Structural Topography Optimization for Thin-plate Based on Isogeometric Analysis

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Abstract: To satisfy high performance requirements for some structures, it is desirable for structure to be designed smaller, thinner and lighter in automobile and aerospace industries, etc. One prominent example is the application of topology stiffened structures. The stiffness of a structure can be increased obviously by stiffeners under a given amount of material. This paper gives an approach to optimize the structural topography design of thin-walled structures automatically based on isogeometric analysis. Optimization model, sensitivity analysis and optimization process for structural topography design are described and studied in details. The objective function is minimum compliance of structure, and the y-value(vertical coordinate value) of control points as design variables. Then structural topography is formed with changing the y-value of control points in design domain according to the specific algorithm. Several examples illustrate that the proposed method works well.

1. Introduction

Structural optimization is aimed at obtaining high performance of structures through changing the initial size, shape and topology of structure, or material properties within a series of prescribed objectives and constraints[1-5]. In automobile and aerospace etc. the optimization of stiffened plate structure has significant applications, due to stiffeners increasing small weight but the total stiffness improving obviously. However, what the best stiffentopography is and how to get the optimum size, shape and layout of structural topography are important issues.

During the past decades, the optimal design for stiffened structure has received great attention. Through optimizing thickness distribution, CHENG and OLHOFF[6] considered the problem of improving stiffness of solid elastic plates. CHUNG and LEE[7] studied the optimal shape and location of rib by topology optimization based on variable density approach. In this research, many discrete shell elements are first added to ground plate, which densities are determined afterwards by the optimization procedure to minimizing compliance of structure. Finally, the shell elements with higher densities constitute the structural rib. To get the optimized configuration of the ribs, vertical shell elements have to be located densely enough to the existing structure in advance.

LAM and SANTHIKUMAR[2] investigated the optimum of the stiffeners. In this research, element thickness is changed to achieve uniform strain energy density distribution. DING and YAMAZAKI[8; 9] suggested the growing and branching tree model to optimize stiffener patterns of plate structures. Stiffeners in the ground structure start from seeds specified in advance, and extend according to the growing and branching rules. The positions of seeds have critical influence on the final stiffener layout. So, how to preset them is full of challenge.
However, up to now, the structural topography optimization is specifically research about stiffer location based on the traditional finite element method, and rarely to material distribution of structural surface. Presented optimal model by FEM will lead to a large number of design variables to achieve the optimum stiffener shape and layout. Moreover, frequent transform between analysis model and design model is inconvenient. This paper presents a new method to deal with structural topography generation on a plate, which is based on IGA (Isogeometric Analysis). The analysis model is unified with the design model (i.e., geometry model) represented by NURBS and the corresponding control points, which avoids the difficulties in transform between these two models. Because the optimal result is described by NURBS, it can be returned to CAD system directly.

The outline of this paper is as follows: In section 2, the structural topography optimization problem is defined, optimization model and process are described, and analytical sensitivities on linear elasticity problem are formulated. In section 3, several numerical examples of structural topography optimization are presented. Finally, a summary and outlook are given in section 4.

2. Problem definition of Structural topography optimization

The goal of this paper is to generate optimum structural topography on plate automatically to increase the stiffness of plate structure. Assuming that only a prescribed amount of material can be added to the ground plate to generate structural topography. The numerical models of plate are defined using NURBS, in which, two quadratic elements with four control points are employed in the through-thickness direction and biquadratic NURBS basis functions are utilized for constructing the surface.

In the isogeometric method, analysis model and design model are represented by the NURBS, and changed with control points movement. So, this paper chose the coordinate of control points as the design variables. Since the structural topography is usually distributed on the surface of the plate, the design variables are further limited to y-value (vertical coordinate value) of the control points. As shown in Figure. 1, the control points on the lower surface are fixed, while those on the upper surface can move upward, which results in the topography growing vertically on the ground plate. To homogenize the meshes, the middle control points are always distributed evenly in the thickness direction during the optimization process. With change of y-values of the control points, the topography shape on the upper surface is changed automatically. When several adjacent control points move upward together, the stiffened plate is produced as depicted in Figure. 2. The distribution of structural topography is determined automatically by the optimization algorithm.

