Reduction of Couplings in Quantum Field Theories with applications in Finite Theories and the MSSM

S. Heinemeyer1*, M. Mondragón2†, N. Tracas3‡ and G. Zoupanos3,4§

1Instituto de Física de Cantabria (CSIC-UC), E-39005 Santander, Spain,
2Instituto de Física, Universidad Nacional Autónoma de México, A.P. 20-364, México 01000
3Physics Department, Nat. Technical University, 157 80 Zografou, Athens, Greece
4Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, D-69120 Heidelberg, Germany

Abstract

We apply the method of reduction of couplings in a Finite Unified Theory and in the MSSM. The method consists on searching for renormalization group invariant relations among couplings of a renormalizable theory holding to all orders in perturbation theory. It has a remarkable predictive power since, at the unification scale, it leads to relations between gauge and Yukawa couplings in the dimensionless sectors and relations involving the trilinear terms and the Yukawa couplings, as well as a sum rule among the scalar masses and the unified gaugino mass in the soft breaking sector. In both the MSSM and the FUT model we predict the masses of the top and bottom quarks and the light Higgs in remarkable agreement with the experiment. Furthermore we also predict the masses of the other Higgses, as well as the supersymmetric spectrum, both being in very confortable agreement with the LHC bounds on Higgs and supersymmetric particles.

1 Introduction

The discovery of a Higgs boson [1–4] at the LHC completes the search for the particles of the Standard Model (SM), and confirms the existence of a Higgs field and the spontaneous electroweak symmetry breaking mechanism as the way to explain the masses of the fundamental particles. The over twenty free parameters of the SM, the hierarchy problem, the existence of Dark Matter, the very small masses of the neutrinos, among others, point towards a more fundamental theory, whose goal among others should be to explain at least some of these facts.
The main achievement expected from a unified description of interactions is to understand the large number of free parameters of the Standard Model (SM) in terms of a few fundamental ones. In other words, to achieve *reduction of couplings* at a more fundamental level. To reduce the number of free parameters of a theory, and thus render it more predictive, one is usually led to introduce more symmetry. Supersymmetric Grand Unified Theories (GUTs) are very good examples of such a procedure [5–11].

For instance, in the case of minimal SU(5), because of (approximate) gauge coupling unification, it was possible to reduce the gauge couplings to one. LEP data [12] seem to suggest that a further symmetry, namely $N = 1$ global supersymmetry [10, 11] should also be required to make the prediction viable. GUTs can also relate the Yukawa couplings among themselves, again SU(5) provided an example of this by predicting the ratio $M_\tau/M_b$ [13] in the SM. Unfortunately, requiring more gauge symmetry does not seem to help, since additional complications are introduced due to new degrees of freedom and in the ways and channels of breaking the symmetry.

A natural extension of the GUT idea is to find a way to relate the gauge and Yukawa sectors of a theory, that is to achieve Gauge-Yukawa Unification (GYU) [14–16]. A symmetry which naturally relates the two sectors is supersymmetry, in particular $N = 2$ supersymmetry [17]. It turns out, however, that $N = 2$ supersymmetric theories have serious phenomenological problems due to light mirror fermions. Also in superstring theories and in composite models there exist relations among the gauge and Yukawa couplings, but both kind of theories have other phenomenological problems, which we are not going to address here.

A complementary strategy in searching for a more fundamental theory, consists in looking for all-loop renormalization group invariant (RGI) relations [24, 25] holding below the Planck scale, which in turn are preserved down to the unification scale [14–16, 18–23]. Through this method of reduction of couplings [24, 25] it is possible to achieve Gauge-Yukawa Unification [14–16, 26]. Even more remarkable is the fact that it is possible to find RGI relations among couplings that guarantee finiteness to all-orders in perturbation theory [27, 29].

Although supersymmetry seems to be an essential feature for a successful realization of the above programme, its breaking has to be understood too, since it has the ambition to supply the SM with predictions for several of its free parameters. Indeed, the search for RGI relations has been extended to the soft supersymmetry breaking sector (SSB) of these theories [23, 30, 32], which involves parameters of dimension one and two.

### 2 The Method of Reduction of Couplings

In this section we will briefly outline the reduction of couplings method. Any RGI relation among couplings (i.e. which does not depend on the renormalization scale $\mu$ explicitly) can be expressed, in the implicit form $\Phi(g_1, \cdots, g_A) = \text{const.}$, which has to satisfy the partial differential equation (PDE)

$$
\frac{d\Phi}{dt} = \sum_{a=1}^{A} \frac{\partial \Phi}{\partial g_a} \frac{dg_a}{dt} = \sum_{a=1}^{A} \frac{\partial \Phi}{\partial g_a} \beta_a = \vec{\nabla} \Phi \cdot \vec{\beta} = 0,
$$

(1)
where $t = \ln \mu$ and $\beta_a$ is the $\beta$-function of $g_a$. This PDE is equivalent to a set of ordinary differential equations, the so-called reduction equations (REs) \[24, 25, 31\],

$$
\beta_g \frac{dg_a}{dg} = \beta_a , \ a = 1, \cdots, A ,
$$

(2)

where $g$ and $\beta_g$ are the primary coupling and its $\beta$-function, and the counting on $a$ does not include $g$. Since maximally $(A - 1)$ independent RGI “constraints” in the $A$-dimensional space of couplings can be imposed by the $\Phi_a$’s, one could in principle express all the couplings in terms of a single coupling $g$. The strongest requirement in the search for RGI relations is to demand power series solutions to the REs,

$$
g_a = \sum_{n=0}^{\infty} \rho_a^{(n)} g^{2n+1} ,
$$

(3)

which formally preserve perturbative renormalizability. Remarkably, the uniqueness of such power series solutions can be decided already at the one-loop level \[24, 25, 31\].

Searching for a power series solution of the form (3) to the REs (2) is justified since various couplings in supersymmetric theories have the same asymptotic behaviour, thus one can rely that keeping only the first terms in the expansion is a good approximation in realistic applications.

### 3 Reduction of Couplings in Soft Breaking Terms

The method of reducing the dimensionless couplings was extended \[23, 30, 32\] to the soft supersymmetry breaking (SSB) dimensionful parameters of $N = 1$ supersymmetric theories. In addition it was found \[33, 34\] that RGI SSB scalar masses in Gauge-Yukawa unified models satisfy a universal sum rule.

