GEPNER TYPE STABILITY CONDITION VIA
ORLOV/KUZNETSOV EQUIVALENCE

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Abstract. We show the existence of Gepner type Bridgeland stability conditions on the triangulated categories of graded matrix factorizations associated with homogeneous polynomials which define general cubic fourfolds containing a plane. The key ingredient is to describe the grade shift functor of matrix factorizations in terms of sheaves of Clifford algebras on the projective plane under Orlov/Kuznetsov equivalence.

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1. Introduction

1.1. Motivation and results. This paper is a continuation of the previous papers [Todb], [Toda]. For a homogeneous polynomial

\[ W \in A := \mathbb{C}[x_1, x_2, \cdots, x_n] \]

of degree \( d \), let \( \text{HMF}^{gr}(W) \) be the triangulated category of graded matrix factorizations of \( W \). It consists of objects given by data

\[ P^0 \xrightarrow{\nu^0} P^1 \xrightarrow{\nu^1} P^0(d) \]

where \( P^i \) are graded free \( A \)-modules of finite rank, \( \nu^i \) are homomorphisms of graded \( A \)-modules, \( P^i \mapsto P^i(1) \) is the shift of the grading, satisfying \( \nu^1 \circ \nu^0 = \nu^0 \circ \nu^1 = -W \). The category \( \text{HMF}^{gr}(W) \) is considered to be the category of \( B \)-branes of a graded Landau-Ginzburg model with superpotential \( W \), and studying Bridgeland stability conditions [Bri07] on it is an interesting problem both in mathematics and string theory. Among them, we focus on a specific type of a stability condition, which has a symmetric property with respect to the grade shift functor \( \tau \) on \( \text{HMF}^{gr}(W) \) sending \( P^* \) to \( P^*(1) \). The existence of such specific type of stability conditions is conjectural, and formulated as follows (cf. [Todb, Conjecture 1.2]):
Conjecture 1.1. There is a Bridgeland stability condition
\[ \sigma_G = (Z_G, \{ P_G(\phi) \}_{\phi \in \mathbb{R}}) \in \text{Stab}(\text{HMF}^{\text{gr}}(W)) \]
where the central charge \( Z_G \) is given by
\[ Z_G(P^\bullet) = \text{str}(e^{2\pi \sqrt{-1}/d} : P^\bullet \to P^\bullet) \]
and the set of semistable objects satisfy \( \tau P_G(\phi) = P_G(\phi + 2/d) \).

A Bridgeland stability condition in Conjecture 1.1 was called Gepner type with respect to \((\tau, 2/d)\) in [Todb], since if \(W\) is a quintic polynomial with five variables such a stability condition is expected to correspond to the Gepner point in the stringy Kähler moduli space of the quintic 3-fold defined by \(W\), under mirror symmetry and Orlov’s result relating \(\text{HMF}^{\text{gr}}(W)\) with the derived category of coherent sheaves on the hypersurface
\[ X := (W = 0) \subset \mathbb{P}^{n-1}. \]

While the above conjecture is motivated by string theory [Wal05], this is also an interesting mathematical problem as the resulting \(\sigma_G\) is an analogue of a Gieseker stability for coherent sheaves on polarized varieties. The Donaldson (Thomas) type theory counting \(\sigma_G\)-semistable graded matrix factorizations should be an analogue of Fan-Jarvis-Ruan-Witten theory [FJR] in Gromov-Witten theory. The symmetric property of \(\sigma_G\) may yield an interesting relationship among the classical Donaldson (Thomas) type invariants on the hypersurface (1) via the above mentioned Orlov’s result [Orl09] together with wall-crossing arguments [JS12, KS]. On the other hand, constructing \(\sigma_G\) in Conjecture 1.1 turns out to be a hard problem, due to the absence of a natural \(t\)-structure on \(\text{HMF}^{\text{gr}}(W)\). So far Conjecture 1.1 is proved in the following cases: \(n = 1\) [Tak], \(d < n = 3\) [KST07], \(n \leq d \leq 4\) [Todb], and some other weighted cases [KST07, Todb]. The strategy in [Todb] was to apply Orlov’s result [Orl09] and construct desired stability conditions in the geometric side.

Let us focus on the low degree cases of Conjecture 1.1. It is almost trivial to prove it in the \(d \leq 2\) cases for any \(n\) (cf. Remark 2.8), so the \(d = 3\) case is the non-trivial lowest degree case. The purpose of this paper is to prove Conjecture 1.1 for one of the \(d = 3\) cases in which there is an interesting geometric background: that is when \(X\) is a cubic fourfold. It has been long observed that the geometry of cubic fourfolds is very similar to that of K3 surfaces [BD85, Voi86]. That observation inspired some conjectures relating the rationality of cubic fourfolds with the existence of the corresponding K3 surfaces: the Hodge theoretic one is due to Hassett [Has03], the derived categorical one is due to Kuznetsov [Kuz10], and an equivalence of these conjectures (under a certain genericity condition) is due to Addington-Thomas [AT]. The main result of this paper, presented as follows, is an application of the above relationship between cubic fourfolds and K3 surfaces to the study of Conjecture 1.1:

**Theorem 1.2.** Conjecture 1.1 is true when \((d, n) = (3, 6)\) and the hypersurface \((W = 0) \subset \mathbb{P}^5\) is a general cubic fourfold containing a plane.

Our strategy is to combine Orlov’s work [Orl09] relating \(\text{HMF}^{\text{gr}}(W)\) with \(\text{D}^b\text{Coh}(X)\), with Kuznetsov’s work [Kuz10] relating the latter category with
the derived category of coherent sheaves on a twisted K3 surface. The result of Theorem 1.2 immediately follows from the corresponding result for the twisted K3 surfaces, given in Theorem 1.6 below. Our genericity condition and more detail on the proof of Theorem 1.2 will be given in the next subsection.

Here we should mention the case of cubic polynomials with lower number of variables, i.e. $d = 3$ and $n \leq 4$. In this case, Conjecture 1.1 is easier to prove than Theorem 1.2 and we give some detail in Appendix B:

**Theorem 1.3.** (Theorem 0.6, Theorem 0.9) Conjecture 1.1 is true when $d = 3$ and $n \leq 5$.

1.2. **Strategy for Theorem 1.2 via Orlov/Kuznetsov equivalence.**

Let

$$X = (W = 0) \subset \mathbb{P}^5$$

be a smooth cubic fourfold containing a plane $P$. The bounded derived category of coherent sheaves on $X$ has the following semiorthogonal decomposition

$$\mathcal{D}^b \text{Coh}(X) = \langle \mathcal{D}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle.$$

There are two ways to relate the semiorthogonal summand $\mathcal{D}_X$ with other triangulated categories. The first one is due to Orlov [Orl09], which provides an equivalence

$$\Phi_1: \text{HMF}^{gr}(W) \xrightarrow{\sim} \mathcal{D}_X.$$

The second one is due to Kuznetsov [Kuz10], which provides an equivalence

$$\Theta: \mathcal{D}^b \text{Coh}(B_0) \xrightarrow{\sim} \mathcal{D}_X.$$

Here $B_0$ is an even part of a sheaf of Clifford algebras on $\mathbb{P}^2$, which is constructed in [Kuz04] from a quadric fibration $\tilde{X} \rightarrow \mathbb{P}^2$ for a blow-up $\tilde{X} \rightarrow X$ at $P$. The construction also defines an odd part $B_1$, which is $B_0$-bimodule, and other $B_i$ are defined by $B_{i+2} = B_i(1)$. Our first step is to relate the grade shift functor $\tau$ with the autoequivalence $F_B$ of $\mathcal{D}^b \text{Coh}(B_0)$, given as

$$F_B := \text{ST}^{-1}_{B_1} \circ \otimes_{B_0} B_{-1}[1].$$

Here $\text{ST}_{B_1}$ is the Seidel-Thomas twist [ST01] associated to $B_1$. We show the following proposition:

**Proposition 1.4.** (Corollary 3.4) The following diagram commutes:

$$\begin{array}{ccc}
\mathcal{D}^b \text{Coh}(B_0) & \xrightarrow{\Phi_1^{-1} \circ \Theta} & \text{HMF}^{gr}(W) \\
F_B \downarrow & & \downarrow \tau \\
\mathcal{D}^b \text{Coh}(B_0) & \xrightarrow{\Phi_1^{-1} \circ \Theta} & \text{HMF}^{gr}(W).
\end{array}$$

The above proposition is proved by combining Ballard-Favero-Katzarkov’s work [BFK12] describing $\tau$ in terms of $\mathcal{D}_X$, with an explicit computation of $\Theta(B_1)$ and Lahoz-Macri-Stellari’s work [LMS] describing the image of point like objects in $\mathcal{D}^b \text{Coh}(B_0)$ under $\Theta$. 
The next step is to describe the central charge $Z_G$ in terms of twisted sheaves on a K3 surface. By [Kuz08], if $X$ is a general cubic fourfold containing a plane, the sheaf of non-commutative algebras $B_0$ is a push-forward of a sheaf of Azuyama algebras $B_S$ on a smooth K3 surface obtained as a double cover $S \to \mathbb{P}^2$. The category of right $B_S$-modules is equivalent to $\text{Coh}(S, \alpha)$, which is the category of coherent sheaves on $S$ twisted by an element in the Brauer group $\alpha \in \text{Br}(S)$ with $\alpha^2 = 1$. This provides another equivalence

$$\Upsilon : D^b \text{Coh}(S, \alpha) \xrightarrow{\sim} D^b \text{Coh}(B_0).$$

As a summary, there is a sequence of equivalences

$$D^b \text{Coh}(S, \alpha) \xrightarrow{\Upsilon} D^b \text{Coh}(B_0) \xrightarrow{\Theta} D_X \xrightarrow{\Phi} \text{HMF}^{gr}(W).$$

We compute the pull-back of the central charge $Z_G$ on $\text{HMF}^{gr}(W)$ by the above sequence of equivalences, using the result of Proposition 1.4. The resulting central charge on $D^b \text{Coh}(S, \alpha)$ coincides with an integral over $S$ which appeared in Bridgeland’s paper [Bri08]:

Proposition 1.5. (Proposition 4.7.) There is an element $\mathcal{B} \in H^{1,1}(S, \mathbb{Q})$ and $c \in \mathbb{C}^*$ such that we have

$$Z_G \circ \Phi^{-1} \circ \Theta \circ \Upsilon(E) = c \cdot \int_S e^{\mathcal{B} - \frac{1}{\sqrt{-1}} h \cdot \text{ch}(E) \sqrt{\text{td}S}}$$

for any $E \in D^b \text{Coh}(S, \alpha)$. Here $h$ is a hyperplane in $\mathbb{P}^2$ pulled back to $S$.

The Chern character on $D^b(S, \alpha)$ is the untwisted Chern character, defined to be the twisted Chern character by Huybrechts-Stellari [HS05], multiplied by the exponential of the minus of the B-field (cf. Subsection 4.2) to get back to the untwisted one. Although it takes values in algebraic classes, it is no longer defined in the integer coefficient.

The final step is to construct a corresponding Gepner type stability condition on $D^b \text{Coh}(S, \alpha)$, using the above descriptions of the grade shift functor and the central charge. In this step, we need a further genericity assumption: the Brauer class $\alpha$ is non-trivial. This condition is not satisfied only if $X$ lies in a union of countable many hypersurfaces in the moduli space of cubic fourfolds containing a plane. The following result obviously implies Theorem 1.2 as desired:

Theorem 1.6. (Theorem 4.13.) Suppose that $\alpha \neq 1$. Then there is a Gepner type stability condition on $D^b \text{Coh}(S, \alpha)$ with respect to $(\Upsilon^{-1} \circ F_B \circ \Upsilon, 2/3)$, whose central charge is given by $Z_G \circ \Phi^{-1} \circ \Theta \circ \Upsilon$.

The $\alpha \neq 1$ condition puts a strong constraint on the image of $Z_G$, due to the non-existence of twisted line bundles, which makes it possible to prove Theorem 1.6.

The outline of the paper is as follows: in Section 2, we review some background on stability conditions, graded matrix factorizations, Orlov/Kuznetsov equivalence, etc. In Section 3, we prove Proposition 1.4. In Section 4, we prove Proposition 1.5 and Theorem 1.6. In Appendix A, we review Chern characters on graded matrix factorizations by Polishchuk-Vaintrob [PV12], and shows that the central charge $Z_G$ is numerical. In Appendix B, we use...
the work by Bernardara-Macri-Mehrotra-Stellari [BMMS12] to prove Theorem 1.6.

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2. Preliminary

This section is devoted to giving some preliminary background required in this paper.

2.1. Numerical Bridgeland stability condition. Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category, satisfying

$$
\sum_{i \in \mathbb{Z}} \dim \text{Hom}(E, F[i]) < \infty
$$

for any $E, F \in \mathcal{D}$. Let $K(\mathcal{D})$ be the Grothendieck group of $\mathcal{D}$. There is an Euler bilinear pairing $\chi$ on $K(\mathcal{D})$, defined by

$$
\chi(E, F) := \sum_i (-1)^i \dim \text{Hom}(E, F[i]).
$$

The numerical Grothendieck group of $\mathcal{D}$ is defined to be

$$
N(\mathcal{D}) := K(\mathcal{D})/\equiv
$$

where $E \equiv E'$ if $\chi(E, F) = \chi(E', F)$ for any $F \in \mathcal{D}$. In what follows, we always assume that $N(\mathcal{D})$ is finitely generated. This condition is satisfied when $\mathcal{D} = D^b \text{Coh}(X)$ for a smooth projective variety $X$, and $N(\mathcal{D})$ is denoted by $N(X)$ in this case. Let us recall numerical Bridgeland stability conditions on $\mathcal{D}$.

**Definition 2.1.** (Bri07) A numerical stability condition on $\mathcal{D}$ consists of data

$$
Z : N(\mathcal{D}) \to \mathbb{C}, \quad \mathcal{A} \subset \mathcal{D}
$$

where $Z$ is a group homomorphism (called central charge) and $\mathcal{A}$ is the heart of bounded t-structure on $\mathcal{D}$, satisfying the following conditions:

- For any $0 \neq E \in \mathcal{A}$, we have

$$
Z(E) \in \mathbb{H} := \{ r \exp(\sqrt{-1}\pi\phi) : r > 0, 0 < \phi \leq 1 \}.
$$

- Any object $E \in \mathcal{A}$ admits a filtration (called Harder-Narasimhan filtration)

$$
0 = E_0 \subset E_1 \subset \cdots \subset E_N = E
$$

such that each subquotient $F_i = E_i / E_{i-1}$ is $Z$-semistable satisfying $\arg Z(F_i) > \arg Z(F_{i+1})$ for all $i$.

Here an object $F \in \mathcal{A}$ is $Z$-(semi)stable if for any non-zero $F' \subset F$, we have the inequality $\arg Z(F') \leq (<) \arg Z(F)$ in $(0, \pi]$.

Remark 2.2. By the construction, the Euler pairing $\chi$ descends to the perfect pairing
\[ \chi : N(D) \times N(D) \to \mathbb{Z}. \]
Therefore the central charge $Z$ is always written as $\chi(u, -)$ for some $u \in \mathbb{C}$.

Another way to give a numerical stability condition is to giving data
\begin{equation}
(Z, \{ P(\phi) \}_{\phi \in \mathbb{R}})
\end{equation}
where $Z : N(D) \to \mathbb{C}$ is a group homomorphism, $P(\phi) \subset D$ are full subcategories (called semistable objects with phase $\phi$) satisfying some axiom [Bri07, Definition 1.1]. Given data (5), the subcategories $P(\phi)$ for $0 < \phi \leq 1$ are defined to be $Z$-semistable objects $E \in A$ such that the argument of $Z(E)$ coincides with $\pi\phi$. Other $P(\phi)$ are defined by the rule $P(\phi + 1) = P(\phi)[1]$. Conversely given data (5), the heart $A$ is given by the extension closure of $P(\phi)$ for $0 < \phi \leq 1$. For the detail, see [Bri07, Proposition 5.3]. The space of numerical stability conditions is defined as follows:

Definition 2.3. The set $\text{Stab}(D)$ is defined to be the set of numerical stability conditions (3) on $D$, satisfying the support property:
\[ \sup \left\{ \frac{\|E\|}{Z(E)} \right\} : 0 \neq E \in A \text{ is } Z \text{-semistable} < \infty. \]
Here $\|*\|$ is a fixed norm on $N(D)_{\mathbb{R}}$.

In [Bri07] (also see [KS]), Bridgeland shows that there is a natural topology on $\text{Stab}(D)$ such that the forgetting map
\begin{equation}
Z : \text{Stab}(D) \to N(D)^{\vee}_{\mathbb{C}}
\end{equation}
sending a stability condition to its central charge is a local homeomorphism. In particular, $\text{Stab}(D)$ is a complex manifold.

