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Yours sincerely,

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Mass neutrino flavor evolution in spacetime with torsion

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(ricevuto ?; approvato ?)

Summary. — In the framework of the spacetime with torsion, we obtain the flavor evolution equation of the mass neutrino oscillation in vacuum. A comparison with the result of general relativity case, it shows that the flavor evolutionary equations in Riemann spacetime and Weitzenböck spacetimes are equivalent in the spherical symmetric Schwarzschild spacetime, but turns out to be different in the case of the axial symmetry.

PACS 04.50, 04.90 – .

1. – Introduction

On account of the Super-Kamiokande atmospheric neutrino experiment, which confirmed a nonvanishing mass for the neutrino [1], the neutrino oscillation problem became an even hotter topic in high energy physics, both from the experimental and from the theoretical points of view [2, 3, 4]. As a natural extension to this problem, many authors consider the neutrino oscillation in the presence of gravitation, that is, in a curved spacetime. This means to extend the physics related to the neutrino oscillation in a Minkowski spacetime with Lorentz invariance, to a Riemannian spacetime with the usual invariance under general coordinate transformation. The gravitational effect on the neutrino oscillation has attracted much attention recently [5, 6, 7, 8, 9, 10, 11, 12, 13] in the framework of general relativity, although a lot of problems concerning the understanding of the gravitationally induced neutrino oscillation still persists. However many alternative mechanisms have been proposed to account for the gravitational effect on the neutrino oscillation; e.g. the equivalence principle and neutrino oscillation [14, 15]. As a further theoretical exploration, more recently torsion induced neutrino oscillation in $U_4$ spacetime [18] with both curvature and torsion is also proposed [16, 17]. In this article, we extend the neutrino oscillation problem into the spacetime with torsion but without curvature, i.e. in a Weitzenböck spacetime $A_4$ [19, 20, 21, 22], in the framework of new general relativity (NGR)[19].
The paper is organized as follows. In Sec. II we briefly introduce the gravitational theory in Weitzenböck spacetime. In Sec. III we compare Dirac equations in Riemannian spacetime and in Weitzenböck spacetime, and in Sec. IV. we derive the evolutionary equation of the neutrino oscillation amplitude in Weitzenböck spacetime. We set $G = \hbar = c = 1$ throughout this article.

2. – A brief review of the NGR

The new general relativity is a parallel tetrad gravitational theory, which is formulated on a Weitzenböck spacetime [19, 22, 23]. It is characterized by the vanishing curvature tensor and by the torsion tensor formed of four parallel tetrad vector fields. Namely, the gravitational field appears as the nontrivial part of the tetrad field. We will use the greek alphabet $(\mu, \nu, \rho, \cdots = 1, 2, 3, 4)$ to denote tensor indices, that is, indices related to spacetime. The latin alphabet $(a, b, c, \cdots = 1, 2, 3, 4)$ will be used to denote local Lorentz (or tangent space) indices. Of course, being of the same kind, tensor and local Lorentz indices can be changed into each other with the use of the tetrad, denoted by $h^a_{\mu}$, and supposed to satisfy

$$h^a_{\mu} h^a_{\nu} = \delta^a_{\mu} \delta^a_{\nu}, \quad h^a_{\mu} h^b_{\mu} = \delta^a_{b}.$$ (1)

As is known, curvature and torsion are properties of a connection [20, 22], and many different connections may be defined on the same space. For example, denoting by $\eta_{ab}$ the metric tensor of the tangent space, a nontrivial tetrad field can be used to define the riemannian metric

$$g_{\mu\nu} = \eta_{ab} h^a_{\mu} h^b_{\nu},$$ (2)

in terms of which the Levi–Civita connection

$$\Gamma^a_{\beta \gamma} = \frac{1}{2} g^{a\rho} \left[ \partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right]$$ (3)

can be introduced. Its curvature

$$\bar{R}^a_{\beta \mu \nu} = \partial_\beta \bar{\Gamma}^a_{\gamma \mu \nu} + \bar{\Gamma}^a_{\beta \gamma} \Gamma^a_{\rho \mu \nu} - (\mu \leftrightarrow \nu),$$ (4)

according to general relativity, accounts exactly for the gravitational interaction. Owing to the universality of gravitation, which means that all particles feel $\bar{\Gamma}^a_{\rho \mu \nu}$ the same, it turns out possible to describe the gravitational interaction by considering a Riemann spacetime with the curvature of the Levi–Civita connection, in which all particles will follow geodesics. This is the stage set of Einstein’s General Relativity, the gravitational interaction being mimicked by a geometrization of spacetime.

