over $O_j\cap O_i$. Now consider an index $i$ such that $\phi_{g^{-1}}(O_j)\cap O_i \neq \emptyset$. Then we claim that $\Phi_g^*T_{g^{-1}}(O_j)\cap O_i = id$ over $\phi_{g^{-1}}(O_j)\cap O_i$. To prove this we may assume without loss of generality that $\phi_{g^{-1}}(O_j)\cap O_i$ is contractible since in case it were not contractible we could cover it by open contractible subsets and use the following argument for each element of this covering. By $G$-invariance of $B'$ we have $d(\phi_g^*(A^j) - A^i) = 0$. Hence there exist formal functions $b_{ij}$ over $\phi_{g^{-1}}(O_j)\cap O_i$ such that $\phi_g^*A^j - A^i = db_{ij}$. Then $\Phi^*_gA^0_{ij} - 1\Phi^*_gA^0_i$ turns out to be the local automorphism of $\star_0$ generated by $\frac{1}{\nu}\text{ad}_{a_{ij}}(\pi^*b_{ij})$. Like in the argument which showed that $T$ is well-defined one concludes that $S\Phi_g^*(A^0_{ij})^{-1}S^{-1}$ is the local automorphism of $\star_0$ generated by $\frac{1}{\nu}\text{ad}_{a_{ij}}(\pi^*b_{ij})$. On the other hand $\Phi^*_g(A^0_{ij})^{-1}\Phi^*_gA^0_i$ coincides with this local automorphism, since by $G$-invariance of $\beta$ and the choice of the local potentials $A^j$ and $A^j\tilde{T}$ the equation $\phi_g^*A^j - A^i = \phi_g^*A^j - A^i = db_{ij}$ is valid. But this implies that $\Phi^*_gT_{g^{-1}}\Phi_g^* = id$ over $\phi_{g^{-1}}(O_j)\cap O_i$. Hence $T$ is $G$-invariant. So we have shown that $\pi^B_0$ and $\pi^B_0'$ are $G$-equivalent, if $\frac{1}{\nu}B^+_{ij}$ and $\frac{1}{\nu}B^+_{ij}'$ are $G$-cohomologous. For the proof of the converse statement assume that $\pi^B_0$ and $\pi^B_0'$ are $G$-equivalent and that $l \geq 1$ is the smallest index such that $B_{ij}$ and $B_{ij}'$ are not $G$-cohomologous. As usual we have set hereby $B = B_0 + \sum_{r=1}^{\infty} \nu^r B_r$ and $B' = B_0 + \sum_{r=1}^{\infty} \nu^r B_r'$. Now consider $B'' := B_0 + \sum_{r=0}^{l-1} \nu^r B_r + \sum_{r=1}^{\infty} \nu^r B_r'$. Then $B''_{ij}$ is $G$-cohomologous to $B_{ij}'$. Consequently, we know from above that $\pi^B_0$ is $G$-equivalent to $\pi^B_0''$. But this implies that $\pi^B_0$ and $\pi^B_0''$ are also $G$-equivalent. It is now immediate to deduce from Lemma 4.11 that the describing bidiformal operators $C^B_0 r$ and $C^B_0''$ coincide for $r = 0, \ldots, l$ and that 

$$
C^B_{l+1}(f,f') - C^B_{l+1}(f',f) = C^B''_{l+1}(f,f') + C^B''_{l+1}(f',f) = (\pi^*B_{l} - \pi^*B_{l})(X_{f^{B_{l}}}X_{f^{B_{l}}}).
$$

But then the $G$-equivalence of $\pi^B_0$ and $\pi^B_0''$ implies that $\pi^*B_{l} - \pi^*B_{l}$ is $G$-exact (cf. 3 Thm. 2.1)), i.e. there is a $G$-invariant one-form $\Xi$ on $T^*Q$ such that $\pi^*B_{l} - \pi^*B_{l} = d\Xi$. Thus $B_{l} = d\iota^*\Xi + B_{l}$, where $\iota^*\Xi$ is $G$-invariant. Hence $B_{l}$ and $B_{l}'$ are $G$-cohomologous, which is a contradiction, proving the other direction of the statement of the proposition.

Using the construction in the proof of Proposition 4.13 and the preceding proposition we can state the following classification result:

**Theorem 4.17** i.) To every star product $\star$ on $(T^*Q, \omega_{B_0})$ which is invariant with respect to the lifted action of a $G$-action on $Q$ one can assign a formal series in the second $G$-invariant de Rham cohomology of $Q$ by

$$
c_G : \star \mapsto c_G(\star) := \frac{1}{\nu}[B_{+}]_{G} \in H^2_{dR,G}(Q)[[\nu]],
$$

where $\frac{1}{\nu}B_{+}$ denotes a formal series of $G$-invariant closed two-forms on $Q$ as in Proposition 4.13.
ii.) The map $c_G$ in Eq. (4.10) is independent of the chosen $G$-invariant torsion free connection on $Q$ which was used to define $\frac{1}{\nu}B_+$. Moreover, $c_G$ induces by $[\nu]_G \mapsto c_G(\nu)$ a bijection between the set of $G$-equivalence classes of $G$-invariant star products as in i.) and $H^2_{dR,G}(Q)[[\nu]]$.

PROOF: For the proof of i.) we just have to verify that $c_G$ is well-defined. To this end consider two $G$-equivariant systems $T, T'$ and two formal series $B, B'$ of closed $G$-invariant two-forms on $Q$ starting with $B_0$ such that $T \ast = \ast^B_0$ and $T' \ast = \ast^{B'}_0$ as in Proposition 4.13. Then $\ast^B_0$ and $\ast^{B'}_0$ are $G$-equivariant, clearly. Proposition 4.15 implies that $\frac{1}{\nu}B_+$ and $\frac{1}{\nu}B'_+$ are $G$-cohomologous, therefore $c_G$ is well-defined. For the proof of ii.) consider $G$-equivacuities $T, T'$ and two formal series $B, B'$ of closed $G$-invariant two-forms on $Q$ starting with $B_0$ such that $T \ast = \ast^B_0$ and $T' \ast = \ast^{B'}_0$, where $\ast^B_0$ and $\ast^{B'}_0$ are obtained from different connections $\nabla$ and $\nabla'$. Then $\ast^B_0$ and $\ast^{B'}_0$ are $G$-equivariant. Proposition 4.15 implies again that $\frac{1}{\nu}B_+$ and $\frac{1}{\nu}B'_+$ are $G$-cohomologous which shows that $c_G$ is independent of the connection used to construct $\frac{1}{\nu}B_+$. Furthermore, observe that for $G$-equivariant star products $\ast$ and $\ast'$ one has $c_G(\ast) = c_G(\ast')$, since there exists a $G$-equivariance from $\ast$ to $\ast^B_0$, hence we obtain a $G$-equivariance from $\ast'$ to $\ast^B_0$. This implies that $c_G$ induces a mapping from the set of $G$-equivariant star products as in i.) to $H^2_{dR,G}(Q)[[\nu]]$ as given above.

To prove surjectivity of this map consider $\ast^B_0$, where $B = B_0 + \nu\beta$ and $\beta \in Z^2_{dR}(Q)^G[[\nu]]$ is an arbitrary formal series of closed $G$-invariant two-forms on $Q$. By definition of $c_G$ and choosing id as $G$-equivariance we obtain $c_G(\ast^B_0) = [\beta]_G$. Since $\beta$ is arbitrary this proves surjectivity. To prove injectivity let $\ast, \ast'$ be star products with $c_G(\ast) = c_G(\ast')$. By Proposition 4.15 the corresponding star products $\ast^B_0$ and $\ast^{B'}_0$ are $G$-equivariant which implies that $\ast$ and $\ast'$ are $G$-equivariant. \[\square\]

5 The Characteristic Class of the Reduced Star Products $\ast^{J,\mu}$

In this section we want to compute the characteristic class of the reduced star products $\ast^{J,\mu}$ in order to clarify how the equivalence classes of these products depend on the initially chosen parameters of the reduction scheme. To this end we proceed in two steps. Under the general assumption of a proper and free $G$-action on $Q$ we first consider a star product $\ast^B_0$ on $(T^*Q, \omega_{B_0})$ constructed from a $G$-invariant torsion free connection $\nabla$ on $Q$ and a formal series $B$ of $G$-invariant closed two-forms on $Q$. Recall that due to the properness of the $G$-action such a $G$-invariant connection exists on $Q$ by Palais’ Theorem and that the resulting $\ast^B_0$ is $G$-invariant by the results of the preceding section. Additionally, we assume that $j \in C^1(G, C^\infty(Q))[[\nu]]$ satisfies the conditions given in Eq. (4.14) so that we can use $J^B$ with $J^B(\xi) = P(\xi + \pi^B \ast^j(\xi))$ as $G$-equivariant quantum momentum map. Then it is possible to compute the characteristic class of the reduced star product $\ast_0^{B^{J,\mu}}$ explicitly in terms of $B, j, \mu$ and the connection on $p : Q \to \overline{Q}$. In a second step we use the relation between a star product $\ast$ that satisfies all necessary assumptions for the applicability of our reduction procedure and a $G$-equivariant star product $\ast^B_0$ to relate the characteristic class of $\ast^{J,\mu}$ to the one of $\ast_0^{B^{J,\mu}}$.

Let us now provide a few results needed in the sequel for the computation of characteristic class of star products. For more details we refer the reader to [17,24] which treat the case of arbitrary symplectic manifolds and to [8], where the special case of cotangent bundles is considered.

Recall that the characteristic class of a star product $\ast$ on $(M, \omega)$ is an element of $[\omega]_\nu + H^2_{dR}(M)[[\nu]]$. For its computation one first has to find local derivations of $(C^\infty(O_j))[[\nu]], \ast)$, so-called local $\nu$-Euler derivations, where $\{O_j\}_{j \in I}$ is a good open cover of $M$. These local $\nu$-Euler derivations are of form

$$E_j = \nu \partial_\nu + L_{\xi_j} + \sum_{r=1}^\infty \nu^r D_{j,r},$$

(5.1)
where $\xi_j$ is a local conformally symplectic vector field (i.e. $L_{\xi_j}\omega|_{O_j} = \omega|_{O_j}$), and the $D_{j,r}$ are locally defined differential operators on $C^\infty(O_j)$. With the help of these the characteristic class can be determined except for the part of order zero in the formal parameter. For the computation of that term one additionally needs an explicit expression for the anti-symmetric part $C^*_{ij}(f, f') = \frac{1}{2}(C_2(f, f') - C_2(f', f))$ of the bidifferential operator describing the considered star product in the second order of the formal parameter. More explicitly, to determine the characteristic class from the $E_j$ one considers $E_i - E_j$ over $O_i \cap O_j$, which is a quasi-inner derivation, i.e. there are local formal functions $d_{ij} \in C^\infty(O_i \cap O_j)[[\nu]]$ such that $E_i - E_j = \frac{1}{2}\text{ad}_\nu(d_{ij})$. Now, whenever $O_i \cap O_j \cap O_k \neq \emptyset$ the sums $d_{ijk} = d_{jk} - d_{ik} + d_{ij}$ lie in $\mathbb{C}[[\nu]]$ and define a 2-cocycle whose Čech class $[d_{ijk}] \in H^2(M, \mathbb{C})[[\nu]]$ is independent of the choices made. The corresponding class $d(\ast) \in H^2_{\text{dir}}(M)[[\nu]]$ is called Deligne's derivation-related class of $\ast$. In addition, let $C_2^{-\ast}$ denote the image of $C_2$ under the projection onto the second component of the decomposition $H^2_{\text{Chev, nc}}(C^\infty(M), C^\infty(M)) = \mathbb{C} \oplus H^2_{\text{dir}}(M)$ which describes the second cohomology group of the null-on-constants Chevalley cohomology of $(C^\infty(M), \{, , \})$, taken with respect to the adjoint representation. Together, $d(\ast)$ and $C_2^{-\ast}$ define the characteristic class $c(\ast)$ of $\ast$ by

$$c(\ast)_0 = -2C_2^{-\ast} \quad \text{and} \quad \partial_\nu c(\ast) = \frac{1}{\nu^2} d(\ast).$$

(5.2)

The so-defined element of $[[\nu]] + H^2_{\text{dir}}(M)[[\nu]]$ classifies the equivalence classes of star products on a symplectic manifold $(M, \omega)$ in a functorial way (cf. [17 Thm. 6.4]).

In the following $\{O_i\}_{i \in I}$ denotes a $G$-invariant good open cover of $Q$ which projects via $p$ to a good open cover $\{\overline{O}_i\}_{i \in I} = \{p(O_i)\}_{i \in I}$ of $\overline{Q}$. Such a cover exists due to the fact that the action on $Q$ is proper. Our first goal is to define local $\nu$-Euler derivations of $*_{0}^{B^{jB,\mu}}$ on every $\overline{O}_i$ using certain local $\nu$-Euler derivations of $*_{0}^{B}$ on $O_i$. Unfortunately, an arbitrary $\nu$-Euler derivation of $*_{0}^{B}$ cannot be projected down to such a derivation of $*_{0}^{B^{jB,\mu}}$ in general, since such derivations usually neither preserve $\mathcal{P}(Q)[[\nu]]$ nor are $G$-invariant and even then do not preserve the ideal of $G$-invariant formal functions generated by the $G$-equivariant quantum momentum map. Therefore, we have to find appropriately modified $\nu$-Euler derivations of $*_{0}^{B}$, where we let us lead by intuition rather than by a deductive procedure. Actually, the relation between $b_0$ and $B_0$ in the lowest order of the characteristic classes of $*_{0}^{B^{jB,\mu}}$ and $*_{0}^{B}$ suggests that a similar relation might also hold in higher orders. In analogy to classical phase space reduction we therefore consider the following formal series of closed two-forms on $Q$:

$$B + d\Gamma_{\ast} - \mu,$$

(5.3)

where $\Gamma$ has been defined by Eq. (5.4) and has been extended by $\mathbb{C}[[\nu]]$-linearity. A straightforward argument which is completely analogous to the computation in the proof of Theorem 3.2 now shows by $G$-equivariance of $j$ that $B + d\Gamma_{\ast} - \mu$ is a formal series of $G$-invariant closed horizontal two-forms on $Q$. Hence there exists a uniquely defined formal series $b$ of closed two-forms on $\overline{Q}$ such that

$$B + d\Gamma_{\ast} - \mu = p^* b.$$

(5.4)

