A topological invariant for continuous fields of Cuntz algebras II

Taro Sogabe
Graduate School of Science, Kyoto University, Japan
staro@math.kyoto-u.ac.jp

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Abstract

We investigate an invariant for continuous fields of the Cuntz algebra $O_{n+1}$ introduced in [18], and find a way to obtain a continuous field of $M_n(O_\infty)$ from that of $O_{n+1}$ using the construction of the invariant. By Brown’s representability theorem, this gives a bijection from the set of the isomorphism classes of continuous fields of $O_{n+1}$ to those of $M_n(O_\infty)$.

1 Introduction

Our purpose is to investigate the invariant for continuous fields of the Cuntz algebra $O_{n+1}$ introduced in [18]. The Cuntz algebra $O_{n+1}$ is a typical example of a Kirchberg algebra, and its continuous fields over a finite CW-complex $X$ are classified in [5] when the cohomology groups $H^*(X,\mathbb{Z})$ do not admit $n$-torsion. All continuous fields classified in [5] are constructed via the Cuntz–Pimsner algebras, and it is proved in [5] that, for general $X$, not every continuous field of $O_{n+1}$ is given by the Cuntz–Pimsner construction.

Recently, M. Dadarlat and U. Pennig introduced a generalized cohomology $E_D^*$ in [9] for every strongly self-absorbing C*-algebra $D$ satisfying the UCT including the infinite Cuntz algebra $O_\infty$. In [18], using the reduced cohomology $\tilde{E}_{O_\infty}^*$, we define an invariant $b_{O_\infty}$ of continuous fields of $O_{n+1}$, and show that a continuous field $O$ over a finite CW-complex $X$ is given via the Cuntz–Pimsner algebra if and only if $b_{O_\infty}([O]) = 0 \in \tilde{E}_{O_\infty}^1(X)$. By [6], the set of the isomorphism classes of continuous fields of $O_{n+1}$ over a finite CW-complex $X$ is identified with the homotopy set $[X, B\text{Aut}(O_{n+1})]$, and the invariant is a map $b_{O_\infty} : [X, B\text{Aut}(O_{n+1})] \to \tilde{E}_{O_\infty}^1(X)$.

In this paper, we construct a natural transformation

$$T_X : [X, B\text{Aut}(O_{n+1})] \to [X, B\text{Aut}(M_n(O_\infty))]$$

which turns out to be bijective by Brown’s representability theorem (Theorem 3.3). Since the homotopy set $[X, B\text{Aut}(M_n(O_\infty))]$ is identified with the set of the isomorphism classes of locally trivial continuous fields of $M_n(O_\infty)$, the map $B(\eta)_* : [X, B\text{Aut}(M_n(O_\infty))] \ni [\mathcal{B}] \mapsto [K \otimes \mathcal{B}] \in \tilde{E}_{O_\infty}^1(X)$ is defined (see Section 2.2), and we have $-b_{O_\infty} = B(\eta)_* \circ T_X$. 
Since the inverse image $b^{-1}_{\infty}(0)$ is equal to the set of the Cuntz–Pimsner algebras of vector bundles classified in [5] (see [18 Sec. 4]), M. Dadarlat’s classification result [5 Th. 5.3], which enables us to count the cardinality of the set of the isomorphism classes of those Cuntz–Pimsner algebras, can be proved by counting the cardinality of the set of the isomorphism classes of the continuous fields of $\mathcal{M}_n(\mathcal{O}_\infty)$ stably isomorphic to the trivial field (see Remark 2.4 Corollary 3.10).

We also investigate the map $T_{\Sigma X}$ for the reduced suspension $SX$, and show that the map gives a group isomorphism $T_{\Sigma X} : [X, \text{Aut}(\mathcal{O}_{n+1})] \to [X, \text{Aut}(\mathcal{M}_n(\mathcal{O}_\infty))]$ (Corollary 3.9).

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2 Preliminaries

2.1 Notation

Let $\mathbb{K}$ be the C*-algebra of compact operators on the separable infinite dimensional Hilbert space, and let $\mathcal{M}_n$ be the n by n matrix algebra. For a C*-algebra $A$, we denote by $K_i(A)$ the i-th K-group and denote by $[p]_0 \in K_0(A)$ (resp. $[u]_1 \in K_1(A)$) the class of the projection $p$ (resp. the unitary $u$). If $A$ is unital, we denote by $1_A$ the unit and by $U(A)$ the group of unitary elements. Let $C(X)$ be the C*-algebra of all continuous functions on $X$. We write $K^i(X) = K_i(C(X)), K^i(X) = K_i(C_0(X, x_0))$ where $C_0(X, x_0)$ is the set of functions vanishing at $x_0 \in X$. We refer to [1] for the K-groups.

Let $(X, x_0)$ and $(Y, y_0)$ be two pointed finite CW-complexes, and let $[X, Y]$ (resp. $[X, Y]_0$) be the set of the homotopy classes of the continuous maps (resp. the set of the base point preserving homotopy classes of the base point preserving continuous maps). The i-th homotopy group of $X$ is $\pi_i(X) := [S^i, X]_0$, and, for $Y$ with $\pi_0(Y) = \pi_1(Y) = 0$, the natural map $[X, Y]_0 \to [X, Y]$ is bijective (see [11 Th. 6.57]).

We refer to [17] [6] for the definition of the continuous $C(X)$-algebras. Let $\mathcal{P} \to X$ be a principal Aut$(A)$ bundle, and let $\mathcal{A}$ be the associated bundle $\mathcal{P} \times_{\text{Aut}(A)} A$. Then, the section algebra $\Gamma(X, \mathcal{A})$ is a locally trivial continuous $C(X)$-algebra. We identify the section algebra with the associated bundle, and write $\mathcal{A}$ by abuse of notation. Let $\mathcal{A}_x$ denote the image of the evaluation map $ev_x : \mathcal{A} \to \mathcal{A}/C_0(X, x)A \cong A$. We always assume that the fiber $\mathcal{A}_x$ is nuclear. Since those principal Aut$(A)$ bundles are classified by $[X, BAut(A)]$, we denote the $C(X)$-linear isomorphism class of the $C(X)$-algebra by $[\mathcal{A}] \in [X, BAut(A)]$. For two locally trivial continuous $C(X)$-algebras $\mathcal{A}, \mathcal{B}$, we can define the tensor product $\mathcal{A} \otimes_{C(X)} \mathcal{B}$ (see [2]).

