Abstract

Streamlines of a relativistic perfect isentropic fluid are geodesics of a Riemannian space whose metric is defined by enthalpy of the fluid. This fact simplifies the solution of some problems, as well as is of interest from the point of view of fundamental physics.

1 Introduction

The standard method for studying ideal fluid based on solutions of partial differential equations (Euler equations). However, at least in the case of isentropic fluids there is another approach that is of interest not only because of greater simplicity, but also because it demonstrates that not only the gravity can be interpreted as the curvature of space-time.

This approach is inspired by the existence of an effective numerical solution of problems of hydrodynamics [1, 2], known as Smoothed Particle Hydrodynamics (SPH). In this method, a fluid is considered as composed by finite number of particle. These particles move under the action of inter-particle forces which mimic effects of pressure, viscosity, and so on. Due to the replacement of integration by summation over number of particles, continual derivatives become the time derivative along the particle trajectory, and as a result, the motion of particles governed by ordinary differential equations of classical mechanics.
2 Equations of motion

Consider generally accepted equations of relativistic hydrodynamics in a manifold \( \mathcal{M} \) in which defined the structure of the Minkowski space-time \( E \) with line element \( ds^2 = \eta_{\alpha\beta}(x)dx^\alpha dx^\beta \), where \( \eta_{\alpha\beta}(x) \) is the metric tensor in the used coordinate system. In inertial reference frame these equations follow from the law of stress-energy tensor conservation\[3]\ :

\[
T^\beta_{\alpha;\beta} = 0. \tag{1}
\]

In (1) the tensor \( T^\alpha_{\beta} = w w^\alpha u^\beta - \rho n^\alpha_{\beta} \), a semicolon denotes a covariant derivative in \( E \), \( w = mc^2 + \varepsilon + p \) is the enthalpy per unit volume, \( mc^2 + \varepsilon \) is the rest internal energy, \( m \) is the rest mass of particle, \( p \) is the pressure of the fluid, \( c \) is the speed of light, and \( u^\alpha = du^\alpha/d\sigma \). These equations yield:

\[
u_{\alpha} \frac{\partial(ww^\beta)}{\partial x^\beta} + wu^\beta \frac{\partial u_{\alpha}}{\partial x^\beta} - \frac{\partial p}{\partial x^\alpha} = 0. \tag{2}\]

The expression in brackets can be written as \( (nu^\beta)(w/n) \) where \( n \) is particles number density. Then, taking into account the continuity equation \( (\rho u)_{;\beta} = 0 \), were \( \rho = mn \), eqs. (2) can be written as

\[
u_{\alpha}nu^\beta \frac{\partial(w/n)}{\partial x^\beta} + wu^\beta \frac{\partial u_{\alpha}}{\partial x^\beta} - \frac{\partial p}{\partial x^\alpha} = 0. \tag{3}\]

It follows from these equations that along any world line of a fluid element the following equations hold

\[
u w \frac{du_{\alpha}}{d\sigma} + u_{\alpha}n \frac{d}{d\sigma} \left( \frac{w}{n} \right) - \frac{\partial p}{\partial x^\alpha} = 0, \tag{4}\]

where \( d/d\sigma = u^\beta \partial/\partial x^\beta \), or more shortly as

\[
\frac{d}{d\sigma} \left( \frac{w}{n} u_{\alpha} \right) = \frac{1}{n} \frac{\partial p}{\partial x^\alpha}. \tag{5}\]

By using thermodynamic identity \[3]\n
\[
\frac{1}{n} dP = d \left( \frac{w}{n} \right) - T d \left( \frac{\sigma}{n} \right), \tag{6}\]

where \( \sigma \) is the entropy per unit volume, equation of the motion \[5]\ can be written as

\[
\frac{d}{d\sigma} \left( \frac{w}{n} u_{\alpha} \right) = \frac{\partial}{\partial x^\alpha} \left( \frac{w}{n} \right) - T \frac{\partial}{\partial x^\alpha} \left( \frac{\sigma}{n} \right). \tag{7}\]
This equation contains only enthalpy and entropy per particle.

