Steady state correlations and induced trapping of an inertial AOUP particle

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We study the dynamics of an inertial active Ornstein-Uhlenbeck particle self-propelling in a confined harmonic well. The transport behaviour of the particle is investigated by analyzing the particle trajectories, steady state correlations, and mean square displacement (MSD). The steady state correlation functions for the position as well as velocity are exactly calculated using different methods. We explore how the inertia affects the dynamical behaviour, when the particle is confined in a harmonic trap as well as when it is set free. From the exact calculation of MSD, it is observed that the initial time regimes are ballistic for both harmonically confined particle and free particle, whereas the long time regimes are diffusive for a free particle and non diffusive for a harmonically confined particle. One of our interesting observation is that the harmonically confined particle gets more and more confined with increase in the self propulsion time or activity time of the dynamics and finally it gets trapped for very large self propulsion time. For a free particle, the velocity correlation decays by the complex interplay between the inertial time scale and the self propulsion time scale of the dynamics. Moreover, decorrelation in velocity happens only when these two time scales are of equal order.

I. INTRODUCTION

In recent years, there is an immense interest in the study of systems which are driven far away from equilibrium. Active matter is one among these systems, which is inherently driven from equilibrium. The term 'active' refers to the development of directed motion generated by the constituents of the system by consuming energy from the local environment. Examples of such active systems extend from motile cellular microorganisms like bacteria or unicellular protozoa, artificially synthesized microswimmers like Janus particle, micro robots, hexbugs, the flocking of birds, to school of fishes. One of the simplest and most extensively used standard model, namely active Brownian particle (ABP) model has been developed to treat the dynamical behaviour of particles in such active matter systems in both single particle level as well as collective level. This term was first introduced in Ref. [11], where both the rotational as well as translational motion of the particles are taken into account. However, recently another model namely Active Ornstein-Uhlenbeck particle (AOUP) model [12–14] is also used for studying the dynamical behaviour and steady state correlations of particles in such systems. The degree of irreversibility in the dynamical behaviour of an AOUP particle is also discussed in Ref. [15]. This model was initially proposed by Uhlenbeck and Ornstein to study the velocity distribution of passive particles [16]. The advantage of AOUP model is that it helps in the exact computation of analytical results [17–22]. Recently, the use of AOUP model has resulted many of the important features of active matters such as accumulation near boundary [23], motility induced phase separation (MIPS) [24–27] etc. However, in most of the studies, inertia is not taken into account. Systems which are macroscopic in nature, while self-propelling in a dilute/gaseous or lower viscous medium, the inertia becomes prominent and plays an important role in the dynamics. The inertial influence in such systems challenge the theoretical modelling of the dynamical behaviour. Macroscopic swimmers [28–31], flying insects [32], etc. are some examples of this category, where inertia plays an important role. Hence the introduction of inertia to the standard models is necessary for treating the dynamics of such systems. Recently, inclusion of inertia in the ABP model results some modification in the fundamental properties of active systems, such as inertial delay [33], motility induced phase transitions [34] and so on. So far the existing results on active matter systems are mostly on the overdamped regime of the particle dynamics. The inertial effects of active matter systems have received only limited attention to the steady state correlation studies. Without the knowledge of inertia and inertial effects, the exact transport properties of a system will remain unexplored. Hence in this work, we are interested the dynamics of an inertial active Ornstein-Uhlenbeck particle moving in a viscous medium. We are mainly interested in steady state correlation functions and the mean square displacement of the dynamics. The exact computation of steady state correlation functions can provide the complete information about the dynamical behaviour or transport properties of the system. The analytical behaviour of steady state correlation functions of an overdamped active Brownian particle as well as AOUP particle are discussed in [18, 35]. In this paper, we consider the inertia in the AOUP model and exactly compute the steady state correlation functions using various methods which is discussed in the following sections.

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Substituting potential, with $k$ can be obtained as $U(x, t) = 2mω^2 x^2 + \frac{1}{2} k x^2$, with $t_c$ being the noise correlation time or activity time in the dynamics. It is the time up to which there exists a finite correlation or activity in the dynamics. As a result the particle persists to self-propel up to time $t_c$. A finite $t_c$ especially quantifies the activity or correlation in the medium, which decays exponentially with $t_c$. The angular brackets $\langle \rangle$ represent the ensemble average over the noise. $D$ is the parameter which remains constant throughout the dynamics and represents strength of the Ornstein-Uhlenbeck noise \[22, 36–39\]. For a finite $t_c$, the system is in non-equilibrium and can’t be mapped to the thermal equilibrium limit of the dynamics. However, in the white noise limit, i.e., $t_c \to 0$ limit, we consider $D = 2γ k_B T$, with $T$ as the temperature of the system, to have the thermal equilibrium limit of the dynamics.

