Bulk Casimir densities and vacuum interaction forces in higher dimensional brane models

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Abstract

Vacuum expectation value of the energy-momentum tensor and the vacuum interaction forces are evaluated for a massive scalar field with general curvature coupling parameter satisfying Robin boundary conditions on two codimension one parallel branes embedded in $(D+1)$-dimensional background spacetime $AdS_{D+1} \times \Sigma$ with a warped internal space $\Sigma$. The vacuum energy-momentum tensor is presented as a sum of boundary-free, single brane induced, and interference parts. The latter is finite everywhere including the points on the branes and is exponentially small for large interbrane distances. Unlike to the purely AdS bulk, the part induced by a single brane, in addition to the distance from the brane, depends also on the position of the brane in the bulk. The asymptotic behavior of this part is investigated for the points near the brane and for the position of the brane close to the AdS horizon and AdS boundary. The contribution of Kaluza-Klein modes along $\Sigma$ is discussed in various limiting cases. The vacuum forces acting on the branes are presented as a sum of the self-action and interaction terms. The first one contains well known surface divergences and needs a further renormalization. The interaction forces between the branes are finite for all nonzero interbrane distances and are investigated as functions of the brane positions and the length scale of the internal space. We show that there is a region in the space of parameters in which these forces are repulsive for small distances and attractive for large distances. As an example the case $\Sigma = S^{D_2}$ is considered. An application to the higher dimensional generalization of the Randall-Sundrum brane model with arbitrary mass terms on the branes is discussed. Taking the limit with infinite curvature radius for the AdS bulk, from the general formulae we derive the results for two parallel Robin plates on background of $R^{(D_1,1)} \times \Sigma$ spacetime.

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1 Introduction

Recent proposals of large extra dimensions use the concept of brane as a sub-manifold embedded in a higher dimensional spacetime, on which the Standard Model particles are confined.
Braneworlds naturally appear in string/M-theory context and provide a novel setting for discussing phenomenological and cosmological issues related to extra dimensions. The model introduced by Randall and Sundrum [1] is particularly attractive. The corresponding background solution consists of two parallel flat 3-branes, one with positive tension and another with negative tension embedded in a five dimensional AdS bulk. The fifth coordinate is compactified on $S^1/Z_2$, and the branes are on the two fixed points. It is assumed that all matter fields are confined on the branes and only the gravity propagates freely in the five dimensional bulk. In this model, the hierarchy problem is solved if the distance between the branes is about 37 times the AdS radius and we live on the negative tension brane. More recently, alternatives to confining particles on the brane have been investigated and scenarios with additional bulk fields have been considered.

From the point of view of embedding the Randall-Sundrum model into a more fundamental theory, such as string/M-theory, one may expect that a more complete version of this scenario must admit the presence of additional extra dimensions compactified on an internal manifold. From a phenomenological point of view, higher dimensional theories with curved internal manifolds offer a richer geometrical and topological structure. The consideration of more general spacetimes may provide interesting extensions of the Randall-Sundrum mechanism for the geometric origin of the hierarchy. Spacetimes with more than one extra dimension can allow for solutions with more appealing features, particularly in spacetimes where the curvature of the internal space is nonzero. More extra dimensions also relax the fine-tunings of the fundamental parameters. These models can provide a framework in the context of which the stabilization of the radion field naturally takes place. In addition, a richer topological structure of the field configuration in transverse space provides the possibility of more realistic spectrum of chiral fermions localized on the brane. Several variants of the Randall–Sundrum scenario involving cosmic strings and other global defects of various codimensions have been investigated in higher dimensions (see, for instance, [2]-[14] and references therein). In particular, much work has been devoted to warped geometries in six dimensions.

Motivated by the problems of the radion stabilization and the generation of cosmological constant, the role of quantum effects in braneworlds has attracted great deal of attention (see, for instance, references given in [15]). In this paper we continue the investigation of the local quantum effects induced by two codimension one parallel branes embedded in the background spacetime $AdS_{D_1+1} \times \Sigma$ with a warped internal space $\Sigma$ (for the investigation of the vacuum energy-momentum tensor in the geometry with $AdS_{D+1}$ bulk see Refs. [16] [17]). The quantum effective potential and the problem of moduli stabilization in the orbifolded version of this model with zero mass parameters on the branes are discussed recently in Ref. [18]. In particular, it has been shown that one loop-effects induced by bulk scalar fields generate a suitable effective potential which can stabilize the hierarchy without fine tuning. In the previous paper [15] we have studied the Wightman function and the vacuum expectation value of the field square for a scalar field with an arbitrary curvature coupling parameter obeying Robin boundary conditions on the branes. For an arbitrary internal space $\Sigma$, the application of the generalized Abel-Plana formula allowed us to extract form the vacuum expectation values the part due to the bulk without branes and to present the brane induced parts in terms of exponentially convergent integrals for the points away the branes. An application of the general results to the model with $\Sigma = S^N$ was discussed. Here we consider the vacuum expectation values of the energy-momentum tensor and the vacuum interaction forces between the branes. In addition to describing the physical structure of the quantum field at a given point, the energy-momentum tensor acts as the source in the Einstein equations and therefore plays an important role in modelling a self-consistent dynamics involving the gravitational field.

The paper is organized as follows. In the next section, by using the expression for the Wightman function from [15], we derive formula for the expectation values of the energy-momentum...
tensor for a single brane geometry in both regions on the left and on the right of brane. Various limiting cases are discussed in the general case of the internal manifold. The corresponding quantities for the geometry of two branes are investigated in section 3. The case of different Robin coefficients on separate branes is considered and an application to the higher dimensional version of the Randall-Sundrum braneworld is discussed. In section 4 we study the vacuum interaction forces between the branes as functions on brane positions and AdS radius. The application of the general formulae to the example with the internal space \( \Sigma = S^1 \) with the further generalization to the case \( S^{D_2} \) is discussed in section 5. Section 6 contains a summary of the work and some suggestions for further research.

2 Casimir densities for a single brane

In this paper we are interested in local quantum effects arising from a scalar field \( \varphi(x) \) propagating on background of \((D + 1)\)-dimensional spacetime with topology \( AdS_{D_1 + 1} \times \Sigma \) and the line element (we adopt the conventions of Ref. [19] for the metric signature and the curvature tensor)

\[
ds^2 = g_{MN} dx^M dx^N = e^{-2k_D y} \eta_{\mu\sigma} dx^\mu dx^\sigma - e^{-2k_D y} \gamma_{ik} dx^i dx^k - dy^2, \tag{2.1}
\]

where \( \eta_{\mu\sigma} = \text{diag}(1, -1, \ldots, -1) \) is the metric tensor for \( D_1 \)-dimensional Minkowski spacetime \( R^{(D_1-1,1)} \), \( k_D \) is the inverse AdS radius, and the coordinates \( X^i, i = 1, \ldots, D_2 \), cover the manifold \( \Sigma, D = D_1 + D_2 \). Here and below the upper-case latin indices \( M, N \) range from 0 to \( D \) and \( \mu, \sigma = 0, 1, \ldots, D_1 - 1 \). In addition to the radial coordinate \( y \) we will also use the coordinate \( z = e^{k_D y}/k_D \), in terms of which line element (2.1) is written in the form conformally related to the metric in the direct product spacetime \( R^{(D_1,1)} \times \Sigma \) by the conformal factor \((k_D z)^{-2}\). The solutions to Einstein equations of type (2.1) have been considered in Refs. [5, 6, 9]. For the scalar field with curvature coupling parameter \( \zeta \) the field equation has the form

\[
(\nabla^M \nabla_M + m^2 + \zeta R) \varphi(x) = 0, \tag{2.2}
\]

with \( R \) being the scalar curvature for the background spacetime and \( \nabla_M \) is the covariant derivative operator associated with the metric tensor \( g_{MN} \). For minimally and conformally coupled scalars one has \( \zeta = 0 \) and \( \zeta = \zeta_D \equiv (D-1)/(4D) \), respectively. In the case of the bulk under consideration, the corresponding Ricci scalar is given by formula \( R = -D(D+1)k_D^2 - e^{2k_D y} R_\gamma \), where \( R_\gamma \) is the scalar curvature for the metric tensor \( \gamma_{ik} \). In the discussion below we will assume that the field obeys Robin boundary conditions

\[
\left( \tilde{A}_y + \tilde{B}_y \partial_y \right) \varphi(x) = 0, \quad y = a, b, \tag{2.3}
\]

with constant coefficients \( \tilde{A}_y, \tilde{B}_y \), on two parallel branes of codimension one, located at \( y = a \) and \( y = b, a < b \). The expressions for these coefficients in a higher dimensional generalization of the Randall-Sundrum brane model will be given below. The \( z \)-coordinates of the branes we denote by \( z_j = e^{k_D j}/k_D, j = a, b \). Robin type conditions are an extension of Dirichlet and Neumann boundary conditions and appear in a variety of situations, including the considerations of vacuum effects for a confined charged scalar field in external fields, spinor and gauge field theories, quantum gravity and supergravity. These boundary conditions naturally arise for scalar and fermion bulk fields in braneworld models. The Wightman function and the vacuum expectation value (VEV) of the field square for the geometry of two branes in the bulk with line element (2.1) are investigated in our previous paper [15]. Here we are interested in the VEV of the energy-momentum tensor and the vacuum interaction forces between the branes.
The VEV of the energy-momentum tensor can be evaluated by substituting the Wightman function and the VEV of the field square into the formula

\[ \langle 0|T_{MN}(x)|0 \rangle = \lim_{x' \to x} \partial_M \partial_N \langle 0|\phi(x)\phi(x')|0 \rangle \\
+ \left[ \left( \frac{\zeta - 1}{4} \right) g_{MN} \nabla_L \nabla^L - \zeta \nabla_M \nabla_N - \zeta R_{MN} \right] \langle 0|\phi^2(x)|0 \rangle, \quad (2.4) \]

with the components of the Ricci tensor (here and below the components of tensors are given in the coordinate system with the radial coordinate \( y \))

\[ R_{\mu\sigma} = -Dk_d^2 g_{\mu\sigma}, \quad R_{ik} = R_{(\gamma)ik} - Dk_d^2 g_{ik}, \quad R_{DD} = Dk_D^2, \quad (2.5) \]

where \( R_{(\gamma)ik} \) is the Ricci tensor for the metric \( \gamma_{ik} \). This corresponds to the point-splitting regularization technique for the VEVs. First let us consider the geometry of a single brane located at \( y = a \). In the region \( y > a \) the corresponding Wightman function is determined by the formula [15]

\[ \langle \varphi(x)\varphi(x') \rangle = \frac{k_D^{-1}(zz')^{D/2}}{2^D \pi^{D-1}} \sum_{\beta} \psi_{\beta}(X)\psi_{\beta}(X') \int dk e^{ik(x-x')} \]

\[ \times \left\{ \int_0^\infty du \frac{e^{i(t-t')\sqrt{u^2+k^2}}}{\sqrt{u^2+k^2}} J_\nu(uz)J_\nu(uz') - 2 \int k_\beta \frac{I_\beta(a)}{K_\beta(a)} \right\}, \quad (2.6) \]

with \( k_\beta = \sqrt{k^2 + \lambda_\beta^2}, \quad k = |k|, \quad \text{and} \quad k \) is the wave vector in the subspace \( R^{(D-1,1)} \) with spatial coordinates \( x = (x^1, \ldots, x^{D-1}) \). In formula (2.6), \( J_\nu(x) \) is the Bessel function, \( I_\nu(x) \), \( K_\nu(x) \) are the Bessel modified functions with the order

\[ \nu = \sqrt{D^2/4 - D(D + 1)\zeta + m^2/k_D^2}, \quad (2.7) \]

