Quantum informational properties of the Landau–Streater channel

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We study the Landau–Streater quantum channel \( \Phi : \mathcal{B}(\mathcal{H}_d) \rightarrow \mathcal{B}(\mathcal{H}_d) \), whose Kraus operators are proportional to the irreducible unitary representation of \( SU(2) \) generators of dimension \( d \). We establish \( SU(2) \) covariance for all \( d \) and \( U(3) \) covariance for \( d = 3 \). Using the theory of angular momentum, we explicitly find the spectrum and the minimal output entropy of \( \Phi \). Negative eigenvalues in the spectrum of \( \Phi \) indicate that the channel cannot be obtained as a result of Hermitian Markovian quantum dynamics. Degradability and antidegradability of the Landau–Streater channel is fully analyzed. We calculate classical and entanglement-assisted capacities of \( \Phi \). Quantum capacity of \( \Phi \) vanishes if \( d = 2, 3 \) and is strictly positive if \( d \geq 4 \). We show that the channel \( \Phi \otimes \Phi \) does not annihilate entanglement and preserves entanglement of some states with Schmidt rank 2 if \( d \geq 3 \).

I. INTRODUCTION

The state of a finite dimensional quantum system is described by the density operator \( \varrho \in \mathcal{B}(\mathcal{H}_d) \) acting on the Hilbert space \( \mathcal{H}_d \), \( d = \dim \mathcal{H}_d \). The density operator is Hermitian, positive semidefinite, and has unit trace.

In the theory of open quantum systems, the most general form of the density operator transformation due to its own evolution and interaction with some environment (initially uncorrelated from the system) is given by a quantum channel \( \Phi : \mathcal{B}(\mathcal{H}_d) \rightarrow \mathcal{B}(\mathcal{H}_d) \), which is a completely positive and trace preserving linear map (Ref.1 section 6.3). In what follows, we consider the case when the system dimension does not change, i.e., \( d_1 = d_2 = d \).

There is a special class of unital quantum channels, which preserve the maximally mixed state \( \frac{1}{d} I \), where \( I \) is the identity operator, i.e. \( \Phi[\frac{1}{d} I] = \frac{1}{d} I \). The seminal result of Landau and Streater\(^{2} \) is that all unital channels \( \Phi : \mathcal{B}(\mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_2) \) are random unitary, i.e. \( \Phi[\varrho] = \sum_i p_i U_i \varrho U_i^\dagger \) for some probability distribution \( \{p_i\} \) and unitary operators \( U_i \) acting on \( \mathcal{H}_2 \); whereas for larger dimensions this is not the case\(^{2} \). Moreover, Landau and Streater\(^{2} \) provided an example of quantum channel \( \Phi : \mathcal{B}(\mathcal{H}_d) \rightarrow \mathcal{B}(\mathcal{H}_d) \), which is unital but not random unitary for all \( d \geq 3 \).

The main goal of this paper is to explore quantum informational properties of the Landau–Streater map

\[
\Phi[\varrho] = \frac{1}{j(j+1)} (J_x \rho J_x + J_y \rho J_y + J_z \rho J_z)
\]

(1)
defined through the \( SU(2) \) generators \( J_x, J_y, J_z \) acting on a \( (2j+1) \)-dimensional Hilbert space \( \mathcal{H}_{2j+1} \). Physically, this space corresponds to the state space of a spin-\( j \) particle. Hermitian operators \( J_x, J_y, J_z \) are spin projection operators onto axes \( x, y, z \), respectively. Hereafter, we will use indices \( x, y, z \) and 1, 2, 3 interchangeably. The map \( \Phi \) is completely positive as it has diagonal sum (Kraus) representation (Ref.4, Corollary 6.13), and is trace preserving as \( \sum_{k=1}^{3} J_k^2 = j(j+1)I \) (Ref.4 section 6.1.2, formula (5)). The latter formula is also responsible for unitality of the map \( \Phi \).

If \( j = \frac{1}{2} \), then \( J_k = \frac{1}{2} \sigma_k \), where \( \{\sigma_k\}_{k=1}^{3} \) is the conventional set of Pauli operators. In this case \( d = 2 \) and \( \Phi \) is random unitary\(^{2} \). Such a channel \( \Phi \) is also referred to as the best physical approximation of the universal NOT gate for qubits\(^{2} \).
If \( j \geq 1 \), then the map (1) is an extremal channel in the set of all channels, and therefore it cannot be random unitary. Also, in contrast to the case \( j = \frac{1}{2} \), the channel (1) differs from the spin polarization-scaling channels if \( j \geq 1 \).

Since the Landau–Streater channel (1) is essentially unexplored, the goal of this paper is to fill the gap in both the fundamental properties (such as covariance and spectrum of the output states) and more specific quantum informational properties (such as classical and quantum capacities, entanglement dynamics).

The paper is organized as follows.

In section II, we study covariance properties of the Landau–Streater channel. In section III A we explicitly find the spectrum of \( \Phi \) for all \( j \). In section III B we analyze the spectrum of the output operator \( \Phi[X] \) and reveal its peculiarities in the case \( j = 1 \). Such peculiarities are attributed to the fact that the Landau–Streater channel for \( j = 1 \) reduces to the Werner–Holevo channel. In section III C we explicitly find the maximal \( p \)-norm and the minimal output entropy of the general Landau–Streater channel. In section IV, physical realization of the Landau–Streater channel and its complementary version is discussed. In section V, different capacities of the Landau–Streater channel are evaluated. In section VI, we examine the entanglement annihilation property of the channel \( \Phi \otimes \Phi \). In section VII, brief conclusions and outlook are presented.

II. COVARIANCE

Following Refs. 7, consider a group \( G \) and a unitary representation \( g \rightarrow U_g \), \( g \in G \), in \( \mathcal{H}_d \). The channel \( \Phi : \mathcal{B}(\mathcal{H}_d) \rightarrow \mathcal{B}(\mathcal{H}_d) \) is called covariant with respect to representation \( \Phi \) if there exists a unitary representation \( g \rightarrow V_g \), \( g \in G \), in \( \mathcal{H}_d \) such that

\[
\Phi[U_gXU_g^\dagger] = V_g\Phi[X]V_g^\dagger
\]

for all \( g \in G \) and \( X \in \mathcal{B}(\mathcal{H}_d) \). Covariance of a channel has many implications, e.g. the strong converse property of the entanglement-assisted classical capacity. Many other properties and the structure of irreducibly covariant quantum channels are reviewed in Ref. 10.

We start with the simple observation that the Landau–Streater channel (1), by construction, is endowed with the \( SU(2) \) covariance.

**Proposition 1** The Landau–Streater channel \( \Phi : \mathcal{B}(\mathcal{H}_{2j+1}) \rightarrow \mathcal{B}(\mathcal{H}_{2j+1}) \) is covariant with respect to the unitary representation of \( SU(2) \) for all spins \( j \), namely, \( \Phi[U_gXU_g^\dagger] = U_g\Phi[X]U_g^\dagger \) for all \( g \in SU(2) \) and \( X \in \mathcal{B}(\mathcal{H}_{2j+1}) \).

**Proof.** Up to an irrelevant phase, any \( U_g, g \in SU(2) \) can be expressed through the \( SU(2) \) generators \( \{ J_\alpha \}_{\alpha \in \{ x, y, z \}} \) as follows:

\[
U_g = \exp \left( -i\theta \sum_\alpha n_\alpha J_\alpha \right),
\]

where \( n = (n_x, n_y, n_z) \) is a unit vector in \( \mathbb{R}^3 \), which defines the rotation axis, and \( \theta \in \mathbb{R} \) is the rotation angle. The operators \( J_x, J_y, J_z \) satisfy the commutation relation \( [J_\alpha, J_\beta] = i\sum_\gamma e_{\alpha\beta\gamma} J_\gamma \), where \( e_{\alpha\beta\gamma} \) is the Levi-Civita symbol (Ref. 2, section 2.1.2, formula (7)). Using such a commutation relation, it is not hard to see that (Ref. 2, section 3.1.3, formula (11); section 3.1.6, item (a); section 1.4.5, formula (33))

\[
U_g^\dagger J_\alpha U_g = \sum_\beta Q_{\alpha\beta} J_\beta,
\]

where \( Q_{\alpha\beta} = e_{\alpha\beta\gamma} n_\gamma \).
where \((Q_{\alpha\beta})\) is the orthogonal matrix describing the rotation in \(\mathbb{R}^3\) around the axis \(n\) by angle \(\theta\) (Ref.\(^3\), section 1.4.2). Finally, we get

\[
\Phi[U_g X U_g^\dagger] = \frac{1}{j(j+1)} \sum_{\alpha} J_{\alpha} U_g X U_g^\dagger J_{\alpha}
\]

\[
= \frac{1}{j(j+1)} \sum_{\alpha} U_g (U_g^\dagger J_{\alpha} U_g) X (U_g^\dagger J_{\alpha} U_g) U_g^\dagger
\]

\[
= \frac{1}{j(j+1)} \sum_{\alpha,\beta,\gamma} Q_{\alpha\beta} Q_{\alpha\gamma} U_g J_{\beta} X J_{\gamma} U_g^\dagger
\]

\[
= \frac{1}{j(j+1)} \sum_{\beta,\gamma} \delta_{\beta\gamma} U_g J_{\beta} X J_{\gamma} U_g^\dagger
\]

\[
= U_g \Phi[X] U_g^\dagger,
\]

where we have taken into account that \(\sum_{\alpha} Q_{\alpha\beta} Q_{\alpha\gamma} = \delta_{\beta\gamma}\), the Kronecker delta. ■

Since the Landau–Streater channel is extreme in the set of all channels\(^2\) it follows that it is also an extreme point of \(SU(2)\) irreducibly covariant channels (abbreviated as an EPOSIC channel). The general properties of an EPOSIC channel such as the Kraus representation, the Choi matrix, and the dual channel are reviewed in Ref.\(^11\).

It turns out that in the case \(j = 1\) the Landau–Streater channel is not only \(SU(2)\) covariant, but also globally unitarily covariant. It means that in dimension \(d = 3\) the channel \(\Phi\) possesses \(U(3)\) covariance.

\section*{Proposition 2}

In the case \(j = 1\), the Landau–Streater channel is globally unitarily covariant, namely, for all \(X \in \mathcal{B}(\mathcal{H}_3)\) and unitary operators \(U\) on \(\mathcal{H}_3\) the following holds

\[
\Phi[UXU^\dagger] = V \Phi[X] V^\dagger,
\]

where the unitary operator \(V\) is expressed through \(U\) in the orthonormal basis of eigenvectors of \(J_z\) via formula

\[
U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}, \quad V = \begin{pmatrix} u_{33} & -u_{32} & u_{31} \\ -u_{23} & u_{22} & -u_{21} \\ u_{13} & -u_{12} & u_{11} \end{pmatrix}.
\]

\section*{Proof.}

Note that \(V\) is unitary if \(U\) is unitary. In the basis of eigenvectors of \(J_z\), the operators \(\{J_{\alpha}\}_{\alpha=x,y,z}\) have the following form if \(j = 1\):

\[
J_x = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

Substituting these operators in equation \((\text{1})\), the direct calculation justifies the validity of formulas \((\text{6}), (\text{7})\). ■

The global unitary covariance is known to be a peculiar property of the tracing, transposition, and identity maps. This feature allowed one to find specific results for the Werner–Holevo channel\(^{12}\) and transpose-depolarizing channels\(^{13}\) as well as to prove additivity of classical capacity for depolarizing quantum channels\(^{14}\). We have just found out that the Landau–Streater map for \(j = 1\) is globally unitarily covariant too. However, as we show below, in the case \(j > 1\) the Landau–Streater channel loses the property of global unitary covariance.

