The $\pi_0 \rightarrow \gamma\gamma$ decay and the chiral anomaly in the quark-composites approach to QCD

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Abstract

We evaluate the $\pi_0 \rightarrow \gamma\gamma$ decay amplitude by an effective action derived from QCD in the quark composites approach, getting the standard value. We also verify that our effective action correctly reproduces the chiral anomaly.

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1 Introduction

Unlike previous field theories, QCD does not contain the fields which describe the particles observed in the experiments. This motivated the introduction of effective Lagrangians like the chiral Lagrangians, written in terms of the phenomenological fields and based solely on general principles of invariance.

The chiral Lagrangians encode the spontaneous breaking of the chiral symmetry which is generally believed to occur in QCD and the old results of current algebra and PCAC \[1, 2\]. But while these descriptions have been put on the safe ground of a consistent and predictive field theory, the determination from QCD of the parameters appearing in them as well as the proof that chiral symmetry is spontaneously broken in this theory are still lacking. The foundations of the chiral Lagrangians could, in principle, be completed by the use of numerical simulations in the lattice formulation of QCD, but, of course, an analytical approach is desirable also for a better understanding and a more clever way of using the numerical recipes. There have been, indeed, attempts in this direction, but they have been so far restricted to the strong coupling region in the gauge coupling constant \[3–7\], which is unfortunately far from the interesting continuum limit.

It has also been proposed to extend QCD via the introduction of extra degrees of freedom \[8, 9\]. The corresponding fields, even though carry the quantum numbers of the chiral mesons, are supposed to decouple in the continuum limit, in order to avoid double counting.

The idea behind the quark composites approach is that it should be possible to recover the interactions of the phenomenological fields by a change of variables in the partition function of QCD whereby the quark composites with the quantum numbers of the phenomenological fields of interest are assumed as new integration variables. The final goal is to unify the description of the spectrum properties and scattering processes in a framework consistent with the confinement of quarks.

For technical reasons this program can be realized in this form only for the baryons \[10, 11\]. For the mesons instead of a change of variables we make recourse to auxiliary fields \[12\]. This latter procedure, however, should not be confused with the quoted use of additional phenomenological fields in QCD.

In practice, to implement our approach in a perturbative framework, is also necessary that, after our manipulations, the free actions of the composites emerge in the effective action. To this aim we make use of the arbi-
trariness in the definition of the regularized action and perform a suitable choice of irrelevant terms which help to construct the effective action of the composites.

In our previous work [12,13] we got a proof of the spontaneous breaking of the chiral symmetry in QCD in the framework of our perturbative approach. Exploratory applications of this approach have been performed in the study of the pion-nucleon interaction [11] and the high temperature QCD phase transition [14]. These works are limited by the fact that in the broken phase the chiral symmetry is not realized nonlinearly. This property has been included in [13] where we proved that, in the absence of an explicit breaking due to the regularization, our approach generates the usual expansion of the chiral models in momenta and masses.

In the present paper we evaluate the amplitude of the decay of the $\pi_0$ into two photons and show that our effective action correctly reproduces the chiral anomaly. These results are per se interesting, but they have in the present context a special relevance, related to the fact that in our effective action there is an explicit breaking of the chiral symmetry due to the regularization. This is because we are at the moment unable to treat the gluon composites analytically in analogy to the quark composites and therefore, to have access to the non-perturbative (in the gauge coupling constant) regime of the gluon field we are forced to define the theory on a lattice (but we emphasize that our approach is otherwise general). It is likely that in the near future we will have a chirally invariant form of this regularization also in the presence of non-Abelian fields [15–17]. But for the time being to get rid of the spurious states of the fermions on a lattice we must introduce the so called Wilson term [18], which explicitly breaks the chiral invariance of our effective action. It is therefore very important to ascertain whether the Wilson parameter can be taken arbitrarily small in order to avoid this unpleasant consequence, but at the same time reproducing the chiral anomaly of QCD.

This problem is made somewhat more intriguing by the fact that in our approach the quarks are perturbatively confined, because in the broken vacuum they acquire a large mass which does not allow their propagation to any finite order of our perturbative expansion. But if the quarks do not have any poles, why should we worry about the spurious ones and introduce the Wilson term? At the same time, if we omit this term, how can we reproduce the chiral anomaly?

