ON POSITIVE SOLUTIONS OF THE HOMOGENEOUS HAMMERSTEIN
INTEGRAL EQUATION

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Abstract. In this paper the existence and uniqueness positive fixed points of the one non-
linear integral operator are discussed. We prove that existence finite positive solutions of the
integral equation of Hammerstein type. Obtained results applied to study Gibbs measures
for models on a Cayley tree.

Key words. integral equation of Hammerstein type, fixed point of operator, Gibbs mea-
sure, Cayley tree.

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1. Introduction

It is well known that integral equations have wide applications in engineering, mechanics,
physics, economics, optimization, vehicular traffic, biology, queuing theory and so on (see
[15], [16], [19], [2], [17]). The theory of integral equations is rapidly developing with the help
of tools in functional analysis, topology and fixed point theory. Therefore, many different
methods are used to obtain the solution of the nonlinear integral equation. Moreover, some
methods can be found in Refs. [6], [10], [1], [21], [4], [7], [5], [3], to discuss an-
d and obtain the
solution of Hammerstein integral equation. In [7] J.Appell and A.S. Kalitvin used fixed
point methods and methods of nonlinear spectral theory to obtain the solution of integral
equations of Hammerstein or Uryson type. The existence of positive solutions of abstract
integral equations of Hammerstein type is discussed in [3]. In [3] M.A. Abdou, M.M. El-Borai
and M.M. El-Kojok the existence and uniqueness solution of the nonlinear integral equation
of Hammerstein type with discontinuous kernel are discussed.

This present paper, we study solvability homogeneous integral equation of Hammerstein
type. An integral equation of the form

\[ \int_{0}^{1} K(t, u)\Psi(t, f(u))du = f(t) \]  

(1.1)

is called the homogeneous Hammerstein integral equation, where \( K(t, u) \) is continuous real-
valued function defined on \( 0 \leq t \leq 0, \ 0 \leq u \leq 1, \ \Psi : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function
and \( f(t) \) is unknown function from \( C[0, 1] \).

Let \( \Psi(t, z), \frac{\partial}{\partial z}\Psi(t, z) \) be continuous and bounded for \( t \in [0, 1] \) and for all \( z \). Then [20]
the Hammerstein integral equation (1.1) has a solution. Assume that \( \Psi(t, z) \) is a bounded
continuous function for \( t \in [0, 1] \) and \( z \in \mathbb{R} \). In this case, also the Hammerstein integral
equation (1.1) has [18] a solution. For the necessary details of this theorem and for more
results on the Hammerstein integral equation, we refer to Petryshyn and Fitzpatrick \[22\], Browder \[9\], Brezis and Browder \[8\].

Recently, in \[11\] consider the case $\Psi(t, z) = \Psi(z)$. Let $\Psi(z)$ is monotone left-continuous function on $[0, +\infty)$ and $\lim_{x \to 0} \frac{\Psi(x)}{x} = +\infty$, $\lim_{x \to +\infty} \frac{\Psi(x)}{x} = 0$. Then \[11\] the integral equation of Hammerstein type \[1.1\] has a solution.

In this work, we will consider the following integral equation of Hammerstein type (i.e. in \[1.1\] $\Psi(t, z) = \Psi(z) = z$):

$$\int_{0}^{1} K(t, u)f^{\theta}(u)du = f(t), \quad \theta > 1$$

(1.2)

on the $C[0, 1]$, where $K(t, u)$ is strictly positive continuous function.

By the Theorem 44.8 from \[17\] follows the existence of nontrivial positive solution of the Hammerstein equation \[1.2\]. We study the problem of existence finite number positive solutions of the integral equation of Hammerstein type \[1.2\].

Consider the nonlinear operator $R_\alpha$ on the cone of positive continuous functions on $[0, 1]$:

$$(R_\alpha f)(t) = \left( \frac{\int_{0}^{1} K(t, u)f^{\theta}(u)du}{\int_{0}^{1} K(0, u)f^{\theta}(u)du} \right)^{\alpha},$$

(1.3)

where $K(t, u)$ is given in the integral equation of Hammerstein type \[1.1\] and $\alpha > 0$. Operator of the form \[1.3\] arising in the theory of Gibbs measures (see \[13\], \[12\], \[25\]). Positive fixed points of the operator $R_k$, $k \in \mathbb{N}$ and they numbers is very important to study Gibbs measures for models on a Cayley tree.

