A beam theory consistent with three-dimensional thermo-elasticity

Antonino Favata
Department of Civil, Environmental and Mechanical Engineering, University of Trento, Trento, Italy

Received 27 September 2013; accepted 30 January 2014

Abstract
We propose a model for thermo-elastic beams, consistent with the theory of linear three-dimensional thermo-elasticity and deduced by a suitable version of the principle of virtual powers. Dimensional reduction is achieved by postulating convenient a priori representations for mechanical and thermal displacements, the latter playing the role of an additional kinetic variable. Such representations are regarded as internal constraints, some involving the first, others, the second, gradient of deformation and thermal displacements; these constraints are maintained by reactive stresses and hyper-stresses of the type occurring in non-simple elastic materials of grade two, and by reactive entropy influxes and hyper-influxes.

Keywords
Beam theory, thermo-elasticity, principle of virtual powers, second-gradient materials

1. Introduction
In continuum mechanics, the principle of virtual powers (PVP) is a standard tool to obtain all balance laws that apply to a given material class, be it the class of simple materials or, as suggested by Germain [1], the non-simple class of the so-called higher-gradient materials. In the same line of research, modern applications of the principle can be found in [2–4].

Two uses of the principle are non-standard: one is to derive lower-dimensional theories for thin structures, consistent with a three-dimensional parent theory [5–8]; the other is to deduce the balance equations of a multiphysics theory [9], that is, a theory that allows for the composition of two or more material body structures, as is the case for the composition of mechanical and thermal structures considered in [9].

In this paper, we combine such non-standard uses of the principle to derive a thermo-elastic beam theory consistent with three-dimensional thermo-elasticity. Several models for beam thermomechanics have been proposed in the literature, some of them direct [10–16], others induced from three-dimensional thermomechanics [17, 18]. We here follow a different path from these valuable deductions; our goal is to achieve a consistency with three-dimensional thermo-elasticity, by using a generalization accommodating thermal phenomena of the method of internal constraints introduced in [19] in a purely mechanical context. According to that method, lower-dimensional structures are regarded as three-dimensional bodies, having a special shape and partitionability, and whose kinematics is restricted.

In our generalization to thermo-elasticity of the method of internal constraints, an essential role is played by the notion of thermal displacement, a non-standard thermal state variable first considered by Helmholtz. In the spirit of the method, internal constraints have to be considered both on the mechanical and the thermal kinetic
variables, constraints that are chosen so as to capture those characters of the lower-dimensional theory one considers most significant. It may happen, as in the present case, that meaningful lower-dimensional kinetics (mechanical and thermal) emerge from some internal constraint formulated not only on the first gradient of the mechanical and/or thermal displacements, but also on second gradients. We are then led to consider non-simple thermo-elastic materials.

The paper is organized as follows. In Section 2, we introduce our notation and define what we mean by a three-dimensional beam-like body. In Section 3, we recapitulate the extended version of PVP proposed in [9] and we lay down the restrictions on the mechanical and thermal kinetic variables. Having done this, we find ourselves in a position to define the dynamical descriptors, to arrive at a one-dimensional version of PVP, and, finally, to obtain the two balance laws of our beam theory, one for momentum, the other for entropy. Section 4 is devoted to constitutive assumptions, consistent with a dissipation axiom leading to a reduced dissipation inequality.

We observe that the balance equations obtained in Section 3 hold the same for a number of standard and non-standard beam theories, linear or non-linear, elastic or not. In Section 5, we fix our attention on three-dimensional linear thermo-elasticity and wonder whether constitutive assumptions could be found such as to guarantee that our one-dimensional theory yields a sufficiently approximated description of the three-dimensional phenomenology. With this purpose, we introduce the notion of non-simple thermo-elastic materials and discuss the role of the modified dynamical descriptors, those which are constitutively specified and those that serve to maintain the internal constraints.

The last two sections are devoted to exemplify the use of our theory. In Section 6, we show how to recover the standard equations of thermo-elastic beams, with some additional features; in particular, we consider the longitudinal-vibrations problem, and we discuss the relations between one- and three-dimensional constitutive parameters. A second example, bending vibrations, is considered in Section 7.

2. Geometry

We restrict our attention to beam-like solids in the form of right cylinders $B$, of constant cross-section $S$ and length $l$, with $S$ a flat, open, bounded and connected region with a smooth boundary $\partial S$, whose diameter is much smaller than $l$ (Figure 1).

For convenience, we introduce a Cartesian frame $\{o; c_1, c_2, z\}$ and we set

$$B \ni p = x + \zeta z, \quad x \in S, \quad \zeta \in (-h, +h), \quad l = 2h;$$

moreover, $x - o = x_\beta c_\beta$. Region $B$ can be identified point-wise with the set $S \times (-h, +h)$; moreover, we denote by

$$L(x) := \{p \in B \mid (p - x) \cdot z \in (-h, +h)\}$$

the straight fibre of $B$ through a point $x \in S$. We fix once and for all the origin $o$ at the centroid of $B$, so that $L(0)$ is the axis, which we denote by $L$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{beam.png}
\caption{A beam-like three-dimensional body.}
\end{figure}
3. A non-standard use of the Principle of Virtual Powers

Given a material body occupying a three-dimensional open and bounded region $\Omega$ and a test velocity field $\delta \mathbf{u}$ over $\Omega$, the internal virtual power expenditure over a part $\mathcal{P}$ of $\Omega$ associated with $\delta \mathbf{u}$ is

$$\delta W^{(i)} := \int_{\mathcal{P}} \mathbf{S} \cdot \nabla \delta \mathbf{u},$$

where $\mathbf{S}$ denotes the stress field in $\Omega$, and the external virtual power expenditure over $\mathcal{P}$ is

$$\delta W^{(e)} := \int_{\mathcal{P}} \mathbf{d} \cdot \delta \mathbf{u} + \int_{\partial \mathcal{P}} \mathbf{c} \cdot \delta \mathbf{u},$$

where $(\mathbf{d}, \mathbf{c})$ denote, respectively, the distance force per unit volume and the contact force per unit area exerted on $\mathcal{P}$ by its own complement with respect to $\Omega$ and by the environment of the latter. These representations of power expenditure are those typical in the theory of the so-called simple material bodies. In that theory, a part is customarily a subset of the non-null volume of $\Omega$ which makes it for a part collection deemed sufficiently rich, and a virtual velocity field is a smooth vector field whose support is a part; the Principle of Virtual Powers (PVP) is the stipulation that

$$\delta W^{(i)} = \delta W^{(e)}$$

for all parts $\mathcal{P}$ of $\Omega$ and for all velocity fields $\delta \mathbf{u}$ consistent with the admissible motions (see [1, 20]).

In the form (1), PVP is interpreted as a balance statement for the internal and external fields $\mathbf{S}$ and $\mathbf{d}$, $\mathbf{c}$, and as such, it concerns the purely mechanical structure of $\Omega$.

3.1. Extended powers, internal and external

In [9], it is shown that the virtual power format can be generalized so as to deduce the balance laws of three-dimensional thermomechanics. Together with the mechanical displacement $\mathbf{u}$, another kinetic variable is introduced, the thermal displacement $\alpha$; both $\mathbf{u}$ and $\alpha$ are required to be smooth fields over the closure with respect to the product topology of the space-time cylinder $\Omega \times (0, T)$. Their time derivatives are the velocity $\mathbf{v} := \dot{\mathbf{u}}$ and the temperature $\vartheta := \dot{\alpha}$.

