EXTREMAL MEASURES WITH PRESCRIBED MOMENTS

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ABSTRACT. In the approximate integration some inequalities between the quadratures
and the integrals approximated by them are called extremalities. On the other hand, the
set of all quadratures is convex. We are trying to find possible connections between ex-
tremalities and extremal quadratures (in the sense of extreme points of a convex set). Of
course, the quadratures are the integrals with respect to discrete measures and, moreover,
a quadrature is extremal if and only if the associated measure is extremal. Hence the natu-
ral problem arises to give some description of extremal measures with prescribed moments
in the general (not only discrete) case. In this paper we deal with symmetric measures with
prescribed first four moments. The full description (with no symmetry assumptions, and/or
not only four moments are prescribed and so on) is far to be done.

1. INTRODUCTION

The second author considered in [7] so-called extremalities in the approximate integra-
tion.

Let \( P_n \) be the \( n \)-th degree Legendre polynomial given by the Rodrigues formula
\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^2.
\]
Then \( P_n \) has \( n \) distinct roots \( x_1, \ldots, x_n \in (-1, 1) \). The \( n \)-point Gauss–Legendre quadra-
ture is the positive linear functional on \( \mathbb{R}[-1,1] \) given by
\[
G_n[f] = \sum_{i=1}^{n} w_i f(x_i)
\]
with the weights
\[
w_i = \frac{2(1-x_i^2)}{(n+1)^2 P_{n+1}^2(x_i)}, \quad i = 1, \ldots, n.
\]
The \((n+1)\)-point Lobatto quadrature is the functional
\[
L_{n+1}[f] = v_1 f(-1) + v_{n+1} f(1) + \sum_{i=2}^{n} v_i f(y_i),
\]
where \( y_2, \ldots, y_n \in (-1,1) \) are (distinct) roots of \( P_n' \) and
\[
v_1 = v_{n+1} = \frac{2}{n(n+1)}, \quad v_i = \frac{2}{n(n+1) P_{n+1}^2(y_i)}, \quad i = 2, \ldots, n.
\]

For these forms of quadratures as well as for another quadratures appearing in this paper
see for instance [2].

Recall that a continuous function \( f : [-1, 1] \to \mathbb{R} \) is \( n \)-convex (\( n \in \mathbb{N} \)), if and only if \( f \)
is of the class \( C^{n-1} \) and the derivative \( f^{(n-1)} \) is convex (cf. [4, Theorem 15.8.4]). For the
needs of this paper it could be regarded as a definition of \( n \)-convexity.

Let \( T \) be a positive linear functional defined (at least) on a linear subspace of \( \mathbb{R}[-1,1] \)
generated by the cone of \((2n-1)\)-convex functions (i.e. \( T[f] \geq 0 \) for \( f \geq 0 \)). Assume that

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of a convex set, extremalities in the approximate integration, probability measure, quadrature.
In particular, if \( \text{extreme points of a set of positive quadratures, which are exact on polynomials of a given order} \) we will obtain a connection between the extremalities in the approximate integration and \( T \) for some \( \delta \);

\[
G_n[f] \leq T[f] \leq L_{n+1}[f]
\]

holds for any \((2n-1)\)-convex function \( f : [-1, 1] \to \mathbb{R} \). Then the functionals \( G_n \) and \( L_{n+1} \) restricted to the cone of \((2n-1)\)-convex functions are minimal and maximal, respectively, among all positive linear functionals defined (at least) on \((2n-1)\)-convex functions, which are exact on polynomials of order \( 2n-1 \). In [7, Theorem 15] there is a counterpart of the above result for \( 2n \)-convex functions with Radau quadratures in the role of the minimal and maximal operators.

Studying the results of this kind the following problem seems to be natural. Some quadrature operators are extremal in the sense of inequalities like (1). On the other hand, the set of all quadratures which are exact on polynomials of some given order is convex. Then it could be interesting to find its extreme points looking for the possible connections between extremalities in the approximate integration and the extreme points of convex sets. In particular, are \( G_n \) and \( L_{n+1} \) extreme points of the above mentioned set? If the answer is positive, are they the only extreme points, or there exist another ones?