In \[13\], in the case $\alpha = 1$ the uniqueness positive fixed points of the nonlinear operator $R_\alpha \[1.3\]$ is proved. In \[12\], in the case $\alpha = k \in \mathbb{N}, k > 1$ for the nonlinear operator $R_\alpha$ was proved the existence of positive fixed point and the existence Gibbs measure for some mathematical models on a Cayley tree.

The aim of this work is to study the existence finite number positive solutions of the Hammerstein equation \[1.2\] on the space of continuous functions on $[0, 1]$. The plan of this paper is as follows. In the second section using properties of Hammerstein equation \[1.2\] we reduce some statements on the positive fixed point of the operator $R_\alpha$. In the third section we construct the strictly positive continuous kernel $K(t, u)$ such that, the corresponding Hammerstein equation \[1.2\] has $n \in \mathbb{N}$ positive solutions. In the fourth section obtained results for the operator $R_\alpha$ applied to study Gibbs measures for models on a Cayley tree.

2. Existence and uniqueness of positive fixed points of the operator $R_\alpha$

In this section we study the existence and the uniqueness positive fixed points of the nonlinear operator $R_\alpha \[1.3\]$. Put

$$C^+[0, 1] = \{ f \in C[0, 1]: f(x) \geq 0 \}, \quad C^+_0[0, 1] = C^+[0, 1] \ \setminus \ \{ \theta \equiv 0 \}. $$

Well then the set $C^+[0, 1]$ is the cone of positive continuous functions on $[0, 1]$. We define the Hammerstein operator $H_\theta$ on $C[0, 1]$ by the equality

$$H_\theta f(t) = \int_{0}^{1} K(t, u)f^{\theta}(u)du = f(t), \quad \theta > 1.$$
Clearly, that by the Theorem 44.8 from \[17\] we obtained

**Theorem 1.** Let \( \vartheta > 1 \). The equation

\[ H_\vartheta f = f \]  

has at least one solution in \( C^+_0[0, 1] \).

Put

\[ \mathcal{M}_0 = \{ f \in C^+[0, 1] : f(0) = 1 \}. \]

**Lemma 1.** Let \( \alpha > 1 \). The equation

\[ R_\alpha f = f, \ f \in C^+_0[0, 1] \]  

has a positive solution iff the Hammerstein operator has a positive eigenvalue, i.e. the Hammerstein equation

\[ H_\alpha g = \lambda g, \ f \in C^+[0, 1] \]  

has a positive solution in \( \mathcal{M}_0 \) for some \( \lambda > 0 \).

**Proof.** We define the linear operator \( W \) and the linear functional \( \omega \) on the \( C[0, 1] \) by following equalities

\[ (Wf)(t) = \int_0^1 K(t, u)f(u)du, \ \omega(f) = \int_0^1 K(0, u)f(u)du. \]

**Necessariness.** Let \( f_0 \in C^+_0[0, 1] \) be a solution of the equation (2.2). We have

\[ (Wf_0)(t) = \omega(f_0) \sqrt[\alpha]{f_0(t)}. \]

From this equality we get

\[ (H_\alpha h)(t) = \lambda_0 h(t), \]

where \( h(t) = \sqrt[\alpha]{f_0(t)} \) and \( \lambda_0 = \omega(f_0) > 0 \).

It is easy to see that \( h \in \mathcal{M}_0 \) and \( h(t) \) is an eigenfunction of the Hammerstein’s operator \( H_\alpha \), corresponding the positive eigenvalue \( \lambda_0 \).

**Sufficiency.** Let \( h \in \mathcal{M}_0 \) be an eigenfunction of the Hammerstein’s operator \( H_\alpha \). Then there is a number \( \lambda_0 > 0 \) such that \( H_\alpha h = \lambda_0 h \). From \( h(0) = 1 \) we get \( \lambda_0 = (H_\alpha h)(0) = \omega(h^\alpha) \).