Now we choose local potentials $a^i$ of $b$ over $\overline{O}_i$, i.e. $da^i = b|_{\overline{O}_i}$. For $\overline{O}_i \cap \overline{O}_j$ we choose local formal functions $f_{ij}$ with $df_{ij}|_{\overline{O}_i \cap \overline{O}_j} = (a^i - a^j)|_{\overline{O}_i \cap \overline{O}_j}$. Furthermore, we consider the local formal one-forms $A^i$ on $O_i$ defined by $A^i := p^* a^i - \Gamma_{\ast} - \mu$. Then $A^i - A^j = dp^* f_{ij}$ holds true on $O_i \cap O_j$ by construction. Using these $A^i$ we now consider the local isomorphisms $\mathcal{A}^i_0 : (C^\infty(T^*O_i)[[\nu]], *_0) \rightarrow (C^\infty(T^*O_i)[[\nu]], *_{0}^{B})$ from Eq. (2.12) and claim the following:

**Proposition 5.1** With notations from above the following holds true:
i.) The mappings $\mathcal{E}_i : \mathcal{C}^\infty(T^*O_i)[[\nu]] \to \mathcal{C}^\infty(T^*O_i)[[\nu]]$ defined by

$$\mathcal{E}_i := A_0^i \mathcal{H}(A_0^i)^{-1}$$

(5.5)

are $G$-invariant local $\nu$-Euler derivations of $\bullet_0^B$ which preserve $\mathcal{P}(O_i)[[\nu]]$ and $\mathcal{P}(O_i)[[\nu]] \cap \bar{I}_{\nu0}$. 

ii.) The mappings $\mathcal{E}_i : h(\mathcal{P}(O_i)^G)[[\nu]] \to h(\mathcal{P}(O_i)^G)[[\nu]]$ defined by

$$\mathcal{E}_i F := h_{\nu0}^{-\mu_0} \left( \frac{id}{id - \nu \Delta_{\mu_0} \nu_0^B} \mathcal{E}_i F \right), \quad F \in h(\mathcal{P}(O_i)^G)[[\nu]]$$

(5.6)

are local derivations of $(h(\mathcal{P}(O_i)^G)[[\nu]], \bullet_0^{B,\mu_0})$. The corresponding mappings $E_i := l^{-1} \circ E_i \circ l$ are local $\nu$-Euler derivations of $(\mathcal{P}(O_i)[[\nu]], \bullet_0^{B,\mu_0})$ which uniquely extend to such derivations of $(\mathcal{C}^\infty(T^*\mathcal{O}_i)[[\nu]], \bullet_0^{B,\mu_0})$.

iii.) On $\mathcal{C}^\infty(T^*(\mathcal{O}_i \cap \mathcal{O}_j))[\nu]$ one has

$$E_i - E_j = \frac{1}{\nu} \text{ad}_{\bullet_0^{B,\mu_0}}((\nu \partial_\nu - id) \nu^\nu f_{ij}).$$

(5.7)

This implies that the characteristic class $c(\bullet_0^{B,\mu_0})$ is given by

$$c(\bullet_0^{B,\mu_0}) = \frac{1}{\nu} [\pi^\nu b] - [\pi^\nu b_1] + c(\bullet_0^{B,\mu_0})_0.$$ 

(5.8)

PROOF: For the proof of i.) first observe that by definition of $A^i$ one has $dA^i = B_j |_{O_i}$. Therefore, the mappings $A_0^i$ are in fact local isomorphisms from $(\mathcal{C}^\infty(T^*O_i)[[\nu]], \bullet_0)$ to $(\mathcal{C}^\infty(T^*O_i)[[\nu]], \bullet_0^B)$. By their explicit form it is obvious that they preserve $\mathcal{P}(O_i)[[\nu]]$. Moreover, they also turn out to be $G$-invariant due to the $G$-invariance of the $A^i$. Since $\mathcal{H} = L_{\xi_0} + \nu \partial_\nu$ is evidently $G$-invariant and preserves $\mathcal{P}(O_0)[[\nu]]$ these properties hold for $E_i$, as well. In addition, $E_i$ is a local $\nu$-Euler derivation of $\bullet_0^B$, as $\mathcal{H}$ is a global $\nu$-Euler derivation of $\bullet_0$. It remains to show that $E_i$ preserves $\mathcal{P}(O_i)[[\nu]] \cap \bar{I}_{\nu0}$. But this follows from a straightforward proof of $E_i J^B(\xi)|_{T^*O_i} = J^B(\xi)|_{T^*O_i} - \langle \mu, \xi \rangle + \nu \partial_\nu \langle \mu, \xi \rangle$ which uses the explicit shape of $J^B$ and $A_0^i$. Using i.) it is rather evident that $E_i$ defines a local derivation of $\bullet_0^{B,\mu_0}$ on $h(\mathcal{P}(O_i^G))[\nu]$, and we only have to show that $E_i$ is of form provided in Eq. (5.5). To this end recall from [5] Lemma 4.4 that $E_i = \mathcal{H} + F \left( \frac{id - \exp(-\nu D)}{\nu D} (\nu \partial_\nu - id) A^i \right)$ which directly implies that $E_i = l^{-1} \circ E_i \circ l$ has form $\nu \partial_\nu + \sum_{r=0}^\infty \nu^r D_{i,r}$, where the $D_{i,r}$ are locally defined differential operators. Hence, these mappings uniquely extend to $\mathcal{C}^\infty(T^*(\mathcal{O}_i))[\nu]$, since differential operators are completely determined by their values on polynomial functions in the momenta. We only have to show that $D_{i,0} = L_{\xi_i}$ with a locally defined vector field $\xi_i \in \Gamma^\infty(T^*(\mathcal{O}_i))$ satisfying $L_{\xi_i} \omega_{00} = \omega_{00}$. But this follows from an easy computation expanding the exponent in the above given expression for $E_i$. For the proof of iii.) this expression again together with $A^i - A^j = dp^* f_{ij}$ entails

$$(E_i - E_j) F = h_{\nu0}^{-\mu_0} \left( \frac{id}{id - \nu \Delta_{\mu_0} \nu_0^B} \right) F \left( (id - \exp(-\nu D)) (\nu \partial_\nu - id) p^* f_{ij} \right) F = \frac{1}{\nu} \text{ad}_{\bullet_0^{B,\mu_0}}(p^* (\nu \partial_\nu - id) f_{ij}) F,$$
where the second equality follows from Eqs. (2.10) and (2.11) together with Lemma 4.4 i). Conjugation of this result by $l$ gives here the important intermediate steps and omit details of the proof.

To determine the missing part $c(\ast_0^{B_{1B,\mu}})_0$ of the characteristic class of $\ast_0^{B_{1B,\mu}}$ one has to compute the anti-symmetric part of the bidifferential operator describing $\ast_0^{B_{1B,\mu}}$ in the second order of the formal parameter. As this is a rather cumbersome but nevertheless important computation we only give here the important intermediate steps and omit details of the proof.

Lemma 5.2

i.) Writing $f \ast_0^B f' = f f' + \nu C_1^B (f, f') + \nu^2 C_2^B (f, f') + O(\nu^3)$ for $f, f' \in C^\infty(T^*Q)$ we have:

$$C_1^B (f, f') = \frac{1}{2} \left( \{ f, f' \}_B - \Delta_0 (f f') + (\Delta_0 f) f' + f (\Delta_0 f') \right) \quad (5.9)$$

$$C_2^B (f, f') - C_2^B (f', f) = -\frac{1}{2} \left( \Delta_0 \{ f, f' \}_B - \{ \Delta_0 f, f' \}_B - \{ f, \Delta_0 f' \}_B \right) \quad (5.10)$$

where $\Delta_0$ denotes the trace of the curvature tensor of $\nabla$ and $\Delta_0$ the differential operator defined in Eq. (2.6).

ii.) For $s, t \in \Gamma^\infty(\sqrt{T^*Q})$ the anti-symmetric part of the bidifferential operator $C_2^{B,\mu}$ which describes the star product $\ast_0^{B,\mu}$ on $\mathcal{P}(\overline{Q})[[\nu]]$ is given by

$$C_2^{B,\mu} (\overline{P}(s), \overline{P}(t)) = C_2^{B,\mu} (\overline{P}(t), \overline{P}(s))$$

$$= l^{-1} \left( h_{J_0 - \mu_0} \left( \Delta_{\mu, B}^* \{ P \left( s^h \right), P \left( t^h \right) \} \right) B_0 + C_2^{B, \mu} \left( P \left( s^h \right), P \left( t^h \right) \right) - C_2^{B, \mu} \left( P \left( t^h \right), P \left( s^h \right) \right) \right)$$

$$= -\frac{1}{2} \left( \delta_0 (\overline{P}(s), \overline{P}(t)) B_0 - \{ \delta_0 \overline{P}(s), \overline{P}(t) \} B_0 - \{ \overline{P}(s), \delta_0 \overline{P}(t) \} B_0 \right) \quad (5.12)$$

$$+ \pi^* \left( -b_1 - \frac{1}{2} r + \frac{1}{2} \tau \lambda \right) \left( X_{B_0}^{b_0} P(s), X_{B_0}^{b_0} P(t) \right).$$

Hereby, $r$ and $\tau \lambda$ are the unique closed two-forms on $\overline{Q}$ determined by $p^* r = -tr(R) + d\Gamma$ and $p^* \tau \lambda = -tr(ad(\lambda))$, where $d \in C^1(\mathfrak{g}, C^\infty(Q))$ is defined by $d(\xi) := \text{div}(\xi)$. Moreover, the mapping $\delta_0 : \mathcal{P}(\overline{Q}) \rightarrow \mathcal{P}(\overline{Q})$ is given by $\delta_0(P(s)) := l^{-1}(h_{J_0 - \mu_0} (\Delta_0 P(s^h))$).

PROOF: For the proof of i.) first recall [9] Thm. 10 that the star product $\ast_{1/2}$ is of Weyl type. Hence the describing bidifferential operators at order one and two of the formal parameter satisfy

$$C_1^{*_{1/2}} (f, f') = \frac{1}{2} \{ f, f' \} B_0$$

and

$$C_2^{*_{1/2}} (f, f') = -\frac{1}{2} \left( \Delta_0 \{ f, f' \}_B - \{ \Delta_0 f, f' \}_B - \{ f, \Delta_0 f' \}_B \right)$$

a straightforward computation which involves expansion of $N_{1/2}$, $A_0$, and the products $\ast_{1/2}$, $\ast_0$, and $\ast_0^B$ up to the second order in $\nu$ then shows Eqs. (5.9) and (5.10). The first equality stated in ii.) is obtained by an immediate computation using the definition of $\ast_0^{B,\mu}$. In contrast, the proof of the second equality turns out to be more involved but requires nothing more than a consequent
application of the definitions. Last, we provide the argument showing that \( r \) and \( \tau_\lambda \) are well-defined. For \( \tau_\lambda \) this is well known by Chern-Weil theory, and \([\tau_\lambda]\) is a characteristic class of the principal \( G \)-bundle \( p : Q \to \overline{Q} \). To prove that \( r \) is well-defined observe that \(-i_{\xi_\lambda} \text{tr} (R) = dd(\xi) \) and \( \phi_\lambda^* d(\xi) = d(\text{Ad}(g^{-1})\xi) \) for all \( \xi \in \mathfrak{g} \) and repeat the argument used for Theorem 3.2, showing that \( b_0 \) is well-defined.

By definition of the zeroth order of the characteristic class and the above result one directly obtains

\[
c(\star_0^ Bj^B,\mu)_0 = \left[ \mp^{\ast} \left( b_1 + \frac{r}{2} - \frac{\tau_{\lambda}}{2} \right) \right].
\]  

(5.13)

At this point one might expect that the characteristic class of the reduced star product depends on the chosen \( G \)-invariant connection \( \nabla \) and that the geometry of the principal bundle enters the characteristic class \( c(\bullet^ B,\mu) \) via \([\tau_\lambda]\). As we will show in the next lemma none of these dependencies occurs.

**Lemma 5.3**  
i.) Let \( \nabla \) and \( \nabla' \) be two \( G \)-invariant torsion free connections on \( Q \). Then the corresponding closed two-forms \( r \) and \( r' \) constructed as in Lemma 5.2 ii.) are cohomologous.

ii.) With notations from Lemma 5.2 ii.) we have

\[
r - \tau_\lambda = -\text{tr} (\overline{R}) - dw.
\]

(5.14)

**PROOF:** For the proof of i.) write \( \overline{\nabla}' Y = \overline{\nabla} Y + S(X,Y) \) with a symmetric \( G \)-invariant tensor field \( S \in \Gamma^\infty(\gamma^2 T^*Q \otimes TQ) \). Observe that the \( G \)-invariant one-form \( \text{tr} (S) \) defined by \( \text{tr} (S)(X) := dx^i(S(X,\partial_x^i)) \) satisfies \( \text{tr} (R') = \text{tr} (R) + d(\text{tr} (S)) \) and \( \text{div}'(X) = \text{div}(X) + \text{tr} (S)(X) \). Defining \( \sigma \in \mathcal{C}^1(\mathfrak{g}, \Gamma^\infty(\gamma)) \) by \( \sigma(\xi) := \text{tr} (S)(\xi_\lambda) \) it is easy to see that there exists a unique one-form \( s \) on \( Q \) which satisfies \( p^* s = \text{tr} (S) - \Gamma_\phi \). From the above identities relating the traces of the curvature tensors and the covariant divergences it is evident that \( r' = r - ds \). To prove ii.) first verify that \( \overline{\nabla} \) actually defines a torsion free connection on \( \overline{Q} \) and that \( w \) is well-defined. Then the proof consists of an easy computation showing that

\[
(\text{tr} (R) + d\Gamma_\lambda) (t^h, u^h) - \text{tr} \left( \text{ad}(\lambda(t^h, u^h)) \right) = p^* (-\text{tr} (\overline{R}) (t, u) - dw(t, u)),
\]

This computation mainly relies on splitting the definition of \( \text{tr} (R) \) into horizontal and vertical part. But since the trace of the curvature tensor of a torsion free connection is an exact two-form (cf. [3] Lemma 16) this implies \([r] = [\tau_\lambda] \). □

Collecting our results we have shown:

**Theorem 5.4** The characteristic class of the star product \( \star_0^ Bj^B,\mu \) on \( (T^*\overline{Q}, \omega_b) \) is given by

\[
c(\bullet^ B,\mu) = \frac{1}{\nu} [\mp^{\ast} b],
\]

(5.15)

where \( J^B(\xi) = P(\xi_\lambda) + \pi^* j(\xi) \) and \( b \in Z_{2\nu}(\overline{Q})[\nu] \) is determined by \( p^* b = B + d\Gamma_{j-\mu} \).

