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Dimension-free square function estimates for Dunkl operators

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Abstract
Dunkl operators may be regarded as differential-difference operators parameterized by finite reflection groups and multiplicity functions. In this paper, the Littlewood–Paley square function for Dunkl heat flows in \( \mathbb{R}^d \) is introduced by employing the full “gradient” induced by the corresponding carré du champ operator and then the \( L^p \) boundedness is studied for all \( p \in (1, \infty) \). For \( p \in (1, 2] \), we successfully adapt Stein’s heat flows approach to overcome the difficulty caused by the difference part of the Dunkl operator and establish the \( L^p \) boundedness, while for \( p \in [2, \infty) \), we restrict to a particular case when the corresponding Weyl group is isomorphic to \( \mathbb{Z}_2^d \) and apply a probabilistic method to prove the \( L^p \) boundedness. In the latter case, the curvature-dimension inequality for Dunkl operators in the sense of Bakry–Emery, which may be of independent interest, plays a crucial role. The results are dimension-free.

KEYWORDS
curvature-dimension condition, Dunkl heat flow, Dunkl operator, Dunkl process, Littlewood–Paley square function

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1  INTRODUCTION AND MAIN RESULTS

In this section, we first recall some basics on the Dunkl operator initially introduced by Dunkl in [11, 12] and then present the main results of this work. The Dunkl operator has been studied intensively since its introduction. For a general overview, refer to the survey papers [2, 25, 27], as well as the books [8, 13].

Let \( \mathbb{R}^d \) be endowed with the standard inner product \( \langle \cdot, \cdot \rangle \) and the associated Euclidean norm \( |\cdot| \). For \( \alpha \in \mathbb{R}^d \setminus \{0\} \), let \( H_\alpha \) be the hyperplane orthogonal to \( \alpha \), i.e., \( H_\alpha = \{ x \in \mathbb{R}^d : \langle \alpha, x \rangle = 0 \} \), and denote \( r_\alpha \) the reflection with respect to the hyperplane \( H_\alpha \), which is a map from \( \mathbb{R}^d \) to itself such that

\[
r_\alpha x = x - 2 \frac{\langle \alpha, x \rangle}{|\alpha|^2} \alpha, \quad x \in \mathbb{R}^d.
\]

A root system in \( \mathbb{R}^d \) is a finite, nonempty subset of \( \mathbb{R}^d \setminus \{0\} \), denoted by \( \mathcal{R} \), such that for every \( \alpha \in \mathcal{R} \), \( \mathcal{R} \cap \alpha \mathcal{R} = \{ \alpha, -\alpha \} \) and \( r_\alpha (\mathcal{R}) = \mathcal{R} \). Given such a root system \( \mathcal{R} \), denote \( G \) the Weyl group (also called reflection group or Coxeter group) generated by the reflections \( \{ r_\alpha : \alpha \in \mathcal{R} \} \). It is well known that \( G \) is a finite subgroup of the orthogonal group of \( \mathbb{R}^d \).
The Weyl chambers associated to the root system $\mathfrak{R}$ are the connected components of $\{ x \in \mathbb{R}^d : \langle \alpha, x \rangle \neq 0 \text{ for every } \alpha \in \mathfrak{R} \} =: W$. Given $y \in W$, we fix a positive subsystem $\mathfrak{R}_+ := \{ \alpha \in \mathfrak{R} : \langle \alpha, y \rangle > 0 \}$. Then for every $\alpha \in \mathfrak{R}$, either $\alpha \in \mathfrak{R}_+$ or $-\alpha \in \mathfrak{R}_+$. In other words, $\mathfrak{R}$ can be written as the disjoint union of subsystems $\mathfrak{R}_+$ and $-\mathfrak{R}_+$.

Let $\chi : \mathfrak{R} \to \mathbb{C}$ be a $G$-invariant function, i.e., $\chi g \alpha = \chi \alpha$ for every $g \in G$ and every $\alpha \in \mathfrak{R}$. We should mention that, due to the $G$-invariance of $\chi$, the particular choice of $\mathfrak{R}_+$ makes no difference in the definition of Dunkl operators below. So we can fix any subsystem $\mathfrak{R}_+$ from now on.

Without loss of generality, we may normalize the root system such that $|\alpha| = \sqrt{2}$ for every $\alpha \in \mathfrak{R}$. For $\xi \in \mathbb{R}^d$, the Dunkl operator $D_\xi$ along $\xi$ associated to the root system $\mathfrak{R}$ and the function $\chi$ is defined by

$$D_\xi f(x) := \partial_\xi f(x) + \sum_{\alpha \in \mathfrak{R}_+} \kappa_{\alpha}(\alpha, \xi) \frac{f(x) - f(\alpha x)}{\langle \alpha, x \rangle}, \quad f \in C^1(\mathbb{R}^d), \ x \in \mathbb{R}^d,$$

where $\partial_\xi$ denotes the directional derivative along $\xi$.

A remarkable property of Dunkl operators is the commutativity, i.e., for every $\xi, \eta \in \mathbb{R}^d$, $D_\xi D_\eta = D_\eta D_\xi$ (see [12, Theorem 1.9] or [13, Theorem 6.4.8]). Let $\{e_l : l = 1, \ldots, d\}$ be the standard orthonormal basis of $\mathbb{R}^d$. For convenience, we write $D_l = D_{e_l}$, $l = 1, \ldots, d$. We denote $\nabla_\chi = (D_1, \ldots, D_d)$ the Dunkl gradient and $\Delta_\chi = \sum_{l=1}^d D_l^2$ the Dunkl Laplacian. By [12] or [13], we can express specifically that for every $f \in C^2(\mathbb{R}^d)$,

$$\Delta_\chi f(x) = \Delta f(x) + 2 \sum_{\alpha \in \mathfrak{R}_+} \kappa_{\alpha} \left( \frac{\langle \alpha, \nabla f(x) \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\alpha x)}{\langle \alpha, x \rangle^2} \right), \ x \in \mathbb{R}^d,$$

which indicates that $\Delta_\chi$ is a differential-difference operator. It is easy to see that when $\chi = 0$, $D_\xi = \partial_\xi$, and $\nabla_0 = \nabla$ and $\Delta_0 = \Delta$ are the classical gradient operator and the Laplacian, respectively.

A typical example is the rank-one case.

**Example 1.1.** Let $d = 1$. Then the only choice of the root system is $\mathfrak{R} = \{-\sqrt{2}, \sqrt{2}\}$, the corresponding Weyl group is $G = \{ \text{id}, r \}$ with $\text{id}(x) = x$ and $r(x) = -x$ for every $x \in \mathbb{R}$, and the multiplicity function is the constant function. Given a constant $\chi \in \mathbb{C}$, the Dunkl operator $D = D_1$ is expressed as

$$Df(x) = f'(x) + \chi \frac{f(x) - f(-x)}{x}, \quad f \in C^1(\mathbb{R}), \ x \in \mathbb{R},$$

and the Dunkl Laplacian is written as

$$\Delta_\chi f(x) = D^2 f(x) = f''(x) + \frac{\chi}{x^2} \left[ f(-x) - f(x) + 2xf'(x) \right], \quad f \in C^2(\mathbb{R}), \ x \in \mathbb{R}.$$

Another interesting example is the radial Dunkl process. See also [28] and [17, 18] for more details. Here and below, 1 denotes the constant function equal to 1.

**Example 1.2.** Let $C = \{ x \in \mathbb{R}^d : \langle \alpha, x \rangle > 0 \text{ for every } \alpha \in \mathfrak{R}_+ \}$, and let $\overline{C}$ be its closure. The radial Dunkl process is defined as the $\overline{C}$-valued Markov process with continuous path, whose infinitesimal generator is given by

$$\Delta_\chi^W f(x) = \Delta f(x) + 2 \sum_{\alpha \in \mathfrak{R}_+} \kappa_{\alpha} \frac{\langle \alpha, \nabla f(x) \rangle}{\langle \alpha, x \rangle^2}, \ x \in \mathbb{R}^d,$$

where $f$ belongs to $C^4(\overline{C})$ satisfying the boundary condition $\langle \alpha, \nabla f(x) \rangle = 0$ for every $x \in H_\alpha$, $\alpha \in \mathfrak{R}_+$. Note that when the Weyl group $G = S^{d-1}$, the symmetric group, and $\chi = 1$, $\Delta_\chi^W$ is connected with the infinitesimal generator of the $d$-dimensional Dyson Brownian motion. In particular, when $\chi = 0$, $\Delta_\chi^W$ is just the infinitesimal generator of the $d$-dimensional Brownian motion with reflection.
From now on, we assume that $k \geq 0$ and fix it. The natural weight function associated to the Dunkl operator is

$$\prod_{\alpha \in \mathbb{R}^+} |\langle \alpha, x \rangle|^{2\kappa} =: w_k(x), \quad x \in \mathbb{R}^d,$$

which is a homogeneous function of degree $2\gamma$ with $\gamma = \sum_{\alpha \in \mathbb{R}^+} x_\alpha$ and also $G$-invariant. Obviously, $w_0 = 1$. For convenience, set $d\mu_k(x) = w_k(x)dx$. For each $p \in [1, \infty]$, we use $L^p(\mu_k)$ and $\| \cdot \|_{L^p(\mu_k)}$ to denote the classical $L^p$ space $L^p(\mathbb{R}^d, \mu_k)$ and its $L^p$ norm, respectively.

