SPECTRAL THEOREMS FOR POSITIVE ALGEBRA HOMOMORPHISMS

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ABSTRACT. Let \( X \) be a locally compact Hausdorff space, let \( \mathcal{A} \) be a partially ordered algebra, and let \( \pi : \mathcal{C}(X) \to \mathcal{A} \) be a positive algebra homomorphism. Under conditions on \( \mathcal{A} \) that are satisfied in a good number of cases of practical interest, it is shown that \( \pi \) is represented by a unique regular spectral measure \( \mu \) on the Borel \( \sigma \)-algebra of \( X \), taking its values in the positive idempotents in \( \mathcal{A} \). The measure \( \mu \), which is \( \sigma \)-additive in an ordered sense, represents \( \pi \) via the order integral (a generalisation of the Lebesgue integral) that goes back to J.D.M. Wright and which was investigated earlier by the authors.

The positive algebra homomorphism \( \pi \) can be extended from \( \mathcal{C}(X) \) to a positive linear map from the accompanying \( L^1 \)-space of \( \mu \) into \( \mathcal{A} \). It is shown that, quite often, this \( L^1 \)-space is closed under multiplication, so that it is a vector lattice algebra, and that the extended map from \( L^1 \) into \( \mathcal{A} \) is not only an algebra homomorphism but, even when \( \mathcal{A} \) is not a vector lattice, also a vector lattice homomorphism in a sense that is explained in the paper. When \( \mathcal{A} \) has the countable sup property, the image of \( L^1 \) (or of its positive cone) is described in terms of consecutive ups and downs of the image of \( \mathcal{C}(X) \) (or of its positive cone).

The general results are applied in three different contexts, showing how various spectral theorems have a common order-theoretical root.

1. INTRODUCTION AND OVERVIEW

In this paper, we take a direct, order-theoretical, approach to aspects of spectral theory. For comparison, we start by briefly reviewing some known facts.

Let \( X \) be a compact Hausdorff space, and suppose that \( \pi : \mathcal{C}(X; \mathbb{C}) \to \mathcal{B}(H) \) is a representation of its continuous complex-valued functions on a complex Hilbert space \( H \). For \( x, x' \in H \), the Riesz representation theorem furnishes a regular complex Borel measure \( \mu_{x,x'} \) on the Borel \( \sigma \)-algebra \( \mathcal{B} \) of \( X \) such that

\[
\langle \pi(f)x, x' \rangle = \int_X f \, d\mu_{x,x'}
\]

for \( f \in \mathcal{C}(X; \mathbb{C}) \). Turning the tables, equation (1.1) can be used as a definition to extend \( \pi \) to a map \( \pi' : \mathcal{B}(X, \mathcal{B}; \mathbb{C}) \to \mathcal{B}(H) \) from the bounded Borel measurable functions on \( X \) into \( \mathcal{B}(H) \). This extended map can then be shown to be
a representation again, and a spectral measure $\mu$ on $B$ is obtained by setting $\mu(\Delta) := \pi(\chi_\Delta)$ for $\Delta \in B$. It is such that
\begin{equation}
\mu_{x,x'}(\Delta) = \langle \mu(\Delta)x, x' \rangle
\end{equation}
for $x, x' \in H$. In this fashion, the existence of a (unique regular) spectral measure is established that generates the representation $\pi$. It is uniquely determined by equations (1.1) and (1.2), and the validity of these two equations is what is meant by writing
\begin{equation}
\pi(f) = \int_X f \, d\mu
\end{equation}
for $f \in C(X; \mathbb{C})$. We refer to, for example, [7, Section IX.1] or [13, Section 1.4] for this well-known material. A similar approach is used in [10] to find a spectral measure for a positive representation $\pi: C_0(X) \to \mathcal{L}_r(E)$ on a KB-space $E$ of the real-valued functions on a locally compact Hausdorff space $X$ vanishing at infinity.

In the present paper, we take a different, direct, approach. Suppose that $\pi: C_c(X) \to A$ is a positive algebra homomorphism from the real-valued compactly supported continuous functions on a locally compact Hausdorff space into a partially ordered algebra $A$. When $A$ satisfies reasonably mild conditions, which are satisfied in a number of cases of practical interest, we show that there exists a unique regular spectral measure $\mu$ on the Borel $\sigma$-algebra of $X$, taking its values in the positive idempotents in $A$ and $\sigma$-additive in an ordered sense, such that
\begin{equation}
\pi(f) = \int_X f \, d\mu
\end{equation}
for $f \in C_c(X)$. The integral in equation (1.4) is now not a symbolic notation as in equation (1.3), but it is an actual integral that can be defined for measures taking values in (suitable) partially ordered vector spaces. This order integral, which is a generalisation of the Lebesgue integral, is studied in detail in [8]. It goes back to J.D.M. Wright.

The algebra $A$ need not be an algebra of operators but, if it is, then, starting from the measure $\mu$, one can define measures $\mu_{x,x'}$ as in equation (1.2). The validity of equation (1.1) then follows easily from the properties of the order integral. The algebra of self-adjoint operators in a commutative strongly closed $C^*$-subalgebra of $B(H)$ for a complex Hilbert space $H$ satisfies the appropriate conditions, and so does the algebra of regular operators on a KB-space $E$. Consequently, in these cases, our spectral measures for these algebras coincide with the ones that are found via the classical ‘weak’ method summarised above. Our purely order-theoretical method via general partially ordered algebras is, however, essentially different in nature as the spectral measure comes before the Borel functional calculus and the scalar valued measures. The difference (and advantage) becomes especially clear in the context of JBW-algebras. As these

\[1\] As will become apparent in Section 7.1, one can actually do better than that.
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need not be algebras of operators, measures \( \mu_{x,x'} \) as above make no sense then. Yet our results are applicable, and they have the existence of a Borel functional calculus and the spectral resolution for an element of a JBW-algebra as an easy consequence.

The order-theoretical spectral theorems in the current paper are consequences of the Riesz representation theorems in \([8]\) for positive linear maps \( \pi \) from \( C_c(X) \) or \( C_0(X) \) into (suitable) partially ordered vector spaces. These apply, in particular, to the positive algebra homomorphisms from these spaces into \( \mathcal{A} \). The fact that the representing \( \mathcal{A} \)-valued measure is then actually a spectral measure is a consequence of the multiplicativity of \( \pi \) and the explicit formulas from \([8]\) for the representing measure for the positive linear map \( \pi \). As for the special case of the classical Riesz representation theorem, where the partially ordered vector space consists of the real numbers, we have

\[
\mu(V) = \bigvee \{ \pi(f) : f \in C_c(X), 0 \leq f \leq 1, \text{supp} f \subseteq V \}
\]

for an open subset \( V \) of \( X \), and

\[
\mu(K) = \bigwedge \{ \pi(f) : f \in C_c(X), 0 \leq f \leq 1, f(x) = 1 \text{ for } x \in K \}
\]

for a compact subset \( K \) of \( X \). In the classical real case, these formulas tend to fade into the background once they have served their purpose during the proof of the Riesz representation theorem. In the present paper, however, they are very much in the foreground as they underlie the spectral property of \( \mu \) when \( \pi \) is multiplicative. They are also instrumental to the up-down theorems that we shall establish. Even in the well-studied case of representations on complex Hilbert spaces these formulas appear to yield something new. For this, we recall that the existence of the extremum of a monotone net of self-adjoint operators in \( B(H) \) and the existence of its strong operator limit are equivalent and that, when they exist, they are equal. In this case, therefore, the supremum (resp. infimum) in equation (1.5) (resp. equation (1.6)) gives the spectral measures of non-empty open subsets and of compact subsets as explicit strong operator limits. Similar remarks apply to the regular operators on a Banach lattice with an order continuous norm.

This paper is organised as follows.

Section 2 contains the basic notion, definitions, and conventions, as well as a few preparatory results. Of particular relevance are the partially ordered algebras in Proposition 2.11, Corollary 2.12, Proposition 2.13, and Proposition 2.14. These are commonly occurring algebras to which the basic spectral theorem, Theorem 6.4, applies. In Section 2.4, we introduce a terminology to express that a linear map between two partially ordered vector spaces behaves to some extent as a vector lattice homomorphism. We say that such maps preserve moduli. The necessary material from \([3]\) concerning the order integral is summarised in Section 2.5 and Section 2.6 contains the definition of spectral measures.

\[2\] See Remark 7.8 for further comments.
in the general context. We also include an embedding result for \(C_0(X)\) for which we are not aware of a reference; see Proposition 2.31.

Section 3 contains the proof that the representing measure for a positive algebra homomorphism is spectral; see Theorem 3.6. A finitely additive measure on an algebra of sets, with values in an algebra without additive 2-torsion, and with the property that its image consists of commuting idempotents, is automatically a spectral measure; see Proposition 3.1. Based on this, the spectrality of the representing measure is a surprisingly easy of equation (1.5).

Section 4 is mostly intended as a preparation for Section 5 but it also has a value of its own. Its starting point is a measure \(\mu\) with values in a vector lattice or in a partially ordered algebra. There is an associated map \(I_\mu\)—defined by the order integral—from the corresponding \(L^1\)-space into the vector lattice or partially ordered algebra. When is \(I_\mu\) a vector lattice homomorphism, or an algebra homomorphism? After answering these questions, we proceed to show that, when \(I_\mu\) is an algebra homomorphism (which is the case if and only if \(\mu\) is spectral), it is, properly interpreted if necessary, also a vector lattice homomorphism; see Theorems 4.4 to 4.6.

Section 5 is not concerned with a positive algebra homomorphism, but, more generally, with a positive linear map \(\pi\) into a partially ordered vector space. Using equations (1.5) and (1.6) and the regularity of the representing measure, it is not too difficult to see that various images of the associated operator \(I_\mu\) are contained in consecutive ups and downs of the image of (the positive cone of) \(C_0(X)\) under \(\pi\). When the codomain has the countable sup property and \(I_\mu\) preserves moduli, inclusions can be improved to equalities and net ups and downs can be replaced by their sequential counterparts. The monotone convergence theorem for the order integral (see [8, Theorem 6.9]) is the key to this.

In Section 6, we put the pieces together. It contains two spectral theorems for positive algebra homomorphisms. In the first one, the codomain is a Banach lattice algebra with an order continuous norm. Algebras of operators will only rarely fall into this category, but they do tend to be in the range of the second theorem, where the codomain is a partially ordered algebra. We have made some effort to collect all relevant results from [8,9] and the present paper in these two theorems, which are the focal points of this paper.

In Section 7, we apply the general theory to positive representations of \(C_0(X)\) on Banach lattices and, via its restriction to \(C_0(X)\), to representations of the complex algebra \(C_0(X;\mathbb{C})\) on Hilbert spaces. In the first case, a significant extension of the results in [10] is obtained. In the second case, we obtain a spectral theorem for (possibly) degenerate representations of \(C_0(X;\mathbb{C})\) that appears to be more complete than is to be found in the literature. When combined with [11, Theorem 2.4.4], our results on ups and downs from Section 5 imply that, under a condition that is satisfied for separable Hilbert spaces, the image of the Borel functional calculus for a (possibly degenerate) representation of \(C_0(X;\mathbb{C})\) equals the strongly closed subalgebra that is generated by the image

\(^3\text{As Section 2.2 shows, this is, in practice, rather often the case.}\)
of $C_0(X; \mathbb{C})$. Although material in this direction exists, we are not aware of a reference where this result in our generality is actually proved.\footnote{See Remark 7.10 for further comments.}

Section 8 is concerned with JBW-algebras. As mentioned above, the classical ‘weak’ approach to spectral theorems is not applicable here. Still, our methods apply. The introduction of a spectral measure in this context, which appears to be new, simplifies the picture and, as we believe, gives a better understanding.

In the first part of the 20th century, spectral theorems for hermitian and normal operators on a Hilbert space were developed using spectral resolutions. Later, these were seen to be consequences of more general results on representations of unital commutative C$^*$-algebras. Our approach in Section 8 is the analogue of this for JBW-algebras.

2. Preliminaries

In this section, we collect the necessary notation, definitions, conventions, and preliminary results.

All vector spaces are over the real numbers, unless otherwise indicated. Operators between two vector spaces are always supposed to be linear. An algebra homomorphism between two unital associative algebras need not be unital.

When $E$ is a partially ordered set, we shall employ the usual notation in which $a_\lambda \uparrow$ means that $\{a_\lambda\}_{\lambda \in \Lambda}$ is an increasing net in $E$, and in which $a_\lambda \uparrow x$ means that $\{a_\lambda\}_{\lambda \in \Lambda}$ is an increasing net in $E$ with supremum $x$ in $E$. The notations $a_\lambda \downarrow$ and $a_\lambda \downarrow x$ are similarly defined.

Suppose that $S$ is a non-empty subset of a partially ordered set $E$. Then we shall say that $S^\vee$ exists in $E$ when the supremum of finitely many arbitrary elements of $S$ exists in $E$. In that case, we let $S^\vee$ denote the set consisting of all suprema of finitely many arbitrary elements of $S$. There are a similar definition and notation $S^\wedge$ for infima.

When $S$ is a subset of a set $X$, $\chi_S$ denotes its indicator function; we write $0$ for $\chi_\emptyset$ and $1$ for $\chi_X$.

When $X$ is a topological space, we write $C(X)$, $C_0(X)$, and $C_c(X)$ for its (real-valued) continuous functions, its continuous functions that vanish at infinity, and its compactly supported continuous functions, respectively. Their respective complex-valued counterparts are denoted by $C(X; \mathbb{C})$, $C_0(X; \mathbb{C})$, and $C_c(X; \mathbb{C})$. When $S$ is a subset of $X$, we shall write $f \prec S$ to mean that $f \in C_c(X)$, that $0 \leq f \leq 1$, and that $\text{supp} f \subseteq S$; we shall write $S \prec f$ to mean that $f \in C_c(X)$, that $0 \leq f \leq 1$, and that $f(x) = 1$ for $x \in S$. The Borel $\sigma$-algebra of $X$ is the $\sigma$-algebra generated by the open subsets of $X$ and is denoted by $\mathcal{B}$.

When $E$ is a normed space, its norm dual will be denoted by $E^*$. When $H$ is a Hilbert space, its inner product is denoted by $\langle \cdot, \cdot \rangle$. In the complex case, it is linear in the first variable. Its bounded operators are denoted by $\mathcal{B}(H)$.

When $E$ is a non-empty set supplied with an equivalence relation, the set of equivalence classes corresponding to a subset $S$ of $E$ will be denoted by $[S]$.\footnote{See Remark 7.10 for further comments.}
rather than the more customary \( [S] \); this makes the formulas in Section 5 easier to read. When \( x \in E \), we write \( \llbracket x \rrbracket \) for \( \llbracket \{x\} \rrbracket \).

2.1. Partially ordered vector spaces. When \( E \) is a partially ordered vector space, we let \( E^+ \) denote its positive cone. We do not require that \( E^+ \) be generating, i.e., we do not require that \( E \) be directed, but we do require that \( E^+ \) be proper. It is always supposed that \( E \) is Archimedean: for all \( x \in E^+ \), \( r_n x \downarrow 0 \) whenever \( \{r_n\}_{n=1}^\infty \) is a sequence in \( \mathbb{R}^+ \) such that \( r_n \downarrow 0 \).

When \( E \) and \( F \) are vector spaces, \( \mathcal{L}(E,F) \) denotes the operators from \( E \) into \( F \). An operator \( T : E \to F \) between two partially ordered vector spaces is positive when \( T(E^+) \subseteq F^+ \), and regular when it is the difference of two positive operators. We let \( \mathcal{L}_r(E,F) \) denote the vector space of regular operators from \( E \) into \( F \). When \( E^+ \) is generating in \( E \), \( \mathcal{L}(E,F) \) and \( \mathcal{L}_r(E,F) \) are partially ordered vector spaces via their common positive cones \( \mathcal{L}_r(E,F)^+ \).

An operator \( T \in \mathcal{L}(E,F)^+ \) between two partially ordered vector spaces \( E \) and \( F \) is called order continuous (resp. \( \sigma \)-order continuous) if \( T x \downarrow 0 \) in \( F \) whenever \( \{x_\lambda\}_{\lambda \in A} \) is a net in \( E \) such that \( x \downarrow 0 \) in \( E \) (resp. if \( T x_n \downarrow 0 \) in \( F \) whenever \( \{x_n\}_{n=1}^\infty \) is a sequence in \( E \) such that \( x_n \downarrow 0 \) in \( E \)).\(^5\) An operator in \( \mathcal{L}_r(E,F) \) is said to be order continuous (resp. \( \sigma \)-order continuous) if it is the difference of two positive order continuous (resp. \( \sigma \)-order continuous) operators.\(^6\) We let \( \mathcal{L}_{oc}(E,F) \) (resp. \( \mathcal{L}_{\sigma oc}(E,F) \)) denote the order continuous (resp. \( \sigma \)-order continuous) operators from \( E \) into \( F \); they are linear subspaces of \( \mathcal{L}_r(E,F) \). When \( E \) is directed, they are partially ordered vector spaces with the positive order continuous (resp. \( \sigma \)-order continuous) operators as positive cones, which are generating by definition. We write \( \mathcal{L}(E) \) for \( \mathcal{L}(E,E) \), etc.; \( E^\prec \) for \( \mathcal{L}_r(E,\mathbb{R}) \); \( E_{oc}^\prec \) for \( \mathcal{L}_{oc}(E,\mathbb{R}) \); and \( E_{\sigma oc}^\prec \) for \( \mathcal{L}_{\sigma oc}(E,\mathbb{R}) \). When \( E \) is a Banach lattice, \( E^\preceq \) coincides with the norm dual \( E^\vee \) of \( E \).

**Definition 2.1.** A partially ordered vector space \( E \) is called normal when, for \( x \in E \), \( (x,x') \geq 0 \) for all \( x' \in (E_{oc}^\prec)^+ \) if and only if \( x \in E^+ \).\(^7\)

Clearly, when \( E \) is normal, \((E_{oc}^\prec)^+\) separates the points of \( E \).

A partially ordered vector space \( E \) is monotone complete (resp. \( \sigma \)-monotone complete) when every increasing net (resp. sequence) in \( E \) that is bounded from above has a supremum; Dedekind complete when every non-empty subset of \( E \) that is bounded from above has a supremum; and \( \sigma \)-Dedekind complete when every non-empty countable infinite subset of \( E \) that is bounded from above has a supremum. As was observed in [29] Lemma 1.1, every \( \sigma \)-monotone complete partially ordered vector space is automatically Archimedean.

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\(^5\)One can argue that it is better to speak of monotone (\( \sigma \)-)order continuous operators, but we do not want to burden the terminology further than is necessary for our purposes.

\(^6\)When \( E \) and \( F \) are both vector lattices, the present definitions specialise to those in the literature for vector lattices; see [8] Remark 3.5.

\(^7\)For a vector lattice \( E \), this is equivalent to the usual requirement that \( E_{oc}^\prec \) separate the points of \( E \); see [8] Lemma 3.7.
For vector lattices, Dedekind completeness (resp. \(\sigma\)-Dedekind completeness) and monotone completeness (resp. \(\sigma\)-monotone completeness) are equivalent. If \(E\) has a generating positive cone and if \(E\) is \(\sigma\)-Dedekind complete, then, for \(x_1, x_2 \in E\), the subset \(\{x_1, x_2\}\) is bounded from above, so that it has a supremum. Hence \(E\) is then a vector lattice.

When \(E\) and \(F\) are vector lattices, and \(F\) is Dedekind complete, \(L_r(E, F)\) is a Dedekind complete vector lattice. When \(E\) and \(F\) are Banach lattices, every regular operator from \(E\) into \(F\) is continuous. When \(E\) and \(F\) are Banach lattices where \(F\) is Dedekind complete, \(L_r(E, F)\) is a Dedekind complete Banach lattice when supplied with the regular norm \(\|\cdot\|_r\), defined by setting \(\|T\|_r := \|\|T\|\) for \(T \in L_r(E, F)\); see [5, Theorem 4.74], for example.

The condition that \(E\) be monotone complete as well as normal is an important one in this paper. The class of such spaces is, for practical purposes, rather large, and contains many spaces that are not vector lattices. It includes, e.g., the Banach lattices with order continuous norms; the regular operators on such Banach lattices; more generally: every subspace of \(L(E, F)\) that contains \(L_r(E, F)\), where \(E\) and \(F\) are partially ordered vector spaces such that \(E\) is directed and \(F\) is monotone complete and normal; the vector space consisting of all self-adjoint elements of a strongly closed complex linear subspace of \(B(H)\) for a complex Hilbert space \(H\); JBW-algebras; and the regular operators on JBW-algebras. For this, and for more examples, we refer to [8, Section 3].

In Section 6, the finiteness of a representing spectral measure can often be concluded when the codomain is a quasi-perfect partially ordered vector space. This notion was introduced in [9]; we recall its definition.

**Definition 2.2.** A partially ordered vector space \(E\) is *quasi-perfect* when the two following conditions are both satisfied:

1. \(E\) is normal;
2. if an increasing net \(\{x_\lambda\}_{\lambda \in \Lambda}\) in \(E^+\) is such that \(\sup (x_\lambda, x') < \infty\) for each \(x' \in (E^+)\), then this net has a supremum in \(E\).

Clearly, a quasi-perfect partially ordered vector space is monotone complete. The terminology is motivated by an existing characterisation of perfect vector lattices. We recall that a vector lattice is called *perfect* when the natural vector lattice homomorphism from \(E\) into \((E_{\text{oc}}^-)_{\text{oc}}\) is a surjective isomorphism. The following alternate characterisation, which we include for comparison, is due to Nakano; see [5, Theorem 1.71]. It shows that a perfect vector lattice is a quasi-perfect partially ordered vector space.

**Theorem 2.3.** A vector lattice \(E\) is a perfect vector lattice if and only if the following two conditions hold:

1. \(E\) is normal;
2. if an increasing net \(\{x_\lambda\}_{\lambda \in \Lambda}\) in \(E^+\) is such that \(\sup (x_\lambda, x') < \infty\) for each \(x' \in (E_{\text{oc}}^-)^+\), then this net has a supremum in \(E\).
Quite a few spaces of practical interest are quasi-perfect. We give a number of examples in Proposition 2.4 and Proposition 2.7; see also Proposition 2.11 and Corollary 2.12. As a preparation for some of our examples, we recall that the norm on a Banach lattice is said to be a Levi norm when every increasing norm bounded net in the positive cone has a supremum. It follows from the uniform boundedness principle that the norm on a Banach lattice $E$ is a Levi norm precisely when $E$ has the property in part (2) of Definition 2.2.

A Banach lattice is a KB-space when every increasing norm bounded net in the positive cone is norm convergent. KB-spaces have order continuous norms, and a reflexive Banach lattice is a KB-space; see [5, p. 232]. It is not difficult to see that the KB-spaces are precisely the Banach lattices with Levi norms that are order continuous.

The following examples of quasi-perfect vector lattices are taken from [9, Proposition 6.7].

**Proposition 2.4.** The following spaces are quasi-perfect partially ordered vector spaces:

1. perfect vector lattices;
2. normal Banach lattices with a Levi norm, such as KB-spaces and, still more in particular, reflexive Banach lattices;
3. for SOT-closed complex linear subspaces $L$ of $B(H)$, where $H$ is a complex Hilbert space: the real vector spaces $L_{sa}$ consisting of all self-adjoint elements of $L$;
4. JBW-algebras.

Two other classes of quasi-perfect partially ordered vector spaces are in Proposition 2.7. We start with by collecting a few properties of order continuous operators in the following result.

**Proposition 2.5.** Let $E$ be a directed partially ordered vector space, and let $F$ be a monotone complete partially ordered vector space.

1. If $\{T_\lambda\}_{\lambda \in \Lambda}$ is a net in $\mathcal{L}_{oc}(E,F)$ and $T_\lambda \uparrow T$ in $\mathcal{L}(E,F)$ for some $T \in \mathcal{L}(E,F)$, then $T \in \mathcal{L}_{oc}(E,F)$;
2. Let $\{T_\lambda\}_{\lambda \in \Lambda}$ be a net in $\mathcal{L}_{oc}(E,F)$, and let $T \in \mathcal{L}_{oc}(E,F)$. Then $T_\lambda \uparrow T$ in $\mathcal{L}_{oc}(E,F)$ if and only if $T_\lambda \uparrow T$ in $\mathcal{L}(E,F)$;
3. $\mathcal{L}_{oc}(E,F)$ is monotone complete;
4. Suppose, in addition, that $F$ is normal. Then $\mathcal{L}_{oc}(E,F)$ is normal.

**Proof.** Part (1) is well known when $E$ and $F$ are vector lattices. The argument in that case (see [5, Proof of Theorem 1.57], for example) works equally well in the general case.

The parts (2) and (3) follow from the combination of part (1) and the monotone completeness of $\mathcal{L}(E,F)$ (see [8, Proposition 3.1]).

For part (4), we observe that the normality of $F$ implies that of $\mathcal{L}(E,F)$; see [8, Proposition 3.11]. The normality of $\mathcal{L}_{oc}(E,F)$ then follows from part (2). □

Combining [8, Proposition 3.11] and Proposition 2.5, we have the following.
Proposition 2.6. Let $E$ be a directed partially ordered vector space, and let $F$ be a monotone complete and normal partially ordered vector space. Then $\mathcal{L}_c(E, F)$ and $\mathcal{L}_{oc}(E, F)$ are directed, monotone complete, and normal partially order vector spaces.

When $F$ is quasi-perfect, one can do better.

Proposition 2.7. Let $E$ be a directed partially ordered vector space, and let $F$ be a quasi-perfect partially ordered vector space. Then $\mathcal{L}_c(E, F)$ and $\mathcal{L}_{oc}(E, F)$ are directed quasi-perfect partially ordered vector spaces.

