Determining the entanglement of formation for arbitrary two-mode Gaussian states using Einstein-Podolsky-Rosen-like uncertainty

Y. Akbari Kourbolagh and H. Alijanzadeh Boora

Department of Physics, Azarbaijan Shahid Madani University, Tabriz 53741-161, Iran
yakbari@azaruniv.edu, h.alijanzadeh@umz.ac.ir

Abstract
In this article, we have derived a compact form for the entanglement of formation of arbitrary two-mode Gaussian states by using the Duan et al.’s inequality which gives a necessary and sufficient separability condition for any two-mode Gaussian state.

PACS numbers: 03.67.Mn, 03.65.Ud, 42.50.Dv
Key Words: entanglement of formation, EPR correlation, squeezed states
1 Introduction

Entanglement or inseparability is the main concept in many areas of quantum information theory. It is well known that the von Neumann entropy of the reduced state is a convenient entanglement measure for pure states $|\psi\rangle$ of bipartite systems [1, 2, 3]

$$E(\psi) = -\text{Tr}(\rho_A \log_2 \rho_A) = -\text{Tr}(\rho_B \log_2 \rho_B)$$

where $\rho_A = \text{Tr}_B(|\psi\rangle\langle\psi|)$ and $\rho_B = \text{Tr}_A(|\psi\rangle\langle\psi|)$ are reduced states. $E(\psi)$ is also called the entropy of entanglement.

Several entanglement measures have been introduced for mixed states. One of the measures which has an attractive physical motivation is the entanglement of formation (EOF). For a bipartite mixed state, Bennet et al. [4] have defined this measure as the minimal amount of average entanglement for any ensemble of bipartite pure states realizing the state. Explicitly, the EOF of a mixed bipartite state $\rho$ is defined as

$$E_F(\rho) := \inf \left\{ \sum_k p_k E(|\psi_k\rangle\langle\psi_k|) : \rho = \sum_k p_k |\psi_k\rangle\langle\psi_k| \right\}.$$

where the infimum is taken over all possible pure-state decompositions of $\rho$. Gaussian states of infinite dimensional quantum systems play an important role in quantum information: they can be created relatively easily and can be used in quantum cryptography and quantum teleportation [5, 6, 7].

Giedke et al. [8] have determined the EOF of symmetric two-mode Gaussian states by establishing a connection between its EPR-like correlation and entanglement. J. S. Ivan and R. Simon [9] have computed the EOF for arbitrary two-mode Gaussian states (TMGSs). Marians [10] have reformulated the problem of evaluating the EOF for TMGS's in terms of characteristic functions and covariance matrices (CMs). They have shown that the exact EOF of such a state coincides with its Gaussian one.

In this paper, we obtain a compact form for the EOF of arbitrary TMGSs in terms of few parameters of the TMGS standard form introduced by Duan et al. [11]. We show that for special values of parameters, our result reduces to the one of Giedke et al. for the symmetric states. Our result also recover the Theorem 1 of [12] which gives an upper bound for the EOF of TMGS.
2 Entanglement of formation for Gaussian states

