Banana and pizza-slice-shaped mesogens give a new constrained ferromagnet universality class

Xiuqi Ma and Elsen Tjhung
Department of Applied Mathematics and Theoretical Physics,
Centre for Mathematical Sciences, University of Cambridge,
Wilberforce Road, Cambridge CB3 0WA, United Kingdom.

It has been known that at high density, the local orientation of banana-shaped molecules shows a spontaneously bent state, giving rise to interesting liquid-crystalline phases such as splay-bend and twist-bend. This spontaneous bend can be modelled theoretically by allowing the bend elastic constant in the Frank elastic energy to become negative. Here we extend this idea to polar banana and pizza-slice-shaped molecules which can also splay spontaneously. By allowing both splay and bend elastic constants to be negative we discovered two additional new liquid crystalline phases. In particular, using renormalization group technique, we showed that the phase transition belongs to a new constrained ferromagnet universality class.

Liquid crystals are usually made up of rod-shaped and head-tail symmetric molecules. At high enough density (or low enough temperature), the molecules tend to align in the same direction; this is the nematic phase \( \mathbb{N} \). In nematics, we denote the local average orientation of the molecules with a headless unit vector, called the director field \( \hat{n}(r) \), such that in some mesoscopic volume at \( r \) the molecules tend to align parallel or anti-parallel to \( \hat{n}(r) \).

Now what happens if the molecules are not straight? For instance, one can imagine banana or pizza-slice-shaped molecules as shown in Fig. 1(a). What kind of liquid crystalline phases do they form? The case of apolar banana-shaped molecules has been widely studied in literature [2–8]. In particular, at high density, banana-shaped molecules can spontaneously bend locally [2] (see literature [2–8]). In the figure, the molecules can either bend upwards or downwards with equal probability. This is called spontaneous symmetry breaking. At large scale, and in two-dimension, one will also get a bend modulation in the direction parallel to the director field, which is the \( x \)-direction in the figure. The resulting phase is called the splay-bend parallel (or \( SB_\parallel \)) phase [9], shown in Fig. 2(a). In this phase, mesoscopically the system is nematic but the director field \( \hat{n}(r) \) can vary slowly in space. This new \( SB_\parallel \) phase has been verified recently in Monte Carlo simulations [5] but not yet seen experimentally.

In three-dimension, suspensions of banana-shaped molecules can spontaneously bend to create a macroscopic helical pattern, called the twist-bend (or \( TB \)) phase [2]. Physically, the system undergoes a chiral spontaneous symmetry breaking. Although first predicted theoretically in [2], the \( TB \) phase was only discovered experimentally [3–5] and numerically [6] recently. In fact in three-dimension, the \( TB \) phase is competing with the \( SB_\parallel \) phase, depending on the ratio of the twist to the splay elastic constant.

In this paper, we will consider polar pizza-slice-shaped molecules which can spontaneously splay (see Fig. 1(b)) and polar banana molecules which can spontaneously bend and align (see Fig. 2(b)). In particular, we discovered two new additional splay-bend phases, which we call \( SB_\perp \) and \( SB_\infty \). (Other molecular shapes such as flag-shaped have also been considered in [12] and they can also give rise to modulated phases such as cubic and hexagonal phases [11], which are different from ours.)

To model spontaneous bend, one can allow the bend elastic constant \( K_3 \) in the Frank elastic energy \( H[\hat{n}(r)] \) to be negative. To prevent \( H[\hat{n}(r)] \) from going to \( -\infty \), one also has to add higher order terms in gradient to stabilise

\[
\begin{align*}
\text{Apply external field } & E \text{ (flexoelectric)} \\
E & \text{ (flexoelectric)} \\
\end{align*}
\]

Figure 1. (a) Banana-shaped molecules can spontaneously bend at high enough density (bend elastic constant \( K_3 \) becomes negative). (b) Similarly, polar pizza-slice-shaped molecules can spontaneously splay when we decrease the temperature (splay elastic constant \( K_1 \) becomes negative). (c) In flexoelectric liquid crystals, splay is induced by an external field not by spontaneous symmetry breaking. (d) The molecules form columnar stack in the \( z \)-direction and hence the director field can be assumed to be two-dimensional while the spatial dimension can be of dimension \( d = 2 \) or \( 3 \). (Red arrows indicate polarity of the molecules.)
Finally, we will consider polar banana-shaped molecules. In each molecule, we add a tiny electric polarization in the direction perpendicular to the longest molecular axis (red arrows in Fig. 2(b)). Obviously polar banana molecules are also flexoelectric [17] but we do not consider the effects of external field or boundaries here. Instead, we will only consider genuine phase transition from the uniformly nematic phase (where \( \mathbf{n}(r) \) is constant) into a completely new phase, \( SB_{\infty} \), as shown in Fig. 2(b). The director field \( \mathbf{n}(r) \) corresponding to the \( SB_{\infty} \) phase is shown in bottom left of Fig. 3. The polarization field, on the other hand, points in the direction of bend: \( \propto \mathbf{n} \times \nabla \times \mathbf{n} \). Thus the \( SB_{\infty} \) phase acquires a macroscopic polarization in some randomly chosen direction, which is upwards in the figure. However, the phase transition from the uniform nematic (zero net polarization) to the \( SB_{\infty} \) phase (macroscopic polarization) is not simply described by para-ferromagnetic transition due to an additional constraint in the system, which we shall see later.

In this paper, we shall consider an effective field theory for the director field \( \mathbf{n}(r) \) (since the polarization field can be determined from \( \mathbf{n}(r) \), if needed). We shall also restrict to a two-dimensional director field and spatial dimension \( d = 2 \) or 3. In the case of \( d = 3 \), physically, the molecules form a columnar stack in the \( z \)-direction (see Fig. 1(d)) [14].

We will now derive analytically the mean field phase diagram as a function of splay (\( K_1 \)) and bend (\( K_3 \)) elastic constants, as shown in Fig. 3 (Note that \( K_1 \) and \( K_3 \) should be interpreted as effective elastic constants which account for steric repulsions and ferroelectric interactions.) We start from the Frank elastic energy \( H[\mathbf{n}(r)] = \int d^3r f \), where the energy density \( f \) is given by the gradient expansion in \( \mathbf{n}(r) \) [11, 2]:

\[
f = \frac{K_1}{2} (\nabla \cdot \mathbf{n})^2 + \frac{K_3}{2} [\mathbf{n} \times \nabla \times \mathbf{n}]^2 + \frac{C}{2} (\nabla^2 \mathbf{n})^2, \tag{1}
\]

and \( |\mathbf{n}(r)| = 1 \). We require \( C > 0 \) for stability (see Section I [21]), but \( K_1 \) and \( K_3 \) can be negative. In the case of \( K_1 < 0 \) and/or \( K_3 < 0 \), the director field locally acquires a spontaneous splay and/or bend (like pizza-slices or bananas). (We assume there is no twist for simplicity.)

