A new proof for global rigidity of vertex scaling on polyhedral surfaces

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Abstract

The vertex scaling for piecewise linear metrics on polyhedral surfaces was introduced by Luo [18], who proved the local rigidity by establishing a variational principle and conjectured the global rigidity. Luo’s conjecture was solved by Bobenko-Pinkall-Springborn [3], who also introduced the vertex scaling for piecewise hyperbolic metrics and proved its global rigidity. Bobenko-Pinkall-Springborn’s proof is based on their observation of the connection of vertex scaling and the geometry of polyhedra in 3-dimensional hyperbolic space and the concavity of the volume of ideal and hyper-ideal tetrahedra. In this paper, we give an elementary and short variational proof of the global rigidity of vertex scaling without involving 3-dimensional hyperbolic geometry. The method is based on continuity of eigenvalues of matrices and the extension of convex functions.

Keywords: Rigidity; Vertex scaling; Piecewise linear metric; Piecewise hyperbolic metric

1 Introduction

The most important two discrete conformal metrics on polyhedral surfaces are circle packing metrics and vertex scaling of polyhedral metrics on surfaces. There are lots of important works on circle packing metrics, please refer to [1, 2, 4, 5, 6, 10, 19, 21, 25, 28, 39] and others. In this paper, we focus on vertex scaling of polyhedral metrics on surfaces, which is an analogue of the conformal transformation in Riemannian geometry. The vertex scaling of piecewise linear metrics (PL metrics for short in the following) on polyhedral surfaces was introduced physically by Röcek-Williams [27] and mathematically by Luo [18] independently. Luo proved the local rigidity of vertex scaling for PL metrics by establishing a variational principle and conjectured the global rigidity in [18], where Luo also introduced
the corresponding combinatorial Yamabe flow and studied its properties. Luo’s conjecture was solved affirmatively by Bobenko-Pinkall-Springborn in their important work [3] by establishing the connection of vertex scaling and the geometry of ideal tetrahedra in 3-dimensional hyperbolic space and using Rivin’s result on the concavity of the volume of ideal tetrahedra [26]. Bobenko-Pinkall-Springborn [3] further introduced the vertex scaling for piecewise hyperbolic metrics (PH metrics for short in the following) and proved its global rigidity by connecting the hyperbolic vertex scaling to the geometry of hyper-ideal tetrahedra in 3-dimensional hyperbolic space and using Leibon’s result on the concavity of the volume of hyper-ideal tetrahedra [17]. Based on Bobenko-Pinkall-Springborn’s observations, the important discrete uniformization theorems for vertex scaling on closed surfaces were recently established in [11, 12, 24]. Other related work on the vertex scaling could be found in [13, 20, 23, 30, 31, 32, 35, 36, 37, 40]. This paper aims at giving an elementary, direct and short variational proof for the global rigidity of vertex scaling of PL and PH metrics on surfaces without involving 3-dimensional hyperbolic geometry.

Suppose \( M \) is a closed surface with a triangulation \( \mathcal{T} = (V, E, F) \), where \( V, E, F \) represent the sets of vertices, edges and faces respectively. A discrete metric is a map \( l : E \to (0, +\infty) \) such that the triangle inequalities are satisfied for \( l_{ij}, l_{ik}, l_{jk} \) on any triangle \( \triangle v_iv_jv_k \in F \), where \( l_{ij} := l(v_iv_j) \) with \( v_iv_j \in E \). In this case, we can attach a Euclidean metric to each triangle \( \triangle v_iv_jv_k \in F \), which gives rise to a Euclidean triangle, still denoted by \( \triangle v_iv_jv_k \). By gluing the Euclidean triangles isometrically along the edges, we have a PL metric on the triangulated surface \((M, T)\). If we replace the Euclidean metric by hyperbolic metric, then we obtain a PH metric on the triangulated surface \((M, T)\). For PL and PH metrics on \((M, T)\), there may exist cone singularities at the vertices, which could be described by combinatorial curvature. The combinatorial curvature \( K_i \) at the vertex \( v_i \) is \( 2\pi \) less the summation of inner angles of triangles at \( v_i \).

**Definition 1.1** \((3, 18, 27)\). Suppose \((M, T)\) is a triangulated surface and \( u : V \to \mathbb{R} \) is a function defined on the vertices.

1. **(1) If** \( l : E \to (0, +\infty) \) and \( \tilde{l} : E \to (0, +\infty) \) are two PL metrics on \((M, T)\) with
   \[
   l_{ij} = \tilde{l}_{ij} e^{\frac{u_i + u_j}{2}}
   \]
   for any edge \( v_iv_j \in E \), we say \( l \) is a Euclidean vertex scaling of \( \tilde{l} \).

2. **(2) If** \( l : E \to (0, +\infty) \) and \( \tilde{l} : E \to (0, +\infty) \) are two PH metrics on \((M, T)\) with
   \[
   \sinh \frac{l_{ij}}{2} = \sinh \frac{\tilde{l}_{ij} e^{\frac{u_i + u_j}{2}}}{2}
   \]
   for any edge \( v_iv_j \in E \), we say \( l \) is a hyperbolic vertex scaling of \( \tilde{l} \).
The function \( u : V \to \mathbb{R} \) is called a discrete conformal factor.

Bobenko-Pinkall-Spingborn proved the following global rigidity of vertex scaling of polyhedral metrics on surfaces in their important work \[3\].

**Theorem 1.2** (\[3\]). Suppose \((M, T)\) is a closed triangulated surface. Then the discrete conformal factor is uniquely determined by the discrete curvature (up to a vector \( t(1,1,\cdots,1) \), \( t \in \mathbb{R} \), in the PL case).

Let us recall Bobenko-Pinkall-Spingborn’s strategy to prove Theorem 1.2. In the PL case, they considered the Legendre transform of the volume of ideal tetrahedra in 3-dimensional hyperbolic space, which has an explicit formula in dihedral angles obtained by Milnor \[22\]. Based on Rivin’s result \[26\] that the volume of ideal tetrahedra is a concave function of the dihedral angles and could be extended, they extended the definition of Legendre transform of the volume to be a globally defined convex function. By modifying the Legendre transform of the volume by a linear function, they showed that this is a globally defined convex extension of Luo’s action function which is locally convex. Then the global rigidity follows from the convexity of the extended function. In the PH case, the global rigidity is proved similarly with the volume of ideal tetrahedra replaced by the volume of hyper-ideal tetrahedra with one hyper-ideal vertex and three ideal vertices. The explicit formula of such hyper-ideal tetrahedra in terms of dihedral angles was obtained by Leibon in \[17\], where the concavity of the volume was also proved. Bobenko-Pinkall-Spingborn’s approach established the connection of vertex scaling on polyhedral surfaces and the geometry of hyperbolic polyhedra in 3-dimensional hyperbolic space. In this approach they could define the vertex scaling for hyperbolic and spherical polyhedral metrics on surfaces and give the explicit formula of the action functional introduced by Luo, which has lots of applications.

In this paper, we give an elementary, direct and short variational proof for the global rigidity of vertex scaling of PL and PH metrics on polyhedral surfaces, which does not involve the volume of ideal and hyper-ideal tetrahedra in 3-dimensional hyperbolic space and their concavity with respect to dihedral angles. The main idea comes from \[3, 6, 19, 34\].

