Large scale geometry of negatively curved $\mathbb{R}^n \rtimes \mathbb{R}$

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Abstract

We classify all negatively curved $\mathbb{R}^n \rtimes \mathbb{R}$ up to quasiisometry. We show that all quasiisometries between such manifolds (except when they are biLipschitz to the real hyperbolic spaces) are almost similarities. We prove these results by studying the quasisymmetric maps on the ideal boundary of these manifolds.

Keywords. quasiisometry, quasisymmetric map, negatively curved solvable Lie groups.

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1 Introduction

In this paper we study quasiisometries between negatively curved solvable Lie groups of the form $\mathbb{R}^n \rtimes \mathbb{R}$ and quasisymmetric maps between their ideal boundaries.
Given an \( n \times n \) matrix \( A \), we let \( G_A \) be the semi-direct product \( \mathbb{R}^n \rtimes_A \mathbb{R} \), where \( \mathbb{R} \) acts on \( \mathbb{R}^n \) by \( (t, x) \mapsto e^{tA}x \). Then \( G_A \) is a solvable Lie group.

Let \( G_A \) be equipped with any left invariant Riemannian metric such that the \( \mathbb{R} \) direction is perpendicular to the \( \mathbb{R}^n \) factor. When \( A = I_n \), \( G_A \) is isometric to \( \mathbb{H}^{n+1} \). More generally, if the eigenvalues of \( A \) all have positive real parts, then it follows from Heintze’s results \([H]\) that \( G_A \) is Gromov hyperbolic. Hence \( G_A \) has a well defined ideal boundary \( \partial G_A \). The ideal boundary \( \partial G_A \) can be naturally identified with (the one-point compactification of) \( \mathbb{R}^n \).

On the ideal boundary \( \mathbb{R}^n \), there is a parabolic visual (quasi)metric \( D_A \), which is invariant under Euclidean translations and admits a family of dilations \( \{ \lambda_t = e^{tA} \} \). See Section 3 for more details.

Given an \( n \times n \) matrix \( A \), the real part Jordan form of \( A \) is obtained from the Jordan form of \( A \) by replacing each diagonal entry with its real part and reordering to make it canonical. Notice that the real part Jordan form is different from the real Jordan form and the absolute Jordan form. It is related to the absolute Jordan form through matrix exponential.

Here are the main results of the paper. See Theorem 5.12 for a more precise statement of Theorem 1.2. Also see Section 2 for basic definitions.

**Theorem 1.1.** Let \( A \) and \( B \) be \( n \times n \) matrices whose eigenvalues all have positive real parts. Then \( (\mathbb{R}^n, D_A) \) and \( (\mathbb{R}^n, D_B) \) are quasisymmetric if and only if there is some \( s > 0 \) such that \( A \) and \( sB \) have the same real part Jordan form.

**Theorem 1.2.** Let \( A \) and \( B \) be \( n \times n \) matrices whose eigenvalues all have positive real parts. Denote by \( \lambda_1 \) and \( \mu_1 \) the smallest real parts of the eigenvalues of \( A \) and \( B \) respectively, and set \( \epsilon = \lambda_1 / \mu_1 \). If the real part Jordan form of \( A \) is not a multiple of the identity matrix \( I_n \), then for every quasisymmetric map \( F : (\mathbb{R}^n, D_A) \to (\mathbb{R}^n, D_B) \), the map \( F : (\mathbb{R}^n, D_A^\epsilon) \to (\mathbb{R}^n, D_B) \) is biLipschitz.

When \( A = I_n \), the manifold \( G_A \) is isometric to the real hyperbolic space \( \mathbb{H}^{n+1} \). In this case, the ideal boundary is \( \mathbb{R}^n \) with the Euclidean metric, and hence the claim in Theorem 1.2 fails: there are non-biLipschitz quasiconformal maps in the Euclidean space \( \mathbb{R}^n \). More generally, if the real part Jordan form of \( A \) is a multiple of \( I_n \), then it follows from the result of Farb-Mosher (see Section 2) that \( (\mathbb{R}^n, D_A) \) is biLipschitz to \( (\mathbb{R}^n, | \cdot |) \), where \( | \cdot | \) denotes the Euclidean metric and \( \epsilon > 0 \) is some constant. Hence the claim in Theorem 1.2 also fails.

There are several consequences of the main results.

Recall that two geodesic Gromov hyperbolic spaces admitting cocompact isometric group actions are quasiisometric if and only if their ideal boundaries are quasisymmetric with respect to the visual metrics, see \([Pa]\) or \([BS]\). Hence Theorem 1.1 yields the quasiisometric classification of all negatively curved \( \mathbb{R}^n \rtimes \mathbb{R} \).

**Corollary 1.3.** Let \( A \) and \( B \) be \( n \times n \) matrices whose eigenvalues all have positive real parts. Then \( G_A \) and \( G_B \) are quasiisometric if and only if there is some \( s > 0 \) such that \( A \) and \( sB \) have the same real part Jordan form.

The next three results are consequences of Theorem 1.2.
A map $f : X \to Y$ between two metric spaces is called an almost similarity if there are constants $L > 0$ and $C \geq 0$ such that $L d(x_1, x_2) - C \leq d(f(x_1), f(x_2)) \leq L d(x_1, x_2) + C$ for all $x_1, x_2 \in X$ and $d(y, f(X)) \leq C$ for all $y \in Y$.

**Corollary 1.4.** Let $A$ and $B$ be $n \times n$ matrices whose eigenvalues all have positive real parts. Suppose the real part Jordan form of $A$ is not a multiple of the identity matrix $I_n$. Then every quasiisometry $f : G_A \to G_B$ is an almost similarity.

We view the canonical projection $h_A : G_A = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ as the height function for $G_A$. Let $A$ and $B$ be two $n \times n$ matrices. A quasiisometry $f : G_A \to G_B$ is height-respecting if it maps the fibers of $h_A$ to within uniformly bounded Hausdorff distance from the fibers of $h_B$.

**Corollary 1.5.** Let $A$ and $B$ be $n \times n$ matrices whose eigenvalues all have positive real parts. Suppose the real part Jordan form of $A$ is not a multiple of the identity matrix $I_n$. Then every quasiisometry $f : G_A \to G_B$ is height-respecting.

**Corollary 1.6.** Let $A$ be a square matrix whose eigenvalues all have positive real parts. If the real part Jordan form of $A$ is not a multiple of the identity matrix, then $G_A$ is not quasiconformal to any finitely generated group.

A group $G$ of bijections $g : X \to X$ of a quasimetric space is a uniform quasimöbius group if there is some homeomorphism $g : [0, \infty) \to [0, \infty)$ such that every element $g$ of $G$ is $\eta$-quasimöbius. The following result follows from Theorem 1.2 and a theorem of Dymarz-Peng [DP].

**Corollary 1.7.** Let $A$ be a square matrix whose eigenvalues all have positive real parts. Suppose that the real part Jordan form of $A$ is not a multiple of the identity matrix. Let $G$ be a uniform quasimöbius group of $\partial G_A$ (equipped with a visual metric). If the induced action of $G$ on the space of distinct triples of $\partial G_A$ is cocompact, then $G$ can be conjugated by a biLipschitz map of $(\mathbb{R}^n, D_A)$ into the group of almost homotheties of $(\mathbb{R}^n, D_A)$.

When $A$ is a Jordan block, we describe all the quasisymmetric maps on $(\mathbb{R}^n, D_A)$. Consequently, we are able to prove a Liouville type theorem. See Section 7 and Section 8.

Theorem 1.2 was established in the diagonal case in [SX] and in the $2 \times 2$ Jordan block case in [X]. We believe that Theorem 1.2 holds true for most homogeneous manifolds with negative curvature (HMNs), with only a few exceptions. Recall that HMNs were characterized by Heintze in [H]: each such manifold is isometric to a solvable Lie group $S$ with a left invariant Riemannian metric, and the group $S$ has the form $S = N \rtimes \mathbb{R}$, where $N$ is a simply connected nilpotent Lie group, and the action of $\mathbb{R}$ on $N$ is generated by a derivation whose eigenvalues all have positive real parts. An open problem now is to establish Theorem 1.2 for most HMNs, and to construct non-biLipschitz quasisymmetric maps (of the ideal boundary) for the few exceptions. The only exceptions known to the author are (those HMNs that are biLipschitz to) the real and complex hyperbolic spaces: there are quasisymmetric maps in the Euclidean spaces [GV] and the Heisenberg groups [B] that change Hausdorff dimensions (of certain subsets), so they can not be biLipschitz.

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2 Some basic definitions

In this section we recall some basic definitions.

A quasimetric \( \rho \) on a set \( X \) is a function \( \rho : X \times X \to \mathbb{R} \) satisfying the following three conditions:

1. \( \rho(x, y) = \rho(y, x) \) for all \( x, y \in X \);
2. \( \rho(x, y) \geq 0 \) for all \( x, y \in X \), and \( \rho(x, y) = 0 \) if and only if \( x = y \);
3. there is some \( M \geq 1 \) such that \( \rho(x, z) \leq M(\rho(x, y) + \rho(y, z)) \) for all \( x, y, z \in X \).

For each \( M \geq 1 \), there is a constant \( \epsilon_0 > 0 \) such that \( \rho^\epsilon \) is biLipschitz equivalent to a metric for all quasimetric \( \rho \) with constant \( M \) and all \( 0 < \epsilon \leq \epsilon_0 \), see Proposition 14.5. in [Hn].

For any quadruple \( Q = (x, y, z, w) \) of distinct points in a quasimetric space \( X \), the cross ratio \( \text{cr}(Q) \) of \( Q \) is:

\[
\text{cr}(Q) = \frac{\rho(x, w)\rho(y, z)}{\rho(x, z)\rho(y, w)}.
\]

Let \( \eta : [0, \infty) \to [0, \infty) \) be a homeomorphism. A bijection \( F : X \to Y \) between two quasimetric spaces is \( \eta \)-quasimöbius if \( \text{cr}(F(Q)) \leq \eta(\text{cr}(Q)) \) for all quadruples \( Q = (x, y, z, w) \) of distinct points in \( X \), where \( F(Q) = (F(x), F(y), F(z), F(w)) \). A bijection \( F : X \to Y \) between two quasimetric spaces is \( \eta \)-quasisymmetric if for all distinct triples \( x, y, z \in X \), we have

\[
\frac{\rho(F(x), F(y))}{\rho(F(x), F(z))} \leq \eta \left( \frac{\rho(x, y)}{\rho(x, z)} \right).
\]

A map \( F : X \to Y \) is quasisymmetric if it is \( \eta \)-quasisymmetric for some \( \eta \).

Let \( K \geq 1 \) and \( C > 0 \). A bijection \( F : X \to Y \) between two quasimetric spaces is called a \( K \)-quasisimilarity (with constant \( C \)) if

\[
\frac{C}{K} \rho(x, y) \leq \rho(F(x), F(y)) \leq C K \rho(x, y)
\]

for all \( x, y \in X \). When \( K = 1 \), we say \( F \) is a similarity. It is clear that a map is a quasisimilarity if and only if it is a biLipschitz map. The point of using the notion of quasisimilarity is that sometimes there is control on \( K \) but not on \( C \).

3 Negatively curved \( \mathbb{R}^n \rtimes \mathbb{R} \)

In this section we first review some basics about negatively curved \( \mathbb{R}^n \rtimes \mathbb{R} \), then define the parabolic visual (quasi)metric on their ideal boundary and study its properties. We also recall a result of Farb-Mosher and the main results of [X] and [SX].

Let \( A \) be an \( n \times n \) matrix. Let \( \mathbb{R} \) act on \( \mathbb{R}^n \) by

\[
\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n
\]

\[
(t, x) \mapsto e^{tA}x.
\]

We denote the corresponding semi-direct product by \( G_A = \mathbb{R}^n \rtimes_A \mathbb{R} \). Then \( G_A \) is a solvable Lie group. Recall that the group operation in \( G_A \) is given by:

\[
(x_1, t_1) \cdot (x_2, t_2) = (x_1 + e^{t_1A}x_2, t_1 + t_2).
\]
We will always assume that the eigenvalues of \( A \) have positive real parts. An \textit{admissible metric} on \( G_A \) is a left invariant Riemannian metric such that the \( \mathbb{R} \) direction is perpendicular to the \( \mathbb{R}^n \) factor. The \textit{standard metric} on \( G_A \) is the left invariant Riemannian metric determined by the standard inner product on the tangent space of the identity element \((0,0) \in \mathbb{R}^n \times \mathbb{R} = G_A\). We remark that \( G_A \) with the standard metric does not always have negative sectional curvature. However, Heintze’s result (\cite{H}) says that \( G_A \) has an admissible metric with negative sectional curvature. Since any two left invariant Riemannian distances on a Lie group are biLipschitz equivalent, \( G_A \) with any left invariant Riemannian metric is Gromov hyperbolic. From now on, unless indicated otherwise, \( G_A \) will always be equipped with the standard metric.

At a point \((x,t) \in \mathbb{R}^n \times \mathbb{R} \cong G_A\), the tangent space is identified with \( \mathbb{R}^n \times \mathbb{R} \), and the standard metric is given by the symmetric matrix

\[
\begin{pmatrix}
Q_A(t) & 0 \\
0 & 1
\end{pmatrix},
\]

where \( Q_A(t) = e^{-tA^T}e^{-tA} \). Here \( T \) denotes matrix transpose.

For each \( x \in \mathbb{R}^n \), the map \( \gamma_x : \mathbb{R} \to G_A; \gamma_x(t) = (x,t) \) is a geodesic. We call such a geodesic a vertical geodesic. It can be checked that all vertical geodesics are asymptotic as \( t \to +\infty \). Hence they define a point \( \xi_0 \) in the ideal boundary \( \partial G_A \). The sets \( \mathbb{R}^n \times \{ t \} \) \((t \in \mathbb{R})\) are horospheres centered at \( \xi_0 \). For each \( t \in \mathbb{R} \), the induced metric on the horosphere \( \mathbb{R}^n \times \{ t \} \subset G_A \) is determined by the quadratic form \( Q_A(t) \). This metric has distance formula \( d_{A,t}(x,t), (y,t)) = |e^{-tA}(x - y)| \). Here \( | \cdot | \) denotes the Euclidean norm.

Each geodesic ray in \( G_A \) is asymptotic to either an upward oriented vertical geodesic or a downward oriented vertical geodesic. The upward oriented vertical geodesics are asymptotic to \( \xi_0 \) and the downward oriented vertical geodesics are in 1-to-1 correspondence with \( \mathbb{R}^n \). Hence \( \partial G_A \setminus \{ \xi_0 \} \) can be naturally identified with \( \mathbb{R}^n \).

We next define a parabolic visual quasimetric on \( \partial G_A \setminus \{ \xi_0 \} = \mathbb{R}^n \). Given \( x, y \in \mathbb{R}^n = \partial G_A \setminus \{ \xi_0 \} \), the parabolic visual quasimetric \( D_A(x,y) \) is defined as follows: \( D_A(x,y) = e^t \), where \( t \) is the smallest real number such that at height \( t \) the two vertical geodesics \( \gamma_x \) and \( \gamma_y \) are at distance one apart in the horosphere; that is,

\[
d_{A,t}(x,t), (y,t)) = |e^{-tA}(x - y)| = 1.
\]

For each \( g = (x,t) \in G_A \), the Lie group left translation \( L_g \) is an isometry of \( G_A \) and fixes the point \( \xi_0 \). It shifts all the horospheres centered at \( \xi_0 \) in the vertical direction by the same amount. It follows that the boundary map of \( L_g \) is a similarity of \((\mathbb{R}^n, D_A)\). When \( g = (z,0) \), \( L_g \) leaves invariant all the horospheres centered at \( \xi_0 \), and the boundary map is the Euclidean translation by \( z \). Hence Euclidean translations are isometries with respect to \( D_A \):

\[
D_A(x+z,y+z) = D_A(x,y) \text{ for all } x,y,z \in \mathbb{R}^n.
\]

When \( g = (0,t) \), \( L_g \) shifts all the horospheres centered at \( \xi_0 \) by \( t \), and the boundary map is the linear transformation \( e^{tA} \). Hence \( e^{tA} \) is a similarity with similarity constant \( e^t \):

\[
D_A(e^{tA}x,e^{tA}y) = e^tD_A(x,y) \text{ for all } x,y \in \mathbb{R}^n \text{ and all } t \in \mathbb{R}.
\]
We remark that $D_A$ in general is not a metric, but merely a quasimetric. See the remark after the proof of Corollary 3.2.

For any integer $n \geq 2$, let

$$J_n = \begin{pmatrix}
1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}$$

be the $n \times n$ Jordan matrix with eigenvalue 1. We write $J_n = I_n + N$. Here we omit the subscript $n$ for $N$ to simplify the notation. Notice that $e^{-tJ_n} = e^{-tI_n}e^{-tN} = e^{-t}e^{-tN}$. Hence $D_{J_n}(x, y) = e^t$ if and only if $t$ is the smallest real number satisfying $e^t = |e^{-tN}(y - x)|$.

For later use, we notice here that

$$e^{tN} = \begin{pmatrix}
1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-2}}{(n-2)!} & \frac{t^{n-1}}{(n-1)!} \\
0 & 1 & t & \cdots & \frac{t^{n-3}}{(n-3)!} & \frac{t^{n-2}}{n-2} \\
0 & 0 & 1 & \cdots & \frac{t^{n-4}}{(n-4)!} & \frac{t^{n-3}}{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & t \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}. \quad (3.1)$$

Let $P$ be a nonsingular $n \times n$ matrix. Define a map $f : G_A \to G_{P \cdot A \cdot P^{-1}}$ by $f(x, t) = (P \cdot x, t)$. Then it is easy to check that $f$ is a Lie group isomorphism. Hence $f$ is an isometry if $G_{P \cdot A \cdot P^{-1}}$ is equipped with the standard metric and $G_A$ has the admissible metric in which $P^{-1}e_1, \cdots, P^{-1}e_n, e_{n+1}$ is orthonormal at the identity element of $G_A$. Here $e_1, \cdots, e_n$ denote the standard basis of $\mathbb{R}^n$, and $e_{n+1}$ is the standard basis for $\mathbb{R}$. Hence, $G_A$ with any admissible metric is isometric to $G_{P \cdot A \cdot P^{-1}}$ with the standard metric for some nonsingular matrix $P$. By Heintze’s result [H], there is a nonsingular matrix $P$ such that $G_{P \cdot A \cdot P^{-1}}$ with the standard metric has negative sectional curvature.