Then

\[ h(t) = \left( \frac{(H_\alpha h)(t)}{\omega(h^\alpha)} \right)^{1/\alpha}. \]

From this equality we get

\[ R_\alpha f_0 = f_0 \]  

with \( f_0 = h^\alpha \in C^+_0[0, 1] \). This completes the proof. \( \square \)

**Theorem 2.** The equation (2.2) has at least one solution in \( C^+_0[0, 1] \).

Let \( \lambda_0 \) be a positive eigenvalue of the Hammerstein operator \( H_\alpha, \alpha > 1 \). Then there exists \( f_0 \in \mathcal{M}_0 \) such that \( H_\alpha f_0 = \lambda_0 f_0 \). Take \( \lambda \in (0, +\infty), \lambda \neq \lambda_0 \). Define function \( h_0(t) \in C^+_0[0, 1] \) by

\[ h_0(t) = \left( \frac{\lambda}{\lambda_0} \right)^{1/\alpha} f_0(t), \ t \in [0, 1]. \]
Then

\[ H_\alpha h_0 = H_\alpha \left( \frac{\alpha^{-1}}{\lambda} \sqrt{\frac{\lambda}{\lambda_0}} f_0 \right) = \lambda h_0, \]

i.e. the number \( \lambda \) is an eigenvalue of Hammerstein operator \( H_\alpha \) corresponding the eigenfunction \( h_0(t) \). It can be easily checked: if the number \( \lambda_0 > 0 \) is eigenvalue of the operator \( H_\alpha, \alpha > 1 \), then an arbitrary positive number is eigenvalue of the operator \( H_\alpha \). Therefore we have

**Lemma 2.** a) Let \( \alpha > 1 \). The equation \( R_\alpha f = f \) has a nontrivial positive solution iff the Hammerstein equation \( H_\alpha g = g \) has a nontrivial positive solution.

Let \( \alpha > 1 \). Denote by \( N_{\text{fix}.p}(H_\alpha) \) and \( N_{\text{fix}.p}(R_\alpha) \) numbers of nontrivial positive solutions of the equations (2.1) and (2.2), respectively.

**Theorem 3.** Let \( \alpha > 1 \). The equality \( N_{\text{fix}.p}(H_\alpha) = N_{\text{fix}.p}(R_\alpha) \) is held.

Denote

\[
\begin{align*}
m &= \min_{t,u\in[0,1]} K(t,u), & M_0 &= \max_{u\in[0,1]} K(0,u), \\
M &= \max_{t,u\in[0,1]} K(t,u), & m_0 &= \min_{u\in[0,1]} K(0,u).
\end{align*}
\]

**Theorem 4.** Let \( \alpha > 1 \). If the following inequality holds

\[
\left( \frac{M}{m_0} \right)^\alpha - \left( \frac{m}{M_0} \right)^\alpha < \frac{1}{\alpha}
\]

then the homogenous Hammerstein equation (2.1) and the equation (2.2) has unique nontrivial positive solution.

Analogous theorem was proved for \( \alpha = k \in \mathbb{N}, k \geq 2 \) in [12] and proof of the Theorem 4 is analogously to its.

3. **Existence finite positive solutions of homogeneous Hammerstein equation**

In this section we’ll show the existence of \( n \in \mathbb{N} \) positive solutions of homogeneous integral equation of Hammerstein type (1.2).

For all \( p, n \in \mathbb{N} \) we define following matrices:

\[
A^{(p)}_n = \left\{ \frac{1}{2(2p+i+j) - 3} \left( \frac{1}{2} \right)^{2(2p+i+j-2)} \right\}_{i,j=1,n}, \quad n, p \in \mathbb{N}.
\]

\[
B[a_1, \ldots, a_n; b_1, \ldots b_n] = \left( \frac{1}{a_i + b_j} \right)_{i,j=1,n}, \quad a_i, b_j > 0.
\]

\[
C^{(p)}_n = B[4p, 4(p+1), \ldots, 2(p+n-1); 1, 5, \ldots, 4n-3].
\]
Lemma 3. \[24\] Let \( n \geq 2 \). Then
\[
\det B[a_1, \ldots, a_n; b_1, \ldots, b_n] = \frac{\prod_{1 \leq i < j \leq n} [(a_i - a_j)(b_i - b_j)]}{\prod_{i,j=1}^{n} (a_i + b_j)}
\]

Corollary 1. \( \det A_n^{(p)} = \left(\frac{1}{2}\right)^{2n(2p+n-1)} \det C_n^{(p)} \).