This is the starting point for our considerations. We will observe that the extreme points in the set of all quadratures exact on polynomials of prescribed order could be determined with the aid of [3, Theorem 6.1, p. 101]. Next we shall investigate the extreme points the set of all positive linear operators defined (at least) on \( C[-1, 1] \) with prescribed moments.

Our research is far from being complete. Actually we are able to give a full description of the extreme points of the set of symmetric operators with four prescribed moments, i.e. \((m_0, m_1, m_2, m_3) = (1, 0, b^2, 0)\).

2. Extremal Quadratures

Let \( D \) be a convex subset of a linear space. Recall that \( x \in D \) is the extreme point of \( D \), if \( x \) is not the "interior" point of any segment with endpoints in \( D \), i.e. \( x = tu + (1-t)v \) for some \( u, v \in D \) and \( t \in [0,1] \) implies that \( x = u = v \). The set of all extreme points of a set \( D \) will be denoted by \( \text{ext} \, D \).

A quadrature on \([-1,1]\] is the linear functional defined on \( \mathbb{R} [-1,1] \) by the formula

\[
Q[f] = \sum_{k=1}^{n(Q)} w_k^Q f(\xi_k^Q),
\]

where \( n(Q) \in \mathbb{N} \), \( \xi_k^Q \in [-1,1] \) are the nodes and \( w_k^Q \) are the weights of \( Q \) (for \( k = 1, \ldots , n(Q) \)). If all the weights of \( Q \) are positive, then \( Q \) is a positive quadrature, i.e. \( Q[f] \geq 0 \) for \( f \geq 0 \). Positive quadratures are often used in the approximate integration.

Let \( e_k(x) = x^k, k = 0,1,\ldots , n \). Fix a vector \( m = (m_0, m_1, \ldots , m_n) \in \mathbb{R}^{n+1} \). Let \( \text{Quad}_+(m) \) be the set of all positive quadratures \( Q \) with moments \( Q[e_k] = m_k, k = 0,1,\ldots , n \). In this section we determine the extreme points of the (convex) set \( \text{Quad}_+(m) \). In particular, if \( m \) is a vector of integral moments \( m_k = \int_{-1}^{1} x^k dx, k = 0,1,\ldots , n \), we will obtain a connection between the extremalities in the approximate integration and extreme points of a set of positive quadratures, which are exact on polynomials of a given order \( n \).

Every positive quadrature \( Q \) could be written in the form

\[
Q[f] = \int_{[-1,1]} f d\mu_Q \quad \text{for} \quad \mu_Q = \sum_{k=1}^{n(Q)} w_k^Q \delta_{\xi_k^Q},
\]

where \( \delta_x \) stands for a Dirac measure concentrated at \( x \). By the Riesz–Markov Theorem (cf. [6, p. 458]) the measure \( \mu_Q \) in the above representation is uniquely determined. Furthermore, if \( Q \in \text{Quad}_+(m) \), then \( m \) is the moment vector of the measure \( \mu_Q \).
Denote by $\text{Disc}(\mathbf{m})$ the set of all discrete measures $\mu$ on $\mathcal{B}([-1,1])$ with moments

$$
\int_{[-1,1]} x_k d\mu = m_k, \quad k = 0, 1, \ldots, n.
$$

The set $\text{Disc}(\mathbf{m})$ is convex.

**Theorem 1.** A quadrature $\mathcal{Q} \in \text{Quad}_+(\mathbf{m})$ is an extreme point of $\text{Quad}_+(\mathbf{m})$ if and only if $n(\mathcal{Q}) \leq n + 1$.

**Proof.** A quadrature $\mathcal{Q} \in \text{Quad}_+(\mathbf{m})$ is an extreme point of $\text{Quad}_+(\mathbf{m})$ if and only if the measure $\mu_{\mathcal{Q}} \in \text{Disc}(\mathbf{m})$ is the extreme point of $\text{Disc}(\mathbf{m})$. By virtue of [3, Theorem 6.1, p. 101] the extreme measures in $\text{Disc}(\mathbf{m})$ are exactly the measures concentrated on at most $n + 1$ points. This finishes the proof. \[\square\]

For $m_k = \int_{-1}^{1} x^k dx, \ k = 0, 1, \ldots, 2n - 1$, we obtain immediately that the $n$-point Gauss quadrature $\mathcal{G}_n$, as well as the $(n + 1)$-point Lobatto quadrature, are the extreme points of $\text{Quad}_+(\mathbf{m})$. Nevertheless, there are infinitely many other extremal quadratures in this set. For instance, all Gauss quadratures and with $p$ nodes ($p \in \{n, \ldots, 2n\}$), also all Lobatto quadratures with the number of nodes $p \in \{n + 1, \ldots, 2n\}$, are the extreme points of $\text{Quad}_+(\mathbf{m})$.