Let $\text{Aut}(D)$ be the group of autoequivalence on $D$. There is a left $\text{Aut}(D)$-action on $\text{Stab}(D)$. For $\Phi \in \text{Aut}(D)$, it acts on (5) as
\[ \Phi_*(Z, \{ P(\phi) \}_{\phi \in \mathbb{R}}) = (Z \circ \Phi^{-1}, \{ \Phi(P(\phi)) \}_{\phi \in \mathbb{R}}). \]
There is also a right $\mathbb{C}$-action on $\text{Stab}(D)$. For $\lambda \in \mathbb{C}$, it acts on (5) as
\[ (Z, \{ P(\phi) \}_{\phi \in \mathbb{R}}) \cdot (\lambda) = (e^{-\sqrt{-1}\pi\lambda}Z, \{ P(\phi + \text{Re } \lambda) \}_{\phi \in \mathbb{R}}). \]
The notion of Gepner type stability conditions is defined as follows:

Definition 2.4. ([Todb]) A numerical stability condition $\sigma \in \text{Stab}(D)$ is called Gepner type with respect to $(\Phi, \lambda) \in \text{Aut}(D) \times \mathbb{C}$ if the following condition holds:
\[ \Phi_\ast \sigma = \sigma \cdot (\lambda). \]

2.2. Gepner type stability conditions on graded matrix factorizations. Let $W$ be a homogeneous element
\begin{equation}
W \in A := \mathbb{C}[x_1, x_2, \cdots, x_n]
\end{equation}
of degree $d$ such that $(W = 0) \subset \mathbb{C}^n$ has an isolated singularity at the origin. For a graded $A$-module $P$, we denote by $P_i$ its degree $i$-part, and $P(k)$ the graded $A$-module whose grade is shifted by $k$, i.e. $P(k)_i = P_{i+k}$. 
Definition 2.5. A graded matrix factorization of $W$ is data

$$P^0 \xrightarrow{p^0} P^1 \xrightarrow{p^1} P^0(d)$$

where $P^i$ are graded free $A$-modules of finite rank, $p^i$ are homomorphisms of graded $A$-modules, satisfying the following conditions:

$$p^1 \circ p^0 = -W, \quad p^0(d) \circ p^1 = -W.$$

The category $\text{HMF}^\text{gr}(W)$ is defined to be the homotopy category of the dg-category of graded matrix factorizations of $W$ (cf. [Orl09]). The grade shift functor $P^\bullet \mapsto P^\bullet(1)$ induces the autoequivalence $\tau$ of $\text{HMF}^\text{gr}(W)$, which satisfies the following identity:

$$\tau^{\times d} = [2].$$

Remark 2.6. There is a Serre functor on $\text{HMF}^\text{gr}(W)$ given by (cf. [KST09, Theorem 3.8])

$$S_W = \tau^{d-n}[n-2].$$

In particular, $S_W^{\times d} = [nd-2]$, and $\text{HMF}^\text{gr}(W)$ is interpreted as a fractional Calabi-Yau category with dimension $n(1-2/d)$. This fact will be used in Appendix B.

Since $(W = 0) \subset \mathbb{C}^n$ has an isolated singularity at the origin, the triangulated category $\text{HMF}^\text{gr}(W)$ satisfies the condition (2), and it is finitely generated. (For instance, use Orlov’s result in Theorem 2.10 below.) The following is the numerical version of [Todb, Conjecture 1.2]:

Conjecture 2.7. There is a Gepner type stability condition

$$\sigma_G = (Z_G, \{P_G(\phi)\}_{\phi \in \mathbb{R}}) \in \text{Stab}(\text{HMF}^\text{gr}(W))$$

with respect to $(\tau, 2/d)$, whose central charge $Z_G$ is given by

$$Z_G(P^\bullet) = \text{str}(e^{2\pi \sqrt{-1}/d} : P^\bullet \to P^\bullet).$$

The $e^{2\pi \sqrt{-1}/d}$-action on $P^\bullet = P^0 \oplus P^1$ is induced by the $Z$-grading on each $P^i$, and ‘str’ is the supertrace which respects the $\mathbb{Z}/2\mathbb{Z}$-grading on $P^\bullet$. The definition of the central charge $Z_G$ first appeared in [Wal05]. It is more precisely written as follows: since $P^i$ are free $A$-modules of finite rank, they are written as

$$P^i \cong \bigoplus_{j=1}^m A(n^i_j), \quad n^i_j \in \mathbb{Z}.$$ 

Then (11) is written as

$$Z_G(P^\bullet) = \sum_{j=1}^m \left( e^{2n^0_j \pi \sqrt{-1}/d} - e^{2n^1_j \pi \sqrt{-1}/d} \right).$$

Remark 2.8. In the low degree cases of Conjecture 2.7, there is nothing to prove in the $d = 1$ case as $\text{HMF}^\text{gr}(W) = \{0\}$ in this case. In the $d = 2$ case, the Knörrer periodicity [Kno87] allows us to reduce to the case of $n = 1$ or $n = 2$, and Conjecture 2.7 in these cases are checked in [Tak], [Todb].
The following lemma, which is an obvious necessary condition for Conjecture 2.7, is an easy consequence of an interpretation of $Z_G$ in terms of a Chern character of graded matrix factorizations, and Hirzebruch-Grothendieck Riemann-Roch formula [PV12]. The detail will be provided in Appendix A.

**Lemma 2.9.** The central charge $Z_G$ factors through the canonical surjection $K(\text{HMF}^{\text{gr}}(W)) \to N(W)$. In particular, it is written as

$$Z_G(P^\bullet) = \chi(u, P^\bullet)$$

for some $u \in N(W) \subset \mathbb{N}(W)$ with $\tau^{-1}u = e^{2\pi \sqrt{-1}/d}u$.

We now recall Orlov’s theorem [Orl09] which relates $\text{HMF}^{\text{gr}}(W)$ with the derived category of coherent sheaves on the hypersurface $X := (W = 0) \subset \mathbb{P}^{n-1}$ by semiorthogonal decompositions (SOD for short). Since we only use the case of $n > d$, we give a statement in this case.

**Theorem 2.10.** ([Orl09, Theorem 2.5]) If $n > d$, then there is a fully faithful embedding for each $i \in \mathbb{Z}$

$$\Phi_i : \text{HMF}^{\text{gr}}(W) \hookrightarrow D^b \text{Coh}(X)$$

and SOD

$$D^b \text{Coh}(X) = \langle \mathcal{O}_X(-i - n + d + 1), \cdots, \mathcal{O}_X(-i), \Phi_i \text{HMF}^{\text{gr}}(W) \rangle. \quad (12)$$

### 2.3. Cubic fourfolds containing a plane and K3 surfaces.

Let $X$ be a cubic fourfold which contains a plane $P$

$$\mathbb{P}^2 = P \subset X = (W = 0) \hookrightarrow \mathbb{P}^5.$$

We recall a relationship between such cubic fourfolds and K3 surfaces obtained as double covers of $\mathbb{P}^2$. Let

$$p : \tilde{\mathbb{P}}^5 \to \mathbb{P}^5, \quad \sigma : \tilde{X} \to X$$

be the blow-ups at the plane $P \subset \mathbb{P}^5$ and $P \subset X$ respectively. The exceptional divisors of $p, \sigma$ are denoted by $D' \subset \tilde{\mathbb{P}}^5$, $D' \subset \tilde{\mathbb{P}}^5$ respectively. The linear projection from $\tilde{P}$ gives morphisms

$$q : \tilde{\mathbb{P}}^5 \to \mathbb{P}^2, \quad \pi := q \circ j : \tilde{X} \to \mathbb{P}^2$$

where $j$ is the inclusion $\tilde{X} \hookrightarrow \tilde{\mathbb{P}}^5$. The morphism $q$ is the projectivization of the following rank four vector bundle $E$ on $\mathbb{P}^2$.

$$E = \mathcal{O}_{\mathbb{P}^2} e_1 \oplus \mathcal{O}_{\mathbb{P}^2} e_2 \oplus \mathcal{O}_{\mathbb{P}^2} e_3 \oplus \mathcal{O}_{\mathbb{P}^2}(-1) f. \quad (13)$$

We will usually abbreviate the basis elements $e_1, e_2, e_3, f$ unless it is necessary to specify them. Below we denote by $h, H$ (resp. $h', H'$) the classes of hyperplanes in $\mathbb{P}^2$, $\mathbb{P}^5$ pulled back to $\tilde{X}$ (resp. $\tilde{\mathbb{P}}^5$) respectively. Note that we have the following relations:

$$D = H - h, \quad D' = H' - h'. \quad (14)$$

Hence we have

$$\tilde{X} \in |3H' - D'| = |2H' + h'|.$$
In particular, $\pi$ is a quadric fibration, and the defining equation of $\tilde{X}$ gives a morphism
\begin{equation}
(15) \quad s: \mathcal{O}_{\mathbb{P}^2}(-1) \to \text{Sym}^2 E^\vee.
\end{equation}

The morphism $s$ induces the morphism
\begin{equation}
(16) \quad s': E \to E^\vee(1).
\end{equation}

The morphism $\pi$ has degenerated fibers along the zero locus of $\det(s') \in \text{Hom}(\det E, \det(E^\vee(1))) \cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$ which is a sextic $C \subset \mathbb{P}^2$. Let $f: S \to \mathbb{P}^2$ be the double cover branched along $C$. The curve $C$ is non-singular for a general cubic fourfold containing a plane. In what follows, we assume that the cubic fourfold $X$ is general so that $C$ is non-singular. The covering involution of $f$ is denoted by $\iota$, and (by abuse of notation) we denote by $h$ the class of a hyperplane in $\mathbb{P}^2$ pulled back to $S$. The relevant diagram in this subsection is summarized below:

2.4. Sheaves of Clifford algebras and twisted K3 surfaces. Similarly to the classical construction of Clifford algebras, the morphism (15) defines the sheaf of Clifford algebras $B_s$ on $\mathbb{P}^2$ (cf. [Kuz08, Section 3]). It has an even part $B_0$, which is described as
\begin{equation}
(18) \quad B_0 = \mathcal{O}_{\mathbb{P}^2} \oplus (\wedge^2 E \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \oplus (\wedge^4 E \otimes \mathcal{O}_{\mathbb{P}^2}(-2)) \\
\cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-3).
\end{equation}

It also has an odd part $B_1$, given by
\begin{equation}
(19) \quad B_1 = E \oplus (\wedge^3 E \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \\
\cong \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3}.
\end{equation}

We also define $B_i$ for $i \in \mathbb{Z}$ by the rule $B_{i+2} = B_i(1)$. By [Kuz08, Corollary 3.9], every sheaves $B_i$ are flat over $B_0$ and we have
\begin{equation}
(20) \quad B_i \otimes_{B_0} B_j \cong B_{i+j}, \quad \text{for all } i, j \in \mathbb{Z}.
\end{equation}

In particular, for every $i$ there is an equivalence of abelian categories
\begin{equation}
(21) \quad \otimes_{B_0} B_i: \text{Coh}(B_0) \xrightarrow{\sim} \text{Coh}(B_0).
\end{equation}

Here Coh($B_0$) is the abelian category of coherent right $B_0$-modules on $\mathbb{P}^2$. 

\begin{itemize}
\item \text{Diagram image:}
\item \text{Explanation of diagram:}
\item \text{Diagram explanation:}
\item \text{Subsection 2.4:}
\item \text{Sheaves of Clifford algebras and twisted K3 surfaces:}
\item \text{Construction of Clifford algebras:}
\item \text{Morphism (15):}
\item \text{Even part $B_0$:}
\item \text{Odd part $B_1$:}
\item \text{Sheaves $B_i$:}
\item \text{Equivalence:}
\item \text{Abelian category:}
\end{itemize}
Let $S$ be the K3 surface obtained as a double cover \cite{Kuz10}. By Kuznetsov Section 3.5], there exists a sheaf of Azumaya algebras $B_S$ on $S$ such that $f_*B_S = B_0$, and an equivalence
\begin{equation}
(22) \quad f_* : \text{Coh}(B_S) \iso \text{Coh}(B_0).
\end{equation}

The abelian categories $\text{Coh}(B_0)$, $\text{Coh}(B_S)$ are also described in terms of twisted sheaves. There exists an element in the Brauer group
\[ \alpha \in \text{Br}(S) = H^2(S, \mathcal{O}_S^*), \quad \alpha^2 = \text{id} \]
and an $\alpha$-twisted vector bundle $\mathcal{U}_0$ of rank two such that $B_S = \text{End}(\mathcal{U}_0)$ and the functor
\[ \text{Coh}(S, \alpha) \ni F \mapsto \mathcal{U}_0 \otimes F \in \text{Coh}(B_S) \]
is an equivalence. Here $\text{Coh}(S, \alpha)$ is the abelian category of $\alpha$-twisted coherent sheaves on $S$ (cf. \cite{HS05}, Section 1). Combined with the above equivalences, we obtain the equivalence
\begin{equation}
(23) \quad \Upsilon(\cdot) := f_* (\mathcal{U}_0 \otimes \cdot) : D^b \text{Coh}(S, \alpha) \iso D^b \text{Coh}(B_0).
\end{equation}

2.5. Orlov/Kuznetsov equivalence. Let $\mathcal{D}_X$ be the semiorthogonal summand of $D^b \text{Coh}(X)$, defined by
\begin{equation}
(24) \quad D^b \text{Coh}(X) = \langle \mathcal{D}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle.
\end{equation}

In \cite{Kuz10}, Kuznetsov established an equivalence between $\mathcal{D}_X$ and $D^b \text{Coh}(B_0)$. A starting point is the fully faithful functor
\[ \Phi : D^b \text{Coh}(B_0) \to D^b \text{Coh}(\tilde{X}) \]
constructed in \cite{Kuz10}, defined as a Fourier-Mukai transform
\begin{equation}
(25) \quad \Phi(\cdot) = \pi_* (\cdot) \otimes \pi^* B_0 \mathcal{E}.
\end{equation}

Here $\mathcal{E}$ is a sheaf of left $\pi^* B_0$-modules on $\tilde{X}$ constructed as follows: by \cite{Kuz10} Corollary 3.12, there are injections as left $q^* \mathcal{B}_0$-modules for each $i \in \mathbb{Z}$
\begin{equation}
(26) \quad \delta_i : q^* \mathcal{B}_i \to q^* \mathcal{B}_{i+1}(H').
\end{equation}

By an abuse of notation, we will also denote by $\delta_i$ the twist of the above morphism by any line bundle. Then $j_* \mathcal{E}$ is given by the cokernel of the above morphism for $i = 0$
\begin{equation}
(27) \quad 0 \to q^* \mathcal{B}_0(-2H') \xrightarrow{\delta_0} q^* \mathcal{B}_1(-H') \to j_* \mathcal{E} \to 0.
\end{equation}

As $\mathcal{O}_{\tilde{X}}$-module, the sheaf $\mathcal{E}$ is locally free of rank four.

Kuznetsov \cite{Kuz10} performs a sequence of mutations of SOD of $D^b \text{Coh}(\tilde{X})$, and replaces $\Phi$ by another fully faithful functor $\Phi''$
\[ \Phi'' := \text{L}_{\mathcal{O}_{\tilde{X}}(h-H)} \circ \text{R}_{\mathcal{O}_{\tilde{X}}(-h)} \circ \Phi : D^b \text{Coh}(B_0) \to D^b \text{Coh}(\tilde{X}) \]
where $\text{L}_{\mathcal{O}_{\tilde{X}}(h-H)}$ and $\text{R}_{\mathcal{O}_{\tilde{X}}(-h)}$ are defined to be
\begin{align*}
\text{L}_{\mathcal{O}_{\tilde{X}}(h-H)}(\cdot) & := \text{Cone} \left( \text{R} \text{Hom}(\mathcal{O}_{\tilde{X}}(h-H), \cdot) \otimes \mathcal{O}_{\tilde{X}}(h-H) \to - \right) \\
\text{R}_{\mathcal{O}_{\tilde{X}}(-h)}(\cdot) & := \text{Cone} \left( - \to \text{R} \text{Hom}(-, \mathcal{O}_{\tilde{X}}(-h))^\vee \otimes \mathcal{O}_{\tilde{X}}(-h) \right) [-1].
\end{align*}
Then it is shown that the image of $\Phi''$ coincides with the image of the pull-back of the blow-up $\sigma: \tilde{X} \to X$ restricted to $D_X$. Applying $R\sigma_*$, the following result is obtained in [Kuz10]:

**Theorem 2.11.** ([Kuz10]) The functor

$$\Theta := R\sigma_* \circ \Phi'' : D^b \text{Coh}(B_0) \xrightarrow{\sim} D_X$$

is an equivalence.

It is useful to write $\Theta(F)$ for $F \in D^b \text{Coh}(B_0)$ as follows (cf. [Kuz10, Theorem 4.3, Step 7]):

$$\Theta(F) = \{ R\text{Hom}(O_{\tilde{X}}(h-H), \Phi(F)) \otimes I_P \rightarrow R\sigma_* \Phi(F) \\
\rightarrow R \text{Hom}(\Phi(F), O_{\tilde{X}}(-h)) \otimes O_X(-1) \}.$$

Here $I_P \subset O_X$ is the ideal sheaf of $P$, which is easily checked to be an object in $D_X$.

Now we combine $\Theta$ with Orlov equivalence. Note that, since $\omega_X = O_X(-3)$, the SOD (24) induces another SOD

$$D^b \text{Coh}(X) = \langle O_X(-3), O_X(-2), O_X(-1), D_X \rangle.$$

Therefore Theorem 2.10 yields an equivalence

$$\Phi_1 : \text{HMF}_{gr}(W) \xrightarrow{\sim} D_X.$$

We summarize the equivalences obtained so far in the following corollary:

**Corollary 2.12.** There is a sequence of equivalences

$$D^b \text{Coh}(S, \alpha) \xrightarrow{\Upsilon} D^b \text{Coh}(B_0) \xrightarrow{\Theta} D_X \xrightarrow{\Phi_1} \text{HMF}_{gr}(W).$$

Here $\Upsilon$ is given in (23), $\Theta$ is given in Theorem 2.11 and $\Phi_1$ is given in Theorem 2.10.

3. Description of the grade shift functor

The purpose of this section is to prove Proposition 1.4. In what follows, we always assume that $X$ is a cubic fourfold containing a plane $P$, which is general so that the associated K3 surface $S$ is smooth (cf. Subsection 2.3).