Now we suppose both $G_A$ and $G_{P \cdot A \cdot P^{-1}}$ are equipped with the standard metric. Then it is easy to check that for each $t \in \mathbb{R}$, the restricted map

$$f|_{\mathbb{R}^n \times \{t\}} : (\mathbb{R}^n \times \{t\}, d_{A,t}) \to (\mathbb{R}^n \times \{t\}, d_{P \cdot A \cdot P^{-1},t})$$

is $K$-biLipschitz, where $K := \max\{|P|, |P|^{-1}|\}$. Here $||M|| = \sup\{|Mx| : x \in \mathbb{R}^n, |x| = 1\}$ denotes the operator norm of an $n \times n$ matrix $M$. We next recall a more general result by Farb-Mosher [FM].

**Proposition 3.1.** (Proposition 4.1 in [FM]) Let $A$ and $B$ be two $n \times n$ matrices. Suppose there are constants $r, s > 0$ such that $rA$ and $sB$ have the same real part Jordan form. Then there is a height-respecting quasiisometry $f : G_A \to G_B$. To be more precise, there exist an $n \times n$ matrix $M$ and $K \geq 1$ such that for each $t \in \mathbb{R}$, the map $v \to Mv$ is a $K$-biLipschitz homeomorphism from $(\mathbb{R}^n, d_{A,t})$ to $(\mathbb{R}^n, d_{B,t})$; it follows that the map $f : G_A \to G_B$ given by

$$(x, t) \mapsto \left(Mx, \frac{1}{r} \cdot t\right)$$

is biLipschitz with biLipschitz constant $\sup\{K, \frac{s}{r}, \frac{r}{s}\}$. 

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Corollary 3.2. Suppose we are in the setting of Proposition 3.1. Assume further that \( r = 1 \) and \( G_A \) has negative sectional curvature. Then:

1. the boundary map \( \partial f : (\mathbb{R}^n, D_A^\ast) \to (\mathbb{R}^n, D_B) \) is biLipschitz;
2. \( f \) is an almost similarity.

Proof. (1) We observe that the boundary map is given by \( \partial f(x) = Mx \). Let \( x, y \in \mathbb{R}^n \) and assume \( D_A^\ast(x, y) = e^t \). Then \( D_A(x, y) = e^{t/s} \). By the definition of \( D_A \), we have \( d_{A, t/s}((x, t/s), (y, t/s)) = 1 \). Since \( G_A \) has pinched negative sectional curvature, there is a constant \( a \) depending only on the curvature bounds of \( G_A \), such that \( d_{A, t/s}((x, t'), (y, t')) < 1/K \) for \( t' > t/s + a \) and \( d_{A, t/s}((x, t'), (y, t')) > K \) for \( t' < t/s - a \). It now follows from Proposition 3.1 that \( d_B(t', (Mx, t''), (My, t'')) < 1 \) for \( t'' > t + sa \) and \( d_B(t', (Mx, t''), (My, t'')) > 1 \) for \( t'' < t - sa \). By the definition of \( D_B \) we have \( e^{-sa}e^t \leq D_B(Mx, My) \leq e^sa e^t \). Hence \( \partial f : (\mathbb{R}^n, D_A^\ast) \to (\mathbb{R}^n, D_B) \) is biLipschitz with biLipschitz constant \( e^sa \).

(2) Let \( p = (x_1, t_1), q = (x_2, t_2) \in G_A \) be arbitrary. We may assume \( t_1 \leq t_2 \). If \( x_1 = x_2 \), then it is clear from the definition of \( f \) that \( d(f(p), f(q)) = s \cdot d(p, q) \). So we assume \( x_1 \neq x_2 \) and that \( d_{A, 0}((x_1, t_0), (x_2, t_0)) = 1 \) for some \( t_0 \). First assume \( t_0 \leq t_2 \). Then \( d((x_1, t_2), q) < d_{A, t_2}((x_1, t_2), q) \leq 1 \) as \( G_A \) has negative sectional curvature. By the triangle inequality, we have \( |d(p, q) - (t_2 - t_1)| \leq 1 \). By Proposition 3.1 \( d(Mx_1, st_2), f(q) \leq d_B(st_2)(Mx_1, st_2), f(q) \). By the triangle inequality again we have \( |d(f(p), f(q)) - (st_2 - st_1)| \leq K \). Hence \( |d(f(p), f(q)) - s \cdot d(p, q)| \leq s + K \).

Next we assume \( t_0 > t_2 \). By Lemma 6.3 (1) of \[SX\] we have \( |d(p, q) - (t_0 - t_1) - (t_0 - t_2)| \leq C_1 \) for some constant \( C_1 \) depending only on the curvature bounds of \( G_A \). By Lemma 6.2 of \[SX\], the point \( (x_1, t_0) \) is a \( C_2 \)-quasicenter of \( x_1, x_2, \xi_0 \in \partial G_A \) for some constant \( C_2 \) depending only on the curvature bounds of \( G_A \). Since \( f \) is a quasisymmetry, \( f(x_1, t_0) = (Mx_1, st_0) \) is a \( C_3 \)-quasicenter of \( Mx_1, Mx_2, \eta_0 \in \partial G_B \) (here \( \eta_0 \) denotes the point in \( \partial G_B \) corresponding to upward oriented vertical geodesics), where \( C_3 \) depends only on \( C_2 \), the quasiisometry constants of \( f \) and the Gromov hyperbolicity constant of \( G_B \). Similarly, the point \( (Mx_2, st_0) \) is also a \( C_3 \)-quasicenter of \( Mx_1, Mx_2, \eta_0 \in \partial G_B \). Now consider the geodesic triangle consisting of \( \{Mx_1\} \times \mathbb{R}, \{Mx_2\} \times \mathbb{R} \) and a geodesic joining \( Mx_1, Mx_2 \). Notice that \( f(p) \in \{Mx_1\} \times \mathbb{R} \) lies between \( Mx_1 \) and \( (Mx_1, st_0) \) and \( f(q) \in \{Mx_2\} \times \mathbb{R} \) lies between \( Mx_2 \) and \( (Mx_2, st_0) \). It follows that

\[
|d(f(p), f(q)) - (st_2 - st_1) - (st_0 - st_2)| = |d(f(p), f(q)) - d(f(p), (Mx_1, st_0)) - d(f(q), (Mx_2, st_0))| \leq C_4
\]

for some constant \( C_4 \) depending only on \( C_3 \) and the Gromov hyperbolicity constant of \( G_B \). This combined with \( |d(p, q) - (t_0 - t_1) - (t_0 - t_2)| \leq C_1 \) implies \( |d(f(p), f(q)) - s \cdot d(p, q)| \leq C_4 + sC_1 \).

We notice that Corollary 3.2 (1) implies that \( D_A \) is indeed a quasisymmetry: by Heintze’s result, there is some nonsingular \( P \) such that \( G_{PAP^{-1}} \) has pinched negative sectional curvature and hence \( D_{PAP^{-1}} \) is a quasisymmetry (this can be proved by the arguments in [CDP], p.124) or by using the relation between parabolic visual quasimetric and visual quasimetric \[SX\] section 5); since \( (\mathbb{R}^n, D_A) \) and \( (\mathbb{R}^n, D_{PAP^{-1}}) \) are biLipschitz, \( D_A \) is also a quasisymmetry.
Let $A$ be an $n \times n$ matrix in real part Jordan form with positive eigenvalues

$$\lambda_1 < \cdots < \lambda_{k_A}.$$ 

Let $V_i \subset \mathbb{R}^n$ be the generalized eigenspace of $\lambda_i$, and let $d_i = \dim V_i$.

If $k := k_A \geq 2$, we write $A$ in the block diagonal form $A = [A_1, \cdots, A_k]$, where $A_i$ is the block corresponding to the eigenvalue $\lambda_i$; we also denote $A' = [A_1, \cdots, A_{k-1}]$. Correspondingly, $\mathbb{R}^n$ admits the decomposition $\mathbb{R}^n = V_1 \times \cdots \times V_k$. Hence each point $x \in \mathbb{R}^n$ can be written $x = (x_1, \cdots, x_k)$, where $x_i \in V_i$. Observe that, for each $x_k \in V_k$, if we identify $V_1 \times \cdots \times V_{k-1} \times \{x_k\}$ with $V_1 \times \cdots \times V_{k-1}$, then the restriction of $D_A$ to $V_1 \times \cdots \times V_{k-1} \times \{x_k\}$ agrees with $D_A$. It is not hard to check that for all $x_k, y_k \in V_k$, the following holds for the Hausdorff distance with respect to the quasimetric $D_A$:

$$HD(V_1 \times \cdots \times V_{k-1} \times \{x_k\}, V_1 \times \cdots \times V_{k-1} \times \{y_k\}) = D_A(x_k, y_k). \quad (3.2)$$

Also, for any $x = (x_1, \cdots, x_k) \in \mathbb{R}^n$ and any $y_k \in V_k$,

$$D_A(x, V_1 \times \cdots \times V_{k-1} \times \{y_k\}) = D_A(x, y_k). \quad (3.3)$$

When $k = 1$, that is, when $A$ has only one eigenvalue $\lambda := \lambda_1 > 0$, the matrix $A$ also has a block diagonal form $A = [\lambda I_{n_0}, \lambda I_{n_1} + N, \cdots, \lambda I_{n_r} + N]$, where $n_0 \geq 0$ and $\lambda I_{n_i} + N$ is a Jordan block. We allow the case $A = \lambda I_n$. We write a point $p \in \mathbb{R}^n$ as $p = (z, (x_1, y_1), \cdots, (x_r, y_r))^T$, where $T$ denotes matrix transpose, $z \in \mathbb{R}^{n_0}$ corresponds to $\lambda I_{n_0}$ and $(x_i, y_i)^T \in \mathbb{R}^{n_i}$ $(x_i \in \mathbb{R}^{n_i-1}, y_i \in \mathbb{R})$ corresponds to $\lambda I_{n_i} + N$. Set

$$\mathbb{R}^{n_{0+r}} = \{p = (z, (x_1, y_1), \cdots, (x_r, y_r))^T \in \mathbb{R}^n : x_1 = \cdots = x_r = \mathbf{0}\},$$

and let $\pi_A : \mathbb{R}^n \to \mathbb{R}^{n_{0+r}}$ be the projection given by:

$$\pi_A(p) = (z, (0, y_1), \cdots, (0, y_r))^T \quad \text{for} \quad p = (z, (x_1, y_1), \cdots, (x_r, y_r))^T \in \mathbb{R}^n.$$

Set

$$A(1) = [\lambda I_{n_1-1} + N, \cdots, \lambda I_{n_r-1} + N],$$

where $\lambda I_1 + N$ is understood to be $\lambda I_1$.

**Lemma 3.3.** The restriction of $D_A$ to the fibers of $\pi_A$ agrees with $D_{A(1)}$. To be more precise, for all $p = (z, (x_1, y_1), \cdots, (x_r, y_r))^T$, $p' = (z, (x'_1, y_1), \cdots, (x'_r, y_r))^T$ we have

$$D_A(p, p') = D_{A(1)}(x, x'),$$

where $x = (x_1, \cdots, x_r)^T$ and $x' = (x'_1, \cdots, x'_r)^T$.

**Proof.** Assume $D_A(p, p') = e^t$ and $D_{A(1)}(x, x') = e^s$. By the definition, $s$ is the smallest real number such that $|e^{-sA(1)}(x' - x)| = 1$. We calculate

$$e^{-sA(1)}(x' - x) = e^{-\lambda s}(e^{-sN_{n_1-1}}(x'_1 - x_1), \cdots, e^{-sN_{n_r-1}}(x'_r - x_r))^T.$$

Similarly, $t$ is the smallest real number such that $|e^{-tA}(p' - p)| = 1$. We calculate

$$e^{-tA}(p' - p) = e^{-\lambda t}(0, e^{-tN_{n_1-1}}(x'_1 - x_1), 0), \cdots, (e^{-tN_{n_r-1}}(x'_r - x_r), 0))^T.$$

It follows that the two equations $|e^{-sA(1)}(x' - x)| = 1$ and $|e^{-tA}(p' - p)| = 1$ are the same. Hence $s = t$. 

\[\square\]
Lemma 3.4. The following hold for all $y, y' \in \mathbb{R}^{p_0+r}$:

1. the Hausdorff distance $HD_D(\pi_A^{-1}(y), \pi_A^{-1}(y')) = |y - y'|^{\frac{1}{2}}$;
2. for any $p \in \pi_A^{-1}(y)$, we have $D_A(p, \pi_A^{-1}(y')) = |y - y'|^{\frac{1}{2}}$.

Proof. Let $p = (z, (x_1, y_1), \ldots, (x_r, y_r))^T \in \pi_A^{-1}(y)$, $p' = (z', (x'_1, y'_1), \ldots, (x'_r, y'_r))^T \in \pi_A^{-1}(y')$, where $y$ and $y'$ are arbitrary. Assume $D_A(p, p') = e^t$. Then $t$ is a real number such that

$$| (z' - z, e^{-tN_{n_1}}(x'_1 - x_1, y'_1 - y_1)^T, \ldots, e^{-tN_{n_r}}(x'_r - x_r, y'_r - y_r)^T) | = e^t.$$

Notice that the last entry of $e^{-tN_{n_i}}(x'_i - x_i, y'_i - y_i)^T$ is $y'_i - y_i$, which is independent of $t$. It follows that $e^{t\lambda} \geq |(z' - z, y'_1 - y_1, \ldots, y'_r - y_r)| = |y' - y|$, and hence $D_A(p, p') = e^t \geq |y' - y|^{\frac{1}{2}}$.

Set $t_0 = \ln |y' - y|/\lambda$. Then $e^{t_0} = |y' - y|$. Now let $p = (z, (x_1, y_1), \ldots, (x_r, y_r))^T \in \pi_A^{-1}(y)$ be arbitrary. Since the matrix $e^{-t_0N_{n_i}}$ is nonsingular, the equation

$$e^{-t_0N_{n_i}}(u_i, v_i)^T = (0, \ldots, 0, y'_i - y_i)^T$$

has a unique solution $(u_i, v_i)^T$, where $u_i \in \mathbb{R}^{n_i-1}$ and $v_i \in \mathbb{R}$. Notice that $v_i = y'_i - y_i$. Set $x'_i = u_i + x_i$ and $p' = (z', (x'_1, y'_1), \ldots, (x'_r, y'_r))^T$. Then $p' \in \pi_A^{-1}(y')$ and

$$e^{-t_0A}(p' - p) = e^{-t_0\lambda} (z' - z, (0, y'_1 - y_1), \ldots, (0, y'_r - y_r))^T.$$

It follows that $t_0$ is a solution of $|e^{-tA}(p' - p)| = 1$ and so $D_A(p, p') \leq e^{t_0} = |y - y'|^{\frac{1}{2}}$. This together with the first paragraph implies $D_A(p, p') = |y - y'|^{\frac{1}{2}}$. So each point $p \in \pi_A^{-1}(y)$ is within $|y - y'|^{\frac{1}{2}}$ of $\pi_A^{-1}(y')$. Similarly, every point $p' \in \pi_A^{-1}(y')$ is also within $|y - y'|^{\frac{1}{2}}$ of $\pi_A^{-1}(y)$. Therefore, (1) holds.

(2) also follows from the above two paragraphs.

The following two results will be used in the proof of Theorems [1.4] and [1.2]. They are the basic steps in the induction.

Theorem 3.5. (Theorem 4.1 in [SX]) Suppose $A$ is diagonal with positive eigenvalues $\alpha_1 < \alpha_2 < \cdots < \alpha_r$ ($r \geq 2$). Then every $\eta$-quasisymmetry $F : (\mathbb{R}^n, D_A) \to (\mathbb{R}^n, D_A)$ is a $K$-quasisimilarity, where $K$ depends only on $\eta$ and $r$.

Theorem 3.6. (Theorems 4.1 and 5.1 in [X]) Every $\eta$-quasisymmetric map $F : (\mathbb{R}^2, D_{J_2}) \to (\mathbb{R}^2, D_{J_2})$ is a $K$-quasisimilarity, where $K$ depends only on $\eta$. Furthermore, a bijection $F : (\mathbb{R}^2, D_{J_2}) \to (\mathbb{R}^2, D_{J_2})$ is a quasisymmetric map if and only if it has the following form: $F(x, y) = (ax + c(y), ay + b)$ for all $(x, y) \in \mathbb{R}^2$, where $a \neq 0$, $b$ are constants and $c : \mathbb{R} \to \mathbb{R}$ is a Lipschitz map.
4 \( Q \)-variation on the ideal boundary

In this section we introduce the main tool in the proof of the main results: \( Q \)-variation for maps between quasimetric spaces. It is a discrete version of the notion of capacity. The advantage of this notion is that it makes sense for quasimetric spaces and does not require the existence of rectifiable curves. We remark that, while dealing with ideal boundary of negatively curved spaces, very often either one has to work with quasimetric spaces in which the triangle inequality is not available, or one needs to work with metric spaces that contain no rectifiable curves. Both scenarios are unpleasant from the point of view of classical quasiconformal analysis.

The notion of \( Q \)-variation is due to Bruce Kleiner [K].

