Abstract

We review the theory of higher-spin gauge fields in four and three space-time dimensions and present some new results on higher-spin gauge interactions of matter fields in two dimensions.
1 Introduction

The aim of this talk is twofold. We review the previously obtained results in the theory of higher-spin gauge fields in four and three space-time dimensions and present some new results on higher-spin gauge interactions of matter fields in two dimensions.

The problem of existence of consistent gauge invariant theories of interacting massless fields of higher spins is one of fundamental questions in field theory. A very stimulating argument came in the late seventies due to the supergravity theory \[1\] after it was realized that the restriction \( s \leq 2 \) on the spins of the particles in the massless supergravity supermultiplet leads to a famous limitation \( N \leq 8 \) on the number of gravitinos which plays a crucial role in supergravity theory and is a direct consequence of the former restriction exhibiting the inability to work with interacting higher-spin gauge fields at that stage.

Another motivation comes from the superstring theory which is known to describe infinite collections of higher-spin excitations of all spins \[2\]. In string theory all higher-spin excitations are massive. One can speculate however that they can be obtained by virtue of a certain spontaneous breaking mechanism from some symmetric phase of the theory. The crucial question is then what is a fundamental theory of interacting gauge higher-spin fields which underlies this most symmetric phase of the string theory?

Theories of interacting massless fields of all spins indeed exist \[3, 4\] constituting a new class of gauge theories based on certain infinite-dimensional gauge symmetries, higher-spin gauge symmetries \[5, 6, 7\]. An important property of the higher-spin theories is that infinite-dimensional higher-spin gauge symmetries contain lower-spin \((s \leq 2)\) gauge symmetries as (maximal) finite-dimensional subalgebras. As a result higher-spin gauge theories describe infinite collections of higher-spin massless fields of all spins \(0 \leq s < \infty\) and generalize naturally usual lower-spin gauge theories containing them as subtheories (truncations). Thus, higher-spin theories can be thought of as most general gauge theories in \(3 + 1\) space-time dimensions. One can speculate that the fact that higher-spin gauge symmetries are infinite-dimensional offers good perspectives for constructing quantum-mechanically consistent theories unifying gravity with other interactions provided that higher-spin gauge symmetries are powerful enough to ensure the cancellation of divergences. On the other hand, that the theory of higher-spin gauge fields in \(3+1\) dimensions contains infinitely many fields of all spins makes it indeed reminiscent of superstring theory thus giving an additional argument in favor of the relationship of higher-spin theories with an unbroken phase of string theory.

In fact, the infinite-dimensional higher-spin symmetries are closely related to (centerless) \(W_{1+\infty}\) algebra and its further generalizations described below. So, higher-spin gauge theories were shown \[3, 6, 7\] to be gauge theories of \(W_{1+\infty}\) before the name \(W_{1+\infty}\) was invented \[8\]. Since \(W_{1+\infty}\) is a fundamental symmetry which nowadays proves to be important in many physical models such as conformal models, integrable systems etc., one can speculate that a gauge theory of \(W_{1+\infty}\) may be intrinsically related to all these
models.

The property that higher-spin theories describe infinite collections of fields means that one has to develop adequate methods to handle them efficiently. One of such methods we are going to focus on in this report, which we call “unfolded formulation”, allows us to formulate dynamical equations of a system under investigation as some zero-curvature equations supplemented with certain constraints which do not involve any space-time derivatives. This formulation is remarkable on its own right because it allows one to reduce entirely the dynamical content of the theory to the analysis of the constraints. The crucial point here is that such a formulation requires infinitely many auxiliary fields which appear very naturally in the higher-spin gauge theories. In principle one can use analogous formulation in any other relativistic theory that may be useful for the analysis of the standard non-linear field theories like Yang-Mills and Einstein theories.

Generally, the formulation of dynamical equations in the “unfolded form” does not imply automatically that the system is solvable because the aforementioned constraints may be difficult to solve themselves. The remarkable feature of the new 1+1 model we focus on in this talk is that it does not require any constraints at all. As a result, the model of higher-spin gauge interactions for matter fields in two dimensions we present in this talk turns out to be integrable. Let us stress that this model is not conformal, while its integrable (in fact topological) form is a consequence of gauging the higher-spin symmetries of d2 matter fields.

2 Lower-Spin Examples

The characteristic property of gauge theories is invariance under local symmetries, i.e. symmetries with parameters being arbitrary functions of the space-time coordinates.

Historically, the first example of a gauge field theory was provided by the Maxwell theory of electromagnetism. In this case, the gauge field is identified with the vector potential \( A_\nu \) which gives rise to the field strength

\[
F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu, \quad \partial_\nu = \frac{\partial}{\partial x^\nu}, \quad \nu = 0 \div 3
\]

invariant under the gauge (gradient) transformations

\[
\delta A_\nu = \partial_\nu \varepsilon
\]

with an arbitrary gauge parameter \( \varepsilon(x) \). As is well known, the gauge invariant Maxwell action,

\[
S = -\frac{1}{4} \int d^4x \, F_{\nu\mu} F^{\nu\mu}, \quad \delta S = 0
\]

describes spin-1 massless particles, photons.
Maxwell theory can be generalized to Yang-Mills theory [9], by introducing a system of mutually charged spin-1 particles described by matrix-valued potentials $A_{\nu j}^{\mu}$ taking values in some Lie algebra $h$. The corresponding field strengths, gauge transformations and action read, respectively

$$G_{\nu \mu} = \partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu} + g [A_{\nu}, A_{\mu}],$$  \hspace{1cm} (2a)
$$\delta A_{\nu} = \partial_{\nu} \varepsilon + g [A_{\nu}, \varepsilon],$$  \hspace{1cm} (2b)
$$S = -\frac{1}{4} \int d^4x \text{tr}(G_{\nu \mu} G^{\nu \mu}).$$  \hspace{1cm} (2c)

Yang-Mills theory can be thought of as a theory of interacting massless spin−1 particles. In fact, under some reasonable conditions on the orders of derivatives [10], the principle of gauge symmetry fixes spin−1 interactions unambiguously up to a choice of a gauge group.

The second text-book example of a gauge theory is general relativity. Here the role of the gauge field is played by the metric tensor $g_{\mu \nu}$ while gauge transformations are identified with the diffeomorphisms

$$\delta g_{\nu \mu} = \partial_{\nu} (\varepsilon^\rho) g_{\rho \mu} + \partial_{\mu} (\varepsilon^\rho) g_{\rho \nu} + \varepsilon^\rho \partial_{\rho} (g_{\nu \mu}),$$  \hspace{1cm} (1)

where $\varepsilon^\rho(x)$ are infinitesimal parameters. The gauge principle identifies with the Einstein equivalence principle. The invariant Einstein-Hilbert action

$$S = -\frac{1}{4\kappa^2} \int \sqrt{-\det |g|} (R + \Lambda)$$  \hspace{1cm} (2)

depends on two independent coupling constants, the gravitational constant $\kappa$ and the cosmological constant $\Lambda$. To interpret this theory in terms of particles, one implements the expansion procedure $g_{\nu \mu} = \eta_{\nu \mu} + \kappa h_{\nu \mu}$ where $\eta_{\nu \mu}$ is some fixed background metric (flat for $\Lambda = 0$ or (anti-) de Sitter for $\Lambda \neq 0$) and $h_{\nu \mu}$ describes dynamical perturbations. It was shown by Fierz and Pauli [11] for the flat case $\Lambda = 0$ that the linearized action $S$ describes free spin−2 massless particles, gravitons. Once again, under certain reasonable conditions, the Einstein-Hilbert action is the only consistent (gauge invariant) one for a selfinteracting spin−2 massless field [12, 13].

In four dimensions, the only non-trivial modification of the spin−1 and spin−2 gauge theories is supergravity [1], the theory which, in addition to spin−1 and spin−2 gauge fields, describes spin−3/2 massless fields, gravitinos, which are responsible for local supersymmetry transformations with spinorial gauge parameters $\varepsilon_\alpha(x)$. A novel feature characterizing supersymmetry is that it relates interactions for fields carrying different spins and, in particular, for bosons and fermions. Again, pure supergravity is the only consistent theory that describes consistent interactions of spin−3/2 particles.

Thus, the conventional gauge theories are based on spin−1 gauge fields with scalar gauge parameters $\varepsilon(x)$, spin−3/2 gauge fields with spinor gauge parameters $\varepsilon_\alpha(x)$ and
the spin−2 gauge field with the vector gauge parameter \( \varepsilon^\rho(x) \). Needless to say, all these theories are of great physical importance. The natural question then arises whether other possibilities related to higher-spin gauge fields \( s > 2 \) with highest tensors as gauge parameters do lead to fruitful physical models too.

### 3 Free Massless Higher-Spin Fields and the Interaction Problem

The theory of free massless fields of all spins is now developed in full detail due to the efforts of many authors (see e.g. [14, 15] and references therein). It was found that all free massless fields with \( s \geq 1 \) are Abelian gauge fields. In particular, integer-spin massless spin−s gauge fields can be described by totally symmetric tensors \( \phi_{\nu_1...\nu_s} \) subject to the double tracelessness condition \( \phi_{\rho\rho\eta_1...\eta_s} = 0 \) which becomes nontrivial for \( s \geq 4 \).

Quadratic actions \( S_s \) for free higher-spin fields can be fixed unambiguously \( [16] \) (up to an overall factor) by the requirement of gauge invariance under the Abelian transformations

\[
\delta \phi_{\nu_1...\nu_s} = \partial \{ \nu_1 \varepsilon_{\nu_2...\nu_s} \} \tag{3}
\]

with the parameters \( \varepsilon_{\nu_1...\nu_{s-1}} \) which are rank-(\( s - 1 \)) totally symmetric traceless tensors, \( \varepsilon^\rho_{\rho\nu_3...\nu_{s-1}} = 0 \). The final result is \( [14] \)

\[
S_s = \frac{1}{2} (-1)^s \int d^4 x \{ \partial_\nu \phi_{\mu_1...\mu_s} \partial_\nu' \phi^{\mu_1...\mu_s} \}
- \frac{1}{2} s(s-1) \partial_\nu \phi^\rho \phi_{\rho\mu_1...\mu_{s-2}} \partial_\nu' \phi^{\mu_1...\mu_s} + s(s-1) \partial_\nu \phi^\rho \phi_{\rho\mu_1...\mu_{s-2}} \partial_\sigma \phi^{\sigma\mu_1...\mu_{s-2}}
- s \partial_\nu \phi_{\mu_1...\mu_{s-1}} \partial_\rho \phi^{\rho\mu_1...\mu_{s-1}} - \frac{1}{4} s(s-1)(s-2) \partial_\nu \phi^\rho \phi_{\rho\mu_1...\mu_{s-1}} \partial_\sigma \phi^{\eta\sigma\mu_1...\mu_{s-3}} \}
\tag{4}
\]

For \( s \geq 1 \) this action describes spin−s massless particles which possess two independent degrees of freedom in \( d = 3 + 1 \). Quantization of this action leads to a unitary theory free from negative-norm states. For \( s = 1 \) and 2, \( S_s \) reduces to the standard lower-spin actions. Fermionic higher-spin gauge fields can be described analogously in terms of rank-(\( s - 1/2 \)) totally symmetric tensor-spinors \( \psi_{\nu_1...\nu_{s-1/2}(\alpha)} \) ((\( \alpha \)) is a spinor index) subject to the \( \gamma \)-tracelessness condition \( \gamma^{(\alpha)}_{(\beta)} \psi^\rho \rho_{\rho\nu_4...\nu_{s-1/2}(\beta)} = 0 \). This formulation is called formalism of symmetric (spinor-) tensors.

Once the theory of free higher-spin gauge fields is shown to be well defined, the next nontrivial problem is how to construct consistent interactions for higher-spin gauge fields. Consistency of a higher-spin gauge theory demands that it should reduce to some combination of free higher-spin systems at the linearized level with the correct signs of the individual actions respecting unitarity and that a number of gauge symmetries should
be the same for free and interacting theories, \textit{i.e.} the interactions are allowed to deform Abelian gauge symmetries of free theories, as it happens in Yang-Mills and Einstein theories, but not to break them down.

Important indications that nontrivial higher-spin gauge theories do exist were originally obtained in \cite{17, 18} where it was shown that some consistent cubic higher-spin interactions can be constructed. These interactions however do not contain the gravitational interaction of massless fields. On the other hand, the problem of existence of the consistent gravitational interaction is of crucial importance because of the universal role of gravity. The analysis of this issue carried out by several groups \cite{19} indicated that the attempts to introduce higher-spin-gravitational interactions encounter serious difficulties. Technically, the reason is quite simple: in order to introduce interaction with gravity respecting general coordinate invariance, one has to covariantize derivatives, \( \partial \rightarrow D = \partial - \Gamma \). This breaks down the invariance under the higher-spin gauge transformations because it turns out that, in order to prove invariance of the action \( S_\text{s} \), one should commute derivatives, while the commutator of the covariant derivatives is proportional to the Riemann tensor, \( [D \ldots, D \ldots] = R \ldots \ldots \). As a result, one concludes that the gauge variation of the covariantized action \( S_\text{s}^{\text{cov}} \) has the following structure:

\[
\delta S_\text{s}^{\text{cov}} = R_{\ldots}(\varepsilon_{\ldots D\varphi_{\ldots}}) \neq 0 \tag{5}
\]

and that it is not clear how to compensate these terms by any modification of the action or/and transformation laws.

The resolution of this problem is tricky enough. It was shown \cite{3} that consistent cubic higher-spin-gravitational interactions can be constructed if one analyzes the problem in the framework of the expansion near the (anti-)de Sitter background. In other words, gauge invariant and general coordinate covariant higher-spin-gravitational interactions contain some terms proportional to the inverse powers of the cosmological constant which diverge in the flat limit. This result is in agreement with the conclusions of \cite{19} where it was implicitly assumed that one can analyze the problem in the framework of some expansion in powers of the Riemann tensor. The point is that such an expansion makes sense only when the Riemann tensor is small, \textit{i.e.} the geometry is nearly flat.

