Pulsations of spherically symmetric systems in general relativity

Othmar Brodbeck¹, Markus Heusler² and Norbert Straumann¹

¹Institute for Theoretical Physics, University of Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland
²Enrico Fermi Institute, University of Chicago, 5640 S. Ellis Ave., Chicago, IL 60637

The pulsation equations for spherically symmetric black hole and soliton solutions are brought into a standard form. The formulae apply to a large class of field theoretical matter models and can easily be worked out for specific examples. The close relation to the energy principle in terms of the second variation of the Schwarzschild mass is also established. The use of the general expressions is illustrated for the Einstein-Yang-Mills and the Einstein-Skyrme system.
1 Introduction

In recent years the stability properties of numerous new black hole and soliton configurations, arising in selfgravitating nonlinear field theories, have been studied. Early investigations of this kind were concerned with the instabilities of such objects in SU(2) Einstein-Yang-Mills theory. It was shown in [1, 2] that both the particle-like solutions of Bartnik and McKinnon [3], as well as the “colored” non-Abelian black holes [4] are very fragile structures. Some black holes with hair which were found in other matter models turned out to be (at least linearly) stable. This is, for instance, the case for the Skyrme model [5, 6, 7] or when a Higgs triplet is added to the Yang-Mills fields [8, 9, 10]. If, however, a Higgs doublet is coupled, black holes and solitons turn out to be unstable again [11, 12].

Most of these and other investigations are restricted to spherically symmetric configurations. A closer look at them reveals many common features. Hence, one would like to have a more general treatment, encompassing a sufficiently large class of field theoretical models. For any given one, it should then be possible, for example, to write down readily the final pulsation equations, without repeating the numerous intermediate computations.

A save and systematic method for a stability analysis is to linearize the coupled field equations around a given equilibrium solution and to study the frequency spectrum of the resulting (multicomponent) eigenvalue problem. For spherically symmetric situations one expects intuitively, that the resulting fluctuation operator is closely related to the second variation of the Schwarzschild mass. Similar “energy principles” are important in other fields of physics such as, for instance, in plasma physics for ideal magnetohydrodynamic conditions [13]. In classical mechanics the second variation $\delta^2 H$ of the Hamiltonian provides the well-known Lagrange-Dirichlet stability criterion, which applies usually also in infinite dimensions. (A counter example in elasticity theory has been discovered by Ball and Marsden [14].) One should, however, remember that the energy criterion is often of no use. A well-known example is provided by the equilibrium positions of the restricted three-body problem. Here one has to linearize the Hamiltonian vector field in order to investigate linear stability; $\delta^2 H$ is indefinite and gives no information on the fluctuation spectrum.

If the energy principle applies, it provides usually the simplest method to decide on (linear) stability. In this paper we establish the validity of
the energy principle for a large class of matter models coupled to gravity. More precisely: Under certain assumptions on the dimensionally reduced form of the matter Lagrangian, we show that the fluctuation operator of the pulsation equation can be read off from the second variation $\delta^2 M$ of the Schwarzschild mass. Although this is not surprising, we felt that a general demonstration of this is needed, in particular since energy considerations in gravity are notoriously problematic. More important, in deriving this connection we arrived at explicit expressions for the pulsation equations and for $\delta^2 M$. These can readily be worked out for a given field theoretical model, which should considerably simplify future stability studies.

The article is organized as follows. In section 2 we specify the form of the dimensionally reduced Lagrangian of the matter model. The coupled field equations for (time-dependent) spherically symmetric configurations are then expressed entirely in terms of this effective Lagrangian, taken for a diagonal form of the metric on orbit space (see eqs. (2.33) – (2.36)). This involves some nontrivial considerations since, in the time-dependent case, one would otherwise loose one of Einstein’s field equations. The general set up is used in section 3 to bring the pulsation equations into a standard form, involving only the matter perturbations. From these the metric perturbations can be obtained algebraically, without solving any differential equations. In section 4 we demonstrate that the fluctuation operator can be read off from the second variation of the total mass. The Einstein-Yang-Mills system and the Einstein-Skyrme system are taken as illustrations for ready applications of our general formulae. The dimensional reduction of the gravitational Lagrangian and the derivation of the mass variation formula for spherically symmetric configurations are deferred to two appendices.

2 Dimensional Reduction

2.1 The effective Lagrangian

Let us consider a spherically symmetric spacetime $(\tilde{M}, \tilde{g})$. By definition, $(\tilde{M}, \tilde{g})$ is a Lorentz manifold on which SO(3) acts as an isometry group, such that all group orbits are metric two-spheres. This guarantees that spacetime $(\tilde{M}, \tilde{g})$ is locally the warped product of a two-dimensional Lorentz manifold $(M, g)$ (the orbit space $\tilde{M}/\text{SO}(3) = M$ with induced metric $g$) and the two-
sphere $S^2$ with standard metric $d\Omega^2$:

$$\tilde{M} = M \times S^2, \quad \tilde{g} = g + r^2d\Omega^2.$$  \hspace{1cm} (2.1)

The positive function $r: M \rightarrow \mathbb{R}$ (the Schwarzschild coordinate) is assumed to have no critical points on $M$. For the volume form $\tilde{\eta}$ on $\tilde{M}$ there is a corresponding split: $\tilde{\eta} = \eta \wedge r^2d\Omega$, where $\eta$ and $d\Omega$ are the volume forms on $M$ and $S^2$, respectively. In addition, we introduce the intrinsically defined function $N = (dr|dr)$, where $(\cdot | \cdot)$ denotes the inner product on $M$ and the mass fraction $m$ is defined by $N = 1 - 2m/r$.