3.1. Summary of the result. Let us consider the equivalence in Corollary 2.12

$$\Phi_1^{-1} \circ \Theta : D^b \text{Coh}(B_0) \xrightarrow{\sim} \text{HMF}_{gr}(W).$$

We are going to describe the grade shift functor $\tau$ on $\text{HMF}_{gr}(W)$ in terms of $D^b \text{Coh}(B_0)$ under the above equivalence. We first recall the description of $\tau$ in terms of the autoequivalence in $D_X$, given in [BFK12]. Let us consider the functor

$$F_X : D^b \text{Coh}(X) \rightarrow D^b \text{Coh}(X).$$

defined to be

$$F_X(-) := \text{Cone}(R\text{Hom}(O_X, - \otimes O_X(1)) \otimes O_X \rightarrow - \otimes O_X(1)).$$

The functor $F_X$ preserves $D_X$, and by [LMS] Lemma 1.10 it gives an autoequivalence of $D_X$. 
Proposition 3.1. ([BFK12 Proposition 5.8]) The following diagram commutes:

\[
\begin{array}{ccc}
\text{HMF}^{gr}(W) & \xrightarrow{\Phi_1} & D_X \\
\tau & & \downarrow F_X \\
\text{HMF}^{gr}(W) & \xrightarrow{\Phi_1} & D_X.
\end{array}
\]

By the above result, it is enough to describe the functor \( F_X \) in terms of \( D^b \text{Coh}(B_0) \). We observe the following:

Lemma 3.2. For all \( i \in \mathbb{Z} \), we have

\[
\begin{align*}
\mathbf{R} \text{Hom}_{B_0}(B_i, B_i) & \cong \mathbb{C} \oplus \mathbb{C}[-2] \\
\mathbf{R} \text{Hom}_{B_0}(B_i, B_{i+1}) & \cong \mathbb{C}^3 \\
\mathbf{R} \text{Hom}_{B_0}(B_i, B_{i+2}) & \cong \mathbb{C}^6.
\end{align*}
\]

Proof. By the equivalence \([21]\), we may assume that \( i = 0 \). Then the result easily follows from

\[
\mathbf{R} \text{Hom}_{B_0}(B_0, B_k) \cong \mathbf{R} \text{Hom}_{\mathcal{O}_{P^2}}(\mathcal{O}_{P^2}, B_k).
\]

As noted in [SM12 Remark 2.1], the above lemma shows that the objects \( B_i \) are spherical objects [ST01]. The associated spherical twists and their inverses are given by

\[
\begin{align*}
\text{ST}_{B_i}(-) & := \text{Cone} (\mathbf{R} \text{Hom}(B_i, -) \otimes B_i \to -) \\
\text{ST}^{-1}_{B_i}(-) & := \text{Cone} (- \to \mathbf{R} \text{Hom}(-, B_i)^\vee \otimes B_i) [-1].
\end{align*}
\]

The above functors are aut-equivalences of \( D^b \text{Coh}(B_0) \). Combined with \([21]\), we define the following aut-equivalence

\[
F_B := \text{ST}^{-1}_{B_1} \circ \otimes_{B_0} B_{-1}[1].
\]

The following proposition is the main result in this section:

Proposition 3.3. The following diagram commutes:

\[
\begin{array}{ccc}
D^b \text{Coh}(B_0) & \xrightarrow{\Theta} & D_X \\
F_B & & \downarrow F_X \\
D^b \text{Coh}(B_0) & \xrightarrow{\Theta} & D_X.
\end{array}
\]

Combined with Proposition 3.1 and \([9]\), we obtain the following corollary:

Corollary 3.4. The following diagram commutes:

\[
\begin{array}{ccc}
D^b \text{Coh}(B_0) & \xrightarrow{\Phi_1 \circ \Theta} & \text{HMF}^{gr}(W) \\
F_B & & \downarrow \tau \\
D^b \text{Coh}(B_0) & \xrightarrow{\Phi_1 \circ \Theta} & \text{HMF}^{gr}(W).
\end{array}
\]

In particular \( F_B^3 \) is isomorphic to \([2]\).

A proof of Proposition 3.3 will be given in Subsection 3.6.
3.2. Explicit description of $\delta_i$. The purpose of this subsection is to give an explicit description of $\delta_i$ in \eqref{eq:sym} in terms of the cubic polynomial $W$, which will be relevant in some computations required in the proof of Proposition \ref{prop:3.3}. Let

$$[x_1 : x_2 : x_3 : x_4 : x_5 : x_6]$$

be the homogeneous coordinate of $\mathbb{P}^5$. Without loss of generality, we may assume that

$$P = \{x_1 = x_2 = x_3 = 0\} \subset \mathbb{P}^5.$$ 

Since $X$ contains $P$, the homogeneous polynomial $W$ is written as

$$W = W'(x_1, x_2, x_3) + \sum_{1 \leq i, j \leq 3} x_i x_j W_{ij}(x_4, x_5, x_6) + \sum_{1 \leq i \leq 3} x_i W_i(x_4, x_5, x_6)$$

such that

- $W'(x_1, x_2, x_3)$ is a homogeneous cubic polynomial in $x_1, x_2, x_3$.
- $W_{ij}(x_1, x_2, x_3)$ is a linear combination of $x_4, x_5, x_6$.
- $W_i(x_4, x_5, x_6)$ is a homogeneous quadric polynomial in $x_4, x_5, x_6$.

Let us take an element

$$x_7 \in H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(D'))$$

which corresponds to 1 under the natural isomorphism

$$H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}) \xrightarrow{\sim} H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(D')).$$

Since $p$ is a blow-up at $P$, for $1 \leq i \leq 3$ we have

$$x_i = x_7 y_i \text{ for some } y_i \in H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(H' - D')).$$

Here by abuse of notation, the pull-back of $x_i \in H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1))$ to $\mathbb{P}^5$ is denoted by the same symbol $x_i$. We define the following polynomial

$$\tilde{W} := W'(y_1, y_2, y_3) x_7^2 + \sum_{1 \leq i, j \leq 3} y_i y_j x_7 W_{ij}(x_4, x_5, x_6) + \sum_{1 \leq i \leq 3} y_i W_i(x_4, x_5, x_6).$$

By \eqref{eq:3.3}, the above polynomial makes sense as

$$\tilde{W} \in H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2H' + h')).$$

The above polynomial $\tilde{W}$ is the defining equation of $\tilde{X}$ in $\mathbb{P}^5$, and the morphism $s'$ in \eqref{eq:3.3} is given by the Hessian of $\tilde{W}$, i.e. by regarding local sections of $E$, $E'$ as column vectors with respect to the basis \eqref{eq:basis} and its dual basis $e_1^*, e_2^*, e_3^*$, $f^*$, and setting $\partial_i := \partial / \partial x_i$, the morphism $s'$ is written as a matrix

$$s' = \frac{1}{2} \begin{pmatrix}
\partial_4 \partial_1 \tilde{W} & \partial_4 \partial_2 \tilde{W} & \partial_4 \partial_3 \tilde{W} & \partial_4 \partial_7 \tilde{W} \\
\partial_5 \partial_1 \tilde{W} & \partial_5 \partial_2 \tilde{W} & \partial_5 \partial_3 \tilde{W} & \partial_5 \partial_7 \tilde{W} \\
\partial_6 \partial_1 \tilde{W} & \partial_6 \partial_2 \tilde{W} & \partial_6 \partial_3 \tilde{W} & \partial_6 \partial_7 \tilde{W} \\
\partial_7 \partial_1 \tilde{W} & \partial_7 \partial_2 \tilde{W} & \partial_7 \partial_3 \tilde{W} & \partial_7 \partial_7 \tilde{W}
\end{pmatrix}.$$ 

Here we regard $y_i$ as an element in $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ by the relation \eqref{eq:3.3}. 

Hence we have 
\[ M(38) \]
Hence (ii) follows using the right contraction. The morphism (26) is obtained by the composition:

Here we set \( \text{Sym}(37) \)

If \( \text{Proof.} \)

where the right morphism is induced by the tautological surjection \( q^* E' \to \mathcal{O}_{P_5}(H') \), which is the right contraction by the element

We also have the following:

Here \( \wedge \) is taking the right wedge product, \( s' \) is the morphism (16) and \( \wedge \) is the right contraction. The morphism (26) is obtained by the composition

where the right morphism is induced by the tautological surjection \( q^* E' \to \mathcal{O}_{P_5}(H') \), which is the right contraction by the element

Hence the composition (34) is the sum of the right wedge product by (35) and the right contraction by

\[ \frac{1}{2} \left( \partial_4 \bar{W} e_1^* + \partial_5 \bar{W} e_2^* + \partial_6 \bar{W} e_3^* + \partial_7 \bar{W} \right). \]

3.3. Some cohomology computations. This subsection is devoted to do some cohomology computations, which will be used in the next subsection.

**Lemma 3.5.** We set \( M_{k,l} \) to be

\[ M_{k,l} := R\Gamma(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(kH' + lh')). \]

Then we have

(i) \( M_{k,l} = 0 \) if \(-3 \leq k \leq -1, \) and \( M_{0,0} = \mathbb{C}, \) \( M_{0,-3} = \mathbb{C}[-2]. \)

(ii) \( M_{k,l} = 0 \) if \(-2 \leq l \leq 0 \) with \(-6 < k + l < 0. \) Moreover we have \( M_{l,-6-l} = \mathbb{C}[-5] \) for \(-2 \leq l \leq 0. \)

(iii) \( M_{-4,-3} = \mathbb{C}^3[-5]. \)

**Proof.** If \( k \geq -3, \) we can compute \( M_{k,l} \) from

\[ M_{k,l} = R\Gamma(\mathbb{P}^5, \text{Sym}^k E' \langle l \rangle). \]

Here we set \( \text{Sym}^k E' = 0 \) for \( k < 0. \) Hence we immediately obtain (i). We can also describe \( M_{k,l} \) as

\[ M_{k,l} = R\Gamma(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}((k + l)H' - lD')). \]

Hence (ii) follows using

\[ R^p \mathcal{O}_{\mathbb{P}^5}(-lD') = \mathcal{O}_{\mathbb{P}^5}, -2 \leq l \leq 0. \]

Finally since \( K_{\mathbb{P}^5} = -4H' - 2h', \) the Serre duality implies

\[ M_{k,l} = M_{-k-4,-l-2}[-5]. \]

Hence we have \( M_{-4,-3} = M_{0,1}[-5], \) which coincides with \( \mathbb{C}^3[-5] \) by (37). □
We will also use the following computations:

**Lemma 3.6.** We have

\[(39)\] \[\mathbf{R} \text{Hom}(O_{\tilde{X}}(h - H), \Phi(B_0)) \cong \mathbb{C}^3[-2]\]

\[(40)\] \[\mathbf{R} \text{Hom}(O_{\tilde{X}}(h - H), \Phi(B_1)) \cong \mathbb{C} \oplus \mathbb{C}[-2]\]

\[(41)\] \[\mathbf{R} \text{Hom}(\Phi(B_0), O_{\tilde{X}}(-h)) \cong \mathbb{C}^6\]

\[(42)\] \[\mathbf{R} \text{Hom}(\Phi(B_1), O_{\tilde{X}}(-h)) \cong \mathbb{C}^3\].

**Proof.** Since \(\omega_{\tilde{X}} = O_{\tilde{X}}(-2H - h)\), the LHS of \((11), (12)\) are written as

\[\mathbf{R} \text{Hom}(\Phi(B_i), O_{\tilde{X}}(-h)) \cong \mathbf{R} \Gamma(\tilde{X}, \Phi(B_i)(-2H))[4]\]

By the exact sequence \((27)\), we see that \(j_* \Phi(B_0)\) is quasi-isomorphic to the complex

\[(43)\] \[\begin{pmatrix}
O_{\mathbb{P}_3}(-2H') \\
\oplus
O_{\mathbb{P}_3}(-2H' - h')^{\oplus 3} \\
\oplus
O_{\mathbb{P}_3}(-2H' - 2h')^{\oplus 3} \\
\oplus
O_{\mathbb{P}_3}(-2H' - 3h')
\end{pmatrix}
\rightarrow
\begin{pmatrix}
O_{\mathbb{P}_3}(-H'')^{\oplus 3} \\
\oplus
O_{\mathbb{P}_3}(-H' - h')^{\oplus 2} \\
\oplus
O_{\mathbb{P}_3}(-H' - 2h')^{\oplus 3}
\end{pmatrix}\]

Applying \(\otimes q^* B_0 \circ q^* B_1\) to the sequence \((27)\), and using \((20)\), we obtain the exact sequence

\[(44)\] \[0 \rightarrow q^* B_1(-2H') \xrightarrow{\delta_1} q^* B_0(h - H') \rightarrow j_* \Phi(B_1) \rightarrow 0\]

where \(\delta_1\) is the morphism \((26)\). Hence \(j_* \Phi(B_1)\) is quasi-isomorphic to the complex

\[(45)\] \[\begin{pmatrix}
O_{\mathbb{P}_3}(-2H')^{\oplus 3} \\
\oplus
O_{\mathbb{P}_3}(-2H' - h')^{\oplus 2} \\
\oplus
O_{\mathbb{P}_3}(-2H' - 2h')^{\oplus 3}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
O_{\mathbb{P}_3}(-H' + h') \\
\oplus
O_{\mathbb{P}_3}(-H')^{\oplus 3} \\
\oplus
O_{\mathbb{P}_3}(-H' - h')^{\oplus 3} \\
\oplus
O_{\mathbb{P}_3}(-H' - 2h')
\end{pmatrix}\]

Applying \(\otimes O_{\mathbb{P}_3}(H' - h'), \otimes O_{\mathbb{P}_3}(-2H')[4]\) to the complexes \((13), (15)\) and then applying \(\mathbf{R} \Gamma(\mathbb{P}_5, -)\), we see that \((39), (10), (11), (12)\) are quasi-isomorphic to the following complexes respectively:

\[
(M_{-1,-1} \oplus M_{-1,-2}^{\oplus 3} \oplus M_{-1,-3}^{\oplus 3} \oplus M_{-1,-4} \rightarrow M_{0,-1}^{\oplus 3} \oplus M_{0,-2}^{\oplus 2} \oplus M_{0,-3}^{\oplus 3})
\]

\[
(M_{-1,-1} \oplus M_{-1,-2}^{\oplus 3} \oplus M_{-1,-3}^{\oplus 3} \rightarrow M_{0,0} \oplus M_{0,-1}^{\oplus 3} \oplus M_{0,-2}^{\oplus 2} \oplus M_{0,-3})
\]

\[
(M_{-4,-1} \oplus M_{-4,-2}^{\oplus 3} \oplus M_{-4,-3}^{\oplus 3} \rightarrow M_{-3,0}^{\oplus 3} \oplus M_{-3,1}^{\oplus 2} \oplus M_{-3,2}^{\oplus 3})[4]
\]

\[
(M_{-4,-1} \oplus M_{-4,-2}^{\oplus 3} \rightarrow M_{-3,1} \oplus M_{-3,2} \oplus M_{-3,1}^{\oplus 3} \oplus M_{-3,2})[4]
\]

Applying the computation in Lemma 3.5 we obtain the result. \(\square\)
3.4. Computation of $\Theta(B_i)$. The purpose of this subsection is to compute $\Theta(B_i)$ for $i = 0, 1$, using an explicit description of $\delta_i$ in Subsection 3.2 and computations in Subsection 3.3. Let $\Phi$ be the fully faithful embedding given in (25). The following lemma includes the key computation in this subsection:

**Lemma 3.7.** There is an isomorphism

$$R\sigma_* \Phi(B_1) \cong I_P \oplus O_X(-1)^{\oplus 3}.$$

*Proof.* By (14) and (14), the object $j_* \Phi(B_1)$ is quasi-isomorphic to the complex

$$(\begin{array}{c}
O_{\mathbb{P}^5}(-2H')^{\oplus 3} \\
O_{\mathbb{P}^5}(D' - 3H')^{\oplus 2} \\
O_{\mathbb{P}^5}(2D' - 4H')^{\oplus 3}
\end{array}) \xrightarrow{\delta_1}$$

where $I_P \subset O_{\mathbb{P}^5}$ is the ideal sheaf of $P$, we see that $i_* R\sigma_* \Phi(B_1)$ is quasi-isomorphic to the complex

$$R\sigma_* O_{\mathbb{P}^5}(-D') \cong I'_P.$$

We apply $Rp_*$ to the above complex. Using (35) and

$$(46)$$

$$\left(\begin{array}{c}
O_{\mathbb{P}^5}(-2)e_1 \\
O_{\mathbb{P}^5}(-2)e_2 \\
O_{\mathbb{P}^5}(-2)e_3 \\
O_{\mathbb{P}^5}(-3)f \\
O_{\mathbb{P}^5}(-3)e_1 \wedge e_2 \wedge e_3 \\
O_{\mathbb{P}^5}(-4)e_1 \wedge e_2 \wedge f \\
O_{\mathbb{P}^5}(-4)e_2 \wedge e_3 \wedge f \\
O_{\mathbb{P}^5}(-4)e_3 \wedge e_1 \wedge f
\end{array}\right) \xrightarrow{p_* \delta_1}$$

Here we have specified basis elements of both sides of (46) induced from those of $\wedge^* E$. Since $\delta_1$ is injective, the morphism $p_* \delta_1$ is generically injective, hence it is injective. This implies that $R\sigma_* \Phi(B_1)$ is a coherent sheaf on $X$.

