Interaction-induced magnetoresistance: From the diffusive to the ballistic regime

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(March 22, 2022)

We study interaction-induced quantum correction $\delta \sigma_{\alpha\beta}$ to the conductivity tensor of electrons in two dimensions for arbitrary $T\tau$, where $T$ is the temperature and $\tau$ the transport mean free time. A general formula is derived, expressing $\delta \sigma_{\alpha\beta}$ in terms of classical propagators (“ballistic diffusons”). The formalism is used to calculate the interaction contribution to the magnetoresistance in a classically strong transverse field and smooth disorder in the whole range of temperatures from the diffusive ($T\tau \ll 1$) to the ballistic ($T\tau \gtrsim 1$) regime.

PACS numbers: 72.10.-d, 73.23.Ad, 71.10.-w, 73.43.Qt

The magnetoresistance (MR) in a transverse field $B$ is one of the most frequently studied characteristics of the two-dimensional (2D) electron gas [1,2]. Within the Drude-Boltzmann theory, the longitudinal resistivity of an isotropic degenerate system is $B$-independent, $\rho_{xx}(B) = \rho_0 = (e^2/nv_F^2\tau)^{-1}$, where $n$ is the density of states per spin direction, $v_F$ the Fermi velocity, and $\tau$ the transport scattering time. There are several distinct sources of a non-trivial MR, which reflect the rich physics of 2D systems. First, quasiclassical memory effects may lead to a MR [3], which shows no $T$-dependence at low temperatures. Second, weak localization [1] induces a negative MR restricted to the range of very weak magnetic fields. Finally, another quantum correction to MR is generated by the electron–electron interaction. This effect is the subject of the present paper.

It was discovered by Al'tshuler and Aronov [1] that the Coulomb interaction enhanced by the diffusive motion of electrons gives rise to a quantum correction to conductivity, which in 2D the form (we set $k_B = \hbar = 1$)

$$\delta \sigma_{xx} \simeq (e^2/2\pi^2) \ln T\tau, \quad T\tau \ll 1. \quad (1)$$

It is assumed here for simplicity that $\kappa \ll k_F$, where $\kappa = 4\pi e^2\nu$ is the inverse screening length. The condition $T\tau \ll 1$ under which Eq. (1) is derived [1] implies that electrons move diffusively on the time scale $1/T$ and is termed the “diffusive regime”. Subsequent works [4] showed that Eq. (1) remains valid in a strong magnetic field, leading (in combination with $\delta \sigma_{xy} = 0$) to a parabolic interaction–induced quantum MR,

$$\frac{\delta \rho_{xx}(B)}{\rho_0} \simeq \left(\frac{\omega_c}{v_F}\right)^2 \frac{1}{\pi k_F l} \ln T\tau, \quad T\tau \ll 1, \quad (2)$$

where $\omega_c = eB/mc$ is the cyclotron frequency and $l = v_F\tau$ the transport mean free path. Indeed, a $T$-dependent negative MR was observed in experiments [5] and attributed to the interaction effect. However, the experiments [5] cannot be directly compared with the theory [1,4] since they were performed at higher temperatures, $T\tau \gtrsim 1$. (In high-mobility GaAs heterostructures conventionally used in MR experiments, $1/\tau$ is typically $\sim 100$ mK and becomes even smaller with improving quality of samples.) There is thus a clear need for a theory of the MR in the ballistic regime, $T \gtrsim 1/\tau$.

In fact, the effect of interaction on the conductivity at $T \gtrsim 1/\tau$ has attracted a great deal of interest in a context of low-density 2D systems showing a seemingly metallic behavior, $d\rho/dT > 0$ [6]. Recently, Zala, Narozhny, and Aleiner [7] developed a systematic theory of the interaction corrections valid for arbitrary $T\tau$. In the ballistic range of temperatures, this theory (improving earlier calculation of temperature–dependent screening [8]), predicts a linear-in-$T$ correction to conductivity $\sigma_{xx}$ and a $1/T$ correction to the Hall coefficient $\rho_{xy}/B$ at $B \to 0$, and describes the MR in a parallel field.

The consideration of [7] is restricted, however, to classically weak transverse fields, $\omega_c T \ll 1$, and to the white-noise disorder. The latter assumption is believed to be justified for Si-based and some p-GaAs structures, and the results of [7] have been by and large confirmed by most recent experiments [9] on such systems. On the other hand, the random potential in n-GaAs heterostructures is, as a rule, due to remote donors and has a long–range character. Thus, the impurity scattering is predominantly of a small–angle nature and is characterized by two relaxation times, the transport time $\tau$ and the single-particle (quantum) time $\tau_s$, with $\tau \gg \tau_s$.

