The Barabanov norm is generically unique, simple, and easily computed

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Abstract

Every irreducible discrete-time linear switching system possesses an invariant convex Lyapunov function (Barabanov norm), which provides a very refined analysis of trajectories. Until recently that notion remained rather theoretical apart from special cases. In 2015 N.Guglielmi and M.Zennaro showed that many systems possess at least one simple Barabanov norm, which moreover, can be efficiently computed. In this paper we classify all possible Barabanov norms for discrete-time systems. We prove that, under mild assumptions, such norms are unique and are either piecewise-linear or piecewise quadratic. Those assumptions can be verified algorithmically and the numerical experiments show that a vast majority of systems satisfy them. For some narrow classes of systems, there are more complicated Barabanov norms but they can still be classified and constructed. Using those results we find all trajectories of the fastest growth. They turn out to be eventually periodic with special periods. Examples and numerical results are presented.

Keywords: discrete linear switching system, Lyapunov function, Barabanov norm, uniqueness, asymptotic growth, trajectories, invariant polytope, algorithm, positive systems, linear programming

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1. Introduction

Discrete-time linear switching systems of the form

\[
\begin{cases}
x(k + 1) = A(k)x(k); \\
A(k) \in A, \quad k \in \mathbb{Z}_+; \\
x(0) = x_0
\end{cases}
\]

(1)

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are the subject of an extensive literature. The set $\mathcal{A}$ is a compact set of $d \times d$ matrices. In this paper we deal with finite families $\mathcal{A} = \{A_1, \ldots, A_m\}$, $m \geq 1$, and often identify the system with the corresponding family. The sequence of matrices $\{A(k)\}_{k=0}^\infty$ taken from $\mathcal{A}$ (with repetitions permitted) is the switching law. The sequence of points $\{x(k)\}_{k=0}^\infty$ from $\mathbb{R}^d$ satisfying $[1]$ for some switching law $A(\cdot)$ is a trajectory of the system. The switching law, along with the initial point $x_0$, defines the trajectory.

The fastest possible growth of trajectories as $k \to \infty$ is an important issue in many applied problems. It is closely related to the asymptotic stability of linear and non-linear systems [18 28 33], to the regularity exponents of fractal curves and surfaces [6 39], wavelets, and subdivision schemes [11 41], to the growth of special sequences in combinatorics, number systems [18 28 33], to the regularity exponents of fractal curves and surfaces [6 39], wavelets, applied problems. It is closely related to the asymptotic stability of linear and non-linear systems [18 28 33], to the regularity exponents of fractal curves and surfaces [6 39], wavelets, and subdivision schemes [11 41], to the growth of special sequences in combinatorics, number theory, and the theory of formal languages, in the automata theory, etc., see [8 24 32 40] and references therein. If the system is irreducible, i.e., the matrices $A_1, \ldots, A_m$ do not share a nontrivial invariant linear subspace, then the maximal value of $\|x(k)\|$ over all trajectories (with fixed $x_0$) is asymptotically equivalent to $\rho^k$. More precisely, it is between $C_1\|x_0\|\rho^k$ and $C_2\|x_0\|\rho^k$, where $C_1 \leq C_2$ are positive constants and $\rho = \rho(A)$ is the joint spectral radius (JSR) of the family $\mathcal{A}$. We recall the definition of the JSR below. There are efficient methods to estimate the JSR [11 11 22] and, in many cases, even to compute it precisely [12]. So, the exponent of the fastest growth $\rho$ can be computed. However, it is not enough to have a comprehensive information on the growth of trajectories, since $C_1$ can be very small or $C_2$ very large. One needs to estimate the constants $C_1$ and $C_2$. This problem, however, is more difficult. Even their rough estimations are usually hard. Theoretically this problem can be solved by using the invariant convex Lyapunov function also called the Barabanov norm.

**Definition 1** An invariant convex Lyapunov function (Barabanov norm) of a family of matrices $\mathcal{A} = \{A_1, \ldots, A_m\}$ is a norm $f$ in $\mathbb{R}^d$ such that

$$
\max_{i=1,\ldots,m} f(A_i x) = \rho f(x) \quad \text{for every } x \in \mathbb{R}^d,
$$

where $\rho = \rho(A)$ is the joint spectral radius of $\mathcal{A}$.

By iterating equation (2) we obtain $\max_{s_1,\ldots,s_k} f(A_{s_k} \cdots A_{s_1} x) = \rho^k f(x)$ for every $k$. Consequently, for the Barabanov norm, we have $C_1 = C_2 = 1$, which means that this norm is optimal among all possible norms in $\mathbb{R}^d$. For an arbitrary norm $\|\cdot\|$, say, Euclidean, the constants $C_1, C_2$ can be obtained by the maximal and minimal values of $f(x)/\|x\|$, provided the Barabanov norm $f$ is known. Thus, if the Barabanov norm is available, then the problem of estimating the maximal growth of trajectories in every norm is efficiently solved. Moreover, in that case it is possible to find all switching laws $A(\cdot)$ corresponding to the fastest growth of trajectories, for which $\limsup_{k \to \infty} \rho^{-k}\|A(k) \ldots A(1)\| > 0$.

Thus, in the analysis of the trajectories, it is very desirable to have a Barabanov norm. It was shown in [2] that such a norm does exist for every irreducible family $\mathcal{A}$. This is a purely existence result, all of its known proofs are non-constructive. Is it possible to obtain the Barabanov norm in a closed form? There are several arguments saying that the answer should be negative:

1) **The non-uniqueness.** We always consider the uniqueness of norms up to their multiplication by a constant. Simple examples in $\mathbb{R}^2$ already show that the Barabanov norm may
not be unique. Say, if $A$ consists of one $2 \times 2$ matrix that defines a rotation of the plane by the right angle, then the $L_2$-norm and the $L_1$-norm in $\mathbb{R}^2$ are both Barabanov.

2) **Non-convergence of the power sequence** $F^n[g]$ as $n \to \infty$, where $F$ is a map on the set of norms: $F[g](x) = \rho^{-1} \max_{i=1,\ldots,m} g(A_i x)$. The Barabanov norm $g = f$ is a fixed point for $F$. However, the iterations $F^n[g]$ may not converge for some $g$ even if that fixed point is unique. For example, if $A$ consists of one rotation of $\mathbb{R}^2$ by an angle $\alpha$ such that $\alpha/\pi$ is irrational, then there is a unique Barabanov norm $f$, which is the Euclidean norm. However, for the $L_1$-norm $g(x_1, x_2) = |x_1| + |x_2|$, the sequence $F^n[g]$ does not have a limit. This shows that even in case of uniqueness, the Barabanov norm cannot be computed by the power method.

3) **The fractal-like boundary of the unit sphere.** Let $B = \{ x \in \mathbb{R}^d \mid f(x) \leq 1 \}$ be a unit ball of the Barabanov norm and let $G = B' = \{ y \in \mathbb{R}^d \mid \max_{x \in B} (x, y) \leq 1 \}$ be its polar. Then, as it was proved in [37, Theorem 17], the convex hull of images $A_i^T G$, $i = 1, \ldots, m$, is homothetic to $G$ itself:

$$\text{co} \left( \bigcup_{i=1}^m A_i^T G \right) = \rho G,$$

where $A_i^T$ denotes the transpose matrix to $A_i$. The convex body $G$ possessing this property generates the Protasov norm according to the terminology in [37]. Thus, Barabanov’s and Protasov’s norms are dual to each other. The property (3) is closely related to the definition of self-similar fractals by J.Hutchinson [21]. In fact, if we iteratively construct such a convex body $G$ on the plane, we will see that there should be segments on its boundary (because of taking the convex hull) and those segments multiply with iterations. Hence, the boundary should have a structure somewhat similar to the Cantor set. See [44] for more details.

The arguments above suggest that the notion of the Barabanov norm is rather theoretical and can hardly be evaluated for general matrices. This is indeed a common belief among specialists working in discrete-time switching systems. Many interesting theoretical results on Barabanov’s norms can be found in [25, 26, 29, 30, 34, 35, 37]. However, as it was remarked in 2012 by R. Teichner and M. Margaliot: “Although the Barabanov norm was studied extensively, it seems that there are only few examples where it was actually computed in closed form” [45].

Surprisingly, in 2015 N. Guglielmi and M. Zennaro [16] showed that for many matrix families (in particular, for all known families from applications) it is possible to construct at least one Barabanov norm in an explicit form. This form is either piecewise linear or piecewise-quadratic and it can be found within finite time. For constructing that norm they put to good use the invariant polytope algorithm from [12], whose idea also traces back to works of 1996 [38] and of 2005 [14]. An earlier versions of that algorithm appeared in [31, 15], see also [30, 38] for other related algorithms.

The invariant polytope algorithm produces an invariant convex body $G$ possessing the property

$$\text{co} \left( \bigcup_{i=1}^m A_i G \right) = \rho G,$$  \hfill (4)
and thus finds precisely the value of JSR $\rho(\mathcal{A})$. Actually the algorithm finds a dominant product $\Pi = A_{s_n} \cdots A_{s_1}$ (see Definition 4 in Section 2) such that $|\lambda|^{1/n} = \rho(\mathcal{A})$, where $\lambda$ is the leading, i.e., the largest in modulus eigenvalue of $\Pi$. It was proved in [12] that the algorithm halts within finite time if and only if the product $\Pi$ is dominant and its leading eigenvalue $\lambda$ is unique and simple. In this case the obtained invariant body $G$ is either a polytope (if $\lambda \in \mathbb{R}$) or a convex hull of several ellipses (if $\lambda \notin \mathbb{R}$). Then from a result of E.Plischke and F.Wirth [37, Theorem 17] it follows that if we get an invariant body $G^*$ of the transpose family $\mathcal{A}^* = \{A_1^T, \ldots, A_m^T\}$, then the function $f(x) = \max_{y \in G^*}(y, x)$ is a Barabanov norm.

A lot of numerical experiments done in [12, 13, 31] demonstrate that, for a vast majority of matrix families, the invariant polytope algorithm halts and hence produces an invariant body. There are well-known counterexamples [7, 19] but they are absolutely rare in practice. Having applied that algorithm to the transpose family we obtain the Barabanov norm.

**An assumption based on numerical experiments.** Let us clarify our claim on the “vast majority of matrix families in the numerical experiments”. First of all, the invariant polytope algorithm is robust: if it halts for some family of matrices, than it does for all close families performing the same number of iterations. The parameters of robustness are efficiently estimated [12, Section 2.5]. This makes it possible to avoid using exact arithmetics or rational matrices in the numerical computations. All the experiments are performed with rounding using well-defined tolerance parameters. There are several computer implementations applying various software [31, 42]. Several hundreds of numerical tests have been done in dimensions up to 20 with two sorts of matrix families: 1) families from known applications 2) randomly generated matrices. In all these experiments (100 %) the invariant polytope algorithm terminates within finite (usually quite short) time. See [12, 13, 31, 42] for more details. The statistics of a small part of those experiments is demonstrated in Section 10. This allows us to assume that a generic family of matrices possesses this property. By “generic” we mean that for every $m$ and $d$, the property holds for an open set of full Lebesgue measure in the space $\mathbb{R}^{md^2}$ (the space of families of $m$ matrices $d \times d$). We are not aware of any rigorous results approving this claim and we believe this is a challenging theoretical problem. Therefore, we make the assumption that for a generic family the algorithm halts based on numerical experiments. For an arbitrary matrix family, this can be checked directly by running the algorithm.

**Statements of the problems and a summary of main results.** Thus, a generic matrix family has at least one Barabanov norm that can be found in a closed form. Its unit ball is either a polytope (if the leading eigenvalue $\lambda$ of the dominant product is real) or a polar to a convex hull of ellipses (otherwise). A question arises if it has other Barabanov norms and, if so, how many and of what structure? If there are norms with fractal properties, how to find them? And how to find an optimal one among all Barabanov norms? In this paper we answer all those questions. We prove that in most cases the Barabanov norm is unique. This means that for all generic families, Barabanov norms are simple (either piecewise-linear or piecewise-quadratic) and there are no others. They are easily computed as maxima of several linear (respectively, quadratic) functionals. Thus, the invariant polytope algorithm produces not some norm but all possible Barabanov norms. In particular, there are no “fractal-like” norms among them. More precisely, we prove that the uniqueness takes place if the leading
eigenvalue $\lambda$ is either real or complex with an argument $\pi q$ with irrational $q$ (Theorems 1, 2 in Section 2). But what about the known simple examples when the Barabanov norm is not unique? It turns out that all of them belong to the third case: $\lambda$ is complex with an argument $\pi q$, where $q$ is a rational non-integer number. In this case, as we shall see, a family $\mathcal{A}$ has a rich variety if Barabanov norms. Nevertheless, they all can be classified. We do it in Theorem 8, Section 8. Next we extend those results to families with several (more than one) dominant products. This case is important in applications (see Section 6 for details). We prove that in this case there always exist infinitely many Barabanov norms but they are all quite simple and can be found by a modified version of the invariant polytope algorithm. In Section 7 we apply our results to the classification of trajectories of the fastest growth. All of them can be explicitly found: a switching law provides the fastest growth, i.e., $\|A(k) \cdots A(1)\| \geq C \rho^k, k \in \mathbb{N}$, precisely when it is eventually periodic, i.e., $A(k + n) = A(k)$ for all $k > N$, where $n, N$ are some natural numbers, and the period is equal to one of the dominant products. For all other trajectories, we have $\|A(k) \cdots A(1)\| \rho^{-k} \to 0$ as $k \to \infty$ (Theorem 3, Section 8). In Section 9 we turn to positive systems, when the Barabanov norm is always unique and is piecewise-linear (provided a dominant product exists). A modification of the invariant polytope algorithm for positive systems is very efficient: it constructs the Barabanov norm even for very large dimensions $d$ (several thousands). Finally, in Section 10 we present numerical results and discuss the computational issue. We will see that in most cases the time of constructing the Barabanov norm does not exceed that for constructing other Lyapunov functions by algorithms known from the literature.