For $\mathcal{P}$ a sub-body of $\Omega$ and $T = (t_i, t_f)$ a subinterval of $(0, T)$, the internal virtual power is defined to be

$$\delta W^{(i)}(\delta \mathbf{u}, \delta \alpha) := \int_{\mathcal{P} \times T} \mathbf{S} \cdot \nabla \delta \mathbf{u} + h \delta \alpha + \mathbf{h} \cdot \nabla \delta \alpha,$$

where $h$ and $\mathbf{h}$ are measures of the thermal interactions (of, respectively, zeroth and first order) of a material element of the sub-body $\mathcal{P}$ with its immediate adjacencies; $h$ has to be interpreted as the internal dissipation per unit temperature change, and $-\mathbf{h} := h$, as a measure of entropy influx at a point of an oriented surface of normal $\mathbf{n}$ (see Remark 3 in [9]). The ‘augmented’ version of the external power is

$$\delta W^{(e)}(\delta \mathbf{u}, \delta \alpha) := \int_{\mathcal{P} \times T} (\mathbf{d} \cdot \delta \mathbf{u} + \mathbf{p} \cdot \delta \mathbf{u} + d \delta \alpha + \eta \delta \dot{\alpha}) + \int_{\partial \mathcal{P} \times T} (\mathbf{c} \cdot \delta \mathbf{u} + c \delta \alpha) + \int_{\mathcal{P} \times \partial \mathcal{I}} \left[ \mathbf{p} \cdot \delta \mathbf{u} + \eta \delta \alpha \right],$$

where

$$\left[ \mathbf{p} \cdot \delta \mathbf{u} + \eta \delta \alpha \right] := p_j(x) \cdot \delta \mathbf{u}(x, t_f) + \eta_j(x) \delta \alpha(x, t_f) + p_j(x) \cdot \delta \mathbf{u}(x, t_i) + \eta_j(x) \delta \alpha(x, t_i),$$

for all $x \in \Omega$; $\mathbf{p}$ is the momentum and $\eta$ the entropy, both having specific sources $\mathbf{d}$ and $d$ at an interior point of $\mathcal{P}$, and specific fluxes $\mathbf{Sn} = c$ and $\mathbf{h} \cdot \mathbf{n} = c$ at a boundary point of $\partial \mathcal{P}$. When the internal and external powers are taken as just detailed, the requirement (1) yields both momentum and entropy balance equations:

$$\dot{\mathbf{p}} = \text{Div} \mathbf{S} + \mathbf{d}, \quad \dot{\eta} = \text{Div} \mathbf{h} - h + d.$$
3.2. Restrictions on kinetic variables

To achieve dimensional reduction, it is necessary to adopt a restricted version of the three-dimensional statement of PVP. The first restriction has to do with the special shape of the three-dimensional bodies we consider: they must be beam-like, in the sense of Section 2. The second restriction concerns the choice of a special class of admissible body parts: they all must have the same cross-section of \( B \). Third and last, the class of virtual velocities is special, in that it is consistent with the representation of admissible displacements. The representations we choose are

\[
u(x, \zeta, t) = a(\zeta, t) + \varphi(\zeta, t) \times (x - o), \tag{2}\]

for the mechanical displacement, and

\[
u(\zeta, t) \cdot z = \varphi(\zeta, t) \cdot z = 0 \tag{3}\]

for the thermal displacement. The representation (2)–(3) is parametrized by the vector fields \( a, \varphi \) over \( (-h, +h) \times (0, T) \); at a fixed time \( t \), the \( \zeta \)-cross-section is subject to a translation \( v(\zeta, t) \cdot z \) and a small rotation about an axis parallel to \( \varphi(\zeta, t) \). Such a representation is typical of the so-called Timoshenko beam theory \([25, 26]\), which accounts for shear deformation and rotational inertia effects.

The Ansatz (4) on the thermal displacement is coherent with an analogous representation for the temperature distribution:

\[
\begin{align*}
\vartheta(x, \zeta, t) &= \vartheta(\zeta, t) + \vartheta_1(\zeta, t) \cdot (x - o), \\
\vartheta_1(\zeta, t) \cdot z &= 0; \tag{5}
\end{align*}
\]

the representation (5) is parametrized by a scalar field, the average value \( \vartheta(\zeta, t) \) of the mean temperature over the cross-section at abscissa \( \zeta \) and by a deviation \( \vartheta_1(\zeta, t) \), a vector field identified with the temperature gradient over the cross-section. We notice that, if Ansatz (5) is accepted, one is contented with an approximated description, basically ruled by the value of the temperature at the intrados and the extrados of the beam. As has been pointed out by \([17]\):

Since the outer surface of the beam is insulated, it would not be unreasonable to assume a linear variation in temperature as well. For other boundary conditions, such as a given temperature on the surface, this assumption would not necessarily be warranted.

The kinetics of a thermo-elastic beam is specified by a list of twice continuously differentiable mappings,

\[
(\zeta, t) \mapsto \left( a(\zeta, t), \varphi(\zeta, t); \vartheta(\zeta, t), \vartheta_1(\zeta, t) \right); \tag{6}
\]

by time differentiation, a list

\[
(\zeta, t) \mapsto \left( \nu(\zeta, t), \omega(\zeta, t); \vartheta(\zeta, t), \vartheta_1(\zeta, t) \right)
\]

of realizable velocities is obtained from each kinetic process (6).

**Remark 1.** Instead of the Timoshenko beam theory, the simpler Bernoulli–Navier model obtains for

\[
\nu(\zeta, t) \cdot z = 0 \tag{7}
\]

the requirement that cross-section and axis stay mutually orthogonal in all admissible deformations.

3.3. Internal power expenditure: Measures of stress, internal dissipation and entropy influx

The first two restrictions on PVP we mentioned imply that a typical part of \( \Omega \equiv B \) can be identified with the Cartesian product \( \mathcal{P} = \mathcal{S} \times \mathcal{I} \), with \( \mathcal{I} = (a, b) \) an open and connected subset of \( \mathcal{L} \). As for virtual velocities, we choose them in the form (2)–(3) and (4):

\[
\begin{align*}
\delta \nu(x, \zeta, t) &= \delta a(\zeta, t) + \delta \varphi(\zeta, t) \times (x - o), \\
\delta \nu_0(x, \zeta, t) &= \delta a_0(\zeta, t) + \delta \varphi_0(\zeta, t) \cdot (x - o).
\end{align*}
\]
The displacement gradients are

\[
\begin{align*}
\nabla \delta u &= \delta a' \otimes z + (\delta \varphi' \times (x - o)) \otimes z + (\delta \varphi \times c_\beta) \otimes c_\beta, \\
\nabla \delta \alpha &= (\delta a'_o + \delta \alpha'_1 \cdot (x - o)) z + \delta \alpha_1,
\end{align*}
\]

where we left the dependence on place and time tacit. Here and henceforth, \((\cdot)'\) denotes the derivative with respect to \(\zeta\); time derivative will be denoted by \((\cdot)\)\cdot. On recourse to the Fubini–Tonelli theorem, we obtain that

\[
\delta W^{(i)} = \int_{S \times I \times T} S \cdot \nabla \delta u + h \delta \alpha + \overline{h} \cdot \nabla \delta \alpha \\
= \int_{I \times T} \left( \delta a' \cdot \int_S S z + \delta \varphi \cdot c_\beta \right) \cdot \int_S S c_\beta + \delta \varphi' \cdot \int_S (x - o) \times S z + \delta \alpha'_o \cdot \int_S h + \delta \alpha'_1 \cdot \int_S (x - o) h \\
+ \delta a'_o \cdot \int_S \overline{h} \cdot z + \delta \alpha'_1 \cdot \int_S (x - o) \overline{h} \cdot z + \delta \alpha_1 \cdot \int_S \overline{h}.
\]

On making use of the following definitions:

\[
f = f(\xi, t) := \int_S S(x, \xi, t) z, \\
m = m(\xi, t) := \int_S (x - o) \times S(x, \xi) z, \\
\xi_o = \xi_o(\xi, t) := \int_S h(x, \xi, t), \\
\xi_1 = \xi_1(\xi, t) := \int_S (x - o) h(x, \xi, t), \\
\overline{h}_o = \overline{h}_o(\xi, t) := \int_S \overline{h}(x, \xi, t) \cdot z, \\
\overline{h}_1 = \overline{h}_1(\xi, t) := \int_S (x - o) \overline{h}(x, \xi, t) \cdot z, \\
\overline{k} = \overline{k}(\xi, t) := \int_S (\overline{h}(x, \xi, t) \cdot c_\beta) c_\beta,
\]

the internal power expenditure can be given in the following form:

\[
\delta W^{(i)} = \int_{I \times T} f \cdot \delta a' + m \cdot \delta \varphi' + z \times f \cdot \delta \varphi + \int_{I \times T} \xi_o \delta \alpha'_o + \xi_1 \cdot \delta \alpha'_1 + \overline{h}_o \delta \alpha'_o + \overline{h}_1 \cdot \delta \alpha'_1 + \overline{k} \cdot \delta \alpha_1.
\]

