Asymptotics of Clebsch-Gordan Coefficients

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Abstract

Asymptotic expressions for Clebsch-Gordan coefficients are derived from an exact integral representation. Both the classically allowed and forbidden regions are analyzed. Higher-order approximations are calculated. These give, for example, six digit accuracy when the quantum numbers are in the hundreds.

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I. INTRODUCTION

This paper contains a detailed study of the asymptotics of Clebsch-Gordan coefficients and includes the derivation of new results. We use the term “Clebsch-Gordan coefficient” in its colloquial sense, i.e., the vector addition coefficients of SU(2). Thus our results also give the asymptotics of the $3j$-symbols. We consider the case in which all of the quantum numbers get large together. What this means is multiplying all of the quantum numbers by a number and studying the asymptotic behavior of the Clebsch-Gordan coefficient as this multiplier gets large. Such a multiplier is often called $1/\hbar$, so that the limit of large quantum numbers is the limit of small $\hbar$.

The history of this subject dates back to the early days of quantum mechanics and the study of the classical limit of quantum mechanical quantities. Numerous papers have been written in this area. We summarize the literature briefly here. In 1959, Wigner [1] discussed the physical interpretation and classical limits of Clebsch-Gordan coefficients. He described a certain average behavior, and did not analyze the oscillatory nature of the Clebsch-Gordan coefficients. There are references in this work to Edmonds [2] and Brussaard and Tolhoek [3]. In 1968, Ponzano and Regge [4] presented asymptotic expressions that included the oscillations. Their work included an interpretation of certain angles that occur in their results and in ours. Additionally, they discussed the allowed and forbidden regions. However their derivation is, in their words, “rather heuristic.” It was borne out in their comparisons with the exact values. William Miller [5] derived similar expressions using semiclassical methods in 1974, but did not treat the forbidden region. Another work that relates to the present paper is that of Srinivasa Rao and V. Rajeswari [6]. It contains exact expressions for Clebsch-Gordan coefficients and their relationship to certain hypergeometric series. There is more information in the work of Biedenharn and Louck [7].

In this paper, we start by deriving an exact integral representation for the Clebsch-Gordan coefficients. Then the methods of stationary phase are used to approximate this integral. The allowed and forbidden regions are treated separately, and the resulting expressions are related to the literature. These methods are then used to derive higher-order results, that is, the next order in an expansion in $\hbar$. These formulas are accurate to five or six digits when the quantum numbers are in the hundreds.

Possible applications of this work include high-angular momentum calculations and theoretical investigations which contain sums over large numbers of Clebsch-Gordan coefficients.

II. EXACT EXPRESSIONS FOR THE CLEBSCH-GORDAN COEFFICIENT

Our starting point is an exact expression for the Clebsch-Gordan (vector-addition) coefficient, due to Wigner (see, for example, Eq. (3.6.11) of Ref. [2]),

$$\langle j_1 m_1 j_2 m_2 | j m \rangle = [(2j + 1)(j_1 + j_2 - j)(j_1 - j_2 + j)(-j_1 + j_2 + j)!(j_1 + j_2 + j + 1)!]^{1/2} \times [(j_1 + m_1)!/(j_1 - m_1)!](j_2 + m_2)!(j_2 - m_2)!/(j + m)!/(j - m)!\]^{1/2} \times \sum_z \frac{(-1)^z}{z!(j_1 + j_2 - j - z)(j_1 - m_1 - z)(j_2 + m_2 - z)!(j - j_1 - m_2 + z)!(j - j_2 + m_1 + z)!(j - j_1 - m_2 + z)!}.$$ (2.1)
A factor of $\delta_{m,m_1+m_2}$ has been omitted; throughout this paper we will assume that $m$ is equal to $m_1 + m_2$. Also, unless otherwise specified, sums over an index are sums over all integers. It will turn out, though, that the summand is nonzero for only finitely many values of the index.

We begin by deriving the following exact expression for the Clebsch-Gordan coefficient.

$$
(j_1m_1j_2m_2 | jm) = (-1)^{j+m}N_{j_1m_1j_2m_2jm}\frac{1}{(j_1-m_1)!(j_2-m_2)!}
$$

$$
\times \left( \frac{d}{du} \right)^{j_1-m_1} \left( \frac{d}{dt} \right)^{j_2-m_2} \left[ (t-1)^{j+j_2-j_1} (t-u)^{j_1+j_2-j} (u-1)^{j_1+j_2-j_2} \right]_{u=0,t=0}, \tag{2.2}
$$

where $N_{j_1m_1j_2m_2jm}$ is defined to be

$$
N_{j_1m_1j_2m_2jm} = \left[ \frac{(2j+1)(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j+m)!(j-m)!}{(j_1+j_2+j+1)!(j_1+j_2-j)!(j_1-j_2+j)!(j_1+j_2+j)!} \right]^{1/2}. \tag{2.3}
$$

Because the quantity being differentiated in Eq. (2.2) is a polynomial in the variables $u$ and $t$, the operation of differentiating this quantity and then evaluating the result at $u = 0$ and $t = 0$ simply selects a particular coefficient in the polynomial. Thus, Eq. (2.2) expresses the Clebsch-Gordan coefficient as a certain coefficient in a polynomial that can be written in closed form. This equation can be derived from results in the literature [10], but we give here an independent derivation of Eq. (2.2) from Eq. (2.1) to verify that all of the conventions involved are consistent.

In order to prove Eq. (2.2), we start by finding the coefficient of $u^{j_1-m_1}t^{j_2-m_2}$ in the polynomial $(t-1)^{j+j_2-j_1} (t-u)^{j_1+j_2-j} (u-1)^{j_1+j_2-j_2}$. This is equal to the coefficient of $u^{j_1-m_1}$ in the polynomial that is given by $(u-1)^{j+j_1-j_2}$ times the $u$-dependent coefficient of $t^{j_2-m_2}$ in the polynomial $(t-1)^{j+j_2-j_1} (t-u)^{j_1+j_2-j}$. Using the Binomial Theorem, we get

$$
(t-1)^{j+j_2-j_1} = \sum_k \binom{j + j_2 - j_1}{k} t^k (-1)^{j_1+j_2-j_1-k}, \tag{2.4}
$$

and

$$
(t-u)^{j_1+j_2-j} = \sum_\ell \binom{j_1 + j_2 - j}{\ell} t^\ell (-u)^{j_1+j_2-j-\ell}. \tag{2.5}
$$

The coefficient of $t^{j_2-m_2}$ in the product of these is

$$
\sum_k \binom{j + j_2 - j_1}{k} (-1)^{j_1+j_2-j_1-k} \binom{j_1 + j_2 - j}{j_2-m_2-k} (-u)^{j_1+j_2-j-(j_2-m_2-k)}
$$

$$
= (-1)^{j_2+m_2} u^{j_1-j+m_2} \sum_k \binom{j + j_2 - j_1}{k} \binom{j_1 + j_2 - j}{j_2-m_2-k} u^k. \tag{2.6}
$$

As explained above, we need to multiply this polynomial by
\[(u - 1)^{j+j_1-j_2} = \sum_\ell \left( \begin{array}{c} j + j_1 - j_2 \\ \ell \end{array} \right) u^\ell (-1)^{j+j_1-j_2-\ell}. \quad (2.7)\]

and find the coefficient of \(u^{j_1-m_1}\). The result is

\[
(1)^{j_2+m_2} \sum_k \left( \begin{array}{c} j + j_2 - j_1 \\ k \end{array} \right) \left( \begin{array}{c} j_1 + j_2 - j \\ j_2 - m_2 - k \end{array} \right) \left( \begin{array}{c} j + j_1 - j_2 \\ j - m - k \end{array} \right) (-1)^{j_1-j_2+m+k}
= (1)^{j+m} \sum_z (-1)^z \left( \begin{array}{c} j + j_2 - j_1 \\ j_2 - m_2 - z \end{array} \right) \left( \begin{array}{c} j_1 + j_2 - j \\ z \end{array} \right) \left( \begin{array}{c} j + j_1 - j_2 \\ j_1 - m_1 - z \end{array} \right), \quad (2.8)
\]

where we have redefined the index of summation according to \(k = j - j_1 - m_2 + z\) in the final line of this equation and made use of the identity \(\left( \frac{a}{b} \right) = \left( \frac{a}{a-b} \right)\). Since \(j + m\) is always an integer, \((-1)^{2(j+m)}\) is equal to one, and we have shown that the right-hand side of Eq. (2.2) is equal to

\[
N_{j_1 m_1 j_2 m_2 j m} \sum_z (-1)^z \left( \begin{array}{c} j + j_2 - j_1 \\ j_2 - m_2 - z \end{array} \right) \left( \begin{array}{c} j_1 + j_2 - j \\ z \end{array} \right) \left( \begin{array}{c} j + j_1 - j_2 \\ j_1 - m_1 - z \end{array} \right)
= \sum_z \frac{(-1)^z N_{j_1 m_1 j_2 m_2 j m} (j + j_2 - j_1)! (j_1 + j_2 - j)! (j + j_1 - j_2)!}{(j - j_1 - m_2 + z)! (j_2 + m_2 - z)! (j_1 + j_2 - j - z)! z! (j_1 - m_1 - z)! (j - j_2 + m_1 + z)!}, \quad (2.9)
\]

This is the same as the right-hand side of Eq. (2.1) and completes the proof of Eq. (2.2). An alternative proof begins by introducing a factor of \(x^z\) into the sum in Eq. (2.1) and deriving a third-order differential equation for the resulting function of \(x\). This differential equation can be solved using hypergeometric functions, and the result eventually leads to the expression shown in Eq. (2.2).

Equation (2.2) can be used to obtain an exact expression for the Clebsch-Gordan coefficient as an integral. One uses the orthogonality of the functions \(\exp(\imath \theta)\) on the interval \([-\pi, \pi]\) to select the desired coefficients in the polynomials. Thus, we substitute \(\exp(i\theta)\) for \(t\) and \(\exp(i\phi)\) for \(u\) in the polynomial \((t - 1)^{j+j_2-j_1} (t - u)^{j_1+j_2-j} (u - 1)^{j_1+j_2} \) in Eq. (2.2), multiply by \(\exp[-i(j_1 - m_1)\phi - i(j_2 - m_2)\theta]\), and integrate the two variable from \(-\pi\) to \(\pi\). The resulting expression for the Clebsch-Gordan coefficient is

\[
\langle j_1 m_1 j_2 m_2 | j m \rangle = (-1)^{j+m} N_{j_1 m_1 j_2 m_2 j m} \frac{1}{(2\pi)^2} \times \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-i(j_1 - m_1)\phi - i(j_2 - m_2)\theta} (e^{\imath \phi} - 1)^{j+j_2-j_1} (e^{\imath \phi} - e^{\imath \phi})^{j_1+j_2-j} (e^{\imath \phi} - 1)^{j_1+j_2} d\theta d\phi. \quad (2.10)
\]

This may be rewritten using the definition of the sin function, whereupon it becomes natural to redefine the angles by a factor of two. The resulting form is
The selection of the coefficient in the polynomial can also be carried out with an integral. As above, the Clebsch-Gordan coefficient is related to the coefficient of a term in a polynomial in one variable, and thus as a one-dimensional integral. Equation (2.2) shows how the Clebsch-Gordan coefficient relates to the coefficient of $u^{-j_1-m_1} t^{j_2-m_2}$ in the polynomial $(t-1)^{j+j_2-j_1} (t-u)^{j_1+j_2-j} (u-1)^{j_1+j_2-j}$. This is the same as the coefficient of $u^{j_1-m_1} (u^M)^{j_2-m_2}$ in the polynomial $(u^M - 1)^{j+j_2-j_1} (u^M-u)^{j_1+j_2-j} (u-1)^{j_1+j_2-j}$ (that is, $t$ has been replaced by $u^M$) for sufficiently large integers $M$. This can be seen as follows.

