Non-linear Hall response in Bloch oscillations with ultracold atoms

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We propose that the non-linear Hall response with the preserved time-reversal symmetry can be observed in the Bloch oscillation of the ultracold atoms in optical lattices. In the short-time limit of Bloch oscillations with a DC driving field, the non-linear Hall current dominates with time reversal symmetry and is a second-order response of the external field strength. The associated Berry curvature dipole, which is the second order nonlinear coefficient of the driving field, can be extracted from the dynamics. In the presence of an AC driving field, the nonlinear Hall response will double the driving frequency in the case of time reversal symmetry.

\textbf{Introduction.}—The Hall effect plays an important role in the condensed matter physics [1]. It is a common technique to measure the charge of carriers in the conductors. The quantized version, quantum Hall effect, was observed in two-dimensional electron gases in 1980 [2]. The quantized Hall conductance is determined by the fundamental topological properties of the materials, such as the Berry’s curvature, and gapless boundary states, which are robust against disorder and impurity [3–5]. Over the past few decades, there have been intensive studies broadening the quantum Hall family with intrinsic or extrinsic magnetism, from fractional quantum Hall effect [6], quantum anomalous Hall effect [7], to topological insulators [8, 9]. These studies reshaped the understanding of phase transition and phases of matter.

Time reversal symmetry (TRS) is broken in both the Hall effect and anomalous Hall effect, either by external magnetic field or by spontaneous symmetry breaking. The linear-order transverse response of the applied electric field, i.e. the Hall conductance, will vanish in the present of TRS. However, if beyond linear order, for example to the quadratic order of external electric field, even the system with TRS could exhibit non-vanishing Hall response. This non-linear Hall effect (NLHE) was proposed by Sodemann and Fu in the materials with broken inversion symmetry (IS) [10]. This NLHE was intensively studied in condensed matter materials [11–18], and very recently it was observed in multi-layer WTe\textsubscript{2} [19, 20] and topological insulator Bi\textsubscript{2}Se\textsubscript{3} [21], see recent review in Ref. [22].

The NLHE is induced by the absence of inversion symmetry and the non-equilibrium distribution of carriers in the Bloch band. In the condensed matter system, when the back scattering of impurities balances the applied electric field, the carriers reach a current carrying steady state with near equilibrium distribution. In this situation the NLHE can be detected. While in cold atom system, since the carriers, i.e., atoms, are charge neutral, the external electrical field is replaced by a gradient potential, which can be realized either by acceleration of optical lattice or by a gradient magnetic field [23, 24]. Due to the lack of back scattering of impurities, atoms will exhibit continuous, undamped Bloch oscillation [25, 26]. During the oscillation, the distribution of carriers is far from the equilibrium and evolves with time. It is natural to ask if the NLHE can be observed in this far-from-equilibrium Bloch oscillation.

In this work, we investigate non-linear Hall response during Bloch oscillation of ultracold atoms in optical lattice. For DC driving force, the dynamics of transverse current is dominated by the second-order non-linear Hall effect in short time for a system with TRS but without IS. The Berry curvature dipole, which reflects the distribution of Berry curvature in the Brillouin zone, can be extracted from dynamics of the Hall response. In the AC case, the transverse drifts will oscillate with half period of the driving force in a time reversal symmetric band. This is a typical frequency doubling induced by the non-linear Hall effect. While in the system with broken TRS, the period of transverse oscillation is identical to the driving one. The Berry curvature dipole can be also extracted from the amplitude of the oscillating Hall response.

\textbf{Bloch oscillation.}—We consider loading the non-interacting fermionic ultracold atoms onto the lowest band of an optical lattice, and we apply a gradient potential to drive atoms undergoing Bloch oscillation. This can be done by accelerating the optical lattice or applying a gradient magnetic field. At zero temperature and when the gradient potential is weak enough, all atoms are populating on the lowest band during the whole oscillation. What can be easily measured in ultracold atom experiment is the center-of-mass velocity of atoms, being

\begin{equation}
\bar{\mathbf{v}}(t) = \frac{1}{N} \int_{BZ} \frac{d^d k}{(2\pi\hbar)^d} f^p(k, t) \mathbf{v}(k),
\end{equation}

where \(N\) is the total particle number, \(d\) spacial dimension, \(f^p(k, t)\) the non-equilibrium distribution function of fermionic atoms during the oscillation, and \(\mathbf{v}(k)\) is the...
velocity of the atom at quasi-momentum $\mathbf{k}$ on the lowest band. The integral is over the first Brillouin Zone. In the limit of weak and slowly varying gradient potential, one can use the semi-classical approximation to describe the drifts of atoms during the Bloch oscillation \[27–31\]. It results in a set of equations of motion

$$
\mathbf{v}(\mathbf{k}) = \dot{\mathbf{r}} = \nabla_k \epsilon(\mathbf{k}) - \mathbf{k} \times \Omega(\mathbf{k}),
$$

