On the Radius of Spatial Analyticity for the Quartic Generalized KdV Equation

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Abstract. Lower bound on the rate of decrease in time of the uniform radius of spatial analyticity of solutions to the quartic generalized KdV equation is derived, which improves an earlier result by Bona, Grujić and Kalisch.

1. Introduction

Consider the Cauchy problem for the quartic generalized Korteweg–de Vries (KdV) equation

\[
\begin{aligned}
&u_t + u_{xxx} + (u^4)_x = 0, \quad t, x \in \mathbb{R}, \\
&u(x, 0) = u_0(x),
\end{aligned}
\]  

(1.1)

where the unknown \( u(x, t) \) and the datum \( u_0(x) \) are real-valued.

In [6], Grünrock proved that the Cauchy problem (1.1) is locally well posed for data \( u_0 \in H^s(\mathbb{R}) \) with \( s > -1/6 \) and globally well posed for data \( u_0 \in H^s(\mathbb{R}) \) with \( s \geq 0 \). Later, Tao [18, 19] proved that (1.1) is globally well posed for data in the critical space \( \dot{H}^{-\frac{1}{6}}(\mathbb{R}) \) with small norm. For an earlier study of well posedness for (1.1), we refer to [5].

In the present paper, we shall study spatial analyticity of the solutions to the above Cauchy problem motivated by earlier works on this issue for generalized KDV by Bona et al. [2] and a recent one for KDV by Selberg and Da Silva [16]. In particular, we consider a real-analytic initial data \( u_0 \) with uniform radius of analyticity \( \sigma_0 > 0 \), so there is a holomorphic extension to a complex strip

\[ S_{\sigma_0} = \{ x + iy : |y| < \sigma_0 \}. \]

The question is then whether this property persists for all later times \( t \), but with a possibly smaller and shrinking radius of analyticity \( \sigma(t) > 0 \), i.e., is the solution \( u(t, x) \) of (1.1) analytic in \( S_{\sigma(t)} \) for all \( t \)? For short times, it was shown by Grujić and Kalisch in [8] that the radius of analyticity remains at
least as large as the initial radius, i.e., one can take \( \sigma(t) = \sigma_0 \). For large times on the other hand, it was shown by Bona et al. in [2, see Corollary 4] that \( \sigma(t) \) can decay no faster than \( t^{-164} \) as \( t \to \infty \). In this paper, we use the idea introduced in [17] (see also [16]) to improve this result significantly showing \( \sigma(t) \) can decay no faster than \( t^{-2} \) as \( t \to \infty \). For studies on related issues for nonlinear partial differential equations, see, for instance, [1,3,9–12,14,15].

The Gevrey space, denoted \( G^{\sigma,s} = G^{\sigma,s}(\mathbb{R}) \), is a suitable space to study analyticity of solution. This space is defined by the norm

\[
\|f\|_{G^{\sigma,s}(\mathbb{R})} = \left\| e^{\sigma|\xi|} \langle \xi \rangle^s \hat{f}(\xi) \right\|_{L^2_\xi(\mathbb{R})},
\]

where \( \hat{f} \) denotes the Fourier transform given by

\[
\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx
\]

and \( \langle \xi \rangle = \sqrt{1 + |\xi|^2} \). For \( \sigma = 0 \), the Gevrey space coincides with the Sobolev space \( H^s \). We shall write \( G^\sigma = G^{\sigma,0} \). One of the key properties of the Gevrey space is that every function in \( G^{\sigma,s} \) with \( \sigma > 0 \) has an analytic extension to the strip \( S_\sigma \). This property is contained in the following.

**Paley–Wiener Theorem.** Let \( \sigma > 0, s \in \mathbb{R} \). Then the following are equivalent:

(i) \( f \in G^{\sigma,s} \).

(ii) \( f \) is the restriction to the real line of a function \( F \) which is holomorphic in the strip

\[
S_\sigma = \{ x + iy : x, y \in \mathbb{R}, |y| < \sigma \}
\]

and satisfies

\[
\sup_{|y| < \sigma} \|F(x + iy)\|_{H^s_x} < \infty.
\]

The proof given for \( s = 0 \) in [13, p. 209] applies also for \( s \in \mathbb{R} \) with some obvious modifications.

