MILNOR-MOORE CATEGORIES AND MONADIC DECOMPOSITION

ALESSANDRO ARDIZZONI AND CLAUDIA MENINI

Abstract. In this paper Hom-Lie algebras, Lie color algebras, Lie superalgebras and other
type of generalized Lie algebras are recovered by means of an iterated construction, known as
monadic decomposition of functors, which is based on Eilenberg-Moore categories. To this aim
we introduce the notion of Milnor-Moore category as a monoidal category for which a Milnor-
Moore type Theorem holds. We also show how to lift the property of being a Milnor-Moore
category whenever a suitable monoidal functor is given and we apply this technique to provide
examples.

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Introduction

Let \((L : B \to A, R : A \to B)\) be an adjunction with unit \(\eta\) and counit \(\epsilon\). Then \((RL, R\eta L, \eta)\) is
a monad on \(B\) and one can consider the Eilenberg-Moore category \(RLB\) associated to this monad
and the so-called comparison functor \(K : A \to _{RL}B\) which is defined by \(KX := (RX, R\epsilon X)\) and
\(Kf := Rf\). This gives the diagram

\[
\begin{array}{c}
A \\
\downarrow \quad \downarrow K \\
B \\
\end{array}
\begin{array}{cc}
\leftrightarrow & \text{Id}_A \\
\downarrow L & \downarrow R \\
\uparrow R & \uparrow RL \\
\end{array}
\begin{array}{c}
A \\
\end{array}
\begin{array}{c}
\leftrightarrow & RL_B \\
\downarrow \quad \downarrow RL \\
\end{array}
\]

where the undashed part commutes. In the case when \(K\) itself has a left adjoint \(\Lambda\) one can repeat
this construction starting from the new adjunction \((A, K)\). Going on this way one possibly obtains

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a diagram of the form

```
    A ← Id_A ← A ← Id_A ← A ← Id_A ← ...  
    L_0 ↑ R_0   U_{0,1} ↑ L_1 ↑ R_1   U_{1,2} ↑ L_2 ↑ R_2   U_{2,3} ↑ ...  
    B_0 ← B_1 ← B_2 ← ...  
```

where it is more convenient to relabel \((L, R)\) and \((\Lambda, K)\) as \((L_0, R_0)\) and \((L_1, R_1)\) respectively. If there is a minimal \(N \in \mathbb{N}\) such that \(L_N\) is full and faithful, then \(R\) is said to have monadic decomposition of monadic length \(N\). This is equivalent to require that the forgetful functor \(\text{Br}_{\mathcal{M}}\) is a category isomorphism and no \(U_{n,n+1}\) has this property for \(0 \leq n \leq N - 1\) (see e.g. [AGM], Remark 2.4]). In [AGM] Theorem 3.4, we investigated the particular case

```
\begin{array}{cccc}
\mathfrak{M} & \mathfrak{Bialg}_{\mathfrak{M}} & \mathfrak{Bialg}_{\mathfrak{M}} & \mathfrak{Bialg}_{\mathfrak{M}} \\
\mathfrak{T} & \mathfrak{T}_1 & \mathfrak{T}_2 & \mathfrak{T}_3 \\
\mathfrak{M} & \mathfrak{M}_1 & \mathfrak{M}_2 \\
\end{array}
```

where \(\mathfrak{M}\) denotes the category of vector spaces over a fixed base field \(k\), \(\mathfrak{Bialg}_{\mathfrak{M}}\) is the category of k-bialgebras, \(\mathfrak{T}\) is the tensor bialgebra functor (the barred notation serves to distinguish this functor from the tensor algebra functor \(T : \mathfrak{M} \to \text{Alg}_{\mathfrak{M}}\) which goes into k-algebras) and \(P\) is the primitive functor which assigns to each k-bialgebra its space of primitive elements. We proved that this \(P\) has a monadic decomposition of monadic length at most 2. Moreover, when \(\text{char} (k) = 0\), for every \(V_2 = ((V, \mu), \mu_1) \in \mathfrak{M}_2\) one can define \([x, y] := \mu (xy - yx)\) for every \(x, y \in V\). Then \((V, [\cdot, \cdot])\) is an ordinary Lie algebra and \(\mathfrak{T}_2 V_2 = TV/ \langle xy - yx - [x, y] | x, y \in V \rangle\) is the corresponding universal enveloping algebra. This suggests a connection between the category \(\mathfrak{M}_2\) and the category \(\text{Lie}_{\mathfrak{M}}\) of Lie k-algebras. It is then natural to expect the existence of a category equivalence \(\Lambda\) such that the following diagram

```
\begin{array}{cccc}
\mathfrak{M} & \mathfrak{Bialg}_{\mathfrak{M}} & \mathfrak{Bialg}_{\mathfrak{M}} & \mathfrak{Bialg}_{\mathfrak{M}} \\
\mathfrak{T} & \mathfrak{T}_1 & \mathfrak{T}_2 & \mathfrak{T}_3 \\
\mathfrak{M} & \mathfrak{M}_1 & \mathfrak{M}_2 \\
\end{array}
```

commutes in its undashed part, where \(\text{Lie}_{\mathfrak{M}}\) denotes the forgetful functor, \(\mathfrak{U}\) the universal enveloping bialgebra functor and \(\mathfrak{P}\) the corresponding primitive functor.

A first investigation showed that, in order to solve the problem above, it is more natural to work with braided k-vector spaces instead of ordinary k-vector spaces and to replace the categories \(\mathfrak{M}\), \(\mathfrak{Bialg}_{\mathfrak{M}}\) and \(\text{Lie}_{\mathfrak{M}}\) with their braided analogues \(\text{Br}_{\mathfrak{M}}, \text{BrBialg}_{\mathfrak{M}}\) and \(\text{BrLie}_{\mathfrak{M}}\) consisting of braided vector spaces, braided bialgebras and braided Lie algebras respectively. We were further led to substitute \(\mathfrak{M}\) with an arbitrary monoidal category \(\mathcal{M}\). We point out that, in order to produce a braided analogue of the universal enveloping algebra which further carries a braided bialgebra structure, the assumption that the underlying braiding is symmetric is also needed. Thus let \(\text{Br}_{\mathcal{M}}, \text{BrBialg}_{\mathcal{M}}\) and \(\text{BrLie}_{\mathcal{M}}\) be the analogue of \(\text{Br}_{\mathfrak{M}}, \text{BrBialg}_{\mathfrak{M}}\) and \(\text{BrLie}_{\mathfrak{M}}\) consisting of objects with symmetric braiding. Let \(\overline{T}_{\text{Br}} : \text{Br}_{\mathcal{M}} \to \text{BrBialg}_{\mathcal{M}}\) be the symmetric braided tensor bialgebra functor and let \(\overline{P}_{\text{Br}}^*\) be its right adjoint, the primitive functor. We seek for a condition for \(\overline{P}_{\text{Br}}^*\) to have monadic decomposition of monadic length at most 2. On the other hand, in this setting, the functor \(\overline{P}_{\text{Br}}^*\) induces a functor \(\overline{P}_{\text{Br}}^* : \text{BrBialg}_{\mathcal{M}} \to \text{BrLie}_{\mathcal{M}}\) which comes out to have a left adjoint \(\overline{U}_{\text{Br}}^*\), the universal enveloping bialgebra functor.

In view of the celebrated Milnor-Moore Theorem, see Remark [7], we say that a category \(\mathcal{M}\) is a Milnor-Moore category (MM-category for short) whenever the unit of the adjunction \((\overline{U}_{\text{Br}}^*, \overline{P}_{\text{Br}}^*)\) is a functorial isomorphism (plus other conditions required for the existence of the
functors involved). One of the main results in the paper is Theorem 7.2, which ensures that, for an MM-category $\mathcal{M}$, the functor $P_{\mathcal{M}}$, has a monadic decomposition of monadic length at most two. Moreover, in this case, we can identify the category $(\text{Br}^{*}_{\mathcal{M}})_{2}$ with $\text{BrLie}_{\mathcal{M}}^{*}$. Hence MM-categories, besides having an interest in their own, give us an environment where the functor $P_{\mathcal{M}}$ has a behaviour completely analogous to the classical vector space situation we investigated in [AGM, Theorem 3.4]. In the case of a symmetric MM-category $\mathcal{M}$ the connection with Milnor-Moore Theorem becomes more evident. In fact, in this case, we can apply Theorem 7.3 to obtain that the unit of the adjunction $(\mathcal{M}, P)$ is a functorial isomorphism.

The next step is to provide meaningful examples of MM-categories. A first result in this direction is Theorem 8.4, based on a proof by Kharchenko, which states that the category $\mathfrak{M}$ of vector spaces over a field of characteristic 0 is an MM-Category. Note that the Lie algebras involved are not ordinary ones but they depend on a symmetric braiding.

Much of the material developed in the paper (see e.g. Proposition 3.7, Theorem 8.3 and the construction of the adjunctions used therein) is devoted to the proof of our central result namely Theorem 8.4 which allows us to lift the property of being an MM-category whenever a suitable monoidal functor is given. A main tool in this proof is the concept of commutation datum which they obey the $	ext{Pentagon Axiom}$, see Lemma 9.4 (resp. Lemma 9.13). As particular cases of this situation we prove that the category $\mathcal{H}(\mathfrak{M})$, for $\mathcal{M} = \mathcal{H}(\mathfrak{M})$, is nothing but a Hom-Lie algebra. In Remark 9.17, we recover $(\mathcal{H}, R)$-Lie algebras in the sense of [BFM, Definition 4.1] by considering the category of comodules over a co-triangular bialgebra regarded as a co-triangular dual quasi-bialgebra with trivial reassociator. In particular, let $G$ be an abelian group endowed with an anti-symmetric bicharacter $\chi: G \times G \to k\setminus\{0\}$ and extend $\chi$ by linearity to a $k$-linear map $R: k\{G\} \otimes k\{G\} \to k$, where $k\{G\}$ denotes the group algebra. Then $(k\{G\}, R)$ is a co-triangular bialgebra and, as a consequence, we recover $(G, \chi)$-Lie color algebras in the sense of [Mc, Example 10.5.14], in Example 9.18 and in particular Lie superalgebras in Example 9.19.

The appendices contain general results regarding the existence of (co)equalizers in the category of (co)algebras, bialgebras and their braided analogue over a monoidal category. These results are applied to obtain Proposition 8.11 which is used in the proof of Theorem 7.3.

1. Preliminaries

In this section, we shall fix some basic notation and terminology.

**Notation 1.1.** Throughout this paper $k$ will denote a field. All vector spaces will be defined over $k$. The unadorned tensor product $\otimes$ will denote the tensor product over $k$, if not stated otherwise.

1.2. Monoidal Categories. Recall that (see [Ka, Chap. XI]) a monoidal category is a category $\mathcal{M}$ endowed with an object $1 \in \mathcal{M}$ (called unit), a functor $\otimes: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ (called tensor product), and functorial isomorphisms $a_{X,Y,Z}: (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l_{X}: 1 \otimes X \to X$, $r_{X}: X \otimes 1 \to X$, for every $X, Y, Z$ in $\mathcal{M}$. The functorial morphism $a$ is called the associativity constraint and satisfies the Pentagon Axiom, that is the equality

$$(U \otimes a_{V,W,X}) \circ a_{U,V \otimes W,X} \circ (a_{U,V,W} \otimes X) = a_{U,V,W \otimes X} \circ a_{U \otimes V,W,X}$$

holds true, for every $U, V, W, X$ in $\mathcal{M}$. The morphisms $l$ and $r$ are called the unit constraints and they obey the Triangle Axiom, that is $(V \otimes l_{W}) \circ a_{V,1,W} = r_{V} \otimes W$, for every $V, W$ in $\mathcal{M}$.
A **monoidal functor** (also called strong monoidal in the literature)
\[(F, \phi_0, \phi_2) : (\mathcal{M}, \otimes, 1, a, l, r) \rightarrow (\mathcal{M}', \otimes', 1', a', l', r')\]

between two monoidal categories consists of a functor \(F : \mathcal{M} \rightarrow \mathcal{M}'\), an isomorphism \(\phi_2(U, V) : F(U) \otimes' F(V) \rightarrow F(U \otimes V)\), natural in \(U, V \in \mathcal{M}\), and an isomorphism \(\phi_0 : 1' \rightarrow F(1)\) such that the diagram

\[
\begin{array}{c}
(F(U) \otimes' F(V)) \otimes' F(W) & \xrightarrow{\phi_2(U,V) \otimes' F(W)} & F(U \otimes V) \otimes' F(W) & \xrightarrow{\phi_2(U \otimes V, W)} & F(U \otimes V) \otimes W \\
F(U) \otimes' (F(V) \otimes' F(W)) & \xrightarrow{\phi_2(U,V) \otimes' \phi_2(V,W)} & F(U) \otimes' F(V \otimes W) & \xrightarrow{\phi_2(U,V \otimes W)} & F(U \otimes (V \otimes W))
\end{array}
\]

is commutative, and the following conditions are satisfied:

\[
F(l_U) \circ \phi_2(1, U) \circ (\phi_0 \otimes' F(U)) = l'_F(U), \quad F(r_U) \circ \phi_2(U, 1) \circ (F(U) \otimes' \phi_0) = r'_F(U).
\]

The monoidal functor is called **strict** if the isomorphisms \(\phi_0, \phi_2\) are identities of \(\mathcal{M}'\).

The notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra can be introduced in the general setting of monoidal categories.

From now on we will omit the associativity and unity constraints unless needed to clarify the context.

Let \(V\) be an object in a monoidal category \((\mathcal{M}, \otimes, 1)\). Define iteratively \(V^\otimes n\) for all \(n \in \mathbb{N}\) by setting \(V^\otimes 0 := 1\) for \(n = 0\) and \(V^\otimes n := V^\otimes (n-1) \otimes V\) for \(n > 0\).

**Remark 1.3.** Let \(\mathcal{M}\) be a monoidal category. Denote by \(\text{Alg}_{\mathcal{M}}\) the category of algebras in \(\mathcal{M}\) and their morphisms. Assume that \(\mathcal{M}\) has denumerable coproducts and that the tensor products \((\mathcal{M} \otimes \mathcal{M}) : \mathcal{M} \rightarrow \mathcal{M}\) and \((\mathcal{M} \otimes -) : \mathcal{M} \rightarrow \mathcal{M}\) for every object \(M\) in \(\mathcal{M}\) preserve such coproducts.

By [McL Theorem 2, page 172], the forgetful functor

\[
\Omega : \text{Alg}_{\mathcal{M}} \rightarrow \mathcal{M}
\]

has a left adjoint \(T : \mathcal{M} \rightarrow \text{Alg}_{\mathcal{M}}\). By construction \(\Omega TV = \oplus_{n \in \mathbb{N}} V^\otimes n\) for every \(V \in \mathcal{M}\). For every \(n \in \mathbb{N}\), we will denote by

\[
\alpha_n V : V^\otimes n \rightarrow \Omega TV
\]

the canonical injection. Given a morphism \(f : V \rightarrow W\) in \(\mathcal{M}\), we have that \(Tf\) is uniquely determined by the following equality

(1) \[
\Omega Tf \circ \alpha_n V = \alpha_n W \circ f^\otimes n, \quad \text{for every } n \in \mathbb{N}.
\]

The multiplication \(m_{\Omega TV}\) and the unit \(u_{\Omega TV}\) are uniquely determined by

(2) \[
m_{\Omega TV} \circ (\alpha_m V \otimes \alpha_n V) = \alpha_{m+n} V, \quad \text{for every } m, n \in \mathbb{N},
\]

(3) \[
u_{\Omega TV} = \alpha_0 V.
\]

Note that (2) should be integrated with the proper unit constraints when \(m\) or \(n\) is zero.

The unit \(\eta\) and the counit \(\epsilon\) of the adjunction \((T, \Omega)\) are uniquely determined, for all \(V \in \mathcal{M}\) and \((A, m, u_A) \in \text{Alg}_{\mathcal{M}}\), by the following equalities

(4) \[
\eta V := \alpha_1 V \quad \text{and} \quad \Omega \epsilon(A, m, u_A) \circ \alpha_n A := m_{A}^{n-1} \quad \text{for every } n \in \mathbb{N}
\]

where \(m_{A}^{n-1} : A^\otimes n \rightarrow A\) is the iterated multiplication of \(A\) defined by \(m_{A}^{1} := u_A, m_{A}^{0} := \text{Id}_A\) and, for \(n \geq 2\), \(m_{A}^{n-1} = m_{A}(m_{A}^{n-2} \otimes A)\).

**Definition 1.4.** Recall that a **monad** on a category \(\mathcal{A}\) is a triple \((Q, m, u)\), where \(Q : \mathcal{A} \rightarrow \mathcal{A}\) is a functor, \(m : QQ \rightarrow Q\) and \(u : A \rightarrow Q\) are functorial morphisms satisfying the associativity and the unitality conditions \(m \circ mQ = m \circ Qm\) and \(m \circ Qu = \text{Id}_Q = m \circ uQ\). An **algebra** over a monad \(Q\) on \(\mathcal{A}\) (or simply a \(Q\)-algebra) is a pair \((X, \mu)\) where \(X \in \mathcal{A}\) and \(\mu : QX \rightarrow X\) is a morphism in \(\mathcal{A}\) such that \(\mu \circ Q\mu = \mu \circ mX\) and \(\mu \circ uX = \text{Id}_X\). A **morphism** between two \(Q\)-algebras \((X, \mu)\) and \((X', \mu')\) is a morphism \(f : X \rightarrow X'\) in \(\mathcal{A}\) such that \(\mu' \circ Qf = f \circ \mu\). We will denote by \(\mathcal{Q}\) the
category of $Q$-algebras and their morphisms. This is the so-called *Eilenberg-Moore category* of the monad $Q$ (which is sometimes also denoted by $A^Q$ in the literature). When the multiplication and unit of the monad are clear from the context, we will just write $Q$ instead of $\mathbb{Q}$.

A monad $Q$ on $A$ gives rise to an adjunction $(F,U) := (Q,F,QU)$ where $U : QA \to A$ is the forgetful functor and $F : A \to QA$ is the free functor. Explicitly:

\[ U(X,\mu) := X, \quad UF := f \quad \text{and} \quad FX := (QX,mX), \quad Ff := Qf. \]

Note that $UF = Q$. The unit of the adjunction $(F,U)$ is given by the unit $u : A \to UF = Q$ of the monad $Q$. The counit $\lambda : FU \to QA$ of this adjunction is uniquely determined by the equality $U(\lambda(X,\mu)) = \mu$ for every $(X,\mu) \in QA$. It is well-known that the forgetful functor $U : QA \to A$ is faithful and reflects isomorphisms (see e.g. [Bo2, Proposition 4.1.4]).

Let $(L : B \to A, R : A \to B)$ be an adjunction with unit $\eta$ and counit $\epsilon$. Then $(RL,ReL,\eta)$ is a monad on $B$ and we can consider the so-called *comparison functor* $K : A \to RLB$ of the adjunction $(L,R)$ which is defined by $KX := (RX,ReX)$ and $Kf := Rf$. Note that $RLU \circ K = R$.

**Definition 1.5.** An adjunction $(L : B \to A, R : A \to B)$ is called *monadic* ([tripleable in Beck’s terminology [Bc], Definition 3, page 8]) whenever the comparison functor $K : A \to RLB$ is an equivalence of categories. A functor $R$ is called *monadic* if it has a left adjoint $L$ such that the adjunction $(L,R)$ is monadic, see [Bc, Definition 3’, page 8]. In a similar way one defines *comonadic* adjunctions and functors using the Eilenberg-Moore category $LRA$ of coalgebras over the comonad induces by $(L,R)$.

The notion of an idempotent monad is tightly connected with the monadic length of a functor.

**Definition 1.6.** ([M1, page 231]) A monad $(Q, m, u)$ is called *idempotent* whenever $m$ is an isomorphism. An adjunction $(L,R)$ is called *idempotent* whenever the associated monad is idempotent.

The interested reader can find results on idempotent monads in [M1, MS]. Here we just note that the fact that $(L,R)$ is idempotent is equivalent to require that $\eta R$ is a functorial isomorphism.

**Definition 1.7.** (See [AGM, Definition 2.7], [AHW, Definition 2.1] and [MS, Definitions 2.10 and 2.14]) Fix a $N \in \mathbb{N}$. We say that a functor $R$ has a *monadic decomposition of monadic length* $N$ whenever there exists a sequence $(R_n)_{n \leq N}$ of functors $R_n$ such that

1) $R_0 = R$;

2) for $0 \leq n \leq N$, the functor $R_n$ has a left adjoint functor $L_n$;

3) for $0 \leq n \leq N - 1$, the functor $R_{n+1}$ is the comparison functor induced by the adjunction $(L_n,R_n)$ with respect to its associated monad;

4) $L_N$ is full and faithful while $L_n$ is not full and faithful for $0 \leq n \leq N - 1$.

Compare with the construction performed in [M1, 1.5.5, page 49].

Note that for functor $R : A \to B$ having a monadic decomposition of monadic length $N$, we have a diagram

\[
\begin{array}{cccccccccc}
A & \xrightarrow{\text{Id}_A} & A & \xrightarrow{\text{Id}_A} & A & \xrightarrow{\text{Id}_A} & \cdots & \xrightarrow{\text{Id}_A} & A \\
L_0 & \downarrow R_0 & L_1 & \downarrow R_1 & L_2 & \downarrow R_2 & \cdots & \downarrow R_{N-1} & L_N \\
B_0 & \xrightarrow{U_{0,1}} & B_1 & \xrightarrow{U_{1,2}} & B_2 & \xrightarrow{U_{2,3}} & \cdots & \xrightarrow{U_{N-1,N}} & B_N \\
\end{array}
\]

where $B_0 = B$ and, for $1 \leq n \leq N$,

- $B_n$ is the category of $(R_{n-1}L_{n-1})$-algebras $R_{n-1}L_{n-1}B_{n-1}$;

- $U_{n-1,n} : B_n \to B_{n-1}$ is the forgetful functor $R_{n-1}L_{n-1}U$.

We will denote by $\eta_n : \text{Id}_{B_n} \to R_nL_n$ and $\epsilon_n : L_nR_n \to \text{Id}_A$ the unit and counit of the adjunction $(L_n,R_n)$ respectively for $0 \leq n \leq N$. Note that one can introduce the forgetful functor $U_{m,n} : B_n \to B_m$ for all $m \leq n$ with $0 \leq m, n \leq N$.

**Proposition 1.8** ([AGM, Proposition 2.9]). Let $(L : B \to A, R : A \to B)$ be an idempotent adjunction. Then $R : A \to B$ has a monadic decomposition of monadic length at most $1$. 


We refer to [AGM, Remarks 2.8 and 2.10] for further comments on monadic decompositions.

**Definition 1.9.** We say that a functor $R$ is comparable whenever there exists a sequence $(R_n)_{n \in \mathbb{N}}$ of functors $R_n$ such that $R_0 = R$ and, for $n \in \mathbb{N}$,

1) the functor $R_n$ has a left adjoint functor $L_n$;
2) the functor $R_{n+1}$ is the comparison functor induced by the adjunction $(L_n, R_n)$ with respect to its associated monad.

In this case we have a diagram as \( \mathcal{B} \) but not necessarily stationary. Hence we can consider the forgetful functors $U_{m,n} : \mathcal{B}_n \to \mathcal{B}_m$ for all $m \leq n$ with $m, n \in \mathbb{N}$.

**Remark 1.10.** Fix a $N \in \mathbb{N}$. A functor $R$ having a monadic decomposition of monadic length $N$ is comparable, see [AGM, Remark 2.10].

By the proof of Beck’s Theorem [BW, Proof of Theorem 1] one gets the following result.

**Lemma 1.11.** Let $\mathcal{A}$ be a category such that, for any (reflexive) pair $(f,g)$ ([BW, 3.6, page 98]) where $f,g : X \to Y$ are morphisms in $\mathcal{A}$, one can choose a specific coequalizer. Then the comparison functor $K : \mathcal{A} \to RL\mathcal{B}$ of an adjunction $(L, R)$ is a right adjoint. Thus any right adjoint $R : \mathcal{A} \to \mathcal{B}$ is comparable.

**Lemma 1.12.** Let $F : \mathcal{C} \to \mathcal{B}$ be a full and faithful functor which is also injective on objects.

1) Let $G : \mathcal{A} \to \mathcal{B}$ be a functor such that any object in $\mathcal{B}$ which is image of $G$ is also image of $F$. Then there is a unique functor $\hat{G} : \mathcal{A} \to \mathcal{C}$ such that $F\hat{G} = G$.
2) Let $G, G' : \mathcal{A} \to \mathcal{B}$ be functors as in 1). For any natural transformation $\gamma : G \to G'$ there is a unique natural transformation $\hat{\gamma} : \hat{G} \to \hat{G'}$ such that $F\hat{\gamma} = \gamma$.

### 2. Commutation Data

**Definition 2.1.** A functor is called conservative if it reflects isomorphisms.

**Lemma 2.2.** Let $(L, R)$ and $(L', R')$ be adjunctions that fit into the following commutative diagram of functors

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\
\mathcal{B} & \xleftarrow{R} & \mathcal{B}'
\end{array}
\]

Then there is a unique natural transformation $\zeta : L'G \longrightarrow FL$ such that

\[
R'\zeta \circ \eta'G = G\eta
\]

holds, namely

\[
\zeta := \left( L'G \xrightarrow{L'G\eta} L'GRL = L'R'FL \xrightarrow{\epsilon'FL} FL \right).
\]

Moreover we have that

\[
\epsilon'F = F\epsilon \circ \zeta R.
\]

**Definition 2.3.** We will say that $(F, G) : (L, R) \to (L', R')$ is a commutation datum if

1) $(L, R)$ and $(L', R')$ are adjunctions that fit into the commutative diagram \( \mathfrak{B} \);
2) The natural transformation $\zeta : L'G \longrightarrow FL$ of Lemma 2.2 is a functorial isomorphism.

The map $\zeta$ will be called the canonical transformation of the datum.

**Proposition 2.4.** Let $(F, G) : (L, R) \to (L', R')$ and $(F', G') : (L', R') \to (L'', R'')$ be a commutation data. Then $(F'F, G'G) : (L, R) \to (L'', R'')$ is a commutation datum.

**Proposition 2.5.** Let $(F, G) : (L, R) \to (L', R')$ be a commutation datum of functors as in \( \mathfrak{B} \). Assume also that $F$ preserves coequalizers of reflexive pairs of morphisms in $\mathcal{A}$ and that the comparison functors $R_1'$ and $R_1$ have left adjoints $L_1'$ and $L_1$ respectively. Then $G$ lifts to a functor
Moreover \((F,G_1) : (L_1,R_1) \rightarrow (L_1',R_1')\) is a commutation datum.

Furthermore the functor \(G_1\) is conservative (resp. faithful) whenever \(G\) is.

**Proof.** Denote by \(\zeta\) the canonical map of the datum \((F,G) : (L,R) \rightarrow (L',R')\). Set \(\lambda := R' \circ \zeta : (R'L') \rightarrow R'FL = G(RL)\). By Lemma 2.3, \(\zeta\) fulfills (10). By (9), we have \(\lambda \circ \eta'G = G\eta\) and

\[
GR eL \circ \lambda RL \circ R'L' \lambda = GR eL \circ R' \circ RL \circ R'L' \circ \zeta = R' (FeL \circ \zeta RL \circ R'L')
\]

Hence we can apply [Jo1, Lemma 1] to the case "\(K'' = R'L', \ "H'' = RL\) and "\(T'' = G\). Thus we get a functor \(G_1 : B_1 \rightarrow B_1\) such that \(U'G_1 = GU\). Explicitly \(G_1 (B, \mu) := (GB, G\mu \circ R' \circ \zeta B)\), \(G_1 (f) = Gf\). We have

\[
G_1 R_1 A = G_1 (RA, ReA) = (GRA, GR eL \circ \zeta RA)
\]

and \(G_1 R_1 f = GR f = R'F f = R'f F\) so that \(G_1 R_1 = R_1' F\). By the proof of [Be, Theorem 1], if we set \(\pi := \epsilon L_1 \circ LU_1\), we get the following coequalizer of a reflexive pair of morphisms in \(A\).

\[
\begin{array}{c}
LRLB \xrightarrow{\epsilon L_1} \xrightarrow{L' \mu} LB = LU (B, \mu) \xrightarrow{\pi (B, \mu)} L_1 (B, \mu)
\end{array}
\]

Since \(F\) preserves coequalizers of reflexive pairs of morphisms in \(A\), we get the bottom fork in the diagram below is a coequalizer.

(10)

\[
\begin{array}{c}
L'R'L'GB \xrightarrow{L' (G\mu \circ R' \circ \zeta B)} L'GB \xrightarrow{\epsilon L' GB} L'GB \xrightarrow{F\pi (B, \mu) \circ \zeta B} FL_1 (B, \mu)
\end{array}
\]

We compute

\[
F\epsilon LB \circ (\zeta RL \circ L' \circ \zeta B) = \zeta B \circ L'G\mu \circ L' \circ \zeta B = \zeta B \circ L' (G\mu \circ R' \circ \zeta B),
\]

\[
F\epsilon LB \circ (\zeta RL \circ L' \circ \zeta B) = \zeta B \circ \epsilon L'GB
\]

so that diagram (10) serially commutes. Since, in this diagram, the vertical arrows are isomorphisms, we get the upper fork is a coequalizer too. In a similar way, if we set \(\pi' := \epsilon L_1 \circ U' \eta'\), we have the coequalizer

\[
\begin{array}{c}
L'R'L'B' \xrightarrow{L' \mu'} L'B' \xrightarrow{\pi' (B', \mu')} L_1 (B', \mu')
\end{array}
\]

For \((B', \mu') := G_1 (B, \mu)\) we get the coequalizer

\[
\begin{array}{c}
L'R'L'GB \xrightarrow{L' (G\mu \circ R' \circ \zeta B)} L'GB \xrightarrow{\epsilon L' GB} L'GB \xrightarrow{\pi' (G_1 (B, \mu))} L_1 (B, \mu)
\end{array}
\]

By the foregoing, \(F\pi (B, \mu) \circ \zeta B\) coequalizes the pair \((L' (G\mu \circ R' \circ \zeta B), \epsilon L'GB)\). By the universal property of coequalizers, there is a unique morphism \(\zeta_1 (B, \mu) : L_1 G_1 (B, \mu) \rightarrow FL_1 (B, \mu)\) such that \(\zeta_1 (B, \mu) \circ \pi' G_1 (B, \mu) = F\pi (B, \mu) \circ \zeta B\). By the uniqueness of the coequalizers, \(\zeta_1 (B, \mu)\) is an isomorphism.
Let us check that $\zeta_1 (B, \mu)$ is natural. Let $f : (B, \mu) \to (B', \mu')$ in $B_1$. Then
\[
FL_1 f \circ \zeta_1 (B, \mu) \circ \pi' G_1 (B, \mu) = FL_1 f \circ F \pi (B, \mu) \circ \zeta = F \pi (B', \mu') \circ L \pi U f \circ \zeta B
\]
so that $FL_1 f \circ \zeta_1 (B, \mu) = \zeta_1 (B', \mu') \circ L_1 G_1 f$ and hence we get a functorial isomorphism $\zeta_1 : L_1^* G_1 \to FL_1$. We have
\[
\epsilon_1 \circ \pi R_1 = \epsilon_1 \circ \epsilon L_1 R_1 \circ LU \eta_1 R_1 = \epsilon \circ LR \epsilon_1 \circ LU \eta_1 R_1 = \epsilon \circ LU [R_1 \epsilon_1 \circ \eta_1 R_1] = \epsilon,
\]
and similarly for $(R', \mu')$. Hence $\epsilon \circ \pi R_1 = \epsilon$ and $R \pi \circ \eta U = U \eta_1$ and similar equations for $(L', R')$. We compute
\[
U' (R_1' \zeta_1 \circ \eta_1' G_1) = R' \zeta_1 \circ R' \pi' G_1 \circ \eta' U' G_1 \overset{\text{def}}{=} R' F \pi \circ R' \zeta U \circ \eta' G U
\]
so that $R_1' \zeta_1 \circ \eta_1' G_1 = G_1 \eta_1$. Let us check that $G_1$ is conservative whenever $G$ is. Let $f : (B, \mu) \to (B', \mu')$ in $B_1$ be such that $G_1 f$ is an isomorphism. Then $U' G_1 f = GU f$ is an isomorphism. Since $G$ and $U$ are conservative (see [Bo2, Proposition 4.1.4, page 189]), we get that $f$ is an isomorphism.