## 3. Extremal Measures

To find extremal quadratures we needed to know the extreme points of a set of all discrete measures with finite spectrum. In this section we consider all finite symmetric measures on $\mathcal{B}([-1,1])$ with prescribed moments $(1, 0, b^2, 0)$.

Let $\mathcal{M}([-1,1])$ be the set of all finite measures on $\mathcal{B}([-1,1])$. Let $\mathcal{M}^0([-1,1])$ be a subset of $\mathcal{M}([-1,1])$ consisting of measures, which are symmetric with respect to 0, i.e. $\mu \in \mathcal{M}^0([-1,1])$ if and only if $\mu \in \mathcal{M}([-1,1])$ and $\mu(-B) = \mu(B), \ B \in \mathcal{B}([-1,1])$.

Let $\mathcal{P}([-1,1]), \mathcal{P}(\mathbb{R})$ be the sets of probability measures on $\mathcal{B}([-1,1])$ and $\mathcal{B}(\mathbb{R})$, respectively. Denote

$$
\mathcal{P}^0([-1,1]) = \mathcal{M}^0([-1,1]) \cap \mathcal{P}(\mathbb{R}).
$$

For a non-zero measure $\mu \in \mathcal{M}([-1,1])$ define the measure $\tilde{\mu}$ by

$$
\tilde{\mu}(B) = \frac{\mu(B)}{\mu([-1,1])}, \quad B \in \mathcal{B}([-1,1]).
$$

Let $0 < a < 1$ and $\mathcal{M}^0([-1,1], a)$ be the set of all measures $\mu \in \mathcal{M}^0([-1,1])$ satisfying

$$
\int_{-1}^{1} x^2 \mu(dx) = a^2 \mu([-1,1]). \quad (2)
$$

Set

$$
\mathcal{P}^0([-1,1], a) = \mathcal{M}^0([-1,1], a) \cap \mathcal{P}([-1,1]).
$$

Clearly

$$
\int_{-1}^{1} x^2 \mu(dx) = a^2.
$$

for any $\mu \in \mathcal{P}^0([-1,1], a)$. Moreover, $\mu \in \mathcal{M}^0([-1,1], a) \iff \tilde{\mu} \in \mathcal{P}^0([-1,1], a)$, whenever $\mu$ is a non-zero measure.

Obviously, the set $\mathcal{P}^0([-1,1], b)$ is the set consisting of all finite symmetric measures on $\mathcal{B}([-1,1])$ with prescribed moments $(1, 0, b^2, 0)$. 

We start with two lemmas. The proof of the first of them is rather standard and simple, so we omit it.

**Lemma 2.** Let $\mu \in \mathcal{P}^0([-1, 1])$ and $m_2 = \int_{-1}^{1} x^2 \mu(dx)$ for $0 < a < 1$.

a) If $\mu$ is concentrated on $[-a, a]$, then $m_2 \leq a^2$.

b) If $\mu$ is concentrated on $[-1, -a] \cup [a, 1]$, then $m_2 \geq a^2$.

c) If $\mu = \frac{\delta_a + \delta_a}{2}$, i.e. $\mu$ is concentrated on the set $\{-a, a\}$, then $m_2 = a^2$.

d) If $\mu$ is concentrated on $[-a, a]$ and $\mu((-a, a)) > 0$, then $m_2 < a^2$.

e) If $\mu$ is concentrated on $[-1, -a] \cup [a, 1]$ and $\mu((-1, -a) \cup (a, 1)) > 0$, then $m_2 > a^2$.

f) Suppose that $\mu$ is concentrated on $[-a, a]$. Then $m_2 = a^2$ if and only if $\mu$ is concentrated on $\{-a, a\}$.

g) Suppose that $\mu$ is concentrated on $[-1, -a] \cup [a, 1]$. Then $m_2 = a^2$ if and only if $\mu$ is concentrated on $\{-a, a\}$.