Observe that the Gevrey spaces satisfy the following embedding property:

\[
G^{\sigma,s} \subset G^{\sigma',s'} \quad \text{for all } 0 \leq \sigma' < \sigma \text{ and } s, s' \in \mathbb{R}.
\]

In particular, setting \( \sigma' = 0 \), we have the embedding \( G^{\sigma,s} \subset H^s \) for all \( 0 < \sigma \) and \( s, s' \in \mathbb{R} \). As a consequence of this property and the existing well-posedness theory in \( H^s \), we conclude that the Cauchy problem (1.1) has a unique, smooth solution for all time, given initial data \( u_0 \in G^{\sigma_0,s} \) for all \( \sigma_0 > 0 \) and \( s \in \mathbb{R} \). Our main result gives an algebraic lower bound on the radius of analyticity \( \sigma(t) \) of the solution as the time \( t \) tends to infinity.

**Theorem 1.** Assume \( u_0 \in G^{\sigma_0,s} \) for some \( \sigma_0 > 0 \) and \( s \in \mathbb{R} \). Let \( u \) be the global \( C^\infty \) solution of (1.1). Then \( u \) satisfies

\[
u(t) \in G^{\sigma(t),s} \quad \text{for all } t \in \mathbb{R},\]

\[\]

\[1\] We use the notation \( a \pm = a \pm \varepsilon \) for sufficiently small \( \varepsilon > 0 \).
with the radius of analyticity \( \sigma(t) \) satisfying an asymptotic lower bound

\[
\sigma(t) \geq ct^{-2} \quad \text{as} \quad |t| \to \infty,
\]

where \( c > 0 \) is a constant depending on \( ||u_0||_{G^{\sigma_0,s}}, \sigma_0 \) and \( s \).

We note that (1.1) is invariant under the reflection \((t, x) \to (-t, -x)\). Hence, we may from now on restrict ourselves to positive times \( t \geq 0 \). The first step in the proof of Theorem 1 is to show that in a short time interval \( 0 \leq t \leq \delta \), where \( \delta > 0 \) depends on the norm of the initial data, the radius of analyticity remains constant. This is proved by a contraction argument involving energy estimates, Sobolev embedding and a multilinear estimate that is similar to the one proved by Grünrock in [6]. This result is stated in Sect. 2. The next step is to improve the control on the growth of the solution in the time interval \([0, \delta]\), measured in the data norm \( G^{\sigma_0} \). To achieve this, we show that, although the conservation of \( G^{\sigma_0} \)-norm of solution does not hold exactly, it does hold in an approximate sense (see Sect. 3). This approximate conservation law will allow us to iterate the local result and obtain Theorem 1. This will be proved in the last Sect. 4.

2. Preliminaries

2.1. Function Spaces

Define the Bourgain space \( X^{s,b} \) by the norm

\[
||u||_{X^{s,b}} = \left\| \langle \xi \rangle^s \langle \tau - \xi^3 \rangle^b \tilde{u}(\xi, \tau) \right\|_{L^2_{\tau,\xi}},
\]

where \( \tilde{u} \) denotes the space-time Fourier transform given by

\[
\tilde{u}(\tau, \xi) = \int_{\mathbb{R}^2} e^{i(t\tau + x\xi)} u(t, x) \, dt \, dx.
\]

The restriction to time slab \( \mathbb{R} \times (0, \delta) \) of the Bourgain space, denoted \( X^{s,b}_\delta \), is a Banach space when equipped with the norm

\[
||u||_{X^{s,b}_\delta} = \inf \left\{ ||v||_{X^{s,b}} : v = u \text{ on } \mathbb{R} \times (0, \delta) \right\}.
\]

In addition, we also need the Grevey–Bourgain space, denoted \( X^{\sigma,s,b}_\delta \), defined by the norm

\[
||u||_{X^{\sigma,s,b}_\delta} = \left\| e^{\sigma|D_x|} u \right\|_{X^{s,b}},
\]

where \( D_x = -i\partial_x \), which has Fourier symbol \( \xi \). In the case \( \sigma = 0 \), this space coincides with the Bourgain space \( X^{s,b}_\delta \). The restrictions of \( X^{\sigma,s,b}_\delta \) to a time slab \( \mathbb{R} \times (0, \delta) \), denoted \( X^{\sigma,s,b}_\delta \), are defined in a similar way as above.
2.2. Linear Estimates