\section*{Proposition 3}

The Landau–Streater channel is not globally unitarily covariant if \(j > 1\).
Proof. We prove the statement by constructing a counterexample. Suppose the Landau–Streater channel \( \Phi : B(\mathcal{H}_{2j+1}) \mapsto B(\mathcal{H}_{2j+1}) \), \( j > 1 \), is covariant with respect to representation of \( U(2j+1) \). Then for any unitary operator \( U \) acting on \( \mathcal{H}_{2j+1} \), there exists a unitary operator \( V \) such that

\[
\Phi[U \rho U^\dagger] = V \Phi[\rho] V^\dagger
\]

holds true for all density operators \( \rho \). This implies that the output density operators \( \Phi[U \rho U^\dagger] \) and \( \Phi[\rho] \) have identical spectra.

Consider eigenvectors of the spin projection onto z-axis (Ref. 3, section 6.1.2, formula(5)):

\[
J_z |j, m \rangle = m |j, m \rangle, \quad m = j, j - 1, \ldots, -j.
\]

Let \( \rho = |j, j \rangle \langle j, j | \) and

\[
U = |j, j - 1 \rangle \langle j, j | + |j, j \rangle \langle j, j - 1 | + \sum_{k=-j}^{j-2} |j, k \rangle \langle j, k |,
\]

then \( \Phi[|j, j - 1 \rangle \langle j, j - 1 |] \) and \( \Phi[|j, j \rangle \langle j, j |] \) must have the same spectra.

On the other hand, action of the Landau–Streater map on the states \( |j, m \rangle \langle j, m | \) with definite spin projection \( m \) onto z-axis can be expressed explicitly by introducing auxiliary operators \( J_{\pm} = J_x \pm i J_y \) satisfying (Ref. 3, section 2.3.3, formula (7); section 6.1.2, formula (13))

\[
J_{\pm} |j, m \rangle = \sqrt{(j \pm m)(j \mp m + 1)} |j, m \pm 1 \rangle.
\]

Since \( \Phi[X] = [j(j + 1)]^{-1} \left( \frac{1}{2} J_- X J_+ + \frac{1}{2} J_+ X J_- + J_z X J_z \right) \), we get

\[
\Phi[|j, m \rangle \langle j, m |] = \frac{1}{j(j + 1)} \left[ \frac{1}{2} \left( j(j + 1) - m(m - 1) \right) |j, m \rangle \langle j, m - 1 | + \frac{1}{2} \left( j(j + 1) - m(m + 1) \right) |j, m \rangle \langle j, m + 1 | + m^2 |j, m \rangle \langle j, m | \right],
\]

from which we make conclusion about the spectrum of the output state \( \Phi[|j, m \rangle \langle j, m |] \):

\[
\text{Spec} (\Phi[|j, m \rangle \langle j, m |]) = \left\{ \frac{j(j + 1) - m(m + 1)}{2j(j + 1)}, \frac{j(j + 1) - m(m - 1)}{2j(j + 1)}, \frac{m^2}{j(j + 1)}, 0, 0, \ldots \right\}.
\]

If \( j > 1 \), then \( \text{Spec} (\Phi[|j, j \rangle \langle j, j |]) \neq \text{Spec} (\Phi[|j, j - 1 \rangle \langle j, j - 1 |]) \). This contradiction concludes the proof.

III. SPECTRAL PROPERTIES

A. Spectrum of the map

The Landau–Streater channel \( \Phi \) is Hermitian as it coincides with its dual \( \Phi^\dagger \), therefore its spectrum \( \{ \lambda_k \}_{k=0}^{(2j+1)^2-1} \) is real. Hermitian eigenoperators \( X_k \) satisfy \( \Phi[X_k] = \lambda_k X_k \). Due to unitarity of \( \Phi \), the identity operator \( I \) is the eigenoperator, so we can fix the corresponding eigenvalue \( \lambda_0 = 1 \) for all \( j \). By determinant \( \det \Phi \) of the channel \( \Phi \) we will understand the product of its eigenvalues \( \prod_k \lambda_k \).

If \( j = \frac{1}{2} \), then \( J_x, J_y, J_z \) are eigenoperators of \( \Phi \) and \( \lambda_1 = \lambda_2 = \lambda_3 = -\frac{1}{2} \). In this case, \( \det \Phi = \frac{-1}{2^4} < 0 \), so the channel \( \Phi \) is not infinitesimally divisible\( ^{15} \) and cannot be obtained as a result of Markovian evolution, although it can be realized physically, e.g., via collision models\( ^{16} \).
If \( j = 1 \), then \( J_x, J_y, J_z \) are eigenoperators of \( \Phi \) with corresponding eigenvalues \( \lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2} \). Five more eigenoperators have the form \( 3 \left( \sum_{n} n_n^{(k)} J_n \right)^2 - 2I \), \( n^{(k)} \in \mathbb{R}^3 \), \( k = 1, \ldots, 5 \) (Ref.\textsuperscript{2}, formula (8) and text after formula (36)) and correspond to eigenvalues \( \lambda_4 = \ldots = \lambda_8 = -\frac{1}{2} \). Similarly, \( \det \Phi < 0 \), so such a channel cannot be a result of Markovian evolution.

In what follows, we find spectrum of the Landau–Streater map \( \Phi : \mathcal{B}(\mathcal{H}_{2j+1}) \rightarrow \mathcal{B}(\mathcal{H}_{2j+1}) \) for an arbitrary integer or half-integer \( j \). As we show, the eigenoperators of \( \Phi \) are particularly related with the irreducible tensor operator \( T_{LM}^{(j)} \), for the \( SU(2) \) group, which is also known as the polarization operator (Ref.\textsuperscript{2}, section 2.4.2, formula (6); section 8.4.3, formula (10)):

\[
T_{LM}^{(j)} = \sqrt{\frac{2L+1}{2j+1}} \sum_{m_1,m_2=-j}^{j} C_{j,m_1LM}^{m_2} |jm_2\rangle \langle jm_1|,
\]

where \( C_{j,m_1LM}^{m_2} \) is the conventional Clebsch–Gordan coefficient.

**Proposition 4** The spectrum of the Landau–Streater map \( \Phi : \mathcal{B}(\mathcal{H}_{2j+1}) \rightarrow \mathcal{B}(\mathcal{H}_{2j+1}) \) comprises \((2L+1)\)-fold degenerate eigenvalues

\[
\lambda_{L} = 1 - \frac{L(L+1)}{2j(j+1)}, \quad L = 0, \ldots, 2j.
\]

The corresponding eigenoperators are linearly independent operators of the form \( U_g T_{L_0}^{(j)} U_g^\dagger \), where the operators \( U_g \) belong to the unitary representation of the \( SU(2) \) group.

**Proof.** We start with the observation that \( T_{L_0}^{(j)} \) is the eigenoperator of \( \Phi \). To prove this fact we rewrite the Landau–Streater channel in the form \( \Phi[X] = [j(j+1)]^{-1} \left( \frac{1}{2} J_+ X J_- + \frac{1}{2} J_- X J_+ \right) + J_z X J_z \) and use the commutation relations \( [J_\pm,T_{L_0}^{(j)}] = \sqrt{L(L+1)} T_{L_\pm}^{(j)} \) and \( [J_z,T_{L_0}^{(j)}] = 0 \) (see Ref.\textsuperscript{2}, section 2.4.1, formula (1); section 2.3.3, formula (7); for the Clebsch-Gordan coefficients \( C_{L_\pm}^{L_\pm 1} = \mp \frac{1}{\sqrt{2}} \) and \( C_{L_0}^{L_0 0} = 0 \) see section 8.5.1, formula (8)). We get

\[
\Phi \left[ T_{L_0}^{(j)} \right] = \frac{1}{j(j+1)} \left[ \left( \frac{1}{2} J_+ J_+ + \frac{1}{2} J_- J_- + J_z J_z \right) T_{L_0}^{(j)} - \frac{\sqrt{L(L+1)}}{2} \left( J_+ T_{L_1}^{(j)} + J_- T_{L_1}^{(j)} \right) \right].
\]

The first expression in parentheses (\( \cdot \)) is \( j(j+1)L \), whereas the second expression in parentheses (\( \cdot \)) can be simplified because \( J_\pm = \mp \sqrt{\frac{2(j+j+1)(j-j+1)}{3}} T_{L_\pm}^{(j)} \) (Ref.\textsuperscript{2}, section 2.4.2, formula (10); section 2.3.3, formula (7)) and the product \( T_{L_1}^{(j)} T_{L_1}^{(j)} \) is known (Ref.\textsuperscript{2}, section 2.4.4, formula (16)):

\[
J_+ T_{L_1}^{(j)} + J_- T_{L_1}^{(j)} = -\sqrt{\frac{2j(j+1)(2j+1)}{3}} \left( T_{L_1}^{(j)} T_{L_1}^{(j)} - T_{L_1}^{(j)} T_{L_1}^{(j)} \right)
\]

\[
= -\sqrt{\frac{2j(j+1)(2j+1)}{3}} \sum_{L'} (1-2j)(2j+L') \sqrt{3(2L+1)} \left\{ \begin{array}{c} 1 \ L \ L' \\ j \ j \ j \end{array} \right\} \left( C_{L'1L_1}^{L_0} - C_{L'1L_1}^{L_0} \right) T_{L_0}^{(j)}.
\]

The Clebsch-Gordan coefficients \( C_{L'1L_1}^{L_0} \) and \( C_{L'1L_1}^{L_0} \) coincide if \( L' = L \) (Ref.\textsuperscript{2}, section 8.4.3, formula (11)) and vanish if \( L' < L-1 \) or \( L' > L+1 \) (Ref.\textsuperscript{2}, section 8.1.1, formula (1)), so the only contribution to \( \Phi \) makes \( L' = L \), when \( C_{L'1L_1}^{L_0} - C_{L'1L_1}^{L_0} = \sqrt{2} \) (Ref.\textsuperscript{2}, section 8.5.1, formula (8)). The Wigner 6j-symbol \( \left\{ \begin{array}{c} 1 \ L \ L \\ j \ j \ j \end{array} \right\} = \frac{1}{2} (1-2j+L+1) \sqrt{\frac{L(L+1)}{j(j+1)(2j+1)}} \).
Proof. In the basis of eigenvectors of \( \Phi \), the operators \( U_g T_{L_0}^{(j)} U_g^\dagger \) are eigenoperators of \( \Phi \). It is known that there are exactly \( 2L + 1 \) linear independent operators \( U_g T_{L_0}^{(j)} U_g^\dagger \) if \( U_g \) is a representation of the \( SU(2) \) group (see Ref.\(^\text{18} \), formula (11), where \( S_{L}^{(j)} \) is proportional to the operator \( T_{L_0}^{(j)} \), and Ref.\(^\text{17} \), text after formula (36)). Therefore, eigenvalues \( \lambda_L \) are \( (2L + 1) \)-fold degenerate. Since \( \sum_{L=0}^{2j} (2L + 1) = (2j + 1)^2 \), the given eigenvalues are the only ones and constitute the spectrum of \( \Phi \).

In the latter proposition, for the case \( L = 1 \) the generators \( J_x, J_y, J_z \) are exactly three linear independent eigenoperators of \( \Phi : B(H_{2j+1}) \mapsto B(H_{2j+1}) \) corresponding to the eigenvalue \( 1 - \frac{1}{(2j+1)} \).

It is not hard to see that if \( L = 2j \), then \( \lambda_L < 0 \). Negativity of the eigenvalue implies that the Landau–Streater channel cannot be obtained via positive divisible Hermitian evolution\(^\text{19} \) for any \( j \).

B. Spectrum of the output

Let us consider spectral properties of the output operator, i.e., the spectrum of \( \Phi[X] \), where \( X \) is a Hermitian input operator.

