

About proregular sequences and an application to prisms

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ABSTRACT
Let \( x = x_1, \ldots, x_k \) denote an ordered sequence of elements of a commutative ring \( R \). Let \( M \) be an \( R \)-module. We recall the two notions that \( x \) is \( M \)-proregular given by Greenlees and May and Lipman and show that both notions are equivalent. As a main result we prove a cohomological characterization for \( x \) to be \( M \)-proregular in terms of Čech cohomology. This implies also that \( x \) is \( M \)-weakly proregular if it is \( M \)-proregular. A local-global principle for proregularity and weakly proregularity is proved. This is used for a result about prisms as introduced by Bhatt and Scholze.

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1. Introduction
Let \( R \) denote a commutative ring and let \( M \) be an \( R \)-module. Let \( x = x_1, \ldots, x_k \) denote an ordered sequence of elements of \( R \). In their paper Greenlees and May (see [5]) defined \( x \) to be \( M \)-proregular if for all \( i = 1, \ldots, k \) and an integer \( n \geq 1 \) there is an integer \( m \geq n \) such that the multiplication map
\[
(\ldots x_{i-1})^m M : M x_i^m / (x_1, \ldots, x_{i-1})^m M \xrightarrow{\cdot x_i^m} (x_1, \ldots, x_{i-1})^n M : M x_i^n / (x_1, \ldots, x_{i-1})^n M
\]
is zero. In their paper (see [1]) Lipman et al called \( x \) to be \( M \)-proregular if for all \( i = 1, \ldots, k \) and an integer \( n \geq 1 \) there is an integer \( m \geq n \) such that the multiplication map
\[
(x_1, \ldots, x_{i-1})^m M : M x_i^m / (x_1, \ldots, x_{i-1})^m M \xrightarrow{\cdot x_i^m} (x_1, \ldots, x_{i-1})^n M : M x_i^n / (x_1, \ldots, x_{i-1})^n M
\]
is zero (see also the Definition 1.1). The advantage of the second notion of proregularity is its relation to Koszul complexes. Moreover, both definitions are equivalent (see 1.2). The notion of proregular sequences is needed by Greenlees and May for their study of the left derived functors of the completion. This was continued by Lipman et al (see [1] and [2]). Moreover, it turned out that the notion that \( x \) is weakly proregular (see 1.6) is more appropriate for the study of Čech homology and cohomology. A first systematic study of weakly proregular sequences is done in [8]. See also the monograph [10] for a more systematic investigation and its relation to completions. As a first main result we prove a homological characterization of \( M \)-proregular sequences.

Theorem 1.1. Let \( x = x_1, \ldots, x_k \) denote an ordered sequence of a commutative ring \( R \). For an \( R \)-module \( M \) the following conditions are equivalent.
(i) The sequence $\mathbf{x}$ is $M$-proregular.
(ii) $\hat{H}^1_{x_i}(\Gamma_{x_{i-1}}(\text{Hom}_R(M,I))) = 0$ for $i = 1, \ldots, k$ and any injective $R$-module $I$.

For the proof see 2.1. Here $\hat{H}^1_{x_i}(-)$ denotes the Čech homology with respect to $x_i$ (see 1.5); and $\Gamma_{x_{i-1}}(-)$ is the torsion functor with respect to the ideal generated by $x_{i-1} = x_1, \ldots, x_{i-1}$. As an application there is a behavior similar to that of a regular sequence. For a sequence $x = x_1, \ldots, x_k$ and any $k$-tuple $(n_1, \ldots, n_k) \in (\mathbb{N}_+)^{k}$ we write $x^{(n)} = x_1^{n_1}, \ldots, x_k^{n_k}$.

**Corollary 1.2.** With the notation of 0.1 the following conditions are equivalent.

(a) The sequence $\mathbf{x}$ is $M$-proregular.
(b) There is an $n \in (\mathbb{N}_+)^k$ such that the sequence $x^{(n)}$ is $M$-proregular.
(c) The sequence $x^{(n)}$ is $M$-proregular for every $n \in (\mathbb{N}_+)^{k}$.

If the sequence $x$ is $M$-regular it is also $M$-proregular. For a Noetherian $R$-module $M$ any sequence is $M$-proregular. Therefore, the notion of proregularity is important in the non-Noetherian situation. An application of this is the following result inspired by the work of Bhatt and Scholze (see [3]). For the notation we refer to Section 4.

**Corollary 1.3.** Let $(R, I)$ denote a prism. Suppose that $I$ is of bounded $p$-torsion. Then it follows.

i. $(I, p)$ is proregular, i.e. for an integer $n$ there is an $m \geq n$ such that $\mathfrak{I}^{m} : \mathfrak{I}^{m}/\mathfrak{I}^{m-n} \to \mathfrak{I}^{n} : \mathfrak{I}^{n}/\mathfrak{I}^{m}$ is the zero map.
ii. $\Gamma_I(I)/\Gamma_{(I, p)}(I)$ is $p$-divisible for any injective $R$-module $I$, i.e. $\Gamma_I(I) = p\Gamma_I(I) + \Gamma_{(I, p)}(I)$.

Moreover, the conditions (a) and (b) are equivalent.

In Section 1 we summarize the notation and prove some basic results. Section 2 is devoted to the homological approach of proregular sequences and their consequences. A few more results about $M$-weakly proregular sequences are shown in Section 3. This provides a relative version of some results of [10]. In Section 4 we investigate the local-global behavior of proregular and weakly proregular sequences and prove the Corollary 0.3. For results of Commutative Algebra we follow Matsumura’s book [6]. For homological preliminaries and definitions about Čech homology and cohomology we refer to the monograph [10].