In this subsection, we collect linear estimates needed to prove local existence of solution. The $X^{\sigma,s,b}$-estimates given below easily follow by substitution $u \rightarrow e^{\sigma|D_x|}u$ from the properties of $X^{s,b}$-spaces (and its restrictions). In the case $\sigma = 0$, the proofs of the first two lemmas below are found in Sect. 2.6 of [20], whereas the third lemma follows by the argument used to prove Lemma 3.1 of [4] and the fourth lemma is the standard energy estimate in $X^{s,b}_\delta$-spaces.

**Lemma 1.** Let $\sigma \geq 0$, $s \in \mathbb{R}$ and $b > 1/2$. Then $X^{\sigma,s,b} \subset C(\mathbb{R}, G^{\sigma,s})$ and
\[
\sup_{t \in \mathbb{R}} \|u(t)\|_{G^{\sigma,s}} \leq C \|u\|_{X^{\sigma,s,b}},
\]
where the constant $C > 0$ depends only on $b$.

**Lemma 2.** Let $\sigma \geq 0$, $s \in \mathbb{R}$, $-1/2 < b < b' < 1/2$ and $\delta > 0$. Then
\[
\|u\|_{X^{\sigma,s,b}_\delta} \leq C \delta^{b'-b} \|u\|_{X^{\sigma,s,b'}_\delta},
\]
where $C$ depends only on $b$ and $b'$.

**Lemma 3.** Let $\sigma \geq 0$, $s \in \mathbb{R}$, $-1/2 < b < 1/2$ and $\delta > 0$. Then for any time interval $I \subset [0, \delta]$ we have
\[
\|\chi_I u\|_{X^{\sigma,s,b}_\delta} \leq C \|u\|_{X^{\sigma,s,b}_\delta},
\]
where $\chi_I(t)$ is the characteristic function of $I$, and $C$ depends only on $b$.

Next, consider the linear Cauchy problem, for given $g(x,t)$ and $u_0(x)$,
\[
\begin{cases}
  u_t + u_{xxx} = g, \\
  u(0) = u_0.
\end{cases}
\]
Let $W(t) = e^{-t\partial^3} = e^{itD^3_x}$ be the solution group with Fourier symbol $e^{it\xi^3}$. Then we can write the solution using the Duhamel formula
\[
u(t) = W(t)u_0 + \int_0^t W(t-t')g(t') \, dt'.
\]
Then $u$ satisfies the following $X^{\sigma,s,b}$ energy estimate.

**Lemma 4.** Let $\sigma \geq 0$, $s \in \mathbb{R}$, $1/2 < b \leq 1$ and $0 < \delta \leq 1$. Then for all $u_0 \in G^{\sigma,s}$ and $F \in X^{\sigma,s,b-1}_\delta$, we have the estimates
\[
\|W(t)u_0\|_{X^{\sigma,s,b}_\delta} \leq C \|u_0\|_{G^{\sigma,s}},
\]
\[
\left\|\int_0^t W(t-t')g(t') \, dt'\right\|_{X^{\sigma,s,b}_\delta} \leq C \|g\|_{X^{\sigma,s,b-1}_\delta},
\]
where the constant $C > 0$ depends only on $b$. 
2.3. Multilinear Estimates and Local Result

The following multilinear estimates due to Grünrock [6] and Grünrock et al. [7] are key for proving our main result.

