Persistence of normally hyperbolic invariant manifolds in the absence of rate conditions

Maciej J Capiński¹ and Hieronim Kubica²,³

AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland

E-mail: kubica@agh.edu.pl

Received 30 May 2019, revised 24 March 2020
Accepted for publication 4 May 2020
Published 31 July 2020

Abstract

We consider perturbations of normally hyperbolic invariant manifolds, under which they can lose their hyperbolic properties. We show that if the perturbed map which drives the dynamical system preserves the properties of topological expansion and contraction, then the manifold is perturbed to an invariant set. The main feature is that our results do not require the rate conditions to hold after the perturbation. In this case the manifold can be perturbed to an invariant set, which is not a topological manifold. We work in the setting of nonorientable Banach vector bundles, without needing to assume invertibility of the map.

Keywords: normally hyperbolic manifolds, rate conditions, invariant sets, degree theory, intersection number

Mathematics Subject Classification numbers: 34C45, 34D10, 34D35, 55M25.

(Some figures may appear in colour only in the online journal)

1. Introduction

We will be investigating the persistence under perturbations of invariant sets that are associated with normally hyperbolic invariant manifolds (NHIMs). These perturbations will be such that the manifolds lose their hyperbolic properties.

¹Partially supported by the NCN grants 2015/19/B/ST1/01454, 2016/22/A/ST1/00077, 2016/21/B/ST1/02453.
²Partially supported by the NCN grant 2016/21/B/ST1/02453 and by the Faculty of Applied Mathematics AGH UST dean grant for PhD students and young researchers within subsidy of Ministry of Science and Higher Education.
³Author to whom any correspondence should be addressed.
To be more precise, a manifold $\Lambda$ is said to be an NHIM if it is invariant for a dynamical system and there is a splitting of the state space into three invariant subbundles. One is the tangent bundle to $\Lambda$, the second is the unstable bundle and the third is the stable bundle. The dynamics on the stable bundle is contracting and on the unstable bundle—expanding. The key feature for $\Lambda$ to be normally hyperbolic is that the dynamics on the bundle tangent to $\Lambda$ is weaker than the dynamics on the stable and the unstable bundles. The property of the dominance of the dynamics on the stable/unstable bundles over the tangent bundle is formulated in terms the rate conditions, introduced by Fenichel [8–11], Hirsch, Pugh, Shub [19], and later developed by Chaperon [5–7].

The main property of NHIMs is that they persist under perturbations. As long as the rate conditions hold, the manifold is present. There are examples though [15, 16, 20, 27] for which, in the absence of rate conditions, an invariant manifold can be destroyed to a set which is not even a topological manifold. However, this does not mean that the manifold vanishes or that it is completely destroyed.

This problem has been studied by Floer in [12, 13]. He introduced a method, which allowed him to establish continuation of NHIMs to invariant sets which preserve the cohomology ring of the manifold under perturbation. We take a different approach, which is based on a good topological alignment expressed by homotopy conditions. We establish existence of an invariant set whose projection onto the base manifold $\Lambda$ is equal to the whole $\Lambda$. The advantage of our method is that it does not rely on the prior existence of a normally hyperbolic invariant manifold and neither does it use perturbation theory. Moreover, we prove a continuation theorem for invariant sets of continuous one-parameter families of maps under the assumption of correct topological alignment. To be more precise, we show that if we extend the system to include the parameter, then, in such an extended phase space, there exists a compact connected component consisting of points belonging to the invariant sets of maps corresponding to varying parameter values.

Our result does not contradict the work of Mañé [24]. He shows that if a manifold is persistent, then it has to be normally hyperbolic. What we establish though is not persistence of manifolds, but persistence of sets. In fact, we do not need the normally hyperbolic invariant manifold to exist. If we have a family of maps that satisfy our topological assumptions, then we will have persistence of the family of their invariant sets.

The main features of our results are the following. Our work is written in the context of Banach vector bundles, without any orientability assumptions. We establish the existence of non-empty invariant sets for discrete dynamical systems. These sets are not only non-empty, but also have non-empty intersections with each fibre of the vector bundle, meaning that they project surjectively onto the base manifold. We do not need to assume that our map is invertible. We do not need a normally hyperbolic manifold prior to perturbation; our method can be used to establish the existence of invariant sets with ‘topologically normally hyperbolic’ properties. If the assumptions of our theorems are verified, then we obtain the existence of invariant sets within their specific, explicitly given neighbourhood. Verification can be performed using rigorous, interval arithmetic numerics, leading to computer assisted proofs. Our results are written in the context of discrete dynamical systems, but they can also be applied to ODEs by considering a time-shift map.

Our approach is based on the method of covering relations [14, 31, 32]. The following results can be thought of as its generalisation to vector bundles. Covering relations have proven to be a useful tool that, combined with cone conditions, leads to geometric proofs of normally hyperbolic invariant manifold theorems [3, 4]. These results, however, rely also on a form of rate conditions, expressed in terms of cone conditions. Another result in this flavour is [1], which contains another geometric version of the normally hyperbolic invariant manifold theorem.
Although again, it relies on rate conditions and on perturbative methods. Our work is closely related to [2], which can also be applied in the absence of rate conditions. The difference is that in [2] only the case of trivial vector bundles and invertible maps was considered. This paper can be thought of as a generalisation of [2] to the setting of general, possibly nonorientable vector bundles, without the assumption on invertibility of the map. Moreover, in the present work we obtain a continuation result, which states that in the state space extended to include a parameter, the invariant sets for a family of maps contain a connected component which links them together.

The paper is organised as follows. Section 2 contains preliminaries. There we set up our notations used for vector bundles and introduce the notion of an intersection number. The intersection number is a standard tool in differential topology, which can be used to detect intersections of manifolds based on their homotopy properties. In section 3 we state our main results, which are formulated in theorems 8, 13, 15 and 16, we also show that normal hyperbolicity implies the assumptions of the theorems, and give an example of application. Sections 6–9 contain the proofs of the four theorems. Section 10 contains the proof of the fact that normal hyperbolicity implies topological covering. Section 11 contains acknowledgements. To keep the paper self-contained and also since our approach to the intersection number is slightly non-standard (we allow our manifolds to have boundaries), we add the construction of the intersection number in appendix A.

2. Preliminaries

2.1. Notations

For a set $A$ in some topological space we use $\partial A$ to denote its boundary, $\overline{A}$ to denote its closure, and $\text{int } A$ to denote its interior. We write $\# A$ to denote the cardinality of $A$.

For a compact connected manifold $\Lambda$ and a continuous map $f : \Lambda \to \Lambda$ we shall use $\text{deg}_2 f$ to denote the degree modulo 2 of $f$ (see [18] for details).

For two sets $A, B \subset \mathbb{R}^n$ we shall use $\text{dist}(A, B)$ to denote the distance between them. We will use the notation $B_{\mathbb{R}^n}(x, r)$ to stand for an open ball centred at $x$ of radius $r$ in $\mathbb{R}^n$.

2.2. Banach vector bundles

In this section we set up some notations for Banach vector bundles, which will be used throughout the paper.

Let $\Lambda$ be a topological space. We recall that a vector bundle of rank $k$ over $\Lambda$ is a topological space $E$ together with a surjective continuous map $\pi : E \to \Lambda$ satisfying the following conditions:

(a) For all $\theta \in \Lambda$, the fibre $E_\theta := \pi^{-1}(\theta)$ over $\theta$ is a $k$-dimensional vector space.

(b) For every $\theta \in \Lambda$ there exists an open neighbourhood $U_\theta$ of $\theta$ in $\Lambda$ and a homeomorphism

$$\Phi_\theta : \pi^{-1}(U_\theta) \to U_\theta \times \mathbb{R}^k,$$

called a local trivialisation of $E$ over $U_\theta$, such that:

- $\pi_\theta \circ \Phi_\theta = \pi$, where $\pi_\theta : U_\theta \times \mathbb{R}^k \to U_\theta$ is the projection on $U_\theta$.

- For every $\lambda \in U_\theta$ the restriction of $\Phi_\theta$ to the fibre $E_\lambda$

$$\Phi_\theta|_{E_\lambda} : E_\lambda \to \{\lambda\} \times \mathbb{R}^k \cong \mathbb{R}^k$$

is a vector space isomorphism. The set $U_\theta$ is called the base of the local trivialisation $\Phi_\theta$. 
The space $E$ is called the \textit{total space} of the bundle, $\Lambda$ is called its \textit{base}, and $\pi$ is its \textit{projection}. In our paper we will be dealing with smooth vector bundles, meaning that $\Lambda$ and $E$ will be smooth manifolds and the projection will be a smooth map.

When $\Phi_{\theta_1} : \pi^{-1}(U_{\theta_1}) \rightarrow U_{\theta_1} \times \mathbb{R}^k$ and $\Phi_{\theta_2} : \pi^{-1}(U_{\theta_2}) \rightarrow U_{\theta_2} \times \mathbb{R}^k$ are two local trivialisations of $E$ such that $U_{\theta_1} \cap U_{\theta_2} \neq \emptyset$, and $\lambda \in U_{\theta_1} \cap U_{\theta_2}$, the function

$$\left(\pi_{\mathbb{R}^k} \circ \Phi_{\theta_2}|_{E_{\lambda}} \right) \circ \left(\pi_{\mathbb{R}^k} \circ \Phi_{\theta_1}|_{E_{\lambda}} \right)^{-1} : \mathbb{R}^k \rightarrow \mathbb{R}^k$$

is called a \textit{transition function} between local trivialisations.

If we are given a vector bundle $\pi : E \rightarrow \Lambda$ with a fixed collection of local trivialisations $\{ \Phi_\theta : \pi^{-1}(U_\theta) \rightarrow U_\theta \times \mathbb{R}^k \}$ whose bases form an open cover $U = \{ U_\theta \}$ of $\Lambda$, then we call it a \textit{Banach vector bundle} provided that all transition functions between local trivialisations with overlapping bases are isometries.

Henceforth, we shall assume that every vector bundle we work with is a Banach vector bundle even if it is not explicitly pronounced.

For Banach vector bundles we are able to introduce a meaningful notion of a norm on fibres as follows. For every $v \in E$ such that $\pi(v) \in U_\theta$, where $U_\theta$ is trivialised by $\Phi_\theta$, we define

$$\|v\| := \|\pi_{\mathbb{R}^k} \circ \Phi_\theta(v)\|_{\text{Eucl}}$$

where $\|\cdot\|_{\text{Eucl}}$ is the Euclidean norm on $\mathbb{R}^k$. Since all transition functions between local trivialisations with overlapping bases are isometries, we see that $\|v\|$ does not depend on the choice of $\Phi_\theta$.

\textbf{Remark 1.} We use the name \textit{Banach vector bundle} since in our setting the fibres are finite dimensional Banach spaces. By writing \textit{Banach vector bundle} we implicitly assume that the transition functions are isometries, which is somewhat non-standard and needs to be emphasised. Moreover, we do not consider vector bundles with infinite-dimensional fibres, which the prefix ‘Banach’ is often assumed to imply.

\textbf{Remark 2.} For $v \in E$ the notation $\|v\|$ should be understood as the norm on the fibre $E_{\pi(v)}$. (It makes no sense to talk of a norm on $E$, since it is not a vector space).

\subsection{2.3. Whitney sum of Banach vector bundles}

Consider a smooth manifold $\Lambda$, a rank-$u$ smooth Banach vector bundle $\pi^u : E^u \rightarrow \Lambda$ with a fixed collection of local trivialisations

$$\{ \Phi_\theta^u : (\pi^u)^{-1}(U_\theta) \rightarrow U_\theta \times \mathbb{R}^u | \ U_\theta \text{ cover } \Lambda \}$$

inducing a Banach space structure on the fibres of the total space $E^u$ and a rank-$s$ smooth Banach vector bundle $\pi^s : E^s \rightarrow \Lambda$ with fixed

$$\{ \Phi_\theta^s : (\pi^s)^{-1}(V_\theta) \rightarrow V_\theta \times \mathbb{R}^s | \ V_\theta \text{ cover } \Lambda \}$$

inducing a Banach space structure on the fibres of $E^s$.

We combine the two vector bundles in what is called a \textit{Whitney sum} to produce a new vector bundle $E = E^u \oplus E^s$ of rank $u + s$ over $\Lambda$, defined as

$$E = E^u \oplus E^s := \bigsqcup_{\theta \in \Lambda} E^u_\theta \oplus E^s_\theta,$$
where \( \bigcup \) stands for the disjoint union. The fibre \( E_\theta \) of \( E \) over each \( \theta \in \Lambda \) is the direct sum \( E_\theta^0 \oplus E_\theta^1 \). The projection \( \pi : E = E^0 \oplus E^1 \to \Lambda \) is the natural one.

**Notation 3.** To represent a point \( v \in E = E^0 \oplus E^1 \) we shall identify it with a triple \((\theta; x, y)\), where \( \theta = \pi(v) \) and \( v = (x, y) \in E_\theta^0 \oplus E_\theta^1 \). In other words, by writing

\[
v = (\theta; x, y) \in E
\]

we intend to emphasise that \( x \in E_\theta^0 \) and \( y \in E_\theta^1 \).

For \( W_\theta \subset \Lambda \) small enough so that \( \Phi^*_\theta \) and \( \Phi^1_\theta \) are both defined over \( W_\theta \), we define the local trivialisations \( \Phi_\theta : \pi^{-1}(W_\theta) \to W_\theta \times \mathbb{R}^{s+s} \) in the natural way. For any \( \lambda \in W_\theta \) and \( v = (\lambda; x, y) \in E \),

\[
\Phi_\theta(\lambda; x, y) := (\lambda; \pi_2 v \circ \Phi^*_\theta(x), \pi_2 v \circ \Phi^1_\theta(y)).
\]

We will write \( \|x\|_a \) for the norm on the fibre \( E_\sigma^a(x) \) and, similarly \( \|y\|_a \) for the norm on the fibre \( E_\sigma^b(y) \).

### 2.4. Intersection number

In this section we introduce the intersection number, which will be the main tool in our proofs. It is a standard notion in differential topology. (We suggest [17, 18] as references.)

Let \( X \) be a compact set (subset of some smooth manifold). Assume that \( \text{int} X = X \) is a smooth manifold. (We do not need to assume that \( X \) is a manifold with boundary; i.e., we do not need \( \partial X \) to be a smooth manifold.) Let \( Y \) be a boundaryless smooth manifold. Let \( Z \) be an embedded boundaryless smooth submanifold of \( Y \), let \( \bar{Z} \) be its closure in \( Y \), and let \( \partial Z = \bar{Z} \setminus Z \) (which, in general, can be empty). Assume \( X \) and \( Z \) to be of complementary dimension with respect to \( Y \), i.e., \( \dim X + \dim Z = \dim Y \).

We say that a smooth map \( f : X \to Y \) is transversal to \( Z \) if

\[
Df_x(T_xX) + T_{f(x)}Z = T_{f(x)}Y
\]

for all \( x \in f^{-1}(Z) \). \( T_xX \) stands for the tangent space to \( X \) at \( x \); \( Df_x \) denotes the differential of \( f \) at \( x \).

