An enumeration of equilateral triangle dissections

Aleš Drápal*
Department of Mathematics
Charles University
Sokolovská 83
186 75 Praha 8
Czech Republic

Carlo Hämäläinen†
Department of Mathematics
Charles University
Sokolovská 83
186 75 Praha 8
Czech Republic
carlo.hamalainen@gmail.com

April 6, 2010

Abstract

We enumerate all dissections of an equilateral triangle into smaller equilateral triangles up to size 20, where each triangle has integer side lengths. A perfect dissection has no two triangles of the same side, counting up- and down-oriented triangles as different. We computationally prove W. T. Tutte’s conjecture that the smallest perfect dissection has size 15 and we find all perfect dissections up to size 20.

*Supported by grant MSM 0021620839
†Supported by Eduard Cech center, grant LC505.
1 Introduction

We are concerned with the following problem: given an equilateral triangle $\Sigma$, find all dissections of $\Sigma$ into smaller nonoverlapping equilateral triangles. The size of a dissection is the number of nonoverlapping equilateral triangles. An example of such a dissection of size 10 is shown in Figure 1. It is well known that in such a dissection all triangles may be regarded as triangles with sides of integer length. Dissections of squares have been studied earlier [3] as well as dissections of squares into right-angled isosceles triangles [10]. Recently, Laczkovich [15] studied tilings of polygons by similar triangles. The earliest study of dissections of equilateral triangles into equilateral triangles is by Tutte [19]. The problem of dissecting a triangle is different to normal tiling problems where the size of the tiles is known in advance and the tiling area may be infinite. A naive approach to enumerating dissections is to first

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure1.png}
\caption{An example of an equilateral triangle dissection.}
\end{figure}

fix the sizes and number of the dissecting triangles. Observe that in any dissection, some triangles will be oriented in the same way as the triangle $\Sigma$ (these are the up-triangles) while the oppositely oriented triangles are the down-triangles. Let $u_s$ and $d_s$ be the number of up and down triangles of side length $s$, respectively. For any down-triangle the horizontal side is adjacent to the horizontal side of some number of up-triangles. The up-triangles along the bottom of $\Sigma$ are not adjacent to any down-triangle. So if the triangle $\Sigma$ has side length $n \in \mathbb{N}$ then

$$\sum_s s u_s = \sum_s s d_s + n.$$  \hspace{1cm} (1)
A triangle with length \( s \) has height \( \sqrt{3}s/2 \) and so the triangle areas give the relation
\[
\sum_s u_s s^2 + \sum_s d_s s^2 = n^2. \tag{2}
\]

For small values of \( n \) we can solve (1) and (2) for the permissible size and number of up and down triangles, and this data may guide an exhaustive search. We will consider up and down triangles not to be congruent even if they are of the same size. A perfect tiling or perfect dissection has no pair of congruent triangles. This definition of a perfect dissection arises from the fact that it is impossible to have a perfect dissection if orientation is ignored \[3\]. This fact can also easily derived from the results in \[10\]. Tutte conjectured \[3, 19\] that the smallest perfect dissection has size 15 (see also \[18\]). Unfortunately solving (1) and (2) with \( n = 15 \) is computationally intensive and so another approach is needed. Using our enumerative methods we confirm Tutte’s conjecture and provide all perfect dissections up to size 20.

Lines parallel to the outer sides of the main triangle that are induced by a side of a dissecting triangle are dissecting lines. For any dissecting line \( l \), the union of all sides of dissecting triangles that are incident to \( l \) forms one or more contiguous segments. If there are two or more segments, then on \( l \) there exist two dissecting vertices such that all triangles in between are cut by the line into two parts. If such a situation arises for no dissecting line and if no dissecting vertex is incident to six dissecting triangles, then we call the dissection separated. We enumerate all isomorphism classes of separated and nonseparated dissections up to size 20.

