A Fast Fourier Transform for the Johnson Graph

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Abstract
The set $X$ of $k$-subsets of an $n$-set has a natural graph structure where two $k$-subsets are neighbors if and only if the size of their intersection is $k - 1$. This is known as the Johnson graph. The symmetric group $S_n$ acts on the space of complex functions on $X$ and this space has a multiplicity-free decomposition as sum of irreducible representations of $S_n$, so it has a well-defined Gelfand–Tsetlin basis up to scalars. The Fourier transform on the Johnson graph is defined as the change of basis matrix from the delta function basis to the Gelfand–Tsetlin basis. The direct application of this matrix to a generic vector requires $\binom{n}{k}^2$ arithmetic operations. We show that, in analogy with the classical Fast Fourier Transform on the discrete circle, this matrix can be factorized as a product of $n - 1$ orthogonal matrices, each one with at most two nonzero elements in each column. The factorization is based on the construction of $n - 1$ intermediate bases which are parametrized via the Robinson–Schensted insertion algorithm. This factorization shows that the number of arithmetic operations required to apply this matrix to a generic vector is bounded above by $2(n - 1)\binom{n}{k}$. We give an algorithm that constructs all these factors using at most $289(n - 1)\binom{n}{k}$ arithmetic operations. The coefficients of these matrices are rational numbers and the construction does not depend on numerical methods. Instead, they are obtained by solving small linear systems with integer coefficients derived from the Jucys–Murphy operators. In particular we avoid the use of square roots. As a consequence, we show that the problem of computing all the weights of the isotypic components of a given function can be solved in $O(n\binom{n}{k})$ operations, improving the previous bound $O(k^2\binom{n}{k})$ when $k$ asymptotically dominates $\sqrt{n}$. The same improvement is achieved for the problem of computing the isotypic projection onto a single component.

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1 Introduction

The set of all subsets of cardinality $k$ of a set of cardinality $n$ is a basic combinatorial object with a natural metric space structure where two $k$-subsets are at distance $d$ if the size of their intersection is $k - d$. This structure is captured by the Johnson graph $J(n, k)$, whose nodes are the $k$-subsets and two $k$-subsets are neighbors if and only if they are at distance 1.

The Johnson graph is closely related to the Johnson scheme, an association scheme of major significance in classical coding theory (see [7] for a survey on association scheme theory and its application to coding theory). The Johnson graph has played a fundamental role in the breakthrough quasipolynomial time algorithm for the graph isomorphism problem presented in [4] (see [3] for background on the graph isomorphism problem).

Functions on the Johnson graph arise in the analysis of ranked data. In many contexts, agents choose a $k$-subset from an $n$-set, and the data is collected as the function that assigns to the $k$-subset $x$ the number of agents who choose $x$. This situation is considered, for example, in the statistical analysis of certain lotteries (see [8, 10]).

Although in this paper we do not explicitly use the graph structure of the set of $k$-subsets, we strongly rely on a fact that is a consequence of the distance-transitive property of the Johnson graph, namely, that the vector space of functions on the Johnson graph is a representation of the symmetric group that decomposes as a multiplicity-free direct sum of irreducible representations (see [5, 20]). Statistically relevant information about the function is contained in the isotypic projections of the function onto each irreducible component. This approach to the analysis of ranked data was called spectral analysis by Diaconis and developed in [8, 9]. The problem of the efficient computation of the isotypic projections has been studied by Diaconis and Rockmore in [10], and by Maslen et al. in [17].

The classical Discrete Fourier Transform (DFT) on the cyclic group $\mathbb{Z}/2^n\mathbb{Z}$ can be seen as the application of a change of basis matrix from the basis $B_0$ of delta functions to the basis $B_n$ of characters of the group $\mathbb{Z}/2^n\mathbb{Z}$. The direct application of this matrix to a generic vector involves $(2^n)^2$ arithmetic operations. The Fast Fourier Transform (FFT) is a fundamental algorithm that computes the DFT in $O(n2^n)$ operations. This algorithm was discovered by Cooley and Tukey [6] and the efficiency of their algorithm is due to a factorization of the change of basis matrix. Here we denote by $[B]_{B'}$ the change of basis matrix from the basis $B$ to the basis $B'$. The factorization is given by

$$[B_0]_{B_n} = [B_{n-1}]_{B_n} \cdots [B_1]_{B_2} [B_0]_{B_1}$$
Fig. 1 Labels of the intermediate bases in the case $n = 4, k = 2$. The $i$th column parametrize the basis $B_i$ where $B_1, \ldots, B_{n-1}$ are intermediate orthonormal bases such that each matrix $[B_{i-1}]_{B_i}$ has at most two nonzero entries in each column.

In this paper, we show that the same phenomenon occurs in the case of the non-abelian Fourier transform on the Johnson graph. This transform is defined as the application of the change of basis matrix from the basis $B_0$ of delta functions to the basis $B_n$ of Gelfand–Tsetlin functions. The Gelfand–Tsetlin basis, defined in Sect. 4, is well-behaved with respect to the action of the symmetric group $S_n$, in the sense that each irreducible component is generated by a subset of the basis.

The computational model used here counts a single complex multiplication and addition as one arithmetic operation. A direct computation of this Fourier transform involves $\binom{n}{k}^2$ arithmetic operations. We construct intermediate orthogonal bases $B_1, \ldots, B_{n-1}$ such that each change of basis matrix $[B_{i-1}]_{B_i}$ has rational entries and has at most two nonzero entries in each column. Each intermediate basis $B_i$ is parameterized by pairs consisting of a standard Young tableau of height at most two and a word in the alphabet $\{1, 2\}$ as shown in Fig. 1. Our construction of the matrices $[B_{i-1}]_{B_i}$ is based on the Vershik–Okounkov approach [21] to the representation theory of the symmetric groups, which uses the Jucys–Murphy operators as a basic tool.

Our main result is an algorithm that computes all these matrices $[B_{i-1}]_{B_i}$ and its inverses using at most $289(n - 1)\binom{n}{k}$ arithmetic operations (Theorem 6). As a consequence of the factorization and the sparsity of the factors, once these factors have been computed, the application of the Fourier transform or its inverse to a specific vector can be done using at most $2(n - 1)\binom{n}{k}$ operations (Theorem 7).

We apply this result to the problem of computing the projection of a vector onto the space expanded by a specific set of isotypic components. We show that using our
algorithm this projection can be computed in $O(n \binom{n}{k})$ operations. We also show that the same $O(n \binom{n}{k})$ bound is achieved for the problem of computing all the weights of the isotypic components appearing in the decomposition of a function.

We remark that all our algorithms work restricted to the exact arithmetic of sums and multiplications of rational numbers, avoiding the use of numerical methods or the computation of square roots.

2 Previous Work

The present paper is an extension of our previous work [15], which do not included the efficient construction of the matrices $[B_{i-1}]_{B_i}$. The problem of the efficient computation of the isotypic projections has been studied at least by Diaconis and Rockmore in [10], by Driscoll et al. in [11] and by Maslen et al. in [17]. In [11] it is proved that the number of operations needed to compute the projection onto a single isotypic component is at most $N^2$ where $N$ is the size of the homogeneous space. In our case $N = \binom{n}{k}$. The bound given in [10] is not better than this $N^2$ bound. In [17] a numerical algorithm that depends on Lanczos iteration method is given to compute all the isotypic components using $O(k^2 \binom{n}{k})$ operations. We solved the problem of computing the projection onto the space expanded by a specific set of isotypic components in $O(n \binom{n}{k})$ operations and using exact arithmetic of rational numbers. This represents an improvement if $k$ depends on $n$ and it grows asymptotically faster than $\sqrt{n}$.

In [18], Gelfand–Tsetlin bases were defined in the context of semisimple algebras and fast Fourier transforms were given for BMW, Brauer, and Temperley–Lieb algebras. The function space on the Johnson graph is not a semisimple algebra, but it can be viewed as a module over the group ring $\mathbb{C}[S_n]$. Our work is perhaps an indication that the methods in [18] extend to interesting modules over semisimple algebras.

The sparsity of the change of bases matrices $[B_{i-1}]_{B_i}$ was considered in [14] in the more general framework of permutation groups, generalizing in particular the bounds obtained in [15]. Here we solve the problem of the explicit and efficient construction of these matrices in the case of the Johnson module.

Fast Fourier transforms for other homogeneous spaces related to the Johnson graph were consider in [16]. In this paper D. Maslen develops a FFT for $S_n$ and for the homogeneous space $S_n/S_{n-k}$, although there is no mention to the homogeneous space of the Johnson graph which is $S_n/(S_{n-k} \times S_k)$. Maslen builds the FFTs for $S_n$ and $S_n/S_{n-k}$ in the framework of the separation of variables method. We do not apply this technique in our work, but we exploit the particular structure of the Johnson space, more precisely, we rely on the multiplicity-free decomposition of the space of functions on the graph and a certain recurrence property that exists in it. This allows us to obtain a very efficient algorithm for the Fourier transform in this particular space, as well as it allows us to establish a clear relationship between the FFT on the Johnson graph and the Robinson–Schensted algorithm in combinatorics, which we believe it is interesting in itself.
The fast Fourier transform for the Johnson scheme has also been considered in
the quantum computing community [2], where a technique based on the Schur–Weyl
transform is applied. Gelfand–Tsetlin bases for the Johnson scheme also appear in
the work of Srinivasan [19], which presents an inductive construction of all Johnson
schemes at once. Explicit formulas for Gelfand–Tsetlin bases for the Johnson scheme,
as well as a direct proof for the eigenvalues of the Jucys–Murphy elements, were
obtained by Filmus in [12]. An algorithm for computing these bases is given in [1]
that uses $O(k^2 \binom{n}{k}^2)$ serial operations and $O(n)$ parallel time.