If $G$ is faithful, from $U' G_1 = GU$ and the fact that $U$ is faithful, we deduce that $G_1$ is faithful.

Assume $G$ is faithful and full. Let $f \in B_1 (G_1 (B, \mu), G_1 (B', \mu'))$. Then $U' f \in B' (GB, GB')$ so that there is $h \in B (B, B')$ such that $G h = U' f$. We have
\[
G (\mu' \circ RL h) \circ R' \zeta B = G \mu' \circ G R L h \circ R' \zeta B = G \mu' \circ R' FL h \circ R' \zeta B
\]
\[
= G \mu' \circ R' \zeta B' \circ R' L' G h = G \mu' \circ R' \zeta B' \circ R' L' U' f
\]
\[
= U' f \circ G \mu' \circ R' \zeta B = G h \circ G \mu \circ R' \zeta B = G (h \circ \mu) \circ R' \zeta B.
\]
Since $\zeta$ is an isomorphism and $G$ is faithful, we get that $\mu' \circ RL h = h \circ \mu$ so that there is a unique morphism $k \in B_1 ((B, \mu), (B', \mu'))$ such that $U k = h$. Hence $U' f = G h = GU k = U' G_1 k$ and hence $f = G_1 k$. Thus $G_1$ is faithful and full.

Assume $G$ is faithful and injective on objects. If $G_1 (B, \mu) = G_1 (B', \mu')$ i.e. $(GB, G \mu \circ R' \zeta B) = (GB', G \mu' \circ R' \zeta B')$ then $GB = GB'$ and $G \mu \circ R' \zeta B = G \mu' \circ R' \zeta B'$. In view of the assumptions on $G$ and since $\zeta$ is an isomorphism, we get $(B, \mu) = (B', \mu')$ so that $G_1$ is faithful and injective on objects.

**Lemma 2.6.** Let $(L, R)$ and $(L', R')$ be adjunctions of functors as in (4). Assume that $R' \zeta R$ is a functorial isomorphism where $\zeta : L' G \to FL$ is the natural transformation of Lemma 2.4. Assume also that $G$ is conservative.

1) Let $A \in A$ be such that $\eta' R' FA$ is an isomorphism. Then $\eta RA$ is an isomorphism.

2) If the adjunction $(L', R')$ is idempotent then $(L, R)$ is idempotent.

**Proof.** 1) Since $\eta' R' FA = \eta' GRA$ is an isomorphism and $R' \zeta R$ is an isomorphism, we get that $R' \zeta RA \circ \eta' GRA$ is an isomorphism. By (3) this means that $G \eta RA$ is an isomorphism. Since $G$ is conservative, we conclude.

2) $(L, R)$ is idempotent if and only if $\eta R$ is a functorial isomorphism and similarly for $(L', R')$. Thus $(L', R')$ is idempotent if and only if $\eta' R' F$ is a functorial isomorphism. If the letter condition holds then $\eta' R' F$ is a functorial isomorphism and, by 1), so is $\eta R$ and hence $(L, R)$ is idempotent.

**Lemma 2.7.** Let $(F, G) : (L, R) \to (L', R')$ be a commutation datum. If $G$ is conservative and $\eta'$ is an isomorphism so is $\eta$.

**Proof.** By (2), we have $R' \zeta \circ \eta' G = G \eta$.

**Corollary 2.8.** Let $(F, G) : (L, R) \to (L', R')$ be a commutation datum. Assume also that $F$ preserves coequalizers of reflexive pairs of morphisms in $A$ and that $G$ is conservative. Assume that both $R$ and $R'$ are comparable. Let $N \in N$.

1) Let $A \in A$ be such that $\eta_N R_N' FA$ is an isomorphism. Then $\eta_N R_N A$ is an isomorphism.
2) If \((L'_N, R'_N)\) is idempotent so is \((L_N, R_N)\).

**Proof.** Apply Proposition 2.5 and Lemma 2.6. 

Next lemma will be a useful tool to construct new commutation data.

**Lemma 2.9.** Let \((L', R')\) be an adjunction and let \(F\) and \(G\) be full and faithful functors which are also injective on objects and have domain and codomain as in the following diagrams. Assume that any object in \(\mathcal{A}'\) which is image of \(L'G\) is also image of \(F\) and that any object in \(\mathcal{B}'\) which is image of \(R'F\) is also image of \(G\). Set \(L := L'G\) and \(R := R'F\) with notation as in Lemma 1.14. Then \(L\) and \(R\) are the unique functors which make the following diagrams commute:

\[
\begin{array}{ccc}
A & \xrightarrow{F} & A' \\
\downarrow{L} & & \downarrow{L'} \\
B & \xrightarrow{G} & B'
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{F} & A' \\
\downarrow{R} & & \downarrow{R'} \\
B & \xrightarrow{G} & B'
\end{array}
\]

Moreover \((L, R)\) is an adjunction with unit \(\eta : \text{Id}_B \rightarrow RL\) and counit \(\epsilon : LR \rightarrow \text{Id}_A\) which satisfy

\[G\eta = \eta'G \quad \text{and} \quad F\epsilon = \epsilon'F\]

where \(\eta'\) and \(\epsilon'\) are the corresponding unit and counit of \((L', R')\). Moreover \((F, G) : (L, R) \rightarrow (L', R')\) is a commutation datum and the canonical transformation \(\zeta : L'G \rightarrow FL\) is \(\text{Id}_{L'G}\).

**Proof.** Apply Lemma 1.12 once observed that \(RL = R'L'G\), \(LR = L'R'F\), \(\hat{G} = \text{Id}_B\) and \(\hat{F} = \text{Id}_A\). Then define \(\eta := \eta'G\) and \(\epsilon := \epsilon'F\). 

\[
\square
\]

## 3. Braided objects and adjunctions

**Definition 3.1.** Let \((\mathcal{M}, \otimes, 1)\) be a monoidal category (as usual we omit the brackets although we are not assuming the constraints are trivial).

1) Let \(V\) be an object in \(\mathcal{M}\). A morphism \(c = c_V : V \otimes V \rightarrow V \otimes V\) is called a braiding (see [Ka], Definition XIII.3.1 where it is called a Yang-Baxter operator) if it satisfies the quantum Yang-Baxter equation

\[ (c \otimes V)(V \otimes c)(c \otimes V) = (V \otimes c)(c \otimes V)(V \otimes c) \]

on \(V \otimes V \otimes V\). We further assume that \(c\) is invertible. The pair \((V, c)\) will be called a braided object in \(\mathcal{M}\). A morphism of braided objects \((V, c_V)\) and \((W, c_W)\) in \(\mathcal{M}\) is a morphism \(f : V \rightarrow W\) such that \(c_W(f \otimes f) = (f \otimes f)c_V\). This defines the category \(\text{BrM}\) of braided objects and their morphisms.

2) \(\text{Bra}\) A quadruple \((A, m, u, c)\) is called a braided algebra if

- \((A, m, u)\) is an algebra in \(\mathcal{M}\);
- \((A, c)\) is a braided object in \(\mathcal{M}\);
- \(m\) and \(u\) commute with \(c\), that is the following conditions hold:

\[c(m \otimes A) = (A \otimes m)(c \otimes A)(A \otimes c),\]
\[c(A \otimes m) = (m \otimes A)(A \otimes c)(c \otimes A),\]
\[c(u \otimes A)r_A^{-1} = (A \otimes u)r_A^{-1}, \quad c(A \otimes u)r_A^{-1} = (u \otimes A)r_A^{-1}.\]

A morphism of braided algebras is, by definition, a morphism of algebras which, in addition, is a morphism of braided objects. This defines the category \(\text{BrAlgM}\) of braided algebras and their morphisms.

3) A quadruple \((C, \Delta, \varepsilon, c)\) is called a braided coalgebra if

- \((C, \Delta, \varepsilon)\) is a coalgebra in \(\mathcal{M}\);
- \((C, c)\) is a braided object in \(\mathcal{M}\);
- \(\Delta\) and \(\varepsilon\) commute with \(c\), that is the following relations hold:

\[(\Delta \otimes C)c = (C \otimes c)(c \otimes C)(C \otimes \Delta),\]
\[(C \otimes \Delta)c = (c \otimes C)(C \otimes c)(\Delta \otimes C),\]
\[ l_C(\varepsilon \otimes C)c = r_C(C \otimes \varepsilon), \quad r_C(C \otimes \varepsilon)c = l_C(\varepsilon \otimes C). \]

A morphism of braided coalgebras is, by definition, a morphism of coalgebras which, in addition, is a morphism of braided objects. This defines the category \( \text{BrCAlg}_M \) of braided coalgebras and their morphisms.

4) \[ \text{(Definition 5.1)} \] A sextuple \( (B, m, u, \Delta, \varepsilon, c) \) is a called a braided bialgebra if
   \begin{itemize}
   \item \( (B, m, u, c) \) is a braided algebra;
   \item \( (B, \Delta, \varepsilon, c) \) is a braided coalgebra;
   \item the following relations hold:
   \end{itemize}
\[ \Delta m = (m \otimes m)(B \otimes c \otimes B)(\Delta \otimes \Delta), \quad \Delta u = (u \otimes u)\Delta_1, \]
\[ \varepsilon m = m_1(\varepsilon \otimes \varepsilon), \quad \varepsilon u = 1_{\Delta}. \]

A morphism of braided bialgebras is both a morphism of braided algebras and coalgebras. This defines the category \( \text{BrBialg}_M \) of braided bialgebras.

Recall that a braiding \( c \) is called symmetric or a symmetry whenever \( c^2 = \text{Id} \). Denote by \( \text{Br}^s_M, \text{BrAlg}^s_M, \text{BrCAlg}^s_M \) and \( \text{BrBialg}^s_M \) the full subcategories of the respective categories above consisting of objects with symmetric braiding. Denote by
\[ \iota^s_{\text{Br}}, \iota^s_{\text{BrAlg}}, \iota^s_{\text{BrCoalg}}, \iota^s_{\text{BrBialg}} : \text{BrAlg}_M \rightarrow \text{BrCAlg}_M, \]
\[ \iota^s_{\text{BrAlg}}, \iota^s_{\text{BrCoalg}}, \iota^s_{\text{BrBialg}} : \text{BrCoalg}_M \rightarrow \text{BrBialg}_M, \]
the obvious inclusion functors. Note that they are full, faithful, injective on objects and conservative.

**Remark 3.2.** Let \( M \) be a monoidal category. Let \( A \) be one of the following categories \( \text{BrAlg}_M, \text{BrCoalg}_M, \text{BrBialg}_M \), let \( A^s \) be the corresponding full subcategory of objects with symmetric braiding and denote by \( \iota^s_A : A^s \rightarrow A \) the obvious inclusion functor. Let \( \mathbb{D}_A : A \rightarrow M \) be the forgetful functor.

1) Let \( \overline{X} \in A, \overline{Y} \in A^s \) and let \( \sigma : \overline{X} \rightarrow \iota^s_A \overline{Y} \) be a morphism in \( A \) such that \( \alpha := \mathbb{D}_A \sigma \) is a monomorphism. Set \( X := \mathbb{D}_A \overline{X} \) and \( Y := \mathbb{D}_A \iota_A \overline{Y} \). Since \( \alpha \) is braided we have \( (\alpha \otimes \alpha)\xi_X = \xi_Y (\alpha \otimes \alpha) = \alpha \otimes \alpha \) where \( \xi_X \) and \( \xi_Y \) are the braiding of \( X \) and \( Y \) respectively. Assume that \( \alpha \otimes \alpha \) is a monomorphism. Then we obtain \( \overline{X} = \iota^s_A \overline{Y} \) for some \( \overline{X} \in A^s \) and \( \sigma \) is a morphism in \( A^s \). Since \( \mathbb{D}_A \) reflects monomorphisms, we have proved that \( A^s \) is closed in \( A \) for those subobjects in \( A \) which are preserved by \( \mathbb{D}_A \) and by \( (-)^{\otimes 2} \circ \mathbb{D}_A \) where \( (-)^{\otimes 2} : M \rightarrow M : V \mapsto V \otimes V \).

2) Dually \( A^s \) is closed in \( A \) for those quotients in \( A \) which are preserved by \( \mathbb{D}_A \) and by \( (-)^{\otimes 2} \circ \mathbb{D}_A \).

3.3. Let \( M \) and \( M' \) be monoidal categories. Following \[ \text{[AM, Proposition 2.5]}, \] every monoidal functor \( (F, \phi_0, \phi_2) : M \rightarrow M' \) induces in a natural way suitable functors \( F, \text{Alg}F, \text{BrAlg}F \) and \( \text{BrBialg}F \) such that the following diagrams commute
\[
\begin{array}{cccc}
\text{BrAlg}_M & \xrightarrow{\text{BrAlg}} & \text{BrAlg}_M' \\
\text{Alg}_M & \xrightarrow{\text{Alg}} & \text{Alg}_M'
\end{array}
\]
\[
\begin{array}{cccc}
\text{BrAlg}_M & \xrightarrow{\text{BrAlg}} & \text{BrAlg}_M' \\
\text{Alg}_M & \xrightarrow{\text{Alg}} & \text{Alg}_M'
\end{array}
\]
\[
\begin{array}{cccc}
\text{BrAlg}_M & \xrightarrow{\text{BrAlg}} & \text{BrAlg}_M' \\
\text{Alg}_M & \xrightarrow{\text{Alg}} & \text{Alg}_M'
\end{array}
\]
\[
\begin{array}{cccc}
\text{BrAlg}_M & \xrightarrow{\text{BrAlg}} & \text{BrAlg}_M' \\
\text{Alg}_M & \xrightarrow{\text{Alg}} & \text{Alg}_M'
\end{array}
\]
where the vertical arrows denote the obvious forgetful functors. Moreover

1) The functors \( H, \Omega, H_{\text{Alg}}, \Omega_{\text{Br}}, \Omega_{\text{Br}} \) are conservative.

2) \( \text{Br}F, \text{Alg}F, \text{BrAlg}F \) and \( \text{BrBialg}F \) are equivalences (resp. isomorphisms or conservative) whenever \( F \) is.
(3) \( F \) preserves symmetric objects (this follows by definition of the braiding induced by \( F \)).

Thus we can define \( \text{Br}^s F, \text{BrAlg}^s F \) and \( \text{BrBialg}^s F \) such that

\[
\begin{array}{c}
\text{Br}_M \\
\text{BrF} \\
\text{Br}_{M'} \\
\text{BrAlg}_M \\
\text{BrAlg}_{M'} \\
\text{BrAlg}_{M''} \\
\text{BrAlg}^s_M \\
\text{BrAlg}^s_{M'} \\
\text{BrAlg}^s_{M''} \\
\text{BrBialg}^s_M \\
\text{BrBialg}^s_{M'} \\
\text{BrBialg}^s_{M''} \\
\end{array}
\]

Next aim is to recall some meaningful adjunctions that will be investigated in the paper.

3.4. Let \( \mathcal{M} \) be a monoidal category. Assume that \( \mathcal{M} \) has denumerable coproducts and that the tensor functors preserve such coproducts. In view of [AM Proposition 3.1], the functor \( \Omega_{\text{Br}} \) has a left adjoint \( T_{\text{Br}} \) and the following diagrams commute.

\[
\begin{array}{c}
\text{BrAlg}_M \\
\text{BrAlg}_{M'} \\
\text{BrAlg}_{M''} \\
\end{array}
\]

The unit \( \eta_{\text{Br}} \) and the counit \( \epsilon_{\text{Br}} \) are uniquely determined by the following equations

\[
H\eta_{\text{Br}} = \eta H, \quad H\epsilon_{\text{BrAlg}} = \epsilon H_{\text{Alg}},
\]

where \( \eta \) and \( \epsilon \) denote the unit and counit of the adjunction \( (T, \Omega) \) of Remark 2.3. Using Lemma 2.9, one shows that the adjunction \( (T_{\text{Br}}, \Omega_{\text{Br}}) \) induces an adjunction \( (T_{\text{Br}}, \Omega_{\text{Br}}) \) such that the following diagrams commute.

\[
\begin{array}{c}
\text{BrAlg}_M \\
\text{BrAlg}_{M'} \\
\text{BrAlg}_{M''} \\
\end{array}
\]

The lemma can be applied by the following argument. It is clear that any object in the image of \( \Omega_{\text{Br}} \) is in the image of \( \text{Br}_{\text{Alg}} \). Let \( (M, c) \in \text{Br}^s_M \) and set \( (A, m_A, u_A, c_A) := T_{\text{Br}}^s \Omega_{\text{Br}} (M, c) \). Using [AM (42)], we have \( c_A (\alpha_m M \otimes \alpha_n M) = (\alpha_m V \otimes \alpha_m M) c_A^{m,n} \) so that

\[
c_A^2 (\alpha_m M \otimes \alpha_n M) = c_A (\alpha_m V \otimes \alpha_m M) c_A^{m,n} = (\alpha_m M \otimes \alpha_n M) c_A^{m,n}
\]

and \( c_A, c_A^{m,n} \). The latter is proved by induction on \( t = m + n \in \mathbb{N} \) using [AM Proposition 2.7].

Thus \( c_A (\alpha_m M \otimes \alpha_n M) = (\alpha_m M \otimes \alpha_n M) \) for every \( m, n \in \mathbb{N} \) and hence \( c_A^2 = \text{Id}_{\text{BrAlg}} \). Therefore \( (A, m_A, u_A, c_A) \in \text{BrAlg}_M \) and \( T_{\text{Br}}^s \Omega_{\text{Br}} (M, c) = \text{BrAlg}_M (A, m_A, u_A, c_A) \). Hence any object in the image of \( T_{\text{Br}}^{s, s} \) is also in the image of \( \text{BrAlg}^s \). Thus, by Lemma 2.9 we have the desired adjunction

\[
\begin{array}{c}
\text{Id}_{\text{BrAlg}} \eta_{\text{Br}} = \epsilon_{\text{Br}} \eta_{\text{BrAlg}} \quad \text{and} \quad \epsilon_{\text{Br}} \eta_{\text{Br}} = \eta_{\text{Br}} \epsilon_{\text{BrAlg}}
\end{array}
\]

Furthermore \( \left( \eta_{\text{BrAlg}}, \epsilon_{\text{BrAlg}} \right) : (T_{\text{Br}}^{s, s}, \Omega_{\text{Br}}^{s, s}) \rightarrow (T_{\text{Br}}^{s}, \Omega_{\text{Br}}^{s}) \) is a commutation datum with canonical transformation given by the identity.

**Definition 3.5.** Let \( \mathcal{M} \) be a preadditive monoidal category with equalizers. Assume that the tensor functors are additive. Let \( \mathcal{C} := (C, \Delta_C, \varepsilon_C, u_C) \) be a coalgebra \( (C, \Delta_C, \varepsilon_C) \) endowed with a coalgebra morphism \( u_C : 1 \rightarrow C \). In this setting we always implicitly assume that we can choose a specific equalizer

\[
\begin{array}{c}
P (\mathcal{C}) \rightarrow C \rightarrow C \\
\end{array}
\]

We will use the same symbol when \( \mathcal{C} \) comes out to be enriched with an extra structure such as when \( \mathcal{C} \) will denote a bialgebra or a braided bialgebra.
We now investigate some properties of $T_{\text{Br}}$.

3.6. Let $\mathcal{M}$ be a preadditive monoidal category with equalizers and denumerable coproducts. Assume that the tensor functors are additive and preserve equalizers and denumerable coproducts. By $\ref{section}$, the forgetful functor $\Omega_{\text{Br}} : \text{BrAlg}_{\mathcal{M}} \to \mathcal{M}$ has a left adjoint $T_{\text{Br}} : \mathcal{M} \to \text{BrAlg}_{\mathcal{M}}$. In view of $\ref{section}$, $T_{\text{Br}}$ induces a functor $T_{\text{Br}}^* : \mathcal{M} \to \text{BrAlg}_{\mathcal{M}}$ which is constructed in $\ref{section}$. The unit $\eta_{\text{Br}}$ and the counit $\epsilon_{\text{Br}}$ are uniquely determined by the following equalities

$$
\eta_{\text{Br}} : 1_{\mathcal{M}} \to T_{\text{Br}}^* \eta_{\text{Br}},
$$

$$
\epsilon_{\text{Br}} : T_{\text{Br}}^* \epsilon_{\text{Br}} = 1_{\text{BrAlg}_{\mathcal{M}}},
$$

where $(V, c) \in \text{Br}_{\mathcal{M}}$. Let $\mathcal{B} \in \text{BrAlg}_{\mathcal{M}}$ and $\xi : \mathcal{B} \to \Omega_{\text{Br}} \mathcal{B}$ is a monomorphism so that we can apply $\ref{section}$ to get that $(P, c_P) \in \text{Br}_{\mathcal{M}}$. Thus any object in the image of $P_{\text{Br}} : \text{BrAlg}_{\mathcal{M}}$ is also in the image of $\mathcal{B}$. Hence, by $\ref{section}$, we have an adjunction $(T_{\text{Br}}^*, P_{\text{Br}}^*)$ such that the diagrams

$$
\begin{align*}
\text{BrAlg}_{\mathcal{M}} & \xrightarrow{\eta_{\text{Br}}^*} \text{BrAlg}_{\mathcal{M}} & \text{BrAlg}_{\mathcal{M}} & \xrightarrow{\eta_{\text{Br}}^*} \text{BrAlg}_{\mathcal{M}} \\
T_{\text{Br}}^* & \xrightarrow{\epsilon_{\text{Br}}^*} T_{\text{Br}}^* & T_{\text{Br}} & \xrightarrow{\epsilon_{\text{Br}}^*} T_{\text{Br}} \\
\mathcal{M} & \xrightarrow{T_{\text{Br}}^*} \text{BrAlg}_{\mathcal{M}} & \mathcal{M} & \xrightarrow{T_{\text{Br}}^*} \text{BrAlg}_{\mathcal{M}}
\end{align*}
$$

commute and the unit $\eta_{\text{Br}} : 1_{\mathcal{M}} \to P_{\text{Br}}^* T_{\text{Br}}^*$ and the counit $\epsilon_{\text{Br}} : T_{\text{Br}}^* P_{\text{Br}}^* \to 1_{\mathcal{M}}$ are uniquely defined by

$$
\begin{align*}
\eta_{\text{Br}}^* T_{\text{Br}}^* &= T_{\text{Br}}^* \eta_{\text{Br}}^*,
\end{align*}
$$

Moreover, $\left(T_{\text{Br}}^*, P_{\text{Br}}^* \right)$ is a commutation datum with canonical transformation given by the identity. Note that the functor $\mathcal{B}$ induces a functor $\mathcal{B}$ such that the following diagrams commute.

$$
\begin{align*}
\text{BrAlg}_{\mathcal{M}} & \xrightarrow{\eta_{\text{Br}}^*} \text{BrAlg}_{\mathcal{M}} & \text{BrAlg}_{\mathcal{M}} & \xrightarrow{\eta_{\text{Br}}^*} \text{BrAlg}_{\mathcal{M}} \\
\text{BrAlg}_{\mathcal{M}} & \xrightarrow{\epsilon_{\text{Br}}^*} \text{BrAlg}_{\mathcal{M}} & \text{BrAlg}_{\mathcal{M}} & \xrightarrow{\epsilon_{\text{Br}}^*} \text{BrAlg}_{\mathcal{M}}
\end{align*}
$$

Furthermore, by $\ref{section}$, the natural transformation $\xi : P_{\text{Br}} \to \Omega_{\text{Br}} \mathcal{B}$ induces a natural transformation $\xi : \mathcal{B} \to \mathcal{B}$ such that $\eta_{\text{Br}}^* \xi = \xi \eta_{\text{Br}}^*$. 

\begin{align*}
\text{BrAlg}_{\mathcal{M}} & \xrightarrow{\eta_{\text{Br}}^*} \text{BrAlg}_{\mathcal{M}} & \text{BrAlg}_{\mathcal{M}} & \xrightarrow{\eta_{\text{Br}}^*} \text{BrAlg}_{\mathcal{M}} \\
\text{BrAlg}_{\mathcal{M}} & \xrightarrow{\epsilon_{\text{Br}}^*} \text{BrAlg}_{\mathcal{M}} & \text{BrAlg}_{\mathcal{M}} & \xrightarrow{\epsilon_{\text{Br}}^*} \text{BrAlg}_{\mathcal{M}}
\end{align*}
Proposition 3.7. Let $(F, \phi_0, \phi_2) : \mathcal{M} \to \mathcal{M}'$ be a monoidal functor between monoidal categories. Assume that $\mathcal{M}$ and $\mathcal{M}'$ have denumerable coproducts and that $F$ and the tensor functors preserve such coproducts. Then both

$$(\text{Alg}F, F) : (T, \Omega) \to (T', \Omega') \quad \text{and} \quad (\text{BrAlg}F, \text{Br}F) : (T_{\text{Br}}, \Omega_{\text{Br}}) \to (T'_{\text{Br}}, \Omega'_{\text{Br}})$$

are commutation data.

Proof. First we deal with $(\text{Alg}F, F) : (T, \Omega) \to (T', \Omega')$. By 3.3 we have that $\Omega' \circ \text{Alg}F = F \circ \Omega$. By Remark 1.3, we have that $\Omega$ and $\Omega'$ have left adjoints $T$ and $T'$ respectively. The structure morphisms $\phi_0, \phi_2$ induce, for every $n \in \mathbb{N}$, the isomorphism $\phi_nV : (FV)^{\otimes n} \to F(V^{\otimes n})$ given by

$$\hat{\phi}_0V := \phi_0, \quad \hat{\phi}_1V := \text{Id}_{FV}, \quad \hat{\phi}_2V := \phi_2(V, V), \quad \text{and, for } n > 2$$

$$\hat{\phi}_nV := \phi_2 \left( V^{\otimes (n-1)} \right) \circ \left( \hat{\phi}_{n-1} \otimes FV \right).$$

Using the naturality of $\phi_2$ and 3.3 it is straightforward to check, by induction on $n \in \mathbb{N}$, that

$$(36) \quad m_{(\text{Alg}F)TV}^{n-1} \circ (F\alpha_1V)^{\otimes n} = F\alpha_nV \circ \hat{\phi}_nV.$$

Let $\zeta$ be the map of Lemma 2.2 i.e. $\zeta = \epsilon'(\text{Alg}F)T \circ T'F \eta$. We compute

$$\Omega'\zeta V \circ \alpha_n FV = \Omega'\epsilon'(\text{Alg}F)TV \circ \Omega T'F \eta V \circ \alpha_n FV$$

By $\Omega'\epsilon'(\text{Alg}F)TV \circ \Omega T'F \eta V \circ \alpha_n FV = \Omega'\epsilon'(\text{Alg}F)TV \circ \alpha_n FTV \circ (F\alpha_1V)^{\otimes n}$

and by $m_{(\text{Alg}F)TV}^{n-1} \circ (F\alpha_1V)^{\otimes n} \circ F\alpha_nV \circ \hat{\phi}_nV = (\nabla_{t \in \mathbb{N}} F\alpha_tV) \circ \zeta FV \circ \hat{\phi}_nV = (\nabla_{t \in \mathbb{N}} F\alpha_tV) \circ \left( \bigoplus_{t \in \mathbb{N}} \hat{\phi}_tV \right) \circ \alpha_n FV$$

where $j_nV : F(V^{\otimes n}) \to \bigoplus_{t \in \mathbb{N}} F(V^{\otimes t})$ denotes the canonical morphism. Since this equality holds for an arbitrary $n \in \mathbb{N}$, we obtain $\Omega'\zeta V = (\nabla_{t \in \mathbb{N}} F\alpha_tV) \circ \left( \bigoplus_{t \in \mathbb{N}} \hat{\phi}_tV \right)$.

Now $\hat{\phi}_n$ is an isomorphism by construction and $\nabla_{t \in \mathbb{N}} F\alpha_tV : \bigoplus_{n \in \mathbb{N}} F(V^{\otimes n}) \to F(\bigoplus_{n \in \mathbb{N}} V^{\otimes n})$ is an isomorphism as $F$ preserves denumerable coproducts. Hence $\Omega'\zeta V$ is an isomorphism and hence $(\text{Alg}F, F) : (T, \Omega) \to (T', \Omega')$ is a commutation datum.

Now, let us consider $(\text{BrAlg}F, \text{Br}F) : (T_{\text{Br}}, \Omega_{\text{Br}}) \to (T'_{\text{Br}}, \Omega'_{\text{Br}})$. By 3.4, the functor $\Omega_{\text{Br}} : \text{BrAlg}M \to \text{BrM}$ has a left adjoint $T_{\text{Br}} : \text{BrM} \to \text{BrAlg}M$ and the (co)unit of the adjunction obey 2.2. Moreover $H_{\text{Alg}}T_{\text{Br}} = TH$. By 3.3, we have $H'(BrF) = FH, \Omega'(\text{Alg}F) = F\Omega, H'_{\text{Alg}}(\text{BrAlg}F) = (\text{Alg}F)H_{\text{Alg}}$ and $\Omega'_{\text{Br}}(\text{BrAlg}F) = (\text{Br}F)\Omega_{\text{Br}}$. In view of Lemma 2.2, the diagrams

$$(37) \quad \begin{array}{ccc}
\text{BrAlg}_M & \xrightarrow{\text{BrAlg}F} & \text{BrAlg}_M' \\
\text{Br}_M \downarrow & & \text{Br}_M' \downarrow \\
\text{Alg}_M & \xrightarrow{\text{Alg}F} & \text{Alg}_M'
\end{array}$$

are induced the maps $\zeta_{\text{Br}} : T'_{\text{Br}}(\text{Br}F) \to (\text{BrAlg}F)T_{\text{Br}}$ and $\zeta : T'F \to (\text{Alg}F)T$ defined by

$$(38) \quad \zeta_{\text{Br}} = \epsilon'_{\text{Br}}(\text{BrAlg}F)T_{\text{Br}} \circ T'_{\text{Br}}(\text{Br}F) \eta_{\text{Br}} \quad \text{and} \quad \zeta = \epsilon'(\text{Alg}F)T \circ T'F \eta.$$

One easily checks that

$$(39) \quad H'_{\text{Alg}}\zeta_{\text{Br}} = \zeta H.$$

By the first part of the proof, $\zeta$ is a functorial isomorphism so that we get that $H'_{\text{Alg}}\zeta_{\text{Br}}$ is a functorial isomorphism too. Since $H'_{\text{Alg}}$ trivially reflects isomorphisms, we get that $\zeta_{\text{Br}}$ is a functorial isomorphism.

Proposition 3.8. Let $\mathcal{M}$ and $\mathcal{M}'$ be preadditive monoidal categories with equalizers. Assume that the tensor functors are additive and preserve equalizers in both categories. For any monoidal
functor \((F, \phi_0, \phi_2) : \mathcal{M} \to \mathcal{M}'\) which preserves equalizers, the following diagram commutes

\[
\begin{array}{ccc}
\text{BrBialg}_\mathcal{M} & \xrightarrow{\text{BrBialg}F} & \text{BrBialg}_\mathcal{M}' \\
\downarrow P_{\text{Br}} & & \downarrow P'_{\text{Br}} \\
\text{Br}s_\mathcal{M} & \xrightarrow{\text{Br}F} & \text{Br}s_{\mathcal{M}'}
\end{array}
\]  

where \(\text{BrBialg}F\) and \(\text{Br}F\) are the functors of \(\mathcal{B.3}\). Moreover we have

\[
(41) \quad \xi' (\text{BrBialg}F) = (\text{Br}F) \xi.
\]

Assume also that the categories \(\mathcal{M}\) and \(\mathcal{M}'\) have denumerable coproducts and that \(F\) and the tensor functors preserve such coproducts. Then \((\text{BrBialg}F, \text{Br}F) : (\overline{T}_\text{Br}, P_{\text{Br}}) \to (\overline{T}'_\text{Br}, P'_{\text{Br}})\) is a commutation datum.

