Abstract

We propose a rank-$k$ variant of the classical Frank-Wolfe algorithm to solve convex optimization over a trace-norm ball. Our algorithm replaces the top singular-vector computation ($1$-SVD) in Frank-Wolfe with a top-$k$ singular-vector computation ($k$-SVD), which can be done by repeatedly applying $1$-SVD $k$ times. Alternatively, our algorithm can be viewed as a rank-$k$ restricted version of projected gradient descent. We show that our algorithm has a linear convergence rate when the objective function is smooth and strongly convex, and the optimal solution has rank at most $k$. This improves the convergence rate and the total time complexity of the Frank-Wolfe method and its variants.

1 Introduction

Minimizing a convex matrix function over a trace-norm ball, which is: (recall that the trace norm $\|X\|_*$ of a matrix $X$ equals the sum of its singular values)

$$\min_{X \in \mathbb{R}^{m \times n}} \{ f(X) : \|X\|_* \leq \theta \},$$

is an important optimization problem that serves as a convex surrogate to many low-rank machine learning tasks, including matrix completion [2, 10, 17], multiclass classification [4], phase retrieval [3], polynomial neural nets [12], and more. In this paper we assume without loss of generality that $\theta = 1$.

One natural algorithm for Problem (1.1) is projected gradient descent (PGD). In each iteration, PGD first moves $X$ in the direction of the gradient, and then projects it onto the trace-norm ball. Unfortunately, computing this projection requires the full singular value decomposition (SVD) of the matrix, which takes $O(mn \min\{m, n\})$ time in general. This prevents PGD from being efficiently applied to problems with large $m$ and $n$.

Alternatively, one can use projection-free algorithms. As first proposed by Frank and Wolfe [5], one can select a search direction (which is usually the gradient direction) and perform a linear optimization over the constraint set in this direction. In the case of Problem (1.1), performing linear optimization over a trace-norm ball amounts to computing the top (left and right) singular vectors of a matrix, which can be done much faster than full SVD. Therefore, projection-free algorithms become attractive for convex minimization over trace-norm balls.

Unfortunately, despite its low per-iteration complexity, the Frank-Wolfe (FW) algorithm suffers from slower convergence rate compared with PGD. When the objective $f(X)$ is smooth, FW requires $O(1/\varepsilon)$ iterations to convergence to an $\varepsilon$-approximate minimizer, and this $1/\varepsilon$ rate is tight even if the objective is also strongly convex [6]. In contrast, PGD achieves $1/\sqrt{\varepsilon}$ rate if $f(X)$ is smooth (under Nesterov’s acceleration [15]), and $\log(1/\varepsilon)$ rate if $f(X)$ is both smooth and strongly convex.
Recently, there were several results to revise the FW method to improve its convergence rate for strongly-convex functions. The log(1/\varepsilon) rate was obtained when the constraint set is a polyhedron \cite{6, 11}, and the \sqrt{\varepsilon} rate was obtained when the constraint set is strongly convex \cite{8} or is a spectrahedron \cite{6}.

Among these results, the spectrahedron constraint (i.e., for all positive semidefinite matrices \( X \) with \( \text{Tr}(X) = 1 \)) studied by Garber \cite{6} is almost identical to Problem (1.1), but slightly weaker.\footnote{The best of our knowledge, given an algorithm that works for spectrahedron, to solve Problem (1.1), one has to define a function \( g(Y) \) over \((n+m) \times (n+m)\) matrices, by setting \( g(Y) = f(2Y_{1:m,1:m} + \alpha Y_{n:m,n:m}) \) \cite{10} \cite{11}. After this transformation, the function \( g(Y) \) is no longer strongly convex, even if \( f(X) \) is strongly convex. In contrast, most algorithms for trace-norm balls, including FW and our later proposed algorithm, work as well for spectrahedron after minor changes to the analysis.}

When stating the result of Garber \cite{6}, we assume for simplicity that it also applies to Problem (1.1).

Our Question. In this paper, we propose to study the following general question:

Can we design a “rank-k variant” of Frank-Wolfe to improve the convergence rate?

(That is, in each iteration it computes the top \( k \) singular vectors – i.e., \( k\text{-SVD} \) – of some matrix.)

Our motivation to study the above question can be summarized as follows:

- Since FW computes a 1-SVD and PGD computes a full SVD in each iteration, is there a value \( k \leq \min\{n, m\} \) such that a rank-\( k \) variant of FW can achieve the convergence rate of PGD?
- Since computing \( k\text{-SVD} \) costs roughly the same (sequential) time as “computing 1-SVD for \( k \) times” (see recent work \cite{1} \cite{13})\footnote{Using block Krylov \cite{13}. Lanszos \cite{1}, or SVRG \cite{1}, at least when \( k \) is small, the time complexity of (approximately) computing the top \( k \) singular vectors of a matrix is no more than \( k \) times the complexity of (approximately) computing the top singular vector of the same matrix. We refer interested readers to \cite{1} for details.}, if using a rank-\( k \) variant of FW, can the number of iterations be reduced by a factor more than \( k \)? If so, then we can improve the sequential running time of FW.
- \( k\text{-SVD} \) can be computed in a more *distributed* manner than 1-SVD. For instance, using block Krylov \cite{13}, one can distribute the computation of \( k\text{-SVD} \) to \( k \) machines, each in charge of independent matrix-vector multiplications. Therefore, it is beneficial to study a rank-\( k \) variant of FW in such settings.

1.1 Our Results

We propose \texttt{blockFW}, a rank-\( k \) variant of Frank-Wolfe. Given a convex function \( f(X) \) that is \( \beta \)-smooth, in each iteration \( t \), \texttt{blockFW} performs an update \( X_{t+1} \leftarrow X_t + \eta (X_t - X_{t-1}) \), where \( \eta > 0 \) is a constant step size and \( X_t \) is a rank-\( k \) matrix computed from the \( k\text{-SVD} \) of \((-\nabla f(X_t) + \beta \eta X_t)\). If \( k = \min\{n, m\}, \texttt{blockFW} \) can be shown to coincide with PGD, so it can also be viewed as a rank-\( k \) restricted version of PGD.

Convergence. Suppose \( f(X) \) is also \( \alpha \)-strongly convex and suppose the optimal solution \( X^* \) of Problem (1.1) has rank \( k \), then we show that \texttt{blockFW} achieves linear convergence: it finds an \( \varepsilon \)-approximate minimizer within \( O\left( \frac{k \beta}{\alpha} \log \frac{1}{\varepsilon} \right) \) iterations, or equivalently, in

\[
T = O\left( \frac{k \beta}{\alpha} \log \frac{1}{\varepsilon} \right) \text{ computations of 1-SVD.}
\]

We denote by \( T \) the number of 1-SVD computations throughout this paper. In contrast,

\[
T_{\text{FW}} = O\left( \frac{\beta}{\varepsilon} \right) \quad \text{for Frank-Wolfe}
\]

\[
T_{\text{Gar}} = O\left( \min \left\{ \frac{\beta}{\varepsilon}, \left( \frac{\beta}{\alpha} \right)^{1/4} \left( \frac{\varepsilon}{\beta} \right)^{3/4} \sqrt{k}, \left( \frac{\beta}{\alpha} \right)^{1/2} \left( \frac{\varepsilon}{\beta} \right)^{1/2} \frac{1}{\sigma_{\min}(X^*)} \right\} \right) \quad \text{for Garber} \ [6].
\]

Above, \( \sigma_{\min}(X^*) \) is the minimum non-zero singular value of \( X^* \). Note that \( \sigma_{\min}(X^*) \leq \frac{\|X^*\|_F}{\text{rank}(X^*)} \leq \frac{1}{k} \).

We note that \( T_{\text{Gar}} \) is always outperformed by \( \min\{T, T_{\text{FW}}\} \): ignoring the \( \log(1/\varepsilon) \) factor, we have

\[
\min \left\{ \frac{\beta}{\varepsilon}, k^2 \frac{\beta^2}{\alpha} \right\} \leq \left( \frac{\beta}{\alpha} \right)^{1/4} \left( \frac{\varepsilon}{\beta} \right)^{3/4} k^{1/4} \leq \left( \frac{\beta}{\alpha} \right)^{1/4} \left( \frac{\varepsilon}{\beta} \right)^{3/4} \sqrt{k}, \quad \text{and}
\]

\[
\min \left\{ \frac{\beta}{\varepsilon}, k^2 \frac{\beta^2}{\alpha} \right\} \leq \left( \frac{\beta}{\alpha} \right)^{1/2} \left( \frac{\varepsilon}{\beta} \right)^{1/2} k^{1/2} \leq \frac{\beta}{\alpha} \left( \frac{\varepsilon}{\beta} \right)^{1/2} \frac{1}{\sigma_{\min}(X^*)}.
\]

Remark. The low-rank assumption on \( X^* \) should be reasonable: as we mentioned, in most applications of Problem (1.1), the ultimate reason for imposing a trace-norm constraint is to ensure that the optimal
Table 1: Comparison of first-order methods to minimize a \( \beta \)-smooth, \( \alpha \)-strongly convex function over the unit-trace norm ball in \( \mathbb{R}^{m \times n} \). In the table, \( k \) is the rank of \( X^* \), \( \kappa = \frac{\beta}{\alpha} \) is the condition number, \( \nabla = \nabla f(X_t) \) is the gradient matrix, \( \text{nnz}(\nabla) \) is the complexity to multiply \( \nabla \) to a vector, \( \sigma_1(X) \) is the \( i \)-th largest singular value of \( X \), and \( \sigma_{\min}(X) \) is the minimum non-zero singular value of \( X \).

solution is low-rank; otherwise the minimization problem may not be interesting to solve in the first place. Also, the immediate prior work \[6\] also assumes \( X \) to have low rank.

