Duals of Ann-categories

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Abstract

Dual monoidal category $\mathcal{C}^\ast$ of a monoidal functor $F : \mathcal{C} \to \mathcal{V}$ has been constructed by S. Majid. In this paper, we extend the construction of dual structures for an Ann-functor $F : \mathcal{B} \to \mathcal{A}$. In particular, when $F = id_\mathcal{A}$, then the dual category $\mathcal{A}^\ast$ is indeed the center of $\mathcal{A}$ and this is a braided Ann-category.

Mathematics Subject Classification: 18D10, 16D20

Keywords: duals of Ann-categories, braided Ann-category, functored, bi-modules

1 Introduction

Categories with quasi-symmetry appeared under the heading “braided monoidal categories” in a connection with low dimensional topology [5], as well as in the context of quantum groups [6].

The concept “dual of monoidal category” appeared in [9] in the following case. The Hopf algebra can be built via a monoidal category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \text{Vec}$. This event can be generalized as $\text{Vec}$ is replaced by a monoidal category $\mathcal{V}$. Now, if $F$ is a monoidal functor, then $\mathcal{C}$ is called functored on $\mathcal{V}$, or $(\mathcal{C}, F)$ is called a $\mathcal{V}$-category in A. Grothendieck’s terminology [4].

The full subcategory $(\mathcal{C}, F)^\circ$ consists of objects $(V, u_V)$ where $u_{V,X}$ are isomorphisms.

When $\mathcal{V} = \mathcal{C}$ and $F = id$, then $(\mathcal{C}, F)^\circ$ is a braided monoidal category, called the center $Z(\mathcal{C})$ of the monoidal category $\mathcal{C}$. The notion of the center of a monoidal category appeared first in [5], [9]. It was a construction of a braided tensor category from an arbitrary tensor.
category. Then, the center of a category appears as a tool to study categorical groups \[1\] and graded categorical groups \[3\].

The detail proofs of the construction of \((C, F)^*\) have showed in \[10\]. Concurrently, in \[10\], S. Majid enriched the results of the dual categories and established links between dual categories and braided groups.

Monoidal categories were considered in a more general situation due to M. Laplaza with the name *distributivity category* \[7\]. After, A. Fröhlich and C. T. C. Wall \[2\] presented the concept of *ring-like category*. These two concepts are categorifications of the concept of commutative rings, as well as a generalization of the category of modules over a commutative ring \(R\). The overlap of these two concepts has been proved in \[14\].

In order to have descriptions of structures, and a cohomological classification, N. T. Quang \[11\] has introduced the concept of *Ann-categories*, as a categorification of the concept of rings, with requirements of invertibility of objects and morphisms of the underlying category, similar to those of categorical groups (see \[1, 3\]). In \[13\], N. T. Quang proved that each congruence class of an Ann-category \(\mathcal{A}\) is completely defined by three invariants: the ring \(\Pi_0(\mathcal{A})\) of congruence classes of objects of \(\mathcal{A}\), the \(\Pi_0(\mathcal{A})\)-bimodule \(\Pi_1(\mathcal{A})\) of automorphisms of zero object, and an element in the cohomology group \(H^3_{MacL}(R, M)\) due to Mac Lane \[8\]. The concept of *braided Ann-categories* is considered in \[14\], in which authors built the *center* of an Ann-category, an extension of the center construction of a monoidal category presented by A. Joyal and R. Street \[5\]. This motivation leads to the purpose of this paper is to construct a dual Ann-category of an arbitrary Ann-category (in Section 3). This gives us a new framework of the concept of Ann-categories, which is very close to the ring extension problem. We also note that the center of an Ann-category is a dual over \(\mathcal{A}\). Thus, in the duals over \(\mathcal{A}\) there always exist braided Ann-categories.

In this paper, we sometimes denote by \(XY\) the tensor product of two objects \(X, Y\) instead of \(X \otimes Y\).

## 2 Some basic definitions

**Definition 2.1** (\[11\]). An Ann-category consists of:

(i) Category \(\mathcal{A}\) together with two bifunctors \(\oplus, \otimes: \mathcal{A} \times \mathcal{A} \to \mathcal{A}\).