As is shown in appendix A, the dimensional reduction of the Einstein-Hilbert action yields for the effective gravitational Lagrangian $L_G$, after subtracting the standard boundary term,

$$L_G \eta = \frac{1}{N} (dm | dr) \eta. \hspace{1cm} (2.2)$$

Expanding $dm$ with respect to a coordinate basis, $dm = \dot{m} dt + m' dr$, and introducing the metric functions $\beta = -(dt | dr)/N$ and $S = \sqrt{-g}$, the effective Lagrangian takes the form

$$L_G \eta = \mathcal{L}_G dt \wedge dr, \quad \mathcal{L}_G = (m' - \beta \dot{m})S. \hspace{1cm} (2.3)$$

In terms of $N$, $S$ and $\beta$ the metric on $M$ becomes

$$g = -NS^2(dt^2 + 2\beta dt dr) + (\frac{1}{N} - \beta^2NS^2) dr^2. \hspace{1cm} (2.4)$$

The effective Lagrangian $\mathcal{L}_G$ contains, as we shall show below, the entire dynamical information. Requiring that the matter Lagrangian depends only on the metric and its first derivatives, we shall now establish the fact that the Einstein equations agree with the Euler-Lagrange equations for the total effective Lagrangian.

Let us first consider the gravitational part. Defining the Euler-Lagrange operator $E_f^0$ for a dynamical variable $f$, say, according to

$$E_f^0 = \left\{ \frac{\partial}{\partial f} - \partial_r\left(\frac{\partial}{\partial \dot{f}}\right) - \partial_t\left(\frac{\partial}{\partial f'}\right) \right\} \bigg|_{\beta=0}, \hspace{1cm} (2.5)$$

we immediately find from eq. (2.3)

$$E_m^0 \mathcal{L}_G = -S', \quad E_S^0 \mathcal{L}_G = m', \quad E_{\beta}^0 \mathcal{L}_G = -\dot{m}S. \hspace{1cm} (2.6)$$
Using well known expressions for the Einstein tensor $G_{\mu\nu}$ in Schwarzschild coordinates, we see that

$$E^0_m \mathcal{L}_G = -\frac{r}{2} SN^{-1} (G^r_r - G^t_t) , \quad (2.7)$$

$$E^0_s \mathcal{L}_G = -\frac{r^2}{2} G^t_t , \quad (2.8)$$

$$E^0_\beta \mathcal{L}_G = -\frac{r^2}{2} S G^r_r . \quad (2.9)$$

Let us now turn to the effective matter Lagrangian $\mathcal{L}_M$, defined by

$$S_M = G^{-1} \int \mathcal{L}_M \, dt \wedge dr , \quad (2.10)$$

where $S_M$ denotes the matter action. By virtue of the definition of the energy momentum tensor $T_{\mu\nu}$, the variation of $S_M$ with respect to the dynamical variable $f = N, S, \beta$ yields

$$\delta_f S_M = \frac{1}{2} \int T^{ab} \left\{ \frac{\partial g_{ab}}{\partial f} \delta f \right\} \tilde{\eta}$$

$$= \frac{1}{2} \int \left\{ 4\pi r^2 S T^{ab} \frac{\partial g_{ab}}{\partial f} \right\} \delta f \, dt \wedge dr . \quad (2.11)$$

If the matter Lagrangian depends on the metric and its first derivatives only, this implies that ($\kappa = 8\pi G$)

$$E^0_f \mathcal{L}_M = \kappa \frac{r^2}{4} S T^{ab} \frac{\partial g_{ab}}{\partial f} \bigg|_{\beta=0} , \quad f = N, S, \beta . \quad (2.12)$$

Inserting the parametrization (2.4) for the metric now yields the desired result:

$$E^0_m \mathcal{L}_M = \kappa \frac{r}{2} SN^{-1} (T^r_r - T^t_t) , \quad (2.13)$$

$$E^0_s \mathcal{L}_M = \kappa \frac{r^2}{2} T^t_t , \quad (2.14)$$

$$E^0_\beta \mathcal{L}_M = \kappa \frac{r^2}{2} S T^r_r . \quad (2.15)$$
These relations, together with eqs. (2.7) – (2.9), establish our assertion and enable us to write the Einstein equations in the form

\[ m' = -E_0^0 \mathcal{L}_M, \quad S' = E_0^0 \mathcal{L}_M, \quad \dot{m} = E_\beta^0 \mathcal{L}_M. \tag{2.16} \]

The fact that \( \beta \) may be set equal to zero after variation reflects the freedom to diagonalize the metric of the orbit space \( M \). However, it is not surprising that (in a non-static situation) one loses information by using a diagonal metric in the effective action in the first place, that is, before performing the variation.

Before we proceed, let us give two examples for which the above reasoning applies, since, in these cases, the matter action contains no derivatives of the metric.

### 2.2 Examples

As a first example we consider the Einstein-Yang-Mills (EYM) system. The matter action is

\[ S_M = \frac{1}{e^2} \text{Tr} \int F \wedge \tilde{\ast} F, \tag{2.17} \]

where \( e \) is the (dimensionless) gauge coupling, \( F = dA + A \wedge A \) is the field strength assigned to the gauge potential \( A \) and a star \( \ast \) denotes the Hodge dual with respect to the spacetime metric \( \tilde{g} \). For simplicity, we restrict ourselves to the gauge group \( SU(2) \). (The generalization to arbitrary gauge groups is straightforward applying the equations derived in [13].) A spherically symmetric gauge potential \( A \) can then be represented as

\[ A = a \tau_r + \left( \text{Re}(w) - 1 \right) \left\{ \tau_\varphi d\vartheta - \tau_\varphi \sin \vartheta d\varphi \right\} + \text{Im}(w) \left\{ \tau_\theta d\vartheta + \tau_\varphi \sin \vartheta d\varphi \right\}, \tag{2.18} \]

where \( a \) is a one-form on \( M \), \( w \) is a (complex) function on \( M \) and \( \tau_r, \tau_\theta, \tau_\varphi \) denote the spherical generators of \( SU(2) \), normalized such that \([\tau_r, \tau_\theta] = \tau_\varphi\). A gauge transformation with \( U = \exp(\chi \tau_r) \), where \( \chi \) is a function on \( M \), preserves the form of the potential \( A \) and induces the transformations

\[ a \rightarrow a + d\chi, \quad w \rightarrow e^{i\chi} w. \tag{2.19} \]

