A Fast Algorithm for Computing the Fourier Spectrum of a Fractional Period

Jiasong Wang\textsuperscript{1}, Changchuan Yin\textsuperscript{2,*}

\textit{1.} Department of Mathematics, Nanjing University, Nanjing, Jiangsu 210093, China  
\textit{2.} Department of Mathematics, Statistics and Computer Science  
The University of Illinois at Chicago, Chicago, IL 60607-7045, USA  
* Corresponding author, Email: cyin1@uic.edu

Abstract

The Fourier spectrum at a fractional period is often examined when extracting features from biological sequences and time series. It reflects the inner information structure of the sequences. A fractional period is not uncommon in time series. A typical example is the 3.6 period in protein sequences, which determines the $\alpha$-helix secondary structure. Computing the spectrum of a fractional period offers a high-resolution insight into a time series. It has thus become an important approach in genomic analysis. However, computing Fourier spectra of fractional periods by the traditional Fourier transform is computationally expensive. In this paper, we present a novel, fast algorithm for directly computing the fractional period spectrum (FPS) of time series. The algorithm is based on the periodic distribution of signal strength at periodic positions of the time series. We provide theoretical analysis, deduction, and special techniques for reducing the computational costs of the algorithm. The analysis of the computational complexity of the algorithm shows that the algorithm is much faster than traditional Fourier transform. Our algorithm can be applied directly in computing fractional periods in time series from a broad of research fields. The computer programs of the FPS algorithm are available at https://github.com/cyinbox/FPS.

\textit{Keywords:} Fourier transform, periodicity, fractional period spectrum, DNA sequences, protein sequences
1. Introduction

Signal processing approaches have been widely applied in the periodicity analysis of time series, and thus becoming a major research tool for bioinformatics (Anastassiou, 2001; Chen et al., 2003). The main requisite in applying signal processing to symbolic sequences is mapping the sequences onto numerical time series. For example, we previously presented a numerical representation of DNA sequences without degeneracy (Yau et al., 2003). After numerical mapping, mathematical methods can be employed to study features, structures, and functions of the symbolic sequences. The most common signal processing approach is Fourier transform (Welch, 1967), which has been widely used to study periodicity and repetitive regions in symbolic DNA sequences (Silverman and Linsker, 1986; Anastassiou, 2001), as well as in genome comparison (Yin et al., 2014a; Yin and Yau, 2015). We have studied the mathematical properties of the Fourier spectrum for symbolic sequences (Wang et al., 2012, 2014). For a review of mathematical methods for the study of biological sequences, one may refer to our book (Wang and Yau, 2013).

Periods in biological sequences can be categorized into two types: integer periods and fractional periods. For integer periods, the 3-base periodicity of protein-coding regions is often used in gene finding (Tiwari et al., 1997; Yin and Yau, 2005, 2007; Yin, 2015); 2-base periodicity was found in introns of genomes (Arquès and Michel, 1987). 2-period exists in protein sequence regions for beta-sheet structures. As indicated by their name, fractional periods do not correspond to an integer number of cycles in numerical series. Fractional periods are prevalent and important in structures and functions of protein sequences and genomes. For example, the 3.6-period in protein sequences determines the α-helix secondary structure (Eisenberg et al., 1984; Gruber et al., 2005; Leonov and Arkin, 2005; Yin and Yau, 2017). Strong 10.4- or 10.5-base periodicity in genomes are associated with nucleosomes (Trifonov, 1998; Salih and Trifonov, 2015). The 6.5-base periodicity is present in C. elegans introns (Messaoudi et al., 2013). Fractional period spectrum may offer high resolution and precise features of sequences. However, identifying fractional period features in a large genome is challenging. The demand of the periodicity research of DNA and protein sequences motivates us to investigate advanced techniques for fractional periodicity analysis of time series, which correspond to biological symbolic sequences.

With the quickly growing volume of genomic sequences of human and
model organisms, there is a strong demand to characterize not only integer periods but also fractional periods in genomic and protein sequences. We previously proposed a method, periodic power spectrum (PPS), which can directly compute Fourier power spectrum based on periodic distributions of signal strength on periodic positions \cite{Wang et al., 2012, Yin and Wang, 2016}. The main advantage the PPS method is that it can avoid spectral leakage and reduce background noise, which both appear in power spectrum in Fourier transform. Thus, the PPS method can capture all latent integer periodicities in DNA sequences. We have applied this method in detection of latent periodicities in different genome elements, including exons and microsatellite DNA sequences.