Consider the superpotential given by

$$
W = \frac{1}{2} \mu^{ij} \Phi_i \Phi_j + \frac{1}{6} C^{ijk} \Phi_i \Phi_j \Phi_k ,
$$

(4)

where $\mu^{ij}$ (the mass terms) and $C^{ijk}$ (the Yukawa couplings) are gauge invariant tensors and the matter field $\Phi_i$ transforms according to the irreducible representation $R_i$ of the gauge group $G$. The Lagrangian for SSB terms is

$$
- \mathcal{L}_{SSB} = \frac{1}{6} h^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} b^{ij} \phi_i \phi_j + \frac{1}{2} (m^2)_i^j \phi^* i \phi_j + \frac{1}{2} M \lambda \lambda + \text{H.c.},
$$

(5)

where the $\phi_i$ are the scalar parts of the chiral superfields $\Phi_i$, $\lambda$ are the gauginos and $M$ their unified mass, $h^{ijk}$ and $b^{ij}$ are the trilinear and bilinear dimensionful couplings respectively, and $(m^2)_i^j$ the soft scalars masses.

Let us recall that the one-loop $\beta$-function of the gauge coupling $g$ is given by \[35, 39\]

$$
\beta_g^{(1)} = \frac{dg}{dt} = \frac{g^3}{16\pi^2} \left[ \sum_i T(R_i) - 3 C_2(G) \right] ,
$$

(6)

where $C_2(G)$ is the quadratic Casimir of the adjoint representation of the associated gauge group $G$. $T(R)$ is given by the relation $\text{Tr}[T^a T^b] = T(R) \delta^{ab}$, where $T^a$ are the generators of the group in the appropriate representation. Similarly the $\beta$-functions
of \( C_{ijk} \), by virtue of the non-renormalization theorem, are related to the anomalous dimension matrix \( \gamma_i^j \) of the chiral superfields as:

\[
\beta_{ijk}^C = \frac{dC_{ijk}}{dt} = C_{ijkl} \gamma_k^l + C_{ikl} \gamma_j^l + C_{jkl} \gamma_i^l .
\] (7)

At one-loop level the anomalous dimension, \( \gamma_{ijk}^{(1)} \) of the chiral superfield is

\[
\gamma_{ijk}^{(1)} = \frac{1}{32\pi^2} \left[ C_{ikl} C_{jkl} - 2g^2 C_2(R) \delta_i^j \right],
\] (8)

where \( C_2(R) \) is the quadratic Casimir of the representation \( R_i \), and \( C_{ijk} = C_{ijk}^* \). Then, the \( N = 1 \) non-renormalization theorem [40–42] ensures there are no extra mass and cubic-interaction-term renormalizations, implying that the \( \beta \)-functions of \( C_{ijk} \) can be expressed as linear combinations of the anomalous dimensions \( \gamma_i^j \).

Here we assume that the reduction equations admit power series solutions of the form

\[
C_{ijk} = g \sum_{n=0} \rho_{ijk}^{(n)} g^{2n} .
\] (9)

In order to obtain higher-loop results instead of knowledge of explicit \( \beta \)-functions, which anyway are known only up to two-loops, relations among \( \beta \)-functions are required.

The progress made using the spurion technique, [42–46] leads to all-loop relations among SSB \( \beta \)-functions [47–52]. The assumption, following [48], that the relation among couplings

\[
h_{ijk} = -M(C_{ijk})' = -M \frac{dC_{ijk}(g)}{d\ln g} ,
\] (10)

is RGI and furthermore, the use the all-loop gauge \( \beta \)-function of Novikov et al. [53,54]

\[
\beta_{g}^{NSVZ} = \frac{g^3}{16\pi^2} \left[ \sum_i T(R_i)(1 - \gamma_i/2) - 3C_2(G) \right],
\] (11)

lead to the all-loop RGI sum rule [55] (assuming \( (m^2)^{i,j} = m_{ij}^2 \delta^i_j \)),

\[
m_i^2 + m_j^2 + m_k^2 = |M|^2 \left\{ \frac{1}{1 - g^2 C_2(G)/(8\pi^2)} \frac{d \ln C_{ijk}}{d \ln g} + \frac{1}{2} \frac{d^2 \ln C_{ijk}}{d(\ln g)^2} \right\}
\] (12)

The all-loop results on the SSB \( \beta \)-functions lead to all-loop RGI relations (see e.g. [56]). If we assume:

(a) the existence of a RGI surfaces on which \( C = C(g) \), or equivalently that

\[
\frac{dC_{ijk}}{dg} = \frac{\beta_{ijk}^C}{\beta_g}
\] (13)

holds, i.e. reduction of couplings is possible, and

(b) the existence of a RGI surface on which

\[
h_{ijk} = -M \frac{dC(g)^{ijk}}{d\ln g}
\] (14)
holds too in all-orders, then one can prove that the following relations are RGI to all-loops \cite{57,58} (note that in the above assumptions (a) and (b) we do not rely on specific solutions to these equations)

\begin{align}
M &= M_0 \frac{\beta_0}{g}, \quad (15) \\
h^{ijk} &= -M_0 \delta^{ijk}_C, \quad (16) \\
h^{ij} &= -M_0 \delta^{ij}_\mu, \quad (17) \\
(m^2)^i_j &= \frac{1}{2} |M_0|^2 \mu \frac{d\gamma^i_j}{d\mu}, \quad (18)
\end{align}

where \( M_0 \) is an arbitrary reference mass scale to be specified shortly.

Finally we would like to emphasize that under the same assumptions (a) and (b) the sum rule given in Eq.\((12)\) has been proven \cite{55} to be all-loop RGI, which gives us a generalization of Eq.\((18)\) to be applied in considerations of non-universal soft scalar masses, which are necessary in many cases including the MSSM.

As it was emphasized in ref \cite{57} the set of the all-loop RGI relations \((15)-(18)\) is the one obtained in the Anomaly Mediated SB Scenario \cite{59,60}, by fixing the \( M_0 \) to be \( m_3/2 \), which is the natural scale in the supergravity framework. A final remark concerns the resolution of the fatal problem of the anomaly induced scenario in the supergravity framework, which is here solved thanks to the sum rule \((12)\). Other solutions have been provided by introducing Fayet-Iliopoulos terms \cite{61}.

