Prime end rotation numbers of invariant separating continua of annular homeomorphisms

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Abstract. Let $f$ be a homeomorphism of the closed annulus $A$ isotopic to the identity, and let $X \subset \text{Int} A$ be an $f$-invariant continuum which separates $A$ into two domains, the upper domain $U_+$ and the lower domain $U_-$. Fixing a lift of $f$ to the universal cover of $A$, one defines the rotation set $\tilde{\rho}(X)$ of $X$ by means of the invariant probabilities on $X$, as well as the prime end rotation number $\check{\rho}_\pm$ of $U_\pm$. The purpose of this paper is to show that $\check{\rho}_\pm$ belongs to $\tilde{\rho}(X)$ for any separating invariant continuum $X$.

1. Introduction

Let $f$ be a homeomorphism of the closed annulus $A = S^1 \times [-1, 1]$, isotopic to the identity, i.e. $f$ preserves the orientation and each of the boundary components $\partial_\pm A = S^1 \times \{\pm 1\}$. Suppose there is an $f$-invariant partition of $A$: $A = U_- \cup X \cup U_+$, where $U_\pm$ is a connected open set containing the boundary component $\partial_\pm A$ and $X$ is a connected compact set. Let $\pi : \tilde{A} = \mathbb{R} \times [-1, 1] \to S^1 \times [-1, 1]$ be the universal covering map and $T : \tilde{A} \to \tilde{A}$ a generator of the covering transformation group; $T(\xi, \eta) = (\xi + 1, \eta)$. Denote by $p : \tilde{A} \to \mathbb{R}$ the projection onto the first factor.

Fix once and for all a lift $\tilde{f} : \tilde{A} \to \tilde{A}$ of $f$. Then the function $p \circ \tilde{f} - p$ is $T$-invariant and can be looked upon as a function on the annulus $A$. Define the rotation set $\tilde{\rho}(X)$ as the set of values $\mu(p \circ \tilde{f} - p)$, where $\mu$ ranges over the $f$-invariant probability measures supported on $X$. The rotation set is a compact interval (maybe one point) in $\mathbb{R}$, which depends upon the choice of the lift $\tilde{f}$ of $f$.

The first example of an invariant continuum $X$ such that the frontiers of $U_\pm$ satisfy $\text{Fr}(U_+) = \text{Fr}(U_-) = X$ and that the rotation set $\tilde{\rho}(X)$ is not a singleton is constructed by G. D. Birkhoff in his 1932 year paper [B], and is referred to as a Birkhoff attractor. It turns out that the Birkhoff attractor is an indecomposable continuum ([C, L2]). Furthermore it is shown by P. Le Calvez ([L1]) that for...
any rational number between the two prime end rotation numbers is realized by a corresponding periodic point of \( \tilde{f} \).

Let \( \hat{U}_\pm = U_\pm \cup \partial_\infty U_\pm \) be the prime end compactification of \( U_\pm \), where \( \partial_\infty U_\pm \) is the space of the prime ends (\([E, M, MN]\)). The space \( \partial_\infty U_\pm \) is homeomorphic to the circle and \( \hat{U}_\pm \) to the closed annulus. As is well known, the homeomorphism \( f \) restricted to \( U_\pm \) extends to a homeomorphism \( \tilde{f}_\pm : \hat{U}_\pm \to \hat{U}_\pm \). Denoting \( I_+ = [0, 1] \) and \( I_- = [-1, 0] \), define a homeomorphism

\[
\Psi_\pm : \hat{U}_\pm \to S^1 \times I_\pm
\]

such that \( \Psi_\pm(\partial_\infty U_\pm) = S^1 \times 0 \). By some abuse of notations denote by \( \pi : \hat{U}_\pm \to \hat{U}_\pm \) the universal covering map. Thus \( \pi^{-1}(U_\pm) \) is considered to be a subspace of both \( \hat{A} \) and \( \hat{U}_\pm \). Let \( \hat{f}_\pm : \hat{U}_\pm \to \hat{U}_\pm \) be the lift of \( f_\pm \) such that \( \hat{f}_\pm = \tilde{f} \) on \( \pi^{-1}(U_\pm) \). The rotation number of the restriction of \( \hat{f}_\pm \) to \( \pi^{-1}(\partial_\infty U_\pm) \), denoted by \( \hat{\rho}_\pm \), is called the prime end rotation number of \( U_\pm \).

