Coordinate representation of the Lagrange-Poincaré equations for a mechanical system with symmetry on the total space of a principal fiber bundle whose base is the bundle space of the associated bundle

S. N. Storchak*
NRC “Kurchatov Institute” – IHEP, Protvino, Moscow Region, 142281, Russia
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Abstract

Using the dependent coordinates, the local Lagrange-Poincaré equations and equations for the relative equilibria are obtained for a mechanical system with a symmetry describing the motion of two interacting scalar particles on a special Riemannian manifold (the product of the total space of the principal fiber bundle and the vector space) on which a free proper and isometric action of a compact semi-simple Lie group is given. As in gauge theories, dependent coordinates are implicitly determined by means of equations representing the local sections of the principal fiber bundle.

1 Introduction

This note is a continuation of our previous work in which we obtained, in general form, the local Lagrange-Poincaré equations (reduced Lagrange-Euler equations) for a finite-dimensional mechanical system describing the two scalar particles with interaction moving along a special Riemannian manifold represented by the product of two manifolds $\mathcal{P} \times \mathcal{V}$, on which an action

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*E-mail adress: storchak@ihep.ru
of a compact semi-simple group Lie $\mathcal{G}$ is given. The manifold $\mathcal{P}$, is a total space of the principal fiber bundle, and $V$ is a finite-dimensional vector space considered as manifold with an arbitrary constant metric.

The smooth, free, proper, and isometric action action of a group $\mathcal{G}$ leads to the principal fiber bundle $\mathcal{P} \times V \to \mathcal{P} \times_{\mathcal{G}} V$. And, therefore, the original mechanical system can be reduced to the corresponding system given on the orbit space $\mathcal{P} \times_{\mathcal{G}} V$ of this bundle.

In [1] our goal was to obtain the local description of the evolution in the same way as it is done in gauge field theories, i.e., using dependent variables. In gauge theories, these variables, implicitly defined, must satisfy additional constraints (gauges) imposed on the gauge fields. These constraints, given by equations, determine the local gauge surface which is the local cross-section in the principal fiber bundle associated with the dynamical system. Using this surface, one can introduce the local adapted coordinates in the principal fiber bundle. They are given by dependent variables and group variables.

To introduce the adapted coordinates in the principal fiber bundle associated with our mechanical system, we used a local cross-section (the local "gauge surface") of that principal bundle for which the total space is a manifold $\mathcal{P}$. How this can be done was shown in our previous work, where, using the Poincaré variational principle, we obtained the Lagrange-Poincaré equations in general form.

In the present notes, our goal is to get the coordinate representation for these equations. But before proceeding to this, we first briefly recall the main points of our consideration performed in our previous work.

2 Coordinates on the configuration space

As local coordinates of a point $(p, v)$ given on our configuration space, the manifold $\mathcal{P} \times V$, we take $(Q^A, f^n)$, $A = 1, \ldots, N_P$ and $n = 1, \ldots, N_V$ such that $Q^A = \varphi^A(p)$, and $f^n = \varphi^n(v)$, where $(\varphi^A, \varphi^n)$ are the coordinate functions of a chart on the original product manifold.

In these coordinates, the Riemannian metric of the manifold is written as follows:

$$ ds^2 = G_{AB}(Q)dQ^A dQ^B + G_{mn} df^m df^n. \tag{1} $$

The matrix $G_{mn}$ consists of some fixed constant elements.

The right action of the group $\mathcal{G}$, $(p, v)g = (pg, g^{-1}v)$, is written as

$$ \tilde{Q}^A = F^A(Q, g), \quad \tilde{f}^n = \tilde{D}^n_m(g) f^m. $$

Here $\tilde{D}^n_m(g) \equiv D^m_n(g^{-1})$, and $D^m_n(g)$ is the matrix of the finite-dimensional representation of the group $\mathcal{G}$ acting on the vector space $V$. 

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Due to isometry, we have two relations for the metric tensors:

$$G_{AB}(Q) = G_{DC}(F(Q, g)) F^D_A(Q, g) F^C_B(Q, g),$$  \hspace{1cm} (2)

with $F^B_A(Q, g) \equiv \frac{\partial F^a_a(Q, g)}{\partial Q^A}$, and

$$G_{pq} = G_{mn} \bar{D}^m_p(g) \bar{D}^n_q(g).$$  \hspace{1cm} (3)

The Killing vector fields are defined as

$$K^A_{\mu}(Q, g) \partial_{\partial Q^A},$$  \hspace{1cm} (4)

with $K^A_{\mu}(Q, g) \equiv \partial_{\tilde{Q}^A_{\mu}} \big|_{a=e} = \partial_{\bar{D}^A_{\mu}(Q, g)} \big|_{a=e} = (\bar{J}_{\alpha})_{\mu} f^m$. The generators $\bar{J}_{\alpha}$ of the representation $\bar{D}^n_m(a)$ have the following commutation relation: $[\bar{J}_{\alpha}, \bar{J}_{\beta}] = \tilde{c}^\gamma_{\alpha\beta} \bar{J}_{\gamma}$, where the structure constants $\tilde{c}^\gamma_{\alpha\beta} = -c^\gamma_{\alpha\beta}$.

We use the (condensed) notation by which the capital Latin letter with tilde represents two subscripts (or superscripts) that are related with two spaces: $\tilde{A} \equiv (A, p)$. For components of the Killing vector fields, for example, we have

$$K^\tilde{A}_{\mu} = (K^A_{\mu}, K^p_{\mu}).$$

From the general theory \cite{2} it follows that in our case we can regard the original manifold as a total space of the principal fiber bundle

$$\pi' : \mathcal{P} \times V \to \mathcal{P} \times G V,$$

where $\pi' : (p, v) \to [p, v]$, and $[p, v]$ is the equivalence class formed by the equivalence relation $(p, v) \sim (pg, g^{-1}v)$.) This allows us to introduce new coordinates on $\mathcal{P} \times V$ that are related with this principal fiber bundle. Moreover, one can express the coordinates $(Q^A, f^n)$ of the point $(p, v)$ in terms of the bundle coordinates by the well-known procedure \cite{3–10}.

In this procedure, the bundle coordinates are introduced with the help of the local section $\tilde{\sigma}_i$ of the bundle, $\pi' \cdot \tilde{\sigma}_i = \text{id}$, sending the point $[p, v]$ to some element $(\tilde{p}, \tilde{v}) \in \mathcal{P} \times V$. $\tilde{\sigma}_i$ is defined as follows:

$$\tilde{\sigma}_i([p, v]) = (\sigma_i(x), a(p)v),$$

where $\sigma_i$ is a local section of the principal fiber bundle $P(\mathcal{M}, \mathcal{G})$ with the base space $\mathcal{M} = \mathcal{P}/\mathcal{G}$, $\sigma_i : U_i \to \pi_p^{-1}(U_i)$, $x = \pi_p(p)$ and $a(p)$ is the group element defined by $p = \sigma_i(x)a(p)$. Also note that due to

$$(\sigma_i(x), a(p)v) = (pa^{-1}(p), a(p)v) = (p, v)a^{-1}(p),$$

we have

$$\tilde{\sigma}_i([p, v]) = (p, v)a^{-1}(p).$$

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The local sections $\sigma_i$ of the principal fiber bundle $P(M, G)$ in a neighborhood of a point $p \in P$ can be determined by local submanifolds $\Sigma_i$ which have the transversal intersections with the orbits. The section $\sigma_i$ is the map $\sigma_i: U_i \to \Sigma_i$ such that $\pi_{\Sigma_i} \cdot \sigma_i = \text{id}_{U_i}$. In its turn, these submanifolds are given by the equations $\{\chi^\alpha(Q) = 0, \alpha = 1, \ldots, N_G\}$.

The coordinates of the points on the local submanifold $\Sigma_i$ will be denoted by $Q^*A$. Since they satisfy the equations $\{\chi^\alpha(Q^*) = 0\}$, they are called the dependent coordinates. Any point $p$ on the total space $P$ of the principal fiber bundle $P(M, G)$ must have, in addition, a group coordinates $a^\alpha$.

A local isomorphism between trivial principal bundle $\Sigma_i \times G \to \Sigma_i$ and $P(M, G)$ is given in coordinates as

$$\varphi_i: \Sigma_i \times G \to \pi^{-1}(U_i)$$

where $Q^*B$ are the coordinates of a point given on the local surface $\Sigma_i$ and $a^\alpha$ – the coordinates of an arbitrary group element $a$. This element carries the point, taken on $\Sigma_i$, to the point $p \in P$ which has the coordinates $Q^A$.