2 Dissections and latin bitrades

The connection between equilateral triangle dissections and latin bitrades was first studied in \[10\]. The presentation here follows \[11\]. Consider an equilateral triangle \( \Sigma \) that is dissected into a finite number of equilateral triangles. Dissections will be always assumed to be nontrivial so the number of dissecting triangles is at least four. Denote by \( a, b \) and \( c \) the lines induced by the sides of \( \Sigma \). Each side of a dissecting triangle has to be parallel to one of \( a, b, \) or \( c \). If \( X \) is a vertex of a dissecting triangle, then \( X \) is a vertex of exactly one, three or six dissecting triangles. Suppose that there is no vertex
with six triangles and consider triples \((u, v, w)\) of lines that are parallel to \(a\), \(b\) and \(c\), respectively, and meet in a vertex of a dissecting triangle that is not a vertex of \(\Sigma\). The set of all these triples together with the triple \((a, b, c)\) will be denoted by \(T^*\), and by \(T^\triangle\) we shall denote the set of all triples \((u, v, w)\) of lines that are yielded by sides of a dissecting triangle (where \(u\), \(v\) and \(w\) are again parallel to \(a\), \(b\) and \(c\), respectively). The following conditions hold:

(R1) Sets \(T^*\) and \(T^\triangle\) are disjoint;

(R2) for all \((p_1, p_2, p_3) \in T^*\) and all \(r, s \in \{1, 2, 3\}, r \neq s\), there exists exactly one \((q_1, q_2, q_3) \in T^\triangle\) with \(p_r = q_r\) and \(p_s = q_s\); and

(R3) for all \((q_1, q_2, q_3) \in T^\triangle\) and all \(r, s \in \{1, 2, 3\}, r \neq s\), there exists exactly one \((p_1, p_2, p_3) \in T^*\) with \(q_r = p_r\) and \(q_s = p_s\).

Note that (R2) would not be true if there had existed six dissecting triangles with a common vertex. Conditions (R1–3) are, in fact, axioms of a combinatorial object called latin bitrades \([6, \text{p. } 148]\). A bitrade is usually denoted \((T^*, T^\triangle)\). Observe that the bitrade \((T^*, T^\triangle)\) associated with a dissection encodes qualitative (structural) information about the segments and intersections of segments in the dissection. The sizes and number of dissecting triangles can be recovered by solving a system of equations derived from the bitrade (see below).

Dissections are related to a class of latin bitrades with genus 0. To calculate the genus of a bitrade we use a permutation representation to construct an oriented combinatorial surface \([9, 12, 8]\). For \(r \in \{1, 2, 3\}\), define the map \(\beta_r : T^\triangle \to T^*\) where \((a_1, a_2, a_3)\beta_r = (b_1, b_2, b_3)\) if and only if \(a_r \neq b_r\) and \(a_i = b_i\) for \(i \neq r\). By (R1-3) each \(\beta_r\) is a bijection. Then \(\tau_1, \tau_2, \tau_3 : T^* \to T^*\) are defined by

\[
\tau_1 = \beta_2^{-1}\beta_3, \quad \tau_2 = \beta_3^{-1}\beta_1, \quad \tau_3 = \beta_1^{-1}\beta_2. \tag{3}
\]

We refer to \([\tau_1, \tau_2, \tau_3]\) as the \(\tau_i\) representation. To get a combinatorial surface from a bitrade we use the following construction:

**Construction 2.1.** Let \([\tau_1, \tau_2, \tau_3]\) be the representation for a bitrade where
the $\tau_i$ act on the set $\Omega$. Define vertex, directed edge, and face sets by:

$$V = \Omega$$

$$E = \{(x, y) \mid x\tau_1 = y\} \cup \{(x, y) \mid x\tau_2 = y\} \cup \{(x, y) \mid x\tau_3 = y\}$$

$$F = \{(x, y, z) \mid x\tau_1 = y, y\tau_2 = z, z\tau_3 = x\} \cup \{(x_1, x_2, \ldots, x_r) \mid (x_1, x_2, \ldots, x_r) is a cycle of \tau_1 \} \cup \{(x_1, x_2, \ldots, x_r) \mid (x_1, x_2, \ldots, x_r) is a cycle of \tau_2 \} \cup \{(x_1, x_2, \ldots, x_r) \mid (x_1, x_2, \ldots, x_r) is a cycle of \tau_3 \}$$

where $(x_1, x_2, \ldots, x_k)$ denotes a face with $k$ directed edges $(x_1, x_2), (x_2, x_3), \ldots, (x_{k-1}, x_k), (x_k, x_1)$ for vertices $x_1, \ldots, x_k$.

The first set in the definition of $F$ is the set of triangular faces, while the other three are the $\tau_i$ faces. Assign triangular faces a positive (anticlockwise) orientation, and assign $\tau_i$ faces negative (clockwise) orientation. Now glue the faces together where they share a common directed edge $x\tau_i = y$, ensuring that edges come together with opposite orientation.

