LINEAR EQUATIONS WITH TWO VARIABLES
IN PIATETSKI-SHAPIRO SEQUENCES

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Abstract. For every non-integral $\alpha > 1$, the sequence of the integer parts of $n^\alpha$ ($n = 1, 2, \ldots$) is called the Piatetski-Shapiro sequence with exponent $\alpha$, and let $PS(\alpha)$ denote the set of all those terms. For all $X \subseteq \mathbb{N}$, we say that an equation $y = ax + b$ is solvable in $X$ if the equation has infinitely many solutions of distinct pairs $(x, y) \in X^2$. Let $a, b \in \mathbb{R}$ with $a \neq 0$ and $0 \leq b < a$, and suppose that the equation $y = ax + b$ is solvable in $\mathbb{N}$. We show that for all $1 < \alpha < 2$ the equation $y = ax + b$ is solvable in $PS(\alpha)$. Further, we investigate the set of $\alpha \in (s, t)$ so that the equation $y = ax + b$ is solvable in $PS(\alpha)$ where $2 < s < t$. Finally, we show that the Hausdorff dimension of the set is coincident with $2/s$.

1. Introduction

For all $x \in \mathbb{R}$, we define $\lfloor x \rfloor$ as the integer part of $x$, and $\{x\}$ as the fractional part of $x$. For every non-integral $\alpha > 1$, the sequence $(\lfloor n^\alpha \rfloor)_{n=1}^\infty$ is called the Piatetski-Shapiro sequence with exponent $\alpha$, and let $PS(\alpha)$ be the set of all those terms. For all $X \subseteq \mathbb{N}$, and for all polynomials $f(x_1, \ldots, x_n)$ with real coefficients, we say that an equation $f(x_1, \ldots, x_n) = 0$ is solvable in $X$ if the equation has infinitely many solutions $(x_1, \ldots, x_n) \in X^n$ with $\#\{x_1, \ldots, x_n\} = n$. In this article, we discuss the solvability in $PS(\alpha)$ of the equation

\[ y = ax + b \]

for fixed $a, b \in \mathbb{R}$ with $a \notin \{0, 1\}$. Glasscock asserted that if equation (1.1) is solvable in $\mathbb{N}$, then for Lebesgue almost every $\alpha > 1$, it is solvable or not in $PS(\alpha)$, according to $\alpha < 2$ or $\alpha > 2$ [Gla17,Gla20]. In addition, as a corollary, he showed that for Lebesgue almost every $1 < \alpha < 2$, there are infinitely many $(k, \ell, m) \in \mathbb{N}^3$ such that

\[ k, \ell, m, k + \ell, \ell + m, m + k, k + \ell + m \]

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are in $\text{PS}(\alpha)$ [Gla17, Corollary 10]. It is a long-standing open problem whether there exists $(k, \ell, m) \in \mathbb{N}^3$ such that all terms of (1.2) are in $\text{PS}(2)$ which is the set of all squares.

The goal of this article is to put forward the following theorem which is an improvement on Glasscock’s result in the case when $0 \leq b < a$. Here for all $F \subseteq \mathbb{R}$, we let $\dim_H F$ denote the Hausdorff dimension of $F$. We will give this definition in Section 2.

**Theorem 1.1.** Let $a, b \in \mathbb{R}$, with $a \neq 1$ and $0 \leq b < a$. Assume that the equation $y = ax + b$ is solvable in $\mathbb{N}$. Then for all $1 < \alpha < 2$, the equation $y = ax + b$ is solvable in $\text{PS}(\alpha)$. Moreover, for all $s, t \in \mathbb{R}$ with $2 < s < t$, we have

$$\dim_H \{\alpha \in (s, t) : y = ax + b \text{ is solvable in } \text{PS}(\alpha)\} = \frac{2}{s}.$$ 

We avail of two main improvements when $0 \leq b < a$. Firstly, in the case when $1 < \alpha < 2$, we arrive at the same conclusion as Glasscock’s result even if we replace “for Lebesgue almost every” with “for all”. Secondly, in the case when $\alpha > 2$, his result is equivalent to stating that the set $\{\alpha \in (s, t) : y = ax + b \text{ is solvable in } \text{PS}(\alpha)\}$ has Lebesgue measure 0 for all $2 < s < t$. However, from Theorem 1.1 we find that the set has a Hausdorff dimension of exactly $\frac{2}{s}$. Hence we can discern more details concerning the geometric structure of the set. We will show Theorem 1.1 in Section 5.

From the first improvement, we obtain the following:

**Corollary 1.2.** For all $1 < \alpha < 2$, there are infinitely many $(k, \ell, m) \in \mathbb{N}^3$ such that all of $k, \ell, m, k + \ell, \ell + m, m + k, \text{ and } k + \ell + m$ are in $\text{PS}(\alpha)$.