**Lemma 5** [6, Theorem 1 and Corollary 2]. Let \( s > -\frac{1}{6} \). Assume \(-\frac{1}{2} < b' < s - \frac{1}{3}\) if \(-\frac{1}{6} < s \leq 0\) and \(-\frac{1}{2} < b' < -\frac{1}{3}\) if \( s \geq 0\). Then for all \( b > \frac{1}{2} \) we have

\[
\| \partial_x \left( \prod_{j=1}^{4} u_j \right) \|_{X^{s,b'}} \lesssim \prod_{j=1}^{4} \| u_j \|_{X^{s,b}}. \tag{2.1}
\]

**Lemma 6** [7, Lemma 2]. Let \( b > \frac{1}{2} \), \( s_j \leq 0 \) for \( j = 1, \ldots, 4 \) and \( \sum_{j=1}^{4} s_j = -\frac{1}{2} \). Then

\[
\| \partial_x \left( \prod_{j=1}^{4} u_j \right) \|_{X^{0,-b}} \lesssim \prod_{j=1}^{4} \| u_j \|_{X^{s_j,b}}. \tag{2.2}
\]

From Lemma 5 and a simple triangle inequality, we obtain the following.

**Corollary 1.** Let \( s, b \) and \( b' \) be as in Lemma 5. Then for all \( \sigma \geq 0 \) we have

\[
\| \partial_x \left( \prod_{j=1}^{4} u_j \right) \|_{X^{\sigma,s,b'}} \leq C \prod_{j=1}^{4} \| u_j \|_{X^{\sigma,s,b}}, \tag{2.3}
\]

where \( C \) is independent of \( \sigma \).

**Proof.** Let

\( \tilde{v}_j(\tau, \xi) := e^{\sigma|\xi|} \tilde{u}_j(\tau, \xi) \),

then (2.3) is reduced to

\[
\| I \|_{L^2_{\tau,\xi}} \lesssim \prod_{j=1}^{4} \| v_j \|_{X^{s,b}}
\]

where

\[
I(\tau, \xi) = i\xi \langle \xi \rangle^s (\tau - \xi^3)^{b'} \int_{*} e^{\sigma(|\xi|-\sum_{j=1}^{4} |\xi_j|) \tau} \prod_{j=1}^{4} \tilde{v}_j(\tau_j, \xi_j) d\tau_* d\xi_*
\]

where we used the notation

\[
\int_{*} w d\tau_* d\xi_* = \int_{\sum_{j=1}^{4} \xi_j = \xi, \sum_{j=1}^{4} \tau_j = \tau} w \prod_{j=1}^{3} d\tau_j d\xi_j
\]

for a function \( w = w(\tau_j, \xi_j) \). By the triangle inequality, we have \(|\xi| \leq \sum_{j=1}^{4} |\xi_j|\) which implies \( e^{\sigma(|\xi|-\sum_{j=1}^{4} |\xi_j|)} \leq 1 \), and hence

\[
\| I \|_{L^2_{\tau,\xi}} \lesssim \| \partial_x \left( \prod_{j=1}^{4} v_j \right) \|_{X^{s,b'}}.
\]
Thus, (2.3) is reduced to showing
\[ \| \partial_x \left( \prod_{j=1}^{4} v_j \right) \|_{X^{s,b'}} \lesssim \prod_{j=1}^{4} \| v_j \|_{X^{s,b}} \]
which is (2.1). \qed

Then by Picard iteration and Corollary 1, one obtains the following local result (for details, see [16, proof of Theorem 1 therein]).

**Theorem 2.** Let \( \sigma > 0 \) and \( s > -\frac{1}{6} \). Then for any \( u_0 \in G^{\sigma,s} \) there exists a time \( \delta = \delta(\| u_0 \|_{G^{\sigma,s}}) > 0 \) and a unique solution \( u \) of (1.1) on the time interval \((0, \delta)\) such that
\[ u \in C([0, \delta], G^{\sigma,s}). \]
Moreover, the solution depends continuously on the data \( u_0 \), and we have
\[ \delta = c_0(1 + \| u_0 \|_{G^{\sigma,s}})^{-r} \]
for some constants \( c_0 > 0 \) and \( r > 1 \) depending only on \( s \). Furthermore, the solution \( u \) satisfies the bound
\[ \| u \|_{X^{\sigma,s,b}} \leq C \| u_0 \|_{G^{\sigma,s}} \quad \text{for} \quad b > \frac{1}{2}, \] (2.5)
where \( C \) depends only on \( s \) and \( b \).