Proof. The first part is [AM, Proposition 3.6]. Let us prove the last assertion. Assume that the monoidal category \(\mathcal{M}\) has denumerable coproducts and that the tensor functors preserve such coproducts. By \(\mathcal{B.3}\) we have that \(P_{\text{Br}}\) and \(P'_{\text{Br}}\) have left adjoints \(\overline{T}_\text{Br}\) and \(\overline{T}'_\text{Br}\) respectively. By \(\mathcal{B.3}\), we have \(\mathcal{U}'_{\text{Br}}(\text{BrBialg}F) = (\text{BrAlg}F) \mathcal{U}_{\text{Br}}\) and \(\mathcal{O}_{\text{Br}}(\text{BrAlg}F) = (\text{Br}F) \mathcal{O}_{\text{Br}}\). By \(\mathcal{B.1}\), we have \(\mathcal{O}_{\text{Br}} \overline{T}_\text{Br} = T_{\text{Br}}\). The commutative diagrams \(\mathcal{H.0}\) and \(\mathcal{H.1}\)-left induce the natural transformations \(\zeta_{\text{Br}} : \overline{T}_\text{Br} (\text{Br}F) \to (\text{BrBialg}F) \overline{T}_\text{Br}\) and \(\zeta_{\text{Br}} : T_{\text{Br}} (\text{Br}F) \to (\text{BrAlg}F) T_{\text{Br}}\) of Lemma \(\mathcal{H.2}\) i.e.

\[
\zeta_{\text{Br}} = \zeta'_{\text{Br}} (\text{BrBialg}F) \overline{T}_\text{Br} \circ \overline{T}'_\text{Br} (\text{Br}F) \overline{T}_\text{Br} \quad \text{and} \quad \zeta_{\text{Br}} = \zeta'_{\text{Br}} (\text{BrAlg}F) T_{\text{Br}} \circ T'_\text{Br} (\text{Br}F) \eta_{\text{Br}}.
\]

Using \(\mathcal{H.2},\mathcal{H.1}\) and \(\mathcal{H.1}\), one easily checks that \(\mathcal{U}'_{\text{Br}} \zeta_{\text{Br}} = \zeta_{\text{Br}}\). By Proposition \(\mathcal{H.3}\), we know that \(\zeta_{\text{Br}}\) is a functorial isomorphism. Since \(\mathcal{U}'_{\text{Br}}\) is a functorial isomorphism too.

\[\square\]

4. Braided Categories

4.1. A braided monoidal category \((\mathcal{M}, \otimes, 1, a, l, r, c)\) is a monoidal category \((\mathcal{M}, \otimes, 1)\) equipped with a braiding \(c\), that is an isomorphism \(c_{U,V} : U \otimes V \to V \otimes U\), natural in \(U, V \in \mathcal{M}\), satisfying, for all \(U, V, W \in \mathcal{M}\),

\[
\begin{align*}
\alpha_{V,W,U} \circ c_{U,V \otimes W} \circ a_{U,V,W} &= (V \otimes c_{U,W}) \circ a_{V,U,W} \circ (c_{U,V} \otimes W), \\
\alpha_{W,U,V} \circ c_{U \otimes V,W} \circ a_{U,V,W}^{-1} &= (c_{U,W} \otimes V) \circ a_{U,W,V}^{-1} \circ (U \otimes c_{V,W}).
\end{align*}
\]

A braided monoidal category is called symmetric if we further have \(c_{U,U} \circ c_{U,V} = \text{Id}_{U \otimes V}\) for every \(U, V \in \mathcal{M}\).

A (symmetric) braided monoidal functor is a monoidal functor \(F : \mathcal{M} \to \mathcal{M}'\) such that \(F (c_{U,V}) \circ \phi_2(U,V) = \phi_2(V,U) \circ c_{F(U),F(V)}\). More details on these topics can be found in [Ka, Chapter XIII].

Remark 4.2. Given a braided monoidal category \((\mathcal{M}, \otimes, 1, c)\) the category \(\text{Alg}_\mathcal{M}\) becomes monoidal where, for every \(A, B \in \text{Alg}_\mathcal{M}\) the multiplication and unit of \(A \otimes B\) are given by

\[
\begin{align*}
m_{A \otimes B} & : = (m_A \otimes m_B) \circ (A \otimes c_{B,A} \otimes B) : (A \otimes B) \otimes (A \otimes B) \to A \otimes B, \\
u_{A \otimes B} & : = (u_A \otimes u_B) \circ l_1^{-1} : 1 \to A \otimes B.
\end{align*}
\]

Moreover the forgetful functor \(\text{Alg}_\mathcal{M} \to \mathcal{M}\) is a strict monoidal functor, cf. [JS, page 60].

Definition 4.3. A bialgebra in a braided monoidal category \((\mathcal{M}, \otimes, 1, c)\) is a coalgebra \((B, \Delta, \varepsilon)\) in the monoidal category \(\text{Alg}_\mathcal{M}\). Equivalently a bialgebra is a quintuple \((A, m, u, \Delta, \varepsilon)\) where \((A, m, u)\) is an algebra in \(\mathcal{M}\) and \((A, \Delta, \varepsilon)\) is a coalgebra in \(\mathcal{M}\) such that \(\Delta\) and \(\varepsilon\) are morphisms of algebras where \(A \otimes A\) is an algebra as in the previous remark. Denote by \(\text{Bialg}_\mathcal{M}\) the category of bialgebras in \(\mathcal{M}\) and their morphisms, defined in the expected way.
4.4. Let $\mathcal{M}$ be a braided monoidal category. In view of [AM, Proposition 4.4], there are obvious functors $J$, $J_{\text{Alg}}$ and $J_{\text{Bialg}}$ such that the diagrams

\[
\begin{array}{cccccc}
\text{Bialg}_{\mathcal{M}} & \xrightarrow{J_{\text{Bialg}}} & \text{BrBialg}_{\mathcal{M}} & \xrightarrow{\Omega_{\text{Bialg}}} & \text{Alg}_{\mathcal{M}} & \xrightarrow{J_{\text{Alg}}} & \text{BrAlg}_{\mathcal{M}} \\
\text{Alg}_{\mathcal{M}} & \xrightarrow{J_{\text{Alg}}} & \text{BrAlg}_{\mathcal{M}} & \xrightarrow{\Omega_{\text{Alg}}} & \mathcal{M} & \xrightarrow{J} & \text{Br}_{\mathcal{M}} \\
\end{array}
\]

commute. In fact the functors $J$, $J_{\text{Alg}}$ and $J_{\text{Bialg}}$ add the evaluation of the braiding of $\mathcal{M}$ on the object on which they act. Moreover they are full, faithful, injective on objects and conservative.

Assume that $\mathcal{M}$ has denumerable coproducts and that the tensor functors preserve such coproducts. Then, by [AM, Proposition 4.5], the following diagram

\[
\begin{array}{cccccc}
\text{Alg}_{\mathcal{M}} & \xrightarrow{J_{\text{Alg}}} & \text{BrAlg}_{\mathcal{M}} & \xrightarrow{\Omega_{\text{Alg}}} & \mathcal{M} & \xrightarrow{J} & \text{Br}_{\mathcal{M}} \\
\end{array}
\]

is commutative. When $\mathcal{M}$ is symmetric the functors $J$, $J_{\text{Alg}}$ and $J_{\text{Bialg}}$ factors through functors $J^s$, $J^s_{\text{Alg}}$ and $J^s_{\text{Bialg}}$ i.e. the following diagrams commute (apply Lemma 1.12).

\[
\begin{array}{cccccc}
\text{Alg}_{\mathcal{M}} & \xrightarrow{J^s_{\text{Alg}}} & \text{BrAlg}^s_{\mathcal{M}} & \xrightarrow{\Omega_{\text{Alg}}} & \mathcal{M} & \xrightarrow{J^s} & \text{Br}^s_{\mathcal{M}} \\
\end{array}
\]

4.5. Let $\mathcal{M}$ be a preadditive braided monoidal category with equalizers. Assume that the tensor functors are additive and preserve equalizers. Define the functor

$P := H \circ P_{\text{Br}} \circ J_{\text{Bialg}} : \text{Bialg}_{\mathcal{M}} \rightarrow \mathcal{M}$

For any $B := (B, m_B, u_B, \Delta_B, \varepsilon_B) \in \text{Bialg}_{\mathcal{M}}$ one easily gets that $P(B) = P(B, \Delta_B, \varepsilon_B, u_B)$, see [AM, 4.6]. The canonical inclusion $\xi P(B, \Delta_B, \varepsilon_B, u_B) : P(B, \Delta_B, \varepsilon_B, u_B) \rightarrow B$ will be denoted by $\xi B$. Thus we have the equalizer

$P(B) \xrightarrow{\xi B} B \xrightarrow{\Delta_B} B \otimes B$

By [AM, Proposition 4.7], we have a commutative diagram

\[
\begin{array}{cccccc}
\text{Bialg}_{\mathcal{M}} & \xrightarrow{J_{\text{Bialg}}} & \text{BrBialg}_{\mathcal{M}} & \xrightarrow{\xi B} & B \\
\text{Alg}_{\mathcal{M}} & \xrightarrow{J^s_{\text{Alg}}} & \text{BrAlg}^s_{\mathcal{M}} & \xrightarrow{\xi B^s} & B^s \\
\mathcal{M} & \xrightarrow{J} & \text{Br}_{\mathcal{M}} & \xrightarrow{\xi J} & B \\
\end{array}
\]

where the horizontal arrows are the functors of 4.4. Furthermore

$\xi J_{\text{Bialg}} = J_{\xi}.$

Assume further that $\mathcal{M}$ has denumerable coproducts and that the tensor functors preserve such coproducts. By Remark 1.3, the forgetful functor $\Omega : \text{Alg}_{\mathcal{M}} \rightarrow \mathcal{M}$ has a left adjoint $T : \mathcal{M} \rightarrow \text{Alg}_{\mathcal{M}}$. Note that

$T^s_{\text{BrAlg}} T^s_{\text{Br}} J^s \xrightarrow{\xi} T^s_{\text{Br}} J^s T^s_{\text{BrAlg}} J_{\text{Alg}} T^s_{\text{BrAlg}} J^s_{\text{Bialg}} T^s_{\text{BrAlg}}$
and hence, since \( J_{\text{Alg}} \) is both injective on morphisms and objects, we get that the following diagram commutes

\[
\begin{array}{ccc}
\text{Alg}_M & \xrightarrow{J_{\text{Alg}}} & \text{BrAlg}_M \\
\tau & \uparrow & \uparrow J^s_{\text{Br}} \\
\mathcal{M} & \xrightarrow{J^s} & \text{Br}^s_M
\end{array}
\]

In view of [AM, 4.8], there is a functor \( \mathcal{T} : \mathcal{M} \rightarrow \text{Bialg}_\mathcal{M} \) such that the following diagrams commute.

\[
\begin{array}{ccc}
\text{Bialg}_M & \xrightarrow{J_{\text{Bialg}}} & \text{BrBialg}_M \\
\tau & \uparrow \quad & \uparrow \quad J^s_{\text{Br}} \\
\mathcal{M} & \xrightarrow{J^s} & \text{Br}^s_M
\end{array}
\]

\[
\begin{array}{ccc}
\text{Bialg}_M & \xrightarrow{\iota} & \text{Alg}_M \\
\mathcal{T} & \uparrow & \uparrow \tau \\
\mathcal{M} & \xrightarrow{\tau} & \text{Br}^s_M
\end{array}
\]

By [AM, Theorem 4.9], the functor \( \mathcal{T} \) is a left adjoint of the functor \( P : \text{Bialg}_\mathcal{M} \rightarrow \mathcal{M} \). The unit \( \eta \) and counit \( \epsilon \) of the adjunction are uniquely determined by the following equalities

\[
(51) \quad \xi \mathcal{T} \circ \eta = \eta, \quad \epsilon \mathcal{P} \circ T \xi = \mathcal{P}
\]

so that the following diagram commutes.

\[
\begin{array}{ccc}
\text{Bialg}_M & \xrightarrow{J_{\text{Bialg}}} & \text{BrBialg}_M \\
\tau & \uparrow \quad & \uparrow \quad J^s_{\text{Br}} \\
\mathcal{M} & \xrightarrow{J^s} & \text{Br}^s_M
\end{array}
\]

\[
\begin{array}{ccc}
\text{Bialg}_M & \xrightarrow{\iota} & \text{Alg}_M \\
\mathcal{T} & \uparrow & \uparrow \tau \\
\mathcal{M} & \xrightarrow{\tau} & \text{Br}^s_M
\end{array}
\]

**Proposition 4.6.** Let \( \mathcal{M} \) be a preadditive braided monoidal category with equalizers. Assume that the tensor functors are additive and preserve equalizers. Assume further that \( \mathcal{M} \) has denumerable coproducts and that the tensor functors preserve such coproducts. Then the morphism \( \zeta : T_{\text{Br}} J \rightarrow J_{\text{Bialg}} \mathcal{T} \) of Lemma 2.2 is \( \text{Id}_{T_{\text{Br}} J} \). In particular \((J_{\text{Bialg}}, J) : (\mathcal{T}, \mathcal{P}) \rightarrow (T_{\text{Br}}, P_{\text{Br}}) \) is a commutation datum.

**Proof.** Consider the commutative diagram [40]. By Lemma 2.2, then there is a unique natural transformation \( \zeta : T_{\text{Br}} J \rightarrow J_{\text{Bialg}} \mathcal{T} \) such that \( P_{\text{Br}} \zeta \circ \eta_{\text{Br}} J = J_{\mathcal{T}} \). By [AM, Equality (75)], we also have \( \eta_{\text{Br}} J = J \mathcal{T} \). By uniqueness of \( \zeta \), we have \( \zeta = \text{Id}_{T_{\text{Br}} J} \). \( \square \)

**Proposition 4.7.** Let \( \mathcal{M} \) be a preadditive symmetric monoidal category with equalizers. Assume that the tensor functors are additive and preserve equalizers. Assume further that \( \mathcal{M} \) has denumerable coproducts and that the tensor functors preserve such coproducts. Then the morphism \( \zeta^s : T_{\text{Br}} J^s \rightarrow J_{\text{Bialg}} \mathcal{T} \) of Lemma 2.2 is \( \text{Id}_{T_{\text{Br}} J^s} \). In particular \((J_{\text{Bialg}}^s, J^s) : (\mathcal{T}, \mathcal{P}) \rightarrow (T_{\text{Br}}^s, P_{\text{Br}}^s) \) is a commutation datum.
**Proof.** Consider the commutative diagram (46). By Lemma 2.2, then there is a unique natural transformation \( \zeta^* : T_{\text{Br}}^s J^s \to J_{\text{Bialg}}^s \) such that \( P_{\text{Br}}^s \circ \eta_{\text{Br}}^s = J_{\text{Bialg}}^s \). Now

\[
\eta_{\text{Br}}^s \eta_{\text{Br}}^s J^s = \eta_{\text{Br}}^s \eta_{\text{Br}}^s J^s \eta_{\text{Br}}^s = \eta_{\text{Br}}^s \eta_{\text{Br}}^s J^s \eta_{\text{Br}}^s = J_{\text{Bialg}}^s \eta_{\text{Br}}^s
\]

where in (*) we used [\( \text{AM} \), Equality (75)]. Thus \( \eta_{\text{Br}}^s J^s = J_{\text{Bialg}}^s \eta_{\text{Br}}^s \). By uniqueness of \( \zeta^* \), we have \( \zeta^* = \text{Id}_{T_{\text{Br}}^s J^s} \) (note that we are using that the domain and codomain of \( \zeta^* \) coincide by (52)). \( \square \)

4.8. Let \( \mathcal{M} \) and \( \mathcal{M}' \) be braided monoidal categories. Following [\( \text{AM} \), Proposition 4.10], every braided monoidal functor \( (F, \phi_0, \phi_2) : \mathcal{M} \to \mathcal{M}' \) induces in a natural way a functor \( \text{Bialg}_F \) and the following diagrams commute.

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{F} & \mathcal{M}' \\
\mathcal{J} \downarrow & & \downarrow \mathcal{J}' \\
\text{Br}_\mathcal{M} & \xrightarrow{\text{Br}_{\mathcal{M}} F} & \text{Br}_{\mathcal{M}'} \\
\end{array}
\]

Moreover

1) \( \text{Bialg}_F \) is an equivalence (resp. category isomorphism or conservative) whenever \( F \) is.

2) If \( F \) preserves equalizers, the following diagram commutes.

\[
\begin{array}{ccc}
\text{Bialg}_\mathcal{M} & \xrightarrow{\text{Bialg}_F} & \text{Bialg}_{\mathcal{M}'} \\
\mathcal{M} \downarrow & & \downarrow \mathcal{M}' \\
\text{Br}_{\mathcal{M}} & \xrightarrow{\text{Br}_{\mathcal{M}} F} & \text{Br}_{\mathcal{M}'} \\
\end{array}
\]

5. **Lie algebras**

The following definition extends the classical notion of Lie algebra to a monoidal category which is not necessarily braided. We expected this notion to be well-known, but we could not find any reference.

**Definition 5.1.** 1) Given an abelian monoidal category \( \mathcal{M} \) a **braided Lie algebra** in \( \mathcal{M} \) consists of a tern \( (M, c, [-]) : M \otimes M \to M \) where \( (M, c) \in \text{Br}_\mathcal{M} \) and the following equalities hold true:

\[
[-] = - [-] \circ c \quad \text{(skew-symmetry)};
\]

\[
[-] \circ (M \otimes [-]) \circ \text{Id}_{(M \otimes M) \otimes M} + (M \otimes c) (c \otimes M) + (c \otimes M) (M \otimes c) = 0 \quad \text{(Jacobi condition)};
\]

\[
c \circ (M \otimes [-]) = ([M] \otimes c) (c \otimes M) (c \otimes M) ;
\]

\[
c \circ ([M] \otimes c) = (M \otimes [-]) (c \otimes c) (M \otimes c).
\]

Of course one should take care of the associativity constraints, but as we did before, we continue to omit them. A morphism of braided Lie algebras \( (M, c, [-]) \) and \( (M', c', [-']) \) in \( \mathcal{M} \) is a morphism \( f : (M, c) \to (M', c') \) of braided objects such that \( f \circ [-] = [-'] \circ (f \otimes f) \). This defines the category \( \text{BrLie}_\mathcal{M} \) of braided Lie algebras in \( \mathcal{M} \) and their morphisms. Denote by

\[
H_{\text{BrLie}} : \text{BrLie}_\mathcal{M} \to \text{Br}_{\mathcal{M}} : (M, c, [-]) \mapsto (M, c)
\]

the obvious functor forgetting the bracket and acting as the identity on morphisms. Note that \( H_{\text{BrLie}} \) is faithful and conservative.

Denote by \( \text{BrLie}_\mathcal{M}^s \) the full subcategory \( \text{BrLie}_\mathcal{M}^s \) consisting of braided Lie algebras with symmetric braiding. Denote by

\[
\eta_{\text{BrLie}}^s : \text{BrLie}_\mathcal{M}^s \to \text{BrLie}_\mathcal{M}
\]
the inclusion functor. It is clear that, by Lemma 1.12, the functor $H_{\text{BrLie}}$ induces a functor $H_{\text{BrLie}}^s$ such that the diagram

\[
\begin{array}{ccc}
\text{BrLie}^s_M & \xrightarrow{H_{\text{BrLie}}^s} & \text{Br}^s_M \\
\downarrow \text{Id}_{\text{BrLie}} & & \downarrow \text{Id}_{\text{BrLie}} \\
\text{BrLie}_M & \xrightarrow{H_{\text{BrLie}}} & \text{Br}_M
\end{array}
\]

commutes. Since $H_{\text{BrLie}}$ and both vertical arrows are faithful and conservative, the same is true for $H_{\text{BrLie}}^s$.

2) Let $\mathcal{M}$ be an abelian braided monoidal category. A Lie algebra in $\mathcal{M}$ consists of a pair $(M, [-]: M \otimes M \to M)$ such that $(M, c_{M,M}, [-]) \in \text{BrLie}_M^s$, where $c_{M,M}$ is the braiding $c$ of $\mathcal{M}$ evaluated on $M$. A morphism of Lie algebras $(M, [-])$ and $(M', [-])'$ in $\mathcal{M}$ is a morphism $f: M \to M'$ in $\mathcal{M}$ such that $f \circ [-] = [-]' \circ (f \otimes f)$. This defines the category $\text{Lie}_M$ of Lie algebras in $\mathcal{M}$ and their morphisms. Note that there is a full, faithful, injective on objects and conservative functor

\[J_{\text{Lie}}: \text{Lie}_M \to \text{BrLie}_M^s: (M, [-]) \mapsto (M, c_{M,M}, [-])\]

which acts as the identity on morphisms. This notion already appeared in [MM c] page 82, where a Lie algebra in $\mathcal{M}$ is called an $\mathcal{M}$-Lie algebra. Denote by $H_{\text{Lie}}: \text{Lie}_M \to \mathcal{M}$ the obvious functor forgetting the bracket and acting as the identity on morphisms. Note that $H_{\text{BrLie}}J_{\text{Lie}} = JH_{\text{Lie}}$.

3) Let $\mathcal{M}$ be an abelian symmetric monoidal category. Given $(M, [-]) \in \text{Lie}_M$ it is clear that $(M, c_{M,M}, [-]) \in \text{BrLie}_M^s$ so that $J_{\text{Lie}}$ factors through a functor $J_{\text{Lie}}^s$ such that the following diagrams commute.

\[
\begin{array}{ccc}
\text{Lie}_M & \xrightarrow{J_{\text{Lie}}} & \text{BrLie}_M^s \\
\downarrow J_{\text{Lie}} & & \downarrow J_{\text{Lie}}^s \\
\text{BrLie}_M^s & \xrightarrow{H_{\text{BrLie}}^s} & \text{Br}^s_M \\
\downarrow \text{Id}_{\text{BrLie}} & & \downarrow \text{Id}_{\text{BrLie}} \\
\text{Lie}_M^s & \xrightarrow{H_{\text{Lie}}} & \mathcal{M}
\end{array}
\]

**Remark 5.2.** We point out that $\text{BrLie}_M^s = \text{YBLieAlg}(\mathcal{M})$ with the notations of [SV2] Definition 2.5 (note that (53) follows from (52) as we are in the symmetric case).

**Lemma 5.3.** Let $\mathcal{M}$ be an abelian monoidal category. Consider a term $(M, c, [-]: M \otimes M \to M)$ where $(M, c) \in \text{Br}_M$. If $c^2 = \text{Id}$ and (54) holds, then we have that (55) is equivalent to

\[[-] \circ ([-] \otimes M) \circ [\text{Id}_{(V \otimes V) \otimes V} + (M \otimes c) (c \otimes M) + (c \otimes M)(M \otimes c)] = 0.\]

**Proof.** This proof is essentially the same as [SV2] Lemma 2.9. \[\square\]

**Remark 5.4.** In view of Lemma 5.3, in the particular case when $\mathcal{M}$ is the category of vector spaces and $(M, c) \in \text{Br}_M$, conditions (54) and (55) encode the notion of Lie algebra in the sense of Gurevich’s [G].

**Definition 5.5.** Let $\mathcal{M}$ a preadditive monoidal category with equalizers and denumerable coproducts. Let $(M, c) \in \text{Br}_M$. For $\alpha_2 M$ as in of Remark 1.3, we set

\[\theta_{(M, c)} := \alpha_2 M \circ (\text{Id}_{M \otimes M} - c) : M \otimes M \to \text{OTM}.\]

When $\mathcal{M}$ is braided and its braiding is on $c_{M,M}$ we will simply write $\theta_M$ for $\theta_{(M, c_{M,M})}$.

**Definition 5.6.** Let $\mathcal{M}$ be a monoidal category. Let $(A, m_A, u_A)$ be an algebra in $\mathcal{M}$ and let $f: X \to A$ be a morphism in $\mathcal{M}$. We set

\[\Lambda f := m_A \circ (m_A \otimes A) \circ (A \otimes f \otimes A) : A \otimes X \otimes A \to A.\]

When the category $\mathcal{M}$ is also abelian we can consider the two-sided ideal of $A$ generated by $f$ which is defined by $(f), i_f := \text{Im}(\Lambda f)$ and it has the following property (see e.g. [AMS2] Lemma 3.18): for every algebra morphism $g: A \to B$ one has that $g \circ i_f = 0$ if and only if $g \circ f = 0$. 


Remark 5.7. Let $\mathcal{M}$ be an abelian monoidal category. Let $(A, m_A, u_A)$ be an algebra in $\mathcal{M}$.

1) Note that $A_f - A_g = A_{f-g}$ for every $f, g : X \to A$.

2) Assume that the tensor products preserve epimorphisms. Let $f : X \to A$ be a morphism in $\mathcal{M}$ and set $(S, j : S \to A) := \text{Im}(f)$. Define the ideal $(S)$ generated by $S$ by setting $((S), i) := \text{Im}(A_j)$. Write $f = j \circ p$ where $p : X \to S$ is an epimorphism. We compute $\text{Im}(A_f) = \text{Im}(A_{jop}) = \text{Im}(A_j \circ (A \otimes p \otimes A)) = \text{Im}(A_j)$ so that $((f), i_f) := \text{Im}(A_f) = ((S), i)$. Therefore $\langle f \rangle = (\text{Im}(f))$.

Next aim is to construct suitable universal enveloping algebra type functors.

Remark 5.8. Let $\mathcal{M}$ an abelian monoidal category with denumerable coproducts. Assume that the tensor functors preserve denumerable coproducts. Note that $\mathcal{M}$ has also finite coproducts as it has a zero object and denumerable coproduct. Thus, by [St, Proposition 3.3] the tensor functors are additive as they preserve denumerable coproducts.

Proposition 5.9. Let $\mathcal{M}$ an abelian monoidal category with denumerable coproducts. Assume that the tensor functors are right exact and preserve denumerable coproducts.

Let $(M, c, [\cdot]) : M \otimes M \to M) \in \text{BrLie}_\mathcal{M}$ and set

$f := f_{(M, c, [\cdot])} := \alpha_1 \circ \theta_{(M, c)} : M \otimes M \to \Omega TM$.

Let $U_{\text{Br}}(M, c, [\cdot]) := R := \Omega TM/\langle f \rangle$ and let $p_R : \Omega TM \to R$ denote the canonical projection. Then there are morphisms $m_R, u_R, c_R$ such that $(R, m_R, u_R, c_R) \in \text{BrAlg}_\mathcal{M}$ and $p_R$ is a morphism of braided algebras. This way we get a functor

$U_{\text{Br}} : \text{BrLie}_\mathcal{M} \to \text{BrAlg}_\mathcal{M}$,

and the projections $p_R$ define a natural transformation $p : T_{\text{Br}} H_{\text{BrLie}} \to U_{\text{Br}}$. Moreover there is a functor $U_{\text{Br}}^* : \text{BrLie}_\mathcal{M}^* \to \text{BrAlg}_\mathcal{M}^*$ such that the diagram

\[
\begin{array}{ccc}
\text{BrLie}_\mathcal{M}^* & \xrightarrow{i_{\text{BrLie}}} & \text{BrLie}_\mathcal{M} \\
\Uparrow{U_{\text{Br}}^*} & & \Uparrow{U_{\text{Br}}} \\
\text{BrAlg}_\mathcal{M}^* & \xrightarrow{\varpi_{\text{BrAlg}}} & \text{Alg}_\mathcal{M}
\end{array}
\]

commutes and there is a natural transformation $p^* : T_{\text{Br}} H_{\text{BrLie}}^* \to U_{\text{Br}}^*$ uniquely defined by

\[
\varpi^*_{\text{BrAlg}} \circ p^* = p_{\text{BrLie}}^*.
\]

Proof. Set $(A, m_A, u_A, c_A) := T_{\text{Br}} (M, c)$, We will use the equalities for the graded part $c_A^{m,n}$ of the braiding $c_A$ which are in [AM, Proposition 2.7]. Note that, by [AM, (42)], we have that $c_A \circ (\alpha_M \otimes \alpha_M) = (\alpha_M \otimes \alpha_M) \circ c_A^{m,n}$ for every $m, n \in \mathbb{N}$. By induction on $n \in \mathbb{N}$, using (43), one checks that

\[
c_A^{n+1} \circ (M \otimes [-]) = (\alpha_M \otimes M^n) \circ c_A^{n,2}.
\]

If we apply [AM, (32) and (34)], we get

\[
c_A^{l+m} (M^{\alpha l} \otimes c_A^{m,n}) = (c_A^{m,n} \otimes M^{\alpha l}) c_A^{l,m+n}.
\]

If we apply this equality to the case $(l, m, n) = (n, 1, 1)$, we obtain

\[
c_A^{n,2} (M^{\otimes n} \otimes c) = (c \otimes M^{\otimes n}) c_A^{n,2}, \text{ for every } n \in \mathbb{N}.
\]

Since $\langle f \rangle$ is an ideal of $TM$, it is clear that $R$ is an algebra and $p_R$ is an algebra morphism. Consider the exact sequence

\[0 \to \langle f \rangle \xrightarrow{i_f} A \xrightarrow{p_R} R \to 0\]

If we apply to it the functor $A \otimes (-)$, we obtain the exact sequence

\[A \otimes (f) \xrightarrow{A \otimes i_f} A \otimes A \xrightarrow{A \otimes p_R} A \otimes R \to 0\]

We have that $((f), i_f) := \text{Im}(A_f)$ so that we can write $A_f = i_f \circ p_f$ where $p_f : A \otimes X \otimes A \to \langle f \rangle$ is an epimorphism. Since the tensor products preserve epimorphisms, we have that $A \otimes p_f$ is an
epimorphism so that \((p_R \otimes A) c_A (A \otimes i_f) = 0\) if and only if \((p_R \otimes A) A (A \otimes A) = 0\). Using the definition of \(c_A\), one checks that \((p_R \otimes A) A (A \otimes f) (a \otimes M \otimes M) = 0\). Since this holds for every \(n \in \mathbb{N}\) and the tensor products preserve the denumerable coproducts, we get

\[(p_R \otimes A) c_A (A \otimes f) = 0.\]

Now using \([4]\) one gets \((p_R \otimes A) c_A (A \otimes i_f) = 0\). Hence, by the foregoing, we get \((p_R \otimes A) A (A \otimes f) = 0\). Thus there is a unique morphism \(c_{A,R} : A \otimes R \to R \otimes A\) such that \(c_{A,R} (A \otimes p_R) = (p_R \otimes A) c_A\). Consider now the exact sequence

\[
\langle f \rangle \otimes R \xrightarrow{i_f \otimes R} A \otimes R \xrightarrow{pr \otimes R} R \otimes R \to 0.
\]

We will prove that \((R \otimes p_R) c_{A,R} (i_f \otimes R) = 0\).

This equality is equivalent to prove \((R \otimes p_R) c_{A,R} (\Lambda_f \otimes R) = 0\). We have

\[
\begin{align*}
(R \otimes p_R) c_{A,R} (\Lambda_f \otimes R) &= (R \otimes p_R) (A \otimes M \otimes M \otimes A \otimes p_R) c_A (\Lambda_f \otimes A) \\
&= (p_R \otimes R) c_{A,R} (A \otimes p_R) (A \otimes A) c_A (\Lambda_f \otimes A).
\end{align*}
\]

Note that the last term vanishes as \((A \otimes p_R) c_A (\Lambda_f \otimes A) = 0\) by a similar argument to the one used to prove \((p_R \otimes A) c_A (A \otimes \Lambda_f) = 0\). Thus we have a unique morphism \(c_R : R \otimes R \to R \otimes R\) such that \(c_R (R \otimes p_R) = (p_R \otimes R) c_A\).

If we rewrite \([22]\) and \([23]\) in terms of \(c^{-1}\) we get that \((M, c^{-1})\) fulfills \([22]\) and \([23]\). Thus we can repeat the argument above obtaining a morphism \(c'_{R}\) such that

\[
(p_R \otimes p_R) c_{A,R} = (p_R \otimes p_R) c_A\]

is an inverse for \(R \circ p_R\) in \(Br_{Alg}\). By Lemma \([3, 4]\), we get that \((R, c_{R})\) is an object in \(Br_{M}\) and \(p_R\) becomes a morphism in \(Br_{M}\) from \((A, c_{A})\) to this object. We have

\[
(c_R (m_R \otimes R) (p_R \otimes p_R \otimes p_R) = c_R (p_R \otimes p_R) (m_A \otimes A) = (p_R \otimes p_R) c_A (m_A \otimes A)
\]

so that \([22]\) holds for \((R, m_R, c_{R})\). Similarly one proves \([23]\). Moreover

\[
c_R (u_R \otimes R) l^{-1}_A p_R = c_R (u_R \otimes R) (1 \otimes p_R) l^{-1}_A = c_R (p_R u_A \otimes p_R) l^{-1}_A\]

and hence \(c_R (u_R \otimes R) l^{-1}_A = (u_R \otimes R) r^{-1}_A\). Similarly one gets \(c_R (u_R \otimes R) r^{-1}_A = (u_R \otimes R) l^{-1}_A\).

