An inertial upper bound for the quantum independence number of a graph

Pawel Wocjan*  Clive Elphick†

October 19, 2018

Abstract

A well known upper bound for the independence number $\alpha(G)$ of a graph $G$, is that

$$\alpha(G) \leq n^0 + \min\{n^+, n^0, n^-, \}$$

where $(n^+, n^0, n^-)$ is the inertia of $G$. We prove that this bound is also an upper bound for the quantum independence number $\alpha_q(G)$, where $\alpha_q(G) \geq \alpha(G)$. We identify numerous graphs for which $\alpha(G) = \alpha_q(G)$ and demonstrate that there are graphs for which the above bound is not exact with any Hermitian weight matrix, for $\alpha(G)$ and $\alpha_q(G)$. This result complements results by the authors that many spectral lower bounds for the chromatic number are also lower bounds for the quantum chromatic number.

1 Introduction

Wocjan and Elphick [15] proved that many spectral lower bounds for the chromatic number, $\chi(G)$, are also lower bounds for the quantum chromatic number, $\chi_q(G)$. This was achieved using pinching and twirling and a combinatorial definition of $\chi_q(G)$ due to Mancinska and Roberson [9].

In a different paper Mancinska and Roberson [10] defined a quantum independence number $\alpha_q(G)$, using quantum homomorphisms. It is known (see for example Section 6.18 of [12]) that:

$$\alpha(G) \leq \alpha_q(G) \leq [\theta'(G)] \leq [\theta^+(G)] \leq [\theta(G)] \leq \theta^+(G) \leq \theta^+(G) \leq \theta^+(G) \leq [\theta^+(G)] \leq \chi_q(G) \leq \chi(G),$$

where $\theta', \theta, \theta^+$ are the Schrijver, Lovász and Szegedy theta functions.

Analogously to $\chi_q(G)$, $\alpha_q(G)$ is the maximum integer $t$ for which two players sharing an entangled state can convince a referee that the graph $G$ has an independent set of size $t$.

*wocjan@cs.ucf.edu, Department of Computer Science, University of Central Florida, USA
†clive.elphick@gmail.com, School of Mathematics, University of Birmingham, Birmingham, UK
There exist graphs $G$ for which there is an exponential separation between $\alpha(G)$ and $\alpha_q(G)$ \cite{10}. Note that the number of edges in an independent set is zero, but the number of edges in a quantum independent set is greater than zero, when $\alpha_q(G) > \alpha(G)$.

Relationships exist between $\chi_q$ and $\alpha_q$. For example, Theorem 3.4.8 in \cite{13} proves that if $G$ has $n$ vertices and $\chi(G) > \chi_q(G) = k$, then $\alpha(G \square K_k) < \alpha_q(G \square K_k) = n$, where $\square$ denotes the Cartesian product. This result can be used to construct fairly small graphs with $\alpha < \alpha_q$. For example, Mancinska and Roberson \cite{9} identify a graph, $G_{14}$, on 14 vertices with $\chi(G_{14}) > \chi_q(G_{14}) = 4$. Therefore $\alpha(G_{14} \square K_4) < \alpha_q(G_{14} \square K_4) = 14$.

2 Inertial upper bound for the independence number

The best known spectral upper bound for $\alpha(G)$ is as follows:

$$\alpha(G) \leq n^0(W) + \min\{n^+(W), n^-(W)\},$$

where $W$ is a Hermitian weighted adjacency matrix of $G$ and $n^0, n^+, n^-$ are the numbers of zero, positive and negative eigenvalues of $W$ respectively. We also let $A$ denote the adjacency matrix of $G$, and $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$ respectively, where $|V(G)| = n$ and $|E(G)| = m$. Obviously $n = n^0 + n^- + n^+$.