3 Organization of the Paper

In Sect. 4, we review the definition of Gelfand–Tsetlin bases for representations of the
symmetric group. In Sect. 5, we describe the well-known decomposition of the function
space on the Johnson graph and define the corresponding Gelfand–Tsetlin basis. In
Sect. 6 we prove some decomposition theorems for the function space (Theorems 2 and
3) which are central for our subsequent results. In Sect. 7, we introduce the sequence of
intermediate bases of the function space and we prove that the change of basis matrix
between two consecutive bases is a sparse matrix (Theorem 4). In Sect. 8 we point
out the relation of our algorithm with the Robinson–Schensted insertion algorithm.
In Sect. 9 we present some basic tools from the Vershik–Okounkov approach to the
representation theory of the symmetric group, namely, the Jucys–Murphy operators
and the formula for their eigenvalues. In Sect. 10 we give an efficient construction
of the sparse matrices that realize the fast Fourier transform based on the properties
of the Jucys–Murphy operators. In Sect. 11 we present our main results, giving an
upper bound for the number of operations needed for the computation of the change
of bases matrices as well as the application of the Fourier transform to a given vector.
In Sect. 12 we apply our algorithm to the problem of the computation of the isotypic
components of a function on the Johnson graph. In Sect. 13 we illustrate our algorithm
by developing the case $n = 4, k = 2$.

4 Gelfand–Tsetlin Bases

Consider the chain of subgroups of $S_n$

$$S_1 \subset S_2 \subset \cdots \subset S_n$$

where $S_k$ is the subgroup of those permutations fixing the last $n - k$ elements of
{$1, \ldots, n$}. Let $\text{Irr}(n)$ be the set of equivalence classes of irreducible complex rep-
resentations of $S_n$. A fundamental fact in the representation theory of $S_n$ is that if
$V_\lambda$ is an irreducible $S_n$-module corresponding to the representation $\lambda \in \text{Irr}(n)$ and
we consider it by restriction as an $S_{n-1}$-module, then it decomposes as sum of irre-
ducible representations of $S_{n-1}$ in a multiplicity-free way (see for example [21]).
This means that if $V_\mu$ is an irreducible $S_{n-1}$-module corresponding to the represen-
tation $\mu \in \text{Irr}(n - 1)$ then the dimension of the space $\text{Hom}_{S_{n-1}}(V_\mu, V_\lambda)$ is 0 or 1.
The branching graph is the following directed graph. The set of nodes is the disjoint union

$$\bigsqcup_{n \geq 1} \text{Irr}(n).$$

Given representations $\lambda \in \text{Irr}(n)$ and $\mu \in \text{Irr}(n-1)$ there is an edge connecting them if and only if and only if $\mu$ appears in the decomposition of $\lambda$, that is, if the dimension of $\text{Hom}_{S_{n-1}}(V_\mu, V_\lambda)$ is equal to 1. If there is an edge between them we write

$$\mu \nearrow \lambda,$$

so we have a canonical decomposition of $V_\lambda$ into irreducible $S_{n-1}$-modules

$$V_\lambda = \bigoplus_{\mu \nearrow \lambda} V_\mu.$$

Applying this formula iteratively we obtain a uniquely determined decomposition into one-dimensional subspaces

$$V_\lambda = \bigoplus_T V_T,$$

where $T$ runs over all chains

$$T = \lambda_1 \nearrow \lambda_2 \nearrow \cdots \nearrow \lambda_n,$$

with $\lambda_i \in \text{Irr}(i)$ and $\lambda_n = \lambda$. Choosing a nonzero vector $v_T$ of the one-dimensional space $V_T$ we obtain a basis $\{v_T\}$ of the irreducible module $V_\lambda$, which is called a Gelfand–Tsetlin basis.

Observe that if $V$ is a multiplicity-free representation of $S_n$ then there is a uniquely determined, up to scalars, Gelfand–Tsetlin basis of $V$. In effect, if

$$V = \bigoplus_{\lambda \in S \subseteq \text{Irr}(n)} V_\lambda,$$

and $B_\lambda$ is a GT-basis of $V_\lambda$ then a GT-basis of $V$ is given by the disjoint union

$$B = \bigsqcup_{\lambda \in S \subseteq \text{Irr}(n)} B_\lambda.$$

The Young graph is defined as the directed graph where the nodes are the Young diagrams and there is an arrow from $\lambda$ to $\mu$ if and only if $\lambda$ is contained in $\mu$ and their difference consists of a single box. It turns out that there is a bijection between the set
Fig. 2 The Young graph. Each path from the top node to a particular Young diagram \( \lambda \) is identified with a standard Young tableau of shape \( \lambda \).

Fig. 3 The leaves of this tree parametrize the Gelfand–Tsetlin basis of the space \( F \) of functions on the Johnson graph \( J(4, 2) \).

of Young diagrams with \( n \) boxes and \( \text{Irr}(n) \) inducing a graph isomorphism between the Young graph and the branching graph (Theorem 5.8 of [21]).

As a consequence there is a bijection between the Gelfand–Tsetlin basis of \( V_\lambda \), where \( \lambda \) is a Young diagram, and the set of paths in the Young graph starting at the one-box diagram and ending at the diagram \( \lambda \). Each path can be represented by a unique standard Young tableau, so that the Gelfand–Tsetlin-basis of \( V_\lambda \) is parametrized by the set of standard Young tableaux of shape \( \lambda \) (see Figs. 2 and 3). From now on we identify a chain \( \lambda_1 \uparrow \lambda_2 \uparrow \cdots \uparrow \lambda_n \) with its corresponding standard Young tableau.

5 Decomposition of the Function Space on the Johnson Graph

We define a \( k \)-set as a subset of \( \{1, \ldots, n\} \) of cardinality \( k \). Let \( X \) be the set of all \( k \)-sets. Given two \( k \)-sets \( x, y \) the distance \( d(x, y) \) is defined as \( n - |x \cap y| \). The group
$S_n$ acts naturally on $X$ by

$$\sigma\{i_1, \ldots, i_k\} = \{\sigma(i_1), \ldots, \sigma(i_k)\}.$$  

The vector space $\mathcal{F}$ of the complex valued functions on $X$ is a complex representation of $S_n$ where the action is given by $\sigma f = f \circ \sigma^{-1}$.

For each $k$-set $x \in X$ we define the delta function $\delta(x) : X \to \mathbb{C}$ by

$$\delta(x)(z) = \begin{cases} 1 & \text{if } x = z \\ 0 & \text{otherwise} \end{cases}.$$  

Let $(\ , \ )$ be the hermitian inner product on $\mathcal{F}$ such that the set of delta functions form an orthonormal basis.

A Young diagram can be identified with the sequence given by the numbers of boxes in the rows, written top down. For example the Young diagram

\[
\begin{array}{|c|c|c|}
\hline
& & \\
& & \\
\hline
\end{array}
\]

is identified with (5, 4, 2). It can be shown (see [5, 20]) that the decomposition of $\mathcal{F}$ as a direct sum of irreducible representations of $S_n$ is given as follows.

**Theorem 1** Let $s = \min(k, n - k)$. The space $\mathcal{F}$ of functions on the Johnson graph $J(n,k)$ decomposes into $s + 1$ multiplicity-free irreducible representations of the group $S_n$. Moreover, the decomposition is given by

$$\mathcal{F} = \bigoplus_{i=0}^{s} V_{\alpha_i},$$

where $\alpha_i$ is the Young diagram $(n - i, i)$.

For example, if $n = 6$ and $k = 2$ then the irreducible components of $\mathcal{F}$ are in correspondence with the Young diagrams

\[
\begin{array}{|c|c|c|}
\hline
& & \\
\hline
& & \\
\hline
\end{array} \quad
\begin{array}{|c|c|c|}
\hline
& & \\
\hline
& & \\
\hline
\end{array} \quad
\begin{array}{|c|c|c|}
\hline
\hline
\end{array}.
\]

From now on we denote by $s$ the number $\min(k, n - k)$.

### 5.1 Gelfand–Tsetlin Basis of $\mathcal{F}$

From Theorem 1 we see that $\mathcal{F}$ has a well-defined, up to scalars, Gelfand–Tsetlin basis and that there is a bijection between the set of elements of this GT-basis and the set of standard tableaux of shape $(n - a, a)$ where $a$ runs from 0 to $s$.

Let us give a more explicit description of the GT-basis of $\mathcal{F}$. Consider the space $\mathcal{F}$ as an $S_i$-module for $i = 1, \ldots, n$, and let $\mathcal{F}_{i,\lambda}$ be the isotypic component corresponding to the irreducible representation $\lambda$ of $S_i$ so that for each $i$ we have a decomposition

$$\mathcal{F} = \bigoplus_{\lambda \in \text{Irr}(S_i)} \mathcal{F}_{i,\lambda}.$$  

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where $\mathcal{F}_{i, \lambda} \perp \mathcal{F}_{i, \lambda'}$ if $\lambda \neq \lambda'$.