All quantities in (8) are one-dimensional ‘dynamical’ descriptors: \(f\) and \(m\), the force and the moment vectors, are the stress measures; \(\xi_o\) and \(\xi_1\) are the cross-sectional measures of internal dissipation, average and ‘deviational’, respectively; \(-\overline{h}_o =: h_o\), \(-\overline{k} =: k\) and \(-\overline{h}_1 =: h_1\) are the entropy influxes measures, axial, sectional and deviational, respectively.
3.4. External power expenditure: Applied loads, momenta, sources, and entropies

As for the external power, we find

\[ \delta \mathcal{W}^{(e)} = \int_{S \times I \times T} \left( d \cdot \delta u + p \cdot \delta u + d \delta \alpha + \eta \delta \alpha \right) + \int_{\partial(S \times I) \times T} \left( c \cdot \delta u + c \delta \alpha \right) + \int_{S \times I \times \partial T} \left[ p \cdot \delta u + \eta \delta \alpha \right] \]

\[ = \int_{I \times T} \left( \delta a \cdot \int_S d + \delta \varphi \cdot \int_S (x-o) \times d + \delta \dot{a} \cdot \int_S p + \delta \dot{\varphi} \cdot \int_S (x-o) \times p + \delta \alpha_o \int \right. \]

\[ + \delta \alpha_1 \cdot \int_S (x-o) \times d + \delta \dot{\alpha}_o \int \eta + \delta \dot{\alpha}_o \cdot \int_S (x-o) \eta + \delta a \cdot I_S \int_S c + \delta \dot{\varphi} \cdot \int_S (x-o) \times c \]

\[ + \delta \alpha_o \int_{\partial S} c + \delta \alpha_1 \cdot \int_{\partial S} (x-o) c \right) + \int_{I \times \partial T} \left[ (\delta a \cdot \int_S p + \delta \dot{\varphi} \cdot \int_S (x-o) \times p \right. \]

\[ + \delta \alpha_o \int_S \eta + \delta \alpha_1 \cdot \int_S (x-o) \eta \right]. \]

We are now in position to define over \( \mathcal{L} \) applied loads, momenta, sources and entropies, as induced by three-dimensional loads \((d, c)\), momentum \(p\), specific source \(d\), contact thermal interaction \(c\) and entropy \(\eta\). These are:

- **The force and the couple:**
  
  \[ b_o = b_o(\xi, t) := \int_S d(x, \xi, t) + \int_{\partial S} c(x, \xi, t), \]

  \[ m_o = m_o(\xi, t) := \int_S (x-o) \times d(x, \xi, t) + \int_{\partial S} (x-o) \times c(x, \xi, t). \]

- **The linear and rotational momentum:**
  
  \[ l_o = l_o(\xi, t) := \int_S p(x, \xi, t), \]

  \[ r_o = r_o(\xi, t) := \int_S (x-o) \times p(x, \xi, t). \]

- **The average and deviational specific entropy sources:**
  
  \[ d_o = d_o(\xi, t) := \int_S d(x, \xi, t) + \int_{\partial S} c(x, \xi, t), \]

  \[ d_1 = d_1(\xi, t) := \int_S (x-o) d(x, \xi, t) + \int_{\partial S} (x-o) c(x, \xi, t). \]

- **The average and deviational entropies:**
  
  \[ \eta_o = \eta_o(\xi, t) := \int_S \eta(x, \xi, t), \]

  \[ \eta_1 = \eta_1(\xi, t) := \int_S (x-o) \eta(x, \xi, t). \]

With these, the internal power expenditure reads:

\[ \delta \mathcal{W}^{(e)} = \int_{I \times T} b_o \cdot \delta a + m_o \cdot \delta \varphi + l_o \cdot \delta a + r_o \cdot \delta \dot{\varphi} + d_o \cdot \delta \alpha_o + d_1 \cdot \delta \alpha_1 + \eta_o \cdot \delta \alpha_o + \eta_1 \cdot \alpha_1 \]

\[ + \int_{I \times \partial T} \left[ l_o \cdot \delta a + r_o \cdot \delta \dot{\varphi} + \eta_o \cdot \delta \alpha_o + \eta_1 \cdot \alpha_1 \right]. \]

The one-dimensional entropy inflow consists of the list \((h_o, h_1, k; d_o, d_1)\), that we split in two:
We postulate that as many heat influxes exist: \((q_o; r_o), (q_1; j; r_1)\) and, as usually done \([27–29]\), we set the entropy inflow proportional to the heat inflow through the temperature as follows:

\[
\begin{align*}
q_o &= \vartheta o \, h_o, \\
r_o &= \vartheta o \, d_o, \\
q_1 &= \vartheta \beta \, h_1 \cdot c_\beta, \\
\vartheta \beta :&= \vartheta \beta \cdot c_\beta, \\
j &= \vartheta \beta \, k \cdot c_\beta, \\
r_1 &= \vartheta \beta \, d_1.
\end{align*}
\]

(9)

3.5. Momentum and entropy balances

PVP, exploited for each virtual velocity defined over the closure of any subcylinder \(I \times T\) of \(L \times (0, T)\) and such as to vanish at the ends of \(T\) itself, implies that the momentum balances

\[
\begin{align*}
i_o &= f' + b_o, \\
r_o &= m - f \times z + m_o,
\end{align*}
\]

(10)

and the entropy balances

\[
\begin{align*}
\dot{\eta}_o &= \overline{h}_o - \xi_o + d_o, \\
\dot{\eta}_1 &= \overline{h} - \overline{k} - \xi_1 + d_1,
\end{align*}
\]

(11)

holding on \(L\), together with the initial conditions

\[
i_o(x, \zeta, t_i) = i_o(x, \zeta), \quad r_o(x, \zeta, t_i) = r_o(x, \zeta).
\]

As always happens when a dimensional reduction is achieved, a loss of information is to be expected, which is partially mitigated by the appearance of richer dynamical descriptors and new balance equations. It is worth noticing that, for whatever slenderness \(d/L\), our procedure leads to the same balance equations; the choice of Ansätze (2) and (4), typical of those bodies which are inherently slender, as beams are, prescribes the variations and then the format of the resulting one-dimensional balance equations we obtained. For these reasons we can say as the resulting theory that (2) and (4) are suitable for beams. While equations (10) are standard, the entropy balances (11) are not: non-standard thermal descriptors appear and non-standard equations rule the phenomenon.

In order to arrive at one beam model or another, be it elastic or not, some constitutive assumptions are necessary; in the next section, we show which of these assumptions are thermodynamically consistent.

4. Constitutive assumptions

Our constitutive assumptions stem from a dissipation axiom (see [9]), that is the requirement that, whatever kinetic process \((\zeta, t) \mapsto (v, w, \varphi(\zeta, t), \alpha, \beta(\zeta, t))\),

\[
\xi_0 \varphi_0 \leq 0, \quad \xi_1 \cdot \varphi_1 \leq 0,
\]

(12)

over the space-time cylinder \(L \times (0, T)\).