This bound was originally proved by Cvetkovic \cite{2}, using interlacing of the eigenvalues of the empty graph on $\alpha(G)$ vertices. Godsil \cite{4} presented an alternative proof that we generalize to prove the inertial upper bound on the quantum independence number. His proof relies on the following elementary result:

**Lemma 1.** Let $M \in \mathbb{C}^{s \times s}$ be an arbitrary Hermitian matrix. A subspace $U$ of $\mathbb{C}^s$ is called totally isotropic with respect to the Hermitian form defined by $M$ if

$$\langle \Psi | M | \Psi \rangle = 0$$

for all vectors $|\Psi\rangle \in U$. The dimension of all maximal totally isotropic subspaces is equal to

$$n^0(M) + \min\{n^+(M), n^-(M)\}.$$

**Proof.** Using Sylvester’s law of inertia we may assume that $M$ is diagonal and has only eigenvalues $+1$, $0$, and $-1$. It is easy to show that there exists a totally isotropic subspace having dimension as in (2). Let $|\phi^+_a\rangle$, $|\phi^0_b\rangle$, $|\phi^-_c\rangle$ denote the eigenvectors and the corresponding eigenvalues $+1$, $0$ and $-1$, respectively, where $a \in [n^+]$, $b \in [n^0]$, and $c \in [n^-]$. We may assume without loss of generality that $n^+ \geq n^-$. Then the vectors

$$|\phi^0_b\rangle, \quad b \in [n^0],$$

$$|\phi^+_a\rangle + |\phi^-_c\rangle, \quad c \in [n^-]$$

span a totally isotropic subspace of dimension $n^0 + n^-$. 

2
We now show that there cannot exist a totally isotropic subspace whose dimension is larger than \(n^0 + n^-\). Let \(U\) be any totally isotropic subspace. Let \(V\) be the subspace spanned by \(|\varphi_a^+\rangle\) for \(a \in [n^+]\). We have
\[
\begin{align*}
n^+ + n^0 + n^- &= s \\
\geq & \dim(U + V) \\
= & \dim(U) + \dim(V) + \dim(U \cap V) \\
= & \dim(U) + n^+,
\end{align*}
\]
which shows that \(\dim(U)\) cannot be larger than \(n^0 + n^-\).

**Proof.** The inertial upper bound on \(\alpha\) is established as follows. Let \(G\) be a graph with vertex set \(V\) and Hermitian weighted adjacency matrix \(W\). Let \(e_u\) denote the standard basis vector corresponding to vertex \(u\). If \(S\) is an independent set in \(V\) and \(u,v \in S\), then
\[
\langle e_u | W | e_v \rangle = 0.
\]
It follows that the subspace spanned by the orthogonal vectors \(e_u\) for \(u \in S\) is a totally isotropic subspace. The dimension of such a subspace is bounded by the inertia of \(W\), as shown in Lemma 1.

3 Inertial upper bound for the quantum independence number

**Theorem 1.** For any graph \(G\) with quantum independence number \(\alpha_q(G)\) and Hermitian weighted adjacency matrix \(W\):
\[
\alpha_q(G) \leq n^0(W) + \min\{n^+(W), n^-(W)\}.
\]

In order to prove Theorem 1 we need a combinatorial definition of \(\alpha_q\). Laurent and Piovesan provide a combinatorial definition of \(\alpha_q\) in Definition 2.5 of [7], but we use the equivalent definition described in the paragraph after their Definition 2.8.

For matrices \(X, Y \in \mathbb{C}^{d \times d}\), their trace inner product (also called Hilbert-Schmidt inner product) is defined as
\[
\langle X, Y \rangle_{tr} = \text{tr}(X^\dagger Y).
\]

**Definition 1** (Quantum independence number \(\alpha_q\)). For a graph \(G\), \(\alpha_q(G)\) is the maximum integer \(t\) for which there exist orthogonal projectors \(P^{(u,i)} \in \mathbb{C}^{d \times d}\) for \(u \in V(G), i \in [t]\) satisfying the following conditions:
\[
\begin{align*}
\sum_{u \in V(G)} P^{(u,i)} &= I_d \quad \text{for all } i \in [t] \\
\langle P^{(u,i)}, P^{(u,j)} \rangle_{tr} &= 0 \quad \text{for all } i \neq j \in [t], \text{ for all } u \in V(G) \\
\langle P^{(u,i)}, P^{(v,j)} \rangle_{tr} &= 0 \quad \text{for all } i \neq j \in [t], \text{ for all } uv \in E(G).
\end{align*}
\]
We refer to condition (4) as the completeness condition and to conditions (5) and (6) as the orthogonality conditions.