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ is a sequence of Young diagrams where $\lambda_i$ corresponds to a representation of $S_i$ we define

$$
\mathcal{F}_{\lambda_1 \lambda_2 \ldots \lambda_n} = \mathcal{F}_{1, \lambda_1} \cap \mathcal{F}_{2, \lambda_2} \cap \cdots \cap \mathcal{F}_{n, \lambda_n}.
$$

From the branching rule for representations of $S_n$ and from Theorem 1 it turns out that $\mathcal{F}$ has an orthogonal decomposition in one-dimensional subspaces

$$
\mathcal{F} = \bigoplus_{\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n} \mathcal{F}_{\lambda_1 \lambda_2 \ldots \lambda_n}
$$

where $\lambda_n$ runs through all representations of $S_n$ corresponding to Young diagrams $(n - a, a)$ for $a = 0, \ldots, s$ (see Fig. 3).

### 6 Adapted Decompositions of $\mathcal{F}$

Let $X$ be the set of all $k$-sets of $\{1, \ldots, n\}$. We represent a $k$-set by a word in the alphabet $\{1, 2\}$ as follows. The element $i \in \{1, \ldots, n\}$ belongs to the $k$-subset if and only if the place $i$ of the word is occupied by the letter 1. For example,

$$
\{2, 3, 6, 8\} \subseteq \{1, \ldots, 9\} \quad \longrightarrow \quad 211221212
$$

So, from now on, we identify $X$ with the set of words of length $n$ in the alphabet $\{1, 2\}$ such that the letter 1 appears $k$ times. The group $S_n$ acts on $X$ in the natural way.

For $i = 1, \ldots, n$ and $c \in \{1, 2\}$ we define $\mathcal{F}_i^c$ as the subspace of $\mathcal{F}$ generated by the delta functions $\delta(x)$ such that the word $x$ has the letter $c$ in the place $i$. For each $i$ we have a decomposition

$$
\mathcal{F} = \mathcal{F}_i^1 \oplus \mathcal{F}_i^2
$$

with $\mathcal{F}_i^1 \perp \mathcal{F}_i^2$.

**Definition 1** For $q = 1, \ldots, n$ let $c_q c_{q+1} \ldots c_n$ be a word whose letters are in the alphabet $\{1, 2\}$. For $q = n+1$ the word $c_q c_{q+1} \ldots c_n$ denotes the word with no letters. For $q = 1, \ldots, n$ we define $\mathcal{F}^{c_q c_{q+1} \ldots c_n}$ as the subspace of $\mathcal{F}$

$$
\mathcal{F}^{c_q c_{q+1} \ldots c_n} = \mathcal{F}_q^{c_q} \cap \mathcal{F}_{q+1}^{c_{q+1}} \cap \cdots \cap \mathcal{F}_n^{c_n}.
$$

In the case $q = n + 1$ we set $\mathcal{F}^{c_q c_{q+1} \ldots c_n} = \mathcal{F}$

We see that $\mathcal{F}^{c_q c_{q+1} \ldots c_n}$ is nontrivial if and only if the number of letters 1 in the word $c_q \ldots c_n$ is at most $k$.

**Definition 2** For $q = 1, \ldots, n$ we define $X^{c_q c_{q+1} \ldots c_n}$ as the subset of those words $w_1 \ldots w_n$ in $X$ such that $w_q \ldots w_n$ is $c_q \ldots c_n$. For $q = n + 1$ we set $X^{c_q c_{q+1} \ldots c_n} = X$. 
Observe that each subset $X_{cq\ldots cn}$ is stabilized by the action of the subgroup $S_{q-1}$. We have

$$F_{cq\ldots cn} = \bigoplus_{w_1\ldots w_n \in X_{cq\ldots cn}} \mathbb{C} \delta(w_1 \ldots w_n).$$

**Definition 3** For $q = 1, \ldots, n$ we define $X_q$ as the set of words $c_q \ldots c_n$ where the number of letters 1 is at most $k$. For $q = n + 1$ we set $X_q$ to be the set whose only element is the word with no letters.

Then $F$ decomposes as

$$F = \bigoplus_{c_q\ldots cn \in X_q} F_{cq\ldots cn}$$

and each subspace $F_{cq\ldots cn}$ is invariant by the action of $S_{q-1}$.

**Definition 4** for $1 \leq p < q \leq n + 1$ we define

$$F_{\lambda_1\ldots\lambda_p}^{cq\ldots cn} = F_{\lambda_1\ldots\lambda_p} \cap F_{cq\ldots cn}.$$

**Theorem 2** For fixed $1 \leq q \leq n + 1$, the space $F$ decomposes in one-dimensional subspaces as

$$F = \bigoplus_{\lambda_1\ldots\lambda_{q-1}} F_{\lambda_1\ldots\lambda_{q-1}}^{cq\ldots cn}$$

where the direct sum runs through all $\lambda_1 \ldots \lambda_{q-1}$ and $c_q \ldots c_n$ such that if the number of letters 1 in the word $c_q \ldots c_n$ is $k-\mathbb{r}$, where $0 \leq \mathbb{r} \leq k$, then the sequence $\lambda_1 \ldots \lambda_{q-1}$ forms a standard Young tableaux $\lambda_1 \not\supset \lambda_2 \not\supset \ldots \not\supset \lambda_{q-1}$ where $\lambda_{q-1}$ is a Young diagram of the form $(q - 1 - a', a')$ with $a' = 0, \ldots, \min(\mathbb{r}, q - 1 - \mathbb{r})$.

**Proof** Suppose that the letter 1 appears $k - \mathbb{r}$ times in the word $c_q \ldots c_n$, where $0 \leq \mathbb{r} \leq k$. Then the subset $X_{cq\ldots cn}$ consists of those words $w_1 \ldots w_n$ such that $w_1\ldots w_{q-1}$ has exactly $\mathbb{r}$ appearances of the letter 1. This means that $X_{cq\ldots cn}$ has the structure of the Johnson graph $J(q - 1, r)$ and, when acted by the subgroup $S_{q-1}$, the space of $\mathbb{C}$-valued functions on $X_{cq\ldots cn}$ decomposes as an $S_{q-1}$-module in a multiplicity-free way according to the formula of Theorem 1. As a consequence, each subspace $F_{cq\ldots cn}$ has a Gelfand–Tsetlin decomposition

$$F_{cq\ldots cn} = \bigoplus_{\lambda_1 \not\supset \lambda_2 \not\supset \ldots \not\supset \lambda_{q-1}} F_{\lambda_1\ldots\lambda_{q-1}}^{cq\ldots cn},$$

where $\lambda_{q-1}$ runs over all Young diagrams $(q - 1 - a', a')$ with $0 \leq a' \leq \min(\mathbb{r}, q - 1 - \mathbb{r})$. Then the theorem follows from (1). $\square$
**Theorem 3** For fixed $1 \leq p < q \leq n + 1$, the space $\mathcal{F}$ has an orthogonal decomposition

$$
\mathcal{F} = \bigoplus_{\lambda_1 \rightarrow \ldots \rightarrow \lambda_p} \mathcal{F}_{\lambda_1 \rightarrow \ldots \rightarrow \lambda_p}^{c_q \ldots c_n}
$$

where the direct sum runs through all $\lambda_1 \ldots \lambda_p$ and $c_q \ldots c_n$ such that if the number of letters 1 in the word $c_q \ldots c_n$ is $k - r$, where $0 \leq r \leq k$, then the sequence $\lambda_1 \ldots \lambda_p$ forms a standard Young tableaux $\lambda_1 \nearrow \lambda_2 \nearrow \ldots \nearrow \lambda_p$ where $\lambda_p$ is a Young diagram of the form $(p-a, a)$ with $a = 0, \ldots, \min(r, q - 1 - r, p/2)$.

**Proof** For fixed $c_q \ldots c_n$, we group the one dimensional subspaces $\mathcal{F}_{\lambda_1 \rightarrow \ldots \rightarrow \lambda_p}^{c_q \ldots c_n}$ in Eq. (2) according to the $p$ initial Young diagrams defining the standard Young tableaux $\lambda_1 \ldots \lambda_p$ and obtain

$$
\mathcal{F}_{\lambda_1 \rightarrow \ldots \rightarrow \lambda_p}^{c_q \ldots c_n} = \bigoplus_{\lambda_{p+1} \nearrow \lambda_{p+2} \nearrow \ldots \nearrow \lambda_{q-1}} \mathcal{F}_{\lambda_1 \rightarrow \ldots \rightarrow \lambda_p}^{c_q \ldots c_n},
$$

where the direct sum runs through all Young diagrams $\lambda_{p+1} \nearrow \lambda_{p+2} \nearrow \ldots \nearrow \lambda_{q-1}$ such that $\lambda_1 \nearrow \lambda_2 \nearrow \ldots \nearrow \lambda_{q-1}$ is a standard Young tableau where $\lambda_{q-1}$ is a Young diagram of the form $(q-1-a', a')$ with $0 \leq a' \leq \min(r, q - 1 - r)$. Then we obtain the decomposition

$$
\mathcal{F}^{c_q \ldots c_n} = \bigoplus_{\lambda_1 \nearrow \lambda_2 \nearrow \ldots \nearrow \lambda_p} \mathcal{F}_{\lambda_1 \rightarrow \ldots \rightarrow \lambda_p}^{c_q \ldots c_n},
$$

(3)

where $\lambda_1 \ldots \lambda_p$ run through all sequences of Young diagrams for which there is a sequence $\lambda_{p+1} \ldots \lambda_{q-1}$ such that $\lambda_1 \nearrow \lambda_2 \nearrow \ldots \nearrow \lambda_{q-1}$ is a standard Young tableau of the form $(q-1-a', a')$ with $0 \leq a' \leq \min(r, q - 1 - r)$. Such sequences $\lambda_1 \ldots \lambda_p$ are characterized as those sequences such that $\lambda_1 \nearrow \lambda_2 \nearrow \ldots \nearrow \lambda_p$ is a standard Young tableau of the form $(p-a, a)$ with $0 \leq a \leq \min(r, q - 1 - r, p/2)$. From (1) and (3) we obtain the theorem. □

### 7 The Intermediate Bases and the Fast Fourier Transform

Let us describe schematically the Fast Fourier Transform algorithm for the Johnson graph. The input is a vector $f$ in the space $\mathcal{F}$ of functions on the set $X$ of $k$-sets, written in the delta function basis $B_0$, given as a column vector $[f]_{B_0}$. The output of the algorithm is a column vector representing the vector $f$ written in the basis $B_n$, a Gelfand–Tsetlin basis of $\mathcal{F}$. In other words, the objective is to apply the change of basis matrix to a given column vector:

$$
[f]_{B_n} = [B_0]_{B_n} [f]_{B_0}.
$$

Our technique to realize this matrix multiplication is to construct a sequence of intermediate bases $B_1, B_2, \ldots, B_{n-1}$ such that...
\[ [B_0]_{B_n} = [B_{n-1}]_{B_n} \ldots [B_1]_{B_2} [B_0]_{B_1} \]

is a decomposition where each factor is a sparse matrix.