\( \psi_{\beta}(X) \) are the eigenfunctions for the operator \( \Delta_{(\gamma)} + \zeta R_{(\gamma)} \) with eigenvalues \(-\lambda_\beta^2\), and \( \Delta_{(\gamma)} \) is the Laplace-Beltrami operator for the metric \( \gamma_{ik} \). Here and below for a given function \( F(x) \) we use the notation

\[ F^{(j)}(x) = A_j F(x) + B_j x F'(x), \quad (2.8) \]

with the coefficients related to the constants in the boundary conditions by the formulae

\[ A_j = \tilde{A}_j + \tilde{B}_j k_D D/2, \quad B_j = \tilde{B}_j k_D. \quad (2.9) \]

In formula (2.6), the part with the first integral in the figure braces is the Wightman function for \( AdS_{D+1} \times \Sigma \) spacetime without branes, and the part with the second integral is induced in the region \( y > a \) by the presence of the brane. The formula for the Wightman function in the region \( y < a \) is obtained from Eq. (2.6) by the interchange of the Bessel modified functions, \( I_\nu \rightarrow K_\nu \). Substituting the Wightman function and the expression for the VEV of the field square from Ref. [15] into formula (2.4), the VEV of the energy-momentum tensor is presented in the form

\[ \langle 0|T_{MN}^N|0 \rangle = \langle T_{MN}^N \rangle^{(0)} + \langle T_{MN}^N \rangle^{(a)}, \quad (10.10) \]

where

\[ \langle T_{MN}^N \rangle^{(0)} = \frac{k_D^{D+1} z^D}{(4\pi)^2} \Gamma \left( 1 - \frac{D_1}{2} \right) \sum_{\beta} |\psi_{\beta}(X)|^2 \int_0^\infty du (u^2 + \lambda_\beta^2)^{D_1/2 - 1} F_{MN}^{(-)}(u z), \quad (11.1) \]
is the VEV for the energy-momentum tensor in the background without branes, and the term
\[ \langle T_M^N \rangle_{(a)} = \sum_{\beta} |\psi_\beta(X)|^2 \langle T_M^N \rangle_{(a)} \]
(2.12)
is induced by a single brane at \( z = z_a \), with the contribution of the given KK mode \( \beta \) along \( \Sigma \) determined by the formula
\[ \langle T_M^N \rangle_{(a)} = -\frac{2k_D+1}{4\pi} \langle \frac{D}{2} \rangle \int_{\lambda_\beta}^{\infty} du u(u^2 - \lambda_\beta^2) \left\{ \frac{1}{s} \int_{k_{12}}^{\infty} u^k f(u) \left[ \frac{2\lambda_\beta^2}{D} + \frac{1}{2} \right] \left[ -v^2 g^2(v) \right] + D(4\zeta - 1)vg(v)g'(v) + \left( \frac{2m^2}{k_D^2} - v^2 \pm v^2 \right) g^2(v) \right\}, \]
(2.13)
To derive this formula we have used the relation
\[ \int_0^\infty dk \int_{k_{12}}^{\infty} du u^{kn-1} f(u) = \frac{\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n}{2} + 1\right)} \int_{\lambda_\beta}^{\infty} du \frac{uf(u)}{(u^2 - \lambda_\beta^2)^s}, \]
(2.14)
with \( s = -1/2, 1/2 \) and \( n = D_1 - 1 \). For a given function \( g(v) \), the functions \( F_{\beta M}^{(\pm)N}(g(v)) \) in formulae (2.11) and (2.13) are defined by the relations
\[ F_{\beta i}^{(\pm)N}(g(v)) = \frac{k_D}{2} \left[ z^2 g^2(v) \right] \frac{\eta_\beta(X)}{|\psi_\beta(X)|^2} + \frac{1}{2} \delta^k_{\alpha} (1 - 4\zeta) \frac{\partial}{\partial v} F[g(v)], \]
(2.15)
for the components in the AdS part, and by the relations
\[ F_{\beta i}^{(\pm)k}(g(v)) = \frac{t_{\beta i}}{z^2 g^2(v) |\psi_\beta(X)|^2} + \frac{1}{2} \delta^k_{\alpha} (1 - 4\zeta) \frac{\partial}{\partial v} F[g(v)], \]
(2.16)
with \( t_{\beta i}(X) = -\gamma^i t_{\beta i}(X) \), for the components having indices in the internal space. In these expressions we use the following notations
\[ F[g(v)] = v g(v)g'(v) + \left( D + \frac{4\zeta}{4\zeta - 1} \right) g^2(v), \]
(2.17)
\[ \eta_\beta(X) = \frac{\Delta_{(\gamma)} |\psi_\beta(X)|^2}{|\psi_\beta(X)|^2}, \quad \eta_\beta(X) = -\gamma^k \delta_{\alpha} \frac{\partial_k |\psi_\beta(X)|^2}{|\psi_\beta(X)|^2}, \]
(2.18)
\[ t_{\beta i}(X) = \nabla_{(\gamma)} \psi_{\beta}(X) \nabla_{(\gamma)} t_{\beta i} + \left[ \left( \zeta - \frac{1}{4} \right) - \nabla_{(\gamma)} \nabla_{(\gamma)} |\psi_\beta(X)|^2 - \zeta R_{(\gamma)ik} \right] |\psi_\beta(X)|^2, \]
(2.19)
where \( \nabla_{(\gamma)i} \) is the covariant derivative operator associated with the metric tensor \( \gamma_{ik} \). By using the equation for the Bessel modified functions, the derivative in expressions (2.15) and (2.18) can also be presented in the form
\[ v \frac{\partial}{\partial v} F[g(v)] = v^2 g^2(v) + \left( D + \frac{4\zeta}{4\zeta - 1} \right) v g(v)g'(v) + (v^2 + v^2) g^2(v), \]
(2.20)
for \( g(v) = I_{\nu}(v), K_{\nu}(v) \).
By a similar way, for the VEV induced by a single brane in the region \( z < z_a \) one obtains

\[
\langle T^{N_1}_{M \beta} \rangle^{(a)} = - \frac{2k^{D+1}_{D}z^D}{(4\pi)^{\frac{D}{2}} \Gamma \left( \frac{D}{2} \right)} \int_{\lambda_\beta}^{\infty} du (u^2 - \lambda_\beta^2) \frac{\partial}{\partial u} \frac{1}{\lambda_\beta} \left[ \frac{K^{(a)}_{\nu} (uz)}{I^{(a)}_{\nu} (uz)} \right] F^{(+)N}_{\beta M} \left[ I_\nu (uz) \right].
\] (2.23)

In the case of the purely AdS bulk without an internal space, from formulae (2.13) and (2.23) we obtain the results derived in Ref. [17]. As we see from the formulae given above, for the general case of the internal subspace \( \Sigma \), the vacuum energy-momentum tensor is non-diagonal.

In the case of homogeneous internal space the contributions into the energy-momentum tensor coming from the terms containing \( \eta_\beta(X) \) and \( \eta^\dagger_\beta(X) \) vanish. In particular, in this case one has \( \langle T^D_D \rangle^{(a)} = 0 \). Note that, unlike to the AdS\(_{D+1}\) bulk, VEVs (2.13) and (2.23) in addition to the ratio \( z/z_a \) depend also on the absolute position of the brane. For a one-parameter internal space of linear size \( L \) one has \( \lambda_\beta \sim 1/L \). In this case the brane induced part in the vacuum energy-momentum tensor is a function on the ratios \( L/z_a \) and \( z/z_a \). The latter is related to the proper distance of the observation point from the brane by the equation

\[
\frac{z}{z_a} = e^{k_D (y-a)}.
\] (2.24)

From the point of view of an observer residing on the brane, the physical size of the subspace \( \Sigma \) is \( L_a = L e^{-k_D a} \) and the corresponding KK masses are rescaled by the warp factor, \( \lambda_\beta^{(a)} = \lambda_\beta e^{k_D a} \). Now we see that the brane induced part in the vacuum energy-momentum tensor is a function of the proper distance from the brane and on the ratio \( L_a/(1/k_D) \) of the physical size of the internal space \( \Sigma \) for an observer living on the brane to the AdS curvature radius. For a fixed value of the distance of the observation point from the brane, the scaling of \( L \) is equivalent to the shift of the brane position.

Note that if we consider a quantum scalar field with mass \( m \) on background of spacetime \( R \times \Sigma \) with the line element \( ds^2 = dt^2 - \gamma_{ik}dx^i dx^k \), when the VEVs of the spatial components of the energy-momentum tensor are expressed through the tensor \( T^{a}_{ik} \) by the formula

\[
\langle 0_\Sigma | T_{ik} | 0_\Sigma \rangle = \frac{1}{2} \sum_{\beta} \frac{t_{\beta k}(X)}{\sqrt{\lambda_\beta^2 + m^2}},
\] (2.25)

where \( |0_\Sigma\rangle \) is the amplitude for the corresponding vacuum state. The expression for the brane-free part \( (2.11) \) is divergent and needs regularization with further renormalization. As it was discussed in Ref. [15] for the case of the field square, this can be done combining the zeta function technique for the series over \( \beta \) and dimensional regularization for the integral over \( u \). The series over \( \beta \) is expressed in terms of the local zeta function for the operator \( \Delta_{(\gamma)} + \zeta R_{(\gamma)} - m^2 \) and its derivatives. In section 5 to evaluate the brane-free part of the vacuum energy-momentum tensor for the example with \( \Sigma = S^1 \), we will use the Abel-Plana formula which allows to extract the part corresponding to the bulk AdS\(_{D+1}\) and to present the remained finite part in terms of exponentially convergent integrals. The brane-induced parts \( (2.13) \) and \( (2.23) \) are finite for the points away the brane and, hence, the renormalization procedure is needed for the boundary-free part only.

For the comparison with the corresponding results in the case of bulk spacetime AdS\(_{D+1}\) when the internal space is absent, it is useful in addition to the VEVs \( (2.13) \) and \( (2.23) \) to consider the VEV integrated over the subspace \( \Sigma \):

\[
\langle T^{N_1}_{M \beta} \rangle^{\text{integrated}} = \int_{\Sigma} d^2x \sqrt{\gamma} \langle T^{N_1}_{M \beta} \rangle^{(a)}.
\] (2.26)

Note that due to the warp factor in this formula the volume of the extra space \( \Sigma \) exponentially decreases as one moves toward the AdS horizon. Comparing this integrated VEV with the
corresponding formula from Ref. [17], we see that for a homogeneous subspace \( \Sigma \) the contribution of the zero KK mode \( (\lambda_\beta = 0) \) into the integrated \( \partial^\mu \) and \( D^\mu \) -components of the brane induced vacuum energy-momentum tensor differs from the corresponding formulae in the \( AdS_{D+1} \) bulk by the replacement \( D \to D_1 \) in expressions (2.15), (2.16) and in the definition of the order for the modified Bessel functions: for the latter case \( \nu \to \nu_1 \) with \( \nu_1 \) defined by Eq. (2.7) with \( D \) replaced by \( D_1 \). Note that for a scalar field with \( \zeta \leq \zeta_{D+D_1+1} \) one has \( \nu \geq \nu_1 \). In particular, this is the case for minimally and conformally coupled scalars.

It can be checked that the both boundary-free and brane induced parts in the VEV of the energy-momentum tensor obey the continuity equation \( \nabla_N T^N_M = 0 \), which for the geometry under consideration takes the form

\[
(z^{D+1} \frac{\partial}{\partial z} (z^{-D} T^D_i) + D_i T^0_0 + T^i_i + \frac{1}{k_D} \nabla_{(\gamma)} T^i_i) = 0, \tag{2.27}
\]

\[
(z^{D+1} \frac{\partial}{\partial z} (z^{-D} T^D_i) + \frac{1}{k_D} \nabla_{(\gamma)} k^k_i = 0, \tag{2.28}
\]

with \( \nabla_{(\gamma)} T^i_i = \partial_i (\sqrt{\gamma} T^i_i)/\sqrt{\gamma} \). In particular, for a homogeneous internal space the second equation is satisfied trivially and the last term on the left of the first equation vanishes. Note that we also have the relation \( \nabla_{(\gamma)} k^k_{\beta i} = 0 \). By using the equation for the eigenfunctions \( \psi_\beta(X) \), it can be seen that the following trace relation takes place

\[
\frac{t^i_{\beta i}(X)}{|\psi_\beta(X)|^2} = -(D_2 - 1)\zeta_{D_2-1}\eta_\beta(X) - \lambda_\beta^2. \tag{2.29}
\]

On the base of this relation we easily verify that for a conformally coupled massless scalar the brane induced VEVs (2.13) and (2.23) are traceless. The trace anomalies are contained in the boundary-free part only.