Now, let the specific internal action per unit length \(\phi\) be defined by

\[
\begin{align*}
\dot{\phi} &= f \cdot \dot{v} + m \cdot \dot{w} + z \times \dot{f} \cdot \dot{w} - (l_o \cdot \dot{v} + r_o \cdot \dot{w}) + \xi_0 \varphi_0 + \xi_1 \cdot \varphi_1 + \overline{h}_o \varphi_0' + \overline{h}_1 \cdot \varphi_1' + \overline{k} \cdot \varphi_1 \\
&\quad - (\eta_0 \varphi_0 + \eta_1 \cdot \varphi_1),
\end{align*}
\]

(13)

and let \(\kappa\) be the specific kinetic energy per unit length,

\[
\kappa := \frac{1}{2} (l_o \cdot v + r_o \cdot w),
\]

(14)
so that
\[
\mathcal{K}(\mathcal{I}) = \frac{1}{2} \int_{\mathcal{I} \times \mathbb{S}} \mathbf{p} \cdot \mathbf{u} = \frac{1}{2} \int_{\mathcal{I}} \mathbf{I}_o \cdot \mathbf{v} + \mathbf{r}_o \cdot \mathbf{\omega} = \frac{1}{2} \int_{\mathcal{I}} \kappa
\]
is the kinetic energy of the part \(\mathcal{I}\). In classical mechanics, the link between the time rate of the kinetic energy and the power \(\Pi^a\) expended by the inertia forces \(d^a = -\dot{p}\) is such that \(\dot{\mathcal{K}} + \Pi^a = 0\) (see [30, 31]), and then
\[
\dot{\kappa} = \dot{\mathbf{I}}_o \cdot \mathbf{v} + \dot{\mathbf{r}}_o \cdot \mathbf{\omega}.
\]

Let us introduce the specific energy
\[
\tau := \phi + \mathbf{I}_o \cdot \mathbf{v} + \mathbf{r}_o \cdot \mathbf{\omega} + \eta_o \dot{\theta}_o + \eta_1 \cdot \dot{\vartheta}_1 = \phi + 2\kappa + \eta_o \dot{\theta}_o + \eta_1 \cdot \dot{\vartheta}_1
\]
and set
\[
\epsilon = \tau - \kappa, \quad \phi = \psi - \kappa,
\]
with \(\epsilon\) the specific internal energy per unit length and \(\psi\) the specific Helmholtz free energy per unit length, whence
\[
\psi = \epsilon - (\eta_o \dot{\theta}_o + \eta_1 \cdot \dot{\vartheta}_1).
\]

The dissipation axiom (12), combined with (15), (14), (13) yields the dissipation inequality
\[
\psi \leq - (\eta_o \dot{\theta}_o + \eta_1 \cdot \dot{\vartheta}_1) - h_o \dot{\theta}_o - h_1 \cdot \dot{\vartheta}_1 - k \cdot \dot{\vartheta}_1 + f \cdot v' + m \cdot \omega' + z \times f \cdot \omega,
\]
or, rather,
\[
\dot{\psi} \leq - (\eta_o \dot{\theta}_o + \eta_1 \cdot \dot{\vartheta}_1) - h_o \dot{\theta}_o - h_1 \cdot \dot{\vartheta}_1 - k \cdot \dot{\vartheta}_1 + f \cdot v' + m \cdot \omega' + z \times f \cdot \omega. \tag{16}
\]

**Remark 2.** All the quantities defined above can be split into a linear and a rotational part, denoted by \((\cdot)_o\) and \((\cdot)_1\), respectively:

- **Internal action**
  \[
  \dot{\phi}_o := f \cdot v' - \mathbf{I}_o \cdot \dot{v} + \xi_o \dot{\theta}_o + \mathbf{h}_o \dot{\theta}_o' + -\eta_o \dot{\theta}_o,
  \]
  \[
  \dot{\phi}_1 := m \cdot \omega' + z \times f \cdot \omega - \mathbf{r}_o \cdot \dot{\omega} + \xi_1 \cdot \dot{\vartheta}_1 + \mathbf{h}_1 \cdot \dot{\vartheta}_1' + \mathbf{k} \cdot \dot{\vartheta}_1 - \eta_1 \cdot \dot{\vartheta}_1;
  \]

- **Kinetic energy**
  \[
  \kappa_o := \frac{1}{2} \mathbf{I}_o \cdot \dot{v}, \quad \kappa_1 := \frac{1}{2} \mathbf{r}_o \cdot \dot{\omega};
  \]

- **Energy**
  \[
  \tau_o := \phi_o + \mathbf{I}_o \cdot v + \eta_o \dot{\theta}_o = \phi_o + 2\kappa_o + \eta_o \dot{\theta}_o,
  \]
  \[
  \tau_1 := \phi_1 + \mathbf{r}_o \cdot \omega + \eta_1 \cdot \dot{\vartheta}_1 = \phi_1 + 2\kappa_1 + \eta_1 \cdot \dot{\vartheta}_1;
  \]

- **Internal energy**
  \[
  \epsilon_o := \tau_o - \kappa_o, \quad \epsilon_1 := \tau_1 - \kappa_1;
  \]

- **Helmholtz free energy**
  \[
  \psi_o := \phi_o + \kappa_o, \quad \psi_1 := \phi_1 + \kappa_1.
  \]

With these definitions, the dissipation inequality (16) is equivalent to the following two inequalities:
\[
\psi_o \leq -\eta_o \dot{\theta}_o - h_o \dot{\theta}_o' + \mathbf{f} \cdot \mathbf{v}',
\]
\[
\psi_1 \leq -\eta_1 \cdot \dot{\vartheta}_1 - h_1 \cdot \dot{\vartheta}_1' - k \cdot \dot{\vartheta}_1 + m \cdot \omega' + z \times \mathbf{f} \cdot \mathbf{\omega}. \tag{17}
\]

The balance equations (10)–(11), together with one or other set of constitutive assumptions coherent with dissipation inequalities (17), lead to a number of standard and non-standard beams theory linear or non-linear, elastic or not. In the following, we fix our attention on linear thermo-elasticity and wonder if, on adopting constitutive assumptions consistent with the three-dimensional version of that theory, the predictions we obtained for our one-dimensional theory are ‘consistent’ and in what sense ‘approximating’.
5. Reactive stresses and reactive entropy influxes

Once a constitutive choice has been made and a solution of the problem governed by equations (10)–(11) has been found, one wonders how accurately it approximates the three-dimensional thermo-elastic state \( \{\mathbf{u}, E, S; \alpha, h\} \) that exactly solves the parent three-dimensional problem that our lower-dimensional model aims to approximate. As for the mechanical and thermal displacements, we approximate \( \mathbf{u} \) and \( E \) by inserting the solution \( (a, \varphi; \alpha_0, \alpha_1) \) into the representations (2) and (4); consequently, we obtain the approximation for \( E \). As for the stress field \( S^E \) and the entropy influx \( h^E \), there are various approximations, scrutinized, for the stress, in [5].

Firstly, an active (i.e. constitutively determined) stress field \( S^A \) and an active entropy influx field \( h^A \) can be considered. As done in [5, 19] for the stress, a better approximation of \( S^E \) and \( h^E \) can be obtained by adding reactive stress and entropy influx fields, regarded as consequences of the kinematical restrictions. The reactive stresses are specified by the constitutive requirement of doing no work in any admissible deformation (see Section 30 of [32]):

\[
\mathbf{S}^R \cdot \nabla \mathbf{u} = 0
\]

for grade-one of Cauchy materials;

\[
\mathbf{S}^R \cdot \nabla \mathbf{u} + \mathbf{S}^R \cdot \nabla \nabla \mathbf{u} = 0 \tag{18}
\]

(see [5]) for materials of grade two; \( \mathbf{S}^R \) and \( \mathbf{S}^R \) are a reactive stress and a reactive hyper-stress, respectively (see the next subsection). Similarly, in our extended framework, we assume that reactive entropy influxes are specified by the analogous requirement

\[
\mathbf{h}^R \cdot \nabla \alpha + \mathbf{H}^R \cdot \nabla \nabla \alpha = 0, \tag{19}
\]

where \( \mathbf{h}^R \) and \( \mathbf{H}^R \) are a reactive entropy influx and a reactive entropy hyper-influx. We will make this statement more precise in the next subsection.