Hence, \( a \) can be considered a gauge potential and \( w \) a Higgs field on \( M \). With the parametrization (2.18) for the potential \( A \) one finds for the matter
action $S_M,$

$$S_M = -\frac{4\pi}{e^2} \int \left\{ \frac{r^2}{2} (da|da) + \left( Dw | Dw \right) + \frac{(|w|^2 - 1)^2}{2r^2} \right\} \eta ,$$  \hspace{1cm} (2.20)

where we have introduced the covariant derivative $Dw = dw - iaw$. Using the parametrization (2.4) for the metric on $M$ and adopting the temporal gauge $a = a_r dr$, we now easily obtain the following expression for the effective matter Lagrangian $L_M$:

$$L_M = L^{(0)}_M + L^{(\beta)}_M ,$$  \hspace{1cm} (2.21)

with

$$\frac{1}{\alpha^2} L^{(0)}_M = \frac{1}{S} \left\{ \frac{1}{N} |w|^2 + \frac{r^2}{2} a_r^2 \right\}$$

$$- S \left\{ N |w' - ia_r w|^2 + \frac{(|w|^2 - 1)^2}{2r^2} \right\} ,$$  \hspace{1cm} (2.22)

$$\frac{1}{\alpha^2} L^{(\beta)}_M = 2 \beta NS \text{Re} \left\{ \bar{w}(w' + i a_r \bar{w}) \right\}$$  \hspace{1cm} (2.23)

and $\alpha^2 = 4\pi G/e^2$. The Einstein equations are obtained from eq. (2.16). Applying the Euler-Lagrange operators $E_\pi, E_{a_r}$ on the diagonal part $L^{(0)}_M$ of the matter Lagrangian also yields the YM equations: $E_f L^{(0)}_M = E^0_f L_M = 0$ for $f = w, a_r$. (In addition to these equations one has the YM Gauss constraint, which got lost as a consequence of the gauge fixing.)

As a second example we consider the Einstein-Skyrme model, for which the matter action $S_M$ is (see [7] and references therein)

$$S_M = \frac{f^2}{4} \text{Tr} \int A \wedge \bar{\delta} A + \frac{1}{16e^2} \text{Tr} \int dA \wedge \bar{\delta} dA .$$  \hspace{1cm} (2.24)

Here $f$ and $e$ are coupling constants, $A = U^{-1}dU$ and $U$ is a $SU(2)$-valued function on $\hat{M}$ (describing the pion field). The “hedgehog ansatz”

$$U = \cos \chi - 2 \tau_r \sin \chi ,$$  \hspace{1cm} (2.25)

with a function $\chi$ on the orbit space $M$, gives for the effective matter Lagrangian $L_M$,

$$L_M = L^{(0)}_M + L^{(\beta)}_M ,$$  \hspace{1cm} (2.26)
where
\[
\frac{1}{\alpha^2} \mathcal{L}^{(0)}_M = \frac{1}{S} \left( \frac{1}{N} u \dot{\chi}^2 \right) - S (N u \chi^2 + v),
\]
(2.27)
\[
\frac{1}{\alpha^2} \mathcal{L}^{(\beta)}_M = 2 \beta N S u \dot{\chi} \chi',
\]
(2.28)
with \( \alpha^2 = 2 \pi G/e^2 \) and
\[
u = (f e)^2 r^2 + 2 \sin^2 \chi, \quad v = (2(f e)^2 r^2 + \sin^2 \chi) \frac{\sin^2 \chi}{r^2}.
\]
(2.29)
The entire set of field equations (including the Skyrme equation) can be derived quickly from the total effective Lagrangian \( \mathcal{L}_G + \mathcal{L}_M \).

In the examples above, the diagonal part \( \mathcal{L}^{(0)}_M \) of the Lagrangian consists of a term proportional to \( S^{-1} \) and a term proportional to \( S \), which, henceforth, will be called the “kinetic” and the “potential” parts, respectively. Note that the kinetic part is quadratic in the time derivatives of the matter fields and that it only contains terms proportional to \( S^{-1} \) or \((NS)^{-1}\), whereas the potential part consists only of terms proportional to \( S \) or \((NS)\).

### 2.3 Basic equations

Motivated by the previous examples, we assume in the following that the diagonal part \( \mathcal{L} := \mathcal{L}^{(0)}_M \) of the effective matter Lagrangian has the form
\[
\mathcal{L} = \frac{1}{2 S} \left\{ \dot{\chi}^2 \chi^{(0)} + \frac{1}{N} \dot{\chi}^2 \chi^{(1)} \right\} - S (N U + P),
\]
(2.30)
with real matter fields \( \chi^{(0)} \) and \( \chi^{(1)} \) and positive definite quadratic forms \( \dot{\chi}^2 \chi^{(0)} \) and \( \dot{\chi}^2 \chi^{(1)} \):
\[
\dot{\chi}^2 \chi^{(j)} = \langle \dot{\chi}^{(j)}, B^{(j)} \dot{\chi}^{(j)} \rangle, \quad j = 0, 1,
\]
(2.31)
where the inner products for the matter fields are denoted by \( \langle \cdot, \cdot \rangle \). (For the EYM system we have, for instance, \( \chi^{(0)} = a_r, \chi^{(1)} = (\text{Re} w, \text{Im} w) \) and \( B^{(0)} = \alpha^2 r^2, B^{(1)} = 2 \alpha^2 \text{diag}(1, 1) \).) We further require (in view of the following sections) that the function \( P \) does not depend on spatial derivatives of the matter field \( \chi := (\chi^{(0)}, \chi^{(1)}) \):
\[
B^{(j)} = B^{(j)}(\chi, \chi'; r), \quad U = U(\chi, \chi'; r), \quad P = P(\chi; r).
\]
With these assumptions for \( \mathcal{L} \) it is possible to reconstruct the total effective Lagrangian \( \mathcal{L}_G + \mathcal{L}_M \) up to first order in \( \beta \). We will show that

\[
\mathcal{L}_G + \mathcal{L}_M = S(m' - \beta \dot{m}) + \mathcal{L} - \beta \langle \dot{\chi}, \mathcal{L}' \rangle + O(\beta^2),
\]