An inverse map $\varphi_i^{-1}$,

$$\varphi_i^{-1}: \pi^{-1}(U_i) \to \Sigma_i \times G,$$

has the following coordinate representation:

$$\varphi_i^{-1}: Q^A \to (Q^*B(Q), a^\alpha(Q)).$$

Here the group coordinates $a^\alpha(Q)$ of a point $p$ are the coordinates of the group element which connects, by means of its action on $p$, the surface $\Sigma_i$ and the point $p \in P$. These group coordinates are given by the solutions of the following equation:

$$\chi^\beta(F^A(Q, a^{-1}(Q))) = 0.$$  \hspace{1cm} (4)

The coordinates $Q^*B$ are defined by the equation

$$Q^*B = F^B(Q, a^{-1}(Q)).$$ \hspace{1cm} (5)

In the same way as for the principal bundle $P(M, G)$, there exist a local isomorphisms of the principal fiber bundle $P(P \times G, V, G)$ and the trivial principal bundles $\tilde{\Sigma}_i \times G \to \tilde{\Sigma}_i$, where now the local surfaces $\tilde{\Sigma}_i$ are the images of the sections $\tilde{\sigma}_i$. 

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Therefore, we can introduce a new atlas on $\mathcal{P}(\mathcal{P} \times_{\mathcal{G}} V)$. In this atlas, the coordinate functions of the charts $(\tilde{U}_i, \tilde{\varphi}_i)$, where $\tilde{U}_i$ is an open neighborhood of the point $[p, v]$ given on the base space $\mathcal{P} \times_{\mathcal{G}} V$, are such that

$$\tilde{\varphi}_i^{-1} : \pi^{-1}(\tilde{U}_i) \to \tilde{\Sigma}_i \times \mathcal{G}, \text{ or in coordinates,}$$

$$\tilde{\varphi}_i^{-1} : (Q^A, f^m) \to (Q^{*A}(Q), \tilde{f}^n(Q), a^\alpha(Q)).$$

Here $Q^A$ and $f^m$ are the coordinates of a point $(p, v) \in \mathcal{P} \times V$, $Q^{*A}(Q)$ is given by (5) and

$$a(Q) \text{ is defined by (4), and we have used the following property: } D^m_n(a^{-1}) \equiv D^m_n(a).$$

The coordinates $Q^{*A}$, representing a point given on a local surface $\Sigma_i$, satisfy the constraints: $\chi(Q^*) = 0$. That is, they are dependent coordinates.

The coordinate function $\tilde{\varphi}_i$ maps $\tilde{\Sigma}_i \times \mathcal{G} \to \pi^{-1}(\tilde{U}_i)$:

$$\tilde{\varphi}_i : (Q^{*B}, \tilde{f}^n, a^\alpha) \to (F^A(Q^*, a), \bar{D}_n^m(a)\tilde{f}^n).$$

Thus, we have defined the special local bundle coordinates $(Q^{*A}, \tilde{f}^n, a^\alpha)$, also named as adapted coordinates, on the principal fiber bundle $\pi : \mathcal{P} \times V \to \mathcal{P} \times_{\mathcal{G}} V$.

It is not difficult to obtain the representation for the Riemannian metric given on $\mathcal{P} \times V$ in terms of the principal bundle coordinates $(Q^{*A}, \tilde{f}^n, a^\alpha)$. The replacement of the coordinates $(Q^A, f^m)$ of a point $(p, v) \in \mathcal{P} \times V$ for new coordinates

$$Q^A = F^A(Q^{*B}, a^\alpha), \quad f^m = \bar{D}_n^m(a)\tilde{f}^n$$

leads to the following transformation of the local coordinate vector fields:

$$\frac{\partial}{\partial f^m} = D^m_n(a)\frac{\partial}{\partial \tilde{f}^n},$$

$$\frac{\partial}{\partial Q^B} = \frac{\partial Q^{*A}}{\partial Q^B} \frac{\partial}{\partial Q^{*A}} + \frac{\partial a^\alpha}{\partial Q^B} \frac{\partial}{\partial a^\alpha} + \frac{\partial \tilde{f}^n}{\partial Q^B} \frac{\partial}{\partial \tilde{f}^n}$$

$$= \hat{F}_B^C \left( N^A_C(Q^*) \frac{\partial}{\partial Q^{*A}} + \chi^\mu_C(\Phi^{-1})_\mu^\nu(a)\frac{\partial}{\partial a^\nu} - \chi^\mu_C(\Phi^{-1})_\mu^\nu(\hat{J}_\nu)_P^m \frac{\partial}{\partial \tilde{f}^m} \right). (7)$$

Here $\hat{F}_B^C \equiv F_B^C(F(Q^*, a), a^{-1})$ is an inverse matrix to the matrix $F_B^A(Q^*, a)$, $\chi^\mu_C \equiv \frac{\partial \chi^\mu_C(Q^*)}{\partial Q^*} |_{Q=Q^*}$, $(\Phi^{-1})_\mu^\beta(\Phi^*) \equiv (\Phi^{-1})_\mu^\beta(Q^*)$ – the matrix which is inverse to the Faddeev–Popov matrix:

$$(\Phi)^\beta_\mu(Q) = K^A_\mu(Q) \frac{\partial \chi^\beta(Q)}{\partial Q^A}.$$
the matrix $\bar{v}_\beta^\alpha(a)$ is inverse of the matrix $\bar{u}_\beta^\alpha(a)$.\footnote{\text{det} $\bar{v}_\beta^\alpha(a)$ is the density of the right-invariant measure given on the group $G$.}

The operator $N^A_C$, defined as

$$N^A_C(Q) = \delta^A_C - K^A_C(Q)(\Phi^{-1})^\alpha_\beta(Q)\chi_C^\alpha(Q),$$

is the projection operator $(N^A_B N^B_C = N^A_C)$ onto the subspace which is orthogonal to the Killing vector field $K^A_C(Q)\frac{\partial}{\partial Q^\beta}$. $N^A_C(Q^*)$ is the restriction of $N^A_C(Q)$ to the submanifold $\Sigma$:

$$N^A_C(Q^*) \equiv N^A_C(F(Q^*, e)) \quad N^A_C(Q^*) = F_C^B(Q^*, a)N^M_B(F(Q^*, a))F^A_M(Q^*, a)$$

e is the unity element of the group.

We note also that formula (7) is a generalization of an analogous formula from \[5, 7\].

The vector field \(\partial/\partial Q^A\) is determined as an operator using the following rule:\footnote{It can be shown that this rule follows from the approaches developed in \[3\] and \[12\].}

$$\frac{\partial}{\partial Q^A}\varphi(Q^*) = (P_\bot)_A^B(Q^*)\frac{\partial \varphi(Q)}{\partial Q^B}\Bigg|_{Q=Q^*},$$

where the projection operator $(P_\bot)_A^B$ on the tangent plane to the submanifold $\Sigma$ is given by

$$(P_\bot)_B^A = \delta_B^A - \chi_B^\alpha (\chi^{\top})^{-1}_\alpha \chi^{\top}_A.$$

In this formula, $(\chi^{\top})^A_\beta$ is a transposed matrix to the matrix $\chi^\beta_B$:

$$(\chi^{\top})^A_\mu = G^{AB} \gamma_{\mu} \chi_B^\nu \quad \gamma_{\mu\nu} = K^A_B G_{AB} K^B_\nu.$$ Using the above explicit expression for the projection operators, it is easy to derive their multiplication properties:

$$(P_\bot)_B^A N^C_C = (P_\bot)_B^C, \quad N^A_C(P_\bot)_A^B = N^C_C.$$  

In a new coordinate basis $(\partial/\partial Q^A, \partial/\partial \tilde{f}^m, \partial/\partial a^\alpha)$, the metric (11) of the original manifold $\mathcal{P} \times V$ is represented by means of the following tensor:

$$\tilde{G}_{AB}(Q^*, \tilde{f}, a) = \begin{pmatrix}
G_{CD}(P_\bot)_A^D(P_\bot)_B^D & 0 & G_{CD}(P_\bot)_A^C K_D^D \bar{u}_\alpha^\nu \\
G_{BC} K^D_\mu \bar{u}_\beta^\nu & G_{mn} & G_{mn} K^D_\mu \bar{u}_\alpha^\nu \\
0 & G_{mp} K^D_\nu \bar{u}_\beta^\nu & \delta_{\mu\nu} \bar{u}_\alpha^\mu \bar{u}_\beta^\nu
\end{pmatrix}$$ (8)

where $G_{CD}(Q^*) \equiv G_{CD}(F(Q^*, e))$:

$$G_{CD}(Q^*) = F_C^M(Q^*, a) F_D^N(Q^*, a) G_{MN}(F(Q^*, a)),$$
the projection operators $P_\perp$ and the components $K^A_\mu$ of the Killing vector fields depend on $Q^*, \tilde{u}_\beta^\alpha(a)$, $K^a_\mu = K^a_\mu(\tilde{f})$, $d_{\mu\nu}(Q^*, \tilde{f})\tilde{u}_\beta^\alpha(a)\tilde{u}_\gamma^\nu(a)$ is the metric on $G$-orbit through the point $(p, v)$. The components $d_{\mu\nu}$ of this metric are given by

$$
\begin{align*}
d_{\mu\nu}(Q^*, \tilde{f}) &= K^A_\mu(Q^*)G_{AB}(Q^*)K^B_\nu(Q^*) + K^m_\mu(\tilde{f})G_{mn}K^n_\nu(\tilde{f}) \\
&= \gamma_{\mu\nu}(Q^*) + \gamma'_{\mu\nu}(Q^*).
\end{align*}
$$

3 Transformation of the Lagrangian

In terms of initial local coordinates defined on the original manifold $\mathcal{P} \times V$, the Lagrangian for the considered mechanical system can be written as follows:

$$
\mathcal{L} = \frac{1}{2} G_{AB}(Q) \dot{Q}^A \dot{Q}^B + \frac{1}{2} G_{mn} \dot{f}^m \dot{f}^n - V(Q, f).
$$

(9)

By our assumption, the potential $V(Q, f)$ is a $G$-invariant function, that is, $V(Q, f) = V(F(Q, a), \tilde{D}(a)f)$. So the whole Lagrangian is also invariant.