The purpose of this paper is to show the following.

**Theorem 1.** The prime end rotation number \( \hat{\rho}_\pm \) belongs to \( \hat{\rho}(X) \).

This result is already known for \( X = \text{Fr}(U_-) = \text{Fr}(U_+) \) (\([BG]\), and for any \( X \) if the homeomorphism \( f \) is area preserving (Lemma 5.4, \([PL]\)).

It is shown in Theorem 2.2 of [E] that any rational number in \( \hat{\rho}(X) \) is realized by a periodic point if \( X \) consists of nonwandering points. Notice that then \( X \), consisting of chain recurrent points, is chain transitive since it is connected, and thus satisfies the condition of Theorem 2.2. As a corollary we have

**Corollary 2.** If \( X \) consists of nonwandering points and if \( p/q \) lies in the closed interval bounded by \( \hat{\rho}_- \) and \( \hat{\rho}_+ \), then there is a point \( x \in \pi^{-1}(X) \) such that \( \hat{f}^q(x) = \pi^p(x) \).

In what follows we also use the following notation. Let

\[
\hat{\Psi}_\pm : \hat{U}_\pm \to \mathbb{R} \times I_\pm
\]

be a lift of \( \Psi_\pm \), and define \( \hat{\rho}_\pm : \hat{U}_\pm \to \mathbb{R} \) by \( \hat{\rho}_\pm = p \circ \hat{\Psi}_\pm \). The projection \( \hat{\rho}_\pm \) is within a bounded error of \( \hat{\rho} \) on \( \pi^{-1}(C) \) for a compact domain \( C \) of \( U_\pm \). But they may be quite different on the whole \( \pi^{-1}(U_\pm) \).

2. **Proof**

First of all let us state a deep and quite useful theorem of P. Le Calvez ([L3]) which plays a key role in the proof. A fixed point free and orientation preserving homeomorphism \( F \) of the plane \( \mathbb{R}^2 \) is called a Brouwer homeomorphism. A proper oriented simple curve \( \gamma : \mathbb{R} \to \mathbb{R}^2 \) is called a Brouwer line for \( F \) if \( F(\gamma) \subset R(\gamma) \) and \( F^{-1}(\gamma) \subset L(\gamma) \), where \( R(\gamma) \) (resp. \( L(\gamma) \)) is the right (resp. left) side complementary domain of \( \gamma \), which is decided by the orientation of \( \gamma \).

**Theorem 2.1.** Let \( F \) be a Brouwer homeomorphism commuting with the elements of a group \( \Gamma \) which acts on \( \mathbb{R}^2 \) freely and properly discontinuously. Then there is a \( \Gamma \)-invariant oriented topological foliation of \( \mathbb{R}^2 \) whose leaves are Brouwer lines of \( F \).

The proof of Theorem [I] is by absurdity. Assume in way of contradiction that \( \hat{\rho}_- < p/q < \inf \hat{\rho}(X) \). Considering \( \hat{f}^qT^{-p} \) instead of \( \hat{f} \), it suffices to deduce a contradiction under the following assumption.
**Assumption 2.2.** $\hat{\rho}_- < 0 < \inf \hat{\rho}(X)$.