To clarify the dependence of the boundary induced part in the VEV of the energy-momentum tensor for general case of the internal space, it is useful to consider various limiting cases when the corresponding formulae are simplified. In the discussion below for these cases it is convenient to introduce the following functions

\[
F^{(l)}_{\beta D}(u, v) = \left( \zeta - \frac{1}{4} \right) \eta_\beta(X) - u^2 \delta^i_0 \tag{2.30a}
\]

\[
F^{(l)}_{\beta \mu}(u, v) = \delta^\mu_\mu \left[ F^{(1)}_{\beta D}(u, v) + \frac{u^2 - v^2}{D_1} + (4\zeta - 1)u^2 \delta^1_1 \right] \tag{2.30b}
\]

\[
F^{(l)}_{\beta i}(u, v) = k_D \left[ D(\zeta - \zeta_\beta) \delta^j_0 - 2n^{(a)}_u z u \delta^j_1 \left( \zeta - \frac{1}{4} \right) \right] n^i_j(X), \tag{2.30c}
\]

\[
F^{(l)}_{\beta k}(u, v) = (4\zeta - 1)u^2 \delta^k_0 \delta^1_1 - \frac{t^k_{\beta i}(X)}{|\psi_\beta(X)|^2}, \tag{2.30d}
\]

with \( l = 0, 1 \). Here and below we define \( n^{(j)} = 1 \) for the region \( y > j \) and \( n^{(j)} = -1 \) for the region \( y < j \). First of all, as a partial check of the results derived above for the VEV of the energy-momentum tensor let us consider the limit \( k_D \to 0 \). This corresponds to a single Robin plate on the bulk \( R^{(D+1)} \times \Sigma \). For \( k_D \to 0 \), from (2.7) we see that the order \( \nu \) of the cylindrical functions in formulae (2.13) and (2.23) is large. Introducing the new integration variable \( v = u/\nu \), we can use the uniform asymptotic expansions of the Bessel modified functions for large values of the order (see, for instance, [20]). To the leading order this gives:

\[
\langle T^N_M \rangle^{(a)} \approx \langle T^N_M \rangle^{(a)}_{R^{(D+1)} \times \Sigma} = \frac{(4\pi)^{-\frac{D}{2}}}{\Gamma \left( \frac{D}{2} \right)} \sum_\beta |\psi_\beta(X)|^2 \times \int_{v_\beta}^{\infty} du (u^2 - v_\beta^2)^{-\frac{D}{2} - 1} e^{-2u|y-a|/\epsilon_a(u)} F^{(1)}_{\beta M}(u, v_\beta), \tag{2.31}
\]
where \( v_\beta = \sqrt{m^2 + \lambda_\beta^2} \) and in the expression for \( F^{(i)}_{\beta D}(u, z) \) we should take \( k_D z = 1 \) (as we are considering the limit \( k_D \rightarrow 0 \)). In Eq. (2.31) we have introduced the notation

\[
\tilde{c}_j(u) = \frac{\tilde{A}_j - n(j) \tilde{B}_j u}{\tilde{A}_j + n(j) \tilde{B}_j u}, \quad j = a, b.
\] (2.32)

In particular, for a homogeneous internal space, from (2.31) we see that \( D^\beta \)-component of the brane induced part in the VEV of the energy-momentum tensor vanishes. For Dirichlet (Neumann) scalar with \( \zeta \leq \zeta_D \), the corresponding energy density is negative (positive). In the special case of bulk when the internal space \( \Sigma \) is absent, formula (2.31) for the VEV \( \langle T^{N_1(a)}_{M(M)} \rangle_{\Sigma} \) coincides with the result previously derived in Ref. [21].

For large values \( u \), the integrands in formulae (2.13) and (2.23) behave as \( u^{D_1 - 2a|z - z_a|} \) and the integrals converge for the points away the branes. For the points on the branes the integrals are divergent, leading to the divergent VEVs of the energy-momentum tensor. These divergences are well-known in quantum field theory with boundaries and are investigated in general case of boundary geometry. To remove them more realistic model for the brane is needed (see, for instance, the discussion in Ref. [15] and a model with the finite thickness brane in Ref. [22]). To find the asymptotic behavior of the vacuum energy-momentum tensor we note that near the brane the main contribution into the \( u \)-integral comes from large values \( u \), and the Bessel modified functions can be replaced by their asymptotic expressions for large values of the argument [20]. Assuming \( k_D|y - a| \ll 1 \) and \( \lambda_\beta|z - z_a| \ll 1 \) (note that the second condition can also be written in the form \( \lambda_\beta|y - a| \ll 1 \), the distance from the brane is much less than the physical length scale of the KK mode), to the leading order for the contribution of a given KK mode along \( \Sigma \) one finds the following result

\[
\langle T^{N_1(a)}_{M(M)} \rangle \approx \Gamma \left( \frac{D_1 + 1}{2} \right) \frac{\kappa(B_a)(k_D z_a)^{D_1 + 1} F^{N_1}_{\beta M}}{2D_1 \pi^{D_1 + 1} |z - z_a|^{D_1 + 1}},
\] (2.33)

where \( \kappa(B_j) = 2\delta_0B_j - 1 \) and the coefficients for separate components are defined by the formulæ

\[
F^a_{\beta \mu} = D_1 \delta^a_{\mu}(\zeta - \zeta_D), \quad F^{D}_{\beta D} = D(\zeta - \zeta_D) \left( \frac{z}{z_a} - 1 \right),
\] (2.34a)

\[
F^b_{\beta D} = \left( \frac{1}{4} - \zeta \right) k_D z_a(z - z_a) \eta^b_{\beta}(X), \quad F^k_{\beta i} = D_1 \delta^k_{\beta i} \left( \zeta - \frac{1}{4} \right).
\] (2.34b)

As the renormalized boundary-free energy-momentum tensor is finite on the brane, we conclude that near the brane the total vacuum energy-momentum tensor is dominated by the brane induced part and has opposite signs for Dirichlet and non-Dirichlet boundary conditions. Near the brane \( D_D \)- and \( D_B \)-components of this tensor have opposite signs in the regions \( y < a \) and \( y > a \). For \( \zeta > \zeta_D \) (\( \zeta < \zeta_D \)) the energy density is positive (negative) for Dirichlet boundary condition and is negative (positive) for non-Dirichlet boundary condition. Note that for a scalar field conformally coupled in \( D \) spatial dimensions one has \( \zeta = \zeta_D > \zeta_D \). If we denote by \( p_i \) and \( p_D \) the vacuum effective pressures in the subspace \( \Sigma \) and in the radial direction, respectively, when near the brane the equation of state has the form \( p_i = -D_1 \varepsilon/(D_1 - 1) \) for a minimally coupled scalar and \( p_i = D_1 \varepsilon/D_2 \) for a conformally coupled scalar, where \( \varepsilon \) is the vacuum energy density. In both cases one has \( |p/D| \ll 1 \). Note that, unlike to the case of the purely AdS bulk without an internal space, here the vacuum energy-momentum tensor for a conformally coupled massless scalar field diverges on the brane.

Next we consider the behavior of the brane induced VEV at large distances from the brane. For nonzero KK modes along \( \Sigma \) assuming \( z \gg \lambda_\beta^{-1} \) we see that for the region of integration in
Eq. (2.13) one has $uz \gg 1$ and the MacDonald function can be replaced by its asymptotic expression for large values of the argument. The main contribution into the integral comes from values near the lower limit of integration and to the leading order the brane induced part is estimated by the formula

$$
\langle T^N_{M}' \rangle_{\beta} \approx \frac{(k_D z)^{D+1} \frac{D_1}{2} - 1 F_{\beta M}^{(1)N} (\lambda_{\beta}, \lambda_{\beta}) I_{\nu}^{(a)} (z_{\beta} \lambda_{\beta})}{2^{D+1} \pi \frac{D_1}{2} - 1 z^{2 \nu} \epsilon_2 z \lambda_{\beta}} K_{\nu}^{(a)} (z_{\beta} \lambda_{\beta}).
$$

(2.35)

Hence, the contribution of the nonzero KK modes exponentially vanishes when the observation point tends to the AdS horizon, $z \to \infty$. For the zero KK mode ($\lambda_{\beta} = 0$) assuming $z \gg z_a$, we see that the main contribution into the integral in Eq. (2.13) comes from $u$ for which $uz_a \ll 1$ and we can replace the Bessel modified functions with this argument by their asymptotic expressions for small values of the argument. The remained integral is evaluated by the standard formula for the integrals involving the square of the MacDonald function (see, for instance, [23]). For a particular, the contribution of the brane into the total vacuum energy per unit surface on the AdS bulk ($D_2 = 0$) this VEV vanishes on the horizon for $\nu > 0$. For an internal spaces with $D_2 > 2\nu$ the VEV diverges on the horizon. Note that for a conformally coupled massless scalar and $D_2 = 1$ the boundary induced VEV takes nonzero finite value on the horizon. The VEV from the zero mode integrated over the internal space (see Eq. (2.26)) vanishes on the AdS horizon for all values $D_2$ due to the additional warp factor coming from the volume element. In particular, the contribution of the brane into the total vacuum energy per unit surface on the brane in the region $[z, \infty)$, $z > z_a$, is finite.

For large distances from the brane in the region $y < a$ one has $z \ll z_a, 1/\lambda_{\beta}$. The main contribution into the integral in Eq. (2.23) comes from the region $u \lesssim 1/z_a$ in which $uz_a \ll 1$ and we can replace the function $I_{\nu} (uz)$ by the corresponding expression for small values of the argument. As a result to the leading order we obtain the formula

$$
\langle T^N_{M}' \rangle_{\beta} \approx \frac{2^{1-D_1-2\nu} k_D^{D+1}}{\pi D^{\nu+1} \Gamma (D_1 + 2 + \nu)} F_{\beta M}^{(1)N} (\lambda_{\beta}, \lambda_{\beta}) \int_{\lambda_{\beta}}^{\infty} du u^{2\nu+1} (u^2 - \lambda_{\beta}^2 \frac{D_1}{2} - 1 \frac{D_1}{2} - 1 \frac{D_1}{2} - 1 K_{\nu}^{(a)} (uz) I_{\nu}^{(a)} (uz_a)),
$$

(2.39)

with the coefficients defined by the relations

$$
\tilde{F}_\mu^\sigma = 2\nu \delta_\mu^\sigma, \quad \tilde{F}_i^k = 2\nu \delta_i^k, \quad \tilde{F}_D^D = -D, \quad \tilde{F}_i^D = k_D z^2 \eta_\beta (X).
$$

(2.40)
In this limit the brane induced vacuum stresses in the directions parallel to the brane are isotropic. In the case of Dirichlet boundary condition the corresponding energy density is negative for both minimally and conformally coupled scalars. As we see for fixed values $k_D$, $z_a$, $\lambda_\beta$ the brane induced VEV vanishes as $z^{D+2\nu}$ for diagonal components and as $z^{D+2\nu+2}$ for the $i_j$-component when the observation point tends to the AdS boundary. In particular, the contribution of the brane to the total vacuum energy per unit surface on the brane in the region $[0, z]$, $z < z_a$, is finite (near $z = 0$ the integrand in the corresponding integral over $z$ behaves as $z^{2\nu - 1}$).

The integral in Eq. (2.39) is simply estimated in two subcases. For the case $z << \lambda_\beta^{-1} < z_a$ the main contribution comes from the lower limit and to the leading order the integral is proportional to $(\lambda_\beta/z_a)^{D_1/2}\lambda_\beta^{2\nu}e^{-2\lambda_\beta z_a}$. As a result the brane induced vacuum energy-momentum tensor is suppressed by the factor $(\lambda_\beta z)^{D_1/2+2\nu}(z/z_a)^{D_1/2}e^{-2\lambda_\beta z_a}$.

