SUBIDEALS OF OPERATORS II

SASMITA PATNAIK AND GARY WEISS

ABSTRACT. A subideal (also called a $J$-ideal) is an ideal of a $B(H)$-ideal $J$. This paper is the sequel to Subideals of Operators where a complete characterization of principal and then finitely generated $J$-ideals were obtained by first generalizing the 1983 work of Fong and Radjavi who determined which principal $K(H)$-ideals are also $B(H)$-ideals. Here we determine which countably generated $J$-ideals are $B(H)$-ideals, and in the absence of the continuum hypothesis which $J$-ideals with generating sets of cardinality less than the continuum are $B(H)$-ideals. These and some other results herein are based on the dimension of a related quotient space. We use this to characterize these $J$-ideals and settle additional questions about subideals.

A key property in our investigation turned out to be $J$-softness of a $B(H)$-ideal $I$ inside $J$, that is, $IJ = I$, a generalization of a recent notion of softness of $B(H)$-ideals introduced by Kaftal-Weiss and earlier exploited for Banach spaces by Mityagin and Pietsch.

1. INTRODUCTION

In Subideals of Operators [8] we found three types of principal and finitely generated subideals (i.e., $J$-ideals): linear, real-linear and nonlinear subideals. Such types also carry over to general $J$-ideals. The linear $K(H)$-ideals, being the traditional ones, were studied in 1983 by Fong and Radjavi [3]. They found principal linear $K(H)$-ideals that are not $B(H)$-ideals. Herein we take all $J$-ideals $I$ to be linear, but as proved in [8], we expect here also that most of the results and methods apply to the two other types of subideals (real-linear and nonlinear). Also $H$, as in [3], denotes a separable infinite-dimensional complex Hilbert space. One of our main contributions in [8] was to use a modern framework for $B(H)$-ideals to generalize [3] Theorem 2.

We generalized their result on principal $K(H)$-ideals to all principal $J$-ideals by proving that a principal and then a finitely generated $J$-ideal $(S)_J$ generated by the finite set $S \subset J$ is also a $B(H)$-ideal if and only if $(S)$ is $J$-soft, that is, $(S) = (S)_J$ where $(S)$ is the $B(H)$-ideal generated by $S$. Then we used this to characterize the structure of $(S)_J$. $J$-softness is a generalization of a recent notion of $K(H)$-softness of $B(H)$-ideals introduced by Kaftal-Weiss and earlier exploited for Banach spaces by Mityagin and Pietsch (see [8] Remark 2.6), [7], [9]).

Here we further develop the subject by investigating $J$-ideals $I = (S)_J$ generated by arbitrary sets $S$ of varying cardinality, their algebraic structure and when they are $B(H)$-ideals. To add perspective, the reader should keep in mind that all nonzero $J$-ideals have cardinality and Hamel dimension precisely cardinality $\mathfrak{c}$ of the continuum (Remark 5.3), but questions on the cardinalities of their possible generating sets is another matter (Section 6, Questions 1-2), and this we shall see impacts questions on structure. After investigating the cases when $S$ is countable or of cardinality less than $\mathfrak{c}$ (absent the continuum hypothesis CH), we then consider general $J$-ideals $I$ and questions on the possible cardinalities of their generating sets, observing that $I$ is always a generating set for itself but may have generating sets of cardinality less than its cardinality $\mathfrak{c}$. When they do has special implications.

We show $(S)_J$ is a $B(H)$-ideal if and only if $(S)$ is $J$-soft for those $(S)_J$ generated as a $J$-ideal by countable sets and then when generated by sets of cardinality strictly less than cardinality $\mathfrak{c}$ (Theorem 4.1). This will follow from sufficiency of the codimension condition on $(S)^0_J$ in $(S)_J$: $(S)_J/(S)^0_J$ has Hamel dimension strictly less than $\mathfrak{c}$. (For $(S)^0_J$ see Definition 2.2) We then investigate general $J$-ideals to provide an example where softness fails for a $J$-ideal $I$ with codimension of $I^0$ in $I$ equal to $\mathfrak{c}$ (Example 4.5), thereby showing that $J$-ideals that are also $B(H)$-ideals need not be $J$-soft and as a consequence cannot be generated in $J$ by sets of cardinality less than $\mathfrak{c}$. We also answer several questions on $J$-ideals posed in [8] Sections 6-7, provide some additional results and pose new questions.

Date: May 10, 2014.
1991 Mathematics Subject Classification. Primary: 47L20, 47B10, 47B07; Secondary: 47B47, 47B37, 13C05, 13C12.

Key words and phrases. Ideals, operator ideals, principal ideals, subideals, lattices.
In summary the main theorems here are:

For $\mathcal{I}^0 := \text{span}\{IJ + J\mathcal{I}\} + J(\mathcal{I})J$ (see Definition 2.2) where $(\mathcal{I})$ is the $B(\mathcal{H})$-ideal generated by $J$-ideal $\mathcal{I}$,

**Theorem.** (Theorem 4.1) The $J$-ideal $(\mathcal{S})_J$ generated by a set $S$ of cardinality strictly less than $\mathfrak{c}$ is a $B(\mathcal{H})$-ideal if and only if the $B(\mathcal{H})$-ideal $(\mathcal{S})$ is $J$-soft (i.e., $(\mathcal{S}) = (\mathcal{S})_J$). Moreover, for a $J$-ideal $\mathcal{I}$ with Hamel dimension of $\mathcal{I}/\mathcal{I}^0$ strictly less than $\mathfrak{c}$, $\mathcal{I}$ is a $B(\mathcal{H})$-ideal if and only if $(\mathcal{I})$ is $J$-soft.

**Structure Theorem.** (Theorem 4.1) For $(\mathcal{S})_J$ when $|S| < \mathfrak{c}$,

The algebraic structure of the $J$-ideal $(\mathcal{S})_J$ generated by a set $S$ is given by

$$(\mathcal{S})_J = \text{span}\{S + JS + SJ\} + J(\mathcal{S})J$$

$J(\mathcal{S})J$ is a $B(\mathcal{H})$-ideal, span$\{JS + SJ\} + J(\mathcal{S})J$ is a $J$-ideal, and $J(\mathcal{S})J \subset \text{span}\{JS + SJ\} + J(\mathcal{S})J \subset (\mathcal{S})_J$. This inclusion collapses to $J(\mathcal{S})J = (\mathcal{S})_J$ if and only if $(\mathcal{S})$ is $J$-soft.

**Remark.** Although the methods in [8] are quite a bit more analytic, we found here and in [8] a more direct algebraic approach, albeit a key tool [4] used herein is essentially analytic.