The first step is to give a characterization of the admissible space of conformal factors for any given initial discrete metric on a single triangle, which is proved to be simply connected with analytic boundaries by solving a quadratic inequality. The second step is to prove the Jacobian of the combinatorial curvature with respect to the discrete conformal factor is symmetric and positive definite, which could be reduced to the case that the Jacobian of the inner angles with respect to the conformal factors in a triangle is symmetric and negative definite. The symmetry could be proved by direct calculations. For the negative definiteness, we introduce a parameterized admissible space of conformal factors, which is the union of admissible spaces of conformal factors on a single triangle with different initial
discrete metrics. This space is proved to be connected, from which the negativity of the Jacobian of inner angles with respect to the conformal factors in a triangle follows easily by the continuity of eigenvalues and calculating at a good point in the parameterized admissible space. The first step and second step enable us to define a locally convex function on the admissible space of conformal factors for a triangle with fixed initial metric. The third step is to extend the locally convex function to be a globally defined convex function, from which the global rigidity follows. This step is accomplished using Luo’s extension theorem for continuous closed 1-forms [19], which is a development of Bobenko-Pinkall-Spingborn’s extension. We will give the details of the proof of global rigidity for hyperbolic vertex scaling and just sketch the proof for Euclidean vertex scaling.

The paper is organized as follows. In Section 2, we characterize the admissible space of discrete conformal factors for PH metrics on a single triangle. In Section 3, we prove the Jacobian matrix of the inner angles in terms of hyperbolic discrete conformal factors in a triangle is symmetric and negative definite, which enables us to define a locally convex function on the admissible space of conformal factors. In Section 4, we extend the locally convex function to be a globally defined convex function, from which the global rigidity of hyperbolic vertex scaling follows. In Section 5, we sketch the proof for global rigidity of vertex scaling for PL metrics.

2 Admissible space of discrete conformal factors for discrete hyperbolic metrics on a triangle

Suppose $\Delta v_i v_j v_k \in F$ is a triangle and $\tilde{l}_{ij}, \tilde{l}_{ik}, \tilde{l}_{jk}$ is a discrete hyperbolic metric on $\Delta v_i v_j v_k$. The admissible space $\Omega_{ijk}^H(\tilde{l})$ of discrete conformal factors for the triangle $\Delta v_i v_j v_k$ with discrete hyperbolic metric $\tilde{l}_{ij}, \tilde{l}_{ik}, \tilde{l}_{jk}$ is defined to be the set of discrete conformal factors $(u_i, u_j, u_k) \in \mathbb{R}^3$ such that the triangle with edge lengths given by formula (2) exists in 2-dimensional hyperbolic space $\mathbb{H}^2$, i.e.

$$\Omega_{ijk}^H(\tilde{l}) = \{(u_i, u_j, u_k) \in \mathbb{R}^3 | l_i + l_j > l_k, l_i + l_k > l_j, l_j + l_k > l_i\}.$$

Here and in the following, we use $l_i$ to denote $l_{jk}$ for simplicity. The parameterized admissible space of conformal factors for the triangle $\Delta v_i v_j v_k$ is defined to be

$$\Omega_{ijk}^H = \{(\tilde{l}_i, \tilde{l}_j, \tilde{l}_k, u_i, u_j, u_k) \in \mathbb{R}_{>0}^3 \times \mathbb{R}^3 | l_i + l_j > l_k, l_i + l_k > l_j, l_j + l_k > l_i\},$$

which could be taken as the union of the admissible space $\Omega_{ijk}^H(\tilde{l})$ according to the parameters given by the initial discrete metrics $(\tilde{l}_i, \tilde{l}_j, \tilde{l}_k)$.

By formula (2), if the edge lengths $l_i, l_j, l_k$ satisfy the triangle inequalities, there are some restrictions on the discrete conformal factors.
Lemma 2.1. Suppose the triangle $\triangle v_i v_j v_k$ is a triangle with discrete hyperbolic metric $(\tilde{l}_i, \tilde{l}_j, \tilde{l}_k)$, $l_i, l_j, l_k$ are the edge lengths defined by formula (2), then $l_i, l_j, l_k$ satisfy the triangle inequalities if and only if
\[ Q := -S_i^4 \xi_i^2 - S_j^4 \xi_j^2 - S_k^4 \xi_k^2 + 2S_i^2 S_j^2 \xi_i \xi_j + 2S_i^2 S_k^2 \xi_i \xi_k + 2S_j^2 S_k^2 \xi_j \xi_k + 4S_i^2 S_j^2 S_k^2 > 0, \]
where $S_i = \sinh \frac{1}{2} \xi_i$, $\xi_i = e^{-u_i}$.

Proof. $l_i, l_j, l_k$ satisfy the triangle inequalities, i.e. $l_i + l_j > l_k, l_i + l_k > l_j, l_j + l_k > l_i$, which is equivalent to
\[ \sinh \frac{l_i + l_j + l_k}{2} \sinh \frac{l_i + l_j - l_k}{2} \sinh \frac{l_i + l_k - l_j}{2} \sinh \frac{l_j + l_k - l_i}{2} > 0. \]

By direct calculations, we have
\[
\begin{align*}
&= \frac{1}{4} (\cosh(l_i + l_j) - \cosh l_k)(\cosh l_k - \cosh(l_i - l_j)) \\
&= \frac{1}{4} (\cosh^2 l_i - \cosh^2 l_j - \cosh^2 l_k + 2 \cosh l_i \cosh l_j \cosh l_k + 1) \\
&= -\frac{1}{2} \ln \left( \frac{\cosh l_i + \cosh l_j - \cosh l_k}{2} \right) + \frac{1}{2} \ln \left( \frac{\cosh l_i + \cosh l_j + \cosh l_k}{2} \right) \\
&= -\frac{1}{2} \ln \left( \frac{\cosh l_i + \cosh l_j - \cosh l_k}{2} \right) + \frac{1}{2} \ln \left( \frac{\cosh l_i + \cosh l_j + \cosh l_k}{2} \right) \\
&= \frac{1}{4} S_i^2 S_j^2 S_k^2 (-S_i^4 \xi_i^2 - S_j^4 \xi_j^2 - S_k^4 \xi_k^2 + 2S_i^2 S_j^2 \xi_i \xi_j + 2S_i^2 S_k^2 \xi_i \xi_k + 2S_j^2 S_k^2 \xi_j \xi_k + 4S_i^2 S_j^2 S_k^2),
\end{align*}
\]
where the formula (2) is used in the last equality. 

Set
\[
\begin{align*}
h_i &= -S_i^4 \xi_i + S_i^2 S_j^2 \xi_j + S_i^2 S_k^2 \xi_k, \\
h_j &= -S_j^4 \xi_j + S_i^2 S_j^2 \xi_i + S_j^2 S_k^2 \xi_k, \\
h_k &= -S_k^4 \xi_k + S_i^2 S_k^2 \xi_i + S_j^2 S_k^2 \xi_j,
\end{align*}
\]
then we have
\[ Q = \xi_i h_i + \xi_j h_j + \xi_k h_k + 4S_i^2 S_j^2 S_k^2. \]

Lemma 2.1 implies that $(u_i, u_j, u_k) \in \mathbb{R}^3$ is a degenerate hyperbolic discrete conformal factor for a triangle $\triangle v_i v_j v_k$ if and only if
\[ Q = \xi_i h_i + \xi_j h_j + \xi_k h_k + 4S_i^2 S_j^2 S_k^2 \leq 0. \]

Note that $4S_i^2 S_j^2 S_k^2 > 0$ for any $(\tilde{l}_i, \tilde{l}_j, \tilde{l}_k) \in \mathbb{R}^3_{>0}$. If $(u_i, u_j, u_k) \in \mathbb{R}^3$ is a degenerate hyperbolic discrete conformal factor, we have
\[ \xi_i h_i + \xi_j h_j + \xi_k h_k < 0. \]
which implies that at least one of \( h_i, h_j, h_k \) is negative. We further have the following result on the signs of \( h_i, h_j, h_k \).