On the other hand, a nontrivial tetrad field can also be used to define the linear Cartan connection

$$\Gamma^a_{\mu \nu} = h^a_{\sigma} \partial_\nu h^\sigma_{\mu},$$ (5)

with respect to which the tetrad is parallel:

$$\nabla_{\nu} h^a_{\mu} = \partial_\nu h^a_{\mu} - \Gamma^a_{\rho \mu \nu} h^a_{\rho} = 0.$$ (6)
For this reason, tetrad theories have received the name of teleparallelism, or absolute parallelism. Plugging in Eqs. (2) and (3), we get

\[ \Gamma^\sigma_{\mu\nu} = \tilde{\Gamma}^\sigma_{\mu\nu} + K^\sigma_{\mu\nu}, \]

where

\[ K^\sigma_{\mu\nu} = \frac{1}{2} [T^{\sigma}_{\mu,\nu} + T^{\sigma}_{\nu,\mu} - T^{\sigma}_{\mu\nu}] \]

is the contorsion tensor, with

\[ T^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\mu\nu} \]

If now we try to introduce a spacetime with the same properties of the Cartan connection \( \Gamma^\sigma_{\nu\mu} \), we end up with a Weitzenböck spacetime [21], a space presenting torsion, but no curvature. This spacetime is the stage set of the teleparallel description of gravitation. Considering that local Lorentz indices are raised and lowered with the Minkowski metric \( \eta^{ab} \), tensor indices on it will be raised and lowered with the riemannian metric \( g_{\mu\nu} = \eta_{ab} h^{a}_{\mu} h^{b}_{\nu} \) [19]. Universality of gravitation, in this case, means that all particles feel \( \Gamma^\sigma_{\nu\mu} \) the same, which in turn means that they will also feel torsion the same.

From the above considerations, we can infer that the presence of a nontrivial tetrad field induces both, a riemannian and a teleparallel structures in spacetime. The first is related to the Levi–Civita connection, a connection presenting curvature, but no torsion. The second is related to the Cartan connection, a connection presenting torsion, but no curvature. It is important to remark that both connections are defined on the very same spacetime, a spacetime endowed with both a riemannian and a teleparallel structures.

As already remarked, the curvature of the Cartan connection vanishes identically:

\[ R^\theta_{\rho\mu\nu} = \partial_\rho \Gamma^\theta_{\mu\nu} + \Gamma^\theta_{\sigma\mu} \Gamma^\sigma_{\nu\rho} - (\mu \leftrightarrow \nu) \equiv 0. \]

Substituting \( \Gamma^\theta_{\mu\nu} \) from Eq. (7), we get

\[ R^\theta_{\rho\mu\nu} = R^\theta_{\rho\mu\nu} + Q^\theta_{\rho\mu\nu} \equiv 0, \]

The gravitational Lagrangian density in NGR is written in the form

\[ \mathcal{L}_G = \sqrt{-g} \left[ a_1 (t^{\mu\nu\lambda} t_{\mu\nu\lambda}) + a_2 (v^\mu v_\mu) + a_3 (a^\mu a_\mu) \right], \]

where \( a_1, a_2 \) and \( a_3 \) are dimensionless parameters of the theory,

\[ t_{\mu\nu\lambda} = \frac{1}{2} (T_{\mu\nu\lambda} + T_{\nu\mu\lambda}) + \frac{1}{6} (g_{\lambda\mu} v_\nu + g_{\lambda\nu} v_\mu) - \frac{1}{3} g_{\mu\nu} v_\lambda, \]

\[ v_\mu = T^\lambda_{\lambda\mu}. \]
\[ a_\mu = \frac{1}{6} \epsilon_{\mu\nu\rho\sigma} T^{\nu\rho\sigma} \]

with \( \epsilon_{\mu\nu\rho\sigma} \) being the completely antisymmetric tensor normalized as \( \epsilon_{0123} = \sqrt{-g} \). By applying variational principle to the above Lagrangian, we get the field equation:

\[ I^{\mu\nu} = \kappa T^{\mu\nu}, \quad \kappa = 8\pi, \]

with

\[ I^{\mu\nu} = 2\kappa [D_\lambda F^{\mu\nu\lambda} + v_\lambda F^{\mu\nu\lambda} + H^{\mu\nu} - \frac{1}{2} g^{\mu\nu} L_G], \]