(ii) A fixed object \(O \in \mathcal{A}\) together with naturality constraints \(a^+, c^+, g, d\) such that \((\mathcal{A}, \oplus, a^+, c^+, (O, g, d))\) is a symmetric categorical group.

(iii) A fixed object \(I \in \mathcal{A}\) together with naturality constraints \(a, l, r\) such that \((\mathcal{A}, \otimes, a, (I, l, r))\) is a monoidal \(\mathcal{A}\)-category.

(iv) Natural isomorphisms \(\mathcal{L}, \mathcal{R}\)

\[
\mathcal{L}_{A,X,Y} : A \otimes (X \oplus Y) \rightarrow (A \otimes X) \oplus (A \otimes Y),
\]

\[
\mathcal{R}_{X,Y,A} : (X \oplus Y) \otimes A \rightarrow (X \otimes A) \oplus (Y \otimes A),
\]
such that the following conditions are satisfied:

(Ann-1) For each $A \in \mathcal{A}$, the pairs $(L^A, \tilde{L}^A), (R^A, \tilde{R}^A)$ defined by relations:

$$L^A = A \otimes -, \quad R^A = - \otimes A, \quad \tilde{L}^A_{X,Y} = \mathcal{L}_{A,X,Y}, \quad \tilde{R}^A_{X,Y} = \mathcal{R}_{X,Y,A}$$

are $\oplus$-functors which are compatible with $a^+$ and $c^+$.

(Ann-2) The following diagrams commute for all objects $A, B, X, Y \in \mathcal{A}$:

$$
\begin{array}{c}
\xymatrix{(AB)X \oplus (AB)Y \ar[r]^{a_{A,B,X} \oplus a_{A,B,Y}} \ar[d]_{L^{AB}} & A(BX \oplus BY) \ar[d]^{L^A} \\
X(BA) \oplus Y(BA) \ar[r]^{a_{X,B,A} \oplus a_{Y,B,A}} & (XB)A \oplus (YB)A \ar[r]^{\tilde{R}^A} & (X \oplus Y)(BA) \ar[r]^{\tilde{R}^{BA} \otimes id_A} & (X \oplus Y)(BA) \ar[u]_{\tilde{R}^A}
\end{array}
$$

$$
\begin{array}{c}
\xymatrix{(A(X \oplus Y))B \ar[r]^{a_{A,X,Y} \oplus b} \ar[d]_{L^{A \otimes id_B}} & A((X \oplus Y)B) \ar[d]^{L^A} \\
(A \oplus (AY))B \ar[r]^{a \otimes \tilde{R}^A} & A(AY) \oplus A(YB) \ar[r]^{id_A \otimes \tilde{R}^B} & A(AY) \oplus A(YB) \ar[u]_{\tilde{R}^A}
\end{array}
$$

$$
\begin{array}{c}
\xymatrix{(A \oplus B)(X \oplus (A \oplus B))Y \ar[r]^{L} \ar[d]_{R \otimes \tilde{R}^Y} & (A \oplus B)(X \oplus Y) \ar[d]^{L^A \otimes L^B} \\
(AX \oplus (BY \oplus BY)) \ar[r]^{u} & (AX \oplus (AY) \oplus BX \oplus BY) \ar[u]_{L^A \oplus L^B}
\end{array}
$$

where $v = v_{U,V,Z,T} : (U \oplus V) \oplus (Z \oplus T) \rightarrow (U \oplus Z) \oplus (V \oplus T)$ is the unique morphism built from $a^+, c^+, id$ in the symmetric monoidal category $(\mathcal{A}, \oplus)$.