The present findings provide an answer to these questions. We find the standard results subject to a condition on the quark effective mass which
is naturally satisfied in our approach \[19\]. Then we can assume the Wilson parameter \( r \) arbitrarily small and forget the Wilson term in our expansion in the strong sector, even though we must retain it in the electromagnetic amplitude which is non analytic in this parameter.

In the next Section, for the convenience of the reader, we summarize the essential steps of the derivation of the effective action. In Section 3 we evaluate the electromagnetic decay amplitude of the \( \pi_0 \), in Section 4 we evaluate the anomaly and in Section 5 we present our conclusions.

2 The quark-composites approach

We assume the modified partition function

\[
Z = \int [dV][d\lambda] \exp[-S_G - S_Q - S_C],
\]

(2.1)

where \( S_G \) is the action of the gauge fields, that is the Yang-Mills and Maxwell actions, \( S_Q \) is the action of the quark fields and \( S_C \) is a four fermions irrelevant operator which provides the kinetic terms for the quark composites with the quantum numbers of the chiral mesons. Therefore it will not have the form of the Nambu-Jona-Lasinio \[20\] or Gross-Neveu \[21\] models, that is of the so-called chirally extended QCD or \( \chi \)QCD (see also for example \[22\]).

\( \lambda \) is the quark field while the gluon and electromagnetic fields are associated to the link variables \( V_\mu \). Differentials in square brackets are understood to be the product of the differentials over the lattice sites and the internal indices. All the fields live in an euclidean lattice of spacing \( a \).

We introduce the following notation for the sum over the lattice

\[
(f, g) = a^4 \sum_x f(x)g(x).
\]

(2.2)

In this notation the quark action is

\[
S_Q = (\bar{\lambda}, Q\lambda) + m_q(\bar{\lambda}, \lambda).
\]

(2.3)

As already stated we will use the Wilson form of the quark wave operator

\[
Q = \gamma_\mu \nabla_\mu - a \frac{r}{2} \Box.
\]

(2.4)
The symmetric derivative $\nabla_{\mu}$ and the Laplacian $\Box$ are covariant and are defined in terms of the right/left derivatives

$$(\nabla_{\mu})_{xy} = \pm \frac{1}{a} (\delta_{x,x+\mu} V_{\pm\mu}(x) - \delta_{x,y})$$

according to

$$\nabla_{\mu} = \frac{1}{2} (\nabla_{\mu}^+ + \nabla_{\mu}^-)$$

$${\Box} = \sum_{\mu} \nabla_{\mu}^+ \nabla_{\mu}^- = \sum_{\mu} \nabla_{\mu}^- \nabla_{\mu}^+ = \sum_{\mu} \frac{1}{a} (\nabla_{\mu}^+ - \nabla_{\mu}^-)$$

We adopt the standard conventions

$$V_{\mu}(x) = \exp \left[ E a A_{\mu}^{em} \left( x + \frac{1}{2} \mu \right) + g a A_{\mu}^{YM} \left( x + \frac{1}{2} \mu \right) \right]$$

$$V_{-\mu}(x) = V_{\mu}^\dagger(x - \mu),$$

where $E$ is a charge matrix.

The chiral composites are the pions and the sigma

$$\vec{\pi} = i k \pi a^2 \vec{\lambda}, \quad \vec{\sigma} = k \pi a^2 \vec{\lambda}.$$  

$\gamma_5$ is assumed hermitian, the $\vec{\tau}$'s are the Pauli matrices and a factor of dimension (length)$^2$, necessary to give the composites the dimension of a scalar, has been written in the form $a^2 k \pi$.

Since for massless quarks the QCD action is chirally invariant, the action of the chiral mesons must be, apart from a linear breaking term, $O(4)$ invariant. It must then have the form

$$S_C = \frac{1}{4} \langle (\hat{\Sigma}^\dagger, C\hat{\Sigma}) \rangle - \frac{1}{4a^2} \langle (\chi^\dagger, \hat{\Sigma}) + (\hat{\Sigma}^\dagger, \chi) \rangle,$$

where

$$\hat{\Sigma} = \hat{\sigma} - i \vec{\tau} \cdot \vec{\pi}, \quad \chi = s - i \vec{\tau} \cdot \vec{p}, \quad \langle A \rangle = \text{tr}^{\text{isospin}} A.$$  

We introduced the sources $s$ and $\vec{p}$ of the sigma and pion fields (their coupling to the quarks differ by a factor $k \pi$ from the notation of [3]).