Let $E_{n+1}$ be the universal C*-algebra, called the Cuntz–Toeplitz algebra, generated by $n + 1$ isometries with mutually orthogonal ranges. It is known that the unital map $\mathbb{C} \to E_{n+1}$ is a KK-equivalence. Let $\{T_i\}_{i=1}^{n+1}$ be the canonical generators, and let $\varepsilon := 1 - \sum_{i=1}^{n+1} T_i T_i^*$ be the minimal projection which generates the only non-trivial ideal of $E_{n+1}$ isomorphic to $\mathbb{K}$. The quotient algebra $\mathcal{O}_{n+1} := E_{n+1}/\mathbb{K}$ is called the Cuntz algebra. We denote by $\mathcal{O}_\infty$ the universal C*-algebra generated by countably infinite isometries with mutually orthogonal ranges. The inclusion $\mathbb{K} \to E_{n+1}$ gives the map $K_0(\mathbb{K}) = \mathbb{Z} \overset{\sim}{\to}$.
$K_0(E_{n+1}) = \mathbb{Z}$, and one has

$$K_0(\mathcal{O}_{n+1}) = \mathbb{Z}_n, \quad K_0(\mathcal{O}_\infty) = \mathbb{Z}, \quad K_1(\mathcal{O}_{n+1}) = K_1(\mathcal{O}_\infty) = 0$$

(see [3]). One has $\mathcal{O}_{n+1} \cong \mathcal{O}_{n+1} \otimes \mathcal{O}_\infty$ and $\mathcal{O}_{n+1} \cong (E_{n+1} \otimes \mathcal{O}_\infty)/(\mathbb{K} \otimes \mathcal{O}_\infty)$ by the classification result of the Kirchberg algebras.

The homotopy groups of $\text{Aut}(\mathcal{O}_{n+1})$ and $\text{Aut}(\mathcal{M}_n(\mathcal{O}_\infty))$ are given by [8] Th. 5.9:

$$\pi_{2k}(\text{Aut}(\mathcal{O}_{n+1})) = \pi_{2k}(\text{Aut}(\mathcal{M}_n(\mathcal{O}_\infty))) = 0, \quad \pi_{2k+1}(\text{Aut}(\mathcal{O}_{n+1})) = \pi_{2k+1}(\text{Aut}(\mathcal{M}_n(\mathcal{O}_\infty))) = \mathbb{Z}_n, \quad k \geq 0.$$

Thus, two sets $[S^k, \text{BAut}(\mathcal{O}_{n+1})], [S^k, \text{BAut}(\mathcal{M}_n(\mathcal{O}_\infty))]$ have the same cardinality, and one has $[X, \text{BAut}(\mathcal{O}_{n+1})]_0 = [X, \text{BAut}(\mathcal{O}_{n+1})], [X, \text{BAut}(\mathcal{M}_n(\mathcal{O}_\infty))]_0 = [X, \text{BAut}(\mathcal{M}_n(\mathcal{O}_\infty))]$.

### 2.2 The Dadarlat–Pennig theory

We briefly explain the cohomology group $E^1_{\mathcal{O}_\infty}(X)$. Recall that $\mathcal{O}_\infty$ is a strongly self-absorbing C*-algebra, in other words, there is a continuous path of unitary $\{u_t\}_{t \in [0,1]} \subset \mathcal{O}_\infty^\otimes$ and an isomorphism $\phi : \mathcal{O}_\infty \to \mathcal{O}_\infty^\otimes$ satisfying $lim_{t \to 1} \Vert \phi(d) - u_t(1 \otimes d)u_t^* \Vert = 0$. We refer to [19] for the properties of the strongly self-absorbing C*-algebras. The following map

$$\Delta_X : (C(X) \otimes \mathcal{O}_\infty)^{\otimes 2} \ni f_1(x) \otimes f_2(y) \mapsto \phi^{-1}(f_1(x) \otimes f_2(x)) \in C(X) \otimes \mathcal{O}_\infty$$

gives $K_0(C(X) \otimes \mathcal{O}_\infty)$ a ring structure which coincides with the ring structure of $K^0(X)$ coming from the tensor products of vector bundles. Let $K^0(X)^\times := \pm 1 + K^0(X)$ denote the group of the invertible elements.

**Theorem 2.1** ([3] Th. 2.22, 3.8, Lem. 2.8, Cor. 3.9). Let $X$ be a connected compact metrizable space, and let $\text{Aut}_0(\mathbb{K} \otimes \mathcal{O}_\infty)$ be the path component of $\text{Aut}(\mathbb{K} \otimes \mathcal{O}_\infty)$ containing $id_{\mathbb{K} \otimes \mathcal{O}_\infty}$.

1) For two continuous maps $\alpha, \beta : X \to \text{Aut}(\mathbb{K} \otimes \mathcal{O}_\infty)$ which are identified with the $C(X)$-linear isomorphisms of $C(X) \otimes \mathbb{K} \otimes \mathcal{O}_\infty$, one has

$$K_0(\Delta_X) \circ K_0(\alpha \otimes \beta)(([1_{C(X)} \otimes e \otimes 1_{\mathcal{O}_\infty})^{\otimes 2}]_0) = K_0(\alpha \circ \beta)([1_{C(X)} \otimes e \otimes 1_{\mathcal{O}_\infty}]_0)$$

and the following injective map, whose range is $K^0(X)^\times$, is multiplicative:

$$[X, \text{Aut}(\mathbb{K} \otimes \mathcal{O}_\infty)] \ni [\alpha] \mapsto [\alpha(1_{C(X)} \otimes e \otimes 1_{\mathcal{O}_\infty})_0]_0 \in K_0(C(X) \otimes \mathbb{K} \otimes \mathcal{O}_\infty).$$

2) The subgroup $[X, \text{Aut}_0(\mathbb{K} \otimes \mathcal{O}_\infty)] \subset [X, \text{Aut}(\mathbb{K} \otimes \mathcal{O}_\infty)]$ is identified with $1 + K^0(X)$.

3) The homotopy set $E^1_{\mathcal{O}_\infty}(X) := [X, \text{BAut}(\mathbb{K} \otimes \mathcal{O}_\infty)]$ has a group structure defined by the tensor product $\otimes_{C(X)}$ of locally trivial continuous $C(X)$-algebras of $\mathbb{K} \otimes \mathcal{O}_\infty$, and $E^1_{\mathcal{O}_\infty}(X)$ is a subgroup of $E^1_{\mathcal{O}_\infty}(X)$.