The Dunkl Laplacian $\Delta_k$ is essentially self-adjoint in $L^2(\mu_k)$ (see, e.g., [27, Corollary 2.40]). It generates the Dunkl heat flow $(H_k(t))_{t \geq 0}$ in $L^2(\mu_k)$ as

$$H_k(t)f(x) = e^{t\Delta_k}f(x) = \int_{\mathbb{R}^d} h_k(t, x, y)f(y) \, d\mu_k(y), \quad x \in \mathbb{R}^d, \, t > 0,$$

and $H_k(0)f = f$. Here, $h_k(t, x, y)$ is the Dunkl heat kernel, which is symmetric in $x$ and $y$, smooth in $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, positive, stochastically complete, i.e.,

$$\int_{\mathbb{R}^d} h_k(t, x, y) \, d\mu_k(y) = 1, \quad x \in \mathbb{R}^d, \, t > 0,$$

and satisfies the semigroup identity (see [3, Section 4] for more details on the Dunkl heat kernel and its estimates). It turns out that $(H_k(t))_{t \geq 0}$ is a strongly continuous contraction semigroup in $L^2(\mu_k)$, and $H_k(\cdot) : (0, \infty) \to L^2(\mu_k)$ is the unique continuously differentiable map, with values in the domain $D(\Delta_k)$, such that

$$\begin{cases} \frac{d}{dt} H_k(t)f = \Delta_k H_k(t)f, & \text{for } t \in (0, \infty), \\ \lim_{t \to 0^+} H_k(t)f = f, & \text{in } L^2(\mu_k). \end{cases}$$

Moreover, for $1 \leq p < \infty$, $(H_k(t))_{t \geq 0}$ can be extended uniquely to a strongly continuous contraction semigroup in $L^p(\mu_k)$, for which we keep the same notation (see [25, 27, 28] for more details). Furthermore, from [33, Theorem 1 on page 67], we see that $(H_k(t))_{t \geq 0}$ can be extended to an analytic semigroup in $L^p(\mu_k)$ when $1 < p < \infty$, and we also keep the notation the same. Indeed, by [9, Theorem 2.7], $(H_k(t))_{t \geq 0}$ is a symmetric diffusion semigroup in the sense of [33, p. 65].

As in the classical Laplacian case, we introduce the carré du champ operator $\Gamma$ (see, e.g., [4]): for $f, g \in C^2(\mathbb{R}^d)$,

$$\Gamma(f, g) := \frac{1}{2} \left[ \Delta_k(fg) - f\Delta_k g - g\Delta_k f \right].$$

For convenience, set $\Gamma(f) = \Gamma(f, f)$. By a straightforward calculation, we get that, for every $f, g \in C^2(\mathbb{R}^d)$,

$$\Gamma(f, g)(x) = \langle \nabla f(x), \nabla g(x) \rangle + \sum_{\alpha \in \mathbb{R}^+} x_\alpha \frac{(f(x) - f(\alpha x)) (g(x) - g(\alpha x))}{\langle \alpha, x \rangle^2}, \quad x \in \mathbb{R}^d;$$

see also [17, Lemma 4.4] and [34, Lemma 3.1].

For $f \in C^\infty_c(\mathbb{R}^d)$, define the Littlewood–Paley square function $g_\Gamma(f)$ by

$$g_\Gamma(f)(x) = \left( \int_0^\infty \Gamma(H_k(t)f)(x) \, dt \right)^{1/2}, \quad x \in \mathbb{R}^d.$$

The operator $g_\Gamma$, which is obviously nonlinear, is the major study object of the present work. Let $p \in (1, \infty)$. We say that the operator $g_\Gamma$ is bounded in $L^p(\mu_k)$ if, there exists a positive constant $C_p$ such that

$$\| g_\Gamma(f) \|_{L^p(\mu_k)} \leq C_p \| f \|_{L^p(\mu_k)}, \quad f \in L^p(\mu_k).$$
With these preparations in hand, we can present the main result of the paper in the following theorem. Let $|\mathcal{R}_+|$ be the order of $\mathcal{R}_+$.

**Theorem 1.3.** For $p \in (1, 2]$, the operator $g_\Gamma$ is bounded in $L^p(\mu_\kappa)$. For $p \in [2, \infty)$, if, in addition, the Weyl group $G$ is isomorphic to $\mathbb{Z}_2^d$, then the operator $g_\Gamma$ is bounded in $L^p(\mu_\kappa)$. Moreover, the constant depends only on $p$ and $|\mathcal{R}_+|$ in the former case, whereas the constant depends only on $p$ in the latter case.

Some remarks are in order.

**Remark 1.4.**

(i) Let $f \in C_0^\infty(\mathbb{R}^d)$. Define the square function $g_{\nabla_\kappa}(f)$ as

$$g_{\nabla_\kappa}(f)(x) = \left( \int_0^\infty |\nabla_{H_\kappa}(t)f|^2(x) \, dt \right)^{1/2}, \quad x \in \mathbb{R}^d.$$ 

Then $g_{\nabla_\kappa}$ is bounded in $L^p(\mu_\kappa)$ provided that $g_\Gamma$ is bounded in $L^p(\mu_\kappa)$, since $g_{\nabla_\kappa}(f)$ is controlled pointwise by $g_\Gamma(f)$ due to the fact that

$$|\nabla_\kappa f|^2 \leq (1 + 2\gamma)\Gamma(f). \quad (5)$$

Indeed, for every $\alpha \in \mathcal{R}_+$ and $x \in \mathbb{R}^d$, letting $(\delta_\alpha f)(x) = [f(x) - f(r_\alpha x)] / \langle \alpha, x \rangle$, we have

$$|\nabla_\kappa f|^2(x) = \sum_{j=1}^d \left| \partial_j f(x) + \sum_{\alpha \in \mathcal{R}_+} \kappa_\alpha (\delta_\alpha f)(x) \alpha_j \right|^2$$

$$= \sum_{j=1}^d \left( \partial_j f(x)^2 + 2 \sum_{\alpha \in \mathcal{R}_+} \kappa_\alpha \partial_j f(x)(\delta_\alpha f)(x) \alpha_j + \left( \sum_{\alpha \in \mathcal{R}_+} \kappa_\alpha \alpha_j (\delta_\alpha f)(x) \right)^2 \right)$$

$$\leq |\nabla f|^2 + \sum_{j=1}^d \sum_{\alpha \in \mathcal{R}_+} \left( 2\kappa_\alpha (\delta_j f(x))^2 + \frac{\kappa_\alpha}{2} (\delta_\alpha f)^2(x) |\alpha_j|^2 \right)$$

$$+ 2 \left( \sum_{\alpha \in \mathcal{R}_+} \kappa_\alpha \right) \left( \sum_{\alpha \in \mathcal{R}_+} \kappa_\alpha (\delta_\alpha f)^2(x) \right)$$

$$= (1 + 2\gamma)\Gamma(f)(x).$$

Also, if we define the square function $g_\nabla(f)$ as

$$g_\nabla(f)(x) = \left( \int_0^\infty |\nabla_{H_\kappa}(t)f|^2(x) \, dt \right)^{1/2}, \quad x \in \mathbb{R}^d,$$

then $g_\nabla$ is a bounded operator in $L^p(\mu_\kappa)$ provided that $g_\Gamma$ is bounded in $L^p(\mu_\kappa)$, since $|\nabla f|^2(x) \leq \Gamma(f)(x)$ for every $x \in \mathbb{R}^d$, which obviously follows from (4).

(ii) We do not consider square functions defined by the Dunkl Poisson flow (see, e.g., [28]) $(P_\kappa(t))_{t \geq 0}$, where $P_\kappa(t) := e^{-t\sqrt{-\Delta_\kappa}}, t \geq 0$. The reason is that, if for every $f \in C_0^\infty(\mathbb{R}^d)$, define

$$G_\Gamma(f) = \left( \int_0^\infty t\Gamma(P_\kappa(t)f)(x) \, dt \right)^{1/2}, \quad x \in \mathbb{R}^d,$$
then $G_{\Gamma}(f)(x) \leq g_{\Gamma}(f)(x)$ for every $x \in \mathbb{R}^d$, which deduces in particular that the $L^p$ boundedness of $g_{\Gamma}$ implies the $L^p$ boundedness of $G_{\Gamma}$. Indeed, by applying the formula

$$e^{-t\sqrt{-\Delta_\alpha}} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{\frac{r^2}{4u\Delta_\alpha}} e^{-u\frac{1}{2}} du, \quad t \geq 0,$$

we derive that

$$G_{\Gamma}(f)^2(x) = \int_0^\infty t \Gamma \left( \int_0^\infty e^{\frac{r^2}{4u\Delta_\alpha}} f(\cdot)e^{-u\frac{1}{2}} du \right)(x) dt$$

$$\leq \frac{1}{\sqrt{\pi}} \int_0^\infty t \int_0^\infty \Gamma(e^{\frac{r^2}{4u\Delta_\alpha}} f)(x)e^{-u\frac{1}{2}} du dt$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \left( \int_0^\infty t \Gamma(e^{\frac{r^2}{4u\Delta_\alpha}} f)(x) dt \right)e^{-u\frac{1}{2}} du$$

$$= \frac{2}{\sqrt{\pi}} \left( \int_0^\infty e^{-u\frac{1}{2}} du \right) \left( \int_0^\infty \Gamma(e^{\Delta_\alpha} f)(x) ds \right)$$

$$= g_{\Gamma}(f)^2(x),$$

where we also used Jensen’s inequality (see (4) for the explicit expression of the bilinear operator $\Gamma$), Fubini’s theorem, the change-of-variables formula and the facts

$$\int_0^\infty e^{-u\frac{1}{2}} du = \sqrt{\pi}, \quad \int_0^\infty e^{-u\frac{1}{2}} du = \frac{\sqrt{\pi}}{2};$$

hence, $G_{\Gamma}(f) \leq g_{\Gamma}(f)$.