The potential $U(x, t) = \frac{1}{2} k x^2$ represents the harmonic potential, with $k$ as the harmonic stiffness constant. Substituting $k = mω^2_0$ or $U(x, t) = \frac{1}{2} mω^2_0 x^2$ with $ω_0$ as the harmonic frequency, the dynamics [Eq. (1)] can take the form as

$$m \frac{dv}{dt} = -mω^2_0 x - γ v + \sqrt{D} \eta(t). \tag{2}$$

III. RESULTS AND DISCUSSION

By using the Fourier transform method formalism, the solution of the dynamics[Eq. (2)] in Fourier space $x(ω) = \int_{-∞}^{∞} e^{iωt} x(t) dt$ and $v(ω) = \int_{-∞}^{∞} e^{iωt} v(t) dt$ can be obtained as

$$x(ω) = \frac{\sqrt{D} η(ω)}{-mω^2 + iγω + ω^2_0}$$

and

$$v(ω) = \frac{ω\sqrt{D} η(ω)}{imω^2 + γω - iω^2_0 m}.$$
In Fig. 2 we have plotted the spectrum corresponding to the position $S_x(\omega)$ as a function of $\omega$ for various values of $t_c$. For a particular value of $t_c$, the spectrum shows a non monotonic behaviour with $\omega$. For lower $t_c$ values, $S_x(\omega)$ shows a double peaking behaviour. Subsequently with increase in $t_c$ value, the double peaking behaviour gets suppressed and the spectrum becomes single peaked structure across $\omega = 0$. With further increase in $t_c$, the peak gets sharper across $\omega = 0$ as expected. This might be due to the reason that the particle gets more and more confined in the well with increase in $t_c$ value. The same conclusion is drawn from the particle trajectory plot in Fig. 1 for larger $t_c$ value. When the correlation time approaches zero, the active fluctuations become thermal and the particle behaves like a passive particle. In this limit, the analytical expression for $S_x(\omega)$ is consistence with that reported in Ref. [37] in the absence of any external force and the spectrum shows the same behavior as reported in Ref. [37].

The Optimum value of $\omega (\omega_m)$ at which the spectrum shows peaks can be obtained by taking the time derivative of $S_x(\omega)$ with respect to $\omega$ and equating it to zero, i.e., $\frac{dS_x(\omega)}{d\omega}|_{\omega=\omega_m} = 0$. By simplifying this, it is confirmed that the optimum $\omega$ depends on the parameters $t_c$, $\omega_0$ and $\gamma$ values. With increase in $t_c$, $\omega_m$ decreases as a result the peak shifts towards $\omega = 0$ and becomes single peaked at $\omega = 0$. Similarly the spectral density corresponds to velocity $S_v(\omega) [= \langle v(\omega)v^*(\omega) \rangle]$ can be obtained as

$$S_v(\omega) = \frac{D\omega^2}{(1 + \omega^2 t_c^2) \left\{ (m\omega_0^2 - m\omega^2)^2 + \omega^2 \gamma^2 \right\}} \quad (5)$$

In Fig. 3 we have shown the plot for $S_v(\omega)$ as a function of $\omega$ for different values of $t_c$. For a given $t_c$ value, the spectrum is symmetric across $\omega = 0$ and shows a double peaking behaviour with exponential tails. With further increase in $t_c$, the peaks get suppressed as expected. At the same time, the peaks get narrow and shifts towards lower values of $\omega$. For very large $t_c$ value, the influence of the harmonic confinement becomes stronger. As a result the exponential tails also get stiffer. This reflects the stronger effect of the barriers with longer persistence of activity in the dynamics. Therefore, the particle gets more and more confined with $t_c$. The optimum value of $\omega (\omega_m)$ at which the spectrum shows maximum can be obtained by simplifying $\frac{dS_v(\omega)}{d\omega}|_{\omega=\omega_m} = 0$. It is confirmed that the optimum $\omega (\omega_m)$ at which the spectrum shows maximum decreases with $t_c$. Other than $t_c$, it also depend on the parameters $\omega_0$ and $\gamma$.

The position correlation $C_x(t)$ and velocity correlation $C_v(t)$ can be obtained by performing the inverse Fourier transform of $S_x(\omega)$ [Eq. (4)] and $S_v(\omega)$ [Eq. (5)] as $\int_{-\infty}^\infty e^{-i\omega t} S_x(\omega) d\omega$ and $\int_{-\infty}^\infty e^{-i\omega t} S_v(\omega) d\omega$, respectively. Simplifying the integral $\int_{-\infty}^\infty e^{-i\omega t} S_x(\omega) d\omega$, $C_x(t)$ can be calculated as

$$C_x(t) = \frac{t^2}{2} \int_{-1}^1 \int_{-t}^{t} -t \left\{ f_2 \cos(\omega_0 t) + f_3 \sin(\omega_1 t) \right\} e^{2t} \frac{-t}{4\omega_0^2} e^{2t} \left\{ f_2 \cos(\omega_0 t) + f_3 \sin(\omega_1 t) \right\},$$