4 Applications of the Reduction of Couplings Method

In this section we show how to apply the reduction of couplings method in two scenarios, the MSSM and Finite Unified Theories. We will apply it only to the third generation of fermions and in the soft supersymmetry breaking terms. After the reduction of couplings takes place, we are left with relations at the unification scale for the Yukawa couplings of the quarks in terms of the gauge coupling according to Eq.\((9)\), for the trilinear terms in terms of the Yukawa couplings and the unified gaugino mass Eq.\((14)\), and a sum rule for the soft scalar masses also proportional to the unified gaugino mass Eq.\((12)\), as applied in each model.

4.1 RE in the MSSM

We will examine here the reduction of couplings method applied to the MSSM, which is defined by the superpotential,

\[ W = Y_t H_2 Q t^c + Y_b H_1 Q b^c + Y_{\tau} H_1 L \tau^c + \mu H_1 H_2, \quad (19) \]

with soft breaking terms,

\[ -\mathcal{L}_{SSB} = \sum_{\phi} m_{\phi}^2 \phi^* \phi + \left[ m_3^2 H_1 H_2 + \sum_{i=1}^{3} \frac{1}{2} M_i \lambda_i \lambda_i + \text{h.c.} \right] + \left[ h_t H_2 Q t^c + h_b H_1 Q b^c + h_{\tau} H_1 L \tau^c + \text{h.c.} \right], \quad (20) \]
where the last line refers to the scalar components of the corresponding superfield. In general $Y_{t,b,\tau}$ and $h_{t,b,\tau}$ are $3 \times 3$ matrices, but we work throughout in the approximation that the matrices are diagonal, and neglect the couplings of the first two generations.

Assuming perturbative expansion of all three Yukawa couplings in favour of $g_3$ satisfying the reduction equations we find that the coefficients of the $Y_\tau$ coupling turn imaginary. Therefore, we take $Y_\tau$ at the GUT scale to be an independent variable. This “reduced” system holds at all scales, and thus serve as boundary conditions of the RGEs of the MSSM at the unification scale, where we assume that the gauge couplings meet

\[ m_{H_u}^2 + m_Q^2 + m_{H_d}^2 = M_U^2, \quad m_{H_u}^2 + m_Q^2 + m_{H_d}^2 = M_U^2, \]
\[ m_{H_d}^2 + m_Q^2 + m_{H_d}^2 = M_U^2, \]
\[ m_{H_u}^2 + m_Q^2 + m_{H_d}^2 = M_U^2, \]
\[ m_{H_d}^2 + m_Q^2 + m_{H_d}^2 = M_U^2, \]

noting that the sum rule introduces four free parameters.

\[ \text{Note: The second term can be determined once the first term is known.} \]
4.2 Predictions of the Reduced MSSM

With these boundary conditions we run the MSSM RGEs down to the SUSY scale, which we take to be the geometrical average of the stop masses, and then run the SM RGEs down to the electroweak scale ($M_Z$), where we compare with the experimental values of the third generation quark masses. The RGEs are taken at two-loops for the gauge and Yukawa couplings and at one-loop for the soft breaking parameters. We let $M_U$ and $|\mu|$ at the unification scale to vary between $\sim 1$ TeV $\sim 11$ TeV, for the two possible signs of $\mu$. In evaluating the $\tau$ and bottom masses we have taken into account the one-loop radiative corrections that come from the SUSY breaking\cite{62,63}; in particular for large $\tan \beta$ they can give sizeable contributions to the bottom quark mass.

Recall that $Y_\tau$ is not reduced and is a free parameter in this analysis. The parameter $K_\tau$, related to $Y_\tau$ through Eq. (23) is further constrained by allowing only the values that are also compatible with the top and bottom quark masses simultaneously within 1 and 2$\sigma$ of their central experimental value. In the case that $\text{sign}(\mu) = +$, there is no value for $K_\tau$ where both the top and the bottom quark masses agree simultaneously with their experimental value, therefore we only consider the negative sign of $\mu$ from now on. We use the experimental value of the top quark pole mass as

$$m_t^{\text{exp}} = (173.2 \pm 0.9) \text{ GeV}.$$  (28)

The bottom mass is calculated at $M_Z$ to avoid uncertainties that come from running down to the pole mass and, as previously mentioned, the SUSY radiative corrections both to the tau and the bottom quark masses have been taken into account\cite{64}.

$$m_b(M_Z) = (2.83 \pm 0.10) \text{ GeV}.$$  (29)

The variation of $K_\tau$ is in the range $0.33 \sim 0.5$ if the agreement with both top and bottom masses is at the 2$\sigma$ level.

Finally, assuming the validity of Eq. (14) for the corresponding couplings to those that have been reduced before, we calculate the Higgs mass as well as the whole Higgs and sparticle spectrum using Eqs. (24)-(27), and we present them in Fig. (1). The Higgs mass was calculated using a “mixed-scale” one-loop RG approach, which is known to approximate the leading two-loop corrections as evaluated by the full diagrammatic calculation\cite{65,66}. However, more refined Higgs mass calculations, and in particular the results of\cite{67} are not (yet) included.

In the left plot of Fig. (1) we show the full mass spectrum of the model. We find that the masses of the heavier Higgses have relatively high values, above the TeV scale. In addition we find a generally heavy supersymmetric spectrum starting with a neutralino as LSP at $\sim 500$ GeV and comfortable agreement with the LHC bounds due to the non-observation of coloured supersymmetric particles\cite{68,69,70}. Finally note that although the $\mu < 0$ found in our analysis would disfavour the model in connection with the anomalous magnetic moment of the muon, such a heavy spectrum gives only a negligible correction to the SM prediction. We plan to extend our analysis by examining the restrictions that will be imposed in the spectrum by the $B$-physics and Cold Dark Matter (CDM) constraints.