2. Preliminaries

Recall the following standard definitions from [8] with Definition 2.2 evolving from [8].

**Definition 2.1.** Let $J$ be an ideal of $B(\mathcal{H})$ (i.e., a $B(\mathcal{H})$-ideal) and $S \in J$.

- The principal $B(\mathcal{H})$-ideal generated by the single operator $S$ is given by

$$(S) := \bigcap\{I \mid I \text{ is a } B(\mathcal{H})\text{-ideal containing } S\}$$

- The principal $J$-ideal generated by $S$ is given by

$$(S)_J := \bigcap\{I \mid I \text{ is a } J\text{-ideal containing } S\}$$

- As above for principal $J$-ideals, likewise for an arbitrary subset $S \subset J$, $(S)$ and $(S)_J$ respectively denote the smallest $B(\mathcal{H})$-ideal and the smallest $J$-ideal generated by the set $S$. In particular, $(\mathcal{S}) = (\mathcal{S})_{B(\mathcal{H})}$. Denote $(\mathcal{I})$ as the $B(\mathcal{H})$-ideal generated by the $J$-ideal $\mathcal{I}$.

**Definition 2.2.** For a $J$-ideal $\mathcal{I}$, the algebraic $J$-interior of $\mathcal{I}$ is denoted by $\mathcal{I}^0 := \text{span}\{IJ + J\mathcal{I}\} + J(\mathcal{I})J$ where $\mathcal{I} := \{AB \mid A \in \mathcal{I}, B \in J\}$, $\mathcal{J}^0$ is defined similarly, and the ideal product $J(\mathcal{I})J$ is the $B(\mathcal{H})$-ideal given by $J(\mathcal{I})J = \{ES'F \mid E, F \in J, S' \in (\mathcal{I})\}$ (single triple operator products) where equality follows from [2] Lemma 6.3].

**Remark 2.3.** For the $J$-ideal $\mathcal{I} = (\mathcal{S})_J$ generated by a set $S$, $\mathcal{I}^0$ has the simpler form:

$$(\mathcal{S})^0_J = \text{span}\{SJ + JS\} + J(\mathcal{S})J$$

Notice also that $(\mathcal{S})^0_J \subset (\mathcal{S})J$.

Indeed, for $S = \{S_\alpha\}_{\alpha \in \mathcal{A}}$, $(\mathcal{S})_J = \bigcup_{\{\alpha_1, \ldots, \alpha_j\} \subset \mathcal{A}} \{(S_{\alpha_1})_J + (S_{\alpha_2})_J + \cdots + (S_{\alpha_j})_J\}$ where $\mathcal{A}$ is an index set.

By definition, $(\mathcal{S})^0_J = \text{span}\{(S)_J J + J(\mathcal{S})J\} + J((\mathcal{S})_J)J$, and because $(\mathcal{S})_J = (\mathcal{S})$ one has the simplification:

$$(\mathcal{S})^0_J = \text{span}\{(S)_J J + J(\mathcal{S})J\} + J((\mathcal{S})_J)J$$

The algebraic structure for principal $J$-ideals yields $(S_\alpha)_J = CS_\alpha + JS_\alpha + SJ_\alpha + J(S_\alpha)J$ for each $\alpha \in \mathcal{A}$. So span$\{(S_\alpha)_J J + J(S_\alpha)J\} \subset S_\alpha J + JS_\alpha + J(S_\alpha)J$ for each $\alpha \in \mathcal{A}$. Therefore $(\mathcal{S})^0_J \subset \text{span}\{SJ + JS\} + J(\mathcal{S})J$, and since the reverse inclusion is obvious, one has equality.

We recall here the definition of $J$-softness of $B(\mathcal{H})$-ideals [8] Definition 2.5].

**Definition 2.4.**

For $B(\mathcal{H})$-ideals $I$ and $J$, the ideal $I$ is called “$J$-soft” if $IJ = I$.

Equivalently in the language of $s$-numbers:

For every $A \in I$, $s_n(A) = O(s_n(B)s_n(C))$ for some $B \in I, C \in J, m \in \mathbb{N}$.

Because $IJ \subset J$, only $B(\mathcal{H})$-ideals that are contained in $J$ can be $J$-soft.
Remark 2.5. Standard facts on \(B(H)\)-ideals from [8] Remark 2.2.

(i) [2] Sections 2.8, 4.3 (see also [6] Section 4): If \(I, J\) are \(B(H)\)-ideals then the product \(IJ\), which is both associative and commutative, is the \(B(H)\)-ideal given by the characteristic set \(\Sigma(IJ) = \{ \xi \in c_0^* \mid \xi \leq \eta \rho \text{ for some } \eta \in \Sigma(I) \text{ and } \rho \in \Sigma(J) \}\). In abstract rings, the ideal product is defined as the class of finite sums of products of two elements, \(IJ := \{ \sum_{finite} a_i b_i \mid a_i, b_i \in I, J \}\), but in \(B(H)\) the next lemma shows finite sums of operator products defining \(IJ\) can be reduced to single products.

(ii) [2] Lemma 6.3 Let \(I\) and \(J\) be proper ideals of \(B(H)\). If \(A \in IJ\), then \(A = XY\) for some \(X \in I\) and \(Y \in J\).

(iii) [8] Section 1 For \(T \in B(H)\), \(s(T)\) denotes the sequence of \(s\)-numbers of \(T\). Then \(A \in (T)\) if and only if \(s(A) = O(D_m(s(T)))\) for some \(m \in \mathbb{N}\). Moreover, for a \(B(H)\)-ideal \(I\), \(A \in I\) if and only if \(A^* \in I\) if and only if \(|A| \in I\) (via the polar decomposition \(A = U|A|\) and \(U^*A = |A| = (A^*A)^{1/2}\)), with all equivalent to \(\text{diag} s(A) \in I\).

(iv) The lattice of \(B(H)\)-ideals forms a commutative semiring with multiplicative identity \(B(H)\). That is, the lattice is commutative and associative under ideal addition and multiplication (see [2] Section 2.8) and it is distributive. Distributivity with multiplier \(K(H)\) is stated without proof in [6] Lemma 5.6-preceding comments. The general proof is simple and is as follows. For \(B(H)\)-ideals \(I, J, K\), one has \(I \cap J = \{ A+B \mid A \in I, B \in J \}\) and so \(IK, JK \subseteq (I+J)K\), so one has \(IK + JK \subseteq (I+J)K\). The reverse inclusion follows more simply if one invokes (ii) above: \(X \in (I+J)K\) if and only if \(X = (A+B)C\) for some \(A \in I, B \in J, C \in K\). The lattice of \(B(H)\)-ideals is not a ring because, for instance, \(\{0\}\) is the only \(B(H)\)-ideal with an additive inverse, namely, \(\{0\}\) itself, so it is not an additive group. It is also clear that \(B(H)\) is the multiplicative identity but no \(B(H)\)-ideal has a multiplicative inverse.