**2. Definitions and preliminaries**

In the following let $R$ denote a commutative ring. Let $\mathbf{x} = x_1, \ldots, x_k$ denote an ordered sequence of elements of $R$. For a positive integer $n$ we put $\mathbf{x}^{(n)} = x_1^{n_1}, \ldots, x_k^{n_k}$. Let $M$ denote an $R$-module. Note that

$\mathbf{x}^{nk} M \subseteq \mathbf{x}^{(n)} M \subseteq \mathbf{x}^n M$ for all $n \geq 1$

and therefore $\text{Rad} x R = \text{Rad} \mathbf{x}^{(n)} R$ for all $n \geq 1$. The $R$-module $M$ is of bounded $x$-torsion for an element $x \in R$ if the increasing sequence $\{0 :_M x^n\}_{n \geq 1}$ stabilizes, i.e. there is an integer $c$ such that $0 :_M x^n = 0 :_M x^c$ for all $n \geq c$. Note that $M$ is of bounded $x$-torsion for an $M$-regular element $x \in R$. If $N$ is a submodule of $M$ and $M$ is of bounded $x$-torsion, then $N$ is of bounded $x$-torsion too. Moreover, if $M$ is a Noetherian module, then $M$ is of bounded torsion for any element. Let $m \geq n$ denote two positive integers. The multiplication map by $x^{m-n}$ on $M$ induces a map

$0 :_M x^m \to 0 :_M x^n$ for all $m \geq n$. 
Therefore $M$ is of bounded $x$-torsion if and only if the previous inverse system is pro-zero. This leads Greenlees and May to the definition of an $M$-proregular sequence (see [5, Definition 1.8]). We follow here the notion of proregularity as given in [1] and show that it is equivalent to the notion of Greenlees and May (see [5]).

**Definition 2.1.** Let $M$ denote an $R$-module and let $x = x_1, \ldots, x_k$ denote a sequence of elements of $R$. Then $x$ is called $M$-proregular if for $i = 1, \ldots, k$ and any positive integer $n$ there is an integer $m \geq n$ such that the multiplication map

$$(x_1^m, \ldots, x_{i-1}^m)M : Mx_i^m / (x_1^m, \ldots, x_{i-1}^m)M \to (x_1^m, \ldots, x_{i-1}^m)M : Mx_i^m / (x_1^m, \ldots, x_{i-1}^m)M$$

is zero. This is equivalent to saying that for $i = 1, \ldots, k$ and any positive integer $n$ there is an integer $m \geq n$ such that

$$(x_1^m, \ldots, x_{i-1}^m)M : Mx_i^m \subseteq (x_1^n, \ldots, x_{i-1}^n)M : Mx_i^{m-n}.$$  

Note that an element $x \in R$ is $M$-proregular if and only if $M$ is of bounded $x$-torsion.

A first result about the behavior of $M$-proregular sequences is the following.

**Proposition 2.2.** Let $x = x_1, \ldots, x_k$ denote an ordered sequence of elements of $R$. Let $M$ denote an $R$-module. Then the following conditions are equivalent.

(i) The sequence $x$ is $M$-proregular.

(ii) For $i = 1, \ldots, k$ and any positive integer $n$ there is an integer $m \geq n$ such that

$$(x_1, \ldots, x_{i-1})^m M : Mx_i^m \subseteq (x_1^n, \ldots, x_{i-1}^n)M : Mx_i^{m-n}.$$  

(iii) For $i = 1, \ldots, k$ and any integer $n$ there is an integer $m \geq n$ such that the multiplication map

$$(x_1, \ldots, x_{i-1})^m M : Mx_i^m / (x_1, \ldots, x_{i-1})^m M \to (x_1, \ldots, x_{i-1})^n M : Mx_i^n / (x_1, \ldots, x_{i-1})^n M$$

is zero.

**Proof.** We fix $l$ and put $x = x_1, \ldots, x_{l-1}$ and $y = x_l$. The equivalence of (ii) and (iii) is trivially true. Then we prove the implication (i) $\Rightarrow$ (iii). By the assumption and the definition 1.1 for a given $n$ there is an $m \geq n$ such that $\Delta^{(m)} M : M y^m \subseteq \Delta^{(n)} M : M y^{m-n} \subseteq \Delta^n M : M y^{m-n}$. Because of $\Delta^m M : M y^m \subseteq \Delta^{(m)} M : M y^m$, this implies that

$$\Delta^m M : M y^m \subseteq \Delta^{(m)} M : M y^{m-n} \subseteq \Delta^{(n)} M : M y^{m-n} \subseteq \Delta^n M : M y^{m-n}$$

as required. In order to prove (ii) $\Rightarrow$ (i) fix $n$ and choose an integer $m \geq nl$ such that $\Delta^m M : M y^m \subseteq \Delta^{nl} M : M y^{m-nl}$. Then we get the inclusions

$$\Delta^{(m)} M : M y^m \subseteq \Delta^{(m)} M : M y^{m-nl} \subseteq \Delta^{(n)} M : M y^{m-nl} \subseteq \Delta^{(n)} M : M y^{m-nl}$$

which finishes the proof. \[\square\]

The following example by J. Lipman (see [2]) shows that a proregular sequence is not permutable.

**Example 2.3.** Define $R = \prod_{n \geq 1} \mathbb{Z}/2^n \mathbb{Z}$. Let $x = (2 + 2^n)_{n \geq 1}$ and $1 = (1 + 2^n)_{n \geq 1}$ two elements of $R$. Then the sequence $\{1, x\}$ is $R$-regular, in particular $R$-proregular. Moreover, $R$ is not of bounded $x$-torsion. Therefore $\{x, 1\}$ is not a proregular sequence.

Clearly an $M$-regular sequence $x$ is also $M$-proregular. An interesting extension is the following result.
Proposition 2.4. Let $M$ be an $R$-module. Let $\bar{x} = x_1, \ldots, x_k$ denote an $M$-regular sequence. Suppose that $M/\bar{x}M$ is of bounded $y$-torsion for an element $y \in R$. Then $\bar{x}, y = x_1, \ldots, x_k, y$ is an $M$-proregular sequence.