Let \((X, \rho)\) be a quasimetric space and \( L \geq 1 \). A subset \( A \subset X \) is called an \( L \)-quasi-ball if there is some \( x \in X \) and some \( r > 0 \) such that \( B(x, r) \subset A \subset B(x, Lr) \). Here \( B(x, r) := \{y \in X: \rho(y, x) < r\} \).

For any ball \( B := B(x, r) \) and any \( \kappa > 0 \), we sometimes denote \( B(x, \kappa r) \) by \( \kappa B \).

For a subset \( E \) of a quasimetric space \((Y, \rho)\), the \( \rho \)-diameter of \( E \) is \( \text{diam}_\rho(E) := \sup \{\rho(e_1, e_2) : e_1, e_2 \in E\} \).

Let \( u : (X, \rho_1) \to (Y, \rho_2) \) be a map between two quasimetric spaces. For any subset \( A \subset X \), the oscillation of \( u \) over \( A \) is

\[
\text{osc}(u|_A) = \text{diam}_{\rho_2}(u(A)).
\]

Let \( Q \geq 1 \). For a collection of disjoint subsets \( \mathcal{A} = \{A_i\} \) of \( X \), the \( Q \)-variation of \( u \) over \( \mathcal{A} \), denoted by \( V_Q(u, \mathcal{A}) \), is the quantity

\[
\sum_i [\text{osc}(u|_{A_i})]^Q.
\]

For \( \delta > 0 \) and \( K \geq 1 \), set

\[
V^\delta_{Q,K}(u) = \sup \{V_Q(u, \mathcal{A})\},
\]

where \( \mathcal{A} \) ranges over all disjoint collections of \( K \)-quasi-balls in \((X, \rho_1)\) with \( \rho_1 \)-diameter at most \( \delta \). Finally, the \((Q, K)\)-variation \( V_{Q,K}(u) \) of \( u \) is

\[
V_{Q,K}(u) = \lim_{\delta \to 0} V^\delta_{Q,K}(u).
\]

We notice that \( V_{Q,K}(u|_{E_1}) \leq V_{Q,K}(u|_{E_2}) \) whenever \( E_1 \subset E_2 \subset X \).

There are useful variants of this definition, for instance one can look at the infimum over all coverings. Or one can take the infimum over all coverings followed by the sup as the mesh size tends to zero. As a tool, \( Q \)-variation could be compared with Pansu’s modulus [P], but seems slightly easier to work with in our context.

Since quasisymmetric maps send quasi-balls to quasi-balls quantitatively, it is easy to see that \( Q \)-variation is a quasisymmetric invariant. To be more precise, we recall

**Lemma 4.1.** (Lemma 3.1 in [X]) Let \( X \) be a bounded quasimetric space and \( F : X \to Z \) an \( \eta \)-quasisymmetric map. Then for every map \( u : X \to Y \) we have \( V_{Q,K}(u) \leq V_{Q,K}(u \circ F^{-1}) \).
We next calculate the $Q$-variation of certain functions defined on the ideal boundary of negatively curved $\mathbb{R}^n \times \mathbb{R}$. These calculations will be used in the next section to show that certain foliations on the ideal boundary are preserved by quasisymmetric maps.

For later use we recall that, for any $Q > 1$, any integer $k \geq 1$ and any nonnegative numbers $a_1, \ldots, a_k$, Jensen’s inequality states
\[
\frac{\sum_{i=1}^k a_i^Q}{k} \geq \left( \frac{\sum_{i=1}^k a_i}{k} \right)^Q,
\]
and equality holds if and only if all the $a_i$’s are equal. In our applications, the $a_i$’s will be the oscillations of a function $u$ along a “stack” of quasi-balls.

Let $A$ be an $n \times n$ matrix in real part Jordan form with positive eigenvalues
\[
\lambda_1 < \lambda_2 < \ldots < \lambda_k,
\]
and $V_i \subset \mathbb{R}^n$ be the generalized eigenspace of $\lambda_i$, and let $d_i = \dim V_i$. Then $\mathbb{R}^n$ admits the decomposition: $\mathbb{R}^n = V_1 \times \cdots \times V_k$. Since $e^{tA}$ is a linear transformation with $\det(e^{tA}) = e^{t(\sum_i d_i \lambda_i)}$, for any subset $U \subset \mathbb{R}^n$, we have $\text{Vol}(e^{tA}(U)) = e^{t(\sum_i d_i \lambda_i)} \text{Vol}(U)$.

There are constants $C_1, C_2, C_3$ depending only on the dimension $n$ with the following properties. If $B := B(o, 1) \subset \mathbb{R}^n$ is the unit ball (in the Euclidean metric), and $t \leq -1$, then
\[
B(o, C_1 e^{t \lambda_1} |t|^{-n+1}) \subset e^{tA} B \subset B(o, C_2 e^{t \lambda_1} |t|^{n-1}),
\]
while
\[
\text{Vol}(e^{tA} B) = C_3 e^{t(\sum_i d_i \lambda_i)}.
\]

Let $S = \prod_{i=1}^k [0, 1] \subset \mathbb{R}^n$ be the unit cube. We notice that both $S$ and $B$ are $K_0$-quasi-balls with respect to $D_A$ for some $K_0$ depending only on $A$. Hence there is some $r > 0$ such that $B_A(o, r) \subset B \subset B_A(o, K_0 r)$. Here the subscript $A$ refers to $D_A$. Also recall that $D_A$ is a quasimetric: there is a constant $M \geq 1$ such that $D_A(x, z) \leq M(D_A(x, y) + D_A(y, z))$ for all $x, y, z \in \mathbb{R}^n$.

In the following, when we say a subset $E \subset \mathbb{R}^n$ is convex, we mean it is convex with respect to the Euclidean metric. The continuity of a function $u : E \to \mathbb{R}$ is with respect to the topology induced from the usual topology on $\mathbb{R}^n$.

**Lemma 4.2.** Let $E \subset \mathbb{R}^n$ be a convex open subset. If $u : (E, D_A) \to \mathbb{R}$ is a nonconstant continuous function, then $V_{Q,K}(u) = \infty$ for all $Q < \frac{\sum_i d_i \lambda_i}{\lambda_k}$ and all $K \geq K_0$.

**Proof.** Let $p, q \in E$ with $u(p) \neq u(q)$. Let $C \subset E$ be a fixed cylinder with axis $\overrightarrow{pq}$, such that the minimum of $u$ on one cap of $C$ is strictly greater than its maximum on the other cap. We pack $C$ with translates of $e^{tA} B$, for $t \ll 0$, as follows. First pick a maximal set of lines $L = \{L_j\}$ in $\mathbb{R}^n$ satisfying the following conditions:
1. each line is parallel to $\overrightarrow{pq}$;
2. each line intersects $C$;
3. the Hausdorff distance (with respect to $D_A$) between any two of the lines is at least $2MK_0 re^t$. 


The maximality implies that for each $x \in C$, we have $D_A(x, L_j) \leq 2MK_0e^t$ for some $j$. For each $j$, consider a translate $B_j$ of $e^{tA}B$ centered at some point on $L_j$. Then we move $B_j$ along $L_j$ (in both directions) by translations until the translates just touch $B_j$. Repeat this and we obtain a “stack” of $K_0$-quasi-balls centered on $L_j$. Do this for each $j$ and we obtain a packing $\mathcal{P} = \{P\}$ of $C$ by translates of $e^{tA}B$, after removing those that are disjoint from $C$.

We claim that the collection $\mathcal{P}$ covers a fixed fraction of the volume of $C$. To see this, first notice that the $D_A$-distance between the centers $x_1, x_2$ of two consecutive $K_0$-quasi-balls along $L_j$ is at most $M(K_0)e^t + K_0e^t = 2MK_0e^t$, due to the generalized triangle inequality for $D_A$. Assume $D_A(x_1, x_2) = e^t$. Then

$$e^{(n-r-s)}A(x_2 - x_1) \in e^{(n-r-s)}A\overline{B}_A(o, e^t) = \overline{B}_A(o, r) \subset \overline{B} \subset \overline{B}_A(o, K_0r).$$

Since $\overline{B}$ is convex, the line segment joining $o$ and $e^{(n-r-s)}A(x_2 - x_1)$ is contained in $\overline{B} \subset \overline{B}_A(o, K_0r)$. It follows that the segment joining $o$ and $x_2 - x_1$ lies in $e^{s-(n-r)}A\overline{B}_A(o, K_0r) = \overline{B}_A(o, K_0e^t)$. Hence $MK_0e^t \leq K_1 := 2MK_0e^t$ of the center of some $P \in \mathcal{P}$. Now the choice of the lines $\{L_j\}$ and the generalized triangle inequality for $D_A$ imply that $C$ is covered by $D_A$-balls with radius $K_2 := M(2MK_0e^t + K_1)$ and centers the centers of $\{P\}$. Since the volumes of $e^{tA}B$ and $B_A(o, K_2)$ are comparable, the claim follows.

The number of $K_0$-quasi-balls in $\mathcal{P}$ along each line $L_j$ is $\lesssim e^{-\ell_k}|t|^{n-1}$ in view of the estimate (4.1). By Jensen’s inequality, the $Q$-variation of $u$ for the $K_0$-quasi-balls along $L_j$ is at least as large as the $Q$-variation when the oscillations of $u$ on these quasi-balls are equal. This common oscillation is $\gtrsim e^{\ell_k}|t|^{-n+1}$. Since $\mathcal{P}$ covers a fixed fraction of $C$, the cardinality of $\mathcal{P}$ is $\gtrsim e^{-t(\sum \lambda_i)}$. Hence the $Q$-variation of $u$ on $\mathcal{P}$ is

$$\gtrsim e^{-t(\sum \lambda_i)} \left( e^{\ell_k}|t|^{-n+1} \right)^Q$$

$$= e^{t(Q\lambda_k - \sum \lambda_i)|t|^{(-n+1)Q}},$$

which $\to \infty$ as $t \to -\infty$ for $Q < \sum \lambda_i / \lambda_k$. Hence $V_{Q,K}(u) = \infty$.

Notice that $\sum \lambda_i / \lambda_k < n$ if $k \geq 2$ and $\sum \lambda_i / \lambda_k = n$ if $k = 1$. Hence we have the following:

**Corollary 4.3.** Suppose $k = 1$. Let $E \subset \mathbb{R}^n$ be a convex open subset. If $u : (E, D_A) \to \mathbb{R}$ is a nonconstant continuous function, then $V_{Q,K}(u) = \infty$ for all $Q < n$ and all $K \geq K_0$.

**Lemma 4.4.** Let $E \subset \mathbb{R}^n$ be a convex open subset. Let $u : (E, D_A) \to \mathbb{R}$ be a continuous function. Suppose there is an affine subspace $W$ parallel to the subspace $\bigcap_{i \leq l} V_i$ such that $u|_{W \cap E}$ is not constant. Then $V_{Q,K}(u) = \infty$ for all $Q < \sum \lambda_i / \lambda_l$ and all $K \geq K_0$.

**Proof.** Note that in the proof of Lemma 4.2 if $\overline{pq}$ is parallel to the subspace $\bigcap_{i \leq l} V_i$, then the number of quasi-balls in $\mathcal{P}$ along a line $L_j$ is $\lesssim e^{-t\lambda_i}|t|^{n-1}$, so the lower bound on $Q$-variation becomes

$$C e^{t(Q\lambda_i - \sum \lambda_i)|t|^{(-n+1)Q}},$$

which tends to $\infty$ as $t \to -\infty$ if $Q < \sum \lambda_i / \lambda_l$.

□
Let $\pi : \mathbb{R}^n = V_1 \times \cdots \times V_k \to V_k$ be the natural projection.

**Lemma 4.5.** Let $\pi' : V_k \to \mathbb{R}$ be a coordinate function on $V_k$, and set $u = \pi' \circ \pi$. Then $V_{Q,K}(u_{|E}) = 0$ for all $Q > \sum \frac{d_i \lambda_i}{\lambda_k}$, all $K \geq K_0$ and all bounded subsets $E \subset \mathbb{R}^n$.

**Proof.** Let $E$ be a bounded open subset. Let $0 < \delta << 1$ and $\{B_j\}_{j \in I}$ be a packing of $E$ by $K$-quasi-balls with size $< \delta$. Then for each $j$ there is some $x_j \in \mathbb{R}^n$ and some $t_j$ such that

$$B_A(x_j, e^{t_j}) \subset B_j \subset B_A(x_j, K e^{t_j}).$$

Since $B_A(o, r) \subset B \subset B_A(o, K_0 r)$, we have $e^{t_j A} B \subset B_A(o, e^{t_j} r) = B_A(o, e^{t_j} r')$ and $B_A(o, K e^{t_j}) \subset e^{t_j A} B$, where $t_j' = t_j - \ln r - \ln K_0$ and $t_j'' = t_j - \ln r + \ln K$. Set $B_j' = x_j + e^{t_j A} B$ and $B_j'' = x_j + e^{t_j A} B$. Then $B_j' \subset B_j \subset B_j''$. It follows that

$$\text{osc}(u|_{B_j}) \leq \text{osc}(u|_{B_j''}) \lesssim e^{t_j \lambda_k} |t_j'|^{d_k - 1},$$

and

$$(\text{osc}(u|_{B_j}))^Q \lesssim e^{t_j (Q \lambda_k)} |t_j'|^Q (d_k - 1) \lesssim e^{t_j (Q \lambda_k)} |t_j'|^Q (d_k - 1).$$

If $Q > \sum \frac{d_i \lambda_i}{\lambda_k}$, then this will be bounded as $(\text{Vol}(B_j'))^s \leq (\text{Vol}(B_j))^s$ for $s = \frac{Q \lambda_k + \sum d_i \lambda_i}{2 \sum d_i \lambda_i} > 1$, which implies that the $Q$-variation is zero.

For the rest of this section, we will assume $k = 1$ and use the notation introduced before Lemma 3.3.

**Lemma 4.6.** Suppose $A$ has only one eigenvalue $\lambda > 0$. Let $\pi' : \mathbb{R}^{n_0 + r} \to \mathbb{R}$ be a coordinate function and $u = \pi' \circ \pi_A$. Then for any bounded open subset $E$:

1. $V_{Q,K}(u_{|E}) = 0$ for all $Q > n$ and all $K \geq K_0$;
2. $0 < V_{n,K}(u_{|E}) < \infty$ for all $K \geq K_0$.

**Proof.** Let $P$ be a $K$-quasi-ball. Then there is a $D_A$-ball $U$ with $U \subset P \subset KU$. Let $t_0 = \ln(KK_0)$. For some $t \in \mathbb{R}$ there is a translate $S(t)$ of $e^{t A} S$ and a translate $S(t + t_0)$ of $e^{(t + t_0) A} S$ such that $\frac{1}{K_0} U \subset S(t) \subset U$ and $KU \subset S(t + t_0) \subset K K_0 U$. Observe that for any translate $S'$ of $e^{t A} S$, we have $\text{osc}(u|_{S'}) = e^{M}$. It follows that

$$\text{osc}(u|_{P}) \geq \text{osc}(u|_{S(t)}) \geq \frac{1}{(KK_0)^\lambda} \text{osc}(u|_{S(t + t_0)}) \geq \frac{1}{(KK_0)^\lambda} \text{osc}(u|_{P}).$$

Also notice that $\text{osc}(u|_{S(t)}) = \left(\text{Vol}(S(t))\right)^{\frac{1}{n}} \leq \left(\text{Vol}(P)\right)^{\frac{1}{n}}$.

Now let $E$ be a bounded open subset and $\{P_j\}$ a packing of $E$ by a disjoint collection of $K$-quasi-balls with size $< \delta$. For each $P_j$, let $U_j$ be a $D_A$-ball with $U_j \subset P_j \subset KU_j$ and $S_j$ be a translate of some $e^{t A} S$ with $\frac{1}{K_0} U_j \subset S_j \subset U_j$. Then the preceding paragraph implies

$$\sum \text{osc}(u|_{P_j})^Q \leq (KK_0)^{Q \lambda} \sum \text{osc}(u|_{S_j})^Q \leq (KK_0)^{Q \lambda} \sum \text{Vol}(P_j)^{\frac{Q}{n}}.$$
From this it is clear that \( V_{Q,K}(u|E) = 0 \) if \( Q > n \) and \( V_{n,K}(u|E) < \infty \) since \( \{P_i\} \) is a disjoint collection in \( E \).

Now consider a particular packing \( \{P_i\} \) of \( E \) by the images of the integral unit cubes in \( \mathbb{R}^n \) under \( e^{tA} \). Then \( \text{osc}(u|P_i) = e^{\lambda t} \). The cardinality of \( \{P_i\} \) is approximately \( \frac{\text{Vol}(E)}{e^{\lambda t}} \). Hence \( V_{n,K}(u|E) \geq \sum_{i} \text{osc}(u|P_i)^n \approx \text{Vol}(E) \). Hence we have \( 0 < V_{n,K}(u|E) < \infty \).

\[ \square \]

**Lemma 4.7.** Suppose \( A \) has only one eigenvalue \( \lambda > 0 \). Let \( E \subset \mathbb{R}^n \) be a rectangular box whose edges are parallel to the coordinate axes. Let \( u : (E, D_A) \to \mathbb{R} \) be a continuous function. Suppose there is some fiber \( H : \pi_A : \mathbb{R}^n \to \mathbb{R}^{n_0 + r} \) such that \( u|_{H \cap E} \) is not constant. Then \( V_{Q,K}(u) = \infty \) for all \( Q \leq n \) and all \( K \geq K_0 \).