**Novelty.** Our results can be divided into four main groups:

1) The proof of uniqueness of the Barabanov norm provided the dominant product has a leading eigenvalue which is either real or complex with an irrational mod $\pi$ argument. This shows that for a generic family of matrices, the Barabanov norm is unique, has a simple structure, and can be efficiently found (Sections 2 - 4). In the remaining case (the non-real eigenvalue with a rational mod $\pi$ argument), the family of matrices has a large variety of Barabanov norms. We classify them and present an algorithm to find them all (Section 8).

2) In case of finitely many dominant products, we show that there always exists an infinite set of Barabanov norms but all of them have a simple structure and can be explicitly found. An algorithm for their construction is presented (Section 6).

3) For every discrete-time system with finitely many dominant products, all switching laws corresponding to the fastest growth of trajectories are explicitly found. All trajectories of the fastest growth are classified (Section 7).

4) For positive systems (Section 9), we introduce the monotone Barabanov and prove that they are unique and piecewise-linear. They are found by a modification of the invariant polytope algorithm, which works efficiently even in very large dimensions.

**Auxiliary facts and notation.** We use bold letters for vectors and standard letters for numbers, so $\mathbf{x} = (x_1, \ldots, x_d)^T \in \mathbb{R}^d$. We consider a discrete-time system 1 in $\mathbb{R}^d$ with a finite family of matrices $\mathcal{A} = \{A_1, \ldots, A_m\}$ and associate the system with this family. We also assume a basis in $\mathbb{R}^d$ to be fixed and associate matrices with the corresponding linear operators. By $\mathcal{A}^k$ we denote the set of all products of matrices from $\mathcal{A}$ of length $k$. 

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(without ordering and with the repetitions permitted); $\mathcal{A}^N$ denotes the set of all products of lengths $k \geq 1$.

**Definition 2** The joint spectral radius (JSR) of a family $\mathcal{A}$ is

$$\rho(\mathcal{A}) = \lim_{n \to \infty} \max_{\Pi \in \mathcal{A}^n} \|\Pi\|^{1/n}. \quad (5)$$

The limit in (5) always exists and does not depend on the matrix norm \[43\]. For one matrix $\mathcal{A} = \{A\}$, the JSR becomes the usual spectral radius $\rho(A)$, which is the largest modulus of its eigenvalues, i.e., the modulus of a leading eigenvalue. JSR has been studied in the literature due to numerous applications (see bibliography in \[22\]).

As usual, we define a **convex body** in $\mathbb{R}^d$ as a convex compact set with a nonempty interior. If the converse is not stated, we always assume convex bodies and polytopes to be symmetric about the origin.

**Definition 3** A convex body $G \subset \mathbb{R}^d$ is called invariant for a matrix family $\mathcal{A}$ if it satisfies equation (4).

The existence of an invariant body for any irreducible matrix family was proved by A. Dranishnikov and S. Konyagin in 1993 and was first published in 1996 \[38\] with a new proof. Then in \[37\] it was shown that a polar to an invariant body of the transpose family $\mathcal{A}^* = \{A_1^T, \ldots, A_m^T\}$ is a unit ball of the Barabanov norm for $\mathcal{A}$. Thus, there is a one-to-one correspondence between invariant bodies and Barabanov norms. Therefore, we will formulate our results for both those objects.

To a word $s_1 \ldots s_k$ of the alphabet $\{1, \ldots, m\}$, we associate the product $A_{s_k} \ldots A_{s_1} \in \mathcal{A}^k$. Note that the order of multipliers is inverse to the order of letters! A prefix is some left subword of the word and a suffix is a right subword. The product of several words is their concatenation.

We use the trigonometric form of the complex number $z = |z|e^{\varphi i}$, where $\varphi$ is the argument of $z$. If $\frac{\varphi}{\pi} \in \mathbb{Q}$, then we say that $z$ has a rational mod-\(\pi\) argument.

For an arbitrary convex body $G \subset \mathbb{R}^d$ symmetric about the origin, $\|\cdot\|_G$ denotes the Minkowski norm $\|\cdot\|_G = \sup \{\lambda \mid \lambda^{-1}x \in G\}$.

We denote by $\Gamma$ the unit circle on the two-dimensional plane and by $D$ the unit disc. For an arbitrary set $K \subset \mathbb{R}^2$ and arbitrary vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, we denote by $\Phi_{\mathbf{x},\mathbf{y}}(K)$ the image of $K$ under the map $\mathbb{R}^2 \to \mathbb{R}^d$ that takes the basis of $\mathbb{R}^2$ to vectors $\mathbf{x}, \mathbf{y}$. This map is given by the $d \times 2$ matrix composed of two columns $\mathbf{x}, \mathbf{y}$. In particular, $\Phi_{\mathbf{x},\mathbf{y}}(\Gamma)$ is an ellipse. If a matrix $A$ has a complex leading eigenvector $\mathbf{v} = \mathbf{x} + i\mathbf{y}$, then the ellipse $\Phi_{\mathbf{x},\mathbf{y}}(\Gamma)$ will be called leading and its linear span is the leading eigenspace (or leading plane).

As usual, the asymptotic equivalence $\asymp$ means the existence of two positive constants $C_1, C_2$ such that $C_1 \rho^k \leq \|x(k)\| \leq C_2 \rho^k$. 


2. Fundamental theorems

We consider the set of products $A^\mathbb{N} = \{A_{s_k} \ldots A_{s_1} \mid A_{s_i} \in A, k \in \mathbb{N}\}$ of a finite family of matrices $A$. A product $\Pi \in A^\mathbb{N}$ is called primitive if it is not a power of a shorter product. For a given product $\Pi$, we denote its length (the number of multipliers) by $|\Pi|$ and $\nu(\Pi) = [\rho(\Pi)]^{1/|\Pi|}$.

**Definition 4** Let $A$ be a finite family of matrices. A set $P \subset A^\mathbb{N}$ is called a set of dominant products if

1) all products from $P$ are primitive and are all different up to cyclic permutations, i.e., none of them is a cyclic permutation of another;
2) there is a number $q$ such that $\nu(\Pi) = q$ for all $\Pi \in P$;
3) there is $\epsilon > 0$ such that for every $S \in A^\mathbb{N}$, we have $\nu(S) \leq (1 - \epsilon)^{1/|S|} q$ unless $S$ is a power of some product from $P$ or of one of its cyclic permutations.

**Remark 1** At first, this definition may seem impossible to verify within finite time since part (3) involves infinitely many conditions. Nevertheless, this can be done efficiently by the invariant polytope algorithm presented in [12, 13]. Given candidate products are dominant if and only if this algorithm halts. We recall the algorithm and discuss this issue in detail in Section 4.

If $P$ is a set of dominant products for the family $A$, then $q$ is equal to the joint spectral radius $\rho(A)$, see [12]. Moreover, for all matrix products $S \in A^\mathbb{N}$, the value $q^{-|S|}\rho(S)$ is either equal to 1 (if $S$ is a power of a dominant product or of one of its cyclic permutations), or is at most $1 - \epsilon$. Hence, the interval $(1 - \epsilon, 1)$ can be called a spectral gap: no numbers $q^{-|S|}\rho(S)$, $S \in A^\mathbb{N}$, belong to it.

If $q = 1$, then the dominance property can be defined in a simpler way: there is $\epsilon > 0$ such that, for every product $S \in A^\mathbb{N}$, we have $\rho(S) \leq 1 - \epsilon$ unless $S$ is a power of some product from $P$ or of one of its cyclic permutations, in which case $\rho(S) = 1$. Hence, there is an equivalent definition of the dominant set: the set is dominant if 1) and 2) hold and for the normalized family $\tilde{A} = \{\tilde{A}_i = q^{-1}A_1, i = 1, \ldots, m\}$, there is $\epsilon > 0$ such that for every $\tilde{S} \in \tilde{A}^\mathbb{N}$, we have $\rho(\tilde{S}) \leq 1 - \epsilon$ unless the corresponding product $S$ is a power of some product from $P$ or of one of its cyclic permutations. This way the dominance has been defined in [12] for one product and then extended for arbitrary set of products in [13].

We mostly deal with two cases. If $P = \{\Pi\}$ is a one-element set, we say that $\Pi$ is a dominant product (always assuming that it is unique). In this case we say that $A$ has a unique dominant product, although it is actually unique only up to a cyclic permutation. If $P = \{\Pi^{(1)}, \ldots, \Pi^{(r)}\}$ is a finite set, then we say that the family $A$ has finitely many dominant products.

We always make an assumption that each dominant product $\Pi^{(i)}$ has a unique and simple leading eigenvalue $\lambda_i$. This means that $\lambda_i$ is not multiple and all other eigenvalues (except for the complex conjugate $\bar{\lambda}_i$ if $\lambda_i \notin \mathbb{R}$) are strictly smaller than $\lambda_i$ in modulus.

Let us recall that by uniqueness of an invariant body or of a norm we always mean their uniqueness up to multiplication by a constant.
Theorem 1. Let a family of operators $\mathcal{A}$ have a unique dominant product with a unique and simple leading eigenvalue $\lambda$. If $\lambda$ is either real or complex with an irrational $\text{mod} \pi$ argument, then $\mathcal{A}$ possesses a unique invariant body. If $\lambda$ is real, then this invariant body is a polytope, if $\lambda$ is complex with an irrational $\text{mod} \pi$ argument, then this is a convex hull of several ellipses.

At the first sight, the assumption of Theorem 1 is quite restrictive: the family $\mathcal{A}$ must have a unique dominant product whose leading eigenvalue is unique and simple. It turns out, however, that a vast majority of matrix families satisfies it. This observation was made first in [12] and then confirmed in [13, 11, 31] by analysing lots of numerical experiments with random families and with families from applications. Moreover, a dominant product can be efficiently found algorithmically [12] and the same algorithm constructs an invariant body [16]. We analyse this issue in Section 4.

What can be said in the case which is not covered by Theorem 1, when the leading eigenvalue $\lambda$ is non-real but possesses a rational $\text{mod} \pi$ argument? In this case there is still an invariant body as a convex hull of ellipses, but it is never unique: there exist infinitely many invariant bodies of other form.

Proposition 1. Suppose a family of operators $\mathcal{A}$ has a unique dominant product whose leading eigenvalue $\lambda$ is non-real and has a rational $\text{mod} \pi$ argument; then $\mathcal{A}$ has infinitely many invariant bodies, one of which is a convex hull of several ellipses.

In Section 8 we classify all invariant bodies for the case of non-real eigenvalue with a rational $\text{mod} \pi$ argument. Note that the transpose family $\mathcal{A}^*$ possesses the same property of the uniqueness of the dominant product (with the same leading eigenvalue). Applying Theorem 1 and Proposition 1 to the transpose family and taking the polar of the invariant body, we obtain the following theorem that classifies Barabanov norms for generic matrix families.

Theorem 2. Let a family of operators $\mathcal{A}$ have a unique dominant product with a unique and simple leading eigenvalue $\lambda$. If $\lambda$ is real, then $\mathcal{A}$ has a unique Barabanov norm. This norm is piecewise-linear and is given by the formula

$$f(x) = \max_{v^*} \left| (v^*, x) \right|,$$

where the maximum is taken over all vertices $v^*$ of the invariant polytope $G^*$ of the transpose family $\mathcal{A}^*$. If $\lambda$ is complex, then $\mathcal{A}$ has a piecewise-quadratic Barabanov norm given by the formula

$$f(x) = \max_{E^*} \max_{z^* \in E^*} \left| (z^*, x) \right|,$$

where the maximum is taken over all ellipses $E^*$ that form the invariant body of the transpose family $\mathcal{A}^*$. If the argument of $\lambda$ is irrational $\text{mod} \pi$, then this Barabanov norm is unique.

The Barabanov norm (7) can be written in a simpler form (8), see Remark 2 below. Thus, if $\lambda \in \mathbb{R}$, then the unit ball of $f$ is a polyhedron which is a polar to the invariant
polytope of the transpose family $\mathcal{A}^\ast$. In this case $\mathcal{A}$ has no other Barabanov norms. If $\lambda \not\in \mathbb{R}$, then $f$ is piecewise-quadratic; its unit ball is the intersection of right elliptic cylinders with two-dimensional bases, those cylinders are polars to the ellipses forming the invariant body of the transpose family $\mathcal{A}^\ast$. If the argument of $\lambda$ is irrational mod $\pi$, then $\mathcal{A}$ has no other Barabanov norm. If the argument is rational mod $\pi$ but $\lambda$ is non-real, then this norm is not unique and the family $\mathcal{A}$ has infinitely many Barabanov norms. Their complete classification is obtained in Section 8.