For a continuous field $[A] \in E^1_{\mathcal{O}_\infty}(X)$, one has another field denoted by $A^{-1}$ satisfying $\eta[A] = [A^{-1}] \in E^1_{\mathcal{O}_\infty}(X)$ (i.e., a $C(X)$-linear isomorphism $A \otimes_{C(X)} A^{-1} \to C(X) \otimes \mathbb{K} \otimes \mathcal{O}_\infty$ exists). For a locally trivial continuous field $B$ of $\mathcal{M}_n(\mathcal{O}_\infty)$, one has a locally trivial continuous field $\mathbb{K} \otimes B$ of $\mathbb{K} \otimes \mathcal{O}_\infty$. Since $\text{Aut}(\mathcal{M}_n(\mathcal{O}_\infty))$ is path connected, the group homomorphism

$$\eta : \text{Aut}(\mathcal{M}_n(\mathcal{O}_\infty)) \ni \sigma \mapsto id_{\mathbb{K}} \otimes \sigma \in \text{Aut}_0(\mathbb{K} \otimes \mathcal{M}_n(\mathcal{O}_\infty))$$

gives a natural map $B(\eta)_* : [X, \text{BAut}(\mathcal{M}_n(\mathcal{O}_\infty))] \ni [B] \mapsto [\mathbb{K} \otimes B] \in E^1_{\mathcal{O}_\infty}(X)$. 

3
Let $\text{Aut}_X(A)$ be the group of all $C(X)$-linear isomorphisms of a locally trivial continuous $C(X)$-algebra $A$ of $\mathbb{K} \otimes O_\infty$. We fix an isomorphism $A \otimes (\mathbb{K} \otimes O_\infty) \cong f \otimes d \mapsto f \otimes (1_{C(X)} \otimes d) \in A \otimes C(X) (C(X) \otimes \mathbb{K} \otimes O_\infty)$. This gives $K_0(A)$ a $K_0(C(X) \otimes \mathbb{K} \otimes O_\infty)$-module structure by

$$
[q]_0 : K_0(A) \ni [p]_0 \mapsto [p \otimes q]_0 \in K_0(A \otimes C(X) (C(X) \otimes \mathbb{K} \otimes O_\infty)), \quad [q]_0 \in K_0(C(X) \otimes \mathbb{K} \otimes O_\infty).
$$

**Lemma 2.2.** In the above setting, the followings hold:

1) For every $\alpha \in \text{Aut}_X(A)$, there is an element $a \in K^0(X)_\times$ with $K_0(\alpha) = \cdot a$.

2) For every $a \in K^0(X)_\times$, there is an element $\alpha \in \text{Aut}_X(A)$ with $\alpha \cdot a = K_0(\alpha)$.

**Proof.** We prove only 1). Fix an isomorphism $\theta : (A^{-1} \otimes C(X), A) \otimes C(X)^2 \to C(X) \otimes \mathbb{K} \otimes O_\infty$. One has the following commutative diagram

$$
\begin{array}{ccc}
A \otimes (\mathbb{K} \otimes O_\infty) & \xrightarrow{\alpha \otimes \text{id}} & A \otimes (\mathbb{K} \otimes O_\infty) \\
\downarrow & & \downarrow \\
A \otimes C(X) (C(X) \otimes \mathbb{K} \otimes O_\infty) & \xrightarrow{\alpha \otimes \text{id}} & A \otimes C(X) (C(X) \otimes \mathbb{K} \otimes O_\infty) \\
\downarrow \text{id} \otimes \theta & & \downarrow \text{id} \otimes \theta \\
A \otimes C(X) (A^{-1} \otimes A) \otimes C(X) (A^{-1} \otimes A) & \xrightarrow{\alpha \otimes (\text{id}_{A^{-1}} \otimes \text{id}_A) \otimes \theta} & A \otimes C(X) (A^{-1} \otimes A) \otimes C(X) (A^{-1} \otimes A).
\end{array}
$$

Since the flip automorphism $\sigma : (\mathbb{K} \otimes O_\infty)^{\otimes 2} \ni x \otimes y \mapsto y \otimes x \in (\mathbb{K} \otimes O_\infty)^{\otimes 2}$ fixing the minimal projection $(e \otimes 1_{O_\infty})^{\otimes 2}$ is contained in $\text{Aut}_0((\mathbb{K} \otimes O_\infty)^{\otimes 2})$ by Theorem 2.1, 2), one has $K_0(\alpha \otimes (\text{id}_{A^{-1}} \otimes \text{id}_A) \otimes \theta) = K_0(\alpha \otimes (\text{id}_{A^{-1}} \otimes \text{id}_A))$. We have an element

$$
a := [\theta \circ (\text{id}_{A^{-1}} \otimes \alpha \otimes \text{id}_{A^{-1}} \otimes \text{id}_A) \circ \theta^{-1}(1_{C(X)} \otimes e \otimes 1_{O_\infty})]_0 \in K^0(X)_\times
$$

satisfying $K_0(\alpha) = \cdot a$.

Using Theorem 2.1, similar argument proves the statement 2).}

**Proposition 2.3.** Let $(X, x_0)$ be a pointed, path connected, finite CW-complex. For $[A] \in \text{Im}(B(\eta)_*) \subset E^1_{O_\infty}(X)$, we fix an isomorphism $\varphi : A_{x_0} \cong \mathbb{K} \otimes O_\infty$. Then, the following map is a well-defined bijection

$$
B(\eta)_-^{-1}(\{A\}) \ni [B] \mapsto [[\rho_B(e \otimes 1_B)]_0]_0 \in \{[p]_0 \in K_0(A)|K_0(\varphi \circ ev_{x_0})([p]_0) = n\}/\sim,
$$

where $\rho_B : \mathbb{K} \otimes B \to A$ is an isomorphism satisfying $[(\varphi \circ ev_{x_0} \circ \rho_B)(e \otimes 1_B)]_0 = n \in K_0(\mathbb{K} \otimes O_\infty)$. Here, the equivalence relation is defined by $[p]_0 \sim [r]_0 \iff [p]_0 \cdot a = [r]_0$ for some $a \in 1 + K^0(X)$.

**Proof.** First, we show that the map is well-defined. For two continuous fields $B_1, B_2$ with $[B_1] = [B_2] \in \text{B}(\eta)_-^{-1}(\{A\})$ and two $C(X)$-linear isomorphisms $\rho_1 : \mathbb{K} \otimes B_1 \to A$ satisfying $[(\varphi \circ ev_{x_0} \circ \rho_1)(e \otimes 1_{B_1})]_0 = n$, we show $[\rho_1(e \otimes 1_{B_1})]_0 \sim [\rho_2(e \otimes 1_{B_2})]_0$. Since we have an isomorphism $\gamma : B_1 \to B_2$, the following map

$$
\beta := \rho_2 \circ (\text{id}_{\mathbb{K}} \otimes \gamma) \circ \rho_1^{-1} \in \text{Aut}_X(A)
$$

satisfies $\beta \circ \rho_1(e \otimes 1_{B_1}) = \rho_2(e \otimes 1_{B_2})$, and Lemma 2.2 shows $[\rho_1(e \otimes 1_{B_1})]_0 \sim [\rho_2(e \otimes 1_{B_2})]_0$.