Observe that the (classical) independence number $\alpha(G)$ is a special case of $\alpha_q(G)$ when the dimension $d$ is restricted to be 1, that is, the only possible “projectors” are the scalars 1 and 0.

Mancinska et al [11] defined the projective packing number, $\alpha_p(G)$, as follows, and noted that for all graphs $\alpha_q(G) \leq \alpha_p(G)$. For the sake of completeness, we include the simple proof showing that the quantum independence number is bounded from above by the projective packing number. We show afterwards that the projective packing number is bounded from above by the inertia bound.

**Definition 2.** A $d$-dimensional projective packing of a graph $G = (V,E)$ is a collection of orthogonal projectors $P^{(u)} \in \mathbb{C}^{d \times d}$ such that

$$\langle P^{(u)}, P^{(v)} \rangle_{\text{tr}} = 0$$

for all $uv \in E$. The value of a projective packing using projectors $P^{(u)} \in \mathbb{C}^{d \times d}$ is defined as

$$\frac{1}{d} \sum_{u \in V} r^{(u)},$$

where $r^{(u)}$ denote the ranks of the operators $r^{(u)}$. The projective packing number $\alpha_p(G)$ of a graph $G = (V,E)$ is defined as the supremum of the values over all projective packings of the graph $G$.

**Lemma 2.** For all graphs, we have $\alpha_q(G) \leq \alpha_p(G)$.

**Proof.** Let $P^{(u,i)} \in \mathbb{C}^{d \times d}$, $u \in V$, $i \in [t]$, be a collection of orthogonal projectors satisfying the conditions in Definition [11]. Define the operators

$$P^{(u)} = \sum_{i \in [t]} P^{(u,i)}.$$

These operators are orthogonal projectors because of condition [5]. For $uv \in E$, we have

$$\langle P^{(u)}, P^{(v)} \rangle_{\text{tr}} = \sum_{i,j \in [t]} \langle P^{(u,i)}, P^{(v,j)} \rangle_{\text{tr}}$$

$$= \sum_{i \neq j \in [t]} \langle P^{(u,i)}, P^{(v,j)} \rangle_{\text{tr}} + \sum_{i \in [t]} \langle P^{(u,i)}, P^{(v,i)} \rangle_{\text{tr}}$$

$$= 0.$$  

\[\text{(12)}\]

\[\text{\footnotesize{\textsuperscript{1}Deviating slightly from \cite{7}, we present the cases } u = v \text{ and } uv \in E \text{ as separate conditions to simplify the presentation below.}}\]
The sums in (11) are equal to zero because of conditions (6) and (4), respectively. We have

\[ \sum_{u \in V} r^{(u)} = \sum_{i \in [t]} \sum_{u \in V} \text{rank}(P^{(u,i)}) = \sum_{i \in [t]} d = t \cdot d, \]

because the projectors \( P^{(u,i)} \) for each \( i \in [t] \) add up to \( I_d \) due to condition (4). This concludes the proof \( \alpha_q(G) \leq \alpha_p(G) \). \( \square \)

We will use the following result to reformulate the conditions on the orthogonal projectors of a projective packing as conditions on their eigenvectors. We omit the proof of this basic result.