Since $\inf \hat{\rho}(X) > 0$, the map $\tilde{f}$ does not admit a fixed point in $\pi^{-1}(X)$. The overall strategy of the proof is to modify the homeomorphism $f$ away from $X$ to a new one $g$ without creating fixed points in $A$ such that the restrictions of $\tilde{g}$ to the lifts of the both boundary circles $\pi^{-1}(\partial_{\pm} A)$ are nontrivial rigid translations by the same translation number. Then by glueing the two boundary circles we obtain a torus $T^2$ and a homeomorphism on $T^2$. Now we can apply Theorem 2.1 to the lift of the homeomorphism to the universal covering space. This yields a topological foliation on $T^2$, which has long been well understood. The proof will be done by analyzing the foliation. We first prepare a lemma which is necessary for the desired modification. We do not presume Assumption 2.2 in the following.

**Lemma 2.3.** Assume $\tilde{f}$ does not admit a fixed point in $\pi^{-1}(X)$. Then the prime end rotation number $\hat{\rho}_\pm$ is nonzero.

**Proof:** Consider the mapping $\tilde{f} - \text{Id}$ defined on $\tilde{A}$. Since it is $T$-invariant, it yields a mapping from $A$, still denoted by the same letter. Then since there is no fixed point of $\tilde{f}$ in $X$, we have $(\tilde{f} - \text{Id})(X) \subset \mathbb{R}^2 \setminus \{0\}$. Therefore there is an annular open neighbourhood $V$ of $X$ for which we get a mapping $\tilde{f} - \text{Id} : V \to \mathbb{R}^2 \setminus \{0\}$.

Clearly for any positively oriented essential simple closed curve $\gamma$ in $V$, the degree of the map $\tilde{f} - \text{Id} : \gamma \to \mathbb{R}^2 \setminus \{0\}$ must be the same. If the curve $\gamma$ is contained in $U_{\pm}$, then the degree can be studied by considering the map $\tilde{f}_{\pm}$ defined on the lift $\tilde{U}_{\pm}$ of the prime end compactification $\hat{U}_{\pm}$. If the prime end rotation number $\hat{\rho}_{\pm}$ is nonzero, the degree is clearly 0. Notice that our definition of the degree differs from the usual definition of the index.

To analyze the case $\hat{\rho}_{\pm} = 0$, we need the following form of the Cartwright-Littlewood theorem [CL].

**Theorem 2.4.** If $\hat{\rho}_+ = 0$ and if $\text{Fix}(\tilde{f}) \cap \pi^{-1}(X) = \emptyset$, then the map $\hat{f}_+$ on $\partial_{\infty} U_+$ is Morse Smale and the attractors (resp. repellors) of $\hat{f}_+|_{\partial_{\infty} U_+}$ are attractors (resp. repellors) of the whole map $\hat{f}_+$.

This is slightly stronger than the usual version in which it is assumed that $\text{Fix}(f) \cap X = \emptyset$. However the proof works as well under the assumption of Theorem 2.4. See e. g. Sect. 3 of [MN].

Let us complete the proof of Lemma 2.3. Theorem 2.4 enables us to compute the degree of the curve $\delta$ in $U_{\pm}$ when $\hat{\rho}_{\pm} = 0$. The degree is $n$ if $\delta \subset U_-$ and $-n$ if $\delta \subset U_+$, where $n$ is the number of the attractors. Since the degree must be the same in $U_-$ and $U_+$, the conclusion follows.

Now we have $\hat{\rho}_- < 0$ and $\hat{\rho}_+ \neq 0$ by Assumption 2.2 and Lemma 2.3. Let us start the modification of $f$.

**Lemma 2.5.** Under Assumption 2.2, there exists a homeomorphism $g$ of $A$ such that

1. $g = f$ in some neighbourhood of $X$,
2. $\tilde{g}$ does not admit a fixed point in $\tilde{A}$, where $\tilde{g}$ is the lift of $g$ such that $\tilde{g} = \tilde{f}$ on $\pi^{-1}(X)$. 


(3) \( \tilde{g} \) is a negative rigid translation by the same translation number on \( \pi^{-1}(\partial_{+}A) \), and
(4) \( \tilde{p}_{-} \circ \tilde{g}_{-} - \tilde{p}_{-} \leq -c \) on \( \hat{U}_{-} \) for some positive number \( c \).