We present here a general theory of the interaction–induced corrections to the conductivity of 2D electrons valid for arbitrary temperatures, transverse magnetic fields and disorder range. We further apply it to the problem of magnetotransport in a smooth disorder at $\omega_c T \gg 1$ [10]. In the ballistic limit, $T\tau \gg 1$ (where the character of disorder is crucially important), we show that while the correction to $\rho_{xx}$ is exponentially suppressed for $\omega_c \ll T$, a MR arises at stronger $B$ where it scales as $B^2T^{-1/2}$.

To find $\delta \sigma_{\alpha\beta}$, we make use of the “ballistic” generalization of the Matsubara diffusion diagram technique of Ref. [1]. We consider the exchange contribution first and...
will discuss the Hartree term later on. The relevant diagrams are shown in Fig. 1. The shaded blocks in Fig. 1 denote the impurity–line ladders, which we term “ballistic diffusons”. The temperature range of main interest in the present paper is restricted by $T \tau_s \ll 1$, since at higher $T$ the MR will be small in the whole range of the quasiclassical transport $\omega, \tau_s \ll 1$ (see below). In this case the ladders are dominated by contributions with many $\gg s$ impurity lines. Our general formula below is, however, valid irrespective of the value of $T \tau_s$.

![Diagrams](image)

Fig. 1. Diagrams for the interaction correction to $\sigma_{\alpha \beta}$. The wavy (dashed) lines denote the interaction (impurity scattering), the shaded blocks are impurity ladders, and the $+/-$ symbols denote the signs of the Matsubara frequencies. The diagrams obtained by a flip and/or by an exchange $+ \leftrightarrow -$ should also be included.

After the Wigner transformation is performed, the ballistic diffusion takes the form $D(\omega; \mathbf{r}, \mathbf{n}, \mathbf{r}', \mathbf{n}')$ and describes the quasiclassical propagation of an electron in the phase space $[11]$ ($\mathbf{n}$ is the unit vector characterizing the direction of velocity on the Fermi surface). In contrast to the diffusive regime, where $D$ has a universal and simple structure $D(\omega, \mathbf{q}) = 1/(Dq^2 - i\omega)$ determined by the diffusion constant $D$ only, its form in the ballistic regime is much more complicated. We are able, however, to get a general expression for $\delta\sigma_{\alpha \beta}$ in terms of the ballistic propagator $D(\omega, \mathbf{q}; \mathbf{n}, \mathbf{n}')$.

The results reads

$$\delta\sigma_{\alpha \beta} = -2e^2v_F^2\nu \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\partial}{\partial \omega} \left\{ \omega \coth \frac{\omega}{2T} \right\}$$

$$\times \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \text{Im} \left[ U(\omega, \mathbf{q}) B_{\alpha \beta}(\omega, \mathbf{q}) \right],$$

where $U(\omega, \mathbf{q})$ is the interaction potential equal to a constant $V_0$ for point-like interaction and to

$$U(\omega, \mathbf{q}) = \frac{1}{2\nu} \frac{\kappa}{q + [1 + i\omega D(\omega, q)]}$$

for screened Coulomb interaction. For small-angle impurity scattering the tensor $B_{\alpha \beta}(\omega, \mathbf{q})$ in (3) is given by

$$B_{\alpha \beta}(\omega, \mathbf{q}) = \frac{T_{\alpha \beta}}{2} \langle DD \rangle + T_{\alpha \gamma} \frac{\delta_{\beta \delta}}{2} \langle D \rangle - \langle n_\gamma Dn_\delta \rangle \right) T_{\delta \beta}$$

$$- 2T_{\alpha \gamma} \langle n_\gamma Dn_\beta D \rangle - \langle Dn_{\alpha} Dn_{\beta} D \rangle,$$

where $T_{\alpha \beta} = \langle n_{\alpha} Dn_{\beta} \rangle |_{q=0, \omega=0} = \sigma_{\alpha \beta} e^2 v_F^2 \nu$. The angular brackets $\langle \ldots \rangle$ in (4) and (5) denote averaging over velocity directions, e.g. $\langle n_{\gamma} Dn_{\beta} \rangle = (2\pi)^{-2} \int d\phi_1 d\phi_2 \cos \phi_1 D(\omega, \mathbf{q}; \phi_1, \phi_2) \cos \phi_2$, where $\phi$ is the polar angle of $\mathbf{n}$. The first term in (5) originates from the diagrams $a, b, c$ in Fig. 1 (forming together the Hikami box), the second term from $a, f, g \ [12]$, the third term from $h$, and the last one from $d$ and $e$.

