Net-proton number cumulant ratios as function of beam energy from a nonequilibrium chiral Bjorken model

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Abstract

STAR’s beam energy scan program at RHIC provides data on net-proton number fluctuations with the goal to detect the QCD critical point and first-order phase transition. Interpreting these experimental signals requires a vital understanding of the interplay of critical phenomena and the nonequilibrium dynamics of the rapidly expanding fireball. We study these aspects with a fluid dynamic expansion coupled to the explicit propagation of the chiral order parameter sigma via a Langevin equation. Assuming a sigma-proton coupling through an effective proton mass, we relate cumulants of the order parameter and the net-proton number at freeze-out and obtain observable cumulant ratios as a function of beam energy. We emphasize the role of the nonequilibrium first-order phase transition twofold: First, the presence of an unstable phase causes the well-known bending of the trajectories in the space of temperature and baryochemical potential. For these cases at lower beam energies, the system crosses the freeze-out line more than once, allowing us to calculate a wide range of cumulants for each initial condition. Second, the thermodynamic susceptibilities diverge along the spinodal lines in nonequilibrium. Depending on the freeze-out parameters, these divergences can have a dramatic impact on the calculated cumulants and cumulant ratios.

Keywords: heavy-ion collision, hydrodynamics, chiral phase transition, critical point
1 Introduction

Experiments at SPS and STAR have found that nuclear matter at high temperatures undergoes a transition from a hadronic phase to a quark-gluon plasma [1, 2], a state of deconfinement and chiral symmetry restoration. While this transition is a continuous crossover for zero and small baryochemical potential $\mu_B$ [3–5], a critical endpoint (CEP) and first-order phase transition (FOPT) are expected at large $\mu_B$. Hereby, a variety of effective models of quantum chromodynamics (QCD) [6–8] as well as functional techniques [9, 10] yield widely different results regarding existence and location of CEP and FOPT in the space of $T$ and $\mu_B$.

Besides these theoretical efforts, considerable workforce is invested in ongoing experimental programs aiming at understanding the QCD phase diagram, e.g. beam energy scan at STAR [11], NA49/61 [12, 13], HADES [14], or the upcoming facilities NICA [15] and FAIR [16]. Important observables hereby are cumulants of conserved quantities, namely baryon number, strangeness and electric charge. As shown in lattice QCD [17, 18] and various studies of effective models [19–21] and functional techniques [22], these exhibit divergences at the CEP and characteristic behavior in the critical region around the CEP. Even though these calculations are based on equilibrium thermodynamics, it is widely believed that a nonmonotonic behavior of cumulants or cumulant ratios as function of center-of-mass energy indicates the presence of a CEP. Here, however, a thorough understanding of the nonequilibrium dynamics and a design of proper experimental methods are crucial. For an overview of critical point physics at STAR, see [23].

Since cumulants of various order are directly proportional to certain powers of the correlation length, they diverge in an infinitely large equilibrated medium close to the CEP. In a heavy-ion collision, their growth is limited not only by the finite system size but also by critical slowing down [24]. The importance of a proper dynamical description of the critical dynamics has been emphasized by a variety of publications in recent years [25–34]. The direct impact on experimental observables, e.g. the net-proton number fluctuations, has been argued and demonstrated in [35–38]. Equally important than understanding the complicated dynamics near the CEP is understanding and modeling of evolutions passing through a FOPT where spinodal decomposition enforces density inhomogeneities within single events [39–45] and leads to an increased production of entropy [46, 47] which could be observed e.g. in an enhancement of the pion-to-proton ratio.

In this paper, we apply the nonequilibrium chiral fluid dynamics model [48] to a longitudinal Bjorken expansion along the beam axis [47]. This model describes the dynamics of the sigma field as the chiral order parameter with a Langevin equation interacting with a locally thermalized expanding quark fluid. Field and fluid exchange energy-momentum via a source term. We extract cumulants of the sigma field on an event-by-event basis along a parametrized freeze-out curve. As shown in [36], we relate these cumulants to net-proton number fluctuations by assuming a superposition of standard Poisson and critical fluctuations. We correct our results by the effect of volume fluctuations as detailed in [49].

After a description of the model in Section 2, we show our results on the net-proton number cumulant ratios as function of beam energy in comparison to data from STAR and HADES in Section 3, and finally conclude with a summary in Section 4.