where
time and the decay becomes slower. At the same time, further increase in \(\mathcal{C}\) attains a minimum value and then approaches zero. With a finite constant value before decaying to zero. With is negligible, the velocity correlation \(v\) also obtained the same results for both \(C_x(t)\) and \(C_v(t)\), which is shown in appendix-A. The results for both \(C_x(t)\) and \(C_v(t)\) are well agreement with the athermal part of the correlations as reported in Ref. [14]. Similarly simplifying the integral \(\int_0^\infty e^{-i\omega t}S_v(\omega) d\omega\), the velocity correlation \(C_v(t)\) can also be obtained as

\[
f_1 = \frac{D t_c}{(m + m t_c^2 \omega_0^2 - \gamma^2 t_c^2)}, \quad N = \frac{D}{m \gamma \omega_1 \left[ 1 + t_c^2 \left( 2 \omega_0^2 - \frac{1}{t_m^2} + \frac{t_c^2 \omega_0^2}{2 t_m^2} \right) \right]},
\]

\[
f_2 = \omega_1 \left( 2 + 2 \omega_1^2 t_c^2 - \frac{3 t_c^2}{2 t_m^2} \right) \quad \text{and} \quad f_3 = \left[ \frac{1}{t_m} + t_c^2 \left( \frac{3 \omega_1^2}{t_m} \right) \right].
\]

Similarly simplifying the integral \(\int_0^\infty e^{-i\omega t}S_v(\omega) d\omega\), the velocity correlation \(C_v(t)\) can also be obtained as

\[
f_4 = \omega_1 \left( 2 + 2 \omega_1^2 t_c^2 + \frac{t_c^2}{2 t_m^2} \right) \quad \text{and} \quad f_5 = \left[ \frac{1}{t_m} + t_c^2 \left( \frac{\omega_1^2}{t_m} + \frac{1}{4 t_m^2} \right) \right].
\]

In both Eqs. (6) and (7), \(\omega_1 = \sqrt{\omega_0^2 - \frac{1}{n_m^2}}\) and \(t_m = \frac{m}{\gamma}\). Using the correlation matrix formalism method, we have also obtained the same results for both \(C_x(t)\) and \(C_v(t)\), which is shown in appendix-A. The results for both \(C_x(t)\) and \(C_v(t)\) are well agreement with the athermal part of the correlations as reported in Ref. [14]. In Fig. 4(a) both normalized \(C_x(t)\) and \(C_v(t)\) are plotted as a function of \(t\) for lower values of \(\gamma\) in (a) and (b) and for larger values of \(\gamma\) in (c) and (d), respectively. For low viscous medium \((\gamma = 0.1)\), where the inertial influence is prominent, both \(C_x(t)\) and \(C_v(t)\) persists with a finite value for initial time regimes and show an oscillatory behaviour in the long time regime of the dynamics before approaching to zero. For \(C_x(t)\), the amplitude of the oscillation decreases with increase in \(t_c\) values and the curves shift in such a way that for very large \(t_c\), the correlation never approaches negative values (see Fig. 4(a)). However, for \(C_v(t)\), the amplitude of the oscillation decreases with time. At the same time, the minimum value shifts towards right and approaches zero in the long time limit, which can be reflected from Fig. 4(b). For the case of a highly viscous medium (say for \(\gamma = 3.0)\), where the inertial influence is negligible, the \(C_x(t)\) never becomes negative and show a finite constant value before decaying to zero. With increase in \(t_c\), the correlation persists for a longer time before decaying to zero (as in Fig. 4(c)). On the other hand, for very low \(t_c\) value, \(C_v(t)\) shows a damped oscillatory behaviour. It decays faster, becomes negative, attains a minimum value and then approaches zero. With further increase in \(t_c\), the correlation persists for longer time and the decay becomes slower. At the same time, the minimum value shifts towards right and finally approaches zero for large \(t_c\) (see Fig. 4(d)). In \(t_c \to 0\) limit, the result for \(C_v(t)\) is consistent with that reported in Ref. [17] in the absence of any external force. When the particle is not confined by the harmonic potential, it can freely self propel in the environment. The dynamics of the particle [Eq. (2)] in such a case can be reduced as,

\[
m \frac{dv}{dt} = -\gamma v + \sqrt{D} \eta(t),
\]

\[
t_c \eta(t) = -\eta(t) + \zeta(t).
\]

The dynamics [Eq. (3)] and [Eq. (9)] can be written in the form [14],

\[
\dot{w} = -A \cdot w + \sqrt{2} \sigma \cdot \eta,
\]

where,

\[
w = \begin{pmatrix} v \\ \eta \end{pmatrix}, \quad A = \begin{pmatrix} \frac{\gamma}{m} & -\frac{1}{m} \\ 0 & \frac{1}{t_c} \end{pmatrix}, \quad \text{and} \quad \sigma = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{2}t_c} \end{pmatrix}
\]

With the help of correlation matrix method formalism[14], by introducing the correlation matrix \(C\), Eq. (10) can take the form as

\[
A \cdot C + C \cdot A^T = 2M,
\]

where \(M = \sigma \cdot \sigma^T\) is the diffusion matrix and

\[
C = \begin{pmatrix} C_{vv} & C_{vn} \\ C_{nv} & C_{nn} \end{pmatrix}.
\]

Solving Eq. (11), the elements of the correlation matrix can be obtained as \(C_{vv} = \frac{D}{\gamma^2(t_m + t_c)}\), \(C_{vn} = C_{nv} = \frac{1}{t_m} + t_c^2 \left( \frac{3 \omega_1^2}{t_m} \right)\).
for different $\gamma = 10.0$ and (f), respectively. The other common parameters are $D = \omega_0 = m = 1$.