Since \( \mathbf{n}(r) \) is a two-dimensional vector, we can write:

\[
\mathbf{n}(r) = (\cos \theta(r), \sin \theta(r))^T, \quad \text{where } \theta(r) \text{ is the angle between the director field and the } x\text{-axis.}
\]

The energy density \( f \) then becomes:

\[
f = \frac{K_1 + K_3}{4} |\nabla \theta|^2 + \frac{C}{2} (\nabla^2 \theta)^2 + \frac{C}{2} |\nabla^2 \theta|^2 \\
- \frac{K_1 - K_3}{4} \left[ \left( \frac{\partial \theta}{\partial x} \right)^2 - \left( \frac{\partial \theta}{\partial y} \right)^2 \right] \cos(2\theta) \\
- \frac{K_1 - K_3}{2} \left( \frac{\partial \theta}{\partial x} \right) \left( \frac{\partial \theta}{\partial y} \right) \sin(2\theta). \tag{2}
\]

The last two terms in the above equation can also be...
Written as:

\[-\frac{K_1 - K_3}{2} (\nabla \theta)^T \cdot \mathbf{Q} \cdot \nabla \theta,\]

(3)

where

\[\mathbf{Q} = \frac{1}{2} \left( \begin{array}{cc} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{array} \right).\]

(4)

Under some two-dimensional rotation \(\mathbf{R}\), \(\mathbf{Q}\) transforms as \(\mathbf{Q} \rightarrow \mathbf{R} \cdot \mathbf{Q} \cdot \mathbf{R}^T\) and \(\nabla \theta\) transforms as \(\nabla \theta \rightarrow \mathbf{R} \cdot \nabla \theta\).

Therefore (3), and consequently (2), is invariant under two-dimensional rotation as required from \(H[\mathbf{n}(\mathbf{r})]\).

From the mean field phase diagram in Fig. 3, we can identify four distinct phases: uniform nematic, \(SB_\perp\), \(SB_\parallel\), and \(SB_\infty\), separated by critical lines or second order phase transitions (red lines in the figure). The case of \(SB_\parallel\) has been reported before but not \(SB_\perp\) or \(SB_\infty\). In the first quadrant (i.e. \(K_1 > 0\) and \(K_3 > 0\)), we have the uniform phase where \(\theta(\mathbf{r})\) is constant everywhere in space and thus the director field \(\mathbf{n}(\mathbf{r})\) is pointing along some spontaneously-broken direction, which is the \(x\)-direction in the figure. This corresponds to the usual nematic phase formed by rod-shaped molecules.

For \(K_3 < 0\) and \(K_1 \gg |K_3|\), which is approximately the yellow triangular region in Fig. 3, we have the \(SB_\parallel\) phase. This corresponds to apolar banana-shaped molecules, shown in Fig. 1(a) and 2(a). In this phase, the director field \(\mathbf{n}(\mathbf{r})\) oscillates in the direction parallel to the global director \(\hat{\mathbf{n}} = \frac{1}{\lambda_0} \int \mathbf{n}(\mathbf{r}) \, dV\) (see bottom right inset in Fig. 3). In other words, from uniform to \(SB_\parallel\) phase, translational symmetry along the global director \(\hat{\mathbf{n}}\) is broken. In this paper, we choose the spontaneously broken direction to be \(\hat{\mathbf{n}} = \hat{x}\), and thus in the \(SB_\parallel\) phase, \(\mathbf{n}(\mathbf{r})\) oscillates along the \(x\)-axis. Mathematically, the mean field solution to the \(SB_\parallel\) phase can be approximated as \(\theta(\mathbf{r}) = \theta_0 \cos(k_0 x)\), where \(\theta_0\) and \(k_0\) depend on \(K_1\) and \(K_3\).

For \(K_1 < 0\) and \(K_3 \gg |K_1|\), or the blue triangular region in Fig. 3, we have the \(SB_\perp\) phase. This phase is formed by pizza-slice-shaped molecules, shown in Fig. 1(b). In this phase, \(\mathbf{n}(\mathbf{r})\) oscillates in the direction perpendicular to \(\hat{\mathbf{n}}\), which is along \(\hat{y}\) (see top left inset in Fig. 3). Mathematically, the mean field solution to \(SB_\parallel\) can be approximated as \(\theta(\mathbf{r}) = \theta_0 \cos(k_0 y)\).

Finally along the line \(K_1 = K_3 < 0\), or the green line in Fig. 3, we have the \(SB_\infty\) phase. In this phase, \(\mathbf{n}(\mathbf{r})\) tumbles along some spontaneously broken direction, which is \(\hat{z}\) in the figure. Mathematically the mean field solution to this phase is given exactly by \(\theta(\mathbf{r}) = ax + by\) for some constants \(a\) and \(b\) which depend on \(K_1\) and \(K_3\) (\(b = 0\) in Fig. 3). This corresponds roughly to polar banana molecules shown in Fig. 2(b).

First we shall look at the mean field transition from the uniform phase, where \(\theta(\mathbf{r}) = 0\), to the \(SB_\perp\) phase, where \(\theta(\mathbf{r}) = \theta_0 \cos(k_0 y)\). In other words, we fix \(K_3\) to be a positive constant and we decrease \(K_1\) slowly from a positive value to a negative value, crossing the critical line \(K_1 = 0\) (red line on the \(K_3\)-axis in Fig. 3). To characterize this transition, we substitute the solution \(\theta(\mathbf{r}) = \theta_0 \cos(k_0 y)\) to the Hamiltonian density (2).

We then average the Hamiltonian density over one wavelength: \(\bar{f} = \frac{2\pi}{k_0} \int_0^{2\pi/k_0} f \, dy\) and the result is:

\[\bar{f} = \frac{1}{4} K_1 k_0^2 \theta_0^2 + \frac{1}{16} (K_3 - K_1) k_0^2 \theta_0^4 + \frac{1}{4} C k_0^4 \theta_0^4 + O(k_0^6 \theta_0^4).\]

(5)

We then minimize the average Hamiltonian over \(\theta_0\) and \(k_0\): \(\partial f / \partial \theta_0 = \partial f / \partial k_0 = 0\), to obtain the solutions for \(\theta_0\) and \(k_0 = 2\pi/k_0\):

\[\theta_0 = \left\{ \begin{array}{ll} 0 & , K_1 > 0 \\ \sqrt{\frac{2 - K_1}{3 - K_1}} & , K_1 < 0 \end{array} \right.\]

(6)

\[k_0 = \left\{ \begin{array}{ll} 0 & , K_1 > 0 \\ 2\pi \sqrt{\frac{C}{K_1}} & , K_1 < 0 \end{array} \right.\]

(7)

Here \(\lambda_0\) is the wavelength of the splay modulation (see top left inset in Fig. 3). As we approach the critical line from below, the order parameter \(\theta_0\) vanishes as \(\theta_0 \sim |K_1|^{\beta}\) with mean field exponent \(\beta = 1/2\), whereas \(\lambda_0\) becomes longer and longer. Note that since we have neglected a higher order term \(\propto k_0^2 \theta_0^2\) in (5), we require \(k_0\) and \(\theta_0\) to be small. From (7), \(k_0\) and \(\theta_0\) are small as long as \(|K_1|\) is small and \(K_3 \gg |K_1|\). This gives the blue triangular region in the top of Fig. 3. Far from this region, the \(SB_\perp\) phase is no longer accurately represented by \(\theta(\mathbf{r}) = \theta_0 \cos(k_0 y)\) and one may expect higher order
The Hamiltonian (2) then becomes:

$$H[\mathbf{m}] = \int d^d r \left\{ \frac{K}{2} |\nabla \theta|^2 + \frac{C}{2} |\nabla^2 \theta|^2 + \frac{C}{2} |\nabla \theta|^4 \right\}. \tag{8}$$

We then define a vector field $\mathbf{m}(r) = \nabla \theta(r)$ and substituting this to (8), we obtain the Hamiltonian in terms of $\mathbf{m}$:

$$H[\mathbf{m}] = \int d^d r \left\{ \frac{K}{2} |\mathbf{m}|^2 + \frac{C}{2} |\mathbf{m}|^4 + \frac{C}{2} (\nabla \cdot \mathbf{m})^2 \right\}, \tag{9}$$

which looks like a ferromagnet except for the constraint $\nabla \times \mathbf{m} = 0$ [21] (see Table 1).