**Proof:** The modification in \( U_{-} \) will be done in the following way. We identify \( \hat{U}_{-} \) with \( S^{1} \times [-1,0] \) by the homeomorphism \( \Psi_{-} \) and the universal covering space \( \hat{U}_{-} \) with \( \mathbb{R} \times [-1,0] \). Thus \( \hat{p}_{-} \) is just the projection onto the first factor; \( \hat{p}_{-}(\xi,\eta) = \xi \).

Since \( \hat{p}_{-} < 0 \), the lift \( \hat{f}_{-} : \mathbb{R} \times [-1,0] \to \mathbb{R} \times [-1,0] \)
of \( \hat{f}_{-} \) satisfies that \( \hat{p}_{-} \circ \hat{f}_{-}(\xi,0) < \xi - 2c \) for some \( c > 0 \). Therefore changing the coordinates of \([-1,0]\) if necessary, one may assume that \( \hat{p}_{-} \circ \hat{f}_{-}(\xi,\eta) \leq \xi - c \) if \( (\xi,\eta) \in \mathbb{R} \times [-1/2,0] \). Define a homeomorphism \( h \) of \( S^{1} \times [-1,0] \) by
\[
h(\xi,\eta) = (\xi + \varphi(\eta) \mod 1, \eta),
\]
where \( \varphi : [-1,0] \to (-\infty,0] \) is a continuous function such that \( \varphi([-1/2,0]) = 0 \) and
\[
\varphi(\eta) \leq -\sup \{ (\hat{p}_{-} \circ \hat{f}_{-} - \hat{p}_{-})(\xi,\eta) \mid \xi \in S^{1} \} - c.
\]
Define \( g = f \circ h \). Then its lift \( \tilde{g}_{-} \) satisfies
\[
\hat{p}_{-} \circ \tilde{g}_{-} - \hat{p}_{-} \leq -c
\]
on \( \hat{U}_{-} = \mathbb{R} \times [-1,0] \). Clearly condition (3) for \( \pi^{-1}(\partial_{-}A) \) can be established by a further obvious modification.

Now to modify \( f \) in \( U_{+} \), we do the same thing as in \( U_{-} \). If the prime end rotation number \( \hat{p}_{+} \) is negative, then with an auxiliary modification we are done. If it is positive insert a time one map of the Reeb flow. \( \square \)

Consider the torus \( T^{2} \) which is obtained from \( A \) by glueing the two boundary curves \( \partial_{+}A \) and \( \partial_{-}A \). Then the condition (3) above shows that \( g \) induces a homeomorphism of \( T^{2} \), again denoted by \( g \). The universal cover of \( T^{2} \) is \( \mathbb{R} \) and \( \hat{A} = \mathbb{R} \times [-1,1] \) is a subset of \( \mathbb{R}^{2} \). The lift \( \tilde{g} : \hat{A} \to \hat{A} \) can be extended uniquely to a lift \( \tilde{g} : \mathbb{R}^{2} \to \mathbb{R}^{2} \) of \( g : T^{2} \to T^{2} \). The covering transformation group \( \Gamma \) is isomorphic to \( \mathbb{Z}^{2} \), generated by the horizontal translation \( T \) and the vertical translation by \( 2 \), denoted by \( S \). Since \( \tilde{g} \) is a Brouwer homeomorphism which commutes with \( \Gamma \), there is a \( \Gamma \)-invariant oriented foliation on \( \mathbb{R}^{2} \) whose leaves are Brouwer lines for \( \tilde{g} \). This yields an oriented foliation \( \mathcal{F} \) on the torus \( T^{2} \). The proof is divided into several cases according to the topological type of the foliation \( \mathcal{F} \). We are going to deduce a contradiction in each case. But before going into detail we need another lemma.