**k-SVD Complexity.** For theoreticians who are concerned about the time complexity of \( k \)-SVD, we also compare it with the 1-SVD complexity of FW and Garber. If one uses LazySVD \[15\] to compute \( k \)-SVD in each iteration of blockFW, then the per-iteration \( k \)-SVD complexity can be bounded by

\[
k \cdot \tilde{O}(\text{nnz}(\nabla) + k(m + n)\epsilon) \times \min \left\{ \frac{\|\nabla\|_2 + \alpha}{\epsilon}, \frac{\kappa(\|\nabla\|_2 + \alpha)^{1/2}}{\alpha^{1/2}\sigma_{\min}(X^*)} \right\}.
\]

(1.2)

Above, \( \kappa = \frac{\beta}{\alpha} \) is the condition number of \( f, \nabla = \nabla f(X_t) \) is the gradient matrix of the current iteration \( t \), \( \text{nnz}(\nabla) \) is the complexity to multiply \( \nabla \) to a vector, \( \sigma_{\min}(X^*) \) is the minimum non-zero singular value of \( X^* \), and \( \tilde{O} \) hides poly-logarithmic factors.

In contrast, if using Lanczos, the 1-SVD complexity for FW and Garber can be bounded as (see \[6\])

\[
\tilde{O}(\text{nnz}(\nabla)) \times \min \left\{ \frac{\|\nabla\|_2^{1/2}}{\epsilon^{1/2}}, \frac{\|\nabla\|_2^{1/2}}{(\sigma_1(\nabla) - \sigma_2(\nabla))^{1/2}} \right\}.
\]

(1.3)

Above, \( \sigma_1(\nabla) \) and \( \sigma_2(\nabla) \) are the top two singular values of \( \nabla \), and the gap \( \sigma_1(\nabla) - \sigma_2(\nabla) \) can be as small as zero.

We emphasize that our \( k \)-SVD complexity (1.2) can be upper bounded by a quantity that only depends poly-logarithmically on \( 1/\epsilon \). In contrast, the worst-case 1-SVD complexity (1.3) of FW and Garber depends on \( \epsilon^{-1/2} \) because the gap \( \sigma_1 - \sigma_2 \) can be as small as zero. Therefore, if one takes this additional \( \epsilon \) dependency into consideration for the convergence rate, then blockFW has rate \( \text{polylog}(1/\epsilon) \), but FW and Garber have rates \( \epsilon^{-3/2} \) and \( \epsilon^{-1} \) respectively. The convergence rates and per-iteration running times of different algorithms for solving Problem (1.1) are summarized in Table 1.

**Practical Implementation.** Besides our theoretical results above, we also provide practical suggestions for implementing blockFW. Roughly speaking, one can automatically select a different “good” rank \( k \) for each iteration. This can be done by iteratively finding the 1st, 2nd, 3rd, etc., top singular vectors of the underlying matrix, and then stop this process whenever the objective decrease is not worth further increasing the value \( k \). We discuss the details in Section 6.

## 2 Preliminaries and Notation

For a positive integer \( n \), we define \( [n] := \{1, 2, \ldots, n\} \). For a matrix \( A \), we denote by \( \|A\|_F, \|A\|_2 \) and \( \|A\|_r \) respectively the Frobenius norm, the spectral norm, and the trace norm of \( A \). We use \( \langle \cdot, \cdot \rangle \) to

\[\text{In fact, LazySVD is a general framework that says, with a meaningful theoretical support, one can apply a reasonable 1-SVD algorithm \( k \) times in order to compute \( k \)-SVD. For simplicity, in this paper, whenever referring to LazySVD, we mean to apply the Lanczos method \( k \) times.}\]
denote the (Euclidean) inner products between vectors, or the (trace) inner products between matrices (i.e., $\langle A, B \rangle = \text{Tr}(AB^\top)$). We denote by $\sigma_i(A)$ the $i$-th largest singular value of a matrix $A$, and by $\sigma_{\min}(A)$ the minimum non-zero singular value of $A$. We use $\text{nnz}(A)$ to denote the time complexity of multiplying matrix $A$ to a vector (which is at most the number of non-zero entries of $A$). We define the (unit) trace-norm ball $B_{m,n}$ in $\mathbb{R}^{m \times n}$ as $B_{m,n} := \{ X \in \mathbb{R}^{m \times n} : \|X\|_* \leq 1 \}$.

**Definition 2.1.** For a differentiable convex function $f : \mathcal{K} \to \mathbb{R}$ over a convex set $\mathcal{K} \subseteq \mathbb{R}^{m \times n}$, we say
- $f$ is $\beta$-smooth if $f(Y) \leq f(X) + \langle \nabla f(X), Y - X \rangle + \frac{\beta}{2} \|X - Y\|_F^2$ for all $X,Y \in \mathcal{K}$;
- $f$ is $\alpha$-strongly convex if $f(Y) \geq f(X) + \langle \nabla f(X), Y - X \rangle + \frac{\alpha}{2} \|X - Y\|_F^2$ for all $X,Y \in \mathcal{K}$.

For Problem (1.1), we assume $f$ is differentiable, $\beta$-smooth, and $\alpha$-strongly convex over $B_{m,n}$. We denote by $\kappa = \frac{\beta}{\alpha}$ the condition number of $f$, and by $X^*$ the minimizer of $f(X)$ over the trace-norm ball $B_{m,n}$. The strong convexity of $f(X)$ implies:

**Fact 2.2.** $f(X) - f(X^*) \geq \frac{\alpha}{2} \|X - X^*\|_F^2$ for all $X \in \mathcal{K}$.

**Proof.** The minimality of $X^*$ implies $\langle \nabla f(X^*), X - X^* \rangle \geq 0$ for all $X \in \mathcal{K}$. The fact follows then from the $\alpha$-strong convexity of $f$. \qed

**The Frank-Wolfe Algorithm.** We now quickly review the Frank-Wolfe algorithm (see Algorithm 1) and its relation to PGD.

**Algorithm 1 Frank-Wolfe**

**Input:** Step sizes $\{\eta_t\}_{t \geq 1}$ ($\eta_t \in [0, 1]$), starting point $X_1 \in B_{m,n}$

1: for $t = 1, 2, \ldots$ do
2: $V_t \leftarrow \arg\min_{V \in B_{m,n}} \langle \nabla f(X_t), V \rangle$
3: $X_{t+1} \leftarrow X_t + \eta_t (V_t - X_t)$
4: end for

Let $h_t = f(X_t) - f(X^*)$ be the approximation error of $X_t$. The convergence analysis of Algorithm 1 is based on the following relation:

$$
\begin{align*}
 h_{t+1} &= f(X_t + \eta_t (V_t - X_t)) - f(X^*) \\
 &\overset{\Theta}{\leq} h_t + \eta_t \langle \nabla f(X_t), V_t - X_t \rangle + \frac{\beta}{2} \eta_t^2 \|V_t - X_t\|_F^2 \\
 &\overset{\Phi}{\leq} h_t + \eta_t \langle \nabla f(X_t), X^* - X_t \rangle + \frac{\beta}{2} \eta_t^2 \|V_t - X_t\|_F^2 \\
 &\leq (1 - \eta_t) h_t + \frac{\beta}{2} \eta_t^2 \|V_t - X_t\|_F^2.
\end{align*}
$$

(2.1)

Above, inequality $\Theta$ uses the $\beta$-smoothness of $f$, inequality $\Phi$ is due to the choice of $V_t$ in Line 2, and inequality $\Phi$ follows from the convexity of $f$. Based on (2.1), a suitable choice of the step size $\eta_t = \Theta(1/t)$ gives the convergence rate $O(\beta/\epsilon)$ for the Frank-Wolfe algorithm.

If $f$ is also $\alpha$-strongly convex, a linear convergence rate can be achieved if we replace the linear optimization step in Line 2 with a constrained quadratic minimization:

$$
 V_t \leftarrow \arg\min_{V \in B_{m,n}} \langle \nabla f(X_t), V - X_t \rangle + \frac{\beta}{2} \eta_t \|V - X_t\|_F^2.
$$

(2.2)

In fact, if $V_t$ is defined as above, we have the following relation similar to (2.1):

$$
\begin{align*}
 h_{t+1} &= h_t + \eta_t \langle \nabla f(X_t), V_t - X_t \rangle + \frac{\beta}{2} \eta_t^2 \|V_t - X_t\|_F^2 \\
 &\overset{\Phi}{\leq} h_t + \eta_t \langle \nabla f(X_t), X^* - X_t \rangle + \frac{\beta}{2} \eta_t^2 \|X^* - X_t\|_F^2 \\
 &\leq (1 - \eta_t + \kappa \eta_t^2) h_t,
\end{align*}
$$

(2.3)

where the last inequality follows from Fact 2.2. Given (2.3), we can choose $\eta_t = \frac{1}{2\kappa}$ to obtain a linear convergence rate because $h_{t+1} \leq (1 - 1/4\kappa) h_t$. This is the main idea behind the projected gradient descent (PGD) method. Unfortunately, optimizing $V_t$ from (2.2) requires a projection operation onto $B_{m,n}$, and this further requires a full singular value decomposition of the matrix $\nabla f(X_t) - \beta \eta_t X_t$. 4
3 A Rank-\(k\) Variant of Frank-Wolfe

Our main idea comes from the following simple observation. Suppose we choose \(\eta_t = \eta = \frac{1}{\kappa}\) for all iterations, and suppose \(\text{rank}(X^*) \leq k\). Then we can add a low-rank constraint to \(V_t\) in (2.2):

\[
V_t \leftarrow \arg\min_{V \in \mathbb{R}^m \times n, \text{rank}(V) \leq k} \langle \nabla f(X_t), V - X_t \rangle + \frac{\beta}{2} \eta \|V - X_t\|_F^2.
\] (3.1)

Under this new choice of \(V_t\), it is obvious that the same inequalities in (2.3) remain to hold, and thus the linear convergence rate of PGD can be preserved. Let us now discuss how to solve (3.1).

