Local zeta factors and geometries under Spec \( \mathbb{Z} \)

Yu. I. Manin

Abstract. The first part of this note shows that the odd-period polynomial of each Hecke cusp eigenform for the full modular group produces via the Rodriguez-Villegas transform (\([1]\)) a polynomial satisfying the functional equation of zeta type and having non-trivial zeros only in the middle line of its critical strip. The second part discusses the Chebyshev lambda-structure of the polynomial ring as Borger’s descent data to \( \mathbb{F}_1 \) and suggests its role in a possible relation of the \( \Gamma_R \)-factor to ‘real geometry over \( \mathbb{F}_1 \)’ (cf. \([2]\)).

Keywords: cusp forms, period polynomials, local factors.

À Jean-Pierre Serre, en témoignage d’admiration

Introduction

In his influential seminar talk \([3]\), Jean-Pierre Serre stated precise conjectures about the structure of local factors of zeta functions of algebraic varieties over arithmetic rings. In particular, he defined the local factors at complex, resp. real, archimedean completions of the base as multiplicative combinations of gamma functions involving Hodge numbers. (Of course, local factors at finite primes since Weil and Grothendieck have been treated in terms of Galois representations on cohomology as characteristic polynomials of Frobenius.)

In my seminar talks \([4]\) dedicated to the geometry and arithmetics over Jacques Tits’ mythical ‘field with one element \( \mathbb{F}_1 \)’ I suggested the existence of respective local zetas ‘in characteristic one’ and noticed that Riemann’s gamma-factor at the infinite prime looks like such a local factor in characteristic one of infinite-dimensional projective space \( \mathbb{P}_1^{\infty} \) appropriately regularized.

More precisely, in \([4]\) I defined the (inverted) zeta function of \( \mathbb{P}_1^{k} \) as

\[
(2\pi)^{-(k+1)}s(s-1)\cdots(s-k).
\tag{0.1}
\]

On the other hand, Deninger (\([5]\)) represented the basic \( \Gamma \)-factor at (complex) arithmetical infinity as the infinite determinant of the complex Frobenius map and a regularized product

\[
\Gamma_C(s)^{-1} := \left(\frac{2\pi}{\Gamma(s)}\right)^s = \prod_{n \geq 0} \frac{s+n}{2\pi}.
\tag{0.2}
\]
Comparing (0.1) to (0.2), I suggested that this gamma-factor, with changed sign of $s$, might be imagined as the zeta-function of the infinite dimensional projective space over $\mathbf{F}_1$. I did not discuss the problem of a similar interpretation of the real gamma-factor.

Since 1992, there has been a growing body of definitions and studies of $\mathbf{F}_1$-geometries, cf. the surveys and comprehensive bibliographies in [6], [7]. In particular, C. Soulé in [8] put on a firm ground my heuristics about local zeta factors over $\mathbf{F}_1$. In particular, natural factors of zetas of $\mathbf{F}_1$-schemes turned out to be polynomials in $s$, satisfying a functional equation expressing their symmetry with respect to the map $s \mapsto c - s$. In the main text, I will use for such polynomials the generic name ‘zeta polynomials’, complementing their description by the requirement that non-trivial zeros must lie on the vertical line at the middle of critical strip, cf. Theorem 1.1 below.

For other insights about $\mathbf{F}_1$, see [9], [10] and the description of A. Smirnov’s work in [11]. A sophisticated modern approach is developed in [2].

However, the bridges between characteristics zero and one, and in particular the $\mathbf{P}_\mathbf{F}_1$-heuristics about (0.2) still remain to a considerable degree elusive.

In this short note, I contribute additional strokes to this mystery.

In § 1, I show that each cusp form $f$ for $\text{PGL}(2, \mathbf{Z})$ which is eigenform for all Hecke operators, besides the usual $p$-factors of its Mellin transform, produces another polynomial that looks like a ‘local zeta factor in characteristic one’. This polynomial is obtained from the odd-period polynomial of $f$ in the same formal way as the Hilbert polynomial of a graded algebra is produced from its Poincaré series; see [1]. Formulae (1.7) and (1.8) below suggest that this formalism can be considered as a discrete version of the Mellin transform as well.