where

\[ F^{\mu\nu\lambda} = \frac{1}{2} h^{k\mu} \frac{\partial L_G}{\partial h^{k_{\nu\lambda}}} = \frac{1}{\kappa} \left[ a_1 \left( F^{\mu\nu\lambda} - F^{\mu\lambda\nu} \right) + a_2 \left( g^{\mu\nu} v^\lambda - g^{\mu\lambda} v^\nu \right) - \frac{a_3}{3} \epsilon^{\mu\nu\lambda\rho} a_\rho \right] \]

\[ H^{\mu\nu} = T^{\rho\sigma\mu} F_{\rho\sigma}^{\nu} - \frac{1}{2} T^{\rho\sigma\mu} F_{\rho\sigma}^{\mu} = H^{\mu\nu}, \]

\[ L_G = \frac{\mathcal{L}_G}{\sqrt{-g}}, \]

\[ T^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}_M}{\delta h^{k\mu}} h^{k\nu}. \]

Here \( \mathcal{L}_M \) denotes the Lagrangian density of material fields and \( T^{\mu\nu} \) is the material energy-momentum tensor which is nonsymmetric in general. In order to reproduce the correct Newtonian limit, we require the parameters \( a_1 \) and \( a_2 \) to satisfy the condition

\[ a_1 + 4a_2 + 9a_1 a_2 = 0, \]

called the Newtonian approximation condition, which can be solved to give

\[ a_1 = -\frac{1}{3(1-\epsilon)}, \quad a_2 = \frac{1}{3(1-4\epsilon)} \]

with \( \epsilon \) being a dimensionless parameter. The comparison with solar-system experiments shows that \( \epsilon \) should be given by

\[ \epsilon = -0.004 \pm 0.004, \]
3. – Dirac equation in Weitzenböck spacetime

Previous to entering to our main point, we stress that the semiclassical by approximated Dirac particle does not follow a geodesic exactly. However the force aroused by the spin and the curvature coupling has little contribution to the geodesic deviation[26]. So here we take the neutrino as a spinless particle to go along the geodesic. The gravitational effects on the spin are incorporated into Dirac equation through the “spin connection” $\Gamma_\mu$ appearing in the Dirac equation in curved spacetime [24, 25, 26], which is constructed by means of the variation of the covariant Lagrangian of the spinor field. In the parallel tetrad theory of Hayashi and Shirafuji [19], considering the covariant derivative of spinor to coincide with the usual derivative, the Dirac Lagrange density $L_D$ is given by

\[ L_D = \frac{1}{2} h^\mu [\psi \gamma^k \partial_\mu \bar{\psi} - \partial_\mu \bar{\psi} \gamma^k \psi] - m \bar{\psi} \psi. \]  

By taking variation with respect to $\bar{\psi}$, the Dirac equation in Weitzenböck spacetime is given as

\[ [\gamma^a h^\mu_a (\partial_\mu + \Gamma_\mu) + m] \psi = 0, \]

and the spin connection $\Gamma_\mu$ is

\[ \Gamma_\mu = \frac{1}{2} v_\mu \]

where $v_\mu$ is the tetrad vector.

The spin connection $\Gamma_\mu$ is different from that of general relativity because the parallelism of vector in Weitzenböck spacetime makes the covariant derivative of spinor to coincide the usual derivative. However, the Lagrangian of Dirac equation in general relativity is constructed by the covariant derivative and its explicit expression for the spin connection $\Gamma_\mu$ is [9]

\[ \Gamma_\mu = \frac{1}{8}[\gamma^b, \gamma^c] h_b^\nu h_c^\gamma \eta_{\nu\gamma}. \]

We must first simplify the Dirac matrix product in the spin connection term. It can be shown that

\[ \gamma^a [\gamma^b, \gamma^c] = 2\eta^{ab} \gamma^c - 2\eta^{ac} \gamma^b - 2i \epsilon^{abc} \gamma_5, \]

where $\eta^{ab}$ is the metric of flat space and $\epsilon^{abcd}$ is the (flat space) totally antisymmetric tensor, with $\epsilon^{0123} = +1$. With Eq.(30), the contribution from the spin connection in general relativity is

\[ \Gamma_\mu = \frac{1}{2} v_\mu - \frac{3i}{4} a_\mu \gamma_5, \]

where

\[ a_\mu = \frac{1}{6} e_{\mu\nu\lambda\sigma} T^{\nu\lambda\sigma}. \]
$a_\mu$ is the tetrad axial-vector represented the deviation of the axial symmetry from the spherical symmetry [27].