**Lemma 3.** Let \( P, Q \in \mathbb{C}^{d \times d} \) be two arbitrary orthogonal projectors of rank \( r \) and \( s \), respectively. Let

\[ P = \sum_{k \in [r]} |\psi_k\rangle \langle \psi_k| \quad \text{and} \quad Q = \sum_{\ell \in [s]} |\phi_\ell\rangle \langle \phi_\ell| \]

denote their spectral resolutions, respectively. Then, the following two conditions are equivalent:

\[ \langle P, Q \rangle_{\text{tr}} = 0 \quad (13) \]
\[ \langle \psi_k | \phi_\ell \rangle = 0 \quad \text{for all} \quad k \in [r], \ell \in [s]. \quad (14) \]

**Theorem 2.** For any graph \( G \) with projective packing number \( \alpha_p(G) \) and Hermitian weighted adjacency matrix \( W \):

\[ \alpha_p(G) \leq n^0(W) + \min\{n^+(W), n^-(W)\}. \quad (15) \]

**Proof.** Let

\[ P^{(u)} = \sum_{k \in [r^{(u)}]} |\psi^{(u,k)}\rangle \langle \psi^{(u,k)}| \]

denote the spectral resolution of \( P^{(u)} \), where \( r^{(u)} \) is its rank. Let

\[ r = \sum_{u \in V} r^{(u)}. \]

Define the composite vectors

\[ |\Psi^{(u,k)}\rangle = |u\rangle \otimes |\psi^{(u,k)}\rangle. \quad (16) \]

For all \( u, v \in V, k \in [r^{(u)}], \) and \( \ell \in [r^{(v)}], \) we have

\[ \langle \Psi^{(u,k)} | \Psi^{(v,\ell)} \rangle = \delta_{u,v} \cdot \delta_{k,\ell} \quad (17) \]
\[ \langle \Psi^{(u,k)} | (W \otimes I_d) | \Psi^{(v,\ell)} \rangle = 0 \quad (18). \]

The above equalities hold due to the special tensor product structure of the vectors \( |\Psi^{(u)}\rangle \), the orthogonality condition in (17), and Lemma 3. It follows that these vectors span a \( r \)-dimensional isotropic subspace with respect to the quadratic form defined by \( W \otimes I_d \).
Using Lemma 1 and that the inertia of $W \otimes I_d$ is $d$ times the inertia of $W$, we obtain
\[
\frac{r}{d} \leq n^0(W) + \min\{n^+(W) + n^-(W)\},
\]
which completes the proof since this bound holds for all projective packings of $G = (V, E)$ and all Hermitian weighted adjacency matrices $W$ of $G$.

4 Eigenvalue upper bound for $\alpha_q(G)$

Hoffman, in an unpublished paper, proved that for $\Delta$-regular\footnote{We use the unconventional symbol $\Delta$ instead of $d$ for the degree of regular graphs because $d$ is the dimension of the Hilbert space used in the definition of the quantum independence number} graphs:
\[
\alpha(G) \leq \frac{n|\lambda_n|}{\Delta + |\lambda_n|},
\]
where $\lambda_n$ is the smallest eigenvalue of $A$. This result is typically proved using interlacing of the quotient matrix, and is known as the Hoffman bound or ratio bound.

Lovász (Theorem 9 in \cite{Lov}) proved that for $\Delta$-regular graphs:
\[
\theta(G) \leq \frac{n|\lambda_n|}{\Delta + |\lambda_n|}.
\]
It is therefore immediate that the Hoffman bound is an upper bound for $\alpha_q(G)$ for regular graphs.

5 Quantum clique number $\omega_q(G)$

Mancinska and Roberson \cite{ManRob} also define the quantum clique number where $\omega(G) \leq \omega_q(G) = \alpha_q(\overline{G})$. Cvetkovic \cite{Cve} proved the following inertial upper bound for the clique number:
\[
\omega(G) \leq \min\{1 + n^{\leq -1}, n^{\geq -1}\}, \quad (19)
\]
where $n^{\leq -1}$ and $n^{\geq -1}$ denote the numbers of eigenvalues $\leq -1$ and $\geq -1$ respectively. The proof uses interlacing of the eigenvalues of a complete graph of size $\omega(G)$. Hoffman and Howes \cite{HofHow} noted that $\overline{C_7}$ demonstrates that this upper bound for $\omega$ is not an upper bound for $\chi$.