We summarize the main results of [8] generalizing the 1983 work of Fong and Radjavi and characterizing all finitely generated linear \(J\)-ideals. Though not needed here, we note that [8] provided similar results for real-linear and nonlinear \(J\)-ideals.

Theorem 2.6. [8] Theorem 4.5 For \(S := \{S_1, \cdots, S_N\} \subseteq J\), the following are equivalent.

(i) The finitely generated \(J\)-ideal \((S)_J\) is a \(B(H)\)-ideal.

(ii) The \(B(H)\)-ideal \((S)\) is \(J\)-soft, i.e., \((S) = J(S)\) (equivalently, \((S) = (S)J\)).

(iii) For all \(1 \leq j \leq N\), \(S_j = \sum_{n \in (j,k)} \sum_{i=1}^{N} A_{ijk} S_k B_{ijk} \)
for some \(A_{ijk}, B_{ijk} \in J, n(j, k) \in \mathbb{N}\).

(iv) For all \(1 \leq j \leq N\), \(s(S_j) = O(D_m(s(|S_1| + \cdots + |S_N|))s(T))\) for some \(T \in J\) and \(m \in \mathbb{N}\).

Theorem 2.7. [8] Theorem 4.6] (Structure theorem for finitely generated \(J\)-ideal \((S)_J\), for \(S = \{S_1, \cdots, S_N\}\))

The algebraic structure of the finitely generated \(J\)-ideal \((S)_J\) is given by
\[(S)_J = \mathbb{C}S_1 + \cdots + \mathbb{C}S_N + JS_1 + \cdots + JS_N + S_1J + \cdots + S_NJ + J(S)J\]

So one has \(J(S)J \subseteq J(S_1 + \cdots + JS_N + S_1J + \cdots + S_NJ + J(S)J) \subseteq (S)_J \subseteq (S)\)
which first two, \((S)_J\) and \(JS_1 + \cdots + JS_N + S_1J + \cdots + S_NJ + J(S)J\) respectively, are a \(B(H)\)-ideal and a \(J\)-ideal. The inclusions collapse to merely
\[J(S)J = (S)_J = (S)\]
if and only if the finitely generated \(B(H)\)-ideal \((S)\) is \(J\)-soft.
3. THE HAMEL DIMENSION OF IDEALS

In this section we show that the Hamel dimension and the cardinality of every nonzero $J$-ideal is precisely $\mathfrak{c}$. This will impact the codimension of the algebraic $J$-interior $\mathcal{I}^0$ in $\mathcal{I}$ and lead to questions on the possible generating sets for general $J$-ideals (see Question 4.5 and Section 6–Questions 1, 2, 5).

It is straightforward to see that $\mathcal{I}^0 := \text{span}(\mathcal{I}J + J\mathcal{I}) + J(\mathcal{I}J)$ is an ideal of $\mathcal{I}$ and hence is a complex vector subspace of $\mathcal{I}$. The quotient space $\mathcal{I}/\mathcal{I}^0$ is a complex vector space and therefore has a Hamel basis where the Hamel dimension is invariant over all Hamel bases. The key notion used in our results is the cardinality of $\mathcal{I}/\mathcal{I}^0$ relative to its vector space structure. ($\mathcal{I}^0$ being an ideal of $\mathcal{I}$, the quotient space $\mathcal{I}/\mathcal{I}^0$ is also a ring but we will not exploit the ring structure.)

**Proposition 3.1.** For the $J$-ideal $(\mathcal{S})_J$ generated by a set $\mathcal{S}$ and $(\mathcal{S})_J^0 = \text{span}(\mathcal{S}J + J\mathcal{S}) + J(\mathcal{S}J)$, the Hamel dimension of the quotient space $(\mathcal{S})_J/(\mathcal{S})_J^0$ is at most the cardinality of the generating set $\mathcal{S}$.

**Proof.** From general ring theory, for $\mathcal{S} = \{S_\alpha\}_{\alpha \in \mathcal{A}}$, $(\mathcal{S})_J = \bigcup_{\alpha_1,\ldots,\alpha_j \in \mathcal{A}} \{(S_{\alpha_1})_J + (S_{\alpha_2})_J + \cdots + (S_{\alpha_j})_J\}$ where $\mathcal{A}$ is an index set. Combining this with the algebraic structure for principal $J$-ideals implied by Theorem 2.7 (or for principal $J$-ideals in particular, see also [8, Proposition 4.2]), one obtains $(\mathcal{S})_J = \text{span}\mathcal{S} + (\mathcal{S})_J^0$. For $(\mathcal{S})_J^0$ being a linear subspace of $(\mathcal{S})_J$, one can show that the quotient space $(\mathcal{S})_J/(\mathcal{S})_J^0 = \text{span}\{[S_\alpha]\}$ where $\alpha \in \mathcal{A}$, $S_\alpha \in \mathcal{S}$ and $[S_\alpha]$ denotes its quotient space coset. Therefore the Hamel dimension of the vector space $(\mathcal{S})_J/(\mathcal{S})_J^0$ is at most the cardinality of $\mathcal{S}$. □

Finishing up our discussion on the Hamel dimension, the following proposition which we need in Example 4.5 is probably a well-known fact but we include it here for completeness.

**Proposition 3.2.** The Hamel dimension of $F(H)$, the $B(H)$-ideal of finite rank operators, is $\mathfrak{c}$ when $H$ is separable (at least $\mathfrak{c}$ for $H$ non-separable).