In the proof of Corollary 1 in [Gla17], Glasscock applies the result given by Frantzkinakis and Wierdl [FW09, Proposition 5.1], and shows that if $y = 2x$ is solvable in $\text{PS}(\alpha)$ for fixed $1 < \alpha < 2$, then there are infinitely many $(k, m, \ell) \in \text{PS}(\alpha)^3$ such that all terms of (1.2) are in $\text{PS}(\alpha)$. By this proof, we obtain Corollary 1.2 by Theorem 1.1.

We next discuss the solvability in $\text{PS}(\alpha)$ of the equation

$$ax + by = cz$$

for fixed $a, b, c \in \mathbb{N}$. As a corollary of Theorem 1.1, the following holds:

**Corollary 1.3.** Let $a, b, c \in \mathbb{N}$ with $\gcd(a, c)|b$ and $a > b$. Then, for all $1 < \alpha < 2$, the equation $ax + by = cz$ is solvable $\text{PS}(\alpha)$. Further, for all $2 < s < t$, we have

$$\dim_H \{\alpha \in (s, t) : ax + by = cz \text{ is solvable in } \text{PS}(\alpha)\} \geq \frac{2}{s}.$$ 

Indeed, from the condition $\gcd(a, c)|b$, the equation $ax + b = cz$ is solvable in $\mathbb{N}$. By dividing both sides by $c$, we have the equation $z = (a/c)x + (b/c)$ whose coefficients $a/c$ and $b/c$ satisfy the conditions in Theorem 1.1. Moreover, if the equation $ax + b = cz$ is
solvable in $\text{PS}(\alpha)$, then by letting $y = 1 = \lfloor 1^\alpha \rfloor$, we see that the equation $ax + by = cz$ is solvable in $\text{PS}(\alpha)$. Therefore we conclude Corollary 1.3 from Theorem 1.1.

In [MS20], it is proved that for all $a, b, c \in \mathbb{N}$ and $s, t \in \mathbb{R}$ with $2 < s < t$, the left-hand side of (1.4) is greater than or equal to

$$\begin{cases} \left( s + \frac{s^3}{(2 + \{s\} - 2^{1-\{s\}})(2 - \{s\})} \right)^{-1} & \text{if } a = b = c \\ 2 \left( s + \frac{s^3}{(2 + \{s\} - 2^{1-\{s\}})(2 - \{s\})} \right)^{-1} & \text{otherwise.} \end{cases}$$

The lower bounds (1.4) in Corollary 1.4 are better than the above for all $2 < s < t$. In particular, we find that the left-hand side of (1.4) goes to 1 as $s \to 2 + 0$ from Corollary 1.4 if $a, b, c$ are restricted.

**Notation 1.4.** Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{Z}$ be the set of all integers, $\mathbb{Q}$ be the set of all rational numbers, and $\mathbb{R}$ be the set of all real numbers. For all $x \in \mathbb{R}$, let $\lceil x \rceil$ denote the minimum integer $n$ such that $x \leq n$. For all $a, b \in \mathbb{Z}$, we say $a \mid b$ if $a$ is a divisor of $b$, and let $\gcd(a, b)$ denote the greatest common divisor of $a$ and $b$. For all sets $X$, let $\#X$ denote the cardinality of $X$. For all sets $X$ and $\Lambda$, let $X^\Lambda$ denote the set of all sequences $(x_\lambda)_{\lambda \in \Lambda}$ composed of $x_\lambda \in X$ for all $\lambda \in \Lambda$. Define $e(x)$ by $e(2\pi \sqrt{-1} x)$ for all $x \in \mathbb{R}$.

## 2. Preparations

A sequence $(x_n)_{n=1}^{\infty} \in \mathbb{R}^\mathbb{N}$ is called **uniformly distributed modulo 1** if for every $0 \leq a < b \leq 1$, we have

$$\lim_{N \to \infty} \frac{\# \{ n \in \mathbb{N} \cap [1, N] : \{x_n\} \in [a, b) \}}{N} = b - a. \tag{2.1}$$

For further details on uniform distribution theory, see the book written by Kuipers and Niederreiter [KN74]. It is useful to calculate the decay of higher-order derivatives of $f(x)$ in order to verify that a sequence $(f(n))_{n \in \mathbb{N}}$ is uniformly distributed modulo 1. For example, by [KN74, Theorem 3.5], let $k$ be a positive integer, and let $f(x)$ be a function defined for $x \geq 1$, which is $k$ times differentiable for $x \geq x_0$. If $f^{(k)}(x)$ tends monotonically to 0 as $x \to \infty$ and if $\lim_{x \to \infty} x|f^{(k)}(x)| = \infty$, then the sequence $(f(n))_{n \in \mathbb{N}}$ is uniformly distributed modulo 1. By this theorem, we have

**Example 2.1.** For all $A \in \mathbb{R} \setminus \{0\}$ and non-integral $\alpha > 1$, the sequence $(An^\alpha)_{n=1}^{\infty}$ is uniformly distributed modulo 1.
From the definition, the following basic properties hold: let \( U \), we define the Hausdorff dimension as \( \text{diam}(U) \). We can find upper bounds of the discrepancy from evaluating exponential sums by the Erdős-Turán inequality, a proof of which can be found in [KN74, Theorem 2.5 in Chapter 2]. If we can write \( hx_n = f(n) \) in (2.1) for some smooth real function \( f(x) \), then we can evaluate the right-hand side of (2.2) by the following lemma.