The replacement of the local coordinates (6), which introduces the coordinates $(Q^*A, \tilde{f}^m, a^\alpha)$ on $\mathcal{P} \times V$, transforms the Lagrangian into

$$
\mathcal{L} = \frac{1}{2} G_{CD}(\frac{dQ^C}{dt} + K^C_\mu \tilde{u}_\alpha^\mu(a) \frac{da^\alpha}{dt}) \left( \frac{dQ^D}{dt} + K^D_\nu \tilde{u}_\beta^\nu(a) \frac{da^\beta}{dt} \right) \\
+ \frac{1}{2} G_{mn}(\frac{df^m}{dt} + K^m_\beta \tilde{u}_\alpha^\beta(a) \frac{da^\alpha}{dt}) \left( \frac{df^n}{dt} + K^n_\mu \tilde{u}_\nu^\mu(a) \frac{da^\nu}{dt} \right) - V,
$$

(10)

where now $G_{CD}, K^C_\mu$ depend on $Q^*, K^m_\beta = K^m_\beta(\tilde{f})$, and $V = V(Q^*\tilde{f})$.

The Lagrange-Poincaré equations are obtained with the help of a special coordinate basis (the horizontal lift basis) on the total space of the principal fiber bundle. The new basis consists of the horizontal and vertical vector fields and can be determined by using the “mechanical connection” which exists [2] in case of the reduction of mechanical systems with a symmetry.

The connection one-form $\omega^\alpha$ in the principal fiber bundle $\mathcal{P} \times_{\mathcal{G}} \mathcal{V}, \mathcal{G}$ is given by the following formula written in terms of the initial local coordinates defined on the total space $\mathcal{P} \times V$:

$$
\omega^\alpha(Q, f) = d^{\alpha\beta}(Q, f) (K^B_\beta(Q)G_{BA}(Q)dQ^A + K^p_\beta(f)G_{pq}df^q).
$$

(11)

In coordinates $(Q^*A, \tilde{f}^n, a^\alpha)$, the one-form is written as follows:

$$
\tilde{\omega}^\alpha = \tilde{\rho}_{\alpha}^\beta(a) \left( d^{\alpha\mu}K^D_\mu(Q^*)G_{DA}(Q^*)dQ^A + d^{\alpha\mu}K^p_\mu(\tilde{f})G_{pq}d\tilde{f}^n \right) + \tilde{u}_\alpha^\beta(a)da^\alpha,
$$

\footnote{The one-form $\tilde{\omega}$ with the value in the Lie algebra of the group Lie $\mathcal{G}$ is $\tilde{\omega} = \tilde{\omega}^\alpha \otimes \lambda_\alpha$.}
where now $d^{\alpha'}{}^\mu = d^{\alpha'}{}^\mu(Q^*, \tilde{f})$. And the matrix $\tilde{\rho}_\alpha^\mu(a)$ is inverse to the matrix $\rho_{\alpha}^\beta = \bar{\omega}_{\alpha}^{\nu\mu}(a)$ of the adjoint representation of the group $\mathcal{G}$.

Introducing the (gauge) potentials $\mathcal{A}^\alpha_B$, and $\mathcal{A}^\alpha_m$, together with a new notation: $\tilde{\mathcal{A}}_B^\alpha = \tilde{\rho}_\alpha^\nu(a)\mathcal{A}^\nu_B(Q^*, \tilde{f})$, we come to

$$\tilde{\omega}^\alpha = \tilde{\mathcal{A}}_B^\alpha(Q^*, \tilde{f}, a)dQ^B + \tilde{\mathcal{A}}_m^\alpha(Q^*, \tilde{f}, a)d\tilde{f}^m + u_\beta^\alpha(a)da^\alpha. \quad (12)$$

In term of condensed notations of indices, $\tilde{\omega}^\alpha$ is written as

$$\tilde{\omega}^\alpha = \tilde{\mathcal{A}}_B^\alpha(Q^*, \tilde{f}, a)dQ^B + u_\beta^\alpha(a)da^\alpha.$$

We note that the replacement of the coordinates convert the Killing vector field $K_\alpha(Q, f)$, the vertical vector field, $K_\alpha(Q, f) = K_\alpha^B(Q)\frac{\partial}{\partial Q^B} + K_\alpha^p\frac{\partial}{\partial f^p}$, into the vector field $L_\alpha = \nu_\alpha^\mu(a)\frac{\partial}{\partial a^\mu}$ which is the left-invariant vector field given on the group manifold $\mathcal{G}$.

The horizontal vector fields are defined with the help of the horizontal projection operators. These operators must extract the direction which is normal to the orbit: $\Pi^A_E K^E_\alpha = 0$. They are defined as follows:

$$\Pi^A_B = \delta^A_B - K^A_\alpha d^{\alpha\beta} K^D_\beta G^{DB}.$$

By $\Pi^A_B$, we denote the four component operator:

$$\Pi^A_B = (\Pi^A_B, \Pi^A_m, \Pi^A_A, \Pi^A_n).$$

The components are given by the following formulae:

$$\Pi^A_B = \delta^A_B - K^A_\alpha d^{\alpha\beta} K^D_\beta G^{DB},$$
$$\Pi^A_m = -K^A_\mu d^{\mu\nu} K^p_{\nu} G_{pm},$$
$$\Pi^A_A = -K^A_\mu d^{\mu\nu} K^D_{\nu} G_{DA},$$
$$\Pi^A_n = \delta^m_n - K^m_\mu d^{\mu\nu} K^r_{\nu} G_{rn}.$$

The horizontal vector fields are defined as follows:

$$H_A(Q, f) = \Pi^R_A \frac{\partial}{\partial Q^R} + \Pi^q_A \frac{\partial}{\partial f^q} \quad (13)$$
$$H_p(Q, f) = \Pi^R_p \frac{\partial}{\partial Q^R} + \Pi^m_p \frac{\partial}{\partial f^m}. \quad (14)$$
For $\hat{\omega}^\alpha$ from (11), we have

$$\hat{\omega}^\alpha(H_A) = 0, \quad \hat{\omega}^\alpha(H_p) = 0, \quad \hat{\omega}^\alpha(K_\beta) = \delta^\alpha_\beta.$$ 

Performing the replacement of the coordinate, by means of the formulae (7), in the expressions (13) and (14) that represent the horizontal vector fields, we come to the horizontal vector fields

$$H_M(Q^*, \tilde{f}, a) = \left[ N_M^T \left( \frac{\partial}{\partial Q^T} - \hat{\omega}^\alpha T L_\alpha \right) + N_M^m \left( \frac{\partial}{\partial f^m} - \hat{\omega}^\alpha m L_\alpha \right) \right], \quad \text{(15)}$$

and

$$H_m(Q^*, \tilde{f}, a) = \left( \frac{\partial}{\partial f^m} - \hat{\omega}^\alpha m L_\alpha \right). \quad \text{(16)}$$

In equation (15), we have used the components of the projection operator $N_C^\hat{A}$:

$$N_C^\hat{A} = (N_C^A, N_m^A, N_A^m, N_p^m).$$

$N_C^A$ was defined above. The other components are

$$N_m^A = 0, \quad N_A^m = -K_m^\alpha (\Phi^{-1})_\mu^\alpha \chi_A^\mu = -K_m^\alpha \Lambda_A^\alpha, \quad N_p^m = \delta_p^m.$$ 

The operator $N_B^\hat{A}$ satisfy the following properties:

$$N_B^\hat{A} N_C^\hat{B} = N_C^\hat{C}, \quad \Pi_B^\hat{L} N_C^\hat{L} = N_B^\hat{L}, \quad \Pi_L^\hat{L} N_C^\hat{C} = \Pi_C^\hat{C}.$$ 

Thus, a new coordinate basis consists of the horizontal vector fields (15) and (16) together with the left-invariant vector field $L_\alpha$.