We start by imagining the polynomial $(t-1)^{j+j_2-j_1} (t-u)^{j_1+j_2-j} (u-1)^{j_1+j_2-j}$ expanded out into a sum of monomials. If $t$ is replaced by $u^M$, each of the monomials is now just a coefficient times a power of $u$. We do not want any of these terms to have the same power of $u$, otherwise they would combine and the coefficients would change. Thus, we look at the original polynomial $(t-1)^{j+j_2-j_1} (t-u)^{j_1+j_2-j} (u-1)^{j_1+j_2-j}$ and ask what is the highest power of $u$ is. This is $(j_1 + j_2 - j) + (j + j_1 - j_2) = 2j_1$. We therefore select $M$ to be $2j_1 + 1$. The result is that the coefficient of $u^{j_1-m_1} t^{j_2-m_2}$ in the polynomial $(t-1)^{j+j_2-j_1} (t-u)^{j_1+j_2-j} (u-1)^{j_1+j_2-j}$ is the same as the coefficient of $u^{j_1-m_1+(2j_1+1)(j_2-m_2)}$ in the polynomial $(u^{2j_1+1} - 1)^{j+j_2-j_1} (u^{2j_1} - 1)^{j_1+j_2-j} (u-1)^{j_1+j_2-j}$.

We may drop an overall factor of $u^{j_1+j_2-j}$, so this coefficient is the same as the coefficient of $u^{j_1-m_1+2(j_2-m_2)}$ in the polynomial $(u^{2j_1+1} - 1)^{j+j_2-j_1} (u^{2j_1} - 1)^{j_1+j_2-j} (u-1)^{j_1+j_2-j}$. The resulting expression for the Clebsch-Gordan coefficient as a coefficient in a polynomial in one variable is

$$
\langle j_1 m_1 j_2 m_2 | j m \rangle = (-1)^{j+m} N_{j_1 m_1 j_2 m_2 j m} \frac{1}{[j-m+2j_1(j_2-m_2)]!} \times \left( \frac{d}{du} \right)^{j-m+2j_1(j_2-m_2)} [ (u^{2j_1+1} - 1)^{j+j_2-j_1} (u^{2j_1} - 1)^{j_1+j_2-j} (u-1)^{j_1+j_2-j} ]_{u=0}.
$$

As above, the selection of the coefficient in the polynomial can also be carried out with an integral.

$$
\langle j_1 m_1 j_2 m_2 | j m \rangle = (-1)^{j+m} N_{j_1 m_1 j_2 m_2 j m} \frac{1}{2\pi} \times \int_{-\pi}^{\pi} e^{-i(j-m+2j_1(j_2-m_2))\phi} (e^{i(2j_1+1)\phi} - 1)^{j+j_2-j_1} (e^{i2j_1\phi} - 1)^{j_1+j_2-j} (e^{i\phi} - 1)^{j_1+j_2-j} d\phi.
$$

This may be rewritten as

$$
\langle j_1 m_1 j_2 m_2 | j m \rangle = (-1)^{j+m} N_{j_1 m_1 j_2 m_2 j m} \frac{(2i)^{j+j_1+j_2}}{2\pi}.
$$
and this may be simplified to the form

\[
\langle j_1 m_1 j_2 m_2 | j m \rangle = (-1)^{j^+ m} N_{j_1 m_1 j_2 m_2 j m} \pi^{-1} 2^{j^+ j_1 + j_2} \times \int_0^\pi \cos[(2j_1 m_2 + m)\phi + \pi/2 (j + j_1 + j_2)] \times \sin^{j^+ j_2 - j_1}[(j_1 + 1/2)\phi] \sin^{j^+ j_1 - j_2}(\phi \phi / 2) d\phi.
\]

Although this is a one-dimensional integral (as opposed to the two-dimensional integral presented above), it seems to be not as useful for the study of asymptotics because of the presence of the magnetic quantum numbers in the argument of the cosine function.

III. STATIONARY-PHASE APPROXIMATION OF INTEGRAL EXPRESSION FOR THE CLEBSCH-GORDAN COEFFICIENT

In order to carry out a stationary-phase approximation of the integral expression for the Clebsch-Gordan coefficient presented in the previous section, we begin by writing the expression in Eq. (2.11) in the form

\[
\langle j_1 m_1 j_2 m_2 | j m \rangle = (-1)^{j^+ m} N_{j_1 m_1 j_2 m_2 j m} \pi^{-1} 2^{j^+ j_1 + j_2} \times \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \exp(g(\theta, \phi)) d\theta d\phi.
\]

where the function \( g(\theta, \phi) \) is defined to be

\[
g(\theta, \phi) = 2im_1 \phi + 2im_2 \theta + (j + j_2 - j_1) \ln(\sin \theta) + (j_1 + j_2 - j) \ln[\sin(\theta - \phi)] + (j + j_1 - j_2) \ln(\sin \phi).
\]

Note that \( g(\theta, \phi) \) has singularities where it goes to \(-\infty\), but the integral is still well-defined because the integrand is \( \exp(g) \).

To find the stationary-phase points (as explained in Appendix A), we must first compute the first derivatives of the function \( g \).

\[
\frac{\partial g}{\partial \theta} = 2im_2 + (j + j_2 - j_1) \cot \theta + (j_1 + j_2 - j) \cot(\theta - \phi),
\]

\[
\frac{\partial g}{\partial \phi} = 2im_1 - (j_1 + j_2 - j) \cot(\theta - \phi) + (j + j_1 - j_2) \cot \phi.
\]

Setting these first derivatives equal to zero results in a system of two equations in two variables. The identity

\[
\cot(\theta - \phi) = \frac{1 + \cot \theta \cot \phi}{\cot \phi - \cot \theta}
\]

may be used to transform this system to an equivalent system.
\[2im + (j + j_2 - j_1) \cot \theta + (j + j_1 - j_2) \cot \phi = 0 \quad (3.6)\]

\[2im_2 + (j + j_2 - j_1) \cot \theta + (j_1 + j_2 - j) \frac{1 + \cot \theta \cot \phi}{\cot \phi - \cot \theta} = 0. \quad (3.7)\]

In order to be clear on phase conventions, choices of signs and branch cuts we write out the steps involved in solving this system of two equations for \(\cot \theta\) and \(\cot \phi\). We start by multiplying the second equation by \((j + j_1 - j_2)(\cot \phi - \cot \theta)\) and substituting in first one:

\[
[2im_2 + (j + j_2 - j_1) \cot \theta] \{ -[2im + (j + j_2 - j_1) \cot \theta] - (j + j_1 - j_2) \cot \theta \} + (j_1 + j_2 - j)(j + j_1 - j_2) - \cot \theta [2im + (j + j_2 - j_1) \cot \theta] \} = 0. \quad (3.8)
\]

This is a quadratic equation in \(\cot \theta\).

\[
cot^2 \theta [(j + j_2 - j_1)(-2j) - (j_1 + j_2 - j)(j + j_2 - j_1)] + \cot \theta [(j + j_2 - j_1)(-2im) + 2im_2(-2j) + (j_1 + j_2 - j)(j + j_1 - j_2) = 0. \quad (3.9)
\]

Simplifying this results in

\[- \cot^2 \theta (j + j_2 - j_1)(j_1 + j_2 + j)
- 4i \cot \theta (j_2m + m_2j) + 4m_2m + (j_1 + j_2 - j)(j + j_1 - j_2) = 0. \quad (3.10)
\]

The two solutions for the quantities \(\cot \theta\) and \(\cot \phi\) are (the upper choice of sign is one solution and the lower choice of sign is the other).

\[
\cot \theta = \frac{-2i (j_2m + m_2j) \mp \beta}{(j_1 + j_2 + j)(j + j_2 - j_1)},
\]

\[
\cot \phi = \frac{-2i (j_1m + m_1j) \pm \beta}{(j_1 + j_2 + j)(j + j_1 - j_2)}, \quad (3.11)
\]

where \(\beta\) is defined to be

\[
\beta = \sqrt{4m_1m_2j^2 - 4mm_1j_2^2 - 4mm_2j_1^2 + (j_1 + j_2 - j)(j + j_2 - j_1)(j + j_1 - j_2)(j_1 + j_2 + j)}. \quad (3.12)
\]

In this equation, we use the usual choice of branch cut for the square-root function: if the argument is negative, then the result is a positive number times the imaginary unit. As discussed in Sec. [IIA], the quantity \(\beta\) is real for classically allowed sets of quantum numbers, and it is pure imaginary for classically forbidden sets of quantum numbers. It should be noted that this is the same definition for the symbol \(\beta\) as in Ref. [I].

The stationary-phase approximation of the integral \(\int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} e^{i \theta \phi} d\theta d\phi\) that appears in the expression for the Clebsch-Gordan coefficient in Eq. (3.11) is given by a sum of terms of the form
when expressing sine quantities sine-functions appear. Using the identity below the negative imaginary axis, as is usual. The symbol $\frac{\partial^2 g}{\partial(\theta, \phi)^2}$ denotes the $2 \times 2$ Hessian matrix of second-order derivatives of the function $g(\theta, \phi)$, whose entries are given by

\[
\begin{align*}
\frac{\partial^2 g}{\partial^2 \theta} &= - (j + j_2 - j_1) \csc^2 \theta - (j_1 + j_2 - j) \csc^2 (\theta - \phi), \\
\frac{\partial^2 g}{\partial \theta \partial \phi} &= (j_1 + j_2 - j) \csc^2 (\theta - \phi), \\
\frac{\partial^2 g}{\partial \phi^2} &= -(j_1 + j_2 - j) \csc^2 (\theta - \phi) - (j_1 + j_2 - j) \csc^2 \phi.
\end{align*}
\]

Using the identity $\csc^2 \theta = 1 + \cot^2 \theta$ these quantities can be expressed in terms of the cotangents in Eq. (3.11) without addressing the issue of branch cuts of the arc-cotangent function. The value of $\csc^2 (\theta - \phi)$ can be determined from the quantities in Eq. (3.11) using the identity $\sin(\theta - \phi) = \sin \theta \cos \phi - \sin \phi \cos \theta = \sin \theta \sin (\cot \phi - \cot \theta)$. The determinant becomes

\[
\begin{align*}
det \frac{\partial^2 g}{\partial(\theta, \phi)^2} &= (j + j_2 - j_1) \csc^2 \theta (j + j_1 - j_2) \csc^2 \phi \\
&+ (j_1 + j_2 - j) \csc^2 (\theta - \phi) [(j + j_2 - j_1) \csc^2 \theta + (j_1 + j - j_2) \csc^2 \phi], \\
&= (1 + \cot^2 \theta)(1 + \cot^2 \phi) \{(j + j_2 - j_1)(j + j_1 - j_2) \\
&+ \frac{(j_1 + j_2 - j)}{(\cot \phi - \cot \theta)^2} [(j + j_2 - j_1)(1 + \cot^2 \theta) + (j_1 + j - j_2)(1 + \cot^2 \phi)]\}. \quad (3.15)
\end{align*}
\]