**Definition 4.** We shall say that \( f : X \to Y \) is admissible if it is continuous and

\[
f(\partial X) \cap \bar{Z} = \emptyset,
\]

\[
f^{-1}(X) \cap \partial Z = \emptyset.
\]

**Definition 5.** We shall say that a continuous homotopy \( H : [0, 1] \times \overline{X} \to Y \) is admissible if (see figure 1)

\[
H([0, 1] \times \partial X) \cap \bar{Z} = \emptyset,
\]

\[
H([0, 1] \times X) \cap \partial Z = \emptyset.
\]

The *modulo 2 intersection number* for an admissible map \( f : X \to Y \) and \( Z \subset Y \) is defined as a number

\[
I_2(f, X, Z) \in \{0, 1\},
\]

which possesses the following properties:
• (Intersection number for transversal maps). If $f|_X$ is smooth and transversal to $Z$ then
  \[ I_2(f, X, Z) = \# f^{-1}(Z) \mod 2. \]

• (Homotopy property). If $f, g$ are homotopic through an admissible homotopy, then
  \[ I_2(f, X, Z) = I_2(g, X, Z). \]

• (Intersection property). If $I_2(f, X, Z) = 1$ then $f(X) \cap Z$ is nonempty.

• (Excision property). If $V$ is an open subset of $X$ such that $f(X) \cap Z = f(V) \cap Z$ and $f(\partial V) \cap Z = \emptyset$ then
  \[ I_2(f, X, Z) = I_2\left(f|_V, V, Z\right). \]

When $f : \overline{X} \to Y, \overline{V} \subset \overline{X}$ and $f|_\overline{\mathcal{V}} : \overline{\mathcal{V}} \to Y$ is admissible, then we will write $I_2(f, V, Z)$ instead of $I_2\left(f|_V, V, Z\right)$ to simplify notation.

In figure 1 we find the intuition behind the definition. There, while passing through an admissible homotopy, we encounter a tangential intersection, but the number of transversal intersections is either 1 or 3, so the mod 2 intersection number is 1. On the picture the $f(\partial X)$ and $g(\partial X)$ are indicated by dots. These need to be disjoint from $Z$ throughout the admissible homotopy. The $\partial Z$ is depicted with squares. It needs to be disjoint from the image of $\overline{X}$ throughout the homotopy.

In the standard approach $X$ is assumed to be a compact boundaryless manifold and $Z$ is assumed to be a closed boundaryless submanifold of $Y$. Here we allow for $X$ and $Z$ to have boundaries, since this will be convenient in our application. We deal with the boundary by restricting to admissible maps and admissible homotopies, which rule out the intersection for points from the boundaries. In such a case, the existence and properties of the intersection number follow in the same way as the construction for manifolds without boundary [17, 18].

To keep the paper self-contained, and since allowing $X$ and $Z$ to have a boundary is slightly nonstandard, we have added the construction of the intersection number in appendix A.

**Remark 6.** In the same way as above we can also allow $X$ and $Z$ to have boundaries in the case of the oriented intersection number. (See [17, 18] for the definition of the oriented intersection number).

### 3. Main results

Assume that $\Lambda$ is a compact smooth $c$-dimensional manifold without boundary, $E^u, E^s$ are smooth Banach vector bundles over $\Lambda$, and that $E = E^u \oplus E^s$. We define the following sets (below and through the reminder of the paper we use the convention from notation 3).
\[ D := \{ (\theta; x, y) \in E | \theta \in \Lambda, \|x\|_u \leq 1, \|y\|_s \leq 1 \}, \]
\[ D^- := \{ (\theta; x, y) \in E | \theta \in \Lambda, \|x\|_u = 1, \|y\|_s \leq 1 \}, \]
\[ D^+ := \{ (\theta; x, y) \in E | \theta \in \Lambda, \|x\|_u \leq 1, \|y\|_s = 1 \}. \tag{1} \]

For \( \theta \in \Lambda \) and \( U \subset \Lambda \) we define the following subsets of \( E \):
\[ D_0 := D \cap E_\theta, \quad D^-_\theta := D^- \cap E_\theta, \quad D_\Theta := \bigcup_{\theta \in U} D_\theta. \]

We will also use the following notation for a closed unit ball in a fibre \( E^u_\theta \)
\[ B^u_\theta := \{ x \in E^u_\theta | \|x\|_u \leq 1 \}. \]

### 3.1. Existence and continuation of invariant sets

In this section we formulate our four main theorems. We first introduce a definition that is required to express the assumptions of our first main result. This is a generalisation of the notion of ‘covering relations’ which was introduced in \[14, 31, 32\]. There the covering involves a topological expansion of a set in the direction of hyperbolic expansion, and topological contraction of the set in the direction of hyperbolic contraction. Our approach is an extension of the notion to vector bundles that also have central directions associated with the base manifold.

**Definition 7.** Consider a continuous map \( f: D \to E \) (not necessarily invertible). For \( \theta \in \Lambda \) we say that \( D_\theta \) \( f \)-covers \( D \), denoted \( D_\theta \overset{f}{\to} D \), if the following conditions are satisfied:

(a) There exists a homotopy \( h_\theta : [0, 1] \times D_\theta \to E \) such that the following hold true
\[ h_\theta(0, \cdot) = f(\cdot), \]
\[ h_\theta([0, 1] \times D^-_\theta) \cap D = \emptyset, \]
\[ h_\theta([0, 1] \times D_\theta) \cap D^+ = \emptyset. \]

(b) One of the following is satisfied:

1. If \( u > 0 \), then there exists a \( \Theta \in \Lambda \) (which can depend on \( \theta \)) and a linear map \( A_\theta : E^u_\theta \to E^u_\Theta \) such that \( A_\theta(\partial B^u_\theta) \subset E^u_\Theta \setminus B^u_\Theta \) (\( A_\theta \) is expanding) and
\[ h_\theta(1, (\theta; x, y)) = (\Theta; A_\theta x, 0) \in E_\Theta. \]

2. If \( u = 0 \), then there exists a point \( \Theta \in \Lambda \) (which can depend on \( \theta \)), such that
\[ h_\theta(1, (\theta; y)) = (\Theta; 0) \in E_\Theta = E^s_\Theta. \]

(In the above line we have omitted \( x \) from the notation \( (\theta; x, y) \) since \( E^u \) is of dimension zero).

The intuition behind definition 7 is depicted in figure 2. There we consider \( \Lambda \) to be a circle, and \( E^u \) and \( E^s \) to be trivial bundles over \( \Lambda \) with real one dimensional fibres; in short, we consider \( E = S^1 \times \mathbb{R} \times \mathbb{R} \). On the plot, the front and the back sides (i.e. \( D_{\theta=0} \) and \( D_{\theta=2\pi} \)) of the set \( D \) are identified to be the same. For the conditions of definition 7 to hold we need to have topological expansion on the \( x \) coordinates. This means that the ‘exit set’ \( D^-_\theta \) will be mapped outside of
Figure 2. Example of a homotopy from definition 7 of the covering \( D_\theta \rightarrow D \).

In addition, we also need topological contraction on the coordinate \( y \). This ensures that \( f(D_\theta) \) will not intersect with \( D^+ \). We impose quite mild conditions on the dynamics on \( \theta \). It is enough that the correct topological alignment can be pulled by a homotopy to a fibre \( E_\theta \). Note that in definition 7 we do not require the map to carry fibres into fibres, as is the case in the setting of normal hyperbolicity. Such assumption is not needed for any of our results in this paper.

We now formulate our first main result.

**Theorem 8.** If \( f : D \rightarrow E \) is a continuous mapping and \( D_\theta \rightarrow D \) holds for every \( \theta \in \Lambda \), then for any \( \theta \in \Lambda \) there exists a trajectory starting from \( D_\theta \), which remains in \( D \) for all forward iterates, i.e., there exists \( v \in D_\theta \) such that \( f^m(v) \in D \) for all \( m \in \mathbb{N} \).

The proof is given in section 6.

Theorem 8 establishes the existence of points that remain in \( D \) for all iterates of a map when going forwards in time. Now we turn to what happens also backwards in time. For this we make an additional assumption that \( \Lambda \) is a connected manifold.

**Definition 9.** Consider a continuous map \( f : D \rightarrow E \) (not necessarily invertible). We say that \( D \) covers \( D \), denoted \( D \rightarrow D \), if the following conditions are satisfied:

(a) There exists a homotopy \( h : [0, 1] \times D \rightarrow E \) such that the following hold true

\[
\begin{align*}
\text{deg}_2(h) &\neq 0, \\
\text{deg}_2(\eta) &\neq 0,
\end{align*}
\]

(b) There exists a continuous map \( \eta : \Lambda \rightarrow \Lambda \) for which

\[
\text{deg}_2(\eta) \neq 0,
\]

moreover,

1. If \( u > 0 \), then for any \( \theta \in \Lambda \) there exists a linear map \( A_\theta : E_\theta^u \rightarrow E_{\eta(\theta)}^u \) such that \( A_\theta(\partial B_\theta^u) \subset E_{\eta(\theta)}^u \) (\( A_\theta \) is expanding) and

\[
h(1, (\theta; x, y)) = (\eta(\theta); A_\theta x, 0) \in E_{\eta(\theta)}^u.
\]

2. If \( u = 0 \), then

\[
h_\theta(1, (\theta; y)) = (\eta(\theta); 0) \in E_{\eta(\theta)}^s = E_{\eta(\theta)}^s.
\]
The intuition behind definition 9 is similar to what we discussed for definition 7. We need to have topological expansion in $x$ and topological contraction in $y$. In addition, we assume that the dynamics on $\theta$ is homotopic to some map with nonzero degree. Such property is visualised in figure 3.

We make a couple of remarks before we formulate our second main result.

**Remark 10.** Condition $D \xrightarrow{f} D$ implies that $D \xrightarrow{f} D$ holds for any $\theta \in \Lambda$. (This follows by taking $h_\theta = h|_{[0,1] \times D_\theta}$ and $\Theta = \eta(\theta)$.) The implication in the other direction is not always true. For instance, when $\pi \circ f(D) = \Theta$, meaning that $f$ maps to a single fibre $E_{\Theta}$, then we can have $D \xrightarrow{f} D$ for any $\theta \in \Lambda$, but $\eta$ satisfying (4) will not exist.

**Remark 11.** Condition (4) is quite natural for stroboscopic (time-shift) maps of flows. In such setting, if the time shift along the flow used to define the map is small enough, then it is possible to find a homotopy to $\eta$ chosen to be the identity on $\Lambda$. Condition (4) is also automatically fulfilled in the setting of normal hyperbolicity; we show this in lemma 20.

**Remark 12.** In (4) we use the degree modulo 2 of a map. This is because we do not wish to impose any orientability assumptions. If the considered manifolds $\Lambda$ and $E$ are orientable, one could use the Brouwer degree instead. Condition (4) can also be replaced by requiring that the degree computed at every point in $\Lambda$ is nonzero (Brouwer degree computed at every point in $\Lambda$ is nonzero, if $\Lambda$ and $E$ are orientable); for which we do not need $\Lambda$ to be connected. (These generalisations are highlighted in the footnote on page [28] during the proof of theorem 13).

We now formulate our second main result:

**Theorem 13.** If $f: D \to E$ is a continuous mapping and $D \xrightarrow{f} D$, then for every $\theta \in \Lambda$ there exists an orbit in $D$ passing through $D_{\theta}$, i.e., there exists a sequence $\{v_i\}_{i \in \mathbb{Z}} \subset D$, such that $v_0 \in D_\theta$ and $f(v_i) = v_{i+1}$, for all $i \in \mathbb{Z}$.

The proof is given in section 7.

**Remark 14.** In theorems 8 and 13 we obtain sets of points that remain in $D$ for iterates of a single map $f$. We can in fact just as well compose sequences of maps.

To be precise let $f_i : E \to E$ be a sequence of continuous maps and consider a dynamical system

$$v_{i+1} = f_i(v_i).$$

Using mirror arguments to those used for the proof of theorem 8 we can obtain forward trajectories of (5) in $D$ as long as $D \xrightarrow{f_i} D$ for all $i \in \mathbb{N}$ and all $\theta \in \Lambda$. 

4975
Similarly, if \( D := f_{\alpha}^{-1} D \) for all \( i \in \mathbb{Z} \), then using mirror arguments to those used for the proof of theorem 13 we can obtain full trajectories of (5) in \( D \).

The minor modifications needed for these results are highlighted in the footnotes during the course of the proofs of theorems 8 and 13 on pages [20] and [26], respectively.

We also have the following continuation results for continuous families of maps, which satisfy the covering condition.

**Theorem 15.** Assume that we have a family of maps \( f_{\alpha} : D \to E \), which depends continuously on \( \alpha \in [0, 1] \). If for all \( \alpha \in [0, 1] \) and all \( \theta \in \Lambda \), \( D_{\theta} f_{\alpha} = D \), then for any \( \theta \in \Lambda \) there exists a compact connected component \( C \) of \( [0, 1] \times D \) which meets both \( \{0\} \times D_\theta \) and \( \{1\} \times D_\theta \), such that for any \( (\alpha, v) \in C \), \( f^n_{\alpha}(v) \in D \) for all \( n \in \mathbb{N} \).

The proof is given in section 8.

**Theorem 16.** Assume that we have a family of maps \( f_{\alpha} : D \to E \), which depends continuously on \( \alpha \in [0, 1] \). If for all \( \alpha \in [0, 1] \), \( D_{\theta} f_{\alpha} = D \), then for any \( \theta \in \Lambda \) there exists a compact connected component \( C \) of \( [0, 1] \times D_\theta \) which meets both \( \{0\} \times D_\theta \) and \( \{1\} \times D_\theta \), such that for any \( (\alpha, v) \in C \) there exists an orbit of \( f_{\alpha} \) in \( D \) passing through \( v \), i.e., there exists a sequence \( \{v_i\}_{i \in \mathbb{Z}} \subset D \) such that \( v_0 = v \in D_\theta \) and \( f_{\alpha}(v_i) = v_{i+1} \), for all \( i \in \mathbb{Z} \).

The proof is given in section 9.

**Remark 17.** We can not assert that \( C \) from theorems 15 and 16 is path connected. We can see this if we take \( f_0 : \mathbb{R} \to \mathbb{R} \) with \( f_0(\alpha) = x + \alpha \left( x - \frac{1}{2} \sin \frac{1}{\alpha} \right) + g(\alpha) \), where \( g : \mathbb{R} \to \mathbb{R} \) is continuous with \( g([-1/2, 1/2]) = 0 \), \( g(\{x > 1/2\}) > 0 \) and \( g(\{x < -1/2\}) < 0 \) (see figure 4). For \( \alpha \in (0, 1] \) we have a family of hyperbolic fixed points \( x_\alpha := \frac{1}{2} \sin \frac{1}{\alpha} \) of \( f_0 \). Assumptions of theorems 15 and 16 hold, since for all \( \alpha \in [0, 1] \) the maps \( f_0 \) stretch the interval \( D = [-1, 1] \). We can not find a path connecting the set of fixed points \( \left[-\frac{1}{2}, \frac{1}{2}\right] \) of \( f_0 \) with \( x_\alpha \). Nevertheless, we see that \( C = \{0\} \times \left[-\frac{1}{2}, \frac{1}{2}\right] \cup \{(\alpha, x_\alpha) \mid \alpha \in (0, 1]\} \) is connected (but not path connected).