3.1 Solving the Low-Rank Quadratic Minimization (3.1)

Although (3.1) is non-convex, we prove that it can be solved efficiently. To achieve this, we first show that \(V_t\) is in the span of the top \(k\) singular vectors of \(\beta \eta X_t - \nabla f(X_t)\).

**Lemma 3.1.** The minimizer \(V_t\) of (3.1) can be written as \(V_t = \sum_{i=1}^{k} a_i u_i v_i^\top\), where \(a_1, \ldots, a_k\) are nonnegative scalars, and \((u_i, v_i)\) is the pair of the left and right singular vectors of \(A_t := \beta \eta X_t - \nabla f(X_t)\) corresponding to its \(i\)-th largest singular value.

The proof of Lemma 3.1 is given in Appendix B. Now, owing to Lemma 3.1, we can perform \(k\)-SVD on \(A_t\) to compute \(\{(u_i, v_i)\}_{i \in [k]}\), plug the expression \(V_t = \sum_{i=1}^{k} a_i u_i v_i^\top\) into the objective of (3.1), and then search for the optimal values \(\{a_i\}_{i \in [k]}\). The last step is equivalent to minimizing \(-\sum_{i=1}^{k} \sigma_i a_i + \frac{\beta}{2} \sum_{i=1}^{k} a_i^2\) over the simplex \(\Delta := \{a \in \mathbb{R}^k : a_1, \ldots, a_k \geq 0, \|a\|_1 \leq 1\}\), which is the same as projecting the vector \(\frac{1}{\beta \eta}(\sigma_1, \ldots, \sigma_k)\) onto the simplex \(\Delta\). It can be easily solved in \(O(k \log k)\) time (see for instance the applications in [16]).

3.2 Our Algorithm and Its Convergence

We summarize our algorithm in Algorithm 2 and call it \texttt{blockFW}.

**Algorithm 2 \texttt{blockFW}**

**Input:** Rank parameter \(k\), starting point \(X_1 = 0\)

1: \(\eta \leftarrow \frac{1}{\kappa}\)
2: for \(t = 1, 2, \ldots\) do
3: \(A_t \leftarrow \beta \eta X_t - \nabla f(X_t)\)
4: \((u_1, v_1, \ldots, u_k, v_k) \leftarrow \text{k-SVD}(A_t)\)
5: \(a \leftarrow \arg\min_{u \in \mathbb{R}^k, a \geq 0, \|a\|_1 \leq 1} \|a - \frac{1}{\beta \eta} \sigma\|_2\) \(\circ (u_i, v_i)\) is the \(i\)-th largest pair of left/right singular vectors of \(A_t\)
6: \(V_t \leftarrow \sum_{i=1}^{k} a_i u_i v_i^\top\)
7: \(X_{t+1} \leftarrow X_t + \eta(V_t - X_t)\)
8: end for

Since the state-of-the-art algorithms for \(k\)-SVD are iterative methods, which in theory can only give approximate solutions, we now study the convergence of \texttt{blockFW} given approximate \(k\)-SVD solvers.

We introduce the following notion of an approximate solution to the low-rank quadratic minimization problem (3.1).

**Definition 3.2.** Let \(g_t(V) = \langle \nabla f(X_t), V - X_t \rangle + \frac{\beta}{2} \eta \|V - X_t\|_F^2\) be the objective function in (3.1), and let \(g^*_t = g_t(X^*)\). Given parameters \(\gamma \geq 0\) and \(\varepsilon \geq 0\), a feasible solution \(V\) to (3.1) is called \((\gamma, \varepsilon)\)-approximate if it satisfies \(g(V) \leq (1 - \gamma) g^*_t + \varepsilon\).

Note that the above multiplicative-additive definition makes sense because \(g^*_t \leq 0\):

**Fact 3.3.** If \(\text{rank}(X^*) \leq k\), for our choice of step size \(\eta = \frac{1}{\kappa}\), we have \(g^*_t = g_t(X^*) \leq -(1 - \kappa \eta) h_t = -\frac{h_t}{2} \leq 0\) according to (2.3).

The next theorem gives the linear convergence of \texttt{blockFW} under the above approximate solutions to
Theorem 3.4. Suppose \( \text{rank}(X^*) \leq k \) and \( \varepsilon > 0 \). If each \( V_i \) computed in blockFW is a \((\frac{1}{2}, \frac{\varepsilon}{\gamma})\)-approximate solution to \((3.1)\) then for every \( t \), the error \( h_t = f(X_t) - f(X^*) \) satisfies

\[
h_t \leq (1 - \frac{1}{8\varepsilon^2})^{t-1} h_1 + \frac{\varepsilon}{\gamma}.
\]

As a consequence, it takes \( O(\kappa \log \frac{h_1}{\varepsilon}) \) iterations to achieve the target error \( h_t \leq \varepsilon \).

Based on Theorem 3.4 the per-iteration running time of blockFW is dominated by the time necessary to produce a \((\frac{1}{2}, \frac{\varepsilon}{\gamma})\)-approximate solution \( V_t \) to \((3.1)\), which we study in Section 4.

4 Per-Iteration Running Time Analysis

In this section, we study the running time necessary to produce a \((\frac{1}{2}, \varepsilon)\)-approximate solution \( V_t \) to \((3.1)\). In particular, we wish to show a running time that depends only \( \text{poly-logarithmically} \) on \( 1/\varepsilon \). The reason is that, since we are concerning about the linear convergence rate (i.e., \( \log(1/\varepsilon) \)) in this paper, it is not meaningful to have a per-iteration complexity that scales polynomially with \( 1/\varepsilon \).

Remark 4.1. To the best of our knowledge, the Frank-Wolfe method and Garber’s method [6] have their worst-case per-iteration complexities scaling polynomially with \( 1/\varepsilon \). In theory, this also slows down their overall performance in terms of the dependency on \( 1/\varepsilon \).

4.1 Step 1: The Necessary \( k \)-SVD Accuracy

We first show that if the \( k \)-SVD in Line 4 of blockFW is solved sufficiently accurate, then \( V_t \) obtained in Line 6 will be a sufficiently good approximate solution to \((3.1)\). For notational simplicity, in this section we denote \( G_t := \|\nabla f(X_t)\|_2 + \alpha \), and we let \( k^* = \text{rank}(X^*) \leq k \).

Lemma 4.2. Suppose \( \gamma \in [0, 1] \) and \( \varepsilon \geq 0 \). In each iteration \( t \) of blockFW if the vectors \( u_1, v_1, \ldots, u_k, v_k \) returned by \( k \)-SVD in Line 4 satisfy \( u_i^\top A v_i \geq (1 - \gamma) \sigma_i(A_t) - \varepsilon \) for all \( i \in [k^*] \), then \( V_t = \sum_{i=1}^{k} a_i u_i v_i^\top \) obtained in Line 6 is \((\frac{\sqrt{h_1}}{\gamma} + 2, \gamma, \varepsilon)\)-approximate to \((3.1)\).

The proof of Lemma 4.2 is in Appendix C, and is based on our earlier characterization Lemma 3.1.

4.2 Step 2: The Time Complexity of \( k \)-SVD

We recall the following complexity statement for \( k \)-SVD:

Theorem 4.3 ([1]). The running time to compute the \( k \)-SVD of \( A \in \mathbb{R}^{m \times n} \) using LazySVD is\footnote{The first is known as the gap-free result because it does not depend on the gap between any two singular values. The second is known as the gap-dependent result, and it requires a \( k \times k \) full SVD after the \( k \) approximate singular vectors are computed one by one. The \( \tilde{O} \) notation hides poly-log factors in \( 1/\varepsilon, 1/\gamma, m, n, \) and \( 1/\text{gap} \).}

\[
\tilde{O} \left( \frac{knz(A)+k^2(m+n)}{\sqrt{(\kappa + \gamma)^2}} \right) \quad \text{or} \quad \tilde{O} \left( \frac{knz(A)+k^2(m+n)}{\sqrt{\text{gap}}} \right).
\]

In the former case, we can have \( u_i^\top A v_i \geq (1 - \gamma) \sigma_i(A) \) for all \( i \in [k] \); in the latter case, if \( \text{gap} \in \left( 0, \frac{\sigma_{k+1}(A)}{\sigma_{k+1}(A)} \right) \) for some \( k^* \in [k] \), then we can guarantee \( u_i^\top A v_i \geq \sigma_i(A) - \varepsilon \) for all \( i \in [k^*] \).

The First Attempt. Recall that we need a \((\frac{1}{2}, \varepsilon)\)-approximate solution to \((3.1)\). Using Lemma 4.2, it suffices to obtain a \((1 - \gamma)\)-multiplicative approximation to the \( k \)-SVD of \( A_t \) (i.e., \( u_i^\top A_t v_i \geq (1 - \gamma) \sigma_i(A_t) \) for all \( i \in [k] \)), as long as \( \gamma \leq \frac{1}{120\varepsilon^2} \). Therefore, we can directly apply the first running time in Theorem 4.3: \( \tilde{O} \left( \frac{knz(A)+k^2(m+n)}{\sqrt{(\kappa + \gamma)^2}} \right) \). However, when \( h_t \) is very small, this running time can be unbounded. In that case, we observe that \( \gamma = \frac{\sigma_{k+1}(A_t)}{\sigma_{k+1}(A_t)} \) (independent of \( h_t \)) also suffices: since \( \|A_t\|_2 = \|A_t - \nabla f(X_t)\|_2 \leq \frac{\gamma}{\sqrt{\kappa + \gamma^2}} \|\nabla f(X_t)\|_2 \leq G_t \), from \( u_i^\top A_t v_i \geq (1 - \varepsilon/G_t) \sigma_i(A_t) \) we have \( u_i^\top A_t v_i \geq \sigma_i(A_t) - \frac{\varepsilon}{\sqrt{\kappa + \gamma^2}} \sigma_i(A_t) \geq \sigma_i(A_t) - \varepsilon \); then according to Lemma 4.2 we can obtain \((0, \varepsilon)\)-approximation to \((3.1)\) which is stronger than \((\frac{1}{2}, \varepsilon)\)-approximation. We summarize this running time (using \( \gamma = \frac{\sigma_{k+1}(A_t)}{\sigma_{k+1}(A_t)} \) in Claim 4.5) the running time depends polynomially on \( \frac{1}{\varepsilon} \).