It is not hard to see that in the case \( j = \frac{1}{2} \) the spectrum of \( \Phi[X] \) is \( \{ \frac{1}{3}(x_1 + 2x_2), \frac{1}{3}(2x_1 + x_2) \} \). Provided the spectrum of the input density operator \( \rho \) is \( \{ x_1, x_2 \} \).

\[ \text{Corollary 1} \quad \text{The output purity and entropy of the Landau–Streater channel } \Phi : B(H_2) \mapsto B(H_2) \text{ for all pure spin-\( \frac{1}{2} \) input states } |\psi\rangle\langle\psi| \text{ are equal to } \frac{1}{2} \text{ and } \log 3 - \frac{1}{2}, \text{ respectively.} \]

The case \( j = 1 \) is more involved, but in this case the spectrum of the output also depends only on the spectrum of the input, as we show in the following proposition.

\[ \text{Proposition 5} \quad \text{Suppose a Hermitian operator } X \in B(H_3) \text{ with spectrum } \{ x_1, x_2, x_3 \}. \text{ The output operator } \Phi[X] \text{ of the Landau–Streater channel } \Phi : B(H_3) \mapsto B(H_3) \text{ has spectrum } \{ \frac{1}{2}(x_1 + x_2), \frac{1}{2}(x_1 + x_3), \frac{1}{2}(x_2 + x_3) \}. \]

\[ \text{Proof.} \quad \text{In the basis of eigenvectors of } J_z, \text{ the action of the Landau–Streater channel reads} \]

\[
\Phi \left[ \begin{array}{ccc}
X_{11} & X_{12} & X_{13} \\
X_{21} & X_{22} & X_{23} \\
X_{31} & X_{32} & X_{33}
\end{array} \right] = \frac{1}{2} \left[ \begin{array}{ccc}
X_{11} + X_{22} & X_{23} & -X_{13} \\
X_{32} & X_{11} + X_{33} & X_{12} \\
-X_{31} & X_{21} & X_{22} + X_{33}
\end{array} \right].
\]

Since the Landau–Streater channel \( \Phi : B(H_3) \mapsto B(H_3) \) is globally unitarily covariant by proposition\(^\text{2} \), the spectrum of the output operator \( \Phi[X] \) depends only on the spectrum of the input operator \( X \). To find the explicit relation between the spectra we consider the unitary operator \( U \) realizing the transition from the basis of eigenvectors of \( X \) to the basis of eigenvectors of \( J_z \). Then in the basis of eigenvectors of \( J_z \) we have \( UXU^\dagger = \text{diag}(x_1, x_2, x_3) \) and \( \Phi[UXU^\dagger] = \text{diag} \left( \frac{1}{2}(x_1 + x_2), \frac{1}{2}(x_1 + x_3), \frac{1}{2}(x_2 + x_3) \right) \). Thanks to the global unitary covariance, the latter diagonal matrix is exactly the spectrum of \( \Phi[X] \). ■

The spectral property of the Landau–Streater channel \( \Phi : B(H_3) \mapsto B(H_3) \) resembles that of the depolarizing channel, but the Landau–Streater channel is not depolarizing in the case \( j = 1 \). This peculiarity is ascribed to the close relation between the Landau–Streater channel and the Werner–Holevo channel \( \Phi_{\text{WH}} : B(H_d) \mapsto B(H_d) \) defined through transposition \( \top \) in some orthonormal basis via formula \( \Phi_{\text{WH}}[X] = \frac{1}{d-1} (\text{tr}[X] I - X^\top) \), Ref.\(^\text{2} \). It turns out that if \( d = 3 \) and transposition \( \top \) is performed in the basis of eigenstates
of \( J_z \), then the Landau–Streeter channel \( \Phi : \mathcal{B}(\mathcal{H}_3) \to \mathcal{B}(\mathcal{H}_3) \) is merely the Werner–Holevo channel concatenated with a unitary channel:

\[
\Phi[X] = \Phi_{\text{WH}}[W X W^\dagger] = \frac{1}{2} \left( \text{tr}[X | I - W X^\dagger W^\dagger] \right),
\]

where \( W = |1, 1\rangle \langle 1, -1| - |1, 0\rangle \langle 1, 0| + |1, -1\rangle \langle 1, 1| \) is a unitary operator.

Since the spectrum of pure states consists of 1 and zeros, we can make conclusions about the output purity \( \mu_{\text{out}} = \text{tr}[(\Phi[\rho])^2] \) and the output entropy \( S_{\text{out}} = -\text{tr}[\Phi[\rho] \log \Phi[\rho]] \) for pure input states in cases \( j = \frac{1}{2} \) and \( j = 1 \). Hereafter, log is understood as \( \log_2 \) if one measures the entropy and capacity in bits.

**Corollary 2** The output purity and entropy of the Landau–Streeter channel \( \Phi : \mathcal{B}(\mathcal{H}_3) \to \mathcal{B}(\mathcal{H}_3) \) for all pure spin-1 input states \( |\psi\rangle \langle \psi| \) are equal to \( \frac{1}{2} \) and 1, respectively.

In the case \( j > 1 \), the spectrum of \( \Phi[X] \) depends not only on the spectrum of \( X \) but also on the particular form of the operator \( X \). For instance, in the case \( j = \frac{3}{2} \), one can consider two different pure input states \( |\frac{3}{2}, \frac{1}{2}\rangle \langle \frac{3}{2}, \frac{1}{2}| \) and \( |\frac{3}{2}, \frac{1}{2}\rangle \langle \frac{3}{2}, \frac{1}{2}| \) with identical spectra \( \{1, 0, 0, 0\} \). By formula (14) \( \text{Spec}(\Phi[|\frac{3}{2}, \frac{1}{2}\rangle \langle \frac{3}{2}, \frac{1}{2}|]) = \{\frac{3}{2}, \frac{1}{2}\} \) and \( \text{Spec}(\Phi[|\frac{3}{2}, \frac{1}{2}\rangle \langle \frac{3}{2}, \frac{1}{2}|]) = \{\frac{3}{2}, \frac{1}{2}\} \), so spectra of output states may not coincide. Similarly to the case \( j = \frac{3}{2} \), for any \( j > 1 \) one can always take pure input states \( |j, j\rangle \langle j, j| \) and \( |j, j-1\rangle \langle j, j-1| \) and make sure that \( \text{Spec}(\Phi[|j, j\rangle \langle j, j|]) \neq \text{Spec}(\Phi[|j, j-1\rangle \langle j, j-1|]) \).

**C. The maximal \( p \)-norm and the minimal output entropy**

The maximal \( p \)-norm of a channel \( \Phi \) is defined by the formula

\[
\nu_p(\Phi) = \sup_{\rho} \{\|\Phi[\rho]\|_p\},
\]

where \( \|\Phi[\rho]\|_p = \left( \text{tr}[(\Phi[\rho])^p] \right)^{1/p} \) is the Schatten \( p \)-norm of \( \Phi[\rho] \). The maximal 2-norm is merely the square root of the maximal output purity. Before we proceed to the analysis of maximal \( p \)-norm and minimal output entropy of the Landau–Streeter channel, we prove auxiliary results following from the theory of angular momentum.

**Lemma 1** Let \( k \in \mathbb{R}^3 \) be a unit vector, \( |k| = \sqrt{k_1^2 + k_2^2 + k_3^2} = 1 \). The spectrum of operator \( \sum_{\alpha=1}^3 k_\alpha J_\alpha \) is \( \{m\}_{m=-j}^j \).

**Proof.** Physically, the operator \( \sum_{\alpha=1}^3 k_\alpha J_\alpha \) is the spin projection operator onto axis \( k \) and therefore has the same spectrum as any of operators \( J_x, J_y, J_z \). Mathematically, there exists a unitary operator \( U_g : \mathcal{B}(\mathcal{H}_{2j+1}) \to \mathcal{B}(\mathcal{H}_{2j+1}) \), \( g \in SU(2) \), such that \( U_g^\dagger J_\beta U_g = \sum_{\beta=1}^3 k_\beta J_\beta \), cf. formula (11) with \( k_\beta = Q_{3\beta} \), where \( Q \) is orthogonal. Hence, \( \text{Spec} \left( \sum_{\alpha=1}^3 k_\alpha J_\alpha \right) = \text{Spec}(J_z) = \{m\}_{m=-j}^j \).

The eigenvector \( |\psi_k\rangle \) of operator \( \sum_{\alpha=1}^3 k_\alpha J_\alpha \) corresponding to the maximal eigenvalue \( j \) will be referred to as a vector with the maximal spin polarization. Clearly, \( |\psi_k\rangle = U_g|j, j\rangle \), where \( U_g \) is the unitary operator used in the proof of Lemma 1.

**Lemma 2** Let \( k \in \mathbb{R}^3, |k| = 1 \). The maximum of expression \( \| \sum_{\alpha=1}^3 k_\alpha J_\alpha |\psi| \|^2 \) with respect to normalized vectors \( |\psi\rangle \in \mathcal{H}_{2j+1} \) equals \( j^2 \) and is attained at the state \( |\psi_k\rangle \) with the maximal spin polarization.

**Proof.** By Lemma 1 the spectrum of \( \left( \sum_{\alpha=1}^3 k_\alpha J_\alpha \right)^2 \) reads \( \{m^2\}_{m=-j}^j \). Therefore,

\[
\| \left( \sum_{\alpha=1}^3 k_\alpha J_\alpha \right) |\psi| \|^2 = |\langle \psi | \left( \sum_{\alpha=1}^3 k_\alpha J_\alpha \right)^2 |\psi \rangle | \leq j^2 |\langle \psi |\psi \rangle | = j^2 \text{ and }
\]

\[
\langle \psi_k | \left( \sum_{\alpha=1}^3 k_\alpha J_\alpha \right)^2 |\psi_k\rangle = j^2.
\]
Lemma 3 If \( j \geq 1 \), then \( \langle \psi | J_z | \psi \rangle^2 \leq 9 j^2 \frac{j^2 - \langle \psi | J_z^2 | \psi \rangle}{2j - 1} \) for all normalized vectors \( | \psi \rangle \in H_{2j+1} \).

Proof. By Lemma 1 the spectral decomposition of \( J_z^2 \) reads \( J_z^2 = j^2 (P_j + P_{-j}) + \sum_{m=-j+1}^{j-1} m^2 P_m \), where \( P_m = | j, m \rangle \langle j, m | \) and \( J_z | j, m \rangle = m | j, m \rangle \). The average value \( \langle J_z^2 \rangle = j^2 - \epsilon \leq p j^2 + (1 - p)(j - 1)^2 \), where \( p = \langle (P_j + P_{-j}) \rangle \). Therefore, \( 1 - p \leq \frac{\epsilon}{2j - 1} = \frac{j^2 - \langle J_z^2 \rangle}{2j - 1} \).

Let \( | \psi \rangle = c_j | j, j \rangle + c_{-j} | j, -j \rangle + \sum_{m=-j+1}^{j-1} c_m | j, m \rangle \). Note that \( x \langle j, j | J_z | j, j \rangle_x = 0 \), \( x \langle j, j | J_z | j, -j \rangle_x = 0 \), \( x \langle j, j | J_z | -j, j \rangle_x = 0 \) if \( j \geq 1 \), \( p = |c_j|^2 + |c_{-j}|^2 \), and \( 1 - p = \sum_{m=-j+1}^{j-1} |c_m|^2 \). We have

\[
\langle \psi | J_z | \psi \rangle = 2 \text{Re} \left( \sum_{m=-j+1}^{j-1} \overline{c_m} x \langle j, m | J_z (c_j | j, j \rangle_x + c_{-j} | j, -j \rangle_x) \right) + \left( \sum_{m=-j+1}^{j-1} \overline{c_m} x \langle j, m | J_z \sum_{m=-j+1}^{j-1} c_{m'} | j, m' \rangle_x \right) \\
\leq 2 \sqrt{p} \sqrt{1 - p} j + (1 - p) j \leq 3 \sqrt{1 - p} j. \quad (22)
\]

Noticing that \( -J_z \) has the same spectrum as \( J_z \) and arguing as above, we see that \( \langle \psi | J_z | \psi \rangle \geq 2 \sqrt{p} \sqrt{1 - p} (-j) + (1 - p) (-j) \geq -3 \sqrt{1 - p} j \). Thus, \( \langle \psi | J_z | \psi \rangle \leq 3 \sqrt{1 - p} j \). Squaring both sides of this inequality and recalling \( 1 - p \leq \frac{j^2 - \langle J_z^2 \rangle}{2j - 1} \), we get the statement of Proposition concludes the proof. \( \blacksquare \)

Lemma 3 shows that the average value \( \langle J_z \rangle \) cannot be large when \( \langle J_z^2 \rangle \) is close to its maximal value \( j^2 \). Lemma 3 obviously remains valid if one replaces \( J_z \) by \( J_y \).