Remark 2 If an ellipse $E^\ast$ is defined by a pair of vectors $a, b \in \mathbb{R}^d$, i.e., $E^\ast = \Phi_{a,b}(\Gamma)$, then $\max_{z^\ast \in E^\ast} \left| (z^\ast, x) \right| = \sqrt{(a, x)^2 + (b, x)^2}$, hence the formula (7) for the Barabanov norm $f(x)$ can be written as follows:

$$f(x) = \max_{i=1,\ldots,N} \sqrt{(a_i, x)^2 + (b_i, x)^2}, \quad (8)$$

where $(a_i, b_i)$ is the pair of vectors defining the $i$th ellipse $E_i^\ast = \Phi_{a_i,b_i}(\Gamma)$ in the convex hull $G^\ast = \text{co} \{ E_1^\ast, \ldots, E_N^\ast \}$ for the invariant body $G^\ast$ of the transpose family $\mathcal{A}^\ast$.

Proofs to Theorems 1 and 2 are given in Section 5. In Section 4 we address the practical issue: how to prove how to prove that the assumptions of those theorems are satisfied and how to construct the invariant body and the Barabanov norm. Now we give several illustrative examples in dimensions $d = 2$ and $d = 3$, with the corresponding pictures of invariant sets and unit balls of the Barabanov norms. Numerical results for higher dimensions (of course, without pictures) are considered later in Section 10.

Remark 3 Comparison with known results on the uniqueness of the Barabanov norm. Various sufficient conditions for the uniqueness of Barabanov’s norm have been proposed in [30, 34, 35]. In [34] it was shown that if the family $\mathcal{A}$ of matrices satisfies the so-called unbounded agreements and possesses the rank one property, then Barabanov’s norm is unique. Both conditions are hard to verify apart from special cases. However, Theorem 6 proved below in Section 5 implies that if $\mathcal{A}$ possesses a unique dominant product with real and simple leading eigenvalue, then both those conditions are satisfied. So, in this special case the the main result of [34] implies the uniqueness part of Theorem 1. Although this implication is not straightforward and requires a proof using Theorem 6.

Another sufficient uniqueness condition presented in [35], the transitivity property seems to be very particular. For instance, under the assumption of Theorem 1 it is never satisfied for dimensions $d > 2$.

The uniqueness issue was addressed in [30] but no corresponding results have been obtained there.
3. Examples

We consider several low-dimensional examples illustrating Theorems 1 and 2. In all the cases the computation took a few seconds on a standard laptop. Higher dimensions (up to 20 for general matrices and to 2000 for nonnegative matrices) are addressed in Section 10.

Example 1 For the family $\mathcal{A} = \{A_1, A_2\}$, where

$$A_1 = \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} ; \quad A_2 = \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix} ,$$

(9)

the dominant product is $\Pi = A_1^3 A_2$, the leading eigenvalue is real. The invariant convex body $G$ is a 10-gon (Fig. 1, left). Its polar $G' = \{x \in \mathbb{R}^d \mid \max_{y \in G^*} (y, x) \leq 1\}$ is the unit ball for the (unique!) Barabanov norm for the transpose family $\mathcal{A}^* = \{A_1^T, A_2^T\}$, Fig. 1 (right). It is also a 10-gon.

Example 2 The family

$$A_1 = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \\ 2 & -2 & 3 \end{pmatrix} ; \quad A_2 = \begin{pmatrix} -1 & 0 & 3 \\ 0 & -1 & -2 \\ -3 & 2 & 1 \end{pmatrix}$$

(10)

has a dominant product $\Pi = A_1^2 A_2$ with a real leading eigenvalue. The invariant polytope $G$ has 24 vertices and 44 faces (Fig. 2, left).

Its polar $G'$ is the unit ball for the (unique!) Barabanov norm of $\mathcal{A}^*$, Fig. 2 (right). This is a polytope with 44 vertices and 24 faces.
Example 3  The family
\[
A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad ; \quad A_2 = \begin{pmatrix} 0.890 & 0.646 \\ -0.129 & -0.178 \end{pmatrix}
\]
has a dominant product \( \Pi = A_1 \) with a complex leading eigenvalue. The invariant convex body \( G \) is a convex hull of three ellipses (Fig. 3, left). Its polar is the intersection of three ellipses, it is the unit ball for the Barabanov norm for \( A^* \) (Fig. 3, right).

Figure 2: Real case, \( d = 3 \). Left: the invariant polytope \( G \) for the family (10); Right: its polar \( G' \), which is the unit ball for the Barabanov norm of \( A \).

Figure 3: Complex case, \( d = 2 \). Left: the invariant body \( G \) for the family (11); Right: its polar \( G' \).
Example 4  The family

\[
A_1 = \begin{pmatrix}
  0 & 1 \\
-1 & 0 \\
\end{pmatrix}; \quad A_2 = \begin{pmatrix}
  0.340 & 1.046 \\
-0.523 & 0.170 \\
\end{pmatrix}
\]

has a dominant product \( \Pi = A_1 \), with a complex leading eigenvalue. The invariant convex body \( G \) is a convex hull of 9 ellipses (Fig. 4, left). Its polar is the intersection of 9 ellipses, it is the unit ball for the Barabanov norm for \( A^* \) (Fig. 4, right).

![Figure 4](image)

Figure 4: Complex case, \( d = 2 \). Left: the invariant body \( G \) for the family (12); Right: its polar \( G' \).

Example 5  The family of \( 3 \times 3 \) matrices

\[
A_1 = \begin{pmatrix}
  -4436 & -3993 & 887 \\
3045 & -257 & -359 \\
2416 & 1895 & 1338 \\
\end{pmatrix}; \quad A_2 = \begin{pmatrix}
  2598 & 2948 & 682 \\
-1424 & -4331 & 2691 \\
821 & -1390 & -388 \\
\end{pmatrix}
\]

has a dominant product \( \Pi = A_1 \), with a complex leading eigenvalue. The invariant convex body \( G \) is a convex hull of 6 ellipses (Fig. 5).

4. Construction of the invariant body and of the Barabanov norm

Theorems 1 and 2 in Section 2 assert that the Barabanov norm is unique and has a simple form, provided the system has a dominant product with a unique and simple leading eigenvalue. In this section we will see that this assumption is not restrictive and is fulfilled for
Figure 5: Complex case, $d = 3$. The invariant body $G$ for the family $\mathcal{A}$

a vast majority of matrix families. Moreover, this unique Barabanov norm can be constructed in an explicit form. This is done by the invariant polytope algorithm presented in [12] for the computation of JSR. That algorithm does a search of a matrix product $\Pi \in \mathcal{A}^N$ with the biggest value of $\nu(\Pi)$ among all products of some bounded lengths and then rigorously approves that $\rho(\mathcal{A}) \leq \nu(\Pi)$. The opposite equality $\rho(\mathcal{A}) \geq \nu(\Pi)$ holds for all products $\Pi$, this is well-known [13]. This implies that $\rho(\mathcal{A}) = \nu(\Pi)$, and the JSR is found. To prove that $\rho(\mathcal{A}) \leq \nu(\Pi)$ the algorithm constructs either a polytope $G$ or a convex hull of several ellipses (depending on the leading eigenvalue of $\Pi$, which can be either real or complex) such that $\text{co}(\cup_{A_i \in \mathcal{A}} A_i G) \subset \nu(\Pi)G$, which proves that $\rho(\mathcal{A}) = \nu(\Pi)$. Of course, there is no guarantee that the algorithm terminates within finite time. There are examples of matrix families for which such a product $\Pi$ does not exist [7, 19]. Nevertheless, numerical experiments and applications show that for a vast majority of matrix families the algorithm halts within finite time and finds the required product $\Pi$. The implementation details of the algorithm were upgraded in [31, 13]. Now it finds the JSR and the dominant product within a reasonable time for matrices of dimensions up to 20-25. The computation in higher dimensions usually takes too long. The version of the algorithm for non-negative matrices (see Section 9) works much faster and finds the JSR even in dimensions of several thousands.

Later it was observed [16] that the polytope $G$ produced by the algorithm is nothing else but the invariant body of the family $\mathcal{A}$. Moreover, from our Theorem [1] (Section 2) and Theorem [3] below in this section, it follows that $G$ is a unique invariant body. So, having
found it once by the algorithm we can be sure that there are no others. Only if the leading
eigenvalue of Π is non-real and has a rational mod π argument, then there are infinitely
many invariant bodies. We classify them all in Section 8 and modify the algorithm for that
case.

Now we need to briefly recall the invariant polytope algorithm, which will be referred to
as Algorithm 1. In [12] the cases of real and complex eigenvalues were considered separately.
Here we combine them in one algorithm.

Algorithm 1.

I. Choosing the candidate product. We choose a matrix product Π = Asn · · · As1 (a
candidate product) and want to prove that ρ(A) = ν(Π). There are several methods to select
the candidate product. One can just exhaust all matrix products up to some length and
take one which attains the maximal value of ν(Π). There are more sophisticated methods,
using branch-and-bound approach, etc. see [31].

Then we normalize our matrices as follows: ˜Π = [ν(Π)]−1Αj, ˜A = { ˜A1, ..., ˜Am} and ˜Π
is the corresponding product of matrices from ˜A.

II. The routine.

Let ν be the leading eigenvector of ˜Π. We define the set V1 as follows. If the leading
eigenvalue of ˜Π is real, then V1 = {ν, −ν}. If it is complex and, respectively, ν = x + iy,
with x, y ∈ R \ {0}, then V1 = φx,y(Γ) is an ellipse. We have ΠV1 = V1. Define Vj =
Aσj−1 · · · As1V1, j = 2, ..., n. The products ˜Πj = Πσj−1 · · · Πσ1 1, j = 2, ..., n are
cyclic permutations of ˜Π. If we put formally ˜Π1 = Π and Vn+1 = V1, then ˜ΠjVj = Vj and
AσjVj = Vj+1 for all j = 1, ..., n. The set ˜R = {V1, ..., Vn} is called a root. Then we
construct a sequence of finite sets Vj and their subsets Hj ⊂ Vj as follows:

Zero iteration. We set V0 = H0 = R.

kth iteration, k ≥ 1. We have a finite set Vk−1 and its subset Hk−1. We set Vk =
Vk−1, Hk = ∅ and for every V ∈ Hk−1, A ∈ ˜A, check whether ˜A V is in the interior of
c0{V | V ∈ Vk}. If it is, then we omit the set ˜A V and take the next pair (V, ˜A) ∈ Hk−1 × ˜A,
otherwise we add ˜A V to Vk and to Hk. If k ≥ 2, we do this for all pairs from Hk−1 × ˜A. If
k = 1 and hence Hk−1 = H0 = R and V = Vj ∈ R, then we exclude n pairs (V j , ˜A j ), j =
1, ..., n.

When all pairs (V, ˜A) are exhausted, both Vk and Hk are constructed. We define Gk =
c0{V | V ∈ Vk} and have

Vk = Vk−1 ∪ Hk , Gk = c0{ ˜A1Gk−1, ..., ˜AmGk−1}.

Termination. The algorithm halts when Vk = Vk−1, i.e., Hk = ∅ (no new sets V are
added in the kth iteration). In this case Gk−1 = Gk. Hence co{ ˜m j=1 AjGk} = ν(Π)Gk.
Therefore, Gk is an invariant convex body for A and ρ(A) = ν(Π).

End of the algorithm.

Implementation details. In practice the algorithm works not with the sets Vj but with
single points (in case or real leading eigenvalue) or with pairs of points (in case or a complex
leading eigenvalue).
The case of real leading eigenvalue of $\Pi$. We replace $V_1$ by the leading eigenvector $v_1$ of $\Pi$ and then in each iteration, the set $AV_j$ is replaced by $\tilde{A}v_j$. Thus, all $V_i$ and $H_i$ become sets of points. Let us have in some step $V_k = \{\tilde{V}_i\}_{i=1}^\ell$. To decide whether a newly born set $V = \{\pm v\}$ lies in the interior of $G_k = \text{co}\{\pm v_i\}_{i=1}^\ell$ we solve a linear programming problem

$$
\begin{align*}
& t_0 \rightarrow \max \\
& \text{subject to:} \\
& -s_i \leq t_i \leq s_i, \ i = 1, \ldots, \ell \\
& \sum_{i=1}^\ell s_i \leq 1 \\
& t_0v = \sum_{i=1}^\ell t_iv_i
\end{align*}
$$

We have $v \in \text{int}G_k$ if and only if $t_0 > 1$. In practice we fix a small tolerance parameter $\delta > 0$ (usually, $\delta$ is between $10^{-8}$ and $10^{-6}$) and decide that the set $V$ is redundant if $t_0 > 1 + \delta$, otherwise we keep the points $\pm v$ among the vertices of an invariant polytope, although they may actually not be vertices.