For analogies with zetas and geometric interpretations of the latter, cf. also [12].

In § 2, I suggest how an expected gamma-bridge between characteristics zero and one could take into account the fact that in Serre’s picture gamma-factors corresponding to real and complex infinite arithmetic primes are different. To this end, I appeal to J. Borger’s identification of lambda-structures on schemes with descent data to $\mathbf{F}_1$ ([13], [11]), and to the idea suggested in [7] that Habiro rings are lifts to $\text{Spec} \mathbf{Z}$ of ‘rings of analytic functions’ in characteristic one. Then it turns out that two different lambda-structures on the polynomial ring, the toric one and the Chebyshev one, faithfully reflect the difference between complex and real analytic geometry in characteristic one.

Notice that lambda-structures naturally appear in several contexts, related to zetas: see, for example, [2], [14] and [15]. It would be interesting to include Borger’s philosophy in these contexts as well.

§ 1. Zeta polynomials from cusp forms

1.1. Period polynomials and period functions. Here we are considering modular forms with respect to $\text{PSL}(2, \mathbf{Z})$; $k$ is a positive even weight; $w := k - 2$; $S_k$ denotes the space of cusp forms; $M_k$ is the space all modular forms of weight $k$.  
Period polynomials for cusp forms are defined by:

\[ r_f(z) := \int_0^{i\infty} f(\tau)(\tau - z)^{k-2} d\tau, \quad r_f^\pm(z) := \frac{r_f(z) \pm r_f(-z)}{2}. \]

The following more general formula is valid also for Eisenstein series: if

\[ f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n s} \in M_k, \]

define its Eichler integral by

\[ \mathcal{E}_f(z) := \int_z^{i\infty} (f(\tau) - a_0)(\tau - z)^{k-2} d\tau = -\frac{(k-2)!}{(2\pi i)^{k-1}} \sum_{n=1}^{\infty} \frac{a_n}{n^{k-1}} e^{2\pi i n z} \]

and then define its period function by

\[ r_f(z) := \mathcal{E}_f(z) - z^{k-2}\mathcal{E}_f\left(-\frac{1}{z}\right), \quad r_f^\pm(z) := \frac{r_f(z) \pm r_f(-z)}{2}. \]

If \( f \) is not a cusp form, then \( r_f(z) \in z^{-1} \mathbb{C}[z]. \)

1.2. Spaces of period functions. If

\[ g \in \text{PSL}(2, \mathbb{Z}), \quad g(z) = \frac{az + b}{cz + d}, \]

the right action \( |_w \) of \( g \) on the space \( V_w \) of polynomials \( r \) of degree \( \leq w \) is defined by

\[ (r |_w g)(z) := (cz + d)^w r(g(z)). \]

Let

\[ S(z) = -\frac{1}{z}, \quad U(z) = 1 - \frac{1}{z} \]

and

\[ Y_w := \{ r \in V_w \mid r |_w (1 + S) = r |_w (1 + U + U^2) = 0 \}. \quad (1.1) \]

For \( f \in S_k \), we have \( r_f(z) \in Y_w \). Let \( Y_w^\pm \) mean the respective subspaces of even/odd polynomials.

It is well known (Eichler–Shimura) that the map \( r^- : f \mapsto r_f^- (z) \) defines an isomorphism \( S_k \rightarrow Y_w^- \), whereas \( r^+ \) defines an embedding of codimension one, \( S_k \rightarrow Y_w^+ \).

Recently it was proved ([16]) that if \( f \in S_k \) is a Hecke eigenform, then

\[ U_f(z) := \frac{r_f^-(z)}{z(z^2 - 4)(z^2 - 1/4)(z^2 - 1)^2} \quad (1.2) \]

is a polynomial without real zeros whose complex zeros all lie on the unit circle. Clearly, its degree is \( e := w - 10 \).
1.3.

**Theorem 1.1.** Fix an integer \( d > e = w - 10 \) and put

\[
P_f(z) := \frac{U_f(z)}{(1 - z)^d}.
\]

(1.3)

There exists a polynomial \( H_f(x) \in \mathbb{C}[x] \) of degree \( d - 1 \) such that if

\[
P_f(z) = \sum_{n=0}^{\infty} H(n)z^n
\]

for \(|z| < 1\), then \( H(n) = H_f(n) \) for all large \( n \). This polynomial satisfies the functional equation

\[
H_f(x) = (-1)^{d-1}H_f(-d + e - x)
\]

(1.4)

and it vanishes at \( x = -1, \ldots, -d + e + 1 \). All its remaining zeros lie on the vertical line \( \text{Re} x = -(d - e - 1)/2 \).