Proof. We give the proof for $\mathcal{L}_{oc}(E, F)$; the easier argument for $\mathcal{L}_c(E, F)$ is similar. Part (4) of Proposition 2.5 shows that $\mathcal{L}_{oc}(E, F)$ is normal. Suppose that $\{T_\lambda\}_{\lambda \in \Lambda}$ is an increasing net in $\mathcal{L}_{oc}(E, F)^+$, and that $\sup_\lambda (T_\lambda, \varphi) < \infty$ for each $\varphi \in (\mathcal{L}_{oc}(E, F)^-)^+$. In particular, then we have that $\sup_\lambda (T_\lambda x, x') < \infty$ for all $x \in E^+$ and $x' \in (F^-)^+$. Since $F$ is quasi-perfect, this implies that $\sup_\lambda T_\lambda x$ exists in $F$ for each $x \in E$. By [8] Proposition 3.1, the net $\{T_\lambda\}_{\lambda \in \Lambda}$ has a supremum $T$ in $\mathcal{L}_c(E, F)$. According to the parts (1) and (2) of Proposition 2.5, $T$ is also the supremum of $\{T_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{L}_{oc}(E, F)$. \hfill \Box

2.2. The countable sup property. Let $E$ be a partially ordered vector space. Then $E$ is said to have the **countable sup property** when, for every net $\{x_\lambda\}_{\lambda \in \Lambda} \subseteq E^+$ and $x \in E^+$ such that $x_\lambda \uparrow x$, there exists a countable set of indices $\{\lambda_n : n \geq 1\}$ such that $x = \sup_{n \geq 1} x_{\lambda_n}$.\footnote{As in our definition of order continuous operators, we refrain from calling this the monotone countable sup property.} In this case, there also always exist $\lambda_1 \leq \lambda_2 \leq \cdots$ such that $x_{\lambda_n} \uparrow x$. For vector lattices, our countable sup property is equivalent to what is usually called the countable sup property in that context; namely, that every subset that has a supremum contains a countable subset with the same supremum.

The countable sup property is not only relevant to the properties of the $L^1$-spaces that we shall introduce in Section 2.5 but, when combined with the monotone convergence theorem, it is also an essential ingredient to the proof of our main results on ups and downs, Theorems 5.9 and 5.11. It is for this reason that we mention a few facts here to show that this property is not at all uncommon, as it can often be obtained by pulling it back from a codomain via a strictly positive operator.

If $E$ is a $\sigma$-Dedekind complete vector lattice, $F$ is a monotone complete partially ordered vector space with the countable sup property, and $T : E \to F$ is a strictly positive $\sigma$-order continuous operator, then $E$ has the countable sup property, $E$ is Dedekind complete, and $T$ is order continuous; see [8] Proposition 6.16.

For vector lattices, the situation is simpler. Suppose that there exists a strictly positive operator $\pi : E \to F$ between two vector lattices $E$ and $F$. If $F$ has the countable sup property, then so does $E$; see [4] Theorem 1.45. Consequently, a vector sublattice of a vector lattice with the countable sup property has the countable sup property; this also follows from [18] Theorem 23.5. As another
consequence, every vector lattice that admits a strictly positive functional has the countable sup property. Consequently, every separable Banach lattice has the countable sup property (see [5 Exercise 4.1.4]).

Suppose that $E$ is a separable Banach lattice, and that $F$ is a Dedekind complete normed vector lattice that admits a strictly positive continuous functional $\varphi$. Choose a sequence $\{e_n\}_{n=1}^{\infty}$ in $E$ that is dense in the positive part of the unit ball of $E$, and define the functional $T \mapsto \sum_{n=1}^{\infty} 2^{-n}(Te_n, \varphi)$ on $L_r(E, F)$. Then $\varphi$ is strictly positive, so that $L_r(E, F)$ has the countable sup property. In particular, this is true when $E$ and $F$ are separable Banach lattices and $F$ is Dedekind complete.

If $H$ is a separable complex Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$, then $T \mapsto \sum_{n=1}^{\infty} 2^{-n} \langle Te_n, e_n \rangle$ is a strictly positive functional on $\mathcal{B}(H)_{sa}$. Hence every linear subspace of $\mathcal{B}(H)_{sa}$ that is a lattice in the restricted partial ordering has the countable sup property.

Suppose that $E$ and $F$ are vector lattices, that $F$ is Dedekind complete, that $E$ has a weak order unit $e$, and that $F$ admits a strictly positive order continuous functional. Then $T \mapsto \langle Te, \varphi \rangle$ is a strictly positive functional on $L_{oc}(E, F)$, so that $L_{oc}(E, F)$ has the countable sup property.

Finally, a Banach lattice with an order continuous norm has the countable sup property; see [31 Theorem 17.8], for example.

2.3. Partially ordered algebras. A partially ordered algebra $\mathcal{A}$ is an associative algebra that is also a partially ordered vector space such that $ab \in \mathcal{A}^+$ for all $a, b \in \mathcal{A}^+$. We do not suppose that $\mathcal{A}$ has an identity element or, if so, that the identity element is positive. A partially ordered algebra that is also a vector lattice is a vector lattice algebra. A normed vector lattice algebra is a vector lattice algebra that is supplied with a norm satisfying $\|x\| \leq \|y\|$ whenever $x, y \in \mathcal{A}$ are such that $|x| \leq |y|$. A possible identity element need not have norm 1. A Banach lattice algebra is a normed vector lattice algebra with a complete norm.

If $\mathcal{A}$ is a partially ordered algebra, then we shall say that the multiplication in $\mathcal{A}$ is monotone continuous if, whenever $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{A}^+$ and $a \in \mathcal{A}^+$ are such that $a_\lambda \uparrow a$, then $ba_\lambda \uparrow ba$ and $a_\lambda b \uparrow ab$ for all $b \in \mathcal{A}^+$; it is $\sigma$-monotone continuous if, whenever $\{a_n\}_{n=1}^{\infty} \subseteq \mathcal{A}^+$ and $a \in \mathcal{A}^+$ are such that $a_n \uparrow a$, then $ba_n \uparrow ba$ and $a_n b \uparrow ab$ for all $b \in \mathcal{A}^+$. It is routine to verify that the analogous definitions using decreasing nets and sequences in $\mathcal{A}^+$ are equivalent to the above ones.

Remark 2.8. Suppose that $\mathcal{A}$ is a vector lattice algebra. Then the multiplication in $\mathcal{A}$ is monotone continuous (resp. $\sigma$-monotone continuous) in our sense if and only if, for all $b \in \mathcal{A}$, the multiplication operators $a \mapsto ba$ and $a \mapsto ab$ are order continuous (resp. $\sigma$-order continuous) operators on $\mathcal{A}$ in the sense of [30 p. 123]. This is an easy consequence of the fact that $\mathcal{A}^+$ generates $\mathcal{A}$ and the observation that linear combinations of positive operators that are order continuous in the sense of [30 p. 123] are again order continuous in the same sense.
The terminology ‘monotone continuous multiplication’, rather than ‘monotone left and right multiplications’, is justified by the following result, which will be used in the proof of Theorem 3.6 on spectral measures. Its proof is routine.

**Lemma 2.9.** Let \( A \) be a partially ordered algebra with a monotone continuous multiplication.

1. If \( \{ a_{\lambda} \}_{\lambda \in \Lambda} \), \( \{ b_{\lambda} \}_{\lambda \in \Lambda} \subseteq A^+ \), and \( a, b \in A^+ \) are such that \( a_{\lambda} \uparrow a \) and \( b_{\lambda} \uparrow b \), then \( a_{\lambda} b_{\lambda} \uparrow ab \);
2. If \( \{ a_{\lambda} \}_{\lambda \in \Lambda} \), \( \{ b_{\lambda} \}_{\lambda \in \Lambda} \subseteq A^+ \), and \( a, b \in A^+ \) are such that \( a_{\lambda} \uparrow a \) and \( b_{\lambda} \uparrow b \), then \( a_{\lambda} b_{\lambda} \uparrow ab \).

When the multiplication is \( \sigma \)-monotone continuous, the analogous statements for the term-wise products of two sequences hold.

The three analogous statements for decreasing nets and sequences in \( A^+ \) are also true.

The following result will be convenient later on.

**Lemma 2.10.** Let \( A \) be a partially ordered algebra with monotone continuous multiplication. Suppose that \( p \in A^+ \) and that \( p^2 = p \).

1. If \( \{ a_{\lambda} \}_{\lambda \in \Lambda} \) is a net in \( (pA)^+ \) and \( a_{\lambda} \uparrow a \) in \( A \) for some \( a \in A \), then \( a \in (pA)^+ \).
   
   If, in addition, \( A \) is monotone complete, then the following hold.

2. Let \( \{ a_{\lambda} \}_{\lambda \in \Lambda} \) be a net in \( (pA)^+ \) and let \( a \in (pA)^+ \). Then \( a_{\lambda} \uparrow a \) in \( pA \) if and only if \( a_{\lambda} \uparrow a \) in \( A \);
3. The subalgebra \( pA \) of \( A \) is a monotone complete partially ordered algebra with monotone continuous multiplication.;
4. If \( A \) is a normal partially ordered vector space, then so is \( pA \);
5. Suppose, in addition, that \( A \) has the countable sup property. Then so does \( pA \).

Similar statements hold for \( Ap \) and \( pAp \).

**Proof.** We consider only the case \( pA \); the other two are treated similarly.

We prove part \( (1) \). Under the pertinent premises, it follows that \( a_{\lambda} = pa_{\lambda} \uparrow pa \), so that \( a = pa \in pA \).

We prove part \( (2) \). Let the net \( \{ a_{\lambda} \}_{\lambda \in \Lambda} \) in \( (pA)^+ \) and \( a \in (pA)^+ \) be such that \( a_{\lambda} \uparrow a \) in \( pA \). Since \( A \) is monotone complete, there exists an \( a' \in A \) such that \( a_{\lambda} \uparrow a' \) in \( A \). By part \( (1) \), we have \( a' \in pA \). Hence \( a = a' \), so that \( a_{\lambda} \uparrow a = a' \) in \( A \). The converse statement is trivial.

The parts \( (3) \), \( (4) \), and \( (5) \) are now easy consequences of the parts part \( (1) \) and \( (2) \).

The second main spectral theorem in Section 6 Theorem 6.4 is in the context of monotone complete and normal partially ordered algebras with a monotone continuous multiplication. The monotone continuity of the multiplication
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may fail for algebras of general regular operators on partially ordered vector spaces, but for algebras of order continuous operators we have the following consequence of \cite[Proposition 3.1]{3} and Propositions \ref{prop:2.5} to \ref{prop:2.7}, exhibiting two major classes of algebras to which Theorem \ref{thm:6.4} applies.

**Proposition 2.11.** Let $E$ be a directed partially ordered vector space.

1. If $E$ is monotone complete and normal, then $L_{oc}(E)$ is a directed, monotone complete, and normal partially ordered algebra with a monotone continuous multiplication.

2. If $E$ is quasi-perfect, then $L_{oc}(E)$ is a directed quasi-perfect partially ordered algebra with a monotone continuous multiplication.

Proposition \ref{prop:2.11} and Proposition \ref{prop:2.4} have the following consequence for Banach lattices.

**Corollary 2.12.** Let $E$ be a Dedekind complete Banach lattice, and supply $L_{oc}(E)$ with the regular norm. If $E$ is normal, i.e., is such that $E_{oc}^\sim$ separates the points of $E$, then $L_{oc}(E)$ is a Dedekind complete normal Banach lattice algebra with a monotone continuous multiplication. If $E$ is normal and the norm on $E$ is a Levi norm, then $L_{oc}(E)$ is a Dedekind complete quasi-perfect Banach lattice algebra with a monotone continuous multiplication.

Another important case where Theorem \ref{thm:6.4} applies is in the context of Hilbert spaces.

**Proposition 2.13.** Let $H$ be a complex Hilbert space, and let $A \subseteq \mathcal{B}(H)$ be a commutative SOT-closed $C^*$-subalgebra. Let $A_{sa}$ be the real vector space that consists of the self-adjoint elements of $A$, supplied with the partial ordering that is inherited from the usual partial ordering on $\mathcal{B}(H)_{sa}$. Then $A_{sa}$ is a quasi-perfect Banach lattice algebra with a monotone continuous multiplication.

Proof. There exists a locally compact Hausdorff space $X$ such that $A$ is isomorphic to $C_0(X; \mathcal{C}^*)$ as a $C^*$-algebra. It is then clear that $A_{sa}$ is a Banach lattice algebra. We know from part \ref{part:3} of Proposition \ref{prop:2.4} that $A_{sa}$ is quasi-perfect.

We turn to the multiplication. Suppose that $T \in A_{sa}^+$ and that $0 \leq S_\lambda \uparrow T$ in $A_{sa}$. Then $S = \text{SOT-lim}_\lambda S_\lambda$ by \cite[Proposition 3.2]{8}, and this implies that $TS = \text{SOT-lim}_\lambda TS_\lambda$. Since $0 \leq TS_\lambda \uparrow \leq TS$ (this is clear in the $C_0(X; \mathcal{C}^*)$-model), \cite[Proposition 3.2]{8} implies that $TS_\lambda \uparrow TS$ in $A_{sa}$. A similar argument shows that $S_\lambda T \uparrow ST$ in $A_{sa}$. Hence the multiplication in $A_{sa}$ is monotone order continuous.

For an application of Theorem \ref{thm:6.4} to JBW-algebras, we record the following.

**Proposition 2.14.** Let $\mathcal{M}$ be an associative JBW-algebra. Then $\mathcal{M}$ is a directed quasi-perfect partially ordered algebra with a monotone continuous multiplication.

Proof. In view of Proposition \ref{prop:2.4} only the monotone continuity of the (commutative) multiplication needs proof. Suppose that $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq M^+$ is a net and that $a_\lambda \uparrow a$ for some $a \in M^+$. Take $b \in M^+$. According to \cite[Proposition 2.5(ii)]{3},
by \[3\] Proposition 2.4, we also have \(a\lambda b \to ab\) \(\sigma\)-strongly. By \[3,\) Proposition 2.4\], we also have \(a\lambda b \to ab\) \(\sigma\)-strongly. Let \(s\) be its supremum. Then \(a\lambda b \uparrow s\) implies that \(a\lambda b \to s\) \(\sigma\)-strongly by \[3,\) Proposition 2.5(ii)\] again. Hence \(ab = s\) and \(a\lambda b \uparrow ab\). The proof that \(ba\lambda \uparrow ba\) is similar. \(\square\)

2.4. Moduli preserving operators. As we shall see in Section 4, a positive algebra homomorphism often gives rise to a vector lattice homomorphism. The present section provides the necessary terminology and elementary preparatory results for this.

**Definition 2.15.** Let \(E\) and \(F\) be partially ordered vector spaces, and let \(T : E \to F\) be an operator. Then \(T\) preserves moduli when, for all \(x \in E\) such that \(|x|\) exists in \(E\), \(|Tx|\) exists in \(F\) and \(T|x| = |Tx|\).

A moduli preserving operator is positive. The moduli preserving operators between two vector lattices are precisely the vector lattice homomorphisms.

It is well known (and easy to see) that, for elements \(x, y\) of a partially ordered vector space \(E\), the existence in \(E\) of \(x \vee y\), \(x \wedge y\), and \(|x - y|\) are all equivalent. In this case, we have \(x + y = x \vee y + x \wedge y\), \(x \vee y = \frac{1}{2}(x + y + |x - y|)\), and \(x \wedge y = \frac{1}{2}(x + y - |x - y|)\). Using this, the following result is easily established, giving two equivalent definitions of moduli preserving operators.

**Lemma 2.16.** Let \(E\) and \(F\) be partially ordered vector spaces, and let \(T : E \to F\) be an operator. The following are equivalent:

1. \(T\) preserves moduli;
2. whenever \(x, y \in E\) are such that \(x \vee y\) exists in \(E\), \(Tx \vee Ty\) exists in \(F\), and \(T(x \vee y) = Tx \vee Ty\);
3. whenever \(x, y \in E\) are such that \(x \wedge y\) exists in \(E\), \(Tx \wedge Ty\) exists in \(F\), and \(T(x \wedge y) = Tx \wedge Ty\).

**Remark 2.17.** Suppose that \(T : E \to F\) is a moduli preserving operator between two partially ordered vector spaces. Let \(n \geq 3\), and suppose that \(x_1, \ldots, x_n \in E\) are such that \(\text{sup}\{x_1, \ldots, x_n\}\) exists in \(E\). In this case, there is no guarantee that \(\text{sup}\{Tx_1, \ldots, Tx_n\}\) exists in \(F\). In the cases where we shall encounter moduli preserving operators, however, \(E\) is a vector lattice; then an inductive argument shows that \(\text{sup}\{Tx_1, \ldots, Tx_n\}\) exists in \(F\) and that also \(T(\text{sup}\{x_1, \ldots, x_n\}) = \text{sup}\{Tx_1, \ldots, Tx_n\}\) when \(n \geq 3\).

The next result follows easily from the fact that the existence of suprema and infima and their values in a partially ordered vector space are compatible with translations.

**Lemma 2.18.** Let \(E\) and \(F\) be partially ordered vector spaces, where \(E\) is directed, and let \(T : E \to F\) be an operator. The following are equivalent:

1. \(T\) preserves moduli;
2. whenever \(x, y \in E^+\) are such that \(x \vee y\) exists in \(E\), \(Tx \vee Ty\) exists in \(F\), and \(T(x \vee y) = Tx \vee Ty\);
whenever \( x, y \in E^+ \) are such that \( x \land y \) exists in \( E \), \( T x \land T y \) exists in \( F \), and \( T(x \land y) = T x \land T y \).

The following result is a direct consequence of the definitions.

**Lemma 2.19.** Let \( E \) be a vector lattice, let \( F \) be a partially ordered vector space, and let \( T : E \to F \) be a moduli preserving operator. Supplied with the partial ordering inherited from \( F \), the space \( T(E) \) is a vector lattice, and the map \( T : E \to T(E) \) is then a vector lattice homomorphism.

We shall apply Lemma 2.19 quite a few times in the sequel. Note, however, that some information regarding the ordering is lost when passing from its premises to its conclusion. Indeed, for \( x \in E \), the supremum of the set \( \{ T x, -T x \} \) even exists in \( F \); it so happens that this supremum in the full space is already in \( T(E) \). Similar remarks apply to \( T x \lor T y \) and \( T x \land T y \) for \( x, y \in E \).

In the terminology of [11, Definition 5.58], \( T(E) \) is a lattice-subspace of \( F \), but it is more than that.

**Remark 2.20.** A detailed investigation of the possible generalisations of the notion of a vector lattice homomorphism to the context of partially ordered vector spaces, and to that of pre-Riesz spaces in particular, is undertaken in [16]. It is shown in [16, Proposition 39] that a positive operator between two pre-Riesz spaces \( E \) and \( F \) preserves moduli if and only if it preserves disjointness on \( E^+ \). In spite of its naturality, the notion of a moduli preserving operator between general partially ordered vector spaces as in Definition 2.15 and the equivalences in Lemma 2.16 appear to be new.

**2.5. Measures and integrals.** In this section, we shall briefly outline part of the material in [8] on measures with values in the extended positive cones of \( \sigma \)-monotone complete partially ordered vector spaces, and on the associated (order) integrals. This extension of earlier work of Wright generalises the theory of the Lebesgue integral. We refer to [8, Section 7] for a comparison with vector measures where it is argued, that, in the case of a \( \sigma \)-monotone complete partially ordered Banach space, the measures and the (order) integrals as in the current section are a more convenient tool to work with than positive vector measures and their integrals.

Let \( E \) be a \( \sigma \)-monotone complete partially ordered vector space. We adjoin a new element \( \infty \) to \( E \), and extend the partial ordering from \( E \) to \( \overline{E} := E \cup \{ \infty \} \) by declaring that \( x \leq \infty \) for all \( x \in E \). The addition on the extended positive cone \( \overline{E}^+ := E^+ \cup \{ \infty \} \) is canonically defined, as well as the action of \( \mathbb{R}^+ \) on \( \overline{E}^+ \). The elements of \( \overline{E} \) that are in \( E \) are called finite.

The following definition is due to Wright; see [28, p. 111]. It generalises the notion of a measure with values in the extended positive real numbers.
Definition 2.21. Let \((X, \Omega)\) be a measurable space\(^4\), and let \(E\) be a \(\sigma\)-monotone complete partially ordered vector space. An \(\overline{E^+}\)-valued measure is a map 
\(\mu : \Omega \to \overline{E^+}\) such that:

1. \(\mu(\emptyset) = 0\);
2. If \(\{\Delta_n\}_{n=1}^\infty\) is a pairwise disjoint sequence in \(\Omega\), then

\[
\mu\left(\bigcup_{n=1}^\infty \Delta_n\right) = \bigvee \sum_{N=1}^N \mu(\Delta_n)
\]

in \(\overline{E}\).

The quadruple \((X, \Omega, \mu, E)\) is then a measure space. If \(\mu(\Omega) \subseteq \overline{E^+}\) (equivalently: if \(\mu(X) \in \overline{E^+}\)), then \(\mu\) is called finite, in which case we shall speak of an \(E\)-valued measure.

As a prelude to Section \([7]\) we collect a few results from \([8]\). They show how the concept of a measure in an ordered context unites those of \(\sigma\)-additive measures in the strong operator topology for rather diverse spaces. Naturally, the same is true for spectral measures.

Proposition 2.22 (see \([8, Proposition 3.2 and Lemma 4.2]\)). Let \(H\) be a complex Hilbert space, and let \(L\) be a strongly closed complex linear subspace of \(B(H)\). Let \(L_{sa}\) be the real vector space that consists of the self-adjoint elements of \(L\), supplied with the partial ordering that is inherited from the usual partial ordering on \(B(H)_{sa}\). Then \(L_{sa}\) is a monotone complete partially ordered vector space. Let \((X, \Omega)\) be a measurable space, and let \(\mu : \Omega \to \overline{L_{sa}^+}\) be a map such that \(\mu(\emptyset) = 0\).

Then the following are equivalent:

1. \(\mu\) is a finite \(\overline{L_{sa}^+}\)-valued measure in the sense of Definition 2.21;
2. If \(\{\Delta_n\}_{n=1}^\infty\) is a pairwise disjoint sequence in \(\Omega\), then

\[
\mu\left(\bigcup_{n=1}^\infty \Delta_n\right) = \sum_{n=1}^\infty \mu(\Delta_n) x \text{ in the norm topology of } H \text{ for all } x \in \overline{E}.
\]

Proposition 2.23 (see \([8, Lemma 4.3]\)). Let \(E\) be a Banach lattice with an order continuous norm. Then \(L_r(E)\) is a Dedekind complete vector lattice. Let \((X, \Omega)\) be a measurable space, and let \(\mu : \Omega \to \overline{L_r(E)^+}\) be a map such that \(\mu(\emptyset) = 0\). The following are equivalent:

1. \(\mu\) is a finite \(L_r(E)^+\)-valued measure in the sense of Definition 2.21;
2. If \(\{\Delta_n\}_{n=1}^\infty\) is a pairwise disjoint sequence in \(\Omega\), then

\[
\mu\left(\bigcup_{n=1}^\infty \Delta_n\right) = \sum_{n=1}^\infty \mu(\Delta_n) x \text{ in the norm topology of } E \text{ for all } x \in \overline{E}.
\]

In a topological context, we distinguish various regularity properties of measures. Since the terminology in the literature is not entirely uniform, we mention them explicitly.

Definition 2.24. When \(X\) is a locally compact Hausdorff space, we let \(\mathcal{B}\) denote its Borel \(\sigma\)-algebra. Let \(E\) be a monotone complete partially ordered vector space, and let \(\mu : \mathcal{B} \to \overline{E^+}\) be a measure. Then \(\mu\) is called:

\(^9\)In the earlier parts of \([9]\), it was sufficient that \(\Omega\) be an algebra of subsets of \(X\), but here we require it to be a \(\sigma\)-algebra from the outset.
(1) a Borel measure (on $X$) if $\mu(K) \in E$ for all compact subset $K$;
(2) inner regular at $\Delta \in \mathcal{B}$ if $\mu(\Delta) = \bigvee \{\mu(K) : K$ is compact and $K \subseteq \Delta\}$ in $\overline{E}$;
(3) outer regular at $\Delta \in \mathcal{B}$ if $\mu(\Delta) = \bigwedge \{\mu(V) : V$ is open and $\Delta \subseteq V\}$ in $\overline{E}$;
(4) a regular Borel measure (on $X$) if $\mu$ is a Borel measure on $X$ that is inner regular at all open subsets of $X$ and outer regular at all Borel sets.

Let $(X, \Omega, \mu, E)$ be a measure space. A (finite-valued) measurable function $\varphi : X \rightarrow \mathbb{R}^+$ is an elementary function when it takes only finitely many values. It can be written (not generally uniquely) as a finite sum $\varphi = \sum_{i=1}^n r_i \chi_{\Delta_i}$ for some $r_1, \ldots, r_n \in \mathbb{R}^+$ and $\Delta_1, \ldots, \Delta_n \in \Omega$. Here the $r_i$ are all finite, but it is allowed that $\mu(\Delta_i) = \infty$ for some of the $\Delta_i$. We let $\mathcal{E}(X, \Omega; \mathbb{R}^+)$ denote the set of elementary functions.

If $\varphi = \sum_{i=1}^n r_i \chi_{\Delta_i}$ is an elementary function, where the $\Delta_i$ have been chosen pairwise disjoint, then we define its (order) integral, which is an element of $E^+$, by setting

$$\int_X \varphi \, d\mu := \sum_{i=1}^n r_i \mu(\Delta_i).$$

This is well defined. For a measurable function $f : X \rightarrow \overline{\mathbb{R}^+}$, we choose a sequence $\{\varphi_n\}_{n=1}^\infty \subseteq \mathcal{E}(X, \Omega; \mathbb{R}^+)$ such that, for all $x \in X$, $\varphi_n(x) \uparrow f(x)$ in $\overline{\mathbb{R}^+}$. We define the (order) integral of $f$, which is an element of $E^+$, by setting

$$\int_X f \, d\mu := \bigvee_{n=1}^\infty \int_X \varphi_n \, d\mu.$$

The value of the integral does not depend on the choice of the sequence $\{\varphi_n\}_{n=1}^\infty$.

We let $\mathcal{L}^1(X, \Omega, \mu; \mathbb{R})$ denote the set of all (finite-valued) measurable functions $f : X \rightarrow \mathbb{R}$ such that $\int_X |f| \, d\mu$ is finite; its positive cone is denoted by $\mathcal{L}^1(X, \Omega, \mu; \mathbb{R}^+)$. We write $\mathcal{E}(X, \Omega; \mathbb{R}^+)$ for the elementary functions with finite integral. For $f \in \mathcal{L}^1(X, \Omega, \mu; \mathbb{R})$, we define $\int_X f \, d\mu$ by splitting $f$ into its positive and negative parts. $\mathcal{L}^1(X, \Omega, \mu; \mathbb{R})$ is a $\sigma$-Dedekind complete vector lattice, and the positive operator $I_\mu : \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \rightarrow E$, defined by setting

$I_\mu f := \int_X f \, d\mu$ for $f \in \mathcal{L}^1(X, \Omega, \mu; \mathbb{R})$, is a $\sigma$-order continuous operator from $\mathcal{L}^1(X, \Omega, \mu; \mathbb{R})$ into $E$; see [8] Proposition 6.14. We shall use the same notation $I_\mu$ to denote the restrictions of the original map to subspaces of $\mathcal{L}^1(X, \Omega, \mu; \mathbb{R})$, and also to denote the induced maps on quotients of such subspaces.