In the opposite limit for $z_a$, $z << z_a < \lambda_\beta^{-1}$, which corresponds to small KK masses, $\lambda_\beta^{(a)} \ll k_D$, the lower limit in the integral can be replaced by 0 and to the leading order the contribution of the mode with a given $\beta$ does not depend on $\lambda_\beta$ and behaves as $z^{D+2\nu}/z_a^{D_1+2\nu}$.

To see the convergence properties of the series over $\beta$ in Eq. (2.41), consider the contribution to the brane induced VEV of the energy-momentum tensor from the modes with large KK masses, $\lambda_\beta z, \lambda_\beta z_a \gg 1$. Replacing the Bessel modified functions by asymptotic expansions for large values of the argument, to the leading order one finds

\[
\langle T^N_{\beta}\rangle^{(a)} \approx \frac{(k_D z)^{D+1}}{(4\pi)^{D_1/2}} \Gamma\left(\frac{D_1}{2}\right) \int_{\lambda_\beta}^{\infty} du \frac{(u^2 - \lambda_\beta^2)^{D_1/2-1}e^{-2u|z-z_a|}}{c_a(u z_a)} F^{(1)N}_{\beta M}(u, \lambda_\beta),
\]

with the notations defined by Eq. (2.30),

\[
c_j(u) = \frac{A_j - n^{(j)} B_j u}{A_j + n^{(j)} B_j u}, \quad j = a, b,
\]

and the definition for $n^{(j)}$ is given after formulae (2.30). Under the condition $\lambda_\beta z_a \gg |A_a/B_a|$ or $B_a = 0$, we have $c_a(u z_a) \approx \kappa(B_a)$ and the integral can be expressed through the MacDonald function by using the formula

\[
\int_{\lambda_\beta}^{\infty} du u^n (u^2 - \lambda_\beta^2)^{D_1/2-1}e^{-2u\eta} = \lambda_\beta^n \frac{\sqrt{\pi}}{\sqrt{\eta}} \left(\frac{D_1}{2}\right) \left(\frac{\lambda_\beta}{\eta}\right)^{D_1/2} K_{D_1/2-1+n}(2\lambda_\beta \eta),
\]

where $n = 0, 1$. Additionally assuming $\lambda_\beta |z-z_a| \gg 1$, from (2.44) we find that the contribution of KK modes with large masses to the leading order is given by

\[
\langle T^N_{\beta}\rangle^{(a)} \approx \frac{(k_D z)^{D+1}\lambda_\beta^{D_1/2-1}e^{-2\lambda_\beta|z-z_a|}}{2^{D_1+1} \pi^{D_1/2} 

\]

and is exponentially suppressed. In particular, the corresponding conditions are satisfied for all nonzero KK modes if the length scale of the internal space $\Sigma$ is sufficiently small. In this case the main contribution into the vacuum energy-momentum tensor comes from the zero mode. In the opposite limit, when the length scale of the internal space is large, to the leading order the vacuum energy-momentum tensor reduces to the corresponding result for a brane in the bulk $AdS_{D+1}$ given in Ref. [17].

In the limit of strong gravitational fields, corresponding to large values of the AdS energy scale $k_D$ when the values of the other parameters are fixed, for nonzero KK modes along $\Sigma$ one has $\lambda_\beta z_a, \lambda_\beta z \gg 1$. Hence, the behavior of the brane induced VEV of the energy-momentum tensor in this case can be estimated by formula (2.31). If in addition the condition $\lambda_\beta |z-z_a| \gg 1$ is
satisfied, we have formula (2.44) with the exponential suppression of the brane induced part. For the zero KK mode the corresponding behavior of the brane induced vacuum energy-momentum tensor is estimated by using the asymptotic formulae for the Bessel modified functions for small and large values of the argument in the regions \( y > a \) and \( y < a \) respectively. To the leading order the components of this tensor behave like \( k_{D}^{2D_{1}+1}e^{D_{2}k_{D}y}\exp[(D_{1}+2\nu)k_{D}(y-a)] \) in the region \( y < a \) and like \( k_{D}^{2D_{1}+1}e^{D_{2}k_{D}y}\exp[2\nu k_{D}(a-y)] \) in the region \( y > a \). The corresponding quantities integrated over the internal space contain additional factor \( e^{-D_{2}k_{D}y} \) coming from the volume element and are exponentially small in both regions.

Now we turn to the consideration of the limiting cases for the parameter \( z_{a} \) determining the position of the brane. For small values of this parameter, \( z_{a} \ll z, 1/\lambda_{\beta} \), for the dominant contribution into the \( u \)-integral in Eq. (2.13) one has \( uz_{a} \ll 1 \) and, using the asymptotic formulae for the Bessel modified functions for small values of the argument, one finds

\[
\langle T_{M}^{N}(a) \rangle_{\beta} \approx -\frac{2^{2-D_{1}-2\nu}k_{D}^{2D_{1}+1}z^{D_{2}\nu}}{\Gamma\left(\frac{\nu}{2}\right)\nu\Gamma^{2}(\nu) c_{a}(\nu)} \int_{l_{\beta}}^{\infty} du \frac{u^{2\nu+1}(u^{2} - \lambda_{\beta}^{2})^{D_{1}+1}F_{\beta}(N)}{\Gamma_{\beta}(N)K_{\nu}(uz)}.
\]

Note that in this limit the distance of the observation point from the brane is large compared with the AdS curvature radius, \( k_{D}(y-a) \gg 1 \), and the KK masses \( \lambda_{\beta}^{(a)} \) are small with respect to the AdS energy scale, \( \lambda_{\beta}^{(a)} \ll k_{D} \). In particular, from Eq. (2.45) it follows that for fixed values \( k_{D}, y, \lambda_{\beta} \) the brane induced VEV in the region \( z > z_{a} \) vanishes as \( z_{a}^{2\nu} \) when the brane position tends to the AdS boundary, \( z_{a} \to 0 \). Formula (2.45) is further simplified for two subcases. In the case \( z_{a} \ll \lambda_{\beta}^{-1} \ll z \), or equivalently \( k_{D}e^{-k_{D}(y-a)} \ll \lambda_{\beta}^{(a)} \ll k_{D} \), the main contribution into the integral comes from the lower limit and to the leading order we find

\[
\langle T_{M}^{N}(a) \rangle_{\beta} \approx \frac{k_{D}^{2D_{1}+1}z^{D_{2}\nu}(z_{a}\lambda_{\beta})^{2\nu}z^{D_{1}+1}}{2D_{1}+2\nu \pi \Gamma^{2}(\nu) c_{a}(\nu)\lambda_{\beta}^{2}} e^{-2z\lambda_{\beta}}F_{\beta}(N)(\lambda_{\beta}, \lambda_{\beta}).
\]

In the limit \( z_{a} \ll z \ll \lambda_{\beta}^{-1} \) (small KK masses, \( \lambda_{\beta}^{(a)} \ll k_{D}e^{-k_{D}(y-a)} \)) to the leading order we can put 0 in the lower limit of the integral and we obtain formula (2.46).

When the brane position tends to the AdS horizon, \( z_{a} \to \infty \), for massive KK modes along \( \Sigma \) the main contribution into the VEV of the energy-momentum tensor in the region \( z < z_{a} \) comes from the lower limit of the \( u \)-integral. To the leading order we find

\[
\langle T_{M}^{N}(a) \rangle_{\beta} \approx -\frac{k_{D}^{2D_{1}+1}z^{D_{2}\nu}e^{-2\lambda_{\beta}z_{a}}}{2D_{1} \pi \Gamma^{2}(\nu) z_{a}^{D_{1}/2}} F_{\beta}(N)[I_{\nu}(\lambda_{\beta}z)] \frac{c_{a}(\lambda_{\beta}z_{a})}{\lambda_{\beta}^{2}},
\]

and the VEV is exponentially small. For the zero mode in the same limit, introducing a new integration variable \( v = uz_{a} \), we expand the functions \( F_{\beta}(N)[I_{\nu}(uz/z_{a})] \) for small values of the argument. To the leading order this yields

\[
\langle T_{M}^{N}(a) \rangle_{\beta} \approx \frac{k_{D}^{2D_{1}+1}z^{D_{2}\nu}c_{D}}{2D_{1}+2\nu \pi \Gamma^{2}(\nu+1)\Gamma\left(\frac{D_{1}}{2}\right)} \left(\frac{z}{z_{a}}\right)^{D_{1}} \int_{0}^{\infty} du \frac{u^{D_{1}+2\nu-1}K_{\nu}^{(a)}(u)}{\Gamma_{\nu}^{(a)}(u)},
\]

and the suppression is power-law with respect to \( z_{a} \).

### 3 Energy-momentum tensor in the region between two branes

In the geometry of two branes we have three distinct regions of the radial coordinate: \( y < a, a < y < b, \) and \( y > b \). The VEVs in the first and last regions are the same as those for a
single brane. In this section we will discuss the vacuum energy-momentum tensor in the region between two branes. Note that in the orbifolded version of the model this region is employed only. By using formula \(2.4\), the VEV of the energy-momentum tensor is expressed through the corresponding Wightman function and the VEV of the field square investigated in Ref. 15. The Wightman function is presented in the form

\[
\langle 0|\varphi(x)\varphi(x')|0 \rangle = \langle \varphi(x)\varphi(x') \rangle^{(0)} + \langle \varphi(x)\varphi(x') \rangle^{(j)} - \frac{k_{D-1}^{D-1}}{2D-1}\pi D_1} \sum_\beta \psi_\beta(X)\psi_\beta^*(X')
\times \int \frac{dk}{2\pi}e^{-ik(x-x')} \int_{k_\beta}^{\infty} duG_{\nu}^{(j)}(uz_j, uz)G_{\nu}^{(j)}(uz_j, uz')
\times \frac{\Omega_{j\nu}(uz_a, uz_b)}{\sqrt{u^2 - k_{\beta}^2(t - t')}}
\]

where \(k_\beta\) is defined after formula \(2.3\) and \(j = a, b\) provide two equivalent representations. Here and in the formulae below we use the notations

\[
\Omega_{\alpha\nu}(u, v) = \frac{\bar{K}_{\nu}^{(b)}(v) - \bar{K}_{\nu}^{(a)}(v)}{\bar{K}_{\nu}^{(b)}(v)\bar{I}_{\nu}^{(b)}(v) - \bar{K}_{\nu}^{(a)}(v)\bar{I}_{\nu}^{(a)}(v)},
\]

\[
\Omega_{\nu\nu}(u, v) = \frac{\bar{I}_{\nu}^{(a)}(v) - \bar{I}_{\nu}^{(b)}(v)}{\bar{K}_{\nu}^{(a)}(v)\bar{I}_{\nu}^{(b)}(v) - \bar{K}_{\nu}^{(b)}(v)\bar{I}_{\nu}^{(a)}(v)},
\]

and

\[
G_{\nu}^{(j)}(u, v) = I_{\nu}(v)\bar{K}_{\nu}^{(j)}(u) - \bar{I}_{\nu}^{(j)}(u)K_{\nu}(v).
\]