**Remark 18.** In the definition of the set \( D \) we fixed the norms to be less than or equal to one. This does not make it less general, since our results will hold in any setting that is homeomorphic to the above.
3.2. Application in the context of normal hyperbolicity

Below we give a corollary to bridge our results with the theory of NHIMs. Before we proceed, we briefly recall the definition.

**Definition 19.** Let $M$ be a smooth manifold and $f: M \to M$ a diffeomorphism. A manifold $\Lambda \subset M$, invariant under $f$, i.e., $f(\Lambda) = \Lambda$, is said to be *normally hyperbolic* if there exist a constant $C > 0$, rates $0 < \lambda < \mu^{-1} < 1$ and a splitting

$$T_\Lambda M = T_\Lambda \Lambda \oplus E^u \oplus E^s,$$

which is invariant under the action of the differential $df$ and such that for $\theta \in \Lambda$

$$x \in E^u_\theta \iff \|df^\theta(x)\| \leq C\lambda^{|i|}\|x\|, \quad i \leq 0,$$

$$y \in E^s_\theta \iff \|df^\theta(y)\| \leq C\lambda^{|i|}\|y\|, \quad i \geq 0,$$

$$w \in T_\theta \Lambda \iff \|df^\theta(w)\| \leq C\mu^{|i|}\|w\|, \quad i \in \mathbb{Z}.$$  

We have the following lemma which states that normal hyperbolicity implies the covering condition.

**Lemma 20.** Let $\Lambda$ be a compact normally hyperbolic invariant manifold for a diffeomorphism $f : M \to M$, and let $k \in \mathbb{N}$ satisfy $k > \log \frac{1}{\lambda}$. Then there exists a neighbourhood $D$ of $\Lambda$ such that $D f^k \to D$.

The proof is given in section 10.

From theorem 13 and lemma 20 we obtain the following corollary.

**Corollary 21.** Assume that $\Lambda$ is a compact normally hyperbolic invariant manifold for a diffeomorphism $f : M \to M$, and assume that $D$ is such that $D f^k \to D$. Let $f_\alpha : M \to M$ be a family of continuous maps, which depends continuously on $\alpha$. Assume that $f_0 = f$. Then for $\alpha$ for which

$$D f^k \to D,$$  

$\Lambda$ persists as an invariant set of $f_\alpha$. Moreover, this set projects surjectively onto $\Lambda$.

Corollary 21 ensures that for a perturbation of $f$ the NHIM will persist as an invariant set. In [13] Floer proved a similar result. He has shown that if $f_\alpha$ are homeomorphisms which are close enough to $f$, then the NHIM persists along with its cohomology ring. The first difference between our result and Floer’s is that corollary 21 provides a verifiable condition (10) for the persistence of the NHIM, effectively getting rid of the ‘close enough’ part of the Floer’s statement. (For small $\lambda$ (10) will hold, and for a particular system we can explicitly check for which $\lambda$ (10) will be satisfied.) In our setting, the existence of the NHIM is in fact not even necessary, since (10) alone establishes the existence of an invariant sets. Another difference is that in corollary 21 it is enough that $f_\alpha$ are continuous; we do not need them to be homeomorphisms as is required in [13]. Floer proves that the cohomology ring of the invariant set which persists contains the cohomology ring of the original manifold as a subring. We prove that the topology of the original manifold is in a sense ‘preserved’, but in our statement this is expressed by the fact that the invariant set which persists projects surjectively onto the original NHIM.
Moreover, we know by theorem 16 that the invariant manifold ‘continues’ in the sense that the sets for different parameters are linked on each fibre by a compact connected component.

A desirable feature of our result is that the covering condition (10) can be checked using computer assisted techniques, which makes our results applicable in practice.

From remark 14 and lemma 20 we also obtain the following result for random perturbations of NHIMs. (See [23] for a similar persistence result of NHIMs for random perturbations of flows).

Corollary 22. Let $\Lambda$ be a compact normally hyperbolic invariant manifold for a diffeomorphism $f : M \to M$, and let $k \in \mathbb{N}$ satisfy $k > \log \frac{1}{\lambda_1}$. Assume that $D$ is such that $D \xrightarrow{\phi} D$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $T : \Omega \to \Omega$ and let $\phi : \mathbb{Z} \times \Omega \times M \to M$ be a random dynamical system over $T$, i.e. $\phi(0, \omega) = \text{Id}$ and

$$\phi(n + m, \omega) = \phi(n, T^m(\omega))\phi(m, \omega).$$

If $\phi$ is close enough to $f$ so that for any $\omega \in \Omega$, $D \xrightarrow{\phi} D$, then the NHIM persists as a set of trajectories of $\phi$. Moreover, the set projects surjectively onto $\Lambda$.

4. Examples of application

The perturbations of a system with an NHIM can be such that the perturbed maps are no longer normally hyperbolic, but we can still apply our results. Below we give an example of such a system.

4.1. Toy example

We start with an example where the dynamics on the unstable coordinate is decoupled from the rest of the coordinates. The aim is to provide a simple model on which we can demonstrate some features, without having to engage in computations.

Let $\Lambda$ be a one dimensional circle, parameterised by $\theta \in [0, 2\pi)$. Let $E^u$ be a trivial bundle over $\Lambda$ (i.e., $E^u = \Lambda \times \mathbb{R}$), let $E^s$ be a Möbius bundle over $\Lambda$, and let $E = E^u \oplus E^s$. Take $\mu \in \mathbb{R}$, $|\mu| < \frac{1}{2}$, and two maps $f_0, f_1 : E \to E$ defined as

$$f_0 (\theta; x, y) = (3\theta \bmod 2\pi; 4x, \mu y),$$
$$f_1 (\theta; x, y) = \left(3\theta \bmod 2\pi; -3x + 5x^3, \frac{1}{2} \sin \theta + \mu y\right).$$

The maps $f_0$ and $f_1$ expand the Möbius strip along $\theta$, wrapping it around itself three times, and squeeze it along the $y$ coordinate (see figure 6). On the $x$ coordinate we have decoupled dynamics. In this example we will discuss invariant sets for a family of maps $f_\beta : E \to E$, defined as

$$f_\beta := (1 - \beta) f_0 + \beta f_1, \quad \text{for } \beta \in [0, 1].$$

For $\beta = 0$ the set $\{x = y = 0\}$ is invariant, and on it the rate conditions hold; i.e. the dynamics in the hyperbolic directions $x, y$ is stronger than on $\theta$. As we increase $\beta$, the expansion along $x$ becomes weaker than the expansion along $\theta$. This means that the classical tools can not ensure that the manifold survives. If we take though $D = \{(\theta; x, y) \mid |x| \leq 1, |y| \leq 1\}$, fix $\beta \in [0, 1]$, consider a homotopy

$$h(\alpha, (\theta; x, y)) = (1 - \alpha) f_\beta (\theta; x, y) + \alpha (3\theta \bmod 2\pi; 2x, 0),$$

$$4978.$$
and \( \eta(\theta) = 3\theta \mod 2\pi \), then it is a simple exercise to verify that

\[
D \xrightarrow{f_\beta} D.
\]

The reason why (12) holds boils down to the fact that on the \( y \) coordinate we have contraction and the cubic terms on the coordinate \( x \) ensure expansion away from zero. Since \( \eta(\theta) = 3\theta \mod 2\pi \), we see that \( \deg_2(\eta) = 1 \).

Theorem 13 ensures that for any \( \beta \in [0,1] \) there is an invariant set in \( D \), with trajectories in \( D \) passing through each \( \theta \in [0,2\pi) \). Theorem 13 does not claim that the invariant set is a manifold. In fact it is not a manifold, which we can see if we look at the projections in figures 5 and 6. Figure 5 contains the plot of the invariant set of \( x \mapsto \pi^u \circ f_\beta(\theta; x, y) \) for \( \beta \in [0,1] \). (The dynamics of \( f_\beta \) on \( x \) is decoupled from other variables, so the set is independent from the choice of \( \theta, y \).) We see that for \( \beta \) close to 1 our set will be chaotic. This is because the function passes through logistic type bifurcations as we increase \( \beta \). In figure 6 we take the parameter \( \mu = \frac{1}{10} \), fix \( \beta = 1 \) and plot the projections of \( f_\beta(D) \) and \( f_\beta^2(D) \) onto the Möbius strip. We see that if we were to consider \( f_\beta^k(D) \) for higher \( k \), then we would see the emergence of a Cantor structure of our invariant set. Theorem 16 states that the resulting invariant set for different \( \beta \) ‘continues’ as the parameter changes, which we see is the case in our example.

The main feature of this example is that we have started with a manifold which satisfied the rate conditions, and perturbed the system into the parameter range where the rate conditions fail. Nevertheless, our method establishes the existence of an invariant set for all parameters. In our example the dynamics on \( x \) is decoupled from the dynamics on the Möbius bundle. We have done this for simplicity. The assumptions of theorems 13 and 16 are robust under small perturbations, so we will also obtain the results for any map that is appropriately close to \( f_\beta \), for one of the \( \beta \in [0,1] \).

In our example we were able to verify (4) because \( f_\beta \) on coordinate \( \theta \) were given as \( 3\theta \mod 2\pi \). If we were to take \( k\theta \mod 2\pi \) with an even number \( k \), then we would get \( \deg(\eta) = 0 \), and we would not be able to apply theorem 13. We finish by observing that in such a setting we can still use theorem 8 to obtain an invariant set of points that stay in \( D \) for all forward iterations.

The above was just a toy example. Similar features though can be found for instance in the Kuznetsov system (see [21, 30]), where we have a hyperbolic invariant set in \( \mathbb{R}^3 \), which has a Cantor set structure.
4.2. An example with a computer assisted proof

Here we modify our example from section 13 by coupling the dynamics between the coordinates. We consider the following map

$$f(\theta; x, y) = \left(3\theta + xy \sin(\theta) \mod 2\pi; 4x^3 - \frac{8}{5} x + \frac{1}{2} xy, \mu y + \frac{2}{5} \sin \theta + x \cos(\theta)\right),$$

with $\mu = 1/10$. This map results from taking $f_0$ from the previous section with $\beta = 0.8$, and by adding the coupling terms $xy \sin(\theta)$, $\frac{1}{2} xy$ and $x \cos(\theta)$ to the $\theta$-, $x$- and $y$-coordinates, respectively. The choice of such coupling was to a large extent arbitrary. We wanted a nontrivial but simple example, with some interesting features.

In figure 7 we give the plot of a numerically obtained representation of the invariant set we will establish. On the plot, the front face $\theta = 2\pi$ is identified with the back face $\theta = 0$, but should be glued together according to the arrows to take into account the fact that $E_s$ is a Möbius bundle.

For $x = 0$ we see the invariant set from our previous example (compare figures 6 and 7); we have intentionally chosen our coupling to preserve it. The coupling is strong enough to distort the two attracting fixed points of the uncoupled map $f_{\beta = 0.8}$ on $x$ (see figure 5) to become the two 'chaotic clouds' from figure 7.

For this example we provide a computer assisted proof that for $D = \{(\theta; x, y) : \|x\| \leq 1, \|y\| \leq 1.2\}$ we will have $D \overset{\rightarrow}{\rightarrow} D$, which, by theorem 13, implies the existence of an invariant set in $D$. This is done by considering the following homotopy

$$h(\alpha, (\theta; x, y)) := (3\theta + (1 - \alpha) xy \sin(\theta) \mod 2\pi;$$

$$\alpha 2x + (1 - \alpha) \left(4x^3 - \frac{8}{5} x + \frac{1}{2} xy\right),$$

$$(1 - \alpha) \left(\mu y + \frac{2}{5} \sin \theta + x \cos(\theta)\right).$$

(14)
Condition (4) follows directly from the definition of $h$. We validate (2) and (3) by using interval arithmetic. Interval arithmetic involves enclosing numbers in intervals that account for possible round-off errors, and performing arithmetic operation on these intervals. The output of these operations are intervals, which account for the numerical error and enclose the true result.

We give the full code which we have used for our computer assisted proof in appendix B and follow with a number of comments associated to the particular routines. The validation of (2) and (3) is based on subdividing the domains into small sets and checking the correct topological alignment by means of inequalities between intervals. A sample of such bounds obtained by our computer program is depicted in figure 8.

4.3. Finding invariant sets and covering relations

If the system under consideration possesses an NHIM, then it is a natural choice to position $D$ around the NHIM, aligning $D^+$ and $D^-$ with the stable and unstable bundles, respectively. When the perturbation is far from the normally hyperbolic case, or if we want to apply our methods in a setting where no NHIM exists, we can use the following numerical method.
We can select some domain within which we expect to find our invariant object, and subdivide it into cubes. Then, using interval arithmetic we can propagate such cubes and discard those that will leave the domain after some iterate of the map. Those cubes that do not escape are dissected into smaller cubes, and the procedure can be repeated. If some invariant set is within our domain, it will be detected by this method.

The reason for using interval arithmetic is that even if just a single point from a considered cube is an element of the invariant object, then it will not leave the domain, and the cube will not be discarded. (For instance, the discussed methodology works very well in the normally hyperbolic setting, to find an enclosure of the stable manifold).

The positioning of the enclosure that comes out of the algorithm can give an insight into how $D^+$ and $D^-$ should be positioned. In figure 9 we show an outcome of the procedure applied to the map (13) for domains of the form $\{ (\theta, x, y) : x^2 + y^2 < r^2 \}$, for $r = 2$ and $r = 1/2$. (On the plot we depict cubes with $\theta = \pi/3$, because we took a liking to the shape on the right-hand side.) For $r = 2$ we see that $D^+$ should be towards the vertical and $D^-$ towards the horizontal axis. For $r = 1/2$ though we obtain an enclosure that does not give a clear indication how $D^+$ and $D^-$ could be positioned. Finding suitable $D$ in complicated systems is not an easy task and is likely to involve trial and error.

5. Embedding into reals

In this section we shall embed $E$ in $\mathbb{R}^N$. We will then extend the map $f$ so that it is defined on a set with nonempty interior in $\mathbb{R}^N$. Such embedding will be useful for us since to find two points $v_1, v_2 \in D$ such that $v_2 = f(v_1)$ we will be able to do so more easily by embedding $f(v_1)$ and $v_2$ into $\mathbb{R}^N$, computing their difference, and solving for zero. Searching for zeros in $\mathbb{R}^N$ is more tractable than finding two points on a vector bundle that map one into the other.