Heuristic considerations, based on experience with simple, solvable models, lead [12] to the following form of the wave operator of the chiral composites

$$C = \frac{\rho^4}{a^4} \left( \frac{1}{-\Box + \rho^2/a^2} \right),$$
where \( \rho \) is a dimensionless parameter. The irrelevance by power counting of \( S_C \) requires that in the continuum limit \( \rho \) do not to vanish and \( k_\pi \), as well as the product \( k_\pi \rho \), do not diverge. Under these conditions operators of dimension higher than 4 are accompanied by the appropriate powers of the cut-off. The operator \( C \) behaves, as a function of the distance, as the propagator of a particle with a mass divergent at least as the cutoff. Therefore, even though strictly speaking it is nonlocal, its departure from locality is very mild in general, and very small with our present choice of \( \rho \approx 1/a \) which makes the mass appearing in \( C \) divergent as the square of the cutoff. Ultimately, however, its irrelevance can be proven by showing that its local approximation \( C \sim - (\rho^2/a^2) - \Box \) yields the same results (at the cost of more involved calculations). This proof is at present not complete.

We replace the chiral composites by the auxiliary fields

\[
\Sigma = \Sigma_0 - i \vec{\tau} \cdot \vec{\Sigma}
\]  

by means of the Stratonovich-Hubbard transformation \cite{23}. Ignoring, as we will systematically do in the sequel, field independent factors, the partition function can be written

\[
Z = \int [dV][d\lambda d\bar{\lambda}] \left[ \frac{d \Sigma}{\sqrt{2\pi}} \right] \exp \left[ -S_G - S_0 + \overline{\lambda} (D - Q) \lambda \right]
\]

\[
= \int [dV] \left[ \frac{d \Sigma}{\sqrt{2\pi}} \right] \exp \left[ -S_G - S_0 + \text{Tr} \ln(D - Q) \right],
\]

(2.15)

where “Tr” is the trace over both space and internal degrees of freedom and we introduced the functions of the auxiliary fields

\[
S_0 = -\frac{1}{4} \rho^4 \langle (\Sigma^\dagger, (a^4C)^{-1}\Sigma) \rangle,
\]

\[
D = \frac{1 - \gamma_5}{2} k_\pi \left[ \rho^2 \Sigma + \chi \right] + \frac{1 + \gamma_5}{2} k_\pi \left[ \rho^2 \Sigma^\dagger + \chi^\dagger \right].
\]

(2.16)

(2.17)

Eventually we will set \( s = m \), the breaking parameter of the chiral symmetry, which must be distinguished from the quark mass \( m_q \). The scaling with the lattice spacing that we will assume for \( m \) will render the corresponding term irrelevant. The quark mass is absorbed in the breaking parameter \( m \) according to

\[
m \to m - \frac{1}{k_\pi} m_q.
\]

(2.18)
Since \(1/k_\pi\) will play the role of an expansion parameter, the contribution from \(m_q\) will be sub-leading in our expansion.

The derivative nature of the couplings of the pions is exhibited after the transformation

\[
\Sigma = R U
\]

where \(U\) is an element of \(SU(2)\), and

\[
R^2 = \Sigma_0^2 + \Sigma^2.
\]

The volume element

\[
\left[\frac{d\Sigma}{\sqrt{2\pi}}\right] = \left[\frac{dR}{\sqrt{2\pi}}\right][dU] \exp \sum_x 3 \ln R
\]

provides the Haar measure \([dU]\) over the group. We get the effective action

\[
\tilde{S} = \frac{1}{4} \sum_\mu \langle (\nabla_\mu (RU^\dagger), \nabla^{\mu} (RU))\rangle + \frac{\rho^2}{2a^2} \langle R, R \rangle - \text{Tr} \ln R - \sum_x 3 \ln R
\]

\[
\frac{1}{2} \text{Tr} \ln \left(1 - \frac{1}{\rho^2 R} \chi^\dagger U\right) - \frac{1}{2} \text{Tr} \ln \left(1 + \frac{1}{\rho^2 R} U^\dagger \chi\right)
\]

\[
- \text{Tr} \ln \left(1 + D^{-1} Q\right),
\]

where \(\nabla_\mu\) is the right or left covariant derivative.