**Lemma 2.2** (van der Corput’s \( k \)-th derivative test). Let \( f(x) \) be real and have continuous derivatives up to the \( k \)-th order, where \( k \geq 4 \). Let \( \lambda_k \leq f^{(k)}(x) \leq h\lambda_k \) (or the same as for \(-f^{(k)}(x)\)). Let \( b - a \geq 1 \). Then there exists \( C(h, k) > 0 \) such that

\[
\left| \sum_{a < n \leq b} e(f(n)) \right| \leq C(h, k) \left( (b-a)\lambda_k^{1/(2^k-2)} + (b-a)^{1-2^{-k}}\lambda_k^{-1/(2^k-2)} \right).
\]

**Proof.** See the book written by Titchmarsh [Tit86, Theorem 5.13].

We next introduce the Hausdorff dimension. For every \( U \subseteq \mathbb{R} \), state the diameter of \( U \) as \( \text{diam}(U) = \sup_{x, y \in U} |x - y| \). Fix \( \delta > 0 \). For all \( F \subseteq \mathbb{R} \) and \( s \in (0, 1] \), we define

\[
\mathcal{H}_\delta^s(F) = \inf \left\{ \sum_{j=1}^{\infty} \text{diam}(U_j)^s : F \subseteq \bigcup_{j=1}^{\infty} U_j, \ \text{diam}(U_j) \leq \delta \text{ for all } j \in \mathbb{N} \right\},
\]

and \( \mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(F) \) is called the \( s \)-dimensional Hausdorff measure of \( F \). Further, we define the Hausdorff dimension of \( F \) by

\[
\dim_H F = \inf \{ s \in (0, 1] : \mathcal{H}^s(F) = 0 \}.
\]

From the definition, the following basic properties hold:

- (monotonicity) for all \( F \subseteq E \subseteq \mathbb{R} \), \( \dim_H F \leq \dim_H E \);
• (bi-Lipschitz invariance) Let $f : F \to \mathbb{R}$ be a bi-Lipschitz map, that is, there exist $C_1, C_2 > 0$ such that $C_1|x-y| \leq |f(x) - f(y)| \leq C_2|x-y|$ for all $x, y \in F$. Then $\dim_H F = \dim_H f(F)$.

Comprehensive details concerning fractal dimensions can be found in the book by Falconer [Fal14]. By the second property and the mean value theorem, we immediately obtain

**Lemma 2.3.** Let $U \subseteq \mathbb{R}$ be an open set and let $V \subseteq U$ be a compact set. Let $f : U \to \mathbb{R}$ be a continuously differentiable function satisfying $|f'(x)| > 0$ for all $x \in V$. Then for all $F \subseteq V$, $\dim_H f(F) = \dim_H F$.

For all $\gamma \geq 2$ and sets $X \subseteq \mathbb{R}$, define

$$A(X, \gamma) = \{ x \in X : \text{there are infinitely many } (p, q) \in \mathbb{Z} \times \mathbb{N} \text{ such that } |x - \frac{p}{q}| \leq \frac{1}{q^\gamma} \}.$$ 

In particular, if $X = \mathbb{R}$ and $\gamma = 2$, we know that $A(\mathbb{R}, 2) = \mathbb{R}$. This result is called Dirichlet’s approximation theorem. In addition, in the case when $\gamma > 2$ and $X = [0, 1]$, the following result is known:

**Theorem 2.4** (Jarník’s theorem). For all $\gamma > 2$, we have $\dim_H A([0, 1], \gamma) = 2/\gamma$.

**Proof.** See [Fal14, Theorem 10.3].

In Section 4 we will use rational approximations of $a^{1/\alpha}$, and we find infinitely many solutions $(x, y) \in \text{PS}(\alpha)^2$ of the equation $y = ax + b$.

### 3. Lemmas

The goal of this section is to show a series of lemmas so as to evaluate discrepancies and calculate the Hausdorff dimension.

We write $O(1)$ for a bounded quantity. If this bound depends only on some parameters $a_1, \ldots, a_n$, then for instance we write $O_{a_1, a_2, \ldots, a_n}(1)$. As is customary, we often abbreviate $O(1)X$ and $O_{a_1, \ldots, a_n}(1)X$ to $O(X)$ and $O_{a_1, \ldots, a_n}(X)$ respectively for a non-negative quantity $X$. We also state $f(X) \ll g(X)$ and $f(X) \ll_{a_1, \ldots, a_n} g(X)$ as $f(X) = O(g(X))$ and $f(X) = O_{a_1, \ldots, a_n}(g(X))$ respectively, where $g(X)$ is non-negative.