The horizontal coordinate vector fields of this basis do not commute between themselves. They have the following commutation relations:

$$[H_A, H_B] = \mathbb{C}^T_{AB} H_T + \mathbb{C}^p_{AB} H_p + \mathbb{C}^\alpha_{AB} L_\alpha, \quad \text{(17)}$$

where the “structure constants” are given by

$$\mathbb{C}^T_{AB} = (\Lambda_A^R N_B^R - \Lambda_B^R N_A^R) K^T_R,$$ 

$$\mathbb{C}^p_{AB} = -N_A^D N_B^R (\Lambda_{R,D}^\alpha - \Lambda_{D,R}^\alpha) K^p_\sigma - c_{\alpha\beta\gamma} \Lambda_A^\beta \Lambda_B^\alpha K^p_\gamma,$$

and

$$\mathbb{C}^\alpha_{AB} = -N_A^S N_B^P \tilde{F}^\alpha_{SP} - (N_A^E N_B^P - N_B^E N_A^P) \tilde{F}^\alpha_{Ep} + N_A^m N_B^p \tilde{F}^\alpha_{pm}.$$ 

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In $\mathbb{C}^{\mathbb{T}_{AB}}$, we denote the partial derivative of $K^T_T$ with respect to $Q^*_R$ by $K^T_T$. In $\mathbb{C}^{\alpha}_{AB}$, the curvature tensor $\tilde{\nabla}_P^\alpha$ is given by

$$\tilde{\nabla}_P^\alpha = \frac{\partial }{\partial Q^*_S} \tilde{\nabla}_P^\alpha - \frac{\partial }{\partial Q^*_P} \tilde{\nabla}_S^\alpha + \Gamma^\alpha_{\nu\sigma} \tilde{\nabla}_\nu^\sigma \tilde{\nabla}_\sigma^\rho,$$

($\tilde{\nabla}_P^\alpha(Q^*, a) = \tilde{\rho}^\alpha_{\mu}(a) \nabla_{SP}^\mu(Q^*)$). The tensors $\tilde{\nabla}_E^\alpha$ and $\tilde{\nabla}_P^\alpha$ are defined in a similar way.

Next commutation relations are

$$[H_A, H_p] = (\tilde{\nabla}_A^m H_p^m + \tilde{\nabla}_p^m H_A^m)$$

with

$$\tilde{\nabla}_A^m = (\tilde{J}_A)^m_{\Lambda A}, \quad \tilde{\nabla}_p^m = -N^E_{\Lambda A} \nabla_{Ep}^\alpha - N^m_{A \alpha} \nabla_{mp}^\alpha,$$

and

$$[H_p, H_q] = \tilde{\nabla}_p^\alpha L_\alpha$$

with

$$\tilde{\nabla}_p^\alpha = -\tilde{\nabla}_{pq}^\alpha.$$

We notice that the left-invariant vector fields $L_\alpha$ of the new basis commute with the coordinate horizontal vector fields:

$$[H_A, L_\alpha] = 0, \quad [H_p, L_\alpha] = 0.$$

Also, for $L_\alpha$ we have $[L_\alpha, L_\beta] = c_{\alpha\beta}^\gamma L_\gamma$.

In a new coordinate basis $(H_A, H_p, L_\alpha)$, the metric tensor $\tilde{g}$ transforms into the tensor $\tilde{G}_{AB}$ with following components:

$$\tilde{G}_{AB}(Q^*, \tilde{f}, a) = \left( \begin{array}{ccc} \tilde{G}^H_{AB} & 0 & 0 \\ \tilde{G}^H_{nB} & \tilde{G}^H_{nm} & 0 \\ 0 & 0 & \tilde{d}_{\alpha\beta} \end{array} \right) \equiv \left( \begin{array}{ccc} \tilde{G}^H_{AB} & 0 \\ 0 & \tilde{d}_{\alpha\beta} \end{array} \right),$$

where $\tilde{d}_{\alpha\beta} = \tilde{\rho}^\alpha_{\rho} \tilde{\rho}^\beta_{\beta} \tilde{d}_{\alpha\beta}$. The components of the “horizontal metric” $\tilde{G}^H_{AB}$ depending on $(Q^*, \tilde{f}^m)$ are defined as follows:

$$\tilde{G}^H_{AB} = \Pi^A_A \Pi^B_B G_{AB} = G_{AB} - G_{AD} K^D_{\alpha} d_{\alpha\beta} K^R_{RB},$$

because of $\Pi^C_A \Pi^D_B G_{CD} = \Pi^C_A \Pi^D_B G_{CD} + \Pi^A_B \Pi^p_B G_{qp}$.

$$\tilde{G}^H_{Am} = -G_{AB} K^B_{\alpha} d_{\alpha\beta} K^p_{BP} G_{pm}.$$
\[ \tilde{G}_{mn} = \Pi^r_m G_{rn}, \quad \text{or} \]

\[ \Pi^C_m \Pi^n_D G_{CD} = \Pi^C_m \Pi^n_D G_{CD} + \Pi^r_m \Pi^n_q G_{rq} = G_{mn} - G_{mr} K^r_{\alpha \beta} K^p_{\beta} G_{pn}. \]

It worth to note that the metric with components \( \tilde{\mathcal{G}}_{AB} \) is given on the local surface \( \tilde{\Sigma} \) and gives rise the metric on the orbit space \( P \times_G V \), provided that the submanifold \( \tilde{\Sigma} \) is given parametrically.

The pseudoinverse matrix \( \tilde{G}^{AB} \) to the matrix (20) is represented as

\[
\tilde{G}^{AB} = \begin{pmatrix}
G^{EF} N^A_E N^B_F & G^{EF} N^A_E N^q_F & 0 \\
G^{EF} N^p_E N^B_F & G^{pq} + G^{AB} N^p_A N^q_B & 0 \\
0 & 0 & \tilde{d}^{\alpha \beta}
\end{pmatrix}.
\] (21)

This matrix is defined from the following orthogonality condition:

\[
\tilde{G}^{AB} \tilde{G}^{BE} = \begin{pmatrix}
N^A_D & 0 & 0 \\
N^p_D & \delta^o_m & 0 \\
0 & 0 & \delta^o_\beta
\end{pmatrix} \equiv \begin{pmatrix}
N^A_D & 0 \\
0 & \delta^o_\beta
\end{pmatrix},
\]

where

\[
N^A_D = \begin{pmatrix}
N^A_D \\
N^A_D \\
N^A_D
\end{pmatrix}
\]

\( (N^A_m = 0, N^p_m = \delta^p_m). \)

Finally, it can be shown that the expression (10) for the Lagrangian \( \mathcal{L} \) takes the following form in the coordinate basis \( (H_A, H_p, L_\alpha) \):

\[
\hat{\mathcal{L}} = \frac{1}{2} (\tilde{G}^{AB} \omega^A \omega^B + \tilde{G}^{Ap} \omega^A \omega^p + \tilde{G}^{pA} \omega^p \omega^A + \tilde{G}^{pq} \omega^p \omega^q + \tilde{d}^{\mu \nu} \omega^\mu \omega^\nu) - V, \] (22)

where we have introduced the new time-dependent variables \( \omega^A, \omega^p \) and \( \omega^\alpha \) that are related to the velocities:

\[
\omega^A = (P_\perp)^A_B \frac{dQ^B}{dt}, \quad \omega^p = \frac{df^p}{dt}, \quad \omega^\alpha = u^\alpha \frac{dm}{dt} + \omega^{\alpha E} \frac{dQ^E}{dt} + \omega^{\alpha m} \frac{df^m}{dt}. \] (23)

4 The Lagrange-Poincaré equations

The Lagrange-Poincaré equations for the Lagrangian (22) were obtained in [1] by using the Poincaré variational principle. They are given by the following
equations:

\(- \frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial \dot{\omega}^F} \right) + \left( \frac{\partial \hat{L}}{\partial \omega^T} \right) C_{CE}^T \omega^C + \left( \frac{\partial \hat{L}}{\partial \omega^P} \right) (C_{CE}^P \omega^C + C_{qE}^P \omega^q) + \left( \frac{\partial \hat{L}}{\partial \omega^E} \right) (C_{CE}^E \omega^E + C_{mE}^E \omega^m) + H_E(\hat{L}) = 0, \)

\(- \frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial \dot{\omega}^m} \right) + \left( \frac{\partial \hat{L}}{\partial \omega^P} \right) C_{Em}^P \omega^E + \left( \frac{\partial \hat{L}}{\partial \omega^E} \right) (C_{Em}^E \omega^E + C_{pm}^E \omega^p) + H_m(\hat{L}) = 0, \)

\(- \frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial \dot{\omega}^\alpha} \right) + \left( \frac{\partial \hat{L}}{\partial \omega^E} \right) C_{\alpha m}^E \omega^m + L_\alpha(\hat{L}) = 0. \)

The first two equations of this system are the horizontal equations, and the last equation, for the group variable, is the vertical one.