Moreover, we can define $G_{\nabla_\alpha}$ and $G_{\nabla}$ similar as $g_{\nabla_\alpha}$ and $g_{\nabla}$, by employing the Dunkl Poisson flow instead of the Dunkl heat flow. Then similar as in (i), the $L^p$ boundedness of $g_{\Gamma}$ implies the $L^p$ boundedness of both $G_{\nabla_\alpha}$ and $G_{\nabla}$.

(iii) In this work, we do not consider square functions defined in terms of time derivatives such as

$$g_{j}(f) := \left( \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial t} H_\alpha(t)f \right|^2 t^{2j-1} dt \right)^{1/2}, \quad f \in C_c^\infty(\mathbb{R}^d), \; j = 1, 2, \ldots.$$

The reason is that by the general result [33, Corollary 1 on p. 120], it is well known that for each $j = 1, 2, \ldots$, $g_{j}$ is bounded in $L^p(\mu_\alpha)$ for all $1 < p < \infty$ with the constant independent of $d$, since $(H_\alpha(t))_{t>0}$ is a symmetric diffusion semigroup in the sense of E.M. Stein mentioned above. Clearly, the same result holds if $H_\alpha(t)$ is replaced by $P_\alpha(t)$.

As a corollary of Theorem 1.3, we summarize the results in Remark 1.4 (i) and (ii) in the following corollary.

**Corollary 1.5.** For $p \in (1, 2], g_{\nabla_\alpha}, g_{\nabla}, G_{\Gamma}, G_{\nabla_\alpha}$, and $G_{\nabla}$ are bounded in $L^p(\mu_\alpha)$.

For $p \in [2, \infty)$, if $G$ is isometric to $\mathbb{Z}^d_2$, then $g_{\nabla_\alpha}, g_{\nabla}, G_{\Gamma}, G_{\nabla_\alpha}$ and $G_{\nabla}$ are bounded in $L^p(\mu_\alpha)$.

In each case, the constant is dimension-free.

It is well known that square functions are important in harmonic analysis and probability theory. For the classic theory of Littlewood–Paley square functions and its applications in multiplier theory, Sobolev spaces, and Hardy spaces, refer to [31–33]. Despite extensive studies in various settings in literature, we concentrate on square function estimates in the Dunkl setting. For $p \in (1, 2]$, the $L^p$ boundedness of the square function $G_{\nabla}$ for the Dunkl Poisson flow was obtained in [29] on $\mathbb{R}$ and in [30] on $\mathbb{R}^d$, respectively. By establishing Banach space valued singular integral theory, for all $p \in (1, \infty)$, the $L^p$ boundedness of $G_{\nabla_\alpha}$ for the Dunkl poisson flow on $\mathbb{R}^d$ was obtained in [1]. Recently, for all $p \in (1, \infty)$, the $L^p$ boundedness of $G_{\Gamma}$ for the Dunkl poisson flow on $\mathbb{R}$ was obtained in [24], where the approach is based on a deep result from the theory of Hilbert space valued singular integrals. After submission of the present manuscript, in [14], the authors considered square functions in their full generality and proved the $L^p$ boundedness for all $p \in (1, \infty)$ by using vector
valued Caderón–Zygmund theory. In contrast to the $L^p$ boundedness, in a recent work [21], the first named author proved weak $(1,1)$ boundedness of $g_Γ$ by employing estimates of the Dunkl heat kernel and its derivatives. We should point out that all of the above mentioned results are dimension dependent.

Our approach to prove Theorem 1.3 in the case when $p \in (1,2]$ is motivated by the recent paper [22], which deals with $L^p$ boundedness for square functions in the setting of Dirichlet forms of pure jump type in metric measure spaces. However, for $p \in (1,2]$, in general, it is not possible to show the $L^p$ boundedness of the corresponding $g_Γ$ for Dirichlet forms of pure jump type; see [6, Example 2] for a counterexample constructed by the $\alpha$-stable process with $\alpha = 1/2$. In contrast to that the Dunkl setting provides an interesting example of non-local nature such that $g_Γ$ is $L^p$-bounded for all $p \in (1,2]$. For $p \in [2,\infty)$, in general, although the Dunkl operator can be regarded as a non-local operator, it seems that we are not able to prove the $L^p$ boundedness of $g_Γ$ by applying the methods in [6, 22]. Instead, we restrict to the setting when the Weyl group $G$ is isomorphic to $\mathbb{Z}^d_2$. In this particular case, we can deal with the Dunkl process as a diffusion process to some extent.

We should mention that, to prove our results, we do not employ direct estimates of Gauss type for the Dunkl heat kernel and estimates on its time and space derivatives (see, e.g., [14, Theorem 2.2]) and the doubling property. We emphasize that our results are dimension-free.

Potential applications of the results on Dunkl multipliers and weighted $L^p$ boundedness of the above square functions are interesting subjects to be considered in future studies.

The remainder of the paper is organized as follows. The next two sections contain proofs of Theorem 1.3. Section 2 serves to prove the case when $p \in (1,2]$, and Section 3 deals with the case when $p \in [2,\infty)$ and the Weyl group $G$ is isomorphic to $\mathbb{Z}^d_2$, where the curvature-dimension condition (see Proposition 3.2) is employed. In Section 4, details on the proof of Proposition 3.2 are presented. The size of constants appeared in proofs are trackable. For a function $f$, let $f^+ := \max\{f, 0\}$ and $f^- := (-f)^+$.

## 2 $L^p$ BOUNDEDNESS FOR $p \in (1,2]$

In this section, we establish $L^p$ boundedness for $g_Γ$ in $L^p(\mu_\kappa)$ for all $1 < p \leq 2$. We should mention that the idea of proof below is motivated by [22, Section 2] (see also [10] for the graph case), which may be regarded as a development of Stein’s method in [33] for non-local operators.

Let $p \in (1,2]$. We introduce the pseudo-gradient $G_p$ as follows:

$$G_p(f) := \frac{1}{p} \left[ f^{2-p} \Delta_\kappa(f^p) - p f \Delta_\kappa(f) \right],$$

for some suitable nonnegative function $f$ defined on $\mathbb{R}^d$.

The next lemma provides an explicit expression for $G_p(f)$; (see also [17, Lemmas 4.3 and 4.4] for potentially related results). Let $0^0 := 1$.

**Lemma 2.1.** For $p \in (1,2]$, $0 \leq f \in C^\infty_c(\mathbb{R}^d)$, and $x \in \mathbb{R}^d$, there holds that

$$G_p(f)(x) = (p - 1)|\nabla f|^2(x) + 2(p - 1) \sum_{\alpha \in \mathbb{R}^+} \frac{\kappa_\alpha}{\langle \alpha, x \rangle^2} \mathbf{1}_{\{y \in \mathbb{R}^d : f(y) \neq f(r_\alpha y)\}}(x) \times$$

$$[f(r_\alpha x) - f(x)]^2 \int_0^1 \frac{f^{2-p}(x)(1 - s)}{[(1 - s)f(x) + sf(r_\alpha x)]^{2-p}} \, ds.$$

**Proof.** By direct calculations, we get

$$pG_p(f)(x) = [f^{2-p}(x)\Delta_\kappa(f^p)(x) - pf(x)\Delta_\kappa(f)(x)]$$

$$= p(p - 1)|\nabla f|^2(x)$$

$$+ 2 \sum_{\alpha \in \mathbb{R}^+} \frac{\kappa_\alpha}{\langle \alpha, x \rangle^2} f^{2-p}(x) \left[ f^p(r_\alpha x) - f^p(x) \right] - p f^{p-1}(x) \left[ f(r_\alpha x) - f(x) \right].$$
By Taylor’s expression of the function \( t \mapsto t^p \) at the point \( s \), and then by the change-of-variables formula, we have

\[
t^p - s^p - ps^{p-1}(t - s) = p(p - 1) \int_s^t u^{p-2}(t - u) \, du
\]

\[
= p(p - 1)(t - s)^2 \int_0^1 \frac{1 - u}{[(1 - u)s + ut]^{2-p}} \, dv,
\]

for \( s, t \geq 0 \) with \( s \neq t \). Thus, if \( f(x) \neq f(r \alpha x) \), then letting \( s = f(x) \) and \( t = f(r \alpha x) \), we finish the proof.