Lemma 4 Let the vectors \( \mathbf{k}, \mathbf{l} \in \mathbb{R}^3 \) satisfy \( |\mathbf{k}|^2 + |\mathbf{l}|^2 = 1 \), then \( \left\| \sum_{n=1}^{3} (k_\alpha + il_\alpha) J_\alpha |\psi\rangle \right\|^2 \leq \max(j, j^2) \) for all normalized vectors \( |\psi\rangle \in H_{2j+1} \).

Proof. Suppose \( \mathbf{k} \) and \( \mathbf{l} \) are linearly dependent, i.e., \( \mathbf{k} = |\mathbf{k}| \mathbf{n} \) and \( \mathbf{l} = |\mathbf{l}| \mathbf{n} \) for some unit vector \( \mathbf{n} \in \mathbb{R}^3 \). Then \( \left\| \sum_{n=1}^{3} (k_\alpha + il_\alpha) J_\alpha |\psi\rangle \right\|^2 = (|\mathbf{k}|^2 + |\mathbf{l}|^2) \left\| \sum_{n=1}^{3} n_\alpha J_\alpha |\psi\rangle \right\|^2 \leq j^2 \) by Lemma 2.

Suppose \( \mathbf{k} \) and \( \mathbf{l} \) are linearly independent. Note that \( \left\| \sum_{n=1}^{3} (k_\alpha + il_\alpha) J_\alpha |\psi\rangle \right\|^2 = \langle \psi | F |\psi\rangle \), where

\[
F := \sum_{\alpha, \beta} (k_\alpha - il_\alpha) (k_\beta + il_\beta) J_\alpha J_\beta = \left( \sum_{\alpha} k_\alpha J_\alpha \right)^2 + \left( \sum_{\alpha} l_\alpha J_\alpha \right)^2 + \sum_\gamma \mathbf{l} \times \mathbf{k} J_\gamma J_\gamma. \quad (23)
\]

Here, we have used the commutation relation \( [J_\alpha, J_\beta] = i \sum_\gamma e_{\alpha \beta \gamma} J_\gamma \) and the notation \( |\mathbf{l} \times \mathbf{k}| \) for the conventional cross product of vectors \( \mathbf{l} \) and \( \mathbf{k} \). Let the angle between vectors \( \mathbf{k} \) and \( \mathbf{l} \) be \( \theta \). Consider a rotation \( Q \) in \( \mathbb{R}^3 \) such that \( Q \left( \frac{\mathbf{k}}{|\mathbf{k}|} + \frac{\mathbf{l}}{|\mathbf{l}|} \right) \) is aligned with the positive direction of axis \( x \) and \( Q |\mathbf{l} \times \mathbf{k}| \) is aligned with the positive direction of axis \( z \). In other words, the vectors \( Q \mathbf{k} \) and \( Q \mathbf{l} \) belong to the \( xy \)-plane, and the axis \( x \) is a bisector of the angle between vectors \( Q \mathbf{k} \) and \( Q \mathbf{l} \). The vector \( Q |\mathbf{l} \times \mathbf{k}| \) is perpendicular to both \( Q \mathbf{k} \) and \( Q \mathbf{l} \) and has length \( |\mathbf{l} \times \mathbf{k}| = |\mathbf{k}| |\mathbf{l}| \sin \theta \). Therefore, the vector \( Q \mathbf{k} \) has coordinates \( (|\mathbf{k}| \cos \frac{\theta}{2}, |\mathbf{k}| \sin \frac{\theta}{2}, 0) \), the vector \( Q \mathbf{l} \) has coordinates \( (|\mathbf{l}| \cos \frac{\theta}{2}, -|\mathbf{l}| \sin \frac{\theta}{2}, 0) \), and the vector \( Q |\mathbf{l} \times \mathbf{k}| \) has coordinates \( (0, 0, |\mathbf{k}| |\mathbf{l}| \sin \theta) \). The corresponding unitary rotation \( U_Q \in \{ U_g \}_{g \in SU(2)} \) transforms the spin
operators in accordance with formula (14) as follows:

\[
U_Q \left( \sum \kappa \alpha J_{\alpha} \right) U_Q^\dagger = \sum_{\beta} (Q_k)_\beta J_{\beta} = |k| \left( \cos \frac{\vartheta}{2} J_x + \sin \frac{\vartheta}{2} J_y \right),
\]

\[
U_Q \left( \sum \lambda \alpha J_{\alpha} \right) U_Q^\dagger = \sum_{\beta} (Q_l)_{\beta} J_{\beta} = |l| \left( \cos \frac{\vartheta}{2} J_x - \sin \frac{\vartheta}{2} J_y \right),
\]

\[
U_Q \left( \sum \gamma \times [k] \gamma J_{\gamma} \right) U_Q^\dagger = \sum_{\beta} (Q[l \times k])_{\beta} J_{\beta} = |k| |l| \sin \vartheta J_z.
\]

Substituting (24)–(26) in (23) and taking into account that with respect to \( \vartheta \)

Suppose the maximum in the right hand side of (31) is attained at some vector \( Q \)

For \( j = 1 \), then the matrix of operator \( U_Q F U_Q^\dagger \) in the basis \( \{|j, m\}\}_{m=-j}^{j} \) has a rather simple form. Its eigenvalues do not depend on \( \vartheta \) and \( \eta \) and read 1, 1, 0.

For the cases \( j = 3/2 \) and \( j = 2 \) one can find eigenvalues of \( U_Q F U_Q^\dagger \) and maximize them with respect to \( \vartheta \) and \( \eta \) to get upper bounds 9/4 and 4, respectively.

For \( j > 2 \) we develop the following technique.

Note that \( A \cos \eta + B \sin \eta \leq \sqrt{A^2 + B^2} \) for \( A, B, \eta \in \mathbb{R} \), so the average value

\[
\left< \left( \frac{1}{4i} (|k|^2 - |l|^2)(J^2_x - J^2_y) + |k| |l| J_z \right) \right> \leq \frac{1}{4} \sqrt{\left< (J_x^2 - J_y^2)^2 \right> + 4 \langle J_z \rangle^2}.
\]

Similarly, \( C \sin \vartheta + D \cos \vartheta \leq \sqrt{C^2 + D^2} \) for all \( \vartheta, C, D \in \mathbb{R} \), therefore

\[
\langle U_Q F U_Q^\dagger \rangle \leq \frac{1}{4} \sqrt{\langle (J_x^2 + J_y^2)^2 \rangle + \langle (J_x^2 - J_y^2)^2 \rangle + 4 \langle J_z \rangle^2}.
\]

Suppose the maximum in the right hand side of (31) is attained at some vector \( \psi_0 \), then this maximum is also attained at the vector \( \psi_0 = e^{-i J_{\vartheta}} \psi \) due to invariance of (31) with respect to rotations around axis z. On the other hand, \( \langle \psi_0 | J_+ | \psi_0 \rangle = e^{i \vartheta} \langle \psi_0 | J_+ | \psi_0 \rangle \), which means that \( \langle J_+ \rangle \) can always be chosen to be real, so \( \langle J_+ \rangle = \langle J_- \rangle \). In other words,

\[
\langle U_Q F U_Q^\dagger \rangle \leq \max_{\psi : \langle \psi | \psi \rangle = 1} \left\{ \frac{1}{4} \langle \psi \{ (J_x^2 + J_y^2) \} \psi^2 + 4 \langle \psi | J_z \rangle \psi^2 \right\} + \frac{1}{2} \langle \psi_0 | J_+^2 + J_-^2 \rangle \psi_0 \psi_0
\]

Denote \( a = \langle \psi | J_x^2 | \psi \rangle, \ b = \langle \psi | J_y^2 | \psi \rangle, \) and \( c = \langle \psi | J_z^2 | \psi \rangle \). The dispersion of spin projection onto axis z denote \( d = \langle \psi | J_z^2 \psi \rangle - \langle \psi | J_z \psi \rangle^2 = \langle \psi | J_z^2 \psi \rangle - c \). Note that \( 0 \leq a, b, c \leq j^2 \) and \( d \geq 0 \). Since \( J_x^2 + J_y^2 + J_z^2 = (j+1)J_{j+1} \), we have \( a + b + c + d = j(j + 1) \). Finally,
from Lemma 3 it follows that \( c \leq 9j^2\frac{j^2-a}{2j-1} \) and \( c \leq 9j^2\frac{j^2-b}{2j-1} \). Therefore, we simplify (31) as follows:

\[
\langle UQFU^\dagger_Q \rangle \leq \frac{1}{2} \max_{a,b,c,d} \sqrt{(a-b)^2 + c + a + b}. \tag{32}
\]

Using the method of Lagrange multipliers one can readily see that the maximum in the right hand side of (32) is attained on the boundary of region for parameters \( a, b, c, d \). If \( j > 2 \), then the maximum equals \( j^2 \) and is attained at points with \( a = j^2 \) and \( c = 0 \) or \( b = j^2 \) and \( c = 0 \).

Although we have considered the cases \( j = 1, \frac{3}{2}, 2 \) and \( j > 2 \) separately, their results can be unified, namely, \( \langle UQFU^\dagger_Q \rangle \leq j^2 \) if \( j \geq 1 \). Recalling the fact \( \langle UQFU^\dagger_Q \rangle \leq \frac{1}{2} \) if \( j = \frac{1}{2} \), we obtain that \( \langle UQFU^\dagger_Q \rangle \leq \max(j, j^2) \). Since this bound is valid for all normalized states \( |\psi\rangle \), we finally conclude that \( \langle F \rangle \leq \max(j, j^2) \).

**Proposition 6** The maximal \( p \)-norm \( (p \geq 1) \) and the minimal output entropy of the Landau–Strater channel \( \Phi : B(H_{2j+1}) \rightarrow B(H_{2j+1}) \) are equal to

\[
\nu_p(\Phi) = \frac{(j^p + 1)^{1/p}}{j+1} \quad \text{and} \quad S_{\min}(\Phi) = \log(j+1) - \frac{j}{j+1}\log j, \tag{33}
\]

respectively, and are attained at the state \( |j, j\rangle \).