Or, in the spherical case, Schwarzschild spacetime, both Dirac equations in Riemannian spacetime and in Weitzenböck spacetime are equivalent. The difference between them will appear if the spacetime includes the axial symmetric components, Kerr spacetime for instance.

4. – Evolutionary equation for the neutrino oscillation amplitude

As proceeded in ref.[9], in order to incorporate the gravitational effect into the matter effect, we rewrite the spin connection term as

$$\gamma^a h_\mu h^\mu \Gamma_\mu = \gamma^a h_\mu (i A_G \mathcal{P}_L) = \gamma^a h_\mu \left\{ i A_G \left[ -\frac{1}{2\sqrt{-g}} \gamma_5 \right] \right\},$$

where $\mathcal{P}_L = -\frac{1}{2\sqrt{-g}} \gamma_5$ is the left–handed projection operator, and

$$A_G^\mu \equiv 2i(-g)^{1/2} \gamma_5 v_\mu$$

In the above equations, $(-g)^{1/2} = [\det(g_{\mu\nu})]^{1/2}$. Proceeding as in the discussion by Cardal and Fuller [9], we will borrow the three-momentum operator used in the neutrino oscillation, which can be calculated from the mass shell condition obtained by iterating the Dirac equation

$$(P_\mu + A_G \mathcal{P}_L)(P_\nu + A_G^\mu \mathcal{P}_L) = -M_f^2,$$

where we have not included background matter effects. $M_f^2$ is the vacuum mass matrix in favour basis

$$M_f^2 = U \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} U^\dagger,$$

where

$$U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},$$

with $\theta$ the mixing angle between different eigenstates of mass neutrinos. For relativistic neutrinos, ignoring terms of $\mathcal{O}(A^2)$ and $\mathcal{O}(AM)$, and remembering that we employ a null tangent vector $n^\mu$, which is defined as $n^\mu = dx^\mu / d\lambda$, and $x^\mu(\lambda) = \{ x^0(\lambda), x^1(\lambda), x^2(\lambda), x^3(\lambda) \}$, we find

$$P_\mu n^\mu = -\left( M_f^2 / 2 + A_G n^\mu \right).$$

It is convenient to define a column vector of flavor amplitudes. For example, for the mixing between $\nu_e$ and $\nu_\tau$,

$$\chi(\lambda) \equiv \begin{pmatrix} \langle \nu_e | \Psi(\lambda) \rangle \\ \langle \nu_\tau | \Psi(\lambda) \rangle \end{pmatrix}.$$
Eq. (39) can be written as a differential equation for the null world line parameter $\lambda$,

$$i \frac{d\chi}{d\lambda} = \left( \frac{M^2_f}{2} + A_f G_{\mu} n^\mu \right) \chi,$$

where the subscript $f$ denotes “flavor basis”. Eq. (40) can be integrated to yield the neutrino flavor evolution. Similar equations were obtained in Refs. [9, 17] in Riemannian spacetime and in $U_4$ spacetime respectively.

5. – Conclusion and discussion

In this paper, we studied the evolution equation for the neutrino oscillation amplitude in the framework of the new general relativity [19]. We find that our results will be equivalent to that of general relativity in the case of spherical symmetry, and the difference will occur when the axial tetrad vector is not zero.

Acknowledgments

The author would like to thank FAPESP-Brazil for financial support, and J.G. Pereira for helpful discussions. Thanks are also due to the hospitalities from S. Civaram, K. Hayashi, F.W. Hehl and J.M. Nester when he visited their research groups.

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Substituting \( \Gamma^\theta_{\mu\nu} \) from Eq. (7), we get

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The gravitational Lagrangian density in NGR is written in the form

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\mathcal{L}_G = \frac{\sqrt{-g}}{\kappa} \left[ a_1 (\mathcal{D}^\mu \mathcal{D}_\mu A_\lambda) + a_2 (F_{\mu\nu} F^{\mu\nu}) + a_3 (F_{\mu\nu} F^{\mu\nu}) \right] ,
\end{equation}

where \( a_1, a_2 \) and \( a_3 \) are dimensionless parameters of the theory,

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t_{\mu\nu\lambda} = \frac{1}{2} (T_{\mu\nu\lambda} + T_{\nu\mu\lambda}) + \frac{1}{6} (g_{\lambda\mu} v_\nu + g_{\lambda\nu} v_\mu) - \frac{1}{3} g_{\mu\nu} v_\lambda ,
\end{equation}

\begin{equation}
v_\mu = T^\lambda_{\mu\lambda} ,
\end{equation}
\[
a_{\mu} = \frac{1}{6} \epsilon_{\mu \nu \rho \sigma} T^{\nu \rho \sigma}
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with \( \epsilon_{\mu \nu \rho \sigma} \) being the completely antisymmetric tensor normalized as \( \epsilon_{0123} = \sqrt{\text{det} g} \). By applying variational principle to the above Lagrangian, we get the field equation:

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with

\[
I^{\mu \nu} = 2\kappa [D_{\lambda} F^{\mu \nu \lambda} + v_{\lambda} F^{\mu \nu \lambda} + H^{\mu \nu} - \frac{1}{2} g^{\mu \nu} L_G],
\]

where

\[
F^{\mu \nu \lambda} = \frac{1}{2} h^{k \mu} \frac{\partial L_G}{\partial h^{k \nu \lambda}} = \frac{1}{\kappa} \left[ a_1 (g^{\mu \nu} - g^{\mu \nu}_{\lambda}) + a_2 (g^{\mu \nu}_{\lambda} v_{\nu} - g^{\mu \nu}_{\lambda} v^2) - \frac{a_3}{3} \epsilon_{\mu \nu \lambda} \epsilon_{\rho \lambda \rho} \right],
\]

\[
H^{\mu \nu} = T^{\rho \sigma \mu} F_{\rho \sigma \nu} - \frac{1}{2} T^{\rho \sigma \rho} F_{\mu \sigma} = H^{\mu \nu},
\]

\[
L_G = \frac{\mathcal{L}_G}{\sqrt{-g}},
\]

\[
T^{\mu \nu} = \frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}_M}{\delta h^{k \nu}} h^{k \mu}.
\]

Here \( \mathcal{L}_M \) denotes the Lagrangian density of material fields and \( T^{\mu \nu} \) is the material energy-momentum tensor which is nonsymmetric in general. In order to reproduce the correct Newtonian limit, we require the parameters \( a_1 \) and \( a_2 \) to satisfy the condition

\[
a_1 + 4a_2 + 9a_3 a_2 = 0,
\]

called the Newtonian approximation condition, which can be solved to give

\[
a_1 = \frac{1}{3(1 - \epsilon)} \quad a_2 = \frac{1}{3(1 - 4\epsilon)}
\]

with \( \epsilon \) being a dimensionless parameter. The comparison with solar-system experiments shows that \( \epsilon \) should be given by

\[
\epsilon = -0.004 \pm 0.004,
\]
3. Dirac equation in Weitzenböck spacetime

Previous to entering to our main point, we stress that the semiclassical by approximated Dirac particle does not follow a geodesic exactly. However the force aroused by the spin and the curvature coupling has little contribution to the geodesic deviation\(\text{[26]}\). So here we take the neutrino as a spinless particle to go along the geodesic. The gravitational effects on the spin are incorporated into Dirac equation through the “spin connection” \(\Gamma_\mu\) appearing in the Dirac equation in curved spacetime \([24, 25, 26]\), which is constructed by means of the variation of the covariant Lagrangian of the spinor field. In the parallel tetrad theory of Hayashi and Shirafuji \([19]\), considering the covariant derivative of spinor to coincide with the usual derivative, the Dirac Langrange density \(L_D\) is given by

\[
L_D = \frac{1}{2} h^\mu_k \gamma^k \partial_\mu \bar{\psi} - \partial_\mu \bar{\psi} \gamma^k \psi - m \bar{\psi} \psi.
\]

By taking variation with respect to \(\bar{\psi}\), the Dirac equation in Weitzenböck spacetime is given as

\[
[\gamma^a h^\mu_a (\partial_\mu + \Gamma_\mu) + m] \psi = 0,
\]

and the spin connection \(\Gamma_\mu\) is

\[
\Gamma_\mu = \frac{1}{2} v_\mu
\]

where \(v_\mu\) is the tetrad vector.