The orientation of a triangular face is shown in Figure 2 along with the orientation for a $\tau_i$ face. For the sake of concreteness we have illustrated a 6-cycle face due to a 6-cycle of $\tau_1$. Figure 3 shows the rotation scheme for an arbitrary vertex in the surface.

Figure 2: Orienting faces in the combinatorial surface.

For a bitrade $T = (T^*, T^\triangle)$ with representation $[\tau_1, \tau_2, \tau_3]$ on the set $\Omega$, define order($T$) = $z(\tau_1) + z(\tau_2) + z(\tau_3)$, the total number of cycles, and size($T$) = $|\Omega|$, the total number of points that the $\tau_i$ act on. By some basic
counting arguments we find that there are $\text{size}(T)$ vertices, $3 \cdot \text{size}(T)$ edges, and order($T$) + $\text{size}(T)$ faces. Then Euler’s formula $V - E + F = 2 - 2g$ gives

$$\text{order}(T) = \text{size}(T) + 2 - 2g$$

where $g$ is the genus of the combinatorial surface. We say that the bitrade $T$ has genus $g$. A spherical bitrade has genus 0.

**Example 2.2.** The following bitrade is spherical:

$$T^* = \begin{bmatrix}
* & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 2 & 4 \\
1 & 4 & 2 \\
2 & 1 & 3 & 0 & 2 \\
3 & 4 & 1 & 3 
\end{bmatrix} \quad T^\triangle = \begin{bmatrix}
\triangle & 0 & 1 & 2 & 3 & 4 \\
0 & 4 & 1 & 2 & 3 \\
1 & 2 & 4 \\
2 & 3 & 4 \\
3 & 1 & 3 & 4 
\end{bmatrix}$$

Here, the $\tau_i$ representation is

$$\tau_1 = (000, 022, 044)(134, 142)(201, 213, 232, 220)(304, 333, 311)$$

$$\tau_2 = (000, 304, 201)(213, 311)(022, 220)(134, 232, 333)(044, 142)$$

$$\tau_3 = (000, 220)(201, 311)(022, 232, 142)(213, 333)(044, 134, 304)$$

where the triples $ijk$ refer to entries $(i, j, k) \in T^*$. 
We will generally assume that a bitrade is *separated*, that is, each row, column, and symbol is in bijection with a cycle of $\tau_1$, $\tau_2$, and $\tau_3$, respectively.

We now describe how to go from a separated spherical latin bitrade to a triangle dissection (for more details see [11]). Let $T = (T^*, T^\triangle)$ be a latin bitrade. It is natural to have different unknowns for rows, columns and symbols, and so we assume that $a_i \neq b_j$ whenever $(a_1, a_2, a_3), (b_1, b_2, b_3) \in T^*$ and $1 \leq i < j \leq 3$. (If the condition is violated, then $T$ can be replaced by an isotopic bitrade for which it is satisfied.) Fix a triple $a = (a_1, a_2, a_3) \in T^*$ and form the set of equations Eq($T$) consisting of $a_1 = 0, a_2 = 0, a_3 = 1$ and $b_1 + b_2 = b_3$ if $(b_1, b_2, b_3) \neq (a_1, a_2, a_3)$ and $(b_1, b_2, b_3) \in T^*$. The theorem below shows that if $T$ is a spherical latin bitrade then Eq($T, a$) always has a unique solution in the rationals. The pair $(T, a)$ will be called a *pointed* bitrade.

Write $\bar{r}_i$, $\bar{c}_j$, $\bar{s}_k$ for the solutions in Eq($T, a$) for row variable $r_i$, column variable $c_j$, and symbol variable $s_k$, respectively. We say that a solution to Eq($T, a$) is *separated* if $\bar{r}_i \neq \bar{r}_{i'}$ whenever $i \neq i'$ (and similar for columns and symbols).

For each entry $c = (c_1, c_2, c_3) \in T^\triangle$ we form the triangle $\Delta(c, a)$ which is bounded by the lines $y = \bar{c}_1$, $x = \bar{c}_2$, $x + y = \bar{c}_3$. Of course, it is not clear that $\Delta$ is really a triangle, i.e. that the three lines do not meet in a single point. If this happens, then we shall say that $\Delta(c, a)$ degenerates. Let $\Delta(T, a)$ denote the subset of $T^\triangle$ such that $\Delta(c, a)$ does not degenerate.