Note that in para-ferromagnetic transition (left column, Table 1), the dimension of the order parameter is $n$ whereas the spatial dimension is $d$. Using renormalization group, the critical exponent $\nu$ can be given in terms of $\epsilon$-expansion from spatial dimension $d = 4$. Many physical systems fall into this broad universality class. For instance, Ising model and liquid critical point correspond to $n = 1$ (up-down symmetry) whereas XY-model corresponds to $n = 2$ [14].

On the other hand for our constrained Hamiltonian, the dimension of our order parameter $\mathbf{m} = \nabla \theta$ is equal to $d$ (right column, Table 1). Moreover, we also get a different critical exponent $\nu$, indicating a different universality class from that of the unconstrained one. (Note that we started from a director field $\mathbf{n}$ is a two-dimensional vector, and we mapped it to $\mathbf{m}$ which is $d$-dimensional.)

At mean field level, the solution for $\mathbf{m}$ to the Hamiltonian (9) is one which minimizes $H[\mathbf{m}]$: $\delta H / \delta \mathbf{m} = 0$, from which we obtain, in $d = 2$:

$$\mathbf{m}(r) = \begin{cases} (0, 0)^T, & K > 0 \\ (a, b)^T, & K < 0 \end{cases} \tag{10}$$

where $a$ and $b$ satisfy $a^2 + b^2 = -K/2C$. Inverting $\mathbf{m}$, we obtain $\theta$:

$$\theta(r) = \begin{cases} \text{constant}, & K > 0 \\ ax + by, & K < 0 \end{cases} \tag{11}$$

as expected. We can also calculate the fluctuations from the mean field: $\delta \mathbf{m}(r) = \mathbf{m}(r) - \mathbf{m}_0$, and the correlation function $\langle \delta \mathbf{m}(r) \cdot \delta \mathbf{m}(r') \rangle$. From the correlation function, we can extract the correlation length, which is given by $\xi = \sqrt{-C/K}$ for the $SB_\infty$ phase (see Section II of [24]). Thus the correlation length diverges at critical point $K = 0$ with critical exponent $\nu = 1/2$. At mean field level, we cannot distinguish the critical exponent of our constrained Hamiltonian from the unconstrained ferromagnetic transition. Furthermore, mean field calculation also predicts that the critical point is at $K = 0$.

Renormalization group procedure allows us to get higher order correction to the mean field exponent $\nu = 1/2$ (detailed in Section III of [24]). We show that at linear order in $\epsilon = 4 - d$, the critical exponent for the uniform-$SB_\infty$ transition is indeed different from that of unconstrained ferromagnetic transition (see Table 1). It is interesting to investigate if there are other physical systems which belong to the same universality class as ours. Note that in $d = 2$, all these phases (including the uniform nematic) become quasi long-range order. In $d = 2$, the transition from the isotropic to quasi-nematic phase is of Kosterlitz-Thouless type [22, 23], however, it is not clear if the same is true for the quasi-nematic to any of the quasi-$SB$ phases.

In conclusion, using field-theoretic methods, we showed that pizza-slice-shaped molecules and polar bananas can give rise to new exotic liquid-crystalline phases. It might be interesting to generalize the above calculation to full three-dimension not just confined to two-dimensional layers, and hopefully, the existence of these phases can also be confirmed experimentally in the future. It might also be interesting to compare our results to particle-based simulations such as Monte Carlo or Molecular Dynamics.
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SUPPLEMENTARY MATERIAL

I. PROOF THAT THE HAMILTONIAN IS BOUNDED FROM BELOW

We shall show that the energy density
\[ f(n) = \frac{K_1}{2}(\nabla \cdot n)^2 + \frac{K_2}{2}(n \cdot (\nabla \times n))^2 + \frac{K_3}{2}|n \times (\nabla \times n)|^2 + \frac{C}{2}|\nabla^2 n|^2, \]
subject to \(|n|^2 = 1\), is bounded below. We shall only consider the case where \(K_1, K_2, K_3 < 0\) (and \(C > 0\) for stability), since otherwise we could simply drop the non-negative term. The idea is to “decouple” \(f\) into a sum of contributions from \(n_\alpha\), where \(\alpha = 1, 2, 3\) represents Cartesian coordinates. We will need:

**Proposition** (Cauchy-Schwarz inequality). Let \(v, w \in \mathbb{R}^n\). Then
\[ (v \cdot w)^2 \leq |v|^2|w|^2, \]
with equality iff one of \(v, w\) is a multiple of the other.

We bound the first term as follows:
\[
(\nabla \cdot n)^2 = (\partial_1 n_1 + \partial_2 n_2 + \partial_3 n_3)^2
\leq 3 \left( (\partial_1 n_1)^2 + (\partial_2 n_2)^2 + (\partial_3 n_3)^2 \right)
\leq 3 \left( |\nabla n_1|^2 + |\nabla n_2|^2 + |\nabla n_3|^2 \right),
\]
where when going to the second line we used Cauchy-Schwarz with \(v = (\partial_1 n_1, \partial_2 n_2, \partial_3 n_3)\) and \(w = (1, 1, 1)\). Next,
\[
(n \cdot (\nabla \times n))^2 \leq |\nabla \times n|^2
\leq 2 \left( (\partial_2 n_3 - \partial_3 n_2)^2 + (\partial_3 n_1 - \partial_1 n_3)^2 + (\partial_1 n_2 - \partial_2 n_1)^2 \right)
\leq 2 \left( |\nabla n_1|^2 + |\nabla n_2|^2 + |\nabla n_3|^2 \right),
\]
where again in the third line we used Cauchy-Schwarz. Note that we also have \(|n \times (\nabla \times n)|^2 \leq |\nabla \times n|^2\), so exactly the same bound holds for the third term in (12).

Putting all of these together, we have
\[ f \geq \sum_{\alpha=1}^{3} \left[ \left( \frac{3}{2}K_1 + K_2 + K_3 \right) |\nabla n_\alpha|^2 + \frac{1}{2}C(\nabla^2 n_\alpha)^2 \right]. \]

Observe the \(\geq\) sign due to the assumption that \(K_i < 0\) for all \(i\).

Now the trick is to add a constant term \(A|n|^4 = A\) to \(f\) for some constant \(A > 0\). This does not affect whether or not the free energy is bounded below. Using \(|n|^4 \geq n_1^4 + n_2^4 + n_3^4\) and (16), we have
\[ \tilde{f} = f + A \]
\[ = f + A|n|^4 \]
\[ \geq \sum_{\alpha=1}^{3} \left[ An_\alpha^4 + \left( \frac{3}{2}K_1 + K_2 + K_3 \right) |\nabla n_\alpha|^2 + \frac{1}{2}C(\nabla^2 n_\alpha)^2 \right]. \]

Thus we have reduced the problem to showing that the new free energy
\[ g(\phi) = A\phi^4 + \left( \frac{3}{2}K_1 + K_2 + K_3 \right) |\nabla \phi|^2 + \frac{1}{2}C(\nabla^2 \phi)^2 \]
is bounded below. After rescaling \(r\) and \(\phi\), this becomes
\[ g(\phi) = \phi^4 + 2B|\nabla \phi|^2 + (\nabla^2 \phi)^2 \]
for some constant $B$. Note that we started with the constraint $|\mathbf{n}|^2 = 1$, but at this stage we can drop this constraint and take $\phi \in \mathbb{R}$. Assuming that the boundary term is zero, we can integrate by parts and then complete the square:

\[
  g = \phi^4 - 2B\phi \nabla^2 \phi + (\nabla^2 \phi)^2
  = (\nabla^2 \phi - B\phi)^2 - B^2\phi^2 + \phi^4
  = (\nabla^2 \phi - B\phi)^2 + \left(\phi^2 - \frac{B^2}{2}\right)^2 - \frac{B^4}{4},
\]

which is now clearly bounded below.