The Second Attempt. To make our linear convergence rate (i.e., the \( \log(1/\varepsilon) \) rate) meaningful, we
want the $k$-SVD running time to depend \textit{poly-logarithmically} on $1/\varepsilon$. Therefore, when $h_t$ is small, we wish to instead apply the \textit{second} running time in Theorem 4.3.

Recall that $X^*$ has rank $k^*$ so $\sigma_{k^*}(X^*) - \sigma_{k^*+1}(X^*) = \sigma_{\text{min}}(X^*)$. We can show that this implies $A^* := \frac{\kappa}{\varepsilon}X^* - \nabla f(X^*)$ also has a large gap $\sigma_{k^*}(A^*) - \sigma_{k^*+1}(A^*)$. (See Lemma C.1.) Now, according to Fact 2.2 when $h_t$ is small, $X_t$ and $X^*$ are sufficiently close. This means $A_t = \frac{\kappa}{\varepsilon}X_t - \nabla f(X_t)$ is also close to $A^*$, and thus has a large gap $\sigma_{k^*}(A_t) - \sigma_{k^*+1}(A_t)$. Then we can apply the second running time in Theorem 4.3.

4.2.1 Formal Running Time Statements

Fact 4.4. We can store $X_t$ as a decomposition into at most $\text{rank}(X_t) \leq kt$ rank-1 components.\footnote{In Section 5, we show how to ensure that $\text{rank}(X_t)$ is always $O(k\kappa \log \kappa)$, a quantity independent of $t$.} Therefore, for $A_t = \frac{\kappa}{\varepsilon}X_t - \nabla f(X_t)$, we have $\text{nnz}(A_t) \leq \text{nnz}(\nabla f(X_t)) + (m + n)\text{rank}(X_t) \leq \text{nnz}(\nabla f(X_t)) + (m + n)kt$.

If we always use the first running time in Theorem 4.3, then Fact 4.4 implies:

Claim 4.5. The $k$-SVD computation in the $t$-th iteration of $\text{blockFW}$ can be implemented in $\tilde{O}(\left(k \cdot \text{nnz}(\nabla f(X_t)) + k^2(m + n)t\right)\sqrt{\text{nnz}(X_t)/\varepsilon})$ time.

Remark 4.6. As long as $(m + n)kt \leq \text{nnz}(\nabla f(X_t))$, the $k$-SVD running time in Claim 4.5 becomes $\tilde{O}(k \cdot \text{nnz}(\nabla f(X_t))\sqrt{\text{nnz}(X_t)/\varepsilon})$, which roughly equals $k$-times the $1$-SVD running time $\tilde{O}(\text{nnz}(\nabla f(X_t))\sqrt{\|\nabla f\|_2/\varepsilon})$ of FW and Garber [6]. Since in practice, it suffices to run $\text{blockFW}$ and FW for a few hundred $1$-SVD computations, the relation $(m + n)kt \leq \text{nnz}(\nabla f(X_t))$ is often satisfied.

If, as discussed above, we apply the first running time in Theorem 4.3 only for large $h_t$, and apply the second running time in Theorem 4.3 for small $h_t$, then we obtain the following theorem whose proof is given in Appendix C.

Theorem 4.7. The $k$-SVD computation in the $t$-th iteration of $\text{blockFW}$ can be implemented in $\tilde{O}\left((k \cdot \text{nnz}(\nabla f(X_t)) + k^2(m + n)t)\sqrt{\text{nnz}(X_t)/\varepsilon}\right)$ time.

Remark 4.8. Since according to Theorem 3.4 we only need to run $\text{blockFW}$ for $O(\kappa \log(1/\varepsilon))$ iterations, we can plug $t = O(\kappa \log(1/\varepsilon))$ into Claim 4.5 and Theorem 4.7 and obtain the running time presented in (1.2). The per-iteration running time of $\text{blockFW}$ depends \textit{poly-logarithmically} on $1/\varepsilon$. In contrast, the per-iteration running times of Garber [6] and FW depend \textit{polylogarithmically} on $1/\varepsilon$, making their total running times even worse in terms of dependency on $1/\varepsilon$.

5 Maintaining Low-Rank Iterate

One of the main reasons to impose trace-norm constraints is to produce low-rank solutions. However, the rank of iterate $X_t$ in our algorithm $\text{blockFW}$ can be as large as $kt$, which is much larger than $k$, the rank of the optimal solution $X^*$. In this section, we show that by adding a simple modification to $\text{blockFW}$, we can make sure the rank of $X_t$ is $O(k\kappa \log \kappa)$ in all iterations $t$, without hurting the convergence rate much.

We modify $\text{blockFW}$ as follows. Whenever $t - 1$ is a multiple of $S = [8\kappa(\log \kappa + 1)]$, we compute (note that this is the same as setting $\eta = 1$ in (3.1))

\[
W_t \leftarrow \arg \min_{W \in B_{m,n} \text{ rank}(W) \leq k} \langle \nabla f(X_t), W - X_t \rangle + \frac{\beta}{2} \|W - X_t\|_F^2,
\]

and let the next iterate $X_{t+1}$ be $W_t$. In all other iterations the algorithm is unchanged. After this change, the function value $f(X_{t+1})$ may be greater than $f(X_t)$, but can be bounded as follows:

Lemma 5.1. Suppose $\text{rank}(X^*) \leq k$. Then we have $f(W_t) - f(X^*) \leq kh_t$.

Proof. We have the following relation similar to (2.3):

\[
f(W_t) - f(X^*) \leq h_t + \langle \nabla f(X_t), W_t - X_t \rangle + \frac{\beta}{2} \|W_t - X_t\|_F^2 \leq h_t + \langle \nabla f(X_t), X^* - X_t \rangle + \frac{\beta}{2} \|X^* - X_t\|_F^2,
\]
\[
\leq h_t - h_t + \frac{\beta}{2} \cdot \frac{2}{\alpha} h_t = \kappa h_t .
\]

From Theorem 3.4 we know that \( h_{S+1} \leq (1 - \frac{1}{8\kappa})h_1 + \frac{2}{\alpha} \leq (1 - \frac{1}{8\kappa})h_1 + \frac{2}{\alpha} = \frac{1}{16\kappa}h_1 + \epsilon/2 \). Therefore, after setting \( X_{S+2} = W_{S+1} \), we still have \( h_{S+2} \leq \frac{1}{16\kappa}h_1 + \epsilon/2 \) (according to Lemma 5.1). Continuing this analysis (letting the \( \kappa \epsilon \) here be the “new \( \epsilon \)”), we know that this modified version of blockFW converges to an \( \epsilon \)-approximate minimizer in \( O(\kappa \log \kappa \log \frac{2}{\alpha}) \) iterations.

Remark 5.2. Since in each iteration the rank of \( X_t \) is increased by at most \( k \), if we do the modified step every \( S = O(\kappa \log \kappa) \) iterations, we have that throughout the algorithm, \( \text{rank}(X_t) \) is never more than \( O(k \kappa \log \kappa) \). Furthermore we can always store \( X_t \) using \( O(k \kappa \log \kappa) \) vectors, instead of storing all the singular vectors obtained in previous iterations.

6 Preliminary Empirical Evaluation

We conclude this paper with some preliminary experiments to test the performance of blockFW. We first recall two machine learning tasks that fall into Problem (1.1).

Matrix Completion. Suppose there is an unknown matrix \( M \in \mathbb{R}^{m \times n} \) close to low-rank, and we observe a subset \( \Omega \) of its entries – that is, we observe \( M_{i,j} \) for every \( (i,j) \in \Omega \). (Think of \( M_{i,j} \) as user \( i \)’s rating of movie \( j \).) One can recover \( M \) by solving the following convex program:

\[
\min_{X \in \mathbb{R}^{m \times n}} \left\{ \frac{1}{2} \sum_{(i,j) \in \Omega} (X_{i,j} - M_{i,j})^2 \mid \| X \| \leq \theta \right\} .
\]

Although Problem (6.1) is not strongly convex, under certain conditions Fact 2.2 still holds [14].

Polynomial Neural Networks. Polynomial networks are neural networks with quadratic activation function \( \sigma(a) = a^2 \). Livni et al. [12] showed that such networks can express any function computed by a Turing machine, similar to networks with ReLU or sigmoid activations. Following [12], we consider the class of 2-layer polynomial networks with inputs from \( \mathbb{R}^d \) and \( k \) hidden neurons:

\[
P_k = \left\{ x \mapsto \sum_{j=1}^k a_j w_j^\top x^2 \mid \forall j \in [k], w_j \in \mathbb{R}^d, \|w_j\|_2 = 1 \right\} .
\]

If we write \( A = \sum_{i=1}^k a_i w_i w_i^\top \), we have the following equivalent formulation:

\[
P_k = \left\{ x \mapsto x^\top Ax \mid A \in \mathbb{R}^{d \times d}, \text{rank}(A) \leq k \right\} .
\]

Therefore, if we replace the hard rank constraint with trace norm \( \| A \| \leq \theta \), the task of empirical risk minimization (ERM) given training data \( \{(x_1, y_1), \ldots, (x_N, y_N)\} \subset \mathbb{R}^d \times \mathbb{R} \) can be formulated as

\[
\min_{A \in \mathbb{R}^{d \times d}} \left\{ \frac{1}{2} \sum_{i=1}^N (x_i^\top Ax_i - y_i)^2 \mid \| A \| \leq \theta \right\} .
\]

Since \( f(A) = \frac{1}{2} \sum_{i=1}^N (x_i^\top Ax_i - y_i)^2 \) is convex in \( A \), the above problem falls into Problem (1.1). Again, this objective \( f(A) \) may not be strongly convex, but we still perform experiments on it.