Proof. Now $x_i$ is regular on $M/(x_1, \ldots, x_{i-1})^nM$ for all $i = 1, \ldots, k$ and all $n \geq 1$ since $\bar{x} = x_1, \ldots, x_k$ is an $M$-regular sequence (see e.g. [6, Theorem 16.2]). That is, condition (ii) in 1.2 is satisfied for all $i = 1, \ldots, k$. In order to finish the proof we have to show that $M/\bar{x}^nM$ is of bounded $y$-torsion for all $n \geq 1$. Namely, if $\bar{x}^nM : y^n = \bar{x}^nM : My^d$ for all $d \geq c$ choose $m = n + c$ and therefore $\bar{x}^mM : y^m \subseteq \bar{x}^nM : y^{m-n}$. It remains to show the previous claim. Since $\bar{x}$ is an $M$-regular sequence $\bar{x}^nM/\bar{x}^{n+1}M \cong \bigoplus b_nM/\bar{x}M$ with $b_n = \binom{k+n-1}{n}$ (see e.g. [6]). Then the short exact sequence

$$0 \to \bar{x}^nM/\bar{x}^{n+1}M \to M/\bar{x}^{n+1}M \to M/\bar{x}^nM \to 0$$

implies by applying $\Gamma_y$ the exact sequence $0 \to \Gamma_y(\bar{x}^nM/\bar{x}^{n+1}M) \to \Gamma_y(M/\bar{x}^{n+1}M) \to \Gamma_y(M/\bar{x}^nM)$. This proves -- by induction on $n$ -- that $M/\bar{x}^nM$ is of bounded $y$-torsion for all $n \geq 1$. Note that a submodule of $\Gamma_y(M/\bar{x}^nM)$ is of bounded $y$-torsion too.

For our further investigations we need a few more notation and definitions.

Notation 2.5. (A) Let $a \subset R$ denote an ideal of $R$. For an $R$-module $X$ let $\Gamma_a(X) = \{ x \in X | a^n x = 0 \text{ for some } n \geq 1 \}$ the $a$-torsion submodule of $X$. Its right derived functors $H_a^i(X), i \geq 0$, are the local cohomology modules of $X$ with respect to $a$ (see e.g. [4] or [10] for more details).

(B) Let $\bar{x} = x_1, \ldots, x_k$ denote a sequence of elements of $R$. Let $\bar{C}_a$ denote the Čech complex with respect to $\bar{x}$. For an $R$-module $X$ we write $\bar{C}_a(X) = \bar{C}_a \otimes_R X$. We denote the cohomology of $\bar{C}_a(X)$ by $\bar{H}_a^i(X), i \geq 0$.

(C) Let $\bar{x} = x_1, \ldots, x_k$ a sequence of elements and $a = \bar{x}R$. Then it follows easily that $\Gamma_a(X) = H_{\bar{x}^0}^0(X)$. The more general isomorphisms $H_a^i(X) \cong H_{\bar{x}^i}^i(X)$ for all $i \geq 0$ do not hold in general. They hold if and only if $\bar{x}$ is a weakly proregular sequence (see [10] for more details about weakly proregular sequences).

(D) For a sequence of elements $\bar{x} = x_1, \ldots, x_k$ and an $R$-module $M$ we use the Koszul complexes $K_a(\bar{x}; M)$ and $K^*(\bar{x}; M)$. We refer to [10, 5.2] for all the details we need. The Koszul homology and Koszul cohomology are denoted by $H_i(\bar{x}; M)$ and $H^i(\bar{x}; M)$ resp. for $i \in \mathbb{Z}$.

We continue with the a further definition.

Definition 2.6. (see [10, Section 7.3]) Let $\bar{x} = x_1, \ldots, x_k$ denote a sequence of elements of $R$. Let $M$ be an $R$-module. The sequence $\bar{x}$ is called $M$-weakly proregular if for all $i > 0$ and any positive integer $n$ there is an integer $m \geq n$ such that the natural homomorphism

$$H_i(\bar{x}^m; M) \to H_i(\bar{x}^n; M)$$

is zero. A sequence $\bar{x}$ is called weakly proregular if $\bar{x}$ is $R$-weakly proregular.

The notion of weakly proregular sequences plays an essential role in respect to local cohomology and the left derived functors of completion (see [1, 2] and [10]). A characterization of $M$-weakly proregular sequence is the following.

Proposition 2.7. (see [9, Proposition 5.3]) Let $\bar{x}$ denote a sequence of elements of $R$. Let $M$ be an $R$-module. Then the following conditions are equivalent.

(i) $\bar{x}$ is $M$-weakly proregular.

(ii) The inverse system $\{ H_i(\bar{x}^n; M \otimes_R F) \}_{n \geq 1}$ is pro-zero for all $i > 0$ and any flat $R$-module $F$.

(iii) $\lim H^i(\bar{x}^n; \text{Hom}_R(M, I)) = 0$ for all $i > 0$ and any injective $R$-module $I$. 
(iv) $\hat{H}_x^i (\text{Hom}_R(M, I)) = 0$ for all $i > 0$ and any injective $R$-module $I$.

We shall see that an $M$-proregular sequence is also $M$-weakly proregular. The converse is not true as follows by 1.3 since the sequence $1, x$ is $R$-regular and in particular $R$-proregular.

3. A homological approach

At first we will give an interpretation of the definition of an $M$-proregular sequence $x = x_1, \ldots, x_k$ in terms of Koszul complexes. For the basics about Koszul complexes we refer to [10]. We fix $l \in \{1, \ldots, k\}$ and put $x = x_1, \ldots, x_{l-1}$ and $y = x_l$. Then the natural map

$$\begin{align*}
\alpha^{(m)} M : M y^m / \alpha^{(m)} M \to \alpha^{(n)} M : M y^n / \alpha^{(n)} M \quad \text{for all } m \geq n
\end{align*}$$

coincides with the natural map

$$H_1 (y^m; H_0 (\alpha^{(m)} ; M)) \to H_1 (y^n; H_0 (\alpha^{(n)} ; M)) \quad \text{for all } m \geq n.$$

induced by the natural map of Koszul complexes $K_\bullet (\alpha^{(m)} ; M) \to K_\bullet (\alpha^{(n)} ; M)$ (see [10, Section 5.2] for some details). In the following we specify $\xi_i = x_1, \ldots, x_i$ for $i = 1, \ldots, k - 1$.

**Theorem 3.1.** Let $\alpha = x_1, \ldots, x_k$ denote an ordered sequence of elements of $R$. Let $M$ denote an $R$-module. Then the following conditions are equivalent.

(i) The sequence $\alpha$ is $M$-proregular.

(ii) The sequence $\alpha$ is $(M \otimes R F)$-proregular for any flat $R$-module $F$.