### 7.1 Block-Diagonal Matrices

In order to establish the sparsity of a matrix we will rely on the following simple principle given by Lemma 1.

Whenever \( T \) is a linear operator on a vector space \( V \) and \( B \) is a basis of \( V \) we denote by \([T]_B\) the matrix whose columns are the elements of the basis \( B \) transformed by \( T \) and written in the basis \( B \).

**Definition 5** Let \( V \) be a finite-dimensional vector space. For \( i = 1, \ldots, n \) let \( V_i \) be a subspace of \( V \) such that

\[ V = \bigoplus_{i=1}^n V_i . \]

Let \( B \) be a basis of \( V \) and \( T \) a linear endomorphism of \( V \). We say that the basis \( B \) is adapted to the decomposition if every element of \( B \) is in \( V_i \) for some \( i \). We say that \( T \) is adapted to the decomposition if it preserves each subspace \( V_i \).

We will use the following simple fact from linear algebra.

**Lemma 1** Let \( V \) be a finite-dimensional vector space with a direct sum decomposition \( V = \bigoplus_{i=1}^n V_i \). Let \( B \) and \( B' \) be two bases of \( V \), both adapted to this decomposition and let \( T \) be a linear endomorphism of \( V \) adapted to this decomposition. Then

(a) there are orders of the elements of \( B \) and \( B' \) such that the change of basis matrix \([B]_{B'}\) is block-diagonal, with each block of size \( \dim(V_i) \) for each \( i = 1, \ldots, n \).

(b) there is an order of the basis \( B \) such that \([T]_B \) is block-diagonal, with a block of size \( \dim(V_i) \) for each \( i = 1, \ldots, n \).

### 7.2 Definition of the Basis \( B_i \)

Let \( 0 \leq i \leq n \). As observed in the proof of Theorem 3, the set \( X_{c_i+1\ldots c_n} \) has the structure of the Johnson graph \( J(i, r) \) where \( k - r \) is the number of letters 1 in the word \( c_i+1\ldots c_n \), and when acted by the subgroup \( S_i \), the space \( \mathcal{F}^{c_i+1\ldots c_n} \) decomposes as an \( S_i \)-module in a multiplicity-free way according to the formula of Theorem 1. As a consequence, each subspace \( \mathcal{F}^{c_i+1\ldots c_n} \) has a Gelfand–Tsetlin decomposition into one-dimensional subspaces

\[ \mathcal{F}^{c_i+1\ldots c_n} = \bigoplus_{\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_i} \mathcal{F}^{c_i+1\ldots c_n}_{\lambda_1\lambda_2\ldots \lambda_i}, \]

where \( \lambda_i \) runs over all Young diagrams \( (i-a, a) \) with \( 0 \leq a \leq \min(r, i-r) \). Let \( B^{c_i+1\ldots c_n} \) be the unique, up to scalars, basis of \( \mathcal{F}^{c_i+1\ldots c_n} \) adapted to the decomposition (4). Then the space \( \mathcal{F} \) decomposes in one-dimensional subspaces as
Fig. 4 Labels of the intermediate bases in the case $n = 4, k = 2$. The boxes in each column represent the decomposition $B_i = \cup B^{c_{i+1}...c_n}$.

$$F = \bigoplus_{\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_i} \bigoplus_{c_{i+1}...c_n \in X_i} F^{c_{i+1}...c_n}_{\lambda_1...\lambda_i},$$

(5)

**Definition 6** We define the $i$th intermediate basis of $F$ as the unique, up to scalars, basis $B_i$ of $F$ adapted to the decomposition (5).

We have

$$B_i = \bigcup_{c_{i+1}...c_n \in X_i} B^{c_{i+1}...c_n}.$$

From Theorem 1, we see that the basis $B^{c_{i+1}...c_n}$ is parametrized by the set of standard tableaux of shape $(i - a, a)$ with $0 \leq a \leq \min(r, i - r)$. On the other hand, the word $c_{i+1}...c_n$ runs over the set $X_i$. Figure 4 illustrates the structure of the intermediate bases.

**Lemma 2** For $1 \leq p \leq i < q \leq n + 1$, the basis $B_i$ is adapted to the decomposition of Theorem 3

$$F = \bigoplus F^{c_q...c_n}_{\lambda_1...\lambda_p}.$$

**Proof** It is clear from the definition of the subspace $F^{c_q...c_n}_{\lambda_1...\lambda_p}$ that if $1 \leq p \leq p' < q' \leq q \leq n + 1$ then
\[ F_{c_1'...c_n}^{c_1'...c_n} \subseteq F_{\lambda_1...\lambda_p}^{c_1...c_n}. \]

Since every element of the basis \( B_i \) is in one of the subspaces of the form \( F_{\lambda_1...\lambda_i}^{c_1...c_n} \), the Lemma follows from the case \( p' = i, q' = i + 1 \).

### 7.3 Sparsity of the Change of Basis Matrix \([B_i]_{B_{i-1}}\)

In this section we establish the fact that, for all \( i \), each column of the matrix \([B_i]_{B_{i-1}}\) has at most two nonzero entries. In fact, we show that if the bases are properly ordered, the matrix \([B_i]_{B_{i-1}}\) is block-diagonal with blocks of size at most two.

**Theorem 4** There is an order of the basis \( B_i \) and an order of the basis \( B_{i-1} \) such that the change of basis matrices \([B_i]_{B_{i-1}}\) and \([B_{i-1}]_{B_i}\) are block-diagonal with all blocks of size at most two.

**Proof** By Lemma 2, both \( B_i \) and \( B_{i-1} \) are adapted to the decomposition

\[ F = \bigoplus F_{\lambda_1...\lambda_{i-1}}^{c_1...c_n}, \]

that is, the decomposition of Theorem 3 with \( p = i - 1 \) and \( q = i + 1 \). Observe that the subspace \( F_{\lambda_1...\lambda_{i-1}}^{c_1...c_n} \) has dimension at most two, since it is spanned by the subspaces \( F_{\lambda_1...\lambda_{i-1}}^{c_1+1...c_n} \) and \( F_{\lambda_1...\lambda_{i-1}}^{c_2+c_1...c_n} \) and, according to Theorem 2, these subspaces have dimension at most one. Then the theorem follows from Lemma 1. \( \square \)

#### 7.4 Example

Consider the case \( n = 4, k = 2 \). For each vector \( b \) of the basis \( B_i \), there exists a unique word \( c_i+1...c_n \in X_i \) and a unique standard tableau \( \lambda_1 \succ \lambda_2 \succ \cdots \succ \lambda_i \) such that \( b \in F_{\lambda_1...\lambda_i}^{c_i+1...c_n}. \) Then \( b \) is a linear combination of those elements of \( B_{i-1} \) that belong to the spaces \( F_{\lambda_1...\lambda_{i-1}}^{c_1+1...c_n} \) and \( F_{\lambda_1...\lambda_{i-1}}^{2+c_1+1...c_n} \). Then the matrices \([B_i]_{B_{i-1}}\) have the form

\[
[B_1]_{B_2} = \begin{bmatrix}
* & 0 & 0 & 0 & 0 \\
0 & * & 0 & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
0 & 0 & 0 & * & 0 \\
0 & 0 & 0 & 0 & * \\
\end{bmatrix}
\quad [B_2]_{B_3} = \begin{bmatrix}
* & 0 & 0 & 0 & 0 \\
0 & * & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * \\
\end{bmatrix}
\quad [B_3]_{B_4} = \begin{bmatrix}
* & 0 & 0 & 0 & 0 \\
0 & * & 0 & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
* & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
\end{bmatrix}
\]

### 8 Connection with the Robinson–Schensted Insertion Algorithm

In Fig. 5 the vertical order of the labels of the elements of each basis \( B_i \) has been carefully chosen in order to simplify the figure. In fact, the order is such that each...
horizontal line corresponds to a well known process: the Robinson–Schensted (RS) insertion algorithm (see [13]).