This paper extends the PPS method to suggest a novel method for fast and directly calculating Fourier spectrum of a fractional period. This method will be also helpful in directly computing the fractional period spectrum for any time series from science and technology fields outside of bioinformatics.

2. Methods Deduced

2.1. Fourier transform and congruence derivative sequence

For a numerical sequence $x$ of length $m$, consisting of $x_0, x_1, \cdots, x_{m-1}$, its discrete Fourier transform (DFT) at frequency $k$ is defined as \cite{Welch, 1967}

$$X(k) = \sum_{j=0}^{m-1} x_j e^{-i2\pi kj/m}, \quad i = \sqrt{-1}, k = 0, 1, \ldots, m - 1 \quad (2.1)$$

And its Fourier power spectrum at frequency $k$ is defined as \cite{Welch, 1967}

$$PS(k) = X^*(k)X(k)$$

$$= \left( \sum_{j=0}^{m-1} x_j e^{-i2\pi jk/m} \right)^* \left( \sum_{j=0}^{m-1} x_j e^{-i2\pi jk/m} \right) \quad (2.2)$$

where $*$ indicates the complex conjugate. Because our method is applied to researching the periodicity of sequence by using Fourier spectrum, for convenience, the frequency $k/m$ is called period $m/k$. In this paper, we directly define the DFT spectrum by period instead of frequency.
We want to compute Fourier spectrum at period \( l/k \), \( l > k \) and \( l, k \) are smaller than \( m \). Using traditional formula of Fourier transform, we need to calculate Fourier transform

\[
X(k) = \sum_{j=0}^{m-1} x_j e^{-i2\pi jk/l} \quad (2.3)
\]

Then, its spectrum at period \( l/k \) or at frequency \( k \) is

\[
PS(k) = (\sum_{j=0}^{m-1} x_j e^{-i2\pi jk/l})^* (\sum_{j=0}^{m-1} x_j e^{-i2\pi jk/l}) \quad (2.4)
\]

We will introduce a fast method for directly computing the Fourier spectrum at period \( l/k \). We previously demonstrate that the Fourier power spectrum of a numerical sequence is determined by the distribution of signal strength at periodic positions in this sequence (Wang et al., 2012; Yin and Wang, 2016). The distribution can be represented and measured by the congruence derivative sequence of the original sequence, which is defined as follows.

**Definition 2.1.** For a real number sequence \( x \) of length \( m \), if two positive integers \( n \) and \( l \) satisfy \( m = nl \), then the congruence derivative sequence, \( y \), of \( x \) has a length of \( l \) and its element is defined as follows

\[
y_t = \sum_{j=0}^{n-1} x_{jt+t}, \quad t = 0, 1, \ldots l - 1 \quad (2.5)
\]

The following deduced theorems indicate that the congruence derivative sequence of length \( l \) of the numerical sequence \( x \) can be used to compute periodic power spectrum at period \( l \). The Fourier transform of the congruence derivative sequence \( y \) is

\[
Y(k) = \sum_{t=0}^{l-1} y_t e^{-i2\pi tk/l}, \quad k = 0, 1, \ldots l - 1 \quad (2.6)
\]

The relationship between Fourier transform of the original sequence and its congruence derivative sequence was introduced in (Wang et al., 2012; Yin and Wang, 2016). The DFT of sequence \( x \) at frequency \( n = m/l \) is
the same as the DFT of the congruence derivative sequence \( y \) at frequency \( l \).

\[
X(n) = X(m/l) = Y(1)
\]  

(2.7)

This formula is the mathematical ground of the PPS method. The following theorem extends the conclusion to a general case.