In this form the determinant is expressed entirely in terms of the cotangents of $\theta$ and $\phi$. The quantity $e^{i\theta} = e^{g(\theta, \phi)}$ in Eq. (3.13)

\[
e^{2im_1 \phi + 2im_2 \theta} \sin^{j_1+2} \sin^{j_1+2-j_1} \theta \sin^{j_1+2-j_1} (\theta - \phi) \sin^{j_1+2-j_1} \phi
\]

we obtain for the factor $e^{g(\theta, \phi)}$ in Eq. (3.13)

\[
e^{2im_1 \phi + 2im_2 \theta} \sin^{j_1+2} \sin^{j_1+2-j_1} \theta \sin^{j_1+2-j_1} (\theta - \phi) \sin^{j_1+2-j_1} \phi
= (i + \cot \phi)^{2m_1} (i + \cot \theta)^{2m_2} \sin^{j_1+2-j_1+2m_1} \sin^{j_1+2-j_1+2m_2} \theta \sin^{j_1+2-j_1+2m_1} (\theta - \phi) \sin^{j_1+2-j_1+2m_1} \phi
= (i + \cot \phi)^{2m_1} (i + \cot \theta)^{2m_2} \sin^{j_1+2-j_1+2m_1} \theta \sin^{j_1+2-j_1+2m_2} (\theta - \phi) \sin^{j_1+2-j_1+2m_1} \phi
= (i + \cot \phi)^{2m_1} (i + \cot \theta)^{2m_2} (1 + \cot^2 \theta)^{j_1+2-j_1+2m_1} \sin^{j_1+2-j_1+2m_1} \phi
= (i + \cot \phi)^{2m_1} (i + \cot \theta)^{2m_2} (1 + \cot^2 \theta)^{j_1+2-j_1+2m_1} \sin^{j_1+2-j_1+2m_1} \phi
\]

Using this equation and Eq. (3.15), all of the quantities in the expression in Eq. (3.13) can be expressed in terms of the cotangents of $\theta$ and $\phi$, given in Eq. (3.11). It should be noted that all of the exponents in Eq. (3.17) are integers, so choices of branch cuts are not necessary.
A. Allowed region

It is useful to introduce the concepts of a triangle-allowed region and a classically allowed region of the space of values for the quantum numbers. We define the triangle-allowed region to be the set of quantum numbers for which \( j_1, j_2 \) and \( j \) satisfy the triangle inequalities and for which the inequalities \(|m| \leq j\) and \( \{|m_i| \leq j_i, \ i = 1, 2\} \) hold. The Clebsch-Gordan coefficient is zero outside of this region, so it is only within this region that asymptotic expressions are desired. The triangle-allowed region is divided into a classically allowed region and a classically forbidden region. As is usual, we call these the allowed and forbidden regions for brevity. The allowed region is defined to be the set of quantum numbers for which it is possible to define \( j \)-vectors in a three-dimensional space in such a way that their lengths are equal to the \( j \)-values and their \( z \)-components are equal to the \( m \)-values (and, of course, such that \( j = j_1 + j_2 \)). An example of such a construction for a set of allowed quantum numbers is shown in Fig. 1. It follows from the definition that the allowed region is contained in the triangle-allowed region. Examples of classically forbidden points are easily found in extreme cases, such as \( m_1 = j_1 \). In this case, there is only one classically allowed value for \( m_2 \) (assuming a set of triangle-allowed \( j \)-values have been given), because the \( j_1 \)-vector must point in the \( z \)-direction, and thus the \( j \)-triangle lies in a vertical plane.

The allowed region is the same as the region in which the three \( \lambda \)-values defined in Eq. (3.25) satisfy the triangle inequalities. This is because the \( \lambda \)-values are the lengths of the projections of the \( j \)-vectors into the \( xy \)-plane. If the \( \lambda \)-values satisfy the triangle inequalities, then it is possible to draw a triangle in the \( xy \)-plane with sides equal to the \( \lambda \)-values. From this, one can construct the \( j \)-vectors by simply including the \( m \)-values as \( z \)-components. Conversely, if the \( j \)-vectors can be constructed, then their projections into the \( xy \)-plane form a triangle (with the tail of \( j_2 \) at the tip of \( j_1 \)), and the \( \lambda \)-values satisfy the triangle inequalities.

It is explained later in this paper that the \( \lambda \)-values satisfy the triangle inequalities if and only if the quantity \((-\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1 - \lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 - \lambda_3)\) is nonnegative (that is, it is not possible for two of the factors to be negative). This observation together with the fact that the quantity \( \beta \) defined in Eq. (3.12) may be written as

\[
\beta = \sqrt{(\lambda_1 + \lambda_2 + \lambda_3)(-\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1 - \lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 - \lambda_3)}
\]  

leads us to the result that the sign of \( \beta^2 \) distinguishes the allowed and forbidden regions: it is positive in the allowed region, and it is negative in the forbidden region. In the allowed region, \( \beta \) is four times the area of the triangle whose sides are the \( \lambda \)-values. This triangle is the projection of the \( j \)-triangle into the \( xy \)-plane (see Fig. 1). From Eq. (3.12) it is apparent that for fixed values of the \( j \) quantum numbers, \( \beta^2 \) is a quadratic polynomial in the \( m \) quantum numbers. Thus, in the \((m_1, m_2)\) plane the boundary between the allowed and forbidden regions is an ellipse. This is shown shown in Fig. 2 for one choice of values for \( j_1, j_2 \) and \( j \). The boundary of the triangle-allowed region is the irregular hexagon. The forbidden region is composed of six subregions. The points that separate them are indicated in Fig. 2. These are the points where the ellipse that separates the allowed and forbidden regions is tangent to the hexagon that defines the triangle-allowed region. The coordinates of these points can be calculated from the expression for \( \beta \) and the equations for the straight-line sections of the boundary of the triangle-allowed region. The resulting coordinates of
these points are indicated in the figure.

The calculations involved in the stationary-phase approximation of the integral expression for the Clebsch-Gordan coefficient are different in the allowed and forbidden regions. We will treat the allowed region first. The sum over stationary-phase points for the case where the set of quantum numbers is in the allowed region is a sum over both of the two solutions for the cotangents of $\theta$ and $\phi$ given in Eq. (3.11). This is analogous to the behavior demonstrated in Appendix A, and it is also the same as in the calculation of the stationary-phase approximation of the Airy integral, which is the canonical example of a stationary-phase calculation. In the case of the Airy integral, there are allowed and forbidden regions in position space, and in the allowed region the contour of integration is deformed to run over both of the stationary-phase points. We define $c_\theta$ and $c_\phi$ to be the first solution for the cotangents in Eq. (3.11).

\[
c_\theta = \frac{-2i(j_2m + m_2 j) - \beta}{(j + j_2 - j_1)(j_1 + j_2 + j)},
\]
\[
c_\phi = \frac{-2i(j_1m + m_1 j) + \beta}{(j_1 + j_2 + j)(j + j_1 - j_2)},
\] (3.19)

Because of the form of the two solutions given in Eq. (3.11), the second solution is obtained from this one by multiplying by minus one and complex conjugating. Note this is only valid in the allowed region, where the quantity $\beta$ is real. The first term in the sum over stationary-phase points is given by plugging the expression for $\det \frac{\partial^2 g}{\partial(\theta, \phi)}$, given in Eq. (3.15), and the expression for $e^{\theta(\theta, \phi)}$, given in Eq. (3.17), into the quantity in Eq. (3.13), using $c_\theta$ and $c_\phi$ for the cotangents. The second term in the sum over stationary-phase points is the same, except $-c_\theta^*$ and $-c_\phi^*$ are used for the cotangents. The result for the determinant in the second term is obtained by simply complex conjugating the first value, since all of the cotangents in this expression are squared. As for the $e^{\theta(\theta, \phi)}$ factor in the second term, we start by considering the expression for this factor in the first term:

\[
\frac{(i + c_\theta)^{m_1 - j_1}}{(-i + c_\theta)^{j_1 + m_1}} \frac{(i + c_\theta)^{m_2 - j_2}}{(-i + c_\theta)^{j_2 + m_2}} (c_\phi^* - c_\theta^*)^{j_1 + j_2 - j}.
\] (3.20)

The complex conjugate of this is

\[
\frac{(-i + c_\phi^*)^{m_1 - j_1}}{(i + c_\phi^*)^{j_1 + m_1}} \frac{(-i + c_\phi^*)^{m_2 - j_2}}{(i + c_\phi^*)^{j_2 + m_2}} (c_\phi^* - c_\theta^*)^{j_1 + j_2 - j}
\]

\[
= (-1)^{j_1 + j_2 + j} \frac{(i - c_\phi^*)^{m_1 - j_1}}{(-i - c_\phi^*)^{j_1 + m_1}} \frac{(i - c_\phi^*)^{m_2 - j_2}}{(-i - c_\phi^*)^{j_2 + m_2}} (-c_\phi^* + c_\theta^*)^{j_1 + j_2 - j}
\] (3.21)

This shows that the $e^{\theta(\theta, \phi)}$ factor in the second term [which appears after the $(-1)^{j_1 + j_2 + j}$ in the last line of Eq. (3.21)] is $(-1)^{j_1 + j_2 + j}$ times the complex conjugate of the $e^{\theta(\theta, \phi)}$ factor in the first term. Thus the second term is $(-1)^{j_1 + j_2 + j}$ times the complex conjugate of the first term. It is therefore convenient to obtain the sum over stationary-phase points by taking $i^{j_1 + j_2 + j}$ times the first term, adding the complex conjugate of this product, and
then dividing by \(i^{j_1+j_2+j}\). Using the fact that the real part of a quantity \(x\) is given by \(\Re[x] = (x + x^*)/2\), our stationary-phase approximation for the integral expression for the Clebsch-Gordan coefficient in Eq. (3.1) can be written as

\[
\langle j_1 \ m_1 \ j_2 \ m_2 \ | \ j \ m \rangle \approx (-1)^{j+m} (2i)^{j_1+j_2} \pi^{-2} N_{j_1 \ m_1 \ j_2 \ m_2 \ j \ m} \frac{2}{\det \partial^2 g/\partial \theta^2} \sqrt{\det \partial^2 g/\partial \phi^2} \mathcal{C}(\theta,\phi)
\]

\[
\mathcal{C}(\theta,\phi) = \frac{\sqrt{\det \partial^2 g/\partial \theta^2 \partial \phi^2} \mathcal{C}(\theta,\phi)}{\det \partial^2 g/\partial \theta^2 \partial \phi^2}
\]

where the quantities \(\det \partial^2 g/\partial \theta^2\) and \(\mathcal{C}(\theta,\phi)\) are obtained from Eq. (3.15) and Eq. (3.17) using the \(c_\theta\) and \(c_\phi\) given in Eq. (3.14).