**Proof.** Suppose there is some fiber \( H \) of \( \pi_A \) such that \( u|_{H \cap E} \) is not constant. Then there is some Jordan block \( J \) in \( A \) with the following property: if we denote by \( x = (x_1, \ldots, x_m) \) the coordinates corresponding to \( J \), then there is some index \( k \), \( 1 \leq k \leq m - 1 \) such that \( u \) is constant along every line parallel to the \( x_j \)-axis for \( j \leq k - 1 \), but is not constant along some line \( L \) parallel to the \( x_k \)-axis. We write \( \mathbb{R}^n = \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{R}^{n-k} \), where the \( \mathbb{R} \) corresponds to the \( x_k \)-axis and the \( \mathbb{R}^{k-1} \) is spanned by the \( x_j \)-axes \( (j \leq k-1) \). After composing \( u \) with an affine function, we may assume that for some rectangular box \( C = \prod_{i=1}^{n}[a_i, b_i] \subset E \), we have \( u \leq 0 \) on the codimension 1 face \( F_0 := \{x \in C : x_k = a_k\} \) of \( C \) and \( u \geq 1 \) on the codimension 1 face \( F_1 := \{x \in C : x_k = b_k\} \) of \( C \). We will induct on \( k \).

Recall that for a Jordan block \( J = \lambda I_m + N \), we have \( e^{tJ} = e^{\lambda t} e^{tN} \). See (3.1) for an expression of \( e^{tN} \).

We first assume \( k = 1 \). For \( t << 0 \), consider the images of the integral unit cubes under \( e^{tA} \). Let \( \{B_i\} \) be the collection of all those images that intersect the box \( C \). Notice that a vertical stack (i.e. parallel to the \( x_m \)-axis) of integral cubes is mapped by \( e^{tA} \) to a sequences of \( K_0 \)-quasi-balls which is almost parallel to the \( x_1 \)-axis. We divide \( \{B_i\} \) into such sequences which join \( F_0 \) and \( F_1 \). Note that the projection of each \( B_i \) to the \( x_1 \)-axis has length comparable to \( e^{\lambda t}|t|^{m-1} \). Hence the cardinality of each sequence is comparable to \( e^{-\lambda t}|t|^{1-m} \). The \( Q \)-variation of \( u \) along each sequence is at least the \( Q \)-variation of \( u \) when oscillations of \( u \) on the members of the sequence are equal. Since \( u \leq 0 \) on \( F_0 \) and \( u \geq 1 \) on \( F_1 \), this common oscillation is at least comparable to \( e^{\lambda t}|t|^{m-1} \). Since each \( B_i \) has volume \( e^{\lambda t} \), the cardinality of \( \{B_i\} \) is comparable to \( e^{-\lambda t} \). It follows that the \( Q \)-variation of \( u|_C \) is at least comparable to

\[ e^{-n\lambda t} \left( e^{\lambda t}|t|^{m-1} \right)^Q = e^{\lambda t(Q-n)|t|Q(m-1)}, \]

which \( \to \infty \) as \( t \to -\infty \) if \( Q \leq n \). Hence \( V_{Q,K}(u|_C) = \infty \) for \( Q \leq n \).

Now we assume \( m - 1 \geq k \geq 2 \). Then \( u \) is constant along affine subspaces parallel to \( \mathbb{R}^{k-1} \times \{0\} \times \{0\} \subset \mathbb{R}^n \). Let

\[ U = \{x \in F_0 : (3a_i + b_i)/4 \leq x_i \leq (a_i + 3b_i)/4 \text{ for all } i \neq k\} \subset F_0. \]

For \( t << 0 \), denote by \( v(t) = (-1)^{m-k} e^{-\lambda t} e^{tA} \). Notice that the components of \( v(t) \) corresponding to the Jordan block \( J \) is

\[ (-1)^{m-k} \left( \frac{t^{m-1}}{(m-1)!}, \frac{t^{m-2}}{(m-2)!}, \ldots, t, 1 \right) \]
and all other components are 0. Hence for \( t << 0 \), lines parallel to \( v(t) \) travel much faster in the \( x_i \) (\( 1 \leq i \leq m-1 \)) direction than in the \( x_{i+1} \) direction. Let \( Z \subset \mathbb{R}^n \) be the subset given by:

\[
Z = \left\{ f + s v(t) : f \in U, \; 0 \leq s \leq \frac{(m-k)!}{|t|^{m-k}} (b_k - a_k) \right\}.
\]

Notice that for each fixed \( f \), the segment \( \{ f + s v(t) : 0 \leq s \leq \frac{(m-k)!}{|t|^{m-k}} (b_k - a_k) \} \) joins the two hyperplanes \( x_k = a_k \) and \( x_k = b_k \). Also notice that these segments are parallel to the images of vertical stacks (i.e., parallel to the \( x_m \)-axis) of integral cubes under \( e^{tA} \). Hence \( Z \) has a packing \( \mathcal{P} \) that can be divided into sequences such that each sequence joins \( x_k = a_k \) and \( x_k = b_k \) and is the image (under \( e^{tA} \)) of a vertical stack of integral cubes.

For \( p = (x, y, z), q = (x', y', z') \in \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{R}^{n-k} \), define \( p \sim q \) if \( y' = y, z' = z \) and \( x' - x \) is an integral multiple of \( b_i - a_i \) for \( 1 \leq i \leq k-1 \). Set \( Y = \mathbb{R}^n / \sim \) and let \( \pi : \mathbb{R}^n \to Y \) be the natural projection. Also let \( \pi_C : C \to Y \) be the composition of the inclusion \( C \subset \mathbb{R}^n \) and \( \pi \). It is clear that \( \pi_C \) is injective on the interior of \( C \). It is also easy to check that \( \pi|_Z \) is injective. Now the packing \( \mathcal{P} \) of \( Z \) projects onto a packing of \( Y \), which can then be pulled back through \( \pi_C \) to obtain a packing \( \mathcal{P}' \) of \( C \) (since \( \pi(Z) \subset \pi_C(C) \)). A sequence in \( \mathcal{P} \) gives rise to a broken sequence in \( \mathcal{P}' \): the broken sequence will first hit the boundary of \( C \) at a point of \( \partial(\prod_{i=1}^{k-1} [a_i, b_i]) \times \prod_{i=k}^{n} [a_i, b_i] \subset \partial C \), it continues after a translation by an element of the form \( (\sum_{i=1}^{k-1} m_i(b_i - a_i), 0, 0) \in \mathbb{R}^n \), where \( m_i \in \mathbb{Z} \); this can be repeated until the sequence hits \( x_k = b_k \). Note that we can apply Jensen’s inequality to each broken sequence while considering \( Q \)-variations of \( u \) since by assumption \( u \) is constant along affine spaces parallel to \( \mathbb{R}^{k-1} \times \{0\} \times \{0\} \).

Each broken sequence joins \( F_0 \) to \( F_1 \). Since the projection of \( e^{tA}S \) to the \( x_k \)-axis has length comparable to \( e^{\lambda t}|t|^{m-k} \), the cardinality of each sequence is comparable to \( e^{-\lambda t}|t|^{k-m} \). The \( Q \)-variation of \( u \) along the sequence is at least the \( Q \)-variation when the oscillations of \( u \) are the same on all members of the sequence. The common oscillation is at least comparable to \( e^{\lambda t}|t|^{m-k} \). Hence the \( Q \)-variation of \( u \) is at least comparable to

\[
\frac{1}{e^{n\lambda t}} \cdot (e^{\lambda t}|t|^{m-k})^Q = |t|^Q e^{Q(m-k)} e^{(Q-n)\lambda t},
\]

which \( \to \infty \) when \( t \to -\infty \) if \( Q \leq n \). Hence \( V_{Q,k}(u|_C) = \infty \) for \( Q \leq n \).

\[\square\]

5 Proof of the main theorems

In this section we prove the main results of the paper. The main tools are the notion of \( Q \)-variation (Section 4) and the arguments from Section 4 of [X] and [SX]. The main results of [SX] and [X] are the basic steps in the induction.

We first fix the notation. Let \( A \) be an \( n \times n \) matrix in real part Jordan form with positive eigenvalues

\[
\lambda_1 < \cdots < \lambda_{k_A}.
\]

Let \( V_i \subset \mathbb{R}^n \) be the generalized eigenspace of \( \lambda_i \), and set \( d_i = \dim V_i \). If \( k_A \geq 2 \), we write \( A \) in the block diagonal form \( A = [A_1, \cdots, A_{k_A}] \), where \( A_i \) is the block corresponding to
the eigenvalue \( \lambda_i \); we also denote \( A' = [A_1, \ldots, A_{k_A-1}] \). If \( k_A = 1 \), that is, if \( A \) has only one eigenvalue \( \lambda = \lambda_1 \), then we also write \( A = [\lambda I_{m_0}, \lambda I_{n_1} + N, \ldots, \lambda I_{s_r} + N] \) in the block diagonal form, and we let \( \pi_A : \mathbb{R}^n \to \mathbb{R}^{n_0+r} \) be the projection defined before Lemma 4.3.

Similarly, let \( B \) be an \( n \times n \) matrix in real part Jordan form with positive eigenvalues

\[ \mu_1 < \cdots < \mu_k. \]

Let \( W_j \subset \mathbb{R}^n \) be the generalized eigenspace of \( \mu_j \), and set \( e_j = \dim W_j \). If \( k_B \geq 2 \), we write \( B \) in the block diagonal form \( B = [B_1, \ldots, B_{k_B}] \), where \( B_j \) is the block corresponding to the eigenvalue \( \mu_j \); we also denote \( B' = [B_1, \ldots, B_{k_B-1}] \). If \( k_B = 1 \), that is, if \( B \) has only one eigenvalue \( \mu = \mu_1 \), we also write \( B = [\mu I_{m_0}, \mu I_{m_1} + N, \ldots, \mu I_{m_s} + N] \) in the block diagonal form, and we let \( \pi_B : \mathbb{R}^n \to \mathbb{R}^{m_0+s} \) be the projection defined before Lemma 3.3.

If \( k_B = 1 \) and \( s \geq 1 \), we set \( l_B = \max\{m_1, \ldots, m_s\} \).

Suppose there is an \( \eta \)-quasisymmetric map \( F : (\mathbb{R}^n, D_A) \to (\mathbb{R}^n, D_B) \).

**Lemma 5.1.** \( k_A = 1 \) if and only if \( k_B = 1 \).

**Proof.** Suppose \( k_A = 1 \) and \( k_B \geq 2 \). Fix any \( Q \) with \( \sum \frac{n_j e_j}{\mu_{k_B}} < Q < n \). Let \( \pi : (\mathbb{R}^n, D_B) \to W_{k_B} \) be the projection onto \( W_{k_B} \), and \( \pi' : W_{k_B} \to \mathbb{R} \) a coordinate function on \( W_{k_B} \). Set \( u = \pi' \circ \pi \). Then Lemma 4.5 implies \( V_{Q, \eta(K)}(u|_{F(E)}) = 0 \) for all sufficiently large \( K \) and all bounded subsets \( E \subset (\mathbb{R}^n, D_A) \). By Lemma 4.4 \( V_{Q,K}(u \circ F|_E) = 0 \). But this contradicts Corollary 4.3.

\[ \square \]

**Lemma 5.2.** Suppose \( k_A = 1 \). Then \( A = \lambda_1 I_n \) if and only if \( B = \mu_1 I_n \).

**Proof.** Suppose \( B = \mu_1 I_n \). Let \( \pi_i \ (i = 1, 2, \ldots, n) \) be the coordinate functions on \((\mathbb{R}^n, D_B)\). Then by Lemma 4.6 we have \( V_{n, \eta(K)}(\pi_i|_{F(E)}) < \infty \) for all \( i \), all sufficiently large \( K \) and all rectangular boxes \( E \subset (\mathbb{R}^n, D_A) \). Hence \( V_{n,K}(\pi_i \circ F|_E) < \infty \) by Lemma 4.1. Now Lemma 4.7 implies that \( \pi_i \circ F \) is constant on the fibers of \( \pi_A \). Since this is true for all \( 1 \leq i \leq n \), the fibers of \( \pi_A \) must have dimension 0. Hence \( A \) must also be a multiple of \( I_n \).

\[ \square \]

**Lemma 5.3.** Suppose \( k_A = 1 \) and \( r \geq 1 \). Then \( F \) maps each fiber of \( \pi_A \) onto some fiber of \( \pi_B \).

**Proof.** Lemmas 5.1 and 5.2 imply that \( k_B = 1 \) and \( s \geq 1 \). Notice that it suffices to show that each fiber of \( \pi_A \) is mapped by \( F \) into some fiber of \( \pi_B \): by symmetry each fiber of \( \pi_B \) is mapped by \( F^{-1} \) into some fiber of \( \pi_A \) and hence the lemma follows. We shall prove this by contradiction and so assume that there is some fiber \( H \) of \( \pi_A \) such that \( F(H) \) is not contained in any fiber of \( \pi_B \). Then there is some coordinate function \( \pi' : \mathbb{R}^{m_0+s} \to \mathbb{R} \) such that \( u \circ F \) is not constant on \( H \), where \( u := \pi' \circ \pi_B \). Now Lemma 4.6 implies that \( V_{n, \eta(K)}(u|_{F(E)}) < \infty \) for all sufficiently large \( K \) and all rectangular boxes \( E \subset (\mathbb{R}^n, D_A) \). By Lemma 4.1 we have \( V_{n,K}(u \circ F|_E) < \infty \). This contradicts Lemma 4.7 since we can choose \( E \) such that \( u \circ F \) is not constant on \( H \cap E \).

\[ \square \]
It follows from Lemma 5.3 that $F$ induces a map $G : \mathbb{R}^{n_0 + r} \to \mathbb{R}^{m_0 + s}$ such that $F(\pi_A^{-1}(y)) = \pi_B^{-1}(G(y))$ for all $y \in \mathbb{R}^{n_0 + r}$. Define 

$$\tau_A : \mathbb{R}^n = \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r} \longrightarrow \mathbb{R}^n = \mathbb{R}^{n_0 - r} \times \mathbb{R}^{n_0 + r}$$

by

$$\tau_A(z, (x_1, y_1), \cdots, (x_r, y_r)) = ((x_1, \cdots, x_r), (z, y_1, \cdots, y_r)),$$

where $(x_i, y_i) \in \mathbb{R}^{n_i} = \mathbb{R}^{n_1 - 1} \times \mathbb{R}$. Similarly, there is an identification 

$$\tau_B : \mathbb{R}^n = \mathbb{R}^{m_0} \times \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_s} \longrightarrow \mathbb{R}^n = \mathbb{R}^{m_0 - s} \times \mathbb{R}^{m_0 + s}.$$

With the identifications $\tau_A$ and $\tau_B$, we have $\pi_A^{-1}(y) = \mathbb{R}^{n_0 - r} \times \{y\}$, $\pi_B^{-1}(G(y)) = \mathbb{R}^{n_0 - m_0 - s} \times \{G(y)\}$, and $F(\mathbb{R}^{n_0 - m_0 - s} \times \{y\}) = \mathbb{R}^{n_0 - m_0 - s} \times \{\tau_B(y)\}$. Hence for each $y \in \mathbb{R}^{n_0 + r}$, there is a map $H(\cdot, y) : \mathbb{R}^{n_0 - r} \to \mathbb{R}^{n_0 - m_0 - s}$ such that $F(x, y) = (H(x, y), G(y))$ for all $x \in \mathbb{R}^{n_0 - r}$.

**Lemma 5.4.** Suppose $k_A = 1$ and $r \geq 1$. Then:

1. The map $G : (\mathbb{R}^{n_0 + r}, | \cdot |^{1/2}) \to (\mathbb{R}^{m_0 + s}, | \cdot |^{1/2})$ is $\eta$-quasisymmetric;
2. for each $y \in \mathbb{R}^{n_0 + r}$, the map $H(\cdot, y) : (\mathbb{R}^{n_0 - r}, D_A(1)) \to (\mathbb{R}^{n_0 - m_0 - s}, D_B(1))$ is $\eta$-quasisymmetric.

**Proof.** (1) follows from Lemma 3.4 and the arguments on page 10 of [X]. The statement (2) follows from Lemma 3.5.

Suppose $k_A = 1$. Set $\epsilon = \lambda/\mu$ and $\eta_1(t) = \eta(t^{1/2})$. We notice that all the following maps are $\eta_1$-quasisymmetric:

1. $F : (\mathbb{R}^n, D_A) \to (\mathbb{R}^n, D_B)$;
2. $G : (\mathbb{R}^{n_0 + r}, | \cdot |^{1/2}) \to (\mathbb{R}^{m_0 + s}, | \cdot |^{1/2})$;
3. $H(\cdot, y) : (\mathbb{R}^{n_0 - r}, D_A(1)) \to (\mathbb{R}^{n_0 - m_0 - s}, D_B(1))$, for each $y \in \mathbb{R}^{n_0 + r}$.

Let $g : (X_1, \rho_1) \to (X_2, \rho_2)$ be a bijection between two quasimetric spaces. Suppose $g$ satisfies the following condition: for any fixed $x \in X_1$, $\rho_1(x, y) \to 0$ if and only if $\rho_2(g(x), y) \to 0$. We define for every $x \in X_1$ and $r > 0$,

$$L_g(x, r) = \sup \{ \rho_2(g(x), g(x')) : \rho_1(x, x') \leq r \},$$

$$l_g(x, r) = \inf \{ \rho_2(g(x), g(x')) : \rho_1(x, x') \geq r \},$$

and set

$$L_g(x) = \limsup_{r \to 0} \frac{L_g(x, r)}{r}, \quad l_g(x) = \liminf_{r \to 0} \frac{l_g(x, r)}{r}.$$

**Lemma 5.5.** Consider the maps $G : (\mathbb{R}^{n_0 + r}, | \cdot |^{1/2}) \to (\mathbb{R}^{m_0 + s}, | \cdot |^{1/2})$ and $H(\cdot, y) : (\mathbb{R}^{n_0 - r}, D_A(1)) \to (\mathbb{R}^{n_0 - m_0 - s}, D_B(1))$. The following hold for all $y \in \mathbb{R}^{n_0 + r}$, $x \in \mathbb{R}^{n_0 - r}$:

1. $L_G(y, r) \leq \eta_1(1) l_H(\cdot, y)(x, r)$ for any $r > 0$;
2. $\eta_1^{-1}(1) l_H(\cdot, y)(x) \leq L_G(y) \leq \eta_1(1) l_H(\cdot, y)(x)$;
3. $\eta_1^{-1}(1) L_H(\cdot, y)(x) \leq L_G(y) \leq \eta_1(1) L_H(\cdot, y)(x)$.
Proof. The proof is very similar to that of Lemma 4.3 in \[\text{X}\]. Let \(y \in \mathbb{R}^{n_0+r}, x \in \mathbb{R}^{n-n_0-r}\) and \(r > 0\). Let \(y' \in \mathbb{R}^{n_0+r}\) with \(|y - y'|^\frac{1}{r} \leq r\) and \(x' \in \mathbb{R}^{n-n_0-r}\) with \(D_{A(1)}^e(x, x') \geq r\).