**Theorem 2.1.** For a real number sequence \( x \) of length \( m \), suppose \( n = m/l \), then the DFT of the numerical sequence \( x \) at frequency \( kn \) is

\[
X(kn) = Y(k), \, 0 \leq k \leq l - 1
\]  

(2.8)

**Proof.** We know

\[
Y(k) = \sum_{t=0}^{l-1} y_t e^{-i2\pi tk/l} = \sum_{t=0}^{l-1} \sum_{j=0}^{n-1} x_{jt+t} e^{-i2\pi tk/l} = \sum_{t=0}^{l-1} \sum_{j=0}^{n-1} x_{jt+t} e^{-i2\pi (jt+t)k/l} = \sum_{r=0}^{m-1} x_r e^{-i2\pi rkn/m}
\]

\[
= X(kn)
\]

The proof of Theorem 2.1 indicates that the property of conjugate symmetry in DFT of a real sequence is preserved. We thus have the following corollary.

**Corollary 2.1.**

\[
X(kn) = X^*((l - k)n), \, 1 \leq k \leq l - 1
\]  

(2.9)

According to formula 2.2, Fourier power spectrum of sequence \( x \) at frequency \( kn \) is defined as

\[
PS(kn) = X^*(kn)X(kn) = X^*(km/l)X(km/l) = Y^*(k)Y(k).
\]  

(2.10)

**Definition 2.2.** If \( m = nl \), the DFT spectrum of numerical sequence \( x \) at fractional period \( l/k \) is named the fractional period spectrum (FPS) and is written as \( FPS(l/k) = Y^*(k)Y(k), \, 1 \leq k \leq l - 1 \)

**Definition 2.3.** For a real number sequence of length \( l \), \( y_t, \, t = 0, 1, \ldots, l - 1 \), its self \( q \)-shift summation is defined by

\[
z_{t,q} = \sum_{t=0}^{l-1} y_t y_{t+q}
\]  

(2.11)
with \( t + q \) taken modulo \( l \), \( 0 \leq q < l \).

If we write sequence \( y_t, t = 0, 1, \ldots, l-1 \), as a vector, \( y^T = [y_0, y_1, \ldots, y_{l-1}] \), then \( z_{l,q} \) can be realized by the autocorrelation of vector \( y \). For a special case, \( z_{l,0} \) is equal to the inner product of vector \( y \), or \( z_{l,0} = y^T y \).

From the definition of DFT of the congruence derivative sequence (formula 2.6), we have

\[
Y(k) = \sum_{t=0}^{l-1} y_t e^{-i2\pi tk/l} = \sum_{t=0}^{l-1} y_t \cos(2\pi tk/l) - i \sum_{t=0}^{l-1} y_t \sin(2\pi tk/l) \quad (2.12)
\]

From the definition of FPS, \( FPS(l/k) = Y^*(k)Y(k) \), we notice that

\[
FPS(l/k) = (\sum_{t=0}^{l-1} y_t \cos(2\pi tk/l))^2 + (\sum_{t=0}^{l-1} y_t \sin(2\pi tk/l))^2 \quad (2.13)
\]

which is real quadratic form of \( l \) variables, \( y_0, y_1, \ldots, y_{l-1} \), can be written as

\[
FPS(l/k) = y^T A y \quad (2.14)
\]

where the coefficient matrix \( A = a_{rs} \) is of quadratic form, in which

\[
a_{rs} = \cos((r-l)\frac{2k\pi}{l}) \cos((s-l)\frac{2k\pi}{l}) + \sin((r-l)\frac{2k\pi}{l}) \sin((s-l)\frac{2k\pi}{l})
\]

\[
r, s = 1, 2, \ldots, l
\]

After reorganizing formula (2.15), we obtain the following theorem about the coefficient matrix \( A \).

**Theorem 2.2.** The coefficient matrix of the quadratic form, \( A \), is a symmetric matrix.

**Proof.** According to some fundamental identities of trigonometry,

\[
a_{rs} = \cos((r-l)\frac{2k\pi}{l}) \cos((s-l)\frac{2k\pi}{l}) + \sin((r-l)\frac{2k\pi}{l}) \sin((s-l)\frac{2k\pi}{l})
\]

would be equal to \( \cos((r-l-(s-l))\frac{2k\pi}{l}) = \cos((r-s)\frac{2k\pi}{l}) \), i.e., \( a_{rs} = a_{sr} \).

It is clear that \( a_{rs} = a_{sr} \), so the coefficient matrix \( A \) is symmetric. \( \square \)
Theorem 2.3. For a real number sequence \( x \) of length \( m \), if \( m = nl \) and its congruence derivative sequence is \( y_t, t = 0, 1, \ldots, l - 1 \), then \( FPS(l/k) \) of the sequence \( x \) can be expressed as follows.