(Ann-3) For the unit object $I \in \mathcal{A}$ of the operation $\otimes$, we have the following relations for all objects $X, Y \in \mathcal{A}$:

$$l_{X \otimes Y} = (l_X \otimes l_Y) \circ L^A_{X,Y}, \quad r_{X \otimes Y} = (r_X \otimes r_Y) \circ \tilde{R}^A_{X,Y}.$$
respect to the operation $\otimes$, satisfying the two following commutative diagrams for all $X, Y, Z \in \text{Ob}(\mathcal{A})$:

\[
\begin{align*}
F(X(Y \oplus Z)) & \xrightarrow{\tilde{F}} FX.F(Y \oplus Z) \xrightarrow{\text{id} \otimes \tilde{F}} FX(FY \oplus FZ) \\
F(XY \oplus XZ) & \xrightarrow{\tilde{F}} F(XY) \oplus F(XZ) \xrightarrow{\tilde{F} \otimes \text{id}} FX.FY \oplus FX.FZ \\
F((X \oplus Y)Z) & \xrightarrow{\tilde{F}} F(X \oplus Y).FZ \xrightarrow{\tilde{F} \otimes \text{id}} (FX \oplus FY).FZ \\
F(XZ \oplus YZ) & \xrightarrow{\tilde{F}} F(XZ) \oplus F(YZ) \xrightarrow{\tilde{F} \otimes \text{id}} FX.FZ \oplus FY.FZ
\end{align*}
\]

**Definition 2.3.** A braided Ann-category $\mathcal{A}$ is an Ann-category $\mathcal{A}$ together with a braid $c$ such that $(\mathcal{A}, \otimes, a, c, (l, r))$ is a braided tensor category, concurrently $c$ satisfies the following relation:

\[
(c_{A,X} \oplus c_{A,Y}) \circ \tilde{L}^A_{X,Y} = \tilde{R}^A_{X,Y} \circ c_{A,X \oplus Y},
\]

and the condition $c_{O,O} = \text{id}$.

Let us recall a result which has been known of an Ann-category.

**Proposition 2.4 ([11] Proposition 3.1).** In the Ann-category $\mathcal{A}$, there exist uniquely the isomorphisms:

\[
\begin{align*}
\hat{L}^A : A \otimes O & \rightarrow A, \\
\hat{R}^A : O \otimes A & \rightarrow A
\end{align*}
\]

such that $(L^A, \hat{L}^A, \tilde{L}^A), (R^A, \hat{R}^A, \tilde{R}^A)$ are the functors which are compatible with the unit constraints of the operator $\oplus$ (also called $U$-functors).

## 3 Duals of Ann-categories

In this section, we shall build *duals of Ann-categories* based on the construction of duals of monoidal categories by S. Majid [9].

Let $\mathcal{A}$ be an Ann-category. An Ann-category $\mathcal{B}$ is *functored* over $\mathcal{A}$ if there is an Ann-functor $F : \mathcal{B} \rightarrow \mathcal{A}$.

First, let us recall that an Ann-category is called *almost strict* if all its natural constraints, except for the commutativity constraint and the left distributivity constraint, are identities. Each Ann-category is Ann-equivalent to an almost strict Ann-category of the type $(R, M)$ (see [12]). In this category, for each $A \in \text{Ob}(\mathcal{A})$, there exists an object $A' \in \text{Ob}(\mathcal{A})$ such that

\[
A \oplus A' = O. \tag{1}
\]
So, hereafter, we always assume that \( A \) is an almost strict Ann-category and satisfies the condition \( [1] \) and the Ann-functor \( F: \mathcal{B} \to \mathcal{A} \) satisfies the conditions \( F(O) = O, F(I) = I \).

**Definition 3.1.** Let \( \mathcal{A} \) be an Ann-category. Let \( (\mathcal{B}, F) \) be a functored Ann-category over \( \mathcal{A} \). A right \( (\mathcal{B}, F) \)-module is a pair \( (A, u_A) \) consisting of an object \( A \) in \( \mathcal{A} \) and a natural transformation \( u_A: A \otimes F(X) \to F(X) \otimes A \) such that \( u_A, I = id \) and the following diagrams commute:

\[
\begin{align*}
A \otimes (FX \oplus FY) & \xrightarrow{L^\mathcal{F}_{FX, FY}} (A \otimes FX) \oplus (A \otimes FY) \xrightarrow{u_A, X \oplus u_A, Y} (FX \otimes A) \oplus (FY \otimes A) \\
& \xrightarrow{id \otimes F} (A \otimes FX) \oplus (A \otimes FY) \xrightarrow{id} (A \otimes FX) \oplus (A \otimes FY) \\
A \otimes F(X \oplus Y) & \xrightarrow{u_A, X \oplus Y} F(X \oplus Y) \otimes A \xrightarrow{\hat{F} \otimes id} (F \otimes FY) \otimes A \\
& \xrightarrow{id \otimes \hat{F}} (A \otimes FX) \oplus (A \otimes FY) \xrightarrow{id} (A \otimes FX) \oplus (A \otimes FY) \\
A \otimes (FX \otimes FY) & \xrightarrow{u_A, X \otimes id} FX \otimes A \otimes FY \xrightarrow{id \otimes u_A, Y} FX \otimes FY \otimes A \\
& \xrightarrow{id \otimes \hat{F}} (A \otimes FX) \oplus (A \otimes FY) \xrightarrow{id} (A \otimes FX) \oplus (A \otimes FY) \\
A \otimes F(X \otimes Y) & \xrightarrow{u_A, X \otimes Y} F(X \otimes Y) \otimes A \xrightarrow{\hat{F} \otimes id} (F \otimes FY) \otimes A \\
& \xrightarrow{id \otimes \hat{F}} (A \otimes FX) \oplus (A \otimes FY) \xrightarrow{id} (A \otimes FX) \oplus (A \otimes FY)
\end{align*}
\]

A morphism \( f: (A, u_A) \to (B, u_B) \) between right \( (\mathcal{B}, F) \)-modules is a morphism \( f: A \to B \) in \( \mathcal{A} \) such that the following diagram commutes for all \( X \in \mathcal{B} \):

\[
\begin{align*}
A \otimes FX & \xrightarrow{u_A, X} FX \otimes A \\
& \xrightarrow{f \otimes id} B \otimes FX \xrightarrow{u_B, X} FX \otimes B \\
& \xrightarrow{id \otimes f} B \otimes FX \xrightarrow{u_B, X} FX \otimes B
\end{align*}
\]

Let \( (\mathcal{B}, F) \) be a functored Ann-category over \( \mathcal{A} \). We consider the category \( \mathcal{B}^* = (\mathcal{B}, F)^* \) defined as follows. The objects of \( \mathcal{B}^* \) are right \( (\mathcal{B}, F) \)-modules. The morphisms of \( \mathcal{B}^* \) are morphisms between right \( (\mathcal{B}, F) \)-modules.

Now, we shall equip the operators and the structures for \( \mathcal{B}^* \) so that \( \mathcal{B}^* \) becomes an Ann-category.

**Lemma 3.2.** For any two objects \( (A, u_A), (B, u_B) \) in \( \mathcal{B}^* \), \( (A \oplus B, u_{A \oplus B}) \) is an object of \( \mathcal{B}^* \), where \( u_{A \oplus B} \) is defined by:

\[
u_{A \oplus B, X} = \mathcal{L}^{-1}_{FX, A, B} \circ (u_{A, X} \oplus u_{B, X}), \text{ for all } X \in \mathcal{A}.
\]

**Proof.** Since \( u_{A, I} = id, u_{B, I} = id, \mathcal{L}_{F1, A, B} = \mathcal{L}_{1, A, B} = id \), we have \( u_{A \oplus B, I} = id \).

To prove that \( u_{A \oplus B} \) satisfies the diagram (2), we consider the diagram (5) (see page 12). In the diagram (5), the regions (I), (VII) commute thanks to the
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determination of $u_{A \oplus B}$, the region (II) commutes thanks to the naturality of $R = id$, the regions (III), (VI) commute since $A$ is an Ann-category, the region (V) commutes thanks to the naturality of $\mathbb{L}$, the region (VIII) commutes thanks to the naturality of $v$, the perimeter commutes since $(A, u_A), (B, u_B)$ satisfy the diagram (2). Therefore, the region (IV) commutes, i.e., $(A \oplus B, u_{A \oplus B})$ satisfies the diagram (2). 