II. STRUCTURE FACTOR AND CORRELATION LENGTH OF THE $SB_\infty$ PHASE

The mean field solution to the $SB_\infty$ phase is given by:

\[
  m_0 = \sqrt{-K/2C} \hat{x},
\]

where $K < 0$ and we have chosen the spontaneously broken direction to be the $x$-direction. Now let us consider some small fluctuation $\delta m(r)$ around the mean field solution $m_0$. Substituting $m(r) = m_0 + \delta m(r)$ to the Hamiltonian, Eq. (9) in the main text, we obtain:

\[
  H[\delta m] = \int d^d r \left\{ \frac{C}{2} (\nabla \cdot \delta m)^2 - K\delta m_x^2 \right\},
\]

subject to constraint $\nabla \times \delta m = 0$. In Fourier space,

\[
  \delta m(r) = \int \frac{d^d q}{(2\pi)^{d/2}} \delta m(q)e^{iqr},
\]

this Hamiltonian becomes:

\[
  H[\delta m(q)] = \frac{1}{2} \int d^d q \left( Cq_x^2 - 2K \delta m_x(q)\delta m_x(-q) + Cq_y^2 \delta m_y(q)\delta m_y(-q) \right.
  + Cq_z^2 \delta m_z(q)\delta m_z(-q).
\]

In particular, in spatial dimension $d = 2$, this becomes:

\[
  H[\delta m(q)] = \frac{1}{2} \int d^2 q \left( Cq_x^2 - 2K \delta m_x(q)\delta m_x(-q) + Cq_y^2 \delta m_y(q)\delta m_y(-q) \right.
  + Cq_z^2 \delta m_z(q)\delta m_z(-q).
\]

Next we apply the constraint $\delta m_y(q) = \frac{2\pi}{q_x} \delta m_x(q)$ to get:

\[
  H[\delta m(q)] = \frac{1}{2} \int d^2 q \left( Cq_x^2 + 2Cq_y^2 + C\frac{q_y^4}{q_x^2} - 2K \right)|\delta m_x(q)|^2.
\]

This Hamiltonian is positive definite since $K < 0$. Therefore the structure factor is:

\[
  S(q) = \left\langle |\delta m_x(q)|^2 \right\rangle = \frac{1}{Cq_x^2 + 2Cq_y^2 + C\frac{q_y^4}{q_x^2} - 2K}
  = \frac{1}{-K\xi^2q_x^2 + 2\xi^2q_y^2 + \xi^2\frac{q_y^4}{q_x^2} + 2}
  = \frac{1}{-K} F(\xi q)
\]

where $\xi = \sqrt{-C/K}$ is the correlation length and $F$ is some function independent of $C$ or $K$. Therefore the correlation length diverges as $K \to 0^-$ as a power law: $\xi \sim |K|^{-\nu}$, with critical exponent $\nu = 1/2$. 

\[
  \phi + \text{constant.
}\]
The renormalization group procedure is as follows: (1) First we coarse-grain over some volume $(ba)^d$, where $b > 1$. (2) Next we rescale the lattice positions $r \rightarrow r/b$ and the field itself $\phi \rightarrow \phi/z$. At the end of this procedure, the correlation length in the system is rescaled to $\xi \rightarrow \xi/b$.

### III. RENORMALIZATION GROUP ALONG THE DIAGONAL $K_1 = K_3$ LINE

At mean field level, the critical exponent for the correlation length is $\nu = \frac{1}{2}$. Renormalization group allows us to get a more accurate estimate of the critical exponent. Usually, the critical exponent is given in expansion of $\epsilon$:

$$\nu = \frac{1}{2} + \alpha_1 \epsilon + \alpha_2 \epsilon^2 + \ldots$$

(30)

where $\epsilon = 4 - d$ is the distance from the upper critical dimension and $\alpha_i$’s are some constants. For $d \geq 4$, mean field theory is exact and the critical exponent is exactly $\nu = \frac{1}{2}$. However for spatial dimension $d < 4$, we have correction of order $\epsilon$ and higher. In this section we will obtain the first order $\epsilon$ correction for the uniform to $SB_\infty$ transition along the diagonal line $K_1 = K_3$ in the phase diagram.

The order parameter describing the uniform to $SB_\infty$ phase transition is a vector field $m$ plus a constraint $\nabla \times m = 0$. It is rather complicated to enforce this constraint to higher spatial dimension and thus we choose to work with a scalar field $\theta$ where $m = \nabla \theta$. First we write the Hamiltonian for $\theta$, Eq. (8) in the main text, as follows:

$$H[\theta] = \int d^d r \left\{ \frac{K}{2} |\nabla \theta|^2 + \frac{C}{2} (\nabla^2 \theta)^2 + L |\nabla \theta|^4 \right\}$$

Gaussian $H_0$

(31)

where we have introduced a new constant $L$ for convenience. Later we can set $L = C/2$. The Hamiltonian above contains a Gaussian part, which we call $H_0[\theta]$, and a quartic part, which we call $H_1[\theta]$. In Fourier space,

$$\theta(r) = \int \frac{d^d q}{(2\pi)^{d/2}} \theta_\mathbf{q} e^{i\mathbf{q} \cdot \mathbf{r}}$$

(32)

the Hamiltonian can be written as:

$$H_0[\theta_\mathbf{q}] = \frac{1}{2} \int d^d q \left( K q^2 + C q^4 \right) |\theta_\mathbf{q}|^2$$

(33)

$$H_1[\theta_\mathbf{q}] = L \int \frac{d^d q_1 d^d q_2 d^d q_3 d^d q_4}{(2\pi)^d} \delta(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4) \delta_1(q_1, \theta_\mathbf{q}_1, \cdot \mathbf{q}_2 \theta_\mathbf{q}_2) \cdot \cdot \cdot \delta_4(q_3, \theta_\mathbf{q}_3) . \cdot \cdot \cdot \delta_4(q_4, \theta_\mathbf{q}_4)$$

(34)

The Gaussian part is exactly solvable, in particular, the correlation function is given by (in the Gaussian limit $L \rightarrow 0$):

$$\langle \theta_\mathbf{q} \theta_{\mathbf{q}'} \rangle_0 = \frac{\delta(q + q')}{K q^2 + C q^4}.$$  

(35)

The renormalization group (RG) consists of two steps. Suppose our field $\theta(r)$ is confined to a lattice with lattice spacing $a$. This defines an upper cut-off frequency: $\Lambda = 2\pi/a$. The first step of RG is the coarse-graining step: we average the field over some box size $ba$, where $b > 1$ (see Fig. 3). After coarse-graining, we end up with fewer lattice points. The second step of RG is to rescale space $r \rightarrow r/b$ and the field itself $\phi \rightarrow \phi/z$. At the end of this procedure, the correlation length $\xi$ is rescaled into $\xi/b$. We can then repeat this procedure many times. This RG procedure turns out to be useful when we are close to criticality, or 2nd order phase transition, where the correlation length $\xi$ is infinite.
First, in the coarse-graining step, we decompose the field $\theta_q$ into low frequency and high frequency modes:

$$\theta_q = \begin{cases} \tilde{\theta}_q & : 0 < |q| < \frac{\Lambda}{b} \\ \phi_q & : \frac{\Lambda}{b} \leq |q| < \Lambda \end{cases} \tag{36}$$