**Proof.** Cardinal arithmetic applied to matrices when $H$ is separable shows $|F(H)| \leq |B(H)| \leq \mathfrak{c}$, so the Hamel dimension of $F(H)$ is at most $\mathfrak{c}$. Suppose the Hamel dimension of $F(H)$ is strictly less than $\mathfrak{c}$. Let $\mathcal{B} := \{F_\alpha \in F(H) | \alpha \in \mathcal{A}\}$ be a Hamel basis for $F(H)$ with cardinality $|\mathcal{A}| < \mathfrak{c}$. Denote a finite basis for the range of $F_\alpha$ by $B_\alpha$. So $|\bigcup_{\alpha \in \mathcal{A}} B_\alpha| = |\mathcal{A}|$ and from the set $\bigcup_{\alpha \in \mathcal{A}} B_\alpha \subset H$, one can extract a maximal linearly independent set $E$ of cardinality at most $|\mathcal{A}| < \mathfrak{c}$. Since the Hamel dimension of infinite-dimensional Hilbert space is at least $\mathfrak{c}$ [3, Lemma 3.4], there is a $0 \neq f \in H$ for which the set $E \bigcup\{f\}$ forms a linearly independent set. Consider the rank one operator $f \otimes f$. Since $\mathcal{B}$ is a Hamel basis for $F(H)$, $f \otimes f = \sum_{i=1}^{n} a_i F_{a_i}$ for some $a_i \in \mathbb{C}$, $n \in \mathbb{N}$. So, in particular, $f \otimes f(f) = \sum_{i=1}^{n} a_i F_{a_i}(f)$ which implies $0 \neq \langle f, f \rangle f = \sum_{i=1}^{n} a_i F_{a_i}(f) = \sum_{i=1}^{m} b_{\beta_i} e_{\beta_i}$ where $e_{\beta_i} \in E \subset \bigcup_{\alpha \in \mathcal{A}} B_\alpha$, hence $f \in \text{span} E$ contradicting that $E \bigcup\{f\}$ is a linearly independent set. In summary, the assumption that the Hamel dimension of $F(H)$ is strictly less than $\mathfrak{c}$ led to the existence of this $f$ and hence to this contradiction. So the Hamel dimension of $F(H)$ is precisely $\mathfrak{c}$, and consequently the cardinality $|B(H)| = \mathfrak{c}$. □

An unrelated and interesting question on Hamel bases appears in [1].

**Remark 3.3.** The Hamel dimension of every nonzero $J$-ideal $\mathcal{I}$ is precisely $\mathfrak{c}$. Indeed, $F(H) \subset \mathcal{I}$ (see [8, Section 6, (2)]) so Proposition 3.2 implies the Hamel dimension of $\mathcal{I}$ is at least $\mathfrak{c}$. Also, since $\mathcal{I} \subset B(H)$ and since cardinality of $B(H)$ is precisely $\mathfrak{c}$, the Hamel dimension of $\mathcal{I}$ is at most $\mathfrak{c}$. Hence the Hamel dimension of $\mathcal{I}$ is precisely $\mathfrak{c}$. Moreover, because $F(H) \subset \mathcal{I} \subset B(H)$, $\mathfrak{c} = |F(H)| \leq |\mathcal{I}| \leq |B(H)| = \mathfrak{c}$ so the cardinality of $\mathcal{I}$ is also precisely $\mathfrak{c}$. 


As mentioned earlier in Section 1, one of our main contributions in [8] was the generalization of Fong and Radjavi’s result [3, Theorem 2] by showing that the principal $J$-ideals and the finitely generated $J$-ideals that are also $B(H)$-ideals must be $J$-soft. Here we show the same for $J$-ideals generated by countable sets and, absent CH, generated by sets of cardinality strictly less than $\mathfrak{c}$. We also show that if a $J$-ideal $\mathcal{I}$ is generated by sets of cardinality equal to $\mathfrak{c}$, then $\mathcal{I}$ being a $B(H)$-ideal does not necessarily imply that $\mathcal{I}$ is $J$-soft (Example 15). Our main softness theorem is:

**Theorem 4.1.** The $J$-ideal $(S)_J$ generated by sets $S$ of cardinality strictly less than $\mathfrak{c}$ is a $B(H)$-ideal if and only if the $B(H)$-ideal $(S)$ is $J$-soft. Moreover, for a $J$-ideal $\mathcal{I}$ with the Hamel dimension of $I/\mathcal{I}^{\mathfrak{c}}$ strictly less than $\mathfrak{c}$, $\mathcal{I}$ is a $B(H)$-ideal if and only if $(\mathcal{I})$ is $J$-soft, where $(\mathcal{I})$ is the $B(H)$-ideal generated by $\mathcal{I}$.

**Proof.** $\Rightarrow$: Since $(S)J \subset (S)$, it suffices to show $(S) \subset (S)J$. Assume otherwise that there is some $T \in (S) \setminus (S)J$. We claim $(S) = (S)J$ so that $T \in (S)J$. Since every $B(H)$-ideal is also a $J$-ideal, $(S)$ is a $J$-ideal containing $S$. And $(S)_J$ being the smallest $J$-ideal containing $S$, one has $(S)_J \subset (S)$ and $(S)_J$ is a $B(H)$-ideal. Hence $(S)_J = (S)_J$ and therefore $T \in (S)J$ or, equivalently because when $(S)_J = (S)$, it is a $B(H)$-ideal, one has the equivalent condition $\text{diag}(s(T)) \subset (S)_J$.

Using the $s$-number sequence $s(T)$ we now construct a sequence of operators $D_n$ as follows. For each $n \geq 1$, let $D_n$ be the diagonal operator with $s_{2^n-1}(2k-1)\langle T \rangle$ at the $2^{n-1}(2k-1)$ scattered diagonal positions for $k \geq 1$ and with zeros elsewhere. Every positive integer has this unique product decomposition $2^{n-1}(2k-1)$. Notice then that the diagonal sequences of the $D_n$’s have pairwise disjoint support and the form of $\sum_{i=0}^{n-1} D_n$ (which incidentally converges in the operator norm) is precisely $\text{diag}(s(T))$. Recall that $(S)_J$ is a $J$-ideal and a complex vector subspace of $(S)_J$ so their quotient $(S)_J/(S)_J$ is a vector space. We will use these $D_n$’s to imbed isomorphic copies of $\ell^p$ (for every $1 \leq p \leq \infty$) inside the quotient space $(S)_J/(S)_J$. (In fact, we imbed isometric isomorphic copies of $\ell^p$ inside the quotient space $(S)_J/(S)_J$.)

