Passive self-propulsion and enhanced annihilation of $p$–atic defects

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We investigate the effects of hydrodynamics on the motion of topological defects in $p$–atic liquid crystals: i.e. two-dimensional liquid crystals characterised by $p$–fold rotational symmetry. Using numerical simulations and analytical work, we demonstrate the existence of a generic passive self-propulsion mechanism for defects of winding number $s = (p - 1)/p$ and arbitrary $p$ values. Remarkably, whereas this mechanism is not unique to nematics (i.e. $p = 2$), nematics are the only type of liquid crystals where passive self-propulsion is thermodynamically stable. Furthermore, we find that hydrodynamics can accelerate the annihilation dynamics of pairs of $±1/p$ defects and that this effect increases with $p$ before approaching a plateau.

The physics of topological defects in liquid crystals have experienced, in the last decade, a tremendous revival, thanks to a wealth of exciting discoveries at the interface between soft condensed matter and biological physics [1–9]. The most common class of liquid crystals defects, known as disclinations, consists of point or line singularities around which the average orientation of the anisotropic building blocks undergoes one or more complete revolutions, thereby disrupting the local orientational order [10–15]. In $p$–atic liquid crystals – i.e. two-dimensional liquid crystals with $p$–fold rotational symmetry, among which nematics (i.e. $p = 2$) and hexatics (i.e. $p = 6$) are the best known examples – defects can be classified in terms of the winding number or strength $s$, defined as the number of revolutions of the orientation field along an arbitrary loop enclosing the core of the defect: i.e. $s = ±1/p$, $±2/p$ etc.

Whereas the equilibrium physics of liquid crystal defects represents a mature topic across several areas of physics – from cosmology [16], down to condensed matter [11, 17] and particle physics [18–20] – our understanding of their dynamics is still in a phase of rapid expansion, especially in the realm of biological matter, where defects have been suggested to accomplish various vital functions. These include driving the extrusion of apoptotic cells in epithelial layers [21, 22], coordinating large scale cellular flows during wound healing and morphogenetic events [23, 24], and seeding the development of non-planar features, such as tentacles and protrusion in simple organisms, such as Hydra [25–28].

Whilst the biochemical and chemotactic apport to these processes is mostly understood, much less is known about the physical interactions. Nevertheless, their origin can be single-handedly ascribed to the existence of a hydrodynamic phenomenon known as backflow. This effect – whose name directly refers to the “bounce” in the optical transmission of a twisted nematic cell between polarizers after switching off the applied field [29–34] – consists of the hydrodynamic flow resulting from spatial variations of the average microscopic orientation. In passive liquid crystals, departure from the uniformly oriented equilibrium configuration is generally transient and often originates from a sudden change in the environmental conditions, such as the abrupt variation of an external electric or magnetic field in optical devices [35, 36]. Conversely, in active systems, distortions occur spontaneously as a consequence of the internal stresses collectively exerted by the active subunits [1, 4, 37].

Whether passive or active, topological defects play a primary role with respect to backflow phenomena, by virtue of the extended and yet persistence distortion resulting from their appearance. In passive nematic liquid crystals, for instance, this effect is known to affect the annihilation dynamics of neutral pairs of elementary disclinations [38], while in active nematics – such as in vitro mixture of cytoskeletal filaments and motor proteins [1, 2, 39, 40] or certain kind of prokaryotic [41, 42] or eukaryotic [21, 43] cells – backflow drives the propulsion of $s = 1/2$ defects and influences the hydrodynamic stability of active layers with respect to non-planar deformations [2, 44–46]. Importantly, a hydrodynamic theory capable of capturing backflows effects in liquid crystals with generic $p$–atic symmetry has been developed very recently [47, 48]. Hence, the role of hydrodynamics in dynamics of defects in liquid crystalline systems with generic $p$–fold symmetry is mostly unknown.