Let $\mu \in \mathcal{M}([-1, 1])$, $E \in \mathcal{B}([-1, 1])$. Then $\mu_E$ stands for the restriction of $\mu$ to the set $E$, i.e. $\mu_E(B) = \mu(B \cap E)$, $B \in \mathcal{B}([-1, 1])$. Similarly to (2), for any $E \in \mathcal{B}([-1, 1])$ with $\mu(E) > 0$, we put

$$\tilde{\mu}_E(B) = \frac{\mu_E(B)}{\mu(E)}, \quad B \in \mathcal{B}([-1, 1]).$$

**Lemma 3.** Let $0 < a < 1$ and $\mu \in \mathcal{M}^0([-1, 1], a)$ be a non-zero continuous measure, i.e. $\mu\{x\} = 0$ for all $x \in [-1, 1]$. Then

(i) $\mu((-a, a)) > 0$ and $\mu((-1, -a) \cup (a, 1)) > 0$;

(ii) there exists $\xi_0 \in (a, 1)$ such that $\mu((a, \xi_0)) > 0$ and $\mu((\xi_0, 1)) > 0$.

**Proof.** The part (i) follows immediately by Lemma 2, because otherwise either $m_2 > a^2$, or $m_2 < a^2$. To prove (ii) assume, on the contrary, that for all $\xi \in (a, 1)$,

$$(3) \quad \mu([a, \xi]) = 0 \quad \text{or} \quad \mu([\xi, 1]) = 0.$$  

We recursively define the sequence of sets $A_n = [a_n, b_n] \subset [a, 1]$, $n \in \mathbb{N}$ starting with $A_1 = [a, 1]$. Using (i) and taking into account the symmetry of $\mu$, we get $\mu([a, 1]) > 0$. Suppose that we have constructed the sets $A_k$, $k = 1, 2, \ldots, n$ such that $\mu(A_k) = \mu([a, 1])$, $k = 1, 2, \ldots, n$ and $A_k \subset A_{k-1}$, $k = 2, \ldots, n$. If $\xi_n = \frac{a_n + b_n}{2}$, then by (3), two cases are possible. If $\mu([a, \xi_n]) = 0$, then $\mu((\xi_n, 1]) = \mu([a, 1])$ and we take $a_{n+1} = \xi_n$, $b_{n+1} = b_n$. If $\mu([\xi_n, 1]) = 0$, then $\mu([a, \xi_n]) = \mu([a, 1])$ and we take $a_{n+1} = a_n$, $b_{n+1} = \xi_n$. Obviously, for $A_{n+1} = [a_{n+1}, b_{n+1}]$ we have $A_{n+1} \subset A_n$ and $\mu(A_{n+1}) = \mu([a, 1])$.

By the above construction $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu([a, 1]) > 0$ and there exists $x \in [a, 1]$ such that $\bigcap_{n=1}^{\infty} A_n = \{x\}$. Because $\mu$ was continuous, we arrive at the contradiction. This completes the proof. \qed

Below we prove some decomposition-type result.

**Theorem 4.** Let $0 < a < 1$ and $\mu \in \mathcal{P}^0([-1, 1], a)$ be a continuous measure. There exist the sets $E_1, E_2 \in \mathcal{B}([-1, 1])$ such that $E_1 \cap E_2 = \emptyset$, $\mu(E_1) > 0$, $\mu(E_2) > 0$, $\mu(E_1) + \mu(E_2) = 1$ and $\mu|_{E_1}, \mu|_{E_2} \in \mathcal{M}^0([-1, 1], a)$.