The case of complex leading eigenvalue of $\Pi$. We replace the ellipse $V_1$ by the pair of points $x_1, y_1 \in \mathbb{R}^d$ such that $v_1 = x_1 + iy_1$ is the leading eigenvector of $\Pi$. As we know, $V_1 = \Phi_{x_1, y_1}(\Gamma)$. Then in each iteration, the ellipse $AV_j$ is replaced by the pair of points $(\tilde{A}x_j, \tilde{A}y_j)$. Clearly, $V_j = \Phi_{\tilde{A}x_j, \tilde{A}y_j}(\Gamma)$. Thus, all $V_i$ and $H_i$ become sets of pairs of points. Let $(x, y)$ be a newly born pair. To prove that the ellipse $V = \Phi_{x, y}(\Gamma)$ is contained in the interior of $G_k = \text{co}\{V_i\}_{i=1}^\ell$ we solve the following optimization problem

$$
\begin{align*}
& t_0 \rightarrow \max \\
& \text{subject to:} \\
& \sqrt{t_k^2 + u_k^2} \leq s_k, \ k = 1, \ldots, 2\ell \\
& \sum_{k=1}^{2\ell} s_k \leq 1 \\
& t_0x = \sum_{i=1}^\ell (t_{2i-1}x_i - u_{2i-1}y_i) + (t_{2i}x_i + u_{2i}y_i) \\
& t_0y = \sum_{i=1}^\ell (u_{2i-1}x_i + t_{2i-1}y_i) + (u_{2i}x_i - t_{2i}y_i)
\end{align*}
$$

This is a conic programming problem and is solved by the interior point method on Lorentz cones (see www.mosek.com for the corresponding software). If $t_0 > 1$, then $V \subset \text{int}G_k$. So, we remove the pair $(x, y)$ if $t_0 > 1 + \delta$. Otherwise we set $V_{\ell+1} = V$ and add this ellipse (i.e., the pair $(x_{\ell+1}, y_{\ell+1}) = (x, y))$ to both $V_k$ and $H_k$. Note that, in contrast to the real case, here the condition $t_0 > 1$ is only sufficient but not necessary for the inclusion $V \subset \text{int}G_k$. That is why, usually the resulting set $G$ contains many redundant ellipses $V_i$ which are not “vertices” of $G$, i.e., are inside $G$ and could be removed. It slows down the algorithm but not significantly, see Section 10 for numerical results.

Comments and analysis of convergence. Actually, the algorithm works with the sets $V_k$ only, the convex bodies $G_k$ are needed to illustrate the geometric idea. Thus, in each iteration we construct a body $G_k \subset \mathbb{R}^d$, which is either a polytope (in case of real eigenvalue of $\Pi$) or a convex hull of ellipses (the case of complex eigenvalue), store all its vertices (ellipses) $V_i$ in the set $V_k$ and spot the set $H_k \subset V_k$ of newly appeared (after the previous iteration) sets $V_i$. Every time we check whether $\tilde{A}G_k \subset G_k$. If $\tilde{A}G_k \subset G_k$, then
$G_k$ is an invariant body, $∥\tilde{A}_k∥G_k\leq 1$ for all $i$. Otherwise, we update the sets $V_k$ and $H_k$ and continue.

If Algorithm 1 halts within finite time, then the candidate product $Π$ not only gives the precise value of JSR but also is a dominant product.

**Theorem A** [12]. Algorithm 1 applied to a candidate product $Π$ terminates within finite time if and only if $Π$ is a unique dominant product for $A$ and its leading eigenvalue is unique and simple.

Thus, we can always check whether a given product is dominant or not. If it is, then the invariant body of the family $A$ is readily available as the body $G_k$ obtained by the end of the algorithm.

**Example 6** For the matrices (9) from Example 1, the dominant product is $A_3^1A_2$. The root $R = \{v_1,v_2,v_3,v_4\}$ consists of four vertices of the polygon $G$, they are marked in red (Fig. 1, left). The four corresponding sides of the polar $G'$ are also red (Fig. 1, right).

For the matrices (10) from Example 2, the dominant product is $A_2^1A_2$. The root $R = \{v_1,v_2,v_3\}$ consists of three vertices of the polytope $G$, they are marked in red (Fig. 2, left). The three corresponding faces of the polar $(G')^*$ are also red (Fig. 1, right).

For the matrices (11) from Example 3, the dominant product is $A_1$, the root $R = \{V_1\}$, where $V_1$ is a circle (Fig. 3, left). The same is for the matrices (12) from Example 4.

**Theorem 3** If Algorithm 1 terminates after $k$th iteration, then it produces an invariant convex body $G_k$ for the family $A$. If the leading eigenvalue of the product $Π$ is either real or complex with an irrational mod $\pi$ argument, then $G_k$ is a unique invariant body for $A$.

The proof is given in the next section. Passing to the transpose family of operators $A^*$ we obtain the method of construction of Barabanov’s norm presented in [16]. Now we can claim that there are no other Barabanov norms. Applying Theorem 3 to the family $A^*$ we obtain the following

**Theorem 4** If Algorithm 1 applied to the transpose family $A^*$ terminates after $k$th iteration, then it produces Barabanov’s norm for the family $A$.

If the leading eigenvalue $\lambda$ of the candidate product $Π^*$ is real, then this norm is piecewise-linear, and is given by formula (6), where $v^*$ runs over the set of vertices of $G_k^*$. This is a unique Barabanov norm for $A$.

If $\lambda \notin \mathbb{R}$, then this norm is piecewise-quadratic, and is given by formula (8), where we set $a_i = x_i, b_i = y_i$ and $E^*_i = \Phi_{x_i,y_i}(\Gamma), i = 1,\ldots,N$, runs over the set of ellipses generating $G_k^*$. If the argument of $\lambda$ is irrational mod $\pi$, then this is a unique Barabanov norm for $A$.

The remaining case, when $\lambda \notin \mathbb{R}$ and the argument of $\lambda$ is rational mod $\pi$, is considered in Section 8. Now we turn to the proofs of the main results.
5. Proofs of the fundamental theorems

To prove Theorems 1 and 3 we need one auxiliary statement on the structure of trajectories of an irreducible system (Theorem B below). We begin with the following well-known fact. Its proof is given for convenience of the reader.

Lemma 1 There is a continuous function \( \psi(\delta, z) \) on \( \mathbb{R}_+^2 \) such that \( \psi(0, z) = 0 \) for all \( z \) and for every \( d \times d \) matrix \( A \), the following is true: if there is a vector \( x \) such that \( \| Ax - x \| \leq \delta \| x \| \), then \( A \) has an eigenvalue \( \lambda \in \mathbb{C} \) such that \( |\lambda - 1| \leq \psi(\delta, \| A \|) \).

Proof. Without loss of generality it can be assumed that \( \| x \| = 1 \) and that \( x = e_1 \) is the first basis vector. The polynomial \( p(\lambda) = \det (\lambda I - A) \) has the leading coefficient one and other coefficients at most \( 2^d \| A \|^d \) in modulus. Moreover, since \( \| A e_1 - e_1 \| \leq \delta \), it follows that the first column of the matrix \( \lambda I - A \) has all components at most \( \delta \) in modulus. Since the moduli of all other entries of this matrix are at most \( \| A \| + 1 \), we have \( |p(1)| = |\det (I - A)| \leq C \delta \), where \( C \leq 2^d(\| A \|+1)^{d-1} \). Therefore, there exists a root of \( p \) on the distance at most \( C_0(\delta)^1/d \) from the number 1.

Now we turn to the structure of trajectories. Let \( A = \{A_1, \ldots, A_m\} \) be an arbitrary irreducible system. The irreducibility implies that \( \rho(A) > 0 \) and that \( A \) possesses at least one invariant body \( G \). After normalization it can be assumed that \( \rho(A) = 1 \), all invariant bodies stay the same. A point \( x \in \mathbb{R}^d \) is called recurrent if it belongs to the boundary of the invariant body \( G \) and there is a trajectory \( \{x_k\}_{k \geq 0} \) such that \( x_0 = x \) and some subsequence \( \{x_{k_j}\}_{j \in \mathbb{N}} \) tends to \( x \) as \( j \to \infty \).

An orbit of a point \( x \) is the set \( \{nx | n \in \mathbb{N}\} \), i.e., is a union of all trajectories starting at \( x \). Observe that if \( x \) is recurrent, then the points of the trajectory of \( x \) are not necessarily recurrent.

Theorem B. Let \( A \) be an irreducible family with \( \rho(A) = 1 \). Then for every invariant body \( G \) of \( A \), there exists a compact subset \( C \) of the set of recurrent points such that \( G \) is the closed convex hull of orbits of points from \( C \).

Note that the set \( C \) in Theorem B depends on \( G \). That is why Theorem B does not imply the uniqueness of the invariant body. In fact, Theorem B holds also in cases when the invariant body is not unique, for instance, when the leading eigenvalue is non-real and has a rational mod \( \pi \) argument.

If a matrix \( A \) has a unique simple leading eigenvalue, then it has a leading eigenspace which is either one-dimensional (the linear span of the real leading eigenvector) or two-dimensional (the real linear span of the real and complex part of the leading eigenvector).

Proposition 2 If a family of operators has finitely many dominant products and each of them has a unique and simple leading eigenvalue, then every recurrent point of this family belongs to the leading eigenspace of one of these products or of one of its cyclic permutations.
Proof. Let the spectral gap be the interval \((1 - \varepsilon, 1)\), where \(\varepsilon > 0\). Since the family \(\mathcal{A}\) is irreducible, the norms of all products of its matrices are bounded by some constant \(C\). Choose small \(\delta > 0\) so that \(\psi(\delta, C) < \varepsilon\), where the function \(\psi\) is defined in Lemma 1. If \(x \neq 0\) is a recurrent point, then there is a product \(S \in \mathcal{A}^k\) such that \(\|Sx - x\| \leq \delta \|x\|\). By Lemma 1 this implies that \(S\) has an eigenvalue \(\mu\) such that \(|1 - \mu| \leq \psi(\delta, C) < \varepsilon\). Hence, \(\mu \in (1 - \varepsilon, 1 + \varepsilon)\). On the other hand, because of the spectral gap, \(\mu\) can neither be in \((1 - \varepsilon, 1)\) nor bigger than one. Consequently, \(\mu = 1\) and therefore \(S\) is a power of some dominant product or of one of its cyclic permutations. It can be assumed that this is a power of a dominant product, the case of a cyclic permutation is literally the same. Thus, for the point \(x\), there is a dominant product \(\Pi\) and a sequence of integers \(\{j_k\}_{k \in \mathbb{N}}\) such that \(\|\Pi^k x - x\| \to 0\) as \(k \to \infty\). Since the leading eigenvalue of \(\Pi\) is simple, it follows that \(\Pi^k x\) converges to the projection of \(x\) to the line containing the leading eigenvector (to the leading eigenspace in the complex case). Therefore, \(x\) coincides with this projection, and so \(x\) is the leading eigenvector (respectively, belongs to the leading eigenspace).

Proof of Theorem 1. Let \(G\) be an arbitrary invariant body for \(\mathcal{A}\), \(\Pi = A_{j_1} \cdots A_{j_n}\) be the dominant product, \(\lambda\) be its leading eigenvalue, \(\Pi_k\) be its \(k\)th cyclic permutation. After normalization it can be assumed that \(\rho(\Pi) = 1\).

The case \(\lambda \in \mathbb{R}\). In this case \(\lambda = \pm 1\) and we assume \(\lambda = 1\), the other case is considered in the same way. Denote by \(v_1\) the leading eigenvector of \(\Pi\) that belongs to \(\partial G\) (any of the two vectors). Then \(v_k = A_{j_{k-1}} \cdots A_{j_1} v_1\), \(k = 2, \ldots, n\). Clearly, \(v_k\) is the leading eigenvector of \(\Pi_k\). Moreover, \(v_k \in \partial G\) for all \(k\). Indeed, \(\|v_k\|_G = \|A_{j_{k-1}} \cdots A_{j_1} v_1\|_G \leq \|v_1\|_G = 1\), because the norm \(\|\cdot\|_G\), as a Lyapunov function of the system, is non-increasing on any trajectory; \(\|v_1\|_G = 1\) because \(v_1 \in \partial G\). Thus, \(\|v_k\|_G \leq 1\). On the other hand, denoting \(v_{n+1} = v_1\), we get \(1 = \|x_{n+1}\|_G = \|A_{j_1} \cdots A_{j_n} v_1\|_G \leq \|v_k\|_G\) and hence \(\|v_k\|_G \geq 1\). Thus \(\|v_k\|_G = 1\) and consequently \(v_k \in \partial G\) for all \(k\). Theorem B and Proposition 2 imply that \(G\) is the closure of the convex hull of all trajectories starting at the points \(v_k\). Hence it is obtained by Algorithm 1 from the candidate product \(\Pi\).

The case \(\lambda \notin \mathbb{R}\). The leading ellipse is \(E = \Phi_{x,y}(\Gamma)\), where \(x + iy\) is the complex leading eigenvector of \(\Pi\). Normalize this vector so that \(x \in \partial G\). If the argument \(\varphi\) of \(\lambda\) is irrational \(mod \pi\), then the set \(\{\Pi^k x\}_{k \in \mathbb{N}}\) is everywhere dense on \(E\). The sequence of norms \(\{\|\Pi^k x\|_G\}_{k \in \mathbb{N}}\) is non-increasing and it has the number \(\|x\|_G = 1\) as a limit point, hence it is an identical one. Therefore, the ellipse \(E\) lies on the boundary of \(G\) and \(E\) is an intersection of this surface with the leading eigenspace of \(\Pi\). Combining Theorem B and Proposition 2 and taking into account that all points from \(E\) are recurrent (because for every \(z \in E\), the sequence \(\{\Pi^k z\}_{k \in \mathbb{N}}\) has a limit point \(z\)), we conclude that \(G\) is the closure of convex hulls of all trajectories starting at points from \(E\). Therefore, \(G\) is a convex hull of images of \(E\) under the action of all products of operators from \(\mathcal{A}\). Hence it is obtained by Algorithm 1 from the candidate product \(\Pi\).