Now we consider an arbitrary star product \( \star \) satisfying the assumptions for our reduction procedure and present the main result of this section.
Proposition 4.8. Corollary 5.6

Let $\star$ be a $G$-invariant star product on $(T^*Q, \omega_{B_0})$ such that $\mathcal{P}(Q)[[\nu]]$ is a $\star$-subalgebra. Let $J$ denote a $G$-equivariant quantum momentum map for $\star$ and $\mu \in g'^\ast G + \nu g_c'^G[[\nu]]$.

i.) Assume that $T$ is a $G$-equivalence from $\star$ to $\star^B_0$ as in Proposition 4.13 (ii.), where $B = B_0 + B_+$ is an appropriate formal series of $G$-invariant closed two-forms. Then the star product $\star^{J,\mu}$ is equivalent to $\star^B_0$ with $G$-equivariant quantum momentum map $J^B$ given by $J^B(\xi) := T J(\xi)$.

ii.) With notations from i.), the characteristic class of the star product $\star^{J,\mu}$ is given by

$$c(\star^{J,\mu}) = c(\star^B_0) = \frac{1}{\nu} [\pi^* b],$$

where $\pi$ denotes the uniquely determined formal series of two-forms on $\overline{Q}$ such that $\pi^* b = B + d\Gamma_{j-\mu}$ and where $j \in C^1(g, C^\infty(Q))[[\nu]]$ is given by $j(\xi) = i^* T J(\xi)$.

PROOF: i.) is a direct consequence of Proposition 5.7 (i.) and iv.) since a $G$-equivalence $T$ exists due to Proposition 4.13 (i.) and since $J^B$ is a $G$-equivariant quantum momentum map for $\star^B_0$ by Corollary 4.14. By i.) the equality $c(\star^{J,\mu}) = c(\star^B_0)$ follows immediately. By Theorem 5.5 we obtain the claim about the explicit form of the characteristic class of $\star^{J,\mu}$. We only have to verify that $j$ is actually given by $j(\xi) = i^* T J(\xi)$, but this is obvious since $J^B$ satisfies $J^B(\xi) = P(\xi_0) + \pi^* j(\xi)$ by Proposition 4.8.

Corollary 5.6. Let $\gamma$ and $\gamma'$ denote two connection one-forms on $p : Q \to \overline{Q}$ and let $\star^{J,\mu}$ resp. $(\star^{J,\mu}')$ be the corresponding reduced star product on $(T^*\overline{Q}, \omega_{B_0})$ resp. $(T^*\overline{Q}, \omega_{B_0}')$ obtained by reduction of the star product $\star$ on $(T^*Q, \omega_{B_0})$. Then the characteristic classes of $\star^{J,\mu}$ and $(\star^{J,\mu}')$ coincide and there is an isomorphism from $(C^\infty(T^*\overline{Q})[[\nu]], \star^{J,\mu})$ to $(C^\infty(T^*\overline{Q})[[\nu]], (\star^{J,\mu}')$). Moreover, the corresponding star products $\star_{\Psi_{\mu_0}}$ and $(\star^{J,\mu})_{\Psi_{\mu_0}'}$ on $((T^*Q)_{\mu_0}, \omega_{\mu_0})$ (cf. Remark 4.6) are equivalent.

PROOF: Let $\Gamma_{j_0-\mu_0}$ be the one-form defined by $\langle \gamma \rangle = (j_0 - \mu_0, \gamma')$. Then one observes first that the translation $t_{\Gamma_{j_0-\mu_0}}$ along the fibre which maps the zero level set $(T^*Q)^0 = \{\zeta \in T^*Q \mid \zeta_\xi(\xi_0(x)) = 0 \text{ for all } \zeta \in g\}$ to itself clearly passes to the quotient defining a diffeomorphism of $T^*\overline{Q}$. Moreover, it is easy to see that this diffeomorphism consists of a translation $t_{\beta_0}$ along the fibres on $T^*\overline{Q}$, where $\beta_0$ is the unique one-form on $\overline{Q}$ such that $\pi^* \beta_0 = \Gamma_{j_0-\mu_0} - \Gamma_{j_{\mu_0}}$. By definition of $b_0$ and $b_0'$, where $b_0'$ is defined as in Theorem 3.2 using $\Gamma_{j_0-\mu_0}$ instead of $\Gamma_{j_0-\mu_0}$, we have $b_0' = b_0 - d\beta_0$. Hence, this diffeomorphism is in fact a symplectomorphism from $(T^*\overline{Q}, \omega_{B_0})$ to $(T^*\overline{Q}, \omega_{B_0})$. Let $\beta$ be the unique formal series of one-forms on $\overline{Q}$ such that $\pi^* \beta = \Gamma_{j_{\mu_0}} - \Gamma_{j_{\mu_0}}$. Then we analogously find that $\beta' = b - d\beta$, therefore $c((\star^{J,\mu}') = \frac{1}{\nu} [\pi^* b'] = \frac{1}{\nu} [\pi^* b] = c(\star^{J,\mu})$. Now consider the star product $(\star^{J,\mu})' := t_{\beta_0} \star^{J,\mu}$ on $(T^*\overline{Q}, \omega_{B_0})$, which has characteristic class $c((\star^{J,\mu})') = t_{\beta_0} c(\star^{J,\mu}) = \frac{1}{\nu} [\pi^* b] = \frac{1}{\nu} [\pi^* b] = c(\star^{J,\mu})$. But then the composition of an equivalence transformation $T$ from $(\star^{J,\mu})'$ to $(\star^{J,\mu'})'$ with $t_{\beta_0}$ yields an isomorphism from $(C^\infty(T^*\overline{Q})[[\nu]], \star^{J,\mu})$ to $(C^\infty(T^*\overline{Q})[[\nu]], (\star^{J,\mu}')$). One finally concludes that the corresponding star products on $((T^*Q)_{\mu_0}, \omega_{\mu_0})$ are equivalent, since the characteristic class is natural with respect to diffeomorphisms (cf. [17] Thm. 6.4) and since $t_{\beta_0} = \Psi_{\mu_0} \circ (\Psi_{\mu_0})^{-1}$ by the above result, where $\Psi_{\mu_0}$ is defined as in Theorem 3.2 using $\gamma'$ instead of $\gamma$.

Similarly, we find the dependence of the characteristic class on different choices of the $G$-equivariant quantum momentum map and different choices of the momentum value;
Corollary 5.7 \ i.) Let $\star^J,\mu$ and $\star^{J',}\mu'$ denote the star products on $(T^\ast Q, \omega_{\mu_0})$ and $(T^\ast Q, \omega'_{\mu'_0})$ obtained from two possibly different $G$-equivariant quantum momentum maps $J$ and $J'$ satisfying $J - J' = \tilde{\mu} \in \mathfrak{g}^{\ast G} + \nu \mathfrak{g}_{\mathbb{C}}^{\ast G}[[\nu]]$. Then the characteristic classes fulfill
\[ c(\star^J,\mu) - c(\star^{J'},\mu') = \frac{1}{\nu} [\pi* \tilde{\mu}], \] (5.17)
where $\tilde{\mu} \in Z^2_{dR}(\tilde{Q})[[\nu]]$ is determined by $p^* \tilde{\mu} = \langle \tilde{\mu}, \lambda \rangle$.

\[ ii.) Let \star^J,\mu \text{ and } \star^{J',}\mu' \text{ denote the star products on } (T^\ast Q, \omega_{\mu_0}) \text{ and } (T^\ast Q, \omega'_{\mu'_0}) \text{ obtained from two possibly different momentum values } \mu \text{ and } \mu' \text{ and let } \tilde{\mu} = \mu - \mu' \in \mathfrak{g}^{\ast G} + \nu \mathfrak{g}_{\mathbb{C}}^{\ast G}[[\nu]]. \text{ Then the characteristic classes satisfy} \]
\[ c(\star^J,\mu) - c(\star^{J'},\mu') = -\frac{1}{\nu} [\pi* \tilde{\mu}], \] (5.18)
where $\tilde{\mu} \in Z^2_{dR}(\tilde{Q})[[\nu]]$ is determined by $p^* \tilde{\mu} = \langle \tilde{\mu}, \lambda \rangle$.

PROOF: Both claims are direct consequences of Theorem 5.5. For the proof of i.) one just has to observe that for a $G$-equivalence $T$ from $\star$ to some $\star^B$ we have $\mu \in \mathfrak{g}^{\ast G} + \nu \mathfrak{g}_{\mathbb{C}}^{\ast G}[[\nu]]$. For the second claim, the second claim is obvious.

Finally, we are able to recover a relation between the characteristic class of the original star product $\star$ on $(T^\ast Q, \omega_{\mu_0})$ and the characteristic class of the reduced star product $\star^J_{\Psi_{\mu_0}}$ (cf. Remark 3.6) on $((T^\ast Q)\mu_0, \omega_{\mu_0})$ which already has been observed by M. Bordemann [9] (cf. also [10]) for arbitrary symplectic manifolds.

Corollary 5.8 Denote by $i_{\mu_0}$ the inclusion of $J_0^{-1}(\mu_0)$ into $T^\ast Q$ and by $\pi_{\mu_0}$ the projection from $J_0^{-1}(\mu_0)$ to $(T^\ast Q)\mu_0$. Then the characteristic classes of $\star$ and $\star^J_{\Psi_{\mu_0}}$ satisfy
\[ i_{\mu_0}* c(\star) = \pi_{\mu_0}* c(\star^J_{\Psi_{\mu_0}}). \] (5.19)

PROOF: By naturality of characteristic classes with respect to diffeomorphisms we obtain $c(\star^J_{\Psi_{\mu_0}}) = \Psi_{\mu_0}^* c(\star^J,\mu)$. The commutative diagram (3.7) and Theorem 5.5 then entail
\[ \pi_{\mu_0}^* c(\star^J_{\Psi_{\mu_0}}) = \frac{1}{\nu} [\pi_{\mu_0}^* \Psi_{\mu_0}^* \pi* b] = \frac{1}{\nu} [i_{\mu_0}^* \Gamma_{J_0^{-1}(\mu_0)}^\pi \pi* (B + d\pi_{\mu_0})] = i_{\mu_0}^* c(\star), \]
where the last equality follows from $\pi \circ t_{J_0^{-1}(\mu_0)} = \pi$ and the fact that $c(\star) = \frac{1}{\nu}[\pi* B]$. 

To conclude this section we discuss some special cases of Theorem 5.5 which clarify the result and allow for some interesting observations.

First consider the star product $\star_0$ and the $G$-equivariant quantum momentum map $J_0(\xi) = \mathcal{P}(\xi_\Omega)$. Then $c(\star_0^{J_0,\mu}) = -\frac{1}{\nu} [\pi* \mu_{\lambda}]$, where $\mu_{\lambda}$ is the unique formal series of closed two-forms on $\tilde{Q}$ defined by $p^* \mu_{\lambda} = \langle \mu, \lambda \rangle$. By Chern-Weil theory one knows that $[\mu_{\lambda}]$ is independent of the chosen connection and defines a formal series with values in the first characteristic classes of the principal $G$-bundle which only depends on $\mu$. Now choose $\mu_0$ in order to fix the symplectic form on $T^\ast Q$ to be $\omega_0 - \pi^\ast \mu_{\lambda}$. In general, one then cannot obtain star products on $(T^\ast Q, \omega_0 - \pi^\ast \mu_0)$ in every possible characteristic class by reduction of the star product $\star_0$ on the original phase space. The reason for this is that not every de Rham cohomology class in $H^2_{dR}(Q)$ is equal to a characteristic class, which is a direct consequence of the fact that the Chern-Weil homomorphism is not surjective, in general. Moreover, by our result it is clear that in general ‘quantization does not commute with reduction’. Consider for example $\mu_0 = 0$. Then the star product $\star_0^{J_0,\mu_0}$ is equivalent to an
intrinsically defined star product $\mathfrak{T}_0$ (starting from a torsion free connection $\nabla$ on $Q$), if and only if $[\mu, \lambda] = [0]$. Furthermore, we will see in the following section that there are even more conditions which have to be fulfilled in order to achieve that $\star_0^{j_0, \mu_+}$ equals $\mathfrak{T}_0$.

Starting with the star products $\star_0^j$ on $(T^* Q, \omega_{B_0})$ one can actually get representatives for every characteristic class of star products on $(T^* Q, \omega_B)$ by varying $B_+$ and the corresponding mappings $j_+$. This follows from the observation that $[b_+]$, where $b = b_0 + b_+$ with fixed $b_0$, corresponds to the formal series of $G$-equivariant cohomology classes defined by the pair $(B_+, j_+ - \mu_+)$ and the well-known fact that the $G$-equivariant cohomology of $Q$ (which is by definition the cohomology of the complex of basic differential forms on $Q$) is isomorphic to the de Rham cohomology of the quotient $\overline{Q} = Q/G$ (cf. [16]).

6 Applications and Examples

6.1 Reduction of $\star_\kappa$ and $\star_\kappa^B$

In this section we apply the reduction scheme developed in Section 3 to several concrete examples of star products on $(T^* Q, \omega_{B_0})$. As we want to identify the resulting reduced products on $(T^* Q, \omega_B)$ with naturally defined star products we will often make use of the fact that certain star products are determined by their representations. Hence we will construct representations of the reduced star products, which is also of independent interest because of some strong relations to the results of [11] [12] [13], where the quantization is formulated in terms of representations alone neglecting the algebra of observables.

As a first step towards the concrete computation of reduced star products derived from $\star_\kappa$ and $\star_\kappa^B$, we are going to establish a relation between the reduced star products obtained from different momentum values $\mu \in \mathfrak{g}^*G + \nu \mathfrak{g}_c^*G[[\nu]]$. This will allow us to restrict our further considerations to the case $\mu = 0$. In the following we assume that the connection used to define $\star_\kappa$ and $\star_\kappa^B$ is $G$-invariant; since the group action on $Q$ is assumed to be proper such a connection always exists. Moreover, we assume that $B \in Z^2_{dR}(Q)^G[[\nu]]$. In case $\kappa \neq 0, 1$ we finally assume that $D(\phi^*_\kappa \alpha - \alpha) = 0$. Note that one can even achieve $\phi^*_\kappa \alpha = \alpha$ using a different $G$-invariant volume density for the definition of $\alpha$, namely the Riemannian volume corresponding to an invariant Riemannian metric on $Q$: if $\nabla$ is the pertinent Levi Civita connection one even has $\alpha = 0$. For reduction of $\star_\kappa$ we always use the canonical $G$-equivariant classical momentum map with $J_\mu(\xi) = P(\xi_Q)$ as quantum momentum map $J^0$. By Remark [11] this causes no loss of generality. For reduction of the products $\star_\kappa^B$ we use a quantum momentum map of the form $J^B(\xi) = P(\xi_Q) + \pi^* j(\xi)$, where $d j(\xi) = i_{\xi_Q} B$ and $\phi^*_\kappa(\xi) = j(Ad(g^{-1})\xi)$.