(iii) $\hat{H}_{\alpha}^i (\Gamma_{\alpha} (\text{Hom}_R (M, I))) = 0$ for $i = 1, \ldots, k$ and any injective $R$-module $I$.

(iv) $\Gamma_{\alpha} (\text{Hom}_R (M, I)) / \Gamma_{\alpha} (\text{Hom}_R (M, I))$ is $x_i$-divisible for $i = 1, \ldots, k$ and any injective $R$-module $I$.

**Proof.** The equivalence of (i) and (ii) holds trivially. Next we show the equivalence of (iii) and (iv). For an $R$-module $X$ and an element $y \in R$ there is an exact sequence

$$0 \to \Gamma_y (X) \to X \to X_y \to \hat{H}_y^1 (X) \to 0,$$

where $X_y$ denotes the localization of $X$ with respect to the element $y \in R$. Note that $X \to X_y$ is just the Čech complex with respect to the singleton element $y \in R$. Next note that $\hat{H}_y^1 (X) = 0$ if and only if $X \to X_y$ is onto or equivalently $X / \Gamma_y (X)$ is $y$-divisible. Applying this observation to $\Gamma_{\alpha} (\text{Hom}_R (M, I))$ proves the equivalence of (iii) and (iv).

We use the abbreviations of the beginning of this section. Now we prove (i) $\Rightarrow$ (iii). Applying the functor $\text{Hom}_R (\cdot, I)$ to the pro-zero sequence of Koszul homologies (4) at the beginning of this section yields homomorphisms that provide a direct system

$$H^1 (y^n; H^0 (\alpha^{(n)} ; \text{Hom}_R (M, I))) \to H^1 (y^m; H^0 (\alpha^{(m)} ; \text{Hom}_R (M, I))).$$

If the system in (4) is pro-zero it follows that

$$0 = \lim_{\longrightarrow} H^1 (y^n; H^0 (\alpha^{(n)} ; \text{Hom}_R (M, I))) \cong \hat{H}_y^1 (\Gamma_{\alpha} (\text{Hom}_R (M, I))).$$

The previous isomorphism might be checked directly, or see [10, 6.1.11].

For the proof of (iii) $\Rightarrow$ (i) we note that $\alpha^{(n)} M : M y^n / \alpha^{(n)} M \cong \text{Hom}_R (R / y^n R, M / \alpha^{(n)} M)$. Now fix $n$ and choose an injection $f : (\alpha^{(n)} M : M y^n / \alpha^{(n)} M) \to J$ into an injective $R$-module $J$. This defines an element

$$f \in \text{Hom}_R (\text{Hom}_R (R / y^n R, M / \alpha^{(n)} M), J) \cong H^1 (y^n; H^0 (\alpha^{(n)} ; \text{Hom}_R (M, I))).$$
By the assumption $\lim \tilde{H}^1(y; H^0(\Gamma_z(\text{Hom}_R(M, I)))) = 0$. Whence there is an integer $m \geq n$ such that the image of $f$ in $\text{Hom}_R(R/y^nR, M/\tilde{x}^{(m)}M, I)$ vanishes. In other words the composite of the maps

$$\tilde{x}^{(m)}M : M y^m / \tilde{x}^{(m)}M \rightarrow \tilde{x}^{(n)}M : M y^n / \tilde{x}^{(n)}M \rightarrow I$$

is zero. This finishes the proof since $f$ is injective.

As a first application we get extensions of some of the authors’ result in [11] to the case of an $R$-module $M$.

**Corollary 3.2.** Let $x \in R$ denote an element and let $M$ denote an $R$-module. Then the following conditions are equivalent.

(i) $M$ is of bounded $x$-torsion.
(ii) $M \otimes_R F$ is of bounded $x$-torsion for any flat $R$-module $F$.
(iii) $\tilde{H}_i^1(\text{Hom}_R(M, I)) = 0$ for any injective $R$-module $I$.
(iv) $\text{Hom}_R(M, I) / \Gamma_x(\text{Hom}_R(M, I))$ is $x$-divisible for any injective $R$-module $I$.

**Proof.** This is a particular case of 2.1 for $k = 1$. Namely $x$ is $M$-proregular if and only if $M$ is of bounded $x$-torsion.

For an $M$-regular sequence $\underline{x} = x_1, \ldots, x_k$ it follows that $\underline{x}^{(n)} = x_1^{n_1}, \ldots, x_k^{n_k}$ is $M$-regular for any $k$-tupel $(n_1, \ldots, n_k) \in (\mathbb{N}_+)^k$. A corresponding result holds for $M$-proregular sequences.

**Corollary 3.3.** Let $\underline{x} = x_1, \ldots, x_k$ be an ordered sequence of elements of $R$ and let $M$ be an $R$-module. Then the following conditions are equivalent.

(a) The sequence $\underline{x}$ is $M$-proregular.
(b) There is an $\underline{n} \in (\mathbb{N}_+)^k$ such that the sequence $\underline{x}^{(\underline{n})}$ is $M$-proregular.
(c) The sequence $\underline{x}^{(\underline{n})}$ is $M$-proregular for all $\underline{n} \in (\mathbb{N}_+)^k$.

**Proof.** The statements are easy consequences of Theorem 2.1 by condition (iii).

Next we provide an alternative proof of [10, A.2.3] that an $M$-proregular sequence is also $M$-weakly proregular. This is done by the characterization of 2.1

**Theorem 3.4.** Let $\underline{x} = x_1, \ldots, x_k$ be an ordered sequence in $R$. Let $M$ denote an $R$-module. If $\underline{x}$ is $M$-proregular, then $\underline{x}$ is also $M$-weakly proregular.