**Proof.** Since $\mu$ is continuous and symmetric, Lemma 3 implies

$$\mu((0, a)) > 0, \quad \mu((a, 1)) > 0, \quad \mu((0, a) \cup (a, 1)) = \frac{1}{2}.$$
Consider the function \( g : [a, 1] \to \mathbb{R} \) given by
\[
g(x) = \int_{-1}^{1} u^2 \tilde{\mu}_{[-x,x]}(du), \quad a \leq x \leq 1
\]
and the measure \( \nu = a^2 \mu \), which is absolutely continuous with respect to \( \mu \). In particular, \( \nu \) is continuous. Denote by \( F_\mu, F_\nu \) the distribution functions of the measures \( \mu, \nu \), respectively, and rewrite the function \( g \) in the form
\[
(4) \quad g(x) = \left( \int_{-1}^{1} u^2 \mu_{[-x,x]}(du) \right) \cdot \left( \mu([-x,x]) \right)^{-1} = \left( \int_{-x}^{x} u^2 \mu(du) \right) \cdot \left( \mu([-x,x]) \right)^{-1} = \frac{\nu([-x,x])}{\mu([-x,x])} = F_\nu(x) - F_\nu(-x) \quad \mu([-x,x])
\]
The distributions functions \( F_\mu, F_\nu \) are continuous by continuity of the measures \( \mu, \nu \), respectively. Furthermore,
\[
F_\mu(x) - F_\mu(-x) = \mu([-x,x]) \geq \mu([-a,a]) > 0
\]
for all \( x \in [a, 1] \). Hence \( g \) is continuous by (4).

By Lemma 2 we infer that \( g(a) < a^2 \). Since \( \mu \in \mathcal{P}^a([-1,1], a) \), then \( g(1) = a^2 \). Continuity of \( g \) implies that there exists \( b_1 \in (a, 1) \) such that
\[
(5) \quad g(a) < g(b_1) \quad \text{and} \quad g(b_1) < a^2.
\]
Then
\[
(6) \quad \mu((a,b_1)) > 0 \quad \text{and} \quad \mu((b_1,1)) > 0.
\]
Indeed, suppose that \( \mu((a,b_1)) = 0 \). Hence \( \mu|_{(a,b_1)} \) is a zero-measure and consequently
\[
g(b_1) = \left( \int_{-1}^{1} u^2 \mu_{[-b_1,b_1]}(du) \right) \cdot \left( \mu([-b_1,b_1]) \right)^{-1} = \left( \int_{-1}^{1} u^2 \mu_{[-a,a]}(du) \right) \cdot \left( \mu([-a,a]) \right)^{-1} = g(a),
\]
which contradicts (5). Similarly, supposing that \( \mu((b_1,1)) = 0 \), we arrive at \( g(b_1) = g(1) = a^2 \), which also contradicts (5).

Now we define the function \( h : [0, a] \to \mathbb{R} \) by
\[
h(x) = \int_{-1}^{1} u^2 \tilde{\mu}_{[-b_1,-x][x,b_1]}(du), \quad 0 \leq x \leq a.
\]
Writing
\[
h(x) = \frac{\nu([x,b_1])}{\mu([x,b_1])} = \frac{F_\nu(b_1) - F_\nu(x)}{F_\nu(b_1) - F_\nu(x)}
\]
and using once more the continuity of \( F_\mu, F_\nu \), we obtain that \( h \) is continuous. We have \( h(0) = g(b_1) < a^2 \) and
\[
h(a) = \int_{-1}^{1} u^2 \tilde{\mu}_{[-b_1,-a][a,b_1]}(du).
\]
Using Lemma 2 we arrive at \( h(a) > a^2 \). Consequently, by continuity of \( h \) we conclude that there exists \( a_1 \in (0, a) \) such that
\[
(7) \quad h(a_1) = a^2.
\]
Define \( E_1 = [-b_1,-a_1] \cup [a_1, b_1] \), \( E_2 = [-1,1] \setminus E_1 \). By (7) we obtain
\[
(8) \quad \int_{-1}^{1} u^2 \mu_{E_1}(du) = a^2 \mu(E_1).
\]
Taking into account the above equation and the moment condition $\int_{-1}^{1} u^2 \mu(du) = a^2$, we have

\begin{equation}
\int_{-1}^{1} u^2 \mu|_{E_2}(du) = a^2 \mu(E_2).
\end{equation}

The following inequalities are true:

\begin{equation}
\mu((0, a_1)) > 0 \quad \text{and} \quad \mu((a_1, a)) > 0.
\end{equation}

Indeed, suppose that $\mu((0, a_1)) = 0$. Then $h(a_1) = h(0) < a^2$, which contradicts (7). Similarly, if $\mu((a_1, a)) = 0$, then $h(a_1) = h(a) > a^2$, which also contradicts (7).