6.1 Preliminary Evaluation 1: Matrix Completion on Synthetic Data

We consider the following synthetic experiment for matrix completion. We generate a random rank-10 matrix in dimension 1000 \times 1000, plus some small noise. We include each entry into \( \Omega \) with probability 1/2. We scale \( M \) to \( \| M \|_\ast = 10000 \), so we set \( \theta = 10000 \) in (6.1).

We compare blockFW with FW and Garber [6]. When implementing the three algorithms, we use exact line search. For Garber’s algorithm, we tune its parameter \( \eta_i = \frac{c}{\theta} \) with different constant values \( c \), and then exactly search for the optimum \( \eta_i \). When implementing blockFW we use \( k = 10 \) and \( \eta = 0.2 \). We use the MATLAB built-in solver for 1-SVD and k-SVD.

In Figure 1(a) we compare the numbers of 1-SVD computations for the three algorithms. The plot confirms our finding: it suffices to apply a rank-\( k \) variant FW in order to achieve linear convergence.

6.2 Auto Selection of \( k \)

In practice, it is often unrealistic to know \( k \) in advance. Although one can simultaneously try \( k = 1, 2, 4, 8, \ldots \) and output the best possible solution, this can be unpleasant to work with. We propose the following modification to blockFW which automatically chooses \( k \).

\footnote{We consider square loss for simplicity. It can be any loss function \( \ell(x_i^\top Ax_i, y_i) \) convex in its first argument.}
In each iteration $t$, we first run 1-SVD and compute the objective decrease, denoted by $d_t \geq 0$. Now, given any approximate $k$-SVD decomposition of the matrix $A_t = \beta \eta X_t - \nabla f(X_t)$, we can compute its $(k+1)$-SVD using one additional 1-SVD computation according to the LazySVD framework [1]. We compute the new objective decrease $d_{k+1}$. We stop this process and move to the next iteration $t+1$ whenever $d_{k+1} < \frac{d_k}{k+1}$. In other words, we stop whenever it “appears” not worth further increasing $k$. We count this iteration $t$ as using $k+1$ computations of 1-SVD.

All the experiments on real-life datasets are performed using this above auto-$k$ process.

### 6.3 Preliminary Evaluation 2: Matrix Completion on MovieLens

We study the same experiment in Garber [6], the matrix completion Problem (6.1) on datasets MovieLens100K $(m = 943, n = 1862$ and $|\Omega| = 10^5$) and MovieLens1M $(m = 6040, n = 3952$ and $|\Omega| \approx 10^6$). In the second dataset, following [6], we further subsample $\Omega$ so it contains about half of the original entries. For each dataset, we run FW, Garber, and blockFW with three different choices of $\theta$.

We present the six plots side-by-side in Figure 2 in Appendix A.

We observe that when $\theta$ is large, there is no significant advantage for using blockFW. This is because the rank of the optimal solution $X^*$ is also high for large $\theta$. In contrast, when $\theta$ is small (so $X^*$ is of low rank), as demonstrated for instance by Figure 1(b), it is indeed beneficial to apply blockFW.

### 6.4 Preliminary Evaluation 3: Polynomial Neural Network on MNIST

We use the 2-layer neural network Problem (6.2) to train a binary classifier on the MNIST dataset of handwritten digits, where the goal is to distinguish images of digit “0” from images of other digits. The training set contains $N = 60000$ examples each of dimension $d = 28 \times 28 = 784$. We set $y_i = 1$ if that example belongs to digit “0” and $y_i = 0$ otherwise. We divide the original grey levels by 256 so $x_i \in [0, 1]^d$.

We again try three different values of $\theta$, and compare FW, Garber, and blockFW.

We present the three plots side-by-side in Figure 3 in Appendix A.

The performance of our algorithm is comparable to FW and Garber for large $\theta$, but as demonstrated for instance by Figure 1(c), when $\theta$ is small so rank($X^*$) is small, it is beneficial to use blockFW.

### 7 Conclusion

In this paper, we develop a rank-$k$ variant of Frank-Wolfe for Problem (1.1) and show that: (1) it converges in log(1/ε) rate for smooth and strongly convex functions, and (2) its per-iteration complexity scales with polylog(1/ε). Preliminary experiments suggest that the value $k$ can also be automatically selected, and our algorithm outperforms FW and Garber [6] when $X^*$ is of relatively smaller rank.

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7We perform exact line search for all algorithms. For Garber [6], we tune the best $\eta = \frac{1}{4}$ and exactly search for the optimal $\eta_t$. For blockFW, we let $k$ be chosen automatically and choose $\eta = 0.01$ for all the six experiments.

8We perform exact line search for all algorithms. For Garber [6], we tune the best $\eta = \frac{1}{4}$ and exactly search for the optimal $\eta_t$. For blockFW, we let $k$ be chosen automatically and choose $\eta = 0.0005$ for all the three experiments.
We hope more rank-$k$ variants of Frank-Wolfe can be developed in the future.

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References

[1] Zeyuan Allen-Zhu and Yuanzhi Li. LazySVD: Even faster SVD decomposition yet without agonizing pain. In \textit{NIPS}, pages 974–982, 2016.

[2] Emmanuel Candes and Benjamin Recht. Exact matrix completion via convex optimization. \textit{Communications of the ACM}, 55(6):111–119, 2012.

[3] Emmanuel J Candes, Yonina C Eldar, Thomas Strohmer, and Vladislav Voroninski. Phase retrieval via matrix completion. \textit{SIAM review}, 57(2):225–251, 2015.

[4] Miroslav Dudík, Zaid Harchaoui, and Jérôme Malick. Lifted coordinate descent for learning with trace-norm regularization. In \textit{AISTATS}, pages 327–336, 2012.

[5] Marguerite Frank and Philip Wolfe. An algorithm for quadratic programming. \textit{Naval research logistics quarterly}, 3(1-2):95–110, 1956.

[6] Dan Garber. Faster projection-free convex optimization over the spectrahedron. In \textit{NIPS}, pages 874–882, 2016.

[7] Dan Garber and Elad Hazan. A linearly convergent conditional gradient algorithm with applications to online and stochastic optimization. \textit{arXiv preprint arXiv:1301.4666}, 2013.

[8] Dan Garber and Elad Hazan. Faster rates for the frank-wolfe method over strongly-convex sets. In \textit{ICML}, pages 541–549, 2015.

[9] Elad Hazan. Sparse approximate solutions to semidefinite programs. In \textit{Latin American Symposium on Theoretical Informatics}, pages 306–316. Springer, 2008.

[10] Martin Jaggi and Marek Sulovský. A simple algorithm for nuclear norm regularized problems. In \textit{ICML}, pages 471–478, 2010.

[11] Simon Lacoste-Julien and Martin Jaggi. An affine invariant linear convergence analysis for frank-wolfe algorithms. \textit{arXiv preprint arXiv:1312.7864}, 2013.

[12] Roi Livni, Shai Shalev-Shwartz, and Ohad Shamir. On the computational efficiency of training neural networks. In \textit{NIPS}, pages 855–863, 2014.

[13] Cameron Musco and Christopher Musco. Randomized block krylov methods for stronger and faster approximate singular value decomposition. In \textit{NIPS}, pages 1396–1404, 2015.

[14] Sahand Negahban, Bin Yu, Martin J Wainwright, and Pradeep K Ravikumar. A unified framework for high-dimensional analysis of $m$-estimators with decomposable regularizers. In \textit{NIPS}, pages 1348–1356, 2009.

[15] Yurii Nesterov. \textit{Introductory Lectures on Convex Programming Volume: A Basic course}, volume I. Kluwer Academic Publishers, 2004.

[16] Yurii Nesterov. Smooth minimization of non-smooth functions. \textit{Mathematical Programming}, 103(1):127–152, December 2005.

[17] Shai Shalev-Shwartz, Alon Gonen, and Ohad Shamir. Large-scale convex minimization with a low-rank constraint. \textit{arXiv preprint arXiv:1106.1622}, 2011.
Appendix

A Additional Plots

Figure 2: Log objective error vs. the number of 1-SVD computations, for matrix completion.

Figure 3: Log objective error vs. the number of 1-SVD computations, for polynomial neural network.
B Missing Proofs for Section 3

The proof of Lemma 3.1 relies on the following folklore lemma.

**Lemma B.1.** Let \( A \in \mathbb{R}^{m \times n} \) and \( r = \min \{ m, n \} \). Let \( g : \mathbb{R}^{r} \to \mathbb{R} \) be a twice-differentiable convex function. For any two sets of orthonormal vectors \( \{ u_1, \ldots, u_r \} \subset \mathbb{R}^m \) and \( \{ v_1, \ldots, v_r \} \subset \mathbb{R}^n \), there exist a permutation \( \pi \) over \( [r] \) and \( \xi_1, \ldots, \xi_r \in \{-1, 1\} \) such that
\[
g(u_1^T Av_1, u_2^T Av_2, \ldots, u_r^T Av_r) \leq g(\xi_1 \sigma_1(A), \xi_2 \sigma_2(A), \ldots, \xi_r \sigma_r(A)) .
\]

We first prove Lemma 3.1 using Lemma B.1, and then give the proof of Lemma B.1.