Duan et al. [11] have shown that any bipartite separable quantum state $\rho$ satisfies the following inequality:
\[
\langle (\Delta \hat{u})^2 \rangle_\rho + \langle (\Delta \hat{v})^2 \rangle_\rho \geq a^2 + \frac{1}{a^2} \tag{1}
\]
where $\hat{u} = |a|\hat{x}_A + \hat{\hat{x}}_A$ and $\hat{v} = |a|\hat{\hat{p}}_A - \hat{\hat{p}}_A$ are EPR-like operators, $a$ is a real nonzero parameter, $\hat{x}_j, \hat{\hat{p}}_j$ have the commutators $[\hat{x}_j, \hat{\hat{p}}_{j'}] = i\delta_{jj'} (j, j' = A, B)$ and $\langle (\Delta \hat{u})^2 \rangle_\rho = Tr(g\hat{u}^2) - Tr(g\hat{u})$. Motivated by this inequality, we introduce an EPR-like uncertainty
\[
\Delta(\sigma) := \min\{1, \frac{\langle (\Delta \hat{u})^2 \rangle_\sigma + \langle (\Delta \hat{v})^2 \rangle_\sigma}{a^2 + 1/a^2}\} \tag{2}
\]
for any normalized quantum state $\sigma$. By Eq. (1), it is clear that $\Delta(\sigma) \in (0,1]$ and $\Delta(\sigma) < 1$ indicates the inseparability of $\sigma$. Note that the first moments can always be shifted to zero by a sequence of local unitaries. Hence, it is irrelevant for the study of entanglement and without lose of generality we may take it to be zero. Then we have:
\[
\frac{\langle (\Delta \hat{u})^2 \rangle_\sigma + \langle (\Delta \hat{v})^2 \rangle_\sigma}{a^2 + 1/a^2} = \frac{a^2(\langle \hat{x}_A \rangle^2 + \langle \hat{\hat{p}}_A \rangle^2) + \frac{1}{a^2}(\langle \hat{x}_B \rangle^2 + \langle \hat{\hat{p}}_B \rangle^2) + \frac{2|a|}{a^2}(\langle \hat{x}_A \hat{x}_B \rangle - \langle \hat{\hat{p}}_A \hat{\hat{p}}_B \rangle)}{a^2 + 1/a^2} \tag{3}
\]
For a given two-mode Gaussian state $\sigma$, we define its covariance matrix (CM) $\gamma$ as usual:
\[
\gamma_{ij} = Tr[(\hat{\hat{x}}_i \hat{x}_j + \hat{\hat{p}}_i \hat{\hat{p}}_j)\sigma] - 2Tr(\hat{\hat{x}}_i \sigma)Tr(\hat{\hat{p}}_j \sigma) \tag{4}
\]
where $\hat{\hat{x}} = (\hat{x}_A, \hat{\hat{p}}_A, \hat{x}_B, \hat{\hat{p}}_B)$. It turns out that the matrix $\gamma$ is a bona fide CM iff it satisfies the uncertainty relation [13]
\[
\det(\gamma + \frac{i}{2}\Omega) \geq 0 \tag{5}
\]
in which
\[
\Omega = \bigoplus_{i=1}^{N} J \quad \text{with} \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]
In lemma 2 of [11] the following standard form for correlation matrix of TMGS has been introduced:
\[
\gamma = \begin{bmatrix} n_1 & 0 & c_1 & 0 \\ 0 & n_2 & 0 & c_2 \\ c_1 & 0 & m_1 & 0 \\ 0 & c_2 & 0 & m_2 \end{bmatrix} \tag{6}
\]
where the $n_i, m_i,$ and $c_i$ satisfy
\[
\frac{n_1 - 1}{m_1 - 1} = \frac{n_2 - 1}{m_2 - 1}
\]
\[ |c_1| - |c_2| = \sqrt{(n_1 - 1)(m_1 - 1)} - \sqrt{(n_2 - 1)(m_2 - 1)}. \]  

In proposition 2 of aforementioned reference it has been shown that inequality (1) is a necessary and sufficient condition for separability of TMGSs for \( a = -\frac{|c_1|}{c_1} a_0 \) with \( a_0^2 = \sqrt{\frac{m_1 - 1}{n_1 - 1}} = \sqrt{\frac{m_2 - 1}{n_2 - 1}} \). By using the correlation matrix (6), for an entangled TMGS we get:

\[ \Delta(\sigma) = \frac{a_0^2 \frac{m_1 + m_2}{2} + \frac{m_1 + m_2}{2a_0^2} - |c_1| - |c_2|}{a_0^2 + 1/a_0^2} := \delta < 1 \]  

(8)

For a given pure state \(|\psi\rangle\), let us write its Schmidt decomposition as

\[ |\psi\rangle = \sum_{N=0}^{\infty} c_N |u_N\rangle_A \otimes |v_N\rangle_B \]  