A separated dissection with $m$ vertices corresponds to a separated spherical bitrade $(T^*, T^\triangle)$ where $|T^*| = m - 2$. One of the main results of [11] is the following theorem:

**Theorem 2.3 ([11]).** Let $T = (T^*, T^\triangle)$ be a spherical latin bitrade, and suppose that $a = (a_1, a_2, a_3) \in T^*$ is a triple such that the solution to Eq($T, a$) is separated. Then the set of all triangles $\Delta(c, a), c \in T^\triangle$, dissects the triangle $\Sigma = \{(x, y); x \geq 0, y \geq 0 \text{ and } x + y \leq 1\}$. This dissection is separated.

An equilateral dissection can be obtained by applying the transformation $(x, y) \mapsto (y/2 + x, \sqrt{3}y/2)$.

**Example 2.4.** Consider the following spherical bitrade $(T^*, T^\triangle)$:
Let \( a = (a_1, a_2, a_3) = (r_0, c_0, s_4) \). Then the system of equations \( \text{Eq}(T, a) \) has the solution

\[
\begin{align*}
\bar{r}_0 &= 0, \quad \bar{r}_1 = 2/7, \quad \bar{r}_2 = 5/14, \quad \bar{r}_3 = 4/7 \\
\bar{c}_0 &= 0, \quad \bar{c}_1 = 3/14, \quad \bar{c}_2 = 5/14, \quad \bar{c}_3 = 3/7, \quad \bar{c}_4 = 5/7 \\
\bar{s}_0 &= 5/14, \quad \bar{s}_1 = 4/7, \quad \bar{s}_2 = 5/7, \quad \bar{s}_3 = 11/14, \quad \bar{s}_4 = 1.
\end{align*}
\]

The dissection is shown in Figure 4. Entries of \( T^\triangledown \) correspond to triangles in the dissection. For example, \((r_0, c_0, s_0) \in T^\triangledown\) is the triangle bounded by the lines \( y = \bar{r}_0 = 0, x = \bar{c}_0 = 0, x + y = \bar{s}_0 = 5/14\) while \((r_1, c_3, s_2) \in T^*\) corresponds to the intersection of the lines \( y = \bar{r}_1 = 2/7, x = \bar{c}_3 = 3/7, x + y = \bar{s}_2 = 5/7\).

Figure 4: Separated dissection for a spherical bitrade. The labels \( r_i, c_j, s_k \), refer to lines \( y = \bar{r}_i, x = \bar{c}_j, x + y = \bar{s}_k \), respectively. The trade \( T^* \) has 12 entries and the dissection has \( 12 + 2 = 14 \) vertices. Applying the transformation \((x, y) \mapsto (y/2 + x, \sqrt{3}y/2)\) gives an equilateral dissection.
Remark 2.5. Suppose that the dissection $\Sigma$ has no vertex of degree 6. Pick a vertex $X$ of degree 4. If we move to the right along the row segment to the next vertex $X'$ then we have $X_\tau_1 = X'$. Similarly, moving along the diagonal segments gives the action of $\tau_2$ and $\tau_3$. If we identify the three vertices of degree 2 then the dissection encodes, geometrically, the permutation representation of the bitrade.

If a dissection has a vertex of degree 6 then the dissecting triangles do not (uniquely) define a partial latin square and hence do not encode a latin bitrade. However, we can recover a separated bitrade by the following procedure. For each vertex $X$ of degree 6, choose one segment (say, the $r_i$ segment) to stay fixed. Then for the $c_j$ and $s_k$ segments, label the column segment below $X$ with a new name $c'$ and label the symbol segment below $X$ with a new name $s'$. For example, the centre vertex in Figure 5 results in the new labels $c_3$ and $s_3$. The resulting separated bitrade is:

$$
\begin{array}{cccc}
\ast & c_0 & c_1 & c_2 & c_3 \\
r_0 & s_2 & s_3 & s_0 \\
r_1 & s_0 & s_1 & s_2 & s_3 \\
r_2 & s_1 & s_2 & s_3 & s_0 \\
\end{array}
\begin{array}{cccc}
\triangle & c_0 & c_1 & c_2 & c_3 \\
r_0 & s_0 & s_2 & s_3 \\
r_1 & s_1 & s_2 & s_3 & s_0 \\
r_2 & s_2 & s_1 & s_3 \\
\end{array}
$$

This procedure works for any number of vertices of degree 6, as long as care is taken to only relabel column or symbol segments below a vertex of degree 6 and not to relabel a segment more than once.$^{[1]}$

Recently Theorem 2.3 has been strengthened to cover nonseparated as well as separated dissections:

Theorem 2.6 ([7]). Let $T = (T^*, T^{\triangle})$ be a spherical latin bitrade. Then for any $a = (a_1, a_2, a_3) \in T^*$, the set $\Delta(T, a)$ of non-degenerate triangles dissects the triangle $\Sigma = \{(x, y); x \geq 0, y \geq 0$ and $x + y \leq 1\}$. The dissection may not be separated.

