Pseudo-unitary symmetry and the Gaussian pseudo-unitary ensemble of random matrices

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Abstract

Employing the currently discussed notion of pseudo-hermiticity, we define a pseudo-unitary group. Further, we develop a random matrix theory which is invariant under such a group and call this ensemble of pseudo-Hermitian random matrices as the pseudo-unitary ensemble. We obtain exact results for the nearest-neighbour level spacing distribution for \((2 \times 2)\)-matrices which has a novel form, \(s \log \frac{1}{s}\) near zero spacing. This shows a level repulsion in marked distinction with an algebraic form \(s^\beta\) in the Wigner surmise. We believe that this paves way for a description of varied phenomena in two-dimensional statistical mechanics, quantum chromodynamics, and so on.

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Postulates of quantum theory require the observables to be represented by Hermitian operators as only real eigenvalues correspond to measurements. However, it has recently been emphasized that there are certain Hamiltonians describing the quantum systems which possess real eigenvalues even though they are not Hermitian. Many of these systems are invariant under space-time reflection, i.e. invariant under a joint action of parity (\(P\)) and time-reversal (\(T\)) \([1, 2, 3]\). In this context, the concept of pseudo-hermiticity was introduced \([4]\) where it was shown that \(P T\)-symmetry is a special case of pseudo-hermiticity. Pseudo-hermiticity of an operator or a matrix \(O\) is simply defined through the condition : \(O^\dagger = \eta O \eta^{-1}\) with \(\eta\) a metric and \(\dagger\) representing the usual adjoint or conjugate-transpose. Remarkably, it was
subsequently shown that non-$\mathcal{PT}$ invariant systems that possess real eigenvalues are also pseudo-Hermitian \[5\]. Physical situations of great interest belong to the above discussion. This includes two-dimensional statistical mechanics where parity and time-reversal are broken (preserving $\mathcal{PT}$) \[3, 4, 8\], quantum chromodynamics where chiral ensembles are used to describe the statistical properties of lattice Dirac operator \[9\], spin-rotation coupling leading to an anomalous g-value for muon \[10\], and related fields. In this Letter, we present a random matrix theory which describes spectral fluctuations in systems which are pseudo-Hermitian and pseudo-unitarily invariant. The two aspects which are particularly notable are the simplicity of this novel description and the fact that this theory is natural when parity or (and) time-reversal is (are) violated.

The problem of two-dimensional statistical mechanics is obviously connected with anyon physics and hence to the behavior of an electron in an Aharonov-Bohm medium \[11\], i.e. a medium filled with non-quantized magnetic fluxes, reminiscent of the theory of fractional quantum Hall effect \[12\]. Important to note here is also another motivation which stems from a speculation due to Nambu that this might serve as a model for theoretical ideas like the quark confinement in a medium of monopoles \[13\]. In this context, it is known that the spectral fluctuations of an Aharonov-Bohm billiard exhibits an interpolating behavior with respect to the strength of the flux line \[14\]. These billiards are experimentally realized in terms of quantum dots in the presence of flux lines. It is of great interest to find an appropriate random matrix description for such $\mathcal{PT}$-invariant systems. Pseudo-hermiticity appears in several contexts. It is instructive to note that in the mean-field, RPA description of nuclei \[15\], the stability matrix leading to an eigenvalue problem can be checked to be pseudo-unitary. In the context of regularization of quantum field theories, pseudo-hermiticity and the associated improper metric was used by Dirac \[16\], Pauli \[17\], and particularly by Gupta and Bleuler \[18\], and others \[19\]. Let us first establish the pseudo-unitary symmetry.