If \( l \) is an odd number,

\[
FPS(l/k) = z_{l,0} + 2 \sum_{q=1}^{l-1} z_{l,q} \cos((q \frac{2k\pi}{l})).
\]

If \( l \) is an even number,

\[
FPS(l/k) = z_{l,0} + 2 \sum_{q=1}^{l-1} z_{l,q} \cos((q \frac{2k\pi}{l}) + 2 \cos((\frac{l}{2}) \frac{2k\pi}{l}) \sum_{t=0}^{l-1} y_t y_{t+\frac{l}{2}}
\]

Proof. It is clear the diagonal elements of the coefficient matrix \( A \) are unit elements. Due to the symmetry of matrix \( A \), we need to check the lower triangular matrix of \( A \) under the diagonal of \( A \), which is sufficient for proof.

The element of the lower triangular matrix of \( A \),

\[
a_{rs} = \cos((r - s) \frac{2k\pi}{l}), r = 2, 3, \ldots, l, r > s,
\]

also has some kind of symmetry. The result is as follows.

When \( l \) is an odd number, \( a_{21} = a_{32} = a_{43} = \ldots = a_{l(l-1)} = \cos((1) \frac{2k\pi}{l}) \) and \( a_{21}, a_{32}, \ldots, a_{l(l-1)} \) also are equal to \( a_{11} \). Because \( a_{rs} = a_{l-t} r-s-t, r > s, 0 \leq t < r-s, \) similarly, \( a_{31} = a_{42} = \ldots = a_{l(l-2)} = a_{l(l-1)1} = a_{l2} = \cos((2) \frac{2k\pi}{l}) \), then \( a_{l+1}/21 = a_{l+3}/22 = \ldots = a_{l(l+1)/2} = a_{l+3)/21 = \ldots, a_{l(l-1)/2} = \cos(((l-1)/2) \frac{2k\pi}{l}). \)

When \( l \) is an even number, we have \( a_{21} = a_{32} = a_{43} = \ldots = a_{l(l-1)} = a_{l1} = \cos((1) \frac{2k\pi}{l}) \) and \( a_{31} = a_{42} = \ldots = a_{l(l-2)} = a_{l(l-1)1} = a_{l2} = \cos((2) \frac{2k\pi}{l}) \). In general, \( a_{l/21} = a_{l(l+1)/2} = \ldots, a_{l(l/2+1)} = a_{l(l/2+2)} = a_{l(l/2+3)} = \ldots, a_{l(l-2)} = \cos((l/2 - 1) \frac{2k\pi}{l}), \) and \( a_{l(l+1)/2} = a_{l(l+2)/2} = \ldots, a_{l/2} = \cos((l/2) \frac{2k\pi}{l}). \)

We then substitute those results into the coefficient matrix of the quadratic form \( A \) for the \( FPS(l/k) \). The following formulas can be obtained.
If \( l \) is an odd number,

\[
F\!PS(l/k) = y_0^2 + y_1^2 + y_2^2 + \cdots + y_{l-1}^2
\begin{align*}
&+ 2 \cos((1)\frac{2k\pi}{l})(y_0y_1 + y_1y_2 + y_2y_3 + y_3y_4 \cdots y_{l-1}y_0) \\
&+ 2 \cos((2)\frac{2k\pi}{l})(y_0y_2 + y_1y_3 + y_2y_4 + \cdots y_{l-2}y_0 + y_{l-1}y_1) \\
&+ 2 \cos((3)\frac{2k\pi}{l})(y_0y_3 + y_1y_4 + y_2y_5 + \cdots y_{l-3}y_0 + y_{l-2}y_1 + y_{l-1}y_2) \\
& \cdots \\
&+ 2 \cos(((l-1)/2)\frac{2k\pi}{l})(y_0y_{(l-1)/2} + y_1y_{(l+1)/2} + y_2y_{(l+3)/2} + \cdots y_{(l-(l-1)/2)}y_0 + \cdots + y_{(l-1)y(l-3)/2}).
\end{align*}
\]

(2.16)