We let $\mathcal{N}(X, \Omega, \mu; \mathbb{R})$ denote the order ideal of $\mathcal{L}^1(X, \Omega, \mu; \mathbb{R})$ that consists of the measurable functions that vanish $\mu$-almost everywhere, and we let $L^1(X, \Omega, \mu; \mathbb{R})$ denote the resulting quotient vector lattice. According to [8] Theorem 6.17, $L^1(X, \Omega, \mu; \mathbb{R})$ is a $\sigma$-Dedekind complete vector lattice, and the map $I_\mu : L^1(X, \Omega, \mu; \mathbb{R}) \rightarrow E$ is strictly positive and $\sigma$-order continuous. When $E$ is monotone complete and has the countable sup property, $L^1(X, \Omega, \mu; \mathbb{R})$ is a Dedekind complete vector lattice with the countable sup property. Moreover, $I_\mu$ is then order continuous.
We shall write $\mathcal{B}(X, \Omega; \mathbb{R})$ for the bounded measurable functions on $X$; $\mathcal{B}(X, \Omega; \mathbb{R}^+)$ for its positive cone; $B(X, \Omega, \mu; \mathbb{R})$ for its quotient with respect to $\mathcal{M}(X, \Omega, \mu; \mathbb{R}) \cap \mathcal{B}(X, \Omega; \mathbb{R})$; and $B(X, \Omega, \mu; \mathbb{R}^+)$ for the positive cone of $B(X, \Omega, \mu; \mathbb{R})$. The essential supremum norm can be defined on $B(X, \Omega, \mu; \mathbb{R})$ as in the case where $E = \mathbb{R}$, and $B(X, \Omega, \mu; \mathbb{R})$ is then a Banach lattice algebra that is isometrically isomorphic to $C(X)$ for a compact Hausdorff space $X$, unique up to homeomorphism.

In [9, Section 6.2], the monotone convergence theorem, Fatou’s lemma, and (when $E$ is $\sigma$-Dedekind complete) the dominated convergence theorem for the order integral are established. Although the space $L^1(X, \Omega, \mu; \mathbb{R})$ consists of finite-valued functions, the monotone convergence theorem is valid for functions with values in the extended positive real numbers. We shall benefit from this in the proof of Proposition 5.7 on ups and downs.

Finally, for later use in the context of JBW-algebras, we record the following on image measures. The proofs are analogous to those in [6 §19].

**Proposition 2.25.** Let $(X, \Omega)$ and $(X', \Omega')$ be measurable spaces, and let $\Psi: X \to X'$ be $\Omega - \Omega'$-measurable. Let $E$ be a $\sigma$-monotone complete partially ordered vector space, and let $\mu: \Omega \to E^+$ be a measure. Define $\mu \circ \Psi^{-1}: \Omega' \to E^+$ by setting $\mu \circ \Psi^{-1}(\Delta') := \mu(\Psi^{-1}(\Delta'))$ for $\Delta' \in \Omega'$. Then $\mu \circ \Psi^{-1}$ is a measure on $\Omega'$. If $f': X' \to \mathbb{R}^+$ is $\Omega'$-measurable, then

\[
\int_{X'} f' \, d(\mu \circ \Psi^{-1}) = \int_X f' \circ \Psi \, d\mu
\]

in $E^+$. If $f': X' \to \mathbb{R}$ is $\Omega'$-measurable, then $f' \in L^1(X', \Omega', \mu \circ \Psi^{-1}; \mathbb{R})$ if and only if $f \circ \Psi \in L^1(X, \Omega, \mu; \mathbb{R})$, in which case equation (2.2) holds in $E$.

### 2.6 Spectral measures

Of particular interest in this paper are measures that take their values in partially ordered algebras. Motivated by the terminology in the literature for measures of various sorts that take their values in algebras of operators, we introduce the following terminology for measures that take their values in (partially ordered) algebras.

**Definition 2.26.** Let $(X, \Omega)$ be a measurable space, and let $\mathcal{A}$ be a $\sigma$-monotone complete partially ordered algebra. A measure $\mu: \Omega \to \mathcal{A}^+$ is a spectral measure when $\mu(\Delta_1 \cap \Delta_2) = \mu(\Delta_1)\mu(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \Omega$ with finite measure.

**Remark 2.27.**

1. For a spectral measure, the finite elements among the $\mu(\Delta)$ for $\Delta \in \Omega$ form a family of commuting idempotents. Somewhat surprisingly, when an $\mathcal{A}^+$-valued measure is actually finite, this property already implies that the measure is a spectral measure; see Proposition 3.1 for an even larger context in which this equivalence holds. This property may be easier to verify. In fact, we shall do just that in the proof of the key result Theorem 3.6.
(2) Infinite spectral measures exist. As an example, one can take a two-point space \( X = \{x, y\} \), take \( \Omega = 2^X \), take \( \mathcal{A} = \mathbb{R} \), and let \( \mu \) be the measure on \( \Omega \) such that \( \mu(\{x\}) = 1 \) and \( \mu(\{y\}) = \infty \).

(3) As a thought experiment, one can supply \( \mathcal{A}_+ \) with a multiplication in the canonical fashion, and then require that \( \mu(\Delta_1 \cap \Delta_2) = \mu(\Delta_1)\mu(\Delta_2) \) for all \( \Delta_1, \Delta_2 \in \Omega \). When \( \mu(X) = \infty \), this implies that \( \mu(\Delta) = \mu(\Delta \cap X) = \mu(\Delta) \cdot \infty \) for \( \Delta \in \Omega \). It follows from this that \( \mu(\Delta) \in \{0, \infty\} \) for all \( \Delta \in \Omega \). As a consequence, all integrable functions have zero order integral. This means that the only such spectral measures that can lead to anything interesting at all, are actually the finite ones, in which case they are then spectral measures as in Definition 2.26. Hence such a more stringent multiplicativity condition is not imposed.

(4) Suppose that, in the context and notation of Proposition 2.25, \( E \) is a \( \sigma \)-monotone complete partially ordered algebra. If \( \mu \) is a spectral measure, then so is the image measure \( \mu \circ \Psi^{-1} \).

For general measures, finite or infinite, an integrable function need not be essentially bounded. Also for spectral measures, finite or infinite, this still need not be the case, as is shown by the following example.

**Example 2.28.** Let \( X \) be an infinite set, and let \( \text{Fun}(X; \mathbb{R}) \) denote the Dedekind complete vector lattice of all real-valued functions on \( X \). Set \( \mathcal{A} := \mathcal{L}_r(\text{Fun}(X; \mathbb{R})) \). Then \( \mathcal{A} \) is a Dedekind complete vector lattice algebra. Take \( \Omega := 2^X \). For \( \Delta \in \Omega \), let \( \mu(\Delta) \in \mathcal{A} \) be the pointwise multiplication by the characteristic function of \( \Delta \). Then \( \mu: \Omega \to \mathcal{A}_+ \) is a finite spectral measure and \( \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \) consists of all functions on \( X \): for \( f \in \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \), \( \int_X f \, d\mu \in \mathcal{A} \) is the pointwise multiplication by \( f \). Since the empty set is the only subset of \( X \) of measure zero and \( X \) is infinite, there are elements of \( \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \) that are not essentially bounded.

To get an example where the spectral measure is infinite, we fix a point \( x_0 \) in \( X \), and define \( \mu': \Omega \to \mathcal{A}_+ \) by letting \( \mu' \) be the multiplication by the characteristic function of \( \Delta \) when \( x_0 \notin \Delta \), and setting \( \mu'(\Delta) := \infty \) when \( x_0 \in \Delta \). Then \( \mu' \) is an infinite spectral measure, and \( \mathcal{L}^1(X, \Omega, \mu'; \mathbb{R}) \) consists of all functions on \( X \) that vanish at \( x_0 \); for \( f \in \mathcal{L}^1(X, \Omega, \mu'; \mathbb{R}) \), \( \int_X f \, d\mu' \in \mathcal{A} \) is, again, the pointwise multiplication by \( f \). As above, there are elements of \( \mathcal{L}^1(X, \Omega, \mu'; \mathbb{R}) \) that are not essentially bounded.

In view of Example 2.28 the following result for normed algebras is, perhaps, a pleasant surprise. It is the counterpart of \cite[Proposition V.4]{20}, which states that an integrable function with respect to a spectral measure with values in the bounded operators on a Banach space (see \cite[Definition III.2 and Definition III.5]{20} for definitions) is automatically essentially bounded. The proof in our (rather different) context is considerably simpler.

**Lemma 2.29.** Let \( (X, \Omega) \) be a measurable space, and let \( \mathcal{A} \) be a \( \sigma \)-monotone complete partially ordered normed algebra such that \( \|x\| \leq \|y\| \) for all \( x, y \in \mathcal{A} \) with
Let $\mu : \Omega \to \mathbb{A}^+$ be a measure such that $\mu(\Delta)^2 = \mu(\Delta)$ for all $\Delta \in \Omega$ with finite measure. Then every element of $\mathcal{L}^1(\mathbb{R}; \mathbb{R})$ is essentially bounded.

**Proof.** Take an $f \in \mathcal{L}^1(\mathbb{R}; \mathbb{R})$. For $n \geq 1$, set $\Delta_n := \{ x \in X : |f(x)| \geq n \}$. Then the monotonicity of the norm on $\mathcal{A}^+$ implies that
\[
\|\mu(\Delta_n)\| \leq \frac{1}{n} \left\| \int_X |f| \, d\mu \right\|
\]
for $n \geq 1$. As each $\mu(\Delta_n)$ is an idempotent, its norm is either zero or at least one. Hence $\mu(\Delta_n) = 0$ for all sufficiently large $n$.

**Remark 2.30.** When every element of $\mathcal{L}^1(\mathbb{R}; \mathbb{R})$ is essentially bounded, there is a canonically defined embedding of $\mathbb{L}^1(X, \mu; \mathbb{R})$ as a vector sublattice of $\mathbb{B}(X, \mu; \mathbb{R})$. We shall allow ourselves to write $\mathbb{L}^1(X, \mu; \mathbb{R}) \subseteq \mathbb{B}(X, \mu; \mathbb{R})$ in this case.

### 2.7. An embedding as a closed subalgebra

It is known that, for a locally compact Hausdorff space $X$, the supremum norm is the minimal algebra norm on $C_0(X)$ and $C_0(X; \mathbb{C})$. We refer to [17] Theorem 6.2 for this fact; the special case of $C(X; \mathbb{C})$ for compact $X$ is also covered by [24] Theorem 1.2.4. The alternate norm need not be complete, nor need it be unital when $X$ is compact. It is only supposed that $\|fg\| \leq \|f\|\|g\|$ for $f, g \in C_0(X)$. This has the following consequence.

**Proposition 2.31.** Let $X$ be a locally compact Hausdorff space, let $A$ be a real normed algebra, and let $\pi : C_0(X) \to A$ be a continuous algebra homomorphism. Let $q : C_0(X) \to C_0(X)/\ker \pi$ be the quotient map, and let $\overline{\pi} : C_0(X)/\ker \pi \to A$ be the algebra homomorphism such that $\pi = \overline{\pi} \circ q$. Then we have $\|q(f)\| \leq \|\overline{\pi}(q(f))\| \leq \|\pi\|\|q(f)\|$ for $f \in C_0(X)$. Consequently, $\overline{\pi}$ is a topological embedding of $C_0(X)/\ker \pi$ as the subalgebra $\pi(C_0(X))$ of $A$, where $\pi(C_0(X))$ is closed in $A$.

We mention explicitly that $A$ is not required to be complete or unital; when it is unital, it is not required that its identity element have norm 1; when $A$ is unital and $X$ is compact, it is not required that $\pi$ be unital.

**Proof.** All will be clear once we know that $\|q(f)\| \leq \|\overline{\pi}(q(f))\|$ for $f \in C_0(X)$. This is a multiplicative property of $\overline{\pi}$, and $\overline{\pi}$ is a subalgebra homomorphism.

The analogous statement for $C_0(X; \mathbb{C})$ is obviously also valid, with a yet easier proof.
3. Commuting idempotents and spectral measures

As indicated in the introduction, our approach to obtain a representing spectral measure for a positive algebra homomorphism \( \pi \) from \( C_c(X) \) or \( C_0(X) \) into a partially ordered algebra consists of two steps. The first step is to invoke a Riesz representation theorem for positive operators from \([2]\), showing that there is a representing measure for \( \pi \). The second step is to use the multiplicativity of \( \pi \) to show that this representing measure is, in fact, a spectral measure. This second step follows from the key result Theorem 3.6 in this section. Its proof is easier than one might perhaps expect.

We need the following preparatory result for the proof of Theorem 3.6. The, perhaps, somewhat surprising equivalence of its parts (1) and (2) and the conditional validity of part (a) also hold in the absence of a partial ordering. Although we shall apply it in the context of associative algebras, neither the associativity of the multiplication nor the vector space structure is needed for its proof. In view of its rather basic nature, we have formulated it under minimal hypotheses. It applies to what could be called (not necessarily associative) partially ordered \( \mathbb{Z} \)-algebras that have no additive 2-torsion. Its parts (b) and (c) already hint at the moduli preserving properties of algebra homomorphisms in Section 4.

**Proposition 3.1.** Let \( X \) be a non-empty set, and let \( \Omega \) be an algebra of subsets of \( X \). Let \( \mathcal{A} \) be an abelian group that has no elements of order 2. Suppose that \( \mathcal{A} \) is supplied with a translation invariant partial ordering such that \( \mathcal{A}^+ \cap \mathcal{A}^+ \subseteq \mathcal{A}^+ \), and with a bi-additive map \( (a_1, a_2) \rightarrow a_1 a_2 \) from \( \mathcal{A} \times \mathcal{A} \) into \( \mathcal{A} \) such that \( \mathcal{A}^+ \mathcal{A}^+ \subseteq \mathcal{A}^+ \). Let \( \mu : \Omega \rightarrow \mathcal{A}^+ \) be a map such that

- \( \mu(\emptyset) = 0 \);
- \( \mu \left( \bigcup_{i=1}^n \Delta_i \right) = \sum_{i=1}^n \mu(\Delta_i) \) whenever \( \Delta_1, \ldots, \Delta_n \in \Omega \) are pairwise disjoint.

Then the following are equivalent:

1. \( \mu(\Delta)^2 = \mu(\Delta) \) for \( \Delta \in \Omega \) and \( \mu(\Delta_1) \mu(\Delta_2) = \mu(\Delta_2) \mu(\Delta_1) \) for \( \Delta_1, \Delta_2 \in \Omega \);
2. \( \mu(\Delta_1 \cap \Delta_2) = \mu(\Delta_1) \mu(\Delta_2) \) for \( \Delta_1, \Delta_2 \in \Omega \).

Suppose that this is the case. Set \( \mu(X)A := \{ a \in A : \mu(X)a = a \} \) and \( A_{\mu(X)} := \{ a \in A : a\mu(X) = a \} \). Then:

- (a) \( \mu(\Omega) \subseteq \mu(X)A \cap A_{\mu(X)} \);
- (b) for \( \Delta_1, \Delta_2 \in \Omega \), \( \mu(\Delta_1) \lor \mu(\Delta_2) \) exists in \( \mu(X)A, A_{\mu(X)} \); \( \mu(X), A \cap A_{\mu(X)} \); it equals \( \mu(\Delta_1 \cup \Delta_2) \) in all cases;
- (c) for \( \Delta_1, \Delta_2 \in \Omega \), \( \mu(\Delta_1) \land \mu(\Delta_2) \) exists in \( \mu(X)A, A_{\mu(X)} \); it equals \( \mu(\Delta_1 \cap \Delta_2) \) in all cases;
- (d) when supplied with the partial ordering inherited from \( \mathcal{A} \), the set \( \mu(\Omega) \) is a Boolean algebra with largest element \( \mu(X) \) and smallest element 0; the complement of \( a \in \mu(\Omega) \) is \( \mu(X) - a \). The map \( \mu : \Omega \rightarrow \mu(\Omega) \) is a homomorphism of Boolean algebras.
Proof: We prove that part (1) implies part (2). First, take $\Delta_1, \Delta_2 \in \Omega$ such that $\Delta_1 \cap \Delta_2 = \emptyset$. Then
\[
\mu(\Delta_1) + \mu(\Delta_2) = \mu(\Delta_1 \cup \Delta_2)
\]
\[
= (\mu(\Delta_1 \cup \Delta_2))^2
\]
\[
= (\mu(\Delta_1) + \mu(\Delta_2))^2
\]
\[
= \mu(\Delta_1) + 2\mu(\Delta_1)\mu(\Delta_2) + \mu(\Delta_2).
\]
Hence $\mu(\Delta_1)\mu(\Delta_2) = 0$ whenever $\Delta_1, \Delta_2 \in \Omega$ are disjoint. Next, take $\Delta_1, \Delta_2 \in \Omega$ arbitrary. Using what we have just established, we see that
\[
\mu(\Delta_1)\mu(\Delta_2) = (\mu(\Delta_1 \setminus (\Delta_1 \cap \Delta_2)) + \mu(\Delta_1 \cap \Delta_2))\left(\mu(\Delta_2 \setminus (\Delta_1 \cap \Delta_2)) + \mu(\Delta_1 \cap \Delta_2)\right)
\]
\[
+ \mu(\Delta_1 \cap \Delta_2)
\]
\[
= \mu(\Delta_1 \setminus (\Delta_1 \cap \Delta_2))\mu(\Delta_2 \setminus (\Delta_1 \cap \Delta_2))
\]
\[
+ \mu(\Delta_1 \setminus (\Delta_1 \cap \Delta_2))\mu(\Delta_1 \cap \Delta_2)
\]
\[
+ \mu(\Delta_1 \cap \Delta_2)\mu(\Delta_2 \setminus (\Delta_1 \cap \Delta_2)) + \mu(\Delta_1 \cap \Delta_2)\mu(\Delta_1 \cap \Delta_2)
\]
\[
= 0 + 0 + 0 + \mu(\Delta_1 \cap \Delta_2)^2
\]
\[
= \mu(\Delta_1 \cap \Delta_2),
\]
as required. It is clear that part (2) implies part (1).

Suppose that the properties in the parts (1) and (2) are valid.

Then part (a) follows from part (2).

We turn to part (b). We consider the supremum in $\{a \in A : a\mu(X) = a\}$; the other case is handled similarly, and each of these implies the statements in the remaining two cases. Take $\Delta_1, \Delta_2 \in \Omega$. Then $\mu(\Delta_1 \cup \Delta_2) \geq \mu(\Delta_1)$ and $\mu(\Delta_1 \cup \Delta_2) \geq \mu(\Delta_2)$. Suppose that $a \in A$ is such that $a \geq \mu(\Delta_1)$, $a \geq \mu(\Delta_2)$, and $a\mu(X) = a$. For any $\Delta_3 \in \Omega$ such that $\Delta_3 \subseteq \Delta_1$, we then have $a \geq \mu(\Delta_3)$, so that $a\mu(\Delta_3) \geq \mu(\Delta_3)^2 = \mu(\Delta_3)$; and similarly for any $\Delta_3 \subseteq \Delta_2$. Using this, we see that
\[
a = a\mu(X)
\]
\[
= a(\mu(\Delta_1 \setminus (\Delta_1 \cap \Delta_2)) + \mu(\Delta_1 \cap \Delta_2) + \mu(\Delta_2 \setminus (\Delta_1 \cap \Delta_2))
\]
\[
+ \mu(X \setminus (\Delta_1 \cup \Delta_2))
\]
\[
\geq \mu(\Delta_1 \setminus (\Delta_1 \cap \Delta_2)) + \mu(\Delta_1 \cap \Delta_2) + \mu(\Delta_2 \setminus (\Delta_1 \cap \Delta_2))
\]
\[
= \mu(\Delta_1 \cup \Delta_2).
\]
This concludes the proof of part (b).

For part (c), we consider only the infimum in $\{a \in A : a\mu(X) = a\}$. The other case is handled similarly, and each of these implies the statements in the remaining two cases. Take $\Delta_1, \Delta_2 \in \Omega$. It is clear that $\mu(\Delta_1 \cap \Delta_2) \leq \mu(\Delta_1)$ and $\mu(\Delta_1 \cap \Delta_2) \leq \mu(\Delta_2)$. Suppose that $a \in A$ is such that $a \leq \mu(\Delta_1)$, $a \leq \mu(\Delta_2)$, and $a\mu(X) = a$. Then $\mu(X) - a \geq \mu(X) - \mu(\Delta_1) = \mu(X \setminus \Delta_1)$; likewise, $\mu(X) - a \geq \mu(X \setminus \Delta_2)$. Part (b) then implies that $\mu(X) - a \geq \mu(X \setminus (\Delta_1 \cap \Delta_2))$, showing that $a \leq \mu(\Delta_1 \cap \Delta_2)$. This concludes the proof of part (c).
Now that the parts [b] and [c] have been established, it is clear that $\mu(\Omega)$ is a lattice when supplied with the partial ordering inherited from $\mathcal{A}$. It is routine to verify the statements in part [d].

\[ \square \]

**Remark 3.2.** If the multiplication in $\mathcal{A}$ in Proposition 3.1 is associative, then clearly $\mu(X)A = \mu(X)A$, $A\mu(X) = A\mu(X)$, and $\mu(X)A \cap A\mu(X) = \mu(X)A\mu(X)$.

Although we shall not use it in our proofs, before we proceed, we still want to mention the following strengthening of what we believe to be a noteworthy result of Alekhno’s in [2]. Compared to Proposition 3.1 its premises are of a different nature, but its conclusions are in the same vein. In Proposition 3.1 the commutativity of the multiplication on the idempotents is supposed in part (1) and its associativity follows from part (2). In Proposition 3.3 it is just the other way around. As in [2], the ordering is already necessary to establish the purely algebraic statement in part (1) of Proposition 3.3 which, just as Proposition 3.1 applies to (not necessarily associative) partially ordered $\mathcal{Z}$-algebras, but now also when there is additive 2-torsion.

**Proposition 3.3** (Alekhno). Let $\mathcal{A}$ be an abelian group that is supplied with a translation invariant partial ordering such that $\mathcal{A}^+ + \mathcal{A}^+ \subseteq \mathcal{A}^+$, and with a biadditive map $(a_1, a_2) \mapsto a_1a_2$ from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$ such that $\mathcal{A}^+\mathcal{A}^+ \subseteq \mathcal{A}^+$. Take a positive idempotent $e$ in $\mathcal{A}$, and let

$$\text{OI}(\mathcal{A}; e) := \{ p \in \mathcal{A} : 0 \leq p \leq e, p^2 = p, ep = pe = p \}$$

denote the order idempotents relative to $e$. Suppose that $(p_1p_2)p_3 = p_1(p_2p_3)$ whenever $p_1, p_2, p_3 \in \text{OI}(\mathcal{A}; e)$. Set $e\mathcal{A} := \{ a \in \mathcal{A} : ea = a \}$ and $\mathcal{A}_e := \{ a \in \mathcal{A} : ae = a \}$. Then $\text{OI}(\mathcal{A}; e) \subseteq e\mathcal{A} \cap \mathcal{A}_e$, and:

1. $p_1p_2 = p_2p_1$ for all $p_1, p_2 \in \text{OI}(\mathcal{A}; e)$;
2. for $p_1, p_2 \in \text{OI}(\mathcal{A}; e)$, $p_1 \lor p_2$ exists in $e\mathcal{A}$, $\mathcal{A}_e$, $e\mathcal{A} \cap \mathcal{A}_e$, and $\text{OI}(\mathcal{A}; e)$. In all cases, it equals $p_1 + p_2 - p_1p_2 \in \text{OI}(\mathcal{A}; e)$;
3. for $p_1, p_2 \in \text{OI}(\mathcal{A}; e)$, $p_1 \land p_2$ exists in $e\mathcal{A}$, $\mathcal{A}_e$, $e\mathcal{A} \cap \mathcal{A}_e$, and $\text{OI}(\mathcal{A}; e)$. In all cases, it equals $p_1p_2 \in \text{OI}(\mathcal{A}; e)$.

Supplied with the partial ordering inherited from $\mathcal{A}$, $\text{OI}(\mathcal{A}; e)$ is a Boolean algebra with largest element $e$ and smallest element $0$; the complement of $p \in \text{OI}(\mathcal{A}; e)$ is $e - p$.

Suppose that every non-empty upward directed subset of $\text{OI}(\mathcal{A}; e)$ has a supremum in $e\mathcal{A} \cap \mathcal{A}_e$. Take a non-empty (not necessarily directed) subset of $\text{OI}(\mathcal{A}; e)$. Then its supremum and infimum exist in $e\mathcal{A} \cap \mathcal{A}_e$, and they are both elements of $\text{OI}(\mathcal{A}; e)$. Hence $\text{OI}(\mathcal{A}; e)$ is a complete Boolean algebra in this case.

**Remark 3.4.**

1. In the algebra of all operators from a vector lattice to itself, the order idempotents relative to the identity operator are precisely the order projections; see [5] Theorem 1.44, for example. This motivates the terminology.
2. Alekhno’s actual statements in [2] Lemma 2.1 and Corollary 2.2] and their proofs are given in the context of an ordered Banach algebra $\mathcal{A}$.
with a positive identity element $e$. An inspection of his arguments shows that, with a few adaptations, they also suffice to establish Proposition 3.3.

Returning to the main line, we have the following consequence of Proposition 3.1.

**Proposition 3.5.** Let $(X, \Omega)$ be a measurable space, let $\mathcal{A}$ be a $\sigma$-monotone complete partially ordered algebra, and let $\mu : \Omega \to \mathcal{A}^+$ be a measure. Then the following are equivalent:

1. $\mu(\Delta)^2 = \mu(\Delta)$ for all $\Delta \in \Omega$ with finite measure, and $\mu(\Delta_1)\mu(\Delta_2) = \mu(\Delta_2)\mu(\Delta_1)$ for all $\Delta_1, \Delta_2 \in \Omega$ with finite measure;
2. $\mu$ is a spectral measure.