**Proof.** This is a direct application of the proposition in §3 of [1] (due in more general form to Popoviciu), and of its corollary. One condition for the applicability of this proposition is ensured by the theorem about zeros of (1.2) in [16]. We have only to check the functional equation (9) in this proposition, that is, the identity

\[
P_f\left(\frac{1}{z}\right) = (-1)^d z^{d-e} P_f(z).
\]

(1.5)

Rewriting (1.5) as

\[
\frac{U_f(1/z)}{(1 - 1/z)^d} = (-1)^d z^{d-e} \frac{U_f(z)}{(1 - z)^d}
\]

one sees that it is equivalent to

\[
U_f\left(\frac{1}{z}\right) = z^{-e} U_f(z)
\]

that is, in view of (1.2),

\[
\frac{r_f^{-}(1/z)}{z^{-1}(z^{-2} - 4)(z^{-2} - 1/4)(z^{-2} - 1)^2} = \frac{r_f^{-}(z)}{z(z^2 - 4)(z^2 - 1/4)(z^2 - 1)^2}.
\]

(1.6)

Now, from \( r|_w (1 + S) = 0 \) it follows that

\[
r_f^{-}\left(\frac{1}{z}\right) = -r_f^{-}\left(\frac{-1}{z}\right) = z^{-w} r_f^{-}(z).
\]

Inserting this into (1.6), we finally get (1.5).
1.4.

Remark 1.2. In [17], it was proved that all zeros of the full period polynomial of a Hecke cusp form lie on the unit circle. Similarly, all zeros of $zr_f(z)$ for Eisenstein Hecke series lie on the unit circle.

However, I was unable to fit these cases into the framework of the Rodriguez-Villegas construction because the analogue of the functional equation (1.5) seemingly fails for the complete period polynomial.

Remark 1.3. I use the generic catchword ‘zeta polynomials’ for polynomials of one variable satisfying a version of a functional equation such as (1.4) and the ‘Riemann hypothesis’. In [1], it was in particular proved that Hilbert polynomials of certain graded rings are zeta polynomials. Golyshev ([12]) considered rings of homogeneous functions on Fano and Calabi–Yau varieties with respect to anticanonical or related projective embeddings and found interesting geometric correlates of these results.

Moreover, comparing the formula
$$H_f(n) = \frac{1}{2\pi i} \int_\gamma P_f(z)z^{-(n+1)} \, dz$$

(1.7)

(where $\gamma$ is a small contour around zero) with the Mellin transform
$$Z_f(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^{i\infty} f(z)\left(\frac{z}{i}\right)^{s-1} d\left(\frac{z}{i}\right)$$

(1.8)

one sees a considerable formal analogy: morally, $H_f$ is the ‘discrete Mellin transform of $P_f$‘.

In particular, the argument $n$ of $H_f$ corresponds to the classical $-s$: this is consistent with observations in [1] and [12].

However, finding an appropriate geometric living space for zeta polynomials $H_f$ associated with Hecke cusp forms seemingly requires the more general realm of ‘geometries under Spec $\mathbb{Z}$‘. The problem is that in most versions of $\mathbb{F}_1$-geometries those zeta polynomials that appear as zeta functions of motives over $\mathbb{F}_1$ have only integer zeros: cf. for example [18]. On the contrary, our $H_f$ seem to come from some non-Tate motives and geometric objects lying below Spec $\mathbb{Z}$ but not over $\mathbb{F}_1$. I expect that they arise from the levels below Spec $\mathbb{Z}$ to which such moduli stacks as $\mathcal{M}_{1,n}$ can be descended.

Remark 1.4. Notice finally that period polynomials also appear in studies of the Galois action on the Grothendieck–Teichmueller groupoid: see [19]–[21] and the references therein about their role in Hodge realisations. One can guess that the special role of period polynomials of Hecke eigenforms will become clearer in the light of an étale setting.