Set \(t_0 = \ln |y' - y|/\lambda\). Let \((u_i, v_i) \ (u_i \in \mathbb{R}^{n_i-1}, v_i \in \mathbb{R}, 1 \leq i \leq r)\) be the unique solution of \(e^{-t_0N_r}(u_i, v_i)^T = (0, \ldots, 0, y_i - y)^T\). Let \(x''_i = u_i + x_i\) and \(x''_i = (x''_1, \ldots, x''_r)\). Then \(D_{A}^e((x, y), (x'', y')) = |y - y'|^\frac{1}{r} \leq r \leq D_{A}((x, y), (x', y))\). Since \(F : (\mathbb{R}^n, D_A^e) \rightarrow (\mathbb{R}^n, D_B)\) is \(\eta_1\)-quasisymmetric, we have

\[
|G(y) - G(y')|^\frac{1}{r} \leq D_B(F(x'', y'), F(x, y)) \leq \eta_1(1) D_B(F(x, y), F(x', y)) = \eta_1(1) D_B(1)(H(x, y), H(x', y)).
\]

Since \(y'\) and \(x'\) are chosen arbitrarily, (1) follows.

The proofs of (2) and (3) are exactly the same as those for Lemma 4.3 in \[\text{X}\].

\[\square\]

**Lemma 5.6.** Suppose \(k_A = 1\) and \(l_A = 2\). Then \(l_B = 2\) and for \(\epsilon = \lambda/\mu\)

(1) \(A\) and \(B\) have the same real part Jordan form; 
(2) The map \(F : (\mathbb{R}^n, D_A^e) \rightarrow (\mathbb{R}^n, D_B)\) is a \(K\)-quasisimilarity, where \(K\) depends only on \(A, B\) and \(\eta\).

**Proof.** (1) By Lemma 5.4 (2), for each \(y \in \mathbb{R}^{n_0+r}\), the map \(H(\cdot, y) : (\mathbb{R}^{n-n_0-r}, D_{A(1)}) \rightarrow (\mathbb{R}^{n-n_0-s}, D_{B(1)})\) is \(\eta\)-quasisymmetric. Since \(l_A = 2\), all Jordan blocks of \(A\) have size 2 and \(A(1) = \lambda I_r\). Now Lemma 7.2 applied to \(H(\cdot, y)\) implies that \(B(1) = \mu I_s\). It follows that all Jordan blocks of \(B\) also have size 2, and hence \(l_B = 2\) and \(B(1) = \mu I_s\). So we have \(r = s\). That is, \(A\) and \(B\) have the same number of \(2 \times 2\) Jordan blocks. Now (1) follows.

(2) The proof of (2) is very similar to the arguments in Section 4 of \[\text{SX}\] and \[\text{X}\]. We will only indicate the differences here. First we notice that \(G : (\mathbb{R}^{n_0+r}, |\cdot|) \rightarrow (\mathbb{R}^{n_0+s}, |\cdot|)\) is also quasisymmetric, and hence is differentiable a.e. Since \(F : (\mathbb{R}^n, D_A^e) \rightarrow (\mathbb{R}^n, D_B)\) is \(\eta\)-quasisymmetric, the arguments in Section 4 of \[\text{SX}\] and \[\text{X}\] imply that there is a constant \(K_1\) depending only on \(\eta\), such that for every \(y \in \mathbb{R}^{n_0+r}\) where \(G : (\mathbb{R}^{n_0+r}, |\cdot|) \rightarrow (\mathbb{R}^{n_0+s}, |\cdot|)\) is differentiable, we have \(0 < l_G(y) < \infty\) and the map

\[
H(\cdot, y) : (\mathbb{R}^{n-n_0-r}, D_{A(1)}) \rightarrow (\mathbb{R}^{n-n_0-s}, D_{B(1)})
\]

is a \(K_1\)-quasisimilarity with constant \(l_G(y)\).

Now let \(y, y' \in \mathbb{R}^{n_0+r}\) be two points where \(G\) is differentiable. We will show that \(l_G(y)\) and \(l_G(y')\) are comparable. Let \(x \in \mathbb{R}^{n-n_0-r}\) and choose \(x' \in \mathbb{R}^{n-n_0-r}\) so that \(D_{A(1)}(x, x') > |y' - y|^\frac{1}{r}\). Let \((u_i, v_i)\) be as in the proof of Lemma 5.3. Let \(x''_i = x_i + u_i\), \(x''_i = x_i + u_i (1 \leq i \leq r)\), and set \(x'' = (x''_1, \ldots, x''_r)\). Then

\[
D_{A}(x, y), (x'', y') = D_{A}((x, y), (x'', y')) = |y' - y|^\frac{1}{r}.
\]

Now the generalized triangle inequality implies

\[
D_{A}(x'', y'), (x', y) \leq M \left\{ D_{A}(x'', y'), (x, y) + D_{A}(x, y), (x', y) \right\} \leq 2MD_{A}(x, y), (x', y).
\]

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By the quasisymmetry condition we have
\[ D_B(F(x'', y'), F(x', y)) \leq \eta(2M)D_B(F(x, y), F(x', y)) \].
Similarly, \( D_B(F(x'', y'), F(x'', y')) \leq \eta(2M)D_B(F(x'', y'), F(x', y')) \). So we have
\[ D_B(F(x'', y'), F(x'', y')) \leq (\eta(2M))^2D_B(F(x, y), F(x', y)). \]
This together with the quasisimilarity properties of \( H(\cdot, y) \) and \( H(\cdot, y') \) mentioned above implies that
\[ l_G(y')D_A'(x'', x''') \leq K_A^2(\eta(2M))^2l_G(y)D_A'(x, x'). \]
Since \( D_{A(1)}(x'', x''') = D_{A(1)}(x, x') \), we have \( l_G(y') \leq K_A^2(\eta(2M))^2l_G(y) \). By symmetry, we also have \( l_G(y) \leq K_A^2(\eta(2M))^2l_G(y') \). Now fix \( y \) and set \( C = l_G(y) \). Then at every \( y' \) where \( G \) is differentiable, \( H(\cdot, y') \) is a \( K_2 \)-quasisimilarity with constant \( C \), where \( K_2 = K_A^2(\eta(2M))^2 \).
Now a limiting argument shows that this is true for every \( y' \in \mathbb{R}^{m+} \). The arguments in Section 4 of [X] (using Lemma 5.5 from above instead of Lemma 4.3 in [X]) then show that there is a constant \( K_3 = K_3(K_2, \eta_1) \) such that \( G : (\mathbb{R}^{m+}, |\cdot|^\frac{1}{\eta}) \rightarrow (\mathbb{R}^{m+}, |\cdot|^\frac{1}{\eta}) \) and all \( H(\cdot, y) \) are \( K_3 \)-quasisimilarities with constant \( C \).

The final difference is in finding a lower bound for \( D_B(F(x, y), F(x', y')) \). If \( D_A((x, y), (x', y')) \leq (2M)^\epsilon|y' - y|^\frac{1}{\eta} \), then
\[ D_B(F(x, y), F(x', y')) \geq |G(y') - G(y)|^\frac{1}{\eta} \geq \frac{C}{K_3}|y' - y|^\frac{1}{\eta} \]
\[ \geq \frac{C}{(2M)^\epsilon K_3} D_A((x, y), (x', y')). \]

Now assume \( D_A((x, y), (x', y')) \geq (2M)^\epsilon|y' - y|^\frac{1}{\eta} \). Let \( (u_i, v_i) \) be as in the above paragraph. Let \( x'' = x' - u_i \) and set \( x''' = (x''_1, \ldots, x''_{i-1}, x''_{i+1}, \ldots, x''_{m}) \). Then \( D_A((x'', y), (x', y')) = |y' - y|^\frac{1}{\eta} \). The generalized triangle inequality implies
\[ \frac{1}{2M} \leq \frac{D_A((x, y), (x'', y))}{D_A((x, y), (x', y'))} \leq 2M. \]

Now the quasisymmetric condition implies
\[ D_B(F(x, y), F(x', y')) \geq \frac{1}{\eta(2M)}D_B(F(x, y), F(x'', y)) \]
\[ \geq \frac{C}{K_3\eta(2M)} D_A((x, y), (x'', y)) \]
\[ \geq \frac{C}{(2M)^\epsilon K_3\eta(2M)} D_A((x, y), (x', y')). \]

So we have found a lower bound for \( D_B(F(x, y), F(x', y')) \). The rest of the proof is the same as in Section 4 of [X]. We notice that the constant \( M \) depends only on \( A \), and \( \epsilon \) depends only on \( A \) and \( B \). Hence \( F \) is a \( K \)-quasisimilarity with \( K \) depending only on \( A \), \( B \) and \( \eta \).
Lemma 5.7. Suppose \( k_A = 1 \) and \( l_A \geq 2 \). Then for \( \epsilon = \lambda/\mu \):
(1) \( A \) and \( \epsilon B \) have the same real part Jordan form;
(2) The map \( F : (\mathbb{R}^n, D_A) \to (\mathbb{R}^n, D_B) \) is a \( K \)-quasisimilarity, where \( K \) depends only on \( A, B \) and \( \eta \).

Proof. We induct on \( l_A \). The basic step \( l_A = 2 \) is Lemma 5.6. Now assume \( l_A = l \geq 3 \) and that the lemma holds for \( l_A = l - 1 \). For any \( y \in \mathbb{R}^{n_0+r} \), the induction hypothesis applied to the \( \eta \)-quasisymmetric map \( H(\cdot, y) : (\mathbb{R}^{n-n_0-r}, D_{A(1)}) \to (\mathbb{R}^{n-n_0-s}, D_{B(1)}) \) implies that for \( \epsilon = \lambda/\mu \):
(a) \( A(1) \) and \( \epsilon B(1) \) have the same real part Jordan form;
(b) \( H(\cdot, y) : (\mathbb{R}^{n-n_0-r}, D_{A(1)}) \to (\mathbb{R}^{n-n_0-s}, D_{B(1)}) \) is a \( K \)-quasisimilarity with \( K \) depending only on \( A(1), B(1) \) and \( \eta \).
Now (1) follows from (a), and (2) follows from (b), Lemma 5.5 and the arguments in Section 4 of [X] (see the proof of Lemma 5.6 (2)).

Lemma 5.8. Suppose \( k_A \geq 2 \). Then \( k_B \geq 2 \) and \( \sum_{\lambda k_A} d_i \lambda_i > \frac{\sum_j e_j \mu_j}{\mu_{k_B}} \).

Proof. Lemma 5.4 implies \( k_B \geq 2 \). Suppose \( \sum_{\lambda k_A} d_i \lambda_i > \frac{\sum_j e_j \mu_j}{\mu_{k_B}} \). Pick any \( Q \) with \( \sum_{\lambda k_A} d_i \lambda_i > Q > \sum_{\lambda k_A} e_j \mu_j \). Let \( \pi : (\mathbb{R}^n, D_B) \to W_{k_B} \) be the projection onto \( W_{k_B} \), and \( \pi' : W_{k_B} \to \mathbb{R} \) a coordinate function on \( W_{k_B} \). Set \( u = \pi' \circ \pi \). By Lemma 5.3 we have \( V_{Q_1, \eta(K)}(u \mid F(E)) = 0 \) for all sufficiently large \( K \) and all Euclidean balls \( E \subset (\mathbb{R}^n, D_A) \). Lemma 4.1 implies \( V_{Q, K}(u \circ F \mid E) = 0 \). This contradicts Lemma 4.2 since \( Q < \sum_{\lambda k_A} d_i \lambda_i \) and the function \( u \circ F \) is nonconstant. Similarly there is a contradiction if \( \sum_{\lambda k_A} d_i \lambda_i < \frac{\sum_j e_j \mu_j}{\mu_{k_B}} \). The lemma follows.

Recall that (see Section 3), if \( k_A \geq 2 \), then the restriction of \( D_A \) to each affine subspace \( H \) parallel to \( \prod_{i<k_A} V_i \) agrees with \( D_{A'} \), where \( A' = [A_1, \cdots, A_{k_A-1}] \).

Lemma 5.9. Denote \( k = k_A \) and \( k' = k_B \). Suppose \( k \geq 2 \). Then each affine subspace \( H \) of \( \mathbb{R}^n \) parallel to \( \prod_{i<k} V_i \) is mapped by \( F \) onto an affine subspace parallel to \( \prod_{j<k'} W_j \).
Furthermore, \( F \mid H : (H, D_A') \to (F(H), D_B') \) is \( \eta \)-quasisymmetric, and \( F \) induces an \( \eta \)-quasisymmetric map \( G : (V_k, D_A_k) \to (W_{k'}, D_B_{k'}) \) such that \( F((\prod_{i<k} V_i) \times \{y\}) = (\prod_{j<k'} W_j) \times \{G(y)\} \).

Proof. As in the proof of Lemma 5.3 to establish the first claim it suffices to show that each affine subspace parallel to \( \prod_{i<k} V_i \) is mapped into an affine subspace parallel to \( \prod_{j<k'} W_j \).
By Lemma 5.8 we have \( \sum_{\lambda k} d_i \lambda_i = \frac{\sum_j e_j \mu_j}{\mu_{k'}} \). Pick any \( Q \) with

\[
\sum_{\lambda k} d_i \lambda_i < Q \leq \min \left\{ \sum_{\lambda k} d_i \lambda_i, \frac{\sum_j e_j \mu_j}{\mu_{k'-1}} \right\}.
\]

Suppose there is an affine subspace \( H \) parallel to \( \prod_{i<k} V_i \) such that \( F(H) \) is not contained in any affine subspace parallel to \( \prod_{j<k'} W_j \). Let \( \pi : \prod_j W_j \to W_{k'} \) be the canonical projection.
Then there is some coordinate function \( \pi' : W_{k'} \to \mathbb{R} \) such that \( u \circ F \) is not constant on \( H \), where \( u = \pi' \circ \pi \). As \( Q > \sum_{i=1}^{j} \frac{d_i}{\lambda_{k+h}} \), Lemma 4.5 implies \( V_{Q,\eta(K)}(u|_{F(E)}) = 0 \) for all sufficiently large \( K \) and all rectangular boxes \( E \subset (\mathbb{R}^n, D_A) \). By Lemma 4.1 and \( V_{Q,K}(u \circ F|_E) = 0 \). This contradicts Lemma 4.4 since \( Q < \frac{\sum_{i=1}^{j} d_i}{\lambda_{k+h}} \) and we can choose a rectangular box \( E \) such that \( u \circ F \) is not constant on \( H \cap E \).

Since by assumption \( F \) is \( \eta \)-quasisymmetric, it follows from the remark preceding the lemma that \( F|_H : (H, D_{A'}) \to (F(H), D_{B'}) \) is \( \eta \)-quasisymmetric.

The first claim implies that there is a map \( G : V_k \to W_{k'} \) such that \( F((\prod_{i<k} V_i) \times \{y\}) = (\prod_{j<k'} W_j) \times \{G(y)\} \) for any \( y \in V_k \). That \( G : (V_k, D_{A_k}) \to (W_{k'}, D_{B_{k'}}) \) is \( \eta \)-quasisymmetric follows from (3.2), (3.3) and the arguments on page 10 of [X].

\[\square\]

**Lemma 5.10.** Suppose \( k_A = 2 \). Then \( k_B = 2 \) and for \( \epsilon = \lambda_1/\mu_1 \):

1. \( A \) and \( \epsilon B \) have the same real part Jordan form;
2. The map \( F : (\mathbb{R}^n, D_A^\epsilon) \to (\mathbb{R}^n, D_B) \) is a \( K \)-quasisimilarity, where \( K \) depends only on \( A, B \) and \( \eta \).

**Proof.** Let \( H \) be an affine subspace of \( \mathbb{R}^n \) parallel to \( \prod_{i<k_A} V_i \). By Lemma 5.9, \( F(H) \) is an affine subspace parallel to \( \prod_{j<k_B} W_j \) and \( F|_H : (H, D_{A'}) \to (F(H), D_{B'}) \) is \( \eta \)-quasisymmetric. Since \( k_A = 2 \), we have \( k_{A'} = 1 \). Now Lemma 5.1 applied to \( F|_H \) implies \( k_{B'} = 1 \), so \( k_B = k_{B'} + 1 = 2 \). Now the \( \eta \)-quasisymmetric map \( F|_H : (H, D_{A'}) \to (F(H), D_{B'}) \) becomes \( (V_1, D_{A_1}) \to (W_1, D_{B_1}) \), and Lemmas 5.7 and 5.2 imply that \( A_1 \) and \( \epsilon_1 B_1 \) have the same real part Jordan form, where \( \epsilon_1 = \lambda_1/\mu_1 \). By Lemma 5.9, \( F \) induces an \( \eta \)-quasisymmetric map \( G : (V_2, D_{A_2}) \to (W_2, D_{B_2}) \), and hence Lemmas 5.7 and 5.2 again imply that \( A_2 \) and \( \epsilon_2 B_2 \) have the same real part Jordan form, where \( \epsilon_2 = \lambda_2/\mu_2 \). Lemma 5.9 also implies \( d_1 = e_1 \) and \( d_2 = e_2 \). Now Lemma 5.8 implies \( \lambda_1/\mu_1 = \lambda_2/\mu_2 \). Hence (1) holds.