Observe that each horizontal line gives the sequence, reading from left to right, that is obtained by applying the RS insertion algorithm to a word corresponding to an element of the basis $B_0$, which is a word in the alphabet $\{1, 2\}$. The elements of this sequence are triples $(P, Q, \omega)$ where $P$ is a semistandard tableau, $Q$ is a standard tableau and $\omega$ is a word in the alphabet $\{1, 2\}$. In our situation $P$ is filled with letters in $\{1, 2\}$ so its height is at most 2. It turns out that the triple $(P, Q, \omega)$ is determined by the pair $(Q, \omega)$ so $P$ can be ommited.

**Definition 7** For $1 \leq i \leq n$, let $b \in B_{i-1}$ and $b' \in B_i$. We say that $b$ and $b'$ are $S$-related if both belong to the subspace $\mathcal{F}_{\lambda_1, \ldots, \lambda_{i-1}}^{c_{i+1}, \ldots, c_n}$ for some standard Young tableau $\lambda_1 \rightarrow \lambda_2 \rightarrow \cdots \rightarrow \lambda_{i-1}$ and some word $c_{i+1}, \ldots, c_n$.

**Definition 8** Let $b \in B_{i-1}$ and $b' \in B_i$. We say that $b$ and $b'$ are RS-related if the label of $b'$ is obtained by applying the RS insertion step to the label of $b$.

From the definitions it is immediate the following (see Fig. 5 for an illustration).

**Theorem 5** If $b \in B_{i-1}$ and $b' \in B_i$ are RS-related then they are $S$-related.

### 9 Jucys–Murphy Operators

Let $\mathbb{C}(S_n)$ denote the group algebra of the group $S_n$. Let $(ij)$ denote the transposition that interchanges $i$ with $j$. For $i = 1, 2, \ldots, n$, the Jucys–Murphy element $J_i$ is defined

![Fig. 5](image-url) An illustration of the sparsity of the matrices $[B_{i-1}]_{B_i}$. A label of an element $b \in B_{i-1}$ is connected with a label of an element $b' \in B_i$ if and only if they are S-related. Two labels are horizontally adjacent if and only if they are RS-related, that is, each row corresponds to the process given by the Robinson–Schensted insertion algorithm.
as the element of $\mathbb{C}(S_n)$ given by
\[ J_i = (1i) + (2i) + \cdots + ((i-1)i) \]
(in particular $J_1 = 0$). For $i = 1, 2, \ldots, n$ let $Z_i \in \mathbb{C}(S_n)$ be
\[ Z_i = \text{sum of all transpositions in } S_i \]
Observe that for $i \in \{2, \ldots, n\}$
\[ J_i = Z_i - Z_{i-1}. \]
If $V$ is a representation of $S_n$ we consider the canonical $\mathbb{C}(S_n)$-module structure on $V$ and we identify an element $A$ of $\mathbb{C}(S_n)$ with the linear operator $A : V \to V$ sending $v$ to $Av$.

**Proposition 1** Let $V$ be a multiplicity-free representation of $S_n$. Then every element of a Gelfand–Tsetlin basis of $V$ is an eigenvector of $J_i$ for all $i \in \{1, \ldots, n\}$.

**Proof** Let $b$ be an element of a Gelfand–Tsetlin basis of $V$. For $i = 1$ the proposition is trivial so let $i$ be any element of $\{2, \ldots, n\}$. Then the vector $b$ belongs to some isotypic component $H_i$ of the decomposition of $V$ as a representation of $S_i$.

We will use the following characterization of isotypic components. Let $End(V)^{S_i}$ be the ring of interwining operators, that is, linear operators $Z \in End(V)$ such that $ZA = AZ$ for all $A \in \mathbb{C}(S_i)$. Observe that $V$ has a natural $End(V)^{S_i}$-module structure. Then, a subspace of $V$ is an isotypic component of the action of $S_i$ if and only if it is a minimal element in the lattice of subspaces that are simultaneously a $\mathbb{C}(S_i)$-submodule and a $End(V)^{S_i}$-submodule of $V$.

Since $H_i$ is an isotypic component it is a $\mathbb{C}(S_i)$-submodule of $V$. Since $Z_i \in \mathbb{C}(S_i)$ we see that $Z_i(H_i) \subseteq H_i$. Let $H'_i$ be an eigenspace of the restriction $Z_i : H_i \to H_i$ with eigenvalue $\alpha_i$. Since $Z_i \in End(V)^{S_i}$, for any $A \in \mathbb{C}(S_i)$ and any $h \in H'_i$ we have
\[ Z_i(A(h)) = A(Z_i(h)) = \alpha_i A(h). \]

This shows that $H'_i$ is a $\mathbb{C}(S_i)$-submodule of $V$.

On the other hand, since $Z_i$ belongs to $\mathbb{C}(S_i)$ it commutes with every element of $End(V)^{S_i}$. Then for any $Z \in End(V)^{S_i}$ and any $h \in H'_i$ we have
\[ Z_i(Z(h)) = Z(Z_i(h)) = \alpha_i Z(h). \]

This shows that $H'_i$ is a $End(V)^{S_i}$-submodule of $V$. Then $H'_i$ is simultaneously a $\mathbb{C}(S_i)$-submodule and a $End(V)^{S_i}$-submodule of $V$, and it is contained in $H_i$. Since $H_i$ is an isotypic component, it is minimal among subspaces with this property, then $H'_i = H_i$. This proves that $H_i$ is an eigenspace of $Z_i$. We have $b \in H_i$, then $b$ is an eigenvector of $Z_i$ for all $i \in \{2, \ldots, n\}$. Since $J_i = Z_i - Z_{i-1}$, we see that $b$ is also an eigenvector of $J_i$ with eigenvalue $\alpha_i - \alpha_{i-1}$ for all $i \in \{2, \ldots, n\}$.

\[ \square \]
Corollary 1 For $1 \leq i \leq n$, the one-dimensional subspace $\mathcal{F}_{\lambda_1 \ldots \lambda_i}^{c_i+1 \ldots c_n}$ is invariant by the action of $J_j$ for $j = 1, \ldots, i$.

Proof From (2) we have that $\mathcal{F}_{\lambda_1 \ldots \lambda_i}^{c_i+1 \ldots c_n}$ is a multiplicity-free representation of $S_i$ such that the Gelfand–Tsetlin decomposition is given by the subspaces $\mathcal{F}_{\lambda_1 \ldots \lambda_i}^{c_i+1 \ldots c_n}$. From Proposition 1 we have that these subspaces are formed by eigenvectors of $J_j$ for $j = 1, \ldots, i$. \hfill \Box

Let $V$ be a multiplicity-free representation of $S_n$ and $B$ a Gelfand–Tsetlin basis of $V$. By Proposition 1 there is a map $\alpha : B \rightarrow \mathbb{C}^n$

$$\alpha(b) = (\alpha_1(b), \ldots, \alpha_n(b))$$

where $\alpha_i(b)$ is defined by

$$J_i b = \alpha_i(b) b.$$
and the map $\alpha$ is a bijection between the Gelfand–Tsetlin basis $B$ of $V$ and the vectors in $\mathbb{C}^n$ of the form

$$(c(\lambda_1/\lambda_0), \ldots, c(\lambda_n/\lambda_{n-1})).$$

where $\lambda_1 \nearrow \cdots \nearrow \lambda_n$ is a standard Young tableau with $\lambda_n$ in the decomposition of $V$. For example, the map $\alpha$ gives the correspondence

$$
\begin{bmatrix}
1 & 3 & 4 \\
2 & 5 \\
\end{bmatrix}
\quad \leftrightarrow \quad (0, -1, 1, 2, 0)
$$

10 Efficient Computation of the Matrices $[B_i]_{B_{i-1}}$

We have shown that the Fourier transform on the Johnson graph can be performed by the successive applications of the matrices $[B_i]_{B_{i-1}}$ for $i = 1, \ldots, n$, and that the application of each matrix uses $O\left(\binom{n}{k}\right)$ operations. Our aim in this section is to show that the matrix $[B_{i+1}]_{B_i}$ can be computed from $[B_i]_{B_{i-1}}$ using $O\left(\binom{n}{k}\right)$ operations. This will enable to compute the Fourier transform in $O(n\binom{n}{k})$ operations.

Let $D_j$ be the matrix whose $kl$-entry is the scalar product $\langle b_k, b_l \rangle$ of elements of the basis $B_j$. Note that $D_j$ is a diagonal matrix. We will construct the matrices $[B_{i+1}]_{B_i}$, $[B_i]_{B_{i+1}}$ and $D_{i+1}$ assuming we know $[B_i]_{B_{i-1}}$, $[B_{i-1}]_{B_i}$ and $D_i$. We will do this in three steps. First we construct the matrix $[J_{i+1}]_{B_i}$ assuming we know the matrices $[B_i]_{B_{i-1}}$ and $[B_{i-1}]_{B_i}$. In a second step we construct $[B_{i+1}]_{B_i}$ assuming we know $[J_{i+1}]_{B_i}$. Then in a final step we compute $[B_i]_{B_{i+1}}$ and $D_{i+1}$ assuming we know $[B_{i+1}]_{B_i}$ and $D_i$.