We next show that for each $n \geq 1$, $(\mathcal{D}_n) = (T)$ and that $\mathcal{D}_n \notin (S)_J$. Clearly $(\mathcal{D}_n) \subset (T)$ since $\mathcal{D}_n = PT$ for a suitable diagonal projection operator, so it remains to show $(T) \notin (\mathcal{D}_n)$. For each $n \geq 1$, $\mathcal{D}_n$ is explicitly given by $\text{diag}(\underbrace{0, \cdots, 0}_{2^{n-1}-1\text{-times}}, s_{2^n-1}(T), \underbrace{0, \cdots, 0}_{2^n-1\text{-times}}, s_{2^n-1}(T), \underbrace{0, \cdots, 0}_{2^n-1\text{-times}}, s_{2^n-1}(T), \cdots, \cdots)$ so the $2^n$-fold amputation of $\mathcal{D}_n$ is $\mathcal{D}_n \in (s_{2^n-1}(T), \cdots, s_{2^n-1}(T), s_{2^n-1}(T), \cdots, s_{2^n-1}(T), \cdots) \in \Sigma((\mathcal{D}_n))$ and since $(s_{2^n+1-1}(T), s_{2^n+1-2}(T), \cdots) \leq (s_{2^n+1-1}(T), \cdots, s_{2^n+1-1}(T), s_{2^n+1-1}(T), \cdots, s_{2^n+1-1}(T), \cdots)$, so $(s_{2^n+1-k}(T))_{k=1}^{\infty} \subset \Sigma((\mathcal{D}_n))$ and hence $(\underbrace{0, \cdots, 0}_{2^n-1\text{-times}}, s_{2^n+1-1}(T), s_{2^n+1-2}(T), \cdots) \in \Sigma((\mathcal{D}_n))$.

But also the finitely supported sequence $(s_1(T), s_2(T), \cdots, s_{2^n-1}(T), 0, \cdots) \in \Sigma((\mathcal{D}_n))$. Adding both sequences one obtains precisely $\text{diag}(s(T))$. Then since $\Sigma((\mathcal{D}_n))$ is additive, we have $\mathcal{D}_n \subset \Sigma((\mathcal{D}_n))$ and hence $\text{diag}(s(T)) \in (\mathcal{D}_n)$. Therefore $(T) \subset (\mathcal{D}_n)$, and then from the reverse inclusion above one has $(T) = (\mathcal{D}_n)$.

To see that $\mathcal{D}_n \notin (S)_J$, assume otherwise that $\mathcal{D}_n \in (S)_J$. Then $\mathcal{D}_n \in (S)_J \subset (S)J$ so $T \in (\mathcal{D}_n) \subset (S)J$ contradicting the assumption $T \notin (S) \setminus (S)J$. So $\mathcal{D}_n \notin (S)_J$ for all $n \geq 1$. (Other choices of the diagonal sequences for the $D_n$’s are possible. Besides disjoint supports or “almost” disjoint supports, the only feature needed is bounded gaps between their nonzero entries.)

The set $X_p := \{\sum_{i} a_i D_i \mid \|a_i\|_p \leq \infty, a_i \in C \} \subset (S)_J$ for $1 \leq p \leq \infty$. The inclusion is because the $s$-number sequence $s(\sum_{i} a_i D_i) \leq \|a_i\|_\infty s(T)$ and because $(S)_J$ contains diag $s(T)$ and $(S)_J$ being a $B(H)$-ideal, is hereditary. Clearly $X_p$, with its canonical vector space structure, is a linear subspace of $(S)_J$. Under the natural projection map, the set of cosets of elements of $X_p$ in the quotient space $(S)_J/(S)_J$ is given by the linear subspace $X_p := \{\sum_{i} a_i D_i \mid \|a_i\|_p < \infty \}$. We first show that the map $\sum_{i} a_i D_i \to \sum_{i} a_i D_i$ is a one-to-one map, that is, each coset $\sum_{i} a_i D_i$ has a unique element of the form $\sum_{i} a_i D_i$. Indeed, if $\sum_{i} a_i D_i = \sum_{i} a_i D_i$, then $\sum_{i} (a_i - a_i) D_i = 0$. This implies $\sum_{i} (a_i - a_i) D_i \in (S)_J$ and $\sum_{i} (a_i - a_i) D_i \in (S)_J$. Suppose there exist $i_0$ such that $\sum_{i} a_i \neq a_i$. Since $(S)_J$ is a $B(H)$-ideal, multiplying $\sum_{i} (a_i - a_i) D_i$ by a suitable projection it follows that the diagonal operator $(a_i - a_i) D_i \in (S)_J$, and hence $D_i \in (S)_J$. But $(T) = (D_n) \subset (S)_J$ implying $T \in (S)_J$, again contradicting $T \notin (S)_J$. 

**SUBIDEALS OF OPERATORS II** 5
Therefore $a_i = a'_i$ for all $i \geq 1$. So the map $\sum a_i D_i \to [\sum a_i D_i]$ is a one-to-one map which is clearly linear, and therefore the map $(a_i) \to [\sum a_i D_i]$ is an isomorphism from $\ell^p$ onto $X'_p$. (In fact, it is straightforward to show that $||[\sum a_i D_i]|| := ||(a_i)||_p$ is a well-defined complete norm on $X'_p$ which establishes that this linear map is an isometric isomorphism under this induced norm on $X'_p$, but we will not exploit this isometric property.)

$\ell^p$ being an infinite-dimensional Banach space over the complex numbers, the cardinality of a Hamel basis for $\ell^p$ is at least $c$ ([3] Lemma 3.4). Then likewise a Hamel basis $B'$ for $X_p$ is at least $c$, since isomorphisms preserve Hamel bases. Since $X_p$ is a vector subspace of $(S)J/(S)J'$, every Hamel basis of a subspace can be extended to a Hamel basis of the full space and because the cardinality of all Hamel bases of a vector space is invariant, it follows that $|B'| \leq |B|$ for $B$ a Hamel basis of $(S)J/(S)J'$. Also since the generating set $S$ for $(S)J$ has cardinality strictly less than $c$, $|S| < c$ by Proposition 3.1. Therefore $c \leq |B'| \leq |B| < c$, a set theoretic contradiction. To sum up, this contradiction followed from assuming properness of the inclusion $(S)J \subseteq (S)$, that is, $(S)$ is $J$-soft.

Next we prove the first implication of the second assertion of this theorem, that is, if $I$ is a $B(H)$-ideal, then $(I)$ is $J$-soft. Following the same method as used above for $I = (S)J$, notice that the contradiction arose from assuming properness of the inclusion $(I)J \nsubseteq (I)$, that is, we showed there how the assumption of $(I)J \neq (I)$ led to an imbedding of $X_p'$ (an isometric isomorphic copy of $\ell^p$) into $X_p'$ without depending on cardinality of $S$, and yet still violating $\dim \frac{X_p}{X_p'} < c$.