From Lemma 2.1, we derive the following result which implies that \( \Gamma(f) \) and \( G_p(f) \) are comparable in some pointwise sense.

**Lemma 2.2.** For \( p \in (1, 2] \), \( 0 \leq f \in C_0^\infty(\mathbb{R}^d) \), there holds that

\[
\Gamma(f)(x) \geq \frac{1}{p - 1} G_p(f)(x) \geq 0, \quad x \in \mathbb{R}^d,
\]

and

\[
\Gamma(f)(x) \leq \frac{1}{p - 1} \left[ G_p(f)(x) + \sum_{\alpha \in \mathbb{R}^d} G_p(f)(r \alpha x) \right], \quad x \in \mathbb{R}^d.
\]

**Proof.** (1) We prove (6). If \( f(x) > f(r \alpha x) \geq 0 \), then \( (1 - s)f(x) + sf(r \alpha x) \geq f(r \alpha x), s \in [0, 1] \), and hence

\[
\int_0^1 \frac{1 - s}{[(1 - s)f(x) + sf(r \alpha x)]^{2-p}} \, ds \leq f(r \alpha x)^{p-2} \int_0^1 (1 - s) \, ds = \frac{1}{2} f(r \alpha x)^{p-2}.
\]

If \( f(r \alpha x) > f(x) \geq 0 \), then \( (1 - s)f(x) + sf(r \alpha x) \geq f(x), s \in [0, 1] \), and hence

\[
\int_0^1 \frac{1 - s}{[(1 - s)f(x) + sf(r \alpha x)]^{2-p}} \, ds \leq f(x)^{p-2} \int_0^1 (1 - s) \, ds = \frac{1}{2} f(x)^{p-2}.
\]

Thus, together with Lemma 2.1, we derive that

\[
G_p(f)(x) \leq (p - 1) |\nabla f|^2(x) + (p - 1) \sum_{\alpha \in \mathbb{R}^d} \frac{\kappa_\alpha}{\langle \alpha, x \rangle^2} f^{2-p}(x)(f(x) - f(r \alpha x))^2 (f(x) \vee f(r \alpha x))^{p-2}
\]

\[
\leq (p - 1) \Gamma(f)(x),
\]

where \( a \vee b := \max\{a, b\}, a, b \in \mathbb{R} \). It is immediate to see that \( G_p(f) \geq 0 \) from Lemma 2.1. Thus, (6) is proved.

(2) Now, we prove (7). Let

\[
I = \sum_{\alpha \in \mathbb{R}^d} \kappa_\alpha \frac{(f(x) - f(r \alpha x))^2}{\langle \alpha, x \rangle^2} 1_{\{y \in \mathbb{R}^d : f(r \alpha y) < f(y)\}}(x) \quad \text{and}
\]

\[
II = \sum_{\alpha \in \mathbb{R}^d} \kappa_\alpha \frac{(f(x) - f(r \alpha x))^2}{\langle \alpha, x \rangle^2} 1_{\{y \in \mathbb{R}^d : f(r \alpha y) > f(y)\}}(x).
\]

For \( f(r \alpha x) < f(x), (1 - s)f(x) + sf(r \alpha x) \leq f(x), s \in [0, 1] \). Then

\[
\int_0^1 \frac{f^{2-p}(x)(1 - s)}{[(1 - s)f(x) + sf(r \alpha x)]^{2-p}} \, ds \geq \int_0^1 (1 - s) \, ds = \frac{1}{2}.
\]
Hence, by Lemma 2.1,
\begin{equation}
I \leq 2 \sum_{\alpha \in \mathbb{R}_+} \kappa_{\alpha} \frac{(f(x) - f(r_\alpha x))^2}{\langle \alpha, x \rangle^2} \int_0^1 \frac{f^{2-p}(x)(1-s)}{[(1-s)f(x) + sf(r_\alpha x)]^{2-p}} ds
\end{equation}
\begin{equation}
\leq \frac{1}{p-1} \left[ G_p(f)(x) - (p-1)|\nabla f|^2(x) \right].
\end{equation}

For $f(r_\alpha x) > f(x)$, $(1-s)f(x) + sf(r_\alpha x) \leq f(r_\alpha x), s \in [0, 1]$. Then
\begin{equation}
\int_0^1 \frac{f^{2-p}(r_\alpha x)(1-s)}{[(1-s)f(r_\alpha x) + sf(x)]^{2-p}} ds \geq \int_0^1 (1-s) ds = \frac{1}{2}.
\end{equation}

Hence, since $\langle \alpha, r_\alpha x \rangle = -\langle \alpha, x \rangle$ for every $\alpha \in \mathbb{R}_+$, by Lemma 2.1 again, we have
\begin{equation}
II \leq 2 \sum_{\alpha \in \mathbb{R}_+} \kappa_{\alpha} \frac{(f(r_\alpha x) - f(x))^2}{\langle \alpha, r_\alpha x \rangle^2} \int_0^1 \frac{f^{2-p}(r_\alpha x)(1-s)}{[(1-s)f(r_\alpha x) + sf(x)]^{2-p}} ds
\end{equation}
\begin{equation}
\leq \frac{1}{p-1} \sum_{\alpha \in \mathbb{R}_+} \left[ G_p(f)(r_\alpha x) - (p-1)|\nabla f|^2(r_\alpha x) \right].
\end{equation}

Thus, combining (8) and (9), we finally arrive at
\begin{equation}
\Gamma(f)(x) = |\nabla f|^2(x) + I + II
\end{equation}
\begin{equation}
\leq \frac{1}{p-1} \left[ G_p(f)(x) + \sum_{\alpha \in \mathbb{R}_+} G_p(f)(r_\alpha x) \right],
\end{equation}
which is (7). \qed

Recall that $(H_\nu(t))_{t \geq 0}$ is the Dunkl heat flow. For every $0 \leq f \in C^\infty_c(\mathbb{R}^d)$, define the square function $g_p(f)$ by
\begin{equation}
g_p(f)(x) = \left( \int_0^\infty G_p(H_\nu(t)f)(x) dt \right)^{1/2}, \quad x \in \mathbb{R}^d.
\end{equation}

The next result is on the $L^p$ boundedness of the operator $g_p$.

**Proposition 2.3.** Let $p \in (1, 2]$. Then there exists a constant $c_p \in (0, \infty)$, depending only on $p$, such that for all $0 \leq f \in C^\infty_c(\mathbb{R}^d)$,
\begin{equation}
\|g_p(f)\|_{L^p(\mu_\nu)} \leq c_p \|f\|_{L^p(\mu_\nu)},
\end{equation}
and, moreover,
\begin{equation}
\left\| \sqrt{G_p(H_\nu(t)f)} \right\|_{L^p(\mu_\nu)} \leq \frac{c_p}{\sqrt{t}} \|f\|_{L^p(\mu_\nu)}, \quad t > 0.
\end{equation}

**Proof.** Let $p \in (1, 2]$. Assume that $0 \leq f \in C^\infty_c(\mathbb{R}^d)$ and $f$ is not identical to the zero function. Then $H_\nu(t)f \in C^\infty(\mathbb{R}^d) \cap D(\Delta_\nu) \cap L^p(\mu_\nu)$ and $H_\nu(t)f > 0$ for all $t > 0$. For notational simplicity, we set
\begin{equation}
v_1(x) = v(t, x) = H_\nu(t)f(x).
\end{equation}
Then
\[ p u_t^{p-2} G_p(v_t) = \Delta_x(u^p_t) - p u_t^{p-1} \Delta_x v_t \]
\[ = p u_t^{p-1}(\partial_t - \Delta_x)u_t - p u_t^{p-1} \partial_t u_t + \Delta_x(u^p_t) \]
\[ = (\Delta_x - \partial_t)u_t^p =: L_t, \]
where we used the fact that \((\partial_t - \Delta_x)u_t = 0\) in the last equality. Then
\[ G_p(v_t) = \frac{1}{p} v^{2-p}_t L_t. \]  
(12)

(1) We prove (10) first.
\[ g_p(f)^2(x) = \int_0^\infty G_p(v_t)(x) \, dt \]
\[ = \frac{1}{p} \int_0^\infty v^{2-p}(t,x)(\Delta_x - \partial_t)u^p(t,x) \, dt \]
\[ \leq \frac{1}{p} \left( \sup_{t>0} v^{2-p}(t,x) \right) L(x), \]
where we have let \(L(x) = \int_0^\infty (\Delta_x - \partial_t)u^p(t,x) \, dt\), and we see that \(L(x) \geq 0\) since \(G_p(v_t)(x) \geq 0\) by Lemma 2.2. Thus
\[ \int_{\mathbb{R}^d} g_p^p(x) \, d\mu_x \leq \frac{1}{p} \int_{\mathbb{R}^d} \left( \sup_{t>0} v(t,x) \right)^{(2-p)p/2} L(x)^{p/2} \, d\mu_x(x) \]
\[ \leq \frac{1}{p} \left( \int_{\mathbb{R}^d} \left( \sup_{t>0} v(t,x) \right)^p \, d\mu_x(x) \right)^{(2-p)/2} \left( \int_{\mathbb{R}^d} L(x) \, d\mu_x(x) \right)^{p/2} \]
\[ \leq C_p \| f \|_{L^p(\mu_x)}^{(2-p)/2} \left( \int_{\mathbb{R}^d} L(x) \, d\mu_x(x) \right)^{p/2}, \]  
(13)
where we used the fact that \(\| \sup_{t>0} v(t,x) \|_{L^p(\mu_x)} \leq C_p \| f \|_{L^p(\mu_x)}\) for some positive constant \(C_p\), depending only on \(p\) (see, e.g., [33, page 73]).