If \( l \) is an even number,

\[
F\!PS(l/k) = y_0^2 + y_1^2 + y_2^2 + \cdots + y_{l-1}^2
\begin{align*}
&+ 2 \cos((1)\frac{2k\pi}{l})(y_0y_1 + y_1y_2 + y_2y_3 + y_3y_4 \cdots y_{l-1}y_0) \\
&+ 2 \cos((2)\frac{2k\pi}{l})(y_0y_2 + y_1y_3 + y_2y_4 + \cdots y_{l-2}y_0 + y_{l-1}y_1) \\
&+ 2 \cos((3)\frac{2k\pi}{l})(y_0y_3 + y_1y_4 + y_2y_5 + \cdots y_{l-3}y_0 + y_{l-2}y_1 + y_{l-1}y_2) \\
& \cdots \\
&+ 2 \cos((l/2 - 1)\frac{2k\pi}{l})(y_0y_{l/2-1} + y_1y_{l/2} + y_2y_{l/2+1} + \cdots y_{l-3}y_{l/2-2} + y_{l-2}y_{l/2-1} + y_{l-1}y_{l/2}) \\
&+ 2 \cos((l/2)\frac{2k\pi}{l})(y_0y_{l/2} + y_1y_{l/2+1} + y_2y_{l/2+2} + \cdots y_{l/2-1}y_{l-1}).
\end{align*}
\]

(2.17)

The last term in equation (2.17), \( \cos((l/2)\frac{2k\pi}{l})(y_0y_{l/2} + y_1y_{l/2+1} + y_2y_{l/2+2} + \cdots + y_{l/2-1}y_{l-1}) \) can be written as \( \cos((l/2)\frac{2k\pi}{l}) \sum_{t=0}^{l/2-1} y_{lt+l/2} \). Noticing the definition of \( z_{l,q} \), the formulas (2.16) and (2.17) may be written as follows.

If \( l \) is an odd number,

\[
F\!PS(l/k) = z_{l,0} + 2 \sum_{q=1}^{l-1} z_{l,q} \cos((q)\frac{2k\pi}{l})
\]

(2.18)
If \( l \) is an even number,
\[
FPS(l/k) = z_{l,0} + 2 \sum_{q=1}^{\frac{l-1}{2}} z_{l,q} \cos((q) \frac{2k\pi}{l}) + 2 \cos((\frac{l}{2}) \frac{2k\pi}{l}) \sum_{t=0}^{\frac{l-1}{2}} y_{t}y_{t+\frac{1}{2}} \quad (2.19)
\]

Based on Theorem 2.3 and Corollary 2.1, \( FPS(l/k) \) may be re-written by following theorem, which may improve the technique for computing \( FPS(l/k) \).

**Theorem 2.4.**
\[
FPS(l/k) = FPS(l/(l-k),)1 \leq k \leq l-1 \quad (2.20)
\]

According to Theorem 2.4, to compute all of the \( FPS(l/k), 1 \leq k \leq l-1 \), we only require about half of computing costs.

For \( k = 1 \), noticing formulas (2.18) and (2.19) and the symmetry of \( FPS(l/k) \), we need to calculate \((\frac{l-1}{2})/2\) coefficients of quadratics \( y_{0}, y_{1}, \ldots, y_{l-1} \) when \( l \) is an odd number; and \( l/2 \) coefficients when \( l \) is an even number.

It is valuable to notice on the property of a cosine function,
\[
\cos((q) \frac{2k\pi}{l}) = \cos((q) \frac{2(l-k)\pi}{l}), 0 \leq k \leq l-1. \quad (2.21)
\]

Certainly, it is the foundation of Theorem 2.4 and an important formula to prove the following theorem.

**Theorem 2.5.** The coefficient matrix of quadratic form \( A \) when \( k = 1 \) is sufficient for computing \( FPS(l/k) \) when \( k \neq 1 \).

**Proof.** Without loss of generality, we show the case when \( l \) is an odd number. For a fixed \( k, k \neq 1 \), from Theorem 2.4, the range of \( k \) should be in \( 2 \leq k \leq (l-1)/2 \) and the \((l-1)/2\) coefficients of quadratics of \( y_{0}, y_{1}, \ldots, y_{l-1} \) are \( \cos((t) \frac{2k\pi}{l}), t = 1, 2, \ldots, \frac{l-1}{2} \).