If we think about fluid as of a finite collection of particles in spirit of [1, 2], we can interpret this equations as describing the motion of “particles”, having parameters of the real gas particles along their world lines.

3 Lagrangian and geometrization

Let us show that in isentropic fluid, where \( \sigma/n = \text{Const.} \), eqs. (7) are the equations of the motion of a particle along a geodesic of a Riemannian space-time with the line element

\[
ds^2 = (G_{\alpha\beta}dx^\alpha dx^\beta),
\]

where the metric tensor \( G_{\alpha\beta} = \kappa^2 \eta_{\alpha\beta} \),

\[
\kappa = \frac{w}{\rho_0 c^2} = 1 + \frac{\varepsilon}{\rho_0 c^2} + \frac{p}{\rho_0 c^2},
\]

\( w \) is the enthalpy per unit volume, \( \rho_0 = mn \), \( n \) is the particle number density, and \( c \) is the speed of light.

The variational principle \( \delta \int ds = 0 \) can be written as \( \delta \int L dt = 0 \), where the Lagrangian \( L \) is given by

\[
L = - mc \left( G_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right)^{1/2} d\lambda,
\]

where \( \lambda \) is a parameter along the world line.

If to set \( \lambda = \sigma \), the Lagrange equations

\[
\frac{d}{d\sigma} \left( \frac{\partial L}{\partial u^\alpha} \right) - \frac{\partial L}{\partial x^\alpha} = 0
\]

yield

\[
\frac{d}{d\sigma} (\kappa u_\alpha) - \frac{\partial \kappa}{\partial x^\alpha} = 0,
\]

were the condition \( \eta_{\alpha\beta}u^\alpha u^\beta = 1 \) in the Minkowski space-time with the signature (+- - - ) has been used.

\footnote{We can consider a fluid as formed by particles with mass \( m \), and \( n \), satisfying the only condition \( mn = \rho_0 \), where \( \rho_0 \) is the real rest fluid density.}
Eqs. (12) are equivalent to (7) if the enthalpy per particle $\sigma/n$ is a constant which takes place in an isentropic fluid. In this case also (7)

$$\frac{\partial}{\partial x^\alpha} \left( \frac{w}{n} \right) = \frac{1}{n} \frac{\partial p}{\partial x^\alpha}.$$ 

For this reason eqs. (12) are equivalent to

$$w \frac{du_\alpha}{d\sigma} + u_\alpha u_\beta \frac{\partial p}{\partial x^\beta} - \frac{\partial p}{\partial x^\alpha} = 0. \tag{13}$$

Since $d/d\sigma = u^\beta \partial/\partial x^\beta$, these equation are in agreement with the following equations for the velocity field

$$wu^\alpha \frac{\partial u_\alpha}{\partial x^\beta} + u_\alpha u_\beta \frac{\partial p}{\partial x^\beta} - \frac{\partial p}{\partial x^\alpha} = 0 \tag{14}$$

which are the general accepted form of the Euler equations. If $\sigma/n \neq \text{Const}$, the Euler equations do not lead to eqs. (12). However, it is well known (3) that in perfect fluid the enthalpy per one particle preserves along its world line, i.e.

$$\frac{d}{ds} \left( \frac{\sigma}{n} \right) = 0. \tag{15}$$

For this reason, the evolution of $u^\alpha$ along the world line giving by eqs. (12) is correct for any perfect fluid.

Let us consider a simple example of the advantage of using Lagrangian (10). If in (10) to set $\lambda = t$, (where $t$ is coordinate time), a stationary gas flow along the axis $x$ is described by the Lagrangian

$$L = mc\kappa(x) \left( c^2 - v^2 \right)^{1/2}, \tag{16}$$

where $v = dx/dt$. From this it is easy to find that differential equation of the motion is

$$\dot{v} - \frac{\varepsilon}{x} \left( 1 - \frac{v^2}{c^2} \right) = 0.$$ 

where a point and prime denote a differentiation with respect to $t$ and $x$, respectively. Since $L$ does not depend on time $t$, the law of the energy $E$ conservation holds, that is $x dL/dx \dot{L} = E$, which yields the relativistic Bernoulli equation [3]:

$$\frac{\varepsilon c^2}{(c^2 - v^2)^{1/2}} = \text{Const.}$$


Thus, instead to find a velocity field with Euler’s PDF equations, we can observe the motion of some separate tiny elements of the fluid, which gives a complete picture of the motion of the fluid.