To prove that $u_{A \oplus B}$ satisfies the diagram (3), we consider the diagram (6) (see page 13). In the diagram (6), the regions (I), (II) commute thanks to the naturality of $R = id$, the regions (III), (VI), (VIII) commute thanks to the determination of $u_{A \oplus B}$, the regions (IV), (X) commute since $A$ is an Ann-category, the regions (VII), (IX) commute thanks to the naturality of $L$, the perimeter commutes thanks to $u_A, u_B$ satisfy the diagram (3). Therefore, the region (V) commutes, i.e., $u_{A \oplus B}$ satisfies the diagram (3). So, $(A \oplus B, u_{A \oplus B})$ is an object of $B^*$. □

By Lemma 3.2, we can determine the operator “$+$” of $B^*$ where the sum of two objects is defined by

$$(A, u_A) + (B, u_B) = (A \oplus B, u_{A \oplus B}),$$

and the sum of two morphisms is the sum of morphisms in $A$.

**Proposition 3.3.** $B^*$ is a symmetric categorical group where the associativity constraint is strict, the unit constraint is $((O, u_{O,X} = \hat{L}^{-1}_F X), id, id)$, and the commutativity constraint is $c^+_{(A, u_A), (B, u_B)} = c^+_{A, B}$.  

**Proof.** Assume that $f : (A, u_A) \to (B, u_B)$ and $g : (C, u_C) \to (D, u_D)$ are two morphisms in the category $B^*$. We shall prove that

$$f + g = f \oplus g$$

satisfies the diagram (11), so it is a morphism of $B^*$. We consider the diagram:

\[
\begin{array}{cccc}
A FX \oplus C FX & u_{A,X} \oplus u_{C,X} & (FX) A \oplus (FX) C
\\
(A \oplus C) FX & u_{A\oplus C,X} & (FX) (A \oplus C)
\\
(f \otimes id) \oplus (g \otimes id) & (f \otimes g) \otimes id & id \otimes (f \otimes g) & (id \otimes f) \oplus (id \otimes g)
\\
(B \oplus D) FX & u_{B\oplus D,X} & (FX) (B \oplus D)
\\
B FX \oplus D FX & u_{B,X} \oplus u_{D,X} & (FX) B \oplus (FX) D
\\
\end{array}
\]
In this diagram, the region (I) commutes thanks to the naturality of \( R = id \), the region (II) commutes thanks to the determination of \( u_{A \oplus B} \), the region (IV) commutes thanks to the determination of \( u_{B \oplus D} \), the region (V) commutes thanks to the naturality of \( L \); each component of the perimeter commutes since \( f \) and \( g \) are morphisms of \( B^* \). So, the perimeter commutes. Therefore, the region (III) commutes, i.e., \( f + g = f \oplus g \) is a morphism of \( B^* \).

Next, we prove that \( a^+ = id \) is a morphism
\[
((A, u_A) + (B, u_B)) + (C, u_C) \to (A, u_A) + ((B, u_B) + (C, u_C))
\]
in \( B^* \). We consider the following diagram:

In the above diagram, the region (I) commutes thanks to the determination of \( u_{A \oplus B} \), the region (II) commutes thanks to the determination of \( u_{B \oplus C} \), the region (III) commutes thanks to the determination of \( u_{(A \oplus B) \oplus C} \), the region (IV) commutes thanks to the determination of \( u_{A \oplus (B \oplus C)} \); the region (VI) commutes since \( A \) is an Ann-category, the perimeter commutes thanks to the naturality of \( a^+ = id \). Therefore, the region (V) commutes, i.e., \( a^+ = id \) is a morphism of \( B^* \).

To prove that \( c^+ \) is the morphism
\[
(A, u_A) + (B, u_B) \to (B, u_B) + (A, u_A)
\]
in \( B^* \), we consider the following diagram. In this diagram, the region (I) commutes thanks to the determination of \( u_{A \oplus B} \), the regions (II), (IV) commute
since $\mathcal{A}$ is an Ann-category, the region (V) commutes thanks to the determination of $u_{B\oplus A}$; the perimeter commutes thanks to the naturality of $c^+$. Therefore, the region (III) commutes, i.e., $c^+$ is a morphism in $\mathcal{B}^*$.