For every \(t > 0\), we claim that
\[ \int_{\mathbb{R}^d} \Delta_x u_t^p \, d\mu_x \leq 0. \]  
(14)

The proof of (14) is motivated by [22] in the case of nonlocal Dirichlet forms. We may fix \(t > 0\) for the moment. By the analyticity of \(H_x(t)\), we have
\[ \| \Delta_x u_t \|_{L^p(\mu_x)} = \| \Delta_x H_x(t)f \|_{L^p(\mu_x)} \leq \frac{a_p}{t} \| f \|_{L^p(\mu_x)}, \]  
(15)
for some positive constant \(a_p\), depending only on \(p\). Then, by Hölder’s inequality, contraction property of the Dunkl heat flow in \(L^p(\mu_x)\) and (15),
\[ \int_{\mathbb{R}^d} |\partial_t u_t^p| \, d\mu_x = p \int_{\mathbb{R}^d} |v_t|^{p-1} |\Delta_x u_t| \, d\mu_x \]
\[ \leq p \| v_t \|_{L^{p/(p-1)}(\mu_x)} \| \Delta_x u_t \|_{L^p(\mu_x)} \]
\[ \leq \frac{p a_p}{t} \| f \|_{L^p(\mu_x)}, \]
which implies that \(\partial_t u_t^p \in L^1(\mu_x)\). By (12), we get \(L_t \geq 0\). Since
\[ \Delta_x u_t^p = L_t + \partial_t u_t^p, \]
we have
\[ \int_{\mathbb{R}^d} (\Delta v^p_t)^- \, d\mu_\kappa < \infty. \]

Let \( \psi : \mathbb{R} \to [0, 1] \) be an infinitely differentiable function such that \( \text{supp}(\psi) \subset (-\infty, 3) \), \( \psi(t) = 1 \) for every \( t \leq 1 \), and \( |\psi'| + |\psi''| \leq C \) for some positive constant \( C \). Given a sequence \( (r_n)_{n \geq 1} \) such that \( r_n \to \infty \) as \( n \) tends to \( \infty \), define the sequence of cut-off functions \( (\phi_n)_{n \geq 1} \) by
\[
\phi_n(x) := \psi \left( \frac{|x|}{r_n} \right), \quad x \in \mathbb{R}^d, \ n \geq 1.
\]

It is easy to see that \( |\Delta \phi_n| \leq C_d/r_n^2 \) for some positive constant \( C_d \) depending only on \( d \) and \( C \). Note that \( \phi_n \) is radial for each \( n \). Then by the generalized Fatou’s Lemma (see [35, Theorem 3.3.6 (2)]) and the integration-by-parts formula (see, e.g., [25, Proposition 2.1]), we deduce that
\[
\int_{\mathbb{R}^d} \Delta v^p_t \, d\mu_\kappa \leq \liminf_{n \to \infty} \int_{\mathbb{R}^d} \phi_n \Delta v^p_t \, d\mu_\kappa
\]
\[ = \liminf_{n \to \infty} \int_{\mathbb{R}^d} v^p_t \Delta \phi_n \, d\mu_\kappa
\]
\[ \leq \liminf_{n \to \infty} \frac{C_d}{r_n^2} \int_{\mathbb{R}^d} v^p_t \, d\mu_\kappa = 0. \]

Thus, the claim is proved.

By (14), we have
\[
\int_{\mathbb{R}^d} L(x) \, d\mu_\kappa(x) = \int_0^\infty \int_{\mathbb{R}^d} \Delta v^p_t(x) \, d\mu_\kappa(x) \, dt - \int_{\mathbb{R}^d} \int_0^\infty \partial_t v^p_t(x) \, dt \, d\mu_\kappa(x)
\]
\[ \leq \int_{\mathbb{R}^d} f^p(x) \, d\mu_\kappa(x)
\]
\[ = \|f\|_{L^p(\mu_\kappa)}^p. \quad (16)
\]

Combining (13) and (16), we obtain
\[
\int_{\mathbb{R}^d} g_p(f) \, d\mu_\kappa \leq C_p \|f\|_{L^p(\mu_\kappa)}^{(2-p)/2} \|f\|_{L^p(\mu_\kappa)}^{p/2}
\]
\[ = C_p \|f\|_{L^p(\mu_\kappa)}^p.
\]

Thus, we complete the proof of (10).

(2) Now, we prove (11). The argument is similar as above. By (14) and Hölder’s inequality,
\[
\int_{\mathbb{R}^d} L_t \, d\mu_\kappa \leq - \int_{\mathbb{R}^d} v_t^{p-1} \partial_t v_t \, d\mu_\kappa
\]
\[ \leq \|v_t\|_{L^p(\mu_\kappa)}^{p-1} \|\partial_t v_t\|_{L^p(\mu_\kappa)}
\]
\[ \leq \|f\|_{L^p(\mu_\kappa)}^{p-1} \frac{a_P}{t} \|f\|_{L^p(\mu_\kappa)}
\]
\[ = \frac{a_P}{t} \|f\|_{L^p(\mu_\kappa)}^p,
\]

where we used the analyticity of $H_k(t)$ (see (15) above) in the last inequality. Hence, combining this together with (12) and applying Hölder’s inequality, we obtain that

$$
\int_{\mathbb{R}^d} G_p(v_t)^{p/2} \, d\mu_x = \frac{1}{p} \int_{\mathbb{R}^d} v_t^{p(2-p)/2} L_t^{p/2} \, d\mu_x \\
\leq \frac{1}{p} \left( \int_{\mathbb{R}^d} v_t^p \, d\mu_x \right)^{(2-p)/2} \left( \int_{\mathbb{R}^d} L_t \, d\mu_x \right)^{p/2} \\
\leq \frac{1}{p} \left( \frac{a_p}{t} \right)^{p/2} \|f\|_{L_p(\mu_x)}^p.
$$

Thus, we complete the proof of (11).

The main result in this section is presented in the next theorem. Recall that $|\mathcal{R}_+|$ is the order of $\mathcal{R}_+$.

**Theorem 2.4.** Let $p \in (1, 2]$. Then the operator $g_\Gamma$ is bounded in $L^p(\mu_x)$; more precisely, there exists a positive constant $C_p$, depending only on $p$ and $|\mathcal{R}_+|$, such that for all $f \in L^p(\mu_x),$

$$
\|g_\Gamma(f)\|_{L^p(\mu_x)} \leq C_p \|f\|_{L^p(\mu_x)},
$$

and, moreover,

$$
\left\| \sqrt{\Gamma(H_k(t)f)} \right\|_{L^p(\mu_x)} \leq \frac{C_p}{\sqrt{t}} \|f\|_{L^p(\mu_x)}, \quad t > 0.
$$

**Proof.** By standard approximation, it suffices to prove the case when $f \in C_c^\infty(\mathbb{R}^d)$. Assume $f \in C_c^\infty(\mathbb{R}^d)$. It is easy to see that

$$
\Gamma(H_k(t)f)(x) = \Gamma(H_k(t)(f^+ - f^-))(x) \\
\leq 2[\Gamma(H_k(t)f^+)(x) + \Gamma(H_k(t)f^-)(x)].
$$

Then it is enough to assume $f \geq 0$ in addition. By (7) in Lemma 2.2, we have, for every $\alpha \in \mathcal{R}_+$,

$$
g_\Gamma(f)^2(x) = \int_0^{\infty} \Gamma(H_k(t)f)(x) \, dt \\
\leq c_p \int_0^{\infty} \left[ G_p(H_k(t)f)(x) + \sum_{\alpha \in \mathcal{R}_+} G_p(H_k(t)f)(r_\alpha x) \right] \, dt \\
= c_p \left[ g_p(f)^2(x) + \sum_{\alpha \in \mathcal{R}_+} g_p(f)^2(r_\alpha x) \right],
$$

where $c_p = 1/(p - 1)$ is from Lemma 2.2. Applying (10) in Proposition 2.3, we deduce that

$$
\int_{\mathbb{R}^d} g_\Gamma(f)^p \, d\mu_x \leq \tilde{c}_p \int_{\mathbb{R}^d} \left[ g_p(f)^p(x) + \sum_{\alpha \in \mathcal{R}_+} g_p(f)^p(r_\alpha x) \right] \, d\mu_x(x) \\
= \tilde{c}_p (1 + |\mathcal{R}_+|) \int_{\mathbb{R}^d} g_p(f)^p(x) \, d\mu_x(x) \\
\leq C_p \int_{\mathbb{R}^d} f^p \, d\mu_x,
$$
for some positive constant $C_p$ depending only on $p$ and $|\mathcal{R}_+|$, where the equality is due to that $r_\alpha$ is a reflection and $\mu_\xi$ is $G$-invariant. We complete the proof of (17).