5 The Lagrange-Poincaré equations in local coordinates

In this section we show how the resulting Lagrange-Poincaré equations can be expressed in terms of local coordinates. First, consider the horizontal equations (24) and (25).

The first term of equation (24), a term with a time derivative, can be written as follows:

\(- \frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial \dot{\omega}^F} \right) + \left( \frac{\partial \hat{L}}{\partial \omega^T} \right) C_{CE}^T \omega^C + \left( \frac{\partial \hat{L}}{\partial \omega^P} \right) (C_{CE}^P \omega^C + C_{qE}^P \omega^q) + \left( \frac{\partial \hat{L}}{\partial \omega^E} \right) (C_{CE}^E \omega^E + C_{mE}^E \omega^m) + H_E(\hat{L}) = 0, \)

\(- \frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial \dot{\omega}^m} \right) + \left( \frac{\partial \hat{L}}{\partial \omega^P} \right) C_{Em}^P \omega^E + \left( \frac{\partial \hat{L}}{\partial \omega^E} \right) (C_{Em}^E \omega^E + C_{pm}^E \omega^p) + H_m(\hat{L}) = 0, \)

\(- \frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial \dot{\omega}^\alpha} \right) + \left( \frac{\partial \hat{L}}{\partial \omega^E} \right) C_{\alpha m}^E \omega^m + L_\alpha(\hat{L}) = 0. \)

(We recall that \((P_\perp)_S^M \omega^S = \omega^M.\) Note that the last term \(\frac{d}{dt}(\hat{G}_{BT})_{\omega^p}\) in the above bracket can be similarly represented.

The last term \(H_E(\hat{L})\) of the equation (24) is given by

\(H_E(\hat{L}) = \frac{1}{2} N_E^P \left[ \frac{\partial \hat{G}_{AB}^H}{\partial Q^B} \omega^A \omega^B + 2 \frac{\partial \hat{G}_{pA}^H}{\partial Q^B} \omega^p \omega^A + \frac{\partial \hat{G}_{pq}^H}{\partial Q^B} \omega^p \omega^q + \frac{\partial \hat{d}_{ \omega^p \omega^q}}{\partial Q^B} \omega^p \omega^q \right]\)
\[ \begin{aligned} &+ \frac{1}{2} N^p_E \left[ \partial \tilde{G}^H_{A\mu} \omega^A \omega^B + 2 N^p \frac{\partial \tilde{G}^H_{n\alpha} \omega^n \omega^q}{\partial f} + \partial \tilde{G}^H_{\mu\nu} \omega^\mu \omega^\nu + \partial \tilde{d}_{\mu\nu} \omega^\mu \omega^\nu \right] \\
 &- \frac{1}{2} N^D E \tilde{A}_D \omega^\alpha - \frac{1}{2} N^p \frac{\partial \tilde{G}^H_{n\alpha} \omega^n \omega^q}{\partial f} - \frac{1}{2} N^D E \frac{\partial V}{\partial f} - N^p E \frac{\partial V}{\partial f}. \end{aligned} \]

It can be rewritten in the following form:

\[ H_E(\tilde{\mathcal{L}}) = \frac{1}{2} N^D E \left[ \tilde{G}^H_{A\mu,\nu} \omega^A \omega^B + 2 \tilde{G}^H_{p\mu,\nu} \omega^p \omega^\nu + \tilde{d}_{\mu\nu,\nu} \omega^\nu \right] \\
+ \frac{1}{2} N^p \left[ \tilde{G}^H_{A,\nu} \omega^A + 2 N^p \frac{\partial \tilde{G}^H_{n\alpha} \omega^n \omega^q}{\partial f} + \tilde{d}_{\mu\nu} \omega^\mu \omega^\nu \right] \\
- \frac{1}{2} N^D E \tilde{A}_D \omega^\alpha - \frac{1}{2} N^p \frac{\partial \tilde{G}^H_{n\alpha} \omega^n \omega^q}{\partial f} - \frac{1}{2} N^D E \frac{\partial V}{\partial f}. \]

(Here we have used the following property satisfied by our projection operators: \( N^M (P_D) = N^D E. \))

An analogous term of the second equation (25) is given by

\[ H_m(\tilde{\mathcal{L}}) = \frac{1}{2} \left[ \tilde{G}^H_{A,\nu} \omega^A + 2 \tilde{G}^H_{p\mu,\nu} \omega^p \omega^\nu + \tilde{d}_{\mu\nu} \omega^\mu \omega^\nu \right] \\
- \frac{1}{2} \tilde{A}_m \omega^\alpha - V_m. \]

For the following it is convenient to represent the horizontal Lagrange-Poincaré equations as a system of two equations. They can be written in the matrix form:

\[ \begin{pmatrix} \tilde{G}^H_{TB} & \tilde{G}^H_{Tp} \\ \tilde{G}^H_{mB} & \tilde{G}^H_{mp} \end{pmatrix} \begin{pmatrix} \frac{d\omega^B}{dt} \\ \frac{d\omega^p}{dt} \end{pmatrix} = \begin{pmatrix} A_T \\ B_m \end{pmatrix}, \]

where by \( A_T \) and \( B_m \) we denote the potential terms of the first and second equations and as well as terms that are quadratic in pseudo-velocities. Multiplying this matrix equation (from the left) by the matrix

\[ \begin{pmatrix} N^A_E & N^T_E \\ N^F_E & N^T_F \end{pmatrix} \begin{pmatrix} \tilde{G}^E_{EF} \tilde{G}^E_{mF} \\ \tilde{G}^E_{mF} + \tilde{G}^E_{EF} \tilde{G}^m_F \end{pmatrix} \]

we get

\[ \begin{pmatrix} N^A_B & 0 \\ N^T_B & \delta^r_p \end{pmatrix} \begin{pmatrix} \frac{d\omega^B}{dt} \\ \frac{d\omega^p}{dt} \end{pmatrix} = \begin{pmatrix} A^A_T \\ B^r_m \end{pmatrix} = 0 \]

(\( N^r_p = \delta^r_p \)). Thus, we have two equations:
\[
N_B^A \frac{d\omega^B}{dt} - \tilde{G}^{EF} N_E^A \left( N_F^T \cdot A_T + N_F^m \cdot B_m \right) = 0 \quad (27)
\]
and
\[
N_B^r \frac{d\omega^B}{dt} + \frac{d\omega^r}{dt} - \tilde{G}^{EF} N_E^r \left( N_F^T \cdot A_T + N_F^m \cdot B_m \right) - \tilde{G}^{rm} \cdot B_m = 0. \quad (28)
\]

Our goal is to obtain a standard coordinate representation for these equations. This can be achieved by means of combining and rearranging the terms of equations, and will be done as follows. First we consider those terms of the equations that depend on the quasi-velocities \(\omega^A\) and \(\omega^p\). And then, in the next step, we get the terms that depend on the group velocities \(\omega^\alpha\). In addition, at the end of our consideration, a remark will be made about the transformation of the potential terms of these equations.

We begin by studying the expression \(N_F^T \cdot A_T\). Those terms of \(A_T\), that are of interest to us, are given by
\[
-\tilde{G}^{H}_{BT,M} \omega^B M^\omega^q q^\omega^q - \tilde{G}^{H}_{pT,M} \omega^p M^\omega^q q^\omega^q + \frac{1}{2} N^D_T \left( \tilde{G}^{H}_{AB,D} \omega^A B^\omega^A + 2 \tilde{G}^{H}_{p,q,D} \omega^p \omega^q \right.
\]
\[
\left. + \frac{1}{2} N^p_T \left( \tilde{G}^{H}_{AB,B} \omega^A B^\omega^A + 2 \tilde{G}^{H}_{n,q,B} \omega^n \omega^q \right) + (\tilde{G}^{H}_{BM} \omega^B M^\omega^q q^\omega^q + \tilde{G}^{H}_{pB,M} \omega^p M^\omega^q q^\omega^q) \right) \right)
\]
\[
\left. \right) \right)
\]
\[
= (\tilde{G}^{H}_{BM} \omega^B M^\omega^q q^\omega^q + \tilde{G}^{H}_{pB,M} \omega^p M^\omega^q q^\omega^q) (C^M_{CT} \omega^C + \tilde{G}^{H}_{pB,M} \omega^p M^\omega^q q^\omega^q) (C^p_{nT} \omega^n + C^p_{nT} \omega^n).
\]
They are multiplied by the projector \(N_F^T\). First we note that \(N_F^T N_F^p = 0\). Consequently, after this multiplication, the third line of the preceding expression does not contribute to the equation. We note also that \(N_F^T C^M_{CT} \omega^C = 0\).