We have so proved that \((R, m_R, u_R, c_R) \in Br_{Alg_{M}}\). It is clear that \(p_R\) is a morphism of braided algebras.

Let \(\nu : (M, c, [-]) \to (M', c', [-'])\) be a morphism of braided Lie algebras. Consider the morphism of braided Lie algebras \(T_{Br} : T_{Br} (M, c) \to T_{Br} (M', c')\). Set \(R' : = U_{Br} (M', c', [-'])\) and denote by \(p_{R'}\) the corresponding projection and set \(f' := f (M', c', [-'])\). We have

\[
p_{R'} \circ \Omega H_{Alg} T_{Br} H_Br_{Br} \nu \circ f (M', c', [-']) = 0\]

since \(p_{R'} \circ \Omega H_{Alg} T_{Br} H_Br_{Br} \nu \circ i_f = 0\) so that there is a unique morphism \(U_{Br} : U_{Br} (M, c, [-]) \to U_{Br} (M', c', [-'])\) such that \(U_{Br} \nu \circ p_{R} = p_{R'} \circ T_{Br} H_Br_{Br} \nu\). It is easy to check that \(U_{Br} \nu\) is a morphism of braided bialgebroids. Since \(T_{Br}\) is a
functor it is then clear that $U_{\text{Br}}$ becomes a functor as well and that the projections define a natural transformation $p : T_{\text{Br}}H_{\text{BrLie}} \to U_{\text{Br}}$.

Let us construct $U_{\text{Br}}'$ We already observed that the functor $T_{\text{Br}}H_{\text{BrLie}}$ is full, faithful and injective on objects.

Let $(M, c, [-]) \in \text{BrLie}_{M}^s$. Then, by Remark 3.2-2), we get that $R = U_{\text{Br}} (M, c, [-]) \in \text{BrAlg}_{M}^s$ as $R$ is a quotient of $T_{\text{Br}}H_{\text{BrLie}} (M, c, [-])$ which is preserved by the required functors. Hence any object which is image of $U_{\text{Br}}^s$ is also image of $\iota_{\text{BrAlg}}$. By Lemma 1.12 there is a unique functor $U_{\text{Br}}' := U_{\text{Br}}^s_{\text{BrLie}}$ such that (62) commutes. We have

$$T_{\text{Br}}H_{\text{BrLie}} \xi_{\text{BrLie}} T_{\text{Br}}H_{\text{BrLie}} (67) T_{\text{Br}}H_{\text{BrLie}} \iota_{\text{BrAlg}} T_{\text{Br}}H_{\text{BrLie}} = U_{\text{Br}}' \xi_{\text{BrAlg}} T_{\text{Br}}H_{\text{BrLie}}$$

By Lemma 1.12, we have $T_{\text{Br}}H_{\text{BrLie}} \xi_{\text{BrLie}} T_{\text{Br}}H_{\text{BrLie}} = U_{\text{Br}}^s$ and there is a unique natural transformation $p^s := p^s_{\text{BrLie}} : T_{\text{Br}}H_{\text{BrLie}} \to U_{\text{Br}}'$ such that (62) holds.

**Lemma 5.10.** Let $\mathcal{M}$ a preadditive monoidal category with denumerable coproducts. Assume that the tensor functors are additive and preserve such coproducts. Let $(M, c, [-]) : M \otimes M \to M) \in \text{BrLie}_{M}$, set $(A, m_A, u_A, \Delta_A, \varepsilon_A, c_A) := T_{\text{Br}} (M, c)$ and use the notations of (64). Then,

$$\Delta_A \circ \theta_{(M, c)} = [(u_A \otimes A) \circ l_A^{-1} + (A \otimes u_A) \circ r_A^{-1}] \circ \theta_{(M, c)}$$

if $c^2 = \text{Id}_{M \otimes M}$;

$$\Delta_A \circ \alpha_1 M = [(u_A \otimes A) \circ l_A^{-1} + (A \otimes u_A) \circ r_A^{-1}] \circ \alpha_1 M.$$  

**Proof.** Using, in the given order, (64), the multiplicity of $\Delta_A$, (68), the definitions of $\delta^i_M$, and $\tilde{\delta}_M$, the equalities $c_A \circ (\alpha_1 M \otimes \alpha_1 M) = (\alpha_1 M \otimes \alpha_1 M) \circ c_A^0$ for $i, j \in \{1, 2\}$, the equalities $c_A^1_{1,0} = l_M^{-1} r_M c_A^1 c_A^0_{1,0} = c_A^0_{1,0} = l_M^{-1} r_1$ and $c_A^0_{1,1} = r_M^{-1} l_M$, the equalities (64) and (68), the equalities $r_M \otimes M = M \otimes l_M$ and $r_1 \otimes M \otimes M \otimes M$, the equalities $l_M^{-1} \otimes M = l_M^{-1} l_{M \otimes M}$, $M \otimes l_1 = r_1^{-1} \otimes 1$ and $M \otimes r_1^{-1} = r_M^{-1} \otimes 1$, the equalities $m_1 = r_1 = l_1$, the naturality of the unit constraints, $l_M^{-1} \otimes M = l_M^{-1} l_{M \otimes M}$, $M \otimes r_1^{-1} = r_M^{-1} \otimes 1$ and $M \otimes M \otimes M$, the equality (64) and the naturality of the unit constraints one proves that

$$\Delta_A \circ \alpha_2 M = [(u_A \otimes A) \circ l_A^{-1} + (A \otimes u_A) \circ r_A^{-1}] \circ \alpha_2 M + (\alpha_1 M \otimes \alpha_1 M) \circ (\text{Id}_{M \otimes M} + c).$$

From this equality, composing with $\text{Id}_{M \otimes M} - c$ on both sides, we get (68) holds true when $c^2 = \text{Id}_{M \otimes M}$.

On the other hand, (65) follows by (68), the definitions of $\delta^i_M$ and $\tilde{\delta}_M$, the naturality of the unit constraints.

**Proposition 5.11.** Let $(B, m_B, u_B, \Delta_B, \varepsilon_B, c_B) \in \text{BrAlg}_{M}$ be a bialgebra in a monoidal category $\mathcal{M}$. Assume that the category $\mathcal{M}$ is abelian and the tensor functors are additive and right exact. Let $(R, m_R, u_R, c_R) \in \text{BrAlg}_{M}$ and let $p_R : B \to R$ be an epimorphism which is a morphism of braided algebras. Set $(I, i_I : I \to B) := \text{Ker}(p_R)$. Assume that

$$(p_R \otimes p_R) \circ \Delta_B = i_I = 0.$$  

Then there are morphisms $\Delta_R, \varepsilon_R$ such that $(R, m_R, u_R, \Delta_R, \varepsilon_R, c_R) \in \text{BrAlg}_{M}$ and $p_R$ is a morphism of braided algebras.

**Proof.** Since $(R, p_R) = \text{Coker}(i_I)$, by (69), there is a unique morphism $\Delta_R : R \to R \otimes R$ such that $\Delta_R \circ p_R = (p_R \otimes p_R) \circ \Delta_B$ and by (70), there is a unique morphism $\varepsilon_R : R \to 1$ such that $\varepsilon_R \circ p_R = \varepsilon$. The rest of the proof is straightforward and relies on the fact that $p_R \otimes p_R = (p_R \otimes R) (A \otimes p_R)$ is an epimorphism by exactness of the tensor functors.

**Theorem 5.12.** Let $\mathcal{M}$ an abelian monoidal category with denumerable coproducts. Assume that the tensor functors are right exact and preserve denumerable coproducts. Then there is a functor
\[ \mathcal{U}_{Br}^s : \text{BrLie}_{\mathcal{M}}^s \rightarrow \text{BrBialg}_{\mathcal{M}}^s \text{ such that} \]

\[
\begin{array}{c}
\text{BrLie}_{\mathcal{M}}^s \\
\mathcal{U}_{Br}^s
\end{array} \xrightarrow{f_{\mathcal{U}_{Br}^s}} 
\begin{array}{c}
\text{BrBialg}_{\mathcal{M}}^s
\end{array}
\]

Moreover there is a natural transformation \( \mathcal{P} : \mathcal{T}^s_{Br}H_{\text{BrLie}}^s \rightarrow \mathcal{U}_{Br}^s \) uniquely defined by

\[
\mathcal{U}_{Br}^s\mathcal{P}^s = \mathcal{P}^s_{\text{BrBialg}} \quad \text{and} \quad \mathcal{U}_{Br}^s\mathcal{P}^s = p^s
\]

where \( p : T^s_{Br}H_{\text{BrLie}} \rightarrow \mathcal{U}_{Br}^s \) and \( p^s : T^s_{Br}H_{\text{BrLie}}^s \rightarrow \mathcal{U}_{Br}^s \) are the natural transformations of Proposition 5.3.

**Proof.** Let \((M, c, [-]) \in \text{BrLie}_{\mathcal{M}}^s\) and set \((A, m_A, u_A, \Delta_A, \varepsilon_A, c_A) := \mathcal{T}^s_{Br}(M, c)\) and \(f := f_{M, c, [-]}\).

Set \((R, m_R, u_R, c_R) := \mathcal{U}_{Br}(M, c, [-])\) and let \(p_R\) be the morphism in \(\mathcal{M}\) underlying the canonical projection \(p(M, c, [-]) : \mathcal{U}_{Br}(M, c) \rightarrow \mathcal{U}_{Br}(M, c, [-])\). By Proposition 5.9 we know that \(p_R : A \rightarrow R\) is a morphism of braided algebras. Using \((18)\) and \((20)\), we get

\[
\Delta_A \circ f = [u_A \otimes A] \circ (\mathcal{P}^s_{\text{BrBialg}} = \mathcal{P}^s_{\text{BrBialg}} + (A \otimes u_A) \circ r_A^{-1}) \circ f
\]

Since \(p_R\) is an algebra morphism and \(p_R \circ i_f = 0\), we get that \(p_R \circ f = 0\). We want to apply Proposition 5.11 to the case \((f, i_f) = ((f), i_f)\). Since \((p_R \otimes p_R) \circ \Delta_A\) is an algebra morphism as a composition of algebra morphisms (use e.g. \([2, Proposition 2.2-3]\)) to prove that \(\Delta_A\) is an algebra morphism and use \((19)\) to have that \(\Delta_A\) is an algebra morphism, we have that \((70)\) is equivalent to \((p_R \otimes p_R) \circ \Delta_A \circ f = 0\) and the latter holds by \((74)\), unitarity of \(p_R\), naturality of the unit constraints, and the equality \(p_R \circ f = 0\).

Since \(\varepsilon_A\) is an algebra morphism, we have that \((71)\) if and only if \(\varepsilon_A \circ f = 0\) and the latter holds by definition of \(f\) and \((20)\). Then, by Proposition 5.11, there are morphisms \(\Delta_R, \varepsilon_R\) such that \((R, m_R, u_R, \Delta_R, \varepsilon_R, c_R) \in \text{BrBialg}_{\mathcal{M}}^s\) and \(p_R\) is a morphism of braided bialgebras. By Remark 5.2, one easily checks that \((R, m_R, u_R, \Delta_R, \varepsilon_R, c_R) \in \text{BrBialg}_{\mathcal{M}}^s\). We denote this datum by \(\mathcal{U}_{Br}(M, c, [-])\). Let \(\nu : (M, c, [-]) \rightarrow (M', c', [-])\) be a morphism in \(\text{BrLie}_{\mathcal{M}}^s\). We know that \(\mathcal{U}_{Br}^s\mathcal{P}^s = \mathcal{U}_{Br}^s\mathcal{P}^s = \mathcal{U}_{Br}(\mathcal{U}_{Br}^s\mathcal{P}^s) = \mathcal{U}_{Br}(\mathcal{U}_{Br}^s\mathcal{P}^s)\). A similar argument shows that \(\mathcal{U}_{Br}^s\mathcal{P}^s\) is also comultiplicative and hence \(\mathcal{U}_{Br}^s\mathcal{P}^s\) is a morphism in \(\text{BrBialg}_{\mathcal{M}}^s\). This defines a functor \(\mathcal{U}_{Br}^s : \text{BrLie}_{\mathcal{M}}^s \rightarrow \text{BrBialg}_{\mathcal{M}}^s\) such that \(\mathcal{U}_{Br}^s \circ \mathcal{U}_{Br} = \mathcal{U}_{Br}^s\). Since \(p_R\) is a morphism of braided bialgebras and it is natural in \(R\) at the level of \(\text{BrAlg}_{\mathcal{M}}^s\), it is clear that \(\mathcal{P}^s\) such that \(\mathcal{U}_{Br}^s\mathcal{P}^s = \mathcal{P}^s_{\text{BrBialg}}\) exists. Moreover we have

\[
\mathcal{U}_{Br}^s\mathcal{P}^s = \mathcal{P}^s_{\text{BrBialg}} \mathcal{U}_{Br}^s
\]

and hence \(p^s = \mathcal{U}_{Br}^s\mathcal{P}^s = \mathcal{P}^s_{\text{BrBialg}}\).

\[
\square
\]

6. ADJUNCTIONS FOR ENVELOPING FUNCTORS

**Proposition 6.1.** Let \(\mathcal{M}\) an abelian monoidal category with denumerable coproducts. Assume that the tensor functors are right exact and preserve denumerable coproducts. Then the functor \(\mathcal{U}_{Br}^s : \text{BrLie}_{\mathcal{M}}^s \rightarrow \text{BrAlg}_{\mathcal{M}}^s\) has a right adjoint \(\mathcal{L}_{Br}^s : \text{BrAlg}_{\mathcal{M}}^s \rightarrow \text{BrLie}_{\mathcal{M}}^s\) acting as the identity on morphisms and defined on objects by \(\mathcal{L}_{Br}(B, m_B, u_B, c_B) := (B, c_B, [-]_B)\), where \([-]_B := m_B \circ (\text{Id}_{B} \otimes c_B)\). The unit \(\eta_{Br}^s : \text{BrLie}_{\mathcal{M}}^s \rightarrow \mathcal{L}_{Br}^s\mathcal{U}_{Br}^s\) and the counit \(\epsilon_{Br}^s : \mathcal{L}_{Br}^s\mathcal{U}_{Br}^s \rightarrow \text{Id}_{\text{BrLie}_{\mathcal{M}}^s}\) of the adjunction fulfill

\[
e_{\text{BrLie}_{\mathcal{M}}^s} \circ p^s\mathcal{L}_{Br} = \epsilon_{Br}^s\mathcal{L}_{Br} = \mathcal{P}_{\text{BrLie}_{\mathcal{M}}^s}
\]

and

\[
H_{\text{BrLie}_{\mathcal{M}}^s}^s \mathcal{L}_{Br}^s \mathcal{P}_{\text{BrLie}_{\mathcal{M}}^s}^s = \mathcal{P}_{\text{BrLie}_{\mathcal{M}}^s}^s \mathcal{P}_{\text{BrLie}_{\mathcal{M}}^s}^s = \eta_{Br}^s_{\text{BrLie}_{\mathcal{M}}^s} H_{\text{BrLie}_{\mathcal{M}}^s}^s = \eta_{Br}^s_{\text{BrLie}_{\mathcal{M}}^s} H_{\text{BrLie}_{\mathcal{M}}^s}^s.
\]

**Proof.** The construction of the functor \(\mathcal{L}_{Br}^s\) is given in [3, Construction 2.16] where \(\text{BrAlg}_{\mathcal{M}}^s\) plays the role of \(\text{YBA}_{\mathcal{M}}\) therein. Let us check that \((\mathcal{L}_{Br}^s, \mathcal{U}_{Br}^s)\) is an adjunction.

Consider the natural transformation \(p^s : T^s_{Br}H_{\text{BrLie}}^s \rightarrow \mathcal{U}_{Br}^s\) of Proposition 5.3.

Note that \(H_{\text{BrLie}_{\mathcal{M}}^s}^s \mathcal{L}_{Br}^s (B, m_B, u_B, c_B) = H_{\text{BrLie}_{\mathcal{M}}^s}^s (B, c_B, [-]_B) = (B, c_B, \eta_{Br}^s B, m_B, u_B, c_B)\) and \(H_{\text{BrLie}_{\mathcal{M}}^s}^s \mathcal{L}_{Br}^s\) and \(\eta_{Br}^s\) both act as the identity on morphisms so that \(H_{\text{BrLie}_{\mathcal{M}}^s}^s \mathcal{L}_{Br}^s = \Omega_{Br}^s\). Then
we have $p^s \mathcal{L}_{B}^{s} : T_{B}^{s} \Omega_{B}^{s} \to U_{B}^{s} \mathcal{L}_{B}^{s}$. Consider $\epsilon_{B}^{s} : T_{B}^{s} \Omega_{B}^{s} \to \text{Id}_{\text{BrAlg}^{s}_{M}}$. Using the notation of Proposition 5.9 by means of (25), (23), (40) and (4) we get

$$\Omega \text{Alg}^{s}_{\text{BrAlg}} \epsilon_{B}^{s} (B, m_{B}, u_{B}, c_{B}) \circ f_{\mathcal{L}_{B}}^{s} (B, m_{B}, u_{B}, c_{B}) = 0.$$ 

Since $\epsilon_{B}^{s}$ is a morphism of braided algebras, by construction of $U_{B}^{s} \mathcal{L}_{B}^{s}$, the latter equality implies that there is a unique morphism $\epsilon_{B}^{s} : U_{B}^{s} \mathcal{L}_{B}^{s} \to \text{Id}_{\text{BrAlg}^{s}_{M}}$ such that $\epsilon_{B}^{s} \circ p^{s} \mathcal{L}_{B}^{s} = \epsilon_{B}^{s}$.

Consider the morphism $H_{\text{BrAlg}}^{s} \mathcal{P}_{s} \circ \eta_{B}^{s} H_{\text{BrLie}}^{s} : H_{\text{BrLie}}^{s} \mathcal{L}_{B}^{s} \to H_{\text{BrLie}}^{s} \mathcal{L}_{B}^{s} \mathcal{U}_{B}^{s}$. Let $(M, c_{M}, [-]) \in \text{BrLie}^{s}_{M}$ and set $\nu := H_{\text{BrAlg}}^{s} \mathcal{P}_{s} \circ \eta_{B}^{s} H_{\text{BrAlg}}^{s} \mathcal{P}_{s} (M, c_{M}, [-]) \circ H_{\text{BrAlg}}^{s} \mathcal{P}_{s} \eta_{B}^{s} H_{\text{BrAlg}}^{s} (M, c_{M}, [-])$, $(R, m_{R}, u_{R}, c_{R}) := U_{B}^{s} (M, c_{M}, [-])$ and $(A, m_{A}, u_{A}, c_{A}) := T_{B}^{s} (M, c_{M})$. Clearly $\nu : (M, c_{M}) \to (R, c_{R})$ is a morphism of braided objects. Using (27), (29), (23) and (4) and the equality $p_{R} = H_{\text{BrAlg}}^{s} \mathcal{P}_{s} \eta_{B}^{s} H_{\text{BrAlg}}^{s} (M, c_{M}, [-])$ (which follows by definition of $p$ in Proposition 5.9), we obtain that $\nu = p_{R} \circ \alpha_{1} M$. By the latter formula, the fact that $p_{R}$ is a braided morphism, the definition of $c_{A}$ given by (42), the multiplicity of $p_{R}$, using (4) and the formula $p_{R} \circ f_{(M, c_{M}, [-])} = 0$, we obtain $[-]_{R} (\nu \circ \nu) = \nu \circ [-]$. Since $\nu$ is the morphism in $\mathcal{M}$ defining $H_{\text{BrAlg}}^{s} \mathcal{P}_{s} \circ \eta_{B}^{s} H_{\text{BrAlg}}^{s} : H_{\text{BrAlg}}^{s} \mathcal{L}_{B}^{s} \to H_{\text{BrAlg}}^{s} \mathcal{L}_{B}^{s} \mathcal{U}_{B}^{s}$, we get that there is a unique natural transformation $\eta_{B}^{s} : \text{Id}_{\text{BrAlg}^{s}_{M}} \to \mathcal{L}_{B}^{s} \mathcal{U}_{B}^{s}$ such that $H_{\text{BrAlg}}^{s} \mathcal{P}_{s} \circ \eta_{B}^{s} H_{\text{BrAlg}}^{s} = H_{\text{BrAlg}}^{s} \mathcal{L}_{B}^{s} \mathcal{U}_{B}^{s}$. It is straightforward to check that this gives rise to the claimed adjunction. Note that

$$H_{\text{BrAlg}}^{s} \mathcal{L}_{B}^{s} \mathcal{U}_{B}^{s} (M, c_{M}, [-]) = \nu = p_{R} \circ \alpha_{1} M.$$ 

The latter equality will be used elsewhere.

As a consequence of the construction of $U_{B}^{s}$ we can introduce an enveloping algebra functor $U$ in the braided case. We remark that in [GV3, 2.2] such a functor is just assumed to exist and the functor $L : \text{Alg}_{M} \to \text{Lie}_{M}$ in the following result is also considered.

**Theorem 6.2.** Let $\mathcal{M}$ be an abelian symmetric monoidal category with denumerable coproducts. Assume that the tensor functors are right exact and preserve denumerable coproducts. There are unique functors $U$ and $L$ such that the following diagrams commute.

$$\begin{array}{ccc}
\text{Alg}_{M} & \xrightarrow{J_{\text{Alg}}} & \text{BrAlg}_{M}^{s} \\
\downarrow U & & \downarrow \mathcal{L}_{B}^{s} \\
\text{Lie}_{M} & \xrightarrow{J_{\text{Lie}}} & \text{BrLie}_{M}^{s}
\end{array}$$

Moreover $(U, L)$ is an adjunction with unit $\eta_{L} : \text{Id}_{\text{Lie}_{M}} \to LU$ and counit $\epsilon_{L} : UL \to \text{Id}_{\text{Alg}_{M}}$ defined by

$$J_{\text{Alg}}^{s} \epsilon_{L} = \epsilon_{B}^{s} J_{\text{Alg}}^{s} \quad \text{and} \quad J_{\text{Lie}}^{s} \eta_{L} = \eta_{B}^{s} J_{\text{Lie}}^{s},$$

and $(J_{\text{Alg}}^{s}, J_{\text{Lie}}^{s}) : (U, L) \to (U_{B}^{s}, L_{B}^{s})$ is a commutation datum with canonical transformation given by the identity. The functors $U$ and $L$ can be described explicitly by $U := H_{\text{Alg}} U_{B} J_{\text{Lie}}$, while $L : \text{Alg}_{M} \to \text{Lie}_{M}$ acts as the identity on morphisms and is defined on objects by $L (B, m_{B}, u_{B}) := (B, [-B]),$ where $[-B] := m_{B} \circ (\text{Id}_{B} \otimes B - c_{B,B})$.

**Proof.** The existence and uniqueness of $U$ and $L$ as in the statement follows by Lemma 5.9. It remains to prove the last sentence. The equality $U = H_{\text{Alg}} U_{B} J_{\text{Lie}}$ follows by (27), (62) and (68). For $(B, m_{B}, u_{B}) \in \text{Alg}_{M}$, by the foregoing, we have

$$J_{\text{Lie}}^{s} (L (B, m_{B}, u_{B})) = L_{B}^{s} J_{\text{Alg}}^{s} (B, m_{B}, u_{B}) = (B, [-B], c_{B,B})$$

so that $L (B, m_{B}, u_{B}) = (B, [-B]).$ Since $J_{\text{Lie}}^{s} L_{B}^{s}$ and $J_{\text{Alg}}^{s}$ act as the identity on morphisms so does $L$. 

**Proposition 6.3.** Let $\mathcal{M}$ be an abelian monoidal category with denumerable coproducts. Assume that the tensor functors are right exact and preserve denumerable coproducts. Then the functor...
Theorem 6.4. Let \( \mathcal{M} \) be an abelian symmetric monoidal category with denumerable coproducts. Assume that the tensor functors are right exact and preserve denumerable coproducts. Then there are unique functors \( \mathcal{T} \) and \( P \) such that the following diagrams commute

\[
\begin{array}{ccc}
\text{Bialg}_\mathcal{M} & \xrightarrow{J_{\text{Bialg}}} & \text{BrBialg}_\mathcal{M} \\
\mathcal{T} \mid & & \mid \mathcal{P} \\
\text{Lie}_\mathcal{M} & \xrightarrow{J_{\text{Lie}}} & \text{BrLie}_\mathcal{M}
\end{array}
\]
where \( \mathcal{U} \) is the functor of Theorem 6.4. Moreover \((\mathcal{U}, P)\) is an adjunction with unit \( \eta_L : \text{Id}_{\text{Lie}_M} \to \mathcal{U}\mathcal{U} \) and counit \( \epsilon_L : \mathcal{U}\mathcal{P} \to \text{Id}_{\text{Bialg}_M} \) uniquely determined by

\[
J^s_{\text{Lie}} \eta_L = \epsilon_{\text{BrL}_M} J^s_{\text{Lie}} \quad \text{and} \quad J^s_{\text{Bialg}} \epsilon_L = \epsilon_{\text{BrL}_M} J^s_{\text{Bialg}},
\]

and \((J^s_{\text{Bialg}}, J^s_{\text{Lie}}) : (\mathcal{U}, \mathcal{P}) \to (\mathcal{U}_{\text{Br}}, \mathcal{P}_{\text{Br}})\) is a commutation datum with canonical transformation given by the identity. Furthermore there is a natural transformation \( \xi : \mathcal{T}\mathcal{H}_\mathcal{L} \to \mathcal{U} \) such that

\[
\xi J^s_{\text{Lie}} \epsilon_L = J^s_{\text{Bialg}} \xi \quad \text{and} \quad \xi J^s_{\text{Bialg}} \epsilon_L = p J^s_{\text{Lie}}
\]

where \( \xi : T^s_{\text{Br}} H^s_{\text{BrLie}} \to U^s_{\text{Br}} \) is the natural transformation of Theorem 5.12 and \( p : T^s_{\text{Br}} H^s_{\text{BrLie}} \to U^s_{\text{Br}} \) is the natural transformation of Proposition 5.3. The natural transformation \( \xi : P^s \to L^s \) induces a natural transformation \( \xi^s : P \to L^s \) such that \( J^s_{\text{Lie}} \xi^s = \xi J^s_{\text{Bialg}} \).

\( \xi J^s_{\text{Bialg}} \epsilon_L = \eta^s \)

\( \eta^s = \epsilon_{\text{Bialg}}^s \)

\( \mathcal{U} \)

**Proof.** The first part is a consequence of Lemma 2.9. The commutativity of the third diagram of (81) follows by (81), (82) and (83). By Lemma 1.12 there is a natural transformation \( \mathfrak{p} := \mathfrak{p} J^s_{\text{Lie}} : \mathcal{T}\mathcal{H}_\mathcal{L} \to \mathcal{U} \) such that \( J^s_{\text{Bialg}} \mathfrak{p} = \mathfrak{p} J^s_{\text{Lie}} \). Using (14) (88), (73) and (58) we get \( \mathfrak{p} J^s_{\text{Bialg}} \mathfrak{p} = p J^s_{\text{Lie}} \). By Lemma 1.12 there is a natural transformation \( \xi := \xi J^s_{\text{Bialg}} : P \to L^s \) such that \( J^s_{\text{Lie}} \xi = \xi J^s_{\text{Bialg}} \).

**Remark 6.5.** By Lemma 1.12 there is a natural transformation \( \xi^s : P^s \to L^s \) such that

\[
\xi J^s_{\text{Lie}} \epsilon_L = \eta^s \quad \text{and} \quad \eta^s = \epsilon_{\text{Bialg}}^s \quad \text{where} \quad \mathcal{U} \mathcal{P} = H^s_{\text{Br}} p J^s_{\text{Lie}} \quad \text{and} \quad p : T^s_{\text{Br}} H^s_{\text{BrLie}} \to U^s_{\text{Br}} \quad \text{is the morphism of Proposition 5.3}.\]

This means that for every \((M, [-]) \in \text{Lie}_M\) the morphism \( \eta^s \) is really induced by the canonical projection \( p^s : \Omega T^s M \to R := \mathcal{U}^s_{\text{Br}} J^s_{\text{Lie}} (M, [-]) \) defining in this lemma the universal enveloping algebra. Summing up, as a bialgebra in \( M \) we have that \( \mathcal{U} (M, [-]) \) is a quotient of \( \mathcal{T}\mathcal{H}_\mathcal{L} (M, [-]) = \mathcal{T} M \) via \( \eta^s \) and the underlying algebra structure is the original one underlying \( \mathcal{U}^s_{\text{Br}} J^s_{\text{Lie}} (M, [-]) \).

## 7. Stationary monadic decomposition

**Theorem 7.1.** Let \( \mathcal{M} \) be an abelian monoidal category with denumerable coproducts. Assume that the tensor functors are exact and preserve denumerable coproducts.

\[
\begin{align*}
\text{BrBialg}^s_{\mathcal{M}} & \cong & \text{Id}_{\text{BrBialg}_{\mathcal{M}}} & \text{BrBialg}_{\mathcal{M}} & \cong & \text{Id}_{\text{BrBialg}_{\mathcal{M}}} & \text{BrBialg}^s_{\mathcal{M}} \\
T^s_{\text{Br}} & \cong & T^s_{\text{Br}} & \cong & T^s_{\text{Br}} & \cong & T^s_{\text{Br}} \\
\text{Br} & \cong & \text{Br} & \cong & \text{Br} & \cong & \text{Br} \\
\text{BrLie}_{\mathcal{M}} & \cong & \text{BrLie}_{\mathcal{M}} & \cong & \text{BrLie}_{\mathcal{M}} & \cong & \text{BrLie}_{\mathcal{M}}
\end{align*}
\]

The functor \( P^s \) is comparable so that we can use the notation of Definition 5.3. There is a functor \( \Lambda^s_{\text{Br}} : (\text{Br}^s_{\mathcal{M}}) \to \text{BrLie}_{\mathcal{M}} \) such that \( \Lambda^s_{\text{Br}} \circ (P^s_{\text{Br}})_2 = \pi^s_1 \) and \( H^s_{\text{BrLie}} \circ \Lambda^s_{\text{Br}} = \Lambda^s_{\text{Br}} \). Moreover there exists a natural transformation \( \chi^s_{\text{Br}} : \mathcal{T}\mathcal{U} \Lambda^s_{\text{Br}} \to (\mathcal{T}^s_{\text{Br}})_1 U^s_{1,2} \) such that

\[
\chi^s_{\text{Br}} \circ \mathcal{P} \Lambda^s_{\text{Br}} = \pi^s_1 U^s_{1,2}
\]

where \( \mathcal{P}^s \) is the natural transformation of Theorem 5.12 and \( \pi^s_1 : T^s_{\text{Br}} U^s_{0,1} \to (\mathcal{T}^s_{\text{Br}})_1 \) is the canonical natural transformation defining \((\mathcal{T}^s_{\text{Br}})_1 \).

Assume \( \mathcal{P} \) is an isomorphism.

1) The adjunction \((\mathcal{T}^s_{\text{Br}}, \mathcal{P}^s_{\text{Br}})\) is idempotent.
2) The adjunction \((\mathcal{T}^{\bullet}_{Br})_1, (P^n_{Br})_1\) is idempotent, we can choose \((\mathcal{T}^{\bullet}_{Br})_2 := (\mathcal{T}_{Br})_1 U_{1,2}, \pi^n_2 = \text{Id}_{(\mathcal{T}^{\bullet}_{Br})_2}\) and \((\mathcal{T}^{\bullet}_{Br})_2\) is full and faithful i.e. \((\mathcal{T}^{\bullet}_{Br})_2\) is an isomorphism.

3) The functor \(P^n_{Br}\) has a monadic decomposition of monadic length at most two.

4) \((\text{Id}_{Br})_{s,0}, \Lambda_{Br} : ((\mathcal{T}^{\bullet}_{Br})_2, (P^n_{Br})_2) \rightarrow (\mathcal{T}^{\bullet}_{Br}, P^n_{Br})\) is a commutation datum whose canonical transformation is \(\overline{\lambda}_{Br}\).