Although the expression in Eq. (3.22) gives a value that is a real number, it involves intermediate quantities that are complex. It is possible to transform this expression so that only real quantities are involved. This transformation is very lengthy, and it is not practical to describe it in detail here. Instead, we present an expression that is exactly equal to the expression in Eq. (3.22) in the allowed region. This equality can be verified most convincingly by substituting numerical values into the expressions and evaluating the results to high numerical precision (much higher than the level at which discrepancies would occur if order \(\hbar\) terms were dropped). A brief description of the transformation is the following. Every complex quantity \(x + iy\) that occurs in Eq. (3.22) is written as the product of a modulus and a phase, \(\sqrt{x^2 + y^2} \exp[i \tan^{-1}(y/x)]\), where care must be taken that correct branches are used for each \(x + iy\), that is, one must examine the quantities \(x\) and \(y\) to determine the range of phase factors \((x + iy)/\sqrt{x^2 + y^2}\) that can occur in the allowed region, and make branch choices accordingly. At some stages in the calculation, large polynomials are involved, and computer-aided symbol manipulation becomes useful in working with these. Our result may be put in the form

\[
\langle j_1 \ m_1 \ j_2 \ m_2 \ | \ j \ m \rangle \approx 2I_{j_1 \ m_1 \ j_2 \ m_2 \ j \ m} \sqrt{\frac{j}{\pi \beta}} \cos \left[ \chi + \frac{\pi}{4} - \pi(j + 1) \right],
\]

where \(\chi\) is defined to be

\[
\chi = \left(j_1 + \frac{1}{2}\right) \cos^{-1} \left[ \frac{(-m)(j_1^2 + j_2^2 - j^2) - m_2(j_1^2 + j^2 - j_2^2)}{2 \alpha \lambda_1} \right]
\]

\[
+ \left(j_2 + \frac{1}{2}\right) \cos^{-1} \left[ \frac{m_1(j_2^2 + j_2^2 - j_1^2) - (-m)(j_2^2 + j_1^2 - j^2)}{2 \alpha \lambda_2} \right]
\]

\[
+ \left(j + \frac{1}{2}\right) \cos^{-1} \left[ \frac{m_2(j_2^2 + j_2^2 - j_1^2) - m_1(j_2^2 + j_1^2 - j_2^2)}{2 \alpha \lambda_3} \right]
\]

\[
- m_1 \cos^{-1} \left[ \frac{\lambda_1^2 + \lambda_2^2 - \lambda_3^2}{2 \lambda_1 \lambda_3} \right]
\]

\[
+ m_2 \cos^{-1} \left[ \frac{\lambda_1^2 + \lambda_3^2 - \lambda_2^2}{2 \lambda_2 \lambda_3} \right]
\]

(3.24)
and
\[
\alpha = \sqrt{(j + j_1 + j_2)(-j + j_1 + j_2)(j - j_1 + j_2)(j + j_1 - j_2)}
\]
\[
\lambda_i = \sqrt{j_i^2 - m_i^2} \quad i = 1, 2
\]
\[
\lambda_3 = \sqrt{j_2^2 - m_2^2}.
\]

(3.25)

(The quantity \(\alpha\) is four times the area of the \(j\)-triangle shown in Fig. 1.) Note that the \(\cos^{-1}\) functions in Eq. (3.24) are the usual principal branch, whose range is the interval from zero to \(\pi\). The quantity \(I_{j_1 m_1 j_2 m_2 j m}\) is defined to be

\[
I_{j_1 m_1 j_2 m_2 j m} = \sqrt{(j + 1/2)(j + j_1 + j_2)}
\]
\[
\times \frac{f(j_1 + m_1) f(j_1 - m_1) f(j_2 + m_2) f(j_2 - m_2) f(j + m) f(j - m)}{f(j_1 + j_2 + j) f(j_1 + j_2 - j) f(j_1 - j_2 + j) f(-j_1 + j_2 + j)},
\]

(3.26)

where the function \(f\) is defined to be

\[
f(n) = \sqrt{\frac{n!}{\sqrt{2\pi n} n^n e^{-n}}},
\]

(3.27)

that is, \(f(n)\) is the square root of the ratio of \(n!\) to the Stirling approximation of \(n!\). Note that for large \(n\), \(f(n)\) approaches one. Thus, for large quantum numbers, \(I_{j_1 m_1 j_2 m_2 j m}\) approaches one. It differs from unity by a correction that is order \(\hbar\), as can be deduced from the discussion of the Stirling approximation in Appendix A. As mentioned above, we present our approximation in the form given in Eq. (3.23) so that the exact equality of this expression and the complex expression given in Eq. (3.22) can be verified numerically. The factor \(I_{j_1 m_1 j_2 m_2 j m}\) may be dropped without reducing the quality of the approximation, that is, the ratio of our approximation to the exact value differs from unity by a quantity that is order \(\hbar\). Thus, we may write our approximation in the form

\[
\langle j_1 m_1 j_2 m_2 | j m \rangle \approx 2 \sqrt{\frac{j}{\pi \beta}} \cos \left[ \chi - \pi \left( j + \frac{3}{4} \right) \right].
\]

(3.28)

Ponzano and Regge [4] give a geometrical interpretation of the five angles that occur in the expression for \(\chi\) in Eq. (3.24). An equation similar to Eq. (3.28) also appears in Ref. [4], but the \((j + 1/2)\) factors in \(\chi\) are included at the end of the calculation to improve the accuracy, and the \(\pi (j + 1)\) in Eq. (3.23) is missing so that the formula gives the wrong sign for even \(j\)-values and does not give the right magnitude for half-integer \(j\) values.

B. Forbidden region

In the forbidden region, only one of the stationary-phase points is used in the approximation. This is analogous to the situation in the Airy-function problem mentioned above, where there are dominant and subdominant branches, and in the forbidden region only the
subdominant branch exists. Similarly, the model problem in Appendix A shows how for the case of \( m > n \), two stationary-phase points are used, while for the case \( m < n \) only one stationary-phase point is involved. The choice of which of the two roots in Eq. (3.11) is to be used for our approximation of the Clebsch-Gordan coefficient is indicated in Table I. Given the \( m \)-values of a point in the forbidden region in Fig. 2, it is inconvenient to determine which subregion it is in by using nested if-then statements, because the relative ordering of, say, the \( m_2 \)-coordinates of the points on the boundaries between the forbidden subregions changes as the \( j \)-values are changed. A much simpler way to determine which branch to use is to find the sign of a certain polynomial which we describe here. As can be seen from Table I, the choice of branch alternates as one goes around the diagram in Fig. 2. Thus we use the sign of the product of three expressions that flip signs in the right way. Given the coordinates of one of the boundary points in the \((m_1, m_2)\)-plane, a vector perpendicular to it can be constructed by exchanging the coordinates and changing the sign of one of them. The dot-product of this vector and \((m_1, m_2)\) is a function on the \((m_1, m_2)\)-plane that changes sign at the boundary between the two subregions in question. Thus we are led to consider the sign of the function

\[
\left[ (m_1, m_2) \cdot (-2j_2^2, j^2 - j_1^2 - j_2^2) \right] \\
\times \left[ (m_1, m_2) \cdot (-j_2^2 + j_1^2 - 2j_2^2, j_2^2 + j_1^2 - j_2^2) \right] \\
\times \left[ (m_1, m_2) \cdot (-j_2^2 + j_1^2 + j_2^2, 2j_1^2) \right].
\]

If this quantity is positive (negative), then the upper (lower) choice of root in Eq. (3.11) is used. Once the cotangents of the angles at the stationary-phase point are determined, the approximation of the Clebsch-Gordan coefficient can be evaluated from the expression

\[
\langle j_1 m_1 j_2 m_2 | j m \rangle \approx (-1)^{j+m} (2i)^{j_1 + j_2} \pi^{-2} N_{j_1, m_1 j_2, m_2 j m} \frac{2\pi e^{i\theta(\phi)}}{\sqrt{\det \partial^2 g / \partial (\theta, \phi)^2}}. 
\]

(3.29)

All of the quantities needed to evaluate this expression were expressed in terms of the cotangents of the angles in Eqs. (3.15) and (3.17). It may be noted that in the forbidden region the cotangents become pure imaginary, as can be seen from Eq. (3.11). This behavior is similar to that in the model problem in Appendix A, where the angle suddenly jumps in terms of its real part (but the analogy is not perfect because in the model problem the cotangent is pure imaginary in both the region \( m > n \) and the region \( m < n \)).

Evaluating Eq. (3.29) results in a real value, although complex numbers are involved at intermediate steps. As in the case of our analysis in the allowed region, the expression may be transformed to a form that involves only operations with real numbers. This can be done in a way that parallels the previous calculation, with hyperbolic functions playing the role of trigonometric functions. The transformation involves choices of branch cuts and depends on which of the six subregions of the forbidden region one is working in. Thus there are six different all-real expressions for the forbidden region. In the interest of brevity, we will present only one of these here. In subregion \( VI \), the expression in Eq. (3.29) is exactly equal to
\[ (-1)^{j_2+m_2} 2 I_{j_1 m_1, j_2 m_2, j m} \sqrt{\frac{j}{\pi |\beta|}} \exp(-\chi^{(vi)}), \] (3.30)

where \( \chi^{(vi)} \) is defined to be

\[
\chi^{(vi)} = \left( j_1 + \frac{1}{2} \right) \cosh^{-1} \left[ \frac{-m(j_1^2 + j_2^2 - j_2^2) - m_2(j_1^2 + j_2^2 - j_2^2)}{\alpha \lambda_1} \right] \\
- \left( j_2 + \frac{1}{2} \right) \cosh^{-1} \left[ \frac{-m_1(j_2^2 + j_1^2 - j_1^2) - m(j_1^2 + j_2^2 - j_2^2)}{\alpha \lambda_2} \right] \\
- \left( j + \frac{1}{2} \right) \cosh^{-1} \left[ \frac{-m_2(j_1^2 + j_2^2 - j_2^2) + m_1(j_1^2 + j_2^2 - j_2^2)}{\alpha \lambda_3} \right] \\
- m \cosh^{-1} \left[ \frac{\lambda_1^2 + \lambda_2^2 - \lambda_3^2}{2\lambda_1\lambda_3} \right] \\
- m_2 \cosh^{-1} \left[ \frac{\lambda_1^2 + \lambda_2^2 - \lambda_3^2}{2\lambda_2\lambda_3} \right],
\] (3.31)

This all-real expression was derived by a very lengthy calculation, as in the case of the analysis in the allowed region. Again, an exact equality such as the one above can be checked easily by substituting in test numbers and evaluating to sufficient precision. As before, to actually use the approximation, one would drop the factor of \( I_{j_1 m_1, j_2 m_2, j m} \) since it can be approximated by unity, to the order that we are working in this section. All-real expressions for the other subregions of the forbidden region can most easily be obtained by using the symmetries of the Clebsch-Gordan coefficients to related the expressions for the different subregions. If one prefers not to work with six different expressions for the forbidden region, one can use the polynomial discussed above to select the required stationary-phase point and then plug this into the approximation given in Eq. (3.29). This requires operations with complex numbers, but is easier to implement in a computer program. Alternatively, to obtain an approximate value for the Clebsch-Gordan coefficient for a given point in the forbidden region, one could work with only one all-real expression for a particular forbidden subregion and use the symmetries of the Clebsch-Gordan coefficients to map the given point to a point that is within the subregion for which the expression is valid.

It is interesting to compare the all-real expressions obtained in the allowed region, Eq. (3.23), and in the forbidden region, Eq. (3.30). They are similar in form, but the behavior is oscillatory in the allowed region and exponentially decaying in the forbidden region. This is the behavior expected in quantum mechanical problems that have an allowed region and a forbidden region.