A fundamental consequence of the theorem is that any dissection of size $s$ can be derived from a pointed spherical bitrade $(T, a)$ of size $s$. Note that the systems of equations Eq($T, a$) have also other applications: both [5] and [11] use them to show that every spherical latin trade can be embedded into a finite abelian group.

$^{[1]}$For a concrete implementation, see generate_bitrade_via_geometric_data in triangle_dissections.py in [14].
3 Computational results

Cavenagh and Lisoněk [4] showed that spherical bitrades are equivalent to planar Eulerian triangulations. To enumerate triangle dissections we use plantri [1, 2] to enumerate all planar Eulerian triangulations up to size 20 (we also note that in [20] all trades and bitrades have been enumerated up size 19). We wrote a plugin [13] to output the equivalent spherical latin bitrade \((U^*, U^\triangle)\) for each triangulation. For each such \((U^*, U^\triangle)\) we find all solutions Eq\((T, a)\) for all \(a \in T^*\) and compute each dissection (in practise this is a list of triangles \(\Delta(c, a)\) for each \(c = (c_1, c_2, c_3) \in T^\triangle\)). To filter out isomorphic dissections we apply all six elements of the symmetry group for a unit-side equilateral triangle (identity, two rotations, and three reflections). The canonical signature of a dissection \(\Delta(T, a)\) is the ordered list \([(x, y) \mid (x, y)\text{ is a vertex of } \Delta(T, a)]\). We repeat the whole process for the bitrade \((U^\triangle, U^*)\) because there are spherical latin bitrades where \((U^*, U^\triangle)\) is not isomorphic to \((U^\triangle, U^*)\). The final counts for the number of dissections up to isomorphism are found by simply removing duplicate signatures.

While the solutions to Eq\((T, a)\) exist in the rationals, we find it easier to work with the final equilateral dissections instead. We use the SymPy package [17] to perform exact symbolic arithmetic on the canonical form of
3.1 Dissections and automorphism groups

Using Theorems 2.3 and 2.6 we have enumerated the number of isomorphism classes of dissections of size \( n \leq 20 \). We also record \( A(n, k) \), the number of dissections of size \( n \) with automorphism group of order \( k \). See Figures 6 and 7 for the data.

The referee raised the question of asymptotic behaviour. Let us denote by \( d_n \) the number of all dissections of size \( n \). Thus \( d_4 = 1, \ldots, d_{13} = 574, \ldots, d_{20} = 2674753 \). There are some reasons to believe that \( d_n \) can be estimated as \( \sigma(n)^n \), where \( \sigma(n) \) is a slowly growing function. The asymptotic behaviour of \( \sigma(n) \) is not clear yet and is a subject of ongoing research. Note however that \( d_n \geq e_n = (3.43)^{n-8} \) for every \( n \not\in \{18, 19\} \) such that \( 8 \leq n \leq 20 \). If we put \( \mu_n = e_n/d_n \), then in this interval the approximate values of \( \mu_n \) are 0.33, 0.38, 0.51, 0.65, 0.74, 0.83, 0.89, 0.94, 0.97, 0.99, 1.00, 1.00 and 0.99. The fact that \( e_{18} \) and \( e_{19} \) are slightly greater than \( d_{18} \) and \( d_{19} \), while \( e_{20} \) is smaller than \( d_{20} \), seems to be surprising.