5) The pair \((P^n_{Br})_2, \overline{\lambda}_{Br}\) is an adjunction with unit \(\overline{\pi}^n_{Br}\) and counit \((\overline{\pi}^n_{Br})_{2,1} \circ (P^n_{Br})_2 \overline{\lambda}_{Br}\) so that \(\Lambda_{Br}\) is full and faithful. Hence \(\overline{\pi}^n_{Br}\) is an isomorphism if and only if \((P^n_{Br})_2 \overline{\lambda}_{Br}, \Lambda_{Br}\) is an equivalence of categories. In this case \((\mathcal{T}^{\bullet}_{Br})_2, (P^n_{Br})_2\) identifies with \((\mathcal{T}^{\bullet}_{Br}, P^n_{Br})\) via \(\Lambda_{Br}\).

**Proof.** By 3.6 we have an adjunction \((\mathcal{T}^{\bullet}_{Br}, P^n_{Br})\). By Proposition 4.11, the right adjoint functor \(R = P^n_{Br}\) is comparable and we can use the notation of Definition 1.9.

Let \(M_2 = (M_1, \mu_1) \in (Br^s)_{M,2}\). Then we can write \(M_1 = (M_0, \mu_0) \in (Br^s)_{M,1}\) and \(M_0 = (M, c) \in Br^s_{M}\). Let \(\theta_{(M,c)} := \theta_{Br_{(M,c)}} : M \otimes M \rightarrow \Omega T(M)\) be defined as in (80) and set \(A := (s, m_s, u_s, s, s, c) := \overline{s}_{Br_{M}} M_{0} = \overline{T}_{Br_{M}} (M, c)\). Since \(c^2 = \text{Id}_{Br_{M}}\) we have that \(\theta_{(M,c)}\) fulfills (84). Thus there is a unique morphism \(\overline{\theta}_{(M,c)} : M \otimes M \rightarrow P(\overline{T}_{Br_{M}} (M, c))\) such that

\[
(87) \quad \xi_A \circ \overline{\theta}_{(M,c)} = \theta_{(M,c)}.
\]

Set

\[
[-] := H \overline{\pi}^n_{Br} \circ \overline{\theta}_{(M,c)} : M \otimes M \rightarrow M.
\]

Let us check that \((M, c, [-]) \in Br_{Lie}^s_{M}\). Now \(\mu_1 \circ \overline{\theta}^n_{Br}_{1})_1 M_1 = \text{Id}_{M_1}\) so that \((\overline{\theta}^n_{Br}_{1})_1 M_1\) is a split monomorphism. Set \(S := (s, m_s, u_s, s, s, c) := \overline{T}_{Br_{M}} (M_1)\). Thus \(H \overline{\pi}^n_{Br_{M}} U_{0,1} (\overline{\theta}^n_{Br})_1 M_1 : M \rightarrow P(S)\) is a split monomorphism too. Let \(\pi^n_1 : \overline{T}_{Br_{M}} U_{0,1} \rightarrow (\mathcal{T}^{\bullet}_{Br})_1\) be the canonical natural transformation defining \((\mathcal{T}^{\bullet}_{Br})_1\). By construction one has

\[
(88) \quad P^n_{Br_{M}} \pi^n_1 \circ \overline{\theta}_{Br_{(M,c)}} = U_{0,1} (\overline{\theta}^n_{Br})_1.
\]

We have

\[
(89) \quad H \xi_A \circ H \overline{\theta}^n_{Br_{M}} = H (\xi_{\overline{T}_{Br_{M}}} \circ \overline{\theta}^n_{Br_{M}}) \circ H \eta_{Br_{M}} = H \eta_{Br_{M}} = \eta H \overline{\theta}^n_{Br_{M}}.
\]

In particular, we have

\[
(90) \quad \xi_A \circ H \overline{\theta}^n_{Br_{M}} = \alpha_1 M
\]

so that

\[
(91) \quad \xi_A = H \overline{\theta}^n_{Br_{M}} = \alpha_1 M
\]

We compute

\[
(92) \quad \xi_A \circ [-]_{P(A)} \circ (H \overline{\theta}^n_{Br_{M}} M_0 \otimes H \overline{\theta}^n_{Br_{M}} M_0) = [-]_A \circ (\xi_A \circ \xi_A) \circ (H \overline{\theta}^n_{Br_{M}} M_0 \otimes H \overline{\theta}^n_{Br_{M}} M_0) = m_A \circ (\text{Id}_{M} - c_A) \circ \alpha_1 M \circ \alpha_1 M = m_A \circ (\alpha_1 M \circ \alpha_1 M) \circ (\text{Id}_{M} - c_A)
\]

Since \(\xi_A\) is a monomorphism we get

\[
(93) \quad \overline{\theta}_{(M,c)} = [-]_{P(A)} \circ (H \overline{\theta}^n_{Br_{M}} M_0 \otimes H \overline{\theta}^n_{Br_{M}} M_0).
\]
Moreover since \( \pi_i^1 M_1 : A = \overline{T}_{Br} \circ U_0,1 M_1 \to (\overline{T}_{Br})_1 M_1 = S \) is a morphism in BrBialg\(_A\), we have that \( H_{Br}^* P_{Br}^{\pi_1^1 M_1} \) \( H_{Br}^* H_{Br}^* \) \( P_{Br}^{\pi_1^1 M_1} \) commutes with lie brackets i.e.

\[
[-]_{P(S)} \circ (H_{Br}^* P_{Br}^{\pi_1^1 M_1} \otimes H_{Br}^* P_{Br}^{\pi_1^1 M_1}) = H_{Br}^* P_{Br}^{\pi_1^1 M_1} \circ [-]_{P(A)}
\]

Hence we get

\[
[-]_{P(S)} \circ (H_{Br}^* U_{0,1} (\overline{T}_{Br})_1 M_1) = H_{Br}^* U_{0,1} (\overline{T}_{Br})_1 M_1
\]

where in (*) we used that \( P_{Br}^{\pi_1^1 M_1} \otimes U_{0,1} M_1 \circ \mu_0 = P_{Br}^{\pi_1^1 M_1} \) which follows from \( \pi_i^1 M_1 \otimes \overline{T}_{Br} \circ \mu_0 = \pi_i^1 M_1 \circ \overline{T}_{Br} \circ \mu_0 \). We have so proved

\[
[-]_{P(S)} \circ (H_{Br}^* U_{0,1} (\overline{T}_{Br})_1 M_1) = H_{Br}^* U_{0,1} (\overline{T}_{Br})_1 M_1 \circ [-].
\]

Using the fact that \( H_{Br}^* U_{0,1} (\overline{T}_{Br})_1 M_1 \) is a monomorphism in \( A \) and

\[
(P(S), c_{P(S)}, [-]_{P(S)}) = P_{Br}^{\pi_1^1 M_1} \in BrLie_{A}^s,
\]

one easily checks that \( \Lambda_{Br} (M_2) := (M, c, [-]) \in BrLie_{A}^s \) and that \( H_{Br}^* U_{0,1} (\overline{T}_{Br})_1 M_1 : M \to P(S) \) is a morphism in BrBialg\(_A\). Let \( \nu : M_2 \to M'_2 \) be a morphism in \( (Br_{A}^s)_2 \). It is clearly a morphism of braided objects. Since, by [3], we have \( H_{Br}^* T_{Br} = H_{Br}^* H_{Br}^* \nu_B \nu_{Br} \), then \( H_{Br}^* P_{Br}^{\pi_1^1 M_1} \) \( \overline{T}_{Br} \cdot U_{0,2} \nu \) commutes with lie brackets and hence

\[
[-]_{P(A)} \circ (H_{Br}^* \nu_B U_{0,2} \nu \otimes H_{Br}^* U_{0,2} \nu) = H_{Br}^* \nu_B \nu_{Br} \nu_{Br} U_{0,2} \nu \circ [-].
\]

Thus \( \nu \) induces a morphism \( \Lambda_{Br} \nu \in BrLie_{A}^s \). It is clear that this defines a functor \( \Lambda_{Br} : (Br_{A}^s)_2 \to BrLie_{A}^s \) acting as the identity on morphisms. Let \( B := (B, m_B, u_B, \Delta_B, \varepsilon_B, c_B) \in BrBialg_{M} \). Set \( M_2 := (P_{Br}^s B) \). Then

\[
(M_1, (\mu_1) \quad [M_2] = ((P_{Br}^s B) \otimes (P_{Br}^s B) \otimes (\overline{T}_{Br})_1) B),
\]

\[
(M_0, (\mu_0) \quad [M_2] = ((P_{Br}^s B) \otimes (P_{Br}^s B) \otimes (\overline{T}_{Br})_1) B),
\]

\[
(M, c) \quad [M_2] = M_0 = P_{Br}^s B.
\]
Since $H_{BrLie}^s: \Lambda_{Br}$ and $U_{0.2}$ act as the identity on morphisms, we get $H_{BrLie}^s \circ \Lambda_{Br} = U_{0.2}$.

In view of (91), naturality of $\xi$, the equality (*) used above and (87) we obtain

\[ H_{Br}^s \Omega_{Br}^s \pi_1^s M_1 \circ \alpha_{11} M \circ [-] = H_{Br}^s \Omega_{Br}^s \tilde{U}_{Br} \pi_1^s M_1 \circ \theta_{(M,c)}. \]

Thus we get $H_{Br}^s \Omega_{Br}^s \tilde{U}_{Br} \pi_1^s M_1 \circ f_{ABr} M_2 = 0$.

Since $\pi_1^s M_1$ is an algebra map, we have $H_{Br}^s \Omega_{Br}^s \tilde{U}_{Br} \pi_1^s M_1 \circ i_{f_{ABr}} M_2 = 0$ so that, by construction of $\tilde{U}_{Br}$, there is an braided algebra morphism $\chi_{Br}^s M_2 : U_{Br}^s \Lambda_{Br} M_2 \rightarrow \tilde{U}_{Br} \left( T_{Br} \right)_1 M_1$ such

\[ \chi_{Br}^s M_2 \circ p^s \Lambda_{Br} M_2 = U_{Br}^s \pi_1^s M_1 = U_{Br}^s \pi_1^s U_{1.2} M_2. \]

By naturality of the other terms we obtain that also $\chi_{Br}^s M_2$ is natural in $M_2$ so that we get

\[ \chi_{Br}^s \circ p^s \Lambda_{Br} = U_{Br}^s \pi_1^s U_{1.2}. \]

By (73) we get $\chi_{Br}^s \circ \tilde{U}_{Br} \eta_{Br}^s \Lambda_{Br} = U_{Br}^s \pi_1^s U_{1.2}$. Since both $\tilde{U}_{Br} \eta_{Br}^s$ and $\pi_1^s U_{1.2}$ are morphism of braided bialgebras and the underlying morphism in $M$ of $\pi^s$ is $p$ which is an epimorphism, one gets that $\chi_{Br}^s$ is a morphism of braided bialgebras too that will be denoted by $\chi_{Br}^s : U_{Br}^s \Lambda_{Br} \rightarrow \left( T_{Br} \right)_1 U_{1.2}$. Thus $\tilde{U}_{Br} \chi_{Br}^s = \chi_{Br}^s$ and hence

\[ \chi_{Br}^s \circ \tilde{U}_{Br} = \pi_1^s U_{1.2}. \]

A direct computation shows that $\eta_{Br}^s = \tilde{U}_{Br} \circ \tilde{U}_{Br} \eta_{Br}^s$ and hence

\[ \eta_{Br}^s = \tilde{U}_{Br} \circ \tilde{U}_{Br}. \]

Thus, using (73), naturality of $\xi$, (80), (73), (73), (94), again naturality of $\xi$ and (88) in the given order, we get

\[ \xi \left( T_{Br}^s \right)_1 U_{1.2} \circ H_{BrLie}^s \left( P_{Br}^s \tilde{U}_{Br} \Lambda_{Br} \circ \tilde{U}_{Br} \Lambda_{Br} \right) = \xi \left( T_{Br}^s \right)_1 U_{0.1} \circ \left( \tilde{U}_{Br} \right)_1 U_{1.2}. \]

Therefore, we obtain

\[ H_{BrLie}^s \left( P_{Br}^s \tilde{U}_{Br} \Lambda_{Br} \circ \tilde{U}_{Br} \Lambda_{Br} \right) = U_{0.1} \left( \tilde{U}_{Br} \right)_1 U_{1.2}. \]

The latter is a split monomorphism. Since $H_{BrLie}$ is faithful, it follows that we get the evaluation on objects of $P_{Br}^s \tilde{U}_{Br} \circ \tilde{U}_{Br} \Lambda_{Br}$ is a monomorphism.

Assume that $\tilde{U}_{Br} \Lambda_{Br}$ is an isomorphism. Note that $\tilde{U}_{Br} \Lambda_{Br}$ isomorphism implies $\tilde{U}_{Br} \Lambda_{Br} \left( P_{Br}^s \right)_2$ isomorphism. Since $\tilde{U}_{Br} = P_{Br}^s \left( P_{Br}^s \right)^2$ this means that $\tilde{U}_{Br} \Lambda_{Br}$ is an isomorphism and hence the adjunction $(\tilde{U}_{Br}, P_{Br}^s)$ is idempotent, cf. [MS] Proposition 2.8]. Moreover, since $\tilde{U}_{Br} \Lambda_{Br}$ is an isomorphism, then the evaluation of $P_{Br}^s \tilde{U}_{Br} : P_{Br}^s \tilde{U}_{Br} \Lambda_{Br} \rightarrow P_{Br} \left( T_{Br} \right)_1 U_{1.2}$ is a monomorphism.

Let $M_2 \in (Br_{M})_2$ and consider the coequalizer

\[ T_{Br}^s P_{Br}^s \tilde{T}_{Br}^s M_0 \xrightarrow{T_{Br}^s \mu_0} T_{Br}^s M_0 \xrightarrow{\pi_1^s M_1} \left( T_{Br}^s \right)_1 M_1. \]

Then, from $\chi_{Br}^s \circ \tilde{U}_{Br} \Lambda_{Br} = \pi_1^s U_{1.2}$, we get

\[ \chi_{Br}^s M_2 \circ \tilde{T}_{Br}^s \Lambda_{Br} M_2 \circ \tilde{U}_{Br}^s \mu_0 = \chi_{Br}^s M_2 \circ \tilde{U}_{Br} \Lambda_{Br} M_2 \circ \tilde{U}_{Br}^s T_{Br} M_0. \]

If we apply $P_{Br}^s$, from the fact that $P_{Br}^s \Lambda_{Br} M_2$ is a monomorphism, we obtain

\[ P_{Br}^s \left( \tilde{U}_{Br} \Lambda_{Br} M_2 \circ \tilde{T}_{Br}^s \mu_0 \right) = P_{Br}^s \left( \tilde{U}_{Br} \Lambda_{Br} M_2 \circ \tilde{U}_{Br}^s T_{Br} M_0 \right). \]

If we apply on both sides $H_{BrLie}^s$, by (73), we obtain

\[ P_{Br}^s \left( \tilde{U}_{Br} \Lambda_{Br} M_2 \circ \tilde{T}_{Br}^s \mu_0 \right) = P_{Br}^s \left( \tilde{U}_{Br} \Lambda_{Br} M_2 \circ \tilde{U}_{Br}^s \tilde{T}_{Br}^s M_0 \right). \]

Since $(\tilde{U}_{Br}, P_{Br}^s)$ is idempotent, by [MS] Proposition 2.8], we also have that $\tilde{U}_{Br}^s \tilde{T}_{Br}^s$ is an isomorphism. Note that the arguments of $P_{Br}^s$ in the above displayed equality are morphisms of the form $T_{Br} X \rightarrow Y$ for some objects $X, Y$. Given two such morphisms $f, g : T_{Br} X \rightarrow Y$ with $P_{Br}^s f = P_{Br}^s g$ we have

\[ f = f \circ \tilde{T}_{Br}^s X \circ \tilde{U}_{Br}^s \eta_{Br}^s X = \tilde{T}_{Br}^s Y \circ \tilde{T}_{Br}^s P_{Br}^s f \circ \tilde{T}_{Br}^s \eta_{Br}^s X. \]
and hence η \circ \Upsilon_{BrM} = \Upsilon_{\Lambda Br} \circ \eta \circ \Upsilon_{BrM} = g \circ \tau_{Br} \cdot \Upsilon_{BrM} \circ g = g.

In our case we get \( \mathcal{P} \Lambda_{Br} M_2 \circ \Upsilon_{BrM} = \mathcal{P} \Lambda_{Br} M_2 \circ \tau_{Br} \cdot \Upsilon_{BrM} \). By the universal property of the coequalizer above, there is a braided bialgebra morphism \( \tau M_2 : (\Upsilon_{Br})_{1}^{1} \rightarrow \Upsilon_{BrM} \Lambda_{Br} M_2 \) such that

\[ \tau M_2 \circ \pi_1^1 M_1 = \overline{\mathcal{P}} \Lambda_{Br} M_2. \]

Note that, by Proposition \textbf{B.1}, the morphism \( \pi_1^1 M_1 \) can be chosen in such a way to be a coequalizer when regarded as a morphism in \( \mathcal{M} \). We already observed that \( \mathcal{P} \) is also an epimorphism in \( \mathcal{M} \). Using these facts one easily checks that \( \Upsilon_{BrM} M_2 \) and \( \tau M_2 \) are mutual inverses and hence \( \Upsilon_{Br} : \Upsilon_{BrM} \Lambda_{Br} \rightarrow (\Upsilon_{Br})_{1}^{1} \) is an isomorphism.

Therefore \( U_{0,1} (\overline{\mathcal{P}}_{BrL})_{1} U_{1,2} = H_{BrLie}^{s} (\mathcal{P}_{Br} \Upsilon_{Br} \circ \eta_{BrL} \Lambda_{Br}) \) is an isomorphism. Since \( U_{0,1} \) reflects is an isomorphism, we conclude that \( (\overline{\mathcal{P}}_{BrL})_{1} U_{1,2} \) is an isomorphism. We have so proved that the adjunction \( (\Upsilon_{Br})_{1}^{1} , (\mathcal{P}_{Br})_{1}^{1} \) is idempotent. Note that in this case we can choose \( (\Upsilon_{Br})_{1}^{1} U_{1,2} \) (and \( \pi^2 \) to be the identity) and it is full and faithful (cf. \textbf{AGM}, Proposition 2.3) i.e. \( (\overline{\mathcal{P}}_{BrL})_{1} \) is an isomorphism. By the quoted result we also have \( (\overline{\mathcal{P}}_{BrL})_{1} U_{1,2} = U_{1,2} (\overline{\mathcal{P}}_{BrL})_{1} \) so that

\[ H_{BrLie}^{s} (\mathcal{P}_{Br} \Upsilon_{Br} \circ \eta_{BrL} \Lambda_{Br}) = U_{0,1} (\overline{\mathcal{P}}_{BrL})_{1} U_{1,2} \]

and hence \( \mathcal{P}_{Br} \Upsilon_{Br} \circ \eta_{BrL} \Lambda_{Br} = \Lambda_{Br} (\overline{\mathcal{P}}_{BrL})_{1} \). This proves \textbf{[3]} holds i.e. that \( (\mathbb{I}d_{BrBialg_{M}}, \Lambda_{Br}) : \left( (\Upsilon_{Br})_{1}^{1}, (\mathcal{P}_{Br})_{1}^{1} \right) \rightarrow \left( \Upsilon_{BrL}, \Upsilon_{BrM} \right) \) is a commutation datum whose canonical transformation is \( \Upsilon_{BrL} \).

Let us check that \( (\mathcal{P}_{BrM} \Upsilon_{BrL}, \Lambda_{Br}) \) is an adjunction with unit and counit as in the statement. We have

\[ \Lambda_{Br} (\overline{\mathcal{P}}_{BrL})_{1}^{-1} \circ (\mathcal{P}_{BrM} \Upsilon_{BrL} \Lambda_{Br} = \Lambda_{Br} (\overline{\mathcal{P}}_{BrL})_{1}^{-1} \circ \Lambda_{Br} (\mathcal{P}_{BrM} \Upsilon_{BrL} \Lambda_{Br} = \Lambda_{Br} (\overline{\mathcal{P}}_{BrL})_{1}^{-1} \circ \Lambda_{Br} (\overline{\mathcal{P}}_{BrL})_{1} \equiv \Lambda_{Br}. \]

Moreover, by \textbf{[3]} applied to our commutation datum, we have \( (\mathcal{P}_{BrM} \Upsilon_{BrL} \Lambda_{Br})_{2} = \Upsilon_{BrL} \) so that

\[
\left( (\overline{\mathcal{P}}_{BrL})_{2}^{-1} \circ (\mathcal{P}_{BrM} \Upsilon_{BrL} \Lambda_{Br})_{2} \right) = (\mathcal{P}_{BrM} \Upsilon_{BrL} \Lambda_{Br})_{2} = \Upsilon_{BrL} \]

Note that the counit is an isomorphism so that \( \Lambda_{Br} \) is full and faithful.

It is then clear that \( (\mathcal{P}_{BrM} \Upsilon_{BrL}, \Lambda_{Br}) \) is an equivalence of categories if and only if \( \Upsilon_{BrL} \) is an isomorphism (see e.g. \textbf{Boi}, Proposition 3.4.3). \( \Box \)

**Theorem 7.2.** Let \( \mathcal{M} \) be an abelian symmetric monoidal category with denumerable coproducts. Assume that the tensor functors are exact and preserve denumerable coproducts.

\[
\begin{array}{llll}
\mathcal{M} & \xrightarrow{J_2} & (\text{BrLie}_\mathcal{M})_2 & \xrightarrow{\Lambda_2} \\
\Lambda & \xrightarrow{\Lambda^r} & \Lambda_{Br} & \xrightarrow{\Lambda} \\
\text{Lie}_\mathcal{M} & \xrightarrow{J^r_{\text{Lie}}} & \text{BrLie}_\mathcal{M} & \xrightarrow{\Lambda} \\
\end{array}
\]

\[
\begin{array}{llll}
\mathcal{M} & \xrightarrow{\mathcal{P}} & \text{Bialg}_\mathcal{M} & \xrightarrow{\mathcal{P}} \\
\mathcal{M} & \xrightarrow{\mathcal{P}} & \text{Bialg}_\mathcal{M} & \xrightarrow{\mathcal{P}} \\
\text{Bialg}_\mathcal{M} & \xrightarrow{\mathcal{P}} & \text{Bialg}_\mathcal{M} & \xrightarrow{\mathcal{P}} \\
\text{Bialg}_\mathcal{M} & \xrightarrow{\mathcal{P}} & \text{Bialg}_\mathcal{M} & \xrightarrow{\mathcal{P}} \\
\end{array}
\]
The functor $P$ is comparable so that we can use the notation of Definition \ref{def:comparable}. We have $H_{\text{Lie}} P = P$ and there is a functor $\Lambda : \mathcal{M}_2 \to \text{Lie}_M$ such that $\Lambda_{\text{Br}} J_2^s = J_{\text{Lie}}^s \Lambda$, $\Lambda \circ P_2 = P$ and $H_{\text{Lie}} \circ \Lambda = U_{0,2}$. Moreover there exists a natural transformation $\eta : \overline{\Pi} \Lambda \to T_1 U_{1,2}$ such that such that

$$J^s_{\text{Bialg}} \overline{\eta} = \xi_1^1 U_{1,2} \circ \overline{\chi}_{\text{Br}} J_2^s,$$

where $\overline{\eta}$ is the natural transformation of Theorem \ref{thm:main} and $\pi_1 : T U_{0,1} \to T_1$ is the canonical natural transformation defining $T_1$.

Assume $\overline{\eta}_{\text{BrL}} \Lambda_{\text{Br}}$ is an isomorphism.

1) The adjunction $(\overline{T}, P)$ is idempotent.
2) The adjunction $(\overline{T}_1, P_1)$ is idempotent, we can choose $\overline{T}_2 := \overline{T}_1 U_{1,2}$, $\pi_2 = \text{Id}_{\overline{T}_2}$ and $\overline{T}_2$ is full and faithful i.e. $\overline{\eta}_2$ is an isomorphism.
3) The functor $P$ has a monadic decomposition of monadic length at most two.
4) $(\operatorname{Id}_{\text{Bialg}_M}, \Lambda) : (\overline{T}_2, P_2) \to (\overline{U}, P)$ is a commutation datum whose canonical transformation is $\overline{\chi}$.
5) The pair $(P_2 \overline{U}, \Lambda)$ is an adjunction with unit $\overline{\eta}_L$ and counit $(\overline{\eta}_L)^{-1} \circ P_2 \overline{\chi}$ so that $\Lambda$ is full and faithful. Hence $\overline{\eta}_L$ is an isomorphism if and only if $(P_2 \overline{U}, \Lambda)$ is an equivalence of categories. In this case $(\overline{T}_2, P_2)$ identifies with $(\overline{U}, P)$ via $\Lambda$.
6) If $\overline{\eta}_{\text{BrL}}$ is an isomorphism so is $\overline{\eta}_L$.

Proof. We have

$$J^s H_{\text{Lie}} P \cong H^0_{\text{BrL}} J^s_{\text{Lie}} P \oplus H^0_{\text{BrL}} J^s_{\text{Bialg}} \oplus H^0_{\text{BrL}} J^s_{\text{Bialg}} \oplus P^s J^s_{\text{Bialg}} \oplus J^s P$$

so that $H_{\text{Lie}} P = P$. By Proposition \ref{prop:commutation} $(J^s_{\text{Bialg}}, J^s) : (\overline{T}, P) \to (\overline{T}^s_{\text{Br}}, P^s_{\text{Br}})$ is a commutation datum. Moreover, by Lemma \ref{lem:adjunction} $J^s_{\text{Bialg}} : \text{Bialg}_M \to \text{BrBialg}_M$ preserves coequalizers. By Proposition \ref{prop:comparable}, the right adjoint functor $R = P^s_{\text{Br}}$ is comparable and we can use the notation of Definition \ref{def:comparable}. By Lemma \ref{lem:adjunction} and Lemma \ref{lem:comparable} we have that $P$ is also comparable. Applying iteratively Proposition \ref{prop:adjunction} we get functors $J^s_n : M_n \to (\text{BrBialg}_M)_n$, for all $n \in \mathbb{N}$, such that $J^s_n \circ P_n = (P^s_{\text{Br}})_n \circ J^s_{\text{Bialg}}$. Let $M_2 \in \mathcal{M}_2$ and consider $\Lambda_{\text{Br}} J^s_2 M_2$. Note that, by construction we have

$$J^s_2 M_2 = (J^s_1 M_1, J^s_1 \mu_1 \circ (P^s_{\text{Br}}) \xi_1^1 M_1) \quad \text{and} \quad J^s_1 M_1 = (J^s_1 M_0, J^s_1 \mu_0 \circ (P^s_{\text{Br}}) \xi_0^1 M_0)$$

where $\xi_i : (\overline{T}_1)^{\prime} \to J^s_{\text{Bialg}} \overline{T}_1$, for $i = 0, 1$ are the canonical transformations of the respective commutation data. By construction we have $\Lambda_{\text{Br}} J^s_2 M_2 = (M_0, c_{M_0, M_0}, [-])$ where

$$[-] := H^3_{\text{BrL} \times \text{Lie}_M} J^s_0 \circ H^3_{\text{BrL} \times \text{Lie}_M} P^s_{\text{BrL}} \circ \overline{\theta}_{(M_0, c_{M_0, M_0})} = \mu_0 \circ H^3_{\text{BrL} \times \text{Lie}_M} P^s_{\text{BrL}} \circ \overline{\theta}_{M_0}.$$ 

Now $\Lambda_{\text{Br}} J^s_2 M_2 \in \text{BrLie}_M^s$ so that $(M_0, c_{M_0, M_0}, [-]) \in \text{Lie}_M$ i.e. $(M_0, [-]) \in \text{Lie}_M$ and $\Lambda_{\text{Br}} J^s_2 M_2 = J^s_{\text{Lie}} (M_0, [-])$. Thus any object in the image of $\Lambda_{\text{Br}} J^s_2$ is also in the image of $J^s_{\text{Lie}}$. Thus, by Lemma \ref{lem:adjunction} there is a unique functor $\Lambda : \mathcal{M}_2 \to \text{Lie}_M$ such that $\Lambda_{\text{Br}} J^s_2 = J^s_{\text{Lie}} \Lambda$. This equality implies that $\Lambda$ acts as the identity on morphisms and that

$$\Lambda M_2 = (M_0, [-]).$$

Note that, by Proposition \ref{prop:adjunction} we have $\xi_0^1 = \text{Id}_{\text{BrL}^s}$, so that we obtain

$$[-] := \mu_0 \circ \overline{\theta}_{M_0}.$$ 

We have

$$J^s_{\text{Lie}} \Lambda P_2 = \Lambda_{\text{Br}} J^s_2 P_2 = \Lambda_{\text{Br}} (P^s_{\text{Br}}) J^s_{\text{Bialg}} \oplus P^s J^s_{\text{Bialg}} \oplus J^s P$$

Since $J^s_{\text{Lie}}$ is both injective on morphisms and objects, we get $\Lambda P_2 = P$. It is clear that $H_{\text{Lie}} \Lambda = U_{0,2}$. We have

$$J^s_{\text{Bialg}} \overline{\Pi} \Lambda \cong \overline{\Pi} \Lambda \circ \overline{\chi}_{\text{Br}} J_2^s$$

so that $\overline{\Pi} \Lambda \circ \overline{\chi}_{\text{Br}} J_2^s = \overline{\Pi} \Lambda$. Thus, by Lemma \ref{lem:natural_transformation} there is a natural transformation $\overline{\chi} := \xi_1^1 U_{1,2} \circ \overline{\chi}_{\text{Br}} J_2^s : \overline{\Pi} \Lambda \to \overline{T}_1 U_{1,2}$ such that $J^s_{\text{Bialg}} \overline{\chi} = \xi_1^1 U_{1,2} \circ \overline{\chi}_{\text{Br}} J_2^s$. We compute

$$(97) \quad J^s_{\text{Lie}} \overline{\eta}_{\text{BrL}} \Lambda \cong \overline{\eta}_{\text{BrL}} \Lambda_{\text{Br}} J^s_2.$$
so that
\[ J^s H_{\text{Lie}} (P\chi \circ \eta_L A) = J^s H_{\text{Lie}} P\chi \circ J^s H_{\text{Lie}} \eta_L A \]
\[ = P^s Bialg \chi \circ H_{\text{BrL}} \eta_L A \]
\[ = P^s_B r \circ \eta_{\text{BrL}} \circ H_{\text{BrL}} \eta_L A \]
\[ = P^s_B r \circ \eta_{\text{BrL}} \circ H_{\text{BrL}} \eta_L A \]
\[ = P^s_B r \circ \eta_{\text{BrL}} \circ H_{\text{BrL}} \eta_L A \]
\[ \text{We have} \]
\[ J^s_Bialg (\chi \circ \eta \Lambda) = J^s_Bialg \chi \circ J^s_Bialg \eta \Lambda \]
\[ = \chi^s U_{1,2} \circ \eta_{\text{BrL}} J^s \circ \eta \Lambda \]
\[ = \chi^s U_{1,2} \circ \eta_{\text{BrL}} J^s \circ \eta \Lambda \]
\[ = \chi^s U_{1,2} \circ \eta_{\text{BrL}} J^s \circ \eta \Lambda \]
\[ = \chi^s U_{1,2} \circ \eta_{\text{BrL}} J^s \circ \eta \Lambda \]
\[ \text{where} (*) \text{ follows by construction of} \chi^s \text{ (see the proof of Proposition 2.3). Thus we obtain} \]
\[ \text{we deduce} \]
\[ \text{Assume} \]
\[ \text{Moreover, since} \]
\[ \text{Moreover, since} \]
\[ \text{We already observed that} \]
\[ \text{Moreover, from} \]
\[ \text{We know that} J^s_Bialg \text{ is full, faithful and injective on objects. Since} J^s \text{ fulfills the same properties, by Proposition 2.3 applied to the commutation datum} \]
\[ \text{Thus} \]
\[ \text{Since} F \text{ and} G \text{ are both conservative, we get that} \epsilon \text{ and} \eta \text{ are an isomorphism whenever} \]
\[(\eta_{Br})_2 \text{ is an isomorphism by Theorem 7.1-2). Using this equality we compute}
\]
\[
J_2 \left( (\eta_2)^{-1} \circ P_2 \chi \right) = J_2 (\eta_2)^{-1} \circ J_2 P_2 \chi = J_2 (\eta_2)^{-1} \circ (P^s_{Br})_2 J^s_{Bialg} \chi
\]
\[
= J_2 (\eta_2)^{-1} \circ (P^s_{Br})_2 (\zeta^s_1 U_{1,2} \circ \chi J_2)
\]
\[
= ((P^s_{Br})_2 \zeta^s_2 \circ (\eta_{Br})_2) J_2^{-1} \circ (P^s_{Br})_2 \zeta^s_1 U_{1,2} \circ (P^s_{Br})_2 \chi J_2^s
\]
\[
= (\eta_{Br})_2^{-1} \cdot J_2 \circ (P^s_{Br})_2 \zeta^s_2^{-1} \circ (P^s_{Br})_2 \zeta^s_1 U_{1,2} \circ (P^s_{Br})_2 \chi J_2^s.
\]

Now, by construction of \(\zeta^s_2\) (see the proof of Proposition 7.3), the fact that \(\pi_2 : T_1 U_{1,2} \to T_2\) is the identity and that also \(\pi_2^s\) is the identity (see Theorem 7.1-2)), we have that \(\zeta^s_2 = \zeta_1^s U_{1,2}\) and hence
\[
F \left( (\eta_2)^{-1} \circ P_2 \chi \right) = J_2 \left( (\eta_2)^{-1} \circ P_2 \chi \right) = (\eta^s_{Br})_2^{-1} \circ (P^s_{Br})_2 \chi J_2^s = \left( (\eta^s_{Br})_2^{-1} \circ (P^s_{Br})_2 \chi J_2^s \right) J_2^s = \epsilon' F.
\]

Thus, by (11), we get \(\epsilon = (\eta_2)^{-1} \circ P_2 \chi\). \(\square\)

**Definition 7.3.** An **MM-category** (Milnor-Moore-category) is an abelian monoidal category \(\mathcal{M}\) with denumerable coproducts such that the tensor functors are exact and preserve denumerable coproducts and such that the unit functorial isomorphism i.e. the functor \(\eta_{U_{1,2}}\) of the adjunction \((\eta_{Br}, \pi_{Br})\) is a functorial isomorphism i.e. the functor \(\eta_{Br}^s : BrLie^s_{\mathcal{M}} \to BrBialg^s_{\mathcal{M}}\) is full and faithful (see e.g. [Bol], dual of Proposition 3.4.1, page 114)).