**Proof.** We fix $l \in \{1, \ldots, k\}$ and put $x = x_1, \ldots, x_{l-1}$ and $y = x_l$. Then we show by induction on $l$ that $\tilde{H}_i^1(\text{Hom}_R(M, I)) = 0$ for all $i > 0$. If $l = k$ this proves the claim by virtue of Proposition 1.7. If $l = 1$ then $y$ is $M$-proregular if and only if $M$ is of bounded $y$-torsion. Whence the claim is true by 2.3. Let $l > 1$. Then there is the short exact sequence

$$0 \rightarrow \tilde{H}_i^1(\tilde{H}_\Sigma^1(\text{Hom}_R(M, I))) \rightarrow \tilde{H}_i^1(\text{Hom}_R(M, I)) \rightarrow \tilde{H}_i^0(\tilde{H}_\Sigma^1(\text{Hom}_R(M, I))) \rightarrow 0$$

for all $i$ (see [10, 6.1.11]). By the induction hypothesis $\tilde{H}_i^1(\text{Hom}_R(M, I)) = 0$ for all $i > 0$ and therefore $\tilde{H}_i^1(\text{Hom}_R(M, I)) = 0$ for all $i > 1$. The vanishing for $i = 1$ holds since $\tilde{H}_i^1(\tilde{H}_\Sigma^1(\text{Hom}_R(M, I))) = 0$ as it follows by the assumption (see Theorem 2.1). This completes the inductive step and finishes the proof.
4. Čech (co-)complexes

In order to continue with the study of weakly proregular sequences let us recall a few more definitions.

**Notation 4.1.** (A) Let $\mathfrak{x} = x_1, \ldots, x_k$ denote a system of elements of $R$. As in 1.5 we denote by $\tilde{C}_\mathfrak{x}$ the corresponding Čech complex. In [10, 6.2] and [9, Section 3] there are constructions of two bounded complexes of free $R$-modules $L_\mathfrak{x}^-$ and $L_\mathfrak{x}^+$ and quasi-isomorphisms $\tilde{L}_\mathfrak{x}^- \cong L_\mathfrak{x}^- \cong \tilde{C}_\mathfrak{x}^-$. That is there are bounded free resolutions of the Čech complex $\tilde{C}_\mathfrak{x}^-$ that is a complex of flat $R$-modules.

(B) Let $M$ denote an $R$-module. With the previous notation the complex $R\text{Hom}_R(\tilde{C}_\mathfrak{x}^-, M)$ in the derived category has the following two representatives $\text{Hom}_R(L_\mathfrak{x}^-, M) \rightarrow \text{Hom}_R(\tilde{L}_\mathfrak{x}^-, M)$.

(C) For an integer $i \in \mathbb{Z}$ we put

$$\tilde{H}_i^\mathfrak{x}(M) := H_i(\text{Hom}_R(L_\mathfrak{x}^-, M)) \cong H_i(\text{Hom}_R(\tilde{L}_\mathfrak{x}^-, M))$$

for the Čech homology of $M$ with respect to $\mathfrak{x}$.

(D) For an $R$-module $M$ and an ideal $\mathfrak{a} \subset R$ we write $\Lambda^0(M)$ for the $\mathfrak{a}$-adic completion of $M$, i.e. $\Lambda^0(M) = \lim M/\mathfrak{a}^iM$. The left derived functors of $\Lambda^0(M)$ are denoted by $\Lambda^0_i(M), i \in \mathbb{Z}$. Note that in general $\Lambda^0(M) \neq \Lambda^0_0(M)$ since $\Lambda^0$ is not right exact.

In the following we shall prove a relative version of one of the main results of [10]. We shall recover it for the case of $M = R$.

**Theorem 4.2.** Let $\mathfrak{x} = x_1, \ldots, x_k$ denote a sequence of elements of $R$ and $\mathfrak{a} = \mathfrak{x}R$. Let $M$ be an $R$-module. Suppose that $\mathfrak{x}$ is $M$-weakly proregular.

(a) $\tilde{H}_i^\mathfrak{x}(\text{Hom}_R(M, I)) = 0$ for all $i > 0$ and $\tilde{C}_\mathfrak{x} \otimes_R \text{Hom}_R(M, I)$ is a right resolution of $\Gamma_\mathfrak{a}(\text{Hom}_R(M, I))$ for any injective $R$-module $I$.

(b) $\tilde{H}_i^\mathfrak{x}(M \otimes_R F) = 0$ for all $i > 0$ and $\text{Hom}_R(L_\mathfrak{x}^-, M \otimes_R F)$ is a left resolution of $\Lambda^0(M \otimes_R F)$ for any flat $R$-module $F$.

**Proof.** The vanishing result in (a) is shown in 1.7. Moreover $\tilde{H}_i^\mathfrak{x}(\text{Hom}_R(M, I)) \cong \Gamma_\mathfrak{a}(\text{Hom}_R(M, I))$. Now the natural morphism $\Gamma_\mathfrak{a}(\text{Hom}_R(M, I)) \rightarrow \tilde{C}_\mathfrak{x} \otimes_R \text{Hom}_R(M, I)$ proves (a).

Now let us prove (b). By view of [10, 6.3.2 (b)] there are the following short exact sequences

$$0 \rightarrow \lim_{i+1} H_i(\mathfrak{x}^{(n)}; M \otimes_R F) \rightarrow H_i(\text{Hom}_R(\tilde{C}_\mathfrak{x}^-, M \otimes_R F)) \rightarrow \lim_{-} H_i(\mathfrak{x}^{(n)}; M \otimes_R F) \rightarrow 0$$

for all $i \in \mathbb{N}$. By the assumption $\{H_i(\mathfrak{x}^{(n)}; M \otimes_R F)\}_{n \geq 1}$ is prozero for all $i > 0$. This yields that

$$\lim_{-} H_i(\mathfrak{x}^{(n)}; M \otimes_R F) = \lim_{-} H_i(\mathfrak{x}^{(n)}; M \otimes_R F) = 0$$

for all $i > 0$. This proves the vanishing part. For $i = 0$ we get the isomorphisms

$$H_0(\text{Hom}_R(\tilde{C}_\mathfrak{x}^-, M \otimes_R F)) \cong \lim_{-} H_0(\mathfrak{x}^{(n)}; M \otimes_R F) \cong \Lambda^0(M \otimes_R F)$$

as easily seen since $\mathfrak{x}R = \mathfrak{a}$. By [10, 8.2.1] and [9, 4.1 (B)] there is a natural morphism $\text{Hom}_R(L_\mathfrak{x}^-, M \otimes_R F) \rightarrow \Lambda^0(M \otimes_R F)$ which completes the proof of (b).

While the condition (a) in Theorem 3.2 is equivalent to the statement that $\mathfrak{x}$ is $M$-weakly proregular (see 1.7) this does not hold for (b). The following example is motivated by Lipman’s example in [2].