3.2 Perfect dissections

Using our enumeration code we can confirm W. T. Tutte's conjecture \( \[3, 19\] \) that the smallest perfect dissection has size 15 (see Figure 8). The perfect dissections of size 16 and 17 are shown in Figures 9 and 10. The perfect dissections of size up to 20 are available in PDF format \( [14] \). The following table summarises the known number of isomorphism classes of perfect dissections:

| \( n \) | \# perfect dissections |
|--------|-----------------------|
| 15     | 2                     |
| 16     | 2                     |
| 17     | 6                     |
| 18     | 23                    |
| 19     | 64                    |
| 20     | 181                   |

It is an open problem to determine if there exists a nonseparated perfect dissection. If such a dissection exists then it will have size greater than 20.
| $n$ | # dissections | $A(n, 1)$ | $A(n, 2)$ | $A(n, 3)$ | $A(n, 6)$ |
|-----|--------------|-----------|-----------|-----------|-----------|
| 4   | 1            | 0         | 0         | 0         | 1         |
| 6   | 1            | 0         | 1         | 0         | 0         |
| 7   | 2            | 0         | 1         | 0         | 1         |
| 8   | 3            | 2         | 1         | 0         | 0         |
| 9   | 8            | 4         | 4         | 0         | 0         |
| 10  | 20           | 15        | 4         | 0         | 1         |
| 11  | 55           | 47        | 8         | 0         | 0         |
| 12  | 161          | 146       | 15        | 0         | 0         |
| 13  | 478          | 460       | 17        | 0         | 1         |
| 14  | 1496         | 1459      | 37        | 0         | 0         |
| 15  | 4804         | 4746      | 58        | 0         | 0         |
| 16  | 15589        | 15506     | 82        | 0         | 1         |
| 17  | 51377        | 51223     | 154       | 0         | 0         |
| 18  | 172162       | 171923    | 239       | 0         | 0         |
| 19  | 583810       | 583426    | 383       | 0         | 1         |
| 20  | 1998407      | 1997752   | 655       | 0         | 0         |

Figure 6: Number of separated dissections of size $n$, up to isomorphism. For each $n$, the column $A(n, k)$ records the number of dissections of size $n$ with automorphism group of order $k$. 
Figure 7: Number of separated and nonseparated dissections of size $n$, up to isomorphism. For each $n$, the column $A(n,k)$ records the number of dissections of size $n$ with automorphism group of order $k$.

| $n$ | # dissections | $A(n,1)$ | $A(n,2)$ | $A(n,3)$ | $A(n,6)$ |
|-----|---------------|----------|----------|----------|----------|
| 4   | 1             | 0        | 0        | 0        | 1        |
| 6   | 1             | 0        | 1        | 0        | 0        |
| 7   | 2             | 0        | 1        | 0        | 1        |
| 8   | 3             | 2        | 1        | 0        | 0        |
| 9   | 9             | 4        | 4        | 0        | 1        |
| 10  | 23            | 15       | 7        | 0        | 1        |
| 11  | 62            | 51       | 11       | 0        | 0        |
| 12  | 188           | 162      | 25       | 0        | 1        |
| 13  | 574           | 532      | 39       | 0        | 3        |
| 14  | 1826          | 1745     | 81       | 0        | 0        |
| 15  | 5953          | 5795     | 157      | 0        | 1        |
| 16  | 19664         | 19380    | 277      | 2        | 5        |
| 17  | 66049         | 65489    | 560      | 0        | 0        |
| 18  | 224700        | 223625   | 1070     | 0        | 5        |
| 19  | 771859        | 769851   | 1992     | 8        | 8        |
| 20  | 2674753       | 2670755  | 3998     | 0        | 0        |

Figure 8: The two perfect dissections of size 15.
3.3 Other observations

There exist bitrades of size $n \geq 10$ for which there is no separated dissection of size $n$. The following table shows a sample of these bitrades, giving the sizes of all possible dissections for the particular bitrade of size $n$.

| $n$ | size of possible dissections |
|-----|-----------------------------|
| 10  | 4, 7                        |
| 12  | 4, 7, 8, 9                  |
| 12  | 4, 9, 11                    |
| 12  | 4, 11                       |
| 12  | 6, 9                        |
| 12  | 9                           |
| 12  | 11                          |
| 13  | 4, 7                        |
| 13  | 4, 7, 9, 10                 |
| 13  | 4, 7, 10                    |
| 13  | 4, 9, 10, 12                |
| 13  | 4, 11, 12                   |
| 13  | 9, 10, 12                   |
| 13  | 10, 11                      |

A trivial dissection has triangles of only one size. Apart from $n = 4$, all trivial dissections are nonseparated. The following table lists lower bounds on the number of bitrades that give rise to the (unique) separated dissection.
Figure 10: The six perfect dissections of size 17.
of size $n$ (naturally we allow for nonseparated solutions to find these trivial dissections).