(2.32)

where \( \chi := (\chi(0), \chi(1)) \) and \( \mathcal{L}' := \partial \mathcal{L}/\partial \chi' \). For the field equations \( E^0_f \{ \mathcal{L}_G + \mathcal{L}_M \} = 0, \) \( f = m, S, \beta, \chi \), this gives

\[
m' = -\mathcal{L}_S,
\]

(2.33)

\[
S' = \mathcal{L}_m,
\]

(2.34)

\[
\dot{m} = -S^{-1} \langle \dot{\chi}, \mathcal{L}' \rangle,
\]

(2.35)

\[
(L \dot{\chi})' = -(L \mathcal{L}')' + L \mathcal{L},
\]

(2.36)

involving only the diagonal part \( \mathcal{L} = \mathcal{L}^{(0)}_M \) of the effective matter Lagrangian.

To prove our claim, we show that the \( \dot{m} \) equation (2.35) follows from the remaining ones. We do so by rewriting the system in Hamiltonian form. Defining the conjugate momenta \( \pi := (\pi(0), \pi(1)) \) by

\[
\pi(0) = \mathcal{L}_\dot{\chi}(0) = \frac{1}{S} B(0) \dot{\chi}(0), \quad \pi(1) = \mathcal{L}_\dot{\chi}(1) = \frac{1}{NS} B(1) \dot{\chi}(1),
\]

(2.37)

the effective matter Hamiltonian is obtained by a Legendre transformation:

\[
\mathcal{H} = \langle \pi, \dot{\chi} \rangle - \mathcal{L},
\]

\[
= \frac{S}{2} \left\{ \pi^2(0) + N \pi^2(1) \right\} + S(NU + P),
\]

(2.38)

where \( \pi^2(0), \pi^2(1) \) denote the quadratic forms

\[
\pi^2(j) = \langle \pi(j), B^{-1}(j) \pi(j) \rangle, \quad j = 0, 1.
\]

(2.39)

Observing that

\[
\mathcal{L}_m = -\mathcal{H}_m, \quad \mathcal{L}_S = -\mathcal{H}_S = -S^{-1}\mathcal{H}
\]

(2.40)

and

\[
\dot{\chi} = \mathcal{H}_\pi, \quad \mathcal{L}_\chi = -\mathcal{H}_\chi, \quad \mathcal{L}' = -\mathcal{H}'.
\]

(2.41)
the field equations (2.33) – (2.36) are equivalent to

\[ m' = \mathcal{H}_S, \quad (2.42) \]
\[ S' = -\mathcal{H}_m, \quad (2.43) \]
\[ \dot{m} = S^{-1} \langle \mathcal{H}_\pi, \mathcal{H}_\chi' \rangle, \quad (2.44) \]
\[ \dot{\chi} = \mathcal{H}_\pi, \quad (2.45) \]
\[ \dot{\pi} = (\mathcal{H}_\chi')' - \mathcal{H}_\chi. \quad (2.46) \]

Taking advantage of these formulae it is now rather simple to complete the proof. Using the Einstein equations (2.42), (2.43) and, subsequently, the matter equations (2.45), (2.46), we obtain

\[ (\dot{m} S)' = (S^{-1} \mathcal{H}) S - \dot{m} \mathcal{H}_m \]
\[ = \mathcal{H} - \dot{S} \mathcal{H}_S - \dot{m} \mathcal{H}_m \]
\[ = \langle \dot{\pi}, \mathcal{H}_\pi \rangle + \langle \dot{\chi}, \mathcal{H}_\chi \rangle + \langle (\chi')', \mathcal{H}_\chi' \rangle \]
\[ = \langle \dot{\chi}, (\mathcal{H}_\chi')' \rangle, \]

which is the spatial derivative of eq. (2.34). (It would have been considerably less convenient to use the Lagrangian formulation and it would have been even less so to work with the matter equations for an explicitly given model.)

To summarize, the diagonal part (2.30) of the matter Lagrangian uniquely determines the field equations (2.42) – (2.46). The last four of these equations form a complete set of differential equations for \( m, S, \pi, \chi \), and, as is easily seen, imply that the constraint equation (2.42) (the Hamilton constraint) propagates.

It will become clear in the following sections, that the above setup is also well adapted for treating linear perturbations of spherically symmetric selfgravitating configurations.

### 3 The Pulsation Equation

In this section we linearize the field equations (2.33) – (2.36) in the vicinity of a static solution. As we will see, the only metric perturbations which enter
the first order matter equations are the variations of $m$ and $\Delta r$, where $\Delta$
denotes the Laplace operator for the orbit space $(M, g)$. Using (2.33) and
(2.34) one easily finds
\[
\Delta r = \frac{1}{S} (NS)' = \frac{2m}{r^2} - \frac{1}{r} (\pi_0^2 + 2P). 
\] (3.1)
Together with the linearized Einstein equations, these metric perturbations
can then be expressed in terms of the equilibrium solution and the pertur-
bations of the matter fields. This enables one to derive a pulsation equation
for the matter fields only.

In the following, $N, S, \chi$, etc. refer to a static solution of (2.33) – (2.36),
whereas time-dependent perturbations are denoted by $\delta N, \delta S, \delta \chi$, etc.