The differential form ($\mathfrak{M}$) defines in the manifold $\mathcal{M}$ a line element of a Riemannian space-time $V$, so that the Lagrange equations (11) with the parameter $\lambda = s$ give the standard equations of geodesic line in $V$:

$$\frac{du^\alpha}{ds} + \Gamma^\alpha_{\beta \gamma} u^\beta u^\gamma = 0, \quad (17)$$

where

$$\Gamma^\alpha_{\beta \gamma} = \frac{1}{2} G^\alpha_\delta \left( \frac{\partial G_{\delta \beta}}{\partial x^\gamma} + \frac{\partial G_{\delta \gamma}}{\partial x^\beta} - \frac{\partial G_{\beta \gamma}}{\partial x^\delta} \right).$$

For the metric $g_{\alpha \beta} = diag(-1, -1, -1, +1)$ the Christoffel symbols are read:

$$\Gamma^\alpha_{\beta \gamma} = -\frac{1}{\phi} \left( \frac{\partial \varphi}{\partial x^\gamma} \delta^\alpha_\beta + \frac{\partial \varphi}{\partial x^\beta} \delta^\alpha_\gamma - \eta^{\alpha \delta} \frac{\partial \varphi}{\partial x^\delta} \delta^\beta_\gamma \right).$$

The curvature of this space-time is other than zero. For example, for a stationary gas flow along $x-$axis which is described by the Lagrangian (16) the scalar curvature is given by

$$R = 6 \left( \frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} \right),$$

which is other than zero.

The component

$$\Gamma^1_{00} = \frac{1}{\varphi} \frac{\partial \varphi}{\partial x^1}, \quad (18)$$

and in due of eqs. (17), in non-relativistic limit eqs. (17) lead to the equation

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla P. \quad (19)$$

Thus the Lagrangian $L$ describes the motion of the particles both in $E$ and $V$. In the first case $G_{\alpha \beta}$ is some tensor field in $E$, in the second case it is a fundamental tensor of the Riemannian space-time $V$.

Space-time $V$ is real physical space-time because an observer in co-moving reference frame of the fluid can observe deviation of geodesic lines exactly as in gravitational field due the fact that the space-time curvature is other than zero. This means that at least isentropic fluid can be considered not only by conventional manner but also as a manifestation of curvature of space-time with a metric defined by the enthalpy.
4 Fluid in gravitational field

There are two ways to consider an ideal fluid in a gravitational field, from the above point of view.

First, the change of the fluid energy $E$ in in gravitational field is $dE = Td\Sigma - pdV + dE_{gr}$, where $T$ is the temperature, $\Sigma$ is the entropy, $V$ is a volume and the last term is the change of the gravitational potential energy. For this reason the change of the enthalpy $w$ per one particle is given by

$$d\left(\frac{w}{n}\right) = Td\left(\frac{\Sigma}{n}\right) + \frac{1}{n}dp + d\left(\frac{e}{n}\right),$$

where $e$ is the density of gravitational energy.

According to this, at the presence of gravitational field we should set in the Lagrangian (10) $G_{\alpha\beta} = \kappa^2 \eta_{\alpha\beta}$, where

$$\kappa = \frac{w}{nmc^2} = 1 + \frac{e}{\rho_0c^2} + \frac{p}{\rho_0c^2} - \frac{U}{c^2},$$

and $U$ is the gravitational potential. Now in eq. (17) appears an additional term $\nabla U/c^2$. Consequently, in the equilibrium state the ordinary condition $\rho \nabla U = \nabla p$ holds.