The vector bundle $E$ is an $n$-dimensional smooth manifold, $n = c + u + s$. By the Whitney embedding theorem [28] there exists a smooth embedding $\omega : E \rightarrow \mathbb{R}^{2n}$. Let $N_w(\omega(E)) \subset \mathbb{R}^{2n}$ stand for the normal space to the manifold $\omega(E)$ in $\mathbb{R}^{2n}$ at $w \in \omega(E)$. (Since $\omega(E)$ is a manifold
Figure 10. Depiction of our embedding. The set $D$ is a tubular neighbourhood of $\omega(D)$. The map $f$ defined on $D$ contracts in the normal direction onto $\omega(f(D))$. This means that the normal direction is treated as an additional contracting coordinate.

of dimension $n$, the dimension of $N_w(\omega(E))$ is $n$.) We consider the tubular neighbourhood of $\omega(E)$

$$\mathcal{T} := \{ w + z | w \in \omega(E), z \in N_w(\omega(E)), \| z \|_{\mathbb{R}^{2n}} \leq \delta(w) \} \subset \mathbb{R}^{2n}, \quad (15)$$

where $\delta : \omega(E) \to \mathbb{R}^+$ is continuous. Let us abuse the notation slightly by introducing a number $\delta \in \mathbb{R}$ defined as

$$\delta := \min_{w \in \omega(D)} \delta(w). \quad (16)$$

Since $D$ is compact $\delta > 0$ is well defined.

**Notation 23.** For $v \in E$ we shall write a pair $(v, z)$ to represent a point $\omega(v) + z \in \mathcal{T}$. In this convention writing the pair $(v, z)$ implies that $z \in N_{\omega(v)}(\omega(E))$. In the same way by writing $(\theta; x, y, z)$ we mean the point $\omega(\theta; x, y) + z \in \mathcal{T}$, and imply that $z \in N_{\omega(\theta; x, y)}(\omega(E))$.

Using notation 23 we define the following subsets of $\mathbb{R}^{2n}$

$$D := \{ (v, z) \in T | v \in D, \| z \|_{\mathbb{R}^{2n}} \leq \delta \}, \quad (17)$$

$$D_\theta := D \cap \{ (v, z) | \pi(v) = \theta \}, \quad D_U := \bigcup_{\lambda \in U} D_\lambda.$$

We define a map

$$f : D \to \mathbb{R}^{2n},$$

as

$$f(v, z) := (f(v), 0), \quad (18)$$

where the zero on the right-hand side is on the $z$-coordinate. (In other words, $f(\omega(v) + z) = \omega(f(v));$ see figure 10.) If we need to include more detail, using also the convention of notation 23, we can write

$$f(\theta; x, y, z) = (f(\theta; x, y), 0),$$

where the zero on the right-hand side is on the $z$-coordinate. Observe that directly from the definition of $f$ we have the following:

**Lemma 24.** If for $(v_1, z_1), (v_2, z_2) \in T$ we have $(v_2, z_2) = f(v_1, z_1)$, then $v_2 = f(v_1)$. 

4983
Proof. By definition, \((v_2, z_2) = f(v_1, z_1)\) implies \(\omega(f(v_1)) = \omega(v_2) + z_2\). Since \(\mathcal{T}\) is a tubular neighbourhood of \(\omega(E)\), each point in \(\mathcal{T}\) is represented in a unique way as \(\omega(v_2) + z_2\). This means that \(\omega(f(v_1)) = \omega(v_2) + z_2\) implies that \(z_2 = 0\) and \(\omega(f(v_1)) = \omega(v_2)\), which in turn gives \(f(v_1) = v_2\), as required. □

Remark 25. When we looking for two points \(v_1, v_2 \in D\) such that \(v_2 = f(v_1)\), by lemma 24 we can solve

\[
(f(v_1, z_1) - (v_2, z_2) = 0, (19)
\]

for \((v_1, z_1), (v_2, z_2) \in D\). (In (19) we are subtracting two vectors in \(\mathbb{R}^{2n}\).)

6. Proof of theorem 8

Proof. Let us fix \(\theta = \Theta_0 \in \Lambda\). Our objective will be to find a trajectory starting from \(D_{\Theta_0}\), which remains in \(D\) for all forward iterates. We start by finding trajectories of length \(k\).

Let \(0^{2n}\) denote zero in \(\mathbb{R}^{2n}\). For fixed \(k \in \mathbb{N}\) we consider the following sets (recall that \(D\) was defined in (17))

\[
X := D^u_{\Theta_0} \times \underbrace{D \times \cdots \times D}_k, (20)
\]

\[
Y := \underbrace{\mathbb{R}^{2n} \times \cdots \times \mathbb{R}^{2n}}_k \times E, (21)
\]

\[
Z := \underbrace{\{0^{2n}\} \times \cdots \times \{0^{2n}\}}_k \times \{(\theta; 0, y) | \theta \in \Lambda, \|y\|_i < 1\}. (22)
\]

We consider \(X\) as a subset of \(E^u_{\Theta_0} \times \mathcal{T} \times \cdots \times \mathcal{T}\), so

\[
X = \text{int}X = \text{int}D^u_{\Theta_0} \times \underbrace{\text{int}D \times \cdots \times \text{int}D}_k.
\]

Note that since \(\Lambda\) is compact so is \(X, Y\) is a manifold without boundary and \(Z\) is its submanifold, with

\[
Z = \{0^{2n}\} \times \cdots \times \{0^{2n}\} \times \{(\theta; 0, y) | \theta \in \Lambda, \|y\|_i \leq 1\}, (23)
\]

\[
\partial Z = \{0^{2n}\} \times \cdots \times \{0^{2n}\} \times \{(\theta; 0, y) | \theta \in \Lambda, \|y\|_i = 1\}. (24)
\]

The manifolds \(X\) and \(Z\) are of complementary dimension with respect to \(Y\):

\[
\dim X = u + 2kn, \quad \dim Z = c + s, \quad \dim Y = 2kn + n.
\]

To show the existence of an orbit of length \(k\) in \(D\) we consider a map

\[
F = (F_1, \ldots, F_k, F_{k+1}) : X \to Y,
\]

which is defined as follows. For

\[
x = ((\Theta_0; x_0), (\theta_1; x_1, y_1, z_1), \ldots, (\theta_k; x_k, y_k, z_k)) \in X
\]

(25)
we define \(^4\) (recall that \(f\) was defined in (18))

\[
F_1(x) := f(\theta_0; x_0, 0, 0) - (\theta_1; x_1, y_1, z_1),
\]

\[
F_2(x) := f(\theta_1; x_1, y_1, z_1) - (\theta_2; x_2, y_2, z_2),
\]

\[
\vdots
\]

\[
F_k(x) := f(\theta_{k-1}; x_{k-1}, y_{k-1}, z_{k-1}) - (\theta_k; x_k, y_k, z_k),
\]

\[
F_{k+1}(x) := f(\theta_k; x_k, y_k).
\]

Our objective will be to prove that there exists an \(x \in \mathcal{X}\), such that

\[
F(x) \in Z.
\]  

(27)

Observe that by lemma 24, (27) establishes the existence of a trajectory of \(f: D \rightarrow E\), that starts in \(v = (\theta_0; x_0, 0)\) and remains in \(D\) for \(k\) iterates of \(f\):

\[
f^i(v) \in D \quad \text{for} \quad i = 1, \ldots, k.
\]

Our plan is to establish (27) by showing that the intersection number \(I_2(F, X, Z) = 1\); then (27) will follow from the intersection property.

The first thing to show is that \(F\) is admissible (in the sense of definition 4). We shall consider an \(x\) of the form (25) which lies in the boundary \(\partial X\) and show that \(F(x) \notin \mathcal{Z}\). There are several possibilities how \(x\) can lie on \(\partial X\), which we will consider one by one below. In the following argument we make use of the fact that \(f(\theta; x, y) = h_0(0, (\theta; x, y))\), which means that all properties of \(h_0\) from definition 7 hold for \(f\).

The first possibility how \(x\) can lie on \(\partial X\) is that \(\|x_0\|_u = 1\). In this case, \((\theta_0; x_0, 0, 0) \in D_{e_{\theta_0}}\), and by the first condition from definition 7 we know that \(f(\theta_0; x_0, 0) \cap D = \emptyset\), so \(f((\theta_0; x_0, 0) \notin D\), hence \(F_1(x) \neq 0\), so in turn \(F(x) \notin \mathcal{Z}\).

The second way \(x\) can lie on \(\partial X\) is that \(\|x_i\|_u = 1\) for some \(i \in \{1, \ldots, k - 1\}\). Then \((\theta_i; x_i, y_i) \in D_{\theta_i}\), and also from condition one of definition 7 we have \(f(\theta_i; x_i, y_i) \notin D\) so \(f(\theta_i; x_i, y_i) = f(\theta_i; x_i, y_i, 0) \notin D\). This means that \(F_{k+1}(x) \neq 0\), hence \(F(x) \notin \mathcal{Z}\).

If \(x\) lies in \(\partial X\) because \(\|x_k\|_u = 1\), then \((\theta_k; x_k, y_k) \in D_{\theta_k}\) and from definition 7 we see that \(F_{k+1}(x) = f(\theta_k; x_k, y_k) \notin D\), hence \(F(x) \notin \mathcal{Z}\).

Another possibility for \(x\) to be in \(\partial X\) is to have \(\|y_i\|_u = 1\) for some \(i \in \{1, \ldots, k\}\). From definition 7 it follows that \(f(D) \cap D^\perp = \emptyset\). We see that since \(\|y_i\|_u = 1\) we have \(F_1(x) = f(\theta_1; x_1, y_1) = f(\theta_1; x_1, y_1, 0) \neq 0\) (where \(y_0 = 0\)), so \(F(x) \notin \mathcal{Z}\).

The last possibility for \(x\) to be on \(\partial X\) is that \(\|z_i\|_{2 \mathcal{R}} = \delta\) for some \(i \in \{1, \ldots, k\}\), then \(F_i(x) = f(\theta_i; x_i, y_i, z_i) = f(\theta_i; x_i, y_i, z_i) \neq 0\), so \(F(x) \notin \mathcal{Z}\).

Above we have shown that

\[
F(\partial X) \cap \mathcal{Z} = \emptyset.
\]  

(28)

We now need to show that

\[
F(\overline{\mathcal{X}}) \cap \partial \mathcal{Z} = \emptyset.
\]  

(29)

\(^4\)To obtain the generalisation stated in remark 14 here we should use \(f_i\) in the definition of \(F_i(x)\) for \(i = 1, \ldots, k\) and \(f_{k+1}\) in the definition of \(F_{k+1}(x)\); throughout the reminder of the proof we would use homotopies resulting from the coverings \(D_{e_{\theta_i}} = D\) in the respective places that follow.
If \( y \in \partial Z \), then
\[
y = (0, \ldots, 0, (\theta; 0, y)) \quad \text{for some } \theta \in \Lambda, \text{ and } \|y\| = 1.
\] (30)
Since \( F_{k+1}(x) = f(\theta_i; x_i, y_i) \) and from definition 7 it follows that \( f(D) \cap D^+ = \emptyset \), we see that \( F_{k+1}(x) \neq y \). We have shown (29), thus \( F \) is admissible.

Our objective will now be to construct an admissible (in the sense of definition 5) homotopy from \( F \) to some map that is transversal to \( Z \). We will do this in a number of steps, by constructing several admissible homotopies and then gluing them together. A less patient reader might want to take a peek at (45), where we write out the map we make the homotopy to. Looking at (45) will give an idea of our final objective.

Our first homotopy will be denoted as
\[
H^{(1)} = \left( H^{(1)}_1, \ldots, H^{(1)}_k, H^{(1)}_{k+1} \right) : [0, 1] \times X \to Y.
\]
Since \( D_{\Theta_0} \to D \), we can take the homotopy \( h_{\Theta_0} \) from definition 7, and for \( x \) of the form (25) we can define
\[
H^{(1)}_1(\alpha, x) := \left( h_{\Theta_0}(\alpha, (\Theta_0; x_0, 0)), 0 \right) - (\theta_l; x_1, y_1, z_1),
\]
\[
H^{(1)}_i(\alpha, x) := F_i(x) \quad \text{for } i \neq 1.
\]
Our homotopy is such that
\[
H^{(1)}(0, x) = F(x),
\]
and for some \( \Theta_1 \in \Lambda \) and linear \( A_0 : E^{\alpha}_{\Theta_0} \to E^{\alpha}_{\Theta_1} \) (\( \Theta_1 \) and \( A_0 \) follow from definition 7)
\[
H^{(1)}_1(1, x) = (\Theta_1; A_0 x_0, 0, 0) - (\theta_l; x_1, y_1, z_1).
\]

We need to show that \( H^{(1)}_i \) is admissible. This will follow from an analogous argument to the one used to prove (28) and (29). We first need to show that
\[
H^{(1)}(\alpha, x) \cap Z = \emptyset \quad \text{for } x \in \partial X \text{ and } \alpha \in [0, 1].
\] (31)
We have already established (28) and we know that for \( \alpha \neq 1 \), by definition, \( H^{(1)}_i(\alpha, x) = F_i(x) \). This means that to check (31) it is enough to consider three cases. The first is that \( x \in \partial X \) is such that \( \|x_0\| = 1 \). The second \( \|y_1\| = 1 \). The third \( \|z_1\| = \delta \). (For all other \( x \in \partial X \) condition (31) follows from (28).) In the first case \( (\Theta_0; x_0, 0) \in D_{\Theta_0} \) so since \( h_{\Theta_0} \left( \alpha, D_{\Theta_0} \right) \cap D = \emptyset \) we obtain
\[
H^{(1)}_1(\alpha, x) = \left( h_{\Theta_0}(\alpha, (\Theta_0; x_0, 0)), 0 \right) - (\theta_l; x_1, y_1, z_1) \neq 0,
\] (32)

hence \( H^{(1)}(\alpha, x) \notin Z \). For the second case, since \( h_{\Theta_0} \left( \alpha, D_{\Theta_0} \right) \cap D^+ = \emptyset \) also ensures (32), we have \( H^{(1)}(\alpha, x) \notin Z \). If \( \|z_1\| = \delta \), then we also see that (32) holds. We have thus established (31). The fact that
\[
H^{(1)}([0, 1] \times X) \cap \partial Z = \emptyset
\] (33)
follows from (29). (This is because \( H^{(1)}_k \circ F_{k+1} = F_{k+1} \) and \( F_{k+1} \) was used to establish (29).) This means that we have established that \( H^{(1)}_i \) is admissible.
Since $H^{(1)}$ is admissible and $H^{(1)}(0, \cdot) = F$, from the homotopy property of the intersection number we obtain

$$I_2(F, X, Z) = I_2(H^{(1)}(0, \cdot), X, Z) = I_2(H^{(1)}(1, \cdot), X, Z).$$  \hspace{1cm} (34)

Before specifying the next homotopy we shall make use of the excision property. For this we take a closed set $U_{\Theta_1} \subset \Lambda$ such that $\text{int} U_{\Theta_1} \neq \emptyset$ and $\Theta_1 \in \text{int} U_{\Theta_1}$. We can take $U_{\Theta_1}$ small enough so that it is in the domain of some trivialisation of $E$ and so that it is contractible to the point $\Theta_1$. Let us denote such a continuous contraction by $g_{\Theta_1} : [0, 1] \times U_{\Theta_1} \to U_{\Theta_1}$ for which $g_{\Theta_1}(0, \theta) = \theta$ and $g_{\Theta_1}(1, \theta) = \Theta_1$. We now define a set $\overline{X}^{(1)} \subset \overline{X}$ as

$$\overline{X}^{(1)} = D_{\Theta_0}^0 \times D_{U_{\Theta_1}}^0 \times D \times \cdots \times D_{k-1}.$$

We see that

$$X^{(1)} = \text{int} \overline{X}^{(1)} = \text{int} D_{\Theta_0}^0 \times \text{int} D_{U_{\Theta_1}}^0 \times \text{int} D \times \cdots \times \text{int} D_{k-1}.$$

We will use the excision property to restrict $H^{(1)}(1, \cdot)$ from $X$ to $X^{(1)}$. For this we first need to show that

$$H^{(1)}(1, X) \cap Z = H^{(1)}(1, X^{(1)}) \cap Z.$$  \hspace{1cm} (35)

If we take some $x \in X \setminus X^{(1)}$ of the form (25), then $\theta_1 \notin \text{int} U_{\Theta_1}$, so in particular $\theta_1 \neq \Theta_1$. This means that

$$H^{(1)}(1, x) = (\Theta_1; A_{\theta_1} x_0, 0, 0) - (\theta_1; x_1, y_1, z_1) \neq 0,$$

which implies (35). To use the excision property we also need to check that

$$H^{(1)}(1, \partial X^{(1)}) \cap Z = \emptyset.$$  \hspace{1cm} (36)

If $x \in \partial X^{(1)} \cap \partial X$, then (36) follows from (31). If $x \in \partial X^{(1)} \setminus \partial X$, then $\theta_1 \in \partial U_{\Theta_1}$ and $\theta_1 \neq \Theta_1$ so $H^{(1)}(1, x) \neq 0$, which implies (36). We can now apply the excision property.