10.1 First Step: Obtaining $[J_{i+1}]_{B_i}$ from $[B_i]_{B_{i-1}}$, and $[B_{i-1}]_{B_i}$

This step is based on the formula

$$J_{i+1}s_i = s_i J_i + 1$$

where $s_i$ is the the transposition $(i(i+1))$ for $i = 1, \ldots, n - 1$. From this formula we derive

$$[J_{i+1}]_{B_i} = ([s_i]_{B_i} [J_i]_{B_i} + I) [s_i]_{B_i} \quad \quad (7)$$

$$[s_i]_{B_i} = [B_{i-1}]_{B_i} [s_i]_{B_{i-1}} [B_i]_{B_{i-1}}. \quad \quad (8)$$

Equations (7) and (8) show that we can compute $[J_{i+1}]_{B_i}$ from $[B_i]_{B_{i-1}}$ and $[B_{i-1}]_{B_i}$ provided we know the matrices $[s_i]_{B_{i-1}}$ and $[J_i]_{B_i}$. On the one hand we observe that $[s_i]_{B_{i-1}}$ is a permutation matrix which is easily described in terms of the labels of the basis $B_{i-1}$. On the other hand, we can see that the matrix $[J_i]_{B_i}$ is a diagonal matrix whose diagonal entries are the eigenvalues described by formula (6).
Fig. 7  An illustration of the case $n = 5, k = 2, i = 3$. The three bases $B_{i-1}, B_i$ and $B_{i+1}$ are adapted to the decomposition $\bigoplus F_{\lambda_1, \ldots, \lambda_{i-1}}$. Each element of these bases is coloured according to the subspace of the decomposition containing it.

We claim that once $[s_i]_{B_{i-1}}, [J_i]_{B_i}, [B_i]_{B_{i-1}}$ and $[B_i]_{B_i}$ have been computed, the computation of $[J_{i+1}]_{B_i}$ using (7) and (8) can be performed using $O\left(\binom{n}{k}\right)$ operations. The key point is that all the operators and bases involved in Eqs. (7) and (8) are adapted to the decomposition

$$\mathcal{F} = \bigoplus F_{\lambda_1, \ldots, \lambda_{i-1}}^{c_{i+2} \ldots c_n}, \quad (9)$$

that is, the decomposition in Theorem 3 with $p = i - 1$ and $q = i + 2$ (see Fig. 7). Since each subspace in this decomposition has dimension at most 4, we see from Lemma 1 that if $B_i$ and $B_{i-1}$ are properly ordered then all the matrices appearing in (7) and (8) are block-diagonal with each block of size at most 4. Let us prove this fact.

**Lemma 3**  For $i = 1, \ldots, n - 1$ let

$$\mathcal{F} = \bigoplus F_{\lambda_1, \ldots, \lambda_{i-1}}^{c_{i+2} \ldots c_n}$$
be the decomposition in Theorem 3 with $p = i - 1$ and $q = i + 2$. Then
(a) the dimension of each subspace $F_{\lambda_1 \ldots \lambda_{i-1}}^{i+2 \ldots n}$ is at most 4
(b) the operators $s_i$ and $J_i$ are adapted to this decomposition
(c) the bases $B_{i-1}$, $B_i$ and $B_{i+1}$ are adapted to this decomposition

Proof For (a) observe that the subspace $F_{\lambda_1 \ldots \lambda_{i-1}}^{i+2 \ldots n}$ is spanned by the four subspaces
\[ F_{\lambda_1 \ldots \lambda_{i-1}}^{11i+2 \ldots n}, F_{\lambda_1 \ldots \lambda_{i-1}}^{12i+2 \ldots n}, F_{\lambda_1 \ldots \lambda_{i-1}}^{21i+2 \ldots n}, F_{\lambda_1 \ldots \lambda_{i-1}}^{22i+2 \ldots n} \] (10)
and according to Theorem 2 these subspaces have dimension at most one.

For (b) note that the operator $s_i$ acts by interchanging the subspaces in (10), and these subspaces span $F_{\lambda_1 \ldots \lambda_{i-1}}^{i+2 \ldots n}$, so $s_i$ stabilizes it. Similarly, to see that $J_i$ stabilizes it, observe that $F_{\lambda_1 \ldots \lambda_{i-1}}^{i+2 \ldots n}$ is spanned by the subspaces of the form $F_{\lambda_1 \ldots \lambda_i \lambda_{i+1}}^{c_i+2 \ldots n}$ where $\lambda$ is a Young diagram and $c$ is a letter. Then by Corollary 1, the operator $J_i$ stabilizes each one of these last subspaces.

Part (c) is an instance of Lemma 2.

Proposition 2 Let us assume that the matrices $[B_i]_{B_{i-1}}$ and $[B_{i-1}]_{B_i}$ have been computed for some $i$ such that $1 \leq i \leq n - 1$. Then the matrix $[J_{i+1}]_{B_i}$ can be computed in at most $272 \binom{n}{i}$ operations.

Proof The algorithm for computing $[J_{i+1}]_{B_i}$ proceeds by restricting the bases and the operators appearing in (7) and (8) to each subspace $F_{\lambda_1 \ldots \lambda_{i-1}}^{i+2 \ldots n}$ in the decomposition (9), one at a time. Once the operators and the bases have been restricted to $F_{\lambda_1 \ldots \lambda_{i-1}}^{i+2 \ldots n}$, by Lemma 3 the matrices in (7) and (8) are square matrices of size at most 4.

The matrix $[s_i]_{B_{i-1}}$ is a permutation matrix corresponding to the permutation of the subspaces in (10). By Corollary 1, $[J_i]_{B_i}$ is a diagonal matrix whose diagonal entries can be computed using formula (6) for the eigenvalues of the Jucys–Murphy operators.

The formulas (7) and (8) involve four matrix multiplications and one matrix sum. The cost of a matrix multiplication is $4 \cdot 4^2 = 64$ operations and that of a matrix sum is $4^2 = 16$, so the number of arithmetic operations required to compute $[J_{i+1}]_{B_i}$ restricted to $F_{\lambda_1 \ldots \lambda_{i-1}}^{i+2 \ldots n}$ is at most $4 \cdot 64 + 16 = 272$. Since the number of subspaces in the decomposition (9) is at most $\binom{n}{i}$, then the total number of arithmetic operations does not exceed $272 \binom{n}{i}$.

10.2 Second Step: Obtaining $[B_{i+1}]_{B_i}$ from $[J_{i+1}]_{B_i}$

Proposition 3 Let us assume that the matrix $[J_{i+1}]_{B_i}$ have been computed for some $i$ such that $1 \leq i \leq n - 1$. Then the matrix $[B_{i+1}]_{B_i}$ can be computed using at most $4 \binom{n}{i}$ arithmetic operations.

Proof The elements of the basis $B_{i+1}$ are non-zero vectors in the subspaces of the form $F_{\lambda_1 \ldots \lambda_{i+1}}^{i+2 \ldots n}$. By Corollary 1, such vectors are eigenvectors of the operator $J_{i+1}$. In order to find all such eigenvectors written in the basis $B_i$ we must find the column
vectors \([v]_{B_i}\) such that
\[
\left( [J_{i+1}]_{B_i} - \lambda I \right) [v]_{B_i} = 0
\]  \hspace{1cm} (11)
for some eigenvalue \(\lambda\) of \(J_{i+1}\). Once again, the calculation can be simplified by restricting to appropriate subspaces. In effect, the operator \(J_{i+1}\) and the bases \(B_{i+1}\) and \(B_i\) are adapted to the decomposition
\[
\mathcal{F} = \bigoplus \mathcal{F}^{c_{i+2} \cdots c_n}_{\lambda_1 \cdots \lambda_i}
\]  \hspace{1cm} (12)
whose subspaces have dimension at most two. We proceed by restricting to each subspace and obtaining the corresponding blocks of \([B_{i+1}]_{B_i}\), one at a time. When the operator and the bases are restricted to \(\mathcal{F}^{c_{i+2} \cdots c_n}_{\lambda_1 \cdots \lambda_i}\), Eq. (11) is a homogeneous linear system of equations in at most two variables.

If \(\mathcal{F}^{c_{i+2} \cdots c_n}_{\lambda_1 \cdots \lambda_i}\) is one-dimensional, then the corresponding block of \([B_{i+1}]_{B_i}\) is of size one with 1 as the only entry, so we are done. If \(\mathcal{F}^{c_{i+2} \cdots c_n}_{\lambda_1 \cdots \lambda_i}\) is two-dimensional, then there are two different eigenvalues which correspond to the two ways in which a letter can be inserted by the Robinson–Schensted insertion step. If the form of the Young diagram \(\lambda_i\) is \((i - a, a)\), then, by the eigenvalue formula (6), the two eigenvalues of the restriction of \(J_{i+1}\) to \(\mathcal{F}^{c_{i+2} \cdots c_n}_{\lambda_1 \cdots \lambda_i}\) are \(i - a\) and \(a - 1\) (see Fig. 8).

Solving each of the two-variable linear systems
\[
\left( [J_{i+1}]_{B_i} - (i - a) I \right) [v]_{B_i} = 0 \hspace{1cm} (13)
\]
\[
\left( [J_{i+1}]_{B_i} - (a - 1) I \right) [v]_{B_i} = 0 \hspace{1cm} (14)
\]
we obtain the two eigenvectors of \(J_{i+1}\) restricted to \(\mathcal{F}^{c_{i+2} \cdots c_n}_{\lambda_1 \cdots \lambda_i}\). Since each linear system can be reduced to a single equation of the form
\[
n_1v_1 + n_2v_2 = 0
\]
solving each system does not require any arithmetic operation. Setting these two eigenvectors as the vector columns of a matrix of size two, we obtain the matrix corresponding to the restriction of \([B_{i+1}]_{B_i}\) to \(\mathcal{F}^{c_{i+2} \cdots c_n}_{\lambda_1 \cdots \lambda_i}\).