**Lemma 2.2.** Suppose \((u_i, u_j, u_k) \in \mathbb{R}^3\) is a degenerate hyperbolic discrete conformal factor for a triangle \( \triangle v_iv_jv_k \), then one of \( h_i, h_j, h_k \) is negative and the others are positive.

**Proof.** We claim that there exists no subset \( \{r, s\} \subset \{i, j, k\} \) such that \( h_r \leq 0 \) and \( h_s \leq 0 \), from which the conclusion of the lemma follows. Otherwise, without loss of generality, we assume \( h_i \leq 0, h_j \leq 0 \), which is equivalent to \( S_i^2 \xi_i \geq S_j^2 \xi_j + S_k^2 \xi_k, S_j^2 \xi_j \geq S_i^2 \xi_i + S_k^2 \xi_k \). This is impossible.

There is a nice geometric explanation of the result in Lemma 2.2 in the Euclidean case in terms of circumcircle center. Please refer to Remark 4.

**Theorem 2.3.** Given any initial nondegenerate hyperbolic discrete metric \( \tilde{l} = (\tilde{l}_i, \tilde{l}_j, \tilde{l}_k) \) on a triangle \( \triangle v_iv_jv_k \), the admissible space \( \Omega_H^{ijk} (\tilde{l}) \) of hyperbolic discrete conformal factors \((u_i, u_j, u_k) \in \mathbb{R}^3\) for the triangle \( \triangle v_iv_jv_k \) is nonempty and simply connected. Furthermore, the set of degenerate hyperbolic discrete conformal factors is a disjoint union \( \bigcup_{\alpha \in \Lambda} V_{\alpha} \), where \( \Lambda = \{i, j, k\} \) and \( V_{\alpha} \) is a closed region in \( \mathbb{R}^3 \) bounded by an analytic graph on \( \mathbb{R}^2 \).

**Proof.** Suppose \((u_i, u_j, u_k) \in \mathbb{R}^3\) is a degenerate hyperbolic discrete conformal factor for the triangle \( \triangle v_iv_jv_k \), which is equivalent to \( Q \leq 0 \). Then by Lemma 2.2, one of \( h_i, h_j, h_k \) is negative and the others are positive. Without loss of generality, we assume \( h_i < 0, h_j > 0, h_k > 0 \). Note that \( Q \leq 0 \) is equivalent to the following quadratic inequality of \( \xi_i \)

\[
A_i \xi_i^2 + B_i \xi_i + C_i \geq 0, \tag{4}
\]

where

\[
A_i = S_i^4 > 0, \\
B_i = -2S_i^2(S_j^2 \xi_j + S_k^2 \xi_k) < 0, \tag{5}
\]

\[
C_i = S_j^4 \xi_j^2 + S_k^4 \xi_k^2 - 2S_j^2 S_k^2 \xi_j \xi_k - 4S_i^2 S_j^2 S_k^2.
\]

By direct calculations, \( \Delta_i = B_i^2 - 4A_iC_i \) is given by

\[
\Delta_i = 16S_i^4S_j^2S_k^2 \xi_j \xi_k + 16S_i^6S_j^2S_k^2 > 0. \tag{6}
\]

Combining formula (4), (5) with (6), we have

\[
\xi_i \geq \frac{-B_i + \sqrt{\Delta_i}}{2A_i} \quad \text{or} \quad \xi_i \leq \frac{-B_i - \sqrt{\Delta_i}}{2A_i}.
\]
Therefore, $\Omega_H$ implies

and $H$.

The ideal of the proof of Theorem 2.3 comes from [33], where the first author introduced the method of homotopy deformation to prove the admissible space of sphere packing metrics for a single tetrahedron is simply connected. This method is then developed [34] to prove the admissible space of inversive distance packing metrics for a single triangle is simply connected and used to proved the admissible space of Thurston’s sphere packing metrics on a tetrahedron is simply connected [14][15]. This method has some other applications in characterizing admissible spaces of discrete metrics, see [38].

Note that $Q$ is a continuous function of $(\tilde{l}_i, \tilde{u}_j, \tilde{v}_k, u_i, u_j, u_k) \in \mathbb{R}^3_0 \times \mathbb{R}^3$ and the space of hyperbolic discrete metrics $(\tilde{l}_i, \tilde{u}_j, \tilde{v}_k)$ satisfying the triangle inequalities is connected. As a direct corollary of Lemma 2.2 and Theorem 2.3 we have the following result on the parameterized admissible space $\Omega_{ijk}^H$.

**Corollary 2.4.** Suppose $\Delta v_i v_j v_k \in F$. Then the parameterized admissible space $\Omega_{ijk}^H$ is connected.

Denote $\alpha_i, \alpha_j, \alpha_k$ as the inner angles in the triangle $\Delta v_i v_j v_k$ so that $\alpha_i$ is opposite to the edge of length $l_i$. We further have the following property of the inner angles on the admissible space $\Omega_{ijk}^H(\tilde{l})$.

**Lemma 2.5.** The inner angles $\alpha_i, \alpha_j, \alpha_k$ defined for $(u_i, u_j, u_k) \in \Omega_{ijk}^H(\tilde{l})$ could be extended to be continuous functions $\tilde{\alpha}_i, \tilde{\alpha}_j, \tilde{\alpha}_k$ defined on $\mathbb{R}^3$. 

\[ \tilde{\alpha}_i, \tilde{\alpha}_j, \tilde{\alpha}_k \]
Proof. By Theorem 2.23, $\Omega_{ijk}^H(\tilde{\Omega}) = \mathbb{R}^3 \setminus \bigcup_{\alpha \in \Lambda} V_\alpha$, where $\Lambda = \{i, j, k\}$ and $V_i = \{(u_i, u_j, u_k) \in \mathbb{R}^3 | \xi_i \geq -\frac{B_i + \sqrt{B_i^2 + 2A_i} \xi_i}{2A_i}\}$. Then $\partial V_i = \{(u_i, u_j, u_k) \in \mathbb{R}^3 | \xi_i = -\frac{B_i + \sqrt{B_i^2 + 2A_i} \xi_i}{2A_i}\}$. Suppose $(u_i, u_j, u_k) \in \Omega_{ijk}^H(\tilde{\Omega})$ tends to a point $(\overline{\nu}_i, \overline{\nu}_j, \overline{\nu}_k) \in \partial V_i$. By the proof of Lemma 2.21, we have

$$4\xi_i^{-2}e^{-2}\xi_k^{-2}Q = 4\sinh \frac{l_i + l_j + l_k}{2} \sinh \frac{l_i + l_j - l_k}{2} \sinh \frac{l_i + l_k - l_j}{2} \sinh \frac{l_j + l_k - l_i}{2},$$

$$(\cosh(l_i + l_j) - \cosh l_k)(\cosh l_k - \cosh(l_i - l_j))$$

$$= \sinh^2 l_i \sinh^2 l_j - (\cosh l_i \cosh l_j - \cosh l_k)^2$$

$$= \sinh^2 l_i \sinh^2 l_j - \sinh^2 l_i \sinh^2 l_j \cosh^2 \alpha_k$$

$$= \sinh^2 l_i \sinh^2 l_j \sin^2 \alpha_k.$$

As $(u_i, u_j, u_k) \in \Omega_{ijk}^H(\tilde{\Omega})$ tends to $(\overline{\nu}_i, \overline{\nu}_j, \overline{\nu}_k) \in \partial V_i$, we have $Q \to 0$, which implies $\alpha_k \to 0$ or $\pi$. Similarly, we have $\alpha_i, \alpha_j \to 0$ or $\pi$.