Inequalities (1) and (19) can be combined as follows, using the complement for bound (19):
\[
\alpha(G) \leq \min\{n^0(G) + n^+(G), n^0(G) + n^-(G), 1 + n^{\leq -1}(\overline{G}), n^{\geq -1}(\overline{G})\}.
\]
Experimentally bound (1) is significantly better than bound (19). Indeed bound (19) does not outperform bound (1) for any of the named graphs with up to 40 vertices in the Wolfram Mathematica database. So to upper bound the clique number of a graph it is best to upper bound the independence number of its complement using (1). Similarly to upper bound $\omega_q(G)$ it is best to upper bound $\alpha_q(\overline{G})$ using Theorem 1.
6 Implications for $\alpha_q(G)$ and for $\alpha(G)$

It follows from Theorem 1 that any graph with $\alpha(G) = n^0 + \min(n^+, n^-)$, has $\alpha_q = \alpha$. This is the case for numerous graphs, including odd cycles, perfect, folded cubes, Kneser, Andrasfai, Petersen, Desargues, Groetzsch, Heawood, Clebsch and Higman-Sims graphs. Furthermore if the inertia bound is tight with an appropriately chosen weight matrix than again $\alpha_q = \alpha$. This is the case for all bipartite graphs. There are also many graphs, including Chvatal, Hoffman-Singleton, Flower Snark, Dodecahedron, Frucht, Octahedron, Thomsen, Pappus, Gray, Coxeter and Folkman for which $\alpha = \lfloor \theta \rfloor$, so again $\alpha_q = \alpha$. For all such graphs there are no benefits from quantum entanglement for independence.

Elzinga and Gregory [3] asked whether there exists a real symmetric weight matrix $W$ for every graph $G$ such that:

$$\alpha(G) = n^0(W) + \min(n^+(W), n^-(W))? \tag{20}$$

They demonstrated experimentally that this is true for all graphs with up to 10 vertices, and for vertex transitive graphs with up to 12 vertices. Sinkovic [14] subsequently proved that there is no real symmetric weight matrix for which (11) is tight for Paley 17. This leaves open, however, whether there is always a Hermitian weight matrix for which (11) is exact.

It follows from Theorem 1 that every graph with $\alpha < \alpha_q$ is a counter-example to (20) for real symmetric and Hermitian weight matrices. This leads to the question of whether (20) is true for $\alpha_q$ or $\alpha_p$? It follows from Theorem 2 that the answer is no, because for some graphs, such as the line graph of the cartesian product of $K_3$ with itself, the projective packing number is non-integral.

There are also numerous regular graphs for which the Hoffman bound on $\alpha(G)$ is exact, but the unweighted inertia bound is not. Examples include the Shrikhande, Tesseract, Hoffman and Cuboctahedral graphs. There are also many regular graphs where the floor of the Hoffman bound is exact, but the unweighted inertia bound is not. Examples include some circulant, cubic and quartic graphs. For all of these graphs, $\alpha_q = \alpha$.

7 Open questions

Cubitt et al [1] prove (see their Corollary 16 and Conclusion) that:

$$\alpha_q(G) \leq \alpha_{vect}(G) = \lfloor \theta^-(G) \rfloor,$$

where $\alpha_{vect}$ denotes the vectorial independence number. The above inequality raises the following question:

**Question 1.** For any graph $G$ and any Hermitian weight matrix $W$, is the inertia bound

$$n^0(W) + \min\{n^+(W) + n^-(W)\}$$

also an upper bound for $\alpha_{vect}(G)$?
Golubev \[5\] proved that for any graph
\[
\alpha(G) \leq \frac{n(\mu - \delta)}{\mu},
\]
where $\delta$ is the minimum degree and $\mu$ is the largest eigenvalue of the Laplacian matrix of $G$. This bound equals the Hoffman bound for regular graphs. This raises the following question:

**Question 2.** For any graph $G$, is the Golubev bound
\[
\frac{n(\mu - \delta)}{\mu}
\]
also an upper bound for $\alpha_q(G)$ and $\alpha_{\text{vect}}(G)$?