To prove (2), we consider two cases. First assume that \( A_1 = \lambda_1 I \) and \( A_2 = \lambda_2 I \). In this case, (2) follows from (1) and Theorem 5.5. Next we assume that either \( A_1 \neq \lambda_1 I \) or \( A_2 \neq \lambda_2 I \) holds. Then Lemma 5.7 implies that either \( F|_H : (H, D_{A_1}^\epsilon) \to (F(H), D_{B_1}) \) is a \( K_1 \)-quasisimilarity with \( K_1 \) depending only on \( A_1, B_1 \) and \( \eta \), or \( G : (V_2, D_{A_2}^\epsilon) \to (W_2, D_{B_2}) \) is a \( K_2 \)-quasisimilarity with \( K_2 \) depending only on \( A_2, B_2 \) and \( \eta \). Then the arguments similar to those in the proof of Lemma 5.6(2) (also compare with Section 4 of [SX]) show that \( F : (\mathbb{R}^n, D_A^\epsilon) \to (\mathbb{R}^n, D_B) \) is a \( K \)-quasisimilarity with \( K \) depending only on \( A, B \) and \( \eta \).

\[\square\]

**Lemma 5.11.** Suppose \( k_A \geq 2 \). Then for \( \epsilon = \lambda_1/\mu_1 \):

1. \( A \) and \( \epsilon B \) have the same real part Jordan form;
2. The map \( F : (\mathbb{R}^n, D_A^\epsilon) \to (\mathbb{R}^n, D_B) \) is a \( K \)-quasisimilarity, where \( K \) depends only on \( A, B \) and \( \eta \).

**Proof.** We induct on \( k_A \). The basic step \( k_A = 2 \) is Lemma 5.10. Now we assume \( k_A = k \geq 3 \) and that the lemma holds for \( k_A = k - 1 \). For each affine subspace \( H \) of \( \mathbb{R}^n \) parallel to \( \prod_{i<k_A} V_i \), the induction hypothesis applied to \( F|_H : (H, D_{A'}) \to (F(H), D_{B'}) \) implies that
for $\epsilon = \lambda_1/\mu_1$:
(a) $A'$ and $\epsilon B'$ have the same real part Jordan form;
(b) The map $F|_H : (H, D_{A'}) \to (F(H), D_{B'})$ is a $K$-quasisimilarity, where $K$ depends only on $A'$, $B'$ and $\eta$.

The statement (a) implies in particular $k_A - 1 = k_B - 1$ (hence $k_A = k_B$), $\lambda_i = \epsilon \mu_i$ and $e_i = d_i$ for $i < k_A$. Now it follows from Lemma 5.8 that $\lambda_k A = \epsilon \mu_k A$. If $A_k A$ is a multiple of $I$, then (1) follows from Lemmas 5.9 and 5.2. If $A_k A$ is not a multiple of $I$, then Lemma 5.9 and Lemma 5.7 (1) imply that $A_k A$ and $\epsilon B_k B$ have the same real part Jordan form. Hence (1) holds as well in this case.

If $A_k A$ is a multiple of $I$, then (2) follows from the statement (b) above and the arguments in the proof of Lemma 5.6 (2). If $A_k A$ is not a multiple of $I$, then Lemma 5.7 (2) implies that $G : (V_k, D_{A_k}) \to (W_k', D_{B_k})$ is a $K_1$-quasisimilarity with $K_1$ depending only on $A_k A$, $B_k B$ and $\eta$. In this case, (2) follows from this, (b) and the arguments in the proof of Lemma 5.6 (2).

Next we will finish the proofs of the main theorems. So let $A, B$ be two arbitrary $n \times n$ matrices whose eigenvalues have positive real parts. Let $G_A, G_B$ be equipped with arbitrary admissible metrics. Then there are nonsingular matrices $P, Q$ such that $G_A$ is isometric to $G_{PAP^{-1}}$ (equipped with the standard metric) and $G_B$ is isometric to $G_{QBQ^{-1}}$ (equipped with the standard metric). Hence below in the proofs we will replace $(\mathbb{R}^n, D_A)$ and $(\mathbb{R}^n, D_B)$ with $(\mathbb{R}^n, D_{PAP^{-1}})$ and $(\mathbb{R}^n, D_{QBQ^{-1}})$ respectively. There also exist nonsingular matrices $P_0$, $Q_0$ such that $G_{P_0AP_0^{-1}}$ and $G_{Q_0BQ_0^{-1}}$ have pinched negative sectional curvature. We may choose the same $P_0AP_0^{-1}$ for all conjugate matrices $A$. Denote by $J$ and $J'$ the real part Jordan forms of $A$ and $B$ respectively. By Proposition 3.1 there are biLipschitz maps $f_J : G_{P_0AP_0^{-1}} \to G_J$ and $f_P : G_{P_0AP_0^{-1}} \to G_{PAP^{-1}}$. Then Corollary 3.2 implies their boundary maps $\partial f_J : (\mathbb{R}^n, D_{P_0AP_0^{-1}}) \to (\mathbb{R}^n, D_J)$ and $\partial f_P : (\mathbb{R}^n, D_{P_0AP_0^{-1}}) \to (\mathbb{R}^n, D_{PAP^{-1}})$ are also biLipschitz. Similarly, there are biLipschitz maps $f_J' : G_{Q_0BQ_0^{-1}} \to G_{J'}$ and $f_Q : G_{Q_0BQ_0^{-1}} \to G_{QBQ^{-1}}$, whose boundary maps $\partial f_J' : (\mathbb{R}^n, D_{Q_0BQ_0^{-1}}) \to (\mathbb{R}^n, D_{J'})$ and $\partial f_Q : (\mathbb{R}^n, D_{Q_0BQ_0^{-1}}) \to (\mathbb{R}^n, D_{QBQ^{-1}})$ are also biLipschitz.

**Completing the proof of Theorem 1.1** The “if” part follows from Proposition 3.1 since the boundary map of a quasisimmetry between Gromov hyperbolic spaces is quasisymmetric. We will prove the “only if” part. So we suppose $(\mathbb{R}^n, D_{PAP^{-1}})$ and $(\mathbb{R}^n, D_{QBQ^{-1}})$ are quasisymmetric. Since the four maps $\partial f_P, \partial f_J, \partial f_Q$ and $\partial f_{J'}$ are biLipschitz, we see that $(\mathbb{R}^n, D_J)$ and $(\mathbb{R}^n, D_{J'})$ are quasisymmetric. Now it follows from Lemma 5.2, Lemma 5.7 (1) and Lemma 5.11 (1) that $J$ and $\epsilon J'$ have the same real part Jordan form, where $\epsilon = \lambda_1/\mu_1$. Hence $A$ and $\epsilon B$ also have the same real part Jordan form.

**Theorem 5.12.** Let $A$ and $B$ be $n \times n$ matrices whose eigenvalues all have positive real parts, and let $G_A$ and $G_B$ be equipped with arbitrary admissible metrics. Denote by $\lambda_1$ and $\mu_1$ the smallest real parts of the eigenvalues of $A$ and $B$ respectively, and set $\epsilon = \lambda_1/\mu_1$. If the real part Jordan form of $A$ is not a multiple of the identity matrix $I_n$, then for every $\eta$-quasisymmetric map $F : (\mathbb{R}^n, D_A) \to (\mathbb{R}^n, D_B)$, the map $F : (\mathbb{R}^n, D_A) \to (\mathbb{R}^n, D_B)$ is a $K$-quasisimilarity, where $K$ depends only on $\eta, A, B$ and the metrics on $G_A, G_B$. 

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Completing the proof of Theorem 5.12 Let \( F : (\mathbb{R}^n, D_{PAP^{-1}}) \to (\mathbb{R}^n, D_{Q^BQ^{-1}}) \) be an \( \eta \)-quasisymmetric map. Notice that the biLipschitz constant of the map \( \partial f_J \) depends only on \( A \) (actually the conjugacy class of \( A \)) as the same \( P_0AP_0^{-1} \) is chosen for all matrices \( A \) in the same conjugate class. However, the biLipschitz constant of \( \partial f_J \) depends on \( P \) and hence on the admissible metric on \( G_A \). Hence \( \partial f_J \circ \partial f_P^{-1} \) is \( L_1 \)-biLipschitz for some constant \( L_1 \), depending only on \( A \) and the admissible metric on \( G_A \). Similarly, \( \partial f_J' \circ \partial f_Q^{-1} \) is \( L_2 \)-biLipschitz for some constant \( L_2 \), depending only on \( B \) and the admissible metric on \( G_B \). It follows that

\[
G := (\partial f_J' \circ \partial f_Q^{-1}) \circ F \circ (\partial f_J \circ \partial f_P^{-1})^{-1} : (\mathbb{R}^n, D_J) \to (\mathbb{R}^n, D_J')
\]

is \( \eta \)-quasisymmetric, where \( \eta \) depends only on \( L_1, L_2 \) and \( \eta \). Now Lemma 5.11(2) and Lemma 5.11(2) imply that \( G \) is a \( K \)-quasisimilarity, where \( K \) depends only on \( J, J' \) and \( \eta \). Consequently, \( F \) is a \( KL_1L_2 \)-quasisimilarity.

6 Proof of the corollaries

In this section we prove the corollaries from the introduction and also derive a local version of Theorem 1.1.

Let \( M \) be a Hadamard manifold with pinched negative sectional curvature, \( \xi_0 \in \partial M \), and \( x_0 \in M \) a base point. Let \( \gamma \) be the geodesic with \( \gamma(0) = x_0 \) and \( \gamma(\infty) = \xi_0 \). Let \( h_M = -B_\gamma : M \to \mathbb{R} \), where \( B_\gamma \) is the Busemann function associated with \( \gamma \). Set \( H_t = h_M^{-1}(t) \). A parabolic visual quasimetric \( D_{\xi_0} \) on \( \partial M \backslash \{ \xi_0 \} \) is defined as follows. For \( \xi, \eta \in \partial M \backslash \{ \xi_0 \} \), \( D_{\xi_0}(\xi, \eta) = e^t \) if and only if \( \xi \xi_0 \cap H_t \) and \( \eta \xi_0 \cap H_t \) have distance 1 in the horosphere \( H_t \).

Let \( N \) be another Hadamard manifold with pinched negative sectional curvature, and \( f : M \to N \) a quasiisometry. For any \( \xi \in \partial M \) and \( x \in M \), we set \( \xi' = \partial f(\xi) \) and \( x' = f(x) \). Let \( \gamma' \) be the geodesic with \( \gamma'(0) = x_0' \) and \( \gamma'(\infty) = \xi_0' \). Set \( h_N = -B_{\gamma'} \), where \( B_{\gamma'} \) is the Busemann function associated with \( \gamma' \). Denote \( H_t' = h_N^{-1}(t) \). Let \( D_{\xi_0}' \) be the parabolic visual quasimetric on \( \partial N \backslash \{ \xi_0' \} \) with respect to the base point \( x_0' \).

Lemma 6.1. Let \( s > 0 \). Then the following three conditions are equivalent:
(1) there is a constant \( C \geq 0 \) such that the Hausdorff distance \( \text{HD}(f(H_t), H_{t'}) \leq C \) for all \( t \);
(2) the boundary map \( \partial f : (\partial M \backslash \{ \xi_0 \}, D_{\xi_0}^s) \to (\partial N \backslash \{ \xi_0' \}, D_{\xi_0}'^s) \) is biLipschitz;
(3) there exists a constant \( C \geq 0 \) such that \( s \cdot d(x, y) - C \leq d(f(x), f(y)) \leq s \cdot d(x, y) + C \) for all \( x, y \in M \).

Proof. The arguments in the proof of Lemma 6.4 in [SX] shows (2) \( \Rightarrow \) (1), while the arguments at the end of [SX] (proof of Corollary 1.2) yields (1) \( \Rightarrow \) (3). We shall prove (3) \( \Rightarrow \) (1) and (1) \( \Rightarrow \) (2).

(1) \( \Rightarrow \) (2): Suppose (1) holds. Let \( \xi \neq \eta \in \partial M \backslash \{ \xi_0 \} \). Assume \( D_{\xi_0}(\xi, \eta) = e^t \) and \( D_{\xi_0}'(\xi', \eta') = e^{t'} \). Let \( \gamma_\xi \) be the geodesic joining \( \xi \) and \( \xi_0 \) with \( \gamma_\xi(0) = H_0 \) and \( \gamma_\xi(\infty) = \xi_0 \). By Lemma 6.2 of [SX], \( \gamma_\xi(t) \) is a \( C_1 \)-quasicenter of \( \xi, \eta, \xi_0 \), and \( \gamma_\xi(t') \) is a \( C_1 \)-quasicenter of \( \xi', \eta', \xi_0' \), where \( C_1 \) depends only on the curvature bounds of \( M \) and \( N \). Since \( f \) is a quasiisometry,
\[ f(\gamma(t)) \] is a \( C_2 \)-quasicenter of \( \xi', \eta', \xi_0 \), where \( C_2 \) depends only on \( C_1 \), the quasisymmetry constants of \( f \) and the curvature bounds of \( N \). It follows that \( d(f(\gamma(t)), \gamma'(t')) \leq C_3 \), where \( C_3 \) depends only on \( C_1, C_2 \) and the curvature bounds of \( N \). By condition (1), the point \( f(\gamma(t)) \) is within \( C \) of \( H_{st}' \). It follows that \( \gamma'(t') \in H_{st}' \) is within \( C + C_3 \) of \( H_{st}' \) and so \( |t' - st| \leq C + C_3 \). Therefore, \( e^{-(C+C_3)e^{st}} \leq D_{\xi_0}(\xi', \eta') = e^{t'} \leq e^{C+C_3e^{st}}. \)

(3) \( \implies \) (1): Suppose (3) holds. Let \( \omega : \mathbb{R} \to M \) be any geodesic with \( \omega(0) \in H_0 \) and \( \omega(\infty) = \xi_0 \). Then \( f \circ \omega \) is a \( (L_1, C_1) \)-quasigeodesic in \( N \), where \( L_1 \) and \( C_1 \) depend only on \( s \) and \( C \). By the stability of quasigeodesics in a Gromov hyperbolic space, there is a constant \( C_2 \) depending only on \( L_1, C_1 \) and the Gromov hyperbolicity constant of \( N \), and a complete geodesic \( \omega' \) in \( N \) with one endpoint \( \xi'_0 \) such that the Hausdorff distance between \( \omega'(\mathbb{R}) \) and \( f \circ \omega(\mathbb{R}) \) is at most \( C_2 \). Let \( t_1 < t_2 \). Then it follows from condition (3) and the triangle inequality that

\[ |h_N(f(\omega(t_2))) - h_N(f(\omega(t_1)))| \leq C_3, \]

where \( C_3 \) depends only on \( C, C_2 \) and the Gromov hyperbolicity constant of \( N \). In particular, this applied to \( \omega = \gamma, t_2 = t \) and \( t_1 = 0 \) (or \( t_2 = 0 \) and \( t_1 = t \) if \( t < 0 \)) implies \( |h_N(f(\gamma(t))) - st| \leq C_3. \)

Let \( x \in H_t \) be arbitrary. Let \( \omega_1 \) be the geodesic with \( \omega_1(t) = x \) and \( \omega_1(\infty) = \xi_0 \). Pick any \( t_2 \geq t \) with \( d(\gamma(t_2), \omega_1(t_2)) \leq 1 \). By condition (3),

\[ |h_N(f(\gamma(t_2))) - h_N(f(\omega_1(t_2)))| \leq d(f(\gamma(t_2)), f(\omega_1(t_2))) \leq s + C. \]

The discussion from the preceding paragraph implies

\[ |h_N(f(\omega_1(t_2))) - h_N(f(\omega_1(t_1)))| \leq C_3 \]

and

\[ |h_N(f(\gamma(t_2))) - h_N(f(\gamma(t))) - s(t_2 - t)| \leq C_3. \]

These inequalities together with the one at the end of last paragraph imply

\[ |h_N(f(\omega_1(t_1))) - st| \leq C_4 := 3C_3 + s + C. \]

Hence \( f(x) = f(\omega_1(t)) \) is within \( C_4 \) of \( H_{st}' \). This shows \( f(H_t) \) lies in the \( C_4 \)-neighborhood of \( H_{st}' \). By considering a quasi-inverse of \( f \), we see that the Hausdorff distance \( HD(f(H_t), H_{st}') \leq C_5 \), where \( C_5 \) depends only on \( s, C \) and the Gromov hyperbolicity constants of \( M \) and \( N \).

\[ \square \]

A local version of Theorem 1.1 also holds:

**Theorem 6.2.** Let \( A, B \) be \( n \times n \) matrices whose eigenvalues have positive real parts, and let \( G_A \) and \( G_B \) be equipped with arbitrary admissible metrics. Let \( U \subset (\mathbb{R}^n, D_A), V \subset (\mathbb{R}^n, D_B) \) be open subsets, and \( F : (U, D_A) \to (V, D_B) \) an \( \eta \)-quasisymmetric map. Then \( A \) and \( sB \) have the same real part Jordan form for some \( s > 0 \).