**Example 4.3.** Let $R = k[[x]]$ denote the formal power series ring in the variable $x$ over the field $k$. Then define $S = \prod_{n \geq 1} R/x^n R$. By the component wise operations $S$ becomes a commutative ring. The natural map $R \rightarrow S, r \rightarrow (r + x^n)_{n \geq 1}$, is a ring homomorphism and $x \mapsto x := (x + x^n)_{n \geq 1}$. As a direct product of $xR$-complete modules $S$ is an $xR$-complete $R$-module (see [10,
Since \( R \) is a Noetherian ring \( x \) is \( R \)-weakly proregular and \( \hat{H}_i^S(S) \cong H_i(\text{Hom}_R(L_x, S)) = 0 \) for \( i > 0 \) and \( \hat{H}_0^S(S) \cong H_0(\text{Hom}_R(L_x, S)) \cong S \).

Moreover, by the change of rings there is an isomorphism \( \text{Hom}_R(L_x, S) \cong \text{Hom}_S(L_x, S) \). That is, \( \hat{H}_i^S(S) = 0 \) for \( i > 0 \) and \( \hat{H}_0^S(S) \cong S \). Now note that \( S \) is not of bounded \( x \)-torsion as easily seen. It follows also that \( S \) is \( x \)-adic complete and \( S \) is not a coherent ring.

In the following we shall apply the previous result to the situation of a complex of flat \( R \)-modules. It is a relative version of [10, 7.5.16].

**Corollary 4.4.** We fix the notation of 3.2. Let \( F_\bullet \) denote a complex of flat \( R \)-modules. Then the natural map

\[
\text{Hom}_R(L_x, M \otimes_R F_\bullet) \to \Lambda^a(M \otimes_R F_\bullet)
\]

is a quasi-isomorphisms. Moreover, if \( F_\bullet \) is a complex of finitely generated free \( R \)-modules, then the natural map

\[
\text{Hom}_R(L_x, M \otimes_R F_\bullet) \to \Lambda^a(M) \otimes_R F_\bullet
\]

is a quasi-isomorphism.

**Proof.** Let \( F_\bullet = \{F_q, d_q\}_{q \in \mathbb{Z}} \) be the complex of flat \( R \)-modules. Then the natural map

\[
\text{Hom}_R(L_x, M \otimes_R F_q) \to \Lambda^a(M \otimes_R F_q)
\]

is a quasi-isomorphism for all \( q \in \mathbb{Z} \) (see 3.2). Since \( \text{Hom}_R(L_x, M \otimes_R F_\bullet) \) is the single complex associated to the double complex \( \text{Hom}_R(L_x^2, M \otimes_R F_q) \) and because \( L_x^2 \) is bounded the first claim follows (see also [10, 4.1.3]). For the second note that \( F_q, q \in \mathbb{Z}, \) is a finitely generated free \( R \)-module. Its tensor product commutes with inverse limits.

In the following we specify the previous result for a finitely generated \( R \)-module \( N \).

**Corollary 4.5.** We use the assumption of 3.2. Let \( N \) denote a finitely generated \( R \)-module with \( L_\bullet \to N \) a free resolution by finitely generated free \( R \)-modules. Then there are isomorphisms

\[
\hat{H}_i^S(M \otimes_R L_\bullet) \cong \text{Tor}_i^R(\Lambda^a(M), N)
\]

for all \( i \in \mathbb{Z} \) and a spectral sequence

\[
E_{i,j}^2 = \hat{H}_i^S(\text{Tor}_j^R(M, N)) \Rightarrow E_{i+j}^\infty = \text{Tor}_{i+j}^R(\Lambda^a(M), N).
\]

**Proof.** The isomorphisms are an immediate consequence of 3.4. The spectral sequence is just the spectral sequence of the double complex (see e.g. [7]).

In the following we shall recall a particular case of one of the main results of [10].

**Corollary 4.6.** Let \( \bar{x} = x_1, \ldots, x_k \) denote a sequence of elements of \( R \) and \( a = \bar{x}R \). Suppose that \( \bar{x} \) is weakly proregular. Then \( \hat{H}_i^\bar{x}(N) \cong \Lambda_i^a(N) \) for all \( i \in \mathbb{N} \) and any \( R \)-module \( N \).

**Proof.** Let \( F_\bullet \to N \) be a flat resolution of \( N \). Then

\[
\text{Hom}_R(L_{\bar{x}}, N) \to \text{Hom}_R(L_{\bar{x}}, F_\bullet) \to \Lambda^a(F_\bullet).
\]

Since \( \Lambda_i^a(N) = H_i(\Lambda^a(F_\bullet)) \) the claim follows by taking homology.

In fact the previous result holds for any complex \( X \) of \( R \)-modules provided \( \bar{x} \) is weakly proregular (see [10] for these and more related information). Here we will continue with a further characterization of weakly proregular sequences.
Proposition 4.7. Let $\chi = x_1, ..., x_k$ denote a sequence of elements of $R$. Let $E$ denote an injective cogenerator in the category of $R$-modules. Then the following is equivalent.

(i) The sequence $\chi$ is weakly proregular.
(ii) $\tilde{H}_i^f(\text{Hom}_R(I, J)) = 0$ for all $i > 0$ and any two injective $R$-module $I, J$.
(iii) $\tilde{H}_i^f(\text{Hom}_R(I, E)) = 0$ for all $i > 0$ and any injective $R$-module $I$.

Proof. (i) $\Rightarrow$ (ii): By the adjointness there is the following isomorphisms of complexes
\[ \text{Hom}_R(L_\chi, \text{Hom}_R(I, J)) \cong \text{Hom}_R(L_\chi \otimes_R I, J). \]
Therefore (i) implies that $\text{Hom}_R(H^i(L_\chi \otimes_R I), J) = 0$ for all $i > 0$. By the definition it yields (ii).
(ii) $\Rightarrow$ (iii): This holds trivially.
(iii) $\Rightarrow$ (i): By the previous adjointness isomorphism it implies that $\text{Hom}_R(H^i(L_\chi \otimes_R I), E) = 0$ for all $i > 0$. Since $E$ is an injective cogenerator this implies $H^i(L_\chi \otimes_R I) \cong H^i(\chi(I)) = 0$ for all $i > 0$. Now this is equivalent to the fact that $\chi$ is weakly proregular (see 1.7).