Substituting the Wightman function into formula \(2.4\) and using relation \(2.14\), for the components of the vacuum energy-momentum tensor in the region between the branes we obtain the formula

\[
\langle 0|T_M^{N}|0 \rangle = \langle T_M^{N} \rangle^{(0)} + \langle T_M^{N} \rangle^{(j)} - \frac{2k_{D}^{D+1}z^D}{(4\pi)^{D/2}\Gamma(D_1/2)} \sum_\beta |\psi_\beta(X)|^2
\times \int_{k_\beta}^{\infty} du u(u^2 - \lambda_{\beta}^2)^{-1/2} - \Omega_{j\nu}(uz_a, uz_b)F_{\beta\nu}^{(N)}[G_{\nu}^{(j)}(uz_j, uz)],
\]

with the functions \(F_{\beta\nu}^{(N)}[g(v)]\) defined by relations \(2.5\)–\(2.8\), where \(g(v) = G_{\nu}^{(j)}(uz_j, v)\), and with \(j = a, b\). The last term on the right of this formula is finite on the brane at \(z = z_j\) and diverges for the points on the brane \(z = z_j, j' = a, b, j' \neq j\). These divergences are the same as those for a single brane at \(z = z_j\). As a result, if we write the VEV of the energy-momentum tensor in the form

\[
\langle 0|T_M^{N}|0 \rangle = \langle T_M^{N} \rangle^{(0)} + \sum_{j=a, b} \langle T_M^{N} \rangle^{(j)} + \langle T_M^{N} \rangle^{(ab)},
\]

then the interference part \(\langle T_M^{N} \rangle^{(ab)}\) is finite on both branes. In the case of one parameter manifold \(\Sigma\) with the size \(L\) and for a given \(k_D\), the VEV in Eq. \(3.5\) is a function on \(z_b/z_a, L/z_a,\) and \(z/z_a\). By using formula \(3.3\) and the expression for a single brane induced VEV of the energy-momentum tensor, we can present the contribution of the given KK mode along \(\Sigma\) into the interference part of Eq. \(3.5\) in the form

\[
\langle T_M^{N} \rangle^{(ab)} = \frac{2k_{D}^{D+1}z^D}{(4\pi)^{D/2}\Gamma(D_1/2)} \int_{k_\beta}^{\infty} du u(u^2 - \lambda_{\beta}^2)^{-1/2} \left\{ 2F_{\beta\nu}^{(N)}[I_{\nu}(uz), K_{\nu}(uz)] - \frac{\bar{I}_{\nu}^{(a)}(uz_a)}{\bar{K}_{\nu}^{(b)}(uz_a)}F_{\beta\nu}^{(N)}[K_{\nu}(uz)] - \frac{\bar{K}_{\nu}^{(a)}(uz_b)}{\bar{I}_{\nu}^{(b)}(uz_b)}F_{\beta\nu}^{(N)}[I_{\nu}(uz)] \right\}. \tag{3.6}
\]
where \( \langle T_M^N \rangle_{(ab)} \) is defined by the relation similar to Eq. (2.12), and the functions \( F_{\beta M}^{(+)N}[g_1, g_2] \) are defined by formulae (2.13)–(2.15) with the replacements \( g^2 \rightarrow g_1^2 g_2^2, g' \rightarrow (g_1 g_2)'/2 \), and \( \bar{g}^2 \rightarrow g_1 g_2 \). For large values \( u \) the integrand in formula (3.6) behaves as \( e^{-2u(z_b - z_a)} \) and the integral is convergent for all values \( z \) including the points on the branes. For interbrane distances much smaller than the AdS curvature radius, \( (b - a) \ll 1/k_D \), the main contribution into the integral in Eq. (3.6) comes from large values \( u \) and we can replace the Bessel modified functions by asymptotic expressions for large values of the argument. Additionally assuming that the distance is small compared with the KK length scale for an observer living on the brane \( y = a \), \( (b - a) \ll 1/\lambda^2_\beta \), it can be seen that the diagonal components of the interference part with \( M \neq D \) behave as \((b - a)^{-D_1 - 1}\) whereas \( D_2 \)– and \( i_D \)–components behave like \((b - a)^{-D_1}\).

Now let us consider various limiting cases when the general formula for the interference part is simplified. First of all, in the limit \( k_D \rightarrow 0 \) the order of Bessel modified functions is large and we use the uniform asymptotic expansions for these functions. To the leading order the following result is obtained

\[
\langle T_M^N \rangle_{(ab)} \approx \frac{(4\pi)^{-\frac{D_1}{2}}}{\Gamma \left( \frac{D_2}{2} \right)} \sum_{\beta} |\psi_\beta(X)|^2 \int_{v_\beta}^{\infty} du \frac{e^{-2u(y-j)} \sum_{j=a,b} c_j(u) c_j(u) e^{2u(b-a)}}{c_j(u)^2} - \frac{D_1}{2} \left[ F_{\beta M}^{(1)iN}(u, v_\beta) - 2 F_{\beta M}^{(0)iN}(u, v_\beta) \right]_{k_D=0}, \tag{3.7}
\]

with notations defined by Eq. (2.30) and \( v_\beta = \sqrt{m^2 + \lambda^2_\beta} \). In the expression on the right the functions \( F_{\beta M}^{(1)iN}(u, v) \) are defined by the same relations as the functions \( F_{\beta M}^{(0)iN}(u, v) \) from (2.30) with the replacement \( n^{(a)} \rightarrow n^{(j)} \) in formula (2.30), and with \( n^{(j)} \) defined in the paragraph after formula (2.30). This limit corresponds to the geometry of two parallel Robin plates in the bulk \( R^{(D_1,1)} \times \Sigma \) and the expression on the right of formula (3.7) coincides with the interference part \( \langle T_M^N \rangle_{R^{(D_1,1)} \times \Sigma} \) in the VEV of the energy-momentum tensor in the region between two plates. This expression generalizes the corresponding result from Ref. 21 for Robin plates in the Minkowski background.

For large KK masses, \( z_a \lambda_\beta \gg 1 \), the arguments of the Bessel modified functions in Eq. (3.6) are large. By using the corresponding asymptotic expansions, for the contribution of the mode with a given \( \beta \) one finds

\[
\langle T_M^N \rangle_{(ab)} \approx \frac{(k_D z)^{D_1+1}}{(4\pi)^{\frac{D_1}{2}} \Gamma \left( \frac{D_2}{2} \right)} \int_{v_\beta}^{\infty} du \frac{e^{-2u(z-z_j)}}{c_a(u z_a) c_b(u z_b) e^{2u(z_b - z_a)}} - \frac{D_1}{2} \left[ F_{\beta M}^{(1)iN}(u, \lambda_\beta) - 2 F_{\beta M}^{(0)iN}(u, \lambda_\beta) \right]. \tag{3.8}
\]

In particular, for a one parameter internal manifold with the length scale \( L \ll z_a \), this formula is valid for all nonzero KK modes along \( \Sigma \). Under the additional assumption \( \lambda_\beta |z - z_j| \gg 1 \), the main contribution into the integral in this formula comes from the lower limit and we obtain the formula

\[
\langle T_M^N \rangle_{(ab)} \approx \frac{(k_D z)^{D_1+1}}{(4\pi)^{\frac{D_1}{2}}} c_a(\lambda_\beta z_a) c_b(\lambda_\beta z_b) e^{-2\lambda_\beta(z_b - z_a)} F_{\beta M}^{(0)iN}(\lambda_\beta, \lambda_\beta). \tag{3.9}
\]

In particular, it follows from here that in the expressions for the brane induced VEVs the series over KK modes along \( \Sigma \) are exponentially convergent. Comparing with (2.44) we see that for
large KK masses the interference part is exponentially suppressed with respect to the single
brane parts.

Now let us assume the conditions $\lambda_\beta z_b \gg 1$ and $z_a \lambda_\beta \lesssim 1$. This limit corresponds to large
interbrane distances compared with the AdS curvature radius $k_D^{-1}$ and is realized in braneworld
scenarios for the solution of the hierarchy problem. In this limit the main contribution into the
large KK masses the interference part is exponentially suppressed with respect to the single
for the nonzero KK modes along $\Sigma$. In the limit
the interference part in the vacuum energy-momentum tensor vanishes

$$
\left\langle T_{M/\beta}^{N(ab)} \right\rangle \approx \frac{k_D^{D+1} z D \lambda_\beta \frac{D_1}{D} e^{-2 \lambda_\beta z_b}}{2^{D_1} \pi^{D_1+1} \zeta b (\lambda_\beta z_b)} \tilde{I}_\nu^{(a)} (z_a \lambda_\beta) \\
\times \left\{ 2 F^{(4)+N}_{\beta M} [I_\nu (\lambda_\beta z), K_\nu (\lambda_\beta z)] - \frac{I_\nu^{(a)} (z_a \lambda_\beta)}{K_\nu^{(a)} (z_a \lambda_\beta)} F^{(4)+N}_{\beta M} [K_\nu (\lambda_\beta z)] \right\},
$$

(3.10)

for the nonzero KK modes along $\Sigma$. In the limit $z \ll \lambda_\beta^{-1} \ll z_b$ after the replacement of the
Bessel modified functions by their asymptotic expressions for small values of the argument, the
the corresponding formula takes the form

$$
\left\langle T_{M/\beta}^{N(ab)} \right\rangle \approx \frac{k_D^{D+1} z D \lambda_\beta \frac{D_1}{D} e^{-2 \lambda_\beta z_b}}{2^{D_1} \pi^{D_1+1} \zeta b (\lambda_\beta z_b)} \tilde{I}_\nu^{(a)} (z_a \lambda_\beta) \\
\times \left\{ m^2 F^{D N}_{\beta M} \delta_D + \frac{e^{D(N)}}{2c_b (\nu)} \left( \frac{z_a}{z} \right)^{2 \nu} \right\},
$$

(3.11)

where $\tilde{F}_M^N$ is defined by relations (2.40). For $\lambda_\beta z \gg 1$ and $\lambda_\beta z_a \lesssim 1$, using the asymptotic
formulae for the Bessel modified functions for large values of the argument, we obtain the following formula

$$
\left\langle T_{M/\beta}^{N(ab)} \right\rangle \approx \frac{(k_D z)^{D+1} \lambda_\beta \frac{D_1}{D} e^{-2 \lambda_\beta z_b}}{2^{D_1} \pi^{D_1+1} \zeta b (\lambda_\beta z_b)} \tilde{I}_\nu^{(a)} (z_a \lambda_\beta) \\
\times \left\{ F^{(0)+N}_{\beta M} (\lambda_\beta, \lambda_\beta) \right. - \frac{e^{-2 \lambda_\beta (z_b - z)}}{2c_b (\lambda_\beta z_b)} \left. F^{(1)+N}_{\beta M} (\lambda_\beta, \lambda_\beta) \right\}.
$$

(3.12)

In the limit $z_a \ll z_b \ll \lambda_\beta^{-1}$, to the leading order we can put 0 instead of $\lambda_\beta$ in the lower limit of the integral over $u$, and by using the asymptotic formulae for the Bessel modified functions
for small values of the argument, it can be seen that

$$
\left\langle T_{M/\beta}^{N(ab)} \right\rangle \sim (z_a/z_b)^{2\nu} g(z/z_b).
$$

In particular, it follows from here that the interference part in the vacuum energy-momentum tensor vanishes
as $z_b^{2\nu}$ when the left brane tends to the AdS boundary, $z_a \rightarrow 0$. Under the condition $z \ll z_b$ an additional suppression factor appears in the form $(z/z_b)^{D_1}$ for $D$-component and in the form $(z/z_b)^{D_1+2\alpha_1}$ for the other components, where $\alpha_1 = \min(1, \nu)$. As we see for large interbrane distances the interference part of the brane induced VEV of the energy-momentum tensor is mainly located near the brane $z = z_b$.

In the higher dimensional generalization of the Randall-Sundrum braneworld based on the bulk $AdS_{D+1} \times \Sigma$, y coordinate is compactified on an orbifold $S^1/Z_2$ and the orbifold fixed points are the locations of two $D$-dimensional branes. The corresponding VEVs of the energy-momentum tensor are determined by the formulae given in this section with an additional factor $1/2$ and with Robin coefficients (see also [15])

$$
\frac{\dot{A}_a}{B_a} = -\frac{1}{2} (c_1 + 4D\zeta k_D), \quad \frac{\dot{A}_b}{B_b} = -\frac{1}{2} (-c_2 + 4D\zeta k_D),
$$

(3.13)
for untwisted scalar field. Here $c_1$ and $c_2$ are surface mass parameters on the branes $y = a$ and $y = b$, respectively. For twisted scalar Dirichlet boundary conditions are obtained. The one-loop effective potential and the problem of moduli stabilization in this model with zero mass parameters $c_j$ are discussed in Ref. [18]. A scenario is proposed where supersymmetry is broken near the fundamental Planck scale, and the hierarchy between the electroweak and effective Planck scales is generated by a combination of redshift and large volume effects. The corresponding parameters satisfy the relations $L \lesssim 1/k_D$, $z_a \sim L$, and $z_b/z_a \gg 1$. In this case the contribution into the interference part of the energy-momentum tensor from the nonzero KK modes is estimated by formula (3.10). For the zero mode the interference part behaves like $e^{2\nu k_D(a-b)}$ and is exponentially small.