| $n$ | lower bound on number of source bitrades |
|-----|----------------------------------------|
| 4   | 2380591                                |
| 8   | 111890                                 |
| 13  | 1321                                   |

For each size $n$ we collect examples of dissections with the largest relative difference in size between the largest and smallest triangle in the dissection. In all cases the smallest triangle has size 1 and the largest triangle is given in the second column of the table below:

| $n$ | size of largest triangle |
|-----|--------------------------|
| 4   | 1                        |
| 6   | 2                        |
| 7   | 2                        |
| 8   | 3                        |
| 9   | 4                        |
| 10  | 5                        |
| 11  | 7                        |
| 12  | 9                        |
| 13  | 12                       |
| 14  | 16                       |
| 15  | 21                       |
| 16  | 28                       |
| 17  | 37                       |
| 18  | 49                       |
| 19  | 67                       |
| 20  | 91                       |

The dissections that give rise to these maximum ratios are shown below (sorted by dissection size $n$):
References

[1] G. Brinkmann, B. D. McKay, plantri (software), http://cs.anu.edu.au/bdm/plantri.

[2] G. Brinkmann, B. D. McKay, Fast generation of some classes of planar graphs, Electronic Notes in Discrete Mathematics 3 (1999) 28–31. http://cs.anu.edu.au/bdm/plantri.

[3] R. L. Brooks, C. A. B. Smith, A. H. Stone, W. T. Tutte, The dissection of rectangles into squares, Duke Math. J. 7 (1940) 312–340.

[4] N. J. Cavenagh, P. Lisoněk, Planar eulerian triangulations are equivalent to spherical latin bitrades, J. Comb. Theory, Ser. A 115 (1) (2008) 193–197.

[5] N. J. Cavenagh, I. M. Wanless, Latin trades in groups defined on planar triangulations, J. Algebraic Combin. 30 (2009) 323–347.

[6] C. J. Colbourn, J. H. Dinitz, I. M. Wanless, Handbook of combinatorial designs, Chapman & Hall/CRC, Boca Raton, FL, 2007.

[7] A. Drápal, Dissections of equilateral triangles and pointed spherical latin bitrades, submitted.

[8] A. Drápal, Geometrical structure and construction of latin trades, Advances in Geometry 9 (3) (2009) 311–348.

[9] A. Drápal, Geometry of latin trades, manuscript circulated at the conference Loops, Prague (2003).

[10] A. Drápal, Hamming distances of groups and quasi-groups, Discrete Math. 235 (1-3) (2001) 189–197, combinatorics (Prague, 1998).

[11] A. Drápal, V. Kala, C. Hämäläinen, Latin bitrades, dissections of equilateral triangles and abelian groups, Journal of Combinatorial Designs, Volume 18 Issue 1 (2010), 1–24.

[12] C. Hämäläinen, Latin bitrades and related structures, PhD in Mathematics, Department of Mathematics, The University of Queensland, http://carlo-hamalainen.net/phd/hamalainen-20071025.pdf (2007).
[13] C. Hämäläinen, Spherical bitrade enumeration code, http://bitbucket.org/carlohamalainen/spherical
[14] C. Hämäläinen, Triangle dissections code, http://bitbucket.org/carlohamalainen/dissections
[15] M. Laczkovich, Tilings of polygons with similar triangles, Combinatorica 10 (3) (1990) 281–306.
[16] J. D. Skinner, C. A. B. Smith, W. T. Tutte, On the dissection of rectangles into right-angled isosceles triangles, J. Combin. Theory Ser. B 80 (2) (2000) 277–319.
[17] SymPy Development Team, SymPy: Python library for symbolic mathematics (2009). URL http://www.sympy.org
[18] Tiling by similar triangles, http://www.squaring.net/tri/twt.html
[19] W. T. Tutte, The dissection of equilateral triangles into equilateral triangles, Proc. Cambridge Philos. Soc. 44 (1948) 463–482.
[20] Ian M. Wanless. A computer enumeration of small Latin trades. Australas. J. Combin., 39:247–258, 2007.
Appendix A: Triangle dissections

Here we present representatives of the isomorphism classes of triangle dissections of size \( n \in \{4, 6, 7, 8, 9, 10\} \).

\( n = 4 \)

\( n = 6 \)
$n = 9$