FIG. 4. Normalized $C_x(t)\{(a)\}$ as a function of $t$ in (a) and (c) and the normalized $C_y(t)\{(b)\}$ as a function of $t$ in (b) and (d) are shown for a free particle and for a confined harmonic particle, respectively for different $t_c$ values. The common parameters are $D = \omega_0 = m = 1$.

FIG. 5. 2D single particle trajectories of a free particle. $t_c = 0.1$ is considered for (a),(b) and (c) and $t_c = 10$ is for (d),(e) and (f), respectively. The other common parameters are $m = D = 1$.

$$\frac{\sqrt{\gamma t}}{2\gamma t_c}$$ and $C_{yy} = \frac{1}{2c}$ respectively. The time dependent auto correlation function for velocity can be obtained from Eq. (8) by multiplying initial velocity $v_0$ to both the sides of Eq. (8) and taking the time derivative of average of each terms as,

$$\frac{d}{dt} \langle v(t)v(0) \rangle + \frac{\gamma}{m} \langle v(t)v(0) \rangle = \frac{\sqrt{D}}{m} \langle \eta(t)v(0) \rangle.$$ (12)

Similarly, from Eq. (9) the time derivative of $\langle \eta(t)v(0) \rangle$ can be written as

$$\frac{d}{dt} \langle \eta(t)v(0) \rangle = \frac{-1}{t_c} \langle \eta(t)v(0) \rangle.$$ (13)

By solving the above equation, we get

$$\langle \eta(t)v(0) \rangle = C_{yy} \exp \frac{-t}{t_c}.$$ (14)

Now substituting the value for $C_{yy}$ in Eq. (13) and using the expression of $\langle \eta(t)v(0) \rangle$ in Eq. (12), one can solve Eq. (12) for the velocity correlation $\langle v(t)\langle v(t)v(0) \rangle \rangle$ as

$$\langle v(t) \rangle = \left( \frac{D}{2\gamma^2 \left( t_m^2 - t_c^2 \right)} \right) \left[ t_m e^{\frac{-t}{t_m - t_c}} - \frac{t}{t_c} \right].$$ (15)

The single particle trajectory of a freely moving self propelled particle in xy-plane is shown in Fig. 5 for different values of $\gamma$ as well as for different values of $t_c$. For a free particle, the motion is mainly governed by the inertial force and the self propulsion force and hence the time evolution of the dynamics is characterized by the complex interplay of the inertial time scale ($t_m = \frac{\sqrt{\gamma}}{\omega_0}$) and the self propulsion time scale ($t_c$) of the dynamics. It is observed that for small $t_c$, when $\gamma$ is large, the particle shows a random trajectory. With either decrease in $\gamma$ or increase in $t_c$, the trajectory becomes smoother. Further, in Fig. 5 we have plotted the normalized $C_x(t)$ as a function of $t$ for different values of $\gamma$ in (a) and for different values of $t_c$ in (b), respectively. Since the dynamics
is governed by the inertial time and self propulsion time scale, \( C_v(t) \) decays by the complex interplay of these two time scale. For any fixed \( t_c \) or \( \gamma \), it maintains with the same value in the lower time regimes before decaying to zero in the long time regimes. With increase in \( \gamma \), \( C_v(t) \) decays faster (see Fig. 6(a)) for a fixed \( t_c \) and it’s persistence with time decreases whereas with increase in \( t_c \) it persists for longer time before decaying to zero as reflected from Fig. 6(b). It is always found to be positive for any chosen value of \( \gamma \) or \( t_c \). For very lower value of \( t_c \), \( C_v(t) \) is mainly controlled by the inertial time scale \( t_m \). Hence in \( t_c \to 0 \) limit, \( C_v(t) \) displays the same behaviour as in case of an inertial passive particle.