$\iff$ From general ring theory, for $S = \{S_a\}_{a \in A}$: $(S)J = \bigcup_{\{a_1, \ldots, a_j\} \subseteq A} \{(S_{a_1})J + (S_{a_2})J + \cdots + (S_{a_j})J\}$

where $A$ is an index set with $|A| < c$. Using the algebraic structure for principal $J$-ideals implied by Theorem 2.7 (or for principal $J$-ideals in particular, see also [3] Proposition 4.2), one obtains $(S)J = \text{span} S + (S)J'$. Using Remark 2.3 for $(S)J'$, one obtains $(S)J = \text{span} S + \text{span} (SJ + SJ') + J(S)J$. Then $(S) \subseteq (S)J$ because $(S) = (S)J$ and the commutativity of $B(H)$-ideal multiplication implies $(S) = J(S)J$ [2] Sections 2.8, 4.3. But the minimality of $(S)J$ as a $J$-ideal containing $S$ implies $(S)J \subset (S)$, and so combining both inclusions, one obtains $(S) = (S)J$, that is, $(S)J$ is a $B(H)$-ideal.

Finally, we prove the second implication of the second assertion of this theorem, that is, if $(I)$ is $J$-soft, then $I$ is a $B(H)$-ideal. Notice that when $(I)$ is $J$-soft, one obtains $(I) = (I)J = J(I)J$, hence $I \subseteq J(I)J$. And $J(I)J \subset I$ because $I$ is a $J$-ideal. Combining both inclusions one obtains $I = J(I)J$ which is a $B(H)$-ideal.

Remark 4.2. In Theorem 4.4 only one implication (i.e., $(S)J$ is a $B(H)$-ideal $\Rightarrow (S)$ is $J$-soft) requires the cardinality of $S$ to be strictly less than the continuum. The reverse implication holds for arbitrary $S$.

Question 4.3. Is the codimension condition equivalent to the $J$-ideal $I$ possessing a generating set of cardinality less than $c$. (See also Section 6, Question 5.)

As a consequence of Theorem 4.4 we have

Theorem 4.4. (Structure theorem for $(S)J$ when $|S| < c$)
The algebraic structure of the ideal $(S)J$ generated by the set $S$ is given by

$(S)J = \text{span} S + JS + SJ + J(S)J$

$J(S)J$ is a $B(H)$-ideal, $\text{span} JS + SJ + J(S)J$ is a $J$-ideal, and $J(S)J \subset \text{span} JS + SJ + J(S)J \subset (S)J$.

This inclusion collapse to $J(S)J = (S)J$ if and only if $(S)$ is $J$-soft.

The fact that the cardinality of every nonzero $J$-ideal $I$ is $c$ (Remark 3.3) implies that generating sets for $I$ have at most $c$ elements. So in view of Theorem 4.4 the only $J$-softness cases left to investigate are: if $I$ cannot be generated by fewer than $c$ elements or (possibly more general, see Question 4.3) at least if the Hamel dimension of the quotient space $I/I^0$ is equal to $c$, does either of these imply $I$ is $J$-soft? Indeed, we show in Example 4.5 that Theorem 4.4 is the best possible result of its type by giving an example of a $J$-ideal that is also a $B(H)$-ideal which is not $J$-soft. By the contrapositive of Theorem 4.4 this $J$-ideal has no generating sets of cardinality less than $c$ and the Hamel dimension of its quotient $I/I^0$ is precisely $c$.

Example 4.5. Consider $J = K(H)$ and $I = \text{diag} (\frac{1}{n})$ the principal $B(H)$-ideal generated by the diagonal operator $\text{diag} (\frac{1}{n})$. Since every $B(H)$-ideal is a $J$-ideal, $I$ is also a $J$-ideal. We will show that the Hamel
dimension of the quotient space \(I/T^0\) is precisely \(c\), but yet \(I\) is not \(K(H)\)-soft. Indeed, \(T^0 = \text{span}(TK(H) + K(H)I) + K(H)I K(H)\) and one can show that \(\text{diag}(1/n) \notin T^0\) \[3\] Example 3.3]. In the proof of Theorem \[3\] taking \(T = \text{diag}(1/n)\), imbed \(X^p\) into \(I/T^0\), for any \(1 \leq p \leq \infty\). So the Hamel dimension of the quotient space \(I/T^0\) is at least \(c\). Since \(I\) is a nonzero \(J\)-ideal, the Hamel dimension of \(I\) is equal to \(c\) (Remark \[5\]), so the Hamel dimension of the quotient space \(I/T^0\) is at most \(c\). Therefore the Hamel dimension of \(I/T^0\) is equal to \(c\). But we know that \((\text{diag}(1/n))\) is not \((K(H))\)-soft \[3\] Example 3.3], but for completeness we repeat the proof here. If it were \(K(H)\)-soft, then \((\text{diag}(1/n)) = (\text{diag}(1/n))K(H)\) which further implies \(\langle 1/n \rangle \in \Sigma((\text{diag}(1/n))K(H))\), i.e., \(\langle 1/n \rangle = o(D_m(1/n))\) for some \(m \in \mathbb{N}\), contradicting \(\left(\frac{\langle 1/n \rangle}{D_m(1/n)}\right)_k \to \frac{1}{m}\) as \(k \to \infty\) where \(k = mj + r\).

5. Questions and results on \(J\)-ideals

In this section we address some of the questions posed in \[8\] Sections 6-7] and pose new questions.

The algebraic structure of a principal \(J\)-ideal generated by \(S\) is \((S)_J = CS + SJ + JS + J(S)J\). For idempotent \(B(H)\)-ideals \(J\) (i.e., \(J^2 = J\)), \(J(S)J = SJ + JS + J(S)J\) since \((J(S)J = (S)J^2 = (S)J\) and \(SJ, JS \subset J = J(S)\). So the algebraic structure of \((S)_J\) simplifies to \((S)_J = CS + J(S)J\). Is it necessary for \(J\) to be idempotent for \((J(S)J = SJ + JS + J(S)J)\) to hold? (This is related to \[8\] Remark 6.4, Section 7-Question 6): Find a necessary and sufficient condition for \((J(S)J = JS + SJ + J(S)J)\). The following example shows that \(J\) need not be idempotent for the equality to hold.

**Example 5.1.** For \(0 < p < \infty\), the Schatten \(p\)-ideal \(C_p\) is not idempotent because \(C_p^2 = C_{p/2} \neq C_p\). Moreover, for \(S = \text{diag}(1/n) \in C_p\) we claim that \(C_p(S)C_p = SC_p + C_pS + C_pC_p\) which will follow below from the \(C_p\)-softness of \((S)\). Indeed, for \(m > 2\), recall that \((D_m(s(S)))_n = s_{\left(\frac{1}{n}\right)}(S)\), so

\[
\left(\frac{1}{n}\right)_{D_m(s(S))} = \frac{(j+1)^{j+1}}{(mj+r)^j} \leq \left(\frac{2j}{mj}\right)^j \leq \frac{m}{m(m-2)^j} \leq \frac{1}{j^2},
\]

where \(n = mj + r\) and the roof function \(\left[\frac{1}{n}\right] = j\). By the hereditary property of \(\Sigma(C_p)\), \(s(S)_{D_m(s(S))} \in \Sigma(S)\) which further implies that \(s(S) = \frac{s(S)}{D_m(s(S))} = C_p(S)\) is \(C_p\)-softness of \((S)\), so \((S) = C_p(S)\). Because \(C_p^2 \subset C_p\), it follows that \(C_pS \subset C_p^2(S)C_p \subset C_p(S)C_p\) and similarly \(SC_p \subset C_p(S)C_p\). Therefore \(SC_p + C_pS \subset C_p(S)C_p\), hence \(C_p(S)C_p = SC_p + C_pS + C_p(S)C_p\).