In the more general situation, when the scattering is at least partly of the large–angle character, the first term in (5) acquires a slightly more complicated form,

$$\pi\nu T_{\alpha \alpha'} \langle [DWD] S_{\alpha' \beta'} - 2 \langle Dn_{\alpha'} Wn_{\beta'} D \rangle \rangle T_{\beta' \beta},$$

where $W(\mathbf{n}, \mathbf{n}')$ is the scattering cross-section and $S_{xx} = S_{yy} = 1$, $S_{xy} = -S_{yx} = \omega_c \tau_s$. In particular, for the case of purely white-noise disorder (when $\tau = \tau_s$ and $W(\mathbf{n}, \mathbf{n}') = 1/2\pi \nu \tau$) this yields $T_{\alpha \beta} \langle [DWD] \rangle - \tau^{-1} T_{\alpha \alpha'} \langle Dn_{\alpha'} \rangle \langle n_{\beta} D \rangle T_{\beta' \beta}$. At $B = 0$ we then recover (using the explicit form of the ballistic propagator for this case) the result for $\delta\sigma$ obtained in a different way in [7]. Needless to say, in the diffusive limit, we reproduce (for arbitrary $B$ and disorder range) the logarithmic correction (1), (2) determined by the diagrams $a-c$.

Before turning to the analysis of the results for the strong-$B$ regime, we consider briefly the $B = 0$ case assuming the ballistic temperature range $T \tau \gg 1$. The structure of Eqs. (3), (5), (6) implies that the interaction correction is governed by returns of a particle to the original point in a time $t \lesssim T^{-1} \ll \tau$. Such a quick return may be induced by a single back–scattering process, yielding the contribution $\delta\sigma_{xx} \sim e^2 v_F^2 W (2k_F) T \tau$. For the case of white-noise disorder this reduces to $\delta\sigma_{xx} \sim e^2 T \tau$, in agreement with [8,7]. However, in a smooth disorder with a correlation length $d \gg k_F^{-1}$ this contribution is suppressed by the factor $2\pi \nu W (2k_F) \sim e^{-k_F d}$. The probability to return after many small-angle scattering events is also exponentially suppressed for $t \ll \tau$, yielding a contribution $\delta\sigma_{xx} \sim \exp(-\text{const}(T \tau)^{1/2})$. Thus, the interaction correction in the ballistic regime is exponentially small at $B = 0$ for the case of smooth disorder. Moreover, the same argument applies to the case of a non-zero $B$, as long as $\omega_c \ll T$.

The situation changes qualitatively in a strong magnetic field, $\omega_c \tau \gg 1$ and $\omega_c \gg T$. The particle experiences then within the time $t \sim T^{-1}$ multiple cyclotron returns to the region close to the starting point. The corresponding ballistic propagator satisfies the equation

$$\left[ -i\omega + iv_F q \cos \phi + \omega_c \frac{\partial}{\partial \phi} - \frac{1}{\tau} \frac{\partial^2}{\partial \phi^2} \right] D(\omega, q; \phi, \phi') = 2\pi \delta(\phi - \phi').$$

The approximate solution of (7) at $\omega_c \tau \gg 1$ has the form
where $\chi(\phi) = 1 - iqR_c \cos \phi/\omega_c \tau$ and $D \simeq R_c^2/2\tau$ in strong $B$. Since characteristic frequencies in (3) are $\omega \sim T \ll \omega_c$, it is sufficient to keep only the first term in square brackets in (8) to obtain the leading contribution. Then $\langle D \rangle$ in (4) is given by

$$\langle D \rangle = J_0^2(qR_c)/(Dq^2 - i\omega).$$

(9)

Note that Eqs. (9), (10) differ from those obtained in the diffusive regime by the factor $J_0^2(qR_c)$ only. This is related to the fact that the motion of the guiding center is diffusive even on the ballistic time scale $t \ll \tau$ (provided $t \gg \omega_c^{-1}$), while the additional factor corresponds to the averaging over the cyclotron orbit.