By [7] Th. 1.1, 2.7, the element $[\rho_1(A)p]_0$ is sent to $[[p]_0]$ and this map is surjective.

Finally, we prove the injectivity. Suppose $[\rho_1(e \otimes 1_{B_1})]_0 \sim [\rho_2(e \otimes 1_{B_2})]_0$. Then, Lemma 2.2 shows there is a map $\alpha \in \text{Aut}_X(A)$ with $[\alpha(\rho_1(e \otimes 1_{B_1})]_0 = [\rho_2(e \otimes 1_{B_2})]_0$. Now the cancellation of the properly infinite full projections shows $B_1 \cong B_2$.

**Remark 2.4.** For $[A] = 0$, Proposition 2.3 allows us to count the number of the isomorphism classes of the continuous fields of $\mathcal{M}_n(O_\infty)$ stably isomorphic to $C(X) \otimes \mathbb{K} \otimes O_\infty$, which is equal to $(n + \tilde{K}^0(X))/(1 + \tilde{K}^0(X))$.\]
2.3 The invariant \( b_{\mathcal{O}_\infty} \)

An automorphism of \( E_{n+1} \otimes \mathcal{O}_\infty \) induces an automorphism of \( \mathcal{O}_{n+1} \cong (E_{n+1} \otimes \mathcal{O}_\infty)/(\mathbb{K} \otimes \mathcal{O}_\infty) \), and one has a group homomorphism \( q : \text{Aut}(E_{n+1} \otimes \mathcal{O}_\infty) \to \text{Aut}(\mathcal{O}_{n+1}) \).

**Theorem 2.5 ([IN Cor. 3.15, Def. 4.1])** Let \( X \) be a finite CW-complex.

1) The group homomorphism \( q : \text{Aut}(E_{n+1} \otimes \mathcal{O}_\infty) \to \text{Aut}(\mathcal{O}_{n+1}) \) is a weak homotopy equivalence.

2) For every continuous field \( \mathcal{O} \) of \( \mathcal{O}_{n+1} \), one has an exact sequence of \( C(X) \)-algebras

\[
0 \to A \to E \to \mathcal{O} \to 0,
\]

where we denote by \( E \) (resp. \( A \)) a continuous field of \( E_{n+1} \otimes \mathcal{O}_\infty \) (resp. \( \mathbb{K} \otimes \mathcal{O}_\infty \)), and the following map is well-defined:

\[
b_{\mathcal{O}_\infty} : [X, \text{BAut}(\mathcal{O}_{n+1})] \ni [\mathcal{O}] \mapsto [A] \in E_{n+1}^0(X).
\]

One has a bijection \( (\mathbb{B}(q), s)^{-1} : [X, \text{BAut}(\mathcal{O}_{n+1})] \ni [\mathcal{O}] \mapsto [E] \in [X, \text{BAut}(E_{n+1} \otimes \mathcal{O}_\infty)] \) by Theorem 2.5 1), and the group homomorphism \( \text{Aut}(E_{n+1} \otimes \mathcal{O}_\infty) \ni \alpha \mapsto \alpha|_{\mathbb{K} \otimes \mathcal{O}_\infty} \in \text{Aut}_0(\mathbb{K} \otimes \mathcal{O}_\infty) \) induces the map \( [X, \text{BAut}(E_{n+1} \otimes \mathcal{O}_\infty)] \ni [E] \mapsto [A] \in E_{n+1}^0(X) \) (see [IN Lem. 3.7]).

Since \( E_{n+1} \otimes \mathcal{O}_\infty \) is KK-equivalent to \( \mathbb{C} \), the map \( \text{id}_{A_{-1}} \otimes 1 : A_{-1} \ni f \mapsto f \otimes 1_\mathcal{E} \in A_{-1} \otimes C(X) \) gives the isomorphism \( K_0(\text{id}_{A_{-1}} \otimes 1) \) of their \( K_0 \)-groups by [7 Th. 1.1].

**Lemma 2.6.** Let \((X, x_0)\) be a pointed, path connected, finite CW-complex. Let \( A \) and \( E \) be as in Theorem 2.5 2), and let \( \iota : A \to E \) be the inclusion map. We fix isomorphisms \( \varphi : E_{x_0} \cong E_{n+1} \otimes \mathcal{O}_\infty \) and \( \pi : (A_{-1})_{x_0} \cong \mathbb{K} \otimes \mathcal{O}_\infty \). Then, there exists a properly infinite full projection \( p \in A_{-1} \) satisfying \( K_0(\pi \circ ev_x)([p]_0) = n \in K_0(\mathbb{K} \otimes \mathcal{O}_\infty) \) and

\[
[K_0(\pi \circ ev_x)([\psi(1_{C(X)} \otimes e \otimes 1_{\mathcal{O}_\infty})])_0 = 1 \in K_0(\mathbb{K} \otimes \mathcal{O}_\infty)]^2.
\]

By [IN Th. 4.2] and [10 Th. 2.11], the algebra \( A_{-1} \) is isomorphic to a tensor product of a unital \( \mathcal{O}_\infty \)-stable algebra and \( \mathbb{K} \). Therefore, one can find a properly infinite full projection \( p \in A_{-1} \) such that \([-p]_0 \in K_0(A_{-1}) \) is sent to \( K_0(\text{id}_{A_{-1}} \otimes \iota)([\psi(1_{C(X)} \otimes e \otimes 1_{\mathcal{O}_\infty})])_0 \in K_0(A_{-1} \otimes C(X)_\mathcal{E}) \) by the isomorphism \( K_0(\text{id}_{A_{-1}} \otimes 1) \). Thanks to [7 Th. 1.1], two inclusion maps \( p(A_{-1}) \otimes C(X) \mathcal{E} \hookrightarrow A_{-1} \otimes C(X) \mathcal{E} \) and \( p(A_{-1}) \otimes C(X) \mathcal{E} \hookrightarrow A_{-1} \otimes C(X) \mathcal{E} \) give isomorphisms of \( \mathbb{K} \)-groups, and the statement is now proved.

**Remark 2.7.** Since \( K_0(\mathcal{E}) = K^0(X) \), the image of the \( K^0(X) \)-module homomorphism \( K_0(A) \to K_0(\mathcal{E}) \) is an ideal of \( K^0(X) \). In the case of \( \text{Tor}(H^*(X, \mathbb{Z}_n) = 0 \), the ideal is the complete invariant of the continuous field \( \mathcal{E}/A \) (see [13 Sec. 2] and [18 Sec. 4]). The element \([-p]_0 \in K_0(A_{-1}) \) corresponds to \( KK_X(\iota) \in KK_X(A, \mathcal{E}) \cong K_0(A_{-1}) \), and one can identify \( K_0(\iota) \) with the map \( K_0(A) \ni [p]_0 \mapsto [-p \otimes p]_0 \in K_0(A \otimes \mathcal{E}/A) \cong K^0(X) \).