In this article we take a step in bridging this gap. Leveraging on recent progress toward generalizing the classic hydrodynamic theory of hexatic liquid crystals [49, 50] to account for arbitrary $p$–fold symmetry, we construct the velocity field of arbitrary $p$–atic disclinations and demonstrate the existence of a generic passive self-propulsion mechanism for defects of winding number $s = (p - 1)/p$ and arbitrary $p$ values. Remarkably, whereas this mechanism is not unique to nematics, nematics are the only type of liquid crystals where passive self-propulsion is thermodynamically stable. Furthermore, we analyze the effect of hydrodynamics in the annihilation of pairs of elementary $±1/p$ disclinations and find that backflow always accelerates the annihilation dynamics and, contrary to expectations, becomes increasingly more relevant as $p$ increases.

Let us consider an incompressible $p$–atic liquid crystal, whose microscopic orientation is characterized by
the unit vector \( \mathbf{v} = \cos \vartheta \mathbf{e}_x + \sin \vartheta \mathbf{e}_y \) and whose physical properties are invariant under rotations by \( 2\pi/p \). For \( p = 2 \), this is the direction of the rod-like building blocks comprising a nematic liquid crystals, whereas for \( p = 3 \), \( \mathbf{v} \) is either one of the three equivalent directions of 3–legged star, etc (see e.g. Fig. 1). At length scales larger than that of the individual constituents and yet infinitesimal compared to the system size, \( p \)-atic order can be conveniently described in terms of the tensor order parameter \( Q_p = Q_{i_1 i_2 \ldots i_p} e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_p} \), where \( i_n = \{ x, y \} \) and \( n = 1, 2 \ldots p \), constructed upon averaging the \( p \)-fold tensorial power of the local orientation \( \mathbf{v} \). That is: \( Q_p = \sqrt{2^{p-2} [\mathbf{v} \otimes \mathbf{v}]} = |\Psi_p| [n \otimes p] \), where \( n = \cos \vartheta \mathbf{e}_x + \sin \vartheta \mathbf{e}_y \) and \( |\Psi_p| = |\Psi_p|(r) \) and \( \vartheta = \vartheta(r) \) are respectively the magnitude and the phase of the coarse-grained complex order parameter \( |\epsilon| = \sqrt{\rho n \mathbf{v}} \), while the operator \([\ldots]\) renders its argument traceless (i.e. \( Q_{j j i_1 \ldots i_p} = 0 \)) and symmetric with respect to the exchange of any two indices \([47, 48]\). For nematics (i.e. \( p = 2 \)), this construction gives the standard rank–2 tensor \( Q_2 = |\Psi_2|(n \otimes n - 1/2) \), with \( \mathbf{1} \) the identity tensor and \( |\Psi_2| = 2|\mathbf{v} \cdot n|^2 - 1 \) the scalar order parameter. The dynamics of the fluid is then governed by the following set hydrodynamic equations for momentum density \( \rho \mathbf{v} \) and the tensor order parameter \( Q_p \):\
\[
\rho \frac{D \mathbf{v}}{D t} = \nabla \cdot \mathbf{\sigma} ,
\]
\[
\frac{D Q_p}{D t} = \Gamma_p \mathbf{H}_p + p [Q_p \cdot \mathbf{\omega}] + \lambda_p \left[ \nabla \otimes q^{p-2} u \right] + \nu_p \lambda_p \left[ \nabla \otimes q^{p} \mathbf{u} \right] ,
\]
where \( D/Dt = \partial_t + \mathbf{v} \cdot \nabla \) is the material derivative, \( \rho \) is a constant density, \( \mathbf{v} \) is the incompressible velocity field (\( \nabla \cdot \mathbf{v} = 0 \)) and \( \mathbf{\sigma} \) is the total stress tensor, to be defined later. The tensors \( \mathbf{u} = [\nabla \mathbf{v} + (\nabla \mathbf{v})^T]/2 \) and \( \mathbf{\omega} = [\nabla \mathbf{v} - (\nabla \mathbf{v})^T]/2 \), with \( T \) indicating transposition, are respectively the strain rate and vorticity fields and entail the coupling between \( p \)-atic order and flow, with \( \lambda_p \) and \( \nu_p \) material constants. The dot product indicates the contraction between the last index of \( Q_p \) and the first index of \( \mathbf{\omega} \) – i.e. \( (Q_p \cdot \mathbf{\omega})_{i_1 i_2 \ldots i_p} = Q_{i_1 i_2 \ldots i_p} \omega_{i_1 i_2 \ldots i_p} \) and \( (\nabla \otimes q^{p})_{i_1 i_2 \ldots i_p} = \partial_{i_1} \partial_{i_2} \ldots \partial_{i_p} \) where \([\ldots]\) denotes the floor function and \( p \mod 2 = p - 2[p/2] \) is zero for even \( p \) values and one for odd \( p \) values. The quantity \( \mathbf{H}_p = -\delta F/\delta Q_p \) is the \( p \)-atic analog of the molecular tensor, dictating the relaxation dynamics of the order parameter tensor toward the minimum of the orientational free energy \( F = \int dA \left( L_p/2 |\nabla Q_p|^2 + A_p/2 |Q_p|^2 + B_p/4 |Q_p|^4 \right) \), where \([\ldots]\) is the Euclidean norm and is such that \( |Q_p|^2 = |\Psi_p|^2/2 \). The constant \( L_p \) is the order parameter stiffness and \( A_p \) and \( B_p \) are phenomenological constants setting the magnitude of the scalar order parameter at equilibrium: i.e. \( |\Psi_p| = |\Psi_p(0)| = \sqrt{-2A_p/B_p} \), when \( H_{i_1 i_2 \ldots i_p} = 0 \).