**Proof.** Let \( |\psi\rangle \langle \psi| \) be a pure state at which the maximal \( \infty \)-norm is attained. Then \( \|\Phi|\psi\rangle \langle \psi|\|_{\infty} = \lambda \), where \( \lambda \) is the maximal output eigenvalue. On the other hand,

\[
\lambda = \max_{\chi \neq 0} \frac{\langle \chi | \Phi | \psi \rangle \langle \psi | \chi \rangle}{\langle \chi | \chi \rangle} = \max_{\chi \neq 0} \sum_{a=1}^3 \frac{1}{j(j+1)} |\varphi_a| |\chi\rangle \langle \chi |, \quad |\varphi_a\rangle = J_\alpha |\psi\rangle. \tag{34}
\]

The vector \( |\chi\rangle \) maximizing (34) must belong to a linear span of vectors \( \{|\varphi_\alpha\rangle\}_{\alpha=1}^3 \), i.e. \( |\chi\rangle = \sum_{\beta=1}^3 c_\beta |\varphi_\beta\rangle \). Introduce the Hermitian Gram matrix \( G_{\alpha\beta} = \langle \varphi_\alpha | \varphi_\beta \rangle = \langle \psi | J_\alpha J_\beta \psi \rangle \) and the vector \( |c\rangle = (c_1, c_2, c_3)^T \in \mathbb{H}_3 \), then \( \sum_{\alpha=1}^3 |\langle \varphi_\alpha | \chi \rangle|^2 = \sum_{\alpha=1}^3 \sum_{\beta=1}^3 G_{\alpha\beta} c_\beta^2 = |\langle c | G^2 | c \rangle| \) and \( \langle \chi | \chi \rangle = \langle c | c \rangle \). Equation (34) can be further simplified with the use of vector \( |c\rangle \):

\[
\lambda = \frac{1}{j(j+1)} \max_{|c\rangle \neq 0} \frac{\langle c | G^2 | c \rangle}{\langle c | c \rangle} = \frac{1}{j(j+1)} \max_{|c\rangle \neq 0} \frac{\langle c' | G | c' \rangle}{\langle c' | c' \rangle} = \frac{1}{j(j+1)} \|G\|_{\infty}. \tag{35}
\]

On the other hand, the \( \infty \)-norm of \( G \) reads

\[
\|G\|_{\infty} = \max_{u: |u| = 1} \sum_{\alpha, \beta = 1}^3 u_{\alpha 1} \text{Re} G_{\alpha \beta} \sum_{\beta = 1}^3 u_{\beta 1} = \max_{u: |u| = 1} \left\| \sum_{\alpha = 1}^3 u_{\alpha 1} J_\alpha |\psi\rangle \right\|_{\infty}^2. \tag{36}
\]

Since \( u \) is a unitary matrix, the vector \( u = (u_{11} \quad u_{21} \quad u_{31})^T = \mathbf{k} + \mathbf{l}, \) where \( \mathbf{k}, \mathbf{l} \in \mathbb{R}^3 \) and \( \mathbf{u}^T \mathbf{u} = |\mathbf{k}|^2 + |\mathbf{l}|^2 = 1 \). By Lemma 4 for any \( |\psi\rangle \) the maximum in the right hand side of (36) does not exceed \( \max(j, j^2) \). Therefore,

\[
\lambda = \frac{1}{j(j+1)} \|G\|_{\infty} \leq \max \left( \frac{j}{j+1}, \frac{1}{j+1} \right). \tag{37}
\]
On the other hand, if \( |\psi\rangle = |j, j\rangle \), then by formula (14) we have \( \text{Spec}(\Phi(|j, j\rangle\langle j, j|)) = \left\{ \frac{j}{j+1}, \frac{j}{j+1}, 0, \ldots \right\} \). This implies that \( \lambda = \max\left(\frac{j}{j+1}, \frac{j}{j+1}\right) \) and \( \|\Phi(|\psi\rangle\langle\psi|)\|_\infty \) is attained at the vector \( |j, j\rangle \).

Denote \( \lambda = \left\{ \frac{j}{j+1}, \frac{j}{j+1}, 0, \ldots \right\} \). Since \( \lambda \) has only two nonzero components and the largest component is \( \nu_\infty(\Phi) \), then \( \lambda \) majorizes all other output spectra \( \mu \). Here, we use the conventional definition of majorization (Ref.\(^{20}\), Definition 12.1): a sequence of real numbers \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) majorizes another sequence of real numbers \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \) if, after possible renumeration, the terms of the sequences \( \lambda \) and \( \mu \) satisfy conditions \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n, \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n, \lambda_1 + \lambda_2 + \ldots + \lambda_k \geq \mu_1 + \mu_2 + \ldots + \mu_k \) for each \( k, 1 \leq k \leq n - 1 \), and \( \lambda_1 + \lambda_2 + \ldots + \lambda_n = \mu_1 + \mu_2 + \ldots + \mu_n \). In our case, \( \lambda_1 = \nu_\infty(\Phi) \), which guarantees \( \lambda_1 \geq \mu_1 \) for all other output spectra \( \mu \). Moreover, \( \lambda_1 + \lambda_2 = 1 = \mu_1 + \mu_2 + \ldots + \mu_{2j+1} \geq \mu_1 + \mu_2 \), therefore \( \lambda \) majorizes any other output spectrum \( \mu \). Since functions \( y(x) = x \log x \) and \( y(x) = x^p, p \geq 1 \), are convex, the Shannon entropy \( H(x) = -\sum_{k=1}^{n} x_k \log x_k \) is a Schur-concave function of \( x \in [0, 1]^n \) and the output \( p \)-norm \( V_p(\rho) = (\sum_{k=1}^{n} x_k^p)^{1/p} \) is a Schur-convex function of \( x \in [0, 1]^n \) (Ref.\(^{20}\), Definition 12.23, Theorem 12.27). Therefore, \( H(\mu) \geq H(\lambda) \) and \( V_p(\mu) \leq V_p(\lambda) \) for all output spectra \( \mu \). This observation results in formulas (33).

Corollaries 1 and 2 are merely consequences of proposition 6.

Consider the second tensor power \( \Phi \otimes 2 \) of a channel \( \Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \). Suppose a density operator \( \rho \in \mathcal{B}(\mathcal{H}) \). Then for the factorized input \( \rho^\otimes 2 \) we have \( \Phi \otimes 2[\rho^\otimes 2] = (\Phi[\rho])^\otimes 2 \). Obviously, the purity \( \text{tr}\left[ (\Phi \otimes 2[\rho^\otimes 2])^2 \right] \) of the state \( \Phi \otimes 2[\rho^\otimes 2] \) is equal to the square of the purity \( \text{tr}\left[ (\Phi[\rho])^2 \right] \) of the state \( \Phi[\rho] \). However, if one uses entangled input states \( \rho_{\text{ent}} \in \mathcal{B}(\mathcal{H} \otimes 2) \), then in general the purity of the state \( \Phi \otimes 2[\rho_{\text{ent}}] \) can be greater than the purity of all possible factorized states \( \Phi \otimes 2[\rho^\otimes 2] \). Therefore, in general \( \nu_2(\Phi \otimes 2) \geq (\nu_2(\Phi))^2 \). Clearly, if \( \nu_2(\Phi \otimes 2) > (\nu_2(\Phi))^2 \), then the maximal 2-norm for the channel \( \Phi \otimes 2 \) is attained at some entangled state.

Nevertheless, there exist some channels, for which \( \nu_2(\Phi \otimes 2) = (\nu_2(\Phi))^2 \) and the use of entangled inputs does not help to increase the output purity.\(^{21}\) Among such channels, there is a class of unital qubit channel\(^{22}\), so for the Landau–Streater channel \( \Phi : \mathcal{B}(\mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_2) \) the multiplicativity of the maximal 2-norm holds. Since the Landau–Streater channel \( \Phi : \mathcal{B}(\mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_2) \) reduces to the Werner–Holevo channel, then the multiplicativity of the maximal \( p \)-norm for such a channel holds for all \( 1 \leq p \leq 2 \), Ref.\(^{12}\), and is violated for \( p > 4.79 \), Ref.\(^{23}\). The Landau–Streater channel for \( j > 1 \) cannot be analyzed in the same way as the case \( j = 1 \) and does not satisfy the known sufficient criteria of multiplicativity of the maximal 2-norm.\(^{23}\) Despite this fact, if \( p = 2 \), our numerical investigations of the cases \( j = \frac{3}{2} \) and \( j = 2 \) show that the maximal 2-norm is multiplicative within the accuracy of calculations. We can make a conjecture that the maximal 2-norm is multiplicative for all Landau–Streater channels \( \Phi : \mathcal{B}(\mathcal{H}_{2j+1}) \rightarrow \mathcal{B}(\mathcal{H}_{2j+1}) \).

### IV. COMPLEMENTARY CHANNEL

According to the Stinespring’s dilation theorem, the dual channel \( \Phi^\dagger : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \) adopts a representation (Ref.\(^{1}\), Theorem 6.9)

\[
\Phi^\dagger[X] = V^\dagger(X \otimes I_K)V, \tag{38}
\]

where \( I_K \) is the identity operator in some Hilbert space \( K \), \( V : \mathcal{H} \rightarrow \mathcal{H} \otimes K \) is an isometry operator, i.e., \( V^\dagger V = I \).

In the case of the Landau–Streater channel \( \Phi : \mathcal{B}(\mathcal{H}_{2j+1}) \rightarrow \mathcal{B}(\mathcal{H}_{2j+1}) \) the dual channel \( \Phi^\dagger \) coincides with \( \Phi \) and the corresponding Stinespring’s dilation is achieved with the help
of the isometry operator $V$: $\mathcal{H}_{2j+1} \mapsto \mathcal{H}_{2j+1} \otimes \mathcal{H}_{3}$ of the form

$$V = \frac{1}{\sqrt{j(j+1)}} \begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix}. \quad (39)$$

Therefore, $\dim K = 3$ and the Landau–Streater channel $\Phi : \mathcal{B}(\mathcal{H}_{2j+1}) \mapsto \mathcal{B}(\mathcal{H}_{2j+1})$ can be realized via a 3-dimensional environment. In the Schrödinger picture of system-environment interaction (Ref. 24, Theorem 6.9), we have

$$\Phi[\rho] = \text{tr}_K[V\rho V^\dagger] = \text{tr}_K[U(\rho \otimes \xi)U^\dagger], \quad (40)$$

where $\xi \in \mathcal{B}(\mathcal{H}_3)$ is the pure initial environment state, $U : \mathcal{H}_{2j+1} \otimes \mathcal{H}_3 \mapsto \mathcal{H}_{2j+1} \otimes \mathcal{H}_3$ is the unitary evolution operator. The general technique of finding $U$ is described, e.g., in Ref. 24, section 8.2.3.

If one replaces the partial trace over environment $\text{tr}_K$ by the partial trace over system $\text{tr}_\mathcal{H}$ in formula (40), then one obtains a so-called complementary channel $\tilde{\Phi} : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{K})$ (also referred to as conjugate channel$^{25}$): $\tilde{\Phi}[\rho] = \text{tr}_\mathcal{H}[V\rho V^\dagger] = \text{tr}_\mathcal{H}[U(\rho \otimes \xi)U^\dagger]. \quad (41)$

In the case of the Landau–Streater channel $\Phi : \mathcal{B}(\mathcal{H}_{2j+1}) \mapsto \mathcal{B}(\mathcal{H}_{2j+1})$, the complementary channel $\tilde{\Phi} : \mathcal{B}(\mathcal{H}_2j+1) \mapsto \mathcal{B}(\mathcal{H}_3)$ maps spin-$j$ states into 3-dimensional environment states (also known as qutrit states). In what follows, we use the notation $I_d$ to denote the identity operator $I : \mathcal{H}_d \mapsto \mathcal{H}_d$.

**Proposition 7** The channel $\tilde{\Phi} : \mathcal{B}(\mathcal{H}_{2j+1}) \mapsto \mathcal{B}(\mathcal{H}_3)$, which is complementary to the Landau–Streater channel $\Phi : \mathcal{B}(\mathcal{H}_{2j+1}) \mapsto \mathcal{B}(\mathcal{H}_{2j+1})$, transforms the maximally mixed input state $\frac{1}{2j+1} I_{2j+1}$ into the maximally mixed output state $\frac{1}{3} I_3$.