This follows from an explicit representation for \(C^M_{CT}\):
\[
C^M_{CT} = (A^\gamma_T N^R_T - A^\gamma_T N^R_C) K^M_{\gamma,R},
\]
and due to the properties: \(A^\gamma_T \omega^C = 0\) and \(N_F^T A^\gamma_T = 0\).

The last properties also lead to \(N_F^T C^p_{nT} = 0\), since \(C^p_{nT} = -(\tilde{J}_\alpha)_n A^\gamma_T\). And for \(N_F^T C^p_{MT}\), we have
\[
N_F^T C^p_{MT} = -N_F^R N_F^p (A^\alpha_R,D - A^\alpha_D,R) K^p_{\alpha}.
\]
The vanishing of this term can be shown as follows. Taking the partial differential of
\[
A^\alpha_R(Q^*) = (\Phi^{-1})^\alpha_Q(Q^*) \chi^\beta_R(Q^*)
\]

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with respect to dependent variable $Q^*^D$, we get

$$(P_\perp)_{D-S}^T \Lambda_{R,S}^D(Q^*) = (P_\perp)_{D-S}^T \left( (\Phi^{-1})_{\alpha,\beta,S}^{\alpha} \chi_\beta^R + (\Phi^{-1})_{\beta,\alpha,S}^{\beta} \chi_\alpha^R, S \right).$$

(The appearence of $(P_\perp)_{D-S}^T$ in this formula is due to our rule used for differentiation of the functions with dependent variables.) If we multiply the obtained formula by $N^D_M$, then, because of $N^D_M(P_\perp)^T = N^D_S$, we come to

$$N^S_M \Lambda_{R,S}^S = N^S_M \left( (\Phi^{-1})_{\alpha,\beta,S}^{\alpha} \chi_\beta^R + (\Phi^{-1})_{\beta,\alpha,S}^{\beta} \chi_\alpha^R, S \right).$$

Multiplying this expression by $N^R_F$ and taking into account the following property: $N^R_F \chi_\beta^R = 0$, we get the first term of the above representation for $N^T_F C^p_{MT}$:

$$N^R_F N^S_M \Lambda_{R,S}^\alpha = N^S_M N^R_F (\Phi^{-1})_{\beta,\alpha}^{\alpha} \chi_\beta^R, S.$$ 

The expression for the second term of $N^T_F C^p_{MT}$ can be derived in a similar way. As a result of differentiation, we obtain

$$N^R_F \Lambda_{D,R}^\alpha = N^R_F (\Phi^{-1})_{\beta,\alpha}^{\beta} \chi_\beta^D, R.$$ 

which should now be multiplied by $N^D_M$ to get

$$N^D_M N^R_F \Lambda_{D,R}^\alpha = N^D_M N^R_F (\Phi^{-1})_{\beta,\alpha}^{\beta} \chi_\beta^D, R.$$ 

Hence $N^T_F C^p_{MT}$ is equal to the difference of two obtained expressions:

$$N^R_F N^D_M \Lambda_{R,D}^\alpha - N^D_M N^R_F \Lambda_{D,R}^\alpha = N^D_M N^R_F (\Phi^{-1})_{\beta,\alpha}^{\beta} \chi_\beta^R, D, R.$$ 

Because of $\chi_\beta^R, D = \chi_\beta^D, R$, where $\chi_\beta^R, D(Q^*) \equiv \frac{\partial^2 \chi^\beta(Q)}{\partial Q^R \partial Q^D} \bigg|_{Q=Q^*}$, the right-hand side of the preceding expression is zero.

Thus, those terms in $N^T_F A_T$ that depend on $\omega^A$ and $\omega^p$ are given as follows:

$$-\frac{1}{2} N^D_F (\tilde{G}_{BD,M}^H + \tilde{G}_{MD,B}^H - \tilde{G}_{BM,D}^H) \omega^B \omega^M$$

$$- N^T_F (\tilde{G}_{BT,q}^H + \tilde{G}_{qT,B}^H - \tilde{G}_{qB,T}^H) \omega^q \omega^B - \frac{1}{2} N^T_F (\tilde{G}_{pT,q}^H + \tilde{G}_{qT,p}^H - \tilde{G}_{pq,T}^H) \omega^p \omega^q.$$ 

This expression can be written in the following form:

$$- N^T_F \tilde{\Gamma}_{BM}, \omega^B \omega^M - 2 N^T_F \tilde{\Gamma}_{qBT} \omega^q \omega^B - N^T_F \tilde{\Gamma}_{pqT} \omega^p \omega^q, \quad (29)$$
in which $\tilde{\Gamma}_{BMD}$ is define as

$$\tilde{\Gamma}_{BMD} \equiv \frac{1}{2}(\tilde{G}^H_{BD,M} + \tilde{G}^H_{MD,B} - \tilde{G}^H_{BM,D}).$$

And $\tilde{\Gamma}_{qBT}$ and $\tilde{\Gamma}_{pqT}$ have an analogous definitions.

In $B_m$, the terms that depend on $\omega^A$ and $\omega^p$ are given by

$$-\tilde{G}^H_{Bm,R} \omega^B \omega^R - \tilde{G}^H_{Bm,p} \omega^p \omega^R - \tilde{G}^H_{pm,R} \omega^p \omega^R - \tilde{G}^H_{pm,q} \omega^p \omega^q + \tilde{G}^H_{Bq} \omega^B \mathbb{C}^q_{Em} \omega^E + \tilde{G}^H_{pq} \omega^p \mathbb{C}^q_{Em} \omega^E$$

$$+ \frac{1}{2} \tilde{G}^H_{AB,m} \omega^A \omega^B + \tilde{G}^H_{pA,m} \omega^p \omega^A + \frac{1}{2} \tilde{G}^H_{pq,m} \omega^p \omega^q.$$  

Here $\mathbb{C}^q_{Em} = (\tilde{J}_m)^q_{\alpha} \Lambda^\alpha_E$. And from $\Lambda^E_0 \omega^E = 0$, it follows that $\mathbb{C}^q_{Em} \omega^E = 0$.

Hence for terms in $N^m_F B_m$, we have

$$-\frac{1}{2} N^m_F (\tilde{G}^H_{Bm,A} - \tilde{G}^H_{Am,B} - \tilde{G}^H_{AB,m}) \omega^A \omega^B - N^m_F (\tilde{G}^H_{Bm,p} + \tilde{G}^H_{pm,B} - \tilde{G}^H_{pB,m}) \omega^p \omega^B$$

$$- \frac{1}{2} N^m_F (\tilde{G}^H_{pm,q} + \tilde{G}^H_{qm,p} - \tilde{G}^H_{pq,m}) \omega^p \omega^q.$$

This expression can be rewritten as

$$-N^m_F \tilde{\Gamma}_{ABm} \omega^A \omega^B - 2 N^m_F \tilde{\Gamma}_{pBm} \omega^p \omega^B - N^m_F \tilde{\Gamma}_{pmq} \omega^p \omega^q. \quad (30)$$

Summing $\tilde{\Gamma}_{qBT}$ and $\tilde{\Gamma}_{pqT}$, we obtain the terms that belong to $N^F_2 A_T + N^m_F B_m$:

$$-N^F_2 \tilde{\Gamma}_{BMT} \omega^B \omega^M - 2 N^F_2 \tilde{\Gamma}_{qBT} \omega^q \omega^B - N^F_2 \tilde{\Gamma}_{pqT} \omega^p \omega^q.$$

Here we have used the condensed notation for indices, by which summation over the repeated index $T$ means that we have two summation: one is taken for $T$, and the other is performed over some repeated index, for which we use a small Latin letter. That is, in our case, for example, $T \equiv (T, m)$.