Proof of Theorem 3. After the \(k\)th iteration we obtain the body \(G_k\), which is a convex hull of sets \(V \in \mathcal{V}_k\), where \(\mathcal{V}_k = \mathcal{H}_0 \cup \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_k\). Consider the multivalued operator \(\mathcal{\bar{A}}\).
which maps every element $V \in \mathcal{V}_k$ to the set of elements $\hat{A}_1V, \ldots, \hat{A}_mV$. Then $\overline{A}$ maps the root $\mathcal{H}_0 = \mathcal{R}$ to the union $\{\mathcal{H}_0, \mathcal{H}_1\}$ and each set $\mathcal{H}_j, j \geq 2$, to $\mathcal{H}_{j+1}$. Therefore, $\overline{A}\mathcal{V}_{k-1} = \mathcal{V}_k$. However, the algorithm terminates after the $k$th iteration, hence $\mathcal{V}_{k-1} = \mathcal{V}_k$. Thus, $\overline{A}\mathcal{V}_k = \mathcal{V}_k$ and therefore $\text{co}\{\hat{A}_1G_k, \ldots, \hat{A}_mG_k\} = G_k$. So, $G_k$ is an invariant body. By Theorem A, the product $\Pi$ is dominant. Hence, we can use Theorem 1 which implies the uniqueness of the invariant body in cases of real leading eigenvalue and of complex leading eigenvalue with an irrational mod $\pi$ argument.

Thus, we have proved the uniqueness and have established the structure of Barabanov’s norms for general matrix families possessing dominant products with the leading eigenvalue which is either real or complex with an irrational mod $\pi$ argument. The remaining case when the leading eigenvalue is non-real and has a rational mod $\pi$ argument is more delicate; we attack it in Section 8. To this end we need an auxiliary result characterising the growth of trajectories of an arbitrary system with $\rho(A) = 1$. This result is, probably, of some independent interest and we put in a separate section (Section 7).

Now we are going to analyse systems with several different (up to powers and cyclic permutations) dominant products. This case is rather special but it plays an important role in some applications.

6. Systems with finitely many dominant products

According to numerical experiments, the uniqueness of the dominant product takes place for almost all matrix families (at least, randomly generated ones). Nevertheless, in applications it happens that there are several dominant products. It occurs when there are some relations between matrices of the family. For example, in the computation of the Hölder regularity of wavelets and of limit functions of subdivision schemes, one needs to find the JSR of two special matrices $T_0, T_1$, which are sometimes both dominant, see [11, 6, 13] and references therein. A similar situation occurs in some problems of combinatorics, number theory, and formal languages [8, 24, 31, 32, 40].

In fact, in the results of Section 2-4 the uniqueness of the dominant product is not a restriction. Explicit classification and construction of the invariant body and of the Barabanov norms can be realised in a similar way when the system has finitely many dominant products. The only difference is that, as we are going to see, the Barabanov norm is never unique in this case: any system with several dominant products has infinitely many invariant bodies and Barabanov norms, which can, nevertheless, be classified (Corollary II).

The algorithm of computing the JSR for families with several dominant products was elaborated in [13]. It is very similar to Algorithm 1, but it starts with several roots $\mathcal{R}^{(j)}$ (each root is associated to the corresponding dominant product $\Pi^{(j)}$). However, to provide the convergence of the algorithm one needs to balance the roots, i.e., to multiply each of them by a certain positive constant $\alpha_j$, and those constants have to be found. Otherwise, the algorithm does not converge within finite time.
Algorithm 2.

**Choosing the candidate products.** We choose several candidate products $\Pi^{(1)}, \ldots, \Pi^{(r)}$ that are all primitive, different up to cyclic permutations, and having the same values $\nu(\Pi^{(i)})$ for all $i = 1, \ldots, r$. Denote this value by $q$. Then normalise all matrices from $\mathcal{A}$ as $\tilde{A}_j = q^{-1}A_j$.

**The balancing.** Take a vector of positive numbers $\alpha = (\alpha_1, \ldots, \alpha_r)$ called balancing vector. Those numbers are selected in a special way to provide the convergence of the algorithm. For a method of finding a proper balancing vector see [13]. Then we define the sets $V_1^{(i)}, \ldots, V_n^{(i)}$ as in Algorithm 1: they are either symmetric pairs of leading eigenvectors of $\Pi^{(i)}$ (if $\Pi^{(i)}$ has a real leading eigenvalue) or of leading ellipses (the case of complex leading eigenvalue). Then we form the roots $R^{(i)} = \{\alpha_1V_1^{(i)}, \ldots, \alpha_nV_n^{(i)}\}$ (each element $V_k^{(i)}$ is multiplied by $\alpha_i$) that consist of symmetric pairs of leading eigenvectors (the case of real leading eigenvalue) or of leading ellipses (non-real leading eigenvalue).

**The routine.**

**Zero iteration.** We set $V_0 = H_0 = \cup_{i=1}^r \alpha_i R^{(i)}$.

**kth iteration** is literally the same as in Algorithm 1.

**Termination** is the same as in Algorithm 1.

*End of the algorithm.*

If Algorithm 2 terminates after the $k$th iteration, then we obtain a convex body $G_k = \operatorname{co}\{V \mid V \in V_k\}$ which is an invariant body for $\mathcal{A}$. It is a convex hull of several segments and of several ellipses, all centered at the origin.

**Theorem C** [13]. If Algorithm 2 applied to candidate products $\Pi^{(1)}, \ldots, \Pi^{(r)}$ with equal values of $\nu(\Pi^{(i)})$ and to some balancing vector $\alpha$ terminates within finite time, then these products are dominant and each of them has a unique and simple leading eigenvalue. Conversely, if these products are dominant for $\mathcal{A}$ and their leading eigenvalues are unique and simple, then there is a balancing vector for which Algorithm 2 terminates within finite time.

Thus, if $\mathcal{A}$ has finitely many dominant products, then they can be found by Algorithm 2 along with the weights $\{\alpha_i\}_{i=1}^r$. Note that the set of dominant products is unique but the set of weights is not. Let us now show that the same algorithm gives the invariant body and, if the leading eigenvalues of all the dominant products are either real or complex with irrational mod $\pi$ arguments, then all invariant bodies are exhausted by those found with Algorithm 2.

**Theorem 5** If Algorithm 2 halts within finite time making $k$ iterations, then it produces an invariant convex body $G_k$ for the family $\mathcal{A}$.

If the leading eigenvalues of all the dominant products of $\mathcal{A}$ are either real or complex with irrational mod $\pi$ arguments, then every invariant body of $\mathcal{A}$ is obtained by Algorithm 2 with some balancing vector. Different balancing vectors produce different invariant bodies.

**Proof.** If Algorithm 2 halts after $k$th iteration, then $G_k$ is an invariant body. Moreover, in this case all the products $\Pi^{(i)}$, $i = 1, \ldots, r$ are dominant (Theorem C). If all their leading eigenvalues are either real or complex with an irrational mod $\pi$ argument, then every
invariant body of $\mathcal{A}$ is obtained by Algorithm 2 with some balancing vector. This is proved in the same way as Theorem 1 by applying Proposition 2 and Theorem B from Section 5. Also in the same way we show that all the sets from the roots $\mathcal{R}^{(i)}$ lie on the boundary of the invariant body. Hence, changing the multipliers $\alpha_i$ we obtain different invariant sets.

The key difference with the case of one dominant product is that now choosing different weights we get different invariant bodies. This is the reason of non-uniqueness of the invariant body for families with many dominant products.

**Corollary 1** If a family of operators has $r \geq 2$ dominant products (up to cyclic permutations), then it has infinitely many invariant bodies. If, in addition, the leading eigenvalues of all those dominant products are either real or complex with irrational $\mod \pi$ arguments, then all those invariant bodies are convex hulls of finitely many points and ellipses.

**Proof.** By Theorem C, there exists a balancing vector $(\alpha_1, \ldots, \alpha_r)$ for which Algorithm 2 terminates within finite time and gives an invariant body. Fix $\alpha_1$. If we slightly vary other coefficients $\alpha_2, \ldots, \alpha_r$, then Algorithm 2 performs the same iterations as before. Indeed, each iterations is defined by the set of dead vertices, when $\mathbf{AV}_p \in \text{int} \ G_k$. A sufficiently small variation of parameters keeps this inclusion. Hence, after a small variation of $\alpha_2, \ldots, \alpha_r$, Algorithm 2 performs the same iterations. Consequently it terminates within finite time. By Theorem 5, different variations of parameters produce different invariant body.

**Remark 4** Applying Algorithm 2 to the transpose family $\mathcal{A}^*$ we obtain the Barabanov norm for $\mathcal{A}$. Similarly to Theorem 4 in Section 4 one expresses the relation between the invariant body of $\mathcal{A}^*$ and the Barabanov norm for $\mathcal{A}$.

Thus, in case of several dominant products the Barabanov norm is never unique. Nevertheless, if the leading eigenvalue of every dominant product is either real or complex with an irrational $\mod \pi$ argument, then all those Barabanov norms are classified by Theorem 5. They are parametrized by the balancing vectors $\alpha \in \mathbb{R}_+^r$ for which Algorithm 2 halts within finite time. Since each balancing vector can be normalized by the condition $\sum_{i=1}^r \alpha_i = 1$, we see that there exists a $(r - 1)$-parametric family of Barabanov norms.

If at least one of the dominant products has a complex leading eigenvalue with a rational $\mod \pi$ argument, the Barabanov norm can still be computed by the same Algorithm 2. However, there will be other norms that are not obtained by that algorithm. Their classification requires another method, see Section 8. To introduce that method we first need to make a detailed analysis of growth of trajectories of a discrete-time system. This is a subject of the next section.
7. Classification of trajectories of the fastest growth

The results of this section will be applied in characterising Barabanov norms in case of rational mod \( \pi \) arguments of the leading eigenvalue. They are also of an independent interest. We are going to find all trajectories of a linear switching system with the fastest asymptotic growth.

An analysis of asymptotic growth of trajectories is a subject of an extensive literature, see, for example, [9, 20, 28, 36, 47]. If the discrete-time linear switching system (1) is irreducible, then the maximal possible growth of trajectories is \( \|x(k)\| \asymp \rho^k, k \in \mathbb{N} \). How to identify all those “fastest” trajectories?

**Problem 1.** How to characterise all switching laws \( A(\cdot) \) realising the maximal growth of trajectories of the linear switching system?

We are going to show that if the system has finitely many dominant products, then Problem 1 can be explicitly solved:

Suppose a switching system has a finite set of dominant products and each of them has a unique and simple leading eigenvalue; then a switching law \( A(\cdot) \) generates trajectories of the maximal growth precisely when it is eventually periodic with the period equal to a dominant product. For all other laws, we have \( \|A(k) \cdots A(1)\| = o(\rho^k) \) as \( k \to \infty \).

This condition means that there exist numbers \( n \) and \( N \) and an infinite sequence of indices \( s_1s_2\ldots \) such that \( s_k = s_{k+n} \) for all \( k > N \), \( A(j) = A_{s_j}, j \in \mathbb{N} \), and the period \( \Pi = A_{s_{N+n}} \cdots A_{s_{N+1}} \) is a dominant product.

If we normalize the family so that \( \rho(A) = 1 \), then Problem 1 becomes to characterise all switching laws that do not tend to zero as \( k \to \infty \). Since the normalized family has the same set of switching laws of the fastest growth, it suffices to consider the case \( \rho(A) = 1 \).

**Theorem 6** Let a system \( A \) be irreducible, normalized as \( \rho(A) = 1 \), and have finitely many dominant products. Let also the leading eigenvalues of all dominant products be unique and simple. Then all trajectories of the system converge to zero apart from those corresponding to eventually periodic switching laws with a period equal to a dominant product.

Thus, the switching laws of the maximal growth are precisely those eventually periodic ones with the period equal to a dominant product. All other switching laws tend to zero.

**Example 7** For the family [9] from Example [1] the dominant product is \( A_1^3A_2 \). Hence the trajectories of the fastest growth all have the form \( (A_1^3A_2)^j\Pi_0x_0 \), where \( \Pi_0 \) is an arbitrary product of the matrices \( A_1, A_2 \). In particular, the trajectories \( (A_1^3A_2)^jA_2x_0 \) and \( (A_1^3A_2)^jA_2A_1x_0, j \in \mathbb{N} \), are both of the fastest growth. The trajectories of the corresponding normalized family are shown in Fig. 6. The left figure presents these two trajectories of the fastest growth: brown and green broken lines respectively. The points of all trajectories are shown in black. In Fig. 6 (right), one of remaining trajectories (of not the fastest growth) is shown.
Figure 6: Left: Two fastest growth trajectories \((A_1^jA_2^j)A_1x_0\) and \((A_1^jA_2^j)A_0A_1x_0\), \(j \in \mathbb{N}\), (brown and green respectively) for the family \((\bar{A})\); Right: some of other trajectories.