The spin connection \(\Gamma_\mu\) is different from that of general relativity because the parallelism of vector in Weitzenböck spacetime makes the covariant derivative of spinor to coincide the usual dirivative. However, the Lagrangian of Dirac equation in general relativity is constructed by the covariant derivative and its explicit expression for the spin connection \(\Gamma_\mu\) is \([9]\)

\[
\Gamma_\mu = \frac{1}{8} [\gamma^b, \gamma^c] h^\mu_b h_{\nu c \mu}.
\]

We must first simplify the Dirac matrix product in the spin connection term. It can be shown that

\[
\gamma^a [\gamma^b, \gamma^c] = 2 \eta^{ab} \gamma^c - 2 \eta^{ac} \gamma^b - 2 \epsilon^{abcd} \gamma_d,
\]

where \(\eta^{ab}\) is the metric of flat space and \(\epsilon^{abcd}\) is the (flat space) totally antisymmetric tensor, with \(\epsilon^{0123} = +1\). With Eq.\((30)\), the contribution from the spin connection in general relativity is

\[
\Gamma_\mu = \frac{1}{2} v_\mu - \frac{3i}{4} a_\mu \gamma_5,
\]

where

\[
a_\mu = \frac{1}{6} \epsilon_{\mu \nu \lambda \sigma} T^{\nu \lambda \sigma}
\]
\( a_\mu \) is the tetrad axial-vector represented the deviation of the axial symmetry from the spherical symmetry [27].

Or, in the spherical case, Schwarzschild spacetime, both Dirac equations in Riemannian spacetime and in Weyl-\( \text{bc} \) spacetime are equivalent. The difference between them will appear if the spacetime includes the axial symmetric components, Kerr spacetime for instance.

4. Evolutionary equation for the neutrino oscillation amplitude

As proceeded in ref. [9], in order to incorporate the gravitational effect into the matter effect, we rewrite the spin connection term as

\[
\gamma^\alpha h^\mu_\alpha \Gamma_\mu = \gamma^0 h^\mu_0 (iA_{G\mu}P_L) = \gamma^0 h^\mu_0 \left\{ iA_{G\mu} \left[ \frac{1}{2\sqrt{-g}} \gamma_5 \right] \right\},
\]

where \( P_L = -\frac{1}{2\sqrt{-g}} \gamma_5 \) is the left-handed projection operator, and

\[
A^\mu_0 \equiv 2i(-g)^{1/2}\gamma_5 v_\mu.
\]

In the above equations, \((-g)^{1/2} = |\det(g_{\mu\nu})|^{1/2}\). Proceeding as in the discussion by Cardal and Fuller [9], we will borrow the three-momentum operator used in the neutrino oscillation, which can be calculated from the mass shell condition obtained by iterating the Dirac equation

\[
(P_\mu + A_{G\mu}P_L)(P^\mu + A^\mu_0 P_L) = -M_f^2,
\]

where we have not included background matter effects. \( M_f^2 \) is the vacuum mass matrix in flavour basis

\[
M_f^2 = U \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} U^\dagger,
\]

where

\[
U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},
\]

with \( \theta \) the mixing angle between different eigenstates of mass neutrinos. For relativistic neutrinos, ignoring terms of \( \mathcal{O}(A^2) \) and \( \mathcal{O}(AM) \), and remembering that we employ a null tangent vector \( n^\mu \), which is defined as \( n^\mu = dx^\mu/d\lambda \), and \( x^\mu(\lambda) = [x^0(\lambda), x^1(\lambda), x^2(\lambda), x^3(\lambda)] \), we find

\[
P_\mu n^\mu = -\left( M_f^2/2 + A_{G\mu}n^\mu \right).
\]

It is convenient to define a column vector of flavor amplitudes. For example, for the mixing between \( \nu_e \) and \( \nu_\tau \),

\[
\chi(\lambda) \equiv \begin{pmatrix} \langle \nu_e | \Psi(\lambda) \rangle \\ \langle \nu_\tau | \Psi(\lambda) \rangle \end{pmatrix}.
\]
Eq. (39) can be written as a differential equation for the null world line parameter $\lambda$,

\[ i \frac{d\chi}{d\lambda} = \left( \frac{M_f^2}{2} + A_{f\mu}n^\mu \right) \chi, \]

where the subscript $f$ denotes "flavor basis". Eq. (40) can be integrated to yield the neutrino flavor evolution. Similar equations were obtained in Refs. [9, 17] in Riemannian spacetime and in $U_4$ spacetime respectively.

5. Conclusion and discussion

In this paper, we studied the evolution equation for the neutrino oscillation amplitude in the framework of the new general relativity [19]. We find that our results will be equivalent to that of general relativity in the case of spherical symmetry, and the difference will occur when the axial tetrad vector is not zero.

Acknowledgments

The author would like to thank FAPESP-Brazil for financial support, and J.G. Pereira for helpful discussions. Thanks are also due to the hospitalities from S. Civaram, K. Hayashi, F.W. Hehl and J.M. Nester when he visited their research groups.

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