By Lemma 2.6 we constructs an element \([p(A_{-1})p] \in [X, \text{BAut}(\mathcal{M}_n(\mathcal{O}_\infty))] \) from \([\mathcal{O}] \in [X, \text{BAut}(\mathcal{O}_{n+1})] \). In Section 3, we verify that this gives a natural transformation between two functors \([\cdot, \text{BAut}(\mathcal{O}_{n+1})] \) and \([\cdot, \text{BAut}(\mathcal{M}_n(\mathcal{O}_\infty))] \) defined on the category of the pointed connected finite CW-complexes.
2.4 The homotopy sets \([X, \text{Aut}(\mathcal{O}_{n+1})]\) and \([X, \text{Aut}(\mathbb{M}_n(\mathcal{O}_\infty))]\)

We briefly recall \cite{8} Th. 5.9. Let \(B\) be a unital Kirchberg algebra with path connected \(\text{Aut}(B)\), and let \(C_v := \{f \in C[0, 1] \otimes B]f(0) = 0, f(1) \in C_1 B\}\) be the mapping cone of the unital map \(\nu : C \to B\) with the inclusion map \(j : SB := C_0(0, 1) \otimes B \to C_v B\). Let \(\alpha, \lambda : (X, x_0) \to (\text{Aut}(B), \text{id}_B)\) be the base point preserving continuous maps where \(l : x \mapsto \text{id}_B\) is the constant map. These two maps define an element \((\alpha, l) \in KK(C_v, SC_0(X, x_0) \otimes B)\) (see \cite{8} p 123), and the map \([X, \text{Aut}(B)] \ni [\alpha] \mapsto (\alpha, l) \in KK(C_v, SC_0(X, x_0) \otimes B)\) is bijective.

In the case of \(B = \mathbb{M}_n(\mathcal{O}_\infty)\), one has \(KK(C_v, SC_0(X, x_0) \otimes B) = KK(C_v, SC(X) \otimes B)\), and the map \(j^* : KK(C_v, SC(X) \otimes B) \to KK(SB, SC(X) \otimes B)\) maps \((\alpha, l)\) to \(KK(id_{C_0(0, 1)} \otimes \alpha) - KK(id_{C_0(0, 1)} \otimes l) = S(\eta_\nu([\alpha]) - 1)\). Here, the map \(S : K^0(X) = KK(B, C(X) \otimes B) \to KK(SB, SC(X) \otimes B)\) is the suspension isomorphism (see Theorem 2.1 for the definition of \(\eta_\nu\)). For another map \(\beta : (X, x_0) \to (\text{Aut}(B), \text{id}_B)\), one has

\[
(\alpha \circ \beta, l) = (\alpha \circ \beta, \alpha) + (\alpha, l) = (id_{C_0(B)} \otimes \alpha) \cdot ((\beta, l)) + (\alpha, l) = (\alpha, l) + (\beta, l) = (\alpha, l) + (\beta, l) + (\beta, l) \cdot (S^{-1} \circ j^*)((\alpha, l)).
\]

The product \((\beta, l) \cdot (S^{-1} \circ j^*)((\alpha, l))\) is defined by

\[
KK(C_v, SC(X) \otimes B) \times KK(C, C(X) \otimes B) \to KK(C_v, SC(X \times B) \otimes B)
\]

\[
\Delta x \mapsto KK(C_v, SC(X) \otimes B).
\]

**Theorem 2.8** (\cite{8} Th. 6.3, Th. 5.9). We write \(\text{Ad} : U(C(X) \otimes \mathbb{M}_n(\mathcal{O}_\infty)) \ni v \mapsto \text{Ad} v \in \text{Map}(X, \text{Aut}(\mathbb{M}_n(\mathcal{O}_\infty)))\).

1. There is a short exact sequence of groups:

\[
0 \to K_1(C(X) \otimes \mathbb{M}_n(\mathcal{O}_\infty)) \otimes \mathbb{Z}_n \xrightarrow{\text{Ad}} [X, \text{Aut}(\mathbb{M}_n(\mathcal{O}_\infty))] \xrightarrow{\nu} (1 + \text{Tor}(K^0(X), \mathbb{Z}_n))^\times \to 0.
\]

2. The multiplication \(a \ast b := a + b + b \cdot (S^{-1} \circ j^*)(a)\) makes \((KK(C_v, \mathbb{M}_n(\mathcal{O}_\infty)), SC(X) \otimes \mathbb{M}_n(\mathcal{O}_\infty)), \ast\) a group with the following isomorphism

\[
[X, \text{Aut}(\mathbb{M}_n(\mathcal{O}_\infty))] \ni [\beta] \mapsto (\beta, l) \in KK(C_v, \mathbb{M}_n(\mathcal{O}_\infty), SC(X) \otimes \mathbb{M}_n(\mathcal{O}_\infty)).
\]

M. Izumi obtained similar results for \([X, \text{Aut}(\mathcal{O}_{n+1})]\). Let \(\delta : K_1(C(X) \otimes \mathcal{O}_{n+1}) \to K^0(X)\) be the index map coming from the exact sequence \(C(X) \otimes \mathbb{K} \to C(X) \otimes \mathcal{E}_{n+1} \to C(X) \otimes \mathcal{O}_{n+1}\). The map \(u : \text{Aut}(\mathcal{O}_{n+1}) \to U(\mathcal{O}_{n+1})\) defined by \(u(\alpha) := \sum_{i=1}^{n+1} \alpha(S_i)S_i^*\) is a weak homotopy equivalence (see \cite{8}), where the isometries \(S_i\) are the canonical generators of \(\mathcal{O}_{n+1}\).

**Theorem 2.9** (\cite{15} Th. 3.1). We define a multiplication of \(K_1(C(X) \otimes \mathcal{O}_{n+1})\) by \(a \circ b := a + b - a \cdot \delta(b)\).

1. The map \(u_\ast : [X, \text{Aut}(\mathcal{O}_{n+1})] \to (K_1(C(X) \otimes \mathcal{O}_{n+1}), \circ)\) is a group isomorphism.

