**q-Euler Numbers and Polynomials Associated with Basic Zeta Functions**

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**Abstract.** We consider the $q$-analogue of Euler zeta function which is defined by

$$\zeta_{q,E}(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{[n]_q^s}, \quad 0 < q < 1, \quad \Re(s) > 1.$$ 

In this paper, we give the $q$-extension of Euler numbers which can be viewed as interpolating of the above $q$-analogue of Euler zeta function at negative integers, in the same way that Riemann zeta function interpolates Bernoulli numbers at negative integers. Also, we will treat some identities of the $q$-extension of the Euler numbers by using fermionic $p$-adic $q$-integration on $\mathbb{Z}_p$.

**1. Introduction**

Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will respectively denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$.

The $p$-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = \frac{1}{p}$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we normally assume $|q| < 1$, and when $q \in \mathbb{C}_p$, then we normally assume $|q-1|_p < 1$. We use the notation:

$$[x]_q = [x : q] = \frac{1 - q^x}{1 - q}, \quad \text{and} \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}.$$ 

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Note that \( \lim_{q \to 1} [x]_q = x \) for \( x \in \mathbb{Z}_p \) in presented \( p \)-adic case.

Let \( UD(\mathbb{Z}_p) \) be denoted by the set of uniformly differentiable functions on \( \mathbb{Z}_p \).

For \( f \in UD(\mathbb{Z}_p) \), let us start with the expression

\[
\frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} (-q)^j f(j) = \sum_{0 \leq j < p^N} f(j) \mu_{-q}(j + p^N \mathbb{Z}_p)
\]

representing analogue of Riemann’s sums for \( f \), cf.\([1-30]\).

The fermionic \( p \)-adic \( q \)-integral of \( f \) on \( \mathbb{Z}_p \) will be defined as the limit \((N \to \infty)\) of these sums, which it exists. The fermionic \( p \)-adic \( q \)-integral of a function \( f \in UD(\mathbb{Z}_p) \) is defined as

\[
\int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} f(j)(-q)^j, \quad \text{(see [5, 6, 16])}.
\]

For \( d \) a fixed positive integer with \((p, d) = 1\), let

\[
X = X_d = \lim_{N} \mathbb{Z}/dp^N \mathbb{Z}, \quad X_1 = \mathbb{Z}_p,
\]

\[
X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} a + dp \mathbb{Z}_p,
\]

\[
a + dp^N \mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \},
\]

where \( a \in \mathbb{Z} \) lies in \( 0 \leq a < dp^N \), (see [1-30]).

Let \( \mathbb{N} \) be the set of positive integers. For \( m, k \in \mathbb{N} \), the \( q \)-Euler polynomials \( E_{m,k}^{(-m,k)}(x, q) \) of higher order in the variables \( x \) in \( \mathbb{C}_p \) by making use of the \( p \)-adic \( q \)-integral , cf.\([5, 6]\), are defined by

\[
(1) \quad E_{m,q}^{(-m,k)}(x) = \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + x_2 + \cdots + x_k]_q^m
\]

\[
\cdot q^{-x_1(m+1) - x_2(m+2) - \cdots - x_k(m+k)} d\mu_{-q}(x_1) d\mu_{-q}(x_2) \cdots d\mu_{-q}(x_k).
\]

Now, we define the \( q \)-Euler numbers of higher order as follows:

\[
E_{m,q}^{(-m,k)} = E_{m,q}^{(-m,k)}(0).
\]
From (1), we can derive

\[
E_{m,q}^{(-m,k)} = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x_1=0}^{p^N-1} \cdots \sum_{x_k=0}^{p^N-1} [x_1 + \cdots + x_k]_q^{m} (-1)^{x_1 + \cdots + x_k} q^{-x_1 m - \cdots - x_k (m+k-1)} \]

\[
= \frac{[2]^k}{(1-q)^m} \sum_{i=0}^{m} \binom{m}{i} (-1)^i \frac{1}{(1+q^{i-m})(1+q^{i-m-1}) \cdots (1+q^{i-m-k+1})},
\]

where \(\binom{m}{i}\) is binomial coefficient.

Note that \(\lim_{q \to 1} E_{m,q}^{(-m,k)} = E_{m}^{(k)}\) where \(E_{m}^{(k)}\) are ordinary Euler numbers of order \(k\), which are defined as

\[
\left( \frac{2}{e^t + 1} \right)^k = \sum_{n=0}^{\infty} E_{m}^{(k)} \frac{t^n}{n!}.
\]

By (1), (2), it is easy to see that

\[
E_{m,q}^{(-m,1)}(x) = \sum_{i=0}^{m} \binom{m}{i} q^i E_{i,q}^{(-m,1)}[x]^{m-i} = \frac{[2]_q}{(1-q)^m} \sum_{j=0}^{m} q^j x \binom{m}{j} (-1)^j \frac{1}{1+q^{j-m}}.
\]

We define the \(q\)-analogue of Euler zeta function which is defined as

\[
\zeta_{q,E}(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{[n]_q s^n}, \quad q \in \mathbb{R} \text{ with } 0 < q < 1 \text{ and } s \in \mathbb{C}.
\]

The numerator ensures the convergence. In (4), we can consider the following problem:

“Are there \(q\)-Euler numbers which can be viewed as interpolating of \(\zeta_{q,E}(s)\) at negative integers, in the same way that Riemann zeta function interpolates Bernoulli numbers at negative integers”? 