**Lemma 3.1.** For every non-integral $\alpha > 1$, integer $k \geq 4$, and real numbers $\eta > 0$ and $V \geq 1$, if $\eta V^{\alpha-k} < 1$ holds, then we have

$$D((\eta \alpha^k)_{\nu < n \leq 2V}) \ll_{\alpha, k} (\eta V^\alpha)^{1/(2^k-1)} + \eta^{-1/(2^k-2)} V^{(k-\alpha)/(2^k-2)-2^k/k}.$$
Proof. Fix any $\alpha, k, \eta, V$ given in Lemma 3.1 which satisfy $\eta V^{\alpha - k} < 1$. Let $f_h(x) = h\eta x^\alpha$ for every $h \in \mathbb{N}$ and $x > 0$. Then we have $h\eta V^{\alpha - k} \ll_{a,k} f^{(k)}(x) \ll_{a,k} h\eta V^{\alpha - k}$ for all $V < x \leq 2V$. Therefore, the following holds from the Erdős-Turán inequality (2.2) and Lemma 2.2 with $f = f_h$: for all $m \in \mathbb{N}$,

$$D((\eta m^\alpha)_{V < n \leq 2V}) \ll m^{-1} + \sum_{h=1}^{m} \frac{1}{h} \left| \sum_{V < x \leq 2V} e(h\eta x^\alpha) \right|$$

$$\ll_{a,k} m^{-1} + \sum_{h=1}^{m} \frac{1}{h} \left| (h\eta V^{\alpha-k})^{1/(2^k-2)} + V^{-2^{2-k}} (h\eta V^{\alpha-k})^{-1/(2^k-2)} \right|$$

$$\ll_{a,k} m^{-1} + (m\eta V^{\alpha-k})^{1/(2^k-2)} + \eta^{-1/(2^k-2)} V^{(k-a)/(2^k-2)-2^{-k}}.$$

Hence by substituting $m = \lceil (\eta^{-1} V^{k-a})^{1/(2^k-1)} \rceil$, we get the lemma. \qed

Lemma 3.2. Let $\alpha > 1$ be a non-integral real number, $\gamma \in \mathbb{R}$ with $0 < \gamma - \alpha < 1$, and let $A > 0$ be a real number. Then there exist $Q_0 = Q_0(\alpha, \gamma, A) > 0$, $\xi_0 = \xi_0(\alpha, \gamma) > 0$, and $\psi = \psi(\alpha, \gamma) < 0$ such that for all $Q \geq Q_0$ and $0 < \xi \leq \xi_0$, we have

$$D((AQ^\alpha x^\alpha)_{V < x \leq 2V}) \ll_{a,\gamma, A} Q^\psi$$

where $V = Q^{(\gamma - \alpha - \xi)/\alpha}$.

Proof. Take an integer $k \geq 4$ such that

$$(3.1) \quad \frac{\gamma(k-3)}{\gamma + k - 3} < \alpha < \frac{\gamma k}{k + \gamma}.$$ 

Note that there exists such an integer $k$. Let $g(k) = \gamma(k-3)/(\gamma + k - 3)$. Then $g(k)$ is strictly increasing for all $k \geq 4$. Since $g(4) = \gamma/(\gamma + 1)$ and $\lim_{k \to \infty} g(k) = \gamma$, we have

$$\alpha \in (1, \gamma) \subseteq \bigcup_{k=4}^{\infty} (g(k), g(k+3)).$$

Therefore, there exists an integer $k \geq 4$ satisfying (3.1). Let us fix such an integer as $k = k(\alpha, \gamma) \geq 4$. By the condition $V = Q^{(\gamma - \alpha - \xi)/\alpha}$, we observe that

$$AQ^\alpha V^{\alpha - k} = AQ^{\psi_1}$$

where $\psi_1 := \alpha + (\gamma - \alpha - \xi)(\alpha - k)/\alpha$. Then the inequality $\alpha < \gamma k/(k + \gamma)$ yields that

$$\psi_1 < ((\gamma + k)\alpha - \gamma k)/\alpha < 0.$$
Therefore, if $Q_0$ is sufficiently large and $Q \geq Q_0$, then $AQ^\alpha V^{\alpha-k} < 1$. Thus we may apply Lemma 3.1 with $\eta = AQ^\alpha$ and $V = Q^{(\gamma-\alpha-\xi)/\alpha}$ to obtain

$$
\mathcal{D}(\{AQ^\alpha x^\alpha \} V_{x < 2V}) \ll_{\alpha, \gamma, A} (Q^\alpha V^{\alpha-k})^{1/(2k-1)} + Q^{-\alpha/(2k-2)} V^{(k-\alpha)/(2k-2) - 2^{-k}}
$$

where

$$
\psi_2 := -\frac{\alpha}{2^k - 2} + \frac{\gamma - \alpha - \xi}{\alpha} \left( \frac{k - \alpha - 4}{2^k - 2} \right).
$$