The partition function is ($\beta = 1$):

$$Z = \int \mathcal{D}\theta_q e^{-H_0[\theta_q]} e^{-H_1[\theta_q]} \tag{37}$$

Substituting $\theta_q = \tilde{\theta}_q + \phi_q$ and since we have: $H_0[\tilde{\theta}_q + \phi_q] = H_0[\tilde{\theta}_q] + H_0[\phi_q]$, the partition function becomes:

$$Z = \int \mathcal{D}\tilde{\theta}_q e^{-H_0[\tilde{\theta}_q]} Z_\phi \int \mathcal{D}\phi_q e^{-H_1[\phi_q]} Z_\phi = \int \mathcal{D}\tilde{\theta}_q e^{-H_0[\tilde{\theta}_q]} Z_\phi \left\langle e^{-H_1[\tilde{\theta}_q + \phi_q]} \right\rangle_\phi = Z_\phi \int \mathcal{D}\tilde{\theta}_q e^{-H_0[\tilde{\theta}_q]} - \langle H_1 \rangle_\phi + \frac{1}{2} \left( \langle H_1^2 \rangle_\phi - \langle H_1 \rangle_\phi^2 \right) + \ldots \tag{38}$$

where $\left\langle \cdot \right\rangle_\phi$ indicates averaging over high frequency modes and $Z_\phi = \int \mathcal{D}\phi_q e^{-H_0[\phi_q]}$ is a constant. Defining the coarse-grained Hamiltonian to be $Z = \int \mathcal{D}\tilde{\theta}_q e^{-\tilde{H}[\tilde{\theta}_q]}$, we get:

$$\tilde{H}[\tilde{\theta}_q] = H_0[\tilde{\theta}_q] + \left\langle H_1[\tilde{\theta}_q, \phi_q] \right\rangle_\phi - \frac{1}{2} \left( \langle H_1^2 \rangle_\phi - \langle H_1 \rangle_\phi^2 \right) + \ldots \tag{39}$$

We can ignore the constant term $\ln(Z_\phi)$ which does not depend on $\tilde{\theta}_q$. The first term in (39) is just the Gaussian part:

$$H_0[\tilde{\theta}_q] = \frac{1}{2} \int_0^{\Lambda/b} dq \left( K q^2 + C q^4 \right) \left| \tilde{\theta}_q \right|^2 \tag{40}$$

(note that the integration range is from 0 to $\Lambda/b$).

### A. $O(L)$ correction to $\tilde{H}[\tilde{\theta}_q]$

The second term in (39) is the order $O(L)$ correction:

$$\langle H_1 \rangle_\phi = \frac{L}{2} \int \frac{d^d q_1 \ldots d^d q_4}{(2\pi)^d} \delta(q_1 + \ldots + q_4)(q_1 \cdot q_2)(q_3 \cdot q_4) \left\langle \left( \tilde{\theta}_{q_1} + \phi_{q_1} \right) \cdot \left( \tilde{\theta}_{q_2} + \phi_{q_2} \right) \left( \tilde{\theta}_{q_3} + \phi_{q_3} \right) \cdot \left( \tilde{\theta}_{q_4} + \phi_{q_4} \right) \right\rangle_\phi \tag{41}$$

$$= L_1^{(1)} + L_2^{(1)} + L_3^{(1)} \tag{42}$$

Expanding the integrand in (41), we may get terms such as $\left\langle \phi \tilde{\theta} \tilde{\theta} \tilde{\theta} \right\rangle_\phi$, which is zero since $\langle \phi \rangle_\phi = 0$, or $\langle \phi \phi \phi \phi \rangle_\phi$, which is constant. The non-trivial terms are

$$L_1^{(1)} = L \int \frac{d^d q_1 d^d q_2 d^d q_3 d^d q_4}{(2\pi)^d} \delta(q_1 + q_2 + q_3 + q_4)(q_1 \cdot q_2)(q_3 \cdot q_4) \tilde{\theta}_{q_1} \tilde{\theta}_{q_2} \tilde{\theta}_{q_3} \tilde{\theta}_{q_4} \tag{43}$$

$$L_2^{(1)} = 2L \int \frac{d^d k_1 d^d k_2 d^d q_3 d^d q_4}{(2\pi)^d} \delta(k_1 + k_2 + q_3 + q_4)(k_1 \cdot k_2)(q_3 \cdot q_4) \tilde{\theta}_{k_1} \tilde{\theta}_{k_2} \tilde{\theta}_{q_3} \tilde{\theta}_{q_4} \tag{44}$$

$$L_3^{(1)} = 4L \int \frac{d^d k_1 d^d k_2 d^d q_2 d^d q_4}{(2\pi)^d} \delta(k_1 + k_3 + q_2 + q_4)(k_1 \cdot q_2)(k_3 \cdot q_4) \tilde{\theta}_{k_1} \tilde{\theta}_{k_2} \tilde{\theta}_{q_3} \tilde{\theta}_{q_4} \tag{45}$$

where integral over $q$ is from 0 to $\Lambda/b$ and over $k$ is from $\Lambda/b$ to $\Lambda$. 
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$\mathcal{L}_1^{(1)}$ & $\mathcal{L}_2^{(1)}$ & $\mathcal{L}_3^{(1)}$ & $\mathcal{L}_4^{(1)}$ & $\mathcal{L}_5^{(1)}$ & $\mathcal{L}_6^{(1)}$ \\
\hline
$\begin{aligned}
q_1\delta q_1 & \quad q_3\delta q_3 \\
q_2\delta q_2 & \quad q_4\delta q_4 \\
\end{aligned}$ & $\begin{aligned}
k_1\phi k_1 & \quad q_1\delta q_1 \\
k_2\phi k_2 & \quad q_3\delta q_3 \\
\end{aligned}$ & $\begin{aligned}
q_1\delta q_1 & \quad q_3\delta q_3 \\
q_2\delta q_2 & \quad q_4\delta q_4 \\
\end{aligned}$ & $\begin{aligned}
k_1\phi k_1 & \quad q_1\delta q_1 \\
k_2\phi k_2 & \quad q_4\delta q_4 \\
\end{aligned}$ & $\begin{aligned}
k_1\phi k_1 & \quad q_2\delta q_2 \\
k_2\phi k_2 & \quad q_3\delta q_3 \\
k_3\phi k_3 & \quad q_4\delta q_4 \\
\end{aligned}$ & $\begin{aligned}
k_1\phi k_1 & \quad q_2\delta q_2 \\
k_2\phi k_2 & \quad q_3\delta q_3 \\
k_3\phi k_3 & \quad q_4\delta q_4 \\
\end{aligned}$ \\
\hline
$1\times$ & $2\times$ & $4\times$ & $32\times$ & $32\times$ & $32\times$ \\
\hline
\end{tabular}
\end{table}

Table II. Diagrammatic representation of the quartic term in the Hamiltonian: $\langle H_1 \rangle - \frac{1}{4} \langle (H_2^2) - (H_1^2) \rangle$. Solid lines represent low frequency modes, wavy lines represent high frequency modes, dots represent dot products, and short dashed lines between two dots represent a momentum conservation, e.g. $\delta(k_1 + k_2 + q_3 + q_4)$ in $L_2^{(1)}$. (Note that we have followed diagrammatic convention of $[13]$.)