From the excision property it follows that

$$I_2(H^{(1)}(1, \cdot), X, Z) = I_2(H^{(1)}(1, \cdot), X^{(1)}, Z).$$  \hspace{1cm} (37)

We are ready to define our second homotopy. We consider

$$G^{(1)} = \left( G^{(1)}_1, \ldots, G^{(1)}_k, G^{(1)}_{k+1} \right) : [0, 1] \times \overline{X^{(1)}} \to Y,$$

defined as

$$G_2^{(1)}(\alpha, x) := f(g_{\Theta_1}(\alpha, \theta_1; x_1, y_1, z_1) - (\theta_2; x_2, y_2, z_2),$$

$$G_3^{(1)}(\alpha, x) := H^{(1)}_1(1, x) \quad \text{for } i \neq 2.$$  \hspace{1cm} (38)

To show that $G^{(1)}$ is an admissible homotopy we first need that $G^{(1)}([0, 1] \times \partial X^{(1)}) \cap Z = \emptyset$. It is enough to show that for $x$ with $(\theta_1; x_1, y_1, z_1) \in \partial D_{U_{\Theta_1}}$, we have $G^{(1)}(\alpha, x) \notin Z$. (We do not need to consider other $x \in \partial X^{(1)}$ since we have (38) and (31).) If $(\theta_1; x_1, y_1, z_1) \in \partial D_{U_{\Theta_1}}$, then we have three possibilities which we consider below.
The first possibility is that \( \|x_1\|_s = 1 \), so \( (g_{\Theta_1}(\alpha, \theta_1); x_1, y_1) \in D_{g_{\Theta_1}(\alpha, \theta_1)}^- \). Then, since we have \( D_{g_{\Theta_1}(\alpha, \theta_1)} f \Rightarrow D \), we see that \( f \left( D_{g_{\Theta_1}(\alpha, \theta_1)}^- \right) \cap D = \emptyset \), therefore

\[
G_2^{(1)}(\alpha, x) = (f \left( g_{\Theta_1}(\alpha, \theta_1); x_1, y_1 \right), 0) - \left( \theta_2; x_2, y_2, z_2 \right) \neq 0,
\]

which implies that \( G^{(1)}(\alpha, x) \notin Z \).

The second possibility is that \( \|y_1\|_s = 1 \) or \( \|z_1\|_s = \delta \). Then \( x \in \partial X \) and by (31) we obtain that

\[
G_1^{(1)}(\alpha, x) = H_1^{(1)}(1, x) \notin Z.
\]

The third and last possibility is that \( \theta_1 \in \partial U_{\Theta_1} \), but then \( \theta_1 \neq \Theta_1 \), so

\[
G_1^{(1)}(\alpha, x) = H_1^{(0)}(1, x) = (\Theta_1; A_0x_0, 0, 0) - (\theta_1; x_1, y_1, z_1) \neq 0,
\]

hence \( G^{(1)}(\alpha, x) \notin Z \).

We also need to show that \( G^{(1)} \left( [0, 1] \times X^{(1)} \right) \cap \partial Z = \emptyset \). This follows from (29) since \( G_{k+1}^{(1)} = F_{k+1} \). We have thus shown that \( G^{(1)} \) is an admissible homotopy, so from the homotopy property we obtain that

\[
I_2 \left( F^{(0)}(1, \cdot), X^{(1)} \right) = I_2 \left( G^{(1)}(0, \cdot), X^{(1)} \right) = I_2 \left( G^{(1)}(1, \cdot), X^{(1)} \right) .
\]

Combining (39) with (34) and (37) gives

\[
I_2(F, X, Z) = I_2 \left( G^{(1)}(1, \cdot), X^{(1)} \right) .
\]

Observe that

\[
G_2^{(1)}(1, x) = f(\Theta_1; x_1, y_1, z_1) - \left( \theta_2; x_2, y_2, z_2 \right).
\]

What is important for us is that we have the fixed \( \Theta_1 \) on the right-hand side of the above expression. This means that we can use the homotopy \( h_{\Theta_1} \) from \( D_{\Theta_1} \Rightarrow D \) to define

\[
H^{(2)} = \left( H_1^{(2)}, \ldots, H_k^{(2)}, H_{k+1}^{(2)} \right) : [0, 1] \times X^{(1)} \to Y
\]

as

\[
H_2^{(i)}(\alpha, x) := \left( h_{\Theta_1}(\alpha, (\Theta_1; x_1, y_1)), 0 \right) - \left( \theta_2; x_2, y_2, z_2 \right), \quad H_i^{(2)}(\alpha, x) := G_i^{(1)}(1, x) \quad \text{for } i \neq 2.
\]

Showing that \( H^{(2)} \) is an admissible homotopy follows from mirror steps to establishing that \( H^{(1)} \) was admissible. Thus
\[ I_2 \left( \mathcal{G}^{(1)}(1, \cdot), X^{(1)} \right) = I_2 \left( \mathcal{H}^{(2)}(0, \cdot), X^{(1)} \right) = I_2 \left( \mathcal{H}^{(2)}(1, \cdot), X^{(1)} \right), \]

hence by (40) we have
\[ I_2 (F, X, Z) = I_2 \left( \mathcal{H}^{(2)}(1, \cdot), X^{(1)} \right). \] (42)

Observe that
\[
\begin{align*}
\mathcal{H}_2^{(1)}(0, x) &= G^{(1)}(1, x), \\
\mathcal{H}_2^{(2)}(1, x) &= (\Theta_2; A_1; x_1, 0, 0) - (\theta_2; x_2, y_2, z_2),
\end{align*}
\]
where \( \Theta_2 \) and \( A_1 \) : \( \mathcal{U}_\Theta \to \Lambda \) result from the homotopy \( h_{\Theta_1} \) from definition 7. This means that we can take an excision to
\[ X^{(2)} := D_{U_{\Theta_0}} \times D_{U_{\Theta_1}} \times D_{U_{\Theta_2}} \times D \times \cdots \times D. \]

where \( U_{\Theta_1} \subset \Lambda \) is a closure of some small enough open set around \( \Theta_2 \), which is contractible to the point \( \Theta_2 \) via a homotopy \( g_{\Theta_2} (\alpha, \theta) \). Using the same arguments to those that lead to (37) we obtain
\[ I_2 \left( \mathcal{H}^{(2)}(1, \cdot), X^{(1)} \right) = I_2 \left( \mathcal{H}^{(2)}(1, \cdot), X^{(2)} \right), \]

and by (42)
\[ I_2 (F, X, Z) = I_2 \left( \mathcal{H}^{(2)}(1, \cdot), X^{(2)} \right). \]

We can now iterate the above construction step by step by taking, for \( j = 2, \ldots, k \), the sets
\[ X^{(j)} := D_{U_{\Theta_0}} \times D_{U_{\Theta_1}} \times D_{U_{\Theta_2}} \times D \times \cdots \times D, \]
and admissible homotopies
\[
\begin{align*}
\mathcal{H}_j^{(j)} : [0, 1] \times X^{(j-1)} &\to Y, \\
\mathcal{G}_j^{(j)} : [0, 1] \times X^{(j)} &\to Y,
\end{align*}
\]
defined as (compare with (38) and (41))
\[
\begin{align*}
\mathcal{H}_j^{(j)} (\alpha, x) &:= \left( h_{\Theta_{j-1}} (\alpha, \left( \Theta_{j-1}; x_{j-1}, y_{j-1}, z_{j-1} \right)), 0 \right) - (\theta_j; x_j, y_j, z_j), \\
\mathcal{H}_i^{(j)} (\alpha, x) &:= \mathcal{G}_i^{(j-1)} (\alpha, x) \quad \text{for } i \neq j, \\
\mathcal{G}_{j+1}^{(j)} (\alpha, x) &:= \mathcal{G}_{j+1}^{(j)} (\alpha, \theta_j; x_j, y_j, z_j) - (\theta_{j+1}; x_{j+1}, y_{j+1}, z_{j+1}), \\
\mathcal{G}_j^{(j)} (\alpha, x) &:= \mathcal{H}_j^{(j)} (1, x) \quad \text{for } i \neq j + 1.
\end{align*}
\]

We sum up what we have achieved so far:
We finally consider the last homotopy

\[ H^{(k+1)}: [0, 1] \times \overline{X}^{(k)} \rightarrow Y, \]

defined as

\[ H^{(k+1)}_i (\alpha, x) := h_{\Theta_k} (\alpha, (\Theta_k; x_k, y_k)), \]

\[ H^{(k+1)}_i (\alpha, x) := G^{(k)}_i (1, x) \quad \text{for } i \neq k + 1. \]

Showing that \( H^{(k+1)} \) is admissible follows from analogous argument to showing that \( H^{(1)} \) is admissible. We therefore have

\[ I_2 (G^{(k)} (1, \cdot), X^{(k)}, Z) = I_2 (H^{(k+1)} (0, \cdot), X^{(k)}, Z) \]

\[ = I_2 (H^{(k+1)} (1, \cdot), X^{(k)}, Z). \]  

(44)

What is important for us is that at the end of our construction we have achieved:

\[ H^{(k+1)}_1 (1, x) = (\Theta_1; A_{0, x_0}, 0, 0) - (\theta_1; x_1, y_1, z_1), \]

\[ H^{(k+1)}_2 (1, x) = (\Theta_2; A_{1, x_1}, 0, 0) - (\theta_2; x_2, y_2, z_2), \]

\[ \vdots \]

\[ H^{(k+1)}_k (1, x) = (\Theta_k; A_{k-1, x_{k-1}, 0, 0} - (\theta_k; x_k, y_k, z_k), \]

\[ H^{(k+1)}_{k+1} (1, x) = (\Theta_{k+1}; A_{k+1}, x_k, 0). \]  

(45)

Since for \( i = 0, \ldots, k \), \( A_i : E^u_{\Theta_i} \rightarrow E^u_{\Theta_{i+1}} \) are linear and \( A_i \left( \partial B^u_{\Theta_i} \right) \subset E^u_{\Theta_{i+1}} \setminus B^u_{\Theta_{i+1}} \), there is a unique transversal intersection of \( H^{(k+1)} (1, X^{(k)}) \) with \( Z \) at the point \( H^{(k+1)} (1, x^*) \) for
\[ x^* = ((\Theta_0; 0), (\Theta_1; 0, 0, 0), \ldots, (\Theta_k; 0, 0, 0)) \in \mathcal{X}(k). \]

This means that \( I_2 (H^{(k+1)}(1, \cdot), \mathcal{X}(k), Z) = 1 \), hence by (43) and (44)

\[ I_2 (F, X, Z) = I_2 (H^{(k+1)}(1, \cdot), \mathcal{X}(k), Z) = 1. \]

From the intersection property we therefore obtain an \( x \in X \) for which we have (27).

By establishing (27) we have shown that for any \( k \in \mathbb{N} \) there exists a trajectory starting from some \( v_k \in D_{\Theta_0} \) for which \( f^i (v_k) \in D \) for \( i = 1, \ldots, k \). Since \( D_{\Theta_0} \) is compact, the claim of our theorem now simply follows by passing to the limit \( v^* \in D_0 \) of a convergent subsequence of \( \{v_k\}_{k \in \mathbb{N}} \). For such a \( v^* \), by continuity of \( f \), we will have \( f^i (v^*) \in D \) for all \( i \in \mathbb{N} \), as required. \( \square \)

7. Proof of theorem 13

The proof is similar to the one from the previous section. The difference is that we will also keep track of what is happening backwards in time while setting up our maps and homotopies.

**Proof.** Let us fix \( \Theta_0 \in \Lambda \). We start by showing that for a fixed \( k \in \mathbb{N} \) we have a sequence \( \{v_i\}_{i=-k, \ldots, k} \subset D \) such that \( v_0 \in D_{\Theta_0} \) and \( f (v_i) = v_{i+1} \) for \( i = -k, \ldots, k-1 \). We define the sets

\[
\mathcal{X} = D^u \times D \times \cdots \times D \times D_{\Theta_0} \times D \times \cdots \times D \tag{46}
\]

\[
Y = \mathbb{R}^{2n} \times \cdots \times \mathbb{R}^{2n} \times E \tag{47}
\]

\[
Z = \{0^{2n} \times \cdots \times \{0^{2n}\} \times \{(\theta; 0, y) | \theta \in \Lambda, \|y\|_s < 1\} \}. \tag{48}
\]

We see that

\[
\dim X = (c + u) + 2kn + (u + s + n) + 2kn = (2k + 1)2n + u,
\]

\[
\dim Z = c + s,
\]

\[
\dim Y = (2k + 1)2n + n,
\]

therefore \( X \) and \( Z \) are manifolds of complementary dimensions with respect to \( Y \). \( Y \) is a boundaryless manifold and \( Z \) is its submanifold with boundary \( \overline{Z} \) and \( \partial Z \) of the form (23) and (24).