Similarly, by (7) again and (11), we complete the proof of (18). \qed 

3 \  L^p BOUNDEDNESS FOR $p \in [2, \infty)$

In this section, we prove the $L^p$ boundedness for the operator $g_\Gamma$ for all $p \in [2, \infty)$ in the particular case when the Weyl group $G$ is isomorphic to $\mathbb{Z}^d_2 = \{0,1\}^d$. We employ the probabilistic approach which was initially introduced in [7] for Brownian motions and was recently adapted successfully to deal with diffusion processes on RCD$(K,N)$ spaces (see [20] for the case when $K = 0$ and $1 \leq N < \infty$ and [19] for the case when $K \in \mathbb{R}$ and $N = \infty$, as well as for more details on RCD spaces).

The natural stochastic process generated by the Dunkl Laplacian is the so-called Dunkl process, which was studied earlier in [15, 16, 26, 28] for instance. Let $X = (X_t)_{t \geq 0}$ be the Dunkl process with infinitesimal generator $(\Delta_\xi, D(\Delta_\xi))$ in $\mathbb{R}^d$. For each $\alpha \in \mathbb{R}_+$, recall that $H_\alpha$ is the hyperplane orthogonal to $\alpha$. For every subset $I$ of $\mathbb{R}_+$, let

$$U_I := \{\alpha \in \mathbb{R}_+ : \langle \alpha, x \rangle = 0, x \in \bigcap_{\alpha \in I} H_\alpha \}.$$ 

It is known that $X$ is a càdlàg Markov process of jump type with jumping kernel (see [16, Proposition 3.1])

$$J(x, dy) = \begin{cases} 
\sum_{\alpha \in \mathbb{R}_+} \frac{2\nu_\alpha}{\langle \alpha, x \rangle^2} \delta_{r_\alpha x}(dy), & x \in \mathbb{R}^d \setminus (\bigcup_{\alpha \in \mathbb{R}_+} H_\alpha), \\
\sum_{\alpha \in \mathbb{R}_+ \setminus U_I} \frac{2\nu_\alpha}{\langle \alpha, x \rangle^2} \delta_{r_\alpha x}(dy), & x \in \bigcap_{\alpha \notin I} H_\alpha, \\
0, & x = 0,
\end{cases}$$

where $I$ is any subset of $\mathbb{R}_+$, $\delta_z$ denotes the Dirac measure at the point $z \in \mathbb{R}^d$. Due to our purpose, we may assume that the process $X$ does not start from 0 in what follows.

We should mention that although the Dunkl process $X$ is a jump process, the approaches developed mainly for pure jump Lévy processes in recent papers [6, 22] (see also [23] for the non-local Schrödinger case) seem not applicable directly. However, the Dunkl heat flow in the special situation when the Weyl group $G$ is isomorphic to $\mathbb{Z}_2^d$ seems more well-behaved as the diffusion one. Due to this, we may apply the method used in [19].

There are essentially no new ideas in the following arguments. The novelty here maybe is that we can calculate more explicitly in the present Dunkl setting than [22, Section 3] in the general setting of pure jump Dirichlet forms.

Now, fix $f \in C^\infty_c(\mathbb{R}^d)$ and $T > 0$. Let

$$N_t := H_\xi(T - t)f(X_t) - H_\xi(T)f(X_0), \quad t \in [0,T].$$

Let $(B_t)_{t \geq 0}$ be the Brownian motion in $\mathbb{R}^d$ with infinitesimal generator $\Delta$, and denote by $(\mathcal{F}_t)_{t \geq 0}$ the natural filtration of the process $X$.

**Lemma 3.1.** $(N_t, \mathcal{F}_t)_{t \in [0,T]}$ is a martingale starting from 0, and for any $t \in [0,T]$,

$$\langle N \rangle_t = 2 \int_0^t |V H_\xi(T - s)f|^2 ds + 2 \sum_{\alpha \in \mathbb{R}_+} \int_0^t \frac{(H_\xi(T - s)f(X_{s-}) - H_\xi(T - s)f(r_\alpha X_{s-}))^2}{\langle \alpha, X_{s-} \rangle^2} ds,$$

where $\langle N \rangle_t$ is the predictable quadratic variation of $N_t$ and $X_{t-} := \lim_{s \downarrow t} X_s$.

**Proof.** By Itô’s formula (see, e.g., [16, Corollary 3.6]), we have

$$N_t = \int_0^t \langle \nabla H_\xi(T - s)f(X_s), dB_s \rangle + \sum_{\alpha \in \mathbb{R}_+} \int_0^t \sqrt{\nu_\alpha} \frac{H_\xi(T - s)f(X_{s-}) - H_\xi(T - s)f(r_\alpha X_{s-})}{\langle \alpha, X_{s-} \rangle} dM^\xi_s,$$
where \((M_t^2)_{t \geq 0}\) is a one-dimensional martingale with discontinuous paths. Hence \((\mathcal{N}_t)_{t \in [0,T]}\) is a martingale. From [16, Theorem 1], we have \(\langle M^2 \rangle_t = 2t\). Thus, we immediately get (19). □

For \(f, g \in C^4(\mathbb{R}^d)\), we define as in the classical Laplacian case (see, e.g., [4]) that
\[
\Gamma_2(f, g) := \frac{1}{2} [\Delta_\xi \Gamma(f, g) - \Gamma(\Delta_\xi f, g) - \Gamma(f, \Delta_\xi g)].
\] (20)

For convenience, we set \(\Gamma_2(f) = \Gamma_2(f, f)\).

The following result is the key to apply the approach in [19] mentioned above, and it may be of independent interest. Since the proof is a little bit long, we present the details in the next section.

**Proposition 3.2.** Let \(G\) be isomorphic to \(\mathbb{Z}^d_2\). Then for every \(f \in C^4(\mathbb{R}^d)\),
\[
\Gamma_2(f) \geq \|\text{Hess}(f)\|_{\text{HS}}^2.
\]
where \(\text{Hess}(f)\) is the Hessian of \(f\) and \(\| \cdot \|_{\text{HS}}\) is the Hilbert–Schmidt norm.

**Remark 3.3.** It should be regarded as that in the particular setting of Proposition 3.2, the Dunkl Laplacian \(\Delta_\xi\) satisfies the curvature-dimension condition \(\text{CD}(0, \infty)\) in the sense of Bakry–Emery [4].

Applying Proposition 3.2, we immediately derive (21) below, which may be regarded as the gradient estimate for Dunkl heat flows in the sense of Bakry–Ledoux (see, e.g., [5] for the diffusion heat flow case). The proof follows from the standard heat flow interpolation approach.

**Corollary 3.4.** The assertion that
\[
\Gamma(H_\xi(t)f) \leq H_\xi(t)\Gamma(f), \quad f \in C^\infty_c(\mathbb{R}^d), \quad t > 0,
\] (21)
is equivalent to
\[
\Gamma_2(f) \geq 0, \quad f \in C^\infty(\mathbb{R}^d).
\]

**Proof.** Let \(0 \leq s \leq t\). Then for every \(f \in C^\infty_c(\mathbb{R}^d)\),
\[
H_\xi(t)\Gamma(f) - \Gamma(H_\xi(t)f) = \int_0^t \frac{d}{ds}H_\xi(s)\Gamma(H_\xi(t-s)f) \, ds
\]
\[
= \int_0^t H_\xi(s)\left[\Delta_\xi \Gamma(H_\xi(t-s)f) - 2\Gamma(\Delta_\xi H_\xi(t-s)f, H_\xi(t-s)f)\right] \, ds
\]
\[
= 2 \int_0^t H_\xi(s)\Gamma_2(H_\xi(t-s)f) \, ds,
\]
where the second equality can be checked without much effort. Thus, the equivalence of both assertions is clear. □

Define another square function \(\tilde{g}(f)\) as
\[
\tilde{g}(f)(x) = \left( \int_0^\infty H_\xi(t)\Gamma(H_\xi(t)f) \, dt \right)^{1/2}, \quad x \in \mathbb{R}^d.
\]
Then for every $x \in \mathbb{R}^d$, we have

$$g_t(f)^2(x) = \int_0^\infty \Gamma(H_x(t)f)(x) \, dt$$

$$= \int_0^\infty \Gamma(H_x(t/2)H_x(t/2)f)(x) \, dt$$

$$\leq \int_0^\infty H_x(t/2)\Gamma(H_x(t/2)f)(x) \, dt$$

$$= 2g(f)^2(x), \quad (22)$$

where we used Proposition 3.2 and Corollary 3.4 in the above inequality.

Let

$$\tilde{g}_T(f)(x) := \left( \int_0^T H_x(t)\Gamma(H_x(t)f) \, dt \right)^{1/2}, \quad x \in \mathbb{R}^d,$$

Then it is immediately to see that for every $x \in \mathbb{R}^d$, $\tilde{g}_T(f)(x)$ increases to $\tilde{g}(f)(x)$ as $T$ goes to $\infty$. The key point here is that $\tilde{g}_T(f)(x)$ can be expressed as an integral of the conditional expectation of the predictable quadratic variation $\langle N \rangle_T$ as the next lemma shows. For instance, see [6] for the case of pure jump Lévy processes and [20] for the case of diffusion processes.