2. There is a short exact sequence of groups

\[
0 \to K^1(X) \otimes \mathbb{Z}_n \to [X, \text{Aut}(\mathcal{O}_{n+1})] \xrightarrow{1 - \delta} (1 + \text{Tor}(K^0(X), \mathbb{Z}_n))^\times \to 0
\]

where the map \(1 - \delta\) is defined by \((1 - \delta)([\alpha]) := 1 - \delta([u(\alpha)]_1)\).
Let \( \Sigma X \) be the unreduced suspension of \( X \). Then, one has a bijection \( [\Sigma X, BG] \rightarrow [X, G] \) for every path connected group \( G \) (see [12 Cor. 8.3]), where an element \( [\alpha] \in [X, G] \) is sent to the principal \( G \)-bundle whose clutching function is \( \alpha : X \rightarrow G \). We denote by
\[
\Gamma_\alpha(\Sigma X)_A := \{(F_1, F_2) \in (C([0, 1] \times X) \otimes A)^{\otimes 2} | F_1(0) \in 1_{C(X) \otimes A}, F_1(1) = \alpha(F_2(1)) \in C(X) \otimes A\}
\]
the \( C(\Sigma X) \)-algebra whose isomorphism class in \( [\Sigma X, BAut(A)] \) corresponds to the element \([\alpha] \in [X, Aut(A)]\). We use another \( C^* \)-algebra defined by
\[
M_{\alpha}(X)_A := \{ f \in C([0, 1] \times X) \otimes A | f(0) \in 1_{C(X) \otimes A}, f(1) = \alpha(f(0)) \in C(X) \otimes A \}
\]
with the unital map \( M_{\alpha}(X)_A \ni f \mapsto (f(t), f(0)) \in \Gamma_\alpha(\Sigma X)_A \).

For every \([\alpha] \in K^1(X) \otimes \mathbb{Z}_n \subset [X, Aut(O_{n+1})] \), we have a unitary \( U_{\alpha} \in U(C(X) \otimes E_{n+1}) \) with \( \pi(U_{\alpha}) = u(\alpha) \in C(X) \otimes O_{n+1} \), where \( \pi : C(X) \otimes E_{n+1} \rightarrow C(X) \otimes O_{n+1} \) is the quotient map. We need the following lemma in Section 3.

**Lemma 2.10** ([13]). Let \( X \) be a connected finite CW-complex, and let \( \alpha, U_{\alpha} \) be as above. Let \( v \in U(C(X) \otimes M_n(O_{\infty})) \) be a unitary satisfying \([v]_1 = [U_{\alpha}]_1 \in K_1(C(X) \otimes E_{n+1} \otimes M_n(O_{\infty})) \). Then, we have
\[
[1_{M_n \otimes Ad_{\alpha}(X)(O_{n+1} \otimes O_{n+1} \otimes O_{\infty})}]_0 = 0 \in K_0(M_{\alpha \otimes Ad_{\alpha}}(X)(O_{n+1} \otimes O_{n+1} \otimes O_{\infty})).
\]

In particular, one has \([1_{\Gamma_{\alpha}(\Sigma X)(O_{n+1} \otimes O_{n+1} \otimes O_{\infty})}]_0 = 0 \).

**Proof.** Let \( W \) be the following unitary
\[
W := \begin{pmatrix}
O_{n+1} & S^* \\
S & 0_1
\end{pmatrix} \in M_{n+2}(O_{n+1}) \subset C(X) \otimes O_{n+1} \otimes M_n(O_{\infty}),
\]
where we write \( S := (S_1, \ldots, S_{n+1}) \). The unitary \( W \) is self-adjoint and there is a continuous path of unitary \( \{V_t|t \in [0, 1] \subset C(X) \otimes O_{n+1} \otimes M_n(O_{\infty}) \) satisfying \( V_0 = W, V_1 = \alpha \otimes Ad(v \oplus 1)\) \). The unitary \( V \) is an element of \( M_{\alpha \otimes Ad(v \oplus 1)}(X)(O_{n+1} \otimes O_{n+1} \otimes O_{\infty}) \). It is easy to check that the following inclusion gives isomorphisms of K-groups
\[
M_{\alpha \otimes Ad_{\alpha}}(X)(O_{n+1} \otimes O_{n+1} \otimes O_{\infty}) \ni f \mapsto f \oplus O_{2} \in M_{\alpha \otimes Ad(v \oplus 1)}(X)(O_{n+1} \otimes O_{n+1} \otimes O_{\infty}),
\]
and one has
\[
[1_{M_n \otimes Ad_{\alpha}(X)(O_{n+1} \otimes O_{n+1} \otimes O_{\infty})}]_0 = \begin{bmatrix}
1_{n+1} & 0 \\
0 & 0_1
\end{bmatrix} \otimes \begin{bmatrix}
1_{n+1} & 0 \\
0 & 0_1
\end{bmatrix} \otimes \begin{bmatrix}
O_{n+1} & 0 \\
0 & 1_1
\end{bmatrix} = 0.
\]

One has
\[
V \begin{pmatrix}
1_{n+1} & 0 \\
0 & 0_1
\end{pmatrix} V^* = WV \begin{pmatrix}
O_{n+1} & 0 \\
0 & 1_1
\end{pmatrix} WV^*,
\]
and the direct computation yields
\[
V_1 W = \begin{pmatrix}
v \oplus 1_1 & 0 \\
0 & 1_1
\end{pmatrix} \begin{pmatrix}
O_{n+1} & \alpha(S)^* \\
\alpha(S) & 0_1
\end{pmatrix} \begin{pmatrix}
v \oplus 1_1 & 0 \\
0 & 1_1
\end{pmatrix} \begin{pmatrix}
O_{n+1} & S^* \\
S & 0_1
\end{pmatrix} = \begin{pmatrix}
S^*(v \oplus 1_1)S^* u(\alpha)^* S \\
0 & u(\alpha) S (v \oplus 1_1) S^*
\end{pmatrix}.
\]

Using the following exact sequence
\[
K_0(SC(X) \otimes O_{n+1} \otimes M_{n+2}(O_{\infty})) \rightarrow K_0(M_{\alpha \otimes Ad(v \oplus 1)}(X)(O_{n+1} \otimes O_{n+1} \otimes O_{\infty})))
\]
3 The main theorem

Let \([\mathcal{E}] \in [X, \text{BAut}(E_{n+1} \otimes O_\infty)]\) (resp. \([\mathcal{B}] \in [X, \text{BAut}(M_n(O_\infty))]\)) denote the isomorphism class of a locally trivial continuous \(C(X)\)-algebra \(\mathcal{E}\) (resp. \(\mathcal{B}\)) whose fiber is \(E_{n+1} \otimes O_\infty\) (resp. \(M_n(O_\infty)\)). There is a locally trivial continuous \(C(X)\)-algebra \(\mathcal{A} \subset \mathcal{E}\) whose fiber is \(K \otimes O_\infty\), and we have \(b_{O_\infty}([\mathcal{E}]) := [\mathcal{A}] \in E_{O_\infty}^1(X)\) (see Theorem 2.5).