We define

\[
F = (F_{-k}, \ldots, F_{k}, F_{k+1}) : \mathcal{X} \rightarrow Y \tag{49}
\]

as follows. For

\[
x = ((\theta_{-k-1}; x_{-k-1}), (\theta_{-k}; x_{-k}, y_{-k}, z_{-k}), \ldots, (\theta_{-1}; x_{-1}, y_{-1}, z_{-1}),
\]

\[
(\Theta_0; x_0, y_0, z_0), (\theta_1; x_1, y_1, z_1), \ldots, (\theta_{k}; x_k, y_k, z_k))
\]
we define \(^5\)

\[
F_{-k}(x) := f(\theta_{-k-1}; x_{-k-1}, 0, 0) - (\theta_{-k}; x_{-k}, y_{-k}, z_{-k}),
\]

\[
F_{-k+1}(x) := f(\theta_{-k}; x_{-k}, y_{-k}, z_{-k}) - (\theta_{-k+1}; x_{-k+1}, y_{-k+1}, z_{-k+1}),
\]

\[\vdots\]

\[
F_{-1}(x) := f(\theta_{-2}; x_{-2}, y_{-2}, z_{-2}) - (\theta_{-1}; x_{-1}, y_{-1}, z_{-1}),
\]

\[
F_0(x) := f(\theta_{-1}; x_{-1}, y_{-1}, z_{-1}) - (\Theta_0; x_0, y_0, z_0),
\]

\[
F_1(x) := f(\Theta_0; x_0, y_0, z_0) - (\theta_1; x_1, y_1, z_1),
\]

\[
F_2(x) := f(\theta_1; x_1, y_1, z_1) - (\theta_2; x_2, y_2, z_2),
\]

\[\vdots\]

\[
F_k(x) := f(\theta_{k-1}; x_{k-1}, y_{k-1}, z_{k-1}) - (\theta_k; x_k, y_k, z_k),
\]

\[
F_{k+1}(x) := f(\theta_k; x_k, y_k).
\]

If we find a point \( x \in \overline{X} \) for which \( F(x) \in Z \), then, by lemma 24, we will obtain a finite trajectory (of length \( 2k + 1 \)) of \( f \) which remains in \( D \). The way in which we have chosen \( F_0 \) and \( F_1 \) has a special role. The condition that \( F_0 = 0 \) ensures that the trajectory of \( f \) reaches \( D_{\Theta_0} \).

In \( F_1 \) we also find \( \Theta_0 \); this ensures that the trajectory that reached \( D_{\Theta_0} \) (because of \( F_0 = 0 \)) exits \( D_{\Theta_0} \) in the next iterate.

Our objective is to show that \( F(X) \cap Z = \emptyset \). We will show this by proving that \( I_2(F, X, Z) = 1 \). For this we construct a sequence of admissible homotopies to a map for which it is easy to compute the intersection number directly.

Our first homotopy \( H^{(0)}: [0, 1] \times \overline{X} \to Y \) is defined as

\[
H^{(0)}(\alpha, x) := (h(\alpha, (\theta_{-k-1}; x_{-k-1}, 0)), 0) - (\theta_{-k}; x_{-k}, y_{-k}, z_{-k}),
\]

\[
H^{(0)}_{k+1}(\alpha, x) := (h(\alpha, (\theta_{-k}; x_{-k}, y_{-k})), 0) - (\theta_{-k+1}; x_{-k+1}, y_{-k+1}, z_{-k+1}),
\]

\[\vdots\]

\[
H^{(0)}_1(\alpha, x) := (h(\alpha, (\theta_{-2}; x_{-2}, y_{-2})), 0) - (\theta_{-1}; x_{-1}, y_{-1}, z_{-1}),
\]

\[
H^{(0)}_0(\alpha, x) := (h(\alpha, (\theta_{-1}; x_{-1}, y_{-1})), 0) - (\Theta_0; x_0, y_0, z_0),
\]

\[
H^{(0)}_i(\alpha, x) := F_i(x) \quad \text{for } i > 0.
\]

The fact that this homotopy is admissible follows from \( D \xrightarrow{\phi} D \) (by using analogous arguments to those used to show that the homotopies considered in the proof of theorem 8).

We now take the sequence of admissible homotopies and excisions \( H^{(1)}, G^{(1)} \ldots H^{(k)}, G^{(k)}, H^{(k+1)} \) defined as in the proof of theorem 8, leaving the coordinates \(-k, \ldots, 0\) without any

\(^5\) To obtain the generalisation stated in remark 14 here we should use \( f \) in the definition of \( F_i(x) \) for \( i = -k, \ldots, k \) and \( f_{k+1} \) in the definition of \( F_{k+1}(x) \); throughout the reminder of the proof we would use homotopies resulting from the coverings \( D \xrightarrow{\phi} D \) in the respective places that follow.

4992
changes. While making the excisions, we make them to sets of the form
\[ X^{(j)} := D^n \times D \times \cdots \times D \times D_{e_{i-j}} \times D_{e_{i-1}} \times \cdots \times D_{e_{i-j}} \times D \times \cdots \times D, \]
for \( j = 1, \ldots, k \). We thus find an admissible homotopy of \( F \) to
\[
H_{i-k}^{(k+1)}(1, x) = (\eta \circ (\theta_{i-k}; A_{\theta_{i-k-1}}x_{i-k-1}, 0, 0, 0) - (\theta_{i-k}; x_{i-k}, y_{i-k}, z_{i-k})],
\]

\[
H_{i-k+1}^{(k+1)}(1, x) = (\eta (\theta_{i-k}; A_{\theta_{i-k}}x_{i-k}, 0, 0) - (\theta_{i-k+1}; x_{i-k+1}, y_{i-k+1}, z_{i-k+1})]],
\]

\[
\vdots
\]
\[
H_{i}^{(k+1)}(1, x) = (\eta_{i-k}; \cdots ; \eta_{i-k+1}; A_{\theta_{i-k}}x_{i-k}, x_{i-k}, \cdots, y_{i-k}, z_{i-k}).
\]

We need to show that \( I_2 \{ H^{(k+1)}(1, \cdot), X^{(j)}; Z \} = 1 \).

Since \( \deg_2 (\eta) \neq 0 \), we have a smooth map \( \hat{\eta}_0 : \Lambda \to \Lambda \), homotopic to \( \eta \), so that \( \Theta_0 \) is a regular value of \( \hat{\eta}_0 \) for which the set \( \hat{\eta}_0^{-1}(\Theta_0) \) has an odd number of points\(^6\). For the same reason we have a smooth \( \hat{\eta}_1 : \Lambda \to \Lambda \) homotopic to \( \eta \), for which each point in \( \hat{\eta}_1^{-1}(\Theta_0) \) is regular and again the number of points in \( \hat{\eta}_1^{-1}(\Theta_0) \) is odd. Proceeding inductively we find smooth \( \hat{\eta}_{i-1} \), homotopic and arbitrarily close to \( \eta \), such that the points in \( \hat{\eta}_{i-1}^{-1}(\Theta_0) \) are regular for \( \hat{\eta}_{i-1} \), and that their number is odd; we find such maps for \( i = 0, \ldots, k \). This means that \( H^{(k+1)}(1, \cdot) \) is homotopic through an admissible map to \( H : X^{(k)} \to Y \) defined as
\[
H_{i-k}(x) = (\hat{\eta}_{i-k}(\theta_{i-k}; A_{\theta_{i-k-1}}x_{i-k-1}, 0, 0, 0) - (\theta_{i-k}; x_{i-k}, y_{i-k}, z_{i-k}),
\]
\[
H_{i-k+1}(x) = (\hat{\eta}_{i-k+1}(\theta_{i-k}; A_{\theta_{i-k}}x_{i-k}, 0, 0) - (\theta_{i-k+1}; x_{i-k+1}, y_{i-k+1}, z_{i-k+1}),
\]

\[
\vdots
\]
\[
H_{i}(x) = H_{i}^{(k+1)}(1, x), \quad \text{for } i > 0.
\]

\(^6\) As highlighted in remark \( 12 \), we could use alternative assumptions for this part of the argument. It would be enough if the degree was not zero at each point in \( \Lambda \), instead of assuming that the (global) degree is not zero. Also, in the setting of orientable manifolds we could use the Brouwer degree for this part of the argument. Then, instead of the mod 2 intersection number, we would use the oriented intersection number throughout the proof.
We see that $H$ intersects transversely with $Z$ at $H(x)$ for points of the form
\[
x = ((\lambda_{-k-1}; 0), (\lambda_{-k}; 0, 0, 0), \ldots, (\lambda_{-1}; 0, 0, 0),
(\Theta_0; 0, 0, 0), (\Theta_1; 0, 0, 0), \ldots, (\Theta_2; 0, 0, 0)),
\]
where $\lambda_{-k-1} \in \tilde{\eta}_{-k}^{-1} \circ \cdots \circ \tilde{\eta}_0^{-1}(\Theta_0)$ and $\lambda_{-i} = \tilde{\eta}_{-i} \circ \cdots \circ \tilde{\eta}_0(\lambda_{-k-1})$ for $i = 1, \ldots, k$. The number of the points of the form (50) is equal to $\#\tilde{\eta}_{-1}^{-1} \circ \cdots \circ \tilde{\eta}_0^{-1}(\Theta_0)$, which is odd, and so $I_2(H, X^{(k)}, Z) = 1$. Since
\[
I_2(F, X, Z) = I_2(H^{(k+1)}(1, \cdot), X^{(k)}, Z) = I_2(H, X^{(k)}, Z),
\]
this implies that $I_2(F, X, Z) = 1$.

Since $I_2(F, X, Z) = 1$, we have established the existence of a trajectory $\{v_i\}_{i=k}^k$ in $D$, for which $v_0 \in D_{\theta_0}$. Because this holds for any $k \in \mathbb{N}$, we obtain a sequence of such $v_0$’s lying in $D_{\theta_0}$ which depend on $k$. Our claim now follows by passing to the limit of a convergent subsequence, by the virtue of compactness of $D_{\theta_0}$, to obtain a point $v_0 \in D_{\theta_0}$ for which the full trajectory is contained in $D$.

8. Proof of theorem 15

Before we proceed with the proof, we shall need two auxiliary results. The first is a classical lemma:

**Lemma 26** [29, (9.3) p 12] (Whyburn’s lemma). Assume that $K$ is a compact metric space and $K_0$, $K_1$ two closed disjoint subsets of $K$. Then either

(a) There exists a component (maximal closed connected subset) of $K$ meeting $K_0$ and $K_1$,
(b) Or there exist two disjoint compact sets $K_0$ and $K_1$ such that $K = K_0 \cup K_1$ and $K_i \subset \hat{K}_i$ for $i = 1, 2$.

The second result is a generalisation of the homotopy property of the intersection number. Let $X, Y, Z$ be as in section 2.4. On $[0, 1] \times \overline{X}$ consider the topology induced from $\mathbb{R} \times \overline{X}$. (This means in particular that $\partial ([0, 1] \times \overline{X}) = [0, 1] \times \partial \overline{X}$.) Let $V \subset [0, 1] \times \overline{X}$ be open and for $\alpha \in [0, 1]$ let $V_\alpha = \{x | (\alpha, x) \in V\}$.

**Lemma 27.** If $H : [0, 1] \times \overline{X} \to Y$ is continuous, $H(\partial V) \cap Z = \emptyset$ and $H(\overline{V}) \cap \partial Z = \emptyset$, then
\[
I_2(H(0, \cdot), V_0, Z) = I_2(H(1, \cdot), V_1, Z).
\]

**Proof.** The proof follows from mirror arguments to the proof of the homotopy property of the intersection number (see lemma 29 in appendix A). The intuition behind the proof is given in figure 11.

By performing an arbitrarily small modification of $H$ we can obtain $\tilde{H}$ for which $\tilde{H}(0, \cdot)$ and $\tilde{H}(1, \cdot)$ are transversal to $Z$ and that $H|_{V}$ is smooth and transversal to $Z$. We can make the modification small enough so that for $\beta \in [0, 1]$,
\[
\left((1 - \beta)H + \beta\tilde{H}\right)(\partial V) \cap \overline{Z} = \emptyset,
\]
\[
\left((1 - \beta)H + \beta\tilde{H}\right)(\overline{V}) \cap \partial Z = \emptyset.
\]
This in particular implies that for $d = 0, 1$, $H(d, \cdot)$ and $\tilde{H}(d, \cdot)$ are homotopic through an admissible homotopy, so

$$I_2(H(d, \cdot), V_d, Z) = I_2(\tilde{H}(d, \cdot), V_d, Z) \quad \text{for } d = 0, 1.$$  \hspace{1cm} (51)

Since $\tilde{H}|_V$ is transversal to $Z$, we have that $\tilde{H}^{-1}(Z)$ is a one-dimensional submanifold with boundary of $V$, the boundary being (see figure 11)

$$\partial \tilde{H}^{-1}(Z) = \{0\} \times \tilde{H}(0, \cdot)^{-1}(Z) \cup \{1\} \times \tilde{H}(1, \cdot)^{-1}(Z).$$

By the classification of one-manifolds [17], $\partial \tilde{H}^{-1}(Z)$ consists of an even number of points, hence

$$\# \tilde{H}(0, \cdot)^{-1}(Z) \equiv \# \tilde{H}(1, \cdot)^{-1}(Z) \mod 2.$$

This by the intersection property for transversal maps means that

$$I_2\left(\tilde{H}(0, \cdot), V_0, Z\right) = I_2\left(\tilde{H}(1, \cdot), V_1, Z\right),$$

which combined with (51) concludes our proof. \hspace{3cm} \square

The proof of theorem 15 is based on the classical ideas that stem from the Leray–Schauder continuation theorem [22]. This is a standard technique (see [25] for an overview of related results). We adopt it to be combined with the intersection number in our particular setting.

**Proof of theorem 15.** Let us fix $\theta = \Theta_0$. We will look for a connected component $C$ in the set $[0, 1] \times E_{\Theta_0}$. In fact it will turn out that we can find $C$ in $[0, 1] \times \left( E_{\Theta_0} \oplus \{0\} \right)$. Let $D_{\Theta_0}^\infty := \left\{ x \in E_{\Theta_0} | \|x\|_u \leq 1 \right\}$. The set $D_{\Theta_0}^\infty$ is a compact metric space, with the metric defined by the norm on the bundle $E_{\Theta_0}^\infty$. Let us equip $[0, 1] \times D_{\Theta_0}^\infty$ with a metric

$$m((\alpha_1, x_1), (\alpha_2, x_2)) := \max\{|\alpha_1 - \alpha_2|, \|x_1 - x_2\|_u\},$$  \hspace{1cm} (52)

and define a set

$$K := \left\{ (\alpha, x) | x \in D_{\Theta_0}^\infty, f_{\alpha}^i(\Theta_0; x, 0) \in D \quad \text{for all } i \in \mathbb{N} \text{ and } \alpha \in [0, 1] \right\}.$$
From the covering $D_{\epsilon_0} \ni D$ it follows that any point from $D_{\epsilon_0}$ exits $D$, which implies
\[
K \cap ([0,1] \times \partial D_{\epsilon_0}) = \emptyset. \quad (53)
\]

Since the family $f_\alpha$ is continuous and $D$ is closed, if we take a convergent sequence $(\alpha_j, x_j) \in K$, then $\lim_{j \to \infty} f_{\alpha_j}^i (\Theta_0; x_j, 0) \in D$, so $K$ is a compact metric space with the metric (52). Let $L_{\alpha}, L_{\alpha}^k \subset D_{\epsilon_0}^\alpha$ be compact sets defined as
\[
\begin{align*}
L_{\alpha} := \{ x \in D_{\epsilon_0}^\alpha | f_\alpha^i (\Theta_0; x, 0) \in D \quad \text{for all } i \in \mathbb{N} \}, \\
L_{\alpha}^k := \{ x \in D_{\epsilon_0}^\alpha | f_\alpha^i (\Theta_0; x, 0) \in D \quad \text{for } i = 0, \ldots, k \},
\end{align*}
\]
for $\alpha \in [0,1]$ and $k \in \mathbb{N}$. Note that $L_{\alpha} \subset L_{\alpha}^k$ and $L_{\alpha}^{k+1} \subset L_{\alpha}^k$ for $\alpha \in [0,1]$ and $k \in \mathbb{N}$.