In the forbidden region, writing the approximation in an all-real form is illuminating because it makes it apparent that sign functions exist. We call the factor \((-1)^{j_2+m_2}\) in Eq. (3.30) a sign function. The remaining factors in that equation are all positive, so the sign function gives the sign of the result. However, since the result is an approximation of a Clebsch-Gordan coefficient, the sign function also gives the sign of the Clebsch-Gordan coefficient, at least in the asymptotic regime. Thus, the sign functions are actually properties of the Clebsch-Gordan coefficients themselves, for a given choice of phase conventions. We are using the conventions defined by Eq. (2.1). The existence of sign functions was not clear
from Eq. (2.1), which was our starting point. The sign functions for each of the six forbidden subregions are given in Table I.

The angle $\chi$, given in Eq. (3.24), that appears in our approximation in the allowed region can be rewritten in several different ways. The reason is that the angles that multiply the $m$’s in the equation for $\chi$ are two of the interior angles in the triangle formed by the three $\lambda$-values. If we call these angles $\alpha_1$, $\alpha_2$ and $\alpha_3$ (where $\alpha_i$ is the angle opposite the side of length $\lambda_i$), then we have

$$m_1 + m_2 = m,$$

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi.$$  \hspace{1cm} (3.32)

Thus, the vectors $(m_1, m_2, -m)$ and $(\alpha_1, \alpha_2, \alpha_3 - \pi)$ are both perpendicular to $(1, 1, 1)$, and their cross-product is parallel to $(1, 1, 1)$. Each of the three components of their cross product are thus equal, and each one could be used as part of $\chi$ in the allowed region,

$$m_1 \alpha_2 - m_2 \alpha_1 = m_2(\alpha_3 - \pi) + m\alpha_2 = -m\alpha_1 - m_1(\alpha_3 - \pi).$$  \hspace{1cm} (3.33)

In the forbidden region there is no such flexibility in how to write the corresponding terms, because there do not exist three angles corresponding to $\alpha_1$, $\alpha_2$ and $\alpha_3$. This is because it is not possible to form a triangle using the three $\lambda$-values. Three different quantities like

$$\frac{\lambda_3^2 + \lambda_1^2 - \lambda_2^2}{2\lambda_3\lambda_1}$$  \hspace{1cm} (3.34)

can be written down by cyclically permuting the indices, but only two of these can be used as arguments of the $\cosh^{-1}$ function in an all-real expression. This can be seen in the following way. The $\lambda$’s are nonnegative and if three nonnegative numbers fail to satisfy the triangle inequalities, exactly one triangle inequality is violated. [Proof: Let $\lambda_{\text{max}}$ be the largest value, $\lambda_{\text{mid}}$ be the middle value, and $\lambda_{\text{min}}$ be the smallest. Then $-\lambda_{\text{min}} + \lambda_{\text{mid}} + \lambda_{\text{max}} \geq 0$ and $\lambda_{\text{min}} - \lambda_{\text{mid}} + \lambda_{\text{max}} \geq 0$, so we must have $\lambda_{\text{min}} + \lambda_{\text{mid}} - \lambda_{\text{max}} < 0$.] Now we consider rewriting the expression

$$\frac{\lambda_3^2 + \lambda_1^2 - \lambda_2^2}{2\lambda_3\lambda_1} = 1 + \frac{(\lambda_3 - \lambda_1)^2 - \lambda_2^2}{2\lambda_3\lambda_1} = 1 - \frac{(\lambda_1 + \lambda_2 - \lambda_3)(\lambda_2 + \lambda_3 - \lambda_1)}{2\lambda_3\lambda_1}.$$  \hspace{1cm} (3.35)

This shows that of the three permutations of the expression in Eq. (3.34), exactly two will be greater than unity. It is these two that must be used as arguments of the $\cosh^{-1}$ function in an all-real expression. Thus there is no flexibility in ways to write the $m$-terms in $\chi$ as in the allowed region. Throughout each one of the six subregions of the forbidden region, a single triangle inequality for the $\lambda$’s is violated. It is not possible that one triangle inequality is violated in one part of a subregion and another triangle inequality is violated in another part of the same subregion because at the boundary between these two parts $\beta$ would be zero, as can be seen from Eq. (3.18). But $\beta^2$ is a quadratic polynomial in $m_1$ and $m_2$ [see Eq. (3.12)], the zero-contour of which is the ellipse in Fig. 2, so it is not possible for it to be zero along a curve in the forbidden region. The $\lambda$ that is largest in each subregion is indicated in Table I. The forms of all-real expressions in each of the forbidden subregions will reflect the fact that in each one of the subregions one of the $\lambda$’s is larger than the sum of the other two.
It remains to discuss the case of points that are on the boundary between the allowed and forbidden regions. The quantity $\beta$ in Eq. (3.12) is zero on this boundary, and since $\beta$ is invariant under the full 72-element symmetry group of the 3-j symbol \[^10\], the Clebsch-Gordan coefficient cannot be approximated on the boundary with the formulas presented in this paper. The reason is that $\beta$ occurs in the denominator in Eqs. (3.28) and (3.30). Since $\beta^2$ is a homogeneous polynomial in the quantum numbers, it will be zero for sets of quantum numbers equal to any multiple of a set of quantum numbers for which $\beta$ is zero. The behavior of the Clebsch-Gordan coefficients in the direction transverse to the boundary should be similar to that of the Airy function (see Ref. \[^4\]).

The invariance of $\beta$ under the 72-element symmetry group of the 3-j symbols may be shown as follows. We begin by constructing the $3 \times 3$ Regge array of linear combinations of quantum numbers, given in Ref. \[^10\]. For any integer $n$, we define the polynomial $p_n$ to be the sum of the $n$-th powers of the nine elements of this matrix. These polynomials are invariant under the symmetry group, because if two sets of quantum numbers are related by a Regge symmetry, we can construct the $3 \times 3$ Regge array for each set and compute $p_n$. The results are the same because of the commutativity of addition. It is possible to write $\beta^2$ in terms of the $p_n$.

\begin{equation}
\beta^2 = \frac{(p_4^2 - 6p_2 p_2^2 - 27p_2^2 + 108p_4)}{324}.
\end{equation}

(3.36)

The coefficients in this equation may be simplified slightly by using the relation $p_1 = 3(j_1 + j_2 + j)$. This equation proves the invariance of $\beta$ under the symmetry group.

An example of quantum numbers for which $\beta$ is zero is

\begin{equation}
(j_1, m_1, j_2, m_2, j, m) = (3, -2, 6, 4, 7, 2).
\end{equation}

(3.37)

This point is not on the edge of the triangle-allowed region. Points for which $\beta$ is zero and which are on the edge of the triangle-allowed region are easier to find. For example, one can choose $(j_1, m_1) = (j_2, m_2) = \frac{1}{2}(j, m)$. For such a point, $j_1 + j_2 - j$ is zero.

The reason we are unable to approximate the Clebsch-Gordan coefficient for cases in which $\beta$ is zero is that the determinant of the $2 \times 2$ matrix of second derivatives of $g(\theta, \phi)$ is zero at the stationary-phase points. This can be shown by plugging the solutions for the cotangents of $\theta$ and $\phi$ at a stationary-phase point into Eq. (3.15) for the determinant; the result has an overall factor of $\beta$ after being simplified [see Eq. (B7)]. This determinant appears in the denominator of Eq. (3.13), so our method cannot be applied. Note that when $\beta$ is zero, the two solutions for the cotangents at the stationary-phase point are the same [see Eq. (3.11)]. Also, it should be noted that if any of the $m$-values has its absolute value close to the corresponding $j$, then the corresponding $\lambda$ will be small [see Eq. (3.23)], and the area of the $\lambda$-triangle will be small. Thus, $\beta$ will be small, and the set of quantum numbers is close to the boundary. In contrast to this, there is no difficulty with the approximation if the $m$ values are close to zero. These considerations are mirrored in the approximation (using Stirling’s formula) of $N_{j_1 m_1 j_2 m_2 j m}$, defined in Eq. (2.3); no factorials of $m$-values appear, only factorials of $j - m, j + m$, etc.
IV. HIGHER-ORDER APPROXIMATION

The methods used in the previous sections can be extended to higher order. In this section, we derive the next correction to the previous results. The approximation that is obtained in this way gives results that are accurate to six digits, for example, when the quantum numbers are in the hundreds.

Let \((\theta_0, \phi_0)\) be a stationary-phase point, i.e. a point at which \(\frac{\partial g}{\partial \theta} = \frac{\partial g}{\partial \phi} = 0\). We write the Taylor expansion of the function \(g(\theta, \phi)\) about the point \((\theta_0, \phi_0)\) as a sum of homogeneous polynomials,

\[
g(\theta_0 + x, \phi_0 + y) = g_0 + g_2 + g_3 + g_4 + \ldots, \quad (4.1)
\]

where

\[
g_0 = g(\theta_0, \phi_0),
\]

\[
g_2 = g_{00}x^2/2 + g_{0\phi}xy + g_{\phi\phi}y^2/2,
\]

\[
g_3 = g_{000}x^3/6 + g_{0\phi\phi}x^2y + g_{\phi\phi\phi}xy^2 + g_{\phi\phi\phi}y^3/6,
\]

\[
g_4 = g_{0000}x^4/24 + g_{0\phi\phi\phi}x^3y + g_{\phi\phi\phi\phi}x^2y^2/4 + g_{\phi\phi\phi\phi}xy^3 + g_{\phi\phi\phi\phi}y^4/24, \quad (4.2)
\]

where, for example, \(g_{0000}\) is defined to be \(\frac{\partial^3 g}{\partial \theta^2 \partial \phi} \) at the stationary-phase point.

To obtain the next higher stationary-phase approximation for the Clebsch-Gordan coefficient, we terminate the series in Eq. (4.1) at the fourth-order term. The reason for this is explained below. Thus, the approximation of the function \(g\) has derivatives at the stationary-phase point \((\theta_0, \phi_0)\) that agree with those of \(g\) through fourth order. Our approximation of the integrand \(\exp(g)\) is

\[
\exp[g(\theta_0 + x, \phi_0 + y)] \approx \exp(g_0) \exp(g_2) \exp(g_3) \exp(g_4) \\
= \exp(g_0) \exp(g_2)(1 + g_3 + g_4^2/2 + \ldots)(1 + g_4 + g_4^2/2 + \ldots). \quad (4.3)
\]

When this is multiplied out, each of the terms may be integrated over the \(xy\)-plane in closed form. We are interested in the asymptotic behavior of the resulting terms. The question is how the terms behave when all of the quantum numbers \((j_1, m_1, j_2, m_2, j, m)\) are multiplied by the same factor (such a factor is called \(1/\hbar\), as explained in the introduction). The stationary-phase point \((\theta_0, \phi_0)\) is independent of the factor, i.e. \((\theta_0, \phi_0)\) is order \(\hbar^0\), as can be seen from Eq. (3.11). The second derivatives of \(g\) at \((\theta_0, \phi_0)\) are order \(1/\hbar\), as can be seen from Eq. (3.14), so to see how the integral of \(\exp(g_2)\) depends on \(\hbar\), we define new variables of integration to be the old variables times \(\hbar^{-1/2}\). From this it follows that the integral of \(\exp(g_2)\) is order \(\hbar\). By the same reasoning, the integral of a homogeneous quartic polynomial times \(\exp(g_2)\) is order \(\hbar^3\), and the integral of a homogeneous sixth-order polynomial times \(\exp(g_2)\) is order \(\hbar^4\). The polynomials \(g_4\) and \(g_3^2\) have coefficients that are order \(1/\hbar\) and \(1/\hbar^2\), respectively, so the integrals of these times \(\exp(g_2)\) are both order \(\hbar^2\). This is one order of \(\hbar\) smaller than the integral of \(\exp(g_2)\). The integral of a homogeneous polynomial of odd degree times \(\exp(g_2)\) vanishes due to antisymmetry. Thus, the next higher order approximation of the integral of \(\exp(g)\) is obtained by integrating

\[
\exp(g_0) \exp(g_2)(1 + g_4 + g_4^2/2). \quad (4.4)
\]
Terms coming from $g_5$, etc, contribute at higher orders.