**Theorem 3.1.** Let \((X, x_0)\) be a pointed, path connected, finite CW-complex. For every \([\mathcal{E}] \in [X, \text{BAut}(E_{n+1} \otimes O_\infty)]\), there is a unique element \([\mathcal{B}] \in [X, \text{BAut}(M_n(O_\infty))]\) satisfying

\[
\{K \otimes O_\infty = -[\mathcal{A}] = -b_{O_\infty}([\mathcal{E}])\} \in E_{O_\infty}^1(X),
\]

\[
[1_{B \otimes \mathcal{E}}]_0 \in \text{Im}(K_0(B \otimes C(X), A) \xrightarrow{K_0(id \otimes \gamma)} K_0(B \otimes C(X), \mathcal{E})).
\]

Here, we denote by \(\iota : \mathcal{A} \rightarrow \mathcal{E}\) the inclusion map.

Lemma 2.6 implies \([\mathcal{B}] = [p(A^{-1})p]\), and the second condition is equivalent to

\[
[1_{B \otimes O}]_0 = 0 = K_0(B \otimes C(X), O).
\]

**Lemma 3.2.** Fix a continuous field \(\mathcal{E}\) of \(E_{n+1} \otimes O_\infty\). For two continuous fields \(B_1, B_2\) of \(M_n(O_\infty)\) satisfying two conditions in Theorem 3.1 with respect to \(\mathcal{E}\), we have \([B_1] = [B_2]\).

**Proof.** One can find a \(C(X)\)-linear isomorphism \(\gamma : K \otimes B_1 \rightarrow K \otimes B_2\) satisfying \([e \otimes 1_{B_2}]_0 = n \in K_0(K \otimes B_2, x_0)\). Let \(\iota : \mathcal{A} \rightarrow \mathcal{E}\) be as in Theorem 2.5. Since the inclusion \(K \otimes O_\infty \rightarrow E_{n+1} \otimes O_\infty\) gives a map \(\mathbb{Z} \rightarrow \mathbb{Z}\) of \(K_0\)-groups, the preimage of \([1_{B_2 \otimes \mathcal{E}}]_0\) should be a “rank \(-1\)” projection in \(K_0(B_1 \otimes C(X), A) = K_0(X)\). Now one has an element \(a_1 \in -1 + \mathbb{K}^0(X) \subset K_0(K \otimes B_1 \otimes C(X), A)\) which is sent to \([e \otimes 1_{B_2}]_0 \in K_0(K \otimes B_1 \otimes C(X), \mathcal{E})\) by the map \(K_0(id \otimes \gamma)\). The following commutative diagram and Lemma 2.2 show that there is a map \(\alpha \in \text{Aut}_X(K \otimes B_2)\) satisfying \([\gamma(e \otimes 1_{B_1})]_0 = [\alpha(e \otimes 1_{B_2})]_0\), \((-K_0(\gamma \otimes id)(a_1) \cdot a_2^{-1}) = K_0(\alpha)\):

\[
\begin{array}{ccc}
K \otimes B_1 \otimes C(X, \mathcal{A}) & \xrightarrow{id \otimes \gamma} & K \otimes B_1 \otimes C(X) \mathcal{E} & \xrightarrow{id \otimes 1} & K \otimes B_1 \\
\gamma \otimes id & & & & \\
K \otimes B_2 \otimes C(X, \mathcal{A}) & \xrightarrow{id \otimes \gamma} & K \otimes B_2 \otimes C(X) \mathcal{E} & \xrightarrow{id \otimes 1} & K \otimes B_2.
\end{array}
\]

Therefore, we have \(B_1 \cong B_2\). \(\square\)

**Proof of Theorem 3.1.** Let \((\mathcal{E}_i, A_i, B_i), i = 1, 2\) be continuous fields satisfying the conditions of Theorem 3.1. Assume that there is a \(C(X)\)-linear isomorphism \(\phi : \mathcal{E}_1 \rightarrow \mathcal{E}_2\) (i.e., \([\mathcal{E}_1] = [\mathcal{E}_2]\)). We show \([B_1] = [B_2]\). Since the following diagram commutes,

\[
B_1 \otimes C(X) \mathcal{A}_1 \xrightarrow{id \otimes 1} B_1 \otimes C(X) \mathcal{E}_1 \\
\downarrow{id \otimes \phi} & \downarrow{id \otimes \phi} \\
B_1 \otimes C(X) \mathcal{A}_2 \xrightarrow{id \otimes 1} B_1 \otimes C(X) \mathcal{E}_2,
\]

Therefore, we have \(B_1 \cong B_2\). \(\square\)
the pair \((\mathcal{E}_2, \mathcal{A}_2, \mathcal{B}_1)\) also satisfies the conditions. Now Lemma 3.2 proves the statement.

By Theorem 3.1, the map \(t_X : [X, \text{BAut}(E_n \otimes O_\infty)] \ni [\mathcal{E}] \mapsto [\mathcal{B}] \in [X, \text{BAut}(M_n(\mathcal{O}_\infty))]\) is well-defined. For a base point preserving continuous map \(f : (Y, y_0) \to (X, x_0)\) and a continuous field \(\mathcal{E}\) over \(X\), one has the pull-back of the continuous field \(f^* \mathcal{E} := C(Y) \otimes C(X) \mathcal{E}\) with a natural homomorphism \(\mathcal{E} \ni d \mapsto 1 \otimes d \in C(Y) \otimes C(X) \mathcal{E} = f^* \mathcal{E}\). Since one has \(f^* (\mathcal{B} \otimes C(X) \mathcal{E}) \cong f^* \mathcal{B} \otimes C(Y) f^* \mathcal{E}\), it is easy to check that the map \(t_X\) is natural with respect to \(X\).

**Definition 3.3.** Let \(q : \text{Aut}(E_n \otimes O_\infty) \to \text{Aut}(O_{n+1})\) be the group homomorphism which is a weak homotopy equivalence. Let \(\mathcal{C}_0\) be the category whose objects are pointed, path connected, finite CW-complexes, and morphisms are the base point preserving continuous maps. Let \(\mathcal{S}\) be the category of sets with a distinguished element and maps preserving the distinguished elements. For \((X, x_0) \in \mathcal{C}_0\), we regard \((X, x_0) \mapsto [X, \text{BAut}(O_{n+1})]\) and \((X, x_0) \mapsto [X, \text{BAut}(M_n(\mathcal{O}_\infty))]\) as contravariant functors from \(\mathcal{C}_0\) to \(\mathcal{S}\) where the distinguished element is the homotopy class of the constant map. We define a natural transformation by

\[
T_X := t_X \circ (B(q)_*)^{-1} : [X, \text{BAut}(O_{n+1})] \to [X, \text{BAut}(M_n(\mathcal{O}_\infty))].
\]

**Proposition 3.4.** One has \(T_X([\mathcal{O}]) = [\mathcal{B}]\) if and only if \([1_{\mathcal{B} \otimes \mathcal{O}}]_0 = 0 \in K_0(\mathcal{B} \otimes C(X) \mathcal{O})\).