2 Model description

The model is based on the Lagrangian of the widely studied quark-meson model [6, 7, 50] with a CEP at $(T_{\text{CP}}, \mu_{\text{CP}}) = (100, 200)$ MeV. Hereby, and in the following, $\mu$ denotes the quark chemical potential, thus $\mu = \mu_B/3$. The Lagrangian for light quarks $q = (u, d)$ and the chiral order parameter $\sigma$ reads

$$\mathcal{L} = \bar{q} (i\gamma^\mu \partial_\mu - g\sigma) q + \frac{1}{2} (\partial_\mu \sigma)^2 - U(\sigma),$$

with standard parameters $f_\pi = 93$ MeV, $m_\pi = 138$ MeV and $U_0$ such that the potential $U(\sigma) = 0$ in the ground state. Hereby, the value of the pion fields has already been set to its vacuum expectation value of zero. The quark-sigma coupling

$$U(\sigma) = \frac{\lambda^2}{4} (\sigma^2 - f_\pi^2)^2 - f_\pi m_\pi^2 \sigma + U_0,$$

is used to ensure the vacuum recovery in the ground state and an appropriate scaling of critical temperature. The model is used to extract the equation of state $\sigma(T, \mu)$ from lattice calculations [20, 48] and parametrizes the resulting dependence on $T$ and $\mu$ via a functional form [20]. If $\sigma(T, \mu)$ is set to a vacuum value, $\sigma(0, \mu)$, the model is reduced to a longitudinal Bjorken expansion along the beam axis. This setup is selected for the present analysis due to its simplicity and the well-defined state of the medium close to the CEP. It also allows a straightforward comparison with experimental observables, e.g. beam energy scan at STAR [11].
constant $g$ is fixed requiring that $g\sigma$ equals the nucleon mass of 940 MeV in vacuum.

The grand potential $\Omega$ in mean-field approximation resembles that of a Fermi gas of quarks with energies $E = \sqrt{p^2 + m^2}$ and is evaluated as

$$\Omega_{q\bar{q}} = -2N_f N_c T \int \frac{d^3p}{(2\pi)^3} \left[ \log \left(1 + e^{-\frac{E - \mu}{T}}\right) + \log \left(1 + e^{-\frac{E + \mu}{T}}\right) \right].$$

(3)

Here, $N_f = 2$, $N_c = 3$ denote the number of light quark flavors and the number of colors.

2.1 Equations of motion

We evolve the zero mode or volume-averaged sigma field defined as $\sigma(\tau) = \frac{1}{T} \int d^3x \sigma(\tau, x)$ using a Langevin equation of motion,

$$\ddot{\sigma} + \left(\frac{D}{\tau} + \eta\right) \dot{\sigma} + \frac{\delta \Omega}{\delta \sigma} = \xi,$$

(5)

neglecting any spatial fluctuations. Since we describe the expanding fluid using a Bjorken model, we use proper time $\tau$ rather than coordinate time $t$, starting from an initial thermalization time $\tau_0 = 1$ fm. Consequently, the dots in (5) denote derivatives with respect to $\tau$. For our case of purely longitudinal hydrodynamic flow, we set $D = 1$ in the Hubble term. The full and proper nonequilibrium dynamics of sigma is encoded in the dissipation coefficient $\eta$ and the stochastic noise $\xi$ which are related by a dissipation-fluctuation relation,

$$\langle \xi(t)\xi(t') \rangle = \frac{m_\sigma^2 \eta}{V} \coth \left(\frac{m_\sigma^2}{2T}\right) \delta(t - t').$$

(6)

Here, $\xi$ is assumed Gaussian and white, i.e. it is not correlated over time. The coefficient $\eta$ includes effects from various processes:

- Mesonic interactions, i.e. scattering of a condensed sigma meson with a thermal sigma, $\sigma\sigma \leftrightarrow \sigma\sigma$, (scattering of condensed a sigma with a thermal sigma), and $\sigma \leftrightarrow \pi\pi$ (two-pion decay) [51], described by a phenomenological damping coefficient of $\eta = 2.2$/fm [52] whenever kinematically allowed.