$L_1^{(1)}$ can be represented as a Feynman diagram in Table II. Here the solid lines indicate the low frequency modes. The dots represent dot product, i.e. $q_1\tilde{\theta} q_1$ is dotted with $q_2\tilde{\theta} q_2$ and $q_3\tilde{\theta} q_3$ is dotted with $q_4\tilde{\theta} q_4$. Finally the tiny dashed line between the two dots represents a momentum conservation $\delta(q_1 + q_2 + q_3 + q_4)$.

Next, the $L_2^{(1)}$ term can be represented diagrammatically as in Table II. Here the wavy lines represent high frequency modes ($k_1\phi k_1$ and $k_2\phi k_2$). These two wavy lines are connected to represent the correlation $\langle \phi k_1\phi k_2 \rangle$ (which has another momentum conservation $\delta(k_1 + k_2)$ inside). Finally we can have 2 different permutations of diagram $L_2^{(1)}$ in Table II (with wavy loop on the left or on the right hand side) and thus we have a prefactor of 8 in Eq. (44). We can now calculate $L_2^{(1)}$ term explicitly

$$L_2^{(1)} = 2 \times L \int \frac{d^d k_1 d^d k_2 d^d q_3 d^d q_4}{(2\pi)^d} \delta(k_1 + k_2 + q_3 + q_4)(k_1 \cdot k_2)(q_3 \cdot q_4)$$

$$\times \frac{\delta(k_1 + k_2)}{K k_1^2 + C k_2^2} \tilde{\theta} q_1 \tilde{\theta} q_4$$

(46)

where we have substituted $\langle \phi k_1 \phi k_2 \rangle = \frac{\delta(k_1 + k_2)}{K k_1^2 + C k_2^2}$. We next perform integral over $k_2$ and then over $q_4$ to eliminate the delta functions. Finally we obtain (after relabelling $k_1 \rightarrow k$ and $q_4 \rightarrow q$):

$$L_2^{(1)} = 2L \int_{0}^{\Lambda/b} \frac{d^d q}{(2\pi)^d} q^2$$

$$\left| \tilde{\theta} q \right|^2 \int_{\Lambda/b}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{k^2}{K k^2 + C k^4}$$

(47)

Similarly, the $L_3^{(1)}$ term can be represented diagrammatically as in Table II. Here we have 4 different permutations and hence a prefactor of 4 in Eq. (45). Again this integral can be computed as:

$$L_3^{(1)} = 4 \times L \int \frac{d^d k_1 d^d k_2 d^d q_3 d^d q_4}{(2\pi)^d} \delta(k_1 + k_3 + q_2 + q_4)(k_1 \cdot q_2)(k_3 \cdot q_4)$$

$$\times \frac{\delta(k_1 + k_3)}{K k_1^2 + C k_3^2} \tilde{\theta} q_2 \tilde{\theta} q_4.$$ 

(48)

First we do integral over $k_3$ then over $q_4$ to eliminate the delta functions and then relabel $k_1 \rightarrow k$ and $q_2 \rightarrow q$ to obtain:

$$L_3^{(1)} = 4 \times L \int_{0}^{\Lambda/b} \frac{d^d q}{(2\pi)^d} q_{\alpha} q_{\beta}$$

$$\left| \tilde{\theta} q \right|^2 \int_{\Lambda/b}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{k_{\alpha} k_{\beta}}{K k^2 + C k^4}.$$ 

(49)

We observe that the integral over $k$ is isotropic and symmetric under swapping the indices $\alpha \leftrightarrow \beta$ and thus:

$$\int_{\Lambda/b}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{k_{\alpha} k_{\beta}}{K k^2 + C k^4} = A \delta_{\alpha \beta}$$

(50)
for some constant A. Contracting the index α and β, we can get A. Therefore

\[ L_3^{(1)} = 4L \int_0^{\Lambda/b} d^{d}q q^2 |\tilde{q}_1|^2 \frac{1}{d} \int_{\Lambda/b}^{\Lambda} \frac{d^{d}k}{(2\pi)^d} \frac{1}{K + Ck^2}. \] (51)

Finally, the \( O(L) \) correction is

\[ \langle H_1 \rangle_\phi = L_1^{(1)} + L_2^{(1)} + L_3^{(1)} \]

\[ = \frac{1}{2} \int_0^{\Lambda/b} d^{d}q q^2 |\tilde{q}_1|^2 \left\{ 2L \left( 2 + \frac{4}{d} \right) \int_{\Lambda/b}^{\Lambda} \frac{d^{d}k}{(2\pi)^d} \frac{1}{K + Ck^2} \right\} \]

\[ + L \int_0^{\Lambda/b} \frac{d^{d}q_1 \cdots d^{d}q_4}{(2\pi)^d} \delta(q_1 + \cdots + q_4)(q_1 \cdot q_2)(q_3 \cdot q_4)\tilde{q}_1\tilde{q}_2\tilde{q}_3\tilde{q}_4. \] (53)

**B. \( O(L^2) \) correction to \( \tilde{H}[\tilde{q}_n] \)**

Next we calculate the \( O(L^2) \) correction to the coarse-grained Hamiltonian, i.e. the second term in Eq. [39]:

\[ -\frac{1}{2} \left( \langle H_1^2 \rangle_\phi - \langle H_1 \rangle_\phi^2 \right) = L_4^{(2)} + L_5^{(2)} + L_6^{(2)} \] (54)

The non-trivial terms are:

\[ L_4^{(2)} = 8 \times \frac{-L^2}{2} \int \frac{d^{d}q_1d^{d}q_2d^{d}k_3d^{d}k_4d^{d}k_5d^{d}k_6d^{d}q_7d^{d}q_8}{(2\pi)^{2d}} \delta(q_1 + q_2 + k_3 + k_4)\delta(k_5 + k_6 + q_7 + q_8) \]

\[ (q_1 \cdot q_2)(k_3 \cdot k_4)(k_5 \cdot k_6)(q_7 \cdot k_8)\tilde{q}_1\tilde{q}_2\tilde{q}_3\tilde{q}_4\tilde{q}_5\tilde{q}_6\tilde{q}_7\tilde{q}_8 \] (55)

\[ L_5^{(2)} = 32 \times \frac{-L^2}{2} \int \frac{d^{d}q_1d^{d}q_2d^{d}k_3d^{d}k_4d^{d}k_5d^{d}k_6d^{d}q_7d^{d}k_8}{(2\pi)^{2d}} \delta(q_1 + q_2 + k_3 + k_4)\delta(q_5 + k_6 + q_7 + k_8) \]

\[ (q_1 \cdot q_2)(k_3 \cdot k_4)(k_5 \cdot k_6)(q_7 \cdot k_8)\tilde{q}_1\tilde{q}_2\tilde{q}_3\tilde{q}_4\tilde{q}_5\tilde{q}_6\tilde{q}_7\tilde{q}_8 \] (56)

\[ L_6^{(2)} = 32 \times \frac{-L^2}{2} \int \frac{d^{d}k_1d^{d}q_3d^{d}k_4d^{d}q_5d^{d}k_6d^{d}q_7d^{d}k_8}{(2\pi)^{2d}} \delta(k_1 + q_2 + k_3 + q_4)\delta(q_5 + k_6 + q_7 + k_8) \]