(9)

where \( c = (c_0, c_1, \ldots) \) are non-negative Schmidt coefficients such that \( c_N \leq \frac{2}{a^2 + 1/a^2} c_{N-1} \) with \( \sum_{N=0}^{\infty} c_N^2 = 1 \) and \( \{|u_N\} \) and \( \{|v_N\} \) are orthonormal bases in \( \mathcal{H}_{A,B} \), respectively. Then the entropy of entanglement is

\[ E(\psi) = e(c) := -\sum_{N=0}^{\infty} c_N^2 \log(c_N^2) \]  

(10)

Let \(|\psi_r\rangle\) denote the standard two-mode squeezed vacuum state with squeezing parameter \( r > 0 \). In the standard Fock basis, it takes the Schmidt form [8]:

\[ |\psi_r\rangle = \sum_{n=0}^{\infty} c_n |n\rangle_A \otimes |n\rangle_B = \sum_{n=0}^{\infty} c_n |n, n\rangle, \quad c_n = \frac{\tanh^n r}{\cosh^r} \]  

(11)

Here \(|n\rangle\) denotes the \( n \)th Fock state, i.e., \( A^\dagger A |n\rangle_A = n |n\rangle_A \) and \( B^\dagger B |n\rangle_B = n |n\rangle_B \) where \( A = \frac{x_A + ip_A}{\sqrt{2}} \) and \( B = \frac{x_B + ip_B}{\sqrt{2}} \) are annihilation operators. The entropy of entanglement and the \( \Delta \) value for the \(|\psi_r\rangle\) are as follows:

\[ E(\psi_r) = \cosh^2 r \log_2(\cosh^2 r) - \sinh^2 r \log_2(\sinh^2 r) \]  

\[ \Delta(\psi_r) = \cosh(2r) + \frac{2|a|}{a} \frac{\sinh 2r}{a^2 + 1/a^2} \]  

(13)

where \( \Delta(\psi_r) \in (0, 1) \), i.e.,

\[ 0 < \tanh r < -\frac{2|a|}{a} \frac{1}{a^2 + 1/a^2} < 1 \]  

(14)

reveals the existence of entanglement. Eq. (14) implies that \( a < 0 \). Any value of \( \Delta \in (0, 1) \) can be achieved by a two-mode squeezed state with squeezing parameter \( r_\Delta \) satisfying the constraint (14) for a given value of the parameter \( a \).
By plotting $\Delta(\psi_r)$ in the range $r \in (0, \tanh^{-1}(2/(a^2 + 1/a^2)))$ for a given $a < 0$ it can be seen that in small $r$, $\Delta$ decreases with $r$, while in large $r$, it grows up. In what follows, we will use the behavior of $\Delta$ in small $r$ in proving the proposition 1. So from two solutions of Eq. (13) for a given value of $\Delta$, the smaller solution for $r$ is the desired one. Hence, we have:

$$e^{-2r\Delta} = \frac{\Delta + \sqrt{\Delta^2 - 1 + 4/(a^2 + 1/a^2)^2}}{1 + \frac{2}{a^2+1/a^2}} := \Delta'$$

(15)

Figure 1: $\Delta(\psi_r)$ in terms of $r$ for various values of $a$.

With these definitions at hand, we can state the following proposition which is a generalized form of proposition 1 of Giedke et al. [8].

**Proposition 1**: For all $\psi \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, we have $E(\psi) \geq E(\psi_{r,\Delta(\psi)})$.

Like Giedke et al., we prove this proposition by two lemmas and the following definition. For a given $c$ we define

$$\delta(c) = 1 + 2 \sum_{N=0}^{\infty} (c_N^2 - \frac{2}{a^2+1/a^2} c_N c_{N-1}) N$$

(16)

By definition, it is clear that $\delta(c) \leq 1$. We have $\delta(c) = \Delta(\psi)$ whenever the Schmidt vectors coincides with Fock states.