One can verify that $((O, u_{O,X} = \hat{L}^{-1}_{FX}), id, id)$ is the unit constraint of $\mathcal{B}^*$. Finally, we shall prove that each object of $\mathcal{B}^*$ is invertible.

Let $(A, u_A)$ be an object of $\mathcal{B}^*$. By the condition (I), there exists an object $A' \in \text{Ob}(\mathcal{A})$ such that

$$A \oplus A' = O.$$ 

The family of natural transformations $u_{A',X} : A' \otimes FX \rightarrow FX \otimes A'$ is defined by:

$$u_{A,X} \oplus u_{A',X} = \mathcal{L}_{FX,A,A'} \circ u_{O,X}.$$ 

One can prove that $(A', u_{A'})$ is the invertible object of the object $(A, u_A)$ in the category $\mathcal{B}^*$.

**Lemma 3.4.** For any two objects $(A, u_A), (B, u_B)$ of $\mathcal{B}^*$, $(A \otimes B, u_{A\otimes B})$ is an object of $\mathcal{B}^*$, where $u_{A\otimes B}$ is defined by:

$$u_{A\otimes B,X} = (u_{A,X} \otimes id_B) \circ (id_A \otimes u_{B,X}), \text{ for all } X \in \mathcal{A}.$$ 

**Proof.** Let $(A, u_A), (B, u_B)$ be two objects of $\mathcal{B}^*$. Since $u_{A,I} = id$ and $u_{B,I} = id$, we have $u_{A\otimes B,I} = id$. Moreover, by Theorem 3.3 [9], $u_{A\otimes B}$ satisfies the diagram (3).

Finally, to prove that $u_{A\otimes B}$ satisfies the diagram (1), we consider the diagram (7) (see page 14). In the diagram (7), the region (I) commutes since $(B, u_B)$ satisfies the diagram (2), the regions (II), (VII) and (IX) commute thanks to the naturality of $a^+ = id$, the region (III) commutes thanks to the naturality of $\mathcal{L}$, the regions (IV), (XI) and the perimeter commutes since $\mathcal{A}$ is an Ann-category, the regions (VI), (VIII) commute thanks to the determination of $u_{AB}$, the region (X) commutes since $(A, u_A)$ satisfies the diagram...
By Lemma 3.4, we can determine the operator \( \times \) of \( B^\ast \) where the product of two objects is defined by

\[
(A, u_A) \times (B, u_B) = (A \otimes B, u_{A \otimes B}),
\]

and the tensor product of two morphisms is the tensor product of two morphisms in \( A \).

**Proposition 3.5.** \( B^\ast \) is a strict monoidal category.

**Proof.** Assume that \( f : (A, u_A) \to (B, u_B) \) and \( g : (C, u_C) \to (D, u_D) \) are two morphisms in the category \( B^\ast \). By Theorem 3.3 [9], the morphism

\[
f \times g = f \otimes g : (A, u_A) \times (C, u_C) \to (B, u_B) \times (D, u_D)
\]

satisfies the diagram (4), i.e., \( f \times g \) is a morphism in \( B^\ast \).

The composition of two morphisms in \( B^\ast \) is the normal composition. By Theorem 3.3 [9], \( B^\ast \) has the associativity constraint be strict. One can easily prove that \((I, id)\) is an object in \( B^\ast \) and it together with the strict constraints \( l = id, r = id \) is the unit constraint of the operator \( \times \) in \( B^\ast \).

**Theorem 3.6.** \( B^\ast \) is an Ann-category with the distributivity constraints are given by

\[
\mathcal{L}_{(A, u_A), (B, u_B), (C, u_C)} = \mathcal{L}_{A, B, C}, \quad \mathcal{R}_{(A, u_A), (B, u_B), (C, u_C)} = id.
\]

**Proof.** By Proposition 3.3 \((B^\ast, +)\) is a symmetric categorical group. By Proposition 3.5 \((B^\ast, \times)\) is a monoidal category. One can prove that

\[
\mathcal{L} : (A, u_A) \times ((B, u_B) + (C, u_C)) \to (A, u_A) \times (B, u_B) + (A, u_A) \times (C, u_C),
\]

\[
\mathcal{R} = id : ((A, u_A + (B, u_B)) \times (C, u_C) \to (A, u_A) \times (C, u_C) + (B, u_B) \times (C, u_C)
\]

are morphisms in \( B^\ast \).