![FIG. 1. Examples of elementary \( s = 1/p \) (red) and \( s = -1/p \) (blue) defects in \( p \)-atics and their associated backflow (black and white) for \( p = 2, 3 \ldots 6 \). The streamlines are obtained from Eq. (4).](image)

The stress tensor on the left-hand side of Eq. (1a) is customarily decomposed into a static and a dynamic contribution: i.e. \( \mathbf{\sigma} = \mathbf{\sigma}^{(s)} + \mathbf{\sigma}^{(d)} \). The static stress tensor is given by \( \mathbf{\sigma}^{(s)} = -P \mathbf{1} + \mathbf{\sigma}^{(e)} \), where \( P \) is the pressure and \( \mathbf{\sigma}^{(e)} = -L_p \partial_i Q_p \otimes \partial_i Q_p \) is the elastic stress resulting from a static distortion of the \( p \)-atic orientation, with the symbol \( \otimes \) indicating a contraction of all matching indices of the two operands yielding a tensor whose rank equates the number of unmatched indices (two in this case). The dynamic stress, on the other hand, can be further decomposed into a viscous or energy dissipating part and a reactive or energy preserving part: i.e. \( \mathbf{\sigma}^{(d)} = \mathbf{\sigma}^{(v)} + \mathbf{\sigma}^{(r)} \). The former given by \( \mathbf{\sigma}^{(v)} = 2\eta |\mathbf{u}| \), with \( \eta \) the shear viscosity; while the latter takes the form
\[
\mathbf{\sigma}^{(r)} = \lambda_p (1/p - 1) |\nabla \otimes q^{p-2}\mathbf{H}_p + \frac{p}{2} (Q_p \cdot \mathbf{H}_p - \mathbf{H}_p \cdot Q_p) .
\]
Both this terms result from a departure from the lowest free energy state and, together with the elastic stress \( \mathbf{\sigma}^{(e)} \), can drive backflow (see e.g. Ref. [51]). To investigate the role of backflow in the dynamics of topological defects, we next consider the case of an isolated disclination of strength \( s = \pm 1/p, \pm 2/p \ldots \) at the origin of an unbounded domain and take \( \vartheta = s\vartheta_0 + \vartheta_1 \), with \( \varphi = \arctan(y/x) \). The constant angle \( \vartheta_0 \) determines
the overall orientation of the defect [52] and can be set to zero without loss of generality: i.e. \( \theta_0 = 0 \). The order parameter \( \Psi | \Psi | \) is assumed uniformly equal to its equilibrium value \( \Psi (0) \) outside the core of the defects – i.e. for \( |r| > a \) with \( a \sim O(\sqrt{-A_p/L_p}) \) the defect core radius – and vanishing inside. Following Refs. [53, 54], we assume flow alignment effects to be negligible and compute the stationary solution of Eq. (1a). Remarkably, of the three backflow-driving terms entering in the total stress tensor, only one contributes to the flow in the surrounding of a defect – i.e. the second term in Eq. (2) – resulting from the interplay between \( p \)-atic order and flow. By contrast, the second term in Eq. (2), originating from the corotational derivative of the order parameter tensor, is proportional to \( \nabla^2 \theta \) when \( |\Psi| = \text{const} \) (see e.g. Ref. [48]), thus it vanishes identically in this case. The elastic stress, on the other hand, yields the isotropic force density \( \nabla \cdot \sigma^{(e)} = (ps|\Psi|)^2/L_p r/(2|r|^4) \), which, in turn, leads to a local pressure variation – i.e. \( P - P - (ps|\Psi|)^2/L_p/(2|r|^2) \) – without influencing the flow. The backflow sourced by isolated defects, can then be found by convoluting the two-dimensional Oseen-Green tensor [55] with the body force \( f = \nabla \cdot \sigma^{(i)} = c_p/|r|^{p+1}[\cos(n\phi) e_x + \sin(n\phi) e_y] \), where \( c_p \) and \( n \) are given by