**Remark 1.** Theorem 2 shows that if the initial data \( u_0 \) is analytic on the strip \( S_{\sigma} \) so is the solution \( u(t) \) on the same strip as long as \( t \in [0, \delta] \). Note also that in view of embedding (1.2) we can allow \( s \leq -\frac{1}{6} \) in Theorem 2, but then the solution will be analytic only on a slightly smaller strip \( S_{\sigma-} \).

### 3. Almost Conservation Law

For a given \( u(0) \in G^{\sigma} \), we have by Theorem 2 a solution \( u(t) \in G^{\sigma} \) for \( 0 \leq t \leq \delta \) satisfying the bound
\[ \sup_{t \in [0, \delta]} \| u(t) \|_{G^{\sigma}} \leq C \| u(0) \|_{G^{\sigma}}, \] (3.1)
where we also used (2.5) and Lemma 1; the constant \( C \) in (3.1) comes from these estimates and is independent of \( \delta \) and \( \sigma \). The question is then whether we can improve on estimate (3.1). In what follows we will use Eq. (1.1) and Theorem 2 to obtain the approximate conservation law
\[ \sup_{t \in [0, \delta]} \| u(t) \|_{G^{\sigma}}^2 = \| u(0) \|_{G^{\sigma}}^2 + E_{\sigma}(0), \]
where \( E_{\sigma}(0) \) satisfies the bound \( E_{\sigma}(0) \leq C \sigma^{\frac{1}{2}} \| u(0) \|_{G^{\sigma}}^{\frac{5}{2}} \). The quantity \( E_{\sigma}(0) \) can be considered an error term since in the limit as \( \sigma \to 0 \), we have \( E_{\sigma}(0) \to 0 \), and hence recovering the well-known conservation of \( L^2 \)-norm of solution:
\[ \| u(t) \|_{L^2} = \| u(0) \|_{L^2} \quad \text{for all} \quad t \in [0, \delta]. \]
Theorem 3. Let $b > \frac{1}{2}$ and $\delta$ be as in Theorem 2. Then there exists $C > 0$ such that for any $\sigma > 0$ and any solution $u \in X_{\delta}^{\sigma,0,b}$ to the Cauchy problem (1.1) on the time interval $[0, \delta]$, we have the estimate

$$\sup_{t \in [0, \delta]} \|u(t)\|_{C^\sigma}^2 \leq \|u(0)\|_{C^\sigma}^2 + C\sigma^{\frac{1}{2}}\|u\|_{X_{\delta}^{\sigma,0,b}}^5.$$  \hspace{1cm} (3.2)

Moreover, we have

$$\sup_{t \in [0, \delta]} \|u(t)\|_{C^\sigma}^2 \leq \|u(0)\|_{C^\sigma}^2 + C\sigma^{\frac{1}{2}}\|u(0)\|_{C^\sigma}^5.$$  \hspace{1cm} (3.3)

Proof. Estimate (3.3) follows from (3.2) and (2.5). Thus, it remains to prove (3.2).

Let $v(t, x) = e^{\sigma|D_x|}u(t, x)$ which is real-valued since the multiplier $e^{\sigma|D_x|}$ is even and $u$ is real-valued. Applying $e^{\sigma|D_x|}$ to (1.1), we obtain

$$v_t + v_{xxx} + \partial_x (v^4) = f,$$  \hspace{1cm} (3.4)

where

$$f = \partial_x \left\{ (e^{\sigma|D|}u)^4 - e^{\sigma|D|} (u^4) \right\}.$$  

Multiplying (3.4) by $v$ and integrating in space, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} v^2 dx + \int_{\mathbb{R}} \partial_x \left( vv_{xx} - \frac{1}{2}v_x^2 + \frac{4}{5}v^5 \right) dx = \int_{\mathbb{R}} vf dx.$$  

We may assume $v, v_x$ and $v_{xx}$ decays to zero as $|x| \to \infty$. This in turn implies

$$\frac{d}{dt} \int_{\mathbb{R}} v^2 dx = 2 \int_{\mathbb{R}} vf dx.$$  

Now integrating in time over the interval $[0, \delta]$, we obtain

$$\int_{\mathbb{R}} v^2(\delta, x) dx = \int_{\mathbb{R}} v^2(0, x) dx + 2 \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{[0, \delta]}(t) vf dx dt.$$  