**Lemma 1**: For all $\psi$ with Schmidt decomposition (9), $\Delta(\psi) \geq \delta(c)$.

Proof: Since $\delta(c) \leq 1$ we need only to consider $\psi$ with $\Delta(\psi) < 1$. In this case, we have:

$$\Delta(\psi) = 1 + \frac{2}{1 + 1/a^4} \sum_{N=0}^{\infty} c_N^2 \langle u_N|A^\dagger A|u_N \rangle + \frac{2}{1 + a^4} \sum_{N=0}^{\infty} c_N^2 \langle v_N|B^\dagger B|v_N \rangle$$
\[-\frac{2}{a^2 + 1/a^2} \sum_{N,M=0}^{\infty} c_N c_M \{ \langle u_M | A | u_N \rangle \langle v_N | B | v_M \rangle + c.c. \} \quad (17)\]

We note that \(\Delta(\psi) \geq \min\{Z(u), Z(v)\} := Z\) where:

\[Z = 1 + 2 \sum_{N=0}^{\infty} c_N^2 \langle u_N | A^\dagger A | u_N \rangle - \frac{4}{a^2 + 1/a^2} \sum_{N,M=0}^{\infty} c_N c_M |\langle u_M | A | u_N \rangle|^2 \quad (18)\]

It can be rewritten as

\[Z = \sum_{N=0}^{\infty} \sum_{M=N+1}^{\infty} (c_N^2 + c_M^2) - \frac{4}{a^2 + 1/a^2} c_N c_M) X_{N,M} + \sum_{N=0}^{\infty} c_N^2 (1 - \frac{2}{a^2 + 1/a^2}) X_{N,N} \quad (19)\]

where

\[X_{N,M} = |\langle u_N | A^\dagger | u_M \rangle|^2 + |\langle u_M | A^\dagger | u_N \rangle|^2. \quad (20)\]

To find an upper bound for \(Z\), we write:

\[c_N^2 + c_M^2 - \frac{4}{a^2 + 1/a^2} c_N c_M = (1 - \frac{4}{(a^2 + 1/a^2)^2}) c_N^2 + \frac{4}{(a^2 + 1/a^2)^2} c_N^2 + c_M^2 - \frac{4}{a^2 + 1/a^2} c_N c_M
\]

On the other hand, we have:

\[(\frac{2}{a^2 + 1/a^2} c_N - c_M)^2 = \left[ \frac{2}{a^2 + 1/a^2} c_N - c_N + \frac{2}{a^2 + 1/a^2} c_N + \frac{2}{a^2 + 1/a^2} c_N + \cdots
\]

\[+ \frac{2}{a^2 + 1/a^2} c_M - c_M + c_N + \cdots (1 - \frac{2}{a^2 + 1/a^2}) + c_N + 2(1 - \frac{2}{a^2 + 1/a^2})
\]

\[+ \cdots + c_M - (1 - \frac{2}{a^2 + 1/a^2})^2 \geq \sum_{R=N}^{\infty} \left( \frac{2}{a^2 + 1/a^2} c_R - c_R + 1\right)^2
\]

\[+ (1 - \frac{2}{a^2 + 1/a^2})^2 \sum_{R=N+1}^{\infty} c_R^2 = \sum_{R=N}^{\infty} \left( \frac{2}{a^2 + 1/a^2} c_R - c_R + 1\right)^2
\]

So we have:

\[c_N^2 + c_M^2 - \frac{4}{a^2 + 1/a^2} c_N c_M \geq \frac{4}{a^2 + 1/a^2} (1 - \frac{2}{a^2 + 1/a^2}) c_N^2 + (1 - \frac{2}{a^2 + 1/a^2})^2 \sum_{R=N}^{\infty} c_R^2
\]

\[+ \sum_{R=N}^{\infty} \left( \frac{2}{a^2 + 1/a^2} c_R - c_R + 1\right)^2 \geq \sum_{R=N}^{\infty} \sum_{M=N+1}^{\infty} X_{N,M} + X_{N,N}
\]

Therefore:

\[Z \geq (1 - \frac{2}{a^2 + 1/a^2}) \sum_{N=0}^{\infty} c_N \left[ \frac{4}{a^2 + 1/a^2} \sum_{M=N+1}^{\infty} X_{N,M} + X_{N,N}\right]
\]

\[+ \sum_{R=0}^{\infty} \left( \frac{2}{a^2 + 1/a^2} c_R - c_R + 1\right)^2 \sum_{N=0}^{\infty} \sum_{M=R+1}^{\infty} X_{N,M}
\]

\[+ (1 - \frac{2}{a^2 + 1/a^2})^2 \sum_{R=0}^{\infty} c_R^2 \sum_{N=0}^{\infty} \sum_{M=R+1}^{\infty} X_{N,M}
\]