Remark 4.8. (A) If $R$ is a coherent ring (see e.g. [10, 1.4.2]), then $\text{Hom}_R(I, J)$ is a flat $R$-module for any two injective $R$-modules (see [10, 1.4.5]).
(B) Let $R$ be a coherent ring. Then the conditions in 3.7 are equivalent to
(iv) $\tilde{H}_i^F(F) = 0$ for all $i > 0$ and any flat $R$-module $F$.
This follows because $\text{Hom}_R(I, E)$ is $R$-flat for any injective $R$-module $I$ (see [10, 7.5.15] for more details).
(C) There are examples of rings $R$ and injective $R$-modules $I, J$ such that $\text{Hom}_R(I, J)$ is not a flat $R$-module (see [10, A.5.7]).

5. Local conditions

Let $R$ denote a commutative ring. For an element $r \in R$ we write $D(f) = \text{Spec}R \setminus V(f)$. Note that $D(f)$ is an open set in the Zariski topology of $\text{Spec}R$. For $f \in R$ there is a natural map $\text{Spec}R_f \to \text{Spec}R$ that induces a homeomorphism between $\text{Spec}R_f$ and $D(f)$. Since $\text{Spec}R = \bigcup_{f \in R} D(f)$ and since $\text{Spec}R$ is quasi-compact there are finitely many $f_1, ..., f_r \in R$ such that $\text{Spec}R = \bigcup_{i=1}^r D(f_i)$.
This provides the following definition.

Definition 5.1. (see [12]) A sequence $f = f_1, ..., f_r$ of elements of $R$ is called a covering sequence if $\text{Spec}R = \bigcup_{i=1}^r D(f_i)$. This is equivalent to saying that $R = fR$. Moreover, if $f$ is a covering sequence then the natural map $M \to \bigoplus_{i=1}^r M_{f_i}$ is injective for any $R$-module $M$ as easily seen.

Next we show preregularity as well as weakly preregularity are local properties. For a sequence $\chi = x_1, ..., x_k$ of elements of $R$ we denote by $\chi/1 = x_1/1, ..., x_k/1$ its image in a localization of $R$.

Proposition 5.2. Let $f = f_1, ..., f_r$ denote a covering sequence of $R$. Let $\chi = x_1, ..., x_k$ be an ordered sequence of elements of $R$. For an $R$-module $M$ the following conditions are equivalent.

i. $\chi$ is $M$-weakly proregular (resp. proregular).
ii. $\chi/1$ in $R_{f_i}$ is $M_{f_i}$-weakly proregular (resp. proregular) for all $i = 1, ..., r$.

Proof. At first we consider the case of weakly proregular sequences. Then for all integers $m \geq n$ and any integer $i > 0$ there is a commutative diagram
\[
H_i(\mathcal{X}^{(m)}; M) \rightarrow \bigoplus_{j=1}^{r} H_i(\mathcal{X}/1^{(m)}; M_f^j)
\]
\[
H_i(\mathcal{X}^{(n)}; M) \rightarrow \bigoplus_{j=1}^{r} H_i(\mathcal{X}/1^{(n)}; M_f^j)
\]

where the horizontal maps are injective since \( f \) is a covering sequence. If (i) holds the vertical maps at the left are zero for a given \( n \) and an appropriate \( m \geq n \). Since localization is flat and commutes with homology the vertical maps at the right are zero too. For the converse fix \( n \) and \( i \) and choose \( m \geq n \) such that \( H^i(\mathcal{X}/1^{(m)}; M_f^j) \rightarrow H^i(\mathcal{X}/1^{(n)}; M_f^j) \) is zero for all \( j = 1, \ldots, r \). Then the vertical map at the right is zero. Since the horizontal maps are injective it follows that the vertical map at the left is zero.

The proof for the case of proregular sequence follows similar arguments by an inspection of the natural map

\[
(x_1^n, \ldots, x_{n-1}^n)_R : M x_1^n/(x_1^n, \ldots, x_{n-1}^n)M x_1^n \rightarrow (x_1^n, \ldots, x_{n-1}^n)_R : M x_1^n/(x_1^n, \ldots, x_{n-1}^n)M
\]

and the direct sum of the localizations with respect to \( R_f^i, i = 1, \ldots, r \).

A corresponding local global principle is the following.

**Proposition 5.3.** Let \( \mathcal{X} = x_1, \ldots, x_k \) be an ordered sequence of elements of \( R \). For an \( R \)-module \( M \) the following conditions are equivalent.

a. \( \mathcal{X} \) is \( M \)-weakly proregular (resp. proregular).

b. \( \mathcal{X}/1 \) in \( R_\mathfrak{p} \) is \( M_\mathfrak{p} \)-weakly proregular (resp. proregular) for all \( \mathfrak{p} \in \text{Spec}R \).

c. \( \mathcal{X}/1 \) in \( R_\mathfrak{m} \) is \( M_\mathfrak{m} \)-weakly proregular (resp. proregular) for all maximal ideals \( \mathfrak{m} \in \text{Spec}R \).

**Proof.** The proof follows easily by [6, Theorem 4.6, p. 27]. We omit the details here. □

By view of [12] we recall the following definition and extend it with the notion of proregularity.

**Definition 5.4.** (A) (see [12]) Let \( R \) denote a commutative ring. An ideal \( \mathcal{I} \subseteq R \) is called an effective Cartier divisor if there is a covering sequence \( f = f_1, \ldots, f_r \) such that \( \mathcal{I} R_{f_i} = x_i R_{f_i}, i = 1, \ldots, r \), for non-zerodivisors \( x_i/1 \) of \( R_{f_i} \) with \( x_i \in R \). It follows that \( \mathcal{I} \subseteq (x_1, \ldots, x_r)_R \).