Substituting (10) in (3), and rescaling the momentum $q \to qR_c \equiv z$, we see that all the $B$-dependence drops out from $\delta\sigma_{xx}$, and the exchange contribution in the case of point-like interaction reads

$$\delta\sigma_{xx} = -\langle D^2 \rangle / \omega_\tau G_0(T\tau),$$

(11)

$$G_0(x) = \pi^2 x^2 \int_0^\infty du \frac{\exp(-u)}{u^3 \sin^2(\pi x/u)} [I_0(u)(1 - u) + uI_1(u)].$$

The Hartree term in this case is of the opposite sign and twice larger due to the spin summation (we neglect the Zeeman splitting). Since the relative correction to the Hall conductivity turns out to be smaller by the factor $\sim (\omega_\tau)^{-2}$ compared to (11), $\delta\sigma_{xy}/\sigma_{yy} \ll \delta\sigma_{xx}/\sigma_{xx}$, the MR is given by $\delta\rho_{xx}/\rho_0 = (\omega_\tau)^2 \delta\sigma_{xx}/\sigma_0$. The MR is thus quadratic in $\omega_c$, with the temperature dependence determined by the function $G_0(T\tau)$, which is shown in Fig. 2a. It has the asymptotics $G_0(x) \simeq -\ln x + \text{const}$ for $x \ll 1$ (diffusive regime) and $G_0(x) \simeq c_0 x^{-1/2}$ with $c_0 = 3\zeta(3/2)/16\sqrt{\pi} \simeq 0.276$ at $x \gg 1$ (ballistic regime). Let us note that the crossover between the two limits takes place at numerically small values $T\tau \sim 0.1$.

For the case of the Coulomb interaction the result turns out to be qualitatively similar. Substituting (4), (9), and (10) in (3), we get the exchange (Fock) contribution

$$\delta\rho_{xx}^{F}(B) = -\frac{(\omega_\tau)^2}{\pi^2 k_F} G_F(T\tau),$$

(12)

$$G_F(x) = 32\pi^2 x^2 \int_0^\infty dz z^3 J_0^2(z) G_{1,3,2}(z),$$

$$G_{jkl}(z) = \sum_{n=1}^{\infty} \frac{n(12\pi x n [1 - J_0^2(z)] + 3 - 3 J_0^2(z) z^2)}{(4\pi x n + z^2)^k (4\pi x n [1 - J_0^2(z)] + z^2)^l},$$

with $G_F(x < 1) \simeq -\ln x + \text{const}$ and $G_F(x > 1) \simeq (c_0/2) x^{-1/2}$, see Fig. 2b.

![FIG. 2. Functions $G_0(T\tau)$ (a) and $G_F(T\tau)$ (b) determining the $T$-dependence of the exchange term for point-like, Eq. (11), and Coulomb, Eq. (12), interaction, respectively.](image-url)

We turn now to the Hartree term, assuming first $\kappa \ll k_F$. The expression for its triplet part is analogous to (3) with the replacement of $U(\omega, q)$ by $-\frac{1}{2}U(0, 2k_F \sin(\phi - \phi'))/2$, where $\phi$ and $\phi'$ are starting and final angles of the electron velocity. As to the singlet part, it is renormalized by mixing with the exchange term, yielding

$$U(\omega, q) \to \frac{U(0, 2k_F \sin \frac{\phi - \phi'}{x}) - U(0, 2k_F \sin \frac{\phi - \phi'}{x})}{2[1 + i\omega D(\omega, q)]^2}.$$ 

(Note that the zero angular harmonic governing the diffusive limit [1] is completely suppressed in the singlet part.) After the angle integration, $J_0^2(z)$ in (10) is replaced by $-\frac{1}{2}U(0, 2k_F \sin(\phi - \phi'))(y + 2\sin \phi)$ for the triplet, and by $\mathcal{J}(y, z) = -\frac{1}{2}U(0, 2k_F \sin(\phi - \phi'))(y + 2\sin \phi)$ for the singlet term ($y = \kappa/k_F$). This yields for the total Hartree contribution

$$\delta\rho_{xx}^{H}(B) = \frac{(\omega_\tau)^2}{\pi^2 k_F} [G_{H1}(T\tau, y) + 3G_{H2}(T\tau, y)]$$