Proof. Let \( i, j = \{1, n\} \). We multiply by \( 2^{2(p+j-1)} \) the column \( j \) of the matrix \( A_n^{(p)} \) and then multiply by \( 2^{2(i-1)} \) the row \( i \) of the matrix obtained. As a result we get \( C_n^{(p)} \).

Lemma 4. Let \( B^{-1}[a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n] = \{\beta_{ij}\}_{i,j=1,n} \) is an inverse matrix of \( B[a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n] \). Then
\[
\beta_{ji} = \frac{\prod_{s=1}^{n} (a_s + b_j) \prod_{s=1, s \neq i}^{n} (a_s + b_i)}{\prod_{s=1, s \neq j}^{n} (b_j - b_s) \prod_{s=1, s \neq i}^{n} (a_i - a_s)}
\]

Proof. Subtracting the \( j \)th column of \( B[a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n] \) from every other column we get a following equality
\[
\det B[a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n] = \frac{\prod_{s=1, s \neq j}^{n} (b_j - b_s)}{\prod_{s=1}^{n} (a_s + b_j)} \left( \begin{array}{cccc}
\frac{1}{a_1 + b_1} & \ldots & \frac{1}{a_1 + b_{j-1}} & 1 \\
\frac{1}{a_2 + b_1} & \ldots & \frac{1}{a_2 + b_{j-1}} & 1 \\
\vdots & & \vdots & \vdots \\
\frac{1}{a_n + b_1} & \ldots & \frac{1}{a_n + b_{j-1}} & 1
\end{array} \right) .
\]

Now we subtract from the \( j \)th row the \( i \)th row for every \( j \in \{1, 2, \ldots, i-1, i+1, \ldots, n\} \). Then
\[
\det B[a_1, \ldots, a_n; b_1, \ldots, b_n] = \frac{\prod_{s=1, s \neq j}^{n} (b_j - b_s) \prod_{s=1, s \neq i}^{n} (a_i - a_s)}{\prod_{s=1}^{n} (a_s + b_j) \prod_{s=1, s \neq i}^{n} (a_i + b_s)} \times \det B^{(i,j)}[a_1, \ldots, a_n; b_1, \ldots, b_n]
\]

where \( B^{(i,j)}[a_1, \ldots, a_n; b_1, \ldots, b_n] \) is the cofactor of the element \( \frac{1}{a_i + a_j} \) in \( B[a_1, \ldots, a_n; b_1, \ldots, b_n] \). Since
\[
\beta_{ji} = \frac{\det B^{(i,j)}[a_1, \ldots, a_n; b_1, \ldots, b_n]}{\det B[a_1, \ldots, a_n; b_1, \ldots, b_n]}
\]

This completes the proof.

Let be
\[
(A_n^{(p)})^{-1} = \{a_{ij}\}_{i,j=1,n}.
\]
Remark 1. For each \( \alpha_{ji} \) element of \((A_n^{(p)})^{-1}\) the following equality holds

\[
\alpha_{ji} = 4^{2p+i+j-n+1} \frac{\prod_{s=1}^{n}(4p + 2s + 2j - 3) \prod_{s=1, s \neq j}^{n}(4p + 2s + 2j - 3)}{\prod_{s=1, s \neq j}^{n}(j - s) \prod_{s=1, s \neq i}^{n}(i - s)}
\]

Proof. By Corollary 1 and Lemma 4 we get

\[
\alpha_{ji} = 4^{2p+i+j} \frac{\det B^{(i,j)}[4p, 4p + 2, ...4p + 2(n-1); 1, 3, ...2n-1]}{\det B[4p, 4p + 2, ...4p + 2(n-1); 1, 3, ...2n-1]} = 4^{2p+i+j} \frac{\prod_{s=1}^{n}(4p + 2s + 2j - 3) \prod_{s=1, s \neq i}^{n}(4p + 2s + 2i - 3)}{\prod_{s=1, s \neq j}^{n}(2i - 2s) \prod_{s=1, s \neq i}^{n}(2i - 2s)}.
\]