By formula (3), we have

$$\frac{\partial \alpha_i}{\partial u_i} = \frac{\cosh l_i + \cosh l_j - \cosh l_k - 1}{A(\cosh l_k + 1)}$$

$$= \frac{\sinh^2 \frac{l_i}{2} + \sinh^2 \frac{l_j}{2} - \sinh^2 \frac{l_k}{2}}{A(\sinh^2 \frac{l_k}{2} + 1)}$$

$$= \frac{\xi_i^{-1}\xi_j^{-1}\xi_k^{-1}}{A(S_k^2\xi_i^{-1}\xi_j^{-1} + 1)}(S_i^2\xi_i + S_j^2\xi_j - S_k^2\xi_k)$$

$$= \frac{\xi_i^{-1}\xi_j^{-1}\xi_k^{-1}}{A S_k^2(S_k^2\xi_i^{-1}\xi_j^{-1} + 1)}.$$

where $A = \sinh l_i \sinh l_k \sin \alpha_i$, $S_i = \sinh \frac{l_i}{2}$, $\xi_i = e^{-u_i}$, and $h_k$ is defined by formula (3). Note that for $(\overline{\nu}_i, \overline{\nu}_j, \overline{\nu}_k) \in \partial V_i$, by Lemma 2.22 and the proof of Theorem 2.23, we have $h_i < 0$, $h_j > 0$, $h_k > 0$ at $(\overline{\nu}_i, \overline{\nu}_j, \overline{\nu}_k)$. By formula (7), we have $\frac{\partial \alpha_j}{\partial u_i} > 0$ for $(u_i, u_j, u_k) \in \Omega_{ijk}^H(\tilde{\Omega})$ around $(\overline{\nu}_i, \overline{\nu}_j, \overline{\nu}_k) \in \partial V_i$. This implies $\alpha_j \to 0$ as $(u_i, u_j, u_k) \to (\overline{\nu}_i, \overline{\nu}_j, \overline{\nu}_k) \in \partial V_i$. Otherwise, we have $\alpha_j \to \pi$ as $(u_i, u_j, u_k) \to (\overline{\nu}_i, \overline{\nu}_j, \overline{\nu}_k) \in \partial V_i$ and then $\frac{\partial \alpha_j}{\partial u_i} > 0$ implies $\alpha_j > \pi$ for some $(u_i, u_j, u_k) \in \Omega_{ijk}^H(\tilde{\Omega})$ around $(\overline{\nu}_i, \overline{\nu}_j, \overline{\nu}_k) \in \partial V_i$, which is impossible for hyperbolic triangles. Similarly, we have $\alpha_k \to 0$ as $(u_i, u_j, u_k) \to (\overline{\nu}_i, \overline{\nu}_j, \overline{\nu}_k) \in \partial V_i$.

Furthermore, we have the following formula for the area $S$ of the hyperbolic triangle in terms of the edge lengths (29 page 66)

$$\tan^2 \frac{S}{4} = \tanh \frac{p}{2} \tanh \frac{p - l_i}{2} \tanh \frac{p - l_j}{2} \tanh \frac{p - l_k}{2}$$

$$= \frac{\xi_i^{-2}\xi_j^{-2}\xi_k^{-2}Q}{64 \cosh^2 \frac{p}{2} \cosh^2 \frac{p - l_i}{2} \cosh^2 \frac{p - l_j}{2} \cosh^2 \frac{p - l_k}{2}}.$$
where \( p = \frac{1}{2}(l_i + l_j + l_k) \). Note that \( Q \to 0 \) as \((u_i, u_j, u_k) \to (\pi_i, \pi_j, \pi_k) \in \partial V_i\), we have \( S \to 0 \). Then we have \( \alpha_i \to \pi \) as \((u_i, u_j, u_k) \to (\pi_i, \pi_j, \pi_k) \in \partial V_i\) by \( S = \pi - \alpha_i - \alpha_j - \alpha_k \) and \( \alpha_j, \alpha_k \to 0 \). The case for the boundary \( \partial V_j \) and \( \partial V_k \) could be discussed similarly.

Therefore, we can extend \( \alpha_i, \alpha_j, \alpha_k \) defined on \( \Omega_{ijk}(\tilde{l}) \) to be continuous functions defined on \( \mathbb{R}^3 \) by setting

\[
\tilde{\alpha}(u_i, u_j, u_k) = \begin{cases} 
\alpha_i, & \text{if } (u_i, u_j, u_k) \in \Omega_{ijk}^H, \\
\pi, & \text{if } (u_i, u_j, u_k) \in V_i, \\
0, & \text{if } (u_i, u_j, u_k) \in V_j \text{ or } V_k.
\end{cases}
\]

This completes the proof of the lemma.

Remark 2. By the proof of Lemma 2.5, we have \( \frac{\partial \alpha_i}{\partial u_i} \to +\infty \) and \( \frac{\partial \alpha_k}{\partial u_i} \to +\infty \) as \((u_i, u_j, u_k) \to (\pi_i, \pi_j, \pi_k) \in \partial V_i\). Recall the following formula obtained by Glickenstein-Thomas ([10] Proposition 9)

\[
\frac{\partial S}{\partial u_i} = \frac{\partial \alpha_j}{\partial u_i} \cosh l_k - 1 + \frac{\partial \alpha_k}{\partial u_i} (\cosh l_j - 1),
\]

where \( S \) is the area of \( \triangle v_i v_j v_k \), which could also be proved using Lemma 3.1 directly. For hyperbolic vertex scaling, we have \( \frac{\partial S}{\partial u_i} \to +\infty \), which implies

\[
\frac{\partial \alpha_i}{\partial u_i} = -\frac{\partial S}{\partial u_i} - \frac{\partial \alpha_j}{\partial u_i} - \frac{\partial \alpha_k}{\partial u_i} \to -\infty
\]
as \((u_i, u_j, u_k) \to (\pi_i, \pi_j, \pi_k) \in \partial V_i\).

3 Negative definiteness of Jacobian matrix

Lemma 3.1. For any triangle \( \triangle v_i v_j v_k \), let \( l_i, l_j, l_k \) be edge lengths of a hyperbolic triangle and \( \alpha_i, \alpha_j, \alpha_k \) be the opposite angles so that \( \alpha_i \) is facing the edge of length \( l_i \), then

\[
\frac{\partial \alpha_i}{\partial u_j} = \frac{\cosh l_i + \cosh l_j - \cosh l_k - 1}{A(\cosh l_k + 1)}, \tag{8}
\]

\[
\frac{\partial \alpha_i}{\partial u_k} = \frac{\cosh^2 l_j + \cosh^2 l_k - 2 \cosh l_j \cosh l_k \cosh l_k + (1 - \cosh l_j)(\cosh l_j + \cosh l_k)}{A(1 + \cosh l_j)(1 + \cosh l_k)},
\]

where \( A = \sinh l_j \sinh l_k \sin \alpha_i \).