5.1. Simple and non-simple reactive stresses and entropy influxes

In classical constitutive theories, the stress of a body particle is decided by the deformation history of a neighborhood of that particle; in a simple material the stress is a function of the history of the deformation gradient \( F \), the first-order approximation of the deformation; in a simple and elastic material, the stress depends on the present value of \( F \). In general, the deformation can be better approximated considering deformation gradients of order \( N > 1 \), leading to non-simple elastic materials of grade \( N > 1 \). For these kinds of materials, the concept of stress has to be reconsidered, by introducing stresses of high order. We focus on materials of grade two, for which the standard stress \( S \) has to be accompanied by the hyper-stress \( \mathbf{S} \), a third-order tensor, in that the internal virtual power becomes

\[
\int_{\mathcal{P}} S \cdot \nabla \delta \mathbf{u} + \mathbf{S} \cdot \nabla \nabla \delta \mathbf{u}.
\]

So far, we were concerned with the purely mechanical structure of the body; within our framework, it is not unreasonable to think of a heat conductor, for which it is possible to give a convenient definition of non-simple conductor material of grade \( M > 1 \): a material whose entropy influxes depend on the \( M \)th gradient of the thermal displacement; together with the notion of hyper-stress, an entropy hyper-influx comes out. Accordingly, thermo-elastic grade-(\( N - M \)) materials are those for which the stresses and entropy influxes depend on the \( N \)th and \( M \)th gradients of the mechanical and thermal displacements, respectively; in principle, there is no reason to assume \( N = M \). We here consider the case when \( N = M = 2 \), whose corresponding extended internal virtual power reads

\[
\int_{\mathcal{P} \times T} S \cdot \nabla \delta \mathbf{u} + \mathbf{S} \cdot \nabla \nabla \delta \mathbf{u} + h \delta \alpha + \mathbf{H} \cdot \nabla \delta \alpha + \mathbf{H} \cdot \nabla \nabla \delta \alpha,
\]

where \( -\mathbf{H} =: \mathbf{H} \) is the entropy hyper-influx, a second-order tensor.

It is not difficult to see that the modified stress

\[
\mathbf{S} := S - \text{Div} \mathbf{S},
\]
and the modified entropy influx
\[ \tilde{h} := h - \text{Div} H, \]
enter the equilibrium and the entropy balance equations. In general, each of \( S, \dot{S}, h \) and \( H \) consists of an active (that is, constitutively determined) and a reactive part, arising in presence of internal constraints. If they are expressed by scalar equations
\[
\begin{align*}
\gamma(E(u)) &= 0, \\
\Gamma(\nabla\nabla u) &= 0, \\
\sigma(\nabla \alpha) &= 0, \\
\Sigma(\nabla\nabla \alpha) &= 0,
\end{align*}
\]
then conditions (18)–(19) yield the following representation for the associated reactive stress, hyper-stress, entropy influx and entropy hyper-influx:
\[
\begin{align*}
S^R &= S^R(\partial_E \gamma), \\
\dot{S}^R &= \dot{S}^R(\partial_{\nabla\nabla \alpha} \Gamma), \\
h^R &= h^R(\partial_{\nabla \alpha} \sigma), \\
H^R &= H^R(\partial_{\nabla\nabla \alpha} \Sigma),
\end{align*}
\]
(20)
with \( S^R, \dot{S}^R, h^R, H^R \) four arbitrary scalars.

It is possible to show [5] that the kinematic assumptions (2)–(3) are the general solutions to the following linear system of partial differential equations:
\[
\begin{align*}
E \cdot c_\beta \otimes c_\gamma &= 0, \\
E_{\gamma\gamma} \cdot z \otimes c_\beta &= 0,
\end{align*}
\]
(21)
where
\[
E := \text{sym} \nabla u = \frac{1}{2}(\nabla u + \nabla u^T)
\]
is the infinitesimal strain tensor. Equations (21) are regarded as internal constraints, because they imply that not all states of strain are admissible; in particular, they imply that the cross-section fibres neither lengthen nor change their mutual angle and that, given a fibre on the cross-section, the change in angle between that fibre and an axial fibre has a constant value. Analogously, it is not difficult to see that condition (4) is the solution of the following internal constraints:
\[
\nabla \nabla \alpha \cdot c_\beta \otimes c_\gamma = 0.
\]
(22)
The four scalar conditions equivalent to (22) are in fact
\[
\alpha_{\iota \iota} (x_1, x_2, \xi) = \alpha_{\iota \iota} (x_1, x_2, \xi) = \alpha_{\iota \iota} (x_1, x_2, \xi) = 0,
\]
(23)
and an easy computation shows that the field solving the three first-order PDEs (23) is (4). The simple constraints (21) produce an admissible reactive stress having the form
\[
S = \sum_{i=1}^{3} S_i V_i, \\
V_1 := c_1 \otimes c_1, V_2 := c_2 \otimes c_2, \\
V_3 := c_1 \otimes c_2 + c_2 \otimes c_1.
\]
Moreover, when the second-order internal constraints (21) hold, the constraint equations are
\[
\Gamma_{\beta\delta} = \nabla \nabla u \cdot \left( z \otimes \text{sym}(c_\beta \otimes c_\delta) + c_\beta \otimes \text{sym}(z \otimes c_\delta) \right) = 0;
\]
the representation (20) becomes
\[
\tilde{\dot{S}} = \Sigma_{\beta\delta} \left( z \otimes \text{sym}(c_\beta \otimes c_\delta) + c_\beta \otimes \text{sym}(z \otimes c_\delta) \right).
\]
From the simple constraint (21) follows that
\[
\gamma_{\beta\delta}(\nabla u) = E_{\beta\delta, \kappa} = 0, \quad k = 1, 2, 3,
\]
whose corresponding non-simple constraint is
\[ \nabla \nabla u \cdot \left( c_\beta \otimes \text{sym}(c_\delta \otimes c_\kappa) + c_\delta \otimes \text{sym}(c_\beta \otimes c_\kappa) \right) = 0, \]
where we mean \( c_3 \equiv z \); the corresponding hyper-stress then is
\[ \mathbf{S}^R = \tau_{\beta\delta\kappa} (c_\beta \otimes \text{sym}(c_\delta \otimes c_\kappa) + c_\delta \otimes \text{sym}(c_\beta \otimes c_\kappa)), \quad \tau_{\beta\delta\kappa} = \tau_{\delta\beta\kappa}. \]
All in all, the reactive hyper-stress has the form
\[ \mathbf{S}^R = \tilde{\mathbf{S}}^R + \mathbf{S}^R = (\tau_{\beta\delta\mu} + \tau_{\mu\beta\delta}) c_\beta \otimes c_\delta \otimes c_\mu + \frac{1}{2} (\Sigma_{\beta\delta} + \Sigma_{\delta\beta}) z \otimes c_\beta \otimes c_\delta + (\Sigma_{\beta\delta} + 2 \tau_{\beta\delta\kappa}) c_\beta \otimes \text{sym}(c_\delta \otimes z) \]
(see [5]).
Concerning the thermal structure, we do not have any constraint on \( \nabla \alpha \), so that \( \mathbf{h}^R = 0 \); as for \( \nabla \nabla \alpha \), the constraint equations are
\[ \Sigma_{\beta\delta}(\nabla \nabla \alpha) = \nabla \nabla \alpha \cdot c_\beta \otimes c_\delta = 0 \]
(see (22)); the representation (20)\(_4\) becomes
\[ \mathbf{H}^R = \sum_{i=1}^{3} \mathbf{H}_{i}^R \mathbf{V}_i. \]
We conclude noticing that the beam we are considering can be then regarded as a grade-(2 \(-\) 2) thermo-elastic body, in which the hyper-stress and entropy hyper-influx are completely reactive; the stress is partially active and partially reactive; the entropy influx is entirely active.

**Remark 3.** To the best of our knowledge, this is the first time that the notion of non-simple conductor material of grade \( M > 1 \) is introduced, together with the modified entropy influx. Nevertheless, ‘augmented’ constitutive choices for the entropy influx have been proposed for instance by Green and Naghdi in their so-called Type II and Type III theories. In the case of the latter [33], they propose taking
\[ \tilde{h}(\alpha, \vartheta, \nabla \alpha, \nabla \vartheta) = -\kappa \vartheta^{-1} \nabla \vartheta - (\kappa^* + \kappa^{**}) \vartheta^{-1} \nabla \alpha, \]
where \( \kappa^* \) and \( \kappa^{**} \) are two new conductivity moduli; the corresponding assumption for the heat influx is
\[ \tilde{q}(\alpha, \vartheta, \nabla \alpha, \nabla \vartheta) = -\kappa \nabla \vartheta - (\kappa^* + \kappa^{**}) \nabla \alpha. \]
The first term of the right-hand side of (24) is the standard Fourier assumption, leading to the classical heat conduction. We can interpret (24) as the expression of a grade-two modified entropy influx \( \tilde{\mathbf{h}} \), for which the following constitutive assumptions have been made:
\[ \mathbf{h} = \tilde{\mathbf{h}}(\alpha, \vartheta, \nabla \alpha, \nabla \vartheta) = -\kappa \vartheta^{-1} \nabla \vartheta \]
for the entropy influx, and
\[ \text{Div} \mathbf{H} = \text{Div} \tilde{\mathbf{H}}(\alpha, \vartheta, \nabla \alpha, \nabla \vartheta) = (\kappa^* + \kappa^{**}) \vartheta^{-1} \nabla \alpha \]
for the entropy hyper-influx.
6. A first example: longitudinal vibrations

In this section, we consider an elementary problem, which allows us to recover the standard equations of thermoelastic beams. First of all, here and henceforth we restrict our attention to Euler–Bernoulli beams (see Remark 1); moreover, we strengthen the kinetic restrictions on $\alpha$, assuming that the following constraint holds:

$$\sigma(\nabla \alpha) = \nabla \alpha \cdot c_\beta = 0. \quad (25)$$

According to representation (20), this further assumption produces reactive entropy influxes

$$h^R_\beta = h^R_\alpha c_\beta,$$

while the constitutively determined part of $h$ is just the one directed as $z$. Moreover, (25) implies that $\alpha_1 = 0$.