Similarly the steady state correlation \( C_x(t) \) corresponding to position of a free particle can be calculated as

\[
C_x(t) = x_0^2 + 2x_0 t_m v_0 + (t_m v_0)^2 + C'_x(t),
\]

where

\[
C'_x(t) = \frac{D}{\gamma^2} \left[ t - t_m - t_c^2 + \frac{t_m^3}{2 t_c^2 - t_m^2} \left( t_m - \frac{t_c}{2} \right) \left( 1 + t'_m \right) \right]
\]

with \( t'_m = \frac{t_m}{t_c + t_m} \). It is worth to note that \( C_x(t) \) depends on the initial velocity of the particle. Although \( C_x(t) \) is part of the correlation which depends on both \( t_m \) and \( t_c \) values, it is mainly controlled by the strength of the Aoup noise, \( D \). Further using Kubo’s theorem, we have calculated the diffusion coefficient as

\[
D_f = \int_0^\infty \langle v(t) v(0) \rangle dt = \frac{D}{2\gamma^2}.
\]

The diffusion coefficient depends only on the strength of noise as well as on the viscous drag of the medium. However, it is independent of the mass of the particle as reported in Ref. 10.

Next we are interested in the exact computation of MSD \((\langle \Delta x^2(t) \rangle)\) for both harmonically confined particle and free particle. The MSD for a harmonically confined particle \((\langle \Delta x^2(t) \rangle)_h\) is calculated as

\[
\langle \Delta x^2(t) \rangle_h = \langle (x(t) - x_0)^2 \rangle
\]

with

\[
<\Delta x^2(t)>_h = \frac{e^{-t/m}}{4 \omega^2 t_m^2} \left[ -2 t_m x_0 \omega_1 e^{+t/m} + 2 t_m x_0 \omega_1 \cos(\omega_1 t) + (2 t_m v_0 + x_0) \sin(\omega_1 t) \right]^2 - \frac{D}{\gamma^2 t_m \omega_1^2} \left[ \frac{2 - \frac{t}{t_m}}{\left( \frac{t}{t_m} - 2 \right)^2 + 4 t_m^2 \omega_1^2} \right]
\]

\[
+ \frac{D}{\gamma^2 t_m \omega_1^2} \left[ \frac{8 \omega_1^2}{4 \omega^2 + \frac{\omega_1}{t_m}} \left( \frac{2 + \frac{t}{t_m}}{2 + \frac{t}{t_m} + 4 t_m^2 \omega_1^2} \right) + 8 t_c \omega_1 \left( \frac{2 + \frac{t}{t_m}}{2 + \frac{t}{t_m} + 4 t_m^2 \omega_1^2} \right) \right]
\]

\[
+ \frac{D}{2 \gamma^2 t_m \omega_1^2} \left[ \frac{e^{-t/m}}{4 \omega^2 + \frac{\omega_1}{t_m}} \left( -2 + \frac{t}{t_m} \right)^2 + 4 t_m^2 \omega_1^2 \right]
\]

\[
= \frac{e^{-t/m}}{4 \omega^2 t_m^2} \left[ -2 t_m x_0 \omega_1 e^{+t/m} + 2 t_m x_0 \omega_1 \cos(\omega_1 t) + (2 t_m v_0 + x_0) \sin(\omega_1 t) \right]^2 - \frac{D}{\gamma^2 t_m \omega_1^2} \left[ \frac{2 - \frac{t}{t_m}}{\left( \frac{t}{t_m} - 2 \right)^2 + 4 t_m^2 \omega_1^2} \right]
\]

\[
+ \frac{D}{\gamma^2 t_m \omega_1^2} \left[ \frac{8 \omega_1^2}{4 \omega^2 + \frac{\omega_1}{t_m}} \left( \frac{2 + \frac{t}{t_m}}{2 + \frac{t}{t_m} + 4 t_m^2 \omega_1^2} \right) + 8 t_c \omega_1 \left( \frac{2 + \frac{t}{t_m}}{2 + \frac{t}{t_m} + 4 t_m^2 \omega_1^2} \right) \right]
\]

\[
+ \frac{D}{2 \gamma^2 t_m \omega_1^2} \left[ \frac{e^{-t/m}}{4 \omega^2 + \frac{\omega_1}{t_m}} \left( -2 + \frac{t}{t_m} \right)^2 + 4 t_m^2 \omega_1^2 \right]
\]

\[
(18)
\]
By expanding equation \( [18] \) in the powers of \( t \), it can be expressed as,

\[
\langle \Delta x^2(t) \rangle_h = v_0^2 t^2 - v_0 \left( \frac{v_0}{t_m} + x_0 \omega_0^2 \right) t^3 + \frac{1}{12} \left[ \frac{3D}{2t_m^2 t_c \gamma} \right]^2 + \frac{7v_0^2}{t_m^2} - 4v_0^2 \omega_0^2 + \frac{10v_0 x_0 \omega_0^2}{t_m} + 3x_0^2 \omega_0^4 \right] t^4 + O \left( t^5 \right) \tag{19}
\]

In the long time limit (i.e., \( t \to \infty \) limit), \( \langle \Delta x^2(t) \rangle_h \) is found to be

\[
\langle \Delta x^2(t) \rangle_h = x_0^2 + \frac{D \left( 1 + \frac{t_c}{t_m} \right)}{4\gamma^2 t_m^2 \omega_0^2} \left( \frac{1}{t_m} + \frac{t_c}{t_m} \left( t_c \omega_0^2 + \frac{1}{t_m} \right) \right) \tag{20}
\]