Let us first discuss the metric perturbations $\delta m$ and $\delta \Delta r$. Since both $\dot{m}$
and $\dot{\chi}$ are quantities of first order, the $\dot{m}$ equation (2.35) yields
\[
(S \delta m)' = -\langle L' \chi', \delta \chi \rangle. 
\] (3.2)
On the other hand, the Einstein equations (2.33) and (2.34) imply that
\[
(S \delta m)' = \mathcal{L} N \delta N - S \delta \mathcal{L} S
= S (N \delta U + \delta P). 
\]
Using the matter equation (2.36) for the equilibrium solution, we see that
\[
(S \delta m)' = -\langle L' \chi', \delta \chi \rangle'. 
\] (3.3)
Hence,
\[
d (S \delta m) = -d \langle L' \chi', \delta \chi \rangle. 
\] (3.4)
By assumption, $P$ does not depend on derivatives of the matter fields. Equation (3.4) thus yields
\[
\delta m = N \langle U' \chi', \delta \chi \rangle. 
\] (3.5)
Here the integration constant has been set equal to zero. In the soliton case
this is a consequence of the regularity requirement for the center. For black
holes, this is the correct choice since we restrict our attention to variations
with fixed position of the horizon. (The relation between eq. (3.4) and the
The variation of eq. (3.1) is now immediately found:

\[
\delta \Delta r = \frac{2}{r^2} \delta m - \frac{2}{r} \delta P
\]

\[
= \frac{2N}{r^2} \langle U_\chi', \delta \chi \rangle - \frac{2}{r} \langle P_\chi, \delta \chi \rangle .
\]

(3.6)

Next, let us consider the matter equation (2.36). Since, as will be shown below, the metric perturbations can be eliminated from the linearized matter equation, the latter can be written in the form

\[
T \delta \ddot{\chi} + \{ U_M + U_G \} \delta \chi = 0 ,
\]

(3.7)

where we have introduced the operators

\[
T \delta \chi = NS \delta (L_\chi) ,
\]

(3.8)

\[
U_M \delta \chi = NS \delta \chi \{ (L_\chi)' - L_\chi \} ,
\]

(3.9)

\[
U_G \delta \chi = NS \delta g \{ (L_\chi)' - L_\chi \} .
\]

(3.10)

Here, \( \delta_\chi \) and \( \delta_g \) denote the variations with respect to the matter fields \( \chi \) and the metric \( g \), respectively. The first two operators are immediately obtained from the definitions (3.8) and (3.9):

\[
T = \begin{pmatrix}
NB(0) & 0 \\
0 & B(1)
\end{pmatrix} ,
\]

(3.11)

\[
U_M = p^* U_\chi' p^* - i [p^*, NS U_\chi'] + NS^2 (NU_\chi + P_\chi) ,
\]

(3.12)

with the differential operator \( p^* = -i NS \partial / \partial r \). For the operator \( U_G \) we find from eq. (3.10)

\[
U_G \delta \chi = NS^2 \delta g \left\{ \frac{1}{S} \left\{ (L_\chi)' - L_\chi \right\} \right\} ,
\]

\[= -NS^2 \left\{ (U_\chi)' - U_\chi \right\} \delta N - NS^2 U_\chi' \delta \Delta r .
\]

(3.13)

The curly bracket in the last expression for \( U_G \delta \chi \) can be further simplified:

\[
N \left\{ (U_\chi)' - U_\chi \right\} = (N + 2P - 1) \frac{1}{r} U_\chi' + P_\chi ,
\]

(3.14)
as can be seen from eqs. (2.36) and (3.4) for the unperturbed solution. Eventually we can use the expressions (3.5) and (3.6) for $\delta m$ and $\delta \Delta r$ to obtain the operator $U_G$ from eq. (3.13),

$$U_G = \frac{2NS^2}{r} \left\{ U_{x'} \langle P_{x'}, \cdot \rangle + P_x \langle U_{x'}, \cdot \rangle + (2P - 1) \frac{1}{r} U_{x'} \langle U_{x'}, \cdot \rangle \right\}.$$  

(3.15)

Here we do not discuss mathematical properties, such as domains of definition or (essential) self-adjointness for the operators in eq. (3.7). This was done by some of us for the EYM system in a previous publication [16].

## 4 Mass Variation

In this section we demonstrate that the second variation of the total mass yields the same expression for the fluctuation operator as the one previously derived by means of linearizing the field equations. First of all, we recall that the Komar formula provides one with an expression for the total mass $M$ in terms of the (asymptotically) timelike Killing field $k$. Defining the “local” mass $M(r)$ by the Komar expression over a 2-sphere with coordinate radius $r$ ([17]),

$$M(r) = -\frac{1}{8\pi} \int_{S_r^2} *dk = \frac{r^2}{2S} (NS^2)' = mS + r^2NS' - rSm', \quad (4.1)$$

one has $M = \lim_{\infty} M(r) = \lim_{\infty} mS$. (Note that asymptotic flatness implies $\lim_{\infty} r^2S' = \lim_{\infty} rm' = 0$). Thus, in a gauge where $\lim_{\infty} S(r) = 1$, we obtain for configurations with $\delta \chi(\infty) = 0$

$$\delta M = \lim_{\infty} S \delta m = \lim_{\infty} SN \langle U_{x'}, \delta \chi \rangle = 0,$$

(4.2)

where we have also used eq. (3.5) with $\delta m(0) = 0$ in the soliton, and $\delta m(r_H) = 0$ in the black hole case. Since $\delta M = 0$ for static solutions, we obtain the second order formula

$$E = M + \frac{1}{2} \delta^2 M = M + \frac{1}{2} \int_{r_0}^{\infty} (S \delta^2 m)' dr,$$