By (6) and (10) we infer that $\mu(E_1) > 0$ and $\mu(E_2) > 0$. Using (8) and (9) we arrive at $\mu|_{E_1}, \mu|_{E_2} \in M^0([-1, 1], a)$. The proof is now complete. \[\square\]

Our next result offers some decomposition of a continuous measure.

**Theorem 5.** Let $0 < a < 1$ and $\mu \in P^0([-1, 1], a)$ be a continuous measure.

(i) There exist non-zero measures $\mu_1, \mu_2 \in M^0([-1, 1], a)$ such that $\mu = \mu_1 + \mu_2$ and $\mu_1 \neq c_1 \mu, \mu_2 \neq c_2 \mu$ for any $c_1, c_2 > 0$.

(ii) There exist measures $\nu_1, \nu_2 \in P^0([-1, 1], a)$ and $\alpha \in (0, 1)$ such that $\nu_1 \neq \mu, \nu_2 \neq \mu$ and

\begin{equation}
\mu = \alpha \nu_1 + (1 - \alpha) \nu_2.
\end{equation}

**Proof.**

(i) Take $E_1, E_2 \in B([-1, 1])$ given by Theorem 4 and denote $\mu_1 = \mu|_{E_1}, \mu_2 = \mu|_{E_2}$. Since the measures $\mu_1, \mu_2$ are concentrated on disjoint sets and $\mu = \mu_1 + \mu_2$, we conclude that $\mu_1 \neq c_1 \mu, \mu_2 \neq c_2 \mu$ for any $c_1, c_2 > 0$.

(ii) Put $\nu_1 = \tilde{\mu}_1, \nu_2 = \tilde{\mu}_2$, where $\mu_1, \mu_2$ are defined in (i). Since $\mu_1 \neq c_1 \mu$ and $\mu_2 \neq c_2 \mu$ for any $c_1, c_2 > 0$, then $\nu_1 \neq \mu, \nu_2 \neq \mu$. Setting $\alpha = \mu(E)$, we get (11). This finishes the proof. \[\square\]

The corollary below follows immediately by Theorem 5.

**Corollary 6.** Let $0 < a < 1$ and $\mu \in P^0([-1, 1], a)$. If $\mu$ is a continuous measure, then $\mu$ is not the extreme point of $P^0([-1, 1], a)$.

Let $0 < b < 1$. By $K(b)$ we denote the set of all discrete symmetric probability measures $\mu$ on $B([-1, 1])$ with prescribed moments $(1, 0, b^2, 0)$ and admitting at most four mass points. Now we state for the symmetric probability measure $\mu$ a necessary condition to be the extreme point of $P^0([-1, 1], b)$.

**Theorem 7.** Let $0 < b < 1$. Then $\text{ext} \left( P^0([-1, 1], b) \right) \subset K(b)$.

**Proof.** Let $\sigma \in P^0([-1, 1], b)$ be the extreme point of $P^0([-1, 1], b)$. Then $\sigma$ can be uniquely represented as the sum of a continuous measure and a discrete measure:

\begin{equation}
\sigma = \beta \lambda_1 + (1 - \beta) \lambda_2,
\end{equation}

where $\beta \in [0, 1], \lambda_1, \lambda_2 \in P([-1, 1]), \lambda_1$ is a continuous measure, while $\lambda_2$ is a discrete measure.

Observe that $0 \leq \beta < 1$, because if $\beta = 1$, then $\sigma$ was continuous and, by Corollary 6, $\sigma$ was not the extreme point. If $\beta = 0$, then $\sigma$ is a discrete measure and the assertion follows by [3, Theorem 6.1, p. 101].

Suppose now that $0 < \beta < 1$. We claim that in this case $\sigma$ is not the extreme point. If we show it, the proof is finished.

It is not difficult to check that both $\lambda_1$ and $\lambda_2$ are symmetric with respect to 0. Indeed, if $\sigma(\{x\}) > 0$ for some $x \in [-1, 1]$, then $\sigma(\{-x\}) > 0$. Therefore a discrete part of
σ, i.e. \((1 - \beta)\lambda_2\), is symmetric, which implies that \(\lambda_2\) is symmetric. Hence also \(\lambda_1\) is symmetric.