Similarly, the MSD for a free particle is obtained as

\[
\langle \Delta x^2(t) \rangle_f = \frac{D}{\gamma^2} \left[ t - 2t_m - t_c e^{\frac{t_m}{t_c}} \right] + v_0^2 t_m^2 \left( e^{\frac{t_m}{t_c}} - 1 \right) + \frac{Dt_c t_m}{2\gamma^2 (t_c^2 - t_m^2)} \left[ t_m e^{\frac{t_m}{t_c}} + 3t_c e^{\frac{t_m}{t_c}} + t_c - t_c e^{-\left( \frac{t_m}{t_c} + \frac{1}{t_m} \right)} \right] - \frac{Dt_m^2}{4\gamma^2 (t_c^2 - t_m^2)} \left[ t_m + t_c e^{\frac{t_m}{t_c}} + 3t_m e^{\frac{1}{t_m}} \right] \tag{21}
\]

The power series expansion of \( [21] \) in the powers of \( t \), can be written as

\[
\langle \Delta x^2(t) \rangle_f = t^2 v_0^2 - \frac{t_c^2 v_0^2}{t_m} + t^4 \left( \frac{D}{8m^2 t_c} + \frac{7v_0^2}{12t_m^2} \right) + O \left( t^5 \right) \tag{22}
\]

In Fig. 7, we have shown the \( \langle \Delta x^2(t) \rangle_f \) as a function of \( t \) for both a free particle(Fig. 7(a)) and a confined harmonic particle (Fig. 7(b)) for different values of \( t_c \). The calculation results for \( \langle \Delta x^2(t) \rangle \) for both free and harmonically confined particle confirms that in the \( t \to 0 \) limit, \( \langle \Delta x^2(t) \rangle \) is proportional to \( t^2 \) and hence the interplay of the inertial time and the self propulsion time regimes are ballistic in nature. The expression of MSD for a free particle in the long time limit (i.e., \( t \to \infty \) limit) is obtained as \( \langle \Delta x^2(t) \rangle_f = \frac{D}{\gamma^2} \), i.e., proportional to \( t \). However, it approaches a constant value for a harmonically confined particle. Hence, the long time regimes for a free particle are diffusive where as for a confined harmonic particle, it is non diffusive in nature. Moreover, for a free particle the initial time regime as well as the long time regimes are not dependent on \( t_c \), but there is a dependence of \( t_c \) in the intermediate time regimes. The MSD for a harmonically confined particle in the time asymptotic limit depends on \( t_c \) and gets suppressed with increase in \( t_c \) value and finally approaches zero in \( t_c \to \infty \) limit. This confirms that the particle gets more and more confined around the mean position of the well for longer persistence of activity in the medium.

**IV. CONCLUSION**

In this work, we have studied the dynamics of a harmonically confined particle following the AOUP process. With the help of both numerical simulation and exact analytical calculations, we have explored the behaviour of particle trajectories, the steady state correlation functions, and mean square displacement both for a free particle and a harmonically confined particle. From the simulated trajectories, it is confirmed that the dynamics of a harmonic particle is mainly characterized by the inertial time, self propulsion time, and harmonic time scale. Similarly, for a free particle it is governed by the complex interplay of the inertial time and the self propulsion time.
time. One of our important observations is that for a harmonic particle, in lower viscous medium, the particle makes circular trajectories due to strong influence of inertia. Further with increase in the self propulsion time of the dynamics, the circular trajectories get squeezed. This confirms that the particle gets a stronger confinement with the longer persistence of activity in the medium. The same conclusion is drawn from the behaviour of mean square displacement, which gets suppressed with increase in the self propulsion time. The steady state velocity correlation shows a damped oscillatory behaviour in the over damped regime. In contrast to that for a free particle, the dynamics, the circular trajectories get squeezed. This increase in the self propulsion time. One of our important observations is that for a harmonic particle, in lower viscous medium, the particle and the UGC Faculty recharge research grant for the recent .

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Appendix A

For the confined harmonic particle, the dynamics [Eq. (2)] can be expressed in the form of a vector equation as [14],

$$\dot{n} = -A' \cdot n + \sqrt{2} \sigma' \cdot \eta' \tag{A1}$$

where,

$$n = \begin{pmatrix} x \\ v \\ \eta \end{pmatrix}, \quad A' = \begin{pmatrix} 0 & -1 & 0 \\ \frac{k}{m} & \frac{\gamma}{m} & -\frac{\sqrt{2} c}{\sqrt{m} \cdot t_c} \\ 0 & 0 & \frac{1}{\sqrt{m} \cdot t_c} \end{pmatrix} \quad \text{and} \quad \sigma' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{m} \cdot t_c} \end{pmatrix}$$