4 Interaction forces

Now we turn to the vacuum forces acting on the branes. The corresponding effective pressure $p^{(j)}$ acting on the brane at $z = z_j$ is determined by $P_D$-component of the vacuum energy-momentum tensor evaluated at the point of the brane location: $p^{(j)} = -(T^D_D)_{z=z_j}$. For the region between two branes it can be presented as a sum of two terms:

$$p^{(j)} = p_1^{(j)} + p_{(\text{int})}^{(j)}, \quad j = a, b. \quad (4.1)$$

The first term on the right is the pressure for a single brane at $z = z_j$ when the second brane is absent. This term is divergent due to the surface divergences in the VEVs and needs additional renormalization. This can be done, for example, by applying the generalized zeta function technique to the corresponding mode sum. This procedure is similar to that used in Ref. [18] for the evaluation of the effective potential. The corresponding calculation lies on the same line with the evaluation of the surface Casimir densities and will be presented in the forthcoming paper. Here we note that the single brane term in the vacuum effective pressure will contain a part which depends on the renormalization scale. This part will change under the change of the renormalization scale and can be fixed by imposing suitable renormalization conditions which relates it to observables. Below we will be concentrated on the second term in the right of Eq. (4.1). This term is the additional vacuum pressure induced by the presence of the second brane, and can be termed as an interaction force. It is determined by the last term on the right of formulae (3.4) evaluated at the brane location $z = z_j$. It is finite for all nonzero interbrane distances and is not changed by the regularization and renormalization procedure. In particular, it does not depend on the type of the regularization procedure used. All regularization ambiguities are involved in $p_1^{(j)}$. Substituting $z = z_j$ into the second term on the right of formula (3.4) and using the relations

$$G^{(j)}_\nu(u, u) = -B_j, \quad \frac{dG^{(j)}_\nu(u, v)}{dv} \bigg|_{v=u} = \frac{A_j}{u}, \quad (4.2)$$

for the interaction part of the vacuum effective pressure one has

$$p_{(\text{int})}^{(j)} = \frac{k_D^{D+1}z_j^D}{(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)} \sum_\beta |\psi_\beta(X)|^2 \int_{\lambda_\beta}^\infty duu(u^2 - \lambda_\beta^2)^\nu_1 - 1 \Omega_{j\nu}(uz_a, uz_b)F^{(j)}_\beta(uz_j), \quad (4.3)$$

where we have introduced the notation

$$F^{(j)}_\beta(u) = \left(u^2 - \nu^2 + \frac{m^2}{k_D^2}\right)B_j^2 - D(4\zeta - 1)A_jB_j - A_j^2 - 2\left(\zeta - \frac{1}{4}\right)z_j^2B_j^2\eta_\beta(X). \quad (4.4)$$

Below we will show that for small interbrane distances the interaction part dominates the single brane parts in Eq. (4.1). For a Dirichlet scalar $B_j = 0$ and one has $F^{(j)}_\beta(u) = -A_j^2$. By using the
properties of the modified Bessel function it can be seen that in this case \( \Omega_{j\nu}(uz_a, uz_b) > 0 \) and, hence, the vacuum interaction forces are attractive. For a given value of the AdS energy scale \( k_D \) and one parameter manifold \( \Sigma \) with size \( L \), the vacuum interaction forces are functions on the ratios \( z_b/z_a \) and \( L/z_a \). The first ratio is related to the proper distance between the branes and the second one is the ratio of the size of the internal space measured by an observer residing on the brane at \( y = a \) to the AdS curvature radius \( k_D^{-1} \). Note that the term ‘interaction’ in the discussion here and below should be understood conditionally. The quantity \( p^{(j)}_{\text{int}} \) determines the force by which the scalar vacuum acts on the brane due to the modification of the spectrum for the zero-point fluctuations by the presence of the second brane. As the vacuum properties depend on the coordinate \( y \), there is no a priori reason for the interaction terms (and also for the total pressures \( p^{(j)} \)) to be equal for the branes \( j = a \) and \( j = b \), and the corresponding forces in general are different even in the case of the same Robin coefficients in the boundary conditions.

Now we turn to the limiting cases when the expressions for the interaction forces between the branes are simplified. First of all we consider the limit \( k_D \to 0 \). By the way similar to that used before for the vacuum energy-momentum tensor, to the leading order we find

\[
p^{(j)}_{\text{int}} \approx -2 \frac{(4\pi)^{\frac{D_1}{2}}}{\Gamma(\frac{D_1}{2})} \sum_\beta |\psi_\beta(X)|^2 \int_{v_\beta}^\infty du \frac{u^2(u^2 - v_\beta^2)\frac{D_1}{2} - 1}{c_a(u)c_b(u)e^{2u(b-a)} - 1} \times \left[ 1 + \frac{(2\zeta - 1/2)B_j^2}{A_j^2 - u^2B_j^2} \eta_\beta(X) \right],
\]

where \( v_\beta \) is defined after formula (2.31). The expression on the right of this formula presents the corresponding force acting per unit surface on the brane in the bulk geometry \( R^{(D_1-1,1)} \times \Sigma \). Note that in this case the vacuum effective pressures are the same for both branes if the coefficients in the boundary conditions are the same. Moreover, for a homogeneous internal space, the contribution of the second term in the square brackets on the right of Eq. (4.5) vanishes, and the interaction forces are the same even in the case of different Robin coefficients for separate branes.

For large values of KK masses along \( \Sigma \), \( \lambda_\beta z_j \gg 1 \), we can replace Bessel modified functions by their asymptotic expansions for large values of the argument. For the contribution of a given KK mode to the leading order this gives

\[
p^{(j)}_{\text{int} \beta} \approx -2 \frac{(k_D z_j)^{D_1+1}}{(4\pi)^{\frac{D_1}{2}} \Gamma(\frac{D_1}{2})} \int_{\lambda_\beta}^\infty du \frac{u^2(u^2 - \lambda_\beta^2)\frac{D_1}{2} - 1}{c_a(uz_a)c_b(uz_b)e^{2u(b-a)} - 1} \times \left[ 1 + \frac{(2\zeta - 1/2)z_j^2B_j^2}{A_j^2 - (uz_jB_j)^2} \eta_\beta(X) \right],
\]

where \( p^{(j)}_{\text{int} \beta} \) is determined by the relation similar to Eq. (2.12). If in addition one has the condition \( \lambda_\beta(z_b - z_a) \gg 1 \), the main contribution into the \( u \)-integral comes from the lower limit and we have the formula

\[
p^{(j)}_{\text{int} \beta} \approx - \frac{(k_D z_j)^{D_1+1}}{(4\pi)^{\frac{D_1}{2}}} \frac{\lambda_\beta^{D_1/2+1} e^{-2\lambda_\beta(z_b - z_a)}}{c_a(\lambda_\beta z_a)c_b(\lambda_\beta z_b)(z_b - z_a)\frac{D_1}{2}} \times \left[ 1 + \frac{(2\zeta - 1/2)z_j^2B_j^2}{A_j^2 - (\lambda_\beta z_jB_j)^2} \eta_\beta(X) \right].
\]

In particular, for sufficiently small length scale of the internal space this formula is valid for all nonzero KK masses and the main contribution to the interaction forces comes from the zero
KK mode. In the opposite limit of large internal space, to the leading order we obtain the corresponding result for parallel branes in $AdS_{D+1}$ bulk [17].

For small interbrane distances, $k_D(b-a) \ll 1$, which is equivalent to $z_b/z_a - 1 \ll 1$, the main contribution into the integral in Eq. (4.3) comes from large values $u$ and to the leading order we obtain formula (4.6). If in addition one has $\lambda_\beta(z_b - z_a) \ll 1$ or equivalently $\chi^{(a)}(b-a) \ll 1$, and assuming $(b-a) \ll |\tilde{B}_j/A_j|$ or $\tilde{B}_j = 0$, we can put in this formula $\lambda_\beta = 0$, and to the leading order one finds

$$p_{(int)\beta}^{(j) \ (a)} \approx a_{D_1} D_1 \Gamma \left( \frac{D_1+1}{2} \right) \zeta_R(D_1+1) e^{D_2 k_D j},$$

(4.8)

where $a_{D_1} = 1 - 2^{-D_1}$ for $\kappa(B_a)\kappa(B_b) = -1$ and $a_{D_1} = -1$ for $\kappa(B_a)\kappa(B_b) = 1$. It follows from here that for small interbrane distances the interaction forces are repulsive for Dirichlet boundary condition on one brane and non-Dirichlet boundary condition on the other and are attractive for other cases. As in the limit $a \rightarrow b$ the renormalized values of the single brane parts $p_1^{(j)}$, $j = a, b$, are finite, for small interbrane distances the main contribution into the vacuum effective pressure $p^{(j)}$ comes from the interaction part.

Now we consider the limit $\lambda_\beta z_a \gg 1$ assuming that $\chi^{(a)} z_a \ll 1$. Using the asymptotic formulæ for the Bessel modified functions containing in the argument $z_b$, we find the following result

$$p_{(int)\beta}^{(a)} \approx \frac{k_D^{D+1} z_a^D}{2^{D+1} 1^{1-rac{D_1}{2}}} \frac{F^{(a)}_\beta(0)}{\kappa_\beta^2 (\lambda_\beta z_a)^2} e^{2 \lambda_\beta z_b} F^{(a)}_\beta(\lambda_\beta z_a),$$

(4.9)

$$p_{(int)\beta}^{(b)} \approx \frac{(k_D z_b)^{D+1}}{2^{D-1} 1^{1-rac{D_1}{2}}} \frac{I^{(a)}_\beta(\lambda_\beta z_b)}{\kappa_\beta^2 (\lambda_\beta z_a)^2} F^{(b)}_\beta(\lambda_\beta z_b).$$

(4.10)

This limit corresponds to the interbrane distances much larger compared with the AdS curvature radius and inverse KK masses, measured by an observer on the left brane: $b-a \gg 1/k_D, 1/\lambda_\beta^{(a)}$. For a single parameter manifold $\Sigma$ with length scale $L$ and $(b-a) \gg L_a$ these conditions are satisfied for all nonzero KK modes.

In the limit $z_a \lambda_\beta \ll 1$ for fixed $z_a \lambda_\beta$, by using the asymptotic formulæ for the Bessel modified functions for small values of the argument $z_b$, we find the following result

$$p_{(int)\beta}^{(a)} \approx \frac{k_D^{D+1} z_a^{D+2\nu}}{2^{D+2\nu} 1^{-\frac{D_1}{2}}} \frac{F^{(a)}_\beta(0)}{\kappa_\beta^2 (\lambda_\beta z_a)^2} \int_{\lambda_\beta}^{\nu} du \, u^{2\nu+1}(u^2 - \lambda_\beta^2)^{\frac{D_1}{2} - 1} \frac{F^{(b)}_\beta(u z_b)}{F^{(b)}_\beta(u z_b)},$$

(4.11)

$$p_{(int)\beta}^{(b)} \approx \frac{k_D^{D+1} z_a^{D+2\nu}}{(D_1+1)} \frac{2^{D_1+2\nu} 1^{1-rac{D_1}{2}}} \Gamma \left( \frac{D_1}{2} \right) \Gamma(2(\nu)) \int_{\lambda_\beta}^{\nu} du \, u^{2\nu+1}(u^2 - \lambda_\beta^2)^{\frac{D_1}{2} - 1} \frac{F^{(b)}_\beta(u z_b)}{F^{(b)}_\beta(u z_b)}. $$

(4.12)

In this case the KK masses measured by an observer on the brane at $y = a$ are much less than the AdS energy scale, $\frac{\lambda_\beta^{(a)}}{a} \ll k_D$, and the interbrane distance is much larger than the AdS curvature radius. In particular, substituting $\lambda_\beta = 0$, from these formulæ we obtain the asymptotic behavior for the contribution of the zero mode to the interaction forces between the branes in the limit $z_a/z_b \ll 1$. From formulæ (4.11) and (4.12) it follows that, in dependence of values of the coefficients in Robin boundary conditions, for large distances the interaction forces can be either attractive or repulsive. In particular, for Dirichlet boundary condition on the brane at $y = b$, one has $p_{(int)\beta}^{(a)} < 0$ for $F^{(a)}_\beta(0) < 0$ and $p_{(int)\beta}^{(b)} < 0$ for $|A_a/B_a| > \nu$, and under these conditions for $B_a \neq 0$ we have an interesting situation when the interaction forces are repulsive for small distances (see Eq. (4.8)) and are attractive for large distances.