In relativistic case it is easy to use more traditional geometrical approach. The presence of gravity can be accounted by considering the line element (1) in a Riemannian rather than pseudo-Euclidean space-time, so that the metric tensor at the presence gravity is given by

$$G_{\alpha\beta} = \kappa^2 g_{\alpha\beta},$$

where $g_{\alpha\beta}(x)$ is the metric tensor of space-time at presence gravity.

That is, each small element of a relativistic perfect isentropic fluid moves along a geodesic of the Riemannian space whose line element is (20).

The equation of the motion of the fluid element are of the form of standard geodesic equations in the Riemannian space-time with the metric tensor $G_{\alpha\beta}$. If we are located very far from the source of gravity, where space-time is, in fact, Minkowskian, it is naturally to use the timing coordinate $t = x^0/c$ as a parameter along the line, and the equations take the form

$$\ddot{x}^\alpha + \left(\Gamma^\alpha_{\beta\gamma} - c^{-1}\Gamma^\alpha_{\beta\gamma}x^0\right)\dot{x}^\beta\dot{x}^\gamma = 0$$

where $\dot{x}^\alpha = \dot{x}^\alpha = dx^\alpha/dt$. Zero component of these equations is satisfied identically, and rest equations are the ones for the 3-spacial velocity.
To verify this, consider for example a fluid in a gravitational field in state of equilibrium in a spherically-symmetric gravitational field.

Since $\dot{x}^\alpha = \ddot{x}^\alpha = dx^\alpha / dt$, the conditions of the equilibrium in spherical coordinates is

$$\Gamma^1_{00} = \frac{1}{2} G (G_{00})' = 0$$

where a prime denotes derivative with respect to the radial distance $r$. It means that $(\kappa^2 g_{00})' = 0$ or

$$\frac{\kappa'}{\kappa} = -\frac{(g_{00})'}{2 g_{00}}.$$

Due to the equality $\kappa' / \kappa = p' / \rho c^2 \kappa$, and taking into account (9), we obtain for the fluid with $\varepsilon = 0$ the standard equation of the equilibrium in General Relativity [5]

$$\frac{(g_{00})'}{g_{00}} = -\frac{2p'}{\rho c^2 + p}. \quad (22)$$

Thus, the motion of fluid particles in pseudo-Euclidean space-time are at the same time some equations of the motion of these particles along geodesic lines of the Riemannian space-time $V$.

This fact allows us to take advantage of knowledge of the Lagrangian, in particular in the case of the existence of symmetries. As an example, by using the symmetry of the Lagrangian, we consider the classification of gas flows at stationary gas accretion onto the compact object like a neutron star or a supermassive compact object at the center of our Galaxy.

5 Example

The knowledge of the Lagrangian allows to obtain a simple classification of gas flow at the spherically-symmetric accretion. Because we observe the motion of separate elements of fluid, this analysis is very like the same problem for study of the motion of test particles in relativistic mechanics [6].

According to the state above in the previous section, we start from the Lagrangian $L = -mc \left( G_{\alpha\beta}(x) \, d\dot{x}^\alpha d\dot{x}^\beta \right)^{1/2}$, where $G_{\alpha\beta} = \kappa^2 g_{\alpha\beta}$, and $g_{\alpha\beta}(x)$ is the classical Schwarzschild solution of Einstein’s gravitation equations, that is

$$L = -mc \kappa \left[ a(r) c^2 - \frac{1}{a(r)} r^2 - r^2 \dot{\varphi}^2 \right]^{1/2}, \quad (23)$$
where \( a(r) = 1 - r_g / r \), and \( r_g = 2\gamma M / c^2 \) is the Schwarzschild radius (\( M \) is the central mass and \( \gamma \) is the gravitational constant).