(4.3)

where $r_0 = 0$ for solitons and $r_0 = r_H$ for black holes.
Our aim is to establish the relation
\[ \int_{r_0}^{\infty} (S \delta^2 m)' dr = \int_{r_0}^{\infty} \left\{ \langle \delta \dot{\chi}, T \delta \dot{\chi} \rangle + \langle \delta \chi, (U_M + U_G) \delta \chi \rangle \right\} \frac{dr}{NS}, \] (4.4)
where the operators $T$ and $U_M + U_G$ in the kinetic and the potential parts of the fluctuation operator are the same as in formula (3.7). This is most easily achieved by applying the Hamiltonian formulation presented above. Using the Hamiltonian (2.38), the $m'$ equation (2.42) becomes
\[ m' = \frac{1}{2} \left\{ \pi^2(0) + N \pi^2(1) \right\} + NU + P. \] (4.5)
Since $\dot{\chi}$ is of first order, we have $\delta \pi(0) = S^{-1} B(0) \delta \dot{\chi}(0)$ and $\delta \pi(1) = (SN)^{-1} B(1) \delta \dot{\chi}(1)$, from which we obtain
\[ \delta^2 m' = S^{-2} \langle \delta \dot{\chi}(0), B(0) \delta \dot{\chi}(0) \rangle + S^{-2} \langle \delta \dot{\chi}(1), N^{-1} B(1) \delta \dot{\chi}(1) \rangle + \delta^2 (NU + P). \]
Recalling the definition (3.11) for $T$ and using eq. (2.34) for $S'$, $S' = 2SU/r$, and $\delta^2 m = -(r/2) \delta^2 N$, this already yields the kinetic term in the expression (4.4) for the second variation of $m$,
\[ (S \delta^2 m)' = \frac{1}{NS} \langle \delta \dot{\chi}, T \delta \dot{\chi} \rangle + S (N \delta^2 U + \delta^2 P) + 2S \delta N \delta U. \] (4.6)
Note that the terms involving the second variation of $N$ cancel, since
\[ (S \delta^2 m)' = -S U \delta^2 N + S (\delta^2 m)'. \]
It remains to show that the integrals of the second and third term in this formula coincide with the corresponding expressions in eq. (4.4). Using the definition (3.9) for $U_M$, the second term can be rewritten as follows:
\[ S(N \delta^2 U + \delta^2 P) = -\delta^2 \mathcal{L} \]
\[ = -\langle \delta \chi \mathcal{L} \chi', \delta \chi \rangle' + \langle \delta \chi \{ (\mathcal{L} \chi')' - \mathcal{L} \chi \}, \delta \chi \rangle \]
\[ = \frac{1}{NS} \langle \delta \chi, U_M \delta \chi \rangle. \]
Here and in the following "$\doteq$" stands for equal up to terms whose spatial integration vanishes. (The integral over the last term in the first line of the
above formula does not contribute, since the variations of the matter fields are required to vanish at \( r = r_0 \) and for \( r \to \infty \).) Hence, the integral of the second term in eq. (4.6) assumes the desired form.

Our final task is to establish the equivalence between the last terms in eqs. (4.6) and (4.4). We do so by writing \( \delta U = U_\chi \delta \chi + U_\chi' \delta \chi' \) in order to integrate \( S \delta N \delta U \) by parts,

\[
S \delta N \delta U = \langle \delta \chi, \{ U_\chi - (U_\chi') \} \rangle S \delta N - \langle \delta \chi, U_\chi' \rangle (S \delta N)'.
\]

The first term on the r.h.s. of this equation is already of the desired form (cf. the first term in eq. (3.13)). In order to obtain the correct expression for the last term in eq. (4.6), we have to add an additional \( S \delta U \delta N \) to the above formula. Hence, it remains to compute \( S \delta U \delta N - \langle \delta \chi, U_\chi' \rangle (S \delta N)' \). Using again \( S' = 2SU/r \) and eq. (3.5) to express \( \delta N \) in terms of \( \delta \chi \), we find

\[
S \delta U \delta N - \langle \delta \chi, U_\chi' \rangle (S \delta N)' = -\langle \delta \chi, U_\chi' \rangle \left\{ NS \delta (2U/r) + (S \delta N)' \right\}
= -\langle \delta \chi, U_\chi' \rangle \left\{ NS \delta (S'/S) + (S \delta N)' \right\}
= -\langle \delta \chi, U_\chi' \rangle \delta \left\{ (NS)'/S \right\} S
= -\langle \delta \chi, U_\chi' \rangle \delta \Delta r S,
\]

which is the desired expression, corresponding to the second term in eq. (3.13). Thus, the preceding two formulae together imply

\[
2 S \delta U \delta N = S \left\langle \delta \chi, \left\{ (U_\chi')' - U_\chi \right\} \delta N - NS^2 U_\chi' \delta \Delta r \right\rangle
= \frac{1}{NS} \langle \delta \chi, U_G \delta \chi \rangle,
\]

where we have also used the expression (3.13) for the operator \( U_G \). This completes the proof of the variation formula (4.4).

For the examples in section 2.2, with Lagrangians (2.22) and (2.27), one can immediately read off the operators \( T, U_M \) and \( U_G \), given in (3.11), (3.12) and (3.13), and thus write down the explicit form of the pulsation equations (3.7), as well as the second variation of the energy (4.4). The results and their discussion can be found in [18] and [5].
Acknowledgments

We gratefully acknowledge financial support from the Swiss National Science Foundation. One of us (MH) also wishes to thank the Enrico Fermi Institute for its hospitality.

Appendix

A The gravitational Lagrangian

This appendix is devoted to the dimensional reduction of the spherically symmetric Einstein-Hilbert action. Our aim is to derive the expression (2.2) for the effective Lagrangian in terms of the intrinsically defined function $r: M \rightarrow \mathbb{R}$. Recall that $M = \tilde{M}/SO(3)$ is the orbit space with induced metric $g$, and spacetime $(\tilde{M}, \tilde{g})$ is a warped product of $M$ and $S^2$ with metric $\tilde{g} = g + r^2 \hat{g}$, where $\hat{g} = d\Omega^2$. In what follows, we will use a tilde for spacetime quantities and a hat for quantities on $S^2$.