**Proof.** We prove that part (1) implies part (2); the converse is trivial. Take $\Delta_1, \Delta_2 \in \Omega$ with finite measure. Set $\Omega' = \{ \Delta \cap (\Delta_1 \cup \Delta_2) : \Delta \in \Omega \}$, and define $\mu' : \Omega' \to \mathcal{A}^+$ by setting $\mu'(\Delta') = \mu(\Delta')$ for $\Delta' \in \Omega'$. Then Proposition 3.1 applies to the finite measure $\mu'$ on $\Omega'$ and shows that, in particular, $\mu(\Delta_1 \cap \Delta_2) = \mu(\Delta_1)\mu(\Delta_2)$.

We now come to one of our key results.

**Theorem 3.6.** Let $X$ be a locally compact Hausdorff space, let $\mathcal{A}$ be a monotone complete partially ordered algebra with a monotone continuous multiplication, and let $\mu : \mathcal{B} \to \mathcal{A}^+$ be a regular Borel measure. Let $\pi : C_c(X) \to \mathcal{A}$ be a positive algebra homomorphism.

1. Suppose that $\mu(V) = \sup\{\pi(f) : f \prec V\}$ for every open subset $V$ of $X$ with finite measure. Then $\mu$ is a spectral measure.
2. Suppose that $\mu(K) = \inf\{\pi(f) : K \prec f\}$ for every compact subset $K$ of $X$, and that $\mu$ is inner regular at all Borel subsets with finite measure. Then $\mu$ is a spectral measure.

**Proof.** We prove part (1). The first step is to show that every $\mu(\Delta)$ for $\Delta \in \mathcal{B}$ with finite measure is an idempotent. We start with a special case. Take an open subset $V$ of $X$ with finite measure. If $f_1, f_2 \prec V$, then also $f_1 f_2 \prec V$. If $f \prec V$, then also $\sqrt{f} \prec V$. Using the first statement in Lemma 2.9, we thus see that

$$\mu(V) = \sup\{\pi(f) : f \prec V\}$$
$$= \sup\{\pi(f_1)\pi(f_2) : f_1, f_2 \prec V\}$$
$$= \sup\{\pi(f_1)\pi(f_2) : f_1, f_2 \prec V\}$$
$$= \sup\{\pi(f) : f \prec V\} \cdot \sup\{\pi(f) : f \prec V\}$$
$$= \mu(V)^2.$$

For the general case, take a Borel subset $\Delta$ of $X$ with finite measure. Using the outer regularity of $\mu$, what he have just established, and the second statement in Lemma 2.9, we have

$$\mu(\Delta) = \inf\{\mu(V) : V \text{ is open, } \Delta \subseteq V, \text{ and } \mu(V) < \infty\}$$
\[
= \inf \{ \mu(V) : V \text{ is open, } \Delta \subseteq V, \text{ and } \mu(V) < \infty \}
= \inf \{ \mu(V) : V \text{ is open, } \Delta \subseteq V, \text{ and } \mu(V) < \infty \}^2
= \mu(\Delta)^2.
\]

In the second step, we proceed to show that the \( \mu(\Delta) \) for \( \Delta \in \mathcal{B} \) with finite measure all commute. Again, we start with a special case. Take open subsets \( V_1, V_2 \) of \( X \) with finite measure. Using the first statement in Lemma 2.9 we have
\[
\mu(V_1)\mu(V_2) = \sup \{ \pi(f_1) : f_1 < V_1 \} \cdot \sup \{ \pi(f_2) : f_2 < V_2 \}
= \sup \{ \pi(f_1)\pi(f_2) : f_1 < V_1, f_2 < V_2 \}
= \sup \{ \pi(f_2)\pi(f_1) : f_2 < V_2, f_1 < V_1 \}
= \sup \{ \pi(f_2) : f_2 < V_2 \} \cdot \sup \{ \pi(f_1) : f_1 < V_1 \}
= \mu(V_1)\mu(V_2).
\]

Using this, the outer regularity of \( \mu \) at all Borel subsets of \( X \) and the decreasing analogue of the first statement in Lemma 2.9 show that \( \mu(\Delta_1)\mu(\Delta_2) = \mu(\Delta_1)\mu(\Delta_2) \) for \( \Delta_1, \Delta_2 \in \mathcal{B} \) with finite measure. We can now invoke Proposition 3.5 to conclude that \( \mu \) is a spectral measure.

The proof of part [2] is similar. \( \square \)

4. Relations between measures, algebra homomorphisms, and vector lattice homomorphisms

Let \((X, \Omega, \mu, E)\) be a measure space, and consider the associated integral operator \( I_\mu : \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \to E \). If \( E \) is a vector lattice, for which measures \( \mu \) is \( I_\mu \) a vector lattice homomorphism? If \( E \) is a partially ordered algebra, for which measures \( \mu \) is \( I_\mu \) an algebra homomorphism? In this section, we shall consider these and other relations between measures, algebra homomorphisms, and vector lattice homomorphisms. The notion of moduli preserving operators from Section 2.4 is a convenient one to formulate some of the results in this section with.

The two questions just raised are easily answered; see Proposition 4.1 and Proposition 4.2 Somewhat surprisingly, when \( E \) is a partially ordered algebra, \( \mu \) is finite, and \( I_\mu : \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \to E \) is an algebra homomorphism, there is always a vector lattice homomorphism associated with \( I_\mu \). Indeed, as Theorem 4.4 shows, the image of \( \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \) under \( I_\mu \) is then a vector lattice, and \( I_\mu : \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \to I_\mu(\mathcal{L}^1(X, \Omega, \mu; \mathbb{R})) \) is a vector lattice algebra homomorphism; see also Theorem 4.6 The results are particularly nice when \( E \) is a vector lattice algebra to begin with; see Theorem 4.5 Various equivalences and (conditional) implications between the properties of finite measures with values in vector lattice algebras and those of the associated integral operators are collected in Theorem 4.6

**Proposition 4.1.** Let \((X, \Omega)\) be a measurable space, let \( E \) be a \( \sigma \)-Dedekind complete vector lattice, and let \( \mu : \Omega \to E^+ \) be a measure. Then the following are equivalent:
Proof. It is clear that part (1) implies part (2). We show that part (2) implies part (1). Take integrable elementary functions \( \varphi_1 \) and \( \varphi_2 \). There exist mutually disjoint elements \( \Delta_1, \ldots, \Delta_n \) of \( \Omega \) with finite measure, \( \alpha_1, \ldots, \alpha_n \geq 0 \), and \( \beta_1, \ldots, \beta_n \geq 0 \) such that \( \varphi_1 = \sum_{i=1}^n \alpha_i \chi_{\Delta_i} \) and \( \varphi_2 = \sum_{i=1}^n \beta_i \chi_{\Delta_i} \). As a consequence of the assumption, \( \mu(\Delta_i) \land \mu(\Delta_j) = 0 \) for all \( k, l = 1, \ldots, n \) such that \( k \neq l \). Using [4] Theorem 1.7(4), this implies that \( I_\mu(\varphi_1) \land I_\mu(\varphi_2) = \sum_{i=1}^n \alpha_i \mu(\Delta_i) \land \sum_{i=1}^n \beta_i \mu(\Delta_i) = \sum_{i=1}^n (\alpha_i \land \beta_i) \mu(\Delta_i) = I_\mu(\varphi_1 \land \varphi_2) \). It then follows from the definition of the order integral and the (sequential) order continuity of the lattice operations in a vector lattice that \( I_\mu(f_1 \land f_2) = I_\mu(f_1) \land I_\mu(f_2) \) for \( f_1, f_2 \in \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}^+) \). This implies that \( I_\mu \) is a vector lattice homomorphism from \( \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \) into \( E \).

The equivalence of the parts (2) and (3) follows from the fact that \( \mu(\Delta_1) + \mu(\Delta_2) = \mu(\Delta_1 \cup \Delta_2) + \mu(\Delta_1 \land \Delta_2) \) for all \( \Delta_1, \Delta_2 \in \Omega \) with finite measure.

Suppose that \( \mu \) is finite. Then it is clear that part (4) implies part (2). Evidently, part (1) then implies part (4).

The final statements follow from [8] Theorem 6.17. \( \square \)

Proposition 4.2. Let \( (X, \Omega) \) be a measurable space, let \( \mathcal{A} \) be a \( \sigma \)-monotone complete partially ordered algebra with a \( \sigma \)-monotone continuous multiplication, and let \( \mu: \Omega \to \mathbb{A}^+ \) be a measure. Then the following are equivalent:

(1) \( \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \) is a commutative algebra, and \( I_\mu: \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \to \mathcal{A} \) is an algebra homomorphism;

(2) \( \mu \) is a spectral measure.

When \( \mu \) is finite, these are also equivalent to each of:

(3) \( I_\mu: \mathcal{B}(X, \Omega; \mathbb{R}) \to \mathcal{A} \) is an algebra homomorphism;

(4) \( \mu(\Delta)^2 = \mu(\Delta) \) for \( \Delta \in \Omega \) and \( \mu(\Delta_1)\mu(\Delta_2) = \mu(\Delta_1)\mu(\Delta_2) \) for \( \Delta_1, \Delta_2 \in \Omega \).

Proof. It is obvious that part (1) implies part (2). We show that part (2) implies part (1). For this, it is sufficient to show that \( fg \in \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \) whenever
Let \( f, g \in \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}^+) \), and that then \( I_\mu(fg) = I_\mu(f)I_\mu(g) \). These are both easily seen to hold when \( f \) and \( g \) are integrable elementary functions. For the general case, we choose integrable elementary functions \( \varphi_1, \varphi_2, \ldots \) and \( \psi_1, \psi_2, \ldots \) such that \( \varphi_m \uparrow f \) and \( \psi_n \uparrow g \) pointwise. Then \( I_\mu(\varphi_m \psi_n) = I_\mu(\varphi_m)I_\mu(\psi_n) \) for \( m, n = 1, 2, \ldots \). For fixed \( m \), we let \( n \) tend to infinity. The definition of the order integral and the \( \sigma \)-monotone continuity of the multiplication in \( \mathcal{A} \) then show that \( \varphi_m g \in \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}^+) \), and that \( I_\mu(\varphi_m g) = I_\mu(\varphi_m)I_\mu(g) \) for \( m = 1, 2, \ldots \). Now we let \( m \) tend to infinity, and use the monotone convergence theorem (see [8, Theorem 6.9]) and the \( \sigma \)-monotone continuity of the multiplication to conclude that \( fg \in \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \), and that \( I_\mu(fg) = I_\mu(f)I_\mu(g) \).

Suppose that \( \mu \) is finite. Then it is clear that part (3) implies part (2). Evidently, part (1) then implies part (3). Proposition 3.1 shows that the parts (2) and (4) are equivalent.

**Remark 4.3.** In the context of spectral measures which take their values in the bounded operators on a Banach space in the sense of [20, Definition III.2], it is also true that the integrable functions (see their ‘weak’ definition in [20, Definition III.5]) form an algebra, and that the pertinent integral operator is an algebra homomorphism; see [20, Proposition V.3]. Since it is established a little later that the integrable functions are precisely the essentially bounded ones (see [20, Proposition V.4]), the fact that they form an algebra becomes somewhat less surprising. Example 2.28 shows, however, that, for the spectral measures in our sense, integrable functions need not be essentially bounded. The fact that they still always form an algebra is, therefore, a longer lasting surprise than the corresponding statement in [20, Proposition V.3].

In the context of Proposition 4.2, suppose that \( \mu \) is finite, and that the four then equivalent statements in it hold. Then the parts (1) and (2) of Proposition 3.1 apply. Its parts (b) and (c) then show that, when appropriately interpreted, \( I_\mu \) preserves the supremum and infimum of the characteristic functions of two measurable subsets of \( X \). This is actually true for two arbitrary elements of \( \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \), as is shown by part (2) of the following result.

**Theorem 4.4.** Let \( (X, \Omega) \) be a measurable space, let \( \mathcal{A} \) be a \( \sigma \)-monotone complete partially ordered algebra with a \( \sigma \)-monotone continuous multiplication, and let \( \mu: \Omega \rightarrow \mathcal{A}^+ \) be a finite spectral measure. Then:

1. \( I_\mu(f) \in \mu(X)A \mu(X) \) for \( f \in \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \);
2. the maps \( I_\mu: \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \rightarrow \mu(X)A \), \( I_\mu: \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \rightarrow \mathcal{A} \mu(X) \), and \( I_\mu: \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \rightarrow \mu(X) \mu(\mathcal{A} \mu(X)) \) preserve moduli;
3. \( \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \) is a commutative unital vector lattice algebra;
4. when \( I_\mu(\mathcal{L}^1(X, \Omega, \mu; \mathbb{R})) \) is supplied with the partial ordering inherited from \( \mathcal{A} \), it is a commutative unital vector lattice algebra with \( \mu(X) \) as its positive multiplicative identity element. The map \( I_\mu: \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \rightarrow I_\mu(\mathcal{L}^1(X, \Omega, \mu; \mathbb{R})) \) is then a surjective unital vector lattice algebra homomorphism.
Part (1) follows from Proposition 4.2 and the fact that the constant 1 function is integrable. For the moduli preserving properties of the two maps in part (2), we consider the case with codomain \( \mu(X)A \). The other can be treated similarly, and clearly each of these implies the statement for their subset \( \mu(X)A_\mu(X) \). Take \( f, g \in \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \). According to Lemma 2.18 the proof will be complete once we show that \( I_\mu(f) \lor I_\mu(g) \) exists in \( \mu(X)A \), and that it equals \( I_\mu(f \lor g) \). Certainly, \( I_\mu(f \lor g) \) is an upper bound of \( \{I_\mu(f), I_\mu(g)\} \). Let \( a \in \mu(X)A \) also be an upper bound of this set.

As a preparation, suppose that \( \varphi \) and \( \psi \) are integrable elementary functions such that \( a \geq I_\mu(\varphi) \) and \( a \geq I_\mu(\psi) \). There exist mutually disjoint \( \Delta_1, \ldots, \Delta_n \in \Omega \) such that \( X = \bigcup_{i=1}^n \Delta_i \), \( \alpha_1, \ldots, \alpha_n \geq 0 \), and \( \beta_1, \ldots, \beta_n \geq 0 \), such that \( \varphi = \sum_{i=1}^n \alpha_i \chi_{\Delta_i} \) and \( \psi = \sum_{i=1}^n \beta_i \chi_{\Delta_i} \). We have \( a \geq I_\mu(\varphi) = \sum_{i=1}^n \alpha_i \mu(\Delta_i) \geq \alpha_j \mu(\Delta_j) \) for \( j = 1, \ldots, n \). Hence \( \mu(\Delta_j)a \geq \alpha_j \mu(\Delta_j)^2 = \alpha_j \mu(\Delta_j) \) for \( j = 1, \ldots, n \). A similar argument applies to \( \psi \), and we conclude that \( \mu(\Delta_j)a \geq (\alpha_j \lor \beta_j) \mu(\Delta_j) \) for \( j = 1, \ldots, n \). Then

\[
a = \mu(X)a = \sum_{i=1}^n \mu(\Delta_i)a \\
\geq \sum_{i=1}^n (\alpha_i \lor \beta_i) \mu(\Delta_i) = I_\mu(\varphi \lor \psi).
\]

After this preparation, we choose integrable elementary functions \( \varphi_1, \varphi_2, \ldots \) and \( \psi_1, \psi_2, \ldots \) such that \( \varphi_m \uparrow f \) and \( \psi_n \uparrow g \) pointwise. Then \( a \geq I_\mu(f) \geq I_\mu(\varphi_m) \) and \( a \geq I_\mu(g) \geq I_\mu(\psi_n) \) for \( m, n = 1, 2, \ldots \), so that \( a \geq I_\mu(\varphi_m \lor \psi_n) \) for \( m, n = 1, 2, \ldots \). On letting \( n \) tend to infinity, we see from the definition of the order integral that \( a \geq I_\mu(f \lor \psi_n) \) for \( n = 1, 2, \ldots \). Now we let \( n \) tend to infinity, and invoke the monotone convergence theorem (see [8] Theorem 6.9) to conclude that \( a \geq I_\mu(f \lor g) \), as required.

The parts (3) and (4) follow from part (1), Lemma 2.19, and Proposition 4.2. Part (5) follows from [8] Theorem 6.17. □

When \( A \) is a vector lattice algebra to begin with, Theorem 4.4 can sometimes be improved.
Theorem 4.5. Let \((X, \Omega)\) be a measurable space, and let \(A\) be a \(\sigma\)-Dedekind complete vector lattice algebra with a positive identity element \(e\) and \(\sigma\)-monotone continuous multiplication. Let \(\mu : \Omega \to A^+\) be a finite spectral measure. Suppose that \(\mu(X) \leq e\). Then:

1. \(\mu(X)A\mu(X)\) is a vector lattice subalgebra of \(A\) that is also a projection band in \(A\);
2. \(L^1(X, \Omega, \mu; \mathbb{R})\) is a unital vector lattice algebra, and \(I_\mu : L^1(X, \Omega, \mu; \mathbb{R}) \to A\) is a vector lattice algebra homomorphism;
3. (a) the image of \(\mathcal{B}(X, \Omega; \mathbb{R})\) under \(I_\mu\) is contained in the order ideal of \(\mu(X)A\mu(X)\) that is generated by \(\mu(X)\);
   (b) the image of \(L^1(X, \Omega, \mu; \mathbb{R})\) under \(I_\mu\) is contained in the order ideal of \(\mu(X)A\mu(X)\) that is generated by \(\mu(X)\);
4. the kernel of \(I_\mu : L^1(X, \Omega, \mu; \mathbb{R}) \to A\) is \(\mathcal{N}(X, \Omega, \mu; \mathbb{R})\), so that the quotient \(L^1(X, \Omega, \mu; \mathbb{R}) / \mathcal{N}(X, \Omega, \mu; \mathbb{R})\) is \(\sigma\)-Dedekind complete, and \(I_\mu : L^1(X, \Omega, \mu; \mathbb{R}) \to A\) and \(I_\mu : L^1(X, \Omega, \mu; \mathbb{R}) \to \mu(X)A\mu(X)\) are both \(\sigma\)-order continuous. If, in addition, \(A\) is monotone complete and has the countable sup property, then \(L^1(X, \Omega, \mu; \mathbb{R})\) is Dedekind complete, it has the countable sup property, and \(I_\mu : L^1(X, \Omega, \mu; \mathbb{R}) \to A\) and \(I_\mu : L^1(X, \Omega, \mu; \mathbb{R}) \to \mu(X)A\mu(X)\) are both order continuous.

Proof. We prove part (1). It is clear that \(\mu(X)A\mu(X)\) is a subalgebra of \(A\). Define the idempotent map \(P : A \to A\) by setting \(Pa := \mu(X)a\mu(X)\) for \(a \in A\). Since \(\mu(X) \leq e\), \(P\) lies between the zero map and the identity map on \(A\). By [5] Theorem 1.44, \(P\) is an order projection on \(A\). Hence its range \(\mu(X)A\mu(X)\) is a projection band in \(A\).

For part (2), we apply Theorem 4.4. It shows that \(L^1(X, \Omega, \mu; \mathbb{R})\) is a commutative unital vector lattice algebra, and also that \(I_\mu : L^1(X, \Omega, \mu; \mathbb{R}) \to \mu(X)A\mu(X)\) preserves moduli. Since we know here that \(\mu(X)A\mu(X)\) is a vector sublattice of \(A\), \(I_\mu : L^1(X, \Omega, \mu; \mathbb{R}) \to A\) is a vector lattice homomorphism.

The first statement of part (3) follows from the triangle inequality for the order integral; see [8] Lemma 6.7. The second is clear from the definition of that integral.

Part (4) follows from [8] Theorem 6.17, except the \(\sigma\)-order continuity of \(I_\mu : L^1(X, \Omega, \mu; \mathbb{R}) \to \mu(X)A\mu(X)\). Since \(\mu(X)A\mu(X)\) is a projection band in \(A\), this follows from the \(\sigma\)-order continuity of \(I_\mu : L^1(X, \Omega, \mu; \mathbb{R}) \to A\).

In the following overview result, we collect various equivalences and conditional implications that are valid for finite measures when \(A\) is a vector lattice algebra. As Theorem 4.4 shows, vector lattice algebras can enter the picture as a codomain even when the initial codomain is only a partially ordered algebra. Regarding the condition in part (b) of Theorem 4.6 we recall that a vector lattice algebra \(A\) is called an \(f\)-algebra when the left and right multiplications preserve disjointness. For this, it is necessary and sufficient that, for \(x, y, z \in A^+\),
\[(zx) \land y = (xz) \land y = 0\] whenever \(x \land y = 0\). A vector lattice algebra with an identity element is an \(f\)-algebra if and only if its squares are positive. We refer to [26 Corollary 1] for this; additional equivalent characterisations of \(f\)-algebras among the vector lattice algebras with a positive identity element can be found in [14, Theorem 2.3]. An \(f\)-algebra is commutative; see [5 Theorem 2.56]. An \(f\)-algebra is called \textit{semiprime} when 0 is its only nilpotent element. Every \(f\)-algebra with an identity element is semiprime; see [11 Theorem 10.4]. The orthomorphisms on a vector lattice form an \(f\)-algebra with an identity element (see [5 Theorem 2.59]) which is then semiprime. When they are algebras, our spaces \(\mathcal{L}^1(X, \Omega, \mu; \mathbb{R})\) and \(L^1(X, \Omega, \mu; \mathbb{R})\) are semiprime \(f\)-algebras. When \(\mathcal{A}\) is a commutative complex \(C^\ast\)-algebra, its self-adjoint part is a semiprime \(f\)-algebra; this is clear from its realisation as a \(C_0(X)\)-space.

\textbf{Theorem 4.6.} Let \((X, \Omega)\) be a measurable space, let \(\mathcal{A}\) be a \(\sigma\)-Dedekind complete vector lattice algebra with a \(\sigma\)-monotone continuous multiplication, and let \(\mu : \Omega \to \mathcal{A}^+\) be a finite measure.

The following are equivalent:

1. \(I_\mu : \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \to \mathcal{A}\) is a vector lattice homomorphism;
2. \(I_\mu : \mathcal{B}(X, \Omega; \mathbb{R}) \to \mathcal{A}\) is a vector lattice homomorphism;
3. \(\mu(\Delta_1 \cap \Delta_2) = \mu(\Delta_1) \land \mu(\Delta_2)\) in \(\mathcal{A}\) for \(\Delta_1, \Delta_2 \in \Omega\);
4. \(\mu(\Delta_1 \cup \Delta_2) = \mu(\Delta_1) \lor \mu(\Delta_2)\) in \(E\) for \(\Delta_1, \Delta_2 \in \Omega\).

The following are equivalent:

5. \(\mathcal{L}^1(X, \Omega, \mu; \mathbb{R})\) is a commutative algebra, and \(I_\mu : \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \to \mathcal{A}\) is an algebra homomorphism;
6. \(I_\mu : \mathcal{B}(X, \Omega; \mathbb{R}) \to \mathcal{A}\) is an algebra homomorphism;
7. \(\mu\) is a spectral measure;
8. \(\mu(\Delta)^2 = \mu(\Delta)\) for \(\Delta \in \Omega\) and \(\mu(\Delta_1)\mu(\Delta_2) = \mu(\Delta_2)\mu(\Delta_1)\) for \(\Delta_1, \Delta_2 \in \Omega\).

Suppose that at least one of the following is satisfied:

(a) \(\mathcal{A}\) has a positive identity element \(e\) and \(\mu(X) \leq e\);
(b) \(\mathcal{A}\) is a semiprime \(f\)-algebra.

Then each of the equivalent parts \(\text{(5)} - \text{(8)}\) implies each of the equivalent parts \(\text{(1)} - \text{(4)}\).

\textbf{Proof.} The equivalence of the parts \(\text{(1)} - \text{(4)}\) follows from Proposition 4.1. The proofs that the parts \(\text{(5)} - \text{(8)}\) follows from Proposition 4.2.

Suppose that \(\mathcal{A}\) has a positive identity element \(e\), that \(\mu(X) \leq e\), and that part \(\text{(7)}\) holds. Then Theorem 4.5 shows that part \(\text{(1)}\) holds.

Suppose that \(\mathcal{A}\) is a semiprime \(f\)-algebra and that part \(\text{(6)}\) holds. Since \(\mathcal{B}(X, \Omega; \mathbb{R})\) is a semiprime \(f\)-algebra, and since a positive algebra homomorphism between two semiprime \(f\)-algebras is automatically a vector lattice homomorphism (see [11 p. 96]), we see that part \(\text{(2)}\) holds. \(\square\)
5. UPS AND DOWNS

Suppose that $E$ is a partially ordered vector space, that $X$ is a locally compact Hausdorff space, and that $\pi: \mathbb{C}_c(X) \to E$ is a positive operator. The earlier paper [9] contains a number of results to the extent that, under appropriate conditions, there is unique regular $E^+$-valued Borel measure $\mu$ on $X$ such that

$$\pi(f) = \int_X f \, d\mu$$

for all $f \in \mathbb{C}_c(X)$. Moreover, if $V$ is a non-empty open subset of $X$, then

$$\mu(V) = \bigvee \{\pi(f) : f \prec V\}$$

in $E^+$; and if $K$ is a compact subset of $X$, then

$$\mu(K) = \bigwedge \{\pi(f) : K \prec f\}$$

in $E^+$. The original operator $\pi$ can be extended to $I_\mu: \mathcal{L}^1(X, \Omega; \mathbb{R}) \to E$, and to $I_\mu: \mathcal{B}(X, \Omega; \mathbb{R}) \to E$ when $\mu$ is finite. Can we then describe the images $I_\mu$ more directly in terms of $\pi(\mathbb{C}_c(X))$? It turns out that this can often be done. The underlying reason is that (as Section 4 shows) $I_\mu$ often preserves moduli; if $E$ has the countable sup property, this fact is then sufficient to make such a description possible. The present section is devoted to this.

We shall actually work in a more general context, where a Borel measure and a positive operator are supposed to be related in a certain way. This allows us to obtain our results without unnecessary restrictions on, in particular, $E$. The Riesz representation theorems in [9] for positive operators $\pi: \mathbb{C}_c(X) \to E$ then guarantee that, under appropriate conditions on, in particular, $E$, these hypotheses are indeed satisfied, enabling us to apply the results in the present section to Section 7 where—this is not needed in the present section—$\mu$ will even be a spectral measure.