**Proof.** Assume \([1_{\mathcal{B} \otimes \mathcal{O}}]_0 = 0\). Since there is an exact sequence \(\mathcal{A} \to \mathcal{E} \to \mathcal{O}\) as in Theorem 2.3, we have an element \(a \in K_0(\mathcal{B} \otimes C(X) \mathcal{A})\) which is sent to \([1_{\mathcal{B} \otimes \mathcal{E}}]_0 \in K_0(\mathcal{B} \otimes C(X) \mathcal{E})\). Therefore, the element \(a\) is a “rank \(-1\)” projection, and [9, Th. 4.2] implies \(-[\mathcal{A}] = [K \otimes \mathcal{B}] \in E^1_{\infty}(X)\) (i.e., \(T_X([\mathcal{O}]) = [\mathcal{B}]\)).

**Remark 3.5.** One has \([X, \text{BAut}(O_{n+1})]_0 = [X, \text{BAut}(O_{n+1})]\) and \([X, \text{BAut}(M_n(\mathcal{O}_\infty))]_0 = [X, \text{BAut}(M_n(\mathcal{O}_\infty))]\) as mentioned in Section 2.1. The map \(T_X\) is bijective for every \((X, x_0) \in \mathcal{C}_0\) if and only if \(T_{S_k}\) is bijective for every \(k \geq 1\) by Brown’s representability theorem [3, Lem. 1.5].

**Lemma 3.6.** Let \(SX\) be the reduced suspension of a connected finite CW-complex \((X, x_0)\). For a path connected group \(G\), we have a natural group isomorphism \([SX, BG]_0 \to [X, G]\).

**Proof.** Since \(G\) is path connected, one has \([SX, BG]_0 = [SX, BG]\). For a CW-complex, the quotient map \(\Sigma X \to X\) is a homotopy equivalence, and the map \([SX, BG] \to [\Sigma X, BG]\) is bijective. By [12, Cor. 8.3], we have a bijection \([\Sigma X, BG] \to [X, G]\), where the homotopy class of the map \(\alpha : X \to G\) is sent to the isomorphism class of the principal \(G\)-bundle on \(\Sigma X\) whose clutching function over \(X\) is the map \(\alpha\). It is easy to check that the composition of the above maps is a group homomorphism.

Since two classifying spaces \(\text{BAut}(O_{n+1})\) and \(\text{BAut}(M_n(\mathcal{O}_\infty))\) have countable homotopy groups, there are CW-complexes \(Y_1, Y_2\) with countably many cells and two weak homotopy equivalences \(Y_1 \to \text{BAut}(O_{n+1}), Y_2 \to \text{BAut}(M_n(\mathcal{O}_\infty))\) (see [11] p 188).

**Corollary 3.7.** The map \(T_{SX} : [SX, \text{BAut}(O_{n+1})] \to [SX, \text{BAut}(M_n(\mathcal{O}_\infty))]\) gives a group homomorphism \([X, \text{Aut}(O_{n+1})] \to [X, \text{Aut}(M_n(\mathcal{O}_\infty))]\), and, for \(X = S^{k-1}\), the map \(T_{S^k} : [S^k, \text{BAut}(O_{n+1})] \to [S^k, \text{BAut}(M_n(\mathcal{O}_\infty))]\) is identified with \(\text{id}_{K_1(S^{k-1} \otimes \mathbb{Z})}\).

**Proof.** First, we show that the map \(T_{SX}\) is a group homomorphism. By Lemma 3.6, it suffices to construct a map \(f : Y_1 \to Y_2\) representing the natural transformation

\[
[SX, Y_1]_0 \to [SX, \text{BAut}(O_{n+1})]_0 \xrightarrow{T_{SX}} [SX, \text{BAut}(M_n(\mathcal{O}_\infty))]_0 \to [SX, Y_2]_0.
\]
Now [3, Lem. 1.7] shows the existence of such a map $f$.

Next, we show that the map $T_{S^k}$ is bijective. Fix $v \in U(C(X) \otimes M_n(O_\infty))$ and $[\alpha] \in [S^{k-1}, \text{Aut}(O_{n+1})]$ with $[v_1] \otimes \mathbb{I} = [U_{\alpha}]_1 \otimes \mathbb{I} \in K^1(S^{k-1}) \otimes \mathbb{Z}_n = [S^{k-1}, \text{Aut}(O_{n+1})]$ as in Lemma 2.10. Lemma 2.10 and Proposition 3.4 implies $T_{S^k}([\alpha]) = [\Gamma_{\text{Ad}}(\Sigma S^{k-1})_{M_n(O_\infty)}]$ (i.e., $T_{S^k}([\alpha]) = [\text{Ad}]$). Therefore, the map $T_{S^k}$ is identified with

$$id : K^1(S^{k-1}) \otimes \mathbb{Z}_n \ni [U_{\alpha}]_1 \otimes \mathbb{I} \mapsto [v_1] \otimes \mathbb{I} \in K^1(S^{k-1}) \otimes \mathbb{Z}_n.$$  

Now we have shown our main theorem.

**Theorem 3.8.** For every path connected finite CW-complex $X$, the map $T_X$ is bijective.

**Corollary 3.9.** We have $-b_{O_\infty} = B(\eta)_* \circ T_X$ and $-\text{Im}(b_{O_\infty}) = \text{Im}(B(\eta)_*)$. The map $T_{S^k}$ gives a group isomorphism, and the following diagram commutes:

$$
\begin{array}{ccc}
K^1(X) \otimes \mathbb{Z}_n & \xrightarrow{1 - \delta} & (1 + \text{Tor}(K^0(X), \mathbb{Z}_n))^x \\
\downarrow \text{id} & & \downarrow \text{Tor} \\
K^1(X) \otimes \mathbb{Z}_n & \xrightarrow{\text{Ad}} & [X, \text{Aut}(O_{n+1})] \\
\end{array}
$$

where the right vertical map sends an invertible element of the K-theory ring to its inverse.

Thanks to Remark 2.4, now we have another proof of the result [5, Th. 5.3].

**Corollary 3.10 ([5, Th. 5.3], [18, Th. 14, Rem. 13, 14]).** The cardinality of $b_{O_\infty}^{-1}(0)$ is equal to $|(n + K^0(X))/(1 + K^0(X))|$. In particular, if $\text{Tor}(H^{2k+1}(X), \mathbb{Z}_n) = 0$ for $k \geq 1$, one has $|[X, B\text{Aut}(O_{n+1})]| = |(n + K^0(X))/(1 + K^0(X))|.$

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