A natural basis on the 2-dimensional orbit space is provided by the 1-forms $dr$ and $*dr$, which are used to introduce the orthonormal diade $\{\theta^0, \theta^1\}$,

$$\theta^0 = -*\theta^1, \quad \theta^1 = N^{-1/2}dr, \quad N = (dr|dr), \quad (A1)$$

where $g = -\theta^0 \otimes \theta^0 + \theta^1 \otimes \theta^1$. Here $*$ and $(\ |\ )$ denote the Hodge dual and the inner product with respect to $g$, respectively. Using $\Delta r \eta = * \Delta r = d*dr$, we immediately find

$$d\theta^0 = -\frac{1}{N} \left\{ \frac{1}{2} dN - \Delta r \, dr \right\} \wedge \theta^0, \quad d\theta^1 = -\frac{1}{2N} dN \wedge \theta^1. \quad (A2)$$

Comparing this with the structure equations and taking advantage of $dN \wedge \theta^0 = (*dN) \wedge \theta^1$ and $dN \wedge \theta^1 = (*dN) \wedge \theta^0$, enables one to read off the expression for the connection form $\omega = \omega^0_1 = \omega^1_0$ of $M$,

$$\omega = \frac{1}{N} * \left( \frac{1}{2} dN - \Delta r \, dr \right). \quad (A3)$$
Using $N = 1 - 2m/r$, this also yields
\[
    dr \wedge \omega = \left\{ \frac{m}{r^2} - \Delta r - \frac{(dm|dr)}{rN} \right\} \eta ,
\] (A4)
which will be used below.

The general expression for the Ricci scalar of a product manifold with warping function $r$ (with norm $N = (dr|dr)$) is
\[
    \tilde{R} = R + \frac{2}{r^2} (1 - N) - \frac{4}{r} \Delta r .
\] (A5)
Since the second structure equation for a 2-dimensional pseudo-Riemannian manifold reduces to $\Omega_1^0 = d\omega_1 = \frac{1}{2} R \eta$, the Ricci scalar of $(M, g)$ is obtained from
\[
    R \eta = 2 d\omega .
\] (A6)
Writing the volume form as $\tilde{\eta} = \eta \wedge r^2 d\Omega$ we thus have
\[
    \tilde{R} \tilde{\eta} = 2 \left\{ r^2 d\omega + 2 \left( \frac{m}{r} - r \Delta r \right) \eta \right\} \wedge d\Omega .
\] (A7)
Since we are interested in an effective Lagrangian involving no second derivatives of the metric fields, we integrate the first term by parts and subsequently use the expression (A4) for $dr \wedge \omega$. The Einstein-Hilbert action then becomes
\[
    16\pi S_G = \int_M \tilde{R} \tilde{\eta} = 2 \int_M d (r^2 \omega \wedge \delta \Omega) + 4 \int_M \frac{(dm|dr)}{N} \eta \wedge d\Omega ,
\] (A8)
where the terms involving the Laplacian of $r$ have canceled each other.

In order to obtain the correct effective Lagrangian after dimensional reduction, it remains to subtract a boundary term $B$, involving the trace of the extrinsic curvature, from the above formula. In terms of the connection forms $\mathring{\omega}_{\mu\nu}$ of spacetime, $B$ is given by (see, e.g., [19])
\[
    16\pi B = \int_{\partial \tilde{M}} \mathring{\omega}_{\mu\nu} \wedge \mathring{\omega} (\theta^\mu \wedge \theta^\nu) ,
\] (A9)
where $\partial \tilde{M} = \partial M \times S^2$ and it is assumed that the induced metric on $\partial M$ is kept fixed. Using the connection forms
\[
    \mathring{\omega}^a_b = \omega^a_b , \quad \mathring{\omega}^A_B = \omega^A_B , \quad \mathring{\omega}^A_b = \frac{r_A b}{r} \theta^A ,
\] (A10)
and the expressions $\tilde{\ast}(\theta^a \wedge \theta^b) = -\epsilon_{ab} r^2 d\Omega^2$, $\tilde{\ast}(\theta^A \wedge \theta^B) = \epsilon_{AB} \eta$ and $\tilde{\ast}(\theta^a \wedge \theta^B) = - \ast \theta^a \wedge \hat{\ast} \theta^B$, we obtain

$$\tilde{\omega}_{\mu \nu} \wedge \ast(\theta^\mu \wedge \theta^\nu) = 2 r^2 \omega \wedge d\Omega^2 - 4 r \ast dr \wedge d\Omega^2 + 2 \hat{\omega}_{23} \wedge \eta.$$  \hfill (A11)

The last 3-form in the above expression does not contribute to the boundary integral, since it contains no volume-form $d\Omega$. In addition, the second term needs not be taken into account neither, since we do not consider variations with respect to $r$. (If one allows for variations with respect to $r$, this term contributes to the effective Lagrangian. As a matter of fact, in this way one obtains an effective Lagrangian which, in addition to the $(ab)$-components, yields also the angular components of the Einstein tensor.)

Hence, considering $r$ as a coordinate, only the first term in eq. (A11) contributes under variations,

$$16\pi \delta B = 2 \delta \int_{\partial M} r^2 \omega \wedge d\Omega.$$  \hfill (A12)

Applying Stokes’ theorem, we observe that this term cancels the first term in the variation of the dimensionally reduced Einstein-Hilbert action (A8). We thus obtain the desired result

$$16\pi \delta (S - B) = 4\delta \int_M \frac{(dm|dr)}{(dr|dr)} \eta \wedge d\Omega = 16\pi \delta \int_M \frac{(dm|dr)}{N} \eta,$$  \hfill (A13)

completing the derivation of eq. (2.2).