Then we have

$$
\psi_2 = -\frac{\alpha^2 2^k + (\gamma - \alpha)(k-\alpha)2^k - 4(\gamma - \alpha)(2^k - 2)}{2^k(2^k - 2) \alpha} + O_{\alpha, \gamma}(\xi)
$$

and

$$
\psi_2 = -\frac{(\gamma k - (\gamma + k - 4)\alpha - 4\gamma)2^k + 8(\gamma - \alpha)}{2^k(2^k - 2) \alpha} + O_{\alpha, \gamma}(\xi).
$$

Therefore the inequalities $\alpha > \gamma(k-3)/(\gamma + k - 3)$ and $\gamma - \alpha < 1$ imply that for sufficiently small $\xi > 0$,

$$
\psi_2 < -\frac{\alpha^2 2^k}{2^k(2^k - 2) \alpha} + 8 + O_{\alpha, \gamma}(\xi)
$$

Therefore, there exists $\psi = \psi(\alpha, k) < 0$ so that $\mathcal{D}(\{AQ^\alpha x^\alpha \} V_{x < 2V}) \ll_{\alpha, \gamma, A} Q^\psi$. □

We next present lemmas on the Hausdorff dimension.

**Lemma 3.3.** For all non-empty and bounded open intervals $J \subseteq \mathbb{R}$, we have

$$
\dim_{\mathcal{H}} \mathcal{A}(J, \gamma) = 2/\gamma.
$$

**Proof.** There exist $m \in \mathbb{Z}$ and $h \in \mathbb{N}$ such that $J \subseteq [m, m+h]$. When $x \in \mathcal{A}(J, \gamma)$, there are infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that $|x - p/q| \leq q^{-\gamma}$. Then for all $\varepsilon > 0$, and for infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}$,

$$
\left| \frac{x - m}{h} - \frac{p - mq}{qh} \right| \leq \frac{1}{hq^\varepsilon} \leq \frac{1}{q^{\gamma - \varepsilon}}.
$$

Thus $f(\mathcal{A}(J, \gamma)) \subseteq \mathcal{A}([0, 1], \gamma - \varepsilon)$ where $f(x) = (x - h)/m$. By the bi-Lipschitz invariance and monotonicity of the Hausdorff dimension and Theorem 2.3, we obtain

$$
\dim_{\mathcal{H}} \mathcal{A}(J, \gamma) = \dim_{\mathcal{H}} f(\mathcal{A}(J, \gamma)) \leq \dim_{\mathcal{H}} \mathcal{A}([0, 1], \gamma - \varepsilon) = \frac{2}{\gamma - \varepsilon}.
$$

By taking $\varepsilon \to +0$, $\dim_{\mathcal{H}} \mathcal{A}(J, \gamma) \leq 2/\gamma$. 


We next show that \( \dim_H A(J, \gamma) \geq 2/\gamma \). There exist \( \ell \in \mathbb{Z} \) and \( M \in \mathbb{N} \) such that \( J \supseteq [\ell/M, (\ell + 1)/M] \). Take such \( \ell \) and \( M \). Then for all \( x \in A([0, 1], \gamma + \varepsilon) \), there are infinitely many \( (p, q) \in \mathbb{Z} \times \mathbb{N} \) such that \( |x - p/q| \leq q^{-\gamma - \varepsilon} \). Then for all \( \varepsilon > 0 \), and for infinitely many \( (p, q) \in \mathbb{Z} \times \mathbb{N} \), we have

\[
\frac{\ell + x}{M} - \frac{\ell q + p}{qM} \leq \frac{1}{M q^{\gamma+\varepsilon}} < \frac{1}{q^\gamma}.
\]

This inequality and \((\ell + x)/M \in J\) imply that \( g(A([0, 1], \gamma + \varepsilon)) \subseteq A(J, \gamma) \) where \( g(x) = (x + \ell)/M \). By the monotonicity and bi-Lipschitz invariance of the Hausdorff dimension and Theorem 2.4, we have \( \dim_H A(J, \gamma) \geq 2/(\gamma + \varepsilon) \) for all \( \varepsilon > 0 \). Hence by taking \( \varepsilon \to +0 \), we conclude \( \dim_H A(J, \gamma) \geq 2/\gamma \).

**Lemma 3.4.** Let \( I \subseteq (1, \infty) \) be a non-empty and bounded open interval, and let \( \gamma > 2 \) and \( a > 0 \) be real numbers with \( a \neq 1 \). Define

\[
E(I, \gamma; a) = \{ \alpha \in I : \text{there are infinitely many } (p, q) \in \mathbb{Z} \times \mathbb{N} \text{ such that } |a^{1/\alpha} - \frac{p}{q}| \leq \frac{1}{q^{\gamma}} \}.
\]

Then we have \( \dim_H E(I, \gamma; a) = 2/\gamma \).