After our manipulations the partition function becomes

\[
Z = \int [dV] \left[\frac{dR}{\sqrt{2\pi}}\right][dU] \exp \left[-S_G - \tilde{S}\right].
\]

The minimum of \(\tilde{S}\) is obtained for

\[
U = 1, \quad R = \sqrt{\Omega} \frac{1}{a\rho} \left[1 - \frac{am}{2\rho \sqrt{\Omega}}\right],
\]

where \(\Omega\) is the total number of quark components. In our case, by collecting the spinorial, colour and flavour indices we get

\[
\Omega = 24.
\]

The expansion around the minimum is naturally organized as a series in \(1/\sqrt{\Omega}\). In this framework we have a realization of the spontaneous breaking of the chiral invariance in QCD.
The dominant part of the Lagrangian density, neglecting the fluctuations of $R$ and terms arising from the expansion of $R$ with respect to $m$ is identical to that of the chiral models \[1, 2\]

\[L_2 = \frac{1}{4} f_\pi^2 \langle \nabla_\mu U^\dagger \nabla^\mu U - 2 B (\chi^\dagger U + U^\dagger \chi) \rangle, \tag{2.26}\]

with the identifications

\[f_\pi = \frac{\sqrt{\Omega}}{a \rho}, \quad B = \frac{1}{2 a^2 f_\pi}. \tag{2.27}\]

If we confine ourselves to this leading term, we must assume $f_\pi = 92$ Mev. After these positions we recognize that the expansion in $1/\sqrt{\Omega}$ is equivalent to that one in $1/f_\pi$. Note that the above definition implies that $\rho \sim 1/a$. Other scalings with the lattice spacing are possible, but will not be considered here.

If we introduce the pion field $\pi$ according to

\[U = \exp \left( \frac{i}{f_\pi} \vec{r} \cdot \vec{\pi} \right), \tag{2.28}\]

we have for the pion mass $m_\pi$ and the chiral condensate the relations

\[m_\pi^2 = 2mB, \quad k_\pi \langle 0 | \overline{\lambda} \lambda | 0 \rangle = 2 f_\pi^2 B. \tag{2.29}\]

The presence of the factor $k_\pi$ is due to the fact that the source $s$ has a coupling to the quark fields that differs by this factor from the conventions of \[2\]. It should also be noted that in the present case $m$ vanishes while $B$ diverges in the continuum limit.

Let us examine the mass of the quarks and the $\sigma$ in the broken vacuum. According to eq. (2.15) the quark effective mass

\[M_Q = k_\pi \rho^2 \tilde{R} = k_\pi \rho^2 f_\pi, \tag{2.30}\]

is $O(k_\pi f_\pi)$, and therefore the quarks are perturbatively confined. Whether their mass is or is not divergent in the continuum limit, depends on how the product $k_\pi \rho$ scales with the lattice spacing. The $\sigma$ instead has a mass $\sqrt{2}\rho/a$ which is always divergent in the continuum limit.
3 The $\pi_0 \to \gamma \gamma$ decay

In this Section all the functions are in momentum space. We will perform the calculations to leading order in all the couplings, namely to order $1/f_\pi$, $e^2/(2\pi)^2$, while the Yang-Mills fields will be suppressed. The gauge fields $A$ therefore, are only photon fields.

The amputated amplitude $I_{\alpha\beta}$ for the decay of the $\pi_0$ into two photons is related to the three-point function $\langle \pi_0(q)A_\alpha(k_1)A_\beta(k_2) \rangle$ according to

$$\langle \pi_0(q)A_\alpha(k_1)A_\beta(k_2) \rangle = \frac{\delta^2}{\delta A_\alpha(k_1)\delta A_\beta(k_2)} \left[ \frac{1}{D - Q\delta\pi_3(q)} \right]_{\vec{\pi} = A = 0}$$

where the $G$ are the free propagators of the pion and the photons. $I_{\alpha\beta}$ can be obtained by taking functional derivatives of the effective action with respect to the Maxwell and the pion fields

$$\langle \pi_0(q)A_\alpha(k_1)A_\beta(k_2) \rangle = \frac{\delta^2}{\delta A_\alpha(k_1)\delta A_\beta(k_2)} \left[ \frac{1}{D - Q\delta\pi_3(q)} \right]_{\vec{\pi} = A = 0} \delta_{k_1, k_2} S(k_1).$$