\[
\begin{align*}
    c_p &= \frac{(-1)^p(ps)^2\lambda_p L_p}{2p^2} \prod_{k=1}^{p-1} [ps - 2(p - k)] , \\
    n &= p(s - 1) + 1 . \tag{3a} \\
\end{align*}
\]

We stress that Eqs. (3) holds for arbitrary \( p \) values, thus also for nematics (i.e. \( p = 2 \)), where, despite the hydrodynamics of topological defects having being thoroughly investigated [38, 56–59], the origin of defect-induced backflow had, to the best of our knowledge, never been clarified in these terms.

Eqs. (3) allows one to approximatively compute the velocity field in the surrounding of isolated defects. This is given by

\[
\begin{align*}
    \frac{v}{c_p/(4n)} &= \frac{\delta_n}{(p - 1)^2} \left[ \left( \frac{|r|}{R} \right)^{1-p} + (p - 1) \left( \frac{a}{R} \right)^{1-p} \log \left( \frac{|r|}{R} + \frac{3}{2} \right) - (p + 1) \right] R^{1-p} e_x + \frac{\delta_n}{p - 2} \left[ \left( \frac{a}{R} \right)^{2-p} - 1 \right] \frac{r}{|r|^p} \\
    &- \frac{1 - \delta_n}{n} \left\{ \frac{1}{n + p - 1} \left( \frac{|r|}{a} \right)^{n+p-1} + \frac{1}{n - p + 1} \left( \frac{|r|}{R} \right)^{n+1-p} - 1 \right\} |r|^{1-p} (\cos n\phi e_x + \sin n\phi e_y) \\
    &- (1 - \delta_n) \cos[(n - 1)\phi] \left\{ \frac{1}{n + p - 1} \left( \frac{|r|}{a} \right)^{n+p-1} + \frac{1}{n - p + 1} \left[ 2 \frac{2}{n + p - 1} - \left( \frac{|r|}{a} \right)^{n+p-3} \right] \right\} \frac{r}{|r|^p} \tag{4a} \end{align*}
\]

where \( R \) is a large-distance cut-off, analogous to the system size. A plot of the flow associated with this velocity field is given in Fig. 1 for \( p = 1, 2 \ldots 6 \) and \( s = \pm 1/p \). As intuitive, these flows are characterized by the same \((|n| + 1)\)-fold rotational symmetry of the driving force \( f \), with \( n \) as given in Eq. (3b). Thus, a \( s = 1/3 \) disclination in nematics, sources a typical Stokeslet-like flow consisting of two counter-rotating vortices meeting along the defect’s longitudinal direction (i.e. the \( x \)-direction in this case), whereas a \( s = -1/2 \) disclinations gives rise to a \( 3 \)-fold symmetric flow consisting of six vortices with alternating positive and negative vorticity. Similarly, a \( s = 1/3 \) (\( s = -1/3 \)) disclinations in trivacitcs drives a \( 2 \)-fold (4-fold) symmetric flow, etc. Because of this rotational symmetry, and with only exception for \( p = 2 \), these flows have the effect of stirring the fluid around a defect and stabilize the core by trapping this at the central stagnation point. By contrast, the Stokeslet-like flow sourced by a \( s = 1/2 \) disclination yields a net momentum current, whose effect is to propel the defect forward.