To find the ratio of the integral of $g_4 \exp(g_2)$ to the integral of $\exp(g_2)$ the following integrals are necessary.

\[
i_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(g_2) \, dx \, dy,
\]
\[
i_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^4 \exp(g_2) \, dx \, dy,
\]
\[
i_3 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^3 y \exp(g_2) \, dx \, dy,
\]
\[
i_4 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 y^2 \exp(g_2) \, dx \, dy.
\] (4.5)

We will need the ratios $i_2/i_1$, $i_3/i_1$ and $i_4/i_1$. The integrals are tabulated and these ratios can be worked out without the use of any information about relationships between the various derivatives of the function $g$ at the stationary-phase point. The results are

\[
i_2/i_1 = \frac{3g_{\theta\phi}^2}{(g_{\theta\theta} g_{\phi\phi} - g_{\theta\phi}^2)^2},
\]
\[
i_3/i_1 = \frac{-3g_{\theta\phi} g_{\phi\theta}}{(g_{\theta\theta} g_{\phi\phi} - g_{\theta\phi}^2)^2},
\]
\[
i_4/i_1 = \frac{2g_{\theta\theta}^2 + g_{\theta\phi} g_{\phi\phi}}{(g_{\theta\theta} g_{\phi\phi} - g_{\theta\phi}^2)^2}.
\] (4.6)

It may be noted that $g_{\theta\theta} g_{\phi\phi} - g_{\theta\phi}^2$ is the determinant of the $2 \times 2$ matrix of second partial derivatives of the function $g$.

The ratio, which we denote by $\delta_4$, of the integral of $g_4 \exp(g_2)$ to the integral of $\exp(g_2)$ works out to be

\[
\delta_4 = \frac{g_{\theta\theta\theta\theta} g_{\phi\phi}^2 - 4g_{\theta\theta\theta\phi} g_{\theta\phi} g_{\phi\phi} + 2g_{\theta\theta\phi\phi}(2g_{\theta\theta}^2 + g_{\theta\phi} g_{\phi\phi}) - 4g_{\theta\theta\phi\phi} g_{\theta\phi} g_{\phi\phi} + g_{\phi\phi\phi\phi} g_{\theta\theta}^2}{8(g_{\theta\theta} g_{\phi\phi} - g_{\theta\phi}^2)^2}
\] (4.7)

Next, we move on to the $g_3^2$ term in Eq. (4.4). To find the ratio of the integral of $\frac{1}{2}g_3^2 \exp(g_2)$ to the integral of $\exp(g_2)$ the following integrals are necessary.

\[
i_5 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^6 \exp(g_2) \, dx \, dy,
\]
\[
i_6 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^5 y \exp(g_2) \, dx \, dy,
\]
\[
i_7 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^4 y^2 \exp(g_2) \, dx \, dy,
\]
\[
i_8 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^3 y^3 \exp(g_2) \, dx \, dy.
\] (4.8)

We will need the ratios $i_5/i_1$, $i_6/i_1$, $i_7/i_1$ and $i_8/i_1$. As in the case of the $g_4$ calculation, the integrals are tabulated and these ratios can be worked out without the use of any information
about relationships between the various derivatives of the function \( g \) at the stationary-phase point. The results are

\[
\begin{align*}
i_5/i_1 &= \frac{-15g_{\phi\phi}^3}{(g_{\phi\phi}g_{\phi\phi} - g_{\phi\phi}^2)^3}, \\
i_6/i_1 &= \frac{15g_{\phi\phi}g_{\phi\phi}^2}{(g_{\phi\phi}g_{\phi\phi} - g_{\phi\phi}^2)^3}, \\
i_7/i_1 &= \frac{-3g_{\phi\phi}(4g_{\phi\phi}^2 + g_{\phi\phi}g_{\phi\phi})}{(g_{\phi\phi}g_{\phi\phi} - g_{\phi\phi}^2)^3}, \\
i_8/i_1 &= \frac{3g_{\phi\phi}(2g_{\phi\phi}^2 + 3g_{\phi\phi}g_{\phi\phi})}{(g_{\phi\phi}g_{\phi\phi} - g_{\phi\phi}^2)^3}.
\end{align*}
\]  

(4.9)

The ratio, which we denote by \( \delta_6 \), of the integral of \( \frac{1}{2}g_3^2 \exp(g_2) \) to the integral of \( \exp(g_2) \) works out to be

\[
\delta_6 = \left[ 2g_{\phi\phi}(3g_{\phi\phi}g_{\phi\phi} + 2g_{\phi\phi}^2)(g_{\phi\phi\phi\phi}g_{\phi\phi\phi} + 9g_{\phi\phi\phi\phi}g_{\phi\phi\phi}\right] \\
- 3(g_{\phi\phi}g_{\phi\phi} + 4g_{\phi\phi}^2)(2g_{\phi\phi\phi\phi}g_{\phi\phi\phi} + 2g_{\phi\phi\phi\phi}g_{\phi\phi\phi}g_{\phi\phi} + 3g_{\phi\phi\phi\phi}g_{\phi\phi}g_{\phi\phi} + 3g_{\phi\phi\phi\phi}g_{\phi\phi}g_{\phi\phi}g_{\phi\phi}) \\
+ 30g_{\phi\phi}(g_{\phi\phi\phi\phi}g_{\phi\phi\phi} g_{\phi\phi\phi}\phi_{\phi\phi} + g_{\phi\phi\phi\phi}g_{\phi\phi\phi}g_{\phi\phi\phi}\phi_{\phi\phi}) \\
- 5(g_{\phi\phi\phi\phi}g_{\phi\phi\phi\phi} + g_{\phi\phi\phi\phi}^2 g_{\phi\phi\phi\phi})/[24(g_{\phi\phi}g_{\phi\phi} - g_{\phi\phi}^2)^3].
\]  

(4.10)

The only remaining matter is to find the necessary values of the higher derivatives of the function \( g \) at the stationary-phase point. The definition of \( g \), given in Eq. (3.3) can be used to find its second derivatives, given in Eq. (3.14), its third derivatives,

\[
\begin{align*}
\frac{\partial^3 g}{\partial \theta^3} &= 2(j + j_2 - j_1) \csc^2 \theta \cot \theta + 2(j_1 + j_2 - j) \csc^2(\theta - \phi) \cot(\theta - \phi), \\
\frac{\partial^3 g}{\partial \theta^2 \partial \phi} &= -2(j_1 + j_2 - j) \csc^2(\theta - \phi) \cot(\theta - \phi), \\
\frac{\partial^3 g}{\partial \theta \partial \phi^2} &= 2(j_1 + j_2 - j) \csc^2(\theta - \phi) \cot(\theta - \phi), \\
\frac{\partial^3 g}{\partial \phi^3} &= 2(j_1 + j_2 - j) \csc^2(\theta - \phi) \cot(\phi - \theta) + 2(j_1 + j_2 - j) \csc^2 \phi \cot \phi,
\end{align*}
\]  

(4.11)

and its fourth derivatives,

\[
\begin{align*}
\frac{\partial^4 g}{\partial \theta^4} &= -2(j + j_2 - j_1) \csc^2 \theta (3 \cot^2 \theta + 1) \\
&\quad - 2(j_1 + j_2 - j) \csc^2(\theta - \phi)[3 \cot^2(\theta - \phi) + 1], \\
\frac{\partial^4 g}{\partial \theta^3 \partial \phi} &= 2(j_1 + j_2 - j) \csc^2(\theta - \phi) [3 \cot^2(\theta - \phi) + 1], \\
\frac{\partial^4 g}{\partial \theta^2 \partial \phi^2} &= -2(j_1 + j_2 - j) \csc^2(\theta - \phi) [3 \cot^2(\theta - \phi) + 1], \\
\frac{\partial^4 g}{\partial \theta \partial \phi^3} &= 2(j_1 + j_2 - j) \csc^2(\theta - \phi) [3 \cot^2(\theta - \phi) + 1],
\end{align*}
\]
\[
\frac{\partial^4 g}{\partial \phi^4} = -2(j_1 + j_2 - j) \csc^2(\theta - \phi) [3 \cot^2(\theta - \phi) + 1] \\
- 2(j + j_1 - j_2) \csc^2 \phi (3 \cot^2 \phi + 1).
\] (4.12)

Given the values of the cotangents of \( \theta \) and \( \phi \) at a stationary-phase point, these derivatives can be evaluated without having to find the angles, i.e. without having to make any choices of branch cuts. One way to do this is to use Eq. (3.13) to evaluate \( \cot(\theta - \phi) \), and the identity \( \csc^2 \theta = 1 + \cot^2 \theta \) to evaluate the squared cosecants.

The relationship between the Clebsch-Gordan coefficient and the integral of \( \exp(g) \) is given in Eq. (3.1). Combining this with our higher-order approximation for the integral results in the following higher-order approximation for the Clebsch-Gordan coefficient. Each stationary-phase point contributes

\[
(-1)^{j+m} (2i)^{j_1+j_2} \pi^{-2} N_{j_1,m_1,j_2,m_2,j,m} \frac{2\pi e^{g(\theta_0,\phi_0)}}{\sqrt{g_{\theta\theta} g_{\phi\phi} - g_{\theta\phi}^2}} (1 + \delta_4 + \delta_6). \tag{4.13}
\]

As explained in the previous sections, for points in the allowed region the sum over stationary-phase points is a sum over both of the solutions for the cotangents of \( \theta \) and \( \phi \) given in Eq. (3.11), and for points in the forbidden region only one of these solutions contributes. Given values for \( \cot \theta \) and \( \cot \phi \), Eqs. (3.14), (4.11) and (4.12) are used to evaluate the higher derivatives of the function \( g \) at the stationary-phase point. Then Eqs. (4.7) and (4.10) are used to obtain \( \delta_4 \) and \( \delta_6 \). As shown in Eq. (3.17), the quantity \( e^{g(\theta_0,\phi_0)} \) can also be evaluated using the values of the cotangents of \( \theta \) and \( \phi \). The quantity \( N_{j_1,m_1,j_2,m_2,j,m} \) can be approximated to sufficient accuracy using the next correction to Stirling’s approximation for the factorials.

Finally, we present some numerical examples. We begin with the allowed region.