\[ (k_1 \cdot q_2)(k_3 \cdot q_4)(k_5 \cdot k_6)(q_7 \cdot k_8)\tilde{q}_1\tilde{q}_2\tilde{q}_3\tilde{q}_4\tilde{q}_5\tilde{q}_6\tilde{q}_7\tilde{q}_8. \] (57)

where the integral over \( q \) is from 0 to \( \Lambda/b \) and the integral over \( k \) is from \( \Lambda/b \) to \( \Lambda \). They are represented as Feynman diagrams in Table. III. The prefactors 8, 32, and 32 represent the number of permutations of these diagrams. Note that for every pair of two connected wavy lines (e.g. \( \phi_{k_3} \) and \( \phi_{k_5} \) in \( L_4^{(2)} \)) represents a single pair correlation \( \langle \phi_{k_3}\phi_{k_5} \rangle_\phi \). Feynman diagrams already take into account of Wick’s theorem: \( \langle \phi_1\phi_2\phi_3\phi_4 \rangle_\phi = \langle \phi_1\phi_2 \rangle_\phi \langle \phi_3\phi_4 \rangle_\phi + \langle \phi_1\phi_3 \rangle_\phi \langle \phi_2\phi_4 \rangle_\phi + \langle \phi_4\phi_3 \rangle_\phi \langle \phi_2\phi_3 \rangle_\phi \). The results over the integrals [55,57] are:

\[ L_4^{(2)} = -4L^2 \int_0^{\Lambda/b} \frac{d^{d}q_1 \cdots d^{d}q_4}{(2\pi)^d} \delta(q_1 + \cdots + q_4)(q_1 \cdot q_2)(q_3 \cdot q_4)\tilde{q}_1\tilde{q}_2\tilde{q}_3\tilde{q}_4 \left( \int_{\Lambda/b}^{\Lambda} \frac{d^{d}k}{(2\pi)^d} \frac{1}{K + Ck^2} \right)^2 \] (58)

\[ L_5^{(2)} = -16L^2 \int_0^{\Lambda/b} \frac{d^{d}q_1 \cdots d^{d}q_4}{(2\pi)^d} \delta(q_1 + \cdots + q_4)(q_1 \cdot q_2)(q_3 \cdot q_4)\tilde{q}_1\tilde{q}_2\tilde{q}_3\tilde{q}_4 \left( \int_{\Lambda/b}^{\Lambda} \frac{d^{d}k}{(2\pi)^d} \frac{1}{K + Ck^2} \right)^2 \] (59)

\[ L_6^{(2)} = -\frac{48}{d(d+2)}L^2 \int_0^{\Lambda/b} \frac{d^{d}q_1 \cdots d^{d}q_4}{(2\pi)^d} \delta(q_1 + \cdots + q_4)(q_1 \cdot q_2)(q_3 \cdot q_4)\tilde{q}_1\tilde{q}_2\tilde{q}_3\tilde{q}_4 \left( \int_{\Lambda/b}^{\Lambda} \frac{d^{d}k}{(2\pi)^d} \frac{1}{K + Ck^2} \right)^2 \] (60)

Here we will only show the result of [60] since the results of [58,59] are similar. From [57], we have:

\[ L_6^{(2)} = -16L^2 \int \frac{d^{d}k_1d^{d}q_2d^{d}k_3d^{d}q_4d^{d}k_5d^{d}k_6d^{d}q_7d^{d}k_8}{(2\pi)^{2d}} \delta(k_1 + q_2 + k_3 + q_4)\delta(q_5 + k_6 + q_7 + k_8) \]

\[ (k_1 \cdot q_2)(k_3 \cdot q_4)(q_5 \cdot k_6)(q_7 \cdot k_8)\tilde{q}_1\tilde{q}_2\tilde{q}_3\tilde{q}_4 \left( \int_{\Lambda/b}^{\Lambda} \frac{d^{d}k}{(2\pi)^d} \frac{1}{K + Ck^2} \right)^2 \] (61)
Next, we integrate over $k_6$ and $k_8$ to eliminate $\delta(k_1 + k_6)$ and $\delta(k_3 + k_8)$ and then over $k_3$ to eliminate another delta function. The result is (after relabelling $k_1 \to k$):

$$L_6^{(2)} = -16L^2 \int \frac{d^d q_2 d^d q_4 d^d q_6 d^d q_7}{(2\pi)^d} \delta(q_2 + q_4 + q_5 + q_7) \bar{\theta}_{q_6} \bar{\theta}_{q_7} \bar{\theta}_{q_6} \bar{\theta}_{q_7} \int \frac{d^d k}{(2\pi)^d} (q_2 \cdot k)(q_5 \cdot k) \frac{q_1 \cdot (k - q_5 - q_7)}{K k^2 + C k^4} \frac{q_7 \cdot (k - q_5 - q_7)}{K |k - q_5 - q_7|^2 + C |k - q_5 - q_7|^4}$$

(62)

Then we can assume $|k| \gg |q_5 + q_7|$ and thus we obtain:

$$L_6^{(2)} = -16L^2 \int \frac{d^d q_2 d^d q_4 d^d q_6 d^d q_7}{(2\pi)^d} \delta(q_2 + q_4 + q_5 + q_7) q_2 q_4 q_5 q_7 \bar{\theta}_{q_6} \bar{\theta}_{q_7} \bar{\theta}_{q_6} \bar{\theta}_{q_7} \int \frac{d^d k}{(2\pi)^d} k_1 k_3 k_4 k_5$$

(63)

Next, we observe that the $k$-integral is isotropic and symmetric under swapping any indices $\alpha \leftrightarrow \beta, \alpha \leftrightarrow \gamma$, etc. Therefore the $k$-integral can be written as:

$$\int \frac{d^d k}{(2\pi)^d} k_1 k_3 k_4 k_5 = B(\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma} + \delta_{\alpha\gamma} \delta_{\beta\delta})$$

(64)

for some constant $B$. Contracting all the indices, we can obtain this constant $B$ and the result is $[60]$. Therefore the coarse-grained Hamiltonian Eq. (39), after averaging out the high-frequency modes, is:

$$\tilde{H} = H_0[\tilde{\theta}_q] + L_1^{(1)} + L_2^{(1)} + L_3^{(1)} + L_4^{(2)} + L_5^{(2)} + L_6^{(2)}$$

(65)

$$= \frac{1}{2} \int_{0}^{\Lambda/b} d^d q \left( \tilde{K} q^2 + C q^4 \right) |\tilde{\theta}_q|^2 + \tilde{L} \int_{0}^{\Lambda/b} d^d q_1 d^d q_2 d^d q_3 d^d q_4 \delta(q_1 + q_2 + q_3) [q_1 \cdot q_2] [q_3 \cdot (-q_1 - q_2 - q_3)] \tilde{\theta}_{q_1} \tilde{\theta}_{q_2} \tilde{\theta}_{q_3} \tilde{\theta}_{q_4} \tilde{\theta}_{q_1'} \tilde{\theta}_{q_2'} \tilde{\theta}_{q_3'} \tilde{\theta}_{q_4'}$$

(66)

where

$$\tilde{K} = K + 4L \left( 1 + \frac{2}{d} \right) \int_{0}^{\Lambda/b} \frac{d^d k}{(2\pi)^d} \frac{1}{K + C k^2}$$

(67)

$$\tilde{L} = L - 4L^2 \left( 1 + \frac{4}{d} + \frac{12}{d(d + 2)} \right) \int_{0}^{\Lambda/b} \frac{d^d k}{(2\pi)^d} \frac{1}{K + C k^2}$$