Denote

\[
\varphi_{(s, n, p)}(u) = \alpha_{s1}u^{2p-1} + ... + \alpha_{sn}u^{2(n+p)-3}, \quad s, n, p \in \mathbb{N}, \ u \in [0, 1].
\]

\[
K_{(n, p)}(t, u; k) = 1 + \sum_{s=1}^{n} \left( \frac{\sqrt[4]{1 + t^{2(p+s)-1} - 1}}{\varphi_{(s, n, p)}(u)} \right), \quad k \in \mathbb{N}, k \geq 2, \ t, u \in [0, 1].
\]

Remark 2. For the given \( k \in \mathbb{N}, k \geq 2 \) the following inequality holds

\[
K_{(n, p)}\left( t - \frac{1}{2}, u - \frac{1}{2}; k \right) \leq K_{(n, 1)}\left( t - \frac{1}{2}, u - \frac{1}{2}; k \right), \quad (t, u) \in [0, 1]^2, n, p \in \mathbb{N}.
\]

Put

\[
\zeta_0(n) = \frac{64}{9} \cdot \frac{4^n - 1}{4n + 1} \left( \frac{(4n + 1)!!}{(n - 1)!(2n + 1)!!} \right)^2.
\]

Lemma 5. Let \( n \in \mathbb{N} \). If \( k \geq \zeta_0(n) \) then the following inequality holds

\[
K_{(n, p)}\left( t - \frac{1}{2}, u - \frac{1}{2}; k \right) > 0, \quad (t, u) \in [0, 1]^2, p \in \mathbb{N}.
\]

Proof. For \( p = 1 \) from Remark 1 we have

\[
\alpha_{ij} = 4^{i+j-n+3} \frac{\prod_{s=1}^{n}(2i + 2s + 1) \prod_{s=1, s \neq j}^{n}(2j + 2s + 1)}{\prod_{s=1, s \neq i}^{n}(i - s) \prod_{s=1, s \neq j}^{n}(j - s)}.
\]

Then

\[
\left| \frac{\alpha_{i,j+1}}{\alpha_{i,j}} \right| = \frac{4(4j + 1)(2j + 2n + 3)}{(2j + 3)(4j + 5)}, \quad i = \overline{1, n}, \ j = \overline{1, n - 1}
\]

and
Hence that Hammerstein integral equation (1.2) has $n$ positive solutions.

**Theorem 5.** Let $g$ be a positive continuous kernel. If $k \geq \zeta_0(n)$ one get $K_{(n,p)}(t - \frac{1}{2}, u - \frac{1}{2}, k) > 0$. This completes the proof.

**Proposition 1.** Let $n \in \mathbb{N}$. If $k \geq \zeta_0(n)$ then the Hammerstein's nonlinear operator $H_k$ with the kernel $K_{(n,p)}(t - \frac{1}{2}, u - \frac{1}{2}, k) \ (p \in \mathbb{N})$ has at least $n$ positive fixed points.

**Proof.** Let $f_j(u) = \sqrt{1 + u^{2(p+j)-1}}, \ j = 1, n$ and $u_1 = u - \frac{1}{2}, t_1 = t - \frac{1}{2}$. Put $g_j(t) = f_j(t - \frac{1}{2})$. We are showing functions $g_j(t)$ are fixed points of the Hammerstein operator $H_k$ with the kernel $K_{(n,p)}(t - \frac{1}{2}, u - \frac{1}{2}, k)$:

$$
\int_0^1 K_{(n,p)}(t - \frac{1}{2}, u - \frac{1}{2}, k) g_j^k(u) du =
$$

$$
= \int_0^1 K_{(n,p)}(t - \frac{1}{2}, u - \frac{1}{2}, k) f_j^k(u - \frac{1}{2}) du
= \int_{-\frac{1}{2}}^{\frac{1}{2}} K_{(n,p)}(t_1, u_1; k) f_j^k(u_1) du_1 =
$$