Using the correlation matrix method formalism, by introducing a correlation matrix \((C')\), Eq. (A1) can take a new form as,

$$A' \cdot C' + C' \cdot A'^T = 2M', \tag{A2}$$

where \(M' = \sigma' \cdot \sigma'^T\) is the diffusion matrix for confined particle and

$$C' = \begin{pmatrix} C'_{xx} & C'_{xv} & C'_{x\eta} \\ C'_{vx} & C'_{vv} & C'_{v\eta} \\ C'_{\eta x} & C'_{\eta v} & C'_{\eta\eta} \end{pmatrix}.$$ 

Solving Eq. (A2), the elements of correlation matrix can be obtained as, \(C'_{xx} = 0\), \(C'_{xv} = \frac{(m + \gamma t_c)}{2\gamma^2(\gamma + \gamma t_c + k t_c^2)} = -C'_{vv}\), \(C'_{x\eta} = C'_{\eta x} = \frac{\gamma^2 t_c}{2(\gamma + \gamma t_c + k t_c^2)}\), \(C'_{vv} = \frac{D t_c}{2(\gamma + \gamma t_c + k t_c^2)}\), \(C'_{\eta\eta} = C'_{\eta v} = \frac{D t_c}{2(\gamma + \gamma t_c + k t_c^2)}\), and \(C'_{v\eta} = \frac{D t_c}{2(\gamma + \gamma t_c + k t_c^2)}\). The time dependent equation for velocity auto correlation function \(\langle v(t)v(0)\rangle\) can be obtained from the Eq. (A2) as,

$$\frac{d^2}{dt^2} \langle v(t)v(0)\rangle + \frac{\gamma}{m} \frac{d}{dt} \langle v(t)v(0)\rangle + \frac{k}{m} \langle v(t)v(0)\rangle = \frac{\sqrt{D}}{m} \langle \eta(t)v(0)\rangle.$$ \tag{A3}

Solving the time dependent equation for \(\langle \eta(t)v(0)\rangle\), one can obtain

$$\langle \eta(t)v(0)\rangle = C'_{v\eta} \exp -\frac{t}{t_c}.$$ 

Now substituting the value of \(C'_{v\eta}\) in the above equation, one can obtain

$$\langle \eta(t)v(0)\rangle = \frac{\sqrt{D}}{2(m + \gamma t_c + k t_c^2)} \exp -\frac{t}{t_c} \tag{A4}$$

Further substituting \(\langle \eta(t)v(0)\rangle\) and solving Eq. (A3), the correlation corresponds to velocity \(C_v(t) = \langle v(t)v(0)\rangle\) can be obtained as

$$C_v(t) = -\frac{f_1}{2m^2} e^{-t/t_c} + \frac{N}{4} e^{-2t/m} (f_4 \cos(\omega_1 t) + f_5 \sin(\omega_1 t)),$$ \tag{A5}

which is same as what we get using the Fourier transform method.

[1] C. Bechinger, R. Di Leonardo, H. Löwen, C. Reichhardt, G. Volpe, and G. Volpe, Active particles in complex and crowded environments, Reviews of Modern Physics 88, 045006 (2016).
[2] P. Romanczuk, M. Bär, W. Ebeling, B. Lindner, and L. Schimansky-Geier, Active brownian particles, The European Physical Journal Special Topics 202, 1 (2012).
[3] H. U. Boedeker, C. Beta, T. D. Frank, and E. Bodenschatz, Quantitative analysis of random ameoboid motion, EPL (Europhysics Letters) 90, 28005 (2010).
[4] A. Walther and A. H. Müller, Janus particles: synthesis, self-assembly, physical properties, and applications,
S. Palagi and P. Fischer, Bioinspired microrobots, Nature Reviews Materials 3, 113 (2018).

C. Tapia-Ignacio, L. L. Gutierrez-Martinez, and M. Sandova, Trapped active toy robots: theory and experiment, Journal of Statistical Mechanics: Theory and Experiment 2021, 053404 (2021).

A. Cavagna, L. Del Castello, I. Giardina, T. Grigera, A. Jelic, S. Melillo, T. Mora, L. Parisi, E. Silvestri, M. Viale, et al., Flocking and turning: a new model for self-organized collective motion, Journal of Statistical Physics 158, 601 (2015).

J. Jhawar, R. G. Morris, U. Amith-Kumar, M. Danny Raj, T. Rogers, H. Rajendran, and V. Guttal, Noise-induced schooling of fish, Nature Physics 16, 488 (2020).

V. Lobaskin and M. Romenskyy, Collective dynamics in systems of active brownian particles with dissipative interactions, Physical Review E 87, 052135 (2013).

H. Löwen, Inertial effects of self-propelled particles: From active brownian to active langevin motion, The Journal of chemical physics 152, 040901 (2020).

L. Schimansky-Geier, M. Mieth, H. Rossé, and H. Malchow, Structure formation by active brownian particles, Physics Letters A 207, 140 (1995).