**Proof.** Denote by $V_\alpha = [(j+1)^{-1/2} J_\alpha]_{\alpha = 1, 2, 3}$, the Kraus operators of $\Phi$ and by $\tilde{V}_i$, $i = 1, \ldots, 2j+1$, the Kraus operators of $\tilde{\Phi}$. These Kraus operators are mutually related with each other by formula (Ref. 24, formula (12))

$$|\alpha_K \rangle \langle \alpha_K | \tilde{V}_i = | i_\mathcal{H} \rangle \langle i_\mathcal{H} | V_\alpha,$$  

where $\{i_\mathcal{H}\}_{i = 1}^{2j+1}$ is the orthonormal basis in $\mathcal{H}$ (input) and $\{\alpha_K\}_{\alpha = 1}^3$ is the orthonormal basis in $\mathcal{K}$ (output). Multiplying (42) from the left by $|\alpha_K \rangle$ and summing over $\alpha$, we get

$$\tilde{V}_i = \sum_{\alpha = 1}^{3} |\alpha_K \rangle \langle \alpha_K | \tilde{V}_i = \sum_{\alpha = 1}^{3} |\alpha_K \rangle \langle i_\mathcal{H} | V_\alpha = \frac{1}{\sqrt{j(j+1)}} \sum_{\alpha = 1}^{3} |\alpha_K \rangle \langle i_\mathcal{H} | J_\alpha. \quad (43)$$

Action of the complementary channel $\tilde{\Phi}$ on the maximally mixed state $(2j+1)^{-1} I_{2j+1}$ reads

$$\frac{1}{2j+1} \tilde{\Phi}[I_{2j+1}] = \frac{1}{2j+1} \sum_{i = 1}^{2j+1} \tilde{V}_i \tilde{V}_i^\dagger = \frac{1}{j(j+1)(2j+1)} \sum_{i = 1}^{2j+1} \sum_{\alpha, \beta = 1}^{3} |\alpha_K \rangle \langle i_\mathcal{H} | J_\alpha J_\beta^\dagger | i_\mathcal{K} \rangle \langle \beta_K |$$

$$= \frac{1}{j(j+1)(2j+1)} \sum_{\alpha, \beta = 1}^{3} \text{tr} \left[ J_\alpha J_\beta \right] |\alpha_K \rangle \langle \beta_K |. \quad (44)$$

Since $SU(2)$ generators $J_\alpha$, $\alpha = 1, 2, 3$, are Hermitian and satisfy the relation $\text{tr}[J_\alpha J_\beta] = \frac{1}{3} j(j+1)(2j+1) \delta_{\alpha\beta}$ (Ref. 2, section 2.3.4, formula (11)), then $\tilde{\Phi}[\frac{1}{2j+1} I_{2j+1}] = \frac{1}{3} I_3$. \hfill \blacksquare
A channel $\Phi$ is called degradable if there exists a channel $T$ such that $\tilde{\Phi} = T \circ \Phi$. Conversely, a channel $\Phi$ is called antidegradable if there exists a channel $T'$ such that $\Phi = T' \circ \tilde{\Phi}$. The structure of degradable and antidegradable quantum channels is studied in Ref. Further, we explore degradability and antidegradability of the Landau–Streater channel for various values of $j$.

A. The case $j = 1/2$

If $j = \frac{1}{2}$, then the Landau–Streater channel $\Phi$ is antidegradable but not degradable. In fact, in this case $\Phi$ is a qubit depolarization channel with depolarization parameter $-\frac{1}{3}$, so it is entanglement breaking and, consequently, antidegradable. The Kraus operators of the complementary channel $\tilde{\Phi}$ are calculated via formula (43), which results in the following form of $\tilde{\Phi}$:

$$\tilde{\Phi}[X] = \frac{1}{3} \left( \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} X \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ i & 0 \\ 0 & -1 \end{pmatrix} X \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & -1 \end{pmatrix} \right).$$ (45)

The factoring map $T = \tilde{\Phi} \circ \Phi^{-1}$ is well defined, and its normalized Choi matrix $\Omega_T = T \otimes \text{Id}_2[|\psi_+\rangle\langle\psi_+|]$, $|\psi_+\rangle = \frac{1}{\sqrt{2}} \sum_{i=1}^{2} |i\rangle \otimes |i\rangle$, reads

$$\Omega_T = \frac{1}{6} \begin{pmatrix} 1 & 0 & 3i & 0 & 0 & 3 \\ 0 & 1 & 0 & -3i & -3 & 0 \\ -3i & 0 & 1 & 0 & 0 & 3i \\ 0 & 3i & 0 & 1 & 3i & 0 \\ 0 & -3 & 0 & -3i & 1 & 0 \\ 3 & 0 & -3i & 0 & 0 & 1 \end{pmatrix}. \quad (46)$$

Since $\Omega_T$ has negative eigenvalues, $T$ is not completely positive, and $\Phi$ is not degradable.

B. The case $j = 1$

If $j = 1$, then the Landau–Streater channel $\Phi$ is both degradable and antidegradable. This follows from the fact that $\Phi : B(H_3) \mapsto B(H_3)$ is unitarily equivalent to the Werner–Holevo channel, which is both degradable and antidegradable (Ref, section 2.2). For the sake of completeness, we list the Kraus operators of the complementary channel in this case:

$$\tilde{\Phi} = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & -1/2 & 0 \\ 1/\sqrt{2} & 0 & 0 \end{pmatrix}, \quad \tilde{\Phi} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\Phi} = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}. \quad (47)$$

C. The case $j \geq 3/2$

Proposition 8 The Landau–Streater channel $\Phi : B(H_{2j+1}) \mapsto B(H_{2j+1})$ is not antidegradable if $j \geq \frac{3}{2}$.

Proof. Since doubly complementary channel $\tilde{\Phi}$ is unitarily equivalent to $\Phi$ (Ref, Exercise 6.29), it is enough to demonstrate that $\tilde{\Phi} : B(H_{2j+1}) \mapsto B(H_3)$ is not degradable. The output space for the complementary channel is 3-dimensional, so we use Theorem 10 in Ref, which states that if $\tilde{\Phi} : B(H_d) \mapsto B(H_3)$ is degradable, then the Choi rank of $\tilde{\Phi}$ is at most 3. Choi rank is defined as the rank of the Choi matrix $\Omega_{\Phi} = \tilde{\Phi} \otimes \text{Id}_d[|\psi_+\rangle\langle\psi_+|]$,
Proposition 9

To verify negativity of some diagonal element of the Choi matrix $\Omega_{\tilde{\Phi}}$ we get $\text{rank } \Omega_{\tilde{\Phi}} = 3$ if $j \geq 1$. We can prove that the factoring map $\Phi$ is not antidegradable.

Proof. To prove that the factoring map $T = \tilde{\Phi} \circ \Phi^{-1}$ is not completely positive, it suffices to verify negativity of some diagonal element of the Choi matrix $\Omega_T = T \otimes \text{Id}_{2j+1} \otimes |\psi_+\rangle\langle \psi_+|$, $|\psi_+\rangle = \frac{1}{\sqrt{2j+1}} \sum_{j, m' = -j}^{j} |j, m'\rangle \otimes |j, m\rangle$. Consider the diagonal element

$$
\langle \alpha | \otimes \langle j, m | \Omega_T | \alpha \rangle \otimes \langle j, m | = \frac{1}{2j+1} \langle \alpha | \tilde{\Phi} \circ \Phi^{-1} | \alpha \rangle \langle \alpha | j, m \rangle \langle j, m | \langle \alpha |
$$

$$
= \frac{1}{2j+1} \sum_{i=1}^{2j+1} \langle \alpha | V_i \Phi^{-1} | j, m \rangle \langle j, m | V_i^\dagger | \alpha \rangle = \frac{1}{2j+1} \sum_{i=1}^{2j+1} \langle i | V_\alpha \Phi^{-1} | j, m \rangle \langle j, m | V_\alpha^\dagger | i \rangle
$$

$$
= \frac{1}{j(j+1)(2j+1)} \text{tr} (J_\alpha \Phi^{-1} | j, m \rangle \langle j, m | J_\alpha) = \frac{1}{j(j+1)(2j+1)} \text{tr} (J_\alpha^2 \Phi^{-1} | j, m \rangle \langle j, m |)
$$

$$
= \frac{1}{j(j+1)(2j+1)} \text{tr} (j, m \rangle \langle j, m | \Phi^{-1} | J_\alpha^2 | j, m \rangle) = \frac{1}{j(j+1)(2j+1)} \langle j, m | \Phi^{-1} | J_\alpha^2 | j, m \rangle. \quad (49)
$$

In derivation of formula (49) we have taken into account that $\Phi$ is a self-dual map and $\text{tr}(X \Phi^{-1} [Y]) = \text{tr} (\Phi [\Phi^{-1} [X] \Phi^{-1} [Y]]) = \text{tr} (\Phi^{-1} [X] \Phi^{-1} [\Phi^{-1} [Y]]) = \text{tr} (\Phi^{-1} [X] Y)$.

Let us fix $\alpha = 3$ and calculate the operator $\Phi[J_\alpha^2]$ by using formula (15) and the spectral...
decomposition $J_z = \sum_{m' = -j}^j m'|j, m'|\langle j, m'|$:

$$\Phi[J_z^2] = \sum_{m' = -j}^j (m')^2 \Phi[|j, m'|\langle j, m'|] = \frac{1}{j(j + 1)} \sum_{m' = -j}^j (m')^2 \left[ (m')^2 |j, m'|\langle j, m'| \\
+ \frac{1}{2} (j(j + 1) - m'(m' - 1)) |j, m' - 1\rangle\langle j, m' - 1| \\
+ \frac{1}{2} (j(j + 1) - m'(m' + 1)) |j, m' + 1\rangle\langle j, m' + 1| \right]$$

$$= \frac{1}{j(j + 1)} \sum_{m' = -j}^j \left[ (j(j + 1) - 3)(m')^2 + j(j + 1) \right] |j, m'|\langle j, m'|$$

$$= \frac{j(j + 1) - 3}{j(j + 1)} J_z^2 + I_{2j+1}. \quad (50)$$

This implies that $\frac{j(j + 1)}{j(j + 1) - 3} \Phi[J_z^2 - I] = J_z^2$ and $\Phi^{-1}[J_z^2] = \frac{j(j + 1)}{j(j + 1) - 3} (J_z^2 - I)$. Substituting the obtained result into formula (49) yields

$$\langle 3| \otimes |j, m| \Omega_T |3\rangle \otimes |j, m| = \frac{1}{j(j + 1)(2j + 1)} \langle j, m| \Phi^{-1}[J_z^2]|j, m|$$

$$= \frac{m^2 - 1}{2j + 1} (2j + 3). \quad (51)$$

If $j$ is a half-integer and $j \geq \frac{3}{2}$, then $\langle 3| \otimes |j, \frac{1}{2}| \Omega_T |3\rangle \otimes |j, \frac{1}{2}| < 0$. If $j$ is an integer and $j \geq 2$, then $\langle 3| \otimes |j, 0| \Omega_T |3\rangle \otimes |j, 0| < 0$. Therefore, the Choi matrix $\Omega_T$ is not positive semidefinite and $T$ is not a channel.

**V. CAPACITIES**

**A. Classical capacity**

Classical capacity\textsuperscript{29,30} $C$ of a quantum channel $\Phi$ is known to be equal to the regularized $\chi$-capacity $C_\chi$, i.e., $C(\Phi) = \lim_{n \to \infty} \frac{1}{n} C_\chi(\Phi^{\otimes n})$, where $\chi$-capacity is defined by the expression

$$C_\chi(\Phi) = \sup_{\{p_i, \rho_i\}} \left[ S \left( \sum_i p_i \Phi[\rho_i] \right) - \sum_i p_i S(\Phi[\rho_i]) \right] \quad (52)$$

and $\{p_i, \rho_i\}$ is an ensemble of quantum states, in which the state $\rho_i$ is presented with the probability $p_i$.