**Lemma 3.1 (rephrased).** The minimizer \( V_\ell \) of (3.1) can be written as \( V_\ell = \sum_{i=1}^{k} a_i u_i v_i^T \), where \( a_1, \ldots, a_k \) are nonnegative reals, and \( (u_i, v_i) \) is the pair of the left and right singular vectors of \( A := \beta \eta X_t - \nabla f(X_t) \) corresponding to its \( i \)-th largest singular value.

**Proof.** The quadratic objective in (3.1) can be written as \( \langle \nabla f(X_t) - \beta \eta X_t, V \rangle + \frac{\lambda}{2} \eta \| V \|^2_F - \langle \nabla f(X_t), X_t \rangle + \frac{\beta}{2} \eta \| X_t \|^2_F \), where the last two terms do not depend on \( V \). Hence, (3.1) is equivalent to
\[
V_\ell \leftarrow \arg \min \limits_{V \in \mathcal{B}_{m,n}, \operatorname{rank}(V) \leq k} \langle A_t, V \rangle - \frac{\beta}{2} \eta \| V \|^2_F . \tag{B.1}
\]

Since we have \( \operatorname{rank}(V) \leq k \), we can write its SVD as \( V = \sum_{i=1}^{k} a_i u_i v_i^T \), where \( \{ u_1, \ldots, u_k \} \subset \mathbb{R}^m \) and \( \{ v_1, \ldots, v_k \} \subset \mathbb{R}^n \) are two sets of orthonormal vectors. To prove Lemma 3.1, it suffices to show for every fixed set of singular values \( \{ a_i \}_{i=1}^{k} \) where \( a_1 \geq \cdots \geq a_k \geq 0 \), the optimal choices of \( u_i \)'s and \( v_i \)'s must coincide with the top \( k \) left and right singular vectors of \( A_t \).

When \( a_1, \ldots, a_k \) are fixed, \( \| V \|_F^2 = \sum_{i=1}^{k} a_i^2 \) is also fixed, so (B.1) becomes maximizing \( \langle A_t, V \rangle = \sum_{i=1}^{k} a_i u_i^T A_t v_i \) with \( A = A_t \) and \( g(x_1, \ldots, x_r) = \sum_{i=1}^{k} a_i x_i \), we get \( \sum_{i=1}^{k} a_i u_i^T A_t v_i \leq \sum_{i=1}^{k} a_i \xi_i \sigma(\pi(i)) \) for some permutation \( \pi \) over \( [r] \) and some \( \xi_i \)'s in \( \{-1, 1\} \). Then by the non-negativity of singular values and the rearrangement inequality we have \( \sum_{i=1}^{k} a_i \xi_i \sigma(\pi(i)) (A_t) \leq \sum_{i=1}^{k} a_i \sigma(\pi(i)) (A_t) \leq \sum_{i=1}^{k} a_i \sigma(i) (A_t) \).

In sum, we have shown \( \langle A_t, V \rangle = \sum_{i=1}^{k} a_i u_i^T A_t v_i \leq \sum_{i=1}^{k} a_i \sigma(i) (A_t) \) for any \( u_i \)'s and \( v_i \)'s. It is easy to see that the equality can be attained if \( u_i A_t v_i = \sigma(i) (A_t) \) for all \( i \in [k] \), or equivalently, if \( u_i \) and \( v_i \) are the left and right singular vectors of \( A_t \) corresponding to its \( i \)-th largest singular value.

In order to prove Lemma B.1 (which we believe is a folklore result), we first prove its special case where \( m = n \).

**Lemma B.2.** Let \( A \in \mathbb{R}^{n \times n} \) and \( g : \mathbb{R}^{n} \to \mathbb{R} \) be a twice-differentiable convex function. For any two orthonormal bases \( \{ u_1, \ldots, u_n \} \) and \( \{ v_1, \ldots, v_n \} \) of \( \mathbb{R}^n \), there exists a permutation \( \pi \) over \( [n] \) and \( \xi_1, \ldots, \xi_n \in \{-1, 1\} \) such that
\[
g(u_1^T Av_1, u_2^T Av_2, \ldots, u_n^T Av_n) \leq g(\xi_1 \sigma_1(A), \xi_2 \sigma_2(A), \ldots, \xi_n \sigma_n(A)) .
\]

**Proof.** We denote by \( g_i'(x) \) the \( i \)-th partial derivative of \( g \) at \( x \in \mathbb{R}^n \), and by \( g''_{ij}(x) \) the \((i,j)\)-th second-order partial derivative. WLOG we assume that \( g \) is strictly convex, which implies that \( g''_{ii}(x) + g''_{jj}(x) > 2 |g''_{ij}(x)| \) holds for all \( i, j \) and \( x \). In fact, if \( g \) is not strictly convex, we can add \( \varepsilon \|x\|^2 \) to \( g(x) \) to make it strictly convex, which will add small perturbations to both sides of the desired inequality. Then we can let \( \varepsilon \to 0 \) to make the perturbations arbitrarily small, and the desired inequality follows.

We now fix a pair of orthonormal bases \( \{ u_1, \ldots, u_n \} \) and \( \{ v_1, \ldots, v_n \} \) that maximize \( g(u_1^T Av_1, u_2^T Av_2, \ldots, u_n^T Av_n) \) over all orthonormal bases of \( \mathbb{R}^n \), and consider a pair of indices \( i \neq j \). For \( \varphi \in \mathbb{R} \), define the rotation matrix \( R_\varphi = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix} \). We apply the rotation \( R_\varphi \) to vectors \( u_i \) and \( u_j \), i.e., we let \( u_i(\varphi) = R_\varphi u_i \) and \( u_j(\varphi) = R_\varphi u_j \). Similarly define \( v_i(\varphi) = R_\varphi v_i \) and \( v_j(\varphi) = R_\varphi v_j \). Note that if we replace \( u_i \) and \( u_j \) by \( u_i(\varphi) \) and \( u_j(\varphi) \), \( \{ u_1, u_i(\varphi), \ldots, u_j(\varphi), \ldots, u_n \} \) is still an orthonormal basis. Similarly, \( \{ v_1, v_i(\varphi), \ldots, v_j(\varphi), \ldots, v_n \} \) is also an orthonormal basis.
Let $x(\phi) = (u_1^T A v_1, \ldots, u_i(\phi)^T A v_i(\phi), \ldots, u_j(\phi)^T A v_j(\phi), \ldots, u_n^T A v_n) \in \mathbb{R}^n$ and consider the following function $h$ defined on $\mathbb{R}$:

$$h(\phi) := g(x(\phi)) = g(u_1^T A v_1, \ldots, u_i(\phi)^T A v_i(\phi), \ldots, u_j(\phi)^T A v_j(\phi), \ldots, u_n^T A v_n).$$

By the optimality of $\{u_i\}$ and $\{v_i\}$, we know that $h(\phi)$ achieves its maximum at $\phi = 0$. Since $h$ is twice-differentiable, this means $h'(0) = 0$ and $h''(0) \leq 0$. We can directly calculate $h'(0)$ and $h''(0)$:

$$h'(0) = (g_i' - g_j') \cdot (u_j^T A v_j + u_i^T A v_i),$$

$$h''(0) = 2(g_i'' + g_j'' + 2g_{i,j}'' - 2g_{i,j}'') (u_i^T A v_j + u_j^T A v_i)^2,$$

where all the partial derivatives of $g$ are at point $x(0)$.

Assume that $u_i^T A v_j + u_j^T A v_i \neq 0$. Then from $h'(0) = 0$ we know that $g_i' - g_j' = 0$, which implies $0 \geq h''(0) = 0 + (g_i'' + g_j'' - 2g_{i,j}'') (u_i^T A v_j + u_j^T A v_i)^2 > 0$ (recall that $g_{i,i}'' + g_{j,j}'' - 2g_{i,j}'' > 0$), a contradiction. Therefore we must have $u_i^T A v_j + u_j^T A v_i = 0$.

Next, we apply the rotation $R_\pi$ on \[
\begin{bmatrix}
u_1 \\
u_2 \\
\end{bmatrix}
\]
and the rotation $R_{-\varphi}$ on \[
\begin{bmatrix}
u_1 \\
u_2 \\
\end{bmatrix}
\]
instead. Repeating the same analysis as above, we can obtain $u_i^T A v_j = u_j^T A v_i = 0$. Combining this with $u_i^T A v_j + u_j^T A v_i = 0$, we know that $u_i^T A v_i = 0$. This holds for all $i \neq j$. Since $\{v_1, \ldots, v_n\}$ is an orthonormal basis and $A v_j$ is orthogonal to $u_i$ for every $i \neq j$, we must have $A v_j = \lambda_j u_j (\lambda_j \in \mathbb{R})$ for all $j$. Hence $u_i$'s and $v_j$'s are (left and right) singular vectors of $A$ (up to sign flips), and $u_i$ and $v_i$ correspond to the same singular value. This completes the proof of Lemma B.2.

Now we prove Lemma B.1.

Proof of Lemma B.1 The case where $m = n$ is proved in Lemma B.2. Now WLOG we assume $m < n$. We pad $(n - m) \times n$ zeros to the bottom of $A$ to make it $n \times n$: $\tilde{A} = \begin{bmatrix} A \\ 0 \end{bmatrix}$, and we also pad each $u_i$ with $(n - m)$ zeros: $\tilde{u}_i = \begin{bmatrix} u_i \\ 0 \end{bmatrix} \in \mathbb{R}^n$. We extend $\{\tilde{u}_1, \ldots, \tilde{u}_m\}$ and $\{v_1, \ldots, v_n\}$ to two orthonormal bases $\{\tilde{u}_1, \ldots, \tilde{u}_n\}$ and $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$.