- Meson-quark interactions, $\sigma \leftrightarrow q\bar{q}$, leading to a $T$- and $\mu$-dependent coefficient [48],

$$\eta = \frac{12g^2}{\pi} \left[1 - 2n_F \left(\frac{m_\sigma}{2}\right)\right] \frac{1}{m_\sigma^2} \left(\frac{m_\sigma^2}{4} - m_q^2\right)^{3/2} T_{q\bar{q}}^{\mu\nu}.$$  

(7)

We assume an ideal fluid of quarks and antiquarks described by the energy-momentum tensor $T_q^{\mu\nu} = (e + p)u^\mu u^\nu - p\eta^{\mu\nu}$. Due to energy-momentum conservation the divergence of the total energy-momentum tensor $T_q^{\mu\nu} + T_\sigma^{\mu\nu}$ vanishes, which leads to

$$\partial_\mu T_{q\bar{q}}^{\mu\nu} = \left[\frac{\delta \Omega_{q\bar{q}}}{\delta \sigma} + \left(\frac{D}{\tau} + \eta\right) \dot{\sigma}\right] \delta^{\nu\sigma}.$$  

(8)

A contraction of (8) with the four-velocity $u^\sigma$ yields the evolution equation for the energy density,

$$\dot{e} = -\frac{e + p}{\tau} + \left[\frac{\delta \Omega_{q\bar{q}}}{\delta \sigma} + \left(\frac{D}{\tau} + \eta\right) \dot{\sigma}\right] \delta^{\nu\sigma},$$

(9)

while the net-baryon density simply follows

$$\dot{n} = -\frac{n}{\tau}.$$  

(10)

In equations (5) and (9) the pressure is given by $p = -\Omega_{q\bar{q}}$. It is an explicit function of $\sigma$ which during the evolution is not fixed to its equilibrium value. The fireball volume which appears in equation (6) and also later in the description of the freeze-out cumulants, is given by $V = \pi R^2 \tau$, with a gold nucleus radius of $R = 7.3$ fm, assuming a central collision.

2.2 Freeze-out and mapping to beam energies

We investigate higher order cumulant ratios of the net-proton number along a freeze-out curve which has been obtained from thermal model fits to experimental data for a wide range of beam energies [53]. The proposed parametrization reads

$$T_{f.o.}(\mu_B) = a - b\mu_B^2 - c\mu_B^4,$$

(11)

with $a = 0.166$ GeV, $b = 0.139$ GeV$^{-1}$ and $c = 0.053$ GeV$^{-3}$. Since the phase boundary of the quark-meson model would lie below the
thus obtained freeze-out line, we scale $T$ and $\mu$ in this parametrization with a common factor $T_{\text{crossover}}(\mu = 0)/T_{\text{freeze}}(\mu = 0)$ yielding a freeze-out curve which corresponds with the crossover temperature at $\mu = 0$ and consistently lies below the crossover and phase transition for $\mu > 0$. Hereby, $T_{\text{crossover}}(\mu = 0) = 145$ MeV is determined by a maximum of the quark number susceptibility.

We define initial conditions for the evolution by choosing initial values $T_i$ and $\mu_i$ similar to previous studies [47]. During the evolution of the fluid according to eqs. (5), (9), (10), the trajectory in $T$-$\mu$ space will hit the freeze-out curve. This hit point is then used to map the evolution to a corresponding beam energy via:

$$\mu_B(\sqrt{s}) = \frac{d}{1 + e^{\sqrt{s}}} ,$$  \hspace{1cm} (12)

with parameters $d = 1.308$ GeV and $e = 0.273$ GeV$^{-1}$ [53].

### 2.3 Sigma and net-proton number cumulants

To relate the fluctuations in the chiral order parameter $\sigma$ to fluctuations or cumulants of the net-proton number, we follow the strategy outlined in [36]. Consider an infinitesimal change of the chiral field, $\delta \sigma$, leading to a change of the effective proton mass by $\delta m = g \delta \sigma$. Assuming a sigma-proton coupling $g \sigma \bar{p}p$, we may write fluctuations of the momentum space distribution function for protons, $f_k$, as

$$\delta f_k = \delta f_k^0 + \frac{\partial n}{\partial m} g \delta \sigma .$$ \hspace{1cm} (13)

The first term $\delta f_k^0$ is the purely statistical fluctuation, and in the second term, $n$ denotes the order parameter $\sigma$ to fluctuations or cumulants of the net-proton number, we follow the strategy outlined in [36]. Consider an infinitesimal change of the chiral field, $\delta \sigma$, leading to a change of the effective proton mass by $\delta m = g \delta \sigma$. Assuming a sigma-proton coupling $g \sigma \bar{p}p$, we may write fluctuations of the momentum space distribution function for protons, $f_k$, as

$$\delta f_k = \delta f_k^0 + \frac{\partial n}{\partial m} g \delta \sigma .$$ \hspace{1cm} (13)