Further, we find $C_\chi(\Phi)$ for the Landau-Streater channel $\Pi$.

**Proposition 10** $\chi$-capacity $C_\chi(\Phi)$ of the Landau–Streater channel $\Phi : \mathcal{H}_{2j+1} \mapsto \mathcal{H}_{2j+1}$ equals

$$C_\chi(\Phi) = \log \frac{2j + 1}{j + 1} + \frac{j}{j + 1} \log j. \quad (53)$$

If $j = 1/2$, then $C(\Phi) = C_\chi(\Phi) = \frac{1}{2} - \log 3$.

**Proof.** We exploit the fact that the Landau–Streater channel is $SU(2)$ covariant. Since the representation $U_g$ of $SU(2)$ group is irreducible, it follows from Refs.\textsuperscript{31} that

$$C_\chi(\Phi) = S \left( \Phi \left[ \frac{1}{2j + 1} I_{2j+1} \right] \right) - \min_{\psi} S(\Phi[|\psi\rangle\langle\psi|]) = \log(2j + 1) - \min_{\psi} S(\Phi[|\psi\rangle\langle\psi|]). \quad (54)$$
The minimal output entropy of $\Phi$ is given by proposition $\text{[1]}$. Substituting $\text{[33]}$ into $\text{[51]}$, we get formula $\text{[53]}$.

In the case $j = \frac{1}{2}$, $\chi$-capacity is known to be additive, so $C(\Phi) = C_\chi(\Phi) = \frac{1}{2}\log 2 - \log 3$.

B. Entanglement assisted capacity

The entanglement assisted capacity $C_{ea}$ quantifies the maximal communication rate of classical information transmission through a quantum channel $\Phi$ with the help of preshared entanglement between the sender and receiver. $\Phi$ is unital and the complementary channel $\tilde{\Phi}$ is defined as $\Phi$ being the complementary channel with respect to $\Phi$. We find the explicit form of the entanglement assisted capacity for the Landau–Streater channel.

Proposition 11 The Landau–Streater channel $\Phi : \mathcal{B}(\mathcal{H}_{2j+1}) \mapsto \mathcal{B}(\mathcal{H}_{2j+1})$ has the entanglement-assisted capacity $C_{ea}(\Phi) = 2\log(2j + 1) - \log 3$.

Proof. Since $\Phi$ is irreducibly covariant by proposition $\text{[1]}$, then it follows that the maximum in $\text{[55]}$ is attained on the maximally mixed input state $\rho = \frac{1}{2j+1}I_{2j+1}$ (Ref. $\text{[8]}$, Proposition 9.3). Recalling that $\Phi$ is unital and the complementary channel $\tilde{\Phi}$ transforms the maximally mixed state $\frac{1}{2j+1}I_{2j+1}$ into the maximally mixed qutrit state $\frac{1}{3}I_3$ by proposition $\text{[9]}$, we get $C_{ea} = 2S(\frac{1}{2j+1}I_{2j+1}) - S(\frac{1}{3}I_3) = 2\log(2j + 1) - \log 3$.

C. Quantum capacity

The coherent information $\text{[36]}$ for a channel $\Phi : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ and state $\rho \in \mathcal{B}(\mathcal{H})$ is defined through $I_c(\rho, \Phi) = S(\rho) - S(\Phi(\rho))$. Maximizing coherent information over states $\rho$ we get a “single-letter” quantum capacity $Q_1(\Phi) = \max_\rho I_c(\rho, \Phi)$. Quantum capacity is known to be a regularized version of $Q_1$, namely, $Q(\Phi) = \lim_{n \to \infty} \frac{1}{n}Q_1(\Phi^\otimes n)$. If $\Phi$ is degradable, then $Q(\Phi) = Q_1(\Phi)$, Ref. $\text{[35]}$. If $\Phi$ is antidegradable, then $Q(\Phi) = 0$, Ref. $\text{[30]}$. If $j = \frac{1}{2}$ or $j = 1$, then the Landau–Streater channels $\Phi : \mathcal{B}(\mathcal{H}_2) \mapsto \mathcal{B}(\mathcal{H}_2)$ and $\Phi : \mathcal{B}(\mathcal{H}_3) \mapsto \mathcal{B}(\mathcal{H}_3)$ are antidegradable, so $Q(\Phi) = 0$. Since the Landau–Streater channel is not antidegradable if $j \geq \frac{1}{2}$, one can expect that $Q(\Phi) > 0$ if $j \geq \frac{1}{2}$. Note that $I_c(\rho^\otimes n, \Phi^\otimes n) = nI_c(\rho, \Phi)$, Ref. $\text{[35]}$, and therefore $Q_1(\Phi^\otimes n) = \max_{\rho, \rho^\otimes n} I_c(\rho, \Phi^\otimes n) \geq \max_{\rho, \rho^\otimes n} I_c(\rho^\otimes n, \Phi^\otimes n) = nQ_1(\Phi)$. Consequently, $Q(\Phi) \geq Q_1(\Phi) \geq I_c(\rho_0, \Phi)$ for any density operator $\rho_0$. This means that one can estimate the quantum capacity of the Landau–Streater channel $\Phi : \mathcal{B}(\mathcal{H}_{2j+1}) \mapsto \mathcal{B}(\mathcal{H}_{2j+1})$ from below by $I_c(\rho_0, \Phi)$. In fact, if we fix the state $\rho_0 = \frac{1}{2j+1}I_{2j+1}$, then $Q(\Phi) \geq I_c(\frac{1}{2j+1}I_{2j+1}, \Phi) = 2\log(2j + 1) - \log 3$. Thus, we have just proved the following result.

Proposition 12 $Q(\Phi) = 0$ for the Landau–Streater channels $\Phi : \mathcal{B}(\mathcal{H}_2) \mapsto \mathcal{B}(\mathcal{H}_2)$ and $\Phi : \mathcal{B}(\mathcal{H}_3) \mapsto \mathcal{B}(\mathcal{H}_3)$. If $j > \frac{1}{2}$, then $Q(\Phi) \geq Q_1(\Phi) \geq \log(2j + 1) - \log 3$ for the Landau–Streater channel $\Phi : \mathcal{B}(\mathcal{H}_{2j+1}) \mapsto \mathcal{B}(\mathcal{H}_{2j+1})$.

VI. ENTANGLEMENT ANNIHILATION AND PRESERVATION

A state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is called separable with respect to bipartition $A|B$ if it can be represented as the closure of convex combination $\sum_i p_i \rho_i^A \otimes \rho_i^B$, where $\{p_i\}$ is a probability
distribution, $\rho^A_1 \in \mathcal{B}(\mathcal{H}^A)$, and $\rho^B_1 \in \mathcal{B}(\mathcal{H}^B)$. Ref.\textsuperscript{40}. The channel $\Phi : \mathcal{B}(\mathcal{H}^A) \mapsto \mathcal{B}(\mathcal{H}^A)$ is called entanglement breaking if $\Phi \otimes \text{Id}^B_2[\rho]$ is separable for all input states $\rho \in \mathcal{B}(\mathcal{H}^A \otimes \mathcal{H}^B)$, with $k = \dim \mathcal{H}^B$ being arbitrary.\textsuperscript{41} Entanglement-breaking channels are exactly measure-and-prepare ones, and their structure is well known.\textsuperscript{42} The channel $\Lambda : \mathcal{B}(\mathcal{H}^A \otimes \mathcal{H}^B) \mapsto \mathcal{B}(\mathcal{H}^A \otimes \mathcal{H}^B)$ is called entanglement annihilating if $\Lambda[\rho]$ is separable for all input states $\rho \in \mathcal{B}(\mathcal{H}^A \otimes \mathcal{H}^B)$.\textsuperscript{43} The structure of entanglement annihilating channels is fully studied for local qubit channels $\Lambda = \Phi_1 \otimes \Phi_2$ and partially studied for other classes of channels.\textsuperscript{44}

We focus on entanglement-annihilating properties of the map $\Lambda = \Phi \otimes \Phi$, where $\Phi$ is the Landau–Streater channel. As we show below, $\Phi \otimes \Phi$ is not entanglement annihilating if $j \geq 1$, from which it will follow that $\Phi$ is not entanglement breaking and $\Phi \otimes \Phi$ is not absolutely separating.\textsuperscript{45}

**Proposition 13** The second tensor power of the Landau–Streater channel $\Phi : \mathcal{B}(\mathcal{H}_{2j+1}) \mapsto \mathcal{B}(\mathcal{H}_{2j+1})$, $\Phi \otimes \Phi$, is entanglement annihilating if $j = \frac{1}{2}$ and is not entanglement annihilating for all $j \geq 1$.

**Proof.** The case $j = \frac{1}{2}$ corresponds to the qubit depolarizing channel with depolarization parameter $q = -\frac{1}{2}$. Entanglement annihilation by $\Phi \otimes \Phi$ in this case is proved in Ref.\textsuperscript{42}.

Let $j \geq 1$. In what follows, we prove that $\Phi \otimes \Phi$ is not entanglement annihilating by presenting a bipartite entangled state, which remains entangled after the action of $\Phi \otimes \Phi$. Consider the vector $|\phi\rangle \in \mathcal{H}_{2j+1} \otimes \mathcal{H}_{2j+1}$ of the form

$$|\phi\rangle = \frac{1}{\sqrt{2}} (|j,j\rangle |j,j\rangle + |j,-j\rangle |j,-j\rangle),$$

where $|j,m\rangle$ denotes the spin-$j$ state vector corresponding to the definite spin projection $m$ onto $z$ axis, $J_z |j,m\rangle = m |j,m\rangle$, $m = j, j - 1, \ldots, -j$. Let $\top$ be the transposition in the basis $\{|j,m\rangle\}$. Since $\top \circ \Phi[X] = \Phi[X^{\top}]$ for the Landau–Streater channel $\Phi$, then the partially transposed output state $\Phi \otimes (\top \circ \Phi)|\phi\rangle$ is given by formula

$$2 \Phi \otimes (\top \circ \Phi) |\phi\rangle = \Phi(|j,j\rangle |j,j\rangle \otimes \Phi(|j,j\rangle |j,j\rangle) + \Phi(|j,-j\rangle |j,-j\rangle \otimes \Phi(|j,-j\rangle |j,-j\rangle) + \Phi(|j,j\rangle |j,-j\rangle \otimes \Phi(|j,j\rangle |j,-j\rangle) + \Phi(|j,-j\rangle |j,j\rangle \otimes \Phi(|j,-j\rangle |j,j\rangle))$$

Using the channel representation $\Phi[X] = (j(j+1))^{-1/2} (\frac{1}{2} I_{2j+2} + \frac{1}{2} J_{x} J_{x} + \frac{1}{2} J_{y} J_{y} + \frac{1}{2} J_{z} J_{z})$ and formula (12), we get

$$\Phi(|j,\pm j\rangle |j,\pm j\rangle) = \frac{j}{j+1} |j,\pm j\rangle |j,\pm j\rangle + \frac{1}{j+1} |j,\pm j\rangle |j,\pm j\rangle,$$

$$\Phi(|j,\pm j\rangle |j,\mp j\rangle) = -\frac{j}{j+1} |j,\pm j\rangle |j,\mp j\rangle.$$  

If $j \geq 1$, then the supports of operators $\Phi(|j,j\rangle |j,j\rangle \otimes \Phi(|j,j\rangle |j,j\rangle) + \Phi(|j,-j\rangle |j,-j\rangle \otimes \Phi(|j,-j\rangle |j,-j\rangle)$ and $\Phi(|j,j\rangle |j,-j\rangle \otimes \Phi(|j,j\rangle |j,-j\rangle) + \Phi(|j,-j\rangle |j,j\rangle \otimes \Phi(|j,-j\rangle |j,j\rangle)$ are orthogonal. Moreover, the operator

$$\Phi(|j,j\rangle |j,-j\rangle \otimes \Phi(|j,-j\rangle |j,j\rangle)$$

$$= \frac{j^2}{(j+1)^2} (|j,j\rangle |j,-j\rangle \otimes |j,-j\rangle |j,j\rangle + |j,-j\rangle |j,j\rangle \otimes |j,j\rangle |j,-j\rangle)$$

is not positive semidefinite as it has a negative eigenvalue $-\frac{j^2}{(j+1)^2}$. Therefore, the partially transposed state $\Phi \otimes (\top \circ \Phi)|\phi\rangle$ is not positive semidefinite and $\Phi \otimes \Phi|\phi\rangle$ is entangled by the Peres–Horodecki criterion.\textsuperscript{16,17}

**Proposition 14** The Landau–Streater channel $\Phi : \mathcal{B}(\mathcal{H}_{2j+1}) \mapsto \mathcal{B}(\mathcal{H}_{2j+1})$ is entanglement breaking if $j = \frac{1}{2}$ and is not entanglement breaking if $j \geq 1$. 

\[\Box\]
Proof. If \( j = \frac{1}{2} \), then the Landau–Streater channel reduces to a depolarizing qubit channel with depolarization parameter \( q = -\frac{1}{3} \). Such a channel is known to be entanglement breaking\(^{27}\).