Now we give an explicit description of $p_* \delta_1$ using the notation in Subsection 3.2. Since $p_* x_T = 1$, the right wedge product by (33) pushes down via $p_*$ to the right wedge product by the element

$$(47)$$

$$x_4 e_1 + x_5 e_2 + x_6 e_3 + f.$$
Also by setting $\partial_i' := \partial_i/2$ and
\[ W'' := W'(x_1, x_2, x_3) + \frac{1}{2} \sum_{1 \leq i, j \leq 3} x_i x_j W_{ij}(x_4, x_5, x_6) \]
we have the following relations:
\[ x_7 \partial_i' W = \partial_i W, \quad 4 \leq i \leq 6, \quad x_2^2 \partial_i' W = W''. \]
Here we have used the relation (32). Then the right contraction by the element (36) pushes down via $p_*$ to the right contraction by the element (48)
\[ \partial_4' W e_1 + \partial_5' W e_2 + \partial_6' W e_3 + W'' f_r. \]
The morphism $p_* \delta_1$ is the sum of the right wedge product by (47) and the right contraction by (48). Therefore if we regard local sections of both sides of (49) as column vectors, we see that $p_* \delta_1$ in (49) is given by the matrix
\[
M = \begin{pmatrix}
\partial_4' W & \partial_5' W & \partial_6' W & W'' & 0 & 0 & 0 & 0 \\
x_5 & -x_4 & 0 & 0 & \partial_5' W & W'' & 0 & 0 \\
0 & x_6 & -x_5 & 0 & \partial_6' W & 0 & W'' & 0 \\
-x_6 & 0 & x_4 & 0 & \partial_5' W & 0 & 0 & W'' \\
1 & 0 & 0 & -x_4 & 0 & -\partial_6' W & 0 & \partial_6' W \\
0 & 1 & 0 & -x_5 & 0 & \partial_4' W & -\partial_6' W & 0 \\
0 & 0 & 1 & -x_6 & 0 & 0 & \partial_5' W & -\partial_4' W \\
0 & 0 & 0 & 0 & 1 & -x_6 & -x_4 & -x_5
\end{pmatrix}.
\]
Now we define the matrices $N_1, N_2$ to be
\[
N_1 := \begin{pmatrix}
0 & 1 & 0 & 0 & -x_5 & x_4 & 0 & -\partial_6' W \\
0 & 0 & 1 & 0 & 0 & -x_6 & x_5 & -\partial_5' W \\
0 & 0 & 0 & 1 & x_6 & 0 & -x_4 & -\partial_5' W
\end{pmatrix}
\]
\[
N_2 := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
and define $N_3, N_4$ to be
\[
N_3 := \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}, \quad N_4 := \begin{pmatrix}
x_4 \\
x_5 \\
x_6 \\
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]
Then noting that
\[ W'' + x_4 \partial_4' W + x_5 \partial_5' W + x_6 \partial_6' W = W \]
the above matrices satisfy the following relations
\[ N_1 M = W \cdot N_2, \quad M N_4 = W \cdot N_3, \quad N_2 N_4 = N_1 N_3 = 0. \]
This implies that we have the commutative diagram of sheaves on $\mathbb{P}^5$

$$
\begin{array}{ccc}
\mathcal{O}(-3) & \xrightarrow{-W} & \mathcal{O}'_P \\
\downarrow N_4 & & \downarrow N_3 \\
\mathcal{O}(-2)^{\oplus 3} \oplus \mathcal{O}(-3)^{\oplus 2} \oplus \mathcal{O}(-4)^{\oplus 3} & \xrightarrow{M} & \mathcal{O}'_P \oplus \mathcal{O}(-1)^{\oplus 3} \oplus \mathcal{O}(-2)^{\oplus 3} \oplus \mathcal{O}(-3)^{\oplus 2} \\
\downarrow N_2 & & \downarrow N_1 \\
\mathcal{O}(-4)^{\oplus 3} & \xrightarrow{-W} & \mathcal{O}(-1)^{\oplus 3}
\end{array}
$$

such that the induced sequence of sheaves on $X$

$$(49) \quad 0 \to I_P \to R\sigma_* \Phi(B_1) \to \mathcal{O}_X(-1)^{\oplus 3} \to 0$$

is a complex. The above sequence is right exact since $N_1$ is surjective, and left exact since $N_3$ is injective and the cokernel of $N_4$ is locally free. Furthermore the middle cohomology of (49) is quasi-isomorphic to the complex

$$
\mathcal{O}_{\mathbb{P}^5}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^5}(-3) \to \mathcal{O}_{\mathbb{P}^5}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^5}(-3)
$$

which must be quasi-isomorphic to zero since it is a sheaf. Therefore (49) is a short exact sequence in Coh($X$). Since $H^1(X, I_P(1)) = 0$ as $I_P \in D_X$, the exact sequence (49) splits and we obtain a desired isomorphism. □

**Proposition 3.8.** There is an isomorphism

$$\Theta(B_1) \cong I_P[-1].$$

**Proof.** By (29), Lemma 3.6 and Lemma 3.7, the object $\Theta(B_1)$ is written as

$$(50) \quad \{I_P \oplus I_P[-2] \to I_P \oplus \mathcal{O}_X(-1)^{\oplus 3} \to \mathcal{O}_X(-1)^{\oplus 3}\}.$$

Since $I_P \in D_X$, $\Theta(B_1) \in D_X$ and $\mathcal{O}_X(-1) \notin D_X$, the $\mathcal{O}_X(-1)^{\oplus 3}$-component of the right morphism in (50) must be an isomorphism. Hence we have

$$\Theta(B_1) \cong \text{Cone} \left( I_P \oplus I_P[-2] \xrightarrow{\theta, \theta'} I_P \right).$$

The morphism $\theta: I_P \to I_P$ must be non-zero, hence an isomorphism, since otherwise $\Theta(B_1)$ is decomposable which contradicts to that $B_1$ is indecomposable (cf. Lemma 3.2) and $\Theta$ is an equivalence. Therefore $\theta$ is an isomorphism and we have $\Theta(B_1) \cong I_P[-1]$. □

We will also need some computations of $\Theta(B_0)$. We first show the following:

**Lemma 3.9.** There is an isomorphism

$$R\sigma_* \Phi(B_0) \cong I'_P(-2) \oplus \mathcal{O}_X(-1)^{\oplus 3}.$$  

Here $-^\vee$ is the derived dual.

**Proof.** By the Grothendieck duality, we have

$$(51) \quad i_* R\text{Hom}_{\mathbb{P}^5}(R\sigma_* \Phi(B_0), i^! \mathcal{O}_{\mathbb{P}^5}) \cong R\text{Hom}_{\mathbb{P}^5}(i_* R\sigma_* \Phi(B_0), \mathcal{O}_{\mathbb{P}^5})
\cong R\text{Hom}_{\mathbb{P}^5}(R\sigma_* \Phi(B_0), \mathcal{O}_{\mathbb{P}^5})
\cong R\sigma_* R\text{Hom}_{\mathbb{P}^5}(j_* \Phi(B_0), p^! \mathcal{O}_{\mathbb{P}^5}).$$
We apply $R\mathcal{H}om_{\mathbb{P}^5}(-, p^! \mathcal{O}_{\mathbb{P}^5})$ to the complex \cite{LMS}, and push it down via $R\pi_*$. Noting $p^! \mathcal{O}_{\mathbb{P}^5} = \mathcal{O}_{\mathbb{P}^5}(2D')$, the resulting complex becomes

$$
\begin{pmatrix}
\mathcal{O}_{\mathbb{P}^5}(1)e_1^1 \\
\mathcal{O}_{\mathbb{P}^5}(1)e_2^2 \\
\mathcal{O}_{\mathbb{P}^5}(1)e_3^3 \\
\mathcal{O}_{\mathbb{P}^5}(2)f^* \\
\mathcal{O}_{\mathbb{P}^5}(3)e_1^1 \wedge e_2^2 \wedge e_3^3 \\
\mathcal{O}_{\mathbb{P}^5}(3)e_1^1 \wedge e_2^2 \wedge f^* \\
\mathcal{O}_{\mathbb{P}^5}(3)e_2^2 \wedge e_3^3 \wedge f^* \\
\mathcal{O}_{\mathbb{P}^5}(3)e_3^3 \wedge e_1^1 \wedge f^*
\end{pmatrix} \\
\oplus
\begin{pmatrix}
\mathcal{O}_{\mathbb{P}^5}(2) \\
\mathcal{O}_{\mathbb{P}^5}(3)e_1^1 \wedge e_2^2 \\
\mathcal{O}_{\mathbb{P}^5}(3)e_2^2 \wedge e_3^3 \\
\mathcal{O}_{\mathbb{P}^5}(3)e_3^3 \wedge e_1^1 \\
\mathcal{O}_{\mathbb{P}^5}(4)e_1^1 \wedge f^* \\
\mathcal{O}_{\mathbb{P}^5}(4)e_2^2 \wedge f^* \\
\mathcal{O}_{\mathbb{P}^5}(4)e_3^3 \wedge f^* \\
I'_p(5)e_1^1 \wedge e_2^2 \wedge e_3^3 \wedge f^*
\end{pmatrix}
$$

The morphism $p_*\delta^0_\pi$ is the sum of the right contraction by \cite{LMS} and the right wedge product by \cite{LMS}. Therefore we can apply the exactly same computation in Lemma 3.7 and show that

$$
R\pi_* R\mathcal{H}om_{\mathbb{P}^5}(j_* \Phi(B_0), p^! \mathcal{O}_{\mathbb{P}^5})[1] \cong I_p(5) \oplus \mathcal{O}_X(4)^{\oplus 3}.
$$

Then noting $i^! \mathcal{O}_{\mathbb{P}^5} = \mathcal{O}_X(3)[-1]$, the above isomorphism together with \cite{LMS} yield a desired result. \hfill \Box

As for $\Theta(B_0)$, we only have to compute its numerical class as follows:

**Lemma 3.10.** We have the identity in $N(X)$:

$$
[\Theta(B_0)] = [I_P^*(2)] - 3[I_P] - 3[\mathcal{O}_X(-1)].
$$

*Proof.* The claim follows from Lemma \cite{LMS}, \cite{LMS}, \cite{LMS} and \cite{LMS}. \hfill \Box

3.5. **Evaluations at skyscraper sheaves.** Let $S$ be the K3 surface \cite{LMS}. For a point $x \in S$, the skyscraper sheaf $\mathcal{O}_x$ determines an object in $\text{Coh}(S, \alpha)$. In the notation of Subsection 2.4 we set

$$
L_x := \mathcal{Y}(\mathcal{O}_x) \in \text{Coh}(B_0).
$$

The object $\Theta(L_x) \in \mathcal{D}_X$ is studied by Lahoz-Macri-Stellari \cite{LMS} Section 4]. They show that there is an isomorphism

$$
\Theta(L_x) \cong M_x[1]
$$

where $M_x$ is a rank four Gieseker stable ACM bundle on $X$ with Chern character given by

$$
\text{ch}(M_x) = (4, -2H, -P, l, 1/4).
$$

Here $l$ is a class of a line in $X$. Furthermore there are exact sequences (cf. \cite{LMS} Proposition 4.2, Step 4)

$$
(53) \quad 0 \rightarrow \mathcal{O}_X(-1)^{\oplus 2} \rightarrow M_x \rightarrow K_x \rightarrow 0,
$$

$$
(54) \quad 0 \rightarrow K_x \rightarrow I_p^{\oplus 2} \rightarrow \mathcal{I}_{L_x,Q(x)} \rightarrow 0.
$$
Here $Q_f(x)$ is the quadric defined to be $\sigma(\pi^{-1}f(x))$, and $l_x \subset Q_f(x)$ is a line determined by $x$, and $I_x, Q_f(x)$ is the ideal sheaf of $l_x$ in $Q_f(x)$. The purpose of this subsection is to compare the following objects

\[(55) \quad F_X(M_x), \quad ST_{I_x}^{-1}(M_x)[1].\]

Here $F_X$ is defined by (50), and $ST_{I_x}^{-1}$ is the inverse of the Seidel-Thomas twist associated to $I_x$, which is spherical by Proposition 3.8. We first investigate the LHS of (55).

**Lemma 3.11.** There is an isomorphism

\[(56) \quad F_X(M_x) \cong \text{Cone}(\mathcal{O}_X^{\oplus 4} \xrightarrow{\cong} K_x(1)).\]

**Proof.** Applying $\otimes \mathcal{O}_X(1)$ to the exact sequence (53), we obtain the exact sequence

\[(57) \quad 0 \to \mathcal{O}_X^{\oplus 2} \to M_x(1) \to K_x(1) \to 0.\]

Applying $R\Gamma(X, -)$, we obtain the distinguished triangle

\[(58) \quad \mathbb{C}^2 \to R\Gamma(X, M_x(1)) \to R\Gamma(X, K_x(1)).\]

It is easy to see that

\[
R\Gamma(X, I_P(1)) = \mathbb{C}^3, \quad R\Gamma(X, I_{l_x, Q_f(x)}(1)) = \mathbb{C}^2.
\]

From (54), we obtain the distinguished triangle

\[R\Gamma(X, K_x(1)) \cong \mathbb{C}^6 \to \mathbb{C}^2.
\]

By [LMS, Proposition 4.4, Step 1], we have $H^1(X, K_x(1)) = 0$, hence we obtain $R\Gamma(X, K_x(1)) = \mathbb{C}^4$. Combined with (58), we obtain $R\Gamma(X, M_x(1)) = \mathbb{C}^6$ and

\[F_X(M_x) \cong \text{Cone}(\mathcal{O}_X^{\oplus 6} \to M_x(1)).\]

Then (56) follows from the above isomorphism and taking account of the exact sequence (57). \qed

**Lemma 3.12.** The object $F_X(M_x)$ is isomorphic to the following object:

\[
\text{Cone}(I_{l_x, Q_f(x)} \xrightarrow{\cong} \text{Ext}^2(I_{l_x, Q_f(x)}, \mathcal{O}_X(-1))^\vee \otimes \mathcal{O}_X(-1)[2][-1].
\]

**Proof.** By [LMS, Lemma 4.3], the sheaf $K_x(1)$ fits into the exact sequence

\[0 \to I_{P \cup Q_f(x)}^{\oplus 2}(1) \to K_x(1) \to I_{l_x, Q_f(x)} \to 0.
\]

Here $I_{P \cup Q_f(x)}$ is the ideal sheaf of $P \cup Q_f(x)$ in $X$, which is a complete intersection of two hyperplanes in $X$. Since $\text{Hom}(\mathcal{O}_X, I_{l_x, Q_f(x)}) = 0$, the evaluation morphism in the RHS of (50) factors through $I_{P \cup Q_f(x)}^{\oplus 2}(1)$. Moreover, the Koszul resolution of $\mathcal{O}_{P \cup Q_f(x)}$ yields the exact sequence

\[0 \to \mathcal{O}_X(-1)^{\oplus 2} \to \mathcal{O}_X^{\oplus 4} \to I_{P \cup Q_f(x)}^{\oplus 2}(1) \to 0.
\]

Hence we obtain the distinguished triangle

\[(59) \quad \mathcal{O}_X(-1)^{\oplus 2}[1] \to F_X(M_x) \to I_{l_x, Q_f(x)} \to \mathcal{O}_X(-1)^{\oplus 2}[2].
\]
On the other hand, using Serre duality, we have
\[
\text{Ext}^2(I_{l,(x),Q(x)}, \mathcal{O}_X(-1))^\vee \cong H^2(Q(x), I_{l,(x),Q(x)}(-2)) \cong \mathbb{C}^2.
\]
Therefore it is enough to show the morphism \( \theta \) in (59) is identified with the evaluation morphism. This easily follows from the vanishing
\[
\text{Hom}(F_X(M_x), \mathcal{O}_X(-1)[1]) = 0
\]
due to \( F_X(M_x) \in \mathcal{D}_X \). \qedhere

We next investigate the RHS of (55).

**Lemma 3.13.** There is an isomorphism
\[
(60) \quad \text{ST}^{-1}_{I_P}(M_x)[1] \cong \text{Cone}(M_x \to I_P^{\oplus 2}).
\]

**Proof.** Since \( \Theta(L_x) = M_x[1] \) and \( \Theta(B_1) = I_P[-1] \) by Proposition 3.8, and \( \Theta \) is an equivalence, we have
\[
(61) \quad \text{R Hom}(M_x, I_P^{\oplus 2}) \cong \text{R Hom}(L_x, B_1)[-2].
\]
Under the equivalence (22), the object \( L_x \) corresponds to \( \mathcal{O}_x \) and \( B_1 \) corresponds to a rank two \( \alpha \)-twisted vector bundle on \( S \). Therefore we see that (61) is isomorphic to \( \mathbb{C}^2 \). Hence (60) follows from the definition of \( \text{ST}^{-1}_{I_P} \). \qedhere

**Lemma 3.14.** The object \( \text{ST}^{-1}_{I_P}(M_x)[1] \) is isomorphic to the following object
\[
(62) \quad \text{Cone}(I_{l,x,Q(x)} \to \text{Ext}^2(I_{l,x,Q(x)}, \mathcal{O}_X(-1))^\vee \otimes \mathcal{O}_X(-1)[2])[1] \to \mathcal{O}_X(-1)[2].
\]

**Proof.** From the exact sequences (53) and (54), we have the morphisms
\[
M_x \to K_x \to I_P^{\oplus 2}.
\]
The above composition must coincide with the evaluation morphism in the RHS of (60) up to a base change of \( I_P^{\oplus 2} \), since otherwise its image has rank less than or equal to one which contradicts to that \( K_x \) has rank two. Therefore we obtain a commutative diagram
\[
\begin{array}{ccc}
\text{ST}^{-1}_{I_P}(M_x) & \to & I_{l,x,Q_x}[-1] \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{O}_X(-1)^{\oplus 2} \\
\downarrow \text{ev} & & \downarrow \\
I_P^{\oplus 2} & \cong & I_P^{\oplus 2}
\end{array}
\]
As a result, we obtain the distinguished triangle
\[
(62) \quad \mathcal{O}_X(-1)^{\oplus 2}[1] \to \text{ST}^{-1}_{I_P}(M_x)[1] \to I_{l,x,Q_x} \to \mathcal{O}_X(-1)^{\oplus 2}[2].
\]
Similarly to the proof of Lemma 3.12, the morphism \( \theta' \) is identified with the evaluation morphism because of the vanishing
\[
\text{Hom}(\text{ST}^{-1}_{I_P}(M_x)[1], \mathcal{O}_X(-1)[1]) = 0
\]
due to \( \text{ST}^{-1}_{I_P}(M_x) \in \mathcal{D}_X \). \qedhere
As a corollary of Lemma 3.12 and Lemma 3.14 we obtain the following:

**Corollary 3.15.** For any $x \in S$, there is an isomorphism

$$F_X(M_x) \cong \text{ST}_{I_p}^{-1}(M_{i(x)})[1].$$

3.6. **Proof of Proposition 3.3.**

**Proof.** Let us consider the autequivalence $\Psi$ of $D^b \text{Coh}(S, \alpha)$ obtained as the composition

$$\Psi: D^b \text{Coh}(S, \alpha) \rightarrow D^b \text{Coh}(B_0) \rightarrow D^b \text{Coh}(B_0) \rightarrow D^b \text{Coh}(S, \alpha).$$

It is enough to show that there is an isomorphism of functors

$$\Psi(-) \cong \text{id}_{D^b \text{Coh}(S, \alpha)}.$$  \hspace{1cm} (63)

By applying the results so far, we show the following:

**Lemma 3.16.** There is $L \in \text{Pic}(S)$ and an isomorphism of functors

$$\Psi \cong \otimes L.$$

**Proof.** For $x \in S$, we have

$$\Psi(O_x) = \Theta^{-1} \circ \otimes B_1 \circ \text{ST}_{B_1} \circ \Theta^{-1} \circ F_X \circ \Theta (L_x)[-1]$$

$$\cong \Theta^{-1} \circ \otimes B_1 \circ \Theta^{-1} \circ \text{ST}_{I_p[-1]} \circ F_X (M_x)$$

$$\cong \Theta^{-1} (L_{i(x)} \otimes B_1)$$

$$\cong O_x.$$ \hspace{1cm} (64) \hspace{1cm} (65) \hspace{1cm} (66)

Here we have used (52) and Proposition 3.8 in (64), Proposition 3.15 and an obvious fact $\text{ST}_{I_p} = \text{ST}_{I_p[-1]}$ in (65), and the fact that $\otimes B_1$ takes $L_x$ to $L_{i(x)}$ in (66) which is a well-known property of representations of Clifford algebras [Mor96, Corollary 2.4.5]. On the other hand, by [CS07], there is an object $P$ in $D^b \text{Coh}(S \times S, \alpha^{-1} \boxtimes \alpha)$ such that the equivalence $\Psi$ is of Fourier-Mukai type with kernel $P$. By a spectral sequence argument as in [Bri99, Lemma 4.3], the condition $\Psi(O_x) \cong O_x$ for any $x \in S$ implies that $P \in \text{Coh}(S \times S, \alpha^{-1} \boxtimes \alpha)$, which is flat over the first factor and supported on the diagonal. In particular, $\Psi$ preserves $\text{Coh}(S, \alpha)$, hence the proof of [CS07, Corollary 5.3] shows that $\Psi$ is written as a desired form. \qed

By Lemma 3.16, an isomorphism (63) follows once we show an isomorphism $L \cong O_X$. Since $S$ is a K3 surface, the isomorphism classes of line bundles are determined by their first Chern classes. Hence it is enough to find a twisted vector bundle $U \in \text{Coh}(S, \alpha)$ such that $\Psi(U)$ and $U$ have the same numerical classes. We check this for a rank two twisted vector bundle which corresponds to $B_1$ under $\Upsilon$, using the following lemma:

**Lemma 3.17.** We have the identity in $N(D_X)$

$$[F_X(I_p)] = [\Theta \circ \text{ST}_{B_1}^{-1}(B_0)].$$  \hspace{1cm} (67)
Proof. It is easy to see that $R\text{Hom}(O_X, I_P(1))$ is isomorphic to $\mathbb{C}^3$, hence the LHS of (67) is given by

$$[F_X(I_P)] = [I_P(1)] - 3[O_X]$$

in $N(X)$. Using Lemma 3.2, Proposition 3.8 and Lemma 3.10 the RHS of (67) is computed in $N(X)$ as

$$[\Theta \circ ST^{-1}_B(B_0)] = [\Theta(B_0)] - 3[\Theta(B_1)]$$

$$= [I_P(-2)] - 3[O_X(-1)].$$

(69)

By a standard calculation, the RHS of (68) and (69) have the same Chern characters given by

$$(-2, H, 2H^2 - P, -2L, -1/8) \in H^4(X, \mathbb{Q}).$$

Here $l$ is a line in $X$. By the Riemann-Roch theorem on $X$, we obtain the identity (67). □

Let $U_1 \in \text{Coh}(S, \alpha)$ be the rank two twisted vector bundle which corresponds to $B_1$ under $\Upsilon$. Applying Proposition 3.8 and Lemma 3.17, we have the identities of the numerical classes of objects in $D^b \text{Coh}(S, \alpha)$

$$[\Psi(U_1)] = -[\Upsilon^{-1} \circ \otimes B_0 B_1 \circ \Theta^{-1} \circ F_X \circ \Theta(B_1)]$$

$$= [\Upsilon^{-1} \circ \otimes B_0 B_1 \circ \Theta^{-1} \circ F_X(I_P)]$$

$$= [U_1].$$

Therefore the first Chern class of the line bundle $L$ in Lemma 3.16 is trivial. Hence $L$ is trivial, and we obtain a desired isomorphism (63). □

4. Construction of a Gepner type stability condition

In this section, we prove Proposition 1.5 and Theorem 1.6. We assume that we are in the same situation as in the previous section.

4.1. Description of $Z_G$ in terms of sheaves of Clifford algebras. In this subsection, we investigate the numerical Grothendieck group $N(B_0)$ of $D^b \text{Coh}(B_0)$, and describe the central charge $Z_G$ on $HMF^P(W)$ in terms of $N(B_0)$. Let

$$V \subset N(B_0)_{\mathbb{Q}}$$

be the $\mathbb{Q}$-vector subspace generated by all $[B_k]$ for $k \in \mathbb{Z}$. Let us consider the autoequivalence $F_B$ in (31), and its action $F_{B^*}$ on $N(B_0)$. Obviously $F_{B^*}$ preserves the subspace $V$. We compute the action of

$$F_{B^*}^{-1} = - (\otimes B_0 B_1)\circ ST_{B^*} : V \to V.$$ 

The following lemma is obvious from (20):

Lemma 4.1. The action $F_{B^*}^{-1}$ on $V$ is given by

$$F_{B^*}^{-1}([B_k]) = -[B_{k+1}] + \chi(B_1, B_{k+1})[B_2].$$

We are going to describe the action $F_{B^*}^{-1}$ more precisely by finding a basis of $V$. 

Proposition 4.2. The vector space $V$ is three dimensional, and
$$V = \mathbb{Q}[B_0] \oplus \mathbb{Q}[B_1] \oplus \mathbb{Q}[B_2].$$

Proof. We divide the proof into four steps.

Step 1. We have $3 \leq \dim V \leq 6$.

Because $B_{i+2} = B_i(1)$ and $N(\mathbb{P}^2)$ is generated by $O_{\mathbb{P}^2}(i)$ for $0 \leq i \leq 2$, the vector space $V$ is at least generated by $[B_i]$ for $0 \leq i \leq 5$. In particular, we have $\dim V \leq 6$. On the other hand, the Chern characters of $B_i$ for $0 \leq i \leq 2$ as $O_{\mathbb{P}^2}$-modules are given as follows:

$$\text{ch}(B_0) = (8, -12, 12)$$
$$\text{ch}(B_1) = (8, -8, 7)$$
$$\text{ch}(B_2) = (8, -4, 4).$$

They are linearly independent, so $[B_i]$ for $0 \leq i \leq 2$ are also linearly independent in $N(B_0)\mathbb{Q}$. In particular, we have $\dim V \geq 3$. Below we reduce the number of generators by finding three more relations among $[B_i]$ for $0 \leq i \leq 5$.

Step 2. First relation.

For $x \in \mathbb{P}^2$, the objects $B_i|x$ do not depend on $i$ since they correspond to $O_{f^{-1}(x)}$ under the equivalence (23). Therefore by taking the Koszul resolution

$$0 \to B_i \to B_i(1)_{\mathbb{P}^2} \to B_i(2) \to B_i|x \to 0$$

we obtain the following relation

$$[B_i] - 2[B_2] + [B_0] = [B_5] - 2[B_3] + [B_1].$$

Step 3. Second relation.

Let us take a general line $l \subset \mathbb{P}^2$, and consider the non-commutative scheme $(l, B_0|l)$. Similarly to (23), there is an equivalence

$$\text{Coh}(B_0|l) \cong \text{Coh}(B_0|f^{-1}(l)).$$

Since $f^{-1}(l)$ is a curve, the Azuyama algebra $B_0|f^{-1}(l)$ splits and the RHS is equivalent to $\text{Coh}(f^{-1}(l))$. This fact easily implies that the numerical class of an object in $D^b\text{Coh}(B_0|l)$ is determined by its rank and degree as $O_l$-module. Note that $B_i|l$ all have rank eight, and

$$\deg(B_{i+1}|l) - \deg(B_i|l) = 4.$$

Also the object $B_i|x$ has rank zero and degree eight for $x \in l$. Therefore we have the following relation in $N(B_0|l)$:

$$2([B_3|l] - [B_2|l]) = [B_4|x].$$

By pushing forward to $N(B_0|l)$, and taking the Koszul resolution (71) and the exact sequence

$$0 \to B_i(-1) \to B_i \to B_i|l \to 0$$

we obtain the relation

$$2([B_3] - [B_4]) - 2([B_2] - [B_0]) = [B_4] - 2[B_2] + [B_0].$$
The above relation is equivalent to
\[(73) \quad [B_4] = [B_0] - 2[B_1] + 2[B_3].\]

**Step 4. Third relation.**

We now apply Lemma 4.1 to describe $F_{B_1}^{-1}$ in terms of $[B_i]$ for $0 \leq i \leq 3$. Using the computation in Lemma 3.2, it is straightforward to deduce that
\[(74) \quad F_{B_1}^{-1}([B_0]) = -[B_1] + 3[B_2]
F_{B_1}^{-1}([B_1]) = [B_2]
F_{B_1}^{-1}([B_2]) = 3[B_2] - [B_3]
F_{B_1}^{-1}([B_3]) = -[B_0] + 2[B_1] + 6[B_2] - 2[B_3].\]

Here we have used the relation (73) in the last equation. Applying the above formulas three times, a little computation shows that:
\[(75) \quad F_{B_1}^{-1}([B_0]) = -[B_1] + 3[B_2]
F_{B_1}^{-1}([B_1]) = [B_2]
F_{B_1}^{-1}([B_2]) = 3[B_2] - [B_3].\]

By Corollary 3.4, the above class should coincide with $[B_0]$. Therefore we obtain the relation
\[(76) \quad [B_3] = [B_0] - 3[B_1] + 3[B_2].\]

The relations (72), (73) and (75) show that $V$ is spanned by $[B_i]$ for $0 \leq i \leq 2$.  

The proof of the above proposition also specifies the action of $F_{B_1}^{-1}$ on $V$:

**Corollary 4.3.** The action of $F_{B_1}^{-1}$ on $V$ is given as follows:
\[(76) \quad F_{B_1}^{-1}([B_0]) = -[B_1] + 3[B_2]
F_{B_1}^{-1}([B_1]) = [B_2]
F_{B_1}^{-1}([B_2]) = -[B_0] + 3[B_1].\]

**Proof.** The action (76) is given by substituting (75) into (74).  

The following corollary will be useful in a later computation:

**Corollary 4.4.** The following relation holds in $V$:
\[(77) \quad [B_4] = \frac{3}{8}[B_0] + \frac{3}{4}[B_2] - \frac{1}{8}[B_4].\]

**Proof.** The claim follows from relations (73) and (75).  

Now we describe the central charge $Z_G$ on $\text{HMF}^\#(W)$ in terms of $D^b \text{Coh}(B_0)$. Let $Z'_G$ be the central charge on $D^b \text{Coh}(B_0)$, defined to be the composition
\[Z'_G: N(B_0) \xrightarrow{\Theta} N(D_X) \xrightarrow{\phi_{B_1}^{-1}} N(\text{HMF}^\#(W)) \xrightarrow{Z_G} \mathbb{C}.\]

We compute $Z'_G$ using Corollary 3.4. Below, we set
\[\omega := e^{2\pi \sqrt{-1}/3} \in \mathbb{C}^\ast.\]
Proposition 4.5. There is a non-zero constant \( c \in \mathbb{C}^* \) such that the central charge \( Z'_G \) is written as

\[
Z'_G(E) = c \cdot \chi(u, E)
\]

where \( u \in V \) is given by

\[
u := [B_0] + (\omega - 2)[B_1] - \omega[B_2].\]

Proof. By Lemma 2.9 and Corollary 3.4, the central charge \( Z'_G \) is written as (77) for some \( u \in N(B_0) \) satisfying \( F^{-1}B_* u = \omega \cdot u \). We first show that \( u \in V \) holds. Let us consider the decomposition \( N(B_0) = V \oplus V^\perp \). Here \( V^\perp \) is the orthogonal complement of \( V \) with respect to \( \chi(\cdot, \cdot) \). Obviously, the action of \( F^{-1}B_* \) preserves both of \( V \) and \( V^\perp \). Suppose by a contradiction that there is \( u' \in V^\perp \) with \( F^{-1}B_* u' = \omega \cdot u' \). Then we have \( \chi(u', B_1) = 0 \), which implies that

\[
F^{-1}B_* (u') = -u' \otimes B_0 B_1.
\]

By applying the above identity twice, we obtain the identity

\[
u' \otimes B_0 B_0(1) = \omega^2 \cdot u'.
\]

On the other hand, the equivalence \( \otimes B_0 B_0(1) \) corresponds to tensoring \( \mathcal{O}_S(h) \) on \( D^b \text{Coh}(S) \) in Subsection 2.4, which acts on \( H^*(S, \mathbb{Z}) \) by multiplying \( e^h \). Since this action is unipotent, there is no non-zero eigenvector in \( H^*(S, \mathbb{C}) \) with eigenvalue \( \omega^2 \), which is a contradiction.

By the above argument and Lemma 4.5, \( u \) is written as

\[
u = x_0[B_0] + x_1[B_1] + x_2[B_2]
\]

for some \( x_i \in \mathbb{C} \). By (76), the condition \( F^{-1}B_* u = \omega \cdot u \) is given by

\[
\begin{pmatrix}
0 & 0 & -1 \\
-1 & 0 & 3 \\
3 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x_1 \\
x_2
\end{pmatrix}
= \omega
\begin{pmatrix}
x_0 \\
x_1 \\
x_2
\end{pmatrix}.
\]

It has the one dimensional solution space, spanned by

\[
x_0 = 1, \ x_1 = \omega - 2, \ x_2 = -\omega.
\]

□

4.2. Description of \( Z_G \) in terms of twisted K3 surfaces. In this subsection, we describe the central charge \( Z_G \) in terms of \( \alpha \)-twisted sheaves on the K3 surface \( S \). We first recall the twisted Chern character theory on \( D^b \text{Coh}(S, \alpha) \), developed by [HS05]. In our situation (cf. [Kuz10, Section 6]), it depends on an additional choice of the following data

\[
B \in H^2(S, \frac{1}{2} \mathbb{Z}), \ \alpha = \exp(B^{0,2}).
\]

Here \( B^{0,2} \) means the \((0,2)\)-part in the Hodge decomposition of \( H^2(S, \mathbb{C}) \). By [HS05 Corollary 2.4], there exists a map

\[
\text{ch}^B : D^b \text{Coh}(S, \alpha) \to H^*(S, \mathbb{Z})
\]
whose image coincides with
\[ \tilde{H}^{1,1}(S, B, \mathbb{Z}) := e^B \left( \bigoplus_{i=0}^{2} H^{i,i}(S, \mathbb{Q}) \right) \cap H^{*}(S, \mathbb{Z}). \]

satisfying the Riemann-Roch theorem: for any \( E, F \in D^b \text{Coh}(S, \alpha) \), we have the formula
\[ \chi(E, F) = -\langle v^B(E), v^B(F) \rangle. \]

Here \( v^B(E) \) is the twisted Mukai vector
\[ v^B(E) := \text{ch}^B(E) \sqrt{\text{td}_S} \in \tilde{H}^{1,1}(S, B, \mathbb{Z}) \]
and \( \langle -, - \rangle \) is the Mukai pairing on \( \tilde{H}^{1,1}(S, B, \mathbb{Z}) \),
\[ \langle (\xi_0, \xi_1, \xi_2), (\xi'_0, \xi'_1, \xi'_2) \rangle = \xi_1 \xi'_1 - \xi_0 \xi'_2 - \xi_2 \xi'_0. \]

In particular, the map \( \text{ch}^B \) induces the isomorphism
\[ \text{ch}^B : N(S, \alpha) \cong \tilde{H}^{1,1}(S, B, \mathbb{Z}). \]

Here \( N(S, \alpha) \) is the numerical Grothendieck group of \( D^b \text{Coh}(S, \beta) \).

Let \( U_i \in \text{Coh}(S, \alpha) \) be the twisted sheaves which correspond to \( B_i \) under the equivalence \( \Upsilon \). We prepare the following lemma:

**Lemma 4.6.** We can write \( v^B(U_i) \) as
\[ v^B(U_i) = e^{hi/2} \left( 2, \beta, \frac{1}{4} \beta^2 + \frac{1}{2} \right) \]
for some \( \beta \in H^2(S, \mathbb{Z}) \) with \( \beta - 2B \in H^{1,1}(S, \mathbb{Z}) \).

**Proof.** Since \( B_{i+2} = B_i(1) \), we have \( U_{i+2} = U_i(h) \). Therefore it is enough to show the case of \( i = 0 \) and \( i = 1 \). In the case of \( i = 0 \), we have \( \chi(U_0, U_0) = 2 \) by Lemma 3.2. Hence we obtain a desired form (80) by the Riemann-Roch theorem (78). Moreover the class \( \beta - 2B \) is algebraic since \( e^{-B} \text{ch}^B(U_0) \) is algebraic. In the case of \( i = 1 \), using Corollary 4.4 we have
\[ v^B(U_1) = \frac{3}{8} v^B(U_0) + \frac{3}{4} v^B(U_2) - \frac{1}{8} v^B(U_4) \]
\[ = \left( \frac{3}{8} + \frac{3}{4} e^{h} - \frac{1}{8} e^{2h} \right) v^B(U_0) \]
\[ = e^{h/2} v^B(U_0). \]

Now we define the **untwisted** Chern character on \( \alpha \)-twisted sheaves to be
\[ \text{ch}(-) := e^{-B} \text{ch}^B(-) : N(S, \alpha) \rightarrow \bigoplus_{i=0}^{2} H^{i,i}(S, \mathbb{Q}). \]

The above untwisted Chern character may depend on a choice of \( B \). If \( \alpha = 1 \), we can take \( B = 0 \) and then it coincides with the usual Chern character. The benefit of the untwisted Chern character is that it takes values in algebraic classes, although it may not be defined in the integer coefficient. We describe \( Z_G \) in terms of \( D^b \text{Coh}(S, \alpha) \) as an integral which
appeared in [Bri08], using the untwisted Chern character and an algebraic \nfield.