(2a)

$$
\mathbf{k}(t) = \mathbf{F}(t),
$$

(2b)

where $\epsilon(\mathbf{k})$ is the energy dispersion of the lowest band, its derivative gives the group velocity; while $\Omega(\mathbf{k})$ is the Berry curvature of the lowest band, which contributes to the anomalous part of the velocity.

We first consider case of time independent driving force. In this situation, the solution of Eq. (2b) is $\mathbf{k}(t) = \mathbf{k}(0) + \mathbf{F}t$. Note that the momentum of atom is drifting in the Brillouin zone at a constant speed. As a consequence, the distribution function evolves as $f^0_\gamma(\mathbf{k}, t) = f^0_\gamma(\mathbf{k} - \mathbf{F}t)$ where $f^0_\gamma(\mathbf{k})$ is the equilibrium distribution function of atoms at $t = 0$. Thus the center-of-mass velocity of atoms during the Bloch oscillation can be obtained as

$$
\bar{\mathbf{v}}(t) = \frac{1}{N} \int \frac{d^d k}{(2\pi \hbar)^d} f^0_\gamma(\mathbf{k} - \mathbf{F}t) \left\{ \nabla_k \epsilon - \mathbf{F} \times \Omega(\mathbf{k}) \right\}.
$$

(3)

**DC driving force.**—In the short-time limit, the distribution function can be expanded to the linear order of time as $f^0_\gamma(\mathbf{k} - \mathbf{F}t) = f^0_\gamma(\mathbf{k}) - \nabla_k f^0_\gamma \cdot \mathbf{F}t + \cdots$. Then the mean velocity becomes

$$
\bar{v}_\alpha^{\text{Hall}}(t) = -\frac{1}{N} \int \frac{d^d k}{(2\pi \hbar)^d} \frac{\partial}{\partial k} f^0_\gamma(\mathbf{k} - \mathbf{F}t) \mathbf{F} \times \Omega(\mathbf{k}),
$$

(4)

where

$$
D^{\alpha\beta}_{\gamma\eta}(\mathbf{k}) = \int \frac{d^d k}{(2\pi \hbar)^d} \frac{\partial}{\partial k} \frac{\partial}{\partial \omega} \Omega(\mathbf{k}) \frac{\partial}{\partial k} f^0_\gamma(\mathbf{k}) = (-1)^\alpha \int \frac{d^d k}{(2\pi \hbar)^d} \frac{\partial^\alpha}{\partial \omega^\alpha} f^0_\gamma(\mathbf{k}).
$$

Note that $D^{\alpha\beta}_{\gamma\eta}(\mathbf{k})$ is the coefficient of the linear Hall response, while $D^{(1)}_{\gamma\eta}(\mathbf{k})$ is the so-called Berry curvature dipole (BCD) tensor, which is the second-order coefficient of non-linear Hall response. Therefore, by looking the oscillating dynamics of ultracold fermions, one can identify this non-linear Hall response. More precisely, by measuring the early growing rate of transverse Hall velocity, we can obtain BCD tensor, which reflects the distribution of Berry curvature in the Brillouin zone.

If the band has IS, we have $\Omega(-\mathbf{k}) = \Omega(\mathbf{k})$; while if the system has TRS, we have $\Omega(-\mathbf{k}) = -\Omega(\mathbf{k})$. Thus if the system has both, that indicates the Berry curvature is zero in the whole Brillouin zone, $\Omega(\mathbf{k}) = 0$. As a result the transverse current vanishes at arbitrary order. If TRS is broken, there is a Hall response in the linear order, i.e., $D^{(0)}_{\gamma\eta}(\mathbf{k}) \neq 0$. When system has TRS but breaks IS, the linear Hall response vanishes, $D^{(0)}_{\gamma\eta}(\mathbf{k}) = 0$, since $f^0_\gamma(\mathbf{k}) \Omega(\mathbf{k})$ is an odd function in the Brillouin zone. However the second order non-linear Hall response may manifest itself nonzero $D^{(1)}_{\gamma\eta}(\mathbf{k}) \neq 0$, since $\frac{\partial}{\partial \omega} f^0_\gamma(\mathbf{k})$ becomes even.