2.1. Optimization model

In this paper, structural topography, by adding a prescribed amount of material on plate, is optimized to minimize the compliance of the structure. As y-values of the upper control points are assigned to design variables, the optimal model is expressed as:

![Figure 1. Cross-section change in optimization](image)

![Figure 2. Structural topography plate](image)
minimize \[ f = \mathbf{F}^T \mathbf{u} \]
subject to
\[
\begin{align*}
g &= V - V_{\text{limit}} \leq 0 \\
\varphi_i &\leq \varphi_i \leq \bar{\varphi}_i, \quad i = 1, \ldots, n
\end{align*}
\]
where \( \mathbf{F} \) is the force vector, \( \mathbf{u} \) is the displacement vector, \( g \) is the constraint function, \( V \) is the final volume of stiffened plate, \( V_{\text{limit}} \) is the upper limit of the final volume prescribed in advance, \( s = (\varphi_1, \ldots, \varphi_n)^T \) is the vector of design variables, \( \underline{\varphi} \) and \( \bar{\varphi} \) are given lower and upper bounds on the design variables.

2.2. Sensitivity analysis
The gradient-based MMA [10] is used, which requires both structural response and sensitivity information. In MMA, the optimization problem is replaced by a strictly convex approximating subproblem in each step of the iterative process, which is mainly based on the sensitivity information at the current iteration points. Both WALL et al.[11] and CHO et al.[12] have formulated the analytical sensitivities for positions of control points. QIAN[13] has further investigated the full analytical sensitivity method, in which both the positions and weights of NURBS control points are served as design variables in structural shape optimization. Based on previous research work, sensitivity analysis of elasticity problem is derived in this section.

For the objective function stated in equation(1), the derivative can be calculated as
\[
\frac{df}{d\varphi} = \left( \frac{d\mathbf{F}}{d\varphi} \right)^T \mathbf{u} + \mathbf{F}^T \left( \frac{d\mathbf{u}}{d\varphi} \right)
\]
where \( d\mathbf{F}/d\varphi \) can be assembled from the derivatives of the components of the local force vectors in the \( i^{th} \) direction, \( d\mathbf{F}_i/d\varphi \), similarly with the assembling of element force vector \( \mathbf{F} \). To express concisely, we simplify the notation by defining \( d() / d\varphi = ()_{\varphi} \).

\[
\mathbf{F}_{i,\varphi} = \int_{\hat{\Omega}} \left( \mathbf{R}^e_{i,\varphi} \mathbf{f}_i J + \mathbf{R}^e_{i,\varphi} \mathbf{f}_i J_{\varphi} + \mathbf{R}^e_{i} J_{\varphi} \right) \mathrm{d}\hat{\Omega} \\
+ \int_{\hat{\Omega}} \left( \mathbf{R}^e_{i,\varphi} \mathbf{h}_i J + \mathbf{R}^e_{i,\varphi} \mathbf{h}_i J_{\varphi} + \mathbf{R}^e_{i} J_{\varphi} \right) \mathrm{d}\hat{\Omega}
\]

Specifically, for a concentrated load, it can be derived as:
\[
\mathbf{F}_{i,\varphi} = \int_{\hat{\Omega}} \left( \mathbf{R}^e_{i,\varphi} \mathbf{f}_i J + \mathbf{R}^e_{i,\varphi} \mathbf{f}_i J_{\varphi} + \mathbf{R}^e_{i} J_{\varphi} \right) \mathrm{d}\hat{\Omega} \\
+ \mathbf{R}^e_{i,\varphi} J_{\varphi} p_i + \mathbf{R}^e_{i} J_{\varphi} p_i
\]
The derivative \( d\mathbf{u}/d\varphi \) can be achieved according to the following function as
\[
\mathbf{K}_{\varphi}\mathbf{u} + \mathbf{Ku}_{\varphi} = \mathbf{F}_{\varphi}
\]
For further details, see [13]. With all the derivations stated above, the differentiation equation(2) can be solved. For the constraint function given by equation(1), the differentiation can be calculated as
\[
g_{\varphi} = V_{\varphi} = \iiint_{\hat{\Omega}} J_{\varphi} \mathrm{d}\hat{\Omega}
\]
Where