Proof. By the derivative cosine law (see Lemma A1 in [5] for example), we have

\[
\frac{\partial \alpha_i}{\partial l_i} = \frac{\sinh l_i}{A}, \quad \frac{\partial \alpha_i}{\partial l_j} = -\frac{\sinh l_i \cos \alpha_k}{A}, \quad \frac{\partial \alpha_i}{\partial l_k} = -\frac{\sinh l_i \cos \alpha_j}{A},
\]
where \( A = \sinh l_j \sinh l_k \sin \alpha_i \). By formula (2), we have
\[
\frac{\partial l_i}{\partial u_i} = 0, \quad \frac{\partial l_i}{\partial u_j} = \frac{\partial l_i}{\partial u_k} = \tanh \frac{l_i}{2}.
\]
Then according to the chain rules, we have
\[
\frac{\partial \alpha_i}{\partial u_j} = \frac{\partial \alpha_i}{\partial l_i} \cdot \frac{\partial l_i}{\partial u_j} + \frac{\partial \alpha_i}{\partial l_j} \cdot \frac{\partial l_j}{\partial u_j} + \frac{\partial \alpha_i}{\partial l_k} \cdot \frac{\partial l_k}{\partial u_j}
= \frac{\sinh l_i}{A} \tanh \frac{l_i}{2} \sin l_j \cos \alpha_j \tanh \frac{l_k}{2} \sin l_j \sinh l_i
= \frac{\sinh l_i}{A} \frac{1}{1 + \cosh l_i} \frac{A}{2} \frac{1}{1 + \cosh l_k} \sinh l_i \cos \alpha_j
= \frac{1}{A(\cosh l_i + 1)} \sinh l_i \cosh l_j - \cosh l_k - 1,
\]
which implies \( \frac{\partial \alpha_i}{\partial u_j} = \frac{\partial \alpha_j}{\partial u_i} \). Similarly, we have
\[
\frac{\partial \alpha_i}{\partial u_i} = \frac{\partial \alpha_i}{\partial l_i} \cdot \frac{\partial l_i}{\partial u_i} + \frac{\partial \alpha_i}{\partial l_j} \cdot \frac{\partial l_j}{\partial u_i} + \frac{\partial \alpha_i}{\partial l_k} \cdot \frac{\partial l_k}{\partial u_i}
= \frac{\cosh^2 l_j + \cosh^2 l_k - 2 \cosh l_i \cosh l_j \cosh l_k + (1 - \cosh l_i)(\cosh l_j + \cosh l_k)}{A(1 + \cosh l_j)(1 + \cosh l_k)}.
\]

Lemma 3.1 shows that the matrix
\[
\Lambda^H_{ijk} = \frac{\partial (\alpha_i, \alpha_j, \alpha_k)}{\partial (u_i, u_j, u_k)} = \begin{pmatrix} \frac{\partial \alpha_i}{\partial u_i} & \frac{\partial \alpha_i}{\partial u_j} & \frac{\partial \alpha_i}{\partial u_k} \\ \frac{\partial \alpha_j}{\partial u_i} & \frac{\partial \alpha_j}{\partial u_j} & \frac{\partial \alpha_j}{\partial u_k} \\ \frac{\partial \alpha_k}{\partial u_i} & \frac{\partial \alpha_k}{\partial u_j} & \frac{\partial \alpha_k}{\partial u_k} \end{pmatrix}
\]
is symmetric on \( \Omega^H_{ijk} \). Furthermore, one has the following result for the matrix \( \Lambda^H_{ijk} \).

**Theorem 3.2** (3). The matrix \( \Lambda^H_{ijk} \) is symmetric, negative definite on \( \Omega^H_{ijk} \).

**Proof.** By the chain rules, we have
\[
\Lambda^H_{ijk} = \frac{\partial (\alpha_i, \alpha_j, \alpha_k)}{\partial (u_i, u_j, u_k)} = \frac{\partial (\alpha_i, \alpha_j, \alpha_k)}{\partial (l_i, l_j, l_k)} \cdot \frac{\partial (l_i, l_j, l_k)}{\partial (u_i, u_j, u_k)}.
\]
By the calculations in the proof of Lemma 3.1, we have
\[
\frac{\partial (\alpha_i, \alpha_j, \alpha_k)}{\partial (l_i, l_j, l_k)} = -\frac{1}{A} \begin{pmatrix} \sinh l_i & 0 & 0 \\ 0 & \sinh l_j & 0 \\ 0 & 0 & \sinh l_k \end{pmatrix} \begin{pmatrix} -1 & \cos \alpha_k & \cos \alpha_j \\ \cos \alpha_k & -1 & \cos \alpha_i \\ \cos \alpha_j & \cos \alpha_i & -1 \end{pmatrix}
\]
\[
\frac{\partial(l_i, l_j, l_k)}{\partial(u_i, u_j, u_k)} = \begin{pmatrix}
\tanh \frac{l_i}{2} & 0 & 0 \\
0 & \tanh \frac{l_j}{2} & 0 \\
0 & 0 & \tanh \frac{l_k}{2}
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}.
\] (10)

Denote the last matrices in (9) and (10) as \( \Phi \) and \( R \) respectively. By direct calculations, we have
\[
\det \Phi = -1 + \cos \alpha_i^2 + \cos \alpha_j^2 + \cos \alpha_k^2 + 2 \cos \alpha_i \cos \alpha_j \cos \alpha_k
\]
\[
= 4 \cos \frac{\alpha_i + \alpha_j - \alpha_k}{2} \cos \frac{\alpha_i - \alpha_j + \alpha_k}{2} \cos \frac{\alpha_i + \alpha_j + \alpha_k}{2} \cos \frac{\alpha_i - \alpha_j - \alpha_k}{2} > 0,
\]
\[
\det R = 2 > 0
\]
for any \((l_i, l_j, l_k, u_i, u_j, u_k) \in \Omega_{ijk}^H\), which implies \( \det \Lambda_{ijk}^H < 0 \) and then the Jacobian matrix \( \Lambda_{ijk}^H \) is non-singular. Therefore, the eigenvalues of \( \Lambda_{ijk}^H \) are non-zero. Combining with the continuity of the eigenvalues and the connectivity of the parameterized admissible space \( \Omega_{ijk}^H \) in Corollary 2.3, the eigenvalues of \( \Lambda_{ijk}^H \) never change signs. So we just need to calculate at one point in \( \Omega_{ijk}^H \) to prove that the eigenvalues of \( \Lambda_{ijk}^H \) are negative and then \( \Lambda_{ijk}^H \) is negative definite. Note that \( p = (1, 1, 1, 0, 0, 0) \in \Omega_{ijk}^H \). By Lemma 3.1 we have
\[
\Lambda_{ijk}^H(p) = -\frac{(\cosh 1 - 1)}{A(1 + \cosh 1)} \begin{pmatrix}
2 \cosh 1 & -1 & -1 \\
-1 & 2 \cosh 1 & -1 \\
-1 & -1 & 2 \cosh 1
\end{pmatrix},
\]
which is negative definite. Therefore, the eigenvalues of the Jacobian matrix \( \Lambda_{ijk}^H \) at \( p = (1, 1, 1, 0, 0, 0) \) are negative. This completes the proof of the theorem. \( \square \)

Remark 3. Theorem 3.2 was first obtained by Bobenko-Pinkall-Springborn in their important work [3] by taking the Jacobian matrix \( \Lambda_{ijk}^H \) as the Hessian matrix of Legendre transform of the volume of hyper-ideal tetrahedra in 3-dimensional hyperbolic space with prescribed metric. The negativity of \( \Lambda_{ijk}^H \) follows from Leibon’s concavity of the volume of hyper-ideal tetrahedra with one hyper-ideal vertex and three ideal vertices [17], which depends on the explicit form of the volume formula in terms of dihedral angles. The proof of Theorem 3.2 presented here involves only the cosine law and the continuity of the eigenvalues.