Thus,

$$\|u(\delta)\|_{C^\sigma}^2 = \|u(0)\|_{C^\sigma}^2 + 2 \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{[0, \delta]}(t) vf dx dt.$$  

We now use Plancherel, Hölder, Lemmas 3 and 7 to estimate the integral on the right-hand side as

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{[0, \delta]}(t) vf dx dt \right| \leq \|v\|_{X_{\delta}^{\sigma,0,b}} \|f\|_{X_{\delta}^{\sigma,0,-b}} \leq \|v\|_{X_{\delta}^{\sigma,0,b}} \cdot C\sigma^{\frac{1}{2}} \|v\|_{X_{\delta}^{\sigma,0,b}}^4 = C\sigma^{\frac{1}{2}} \|u\|_{X_{\delta}^{\sigma,0,b}}^5.$$  

$\square$

\footnote{In general, this property holds by approximation using the monotone convergence theorem and the Riemann–Lebesgue Lemma whenever $u \in X_{\delta}^{\sigma,0,b}$ (see the argument in [16, pp. 9]).}
Lemma 7. Let
\[ f = \partial_x \left\{ (e^\sigma |D_x|^4 - e^\sigma |D_x|^4)u^4 \right\}. \]

For \( b > \frac{1}{2} \), we have
\[ \| f \|_{X^{0,-b}_s} \lesssim \sigma^{\frac{1}{2}} \| v \|^4_{X^{0,b}_s}, \]
where \( v = e^\sigma |D_x|^4 u \).

The following estimate is needed to prove Lemma 7.

Lemma 8. Let \( \xi_{\text{min}}, \xi_{\text{nd}}, \xi_{\text{rd}} \) and \( \xi_{\text{max}} \) denote the minimum, second largest, third largest and maximum of \( \{ |\xi_1|, |\xi_2|, |\xi_3|, |\xi_4| \} \). Then for \( \theta \in [0,1] \) we have the estimate
\[ e^\sigma \sum_{j=1}^4 |\xi_j| - e^\sigma |\sum_{j=1}^4 \xi_j| \leq 24 \sigma \xi_{\text{rd}} \theta e^\sigma \sum_{j=1}^4 |\xi_j|. \] (3.5)

Proof. First note that for \( x \geq 0 \) we have
\[ e^x - 1 \leq e^x \quad \text{and} \quad e^x - 1 \leq xe^x. \]

Hence
\[ e^x - 1 \leq x^\theta e^x \quad \text{for} \quad \theta \in [0,1]. \]

This in turn implies
\[ \text{LHS (3.5)} = \left\{ e^\sigma (\sum_{j=1}^4 |\xi_j| - |\sum_{j=1}^4 \xi_j|) - 1 \right\} e^\sigma |\sum_{j=1}^4 \xi_j| \]
\[ \leq \sigma^\theta \left( \sum_{j=1}^4 |\xi_j| - \sum_{j=1}^4 \xi_j \right)^\theta e^\sigma |\sum_{j=1}^4 \xi_j|. \]

Then (3.5) follows from the following estimate:
\[ \sum_{j=1}^4 |\xi_j| - \sum_{j=1}^4 \xi_j = \frac{(\sum_{j=1}^4 |\xi_j|)^2 - |\sum_{j=1}^4 \xi_j|^2}{\sum_{j=1}^4 |\xi_j| + |\sum_{j=1}^4 \xi_j|} \]
\[ = \frac{\sum_{j=1}^4 \sum_{k=1}^4 (|\xi_j||\xi_k| - \xi_j \xi_k)}{\sum_{j=1}^4 |\xi_j| + |\sum_{j=1}^4 \xi_j|} \]
\[ \leq 24 \frac{\xi_{\text{rd}} \cdot \xi_{\text{max}}}{\xi_{\text{max}}^2} = 24 \xi_{\text{rd}}. \]