For \((j_1, m_1, j_2, m_2, j, m) = (200, 100, 300, 150, 400, 250)\), the values are

\[
\text{exact} = 0.0703499, \\
\text{approx} = 0.0703496. \tag{4.14}
\]

For \((j_1, m_1, j_2, m_2, j, m) = (200, 100, 300 + 1/2, 150 + 1/2, 400 + 1/2, 250 + 1/2)\), the values are

\[
\text{exact} = 0.0730636, \\
\text{approx} = 0.0730633. \tag{4.15}
\]

In the forbidden region, the Clebsch-Gordan coefficients are much smaller. The following examples are from subregion I.

For \((j_1, m_1, j_2, m_2, j, m) = (200, 150, 300, -250, 400, -100)\), the values are

\[
\text{exact} = 3.08961 \times 10^{-19}, \\
\text{approx} = 3.08958 \times 10^{-19}. \tag{4.16}
\]

For \((j_1, m_1, j_2, m_2, j, m) = (200, 150, 300 + 1/2, -250 + 1/2, 400 + 1/2, -100 + 1/2)\), the values are
Further examples of results from the higher-order approximation are discussed in Appendix B.

V. CONCLUSION

The methods presented in this paper provide simple formulas for calculating first-order approximations to Clebsch-Gordan coefficients in the allowed region and in all of the forbidden subregions. Additionally, a higher-order approximation is derived, although the expressions are more complicated. We do not know if the quantity \( \delta_4 + \delta_6 \) in Eq. (4.13) can be simplified when expressed in terms of the quantum numbers (see Appendix B for a special case). It appears to be complicated, as is often the case for higher-order approximations. The geometrical structure is not as clear.

Our higher-order approximation provides the only known way to compute certain digits of some Clebsch-Gordan coefficients. By this we mean that given any computer, we can always find quantum numbers large enough so that the exact calculation is not feasible. The beginning digits may be calculated using first-order approximations; the higher-order approximation makes it possible to compute further digits.

The methods of this paper could also be used to derive asymptotic expressions for the \( 6j \)-symbols, etc. The starting point would again be an exact expression for the quantity of interest. One would then have to construct a polynomial with the property that the coefficient of one of its terms is this exact expression. Then an integral expression would be obtained, and finally this integral would be approximated using the stationary-phase method.

As mentioned in the introduction, this work could have applications in high-angular momentum calculations and theoretical investigations which contain sums over large numbers of Clebsch-Gordan coefficients [8,9].

Our analysis in the forbidden region led us to the realization that simple sign functions exist there that give the sign of the exact Clebsch-Gordan coefficients. These are summarized in Table I.

A subject for future work is the approximation of Clebsch-Gordan coefficients and \( 6j \)-symbols near the boundary between the allowed and forbidden regions. Ponzano and Regge [4] have conjectured and supplied numerical evidence for a typical Airy-function caustic behavior. Also, of course, it should be possible to extend the present calculations to even higher orders.

APPENDIX A: A ONE-DIMENSIONAL EXAMPLE

In this appendix we consider a one-dimensional example of an integral that gives a Fourier coefficient of a function which is an integer power of a fixed function. We are interested in the asymptotics of the result for large values of the two integers involved.

We define the function \( F(m, n) \) for positive integers \( m \) and \( n \) by

\[
\begin{align*}
\text{exact} &= 5.32718 \times 10^{-19}, \\
\text{approx} &= 5.32712 \times 10^{-19}.
\end{align*}
\]
\[ F(m, n) = \int_{-\pi/2}^{\pi/2} \cos^n x \, e^{imx} \, dx. \]  

(A1)

It is possible to evaluate this integral exactly in closed form:

\[
F(m, n) = \begin{cases} 
2^{-n} \pi \left( \frac{n}{(n-m)/2} \right) & n - m \text{ even} \\
(-1)^{(n+1-m)/2} 2^{n+2} n! \frac{[(m+n+1)/2]! (m-n-1)!}{(m+n+1)! [(m-n-1)/2]!} & n - m \text{ odd, } n < m \\
2^{n+2} n! \frac{[(n+1-m)/2]! [(n+1+m)/2]!}{(n+1-m)! (n+1+m)!} & n - m \text{ odd, } n > m
\end{cases}
\]  

(A2)

In deriving these results, one uses the definition of the beta function, \( B(z + 1, w + 1) = \int_0^1 t^z (1 - t)^w \, dt \), and the relation between the beta function and the gamma function, \( B(z, w) = \Gamma(z) \Gamma(w) / \Gamma(z + w) \). One also uses the results that for integers \( k \geq 0 \),

\[
\Gamma(k + \frac{1}{2}) = \frac{\sqrt{\pi} (2k)!}{2^{2k} k!}, \\
\Gamma(-k + \frac{1}{2}) = (-1)^k \frac{2^{2k} \sqrt{\pi} k!}{(2k)!}.
\]  

(A3)

1. Asymptotics of the exact expressions

In order to compare the stationary-phase approximations derived in the following subsection with the exact value of \( F(m, n) \), we will use Stirling’s approximation for the factorials in the exact expressions in Eq. (A2). The accuracy to which we will work is that the ratio of the exact value to the approximation should go to unity as \( n \) and \( m \) go to infinity, holding the ratio of \( n \) to \( m \) fixed. The difference between the logarithm of the exact expression and the logarithm of the approximation thus goes to zero as the two integers get large (the errors are of order \( 1/n \)).

Stirling’s approximation, through order unity (for the logarithms), is

\[ x! \approx \sqrt{2\pi x} \, x^x \, e^{-x}. \]  

(A4)

The next correction to this is a multiplicative factor of \( e^{1/(12x)} \). Thus, the ratio of \( x! \) to the approximation given in Eq. (A4) approaches unity as \( x \) goes to infinity.

Our approximation of the exact expression for \( F(m, n) \) works out to be

\[
F(m, n) \approx \begin{cases} 
\sqrt{\frac{2\pi}{n}} \left( \frac{1 - \frac{m}{n}}{1 + \frac{m}{n}} \right)^{m/2} \left[ 1 - \left( \frac{m}{n} \right)^2 \right]^{-(n+1)/2} & n > m \\
0 & n - m \text{ even, } n < m \\
(-1)^{(n+1-m)/2} \sqrt{\frac{2\pi}{n}} \left( \frac{m}{n+1} \right)^{m/2} \left[ \left( \frac{m}{n} \right)^2 - 1 \right]^{-(n+1)/2} & n - m \text{ odd, } n < m
\end{cases}
\]  

(A5)
In deriving this result, we have used the fact that the inequality $n > m$ implies $n - m >> 1$. This is true because we are holding the ratio of the two integers fixed while letting them become large. In other words, errors of order $1/n$ are the same order as errors of order $1/(n - m)$. Thus the Stirling approximation is used for quantities such as $(n - m)!$. Similar remarks apply to the inequality $n < m$.

In Eq. (A5), only positive quantities are raised to powers that could be non-integer. Thus there are no phase ambiguities. If one is sloppy about phases, the last expression appears to be the same as the first, differing only by a factor of two. The origin of this factor of two has a simple interpretation in the stationary-phase approximation, described in the next subsection.

It is remarkable that the first expression (for the case $n > m$, $n - m$ even) and the third expression (for the case $n > m$, $n - m$ odd) in Eq. (A2) have the same asymptotics to the order at which we are working. A calculation is involved in showing this. The result that comes from applying the Stirling approximation to the third expression is

$$
\sqrt{\frac{2\pi}{n}} \left( \frac{1 - \frac{m}{n+1}}{1 + \frac{m}{n+1}} \right)^{m/2} \left[ 1 - \left( \frac{m}{n+1} \right)^2 \right]^{-(n+1)/2}.
$$

To the accuracy to which we are working, this turns out to be the same as the first expression in Eq. (A3), although some work is required to show this.

2. Stationary-phase approximation

To do a stationary-phase approximation for the function $F(m, n)$, we write the function as

$$
F(m, n) = \int_{-\pi/2}^{\pi/2} e^{n g(x)} \, dx,
$$

where the function $g$ is defined by

$$
g(z) = \ln \cos z + i \frac{m}{n} z.
$$

With the usual choice of branch cut for the logarithm function, the function $g(z)$ is analytic everywhere in the complex plane except for vertical lines that intersect the real axis at odd multiples of $\pi$, and at the intervals on the real axis where $\cos z$ is nonpositive. The identity

$$
\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y
$$

is useful in showing this. Knowledge of the region of analyticity of $g$ allows us to deform the contour of integration in Eq. (A6) without changing the value of the integral. We would like to deform the contour so that the phase of the integrand $e^{n g(z)}$ is constant. To do this, we need to know the imaginary part of $g(z)$. With the help of Eq. (A8) we find that this is

$$
\Im[g(x + iy)] = -\tan^{-1}(\tan x \tanh y) + \frac{m}{n} x.
$$
If a contour is selected in such a way that this function is a constant, then \( g(x + iy) \) will equal an imaginary constant plus a real-valued function along the contour. The integrand \( e^{ng(z)} \) will then equal a constant phase factor raised to the \( n \)-th power times a fixed real-valued function raised to the \( n \)-th power. This fixed real-valued function may be approximated by a Gaussian, and the integral may then be evaluated. Stationary-phase points \( z_s \) satisfy the condition

\[
g'(z_s) = 0. \tag{A10}
\]

This is equivalent to the condition

\[
\tan z_s = i \frac{m}{n}. \tag{A11}
\]

We note that the value of \( g'' \) at a stationary-phase point is

\[
g''(z_s) = -\sec^2 z_s = \left(\frac{m}{n}\right)^2 - 1. \tag{A12}
\]

It is necessary to distinguish two cases, the case \( m < n \) and the case \( m > n \) (recall that \( m \) and \( n \) are both positive by assumption). We first consider the case \( m < n \). In this case, it follows from a study of Eq. (A9) that a constant phase contour exists that connects the endpoints of the integral and passes through the stationary-phase point

\[
z_s = i \tanh^{-1} \frac{m}{n} \tag{A13}
\]

in a direction that is parallel to the real axis. The integrand is approximated by

\[
e^{n g(z_s) + n g''(z_s)(z-z_s)^2/2},
\]

and the result for the integral is

\[
\sqrt{\frac{2\pi}{-n g''(z_s)}} e^{n g(z_s)} = \sqrt{\frac{2\pi}{n}} \left(\frac{1 - \frac{m}{n}}{1 + \frac{m}{n}}\right)^{m/2} \left[1 - \left(\frac{m}{n}\right)^2\right]^{-(n+1)/2}, \tag{A14}
\]

which agrees with the result in Eq. (A5).

We now move on to the case \( m > n \). In this case, no single contour exists with the properties that it connect the endpoints of the integral and that the quantity in Eq. (A9) be constant. Instead, we choose a contour consisting of three straight-line pieces. The first part, \( C_1 \), is defined to start at \(-\pi/2\) and go vertically upwards to \(-\pi/2 + iY\), \( Y \) being a large positive real number. The second part, \( C_2 \), is defined to go from \(-\pi/2 + iY\) to \(\pi/2 + iY\), and the third part, \( C_3 \), goes straight down to the \(\pi/2\) endpoint of the integral. The parts \( C_1 \) and \( C_3 \) contain stationary-phase points, which we call \( z_{s-} \) and \( z_{s+} \), and which are given by

\[
z_{s\pm} = \pm \frac{\pi}{2} + i \tanh^{-1} \frac{n}{m}. \tag{A15}
\]

For the integral along \( C_1 \) the integrand is approximated by
We parametrize the curve $C_1$ by $z = z_s + i t$, where $t$ is a real parameter. Then $dz$ is $i dt$ and the resulting approximation for the integral along $C_1$ is

$$i \sqrt{\frac{2\pi}{+n g''(z_s-)}} e^{ng(z_s-)}.$$

Similarly, the approximation for the integral along $C_3$ is

$$-i \sqrt{\frac{2\pi}{+n g''(z_s+)}} e^{ng(z_s+)}.$$

Because of the condition $m > n$ the integral along the curve $C_2$ goes to zero as $Y$ goes to infinity. The approximation for $F(m,n)$ is thus the sum of the expressions given in Eqs. (A16) and (A17). If $m - n$ is odd the sum is zero, in agreement with the exact value. If $m - n$ is even, the sum agrees with the approximation of the exact result, given in Eq. (A5).