**Proof.** For all \( u > 0 \), let \( f(u) = a^{1/u} \). Fix a compact set \( V \subseteq \mathbb{R} \) with \( I \subseteq V \). Clearly, \( f \) is continuously differentiable and \( |f'(u)| > 0 \) for all \( u \in V \). By the definitions, \( f(E(I, \gamma; a)) = A(f(I), \gamma) \). Since \( f(I) \) is also a bounded open interval, Lemma 2.3 and Lemma 3.3 imply that

\[
\dim_H E(I, \gamma; a) = \dim_H f(E(I, \gamma; a)) = \dim_H A(f(I), \gamma) = \frac{2}{\gamma}.
\]

4. Key Propositions

In this section, we show two key propositions by applying rational approximations.

**Proposition 4.1.** Let \( a, b \in \mathbb{R} \) with \( a \neq 1 \) and \( a > 0 \). For all \( 1 \leq \beta < \gamma \), we have

\[
\{ \alpha \in (\beta, \gamma) : y = ax + b \text{ is solvable in } PS(\alpha) \} \subseteq E((\beta, \gamma), \beta; a).
\]

**Proof.** Fix \( \beta, \gamma \in \mathbb{R} \) with \( 1 \leq \beta < \gamma \). Take any \( \alpha \in (\beta, \gamma) \) such that the equation \( y = ax + b \) is solvable in \( PS(\alpha) \). Then there are infinitely many \( (p, q) \in \mathbb{N} \times \mathbb{N} \) such that \( |p^\alpha| = a|q^\alpha| + b \), which implies that

\[
\frac{p}{q} = \left( a + b + \frac{p^\alpha}{q^\alpha} - a q^\alpha \right)^{1/\alpha} = a^{1/\alpha} + O_{a,b}(q^{-\alpha}).
\]
Hence, there exist $C = C(a, b) > 0$ such that for infinitely many $(p, q) \in \mathbb{N}^2$,\[ |a^{1/\alpha} - \frac{p}{q}| \leq \frac{C}{q^\beta} \leq \frac{1}{q^\beta}. \]

This yields that $\alpha \in \mathcal{E}((\beta, \gamma), \beta; a)$.

**Proposition 4.2.** Let $a, b \in \mathbb{R}$ with $a \neq 1$ and $0 \leq b < a$. Suppose that $y = ax + b$ is solvable in $\mathbb{N}$. Then for all $1 \leq \beta < \gamma$ with $|\beta| < \beta < \gamma < |\beta| + 1$, we have\[ \mathcal{E}((\beta, \gamma), \gamma; a) \subseteq \{ \alpha \in (\beta, \gamma) : y = ax + b \text{ is solvable in } \text{PS}(\alpha) \}. \]

**Proof.** Since the equation $y = ax + b$ is solvable in $\mathbb{N}$, there exist distinct solutions $(x_1, y_1), (x_2, y_2) \in \mathbb{N}^2$ of the equation. Then since $y_2 - y_1 = a(x_2 - x_1)$ and $(x_1, y_1) \neq (x_2, y_2)$, we have $a \in \mathbb{Q}$. In addition, $b \in \mathbb{Q}$ holds from $b = y_1 - ax_1$. Thus we may let $a = a_1/a_2$, $b = b_1/b_2$, $a_1, a_2, b_1, b_2 \in \mathbb{N}$, $b_1 \in \mathbb{N} \cup \{0\}$. By letting $c = a_2b_2, d = a_1b_2, e = a_2b_1$, a pair $(x, y) \in \mathbb{N}^2$ satisfies the equation $cy - dx = e$ if and only if $(x, y)$ satisfies the equation $y = ax + b$. Therefore we now discuss the solvability in $\text{PS}(\alpha)$ of the equation $cy - dx = e$. Take any $\alpha \in \mathcal{E}((\beta, \gamma), \gamma; a) = \mathcal{E}((\beta, \gamma), \gamma; d/c)$. Let us show that the equation $cy - dx = e$ is solvable in $\text{PS}(\alpha)$.

By the definition, there is a sequence $((p_n, q_n))_{n=1}^{\infty} \in (\mathbb{Z} \times \mathbb{N})^\mathbb{N}$ such that for all $n \in \mathbb{N}$,\[ |(d/c)^{1/\alpha} - p_n/q_n| < q_n^{-\gamma}, \]
where $q_1 < q_2 < \cdots$. Since $(d/c)^{1/\alpha} > 0$ and $d/c \neq 1$, there exists $n_0 = n_0(d, c) \in \mathbb{N}$ such that for all $n \geq n_0$, we obtain $p_n > 0$ and $p_n \neq q_n$.