By [8], we have \(\sum_{R=0}^{\infty} c_R^2 \geq R + 1\). Since \(2(1 - \frac{2}{a^2 + 1/a^2}) \geq \frac{4}{a^2 + 1/a^2} (1 - \frac{2}{a^2 + 1/a^2})\)

we can write:

\[Z \geq \frac{4}{a^2 + 1/a^2} (1 - \frac{2}{a^2 + 1/a^2}) \left[ \sum_{N=0}^{\infty} (c_N^2 |\langle u_N | A^\dagger | u_N \rangle|^2 + c_N^2 \sum_{M=N+1}^{\infty} X_{N,M})\right]
\]

\[+ (1 - \frac{2}{a^2 + 1/a^2})^2 \sum_{R=0}^{\infty} c_R^2 (R + 1)
\]

\[+ \sum_{R=0}^{\infty} c_R^2 \sum_{N=0}^{\infty} c_R (R + 1)
\]

\[+ \sum_{R=0}^{\infty} c_R^2 (R + 1) - \frac{4}{a^2 + 1/a^2} \sum_{R=0}^{\infty} c_R (R + 1)
\]
We conjecture that the following inequality holds:

\[
\sum_{N=0}^{\infty} c_N^2 \left[ |\langle u_N | A^† | u_N \rangle|^2 + \sum_{M=N+1}^{\infty} X_{N,M} \right] \geq \sum_{N=0}^{\infty} c_N^2 (N + 1)
\]

To confirm this conjecture, we give an example in Appendix. Accepting the conjecture completes the proof of the Lemma 1.

So for a given set of Schmidt coefficients \( c \), EPR- correlations are maximized if the Schmidt vectors are Fock states in the right order, i.e. \( |u_N \rangle = |v_N \rangle = |N \rangle \).

**Lemma 2**: For \( \Delta \in (0, 1) \), and any sequence \( c \) with \( \delta(c) = \Delta \), we have \( e(c) \geq e(c^{\Delta}) \equiv E(\psi_{r\Delta}) \), where \( c^{\Delta}_N \propto e^{-Nr\Delta} \) is the unique geometric sequence in \( C \) with \( \delta(c^{\Delta}) = \Delta \).

As in [8], the method of Lagrange multipliers is used. The Lagrange functional is:

\[
F(c, \lambda, \mu) := e(c) + \frac{\lambda}{2 \ln(2)}[\delta(c) - \Delta] + \frac{\mu + 1}{\ln(2)}(\|c\| - 1).
\]

(21)

where \( \mu \) and \( \lambda \) are positive Lagrange multipliers. Putting from (10) and (16), we obtain:

\[
c_N[N\lambda + \mu - \ln(c_N^2)] = -\lambda \frac{|a|/a}{a^2 + 1/a^2} \left[ Nc_{N-1} + (N + 1)c_{N+1} \right]
\]

(22)

We can divide (22) by \( c_N \) and subtract the same expression but for \( N + 1 \). Defining \( x_N := c_{N+1}/c_N =: \exp(-2r_N) \in (0, 1] \) for \( N = 0, 1, ... \) and writing \( \lambda = 2r/\sinh^2(r) \) for some \( r > 0 \) we find:

\[
x_{N+1} = x_N - A_N - B_N
\]

(23)

where

\[
A_N = \frac{4}{N + 2} \left[ \sinh^2(r_N) - \frac{a^2 + 1/a^2 r_N}{-|a|/a} \sinh^2(r) + 1/2 + 1/4 \frac{a^2 + 1/a^2}{|a|/a} \right],
\]

(24)

\[
B_N = \frac{N}{N + 2} \left[ \frac{1}{x_N} - \frac{1}{x_{N-1}} \right].
\]

(25)