**Proof.** By Corollary 3.2 and the discussion before the proof of Theorem 1.1, we may assume \( A \) and \( B \) are in real part Jordan form. Fix a base point \( x \in U \). We may assume both \( x \) and
$F(x)$ are the origin $o$. Then there is some constant $a > 1$ and a sequence of distinct triples $(x_k, y_k, z_k)$ from $U$ satisfying $x_k = o$, $D_A(x_k, y_k) \to 0$ and

$$
\frac{D_A(x_k, y_k)}{D_A(y_k, x_k)} \frac{D_A(y_k, x_k)}{D_A(z_k, x_k)} \frac{D_A(z_k, x_k)}{D_A(z_k, y_k), D_A(z_k, y_k)} \in (1/a, a).
$$

Such a triple can be chosen from the eigenspace of $\lambda_1$ (the smallest eigenvalue of $A$) so that $x_k = o$ is the middle point of the segment $y_k, z_k$. Since $F$ is $\eta$-quasisymmetric, there is a constant $b > 0$ depending only on $a$ and $\eta$ such that:

$$
\frac{D_B(F(x_k), F(y_k))}{D_B(F(x_k), F(z_k))} \frac{D_B(F(y_k), F(x_k))}{D_B(F(z_k), F(x_k))} \frac{D_B(F(z_k), F(x_k))}{D_B(F(z_k), F(y_k))} \in (1/b, b).
$$

Assume $D_A(x_k, y_k) = e^{-t_k}$ and $D_B(F(x_k), F(y_k)) = e^{-t_k}$. Then $e^{t_k} : (U, e^{t_k}D_A) \to \eta$ is an isometry. Hence the sequence of pointed metric spaces $(U, e^{t_k}D_A, o)$ converges (as $k \to \infty$) in the pointed Gromov-Hausdorff topology towards $(\mathbb{R}^n, D_A)$. Similarly, the sequence of pointed metric spaces $(V, e^{t_k}D_B, o)$ converges (as $k \to \infty$) in the pointed Gromov-Hausdorff topology towards $(\mathbb{R}^n, D_B)$. On the other hand, the sequence of maps $F_k = F : (U, e^{t_k}D_A) \to (V, e^{t_k}D_B)$ are all $\eta$-quasisymmetric, and the triples $(x_k, y_k, z_k) \in (U, e^{t_k}D_A)$ and $(F(x_k), F(y_k), F(z_k)) \in (V, e^{t_k}D_B)$ are uniformly separated and uniformly bounded. Now the compactness property of quasisymmetric maps implies that a subsequence of $\{F_k\}$ converges in the pointed Gromov-Hausdorff topology towards an $\eta$-quasisymmetric map $F' : (\mathbb{R}^n, D_A) \to (\mathbb{R}^n, D_B)$. Now the theorem follows from Theorem [X].

\[ \Box \]

**Lemma 6.3.** Let $F : \partial G_A \to \partial G_B$ be a quasisymmetric map, where $\partial G_A$ and $\partial G_B$ are equipped with visual metrics. Let $\xi_0 \in \partial G_A, \xi'_0 \in \partial G_B$ be the points corresponding to upward oriented vertical geodesic rays. If the real part Jordan form of $A$ is not a multiple of the identity matrix, then $F(\xi_0) = \xi'_0$.

**Proof.** The proof is similar to that of Proposition 3.5, [X]. Suppose $F(\xi_0) \neq \xi'_0$. By the relation between visual metrics and parabolic visual metrics (see Section 5 of [SX]), the map

$$
F : (\mathbb{R}^n \setminus \{F^{-1}(\xi'_0)\}, D_A) \to (\mathbb{R}^n \setminus \{F(\xi_0), D_B\})
$$

is locally quasisymmetric. By Theorem 6.2, $A$ and $sB$ have the same real part Jordan form for some $s > 0$. In particular, we have $k_B = k_A$; the fibers of $\pi_A$ and $\pi_B$ have the same dimension if $k_A = 1$, and the subspaces $\prod_{i<k_A} V_i$ and $\prod_{j<k_B} W_j$ have the same dimension if $k_A \geq 2$. If $k_A = 1$, let $H$ be a fiber of $\pi_A$ not containing $F^{-1}(\xi'_0)$; if $k_A \geq 2$, then let $H$ be an affine subspace parallel to $\prod_{i<k_A} V_i$ and not containing $F^{-1}(\xi'_0)$. Let $m$ be the topological dimension of $H$. Then $H \cup \{\xi_0\} \subset \partial G_A$ is an $m$-dimensional topological sphere. Since $F(\xi_0) \neq \xi'_0$ and $F^{-1}(\xi'_0) \notin H$, the image $F(H \cup \{\xi_0\})$ is an $m$-dimensional topological sphere in $\mathbb{R}^n = \partial G_B \setminus \{\xi'_0\}$. In particular, $F(H \cup \{\xi_0\})$ (and hence $F(H)$) is not contained in any fiber of $\pi_B$ (if $k_A = 1$) or any affine subspace parallel to $\prod_{j<k_B} W_j$ (if $k_A \geq 2$). Now the arguments of Lemma 5.3 and Lemma 6.9 yield a contradiction. Hence $F(\xi_0) = \xi'_0$.

\[ \Box \]
Now Corollary 1.3 follows from Proposition 3.1, Lemma 6.3, Theorem 1.1 and the fact that a quasiisometry between Gromov hyperbolic spaces induces a quasisymmetric map between the ideal boundaries.

Proofs of Corollary 1.4 and Corollary 1.5. We use the notation introduced before the proof of Theorem 1.1. Let \( f : G_{PAP} \to G_{QBQ} \) be a quasiisometry. By Lemma 6.3, \( f \) induces a boundary map \( \partial f : (\mathbb{R}^n, D_{PAP}) \to (\mathbb{R}^n, D_{QBQ}) \), which is quasisymmetric. By Theorem 1.2, there is some \( s \geq 0 \) such that \( \partial f : (\mathbb{R}^n, D_{PAP}^s) \to (\mathbb{R}^n, D_{QBQ}^s) \) is biLipschitz. Since \( \partial f_P \) and \( \partial f_Q \) are also biLipschitz, the boundary map \( \partial (f_Q^{-1} \circ f \circ f_P) : (\mathbb{R}^n, D_{P0AP}^s) \to (\mathbb{R}^n, D_{Q0BQ}^s) \) of \( f_Q^{-1} \circ f \circ f_P : G_{P0AP} \to G_{Q0BQ} \) is biLipschitz. Since \( G_{P0AP} \) and \( G_{Q0BQ} \) have pinched negative sectional curvature, Lemma 6.3 implies the map \( f_Q^{-1} \circ f \circ f_P \) is height-respecting and is an almost similarity. By Proposition 3.1 and Corollary 3.2, the two maps \( f_P \) and \( f_Q \) are height-respecting and are almost similarities. Hence \( f \) is height-respecting and is an almost similarity.

The proof of Corollary 1.6 is the same as in [SX] (Corollary 1.3).

Next we give a proof of Corollary 1.7. Recall that a group \( G \) of quasisimilarity maps of \((\mathbb{R}^n, D_A)\) is a uniform group if there is some \( K \geq 1 \) such that every element of \( G \) is a \( K \)-quasisimilarity. Dymarz and Peng have established the following (see [DP] for the definition of almost homotheties):

**Theorem 6.4.** ([DP]) Let \( A \) be a square matrix whose eigenvalues all have positive real parts, and \( G \) be a uniform group of quasisimilarity maps of \((\mathbb{R}^n, D_A)\). If the induced action of \( G \) on the space of distinct couples of \( \mathbb{R}^n \) is cocompact, then \( G \) can be conjugated by a biLipschitz map into the group of almost homotheties.

Proof of Corollary 1.7. Let \( G \) be a group of quasimöbius maps of \((\partial G_A, d)\) such that every element of \( G \) is \( \eta \)-quasimöbius, where \( d \) is a fixed visual metric on \( \partial G_A \). Let \( \xi_0 \in \partial G_A \) be the point corresponding to vertical geodesic rays. Since the real part Jordan form of \( A \) is not a multiple of the identity matrix, Lemma 6.3 implies that the point \( \xi_0 \) is fixed by all quasisymmetric maps \( \partial G_A \to \partial G_A \). Hence \( G \) restricts to a group of quasisymmetric maps of \((\mathbb{R}^n, D_A)\). For any three distinct points \( \xi_1, \xi_2, \xi_3 \in \mathbb{R}^n = \partial G_A \setminus \{\xi_0\} \), the quasimöbius condition applied to the quadruple \( Q = (\xi_1, \xi_2, \xi_3, \xi_0) \) implies that every element of \( G \) is an \( \eta \)-quasisymmetric map of \((\mathbb{R}^n, D_A)\). Now Theorem 1.2 implies that there is some \( K \geq 1 \) such that every element of \( G \) is a \( K \)-quasisimilarity. In other words, \( G \) is a uniform group of quasisimilarities of \((\mathbb{R}^n, D_A)\).

Since the induced action of \( G \) on the space of distinct triples of \((\partial G_A, d)\) is cocompact, there is some \( \delta > 0 \) such that for any distinct triple \((\xi_1, \xi_2, \xi_3)\), there is some \( g \in G \) such that \( d(g(\xi_i), g(\xi_j)) \geq \delta \) for all \( 1 \leq i \neq j \leq 3 \). Now let \( \xi \not\in \xi_2 \in \mathbb{R}^n = \partial G_A \setminus \{\xi_0\} \) be any distinct couple. Then there is an element \( g \in G \) as above corresponding to the triple \((\xi_0, \xi_1, \xi_2)\). Since \( g(\xi_0) = \xi_0 \), there are two constants \( a, b > 0 \) depending only on \( \delta \) such that \( D_A(g(\xi_1), o) \leq b, D_A(g(\xi_2), o) \leq b \) and \( D_A((g(\xi_1), g(\xi_2)) \geq a \). This shows that \( G \) acts cocompactly on the space of distinct couples of \((\mathbb{R}^n, D_A)\).

Now the corollary follows from the theorem of Dymarz-Peng.

\[ \square \]
7 QS maps in the Jordan block case

In this section we describe all the quasisymmetric maps on the ideal boundary in the case when $A$ is a Jordan block.

**Theorem 7.1.** Let $J_n = I_n + N$ be the $n \times n$ ($n \geq 2$) Jordan block with eigenvalue 1. Then a bijection $F : (\mathbb{R}^n, D_{J_n}) \to (\mathbb{R}^n, D_{J_n})$ is a quasisymmetric map if and only if there are constants $a_0 \neq 0, a_1, \cdots, a_{n-2} \in \mathbb{R}$, a vector $v \in \mathbb{R}^n$ and a Lipschitz map $C : \mathbb{R} \to \mathbb{R}$ such that

$$F(x) = (a_0 I_n + a_1 N + \cdots + a_{n-2} N^{n-2})x + v + \tilde{C}(x)$$

for all $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$, where $\tilde{C}(x) = (C(x_n), 0, \cdots, 0)^T$. Here $T$ indicates matrix transpose.

We first prove that every map of the indicated form is actually biLipschitz. Notice that the map $F$ described in the theorem decomposes as $F = F_1 \circ F_2 \circ F_3$, with $F_1(x) = x + v$, $F_2(x) = x + \tilde{C}(x)$ and $F_3(x) = (a_0 I_n + a_1 N + \cdots + a_{n-2} N^{n-2})x$, where $C_1 : \mathbb{R} \to \mathbb{R}$ is defined by $C_1(t) = C(t/a_0)$. Since $D_{J_n}$ is invariant under Euclidean translations, $F_1$ is an isometry. We shall prove that $F_2$ and $F_3$ are biLipschitz in the next two lemmas.

For an $n \times n$ matrix $M = (m_{ij})$, set $Q(M) = \sum_{i,j} m_{ij}^2$. We will use the fact $||M|| \leq Q(M)^{\frac{1}{2}}$, where $||M||$ denotes the operator norm of $M$.

**Lemma 7.2.** Suppose $C : \mathbb{R} \to \mathbb{R}$ is $L$-Lipschitz for some $L > 0$. Then $F_2 : (\mathbb{R}^n, D_{J_n}) \to (\mathbb{R}^n, D_{J_n})$, $F_2(x) = x + \tilde{C}(x)$ is $L'$-biLipschitz, where $L'$ depends only on $L$ and the dimension $n$.

**Proof.** Let $x = (x_1, \cdots, x_n)^T$ and $x' = (x'_1, \cdots, x'_n)^T$ be two arbitrary points in $\mathbb{R}^n$. Then $F_2(x) = (x_1 + C(x_n), x_2, \cdots, x_n)^T$ and $F_2(x') = (x'_1 + C(x'_n), x'_2, \cdots, x'_n)^T$. Assume $D_{J_n}(x, x') = e^t$ and $D_{J_n}(F_2(x), F_2(x')) = e^s$. We need to show that there is some constant $a$ depending only on $L$ and $n$ such that $|t - s| \leq a$.

Since $D_{J_n}(x, x') = e^t$, we have $e^t = |e^{-sN}(x' - x)|$, see Section 3. Similarly, $D_{J_n}(F_2(x), F_2(x')) = e^s$ implies $e^s = |e^{-sN}(F_2(x') - F_2(x))|$. Notice $F_2(x') - F_2(x) = (x' - x) + w$, where $w = (C(x'_n) - C(x_n), 0, \cdots, 0)^T$. The only nonzero entry in $e^{-sN}w$ is $C(x'_n) - C(x_n)$. So we have

$$|e^{-sN}w| = |C(x'_n) - C(x_n)| \leq L|x'_n - x_n|.$$ 

On the other hand, the last entry in $e^{-sN}(x' - x)$ is $(x'_n - x_n)$, hence

$$|e^{-sN}w| \leq L|x'_n - x_n| \leq L|e^{-sN}(x' - x)| = Le^t.$$ 

We write

$$e^{-sN}(F_2(x') - F_2(x)) = e^{(t-s)N} e^{-tN}[(x' - x) + w] = e^{(t-s)N}[e^{-tN}(x' - x) + e^{-tN}w].$$

Now

$$e^s = |e^{-sN}(F_2(x') - F_2(x))| = |e^{(t-s)N}[e^{-tN}(x' - x) + e^{-tN}w]|$$

$$\leq ||e^{(t-s)N}|| \cdot |e^{-tN}(x' - x) + e^{-tN}w| \leq ||e^{(t-s)N}|| \cdot \{ |e^{-tN}(x' - x)| + |e^{-tN}w| \}$$

$$\leq ||e^{(t-s)N}|| \cdot \{ e^t + Le^t \} \leq e^t(1 + L) \sqrt{Q(e^{(t-s)N})}.$$
From this we derive \( e^{s-t} \leq (1 + L)\sqrt{Q(e^{(t-s)N})} \). Notice that \( Q(e^{(t-s)N}) \) is a polynomial of degree \( 2(n-1) \) in \( t-s \) that depends only on \( n \). It follows that there is a constant \( a \) depending only on \( n \) and \( L \) such that \( s-t \leq a \). Since the inverse of \( F_2 \) is \( F_2^{-1}(x) = x + (-C(x_n), 0, \cdots, 0)^T \), the above argument applied to \( F_2^{-1} \) yields \( t-s \leq a \). Hence \( |s-t| \leq a \), and we are done.

\[ \square \]

**Lemma 7.3.** Let \( F_3 : (\mathbb{R}^n, D_{J_n}) \to (\mathbb{R}^n, D_{J_n}) \) be given by

\[
F_3(x) = (a_0 I_n + a_1 N + \cdots + a_{n-1} N^{n-1}) x,
\]

where \( a_0 \neq 0, a_1, \cdots, a_{n-1} \in \mathbb{R} \) are constants. Then \( F_3 \) is \( L \)-biLipschitz for some \( L \) depending only on \( n \) and \( a_0, a_1, \cdots, a_{n-1} \).

**Proof.** The proof is similar to that of Lemma 7.2. Let \( x, x' \in \mathbb{R}^n \) be arbitrary. Assume \( D_{J_n}(x, x') = e^t \) and \( D_{J_n}(F_3(x), F_3(x')) = e^s \). Then we have \( e^t = |e^{-tN}(x' - x)| \) and \( e^s = |e^{-sN}(F_3(x') - F_3(x))| \). We need to find a constant \( a \) that depends only on \( n \) and the numbers \( a_0, \cdots, a_{n-1} \) such that \( |s-t| \leq a \).

Set \( B_1 = e^{(t-s)N} \) and \( B_2 = a_0 I_n + a_1 N + \cdots + a_{n-1} N^{n-1} \). Notice that \( B_2 \) commutes with \( N \). We have

\[
e^s = |e^{-sN}(F_3(x') - F_3(x))| = |e^{(t-s)N} e^{-tN} B_2 (x' - x)|
\]

\[
= |B_1 B_2 e^{-tN}(x' - x)| \leq ||B_1|| \cdot ||B_2|| \cdot |e^{-tN}(x' - x)|
\]

\[
\leq \sqrt{Q(B_1)} \sqrt{Q(B_2)} e^t.
\]

Hence \( e^{s-t} \leq \sqrt{Q(B_1)Q(B_2)} \). Since \( Q(B_1)Q(B_2) \) is a polynomial in \( t-s \) that depends only on \( n \) and the numbers \( a_0, \cdots, a_{n-1} \), there is some constant \( a > 0 \) depending only on \( n \) and \( a_0, \cdots, a_{n-1} \) such that \( s-t \leq a \).

Notice that \( F_3^{-1}(x) = B_2^{-1} x \). Set

\[
\beta = - \left( \frac{a_1}{a_0} N + \cdots + \frac{a_{n-1}}{a_0} N^{n-1} \right)
\]

Then \( \beta a = 0 \). We have \( B_2 = a_0(I - \beta) \) and \( B_2^{-1} = a_0^{-1}(I + \beta + \beta^2 + \cdots + \beta^{n-1}) \). It follows that \( B_2^{-1} \) has the expression \( B_2^{-1} = a_0^{-1} I + b_1 N + \cdots + b_{n-2} N^{n-2} + b_{n-1} N^{n-1} \), where \( b_1, \cdots, b_{n-1} \) are constants depending only on \( a_0, \cdots, a_{n-1} \). Now the preceding paragraph implies that \( t-s \leq a' \) for some constant \( a' \) depending only on \( n \) and \( a_0^{-1}, b_1, \cdots, b_{n-1} \), hence only on \( n \) and \( a_0, \cdots, a_{n-1} \). Therefore \( |s-t| \leq \max\{a, a'\} \), and the proof of Lemma 7.3 is complete.

\[ \square \]

To prove that every quasismetric map has the described type, we induct on \( n \). The basic step \( n = 2 \) is given by Theorem 3.6. Now we assume \( n \geq 3 \) and that Theorem 7.1 holds for \( J_{n-1} \).