Proof of Lemma 7. Taking the space-time Fourier Transform of \( f \) and using the notation in (2.4) we have
\[ |\tilde{f}(\tau, \xi)| \leq |\xi| \int |e^\sigma \sum_{j=1}^4 |\xi_j| - e^\sigma |\sum_{j=1}^4 \xi_j|| \prod_{j=1}^4 |\tilde{u}(\tau_j, \xi_j)| d\tau_* d\xi_. \]

Now we use (3.5) with \( \theta = \frac{1}{2} \) to obtain
\[ |\tilde{f}(\tau, \xi)| \lesssim |\xi| \int (\sigma \xi_{\text{rd}})^{\frac{1}{2}} \prod_{j=1}^4 |\tilde{v}(\tau_j, \xi_j)| d\tau_* d\xi_. \]
Depending on the relative sizes of $|\xi_j|$, $j = 1, \ldots, 4$, the quantity $\xi_{rd}$ is either $|\xi_1|$, $|\xi_2|$, $|\xi_3|$ or $|\xi_4|$. So we obtain
\[
\|f\|_{X^{0,-b}} = \|(\tau - \xi^3)^{-b} \tilde{f}(\tau, \xi)\|_{L^2_{\tau,\xi}} \\
\leq C\sigma^{\frac{1}{2}} \|\xi\|(\tau - \xi^3)^{-b} \int_{x} \xi_{rd}^\frac{1}{2} \prod_{j=1}^{4} |\tilde{v}(\tau_j, \xi_j)| d\tau d\xi \|_{L^2_{\tau,\xi}} \\
= C\sigma^{\frac{1}{2}} \|\partial_x \left\{ v^3 \cdot |D_x|^\frac{1}{2} v \right\} \|_{X^{0,-b}} \\
\leq C' \sigma^{\frac{1}{2}} \|v\|_{X^{0,b}}^3 \|D_x|^\frac{1}{2} v\|_{X^{-\frac{1}{2},b}} \\
\leq C' \sigma^{\frac{1}{2}} \|v\|_{X^{0,b}}^4,
\]
where in the fourth line we used Lemma 6.

\[\Box\]

4. Proof of Theorem 1

We closely follow the argument in [16]. First we consider the case $s = 0$. The general case, $s \in \mathbb{R}$, will essentially reduce to $s = 0$ as shown in the next subsection.

4.1. Case $s = 0$

Let $u_0 \in G^{\sigma_0}$ for some $\sigma_0 > 0$. Then to construct a solution on $[0, T]$ for arbitrarily large $T$, we will apply the approximate conservation law in Theorem 3 so as to repeat the local result on successive short time intervals to reach $T$, by adjusting the strip width parameter $\sigma$ according to the size of $T$. By employing this strategy, we will show that the solution $u$ to (1.1) satisfies
\[
u(t) \in G^{\sigma(t)} \text{ for all } t \in [0, T],
\]
with
\[
\sigma(t) \geq cT^{-2},
\]
where $c > 0$ is a constant depending on $\|u_0\|_{G^{\sigma_0}}$, $\sigma_0$ and $s$.

To this end, define
\[
A_{\sigma}(t) = \|u(t)\|_{G^{\sigma}},
\]
where $\sigma \in (0, \sigma_0]$ is a parameter to be chosen later. By Theorem 2, there is a solution $u$ to (1.1) satisfying
\[
u \in C([0, \delta]; G^{\sigma_0}),
\]
where
\[
\delta = c_0(1 + A_{\sigma_0}(0))^{-r} \text{ for some } r > 1.
\]
Now fix $T$ arbitrarily large. We shall apply the above local result and Theorem 3 repeatedly, with a uniform time step $\delta$ as in (4.3), and prove
\[
\sup_{t \in [0, T]} A_{\sigma}^2(t) \leq 2A_{\sigma_0}^2(0)
\]