Let us stress that the nonanalyticity of the higher-spin interactions in the cosmological constant is a consequence of the requirement that the higher-spin gauge symmetries are unbroken. On the other hand, higher-spin-gauge symmetries are expected to be broken in an appropriate physical phase to make all originally massless fields massive. A value of the cosmological constant in this physical phase is expected to be modified too. Thus, the nonanalyticity of the higher-spin-gravitational interactions in the cosmological constant in the symmetric phase, does not prevent one from building realistic models based on higher-spin gauge theories with broken higher-spin symmetries.
4 Geometric Formulation of Einstein Gravity

To illustrate some of the features of the formulation of higher-spin gauge theories described below, let us first remind the reader relevant facts about the “geometric formulations” of gravity.

It is well known [20, 21] that gravity can be interpreted to some extent as a gauge theory corresponding to an appropriate space-time symmetry algebra $g$. Vierbein $h_\nu^a$ and Lorentz connection $\omega_\nu^{ab}$ can be identified with the connection 1-forms of $g$. For example, in the four-dimensional space-time one can choose [21] $g$ to be the anti-de Sitter (AdS) algebra $o(3,2)$, which gives rise to the gauge fields $A_\nu^{\hat{a}\hat{b}} = -A_\nu^{\hat{b}\hat{a}}$ with $\hat{a}, \hat{b} = 0 \div 4$, and one can set $\omega_\nu^{ab} = A_\nu^{ab}$ and $h_\nu^a = \lambda^{-1} A_\nu^{a4}$ with $a, b = 0 \div 3$. The $o(3,2)$—Yang-Mills strengths read in these terms

$$ R_{\nu\mu}^{ab} = \partial_\nu \omega_\mu^{ab} + \omega_\nu^c \omega_\mu^{cb} + \lambda^2 h_\nu^a h_\mu^b - \nu \leftrightarrow \mu , \quad (6) $$

$$ R_{\nu\mu}^a = \partial_\nu h_\mu^a + \omega_\nu^c h_\mu^c - (\nu \leftrightarrow \mu) . \quad (7) $$

From (7) one recognizes that $R_{\nu\mu}^a$ has a form of the torsion tensor in the vierbein formulation of gravity. The constraint $R_{\nu\mu}^a = 0$ expresses the Lorentz connection $\omega_\nu^{ab}$ in terms of (derivatives of) the vierbein $h_\nu^a$ provided that $h_\nu^a$ is a non-degenerate matrix. Substituting these expressions back into the Lorentz components of the field strength (6), one can make sure that, up to the cosmological-type terms $\lambda^2 hh$, $R_{\nu\mu}^{ab}$ coincides with the Riemann tensor in gravity.

Then one observes that the equations $R_{\nu\mu}^{ab} = 0$ and $R_{\nu\mu}^a = 0$ describe anti-de Sitter space of radius $\lambda^{-1}$. In fact, this is the way how AdS space appears as a vacuum solution of the higher-spin equations considered below.

A remarkable observation by MacDowell and Mansouri [21] is that Einstein-Hilbert action with the cosmological term can be formulated in terms of the curvatures (6) in the form

$$ S^{MM} = -\frac{1}{4\kappa^2 \lambda^2} \int d^4x \, \epsilon^{\nu\rho\sigma\tau} \epsilon^{abcd} R_{\nu\rho\mu\sigma} R_{\rho\sigma\nu\tau} . \quad (8) $$

Let us note that the terms proportional to $\lambda^{-2}$ in $S^{MM}$, which involve higher derivatives, combine into a topological term and do not affect the equations of motion. The $\lambda$—independent term and the term proportional to $\lambda^2$ reduce to the scalar curvature and the cosmological term, respectively.

Another version of this action is due to Stelle and West [23] who observed that there is the following $so(3,2)$ covariant version of the MacDowell-Mansouri action

$$ S^{SW} = -\frac{1}{4\kappa^2 \lambda^2} \int_M \epsilon^{\hat{a}\hat{b}\hat{c}\hat{d}} \phi_{\hat{e}} R_{\hat{a}\hat{b}} R_{\hat{c}\hat{d}} , \quad (9) $$

where we made use of the exterior algebra formalism considering the field strength $R$ as a 2-form, and $\phi_{\hat{a}}$ is an additional auxiliary $o(3,2)$ vector 0-form subject to the constraint.
\( \phi_\alpha \phi^{\dot{\alpha}} = 1 \). The MacDowell-Mansouri formulation can be recognized as a spontaneously broken version of the Stelle-West formulation in a particular gauge \( \phi_a = 0 \) which breaks \( \mathfrak{o}(3, 2) \) down to \( \mathfrak{o}(3, 1) \).

The situation in 2+1 gravity is even simpler. The relevant AdS group is \( \mathfrak{o}(2, 1) = \mathfrak{o}(2, 1) \oplus \mathfrak{o}(2, 1) \). The gravitational action proposed by Witten \cite{22} is the Chern-Simons action for this group

\[
S^W = \int_{M^3} \text{str}(w \wedge dw + \frac{2}{3} w \wedge w \wedge w),
\]

where \( w \) is the \( \mathfrak{o}(2, 1) \) connection 2-form.

A version of the two-dimensional gravity action used in \cite{24} can be formulated by analogy with the Stelle-West action as

\[
S = -\frac{1}{4\kappa^2\lambda^2} \int_{M^4} \epsilon^{\hat{a}\hat{b}\hat{c}\hat{d}} \phi \epsilon R_{\hat{a}\hat{b}} \phi^{\dot{a}} = 1,
\]

where \( R \) is the curvature 2-form of the \( d=2 \) AdS group \( \mathfrak{o}(2, 1) \) and \( \phi^{\dot{a}} \) is a 0-form in the adjoint representation of \( \mathfrak{o}(2, 1) \).

A role of the space-time symmetry algebra \( \mathfrak{o}(d-1, 2) \) in these examples is twofold. On the one hand, connection 1-forms of this algebra are identified with the dynamical fields of the theory. On the other hand, \( \mathfrak{o}(d-1, 2) \) serves as the symmetry algebra of the most symmetric vacuum space.

It is then natural to look for an appropriate generalization of this approach which would lead to the description of the higher-spin dynamics. To this end it is instructive to use the formalism of two-component spinors which works in the cases \( d = 2, 3 \) and 4 due to the isomorphisms \( \mathfrak{o}(2, 1) \sim \mathfrak{sp}(2) \), \( \mathfrak{o}(2, 2) \sim \mathfrak{sp}(2) \oplus \mathfrak{sp}(2) \) and \( \mathfrak{o}(3, 2) \sim \mathfrak{sp}(4) \). The gravitational gauge fields now take the form \( \omega_{\nu}^{\alpha\beta}, \omega_{\nu}^{\dot{\alpha}\dot{\beta}} \) in four dimensions, \( h_{\nu}^{\alpha\beta}, \omega_{\nu}^{\alpha\beta} \) in three dimensions, and \( h^{\pm} \) and \( \omega^{\pm} \) in two dimensions (here \( \alpha, \beta, \ldots = 1, 2 \) and \( \dot{\alpha}, \dot{\beta}, \ldots = 1, 2 \) are spinor indices).

The key observation then is that the generators of \( \mathfrak{sp}(2) \) and \( \mathfrak{sp}(4) \) admit the so-called oscillator realization. Namely, \( \mathfrak{sp}(4) \) can be realized in terms of bilinears

\[
L_{\alpha\beta} = \frac{1}{2}\{\hat{y}_\alpha, \hat{y}_\beta\}, \quad \bar{L}_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}\{\hat{\bar{y}}_{\dot{\alpha}}, \hat{\bar{y}}_{\dot{\beta}}\}, \quad P_{\alpha\dot{\beta}} = \hat{y}_\alpha \hat{\bar{y}}_{\dot{\beta}}
\]

constructed from mutually conjugated bosonic oscillators \( \hat{y}_\alpha \) and \( \hat{\bar{y}}_{\dot{\alpha}} \) obeying the commutation relations

\[
[\hat{y}_\alpha, \hat{y}_\beta] = 2i\epsilon_{\alpha\beta}, \quad [\hat{\bar{y}}_{\dot{\alpha}}, \hat{\bar{y}}_{\dot{\beta}}] = 2i\epsilon_{\dot{\alpha}\dot{\beta}}, \quad [\hat{y}_\alpha, \hat{\bar{y}}_{\dot{\beta}}] = 0,
\]

\( \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \epsilon_{12} = 1, (\hat{y}_\alpha)^\dagger = \hat{\bar{y}}_{\dot{\alpha}} \). The algebra \( \mathfrak{sp}(2) \) is realized by the generators \( L_{\alpha\beta} \) (\( \bar{L}_{\dot{\alpha}\dot{\beta}} \)) constructed from only undotted (dotted) indices.

Equivalently, one can say that the gravitational fields are 1-forms bilinear in the auxiliary oscillator variables,

\[
\omega_{\nu} = \omega_{\nu}^{\alpha\beta} \hat{y}_\alpha \hat{y}_\beta + \hat{\omega}_{\nu}^{\dot{\alpha}\dot{\beta}} \hat{\bar{y}}_{\dot{\alpha}} \hat{\bar{y}}_{\dot{\beta}} + h_{\nu}^{\alpha\beta} \hat{y}_\alpha \hat{\bar{y}}_{\dot{\beta}} \hat{\bar{y}}_{\dot{\beta}}, \quad d = 3 + 1;
\]

\( \phi^{\dot{a}} = 1 \).
\[ \omega_\nu = \omega_\nu^{\alpha\beta} \hat{y}_\alpha \hat{y}_\beta + h_\nu^{\alpha\beta} \psi \hat{y}_\alpha \hat{y}_\beta, \quad d = 2 + 1; \]
\[ \omega_\nu = \omega_\nu \frac{1}{2} \{ \hat{y}_+, \hat{y}_- \} + h_\nu^\pm \psi \hat{y}_\pm, \quad d = 1 + 1, \]

where we have introduced the independent involutive element \( \psi \), \( \psi^2 = 1 \), for the case of \( d = 2 + 1 \) and use the convention that \( \alpha = \pm \) for the case of \( d = 1 + 1 \).

The instructive observation then is that the action (8) can be generalized [21] to the case of supergravity via extension of \( sp(4) \) to the \( N = 1 \) anti-de Sitter superalgebra \( osp(1; 4) \). In terms of gauge fields, this results in adding spin-3/2 gravitino fields linear in oscillators, \( \omega_\nu \hat{y}_\alpha \) and \( \bar{\omega}_\nu \hat{\bar{y}}_\dot{\alpha} \). For the case of \( d = 2 + 1 \) it was shown in [25] that the analogous extension of \( sp(2) \) to \( osp(1; 2) \) leads to d3 supergravity.

5 Higher-Spin Algebras and Star Product

A natural generalization of the above construction to higher spins consists of allowing all powers of the spinor oscillators \( \hat{y} \) and \( \hat{\bar{y}} \).

Let us consider the infinite-dimensional associative algebra \( A(2) \) spanned by all polynomials of \( \hat{y}_\alpha \). Its general element \( \hat{P} \) has a form
\[ \hat{P}(\hat{y}) = \sum_{n=0}^{\infty} \frac{1}{2 n!} P^{\alpha_1 \cdots \alpha_n} \hat{y}_{\alpha_1} \cdots \hat{y}_{\alpha_n}. \]

The coefficients \( P^{\alpha_1 \cdots \alpha_n} \) are supposed to be totally symmetric in the indices \( \alpha_i \), that implies Weyl ordering of \( \hat{y}_{\alpha_i} \). \( A(2) \) is called Heisenberg-Weyl algebra. Analogously one defines the algebras \( A(2n) \) with the generating elements \( \hat{y}_\alpha, \alpha = 1 \ldots 2n \), obeying the commutation relations
\[ [\hat{y}_\alpha, \hat{y}_\beta] = 2i C_{\alpha\beta}, \]

where \( C_{\alpha\beta} \) is some non-degenerate antisymmetric matrix considered as as a symplectic form with the conventions
\[ C^{\alpha\gamma} C_{\alpha\beta} = \delta_\gamma^\beta, \quad A^\alpha = C^{\alpha\beta} A_\beta, \quad A_\alpha = A^\beta C_{\beta\alpha}. \]

An important property of the algebra \( A(2n) \) is that it admits [3] the following unique supertrace operation
\[ str(\hat{P}_W(\hat{y})) = \hat{P}_W(0) \]
(\( \hat{P}_W(\hat{y}) \) is Weyl ordered) such that
\[ str[\hat{P}_1, \hat{P}_2] = 0, \quad \forall \hat{P}_{1,2} \in A(2n) \]
with the bracket \([,]\) defined in the following way
\[ [\hat{P}_1, \hat{P}_2] = \hat{P}_1 \hat{P}_2 - (-1)^{\pi_1 \pi_2} \hat{P}_2 \hat{P}_1, \]

\( \pi_i \) being the parities of \( \hat{y}_\alpha \) and \( \hat{\bar{y}}_\dot{\alpha} \).
where $\pi_{1,2}$ are the “boson-fermion” parities introduced in the standard fashion $\hat{P}(-\hat{y}) = (-1)^\pi \hat{P}(\hat{y})$.

Higher-spin algebras $shs(2n)$ are the Lie superalgebras of (super)commutators (22) constructed from the associative algebras $A(2n)$. The supertrace (20) allows one to build invariants of $shs(2n)$ by taking supertraces of products of its elements. The existence of the supertrace implies that the elements with vanishing supertrace form an ideal of this algebra. We will use the notation $shs(2n)$ for this ideal, which is a simple algebra spanned by all elements (17) such that $\hat{P}(\hat{y}) = 0$.