The equations of the motion of test particles in the plane \( \theta = \pi / 2 \) can be obtained by the law of conservation of the energy \( \mathcal{E} \) and the angular moment \( J \), which follow from the fact that \( L \) does not depend on time \( t \) and \( \phi \):

\[
\dot{r} \frac{\partial L}{\partial \dot{r}} + \dot{\phi} \frac{\partial L}{\partial \dot{\phi}} - L = \mathcal{E} \tag{24}
\]

and

\[
\frac{\partial L}{\partial \dot{\phi}} = J. \tag{25}
\]

It yields the following equations of the motion of the gas element

\[
\dot{r}^2 = a^2 c^2 \left[ 1 - \frac{a \kappa c^2}{\mathcal{E}^2} \left( 1 + \frac{\bar{J}^2}{\kappa r^2} \right) \right] \tag{26}
\]

and

\[
\dot{\phi} = \frac{c^2 a J}{r^2 \mathcal{E}}, \tag{27}
\]

where \( \bar{\mathcal{E}} = \mathcal{E} / \rho_0 \) and \( \bar{J} = J / r_g \rho_0 \). The magnitude \( \mathcal{E} = \mathcal{E}(r, \dot{r}) \) is the total energy, it includes both the “kinetic” and “potential” one. Evidently, the value of \( \Upsilon = \mathcal{E}(r, \dot{r}) \) at \( \dot{r} = 0 \) is an effective potential energy of a gas element. It follows from (26) that

\[
\Upsilon(r) = c^2 \left( 1 - r_g / r \right) \left( \kappa(r) + \frac{\bar{J}^2}{r^2} \right).
\]

The effective potential for particles in vacuum can be obtained from this equation by setting \( \kappa(r) = 1 \).

Fig. 1 and 2 show the effective potentials for a test particle freely falling to a compact object, and for a gas particle, respectively, for the same parameters \( \bar{J} \) and \( \bar{\mathcal{E}} = 1 \). For an illustrative purpose the gas is supposed to be one-atomic with \( \varepsilon / n = 3/2kT \) and \( P = nkT \), and the temperature dependence on \( \bar{r} \) is \( \sim 1 / \bar{r} \), so that we set \( \kappa(\bar{r}) = 0.8 / \bar{r} \).

We see that similarly the case of free particles the flow gas can have finite motion due to the existence of a potential well where \( \Upsilon(\bar{r}) \) has a minimum. Positions of these minimums (\( \bar{r}_{\text{min}} \)) for high-temperature gas are very different from this for test particles in vacuum. Finite gas motion (in particular, accretion disks) play important pole in astrophysics. This method makes it easy to find the relation between the distance of the circular motion of the gas from the central object and physical conditions in the accreting gas.
Figure 1: The Effective potential for a free falling test particle near the attractive point mass for $\mathcal{E} = 1$. The potential has minimum $\bar{r}_{min} = 10.7$ for $\bar{J} = 2.5$ and $\bar{r}_{min} = 4.1$ for $\bar{J} = 1.8$.

Figure 2: The Effective potential for an accretion gas particle near the attractive point mass for $\mathcal{E} = 1$. The potential has minimum $\bar{r}_{min} = 52.7$ for $\bar{J} = 2.5$ and $\bar{r}_{min} = 22.2$ for $\bar{J} = 1.8$.

Figure 3: The obit of a free falling test particle in vacuum from the distance $\bar{r} = 100$.

Figure 4: The orbit of a gas particle free falling to an attractive point mass from the distance $\bar{r} = 100$. 
Figures 3 and 4 show the motion of gas particles at accretion in comparison with free motion of particles in vacuum. It is a good illustration of fate of gas at accretion for the $\vec{J} = 2$. The accreting gas concentrates at the distances where $\Upsilon(\vec{r})$ has minimum. For high-temperature gas such places can be very different from this for free falling particles in vacuum.

6 Open questions

Equations considered here are generalizations of equations of motion of a test particle to case of small elements of isentropic fluid.

The fact that isentropic fluid can be considered as a curvature of space-time generates many questions from the point of view of fundamental physics. The most interesting question is: Is it possible to geometrize fluid without the limitation "for isentropic fluid"? If - "no", then why not?

References

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