Consider vectors $\mathbf{x}$ and $\mathbf{y}$ residing in a vector space $\mathcal{V}$ and a fixed metric $\eta$. In this vector space, we define a pseudo-inner product ($\eta$-norm), which can be written in the usual quantum mechanical notation as $\langle \mathbf{x} | \eta | \mathbf{y} \rangle$. We shall consider symmetry transformations which preserve the $\eta$-norm between the vectors. We consider the Cayley form, $\mathbf{D} = e^{i\mathbf{G}}$ as a symmetry transformation acting on $\mathbf{x}$, $\mathbf{y}$ where $\mathbf{G}$ is pseudo-Hermitian in accordance with $\eta \mathbf{G} \eta^{-1} = \mathbf{G}^\dagger$. Noting an interesting feature of $\mathbf{D}$:

$$
\mathbf{D}^\dagger = e^{-i\mathbf{G}} = e^{-i\eta \mathbf{G} \eta^{-1}} = \eta e^{-i\mathbf{G} \eta^{-1}} = \eta \mathbf{D}^{-1} \eta^{-1},
$$

let us call $\mathbf{D}$ as pseudo-unitary with respect to $\eta$. $\eta$ equal to unity makes
To establish that $D$ is indeed a symmetry transformation, we need to show that the transformation preserves the $\eta$-norm and a consistently-defined matrix element.

Let us assume that $x (y) \rightarrow x' (y') = Dx (Dy)$. Then, the pseudo-unitary symmetry is defined by preserving the pseudo-norm:

$$\langle x' | \eta y' \rangle = \langle Dx | \eta Dy \rangle = \langle x | \eta y \rangle \quad (2)$$

In proving (2), use $e^{-iG^\dagger} \eta e^{iG} = e^{-iG^\dagger} \eta e^{iG} \eta^{-1} \eta = e^{-iG^\dagger} e^{iG} \eta = \eta$. Under the same pseudo-unitary transformation, the matrix element of an arbitrary operator, $A$, transforms as

$$\langle x' | \eta A' | y' \rangle = \langle x | \eta A | y \rangle \quad (3)$$

if $D A D^{-1} = A'$.

Let us now prove that pseudo-unitary matrices form a group under matrix multiplication. For closure, let $D_1$ and $D_2$ be two pseudo-unitary matrices. $D_1 D_2$ is pseudo-unitary because $\eta^{-1} (D_1 D_2)^\dagger \eta = \eta^{-1} D_2^\dagger \eta \eta^{-1} D_1^\dagger \eta = (D_1 D_2)^{-1}$. It easily follows that $D^{-1}$ is pseudo-unitary with respect to $\eta$ if $D$ is pseudo-unitary: $\eta^{-1} (e^{-iG})^\dagger \eta = e^{\eta^{-1} G} \eta = e^G$. The identity matrix acts as the unit element of the symmetry transformation. Finally, since the associativity is guaranteed, the $N \times N$ pseudo-unitary matrices form a pseudo-unitary group of order $N$, $PU(N)$.

In the following, to keep the proceedings simple and explicit, we consider Hamiltonians in their matrix representations. Also, in the spirit of the original work of Wigner [20], we consider $(2 \times 2)$ matrices as they bring out most of the essence. In this context, there is a recent generalization of Wigner surmise for $2 \times 2$ matrices [21]. Thus, we concentrate on $PU(2)$ and consider the following pseudo-Hermitian matrix,

$$H = \begin{bmatrix} a & -ib \\ ic & a \end{bmatrix}, \quad (4)$$

$a, b, c$ being real. Consequently, $e^{iH}$ will be a pseudo-unitary matrix. For the above matrices, a metric is

$$\delta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (5)$$

This metric may be interpreted as the parity operator $P$, and the complex conjugation, $K_0$ as time-reversal operator $T$. With these operations, it may be verified that $H$ is $PT$-invariant in addition to being $P$-pseudo-Hermitian. Besides these commuting $P$ and $T$ operators, if we choose $T$ as the Pauli
matrix times the complex conjugation, $\sigma_x K_0$, they do not commute, however preserving other conclusions.