In the right plot of Fig. (1) we show the results for the light Higgs boson mass as a function of $K_\tau$. The results are in the range 123.7 - 126.3 GeV, where the uncertainty is due to the variation of $K_\tau$, the gaugino mass $M_U$ and the variation of the scalar...
soft masses, which are however constrained by the sum rules \[ [27] \]. The gaugino mass \( M_U \) is in the range \( \sim 1.3 \text{ TeV} \sim 11 \text{ TeV} \), the lower values having been discarded since they do not allow for radiative electroweak symmetry breaking. To the lightest Higgs mass value one has to add at least \( \pm 2 \text{ GeV} \) coming from unknown higher order corrections \[ [71] \]. Therefore it is in excellent agreement with the experimental results of ATLAS and CMS \[ [1–4] \].

4.3 Finiteness

Finiteness can be understood by considering a chiral, anomaly free, \( N = 1 \) globally supersymmetric gauge theory based on a group \( G \) with gauge coupling constant \( g \). Consider the superpotential Eq. (4) together with the soft supersymmetry breaking Lagrangian Eq. (5). All the one-loop \( \beta \)-functions of the theory vanish if the \( \beta \)-function of the gauge coupling \( \beta_g^{(1)} \), and the anomalous dimensions of the Yukawa couplings \( \gamma_i^{(1)} \), vanish, i.e.

\[
\sum_i \ell(R_i) = 3C_2(G), \quad \frac{1}{2}C_{ijpq}C^{jpq} = 2\delta_i^j g^2 C_2(R),
\]

where \( \ell(R_i) \) is the Dynkin index of \( R_i \), and \( C_2(G) \) and \( C_2(R) \) are the quadratic Casimir invariants of the adjoint representation of \( G \) and the representation \( R_i \), respectively. These conditions are also enough to guarantee two-loop finiteness \[ [72] \]. A striking fact is the existence of a theorem \[ [27–29] \], that guarantees the vanishing of the \( \beta \)-functions to all-orders in perturbation theory. This requires that, in addition to the one-loop finiteness conditions \[ (30) \], the Yukawa couplings are reduced in favour of the gauge coupling to all-orders (see \[ [73] \] for details). Alternatively, similar results can be obtained \[ [74–76] \] using an analysis of the all-loop NSVZ gauge beta-function \[ [53,77] \].
Since we would like to consider only finite theories here, we assume that the gauge group is a simple group and the one-loop $\beta$-function of the gauge coupling $g$ vanishes. We also assume that the reduction equations admit power series solutions of the form Eq. (9). According to the finiteness theorem of ref. [27–29, 80], the theory is then finite to all orders in perturbation theory, if, among others, the one-loop anomalous dimensions $\gamma_i^{(1)}$ vanish. The one- and two-loop finiteness for $h^{ijk}$ can be achieved through the relation [81]

$$h^{ijk} = -MC^{ijk} + \cdots = -M\rho_{(0)}^{ijk} g + O(g^5),$$

(31)

where $\cdots$ stand for higher order terms.

In addition it was found that the RGI SSB scalar masses in Gauge-Yukawa unified models satisfy a universal sum rule at one-loop [33]. This result was generalized to two-loops for finite theories [34], and then to all-loops for general Gauge-Yukawa and finite unified theories [55]. From these latter results, the following soft scalar-mass sum rule is found [34]

$$\frac{m_i^2 + m_j^2 + m_k^2}{MM^i} = 1 + \frac{g^2}{16\pi^2} \Delta^{(2)} + O(g^4)$$

(32)

for $i, j, k$ with $\rho_{(0)}^{ijk} \neq 0$, where $m_{i,j,k}^2$ are the scalar masses and $\Delta^{(2)}$ is the two-loop correction which vanishes for the universal choice, i.e. when all the soft scalar masses are the same at the unification point, as well as in the model considered here.

### 4.4 An $SU(5)$ Finite Unified Theory

We examine an all-loop Finite Unified theory with $SU(5)$ as gauge group, where the reduction of couplings has been applied to the third generation of quarks and leptons. The particle content of the model we will study, which we denote $\text{FUT}$, consists of the following supermultiplets: three $\mathbf{(5 + 10)}$, needed for each of the three generations of quarks and leptons, four $\mathbf{(5 + 5)}$ and one $\mathbf{24}$ considered as Higgs supermultiplets.

When the gauge group of the finite GUT is broken the theory is no longer finite, and we will assume that we are left with the MSSM [15,18–21].

A predictive Gauge-Yukawa unified $SU(5)$ model which is finite to all orders, in addition to the requirements mentioned already, should also have the following properties:

1. One-loop anomalous dimensions are diagonal, i.e., $\gamma_i^{(1)} \propto \delta_i^j$.

2. Three fermion generations, in the irreducible representations $\mathbf{5_i, 10_i}$ ($i = 1, 2, 3$), which obviously should not couple to the adjoint $\mathbf{24}$.

3. The two Higgs doublets of the MSSM should mostly be made out of a pair of Higgs quintet and anti-quintet, which couple to the third generation.

After the reduction of couplings the symmetry is enhanced, leading to the following

\[^{2}\text{Realistic Finite Unified Theories based on product gauge groups, where the finiteness implies three generations of matter, have also been studied [78,79].}\]
superpotential \[ W = \sum_{i=1}^{3} \left[ \frac{1}{2} g_i^u 10_i 10_i H_i + g_i^d 10_i \bar{5}_i \bar{1}_i \right] + g_{23}^u 10_2 10_3 H_4 \]
\[ + g_{23}^d 10_2 \bar{5}_3 \bar{1}_4 + g_{32}^d 10_3 \bar{5}_2 \bar{1}_4 + g_2^u H_2 24 \bar{1}_2 + g_3^u H_3 24 \bar{1}_3 + \frac{g_\lambda}{3} (24)^3. \]

The non-degenerate and isolated solutions to \( \gamma_i^{(1)} = 0 \) give us:

\[ (g_1^u)^2 = \frac{8}{5} g^2, \quad (g_1^d)^2 = \frac{6}{5} g^2, \quad (g_2^u)^2 = (g_2^d)^2 = \frac{4}{5} g^2, \]
\[ (g_3^u)^2 = \frac{3}{5} g^2, \quad (g_3^d)^2 = \frac{4}{5} g^2, \quad (g_{23}^u)^2 = (g_{23}^d)^2 = \frac{3}{5} g^2, \]
\[ (g_\lambda)^2 = \frac{15}{7} g^2, \quad (g_2^d)^2 = (g_3^d)^2 = \frac{1}{2} g^2, \quad (g_2^u)^2 = 0, \quad (g_3^u)^2 = 0, \]

and from the sum rule we obtain:

\[ m_{H_u}^2 + 2 m_{10}^2 = M^2, \quad m_{H_d}^2 - 2 m_{10}^2 = -\frac{M^2}{3}, \quad m_{\tilde{u}}^2 + 3 m_{10}^2 = \frac{4M^2}{3}, \]

i.e., in this case we have only two free parameters \( m_{10} \) and \( M \) for the dimensionful sector.