(B) Let \( \mathcal{I} \) denote a Cartier divisor and \( x \in R \). The pair \( \langle \mathcal{I}, x \rangle \) is called proregular if for any integer \( n \) there is an integer \( m \geq n \) such that \( \mathcal{I}^m : x^n \subseteq \mathcal{I}^n : x^{m-n} \). This is in consistence with the definition in [5] (see 1.1) and is equivalent to the fact that for each \( n \) there is an integer \( m \geq n \) such that the multiplication map \( \mathcal{I}^m : R x^n / \mathcal{I}^m x^n \rightarrow \mathcal{I}^n : R x^n / \mathcal{I}^n \) is the zero map.

In the following we shall consider a local global principle for proregular effective Cartier divisors. We have to modify our previous arguments slightly since \( \mathcal{I} \) is in general not singly generated.

**Theorem 5.5.** Let \( \mathcal{I} \subseteq R \) an effective Cartier divisor with the covering sequence \( f = f_1, \ldots, f_r \) such that \( \mathcal{I} R_{f_i} = x_i R_{f_i}, i = 1, \ldots, r \), for non-zerodivisors \( x_i \) of \( R \). Suppose that \( R/\mathcal{I} \) is of bounded x-torsion for some \( x \in R \). Then the following equivalent conditions hold:

(a) \( \langle \mathcal{I}, x \rangle \) is proregular, i.e. for each integer \( n \) there is an \( m \geq n \) such that the multiplication map

\[
\mathcal{I}^m : R x^n / \mathcal{I}^m x^n \rightarrow \mathcal{I}^n : R x^n / \mathcal{I}^n
\]

is the zero map.

(b) \( \Gamma_\mathcal{I}(L)/\Gamma_{\langle \mathcal{I}, x \rangle}(L) \) is \( x \)-divisible for any injective \( R \)-module \( L \).
Proof. At first we prove (a). Since $R/I$ is of bounded $x$-torsion there is an integer $c$ such that $\mathcal{I}^m : R^m = \mathcal{I} : R^x$ for all $m \geq c$. By localization at $R_f$ it follows that $x_i R_f : R_f x^m / 1 = x_i R_f : R_f x^i / 1$ for all $m \geq c$ and $i = 1, \ldots, r$ with $R_f$-regular elements $x_i / 1 \in R_f$. By view of 1.2 it follows $x_i / 1, x_i / 1$ is an $R_f$-proregular sequence for all $i = 1, \ldots, r$. For a given $n$ and $m \geq n$ there is the following commutative diagram

$$
\begin{align*}
\mathcal{I}^m : R^{x^m} / I^m &\quad \to \quad \bigoplus_{j=1}^{r} (x_i^m R_f^j : R_f x^m / 1) / x_i^m R_f^j \\
\mathcal{I}^n : R^{x^n} / I^n &\quad \to \quad \bigoplus_{j=1}^{r} (x_i^n R_f^j : R_f x^n / 1) / x_i^n R_f^j
\end{align*}
$$

Now choose $m$ such that multiplication at the vertical maps at the right are all zero. Then the multiplication at the left is the zero map too since the horizontal maps are injective as follows by the localization (see 4.1).

For the proof of (a) $\Rightarrow$ (b) recall that the natural map $\mathcal{I}^m : R^{x^m} / I^m \to \mathcal{I}^n : R^{x^n} / I^n$ coincides with the map induced by the Koszul complexes, i.e.,

$$H_1 (x^m ; R/I^m) \to H_1 (x^n ; R/I^n).$$

Now apply $\text{Hom}_R (\cdot, I)$ with an arbitrary injective $R$-module. Then the induced map $H_1 (x^n ; \text{Hom}_R (R/I^n, I)) \to H_1 (x^m ; \text{Hom}_R (R/I^m, I))$ is zero and $0 = \lim H_1 (x^n ; \text{Hom}_R (R/I^m, I)) \cong \tilde{H}_1 (\Gamma (I))$ which proves the statement as in the proof of 2.1.

For the implication (b) $\Rightarrow$ (a) we fix $n$ and choose an injection $H_1 (x^n ; \text{Hom}_R (R/I^n, I)) \cong \mathcal{I}^n : R^{x^n} / I^n$, $\to I$ into an injective $R$-module $I$. By the vanishing of the direct limit there is, as in the proof of \textit{Theorem 2.1}, an integer $m \geq n$ such that (a) holds.

In the following we shall give a comment of the previous investigations to the recent work of Bhatt and Scholze (see [3]). To this end let $p \in \mathbb{N}$ denote a prime number and let $\mathbb{Z}_p := \mathbb{Z}_p$ the localization at the prime ideal $(p) = p \in \text{Spec} \mathbb{Z}$. In the following let $R$ be a $\mathbb{Z}_p$-algebra.

\textbf{Definition 5.6.} (see [3, Definition 1.1]) A prism is a pair $(R, \mathcal{I})$ consisting of a $\delta$-ring $R$ (see [3, Remark 1.2]) and a Cartier divisor $\mathcal{I}$ on $R$ satisfying the following two conditions.

(a) The ring $R$ is $(p, \mathcal{I})$-adic complete.
(b) $p \in \mathcal{I} + \phi_R (\mathcal{I}) R$, where $\phi_R$ is the lift of the Frobenius on $R$ induced by its $\delta$-structure (see [3, Remark 1.2]).

With the previous definition there is the following application of our results.

\textbf{Corollary 5.7.} Let $(R, \mathcal{I})$ denote a prism. Suppose that $\mathcal{I}$ is of bounded $p$-torsion. Then

(a) $(\mathcal{I}, p)$ is proregular in the sense of 4.4,
(b) $\Gamma (I) / \Gamma (\mathcal{I}, p) (I)$ is $p$-divisible for any injective $R$-module $I$, i.e. $\Gamma (I) = p \Gamma (I) + \Gamma (\mathcal{I}, p) (I)$, and the conditions in (a) and (b) are equivalent.

\textbf{Proof.} This is an immediate consequence of 4.5.

We note that Yekutieli (see [13, Theorem 7.3]) has slightly modified the notion of weakly proregularity and has shown that $(\mathcal{I}, p)$ is weakly proregular under the assumption of 4.7. It should
be mentioned that proregularity is more strong than weakly proregularity as shown by the example in [2].

While the notion of weakly proregularity plays an essential role in the study of local (co-)homology (see [10] and the references there) the previous statements seem to be a further application of the notion of proregularity as introduced by Greenlees and May (see [5]) and by Lipman (see [1]).

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