**Remark 5** Theorem 6 characterises all switching laws for which \(\|A(k) \cdots A(1)\| \geq C \rho^k\). Not all trajectories of those switching laws have the maximal growth. This property may depend on the initial point \(x_0\). Nevertheless, all trajectories of the maximal growth can be explicitly characterized. For the sake of simplicity, assume again that we have a normalized family, for which \(\rho(A) = 1\). Since \(\|x(k)\| \leq \|A(k) \cdots A(1)\| \cdot \|x_0\|\), we see that if the switching law is not eventually periodic with a dominant period, then \(\|x(k)\| \to 0\) as \(k \to \infty\), hence this trajectory is not of the maximal growth. If it is eventually periodic with a dominant product \(\Pi\) of length \(|\Pi| = n\) as a period, then the product \(A(k) \cdots A(1)\) for \(k = jn + N\) has the form \(\Pi \Pi_0\), where \(\Pi_0\) is a product of length \(N\). The leading eigenvalue of \(\Pi\) is equal to \(\rho(A) = 1\) and, by the assumption, this eigenvalue is unique and simple. Denote by \(L\) the subspace of \(\mathbb{R}^k\) of dimension \(d - 1\) spanned by all vectors of the Jordan basis of \(\Pi\) except for the leading eigenvector. Actually, \(L\) is an orthogonal complement of the leading eigenvector of the transpose matrix \(\Pi^T\). Then, if \(\Pi_0x_0 \in L\), then \(\Pi^j \Pi_0x_0 \to 0\) as \(j \to \infty\), and hence \(x(k) \to 0\) as \(k \to \infty\). Otherwise, \(\|\Pi^j \Pi_0x_0\| \geq C\) for all \(j\) and hence the norms \(\|x(k)\|\) are bounded below by a positive constant for all \(k \in \mathbb{N}\). Thus, the trajectory \(\{x(k)\}_{k=0}^{\infty}\) has the maximal growth if and only if the switching law is periodic with a dominant period \(\Pi\) and \(\Pi_0x_0 \notin L\). This gives the complete classification of all trajectories of the fastest growth.

Before giving a proof of Theorem 6 we need to introduce some more notation.

The **cyclic tree of matrix products.** To a family of operators \(\bar{A} = \{\bar{A}_1, \ldots, \bar{A}_m\}\) and to some product \(\bar{\Pi} = \bar{A}_{d_n} \cdots \bar{A}_{d_1}\) with the spectral radius 1 we associate the cyclic tree \(T\)
generated by the word \(d_1 \ldots d_n\) (or, which is the same, by the product \(\Pi\)). It is defined as follows. The root is formed by a cycle \(\mathcal{R}\) of \(n\) nodes \(V_1, \ldots, V_n\). They are, by definition, the nodes of zero level. For every \(i \leq n\) an edge (all edges are directed) goes from \(V_i\) to \(V_{i+1}\), where we set \(V_{n+1} = V_1\). At each node of the root \(m-1\) edges start to nodes of the first level. So, there are \(n(m-1)\) different nodes on the first level. The sequel is by induction: there are \(n(m-1)m^{k-1}\) nodes of the \(k\)th level, \(k \geq 1\), from each of them \(m\) edges ("children") go to \(m\) different nodes of the \((k+1)\)st level.

Each index (letter) \(q\) belongs to the alphabet \(\{1, \ldots, m\}\) and is associated to the matrix \(A_q \in \mathcal{A} = \{A_1, \ldots, A_m\}\). Let us recall that we write products in the inverse order: from the right to the left. We assume that the root \(\mathcal{R}\) is primitive, i.e., is not a power of a shorter word. To every edge of the tree \(\mathcal{T}\) we associate a letter \(q\) (or the corresponding matrix \(A_q\)), as follows: the edge \(V_iV_{i+1}\) of the root corresponds to \(d_i\), \(i = 1, \ldots, n\); at each node \(V_i\) of the root \(m-1\) edges start to the first level associated to all the \(m\) letters except for \(i\). From each node \(V\) of level \(k \geq 1\) exactly \(m\) edges start associated to all the letters \(1, \ldots, m\).

We identify the words with the corresponding products of matrices from \(\mathcal{A}\). To a given point \(V_i \in \mathcal{R}\) and to a given finite word \(q_1 \ldots q_k\) we associate the node \(A_{q_k} \ldots A_{q_1}V_j\), which is the end of the path from \(V_i\) along the edges \(q_1, q_2, \ldots, q_k\) respectively. The empty word corresponds to \(V_i\). To an infinite word and to a node \(V_i \in \mathcal{R}\) we associate an infinite path \(V_i = V^{(0)} \rightarrow V^{(1)} \rightarrow V^{(2)} \rightarrow \cdots\) on the tree (all the paths are without backtracking) starting at \(V_i\). This path corresponds to the starting node \(V_i\) and to an infinite word \(s_1 \ldots s_k\). A node \(V^{(k)}\) on this path on \(k\)th level is \(V^{(k)} = A_{s_k} \ldots A_{s_1}V_i\).

The routine of Algorithm 1 can be described in terms of the tree \(\mathcal{T}\). First, we have a root \(\mathcal{R} = \{V_1, \ldots, V_n\}\). At the first step we take any node \(V_i \in \mathcal{R}\) and consider successively its \((m-1)\) children from the first level. For each child \(V = AV_i\), where \(A \in \mathcal{A} \setminus \{A_{d_i}\}\) we determine, whether or not \(V\) belongs to the interior of \(G_1 = \co V_1\). If it does, then \(V\) is a dead node or dead leaf generating a dead branch: we will never come back to \(V\), nor to nodes of the branch starting at \(V\) (so, this branch is cut off). If it does not, then \(V\) is an alive leaf, and we add this element \(V\) to the set \(V_1\) and to the set \(H_1\). After the first iteration all alive nodes of the first level form the set \(H_1\). At the second step we deal with the nodes from \(H_1\) only and obtain the next set of alive nodes of the second level \(H_2\), etc. Thus, after the \(k\)th iteration we have a family \(H_k\) of alive nodes from the \(k\)th level, and a set \(V_k = \bigcup_{j=0}^{k} H_j\). A node \(V\) belongs to the set \(V_k\) if and only if its level does not exceed \(k\) and it belongs to an alive branch starting from the root. The convex body \(G_k\) is the convex hull \(\co V_k\). The convex body \(G_{k-1}\) is invariant if \(H_k = \emptyset\), i.e., the \(k\)th iteration produces no alive leaves (only dead ones). This means that there are no alive paths of length \(k\) from the root. Therefore \(G_k = G_{k-1}\). Otherwise, if \(H_k\) is nonempty, the algorithm makes the next iteration and goes to the \((k+1)\)st level: we take children of each element of \(H_k\), determine whether they alive of dead, etc.

Algorithm 2 works simultaneously with \(r\) cyclic trees. Each cyclic tree \(\mathcal{T}^{(i)}\) is generated by the \(i\)th candidate product \(\hat{\Pi}^{(i)}\), \(i = 1, \ldots, r\). On \(k\)th iteration we run over the set \(H_{k-1}\) that consists of nodes of the \((k-1)\)st level of all the trees added in the \((k-1)\)st iteration. The alive leafs of every node are added to \(H_k\), the dead leafs are omitted together with edges growing from them. When the whole set \(H_{k-1}\) is exhausted, we set \(V_k = V_{k-1} \cup H_k\) and go
to the next iteration.

**Proof of Theorem 6** For the sake of simplicity, we assume that all the dominant products have real eigenvalues. Otherwise we replace the leading eigenvectors of some products by leading ellipses.

By Theorem 5 (Section 6), for every family with finitely many dominant products, Algorithm 2 converges within finite time. Let it perform \( k \) iterations. Denote by \( N \) the sum of the number \( k \) and of maximal length of all dominant products of \( \mathcal{A} \). Take an arbitrary dominant product \( \Pi \) of length \( n \), denote the corresponding word by \( \pi \) and denote the leading eigenvector of \( \Pi \) by \( v \). Every infinite switching law which does not have period \( \pi \) has the form \( \pi^\ell q s \), where \( \ell \geq 0 \) is an integer, \( q \) is a word of length \( N \) whose prefix of length \( n \) is different from \( \pi \), and \( s \) is an infinite word. Let \( Q \in \mathcal{A}^N \) be the product corresponding to the word \( q \). We have \( Q\Pi^\ell v = Qv \). By Algorithm 1, the trajectory starting at \( v \) with the switching law \( q \) has a point with G-norm (i.e., with the norm that has a unit ball \( G \)) strictly less than one. This point is a dead node on the path along the cyclic tree \( T \) generated by the product \( q \) and starting at \( v \). Denote by \( \mu \) the maximal G-norm of dead vertices of the trees generated by the dominant products. Since this set of vertices is finite, it follows that \( \mu < 1 \). At every path starting from the root, a dead node has to appear by the \( k \)th iteration, hence it corresponds to a product of length at most \( k + n \leq N \). Since the G-norm is non-increasing along any trajectory, it follows that for every prefix \( q' \) of the word \( q \) of length at least \( N \), we have \( \|Q\Pi^\ell v\|_G \leq \mu \). Thus, for every switching law, unless it is eventually periodic with a dominant period, there is a number \( N_1 \) such that for every its prefix \( s \) of lengths bigger than \( N_1 \), we have \( \|Sv\|_G \leq \mu \). Choosing the maximum of those numbers \( N_1 \) over all vertices of \( G \) and taking into account that the G-norm of every linear operators is achieved at one of the vertices of \( G \) we conclude that for every switching law, unless it is eventually periodic with a dominant period, every sufficiently long its prefix \( s \) satisfies \( \|Sv\|_G \leq \mu \). Hence, \( v_j \) tends to zero as \( j \to \infty \).

\( \square \)

8. The case of rational \( \mod \pi \) argument

By the results of Sections 2 and 4, if a system has a dominant product, then its Barabanov norm is unique, provided the leading eigenvalue of the dominant product is either real or complex with a rational \( \mod \pi \) argument. In the former case the unique norm is piecewise-linear, in the latter case it is piecewise-quadratic. In both cases the norm has a simple structure and can be constructed with Algorithm 1. What can be said in the last case when a dominant product has a non-real leading eigenvalue with a rational \( \mod \pi \) argument? We are going to see that in this case the set of Barabanov norms is much richer and more complicated. Nevertheless, we will classify all those norms (Theorem 8).
First of all, the uniqueness may fail. For example, if \( A_1 \) is a rotation of the plane \( \mathbb{R}^2 \) by \( 90^\circ \) and \( A_2 \) is an arbitrary operator with the (Euclidean) norm at most \( \frac{1}{2} \), then the pair \( \{A_1, A_2\} \) has infinitely many invariant bodies: every regular \( 4n \)-gon is invariant, \( n \in \mathbb{N} \). The operator \( A_1 \) is dominant and all other conditions of Theorem \( \PageIndex{1} \) (Section 2) are fulfilled, but the argument of the leading eigenvalue \( \lambda = e^{\frac{i\pi}{4}} \) is rational mod \( \pi \).

In spite of the non-uniqueness, one may hope that all invariant bodies can still be constructed by Algorithm 1 with a proper choice of the root sets \( \mathcal{R} = \{V_j\}_{j=1}^n \). To introduce the idea we need one more notation generalizing leading eigenvectors or leading ellipses. In this section we deal with solid ellipses \( E = \Phi_{x,y}(D) \), where \( D \subset \mathbb{R}^2 \) is a unit disc, and use for them the same notation as for the curves \( E = \Phi_{x,y}(\Gamma) \), where \( \Gamma \subset \mathbb{R}^2 \) is a unit circle.

**Definition 5** For a given product \( \Pi \in \mathcal{A}^N \) with a simple complex leading eigenvalue \( \lambda \), a convex compact subset \( V \) of the leading eigenspace of \( \Pi \) is called an admissible set if \( V \neq \{0\}, V = -V \), and \( \Pi V = |\lambda| V \).

If \( \lambda \in \mathbb{R} \), then an admissible set is a segment parallel to the leading eigenvector. If \( \lambda \) is complex with an irrational mod \( \pi \) argument, then an admissible set is a leading ellipse \( E = \Phi_{x,y}(D) \), where \( x + iy \) is the leading eigenvector and \( D \) is a unit disc on the plane. If the argument of \( \lambda \) is rational mod \( \pi \), then there are infinitely many, up to homothety, admissible sets. The proof of the following lemma is omitted since it is simple.

**Lemma 2** Let a product \( \Pi \in \mathcal{A}^N \) has a complex leading eigenvalue with an incommensurable with \( \pi \) argument \( \varphi \); then a set \( V \) is admissible if and only if \( V = \Phi_{x,y}(M) \), where \( M \subset \mathbb{R}^2 \) is a convex body symmetric about the origin and mapped to itself by the rotation by the angle \( \varphi \).