To construct the linear systems (13) and (14) it suffices to sum to the matrix of size two \([J_{i+1}]_{B_i}\) the diagonal matrices \((i - a) I\) and \((a - 1) I\) respectively, so four arithmetic operations are sufficient. Since there are at most \(\binom{n}{k}\) subspaces in the decomposition (12), then the number of arithmetic operations involved in the second step does not exceed \(4\binom{n}{k}\).

\[\square\]

10.3 Third Step: Obtaining \(D_{i+1}\) and \([B_i]_{B_{i+1}}\) from \([B_{i+1}]_{B_i}\) and \(D_i\)

**Proposition 4** Let us assume that the matrices \([B_{i+1}]_{B_i}\) and \(D_i\) have been computed for some \(i\) such that \(1 \leq i \leq n - 1\). Then the matrices \([B_i]_{B_{i+1}}\) and \(D_{i+1}\) can be computed using at most \(13\binom{n}{k}\) arithmetic operations.
Fig. 8 The number in the boxes indicates their contents. The content of the inserted box is the eigenvalue of the operator $J_{i+1}$, which can be $i - a$ (left) or $a - 1$ (right).

\[ \begin{array}{c|ccccc} \hline & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & -1 & 0 & 1 \\ \hline \end{array} \quad \begin{array}{c|ccccc} \hline & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -1 & 0 & 1 & 2 \\ \hline \end{array} \]

Proof Let us denote by $b_k^i$ the elements of the basis $B_i$, and let $a_{jk}$ be the $jk$-entry of the matrix $[B_{i+1}]_{B_i}$. That is

\[ b_k^{i+1} = \sum_j a_{jk} b_k^j. \]

Then the $rs$-entry of the matrix $D_{i+1}$ is

\[ \langle b_r^{i+1}, b_s^{i+1} \rangle = \sum_j \sum_{j'} a_{jr} \bar{a}_{j's} \langle b_j^i, b_{j'}^i \rangle. \]

Since the bases $B_i$ and $B_{i+1}$ are orthogonal, this entry is zero if $r \neq s$ and it is

\[ \|b_r^{i+1}\|^2 = \sum_j \|a_{jr}\|^2 \|b_j^i\|^2 \]

if $r = s$. All numbers involved are rational, then this is equivalent to

\[ (b_r^{i+1})^2 = \sum_j (a_{jr})^2 (b_j^i)^2. \]

Since there are at most two terms in this sum, the number of arithmetic operations needed to obtain $D_{i+1}$ from $[B_{i+1}]_{B_i}$ and $D_i$ is at most $5 \binom{n}{k}$.

Since the basis $B_i$ is orthogonal for all $i$, then $[B_i]_{B_{i+1}}$ can be easily computed from the transpose of $[B_{i+1}]_{B_i}$ by

\[ [B_i]_{B_{i+1}} = D_{i+1}^{-1} ([B_{i+1}]_{B_i})^t D_i. \]  \hfill (15)

This matrix equation can be restricted to each subspace $\mathcal{F}_{\lambda_1, \ldots, \lambda_i}^{c_i, \ldots, c_n}$ so that it is an equation of matrices of size two. The number of arithmetic operation to compute $[B_i]_{B_{i+1}}$ restricted to $\mathcal{F}_{\lambda_1, \ldots, \lambda_i}^{c_i, \ldots, c_n}$ using (15) is at most 8. Since there are at most $\binom{n}{k}$ subspaces in the decomposition (12), then the number of arithmetic operations needed to compute $[B_i]_{B_{i+1}}$ does not exceed $8 \binom{n}{k}$, and the overall number of operations involved in the third step does not exceed $13 \binom{n}{k}$. \hfill \square
10.4 Collecting All the Steps

Putting together the three steps we obtain an algorithm to construct the matrices \([B_{i+1}]_{B_i}, [B_i]_{B_{i+1}}\) and \(D_{i+1}\) assuming we know \([B_i]_{B_{i-1}}, [B_{i-1}]_{B_i}\) and \(D_i\).

Proposition 5  Let us assume that the matrices \([B_i]_{B_{i-1}}, [B_{i-1}]_{B_i}\) and \(D_i\) have been computed for some \(i\) such that \(1 \leq i \leq n - 1\). Then the matrices \([B_{i+1}]_{B_i}, [B_i]_{B_{i+1}}\) and \(D_{i+1}\) can be computed using at most \(289\binom{n}{k}\) arithmetic operations.

11 Main Results

Theorem 6  The set of factors \([[B_i]_{B_{i-1}}, i \in \{2, \ldots, n\}]\) of the Fourier transform on the Johnson graph \(J(n, k)\) and the set of factors \([[B_{i-1}]_{B_i}, i \in \{2, \ldots, n\}]\) of the inverse Fourier transform can be computed using at most \(289(n - 1)\binom{n}{k}\) arithmetic operations. These sets of matrices can be stored as a set \(S\) of rational numbers such that \(|S| \leq 16(n - 1)\binom{n}{k}\).

Proof for \(i = 1\) we have \([B_1]_{B_0} = [B_0]_{B_1} = D_1 = Id\). This is the initial step of the recursion given by Proposition 5. Applying the recursion step \(n - 1\) times we obtain both sets of matrices and the number of operations does not exceed \(289(n - 1)\binom{n}{k}\).

Each entry of a matrix in the set \([[B_i]_{B_{i-1}}, i \in \{2, \ldots, n\}]\) can be codified by a sequence \((i, r, s, c)\) of rational numbers indicating that the \(rs\)-entry of the matrix \([B_i]_{B_{i-1}}\) is the number \(c\). Since each matrix has at most two nonzero entries in each column (Theorem 4), there are at most \(2\binom{n}{k}\) such sequences to store for each matrix. Since there are \(n - 1\) matrices in the set, and we have to repeat the same with the set \([[B_{i-1}]_{B_i}, i \in \{2, \ldots, n\}]\), it follows that \(16(n - 1)\binom{n}{k}\) numbers are sufficient to store both sets of matrices.

Theorem 7  Let \(B_0\) be the delta function basis of \(\mathcal{F}\) and let \(B_n\) be a Gelfand–Tsetlin basis of \(\mathcal{F}\). We assume that the matrices \([B_{i-1}]_{B_i}\) and \([B_i]_{B_{i-1}}\) for \(i = 2, 3, \ldots, n\) have been computed. Then, given a column vector \([f]_{B_n}\) with \(f \in \mathcal{F}\), the column vector \([f]_{B_n}\) given by

\[ [f]_{B_n} = [B_0]_{B_n} [f]_{B_0} \]

can be computed using at most \(2(n - 1)\binom{n}{k}\) operations. Similarly, given a column vector \([f]_{B_n}\) with \(f \in \mathcal{F}\), the column vector \([f]_{B_0}\) given by

\[ [f]_{B_0} = [B_n]_{B_0} [f]_{B_n} \]

can be computed using at most \(2(n - 1)\binom{n}{k}\) operations.

Proof  By Theorem 4 we see that each row of the matrix \([B_{i-1}]_{B_i}\) has at most two non-zero elements. Then the application of the matrix \([B_{i-1}]_{B_i}\) to a generic column vector can be done using at most \(2\binom{n}{k}\) operations. Observe that \([B_0]_{B_i}\) is the identity.
matrix. The Fourier transform can be factorized as
\[
[B_0]_{B_n} = [B_{n-1}]_{B_n} \cdots [B_1]_{B_2} [B_0]_{B_1} = [B_{n-1}]_{B_n} \cdots [B_1]_{B_2}.
\]
Then the successive applications of the \(n-1\) matrices can be done in at most \(2(n-1)\binom{n}{k}\) operations. The same argument can be applied to the inverse Fourier transform considering the factorization
\[
[B_n]_{B_0} = [B_1]_{B_0} \cdots [B_{n-2}]_{B_{n-2}} [B_n]_{B_{n-1}}.
\]

\[\Box\]

### 12 Application to the Computation of Isotypic Components

The upper bound we obtained for the algebraic complexity of the Fourier transform can be applied to the problem of computing the isotypic projections of a given function on the Johnson graph.

For \(a = 0, \ldots, s\), let \(F_a\) be the isotypic component of \(F\) corresponding to the Young diagram \((n-a, a)\) under the action of the group \(S_n\). Since these components are orthogonal and expand the space \(F\), given a function \(f \in F\) there are uniquely determined functions \(f_a \in F_a\) such that
\[
f = \sum_{a=0}^{s} f_a.
\]
For \(H \subseteq \{0, \ldots, s\}\) let \(f_H\) be defined by
\[
f_H = \sum_{a \in H} f_a.
\]

**Theorem 8** Assume that the matrices \([B_{i-1}]_{B_i}\) for \(i = 2, 3, \ldots, n\) have been computed. Given a column vector \([f]_{B_0}\) with \(f \in F\), the column vector \([f_H]_{B_0}\) can be computed using at most \(4(n-1)\binom{n}{k}\) operations.