Such an interesting example of passive self-propulsion is not unique of nematics (Fig. 2a-c), but nematics are the only type of liquid crystal where passive self-propulsion is thermodynamically stable. To substantiate the latter statement we express the defects strength in terms of the hydrodynamics of topological defects having being explicitly broken, ultimately leading to the propulsion of the defect and stablize the core by trapping this at the central stagnation point. By contrast, the Stokeslet-like flow sourced by a \( s = 1/2 \) disclination yields a net momentum current, whose effect is to propel the defect forward.
propelled defects features an elementary winding number \( s = \pm 1/p \) and when let free to evolve it rapidly splits into \( p - 1 \) elementary defects. Finally, the proportionality between the speed of isolated \( s = 1/2 \) disclinations and the flow alignment parameter, brings to light an exciting opportunity for estimating the flow alignment parameter \( \lambda_2 \) — a notoriously elusive material parameter in liquid crystals (see e.g. Ref. [60]) — from measurements of the self-propulsion speed of elementary nematic defects.

Next we focus on the annihilation dynamics of pairs of elementary \( s = \pm 1/p \) defect pairs. In the absence of hydrodynamics effects, two-dimensional disclinations of opposite winding numbers are known to attract each other via a Coulomb-like force and eventually annihilate. In nematics, Tóth et al. showed that hydrodynamic affects this process in a two-fold way [38]: on the one hand, advection by the backflow causes the defects to move faster, thereby speeding up their annihilation dynamics; on the other hand, the different configuration of the velocity field in the surrounding of positive and negative defects introduces an asymmetry in the annihilation trajectory (Fig. 3a), which is then no longer symmetric about the mid-plane separating the defects at \( t = 0 \). Although tempting to explain this phenomenology in the light of the aforementioned passive-self propulsion of \( s = 1/2 \) defects, in the following we demonstrate that both effects arise instead from the antisymmetric part of the dynamic stress tensor: i.e. the second term on the right-hand side of Eq. (2).

To demonstrate the latter statement we show in Fig. 3 the power \( \mathcal{P} = \int dA (\nabla \cdot \mathbf{\sigma}) \cdot \mathbf{v} \) delivered by each and every contribution to the total stress, before, during and after an annihilation event [61, 62]. Data are generated by a numerical integration of Eqs. (1) on a square domain with periodic boundary and whose initial configuration consists of a neutral pair of elementary \( p \)–atic defects [55].

As evident from our numerical data, for all \( p \) values the annihilation dynamics is dominated by the antisymmetric part of the stress tensor (yellow tones). This converts the energy stored in the distorted configuration of the \( p \)–atic director into kinetic energy, which in turn dissipated by viscous stresses (red tones). By contrast the stresses originating from flow alignment, which source the propulsion of isolated defects, contributes to the annihilation dynamics only for \( p = 2 \) — i.e. when the second and third terms in Eq. (2) have the same differential order — and only in close proximity of annihilation (Fig. 3 inset).

In addition to highlighting the origin of the hydrodynamic enhancement of pair annihilation, Fig. 3 illustrates another surprising result: annihilation occurs more rapidly as \( p \) increases. To further clarify this phenomenon we focus on the trajectories of the annihilating defects, with and without backflow (Fig. 4a). In the absence of hydrodynamics, when annihilation is solely driven by elastic forces, the trajectories are symmetric about the \( y \)–axis and the defects annihilate at the origin (Fig. 4b). Conversely, switching on hydrodynamics causes the positive defect to move faster toward the positive \( x \)–direction and annihilation occurs in the half-plane that was initially occupied by the negative defect. This effect is visibly more pronounced in hexatics, thus confirming our previous observation that annihilation occurs...
more rapidly as \( p \) increases. Comparing the annihilation trajectories of all \( p \) values in the range \( p = 2, 3 \ldots, 6 \), however, we find that, even though the annihilation time decreases with \( p \), it does so at a decreasing rate and eventually plateaus as \( p \to \infty \) (4b). This behavior can be rationalized by requesting all forces experienced by each of the defect to balance each other along the \( x \)-direction. Calling \( x_\pm \) the \( x \)-coordinate of the positive and negative defect and \( v_\pm \) the contribution to their velocity due to backflow, this gives