Consider a function $h(x_1, \ldots, x_n) := g(x_1, \ldots, x_m)$ defined on $\mathbb{R}^n$. Using Lemma B.2, we have

$$g(u_1^T A v_1, u_2^T A v_2, \ldots, u_m^T A v_m) = g(\tilde{u}_1^T \tilde{A} v_1, \ldots, \tilde{u}_m^T \tilde{A} v_m) = h(\tilde{u}_1^T \tilde{A} v_1, \ldots, \tilde{u}_m^T \tilde{A} v_m)$$

for some permutation $\pi$ over $[n]$ and $\xi_i$’s in $\{-1, 1\}$. Note that the $n$ singular values of $\tilde{A}$ include all the $m$ singular values of $A$ and $(n - m)$ zeros, i.e., $\sigma_i(\tilde{A}) = \begin{cases} \sigma_i(A) & i \in [m] \\ 0 & i \in \{m + 1, \ldots, n\} \end{cases}$. Thus the remaining thing to prove is that we can choose the permutation $\pi$ such that $\tilde{A} \tilde{x}_i = \tilde{A} \tilde{x}_j = \sigma_i(\tilde{A})$ (for all $i < j$) and $\tilde{A} \tilde{x}_i = \tilde{A} \tilde{x}_j = \sigma_j(\tilde{A})$ (for all $i > j$). Then, by the convexity of $g$ we have

$$g(\xi_1 \sigma_1(\tilde{A}), \ldots, \xi_m \sigma_m(\tilde{A})) = g(\tilde{x}_1, \ldots, \tilde{x}_n).$$

Repeating this procedure, we can eventually make $\sigma(1), \ldots, \sigma(m)$ a permutation of $[m]$ without decreasing the right hand side of $\tilde{A}$. (The values of $\xi_i$’s may change.) This completes the proof of Lemma B.1.

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Theorem 3.4 (restated). Suppose \( \text{rank}(X^*) \leq k \) and \( \varepsilon > 0 \). If each \( V_t \) computed in blockFW is a \((1/2, \varepsilon/8)\)-approximate solution to (3.1) then for every \( t \), the error \( h_t = f(X_t) - f(X^*) \) satisfies

\[
h_t \leq (1 - \frac{1}{8\kappa})^{t-1} h_1 + \frac{\varepsilon}{2}.
\]

As a consequence, it takes \( O(\kappa \log \frac{h_1}{\varepsilon}) \) iterations to achieve the target error \( h_t \leq \varepsilon \).

Proof. The proof only requires a simple modification to our previous analysis (2.3). Using that \( V_t \) is a \((1/2, \varepsilon/8)\)-approximate solution to (3.1) we have

\[
h_{t+1} \leq h_t + \eta(\nabla f(X_t), V_t - X_t) + \frac{\beta}{2} \eta^2 \|V_t - X_t\|_F^2 = h_t + \eta g_t(V_t)
\]

\[
\leq h_t + \eta(g_0^t/2 + \varepsilon/8) = h_t + \eta(-h_t/4 + \varepsilon/8) = \left(1 - \frac{1}{8\kappa}\right) h_t + \frac{\varepsilon}{16\kappa}.
\]

Repeatedly applying the above inequality for \( t = 1, 2, \ldots, \) we get

\[
h_t \leq \left(1 - \frac{1}{8\kappa}\right)^{t-1} h_1 + \frac{\varepsilon}{16\kappa} \left[1 + \left(1 - \frac{1}{8\kappa}\right) + \cdots + \left(1 - \frac{1}{8\kappa}\right)^{-2}\right]
\]

\[
\leq \left(1 - \frac{1}{8\kappa}\right)^{t-1} h_1 + \frac{\varepsilon}{16\kappa} \cdot \frac{1}{1 - (1/(8\kappa))} = \left(1 - \frac{1}{8\kappa}\right)^{t-1} h_1 + \frac{\varepsilon}{2}.
\]

C Missing Proofs for Section 4

Recall that in Section 4 we let \( G_t = \|\nabla f(X_t)\|_2 + \alpha \) and \( k^* = \text{rank}(X^*) \leq k \).

Lemma 4.2 (restated). Suppose \( \gamma \in [0, 1] \) and \( \varepsilon \geq 0 \). In each iteration \( t \) of blockFW, if the vectors \( u_1, v_1, \ldots, u_k, v_k \) returned by \( k \)-SVD in Line 4 satisfy \( u_i^\top A_i v_i \geq (1 - \gamma)\sigma_i(A_i) - \varepsilon \) for all \( i \in [k^*] \), then \( V_t = \sum_{i=1}^k a_i u_i v_i^\top \) obtained in Line 6 is \((\frac{6\kappa\gamma}{\eta^2} + 2\gamma, \varepsilon)\)-approximate to (3.1).

Proof. We rewrite the objective \( g_t(V) \) in (3.1) as \( g_t(V) = -\langle \nabla f(X_t), V - X_t \rangle + \frac{\beta}{2} \eta \|V - X_t\|_F^2 \). Since the rank of \( X^* \) is \( k^* \leq k \), we call define a rank \( k^* \) version of the minimum as

\[
g'_t = \min_{V \in B_{m,n}, \text{rank}(V) \leq k^*} -\langle \nabla f(X_t), V - X_t \rangle + \frac{\beta}{2} \eta \|V - X_t\|_F^2.
\]

We know that \( g'_t \leq g_t(X^*) = g^*_t \). Moreover, According to Lemma 3.1 and the discussion thereafter, we know that

\[
g'_t = \min_{a \in \Delta_{k^*}} \left\{ -\sum_{i=1}^{k^*} \sigma_i(A_i) a_i + \frac{\beta}{2} \eta \sum_{i=1}^{k^*} a_i^2 \right\} + s_t,
\]

where \( \Delta_{k^*} = \{a \in \mathbb{R}^{k^*} : a_1, \ldots, a_{k^*} \geq 0, \sum_{i=1}^{k^*} a_i \leq 1\} \). Denote by \( a^* \in \Delta_{k^*} \) the minimizer in (C.1), i.e., we have

\[
g'_t = -\sum_{i=1}^{k^*} \sigma_i(A_i) a_i^* + \frac{\beta}{2} \eta \sum_{i=1}^{k^*} (a_i^*)^2 + s_t.
\]

Now our algorithm uses approximate singular vectors \( u_i \)'s and \( v_i \)'s which satisfy \( \sigma_i = u_i^\top A_i v_i \geq (1 - \gamma)\sigma_i(A_i) - \varepsilon \) for all \( i \in [k^*] \). Then the \( V_t \) produced by the algorithm should satisfy for \( \Delta_k = \{a \in \mathbb{R}^k : a_1, \ldots, a_k \geq 0, \sum_{i=1}^{k} a_i \leq 1\} \):

\[
g_t(V_t) = \min_{a \in \Delta_k} \left\{ -\sum_{i=1}^{k} \sigma_i a_i + \frac{\beta}{2} \eta \sum_{i=1}^{k} a_i^2 \right\} + s_t.
\]
\[
\min_{a \in \Delta_{k^*}} \left\{ -\sum_{i=1}^{k^*} \sigma_i a_i + \frac{\beta}{2} \eta \sum_{i=1}^{k^*} a_i^2 \right\} + s_t
\]
\[
\leq \min_{a \in \Delta_{k^*}} \left\{ -\sum_{i=1}^{k^*} ((1 - \gamma) \sigma_i (A_t) - \varepsilon) a_i + \frac{\beta}{2} \eta \sum_{i=1}^{k^*} a_i^2 \right\} + s_t
\]
\[
\leq \min_{a \in \Delta_{k^*}} \left\{ -(1 - \gamma) \sum_{i=1}^{k^*} \sigma_i (A_t) a_i + \frac{\beta}{2} \eta \sum_{i=1}^{k^*} a_i^2 \right\} + s_t + \varepsilon .
\]

Since \((1 - \gamma) a^* \in \Delta_{k^*}\), we can choose \(a = (1 - \gamma) a^*\) on the right hand side and obtain
\[
g_t(V_t) \leq - (1 - \gamma) \sum_{i=1}^{k^*} \sigma_i (A_t) (1 - \gamma) a_i^* + \frac{\beta}{2} \eta \sum_{i=1}^{k^*} ((1 - \gamma) a_i^*)^2 + s_t + \varepsilon
\]
\[
= (1 - \gamma)^2 (g_t' - s_t) + s_t + \varepsilon
\]
\[
\leq (1 - \gamma)^2 (g_t' - s_t) + s_t + \varepsilon
\]

Above, the only equality is due to \((C.2)\). Finally, from \(s_t \leq G_t\) and \(g_t^* \leq -\frac{\beta \sigma_{k^*}}{2} \leq 0\) (see Fact 3.3), we have \(s_t \leq -\frac{3G_t}{\eta} g_t^*\). Then it follows from \((C.3)\) that
\[
g_t(V_t) \leq \left( (1 - \gamma)^2 \left( 1 + \frac{3G_t}{\eta} \right) - \frac{3G_t}{\eta} \gamma \right) g_t^* + \varepsilon = \left( 1 - (2\gamma - \gamma^2) \left( 1 + \frac{3G_t}{\eta} \right) \right) g_t^* + \varepsilon
\]
\[
\leq \left( 1 - 2\gamma \left( 1 + \frac{3G_t}{\eta} \right) \right) g_t^* + \varepsilon
\]

By definition, we know that \(V_t\) is \(\left( \frac{3G_t}{\eta} + 2 \right) \gamma, \varepsilon\)-approximate to \((3.1)\). \(\square\)

The following lemma is used to prove Theorem 4.7.

**Lemma C.1.** The matrix \(A^* := \frac{\sigma}{2} X^* - \nabla f(X^*)\) satisfies \(\sigma_{k^*}(A^*) - \sigma_{k^*+1}(A^*) \geq \frac{\sigma}{2} \sigma_{\min}(X^*)\).