Let \( j \geq 1 \). Suppose that the Landau–Streater channel \( \Phi : B(\mathcal{H}_{2j+1}) \to B(\mathcal{H}_{2j+1}) \) is entanglement breaking, then \( \Phi \otimes \Phi \) must be entanglement annihilating by construction\(^{42}\).

By proposition \( 13 \) it is not the case. This contradiction implies that \( \Phi \) is not entanglement breaking. \( \qed \)

VII. CONCLUSIONS

The channel \( \Phi \) has been originally constructed as an example of a unital completely positive and trace preserving map, which is extremal in the set of channels if \( j \geq 1 \) and, consequently, is not random unitary. By construction, the example of Landau and Streater is \( SU(2) \) covariant for all \( j \) and, surprisingly, is globally unitarily covariant if \( j = \frac{1}{2} \) and \( j = 1 \).

We have proved that for \( j > 1 \) the Landau–Streater channels is not \( U(2j+1) \) covariant, so global unitary covariance is a peculiar property of spin-\( \frac{1}{2} \) and spin-1 maps. Using the theory of angular momentum, we have explicitly found the spectrum of the Landau–Streater map in proposition \( 4 \) and pointed out that \( \Phi \) always has negative eigenvalues. Negativity of those eigenvalues indicates that \( \Phi \) cannot be obtained as a result of Hermitian Markovian quantum dynamics.

We have found the Stinespring dilation of the Landau–Streater channel, which reveals its physical realization. The Landau–Streater channel can be implemented as a result of the controlled interaction between a spin-\( j \) particle (system) and a spin-1 particle (environment). The partial trace over environment results in the Landau–Streater channel \( \Phi \), whereas the partial trace over system results in the complementary channel \( \tilde{\Phi} \). The important property of the complementary channel is its action on the maximally mixed input state, which we have established in proposition \( 7 \). If \( j = \frac{1}{2} \), then the Landau–Streater channel is antidegradable but not degradable. If \( j = 1 \), the Landau–Streater channel is unitary equivalent to the Werner–Holevo channel, so in this case \( \Phi \) is both degradable and antidegradable. For larger spins (\( j \geq \frac{3}{2} \)) the Landau–Streater channel is neither degradable nor antidegradable.

Using the theory of angular momentum, we find the minimal output entropy of the Landau–Streater channel in proposition \( 6 \). Combining this result with \( SU(2) \) covariance, we have managed to calculate the \( \chi \)-capacity (proposition \( 10 \)) and the entanglement-assisted capacity (proposition \( 11 \)). Also, we have estimated the lower bound on quantum capacity of the Landau–Streater channel (proposition \( 12 \)).

We have explored the entanglement dynamics induced by the Landau–Streater channel. The channel is shown to be entanglement breaking if and only if \( j = \frac{1}{2} \). The channel’s second tensor power \( \Phi \otimes \Phi \) does not annihilate entanglement for any \( j \geq 1 \). We have constructed the state with Schmidt rank 2, formula \( 56 \), which remains entangled when affected by \( \Phi \otimes \Phi \).

Finally, we have discussed the multiplicativity property of the maximal \( p \)-norms for the Landau–Streater channel and conjectured multiplicativity of the maximal \( 2 \)-norms with respect to the second tensor power of the channel.

ACKNOWLEDGMENTS

The authors thank the anonymous referee for helpful suggestions to improve the quality of the paper, pointing out misprints, and proposing a proof for the fact that \( \text{rank} \Omega \tilde{\Phi} = 2j+1 \) in Proposition 8. The study in Sec. II was supported by Russian Science Foundation under Project No. 16-11-00084. The results of Secs. III–VI were obtained by S.N.F., supported by Russian Science Foundation under Project No. 17-11-01388, and performed at the Steklov Mathematical Institute of Russian Academy of Sciences.
1. A. S. Holevo, *Quantum Systems, Channels, Information* (Walter de Gruyter, Berlin, 2012).
2. L. J. Landau and R. F. Streater, On Birkhoff’s theorem for doubly stochastic completely positive maps of matrix algebras, Lin. Algebra Appl. 193, 107 (1993).
3. D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Theory of Angular Momentum* (World Scientific, Singapore, 1988).
4. V. Bužek, M. Hilery and R. F. Werner, Optimal manipulations with qubits: Universal-NOT gate, Phys. Rev. A 60, R2626 (1999).
5. S. N. Filippov and K. Y. Magadov, Spin polarization-scaling quantum maps and channels, Lobachevskii Journal of Mathematics 39, 65 (2018).
6. R. F. Werner and A. S. Holevo, Counterexample to an additivity conjecture for output purity of quantum channels, J. Math. Phys. 43, 4353 (2002).
7. A. S. Holevo, Covariant dynamical semigroups, Rev. Math. Phys. 12, 211 (1990).
8. A. S. Holevo, Covariant quantum Markovian evolutions, J. Math. Phys. 37, 1812 (1996).
9. N. Datta, M. Tomamichel, and M. M. Wilde, On the second-order asymptotics for entanglement-assisted communication, Quantum Inf. Process. 15, 2569 (2016).
10. M. Mozrzymas, M. Studziński, and N. Datta, Structure of irreducibly covariant quantum channels for finite groups, J. Math. Phys. 58, 052204 (2017).
11. M. Al Nuwairan, The extreme points of SU(2)-irreducibly covariant channels, Int. J. Math. 25, 1450048 (2014).
12. N. Datta, Multiplicativity of maximal p-norms in Werner-Holevo channels for 1 ≤ p ≤ 2, [arXiv:quant-ph/0501064].
13. N. Datta, A. S. Holevo, and Y. Suhov, Additivity for transpose depolarizing channels, Int. J. Quantum Inform. 04, 85 (2006).
14. C. King, The capacity of the quantum depolarizing channel, IEEE Trans. Inf. Theory 49, 221 (2003).
15. M. M. Wolf and J. I. Cirac, Dividing Quantum Channels, Commun. Math. Phys. 279, 147 (2008).
16. S. N. Filippov, J. Piilo, S. Maniscalco, and M. Ziman, Divisibility of quantum dynamical maps and collision models, Phys. Rev. A 96, 032111 (2017).
17. S. N. Filippov and V. I. Man’ko, Inverse spin-s portrait and representation of qubit states by single probability vectors, J. Russ. Laser Res. 31, 32 (2010).
18. S. N. Filippov and V. I. Man’ko, Spin tomography and star-product kernel for qubits and qudits, J. Russ. Laser Res. 30, 129 (2009).
19. D. Chruściński, C. Macchiavello, and S. Maniscalco, Detecting non-Markovianity of quantum evolution via spectra of dynamical maps, Phys. Rev. Lett. 118, 080404 (2017).
20. J. E. Pečaric, F. Proschan, and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Mathematics in Science and Engineering, vol. 187 (Academic Press, Boston, 1992).
21. S. Michalakis, Multiplicativity of the maximal output 2-norm for depolarized Werner-Holevo channels, J. Math. Phys. 48, 122102 (2007).
22. C. King, Additivity for unital qubit channels, J. Math. Phys. 43, 4641 (2002).
23. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
24. A. S. Holevo, Complementary channels and the additivity problem, Theory Probab. Appl. 51, 92 (2007).
25. C. King, K. Matsumoto, M. Nathanson, and M. B. Ruskai, Properties of conjugate channels with applications to additivity and multiplicativity, Markov Process and Related Fields 13, 391 (2007).
26. T. S. Cubitt, M. B. Ruskai, and G. Smith, The structure of degradable quantum channels, J. Math. Phys. 49, 102104 (2008).
27. M. B. Ruskai, Qubit entanglement breaking channels, Rev. Math. Phys. 15, 643 (2003).
28. D. Choi, Completely positive linear maps on complex matrices, Lin. Algebra Appl. 10, 285 (1975).
29. B. Schumacher and M. D. Westmoreland, Sending classical information via noisy quantum channels, Phys. Rev. A 56, 131 (1997).
30. A. S. Holevo, The capacity of quantum channel with general signal states, IEEE Trans. Inform. Theory 44, 269 (1998).
31. A. S. Holevo, Remarks on the classical capacity of quantum channel, [arXiv:quant-ph/0212025].
32. C. H. Bennett, P. W. Shor, J. A. Smolin, A. V. Thapliyal, Entanglement-assisted classical capacity of noisy quantum channel, Phys. Rev. Lett. 83, 3081 (1999).
33. C. H. Bennett, P. W. Shor, J. A. Smolin and A. V. Thapliyal, Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem, IEEE Trans. Inf. Theory 48, 2637 (2002).
34. A. S. Holevo, On entanglement-assisted classical capacity, J. Math. Phys. 43, 4326 (2002).
35. H. Barnum, M. A. Nielsen, and B. Schumacher, Information transmission through a noisy quantum channel, Phys. Rev. A 57, 4153 (1998).
36. B. Schumacher and M. A. Nielsen, Quantum data processing and error correction, Phys. Rev. A 54, 2629 (1996).
37. I. Devetak, The private classical capacity and quantum capacity of a quantum channel, IEEE Trans. Inf. Theory 51, 44 (2005).
38. I. Devetak and P. W. Shor, The capacity of a quantum channel for simultaneous transmission of classical and quantum information, Commun. Math. Phys. 256, 287 (2005).
39. V. Giovannetti and R. Fazio, Information-capacity description of spin-chain correlations, Phys. Rev. A 71, 032314 (2005).
R. F. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, Phys. Rev. A **40**, 4277 (1989).

M. Horodecki, P. W. Shor, and M. B. Ruskai, Entanglement breaking channels, Rev. Math. Phys. **15**, 629 (2003).

L. Moravčíková and M. Ziman, Entanglement-annihilating and entanglement-breaking channels, J. Phys. A: Math. Theor. **43**, 275306 (2010).

S. N. Filippov, V. V. Frizen, and D. V. Kolobova, Ultimate entanglement robustness of two-qubit states against general local noises, Phys. Rev. A **97**, 012322 (2018).

L. Lami and M. Huber, Bipartite depolarizing maps, J. Math. Phys. **57**, 092201 (2016).

S. N. Filippov, K. Y. Magadov, and M. A. Jivulescu, Absolutely separating quantum maps and channels, New J. Phys. **19**, 083010 (2017).

A. Peres, Separability criterion for density matrices, Phys. Rev. Lett. **77**, 1413 (1996).

M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed states: necessary and sufficient conditions, Phys. Lett. A **223**, 1 (1996).