As discussed in [8], for a fixed \( r > 0 \) and \( x_0 \), this implies that \( x_N = \exp(-2r) \) and hence \( c_N = c_0 e^{-2Nr} \) for all \( N \). By applying \( \sum_{N=0}^{\infty} c_N^2 = 1 \) we get:

\[
c_0^2 = \frac{1}{\sum_{N=0}^{\infty} \exp(-4Nr)} = 1 - \exp(-4r)
\]

(26)

\[
\delta(c) = 1 + 2c_0^2 \left[ \sum_N N \exp(-4Nr) \right][1 - \frac{2\exp(2r)}{a^2 + 1/a^2}]
\]

(27)

\[
\Delta = 1 + 2\left( 1 - \frac{2\exp(2r_\Delta)}{a^2 + 1/a^2} \right) \frac{\sum_N N \exp(-4Nr_\Delta)}{\sum_N \exp(-4Nr_\Delta)}
\]

(28)
Taking derivative of both sides of (26) with respect to \( r \), it follows:

\[
\sum_{N=0}^{\infty} N \exp(-4Nr) \left/ \sum_{N=0}^{\infty} \exp(-4Nr) \right. = \frac{\exp(-4r)}{(1 - \exp(-4r))}
\]

Replacing this into (27) and (28) and then equating them, we obtain:

\[
\exp(-2r) = \frac{1}{\tanh(r_\Delta)}
\]

So \( c_0 = \frac{1}{\cosh r_\Delta} \). Hence, for a given EPR-like uncertainty the squeezed states has minimal entanglement.

Finally, with the help of Lemmas 1 and 2, Proposition 1 is proved as in [8].

So far, we have shown that for a given value of EPR-like uncertainty, the pure two-mode squeezed state has a minimum entanglement compared to other two-mode pure states. Hence among pure-state decompositions of a TMGS, the optimal decomposition is one in which all the constituent pure states are two-mode squeezed states. As Wootters [2] has shown, the optimal decomposition also has the property that in which all constituent pure states have the same entanglement. For a TMGS \( \sigma \) with EPR-like uncertainty \( \delta \) given by (8), a decomposition having both properties consists of the standard two-mode squeezed state with squeezing parameter \( r_\delta \) and all of its displaced versions

\[
\sigma = \int d^2x d^2p D(\xi) |\psi_{r_\delta}\rangle \langle \psi_{r_\delta}| D^\dagger(\xi) g(\xi)
\]

(29)

Here \( D(\xi) = e^{i\xi^T \sigma_z \xi} \) is the Weyl displacement operator with \( \hat{\xi} = (\hat{x}_A, \hat{p}_A, \hat{x}_B, \hat{p}_B), \xi = (x_A, p_A, x_B, p_B) \) and \( \sigma_z \) is the Pauli matrix. To calculate unknown function \( g(\xi) \), we multiply both sides of (29) by \( D(\eta) \) and then take the trace

\[
\text{Tr} (D(\eta)\sigma) = \int d^2x d^2p \text{Tr} \left( D(\eta)D(\xi)|\psi_{r_\delta}\rangle \langle \psi_{r_\delta}| D^\dagger(\xi) g(\xi) \right)
\]

(30)

In this equation, \( \text{Tr} (D(\eta)\sigma) := \chi_\sigma(\eta) \) is the characteristic function of \( \sigma \). Using the identity \( D(\eta)D(\xi) = D(\xi)D(\eta) \exp \left( i \sum_{j=1}^{2} (x_j \hat{p}_j - p_j \hat{x}_j) \right) \) and the cyclic property of trace, we have:

\[
\chi_\sigma(\eta) = \int d^2x d^2p \exp \left( i \sum_{j=1}^{2} (x_j \hat{p}_j - p_j \hat{x}_j) \right) \text{tr}(D(\eta)|\psi_r\rangle \langle \psi_r|) g(\xi)
\]

(31)

Now, multiplying both sides by \( \exp(-i\eta^T J \xi') \), taking the integral with respect to \( r \) and using the following integral relation [14]

\[
\int d^n \lambda \exp \left( -\frac{1}{2} \lambda^T Q \lambda + i \lambda^T x \right) = \frac{(2\pi)^n \exp(-\frac{1}{2}x^T Q^{-1} X)}{\sqrt{\det(Q)}}
\]