**Proof** First we apply the Fourier transform to the function \(f\), so that we obtain the column vector \([f]_{B_n}\) using \(2(n-1)\binom{n}{k}\) operations. The basis \(B_n\) is parametrized by all Young tableaux of shape \((n-a, a)\) for \(a = 0, \ldots, s\). Then we substitute by 0 the values of the entries of the vector \([f]_{B_n}\) that correspond to Young tableaux of shape \((n-a, a)\) with \(a\) not in \(H\). The resulting column vector is \([f_H]_{B_n}\). Finally we apply the inverse Fourier transform to \([f_H]_{B_n}\) so that we obtain \([f_H]_{B_0}\) using \(2(n-1)\binom{n}{k}\) more operations. \(\Box\)

**Theorem 9** Assume that the matrices \([B_{i-1}]_{B_i}\) for \(i = 2, 3, \ldots, n\) have been computed. Given a column vector \([f]_{B_0}\) with \(f \in F\), all the weights \(\|f_a\|^2\), for \(a = 0, \ldots, s\), can be computed using at most \((2n-1)\binom{n}{k}\) operations.
**Proof** Observe that \( \| f_a \|^2 = \| [f_a]_{B_n} \|^2 \). To obtain the column vector \([f_a]_{B_n}\), we apply the Fourier transform to the function \( f \), so that we obtain the column vector \([f]_{B_n}\) using \( 2(n - 1)\binom{n}{2} \) operations. Then we select the entries of the vector \([f]_{B_n}\) that correspond to Young tableaux of shape \((n - a, a)\), and we compute the sum of the squares of these entries. Doing this for all the values of \( a \) can be accomplished using at most \( \binom{n}{k} \) operations. \qed

13 Example

In this section we illustrate the algorithm given in Sect. 10 for the construction of the change of bases matrices by developing the case \( n = 4, k = 2 \).

13.1 Computation of \([B_1]_{B_2}\) and \([B_2]_{B_1}\)

13.1.1 First Step

Since \([B_1]_{B_0} = I\) then \([s_1]_{B_1} = [s_1]_{B_0}\). From \([J_0]_{B_1} = 0\), using the recurrence formula (7) we obtain

\[ [J_2]_{B_1} = ([s_1]_{B_1} [J_1]_{B_1} + I) [s_1]_{B_1} = [s_1]_{B_1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \]

13.1.2 Second Step

We consider the decomposition

\[ \mathcal{F} = \mathcal{F}_{11} \oplus \mathcal{F}_{21} \oplus \mathcal{F}_{12} \oplus \mathcal{F}_{22}. \]

The subspaces \( \mathcal{F} = \mathcal{F}_{11} \) and \( \mathcal{F}_{22} \) have dimension 1 and each one is generated by an eigenvector of \( J_2 \) that belongs to the bases \( B_1 \) and \( B_2 \). The subspaces \( \mathcal{F}_{21} \) and \( \mathcal{F}_{12} \) have dimension 2, and they are generated by two eigenvectors of \( J_2 \) associated to the eigenvalues \(-1\) and \( 1 \) respectively. In order to determine the coordinates of these eigenvectors in the basis \( B_1 \) we restrict the operators to these subspaces and solve the following systems of equations:

\[ ([J_2]_{B_1} + I) [b_2^2]_{B_1} = 0 \implies \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} [b_2^2]_{B_1} = 0 \implies [b_2^2]_{B_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} , \]

\[ ([J_2]_{B_1} - I) [b_3^2]_{B_1} = 0 \implies \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} [b_3^2]_{B_1} = 0 \implies [b_3^2]_{B_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \]
\[
([J_2]_{B_1} + I) [b_4^2]_{B_1} = 0 \implies \left[ \begin{array}{r} 1 & 1 \\ 1 & 1 \end{array} \right] [b_4^2]_{B_1} = 0 \implies [b_4^2]_{B_1} = \left[ \begin{array}{r} -1 \\ 1 \end{array} \right].
\]

\[
([J_2]_{B_1} - I) [b_5^2]_{B_1} = 0 \implies \left[ \begin{array}{r} -1 & 1 \\ 1 & -1 \end{array} \right] [b_5^2]_{B_1} = 0 \implies [b_5^2]_{B_1} = \left[ \begin{array}{r} 1 \\ 1 \end{array} \right].
\]

Then

\[
[B_2]_{B_1} = \left[ \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right].
\]

### 13.1.3 Third Step

In order to construct \( D_2 \) we compute:

\[
\|b_1^2\|^2 = \|b_1^1\|^2 = 1,
\]

\[
\|b_2^2\|^2 = \|b_2^1\|^2 + (-1)^2 \|b_3^1\|^2 = 2,
\]

\[
\|b_3^2\|^2 = \|b_2^1\|^2 + \|b_3^1\|^2 = 2,
\]

\[
\|b_4^2\|^2 = (-1)^2 \|b_4^1\|^2 + \|b_5^1\|^2 = 2,
\]

\[
\|b_5^2\|^2 = \|b_4^1\|^2 + \|b_5^1\|^2 = 2,
\]

\[
\|b_6^2\|^2 = \|b_6^1\|^2 = 1.
\]

Then

\[
D_2 = \left[ \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right].
\]
The matrix \([B_1]_{B_2}\) is given by

\[
[B_1]_{B_2} = D_2^{-1} ([B_2]_{B_1})' D_1 = \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & \frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

13.2 Computation of \([B_2]_{B_3}\) and \([B_3]_{B_2}\)

13.2.1 First Step

Since

\[
[s_2]_{B_2} = \\
\begin{bmatrix}
0 & -1 & 1 & 0 & 0 & 0 \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 1 \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \\
0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
\]

and

\[
[J_2]_{B_2} = \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

using the recurrence formula we obtain

\[
[J_3]_{B_2} = ((s_2)_{B_2} [J_2]_{B_2} + I) [s_2]_{B_2} = \\
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}.
\]

13.2.2 Second Step

Let us consider the decomposition

\[
\mathcal{F} = \mathcal{F}^1_{B_2} \oplus \mathcal{F}^2_{B_2} \oplus \mathcal{F}^1_{B_3} \oplus \mathcal{F}^2_{B_3}.
\]

The subspaces \(\mathcal{F}^1_{B_2}\) and \(\mathcal{F}^2_{B_2}\) have dimension 1 and each one is generated by an eigenvector of \(J_3\) that belongs to both bases \(B_2\) and \(B_3\). The subspaces \(\mathcal{F}^1_{B_3}\) and \(\mathcal{F}^2_{B_3}\) have dimension 2, and each one is generated by two eigenvectors of \(J_3\) associated to the eigenvalues \(-1\) and \(2\) respectively. In order to determine the coordinates of these eigenvectors in the basis \(B_2\) we restrict the operators to these subspaces and we solve the systems of linear equations. Then
\[ [B_3]_{B_2} = \begin{bmatrix}
-2 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & -2 & 1 \\
\end{bmatrix}. \]

13.2.3 Third Step

Now we compute \( D_3 \) from \([B_3]_{B_2}\) and \( D_2 \).

\[ D_3 = \begin{bmatrix}
6 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 \\
\end{bmatrix}. \]

The matrix \([B_2]_{B_3}\) is given by

\[ [B_2]_{B_3} = D_3^{-1}([B_3]_{B_2})^t D_2 = \begin{bmatrix}
-\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\
\end{bmatrix}. \]

13.3 Computation of \([B_3]_{B_4}\) and \([B_4]_{B_3}\)

13.3.1 First Step

Since

\[ [s_3]_{B_3} = \begin{bmatrix}
\frac{2}{3} & 0 & -\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 0 & 1 & 0 & 0 \\
-\frac{2}{3} & 0 & \frac{1}{3} & 0 & \frac{2}{3} & \frac{2}{3} \\
0 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{3} \\
\frac{3}{3} & 0 & \frac{3}{3} & 0 & -\frac{2}{3} & \frac{1}{3} \\
\end{bmatrix} \quad \text{and} \quad [J_3]_{B_3} = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 \\
\end{bmatrix}. \]
then, using the recurrence formula, we obtain

\[
\begin{bmatrix} J_4 & B_3 \end{bmatrix} = \begin{bmatrix} s_3 & J_3 & B_3 & I \end{bmatrix} \begin{bmatrix} s_3 & B_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}.
\]

### 13.3.2 Second Step

We consider the decomposition

\[
\mathcal{F} = \mathcal{F}_{123} \oplus \mathcal{F}_{123} \oplus \mathcal{F}_{132}.
\]

All these subspaces have dimension 2. \(\mathcal{F}_{123}\) is generated by two eigenvectors of \(J_4\) associated with the eigenvalues \(-1\) and \(3\); \(\mathcal{F}_{123}\) and \(\mathcal{F}_{132}\) are generated by two eigenvectors of \(J_4\) associated to the eigenvalues \(0\) and \(2\). In order to determine the coordinates of these eigenvectors in the basis \(B_3\) we restrict the operators to these subspaces and we solve the systems of linear equations. Then

\[
[B_4]_{B_3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}.
\]

### 13.3.3 Third Step

We obtain \(D_4\) from \([B_3]_{B_2}\) and \(D_2\).

\[
D_4 = \begin{bmatrix} 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}.
\]
The matrix $[B_3]_{B_4}$ is given by

$$
[B_3]_{B_4} = D_4^{-1}([B_4]_{B_3})^t D_3 = \begin{bmatrix}
\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -\frac{1}{2} \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2}
\end{bmatrix}.
$$

13.4 Computation of the Fourier Transform

Finally we compute the Fourier transform for $J(4, 2)$

$$
[B_1]_{B_4} = [B_3]_{B_4} [B_2]_{B_3} [B_1]_{B_2}
$$

and its inverse

$$
[B_4]_{B_1} = [B_2]_{B_1} [B_3]_{B_2} [B_4]_{B_3}.
$$

We obtain

$$
[B_1]_{B_4} = \begin{bmatrix}
-\frac{1}{6} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{6} \\
0 & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\
0 & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 \\
-\frac{1}{6} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6}
\end{bmatrix}
$$

and

$$
[B_4]_{B_1} = \begin{bmatrix}
-2 & 0 & 1 & 0 & -2 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 \\
-2 & 0 & -1 & 0 & 2 & 1
\end{bmatrix}.
$$

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