**Lemma 2.6.** For any \( C > 0 \) there is \( n > 0 \) such that \( p \circ \tilde{g}^{n} - p \geq C \) on \( X \).

**Proof:** If not, there would be a point \( x_{n} \in X \) for any \( n > 0 \) such that
\[
(p \circ \tilde{g}^{n} - p)(x_{n}) = \sum_{j=0}^{n-1} (p \circ \tilde{g} - p)(g^{j}(x_{n})) < C
\]
for some \( C > 0 \), and the averages of Dirac masses
\[
\mu_{n} = \frac{1}{n} \sum_{j=0}^{n-1} g_{*}^{j} \delta_{x_{n}}
\]
would satisfy \( \mu_n(p \circ \hat{g} - p) < \frac{C}{n} \). Therefore an accumulation point \( \mu \) of \( \mu_n \) would have the property that \( \mu(p \circ \hat{g} - p) \leq 0 \), contradicting the assumption \( \inf \hat{\rho}(X) > 0 \).

\[ \square \]

**Case 1.** The foliation \( \mathcal{F} \) does not admit a compact leaf. Then \( \mathcal{F} \) is conjugate either to a linear foliation or to a Denjoy foliation, both of irrational slope. The lift \( \hat{\mathcal{F}} \) of \( \mathcal{F} \) to the open annulus \( \mathbb{R}^2/\langle T \rangle \) is conjugate to a foliation by vertical lines. The space of leaves of \( \hat{\mathcal{F}} \) is homeomorphic to \( S^1 \) and there is a projection from \( \mathbb{R}^2/\langle T \rangle \) to \( S^1 \) along the leaves of the foliation. This lifts to a projection \( q : \mathbb{R}^2 \to \mathbb{R} \).

Now \( q \) restricted to \( \hat{A} \) is within a bounded error of the first factor projection \( p : \hat{A} \to \mathbb{R} \) that we have used for the definition of the rotation set \( \hat{\rho}(X) \). In fact both \( p \) and \( q \) are lifts of degree one maps from \( \mathbb{R}^2/\langle T \rangle \) to \( S^1 \) and their difference is bounded on the preimage \( \hat{A} = \pi^{-1}(A) \) of a compact subset \( A \). Thus Lemma 2.7 shows that \( q \circ \hat{g}^n(x) \to \infty (n \to \infty) \) for \( x \in \pi^{-1}(X) \). That is, the foliation \( \hat{\mathcal{F}} \) is oriented upward. But this shows that \( q \circ \hat{g}(x) \geq q(x) \) even for a point \( x \in \pi^{-1}(\partial_- A) \). On the other hand by condition (3) of Lemma 2.7 \( \hat{g} \) is a negative translation on \( \pi^{-1}(\partial_- A) \). A contradiction.

**Case 2.1.** The foliation \( \mathcal{F} \) admits a compact leaf \( L \) of nonzero slope and does not admit a Reeb component. In this case the lifted foliation \( \hat{\mathcal{F}} \) is also conjugate to the vertical foliation and the argument of Case 1 applies.

**Case 2.2.** The foliation \( \mathcal{F} \) admits a Reeb component \( R \) of nonzero slope. The Brouwer property of leaves implies that \( g(R) \subset \text{Int}(R) \) or \( g^{-1}(R) \subset \text{Int}(R) \). That is, a point of the boundary of \( R \) is wandering under \( g \). Therefore \( \partial_- A \), consisting of nonwandering points of \( g \) according to (3) of Lemma 2.3 cannot intersect the boundary of \( R \), which is however impossible since the slope of \( R \) is nonzero.

**Case 2.3.** The foliation \( \mathcal{F} \) admits a compact leaf of slope 0. Hereafter we only consider the dynamics and the foliation on the open annulus \( \mathbb{R}^2/\langle T \rangle \). Recall that \( A \) is a subset of \( \mathbb{R}^2/\langle T \rangle \), and the homeomorphism \( g \) on \( A \) is extended to the whole \( \mathbb{R}^2/\langle T \rangle \), again denoted by \( g \), in such a way that \( g \) commutes with the vertical translation \( S \), while the foliation is denoted by \( \hat{\mathcal{F}} \) as before.