Moreover, the constraints \( a^+ = id, c^+, a = id, \mathcal{L}, \mathcal{R} = id \) of the Ann-category \( A \) satisfy the conditions (Ann-1), (Ann-2), (Ann-3), so, in the category \( B^\ast \), they also satisfy these conditions. Thus \( B^\ast \) is an Ann-category. \( \square \)
The following proposition is obvious.

**Proposition 3.7.** $B^*$ is funtored over $A$ with the forgetful Ann-functor $F^* : B^* \to A$.

**Example 1. The center of an Ann-category $A$**

Let $A$ be an Ann-category. Let $B = A$ and $F = id$. Then $B^* = C_A$, where $C_A$ is the center of the Ann-category $A$ which is built in [14]. This is a braided Ann-category with the quasi-symmetric $c(A,u_A,(B,u_B)) = u_{A,B} : A \otimes B \to B \otimes A$.

Next, we shall apply above results to build the dual Ann-category of the pair $(B,F)$, where $B = (R',M',f')$, $A = (R,M,f)$ are Ann-categories.

**Example 2. Duals of an Ann-category of the type $(R,M)$**

Let $R$ be a ring and $M$ be a $R$-bimodule. An Ann-category of the type $(R,M)$ is a category $I$ whose objects are elements of $R$, and whose morphisms are automorphisms, $(x,a) : x \to x, \forall a \in M$. The composition of morphisms is the addition in $M$. The two operators $\oplus$ and $\otimes$ of $I$ are given by

$$x \oplus y = x + y, \quad (x,a) \oplus (y,b) = (x + y, a + b),$$

$$x \otimes y = xy, \quad (x,a) \otimes (y,b) = (xy, xb + ay).$$

All constraints of $I$ are strict, except for the left distributivity constraint and the commutativity constraint given by

$$c_{x,y,z} = (\bullet, \lambda(x,y,z)) : x(y + z) \to xy + xz,$$

$$c_{x,y}^+ = (\bullet, \eta(x,y)) : x + y \to y + x,$$

where $\lambda : R^3 \to M, \eta : R^2 \to M$ are functions satisfying the some certain coherence conditions (for detail, see [12], [13]).

Let $A$ be an almost strict Ann-category of the type $(R,M)$ and $B$ be an almost strict Ann-category of the type $(R',M')$. Let $(F,\tilde{F},\check{F}) : B \to A$ be an Ann-functor. Then, by Theorem 4.3 [15], $F$ is a functor of the type $(p,q)$, i.e.,

$$F(x) = p(x), F(x,a) = (p(x),q(a)),$$

where $p : R' \to R$ is a ring homomorphism and $q : M' \to M$ is a group homomorphism and

$$q(xa) = p(x)q(a), \quad q(ax) = q(a)p(x), \text{ for all } x \in R, a \in M.$$

Moreover, $\tilde{F}, \check{F}$ are associated, respectively, to $\mu, \nu$ which satisfy some certain coherence conditions (for detail, see Theorem 4.4 [15]).
According to the above steps, each object of $B^*$ is a pair $(r, u_r)$, where $r$ is in the centerization of $\text{Imp} = p(R')$ in the ring $R$, (i.e., $rp(x) = p(x)r \forall x \in R'$) and $u_r : R' \to M$ is a function satisfying the condition $u_{r,1} = 0$ and the two following conditions for all $x, y \in R'$:

\[
\begin{align*}
    u(r, x) - u(r, x + y) + u(r, y) &= \mu(x, y)r + r\mu(x, y) - \lambda(r, px, py), \\
    xu(r, y) - u(r, xy) + u(r, x)y &= r\nu(x, y) - \nu(x, y)r.
\end{align*}
\]

We now describe a morphism $f : (r, u_r) \to (s, u_s)$ of $B^*$. Since $f : r \to s$ is a morphism in the Ann-category $A$, $s = r$, and $f = (r, a)$ with $a \in M$.