(32)
with $Q$ a real positive definite symmetric matrix, $g(\xi)$ is achieved as

$$g(\xi) \propto \exp(\frac{1}{2} \xi^T (\gamma - \gamma_{\psi_{r\delta}})^{-1} \xi)$$  \hspace{1cm} (33)

Finally we have:

$$\sigma \propto \int d^2 \xi D(\xi) \langle \psi_{r\delta} | \psi_{r\delta} \rangle D(\xi) \exp\{\frac{1}{2} \xi^T (\gamma - \gamma_{\psi_{r\delta}})^{-1} \xi\}$$  \hspace{1cm} (34)

in which $\gamma - \gamma_{\psi_{r\delta}} \geq 0$ and $\gamma_{\psi_{r\delta}}$ is the CM of the two-mode squeezed vacuum state with squeezing parameter $r_{\delta}$.

Since $D(\xi)$ are local unitary operators, the average entanglement of the decomposition (34) is equal to $E[\psi_{r\delta}]$. We also introduce the auxiliary function $f : (0, 1] \rightarrow [0, \infty)$ as in [8]

$$f(\Delta) = c_+(\Delta) \log[c_+(\Delta)] - c_-(\Delta) \log[c_-(\Delta)]$$  \hspace{1cm} (35)

where $c_\pm = (\Delta^{-1/2} \pm \Delta^{1/2})^2/4$ and $f$ is a convex and decreasing function of $\Delta$ such that

$$E(\psi_{r\Delta}) = f(\Delta).$$  \hspace{1cm} (36)

**Proposition 2**: For a two-mode Gaussian state $\sigma$, we have

$$E_F(\sigma) = f(\delta')$$  \hspace{1cm} (37)

in which

$$\delta' = \frac{\delta + \sqrt{\delta^2 - 1 + 4/(a_0^2 + 1/a_0^2)^2}}{1 + a_0^2 + 1/a_0^2}$$  \hspace{1cm} (38)

with $\delta$ given by (8).

The proof of Proposition 2 is the same as the one of [8].

For a symmetric two-mode Gaussian state, we have $a_0 = 1$ and the result of [8] is reproduced. Our result also recover the Theorem 1 of [12] which states that the EOF of all bipartite Gaussian states with given EPR uncertainty is greater than or equal to the EOF of a symmetric Gaussian mixed state with the same EPR uncertainty.

### 3 Discussion and Conclusion:

We have determined the EOF of arbitrary two-mode Gaussian states by using Duan et al. inequality which gives a neccessary and sufficient separability condition for all two-mode Gaussian states. Motivated by this inequality and the work of Giedke et al. on symmetric two-mode Gaussian states, we have defined an EPR-like uncertainty. The EOF has been given as a function of EPR-like uncertainty and a real parameter. We expect that this work will provide new insight into the subject of Gaussian states entanglement.
4 Appendix

4.1 Example for conjecture of Lemma 1

Let
\[ |u_0\rangle = |0\rangle \]
\[ |u_1\rangle = \alpha_1|1\rangle + \alpha_2|2\rangle \quad |\alpha_1|^2 + |\alpha_2|^2 = 1 \]
\[ |u_2\rangle = \alpha_2^*|1\rangle - \alpha_1^*|2\rangle \]
\[ |u_N\rangle = |N\rangle \quad N \geq 3 \]

Then we have
\[
\sum_{N=0}^{\infty} c_N^2 \left[ |\langle u_N|A^\dagger|u_N\rangle|^2 + \sum_{M=N+1}^{\infty} X_{N,M} \right] = c_0^2 \left[ |\langle u_0|A^\dagger|u_0\rangle|^2 + \sum_{m=1}^{\infty} X_{0,m} \right] \\
+ c_1^2 \left[ |\langle u_1|A^\dagger|u_1\rangle|^2 + \sum_{m=2}^{\infty} X_{1,m} \right] + c_2^2 \left[ |\langle u_2|A^\dagger|u_2\rangle|^2 + \sum_{m=3}^{\infty} X_{2,m} \right] \\
+ \sum_{N=3}^{\infty} c_N^2 \left[ |\langle u_N|A^\dagger|u_N\rangle|^2 + \sum_{m=N+1}^{\infty} X_{N,M} \right]
\]