4 Proof of the global rigidity of hyperbolic vertex scaling

By Theorem 2.3 and Theorem 3.2, the following function
\[
F_{ijk}(u_i, u_j, u_k) = \int_{(\overline{u}_i, \overline{u}_j, \overline{u}_k)} \alpha_i du_i + \alpha_j du_j + \alpha_k du_k
\]
is a well-defined locally strictly concave function of \((u_i, u_j, u_k) \in \Omega_{ijk}(\tilde{l})\). We need to extend \(F_{ijk}\) to be a globally defined concave function on \(\mathbb{R}^3\). Recall the following definition of closed continuous 1-form and extension of locally convex function of Luo [19], which is a development of Bobenko-Pinkall-Spingborn’s extension in [4].

**Definition 4.1** ([19], Definition 2.3). A differential 1-form \(w = \sum_{i=1}^{n} a_i(x)dx^i\) in an open set \(U \subset \mathbb{R}^n\) is said to be continuous if each \(a_i(x)\) is continuous on \(U\). A continuous differential 1-form \(w\) is called closed if \(\int_{\partial \tau} w = 0\) for each triangle \(\tau \subset U\).

**Theorem 4.2** ([19], Corollary 2.6). Suppose \(X \subset \mathbb{R}^n\) is an open convex set and \(A \subset X\) is an open subset of \(X\) bounded by a real analytic codimension-1 submanifold in \(X\). If \(w = \sum_{i=1}^{n} a_i(x)dx^i\) is a continuous closed 1-form on \(A\) so that \(F(x) = \int_{u}^x a w\) is locally convex on \(A\) and each \(a_i\) can be extended continuous to \(X\) by constant functions to a function \(\tilde{a}_i\) on \(X\), then \(\tilde{F}(x) = \int_{a}^{x} \sum_{i=1}^{n} \tilde{a}_i(x)dx^i\) is a \(C^1\)-smooth convex function on \(X\) extending \(F\).

By Lemma 2.5 and Theorem 4.2, \(F_{ijk}(u_i, u_j, u_k)\) defined on \(\Omega_{ijk}(\tilde{l})\) could be extended to be the following function

\[
\tilde{F}_{ijk}(u_i, u_j, u_k) = \int_{(u_i, u_j, u_k)} \tilde{\alpha}_i du_i + \tilde{\alpha}_j du_j + \tilde{\alpha}_k du_k,
\]

which is a \(C^1\)-smooth concave function defined on \(\mathbb{R}^3\) with \(\nabla_{u} \tilde{F}_{ijk} = (\tilde{\alpha}_i, \tilde{\alpha}_j, \tilde{\alpha}_k)^T\). Set

\[
\tilde{F}(u_1, \cdots, u_{|V|}) = - \sum_{\Delta_{v_j} v_j v_k \in F} \tilde{F}_{ijk}(u_i, u_j, u_k) + \int_{u}^{u_{|V|}} 2\pi \sum_{i=1}^{u_{|V|}} du_i,
\]

where \(|V|\) is the number of vertices. Then \(\tilde{F}(u_1, \cdots, u_{|V|})\) is a \(C^1\) smooth convex function on \(\mathbb{R}^{|V|}\) with

\[
\nabla_{u_i} \tilde{F}(u_1, \cdots, u_{|V|}) = - \sum_{\Delta_{ijkl} \in F} \tilde{\alpha}_i + 2\pi = \tilde{K}_i,
\]

where \(\tilde{K}_i = 2\pi - \sum_{\Delta_{v_j} v_j v_k \in F} \tilde{\alpha}_i\) is an extension of \(K_i\). Then the global rigidity of hyperbolic vertex scaling follows from the convexity of \(\tilde{F}\) on \(\mathbb{R}^{|V|}\) and the locally strict convexity of \(\tilde{F}\) on \(\cap_{\Delta_{v_j} v_j v_k \in F} \Omega_{ijk}(\tilde{l})\). This completes the proof of Theorem 1.2 in the hyperbolic case.

**5 Rigidity for vertex scaling of PL metrics**

As the main steps for the proof of global rigidity of Euclidean vertex scaling is paralleling to the hyperbolic case, we just list the main steps here.
Given any initial discrete Euclidean metric $\tilde{h}_{ij}, \tilde{l}_{ik}, \tilde{l}_{jk}$ on the triangle $\Delta v_i v_j v_k$, the admissible space $\Omega^E_{i j k}(\tilde{t})$ of Euclidean conformal factors is defined to be

$$\Omega^E_{i j k}(\tilde{t}) = \{(u_i, u_j, u_k) \in \mathbb{R}^3 \mid l_i + l_j > l_k, l_i + l_k > l_j, l_j + l_k > l_i\},$$

where the edge lengths are given by formula (1) and we use $l_i$ to denote $l_{jk}$ for simplicity. The Euclidean parameterized admissible space of conformal factors for the triangle $\Delta v_i v_j v_k$ is defined to be

$$\Omega^E_{i j k} = \{(l_i, l_j, l_k, u_i, u_j, u_k) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid l_i + l_j > l_k, l_i + l_k > l_j, l_j + l_k > l_i\}.$$

**Lemma 5.1.** Suppose the triangle $\Delta v_i v_j v_k$ is a topological triangle, $l_i, l_j, l_k$ are the edge lengths defined by (1), then the triangle inequalities are satisfied if and only if

$$Q := -\tilde{l}_i^4 \xi_i^2 - \tilde{l}_j^4 \xi_j^2 - \tilde{l}_k^4 \xi_k^2 + 2\tilde{l}_i l_j \xi_i \xi_j + 2\tilde{l}_i l_k \xi_i \xi_k + 2\tilde{l}_j l_k \xi_j \xi_k > 0,$$

where $\xi_i = e^{-u_i}$.

Set

$$h_i = -\tilde{l}_i^4 \xi_i + \tilde{l}_j^4 \xi_j + \tilde{l}_k^4 \xi_k,$$

$$h_j = -\tilde{l}_j^4 \xi_j + \tilde{l}_i^4 \xi_i + \tilde{l}_k^4 \xi_k,$$

$$h_k = -\tilde{l}_k^4 \xi_k + \tilde{l}_i^4 \xi_i + \tilde{l}_j^4 \xi_j.$$  \hfill (11)

Then we have $Q = \xi_i h_i + \xi_j h_j + \xi_k h_k$. $(u_i, u_j, u_k) \in \mathbb{R}^3$ is a degenerate Euclidean discrete conformal factor if and only if $Q = \xi_i h_i + \xi_j h_j + \xi_k h_k \leq 0$, which implies at least one of $h_i, h_j, h_k$ is nonpositive. Similar to Lemma 2.2 in the hyperbolic case, we have the following result on the signs of $h_i, h_j, h_k$ in the Euclidean case.