**Remark 7.4.** 1) The celebrated Milnor-Moore Theorem, cf. [MM] Theorem 5.18 states that, in characteristic zero, there is a category equivalence between the category of Lie algebras and the category of primitively generated bialgebras. The fact that the counit of the adjunction involved is an isomorphism just encodes the fact that the bialgebras considered are primitively generated. On the other hand the crucial point in the proof is that the unit of the adjunction is an isomorphism.

In our wider context this translates to the unit of the adjunction \((\eta_{Br}^s, \pi_{Br}^s)\) being a functorial isomorphism. From this the definition of MM-category stems. Note that for an MM-category \(\mathcal{M}\) we can apply Theorem 7.1 to obtain that the functor \(\pi_{Br}^s\) has a monadic decomposition of monadic length at most two. Moreover we can identify the category \((Br^s_{\mathcal{M}})_2\) with \(BrLie^s_{\mathcal{M}}\).

2) In the case of a symmetric MM-category \(\mathcal{M}\) the connection with Milnor-Moore Theorem becomes more evident. In fact, in this case, we can apply Theorem 7.2 to obtain that the unit of the adjunction \((\eta, \pi)\) is a functorial isomorphism.

**8. Lifting the structure of MM-category**

We first prove a crucial result for braided vector spaces.

**Theorem 8.1.** The category of vector spaces over a fixed field \(k\) of characteristic zero is an MM-category.

**Proof.** Let \(\mathcal{M} = \mathfrak{M}\) be the category of vector spaces over \(k\). We have just to prove that \(\eta_{BrL}^s\) is an isomorphism. Let \((M, c, [-]) \in BrLie^s_{\mathcal{M}}\). Since we are working on vector spaces, we can express explicitly the universal enveloping algebra \(\eta_{Br}^s (M, c, [-])\) with elements as follows

\[
\eta_{Br}^s (M, c, [-]) = \frac{\mathfrak{T}_c^s (M, c)}{\langle [x \otimes y] - x \otimes y + c (x \otimes y) \mid x, y \in M \rangle}.
\]

By Lemma 5.3 \((M, [-])\) is a Lie \(c\)-algebra and \(\eta_{Br}^s (M, c, [-])\) coincides with the corresponding universal enveloping algebra in the sense of [Kh], Section 2.5. Hence we can apply [Kh], Lemma 6.2] to conclude that the canonical map from \(M\) into the primitive part of \(\eta_{Br}^s (M, c, [-])\) is an isomorphism. In our notation this means that

\[
H_{BrLie}^s H_{BrLie}^s BrLie^s_{\mathcal{M}} (M, c, [-]) : M \to H_{BrLie}^s H_{BrLie}^s BrLie^s_{\mathcal{M}} (M, c, [-])
\]

is bijective. Note that \(H_{BrLie}^s \) and \(H_{BrLie}^s \) are conservative by [3.3], Definition 3.3, and Definition 5.3 respectively. Thus \(H_{BrLie}^s \) is conservative and hence we get that \(\eta_{BrL}^s (M, c, [-])\) is an isomorphism for all \((M, c, [-]) \in BrLie^s_{\mathcal{M}}\). We have so proved that \(\eta_{BrL}^s\) is an isomorphism. \(\square\)
In the rest of this section we will deal with symmetric braided monoidal categories $\mathcal{M}$ endowed with a faithful monoidal functor $W : \mathcal{M} \to \mathcal{M}^H$ which is not necessarily braided. The examples we will treat take $\mathcal{M} = \mathcal{M}^H$ for a dual quasi-bialgebra $H$ or $\mathcal{M} = \mathcal{M}^H$ for a quasi-bialgebra case. Note that in general the obvious forgetful functors need not to be monoidal, see e.g. [Ma] Example 9.1.4 so that further conditions will be required on $H$. Note that the results on $\mathcal{M}^H$ and $\mathcal{M}^H$ are not dual each other, unless $H$ is finite-dimensional.

**Lemma 8.2.** Let $\mathcal{M}$ and $\mathcal{N}$ be monoidal categories. Any monoidal functor $(F, \phi_0, \phi_2) : \mathcal{M} \to \mathcal{N}$ induces a functor $\text{BrLie}^s F : \text{BrLie}_\mathcal{M} \to \text{BrLie}_\mathcal{N}$ which acts as $F$ on morphisms and such that $\text{BrLie}^s F (M, c_M, [-]_M) := (FM, c_{FM}, [-]_{FM})$ where $(FM, c_{FM}) = \text{BrF} (M, c_M)$ and $[-]_{FM} := F [-]_M \circ \phi_2 (M, M) : FM \otimes FM \to F (M)$.

Moreover the first diagram below commutes and there is a unique functor $\text{BrLie}^s F$ such that the second diagram commutes.

\[
\begin{array}{cccc}
\text{BrLie}_\mathcal{M} & \text{BrLie}_\mathcal{N} \\
\text{BrF} & \text{BrF} \\
\text{Br}_{\mathcal{M}} & \text{Br}_{\mathcal{N}}
\end{array}
\]

Furthermore the functors $\text{BrLie}^s F$ and $\text{BrLie}^s F$ are conservative whenever $F$ is.

**Proof.** It is straightforward.

**Theorem 8.3.** Let $\mathcal{M}$ and $\mathcal{N}$ be monoidal categories. Assume that both $\mathcal{M}$ and $\mathcal{N}$ are abelian with denumerable coproducts, and that the tensor functors are exact and preserves denumerable coproducts. Assume that there exists an exact monoidal functor $(F, \phi_0, \phi_2) : \mathcal{M} \to \mathcal{N}$ which preserves denumerable coproducts. Then we have the following commutation data with the respective canonical transformations

\[
\begin{array}{l}
\text{(BrAlg}^s F, \text{BrLie}^s F) : (U_{Br}^s, \mathcal{L}_{Br}^s) \to (U_{Br}^s, \mathcal{L}_{Br}^s), \quad \zeta_{\text{BrL}} : U_{Br}^s (\text{BrLie}^s F) \to (\text{BrAlg}^s F) U_{Br}^s, \\
\text{(BrBialg}^s F, \text{BrLie}^s F) : (\mathcal{U}_{Br}^s, \mathcal{P}_{Br}^s) \to (\mathcal{U}_{Br}^s, \mathcal{P}_{Br}^s), \quad \zeta_{\text{BrL}} : \mathcal{U}_{Br}^s (\text{BrLie}^s F) \to (\text{BrBialg}^s F) \mathcal{U}_{Br}^s.
\end{array}
\]

**Proof.** A direct computation using (99) shows that

\[
\begin{array}{l}
\text{BrLie}_\mathcal{M} (\text{BrLie}^s F) \mathcal{L}_{Br}^s (B, mB, uB, cB) = \text{BrLie}_\mathcal{M} (\text{BrAlg}^s F) (B, mB, uB, cB). \\
\end{array}
\]

Since both functors act as $F$ on morphisms, we get $\text{BrLie}_\mathcal{M} (\text{BrLie}^s F) \mathcal{L}_{Br}^s = \text{BrLie}_\mathcal{M} (\text{BrAlg}^s F)$.

Since $\text{BrLie}_\mathcal{M}$ is both injective on morphisms and objects we obtain

\[
\begin{array}{l}
(\text{BrLie}^s F) \mathcal{L}_{Br}^s = (\text{BrAlg}^s F). \\
\end{array}
\]

Now, using in the given order (5), (9), again (5), (11) and again (9), we get the equality $H_{\text{BrLie}} (\text{BrLie}^s F) \xi = H_{\text{BrLie}} (\text{BrAlg}^s F)$. Then one shows that $(\text{BrAlg}^s F) \xi$ and $(\text{BrBialg}^s F)$ has the same domain and codomain. Thus, from $H_{\text{BrLie}} (\text{BrAlg}^s F) \xi = H_{\text{BrLie}} (\text{BrBialg}^s F)$ we deduce that

\[
(\text{BrAlg}^s F) \xi = (\text{BrBialg}^s F) .
\]

Consider the natural transformation $\zeta_{\text{BrL}} : \mathcal{U}_{Br}^s (\text{BrLie}^s F) \to (\text{BrBialg}^s F) \mathcal{U}_{Br}^s$ of Lemma 2.2. By definition

\[
\zeta_{\text{BrL}} := \zeta_{\text{BrL}} (\text{BrBialg}^s F) \mathcal{U}_{Br}^s \circ \mathcal{U}_{Br}^s (\text{BrLie}^s F) \mathcal{U}_{Br}^s.
\]

It is straightforward to check that

\[
\begin{array}{l}
\text{BrLie}_\mathcal{M} (\text{BrLie}^s F) = \zeta_{\text{BrL}}.
\end{array}
\]

where $\zeta_{\text{BrL}} : \mathcal{U}_{Br}^s (\text{BrLie}^s F) \to (\text{BrAlg}^s F) U_{Br}^s$ is the canonical morphism of Lemma 2.2 and also

\[
\zeta_{\text{BrL}} \circ \mathcal{P}_{Br}^s (\text{BrLie}^s F) = (\text{BrAlg}^s F) \mathcal{P}_{Br}^s \circ \zeta_{\text{BrL}} H_{\text{BrLie}}^s.
\]
Let \((M, cM, [-]_M)\) ∈ \(\text{BrLie}^s_M\). Then we have that \((M \otimes M, cM \otimes M)\) ∈ \(\text{BrLie}^s_M\) where \(cM \otimes M \) := \((M \otimes cM) \otimes M \) \((M \otimes cM) \otimes M\). It is easy to check that \([-] : M \otimes M \to M\) and \(\theta(M, cM) : M \otimes M \to \Omega^s_{\text{Br}} T^s_{\text{Br}} (M, cM)\) such that

\[-]^s : (M \otimes M, cM \otimes M) \to (M, cM) \quad \text{and} \quad \theta^s(M, cM) : (M \otimes M, cM \otimes M) \to \Omega^s_{\text{Br}} T^s_{\text{Br}} (M, cM)

Let us check that the following is a coequalizer in \(\text{BrAlg}^s_M\)

\[(101) \quad T^s_{\text{Br}} (M \otimes M, cM \otimes M) \xrightarrow{T^s_{\text{Br}} [-]^s} T^s_{\text{Br}} (M, cM) \xrightarrow{\theta^s(M, cM)} T^s_{\text{Br}} (M, cM) \otimes \Omega^s_{\text{Br}} T^s_{\text{Br}} (M, cM).

Apply \(H_{\text{Alg}}^s \Pi^s_{\text{BrAlg}}\) to this diagram we get the diagram

\[(102) \quad T(M \otimes M) \xrightarrow{T[-]} T(M) \xrightarrow{H_{\text{Alg}}^s \Pi^s_{\text{BrAlg}}(M, cM \otimes [-]_M)} H_{\text{Alg}}^s \Pi^s_{\text{BrAlg}} \Pi^s_{\text{BrLie}} (M, cM, [-]_M).

which can be checked to be a coequalizer in \(\text{Alg}^s_M\). By Lemma 3.6 we have that \(H_{\text{Alg}}^s\) reflects coequalizers and by [Bo1, Proposition 2.9.9], we have that \(\Pi^s_{\text{BrAlg}}\) reflects coequalizers. Thus \(101\) is also a coequalizer. By Lemma 3.10 since \(F\) preserves coequalizers, we get that \(\text{Alg} F\) preserves the coequalizer \(102\). Denote by \(\text{Alg} F(102)\) the coequalizer obtained in this way. Now, with the same notation, \(\text{Alg} F(102)\) can also be obtained as \(H_{\text{Alg}}^s \Pi^s_{\text{BrAlg}}(\text{BrAlg}^s F)(101)\) (this is straightforward). Since we already observed that both \(H_{\text{Alg}}^s\) and \(\Pi^s_{\text{BrAlg}}\) reflect coequalizers, we deduce that \((\text{BrAlg}^s F)(101)\) is a coequalizer too. This coequalizer appears in the second line of the diagram

\[
\begin{align*}
\begin{array}{c}
\xrightarrow{T^s_{\text{Br}} (F M \otimes F M, c_{F M \otimes F M})} \xrightarrow{T^s_{\text{Br}} [-]^s} \xrightarrow{\theta^s(M, cM, [-]_M)} \\
\xrightarrow{\pi^s(M, cM, [-]_M)} \xrightarrow{\phi^s(M, cM, [-]_M)} \xrightarrow{U^s_{\text{Br}} (\text{BrAlg}^s F)} \xrightarrow{\xi^s_{\text{BrAlg}}}
\end{array}
\end{align*}
\]

where, for sake of shortness, we set \(\overline{M} := (M, cM, [-]_M)\) and \(\overline{F} := \text{BrAlg}^s F\). One proves that the morphism

\[
\begin{align*}
\xrightarrow{T^s_{\text{Br}} (F M \otimes F M, c_{F M \otimes F M})} \xrightarrow{T^s_{\text{Br}} [-]^s} \xrightarrow{T^s_{\text{Br}} (F M \otimes M, c_{F M \otimes M})} \\
\xrightarrow{T^s_{\text{Br}} (\text{BrAlg}^s F)} \xrightarrow{T^s_{\text{Br}} (M \otimes M, c_{M \otimes M})}
\end{align*}
\]

is an isomorphism (we just point out that, as one easily checks, the morphism \(\phi^s(M, M)\) is a braided morphism so that the morphism above is well-defined) and it completes the diagram above on the left making it a serially commutative diagram. The fact it is serially commutative depends on the following equality that can be easily checked

\[(103) \quad \Pi^s_{\text{BrAlg}} \xi^s_{\text{Br}} = \xi^s_{\text{Br}} \Pi^s_{\text{Br}}.
\]

Now, by \(103\) we have \(\Pi^s_{\text{BrAlg}} \xi^s_{\text{Br}} = \xi^s_{\text{Br}} \Pi^s_{\text{Br}}\). On the other hand, by Proposition 3.1 (here we use the fact that \(F\) preserves denumerable coproducts), we know that \(\xi^s_{\text{Br}}\) is a functorial isomorphism. Since \(\Pi^s_{\text{BrAlg}}\) is conservative, we deduce that \(\xi^s_{\text{Br}}\) is a functorial isomorphism. Thus, by the uniqueness of coequalizers (note that the first line in the diagram above is just \(101\) applied to \((\text{BrAlg}^s F)(M, cM, [-]_M) = (F M, c_{F M}, [-]_{F M})\) instead of \((M, cM, [-]_M)\)), we get that \(\xi^s_{\text{Br}}\) is a functorial isomorphism too. Thus \(\xi^s_{\text{Br}}\) is a functorial isomorphism.

By \(100\) we have \(\Pi^s_{\text{Br}} \xi^s_{\text{BrAlg}} = \xi^s_{\text{Br}}\) so that \(\xi^s_{\text{Br}}\) is a functorial isomorphism too. \(\square\)
Theorem 8.4. Let $\mathcal{M}$ be an abelian monoidal category with denumerable coproducts and such that the tensor functors are exact and preserve denumerable coproducts. Let $\mathcal{N}$ be an MM-category and assume that there exists a conservative (see [24]) and exact monoidal functor $(F, \phi_0, \phi_2) : \mathcal{M} \to \mathcal{N}$ which preserves denumerable coproducts. Then $\mathcal{M}$ is an MM-category.

Proof. By Theorem 8.3, we have the following commutation datum

$$\bigl(\mathcal{Br} \text{Bialg}^s F, \mathcal{Br} \text{Lie}^s F\bigr) : \bigl(\mathcal{P}_{Br}^s, \mathcal{P}_{Br}^s\bigr) \to \bigl(\mathcal{P}_{Br}^s, \mathcal{P}_{Br}^s\bigr).$$

By Lemma 8.2, we know that $\mathcal{Br} \text{Lie}^s F$ is conservative as $F$ is. By Lemma 8.3, we have that the unit $\eta : \mathcal{Id}_{\text{Br} \text{Lie}^s F, \phi} \to \mathcal{P}_{Br}^s \mathcal{U}_{Br}$ is a functorial isomorphism. □

Theorem 8.5. Let $\mathfrak{M}$ be the category of vector spaces over a field $k$ with $\text{char}(k) = 0$. Let $\mathcal{M}$ be an abelian monoidal category with denumerable coproducts, such that the tensor functors are exact and preserve denumerable coproducts. Assume that there exists a conservative and exact monoidal functor $(F, \phi_0, \phi_2) : \mathcal{M} \to \mathfrak{M}$ which preserves denumerable coproducts. Then $\mathcal{M}$ is an MM-category.

Proof. By Theorem 8.4 we have $\mathfrak{M}$ is an MM-category. We conclude by Theorem 8.4. □

9. Examples of MM-categories

Example 9.1. Let $k$ be a field with $\text{char}(k) = 0$. Let $H$ be any Hopf algebra over $k$ of and consider the monoidal category of Yetter-Drinfeld modules $(\mathcal{YD}, \otimes_k, k)$. Then the forgetful functor $\mathcal{F} : (\mathcal{YD}, \otimes_k, k) \to (\mathfrak{M}, \otimes_k, k)$ is monoidal. One can prove by hand that $(\mathcal{YD}, \otimes_k, k)$ is abelian with denumerable coproducts. The tensor functors are clearly exact and preserve denumerable coproducts in $\mathcal{YD}$ as this is the case in $\mathfrak{M}$. Furthermore $\mathcal{F}$ is clearly conservative and exact and preserves denumerable coproducts. By 8.4, we conclude that $(\mathcal{YD}, \otimes_k, k)$ is an MM-category. Note that, by Theorem 8.2, this category, with respect to its standard pre-braiding, is not symmetric unless $H = k$.

9.1. Quasi-Bialgebras. The following definition is not the original one given in [24], page 1421]. We adopt the more general form of [24, Remark 1, page 1423] (see also [23, Proposition XV.1.2]) in order to comprise the case of Hom-Lie algebras. Later on, for dual quasi-bialgebras, we will take the simplified respective definition from the very beginning having no meaningful example to treat in the full generality.

Definition 9.2. A quasi-bialgebra is a datum $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ where $(H, m, u)$ is an associative algebra, $\Delta : H \to H \otimes H$ and $\varepsilon : H \to k$ are algebra maps, $\lambda, \rho \in H$ are invertible elements, $\phi \in H \otimes H \otimes H$ is a counital 3-cocycle i.e. it is an invertible element and satisfies

$$(H \otimes H \otimes \Delta)(\phi) \cdot (\Delta \otimes H \otimes H)(\phi) = (1_H \otimes \phi) \cdot (H \otimes \Delta \otimes H)(\phi) \cdot (\phi \otimes 1_H),$$

$$(H \otimes \varepsilon \otimes H)(\phi) = \rho \otimes \lambda^{-1}.$$ \smallskip

Moreover $\Delta$ is required to be quasi-coassociative and counitary i.e. to satisfy

$$(H \otimes \Delta)(h) = \phi \cdot (\Delta \otimes H)(\Delta)(h) \cdot \phi^{-1},$$

$$(\varepsilon \otimes H)(\Delta)(h) = \lambda^{-1} h \lambda, \quad (H \otimes \varepsilon)(\Delta)(h) = \rho^{-1} h \rho.$$ \smallskip

A morphism of quasi-bialgebras $\Xi : (H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho) \to (H', m', u', \Delta', \varepsilon', \phi', \lambda', \rho')$ (see [23, page 371]) is an algebra homomorphism $\Xi : (H, m, u) \to (H', m', u')$ such that $(\Xi \otimes \Xi)(\Delta) = \Delta'(\Xi) \otimes \Xi$, $\varepsilon' \Xi = \varepsilon$, $(\Xi \otimes \Xi \otimes \Xi)(\phi) = \phi'$, $\Xi(\lambda) = \lambda'$ and $\Xi(\rho) = \rho'$. It is an isomorphism of quasi-bialgebras if, in addition, it is invertible. We will adopt the standard notation

$$\phi^1 \otimes \phi^2 \otimes \phi^3 := \phi \, \, \text{(summation understood)}.$$ \smallskip

In the case when $\phi$ is not trivial and $\lambda = \rho = 1_H$, we call $H$ an ordinary quasi-bialgebra. If further $\phi$ is trivial we then land at the classical concept of bialgebra.

A quasi-subbialgebra of a quasi-bialgebra $H'$ is a quasi-bialgebra $H$ such that $H$ is a vector subspace of $H'$ and the canonical inclusion is a morphism of dual quasi-bialgebras.
Let \((H,m,u,\Delta,\varepsilon,\phi,\lambda,\rho)\) be a quasi-bialgebra. It is well-known, see [Ka] page 285 and Proposition XV.1.2, that the category \(\mathcal{HM}\) of left \(H\)-modules becomes a monoidal category as follows. Given a left \(H\)-module \(V\), we denote by \(\mu = \mu_1 : H \otimes V \to V, \mu(h \otimes v) = hv\), its left \(H\)-action. The tensor product of two left \(H\)-modules \(V\) and \(W\) is a module via diagonal action i.e. \(h(v \otimes w) = h_1v \otimes h_2w\). The unit is \(k\), which is regarded as a left \(H\)-module via the trivial action i.e. \(hk = \varepsilon(h)k\), for all \(h \in H, k \in k\). The associativity and unit constraints are defined, for all \(V,W,Z \in \mathcal{HM}\) and \(v \in V, w \in W, z \in Z\), by \(\Delta_V : (v \otimes w) \otimes z =: \phi^1 v \otimes (\phi^2 w \otimes \phi^3 z)\), \(\lambda_V (1 \otimes v) := \lambda v\) and \(\rho_V (v \otimes 1) := \rho v\). This monoidal category will be denoted by \((\mathcal{HM}, \otimes, k, a, l, r)\). Given an invertible element \(\alpha \in H \otimes H\), we can construct a new quasi-bialgebra \(H_\alpha = (H,m,u,\Delta_\alpha,\varepsilon,\phi_\alpha,\lambda_\alpha,\rho_\alpha)\) where

\[
\Delta_\alpha(h) = \alpha \cdot \Delta(h) \cdot \alpha^{-1}, \quad \lambda_\alpha = \lambda \cdot (\varepsilon_H \otimes H)(\alpha^{-1}), \quad \rho_\alpha = \rho \cdot (H \otimes \varepsilon_H)(\alpha^{-1}),
\]

\[
\phi_\alpha = (1_H \otimes \alpha) \cdot (H \otimes \Delta)(\alpha) \cdot (\Delta \otimes H)(\alpha^{-1}) \cdot (\alpha^{-1} \otimes 1_H).
\]

**Definition 9.3.** We refer to [Ka] Proposition XV.2.2 but with a different terminology (cf. [Dr], page 1439)). A quasi-bialgebra \((H,m,u,\Delta,\varepsilon,\phi,\lambda,\rho)\) is called quasi-triangular whenever there exists an invertible element \(R \in H \otimes H\) such that, for every \(h \in H\), one has

\[
(\Delta \otimes H)(R) = \begin{bmatrix}
(\phi^2 \otimes \phi^3 \otimes \phi^1)(R^1 \otimes 1 \otimes R^2)(\phi^3 \otimes \phi^3 \otimes \phi^2)^{-1} \\
1 \otimes R^1 \otimes R^2
\end{bmatrix}
\]

\[
(H \otimes \Delta)(R) = \begin{bmatrix}
(\phi^3 \otimes \phi^1 \otimes \phi^2)^{-1}(R^1 \otimes 1 \otimes R^2)(\phi^2 \otimes \phi^3 \otimes \phi^2)^{-1} \\
R^1 \otimes R^2 \otimes 1
\end{bmatrix}
\]

\[
\Delta^{opp}(h) = R\Delta(h)R^{-1}
\]

where \(\phi := \phi^1 \otimes \phi^2 \otimes \phi^3, R = R^1 \otimes R^2\). A morphism of quasi-triangular quasi-bialgebras is a morphism \(\Xi : H \to H'\) of quasi-bialgebras such that \((\Xi \otimes \Xi)(R) = R'\).

By [Ka] Proposition XV.2.2, \(h\mathcal{M} = (h\mathcal{M}, \otimes, k, a, l, r)\) is braided if and only if there is an invertible element \(R \in H \otimes H\) such that \((H,m,u,\Delta,\varepsilon,\phi,\lambda,\rho, R)\) is quasi-triangular. Note that the braiding is given, for all \(X,Y \in h\mathcal{M}\), by

\[
e_{X,Y} : X \otimes Y \to Y \otimes X : x \otimes y \mapsto R^2 y \otimes R^1 x.
\]

Moreover \(h\mathcal{M}\) is symmetric if and only if we further assume \(R^2 \otimes R^1 = R^{-1}\). Such a quasi-bialgebra will be called a triangular quasi-bialgebra. A morphism of triangular quasi-bialgebras is just a morphism of the underlying quasi-triangular quasi-bialgebras structures.

Given an invertible element \(\alpha \in H \otimes H\), if \(H\) is (quasi-)triangular so is \(H_\alpha\) with respect to \(\alpha R = (\alpha^2 \otimes \alpha)(R)\), where \(\alpha := \alpha^1 \otimes \alpha^2\).

Let \((H,m,u,\Delta,\varepsilon,\phi,\lambda,\rho)\) be a quasi-bialgebra. We want to apply Theorem 8.5 to the case \(\mathcal{M} = h\mathcal{M}\). Let \(F : h\mathcal{M} \to \mathcal{M}\) be the forgetful functor. We need a monoidal \((F, \psi_0, \psi_2) : (h\mathcal{M}, \otimes, k, a, l, r) \to \mathcal{M}\).

**Lemma 9.4.** Let \((H,m,u,\Delta,\varepsilon,\phi,\lambda,\rho)\) be a quasi-bialgebra. Let \(F : h\mathcal{M} \to \mathcal{M}\) be the forgetful functor. The following are equivalent.

1. There is a natural transformation \(\psi_2\) such that \((F, \text{Id}_k, \psi_2) : (h\mathcal{M}, \otimes, k, a, l, r) \to \mathcal{M}\) is monoidal.
2. There is an invertible element \(\alpha \in H \otimes H\) such that \(H_\alpha\) is an ordinary bialgebra.
3. There is an invertible element \(\alpha \in H \otimes H\) such that

\[
\phi = (H \otimes \Delta)(\alpha^{-1}) \cdot (1_H \otimes \alpha^{-1}) \cdot (\alpha \otimes 1_H) \cdot (\Delta \otimes H)(\alpha),
\]

\[
(\varepsilon_H \otimes H)(\alpha) = \lambda, \quad (H \otimes \varepsilon_H)(\alpha) = \rho.
\]

Moreover, if (2) holds, we can choose \(\psi_2(V,W)(v \otimes w) := \alpha^{-1}(v \otimes w)\).

**Proof.** (1) \(\Leftrightarrow\) (2) Cf. [ABM] Proposition 1. (2) \(\Leftrightarrow\) (3) We have that \(H_\alpha\) is an ordinary bialgebra if and only if \(\phi_\alpha = 1_H \otimes 1_H \otimes 1_H, \lambda_\alpha = 1_H\) and \(\rho_\alpha = 1_H\), if and only if \(\alpha\) fulfills the equations in (3). \(\square\)
Note that $F : \mathcal{M} \rightarrow \mathcal{M}$ is clearly conservative and preserves equalizers, epimorphisms and coequalizers. Furthermore we need $\mathcal{M}$ to be braided.

**Theorem 9.5.** Let $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ be a quasi-bialgebra such that (104) and (105) hold for some invertible element $\gamma \in H \otimes H$. Let $\mathcal{M}$ be the monoidal category $(\mathcal{M}, \otimes, k, a, l, r)$ of left modules over $H$. Assume $\text{char} k = 0$. Then $\mathcal{M}$ is an MM-category. In particular, if $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ is endowed with a triangular structure, then $\mathcal{M}$ is a symmetric MM-category.

**Proof.** First note that $\mathcal{M}$ is a Grothendieck category. In $\mathcal{M}$ the tensor products are exact and preserve denumerable coproducts. We can apply Theorem 8.5 to the monoidal functor $(F, \text{Id}_k, \phi_2) : (\mathcal{M}, \otimes, k, a, l, r) \rightarrow \mathcal{M}$ of Lemma 1.4. Then $\mathcal{M}$ is an MM-category.

If $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ is endowed with a triangular structure, by the foregoing $\mathcal{M}$ is also symmetric monoidal. □

**Example 9.6.** Let $H$ be a bialgebra over a field $k$ of characteristic zero. Then $H$ is a quasi-bialgebra with $\phi, \lambda, \rho$ trivial. Note that (104) and (105) hold for $\gamma = \varepsilon_H \otimes \varepsilon_H$. Thus, by Theorem 9.5, the monoidal category $\mathcal{M}$ is an MM-category. In particular, if $\text{char} k = 0$, we can apply Theorem 8.5 to get that $\mathcal{M}$ is an MM-category.

**Example 9.7.** Examples of triangular quasi-bialgebra structures on the group algebra $k[G]$ over a torsion-free abelian group $G$ are investigated in [ABM, Proposition 3]. Consider the particular case when $G = \langle g \rangle$ is the group $\mathbb{Z}$ in multiplicative notation, where $g$ is a generator. Let $(q, a, b) \in (k \setminus \{0\}) \times \mathbb{Z} \times \mathbb{Z}$. In view of [ABM, Proposition 3], we have the triangular quasi-bialgebra

$$k[[g]]^{a,b} = \langle k[[g]] \rangle, \Delta, \varepsilon, \phi, \lambda, \rho, R$$

on the group algebra $k[[g]]$ which is defined by

$$\begin{align*}
\Delta (g) &= g \otimes g, \\
\varepsilon (g) &= 1, \\
\phi &= g^a \otimes 1_H \otimes g^b, \\
\lambda &= q^{-a} g^{-b}, \\
\rho &= g^a, \\
R &= g^{a+b} \otimes g^{-a-b}.
\end{align*}$$

In order to apply Theorem 9.5 in case $H = k[[g]]^{a,b}$, we must check that (104) and (105) hold for some invertible element $\gamma \in H \otimes H$. By [ABM, Theorem 2], one has $k[[g]]^{a,b} = k[[g]]_a$ where $\alpha := q^{-a} g^{-a} \otimes g^a$ and $k[[g]]$ is the usual bialgebra structure on the group algebra regarded as a triangular quasi-bialgebra (i.e., $\phi, \lambda, \rho$ are all trivial). Set $\gamma := \alpha^{-1} = q^{-a} g^{-a} \otimes g^a$. Then $H = k[[g]]$ which is an ordinary bialgebra so that, by Lemma 9.4 we have that (104) and (105) hold for our $\gamma$. Hence by Theorem 9.5 the symmetric monoidal category $(\mathcal{M}, \otimes, k, a, l, r)$ of left modules over $H$ is an MM-category.