The oscillators $\hat{y}_\alpha$ admit the standard differential realization. For example, for $n=2$

$$\hat{y}_1 = 2i \frac{\partial}{\partial z}, \quad \hat{y}_2 = z.$$  \hspace{1cm} (23)

As a result, the algebra $shs(2)$ turns out to be isomorphic to the centerless version of $W_{1+\infty}$ \cite{8, 26} which is often denoted as $W_{1+\infty}$ too. Note that the supertrace (20) is defined only for this centerless part $shs(2)$ of $W_{1+\infty}$ and cannot be extended to the full algebra of differential operators on the circle which allows negative powers of $z$.

For practical manipulations with the higher-spin algebras it is convenient to use the language of symbols of operators \cite{27} instead of the operator one we started with. The idea is simple. Given $\hat{P}(\hat{y}) \in A(2n)$ of the form (17) one introduces its symbol $P(y)$ which, by definition, is a function of the commuting variables $y_\alpha$ of the same form as $\hat{P}(\hat{y})$, i.e.

$$P(y) = \sum_{n=0}^\infty \frac{1}{2i^n n!} P_{\alpha_1 \cdots \alpha_n} y_{\alpha_1} \cdots y_{\alpha_n},$$  \hspace{1cm} (24)

and then defines star-product $\star$ in such a way that $P_1 \star P_2$ be a symbol of the operator product $\hat{P}_1 \hat{P}_2$. For the case of the Weyl symbols under consideration one can derive by virtue of the Campbell-Hausdorff formula the following star-product formula

$$(P \star Q)(y) = (2\pi)^{-2n} \int d^{2n} u d^{2n} v P(y + u)Q(y + v) \exp[iu_\alpha v_\alpha],$$  \hspace{1cm} (25)

where $u_\alpha$ and $v_\alpha$ are integration variables. By its definition the star-product is associative

$$f \star (g \star h) = (f \star g) \star h$$  \hspace{1cm} (26)

but non-commutative. One can easily check with the aid of (25) that

$$[y_\alpha, y_\beta]_\star = 2i C_{\alpha\beta},$$  \hspace{1cm} (27)

where $[a, b]_\star = a \star b - b \star a$. Another important property which follows from the definition of the star-product is that given two polynomials $P(y)$ and $Q(y)$, $(P \star Q)(y)$ is some polynomial too.
Now one defines the symbol version of the connection 1-form of the algebra \( shs(2n) \) as
\[
\omega(y \mid x) = dx^\nu \omega_\nu(y \mid x) = \sum_{n=0}^{\infty} \frac{1}{2i^n n!} \omega^{\alpha_1 \ldots \alpha_n}(x)y_{\alpha_1} \cdots y_{\alpha_n},
\]
where \( x^\nu \) are space-time coordinates and the gauge field components \( \omega^{\alpha_1 \ldots \alpha_n}(x) \) are supposed to carry additional Grassman grading for odd \( n \), i.e. the fermion fields are anti-commuting in accordance with the standard relationship between spin and statistics.

The curvature 2-form has the standard form
\[
R(y \mid x) = d\omega(y \mid x) + \omega(y \mid x) \wedge \star \omega(y \mid x),
\]
where \( d = dx^\nu(\partial/\partial x^\nu) \) is the space-time exterior differential. Note that the second term on the r.h.s. of (29) does not vanish because of the noncommutativity of the star-product and automatically contains the supercommutators (22) due to the anticommutativity of the fermionic gauge fields. One can expand the curvature 2-form (15) in powers of the auxiliary variables \( y \)
\[
R(y \mid x) = \sum_{n=0}^{\infty} \frac{1}{2i^n n!} y_{\alpha_1} \cdots y_{\alpha_n} R^{\alpha_1 \ldots \alpha_n}. \tag{30}
\]
The explicit form of the coefficients \( R^{\alpha_1 \ldots \alpha_n} \) can be obtained by virtue of the formula (25). We do not need it in this talk however and refer the reader for more details to [6]. Let us note that originally a form of these curvatures has been derived from the detailed analysis of the higher-spin dynamics [5] while the operator realization of the higher-spin algebras described above was found afterwards [6].

### 6 Higher-Spin Action in 2+1 Dimensions

The algebra of global higher-spin symmetries of the 2+1 problem is \( shs(2) \oplus shs(2) \) which is a sum of two simple algebras by analogy to the case of pure gravity with \( o(2, 2) \sim sp(2) \oplus sp(2) \). Such a doubling can be introduced with the aid of the additional generating element \( \psi \) such that \( \psi^2 = 1 \). A general element of this algebra then has a form
\[
P(\hat{y}, \psi) = \sum_{n=0}^{\infty} \frac{1}{2i^n n!} P^{A \alpha_1 \ldots \alpha_n}(\psi)^A y_{\alpha_1} \cdots y_{\alpha_n}. \tag{31}
\]

A higher-spin counterpart of the Witten gravity action was introduced by Blencowe [28] as the Chern-Simons action for \( shs(2) \oplus shs(2) \),
\[
S^{2+1} = \int_{M_3} str(\omega \wedge \star d\omega + \frac{2}{3} \omega \wedge \star \omega \wedge \star \omega) \tag{32}
\]
Higher-spin gauge fields become propagating for \( d \geq 4 \). The simplest non-trivial case therefore is 3+1-dimensional space-time. For simplicity let us focus on the purely bosonic version of the higher-spin superalgebra in \( d = 3+1 \) which is the even (bosonic) subalgebra \( hs(4) \) of \( shs(4) \). It is convenient to use the two-component spinor notations splitting the full \( sp(4) \) spinor into the complex \( sp(2) \) two-component spinor \( y_\alpha \) and its complex conjugate \( \bar{y}_\dot{\alpha} \). The algebra \( hs(4) \) is thus spanned by even power polynomials in \( y \) and \( \bar{y} \), 

\[
P(y, \bar{y}) = P(-y, -\bar{y}).
\]

The gauge fields of \( hs(4) \) are described by the generating function

\[
\omega(y, \bar{y} | x) = \sum_{n,m = 0; \ n+m - \text{even}}^{\infty} \frac{1}{2i n! m!} y_{\alpha_1} \cdots y_{\alpha_n} \bar{y}_{\dot{\beta}_1} \cdots \bar{y}_{\dot{\beta}_m} \omega^{\alpha_1 \cdots \alpha_n, \dot{\beta}_1 \cdots \dot{\beta}_m}(x),
\]

where the multispinor space-time 1-form coefficients \( \omega^{\alpha_1 \cdots \alpha_n, \dot{\beta}_1 \cdots \dot{\beta}_m}(x) \) are identified with the physical higher-spin fields. It was argued in section 3 that the massless spin–2 field is described by \( \omega^{\alpha_1 \cdots \alpha_n, \dot{\beta}_1 \cdots \dot{\beta}_m} \) with \( n + m = 2 \). This relation is generalized to an arbitrary spin \( s \) as follows

\[
n + m = 2(s - 1),
\]

\( i.e. \) a spin–\( s \) massless particle is described with the aid of the collection of all 1–forms \( \omega^{\nu_{\alpha_1 \cdots \alpha_n, \dot{\beta}_1 \cdots \dot{\beta}_m}} \) with the overall number of spinor indices fixed according to (34). It is worth mentioning that these collections of gauge fields form the irreducible rank - \( 2(s - 1) \) tensor representations with respect to the adjoint action of the full anti-de Sitter algebra \( sp(4) \).

The fact that such a set of fields properly describes spin–\( s \) massless fields is a consequence of the explicit analysis of the higher-spin dynamics based on the following action principle

\[
S = -\frac{1}{4\kappa^2\lambda^2} \sum_{n,m = 0; \ n+m - \text{even}}^{\infty} \frac{1}{n! m!} \epsilon(n - m) \int_{M^4} R^{\alpha_1 \cdots \alpha_n, \dot{\beta}_1 \cdots \dot{\beta}_m} \wedge R^{\alpha_1 \cdots \alpha_n, \dot{\beta}_1 \cdots \dot{\beta}_m},
\]
where \( R^{\alpha_1...\alpha_n, \beta_1...\beta_m} \) are the components of the full \( hs(4) \) curvature tensor (29),

\[
R(y, \bar{y} | x) = \sum_{n,m=0; \ n+m-{\text{even}}}^{\infty} \frac{1}{2^n n! m!} y_{\alpha_1} \cdots y_{\alpha_n} \bar{y}_{\dot{\beta}_1} \cdots \bar{y}_{\dot{\beta}_m} R^{\alpha_1...\alpha_n, \beta_1...\beta_m}(x) \cdot \tag{36}
\]

The explicit analysis of the quadratic part of the action (35) shows [29] that its variation with respect to the “extra fields”, \( \omega_{\alpha_1...\alpha_n, \beta_1...\beta_m} \) with \( |n-m| > 2 \), vanishes identically, while the variation with respect to the dynamical fields, \( \omega_{\alpha_1...\alpha_n, \beta_1...\beta_m} \) with \( |n-m| \leq 2 \), is non-trivial and leads to the correct free equations for massless fields.

Since extra fields contribute to the interaction terms one has to express them in terms the dynamical fields to have a well defined non-linear action. The appropriate constraints have the form [29]

\[
h_{\alpha_1...\alpha_n, \beta_1...\beta_m} = 0 \quad (n > m \geq 0), \tag{37}
\]

\[
h_{\gamma \alpha_1...\alpha_n, \beta_1...\beta_m} = 0 \quad (m > n \geq 0). \tag{38}
\]

These constraints express successively all “extra fields” \( \omega_{\alpha_1...\alpha_n, \beta_1...\beta_m} \) with \( |n-m| > 2 \) in terms of the dynamical fields \( \omega_{\alpha_1...\alpha_n, \beta_1...\beta_m} \) with \( |n-m| \leq 2 \). It is important that these constraints are algebraic with respect to the extra fields expressing the latter in terms of derivatives of the dynamical fields. The constraints (37) and (38) play a crucial role in the description of the higher-spin dynamics, governing a form of the interactions of the dynamical higher–spin fields in the action (35). A consequence of this mechanism is that higher-spin interactions for the dynamical fields contain higher derivatives. The same mechanism leads to the non-analyticity of the interaction terms in the cosmological constant.

The action (35) possesses the following basic properties:

(i) \( S \) is explicitly general coordinate invariant due to the exterior algebra formalism;

(ii) in the spin–2 sector \( (n + m = 2) \) \( S \) reduces to the Einstein-Hilbert action in the MacDowell-Mansouri form (8);

(iii) on the linearized level, \( S \) amounts to the sum of free actions for all massless bosonic fields with \( s \geq 2 \) in the formalism of nonsymmetric tensors [15, 29] which is dynamically equivalent to the formalism of symmetric tensors sketched in Introduction;

(iv) \( S \) is gauge invariant in the cubic order provided that the constrains (37) and (38) are imposed [3].

Thus, the action \( S \) supplemented with the constraints (37), (38) solves the higher-spin problem in the lowest order in interactions [3]. A non-trivial problem which still remains unsolved is how to generalize this result to highest orders in interactions. To construct a closed action one has to develop a formalism based on appropriate generating functions of auxiliary spinor variables. This is expected to lead to a certain higher-spin generalization of the Stelle-West formulation of gravity which requires an appropriate counterpart of the
auxiliary field $\phi^a$. At the moment this problem is not solved at the action level. However an analogous program is completed for the equations of motion. For this reason in the sequel we focus mainly on the formulation of equations of motion.

8 Unfolded Formulation

Before going into details of the full formulation of the higher-spin equations let us discuss some general features of the “unfolded formulation” \cite{31} we are going to implement and give some simple examples.

In the context of applications to the higher-spin problem we will use a particular case of unfolded formulation where the dynamics is described in terms of a set of 1-forms $\omega(x) = dx^i \omega_i^a(x) T_i$ taking values in some Lie superalgebra $l$ ($T_i \in l$) and a set of 0-forms $B^A(x)$ which takes values in a representation space of some representation $(t_i)^B_A$ of $l$. The dynamical equations are then formulated in the form

$$d\omega = \omega \wedge \omega,$$  \hspace{1cm} (39)

$$dB^A = \omega^i t_i^A B^B$$ \hspace{1cm} (40)

and

$$\chi(B) = 0,$$ \hspace{1cm} (41)

where $\chi(B)$ are some constrains which do not contain the space-time differential $d = dx^\nu \frac{\partial}{\partial x^\nu}$ and are invariant under the gauge transformations

$$\delta \omega = d\epsilon - [\omega, \epsilon],$$ \hspace{1cm} (42)

$$\delta B^A = \epsilon^i (t_i)^A_B B^B$$ \hspace{1cm} (43)

that guarantees the invariance of the full system of equations (39)-(41) under the transformations (42) and (43).

Dynamical content of the equations (39)-(41) is screened in the constraints (41). Indeed, locally one can integrate out explicitly the first two equations to a pure gauge solution

$$\omega = d(g^{-1}(x))g(x),$$ \hspace{1cm} (44)

$$B(x) = t_{g(x)}(B_0),$$ \hspace{1cm} (45)

where $B_0$ is an arbitrary x-independent quantity and $t_{g(x)}$ is the exponential of the representation $t$ of $l$. Since the constraints $\chi(B)$ are gauge invariant one is left with the only condition

$$\chi(B_0) = 0.$$ \hspace{1cm} (46)

Suppose that $g(x_0) = I$ for some space-time point $x_0$. From (45) it follows then that $B_0 = B(x_0)$. One can wonder how any restrictions imposed on values of some 0-forms in
a fixed point of space-time can lead to a non-trivial dynamics. The answer is that this is possible if the set of 0-forms $B$ is reach enough to describe all space-time derivatives of the dynamical fields in a fixed point of space-time provided that the constraints (41) just single out those values of the derivatives which are compatible with the dynamical equations of the system under consideration. By knowing any solution of (45) one knows all derivatives of the dynamical fields compatible with the field equations and can therefore reconstruct these fields by analyticity in some neighborhood of $x_0$. The crucial point here is that in order to proceed along these lines one necessarily has to use some infinite-dimensional representation $t$ for 0-forms. From this point of view the special feature of higher-spin theories is that they contain infinite collections of dynamical fields from the very beginning so that it is natural to introduce infinitely many auxiliary 0-forms in these theories.