$S$ is the free propagator of the quark in the broken vacuum

$$S(k) = \left(-i \bar{k} + W(k)\right)^{-1}$$

with

$$W(k) = \frac{r}{2} \hat{k}^2 + M_Q$$

$$\bar{k}_\alpha = \frac{1}{a} \sin a k_\alpha$$

$$\hat{k}_\alpha = \frac{1}{a} 2 \sin a \frac{k_\alpha}{2}.$$
Having in mind the continuum limit, we neglected $\sqrt{\Omega k_a m}$ with respect to $M_Q$.

Next we identify the vertices. The electromagnetic vertex is

$$\frac{\delta Q(k_1, k_2)}{\delta A_\alpha(k)} \bigg|_{\pi = A = 0} = E V_\alpha(k_1, k_2) (2\pi)^4 \delta^4(-k_1 + k_2 + k) \quad (3.8)$$

with

$$V_\alpha(k_1, k_2) = V_\alpha\left(\frac{k_1 + k_2}{2}\right) \quad (3.9)$$

and

$$V_\alpha(k) = \frac{\partial}{\partial k_\alpha} S^{-1}(k). \quad (3.10)$$

The quark-charge matrix $E$ is defined as

$$E = \frac{e}{6}(1 + 3\tau_3) \quad (3.11)$$

where $e$ is the electric charge of the proton.

The anomalous vertex is

$$\frac{1}{2} \left\{ Q, \frac{1}{M_Q} \frac{\partial D(k_1, k_2)}{\partial \pi_3(q)} \right\} \bigg|_{\pi = A = 0} = i \frac{1}{f_\pi} \gamma_5 \tau_3 W(q)(2\pi)^4 \delta^4(-k_1 + k_2 + q). \quad (3.12)$$

Using the above equations the decay amplitude becomes

$$\mathcal{I}_{\alpha\beta}(k_1, k_2) = i \frac{1}{f_\pi} \int \left(\frac{dk}{2\pi}\right)^4 \text{tr} \left[ E^2 \tau_3 \gamma_5 W(k) S(k + k_1) V_\alpha(k + k_1, k) S(k) V_\beta(k, k - k_2) S(k - k_2) \right], \quad (3.13)$$

and after the sum over colour and isospin indices

$$\mathcal{I}_{\alpha\beta}(k_1, k_2) = -2i \frac{e^2}{f_\pi} \int \left(\frac{dk}{2\pi}\right)^4 W(k) \quad \text{tr}^{\text{spin}} [\gamma_5 S(k + k_1) V_\alpha(k + k_1, k) S(k) V_\beta(k, k - k_2) S(k - k_2)]. \quad (3.14)$$

Let us develop the expression (3.14) in series of the photon momenta $k_1$ and $k_2$. By using the Ward identity (3.10) in the form

$$SV_\alpha S = S \partial_\alpha S^{-1} S = -\partial_\alpha S, \quad (3.15)$$
where all the functions are evaluated at the same momentum \( k \), one can reduce the number of \( \gamma \)-matrices appearing in the trace. It is easy to show in this way that the first non-vanishing contribution comes at order \( k_1k_2 \) and is given by

\[
I_{\alpha\beta}(k_1, k_2) = i \frac{e^2}{f_\pi} k_{1\mu} k_{2\nu} \int \left( \frac{dk}{2\pi} \right)^4 \text{tr}^{\text{spin}} \left\{ \gamma_5 W \left[ \frac{1}{2} S \partial_\mu V_\alpha S \partial_\nu V_\beta S + 2 \partial_\mu SV_\alpha SV_\beta \partial_\nu S \\
+ \partial_\mu SV_\alpha \partial_\nu V_\beta S + S \partial_\mu V_\alpha SV_\beta \partial_\nu S \right] \right\}
\]