§ 2. Habiro Lambda-Rings

2.1. Habiro rings. The Habiro ring $\mathcal{H}$ of one variable over $\mathbb{Z}$ is defined as the projective limit of quotient rings $\mathbb{Z}[q]/(f(q))$, where $f(q)$ runs over the multiplicative set of monic polynomials all of whose roots are roots of unity. This ring was
introduced and studied in [22], and in [7] it was suggested to consider it as ‘the ring of analytic functions on $G_m$ lifted from $F_1$’. In fact, $\mathbb{Z}[q]$ is naturally embedded into the Habiro completion $\mathcal{H}$, and $q$ becomes invertible there, so that $\mathcal{H}$ can be also defined as a completion of $\mathbb{Z}[q, q^{-1}]$. One can extend this definition to the case of several invertible variables that is, functions on tori.

2.2. Lambda-rings. J. Borger developed in [13] the idea of interpreting Grothendieck lambda-structures on schemes as general descent data to $F_1$. It is therefore natural to expect that the Habiro ring admits a natural lambda-structure. Here we will be concerned only with commutative rings $A$ flat over $\mathbb{Z}$, in which case a lambda-structure can be considered simply as a system of commuting lifts of Frobenius: ring homomorphisms $\psi^p: A \to A$ for each prime $p$ such that

$$\psi^p(x) \equiv x^p \mod pA \quad \forall x \in A$$

and $\psi^{p_1} \psi^{p_2} = \psi^{p_2} \psi^{p_1}$. In particular, we can define $\psi^k: A \to A$ for all positive integers $k$ by multiplicativity.

The most natural lambda-structure on $\mathbb{Z}[q]$ and $\mathbb{Z}[q, q^{-1}]$ is determined by $\psi^k(q) = q^k$, and since it is compatible with the projective limit over the system of cyclotomic polynomials in $q$, it is inherited by the Habiro ring. We will call this structure the toric one.

However, the polynomial ring $\mathbb{Z}[r]$ admits another lambda-structure discovered by Clauwens ([23]). In this structure,

$$\psi^k(r) := T_k(r)$$

where $T_k$ is the $k$-th Chebyshev polynomial. Our next result describes a subring $\mathcal{H}_0 \subset \mathcal{H}$ which can be endowed with Chebyshev lambda-structure.

2.3. Proposition 2.1. (i) Consider in the Habiro ring $\mathcal{H}$ the subring $\mathcal{H}_0$ defined as the completion of the polynomial subring $\mathbb{Z}[r]$, where

$$r := 1 + q + \sum_{n=1}^{\infty} q^n (1 - q) \cdots (1 - q^n). \quad \text{(2.1)}$$

This subring is invariant with respect to the standard lambda-structure $\psi^k$, which induces on this subring, in terms of the coordinate $r$, the Chebyshev lambda-structure.

(ii) $\mathcal{H}_0$ is strictly smaller than $\mathcal{H}$.

Proof. (i) In $\mathcal{H}$, we have the convergent expression for $q^{-1}$ (see [22], Proposition 7.1):

$$q^{-1} = 1 + \sum_{n=1}^{\infty} q^n (1 - q) \cdots (1 - q^n).$$

Hence $r = q + q^{-1}$. Moreover, using one of the definitions of Chebyshev polynomials, we see that

$$\psi^k(r) = q^k + q^{-k} = T_k(q + q^{-1}) = T_k(r).$$
(ii) In order to see that $H_0$ is strictly smaller than $H$, we can use the following result due to Habiro. Any element of $H$ determines a function on the set of roots of unity $\mu_\infty$ with values in $\mathbb{Z}[\mu_\infty]$, and the resulting map

$$H \to \text{Map}(\mu_\infty, \mathbb{Z}[\mu_\infty])$$

is an embedding (see [22]). The element $q$ corresponds to the tautological map $\mu_\infty \to \mathbb{Z}[\mu_\infty]$.

Then all elements of $\mathbb{Z}[r]$ become functions invariant under the involution $\zeta \to \zeta^{-1}$ of $\mu_\infty$ and their values are invariant as well. This property holds after the completion. Hence $q \notin H_0$.