Multiplying the resulting expression by $-\hat{\mathbb{G}}^F_E N^A_F$, one can obtain the following terms of the first Lagrange-Poincaré equations:

$$\hat{\mathbb{G}}^F_E N^A_F \left( \tilde{\Gamma}_{BMT} \omega^B \omega^M + 2 \tilde{\Gamma}_{qBT} \omega^q \omega^B + \tilde{\Gamma}_{pqT} \omega^p \omega^q \right).$$

We may also introduce the Christoffel symbols $\hat{\Gamma}_{BM}^R$, $\hat{\Gamma}_{qB}^R$ and $\hat{\Gamma}_{pq}^R$ for the horizontal (degenerate) metric $\hat{G}^H_{RT}$. They are defined by means of the equalities:

$$\hat{\Gamma}_{BMT} = \hat{G}^H_{RT} \hat{\Gamma}_{BM}^R, \quad \hat{\Gamma}_{qBT} = \hat{G}^H_{RT} \hat{\Gamma}_{qB}^R \quad \text{and} \quad \hat{\Gamma}_{pqT} = \hat{G}^H_{RT} \hat{\Gamma}_{pq}^R.$$
Then, taking into account the following properties of the projectors:
\[ N^A_R \hat{G}^H_{RR} = \hat{C}^H_{FR}, \quad \hat{G}^{EF} \hat{C}^H_{FR} = \Pi^E_R, \quad N^A_E \Pi^E_R = N^A_R \quad \text{(if } N^E_r = 0), \]
it becomes possible to rewrite the obtained expression for the discussed terms of the first horizontal Lagrange-Poincaré equations as
\[ N^A_R \left( H^{FR} B_M \omega^B = 2 H^{FR} q_b \omega^b + H^{FR} p_\rho \omega^\rho \right) \equiv N^A_R H^{FR} B_M \omega^B \omega^M. \quad (31) \]

In the second horizontal Lagrange-Poincaré equation, we are interested in by those terms of
\[ -\hat{G}^{EF} N^T_E (N^T_F \cdot A_T + N^m_F \cdot B_m) - \hat{G}^{rm} \cdot B_m \]
that depend on \( \omega^A \) and \( \omega^p \). Here we proceed in the same way as in the case of the first horizontal equation. As a result, we arrive at
\[ (N^E_F \Pi^E_R + \Pi^E_R) H^{FR} B_M \omega^B \omega^M \equiv N^A_R H^{FR} B_M \omega^B \omega^M. \quad (32) \]
(Note that \( N^r_p = \delta^r_p \).)

Now we will consider the terms of the first horizontal Lagrange-Poincaré equations, which depend on \( \omega^\mu \). We begin by studying such terms in \( N^T_F A_T \). In \( A_T \), they are represented by the following expression:
\[ H_T (\hat{\mathcal{L}})(\omega^\mu) + \left( \frac{\partial \hat{\mathcal{L}}}{\partial \omega^\mu} \right) (\mathcal{C}^e_{CT} \omega^C + \mathcal{C}^e_{\sigma T} \omega^\sigma) = \]
\[ \frac{1}{2} \left[ N^D_F \frac{\partial \tilde{d}_{\mu \nu}}{\partial Q^{BD}} + N^T_F \frac{\partial \tilde{d}_{\mu \nu}}{\partial f^T} - N^D_F \alpha_D \tilde{d}_{\mu \nu} - N^T_F \alpha_T \tilde{d}_{\mu \nu} \right] \omega^\nu \omega^\nu \]
\[ + \tilde{d}_{\mu \nu} \omega^\mu (\mathcal{C}^e_{CT} \omega^C + \mathcal{C}^e_{\sigma T} \omega^\sigma), \]
in which by \( H_T (\hat{\mathcal{L}})(\omega^\mu) \) we denote the \( \omega^\mu \)-dependent part of \( H_T (\hat{\mathcal{L}}) \). Multiplying this expression by \( N^T_F \) and discarding the corresponding terms due to \( N^T_F N^p_F = 0 \), we get
\[ \frac{1}{2} N^D_F \left[ \frac{\partial \tilde{d}_{\mu \nu}}{\partial Q^{BD}} - \alpha_D \tilde{d}_{\mu \nu} \right] \omega^\nu + \tilde{d}_{\mu \nu} \omega^\mu (\mathcal{C}^e_{CT} \omega^C + \mathcal{C}^e_{\sigma T} \omega^\sigma) N^T_F = \]
\[ \frac{1}{2} \left[ N^D_F \hat{\omega}_D \tilde{d}_{\mu \nu} \right] \omega^\mu \omega^\nu + N^T_F \tilde{d}_{\mu \nu} \omega^\mu (\mathcal{C}^e_{CT} \omega^C + \mathcal{C}^e_{\sigma T} \omega^\sigma), \]
where the covariant derivative is defined as
\[ N^D_F (\hat{\omega}_D \tilde{d}_{\mu \nu}) = N^D_F \left[ \frac{\partial}{\partial Q^{BD}} \tilde{d}_{\mu \nu} - \alpha_{\omega \nu} \tilde{d}_{\sigma \nu} - \alpha_{\omega \mu} \tilde{d}_{\sigma \mu} \right], \]
\[ 17 \]
\[ \hat{d}_{\mu \nu} = \rho^\mu_{\nu}(a) \rho^\nu_{\nu}(Q^*, \hat{f}) \] and \[ \mathcal{A}_B = \bar{p}^\mu_{\nu}(a) \mathcal{A}_D^\mu(Q^*, \hat{f}) . \]

Recalling that
\[ \mathbb{C}_{CT}^e = -N^R_C \mathcal{F}_R - N^m_{CT} \mathcal{F}_pm, \]
\[ \mathbb{C}_{CT}^e = -N^R_C N^Q_T \mathcal{F}_{RQ} - (N^R_C N^T_{PV} - N^R_C N^P_T) \mathcal{F}_{R} + N^m_{CT} \mathcal{F}_pm, \]
and using the following equalities: 
\[ N^R_C \omega^C = \omega^R, \quad N^m_{CT} \omega^C = -K^m_{CT} \omega^C = 0, \]

it is not difficult to show that the expression for \( \omega^\mu \)-terms in \( N_T^m A_T \) is given by
\[ -N_F^T \hat{d}_{\mu \nu} \omega^\mu (\mathcal{F}_Q Q^\nu + \mathcal{F}_Q Q^\nu) + \frac{1}{2} (N_F^T \mathcal{D}_T \hat{d}_{\mu \nu}) \omega^\mu \omega^\nu \equiv \]
\[ -N_F^T \hat{d}_{\mu \nu} \mathcal{F}_Q Q^\nu \omega^\nu + \frac{1}{2} (N_F^T \mathcal{D}_T \hat{d}_{\mu \nu}) \omega^\mu \omega^\nu . \]

In \( B_m \), \( \omega^\mu \)-terms are represented by the following expression:
\[ H_m(\mathcal{L})(\omega^\mu) + \left( \frac{\partial \mathcal{L}}{\partial \omega^\nu} \right) \left( \mathbb{C}_{Em}^e \omega^E + \mathbb{C}_{pm}^e \omega^p \right) = \]
\[ \frac{1}{2} \mathcal{D}_m(\hat{d}_{\mu \nu}) \omega^\mu \omega^\nu + \hat{d}_{\mu \nu} (\mathbb{C}_{Em}^e \omega^E + \mathbb{C}_{pm}^e \omega^p) , \]
in which the covariant derivative is defined as
\[ \mathcal{D}_m(\hat{d}_{\mu \nu}) = \partial \hat{d}_{\mu \nu} / \partial f^m - e_{\beta \mu}^\sigma \mathcal{A}_m^\beta \hat{d}_{\sigma \nu} - e_{\beta \nu}^\sigma \mathcal{A}_m^\beta \hat{d}_{\sigma \mu} . \]

Therefore in \( N_T^m B_m \), they are given by
\[ \frac{1}{2} N_F^m \mathcal{D}_m(\hat{d}_{\mu \nu}) \omega^\mu \omega^\nu - N_F^m \hat{d}_{\mu \nu} \omega^\mu (\mathcal{F}_R \omega^R + \mathcal{F}_R \omega^R) \equiv \]
\[ N_F^m \left( \frac{1}{2} \mathcal{D}_m(\hat{d}_{\mu \nu}) \omega^\mu \omega^\nu - \hat{d}_{\mu \nu} \mathcal{F}_Q Q^\nu \omega^\mu \right) . \]

Hence \( \omega^\mu \)-terms of \( N_T^m A_T + N_T^m B_m \) are as follows:
\[ -N_F^m \hat{d}_{\mu \nu} \mathcal{F}_Q Q^\nu \omega^\mu + \frac{1}{2} N_F^m (\mathcal{D}_R \hat{d}_{\mu \nu}) \omega^\mu \omega^\nu . \quad (33) \]

The obtained expressions (31), (32) and (33), when used them in the equations (27) and (25), lead to new coordinate representations of the horizontal Lagrange-Poincaré equations:
\[ N_B^A d^B_T dt + N_R^A H^R_{BA} \omega^R \omega^A + \]
\[ G_{EE} N_F^A N_F^R \left[ \hat{d}_{\mu \nu} \mathcal{F}_R Q^\nu \omega^\mu - \frac{1}{2} (\mathcal{D}_R \hat{d}_{\mu \nu}) \omega^\mu \omega^\nu + V_R \right] = 0 . \]
and
\[
N_B^r \frac{d \omega^B}{dt} + \frac{d \omega^r}{dt} + N_R^A \mathcal{H} \Gamma^A_B \omega \dot{\omega}^B + G^{EF} N_E^F N^R_E \left[ \dot{N}^\mu \mathcal{F}_{Q^R \omega} \dot{Q} \omega^\mu - \frac{1}{2} (\mathcal{D}^A \mathcal{D}^B) \omega^\mu \omega^\nu + V_R \right] + G^{EF} N^A_E N_F^R \left[ \mathcal{F}_{Q^M \omega} \dot{Q} \omega^\mu + \frac{1}{2} (\mathcal{D}_m \mathcal{D}^m) \omega^\mu \omega^\nu + V_m \right] = 0.
\]

We note that because of the invariance of the potential \( V(Q^*, \tilde{\omega}) \) under the action of the group \( G \), in these equations, we have, in fact, \( N^ \tilde{R}^F V^\tilde{R} \).