From the formulæ given above it follows that in the limit when the right brane tends to the AdS horizon, $z_b \rightarrow \infty$, the force $p_{(int)\beta}^{(a)}$ vanishes as $e^{-2 \lambda_\beta z_b / z_b^{D_1/2}}$ for the nonzero KK mode along
$L$ and $\kappa$, formulae (4.9) and (4.10), and the suppression is stronger compared with the previous case. For $y$ on the brane at $\Sigma$ and as $\nu$ tends to the AdS boundary the contribution of a given KK mode vanishes as $z_\alpha^{D+2\nu}$ for the pressure $P^{(a)}_{\text{(int)}}$ and as $z_\alpha^{2\nu}$ for the pressure $P^{(b)}_{\text{(int)}}$. For small values of the AdS curvature radius corresponding to strong gravitational fields, assuming $\lambda_\beta z_\alpha \gg 1$ and $\lambda_\beta (z_b - z_\alpha) \gg 1$, we can estimate the contribution of the nonzero KK modes to the vacuum interaction forces by formula (5.7). In particular, for the case of a single parameter internal space with the length scale $L$, under the assumed conditions the length scale of the internal space measured by an observer on the brane at $y = a$ is much smaller compared to the AdS curvature radius, $L_a \ll k_D^{-1}$. If $L_a \gtrsim k_D^{-1}$ one has $\lambda_\beta z_\alpha \lesssim 1$ and to estimate the contribution of the interference part we can use formulae (5.9) and (5.10), and the suppression is stronger compared with the previous case. For the zero KK mode, under the condition $k_D^2 (b - a) \gg 1$ we have $z_\alpha / z_b \ll 1$ and to the leading order the corresponding interaction forces are described by relations (5.11) and (5.12). From these formulae it follows that the interaction forces integrated over the internal space behave as $k_D^{D+1} \exp[(D_1 \delta_j^a + 2\nu) k_D (a - b)]$ for the brane at $y = j$ and are exponentially suppressed. Note that in the model without the internal space we have similar behavior with $\nu$ replaced by $\nu_1$ and for a scalar field with $\zeta < \zeta_D + D_1$ the suppression is relatively weaker.

5 An example: $\Sigma = S^1$

To make our discussion concrete, here we consider a simple example with $\Sigma = S^1$. In this case the bulk corresponds to the $AdS_{D+1}$ spacetime with one compactified dimension $X$. The corresponding normalized eigenfunctions and eigenvalues are as follows

$$\psi_{\beta}(X) = \frac{1}{\sqrt{L}} e^{\pi i \beta X/L}, \quad \lambda_\beta = \frac{2\pi}{L} |\beta|, \quad \beta = 0, \pm 1, \pm 2, \ldots, \quad (5.1)$$

where $L$ is the length of the compactified dimension. The boundary-free part of the vacuum energy-momentum tensor can be evaluated on the base of formula (2.4) by using the Wightman function and the VEV of the field square from (13). The Wightman function is determined by the term in formula (2.6) with the first integral in the figure braces. The application of the Abel-Plana formula to the series over $\beta$ allows to present the boundary-free part of the energy-momentum tensor in the form (no summation over $M$)

$$\langle T^{(0)}_{M} \rangle = \langle T^{(0)}_{M/AdS_{D+1}} \rangle + \delta^N_M \int_0^\infty du \sum_{l=0}^1 S^{(l)}_{(M)} f^{(l)}_{\nu}(u, z/L), \quad (5.2)$$

where we use the following notations

$$S^{(0)}_{(\mu)} = \left( \frac{1}{4} - \zeta \right) \frac{e^u (u - D - 1) e^u + u + D + 1}{(e^u - 1)^2} - \frac{\zeta e^u}{e^u - 1} + D \zeta, \quad S^{(1)}_{(\mu)} = -\frac{2^2 u^2}{2L^2}, \quad (5.3)$$

$$S^{(0)}_{(D-1)} = S^{(0)}_{(\mu)} + \frac{2^2 u^2}{L^2}, \quad S^{(1)}_{(D-1)} = 0, \quad S^{(1)}_{(D)} = -D S^{(1)}_{(\mu)}, \quad (5.4)$$

$$S^{(0)}_{(D)} = -\frac{u}{4} e^u (u - D - 1) e^u + u + D + 1 \left( e^u - 1 \right)^2 + \frac{D \zeta e^u}{e^u - 1} - D^2 \zeta + \frac{m^2}{k_D^2} - \frac{2^2 u^2}{2L^2}, \quad (5.5)$$

and

$$f^{(l)}_{\nu}(u, z/L) = \frac{2 k_D^{D+1}}{\pi^2} _2 F_2 \left( \nu + \frac{1}{2}; \nu + l + \frac{D+1}{2}, 2\nu + 1; -\frac{z^2 u^2}{L^2} \right) \frac{u z \nu + \frac{1}{2} l + \frac{D+1}{2}}{2L}, \quad (5.6)$$
with the hypergeometric function $1F_2$. The first term on the right of formula (5.2) is the corresponding tensor in $AdS_{D+1}$ bulk without boundaries and the second term is induced by the compactness of $X$ direction. The latter is finite and the renormalization procedure is needed for the term $\langle T^N_M \rangle_{AdS_{D+1}}$ only. The renormalized value of this tensor does not depend on the spacetime point and is well-investigated in literature. For this reason below we will be concentrated on the second term. For $z \ll L$ to the leading order one has

$$\langle T^N_M \rangle(0) \approx \langle T^N_M \rangle_{AdS_{D+1}} - 2\delta_M^N \sum_{k_D^2} \frac{k_D^{D+1} \zeta^{(+)}(D+2\nu)}{\pi^2 \Gamma(\nu)} \right| \left. \zeta_R(D+2\nu) \Gamma \left( \frac{D}{2} + \nu \right) \left( \frac{z}{L} \right)^{D+2\nu}, \right) \label{5.7}$$

for $M = 0, 1, \ldots, D - 1$ and $\zeta_R(z)$ being the Riemann zeta function. The corresponding formula for the component $\langle T^D_D \rangle(0)$ differs from (5.7) by an additional coefficient $-D/2\nu$ in the second term on the right of this formula. This directly follows from the continuity equation (2.27). Hence, the second term on the right of formula (5.2) vanishes at the AdS boundary as $z^{D+2\nu}$. For both minimally and conformally coupled cases the energy density corresponding to this term is positive. Note that in this limit the vacuum stresses along compactified and uncompactified directions are isotropic. Of course, we could expect the vanishing of the second term on the right hand side of Eq. (5.2) when $z/L \rightarrow 0$, as this corresponds to the decompactification limit for the internal space. For $z \gg L$, by using the asymptotic formula for the hypergeometric function for large values of the argument, to the leading order one finds

$$\langle T^N_M \rangle(0) \approx \langle T^N_M \rangle_{AdS_{D+1}} - \delta_M^N \sum_{k_D^2} \frac{\zeta_R(D+1)}{\pi^2 \Gamma(D+1/2)} \right| \left. \frac{z}{L} \right| \left. \frac{k_D^2}{L} \right|^{D+1}, \right) \label{5.8}$$

for $M \neq D - 1$. The corresponding formula for $D-1$-component differs from (5.8) by an additional coefficient $-D$ in the second term on the right. In particular, from here it follows that the energy density corresponding to the second term on the right of Eq. (5.2) is negative near the AdS horizon and diverges as $z^{D+1}$. The same is the case for $\langle T^0_0 \rangle(0)$ as the energy density corresponding to the first term does not depend on $z$ and the total energy density is dominated by the second term. The limit $z \gg L$ is realized, in particular, when $k_D \rightarrow 0$ for fixed $x$ and $L$. In this case the first term on the right of Eq. (5.8) vanishes and from this formula we obtain the standard result for the Casimir energy-momentum tensor in $R^{(D-1,1)} \times S^1$. Combining the asymptotic formulae (5.7) and (5.8), we see that for a scalar field with $\zeta < \zeta^{(+)}_D$ (in particular, this is the case for minimally and conformally coupled scalars) the energy density corresponding to the second term on the right of formula (5.2) tends to zero for small values of the ratio $z/L$ being positive and tends to $-\infty$ for large values of this ratio. Hence, it has a maximum at some intermediate value of $z/L$. As the first term on the right of Eq. (5.2) is constant, the same is true for the boundary-free total energy density.

For the internal space $S^1$ the brane induced VEV of the energy-momentum tensor and the vacuum interaction forces between the branes are obtained from general formulae given in previous sections by the replacements

$$\sum_{\beta} |\psi_\beta(X)|^2 \rightarrow \frac{2}{L} \sum_{\beta = 0}^\infty \lambda_\beta \rightarrow \frac{2\pi}{L} |\beta|, \quad D_1 \rightarrow D - 1, \right) \label{5.9}$$

where the prime means that the summand $\beta = 0$ should be taken with the weight $1/2$. From the general analysis given above it follows that under the conditions $z \gg L, z_a$, the main contribution into the part of the vacuum energy-momentum tensor induced by a single brane comes from the zero mode and this part behaves as $z^{1-2\nu}$. Comparing with (5.8), we see that for the geometry of a single brane, near the AdS horizon the total vacuum energy-momentum tensor is dominated by the second term on the right of Eq. (5.2). Near the AdS boundary, $z \ll L, z_a$, the single brane
induced term in the vacuum energy-momentum tensor behaves like \( z^{D+2\nu} \) (see Eq. (2.39)). By taking into account the estimate (5.7), we see that in this case the vacuum energy-momentum tensor is dominated by the first term on the right of (5.2). For the points near the brane the main contribution into the energy-momentum tensor comes from the brane induced term. As an illustration, in figure 1 we have plotted single brane induced VEVs for the vacuum energy density and \( D^{-}\)-stress as functions on \( z/z_a \) for the internal space with \( L/z_a = 1 \) (left panel) and \( L/z_a = 2 \) (right panel) in the case of \( D = 5 \) minimally coupled massless scalar field with the ratio of Robin coefficients \( B_a/\tilde{A}_a = 0.15 \). In this case for large values \( z/z_a \) the brane induced part of the energy-momentum tensor behaves as \( (z/z_a)^{-4} \). For small values of this ratio one has the behavior \( (z/z_a)^{10} \).

In this case for large values \( z/z_a \) the brane induced part of the energy-momentum tensor behaves as \( (z/z_a)^{-4} \). For small values of this ratio one has the behavior \( (z/z_a)^{10} \).

Figure 1: Brane induced vacuum energy density \( \langle T_0^0 \rangle^{(a)}/k_{D+1}^D \) and stress \( \langle T_D^D \rangle^{(a)}/k_{D+1}^D \) in units of \( k_{D+1}^D \) as functions on \( z/z_a \) for a minimally coupled massless scalar in \( D = 5 \). The left panel is for \( L/z_a = 1 \) and the right one is for \( L/z_a = 2 \).

In figure 2 we present the vacuum interaction forces in the geometry of two branes as functions on the size of the internal space and interbrane distance for a \( D = 5 \) minimally coupled massless scalar field with the Robin coefficients \( B_a = 0 \) and \( B_b/\tilde{A}_b = 0.15 \). In this example the effective pressures \( p^{(j)}_{(\text{int})} \) are positive for small interbrane distances and are negative for large distances leading to the repulsive and attractive interaction forces respectively.