The generalization of Algorithm 1 to an arbitrary admissible starting set \( V_1 = \Phi_{x,y}(M) \) is the following. We take an admissible set \( V_1 \) and define as usual the sets \( V_j = A_{d_{j-1}} \cdots A_{d_1} V_1 \), \( j = 2, \ldots, n \). Since the restriction of the operator \( \Pi = A_{d_n} \cdots A_{d_1} \) is a composition of rotation by the angle \( \varphi \) and of multiplication by \( |\lambda| = \nu(\Pi) \), we see that the normalized operator \( \tilde{\Pi} \) is a rotation by the angle \( \varphi \). Hence \( \Pi V_1 = V_1 \), so the sets \( \{V_j\}_{j=1}^n \) indeed form a cycle \( V_1 \to \cdots \to V_n \to V_1 \) with the edges (operators) \( \tilde{A}_{d_1}, \ldots, \tilde{A}_{d_n} \). Thus, Algorithm 1 with the candidate product \( \Pi \) and with the root \( \mathcal{R} = \{V_j\}_{j=1}^n \) produces an invariant body \( G_k \) whenever it halts after \( k \)th iteration.

Thus, taking an arbitrary admissible set \( V_1 \) we define the root \( \mathcal{R} \) and start Algorithm 1. If \( M = D \) is a disc, then \( V_1 \) is the leading ellipsoid. In this case, as it follows from Theorem A, Algorithm 1 halts within finite time provided the product \( \Pi \) is dominant. However, this is not true for some other admissible sets \( V_1 \) as Example \( \PageIndex{8} \) below demonstrates. Therefore, this direct generalization of Algorithm 1 may not be applicable for admissible sets other than ellipses.

**Example 8** We are going to construct a pair of \( 4 \times 4 \) matrices with one dominant product and a complex leading eigenvalue with a rational mod \( \pi \) argument and an admissible set \( V_1 \) for which Algorithm 1 does not terminate within finite time.
We consider the space $\mathbb{R}^4$ and its two-dimensional subspace $L = \{(x_1, x_2, 0, 0)^T \in \mathbb{R}^4\}$. Sometimes we denote a point from $L$ as $x = (x_1, x_2)^T$. An orthogonal projection of a point $y$ to $L$ is denoted by $\tilde{y}$. Consider a regular hexagon $H$ on $L$ centered at the origin with one vertex at the point $v = (0, 1, 0, 0)^T$. Its side is equal to one. Take small $\tau > 0$ and a vector $b = (\tau, 1 - 4\tau^2, \tau, \tau)^T \in \mathbb{R}^4$. For all sufficiently small $\tau$, we have $\|b\|_2 < 1$. Consider a pair of matrices $A = \{A_1, A_2\}$ with

$$A_1 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad A_2 = b b^T. \quad (16)$$

Thus, the matrix $A_1$ consists of three diagonal blocks. The first $2 \times 2$ block (we call it $B$) is the rotation of the plane $L$ by $120^\circ$, the other two one-dimensional blocks are $\frac{1}{2}$ and $\frac{1}{4}$. The rank-one matrix $A_2$ defines the operator $A_2 x = (x, b) b$. This is an orthogonal projection to the direction of vector $b$ multiplied by $\|b\|_2$. Clearly, $\|A_2\| = \|b\|_2 < 1$. On the other hand, $\|A_1\| = \rho(A_1) = 1$. Hence, for every product $S$ of matrices $A_1, A_2$, we have $\|S\| \leq \|b\|_2$, unless $S$ is a power of $A_1$. Therefore, $A_1$ is a (unique!) dominant product of the family $A = \{A_1, A_2\}$. Consequently, $A$ has a spectral gap $(\|b\|_2, 1)$ and $\rho(A) = \rho(A_1) = 1$.

**Proposition 3** For every small $\tau > 0$, Algorithm 1 applied to the pair $\{A_1, A_2\}$ with the initial admissible set $V_1 = H$, makes infinitely many iterations and produces an invariant body with infinite discrete set of extreme points.

**Remark 6** Algorithm 1 applied to the pair $\{v, -v\}$ gives the same result as being applied to the hexagon $H$.

**Proof of Proposition 3** For every $k$, the points $\pm A_1^k v$ are vertices of $H$. Denote $c = A_2 v = (1 - 4\tau^2) b$. For small $\tau$, the point $\tilde{c} = \left((1 - 4\tau^2)\tau, (1 - 4\tau^2)^2\right)^T$ (the projection of $c$ to $L$) is out of $H$. Denote by $H_\tau$ the regular hexagon in $L$ centered at the origin and having one of the vertices at the point $\tilde{c}$. All vertices of $H_\tau$ are out of $H$. Clearly, all points $\pm B^k \tilde{c}$, $k \in \mathbb{N}$, are also vertices of $H_\tau$. Finally, if $\tau$ is small enough, then the point $v$ has the biggest in modulus scalar product with the vector $b$ among all vertices of the hexagons $H$ and $H_\tau$ and the vector $b$. So, the maximum of the functional $F(x) = (b, x)$ on the set $\text{co}\{H, H_\tau, b\}$ is attained at a unique point $v$. Therefore, projections of all points generated by Algorithm 1 to the plane $L$ are in the set $\text{co}\{H, H_\tau\}$. Consider the sequence

$$A_{3k}^2 A_2 v = (B^{3k} \tilde{c}, 2^{-k} \tau, 4^{-k} \tau)^T = (\tilde{c}, 2^{-k} \tau, 4^{-k} \tau)^T, \quad k \geq 0.$$ 

All these points are convex independent (none of them is in the convex hull of others) since so are the points $(2^{-k} \tau, 4^{-k} \tau)^T$, $k \in \mathbb{N}$, because they all lie on the positive part of the parabola $y = \frac{1}{2} x^2$, $x \geq 0$. If some point $A_{3j}^2 A_2 v$ is not an extreme point of the body $G$ generated by the algorithm, then by the Minkowski theorem it must be a convex combination of other extreme points. However, the projection of $A_{3j}^2 A_2 v$ to $L$, which is the point $\tilde{c}$, is extreme.
for the projection of the set $G$ to $L$, which is the set $\text{co} \{H, H_{\tau}\}$. Hence, $\tilde{c}$ must be a convex combination of points generated by Algorithm 1 whose projection to $L$ coincides with $\tilde{c}$, i.e., points from the sequence $\{A_1^{3k}A_2\mathbf{v}\}_{k \geq 0}$. This is impossible due to convex independence of this sequence.

Thus, Algorithm 1 starting with the set $V_1 = \mathcal{H}$ produces a sequence of extreme points $\{A_1^{3k}A_2\mathbf{v}\}_{k \in \mathbb{N}}$. Other sequences are $\{A_1^{3k+1}A_2\mathbf{v}\}_{k \geq 0}$, $\{A_1^{3k+2}A_2\mathbf{v}\}_{k \geq 0}$ and the sequences symmetric to them about the origin. Those six sequences converge to vertices of the hexagon $H_{\tau}$.

The convex hull of these six sequences and of vertices of $H$ and of $H_{\tau}$ is the invariant body $G$ produced by Algorithm 1. This body has an infinite discrete set of extreme points.

Thus, a direct application of Algorithm 1 to an arbitrary admissible set $V_1$ may lead to divergence. Therefore, a classification of invariant sets in the case of complex leading eigenvalue with a rational mod $\pi$ argument requires a different procedure. This can be done by modifying Algorithm 1 as stated below. We describe the modified algorithm as Algorithm 3. Each iteration of the new algorithm deals with infinite sets of points, therefore it cannot be considered as a finite procedure and its significance is rather theoretical. Nevertheless, it establishes a complete classification of invariant sets and of Barabanov's norms in the case of rational mod $\pi$ argument. It is realised in the same way as Algorithms 1 but with two differences:

1) The starting set $V_1$ is an arbitrary admissible set for the candidate product $\Pi$. In particular, for $V_1 = E_1$, we obtain Algorithm 1 in case of complex leading eigenvalue.

2) Every node $V$ of the cyclic tree $\mathcal{T}$ is either an element of the root $V_j \in \mathcal{R} = \{V_1, \ldots, V_n\}$ or the end of a finite path starting at some node $V_j \in \mathcal{R}$. Denote by $\tilde{\Pi}_j$ the $j$th cyclic permutation of $\Pi$, which sends $V_j$ to itself. Let $\pi_j$ be the word corresponding to the product $\tilde{\Pi}_j$ and $\pi_\infty^j = \pi_j \pi_j \ldots$ be the corresponding infinite word. For a node $V \in \mathcal{T}$, we denote by $\pi_\infty^j(V)$ the corresponding infinite path $\pi_\infty^j$ along $\mathcal{T}$ starting at the node $V$.

In one step we add the following sets to $\mathcal{H}_{k+1}$:

a) all nodes of the infinite path $\pi_\infty^j(V)$;

2) the $m - 1$ children $\tilde{A}_k V$, $k \in \{1, \ldots, m\}$, $A_k$ is not the first matrix in the product $\Pi_j$ (i.e., the child does not belong to this path).

So, each step adds infinitely many nodes. We check all those new nodes. A node $V'$ is dead if and only if it is (all its points) is in the interior of the current set $G_k$. In contrast to Algorithm 1, here the set of new vertices $\mathcal{H}_k$ may contain an infinite set of nodes and is not necessarily located in one level.

Now write the formal routine.

**Algorithm 3.**

I. **Choosing the candidate product.** The same as in Algorithm 1.

II. **The routine.**

Choose $V_1$ is an arbitrary subset of leading eigenspace $L_1$ of $\tilde{\Pi}$ such that $\tilde{\Pi}V_1 = V_1$ and $V_1$ is symmetric about the origin. Then define the root $\mathcal{R} = \{V_1, \ldots, V_n\}$ from the set $V_1$ as
in Algorithm 1. Then we construct a sequence of sets \( V_i \) of nodes and their subsets \( \mathcal{H}_i \subset V_i \) (may be infinite) as follows:

**Zero iteration.** We set \( V_0 = \mathcal{H}_0 = \mathcal{R} \).

**kth iteration, \( k \geq 1 \).** We have a set of nodes \( V_{k-1} \) and its subset \( \mathcal{H}_{k-1} \). We set \( V_k = V_{k-1} \setminus \mathcal{H}_{k-1} \). Take an arbitrary node \( V \in \mathcal{H}_{k-1} \). It is the end of a finite path starting at a node \( V_j \) of the root. Denote by \( \Pi_j \) the \( j \)th cyclic permutation of \( \Pi \), which sends \( V_j \) to itself. For every \( \Pi_j \in \Pi \), consider the infinite path \( \pi_j \), which belongs to \( \mathcal{H}_k \). We remove this node and the corresponding branch of the tree growing from it, including the remainder of this path. All the nodes of this path higher than \( V' \) are added to both \( V_k \) and to \( \mathcal{H}_k \). If such a node \( V' \) does not exist, then all nodes of \( \pi_j \) are added to \( V_k \) and to \( \mathcal{H}_k \).

When all proper pairs \( (V, A) \) are exhausted, both \( V_k \) and \( \mathcal{H}_k \) are constructed. We define \( G_k = \text{co}\{V \mid V \in V_k\} \) and have

**Termination.** The algorithm halts when \( V_k = V_{k-1} \), i.e., \( \mathcal{H}_k = \emptyset \). In this case \( G_k \) is an invariant convex body for \( \mathcal{A} \).

**End of the algorithm.**

**Remark 7** Algorithm 3 is rather theoretical because each iteration assumes infinite number of steps: verifying the assertion \( AV \in \text{int} G_{k-1} \) for infinitely many nodes \( V \in \mathcal{H}_{k-1} \). Nevertheless, it shows the theoretical way to find the invariant convex body generated by an arbitrary admissible set \( V_1 \). On the other hand, if \( V_1 = E_1 \) (the leading ellipsoid), then Algorithm 1 converges within finite time provided \( \Pi \) is dominant. Therefore, in this case there is no need to apply Algorithm 3. Moreover, it is not reasonable to apply Algorithm 3 for computing the joint spectral radius either, because the JSR can always be computed with Algorithm 1 for \( V_1 = E_1 \).

**Theorem 7** Let a family \( \mathcal{A} \) possess a unique dominant product \( \Pi \) and let \( \Pi \) have a unique and simple complex leading eigenvalue with a rational \( \mod \pi \) argument. Then for every admissible set \( V_1 \), Algorithm 3 terminates within finite number of iterations and produces an invariant body.

**Proof.** If the algorithm does not converge within finitely many iterations, then there is an infinite path on the cyclic tree \( \mathcal{T} \) starting at the root that consists of alive nodes and is constructed by infinitely many iterations. Denote the node of this path on \( k \)th level by \( V^{(k)} \). Since \( V^{(0)} \) belongs to the root, it can be assumed that \( V^{(0)} = V_1 \). The set \( V^{(k)} \), which is the image of \( V_1 \) by the corresponding matrix product of length \( k \), is alive if it is not
the interior of the body \( G_k \) constructed in the \( k \)th iteration. Therefore, the diameter of the set \( V^{(k)} \) cannot converge to zero as \( k \to \infty \). In view of Theorem 6 from Section 7, this means that the path \( V^{(0)} \to V^{(1)} \to \cdots \) corresponds to an eventually periodic switching law with the period \( \pi \) (the word associated to the product \( \Pi \)). Suppose the periodic part starts after \( j \)th iteration, at the node \( V^{(j)} \). Then this is the infinite path \( \pi^\infty(V^{(j)}) \). However, all nodes of this path are added at once in the \( j \)th iterations. Thus, the whole path \( V^{(0)} \to V^{(1)} \to \cdots \) is constructed in the first \( j \) iterations, which contradicts to the assumption.