Let us now illustrate how this mechanism works for the simplest field-theoretical model of free massless spin-0 equations in the flat space-time of arbitrary dimension $d$. In this example $l$ is identified with the Poincare algebra $iso(d - 1, 1)$

$$\omega_\nu = (h_\nu^a, \omega_\nu^{ab})$$

(47)

$(a, b = 0 - (d - 1))$. The vanishing curvature conditions of $iso(d - 1, 1)$

$$R_{\nu \mu}^a = 0, \quad R_{\nu \mu}^{ab} = 0$$

(48)

then imply that the vierbein $h_\nu^a$ and Lorentz connection $\omega_\nu^{ab}$ describe the flat geometry. Fixing the local Poincare gauge transformations one can set

$$h_\nu^a = \delta_\nu^a, \quad \omega_\nu^{ab} = 0.$$  

(49)

Let us note that the ambiguity in local Poincare gauge transformations is equivalent to the general coordinate transformations provided that the zero-curvature conditions (48) are true and the vierbein $h_\nu^a$ is invertible. As a result, the gauge fixing (49) is equivalent to choosing the Cartesian coordinate frame.

To describe dynamics of the spin zero massless field $\phi(x)$ let us introduce the infinite collection of 0-forms $\phi_{a_1...a_n}(x)$ which are totally symmetric traceless tensors

$$\eta^{bc}\phi_{bca_3...a_n} = 0,$$

(50)

where $\eta^{bc}$ is the flat Minkowski metrics. The “unfolded” version of the Klein-Gordon equation has a form of the following infinite chain of equations

$$\partial_\nu \phi_{a_1...a_n}(x) = h_\nu^b \phi_{a_1...a_nb}(x),$$

(51)

where we have used the opportunity to replace the Lorentz covariant derivative by the ordinary flat derivative $\partial_\nu$ due to the flatness condition (49) (in any other gauge one has
to replace the flat derivative $\partial$ by the Lorentz covariant derivative). The condition \((50)\) is a specific realization of the constraints \((\text{II})\) while the system of equations \((51)\) is a particular realization of the equations \((\text{III})\). It is easy to see that this system is formally consistent, i.e. the repeated $\partial_{\mu}$ differentiation of \((51)\) does not lead to any new conditions after antisymmetrization $\nu \leftrightarrow \mu$. This property is equivalent to the fact that the set of zero forms $\phi_{a_1...a_n}$ spans some representation of the Poincare algebra.

To show that this system of equations is indeed equivalent to the free massless field equation $\Box \phi(x) = 0$ let us identify the scalar field $\phi(x)$ with the member of the family of 0-forms $\phi_{a_1...a_n}$ at $n = 0$. Then the first two members of the system \((51)\) read

$$\partial_{\nu} \phi = \phi_{\nu}, \quad (52)$$
$$\partial_{\nu} \phi_{\mu} = \phi_{\mu \nu}, \quad (53)$$

where we have identified the world and tangent indices taking into account the gauge condition \((49)\).

The first of these equations just tells us that $\phi_{\nu}$ is a first derivative of $\phi$. The second one tells us that $\phi_{\nu \mu}$ is a second derivative of $\phi$. However, because of the tracelessness condition \((50)\) it imposes the Klein-Gordon equation $\Box \phi = 0$. It is easy to see that all other equations in \((51)\) express highest tensors in terms of the higher-order derivatives

$$\phi_{\nu_1...\nu_n} = \partial_{\nu_1} \ldots \partial_{\nu_n} \phi$$

and impose no new conditions on $\phi$. The tracelessness conditions \((50)\) are all satisfied once the Klein-Gordon equation is true.

Let us note that the system \((51)\) without the constraints \((50)\) remains formally consistent but is dynamically empty just expressing all highest tensors in terms of derivatives of $\phi$ according to \((54)\). This simple example illustrates how constraints can be equivalent to the dynamical equations.

The above consideration can be simplified further by means of introducing the auxiliary coordinate $u^a$ and the generation function

$$\Phi(x, u) = \sum_{n=0}^{\infty} \frac{1}{n!} \phi_{a_1...a_n}(x)u^{a_1} \ldots u^{a_n} \quad (55)$$

with the convention that

$$\Phi(x, 0) = \phi(x). \quad (56)$$

This generating function accounts for all tensors $\phi^{a_1...a_n}$ provided that the tracelessness condition is imposed which in these terms implies that

$$\Box_u \Phi(x, u) \equiv \frac{\partial}{\partial u^a} \frac{\partial}{\partial u^a} \Phi = 0. \quad (57)$$
The equations (51) then acquire the simple form
\[ \frac{\partial}{\partial x^\nu} \Phi(x, u) = \frac{\partial}{\partial u^\nu} \Phi(x, u). \] (58)

From this realization one concludes, first, that the translation generators in the infinite-dimensional representation of the Poincare algebra formed by the higher tensors \( \phi^{a_1 \ldots a_n} \) are realized as translations in the \( u \)-space and, second, that one can indeed find a general solution of the equation (58) in the form
\[ \Phi(x, u) = \Phi(x + u, 0) = \Phi(0, x + u) \] (59)
from which it follows in particular that
\[ \phi(x) = \Phi(0, x) = \sum_{n=0}^{\infty} \frac{1}{n!} \phi_{\nu_1 \ldots \nu_n}(0)x^{\nu_1} \ldots x^{\nu_n}. \] (60)

From (50) and (54) one can see that this is indeed the Taylor expansion for any solution of the Klein-Gordon equation which is analytic in \( x = 0 \). Moreover one can recognize the equation (60) as a particular realization of the pure gauge solution (45) with the gauge function \( g(x) \) of the form
\[ g(x) = \exp(x^\nu h^a_{\nu} \frac{\partial}{\partial u^a}). \] (61)

The example of the scalar field considered here is so simple that it tends to be trivial after introducing the auxiliary variables \( u^a \). Remarkably, a proper generalization of this approach to non-trivial higher-spin dynamics is at the moment the only known one working for non-linear higher-spin equations. Let us note that the described formalism has some similarities with the non-linear realization approach developed for the Yang-Mills case in [31] where the relevance of the bilocal fields analogous to \( \Phi(x, u) \) subject to the equations analogous to (58) was demonstrated.

9 Higher-Spin Equations of Motion in 3+1 Dimensions

Let us now explain how one can formulate non-linear higher-spin equations in 3 + 1 dimensions in the unfolded form. For simplicity we confine ourselves to the bosonic case. A general treatment which allows one to include fermions can be found in [3].

To describe on-mass-shell higher-spin dynamics in \( d = 3 + 1 \), we introduce the following set of generating functions
\[ W(Z; Y \mid x) = dx^\nu W_\nu(Z; Y \mid x), \] (62)
\[ B(Z; Y \mid x), \] (63)
where $Z = (z_\alpha, \bar{z}_\dot{\alpha})$ and $Y = (y_\alpha, \bar{y}_\dot{\alpha})$ are two independent sets of auxiliary spinor variables while $x$ denotes space-time coordinates as before. A physical meaning of the generating functions (62)-(64) is as follows. The space-time 1-form $W$ is the generating function for higher-spin gauge potentials. The 0-form $B$ serves as a generating function for lower-spin fields (i.e., a spin–0 scalar) and for on-mass-shell nontrivial higher-spin curvatures generalizing the gravitational Weyl-tensor (see below). The space-time 0-form $s$ can be interpreted as a 1-form with respect to auxiliary anticommuting spinor differentials $dz^\alpha$ and $d\bar{z}^{\dot{\alpha}}$,

$$\{dz^\alpha, dz^\beta\} = 0, \quad \{d\bar{z}^{\dot{\alpha}}, d\bar{z}^{\dot{\beta}}\} = 0, \quad \{dz^\alpha, d\bar{z}^{\dot{\beta}}\} = 0, \quad (65)$$

which commute with all other variables. The field $s$ is auxiliary in nature, describing no independent degrees of freedom. It serves as a differential operator shifting along the auxiliary spinorial variables $Z$. That we confine ourselves to the purely bosonic theory in this section means that $G(-dZ; -Z; -Y) = G(dZ; Z; Y)$ for $G = W$, $B$ and $s$.

To formulate higher-spin equations of motion, we endow the linear space of functions $f(Z; Y)$ with a structure of associative algebra with the $*$ product law,

$$(f * g)(Z; Y) = (2\pi)^{-4} \int d^4 U d^4 V \ f(Z + U; Y + U) \ g(Z - V; Y + V) \ \exp i(u_\alpha v^\alpha + \bar{u}_{\dot{\alpha}} \bar{v}^{\dot{\alpha}}), \quad (66)$$

where $U = (u_\alpha, \bar{u}_{\dot{\alpha}})$ and $V = (v_\alpha, \bar{v}^{\dot{\alpha}})$ are integration variables. This product law is some particular symbol version of the Heisenberg–Weyl algebra,

$$[y_\alpha, y_\beta]_* = -[z_\alpha, z_\beta]_* = 2i \epsilon_{\alpha\beta}, \quad [\bar{y}_{\dot{\alpha}}, \bar{y}_{\dot{\beta}}]_* = -[\bar{z}_{\dot{\alpha}}, \bar{z}_{\dot{\beta}}]_* = 2i \epsilon_{\dot{\alpha}\dot{\beta}} \quad (67)$$

(all other commutators vanish). The product law (66) is associative, $(f * g) * h = f * (g * h)$, and regular: given two polynomials $f(Z; Y)$ and $g(Z; Y)$, $(f * g)(Z; Y)$ is some polynomial too. The latter property guarantees that the formulae containing star products make sense for the coefficients of the power series expansions of the generating functions $W$, $B$ and $s$.

The totally consistent system of higher-spin equations reads [4]

$$dW = W * \wedge W, \quad (68)$$

$$dB = W * B - B * \bar{W}, \quad (69)$$

$$ds = W * s - s * W, \quad (70)$$

$$s * B = B * \bar{s}, \quad (71)$$

$$s * s = dz^\alpha dz_\alpha (i + B * \kappa) + d\bar{z}^{\dot{\alpha}} d\bar{z}_{\dot{\alpha}} (i + B * \bar{\kappa}), \quad (72)$$

where

$$\kappa = \exp(i z_\alpha y^\alpha), \quad \bar{\kappa} = \exp(i \bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}}) \quad (73)$$
and ∼ changes a sign of all undotted spinors,
\[ f(dz, d\bar{z}; z, \bar{z}; y, \bar{y}) = f(-dz, d\bar{z}; -z, -y, \bar{y}). \] (74)

The system of equations (68)-(72) has “unfolded form”. The equation (68) is a particular case of the equation (39), the equations (69) and (70) have the form (40), and the equations (71) and (72) serve as some constraints (41). It is important that these constraints are gauge invariant so that the equations (68)-(72) are explicitly invariant under the higher-spin gauge transformations
\[ \delta W = d\varepsilon - W \ast \varepsilon + \varepsilon \ast W, \] (75)
\[ \delta B = \varepsilon \ast B - B \ast \bar{\varepsilon}, \] (76)
\[ \delta s = \varepsilon \ast s - s \ast \varepsilon. \] (77)

Also this system of equations is explicitly general coordinate covariant due to the exterior algebra formalism.

What is less straightforward to see is that the system of equations (68)-(72) indeed describes the dynamics of massless higher-spin fields. The detailed analysis of this issue is presented in [4, 32] (in [4] also a more general form of higher-spin interactions has been considered). Here, we only outline the main steps and basic ideas.

The crucial point is to show that the equations (68)-(72) describe correct free field dynamics at the linearized level. The relevant perturbative procedure consists of the order by order analysis of the equations (68)-(72) in the framework of the expansion in powers of $B$ which physically is equivalent to the expansion in powers of higher-spin curvatures generalizing the Weyl tensor. One starts with the following ansatz for $s$
\[ s = (dz^\alpha z_\alpha + d\bar{z}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}}) + O(B), \] (78)
which can be easily verified to solve (72) in the lowest order in $B$. From (78) and (67), it follows that
\[ s \ast f - f \ast s = -i (dz^\alpha \frac{\partial}{\partial z^\alpha} + d\bar{z}^{\dot{\alpha}} \frac{\partial}{\partial \bar{z}^{\dot{\alpha}}}) f(Z; Y) + O(B) f. \] (79)

As a result, the equations (70)-(72) reduce to some differential equations with respect to \( \partial/\partial z^\alpha \) and \( \partial/\partial \bar{z}^{\dot{\alpha}} \), which determine $W(Z; Y), B(Z; Y)$ and $s(Z; Y)$ itself in terms, of the initial data $\omega(Y) = W(0; Y)$ and $C(Y) = B(0; Y)$ up to some pure gauge transformations (71). In fact, these initial data serve as the generating functions for physical higher-spin fields. In particular, $\omega(Y)$ can be identified with the higher-spin generating function (33). The doubling of spinor variables, $Y \rightarrow (Z, Y)$, serves as a sort of a technical trick which enables one to describe complicated expressions as solutions of certain simple nonlinear differential equations with respect to $Z$. 
At the second stage, inserting the expressions for $W$, $B$ and $s$ in terms of the initial data $\omega$ and $C$ back into the equations (88) and (89), one gets the dynamical equations for physical fields of all spins provided that the background gravitational field is introduced as a vacuum value $\omega_0(y|x)$ of $\omega(y|x)$ such that the zero-curvature equation (88) is true for $\omega_0(Y|x)$. The AdS geometry then arises as a solution with $\omega_0(Y|x)$ of the form (14). The form of the resulting dynamical equations is analogous to that of the spin 0 example considered in section 8. The only distinctions are that now we use the formalism of two-component spinors and the analysis is carried out in the anti-de Sitter background.

