Negation-Closure for JSON Schema

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Abstract. JSON Schema is an evolving standard for describing families
of JSON documents. It is a logical language, based on a set of assertions
that describe features of the JSON value under analysis and on logical or
structural combinators for these assertions, including a negation operator.
Most logical languages with negation enjoy negation closure, that is,
for every operator they have a negation dual that expresses its negation.
We show that this is not the case for JSON Schema, we study how that
changed with the latest versions of the Draft, and we discuss how the
language may be enriched accordingly. In the process, we define an alge-
braic reformulation of JSON Schema, which we successfully employed in
a prototype system for generating schema witnesses.

Keywords: JSON Schema, Negation Closure, Schema Languages

1 Introduction

JSON is a simple data language whose terms represent trees constituted by
nested records and arrays, with atomic values at the leaves, and is now widely
used for data exchange on the Web. JSON Schema \[13\] is an ever evolving specifi-
cation for describing families of JSON terms, and heavily used for specifying web
applications and REST services \[8\], compatibility of operators in data science
pipelines \[1\], as well as schemas in NoSQL systems (e.g., MongoDB \[12\]).

JSON Schema is a logical language based on a set of assertions that describe
features of JSON values, and on boolean and structural combinators for these
assertions, including negation and recursion. The expressive power of this com-
plex language, and the complexity of validation and satisfiability have recently
been studied: Pezoa et al. \[14\] relied on tree automata and MSO to study the
expressive power, while Bourhis et al. \[6\] mapped JSON Schema onto an equiva-
 lent modal logic, called JSL, to investigate the complexity of validation and
satisfiability, and proved that satisfiability is in 2EXPTIME in the general case.
In this context, we are working on tools for the simplification and manipulation of JSON Schema, and we have studied the problem of negation-closure, that is the property that every negated assertion can be rewritten into a negation-free one. Most logical languages with negation enjoy negation-closure because, for every operator, they have a “dual” that allows negation to be pushed to the leaves of any logical formula, as happens for the pairs and-or and for-all-exists in first order logics, and for the modal-logic pair ♦-□ used in JSL to encode JSON Schema. Negation-closure is an important design principle for a logical system, since it ensures that, for every algebraic property that involves an operator, a “symmetric” algebraic property holds for its dual operator. This facilitates both reasoning and automatic manipulation.

In this paper, we prove that JSON Schema, despite having the same expressive power as JSL, does not enjoy negation-closure, by showing that both objects and arrays are described by pairs of operators that are “almost” able to describe the negation of the other one, but not “exactly”. We mostly focus here on Draft-06 [15] of the JSON Schema language. Indeed, we have built a repository of JSON Schema documents (available at [2]), by harvesting from GitHub all JSON documents with a schema attribute, retrieving over 91K documents; we found that almost all of them adhere to Draft-06 or to a preceding one.

We show that, in the most common use-cases, negation can indeed be pushed through these operators, and we exactly characterize the specific cases when this is not possible. Moreover, we show that Draft 2019-09 introduces another twist: a new version of the contains operator allows not-elimination for array schemas in the general case, but only through a complex encoding.

In addition, based on these results, we define and present here a negation-closed extension of JSON Schema, with the same expressive power as the original language, but where all operators have a negation dual, together with a simple and complete not-elimination algorithm for the extended language.

Our recent witness generation tool for JSON Schema [1] is capable of generating a valid instance, given a satisfiable schema. That approach is able to deal with almost the totality of JSON Schema (the only mechanism we rule out is uniqueItems), and to this end it proceeds in an inductive fashion: first it generates witnesses for subexpressions and then these are used for witness generation of surrounding expressions. In this context, one problem to solve is negation: there is no way to generate a witness for ¬S starting from a witness for an arbitrarily complex schema S. To address this issue, our witness generation approach heavily relies on the not-elimination algorithm we present here, and we believe that the present study sheds light on aspects of JSON Schema that have not been studied in previous works [14,6], and that may be useful for the development of tools manipulating JSON Schemas.