As for the inertial forces, considered for the three-dimensional beam-like body $B$, we make the usual assumptions, that is,

$$d_\rho^o(x, \zeta, t) = -\dot{p}(x, \zeta, t) = -\rho(x, \zeta) \ddot{u}(x, \zeta, t),$$

where $u$ is consistent with (2)–(3) and (7), and $\rho(x, \zeta)$ is the mass density per unit volume. Both sources and non-inertial forces are assumed to be null.

It is not difficult to see that balance equations (10) and (11) reduce to

$$\begin{aligned}
-\rho_o \ddot{w} &= N' \\
\dot{\eta}_o &= -h'_o - \xi_o
\end{aligned} \quad (26)$$

where $N := f \cdot z$ is the normal force and

$$\rho_o = \rho_o(\zeta) := \int_S \rho(x, \zeta)$$

is the mass density per unit length. We aim to find a coupled system of PDEs describing mechanical effects and heat conduction. Constitutive assumptions stem from the use à la Coleman–Noll [34] of inequality (17), which takes the form

$$\dot{\psi}_o \leq -\eta_o \dot{\theta}_o - h_o \theta'_o + N \dot{\varepsilon}, \quad (27)$$

with $\varepsilon := w'$ the axial strain of the beam. As is customary, we assume that the quantities in need of a constitutive prescription depend on one and the same list of variables:

$$\psi_o = \hat{\psi}_o(\varepsilon, \theta_o, \vartheta'_o), \quad \eta_o = \hat{\eta}_o(\varepsilon, \theta_o, \vartheta'_o),$$

$$N = \hat{N}(\varepsilon, \theta_o, \vartheta'_o).$$

Thus, inequality (27) reads:

$$(\partial_{\theta_o} \psi_o + \eta_o) \dot{\theta}_o + (\partial_{\varepsilon} \psi_o - N) \dot{\varepsilon} + \partial_{\vartheta_o} \psi_o \dot{\vartheta}_o' + h_o \theta'_o \leq 0; \quad (28)$$

we now require that (28) be satisfied whatever the local continuation of any conceivable process, that is, whatever $(\dot{\varepsilon}, \dot{\theta}_o, \dot{\vartheta}_o')$ at whatever state $(\varepsilon, \theta_o, \vartheta'_o)$. This requirement is satisfied if and only if

$$\hat{\psi}_o \text{ is independent of } \theta'_o, \quad \hat{\eta}_o(\varepsilon, \theta_o) = -\partial_{\theta_o} \hat{\psi}_o(\varepsilon, \theta_o),$$

$$\hat{N}(\varepsilon, \theta_o) = \partial_{\varepsilon} \hat{\psi}_o(\varepsilon, \theta_o), \quad (29)$$

and moreover,

$$\hat{h}_o(\varepsilon, \theta_o, \vartheta'_o) \vartheta'_o \leq 0,$$

for all $\varepsilon, \theta_o$ and $\vartheta'_o$; on using (9), this latter condition is equivalent to

$$\hat{q}_o(\varepsilon, \theta_o, \vartheta'_o) \vartheta'_o \leq 0. \quad (30)$$

Condition (30) implies that

$$\hat{q}_o(\varepsilon, \theta_o, \vartheta'_o) = -\chi(\varepsilon, \theta_o, \vartheta'_o) \vartheta'_o, $$
where \( \hat{\chi} \) is the conductivity mapping; as is usually done, we assume that \( \hat{\chi} \) has a constant value, whence

\[
h_o = -\chi \vartheta_o^{-1} \vartheta_o' = -\chi (\log \vartheta_o)'.
\]

We now choose the free energy as follows:

\[
\hat{\psi}_o(\varepsilon, \vartheta_o) = \frac{1}{2} s_E \varepsilon^2 + m \varepsilon (\vartheta_o - \bar{\vartheta}_o) - c_o \vartheta_o \log \vartheta_o,
\]

where \( s_E \) is the extensional stiffness of the beam, \( m \) is the stress-temperature modulus, \( \bar{\vartheta}_o \) is a prescribed (constant) value of the mean temperature over the cross-section, and \( c_o \) is the heat capacity per unit length; we will discuss the relation between these moduli and their correspondent three-dimensional version in Section 6.1. With this choice, (29) implies

\[
\hat{\eta}(\varepsilon, \vartheta_o) = c_o (1 + \log \vartheta_o) - m \varepsilon, \quad \hat{N}(\varepsilon, \vartheta_o) = s_E \varepsilon + m (\vartheta_o - \bar{\vartheta}_o).
\]

Note that, if \( s_E \neq 0 \), as we request, then (31) can be inverted to yield

\[
\varepsilon = \frac{N}{s_E} + \delta_o (\vartheta_o - \bar{\vartheta}_o),
\]

where

\[
\delta_o := -\frac{m}{s_E}
\]

is the coefficient of thermal dilation (see equation (8.4), Section 8 in [35]). As for the dissipation \( \xi_o \), as in three-dimensional thermo-elasticity [23, 24], we set

\[
\xi_o = q_o \vartheta_o' = -\chi (\vartheta_o')^2.
\]

All in all, system (26) becomes

\[
\begin{cases}
-\frac{\rho_o}{s_E} \ddot{\vartheta}_o = w'' - \delta_o \vartheta_o', \\
-c_o (\log \vartheta_o)' - \chi (\log \vartheta_o)'' = -\delta_o s_E \dot{w}' + \chi (\vartheta_o')^2.
\end{cases}
\]

If dissipation is neglected because it is a quadratic quantity, we obtain

\[
\begin{cases}
-\frac{\rho_o}{s_E} \ddot{\vartheta}_o = w'' - \delta_o \vartheta_o', \\
-c_o s_E (\log \vartheta_o)' - \chi (\log \vartheta_o)'' = -\delta_o s_E \dot{w}'.
\end{cases}
\]

If we borrow from [23, 24] the small perturbation assumption \( \vartheta_o = \bar{\vartheta}_o + \theta(\xi, t) \), where \( \theta \) is the small perturbation, the system (32) becomes

\[
\begin{cases}
-\frac{\rho_o}{s_E} \ddot{\vartheta}_o = w'' - \delta_o \theta', \\
-c_o \dot{\theta} - \chi \theta'' = -\bar{\vartheta}_o \delta_o s_E \dot{w}'.
\end{cases}
\]

The first equation is well known in technical beam theory; the second one is similar to the classical heat conduction equation, where the time rate of axial deformation \( \dot{\varepsilon} = \dot{w}' \) plays the role of a heat source.

### 6.1. One- and three-dimensional constitutive parameters

Our dimensional reduction produces one-dimensional quantities describing the thermo-elastic behaviour of a beam and the dissipation inequality, restricting the constitutive assumptions. At this point, one-dimensional free energy has to be selected and the work is done. On the other hand, in the spirit of our approach, one may ask if this choice has a three-dimensional counterpart, namely if constants entering the free energy \( (s_E, m_o, c_o) \) are
related to three-dimensional quantities. One way to accomplish this parameter identification consists of equating the one-dimensional energy to the corresponding three-dimensional one [36].