\[ |J|_\varphi = |J| \text{Tr} \left( J^{-1} J_\varphi \right) \]  \hspace{1cm} (7)

and

\[
J_\varphi = \begin{bmatrix}
\frac{\partial x_\varphi}{\partial \xi} & \frac{\partial y_\varphi}{\partial \xi} & \frac{\partial z_\varphi}{\partial \xi} \\
\frac{\partial x_\varphi}{\partial \eta} & \frac{\partial y_\varphi}{\partial \eta} & \frac{\partial z_\varphi}{\partial \eta} \\
\frac{\partial x_\varphi}{\partial \zeta} & \frac{\partial y_\varphi}{\partial \zeta} & \frac{\partial z_\varphi}{\partial \zeta}
\end{bmatrix}
\]  \hspace{1cm} (8)

Because the design variables are the vertical coordinate value of upper control points, so

\[ x_\varphi = y_\varphi = 0 \]
\[ z_\varphi = R_{0,j,k}^{p,q,r} (\xi, \eta, \zeta) \]  \hspace{1cm} (9)

Hence, both the sensitivities of objective and constraint functions required in MMA can be evaluated.

In the process of optimization, the structural topography grows or disappears with the vertical coordinate value of control points changing in prescribed ranges. The optimization process continues until the convergence criterion equation (10) is satisfied.

\[ \varepsilon = \left| \frac{f^{(k)} - f^{(k-1)}}{f^{(0)}} \right| \leq 5 \times 10^{-5} \]  \hspace{1cm} (10)

where \( f^{(k)} \) is the objective function value at the \( k^{\text{th}} \) iteration.

3. Numerical examples

In this section, the feasibility of the proposed method will be demonstrated by three examples. The first example is a cantilever plate, the second is a completely supported square plate and the third is a semi-annular cantilever plate. In three examples, Young’s modulus is \( E = 2.1 \times 10^5 \), Poisson’s ratio \( \nu = 0.3 \).

3.1. Cantilever plate

This example is a cantilever plate with dimensions \( l \times a \times h = 10 \times 7 \times 0.1 \), as shown in Figure 3(a). The initial geometry is constructed from \( 19 \times 10 \times 4 = 760 \) control points and knot vectors are

\[
\Xi = \left\{ 0,0,0,\frac{1}{17},\frac{2}{17},...,\frac{16}{17},1,1,1 \right\} \\
H = \left\{ 0,0,0,\frac{1}{8},\frac{2}{8},...,\frac{7}{8},1,1,1 \right\} \\
Z = \left\{ 0,0,0,0.5,1,1,1 \right\}
\]
The entire upper surface of the plate is chosen as the design domain which includes 19×10=190 control points. Note that in our examples, the optimization variables are the vertical coordinate value of upper control points, and all control weights are set to 1. In this example, the optimization variables are limited in the range of 0.1 ≤ ϕ_i ≤ 0.5.

**Figure 3.** Initial model of cantilever plate and loading conditions

In this example, three loading conditions are considered. In the first situation, a concentrated load $F = 10$ is applied at the midpoint of the free end, depicted in Figure 3(b). The upper limit of the final volume $V_{\text{limit}}$ is restricted to $1.2V_0$, where $V_0$ denotes the original volume. In the second situation, two equal forces $F_1 = F_2 = 5$ are applied at the two sides of the free end respectively, see Figure 3(c), and $V_{\text{limit}}$ is restricted to $1.25V_0$. In the third one, a distributed force $q = 1$ is applied as depicted in Figure 3(d), and $V_{\text{limit}} = 1.5V_0$.

The optimized results for three loading conditions are gotten after 19, 18 and 20 iterations, respectively, as shown in Figure 4(a), 4(b) and 4(c). Convergence processes for both structure compliance and volume for three cases are shown in Figure 5.