Let $K_0 := \{0\} \times L_0$ and $K_1 := \{1\} \times L_1$. By theorem 8, $K_0$ and $K_1$ are nonempty. (Here we in fact used the fact that in the proof we have established that we can take the $v$ from the statement of theorem 8 to be from $E_{\epsilon_0}^\alpha \oplus \{0\} \cap D$.) By lemma 26 we have two possibilities. The first ensures our claim, so we need to rule out the second one, which will conclude our proof.

Suppose that we have two disjoint compact sets $\hat{K}_0$ and $\hat{K}_1$ such that $K = \hat{K}_0 \cup \hat{K}_1$ and $K_i \subset \hat{K}_i$, for $i = 0, 1$. Let us take small $\epsilon$ so that
\[
\epsilon < \frac{1}{2} \operatorname{dist} (\hat{K}_0, \hat{K}_1). \quad (54)
\]
Because of (53), we can take $\epsilon > 0$ small enough so that in addition to (54) we have
\[
U := \{ (\alpha, x) \in [0,1] \times D_{\epsilon_0}^\alpha | \operatorname{dist} ((\alpha, x), \hat{K}_0) < \epsilon \} \subset [0,1] \times \text{int} D_{\epsilon_0}^\alpha.
\]
Clearly $K_0 \subset U$ and also by (54) we see that $K_1 \cap U = \emptyset$. We shall use the notation $\tilde{U}_\alpha = \{ x | (\alpha, x) \in U \}$, so we can rewrite the previous statement as $L_0 \subset \tilde{U}_0$ and $L_1 \cap \tilde{U}_1 = \emptyset$. Since
\[
L_{\alpha} = \bigcap_{k=0}^{\infty} L_{\alpha}^k \quad \text{for } \alpha \in [0,1],
\]
by taking sufficiently large $k$ we will have
\[
L_{\alpha}^k \subset \tilde{U}_0, \quad L_{\alpha}^k \cap U_1 = \emptyset, \quad (55, 56)
\]
and since $\partial U \cap K = \emptyset$, we can also choose $k$ large enough so that
\[
L_{\alpha}^k \cap \partial \tilde{U}_\alpha = \emptyset \quad \text{for } \alpha \in [0,1]. \quad (57)
\]
Consider $Y$ and $Z$ defined in (21) and (22), and take
\[
\nabla := \mathcal{U} \times D \times \cdots \times D.
\]
We shall consider $H = (H_1, \ldots, H_k, H_{k+1}) : \nabla \to Y$ which is defined for points
\[
x = ((\Theta_0; x_0), (\theta_1; x_1, y_1, z_1), \ldots, (\theta_k; x_k, y_k, z_k)) \in \nabla, \quad (58)
\]
as
\[
H_1(\alpha, x) := (f_\alpha(\Theta_0; x_0, 0), 0) - (\theta_1; x_1, y_1, z_1),
\]
\[
H_2(\alpha, x) := (f_\alpha(\theta_1; x_1, y_1), 0) - (\theta_2; x_2, y_2, z_2),
\]
\[
\vdots
\]
\[
H_k(\alpha, x) := (f_\alpha(\theta_{k-1}; x_{k-1}, y_{k-1}), 0) - (\theta_k; x_k, y_k, z_k),
\]
\[
H_{k+1}(\alpha, x) := (f_\alpha(\theta_k; x_k, y_k)).
\]

We will now show that
\[
H(\partial V) \cap \mathbb{Z} = \emptyset. \tag{59}
\]

This follows from mirror arguments to those used to show (28). The only difference is that we also need to consider the case when \((\alpha, x) \in V\) is such that \((\alpha, x_0) \in \partial U\). In such case, due to (57), we see that we cannot have \(f^k_\alpha(\Theta_0; x_0, 0) \in D\) so any point for which \(H(\alpha, x) \in Z\) can not have \((\alpha, x_0) \in \partial U\). We have thus shown (59).

From arguments identical to showing (29) we also obtain
\[
H(\mathbb{V}) \cap \partial Z = \emptyset. \tag{60}
\]

By lemma (27) together with (59) and (60) we obtain
\[
I_2(H(0, \cdot), V_0, Z) = I_2(H(1, \cdot), V_1, Z). \tag{61}
\]

We will now compute \(I_2(H(0, \cdot), V_0, Z)\). Let \(X\) be the set defined in (20). The first coordinate of \(V_0\) is \(U_0\) (see (58)). By (55) \(U_0\) contains all points \(x_0\) such that \(f^k_\alpha(\Theta_0; x_0, 0) \in D\) for every \(i \in \{1, \ldots, k\}\). This means that if \(x \in X \setminus V_0\) then we cannot have \(f^k_\alpha(\Theta_0; x_0, 0) \in D\) for any \(i \in \{1, \ldots, k\}\). Thus, for \(x \in X \setminus V_0\) we cannot have \(H(0, x) \in Z\), hence
\[
H(0, X) \cap Z = H(0, V_0) \cap Z.
\]

Note that from (59) \(H(0, \partial V_0) \cap \mathbb{Z} = \emptyset\), so from the excision property
\[
I_2(H(0, \cdot), X, Z) = I_2(H(0, \cdot), V_0, Z). \tag{62}
\]

In the proof of theorem 8 we have established that \(I_2(H(0, \cdot), X, Z) = 1\), so by (62)
\[
I_2(H(0, \cdot), V_0, Z) = 1. \tag{63}
\]

We will now compute \(I_2(H(1, \cdot), V_1, Z)\). The set \(U_1^k\) contains all points \(x_0\) for which \(f^k_\alpha(\Theta_0; x_0, 0) \in D\) for all \(i \in \{1, \ldots, k\}\). If \(x \in V_1\), then \(x_0 \in U_1\), so by (56) we cannot have \(f^k_\alpha(\Theta_0; x_0, 0) \in D\) for all \(i \in \{1, \ldots, k\}\). This means that for \(x \in V_1\), \(H(1, x) \notin Z\), so by the intersection property
\[
I_2(H(1, \cdot), V_1, Z) = 0. \tag{64}
\]

By (61), (63) and (64) we have obtained a contradiction. This means that we have ruled out the second case of lemma 26 and finished our proof. \(\square\)
9. Proof of theorem 16

Proof. The proof follows along the same lines as the proof of theorem 15.

Let us fix $\theta = \Theta_0$. We will look for a connected component $C$ in $[0, 1] \times E_{\Theta_0}$. The set $D_{\Theta_0}$ is a compact metric space, with the metric defined by the norm on the bundle $E_{\Theta_0}$. We equip $[0, 1] \times D_{\Theta_0}$ with a metric

$$
m((\alpha_1, v_1), (\alpha_2, v_2)) = \max\{|\alpha_1 - \alpha_2|, \|v_1 - v_2\|\},
$$

and define a set

$$K := \{ (\alpha, v) \mid v \in D_{\Theta_0} \text{ and there exists a trajectory of } f_\alpha \text{ in } D \text{ passing through } v \}.$$  

We shall say that a sequence $\{v_i\}$ is a trajectory of $f_\alpha$ of length $k$ in $D$ passing through $v$ if $v_0 = v, v_i \in D$ for $i = -k, \ldots, k$ and $f_\alpha(v_i) = v_{i+1}$ for $i = -k, \ldots, k - 1$.

The set $K$ is a compact metric space with the metric (65). For $\alpha \in [0, 1]$ and $k \in \mathbb{N}$ let $L_\alpha, L_k \subset D_{\Theta_0}$ be compact sets defined as

$$L_\alpha := \{ v \in D_{\Theta_0} \mid \text{there exists a trajectory of } f_\alpha \text{ in } D \text{ passing through } v \},$$

$$L_k := \{ v \in D_{\Theta_0} \mid \text{there exists a trajectory of } f_\alpha \text{ of length } k \text{ in } D \text{ passing through } v \}.$$  

Note that $L_{k+1} \subset L_k$ and $L_\alpha \subset L_k$ for $\alpha \in [0, 1]$ and $k \in \mathbb{N}$.

Let $K_0 := \{0\} \times L_0$ and $K_1 := \{1\} \times L_1$. By theorem 13, $K_0$ and $K_1$ are nonempty. By lemma 26 we have two possibilities. The first ensures our claim, so we need to rule out the second one, which will conclude our proof.

Suppose that we have disjoint compact $K_0, K_1$ such that $K = K_0 \cup K_1$ and $K_0 \subset \hat{K}_0, K_1 \subset \hat{K}_1$. Consider $\varepsilon < \frac{1}{2} \text{dist}(\hat{K}_0, \hat{K}_1)$, chosen sufficiently small so that

$$U := \{ (\alpha, v) \in [0, 1] \times D_{\Theta_0} \mid \text{dist}((\alpha, v), \hat{K}_0) < \varepsilon \} \subset [0, 1] \times \text{int} D_{\Theta_0}.$$  

We shall embed $U$ in $\mathbb{R}^{2n}$ (we use notation 23)

$$U := \{ (\alpha, (v, z)) \in [0, 1] \times \mathcal{T} \mid (\alpha, v) \in U, ||z||_{\mathbb{R}^{2n}} \leq \delta \} \subset \mathbb{R}^{2n}.$$  

Consider $Y$ and $Z$ defined in (47) and (48), and take

$$\nabla := D^n \times D \times \cdots \times D \times \mathcal{U} \times D \times \cdots \times D.$$  

We shall consider $H = (H_{-k}, \ldots, H_{k}, H_{k+1}) : \nabla \to Y$ which is defined for points

$$x = ((\theta_{-k-1}; x_{-k-1}), (\theta_{-k}; x_{-k}, y_{-k}, z_{-k}), \ldots, (\theta_{-1}; x_{-1}, y_{-1}, z_{-1}),$$

$$(\theta_0; x_0, y_0, z_0), (\theta_1; x_1, y_1, z_1), \ldots, (\theta_k; x_k, y_k, z_k))$$

as (compare with (49) used in the proof of theorem 13)

$$H_{-k}(\alpha, x) := (f_\alpha(\theta_{-k-1}; x_{-k-1}, 0), 0) - (\theta_{-k}; x_{-k}, y_{-k}, z_{-k}),$$

$$H_{k+1}(\alpha, x) := (f_\alpha(\theta_k; x_{-k}, y_{-k}), 0) - (\theta_{k+1}; x_{-k+1}, y_{-k+1}, z_{-k+1}),$$

$$\vdots$$
\[ H_{-1}(\alpha, x) := (f_0(\theta_{-2}; x_{-2}, y_{-2}), 0) - (\theta_{-1}; x_{-1}, y_{-1}, z_{-1}), \]
\[ H_0(\alpha, x) := (f_0(\theta_{-1}; x_{-1}, y_{-1}), 0) - (\theta_0; x_0, y_0, z_0), \]
\[ H_1(\alpha, x) := (f_0(\theta_0; x_0, y_0), 0) - (\theta_1; x_1, y_1, z_1), \]
\[ H_2(\alpha, x) := (f_0(\theta_1; x_1, y_1), 0) - (\theta_2; x_2, y_2, z_2), \]
\[ \vdots \]
\[ H_k(\alpha, x) := (f_0(\theta_{k-1}; x_{k-1}, y_{k-1}), 0) - (\theta_k; x_k, y_k, z_k), \]
\[ H_{k+1}(\alpha, x) := f_0(\theta_k; x_k, y_k). \]

From now on we skip the details since they follow along the same lines as in the proof of theorem 15. We just outline the steps: using lemma 27 we can show that

\[ I_2(H(0, \cdot), V_0, Z) = I_2(H(1, \cdot), V_1, Z). \] (66)

Using the excision property, for \( X \) defined in (46), we obtain

\[ I_2(H(0, \cdot), V_0, Z) = I_2(H(0, \cdot), X, Z) = 1. \] (67)

From the fact that \( V_1 \) cannot contain trajectories of length \( k \) in \( D \) passing through \( D_{\theta_0} \) we also obtain

\[ I_2(H(1, \cdot), V_1, Z) = 0. \] (68)

Conditions (66)–(68) lead to a contradiction, which concludes our proof. \( \Box \)

10. Proof of lemma 20

**Proof.** Let \( E = E^u \oplus E^s \) and consider \( l : E \rightarrow E \) defined as

\[ l(\theta; x, y) := (f(\theta); df(\theta)x, df(\theta)y). \]

Note \( l \) is well defined since the splitting (6) is invariant under the action of the differential \( df \).

We shall refer to \( l \) as the ‘linearised map’. Note that

\[ \dot{\mathbf{x}} = (f(\theta); df(\theta)x, df(\theta)y). \]

For \( r > 0 \) we define \( D(r) \subset E \) as

\[ D(r) := \{ (\theta; x, y) | \theta \in \Lambda, x \in E^u_\theta, y \in E^s_\theta, \|x\| \leq r, \|y\| \leq r \}. \]

We will show that for any \( r > 0 \) we have

\[ D(r) \overset{\beta}{\longrightarrow} D(r). \] (69)

The homotopy \( h \) (see definition 9) for the covering (69) can be taken as

\[ h(\beta, (\theta; x, y)) := (f^\beta(\theta); df^\beta(\theta)x, (1 - \beta)df^\beta(\theta)y), \quad \beta \in [0, 1]. \] (70)
Before showing that $h$ satisfies all required conditions we note that since $\lambda < 1$, from $k > \log_\lambda \frac{1}{r}$ it follows that $C\lambda^k < 1$. Using (7) we also see that for any $\theta \in \Lambda$ and $x \in E^u_\theta$

$$
\|x\| = \|d (f^{-k} \circ f^k)(\theta)x\| = \|df^{-k}(f^k(\theta))df^k(\theta)x\| \leq C\lambda^k \|df^k(\theta)x\|. \quad (71)
$$

We will now show that $h$ satisfies conditions from definition 9. If $(\theta; x, y) \in D^-(r)$, meaning that $\|x\| = r$, then by (71), $\|df^k(\theta)x\| > (C\lambda^k)^{-1}\|x\| > r$, hence $h(\beta, (\theta; x, y)) \notin D(r)$, ensuring (2).

For any $(\theta; x, y) \in D(r)$, by (8), $\| (1 - \beta) df^k(\theta)y\| < C\lambda^k \|y\| < r$, so $h(\beta, (\theta; x, y)) \notin D^+(r)$, which means that we have verified (3).

From (70) we see that the map $\eta$ from definition 9 is $\eta = (f|\Lambda)^k$. Since $\Lambda$ is invariant under $f$, and $f$ is a diffeomorphism, $(f|\Lambda)^k$ is also a diffeomorphism, so $\deg_2(\eta) = 1$ ensuring (4). Also from (70), $A_\theta = df^k(\theta)|_{E^s}$. Since $C\lambda^k < 1$, by (71) $A_\theta$ is expanding. We have thus established (69).

For sufficiently small $r$ the linearised dynamics inside $D(r)$ is topologically conjugate to the true dynamics in a neighbourhood of $\Lambda$, i.e. $f^k \circ g = g \circ f^k$ where $g$ is the conjugating homeomorphism [26]. The set $D = g^{-1}(D(r))$, equipped with the structure of the vector bundle $E$ induced by $g$, constitutes the neighbourhood of $\Lambda$ in $M$ which $f^k$-covers itself.