As already mentioned, after the \( SU(5) \) gauge symmetry breaking we assume we have the MSSM, i.e. only two Higgs doublets. This can be achieved by introducing appropriate mass terms that allow to perform a rotation of the Higgs sector \[18, 22, 83, 85\], in such a way that only one pair of Higgs doublets, coupled mostly to the third family, remains light and acquire vacuum expectation values. To avoid fast proton decay the usual fine tuning to achieve doublet-triplet splitting is performed, although the mechanism is not identical to minimal \( SU(5) \), since we have an extended Higgs sector.

Thus, after the gauge symmetry of the GUT theory is broken we are left with the MSSM, with the boundary conditions for the third family given by the finiteness conditions, while the other two families are not restricted.

### 4.5 Predictions of the Finite Model

Since the gauge symmetry is spontaneously broken below \( M_{\text{GUT}} \), the finiteness conditions do not restrict the renormalization properties at low energies, and all it remains are boundary conditions on the gauge and Yukawa couplings \[34\], the \( h = -MC \) \[31\] relation, and the soft scalar-mass sum rule at \( M_{\text{GUT}} \). The analysis follows along the same lines as in the MSSM case.

In Fig.2 we show the FUT predictions for \( m_t \) and \( m_b(M_Z) \) as a function of the unified gaugino mass \( M \), for the two cases \( \mu < 0 \) and \( \mu > 0 \). The bounds on the \( m_b(M_Z) \) and the \( m_t \) mass clearly single out \( \mu < 0 \), as the solution most compatible with these experimental constraints \[86, 87\].

We now analyze the impact of further low-energy observables on the model FUT with \( \mu < 0 \). As additional constraints we consider the flavour observables \( \text{BR}(b \rightarrow s\gamma) \) and \( \text{BR}(B_s \rightarrow \mu^+\mu^-) \).
Figure 2: The bottom quark mass at the $Z$ boson scale (left) and top quark pole mass (right) are shown as function of $M$, the unified gaugino mass.

Figure 3: The lightest Higgs mass, $M_h$, as function of $M$ for the model FUT with $\mu < 0$.

For the branching ratio $\text{BR}(b \to s\gamma)$, we take the value given by the Heavy Flavour Averaging Group (HFAG) is \cite{HFAG}

$$\text{BR}(b \to s\gamma) = (3.55 \pm 0.24^{+0.09}_{-0.10} \pm 0.03) \times 10^{-4}.$$ \hspace{1cm} (36)

For the branching ratio $\text{BR}(B_s \to \mu^+ \mu^-)$, the SM prediction is at the level of $10^{-9}$, while we employ an upper limit of

$$\text{BR}(B_s \to \mu^+ \mu^-) \lesssim 4.5 \times 10^{-9}$$ \hspace{1cm} (37)

at the 95% C.L. \cite{CMS}. This is in relatively good agreement with the recent direct measurement of this quantity by CMS and LHCb \cite{LHCb}. As we do not expect a sizable impact of the new measurement on our results, we stick for our analysis to the simple upper limit.

For the lightest Higgs mass prediction we used the code FeynHiggs \cite{FeynHiggs}. The prediction for $M_h$ of FUT with $\mu < 0$ is shown in Fig. 3, where the constraints from the two $B$ physics observables are taken into account. The lightest Higgs mass ranges in

$$M_h \sim 121 - 126 \text{ GeV},$$ \hspace{1cm} (38)
where the uncertainty comes from variations of the soft scalar masses. To this value one has to add at least ±2 GeV coming from unknown higher order corrections \[\pm 2\text{ GeV}\].

We have also included a small variation, due to threshold corrections at the GUT scale, of up to 5\% of the FUT boundary conditions. The masses of the heavier Higgs bosons are found at higher values in comparison with our previous analyses \[\pm 1\text{ GeV}\]. This is due to the more stringent bound on BR(\(B_s \rightarrow \mu^+\mu^-\)), which pushes the heavy Higgs masses beyond \(\sim 1\text{ TeV}\), excluding their discovery at the LHC.

We impose now a further constraint on our results, which is the value of the Higgs mass

\[M_h \sim 126.0 \pm 1 \pm 2\text{ GeV}\, , \tag{39}\]

where ±3 GeV corresponds to the current theory and experimental uncertainty, and ±1 GeV to a reduced theory uncertainty in the future\[\pm 1\text{ GeV}\]. We find that constraining the allowed values of the Higgs mass puts a limit on the allowed values of the unified gaugino mass, as can be seen from Fig. 3. The red lines correspond to the anticipated future uncertainty and restrict \(2\text{ TeV} \lesssim M \lesssim 5\text{ TeV}\). The blue line includes the current theoretical uncertainty. Taking this uncertainty into account no bound on \(M\) can be placed, but many parameter points can be discarded.

The full particle spectrum of model \textbf{FUT} with \(\mu < 0\), compliant with quark mass constraints and the \(B\)-physics observables is shown in Fig. 4. It can be seen from the figure that the lightest observable SUSY particle (LOSP) is the light scalar tau. In

\footnote{We have not yet taken into account the improved \(M_h\) prediction presented in \cite{67} (and implemented into the most recent version of \texttt{FeynHiggs}), which will lead to an upward shift in the Higgs boson mass prediction.}

\footnote{In this analysis the new \(M_h\) evaluation \cite{67} may have a relevant impact on the restrictions on the allowed SUSY parameter space shown below.}
the left (right) plot we impose $M_h = 126 \pm 3(1)$ GeV. Without any $M_h$ restrictions the coloured SUSY particles have masses above $\sim 1.8$ TeV in agreement with the non-observation of those particles at the LHC [68–70]. Including the Higgs mass constraints in general favours the lower part of the SUSY particle mass spectra, but also cuts away the very low values [97–100]. Going to the anticipated future theory uncertainty of $M_h$ (as shown in the right plot of Fig. 4) still permits SUSY masses which would remain unobservable at the LHC, the ILC or CLIC. On the other hand, large parts of the allowed spectrum of the lighter scalar tau or the lighter neutralinos might be accessible at CLIC with $\sqrt{s} = 3$ TeV.