For \( k = 1 \), we obtain the coefficients \( \cos((q) \frac{2\pi}{l}), q = 1, 2, \ldots, \frac{l-1}{2} \).

If the \((l-1)/2\) coefficients, \( \cos((t) \frac{2k\pi}{l}), t = 1, 2, \ldots, (l-1)/2 \) can be represented by \( \cos((q) \frac{2\pi}{l}), q = 1, 2, \ldots, (l-1)/2 \), then, the proposition is proved.

It is known that the interval of \( k \) is \([2, (l-1)/2]\), so, for \( t = 1, \cos((k) \frac{2\pi}{l}) \), the first coefficient of \( FPS(l/k) \), is one of \( \cos((q) \frac{2\pi}{l}), q = 1, 2, \ldots, (l-1)/2 \).
When $t \neq 1$, we may have one of two results: if $tk$ in the interval $[1, (l-1)/2]$, it is obvious that $\cos((t\frac{2k\pi}{l}))$ is one of $\cos((q\frac{2\pi}{l}))$, $q = 1, 2, \ldots, (l-1)/2$; otherwise, if $tk > (l-1)/2$, based on formula (2.21) since $l-tk$ belongs to $[1, (l-1)/2]$. \hfill \Box

3. Fast Algorithm

3.1. Fast algorithm for computing fractional period spectrum at period $l/k$

Based on Theorem 2.5, the algorithm should do a build-in data in the computer. The data contains the coefficients of quadratics, when $k = 1$ and for periods, $l = 2, 3, \ldots$, to store in the computer. The built-in data save much computing time because the algorithm does not need to repeatedly find the coefficients for next user.

The programs shall design two functions. One function is to obtain the congruence derivative sequence (CDS) ($y$). The other function $Z(l, q)$ is to yield the autocorrelation of the congruence derivative sequence with two parameters $l$ and $q$.

A MATLAB function $\text{rem}(p, l)$ may be used for $p$ and $l$ are integer number if the programming is written by MATLAB, where the $\text{rem}(p, l)$ means the remainder after $p$ divided by $l$.

Given input original sequence $x$, the function CDS ($y$) yields the congruence derivative sequence of the sequence $x$ for a period $l$ in the computation.

Suppose we need to calculate the fractional period $l/k$, we then have the following algorithm.

If $l$ is odd number, complete the loop for ordering to take out $\frac{l-1}{2}$ coefficients of quadratics, $\cos((1)\frac{2\pi}{l})$, $\cos((2)\frac{2\pi}{l})$, $\ldots$, $\cos(((l-1)/2)\frac{2\pi}{l})$, from the build-in data. From $t = 1$ to $(l-1)/2$, do if $r = \text{rem}(tk, l) > (l-1)/2$ then take $\cos((l-r)\frac{2\pi}{l})$ else take $\cos((r)\frac{2\pi}{l})$. At the same time, the $2\cos((r)\frac{2\pi}{l})$ value times the $Z(l, t)$ and to do summation corresponding to $t$ order until $t = (l-1)/2$. The computed summation added with $Z(l, 0)$ is the $FPS(l/k)$.

If $l$ is even number, doing the loop for ordering to take out $\frac{l}{2}$ coefficients of quadratics, $\cos((1)\frac{2\pi}{l})$, $\cos((2)\frac{2\pi}{l})$, $\ldots$, $\cos((l/2)\frac{2\pi}{l})$, from the build-in data, i.e., $t = 1$ to $(l-1)/2$ do if $r = \text{rem}(tk, l) > (l)/2$ then take $\cos((l-r)\frac{2\pi}{l})$ else take $\cos((r)\frac{2\pi}{l})$. At the same time, the $2\cos((r)\frac{2\pi}{l})$ value times the $Z(l, t)$ to do summation corresponding to $t$ order until $t = (l/2 - 1)$. $FPS(l/k)$ will be the computed summation added with two quantities, $2\cos((l/2)\frac{2\pi}{l})$ timing $\sum_{t=0}^{l/2-1} y_t y_{t+l/2}$, and $Z(l, 0)$.
To assess the effectiveness of proposed FPS algorithm in capturing fractional periodicities in digital signals, we compute the FPS spectrum of the simulated sinusoidal signal. The sinusoidal signal of length $N = 300$, consists of sine and cosine signals with fractional periodicities 3.7 and 5.6, respectively. The signal is corrupted by white Gaussian noise (Equation (3.1)).