**Proof.** Since \(\langle \nabla f(X^*), X - X^* \rangle \geq 0\) for all \(X \in B_{m,n}\), we have
\[
\langle \nabla f(X^*), X^* \rangle = \min_{X \in B_{m,n}} \langle \nabla f(X^*), X \rangle = -\sigma_1(\nabla f(X^*)) .
\]

Let the SVD of \(X^*\) be \(X^* = \sum_{i=1}^{k^*} \sigma_i(X^*) u_i v_i^T\). We first show that for all \(i \in [k^*]\), we must have
\[
\langle \nabla f(X^*), u_i v_i^T \rangle = \langle \nabla f(X^*), X^* \rangle .
\]

Assume \((C.5)\) is false, then there exists \(i \in [k^*]\) such that \(\langle \nabla f(X^*), u_i v_i^T \rangle > \langle \nabla f(X^*), X^* \rangle\). Consider \(X' = \frac{X^* - \frac{\sigma_i(X^*)}{2} u_i v_i^T}{1 - \frac{\sigma_i(X^*)}{2}}\). We have
\[
\langle \nabla f(X^*), X' \rangle = \frac{\langle \nabla f(X^*), X^* - \frac{\sigma_i(X^*)}{2} \sigma_i(X^*) (\nabla f(X^*), u_i v_i^T) \rangle}{1 - \frac{\sigma_i(X^*)}{2} \sigma_i(X^*)}
\]
\[
\leq \frac{\langle \nabla f(X^*), X^* - \frac{\sigma_i(X^*)}{2} \sigma_i(X^*) (\nabla f(X^*), X^*) \rangle}{1 - \frac{\sigma_i(X^*)}{2} \sigma_i(X^*)}
\]
\[
= \langle \nabla f(X^*), X^* \rangle .
\]

Since \(\|X'\|_1 = \|X^*\|_1 - \frac{\sigma_i(X^*)}{1 - \frac{\sigma_i(X^*)}{2}} \leq 1\), we have \(X' \in B_{m,n}\) and this contradicts \((C.4)\). Thus, we have proved \((C.5)\).

From \((C.4)\) and \((C.5)\) we know that for each \(i \in [k^*]\) we have \((-\nabla f(X^*), u_i v_i^T) = \sigma_1(\nabla f(X^*)) = \sigma_1(-\nabla f(X^*))\). This implies that the largest \(k^*\) singular values of \(-\nabla f(X^*)\) are all equal to the same value, and all pairs \((u_i, v_i)\) for \(i \in [k^*]\) are left and right singular vectors corresponding to this singular value. Therefore we can write the SVD of \(-\nabla f(X^*)\) as \(-\nabla f(X^*) = \sigma_1(\nabla f(X^*)) \sum_{i=1}^{k^*} u_i v_i^T + \sigma_2(\nabla f(X^*)) \sum_{i=k^*+1}^{m} u_i v_i^T + \cdots\).
\[ \sum_{i=k^*+1}^{r} \sigma_i(\nabla f(X^*))u_i v_i^\top, \text{ where } r = \min\{m, n\}. \] It follows that
\[ A^* = \frac{\alpha}{2} X^* - \nabla f(X^*) = \sum_{i=1}^{k^*} \left( \frac{\alpha}{2} \sigma_i(X^*) + \sigma_1(\nabla f(X^*)) \right) u_i v_i^\top + \sum_{i=k^*+1}^{r} \sigma_i(\nabla f(X^*))u_i v_i^\top, \]
which is the SVD of \( A^* \). Therefore, we have
\[ \sigma_i(A^*) = \begin{cases} \frac{\alpha}{2} \sigma_i(X^*) + \sigma_1(\nabla f(X^*)), & i = 1, 2, \ldots, k^*, \\ \sigma_i(\nabla f(X^*)), & i = k^* + 1, \ldots, r. \end{cases} \]
which implies \( \sigma_{k^*}(A^*) = \frac{\alpha}{2} \sigma_{k^*}(X^*) + \sigma_1(\nabla f(X^*)) - \sigma_{k^*+1}(\nabla f(X^*)) \geq \frac{\alpha}{2} \sigma_{k^*}(X^*). \)

**Theorem 4.7 (restated).** The k-SVD computation in the t-th iteration of blockFW can be implemented in \( \tilde{O}\left((k \cdot \text{nnz}(\nabla f(X_t))) + k^2(m+n)t\frac{\sqrt{G_t/\alpha}}{\sigma_{\min}(X^*)}\right) \) time.

**Proof of Theorem 4.7.** From Theorem 3.4 and Lemma 4.2, we only need to show that the stated running time is enough for LazySVD to ensure \( u_i^\top A_t v_i \geq (1 - \gamma)\sigma_i(A_t) - \frac{\alpha}{2} \) for all \( i \in [k^*] \), where \( \gamma = \frac{1}{\beta^2 G_t + 4} \). We consider two cases.

**Case 1:** \( h_t \leq \frac{\alpha^3(\sigma_{\min}(X^*))^2}{16 \beta^2 G_t^3} \). From Fact 2.2 we know that \( \|X_t - X^*\|_F \leq \sqrt{\frac{2}{\alpha}} h_t \leq \sqrt{\frac{2}{\alpha}} \cdot \frac{\alpha^3(\sigma_{\min}(X^*))^2}{16 \beta^2 G_t^3} = \frac{\sigma_{\min}(X^*)}{9\kappa} \).

Then, \( A_t = \frac{\alpha}{2} X_t - \nabla f(X_t) \) and \( A^* = \frac{\alpha}{2} X^* - \nabla f(X^*) \) satisfy
\[ \|A_t - A^*\|_F = \left\| \frac{\alpha}{2} (X_t - X^*) - (\nabla f(X_t) - \nabla f(X^*)) \right\|_F \leq \frac{\alpha}{2} \|X_t - X^*\|_F + \|\nabla f(X_t) - \nabla f(X^*)\|_F \leq \frac{3\beta}{2} \|X_t - X^*\|_F \leq \frac{3\beta}{2} \cdot \frac{\sigma_{\min}(X^*)}{9\kappa} = \frac{\alpha}{6} \sigma_{\min}(X^*). \]

By Weyl’s inequality we know that \( |\sigma_i(A_t) - \sigma_i(A^*)| \leq \|A_t - A^*\|_2 \leq \|A_t - A^*\|_F \leq \frac{\alpha}{6} \sigma_{\min}(X^*). \)

Since we have \( \sigma_{k^*}(A^*) - \sigma_{k^*+1}(A^*) \geq \frac{\alpha}{2} \sigma_{\min}(X^*) \) by Lemma C.1, we have \( \sigma_{k^*}(A_t) - \sigma_{k^*+1}(A_t) \geq \left( \sigma_{k^*}(A^*) - \frac{\alpha}{6} \sigma_{\min}(X^*) \right) - \left( \sigma_{k^*+1}(A^*) + \frac{\alpha}{6} \sigma_{\min}(X^*) \right) = \sigma_{k^*}(A^*) - \sigma_{k^*+1}(A^*) - \frac{\alpha}{3} \sigma_{\min}(X^*) \geq \frac{\alpha}{3} \sigma_{\min}(X^*). \)

We can now apply the second running time of Theorem 4.3: \( \tilde{O}\left(\frac{k \cdot \text{nnz}(A_t) + k^2(m+n)}{\sqrt{\text{gap}}}\right) \) where \( \text{gap} = \frac{\sigma_{k^*}(A_t) - \sigma_{k^*+1}(A_t)}{\sigma_{k^*}(A_t)} \); in such running time LazySVD can guarantee \( u_i^\top A_t v_i \geq \sigma_i(A_t) - \varepsilon \) for all \( i \in [k^*] \). Note that \( \sigma_{k^*}(A_t) \leq \|A_t\|_2 \leq \frac{\alpha}{2} \|X_t\|_2 + \|\nabla f(X_t)\|_2 \leq G_t \), which implies \( \text{gap} = \frac{\sigma_{k^*}(A_t) - \sigma_{k^*+1}(A_t)}{\sigma_{k^*}(A_t)} \geq \frac{\alpha}{3} \sigma_{\min}(X^*) \). Thus the running time is
\[ \tilde{O}\left(\frac{k \cdot \text{nnz}(A_t) + k^2(m+n)}{\sqrt{\text{gap}}}\right) = \tilde{O}\left(\frac{k \cdot \text{nnz}(A_t) + k^2(m+n)}{\sqrt{\beta^2 G_t}}\right) = \tilde{O}\left(k \cdot \text{nnz}(A_t) + k^2(m+n)\right). \]

**Case 2:** \( h_t > \frac{\alpha^3(\sigma_{\min}(X^*))^2}{16 \beta^2 G_t^3} \). Lemma 4.2 implies that it suffices to ensure \( u_i^\top A_t v_i \geq (1 - \gamma)\sigma_i(A_t) \) for all \( i \in [k^*] \), where \( \gamma = \frac{\alpha^3}{\beta^2 G_t + 4} \). The last step here is because we have \( \alpha^3(\sigma_{\min}(X^*))^2 \leq (\beta^2 G_t + 1)^2 = \beta^2 G_t \).

We apply the first running time of Theorem 4.3:
\[ \tilde{O}\left(\frac{k \cdot \text{nnz}(A_t) + k^2(m+n)}{\sqrt{\gamma}}\right) = \tilde{O}\left(\frac{k \cdot \text{nnz}(A_t) + k^2(m+n)}{\sqrt{\beta^2 G_t}}\right) = \tilde{O}\left(k \cdot \text{nnz}(A_t) + k^2(m+n)\right). \]
Note that the running time for Case 2 is always no smaller than the running time for Case 1 (ignoring logarithmic factors). Combining the two cases and plugging the bound of $\text{nnz}(A_t)$ from Fact 4.4 we have the desired running time statement.