In this formalism the spin-0 matter field is described by the infinite chain of 0–forms $C_{\alpha_1...\alpha_n,\beta_1...\beta_n}$ which are totally symmetric multispinors. This set is the two-component spinor version of the set of totally symmetric traceless tensors considered in section 8. The equations which follow from (89) in this sector read

$$D_i C_{\alpha_1...\alpha_n,\beta_1...\beta_n} = h^{\gamma\delta} C_{\alpha_1...\alpha_n\gamma,\beta_1...\beta_n\delta} - n^2 \lambda^2 h_{\{\alpha_1\beta_1} C_{\alpha_2...\alpha_n,\beta_2...\beta_n\}_\alpha\beta}.$$  

(80)

This system of equations can be shown to be equivalent to the spin 0 massless equations for the field $C$ in the anti-de Sitter space analogously to the flat space example considered in section 8. Again, the infinite chain of 0–forms $C_{\alpha_1...\alpha_n,\beta_1...\beta_n}$ with $n \geq 0$ describes all on-mass-shell nontrivial combinations of the derivatives of the scalar field $C$.

Analogous analysis shows that the fields $C_{\alpha_1...\alpha_n,\beta_1...\beta_m}$ with $|n - m| = 2s$ describe massless fields of spin $s$. Let us illustrate this for the particular case of Einstein gravity, i.e. $s = 2$. As argued in section 2, Lorentz connection 1–forms $\omega_{\alpha\beta}$, $\bar{\omega}_{\bar{\alpha}\bar{\beta}}$ and vierbein 1–forms $h_{\alpha\beta}$ can be identified with the $sp(4)$–gauge fields. The corresponding $sp(4)$–curvatures read in terms of two-component spinors

$$R_{\alpha_1\alpha_2} = d\omega_{\alpha_1\alpha_2} + \omega_{\alpha_1\gamma} \wedge \omega_{\alpha_2\gamma} + \lambda^2 h_{\alpha_1\beta} \wedge h_{\alpha_2\delta},$$  

(81)

$$\bar{R}_{\bar{\alpha}_1\bar{\alpha}_2} = d\bar{\omega}_{\bar{\alpha}_1\bar{\alpha}_2} + \bar{\omega}_{\bar{\alpha}_1\bar{\gamma}} \wedge \bar{\omega}_{\bar{\alpha}_2\bar{\gamma}} + \lambda^2 h_{\bar{\alpha}_1\bar{\gamma}} \wedge h_{\bar{\alpha}_2\bar{\delta}},$$  

(82)

$$r_{\alpha\beta} = dh_{\alpha\beta} + \omega_{\alpha\gamma} \wedge h_{\gamma\beta} + \bar{\omega}_{\bar{\gamma}\bar{\delta}} \wedge h_{\alpha\delta}.  

(83)

The zero-torsion condition $r_{\alpha\beta} = 0$ expresses the Lorentz connection $\omega$ and $\bar{\omega}$ via derivatives of $h$. After that, the $\lambda$–independent part of the curvature 2–forms $R$ (81) and $\bar{R}$ (82) coincides with the Riemann tensor. Einstein equations imply that the Ricci tensor vanishes up to a trace part proportional to the cosmological constant. This is equivalent to saying that only those components of the tensors (81) and (82) are allowed to be non-vanishing which belong to the Weyl tensor. As is well-known [33], Weyl tensor is described by the fourth-rank mutually conjugated totally symmetric multispinors $C_{\alpha_1\alpha_2\alpha_3\alpha_4}$ and $\bar{C}_{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4}$. Therefore, Einstein equations with the cosmological term can be cast into the form

$$r_{\alpha\beta} = 0,$$

(84)

$$R_{\alpha_1\alpha_2} = h_{\gamma\delta} \wedge h_{\gamma\delta} C_{\alpha_1\alpha_2\gamma\delta},$$

(85)
\[ \bar{R}_{\dot{\beta}_1 \dot{\beta}_2} = h^{\dot{\beta}_1} \wedge h_{\eta}^{\dot{\beta}_2} \tilde{C}_{\dot{\beta}_1 \dot{\beta}_2 \dot{\alpha}_1 \dot{\alpha}_2}. \]  

(86)

It is convenient to think of the 0–forms \( C \) and \( \bar{C} \) on the right hand sides of \( (84) \) and \( (85) \) as of independent field variables which turn out to be equivalent to the Weyl tensor due to the equations \( (85) \) and \( (86) \) themselves. From \( (85) \) and \( (86) \) it follows that the 0–forms \( C \) and \( \bar{C} \) should obey certain differential restrictions as a consequence of the Bianchi identities for the curvatures \( R \) and \( \bar{R} \). It is not difficult to make sure that these differential restrictions can be equivalently rewritten in the form

\[ D^L C_{\dot{\alpha}_1 \dot{\alpha}_2 \dot{\alpha}_3 \dot{\alpha}_4} = h^{\dot{\alpha}} C_{\dot{\alpha}_1 \dot{\alpha}_2 \dot{\alpha}_3 \dot{\alpha}_4 \gamma, \dot{\delta}} , \]  

(87)

\[ D^L \bar{C}_{\dot{\beta}_1 \dot{\beta}_2 \dot{\beta}_3 \dot{\beta}_4} = h^{\dot{\beta}} \bar{C}_{\dot{\gamma}, \dot{\beta}_1 \dot{\beta}_2 \dot{\beta}_3 \dot{\beta}_4 \dot{\delta}} , \]  

(88)

where \( C_{\dot{\alpha}_1 \ldots \dot{\alpha}_5} \) and \( \bar{C}_{\dot{\gamma}, \dot{\beta}_1 \ldots \dot{\beta}_5} \) are new arbitrary multispinor field variables which are totally symmetric in spinor indices of each type, while \( D^L \) is the Lorentz-covariant differential

\[ D^L A_{\alpha \dot{\beta}} = dA_{\alpha \dot{\beta}} + \omega_{\alpha \gamma} \wedge A_{\gamma \dot{\beta}} + \bar{\omega}_{\dot{\beta}} \wedge A_{\alpha \dot{\gamma}}. \]  

(89)

Once again, Bianchi identities for the l.h.s.'s of \( (87) \), \( (88) \) impose certain differential restrictions on \( C_{\dot{\alpha}_1 \ldots \dot{\alpha}_5} \) and \( \bar{C}_{\dot{\gamma}, \dot{\beta}_1 \ldots \dot{\beta}_5} \) which can be cast into the form analogous to \( (87) \), \( (88) \) by virtue of introducing new field variables \( C_{\dot{\alpha}_1 \ldots \dot{\alpha}_5, \dot{\beta}_1 \dot{\beta}_2} \) and \( \bar{C}_{\dot{\alpha}_1 \dot{\alpha}_2, \dot{\alpha}_3 \ldots \dot{\alpha}_6} \). Continuation of this process leads to the following infinite chain of differential relations:

\[ D^L C_{\dot{\alpha}_1 \ldots \dot{\alpha}_{n+4}, \dot{\beta}_1 \ldots \dot{\beta}_n} = h^{\dot{\alpha}} C_{\dot{\alpha}_1 \ldots \dot{\alpha}_{n+4} \gamma, \dot{\beta}_1 \ldots \dot{\beta}_n \dot{\delta}} - n(n+4) \lambda^2 h_{\dot{\alpha}_1 \dot{\alpha}_2 \ldots \dot{\alpha}_{n+4}, \dot{\beta}_1 \ldots \dot{\beta}_n} + O(C^2), \]  

(90)

\[ D^L \bar{C}_{\dot{\alpha}_1 \ldots \dot{\alpha}_n, \dot{\beta}_1 \ldots \dot{\beta}_{n+4}} = h^{\dot{\beta}} \bar{C}_{\dot{\alpha}_1 \ldots \dot{\alpha}_n \gamma, \dot{\beta}_1 \ldots \dot{\beta}_{n+4} \dot{\delta}} - n(n+4) \lambda^2 h_{\dot{\alpha}_1 \dot{\alpha}_2 \ldots \dot{\alpha}_n, \dot{\beta}_1 \ldots \dot{\beta}_{n+4}} + O(C^2). \]  

(91)

All these relations contain no new dynamical information in addition to that contained in the original Einstein equations in the form \( (84) \)–\( (86) \). Analogously to the spin 0 case, \( (90) \) and \( (91) \) merely express highest 0–forms \( C_{\dot{\alpha}_1 \ldots \dot{\alpha}_{n+4}, \dot{\beta}_1 \ldots \dot{\beta}_n} \) and \( \bar{C}_{\dot{\alpha}_1 \ldots \dot{\alpha}_{n+4}, \dot{\beta}_1 \ldots \dot{\beta}_n} \) via derivatives of the lowest 0–forms \( \bar{C}_{\dot{\alpha}_1 \ldots \dot{\alpha}_4} \) and \( \bar{C}_{\dot{\beta}_1 \ldots \dot{\beta}_4} \) containing at the same time all consistency conditions for \( (85) \), \( (86) \) and the equations \( (90) \), \( (91) \) themselves. Thus, the system of equations \( (84) \)–\( (86) \), \( (90) \) and \( (91) \) turns out to be dynamically equivalent to the Einstein equations with the cosmological term. It is this form of the equations which one arrives at in the analysis of the higher-spin equations of the previous section in the spin–2 sector.

Let us note that although we know closed equations for higher-spins \( (88) \)–\( (72) \), for the case of pure gravity an explicit form of all terms nonlinear in \( C \) on the r.h.s.'s of \( (90) \) and \( (91) \) is not still known in all orders. The form of \( C^2 \)– type terms was obtained in \( [84] \).

The infinite set of the 0–forms \( C \) and \( \bar{C} \) can be interpreted as a convenient basis in the linear space of all on-mass-shell nontrivial components of curvatures and their covariant derivatives.
The example of pure gravity can be generalized straightforwardly to all higher spins as explained in \[32\]. The general linearized equations read

\[ R_{\alpha_1...\alpha_n, \beta_1...\beta_m} = \delta(m) h^{\gamma_1 \delta} \wedge h^{\gamma_2 \delta} C_{\alpha_1...\alpha_{n_1} \gamma_1} + \delta(n) h^{\eta \delta} \wedge h^{\delta_2 \delta} \bar{C}_{\beta_1...\beta_{m_1} \delta_2}, \] (92)

\[ D^LC_{\alpha_1...\alpha_n, \beta_1...\beta_m} = h^{\gamma \delta} C_{\alpha_1...\alpha_m \gamma, \beta_1...\beta_n \delta} = n m \lambda^2 h_{(\alpha_1 \beta_1} C_{\alpha_2...\alpha_m, \beta_2...\beta_n)_{\alpha, \beta} + O(C^2) \] (93)

Needless to say that it is this form of the linearized equations which one derives from the equation (69).

10 Higher-Spin Equations of Motion in 2+1 Dimensions

The situation for \( d = 2 + 1 \) is very much parallel to that for the 3+1 dimensional case. The full equations are again formulated in terms of the generating functions \( W(z, y; \psi|x) \), \( B(z, y; \psi|x) \) and \( s(z, y; \psi|x) \) which depend on the space-time coordinates \( x^\nu \) (\( \nu = 0 - 2 \)), auxiliary commuting spinors \( z_\alpha \) and \( y_\alpha \) (\( \alpha = 1, 2 \)), and two Clifford elements \( \psi^i \) (\( i = 1, 2 \))

\[ \{ \psi^i, \psi^j \} = 2 \delta^{ij}, \] (94)

where \( W = dx^\nu W_\nu(z, y; \psi|x) \) and \( B = B(z, y; \psi|x) \) are space-time 1-form and 0-form, respectively, while \( s = dz^\alpha s_\alpha(z, y; \psi|x) \) is a space-time 0-form and \( z\)–space 1-form with the auxiliary anticommuting differentials

\[ dz^\alpha dz^\beta = -dz^\beta dz^\alpha \] (95)

which commute with all other variables \( z_\alpha, y_\alpha, \psi^i, x^\nu \) and \( dx^\nu \). Again, \( s \) acts as a differential operator shifting along \( z \)–directions and does not possess its own degrees of freedom. It is expressed entirely (up to a pure gauge part) in terms of the 0-form \( B \) which serves as the generating function for matter fields. The 1-form \( W \) is the generating function for higher-spin gauge fields which do not propagate in \( 2+1 \) dimensions but mediate interactions of the matter fields.

We endow the space of functions \( f(z, y) \) with the structure of the star-product algebra by restricting the formula (56) to the subspace of functions independent of the dotted spinors.

The equations which describe higher-spin interactions of massless matter in \( 2+1 \) dimensions have the form \[35\] similar to that of the equations in \( 3+1 \) dimensions

\[ dW = W \ast W, \quad dB = W \ast B - B \ast W, \quad ds = W \ast s - s \ast W, \] (96)

\[ s \ast s = -dz_\alpha dz^\alpha(i + B), \] (97)
\[ s \ast B = -B \ast s. \]  

These equations have “unfolded form” and therefore possess explicit general coordinate invariance and gauge invariance under the infinitesimal gauge transformations

\[ \delta W = d\varepsilon - W \ast \varepsilon + \varepsilon \ast W, \quad \delta B = \varepsilon \ast B - B \ast \varepsilon, \quad \delta s = \varepsilon \ast s - s \ast \varepsilon, \]

where \( \varepsilon = \varepsilon(z, y; \psi|x) \) is an arbitrary gauge parameter.

To make sure that the system (96)-(98) describes higher-spin interactions of massless matter fields in 2+1 dimensions, one has to analyze it in the linearized approximation. This analysis is in many respects parallel to that carried out for the 3+1-dimensional case.