Let \( F : (\mathbb{R}^n, D_{J_n}) \to (\mathbb{R}^n, D_{J_n}) \) be a quasismetric map. Let \( F_i \) \( (i = 1, \cdots, n-1) \) be the foliation of \( \mathbb{R}^n \) consisting of affine subspaces parallel to the linear subspace

\[
H_i := \{ x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n : x_i+1 = \cdots = x_n = 0 \}.
\]

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Then the proof of Theorem 1.2 shows that the foliation $\mathcal{F}_i$ is preserved by $F$. To be more precise, if $H$ is an affine subspace parallel to $H_i$, then $F(H)$ is also an affine subspace parallel to $H_i$. In particular, $F$ maps every line parallel to the $x_1$-axis (that is, parallel to $H_1$) to a line parallel to the $x_1$-axis, and maps every horizontal hyperplane (that is, parallel to $H_{n-1}$) to a horizontal hyperplane. It follows that there is a map $G : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that for any $y \in \mathbb{R}^{n-1}$, $F(\mathbb{R} \times \{y\}) = \mathbb{R} \times \{G(y)\}$. For each $y \in \mathbb{R}^{n-1}$, there is a map $H(\cdot, y) : \mathbb{R} \to \mathbb{R}$ such that $F(x_1, y) = (H(x_1, y), G(y))$.

Arguments similar to the proofs of Lemmas 3.3 and 3.4 show the following:

1. For each $y \in \mathbb{R}^{n-1}$, the restriction of $D_{J_n}$ to $\mathbb{R} \times \{y\}$ agrees with the Euclidean distance on $\mathbb{R}$;
2. For any two $y_1, y_2 \in \mathbb{R}^{n-1}$, the Hausdorff distance with respect to $D_{J_n}$: $HD(\mathbb{R} \times \{y_1\}, \mathbb{R} \times \{y_2\}) = D_{J_{n-1}}(y_1, y_2)$;
3. For any $p = (x_1, y_1) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and any $y_2 \in \mathbb{R}^{n-1}$, we have $D_{J_n}(p, \mathbb{R} \times \{y_2\}) = D_{J_{n-1}}(y_1, y_2)$.

Hence each $H(\cdot, y) : (\mathbb{R}, | \cdot |) \to (\mathbb{R}, | \cdot |)$ is quasisymmetric, and the arguments on page 10 of [X] shows that $G : (\mathbb{R}^{n-1}, D_{J_{n-1}}) \to (\mathbb{R}^{n-1}, D_{J_{n-1}})$ is also quasisymmetric.

Now the induction hypothesis applied to $G$ shows that there are constants $a_0 \neq 0, a_1, \ldots, a_{n-3}$, $b_i$ ($2 \leq i \leq n$) and a Lipschitz map $g : \mathbb{R} \to \mathbb{R}$ such that

$$G \left( \begin{array}{c} x_2 \\ x_3 \\ \vdots \\ x_n \end{array} \right) = \left( \begin{array}{c} a_0 x_2 + a_1 x_3 + \cdots + a_{n-3} x_{n-1} + b_2 + g(x_n) \\ a_0 x_3 + a_1 x_4 + \cdots + a_{n-3} x_n + b_3 \\ \vdots \\ a_0 x_{n-1} + a_1 x_n + b_{n-1} \\ a_0 x_n + b_n \end{array} \right).$$

Notice that the horizontal hyperplane $\mathbb{R}^{n-1} \times \{x_n\}$ at height $x_n$ is mapped by $F$ to the horizontal hyperplane $\mathbb{R}^{n-1} \times \{a_0 x_n + b_n\}$ at height $a_0 x_n + b_n$. Since the restriction of $D_{J_n}$ to a horizontal hyperplane agrees with $D_{J_{n-1}}$ (Lemma 3.3), the map

$$F : (\mathbb{R}^{n-1} \times \{x_n\}, D_{J_{n-1}}) \to (\mathbb{R}^{n-1} \times \{a_0 x_n + b_n\}, D_{J_{n-1}})$$

is quasisymmetric. Now the induction hypothesis, the fact $F(x_1, y) = (H(x_1, y), G(y))$ and the expression of $G$ imply that

$$H(x_1, y) = a_0 x_1 + a_1 x_2 + \cdots + a_{n-3} x_{n-2} + c_1(x_n) + c_2(x_{n-1}, x_n),$$

where $c_1 : \mathbb{R} \to \mathbb{R}$ and $c_2 : \mathbb{R}^2 \to \mathbb{R}$ are two maps and for each fixed $v$, $c_2(u, v)$ is Lipschitz in $u$. Since $F$ is a homeomorphism, $c_1$ and $c_2$ are continuous. Define $c_3 : \mathbb{R}^2 \to \mathbb{R}$ by $c_3(u, v) = c_1(v) + c_2(u, v)$. After composing $F$ with a map of the described type, we may assume $F$ has the following form

$$F(x_1, x_2, \cdots, x_n) = (x_1 + c_3(x_{n-1}, x_n), x_2 + g(x_n), x_3, \cdots, x_n).$$

We need to show that there are constants $a_{n-2}$, $b_2$ and a Lipschitz map $C : \mathbb{R} \to \mathbb{R}$ such that $g(x_n) = a_{n-2} x_n + b_2$ and $c_3(x_{n-1}, x_n) = a_{n-2} x_{n-1} + C(x_n)$.

**Lemma 7.4.** There is a constant $L$ such that the following holds for all $u, v, v' \in \mathbb{R}$:

$$\left| \left(c_3(u + (v'-v) \ln |v'-v|, v') - c_3(u, v)\right) - \ln |v'-v| \left\{ g(v') - g(v) \right\} \right| \leq L |v'-v|.$$
we write

Notice also that $t$ is the smallest solution for $e^t = |e^{-tN}(x' - x)|$ and so $D_{J^n}(x, x') = e^t$. Suppose $D_{J^n}(F(x), F(x')) = e^s$. Then $e^s = |e^{-sN}(F(x') - F(x))|$. By Theorem 1.2, $F$ is $L_1$-biLipschitz for some $L_1 \geq 1$. Hence $e^t/L_1 \leq e^s \leq L_1e^t$. It follows that $|t - s| \leq \ln L_1$. Now we write

$$e^{-sN}(F(x') - F(x)) = e^{-sN}(x' - x) + e^{-sN}$$

$$= e^{(t-s)N}e^{-tN}(x' - x) + e^{(t-s)N}e^{-tN}$$

$$= e^{(t-s)N}$$

where

$$
\begin{pmatrix}
  c_3(x_{n-1}', x_n') - c_3(x_{n-1}, x_n) \\
  g(x_n') - g(x_n) \\
  \vdots \\
  \vdots \\
  \vdots \\
  0
\end{pmatrix}
$$

$$= e^{(t-s)N}$$

Set

$$\tau = \{c_3(x_{n-1}', x_n') - c_3(x_{n-1}, x_n)\} - t\{g(x_n') - g(x_n)\}.$$ 

The first entry of $e^{-sN}(F(x') - F(x))$ is

$$q := \tau + (t-s)\{g(x_n') - g(x_n)\} + \frac{(t-s)^{n-1}}{(n-1)!} (x_n' - x_n).$$

We have

$$|q| \leq |e^{-sN}(F(x') - F(x))| = e^s \leq L_1e^t = L_1|v' - v|.$$ 

Recall that $g$ is $L_2$-Lipschitz for some $L_2 \geq 0$. Hence,

$$|g(x_n') - g(x_n)| \leq L_2|x_n' - x_n| = L_2|v' - v|.$$
Now it follows from $|t - s| \leq \ln L_1$ and the triangle inequality that

$$|\tau| \leq \left( L_1 + L_2 \ln L_1 + \frac{(\ln L_1)^{n-1}}{(n-1)!} \right) |v' - v|.$$

Recall that the map $g$ is Lipschitz and for each fixed $v$, $c_3(u, v)$ is Lipschitz in $u$. Hence $g$ is differentiable a.e., and for each fixed $v$, the partial derivative $\frac{\partial c_3}{\partial u}$ exists for a.e. $u$.

**Lemma 7.5.** Let $v$ be any point such that $g'(v)$ exists. Then $c_3(u, v) = c_3(0, v) + g'(v)u$ for all $u$.

**Proof.** Fix an arbitrary $u \in \mathbb{R}$. Let $a > 0$. For any positive integer $n$, define $(y_0, z_0) = (u, v)$ and $(y_i, z_i) = (u + \frac{a}{n} \ln \frac{a}{n}, v + \frac{a}{n})$ ($1 \leq i \leq n$). Applying Lemma 7.4 to $y_i-1, z_i-1, z_i$ we obtain:

$$\left| \{c_3(y_i, z_i) - c_3(y_{i-1}, z_{i-1})\} - \frac{L}{n} \{g(z_i) - g(z_{i-1})\} \right| \leq L \frac{a}{n}.$$

Now let $k = k(n)$ be the integer part of $n/\ln \frac{a}{n}$. Then $k \ln \frac{a}{n} \to -1$ as $n \to \infty$. Combining the above inequalities for $1 \leq i \leq k$ and using the triangle inequality, we obtain

$$\left| \{c_3(y_k, z_k) - c_3(u, v)\} - \frac{L}{\ln \frac{a}{n}} \{g(z_k) - g(v)\} \right| \leq L \frac{ak}{n}.$$

Now divide both sides by $\frac{ak}{n} \ln \frac{a}{n}$ (which converges to $a$ as $n \to \infty$), we get:

$$\left| \frac{c_3(y_k, z_k)}{\frac{ak}{n} \ln \frac{a}{n}} - \frac{c_3(u, v)}{\frac{ak}{n}} \right| + \frac{g(z_k) - g(v)}{\frac{ak}{n}} \leq \frac{L}{\ln \frac{a}{n}}.$$

As $n \to \infty$, we have $z_k = v + \frac{ak}{n} \to v$, $y_k \to u - a$. Also, since $g'(v)$ exists, we have

$$\frac{g(z_k) - g(v)}{\frac{ak}{n}} \to g'(v).$$

Consequently,

$$\frac{c_3(u-a, v) - c_3(u, v)}{a} + g'(v) = 0.$$

Hence $c_3(u-a, v) - c_3(u, v) = -ag'(v)$ for all $u \in \mathbb{R}$ and all $a > 0$. It follows that $c_3(u, v) = c_3(0, v) + g'(v)u$ for all $u$.

**Lemma 7.6.** Suppose $g$ is differentiable at $v_1$ and $v_2$. Then $g'(v_1) = g'(v_2)$.

**Proof.** By Lemma 7.5 we have $c_3(u, v_1) = c_3(0, v_1) + ug'(v_1)$ and

$$c_3(u + [v_2 - v_1] \ln |v_2 - v_1|, v_2) = c_3(0, v_2) + (u + [v_2 - v_1] \ln |v_2 - v_1|)g'(v_2)$$

for all $u$. Now Lemma 7.4 applied to $u, v_1, v_2$ implies that $|u(g'(v_2) - g'(v_1))| \leq C$ holds for all $u$, where $C$ is a quantity independent of $u$. It follows that $g'(v_2) - g'(v_1) = 0$. 

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Completing the proof of Theorem 7.1. Lemma 7.6 implies that $g$ is an affine function and hence there are constants $a, b$ such that $g(v) = av + b$. It now follows from Lemma 7.5 that for any $v$ we have $c_3(u, v) = c_3(0, v) + au$. To finish the proof of Theorem 7.1 it remains to show that $c_3(0, v)$ is Lipschitz in $v$. This follows immediately from Lemma 7.4 after plugging in the formulas for $g$ and $c_3$.

Now the proof of Theorem 7.1 is complete.

\[ \square \]

8 A Liouville type theorem

In this section we prove a Liouville type theorem for $G_A$ in the case when $A$ is a Jordan block: every conformal map of the ideal boundary of $G_A$ extends to an isometry of $G_A$.

Let $X$ and $Y$ be quasimetric spaces with finite Hausdorff dimension. Denote by $H_X$ and $H_Y$ their Hausdorff dimensions and by $\mathcal{H}_X$ and $\mathcal{H}_Y$ their Hausdorff measures. We say a quasimetric map $f : X \to Y$ is conformal if:

1. $L_f(x) = l_f(x) \in (0, \infty)$ for $\mathcal{H}_X$-almost every $x \in X$;
2. $L_f^{-1}(y) = l_f^{-1}(y) \in (0, \infty)$ for $\mathcal{H}_Y$-almost every $y \in Y$.

We now describe some isometries of $G_A$. For any $g = (x, t) \in G_A = \mathbb{R}^n \times \mathbb{R}$, the Lie group left translation $L_g$ is an isometry. If $g = (x, 0)$, then the boundary map $\partial L_g : \mathbb{R}^n \to \mathbb{R}^n$ of $L_g$ is translation by $x$. If $g = (0, t)$, then the boundary map of $L_g$ is the similarity $e^{t \mathbf{A}}$. Let $\tau' : G_A \to G_A$ be given by $\tau'(x, t) = (-x, t)$. Then $\tau'$ is an isometry, and its boundary map is $\tau : \mathbb{R}^n \to \mathbb{R}^n$, $\tau(x) = -x$.

**Theorem 8.1.** Let $J_n$ be the $n \times n$ ($n \geq 2$) Jordan matrix with eigenvalue 1. Then every conformal map $F : (\mathbb{R}^n, D_{J_n}) \to (\mathbb{R}^n, D_{J_n})$ is the boundary map of an isometry $G_{J_n} \to G_{J_n}$.

**Proof.** Let $F : (\mathbb{R}^n, D_{J_n}) \to (\mathbb{R}^n, D_{J_n})$ be a quasimetric map. After composing with the boundary maps of isometries described above, we may assume $F$ has the following form

$$F(x) = (I + a_1N + \cdots + a_{n-2}N^{n-2})x + (C(x_n), 0, \cdots, 0)^T,$$

where $C : \mathbb{R} \to \mathbb{R}$ is Lipschitz. We will prove the following statement by inducting on $n$:

If $F$ as above is conformal, then $a_1 = \cdots = a_{n-2} = 0$ and $C$ is constant.

The basic step $n = 2$ is Theorem 6.3 in [X]. Now we assume $n \geq 3$ and that the statement holds for Jordan matrices with sizes $\leq n - 1$. Notice that $F$ maps every horizontal hyperplane $H(x_n) := \mathbb{R}^{n-1} \times \{x_n\}$ to itself. By Lemma 7.3, the restriction of $D_{J_n}$ on $H(x_n)$ agrees with the metric $D_{J_{n-1}}$. It now follows from Fubini’s theorem that for a.e. $x_n \in \mathbb{R}$, the restricted map

$$F|_{H(x_n)} : (H(x_n), D_{J_{n-1}}) \to (H(x_n), D_{J_{n-1}})$$

is also conformal. Now the induction hypothesis applied to $F|_{H(x_n)}$ implies that $a_i = 0$ for $1 \leq i \leq n - 2$. It remains to show $C$ is constant.

Suppose $C$ is not constant. Then there is some $u \in \mathbb{R}$ such that $C'(u) \neq 0$ and $L_F(p) = l_F(p)$ for some $p \in H(u)$. After pre-composing and post-composing with Euclidean
translations, we may assume \( u = 0, \) \( C(0) = 0 \) and \( p \) is the origin \( o \). Notice that the restriction of \( F \) to the \( x_1 \)-axis is the identity, so \( L_F(o) = l_F(o) = 1 \). Now for any \( x_n > 0 \), choose \( x_1, \cdots, x_{n-1} \) such that \( x = (x_1, \cdots, x_n)^T \) satisfies \( e^{-tN} x = (0, \cdots, 0, x_n)^T \), where \( t = \ln x_n \). It follows that \( D_{J_n}(o, x) = e^t = x_n \). Suppose \( D_{J_n}(F(o), F(x)) = e^s \). Then \( e^s = |e^{-sN} F(x)| \).

We calculate as before that

\[
e^{-sN} F(x) = \left( C(x_n) + \frac{(t-s)^{n-1}}{(n-1)!} x_n, \frac{(t-s)^{n-2}}{(n-2)!} x_n, \cdots, (t-s), 1 \right)^T.
\]

Since \( L_F(o) = l_F(o) = 1 \), we must have \( \frac{e^s}{e^t} = \frac{D_{J_n}(F(x), F(o))}{D_{J_n}(x, o)} \) as \( x_n \to 0 \) and hence \( t-s \to 0 \). Now

\[
e^s = |e^{-sN} F(x)| = x_n \left| \left( C(x_n) + \frac{(t-s)^{n-1}}{(n-1)!} x_n, \frac{(t-s)^{n-2}}{(n-2)!} x_n, \cdots, (t-s), 1 \right) \right|^T.
\]

Since \( x_n = e^t \), we have

\[
e^{s-t} = \left| \left( C(x_n) + \frac{(t-s)^{n-1}}{(n-1)!} x_n, \frac{(t-s)^{n-2}}{(n-2)!} x_n, \cdots, (t-s), 1 \right) \right|^T.
\]

Now as \( x_n \to 0 \), the right hand side converges to

\[
\left| \left( C'(0), 0, \cdots, 0, 1 \right) \right|^2 = \sqrt{1 + (C'(0))^2},
\]

which is > 1 since \( C'(0) \neq 0 \). However, the left hand side converges to 1. The contradiction shows \( C \) must be a constant function.

\[\square\]

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