$$x(n) = \sin(2\pi \frac{n}{300} + \frac{\pi}{4}) + \cos(2\pi \frac{n}{300} + \frac{3\pi}{4}) + \text{noise}$$  
$n = 1, 2, \ldots, 300$ (3.1)

Since the periodic signal is buried by the white Gaussian noise, two periodicities 3.7 and 5.6 in the original signal are hidden by noise (Fig.1 (a)). The proposed FPS spectrum of the signal can clearly discover two pronounced peaks at positions 3.7 and 5.6 (Fig.1 (b)), corresponding to two periodicities 3.7 and 5.6 in the original signal. The power spectrum of this simulation sequence by the proposed method agrees with the DFT (Fig.1 (b) and (c)). This result demonstrates the effectiveness of the FPS algorithm.

3.2. Computational complexity of the FPS algorithm

Recalling the process for computing $FPS(l/k)$ of a time series by using the new algorithm, the first step is to yield the congruence derivative sequence $y_j$, $j = 0, 1, \ldots, l - 1$, from the original sequence $x$ of length $m$. This computing process only needs plus operation and do not use multiplication. The second step in the algorithm is to compute $z_{l,q}$ when $l$ is an even or odd number, the algorithm needs approximate $l/2(l + 1)$ multiplications. Finally, using formulas (2.18, 2.19) needs $l/2$ multiplications to determine the value of $z_{l,q}\cos((q)2k\pi/l)$. So the total computational quantity for the FPS algorithm involves $l/2(l + 1) + l/2 = l^2/2 + l$ multiplications, i.e., the computational complexity is then $O(l^2 + l)$.

Using the traditional Fourier transform to compute $FPS(l/k)$, we need to calculate the Fourier transform at period $l/k$ by the formulas 2.3 and 2.4. From (2.3), we must calculate $m$ exponents, $-i2\pi jk/l$, $j = 0, 1, 2, \ldots, m - 1$, and call exp function $m$ times to figure out $e^{-i2\pi jk/l}$, $j = 0, 1, 2, \ldots, m - 1$, if using MATLAB. It is known that to compute function $e^{-i2\pi jk/l}$ costs much time. Then to figure out $\sum_{j=0}^{m-1} e^{-i2\pi jk/l}$ needs $2m$ multiplications and the time for calling $m$ exp functions. So, according to formula (2.4) calculating out the Fourier spectrum at period $l/k$ needs $2m + 1$ multiplications, i.e., $O(2m + 1)$. If $FPS(l/k)$ is computed for whole fractional periods, traditional Fourier transform needs $l/2O(2m + 1)$ operations, i.e. $O(lm + l)$. 

11
Figure 1: Fractional period spectrum analysis of sinusoidal signal collapsed with white noise. (a) original sinusoidal signal. (b) Fractional period spectrum of the sinusoidal signal by the FPS algorithm. (c) Fractional period spectrum of the sinusoidal signal by DFT.
In summary, the analysis of computational complexity indicates that the FPS algorithm is much faster than traditional Fourier transform for computing fractional periodic spectrum because of $O(l^2 + l) << O(lm + l)$.

4. Conclusions

A fast algorithm for computing the FPS of biological sequences is introduced in this paper. First, the mathematical deduction and proof are presented. Then, based on the mathematical theory, the technique of built-in data is suggested and some functions are introduced, so that a simple program can be designed. The analysis of computational complicity shows that this fast algorithm is much faster than traditional Fourier transform.

The fast algorithm will be also helpful in computing the FPS for any time series from the science and technology field other than bioinformatics.

In addition, the congruence derivative sequence of original sequences can be used to yield Ramanujan transform easily (Hua et al., 2014; Yin et al., 2014b), which do not have any computing cost. The computation for identifying periods of the biological sequence or other time series would be done by Fourier transform and Ramanujan transform simultaneously.
References

Anastassiou, D., 2001. Genomic signal processing. Signal Processing Magazine, IEEE 18 (4), 8–20.

Arquès, D. G., Michel, C. J., 1987. Periodicities in introns. Nucleic Acids Research 15 (18), 7581–7592.

Chen, J., Li, H., Sun, K., Kim, B., 2003. How will bioinformatics impact signal processing research? Signal Processing Magazine, IEEE 20 (6), 106–206.