From the solvability in $\mathbb{N}$, there exist $u, v \in \mathbb{N}$ such that $cu - dv = e$. By the division algorithm, there exist $r, v' \in \mathbb{Z}$ such that $v = cr + v', 0 \leq v' < c$. Hence by replacing $u - dr$ and $v'$ with $u$ and $v$ respectively, we obtain\[ cu - dv = e, \quad 0 \leq v < c. \]

Take sufficiently small parameters $\xi = \xi(\alpha, \gamma) > 0$ and $\varepsilon \in (0, 1 - e/d)$, and take a sufficiently large parameter $n_1 = n_1(\alpha, \gamma, c, d, \varepsilon) \in \mathbb{N}$. Note that we verify the existence of $\varepsilon$ since $e < d$. Take $n \in \mathbb{N}$ with $n \geq n_1$. Let $V_n = q_n^{(\gamma - \alpha - \xi)/\alpha}$. Define\[ I = \left[ \frac{v}{c}, \frac{v + 1}{c} \right] \cap \left[ \frac{u}{d} + \frac{\varepsilon}{d}, \frac{u + 1 - \varepsilon}{d} \right], \quad B_n = \left\{ x \in \mathbb{N} : \left\{ \frac{(q_n x)^{\alpha}}{\gamma} \right\} \in I \right\}. \]
If $n_1$ is large enough and $\xi$ is small enough, then by the definition of the discrepancy and Lemma 3.2 with $V = V_n, Q = q_n, A = 1/c$, there exists $\psi = \psi(\alpha, \gamma) < 0$ such that\[ \#(B_n \cap (V_n, 2V_n])/V_n = \text{diam}(I \cap [0, 1]) + O_{\alpha, \gamma, c}(q_n^\psi). \]
Here we show \( \text{diam}(I \cap [0, 1]) > 0 \). Indeed, by (4.1), we obtain

\[
\frac{u}{d} + \frac{\varepsilon}{d} = \frac{v}{c} + \frac{e}{cd} + \frac{\varepsilon}{d} > \frac{v}{c} \geq 0.
\]

Moreover, the inequality \( \varepsilon < 1 - e/d \) yields that

\[
\frac{u}{d} + \frac{\varepsilon}{d} = \frac{v}{c} + \frac{1}{c} + \frac{e - d}{cd} + \frac{\varepsilon}{c} < \frac{v}{c} + \frac{1}{c} \leq 1.
\]

Hence \( \text{diam}(I \cap [0, 1]) > 0 \). Therefore there exists a large enough \( n_1 = n_1(\alpha, \gamma, c, d, e, \varepsilon) \in \mathbb{N} \) such that for all \( n \geq n_1 \), we get \( \#(B_n \cap (V_n, 2V_n])/V_n \geq \text{diam}(I \cap [0, 1])/2 \), which means that \( B_n \cap (V_n, 2V_n] \) is non-empty.

Hence we may take \( x \in B_n \cap (V_n, 2V_n] \) where \( n \geq n_1 \). Then

\[
(q_n x)^\alpha = c \left( \frac{(q_n x)^\alpha}{c} \right) + c \left( \frac{(q_n x)^\alpha}{c} \right) + c \left( \frac{(q_n x)^\alpha}{c} \right) - v.
\]

The first term on the most right-hand side is an integer, and the second is in \([0, 1]\) from the definition of \( B_n \). Therefore we have

\[
\lceil(q_n x)^\alpha \rceil = c \left( \frac{(q_n x)^\alpha}{c} \right) + v.
\]

Let \( \theta = p_n/q_n - (d/c)^{1/\alpha} \). By the mean value theorem, there exist \( C = C(c, d, \alpha) > 0 \) and \( \theta' \in \mathbb{R} \) with \( |\theta'| \leq |\theta| \) such that \( (p_n/q_n)^\alpha = ((d/c)^{1/\alpha} + \theta)^\alpha = d/c + C\theta' \). Therefore,

\[
(p_n x)^\alpha = \left( \frac{p_n}{q_n} \right)^\alpha (q_n x)^\alpha = d \left( \frac{(q_n x)^\alpha}{c} \right) + d \left( \frac{(q_n x)^\alpha}{c} \right) + C\theta'(q_n x)^\alpha
\]

\[
= d \left( \frac{(q_n x)^\alpha}{c} \right) + u + d \left( \frac{(q_n x)^\alpha}{c} \right) - u + C\theta'(q_n x)^\alpha.
\]

The first term on the most right-hand side is an integer, and the second term is in \([\varepsilon, 1 - \varepsilon]\) by \( x \in B_n \). Further, if necessary, we replace \( n_1 \) with a larger one, and by \( x \in (V_n, 2V_n] \), the third term is evaluated by

\[
|C\theta'(q_n x)^\alpha| \leq 2^\alpha C \frac{q_n^\alpha q_n^{-\alpha - \xi}}{q_n^\alpha} \leq 2^\alpha C q_n^{-\xi} \leq 2^\alpha C q_{n_1}^{-\xi} < \varepsilon.
\]

Hence we obtain

\[
\lceil(p_n x)^\alpha \rceil = d \left( \frac{(q_n x)^\alpha}{c} \right) + u.
\]