The first term $\delta f_k^0$ is the purely statistical fluctuation, and in the second term, $n$ denotes the Fermi-Dirac distribution for a particle of a given mass. The fluctuation of the net-proton multiplicity $N = V d \int \frac{d^3k}{(2\pi)^3} f_k$ is then given by

$$\delta N = \delta N^0 + V g \delta \sigma d \int \frac{d^3k}{(2\pi)^3} \frac{\partial n}{\partial m} ,$$ \hspace{1cm} (14)

where $d = 2$ is the spin degeneracy factor. The first term $\delta N^0$ can be assumed Poisson distributed given that $n_p \ll 1$. Consequently, all of its cumulants are equal to the expectation value $\langle N \rangle$. In leading order and assuming no correlations between $\delta N_0$ and $\delta \sigma$, we can express cumulants of order $n$ as

$$\langle \delta N^n \rangle_c = \langle N \rangle + \langle \delta \sigma \rangle_c^n \left( g d \int \frac{d^3k}{(2\pi)^3} \frac{\partial n}{\partial m} \right) .$$ \hspace{1cm} (15)

In this notation, $\sigma_V = \int d^3x \sigma = \sigma V$ as we neglect spatial fluctuations, and $\langle \cdot \rangle_c$ is the respective central moment, which is equal to the corresponding cumulant, which is also commonly used in experimental studies of recent years.

### 3 Research procedure and results

We initialize the fluid at a set of fixed values $(T_i, \mu_i)$ and define initial sigma field, energy density, quark number density, and pressure as the corresponding equilibrium values. Then the coupled system is evolved according to the equations of motion until the freeze-out curve is hit. Notably, for expansions at high baryochemical potential, the freeze-out curve can be hit more than once due to the nonequilibrium evolution of the expanding plasma. In contrast to that, an equilibrated hydrodynamic system with constant $S/A$ would pass...
along the phase boundary for a certain amount of time [54]. To take into account this effect of a possible mixed phase, we evolve the system for these cases until the second crossing of the freeze-out curve and subsequently calculate cumulants at both points, leading to a range of possible values. This plays an important role for evolutions near the CP in the phase diagram or crossing the first-order phase transition line, see figure 1 for some examples. We can see three trajectories starting from the same initial condition and evolving differently mainly due to stochastic fluctuations in the equation of motion. The characteristic back-bending after passing the phase boundary is visible here, an effect that becomes much stronger for evolutions passing the FOPT, where it could manifest itself e.g. in an enhanced dilepton production [55]. The freeze-out points for the three trajectories in figure 1, and in general for different events, are not the same. For matching with $\sqrt{s}$ according to eq. (12), we use the event-averaged values of $T$ and $\mu$ at the various points where the curves hit the freeze-out line for the first time.

We simulate $N = 10^7$ events and calculate event-by-event fluctuations in terms of cumulants of $\sigma_V$. Since these are subject to significant fluctuations of the freeze-out volume, we include the corresponding corrections as derived in [49]. Then, eq. (15) allows us to determine the corresponding net-proton number cumulants. The such obtained values are compared with a Poisson baseline. For the net-proton number ($p - \bar{p}$), the cumulants assuming Poisson distributed proton and antiproton numbers are calculated by

$$C_{n,p-\bar{p}} = C_{n,p} + (-1)^n C_{n,\bar{p}}, \quad (17)$$

where $C_{n,p}$, $C_{n,\bar{p}}$ are equal to the expectation values of the Poisson distribution for all orders $n$, cf. [11]. We furthermore provide comparison to the equilibrium net-baryon number susceptibilities which are obtained as