Our final transformation of these equations is the replacement of the variable \( \omega^\mu \) by the new variable \( p_\alpha = d_{\alpha \mu} \rho^\mu \omega^\mu \). As a result, we obtain
\[
N_B^r \frac{d \omega^B}{dt} + \frac{d \omega^r}{dt} + N_R^A \mathcal{H} \Gamma^A_B \omega \dot{\omega}^B + G^{EF} N_E^F N^R_E \left[ \mathcal{F}_{Q^R \omega} \dot{Q} p_\alpha + \frac{1}{2} (\mathcal{D}^A \mathcal{D}^B) p_\alpha p_\nu + V_R \right] = 0, \tag{34}
\]
\[
N_B^r \frac{d \omega^B}{dt} + \frac{d \omega^r}{dt} + N_R^A \mathcal{H} \Gamma^A_B \omega \dot{\omega}^B + G^{EF} N_E^F N^R_E \left[ \mathcal{F}_{Q^M \omega} \dot{Q} p_\alpha + \frac{1}{2} (\mathcal{D}_m \mathcal{D}^m) p_\alpha p_\nu + V_m \right] = 0. \tag{35}
\]

Now consider the third Lagrange-Poincaré equation. For the Lagrangian \( \{22\}, \) this vertical Lagrange-Poincaré equation \( \{26\} \) is as follows:
\[
-d \frac{d}{dt}(\tilde{d}_\mu \omega^\mu) + \tilde{d}_\mu \omega^\mu c^\mu_{\nu \alpha} \omega^\nu + \frac{1}{2} L_\alpha (\tilde{d}_\mu \omega^\mu) \omega^\mu \omega^\nu = 0.
\]

Since
\[
\frac{1}{2} L_\alpha (\tilde{d}_\mu \omega^\mu) = \frac{1}{2} L_\alpha (\rho^\mu_{\alpha} \rho^\nu_{\alpha}) d_{\nu \mu} = \frac{1}{2} (c^\gamma_{\alpha \mu} \rho^\mu_{\alpha} \rho^\nu_{\gamma} + c^\gamma_{\alpha \nu} \rho^\nu_{\alpha} \rho^\mu_{\gamma}) d_{\mu \nu},
\]
we have
\[
\frac{1}{2} L_\alpha (\tilde{d}_\mu \omega^\mu) = \frac{1}{2} (c^\gamma_{\alpha \mu} \tilde{d}_{\gamma \mu} + c^\gamma_{\alpha \nu} \tilde{d}_{\gamma \nu}).
\]

This means that the second and third terms of the Lagrange-Poincaré equation cancel each other out, and we get
\[
\frac{d}{dt}(\tilde{d}_\mu \omega^\mu) = 0.
\]

The resulting equation, resembling the conservation law of the ”color charge”, can be rewritten in terms of the dual variable \( p_\alpha \), which we introduced above.
Replacing $\omega^\mu$ by $p_\alpha = d_{\alpha\mu} \rho_\nu \omega^\nu$, we obtain
\[
\frac{d}{dt} (\rho_\alpha p_\nu) = \rho_\alpha \frac{d}{dt} p_\nu + \frac{\partial \rho_\alpha}{\partial \omega^\mu} \frac{d\omega^\mu}{dt} p_\nu = 0.
\]
But since
\[
\frac{da_\beta}{dt} = v_\alpha (\omega^\alpha - \omega^E A_\alpha - \omega^p \tilde{E}_\alpha - \omega^g \tilde{E}_g)\quad\text{with}\quad \tilde{E}_E = \tilde{r}_\alpha \omega^g_{\tilde{E}_g},
\]
and $L_\alpha \rho_\mu^\gamma = c_\mu^\nu \rho_\mu^\gamma$, after using this substitution it can be shown that the vertical Lagrange-Poincaré equation is as follows:
\[
\frac{dp_\beta}{dt} + c_\mu^\nu d_{\mu\sigma} p_\sigma p_\nu - c_\mu^\nu \omega^E \tilde{E}_g p_\nu = 0. \quad (36)
\]
Thus, this equation, together with the equations (34) and (35), are the local Lagrange-Poincaré equations written in terms of the local coordinates.

6 The equations for relative equilibrium

Having derived the local Lagrange-Poincaré equations, we will make a brief remark about the equation for relative equilibria.

We recall that for mechanical systems with symmetry, the relative equilibrium is a special movement of the original system, which in the case of a projection onto the reduced manifold becomes the equilibrium of the reduced mechanical system. From the theory of dynamical systems with the symmetry [13] it is known that in a relative equilibrium the system performs the stationary motion. In addition, in this motion the shape of the system does not change. So to get the equations for finding the relative equilibria, we need to put $\omega^A = 0$ ($\omega^A = 0$, $\omega^p = 0$) in the Lagrange-Poincaré equations. The horizontal equations (34) and (35) become as follows:
\[
G^{EF} N_E^A N_F^R \left[ \frac{1}{2} (\mathcal{D}_{\hat{R}} d^{\rho\sigma}) p_\mu p_\sigma + V_{\hat{R}} \right] = 0
\]
and
\[
G^{EF} N_F^E N_E^R \left[ \frac{1}{2} (\mathcal{D}_{\hat{R}} d^{\rho\sigma}) p_\mu p_\sigma + V_{\hat{R}} \right] + G^{rm} \left[ \frac{1}{2} (\mathcal{D}_{m} d^{\rho\sigma}) p_\mu p_\sigma + V_{m} \right] = 0.
\]
They can be rewritten as
\[
\left\{ \begin{array}{c}
G^{EF} N_E^A N_F^R \cdot (1)_R + G^{EF} N_E^A N_m^m \cdot (2)_m = 0 \\
G^{EF} N_F^E N_E^R \cdot (1)_R + (G^{rm} + G^{EF} N_F^E N_E^m) \cdot (2)_m = 0,
\end{array} \right.
\]
where
\[(1)_R = \frac{1}{2}(\mathcal{D}_R d^{\alpha \sigma}) p_\alpha p_\sigma + V_R\]
and
\[(2)_m = \frac{1}{2}(\mathcal{D}_m d^{\alpha \sigma}) p_\alpha p_\sigma + V_m.\]

In the matrix form this system of equations looks like
\[
\begin{pmatrix}
G^{EF} N_E^A N_F^R & G^{EF} N_E^A N_F^m \\
G^{EF} N_F^E N_E^R & G^{mn} + G^{EF} N_F^E N_F^m
\end{pmatrix}
\begin{pmatrix}
(1)_R \\
(2)_m
\end{pmatrix} = 0.
\]

Multiplying it from the left by the matrix
\[
\begin{pmatrix}
G^{HA} & G^{HR} \\
G^{pA} & G^{pr}
\end{pmatrix},
\]
we obtain
\[
\begin{pmatrix}
N_B^R & N_B^m \\
0 & \delta^m_p
\end{pmatrix}
\begin{pmatrix}
(1)_R \\
(2)_m
\end{pmatrix} = 0.
\]

That is, we have
\[
\begin{cases}
N_B^R \cdot (1)_R + N_B^m \cdot (2)_m = 0 \\
\delta^m_p \cdot (2)_m = 0.
\end{cases}
\]

But this means that
\[
\begin{cases}
N_B^R \cdot (1)_R = 0 \\
(2)_m = 0.
\end{cases}
\]

In other words,
\[
\begin{cases}
N_B^R \left( \frac{1}{2}(\mathcal{D}_R d^{\alpha \sigma}) p_\alpha p_\sigma + V_R \right) = 0 \\
\frac{1}{2}(\mathcal{D}_m d^{\alpha \sigma}) p_\alpha p_\sigma + V_m = 0.
\end{cases}
\]

(37)

It should be noted that although these resulting equations look as if they are independent, but really it is not so. They are interrelated, since the matrix \(d^{\mu \sigma}\) is inverse to the matrix representing the sum of two orbital metrics \(\gamma_{\mu \nu}\) and \(\gamma'_{\mu \nu}\).

For solvability of the equations (37) it is required (in the standard approach) that \(p_\alpha = \text{const.}\) Taking this condition into account, the vertical Lagrange-Poincare equation (36) is transformed into
\[
c^\nu_{\mu \beta} d^{\mu \sigma} p_\alpha p_\nu = 0.
\]

(38)

Thus, (37) and (38) are the equations for determining the relative equilibria of the mechanical system under consideration.

Note that (38) can be solved using the eigenvectors of the matrix \(d^{\mu \sigma}\) [14]. Namely, \(p_\alpha\) is assumed to be proportional to the eigenvector of this matrix.
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