In order to relate the reduced star products $\star_\kappa^{J^0, \mu}$ and $\star_\kappa^{B, \mu}$ for different momentum values with each other we are going to construct local isomorphisms between them. For $\mu \in \mathfrak{g}^*G + \nu \mathfrak{g}_c^*G[[\nu]]$ consider the formal series of one-forms $\Gamma_\mu = \langle \mu, \gamma \rangle \in \Gamma^\infty(T^*Q)^G[[\nu]]$. Clearly, there is a uniquely defined $b_\mu \in Z^2_{dR}(Q)[[\nu]]$ such that $p^* b_\mu = -d \Gamma_\mu = -\langle \mu, \lambda \rangle$. Now let $\{O_i\}_{i \in I}$ denote a good open cover of $Q$ such that $\overline{O_i} = p(O_i)$ is a good open cover of $\overline{Q}$. Over every $\overline{O_i}$ choose a local potential $a^i_\mu$ of $b_\mu$ which means $b_\mu|_{\overline{O_i}} = d a^i_\mu$ and consider the formal series of locally defined one-forms $A^i_\mu := \Gamma_\mu + p^* a^i_\mu \in \Gamma^\infty(T^*O_i)[[\nu]]$. Then the $A^i_\mu$ turn out to be closed, hence the operator $A^i_{\mu, \kappa}$ defined by Eq. (2.12) using $A^i_\mu$ instead of $A$ is a local automorphism of $\star_\kappa$ and also of $\star_\kappa^B$ by Lemma 2.4 ii.). Furthermore, $A^i_{\mu, \kappa}$ is $G$-invariant due to the invariance of the connection and the invariance of $A^i_\mu$. By its form it is clear that $A^i_{\mu, \kappa}$ preserves $P(O_i)[[\nu]]$. Moreover, an easy computation shows that $A^i_{\mu, \kappa}(P(\xi_Q)|_{T^*O_i}) = P(\xi_Q)|_{T^*O_i} - \langle \mu, \xi \rangle$ and $A^i_{\mu, \kappa}(P(\xi_Q) + \pi^* j(\xi))|_{T^*O_i} = (P(\xi_Q) + \pi^* j(\xi))|_{T^*O_i} - \langle \mu, \xi \rangle$ for all $\xi \in \mathfrak{g}$. By these observations and
Proposition 3.7 ii.) we obtain:

**Lemma 6.1** With notations from above, the mapping \( A^i_{\mu,\kappa} : h(\mathcal{P}(O_i))^G[[\nu]] \to h(\mathcal{P}(O_i))^G[[\nu]] \) defined by

\[
A^i_{\mu,\kappa} F := \mu_0 \left( \frac{id}{id - \nu \Delta_{\mu,\kappa}} A^i_{\mu,\kappa} F \right), \quad F \in h(\mathcal{P}(O_i))^G[[\nu]],
\]

(6.1)
yields a local isomorphism from \( h(\mathcal{P}(O_i))^G[[\nu]] \to h(\mathcal{P}(O_i))^G[[\nu]] \). Moreover, the corresponding operator \( A^i_{\mu,\kappa} := l^{-1} \circ A^i_{\mu,\kappa} \circ l \) is a local isomorphism from \( (\mathcal{P}(O_i))^G[[\nu]] \) to \( \mathcal{P}(O_i))^G[[\nu]] \) which coincides with \( A^i_{\mu,\kappa} \). This operator extends uniquely to a local isomorphism, also denoted by \( A^i_{\mu,\kappa} \), from \( C^\infty(T^*\mathcal{P}(O_i))^G[[\nu]] \) to \( C^\infty(T^*\mathcal{P}(O_i))^G[[\nu]] \) and \( C^\infty(T^*\mathcal{P}(O_i))^G[[\nu]] \) to \( C^\infty(T^*\mathcal{P}(O_i))^G[[\nu]] \). Together with the product \( \ast_{\mu,\kappa}^{J,0} \) resp. \( \ast_{\mu,\kappa}^{J,0} \) the local isomorphisms \( \{ A^i_{\mu,\kappa} \}_{i \in I} \) completely determine the product \( \ast_{\mu,\kappa}^{J,0} \) resp. \( \ast_{\mu,\kappa}^{J,0} \).

**Proof:** First we observe that translations along the fibre and the operators defined by Eq. (2.1) preserve \( h(\mathcal{P}(Q))^G[[\nu]] \). This implies that \( A^i_{\mu,\kappa} \) preserves \( h(\mathcal{P}(Q))^G[[\nu]] \), therefore the contributions involving \( \Delta_{\mu,\kappa} \) vanish. Observing that \( \mu_0 F = F' \) for all \( F' \in h(\mathcal{P}(Q))^G[[\nu]] \) implies \( A^i_{\mu,\kappa} F = A^i_{\mu,\kappa} F \) for \( F \in h(\mathcal{P}(Q))^G[[\nu]] \). A similar argument shows that the operator analogous to \( A^i_{\mu,\kappa} \) which is obtained by replacing \( \ast_{\mu,\kappa} \) by \( \ast_{\mu,\kappa}^{J,0} \) by \( J^B \), and \( -\mu_0 \) by \( J^0 + \mu_0 \) also coincides with \( A^i_{\mu,\kappa} \). Last, it remains to show that the product \( \ast_{\mu,\kappa}^{J,0} \) resp. \( \ast_{\mu,\kappa}^{J,0} \) is completely determined by \( \ast_{\mu,\kappa}^{J,0} \) resp. \( \ast_{\mu,\kappa}^{J,0} \) and the local isomorphisms \( \{ A^i_{\mu,\kappa} \}_{i \in I} \). In order to check this one just has to observe that for \( O_i \cap O_k \neq \emptyset \) the compositions \( (A^i_{\mu,\kappa})^{-1} \) define automorphisms of \( h(\mathcal{P}(O_i \cap O_k))^G[[\nu]] \). Therefore, \( \ast_{\mu,\kappa}^{J,0} \) resp. \( \ast_{\mu,\kappa}^{J,0} \) is globally defined by

\[
f_{\ast_{\mu,\kappa}^{J,0}} f'_{|T^*\mathcal{O}_i} = A^i_{\mu,\kappa}((A^i_{\mu,\kappa})^{-1} f_{|T^*\mathcal{O}_i}) \]

(6.2)

Clearly, the choice of \( J^0 \) and \( J^B \) in the above lemma causes no loss of generality since \( \ast_{\mu,\kappa}^{J,0} = \ast_{\mu,\kappa}^{J,0} \) if \( J'(\xi) = J'(\xi) + \tilde{\mu}, \xi, \) and \( \ast_{\mu,\kappa}^{J,0} = \ast_{\mu,\kappa}^{J,0} \) if \( J''(\xi) = J''(\xi) + \hat{\mu}, \xi, \) (cf. Remark 3.11). Therefore, the above result allows us to compute the star products obtained from \( \ast_{\mu,\kappa}^{J,0} \) by our reduction scheme in case we have at least determined one reduced star product explicitly for a special choice of the quantum momentum map and a special choice of the momentum value.

Now let us consider the standard ordered star product \( \ast_0 \) and the reduced product \( \ast_0^{J,0} \) more closely. Our goal is to find out whether the reduced star product \( \ast_0^{J,0} \) is again a standard ordered star product corresponding to a certain torsion free connection \( \nabla \) on \( \mathcal{Q} \).

**Lemma 6.2** i.) Let us assign to every \( f \in \mathcal{P}(\mathcal{Q})[[\nu]] \) a formal series \( \tilde{g}_0(f) \) of differential operators on \( \mathcal{C}^\infty(\mathcal{Q}) \) by the following relation:

\[
p^* \tilde{g}_0(f) \chi = g_0(l(f)) p^* \chi, \quad \chi \in \mathcal{C}^\infty(\mathcal{Q}).
\]

(6.3)

Then \( \tilde{g}_0 \) defines a representation of \( \mathcal{P}(\mathcal{Q})[[\nu]], \ast_0^{J,0} \) on \( \mathcal{C}^\infty(\mathcal{Q})[[\nu]] \) by \( \mathbb{C}[[\nu]] \)-linear extension.
ii.) If $\tilde{\varrho}_0$ coincides with the standard ordered representation $\overline{\varrho}_0$ with respect to some torsion free connection on $Q$, then $\star_0^{j_0,0}$ coincides with the standard ordered star product $\overline{\tau}_0$ corresponding to the connection $\nabla$ determined by

$$Tp \nabla_{\phi^h} = \nabla_{\phi} \circ p, \quad s,t \in \Gamma^\infty(TQ). \quad (6.4)$$

iii.) If $\star_0^{j_0,0}$ coincides with the standard ordered star product $\overline{\tau}_0$ on $(T^*Q, \varrho_0)$ corresponding to some torsion free connection on $Q$, then $\tilde{\varrho}_0$ coincides with the standard ordered representation $\overline{\varrho}_0$ with respect to the connection $\nabla$ determined by Eq. (6.4).

PROOF: By equivariance of $\varrho_0$ (see Eq. (4.1)) and by the fact that $l(f) \in h(\mathcal{P}(Q)^G) [[\nu]]$ the right-hand side of Eq. (6.3) is a $G$-invariant element of $C^\infty(Q) [[\nu]]$, therefore $\tilde{\varrho}_0(0) \in C^\infty(Q) [[\nu]]$ is well-defined by this equation, indeed. By the form of $\varrho_0(l(f))$ one concludes that $\tilde{\varrho}_0(0)$ is a formal series of differential operators, hence it can be extended to $C^\infty(Q) [[\nu]]$ by $\mathbb{C}[[\nu]]$-linearity. To prove that $\tilde{\varrho}_0$ is a representation of $(\mathcal{P}(Q), l_0, \star_0^{j_0,0})$ we let first note that $\varrho_0(F)p^*\chi = 0$ for all $F \in T^*_{0,x_0}$, since $\varrho_0$ is a representation of $\star_0$ and since $\varrho_0(J_0(J)) p^*\chi = -\nu L_{\xi_0} p^*\chi = 0$ for all $\xi \in \mathfrak{g}$. Using this observation and the definition of $\star_0^{j_0,0}$ one immediately verifies that $\tilde{\varrho}_0$ is a representation. This proves i.). Let us consider ii.). In case $\tilde{\varrho}_0$ coincides with the standard ordered representation $\overline{\varrho}_0$ with respect to some torsion free connection on $Q$ the star product $\star_0^{j_0,0}$ coincides with the corresponding standard ordered star product $\overline{\tau}_0$, since the representation completely determines the star product (cf. Section 2.1.). Moreover, considering $\tilde{\varrho}_0(0, (s \vee t)) \chi = \overline{\varrho}_0(0, (s \vee t)) \chi$ for $s,t \in \Gamma^\infty(TQ)$ one finds that the torsion free connection used to define $\overline{\varrho}_0$ is uniquely determined and is given by $\overline{\nabla}$ as in Eq. (6.3). For the proof of iii.) assume that $\star_0^{j_0,0} = \overline{\tau}_0$, where $\overline{\tau}_0$ is obtained from some torsion free connection on $Q$. Using the definition of $\tilde{\varrho}_0$ it is immediate to verify that $\tilde{\varrho}_0(f) = \overline{\varrho}_0(f)$ for all $f \in \mathcal{P}^0(Q) \oplus \mathcal{P}_1(Q)$; note that the torsion free connection used to define $\overline{\varrho}_0$ is of no importance, hereby. Let us now assume that $\tilde{\varrho}_0(f) = \overline{\varrho}_0(f)$ for all $f \in \bigoplus_{k=0}^{\infty} \mathcal{P}^k(Q)$ and consider $\mathcal{P}(x_1 \vee \ldots \vee x_{r+1})$ with $x_j \in \Gamma^\infty(TQ)$. By Lemma 3.10 we know that $\star_0^{j_0,0}$ is a homogeneous star product. Therefore $\mathcal{P}(x_1) \star_0^{j_0,0} \ldots \star_0^{j_0,0} \mathcal{P}(x_{r+1}) = \mathcal{P}(x_1 \vee \ldots \vee x_{r+1}) + \sum_{l=1}^{r+1} \nu f_l = \mathcal{P}(x_1) \overline{\varrho}_0 \ldots \overline{\varrho}_0 \mathcal{P}(x_{r+1})$, where $f_l \in \mathcal{P}^{r+1-l}(Q)$. Furthermore, we have $\tilde{\varrho}_0(\mathcal{P}(x_1) \star_0^{j_0,0} \ldots \star_0^{j_0,0} \mathcal{P}(x_{r+1})) = \overline{\varrho}_0(\mathcal{P}(x_1) \overline{\varrho}_0 \ldots \overline{\varrho}_0 \mathcal{P}(x_{r+1}))$ by the representation properties and the fact that $\tilde{\varrho}_0(\mathcal{P}(x_j)) = \overline{\varrho}_0(\mathcal{P}(x_j))$. Using the above expression for $\mathcal{P}(x_1) \star_0^{j_0,0} \ldots \star_0^{j_0,0} \mathcal{P}(x_{r+1})$, this equation implies that $\tilde{\varrho}_0(\mathcal{P}(x_1 \vee \ldots \vee x_{r+1})) = \overline{\varrho}_0(\mathcal{P}(x_1 \vee \ldots \vee x_{r+1}))$. By an induction argument we then conclude that $\tilde{\varrho}_0$ coincides with $\overline{\varrho}_0$ on $\mathcal{P}(Q)$, hence $\tilde{\varrho}_0 = \overline{\varrho}_0$. Like for ii.) it then follows that the connection used to define $\overline{\varrho}_0$ is given by $\overline{\nabla}$ as in Eq. (6.4). \[ \square \]

Unfortunately, it is in general not true that $\tilde{\varrho}_0$ equals $\overline{\varrho}_0$ by the following equality:

$$p^*(\tilde{\varrho}_0 - \overline{\varrho}_0)(\mathcal{P}(x_1 \vee x_2 \vee x_3)) \chi = \frac{(-\nu)^3}{3} \sum_{\sigma \in S_3} \sum_{i=1}^{\dim(G)} \Gamma_{\sigma}(\nabla_{x_1^{h_{\sigma(1)}}} x_2^{h_{\sigma(2)}}) (dp^* \chi)(\nabla_{x_3^{h_{\sigma(3)}}} e_{iQ}).$$

In fact, the analysis of the condition $\tilde{\varrho}_0 = \overline{\varrho}_0$ turns out to be rather involved, but at least we can give two conditions which guarantee that this equality holds true and then $\star_0^{j_0,0} = \overline{\tau}_0$.