(4.4)
for \( \sigma \) satisfying (4.2). Hence, we have

\[ A_\sigma(t) < \infty \quad \text{for} \quad t \in [0, T], \]

and this completes the proof of (4.1)–(4.2). It remains to prove (4.4), and this is done as follows. Choose \( n \in \mathbb{N} \) so that \( T \in [n\delta, (n+1)\delta) \). Using induction we can show for any \( k \in \{1, \ldots, n\} \) that

\[
\sup_{t \in [0,k\delta]} A_\sigma^2(t) \leq A_\sigma^2(0) + C_1 \frac{C_2}{A^{1/2}_0}(0), \tag{4.5}
\]

\[
\sup_{t \in [0,k\delta]} A_\sigma^2(t) \leq 2A_\sigma^2(0), \tag{4.6}
\]

provided \( \sigma \) satisfies

\[
2T \frac{C_1}{\delta} 2^{5/2} A_\sigma^3(0) \leq 1. \tag{4.7}
\]

Indeed, for \( k = 1 \), we have from Theorem 3 that

\[
\sup_{t \in [0,\delta]} A_\sigma^2(t) \leq A_\sigma^2(0) + C_1 \frac{C_2}{A^{1/2}_0}(0) \leq A_\sigma^2(0) + C_1 \frac{C_2}{A^{1/2}_0}(0),
\]

where we used \( A_\sigma(0) \leq A_\sigma(0) \). This in turn implies (4.6) provided \( C_1 \frac{C_2}{A^{1/2}_0}(0) \leq 1 \) which holds by (4.7) since \( T > \delta \).

Now assume (4.5) and (4.6) hold for some \( k \in \{1, \ldots, n\} \). Then by Theorem 3, (4.5) and (4.6), we have

\[
\sup_{t \in [k\delta,(k+1)\delta]} A_\sigma^2(t) \leq A_\sigma^2(k\delta) + C_1 \frac{C_2}{A^{1/2}_0}(k\delta)
\]

\[
\leq A_\sigma^2(k\delta) + C_1 \frac{C_2}{A^{1/2}_0}(2^{5/2} A_\sigma^3(0))
\]

\[
\leq A_\sigma^2(0) + kC_1 \frac{C_2}{A^{1/2}_o}(2^{5/2} A_\sigma^3(0)) + C_1 \frac{C_2}{A^{1/2}_0}(2^{5/2} A_\sigma^3(0)).
\]

Combining this with the induction hypothesis (4.5) (for \( k \)), we obtain

\[
\sup_{t \in [0,(k+1)\delta]} A_\sigma^2(t) \leq A_\sigma^2(0) + (k+1)C_1 \frac{C_2}{A^{1/2}_0}(2^{5/2} A_\sigma^3(0))
\]

which proves (4.5) for \( k + 1 \). This also implies (4.6) for \( k + 1 \) provided

\[
(k+1)C_1 \frac{C_2}{A^{1/2}_0}(2^{5/2} A_\sigma^3(0)) \leq 1.
\]

But the latter follows from (4.7) since

\[
k + 1 \leq n + 1 \leq \frac{T}{\delta} + 1 \leq \frac{2T}{\delta}.
\]

Finally, condition (4.7) is satisfied for \( \sigma \) such that

\[
2T \frac{C_1}{\delta} 2^{5/2} A_\sigma^3(0) = 1.
\]

Thus,

\[
\sigma = c_1 T^{-2}, \quad \text{where} \quad c_1 = \left( \frac{c_0}{C 2^{7/2} A_\sigma^3(0)(1 + A_\sigma(0))^r} \right)^{1/2}
\]

which gives (4.2) if we choose \( c \leq c_1 \).
4.2. The General Case: $s \in \mathbb{R}$

For any $s \in \mathbb{R}$, we use embedding (1.2) to get

$$u_0 \in G^{\sigma_0, s} \subset G^{\sigma_0/2}.$$

From the local theory, there is a $\delta = \delta \left(A_{\sigma_0/2}(0)\right)$ such that

$$u \in C \left([0, \delta], G^{\sigma_0/2}\right).$$

Fix an arbitrarily large $T$. From the case $s = 0$ in the previous subsection, we have

$$u(t) \in G^{2c_*, T^{-2}}$$

for $t \in [0, T]$, where $c_* > 0$ depends on $A_{\sigma_0/2}(0)$ and $\sigma_0$. Applying again embedding (1.2), we conclude that

$$u(t) \in G^{c_*, T^{-2}, s}$$

for $t \in [0, T]$, completing the proof of Theorem 1.

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