$$
\int_{\frac{1}{2}}^{\frac{1}{2}} \left[ 1 + \sum_{s=1}^{n} \left( \sqrt{1 + t_1^{2(p+s)-1}} - 1 \right) \varphi(s,n,p)(u_1) \right] \left( 1 + u_1^{2(p+j)-1} \right) du_1 =
$$

$$
1 + \sum_{s=1}^{n} \left( \sqrt{1 + t_1^{2(p+s)-1}} - 1 \right) \int_{\frac{1}{2}}^{\frac{1}{2}} \left( \alpha_{s1} u_1^{4(p-1)+2s+2j} + \ldots + \alpha_{sn} u_1^{4(p+s+j+n)-6} \right) du_1 =
$$

$$
1 + \sum_{s=1}^{n} \left( \sqrt{1 + t_1^{2(p+s)-1}} - 1 \right) \left( \alpha_{s1} \beta_{s1} + \ldots + \alpha_{nj} \beta_{sn} \right) = \sqrt{1 + t_1^{2(p+j)-1}}.
$$

Hence

$$
\int_0^1 K_{(n,p)}(t - \frac{1}{2}, u - \frac{1}{2}; k) g_j^k(u) du = g_j(t), \ j \in \{1, 2, ..., n\}.
$$

**Theorem 5.** For each $n \in \mathbb{N}$ there exists $\vartheta > 1$ and a positive continuous kernel $K(t, u)$ such that Hammerstein integral equation (1.2) has $n$ positive solutions.
4. **Gibbs measures for models on Cayley tree** $\Gamma^k$

In this section we study Gibbs measures for models on Cayley tree. You may be acquainted with definitions and properties of Gibbs measures in books [14], [23], [25]. A Cayley tree (Bethe lattice) $\Gamma^k$ of order $k \in \mathbb{N}$ is an infinite homogeneous tree, i.e., a graph without cycles, such that exactly $k + 1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where $V$ is the set of vertices and $L$ that of edges (arcs). Two vertices $x$ and $y$ are called nearest neighbors if there exists an edge $l \in L$ connecting them. We will use the notation $l = (x, y)$. A collection of nearest neighbor pairs $(x, x_1), (x_1, x_2), \ldots (x_{d-1}, y)$ is called a path from $x$ to $y$. The distance $d(x, y)$ on the Cayley tree is the number of edges of the shortest path from $x$ to $y$.

For a fixed $x^0 \in V$, called the root, we set

$$W_n = \{x \in V | d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^{n} W_n$$

and denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, x \in W_n,$$

the set of direct successors of $x$.

Consider models where the spin takes values in the set $[0, 1]$, and is assigned to the vertexes of the tree. For $A \subset V$ a configuration $\sigma_A$ on $A$ is an arbitrary function $\sigma_A : A \to [0, 1]$. Denote $\Omega_A = [0, 1]^A$ the set of all configurations on $A$ and $\Omega = [0, 1]^V$. The Hamiltonian on $\Gamma^k$ of the model is

$$H(\sigma) = -J \sum_{(x, y) \in L} \xi(\sigma(x), \sigma(y)), \quad \sigma \in \Omega$$  \hspace{1cm} \text{(4.1)}$$

where $J \in \mathbb{R} \setminus \{0\}$ and $\xi : (u, v) \in [0, 1]^2 \to \xi_{u,v} \in \mathbb{R}$ is a given bounded, measurable function.

Let $\lambda$ be the Lebesgue measure on $[0, 1]$. On the set of all configurations on $A$ the a priori measure $\lambda_A$ is introduced as the $|A|$ fold product of the measure $\lambda$. Here and further on $|A|$ denotes the cardinality of $A$. We consider a standard sigma-algebra $\mathcal{B}$ of subsets of $\Omega = [0, 1]^V$ generated by the measurable cylinder subsets.