## B The First Law

In this appendix we establish that the first law for spherically symmetric, static black hole configurations reads

$$\delta M - \frac{1}{8\pi} \kappa \delta A = \int_{r_H}^\infty (S \delta m)' dr,$$  \hfill (B1)

where $\kappa$ and $A$ denote the surface gravity and the area of the horizon, respectively. This expression is valid for arbitrary matter models. For the theories under consideration in this article the term on the r.h.s. generically does not contribute. This is due to the fact that the integral is, by (3.4), equal to
\[ \lim_{r \to \infty} [SN(U^\chi, \delta \chi)], \] which usually vanishes as a consequence of asymptotic flatness. The above formula can, of course, be obtained from evaluating the general mass variation formula \[20\]

\[ \delta M - \frac{1}{8\pi} \kappa \delta A = \frac{1}{16\pi} \int_{\Sigma} G^{\mu\nu} \delta g_{\mu\nu} * k - \frac{1}{8\pi} \delta \int_{\Sigma} *G(k) \] (B2)

(with \(G(k)_{\mu} = G_{\mu\nu}k^\nu\)) in the spherically symmetric metric used throughout this article. Here we shall, however, present a direct derivation of eq. (B1), which is adapted to the spherical symmetry.

Using the expression (4.1) for the “local” Komar mass and the requirement of asymptotic flatness, \(M = \lim_{r \to \infty} M(r) = \lim_{r \to \infty} (mS)\), we have

\[ \delta M = \lim_{r \to \infty} (S \delta m) = \int_{r_H}^{\infty} (S \delta m)' dr + (S \delta m)(r_H), \]

where we have also used \(\lim_{r \to \infty} S = 1\) and \(\lim_{r \to \infty} \delta m = \delta \lim_{r \to \infty} m\). Hence, in order to establish the desired result, we have to show that

\[ \frac{1}{8\pi} \kappa \delta A = (S \delta m)(r_H). \] (B4)

To see this, we first note that \(2m(r_H) = r_H\) yields

\[ (\delta m)(r_H) = \frac{1}{2} (1 - 2m'(r_H)) \delta r_H. \] (B5)

In order to complete the derivation, we recall the general result \(M_H = \frac{1}{4\pi} \kappa A\) for the evaluation of the Komar integral over the Horizon. Comparing this with the expression \(M_H = M(r_H) = [S(m - rm')](r_H)\) which is obtained from eq. (4.1), the surface gravity of the horizon becomes

\[ \kappa = \frac{S_H}{2r_H} (1 - 2m'(r_H)). \] (B6)

Using this and \(\delta A = 8\pi r_H \delta r_H\) in eq. (B5) yields the desired result (B4). Together with eq. (B3), this eventually establishes the variation formula (B1).

References

[1] N. Straumann and Z.-H. Zhou, Phys. Lett. B 237, 353 (1990); Phys. Lett. B 243, 33 (1990).
[2] Z.-H. Zhou and N. Straumann, Nucl. Phys. B 369, 180 (1991).

[3] R. Bartnik and J. McKinnon, Phys. Rev. Lett. 61, 141 (1988).

[4] M. S. Volkov and D.V. Gal’tsov, Pis’ma Zh. Eksp. Teor. Fiz. 50, 312 (1989) [JETP Lett. 50, 345 (1990)]; Sov. J. Nucl. Phys. 51, 747 (1990); H.P. Künzle and A.K.M. Masood-ul-Alam, J. Math. Phys. 31, 928 (1990); P. Bizon, Phys. Rev. Lett. 64, 2644 (1990).

[5] S. Droz, M. Heusler and N. Straumann, Phys. Lett. B 268, 371 (1991).

[6] M. Heusler, S. Droz and N. Straumann, Phys. Lett. B 271, 61 (1991); Phys. Lett. B 285, 21 (1992).

[7] M. Heusler, N. Straumann and Z.-H. Zhou, Helv. Phys. Acta 66, 614 (1993).

[8] K.-Y. Lee, V. P. Nair and E. Weinberg, Phys. Rev. Lett. 68, 1100 (1992); Phys. Rev. D 45, 2751 (1992); M. E. Ortiz, Phys. Rev. D 45, R2586 (1992).

[9] P. Breitenlohner, P. Forgács and D. Maison, Nucl. Phys. B 383, 357 (1992); *Gravitating monopole solutions II*, Preprint MPI-PhT/94-87, gr-qc/9412039.

[10] P. C. Aichelburg and P. Bizon, Phys. Rev. D 48, 607 (1993).

[11] P. Boschung, O. Brodbeck, F. Moser, N. Straumann, and M. S. Volkov, Phys. Rev. D 50 (1994) 3842;

[12] E. Winstanley and N. E. Mavromatos, *Instability of hairy black holes in spontaneously-broken EYMH systems*, Preprint ENSLAPP-A-508/95, OUTP-95-08P.

[13] S. Chandrasekhar, *Hydrodynamic and hydrodynamic stability*, Clarendon Press, 1961, Chapter XIV.

[14] J. M. Ball and J. E. Marsden, Arch. Rat. Mech. Anal. 86, 251-277.

[15] O. Brodbeck and N. Straumann, J. Math. Phys. 6, 2412 (1993).
[16] O. Brodbeck and N. Straumann, *Instability proof for EYM solitons and black holes with arbitrary gauge groups*, Preprint ZU-TH 38/94, gr-qc/9411058.

[17] M. Heusler, Class. Quantum Grav. **12**, 779 (1995).

[18] M. S. Volkov, O. Brodbeck, G. Lavrelashvili and N. Straumann, *The number of sphaleron instabilities of the Bartnik-McKinnon solitons and non-Abelian black holes*, Preprint ZU-Th 3/95, hep-th/9502045, to appear in Phys. Lett. B.

[19] N. Straumann, *General relativity and relativistic astrophysics*, Springer-Verlag, Berlin 1984, Chapter 2.5.

[20] B. Carter, *General relativity: An Einstein centenary survey*, Cambridge University Press, Cambridge 1979; M. Heusler and N. Straumann, Class. Quantum Grav. **10**, 1299 (1993).