**Definition 9.8.** Let $C$ be an ordinary category. Following [CG, Section 1], we associate to $C$ a new category $\mathcal{HC}$ as follows. Objects are pairs $(M, f_M)$ with $M \in C$ and $f_M \in \text{Aut}_C(M)$. A morphism $\xi : (M, f_M) \rightarrow (N, f_N)$ is a morphism $\xi : M \rightarrow N$ in $C$ such that $f_M \circ \xi = \xi \circ f_N$. The category $\mathcal{HC}$ is called the Hom-category associated to $C$.

**Example 9.9.** Take $C := \mathcal{M}$. In view of [ABM, Theorem 4], to each datum $(q, a, b) \in (k \setminus \{0\}) \times \mathbb{Z} \times \mathbb{Z}$ one associates a monoidal category

$$\mathcal{H}_{q}^{a,b}(\mathcal{M}) = (\mathcal{M}(\mathcal{M}), \otimes, (k, f_k), a, l, r)$$

which consists of the category $\mathcal{H}(\mathcal{M})$ equipped with a suitable braided (actually symmetric) monoidal structure. By [ABM, Theorem 4] there is a strict symmetric monoidal category isomorphism

$$(W, w_0, w_2) : k[[g]]^{a,b} \mathcal{M} \rightarrow \mathcal{H}_{q}^{a,b}(\mathcal{M}).$$

The underlying functor $W : k[[g]] \mathcal{M} \rightarrow \mathcal{H}(\mathcal{M})$ is given on objects by

$$W (X, \mu_X : k[[g]] \otimes X \rightarrow X) = (X, f_X : X \rightarrow X),$$

where $f_X (x) := \mu_X (g \otimes x)$, for all $x \in X$, and on morphisms by $W \xi = \xi$.

Composing $W^{-1}$ with the forgetful functor $k[[g]]^{a,b} \mathcal{M} \rightarrow \mathcal{M}$ we get a monoidal functor $\mathcal{H}_{q}^{a,b}(\mathcal{M}) \rightarrow \mathcal{M}$ to which we can apply Theorem 9.5 to get that $\mathcal{H}_{q}^{a,b}(\mathcal{M})$ is an MM-category.
Remark 9.10. By [ABM, Proposition 5], \(\mathcal{M} := \mathcal{H}^{1}_{\mathcal{T}}(\mathcal{M})\) is the symmetric braided monoidal category \(\mathcal{H}(\mathcal{M})\) of [CG Proposition 1.1]. Thus, by the foregoing, \(\mathcal{H}(\mathcal{M})\) is an MM-category. By [CG, page 2236], an object in \((\mathcal{M}, [-])\) \(\in \text{Lie}_{\mathcal{M}}\) is nothing but a Hom-Lie algebra. By Remark 5.2, \(\mathcal{H}(\mathcal{M}, [-])\) as a bialgebra is a quotient of \(\mathcal{T}\mathcal{M}\). The morphism giving the projection is induced by the canonical projection \(P_{H} : \mathcal{T}\mathcal{M} \to R := \mathcal{H}_{\text{Lie}}(\mathcal{M}, [-])\) defining the universal enveloping algebra. At algebra level we have

\[
\mathcal{O}\mathcal{H}(\mathcal{M}, [-]) = \frac{\mathcal{T}\mathcal{M}}{(f_{\mathcal{H}_{\mathcal{M}}}(\mathcal{M}, [-]) (x \otimes y) | x, y \in \mathcal{M})} = \frac{\mathcal{T}\mathcal{M}}{([x, y] - x \otimes y + c_{\mathcal{M}, \mathcal{M}} (x \otimes y) | x, y \in \mathcal{M})}
\]

which is the Hom-version of the universal enveloping algebra, see [CG, Section 8]. Note that, as a by-product, we have that \(\pi_{H} : \text{Id}_{\mathcal{H}_{\mathcal{M}}r} \to \mathcal{O}\mathcal{H}(\mathcal{M}, [-])\) is an isomorphism so that \((\mathcal{M}, [-]) \cong \mathcal{O}\mathcal{H}(\mathcal{M}, [-])\).

9.2. Dual Quasi-Bialgebras. First, observe that dual quasi-bialgebras can be understood as a dual version of quasi-bialgebras just in the finite-dimensional case. In fact, for an infinite-dimensional quasi-bialgebra \(H\) (as in the case for \(H = k\mathbb{Z}\) considered above) it is not true that the dual is a dual quasi-bialgebra so that the results in the two settings are independent, in general.

Definition 9.11. A dual quasi-bialgebra is a datum \((H, m, u, \Delta, \varepsilon, \omega)\) where \((H, \Delta, \varepsilon)\) is a coassociative coalgebra, \(m : H \otimes H \to H\) and \(u : k \to H\) are coalgebra maps called multiplication and unit respectively, we set \(1_{H} := u(1_{k})\), \(\omega : H \otimes H \otimes H \to k\) is a unital 3-cocycle i.e. it is convolution invertible and satisfies

\[
\omega(H \otimes H \otimes m) * \omega(m \otimes H \otimes H) = (\varepsilon \otimes \omega) * \omega(H \otimes m \otimes H) * (\omega \otimes \varepsilon)
\]

\[
\text{and} \quad \omega(h \otimes k \otimes l) = \varepsilon(h) \varepsilon(k) \varepsilon(l) \quad \text{whenever} \quad 1_{H} \in \{h, k, l\}.
\]

Moreover \(m\) is quasi-associative and unitary i.e. it satisfies

\[
m(H \otimes m) * \omega = \omega * m(m \otimes H),
\]

\[
m(1_{H} \otimes h) = h \quad \text{and} \quad m(h \otimes 1_{H}) = h, \quad \text{for all} \ h \in H.
\]

The map \(\omega\) is called the reassociator of the dual quasi-bialgebra.

A morphism of dual quasi-bialgebras \(\Xi : (H, m, u, \Delta, \varepsilon, \omega) \to (H', m', u', \Delta', \varepsilon', \omega')\) is a coalgebra homomorphism \(\Xi : (H, \Delta, \varepsilon) \to (H', \Delta', \varepsilon')\) such that \(m'\Xi \otimes \Xi \Xi = \Xi m, \Xi u = u'\) and \(\omega'\Xi \Xi \Xi = \omega\). It is an isomorphism of dual quasi-bialgebras if, in addition, it is invertible.

A dual quasi-subbialgebra of a dual quasi-bialgebra \(H^{\prime}\) is a quasi-bialgebra \(H\) such that \(H\) is a vector subspace of \(H^{\prime}\) and the canonical inclusion is a morphism of dual quasi-bialgebras.

Let \((H, m, u, \Delta, \varepsilon, \omega)\) be a dual quasi-bialgebra. It is well-known that the category \(\mathfrak{H}^{H}\) of right \(H\)-comodules becomes a monoidal category as follows. Given a right \(H\)-comodule \(V\), we denote by \(\rho = \rho(V) : V \to V \otimes H\), \(\rho(v) = v_{0} \otimes v_{1}\), its right \(H\)-coaction. The tensor product of two right \(H\)-comodules \(V\) and \(W\) is a comodule via diagonal coaction i.e. \(\rho(v \otimes w) = v_{0} \otimes v_{1} \otimes w_{0} \otimes w_{1}\). The unit is \(k\), which is regarded as a right \(H\)-comodule via the trivial coaction i.e. \(\rho(k) = k \otimes 1_{H}\). The associativity and unit constraints are defined, for all \(U, V, W \in \mathfrak{H}^{H}\) and \(u \in U, v \in V, w \in W, k \in k\), by \(a_{U, V, W}((u \otimes v) \otimes w) := u_{0} \otimes (v_{0} \otimes w_{0})\omega(u_{1} \otimes v_{1} \otimes w_{1}), l_{U}(k \otimes u) := ku\) and \(r_{U}(u \otimes k) := uk\).

This monoidal category will be denoted by \((\mathfrak{H}^{H}, \otimes, k, a, l, r)\).

Let \((H, m, u, \Delta, \varepsilon, \omega)\) be a dual quasi-bialgebra. Let \(v : H \otimes H \to k\) be a gauge transformation i.e. a convolution invertible map such that \(v(1_{H} \otimes h) = \varepsilon(h) = v(h \otimes 1_{H})\) for all \(h \in H\). Then \(H^{v} := (H, m^{v}, u, \Delta, \varepsilon, \omega^{v})\) is also a dual quasi-bialgebra where

\[
m^{v} := v * m * v^{-1}
\]

\[
\omega^{v} := (\varepsilon \otimes v) * v(H \otimes m) * \omega * v^{-1}(m \otimes H) * (v^{-1} \otimes \varepsilon).
\]

Definition 9.12. A dual quasi-bialgebra \((H, m, u, \Delta, \varepsilon, \omega)\) is called quasi-co-triangular whenever there exists \(R \in \text{Reg}(H \otimes^{2}, k)\) such that

\[
\begin{align*}
R(m \otimes H) &= \left[ \omega_{R}H_{H}H_{H} * R(H \otimes l_{H})(H \otimes \varepsilon \otimes H) * \omega^{-1}(H \otimes \tau_{H,H}) * m_{k}(\varepsilon \otimes R) * \omega \right].
\end{align*}
\]
Let \( \mathcal{M} = \mathfrak{M}^H \) be the forgetful functor. Note that \( \mathcal{M} \) is clearly conservative and preserves equalizers, epimorphisms and coequalizers. Note also that we will further need \( \mathfrak{M}^H \) to be braided.

**Lemma 9.13.** Let \( (H, m, u, \Delta, \varepsilon, \omega) \) be a dual quasi-bialgebra. Let \( F : \mathfrak{M}^H \to \mathcal{M} \) be the forgetful functor. The following are equivalent.

1. There is a natural transformation \( \psi_2 \) such that \( (F, \text{Id}_k, \phi_2) : (\mathfrak{M}^H, \otimes, k, a, l, r) \to \mathcal{M} \) is monoidal.
2. There is a gauge transformation \( \nu : H \otimes H \to k \) such that \( H^\nu \) is an ordinary bialgebra.
3. There is a gauge transformation \( \phi : H \otimes H \to k \) such that

\[
\omega = \nu^{-1} (H \otimes m) * (\varepsilon \otimes \nu^{-1}) * (\nu \otimes \varepsilon) * \nu (m \otimes H)
\]

Moreover, if (2) holds, we can choose \( \phi_2 (V, W) (x \otimes y) = x_0 \otimes y_0 \nu^{-1} (x_1 \otimes y_1) \).

**Proof.** It is similar to the one of Lemma 9.4. \( \square \)

**Lemma 9.14.** Let \( (H, m, u, \Delta, \varepsilon, \omega) \) be a quasi-co-triangular dual quasi-bialgebra. Then \( R \) is unital i.e. \( R (1_H \otimes h) = \varepsilon (h) = R (h \otimes 1_H) \) for all \( h \in H \).

**Theorem 9.15.** Let \( (H, m, u, \Delta, \varepsilon, \omega) \) be a dual quasi-bialgebra such that \( \omega \) fulfills \( (\text{I}) \) for some gauge transformation \( \gamma : H \otimes H \to k \). Let \( \mathcal{M} \) be the monoidal category \( (\mathfrak{M}^H, \otimes, k, a, l, r) \) of right comodules over \( H \). Assume \( \text{ch}(k) = 0 \). Then \( \mathcal{M} \) is an MM-category. In particular, if \( (H, m, u, \Delta, \varepsilon, \omega) \) is endowed with a co-triangular structure, then \( \mathcal{M} \) is a symmetric MM-category.

**Proof.** It is analogous to the proof of Theorem 9.3 but using, from Lemma 9.13, the functor \( (F, \text{Id}_k, \phi_2) : (\mathfrak{M}^H, \otimes, k, a, l, r) \to \mathcal{M} \). \( \square \)

**Example 9.16.** Let \( H \) be a bialgebra over a field \( k \) of characteristic zero. Then \( H \) is a dual quasi-bialgebra with reassociator \( \omega = \varepsilon_H \otimes \varepsilon_H \otimes \varepsilon_H \). Note that \( \omega \) fulfills \( (\text{I}) \) for \( \gamma = \varepsilon_H \otimes \varepsilon_H \). Thus, by Theorem 9.15, the monoidal category \( \mathfrak{M}^H \) of right comodules over \( H \) is an MM-category. In particular, for \( H = k [\mathbb{N}] \), the monoid bialgebra over the naturals, defined by taking \( \Delta n = n \otimes n \) and \( \varepsilon (n) = 1 \) for every \( n \in \mathbb{N} \), then the category \( \mathfrak{M}^H \) is the category of \( \mathbb{N} \)-graded vector spaces \( V = \oplus_{n \in \mathbb{N}} V_n \) with monoidal structure having tensor product given by \( (V \otimes W)_n = \oplus_{i+j=n} (V_i \otimes W_{n-i}) \) and unit \( k \) concentrated in degree 0. The constraints are the same of vector spaces. The category \( \mathfrak{M}^H \) is braided with respect to the canonical flip (this can be seen by showing that \( R = \varepsilon_H \otimes \varepsilon_H \) turns \( H \) into a co-triangular bialgebra, see remark below).
Remark 9.17. Let \((H, m, u, \Delta, \varepsilon, \omega, R)\) be a co-triangular dual quasi-bialgebra. Assume that \(\omega\) fulfills (11) for \(\gamma = \varepsilon_H \otimes \varepsilon_H\). This means \(\omega = \varepsilon_H \otimes \varepsilon_H \otimes \varepsilon_H\) and \((H, m, u, \Delta, \varepsilon, R)\) is a co-triangular bialgebra i.e. for every \(x, y, z \in H\) we have
\[
R(xy \otimes z) = R(x \otimes z_1)R(y \otimes z_2), \quad R(x \otimes yz) = R(x_1 \otimes z)R(x_2 \otimes y),
\]
\[
y_1x_1R(x_2 \otimes y_2) = R(x_1 \otimes y_1)x_2y_2.
\]
Let \((M, [-]) \in \text{Lie}_M\). Then \(\mathfrak{g}_3\), \(\mathfrak{g}_4\), \(\mathfrak{g}_5\), \(\mathfrak{g}_6\) and \(\mathfrak{g}_7\) become
\[
[x, y] = - \sum [y(0), x(0)] R(x(1) \otimes y(1)),
\]
\[
\sum [[x, y], z] + \sum [[y(0), z(0)], x(0)] R(x(1) \otimes y(1)z(1)) + \sum [[z(0), x(0)], y(0)] R(x(1)y(1) \otimes z(1)) = 0.
\]
This means that \((M, [-])\) is an \((H, R)\)-Lie algebra in the sense of [BFM, Definition 4.1]. By Remark 1.3, \(\overline{U}(M, [-])\) as a bialgebra is a quotient of \(\overline{T}M\). The morphism giving the projection is induced by the canonical projection \(p_R : \Omega TM \to R := \overline{U}_{\text{Lie}}(M, [-])\) defining the universal enveloping algebra. At algebra level we have
\[
\mathcal{U}(M, [-]) \cong \mathfrak{g}_7 : \frac{TM}{\theta_{\mathfrak{g}_2,\mathfrak{g}_7}(x \otimes y)\vert x, y \in M} = \frac{TM}{\theta_{\mathfrak{g}_2,\mathfrak{g}_7}(x \otimes y)\vert x, y \in M} = \frac{TM}{\theta_{\mathfrak{g}_2,\mathfrak{g}_7}(x \otimes y + c_{\mathfrak{g}_2,\mathfrak{g}_7}(x \otimes y)\vert x, y \in M)}
\]
which is the universal enveloping algebra of our \((H, R)\)-Lie algebra, see e.g. [FM, (2.6)]. Note that, as a by-product, we have that \(\mathfrak{g}_6 : \text{Id}_{\text{Lie}_M} \to \mathfrak{g}_7\) is an isomorphism so that \((M, [-]) \cong \mathfrak{g}_7(M, [-])\).

Example 9.18. Let \(k\) be a field with \(\text{char}(k) = 0\) and let \(G\) be an abelian group endowed with an anti-symmetric bicharacter \(\chi : G \times G \to k \setminus \{0\}\), i.e. for all \(g, h, k \in G\), we have:
\[
\chi(g, hk) = \chi(g, h)\chi(h, k) = 1.
\]
Extend \(\chi\) by linearity to a \(k\)-linear map \(R : k[G] \otimes k[G] \to k[G]\), where \(k[G]\) denotes the group algebra. Then \((k[G], R)\) is a co-triangular bialgebra, cf. [Ma] Example 2.2.5). Hence, we can apply Theorem 1.15 and Remark 9.17 to \(H = k[G]\). Note that the category \((\mathfrak{g}_2, \otimes, k, k, a, l, r, c)\) consists of \(G\)-graded modules \(V = \bigoplus_{g \in G} V_g\). Given \(G\)-graded modules \(V\) and \(W\), their tensor product \(V \otimes W\) is graded with \(V \otimes W = \bigoplus_{g \in G} (V_g \otimes W_g)\). The braiding is given on homogeneous elements by
\[
c_{V, W} : V \otimes W \to W \otimes V, \quad c_{V, W}(v \otimes w) = w \otimes v \chi(|v|, |w|),
\]
where \(|v|\) denotes the degree of \(v\). In this case a \((H, R)\)-Lie algebra \((V, [-, -])\) in the sense of [BFM, Definition 4.1] means
\[
[[x, y], z] + \sum [[y, z], x] \chi(|x|, |y| |z|) + \sum [[z, x], y] \chi(|x|, |y| |z|) = 0.
\]
Multiplying by \(\chi(|z|, |x|)\) the two sides of the second equality, we get the equivalent
\[
[[x, y], z] \chi(|z|, |x|) + \sum [[y, z], x] \chi(|x|, |y|) + \sum [[z, x], y] \chi(|y|, |z|) = 0.
\]
This means that \((V, [-, -])\) is a \((G, \chi)\)-Lie color algebra in the sense of [Ma] Example 10.5.14]. Note that the braiding defined in [Ma, page 200] is \(c^*_{V, W}(v \otimes w) = w \otimes v \chi(|w|, |v|) = w \otimes v \chi^{-1}(|v|, |w|)\) so that we should say more precisely that \((V, [-, -])\) is a \((G, \chi^{-1})\)-Lie color algebra. The corresponding enveloping algebra is
\[
\mathcal{U}(M, [-]) = \frac{TM}{\theta_{\mathfrak{g}_2,\mathfrak{g}_7}(x \otimes y + y \otimes x \chi(|x|, |y|)\vert x, y \in V \text{ homogeneous})}
\]

Example 9.19. Lie superalgebras are a particular instance of the construction above. One has to take \(G = \mathbb{Z}_2\) and consider the anti-symmetric bicharacter \(\chi : G \times G \to k \setminus \{0\}\) defined by \(\chi(a, b) := (-1)^{ab}\) for all \(a, b \in \mathbb{Z}\).
Example 9.20. Let $G := (\mathbb{Z}, +, 0)$. Let $k$ be a field and let $q \in k \setminus \{0\}$. Then it is easy to check that $(\cdot, \cdot) : G \times G \to k, (a, b) := q^{ab}$ is a bicharacter of $G$.

Remark 9.21. Let $k$ be a field with char ($k$) = 0. Let $H$ be a finite-dimensional Hopf algebra. By [RT, Proposition 6], the category of Yetter-Drinfeld modules $HYD$ and $HYD^H$ are isomorphic. Moreover, by [Mo, Proposition 10.6.16], the category $M$ has coequalizers. Hence, if the tensor products preserve the coequalizers in $M$, then $H$ has coequalizers and the tensor functors preserve coequalizers. We just say that $H$ creates limits for a functor $\Omega : Bialg \to M$ and for all small category $\mathcal{I}$. Similarly one defines creation of colimits.

Lemma A.2. Let $\mathcal{M}$ be a monoidal category. Then the functor $\Omega : Alg_{\mathcal{M}} \to \mathcal{M}$ creates limits and the functor $\Omega : Coalg_{\mathcal{M}} \to \mathcal{M}$ creates colimits.

Proof. It is straightforward.

Appendix A. (Co)equalizers and (co)monadicity

Definition A.1. [McL, page 112] Let $\mathcal{I}$ be a small category. Recall that a functor $V : A \to B$ creates limits for a functor $F : \mathcal{I} \to A$ if in case $VF$ has a limit $(X, (\tau_I : X \to VF_I)_{I \in \mathcal{I}})$, then there is exactly one pair $(L, (\sigma_I : L \to FI)_{I \in \mathcal{I}})$ which is a limit of $F$ and such that $VL = X$, $V\sigma_I = \tau_I$ for every $I \in \mathcal{I}$. We just say that $V : A \to B$ creates limits if it creates limits for all functors $F : \mathcal{I} \to A$ and for all small category $\mathcal{I}$. Similarly one defines creation of colimits.

Lemma A.3. Let $\mathcal{M}$ be a monoidal category. Assume that $\mathcal{M}$ has coequalizers and that the tensor functors preserve them. It is well-known that $Alg_{\mathcal{M}}$ has coequalizers, see e.g. [AEM, Proposition 2.15]. Given an algebra morphism $\alpha : E \to A$, consider $\Lambda_\alpha := m_A^2 \circ (\alpha \otimes \alpha : A \otimes A \to A)$, where $m_A^2 : A \otimes A \to A$ is the iterated multiplication. The coequalizer of algebra morphisms $\alpha, \beta : E \to A$ is, as an object in $\mathcal{M}$, the coequalizer $(B, \pi : A \to B)$ of $(\Lambda_\alpha, \Lambda_\beta)$ in $\mathcal{M}$ and the algebra structure is the unique one making $\pi$ an algebra morphism.

Lemma A.4. Let $\mathcal{M}$ be a monoidal category.

1) If $\mathcal{M}$ has coequalizers then $Coalg_{\mathcal{M}}$ has coequalizers, and $\Omega : Coalg_{\mathcal{M}} \to \mathcal{M}$ preserves coequalizers. Moreover if the tensor products preserve the coequalizers in $\mathcal{M}$, then $Alg_{\mathcal{M}}$ has coequalizers.

2) If $\mathcal{M}$ has equalizers then $Alg_{\mathcal{M}}$ has equalizers, and $\Omega : Alg_{\mathcal{M}} \to \mathcal{M}$ preserves equalizers. Moreover if the tensor products preserve the equalizers in $\mathcal{M}$, then $Coalg_{\mathcal{M}}$ has equalizers.

3) If $\mathcal{M}$ is braided, it has coequalizers and the tensor products preserve them, then $Bialg_{\mathcal{M}}$ has coequalizers and $\Omega : Bialg_{\mathcal{M}} \to Alg_{\mathcal{M}}$ preserves coequalizers.

4) If $\mathcal{M}$ is braided, it has equalizers and the tensor products preserve them, then $Bialg_{\mathcal{M}}$ has equalizers and $\Omega : Bialg_{\mathcal{M}} \to Coalg_{\mathcal{M}}$ preserves equalizers.

Proof. 1) The first part follows by Lemma A.2 and uniqueness of coequalizers in $Coalg_{\mathcal{M}}$. By A.1, $Alg_{\mathcal{M}}$ has coequalizers. 2) is dual to 1).

3) Note that $Bialg_{\mathcal{M}} = Coalg_{\mathcal{N}}$ for $\mathcal{N} := Alg_{\mathcal{M}}$. By 1) we have that $\mathcal{N}$ has coequalizers and then $Coalg_{\mathcal{N}}$ has coequalizers, and $\Omega : Coalg_{\mathcal{N}} \to \mathcal{N}$ preserves coequalizers. 4) is dual to 3).

Lemma A.5. Let $\mathcal{M}$ be a braided monoidal category. Assume that $\mathcal{M}$ is abelian and that the tensor functors preserves equalizers, coequalizers.

1) Let $\alpha : J_{Bialg}D \to E$ be a morphism in $BrBialg_{\mathcal{M}}$. Then there is a bialgebra $Q \in Bialg_{\mathcal{M}}$ and a morphism $\pi : D \to Q$ in $Bialg_{\mathcal{M}}$ and a morphism $\sigma : J_{Bialg}Q \to E$ in $BrBialg_{\mathcal{M}}$ such that $\alpha = \sigma \circ J_{Bialg}(\pi)$ and $\sigma$ and $\pi$ are a monomorphism and an epimorphism respectively when regarded as morphism in $\mathcal{M}$.

2) The functor $J_{Bialg} : Bialg_{\mathcal{M}} \to BrBialg_{\mathcal{M}}$ preserves coequalizers.

3) Assume that $\mathcal{M}$ is symmetric. Then $J^s_{Bialg} : Bialg_{\mathcal{M}} \to BrBialg_{\mathcal{M}}$ preserves coequalizers.
Lemma A.4) so that $F$ with exact rows is abelian) we have that $(\Delta D, \varepsilon_p, \varepsilon_d, \varepsilon_D)$ be a pair of morphisms in $\text{Bialg}_M$. Assume that this pair has coequalizer $(E, p)$ in $\text{Bialg}_M$.

$$\begin{array}{ccc} B & \xrightarrow{\pi_0} & D \xrightarrow{p} E \\ \varepsilon_1 \end{array}$$

Let us check that $J_{\text{Bialg}}$ preserves this coequalizer. Let $\alpha : J_{\text{Bialg}}D \to Z$ be a morphism in $\text{BrBialg}_M$ such that $\sigma_0 = \alpha e_1$. Since $\sigma$ is a monomorphism in $M$, we have that $\pi_0 = \varepsilon_1$. Since the coequalizer $(E, p)$ is in $\text{Bialg}_M$, there is a unique morphism $\pi : E \to Z$ in $\text{Bialg}_M$ such that $\pi \circ p = \pi$. Set $\pi := \pi \circ : E \to Z$ as morphisms in $M$. Then $\pi_0 = \sigma_0 \pi p = \sigma_0 \pi = \alpha$. Therefore $\sigma$ and $\pi$ commute with (co)multiplications and (co)units and

$$(\varepsilon_0 \otimes \varepsilon_0) c_{E, E} = (\pi_0 \otimes \pi_0) c_{E, E} = (\sigma_0 \otimes \sigma_0) c_{Q, Q} (\pi_0 \otimes \pi_0) = c_Z (\sigma_0 \otimes \sigma_0) (\pi_0 \otimes \pi_0) = c_Z (\pi_0 \otimes \pi_0).$$

We have so proved that $\tau$ is a morphism in $\text{BrBialg}_M$ from $J_{\text{Bialg}}E$ to $Z$.

Let $\beta : J_{\text{Bialg}}E \to Z$ be such that $\beta p = \alpha$ as morphisms in $\text{BrBialg}_M$. Then $\beta p = \pi_0 \pi p = \sigma_0 \pi = \alpha$. Moreover $\sigma$ and $\pi$ commute with (co)multiplications and (co)units and

$$\tau = \pi_0 \pi.$$  

Since $J_{\text{Bialg}} = \text{Bialg} \circ J_{\text{Bialg}}$ preserves coequalizers, we get that $\tau_0$ is a morphism in $\text{Bialg}_M$. Since $\tau_0$ is both full and faithful, it reflects coequalizers (see the dual of [Bo2, Proposition 2.9.9]) so that $J_{\text{Bialg}}$ preserves coequalizers.

The following result can be obtained mimicking the proof of (1) $\Rightarrow$ (2) in [Bo2, Theorem 4.6.2]. For the reader’s sake we write here a proof in the specific case we are concerned.

**Theorem A.6.** Let $M$ be a monoidal category.

1. If the forgetful functor $\Omega : \text{Alg}_M \to M$ has a left adjoint, then $\Omega$ is monadic. In fact the comparison functor is a category isomorphism.
2. If $\Omega$ is monadic. In fact the comparison functor is a category isomorphism.

**Proof.**

1) We will apply Theorem [BLV, Theorem 2.1] (which is a form of Beck’s Theorem). First, in order to prove that $\Omega$ is monadic, we have to check that $\Omega$ is conservative and that for any reflexive pair of morphisms in $\text{Alg}_M$ whose image by $\Omega$ has a split coequalizer has a coequalizer which is preserved by $\Omega$. Clearly if $f$ is a morphism in $\text{Alg}_M$ such that $\Omega f$ is an isomorphism then the inverse of $\Omega f$ is a morphism of monoids whence it gives rise to an inverse of $f$ in $\text{Alg}_M$. Thus $\Omega$ is conservative.

Let $(d_0, d_1)$ from $A$ to $A'$ be a reflexive pair as above. Then there exists $C \in M$ and a morphism $c : \Omega A' \to C$ such that

$$\Omega A \xrightarrow{\Omega d_0} \Omega A' \xrightarrow{c} C$$

is a split coequalizer, whence preserved by any functor in particular by $F_n : M \to M$, the functor defined by $F_n := (-)^{\otimes n}$ i.e. the $n$th tensor power functor. Then we have a commutative diagram with exact rows

$$\begin{array}{ccc} \Omega A \otimes \Omega A & \xrightarrow{\Omega d_0 \otimes \Omega d_0} & \Omega A' \otimes \Omega A' \xrightarrow{c \otimes c} C \otimes C \\ m_{\Omega A} \downarrow & & m_{\Omega A'} \downarrow \\ \Omega A & \xrightarrow{\Omega d_0} & \Omega A' \xrightarrow{c} C \end{array}$$
By the universal property of coequalizers there is a unique morphism \( m_C : C \otimes C \to C \) in \( \mathcal{M} \) such that \( m_C \circ (c \otimes c) = co \Omega_{\mathcal{M}}. \) One easily checks that \( Q := (C, m_C, u_C) \in \text{Alg}_{\mathcal{M}} \) where \( u_C := co \Omega_{\mathcal{M}}. \) Moreover \( c \) gives rise to a morphism \( q : A' \to Q \) in \( \text{Alg}_{\mathcal{M}} \) such that \( \Omega q = c. \) Since \( \Omega \) is faithful, it is straightforward to check that \( (Q, q) \) is the coequalizer of \((d_0, d_1)\) in \( \mathcal{M}_m. \) Thus \( \Omega \) is monadic.

Let us check that the comparison functor is indeed a category isomorphism. It suffices to check that for any isomorphism \( f, g \) in \( \mathcal{M} \) in \( \text{Alg}_{\mathcal{M}} \) in \( \mathcal{M} \), it becomes an algebra morphism from \((f, g)\) to \((Q, q)\). See \([\text{Va}, \text{Proposition 2}]\) (where we can drop the assumption on abelianness as the result follows by \([\text{Jo2}, \text{Lemma 0.17}]\) where this condition is not used).

\[ \text{Proposition A.9:} \text{ Let } \mathcal{M} \text{ be a monoidal category. Assume that the tensor functors preserve coequalizers of reflexive pairs in } \mathcal{M}. \text{ Then the forgetful functor } \Omega : \text{Alg}_{\mathcal{M}} \to \mathcal{M} \text{ creates coequalizers of those pairs } (f, g) \text{ in } \text{Alg}_{\mathcal{M}} \text{ for which } (\Omega f, \Omega g) \text{ is a reflexive pair}. \]

\[ \text{Proof:} \text{ Let } f, g : (A, m_A, u_A) \to (B, m_B, u_B) \text{ be a pair of morphism in } \text{Alg}_{\mathcal{M}} \text{ that fits into a coequalizer} \]

\[ A \xrightarrow{\Omega f} B \xrightarrow{p} C \]

in \( \mathcal{M} \) such that \( (\Omega f, \Omega g) \) is a reflexive pair. By Lemma A.8, we have the following coequalizer

\[ A \otimes A \xrightarrow{\Omega f \otimes \Omega f} B \otimes B \xrightarrow{p \otimes p} C \otimes C \]

We have

\[ p \circ m_B \circ (\Omega f \otimes \Omega f) = p \circ \Omega f \circ m_A = p \circ \Omega g \circ m_A = p \circ m_B \circ (\Omega g \otimes \Omega g). \]

The universal property of the latter coequalizer entails there is a unique morphism \( m_C : C \otimes C \to C \) such that \( m_C \circ (p \otimes p) = p \circ m_B. \) Set \( u_C := p \circ u_B. \) It is easy to check that \((C, m_C, u_C) \in \text{Alg}_{\mathcal{M}}, \text{ that } p \text{ becomes an algebra morphism from } (B, m_B, u_B) \text{ to } (C, m_C, u_C) \text{ and that} \]

\[ (A, m_A, u_A) \xrightarrow{f} (B, m_B, u_B) \xrightarrow{p} (C, m_C, u_C) \]

is a coequalizer in \( \text{Alg}_{\mathcal{M}}. \)
Corollary A.10. Let $\mathcal{M}$ be a monoidal category with coequalizers of reflexive pairs. Assume these coequalizers are preserved by the tensor functors in $\mathcal{M}$. Then $\text{Alg}\mathcal{M}$ has coequalizers of reflexive pairs and they are preserved by the forgetful functor $\Omega : \text{Alg}\mathcal{M} \to \mathcal{M}$.