Thus, for every admissible set \( V_1 \), Algorithm 3 converges and produces an invariant body. The next result shows that every invariant body is obtained this way.

**Theorem 8** Let a family \( \mathcal{A} \) possess a unique dominant product \( \Pi \) whose leading eigenvalue has a rational mod \( \pi \) argument. Then every its invariant body is constructed by Algorithm 3 with some admissible subset \( V_1 \) of the leading plane.

**Proof.** Assume after possible normalization that \( \rho(\Pi) = 1 \). Let \( G \) be an invariant body and \( V \) be its intersection with the leading eigenspace \( L \) of \( \Pi \). Then \( \Pi V \subset V \). On the other hand, since the two-dimensional restriction of the operator \( \Pi \) to \( L \) has both its eigenvalues equal to one in modulus, it preserves the two-dimensional volume. Hence \( \Pi V = V \) and so \( V \) is admissible. Since both eigenvalues of \( \Pi|_V \) have rational mod \( \pi \) arguments, it follows that each point \( x \in \partial V \) is recurrent for the family \( \mathcal{A} \). On the other hand, by Proposition 2 from Section 5, all recurrent points are on \( \partial V \). Therefore, \( \partial V \) is the locus of recurrent points. Invoking now Theorem B (Section 5) we conclude that \( G \) is the closure of convex hulls of all trajectories starting from \( \partial V \). Hence \( G \) it is obtained by Algorithm 3 from the candidate product \( \Pi \) and the admissible set \( V_1 = V \).

**Remark 8** Note that if we need one invariant set/Barabanov norm, Algorithm 3 is not necessary. This can be done by Algorithm 1 with the admissible set \( V_1 \) being the leading ellipse \( E_1 \). Algorithm 3 is needed only to obtain all invariant sets/Barabanov norms.

Theorem 8 classifies all invariant sets of the family \( \mathcal{A} \) and explains why it may not be unique: every admissible set \( V_1 \) generates an invariant body. If the leading eigenvalue of \( \Pi \) has an irrational mod \( \pi \) argument, then there is a unique (up to multiplication by a constant) admissible set, which is the leading ellipse \( E_1 \). In the case of rational mod \( \pi \) arguments there are many admissible sets.

The transfer to Barabanov’s norm is realised in the standard way: we take an arbitrary admissible set \( V_1^* \) for the dual family \( \mathcal{A}^* \) and generate an invariant body \( G^* \) applying Algorithm 3. Then the Barabanov norm is \( f(x) = \max_{y^* \in G^*}(x, y^*) \).

**Remark 9** If a family \( \mathcal{A} \) has several dominant products, then if all of them have leading eigenvalues which are either real or complex with irrational mod \( \pi \) arguments, then all invariant bodies of \( \mathcal{A} \) are convex hulls of finitely many points and ellipses (Corollary 1). If at least one of dominant products, say, \( \Pi^{(j)} \) has a non-real leading eigenvalue with a
rational mod π argument, then there are more complicated invariant bodies. Namely, the corresponding root \( R_j = \{ V_{j1}, \ldots, V_{jn} \} \) can be generated by an arbitrary admissible set \( V_j \) of the product \( \Pi^{(j)} \).

9. Barabanov norms for positive systems

A linear switching system is called positive if all matrices of the family \( A \) are (entrywise) non-negative. If a positive system starts at a non-negative point \( x_0 \in \mathbb{R}^d_+ \), then the whole trajectory is in \( \mathbb{R}^d_+ \). For positive systems, the invariant polytope algorithm is extremely efficient even in dimension of 5000 and higher \([13, 31]\). However, to reach this efficiency we need to modify the concepts of invariant body and of Barabanov norm. We recall that inequalities \( x \geq 0 \), \( x \geq y \), \( A \geq 0 \), \( A \geq B \) are understood entrywise. The positive orthant is \( \mathbb{R}^d_+ = \{ x \in \mathbb{R}^d \mid x \geq 0 \} \). For positive systems, we usually work only with norms defined in \( \mathbb{R}^d_+ \). Moreover, it suffices to consider only monotone norms \( f \) for which \( f(x) \geq f(y) \) whenever \( x \geq y \geq 0 \). Respectively, we can consider monotone convex bodies \( G \) which lie in \( \mathbb{R}^d_+ \) and possess the following property: if \( x \in G \), then \( y \in G \) whenever \( y \leq x \). Similarly one defines the monotone convex hull of a set \( K \subset \mathbb{R}^d_+ \):

\[
\text{co}_- K = \left\{ y \in \mathbb{R}^d_+ \mid \exists x \in \text{co} K : x \geq y \right\}.
\]

Thus, the monotone convex hull contains the usual convex hull plus all points majorated by it. A monotone convex hull of a finite set is a monotone polytope. In contrast to the usual polytope, a monotone polytope can have less than \( d \) vertices. For example, it can have only one vertex \( a \), in which case it is a parallelepiped \( \{ x \in \mathbb{R}^d \mid 0 \leq x \leq a \} \).

A monotone norm is Barabanov if \( \lambda f(x) = \max_{A_j \in A} f(A_j x) \) for all \( x \in \mathbb{R}^d_+ \). A monotone convex body \( G \) is invariant for \( A \) if \( \lambda G = \text{co}_- \left\{ \bigcup_{A_j \in A} A_j G \right\} \). The monotone invariant body and the monotone invariant norm are related by the monotone polar transform. The monotone polar to a set \( G \subset \mathbb{R}^d_+ \) is

\[
G_*^\perp = \{ x \in \mathbb{R}^d_+ \mid \sup_{y \in G}(x, y) \leq 1 \}.
\]

Note that for \( y \geq 0 \) the relation \( x_1 \leq x_2 \) implies that \( (x_1, y) \leq (x_2, y) \). Therefore, the sets \( G \) and \( \text{co}_+ G \) have the same monotone polar. If \( f \) is a monotone invariant norm for \( A \), then the monotone polar to its unit ball is a monotone invariant body for \( A^* \) \([12]\).

Finally, the irreducibility assumption for positive systems is weakened to positive irreducibility: the matrices from \( A \) do not share an invariant coordinate subspace i.e., subspace of the form \( L_S = \{ x \in \mathbb{R}^d \mid x_i = 0, i \notin S \} \), where \( S \subset \{1, \ldots, d\} \).

**Theorem D** \([12]\). A positively irreducible system \( A \) possesses a monotone Barabanov norm and a monotone invariant body. The unit ball of the monotone Barabanov norm is a polar to the invariant body of the transpose system \( A^* \).
As for the structure of invariant bodies, the Perron-Frobenius theorem reduces the three possible cases of leading eigenvalues (real, complex with an irrational \( \mod \pi \) argument, and non-real with a rational \( \mod \pi \) argument) to one case. Indeed, since a non-negative matrix always has a non-negative leading eigenvalue, the cases of complex leading eigenvalues become impossible. Hence, Theorems 1 and 2 from Section 2 get the following simple form:

**Corollary 2** If a family of non-negative matrices \( \mathcal{A} \) has a unique dominant product with a unique and simple leading eigenvalue, then it possesses a unique invariant body and a unique Barabanov norm.

Note that this unique invariant body may not be monotone. For the corresponding example, see, for instance [15, Figure 4]. However, a monotone invariant body does exist.

**Theorem 9** If a family of non-negative matrices \( \mathcal{A} \) has a unique dominant product with a unique and simple leading eigenvalue, then it possesses a unique monotone invariant body and a unique monotone Barabanov norm. The invariant body is a monotone polytope. The monotone Barabanov norm is given by the formula

\[
f(x) = \max_{v^*} (v^*, x), \quad x \in \mathbb{R}^d_+ ,
\]

where the maximum is taken over all vertices \( v^* \) of the monotone invariant polytope \( G^* \) of the dual family \( \mathcal{A}^* \).

The algorithm for construction of the monotone invariant polytope works in the same way as Algorithm 1 with the only difference: for each \( k \), the polytope \( G_k \) is a monotone convex hull of \( V_k \) (not just a convex hull as in Algorithm 1). The proof of Theorem 9 is realized in the same way as for Theorem 1. We only remark that if \( G \) is the (usual) invariant body, then the monotone invariant body is the monotone convex hull of the set \( G \cap \mathbb{R}^d_+ \). Since a set and its monotone convex hull have the same monotone polar, it follows that the Barabanov norm restricted to \( \mathbb{R}^d_+ \) is monotone. In particular, the Barabanov norm on \( \mathbb{R}^d_+ \) coincides with the monotone Barabanov norm.

As a rule, a monotone invariant polytope has much less vertices. In practice, even in very high dimensions, the number of vertices of an invariant monotone polytope do not exceed several dozens. That is why in dimensions of several thousands the algorithm constructs the invariant body and the Barabanov norm within a few iterations. We report the numerical results in the next section, Table 2.

\[\square\]

## 10. Numerical results

We report the results of performing Algorithms 1 and 2 for randomly generated matrices. Many results for matrices taken from practical applications can be found in [12, 13, 31] and they are either similar or better than those for random matrices. The numerical results
presented here are done by the most recent version of Algorithms 1 and 2 from [31]. Table 1 shows the results of Algorithm 1 for arbitrary matrices with the case $\lambda \in \mathbb{R}$ (real leading eigenvalue of the dominant product). For even dimensions $d$ from 2 to 20, we took pairs of random $d \times d$ matrices $\mathcal{A} = \{A_1, A_2\}$ and normalise them either as $\|A_1\| = \|A_2\|$ (the first column) or as $\rho(A_1) = \rho(A_2)$ (the second column). This normalization makes the problem more complicated, otherwise in most cases the dominant product has length 1, i.e., either $A_1$ dominates $A_2$ or vice versa. For every dimension $d$ in each case 20 experiments have been made in a standard laptop and the median value of the computer time and of the number of vertices of the invariant polytope $G$ is put in the table. The symbol $\#V$ denotes the number of pairs of vertices, so the invariant polytope has twice as many vertices. Note that we did not remove the redundant vertices, so the real number of vertices is usually much smaller.

In the case $\lambda \notin \mathbb{R}$ (complex leading eigenvalue of the dominant product), Algorithm 1 works slower. In our experiments it is mostly applicable for dimensions $\leq 13$, for higher dimensions, the running time often exceeds reasonable limits. This can be explained by the fact that the conic programming problem (15) takes more time than the linear programming problem (15) in the real case. The total number or vertices (ellipses in this case) in the invariant body slightly exceeds the number if vertices in Table 1 for the real case.

Table 1: Computation of the Barabanov norm, arbitrary matrices

| dim | $\|A_1\| = \|A_2\|$ | $\rho(A_1) = \rho(A_2)$ |
|-----|-----------------|-----------------|
|     | $\text{time}$   | $\#V$           | $\text{time}$   | $\#V$           |
| 2   | 1.1 s           | 5 · 2           | 1.2 s           | 6 · 2           |
| 4   | 1.4 s           | 17 · 2          | 1.8 s           | 77 · 2          |
| 6   | 2.0 s           | 47 · 2          | 2.5 s           | 130 · 2         |
| 8   | 2.5 s           | 100 · 2         | 3.9 s           | 220 · 2         |
| 10  | 4.9 s           | 270 · 2         | 5.1 s           | 320 · 2         |
| 12  | 4.7 s           | 280 · 2         | 11 s            | 770 · 2         |
| 14  | 8.4 s           | 510 · 2         | 21 s            | 1100 · 2        |
| 16  | 25 s            | 1100 · 2        | 33 s            | 1400 · 2        |
| 18  | 90 s            | 2100 · 2        | 200 s           | 2500 · 2        |
| 20  | 295 s           | 3100 · 2        | 5000 s          | 6200 · 2        |

Table 2 shows the results for non-negative matrices. In the first column the matrices are positive and in the second they are sparse with 90% zero entries. We see that in the non-negative case the algorithm is extremely efficient. Usually it constructs the Barabanov norm within 3 – 4 iterations and this seems not to depend on the dimension. The number of vertices is usually around 8 since we did not remove redundant vertices. For every dimension $d$, in each case 20 experiments have been made and the median values are reported. The algorithm always halted within finite time.

We see that for arbitrary matrices, the construction of Barabanov’s norm in dimensions less than 15 takes more or less the same time as for constructing other Lyapunov functions by known methods, which give only approximate values of JSR. For positive systems, Barabanov’s norm is constructed much faster even for very large dimensions.
Table 2: Computation of the monotone Barabanov norm, non-negative matrices

| dim  | 0% sparsity | 90% sparsity |
|------|-------------|--------------|
|      | time #V     | time #V      |
| 20   | 0.3 s 7     | 1.7 s 42     |
| 50   | 0.3 s 8     | 1.6 s 50     |
| 100  | 0.4 s 8     | 0.8 s 25     |
| 200  | 0.5 s 8     | 1.0 s 23     |
| 500  | 1.2 s 8     | 1.8 s 16     |
| 1000 | 6.3 s 8     | 11 s 16      |
| 2000 | 35 s 8      | 72 s 16      |

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