For the last term, we have
\[
+ \sum_{N=3}^{\infty} c_N^2 \left[ |\langle u_N|A^\dagger|u_N\rangle|^2 + \sum_{m=N+1}^{\infty} X_{N,M} \right] = \sum_{N=3}^{\infty} c_N^2 (N + 1)
\]

For the other terms, we have
\[
c_0^2 \left[ |\langle u_0|A^\dagger|u_0\rangle|^2 + \sum_{m=1}^{\infty} X_{0,m} \right] + c_1^2 \left[ |\langle u_1|A^\dagger|u_1\rangle|^2 + \sum_{m=2}^{\infty} X_{1,m} \right] \\
+ c_2^2 \left[ |\langle u_2|A^\dagger|u_2\rangle|^2 + \sum_{m=3}^{\infty} X_{2,m} \right] = c_0^2 (|\alpha_1|^2 + |\alpha_2|^2) \\
+ c_1^2 (2|\alpha_1|^2|\alpha_2|^2 + 2|\alpha_1|^4 + 2|\alpha_2|^4 + 3|\alpha_2|^2) + c_2^2 (2|\alpha_1|^2|\alpha_2|^2 + 3|\alpha_1|^2) \\
= c_0^2 + c_1^2 (2 - 2|\alpha_1|^2|\alpha_2|^2 + 3|\alpha_2|^2) + c_2^2 (2|\alpha_1|^2|\alpha_2|^2 + 3|\alpha_1|^2) \\
= c_0^2 + c_1^2 (2 + 2|\alpha_2|^2(1 - |\alpha_1|^2) + |\alpha_2|^2) + c_2^2 (2|\alpha_1|^2|\alpha_2|^2 + 3|\alpha_1|^2) \\
= c_0^2 + 2c_1^2 + c_1^2 (2|\alpha_2|^4 + |\alpha_2|^2) + c_2^2 (2(1 - |\alpha_2|^2)|\alpha_2|^2 + 3|\alpha_1|^2) \\
\geq c_0^2 + 2c_1^2 + c_2^2 \left[ 2|\alpha_2|^4 + |\alpha_2|^2 + 2|\alpha_2|^2 - 2|\alpha_2|^4 + 3|\alpha_1|^2 \right] = c_0^2 + 2c_1^2 + 3c_2^2
\]
References

[1] Charles H. Bennett, Gilles Brassard, Sandu Popescu, Benjamin Schumacher, John A. Smolin, and William K. Wootters, *Phys. Rev. Lett.* **76**, 722 (1996).

[2] William K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).

[3] V. Vedral and M. B. Plenio, *Phys. Rev. A* **57**, 1619 (1998).

[4] Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters, *Phys. Rev. A* **54**, 3824 (1996).

[5] A. Ferraro, S. Olivares, and M. G. A. Paris, *Gaussian States in Quantum Information* (Bibliopolis, Napoli, 2005).

[6] J. Eisert and M. B. Plenio, *Int. J. Quantum Inform.* **1**, 479 (2003).

[7] S. L. Braunstein and P. Van Loock, *Rev. Mod. Phys.* **77**, 513 (2005).

[8] G. Giedke, M. M. Wolf, O. Krüger, R. F. Werner, and J. I. Cirac, *Phys. Rev. Lett.* **91**, 107901 (2003).

[9] J. Solomon Ivan and R. Simon, arXiv: 0808.1658v1.

[10] Paulina Marian and Tudor A. Marian, *Phys. Rev. Lett.* **101**, 220403 (2008).

[11] Lu-Ming Duan, G. Giedke, J. I. Cirac, and P. Zoller, *Phys. Rev. Lett.* **84**, 2722 (2000).

[12] G. Rigolin and C.O. Escobar, *Phys. Rev. A* **69**, 012307 (2004).

[13] R. Simon, *Phys. Rev. Lett.* **84**, 2726 (2000).

[14] S. Olivares, *Eur. Phys. J. Special Topics*, **203**, 3 (2012).