\[
\frac{dx_\pm}{dt} = v_\pm - \frac{ks^2}{x_\pm - x_\mp},
\]

(6)

The first term on the right-hand side embodies the effective drag force acting on the defects and originating from the dissipation of the defects elastic energy [63], whereas the last term arises from the elastic Coulomb attraction between defects, with \( k \sim \Gamma_p L_p \) a constant [53]. From Eq. (6), taking \( \Delta v = v_+ - v_- \geq 0 \), solving Eq. (6) and imposing \( x_+(t_a) = x_-(t_a) \) allows one to compute the annihilation time \( t_a \) in the form

\[
t_a = \frac{|\Delta x(0)|}{\Delta v} - \frac{2ks^2}{\Delta v^2} \log \left[ 1 + \frac{|\Delta x(0)|}{2ks^2} \left| \frac{\Delta x(0)}{\Delta v} \right| \right],
\]

(7)

where \( \Delta x(0) = x_+(0) - x_-(0) \). Now, in the absence of backflow, \( \Delta v = 0 \) and Eq. (7) reduces to \( t_a = \frac{|\Delta x(0)|^2}{4ks^2} \). For finite \( \Delta v \) values, on the other hand, \( t_a \) decreases monotonically with \( \Delta v \) and approaches \( t_a \approx |\Delta x(0)|/\Delta v \) for large \( \Delta v \) values. In turn, \( \Delta v \) increase with \( p \) for small \( p \) values [55], but vanishes for \( p \to \infty \) when isotropy is restored at the macroscopic scale and the defects themselves disappear: i.e. \( s = \pm 1/p \to 0 \).

In conclusion, we demonstrated that hydrodynamic backflow profoundly affects the dynamics of \( p \)-atic defects. In the case of isolated defects, hydrodynamics can either stabilize defects, by trapping them at the stagnation point of the resulting backflow, or propel them forward, by means of a Stokeslet-type backflow consisting of two counterrotating vortices meeting along the defect’s longitudinal direction. The latter phenomenon, however, requires \( s = (p - 1)/p \), this is thermodynamically stable only in nematics (i.e. \( p = 2 \)), where this coincides with the elementary winding number \( s = 1/2 \). In the case of annihilating defect pairs, we have established that hydrodynamic effects become progressively more relevant as \( p \) increases and we have identified the precise origin of this effect. Our work paves the way for a deeper understanding of cell intercalation and other remodelling events in epithelial layers, where small-scale hexatic order (i.e. \( p = 6 \)) has been recently discovered [64, 65].

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Supplemental Information for: Passive self-propulsion and enhanced annihilation of $p$–tic defects

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This supplemental information contains details on the analytical calculation of the velocity field generated by isolated topological defect of arbitrary strength. Additionally, it contains a short description of the numerical method behind our defect annihilation experiments.

S1. ISOLATED DEFECT VELOCITY FIELD CALCULATION

To calculate the velocity of isolated defects we first calculate the force density on the liquid crystal sourced by the reactive stress in presence of a defect of strength $s$ located at the origin of the reference system. Then using the Oseen solution we obtain the velocity field generated by that force. The full stress tensor for an incompressible $p$–atic liquid crystal, excluding pressure, are

$$\sigma = -L_p \partial_i Q_p \otimes \partial_j Q_p + \frac{p}{2} (Q_p \cdot H_p - H_p \cdot Q_p) + \lambda_p (-1)^{p-1} \nabla \otimes p^{-2} \otimes H_p ,$$

where the first term is the elastic contribution to the stress, the second is the antisymmetric one, and the last term is the flow alignment stress. For isolated defects and outside the defect core, the $p$-atic director angle is given by $\theta = s \phi$ and the magnitude of the order parameter can be assumed to be constant, i.e. $|\Psi_p| = \text{constant}$. In this regime, the antisymmetric stress, i.e. the second term of Eq. (S1) $K_p \epsilon_{ij} \nabla^2 \theta / 2 = 0$, with $K_p = p^2 |\Psi_p|^{2} L_p / 2$ vanishes identically.