**Lemma 5.2.** Suppose $(u_i, u_j, u_k) \in \mathbb{R}^3$ is a degenerate Euclidean discrete conformal factor for a triangle $\Delta v_i v_j v_k$, then one of $h_i, h_j, h_k$ is negative and the others are positive.

**Remark 4.** Lemma 5.2 has the following interesting geometrical explanation. For a Euclidean triangle $\Delta v_i v_j v_k$ with a nondegenerate discrete conformal factor $(u_i, u_j, u_k)$, there exists a geometric center $C_{ijk}$ (2 Proposition 4) of the triangle $\Delta v_i v_j v_k$ with the same Euclidean distance from $C_{ijk}$ to each vertex of the triangle, which is in fact the circumcircle center for vertex scaling of PL metrics. $h_i$ in formula (11) is positive multiplication of the signed distance $h_{jk,i}$ from $C_{ijk}$ to the edge $\{jk\}$, which is defined to be positive if $C_{ijk}$ is on the same side of the line determined by $\{jk\}$ as the triangle $\Delta v_i v_j v_k$ and negative otherwise (or zero if $C_{ijk}$ is on the edge). By direct calculations, we have the following relationship for $h_i$ and $h_{jk,i}$

$$h_{jk,i} = \frac{\xi_i^{-\frac{1}{2}} \xi_j^{-\frac{1}{2}} \xi_k^{-\frac{3}{2}}}{8S_{ij}^2} h_i,$$
where $S$ is the area of the Euclidean triangle $\triangle v_i v_j v_k$. See [8] for more general cases.

For degenerate conformal factors for Euclidean vertex scaling, Lemma [5.2] implies that the circumcircle center lies in some special regions in the plane relative to the triangle $\triangle v_i v_j v_k$.

**Theorem 5.3** ([18]). Given any initial nondegenerate Euclidean discrete metric $\tilde{\ell} = (\tilde{l}_i, \tilde{l}_j, \tilde{l}_k)$ on a triangle $\triangle v_i v_j v_k$, the admissible space $\Omega_{ij}^E(\tilde{l})$ of Euclidean discrete conformal factors $(u_i, u_j, u_k) \in \mathbb{R}^3$ for the triangle $\triangle v_i v_j v_k$ is nonempty and simply connected. Furthermore, the set of degenerate Euclidean discrete conformal factors is a disjoint union $\bigcup_{\alpha \in \Lambda} V_{\alpha}$, where $\Lambda = \{i, j, k\}$ and $V_{\alpha}$ is bounded by an analytic graph on $\mathbb{R}^2$ with

$$V_i = \{(u_i, u_j, u_k) \in \mathbb{R}^3 | u_i \leq -\ln(\tilde{l}_j^2 e^{-u_j} + \tilde{l}_k^2 e^{-u_k}) + 2 \ln \tilde{l}_i\}.$$ 

As a corollary, $\Omega_{ij}^E$ is connected.

Following the hyperbolic case, as an application of Theorem [5.3] we have the following result, which was obtained by Luo [18] by direct calculations.

**Theorem 5.4** ([18]). The matrix $\Lambda_{ijk}^E = [\partial_{u_i}]_{3 \times 3}$ is symmetric, semi-negative definite on $\Omega_{ij}^E(\tilde{l})$ with null space $\{(t, t, t) \in \mathbb{R}^3 | t \in \mathbb{R}\}$.

**Remark 5.** In fact, by the derivative cosine law (see [5] for example), we have $\frac{\partial a}{\partial l_i} = \frac{l}{2S}$, $\frac{\partial a}{\partial l_j} = -\frac{l \cos \alpha_k}{2S}$, $\frac{\partial a}{\partial l_k} = -\frac{l \cos \alpha_j}{2S}$, where $S$ is the area of the Euclidean triangle $\triangle v_i v_j v_k$. According to formula (1), we have $\frac{\partial a}{\partial u_i} = 0$, $\frac{\partial a}{\partial u_j} = \frac{\partial a}{\partial u_k} = \frac{l}{2}$. By direct calculations with the chain rules, we have $\frac{\partial a}{\partial u_j} = \frac{l_1 l_j \cos \alpha_k}{4S}$, $\frac{\partial a}{\partial u_k} = -\frac{l_1^2}{4S}$, which implies $\Lambda_{ijk}^E = [\partial_{u_i}]_{3 \times 3}$ is symmetric. Luo [18] proved the semi-negative definiteness of $\Lambda_{ijk}^E$ by direct calculations for any nondegenerate Euclidean conformal factor. If we use the connectivity of $\Omega_{ij}^E$, we just need to check the signs of the eigenvalues of $\Lambda_{ijk}^E$ at the point $p = (1, 1, 1, 0, 0, 0) \in \Omega_{ij}^E$. By direct calculations, we have

$$\Lambda_{ijk}^E(p) = -\frac{\sqrt{3}}{6} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

which has two negative eigenvalues and one zero eigenvalue. This also implies semi-negative definiteness of $\Lambda_{ijk}^E$.

**Lemma 5.5** ([3]). Suppose $(u_i, u_j, u_k) \in \mathbb{R}^3$ is a nondegenerate Euclidean discrete conformal factor for a triangle $\triangle v_i v_j v_k$, denote $\alpha_i$ as the angle at vertex $v_i$. Then $\alpha_i, \alpha_j, \alpha_k$ defined for $(u_i, u_j, u_k) \in \Omega_{ij}^E(\tilde{l})$ could be extended by constants to be continuous functions $\tilde{\alpha}_i, \tilde{\alpha}_j, \tilde{\alpha}_k$ defined on $\mathbb{R}^3$. 

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By Theorem 5.3 and Theorem 5.4, the following function
\[ F_{ijk}(u_i, u_j, u_k) = \int_{(\alpha, \beta, \gamma)} \alpha_i du_i + \alpha_j du_j + \alpha_k du_k \]
is a well-defined locally concave function of \((u_i, u_j, u_k) \in \Omega_{ij}^E(\tilde{l})\) with \(F_{ijk}(u_i + t, u_j + t, u_k + t) = F_{ijk}(u_i, u_j, u_k) + t\pi\). By Lemma 5.5 and Theorem 4.2, \(F_{ijk}(u_i, u_j, u_k)\) defined on \(\Omega_{ij}^E(\tilde{l})\) could be extended to the following function
\[ \tilde{F}_{ijk}(u_i, u_j, u_k) = \int_{(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})} \tilde{\alpha}_i du_i + \tilde{\alpha}_j du_j + \tilde{\alpha}_k du_k, \]
which is a \(C^1\)-smooth concave function defined on \(\mathbb{R}^3\) with \(\nabla_u \tilde{F}_{ijk} = (\tilde{\alpha}_i, \tilde{\alpha}_j, \tilde{\alpha}_k)^T\). Then the following of the proof for the global rigidity for Euclidean vertex scaling is almost the same as the hyperbolic case. We omit the details here.