This group admits three generators and an identity, viz.,

\[
\rho_1 = \begin{bmatrix} 1 & 0 \\ i & -1 \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} 1 & -i \\ 0 & -1 \end{bmatrix}, \\
\rho_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

(6)

Note that \( H = aI + b\rho_1 + c\rho_2 - (b + c)\rho_3 \). It is interesting to see that \( \rho_1 \) and \( \rho_2 \) are pseudo-hermitian and pseudo-unitary, possessing eigenvalues \( \pm 1 \). It may be recalled that the Pauli matrices \( \sigma_x \) and \( \sigma_y \) are Hermitian and unitary. Further, the generators satisfy the following important properties,

\[
\rho_1^2 = \rho_2^2 = \rho_3^2 = I, \\
[\rho_i, \rho_j] = \sum_k C_{ij}^k \rho_k,
\]

with \( C_{12} = C_{22} = C_{23} = C_{31} = C_{31} = 2 \) and \( C_{12} = 5 \). All the structure constants can be found with the help of commutation relations and symmetry properties, and they turn out to be \( \pm 5, \pm 2, \) or 0. Interestingly, the following relations between the structure constants hold :

\[
C_{ij}^{kl} = -C_{ji}^{lk} \\
\sum_{m=1}^{3} \left[ C_{kl}^{mj} C_{jm}^{is} + C_{lj}^{mk} C_{km}^{is} + C_{jk}^{ml} C_{lm}^{is} \right] = 0,
\]

(8)

thus making it a Lie group and defining a Lie algebra \[23\].

We now consider a Hamiltonian \( H \) which is diagonalizable by \( D \), i.e.,

\[
H = D \begin{bmatrix} E_+ & 0 \\ 0 & E_- \end{bmatrix} D^{-1}.
\]

(9)

The eigenvalues of \( H \) are \( a \pm \sqrt{bc} \) (\( bc \geq 0 \)). The corresponding matrix, \( D \),

\[
D = \begin{bmatrix} 1 & i/r \\ i/r & 1 \end{bmatrix},
\]

(10)

is pseudo-unitary under the metric,

\[
\eta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

(11)
The eigenvalues are
\[
E_{\pm} = a \pm \frac{c}{2r} + \frac{br}{2}
\]  
(12)
where \( r = \sqrt{c/b} \) (0 \( \leq r \leq \infty \)).

Consider that the matrix \( \mathbf{H} \) is drawn from an ensemble of random matrices with a Gaussian distribution given by

\[
P(\mathbf{H}) = \mathcal{N} e^{-\frac{1}{2\sigma^2} \text{tr} \mathbf{H}^\dagger \mathbf{H}}.
\]  
(13)

Accordingly, the joint probability distribution of \( a, b, c \) is

\[
P(a, b, c) = \frac{1}{2(\pi\sigma^2)^{3/2}} e^{-\frac{1}{2\pi^2} [2a^2 + b^2 + c^2]}.
\]  
(14)

From (4) and (9), we have the following relations:

\[
a = \frac{E_+ + E_-}{2}, \quad b = \frac{E_+ - E_-}{2r}, \quad c = \frac{r(E_+ - E_-)}{2}.
\]  
(15)

The Jacobian, \( J \) connecting \( (a, b, c) \) and \( (E_+, E_-, r) \) is \( \frac{|E_+ - E_-|}{2r} \). With these, the joint probability distribution function (j.p.d.f.) of eigenvalues is

\[
P(E_+, E_-) = \frac{|E_+ - E_-|}{2(\pi\sigma^2)^{3/2}} K_0 \left( \frac{(E_+ - E_-)^2}{4\sigma^2} \right) e^{-\frac{(E_+ + E_-)^2}{4\sigma^2}}.
\]  
(16)

Following the Dyson Coulomb gas analogy, this j.p.d.f. can be written as an equilibrium distribution of two interacting particles with a partition function \( P(E_+, E_-) \rightarrow Z(x_1, x_2) = e^{-\beta \mathcal{H}(x_1, x_2)} \). It is interesting to note that the \( \mathcal{H} \) has a potential term involving the logarithm of the modified Bessel function along with the familiar harmonic confinement and the two-dimensional Coulomb potential. \( 4\sigma^2 \) plays the role of inverse scaled temperature.

Integrating with respect to \( E_- \) gives the average density, shown in Fig. 1. This is not amenable to an analytically closed form.