Finishing this discussion on \(JS + SJ + J(S)J\), recall that in the case of a principal \(J\)-ideal \((S)_J\), \(JS + SJ + J(S)J\) is always a maximal \(J\)-ideal in \((S)_J\) \[8\] Remark 6.3]. The following example gives a partial answer to \[8\] Section 7, Question 3]: is \(JS + SJ + J(S)J\) always a principal \(J\)-ideal or is it always a non-principal \(J\)-ideal?

**Example 5.2.** For \(J = K(H)\) and \(S = \text{diag}(1/n)\),

\[
JS + SJ + J(S)J = K(H)\text{diag}(1/n) + \text{diag}(1/n)K(H) + K(H)(\text{diag}(1/n))K(H)
\]

We claim that this is not a principal \(K(H)\)-ideal.

Suppose \(JS + SJ + J(S)J = (T)_{K(H)}\). Since \((B(H)\)-ideals commute and \((K(H))^2 = K(H)\), one has \(K(H)(\text{diag}(1/n))K(H) = (\text{diag}(1/n))K(H)\) and \(K(H)\text{diag}(1/n) + \text{diag}(1/n)K(H) \subset (\text{diag}(1/n))K(H)\). Therefore \((T)_{K(H)} = (\text{diag}(1/n))K(H)\) hence \((T)_{K(H)}\) is the \((B(H)\)-ideal \(T\) \[8\] Theorem 2.6, for principal \(J\)-ideal see also Theorem 1.2]. Since \(T \in (\text{diag}(1/n))K(H)\), \(s(T) = O(D_m(1/n)\rho)\) for some \(\rho = \langle p_n \rangle \in c_0^*\) \[6\] Section 1, \[4\]1], and without loss of generality one can assume \(\rho_1 = 1\). Observe that the sequence

\[
\sum_{l=1}^{\infty} \frac{1}{2^l}(D_l(\rho^{1/l})) \in \Sigma(K(H)) = c_0^* (the cone of positive sequences decreasing to 0) because it is the c_0-norm limit of the sequences \(\sum_{l=1}^{n} \frac{1}{2^l}(D_l(\rho^{1/l})) as n \to \infty \) and \(c_0\) (the sequence space of complex numbers tending to 0) is norm-closed. So in particular, \(\sum_{l=1}^{\infty} \frac{1}{2^l}(D_l(\rho^{1/l})) \cdot \langle 1/n \rangle \in \Sigma(K(H)(\text{diag}(1/n))) = \Sigma((T)).\)
So \( \sum_{i=1}^{\infty} \frac{1}{2^i} (D_i(\rho^{1/i})) \cdot \left( \frac{1}{n} \right) \in \Sigma((T)) \), and expressing this in terms of s-numbers, for some \( k, m \in \mathbb{N} \) and all \( j = km + r \) for some \( i \) and some \( 0 \leq r < km \),

\[
\sum_{i=1}^{\infty} \frac{1}{2^i} (D_i(\rho^{1/i})) \cdot \left( \frac{1}{n} \right) = O(D_k s(T)) = O(D_k (D_m \left( \frac{1}{n} \right) \rho)) = O(D_{km} \left( \frac{1}{n} \right) D_k(\rho)),
\]

thereby contradicting, after setting arbitrary \( j = km + r \), for some \( i \) and \( 0 \leq r < km \),

\[
\left( \sum_{i=1}^{\infty} \frac{1}{2^i} (D_i(\rho^{1/i})) \right) \cdot \frac{1}{(D_{km} \left( \frac{1}{n} \right) D_k(\rho))} \geq \frac{1}{4} \frac{1}{D_{km} \left( \frac{1}{n} \right) D_k(\rho)} = \frac{1}{k m + r + 1} \rho_{i+1},
\]

which diverges to \( \infty \) as \( j \to \infty \) since \( km > 1 \) and \( \rho_i \to 0 \).

For principal \( B(H) \)-ideals \((S), (T)\), \((S) = (T)\) if and only if \( s(S) = O(D_m s(T)) \) and \( s(T) = O(D_k s(S)) \) for some \( m, k \in \mathbb{N} \). When \( s(S) \) or \( s(T) \) satisfies the \( \Delta_{1/2} \) condition, a simpler condition is: \((S) = (T)\) if and only if \( s(S) = O(s(T)) \) and \( s(T) = O(s(S)) \). (See \[8\] Section 7, Question 4.) The following proposition gives a necessary and sufficient condition for two principal \( J \)-ideals to be equal \[8\] Section 7, Question 4. We hope for a simpler condition.

**Proposition 5.3.** For \( S, T \in J \),

\[(S)_J = (T)_J \quad \text{if and only if} \quad aS + bT \in \{SJ + JS + J(S)J\} \cap \{TJ + JT + J(T)J\}\]

for some nonzero \( a, b \in \mathbb{C} \).

**Proof.** \( \Rightarrow \): Based on the algebraic structure of principal \( J \)-ideals implied by Theorem 2.7 (or for principal \( J \)-ideals in particular, see also \[8\] Proposition 2.3), \((S)_J = (T)_J\) if and only if

\[ S = \alpha T + AT + TB + \sum_{i=1}^{n} A_iTB_i \quad \text{and} \quad T = \beta S + CS + SD + \sum_{j=1}^{m} C_jSD_j \]