5 Conclusions

The serious problem of the appearance of many free parameters in the SM of Elementary Particle Physics, takes dramatic dimensions in the MSSM, where the free parameters are proliferated by at least hundred more, while it is considered as the best candidate for Physics Beyond the SM. The idea that the Theory of Particle Physics is more symmetric at high scales, which is broken but has remnant predictions in the much lower scales of the SM, found its best realisation in the framework of the MSSM assuming further a GUT beyond the scale of the unification of couplings. However, the unification idea, although successful, seems to have exhausted its potential to reduce further the free parameters of the SM.

A new interesting possibility towards reducing the free parameters of a theory has been put forward in refs. [24, 25] which consists on a systematic search on the RGI relations among couplings. This method might lead to further symmetry, however its scope is much wider. After several trials it seems that the basic idea found very nice realisations in a Finite Unified Theory and the MSSM. In the first case one is searching for RGI relations among couplings holding beyond the unification scale, which moreover guarantee finiteness to all-orders in perturbation theory. In the second, the search of RGI relations among couplings is concentrated within the MSSM itself and the assumption of GUT is not necessarily required. The results in both cases are indeed impressive as we have discussed. Certainly one can add some more comments on the Finite Unified Theories. These are related to some fundamental developments in Theoretical Particle Physics based on reconsiderations of the problem of divergencies and serious attempts to solve it. They include the motivation and construction of string and noncommutative theories, as well as $N = 4$ supersymmetric field theories [101–102], $N = 8$ supergravity [103–107] and the AdS/CFT correspondence [108]. It is a thoroughly fascinating fact that many interesting ideas that have survived various theoretical and phenomenological tests, as well as the solution to the UV divergencies problem, find a common ground in the framework of $N = 1$ Finite Unified Theories, which have been discussed here. From the theoretical side they solve the problem of UV divergencies in a minimal way. On the phenomenological side in both cases of reduction of couplings discussed here the celebrated success of predicting the top-quark mass [18,19] is now extended to the predictions of the Higgs masses and the supersymmetric spectrum of the MSSM, which so far have been confronted very successfully with the findings and bounds at the LHC.

The various scenarios will be refined/scrutinized in various ways in the upcoming years. Important improvements in the analysis are expected from progress on the
theory side, in particular in an improved calculation of the light Higgs boson mass. The corrections introduced in [67] not only introduce a shift in $M_h$ (which should to some extent be covered by the estimate of theory uncertainties). They will also reduce the theory uncertainties, see [67,109], and in this way refine the selected model points, leading to a sharper prediction of the allowed spectrum. One can hope that with even more higher-order corrections included in the $M_h$ calculation an uncertainty below the 0.5 GeV level can be reached.

The other important improvements in the future will be the continuing searches for SUSY particles at collider experiments. The LHC will re-start in 2015 with an increased center-of-mass energy of $\sqrt{s} \lesssim 14$ TeV, largely extending its SUSY search reach. The lower parts of the currently allowed/predicted colored SUSY spectra will be tested in this way. For the electroweak particles, on the other hand, $e^+e^-$ colliders might be the better option. The ILC, operating at $\sqrt{s} \lesssim 1$ TeV, has only a limited potential for our model spectra. Going to higher energies, $\sqrt{s} \lesssim 3$ TeV, that might be realized at CLIC, large parts of the predicted electroweak model spectra can be covered.

All spectra, however, (at least with the current calculation of $M_h$ and its corresponding uncertainty), contain parameter regions that will escape the searches at the LHC, the ILC and CLIC. In this case we would remain with a light Higgs boson in the decoupling limit, i.e. would be undistinguishable from a SM Higgs boson. The only hope to overcome this situation is that an improved $M_h$ calculation would cut away such high spectra.

Acknowledgements

G.Z. thanks the Institut für Theoretische Physik, Heidelberg, for its generous support and warm hospitality. The work of G.Z. was supported by the Research Funding Program ARISTEIA, Higher Order Calculations and Tools for High Energy Colliders, HOCTools (co-financed by the European Union (European Social Fund ESF) and Greek national funds through the Operational Program Education and Lifelong Learning of the National Strategic Reference Framework (NSRF)). G.Z and N.T. acknowledge also support from the European Union’s ITN programme HIGGSTOOLS. The work of M.M. was supported by mexican grants PAPIIT IN113712 and Conacyt 132059. The work of S.H. was supported in part by CICYT (grant FPA 2010–22163-C02-01) and by the Spanish MICINN’s Consolider-Ingenio 2010 Program under grant MultiDark CSD2009-00064.