From the commutation of the diagram (4), we have

\[ p(x)a = ap(x), \text{ for all } x \in R'. \]

Now, $B^*$ is an Ann-category with the two operators given by

\[
\begin{align*}
    (r, u_r) + (s, u_s) &= (r + s, u_{r+s}), \\
    (r, u_r) \times (s, u_s) &= (rs, u_{rs}),
\end{align*}
\]

where

\[
\begin{align*}
    u_{r+s,x} &= u_{r,x} + u_{s,x} - \lambda(px, r, s), \\
    u_{rs,x} &= u_{r,x}s + r, u_{s,x},
\end{align*}
\]

and $f + g = f \oplus g$, $f \times g = f \otimes g$ where $f : (r, u_r) \to (r, u_r)$, $g : (s, u_s) \to (s, u_s)$.

All constraints of $B^*$ are strict, except for the commutativity constraint and the left distributivity constraint given by

\[
\begin{align*}
    c^+_{(r, u_r)(s, u_s)} &= c^+_{r,s} = (\bullet, \eta(r, s)), \\
    \mathfrak{L}_{(r, u_r)(s, u_s), (t, u_t)} &= \mathfrak{L}_{r,s,t} = (\bullet, \lambda(r, s, t)).
\end{align*}
\]

The invertible object of the object $(r, u_r)$ respect to the operator $+$ is $(-r, u_{-r})$, where $-r$ is the opposite element of $r$ in the group $(R, +)$ and $u_{-r} : R' \to M$ is given by

\[ u_{-r,x} = \lambda(px, r, -r) - u_{r,x}. \]
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\[
\begin{align*}
& AF(X \oplus Y) \oplus BF(X \oplus Y) \\
& (id \otimes \tilde{F}) \oplus (id \otimes \tilde{F}) \\
& A(FX \oplus FY) \oplus B(FX \oplus FY) \\
& (AFX \oplus AFY) \oplus (BFX \oplus BFY)
\end{align*}
\]

Diagram (5)

\[
\begin{align*}
& (I) \\
& (A \oplus B)F(X \oplus Y) \\
& (F(X \oplus Y))(A \oplus B) \\
& (F(X \oplus Y))A \oplus (F(X \oplus Y))B
\end{align*}
\]

where

\[
\begin{align*}
t_1 &= u_{A,X \oplus Y} \oplus u_{B,X \oplus Y} \\
t_2 &= (u_{A,X} \oplus u_{B,X}) \oplus (u_{A,Y} \oplus u_{B,Y}) \\
t_3 &= (u_{A,X} \oplus u_{A,Y}) \oplus (u_{B,X} \oplus u_{B,Y})
\end{align*}
\]
\[ (AF X)F Y \oplus (BF X)F Y \xrightarrow{(u_{A,X} \otimes id) \oplus (u_{B,X} \otimes id)} ((F X)A)F Y \oplus ((F X)B)F Y \xrightarrow{id \otimes u_{A,Y} \oplus (id \otimes u_{B,Y})} (F X)((F Y)A) \oplus (F X)((F Y)B) \]

\[ (A \oplus B)(F X)F Y \xrightarrow{id \otimes u_{A\oplus B,X}} ((A \oplus B)(F X)F Y \xrightarrow{u_{A\oplus B,X} \otimes id} (F X)(A \oplus B)F Y \xrightarrow{id \otimes u_{A\oplus B,Y} \oplus \tilde{L}_{A,B}} (F X)((A \oplus B)F Y) \]

where \( t_4 = (id_A \otimes \tilde{F}_{X,Y}) \oplus (id_A \otimes \tilde{F}_{X,Y}) \)

\[ t_5 = (\tilde{F}_{X,Y} \otimes id_A) \oplus (\tilde{F}_{X,Y} \otimes id_B) \]
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