**Remark 6.** In the Euclidean case, similar idea to use Luo’s extension theorem 4.2 to extend \(F_{ijk}(u_i, u_j, u_k)\) appears in [7], where the extension depends on the simply connectivity of the admissible space \(\Omega_{ij}^E(\tilde{l})\) and negative semi-definiteness of \(\Lambda_{ij}^E\) obtained by Luo [18]. Here we provide a unified approach to prove the simply connectivity of the admissible space of conformal factors and the negative definiteness of the Jacobian matrix \(\left[ \frac{\partial \alpha}{\partial u_s} \right]_{3 \times 3}\) for a triangle in the Euclidean and hyperbolic cases.

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**References**

[1] E. M. Andreev, *Convex polyhedra in Lobachevsky spaces*. (Russian) Mat. Sb. (N.S.) 81 (123) 1970 445-478.
[2] E. M. Andreev, *Convex polyhedra of finite volume in Lobachevsky space*. (Russian) Mat. Sb. (N.S.) 83 (125) 1970 256-260.
[3] A. Bobenko, U. Pinkall, B. Springborn, *Discrete conformal maps and ideal hyperbolic polyhedra*. Geom. Topol. 19 (2015), no. 4, 2155-2215.
[4] A. Bobenko, B. Springborn, *Variational principles for circle patterns and Koebe’s theorem*, Trans. Amer. Math. Soc. 356 (2004) 659-689.
[5] Bennett Chow, Feng Luo, *Combinatorial Ricci flows on surfaces*, J. Differential Geom. 63 (2003) 97-129.
[6] Y. C. de Verdière, *Un principe variationnel pour les empilements de cercles*, Invent. Math. 104(3) (1991) 655-669.

[7] H. Ge, W. Jiang, *On the deformation of discrete conformal factors on surfaces*. Calc. Var. Partial Differential Equations 55 (2016), no. 6, Art. 136, 14 pp.

[8] D. Glickenstein, *Discrete conformal variations and scalar curvature on piecewise flat two and three dimensional manifolds*, J. Differential Geom. 87 (2011), no. 2, 201-237.

[9] D. Glickenstein, *Geometric triangulations and discrete Laplacians on manifolds*, arXiv:math/0508188 [math.MG].

[10] D. Glickenstein, J. Thomas, *Duality structures and discrete conformal variations of piecewise constant curvature surfaces*, Adv. Math. 320 (2017), 250-278.

[11] X. D. Gu, R. Guo, F. Luo, J. Sun, T. Wu, *A discrete uniformization theorem for polyhedral surfaces II*, J. Differential Geom. 109 (2018), no. 3, 431-466.

[12] X. D. Gu, F. Luo, J. Sun, T. Wu, *A discrete uniformization theorem for polyhedral surfaces*, J. Differential Geom. 109 (2018), no. 2, 223-256.

[13] X. D. Gu, F. Luo, T. Wu, *Convergence of discrete conformal geometry and computation of uniformization maps*. Asian J. Math. 23 (2019), no. 1, 21-34.

[14] X. He, X. Xu, *Thurston’s sphere packing on 3-dimensional manifolds, I*, arXiv:1904.11122v3 [math.GT].

[15] X. He, X. Xu, *Thurston’s sphere packing on 3-dimensional manifolds, II*, In preparation.

[16] P. Koebe, *Kontaktprobleme der konformen Abbildung*. Ber. Sächs. Akad. Wiss. Leipzig, Math. Phys. Kl. 88 (1936), 141-164.

[17] G. Leibon, *Characterizing the Delaunay decompositions of compact hyperbolic surfaces*, Geom. Topol. 6 (2002), 361-391.

[18] F. Luo, *Combinatorial Yamabe flows on surfaces*, Commun. Contemp. Math. 6 (2004), no. 5, 765-780.

[19] F. Luo, *Rigidity of polyhedral surfaces, III*, Geom. Topol. 15 (2011), 2299-2319.

[20] F. Luo, T. Wu, *Koebe conjecture and the Weyl problem for convex surfaces in hyperbolic 3-space*, arXiv:1910.08001v2 [math.GT].

[21] A. Marden, B. Rodin, *On Thurston’s formulation and proof of Andreev’s theorem*. Computational methods and function theory (Valparaíso, 1989), 103-116, Lecture Notes in Math., 1435, Springer, Berlin, 1990.

[22] J. Milnor, *Hyperbolic geometry: The first 150 years*, Bull. Amer. Math. Soc. 6 (1982) 9-24.

[23] J. Sun, T. Wu, X. D. Gu, F. Luo, *Discrete conformal deformation: algorithm and experiments*, SIAM J. Imaging Sci. 8 (2015), no. 3, 1421-1456.

[24] B. Springborn. *Ideal hyperbolic polyhedra and discrete uniformization*. Discrete Comput. Geom. 64 (2020), no. 1, 63-108.

[25] K. Stephenson, *Introduction to circle packing: The theory of discrete analytic functions*, Cambridge Univ. Press (2005)

[26] I. Rivin, *Euclidean structures of simplicial surfaces and hyperbolic volume*. Ann. of Math. 139 (1994), 553-580.
[27] M. Růcek, R. M. Williams, *The quantization of Regge calculus*. Z. Phys. C 21 (1984), no. 4, 371-381.

[28] W. Thurston, *Geometry and topology of 3-manifolds*, Princeton lecture notes 1976, [http://www.msri.org/publications/books/gt3m](http://www.msri.org/publications/books/gt3m).

[29] E.B. Vinberg, *Geometry. II*, Encyclopaedia of Mathematical Sciences, 29, Springer-Verlag, New York, 1988.

[30] T. Wu, *Finiteness of switches in discrete Yamabe flow*, Master Thesis, Tsinghua University, Beijing, 2014.

[31] T. Wu, X. D. Gu, J. Sun, *Rigidity of infinite hexagonal triangulation of the plane*. Trans. Amer. Math. Soc. 367 (2015), no. 9, 6539-6555.

[32] T. Wu, X. Zhu, *The convergence of discrete uniformizations for closed surfaces*, [arXiv:2008.06744v2 [math.GT]].

[33] X. Xu, *On the global rigidity of sphere packings on 3-dimensional manifolds*, J. Differential Geom. 115 (2020), no. 1, 175-193.

[34] X. Xu, *A new proof of Bowers-Stephenson conjecture*, Math. Res. Lett. 28 (2021), no. 4, 1283-1306.

[35] X. Xu, *Parameterized discrete uniformization theorems and curvature flows for polyhedral surfaces, I*, [arXiv:1806.04516 [math.GT]].

[36] X. Xu, C. Zheng, *Prescribing discrete Gaussian curvature on polyhedral surfaces*. Calc. Var. Partial Differential Equations 61 (2022), no. 3, Paper No. 80.

[37] X. Xu, C. Zheng, *Parameterized discrete uniformization theorems and curvature flows for polyhedral surfaces, II*. Trans. Amer. Math. Soc. 375 (2022), no. 4, 2763-2788.

[38] X. Xu, *Combinatorial Ricci flow on cusped 3-manifolds*, [arXiv:2009.05477 [math.GT]].

[39] Z. Zhou, *Circle patterns with obtuse exterior intersection angles*, [arXiv:1703.01757v3 [math.GT]].

[40] X. Zhu, X. Xu, *Combinatorial Calabi flow with surgery on surfaces*, Calc. Var. Partial Differential Equations 58 (2019), no. 6, Paper No. 195, 20 pp.

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