Lemma 6.3 If the connection $\nabla$ satisfies either

$$\nabla_X V \in \Gamma^\infty(VQ)^G \quad \text{for all } X \in \Gamma^\infty(TQ)^G, \quad V \in \Gamma^\infty(VQ)^G, \quad (6.5)$$

or

$$\nabla_{x^{h}} x^{h} \in \Gamma^\infty(HQ)^G \quad \text{for all } x, y \in \Gamma^\infty(TQ), \quad (6.6)$$

the representation $\tilde{\varrho}_0$ coincides with the standard ordered representation $\overline{\varrho}_0$ with respect to the torsion free connection $\overline{\nabla}$ defined by Eq. (6.4).
The claim follows immediately from Lemma 6.2 and Lemma 6.3.

Standard ordered quantization commutes with reduction. Symbolically we have the following commu-
tation and

\[ \rho_i(x_1) \ldots \rho_i(x_k) = \left(\begin{array}{c} (-\nu)^k \k! \rho_i(x_1) \ldots \rho_i(x_k) D^k \rho_i \end{array}\right). \]

This implies that \( \tilde{\rho}_0(\mathcal{P}(x_1 \vee \ldots \vee x_k)) = \left(\begin{array}{c} (-\nu)^k \k! \rho_i(x_1) \ldots \rho_i(x_k) D^k \rho_i \end{array}\right). \) The last expression now is the standard ordered representation of \( \mathcal{P}(x_1 \vee \ldots \vee x_k) \) with respect to the connection \( \nabla \). This proves the lemma. Finally, let us note that if Eq. (6.5) is satisfied, we moreover have \( (\nabla^k \rho_i)(Y_1, \ldots, Y_k) = 0 \) for \( Y_1, \ldots, Y_k \in \Gamma^\infty(TQ)^G \) in case at least one \( Y_i \) is vertical.

As a direct consequence of the lemma we obtain:

**Proposition 6.4** Let \( *_0 \) be the standard ordered star product on \( (T^*Q, \omega_0) \) obtained from a \( G \)-invariant torsion free connection \( \nabla \) on \( Q \) which satisfies one of the conditions (6.5), (6.6). Then the reduced star product \( *_0^{j^0,0} \) on \( (T^*\overline{Q}, \overline{\rho}_0) \) coincides with the standard ordered star product \( \overline{\rho}_0 \) corresponding to the connection \( \overline{\nabla} \) on \( \overline{Q} \) defined by Eq. (6.4).

**Proof:** The claim follows immediately from Lemma 6.2 and Lemma 6.3.

In other words the preceding proposition just means that, using appropriate connections, standard ordered quantization commutes with reduction. Symbolically we have the following commutative diagram:

\[
\begin{array}{ccc}
(C^\infty(T^*Q), \{, \})_{q_0(\nabla)} & \to & (C^\infty(T^*Q)[[\nu]], *_0) \\
\downarrow \mathcal{R}(j^0,0,\cdot) & & \downarrow \mathcal{R}(j^0,0,*_0) \\
(C^\infty(T^*\overline{Q}), \{, \})_{q_0(\overline{\nabla})} & \to & (C^\infty(T^*\overline{Q})[[\nu]], *_0^{j^0,0} = \overline{\rho}_0).
\end{array}
\] (6.7)

Hereby, \( q_0(\nabla) \) resp. \( q_0(\overline{\nabla}) \) denotes the standard ordered quantization using the respective connection and \( \mathcal{R}(j^0,0,\cdot) \) resp. \( \mathcal{R}(j^0,0,*_0) \) denotes classical resp. quantum reduction using the indicated momentum map, momentum value and associative product. Note that the condition expressed by Eq. (6.5) is rather restrictive, since it particularly implies, by using that \( \nabla \) is torsion free, that the horizontal distribution has to be integrable, hence the principal connection corresponding to \( \gamma \) has to be flat. In contrast, the next lemma shows that given a \( G \)-invariant torsion free connection \( \nabla \), we can always find another \( G \)-invariant torsion free connection \( \nabla' \) which satisfies Eq. (6.5) and even induces the same connection \( \nabla \) on \( Q \).

**Lemma 6.5** Let \( \nabla \) denote a torsion free \( G \)-invariant connection on \( Q \). Define \( \nabla' \) by

\[
\nabla_H H' := \nabla_H H', \quad \nabla_V H := \nabla_V H - H(\nabla_H V), \quad \nabla_H V := \nabla_H V - H(\nabla_V V'), \quad \nabla_V V' := \nabla_V V' - H(\nabla_V V'), \tag{6.8}
\]

where \( H, H' \in \Gamma^\infty(H^0) \) and \( V, V' \in \Gamma^\infty(V^0) \). Then \( \nabla' \) is a torsion free \( G \)-invariant connection on \( Q \) such that the induced connection on \( \overline{Q} \) coincides with the one induced by \( \nabla \) and such that \( \nabla_X V \in \Gamma^\infty(V^0) \) for all \( X \in \Gamma^\infty(TQ), V \in \Gamma^\infty(V^0) \). Any other connection \( \hat{\nabla} \) satisfying these conditions as well is of form \( \hat{\nabla}_X Y = \hat{\nabla}_X Y + S(X, Y) \) with \( S \in \Gamma^\infty(V^0 Q \otimes V^0 Q)^G \).
γ is a torsion free ∇ product

Lemma 6.6 Let ∇ denote a torsion free G-invariant connection on Q and assume that the principal connection corresponding to γ is flat. Define ∇ by

\[ \tilde{\nabla}_H H' := H(\nabla_H H'), \quad \tilde{\nabla}_Y Y' := \nabla_Y Y', \]

where \( H, H' \in \Gamma^\infty(HQ) \) and at least one of the vector fields \( Y, Y' \in \Gamma^\infty(TQ) \) is vertical. Then \( \tilde{\nabla} \) is a torsion free G-invariant connection on \( Q \) such that the induced connection on \( Q' \) coincides with the one induced by \( \nabla \) and such that \( \tilde{\nabla}_H H' \in \Gamma^\infty(HQ) \) for all \( H, H' \in \Gamma^\infty(HQ) \). Any other connection \( \tilde{\nabla} \) satisfying these conditions as well is of form \( \tilde{\nabla}_X Y = \nabla_X Y + S(X,Y) \) with \( S \in \Gamma^\infty(\sqrt{2} T^*Q \otimes TQ)^G \) satisfying \( S(H,H') = 0 \) for all \( H, H' \in \Gamma^\infty(HQ) \).

Proof: Again the proof is straightforward; the only crucial point to observe is that the flatness of the principal connection implies \( \tilde{\nabla}_H H' = \nabla_H H' \). Moreover, the condition that \( \tilde{\nabla} \) induces the same connection like \( \tilde{\nabla} \) and that it satisfies \( \tilde{\nabla}_H H' \in \Gamma^\infty(HQ) \) for all \( H, H' \in \Gamma^\infty(HQ) \) entails that \( S(H,H') \) has to be both vertical and horizontal. Thus \( S(H,H') \) has to vanish for all \( H, H' \in \Gamma^\infty(HQ) \).

To conclude our study of the reduced star product \(*_0^{J_0,0} \) let us mention that one can also use the relation between the standard ordered representation \( g_0 \) and a symbolic calculus for pseudo-differential operators on \( C^\infty(Q) \) (see [7, Sect. 6] and [8, Sect. 10]) to obtain the above ‘reduction commutes with quantization’ result. For details about reduction of star products in this framework we refer the interested reader to the thesis [10].

Now we consider the reduction of the products \(*_\kappa \) with \( \kappa \neq 0 \). These investigations will turn out to be slightly more involved. First of all let us recall that the reduction scheme introduced in Section 3 works for all the star products \(*_\kappa \) under the general assumption that the connection ∇ is G-invariant and that \( D(\phi^*_g \alpha - \alpha) = 0 \) holds true additionally, if \( \kappa \neq 0,1 \). We will now show that without additional assumptions the reduced star products \(*_\kappa^{J_0,0} \) for \( \kappa \neq 0 \) are in general not even equivalent to \(*_0^{J_0,0} \). This destroys the expectation that \(*_\kappa^{J_0,0} \) could coincide with some naturally defined star product \( \tilde{r}_\kappa \) (cf. Section 6.2 for a concrete example in case \( \kappa = 1/2 \)).

Lemma 6.7 For all \( \kappa \in [0,1] \) the G-invariant star product \(*_\kappa \) is G-equivalent to \(*_0^{B_\kappa} \) with \( B_\kappa = k\nu tr(R) \). Hence, \(*_\kappa \) is G-equivalent to \(*_0 \), if and only if \([B_\kappa]_G = [0]_G \). Consequently, the characteristic class of \(*_\kappa^{J_0,0} \) is given by

\[ c(*_\kappa^{J_0,0}) = -\kappa[\overline{\nu}]_r = -\kappa[\overline{\nu}]_\lambda, \]

where we have used the notation of Lemma 7.5.

Proof: Consider the operator defined by Eq. (2.42) using \( A_\kappa = -\kappa\nu \alpha \) instead of \( A \) and denote it by \( A_0^{B_\kappa} \). Clearly, \( A_0^{B_\kappa} \) is an equivalence transformation from \(*_0 \) to \(*_0^{B_\kappa} \), where \( B_\kappa = -\kappa\nu \alpha = k\nu tr(R) \). Therefore, \( A_0^{B_\kappa} N_\kappa \) defines an equivalence transformation from \(*_{\kappa} \) to \(*_0^{B_\kappa} \). Using Eq. (2.45), a straightforward computation shows \( A_0^{B_\kappa} N_\kappa = \exp\left( F \sum_{r=2}^{\infty} \frac{1}{r}(\kappa^r - \kappa)(-\nu)^{r-1}\lambda \right) \exp(-\kappa\nu\Delta_0) \). For \( \kappa = 0 \) and \( \kappa = 1 \) we have \( A_0^{B_0} N_0 = id \) and \( A_0^{B_1} N_1 = \exp(-\nu\Delta_0) \), which are both G-invariant operators. For \( \kappa \neq 0,1 \) the operator \( A_0^{B_\kappa} N_\kappa \) is G-invariant as well due to the additional condition \( D(\phi^*_g \alpha - \alpha) = 0 \). This proves that \(*_\kappa \) is G-equivalent to \(*_0^{B_\kappa} \). Thus, by Proposition 14.13 the star product \(*_\kappa \) is G-equivalent to \(*_0 \), if and only if \([B_\kappa]_G = [0]_G \). The result for the characteristic
class of $\star_{\kappa}^{J_0,0}$ is an immediate consequence of Theorem 5.3 and the observation that the equations $J_\kappa(\xi) = i^* A_0^R N_\kappa J^0(\xi) = -\kappa \nu \div (\xi_\kappa)$ and $p^* b_\kappa = B_\kappa + d\Gamma_\kappa$ with $b_\kappa = -\kappa \nu r$ hold true.

Clearly, one could now change the momentum value to the one defined by $(\mu, \xi) = -\kappa \nu r (\ad(\xi))$. Thus one could achieve $\mathrm{cl}(\star_{\kappa}^{J_0,0}) = [0]$, but the result would not be a naturally defined star product $\star_\kappa$ for $\kappa \neq 0$. Instead, we remain at the choice of 0 momentum value and merely adjust the parameters entering the construction of $\star_\kappa$ in order to obtain a star product in the desired equivalence class. Henceforth, we thus assume that the volume density $v$ is $G$-invariant. Consequently, the one-form $\alpha$ defined in Eq. (2.4) is also $G$-invariant. Then $N_\kappa$ is a $G$-equivalence from $\star_\kappa$ to $\star_0$, implying that $\star_{\kappa}^{J_0,0}$ is equivalent to $\star_0^{J_0,0}$, where $J_\kappa(\xi) = N_\kappa J^0(\xi) = J^0(\xi) - \kappa \nu r \star (\div (\xi_\kappa) + \alpha(\xi_\kappa))$.

But by $G$-invariance of $v$ we have $L_{\xi_\kappa} v = 0$, which entails $\div (\xi_\kappa) + \alpha(\xi_\kappa) = 0$ by definition of $\alpha$. Hence $\star_0^{J_0,0} = \star_0^{J_0,0}$. Analogously to Lemma 6.2 we now obtain:

**Lemma 6.8** Assume that $\kappa \neq 0$ and that the volume density $v$ is $G$-invariant. Then assign to every $f \in \mathcal{P}(\Omega)([\nu])$ a formal series $\tilde{\varrho}_\kappa(f)$ of differential operators on $C^\infty(\Omega)$ by

$$p^* \tilde{\varrho}_\kappa(f) \chi = \varrho_\kappa(l(f)) p^* \chi \quad \text{for all } \chi \in C^\infty(\Omega).$$

Then $\tilde{\varrho}_\kappa$ gives rise to a representation of $\mathcal{P}(\Omega)([\nu])$ on $C^\infty(\Omega)([\nu])$ by $C[[\nu]]$-linear extension.

**Proof:** The proof of the claim is along the lines of the proof of Lemma 6.2. The only additional relation which should be noted for the proof of the representation property is the equality $\varrho_\kappa(J^0(\xi)) = \varrho_0(N_\kappa J^0(\xi)) = -\nu L_{\xi_\kappa}$ which holds by $G$-invariance of $v$.

In order to interpret certain star products $\star_{\kappa}^{J_0,0}$ as naturally defined star products $\varpi_\kappa$ we need a further condition which guarantees that the volume density $v$ induces a volume density $\varpi$ on $\Omega$. The function $v(e_{1Q}, \ldots, e_{\dim(G)} x_1^h, \ldots, x_{n-\dim(G)}^h)$ turns out to be $G$-invariant, if the group $G$ is unimodular. Unimodularity hereby means that $|\det(\Ad(g))| = 1$ for all $g \in G$, whence $\tr (\ad(\xi)) = 0$ for all $\xi \in \mathfrak{g}$. With the additional assumption of $G$ to be unimodular we can define a volume density $\varpi$ on $\Omega$ by

$$p^* (\varpi(x_1, \ldots, x_{n-\dim(G)})) = v(e_{1Q}, \ldots, e_{\dim(G)} x_1^h, \ldots, x_{n-\dim(G)}^h).$$

Evidently, the so-defined volume density $\varpi$ depends on the chosen basis $\{e_i\}_{1 \leq i \leq \dim(G)}$ of $\mathfrak{g}$. But the choice of a different basis $\{e'_i\}_{1 \leq i \leq \dim(G)}$ yields a volume density $\varpi' = a \varpi$ with $a \in \mathbb{R}^+$. Therefore, the one-forms $\varpi, \varpi'$ defined by $\nabla_x \varpi = \varpi(x) \varpi$ and $\nabla_x \varpi' = \varpi'(x) \varpi'$ with $x \in \Gamma^\infty(T\Omega)$ coincide. Hence, the corresponding $\kappa$-ordered star products also coincide and are independent of the above choice of a basis of $\mathfrak{g}$.