The elastic stress takes the form $-K_p \partial_i \theta \partial_j \theta$ so that the elastic force density given by $f = \nabla \cdot \sigma^{(e)} = \frac{c_p}{|r|^{p+1}} [\cos(n \phi) e_x + \sin(n \phi) e_y]$, (S4)

where

$$c_p = (-1)^p (ps)^2 \lambda_p L_p \prod_{k=1}^{p-1} [ps - 2(p - k)] ,$$

S5a

$$n = p(s - 1) + 1 .$$

S5b

Finally, to calculate the components of the hydrostatic velocity field, we turn to the Oseen formal solution

$$v(r) = \int dA' G(r - r') \cdot f(r') ,$$

where

$$G(r - r') = \frac{1}{4\pi \eta} \left[ \left( \log \frac{L}{|r - r'|} - 1 \right) 1 + \frac{rr}{|r - r'|^2} \right] ,$$

S7
is the two-dimensional Oseen tensor (see e.g. Ref. [54]), \( R \) a large distance cut-off, which, without loss of generality, is taken to be \( \mathcal{L} = R\sqrt{e} \). To calculate the velocity field integrals we make use of the logarithmic expansion

\[
\log \frac{|\mathbf{r} - \mathbf{r}'|}{\mathcal{L}} = \log \frac{R_p}{\mathcal{L}} - \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{R_2}{R_1} \right)^m \cos \left[ m(\phi - \phi') \right], \tag{S8}
\]

and the orthogonality of trigonometric functions

\[
\int_0^{2\pi} d\phi' \cos \left[ m(\phi - \phi') \right] \cos n\phi' = \pi \cos n\phi \delta_{mn}. \tag{S9}
\]

As explained in the main text, the elastic stress \( \sigma^{(e)} \) only contributes to the pressure, thus it does not affect the flow under the assumption of incompressibility. For the passively self-propelled defects, with \( s = (p-1)/p \), the only non-vanishing contributions to the velocity field are those arising from the zero-th order terms in the expansion Eq. (S8).

### S2. DEFECT ANNIHILATION NUMERICAL EXPERIMENTS

After studying the hydrostatic velocity field generated by isolated defects, we turn to the dynamics multiple defects. In this regime, the assumptions used to make the previous calculation analytically tractable can no longer be trusted; namely the director field \( \theta \) is no longer a simple linear function of \( \phi \), the order parameter can no longer be assumed uniform outside the core, and finally the velocity is now a function of time. To obtain the velocity field generated by multiple defects, sourced by Eq. (S1), we numerically solve Eqs. (1).

All numerics are performed using a C-code developed in our group. For the numerical integration, we use a one-step Euler Method, and for the derivatives we are use the symmetric difference quotient formula. The liquid crystal is confined in a two-dimensional grid of size \( 256 \times 256 \) with doubly periodic boundary conditions. To study the dynamics of the defects, we perform defect annihilation (numerical) experiments, for pair of defects of strength \( s = 1/p \) and \( s = -1/p \), for \( p = 2, \ldots 6 \). The model parameters, expressed in the lattice units used for the numerical simulations are: \( \Delta t = 1 \), \( \rho = 1 \), \( \eta = 1 \), \( L_p = 0.5 \), \( A_p = -0.2 \), \( B_p = 0.4 \), \( \Gamma_p = 1 \) and \( \lambda_p = 1.11 \). The initial configuration consists of a pair of isolated defects on the \( x \)--direction equidistant from the origin, in such a way that they are far enough from each-other and from the boundaries of the periodic grid. Next, we let the system thermalize by solving Eq. (1b) in a punctured domain featuring two disks of radius \( 2a \) centred at the defect cores. Finally we let the system evolve under by solving the full set of Eqs. (1) and track the position of the defects.