Perhaps the most well-studied characterizer is the nearest-neighbour level spacing distribution, \( P(S) \). This gives the frequency with which a certain spacing between adjacent levels occurs. For the Wigner-Dyson ensembles, \( P(S) \sim S^{\beta_0} e^{-\gamma S^2} \) where \( \beta_0 \) is 1, 2, and 4 for the orthogonal, unitary, and symplectic ensembles. A wide variety of systems display universal properties possessed by random matrix ensembles as can be seen in \( [20, 24, 25] \). However, there are systems that display intermediate statistics \( [26, 27, 28] \).
These systems range from examples of billiards in polygonal enclosures, three-dimensional Anderson model at the metal-insulator transition point, and so on. On the other hand, there have been important developments on non-Hermitian ensembles since long where the eigenvalues are complex [20, 25], and where an ensemble of unstable states are considered [29]. Clearly, the ensemble developed here does not fall into any of the known categories and, indeed, displays some novel features as shown below.

The spacing distribution, \( P(S) \), is given in terms of the j.p.d.f. by

\[
P(S) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(E_+, E_-) \delta(S - |E_+ - E_-|) dE_+ dE_-
\]

\[
= \frac{|S|}{\pi \sigma^2} K_0 \left( \frac{S^2}{4\sigma^2} \right).
\]

This result is distinctly different and very interesting (Fig. 2), particularly for its behaviour near zero spacing. Near \( S = 0 \), the probability distribution varies as \( S \log \frac{1}{S} \). This follows from the asymptotic properties of the modified Bessel function.

We present the form of two-time correlation function for a complex system with spectral properties given by Gaussian pseudo-unitary ensembles (GPUE). In this context, we consider a system with a Hamiltonian \( H \in \text{GPUE} \), and an observable given by an operator, \( V \in \text{an another GPUE} \).

Imagine the system to be in thermodynamic equilibrium with a canonical density matrix, \( \rho = e^{-\beta H} \). Following an extensive study along the lines of [30], the time correlation function, \( C(t) = Z^{-1}(\beta) \text{ tr } \rho V(0)V(t) \) decays over long times \( \sim \left( \frac{\log 1/t}{t} \right)^3 \).

Finally, we wish to point out an aspect of general importance, encountered on many occasions in many-body theory. To give one concrete example, in the theory of collective excitations of Fermionic systems, a mean-field description is used where a collective state is first expressed in terms of particle-hole excitations [15]. Here, one generally encounters a matrix equation like \( H\Psi = \lambda \Phi \) with \( H \) a Hermitian or unitary operator. The above problem may be transformed into an eigenvalue problem for \( H' \), i.e., \( H'\Phi = \lambda \Phi \) with \( H' \) a pseudo-Hermitian or pseudo-unitary operator. With this, there are many results immediately possible. First of all, the eigenvalues will either be real, complex-conjugate-pairs, unimodular, or they occur in pairs such that product of eigenvalues is unimodular [22]. Secondly, the statistical properties of the eigenvalues related to collective excitations will be distributed in accordance with the results obtained for GPUE above. Thirdly, there will be long-time tail for relaxation in the same spirit as the time correlation.

The above results are found for \( 2 \times 2 \) matrices. Although for \( N \times N \) matrices, invariant under \( PU(N) \), the results are not known, we conjecture
that the fluctuation properties will have a similar form as above. As discussed earlier, the result found here gives a new universality corresponding to systems which are pseudo-unitarily invariant. In such systems, parity and time-reversal may be individually broken, preserving their joint action. This universality also includes those pseudo-Hermitian quantum systems where $\mathcal{PT}$ is broken. The examples discussed include quantum chromodynamics, two-dimensional statistical mechanics, and so on.
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FIGURE CAPTIONS

1. The average level density of an ensemble of $2 \times 2$ random Gaussian pseudo-unitary ensemble is shown here.

2. The nearest-neighbor level-spacing distribution is shown here. For comparison, the results corresponding to the Wigner-Dyson ensembles corresponding to orthogonal and unitary symmetries are also shown. Whereas the level repulsion is linear and quadratic in the orthogonal and unitary ensembles, here it is of the form $s\log(1/s)$, as shown in the inset. This then is a new universality.
Fig. 2