In this paper, we give the value \(\zeta_{q,E}(-m)\), for \(m \in \mathbb{N}\), which is a answer of the above problem and construct a new complex \(q\)-analogue of Hurwitz’s type Euler zeta function and \(q\)-L-series related to \(q\)-Euler numbers. Also, we will treat some interesting identities of \(q\)-Euler numbers.
2. SOME IDENTITIES OF \(q\)-EULER NUMBERS \(E_{m,q}^{(-m,1)}\).

In this section, we assume \(q \in \mathbb{C}_p\) with \(|1 - q|_p < 1\). By (1), we see that

\[
E_{n,q}^{(-n,1)}(x) = \int_X q^{-(n+1)t} [x + t]^n d\mu_q(t)
= \frac{[2]_q}{[2]_q^n} [d]_q^n \sum_{i=0}^{d-1} (-1)^i q^{-ni} \int_{\mathbb{Z}_p} q^{(n+1)dt} \left[\frac{x + t}{d}\right]^n d\mu_q(t).
\]

Thus we have

\[
E_{n,q}^{(-n,1)}(x) = \frac{[2]_q}{[2]_q^n} [d]_q^n \sum_{i=0}^{d-1} (-1)^i \int_{\mathbb{Z}_p} q^{(n+1)dt} \left[\frac{x + t}{d}\right]^n d\mu_q(t),
\]

where \(d, n\) are positive integers with \(d \equiv 1(\mod 2)\).

If we take \(x = 0\), then we have

\[
E_{m,q}^{(-m,1)} = \frac{[2]_q}{[2]_q^n} \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array}\right) [n]_q^k E_{k,q^n}^{(-m,1)} \sum_{j=0}^{n-1} (-1)^j q^{-(m-k)j} [j]_q^{m-k}, \quad \text{where } n \equiv 1(\mod 2).
\]

From (6), we can easily derive the following equation (7).

\[
E_{m,q}^{(-m,1)} - \frac{[2]_q}{[2]_q^n} [n]_q^m E_{m,q^n}^{(-m,1)} = \frac{[2]_q}{[2]_q^n} \sum_{k=0}^{m-1} \left(\begin{array}{c} m \\ k \end{array}\right) [n]_q^k E_{k,q^n}^{(-m,1)} \sum_{j=1}^{n-1} (-1)^j q^{-(m-k)j} [j]_q^{m-k}.
\]

It is easy to see that \(\lim_{q \to 1} E_{m,q}^{(-m,1)} = E_m\), where \(E_m\) are the \(m\)-th ordinary Euler numbers, cf. [5]. From (7), we note that

\[
(1 - n^m)E_m = \sum_{k=0}^{m-1} \left(\begin{array}{c} m \\ k \end{array}\right) n^k E_k \sum_{j=1}^{n-1} (-1)^j j^{m-k}.
\]

Let \(F_q(t, x)\) be generating function of \(E_{n,q}^{(-n,1)}\) as follows:

\[
F_q(t, x) = \sum_{k=0}^{\infty} E_{k,q}^{(-k,1)}(x) \frac{t^k}{k!}.
\]
By (3), (8), we easily see:

$$F_q(t, x) = [2]_q \sum_{k=0}^{\infty} \left( \frac{1}{(q-1)k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{q^{jx}}{1+q^{j-k}} \right) \frac{t^k}{k!}$$

(9)

$$= \sum_{k=0}^{\infty} \left( [2]_q \sum_{n=0}^{\infty} (-1)^n q^{-kn} [n+x]_q^k \right) \frac{t^k}{k!}$$

Differentiating both sides with respect to $t$ in (5), (6) and comparing coefficients, we obtain the following:

**Theorem 1.** For $m \geq 0$, we have

$$E_{m,q}^{(-m,1)}(x) = [2]_q \sum_{n=0}^{\infty} q^{-nm} [n+x]_q^m (-1)^n.$$  

(10)

**Corollary 2.** Let $m \in \mathbb{N}$. Then there exists

$$E_{m,q}^{(-m,1)} = [2]_q \sum_{n=1}^{\infty} q^{-nm}[n]_q^m (-1)^n, \text{ and } E_{0,q}^{(0,1)} = \frac{[2]_q}{2}.$$  