Now the foliation \( \hat{\mathcal{F}} \) yields a partition \( \mathcal{P} \) of the open annulus \( \mathbb{R}^2/\langle T \rangle \) into compact leaves, interiors of Reeb components and foliated \( I \)-bundles. The set \( \mathcal{P} \) is totally ordered by the height. The minimal element which intersects \( X \) cannot be a compact leaf by the Brouwer line property. Let \( R \) be the closure of the minimal element. Thus \( R \) is either a Reeb component or a foliated \( I \)-bundle such that \( \text{Int}(R) \cap X \neq \emptyset \) and \( \partial_- R \cap X = \emptyset \), where \( \partial_- R \) is the lower boundary curve of \( R \).

Assume for a while that \( \partial_- R \) is oriented from the right to the left. Thus the homeomorphism \( g \) carries \( \partial_- R \) into the upper complement of \( \partial_- R \).

**Case 2.3.1** \( R \) is a Reeb component. First notice that \( g(R) \subset \text{Int}R \) and that the interior leaves of \( R \) are oriented upwards by the assumption \( \inf \hat{\rho}(X) > 0 \) and the fact that \( g(X \cap R) \subset X \cap R \). Choose a simple arc

\[
\alpha : [0, 1] \to \pi^{-1}(R)
\]

such that \( \alpha(0) \in \pi^{-1}(\partial_- R) \), \( \alpha(1) = \hat{g}(\alpha(0)) \), and \( \alpha((0, 1)) \subset \text{Int}(\pi^{-1}(R)) \setminus \hat{g}(\pi^{-1}(R)) \). Since \( g^{-1}(\pi(\alpha)) \) is below \( \text{Int}R \), \( \hat{g}^{-1}(\alpha) \), and hence \( \alpha \), is contained in \( \pi^{-1}(U_-) \).

Concatenating nonnegative iterates of \( \alpha \), we obtain a simple path \( \gamma : [0, \infty) \to \pi^{-1}(R \cap U_-) \) such that \( \hat{g} \circ \gamma(t) = \gamma(t + 1) \) for any \( t \geq 0 \). Let \( q : \pi^{-1}(\text{Int}(R)) \to \mathbb{R} \)
be the lift of the projection along the leaves. Since $\gamma([1, \infty))$ is contained in the lift of a compact subset $\tilde{g}(R) \subset \text{Int}(R)$ and the leaves in $\text{Int}(R)$ is oriented upward, we have $q \circ \gamma(t) \to -\infty$ as $t \to \infty$. We also have $p \circ \gamma(t) \to \infty$ because $q$ is within bounded error of $p$ on $\gamma([1, \infty))$.

On the other hand by condition (4) of Lemma 2.5, we have $\tilde{p} \circ \gamma(t) \to -\infty$ as $t \to \infty$. In particular the curve $\gamma$ is proper both in $\tilde{A}$ and in $\tilde{U}_-$ pointing toward the opposite direction. By joining the point $\gamma(0)$ to an appropriate point in $\pi^{-1}(\partial_- A)$, we obtain a simple curve $\delta$ in $\pi^{-1}(U_-)$ starting at a point on $\pi^{-1}(\partial_- A)$ which extends $\gamma$.

Notice that there is a point of $\pi^{-1}(X)$ on the left of a proper oriented curve $\delta$ in $\tilde{A}$, because the map $p$ is bounded from below on $\delta$ and a high iterate of $T^{-1}$ carries a point in $\pi^{-1}(X)$ beyond that bound. (There might be a point of $\pi^{-1}(X)$ on the right of $\delta$ however.)