L. Caprini and U. M. B. Marconi, Time-dependent properties of interacting active matter: Dynamical behavior of one-dimensional systems of self-propelled particles, Physical Review Research 2, 033518 (2020).

L. L. Bonilla, Active ornstein-uhlenbeck particles, Physical Review E 100, 022601 (2019).

L. Caprini and U. Marini Bettolo Marconi, Inertial self-propelled particles, The Journal of Chemical Physics 154, 024902 (2021).

L. Dabelow and R. Eichhorn, Irreversibility in active matter: General framework for active ornstein-uhlenbeck particles, Frontiers in Physics, 516 (2021).

G. E. Uhlenbeck and L. S. Ornstein, On the theory of the brownian motion, Physical review 36, 823 (1930).

C. Sandford and A. Y. Grosberg, Memory effects in active particles with exponentially correlated propulsion, Physical Review E 97, 012602 (2018).

F. J. Sevilla, R. F. Rodríguez, and J. R. Gomez-Solano, Generalized ornstein-uhlenbeck model for active motion, Physical Review E 100, 032123 (2019).

P. Singh and A. Kundu, Crossover behaviours exhibited by fluctuations and correlations in a chain of active particles, Journal of Physics A: Mathematical and Theoretical 54, 305001 (2021).

M. Bothe and G. Pruessner, Doi-peliti field theory of free active ornstein-uhlenbeck particles, Physical Review E 103, 062105 (2021).

L. Caprini, A. Puglisi, and A. Sarracino, Fluctuation–dissipation relations in active matter systems, Symmetry 13, 81 (2021).

M. Muhsin, M. Sahoo, and A. Saha, Orbital magnetism of an active particle in viscoelastic suspension, Physical Review E 104, 034613 (2021).

W.-D. Tian, Y. Gu, Y.-K. Guo, and K. Chen, Anomalous boundary deformation induced by enclosed active particles, Chinese Physics B 26, 100502 (2017).

M. E. Cates and J. Tailleur, Motility-induced phase separation, Annu. Rev. Condens. Matter Phys. 6, 219 (2015).

T. Speck, A. M. Menzel, J. Bialké, and H. Löwen, Dynamical mean-field theory and weakly non-linear analysis for the phase separation of active brownian particles, The Journal of chemical physics 142, 224109 (2015).

A. Patch, D. Yllanes, and M. C. Marchetti, Kinetics of motility-induced phase separation and swim pressure, Physical Review E 95, 012601 (2017).

R. Wittkowski, A. Tiribocchi, J. Stenhammar, R. J. Allen, D. Marenduzzo, and M. E. Cates, Scalar ϕ4 field theory for active-particle phase separation, Nature communications 5, 1 (2014).

M. Leoni, M. Paoluzzi, S. Eldeen, A. Estrada, L. Nguyen, M. Alexandrescu, K. Sherb, and W. W. Ahmed, Surfing and crawling macroscopic active particles under strong confinement: Inertial dynamics, Physical Review Research 2, 043209 (2020).

M. Gazzola, M. Argentina, and L. Mahadevan, Scaling macroscopic aquatic locomotion, Nature Physics 10, 758 (2014).

M. Saadat, F. E. Fish, A. Domel, V. Di Santo, G. Lauder, and H. Haj-Hariri, On the rules for aquatic locomotion, Physical Review Fluids 2, 083102 (2017).

H. Shahsav, A. Aghakhani, H. Zeng, Y. Guo, Z. S. Davidson, A. Primagi, and M. Sitti, Bioinspired underwater locomotion of light-driven liquid crystal gels, Proceedings of the National Academy of Sciences 117, 5125 (2020).

S. P. Sane, The aerodynamics of insect flight, Journal of experimental biology 206, 4191 (2003).

C. Schoz, S. Jahanshahi, A. Ldev, and H. Löwen, Inertial delay of self-propelled particles, Nature communications 9, 1 (2018).

J. Su, H. Jiang, and Z. Hou, Inertia-induced nucleation-like motility-induced phase separation, New Journal of Physics 23, 013005 (2021).

G. Szamel, Self-propelled particle in an external potential: Existence of an effective temperature, Physical Review E 90, 02111 (2014).

J. Tóthová, G. Vasziová, L. Gld, and V. Lisý, Langevin theory of anomalous brownian motion made simple, European journal of physics 32, 645 (2011).

A. Noushad, S. Shajahan, and M. Sahoo, Velocity auto correlation function of a confined brownian particle, The European Physical Journal B 94, 1 (2021).

A. M. Jayannavar and M. Sahoo, Charged particle in a magnetic field: Jarzynsky equality, Physical Review E 75, 032102 (2007).

M. Muhsin and M. Sahoo, Inertial active ornstein–uhlenbeck particle in the presence of magnetic field, arXiv preprint arXiv:2206.14715 (2022).

G. P. Nguyen, R. Wittmann, and H. Löwen, Active ornstein–uhlenbeck model for self-propelled particles with inertia, Journal of Physics: Condensed Matter 34, 035101 (2021).