It is an interesting question to determine the smallest graph $G$, that is, the graph with the smallest number of vertices, that exhibits a separation between $\alpha(G)$ and $\alpha_q(G)$:

**Question 3.** What is the smallest graph $G$ with $\alpha(G) < \alpha_q(G)$?

Such a graph must have at least 11 vertices (given the experimental results due to Elzinga and Gregory). Scarpa \[13\] has shown how starting from any graph $G$ (with $n > 2$), a graph $H(G)$ can be constructed which has $\alpha_q(H) > \alpha(H)$. Using this approach Figure 3.1 in \[13\] illustrates $H(P_3)$ which has two connected components and 22 vertices with $\alpha_q(H) = 2^n - 1 = 2^3 - 1 = 7$ and $\alpha(H) = 6$.

**Acknowledgements**

We would like to thank David Roberson for helpful comments on an earlier version of this paper, in particular in regard to the projective packing number.

This research has been supported in part by National Science Foundation Award 1525943.

**References**

[1] T. Cubitt, L. Mancinska, D. E. Roberson, S. Severini, D. Stahlke and A. Winter, *Bounds on Entanglement-Assisted Source-Channel Coding via the Lovász $\theta$ number and its variants*, IEEE Translations on Inf. Theory, 60, 11 (2014), 7330 - 7344.

[2] D. M. Cvetkovic, *Inequalities obtained on the basis of the spectrum of the graph*, Studia Sci. Math. Hungar. 8, (1973), 433 - 436.

[3] R. J. Elzinga and D. A. Gregory, *Weighted matrix eigenvalue bounds on the Independence Number*, Elec. J. Linear Algebra, 20, (2010), 468 - 489.

[4] C. Godsil, *Interesting graphs and their colourings*, (2006).
[5] K. Golubev, *On the chromatic number of a simplicial complex*, Combinatorica, 37, 5, (2017), 953 - 964.

[6] A. J. Hoffman and L. Howes, *On eigenvalues and colourings of graphs II*, Annals of New York Academy of Sciences, (1972), 238 - 242.

[7] M. Laurent and T. Piovesan, *Conic approach to quantum graph parameters using linear optimization over the completely positive semidefinite cone*, SIAM J. Optimization, 25(4), (2015), 2461 - 2493.

[8] L. Lovász, *On the Shannon capacity of a graph*, IEEE Trans. Inform. Th. 25, (1979), 1 - 7.

[9] L. Mancinska and D. E. Roberson, *Oddities of Quantum Colorings*, Baltic J. Modern Computing, 4, (2016), 846 - 859.

[10] L. Mancinska and D. E. Roberson, *Graph homomorphisms for quantum players*, J. Combinatorial Theory Ser. B, 118, (2016), 228 - 267.

[11] L. Mancinska, D. E. Roberson and A. Varvitsiotis, *On deciding the existence of perfect entangled strategies for nonlocal games*, Chicago J. Theoretical Computer Science, 5, (2016), 1 - 16.

[12] D. E. Roberson, *Variations on a Theme: Graph Homomorphisms*, PhD thesis, University of Waterloo, (2013).

[13] G. Scarpa, *Quantum entanglement in non-local games, graph parameters and zero-error information theory*, University of Amsterdam, (2013).

[14] J. Sinkovic, *A graph for which the inertia bound is not tight*, J. Algebraic Combinatorics, 47, 1 (2018), 39 - 50.

[15] P. Wocjan and C. Elphick, *Spectral lower bounds for the quantum chromatic number of a graph*, math arxiv: 1805.08334, (2018).