Eisenberg, D., Weiss, R., Terwilliger, T., 1984. The hydrophobic moment detects periodicity in protein hydrophobicity. Proc. Natl. Acad. Sci. 81, 140–144.

Gruber, M., Söding, J., Lupas, A. N., 2005. Repper repeats and their periodicities in fibrous proteins. Nucleic Acids Research 33 (suppl 2), W239–W243.

Hua, W., Wang, J., Zhao, J., 2014. Discrete ramanujan transform for distinguishing the protein coding regions from other regions. Molecular and cellular probes 28 (5), 228–236.

Leonov, H., Arkin, I. T., 2005. A periodicity analysis of transmembrane helices. Bioinformatics 21 (11), 2604–2610.

Messaoudi, I., Oueslati, A. E., Lachiri, Z., 2013. Detection of the 6.5-base periodicity in the c. elegans introns based on the frequency chaos game signal and the complex morlet wavelet analysis. International Journal of Scientific Engineering and Technology 2 (12), 1247–1251.

Salih, B., Trifonov, E. N., 2015. Strong nucleosomes reside in meiotic centromeres of c. elegans. Journal of Biomolecular Structure and Dynamics 33 (2), 365–373.

Silverman, B., Linsker, R., 1986. A measure of DNA periodicity. Journal of theoretical biology 118 (3), 295–300.

Tiwari, S., Ramachandran, S., Bhattacharya, A., Bhattacharya, S., Ramaswamy, R., 1997. Prediction of probable genes by Fourier analysis of genomic sequences. Bioinformatics 13 (3), 263–270.
Trifonov, E. N., 1998. 3-, 10.5-, 200-and 400-base periodicities in genome sequences. Physica A: Statistical Mechanics and its Applications 249 (1), 511–516.

Wang, J., Dang, C., Yin, C., 2014. A comparative study of the signal-to-noise ratios of different representations for symbolic sequences. Journal of Applied Mathematics and Bioinformatics 372, 135–145.

Wang, J., Liu, G., Zhao, J., 2012. Some features of Fourier spectrum for symbolic sequences. Numerical Mathematics A Journal of Chinese Universities (4), 341–356.

Wang, J., Yan, M., 2013. Numerical Methods in Bioinformatics: An Introduction. Science Press.

Welch, P., 1967. The use of fast fourier transform for the estimation of power spectra: a method based on time averaging over short, modified periodograms. IEEE Transactions on Audio and Electroacoustics 15 (2), 70–73.

Yau, S. S.-T., Wang, J., Niknejad, A., Lu, C., Jin, N., Ho, Y.-K., 2003. DNA sequence representation without degeneracy. Nucleic Acids Research 31 (12), 3078–3080.

Yin, C., 2015. Representation of DNA sequences in genetic codon context with applications in exon and intron prediction. Journal of Bioinformatics and Computational Biology 13 (2), 1550004–2.

Yin, C., Chen, Y., Yau, S. S.-T., 2014a. A measure of DNA sequence similarity by Fourier transform with applications on hierarchical clustering. Journal of Theoretical Biology 359, 18–28.

Yin, C., Wang, J., 2016. Periodic power spectrum with applications in detection of latent periodicities in DNA sequences. Journal of Mathematical Biology 73 (5), 1053–1079.

Yin, C., Yau, S. S.-T., 2005. A Fourier characteristic of coding sequences: origins and a non-Fourier approximation. Journal of Computational Biology 12 (9), 1153–1165.
Yin, C., Yau, S. S.-T., 2007. Prediction of protein coding regions by the 3-base periodicity analysis of a DNA sequence. Journal of Theoretical Biology 247 (4), 687–694.

Yin, C., Yau, S. S.-T., 2015. An improved model for whole genome phylogenetic analysis by Fourier transform. Journal of Theoretical Biology 359 (21), 18–28.

Yin, C., Yau, S. S.-T., 2017. A coevolution analysis for identifying protein-protein interactions by Fourier transform. PloS ONE 12 (4), e0174862.

Yin, C., Yin, X. E., Wang, J., 2014b. A novel method for comparative analysis of DNA sequences by Ramanujan-Fourier transform. Journal of Computational Biology 21 (12), 867–879.