(3.16)

which, after a few integrations by parts, becomes

\[
I_{\alpha\beta}(k_1, k_2) = -i \frac{e^2}{f_\pi} k_{1\mu} k_{2\nu} \int \left( \frac{dt}{2\pi} \right)^4 \text{tr}^{\text{spin}} \left\{ \gamma_5 W \left[ \partial_\mu SV_\alpha \partial_\nu V_\beta S + SV_\alpha \partial_\mu SV_\beta \partial_\nu S \right] \right\}.
\]

(3.17)

By using the explicit expressions of \( S \) and \( V \) we get

\[
I_{\alpha\beta}(k_1, k_2) = -2i \frac{e^2}{f_\pi} k_{1\mu} k_{2\nu} \int \left( \frac{dt}{2\pi} \right)^4 \text{tr}^{\text{spin}} \left[ \gamma_5 \gamma_\mu \gamma_\alpha \gamma_\nu \gamma_\beta \right] \frac{W}{d^3} \cos t_\alpha \cos t_\nu \cos t_\beta \left[ W \cos t_\mu - 4 \partial_\mu W \sin t_\mu \right]
\]

(3.18)

where

\[
d = \bar{t}^2 + W^2(t).
\]

(3.19)

We assume, as usual, that in the continuum limit

\[
\lim_{a \to 0} \frac{a M_Q}{r} = 0,
\]

(3.20)

a condition which can be naturally satisfied in our approach \[19\]. Then we see that the integral takes its contribution only from the pole and gives the well known result \[24\]

\[
I_{\alpha\beta}(k_1, k_2) = -8i \frac{e^2}{f_\pi} \epsilon_{\mu\nu\alpha\beta} k_{1\mu} k_{2\nu} \int \left( \frac{dt}{2\pi} \right)^4 \cos t_2 \cos t_3 \cos t_4 \partial_1 \frac{\sin t_1}{d^2}
\]

\[
= -i \frac{1}{f_\pi} \left( \frac{e^2}{2\pi} \right)^2 \epsilon_{\mu\nu\alpha\beta} k_{1\mu} k_{2\nu}.
\]

(3.21)
4 The chiral anomaly

Apart from the explicit breakings, that is the source and the Wilson terms, our partition function is exactly invariant under the symmetry transformations

$$U \rightarrow g_R U g_L^†,$$  \hspace{1cm} (4.1)

with \((g_R, g_L) \in SU(2) \times SU(2)\). While the presence of the source term is necessary to provide a mass term to the pions, the Wilson term is a residue of the doubling problem of lattice fermions which induces a departure from the general structure of the chiral Lagrangians. Nonetheless this term is responsible for the correct chiral anomaly in lattice QCD, which, as it is well known, is deeply related to the electromagnetic decay of the \(\pi_0\).

Let us derive the general Ward identities. For the sake of simplicity hereafter we shall restrict the field \(R\) at his extremal value and we neglect its fluctuations.

For an arbitrary infinitesimal transformation

$$U \rightarrow U + \delta U$$  \hspace{1cm} (4.2)

with

$$\delta U = w_R U - U w_L.$$  \hspace{1cm} (4.3)

The variation of the first term of \(\tilde{S}\) is

$$\frac{1}{2} \delta \text{Tr} \sum_\mu \left[ \nabla_\mu^+ U^\dagger \nabla_\mu^+ U \right] =$$

$$= -\delta \text{Tr} \sum_\mu \left[ U^\dagger_{x+\mu} U_x + U^\dagger_x U_{x+\mu} \right]$$

$$= -\text{Tr} \sum_\mu \left[ -U^\dagger_{x+\mu} \delta U_{x+\mu} U_x + U^\dagger_x \delta U_x - U^\dagger_x \delta U_x U^\dagger_x U_{x+\mu} + U^\dagger_x U_{x+\mu} \delta U_{x+\mu} \right]$$

$$= -\text{Tr} \sum_\mu \left[ -U^\dagger_\mu U_{x+\mu} + U_{x+\mu} U^\dagger_\mu \right] \left[ \delta U_{x+\mu} U^\dagger_x - \delta U_x U^\dagger_x \right]$$

$$= -\text{Tr} \sum_\mu \left[ -U^\dagger_\mu U_{x+\mu} + U_{x+\mu} U^\dagger_\mu \right] \nabla_\mu^+ \left[ \delta U_x U^\dagger_x \right]$$