Notice that for each complex embedding of $\mu_\infty$ and any $\eta \in \mu_\infty$, $\eta + \eta^{-1}$ is real. This is why we referred to ‘real analytic geometry over $F_1$’.

Bibliography

[1] F. Rodriguez-Villegas, “On the zeros of certain polynomials”, Proc. Amer. Math. Soc. 130:8 (2002), 2251–2254.
[2] A. Connes and C. Consani, Cyclic homology, Serre’s local factors and the $\lambda$-operations, arXiv:1211.4239.
[3] J.-P. Serre, “Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures)”, Séminaire Delange–Pisot–Poitou. Théorie des nombres, 11e année, 1969/70, exp. No. 19, Secrétariat Math., Paris 1970.
[4] Yu. Manin, “Lectures on zeta functions and motives (according to Deninger and Kurokawa)”, Columbia University number theory seminar (New York 1992), Astérisque, vol. 228, Soc. Math. France, Paris 1995, pp. 121–163.
[5] C. Deninger, “On the $\Gamma$-factors attached to motives”, Invent. Math. 104:1 (1991), 245–261.
[6] O. Lorscheid, A blueprinted view on $\mathbb{F}_1$-geometry, arXiv:1301.0083.
[7] Yu. I. Manin, “Cyclotomic and analytic geometry over $\mathbb{F}_1$”, Quanta of maths, Clay Math. Proc., vol. 11, Amer. Math. Soc., Providence, RI 2010, pp. 385–408; arXiv: 0809.1564.
[8] C. Soulé, “Les variétés sur le corps à un élément”, Mosc. Math. J. 4:1 (2004), 217–244.
[9] M. Kapranov and A. Smirnov, Cohomology determinants and reciprocity laws: number field case, Unpublished manuscript, 1996.
[10] A. Connes and C. Consani, “Schemes over $\mathbb{F}_1$ and zeta functions”, Compos. Math. 146:6 (2010), 1383–1415.
[11] L. Le Bruyn, Absolute geometry and the Habiro topology, arXiv:1304.6532.
[12] V. V. Golyshin, The canonical strip. I, arXiv:0903.2076.
[13] J. Borger, $\Lambda$-rings and the field with one element, arXiv:0906.3146.
[14] N. Naumann, “Algebraic independence in the Grothendieck ring of varieties”, Trans. Amer. Math. Soc. 359:4 (2007), 1653–1683.
[15] N. Ramachandran, “Zeta functions, Grothendieck groups, and the Witt ring”, Bull. Sci. Math. 139:6 (2015), 599–627; arXiv: 1407.1813.
[16] J. B. Conrey, D. W. Farmer, and Ö. Imamoglu, “The nontrivial zeros of period polynomials of modular forms lie on the unit circle”, Int. Math. Res. Not. IMRN 2013:20 (2013), 4758–4771; arXiv: 1201.2322.
[17] A. El-Guindy and W. Raji, “Unimodularity of zeros of period polynomials of Hecke eigenforms”, Bull. Lond. Math. Soc. 46:3 (2014), 528–536.
[18] O. Lorscheid, “Functional equations for zeta functions of $F_1$-schemes”, C. R. Math. Acad. Sci. Paris 348:21-22 (2010), 1143–1146; arXiv: 1010.1754.
[19] L. Schneps, “On the Poisson bracket on the free Lie algebra in two generators”, J. Lie Theory 16:1 (2006), 19–37.
[20] R. Hain, The Hodge–de Rham theory of modular groups, arXiv: 1403.6443.
[21] A. Pollack, Relations between derivations arising from modular forms, http://dukespace.lib.duke.edu/dspace/handle/10161/1281.
[22] K. Habiro, “Cyclotomic completions of polynomial rings”, Publ. Res. Inst. Math. Sci. 40:4 (2004), 1127–1146.
[23] F. J.-B. J. Clauwens, “Commuting polynomials and $\lambda$-ring structures on $\mathbb{Z}[x]$”, J. Pure Appl. Algebra 95:3 (1994), 261–269.

Yurii I. Manin
Max-Planck-Institut für Mathematik, Bonn, Germany
E-mail: manin@mpim-bonn.mpg.de

Received 20/ APR/ 15
1/ SEP/ 15