By the above discussion, if \( x \in B_n \cap (V_n, 2V_n) \) and \( n \geq n_1 \), then

\[
c \lceil(p_n x)^\alpha \rceil - d \lceil(q_n x)^\alpha \rceil = cd \left( \frac{(q_n x)^\alpha}{c} \right) + cu - dc \left( \frac{(q_n x)^\alpha}{c} \right) - dv = cu - dv = e,
\]
which means that $((q_n x)^\alpha, (p_n x)^\alpha) \in \mathbb{N}^2$ is a solution of the equation $cy - dx = e$. Therefore the equation $cy - dx = e$ is solvable in $\text{PS}(\alpha)$ since $B_n \cap (V_n, 2V_n]$ is non-empty for all $n \geq n_1$. \hfill \Box

5. Proof of Theorem 1.1

Fix $a, b \in \mathbb{R}$ with $a \neq 1$ and $0 \leq b < a$. In the case $\alpha \in (1, 2)$, we apply Proposition 4.2 with $\beta = 1$ and $\gamma = 2$. Then

$$E((1, 2), 2; a) \subseteq \{\alpha \in (1, 2) : y = ax + b \text{ is solvable in } \text{PS}(\alpha)\}.$$ 

By Dirichlet’s approximation theorem, $E((1, 2), 2; a) = (1, 2)$. Therefore the equation $y = ax + b$ is solvable in $\text{PS}(\alpha)$ for all $\alpha \in (1, 2)$.

We next discuss the case when $\alpha > 2$. Fix $s, t \in \mathbb{R}$ with $2 < s < t$. By applying Proposition 4.1 with $\beta = s$ and $\gamma = t$, and applying Lemma 3.4, we have

$$(5.1) \quad \dim H\{\alpha \in (s, t) : y = ax + b \text{ is solvable in } \text{PS}(\alpha)\} \leq \dim H E((s, t), t; a) = \frac{2}{s}.$$ 

Further, let $\delta > 0$ be an arbitrarily small parameter. By applying Proposition 4.2 with $\beta = s$ and $\gamma = \min\{s + \delta, \lfloor s \rfloor + 1, t\}$, and applying Lemma 3.4 we obtain

$$\dim H\{\alpha \in (s, t) : y = ax + b \text{ is solvable in } \text{PS}(\alpha)\} \geq \dim H E((s, \gamma), \gamma; a) = \frac{2}{s + \delta}$$

for every small enough $\delta > 0$. Therefore we get the theorem by taking $\delta \to 0$.

Remark 5.1. Let $\alpha \in (\beta, \gamma)$ where $\beta$ and $\gamma$ satisfy $1 \leq \beta < \gamma$ and $|\beta| < \beta < |\beta| + 1$. If $a^{1/\alpha} \in \mathbb{Q}$, then it is clear that for infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}$ we have $|a^{1/\alpha} - p/q| \leq q^{-\gamma}$. By Proposition 4.2 the equation $y = ax + b$ is solvable in $\text{PS}(\alpha)$. Therefore, for all $a, b \in \mathbb{R}$ with $a \neq 1$ and $0 \leq b < a$, and for all non-integral $\alpha > 1$ satisfying $a^{1/\alpha} \in \mathbb{Q}$, the equation $y = ax + b$ is solvable in $\text{PS}(\alpha)$.

Remark 5.2. We apply Proposition 4.1 and Lemma 3.4 to show the inequality (5.1). Note that the condition $0 \leq b < a$ is not required in Proposition 4.1 and Lemma 3.4. Hence, for all $a, b \in \mathbb{R}$ with $a \neq 1$ and $a > 0$, and for all $s, t \in \mathbb{R}$ with $2 < s < t$, we obtain

$$\dim H\{\alpha \in (s, t) : y = ax + b \text{ is solvable in } \text{PS}(\alpha)\} \leq \frac{2}{s}.$$

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REFERENCES

[Fal14] Kenneth Falconer, *Fractal geometry*, third ed., John Wiley & Sons, Ltd., Chichester, 2014, Mathematical foundations and applications. MR 3236784

[FW09] Nikos Frantzikinakis and Máté Wierdl, *A Hardy field extension of Szemerédi’s theorem*, Adv. Math. 222 (2009), no. 1, 1–43. MR 2531366

[Gla17] Daniel Glasscock, *Solutions to certain linear equations in Piatetski-Shapiro sequences*, Acta Arith. 177 (2017), no. 1, 39–52. MR 3589913

[Gla20] , *A perturbed Khinchin-type theorem and solutions to linear equations in Piatetski-Shapiro sequences*, Acta Arith. 192 (2020), no. 3, 267–288. MR 4048606

[KN74] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974, Pure and Applied Mathematics. MR 0419394

[MS20] Toshiki Matsusaka and Kota Saito, *Linear Diophantine equations in Piatetski-Shapiro sequences*, preprint, available at https://arxiv.org/abs/2009.12899.

[Tit86] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, second ed., The Clarendon Press, Oxford University Press, New York, 1986, Edited and with a preface by D. R. Heath-Brown. MR 882550

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