Proof. It follows by Proposition A.9 and uniqueness of coequalizers in $\text{Alg}\mathcal{M}$.

Appendix B. Braided (co)equalizers

Lemma B.1. Let $\mathcal{M}$ be a monoidal category. We have functors

\[ \text{Br}_{\mathcal{M}} \to \text{Br}_{\mathcal{M}} : (V, c) \to (V, c^{-1}) : f \mapsto f \]

\[ \text{Br}_{\text{Alg}\mathcal{M}} \to \text{Br}_{\text{Alg}\mathcal{M}} : (A, m, u, c) \to (A, m, u, c^{-1}) : f \mapsto f \]

Proof. It is straightforward.

Lemma B.2. Let $\mathcal{M}$ be a monoidal category and let $(V, c_V)$ be an object in $\text{Br}_{\mathcal{M}}$. Assume there is a morphism $d : D \to V$ in $\mathcal{M}$ and a morphism $c_D : D \otimes D \to D \otimes D$ such that $(d \otimes d) c_D = c_V (d \otimes d)$ and $d \otimes d \otimes d$ is a monomorphism.

1) Assume that $c_D$ is an isomorphism. Then $(D, c_D)$ is an object in $\text{Br}_{\mathcal{M}}$ and $d$ becomes a morphism in $\text{Br}_{\mathcal{M}}$ from $(D, c_D)$ to $(V, c_V)$.

2) Assume that $d \otimes d$ is a monomorphism. If $(V, c_V) \in \text{Br}^s_{\mathcal{M}}$ then $(D, c_D) \in \text{Br}^s_{\mathcal{M}}$.

Proof. Using $(d \otimes d) c_D = c_V (d \otimes d)$ and the quantum Yang-Baxter equation for $c_V$ one gets

\[ (d \otimes d \otimes d) (c_D \otimes D) (D \otimes c_D) (c_D \otimes D) = (d \otimes d \otimes d) (D \otimes c_D) (c_D \otimes D) (D \otimes c_D) \]

Since $d \otimes d \otimes d$ is a monomorphism we get that $c_D$ satisfies the quantum Yang-Baxter equation.

1) Since $c_D$ is an isomorphism it is clear that $(D, c_D) \in \text{Br}_{\mathcal{M}}$ and that $d : (D, c_D) \to (V, c_V)$ is a morphism in $\text{Br}_{\mathcal{M}}$.

2) Since $(d \otimes d) c_D^2 = c_V^2 (d \otimes d) = d \otimes d$ and $d \otimes d$ is a monomorphism we get $c_D^2 = \text{Id}_{D \otimes D}$ so that we can apply 1).

Lemma B.3. Let $\mathcal{M}$ be a monoidal category and let $H : \text{Br}_{\mathcal{M}} \to \mathcal{M}$ be the forgetful functor. Let $(e_0, e_1)$ be a pair of morphisms in $\text{Br}_{\mathcal{M}}$ such $(H e_0, H e_1)$ is a coreflexive pair of morphisms in $\mathcal{M}$. Assume that $(H e_0, H e_1)$ has an equalizer which is preserved by the tensor functors. Then $(e_0, e_1)$ has an equalizer in $\text{Br}_{\mathcal{M}}$ which is preserved by $H$. The same statement holds when we replace $\text{Br}_{\mathcal{M}}$ by $\text{Br}^s_{\mathcal{M}}$ and $H$ by the corresponding forgetful functor.

Proof. Let $(e_0, e_1)$ from $(V, c_V)$ to $(W, c_W)$ a coreflexive pair of morphisms in $\text{Br}_{\mathcal{M}}$. We denote $(H e_0, H e_1)$ by $(e_0, e_1)$ to simplify the notation. By definition, there exists a morphism $p : W \to V$ in $\mathcal{M}$ such that $p \circ e_0 = \text{Id}_V = p \circ e_1$. Consider the equalizer

\[ \begin{array}{ccc} D & \xrightarrow{d} & V \\ \text{e}_0 \downarrow & & \text{e}_1 \downarrow \\ W & & W \end{array} \]

By the dual version of Lemma A.8, we have the following equalizer

\[ \begin{array}{ccc} D \otimes D & \xrightarrow{d \otimes d} & V \otimes V \\ \text{e}_0 \otimes \text{e}_0 \downarrow & & \text{e}_1 \otimes \text{e}_1 \downarrow \\ W \otimes W & & W \otimes W \end{array} \]

We have

\[ (e_0 \otimes e_0) c_V (d \otimes d) = c_W (e_0 \otimes e_0) (d \otimes d) = c_W (e_1 \otimes e_1) (d \otimes d) = (e_1 \otimes e_1) c_V (d \otimes d). \]

Hence there is a unique morphism $c_D : D \otimes D \to D \otimes D$ such that $(d \otimes d) c_D = c_V (d \otimes d)$.

Since $(V, c_V^{-1})$ and $(W, c_W^{-1})$ are also braided objects, and $e_0, e_1$ are also morphisms from $(V, c_V^{-1})$ to $(W, c_W^{-1})$, as above we can construct a morphism $\gamma_D : D \otimes D \to D \otimes D$ such that $(d \otimes d) \gamma_D = c_V^{-1} (d \otimes d)$. We have $(d \otimes d) c_D \gamma_D = c_V (d \otimes d) \gamma_D = c_V c_V^{-1} (d \otimes d) = d \otimes d$ and hence $c_D \gamma_D = \text{Id}_{D \otimes D}$. Similarly $\gamma_D c_D = \text{Id}_{D \otimes D}$. Thus $c_D$ is invertible. Since $d \otimes d \otimes d = (d \otimes V \otimes V) (D \otimes d \otimes V) (D \otimes D \otimes d)$ we have that $d \otimes d \otimes d$ is a monomorphism. Thus we can
apply Lemma B.2 to get that \((D,c_D)\) is an object in \(Br_\mathcal{M}\) and \(d:(D,c_D) \to (V,c_V)\) is a morphism in \(Br_\mathcal{M}\). It is straightforward to check that
\[
(D,c_D) \xrightarrow{d} (V,c_V) \xrightarrow{e_0} (W,c_W)
\]
is an equalizer in \(Br_\mathcal{M}\). Consider now the case of \(Br_\mathcal{L}\) so that \((e_0,e_1)\) as above is a pair in \(Br_\mathcal{L}\). Since \(d\) is a monomorphism, by Remark 3.2, we get that \((D,c_D)\in Br_\mathcal{L}\) and \(d\) becomes a morphism in this category. Since \(Br_\mathcal{L}\) is a full subcategory of \(Br_\mathcal{M}\) we have that \(\pi_{Br_\mathcal{L}}: Br_\mathcal{L} \to Br_\mathcal{M}\) is full and faithful and hence it reflects equalizers (see [Bo1, Proposition 2.9.9]) so that the above equalizer obtained in \(Br_\mathcal{M}\) is indeed an equalizer in \(Br_\mathcal{L}\).

\[\square\]

**Lemma B.4.** Let \(\mathcal{M}\) be a monoidal category and let \((W,c_W)\) be an object in \(Br_\mathcal{M}\). Assume there is a morphism \(d: W \to D\) in \(\mathcal{M}\) and a morphism \(c_D: D \otimes D \to D\) such that \((c_D(d \otimes d)) = (d \otimes d)\) and \(d \otimes d \otimes d\) is an epimorphism.

1) Assume that \(c_D\) is an isomorphism. Then \((D,c_D)\) is an object in \(Br_\mathcal{M}\) and \(d\) becomes a morphism in \(Br_\mathcal{M}\) from \((W,c_W)\) to this object.

2) Assume that \(d \otimes d\) is an epimorphism. If \((V,c_V)\in Br_\mathcal{M}\) then \((D,c_D)\in Br_\mathcal{M}\).

**Proof.** It is dual to Lemma B.3. \[\square\]

**Lemma B.5.** Let \(\mathcal{M}\) be a monoidal category and let \(H: Br_\mathcal{M} \to \mathcal{M}\) be the forgetful functor. Let \((e_0,e_1)\) be a pair of morphisms in \(Br_\mathcal{M}\) such that \((H e_0,H e_1)\) is a reflexive pair of morphisms in \(\mathcal{M}\). Assume that \((H e_0,H e_1)\) has a coequalizer which is preserved by the tensor functors. Then \((e_0,e_1)\) has an coequalizer in \(Br_\mathcal{M}\) which is preserved by \(H\).

The same statement holds when we replace \(Br_\mathcal{M}\) by \(Br_\mathcal{L}\) and \(H\) by the corresponding forgetful functor.

**Proof.** It is dual to Lemma B.3. \[\square\]

**Lemma B.6.** Let \(\mathcal{M}\) be a monoidal category. Assume that \(\mathcal{M}\) has coequalizers and that the tensor functors preserve them. Then the functor \(H_{\mathcal{M}}: Br_{\mathcal{M}} \to \mathcal{M}\) reflects coequalizers.

**Proof.** Let
\[
\begin{array}{c}
(A,c_A) \\
\alpha \\
\beta \\
B,c_B
\end{array} 
\xrightarrow{p} 
\begin{array}{c}
(D,c_D) \\
\end{array}
\]
be a diagram of morphisms and objects in \(Br_{\mathcal{M}}\) which is sent by \(H_{\mathcal{M}}\) to a coequalizer in \(\mathcal{M}\). Since \(H_{\mathcal{M}}\) is faithful we have that \(p \alpha = p \beta\) as morphisms in \(Br_{\mathcal{M}}\). Let \(\lambda: (B,c_B) \to (E,c_E)\) be a morphism in \(Br_{\mathcal{M}}\) such that \(\lambda \alpha = \lambda \beta\). Then \(H_{\mathcal{M}} \lambda \circ H_{\mathcal{M}} \alpha = H_{\mathcal{M}} \lambda \circ H_{\mathcal{M}} \beta\) so that there is a unique algebra morphism \(\lambda': D \to E\) such that \(\lambda' \circ H_{\mathcal{M}} p = H_{\mathcal{M}} \lambda\). We have
\[
c_E (\Omega \lambda' \otimes \Omega \lambda') (\Omega H_{\mathcal{M}} p \otimes \Omega H_{\mathcal{M}} p) = c_E (\Omega H_{\mathcal{M}} \lambda \otimes \Omega H_{\mathcal{M}} \lambda) = (\Omega H_{\mathcal{M}} \lambda \otimes \Omega H_{\mathcal{M}} \lambda) c_B = (\Omega \lambda' \otimes \Omega \lambda') c_D (\Omega H_{\mathcal{M}} p \otimes \Omega H_{\mathcal{M}} p)
\]
By A.3, we have that \((H_{\mathcal{M}} \alpha,H_{\mathcal{M}} \beta)\) has a coequalizer in \(\mathcal{M}\) which is a regular epimorphism in \(\mathcal{M}\). By the uniqueness of coequalizers in \(\mathcal{M}\), we get that \(\Omega H_{\mathcal{M}} p\) is also regular epimorphism in \(\mathcal{M}\). By the assumption on the tensor products, we get that \(\Omega H_{\mathcal{M}} p \otimes \Omega H_{\mathcal{M}} p\) is an epimorphism in \(\mathcal{M}\). Thus the computation above implies \(c_E (\Omega \lambda' \otimes \Omega \lambda') (\Omega \lambda' \otimes \Omega \lambda') c_D\) so that there is a morphism \(\lambda'': (D,c_D) \to (E,c_E)\) in \(Br_{\mathcal{M}}\) such that \(H_{\mathcal{M}} \lambda'' = \lambda'.\) Since \(H_{\mathcal{M}}\) is faithful we get \(\lambda'' \circ p = \lambda\). Also the uniqueness follows by the fact that \(H_{\mathcal{M}}\) is faithful. \[\square\]

**Lemma B.7.** Let \(\mathcal{M}\) be a monoidal category. Let \((e_0,e_1): A \to B\) be a pair of morphisms in \(Br_{\mathcal{M}}\) such that \((\Omega H_{\mathcal{M}} e_0,\Omega H_{\mathcal{M}} e_1)\) is a reflexive pair of morphisms in \(\mathcal{M}\). Assume that \(\mathcal{M}\) has coequalizers and that the tensor functors preserve them. Then \((e_0,e_1)\) has a coequalizer \((C,p): B \to C\) in \(Br_{\mathcal{M}}\) which is preserved both by the functors \(H_{\mathcal{M}}: Br_{\mathcal{M}} \to \mathcal{M}\) and the functor \(\Omega H_{\mathcal{M}}\) (in particular \(\Omega H_{\mathcal{M}} p\) is a regular epimorphism in \(\mathcal{M}\) in the sense of [Bo1, Definition 4.3.1]).

The same statement holds when we replace \(Br_{\mathcal{M}}\) by \(Br_{\mathcal{L}}\) and \(H_{\mathcal{M}}\) by the corresponding forgetful functor.
Proof. Let \((A, m_A, u_A, c_A) := \mathbb{A}\) and let \((B, m_B, u_B, c_B) := \mathbb{B}\). By Proposition \(\text{AM}, 3\), we have that 
\((H_{\text{Alg}} e_0, H_{\text{Alg}} e_1)\) has a coequalizer \(((C, m_C, u_C), p : (B, m_B, u_B) \to (C, m_C, u_C))\) in \(\text{Alg}_M\) and it is preserved by \(\Omega\). Thus, we have the following coequalizer in \(M\)

\[
\begin{array}{ccc}
A & \xrightarrow{e_0} & B \\
& \searrow^{e_1} & \downarrow^{p} \\
& & C
\end{array}
\]

where \(e_0, e_1\) and \(p\) denotes the same morphisms regarded as morphisms in \(M\)(hence, by construction, \(p\) is a regular epimorphism in \(M\)). By Lemma \(\text{B.8}\), we have the following coequalizer

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{e_0 \otimes e_0} & B \otimes B \\
& \searrow^{e_1 \otimes e_1} & \downarrow^{p \otimes p} \\
& & C \otimes C
\end{array}
\]

We have

\[(p \otimes p) c_B (e_0 \otimes e_0) = (p \otimes p) (e_0 \otimes e_0) c_A = (p \otimes p) (e_1 \otimes e_1) c_A = (p \otimes p) c_B (e_1 \otimes e_1)\]

so that there is a unique morphism \(c_C : C \otimes C \to C \otimes C\) such that

\[c_C (p \otimes p) = (p \otimes p) c_B.\]

Now, by Lemma \(\text{B.1}\), we have that \(e_0, e_1 : (A, m_A, u_A, c_A^{-1}) \to (B, m_B, u_B, c_B^{-1})\) are morphisms of braided objects. Hence the same argument we used above proves that there is a unique morphism \(\tilde{c}_C : C \otimes C \to C \otimes C\) such that \(\tilde{c}_C (p \otimes p) = (p \otimes p) c_B^{-1}\). We have \(\tilde{c}_C c_C (p \otimes p) = \tilde{c}_C (p \otimes p) c_B = (p \otimes p) c_B = (p \otimes p) \tilde{c}_B = p \otimes p\) and hence \(\tilde{c}_C c_C = \text{Id}_{C \otimes C}\). Similarly \(c_C \tilde{c}_C = \text{Id}_{C \otimes C}\) so that \(c_C\) is invertible.

By Lemma \(\text{B.4}\), \((C, c)\) is an object in \(\text{Br}_M\) and \(p : (B, c_B) \to (C, c)\) is a morphism in \(\text{Br}_M\). It is straightforward to check that \((C, m_C, u_C, c_C), p : (B, m_B, u_B, c_B) \to (C, m_C, u_C, c_C)\) is the coequalizer of \((e_0, e_1)\).

Consider now the case of \(\text{BrAlg}_M^s\) so that \((e_0, e_1)\) as above is a pair in \(\text{BrAlg}_M^s\). Since \(p\) is an epimorphism, by Remark \(\text{B.2}\), we get that \((C, m_C, u_C, c_C) \in \text{BrAlg}_M^s\) and \(p\) becomes a morphism in this category. Since \(\text{BrAlg}_M^s\) is a full subcategory of \(\text{BrAlg}_M\) we have that \(\text{inj}_{\text{BrAlg}} : \text{BrAlg}_M^s \to \text{BrAlg}_M\) is full and faithful and hence it reflects coequalizers (dual to \(\text{Boe}\). Proposition 2.9.9)) so that the above coequalizer obtained in \(\text{BrAlg}_M\) is indeed a coequalizer in \(\text{BrAlg}_M^s\).

**Proposition B.8.** Let \(M\) be a monoidal category such that the tensor functors preserve coequalizers. Let \((e_0, e_1) : \mathbb{A} \to \mathbb{B}\) be a pair of morphisms in \(\text{BrBialg}_M\) such \((\tilde{u}_{\text{Br}}, e_0, \tilde{u}_{\text{Br}}, e_1)\) has a coequalizer in \(\text{BrAlg}_M^s\) which is preserved by the functor \(H_{\text{Alg}} : \text{BrAlg}_M \to \text{Alg}_M\) and which is a regular epimorphism when regarded in \(M\). Then \((e_0, e_1)\) has a coequalizer in \(\text{BrBialg}_M\) which is preserved by the functor \(\tilde{u}_{\text{Br}} : \text{BrBialg}_M \to \text{BrAlg}_M\).

The same statement holds when we replace \(\text{BrBialg}_M\), \(\text{BrAlg}_M\) and \(\tilde{u}_{\text{Br}}\) by \(\text{BrAlg}_M^s\), \(\text{BrAlg}_M^s\) and \(\tilde{u}_{\text{Br}}^s\) respectively and we replace \(H_{\text{Alg}}\) by the corresponding forgetful functor.

**Proof.** Let \((A, m_A, u_A, \Delta_A, \varepsilon_A, c_A)\) be the domain of \(e_0\) and let \((B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)\) be its codomain. Now, the pair \((\tilde{u}_{\text{Br}}, e_0, \tilde{u}_{\text{Br}}, e_1)\) has a coequalizer in \(\text{BrAlg}_M\), say

\[((C, m_C, u_C, c_C), p : (B, m_B, u_B, c_B) \to (C, m_C, u_C, c_C)),\]

which is preserved by the functor \(H_{\text{Alg}} : \text{BrAlg}_M \to \text{Alg}_M\) and such that \(p\) is a regular epimorphism in \(M\). Denote by \(e_0, e_1\) and \(p\) the same morphisms regarded as morphisms in \(\text{Alg}_M\). By \([\text{AM}]\), Lemma 2.3, \((A \otimes A, m_{A \otimes A}, u_{A \otimes A}) \in \text{Alg}_M\), where \(m_{A \otimes A} := (m_A \otimes m_A) (A \otimes c_A \otimes B)\) and \(u_{A \otimes A} := (u_A \otimes u_A) \Delta_1\). Similarly, \((C \otimes C, m_{C \otimes C}, u_{C \otimes C}) \in \text{Alg}_M\). Since \((B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)\) is a braided bialgebra, we have that \(\Delta_B : (B, m_B, u_B) \to (A \otimes A, m_{A \otimes A}, u_{A \otimes A})\) is an algebra map. Moreover, by Proposition \(\text{AM}, 3\) of Proposition 2.2, we have that \(p \otimes p\) is a morphism in \(\text{Alg}_M\). Thus \((p \otimes p) \Delta_B\) is an algebra map. Since \(H_{\text{Alg}} : \text{BrAlg}_M \to \text{Alg}_M\) preserves the coequalizer of
Proof. 1) In view of A.3, the coequalizer of the pair \((\Omega(B)e_0, \Omega(B)e_1)\) the first row in the following diagram is a coequalizer in \(\text{Alg}_M\).

\[
\begin{array}{ccc}
A & \xrightarrow{e_0} & B \\
\Delta_A & & \Delta_B \\
A \otimes A & \xrightarrow{e_0 \otimes e_0} & B \otimes B & \xrightarrow{p \otimes p} & C \otimes C \\
\end{array}
\]

Since the same diagram serially commutes, by the universal property of the coequalizer in \(\text{Alg}_M\), we get that there is a unique algebra morphism \(\Delta_C : (C, m_C, u_C) \rightarrow (C \otimes C, m_{C \otimes C}, u_{C \otimes C})\) such that \(\Delta_C p = (p \otimes p) \Delta_B\). Denote by \(\Delta_C\) the same morphism regarded as a morphism in \(M\). Since \(p\) is an epimorphism in \(M\), one easily checks that \(\Delta_C\) is a coalgebra map. Hence \((C, \Delta_C, \varepsilon_C)\) is a coalgebra in \(M\) and \(p : (B, \Delta_B, \varepsilon_B) \rightarrow (C, \Delta_C, \varepsilon_C)\) is a coalgebra map.

Since \(p\) is a regular epimorphism in \(M\) we have \(p \otimes p\) is an epimorphism too by the assumption on the tensor products. Using this fact, that \((B, \Delta_B, \varepsilon_B)\) is a braided coalgebra and that \(p\) is a coalgebra morphism compatible with the braiding, one easily checks that \((C, \Delta_C, \varepsilon_C, \varepsilon_C)\) is a braided coalgebra too and hence \(p\) a morphism of these braided coalgebras. Summing up \(p : (B, m_B, u_B, \Delta_B, \varepsilon_B, \varepsilon_B, \varepsilon_C) \rightarrow (C, m_C, u_C, \Delta_C, \varepsilon_C, \varepsilon_C, \varepsilon_C)\) is a morphism of braided bialgebras in \(M\). Using the fact that \(p\) is an epimorphism in \(M\), one easily checks it is the searched coequalizer. The symmetric case can be treated analogously. \(\square\)

Corollary B.9. Let \(M\) be a monoidal category. Let \((e_0, e_1)\) be a pair of morphisms in \(\text{BrBialg}_M\) such that \((\Gamma e_0, \Gamma e_1)\) is a reflexive pair of morphisms in \(M\) where \(\Gamma := \Omega H_{\text{Alg}} \Omega_{\text{Br}} : \text{BrBialg}_M \rightarrow M\) denotes the forgetful functor. Assume that \(M\) has coequalizers and that the tensor functors preserve them. Then \((e_0, e_1)\) has a coequalizer in \(\text{BrBialg}_M\) which is preserved by the functors \(\Omega_{\text{Br}} : \text{BrBialg}_M \rightarrow \text{BrAlg}_M\), \(H_{\text{Alg}} \Omega_{\text{Br}} : \text{BrBialg}_M \rightarrow \text{Alg}_M\) and \(\Gamma\), and which is a regular epimorphism when regarded in \(M\).

The same statement holds when we replace \(\text{BrBialg}_M\), \(\text{BrAlg}_M\) and \(\Omega_{\text{Br}}\) by \(\text{BrAlg}_M^s\), \(\text{BrAlg}_M^s\) and \(\Omega_{\text{Br}}^s\) respectively and we replace \(H_{\text{Alg}}\) by the corresponding forgetful functor.

Proof. The pair \((\Omega(B)e_0, \Omega(B)e_1)\) fulfills the requirements of Lemma B.7 so that \((\Omega(B)e_0, \Omega(B)e_1)\) has a coequalizer in \(\text{BrAlg}_M\) which is preserved by the functors \(H_{\text{Alg}} : \text{BrAlg}_M \rightarrow \text{Alg}_M\) and \(\Omega H_{\text{Alg}}\) (and which is a regular epimorphism when regarded in \(M\)). Hence we can apply Proposition B.8 to conclude. \(\square\)

Lemma B.10. Let \(M\) and \(N\) be monoidal categories. Assume that both \(M\) and \(N\) have coequalizers and that the tensor functors preserve them. Assume that there exists a monoidal functor \((F, \phi_0, \phi_2) : M \rightarrow N\) which preserves coequalizers. Then

1) \(\text{Alg}_F : \text{Alg}_M \rightarrow \text{Alg}_N\) preserves coequalizers.

2) \(\text{BrBialg}_F : \text{BrBialg}_M \rightarrow \text{BrBialg}_N\) preserves coequalizers of reflexive pairs of morphisms. The same statement holds when we replace \(\text{BrBialg}\) by \(\text{BrBialg}^s\) everywhere.

Proof. 1) In view of A.3 the coequalizer of the pair \((\alpha, \beta)\) of algebra morphisms \(E \rightarrow A\) is, as an object in \(M\), the coequalizer \((B, \pi : A \rightarrow B)\) of \((\Lambda_\alpha, \Lambda_\beta)\) in \(M\) and the algebra structure is the unique one making \(\pi\) an algebra morphism. Since \(F\) preserves coequalizers, we get the coequalizer in \(N\)

\[
\begin{array}{ccc}
F(A \otimes E \otimes A) & \xrightarrow{F(\Lambda_\alpha)} & FA & \xrightarrow{F\pi} & FB \\
\end{array}
\]
Note that, since Alg\(F\) is a functor, we have that \(FA, FB\) are algebras and \(F\pi\) is an algebra morphism.

Using the definition of \(\Lambda\), the naturality of \(\phi_2\), the equality \(m_{FA} = Fm_A \circ \phi_2 (A, A)\) and the definition of \(\Lambda_{F\alpha}\), one proves that \(F (\Lambda_{\alpha}) \circ \phi_2 (\hat{A} \otimes E, A) \circ (\phi_2 (A, E) \otimes FA) = \Lambda_{F\alpha}\) and similarly with \(\beta\) in place of \(\alpha\). Since \(\phi_2 (A \otimes E, A) \circ (\phi_2 (A, E) \otimes FA)\) is an isomorphism, we get the coequalizer

\[
FA \otimes FE \otimes FA \xrightarrow{\Lambda_{F\alpha}} FA \xrightarrow{F\pi} FB.
\]

By construction we get that \((FB, F\pi)\) is the coequalizer of \((\Lambda_{F\alpha}, \Lambda_{F\beta})\) in \(N\). Since, as observed, \(FA\) and \(FB\) are algebras and \(F\pi\) is an algebra morphism, we conclude that \((FB, F\pi)\) is the coequalizer of \((F\alpha, F\beta)\) in \(N\) (apply \(L\)) again.

2) Consider a coequalizer of a reflexive pair in \(\text{BrBialg}_M\)

\[
B \xrightarrow{e_0} D \xrightarrow{d} E
\]

If we apply the forgetful functor \(\Gamma := \Omega H_{\text{Alg}B} : \text{BrBialg}_M \to M\) to the pair, we get a reflexive pair in \(M\). By Corollary B.9, \((e_0, e_1)\) has a coequalizer in \(\text{BrBialg}_M\) (different from (112), in principle) which is preserved by the functor \(H_{\text{Alg}B} : \text{BrBialg}_M \to \text{Alg}_M\). By uniqueness of coequalizers, we get that the coequalizer \((112)\) is preserved by \(H_{\text{Alg}B}\), and hence, by 1), it is preserved by \((\text{Alg}F) H_{\text{Alg}B} : \text{BrBialg}_M \to \text{Alg}_N\). Hence \((FE, Fd)\) is the coequalizer of \((Fe_0, Fe_1)\) in \(\text{Alg}_N\).

Note that \((Fe_0, Fe_1)\) is a reflexive pair of morphisms in \(\text{BrBialg}_N\). By Corollary B.9, \((Fe_0, Fe_1)\) has a coequalizer \((E', \pi : FD \to E')\) in \(\text{BrBialg}_N\) which is preserved by the functor \(H_{\text{Alg}B} : \text{BrBialg}_N \to \text{Alg}_N\). By uniqueness of coequalizers in \(\text{Alg}_N\), there is an algebra isomorphism \(\xi : E' \to FE\) such that \(\xi \circ \pi = Fd\). Since \(\text{BrBialg}F\) is a functor we have that \(FE\) is a braided bialgebra and \(Fd\) is a morphism in \(\text{BrBialg}_N\). Now, by construction \(\pi\) is a suitable coequalizer in \(N\) (which further inherits a proper braided bialgebra structure) so that, by assumption it is preserved by the tensor functors. Hence both \(\pi\) and \(\pi \circ \pi\) are epimorphisms in \(N\). Using these properties one proves that \(\xi : E' \to FE\) is a morphism in \(\text{BrBialg}_N\).

Since \(\xi\) is invertible, we obtain that \((FE, Fd)\) is the coequalizer of \((Fe_0, Fe_1)\) in \(\text{BrBialg}_N\) i.e. \(\text{BrBialg}F : \text{BrBialg}_M \to \text{BrBialg}_N\) preserves coequalizers of reflexive pairs of morphisms. The symmetric case follows analogously once observed that \(F\) preserves symmetric objects, see B.3 \(\square\)

**Proposition B.11.** Let \(M\) be a monoidal category. Assume that \(M\) has a coequalizers and that the tensor functors preserve them. Consider a right adjoint functor \(R : \text{BrBialg}_M \to B\) into an arbitrary category \(B\). Then the comparison functor \(R1\) has a left adjoint \(L1\) which is uniquely determined by the following properties.

1) For every object \((B, \mu) \in RL B\), there is a morphism \(\pi (B, \mu) : LB \to L1 (B, \mu)\) such that

\[
\Gamma LRLB \xrightarrow{\Gamma L\mu} \Gamma LB \xrightarrow{\Gamma \pi (B, \mu)} \Gamma L1 (B, \mu)
\]

is a coequalizer in \(M\), where \(\Gamma := \Omega H_{\text{Alg}B} : \text{BrBialg}_M \to M\) denotes the forgetful functor.

2) The bialgebra structure of \(\Gamma L1 (B, \mu)\) is uniquely determined by the fact that \(\Gamma \pi (B, \mu)\) is a morphism of braided bialgebras in \(M\).

3) \(R\) is comparable.

4) The statements above still hold true when \(\text{BrBialg}_M^*\) replaces \(\text{BrBialg}_M\).

**Proof.** By Beck’s Theorem, it suffices to check that for every \((B, \mu) \in RL B\) the fork \((L\mu, \epsilon LB)\) has a coequalizer in \(\text{BrBialg}_M\), where \(L\) denotes the left adjoint of \(R\) and \(\epsilon : LR \to \text{Id}_{\text{BrBialg}_M}\) the counit of the adjunction. Now \(L\mu \circ L\eta B = \text{Id}_{LB} = \epsilon LB \circ L\eta B\) where \(\eta : \text{Id}_B \to RL\) is the unit of the adjunction. Thus \((L\mu, \epsilon LB)\) is a reflexive pair of morphisms in \(\text{BrBialg}_M\). Therefore \((\Gamma L\mu, \Gamma \epsilon LB)\) is a reflexive pair of morphisms in \(M\). By Corollary B.9, the pair \((L\mu, \epsilon LB)\) has a coequalizer in \(\text{BrBialg}_M\) which is preserved both by the functors \(\Omega_{\text{Br}} : \text{BrBialg}_M \to \text{BrAlg}_M\),...
$H_{\text{Alg} \hat{\Omega}_B} : \text{BrBialg}_M \to \text{Alg}_M$ and $\Gamma := \Omega H_{\text{Alg} \hat{\Omega}_B} : \text{BrBialg}_M \to \mathcal{M}$. By construction the coequalizer of $(L_\mu, \epsilon L_B)$ is $(L_1 (B, \mu), \pi (B, \mu) : LB \to L_1 (B, \mu))$. Furthermore, $(13)$ is a coequalizer in $\mathcal{M}$ and the bialgebra structure of $\Gamma L_1 (B, \mu)$ is uniquely determined by the fact that $\Gamma \pi (B, \mu)$ is a morphism of braided bialgebras in $\mathcal{M}$. By Lemma 1.11, $R$ is comparable.

The symmetric case follows analogously. □

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University of Turin, Department of Mathematics “G. Peano”, via Carlo Alberto 10, I-10123 Torino, Italy

E-mail address: alessandro.ardizzoni@unito.it
URL: www.unito.it/persone/alessandro.ardizzoni

University of Ferrara, Department of Mathematics, Via Machiavelli 35, Ferrara, I-44121, Italy

E-mail address: men@unife.it
URL: web.unife.it/utenti/claudia.menini