Let $\sigma_n : x \in V_n \mapsto \sigma_n(x)$ is a configuration in $V_n$ and $h : x \in V \mapsto h_x = (h_{t,x}, t \in [0, 1]) \in \mathbb{R}^{[0,1]}$ be mapping of $x \in V \setminus \{x^0\}$. Given $n = 1, 2, \ldots$, consider the probability distribution $\mu^{(n)}$ on $\Omega_{V_n}$ defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp \left( -\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x} \right).$$ \hspace{1cm} \text{(4.2)}$$

Here, as before, $\sigma_n : x \in V_n \mapsto \sigma(x)$ and $Z_n$ is the corresponding partition function:

$$Z_n = \int_{\Omega_{V_n}} \exp \left( -\beta H(\bar{\sigma}_n) + \sum_{x \in W_n} h_{\bar{\sigma}(x),x} \right) \lambda_{V_n}(\bar{\sigma}_n).$$ \hspace{1cm} \text{(4.3)}$$

where $\beta = T^{-1}, T > 0$—temperature. The probability distributions $\mu^{(n)}$ are compatible \cite{13} if for any $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$: 
\[ \int_{\Omega_{V_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n)\lambda_{W_n}(d(\omega_n)) = \mu^{(n-1)}(\sigma_{n-1}). \] (4.4)

Here \( \sigma_{n-1} \vee \omega_n \in \Omega_{V_n} \) is the concatenation of \( \sigma_{n-1} \) and \( \omega_n \). In this case there exists \[13\] a unique measure \( \mu \) on \( \Omega_V \) such that, for any \( n \) and \( \sigma_n \in \Omega_{V_n} \), \( \mu\left(\sigma|_{V_n} = \sigma_n\right) = \mu^{(n)}(\sigma_n) \).

The measure \( \mu \) is called splitting Gibbs measure corresponding to Hamiltonian \((4.1)\) and function \( x \mapsto h_x, x \neq x^0 \).

The following statement describes conditions on \( h_x \) guaranteeing compatibility of the corresponding distributions \( \mu^{(n)}(\sigma_n) \).

**Proposition 2.** \[13\] The probability distributions \( \mu^{(n)}(\sigma_n), n = 1, 2, \ldots, \) in \((4.2)\) are compatible iff for any \( x \in V \setminus \{x^0\} \) the following equation holds:

\[ f(t, x) = \prod_{y \in S(x)} \int_0^1 \exp(J \beta \xi_{t,u}) f(u, y) du \int_0^1 \exp(J \beta \xi_{0,u}) f(u, y) du. \] (4.5)

Here, and below \( f(t, x) = \exp(h_{t,x} - h_{0,x}), t \in [0, 1] \) and \( du = \lambda(du) \) is the Lebesgue measure.

We consider \( \xi_{tu} \) as a continuous function and we are going to solve equation \((4.5)\) in the class of translation-invariant functions \( f(t, x) \) (i.e. \( f(t, x) = f(t) \) for all \( x \in \Gamma_k \setminus \{x_0\} \)) and we’ll show that there exists a finite number of translation-invariant Gibbs measures for model \((4.1)\).

Then for translation-invariant functions the equation \((4.5)\) can be written as

\[ (R_k f)(t) = f(t), \quad k \in \mathbb{N} \] (4.6)

where \( K(t, u) = Q(t, u) = \exp(J \beta \xi_{tu}), \quad f(t) \in C_0^+[0, 1], \quad t, u \in [0, 1] \) (see \[13\], \[12\]).

Consequently, for each \( k \in \mathbb{N}, k \geq 2 \) Hammerstein integral equation corresponding to the equation \((4.6)\) has the following form:

\[ \int_0^1 Q(t, u)f^k(u)du = f(t). \] (4.7)

By Theorem \[3\] and Propositions \[1\], \[2\] we’ll get following Theorem.

**Theorem 6.** Let \( n \in \mathbb{N} \). If \( k \geq \zeta_0(n) \) then the model

\[ H(\sigma) = -\frac{1}{\beta} \sum_{<x,y>} \ln \left( K_{(n,p)} \left( \sigma(x) - \frac{1}{2}, \sigma(y) - \frac{1}{2}; k \right) \right), \quad \sigma \in \Omega, (p \in \mathbb{N}) \]

on the Cayley tree \( \Gamma^k \) has at least \( n \) translation-invariant Gibbs measures.
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