**Lemma 3.5.** Let $T > 0$. For every $f \in C^\infty_c(\mathbb{R}^d)$ and every $x \in \mathbb{R}^d$,

$$\tilde{g}_T(f)(x) = \left( \int_0^T \mathbb{E}_y \left[ \left( \sum_{\alpha \in \mathbb{N}^+} \kappa_\alpha (H_x(T-t)f(X_{t-}) - H_x(T-t)f(r_\alpha X_{t-}))^2 \langle \alpha, X_{t-} \rangle^2 \right. \right. \right.$$

$$\left. \left. + |\nabla H_x(T-t)f(X_{t-})|^2 \right) dt \right| X_T = x \right) h_x(T, x, y) \, d\mu_x(y)\right)^{1/2}, \quad (23)$$

where $\mathbb{E}_y$ denotes the expectation of the process $(X_t)_{t \geq 0}$ starting from $y$.

**Proof.** Indeed, by the change-of-variables formula, the stochastic completeness (3) and (19), we have

$$\tilde{g}_T(f)^2(x) = \int_0^T \int_{\mathbb{R}^d} h_x(t,x,z)\Gamma(H_x(t)f)(z) \, d\mu_x(z) \, dt$$

$$= \int_0^T \int_{\mathbb{R}^d} h_x(T-t,z,x)\Gamma(H_x(T-t)f)(z) \, d\mu_x(z) \, dt$$

$$= \int_0^T \int_{\mathbb{R}^d} h_x(T-t,z,x)\Gamma(H_x(T-t)f)(z) \left( \int_{\mathbb{R}^d} h_x(t,z,y) \, d\mu_x(y) \right) \, d\mu_x(z) \, dt$$

$$= \int_{\mathbb{R}^d} \left( \int_0^T \int_{\mathbb{R}^d} \frac{h_x(t,y,z)}{h_x(T,y,x)} \Gamma(H_x(T-t)f)(z) \, d\mu_x(z) \, dt \right) h_x(T,y,x) \, d\mu_x(y)$$

$$= \int_{\mathbb{R}^d} \mathbb{E}_y \left[ \int_0^T \left( \sum_{\alpha \in \mathbb{N}^+} \kappa_\alpha (H_x(T-t)f(X_{t-}) - H_x(T-t)f(r_\alpha X_{t-}))^2 \langle \alpha, X_{t-} \rangle^2 \right.$$

$$\left. + |\nabla H_x(T-t)f(X_{t-})|^2 \right) dt \right| X_T = x \right) h_x(T, x, y) \, d\mu_x(y)\right)^{1/2}.$$

Now, we are ready to present the main result in this section.
Theorem 3.6. Let $p \in [2, \infty)$. Suppose the Weyl group $G$ is isomorphic to $\mathbb{Z}_2^d$. Then the operator $g_G$ is bounded in $L^p(\mu_\xi)$; more precisely, there exists a positive constant $C(p)$ such that

$$\|g_G(f)\|_{L^p(\mu_\xi)} \leq C(p)\|f\|_{L^p(\mu_\xi)}, \quad f \in L^p(\mu_\xi);$$

moreover, $C(p)$ depends only on $p$.

The proof is the same as the one for [19, Theorem 4.4] by combining (22) and (23) together, applying the Burkholder–Davis–Gundy inequality and the monotone convergence theorem, and by standard approximation. We omit the details here.

Finally, we give a remark on Theorem 3.6, pointed out by an anonymous referee.

Remark 3.7. The analytic proof in [33, pp. 52–55] can be possibly adapted to prove that $g_G$ is bounded in $L^p(\mu_\xi)$ for all $2 < p < \infty$ when $G$ is isomorphic to $\mathbb{Z}_2^d$.

4 | PROOFS OF PROPOSITION 3.2

In this section, we present the details on the proof of Proposition 3.2. As an illustration, we first consider the rank-one case following the notations of Example 1.1 introduced in Section 1.

Corollary 4.1. In the case of Example 1.1 with $\kappa \in \mathbb{R}$, for every $f \in C^4(\mathbb{R})$ and every $x \in \mathbb{R}$,

$$\Gamma_2(f)(x) = f''(x)^2 + \frac{\kappa}{x^2} \left[ f'(x) + f'(-x) - \frac{f(x) - f(-x)}{x} \right] + \frac{1}{2x^2} \left[ 2f''(x) - \frac{f(x) - f(-x)}{x} \right]^2.$$  (24)

Proof. By (20), we see

$$\Gamma_2(f)(x) = \frac{1}{2} \Delta_\kappa \Gamma(f)(x) - \Gamma(\Delta_\kappa f, f)(x).$$  (25)

Next, we calculate the two terms on the right hand side of (25). By (4), direct calculation leads to

$$\Delta_\kappa \Gamma(f)(x) = \Gamma(f)'(x) + \frac{\kappa}{x^2} \left[ 2x \Gamma(f)'(x) + \Gamma(f)(-x) - \Gamma(f)(x) \right]$$

$$= 2f''(x)^2 + 2f'(x)f'''(x) + \frac{4\kappa}{x} f'(x)f''(x)$$

$$+ \frac{\kappa}{x^2} \left\{ \left[ f'(-x)^2 - f'(x)^2 \right] + \left[ f(x) - f(-x) \right] \left[ f''(x) - f''(-x) \right] \right\}$$

$$+ 2 \frac{\kappa^2 - 2\kappa}{x^3} \left[ f(x) - f(-x) \right] \left[ f'(x) + f'(-x) \right] + \frac{3\kappa - 2\kappa^2}{x^4} \left[ f(x) - f(-x) \right]^2.$$  (26)

Similarly, by (4), (1), and direct calculation, we have

$$\Gamma(\Delta_\kappa f, f)(x) = (\Delta_\kappa f)'(x)f'(x) + \frac{\kappa}{2x^2} \left[ f(x) - f(-x) \right] \left[ (\Delta_\kappa f)(x) - (\Delta_\kappa f)(-x) \right]$$

$$= f'(x)f'''(x) + \frac{2\kappa}{x} f'(x)f''(x)$$

$$+ \frac{\kappa}{2x^2} \left\{ -6f''(x)^2 - 2f'(x)f'''(-x) + \left[ f(x) - f(-x) \right] \left[ f''(x) - f''(-x) \right] \right\}$$

$$+ \frac{1}{x^3} \left\{ \kappa^2 \left[ f(x) - f(-x) \right] \left[ f'(x) + f'(-x) \right] + 2xf'(x) \left[ f(x) - f(-x) \right] \right\} - \frac{\kappa^2}{x^4} \left[ f(x) - f(-x) \right]^2.$$  (27)
Thus, combining (25), (26), and (27), we obtain

\[
\Gamma_2(f)(x) = f''(x)^2 + \kappa \left\{ \frac{[f'(x) + f'(-x)]^2}{x^2} + \frac{[f(x) - f(-x)]^2}{x^4} - 2 \frac{[f(x) - f(-x)] [f'(x) + f'(-x)]}{x^3} \right\} \\
+ \kappa \left\{ \frac{[f(x) - f(-x)]^2}{x^4} + 4 \frac{f'(x)^2}{x^2} - 4 \frac{f'(x) [f(x) - f(-x)]}{x^3} \right\} \\
= f''(x)^2 + \kappa \left\{ \frac{2f'(x) - f(x) - f(-x)}{x} \right\}^2 + \kappa \left\{ 2f'(x) - f(x) - f(-x) \right\}^2.
\]

We finish the proof. \(\square\)

Now, we are ready to present the proof for Proposition 20, which is much more complicated than the one for Corollary 4.1. Assume that the Weyl group \(G\) is isomorphic to \(\mathbb{Z}_2^d\). The crucial fact is that in this case, we have

\[
\langle \alpha, \beta \rangle = 0, \quad \text{for every } \alpha, \beta \in \mathbb{R}_+ \text{ with } \alpha \neq \beta. \tag{28}
\]

**Proof of Proposition 3.2.** Let \(x \in \mathbb{R}^d, \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}_+^d\) and \(y = (y_1, \ldots, y_d) \in \mathbb{R}^d\). We need the following elementary facts, which can be verified directly.

\[
\Delta \left[ \frac{1}{\langle \alpha, \cdot \rangle^2}(x) \right] = \frac{12}{\langle \alpha, x \rangle^4}, \quad \langle \alpha, x \rangle \neq 0;
\]
\[
\Delta \left[ \frac{1}{2\langle \alpha, \cdot \rangle^2}(x) \right] = \frac{1}{\langle \alpha, x \rangle^2}, \quad \langle \alpha, x \rangle \neq 0;
\]
\[
\nabla [f(\cdot) - f(r_\alpha \cdot)](x) = \nabla f(x) - \nabla f(r_\alpha x) + \langle \nabla f(r_\alpha x), \alpha \rangle \alpha;
\]
\[
\Delta [f(\cdot) - f(r_\alpha \cdot)](x) = \Delta f(x) - \Delta f(r_\alpha x);
\]
\[
\langle \nabla (\|\nabla f(\cdot)\|^2)(x), y \rangle = 2 \sum_{i,j=1}^{d} \partial_{x_i} f(x) \partial^2_{x_i x_j} f(x) y_j;
\]
\[
\langle \nabla (\langle \nabla f(\cdot), \alpha \rangle)(x), y \rangle = \sum_{i,j=1}^{d} \alpha_i \partial^2_{x_i x_j} f(x) y_j.
\]