$$\chi_n = \frac{\partial^n(p/T^4)}{\partial(\mu_B/T)^n}. \quad (18)$$

Figure 2 (left) shows the ratio $C_2/C_1$ of the net-proton number compared to results from STAR for energies $\sqrt{s_{NN}} \geq 7.7$ GeV [11] and HADES for $\sqrt{s_{NN}} = 2.4$ GeV [14]. Since the HADES collaboration reported a strong dependence of cumulant ratios on the chosen rapidity window, we depict results for both $y < 0.4$ and $y < 0.5$ for comparison, the latter one also being applied to STAR data. In [14], a reliable rapidity window of $y < 0.46$ is quoted. We see that as for high energies, STAR data lie close to the Poisson baseline, our model yields values that are gradually enhanced as the energy decreases until a maximum at around 5 GeV for the evolution that passes through the CP and freezes out precariously close to a spinodal line, where all susceptibilities diverge in nonequilibrium [43, 56]. This is clearly reflected in the peak of the corresponding susceptibility ratio visible on the right hand side of figure 2. Further lowering the energy results in a gradual return to the baseline for the first hit of the freeze-out curve, while for the second hit, significantly larger fluctuations are observed, possibly due to an enhancement of spinodal instabilities. Notably, the data from HADES
lies within the thus obtained range of cumulant ratios at the lowest beam energy. Obviously, a significant gap in the experimental data will have to be filled by future experiments such as FAIR and NICA that aim at exploring the high-$\mu_B$ region of the QCD phase diagram. The susceptibility ratio on the right hand side of figure 2 shows a similar trend for high energies. The freeze-out close to the spinodal line, where susceptibilities in the presence of spinodal instabilities diverge and change sign [43, 56], results in the strongly negative value of $\chi_2/\chi_1$, which is reflected in the peak of $C_2/C_1$, similar to what we found for the other cumulant ratios as discussed below.

All cumulants presented here have been corrected for volume fluctuations as detailed in [49] which occur due to variations in the time at which different evolutions from the same initial condition hit the freeze-out curve.

Fig. 3 Cumulant ratios of the net-proton number are enhanced around the critical point (left). Susceptibility ratios for comparison (right).

The cumulant ratio $C_3/C_2$ is shown and compared to experimental data in figure 3 (left). The most notable feature is, again, the strong impact of the critical point around $\sqrt{s} = 5$ GeV. Besides that, the obtained points from our model are close to the baseline for high energies and the second hit at the lowest energy is close to the data point from HADES, where a suppression of $C_3/C_2$ was found, possibly a result of the dynamics at the first-order phase transition. This is also found in the net-baryon number susceptibility ratio on the right hand side of the same figure, approaching zero for the first hit and remaining negative for the second hit of the parametrized freeze-out curve. The most apparent difference is the sign at the critical point evolution which is positive for the susceptibilities, but negative for the cumulants. As mentioned before, the freeze-out happens very close to the spinodal line for these evolutions where the same susceptibilities change sign. Therefore,
finite-time effects can here dramatically influence the outcome.

Finally, the ratio \( C_4/C_2 \) is depicted in figure 4, on the right hand side we see at least qualitative similarities between our model results and the experimental data. A slight suppression in the STAR data around 20 GeV is also present in the model calculations where the cumulant ratio lies below the Poisson baseline, although with smaller significance. Then, as the beam energy is lowered, the notable point at 7.7 GeV where \( C_4/C_2 \) is enhanced is reflected in an enhancement, albeit orders of magnitude larger, of our calculation. Hereby, it is necessary to mention that the aforementioned freezing out near the spinodal line leads to larger and larger cumulants at higher orders. Lowering the beam energy even further, our calculations approach the HADES results which for this cumulant ratio show the strongest dependence on the applied experimental cut. Within error bars, both points lie within our range defined by the first and second hit of \( C_4/C_2 \sim 1 - 10 \). The susceptibility ratios, shown on the right hand side, are close to zero at these low energies which could indicate a enhancement through prolonged evolution in the mixed-phase region for the first-order phase transition. The positive peak for the CP evolution is also found in the susceptibilities, however, for larger energies, the values are slightly negative, which is only partly reflected in the model data.

4 Summary

We have studied cumulant ratios \( C_2/C_1, C_3/C_2, C_4/C_2 \) of the net-proton number at STAR and HADES energies within a nonequilibrium chiral Bjorken expansion. Hereby, a sigma model served as input for chiral the phase structure and net-proton cumulants have been calculated event-by-event from cumulants of the sigma field at a parametrized freeze-out curve. Hereby, volume fluctuations have been accounted for and were properly corrected. Although admittedly crude and neglecting effects of an inhomogeneous medium, the dynamical description nevertheless shows some qualitative resemblance to the experimental data, in the approach of the Poisson baseline for high energies far away from the critical region, but also the enhancement or suppression of certain cumulant ratios at a speculated CP or first-order phase transition. We have also seen that the gap in beam energies from 2.4 for 7.7 GeV requires filling. If indeed the CP is found in this range, then more dramatic effects can be expected.

In the future, we are going to use a more sophisticated approach for determining the initial beam energy and are going to extend the study to full (3+1) dimensional hydrodynamics.

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