The three-dimensional constitutive equation for an isotropic linearly thermo-elastic material (see [35]) is

\[
S = 2\mu E + \lambda(\text{tr} E) I + m(\vartheta - \bar{\vartheta}) I,
\]

whose inverse is

\[
E = \frac{1}{2\mu} \left( S - \frac{\lambda}{3\lambda + 2\mu}(\text{tr} S) I \right) + a(\vartheta - \bar{\vartheta}) I,
\]

where

\[
a := -\frac{m}{3\lambda + 2\mu}
\]

is the coefficient of thermal expansion and \(\lambda\) and \(\mu\) are the Lamé coefficients. The deformation tensor \(E\) then splits into two parts \(E = E_m + E_t\), where

\[
E_m := \frac{1}{2\mu} \left( S - \frac{\lambda}{3\lambda + 2\mu}(\text{tr} S) I \right), \quad E_t := a(\vartheta - \bar{\vartheta}) I
\]

are the purely mechanical and purely thermal parts, respectively. When expressed in terms of stress components, the stored-energy density per unit volume is

\[
w^{3D}(S) = w^{3D}_m(S) + w^{3D}_t(S),
\]

with

\[
w^{3D}_m(S) := \frac{1}{2} S \cdot E_m = \frac{1}{4\mu} \left( |S|^2 - \frac{\lambda}{3\lambda + 2\mu}(\text{tr} S)^2 \right),
\]

\[
w^{3D}_t(S) := \frac{1}{2} S \cdot E_t = \frac{1}{2} a(\vartheta - \bar{\vartheta}) \text{tr} S.
\]  

(33)

Their one-dimensional counterparts, on following (31), are

\[
w^{1D}_m(N) = \frac{1}{2} \frac{N^2}{s_E}, \quad w^{1D}_t(N) = \frac{1}{2} \delta_o N(\vartheta_o - \bar{\vartheta}_o).
\]

For a purely axial problem \(S = (S \cdot z \otimes z) z \otimes z =: S_{zz} z \otimes z\), and then (33) reduces to

\[
w^{3D}_m = \frac{1}{2} E S_{zz}^2, \quad w^{3D}_t = \frac{1}{2} a(\vartheta - \bar{\vartheta}) S_{zz},
\]

where

\[
E := \frac{\mu(3\lambda + 2\mu)}{\lambda + 2\mu}
\]

is the Young modulus. Following [36], we first identify the extensional stiffness \(s_E\) by imposing that

\[
\frac{1}{2} \frac{N^2}{s_E} = w^{1D}_m = \int_S w^{3D}_m = \frac{1}{2E} \int_S S_{zz}^2 ;
\]

hence,

\[
s_E := E \frac{N^2}{\int_S S_{zz}^2}.
\]

Next, since the factor multiplying the Young modulus \(E\) has the dimensions of an area, we give \(s_E\) the following form:

\[
s_E = \frac{EA}{\chi_e}, \quad \chi_e := A \int_S \frac{S_{zz}^2}{N^2}, \quad A := \int_S dA.
\]
Analogously, we identify the stress-temperature modulus \( m_o \) by imposing that
\[
\frac{1}{2} \delta_o N (\vartheta_o - \bar{\vartheta}_o) \left( = w_t^{1D} = \int_S w_{t}^{3D} \right) = \frac{1}{2} a \int_S (\vartheta - \bar{\vartheta}) Szz;
\]
hence,
\[
\delta_o := a \frac{\int (\vartheta - \bar{\vartheta}) Szz}{N (\vartheta_o - \bar{\vartheta}_o)},
\]
that we rewrite as
\[
\delta_o = a A \chi_t, \quad \chi_t := A \frac{N (\vartheta_o - \bar{\vartheta}_o)}{\int (\vartheta - \bar{\vartheta}) Szz}.
\]
Note that ‘mechanical shape factor’ \( \chi_e \) equals 1 whatever the shape of the cross-section \( S \) if the field \( Szz \) is constant-valued over \( S \), just as it happens to be in Saint-Venant’s case of normal force; analogously, the ‘thermal shape factor’ \( \chi_t = 1 \) if there is a constant temperature over the cross-section and the field \( Szz \) is constant-valued or linear in \( x_\beta \). In order to achieve a similar identification for the heat capacity, let us consider a rigid three-dimensional conductor, whose classical free energy is
\[
\psi^{3D} = -c \vartheta \log \vartheta,
\]
where \( c \) is the heat capacity; the corresponding one-dimensional energy we have picked is
\[
\psi^{1D} = -c_o \vartheta_o \log \vartheta_o.
\]
We conclude that
\[
c_o = c A \chi_c, \quad \chi_c := \frac{A \vartheta_o \log \vartheta_o}{\int \vartheta \log \vartheta};
\]
note that \( \chi_c = 1 \) if there is a constant temperature over the cross-section.

7. A second example: Bending vibrations

In this section we assume that \( h = h_o z \); moreover, we restrict our attention to plane beams lying in the plane spanned by \( z \) and \( c_1 \), so that \( x - o = x_1 c_1 \). In order to take into account just bending effects, we consider the following displacement ansatz:
\[
u(x, \zeta, t) = v(\zeta, t) c_1 - x_1 v'(\zeta, t) c_1, \\
\alpha(x, \zeta, t) = (x - o) \cdot \alpha_1 = x_1 \alpha_1, \quad \alpha_1 := \alpha_1 \cdot c_1.
\]

With these assumptions, it is not difficult to see that the pure (without the reactive part) balance equations are
\[
\begin{aligned}
M'' &= -\rho_o \dot{\vartheta} \\
\dot{\eta}_1 &= \dot{h}_1 - \xi_1,
\end{aligned}
\]
where we set
\[
\eta_1 := \eta_1 \cdot c_1, \quad \xi_1 := \xi_1 \cdot c_1.
\]
The dissipation inequality (17)_2 becomes
\[
\psi_1 \leq -\eta_1 \dot{\vartheta}_1 - h_1 \vartheta_1' + M \kappa,
\]
with \( M := m \cdot c_1 \) the bending moment, \( \vartheta_1 = \vartheta_1 \cdot c_1 \) and \( \kappa := -v'' \) the curvature. The quantities in need of a constitutive prescription depend on the same list of variables:
\[
\psi_1 = \hat{\psi}_1(\kappa, \vartheta_1, \vartheta_1'), \quad \eta_1 = \hat{\eta}_1(\kappa, \vartheta_1, \vartheta_1'),
\]
\[
M = \hat{M}(\kappa, \vartheta_1, \vartheta_1').
\]
The Coleman–Noll procedure yields
\[ \hat{\psi}_1 \text{ is independent of } \vartheta'_1, \quad \hat{\chi}_1(\kappa, \vartheta_1) = -\partial_{\vartheta_1} \hat{\psi}_1(\kappa, \vartheta_1), \]
and moreover,
\[ \hat{h}_1(\kappa, \vartheta_1, \vartheta'_1) \vartheta'_1 \leq 0. \]
We now choose the following free energy:
\[ \hat{\psi}_1(\kappa, \vartheta_1) = \frac{1}{2} s_B \kappa^2 + m_1 \kappa (\vartheta_1 - \bar{\vartheta}_1) - c_1 \vartheta_1 \log \vartheta_1, \]
where \( s_B \) is the bending stiffness, \( m_1 = -\delta_o s_B \) is the stress-temperature modulus, \( \vartheta_1 \) is a prescribed (constant) value of the deviational temperature, and \( c_1 \) is the heat capacity. If we neglect the dissipation and make the small perturbation assumption \( \vartheta_1 = \bar{\vartheta}_1 + \theta_1(\xi, t) \), as done in the previous section, we obtain the following balance equations:
\[
\begin{align*}
\dddot{v} + \delta_o \theta''_1 &= -\frac{\rho}{\chi_b} \ddot{v}, \\
c_1 \dot{\theta} - \chi_c \theta''_1 &= \bar{\vartheta}_1 s_B \ddot{v}''.
\end{align*}
\]
\[ \text{Remark 4.} \]
An easy computation, analogous to the one carried out in the previous section, shows that
\[ s_B = \frac{EJ}{\chi_b}, \quad \chi_b := J \int_S \frac{S_{zz}^2}{M^2}, \]
\[ c_1 = \frac{cA}{\chi_c_1}, \quad \chi_c_1 := A \int_S \frac{\vartheta_1 \log \vartheta_1}{\partial \log \vartheta} \]
for
\[ M = \int_S \chi_1 S_{zz}, \quad J = \int_S \chi^2_1; \]
note that \( \chi_b = 1 \) if \( S_{zz} \) is constant over \( S \), just as it happens to be for \( \chi_e \) and \( \chi_c_1 = 1 \) if the temperature over the cross-section has null average.