**Figure 4.** Optimized designs of cantilever plate under two loading conditions by suggested method
3.2. **Clamped square plate**

This example is a square plate with four edges clamped and a concentrated force applied at the center point of the upper surface, as shown in Figure 6. The ratio of length to thickness of the initial clamped square plate is 100.

![Initial model of clamped square plate and loading condition](image)

**Figure 6.** Initial model of clamped square plate and loading condition

The NURBS solid is defined by the following open knot vectors and the corresponding $13 \times 13 \times 4 = 676$ control points. All control weights are set to 1, and the NURBS solid degenerates to a B-spline solid.

Analogously, the vertical coordinate value of the upper 169 control points are chosen as the optimization variables. The upper limit of the final volume $V_{\text{limit}}$ is restricted to $1.4V_0$ and the allowable vertical positions of the control points are restricted in the range of $0.1 \leq \varphi \leq 0.5$. The optimized solution is gotten after 19 iterations, as shown in Figure 7(a). The convergence processes of the structure compliance and volume are depicted in Figure 8.

$$\Xi = \left\{ 0, 0, 0, \frac{1}{11}, \frac{2}{11}, \ldots, \frac{10}{11}, 1, 1, 1 \right\}$$

$$H = \left\{ 0, 0, 0, \frac{1}{11}, \frac{2}{11}, \ldots, \frac{10}{11}, 1, 1, 1 \right\}$$

$$Z = \{ 0, 0, 0, 0.5, 1, 1, 1 \}$$

The result of topology optimization of square plate with four edges clamped has been given by CHUNG and LEE [7] using variable density method. The topology optimal result is shown as in Figure 7(b), which is also similar to the distribution of structural topography in Figure 7(a).
Although the topology optimization method by reducing material is conversely with the proposed method by adding materials, both of them can obtained the similar layout of material. However, the suggested method has several advantages. Firstly, a distinct distribution of structural topography can be gotten by using the suggested method. Secondly, the computational cost of the suggested method is relatively lower, as the plate does not need to be discretized into a very large number of elements, as in the topology optimization method.

(a) Result from suggested method  
(b) Result from variable density method by CHUNG

Figure 7. Optimized design of completely supported square plate

(a) Convergence process for compliance  
(b) Convergence process for volume

Figure 8. Convergence process for compliance and volume

3.3. Semi-annular cantilever plate

This example is a semi-annular plate with its two ends clamped and a distributed force applied along the outer edge of the upper surface, as illustrated in Figure 9. The initial geometry is build from 22×14×4=1232 control points and the corresponding knot vectors

\[ \Xi = \left\{ 0,0,0, \frac{1}{18}, \frac{2}{18}, \ldots, \frac{9}{18}, \frac{9}{18}, \frac{16}{18}, \frac{17}{18}, 1,1,1 \right\} \]

\[ H = \left\{ 0,0,0, \frac{1}{12}, \frac{2}{12}, \ldots, \frac{11}{12}, 1,1,1 \right\} \]

\[ Z = \{ 0,0,0,0.5,1,1,1 \} \]
As the preceding examples, the vertical coordinates of upper control points are chosen as the design variables. The upper limit of the final volume $V_{\text{limit}}$ equals to $1.2V_0$ and the permissible range of design variables is limited to $0.1 \leq \varphi \leq 0.3$. The optimized result is reached in 19 iterations, as depicted in Figure 10. The convergence of both compliance and volume are shown in Figure 11.

**Figure 9.** Initial boundary condition of semi-annular cantilever

**Figure 10.** Optimized design of semi-annular cantilever plate

**Figure 11.** Convergence process for compliance and volume

4. Conclusions
In this paper, a new structural topography optimization technique based on isogeometric analysis is proposed. Since the analysis model is described by NURBS, this method can generate smooth and continuous structural topography without any post-processing because of modeling based on IGA. The integration of design and analysis model based on IGA eliminates the connecting problem between both analysis model and CAD model. So, the optimized structure can be directly delivered to the CAD system. This method can also be used to improve the dynamic characteristics of structure, such as natural frequency and frequency response function, etc.
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