As a probability measure, \(\lambda_1\) is non-zero. Then \(\lambda_1 \in \mathcal{P}^0([-1,1],a)\), where \(a^2 = \int_{-1}^{1} x^2 \lambda_1(dx) \) and \(0 < a < 1\). Now we apply Theorem 5 (ii) to the measure \(\lambda_1\). There exist the measures \(\nu_1, \nu_2 \in \mathcal{P}^0([-1,1],a)\) and \(0 < \alpha < 1\) such that \(\nu_1 \neq \nu_2\) and

\[
\lambda_1 = \alpha \nu_1 + (1 - \alpha) \nu_2. 
\]

We have also \(\lambda_2 \in \mathcal{P}^0([-1,1],c)\), where \(c^2 = \int_{-1}^{1} x^2 \lambda_2(dx)\) and \(0 < c < 1\). Write

\[
\lambda_2 = \alpha \lambda_2 + (1 - \alpha) \lambda_2.
\]

Using the properties of the measures \(\sigma, \lambda_1, \lambda_2\) we get

\[
b^2 = \int_{-1}^{1} x^2 \sigma(dx) = \beta \int_{-1}^{1} x^2 \lambda_1(dx) + (1 - \beta) \int_{-1}^{1} x^2 \lambda_2(dx) = \beta a^2 + (1 - \beta) c^2.
\]

By (12), (13), (14), \(\sigma\) can be written as

\[
\sigma = \alpha \sigma_1 + (1 - \alpha) \sigma_2
\]

with \(\sigma_1 = \beta \nu_1 + (1 - \beta) \nu_2\) and \(\sigma_2 = \beta \nu_2 + (1 - \beta) \lambda_2\). Since \(\nu_1 \neq \nu_2\), then \(\sigma_1 \neq \sigma_2\). Using (15) we arrive at \(\sigma_1, \sigma_2 \in \mathcal{P}^0([-1,1],b),\) which implies that \(\sigma\) is not the extreme point of \(\mathcal{P}^0([-1,1],b)\) and completes the proof.

Notice that every probability measure \(\mu \in \mathcal{K}(b)\) (with \(0 < b < 1\)) can be written in the form

\[
\mu_{(x,y)} = \frac{1}{2} p (\delta_x + \delta_{-x}) + \frac{1}{2} q (\delta_y + \delta_{-y})
\]

for some \(0 \leq x \leq y \leq 1\) and \(p, q > 0\) with \(p + q = 1\).

Observe that if

\[
\int_{-1}^{1} u^2 \mu_{(x,y)}(du) = b^2,
\]

then

\[
px^2 + qy^2 = b^2.
\]

It is easy to see that if \(0 \leq x \leq y \leq 1\) and \(p, q > 0\) with \(p + q = 1\) satisfying the above equation, then \(0 \leq x \leq b \leq y \leq 1\).

If additionally \(x \neq y\), then the numbers \(p, q\) could be computed by

\[
p = \frac{y^2 - b^2}{y^2 - x^2}, \quad q = \frac{b^2 - x^2}{y^2 - x^2}.
\]

**Remark 8.** The set \(\mathcal{K}(b)\) consists of all probability measures \(\mu_{(x,y)}\) given by (16), where \(0 \leq x \leq b \leq y \leq 1, p, q > 0\) with \(p + q = 1\) satisfying (17).

One could easily show the lemma.

**Lemma 9.** Let \(0 < b < 1\). Then \(\mathcal{K}(b) \subset \text{ext} \left( \mathcal{P}^0([-1,1],b) \right)\).

Thus, we derive from Theorem 7 and Lemma 9 the main result of this paper.

**Theorem 10.** Let \(0 < b < 1\). Then \(\text{ext} \left( \mathcal{P}^0([-1,1],b) \right) = \mathcal{K}(b)\).

Of course the extreme points of \(\mathcal{P}^0([-1,1],b)\) are the measures admitting 2, 3 or 4 mass points. The only two-point extreme measure is

\[
\mu_{(b,b)} = \frac{1}{2} (\delta_{-b} + \delta_b).
\]
All three-point extreme measures have the form

\[ \mu_{(0,y)} = p\delta_0 + \frac{1}{2y} (\delta_{-y} + \delta_y), \]

where \( b \leq y \leq 1 \). In particular, for \( y = 1 \) we get

\[ \mu_{(0,1)} = (1 - b^2)\delta_0 + \frac{b^2}{2} (\delta_{-1} + \delta_1). \]

4. Integral representation of probability measures

As an application of results obtained in the previous section concerning the extreme measures we shall give a theorem on integral representation of probability measures from the set \( \mathcal{P}^0([-1,1], b) \).