From the point of view of embedding the Randall-Sundrum model into the string theory and also in the discussions of the holographic principle, the case of the internal space \( \Sigma = S^{D_2} \) is of special interest. The corresponding eigenfunctions \( \psi_\beta(X) \) are expressed in terms of spherical harmonics of degree \( l \), \( l = 0,1,2,\ldots \). For the internal space with radius \( R_0 \) the VEVs of the energy-momentum tensor and the vacuum interaction forces are obtained from the general formulae in previous sections by the replacements

\[
\sum_\beta |\psi_\beta(X)|^2 \rightarrow \frac{\Gamma(D_2+1)}{2\pi^{D_2+1} R_0^{D_2}} \sum_{l=0}^{\infty} (2l + D_2 - 1) \frac{\Gamma(l + D_2 - 1)}{l!\Gamma(D_2)}, \quad (5.10)
\]

\[
\lambda_\beta \rightarrow \frac{1}{R_0} \sqrt{l(l + D_2 - 1) + \zeta D_2(D_2 - 1)}, \quad (5.11)
\]

with the factor under the summation sign on the right in Eq. (5.10) being the degeneracy of the angular mode with a given \( l \).
Figure 2: The $D = 5$ vacuum effective pressures $p^{(j)}_{\text{int}}/k^{D+1}_D$, $j = a, b$, in units of $k^{D+1}_D$ for a massless minimally coupled scalar field as functions on $L/z_b$ and $z_a/z_b$. The values for the Robin coefficients are $B_a = 0$ and $B_b/\tilde{A}_b = 0.15$. Left panel corresponds to $j = a$ and right panel corresponds to $j = b$.

6 Conclusion

In this paper we continue the investigation of local quantum effects in higher dimensional brane models on the bulk of topology $AdS_{D_1+1} \times \Sigma$ with a warped internal subspace $\Sigma$ and the line element (2.1). The case of bulk scalar field with general curvature coupling parameter and satisfying Robin boundary conditions on two codimension one branes is considered. In section 2 the model with a single brane is discussed. By using the Wightman function and the VEV of the field square from Ref. [15], in the general case of the extra space we derived formulae for the VEV of the energy-momentum tensor in both regions on the right and on the left from the brane, given by expressions (2.13) and (2.23) respectively. Unlike to the case of purely AdS bulk, here the VEVs in addition to the distance from the brane depend also on the position of the brane in the bulk. In the limit when the AdS curvature radius tends to infinity we derive the formula for the vacuum energy-momentum tensor for parallel plates on the background spacetime with topology $R^{(D_1,1)} \times \Sigma$. In this limit for a homogeneous internal space $D_2$-component of the brane induced part in the VEV of the energy-momentum tensor vanishes. Further we have investigated various limiting cases when the expression for the brane induced VEV is simplified. For the points on the brane the vacuum energy-momentum tensor diverges. The leading term in the corresponding asymptotic expansion near the brane is given by formula (2.33). Near the brane the total vacuum energy-momentum tensor is dominated by the brane induced part and has opposite signs for Dirichlet and non-Dirichlet boundary conditions. Near the brane $D_2$- and $D_1$-components of this tensor have opposite signs in the regions $y < a$ and $y > a$. For $\zeta > \zeta_{D_1}$ ($\zeta < \zeta_{D_1}$) the energy density is positive (negative) for Dirichlet boundary condition and is negative (positive) for non-Dirichlet boundary condition. For large distances from the brane in the region $z > z_a$ the contribution of a given mode along $\Sigma$ with nonzero KK mass is exponentially suppressed by the factor $e^{-2\lambda_D z}$. For the zero mode the brane induced VEV near the AdS horizon behaves as $z^{D_2-2\nu}$. In the purely AdS bulk ($D_2 = 0$) this VEV vanishes on the horizon for $\nu > 0$. For an internal spaces with $D_2 > 2\nu$ the VEV diverges on the horizon. The VEV integrated over the internal space (see Eq. (2.20)) vanishes on the AdS horizon for all values $D_2$ due to the additional warp factor coming from the volume element. For the points
near the AdS boundary, the brane induced VEV vanishes as $z^{D+2\nu}$ for diagonal components and as $z^{D+2\nu+2}$ for the $D$-component. For small values of the length scale for the internal space, the contribution of nonzero KK masses is exponentially suppressed and the main contribution into the brane induced energy-momentum tensor comes from the zero mode. In the opposite limit, when the length scale of the internal space is large, to the leading order the vacuum energy-momentum tensor reduces to the corresponding result for a brane in the bulk $AdS_{D+1}$ given in Ref. [17]. For strong gravitational fields corresponding to small values of the AdS curvature radius, the contribution from nonzero KK modes along $\Sigma$ is exponentially suppressed by the factor $e^{-2\lambda|z-z_a|}$. For the zero KK mode the components of the brane induced vacuum energy-momentum tensor behave like $k_D^{D+1} e^{DkDy} \exp[(D_1 + 2\nu)k_D(y-a)]$ in the region $y < a$ and like $k_D^{D+1} e^{DkDy} \exp[2\nu k_D(a-y)]$ in the region $y > a$. The corresponding quantities integrated over the internal space contain additional factor $e^{-DkDy}$ coming from the volume element and are exponentially small in both regions. For fixed values of the other parameters, the brane induced VEV in the region $z > z_a$ vanishes as $z_a^{2\nu}$ when the brane position tends to the AdS boundary. When the brane position tends to the AdS horizon, $z_a \to \infty$, for massive KK modes along $\Sigma$ the VEV of the energy-momentum tensor in the region $z < z_a$ is suppressed by the factor $e^{-2z_a\lambda\beta}$. For the zero mode in the same limit the suppression is power-law with respect to $z_a$.

The geometry of two branes we consider in section 3. The VEVs in the region between the branes is presented in the form (3.3) with separated boundary-free, single branes and interference parts. The latter is given by formula (3.6) and is finite everywhere including the points on the branes. The surface divergences are contained in the single brane parts only. We have explicitly checked that the both single brane and interference parts separately satisfy the continuity equation and are traceless for a conformally coupled massless scalar. The possible trace anomalies are contained in the boundary-free parts. In the limit $k_D \to 0$ we derive the corresponding results for two parallel Robin plates in the bulk $R^{(D_1+1)} \times \Sigma$. For small values of the length scale of the internal space corresponding to large KK masses, the interference part in the VEV of the energy-momentum tensor is estimated by formula (3.8) and is exponentially suppressed. For large interbrane distances compared with the AdS curvature radius $k_D^{-1}$ we have approximate formula (3.10) for the nonzero KK modes along $\Sigma$. This limit is realized in braneworld scenarios for the solution of the hierarchy problem. The interference part vanishes as $z_a^{2\nu}$ when the left brane tends to the AdS boundary. Under the condition $z < z_a$ an additional suppression factor appears in the form $(z/z_b)^{D_1}$ for $D_1$-component and in the form $(z/z_b)^{D_1+2\alpha_1}$ for the other components, where $\alpha_1 = \min(1,\nu)$. In a higher dimensional generalization of the Randall-Sundrum braneworld based on the bulk $AdS_{D_1+1} \times \Sigma$ with orbifolded $y$-direction the VEV of the energy-momentum tensor is obtained from the formulae in this paper with an additional factor $1/2$. For untwisted scalar field the corresponding Robin coefficients are related to the surface mass parameters by formulae (3.13). For twisted scalar Dirichlet boundary conditions are obtained. Note that in the present paper we consider the VEV of the bulk energy-momentum tensor. On manifolds with boundaries in addition to this part, the energy-momentum tensor contains a contribution located on on the boundary (for the expression of the surface energy-momentum tensor in the case of arbitrary bulk and boundary geometries see Ref. [24]). As it has been discussed in Refs. [21, 24, 25, 26, 27, 28], the surface part of the energy-momentum tensor is essential in considerations of the relation between local and global characteristics in the Casimir effect. The vacuum expectation value of the surface energy-momentum tensor for the geometry of two parallel branes in $AdS_{D_1+1}$ bulk is evaluated in Ref. [29]. In particular, it has been shown that for large distances between the branes the induced surface densities give rise to an exponentially suppressed cosmological constant on the brane. The investigation of the surface densities and induced cosmological constant on the branes for the bulk $AdS_{D_1+1} \times \Sigma$ will be reported in the forthcoming paper.

The vacuum effective pressure acting on the branes is determined by the $D$-component of
the energy-momentum tensor and can be separated into single brane and second brane induced parts. The first one is divergent and needs additional renormalization. The corresponding procedure lies on the same line as the evaluation of the surface energy-momentum tensor and will be discussed in the forthcoming paper. Here we concentrate on the second brane induced part which is finite for all nonzero interbrane distances and is not changed by the regularization and renormalization procedure. For the brane at \( z = z_j \) this term is determined by formula (4.3). For Dirichlet scalar the corresponding vacuum forces are attractive for all interbrane distances. Taking the limit \( k_D \rightarrow 0 \) we obtained the result for the bulk \( R(D_1 - 1, 1) \times \Sigma \), given by the right hand side of Eq. (4.5). In this case, for a homogeneous internal space the interaction forces are the same even in the case of different Robin coefficients for separate branes. For the modes along \( \Sigma \) with large KK masses, the interaction forces are exponentially small. In particular, for sufficiently small length scale of the internal space this is the case for all nonzero KK modes and the main contribution to the interaction forces comes from the zero mode. For small interbrane distances, to the leading order the interaction forces are given by formula (4.8). In this limit the contribution of the interaction term dominates the single brane parts, and the same is the case for the zero mode. In the same limit the corresponding force acting on the right brane behaves as \( e^{-2\lambda_\beta z_b} z_b^{D_1}/2 \) for the nonzero KK mode and like \( z_b^{-D_1-2\nu} \) for the zero mode. In the same limit the corresponding force acting on the right brane behaves as \( z_b^{D_2+D_1/2+1} e^{-2\lambda_\beta z_b} \) for the nonzero KK mode and like \( z_b^{D_2-2\nu} \) for the zero mode. In the limit when the left brane tends to the AdS boundary the contribution of a given KK mode into the vacuum interaction force vanishes as \( z_a^{D+2\nu} \) and as \( z_b^{2\nu} \) for the left and right branes, respectively. For small values of the AdS curvature radius corresponding to strong gravitational fields, for nonzero KK modes under the conditions \( \lambda_\beta z_a \gg 1 \) and \( \lambda_\beta (z_b - z_a) \gg 1 \), the contribution to the interaction forces is suppressed by the factor \( e^{-2\lambda_\beta (z_b - z_a)} \). For the zero KK mode, the corresponding interaction forces integrated over the internal space behave as \( k_D^{D_1+1} \exp[(D_1 \delta_j^a + 2\nu) k_D (a - b)] \) for the brane at \( y = j \) and are exponentially small. In the model without the internal space we have similar behavior with \( \nu \) replaced by \( \nu_1 \) and for a scalar field with \( \zeta < \zeta_{D+D_1+1} \) the suppression is relatively weaker.

As an example of application of the general results, in section 4, we consider a special case of the internal space with \( \Sigma = S^1 \). By using the Abel-Plana summation formula, the boundary-free part of the vacuum energy-momentum tensor is presented in the form (5.2), where the first term on the right is the corresponding tensor in \( AdS_{D+1} \) bulk without boundaries. The second term is induced by the compactness of the \( X \) direction and behaves as \( z^{D+2\nu} \) for \( z \ll L \) and like \( z^{D+1} \) for \( z \gg L \). For minimally and conformally coupled scalar the corresponding energy density is positive in the first case and negative in the second case and has a maximum at some intermediate value for \( z/L \). The total energy-momentum tensor is dominated by the first term on the right of Eq. (5.2) near the AdS boundary and by the second term near the horizon. For the points near the branes the brane induced parts dominate. In the case of internal space \( \Sigma = S^1 \) the latter are obtained from the general formulae by replacements (5.9). We also comment on the generalization to the case of the internal space \( \Sigma = S^D_2 \).

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