**Lemma 6.9** Assume that $\kappa \neq 0$, that the volume density $v$ is $G$-invariant and that $G$ is unimodular. Then $\tilde{\varrho}_\kappa$ coincides with the $\kappa$-ordered representation $\varpi_\kappa$ induced by the connection $\nabla$ defined by Eq. (6.4) and the volume density $\varpi$ defined by Eq. (6.12), if and only if $\star_{\kappa}^{J_0,0}$ coincides with the $\kappa$-ordered star product $\varpi_\kappa$ corresponding to $\nabla$ and the volume density $\varpi$.

**Proof:** First we note that $\tilde{\varrho}_\kappa(f) = \varpi_\kappa(f)$ for all $f \in \mathcal{P}(\Omega) \oplus \mathcal{P}(\Omega)$, where $\varpi_\kappa$ is defined using the connection $\nabla$ and the one-form $\varpi = \Gamma^\infty(T\Omega)$ determined by $p^* \varpi = H(\alpha + W)$. Here, the one-form $W$ is given by $W(X) = \sum_{i=1}^{\dim(G)} \Gamma_{e_i}(\nabla_X e_i)$ (cf. Lemma 5.3). In order to interpret $\varpi$ as the one-form defined by $\nabla$ and some volume density $\varpi$ we must necessarily have $d(\varpi) = -\tr(R)$. Using the results and notation of Lemma 5.3 it is easy to compute that $d(\varpi) = \tau_\lambda - \tr(R)$, since $r = d(\varpi - w)$. Actually, this relation suggests to assume that the Lie group $G$ is unimodular, since then $\tau_\lambda = 0$. In this case, one finds $\nabla_x \varpi = \varpi(x) \varpi$ for all $x \in \Gamma^\infty(T\Omega)$. Furthermore, it turns out
that $\tilde{\partial}_\kappa(f) = \overline{\partial}_\kappa(f)$ also holds for all $f \in \mathcal{P}^2(Q)$ without any further conditions on the connection $\nabla$. With these observations, the proof of the claim is completely analogous to the one of Lemma 6.2 ii.) and iii.).

Like in the case $\kappa = 0$, the operator $\tilde{\partial}_\kappa(f)$ coincides with $\overline{\partial}_\kappa(f)$ without any further conditions on $\nabla$, if $f \in \bigoplus_{k=0}^2 \mathcal{P}^k(Q)$, but for polynomials in the momenta of higher degree one does not have $\tilde{\partial}_\kappa(f) = \overline{\partial}_\kappa(f)$, in general. Fortunately, the conditions imposed on $\nabla$ in case $\kappa = 0$ turn out to be also guarantee that the reduced star product $\ast_{\kappa}^{j,0}$ coincides with $\overline{\partial}_\kappa$ defined by $\nabla$ and $\overline{\nabla}$ resp. $\overline{\kappa}$.

**Proposition 6.10** For $\kappa \neq 0$ let $\ast_{\kappa}$ be the $\kappa$-ordered star product on $(T^*Q, \omega)$ obtained from a $G$-invariant volume density $\nu$ and a $G$-invariant torsion free connection $\nabla$ on $Q$ which satisfies one of the conditions (6.7), (6.6). If $G$ is unimodular, then the reduced star product $\ast_{\kappa}^{j,0}$ on $(T^*\overline{Q}, \overline{\omega})$ coincides with the $\kappa$-ordered star product $\overline{\partial}_\kappa$ which corresponds to the connection $\nabla$ on $\overline{Q}$ defined by Eq. (6.4) and to the volume density $\overline{\nu}$ determined by Eq. (6.12).

**Proof:** By Lemma 6.9 we only have to prove that each of the conditions (6.7), (6.6) implies $\dot{\partial}_\kappa = \overline{\partial}_\kappa$. First, we consider the case, where condition (6.7) is satisfied. We have to determine $\dot{\partial}_\kappa (l(\overline{\nabla}(x_1 \ldots \ldots x_k)))$ for all $\nu \in \mathcal{P}(Q)^G$.

Using the explicit form of the operator $\Delta$ it is easy to find (cf. [3, Eq. (1.3)])

$$\Delta P \left( x_1^h \ldots \ldots x_k^h \right) = P \left( \sum_{l=1}^k x_1^h \ldots \ldots x_{l-1}^h (\text{div}(x_1^h) + \alpha(x_1)) x_{l+1}^h \ldots \ldots x_k^h \right)$$

where $\ldots$ denotes omission of the $j$th term. Using the definition of $\overline{\nabla}$ and $\overline{\kappa}$, it is straightforward to show that $\text{div}(x_1^h) + \alpha(x_1) = p^* \partial(x_1) + \overline{\alpha}(x_1))$. Furthermore, the assumption about the connection $\nabla$ implies $\nabla_{x_j} x_i^h = (\overline{\nabla}_{x_j} x_i)^h$. Putting these formulas together we get $\Delta(l(\overline{\nabla}(x_1 \ldots \ldots x_k))) = l(\overline{\nabla}(\overline{\nabla}(x_1 \ldots \ldots x_k)))$. Where the differential operator $\overline{\nabla}$ on $C^\infty(T^*\overline{Q})$ is defined completely analogously to $\Delta$ using the connection $\nabla$ and the one-form $\overline{\kappa}$. Defining $\overline{\nu} := \exp(-\kappa/\overline{\nabla})$ we obtain by induction that $\overline{\nabla}_\kappa := \exp(-\kappa/\overline{\nabla})$ for all $f \in \mathcal{P}(Q)^G$. Using Lemma 6.3 we then get $p^* \tilde{\partial}_\kappa(f) \chi = q_0 (N_\kappa(l(f))) \chi = q_0 \left( \left( \frac{\overline{\nabla}_{x_j} x_i^h}{(\nabla_{x_j} x_i)^h} \right) \right) \chi$, which says that $\tilde{\partial}_\kappa$ coincides with $\overline{\partial}_\kappa$. Now we consider the case, where condition (6.6) is satisfied. As a first step we will show that $h(\Delta F) = l(\overline{\Delta}(l^{-1}(h(F))))$ for all $F \in \mathcal{P}(Q)^G$. To end this consider $F = P \left( V_1 \ldots \ldots V_r \right) \in \Gamma^\infty(VQ)^G$. After application of $\Delta$ to $F$ several types of terms appear according to the above formula. The terms involving $\text{div}(V_i) + \alpha(V_i)$ vanish, since $\text{div}(V_i) + \alpha(V_i) = 0$ by unimodularity of $G$ and the fact that $\text{div}(\xi_0) + \alpha(\xi_0) = 0$. Moreover, as in the first part of the proof we have $\text{div}(x_1^h) + \alpha(x_1^h) = p^* \partial(x_1^h)$. From the second sum in the above formula four types of terms arise, namely those involving $\nabla_{x_j} V_j$, $\nabla_{x_i} V_j$, $\nabla_{x_i} x_j^h$, and $\nabla_{x_j} x_i^h$. By the assumption on the connection it is evident that $\nabla_{x_j} V_j$, $\nabla_{x_i} V_j$, and $\nabla_{x_i} x_j^h$ are all vertical, hence these contributions vanish after projection to the total horizontal part. Projecting to the horizontal part we may also replace $\nabla_{x_j} x_i^h$ by $\overline{\nabla}_{x_j} x_i^h$. Combining these results we get $h(\Delta F) = 0$, if $r \geq 1$, and $h(\Delta F) = l(\overline{\Delta}(l^{-1}(h(F))))$, if $r = 0$. But this implies $h(\Delta F) = l(\overline{\Delta}(l^{-1}(h(F))))$ for all $F \in \mathcal{P}(Q)^G$, since these are sums of terms of form $P \left( V_1 \ldots \ldots V_r \right) \in \Gamma^\infty(VQ)^G$. By induction, this implies that $h(\Delta^k F) = l(\overline{\Delta}(l^{-1}(h(F))))$ for all $k \in \mathbb{N}$. Finally, one has to observe that $q_0 (P \left( Y_1 \ldots \ldots Y_r \right) \ast_{\kappa}^{j,0}) p^* \chi = 0$ for $Y_1, \ldots, Y_r \in \Gamma^\infty(TQ)^G$ in case at least one of the $Y_i$ is
vertical (cf. proof of Lemma 6.3). But then one finds  
\[ p^* \tilde{g}_\kappa(f) \chi = \sum_{k=0}^{\infty} (-\kappa \nu)^k \varrho_0 \left( h(\Delta^k l(f)) \right) p^* \chi = \sum_{k=0}^{\infty} (-\kappa \nu)^k \varrho_0 \left( l(\Delta^k f) \right) p^* \chi = \sum_{k=0}^{\infty} (-\kappa \nu)^k \varrho_0 \left( l(\Delta^k f) \right) p^* \chi = p^* \tau_0 (N_\kappa f) \chi = p^* \tau_\kappa (f) \chi. \] 
This shows that  \( \tilde{g}_\kappa = \tau_\kappa. \)  
\[ \square \]

Symbolically, the above proposition can be expressed by a commutative diagram analogous to the one for  \( \kappa = 0. \) More precisely, for  \( \kappa \in (0,1] \) and the appropriate connections and volume densities the  \( \kappa \)-ordered quantization commutes with reduction, if  \( G \) is unimodular:

\[ (C^\infty(T^*Q), \{ \nu \}) \xrightarrow{Q_\kappa(\nabla,v)} (C^\infty(T^*Q)[[\nu]], \star_\kappa) \]
\[ \downarrow R(J_0,0,\cdot) \]
\[ (C^\infty(T^*Q), \{ \nu \}) \xrightarrow{Q_\kappa(\nabla,v)} (C^\infty(T^*Q)[[\nu]], \star_{\kappa J_0,0} = \tau_\kappa). \]

Hereby,  \( Q_\kappa(\nabla,v) \) resp.  \( Q_\kappa(\nabla,v) \) denotes the  \( \kappa \)-ordered quantization using the respective connection and the respective volume density, and  \( R(J_0,0,\cdot) \) resp.  \( R(J_0,0,\kappa) \) denotes classical resp. quantum reduction using the indicated momentum map, momentum value and associative product.

**Remark 6.11** At this point let us mention that the result of Proposition 6.16 does not contradict Thm. 4, where it was shown that for the special case of a certain  \( G \)-invariant Riemannian connection  \( \nabla^g \) the horizontal distribution has to be integrable in order to achieve  \( \tilde{g}_{1/2} = \tilde{v}_{1/2}. \) In contrast, our result shows that the, in general, weaker condition [6.3] suffices for the equality  \( \tilde{g} = \tilde{v} \) to hold. The reason for this peculiarity is that in the special case, where the  \( G \)-invariant Riemannian metric  \( g \) has been chosen such that  \( g(V,H) = 0 \) for all  \( V \in \Gamma^\infty(VQ) \) and all  \( H \in \Gamma^\infty(HQ) \), condition [6.3] implies that [6.4] has to be satisfied, too. To check this, one observes  \( g(\nabla^g_{x,y} V, y^h) = -g(V, \nabla^g_{x,y} y^h). \) But this term has to vanish for all  \( V \in \Gamma^\infty(VQ)^G \) due to Eq. (6.5) and the orthogonality of vertical and horizontal vector fields. This implies that  \( \nabla^g_{x,y} V \in \Gamma^\infty(HQ)^G \) for all  \( x,y \in \Gamma^\infty(T\overline{Q}), \) i.e. (6.4) holds true.

After the investigation of the reduced star products obtained from  \( \star_\kappa \), we finally consider the reduced star products  \( \star_{\kappa J_0,0} \) obtained from  \( \star^B \). Having identified the products  \( \star_{\kappa J_0,0} \) we now want to relate these products to  \( \star_{\kappa J_0,0} \) by means of local isomorphisms. In fact, the main steps for the construction of these isomorphisms have already been achieved in Section 5 where we have considered the case  \( \kappa = 0. \) Like in Eq. (5.14) let us now define  \( b_{B,j} \in Z^2_{dR}(\overline{Q})[[\nu]] \) by  \( p^* b_{B,j} = B + \Gamma_{j}. \) Denote by  \( \{ O_i \}_{i \in I} \) a good open cover of  \( Q \) as in Section 5 and by  \( \{ O_i \}_{i \in I} \) the corresponding good open cover of  \( \overline{Q} \). Then we obtain formal local one-forms  \( A^i_{B,j} = p^* a^i_{B,j} \) on the  \( O_i \), where  \( a^i_{B,j} \) denotes a local potential of  \( b_{B,j} \) on  \( O_i \). These formal local one-forms induce local isomorphisms  \( A^i_{B,j,\kappa} : (C^\infty(T^*O_i)[[\nu]], \star_\kappa) \to (C^\infty(T^*O_i)[[\nu]], \star^B_{\kappa}) \) as defined by Eq. (2.12). With these preparations we now get:

**Lemma 6.12** With notations from above, the mapping  \( A^i_{B,j,\kappa} : h(\mathcal{P}(O_i)^G)[[\nu]] \to h(\mathcal{P}(O_i)^G)[[\nu]] \) defined by  
\[ A^i_{B,j,\kappa} F := h^\nu \left( \frac{id}{id - \nu \Delta_{0,\kappa}} A^i_{B,j,\kappa} F \right), \quad F \in h(\mathcal{P}(O_i)^G)[[\nu]], \]
\[ \text{yields a local isomorphism from } (h(\mathcal{P}(O_i)^G)[[\nu]], \star_{\kappa J_0,0}) \to (h(\mathcal{P}(O_i)^G)[[\nu]], \star^B_{\kappa J_0,0}) \text{ which fulfills } \]
\[ A^i_{B,j,\kappa} F = A^i_{B,j,\kappa} F \quad \text{for all } F \in h(\mathcal{P}(O_i)^G)[[\nu]]. \]
The induced mapping $A^i_{B,j,k} := l^{-1} \circ A^i_{B,j,k} \circ l$ is a local isomorphism from $(P(\Omega))[[\nu]], \ast^B_i, j^B_0, 0)$ to $(P(\Omega))[[\nu]], \ast^B_{i,j,k})$. It extends uniquely to a local isomorphism from $(C^\infty(T^*\Omega))[[\nu]], \ast^B_i, j^B_0, 0)$ to $(C^\infty(T^*\Omega))[[\nu]], \ast^B_{i,j,k})$ which will be denoted by $A^i_{B,j,k}$ as well.