$$= \text{Tr} \sum_\mu \left\{ \nabla_\mu^- \left[ -U^\dagger_\mu U_{x+\mu} + U_{x+\mu} U^\dagger_\mu \right] \right\} \left[ \delta U_x U^\dagger_x \right]$$

$$= \text{Tr} \sum_\mu \left\{ \nabla_\mu^- \left[ -U \left( \nabla_\mu^+ U^\dagger \right) + \left( \nabla_\mu^+ U \right) U^\dagger \right] \right\} \left[ \delta U U^\dagger \right]$$

12
\[
\begin{align*}
\sum_{\mu} \{ \nabla_{\mu}^- \left[ -U \left( \nabla_{\mu}^+ U^\dagger \right) + \left( \nabla_{\mu}^+ U \right) U^\dagger \right] \} \{ w_R - U w_L U^\dagger \} = \\
\sum_{\mu} \{ \nabla_{\mu}^- \left[ -U \left( \nabla_{\mu}^+ U^\dagger \right) + \left( \nabla_{\mu}^+ U \right) U^\dagger \right] \} w_R \\
+ \sum_{\mu} \{ \nabla_{\mu}^- \left[ -U^\dagger \left( \nabla_{\mu}^+ U \right) + \left( \nabla_{\mu}^+ U^\dagger \right) U \right] \} w_L
\end{align*}
\]

which has the form of the divergence of a current. More precisely for a vectorial transformation, that is when

\[
g_L = g_R \quad \text{(4.5)}
\]

\[
w_L = w_R = w_V \quad \text{(4.6)}
\]

we get the vector current

\[
\vec{V}_\mu = \frac{1}{2} \left\langle \vec{\tau} \left\{ \left[ \nabla_{\mu}^+ U, U^\dagger \right] + \left[ \nabla_{\mu}^+ U^\dagger, U \right] \right\} \right\rangle. \quad \text{(4.7)}
\]

While for an axial transformation

\[
g_L = g_{R}^\dagger \quad \text{(4.8)}
\]

\[
w_L = -w_R = -w_A \quad \text{(4.9)}
\]

we get

\[
\vec{A}_\mu = \frac{1}{2} \left\langle \vec{\tau} \left\{ \left[ \nabla_{\mu}^+ U, U^\dagger \right] - \left[ \nabla_{\mu}^+ U^\dagger, U \right] \right\} \right\rangle. \quad \text{(4.10)}
\]

Let us consider now the explicit breaking at linear level

\[
\delta \text{Tr} \left( \chi U^\dagger + U^\dagger \chi \right) = \text{Tr} \left[ \left( U \chi^\dagger - \chi U^\dagger \right) w_R - \left( \chi^\dagger U - U^\dagger \chi \right) w_L \right]. \quad \text{(4.11)}
\]

From this expression in the case of a vector transformation we get

\[
\delta \text{Tr} \left( \chi U^\dagger + U^\dagger \chi \right) = \text{Tr} \left\{ \left[ U, \chi^\dagger \right] + \left[ U^\dagger, \chi \right] \right\} w_V \quad \text{(4.12)}
\]

while for an axial transformation

\[
\delta \text{Tr} \left( \chi U^\dagger + U^\dagger \chi \right) = \text{Tr} \left\{ \left[ U, \chi^\dagger \right] - \left[ U^\dagger, \chi \right] \right\} w_A. \quad \text{(4.13)}
\]

From these expressions we get, in the absence of the fermionic determinant, when \( \chi = m \)

\[
\begin{align*}
\sum_{\mu} \nabla_{\mu}^- \vec{V}_\mu &= 0 \quad \text{(4.14)} \\
\sum_{\mu} \nabla_{\mu}^- \vec{A}_\mu &= m^2 \left\langle \vec{\tau} \left( U - U^\dagger \right) \right\rangle \quad \text{(4.15)}
\end{align*}
\]
which is nothing but the classical equation of motion.