**Proposition 4.7.** There is an element \(\mathfrak{B} \in H^{1,1}(S, \mathbb{Q})\) such that the com-
position

\[
N(S, \alpha) \overset{\Upsilon}{\to} N(B_0) \overset{\Theta}{\to} N(D_X) \overset{\Phi^{-1}}{\to} N(\text{HMF}^\sigma(W)) \overset{Z_G}{\to} \mathbb{C}
\]

coincides with the following integral

\[
Z''_G(E) := -\int_S e^{\mathfrak{B} - \frac{3}{4} h} \text{ch}(E) \sqrt{\text{td}_S}.
\]

up to a non-zero scalar multiplication.

**Proof.** For \(E \in D^b \text{Coh}(S, \alpha)\), the composition \(Z''_G(\Upsilon^*(E))\) becomes

\[
Z'_G(\Upsilon^*(E)) = c \cdot \chi([U_0] + (\omega - 2)[U_1] - \omega[U_2], E).
\]

Using Lemma 4.6, we have

\[
v^B([U_0] + (\omega - 2)[U_1] - \omega[U_2]) = \left\{1 + (\omega - 2)e^{h/2} - \omega e^h\right\} v^B(U_0)
\]

\[
= \left(-1, -\left(\frac{1}{2} \omega + 1\right) h, -\frac{3}{4} \omega - \frac{1}{2}\right) \left(2, \beta, \frac{\beta^2 + 1}{4}\right)
\]

\[
= \left(-2, -\beta - (2 + \omega)h, -\frac{\beta^2}{4} - \left(\frac{1}{2} \omega + 1\right) \beta h - \frac{3}{4} \omega - \frac{3}{2}\right)
\]

\[
= -2 \exp\left(\frac{1}{2} \beta + \left(\frac{1}{2} \omega + 1\right) h\right)
\]

Applying the Riemann-Roch theorem (78), we see that the RHS of \(84\) is written as an integral

\[
-2c \cdot \int_S e^{-3h/4 - \beta/2 - \sqrt{-3h/4} v^B(E)}.
\]

We define \(\mathfrak{B}\) to be

\[
\mathfrak{B} := B - \frac{3}{4} h - \frac{1}{2} \beta.
\]

Note that \(\mathfrak{B}\) is an element in \(H^{1,1}(S, \mathbb{Q})\) by Lemma 4.6. Combined with the defini-
tion of the untwisted Chern character (81), we arrive at the desired result. \(\square\)

**4.3. Construction of a Gepner type stability condition.** We finish
the proof of Theorem 1.6 hence Theorem 1.2 in this subsection. For \(E \in D^b \text{Coh}(S, \alpha)\), let \(v^\mathfrak{B}(E)\) be the \(\mathfrak{B}\)-twisted Mukai vector

\[
v^\mathfrak{B}(E) := e^\mathfrak{B} \text{ch}(E) \sqrt{\text{td}_S}.
\]

It is useful to rewrite the integral \(83\) into the following form:

\[
Z''_G(E) = -v_2^\mathfrak{B}(E) + \frac{3}{16} v_0^\mathfrak{B}(E) + \frac{\sqrt{-3}}{4} v_1^\mathfrak{B}(E) h.
\]
Let us consider the following slope function on $\text{Coh}(S, \alpha)$

$$\mu(E) := \frac{v_0^\beta(E) \cdot h}{\text{rank}(E)}.$$ 

Here we set $\mu(E) = \infty$ if $\text{rank}(E) = 0$.

**Definition 4.8.** An object $E \in \text{Coh}(S, \alpha)$ is $\mu$-(semi)stable if for any exact sequence $0 \to F \to E \to G \to 0$ in $\text{Coh}(S, \alpha)$, we have $\mu(F) < (\leq) \mu(G)$.

**Remark 4.9.** When $\alpha = 1$, the above $\mu$-stability coincides with the classical twisted slope stability. It also behaves well even when $\alpha \neq 1$, e.g. the Harder-Narasimhan property. The proof is the same as in the $\alpha = 1$ case.

Following [Bri08], we define the following subcategories in $\text{Coh}(S, \alpha)$

$$T := \langle E \in \text{Coh}(S, \alpha) : E \text{ is } \mu\text{-semistable with } \mu(E) > 0 \rangle_{\text{ex}}.$$

$$F := \langle E \in \text{Coh}(S, \alpha) : E \text{ is } \mu\text{-semistable with } \mu(E) \leq 0 \rangle_{\text{ex}}.$$

Here $(-)_{\text{ex}}$ is the extension closure. The existence of Harder-Narasimhan filtrations in $\mu$-stability shows that the above pair is a torsion pair [HRS96] on $\text{Coh}(S, \alpha)$. The associated tilting is defined to be

$$A_G := (F[1], T)_{\text{ex}} \subset D^b \text{Coh}(S, \alpha).$$

Let $F_S$ be the autoequivalence of $D^b \text{Coh}(S, \alpha)$ defined to be

$$F_S := Y^{-1} \circ F_B \circ Y : D^b \text{Coh}(S, \alpha) \widetilde{\to} D^b \text{Coh}(S, \alpha).$$

We would like to claim that $(Z^G_S, A_G)$ is a Gepner type stability condition with respect to $(F_S, 2/3)$. Unfortunately we are able to prove this only for $\alpha \neq 1$ case. We note that for a general cubic fourfold containing a plane $P$ (e.g. when the numerical classes of codimension two algebraic cycles are spanned by $H^2$ and $P$) the associated Brauer group satisfies $\alpha \neq 1$ (cf. [Kuz10] Proposition 4.8]). The $\alpha \neq 1$ condition is required to show the following lemmas:

**Lemma 4.10.** Suppose that $\alpha \neq 1$. Then for any $E \in D^b \text{Coh}(S, \alpha)$, we have

$$v_0^\beta(E) \in 2\mathbb{Z}, \ 2v_1^\beta(E) \in H^{1,1}(S, \mathbb{Z}), \ 8v_2^\beta(E) \in \mathbb{Z}.$$ 

Furthermore $4v_2^\beta(E) \notin \mathbb{Z}$ if $v_0^\beta(E)/2$ is odd.

**Proof.** Suppose that $v_0^\beta(E) = \text{rank}(E)$ is odd. Then $\alpha$ is also the Brauer class of the twisted line bundle $\text{det}(F)$, whose transition function provides a cocycle which makes $\alpha$ to be trivial. This is a contradiction, hence $v_0^\beta(E)$ is an even number. (Also see [SM12] Corollary 3.2.) By the definition of $v^\beta(E)$, we have

$$v_1^\beta(E) = e^{-3h/4 - \beta/2} \cdot v^R(E).$$

Hence if we write $\xi_i = v^R(E) \in H^{2i}(S, \mathbb{Z})$, then

$$v_1^\beta(E) = \xi_1 - \left(\frac{3}{4}h + \frac{\beta}{2}\right)\xi_0,$$

$$v_2^\beta(E) = \left(\frac{\beta^2}{4} + \frac{3}{4}h + \frac{9}{8}\right)\xi_0 - \left(\frac{3}{4}h + \frac{\beta}{2}\right)\xi_1 + \xi_2.$$
Therefore the claim for \( v^3_1(E) \) and \( v^3_2(E) \) follow from the integrality of \( \beta, \xi \) and \( v^3_0(E) = \xi_0 \in \mathbb{Z} \).

\[ \square \]

**Lemma 4.11.** Suppose that \( \alpha \neq 1 \). Then for any \( E \in D^b \text{Coh}(S, \alpha) \), we have

\[
\begin{align*}
\text{Im } Z''_G(E) &< \frac{\sqrt{3}}{4} \times \mathbb{Z}, & \text{Re } Z''_G(E) &< \frac{1}{4} \times \mathbb{Z}, \\
\text{Re } Z''_G(E) - \frac{1}{\sqrt{3}} \text{Im } Z''_G(E) &< \frac{1}{2} \times \mathbb{Z}.
\end{align*}
\]

**Proof.** Let us write \( \text{ch}^R(E) = (\xi_0, \xi_1, \xi_2) \). By (88) and (89), we have

\[
\text{Im } Z''_G(E) = \frac{\sqrt{3}}{4} \times \left( \xi_1 h - (3 + \beta) \frac{\xi_0}{2} \right),
\]

\[
\text{Re } Z''_G(E) = -\left( \beta^2 + 3\beta h + 3 \right) \frac{\xi_0}{2} + (3h + 2\beta) \frac{\xi_1}{4} - \xi_2,
\]

\[
\text{Re } Z''_G(E) - \frac{1}{\sqrt{3}} \text{Im } Z''_G(E) = -\left( \beta^2 + 4\beta h + 6 \right) \frac{\xi_0}{2} + (h + \beta) \frac{\xi_1}{2} - \xi_2.
\]

Therefore the claim follows since \( \xi_0 = v^3_0(E) \) is even by Lemma 4.10.

\[ \square \]

**Lemma 4.12.** Suppose that \( \alpha \neq 1 \). Then we have

\[ F_S^{-1}(O_x) \in \mathcal{A}_G \]

for any \( x \in S \), and it is \( Z''_G \)-semistable.

**Proof.** By the definition of \( F_S \), we have

\[
\begin{align*}
F_S^{-1}(O_x) &\cong \Upsilon^{-1}\left\{ \text{Cone}(B_1^{\xi_2} \xrightarrow{\text{ev}} L_x) \right\} \otimes B_0 [-1] \\
&\cong \Upsilon^{-1}\left\{ \text{Ker}(B_2^{\xi_2} \xrightarrow{\text{ev}} L_t(x)) \right\} \\
&\cong \text{Ker}(U_2^{\xi_2} \xrightarrow{\text{ev}} O_t(x)).
\end{align*}
\]

Note that every \( U_t \) is \( \mu \)-stable, since there are no rank one subsheaves by Lemma 4.10. The slope \( \mu(U_2) \) can be easily computed to be 1/2 by Lemma 4.10 and (88). Therefore both of \( U_2 \) and \( F_S^{-1}(O_x) \) are objects in \( \mathcal{A}_G \).

Let us show that \( U_2 \) is \( Z''_G \)-stable. Using (81), the complex number \( Z''_G(U_2) \) is computed as

\[
Z''_G(U_2) = -\frac{1}{4} + \frac{\sqrt{-3}}{4}.
\]

By Lemma 4.11 it is enough to check the following: for any \( E \in \mathcal{A}_G \) with \( \text{Im } Z''_G(E) = 0 \), we have \( \text{Hom}(E, U_2) = 0 \). Since such \( E \) is a successive extensions by objects of the form \( O_x \) for \( x \in S \) or \( F[1] \) for \( \mu \)-stable \( F \in \text{Coh}(S, \alpha) \) with \( \mu(F) = 0 \), the vanishing \( \text{Hom}(E, U_2) = 0 \) is obvious.

We next show the \( Z''_G \)-semistability of \( F_S^{-1}(O_x) \). We have

\[
Z''_G(F_S^{-1}(O_x)) = \frac{1}{2} + \frac{\sqrt{-3}}{2}.
\]

Let \( E \in \mathcal{A}_G \) be a subobject of \( F_S^{-1}(O_x) \) in \( \mathcal{A}_G \). We need to check that

\[
\arg Z''_G(E) \leq \arg Z''_G(F_S^{-1}(O_x)).
\]
Similarly to the above argument, the imaginary part of $Z''_G(E)$ should be positive. By Lemma 4.11, we have the two possibilities: $\text{Im} Z''_G(E) = \sqrt{3}/4$ or $\sqrt{3}/2$. In the latter case, the inequality (87) is obvious since $Z''_G(F_S^{-1}(O_x)/E)$ lies in the negative real line.

Suppose that $\text{Im} Z''_G(E) = \text{Im} Z''_G(U_2) = \sqrt{3}/4$. Since there is an exact sequence in $A_G$

$$0 \to F_S^{-1}(O_x) \to U_2^{\oplus 2} \xrightarrow{ev} O_{i(x)} \to 0$$

and $U_2$ is $Z''_G$-stable, we obtain the inequality

$$\arg Z''_G(E) \leq \arg Z''_G(U_2)$$

Furthermore the equality holds in (89) only if $E = U_2$. However the exact sequence (88) shows that $\text{Hom}(U_2, F_S^{-1}(O_x)) = 0$, so this case is excluded.

Therefore the inequality (89) is strict, and Lemma 4.11 shows that $\text{Re} Z''_G(E) \geq \text{Re} Z''_G(U_2) + \frac{1}{4} = \frac{1}{4}$.

The above inequality implies (87). □

By Corollary 3.4 and Proposition 4.7, the result of Theorem 1.2 follows from the following statement:

**Theorem 4.13.** Suppose that $\alpha \neq 1$. Then the pair

$$\sigma_G := (Z''_G, A_G)$$

is a Gepner type stability condition on $D^b \text{Coh}(S, \alpha)$ with respect to $(F_S, 2/3)$.

**Proof.** We divide the proof into three steps.

**Step 1. Checking the axioms.**

The construction of the pair (90) is similar to Bridgeland’s one in [Bri08, Section 6], so almost the same argument in [Bri08, Section 6] is applied to show that $\sigma_G$ is a stability condition. (Also see [SM12, Lemma 5.4].) We may take care of the non-integrality of the untwisted Mukai vector $(r, \Delta, n) = \text{ch}(E)\sqrt{\text{td}S}$. This is not a matter, since we have

$$\Delta^2 - 2rs = (v^B(E), v^B(E)).$$

The above number is an even integer, so the argument of [Bri07, Lemma 6.2] is not affected. Therefore [Bri07, Lemma 6.2] shows that the pair (90) satisfies (1) if the following condition holds: for any spherical twisted sheaf $F \in \text{Coh}(S, \alpha)$ with $\mu(F) = 0$, we have the inequality $\text{Re} Z''_G(F) > 0$. Once this conditions is checked to satisfy, the Harder-Narasimhan property is proved along with the same argument of [Bri08, Proposition 7.1], since the image of $Z''_G$ is discrete. The support property is an easy consequence of the same argument in [Bri08, Lemma 8.1], so the detail is left to the reader.

**Step 2. Non-existence of certain spherical twisted sheaves.**

Suppose by a contradiction that there is a spherical twisted sheaf $F \in \text{Coh}(S, \alpha)$ with $\mu(F) = 0$ and $\text{Re} Z''_G(F) \leq 0$. For simplicity, we write
v_i := v_i^\mathbb{M}(F). The spherical condition of F, together with the Riemann-Roch theorem and the Serre duality implies

\[ v_1^2 = 2v_0v_2 - 2. \]  

(91)

The condition \( \mu(F) = 0 \) implies \( v_1 \cdot h = 0 \). Hence by the Hodge index theorem, we have

\[ 0 = (v_1 \cdot h)^2/h^2 \geq v_1^2 = 2v_0v_2 - 2 \]

which implies \( v_0v_2 \leq 1 \). Combined with \( \text{Re}Z_G(F) = -v_2 + 3v_0/16 \leq 0 \), we obtain the inequalities

\[ \frac{3}{16} v_0 \leq v_2 \leq \frac{1}{v_0} \]

(92)

which show \( v_0^2 \leq 16/3 \). Therefore from Lemma 4.10 we have \( v_0 = 2 \). Also we have \( 3/8 \leq v_2 \leq 1/2 \) from (92), hence \( v_2 = 3/8 \) by Lemma 4.10. By substituting into (91), we obtain \( v_1^2 = -1/2 \). Now by Lemma 4.10 \( v_1 \) is written as \( \gamma/2 \) for some \( \gamma \in H^{1,1}(S, \mathbb{Z}) \). By the above argument, we have \( \gamma \cdot h = 0 \) and \( \gamma^2 = -2 \). However the latter condition implies that \( \gamma \) or \( -\gamma \) is represented by an effective divisor by the Riemann-Roch theorem on \( S \). This contradicts to the former condition, so the property (4) is proved.

**Step 3. Gepner type property.**

We denote by \( \text{Stab}(S, \alpha) \) the space of numerical stability conditions on \( D^b\text{Coh}(S, \alpha) \), and by \( \mathcal{Z} \) the forgetting map

\[ \mathcal{Z} : \text{Stab}(S, \alpha) \to N(S, \alpha)^\vee. \]

By the argument so far, we have shown that \( \sigma_G \in \text{Stab}(S, \alpha) \). Let us consider the following stability condition

\[ \sigma_G' := (-2/3) \cdot F_S \cdot \sigma_G. \]

If we write \( \sigma_G = (Z_G^\nu, \{ \mathcal{P}_G(\phi) \}_{\phi \in \mathbb{R}}) \) as in (5), then \( \sigma_G' \) is written as

\[ \sigma_G' = (Z_G^\nu, \{ \mathcal{P}_G'(\phi) \}_{\phi \in \mathbb{R}}), \quad \mathcal{P}_G'(\phi) = F_S \mathcal{P}_G(\phi - 2/3). \]

By Lemma 4.12 we see that \( \mathcal{O}_x \) for any \( x \in S \) is \( \sigma_{G'} \)-semistable with phase one. Let \( U \subset \text{Stab}(S, \alpha) \) be the open subset in which \( \mathcal{O}_x \) is stable with the same phase. It is easy to see that \( \sigma_G \in U \), and the above argument shows that \( \sigma_G' \in U \). Now we use the same argument of [Bri08 Corollary 11.3], showing that any point in \( U \) is determined by the image of \( \mathcal{Z} \). Since \( \text{Stab}(S, \alpha) \) is Hausdorff, and the map \( \mathcal{Z} \) is a local homeomorphism [Bri07], it follows that \( \sigma_G = \sigma_G' \). \( \square \)

**Remark 4.14.** By the SOD (24) and the gluing method in [CP10 Proposition 3.3], it is possible to construct stability conditions on \( D^b\text{Coh}(X) \) for general cubic fourfolds \( X \) containing a plane, from stability conditions on \( \mathcal{D}_X \), e.g. those constructed in this subsection. This idea was used in [BMMS12 Corollary 3.8] to construct stability conditions on cubic 3-folds.
5. Appendix A: Chern characters on graded matrix factorizations

In this section, we recall the Chern character theory on graded matrix factorizations by Polishchuk-Vaintrob [PV12], [PV], and prove Lemma 2.9.