The description of the images of $I_\mu$ will be in terms of ups and downs. We start with some preparations.

Let $E$ be a partially ordered vector space, and let $S$ be a non-empty subset of $E$. We define

$$S^\uparrow := \{x \in E : \text{there exists a net } \{x_\lambda\}_{\lambda \in \Lambda} \text{ in } S \text{ such that } x_\lambda \uparrow x \text{ in } E\}$$

and its sequential version

$$S^\uparrow := \{x \in E : \text{there exists a sequence } \{x_n\}_{n=1}^{\infty} \text{ in } S \text{ such that } x_n \uparrow x \text{ in } E\}$$

and define $S^\downarrow$ and $S^\downarrow$ similarly. We shall use self-evident notations such as $S^\uparrow$, etc.

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10In [5], our $S^\uparrow$, $S^\downarrow$, $S^\uparrow$, and $S^\downarrow$ are denoted by $S^\uparrow$, $S^\downarrow$, $S^\downarrow$, and $S^\downarrow$, respectively; in [12], they are $S^\uparrow$, $S^\downarrow$, $S^\downarrow$, and $S^\downarrow$, respectively. Our notation may be a little clearer in smaller font.
Remark 5.1. When $F$ is a linear subspace of $E$ and $S$ is a non-empty subset of $F$, the ups and downs of $S$ in $F$ and in $E$ need not be the same. With Theorem [4,4] in mind, we note that, for a monotone complete partially ordered algebra $A$ with monotone continuous multiplication and an idempotent $p \in \mathcal{A}^+$, the ups and downs of a non-empty subset of $pA$ in $pA$ coincide with their counterparts in $A$. This follows from Lemma [2,10]. Similar statements hold for $A^p$ and $pAp$.

We collect a few basic facts in the next three results.

Lemma 5.2. Let $S$ be a non-empty subset of a partially ordered vector space $E$, and let $\lambda \geq 0$.

1. If $S + S \subseteq S$, or if $\lambda S \subseteq S$, then $S^\uparrow, S^\updownarrow, S^\downarrow, \text{ and } S^\uparrow$ all have the same respective property.
2. If $S^\uparrow$ exists in $E$ and $S^\downarrow = S$, then $S^\uparrow = S^\uparrow$ and $S^\downarrow = S^\downarrow$.
3. If $S^\downarrow$ exists in $E$ and $S^\uparrow = S$, then $S^\uparrow = S^\uparrow$ and $S^\downarrow = S^\downarrow$.
4. Suppose that $E$ is a vector lattice. If $S^\downarrow = S$, or if $S^\downarrow = S$, then each of $S^\uparrow, S^\updownarrow, S^\downarrow$, and $S^\downarrow$ has the same respective property.

Proof. The parts [1] and [4] are routine to establish.

We prove part [2]; the proof of part [3] is similar.

Take an $x \in S^\downarrow$. There exists a net $\{x_\lambda\}_{\lambda \in \Lambda}$ in $S^\downarrow$ such that $x_\lambda \uparrow x$. For each $\lambda \in \Lambda$, there exists a net $\{x_{i_\lambda}\}_{i_\lambda \in I_\lambda}$ in $S$ such that $x_{i_\lambda} \uparrow x_\lambda$. Then $x$ is the supremum of the subset $S_0 := \{x_{i_\lambda} : \lambda \in \Lambda, i_\lambda \in I_\lambda\}$ of $S$. This subset is not obviously (the image of) an increasing net in $S$. However, the subset $S_0^\downarrow$ of $S$ still provides a net in $S$ that increases to $x$.

Take an $x \in S^\uparrow$. There exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $S^\uparrow$ such that $x_n \uparrow x$. For each $n \geq 1$, there exists a sequence $\{x_n^m\}_{m=1}^{\infty}$ in $S$ such that $x_n^m \uparrow x_n$. Then $x$ is the supremum of the countable subset $S_0 := \{x_n^m : n, m \geq 1\}$ of $S$. Choose an enumeration $z_1, z_2, z_3, \ldots$ of $S_0$. Then the sequence $z_1, \sup\{z_1, z_2\}, \sup\{z_1, z_2, z_3\}, \ldots$ is a sequence in $S$ that increases to $x$. □

Remark 5.3. For a vector lattices $E$, it is stated on [5, p. 83] that it is clear that $S^\uparrow = S^\uparrow$ and that $S^\downarrow = S^\downarrow$, without including any condition on $S$. This appears to be a mistake.

Lemma 5.4. Let $L$ be a linear subspace of a partially ordered vector space $E$. Then $L^\uparrow \cap L^\downarrow$ and $L^\downarrow \cap L^\downarrow$ are linear subspaces of $E$. In general, intersections such as $L^\uparrow \cap L^\downarrow \cap L^\downarrow \cap L^\uparrow$ of a finite number of consecutive ups or downs of $L$, an arbitrary number of which may be sequential, and the ‘mirrored’ consecutive downs or ups of $L$, with the directions of the arrows reversed, is a linear subspace of $E$.

Proof. As an example, we prove that $L^\uparrow \cap L^\downarrow$ is a linear subspace of $E$. In view of a double application of part [1] of Lemma 5.2, it is sufficient to show that $-x \in L^\uparrow \cap L^\downarrow$ whenever $x \in L^\uparrow \cap L^\downarrow$. The fact that $x \in L^\uparrow \cap L^\downarrow$ implies that $-x \in -(L^\uparrow \cap L^\downarrow) = (L^\downarrow)^\downarrow = (L^\downarrow)^\downarrow = L^\downarrow$. Likewise, it follows from the fact that $x \in L^\downarrow \cap L^\downarrow$ that $-x \in L^\downarrow$. □
Suppose that $S$ is a non-empty subset of the partially ordered vector space $E$. Then we let
\[ \text{Wed}[S] := \left\{ \sum_{i=1}^{n} \alpha_i s_i : n = 1, 2, \ldots, \alpha_i \geq 0 \text{ for } i = 1, \ldots, n \right\} \]
denote the wedge in $E$ that is generated by $S$. The following is clear from Lemma 5.2.

**Lemma 5.5.** Let $E$ be a partially ordered vector space, and let $W$ be a wedge in $E$. Then each of $W^\uparrow$, $W^\downarrow$, $W^\uparrow$, and $W^\downarrow$ is a wedge in $E$. Suppose that $S$ is a non-empty subset of $E$. Then
\[ \text{Wed}[S^\uparrow] \subseteq \text{Wed}[S]^\uparrow. \]
Similar statements hold for $S^\downarrow$, $S^\uparrow$, and $S^\downarrow$.

Suppose that $E$ is a vector lattice. If $W^\vee = W$ or $W^\wedge = W$, then each of $W^\uparrow$, $W^\downarrow$, $W^\uparrow$, and $W^\downarrow$ has the same respective property.

We can now show that the images of canonical positive cones are contained in cones that are built from the image of $C_c(X)^+$ by using ups and downs.

**Proposition 5.6.** Let $X$ be a locally compact Hausdorff space, let $E$ be a monotone complete partially ordered vector space, and let $\pi : C_c(X) \to E$ be a positive operator. Suppose that $\mu : B \to E^+$ is a regular Borel measure such that
\[ \pi(f) = \int_X f \, d\mu \]
for $f \in C_c(X)$;
\[ \mu(V) = \bigvee \{ \pi(f) : f \prec V \} \]
in $E^+$ for every open subset $V$ of $X$ with finite measure; and
\[ \mu(K) = \bigwedge \{ \pi(f) : K \prec f \} \]
in $E^+$ for every compact subset $K$ of $X$. Then
\[ I_\mu(L_1(X, B; \mu; \mathbb{R}^+)) \subseteq \left[ \pi(C_c(X)^+) \right]^\uparrow \downarrow ; \]
if $\mu$ is inner regular at all Borel subsets of $X$ with finite measure, then also
\[ I_\mu(L_1(X, B; \mu; \mathbb{R}^+)) \subseteq \left[ \pi(C_c(X)^+) \right]^{\uparrow \downarrow} . \]
Suppose that $\mu$ is finite. Then
\[ I_\mu(B(X, B; \mathbb{R}^+)) \subseteq \left[ \pi(C_c(X)^+) \right]^\uparrow \downarrow ; \]
if $\mu$ is inner regular at all Borel subsets of $X$, then also
\[ I_\mu(B(X, B; \mathbb{R}^+)) \subseteq \left[ \pi(C_c(X)^+) \right]^{\uparrow \downarrow} . \]

**Proof.** We establish equation (5.3). It is clear that
\[ I_\mu(L_1(X, B; \mathbb{R}^+)) = \text{Wed}\left[ \{ \mu(\Delta) : \Delta \in B, \mu(\Delta) < \infty \} \right] . \]
The outer regularity of \( \mu \) and equation (5.1) imply that
\[
\{ \mu(\Delta) : \Delta \in B, \mu(\Delta) < \infty \} \subseteq \{ \pi(f) : f \prec X \}^{\#}.
\]
Using a twofold application of Lemma 5.5 in the second step, we therefore see that
\[
I_\mu \left( \mathcal{E}^1(X, B; \mathbb{R}^+) \right) \subseteq \text{Wed} \left[ \{ \pi(f) : f \prec X \}^{\#} \right]
\]
(5.7)
\[
\subseteq \left[ \text{Wed} \{ \{ \pi(f) : f \prec X \} \} \right]^{\#} = \left[ \pi(C_\pi(X^+)) \right]^{\#}.
\]
The definition of the order integral now shows that equation (5.3) holds.

The proof of equation (5.4) is similar, but now combines equation (5.2) with
the inner regularity of \( \mu \) at all Borel sets.

Suppose that \( \mu \) is finite. Take an \( f \in \mathcal{B}(X, B; \mathbb{R}^+) \). As for general elements of
\( \mathcal{L}^1(X, B, \mu; \mathbb{R}) \), we have that \( I_\mu(f) \in \left[ I_\mu(\mathcal{E}^1(X, B; \mathbb{R}^+)) \right]^{\dagger} \), but we claim that
now also \( I_\mu(f) \in \left[ I_\mu(\mathcal{E}^1(X, B; \mathbb{R}^+)) \right]^{\dagger} \). To see this, take an \( M \geq 0 \) such that
\( f(x) \leq M \) for all \( x \in X \). Then \( M1 - f \geq 0 \), so there exists a sequence \( \{ s_n \}_{n=1}^\infty \) of
elementary function such that \( s_n \uparrow M1 - f \). Hence \( I_\mu(s_n) \uparrow I_\mu(M1 - f) \), which
implies that \( I_\mu(M1 - s_n) \downarrow I_\mu(f) \). Since \( M1 - s_n \in \mathcal{E}^1(X, B; \mathbb{R}^+) \) for all \( n \), this
establishes our claim. An appeal to equation (5.7) then concludes the proof of
equation (5.5). Similarly, the proof of equation (5.6) uses equation (5.2), the
inner regularity of \( \mu \) at all Borel sets, and our claim.

To improve—under extra conditions—inclusions such as in Proposition 5.6
to equalities, we need the following preparatory result. It is based on the mono-
tone convergence theorem.

**Proposition 5.7.** Let \( (X, \Omega, \mu, E) \) be a measure space. Suppose that \( E \) has the
countable sup property, and that \( I_\mu : \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \to E \) preserves moduli. Let \( S \)
be a non-empty subset of \( \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}^+) \).

1. Suppose that \( S^\vee = S \). Then \( I_\mu(S^\dagger) = [I_\mu(S)]^\dagger = [I_\mu(S)]^{\#} \).
2. Suppose that \( S^\wedge = S \). Then \( I_\mu(S^1) = [I_\mu(S)]^1 = [I_\mu(S)]^{\#} \).

In fact:

3. if \( S^\vee = S \), then the subset of \( E \) that is obtained by taking at least one
   consecutive ups of \( I_\mu(S) \), an arbitrary number of which may be sequential,
   is always equal to \( I_\mu(S^\dagger) \);
4. if \( S^\wedge = S \), then the subset of \( E \) that is obtained by taking at least one
   consecutive downs of \( I_\mu(S) \), an arbitrary number of which may be sequential,
   is always equal to \( I_\mu(S^1) \).

**Proof.** We establish part (1). Already without any further conditions on \( E \) or \( I_\mu \),
it is a consequence of the monotone convergence theorem that \( I_\mu : \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \to E \) is \( \sigma \)-order continuous; see [8] Theorem 6.17. Hence certainly \( I_\mu(S^\dagger) \subseteq
\left[ I_\mu(S) \right]^{\dagger} \subseteq [I_\mu(S)]^{\#} \). We shall now use the extra hypotheses to show that \( [I_\mu(S)]^{\#} \subseteq
Suppose that \( e \in E \) and that \( I_\mu(s_\lambda) \uparrow e \) for some net \( \{s_\lambda\}_{\lambda \in \Lambda} \) in \( S \). Since \( E \) has the countable sup property, there exists a sequence \( \{f_n\}_{n=1}^\infty \) in \( \mathcal{L}^1(X,\Omega,\mu;\mathbb{R}^+) \) such that \( \{\int f_n\} : n = 1, 2, \ldots \} \subseteq \{s_\lambda : \lambda \in \Lambda\} \subseteq S \) and \( e = \sup_{n \geq 1} I_\mu(f_n) \). For \( n = 1, 2, \ldots \), set \( g_n := f_1 \lor \cdots \lor f_n \). Since \( S^\uparrow = S \), we have \( \int g_n = \int f_1 \lor \cdots \lor \int f_n \in S \). Furthermore, since \( I_\mu : \mathcal{L}^1(X,\Omega,\mu;\mathbb{R}) \to E \) preserves moduli, we have (see Remark 2.17) that \( I_\mu(g_n) = I_\mu(f_1) \lor \cdots \lor I_\mu(f_n) \uparrow e \). We define the measurable function \( g : X \to \mathbb{R}^+ \) by setting \( g(x) := \sup_{n \geq 1} g_n(x) \in \mathbb{R}^+ \) for \( x \in X \). Then \( g_n(x) \uparrow g(x) \) in \( \mathbb{R}^+ \) for every \( x \in X \). According to the monotone convergence theorem (see [8] Theorem 6.9), we have that \( I_\mu(g_n) \uparrow I_\mu(g) \) in \( E^\uparrow \). Hence \( I_\mu(g) = e \). Since this is finite, it follows from [8] Lemma 6.4 that \( g \) is almost everywhere finite-valued. When necessary, we can, therefore, redefine \( g \) and the \( g_n \) for \( n \geq 1 \) to be zero on a subset of measure zero, and arrange that \( g \) is finite-valued and that \( g_n(x) \uparrow g(x) \) in \( \mathbb{R} \) for all \( x \in X \). Then \( g_n \uparrow g \) in \( \mathcal{L}^1(X,\Omega,\mu;\mathbb{R}) \). Because the quotient map from \( \mathcal{L}^1(X,\Omega,\mu;\mathbb{R}) \) to \( \mathcal{L}^1(X,\Omega,\mu;\mathbb{R}) \) is \( \sigma \)-order continuous (see [8] Theorem 6.17), this implies that \( \int g_n \uparrow \int g \) in \( \mathcal{L}^1(X,\Omega,\mu;\mathbb{R}) \). Since \( I_\mu(\int g) = e \), we can now conclude that \( e \in I_\mu(S^\uparrow) \), as desired.

The proof of part (2) is similar, using the monotone convergence theorem for decreasing sequences; see [8] Corollary 6.10. It is even slightly easier because the then occurring pointwise limit function is already automatically finite-valued.

We prove part (3). Suppose that \( S^\uparrow = S \). Take \( n \geq 1 \) and take \( n \) consecutive ups of \( I_\mu(S) \), an arbitrary number of which may be sequential, and let \( \Sigma \) denote the resulting subset of \( E \). For \( n = 1 \), part (3) coincides with part (1) for \( n \geq 2 \), we let \( \bar{\Sigma} \) denote the subset of \( E \) that is obtained by \( n \) consecutive net ups \( \uparrow \) of \( I_\mu(S) \). We note that \( I_\mu(S^\uparrow) = [I_\mu(S)]^\uparrow \subseteq \Sigma \subseteq \bar{\Sigma} \). Since \( S^\uparrow = S \) and \( I_\mu \) preserves moduli, we see that \( [I_\mu(S)]^\uparrow \) exists in \( E \) and that \( [I_\mu(S)]^\uparrow = I_\mu(S) \); see Remark 2.17. An \( (n-1) \)-fold application of part (2) of Lemma 5.2 shows that \( \bar{\Sigma} = [I_\mu(S)]^\uparrow \). By part (1) \( [I_\mu(S)]^\uparrow \) equals \( I_\mu(S^\uparrow) \). The proof of part (3) is now complete.

Similarly, part (4) follows from part (2) and part (3) of Lemma 5.2.

**Corollary 5.8.** Let \( (X,\Omega,\mu,\mu) \) be a measure space. Suppose that \( E \) has the countable sup property, and that \( I_\mu : \mathcal{L}^1(X,\Omega,\mu;\mathbb{R}) \to E \) preserves moduli. Let \( S \) be a vector sublattice of \( \mathcal{L}^1(X,\Omega,\mu;\mathbb{R}) \). Then:

1. \( I_\mu(S^\uparrow) = [I_\mu(S)]^\uparrow = [I_\mu(S)]^\uparrow \);
2. \( I_\mu(S^\downarrow) = [I_\mu(S)]^\downarrow = [I_\mu(S)]^\downarrow \).

In fact:

3. the subset of \( E \) that is obtained by taking at least one consecutive ups of \( I_\mu(S) \), an arbitrary number of which may be sequential, is always equal to \( I_\mu(S^\uparrow) \);
(4) the subset of \( E \) that is obtained by taking at least one consecutive downs of \( I_\mu(S) \), an arbitrary number of which may be sequential, is always equal to \( I_\mu(S^\downarrow) \).

**Proof.** We prove part (1); the proof of part (2) is similar. It follows from the monotone convergence theorem that \( I_\mu(S^\downarrow) \subseteq [I_\mu(S)]^\uparrow \), and trivially \([I_\mu(S)]^\uparrow \subseteq [I_\mu(S)]^\uparrow \). We show that \([I_\mu(S)]^\uparrow \subseteq [I_\mu(S)]^\uparrow \). Suppose that \( e \in E \) and that \( I_\mu(s_\lambda) \uparrow e \) for some net \( \{s_\lambda\}_{\lambda \in \Lambda} \in S \). Since \( E \) has the countable sup property, there exists a sequence \( \{f_n\}_{n=1}^\infty \) in \( \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}^+) \) such that \( \{[f_n]\} : n = 1, 2, \ldots \) \( \subseteq \{s_\lambda : \lambda \in \Lambda \} \subseteq S \) and \( e = \sup_{n \geq 1} I_\mu(f_n) \). For \( n = 1, 2, \ldots \), set \( g_n := f_1 \vee \cdots \vee f_n - f_1 \). Then \( [g_n] \in S^+ \), and \( I_\mu([g_n]) \uparrow e - I_\mu([f_1]) \). Part (1) of Proposition 5.7 shows that \( e - I_\mu([f_1]) \in I_\mu([S^\uparrow]^\downarrow) \subseteq I_\mu(S^\downarrow) \). This implies that \( e \in I_\mu(S^\downarrow) \).

The parts (3) and (4) follow from the parts (1) resp. (2) as in the proof of Proposition 5.7.

We can now establish the following result, where inclusions as in Proposition 5.6 are replaced with equalities. As Section 2.2 indicates, the condition in it that \( E \) have the countable sup property is often met. Regarding its final part we recall that, if \( E \) is a \( \sigma \)-monotone complete normed partially ordered algebra with a monotone norm, and \( \mu \) is a finite spectral measure, then Lemma 2.29 shows that it is automatic that \( \mathbb{L}^1(X, B, \mu; \mathbb{R}) = \mathbb{B}(X, B, \mu; \mathbb{R}) \); here we have used our convention as in Remark 2.30.

**Theorem 5.9** (Ups and downs for positive cones). Let \( X \) be a locally compact Hausdorff space, let \( E \) be a monotone complete partially ordered vector space, and let \( \pi : C_c(X) \to E \) be a positive operator. Suppose that \( \mu : B \to E^\uparrow \) is a regular Borel measure such that

\[
\pi(f) = \int_X f \, d\mu
\]

for \( f \in C_c(X) \);

\[
\mu(V) = \sqrt{\{\pi(f) : f \prec V\}}
\]

in \( E^+ \) for every open subset \( V \) of \( X \) with finite measure; and

\[
\mu(K) = \bigwedge_{\{\pi(f) : K \prec f\}}
\]

in \( E^+ \) for every compact subset \( K \) of \( X \). Suppose, furthermore, that \( E \) has the countable sup property, and that \( I_\mu : \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \to E \) preserves moduli. Then:

(1) \( I_\mu(\mathbb{L}^1(X, \mathcal{B}, \mu; \mathbb{R}^+)) = I_\mu(\mathbb{L}^1(X, \mathcal{B}, \mu; \mathbb{R}^+)) \uparrow = I_\mu(\mathbb{L}^1(X, \mathcal{B}, \mu; \mathbb{R}^+)) \downarrow \)

and

(2) \( I_\mu(\mathcal{L}^1(X, \mathcal{B}, \mu; \mathbb{R}^+)) = I_\mu([\pi(C_c(X)^+)]) \uparrow \downarrow = I_\mu([\pi(C_c(X)^+)]) \uparrow \downarrow \).

We have used our convention as in Remark 2.30.
(3) If \( \mu \) is finite and \( L^1(X, \mathcal{B}, \mu; \mathbb{R}) = B(X, \mathcal{B}, \mu; \mathbb{R}) \), then

\[
I_\mu(\mathcal{L}^1(X, \mathcal{B}, \mu; \mathbb{R}^+)) = I_\mu(\mathbb{C}_c(X)^+) = \left[ \pi(\mathbb{C}_c(X)^+) \right]^\uparrow_{\downarrow}.
\]

**Proof.** The equalities in equation (5.8) are immediate from Proposition 5.7. We know from equation (5.8) that

\[
I_\mu(\mathcal{L}^1(X, \mathcal{B}, \mu; \mathbb{R}^+)) = \left[ \pi(\mathbb{C}_c(X)^+) \right]^\uparrow_{\downarrow} = \left[ I_\mu(\mathbb{C}_c(X)^+) \right]^\uparrow_{\downarrow}.
\]

Since \( \mathbb{C}_c(X)^+ \) is closed under the taking of finite suprema (and of finite infima), Proposition 5.7 shows that

\[
\left[ I_\mu(\mathbb{C}_c(X)^+) \right]^\uparrow_{\downarrow} = I_\mu(\mathbb{C}_c(X)^+) = \left[ I_\mu(\mathbb{C}_c(X)^+) \right]^\uparrow_{\downarrow}.
\]

According to Lemma 5.2, \( \mathbb{C}_c(X)^+ \) is still closed under the taking of finite infima (and of finite suprema). Another appeal to Proposition 5.7 followed by one further repetition of the argument, therefore yields that

\[
\left[ I_\mu(\mathbb{C}_c(X)^+) \right]^\uparrow_{\downarrow} = I_\mu(\mathbb{C}_c(X)^+) = \left[ I_\mu(\mathbb{C}_c(X)^+) \right]^\uparrow_{\downarrow}.
\]

Since, trivially,

\[
I_\mu(\mathbb{C}_c(X)^+) \subseteq I_\mu(L^1(X, \mathcal{B}, \mu; \mathbb{R}^+)) = I_\mu(\mathcal{L}^1(X, \mathcal{B}, \mu; \mathbb{R}^+)),
\]

we can now conclude that

\[
I_\mu(\mathcal{L}^1(X, \mathcal{B}, \mu; \mathbb{R}^+)) = I_\mu(\mathbb{C}_c(X)^+) = I_\mu(\mathbb{C}_c(X)^+) = \left[ I_\mu(\mathbb{C}_c(X)^+) \right]^\uparrow_{\downarrow}.
\]

A threefold application of Lemma 5.2 and Proposition 5.7 shows that

\[
I_\mu(\mathbb{C}_c(X)^+) = \left[ I_\mu(\mathbb{C}_c(X)^+) \right]^\uparrow_{\downarrow} = \left[ \pi(\mathbb{C}_c(X)^+) \right]^\uparrow_{\downarrow}.
\]

This completes the proof of equation (5.9).

When \( \mu \) is inner regular at all Borel subsets of finite measure, we use equation (5.4) as a starting point, and show similarly that

\[
I_\mu(\mathcal{L}^1(X, \mathcal{B}, \mu; \mathbb{R}^+)) = I_\mu(\mathbb{C}_c(X)^+) = I_\mu(\mathbb{C}_c(X)^+) = \left[ I_\mu(\mathbb{C}_c(X)^+) \right]^\uparrow_{\downarrow}.
\]

Since \( \mathbb{C}_c(X)^+ \) is closed under the taking of finite suprema by Lemma 5.2, the same Lemma 5.2 shows that \( \mathbb{C}_c(X)^+ = \mathbb{C}_c(X)^+ \). A twofold application of Lemma 5.2 and Proposition 5.7 then completes the proof of equation (5.10).

When \( \mu \) is finite and \( L^1(X, \mathcal{B}, \mu; \mathbb{R}) = B(X, \mathcal{B}, \mu; \mathbb{R}) \), equation (5.5) shows that

\[
I_\mu(\mathcal{L}^1(X, \mathcal{B}, \mu; \mathbb{R}^+)) = I_\mu(\mathcal{L}^1(B(X, \mathcal{B}; \mathbb{R}^+)) = \left[ \pi(\mathbb{C}_c(X)^+) \right]^\uparrow_{\downarrow} = \left[ I_\mu(\mathbb{C}_c(X)^+) \right]^\uparrow_{\downarrow}.
\]

Arguing as before, equation (5.11) follows from this. \( \square \)

**Remark 5.10.** It is known from [12, Theorem 3.10] that the set of components of a positive operator between two Dedekind complete vector lattices \( E \) and \( F \), where the supremum separates the points of \( F \), can be obtained as the \( \uparrow \downarrow \uparrow \downarrow \) and as the \( \downarrow \uparrow \downarrow \downarrow \) of the set of its simple components. When the operator is order

\[\text{[11] According to [5, Theorem 2.6], \( E \) need merely have the principal projection property.}\]
continuous, it suffices to take the $\uparrow \downarrow$ or the $\downarrow \uparrow$ of the set of its simple components; see [12, Theorem 3.11]. Given the similarities, it is an intriguing question whether (special cases of) these results in [12] may be related to (special cases of) those in Theorem 5.9.