To illustrate this non-linear Hall response and its dependence on band symmetry, we numerically investigate the Bloch oscillation in the Haldane model \[32\] (see Fig. 1), which has been realized by shaking a two dimensional honeycomb optical lattice in ultracold atom system \[24, 33\]. The Hamiltonian can be written as

$$
H(\mathbf{k}) = E_0(\mathbf{k}) \mathbb{I} + \mathbf{B}(\mathbf{k}) \cdot \mathbf{\sigma} \quad \text{with elements being} \quad [34]
$$

$$
E_0(\mathbf{k}) = 2J_2 \cos \phi \sum_{\alpha=1,2,3} \cos (\mathbf{k} \cdot \mathbf{a}_\alpha),
$$

(5a)

$$
B_x(\mathbf{k}) = -J_1 \cos (\mathbf{k} \cdot \mathbf{d}_1) - J'_1 \sum_{\alpha=2,3} \cos (\mathbf{k} \cdot \mathbf{d}_\alpha),
$$

(5b)

$$
B_y(\mathbf{k}) = -J_1 \sin (\mathbf{k} \cdot \mathbf{d}_1) - J'_1 \sum_{\alpha=2,3} \sin (\mathbf{k} \cdot \mathbf{d}_\alpha),
$$

(5c)

$$
B_z(\mathbf{k}) = M + 2J_2 \sin \phi \sum_{\alpha} \sin (\mathbf{k} \cdot \mathbf{a}_\alpha).
$$

(5d)

Here $J_1$ ($J'_1$) and $J_2$ denote the nearest-neighbour and the next-nearest-neighbour hopping strengths respectively, $\phi$ is the phase of the next-nearest-neighbour hopping.
the reflection symmetry along the x axis. That gives
\[ \Omega_{\theta}(k_x, k_y) = \Omega_{\theta}(k_x, k_y). \]
The TRS ensures \[ \Omega_{\theta}(-k) = -\Omega_{\theta}(k), \]
giving \[ \Omega_{\theta}(k_x, -k_y) = -\Omega_{\theta}(k_x, k_y). \] Note that, in
this case, \[ \partial \Omega_{\theta}/\partial k_x \] is odd. Thus, \[ D^{(2)}_{xyz} \] vanishes
but \[ D^{(2)}_{zy} \] survives. From now on, we will set external
force only along the y direction and equation (6) can be
simplified into \[ \bar{v}_y(t) = 0 \] and \[ \bar{v}_x(t) = \frac{1}{2} D^{(2)}_{yx} F_y t. \]

In Fig. 2, we calculate the mean Hall velocities \( \bar{v}_x \) by
the semi-classical equation of motions (2) and by the
short-time expansion (4). In both cases with and without
TRS in Figs. 2(a,c), the results from Eq. (4) (grey-dashed
lines with circles) can recover the Hall velocity \( \bar{v}_x \) (solid
lines) obtained from semi-classical dynamics over a short
time. In the presence of TRS, the non-linear contribu-
tion plays a dominant role at short time and \( \bar{v}_x(t) \) becomes
linear at short time. Here we will set external
force only along the y direction and equation (6) can be
simplified into \( \bar{v}_y(t) = 0 \) and \( \bar{v}_x(t) = \frac{1}{2} D^{(2)}_{yx} F_y t. \)

We calculate BCD in the honeycomb lattice with differ-
tent \( J_1/J_1 \) and chemical potential \( \mu \), as shown in Fig. 3.
We note that at \( J_1 = J_1 \), the BCD vanishes, for any given
filling of fermions. This is because at \( J_1 = J_1 \), the sys-

 FIG. 2. Mean velocity \( \bar{v}_x \) calculated from semi-classical dy-
namics (solid line) and from Eq. (4) (gray-dashed line with
circles), (a,c) over a small range of time \( t \) at a small \( F_y \), and
(b,d) as functions of \( F_y \) at a definite short time. Here we have
set hopping ratios \( J_1/J_1 = 0.6, J_2/J_1 = 0.1 \), a mass
term \( M/J_1 = -0.2 \) breaking the inversion symmetry. \( \phi \) is set
to be \( \pi/8 \) (or 0) in (a,b) [or (c,d)] to break (or preserve)
the time-reversal symmetry.