5.1. Chern characters and the central charge. Let \( W \) be a homogeneous polynomial as in (7). The Chern character map on \( \text{HMF}^{\text{gr}}(W) \) takes its value in the Hochschild homology group of \( \text{HMF}^{\text{gr}}(W) \), which we denote by \( \text{HH}_*(W) \):

\[
\text{ch} : K(\text{HMF}^{\text{gr}}(W)) \rightarrow \text{HH}_*(W).
\]

The above Chern character map is a composition of that of the \( \mu_d \)-equivariant matrix factorizations of \( W \) with the forgetting the functor \( \text{HMF}^{\text{gr}}(W) \rightarrow \text{HMF}^{\mu_d}(W) \).

Here \( \mu_d \) acts on \( x_i \) by weight \(-1\). By [PV, Theorem 2.6.1 (i)], \( \text{HH}_*(W) \) is described as

\[
\text{HH}_*(W) \cong \bigoplus_{\gamma \in \mu_d} H(W_\gamma)^{\mu_d}.
\]

(93)

Here \( H(W) \) is defined by

\[
H(W) := (\mathbb{C}[x_1, \ldots, x_n] / (\partial x_1 W, \ldots, \partial x_n W)) dx_1 \wedge \cdots \wedge dx_n
\]

and the space \( H(W_\gamma) \) is given by applying the above construction for \( W_\gamma := W|_{\mathbb{C}^n \gamma} \). Note that we have

\[
H(W_\gamma)^{\mu_d} \cong \mathbb{C}, \quad \gamma \neq 1.
\]

(94)

Since \( \tau^{\times d} = [2] \) on \( \text{HMF}^{\text{gr}}(W) \), we have \( \tau^{\times d}_* = \text{id} \) on \( \text{HH}_*(W) \). This implies that \( \tau_* \) generates the \( \mathbb{Z}/d\mathbb{Z} \)-action on \( \text{HH}_*(W) \). By [PV] Theorem 2.6.1 (ii)], the decomposition (93) coincides with the character decomposition of \( \text{HH}_*(W) \) with respect to the above \( \mathbb{Z}/d\mathbb{Z} \)-action.

For \( P^* \in \text{HMF}^{\text{gr}}(W) \) and \( 0 \leq j \leq d - 1 \), we denote by \( \text{ch}_j(P^*) \) the \( H(W_\gamma^{e^{2\pi j\sqrt{-1}/d}}) \)-component of \( \text{ch}(P^*) \) under the isomorphism (93). By [PV12, Theorem 3.3], we have

\[
\text{ch}_0(P^*) = \text{str}(\partial x_n \delta_{P^*} \circ \cdots \circ \partial x_1 \delta_{P^*}).
\]

Here for a graded matrix factorization (8), the matrix \( \delta_{P^*} \) is given by

\[
\delta_{P^*} := \begin{pmatrix} 0 & p^0 \\ p^1 & 0 \end{pmatrix} : P^* \rightarrow P^*.
\]

Note that \( \text{ch}_0(P^*) \) is always zero if \( n \) is an odd integer.

For \( 1 \leq j \leq d - 1 \), we have (cf. [PV12, Theorem 3.3])

\[
\text{ch}_j(P^*) = \text{str}(e^{2\pi j\sqrt{-1}/d} : P^* \rightarrow P^*).
\]

In particular, the central charge \( Z_G(P^*) \) coincides with \( \text{ch}_1(P^*) \).
5.2. Some computation of the central charge. We set \( R = A/(W) \) and \( D^r_{\text{sg}}(R) \) the triangulated category of singularities in the sense of [Orl09]. Namely \( D^r_{\text{sg}}(R) \) is the quotient category of the bounded derived category of finitely generated graded \( R \)-modules by the subcategory generated by finitely generated projective graded \( R \)-modules. Then by [Orl09 Theorem 3.10], there is an equivalence of triangulated categories

\[
\text{HMF}^r_{\text{gr}}(W) \xrightarrow{\sim} D^r_{\text{sg}}(R)
\]

sending a graded matrix factorization \([8]\) to the cokernel of \( p^0 \). Let \( C(k) \) be the graded \( R = A/(W) \)-module given by

\[
C(k) := (A/m)(k), \quad m = (x_1, \cdots, x_n) \subset A.
\]

The object \( C(k) \) is regarded as an object in \( D^r_{\text{sg}}(R) \). By an abuse of notation, we denote by \( C(k) \) the corresponding object in \( \text{HMF}^r_{\text{gr}}(W) \) under the equivalence \([95]\).

Lemma 5.1. We have \( \text{ch}_0(C(0)) = 0 \), and

\[
\text{ch}_j(C(k)) = -e^{2\pi kj\sqrt{-1}/d}(1 - e^{-2\pi j\sqrt{-1}/d})^n, \quad 1 \leq j \leq d - 1.
\]

Proof. Since \([93]\) is the character decomposition with respect to the \( \mathbb{Z}/d\mathbb{Z} \)-action on \( \text{HH}^*(W) \) generated by \( \tau_\ast \), we have

\[
\text{ch}_j(C(k)) = e^{2\pi kj\sqrt{-1}/d}\text{ch}_j(C(0)).
\]

Therefore we may assume that \( k = 0 \). The computation of \( \text{ch}_1(C(0)) = Z_G(C(0)) \) is given in [Todb Example 2.8], and the same computation is applied for \( \text{ch}_j(C(0)) \) for \( 1 \leq j \leq d - 1 \). It remains to prove that \( \text{ch}_0(C(0)) = 0 \). This is obvious if \( n \) is odd, so we may assume that \( n \) is even. Let \( W' \) be

\[
W' = W(x_1, \cdots, x_{n-1}, 0) \in A' := \mathbb{C}[x_1, \cdots, x_{n-1}]
\]

and set \( R' = A'/(W') \). There is a natural push-forward functor

\[
i_\ast : D^r_{\text{sg}}(R') \to D^r_{\text{sg}}(R)
\]

by regarding a graded \( R' \)-module as a graded \( R \)-module by the surjection \( R \twoheadrightarrow R' \). Combined with the equivalence \([95]\) and the functoriality of Hochschild homologies, we have the commutative diagram (cf. [PV12 Lemma 1.3.2])

\[
\begin{array}{ccc}
\text{HMF}^r_{\text{gr}}(W') & \xrightarrow{i_\ast} & \text{HMF}^r_{\text{gr}}(W) \\
\downarrow{\text{ch}} & & \downarrow{\text{ch}} \\
\text{HH}^*(W') & \xrightarrow{i_{H\ast}} & \text{HH}^*(W).
\end{array}
\]

Let \( \tau' \) be the grade shift functor on \( \text{HMF}^r_{\text{gr}}(W') \). Because \( i_\ast \) commutes with \( \tau' \) and \( \tau \), the morphism \( i_{H\ast} \) commutes with \( \tau'_\ast \) and \( \tau_\ast \) on \( \text{HH}^*(W') \) and \( \text{HH}^*(W) \) respectively (cf. [PV12 Lemma 1.2.1]). Therefore \( i_{H\ast} \) preserves the direct sum decomposition \([93]\). Since \( i_\ast C(0) = C(0) \), the vanishing \( \text{ch}_0(C(0)) = 0 \) follows from the case of \( W' \). \( \square \)
5.3. Proof of Lemma 2.9.

Proof. It is enough to show that, if $P^\bullet \in K(\text{HMF}^\text{gr}(W))$ satisfies $\chi(P^\bullet, Q^\bullet) = 0$ for any $Q^\bullet \in \text{HMF}^\text{gr}(W)$, then we have $\text{ch}_1(P^\bullet) = 0$. Applying the Hirzebruch-Riemann-Roch theorem for matrix factorization s [PV12, Theorem 4.2.1], we have

$$\chi(P^\bullet, C(k)) = \frac{1}{d} \left( \langle \text{ch}_0(P^\bullet), \text{ch}_0(C(k)) \rangle_W + \sum_{j=1}^{d-1} c_\lambda^j \text{ch}_j(P^\bullet) \cdot \text{ch}_{-j}(C(k)) \right).$$

Here $(\ast, \ast)_W$ is the Residue pairing on $H(W)$ (cf. [PV12, Proposition 4.1.2]), $\lambda = e^{-2\pi \sqrt{-1}/d}$ and $c_\gamma$ for $\gamma \in \mu_d$ is given by

$$c_\gamma := \text{det}(1 - \gamma : \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2)^{-1}$$

where $\mathfrak{m}$ is given by (39). Combined with the assumption $\chi(P^\bullet, C(k)) = 0$ and the computation of $\text{ch}_j(C(k))$ in Lemma 5.1, we obtain

$$\sum_{j=1}^{d-1} \lambda^kj \text{ch}_j(P^\bullet) = 0$$

for all $k \in \mathbb{Z}$. Then we have $\text{ch}_j(P^\bullet) = 0$ for all $1 \leq j \leq d - 1$ by

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda & \lambda^2 & \cdots & \lambda^{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{d-2} & \lambda^{2d-2} & \cdots & \lambda^{(d-2)(d-1)} \end{pmatrix} = \prod_{1 \leq i < j \leq d-1} (\lambda^j - \lambda^i) \neq 0.$$

The latter statement follows from the obvious condition

$$Z_G(\tau P^\bullet) = e^{2\pi \sqrt{-1}/d} Z_G(P^\bullet)$$

for any $P^\bullet \in \text{HMF}^\text{gr}(W)$. □

6. Appendix B: the lower dimensional cases

In this section, we prove Theorem 1.3. The case of $n \leq 3$ is treated in [Todh], so we assume $n = 4$ or 5.

6.1. t-structures on non-commutative surfaces. We discuss t-structures on non-commutative surfaces and their semiorthogonal summand. The situation here is applied both in $n = 4$ and 5 cases. Let $S$ be a smooth projective surface and $\mathcal{B}_S$ a sheaf of $\mathcal{O}_S$-algebras on $S$, which is coherent as $\mathcal{O}_S$-module. For an ample divisor $H$ in $S$, it defines the slope stability on $\text{Coh}(\mathcal{B}_S)$ by setting

$$\mu_H(E) = \frac{c_1(\text{Forg}(E)) \cdot H}{\text{rank Forg}(E)}.$$  

Here $\text{Forg} : \text{Coh}(\mathcal{B}_S) \to \text{Coh}(S)$ is forgetting the $\mathcal{B}_S$-module structure, and $\mu_H(E) = \infty$ if $\text{rank Forg}(E) = 0$. We put the following additional assumptions:

- The object $\mathcal{B}_S \in D^b \text{Coh}(\mathcal{B}_S)$ is $\mu_H$-stable and exceptional.
There is a Serre functor $\mathcal{S}_B$ on $D^b\text{Coh}(\mathcal{B}_S)$ given by

$$\mathcal{S}_B(E) = E \otimes_{\mathcal{B}_S} M[2]$$

for some $\mathcal{B}_S$-bimodule $M$.

- For any $\mu_H$-semistable object $E \in \text{Coh}(\mathcal{B}_S)$, the object $E \otimes_{\mathcal{B}_S} M$ is also $\mu_H$-semistable with $\mu_H(E \otimes_{\mathcal{B}_S} M) < \mu_H(E)$.

We define the pair of full subcategories $(\mathcal{T}, \mathcal{F})$ in $\text{Coh}(\mathcal{B}_S)$ as follows:

$$\mathcal{T} := \langle E : \mu_H(E) > \mu_H(\mathcal{B}_S) \rangle_{\text{ex}},$$

$$\mathcal{F} := \langle E : \mu_H(E) \leq \mu_H(\mathcal{B}_S) \rangle_{\text{ex}}.$$

The pair of subcategories $(\mathcal{T}, \mathcal{F})$ forms a torsion pair in $\text{Coh}(\mathcal{B}_S)$, and its tilting is defined to be

$$\mathcal{A} := (\mathcal{F}[1], \mathcal{T})_{\text{ex}} \subset D^b\text{Coh}(\mathcal{B}_S).$$

The category $\mathcal{A}$ is the heart of a bounded t-structure on $D^b\text{Coh}(\mathcal{B}_S)$. By our assumption, we have $\mathcal{B}_S[1] \in \mathcal{A}$. Our assumption that $\mathcal{B}_S$ is exceptional allows us to define the triangulated category $\mathcal{D}$ by the SOD

$$D^b\text{Coh}(\mathcal{B}_S) = \langle \mathcal{B}_S, \mathcal{D} \rangle.$$

The following lemma is a generalization of [BMMS12, Lemma 3.4].

**Lemma 6.1.** The intersection $\mathcal{C} := \mathcal{A} \cap \mathcal{D}$ is the heart of a bounded t-structure on $\mathcal{D}$.

**Proof.** Let us take $E \in \mathcal{D}$. We denote by $\mathcal{H}^i_A(E) \in D^b\text{Coh}(\mathcal{B}_S)$ the $i$-th cohomology of $E$ with respect to the t-structure with heart $\mathcal{A}$. It is enough to show that $\mathcal{H}^i_A(E) \in \mathcal{D}$, i.e. $R\text{Hom}(\mathcal{H}^i_A(E), \mathcal{B}_S) = 0$ for all $i$. To show this, we see the following: for any $P \in \mathcal{A}$, we have

$$\text{Hom}(P, \mathcal{B}_S[j]) = 0, \text{ unless } j = 1, 2.$$  

(98)

Indeed if (98) holds, then the spectral sequence

$$E_2^{p,q} = \text{Hom}(\mathcal{H}_A^{-q}(E), \mathcal{B}_S[p]) \Rightarrow \text{Ext}^{p+q}(E, \mathcal{B}_S) = 0$$

degenerates and we conclude that $\mathcal{H}^i_A(E) \in \mathcal{D}$ for all $i$.

Let us prove (98). We have the exact sequence in $\mathcal{A}$

$$0 \to \mathcal{H}^{-1}(P)[1] \to P \to \mathcal{H}^0(P) \to 0.$$  

Applying $\text{Hom}(\mathcal{A}, \mathcal{B}_S[j])$ and noting that $\text{Hom}(\mathcal{T}, \mathcal{B}_S) = 0$ because $\mathcal{B}_S \in \mathcal{F}$, we see that (98) holds for $j \leq 0$. On the other hand, for $j \geq 3$, we use the duality

$$\text{Hom}(P, \mathcal{B}_S[j]) \cong \text{Hom}(\mathcal{B}_S[j - 2], P \otimes_{\mathcal{B}_S} M)^\vee$$

and show that the RHS is zero. Indeed we have the distinguished triangle

$$\mathcal{H}^{-1}(P) \otimes_{\mathcal{B}_S} M[1] \to P \otimes_{\mathcal{B}_S} M \to \mathcal{H}^0(P) \otimes_{\mathcal{B}_S} M.$$  

By our assumption, every $\mu_H$-semistable factor of $\mathcal{H}^{-1}(P) \otimes_{\mathcal{B}_S} M$ has $\mu_H$-slope less than $\mu_H(\mathcal{B}_S)$. Therefore we have

$$\text{Hom}(\mathcal{B}_S, \mathcal{H}^{-1}(P) \otimes_{\mathcal{B}_S} M) = 0,$$

and this implies the vanishing of the RHS of (100) for $j \geq 3$.  

$\square$
Let Π: \(D^b \text{Coh}(\mathcal{B}_S) \to \mathcal{D}\) be the right adjoint functor of the inclusion \(i: \mathcal{D} \hookrightarrow D^b \text{Coh}(\mathcal{B}_S)\). We have the following another description of \(\mathcal{B}\):

**Lemma 6.2.** We have \(\mathcal{C} = \Pi(\mathcal{A})\).

**Proof.** The inclusion \(\mathcal{C} \subset \Pi(\mathcal{A})\) is obvious. Let \((\mathcal{A}^{\leq 0}, \mathcal{A}^{\geq 0})\) be the t-structure on \(D^b \text{Coh}(\mathcal{B}_S)\) with heart \(\mathcal{A}\). We show that the pair \((\Pi \mathcal{A}^{\leq 0}, \Pi \mathcal{A}^{\geq 0})\) is a bounded t-structure on \(\mathcal{D}\). Indeed if this is true, then we have

\[
\mathcal{C} \subset \Pi(\mathcal{A}) \subset \Pi \mathcal{A}^{\leq 0} \cap \Pi \mathcal{A}^{\geq 0}
\]

hence we obtain \(\mathcal{C} = \Pi(\mathcal{A})\) as both sides are hearts of bounded t-structures on \(\mathcal{D}\). In order to prove the above claim, the only thing to check is the vanishing \(\text{Hom}(\Pi(P), \Pi(Q)) = 0\) for any \(P \in \mathcal{A}^{<0}\) and \(Q \in \mathcal{A}^{\geq 0}\). We have the distinguished triangle

\[
i \circ \Pi(P) \to P \to P' \to \]

with \(P' \in \langle \mathcal{B}_S \rangle\). Using the same notation in the proof of the previous lemma, we have the following exact sequence in \(\mathcal{A}\)

\[
\cdots \to \mathcal{H}^{-1}_\mathcal{A}(P') \to \mathcal{H}^0_\mathcal{A}(i \circ \Pi(P)) \to \mathcal{H}^0_\mathcal{A}(P) \cong 0 \to \cdots
\]

and the isomorphism \(\mathcal{H}^j_\mathcal{A}(P') \cong \mathcal{H}^j_\mathcal{A}(i \circ \Pi(P))\) for all \(j \geq 0\). Because \(P' \in \langle \mathcal{B}_S \rangle\), we have \(\mathcal{H}^j_\mathcal{A}(P') \cong \mathcal{B}_S[1]^{\oplus m_j}\) for some \(m_j \in \mathbb{Z}\). Combined with (102) and noting that \(\mathcal{B}_S[1] \in \mathcal{A}\) is a simple object, which is easy to check, we have \(\mathcal{H}^j_\mathcal{A}(i \circ \Pi(P)) \in \langle \mathcal{B}_S \rangle\) for \(j \geq 0\). This implies that \(i \circ \Pi(P) \in \mathcal{A}^{<0}\) since \(R \text{Hom}(i \circ \Pi(P), \mathcal{B}_S) = 0\). Therefore we have

\[
\text{Hom}(\Pi(P), \Pi(Q)) \cong \text{Hom}(i \circ \Pi(P), Q)
\]

\[
\cong 0.
\]

□

Let \(\mathcal{S}_D\) be the Serre functor of \(\mathcal{D}\). The following lemma will be useful in checking the Gepner type property.