**Notes**
1. Actually, in [37, 38], it is argued that non-local and higher-gradient continuum mechanics was considered already in Piola’s works, by means of a suitable version of PVP; the reader is referred to [39] and other works by the same author cited in [37, 38]. Therein it is possible to find other references to papers using the same spirit and methods as Piola to introduce generalized stress tensors, for example Green and Rivlin [40–43].
2. For a discussion of the different approaches adopted to model thin structures, see [44].
3. A short account of the notion of thermal displacement, along with its history and use, can be found in [9].
4. We use the convention that Latin and Greek indices have the ranges \( \{1, 2, 3\} \) and \( \{1, 2\} \), respectively.
5. One may ask why a zeroth-order interaction has not been taken into account for the mechanical structure; as pointed out in [9], such a term would be cancelled by the requirement of translational invariance of \( \delta \chi^{(0)} \).
6. For a revised exposition of Green–Naghdi theory, the reader is referred to [45].
7. We also made use of the algebraic identity \( c_\beta \times A c_\beta = Az_\times z, \) where \( A \) is a symmetric tensor [8].
8. We used the adjective ‘deviational’ to recall that \( \xi \) is power-conjugated to the cross-sectional temperature deviation.

**Funding**
This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.
Acknowledgement

I thank Professor Paolo Podio-Guidugli for the time he spent in reading my manuscript and for his many fruitful pieces of advice.

Conflict of interest

None declared.

References

[1] Germain, P. La méthode des puissances virtuelles en mécanique des milieux continus, première partie: Théorie du second gradient. J Mec 1973; 12: 235–274.

[2] Alibert, JJ, Seppecher, P, and dell’Isola, F. Truss modular beam with deformation energy depending on higher displacement gradients. Math Mech Solids 2003; 8(1): 51–73.

[3] Dell’Isola, F, and Seppecher, P. Edge contact forces and quasi-balanced power. Meccan 1997; 32: 33–52.

[4] Pideri, C, and Seppecher, P. A second gradient material resulting from the homogenization of an heterogeneous linear elastic medium. Continuum Mech Thermodyn 1997; 9(5): 241–257.

[5] Lembo, M, and Podio-Guidugli, P. Internal constraints, reactive stresses, and the Timoshenko beam theory. J Elast 2001; 65: 131–148.

[6] Favata, A, and Podio-Guidugli, P. What shell theory fits carbon nanotubes? In: Altenbach, H and Eremeyev, VA (eds.) Shell-like structures (Advanced Structured Materials, vol. 15). New York, NY: Springer, 2011; 561–570.

[7] Favata, A, and Podio-Guidugli, P. A new CNT-oriented shell theory. Eur J Mech A Solids 2012; 35: 75–96.

[8] Podio-Guidugli, P. A primer in elasticity. ME l a s t.

[9] Alibert, JJ, Seppecher, P, and dell’Isola, F. Truss modular beam with deformation energy depending on higher displacement gradients. Math Mech Solids 2003; 8(1): 51–73.

[10] Kafadar, CB. On the nonlinear theory of rods. Int J Solids Struct 1972; 10: 369–391.

[11] Simmonds, JG. A simple nonlinear thermodynamic theory of arbitrary elastic beams. J Elast 2005; 81: 51–62.

[12] Maugin, GA. The method of virtual power in continuum mechanics: Application to coupled fields. Acta Mech 1980; 35(1–2): 1–70.

[13] Green, AE, and Naghdi, PM. On thermodynamics and the nature of the second law. Proc R Soc Lond Ser A 1977; 4: 253–270.

[14] Green, AE, and Naghdi, PM. Demonstration of consistency of an entropy balance with balance of energy. ZAMP 1991; 42: 159–168.

[15] Colli, P, Bonetti, E, and Frémond, M. Entropy balance versus energy balance. Application to the heat equation and to phase transitions. In: Frémond, M and Maceri, F (eds.) Mechanical modelling and computational issues in civil engineering (Lecture Notes in Applied and Computational Mechanics, vol. 23). Berlin: Springer, 2005, 379–388.

[16] Frémond, M. Phase change in mechanics (Lecture Notes of the Unione Matematica Italiana, vol. 13). Berlin: Springer, 2012, 5–29.

[17] Timoshenko, SP. On the correction factor for shear of the differential equation for transverse vibrations of bars of uniform cross-section. Philos Mag 1921; 41: 744–746.

[18] Timoshenko, SP. On the transverse vibrations of bars of uniform cross-section. Philos Mag 1922; 43: 125–131.

[19] Bargmann, S, Favata, A, and Podio-Guidugli, P. On energy and entropy influxes in the Green-Naghdi type III theory of heat conduction. Proc R Soc Lond A 2013; 469(2152): .

[20] Muller, I. The coldness, a universal function in thermoelastic bodies. Arch Rat Mech Anal 1971; 41(5): 319–332.

[21] Podio-Guidugli, P. Dissipative entropy makes the heat equation hyperbolic. Att Acc Pelor Peri 2013; 91: A15-1–A15-8.

[22] Favata, A, Podio-Guidugli, P, and Tomassetti, G. Energy splitting theorems for materials with memory. J Elast 2010; 101(1): 59–67.

[23] Podio-Guidugli, P. Inertia and invariance. Ann Mat Pura Appl 1997; 172: 103–124.

[24] Truesdell, C, and Noll, W. The non-linear field theory of mechanics. In: Flügge, S (ed.) Encyclopedia of physics, vol. III/3. New York, NY: Springer, 1965.

[25] Green, AE, and Naghdi, PM. On undamped heat waves in an elastic solid. J Therm Stresses 1992; 15: 253–264.

[26] Coleman, B, and Noll, W. The thermodynamics of elastic materials with heat conduction and viscosity. Arch Rat Mech Anal 1963; 13: 167–178.
Carlson, DE. Linear thermoelasticity. In: Truesdell, C (ed.) Encyclopedia of physics, volume VI a/2. Berlin: Springer, 1972, 297–345.

Favata, A, Micheletti, A, and Podio-Guidugli, P. On shear and torsion factors in the theory of linearly elastic rods. J Elast 2010; 99(2): 203–210.

Auffray, N, dell’Isola, F, Eremeyev, VA, et al. Analytical continuum mechanics à la Hamilton–Piola: Least action principle for second gradient continua and capillary fluids. Math Mech Solids. Epub ahead of print 28 August 2013. DOI: 10.1177/1081286513497616.

Dell’Isola, F, Andreaus, U, and Placidi, L. At the origins and in the vanguard of peri-dynamics, non-local and higher gradient continuum mechanics. An underestimated and still topical contribution of Gabrio Piola. Math Mech Solids. Forthcoming.

Piola, G. Memoria intorno alle equazioni fondamentali del movimento di corpi qualstivogliono considerati secondo la naturale loro forma e costituzione. Modena: Tipi del R.D. Camera, 1846.

Green, AE, and Rivlin, RS. Multipolar continuum mechanics. Arch Rat Mech Anal 1964; 17: 113–147.

Green, AE, and Rivlin, RS. Multipolar continuum mechanics: Functional theory I. Proc R Soc Lond Ser A 1965; 284: 303–324.

Green, AE, and Rivlin, RS. On Cauchy’s equations of motion. ZAMP 1964; 15: 290–292.

Green, AE, and Rivlin, RS. Simple force and stress multipoles. Arch Rat Mech Anal 1964; 16: 325–353.

Podio-Guidugli, P. Concepts in the mechanics of thin structures. In: Morassi, A and Paroni, R (eds.) Classical and advanced theories of thin structures (CISM International Centre for Mechanical Sciences, vol. 503). Vienna: Springer, 2008, 77–109.

Bargmann, S, Favata, A, and Podio-Guidugli, P. A revised exposition of the Green-Naghdi theory of heat propagation. J Elast 2014; 114(2): 143–154.