(13)

$$\simeq \frac{(\omega_\tau)^2}{\pi^2 k_F} \begin{cases} \frac{y \ln y}{2} \ln(\mathcal{T} + y) & T\tau \ll 1, \\ \frac{y \ln y}{2} \mathcal{T}^{1/2}, & 1 \ll T\tau \ll y^{-2}, \\ \pi c_0(\mathcal{T})^{-1/2}, & \mathcal{T} \gg y^{-2}, \end{cases}$$

$$G_{H1}(x, y) = 32\pi^2 x^2 \int_0^\infty dz z^3 \mathcal{J}(y, z) G_{2,3,2}(z),$$

$$G_{H2}(x, y) = \frac{\pi^2 x^2}{4} \int_0^\infty du \frac{\exp(-2u \sin^2 \phi)}{u^3 \sin^2(\pi x/u)} \int_0^\pi d\phi \frac{y}{y + 2\sin \phi}.$$
\[-2\delta \rho_{xx}^{F}, \text{leading to a positive MR with the same } (T\tau)^{-1/2}\text{ temperature-dependence, see Fig. 3a,c.}\]

If \(\kappa/k_F\) is not small, the exchange contribution (12) remains unchanged, while the Hartree term is subject to strong Fermi-liquid renormalization [1,7] and is determined by angular harmonics \(F_{m}^{\sigma}\) of the Fermi-liquid interaction \(F^{\sigma}\). The formula for arbitrary \(T\tau\) becomes then rather cumbersome [13]; here we restrict ourselves to a discussion of limiting cases. In the diffusive regime, \(T \ll 1/\tau\), we reproduce the known result [1,7]

\[G_H(T\tau) = 3[1 - \ln(1 + F_0^\sigma)/F_0^\sigma]\ln T\tau.\]

In the ballistic limit, \(T \gg 1/\tau\), we find for the Hartree contribution

\[G_H(T\tau) = \frac{c_0}{2} \left[\sum_{m \neq 0} \frac{F_m^\sigma}{1 + F_m^\sigma} + 3 \sum_{m} \frac{F_m^\sigma}{1 + F_m^\sigma}\right] \frac{1}{\sqrt{T\tau}}.\]

Finally, within a frequently used (though parametrically uncontrollable) approximation neglecting all \(F_m\) with \(m \neq 0\), the Hartree term takes the form of Eq. (12) with an additional overall factor of 3 and with \(J_g^f(z)\) multiplied by \(F_0^\sigma/(1 + F_0^\sigma)\) everywhere; the result is shown in Fig. 3b for several values of \(F_0^\sigma\).

\[
\begin{align*}
\text{FIG. 3. Hartree contribution, } G_H(T\tau), \text{ for (a) weak interaction, } \kappa/k_F = 0.1, 0.2, 0.3, 0.5, \text{ and (b) strong interaction, } F_0 = -0.3, -0.4, -0.5 \text{ (from bottom to top); (c) schematic plot of MR } \delta \rho_{xx}(B) \text{ in different temperature regimes: 1) } T_1 \ll \tau^{-1}, \text{ 2) } \tau^{-1} \ll T_2 \ll T_3, \text{ 3) } T_3 \gg T_3.\end{align*}
\]

In summary, we have derived a general formula for the interaction-induced quantum correction \(\delta \sigma_{\alpha\beta}\) to the conductivity tensor of 2D electrons valid for arbitrary temperature, magnetic field and disorder range. It expresses \(\delta \sigma_{\alpha\beta}\) in terms of classical propagators in random potential (“ballistic diffusons”). Applying this formalism, we have calculated the interaction contribution to the MR in strong \(B\) in a system with smooth disorder. We have shown that the parabolic MR found earlier in the diffusive limit \(T\tau \ll 1\) persists in the ballistic regime \(T\tau \gtrsim 1\), where it scales as \(T^{-1/2}\). At sufficiently high \(T\) the sign of the MR is changed (see Fig. 3c).

Before closing the paper, we list a few further applications of our formalism [13]. First, we can consider the model of mixed disorder, in which \(\tau_{s}\) is determined by a smooth random potential while \(\tau_{e}\) is governed by rare short-range scatterers. This model is relevant to ultrahigh mobility heterostructures as well as to random antidot arrays [3]. Second, the interaction correction to the MR in a periodically modulated system (lateral superlattice) can be studied. Finally, our results can be generalized to frequency-dependent MR.

After completion of this work, we learnt about a recent experiment [14] supporting our theoretical predictions. We thank A.K. Savchenko for useful discussions and for informing us about results of [14] prior to publication. This work was supported by the Schwerpunktprogramm “Quanten-Hall-Systeme” and the SFB195 der Deutschen Forschungsgemeinschaft, and by the RFBR.