Thus, the BCD after rotation reads
\[
\hat{D}^{(1)}_{yz} = \int_{BZ} \frac{d^2 k}{(2\pi)^2} \frac{\partial \Omega^{(2)}_{\theta}}{\partial k_x} f_0^\theta(k, \mu) \frac{\partial \Omega^{(2)}_{\theta}}{\partial k_y} f_0^\theta(k, \mu) \] 
\[
= \int_{BZ} \frac{d^2 k}{(2\pi)^2} \left( \frac{\partial \Omega^{(2)}_{\theta}}{\partial k_y} \sin \theta + \frac{\partial \Omega^{(2)}_{\theta}}{\partial k_x} \cos \theta \right) f_0^\theta(k, \mu) \] 
\[
= D^{(1)}_{yx} \sin \theta + D^{(1)}_{zy} \cos \theta,
\]
where we already know \( D^{(1)}_{xx} = 0 \). If the system has
C
 symmetry, a rotation by \( \theta = 2\pi/3 \) will keep system
invariant. That indicates \( \Omega^{(2\pi/3)} = \Omega_{\theta} \), thus we have
\( \hat{D}^{(1)}_{yz} = D^{(1)}_{yz} \), i.e., \( D^{(1)}_{yz} \cos \theta = D^{(1)}_{zy} \), leading to \( D^{(1)}_{xz} = 0 \). This
conclusion can be generalized, i.e. any systems with
discrete rotation symmetry have vanishing BCD.

We present BCD as a function of chemical potential in
Fig. 3(a). When chemical potential increases from the
bottom of energy band, BCD first increases with chemi-
cal potential, reaching its maximum. Then it decreases
to zero at half filling when chemical potential exceeds the
lower band top. This phenomena can be understood
in two limits, i.e. low and high fillings. In the low fill-
ing limit, the tight-binding Hamiltonian can be expanded
near the bottom of the lower band, i.e., around \( k = 0 \) as
\[
H(k) \simeq M \sigma_x + \left\{ -(\lambda + 2)J_1' \right\} (2\lambda + 1)k_x^2 + \left\{ 3\lambda k_y^2 \right\} k_x \sigma_y \\
+ (\lambda - 1)J_1' k_x k_y
\]
with \( J_1 = \lambda J_1' \). Then the associated Berry curvature is
\[
\Omega_{\theta}(k) = \frac{1}{2|B(k)|^3} \left( \frac{\partial B_y}{\partial k_x} \frac{\partial B_z}{\partial k_y} - \frac{\partial B_z}{\partial k_x} \frac{\partial B_y}{\partial k_y} \right) \\
= \frac{3M}{4|B(0)|^3} (1 - \lambda) J_1^2 k_y.
\]

FIG. 3. Berry curvature dipole \( D^{(2)}_{yz} \) (symbols) and the one
(lines) calculated via \( \bar{v}_x \) in Eq. (6) as a function of, (a) filling
of the lowest band \( \mu = \min[\epsilon]/\Delta \epsilon \) at \( J_1/J_1 = 0.6, 1.4 \) in
two colors, and (b) hopping ratio \( J_1'/J_1 \) at two values of filling
in different colors. Other parameters are the same as in Fig. 2.
Near \( \mathbf{k} = 0 \), \( \partial \Omega_z / \partial k_y \) becomes a constant. Thus BCD is proportional to the volume \( A_p(\mu) \) of Fermi sea, i.e., \( \Omega^{(1)}_z \propto A_p(\mu) \) in the low filling limit. As \( \mu \) increases, the volume of particle Fermi sea grows from zero, therefore BCD will increase with chemical potential. Besides, depending on the sign of \( 1 - \lambda \), i.e., \( 1 - \lambda_1/\lambda_1' \), BCD will change its sign as shown in Fig. 3.

At high fillings, we can expand the Hamiltonian near the top of energy band, i.e., around two Dirac points,

\[
H^{(1)}, H^{(2)} \simeq M \sigma_z \pm \left( \eta v_F p_y + \frac{1}{4} \lambda v_F (p_x^2 - p_y^2) \right) \sigma_x + \left( \lambda v_F p_x \pm \frac{1}{2} \eta v_F p_x p_y \right) \sigma_y,
\]

where \( \eta = \sqrt{(4 - \lambda^2)/3} \), \( v_F = -3J'/2 \), \( p = k - k_D \), and \( k_D \) is the location of the Dirac points. So the Berry curvature near two Dirac points are

\[
\int_{\mathbb{BZ}} \frac{d^2k}{(2\pi\hbar)^2} \frac{\partial \Omega_z}{\partial k_y} \simeq \frac{\eta \lambda M v_F^2}{4|M|} \chi.
\]