At the first stage, one fixes an appropriate vacuum solution. We assume that \( W \) and \( s \) contain zero-order nontrivial vacuum components \( W_0 \) and \( s_0 \) while \( B \) starts from the first-order terms. Namely, we fix

\[ B_0 = 0, \quad s_0 = dz^\alpha z_\alpha, \quad W_0 = \frac{1}{4i} (\omega_0^{\alpha\beta} y_\alpha y_\beta + h_0^{\alpha\beta} y_\alpha y_\beta \psi_1). \]

One observes that \( s_0 \) acts as \( z \)-differential,

\[ [s_0, f]_\ast = -2i\partial f(z, y), \quad \partial = dz^\alpha \frac{\partial}{\partial z^\alpha} \]

for every \( f = f(z, y) \). As a result, the equations (100) solve the equations (96)-(97) except for the equations for \( W \), which imposes additional restrictions on \( \omega_0 \) and \( h_0 \)

\[ \begin{align*}
\omega_0^{\alpha\beta}(x) &= \omega_0^{\alpha\gamma}(x) \wedge \omega_0^{\gamma\beta}(x) + h_0^{\alpha\delta}(x) \wedge h_0^{\delta\beta}(x), \\
h_0^{\alpha\beta}(x) &= \omega_0^{\alpha\gamma}(x) \wedge h_0^{\gamma\beta}(x) + \omega_0^{\gamma\beta}(x) \wedge h_0^{\alpha\delta}(x).
\end{align*} \]

According to the analysis of section 4 the fields \( \omega_0^{\alpha\beta} \) and \( h_0^{\alpha\beta} \) are identified with the background gravitational Lorentz connection and dreibein, respectively. It is worth mentioning that it is the necessity to have a non-degenerate space-time background metric that forces one to introduce the non-vanishing background 1-form \( W_0 \) since otherwise the equations (96)-(98) cannot be interpreted in terms of particles.

The explicit analysis then shows that the role of nontrivial dynamical variables is played by the “initial data” \( C(y; \psi|x) = B(0, y; \psi|x) \). Expanding \( C \) as

\[ C(y; \psi|x) = C^{aux}(y; \psi_1|x) + \psi_2 C^{mat}(y; \psi_1|x) \]

one finds that \( C^{mat} \) describes massless matter fields while \( C^{aux} \) describes some auxiliary fields which do not carry dynamical degrees of freedom \([30]\). The matter sector contains two massless bosons described by the even functions \( C^{mat}(-y; \psi_1|x) = C^{mat}(y; \psi_1|x) \) and two massless fermions described by the odd functions \( C^{mat}(-y; \psi_1|x) = -C^{mat}(y; \psi_1|x) \). The doubling is due to the dependence on the Clifford element \( \psi_1 \). Note that in the bosonic sector there exists a reduction to a model describing a single massless scalar.
This reduction is not possible however in presence of fermions. Let us note that for analogous reason we confined ourselves to the pure bosonic model in the case of 3+1 theory. The full 3+1 dimensional theory which involves fermions requires some additional variables analogous to $\psi_1$ or matrix algebras analogous to those considered in section 13.

The physical interpretation of the generating function $C^{\text{mat}}(y; \psi_1|x)$ is that the lowest modes of its expansion in powers of $y$ are identified with the dynamical fields, i.e. $C^{\text{mat}}(0; \psi_1|x)$ describes scalar while the linear part of $C^{\text{mat}}(y; \psi_1|x)$ describes spinor. The highest modes describe all on-mass-shell nontrivial derivatives of the matter fields. The important physical distinction of the 2+1 dimensional case from the 3+1 dimensional one is that in the former case there is no room for the Weyl tensors related to the gauge fields. This is in accordance with the well known fact that higher-spin gauge fields do not propagate in 2+1 dimensions. In fact this implies that the dynamics we analyze is of the Chern-Simons type thus generalizing the pure higher-spin dynamics of Blencowe to the case with non-trivial matter fields.

An interesting question which we cannot discuss in full detail in this talk is why the higher-spin equations have this particular form in 3+1 and 2+1 dimensions. This mainly concerns the sector of the constraints (71), (72), (77) and (78) since one can write a lot of other versions of invariant constraints. The point is that these constraints are singled out by the requirement that the full theory must possess the local Lorentz invariance in the physical sector of $z-$ independent fields $C(y; \ldots)$. This property is not straightforward at all. The reason is that the vacuum solution like (100) is not invariant under the Lorentz transformations rotating all spinors $z_\alpha$ and $y_\alpha$ because higher-spin gauge transformations do not affect the differentials $dz_\alpha$. The requirement that Lorentz symmetry must act in the standard way on the physical modes is necessary for the proper relativistic field theory interpretation of the model. This property can be shown to be guaranteed by the constraints described in this section and in section 9 but fail for different choices of the constraints.

11 D=2 Matter

Let us now describe the new results on the higher-spin interactions of d2 matter fields. Originally it was observed in [36, 37] that one can construct Noether current interactions for a massless scalar field in two dimensions in the form

$$S = \frac{1}{2} \int_{M^2} \left( \partial_\nu \phi \partial^{\nu} \phi + \sum_n q A^\nu_1 \ldots \nu_{2n} J_{\nu_1 \ldots \nu_{2n}} \right),$$

where $J_{\nu_1 \ldots \nu_{2n}}$ are some conserved currents which have a form

$$J_{\nu_1 \ldots \nu_{2n}} = \partial_{\nu_1} \ldots \partial_{\nu_n} (\phi) \partial_{\nu_{n+1}} \ldots \partial_{\nu_{2n}} (\phi) + \ldots,$$
where dots denote some trace terms proportional to $\eta_{\nu\nu_i}$. These currents generalize the usual stress tensor

$$ J_{\nu\mu} = T_{\nu\mu} = \partial_{\nu} \phi \partial_{\mu} \phi - \frac{1}{2} \left( \partial_{\rho} \phi \right)^2 \eta_{\nu\mu} $$

(106)
to higher spins. In the light-cone coordinates the currents have only two on–mass–shell nontrivial components

$$ J^+_{\ldots} = \left( \partial^\nu_n \phi \right)^2, \quad J^-_{\ldots} = \left( \partial^-_{\rho_n} \phi \right)^2. $$

(107)

It was argued in [36] that these currents generate some infinite-dimensional algebra later on called $W_{1+\infty}$.

Below we generalize the models of [36, 37] by introducing gauge invariant interactions for the higher-spin gauge fields so that no vanishing current constraints on the matter fields are present in our model. The model is formulated in an explicitly higher-spin gauge invariant and general coordinate invariant fashion. A natural $d=2$ background is AdS spacetime. The presented model is not conformal. Since the full equations of motion have a form of some zero-curvature equations and covariant constantness conditions without any additional constraints, the model turns out to be integrable. This unexpected property is specific for $d=2$ and allows us to formulate a simple $BF-$ type action principle for the model.

Let us first reformulate free equations of motion for matter fields in $d=2$ AdS space in the form of some covariant constantness conditions along the lines of the “unfolded formulation” described in section 8. Here we use light-cone coordinates and consider the AdS background described by the zweibein $h^\pm$ and Lorentz connection $\omega$ obeying the vacuum equations

$$ R^0 = d\omega + h^- \wedge h^+ = 0, \quad R^\pm = dh^\pm \pm 2\omega \wedge h^\pm = 0. $$

(108)

Consider the following system of equations

$$ D\phi_n = \alpha(n)h^-\phi_{n+2} + \beta(n)h^+\phi_{n-2}, $$

(109)

where $D$ is the Lorentz covariant derivative,

$$ D\phi_n = d\phi_n + n\omega\phi_n. $$

(110)

This system is formally consistent (i.e. the Bianchi identities are satisfied) provided that the numerical parameters $\alpha(n)$ and $\beta(n)$ obey the condition

$$ \alpha(n)\beta(n+2) = \mu + \frac{1}{4} n(n+2) $$

(111)

and zero curvature conditions (108) are satisfied. Here $\mu$ is an arbitrary numerical parameter. Note that the ambiguity in the coefficients $\alpha(n)$ and $\beta(n)$, which is not fixed by (111), is irrelevant and reflects a freedom in the rescaling $\phi_n \rightarrow \gamma(n)\phi_n$. 

To make sure that, e.g., the equations (109) with even \( n \) are equivalent to the Klein-Gordon equation let us introduce the inverse zweibein \( h^\nu_\pm \) and rewrite the system of equations (109) in the form

\[
h^\nu_+ D_\nu \phi_n = \beta(n) \phi_{n-2}, \quad h^\nu_- D_\nu \phi_n = \alpha(n) \phi_{n+2}.
\]

(112)

One observes that these equations with \( n = 0 \) express the fields \( \phi_{\pm 2} \) in terms of the first space-time derivatives of \( \phi_0 \). Then the equations (112) with \( n = \pm 2 \) contain the Klein-Gordon equation and express the fields \( \phi_{\pm 4} \) via second space-time derivatives of \( \phi_0 \). (Note that although the Klein-Gordon equation appears twice, i.e. both in the first of the equations (112) with \( n = 2 \) and in the second one with \( n = -2 \), an appropriate combination of these equations vanishes identically due to the Bianchi identities of the original equations (109) so that, effectively, the Klein-Gordon equation appears only once.) Finally, one finds that all higher-\( n \) equations in the system (112) either express the fields \( \phi_m \) with \( m \neq 0 \) via higher derivatives of \( \phi_0 \) or encode all Bianchi identities for these expressions imposing no additional dynamical conditions on the field \( \phi_0 \). This analysis is parallel to that of section 8.

As a result, the system (109) with even \( n \) turns out to be dynamically equivalent to the Klein-Gordon equation supplemented with some constraints which express all highest \( \phi_n \) via higher space-time derivatives of the dynamical field \( \phi_0 \). The situation with fermions (\( n \) is odd) is analogous. A physical meaning of the components \( \phi_n \) is that they describe all on-mass-shell nontrivial derivatives of the dynamical boson and fermion fields, generalizing the flat-space higher derivatives \( (\partial_+)^n \phi \) and \( (\partial_-)^n \phi \) to the AdS case.

By analogy with the analysis of the d=3+1 and d=2+1 cases one can conjecture [39] that the relevant higher-spin algebra is \( shs(2) \), which gives rise to the set of gauge fields

\[
\omega_\nu = \sum_{n,m=0}^\infty \omega_{\nu n,m} (\hat{y}_+)^n (\hat{y}_-)^m
\]

(113)

with the elementary oscillators obeying the relations

\[
[\hat{y}_-, \hat{y}_+] = -2i.
\]

(114)

This algebra contains the AdS subalgebra spanned by the generators

\[
L^\pm = \frac{i}{4}(\hat{y}_\pm)^2, \quad L^0 = \frac{i}{4}\{\hat{y}_+, \hat{y}_-\}
\]

(115)

obeying the \( sp(2) \) commutation relations

\[
[L^0, L^\pm] = \pm 2L^\pm, \quad [L^-, L^+] = L^0.
\]

(116)

Let us emphasize that since higher-spin gauge fields are not propagating in 1+1 dimensions, at this stage a choice of the higher-spin algebra isambiguous enough. The only
important property is that it should contain $sp(2)$ as a subalgebra. A final choice can be
done from the analysis of interactions.

A less trivial problem is how to describe matter fields. From the linearized analysis it
follows that one has to introduce a one-parametric set $\phi_n$ with $-\infty < n < \infty$. Evidently,
it does not work any longer to take a function $\Phi = \sum_{n,m=0}^{\infty} \Phi_{n,m}(\hat{y}^+)^n(\hat{y}^-)^m$ as in $d = 3, 4$ since it involves too many components. The idea to chose Fock (i.e. metaplectic)
representation $|\Phi\rangle = \sum_{n=0}^{\infty} \Phi_n(a^+)^n|0\rangle$ is not working either since it contains only a half
of states.

The way out is tricky enough. One has to start with the tensor prod uct of two Fock
spaces $|\Phi(x)\rangle = \sum_{n,m=0}^{\infty} \Phi_{n,m}(a^+)^n(b^+)^m|0\rangle$ and then to gauge away all operators which contain $(a^+b^+)|\chi\rangle$ for any $\chi$. As a result one
is left just with the appropriate set of matter fields

$$|\Phi(x)\rangle = \left[ \sum_{n=1}^{\infty} \left( \Phi^+_n(x)(a^+)^n + \Phi^-_n(x)(b^+)^n \right) + \Phi_0(x) \right]|0\rangle .$$

In section 12 we show how this idea is realized for interacting d2 matter fields.