The set of probability measures \( \mathcal{P}(\mathbb{R}) \) is metrizable. In metrizing the weak convergence of probability measures on \( \mathcal{B}(\mathbb{R}) \) the Lévy-Prohorov distance (see \([1, \text{Chapter 11, Theorem 11.3.3, p. 395]}) can be used. The set \( \mathcal{P}([-1,1]) \) is a metrizable compact convex subset of \( \mathcal{P}(\mathbb{R}) \). It is not difficult to prove that \( \mathcal{P}^0([-1,1], b) \) is a closed and convex subset of \( \mathcal{P}([-1,1]) \). Consequently, we have the following lemma.

**Lemma 11.** The set \( \mathcal{P}^0([-1,1], b) \) is a metrizable compact convex subset of \( \mathcal{P}([-1,1]) \).

On \( \mathcal{P}^0([-1,1], b) \) consider the topology induced from \( \mathcal{P}(\mathbb{R}) \) and the mapping \( T : [0, b] \times [b, 1] \rightarrow \mathcal{K}(b) \) given by

\[ T(x, y) = \mu_{(x,y)}. \]

Taking into account Remark 8 and the formulae (18), it is not difficult to prove the lemma.

**Lemma 12.** The mapping \( T \) is a homeomorphism between \([0, b] \times [b, 1] \) and \( \mathcal{K}(b) \).

Now we state the main result of this section.

**Theorem 13.** Let \( 0 < b < 1 \). For every probability measure \( \sigma \in \mathcal{P}^0([-1,1], b) \) there exists a probability measure \( \gamma \) on \( \mathcal{B}([0, b] \times [b, 1]) \) such that

\[ \int_{-1}^{1} f(u)\sigma(du) = \int_{-1}^{1} f(u) \left( \int_{[0,b] \times [b,1]} \mu_{(x,y)} \gamma(dx,y) \right)(du) \]

for any continuous function \( f : [-1,1] \rightarrow \mathbb{R} \).

**Proof.** We shall use Choquet’s Representation Theorem ([5, p. 14]). Taking into account Theorem 10 we obtain that for every measure \( \sigma \in \mathcal{P}^0([-1,1], b) \) there exists a probability measure \( m \in \mathcal{K}(b) \) such that

\[ L(\sigma) = \int_{\mathcal{K}(b)} L(\mu)m(d\mu) \]

for any continuous linear functional \( L \) on \( \mathcal{P}([-1,1]) \).

Let \( \gamma \) be the measure on \( \mathcal{B}([0, b] \times [b, 1]) \) defined by \( \gamma = m \circ T \), i.e. \( \gamma(B) = m(TB) \) for \( B \in \mathcal{B}([0, b] \times [b, 1]) \). Taking into account Lemma 12, for any continuous linear functional \( L \) on \( \mathcal{P}([-1,1]) \) we have

\[ L(\sigma) = \int_{\mathcal{K}(b)} L(\mu)m(d\mu) = \int_{T^{-1}(\mathcal{K}(b))} L \circ T(\gamma)(m \circ T)(dx,y) \]

\[ = \int_{[0,b] \times [b,1]} L \circ T(dx,y)\gamma(dx,y). \]

For every continuous function \( f : [-1,1] \rightarrow \mathbb{R} \) the linear functional

\[ L_f(\mu) = \int_{-1}^{1} f(u)\mu(du) \]
is continuous. Hence, for the probability measure $\sigma \in \mathcal{P}^0([-1, 1], b)$ we obtain by (19)
\[
\int_{-1}^{1} f(u) \sigma(du) = \int_{[0,b] \times [b,1]} \int_{-1}^{1} f(u) \mu(x,y)(du) \gamma(d(x,y)) \\
= \int_{-1}^{1} f(u) \left( \int_{[0,b] \times [b,1]} \mu(x,y) \gamma(d(x,y)) \right)(du).
\]

The theorem is proved. $\square$

**Remark 14.** Notice that Theorem 13 is related to [3, Theorem 6.3, p. 103].

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