References

[1] ATLAS Collaboration, G. Aad et al., Phys.Lett. B716, 1 (2012), arXiv:1207.7214.
[2] ATLAS Collaboration, (2013), ATLAS-CONF-2013-014, ATLAS-COM-CONF-2013-025.
[3] CMS Collaboration, S. Chatrchyan et al., Phys.Lett. B716, 30 (2012), arXiv:1207.7235.
[4] CMS Collaboration, S. Chatrchyan et al., (2013), arXiv:1303.4571.
[5] J. C. Pati and A. Salam, Phys. Rev. Lett. 31, 661 (1973).
[6] H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32, 438 (1974).
[7] H. Georgi, H. R. Quinn, and S. Weinberg, Phys. Rev. Lett. 33, 451 (1974).
[8] H. Fritzsch and P. Minkowski, Ann. Phys. 93, 193 (1975).
[9] H. Georgi, Particles and Fields: Williamsburg 1974. AIP Conference Proceedings No. 23, American Institute of Physics, New York, 1974, ed. Carlson, C. E.
[10] S. Dimopoulos and H. Georgi, Nucl. Phys. B193, 150 (1981).
[11] N. Sakai, Zeit. Phys. C11, 153 (1981).
[12] U. Amaldi, W. de Boer, and H. Furstenau, Phys. Lett. B260, 447 (1991).
[13] A. J. Buras, J. R. Ellis, M. K. Gaillard, and D. V. Nanopoulos, Nucl. Phys. B135, 66 (1978).
[14] J. Kubo, M. Mondragon, M. Olechowski, and G. Zoupanos, Nucl. Phys. B479, 25 (1996), arXiv:hep-ph/9512435.
[15] J. Kubo, M. Mondragon, and G. Zoupanos, Acta Phys. Polon. B27, 3911 (1997), arXiv:hep-ph/9703289.
[16] T. Kobayashi, J. Kubo, M. Mondragon, and G. Zoupanos, Acta Phys. Polon. B30, 2013 (1999).
[17] P. Fayet, Nucl. Phys. B149, 137 (1979).
[18] D. Kapetanakis, M. Mondragon, and G. Zoupanos, Z. Phys. C60, 181 (1993), arXiv:hep-ph/9210218.
[19] J. Kubo, M. Mondragon, and G. Zoupanos, Nucl. Phys. B424, 291 (1994).
[20] J. Kubo, M. Mondragon, N. D. Tracas, and G. Zoupanos, Phys. Lett. B342, 155 (1995), arXiv:hep-th/9409003.
[21] J. Kubo, M. Mondragon, M. Olechowski, and G. Zoupanos, (1995), arXiv:hep-ph/9510279.
[22] M. Mondragon and G. Zoupanos, Nucl. Phys. Proc. Suppl. 37C, 98 (1995).
[23] J. Kubo, M. Mondragon, and G. Zoupanos, Phys. Lett. B389, 523 (1996), arXiv:hep-ph/9609218.
[24] W. Zimmermann, Commun. Math. Phys. 97, 211 (1985).
[25] R. Oehme and W. Zimmermann, Commun. Math. Phys. 97, 569 (1985).
[26] J. Kubo, M. Mondragon, S. Shoda, and G. Zoupanos, Nucl. Phys. B469, 3 (1996), arXiv:hep-ph/9512258.
[27] C. Lucchesi, O. Piguet, and K. Sibold, Helv. Phys. Acta 61, 321 (1988).
[28] O. Piguet and K. Sibold, Int. J. Mod. Phys. A1, 913 (1986).
[29] C. Lucchesi and G. Zoupanos, Fortschr. Phys. 45, 129 (1997), arXiv:hep-ph/9604216.
[30] I. Jack and D. R. T. Jones, Phys. Lett. B349, 294 (1995), arXiv:hep-ph/9501395.
[31] R. Oehme, Prog. Theor. Phys. Suppl. 86, 215 (1986).
[32] W. Zimmermann, Commun.Math.Phys. 219, 221 (2001).
[33] Y. Kawamura, T. Kobayashi, and J. Kubo, Phys. Lett. B405, 64 (1997), arXiv:hep-ph/9703320.
[34] T. Kobayashi, J. Kubo, M. Mondragon, and G. Zoupanos, Nucl. Phys. B511, 45 (1998), arXiv:hep-ph/9707425.
[35] A. Parkes and P. C. West, Phys. Lett. B138, 99 (1984).
[36] P. C. West, Phys. Lett. B137, 371 (1984).
[37] D. R. T. Jones and A. J. Parkes, Phys. Lett. B160, 267 (1985).
[38] D. R. T. Jones and L. Mezincescu, Phys. Lett. B138, 293 (1984).
[39] A. J. Parkes, Phys. Lett. B156, 73 (1985).
[40] J. Wess and B. Zumino, Phys. Lett. B49, 52 (1974).
[41] J. Iliopoulos and B. Zumino, Nucl. Phys. B76, 310 (1974).
[42] K. Fujikawa and W. Lang, Nucl. Phys. B88, 61 (1975).
[43] R. Delbourgo, Nuovo Cim. A25, 646 (1975).
[44] A. Salam and J. A. Strathdee, Nucl. Phys. B86, 142 (1975).
[45] M. T. Grisaru, W. Siegel, and M. Rocek, Nucl. Phys. B159, 429 (1979).
[46] L. Girardello and M. T. Grisaru, Nucl. Phys. B194, 65 (1982).
[47] J. Hisano and M. A. Shifman, Phys. Rev. D56, 5475 (1997), arXiv:hep-ph/9705417.
[48] I. Jack and D. R. T. Jones, Phys. Lett. B415, 383 (1997), arXiv:hep-ph/9709364.
[49] L. V. Avdeev, D. I. Kazakov, and I. N. Kondrashuk, Nucl. Phys. B510, 289 (1998), arXiv:hep-ph/9709397.
[50] D. I. Kazakov, Phys. Lett. B449, 201 (1999), arXiv:hep-ph/9812513.
[51] D. I. Kazakov, Phys. Lett. B421, 211 (1998), arXiv:hep-ph/9709465.
[52] I. Jack, D. R. T. Jones, and A. Pickering, Phys. Lett. B426, 73 (1998), arXiv:hep-ph/9712542.
[53] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B229, 407 (1983).
[54] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Phys. Lett. B166, 329 (1986).
[55] T. Kobayashi, J. Kubo, and G. Zoupanos, Phys. Lett. B427, 291 (1998), arXiv:hep-ph/9802267.
[56] M. Mondragón, N. Tracas, and G. Zoupanos, Phys. Lett. B728, 51 (2014), arXiv:1309.0996.
[57] I. Jack and D. Jones, Phys. Lett. B465, 148 (1999), arXiv:hep-ph/9907255.
[58] T. Kobayashi, J. Kubo, M. Mondragon, and G. Zoupanos, AIP Conf. Proc. 490, 279 (1999).
[59] L. Randall and R. Sundrum, Nucl. Phys. B557, 79 (1999), arXiv:hep-th/9810155.
[60] G. F. Giudice, M. A. Luty, H. Murayama, and R. Rattazzi, JHEP 9812, 027 (1998), arXiv:hep-ph/9810442.
[61] R. Hodgson, I. Jack, D. Jones, and G. Ross, Nucl.Phys. B728, 192 (2005), arXiv:hep-ph/0507193.
[62] M. S. Carena, M. Olechowski, S. Pokorski, and C. Wagner, Nucl.Phys. B426, 269 (1994), arXiv:hep-ph/9402253.
[63] J. Guasch, W. Hollik, and S. Penaranda, Phys.Lett. B515, 367 (2001), arXiv:hep-ph/0106027.
[64] Particle Data Group, K. Nakamura et al., J.Phys. G37, 075021 (2010).
[65] M. S. Carena et al., Nucl.Phys. B580, 29 (2000), arXiv:hep-ph/0001002.
[66] S. Heinemeyer, Int.J.Mod.Phys. A21, 2659 (2006), arXiv:hep-ph/0407244.
[67] T. Hahn, S. Heinemeyer, W. Hollik, H. Rzehak, and G. Weiglein, (2013), arXiv:1312.4937.
[68] CMS Collaboration, S. Chatrchyan et al., Phys.Lett. B713, 68 (2012), arXiv:1202.4083.
[69] ATLAS Collaboration, P. Pravalorio, Talk at SUSY2012 (2012).
[70] CMS Collaboration, C. Campagnari, Talk at SUSY2012 (2012).
[71] G. Degrassi, S. Heinemeyer, W. Hollik, P. Slavich, and G. Weiglein, Eur. Phys. J. C28, 133 (2003), hep-ph/0212020.
[72] D. R. T. Jones, L. Mezincescu, and Y. P. Yao, Phys. Lett. B148, 317 (1984).
[73] S. Heinemeyer, M. Mondragon, and G. Zoupanos, (2011), arXiv:1101.2476.
[74] A. V. Ermushev, D. I. Kazakov, and O. V. Tarasov, Nucl. Phys. B281, 72 (1987).
[75] D. I. Kazakov, Mod. Phys. Lett. A2, 663 (1987).
[76] R. G. Leigh and M. J. Strassler, Nucl. Phys. B447, 95 (1995), arXiv:hep-th/9503121.
[77] M. A. Shifman, Int. J. Mod. Phys. A11, 5761 (1996), arXiv:hep-ph/9606281.
[78] E. Ma, M. Mondragon, and G. Zoupanos, JHEP 12, 026 (2004), arXiv:hep-ph/0407236.
[79] S. Heinemeyer, E. Ma, M. Mondragon, and G. Zoupanos, Fortsch.Phys. 58, 729 (2010).
[80] C. Lucchesi, O. Piguet, and K. Sibold, Phys. Lett. B201, 241 (1988).
[81] I. Jack and D. R. T. Jones, Phys. Lett. B333, 372 (1994), hep-ph/9405233.
[82] M. Mondragon and G. Zoupanos, J.Phys.Conf.Ser. 171, 012095 (2009).
[83] J. Leon, J. Perez-Mercader, M. Quiros, and J. Ramirez-Mittelbrunn, Phys. Lett. B156, 66 (1985).
[84] S. Hamidi and J. H. Schwarz, Phys. Lett. B147, 301 (1984).
[85] D. R. T. Jones and S. Raby, Phys. Lett. B143, 137 (1984).
[86] S. Heinemeyer, M. Mondragon, and G. Zoupanos, JHEP 07, 135 (2008), arXiv:0712.3630.
[87] S. Heinemeyer, M. Mondragon, and G. Zoupanos, SIGMA 6, 049 (2010), arXiv:1001.0428.
[88] Heavy Flavour Averaging Group, see: www.slac.stanford.edu/xorg/hfag/.