Proposition 5.6 and Theorem 5.9 are concerned with the images of the positive cones of the integrable and bounded measurable functions. It is also possible to establish versions for the image of the full vector lattice of bounded measurable functions. These are taken together in the following result. Its final statement will be particularly relevant in the sequel.

**Theorem 5.11** (Ups and downs for vector lattices). Let $X$ be a locally compact Hausdorff space, let $E$ be a monotone complete partially ordered vector space, and let $\pi : C_c(X) \to E$ be a positive operator. Suppose that $\mu : B \to E^+$ is a finite regular Borel measure that is inner regular at all Borel subsets of $X$, and such that

$$\pi(f) = \int_X f \, d\mu$$

for $f \in C_c(X)$;

$$\mu(V) = \bigvee \{\pi(f) : f < V\}$$

in $E^+$ for every open subset $V$ of $X$;

$$\mu(K) = \bigwedge \{\pi(f) : K < f\}$$

in $E^+$ for every compact subset $K$ of $X$. Then

$$I_\mu(\mathcal{B}(X; B; \mathbb{R})) \subseteq [\pi(C_c(X))]^{\uparrow \downarrow} \cap [\pi(C_c(X))]^{\downarrow \uparrow}$$

and

$$I_\mu(\mathcal{B}(X; B; \mathbb{R})) \subseteq [\pi(C_c(X))]^{\downarrow \uparrow} \cap [\pi(C_c(X))]^{\uparrow \downarrow}.$$  

If $E$ has the countable sup property and $I_\mu : \mathcal{L}^1(X, B; \mu; \mathbb{R}) \to E$ preserves moduli, then

$$I_\mu(\mathcal{B}(X; B; \mathbb{R})) \subseteq [\pi(C_c(X))]^{\downarrow} \cap [\pi(C_c(X))]^{\uparrow}.$$  

If $E$ has the countable sup property, $I_\mu : \mathcal{L}^1(X, B; \mu; \mathbb{R}) \to E$ preserves moduli, and $L^1(X, B, \mu; \mathbb{R}) = B(X, B, \mu; \mathbb{R})$, then

$$I_\mu(\mathcal{B}(X; B; \mathbb{R})) = [\pi(C_c(X))]^{\uparrow} = [\pi(C_c(X))]^{\downarrow}.$$  

**Proof.** It follows from equations (5.3) and (5.6) that

$$I_\mu(\mathcal{B}(X; B; \mathbb{R}^+)) \subseteq [\pi(C_c(X))^{\uparrow}]^{\uparrow \downarrow} \cap [\pi(C_c(X))^{\downarrow}]^{\downarrow \uparrow} \subseteq [\pi(C_c(X))]^{\uparrow \downarrow} \cap [\pi(C_c(X))]^{\downarrow \uparrow}.$$  

Since $[\pi(C_c(X))]^{\uparrow \downarrow} \cap [\pi(C_c(X))]^{\downarrow \uparrow}$ is a linear subspace of $E$ by Lemma 5.4, the validity of equation (5.12) is now clear. Similarly, equations (5.4) and (5.5) can be used to establish equation (5.13).
Suppose that $E$ has the countable sup property, and that $I_\mu \colon \mathcal{L}^1(X, \mathcal{B}, \mu; \mathbb{R}) \to E$ preserves moduli. Using Corollary 5.8, equation (5.13) implies that
\[
I_\mu(\mathcal{B}(X, \mathcal{B}; \mathbb{R})) \subseteq [\pi(C_c(X))]^{\oplus \downarrow} = [I_\mu(\mathbb{R treatment of vector space, and let $\mu \colon \mathcal{B} \to E^+$ be a finite Borel measure. Then $I_\mu(C_0(X)^+) \subseteq [I_\mu(C_0(X)^+)]^\uparrow$.\\
\textit{Proof:} Take an $f \in C_0(X)$. For $n = 1, 2, \ldots$, there exists a $\varphi_n \in C_c(X)$ such that $0 \leq \varphi_n \leq 1$ and $\varphi_n(x) = 1$ when $|f(x)| \leq 1/n$. Then $\varphi_1 f$, $(\varphi_1 \lor \varphi_2) f$, $(\varphi_1 \lor \varphi_2 \lor \varphi_3) f$ is a sequence in $C_c(X)^+$ that increases pointwise to $f$, so that the image sequence increases to $I_\mu(f)$ by the monotone convergence theorem. \hfill $\square$
6. SPECTRAL THEOREMS FOR POSITIVE ALGEBRA HOMOMORPHISMS

In this section, a number of results from [8] and [9] are combined with those from the present paper to yield two spectral theorems for positive algebra homomorphisms. The statements of the theorems are long, and some parts of them are identical, but we thought it worthwhile to collect all major results from these three papers that are applicable in a particular context in one place. With an eye towards possible further extensions and applications, we mention that the monotone convergence theorem, Fatou’s lemma, and the dominated convergence theorem hold for the order integral that occurs in the results below; see [8, Section 6.2].

It will be a recurring theme to know that a spectral measure for a positive algebra homomorphism from $C_c(X)$ into a partially ordered algebra is finite, as a consequence of the fact that it is the restriction of a positive algebra homomorphism that is defined on $C_0(X)$. This will be possible when the algebra is a quasi-perfect partially ordered vector space. As Proposition 2.4 shows, this class of spaces contains a good number of spaces of practical interest.

The results on ups and downs below are established under the hypothesis that the codomain have the countable sup property. As indicated in Section 2.2, this condition is also satisfied for a variety of spaces of practical interest.

Let $(X, \Omega, \mu, E)$ be a measure space. We recall for the convenience of the reader that $I_\mu: L^1(X, \Omega, \mu; \mathbb{R}) \to E$ denotes the map $f \mapsto \int_X f \, d\mu$, and that we use the same notation for its restriction to subspaces of $L^1(X, \Omega, \mu; \mathbb{R})$ and to quotients of such subspaces. In the results below, it will typically denote an extension of a positive operator $\pi: C_c(X) \to E$, or an operator that is compatible with a canonical map from $C_0(X)$ into the domain of $I_\mu$.

Our first result is for Banach lattice algebras with order continuous norms and monotone continuous multiplications. Banach lattice algebras of operators on infinite dimensional spaces will not often fall into this category—the order continuity of the norm is problematic—but on finite dimensional spaces these requirements are met. Function algebras such as $\ell^p$ for $1 \leq p < \infty$ provide another class of examples to which Theorem 6.1 can be applied.

**Theorem 6.1** (Positive algebra homomorphisms from $C_c(X)$ into Banach lattice algebras with order continuous norms). Let $X$ be a locally compact Hausdorff space, let $A$ be a Banach lattice algebra with an order continuous norm and a monotone continuous multiplication, and let $\pi: C_c(X) \to A$ be a positive algebra homomorphism.

1. There exists a unique regular Borel measure $\mu: \mathcal{B} \to A^+$ on the Borel $\sigma$-algebra $\mathcal{B}$ of $X$ such that

$$\pi(f) = \int_X f \, d\mu$$

for all $f \in C_c(X)$. If $V$ is a non-empty open subset of $X$, then

$$\mu(V) = \bigvee \{ \pi(f) : f \prec V \}$$

$\mu(V) = \bigvee \{ \pi(f) : f \prec V \}$
in $\mathcal{A}^+$. If $K$ is a compact subset of $X$, then

$$\mu(K) = \bigwedge \{ \pi(f) : K < f \}$$

in $\mathcal{A}^+$.

(2) The measure $\mu$ is a spectral measure which is inner regular at all Borel sets of finite measure. It is finite if and only if $\{ \pi(f) : f \in C_c(X)^+, \|f\| \leq 1 \}$ is bounded above in $\mathcal{A}$. This is automatically the case when $X$ is compact, and also when $\mathcal{A}$ is quasi-perfect and $\pi$ is the restriction of a positive algebra homomorphism $\pi : C_0(X) \to \mathcal{A}$. In the latter case, equation (6.1) also holds for $f \in C_0(X)$.

(3) $L^1(X, \mathcal{B}, \mu; \mathbb{R}) \subseteq B(X, \mathcal{B}, \mu; \mathbb{R})$. If $\mu$ is finite, then $L^1(X, \mathcal{B}, \mu; \mathbb{R}) = B(X, \mathcal{B}, \mu; \mathbb{R})$.

(4) The spaces $\mathcal{L}^1(X, \mathcal{B}, \mu; \mathbb{R})$ and $L^1(X, \mathcal{B}, \mu; \mathbb{R})$ are $\sigma$-Dedekind complete vector lattices, and the naturally defined operators $I_{\mu}$ from these spaces into $\mathcal{A}$ are both $\sigma$-order continuous. The operator $I_{\mu} : L^1(X, \mathcal{B}, \mu; \mathbb{R}) \to E$ is strictly positive. When $\mathcal{A}$ has the countable sup property, $L^1(X, \mathcal{B}, \mu; \mathbb{R})$ is a Dedekind complete vector lattice with the countable sup property, and $I_{\mu} : L^1(X, \mathcal{B}, \mu; \mathbb{R}) \to \mathcal{A}$ is order continuous.

(5) Suppose that $\mu$ is finite. Then:

(a) $I_{\mu}(B(X, \mathcal{B}, \mu; \mathbb{R})) \subseteq \mu(X).A\mu(X)$;

(b) $I_{\mu} : B(X, \mathcal{B}, \mu; \mathbb{R}) \to \mu(X).A$, $I_{\mu} : B(X, \mathcal{B}, \mu; \mathbb{R}) \to A\mu(X)$, and $I_{\mu} : B(X, \mathcal{B}, \mu; \mathbb{R}) \to \mu(X).A\mu(X)$ preserve moduli;

(c) $I_{\mu} : B(X, \mathcal{B}, \mu; \mathbb{R}) \to \mathcal{A}$ is a topological embedding of the Banach algebra $B(X, \mathcal{B}, \mu; \mathbb{R})$ as a Banach subalgebra of $\mathcal{A}$;

(d) when $I_{\mu}(B(X, \mathcal{B}, \mu; \mathbb{R}))$ is supplied with the partial ordering inherited from $\mathcal{A}$, it is a unital Banach lattice algebra with identity element $\mu(X)$, and $I_{\mu} : B(X, \mathcal{B}, \mu; \mathbb{R}) \to I_{\mu}(B(X, \mathcal{B}, \mu; \mathbb{R}))$ is an isomorphism of Banach lattice algebras.

(e) if $\mathcal{A}$ has an identity $e$ and $\mu(X) \leq e$, then $I_{\mu}(B(X, \mathcal{B}, \mu; \mathbb{R}))$ is a Banach lattice subalgebra of $\mathcal{A}$.

(6) Suppose that $\mu$ is finite. For $x' \in (\mathcal{A}_c^\infty)^+ = (\mathcal{A}^+)^+$ and $\Delta \in \mathcal{B}$, set $\mu_{x'}(\Delta) := (\mu(\Delta), x')$. Then $\mu_{x'} : \mathcal{B} \to \mathbb{R}^+$ is a regular Borel measure, and we have $f \in \mathcal{L}^1(X, \Omega, \mu_{x'}; \mathbb{R})$ for $f \in \mathcal{L}^1(X, \mathcal{B}, \mu; \mathbb{R})$. For $f \in \mathcal{L}^1(X, \mathcal{B}, \mu; \mathbb{R})$, $I_{\mu}(f) = \int_X f \, d\mu$ is the unique element of $\mathcal{A}$ such that

$$(I_{\mu}(f), x') = \int_X f \, d\mu_{x'}$$

for all $x' \in (\mathcal{A}_c^\infty)^+$.

(7) For $a \in \mathcal{A}$, the following are equivalent:

(a) $a\pi(f) = \pi(f)a$ for all $f \in C_c(X)$;

(b) $a\mu(\Delta) = \mu(\Delta)a$ for all $\Delta \in \mathcal{B}$ with finite measure;

(c) $aI_{\mu}(f) = I_{\mu}(f)a$ for all $f \in \mathcal{L}^1(X, \Omega, \mu; \mathbb{R})$.

When $C_0(X) \subseteq \mathcal{L}^1(X, \Omega, \mu; \mathbb{R})$, these are also equivalent to:

(d) $aI_{\mu}(f) = I_{\mu}(f)a$ for all $f \in C_0(X)$. 
(8) Suppose that $\mu$ is finite and that $\mathcal{A}$ has the countable sup property. Then:

(6.3) \[ I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+)) = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+))^\dagger = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+))^\ddagger; \]

(6.4) \[ I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+)) = I_\mu([\mathbb{C}_c(X)^+] \mathbb{I}) = [\pi(\mathbb{C}_c(X))^{\dagger}]^\uparrow; \]

(6.5) \[ I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+)) = I_\mu([\mathbb{C}_c(X)^+] \mathbb{I}) = [\pi(\mathbb{C}_c(X))]^\uparrow; \]

(6.6) \[ I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+)) = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+))^\dagger = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+))^\ddagger; \]

(6.7) \[ I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+)) = [\pi(\mathbb{C}_c(X))]^\uparrow = [\pi(\mathbb{C}_c(X))]^\dagger. \]

Here the ups and downs of the images of $I_\mu$ and $\pi$ can be taken in $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}_\mu$, $\mathcal{B}_\mu$, $\mathcal{A}_\mu$, or $I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+))$ with equal outcomes.

**Proof.** The parts (1) and (2) are a consequence of [9] Theorems 4.2 and 6.8 for positive operators, and of [9] Proposition 3.6, except the fact that $\mu$ is a spectral measure. Since $\pi$ is now an algebra homomorphism, this follows from Theorem 3.6.

Part (3) follows from Lemma 2.29.

Part (4) follows from [8] Proposition 6.14 and Theorem 6.17.

Part (5) follows from part (3) Proposition 2.31 (which applies as the positive operator $I_\mu$ between two Banach lattices is automatically continuous), Theorem 4.4, and Theorem 4.5.

Part (6) follows from [8] Proposition 6.8.

We turn to part (7) and prove that part (a) implies part (b). Suppose that $a = a^+ - a^-$ in $\mathcal{A}$ commutes with $\pi(f)$ for all $f \in \mathbb{C}_c(X)$. Then, in particular,

(6.8) \[ a^+ \pi(f) + \pi(f) a^- = \pi(f) a^+ + a^- \pi(f) \]

for all $f \in \mathbb{C}_c(X)^+$. Take an non-empty open subset $V$ of $X$ with finite measure. It follows from equation (6.8), equation (6.2), and the monotone continuity of the multiplication in $\mathcal{A}$ that

\[ a^+ \mu(V) + \mu(V) a^- = \mu(V) a^+ + a^- \mu(V). \]

The inner regularity of $\mu$ can now be used to establish the same equality where $\mu(V)$ is replaced with $\mu(\Delta)$ for a Borel subset $\Delta$ of $X$ with finite measure. This shows that part (b) holds. When part (b) holds, the definition of the order integral and the ($\sigma$-)monotone continuity of the multiplication in $\mathcal{A}$ imply that part (c) holds. It is clear that part (c) implies part (a). The statement regarding part (d) is clear.

We turn to part (8). Part (5) shows that $I_\mu$ maps $\mathcal{L}^1(X, \Omega, \mu; \mathbb{R})$ into $\mu(X)\mathcal{A}$, and that $I_\mu : \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \rightarrow \mu(X)\mathcal{A}$ preserves moduli. We can view $\mu$ as a $\mu(X)\mathcal{A}$-valued measure, and $\pi$ as a positive operator from $\mathbb{C}_c(X)$ into $\mu(X)\mathcal{A}$. It then follows from Lemma 2.10 that Theorem 5.9 can be applied to this context. Thus the equations (6.3), (6.4), and (6.5) (with the ups and downs of the images taken in $\mu(X)\mathcal{A}$) follow from the equations (5.8), (5.10), and (5.11), respectively. Similarly, the equations (6.6) and (6.7) (again with the ups and
downs of the images taken in \(\mu(X,A)\) follow from Corollary 5.8 and equation (5.15), respectively. Likewise, one establishes the validity of the equations (6.3)–(6.7) with the ups and downs of the images taken in \(\mu(X,A)\), and in \(\mu(X,A)\mu(X)\). It follows from Remark 5.1 that the outcomes in all three cases agree with those in \(\mathcal{A}\).

The monotone completeness of \(\mathcal{A}\) and the validity of equation (6.6) with the up an down taken in \(\mathcal{A}\) imply that \(\mathcal{A}\) and \(I_\mu(B(X,B;\mu;\mathbb{R}))\) as supersets for ups and downs also give the same result.

**Remark 6.2.** Suppose that, in Theorem 6.1, \(\mathcal{A}\) consists of order continuous operators on a directed normal partially ordered vector space, and that \(\mu\) is finite. For \(x \in E^+\) and \(x' \in (E_\omega^-)^+\), the functional \(a \mapsto (ax,x')\) is a positive order continuous functional on \(\mathcal{A}\), and setting \(\mu_{x,x'}(\Delta) := (\mu(\Delta)x,x')\) for \(\Delta \in \mathcal{B}\) yields a finite regular Borel measure \(\mu_{x,x'} : \mathcal{B} \to \mathbb{R}^+\). Using [8] Proposition 6.8, it is easy to see that \(f \in \mathcal{L}^1(X,\Omega,\mu_{x,x'};\mathbb{R})\) when \(f \in \mathcal{L}^1(X,\mathcal{B};\mu;\mathbb{R})\) and that, for \(f \in \mathcal{L}^1(X,\mathcal{B};\mu;\mathbb{R})\), \(I_\mu(f) = \int_X f \, d\mu\) is the unique element of \(\mathcal{A}\) such that

\[(I_\mu(f)x,x') = \int_X f \, d\mu_{x,x'}\]

for all \(x \in E^+\) and \(x' \in (E_\omega^-)^+\).

**Remark 6.3.** It is not true that the spectral measure \(\mu\) in Theorem 6.1 is always finite, even when \(\pi\) can be extended to a positive algebra homomorphism from \(C_0(X)\) into \(\mathcal{A}\). By way of example, let \(S\) be an infinite set, supplied with the discrete topology. Then \(C_0(S)\) is a Banach lattice algebra with an order continuous norm and monotone continuous multiplication. Consider the identity map \(\pi : C_0(S) \to C_0(S)\). Its representing spectral measure \(\mu\) is given by \(\mu(\Delta) = \chi_\Delta\) when \(\Delta\) is a finite subset of \(S\), and by \(\mu(\Delta) = \infty\) when \(\Delta\) is an infinite subset of \(S\). Hence \(\mu\) is not finite.

Apparently, \(C_0(S)\) is not quasi-perfect. It is normal, as is any Banach lattice with an order continuous norm, but it does not satisfy condition (2) in Definition 2.2. This is easy to see. Take an infinite countable subset \(\{s_1, s_2, \ldots\}\) of \(S\), and consider the sequence \(\{\chi_{\{s_1,\ldots,s_n\}}\}_{n=1}^\infty\) of indicator functions in \(C_c(X)\). Since the dual of \(C_0(S)\) can be identified with \(\ell^1(S)\), it is clear that \(\sup_{n \geq 1} \chi_{\{s_1,\ldots,s_n\}}(x') < \infty\) for each \(x' \in (C_0(S)^-)^+\). Yet the sequence has no supremum in \(C_0(S)\).

Whereas Theorem 6.1 does typically not apply to algebras of operators on infinite dimensional spaces, our next result, Theorem 6.4, often does; see Proposition 2.11 Corollary 2.12 and Proposition 2.13. The order continuous operators on quasi-perfect spaces are an example of quasi-perfect partially ordered algebras with a monotone continuous multiplication to which it applies; another important example is in the context of operators on Hilbert spaces. In Section 7, we shall see its consequences for positive representations of \(C_0(X)\) on Banach lattices and Hilbert spaces. As Remark 6.3 makes clear, Theorem 6.1 and Theorem 6.4 have an independent value of their own.
Theorem 6.4 (Positive algebra homomorphisms from $C_c(X)$ into normal partially ordered algebras with a monotone continuous multiplication). Let $X$ be a locally compact Hausdorff space, let $A$ be a monotone complete and normal partially ordered algebra with a monotone continuous multiplication, and let $\pi : C_c(X) \to A$ be a positive algebra homomorphism such that $\{\pi(f) : f \in C_c(X)^+, \|f\| \leq 1\}$ is bounded above in $A$; the latter condition is automatically satisfied when $X$ is compact, and also when $A$ is quasi-perfect and $\pi$ is the restriction of a positive algebra homomorphism $\pi : C_0(X) \to A$.

1. There exists a unique regular Borel measure $\mu : \mathcal{B} \to \overline{A}^+$ on the Borel $\sigma$-algebra $\mathcal{B}$ of $X$ such that

$$\pi(f) = \int_X f \, d\mu$$

for all $f \in C_c(X)$. If $V$ is a non-empty open subset of $X$, then

$$\mu(V) = \bigvee \{\pi(f) : f \prec V\}$$

in $A^+$. If $K$ is a compact subset of $X$, then

$$\mu(K) = \bigwedge \{\pi(f) : K \prec f\}$$

in $A^+$. When $\pi$ is the restriction of a positive algebra homomorphism $\pi : C_0(X) \to A$, equation (5.1) also holds for $f \in C_0(X)$.

2. The measure $\mu$ is a finite spectral measure which is inner regular at all Borel sets.

3. The spaces $\mathcal{L}^1(X, \mathcal{B}, \mu; \mathbb{R})$ and $L^1(X, \mathcal{B}, \mu; \mathbb{R})$ are both $\sigma$-Dedekind complete vector lattices, and the naturally defined operators $I_{\mu}$ from these spaces into $A$ are both $\sigma$-order continuous. When $A$ has the countable sup property, $L^1(X, \mathcal{B}, \mu; \mathbb{R})$ is a Dedekind complete vector lattice with the countable sup property, and $I_{\mu} : L^1(X, \mathcal{B}, \mu; \mathbb{R}) \to A$ is order continuous.

4. (a) $I_{\mu}(L^1(X, \mathcal{B}, \mu; \mathbb{R})) \subseteq \mu(X)A\mu(X)$;

(b) The operators $I_{\mu} : L^1(X, \mathcal{B}, \mu; \mathbb{R}) \to \mu(X)A$, $I_{\mu} : L^1(X, \mathcal{B}, \mu; \mathbb{R}) \to \mu(X)A\mu(X)$, and $I_{\mu} : L^1(X, \mathcal{B}, \mu; \mathbb{R}) \to \mu(X)A\mu(X)$ preserve moduli.

(c) $L^1(X, \mathcal{B}, \mu; \mathbb{R})$ is a unital vector lattice algebra.

(d) When supplied with the partial ordering inherited from $A$, the image $I_{\mu}(L^1(X, \mathcal{B}, \mu; \mathbb{R}))$ is a unital vector lattice algebra with $\mu(X)$ as identity element, and $I_{\mu} : L^1(X, \mathcal{B}, \mu; \mathbb{R}) \to I_{\mu}(L^1(X, \mathcal{B}, \mu; \mathbb{R}))$ is an isomorphism of vector lattice algebras.

(e) If $A$ is a vector lattice algebra with an identity element $e$ and $\mu(X) \leq e$, then $I_{\mu}(L^1(X, \mathcal{B}, \mu; \mathbb{R}))$ is a vector lattice subalgebra of $A$.

5. Suppose that $A$ is also a normed algebra such that $\|x\| \leq \|y\|$ for all $x, y \in A$ with $0 \leq x \leq y$. Then $L^1(X, \mathcal{B}, \mu; \mathbb{R}) = B(X, \mathcal{B}, \mu; \mathbb{R})$, and the map $I_{\mu} : B(X, \mathcal{B}, \mu; \mathbb{R}) \to A$ is a topological embedding of the Banach algebra $B(X, \mathcal{B}, \mu; \mathbb{R})$ as a closed subalgebra of $A$.

6. For $x' \in (A^*_w)^+$ and $\Delta \in \mathcal{B}$, set $\mu_{x'}(\Delta) := (\mu(\Delta), x')$. Then $\mu_{x'} : \mathcal{B} \to \mathbb{R}^+$ is a regular Borel measure, and $f \in \mathcal{L}^1(X, \Omega, \mu_{x'}; \mathbb{R})$ when $f \in$...
\( \mathcal{L}^1(X, \mathcal{B}, \mu; \mathbb{R}) \). For \( f \in \mathcal{L}^1(X, \mathcal{B}, \mu; \mathbb{R}) \), \( I_\mu(f) = \int_X f \, d\mu \) is the unique element of \( \mathcal{A} \) such that

\[
(I_\mu(f), x') = \int_X f \, d\mu_{x'}
\]
for all \( x' \in (\mathcal{A}^*_\mu)^+ \).

(7) For \( a \in \mathcal{A} \), the following are equivalent:

(a) \( a \pi(f) = \pi(f) a \) for all \( f \in \mathcal{C}_c(X) \);
(b) \( a \mu(\Delta) = \mu(\Delta) a \) for all \( \Delta \in \mathcal{B} \) with finite measure;
(c) \( a I_\mu(f) = I_\mu(f) a \) for all \( f \in \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \).

When \( \mathcal{C}_0(X) \subseteq \mathcal{L}^1(X, \Omega, \mu; \mathbb{R}) \), these are also equivalent to:

(d) \( a I_\mu(f) = I_\mu(f) a \) for all \( f \in \mathcal{C}_0(X) \).

(8) Suppose that \( \mathcal{L}^1(X, \mathcal{B}, \mu; \mathbb{R}) = \mathcal{B}(X, \mathcal{B}, \mu; \mathbb{R}) \), and that \( \mathcal{A} \) has the countable sup property. Then:

\[
I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+)) = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+)) = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+))^\emptyset = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+))^\emptyset; \\
I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+)) = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+))^{\uparrow\downarrow} = I_\mu\left(\left[\pi(\mathcal{C}_c(X)^+)\right]^{\downarrow\uparrow}\right); \\
I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+)) = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+))^{\downarrow\uparrow} = I_\mu\left(\left[\pi(\mathcal{C}_c(X)^+)\right]^{\downarrow\uparrow}\right); \\
I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R})) = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}))^\emptyset = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}))^\emptyset; \\
I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R})) = \left[\pi(\mathcal{C}_c(X))\right]^{\downarrow\downarrow} = \left[\pi(\mathcal{C}_c(X))\right]^{\downarrow\downarrow}.
\]

Here the ups and downs of the images of \( \pi \) and \( I_\mu \) can be taken in \( \mathcal{A} \), \( \mu(X)A, A\mu(X), \mu(X)A\mu(X) \), or \( I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R})) \) with equal outcomes.