**Proof:** The claim is evident by Proposition 3.8 ii.), since $A^i_{B,j,k} \circ (\xi_\Omega) |_{T^*\Omega} = (P(\xi_\Omega) + \pi^*j(\xi)) |_{T^*\Omega}$ and since $A^i_{B,j,k}$ preserves $(P(O) \Gamma)^G[[\nu]]$.

In the following lemma we give sufficient conditions which allow for a concrete computation of the above local isomorphisms $\{A^i_{B,j,k}\}_{i \in I}$.

**Lemma 6.13**

i.) Assume that $\nabla$ satisfies condition 6.5 and that $B \in Z^2_{dr}(Q)^G[[\nu]]$ is horizontal so that we can choose $j = 0$; this means in particular that $\ast^B_i$ is strongly $G$-invariant. Then the local isomorphism $A^i_{B,0,k} : (C^\infty(T^*\Omega))[[\nu]], \ast^B_i, j^B_0, 0) \to (C^\infty(T^*\Omega))[[\nu]], \ast^B_{i,j,k})$ is given by

$$A^i_{B,0,k} = t^*_{-\ast^i_{B,0,0}} \exp \left( -F \left( \frac{\exp(\nu \mathfrak{D}) - \exp((\kappa - 1)\nu \mathfrak{D})}{\nu \mathfrak{D}} a^i_{B,0,0} - (a^i_{B,0,0})_0 \right) \right),$$

(6.16)

where $\mathfrak{D}$ denotes the operator of symmetric covariant derivation with respect to $\nabla$.

ii.) Assume that the $G$-invariant torsion free connection $\nabla$ satisfies condition 6.6. Then the local isomorphism $A^i_{B,j,k} : (C^\infty(T^*\Omega))[[\nu]], \ast^B_i, j^B_0, 0) \to (C^\infty(T^*\Omega))[[\nu]], \ast^B_{i,j,k})$ is given by

$$A^i_{B,j,k} = t^*_{-\ast^i_{B,j,k}} \exp \left( -F \left( \frac{\exp(\nu \mathfrak{D}) - \exp((\kappa - 1)\nu \mathfrak{D})}{\nu \mathfrak{D}} a^i_{B,j,k} - (a^i_{B,j,k})_0 \right) \right).$$

(6.17)

**Proof:** In order to determine $A^i_{B,j,k}$ explicitly, it suffices to compute $A^i_{B,j,k} \circ (x_1 \ldots \nabla x_k)$ for $x_1, \ldots, x_k \in \Gamma^\infty(TQ)$ and arbitrary $k \in \mathbb{N} \setminus \{0\}$, since for $\chi \in C^\infty(Q)$ the equation $A^i_{B,j,k} \pi^* \chi = \pi^* \chi$ is evident. But from the definition of $F$ this means that we have to evaluate terms of form $(\mathfrak{D}^{k-1}(p^*a^i_{B,j,k} - \Gamma_j^i))(x_1^h, \ldots, x_k^h)$. Let us first consider the term involving $p^*a^i_{B,j,k}$. To this end observe that $p^*a^i_{B,j,k}$ is a formal series of sums consisting of terms of form $p^*(\chi d\chi')(x_1^h, \ldots, x_k^h) = (p^*\chi)(d\chi')$ with $\chi, \chi' \in \Gamma^\infty(\Omega)$. Hence it suffices to determine $(\mathfrak{D}^{k-1}(p^*\chi)(d\chi'))(x_1^h, \ldots, x_k^h)$. Now recall that $\mathfrak{D}$ is a derivation and observe that the result is a sum of terms we have already computed in the proof of Lemma 6.3. Since $\mathfrak{D}$ is a derivation as well, it is now straightforward to compute that $(\mathfrak{D}^{k-1}(p^*\chi)(d\chi'))(x_1^h, \ldots, x_k^h) = p^*(\mathfrak{D}^{k-1}(\chi d\chi'))(x_1^h, \ldots, x_k^h)$, if one of the conditions 6.5, 6.6 is satisfied. If $B$ is horizontal and we may choose $j = 0$, this already implies Eq. (6.16). Now we turn to consider the term $(\mathfrak{D}^{k-1}\Gamma_j^i)(x_1^h, \ldots, x_k^h)$ in case $j$ is arbitrary and $\nabla$ satisfies condition 6.6. But in this case a straightforward induction argument shows that $(\mathfrak{D}^{k-1}\Gamma_j^i)(x_1^h, \ldots, x_k^h) = 0$, since $\Gamma_j^i$ vanishes on horizontal vector fields. Therefore, $A^i_{B,j,k}$ then is given by Eq. (6.17).

Using the result of the preceding lemma we can now easily identify the star products $\ast^B_{i,j,k}$ with naturally defined star products on $(T^*Q, \omega_h)$, if certain conditions are satisfied. Moreover, observing that the local isomorphisms $A^i_{B,j,k}$ relating the reduced star products obtained from different momentum values that were constructed in Lemma 6.1 are of the same shape as the operators $A^i_{B,j,k}$ we can – slightly modifying the above proof – obtain the main result of this section.

**Theorem 6.14** Let $\ast^B_i$ and $\ast^B_j$ be the star products obtained from a $G$-invariant torsion free connection $\nabla$ on $Q$ and $B \in Z^2_{dr}(Q)^G[[\nu]]$. For $\kappa \neq 0$, we moreover assume that the volume density $\nu$ used to define $\alpha$ is $G$-invariant and that the Lie group $G$ is unimodular.
i.) If \( \nabla \) satisfies condition (6.2) and \( B \) is horizontal, then the star product \( \star^B_{T QQ} J^0,0 \) on \( (T^*Q, \omega_{(bB)_0}) \) coincides with the naturally defined star product \( \mathcal{T}^Q \) obtained from the connection \( \nabla \), the volume density \( \pi \), and \( b_B \in \mathbb{Z}_2 \mathfrak{a}_\mathfrak{b}(Q)[[\nu]] \) defined by \( p^*b = B \). In particular, the star product \( \star^B_{T QQ} J^0,0 \) on \( (T^*Q, \omega_{(bB)_0}) \) coincides with \( \mathcal{T}^Q \).

ii.) If \( \nabla \) satisfies condition (6.2), then the star product \( \star^B_{T QQ} J^0,0 \) on \( (T^*Q, \omega_{(bB)_0}) \) coincides with the naturally defined star product \( \mathcal{T}^Q \) obtained from the connection \( \nabla \), the volume density \( \pi \), and \( b_B \in \mathbb{Z}_2 \mathfrak{a}_\mathfrak{b}(Q)[[\nu]] \) defined by \( p^*b = B + d\Gamma_{\nu} \). In particular, the star product \( \star^B_{T QQ} J^0,0 \) on \( (T^*Q, \omega_{(bB)_0}) \) coincides with \( \mathcal{T}^Q \), where \( b_B \in \mathbb{Z}_2 \mathfrak{a}_\mathfrak{b}(Q)[[\nu]] \) is defined by \( p^*b = -d\Gamma_{\nu} \).

**Proof:** The claim about \( \star^B_{T QQ} J^0,0 \) in i.) is just a restatement of Proposition 6.4 and Proposition 6.10. Moreover, Lemma 6.13 shows that the local isomorphisms from \( \star^B_{T QQ} J^0,0 \) to \( \star^B_{T QQ} J^0,0 \) are of the same form as the operators \( A_\nu \) of Eq. (2.12) using the connection \( \nabla \) and local potentials of the formal series \( b_B = b_{B,0} \) of closed two-forms on \( Q \) defined by \( p^*b = B \). But this implies that \( \star^B_{T QQ} J^0,0 \) coincides with \( \mathcal{T}^Q \), which is evidently a star product with respect to \( \omega_{(bB)_0} \). For the proof of ii.) we observe that the composition of the operators \( A_{\mu,B} \) and \( A_{\mu,B}^{-1} \), which is a local isomorphism from \( \star^B_{T QQ} J^0,0 \) to \( \star^B_{T QQ} J^0,0 \), is given by \( t_{\mu,B}^{-1} \) \( \exp (-F \left( \frac{\exp((\nu B^{0,0})-\exp((\nu-1)^{0,0}))}{\nu B^{0,0}} \right) \right) \) and that \( d(a_{B,0} + a_{\mu,0}) = b_{B,0} + \mu = b \), where \( p^*b = B + d\Gamma_{\nu} \). But this implies that \( \star^B_{T QQ} J^0,0 \) coincides with \( \mathcal{T}^Q \), using that \( \star^B_{T QQ} J^0,0 = \mathcal{T}^Q \). Finally, the local isomorphism \( A_{\mu,B} \) from \( \star^B_{T QQ} J^0,0 \) to \( \star^B_{T QQ} J^0,0 \) is given by \( t_{\mu,B}^{-1} \) \( \exp (-F \left( \frac{\exp((\nu B^{0,0})-\exp((\nu-1)^{0,0}))}{\nu B^{0,0}} \right) \right) \). Hence \( \star^B_{T QQ} J^0,0 \) equals \( \mathcal{T}^Q \), where \( p^*b = -d\Gamma_{\nu} \). \( \square \)

### 6.2 Comparison to Existing Results

In this section we establish some relations between our construction of reduced star products and known concepts of reduction in deformation quantization. Additionally, we consider the more specific example \( T^*S^n-1 \) which has been discussed in the literature.

**Remarks on Fedosov’s Method**

In order to relate our results to the investigations of Fedosov [13], we will assume in this paragraph that the group acting on \( Q \) is compact. Then we consider the usual Fedosov star product \( \star^F \) on \( (T^*Q, \omega_{B_0}) \) associated to a \( G \)-invariant torsion free symplectic connection \( \nabla^{T^*Q} \) on \( T^*Q \) with \( c(\star^F) = \frac{1}{2}[\pi^\ast B_0] \). In order to achieve that \( \mathcal{P}(Q)[[\nu]] \) of \( \star^F \) is a \( \pi^\ast \)-subalgebra, we restrict our considerations to connections whose Christoffel symbols in a bundle chart are polynomials in the momenta (cf. [13] Appx. A]). Under these general assumptions it is clear that \( \star^F \) is \( G \)-invariant and even strongly \( G \)-invariant, i.e. we can use \( J_0(\xi) = \mathcal{P}(\xi_Q) + \pi^\ast j_0(\xi) \) as \( G \)-equivariant quantum map for reduction. Moreover, Proposition 1.13 tells us that \( \star^F \) is \( G \)-equivariant to some \( G \)-invariant star product \( \star_0^F \) with \( B = B_0 + B_+ \), where \( B_+ \in \mathbb{Z}_2 \mathfrak{a}_\mathfrak{b}(Q)^G[[\nu]] \). But then the characteristic classes of \( \star_0^F \) and \( \star^F \) coincide, therefore \( B_+ \) has to be exact. Since the group acting on \( Q \) is compact, one can therefore find a \( G \)-invariant formal potential \( A_+ \) for \( B_+ \). Hence \( \star_0^F \) is \( G \)-equivariant to \( \star_0^F \) by Proposition 1.13. Together with Theorem 5.5 this implies that the characteristic class of \( \star_0^F J^0,0 \) is given by

\[
c(\star_0^F J^0,0) = \frac{1}{\nu} [\pi^\ast b_0] + \frac{1}{\nu} [\pi^\ast \bar{\mu}_0],
\]

(6.18)
where \( p^*b_0 = B_0 + d\Gamma_{\lambda} \), and \( \mu \in \nu g^*[[\nu]] \) is defined by \( \langle \mu, \xi \rangle := i^*TJ_0(\xi) - j_0(\xi) \) with a \( G \)-equivalence \( T \) from \( *_F \) to \( *_{\rho_{B_0}} \). Here again, \( \mu_{\lambda} \in \nu Z^2_{\text{dR}}(\Omega)[[\nu]] \) is determined by \( p^*\mu_{\lambda} = \langle \mu, \lambda \rangle \). Thus, the reduced star product \( *_{\rho_{B_0}} \) is in general not equivalent to a canonical star product on the reduced phase space with characteristic class \( \frac{1}{2}(\overline{\Omega})b_0 \). In view of this fact – similar to the situation for the star products \( *_\kappa \) – the ‘reduction commutes with quantization’ theorem proved by Fedosov appears to be a consequence of the appropriate choice of the original star product on the large phase space rather than a general principle in deformation quantization. Finally, let us emphasize that the picture changes in case the Lie algebra \( g \) is semi-simple, since then there are no non-zero elements of \( g^* \) vanishing on \([g, g] \) which implies that in this case \( \mu = 0 \). Another example, where the term \( \Pi \mu_{\lambda} \) obviously vanishes, is the case of a trivial principal \( G \)-bundle \( p : Q \to \Omega \), where all the characteristic classes of the bundle are zero.

**The Star Product on \( T^*S^{n-1} \) à la Bayen et al.**

In this paragraph we consider a very concrete example of reduction. The reduced phase space is the cotangent bundle of the \( n-1 \)-sphere, which is obtained from \( T^*(\mathbb{R}^n \setminus \{0\}) \) by classical Marsden-Weinstein reduction. Using our reduction method for star products we will show that the reduced star product obtained from the Weyl-Moyal star product \( *_{1/2} \) on \( T^*(\mathbb{R}^n \setminus \{0\}) \) coincides with the deformation quantization obtained by Bayen et al. in [2].