### S3. HYDROSTATIC CALCULATION OF VELOCITY FIELD FOR PAIR OF DEFECTS

Our findings, show that the main source of the velocity field generated by a pair of \( s = 1/p \) and \( s = -1/p \) (for all \( p > 2 \)) is the antisymmetric stress. As an additional check, we once again use the Oseen formal solution in the presence of the body force generated by a pair of defects located at \( \mathbf{r}_+ = \{x_+, y_+\} \) and \( \mathbf{r}_- = \{x_-, y_-\} \). To this end, we first compute the orientation field

\[
\theta(x, y) = \frac{1}{p} \sum_i \arctan \left( \frac{y-y_i}{x-x_i} \right) + \theta_0, \tag{S10}
\]

where \( \theta_0 \) is a global phase. The \( |\Psi_p| \) that minimizes the \( p \)--atic free energy, far from the defect cores where the condition \( \nabla^2 |\Psi_p| \approx 0 \) is satisfied, is given by

\[
|\Psi_p|^2 = |\Psi^{(0)}|^2 \left[ 1 - (ap)^2 |\nabla \theta(x, y)|^2 \right] \tag{S11}
\]

where \( a = \sqrt{-L_p/A_p} \) is the defect core radius. Given (S10) and (S11), the non-zero components of the antisymmetric stress tensor are

\[
\sigma_{xy} = apL_p |\Psi^{(0)}|^2 \frac{(|\mathbf{r}_-| - |\mathbf{r}_+|)^2 [x(y_+ - y_-) + x_-(y - y_+) + x_+(y_+ - y)]}{(r - |\mathbf{r}_-|)^4(r - |\mathbf{r}_+|)^4} = -\sigma_{xy}, \tag{S12}
\]
and the body force they produce is given by

\[ f = 4a \rho L_p |\Psi(0)|^2 \left( \frac{(|r|-|r_+|)^2}{(|r|-|r_-|)^6} \right) \left\{ \left( |r| - |r_-| \right)^2 \left( |r| - |r_+| \right)^2 (x_- - x_+) + \\
+ 4 \left( |r| - |r_-| \right)^2 (y - y_+) \left[ x(y_- - y_+) + x_-(y_+ - y) + x_+(y - y_-) \right] + \\
+ 4 \left( |r| - |r_+| \right)^2 (y - y_-) \left[ x(y_- - y_+) + x_-(y_+ - y) + x_+(y - y_-) \right] \} e_x + \\
+ 4a \rho L_p |\Psi_0|^2 \left( \frac{(|r_-| - |r_+|)^2}{(|r_-| - |r_+|)^6} \right) \left\{ \left( |r| - |r_-| \right)^2 (r - |r_+|)^2 (y_- - y_+) + \\
+ 4 \left( |r| - |r_-| \right)^2 (x - x_+) \left[ x(y_+ - y_-) + x_-(y - y_+) + x_+(y_+ - y) \right] + \\
+ 4 \left( |r| - |r_+| \right)^2 (x - x_-) \left[ (x(y_+ - y_-) + x_-(y - y_+) + x_+(y_+ - y) \right] \} e_y. \]  

\[ (S13) \]

Finally, to find the velocity field arising from \((S13)\), we calculate the integrals that appear in the Oseen solution numerically in Mathematica. We find that the vortices of the resulting velocity field, match those obtained by the full numerical integration of the dynamical Eqs. (1) sourced by the full stress tensor Eq. (S1) (see Fig. S1).

Figure S1. (a) Velocity field coming from the full numerical solution at an intermediate stage of the annihilation, for a pair of defects with strength \( s = \pm 1/6 \). The color code shows the magnitude of the order parameter \( |\Psi_p| \). The defect cores correspond to the size of blue regions, where the amplitude of the order parameter \( |\Psi_p| \) drops significantly below 1. (b) Velocity field coming from the Oseen solution. The positions of the positive (left) and negative (right) defect are marked by the blue dots.