**Proof.** It is clear that \( \{\pi(f) : f \in \mathcal{C}_c(X)^+, \|f\| \leq 1\} \) is bounded above in \( \mathcal{A} \) when \( X \) is compact. It follows from [9] Theorem 6.8 that this is also true when \( \mathcal{A} \) is quasi-perfect and \( \pi \) is the restriction of a positive algebra homomorphism \( \pi : \mathcal{C}_0(X) \rightarrow \mathcal{A} \).

The statements in the parts [1] and [2] follow from [9] Proposition 3.6, Lemma 6.3, and Theorem 5.4] and Theorem 3.6.

Part [3] follows from [8] Proposition 6.14 and Theorem 6.17.

Part [4] follows from Theorems 4.4 and 4.5.

The first statement in part [5] follows from Lemma 2.29 and the finiteness of \( \mu \). It is easy to see that \( I_\mu : \mathcal{B}(X, \mathcal{B}, \mu; \mathbb{R}) \rightarrow \mathcal{A} \) is continuous, so that the statement on the embedding follows from Proposition 2.31.

The proofs of the parts [6] [7] and [8] are as in the proof of Theorem 6.1.

**Remark 6.5.** Except for the fact that the finiteness of \( \mu \) need not be supposed, Remark 6.2 applies verbatim to the algebra \( \mathcal{A} \) in Theorem 6.4. It yields the same weak characterisation of order integrals of integrable functions as in equation (6.9) in the case where \( \mathcal{A} \) consists of order continuous operators on a directed normal partially ordered vector space.
7. Positive representations of $C_0(X)$ on Banach lattices and Hilbert spaces

We shall now apply Theorem 6.4 to positive representations of $C_0(X)$ on Banach lattices, and to representations of $C_0(X; \mathbb{C})$ on Hilbert spaces.

As a preparation, we recall the following. Suppose that $B$ is a normed algebra, and that $\pi$ is a bounded representation of $B$ on a normed space $E$. Then the representation $\pi$ is non-degenerate when $E$ is the closed linear span of the elements $\pi(b)x$ for $b \in B$ and $x \in E$. When $B$ has a bounded left approximate identity, then it is not difficult to see that there is as largest invariant linear subspace $E_{\text{nd}}$ of $E$ such that the restricted representation of $B$ on it is non-degenerate. This subspace is closed; in fact, it is the closed linear span of the elements $\pi(b)x$ for $b \in B$ and $x \in E$. In our case, we shall apply this with $B = C_0(X)$, so that $E_{\text{nd}} = \text{Span}\{\pi(f)x : f \in C_0(X), x \in E\}$. When working with a positive representation $\pi$ of $C_0(X)$ on a Dedekind complete Banach lattice, the automatic continuity of $\pi$ with respect to the regular norm (and then also the operator norm) implies that then also $E_{\text{nd}} = \text{Span}\{\pi(f)x : f \in C_c(X), x \in E\}$. Similar remarks apply to representations of $C_0(X; \mathbb{C})$ on complex Hilbert spaces.

7.1. Banach lattices. According to Corollary 2.12, the order continuous operators on a Dedekind complete normal Banach lattice form a Dedekind complete normal Banach lattice algebra with a monotone continuous multiplication. Hence Theorem 6.4 can be applied to positive algebra homomorphisms $\pi : C_0(X) \to \mathcal{L}_{\text{oc}}(E)$. If, in addition, the norm on $E$ is a Levi norm, then, according to Corollary 2.12, $\mathcal{L}_{\text{oc}}(E)$ is even a quasi-perfect Banach lattice algebra with a monotone continuous multiplication. In this case, the measure $\mu$ in Theorem 6.4 is automatically finite.

On taking into account Proposition 2.31, the weak characterisation of $I_{\mu}$ in Remark 6.5, the fact that every vector sublattice of $\mathcal{L}_r(E)$ has the countable sup property whenever $E$ is a Dedekind complete separable Banach lattice (see Section 2.2), and the fact that $\mathcal{L}_{\text{oc}}(E)$ is a band in $\mathcal{L}_r(E)$, we obtain the following from Theorem 6.4. In its part (5), the regular operators on the Banach lattice $E$ are supplied with the regular norm.

**Theorem 7.1** (Positive representations of $C_0(X)$ on Dedekind complete normal Banach lattices). Let $X$ be a locally compact Hausdorff space, let $E$ be a Dedekind complete normal Banach lattice such that $E_0^\infty$ separates the points of $E$, and let $\pi : C_0(X) \to \mathcal{L}_{\text{oc}}(E)$ be a positive algebra homomorphism. Suppose that at least one of the following is satisfied:

(i) $X$ is compact;
(ii) the norm on $E$ is a Levi norm.

Then the following hold.
(1) There exists a unique regular Borel measure \( \mu : \mathcal{B} \to L_{\infty}(E)^{+} \) on the Borel \( \sigma \)-algebra \( \mathcal{B} \) of \( X \) such that

\[
\pi(f) = \int_{X} f \, d\mu
\]

for all \( f \in C_{c}(X) \). If \( V \) is a non-empty open subset of \( X \), then

\[
\mu(V) = \sqrt{\{ \pi(f) : f < V \}}
\]

in \( L_{\infty}(E) \) and in \( L_{r}(E) \). If \( K \) is a compact subset of \( X \), then

\[
\mu(K) = \sqrt{\{ \pi(f) : K < f \}}
\]

in \( L_{\infty}(E) \) and in \( L_{r}(E) \).

(2) The measure \( \mu \) is a finite spectral measure which is inner regular at all Borel sets. Equation (6.1) also holds for \( f \in C_{0}(X) \).

(3) We have \( L^{1}(X, \mathcal{B}, \mu; \mathbb{R}) = B(X, \mathcal{B}, \mu; \mathbb{R}) \).

(4) The spaces \( \mathcal{B}(X, \mathcal{B}; \mathbb{R}) \) and \( B(X, \mathcal{B}, \mu; \mathbb{R}) \) are both \( \sigma \)-Dedekind complete vector lattices, and the naturally defined operators \( I_{\mu} \) from these spaces into \( L_{\infty}(E) \) are both \( \sigma \)-order continuous. When \( L_{\infty}(E) \) has the countable sup property (which is the case when \( E \) is separable), \( B(X, \mathcal{B}, \mu; \mathbb{R}) \) is a Dedekind complete vector lattice with the countable sup property, and \( I_{\mu} : B(X, \mathcal{B}, \mu; \mathbb{R}) \to L_{\infty}(E) \) is order continuous.

(5) (a) \( I_{\mu}(B(X, \mathcal{B}, \mu; \mathbb{R})) \subseteq \mu(X)L_{\infty}(E)\mu(X) \).

(b) The operators \( I_{\mu} : B(X, \mathcal{B}, \mu; \mathbb{R}) \to \mu(X)L_{\infty}(E) \) and \( I_{\mu} : B(X, \mathcal{B}, \mu; \mathbb{R}) \to \mu(X)L_{\infty}(E) \) are both order continuous.

(c) The map \( I_{\mu} : B(X, \mathcal{B}, \mu; \mathbb{R}) \to I_{\mu}(B(X, \mathcal{B}, \mu; \mathbb{R})) \) is a topological embedding of the Banach algebra \( B(X, \mathcal{B}, \mu; \mathbb{R}) \) as a Banach subalgebra of \( L_{\infty}(E) \).

(d) When supplied with the partial ordering inherited from \( L_{r}(E) \), the image \( I_{\mu}(B(X, \mathcal{B}, \mu; \mathbb{R})) \) is a unital vector lattice algebra with \( \mu(X) \) as identity element. The map \( I_{\mu} : B(X, \mathcal{B}, \mu; \mathbb{R}) \to I_{\mu}(B(X, \mathcal{B}, \mu; \mathbb{R})) \) is then an isomorphism of vector lattice algebras.

(e) If \( \mu(X) \leq 1 \), then \( I_{\mu}(B(X, \mathcal{B}, \mu; \mathbb{R})) \) is a Banach lattice subalgebra of the center of \( E \), and \( I_{\mu} : B(X, \mathcal{B}, \mu; \mathbb{R}) \to I_{\mu}(B(X, \mathcal{B}, \mu; \mathbb{R})) \) is an isomorphism of Banach lattice algebras.

(f) The image \( \pi(C_{0}(X)) \) is a closed subalgebra of \( L_{r}(E) \).

(6) For \( x \in E^{+} \) and \( x' \in (E_{c}^{-})^{+} \), set \( \mu_{x,x'}(\Delta) := (\mu(\Delta)x, x') \) for \( \Delta \in \mathcal{B} \). Then \( \mu_{x,x'} : \mathcal{B} \to \mathbb{R}^{+} \) is a finite regular Borel measure on \( X \). For \( f \in \mathcal{B}(X, \mathcal{B}; \mathbb{R}) \), \( I_{\mu}(f) = \int_{X} f \, d\mu \) is the unique element of \( L_{r}(E) \) such that

\[
(I_{\mu}(f)x, x') = \int_{X} f \, d\mu_{x,x'}
\]

for all \( x \in E^{+} \) and \( x' \in (E_{c}^{-})^{+} \).

(7) For \( S \in L_{\infty}(E) \), the following are equivalent:

(a) \( S\pi(f) = \pi(f)S \) for all \( f \in C_{c}(X) \);
(b) \( S\mu(\Delta) = \mu(\Delta)S \) for all \( \Delta \in \mathcal{B} \) with finite measure;
(c) \( S I_\mu(f) = I_\mu(f)S \) for all \( f \in \mathcal{B}(X, \Omega; \mathbb{R}) \);
(d) \( S \pi(f) = \pi(f)S \) for all \( f \in C_0(X) \).

(8) If \( L_{oc}(E) \) has the countable sup property (which is the case when \( E \) is separable), then:

\[
I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}_+)) = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}_+))^\delta = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}_+))^\delta; \\
I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}_+)) = I_\mu([C_c(X)^{\uparrow\downarrow}]^{\uparrow\downarrow}) = [\pi(C_c(X^{\uparrow\downarrow})]^{\uparrow\downarrow}; \\
I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}_+)) = I_\mu([C_c(X)^{\uparrow\downarrow}]^{\uparrow\downarrow}) = [\pi(C_c(X^{\uparrow\downarrow})]^{\uparrow\downarrow}; \\
I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R})) = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}))^\delta = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}))^\delta; \\
I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R})) = [\pi(C_c(X))]^{\uparrow\downarrow} = [\pi(C_c(X))]^{\uparrow\downarrow}.
\]

Here the ups and downs of the images of \( I_\mu \) and \( \pi \) can be taken in \( L_{oc}(E), \mu(X)L_{oc}(E), L_{oc}(E)\mu(X), \mu(X)L_{oc}(E)\mu(X), I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R})), \) or \( L_r(E) \) with equal outcomes.

It is easy to see that, for a monotone net of regular operators on a Banach lattice with an order continuous norm, the existence of its order limit and the existence of its strong operator limit are equivalent, and that, when they exist, they are equal. It follows from this that the regular operators are closed in the bounded operators under the taking of monotone strong operator limits. These two observations enable us to extend Theorem 7.1 for Banach lattices with order continuous norms. We recall that the Banach lattices with order continuous Levi norms are precisely the KB-spaces.

**Theorem 7.2** (Positive representations of \( C_0(X) \) on Banach lattices with order continuous norms). Let \( X \) be a locally compact Hausdorff space, let \( E \) be a Banach lattice with an order continuous norm, and let \( \pi : C_0(X) \rightarrow L_r(E) \) be a positive algebra homomorphism. Suppose that at least one of the following is satisfied:

(i) \( X \) is compact;
(ii) \( E \) is a KB-space.

Then all statements in Theorem 7.1 hold, with the additional observations that \( L_{oc}(E) = L_r(E) \) and that \( E_{oc} = E \). Furthermore:

(1) If \( V \) is a non-empty open subset of \( X \), then

\[
\mu(V) = \text{SOT–lim}_{f \downarrow V} \pi(f).
\]

If \( K \) is a compact subset of \( X \), then

\[
\mu(K) = \text{SOT–lim}_{K \downarrow f} \pi(f).
\]

In particular,

\[
\mu(X) = \text{SOT–lim}_{f \downarrow X} \pi(f).
\]
(2) If \( \{ \Delta_n \}_{n=1}^{\infty} \) is a pairwise disjoint sequence in \( B \) then, for all \( x \in E \),
\[
\mu \left( \bigcup_{n=1}^{\infty} \Delta_n \right) x = \sum_{n=1}^{\infty} \mu(\Delta_n)x
\]
in the norm topology of \( E \);

(3) The projection \( \mu(X) \) projects onto the non-degenerate part of \( E \), i.e.,
\[
\mu(X)E = E_{\text{nd}} = \text{Span}\{ \pi(f)x : f \in C_0(X), x \in E \}
\]
\[
= \text{Span}\{ \pi(f)x : f \in C_c(X), x \in E \}.
\]

(4) The SOT-closed linear subspaces of the bounded (not necessarily regular) linear operators on \( E \) that are generated by the following sets are equal:
(a) \( \{ \pi(f) : f \in C_c(X) \} \);
(b) \( \{ \pi(f) : f \in C_0(X) \} \);
(c) \( \{ I_{\mu}(f) : f \in \mathcal{B}(X, B; \mathbb{R}) \} \);
(d) \( \{ \mu(V) : V \text{ is an open subset of } X \} \);
(e) \( \{ \mu(K) : K \text{ is a compact subset of } X \} \);
(f) \( \{ \mu(\Delta) : \Delta \text{ is a Borel subset of } X \} \).

(5) If \( \mathcal{L}_r(E) \) has the countable sup property (which is the case when \( E \) is separable), then the equalities in part (8) of Theorem 7.1 hold when the ups and downs of the images of \( I_{\mu} \) and \( \pi \) in \( \mathcal{L}_r(E) \) are defined in the strong operator topology;

Proof. The first observation preceding the theorem implies that the parts [1] and [2] hold. It also the basis for [9, Theorem 6.10], from which part [4] is taken. For part [5] one uses both observations preceding the theorem.

We turn to part [3]. The final two equalities were already observed in the beginning of this section. Since \( \mu(X)\pi(f) = \pi(f) \) for \( f \in C_0(X) \), it is clear that \( E_{\text{nd}} \subseteq \mu(X)E \). The reverse inclusion follows from part [1]. \( \square \)

Remark 7.3.

(1) It follows from part [4] of Theorem 7.2 that the commutants and then also the bicommutants (both in the bounded, not necessarily regular, operators on \( E \)) of the seven sets in part [4] are equal. Thus part [7] of Theorem 7.1 has been improved. Consequently, in the context of Theorem 7.2 \( \mu \) takes its values in the coinciding bicommutants of these sets. This statement, which is familiar from the representation theory of \( C(X; \mathbb{C}) \) on Hilbert spaces is, however, less precise than part [4].

(2) The positive algebra homomorphisms \( \pi : C_0(X) \to \mathcal{L}_r(E) \) for a KB-space \( E \) were studied in [10]. Theorem 7.2 provides a substantial improvement of the main results in [10].

We conclude this subsection with a discussion of possibly degenerate positive representations of \( C_0(X) \) on KB-spaces.

Let \( E \) be a Banach lattice, and let \( P : E \to E \) be a positive projection. Following ideas of Schaefer's (see [25, p. 214]) it can be shown that the closed subspace \( PE \) is a vector lattice when supplied with the partial ordering inherited from
E. Its modulus is given by $|x|_{PE} = P|x|$ for $x \in PE$. The definition $\|x\|_{PE} := \|x|_{PE}\| = \|P|x\|$ yields a lattice norm on $PE$ that is equivalent to the restriction of the original norm to $PE$. Thus $PE$ is a Banach lattice in the inherited partial ordering and with the norm $\|\cdot\|_{PE}$. We refer to [1] Theorem 5.59 for details and additional information.

If $E$ is a KB-space, then the equivalence of $\|\cdot\|_{PE}$ and the original norm on $PE$ implies that $PE$ is also a KB-space.

Suppose that $\pi : C_0(X) \to \mathcal{L}_r(E)$ is a positive representation on the KB-space $E$, so that $\mu(X)$ projects onto the non-degenerate part $E_{nd}$. Since $\pi(f)\mu(X) = \pi(f)$ for $f \in C_0(X)$, $\pi(C_0(X))$ vanishes on the kernel of $\mu(X)$. We may, therefore, just as well restrict our attention to the representation of $C_0(X)$ on the complementing KB-space $E_{nd}$. We claim that this representation on $E_{nd}$ is non-degenerate. Its non-degenerate part is $\overline{\text{Span}\{\pi(f)x : f \in C_0(X), x \in \mu(X)E\}}$, where the closure is in the lattice norm on $E_{nd}$. Since this lattice norm is equivalent to the restricted original norm, and $E_{nd}$ is closed, this closure is also the closure in $E$. It is not difficult to see that it is $E_{nd}$, establishing our claim.

Theorems 7.1 and 7.2 now apply to the positive representation of $C_0(X)$ on $E_{nd}$. In this case, however, we know from the above that the projection $\mu(X)$ in these theorems is the identity operator (on the Banach lattice-subspace $E_{nd}$ of the original $E$). In particular, part (5) of Theorem 7.1 shows that, for a positive representation of $C_0(X)$ on a KB-space $E$, $\mathcal{B}(X, \mathcal{B}, \mu; \mathbb{R})$ embeds as a Banach lattice subalgebra of the ideal centre of $E_{nd}$\(^{12}\).

7.2. Hilbert spaces. We turn to representations on Hilbert spaces. Suppose that $\pi : C_0(X; \mathbb{C}) \to \mathcal{B}(H)$ is a representation of the complex $C^*$-algebra $C_0(X; \mathbb{C})$ on a complex Hilbert space $H$. It follows from Kaplansky’s density theorem (see [15] Theorem 5.3.5), that $\mathcal{A} := \pi(C_0(X; \mathbb{C}))^{SOT}$ is a commutative SOT-closed $C^*$-subalgebra of $\mathcal{B}(H)$, and that the set $A_{sa}$ of its self-adjoint elements is $\pi(C_0(X))^{SOT}$. Since $A_{sa}$ is a quasi-perfect vector lattice algebra by Proposition 2.13 Theorem 6.4 applies to the positive algebra homomorphism $\pi : C_0(X) \to A_{sa}$ and yields a representing finite regular spectral Borel measure $\mu : \mathcal{B} \to A_{sa}$. It takes its values in the idempotents in $A_{sa}$, which are orthogonal projections on $H$. Before giving the full statement, let us identify in operator algebraic terms the condition that $A_{sa}$ have the countable sup property, which is instrumental to the results on ups and downs. As a preparation for this, we note that $\mu(X)\pi(f) = \pi(f)\mu(X) = \pi(f)$ for $f \in C_0(X)$ because $I_\mu : \mathcal{B}(X, \mathcal{B}; \mathbb{R}) \to A_{sa}$ is an algebra homomorphism. Consequently, $A$ is a unital algebra, so that the following result applies to it.

**Lemma 7.4.** Let $\mathcal{M}$ be a commutative unital SOT-closed $C^*$-subalgebra of $\mathcal{B}(H)$, not necessarily containing the identity operator. Then the vector lattice $M_{sa}$ has

---

\(^{12}\)Some care is in order here: in this statement, $\mu$ is the spectral measure corresponding to the restricted representation, which is—in the obvious sense—the restriction of the original one. However, as these two spectral measures have the same zero sets, the corresponding spaces $\mathcal{B}(X, \mathcal{B}, \mu; \mathbb{R})$ are, in the end, still equal.
the countable sup property if and only if every subset of $M$ that consists of non-zero mutually orthogonal self-adjoint projections is countable.

**Proof.** The unital $C^*$-algebra $M$ is isomorphic to $C(K; C)$ for a Hausdorff space $K$. Since the vector lattice $C(K) \cong M_{sa}$ is Dedekind complete as a consequence of the fact that $M$ is strongly closed, $K$ is extremally disconnected; see [18, Theorem 43.11], for example. We now recall that a vector lattice has the countable sup property if and only if every disjoint system of strictly positive elements that is bounded from above is countable; see [18, Theorem 29.3]. This makes it clear that every subset of $C(K)$ that consists of non-zero disjoint idempotents is countable when $C(K)$ has the countable sup property. Conversely, suppose that every subset of $C(K)$ that consists of non-zero mutually disjoint idempotents is countable, and let $\{ f_i : i \in I \}$ be a system of mutually disjoint non-zero positive elements of $C(K)$ that is bounded above. Using [22, Theorem 2.7], the fact that $K$ is extremally disconnected implies that the interior of the support of each $f_i$ contains a non-empty clopen subset of $K$. As a consequence of the assumption, the characteristic functions of these sets must form an at most countable subset of $C(K)$. Hence the index set $I$ is countable. \hfill $\square$

Hence the countable sup property of $A_{sa}$ is equivalent to $A$ being a $\sigma$-finite von Neumann algebra in the sense of [19, p. 62]. On separable Hilbert spaces, this is obviously always satisfied.

We recall that the existence of the extremum of a monotone net of self-adjoint operators in $B(H)$ and the existence of its strong operator limit are equivalent and that, when they exist, they are equal. Thus part (3) of Theorem 7.5 follows from the corresponding formulas in Theorem 6.4. Its part (8) then follows just as in the proof of Theorem 7.2. Since $A$ is isomorphic to its restriction to $\mu(X)H = H_{nd}$, we see that $A_{sa}$ certainly has the countable sup property when $H_{nd}$ is separable.

This all being said, Theorem 6.4 implies the largest part of the following result. The equality of the SOT-closed linear subspaces that are generated by the sets in part (7) follows from [9, Theorem 6.12]. Furthermore, for $x \in H$, the functional $S \mapsto \langle Sx, x \rangle$ is a positive order continuous functional on $A_{sa}$, and part (4) then follows from [8, Proposition 6.8].

**Theorem 7.5** (Representations of $C_0(X; C)$ on Hilbert spaces). Let $X$ be a locally compact Hausdorff space, let $H$ be a complex Hilbert space, and let $\pi : C_0(X; C) \to B(H)$ be a $^*$-homomorphism. Set $A := \pi(C_0(X; C))^{SOT}$. Then $A$ is a unital commutative $C^*$-algebra, and its algebra $A_{sa}$ of self-adjoint elements is $\pi(C_0(X))^{SOT}$. We supply $A_{sa}$ with the partial ordering that is inherited from the usual partial ordering on $B(H)_{sa}$, so that $A_{sa}$ becomes a quasi-perfect vector lattice algebra.

\[13\text{Recall that, in [19], a von Neumann algebra need not contain the identity operator.}\]
(1) There exists a unique regular Borel measure on $X$ with values in the extended positive cone of $\mathcal{A}_{sa}$ such that

\[ \pi(f) = \int_X f \, d\mu \]

for all $f \in C_0(X)$. The measure $\mu$ is a finite spectral measure which is inner regular at all Borel sets, and equation (7.5) also holds for $f \in C_0(X)$. If $V$ is a non-empty open subset of $X$, then

\[ \mu(V) = \text{SOT-lim}_{f \prec V} \pi(f). \]

If $K$ is a compact subset of $X$, then

\[ \mu(K) = \text{SOT-lim}_{K \prec f} \pi(f). \]

In particular,

\[ \mu(X) = \text{SOT-lim}_{f \prec X} \pi(f). \]

(2) If $\{\Delta_n\}_{n=1}^\infty$ is a pairwise disjoint sequence in $\mathcal{B}$ then, for all $x \in E$,

\[ \mu\left(\bigcup_{n=1}^\infty \Delta_n\right)x = \sum_{n=1}^\infty \mu(\Delta_n)x \]

in the norm topology of $H$.

(3) We have $L^1(X, \mathcal{B}, \mu; \mathbb{R}) = B(X, \mathcal{B}, \mu; \mathbb{R})$.

(4) For $x \in H$, set $\mu_{x,x}(\Delta) := \langle \mu(\Delta)x, x \rangle$ for $\Delta \in \mathcal{B}$. Then $\mu_{x,x}: \mathcal{B} \to \mathbb{R}^+$ is a finite regular Borel measure on $X$. For $f \in \mathcal{B}(X, \mathcal{B}; \mathbb{R})$, $I_\mu(f) = \int_X f \, d\mu$ is the unique element of $\mathcal{B}(H)$ such that

\[ \langle I_\mu(f)x, x \rangle = \int_X f \, d\mu_{x,x} \]

for all $x \in H$.

(5) The spaces $\mathcal{B}(X, \mathcal{B}; \mathbb{R})$ and $B(X, \mathcal{B}, \mu; \mathbb{R})$ are both $\sigma$-Dedekind complete vector lattices, and the naturally defined operators $I_\mu$ from these spaces into $\mathcal{A}_{sa}$ are both $\sigma$-order continuous. If $\mathcal{A}_{sa}$ has the countable sup property (equivalently: when $\mathcal{A}$ is $\sigma$-finite; this is certainly the case when $H_{\text{nd}}$ is separable), then $B(X, \mathcal{B}, \mu; \mathbb{R})$ is a Dedekind complete vector lattice with the countable sup property, and $I_\mu: B(X, \mathcal{B}, \mu; \mathbb{R}) \to \mathcal{A}_{sa}$ is order continuous.

(6) The map $I_\mu: B(X, \mathcal{B}, \mu; \mathbb{R}) \to \mathcal{A}_{sa}$ is a topological embedding of the Banach lattice algebra $B(X, \Omega, \mu; \mathbb{R})$ as a Banach lattice subalgebra of $\mathcal{A}_{sa}$. The map $\pi: C_0(X) \to \mathcal{A}_{sa}$ is a Banach lattice algebra homomorphism with closed range.