## 12 Higher-Spin-Matter Interactions in 1+1 Dimensions

Analogously to the scheme developed for $d=3,4$ the basic algebraic element is the asso-
ciative algebra $A$ of power series in the generating elements $y_\pm$ and $z_\pm$ endowed with the
associative star-product

$$(f * g)(z, y) = (2\pi)^{-4} \int d^2 s d^2 t d^2 p d^2 q \frac{f(z + s, y + p)g(z - t, y + q) \exp[i(s_\alpha t^\alpha + p_\alpha q^\alpha)]}{\alpha, \beta = \pm}.$$  

such that

$$[y_-, z_+]_s = -2i, \quad [z_-, z_+]_s = 2i, \quad [y_\alpha, z_\beta]_s = 0 \quad \alpha, \beta = \pm.$$

It is also convenient to use the following equivalent set of variables

$$u_\alpha = \frac{1}{2}(z_\alpha - y_\alpha), \quad v_\alpha = \frac{1}{2}(y_\alpha + z_\alpha), \quad [v_\pm, u_\mp] = \mp i, \quad [v_\alpha, v_\beta] = [u_\alpha, u_\beta] = 0.$$  

The important property of $A$ is that it contains an element $\Pi$ which is a projection
operator, $\Pi^2 = \Pi$, and behaves as a vacuum vector for the operators $u_\pm$ and $v_\pm$, i.e.
$v_\pm \Pi = 0$ and $\Pi v_\pm = 0$. Its explicit realization is

$$\Pi = \frac{1}{4} \exp(iz_\alpha y^\alpha).$$
To formulate the d2 higher-spin dynamics it is useful to extend the algebra \( A \) to \( \mathcal{A} \) by virtue of the following general procedure. Given associative algebra \( A \) and some projection operator \( \Pi \in A \), one defines the algebra \( \mathcal{A} \) such that its general element \( a \in \mathcal{A} \) is equivalent to a set of four elements of \( A \), \( a = \{ a, |a\rangle, \langle a|, \langle a \rangle \} \), obeying the properties

\[
\{ a \in \mathcal{A} | \ a, |a\rangle, \langle a|, \langle a \rangle \in A ; \ |a\Pi = |a\rangle, \Pi \langle a| = \langle a|, \langle a\Pi = \Pi \langle a \rangle \}
\]  

(123)

The product law \( \circ \) in \( \mathcal{A} \) is defined via the product law in \( A \) as follows

\[
a \circ b = \{ ab + |a\langle b|, a|b\rangle + |a\rangle \langle b|, \langle a|b + \langle a\rangle \langle b|, \langle a||b + \langle a\rangle \langle b \}
\]

(124)

This product law is associative. Note that the supertrace operation \( \text{str}_\mathcal{A} \) in \( \mathcal{A} \) induces the supertrace operation \( \text{str}_A \) in \( A \)

\[
\text{str}_\mathcal{A}(a) = \text{str}_A(a + \langle a \rangle),
\]

(125)

where \( \text{str}_\mathcal{A} \) is the supertrace operation (20).

In the context of the d2 dynamics this construction with the projection operator (122) is used to embed all matter and auxiliary fields into the adjoint representation of \( \mathcal{A} \). Namely, to describe the higher-spin gauge interactions of d2 matter fields we introduce the gauge one-form \( W(x|z_\alpha, y_\alpha) = dx^\nu W_\nu(x|z_\alpha, y_\alpha) \), and the matter field zero-form \( B(x|z_\alpha, y_\alpha) \), both in the adjoint representation of \( \mathcal{A} \), i.e.

\[
W = \{ W, |W\rangle, \langle W|, \langle W \rangle \}, \quad B = \{ B, |B\rangle, \langle B|, \langle B \rangle \}.
\]

(126)

The full system of equations for interacting d2 matter fields has a simple form of zero-curvature conditions:

\[
R \equiv dW + W \circ \wedge W = 0, \quad dB + W \circ B - B \circ W = 0.
\]

(127)

These equations can be derived from the B-F type action principle

\[
S = \int_{M_2} \text{str}_\mathcal{A}(B R).
\]

(128)

The model becomes dynamically non-trivial because the 0-form \( B \) is supposed to have a nonvanishing vacuum value \[1\]

\[
B_{\text{vac}} = \{ N, 0, 0, 0 \}, \quad N = \frac{1}{2i} z_- z_+.
\]

(129)

A physical vacuum value of the gauge 1-form \( W \) is

\[
W_{\text{vac}} = \{ \omega^{gr}, 0, 0, 0 \}, \quad \omega^{gr}(x) = h^+(x)L^+ + h^-(x)L^- + \omega(x)L^0,
\]

(130)

\[1\]Note that the physical vacuum values of the fields \( B \) and \( W \) have nothing to do with the vacuum \( \Pi \) of the algebra \( A \) of auxiliary spinor variables.
where $L^\pm$ and $L^0$ are the $sl_2$ generators.

One-forms $h^\pm(x) = dx^\nu h^\pm_\nu(x)$ and $\omega(x) = dx^\nu \omega_\nu(x)$ describe inverse zweibein and Lorentz connection, respectively. The components of the gravitational field-strength two-form,

$$R^{gr} = d\omega^{gr} + \omega^{gr} \wedge \omega^{gr} = R^+ + R^- + R^0 L^0$$

identify, respectively, with the torsion tensor, $R^+$, $R^-$, and with the Riemann tensor, $R^0$, shifted by a cosmological term $h^- \wedge h^+$. The vacuum gravitational field is supposed to obey the zero-curvature conditions so that the first of the equations is satisfied. The second one is also true because the vacuum value $N$ of $B$ depends only on $z$ and therefore commutes with the background gravitational field due to (120).

Higher-spin gauge fields correspond to higher-order terms of the expansion of $W(x|z_\alpha, y_\alpha)$ in powers of the auxiliary spinor variables. The gauge connection $W$ and the matter field $B$ have the standard transformation laws under the higher-spin gauge transformations with the parameter $\xi(x|z_\alpha, y_\beta)$,

$$\delta W = d\xi + W \circ \xi - \xi \circ W, \quad \delta B = B \circ \xi - \xi \circ B,$$

which leave invariant the equations and the action. General coordinate invariance is explicit too.

A global symmetry subalgebra which acts linearly on physical states is described by the parameters commuting with $B_{vac}$,

$$\xi = (\xi_{vac}, 0, 0, \langle \xi_{vac} \rangle)$$

with an arbitrary Abelian parameter $\langle \xi_{vac} \rangle$ and the parameter $\xi_{vac}$ of the form

$$\xi_{vac} = \sum_{n,m,k=0}^{\infty} \xi_{n,m,k}(x) N^k (y_-)^n (y_+)^m.$$

Since $N$ commutes with the oscillators $y_{\pm}$, the generating elements of the $W_{1+\infty}$ algebra, one is left with the non-negative part of the loop extension $\tilde{W}_{1+\infty}$ of $W_{1+\infty}$.

The topological form of the action is analogous to the topological form of the d2 gravitational action discussed and to the higher-spin action proposed. This analogy is not exact however because in the latter models the zero-curvature equations are true in absence of matter and do not describe propagating degrees of freedom while the equations are shown below to describe interactions of propagating scalar and spinor fields, which phenomenon turns out to be possible because of using infinite multiplets of fields.

Another important point is that the non-vanishing vacuum value of the zero-form $B$ leads effectively to some $\tilde{W}^2$ - type terms in the action that opens a way to a proper diagonalization of the action at the linearized level. Practically, a problem of reducing the
quadratic part of the action (128) to the standard form is highly involved due to presence of infinitely many auxiliary fields.

To analyze the equations (127) perturbatively one considers the fields of the form $W = W_{vac} + w$ and $B = B_{vac} + b$ where $w$ and $b$ denote perturbations. Propagating matter fields belong to the mutually conjugated components $|b\rangle$ and $\langle b|$ of $b$. The linearized equations (127) in the sector of the matter fields $|b\rangle$ read

$$d|b\rangle + w^{gr}|b\rangle = N|w\rangle .$$

This equation implies, first, that $|w\rangle$ expresses via the matter fields $|b\rangle$ and, second, that it imposes some differential equations on those components of the matter fields which are not proportional to $N$. Let us show that the latter differential equations are just the equations for free matter fields analyzed in section 11.

The linearized gauge transformation (132) for the field $|b\rangle$ takes the form $\delta |b\rangle = N|\xi\rangle + O(b)$. This implies that the field $|b\rangle$ contains some Higgs part which can be gauged away and a reminder which is to be shown to describe matter fields. The standard Fock representation for $|b\rangle$ is $|b\rangle = b'(u_+, u_-) \ast \Pi$. Since $N = \frac{1}{2i} z_+ z_- = \frac{1}{2i} (u + v)_+ (u + v)_-$, the Higgs-type component of the transformation law for $|b\rangle$ allows one to get rid of any polynomial in $u_+ u_-$ in $b'$. As a result one can chose a gauge with respect to the transform (132) with

$$b'(u_+, u_-) = b'_+(u_+) + b'_-(u_-) + b'_0 , \quad b'_+(0) = b'_-(0) = 0 .$$

Fields of this form cannot be compensated by virtue of any transformation (132) and therefore can describe some dynamical degrees of freedom. By expanding (136) in powers of $u_\pm$ one observes that, in accordance with the idea sketched in the end of section 11, the structure of the gauge fixed matter field $b'$ (136) is just of the form one expects for d2 matter fields from (109).

To work out explicit form of the field equations one has to substitute (136) into (135), decompose the left-hand-side of (133) into a part proportional to $N$, which is compensated by an appropriate choice of $|\omega\rangle$, and a part depending either only on $u_+$ or only on $u_-$ as in (136), which will impose some equations on $b'$. Let us give the final result for the field equations and the value of the field $|w\rangle = w'(u_\pm) \ast \Pi$:

$$Db' = \frac{1}{4i} h^+ \left( (u_+)^2 (b'_+ + b'_0) + iu_+ b'_-(0) - 4b'_-(u_-) + \int_0^1 ds \left( 3s + 1 \right) b'_-(su_-) \right)$$

$$+ \frac{1}{4i} h^- \left( (u_-)^2 (b'_- + b'_0) - iu_- b'_+(0) - 4b'_+(u_+) + \int_0^1 ds \left( 3s + 1 \right) b'_+(su_+) \right),$$

$$w'(u_{\pm}) = -\omega b'(u_{\pm}) + \frac{i}{2} h^+ \left( iu_+ \int_0^1 ds b'_-(su_-) - \int_0^1 ds \left( 2s + 1 \right) b'_-(su_-) \right)$$

$$+ \frac{i}{2} h^- \left( iu_- \int_0^1 ds b'_+(su_+) + \int_0^1 ds \left( 2s + 1 \right) b'_+(su_+) \right) .$$
where \( \dot{f}(x) = \frac{\partial}{\partial x} f(x) \) and \( D \) is the Lorentz covariant derivative.

One can check directly that the equation (137) is formally consistent thus corresponding to some particular case of the equations (109) with the coefficients of the form (111). By expanding the function \( f^l \) into power series in either \( u_+ \) or \( u_- \) one finds that the coefficients indeed satisfy the condition (111) with \( \mu = 3/16 \). This value is not occasional. It equals to the value of the \( sl_2 \) Casimir operator for the realization (131). There is a possibility to generalize the proposed scheme to an arbitrary mass which we will discuss elsewhere [38]. Let us note that the parameter \( \mu \) is measured here in units of the inverse radius of the background AdS space-time and therefore tends to zero in the flat limit.

Thus it is shown that the linearized equations for \( |b\rangle \) describe properly linearized dynamics for d2 matter fields. Analogously one can analyze the conjugate sector of \( \langle b| \) to show that it describes conjugate matter fields.

An important property which we do not prove explicitly here is that all other components in \( W \) and \( B \) do not carry their own degrees of freedom. This can be shown for example with the aid of the method developed in [30] where it was argued that any system of covariant constantness equations for zero forms cannot describe propagating modes if these zero forms carry some finite-dimensional representations of the space-time symmetry algebra which gives rise to the vacuum gravitational field. Actually, in the model under consideration all components of the zero form \( B \) contained in \( B \) and \( \langle B \rangle \) decompose into a sum of only finite-dimensional representations of the AdS algebra under the adjoint action of the generators (113).

Thus, the matter fields contained in \( |b\rangle \) and the conjugated fields \( \langle b| \) are the only propagating degrees of freedom in the system. All other fields are either auxiliary or mediate interactions of the matter fields. In particular this is the case for the gravitational field which corresponds to the sector of the \( W \) fields quadratic in \( y_\pm \) and for its higher-spin generalizations corresponding to higher powers in \( y_\pm \). Due to the form of the product law (124) the matter fields contribute quadratically to the equations for the gravitational field and its higher-spin analogs in agreement with what one expects from the matter sources for the gravitational field.

The linearized analysis shows that the Lorentz connection occurs only through the standard Lorentz covariant derivative. This is important and not completely trivial property that can be shown to remain valid in all orders in interactions [38] and in fact fixes the form of the d2 higher-spin dynamics.

The remarkable property of the proposed equations (127) is that having a form of some zero curvature conditions they can be integrated explicitly at least locally

\[
W(x) = g^{-1}(x)dg(x), \quad B(x) = g^{-1}(x)B_0 g(x),
\]

(139)

where \( g(x) \) is an arbitrary \( x \)-dependent invertible element of \( \mathcal{A} \) while \( B_0 \) is an arbitrary \( x \)-independent element of \( \mathcal{A} \). The novel feature compared to the theories in higher dimensions is that there are no constraints on \( B_0 \). Thus the presented non-linear model turns out to be integrable due to the specific form of the higher-spin interactions.
13 Extended Higher-Spin Superalgebras

An important property of the higher-spin equations in all examples considered above is that they remain consistent if all field variables \( W, B \) and \( s \) (the latter one in 3+1 and 2+1 dimensions) take values in an arbitrary associative algebra \( A \). The simplest possibility consists of identifying \( A \) with the matrix algebra, \( Mat_N(C) \), in which case field variables possess additional matrix indices, \( W \rightarrow W^i_j, B \rightarrow B^i_j \) with \( i, j = 1, ..., N \). This offers a way for constructing higher-spin systems with nontrivial internal symmetries of Yang-Mills type.

One can address the question what are the most general truncated versions of these extended higher-spin theories which still lead to consistent higher-spin dynamics. A convenient way to classify consistent higher-spin models is to analyze automorphisms of the higher-spin algebras which leave invariant the dynamical equations. In this way the problem was fully analyzed for the 3+1 dimensional theory in [7] and can be analogously analyzed for the cases of 2+1 and 1+1 models. To illustrate this issue let us summarize here the basic results for the 3+1 dimensional model. Following [7] we consider only the vacuum higher-spin symmetries which generalize the symmetry \( hs(4) \) considered in section 5, i.e. those which leave invariant the vacuum solutions of the full field equations and act as true symmetries on the physical states. We focus on the higher-spin algebras which lead to consistent higher-spin systems with finite-dimensional internal symmetries.