Let \( G \) be the group \( \mathbb{R}^+ \) of positive real numbers with the usual multiplication as composition, and consider the action of \( G \) on \( \mathbb{R}^n \setminus \{0\} \) given by \( \phi_g(x) = gx \) for \( g \in \mathbb{R}^+, \ x \in \mathbb{R}^n \setminus \{0\} \). Then the quotient \( (\mathbb{R}^n \setminus \{0\})/\mathbb{R}^+ \) is isomorphic to the sphere \( S^{n-1} \subset \mathbb{R}^n \setminus \{0\} \). Then, the corresponding projection \( p : \mathbb{R}^n \setminus \{0\} \to S^{n-1} \) is given by \( p(x) = \frac{1}{|x|}x \), where \( |x| \) denotes the Euclidean length of \( x \in \mathbb{R}^n \setminus \{0\} \). For \( \xi \in g = \mathbb{R} \), the generating vector field is explicitly given by \( \xi_*e_n \). Hence, the (canonical) \( G \)-equivariant classical momentum map reads \( J_0(\xi)(q, p) = \xi q^i p_i \). It is straightforward to check that \( \gamma \in \Gamma^\infty(T^*(\mathbb{R}^n \setminus \{0\})) \) with \( \gamma(x) = \frac{x^i}{|x|^2}dx^i \) is a connection one-form for the principal \( \mathbb{R}^+ \)-bundle under consideration. Clearly, \( \gamma \) is closed, whence the connection is flat. In order to compute the horizontal lift of a vector field \( t \in \Gamma^\infty(TS^{n-1}) \), it turns out to be convenient to describe \( t \) by a smooth mapping \( \tilde{t} : S^{n-1} \to \mathbb{R}^n \) which satisfies \( \langle \tilde{t}(\frac{1}{|x|}x) | x \rangle = 0 \) for all \( x \in \mathbb{R}^n \setminus \{0\} \), where \( \langle \cdot | \cdot \rangle \) denotes the Euclidean inner product on \( \mathbb{R}^n \). With this mapping, the horizontal lift \( t^h \in \Gamma^\infty(T(\mathbb{R}^n \setminus \{0\})) \) is easily computed and given by \( t^h(x) = |x| \tilde{t}(\frac{1}{|x|}x) \). But from this the following relation is immediate:

\[
(l(\overline{\mathcal{P}}(t)))(q, p) = \mathcal{P} \left( \frac{1}{|q|} \right)(q, p) = \langle \tilde{t}(\frac{1}{|q|}q) | q/p - \frac{\langle q | p \rangle}{|q|^2} q \rangle = (\Pi \overline{\mathcal{P}}(t))(q, p),
\]

where \( \Pi : T^*(\mathbb{R}^n \setminus \{0\}) \to T^*S^{n-1}, (q, p) \mapsto \left( \frac{1}{|q|}q, |q/p - \frac{\langle q | p \rangle}{|q|^2} q \right) \) denotes the canonical projection onto \( T^*S^{n-1} \). Note here, that \( T^*S^{n-1} \) is naturally embedded in \( T^*(\mathbb{R}^n \setminus \{0\}) \) as the submanifold defined by the constraints \( |q| = 1 \) and \( \langle q | p \rangle = 0 \). Clearly, the formula in Eq. (6.19) also holds for all \( t \in \Gamma^\infty(\sqrt{T}S^{n-1}) \), since \( l \) and \( \Pi^* \) are homomorphisms with respect to pointwise multiplication. The inverse \( l^{-1} \) is just given by restriction of \( F \in \mathcal{H}(\mathbb{P}(\mathbb{R}^n \setminus \{0\})^G) \) to \( T^*S^{n-1} \), i.e. \( l^{-1} = I^* \), where \( I : T^*S^{n-1} \to T^*(\mathbb{R}^n \setminus \{0\}) \) denotes the embedding of \( T^*S^{n-1} \) into \( T^*(\mathbb{R}^n \setminus \{0\}) \).

After these preparations recall that the Weyl-Moyal star product \( *_{1/2} \) on \( T^*(\mathbb{R}^n \setminus \{0\}) \) can be written as

\[
f *_{1/2} f' = m \circ \exp \left( \frac{\mu}{2} \left( \partial_{q^i} \otimes \partial_{p_i} - \partial_{p_i} \otimes \partial_{q^i} \right) \right)(f \otimes f'), \quad f, f' \in C^\infty(T^*(\mathbb{R}^n \setminus \{0\}))[\nu],
\]

where \( m \) is defined by \( m(f \otimes f') := ff' \). Given \( \mu \in \mathbb{R} + \nu C[\nu] \), we now want to compute the reduced star product \( *_{1/2}^{Q_0, \mu} \). According to Theorem 3.5 and the above results for \( l \) and \( l^{-1} \) it is
from concrete computation turns out to be much more involved. This is caused by the fact that \( h(\text{T}_
abla) \) which acts on \( \text{T}_\mu \) and therefore not on the particular choice of the \( \ast \) product constructed by Bayen et al. in [2]. There, using a slightly different approach, \( \ast(\Pi_1 \ast) \) simplifies to

\[
\Phi'(g,h)(q,p) = (gq, g^{-1}p + hq).
\]

In this paragraph we do not want to make an attempt to apply the BRST-method in deformation quantization as developped in [5] to the example \((\Pi_1 \ast)\), where \( \ast \) is a derivation of the canonical flat covariant derivative on \( \mathbb{R}^n \). Using the induced connection \( \nabla \) and an appropriate volume density on \( S^{n-1} \) using the induced connection \( \nabla \) and an appropriate volume density on \( S^{n-1} \).

### The Counterexample of the BRST-Method

In this paragraph we do not want to make an attempt to apply the BRST-method in deformation quantization as developped in [5] to the example \((T^*Q,\omega_B)\) in full generality, since this would be far beyond the scope of the present paper but might be an interesting topic for a future project. Instead, we merely consider two concrete intimately related examples also considered in [5], where in the first one the BRST-method can be applied without any problems but turns out to fail in the second example. In the first case we will show in particular that our reduction procedure yields the same result like the BRST-method. Moreover, we want to point out that due to our slightly more restrictive definition of a quantum momentum map the peculiarity occurring in the counterexample of the BRST-method is avoided in our framework.

Let us consider \( T^*(S^1 \times S^1) \cong T^* S^1 \times T^* S^1 \) with the canonical symplectic form. As symmetry group we choose \( S^1 \) which acts on the base of the second factor by group multiplication. Thus the reduced phase space is \( T^* S^1 \). Using the canonical flat covariant derivative on \( S^1 \) we can equip each factor \( T^* S^1 \) with the \( \kappa \)-ordered star product \( \ast_\kappa \) yielding a well-defined star product \( \ast_{\kappa_1,\kappa_2} \) on \( T^*(S^1 \times S^1) \), where we may even use different ordering parameters \( \kappa_1, \kappa_2 \) in each factor. Denoting by \( \Pi_1 : T^* S^1 \times T^* S^1 \to T^* S^1 \) the projection onto the first factor, one clearly has \((\Pi_1^* f) \ast_{\kappa_1,\kappa_2} (\Pi_1^* f') = \)
Thus, the reduced star product just gives back the star product we started from in the first factor
\[ f \star_{\kappa_1, \kappa_2}^{j_0, \mu} f' = f \star_{\kappa_1} f', \quad f, f' \in \mathcal{P}(S^1)[[\nu]]. \] (6.23)

Thus, the reduced star product just gives back the star product we started from in the first factor of
\( T^*S^1 \times T^*S^1 \), which is in perfect agreement with the results of [5].

Following [4], we now consider the \( S^1 \)-invariant equivalence transformation
\( T := \exp(-\nu p_1 \partial_{p_1}) \) and the \( S^1 \)-invariant star product
\( \star'_{\kappa_1, \kappa_2} := T \star_{\kappa_1, \kappa_2} \). According to our results of Proposition 3.7, it is immediately clear that the corresponding reduced star product \( \star'_{\kappa_1, \kappa_2}^{j_0, \mu} \) is equivalent to
\( \star_{\kappa_1, \kappa_2}^{j_0, \mu} \), where \( J'(\xi) = T J_0(\xi) = J_0(\xi) + i \nu \xi \partial_{p_1} \) is used as \( S^1 \)-equivariant quantum momentum map. Actually, \( \star'_{\kappa_1, \kappa_2}^{j', \mu} \) even coincides with \( \star_{\kappa_1, \kappa_2}^{j_0, \mu} \), since \( T^{-1} \circ \Pi_1^* = T \circ \Pi_1^* = \Pi_1^* \), hence
\( (\Pi_1^* f) \star'_{\kappa_1, \kappa_2} (\Pi_1^* f') = \Pi_1^* (f \star_{\kappa_1} f') \) which implies \( \star'_{\kappa_1, \kappa_2}^{j', \mu} = \star_{\kappa_1} \).

Now, the crucial point, which lets the BRST-method fail in this case, is that \( J_0 \) is an allowed ‘quantum momentum map’ for BRST-quantization, since it satisfies
\[ \frac{1}{\nu} (J_0(\xi) \star_{\kappa_1, \kappa_2} J_0(\eta) - J_0(\eta) \star_{\kappa_1, \kappa_2} J_0(\xi)) = 0 = J_0([\xi, \eta]) \]
by \( \dim(S^1) = 1 \). But in contrast to the properties of \( J'(\xi) \) we have
\[ -\frac{1}{\nu} \text{ad}_{\star'_{\kappa_1, \kappa_2}} (J_0(\xi)) \neq L_{\xi T^*(S^1 \times S^1)}, \]
hence the ‘quantum momentum map’ \( J_0(\xi) \) does not generate the classical symmetry via the quasi-inner derivation with respect to \( \star'_{\kappa_1, \kappa_2} \), whereas it does so with respect to \( \star_{\kappa_1, \kappa_2} \). Thus, our slightly more restrictive definition of a quantum momentum map – which of course imposes additional conditions to be satisfied a priori (cf. Section 4) – completely avoids the peculiarity appearing in the BRST-method when using a non-appropriate ‘quantum momentum map’.

A Equivariance Properties of Certain Differential Operators

Throughout this appendix, \( \phi \) will always denote a diffeomorphism of \( Q \) and \( \Phi = T^*(\phi^{-1}) \) the diffeomorphism lifted to \( T^*Q \). Moreover, we continue to use the notation as introduced in Section 2.

Lemma A.1 Let \( \nabla \) be a torsion free connection on \( Q \). Then there is a uniquely defined tensor field \( S_{\phi} \in \Gamma^\infty(\sqrt{2} T^*Q \otimes TQ) \) such that \( (\phi^*\nabla)_X Y - \nabla_X Y = S_{\phi}(X, Y) \) for all \( X, Y \in \Gamma^\infty(TQ) \). Furthermore, for all \( \beta \in \Gamma^\infty(\sqrt{T^*Q}) \) one has
\[ \phi^*D(\phi^{-1})^*\beta = (\phi^*D)\beta, \] (A.1)
where \( \phi^*D \) denotes the operator of symmetric covariant derivation corresponding to the connection \( \phi^*\nabla \) which is explicitly given by
\[ \phi^*D = D - dx^i \vee dx^j \vee i_x(S_{\phi}(\partial_{x^i}, \partial_{x^j})). \] (A.2)
Proof: The claim about the existence and uniqueness of the tensor field $S_{\phi}$ and the fact that it is symmetric follows immediately from the observation that $\phi^*\nabla$ is a torsion free connection. In order to prove Eq. (A.1), it is enough to prove it for $\beta \in \mathcal{C}^\infty(Q)$ and $\beta \in \Gamma^\infty(T^*Q)$, since $\phi^*D(\phi^{-1})^*$ and $\phi^*D$ are derivations with respect to $\nabla$. It is straightforward to verify the formula for these cases. Analogously, it suffices to show that $A(2)$ is satisfied on $\mathcal{C}^\infty(Q)$ and $\Gamma^\infty(T^*Q)$. An easy computation shows this as well. \hfill \blacksquare

Now we briefly recall some well-known basic definitions concerning horizontal and vertical lifts of vector fields and one-forms on $Q$ to vector fields on $T^*Q$. Moreover, we study their behaviour with respect to pull-back by $\Phi$.

**Definition A.2** (cf. [6, Def. 2]) Let $\nabla$ be a connection on $Q$. Consider the connection mapping $K : T(T^*Q) \to T^*Q$ defined by

$$K\left(\frac{d}{dt}\bigg|_{t=0}\zeta(t)\right) := \nabla^c_{\partial_t}\zeta\bigg|_{t=0}$$

for a curve $\zeta$ in $T^*Q$, where $\nabla^c$ denotes the connection pulled-back along the footpoint-curve $c = \pi \circ \zeta$. Then $(T\pi \times K) : T(T^*Q) \to TQ \oplus T^*Q$ is a fibrewise isomorphism. The horizontal and vertical lifts with respect to $\nabla$ then are well-defined and unique by the following. The section $\text{hor}_\nabla(X) \in \Gamma^\infty(T(T^*Q))$ is called the horizontal lift of $X \in \Gamma^\infty(TQ)$, if and only if

$$T\pi \text{hor}_\nabla(X) = X \circ \pi \quad \text{and} \quad K(\text{hor}_\nabla(X)) = 0,$$

and $\text{ver}_\nabla(\beta) \in \Gamma^\infty(T(T^*Q))$ is the vertical lift of $\beta \in \Gamma^\infty(T^*Q)$, if and only if

$$K(\text{ver}_\nabla(\beta)) = \beta \circ \pi \quad \text{and} \quad T\pi \text{ver}_\nabla(\beta) = 0.$$

Working in a local bundle chart one finds that

$$\text{hor}_\nabla(X) = (\pi^*X^i)\partial_q^i + p_j\pi^*(X^k\Gamma^j_{ki})\partial_p^i \quad \text{and} \quad \text{ver}_\nabla(\beta) = (\pi^*\beta_i)\partial_p^i,$$

where the $\Gamma^j_{ki}$ denote the Christoffel symbols of $\nabla$ and $X^i$ resp. $\beta_k$ the components of $X$ resp. $\beta$ in some chart of $Q$. In particular, it turns out that the vertical lift does not depend on the connection, henceforth we will simply denote it by $\text{ver}$.

**Lemma A.3** Let $X \in \Gamma^\infty(TQ)$, $\beta \in \Gamma^\infty(T^*Q)$ and $\beta' \in \Gamma^\infty(\mathcal{V} T^*Q)$. Then one has the following equivariance properties of $\text{hor}_\nabla$, $\text{ver}$ and $F$ with respect to pull-back by $\Phi$:

$$\Phi^*\text{hor}_\nabla(X) = \text{hor}_{\phi^*\nabla}(\phi^*X) \quad \text{and} \quad \Phi^*\text{ver}(\beta) = \text{ver}(\phi^*\beta).$$

Moreover, one has for all $f \in \mathcal{C}^\infty(T^*Q)$

$$\Phi^*F(\beta') f = F(\phi^*\beta') \Phi^* f.$$

Proof: The proof of Eqs. (A.7) consists of a straightforward computation using the definitions of horizontal and vertical lifts. Observe that the second of these identities implies $\Phi^*F(\beta) f = F(\phi^*\beta) \Phi^* f$, since $F(\beta) = L_{\text{ver}(\beta)}$. But since $F$ is compatible with the $\mathcal{V}$-product and since $\Phi^*F(\chi) f = \Phi^*((\pi^*\chi) f) = (\pi^*\phi^*\chi)\Phi^* f = F(\phi^*\chi) \Phi^* f$ for all $\chi \in \mathcal{C}^\infty(Q)$, this also implies (A.8). \hfill \blacksquare
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