(7) The SOT-closed complex linear subspaces of $\mathcal{B}(H)$ that are generated by the following sets are all equal to the algebra $\mathcal{A}$:

(a) $\{ \pi(f) : f \in C_c(X) \}$;
(b) $\{ \pi(f) : f \in C_0(X) \}$;
(c) $\{ I_\mu(f) : f \in \mathcal{B}(X, \mathcal{B}; \mathbb{R}) \}$;
Our approach is from the opposite direction. The spectral measure is found first, valued functions, one obtains the (possibly non-unital) representations of $C^*_{\pi}$ for all $\pi \in \mathcal{P}(X)$. The projection $\mu(X)$ projects onto the non-degenerate part of $H$, i.e.,

$$
\mu(X)H = H_{nd} = \overline{\text{Span}\{\pi(f)x : f \in C_0(X; \mathbb{C}), x \in H\}}
= \overline{\text{Span}\{\pi(f)x : f \in C_c(X; \mathbb{C}), x \in H\}}.
$$

**Remark 7.6.** After extending the order integral in the natural way to complex-valued functions, one obtains the (possibly non-unital) representations of $C^*$-algebras $I_\mu : \mathcal{B}(X, \mathcal{B}; \mathcal{C}) \to B(H)$ and $I_\mu : B(X, \mathcal{B}, \mu; \mathcal{C}) \to B(H)$. Although their definition via the order integral did not involve any topology, the automatic continuity of these representations implies, for example, that, for $f \in \mathcal{B}(X, \mathcal{B}; \mathcal{C})$, $I_\mu(f)$ is the uniform limit of natural linear combinations of the $\mu(\Delta)$ for $\Delta \in \mathcal{B}$.

**Remark 7.7.** In the context of Theorem 7.5 take $x, y \in H$, and set $\mu_{x,y}(\Delta) := \langle \mu(\Delta)x, y \rangle$ for $\Delta \in \mathcal{B}$. Then $\mu_{x,y} : \mathcal{B} \to \mathcal{C}$ is a finite regular Borel measure on $X$, and equation (7.8) implies that, for $f \in C_0(X; \mathcal{C})$, $\pi(f)$ is the unique element of $B(H)$ such that

$$
\langle \pi(f)x, y \rangle = \int_X f \, d\mu_{x,y}
$$

for all $x, y \in H$.

It follows from this that, for compact $X$ and a unital representation of $C(X; \mathcal{C})$, our measure $\mu$ coincides with the spectral measure from the literature. As in [7] Chapter IX.1, for example, this is usually constructed by first using equation (7.2) for $f \in C(X; \mathcal{C})$ to define $\mu_{x,y}$, then using equation (7.9) again to define $\pi(f)$ for $f \in \mathcal{B}(X, \mathcal{B}; \mathcal{C})$, and finally setting $\mu(\Delta) := \pi(\Delta)$ for $\Delta \in \mathcal{B}$. Our approach is from the opposite direction. The spectral measure is found first, and then the relation with the measures $\mu_{x,y}$ is an immediate consequence.

These spectral measures from the literature are such that $\mu(X) = I$. For our measures, which exist in the most general setting of locally compact spaces and possibly non-degenerate representations, this need not be the case.

**Remark 7.8.** Our principled order-theoretical approach to the spectral theorem for representations of commutative $C^*$-algebras on Hilbert spaces appears to be new. For unital representations of $C(X; \mathcal{C})$ with $X$ compact, traits of it appear in the statement of [15] Theorem 5.2.6 and its proof, where equation (7.6) and the outer regularity of $\mu$ can be found. The full underlying picture with $\mu$ as a regular Borel measure that also satisfies equation (7.7), and with the functional calculi as special cases of generalised Lebesgue integrals is, however, not visible there.

Part (8) of Theorem 6.4 does not make an appearance in Theorem 7.5 because, in this particular context, it is an ingredient to the proof of the following strengthening of it.
Theorem 7.9. Let $X$ be a locally compact Hausdorff space, let $H$ be a complex Hilbert space, and let $\pi: C_0(X; \mathbb{C}) \to B(H)$ be a $^*$-homomorphism. Set $A := \pi(C_0(X; \mathbb{C}))$. Suppose that $A_{sa} = \pi(C_0(X))$ has the countable sup property or, equivalently, that $A$ is $\sigma$-finite; this is certainly the case when $H_{nd}$ is separable. Then:

$$A = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{C}));$$

$$A_{sa} = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R})) = [\pi(C(X))]^{\uparrow \downarrow} = [\pi(C_0(X))]^{\uparrow \downarrow} ;$$

$$A^+_{sa} = I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}^+)) = \pi(]\pi(C_0(X))^{\uparrow \downarrow}) = [\pi(C_0(X))]^{\uparrow \downarrow} = I_\mu(]\pi(C_0(X))^{\uparrow \downarrow}) = [\pi(C_0(X))]^{\uparrow \downarrow} .$$

Consequently, $I_\mu: A \to B(X, \mathcal{B}, \mu; \mathbb{C})$ is an isomorphism.

Proof. We establish the first equation. The remaining ones follow from this and part [8] of Theorem 6.4; the final statement is then also clear as $I_\mu$ is injective on $B(X, \mathcal{B}, \mu; \mathbb{C})$. We know that $I_\mu(B(X, \mathcal{B}, \mu; \mathbb{C})) \subseteq A$. We prove the reverse inclusion. Since $\pi(C_0(X; \mathbb{C})) \subseteq I_\mu(B(X, \mathcal{B}, \mu; \mathbb{C}))$, it suffices to show that $I_\mu(B(X, \mathcal{B}, \mu; \mathbb{C}))$ is strongly closed. Since it is a $C^*$-algebra, its being strongly closed is equivalent to its self-adjoint part being closed under the taking of SOT-limits of increasing nets; see [19] Theorem 2.4.4. We now show this. Suppose that $\{S_\lambda\}_{\lambda \in \Lambda} \subseteq I_\mu(\mathcal{B}(X, \mathcal{B}, \mu; \mathbb{R}))$ is a net of self-adjoint operators and that $S_\lambda \uparrow S$ strongly for some $S \in \mathcal{B}(H)_{sa}$. Since $A_{sa}$ is strongly closed, we have $S \in A_{sa}$. Hence $S_\lambda \uparrow S$ in $A$ in order, and then the countability assumption entails that $S \in I_\mu(B(X, \mathcal{B}, \mu; \mathbb{R}))$ by part [8] of Theorem 6.4.

Remark 7.10.

(1) According to Theorem 7.9, if $A$ is $\sigma$-finite (in particular: if $H_{nd}$ is separable), then the SOT-closed subalgebra of $\mathcal{B}(H)$ that is generated by $\pi(C_0(X; \mathbb{C}))$ is equal to the image of the accompanying Borel functional calculus. For a normal operator on a separable Hilbert space, this is [7] Lemma IX.8.7, which is proved by rather different methods [14].

When $H$ is separable, $X$ is compact, and $C(X; \mathbb{C})$ is realised as a $C^*$-subalgebra of $\mathcal{B}(H)$ that contains the identity operator, a proof—again using methods different from ours—is sketched in [13] Theorem 1.57.

We are not aware of a reference for the general result in Theorem 7.9, which, in our approach, is essentially a consequence of the characterisation of SOT-closed $C^*$-subalgebras of $\mathcal{B}(H)$ as the ones that are monotone SOT-closed, combined with the monotone convergence theorem for the order integral.

(2) For a separable Hilbert space, [19] Theorem 2.4.3 shows that $A^+_{sa} = [\pi(C_0(X))^{\uparrow \downarrow}]^{\uparrow \downarrow}$; Theorem 7.9 shows that even $A^+_{sa} = [\pi(C_0(X))^{\uparrow \downarrow}]^{\uparrow \downarrow}$. Suppose that $H_{nd} = H$ and that $A$ is $\sigma$-finite. It then follows from
[27] Theorem 4.2.2] that \( A_{sa} = [\pi(C_b(X))]_{1}^{1} \); Theorem [7,9] yields that \( A_{sa} = [\pi(C_b(X))]_{1}^{1} \). Our result, however, does not without further effort yield any information about norms as in the cited results.

8. JBW-algebras

In this section, we show how the general principles in the present paper lead to stronger and new results in spectral theory for JBW-algebras. Among others, it will become clear how the spectral resolution for an element of a JBW-algebra arises from an underlying spectral measure. The use of the latter in this context appears to be new.

Let \( \mathcal{M} \) be a JBW-algebra with identity element 1. Take \( a \in \mathcal{M} \), and let \( C(a, 1) \) denote the norm closed Jordan subalgebra generated by \( a \) and 1. It is an associative JB-algebra, and by [3, Proposition 1.12] there exist a compact Hausdorff space \( C(X) \) and an isometric unital algebra isomorphism \( b \mapsto \hat{b} \) from \( C(a, 1) \) onto \( C(X) \). Hence \( C(a, 1) \) is a unital Banach lattice algebra in the ordering inherited from \( \mathcal{M} \).

The spectrum \( \sigma(a) \) of \( a \) in \( \mathcal{M} \) is the set of all \( \lambda \in \mathbb{R} \) such that \( \lambda 1 - a \) is not Jordan invertible in \( \mathcal{M} \). By [3, Proposition 1.17], \( \sigma(a) \) equals the Banach algebra spectrum of \( a \) in the Banach algebra \( C(a, 1) \). Take \( b \in \mathcal{M} \) and an associative JB-subalgebra \( \mathcal{M}' \) of \( \mathcal{M} \) containing 1 and \( b \). Since \( C(b, 1) \) is isomorphic as a Banach algebra to \( C(K) \) for a compact Hausdorff space \( K \), and since the Banach algebra spectra of elements of such Banach algebras are easily seen to be stable under passing to Banach superalgebra[s], \( \sigma(b) \) is also equal to the Banach algebra spectrum of \( b \) in the Banach algebra \( \mathcal{M}' \). If \( b \in C(a, 1) \), then \( C(b, 1) \subseteq C(a, 1) \), and we can now conclude that \( \sigma(a) = \hat{a}(X) \). In particular, \( \sigma(a) = \hat{a}(X) \).

Let \( \mathcal{W}(a, 1) \) denote the \( \sigma \)-weakly closed Jordan subalgebra generated by \( a \) and 1. It is an associative JBW-algebra. By [3, Proposition 2.11], it is isomorphic to a \( C(K) \)-space. Hence \( \mathcal{W}(a, 1) \) is also a unital Banach lattice algebra in the ordering inherited from \( \mathcal{M} \).

We view the inverse isomorphism \( \hat{b} \mapsto b \) from \( C(X) \) onto \( C(a, 1) \) as a positive algebra homomorphism from \( C(X) \) into \( \mathcal{W}(a, 1) \). Since Proposition [2,14] shows that \( \mathcal{W}(a, 1) \) is a quasi-perfect partially ordered algebra with a monotone continuous multiplication, we can apply Theorem [6,4]. Hence there exists a unique regular \( \mathcal{W}(a, 1)^+ \)-valued measure \( \mu \) on the Borel \( \sigma \)-algebra of \( X \) such that

\[
\hat{b} = \int_X b\ d\mu
\]

for \( b \in C(a, 1) \). It is a spectral measure with \( \mu(X) = 1 \). If \( V \) is a non-empty open subset of \( X \), then there exists a non-zero positive \( \hat{b} \in C(X) \) with support contained in \( V \). The formula for \( \mu(V) \) in Theorem [6,4] then implies that \( \mu(V) > 0 \).}

\(^{15}\)This ingredient appears to be missing in the proof of the spectral mapping theorem in [3, Proposition 1.21].
We now consider the image \( \mu_a \) of \( \mu \) under the measurable map \( \hat{\alpha} : X \to \sigma(a) \). On setting \( \mu_a(\Delta) := \mu(\hat{\alpha}^{-1}(\Delta)) \) for a Borel subset \( \Delta \) of \( \sigma(a) \), we obtain a spectral measure \( \mu_a \) on the Borel \( \sigma \)-algebra \( \mathcal{B}(\sigma(a)) \) of \( \sigma(a) \) such that \( \mu_a(\sigma(a)) = 1 \). If \( V \) is a non-empty (relatively) open subset of \( \sigma(a) \), then \( \mu_a(V) = \mu(\hat{\alpha}^{-1}(V)) > 0 \).

We claim that \( \mu_a \) is a regular Borel measure. The shortest way to prove this is by noting that \( \hat{\alpha} : X \to \sigma(a) \) is a homeomorphism (see the first part of the proof of \([3, \text{ Corollary 1.19}] \), so that the regularity is inherited from \( \mu \). The regularity is, however, automatic. To see this, take a normal state \( \rho \) on \( \mathcal{W}(a, 1) \). Since every (relatively) open subset of \( \sigma(a) \) is \( \sigma \)-compact, \([23, \text{ Theorem 2.18}] \) implies that \( \rho \circ \mu_a \) is a regular Borel measure on \( \sigma(a) \). It then follows from \([8, \text{ Proposition 3.11}] \) that \( \mu_a \) itself is a regular Borel measure. It is also automatic that \( \mu \) is inner regular at all elements of \( \mathcal{B}(\sigma(a)) \). Indeed, since \( \rho \circ \mu_a \) is inner regular at every \( \Delta \in \mathcal{B}(\sigma(a)) \) by \([13, \text{ Proposition 7.5}] \), we can again use \([8, \text{ Proposition 3.11}] \) to conclude that \( \mu_a \) has the same property.

Using Proposition \(2.25\), we see that

\[
\int_X f \circ \hat{\alpha} \, d\mu = \int_{\sigma(a)} f \, d\mu_a
\]

for \( f \in C(\sigma(a)) \). If \( p \) is a polynomial, then

\[
\int_{\sigma(a)} p \, d\mu_a = \int_X p \circ \hat{\alpha} \, d\mu = \int_X p(\alpha) \, d\mu = p(\alpha).
\]

Equations \(8.1\) and \(8.2\) show that

\[
\left( \int_{\sigma(a)} f \, d\mu_a \right)^\wedge = f \circ \hat{\alpha}
\]

for \( f \in C(\sigma(a)) \). Hence the spectrum of \( \int_{\sigma(a)} f \, d\mu_a \) equals \( (f \circ \hat{\alpha})(\alpha) \).

As a particular case of equation \(8.3\), we have

\[
a = \int_{\sigma(a)} \text{id} \, d\mu_a.
\]

We claim that \( \mu_a \) is the only unital \( \mathcal{W}(a, 1) \)-valued spectral measure on \( \mathcal{B}(\sigma(a)) \) that satisfies equation \(8.4\). To see this, let \( \bar{\mu} \) be another such. Since the associated operators \( I_{\mu_a}, I_{\bar{\mu}} : C(\sigma(a)) \to \mathcal{W}(a, 1) \) are algebra homomorphisms, they agree on all polynomials on \( \sigma(a) \). As they are automatically norm continuous because of their positivity, they are equal. We can now use \([8, \text{ Proposition 6.8}] \) to conclude that

\[
\int_{\sigma(a)} f \, d(\rho \circ \mu_a) = \int_{\sigma(a)} f \, d(\rho \circ \bar{\mu})
\]

for every normal state \( \rho \) on \( \mathcal{W}(a, 1) \) and for all \( f \in C(\sigma(a)) \). Since, again by \([23, \text{ Theorem 2.18}] \), \( \rho \circ \bar{\mu} \) is also automatically regular, we see that \( \rho \circ \mu_a = \rho \circ \bar{\mu} \) for all normal states \( \rho \). Hence \( \mu_a = \bar{\mu} \).
Lemma \[2.29\] shows that \(L^1(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}) = \mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R})
\]

Now that we have the spectral measure \(\mu_a\), we can define the associated operators \(I_{\mu_a}\) from various spaces of (equivalence classes of) measurable functions on \(\sigma(a)\) into \(\mathcal{W}(a, 1)\). Since \(\mu_a(\sigma(a)) = 1\), Theorem \[4.6\] shows that \(I_{\mu_a} : \mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}) \rightarrow \mathcal{W}(a, 1)\) is a unital vector lattice algebra homomorphism. The image is then clearly strictly positive (not necessarily normal) state.

These are both unital Banach lattice algebras in the ordering inherited from vector lattice algebra homomorphism. The image is then clearly strictly positive (not necessarily normal) state. By uniqueness, which, by uniqueness, is \(\mu_a\). Combined with the above, we thus have the following. We recall from Section \[2.2\] that the Banach lattice \(\mathcal{W}(a, 1)\) has the countable sup property when it is separable or, more generally, when it has strictly positive (not necessarily normal) state.

**Theorem 8.1.** Let \(\mathcal{M}\) be a JBW-algebra. Take \(a \in \mathcal{M}\), and let \(\mathcal{C}(a, 1)\) (resp. \(\mathcal{W}(a, 1)\)) be the JB-subalgebra (resp. JBW-subalgebra) that is generated by \(a\). These are both unital Banach lattice algebras in the ordering inherited from \(\mathcal{M}\).

There exists a unique spectral measure \(\mu_a : \mathcal{B} \rightarrow \mathcal{W}(a, 1)^+\) on the Borel \(\sigma\)-algebra \(\mathcal{B}(\sigma(a))\) of \(\sigma(a)\) with \(\mu_a(\sigma(a)) = 1\) such that

\[
a = \int_{\sigma(a)} \text{id} \, d\mu_a
\]

in \(\mathcal{W}(a, 1)\). The measure \(\mu_a\) is a regular Borel measure that is inner regular at all Borel subsets of \(\sigma(a)\). If \(V\) is a non-empty (relatively) open subset of \(\sigma(a)\), then

\[
0 < \mu_a(V) = \bigvee \{I_{\mu_a}(f) : f \in \mathcal{C}(\sigma(a)), f \prec V\}
\]

in \(\mathcal{W}(a, 1)\). If \(K\) is a compact subset of \(\sigma(a)\), then

\[
\mu_a(K) = \bigwedge \{I_{\mu_a}(f) : f \in \mathcal{C}(\sigma(a)), K \prec f\}
\]

in \(\mathcal{W}(a, 1)\).

Furthermore:

1. \(L^1(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}) = \mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R})\)

2. \(\mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}) is a \sigma\)-Dedekind complete unital Banach lattice algebra, and \(I_{\mu_a} : \mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}) \rightarrow \mathcal{W}(a, 1)\) is an isometric \(\sigma\)-order continuous unital vector lattice algebra homomorphism.
(b) If \( \mathcal{W}(a,1) \) has the countable sup property, \( \mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}) \) is Dedekind complete, and \( I_{\mu_a}: \mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}) \to \mathcal{W}(a,1) \) is order continuous.

(3) (a) The natural map from \( C(\sigma(a)) \) into \( \mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}) \) is an isometric unital vector lattice algebra homomorphism.

(b) \( I_{\mu_a}: C(\sigma(a)) \to C(a,1) \) is a surjective isometric unital vector lattice algebra homomorphism which coincides with the continuous functional calculus for \( a \).

(c) \( \sigma(I_{\mu_a}(f)) = f(\sigma(a)) \) for \( f \in C(\sigma(a)) \).

(4) For a normal state \( \rho \) on \( \mathcal{W}(a,1) \) and \( \Delta \in \mathcal{B}(\sigma(a)) \), set \( \mu_{a,\rho}(\Delta) := (\mu_a(\Delta), \rho) \). Then \( \mu_{a,\rho}: \mathcal{B}(\sigma(a)) \to \mathbb{R}^+ \) is a regular Borel measure that is inner regular at all Borel subsets of \( \sigma(a) \). For \( f \in \mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a)); \mathbb{R}) \),

\[
(I_{\mu_a}(f), \rho) = \int_{\sigma(a)} f \, d\mu_{a,\rho}
\]

for all normal states \( \rho \) on \( \mathcal{W}(a,1) \).

(5) Suppose that \( \mathcal{W}(a,1) \) has the countable sup property. Then:

\[
I_{\mu_a}(\mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}) \nonumber ) = \mu_a(\mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}+) \nonumber ) \nonumber )
\]

\[
= \mu_a(\mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}+) \nonumber ) \nonumber ) \nonumber ;
\]

\[
I_{\mu_a}(\mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}+) \nonumber ) = \mu_a(\mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}+) \nonumber ) \nonumber ) \nonumber ;
\]

\[
I_{\mu_a}(\mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}+) \nonumber ) = \mu_a(\mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}+) \nonumber ) \nonumber ) \nonumber ;
\]

\[
I_{\mu_a}(\mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}) \nonumber ) = \mu_a(\mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}) \nonumber ) \nonumber ) \nonumber ;
\]

in \( \mathcal{W}(a,1) \).

Remark 8.2.

(1) The fact that \( \mu_a \) is a measure, the order integral, and the suprema and infima in Theorem 8.1 (with those in \( \mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a), \mu_a; \mathbb{R}) \) in part (5) excepted) are all defined with respect to the ordering in \( \mathcal{W}(a,1) \). However, since these involve only monotone nets, and since bounded monotone nets in \( \mathcal{W}(a,1) \) convergence \( \sigma \)-weakly to their extrema by \( [3] \) Proposition 2.5(ii), the fact that \( \mathcal{W}(a,1) \) is \( \sigma \)-weakly closed in \( \mathcal{M} \) implies that one can equivalently use the ordering in \( \mathcal{M} \).
(2) Using ad hoc methods involving von Neumann algebras, it is shown in \cite{[21]} Theorem 2.1 that there exists a unital algebra homomorphism from $\mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a)); \mathbb{R})$ into $\mathcal{W}(a, 1)$. This also follows from the more precise part (1) of Theorem 8.1.

In \cite{[21]} Theorem 2.1, it is also stated that the pertinent algebra homomorphism is surjective. This can, however, not be concluded from its proof. \cite{[14]} Surjectivity is not stated in part (1) of Theorem 8.1. Given the proof of Theorem 7.9, it seems likely that, for the proof of such a result (presumably under additional hypotheses), an analogue of \cite{[19]} Theorem 2.4.4 for JB- and JBW-algebras is necessary.

(3) If $\{f_n\}_{n=1}^\infty$ is a uniformly bounded sequence in $\mathcal{B}(\sigma(a), \mathcal{B}(\sigma(a)); \mathbb{R})$ converging pointwise to $f$, then $\{I_{\mu_n}(f_n)\}_{n=1}^\infty$ converges $\sigma$-weakly to $I_{\mu_i}(f)$. This fact, which was already observed in \cite{[21]} Theorem 2.1, is immediate from part (3) and the (classical) dominated convergence theorem.

We shall now show how the spectral resolution for $a$ as in \cite{[3]} Theorem 2.20 can be found from the underlying spectral measure $\mu_a$. For $\lambda \in \mathbb{R}$, set $e_\lambda := \mu_a((-\infty, \lambda] \cap \sigma(a))$. We claim that $\{e_\lambda : \lambda \in \mathbb{R}\}$ is the spectral resolution for $a$. First of all, $e_\lambda$ is a projection in the associative algebra $\mathcal{W}(a, 1)$, so that it operator commutes with $a$ by \cite{[3]} Proposition 1.47. Furthermore, $U_{e_\lambda} a = e_\lambda \circ a$. Since $\chi_{(-\infty, \lambda]}(\sigma(a)) \cdot \text{id} \leq \lambda \chi_{(-\infty, \lambda]}(\sigma(a))$ as functions on $\sigma(a)$, we have

$$
U_{e_\lambda} a = e_\lambda \circ a = I_{\mu_a}(\chi_{(-\infty, \lambda]}(\sigma(a))) \circ I_{\mu_a}(\text{id}) = I_{\mu_a}(\chi_{(-\infty, \lambda]}(\sigma(a)) \cdot \text{id}) \leq I_{\mu_a}(\lambda \chi_{(-\infty, \lambda]}(\sigma(a))) = \lambda e_\lambda.
$$

It follows similarly that $U_{1-e_\lambda} \geq \lambda(1-e_\lambda)$. Since $\sigma(a) \subseteq [-\|a\|, \|a\|]$, it is clear that $e_\lambda = 0$ for $\lambda < -\|a\|$ and that $e_\lambda = 1$ for $\lambda > \|a\|$. Certainly, $e_{\lambda_1} \leq e_{\lambda_2}$ when $\lambda_1 < \lambda_2$. It follows from \cite{[3]} Proposition 4.6 that $e_\lambda = \bigwedge_{n \geq 1} e_{\lambda+1/n}$, so that $e_\lambda = \bigwedge_{\lambda' > \lambda} e_{\lambda'}$ by the monotonicity of $\lambda \mapsto e_\lambda$. We have now verified all defining properties for the spectral resolution for $a$ in \cite{[3]} Theorem 2.20.

It is also easy to see why $a$ can be approximated by Riemann–Stieltjes type sums as in \cite{[3]} Theorem 2.20. This is, in fact, true for $I_{\mu_a}(f)$ for every continuous $f : \sigma(a) \to \mathbb{R}$, i.e., for all $b \in C(\sigma(a), 1)$, and then in particular for $a = I_{\mu_a}(\text{id})$. To see this, fix a uniformly continuous extension $f_{\gamma}$ of $f$ to $\mathbb{R}$. One can always find such $f_{\gamma}$ with compact support by \cite{[23]} Theorem 20.4], but for the identity function on $\sigma(a)$ the identity function on $\mathbb{R}$ is also a possible choice. Take a finite increasing sequence $\gamma = \{\lambda_0, \ldots, \lambda_n\}$ such that $\sigma(a) \subseteq (\lambda_0, \lambda_n]$. Write $\|\gamma\| = \max_{i=1, \ldots, n}(\lambda_i - \lambda_{i-1})$. For $i = 1, \ldots, n$, take any $\lambda_i' \in (\lambda_i-1, \lambda_i] \cap \sigma(a)$. Then $\sum_{i=1}^n f_{\gamma}(\lambda_i') \chi_{(\lambda_{i-1}, \lambda_i]}(\sigma(a)) \to f$ uniformly on $\sigma(a)$ as $\|\gamma\| \to 0.$

\footnote{Personal communication by Marten Wortel.}
Applying the continuous operator \( I_{\mu_a} \) yields that \( \sum_{i=1}^{n} f_e'(\lambda_i) (e_{\lambda_i} - e_{\lambda_{i-1}}) \to I_{\mu_a}(f) \) as \( \|\gamma\| \to 0 \). In particular, this is true for \( f = \text{id} \) with as extension \( f_e \) the identity map on \( C(a,1) \), finite increasing sequences \( \gamma = \{\lambda_0, \ldots, \lambda_n\} \) such that \( [-\|a\|,\|a\|] \subset (\lambda_0,\lambda_n] \), and \( \lambda'_i = \lambda_i \). This implies the approximation result for \( a \) by Riemann–Stieltjes type sums for \( a \) in [3, Theorem 2.20].

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As the present argument in the paper shows, it is sufficient to have \( [-\|a\|,\|a\|] \subset (\lambda_0,\lambda_n] \) rather than \( [-\|a\|,\|a\|] \subset (\lambda_0,\lambda_n) \) for \( a \) as in [3, Theorem 2.20].
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