(68)

C. Rescaling $q \to b q$ and $\theta \to \theta/z$

The second and final step in RG (see Fig. 4) is to rescale $q$ and $\theta$. In particular, we define $q' = b q$ and $\theta' = \theta/z$ where $b > 1$ is the rescaling factor. Eq. (66) then becomes:

$$H' = \frac{1}{2} \int_{0}^{\Lambda} d^d q' \left( K' q'^2 + C' q'^4 \right) |\theta_q'|^2$$

(69)

where

$$K' = b^{-d-2} z^2 \left[ K + 4L \left( 1 + \frac{2}{d} \right) \int_{0}^{\Lambda/b} \frac{d^d k}{(2\pi)^d} \frac{1}{K + C k^2} \right]$$

(70)

$$C' = b^{-d-4} z^2 C$$

(71)

$$L' = b^{-3d-4} z^4 \left[ L - 4L^2 \left( 1 + \frac{4}{d} + \frac{12}{d(d + 2)} \right) \int_{0}^{\Lambda/b} \frac{d^d k}{(2\pi)^d} \frac{1}{K + C k^2} \right]$$

(72)
Therefore at the end of RG, we end up with the same form of Hamiltonian but with renormalized coefficients $K'$, $C'$, and $L'$. Now we need to fix $z$ by choosing the coefficient $C$ to be invariant under RG. The reason is because we are more interested in how the coefficients $K$ (which is the control parameter in our model) and $L$ change under RG. If $L$ becomes smaller when we coarse-grain and rescale, the higher order term in $H$ is shown to be irrelevant. Thus we fix $C' = C$ to obtain $z = b'(d+4)/2$. Next we rewrite $b$ as $b = 1 + \delta \ell$ where $\delta \ell$ is small and positive. The integral in Eq. (70) then becomes:

$$\int_{\Lambda}^{\Lambda - d \delta \ell} \frac{1}{\Omega_d} \frac{\Omega_d}{(2\pi)^d} \ell^{d-1}dk \simeq \frac{1}{K + CA^2} \frac{\Omega_d}{(2\pi)^d} \Lambda^{d-1} \Lambda \delta \ell$$

(73)

where $\Omega_d$ is the solid angle in a $d$-dimensional sphere. Similarly, we can compute the integral in Eq. (72) over a thin shell of radius $\Lambda$ and thickness $\Lambda \delta \ell$. We then obtain the RG flow for the coefficients $K$ and $L$:

$$\frac{dK}{d\ell} = 2K + 4 \left(1 + \frac{2}{d}\right) \frac{\Omega_d}{(2\pi)^d} \Lambda^d K + CA^2 L$$

(74)

$$\frac{dL}{d\ell} = (4 - d)L - 4 \left(1 + \frac{4}{d} + \frac{12}{d(d+2)}\right) \frac{\Omega_d}{(2\pi)^d} \Lambda^d (K + CA^2)^2 L^2.$$ 

(75)

The RG flow for the coefficients $K$ and $L$ are plotted in Fig. 5.

For spatial dimension $d \geq 4$, (74-75) have one non-trivial fixed point at $(K = 0, L = 0)$, which is unstable along $\hat{e}_K$-direction and stable along $\hat{e}_L$-direction (see Fig. 5(a)). This means if we are above the critical line $K_c(L)$ (dashed red line in Fig. 5(a)), RG flow will take us to $K \to +\infty$ (uniform nematic phase) and if we are below $K_c(L)$, RG flow will take us to $K \to -\infty$ ($SB_\infty$ phase). Linearizing (74-75) around this fixed point, we obtain:

$$\frac{d}{d\ell} \begin{pmatrix} \delta K \\ \delta L \end{pmatrix} = \begin{pmatrix} 4 \left(1 + \frac{2}{d}\right) \frac{\Omega_d}{(2\pi)^d} \Lambda^d & \frac{\Lambda^d}{CA^2} \\ 0 & 4 - d \end{pmatrix} \begin{pmatrix} \delta K \\ \delta L \end{pmatrix}$$

(76)

with eigenvalue $\lambda_K = 2$ associated with eigendirection $\hat{e}_K$ and $\lambda_L = 4 - d \leq 0$ associated with eigendirection $\hat{e}_L$ (see Fig. 5(a)). Thus close to the fixed point, $\delta K$ diverges as $|\delta K| \sim e^{\lambda_K \ell}$. Now from Fig. 4 under an infinitesimal RG, the correlation length $\xi$ is mapped to $\xi'$, which is given by:

$$\xi' = \frac{\xi}{1 + \delta \ell} \Rightarrow \xi - \delta \ell \xi \Rightarrow \frac{d\xi}{d\ell} = -\xi \Rightarrow \xi \sim e^{-\ell}$$

(77)

Substituting $\ell \sim \ln |\delta K|/\lambda_K$, we obtain:

$$\xi \sim |\delta K|^{-\frac{1}{\lambda_K}}$$

(78)
Therefore for $d \geq 4$ we get the mean field critical exponent $\nu = 1/\lambda_K = 1/2$ as we expect. However, RG calculation also tells us that the critical point $K_c(L)$ is not zero (as shown by the mean field phase diagram in the main text) but shifted by $L$.

Now for spatial dimension $d < 4$, the fixed point at $(0, 0)$ becomes unstable and a new non-trivial fixed point appears at $(K^*, L^*) = \left(-\frac{3}{10}CA^2\epsilon, \frac{4\pi^2}{5}C^2\epsilon\right)$, where $\epsilon = 4 - d > 0$ (see Fig. 5(b)). Linearizing around $(K^*, L^*)$ and expanding for small $\epsilon$, we get:

$$
\frac{d}{d\ell} \left( \begin{array}{c} \delta K \\ \delta L \end{array} \right) = \left( \begin{array}{cc} 2 - \frac{3}{5}\epsilon + \mathcal{O}(\epsilon^2) & \frac{3\Lambda^2}{4\pi C} + \mathcal{O}(\epsilon) \\ \mathcal{O}(\epsilon^2) & -\epsilon + \mathcal{O}(\epsilon^2) \end{array} \right) \left( \begin{array}{c} \delta K \\ \delta L \end{array} \right).
$$

(79)

Thus we identify the eigenvalue $\lambda_K = 2 - \frac{3}{5}\epsilon + \mathcal{O}(\epsilon^2)$ associated with the eigendirection $\hat{e}_K$ (see Fig. 5(b)). Therefore the critical exponent for the correlation length $\xi$ is:

$$
\nu = \frac{1}{\lambda_K} = \frac{1}{2} + \frac{3}{20} \epsilon + \mathcal{O}(\epsilon^2).
$$

(80)

**D. Comparison with ferromagnet**

In comparison, for unconstrained para-ferromagnetic transition, the Hamiltonian is given by

$$
H[m] = \int d^dr \left\{ \frac{K}{2} |m|^2 + \frac{C}{2} |\nabla m|^2 + L |m|^4 \right\},
$$

(81)

and the RG flow equations for $K$ and $L$ are:

$$
\frac{dK}{d\ell} = 2K + 4(n + 2) \frac{\Lambda^d}{K + CA^2} \frac{\Omega_d}{(2\pi)^d} L
$$

(82)

$$
\frac{dL}{d\ell} = (4 - d)L - 4(n + 8) \frac{\Lambda^d}{(K + CA^2)^2} \frac{\Omega_d}{(2\pi)^d} L^2,
$$

(83)

where $n$ is the dimension of $m$ and $d$ is the spatial dimension.