Let $x$ be a point in $\pi^{-1}(\partial_- A)$ left to the initial point of $\delta$. Then there is a simple path $\beta : [0, \infty) \to \pi^{-1}(U_-)$ such that $\beta(0) = x$, $\lim_{t \to \infty} \beta(t) \in \pi^{-1}(X)$, and $\beta$ is disjoint from $\delta$. The path $\beta$, extendable in $\pi^{-1}(A)$ is also extendable in $\tilde{U}_-$, the lift of the prime end compactification. (See e. g. Lemma 2.5 of [MN].) This implies that $\beta$ defines a simple path in $\tilde{U}_-$ joining $x$ to a prime end in $\pi^{-1}(\partial_- \tilde{U}_-)$ without intersecting $\delta$, which is impossible since $\pi^{-1}(\partial_- \tilde{U}_-)$ is contained in the right side of the proper path $\delta$ in $\tilde{U}_-$ since $\tilde{p}(\delta(t)) \to -\infty$, while $x$ is on the left side. A contradiction.

**Case 2.3.2** $R$ is a foliated 1-bundle. Thus the upper boundary curve $\partial_+-R$ of $R$ is also oriented from the right to the left, and its image by $g$ lies on the upper complement of $R$. The interior leaves of $R$ are oriented upward.

Recall that the boundary component $\partial_- A$ consisting of nonwandering points cannot intersect a compact leaf. Moreover $\partial_- A$ lies a Reeb component or a foliated 1-bundle whose interior leaves are oriented downward since $p\tilde{g}^n(x) \to -\infty$ as $n \to \infty$ for $x \in \pi^{-1}(\partial_- A)$. Let $C$ be the annulus in $\mathbb{R}^2/(T)$ bounded by $\partial_- A$ and $\partial_+ R$, the upper boundary curve of $R$. Notice that $\text{Int}(C)$ contains $\partial_- R$.

**Case 2.3.2.1** The intersection $X \cap C$ has a component which separates $\partial_- A$ from $\partial_+ A$. One can derive a contradiction by the same argument as in Case 2.3.1, since the like defined path $\gamma$ cannot evade $R$.

**Case 2.3.2.2** There is a simple path in $U_-$ joining a point in $\partial_- A$ with a point in $\partial_+ R$. Notice first of all that $g^{-1}(C) \subset C$. Let $\mathcal{Y}$ be the family of the connected components of $\pi^{-1}(X \cap C)$. Then any element $Y \in \mathcal{Y}$ is compact, and intersects $\pi^{-1}(\partial_+ R)$ since otherwise $Y$ would be a connected component of $\pi^{-1}(X)$ itself.

Choose a simple curve $\gamma : [0, 1] \to \pi^{-1}(C)$ such that

1. $\gamma(0) \in \pi^{-1}(\partial_- A)$,
2. $\gamma(1) \in \pi^{-1}(X \cap C)$, and
3. $\gamma([0, 1]) \subset \pi^{-1}(U_- \cap C)$.

Let $Y$ be an element of $\mathcal{Y}$ which contains $\gamma(1)$. Then there are two unbounded connected components of the complement $\pi^{-1}(C) \setminus (Y \cup \gamma)$, one $L(Y \cup \gamma)$ on the left, and the other $R(Y \cup \gamma)$ on the right.

Notice that for any $n > 0$, $\tilde{g}^{-n} \gamma$ is a path in $C$, and that $p\tilde{g}^{-n}(\gamma(1)) \to -\infty$ and $p\tilde{g}^{-n}(\gamma(0)) \to \infty$ as $n \to \infty$. That is, for any large $n$, $\tilde{g}^{-n}(\gamma(1)) \in L(Y \cup \gamma)$ and $\tilde{g}^{-n}(\gamma(0)) \in R(Y \cup \gamma)$, showing that $\tilde{g}^{-n}(\gamma)$ intersects $\gamma$. On the other hand in $\tilde{U}_-$, $\gamma$ defines a curve from a point in $\pi^{-1}(\partial_- A)$ to a prime end in $\pi^{-1}(\partial_- \tilde{U}_-)$. 


But by condition (4) of Lemma 2.5, \( \gamma \) cannot intersect \( \tilde{g}^{-n}(\gamma) \) for any large \( n \). A contradiction.

Finally the case where \( \partial_- R \) is oriented from the left to the right can be dealt with similarly by reversing the time. This completes the proof of Theorem 1.

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