Let us now evaluate the variation of the fermionic determinant. Since we have already taken the linear contribution of the breaking term we will set now for simplicity $\chi = 0$. Then

$$D = MQ \left[ \frac{1 - \gamma_5}{2} U + \frac{1 + \gamma_5}{2} U^\dagger \right]$$

and its variation is

$$\delta D = MQ \left[ \frac{1 - \gamma_5}{2} \delta U + \frac{1 + \gamma_5}{2} \delta U^\dagger \right] = MQ \left[ \frac{1 - \gamma_5}{2} \delta U - \frac{1 + \gamma_5}{2} U^\dagger \delta U U^\dagger \right].$$

Therefore

$$\delta \text{Tr} \ln(D - Q) = \text{Tr} \ (D - Q)^{-1} \delta D = \text{Tr} \ (D - Q)^{-1} MQ \left[ \frac{1 - \gamma_5}{2} \delta U - \frac{1 + \gamma_5}{2} U^\dagger \delta U U^\dagger \right]$$

If we specialize to a vector transformation

$$\delta \text{Tr} \ln(D - Q) = -MQ \text{Tr} \left[ (D - Q)^{-1}, \frac{1 - \gamma_5}{2} U + \frac{1 + \gamma_5}{2} U^\dagger \right] w_V$$

While for an axial transformation

$$\delta \text{Tr} \ln(D - Q) = MQ \text{Tr} \left\{ (D - Q)^{-1}, \frac{1 - \gamma_5}{2} U - \frac{1 + \gamma_5}{2} U^\dagger \right\} w_A$$
We arrive at the equations

\[
\sum_{\mu} \nabla_{\mu} \bar{V}_{\mu} = \frac{2}{f_\pi^2} \text{Tr} \bar{\tau} \left[ Q, (D - Q)^{-1} \right] 
\]

\[
\sum_{\mu} \nabla_{\mu} \bar{A}_{\mu} = m_\pi^2 \left\langle \bar{\tau} \left( U - U^\dagger \right) \right\rangle - \frac{2}{f_\pi^2} \text{Tr} \bar{\tau} \left[ \gamma_5 Q, (D - Q)^{-1} \right] 
- \frac{2}{f_\pi^2} \text{Tr} \bar{\tau} (D - Q)^{-1} \{ Q, \gamma_5 \}.
\] (4.22)

where the new terms with respect to (4.15) correspond to the fermionic contributions to the currents. In particular the last term with the anti-commutator originates the anomalous vertex which entered in the computation of the decay rate of the \( \pi_0 \) of the previous section. It is therefore this term which is responsible for the breaking, at the quantum level, of the axial symmetry in the continuum limit. The anomaly of the underlying QCD is correctly reproduced by our effective chiral lagrangian.

5 Conclusions

We evaluated the decay amplitude of the electromagnetic decay of the pion by an effective action derived from QCD in the quark composites approach. This allowed us to treat in a unified way the anomalous and the electromagnetic vertices.

As a consequence of formula (3.21), to leading order the decay rate turns out to take the standard value of 7.63 eV (see for example [25]), surprisingly close to the experimental rate (7.37 ± 1.5) eV. It would be interesting to check whether the strong corrections are sufficiently small in our theory.

The present results allow us to establish a close relation between our expansion for the chiral mesons and the chiral models. Since the chiral anomaly is independent of the value of the Wilson parameter \( r \), provided it is different from zero, we can then assume \( r = O(\Omega^{-n}) \), with \( n \) arbitrarily large. Now there is no reason to believe that the amplitudes in the strong sector are not analytic in \( r \), and therefore studying the strong interactions in the framework of our \( 1/\sqrt{\Omega} \sim 1/f_\pi \) expansion, we can forget the Wilson term. We showed that in this case our theory, under the standard condition (3.20) which is naturally satisfied, generates only terms of the chiral models [13]. In the electromagnetic sector, on the contrary, the amplitudes are not analytic.
in $r$, as we have seen in the previous section, and the Wilson term must be retained to get correct results.

We think that the quark-composites approach might prove useful also in numerical simulations. One can consider the action of eq. (2.15) as an improved lattice QCD action, where the chiral limit is obtained in the simple limit of zero breaking term, where the pion mass vanishes (rather than by a fine tuning). There is a price to pay because of the inclusion of the auxiliary fields, but this should be rewarded by a simpler evaluation of the quark determinant because of the dominance of the diagonal contribution in configuration space. There are indeed indications in this sense in the work presented in [9].

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