It turns out that there exist three types of higher-spin algebras which reduce to unitary, symplectic and orthogonal gauge algebras in the spin\(-1\) Yang-Mills sector. Unitary higher-spin algebras denoted \( hu(n; m|4) \) can be realized as \( (n + m) \times (n + m) \) matrices with the elements depending on operators \( \hat{y}_\alpha \) and \( \hat{\bar{y}}_{\dot{\alpha}} \) for

\[
P^j_i(\hat{y}, \hat{\bar{y}}) = \begin{pmatrix}
\begin{array}{c c}
P^E_{ij}(\hat{y}, \hat{\bar{y}}) & P^O_{ij}(\hat{y}, \hat{\bar{y}}) \\
P^O_{ij}(\hat{y}, \hat{\bar{y}}) & P^E_{ij}(\hat{y}, \hat{\bar{y}})
\end{array}
\end{pmatrix}
\]

(140)

for the conditions that

\[
P^E_{ij}(\hat{y}, \hat{\bar{y}}) = P^E_{ij}(-\hat{y}, -\hat{\bar{y}}),
\]

(141)

\[
P^O_{ij}(\hat{y}, \hat{\bar{y}}) = -P^O_{ij}(-\hat{y}, -\hat{\bar{y}}),
\]

(142)

i.e. the diagonal blocks \( P^E \) are bosonic (the power series coefficients carry even numbers of spinor indices) while the off-diagonal blocks \( P^O \) are fermionic (the power series coefficients carry odd numbers of spinor indices). In addition, it is assumed that the elements of the matrices \( P \) obey the reality conditions

\[
[P^j_i(\hat{y}, \hat{\bar{y}})]^\dagger = -(i)^{\pi(P^j_i)} P^j_i(\hat{y}, \hat{\bar{y}})
\]

(143)
with

\[ (\hat{y}_\alpha)^\dagger = \hat{\bar{y}}_{\dot{\alpha}}, \quad \pi(P^E) = 0, \quad \pi(P^O) = 1. \] (144)

The algebras \( hu(n; m|4) \) contain higher-spin subalgebras of orthogonal and symplectic types denoted \( ho(n; m|4) \) and \( husp(n; m|4) \), respectively, which also give rise to consistent equations of motion for massless fields of all spins via appropriate truncations of the higher-spin equations corresponding to \( hu(n; m|4) \) \[32, 33\]. These subalgebras can be extracted from \( hu(n; m|4) \) by imposing the following conditions

\[ P^l_k(\hat{y}, \hat{\bar{y}}) = -(i)^{\pi(P^l_k)} \eta^{lu} P^v_l(i\hat{y}, i\hat{\bar{y}}) \eta^{-1}_{vk} \] (145)

with some nondegenerate bilinear form \( \eta_{kl} \). If \( \eta_{kl} \) is symmetric, \( \eta_{kl} = \eta_{lk} \), this leads to the orthogonal algebras \( ho(n; m|4) \). The skewsymmetric form, \( \eta_{kl} = -\eta_{lk} \), gives rise to the symplectic algebras \( husp(n; m|4) \) (\( n \) and \( m \) should be even for the latter case).

Spin–1 Yang-Mills subalgebras of the higher-spin algebras defined in this way are spanned by the matrices independent of the operators \( \hat{y}_\alpha \) and \( \hat{\bar{y}}_{\dot{\alpha}} \). As a result, the Yang-Mills subalgebras coincide with \( u(n) \oplus u(m) \), \( o(n) \oplus o(m) \) and \( usp(n) \oplus usp(m) \) for \( hu(n; m|4) \), \( ho(n; m|4) \) and \( husp(n; m|4) \), respectively. Thus, all types of compact Lie algebras which belong to the classical series \( a_n, b_n, c_n \) and \( d_n \) can be realized as spin–1 Yang-Mills symmetries in appropriate higher-spin theories.

The multiplicities of massless spin–\( s \) particles in higher-spin theories based on extended superalgebras are \[7\]

| spin algebra | odd | even | half-integer |
|--------------|-----|------|-------------|
| \( hu(n; m|4) \) | \( n^2 + m^2 \) | \( n^2 + m^2 \) | \( \bar{n} \otimes m + n \otimes \bar{m} \) |
| \( ho(n; m|4) \) | \( \frac{1}{2}(n(n - 1) + m(m - 1)) \) | \( \frac{1}{2}(n(n + 1) + m(m + 1)) \) | \( n \otimes m \) |
| \( husp(n; m|4) \) | \( \frac{1}{2}(n(n + 1) + m(m + 1)) \) | \( \frac{1}{2}(n(n - 1) + m(m - 1)) \) | \( n \otimes m \) |

(146)

Let us note that the fields of all odd spins belong to the adjoint representations of the corresponding Yang-Mills algebras while even spins always belong to a reducible representation which contains a singlet component. This is a highly important property since such a singlet component corresponds to the spin–2 colorless field to be identified with graviton. In other words the finite-dimensional algebra constituted by the elements \( (I \otimes \text{bilinears in } \hat{y}, \hat{\bar{y}}) \) is a proper subalgebra of all higher-spin algebras.

Another important property is that the higher-spin superalgebras are supersymmetric in the standard sense only if \( n = m \). Indeed, one observes that (averaged) numbers of bosons and fermions coincide only for this case. Also, it can be easily verified that all higher-spin superalgebras with \( n = m \) contain the anti-de Sitter superalgebra \( osp(1; 4) \) as a subalgebra. An opposite case with \( n = 0 \) or \( m = 0 \) corresponds to the purely bosonic
higher-spin theories. The simplest version of the higher-spin action and equations of motion discussed in sections 7 and 9 corresponds to the case of $hu(1;0|4) \sim hs(4)$.

From the structure of the higher-spin superalgebras it is clear why consistent interactions for a spin $s \geq 2$ field in $d=3+1$ are only possible in presence of infinite sets of massless fields of infinitely increasing spins. The reason is that any field of spin $s \geq 2$ corresponds to generators of the algebra which are some $deg > 2$ polynomials of $\hat{y}$ and $\hat{\bar{y}}$ so that their successive commutators lead to higher and higher polynomials. The same happens when one attempts to replace the unit matrix $I$ by some non-Abelian matrix algebra for bilinear polynomials in $\hat{y}$ and $\hat{\bar{y}}$, i.e. to introduce spin $-2$ particles possessing a non-Abelian structure.

Let us note that the above construction can be generalized to infinite-dimensional algebras $A$. In particular, one can consider the higher-spin superalgebras $h\ldots(n;m|4)$ with $n \to \infty$ or/and $m \to \infty$, that will lead to theories with infinite numbers of massless particles of every spin. Such theories can be of interest in the context of spontaneous breakdown of higher-spin gauge symmetries and a relationship with string theory.

14 Concluding Remarks

At present, the consistent dynamics of higher-spin gauge theories is formulated in 3+1, 2+1 and 1+1 dimensions. Higher-spin theories generalize quite naturally all conventional massless systems such as spin $-1$ Klein-Gordon field, spin $-1/2$ Weyl field, spin $-1$ Yang-Mills fields, spin $-(3/2)$ (super) gravitational fields containing all of them as subtheories. The analysis of [40, 41] indicates that consistent higher-spin interactions can be formulated in higher dimensions too.

In addition to the arguments in favor of the relationship between higher-spin theories and string theory mentioned in Introduction, there exists a curious parallelism between the two types of theories which consists of the observation that both correspond to certain non-local objects. Actually, it is well-known that nice properties of strings originate from the fact that they are linearly extended non-local objects. Higher-spin gauge theories are much simpler but still non-local in some sense. They correspond to quantum-mechanically non-local point particles in the space of auxiliary variables $\hat{q} = \hat{y}_1$ and $\hat{p} = \hat{\bar{y}}_2$ which non-locality can be traced back to the non-locality of the star-product (25). The important question then is whether this quantum-mechanical non-locality of the classical higher-spin theories is enough to improve quantum behavior after they are quantized as field theories in $d=3+1$ space-time.

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References

[1] P. van Nieuwenhuizen, Phys. Rep. 68 (1981) 189.

[2] M. Green, J. Schwarz and E. Witten, “Superstring Theory”, Vols. 1 and 2, Cambridge Univ. Press, New York, 1987.

[3] E. S. Fradkin and M. A. Vasiliev, Phys. Lett. B189 (1987) 89; Nucl. Phys. B291 (1987) 141.

[4] M.A. Vasiliev, Phys. Lett. B285 (1992) 225.

[5] E. S. Fradkin and M. A. Vasiliev, Ann. of Phys. 177 (1987) 63.

[6] M. A. Vasiliev, Fortschr. Phys. 36 (1988) 33.

[7] S. E. Konstein and M. A. Vasiliev, Nucl. Phys. B331 (1990) 475.

[8] C.N. Pope, L.J. Romans and X. Shen, Phys. Lett. B236 (1990) 173; Phys. Lett. B242 (1990) 401.

[9] C. N. Yang and R. L. Mills, Phys. Rev. 96 (1954) 191.

[10] V. I. Ogievetsky and I. V. Polubarinov, Ann. of Phys. 25 (1963) 358; R. M. Wald, Phys. Rev. D33 (1986) 3613.

[11] M. Fierz and W. Pauli, Proc. R. Soc. A173 (1939) 211.

[12] V. I. Ogievetsky and I. V. Polubarinov, Ann. of Phys. 35 (1965) 167.

[13] D. Boulware and S. Deser, Ann. of Phys. 89 (1975) 193.

[14] C. Fronsdal, Phys. Rev. D18 (1978) 3624; D20 (1979) 848; J. Fang and C. Fronsdal, Phys. Rev. D18 (1978) 3630; D22 (1980) 1361; B. de Wit and D. Z. Freedman, Phys. Rev. D21 (1980) 358.

[15] M. A. Vasiliev, Sov. J. Nucl. Phys. 32 (1980) 855 (p. 439 in English translation); C. Aragone and S. Deser, Nucl. Phys. B170 [FS1] (1980) 329.

[16] T. Curtright, Phys. Lett. B85 (1979) 219.

[17] A. K. Bengtsson, I. Bengtsson and L. Brink, Nucl. Phys. B227 (1983) 31, 41; F. A. Berends, G. J. Burgers and H. van Dam, Z. Phys. C24 (1984) 247; Nucl. Phys. B260 (1985) 295; B271 (1986) 429; A. K. H. Bengtsson and I. Bengtsson, Class. Quant. Grav. 3 (1986) 927; A. K. H. Bengtsson, Class. Quant. Grav. 5 (1988) 437.
[18] R. Metsaev, *Mod. Phys. Lett.* **A6** (1991) 359.

[19] C. Aragone and S. Deser, *Phys. Lett.* **B86** (1979) 161;
F. A. Berends, J. W. van Holten, P. van Niewenhuizen and B. de Wit, *J. Phys.* **A13** (1980) 1643;
B. de Wit and D. Z. Freedman, *Phys. Rev.* **D21** (1980) 358.

[20] R. Utiyama, *Phys. Rev.* **101** (1956) 1597;
T. W. B. Kibble, *J. Math. Phys.* **2** (1961) 212;
A.H. Chamseddine and P. West, Nucl. Phys. **B129** (1977) 39.

[21] S. W. MacDowell and F. Mansouri, *Phys. Rev. Lett.* **38** (1977) 739.

[22] E. Witten, Nucl. Phys. **B311** (1989) 46.

[23] K. Stelle and P. West, Phys. Rev. **D21** (1980) 1466.

[24] R. Jackiw, in: Quantum Theory of Gravity, ed. S. Christensen (Adam Hilger, Bristol 1984) p. 403;
C. Teitelboim, in: Quantum Theory of Gravity, ed. S. Christensen (Adam Hilger, Bristol 1984) p. 327;
A.H. Chamseddine and D. Wyler, Phys. Lett. **B228** (1989) 75; Nucl.Phys. **B340** (1990) 595.

[25] A. Achucarro and P.K. Townsend, Phys. Lett. **B180** (1986) 89.

[26] E. Bergshoeff, B. de Wit and M. A. Vasiliev, *Phys. Lett.* **256** (1991) 199; Nucl. Phys. **B366** (1991) 315.

[27] F.A. Berezin, Mat. Sbornik, **86** (1971) 578; see also F.A. Berezin “The method of Second Quantization”, Nauka, Moscow, 1986 and references therein;
F.A. Berezin and M.S. Marinov, Ann. of Phys. **104** (1977) 336;
F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D.Sternheimer, Ann. of Phys. **110** (1978) 61, 111.

[28] M.P. Blencowe, Class. Quantum Grav. **6** (1989) 443.

[29] M. A. Vasiliev, *Fortschr. Phys.* **35** (1987) 741.

[30] M.A. Vasiliev, Class. Quant. Grav. **11** (1994) 649.

[31] E.A. Ivanov, JETP Letters **30** (1979) 452 (in Russian).

[32] M. A. Vasiliev, *Ann. of Phys.* **190** (1989) 59.
[33] R. Penrose and W. Rindler, *Spinors and space-time* vol.1, Cambridge Univ. Press, Cambridge, 1984.

[34] M.A. Vasiliev, *Nucl. Phys.* B324 (1989) 503.

[35] M.A. Vasiliev, *Mod. Phys. Lett.* A7 (1992) 3689.

[36] A.K.H. Bengtsson and I. Bengtsson, *Phys. Lett.* B174 (1986).

[37] E. Bergshoeff, C.N. Pope, L.J. Romans et al, *Mod. Phys. Lett.* A5 (1990) 1957.

[38] M.A. Vasiliev, in preparation

[39] E.S. Fradkin and V.Ya. Linetsky, Mod. Phys. Lett. A4 (1989) 2635.

[40] V.E. Lopatin and M.A. Vasiliev, *Mod. Phys. Lett.* A3 (1988) 257;
    M.A. Vasiliev, *Nucl. Phys.* B301 (1988) 26.

[41] E.S. Fradkin and R.R. Metsaev, Class. Quantum Grav. 8 (1991) L89;
    R.R. Metsaev, *Phys. Lett.* 309 (1993) 39, Mod. Phys. Lett. A8 (1993) 2413.