Note that in the high filling limit near the Dirac points, \( \partial \Omega_z / \partial k_y \) is also a constant. The Berry curvature dipole can be written into

\[
D^{(2)}_{zy} = \int_{\mathbb{BZ}} \frac{d^2k}{(2\pi\hbar)^2} \frac{\partial \Omega_z}{\partial k_y} f_0^h(k, \mu),
\]

\[
= - \int_{\mathbb{BZ}} \frac{d^2k}{(2\pi\hbar)^2} \frac{\partial \Omega_z}{\partial k_y} f_0^h(k, \mu),
\]

where \( f_0^h(k, \mu) \) is the distribution function of holes in the equilibrium. Here we have used the fact that \( \int_{\mathbb{BZ}} \frac{d^2k}{(2\pi\hbar)^2} \frac{\partial \Omega_z}{\partial k_y} = 0 \). At high filling, there are two Fermi surfaces of holes near two Dirac points, \( f_0^h(k, \mu) = f_0^{(1)}(k, \mu) + f_0^{(2)}(k, \mu) \). Thus,

\[
D^{(1)}_{zy} = - \int_{\mathbb{BZ}} \frac{d^2k}{(2\pi\hbar)^2} \left[ \frac{\partial \Omega_z^{(1)}}{\partial k_y} f_0^{(1)}(k, \mu) + \frac{\partial \Omega_z^{(2)}}{\partial k_y} f_0^{(2)}(k, \mu) \right].
\]

After integrating, we find that \( D^{(1)}_{zy} \propto A_h(\mu) \), where \( A_h(\mu) \) is the volume of Fermi sea of holes. As \( \mu \) approaches to the band top, the Fermi sea of holes decreases, and the BCD decreases towards zero.

**AC driving force.**—We further consider the case that the atoms are driven by an oscillating force, \( \mathbf{F}(t) = F_y e_y \cos(\omega t) \). Therefore, the quasimomentum of atoms is given by \( \mathbf{k}(t) = \mathbf{k}(0) - \frac{e \mathbf{F}}{\omega} \sin(\omega t) \). The transverse mean velocity can be expanded into

\[
\bar{v}_x^{\text{Hall}}(t) = - \frac{\omega}{N} \sum_{n=0}^{\infty} \frac{\sin^n(\omega t) \cos(\omega t)}{n!} D^{(n)}_{zy} \left( \frac{F_y}{\omega} \right)^{n+1},
\]

with \( D^{(n)}_{zy} = (-1)^n \int_{\mathbb{BZ}} \frac{d^2k}{(2\pi\hbar)^2} f_0^h(k, \mu)^n \). If the system has TRS, \( D^{(n)}_{zy} \) vanishes for even \( n \). Thus, in the case of TRS, the period of the oscillating Hall response is half of the period of the driving field \( T_d/2 = \pi/\omega \). If TRS is violated, the period of the Hall response is equal to the period of the driving field. In Fig. 4, we plot the mean Hall velocities in the systems with and without TRS.

In the small \( F_y/\omega \) limit, we can keep to the second order term in Eq. (9) as

\[
\bar{v}_x^{\text{Hall}}(t) = \frac{1}{N} \left( \cos(\omega t) D^{(0)}_{zy} F_y - \sin(2\omega t) D^{(1)}_{zy} \frac{F_y^2}{2\omega} \right).
\]

In Fig. 4, we plot the mean velocity \( \bar{v}_x \) calculated from semi-classical dynamics and from expansion (10). Note that in the small \( F_y/\omega \) limit, the Hall response of the oscillating longitudinal driving is almost harmonic (blue-dotted), and can be well captured by the expansion formula (asterisks). When TRS is broken, \( D^{(0)}_{zy} \neq 0 \), the linear parts dominate the Hall response. The maximum amplitude of oscillating transverse velocity will grow linearly with the driving force, as shown in Fig. 4(d). If the system has TRS, we have \( D^{(0)}_{zy} = 0 \), such that the amplitude of Hall velocity is a parabolic function of the driving strength, as depicted in Fig. 4(c).

Thus the Berry curvature dipole can be extracted directly from the amplitude of the transverse mean velocity.

**Conclusions.**—In summary, we found that the quantum non-linear Hall effect emerges naturally from the Bloch oscillation of ultracold atoms. The semi-classical dynamics features different behaviours with and without
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