Recall that \(|\alpha| = \sqrt{2}\) and \(\delta_{\alpha} f(x) : = [f(x) - f(r_\alpha x)]/\langle \alpha, x \rangle \) (see Remark 1.4(i)). With the above facts, by (20), (4), and (1), we do the similar calculation as the proof of Corollary 4.1. However, due to the mixture of local and non-local terms in (1) and (4), the non-local term in the expression of \(\Gamma_2(f)\) consists of two summations: one is over \(\alpha \in \mathbb{R}_+\) and the other is over \(\alpha, \beta \in \mathbb{R}_+\). Indeed,

\[
\Gamma_2(f)(x) = \|\text{Hess}(f)(x)\|^2_{\text{HS}}
\]
\[
+ \sum_{\alpha \in \mathbb{R}_+} \frac{2\kappa_\alpha}{\langle \alpha, x \rangle^2} \left\{ |\delta_\alpha f(x)\alpha - \nabla [f(x) - f(r_\alpha x)]|^2 + |\langle \nabla f(x), \alpha \rangle - \delta_\alpha f(x)|^2 \right\}
\]
\[
+ \sum_{\alpha, \beta \in \mathbb{R}_+} \frac{\kappa_\alpha \kappa_\beta}{\langle \beta, x \rangle^2} \left\{ - \frac{2|\delta_\alpha f(x)|^2}{\langle \alpha, x \rangle} \langle \alpha, \beta \rangle \langle \beta, x \rangle - [\delta_\alpha f(x)]^2 + [\delta_\alpha f(r_\beta x)]^2 \right\}
\]
\[ +2\delta_{\alpha} f(x) \left( \langle V[f(x) - f(r_{\alpha} x)], \beta \rangle \langle x, x \rangle \right) \]

\[ -2 \left[ \left( \frac{\langle V[f(x), \alpha] \rangle}{\langle x, x \rangle} - \frac{\delta_{\alpha} f(x)}{\langle x, x \rangle} \right) - \left( \frac{\langle V[f(r_{\beta} x), \alpha] \rangle}{\langle x, r_{\beta} x \rangle} - \frac{\delta_{\alpha} f(r_{\beta} x)}{\langle x, r_{\beta} x \rangle} \right) \right] \left[ f(x) - f(r_{\beta} x) \right] \]

\[ =: \| \text{Hess}(f)(x) \|_{2, \text{HS}}^2 + A + B, \]

where \( A \) denotes the summation of terms over \( \alpha \in \mathbb{R}_+ \), and \( B \) denotes the summation of terms over \( \alpha, \beta \in \mathbb{R}_+ \). (In the rank-one case, \( B = 0 \) and \( A \) reduces to the sum of the last two terms in the right hand side of (24).)

For \( B \), we further split it into two summations: one is over \( \alpha, \beta \in \mathbb{R}_+ \) with \( \alpha = \beta \), and the other is over \( \alpha, \beta \in \mathbb{R}_+ \) with \( \alpha \neq \beta \). Using the facts that \( |\alpha| = \sqrt{2}, \langle \alpha, r_{\alpha} x \rangle = -\langle \alpha, x \rangle \), and \( r_{\alpha} r_{\alpha} x = x \) for all \( \alpha \in \mathbb{R}_+ \), it is easy to see that the summation over \( \alpha, \beta \in \mathbb{R}_+ \) with \( \alpha = \beta \) equals 0. For any \( \alpha, \beta \in \mathbb{R}_+ \) with \( \alpha \neq \beta \), by (28), we have \( \langle \alpha, r_{\beta} x \rangle = \langle \alpha, x \rangle \). Thus, \( B = \sum_{\alpha, \beta \in \mathbb{R}_+, \alpha \neq \beta} \kappa_{\alpha} \kappa_{\beta} \left\{ \frac{2\delta_{\alpha} f(x)}{\langle \beta, x \rangle^2} \langle V[f(x) - f(r_{\alpha} x)], \beta \rangle \langle x, x \rangle - \left[ \frac{\langle V[f(x), \alpha] \rangle}{\langle x, x \rangle} - \frac{\delta_{\alpha} f(x)}{\langle x, x \rangle} \right] \left[ f(x) - f(r_{\beta} x) \right] \right\} \]

where we used the symmetry of \( \alpha \) and \( \beta \) in the last equality.

Consider \( B_1 \). For any \( \alpha, \beta \in \mathbb{R}_+ \) with \( \alpha \neq \beta \), again by (28), \( r_{\alpha} r_{\beta} x = r_{\beta} r_{\alpha} x \) and \( \langle \alpha, r_{\beta} x \rangle = \langle \alpha, x \rangle \). Then we obtain

\[ B_1 = \sum_{\alpha, \beta \in \mathbb{R}_+, \alpha \neq \beta} \kappa_{\alpha} \kappa_{\beta} \left\{ \frac{2\delta_{\alpha} f(x)}{\langle \beta, x \rangle^2} \langle V[f(x) - f(r_{\alpha} x)], \beta \rangle \langle x, x \rangle - 2 \left[ \frac{\langle V[f(x), \alpha] \rangle}{\langle x, x \rangle} - \frac{\delta_{\alpha} f(x)}{\langle x, x \rangle} \right] \left[ f(x) - f(r_{\beta} x) \right] \right\} \]

\[ = \sum_{\alpha, \beta \in \mathbb{R}_+, \alpha \neq \beta} \frac{4\kappa_{\alpha} \kappa_{\beta}}{\langle \alpha, x \rangle \langle \beta, x \rangle} \left[ \delta_{\alpha} f(x) - \delta_{\beta} f(x) \langle \beta, x \rangle \right] = 0, \]

where we used the symmetry of \( \alpha \) and \( \beta \) in the last equality.

Consider \( B_2 \). For any \( \alpha, \beta \in \mathbb{R}_+ \) with \( \alpha \neq \beta \), again by (28), \( r_{\alpha} r_{\beta} x = r_{\beta} r_{\alpha} x \) and \( \langle \alpha, r_{\beta} x \rangle = \langle \alpha, x \rangle \). Then we obtain

\[ B_2 = \sum_{\alpha, \beta \in \mathbb{R}_+, \alpha \neq \beta} \kappa_{\alpha} \kappa_{\beta} \left\{ \left[ \delta_{\alpha} f(r_{\beta} x) \right] - \left[ \delta_{\alpha} f(x) \right] \right\} + \left[ f(x) - f(r_{\beta} x) \right] \left[ f(x) - f(r_{\beta} x) \right] \}

\[ = \sum_{\alpha, \beta \in \mathbb{R}_+, \alpha \neq \beta} \frac{\kappa_{\alpha} \kappa_{\beta}}{\langle x, x \rangle^2 \langle \beta, x \rangle^2} \left[ f(x) - f(r_{\alpha} x) \right] \left[ f(x) - f(r_{\beta} x) \right] \]

\[ = \sum_{\alpha, \beta \in \mathbb{R}_+, \alpha \neq \beta} \frac{\kappa_{\alpha} \kappa_{\beta}}{\langle x, x \rangle^2 \langle \beta, x \rangle^2} \left[ f(x) - f(r_{\alpha} x) \right] \left[ f(x) - f(r_{\beta} x) \right] \]

where, in the last equality, we first used the symmetry of \( \alpha \) and \( \beta \) and then took the average.
Therefore, combining (29), (30), (31), and (32), we finally arrive at
\[
\Gamma_2(f)(x) = \|Hess(f)(x)\|_{HS}^2 + \sum_{\alpha \in \mathbb{R}^+} 2\kappa_\alpha \left( |\delta_\alpha f(x) - \nabla f(x) - f(r_\alpha x)|^2 + |\nabla f(x), \alpha - \delta_\alpha f(x)|^2 \right)
\]
\[
+ \sum_{\alpha, \beta \in \mathbb{R}^+, \alpha \neq \beta} \kappa_\alpha \kappa_\beta \left[ f(x) - f(r_\alpha x) - f(r_\beta x) + f(r_\alpha r_\beta x) \right]^2
\]
\[
\geq \|Hess(f)(x)\|_{HS}^2,
\]
which completes the proof of Proposition 3.2.

Remark 4.2. In the above proof, assumption (28) on the Weyl group \( G \) is employed only to show that \( B \geq 0 \). With (28), the terms like
\[
\langle \alpha, \beta \rangle \langle \alpha, x \rangle \langle \beta, x \rangle |f(x) - f(r_\alpha x)|^2, \quad f(r_\alpha x) \left[ f(r_\beta r_\alpha x) - f(r_\alpha r_\beta x) \right]
\]
clearly disappear. However, without (28), it seems difficult to deal with the above terms and hence we do not know the sign of \( B \). It seems interesting to find or construct an example such that \( B_2 \) is negative, which essentially results in the negativity of \( \Gamma_2 \).

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Endnote
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