**Acknowledgments**

We would like to thank Rafael de la Llave for his encouragement and suggestions. In particular, we thank him for pointing us to the Leray–Schauder continuation techniques and for his suggestions that led us to formulating theorems 15 and 16. We would also like to thank the anonymous Reviewers for their comments, suggestions and corrections, which helped us improve our paper.

**Appendix A. Construction of the intersection number**

Here we present a brief overview of the construction of $I_2(f, X, Z)$.

**Lemma 28.** If $f$ is admissible, $|f|_X$ is smooth and transversal to $Z$, then the number $\# f^{-1}(Z)$ is finite.

**Proof.** Since $f$ is admissible, $f^{-1}(Z)$ is separated from $\partial X$ and $f^{-1}(\partial Z) = \emptyset$. From the transversality of $|f|_X$ to $Z$ we obtain that $f^{-1}(Z)$ is a zero-dimensional submanifold of $X$. From transversality, the points in $f^{-1}(Z)$ cannot accumulate, and since they are contained in the compact set $\overline{X}$, their number is finite.

Above we have shown that for $|f|_X$ transversal to $Z$ defining the intersection number as

$$
I_2(f, X, Z) \coloneqq \# f^{-1}(Z) \mod 2 
$$

(A.1)

makes sense since we do not have infinity on the right-hand side of the defining equation. We now show that this number remains constant when passing through an admissible homotopy. The proof of the following lemma is based on the fact that if we have a homotopy $H$ between two smooth maps, both being transversal to some given manifold, then we can make $H$ transversal to that manifold by an arbitrarily small modification (see [17, extension theorem and corollary that follows] for details).
Lemma 29. Assume that $f_0, f_1 : X \to Y$ are two admissible maps and that $f_0 \mid_X$ and $f_1 \mid_X$ are smooth and transversal to $Z$. If $f_0$ and $f_1$ are homotopic via an admissible homotopy, then

$$\# f^{-1}_0 (Z) = \# f^{-1}_1 (Z) \mod 2.$$  

Proof. The intuition behind the proof is given in figure A.1.

Let $F : [0,1] \times X \to Y$ be an admissible homotopy from $f_0$ to $f_1$. By performing an arbitrarily small modification we can arrive at an admissible $F$ such that $F \mid_{[0,1] \times X}$ is smooth and transversal to $Z$. Note that since $F$ is admissible, $F^{-1} (Z)$ does not intersect $[0,1] \times \partial X$. Since $F \mid_{[0,1] \times X}$ is transversal to $Z$, we have that $F^{-1} (Z)$ is a one-dimensional submanifold with boundary of $[0,1] \times X$, the boundary being

$$\partial F^{-1} (Z) = \{0\} \times f^{-1}_0 (Z) \cup \{1\} \times f^{-1}_1 (Z).$$

By the classification of one-manifolds [17], $\partial F^{-1} (Z)$ consists of an even number of points, hence

$$\# f^{-1}_0 (Z) = \# f^{-1}_1 (Z) \mod 2,$$

as required. □

For any admissible map $f$ we can find an admissible map $g$, arbitrarily close to $f$, for which $g \mid_X$ is smooth, and such that $f$ and $g$ are homotopic through an admissible homotopy. If $g \mid_X$ is not transversal to $Z$, then we can again perform an arbitrarily small modification to obtain transversality of $g \mid_X$ to $Z$. We can therefore define

$$I_2 (f, X, Z) := \# g^{-1} (Z) \mod 2,$$  \hspace{1cm} (A.2)

where $f$ and $g$ are as above. By lemma 28 the number $\# g^{-1} (Z)$ is finite and by lemma 29 the number $\# g^{-1} (Z) \mod 2$ does not depend on the choice of $g$, so $I_2 (f, X, Z)$ from (A.2) is well-defined.
What is left is to prove that for $I_2(f, X, Z)$ defined in (A.2) we have the homotopy property, intersection property and the excision property.

**Lemma 30.** If $f_1, f_2$ are homotopic through an admissible homotopy then $I_2(f_1, X, Z) = I_2(f_2, X, Z)$.

**Proof.** Since $f_1, f_2$ are homotopic through an admissible homotopy, they are admissible. As in the construction leading to (A.2) we can find two smooth admissible maps $g_1$ and $g_2$, homotopic through an admissible homotopy to $f_1$ and $f_2$, respectively, for which $g_1|_X$ and $g_2|_X$ are transversal to $Z$. Since $g_1$ and $g_2$ are homotopic through an admissible homotopy (which is a composition of admissible homotopies: $g_1$ to $f_1$, $f_1$ to $f_2$, and $f_2$ to $g_2$) from (A.2) and lemma 29 we obtain

$$I_2(f_1, X, Z) = \#g_1^{-1}(Z) \mod 2 = \#g_2^{-1}(Z) \mod 2 = I_2(f_2, X, Z),$$

as required.

**Lemma 31.** Let $f$ be an admissible map. If $I_2(f, X, Z) \neq 0$ then $f(X) \cap Z$ is nonempty.

**Proof.** We will show that if $f(X) \cap Z = \emptyset$ then $I_2(f, X, Z) = 0$.

Assume that $f(X) \cap Z = \emptyset$. By admissibility $f(\partial X) \cap Z = \emptyset$ and $f(X) \cap \partial Z = \emptyset$, so then $f(X) \cap Z = \emptyset$.

We can approximate $f|_X$ by an arbitrarily close smooth map $g : X \to Y$, homotopic to $f|_X$. We can extend this $g$ to $\overline{X}$ in the natural way to obtain $g : \overline{X} \to Y$. We can take this $g$ close enough so that it is homotopic to $f$ by an admissible homotopy and $g(X) \cap Z = \emptyset$. Since the intersection of $g(X)$ with $Z$ is empty and $g|_X$ is smooth, it is transversal to $Z$ (an empty intersection is by definition transversal), and

$$I_2(f, X, Z) = I_2(g, X, Z) = \#g^{-1}(Z) \mod 2 = 0,$$

as required.

**Lemma 32.** Let $f : X \to Y$ be an admissible map. If $V$ is an open subset of $X$ such that $f(X) \cap Z = f(V) \cap Z$, and $f(\partial V) \cap \partial Z = \emptyset$ then

$$I_2(f, X, Z) = I_2(f|_V, V, Z).$$

**Proof.** We see that $f|_V$ is admissible since

$$f(V) \cap \partial Z \subset f(X) \cap \partial Z = \emptyset.$$

We can find a $g$ arbitrarily close to $f$, homotopic through an admissible homotopy (admissible both for $X$ and $V$) so that $g|_X$ is smooth. If $g|_X$ is not transversal to $Z$, then we can make an arbitrarily small modification of $g|_X$ to make it transversal. We can take $g$ close enough to $f$ so that $g(X) \cap Z = g(V) \cap Z$. Since $g|_X$ is transversal to $Z$, $g|_V$ is also transversal to $Z$. From (A.1) and (A.2)

$$I_2(f, X, Z) = I_2(g, X, Z) = \#g^{-1}(Z) \mod 2 = I_2(g|_V, V, Z) = I_2(f|_V, V, Z),$$

as required.
Appendix B. Code for the computer assisted proof

The program validates that $D \xrightarrow{f} D$. We write out the code and follow with comments.

```c++
#include <iostream>
#include "capd/capdlib.h"
using namespace std; using namespace capd;

const interval nu=interval(1)/interval(10);

interval part(interval x, int N, int k)
{ return x.left()+(x.right()-x.left())/N*(x.left()+k/N); }  

interval hx(interval alpha,interval x,interval y)
{ return alpha*2*x/(1-alpha)*(-@x/5+power(x,3)+x*y/2); }  

interval hy(interval alpha,interval theta,interval x,interval y)
{ return (1-alpha)*(mu_y+2*sin(theta)/5+x*cos(theta)); }  

bool ExitCondition(interval alpha,interval Bu,interval Bs)
{  if(not(hx(alpha,Bu.left()) ,Bs)<Bu)) return 0;
   if(not(hx(alpha,Bu.right()) ,Bs)>Bu)) return 0;
   return 1;
 }

bool EntryCondition(interval alpha,interval theta,interval Bu,interval Bs,int N)
{  for(int i=0;i<N;i++)
    { for(int j=0;j<N;j++)
       { interval x=hx(alpha,part(Bu,N,i),part(Bs,N,j));
         interval y=hy(alpha,theta,part(Bu,N,i),part(Bs,N,j));
         if(not(x<Bu))
           if(not(x>Bu))
             if(not(subsetInterior(y,Bs))) return 0;
       }
    }
    return 1;
  }

int main()
{
   interval alpha=interval(0,0.1,0);
   interval Lambda=interval(2)*interval::pi()*interval(0,0.1,0);
   interval Bu=interval(-1,0.1,0);
   interval Bs=interval(-1.2,1.2,2);
   for(int k=0;k<4;k++)
   {  for(int i=0;i<100;i++)
     {  if(ExitCondition(part(alpha,4,k),Bu,Bs)==0) return 0;
         if(EntryCondition(part(alpha,4,k),part(Lambda,100,1),Bu,Bs,50)==0) return 0;
     }
   }
   cout "proof complete" << endl; return 1;
}
```
The code is based on the CAPD\(^7\) library for C++. To download and install the library follow the instructions found at http://capd.ii.uj.edu.pl.

This routine computes the \(k\)th part out of \(N\) of the interval \(x\). The indexing is \(k = 0, \ldots, N-1\). For example, if \(x = [1.0, 2.0]\) then for \(N = 4\) the 0th part is \([1.0, 1.25]\) and the 3rd part is \([1.75, 2.0]\).

These routines are used for the homotopy (14) along the \(x, y\) coordinates. Condition (4) follows directly from (14) so we need to validate (2) and (3), for which local projections onto \(x, y\) are sufficient.

We check that \(h(\alpha \times D^-) \cap D = \emptyset\) and return 1 if this is validated and 0 otherwise. This function will later be used to check (2).

This function is used to validate that \(h(\alpha \times D_{\theta+}) \cap D^+ = \emptyset\). This is later used to validate (3). The test is performed by subdividing \(D_{\theta+}\) into \(N^2\) cubes and checking that the image of each of them does not intersect \(D^+\).

This is the core of the program, where we validate (2) and (3). We do so by subdividing the parameter interval \([0, 1]\) into four fragments and subdividing \(\Lambda\) into 100 parts.

Once the program reaches this point we are sure that all the needed conditions are validated. The program takes a fraction of a second, running on a standard laptop.

References

[1] Berger P and Bounemoura A 2013 A geometrical proof of the persistence of normally hyperbolic submanifolds Dyn. Syst. 28 567–81
[2] Capiński M J 2009 Covering relations and the existence of topologically normally hyperbolic invariant sets Discrete Continuous Dyn. Syst. 23 705–25
[3] Capiński M J and Zgliczyński P 2011 Cone conditions and covering relations for topologically normally hyperbolic invariant manifolds Discrete Continuous Dyn. Syst. 30 641–70
[4] Capiński M J and Zgliczyński P 2015 Geometric proof for normally hyperbolic invariant manifolds J. Differ. Equ. 259 6215–86
[5] Chaperon M 2002 Invariant manifolds revisited Tr. Mat. Inst. Steklova 236 428–46
[6] Chaperon M 2004 Stable manifolds and the Perron–Irwin method Ergod. Theor. Dynam. Syst. 24 1359–94
[7] Chaperon M 2008 The Lipschitzian core of some invariant manifold theorems Ergod. Theor. Dynam. Syst. 28 1419–41
[8] Fenichel N 1971 Persistence and smoothness of invariant manifolds for flows Indiana Univ. Math. J. 21 193–226
[9] Fenichel N 1974 Asymptotic stability with rate conditions Indiana Univ. Math. J. 23 1109–37
[10] Fenichel N 1974 Asymptotic stability with rate conditions for dynamical systems Bull. Am. Math. Soc. 80 346–9
[11] Fenichel N 1977 Asymptotic stability with rate conditions II Indiana Univ. Math. J. 26 81–93
[12] Floer A 1987 A refinement of the Conley index and an application to the stability of hyperbolic invariant sets Ergod. Theor. Dynam. Syst. 7 93–103
[13] Floer A 1990 A topological persistence theorem for normally hyperbolic manifolds via the Conley index Trans. Am. Math. Soc. 321 647–57
[14] Gidea M and Zgliczyński P 2004 Covering relations for multidimensional dynamical systems II J. Differ. Equ. 202 59–80
[15] Gonzalez F, Drotos G and Jung C 2014 The decay of a normally hyperbolic invariant manifold to dust in a three degrees of freedom scattering system J. Phys. A 47 045101
[16] Gonzalez F and Jung C 2015 Visualizing the perturbation of partial integrability J. Phys. A 48 435101
[17] Guillemin V and Pollack A 1974 Differential Topology (Englewood Cliffs, NJ: Prentice-Hall)

\(^7\) Computer assisted proofs in dynamics.
[18] Hirsch M W 1976 Differential topology Graduate Texts in Mathematics vol 33 (New York: Springer)
[19] Hirsch M W, Pugh C C and Shub M 1977 Invariant manifolds Lecture Notes in Mathematics vol 583 (Berlin: Springer)
[20] Jarýk J and Kurzweil J 1969 On invariant sets and invariant manifolds of differential systems J. Differ. Equ. 6 247–63
[21] Kuznetsov S P 2005 Sep Example of a physical system with a hyperbolic attractor of the Smale-Williams type Phys. Rev. Lett. 95 144101
[22] Leray J and Schauder J 1934 Topologie et équations fonctionnelles Ann. Sci. École Norm. Sup. 51 45–78
[23] Li J, Lu K and Bates P 2013 Normally hyperbolic invariant manifolds for random dynamical systems: Part I—Persistence Trans. Am. Math. Soc. 365 5933–66
[24] Mañé R 1978 Persistent manifolds are normally hyperbolic Trans. Am. Math. Soc. 246 261–83
[25] Mawhin J 2008 Reduction and continuation theorems for Brouwer degree and applications to nonlinear difference equations Opuscula Math 28 541–60
[26] Pugh C and Shub M 1970 Linearization of normally hyperbolic diffeomorphisms and flows Invent Math. 10 187–98
[27] Teramoto H, Toda M and Komatsuzaki T 2015 Breakdown mechanisms of normally hyperbolic invariant manifolds in terms of unstable periodic orbits and homoclinic/heteroclinic orbits in Hamiltonian systems Nonlinearity 28 2677–98
[28] Whitney H 1944 The self-intersections of a smooth n-manifold in 2n-space Ann. Math. 45 220–46
[29] Whyburn G T 1958 Topological Analysis (Princeton Mathematical Series No 23) (Princeton, NJ: Princeton University Press)
[30] Wilczak D 2010 Uniformly hyperbolic attractor of the Smale-Williams type for a Poincaré map in the Kuznetsov system SIAM J. Appl. Dyn. Syst. 9 1263–83
[31] Zgliczyński P 2009 Covering relations, cone conditions and the stable manifold theorem J. Differ. Equ. 246 1774–819
[32] Zgliczyński P and Gidea M 2004 Covering relations for multidimensional dynamical systems J. Differ. Equ. 202 32–58