Thus we see that depending on the ratio $m/n$, different numbers of stationary-phase points must be considered due to fundamental changes in the form of the stationary-phase contours as $m/n$ goes from one side of the critical value of 1 to the other side. On either side of the critical value of 1, a stationary-phase approximation is possible. The behavior near the critical value is discussed briefly in the main part of this paper.

**APPENDIX B: THE CASE OF VANISHING MAGNETIC QUANTUM NUMBERS**

Clebsch-Gordan coefficients for the case of vanishing magnetic quantum numbers ($m_1 = m_2 = m = 0$) are of interest in atomic and nuclear physics. Many of the expressions derived in this paper simplify in this case. Also, comparisons with the asymptotics of the exact closed-form expression, given in Eq. (B1), are possible.

The vanishing of the magnetic quantum numbers implies that $\beta$, defined in Eq. (3.12), is real. The case $\beta = 0$ is simple because one of the $j$ quantum numbers is then equal to the sum of the other two, and the integral in Eq. (2.11) may be evaluated exactly with a small amount of effort. Thus, we will consider the case $\beta > 0$. Two other facts that will be used throughout this appendix are that the set of quantum numbers is in the allowed region (since $\beta$ is real) and that the $j$ quantum numbers are integers (since $j_i - m_i$ is always an integer).

First, we work out the simplifications that occur in the all-real expression in Eq. (3.28). Equation (3.24) becomes

$$\chi = \frac{\pi}{2} \left( j + j_1 + j_2 + \frac{3}{2} \right)$$

and Eq. (3.28) becomes

$$\langle j_1 0 j_2 0 | j 0 \rangle \approx 2 \sqrt{\frac{j}{\pi \beta}} \cos \left[ \frac{\pi}{2} (j_1 + j_2 - j) \right].$$
This agrees with the first-order approximation of the exact expression, which can be obtained from the higher-order approximation [Eq. (B12)] of the exact result, given in Eq. (B11). If \( j_1 + j_2 - j \) is odd, then both of the expressions are zero. (In this case \( j_1 + j_2 + j \) is also odd since \( 2j \) is even.) If \( j_1 + j_2 - j \) is even, then both have a sign of \((-1)^{(j_1+j_2-j)/2}\).

Next, we move on to the higher-order approximation. From Eq. (3.11) it is apparent that the two solutions for the cotangents of \( \theta \) and \( \phi \) are related by simply reversing the signs. It follows from Eqs. (3.14), (4.11) and (4.12) that the values of the second and fourth-order derivatives of \( g(\theta, \phi) \) are unchanged, while the third-order derivatives have their signs flipped. Equations (4.7) and (4.10) imply that the quantities \( \delta_4 \) and \( \delta_6 \) are unchanged.

Finally, Eq. (3.17) implies that the quantity \( e^{g(\theta_0,\phi_0)} \) gets multiplied by \((-1)^{j_1+j_2+j}\) for the second root. Because \( j \) is an integer, this phase factor is the same as \((-1)^{j_1+j_2+j+1}\). Since we are in the allowed region, we must sum over both stationary-phase points. We see that for odd values of \( j_1 + j_2 + j \) the result is zero, while for even values of \( j_1 + j_2 + j \) the result is twice the contribution obtained from one of the stationary-phase points.

The values of the second derivatives of \( g(\theta, \phi) \) at the stationary-phase points are obtained from Eqs. (3.11) and (3.14), and the results simplify quite a bit.

\[
\begin{align*}
  g_{\theta\theta} &= -\frac{4 j_2 (j_1 + j)}{j_1 + j_2 + j}, \\
  g_{\theta\phi} &= \frac{4 j_1 j_2}{j_1 + j_2 + j}, \\
  g_{\phi\phi} &= -\frac{4 j_1 (j_2 + j)}{j_1 + j_2 + j}.
\end{align*}
\]

From these equations results the following expression for the determinant of the \( 2 \times 2 \) Hessian matrix of second derivatives of \( g(\theta, \phi) \).

\[
\det \frac{\partial^2 g}{\partial(\theta, \phi)^2} = \frac{16 j_1 j_2 j}{j_1 + j_2 + j}
\]

We note that this result is nonzero, and it does not vanish in any special cases that have nonzero \( j \) quantum numbers, which seems to contradict the statement made at the end of Sec. III about the vanishing of the determinant when \( \beta \) vanishes. The resolution of this apparent contradiction has to do with the fact that in the stationary-phase analysis of this paper we do not simultaneously consider the cases \( m_1 = m_2 = m = 0 \) and \( \beta = 0 \). These two conditions together would imply \( \alpha \) is zero. The quantity \( \alpha \) is defined in Eq. (3.25). It vanishes when \( j = j_1 + j_2, j_1 = j_2 + j \) or \( j_2 = j + j_1 \). In general, as long as \( \alpha \) is nonzero, the expression for the determinant can be put (after some work) in the form

\[
\det \frac{\partial^2 g}{\partial(\theta, \phi)^2} = \frac{\beta P_1 + \beta^2 P_2}{\alpha^2 (j_1 + j_2)^2},
\]

where \( P_1 \) and \( P_2 \) are (large) polynomials in the quantum numbers. This equation justifies the statement that the determinant is zero in cases where \( \beta \) is zero. On the other hand, in cases where \( m_1 = m_2 = m = 0 \), \( P_1 \) vanishes and \( \alpha \) and \( \beta \) are equal, and the result simplifies to that shown in Eq. (B6). The case of \( m_1 = m_2 = m = 0 \) and \( \beta = 0 \) requires a separate
treatment. It is necessary to go back to the original integral representation for the Clebsch-Gordan coefficient. The integral may be approximated by the methods of stationary phase, but it is simpler just to evaluate it or Eq. (2.13) exactly, which is possible at that point.

Higher-order derivatives of \( g(\theta, \phi) \) at the stationary-phase points simplify as well. Equations (4.7) and (4.10) for \( \delta_4 \) and \( \delta_6 \) yield results that are much simpler than for the general case of nonzero magnetic quantum numbers.

\[
\delta_4 + \delta_6 = (j_1^5 j_2 - 2 j_1^3 j_2^3 + j_1 j_2^5 + j_1^3 j_2^2 j - j_1^2 j_2^3 j + j_2^5 j - j_1^3 j_2 j^2 - 10 j_1^2 j_2^2 j^2 \\
- j_1 j_2^3 j^2 - 2 j_1^3 j^3 - j_1^2 j_2 j^3 - j_1 j_2^2 j^3 - 2 j_2^3 j^3 + j_1 j_5 + j_2 j^5)/(12 j_1 j_2 \beta^2) \quad \text{(B8)}
\]

This expression may be rewritten in a more compact form, as explained after Eq. (B12).

As discussed above, for odd values of \( j + j_1 + j_2 \) the higher-order approximation of the Clebsch-Gordan coefficient vanishes identically. For even values of \( j + j_1 + j_2 \), the result reduces to

\[
\langle j_1 j_2 j | j 0 \rangle \approx 2 (-1)^{j_1 + j_2 - j} \frac{j + j_1 + j_2}{2\pi\beta} \sqrt{\frac{j + j_1 + j_2}{j + j_1 + j_2 + 1}} \left( 1 + \delta_4 + \delta_6 \right) \left[ 1 + \frac{1}{24} \left( \frac{2}{j} \right) \right. \\
+ \frac{2}{j_1 + 2}{j_2} - \frac{1}{j + j_1 + j_2} - \frac{1}{j - j_1 + j_2} - \frac{1}{j + j_1 - j_2} \right], \quad \text{(B9)}
\]

where we have approximated the factorials in \( N_{j_1 m_1 j_2 m_2 j m} \) using the form of Stirling’s approximation that is appropriate for this order,

\[
x! \approx \sqrt{2\pi x x^x e^{-x}} \left( 1 + \frac{1}{12 x} \right), \quad \text{(B10)}
\]

The exact value of the Clebsch-Gordan coefficient is (Ref. [7], p. 87; Ref. [11]),

\[
\langle j_1 j_2 j | j 0 \rangle = \begin{cases} \\
0 & j + j_1 + j_2 \text{ odd}
\end{cases} \quad \text{(B11)}
\]

and this may be approximated using Eq. (B10). The result is

\[
\langle j_1 j_2 j | j 0 \rangle \approx \begin{cases} \\
0 & j + j_1 + j_2 \text{ odd}
\end{cases} \quad \text{(B12)}
\]

To the order that we are working, the higher-order stationary-phase result, given in Eq. (B9), and the corresponding approximation of the exact result, given in Eq. (B12), agree. Equation (B9) contains two factors that have the form of unity plus a small correction. If these are multiplied out and only the first-order terms are kept, the result is the factor of \( \left( 1 - \frac{j_1 j_2}{\beta^2} \right) \) in Eq. (B12). This shows that Eqs. (B9) and (B12) are equivalent, and it also provides an alternative way of writing the expression in Eq. (B8).
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Table I

| forbidden subregion | choice of root | sign function       | largest $\lambda$ |
|---------------------|----------------|---------------------|-------------------|
| I                   | lower          | 1                   | $\lambda_3$       |
| II                  | upper          | $(−1)^{j_1−m_1}$   | $\lambda_2$       |
| III                 | lower          | $(−1)^{j_1+j+m_2}$  | $\lambda_1$       |
| IV                  | upper          | $(−1)^{j_1+j_2−j}$  | $\lambda_3$       |
| V                   | lower          | $(−1)^{j_2−j−m_1}$  | $\lambda_2$       |
| VI                  | upper          | $(−1)^{j_2+m_2}$    | $\lambda_1$       |

Table I: For each forbidden subregion, the choice of root in Eq. (3.11), the sign function as in Eq. (3.30) and the largest $\lambda$ [which determines the form of $\chi$, as in Eq. (3.31)] are given.
Figure Captions.

Fig. 1: An example of a choice of three $j$-vectors, demonstrating that a set of quantum numbers is classically allowed.

Fig. 2: The triangle-allowed region and the classically allowed region, shown in the $m_1$-$m_2$ plane, for the case of $j$-values in the ratio $j : j_1 : j_2 = 4 : 2 : 3$. The six forbidden subregions are labeled with Roman numerals.
An example of a choice of three j-vectors, demonstrating that a set of quantum numbers is classically allowed.
The triangle-allowed region and the classically allowed region, shown in the $m_1$-$m_2$ plane, for the case of $j$-values in the ratio $j:j_1:j_2 = 4:2:3$. The six forbidden subregions are labeled with Roman numerals.