for some \( \alpha, \beta \in \mathbb{C} \), \( A, B, C, D, A_i, B_i, C_i, D_i \in J \). If \( \alpha = 0 \) and \( \beta \neq 0 \), then substituting \( S \) from the first equation in the second equation yields \( T \in (T)_J \) and then substituting \( T \) from the second equation in the first equation yields \( S \in (S)_J \). Therefore in this case \((T) = J(T)_J = (T)_J \) and \((S) = J(S)_J = (S)_J\) implied by Theorem 2.6 (or for principal \( J \)-ideals in particular, see also \[8\] Theorem 1.2). Since \( J(S)_J = (S)_J = (T)_J = J(T)_J \), one has \( S, T \in J(S)_J \cap J(T)_J \), and because \( J(S)_J \cap J(T)_J \) is a \( J \)-ideal, \( aS + bT \in \{SJ + JS + J(S)J\} \cap \{TJ + JT + J(T)J\} \) for all \( a, b \in \mathbb{C} \). The cases \( \alpha \neq 0 \) and \( \beta = 0 \) or \( \alpha = 0 \) and \( \beta \neq 0 \) are handled similarly. Assume finally that \( \alpha, \beta \neq 0 \). Substituting \( T \) from the second equation in the first equation one obtains \( S = \alpha \beta S + X \), where \( X \in (S)_J \). If \( \alpha \beta \neq 1 \), then \( S \in \{SJ + JS + J(S)J\} \), and then, substituting \( S \) from the first equation in the second equation, one obtains \( T = \alpha \beta T + Y \) where \( Y \in (T)_J \), so \( T \in \{TJ + JT + J(T)J\} \). This implies \( J \)-softness of \((S)\) and \((T)\), hence \( J(S)_J = (S)_J = (T)_J = J(T)_J \) and then as above \( aS + bT \in \{SJ + JS + J(S)J\} \cap \{TJ + JT + J(T)J\} \) for all \( a, b \in \mathbb{C} \). When \( \alpha = 1, \beta = \frac{1}{\alpha \beta} \), so substituting \( \beta \) in the second equation yields \( S - \alpha T = AT + TB + \sum_{i=1}^{m} A_iTB_i \)

and \( S - \alpha T = -\alpha (CS + SD + \sum_{j=1}^{m} C_jSD_j) \). Therefore \( S - \alpha T \in \{SJ + JS + J(S)J\} \cap \{TJ + JT + J(T)J\} \) which is the required condition.

\( \Leftarrow \): Suppose \( aS + bT \in \{SJ + JS + J(S)J\} \cap \{TJ + JT + J(T)J\} \) for some nonzero \( a, b \in \mathbb{C} \). Then

\[ aS + bT = AT + TB + \sum_{i=1}^{n} A_iTB_i \quad \text{and} \quad aS + bT = CS + SD + \sum_{j=1}^{m} C_jSD_j \]

for some \( \alpha, \beta \in \mathbb{C} \), \( A, B, C, D, A_i, B_i, C_i, D_i \in J \). From these equalities its clear that \((S)_J = (T)_J\). \( \square \)
Unlike $B(H)$-ideals, $J$-ideals do not necessarily commute as given in the following example.

**Example 5.4.** Consider $J = K(H)$ and with respect to the standard basis take $S$ to be the diagonal matrix $S := \text{diag}(1, 0, 1/2, 0, 1/3, 0, ...)$ and $T$ to be the weighted shift with this same weight sequence. We claim $(S)_{K(H)}(T)_{K(H)} \neq (T)_{K(H)}(S)_{K(H)}$. Indeed, suppose $(S)_{K(H)}(T)_{K(H)} = (T)_{K(H)}(S)_{K(H)}$. Then $TS - ST \in \text{K}(H)(\text{diag} \left( \frac{1}{n} \right))$. Since $TS = T$ and $ST = 0$, one has $T \in K(H)(\text{diag} \left( \frac{1}{n} \right))$. But $T \in K(H)(\text{diag} \left( \frac{1}{n} \right))$ if and only if $\text{diag} \left( \frac{1}{n} \right) \in K(H)(\text{diag} \left( \frac{1}{n} \right)) \subset K(H)(\text{diag} \left( \frac{1}{n} \right))$, the latter inclusion contradicts $\text{diag} \left( \frac{1}{n} \right) \notin K(H)(\text{diag} \left( \frac{1}{n} \right))$ (Example 4.3), so $T \notin K(H)(\text{diag} \left( \frac{1}{n} \right))$, a contradiction. Therefore $(S)_{K(H)}(T)_{K(H)} \neq (T)_{K(H)}(S)_{K(H)}$.

**QUESTIONS**

1. Determine which $J$-ideals have generating sets of cardinality less than $\mathfrak{c}$?

2. Do $J$-ideals have minimal cardinality generating sets?

3. Traces. $B(H)$-ideals are important in part because of the importance of their traces, unitarily invariant functionals on $B(H)$-ideals. Because $B(H)$-ideals $J$ contain no unitaries this concept does not apply. Is there a modification for which one obtains a useful concept of traces for $J$-ideals?

4. The study of commutators in $B(H)$-ideals is directly related to their traces. What can be said about the commutator structure of the $J$-ideals that can be used to motivate notions of traces for $J$-ideals?

5. Is the dimension of $I/I^0$ the cardinality of some generating set for $I$?

At least if dimension $I/I^0 < \mathfrak{c}$, must $I$ possess a generating set of cardinality less than $\mathfrak{c}$? (See also Question 1.3)

**REFERENCES**

[1] Blass, Andreas, *Existence of bases implies the axiom of choice*, Amer. Math. Soc., (1984), 31–33.

[2] Dykema, K., Figiel, T., Weiss, G., and Wodzicki, M., *The commutator structure of operator ideals*, Adv. Math., 185 (1), (2004), 1–79.

[3] Fong, C.K. and Radjavi, H., *On ideals and Lie Ideals of Compact Operators*, Math. Ann. 262, (1983), 23–28.

[4] Halbeisen, Lorenz and Hungerbühler, Norbert, *The cardinality of Hamel bases of Banach spaces*, East-West J. Math., (2000) 153–159.

[5] Kaftal, V and Weiss, G, *B(H) lattices, density and arithmetic mean ideals*, Houston J. Math., (2011), 233–283.

[6] Kaftal, V. and Weiss, G., *Soft ideals and arithmetic mean ideals*, Integral equations and Operator Theory, (2007), 363–405.

[7] Mityagin, B.S., *Normed ideals of intermediate type*, Amer. Math. Soc. Transl., (2) 63, (2000), 180-194.

[8] Patnaik, S and Weiss, G, *Subideals of Operators*, Journal of Operator Theory, to appear.

[9] Pietsch, Albrecht, *Operator ideals*, North-Holland Mathematical Library, North-Holland Publishing Co., 20 (1980).

University of Cincinnati, Department of Mathematics, Cincinnati, OH, 45221-0025, USA

E-mail address: sasmita_19@yahoo.co.in

University of Cincinnati, Department of Mathematics, Cincinnati, OH, 45221-0025, USA

E-mail address: gary.weiss@math.uc.edu