**Lemma 6.3.** The subcategory \(\mathcal{S}_D(\mathcal{C})[-1] \subset \mathcal{D}\) is obtained as a tilting of \(\mathcal{C}\).

**Proof.** The Serre functor \(\mathcal{S}_D\) is related to the Serre functor \(\mathcal{S}_B\) of \(D^b \text{Coh}(\mathcal{B}_S)\) by \(\mathcal{S}_D = \Pi \circ \mathcal{S}_B\). Therefore we have

\[
\mathcal{S}_D(\mathcal{C})[-1] = \Pi(\mathcal{C} \otimes \mathcal{B}_S M)[1]
\]

\[
\subset \Pi(\mathcal{A} \otimes \mathcal{B}_S M)[1]
\]

\[
\subset \Pi(\mathcal{A}, \mathcal{A}[-1]_{\text{ex}})[1]
\]

(103)

\[
\subset \langle \mathcal{C}[1], \mathcal{C} \rangle_{\text{ex}}.
\]

(104)

Here (103) follows from the assumption that \(\ast \otimes \mathcal{B}_S M\) preserves the \(\mu_H\)-stability and decreases the \(\mu_H\)-slope, and (104) follows from Lemma 6.2. Therefore we obtain the assertion. □
6.2. The case of cubic surfaces. In this subsection, we prove Theorem 1.3 for \( n = 4 \). In the setting of the previous subsection, we set \( S = (W = 0) \subset \mathbb{P}^3 \) to be the cubic surface, \( \mathcal{B}_S = \mathcal{O}_S \) and \( H \) is the hyperplane class. Since \( \omega_S = \mathcal{O}_S(-H) \), it satisfies the assumption in the previous subsection. Below we use the same notation in the previous subsection in the above setting.

The cubic surface \( S \) is a blow-up of \( \mathbb{P}^2 \) at six points \( \pi: S \to \mathbb{P}^2 \). We denote by \( C_1, \ldots, C_6 \) the exceptional curves of \( \pi \), and \( h \) the hyperplane of \( \mathbb{P}^2 \) pulled-back to \( S \). By [BO], there is a full strong exceptional collection on \( D^b \text{Coh}(S) \)

\[
D^b \text{Coh}(S) = \langle \mathcal{O}_S, \mathcal{O}_S(h), \mathcal{O}_S(2h), \mathcal{O}_{C_1}, \ldots, \mathcal{O}_{C_6} \rangle.
\]

Therefore the semiorthogonal summand \( D \) in (97) has the SOD

\[
D = \langle \mathcal{O}_S(h), \mathcal{O}_S(2h), \mathcal{O}_{C_1}, \ldots, \mathcal{O}_{C_6} \rangle \subset D^b \text{Coh}(S).
\]

By Orlov’s theorem [12], the functor \( \Phi_0 \) gives an equivalence

\[
\Phi_0: \text{HMF}^\text{gr}(W) \cong D.
\]

On the other hand, by (10) the Serre functor of \( \text{HMF}^\text{gr}(W) \) is given by \( \mathcal{S}_W = \pi^{-1}[2] \). Since the Serre functors are categorical, \( \mathcal{S}_2 \) and \( \mathcal{S}_W \) commute with \( \Phi_0 \). Hence it is enough to construct a Gepner type stability condition on \( D \) with respect to \( (\mathcal{S}_D^{-1}[2], 2/3) \).

Let us compute the central charge \( Z_G \) in terms of \( D \). By the SOD (105), the numerical Grothendieck group of \( D \) decomposes as

\[
N(D) \cong \mathbb{Z}[\mathcal{O}_S(h)] \oplus \mathbb{Z}[\mathcal{O}_S(2h)] \oplus \bigoplus_{i=1}^6 \mathbb{Z}[\mathcal{O}_{C_i}].
\]

By Lemma 2.9 the central charge \( Z_G' := Z_G \circ \Phi_0^{-1} \) on \( D \) is given by

\[
Z_G'(E) = \chi(u, E)
\]

for some \( u \in N(D)_C \) satisfying \( S_D u = \omega \cdot u \) for \( \omega = e^{2\pi \sqrt{-1}/3} \). Below we compute \( u \) by looking at the action of \( \mathcal{S}_D \) on \( N(D) \). Recall that the Serre functor \( S_D \) on \( D \) and the Serre functor \( S_B = \otimes \mathcal{O}_S(-H)[2] \) on \( D^b \text{Coh}(S) \) are related by \( S_D = \Pi \circ S_B \). Hence there is a distinguished triangle for any \( E \in D \)

\[
S_D(E) \to E(-H)[2] \to \mathbb{R} \text{Hom}(E(-H)[2], \mathcal{O}_S)^\vee \otimes \mathcal{O}_S.
\]

Lemma 6.4. We have the following identities in \( N(D) \):

\[
[S_D(\mathcal{O}_S(h))] = 4[\mathcal{O}_S(h)] - 3[\mathcal{O}_S(2h)] + \sum_{i=1}^6 [\mathcal{O}_{C_i}]
\]

\[
[S_D(\mathcal{O}_S(2h))] = 9[\mathcal{O}_S(h)] - 5[\mathcal{O}_S(2h)] + \sum_{i=1}^6 [\mathcal{O}_{C_i}]
\]

\[
[S_D(\mathcal{O}_{C_i})] = 2[\mathcal{O}_S(h)] - [\mathcal{O}_S(2h)] + [\mathcal{O}_{C_i}]
\]

Proof. It is easy to compute the Chern characters on the LHS using (105) and the Riemann-Roch theorem. By comparing them with those on the RHS, we can easily check the result. \( \square \)
Let us write \( u \in N(D)_{\mathbb{C}} \) as

\[
\begin{align*}
u &= x_1[\mathcal{O}_S(h)] + x_2[\mathcal{O}_S(2h)] + \sum_{i=1}^{6} y_i[\mathcal{O}_{C_i}]
\end{align*}
\]

for \( x_1, x_2, y_i \in \mathbb{C} \). By the above lemma, the linear equation \( S_{\mathcal{D}_u}u = \omega \cdot u \) has the one dimensional solution space, spanned by

\[
x_1 = 3\omega, \quad x_2 = -3(\omega + 1), \quad y_1 = \cdots = y_6 = \omega = \omega + 2.
\]

Let us set

\[
\begin{align*}
u_0 &= 3\omega[\mathcal{O}_S(h)] - 3(\omega + 1)[\mathcal{O}_S(2h)] + (\omega + 2)\sum_{i=1}^{6}[\mathcal{O}_{C_i}].
\end{align*}
\]

For \( E \in \mathcal{D} \), we compute \( \chi(u_0, E) \) by using the Riemann-Roch theorem on \( S \). Because \( R\text{Hom}(E, \mathcal{O}_S) = 0 \), we have \( \chi(E, \mathcal{O}_S) = 0 \), which implies the constraint

\begin{equation}
\chi_2(E) = \frac{1}{2} \chi_1(E) \cdot H - \chi_0(E).
\end{equation}

On the other hand, noting that \( H = 3h - \sum_{i=1}^{6} C_i \), we have

\begin{equation}
\chi(u_0) = \left( -3, -(\omega + 2)H, -\frac{3}{2}\omega \right).
\end{equation}

By (107), (108) and noting \( \text{td}_S = (1, H/2, 1) \), a Riemann-Roch computation shows that

\[
\chi(u_0, E) = 3\chi_0(E) + (\omega - 1)\chi_1(E) \cdot H.
\]

As a summary, we obtain the following:

**Lemma 6.5.** There is a constant \( c \in \mathbb{C}^* \) such that \( Z'_G = Z_G \circ \Phi_0^{-1} \) on \( \mathcal{D} \) is written as

\[
\begin{align*}
Z'_G(E) &= c \cdot \left( 3\chi_0(E) - \frac{3}{2} \chi_1(E) \cdot H + \frac{\sqrt{-3}}{2} \chi_1(E) \cdot H \right).
\end{align*}
\]

Now we consider the heart \( \mathcal{C} \subset \mathcal{D} \) constructed in Lemma 6.1. The following statement proves Theorem 1.3 for \( n = 4 \):

**Theorem 6.6.** The pair \( \sigma \mathcal{G} = (Z''_G := Z'_G/c, \mathcal{C}) \) is a Gepner type stability condition on \( \mathcal{D} \) with respect to \( (S_{\mathcal{D}}^{-1}[2], 2/3) \).

**Proof.** We first check that \( \mathcal{H} \) holds. Since \( \mu_H(\mathcal{O}_S) = 0 \), the construction of \( \mathcal{C} \) yields

\[
\text{Im } Z''_G(E) = \frac{\sqrt{-3}}{2} \chi_1(E) \cdot H \geq 0
\]

for any \( 0 \neq E \in \mathcal{C} \). Suppose that \( \text{Im } Z''_G(E) = 0 \). Then \( E \) is a successive extensions in \( D^b\text{Coh}(S) \) by objects of the form \( \mathcal{O}_x \) for \( x \in S \) or \( F[1] \) for \( \mu_H \)-stable sheaf \( F \) on \( S \) with \( \mu_H(F) = 0 \). Since any zero dimensional sheaf is not an object in \( \mathcal{D} \), we have \( \text{Re } Z''_G(E) = 3\chi_0(E) < 0 \). This implies the condition \( \mathcal{H} \). The Harder-Narasimhan property is proved along with the same argument of [Bri08, Proposition 2.4]. The support property is easy to check, and left to the reader.
We show that $\sigma_G$ is Gepner type with respect to $(S_D^{-1}[2], 2/3)$, or equivalently with respect to $(S_D[-2], -2/3)$. Note that the action of $(-2/3)$ on stability conditions changes the corresponding hearts of the t-structures by tilting shifted by $[-1]$. By Lemma 6.3 the heart $S_D(A_G)[-2]$ is a tilting of $A_G$ shifted by $[-1]$. Therefore the desired Gepner type property of $\sigma_G$ follows from Lemma 6.7 below.

We have used the following lemma, whose proof is available in [Toda, Lemma 4.11].

**Lemma 6.7.** Let $D$ be a $\mathbb{C}$-linear triangulated category satisfying (2), $A \subset D$ is the heart of a bounded t-structure on $D$, and $Z: N(D) \to \mathbb{C}$ a group homomorphism. Suppose that there are torsion pairs $(T_k, F_k), k = 1, 2$ on $A$ such that, for $C_k = \langle F_k[1], T_k \rangle$, the associated tilting, both of the pairs $(Z, C_1)$ and $(Z, C_2)$ give numerical stability conditions. Then $C_1 = C_2$.

### 6.3. The case of cubic 3-folds.

In this subsection, we prove Theorem 1.3 for $n = 5$. In this case, the variety $X = (W = 0) \subset \mathbb{P}^4$ is a cubic 3-fold. There is a SOD

$$D^b \text{Coh}(X) = \langle \mathcal{O}_X(-2), \mathcal{O}_X(-1), D_X \rangle$$

hence the functor $\Phi_1$ in (12) gives an equivalence

$$\Phi_1: \text{HMF}^g(W) \cong D_X.$$

In [BMMS12], motivated by Kuznetsov’s work [Kuz08], Bernardara-Macri-Mehrotra-Stellari described the triangulated category $D_X$ in terms of sheaves of Clifford algebras on $\mathbb{P}^2$. Let $B_0$ (resp. $B_1$) be the even (resp. odd) parts of the sheaf of Clifford algebras as in [BMMS12] Section 2, given as follows:

$$B_0 = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2}$$

$$B_1 = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2).$$

In the setting of Subsection 6.1 we set $S = \mathbb{P}^2$, $H$ is the hyperplane class and $B_S = B_0$. All the assumptions in Subsection 6.1 are checked to satisfy in [BMMS12] Section 2.3. Also there is an equivalence of triangulated categories (denoted by $(\sigma_* \circ \Phi')^{-1}$ in [BMMS12])

$$\Psi: D_X \cong D$$

where $D$ is defined by (91). By (10), the Serre functor $S_W$ on $\text{HMF}^g(W)$ is given by $\tau^{-2}[3] = \tau[1]$. Therefore, it is enough to construct a Gepner type stability condition on $D$ with respect to $(S_D[-1], 2/3)$.

The numerical Grothendieck group of $D_X$ is computed in [BMMS12] Section 2, which is rather simpler than the $n = 4$ case. Let $l \subset X$ be a line. We have the isomorphism (cf. [BMMS12] Proposition 2.7)

$$N(D_X) \cong \mathbb{Z}[I_l] \oplus \mathbb{Z}[S_{D_X}(I_l)]$$

where $I_l$ is the ideal sheaf of $l$, $S_{D_X}$ is the Serre functor of $D_X$. The Euler pairing $\chi$ is given by

$$\chi(I_l, I_l) = \chi(S_{D_X}(I_l), S_{D_X}(I_l)) = \left( \begin{array}{cc} -1 & -1 \\ -1 & -1 \end{array} \right).$$
Furthermore the computation in [BMMST12] Proposition 2.7 also shows the identity in \( N(D_X) \):
\[
[S^{-1}_{D_X}(I_1)] = [I] - [S_{D_X}(I_1)].
\]

Let us write \( u \in N(D_X) \) as
\[
u = x_1[I] + x_2[S_{D_X}(I_1)]
\]
for \( x_1, x_2 \in \mathbb{C} \). By the above argument, the linear equation \( -S^{-1}_{D_X} u = \omega \cdot u \)
has the one dimensional solution space spanned by \((x_1, x_2) = (\omega, 1)\). We set
\[
u_0 = \omega[I] + [S_{D_X}(I_1)].
\]

On the other hand, let us consider the composition
\[
\phi: N(D_X) \overset{\Psi_{\omega}}{\rightarrow} N(\mathbb{P}^2) \overset{\text{Forg}_{\omega}}{\rightarrow} \mathbb{P}^2 \quad (\text{rank}_{\omega}c_1) \mathbb{Z}^{\mathbb{P}^2}.
\]

By the computation in [BMMST12] Proposition 2.12, we have
\[
\Psi_*[I] = [B_0] - [B_1], \quad \Psi_*[S_{D_X}(I_1)] = 2[B_0] - [B_{-1}]
\]
where \( B_{-1} := B_1(-1) \). Hence we have
\[
\phi([I_1]) = (0, 2), \quad \phi([S_{D_X}(I_1)]) = (4, -3).
\]

In particular, \( \phi \) induces the isomorphism over \( \mathbb{Q} \), \( \phi: N(D_X) \mathbb{Q} \overset{\cong}{\rightarrow} \mathbb{Q}^\mathbb{P}^2 \). The inverse \( \phi^{-1} \) is given by
\[
\phi^{-1}(r, d) = \left( \frac{3}{8}r + \frac{d}{2} \right) [I] + \frac{1}{4}[S_{D_X}(I_1)].
\]

Using [109], a little computation shows that
\[
\chi(\phi^{-1}(r_1, d_1), \phi^{-1}(r_2, d_2)) = -\frac{19}{64} r_1r_2 - \frac{3}{16} r_1d_2 - \frac{5}{16}r_2d_1 - \frac{1}{4}d_1d_2.
\]

Now let us consider the central charge on \( D \)
\[
Z'_G := Z_G \circ \Phi_{1*}^{-1} \circ \Psi_*^{-1}: N(D) \rightarrow \mathbb{C}.
\]

It differs from \( E \mapsto \chi(\Psi_*u_0, E) \) by multiplying a non-zero scalar constant. Noting that \( \phi(u_0) = (4, 2\omega - 3) \), the above computation yields
\[
\chi(\Psi_*u_0, E) = -\frac{\sqrt{3}}{4} \left\{-\frac{\sqrt{3}}{12}(r + 4d) + \left(d + \frac{5}{4}r\right) \sqrt{-1}\right\}
\]
if \((r, d) = (\text{rank Forg}(E), c_1\text{Forg}(E))\). We have obtained the following:

**Lemma 6.8.** There is a non-zero constant \( c \in \mathbb{C}^* \) such that \( Z'_G = Z_G \circ \Phi_{1*}^{-1} \circ \Psi_*^{-1} \) is written as
\[
Z'_G(E) = c \cdot \left(-\frac{\sqrt{3}}{12}(r + 4d) + \left(d + \frac{5}{4}r\right) \sqrt{-1}\right)
\]
where \((r, d) = (\text{rank Forg}(E), c_1\text{Forg}(E))\). The following statement proves Theorem 1.3 for \( n = 5 \):

**Theorem 6.9.** The pair \( \sigma_G = (Z''_G := Z'_G/c, C) \) is a Gepner type stability condition on \( D \) with respect to \((S_{D[-1]}, 2/3)\).
Proof. We first check that (1) holds. Since $\mu_H(B_0) = -5/4$, the construction of $\mathcal{C}$ yields

$$\text{Im } Z''_G(E) = d - \mu_H(B_0) r \geq 0$$

for any $0 \neq E \in \mathcal{C}$ with $(r, d) = (\text{rank } \text{Forg}(E), c_1 \text{Forg}(E))$. Suppose that $\text{Im } Z''_G(E) = 0$, i.e. $4d + 5r = 0$. Then $E$ is a successive extensions by objects of the form $Q$ with Forg($Q$) zero dimensional or $F[1]$ for a $\mu_H$-stable object $F \in \text{Coh}(B_0)$ with $\mu_H(F) = -5/4$. Since the former object is not an object in $\mathcal{D}$, we have $r < 0$. By substituting $4d + 5r = 0$, we obtain

$$\text{Re } Z''_G(E) = \frac{\sqrt{3}}{3} r < 0.$$

This implies that $\sigma_G$ satisfies (2). The Harder-Narasimhan property is proved along with the same argument of [Bri08, Proposition 2.4], and the support property is easy to check.

We show that $\sigma_G$ satisfies the desired Gepner type property. Note that the action of $(2/3)$ on stability conditions changes the corresponding hearts of the $t$-structures by tilting. By Lemma 6.3, the heart $\mathcal{S}_D(\mathcal{C})[-1]$ is a tilting of $\mathcal{C}$. Therefore the desired Gepner type property of $\sigma_G$ follows from Lemma 6.7.

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