(11)

Note that Corollary 2 is a $q$-analogue of $\zeta_E(m)$, for any positive integer $m$. Let $\chi$ be a primitive Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. For $m \in \mathbb{N}$, we define

$$E_{m,\chi,q}^{(-m,1)} = \int_X q^{-(m+1)x} \chi(x) [x]^m \mu_{-q}(x), \text{ for } m \geq 0.$$  

(12)

Note that

$$E_{m,\chi,q}^{(-m,1)} = \frac{[2]_q}{[2]_q^d [d]_q^m} \sum_{i=0}^{d-1} q^{-mi} (-1)^i \chi(i) \int_{\mathbb{Z}_p} q^{-d(m+1)} \left( \frac{i}{d} + x \right)_q^m \mu_{-q^d}(x)$$

(13)

$$= \frac{[2]_q}{[2]_q^d [d]_q^m} \sum_{i=0}^{d-1} \chi(i) (-1)^i q^{-mi} E_{m,q}^{(-m,1)} \left( \frac{i}{d} \right).$$
3. **q-ANALOGS OF ZETA FUNCTIONS**

In this section, we assume \( q \in \mathbb{R} \) with \( 0 < q < 1 \). Now we consider the \( q \)-extension of the Euler zeta function as follows:

\[
\zeta_{q,E}(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^{ns}}{[n]^s_q}, \quad \text{where } s \in \mathbb{C}.
\]

By (11), we obtain the following theorem.

**Theorem 3.** For \( m \in \mathbb{N} \), we have

\[
\zeta_{q,E}(-m) = E_{m,q}^{(-m,1)}.
\]

From Theorem 1, we can also define the \( q \)-extension of Hurwitz’s type Euler \( \zeta \)-function as follows: For \( s \in \mathbb{C} \), define

\[
\zeta_{q,E}(s, x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^{sn}}{[n + x]^s_q}.
\]

Note that \( \zeta_{q,E}(s, x) \) is an analytic continuation in whole complex \( s \)-plane.

By (14) and Theorem 1, we have the following theorem.

**Theorem 4.** For any positive integer \( k \), we have

\[
\zeta_{q,E}(-k, x) = E_{k,q}^{(-k,1)}(x, q).
\]

For \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2} \), let \( \chi \) be Dirichlet character with conductor \( d \). By (13), the generalized \( q \)-Euler numbers attached to \( \chi \) can be defined as

\[
E_{m,\chi,q}^{(-m,1)} = \frac{[2]_q}{[2]_d} \frac{d}{d_q^m} \sum_{i=0}^{d-1} \chi(i) q^{-mi} (-1)^i E_{m,d_q^i}(\frac{i}{d}).
\]

For \( s \in \mathbb{C} \), we define

\[
L_{q,E}(s, \chi) = [2]_q \sum_{n=1}^{\infty} \frac{\chi(n)(-1)^n q^{sn}}{[n]^s_q}.
\]

It is easy to see that

\[
L_{q,E}(\chi, s) = \frac{[2]_q}{[2]_d} \frac{d}{d_q^s} \sum_{a=1}^{d} \chi(a)(-1)^a q^{sa} \zeta_{q,d,E}(s, \frac{a}{d}).
\]

By (16), (17), (18), we obtain the following theorem.
Theorem 5. Let \( k \in \mathbb{N} \). Then there exists
\[
L_{q,E}(-k, \chi) = E_{k, \chi, q}^{(-k, 1)}.
\]

Let \( a \) and \( F \) be integers with \( 0 < a < F \). For \( s \in \mathbb{C} \), we consider the functions \( H_q(s, a, F) \) as follows:
\[
H_{q,E}(s, a, F) = [2]_q \sum_{m \equiv a(F), m > 0} q^{ns} (-1)^m \left[ \frac{[2]_q}{[2]_{qF}} \right] (-1)^a q^a \zeta_{qF}(s, \frac{a}{F}).
\]

Then we have
\[
H_{q,E}(-n, a, F) = (-1)^a q^a \left[ \frac{[2]_q}{[2]_{qF}} \right] [F]_q^n E_{n, qF}^{(-n, 1)} (\frac{a}{F}),
\]
where \( n \) is any positive integer .

In the recent paper, the \( q \)-analogue of Riemann zeta function related to twisted \( q \)-Bernoulli numbers was studied by Y. Simsek (see [1, 26, 30]). In [30], Y. Simsek have studies the twisted \( q \)-Bernoulli numbers which can be viewed as an interpolating of the \( q \)-analogue of Riemann zeta function at negative integers. In this paper, we have shown that the \( q \)-analogue of Euler zeta function interpolates the \( q \)-Euler numbers at negative integers, in the same way that Riemann zeta function interpolates Bernoulli numbers at negative integers, cf. [5, 6, 30].

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