[89] LHCb collaboration, R. Aaij et al., Phys.Rev.Lett. 108, 231801 (2012), arXiv:1203.4493

[90] CMS and LHCb Collaborations, (2013), http://cdsweb.cern.ch/record/1374913/files/BPH-11-019-pas.pdf

[91] S. Heinemeyer, W. Hollik, and G. Weiglein, Comput. Phys. Commun. 124, 76 (2000), arXiv:hep-ph/9812320

[92] S. Heinemeyer, W. Hollik, and G. Weiglein, Eur. Phys. J. C9, 343 (1999), arXiv:hep-ph/9812472

[93] M. Frank et al., JHEP 02, 047 (2007), hep-ph/0611326

[94] S. Heinemeyer, M. Mondragon, and G. Zoupanos, (2008), arXiv:0810.0727

[95] S. Heinemeyer, M. Mondragon, and G. Zoupanos, J.Phys.Conf.Ser. 171, 012096 (2009).

[96] S. Heinemeyer, M. Mondragon, and G. Zoupanos, (2012), arXiv:1201.5171

[97] S. Heinemeyer, M. Mondragon, and G. Zoupanos, Phys.Lett. B718, 1430 (2013), arXiv:1211.3765

[98] S. Heinemeyer, M. Mondragon, and G. Zoupanos, Int.J.Mod.Phys.Conf.Ser. 13, 118 (2012).

[99] S. Heinemeyer, M. Mondragon, and G. Zoupanos, Phys.Part.Nucl. 44, 299 (2013).

[100] S. Heinemeyer, M. Mondragon, and G. Zoupanos, Fortsch. Phys. 61, 969 (2013), arXiv:1305.5073

[101] S. Mandelstam, Nucl. Phys. B213, 149 (1983).

[102] L. Brink, O. Lindgren, and B. E. W. Nilsson, Phys. Lett. B123, 323 (1983).

[103] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, and R. Roiban, Phys. Rev. Lett. 103, 081301 (2009), arXiv:0905.2326

[104] R. Kallosh, JHEP 09, 116 (2009), arXiv:0906.3495

[105] Z. Bern et al., Phys. Rev. Lett. 98, 161303 (2007), arXiv:hep-th/0702112

[106] Z. Bern, L. J. Dixon, and R. Roiban, Phys. Lett. B644, 265 (2007), arXiv:hep-th/0611086

[107] M. B. Green, J. G. Russo, and P. Vanhove, Phys. Rev. Lett. 98, 131602 (2007), arXiv:hep-th/0611273

[108] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998), arXiv:hep-th/9711200

[109] O. Buchmueller et al., (2013), arXiv:1312.5233