Main Contributions:
(i) We study the problem of negation-closure of JSON Schema, that is, which operators are endowed with a negation dual. We show that JSON Schema structural operators are not negation-closed, and we characterize the schemas whose negation cannot be expressed without negation. In the process, we
present a reformulation of JSON Schema that is *algebraic*, that is, where a subschema can be freely substituted by an equivalent one.

(ii) We then extend the algebraic version of JSON Schema to make it negation-closed, and we define a not-elimination algorithm for this closed algebra. To our knowledge, this is the first algorithm for not-elimination that also deals with negated recursive variables.

(iii) We extend our study to the latest drafts of JSON Schema, which introduced operators that impact negation closure.

**Paper Outline:** In Section 2 we present an algebraic version JSON Schema, and its formal semantics. Section 4 studies negation closure. Section 5 introduces the latest JSON Schema draft. Section 6 presents our experiments. We conclude after a review of related work.

## 2 An algebraic version of JSON Schema

In this work we rely on an algebraic reformulation of JSON Schema, that has a direct correspondence with JSON Schema, but that is more amenable for formal development of schema manipulation algorithms. Also, and importantly, our algebra enjoys substitutability, in the sense that the substitution of every subterm in an algebraic term with a semantically equivalent subterm preserves the semantics of the entire term, independently of the context around the subterm. Actually, current formulations of JSON Schema do not enjoy this property, while it is fundamental in the design of schema manipulation algorithms.

In the remaining part of this section we first define the JSON data model and then our algebraic definitions of JSON Schema.

### 2.1 The data model

JSON values are either basic values, objects, or arrays. Basic values $B$ include the null value, booleans, numbers $m$, and strings $s$. Objects $O$ represent sets of *members*, each member (or *field*) being a name-value pair $(k, J)$, and arrays $A$ represent sequences of values with positional access. We will only consider here objects without repeated names. Objects and arrays may be empty.

In JSON syntax, a name is itself a string, and hence it is surrounded by quotes. Below, we specify the data model syntax, where $n$ is a natural number with $n \geq 0$, and $k_i \in \text{Str}$ for $i = 1, \ldots, n$.

**JSON expressions**

$$
J ::= B \mid O \mid A \\
B ::= \text{null} \mid \text{true} \mid \text{false} \mid m \mid s \quad m \in \text{Num}, s \in \text{Str} \quad \text{Basic values}
$$

$$
O ::= \{k_1 : J_1, \ldots, k_n : J_n\} \quad n \geq 0, \ i \neq j \Rightarrow k_i \neq k_j \quad \text{Objects}
$$

$$
A ::= [J_1, \ldots, J_n] \quad n \geq 0 \quad \text{Arrays}
$$

### 2.2 The algebra

We now introduce our algebraic presentation of JSON Schema. The syntax of the algebra is shown below. In the grammar $n$ is always a natural number starting
from 0, so that we use here indexes going from 1 to \( n+1 \) when at least one element is required. In \( \text{req}(k_1, \ldots, k_n) \), each \( k_i \) is a string. The other metavariables are listed in the first line of the grammar, where \( \mathbb{R} \) indicates “any real number”, \( \mathbb{R}_{>0} \) denotes positive real numbers, \( \mathbb{N} \) are the naturals zero included, and the sets \( \mathbb{R}^{-\infty} \), \( \mathbb{R}^{\infty} \), and \( \mathbb{N}^{\infty} \) stand for the base set enriched with the extra symbols \(-\infty\) or \( \infty \). \( \mathbb{J} \) is the set of all JSON terms. Apart from the syntax, the algebra reflects all operators of JSON Schema, Draft-06.

\[
\begin{align*}
T & ::= \text{Arr} | \text{Obj} | \text{Null} | \text{Bool} | \text{Str} | \text{Num} \\
S & ::= \text{type}(T_1, \ldots, T_{n+1}) | \text{const}(J) | \text{enum}(J_1, \ldots, J_{n+1}) | \text{len}_j^l | \text{betw}_m^M | \text{mulOf}(q) \\
& \quad | \text{pattern}(r) | \text{props}(r_1 : S_1, \ldots, r_n : S_n ; S) | \text{pro}_j^l | \text{req}(k_1, \ldots, k_n) \\
& \quad \quad | \text{pNames}(S) \\
& \quad | \text{items}(S_1, \ldots, S_n ; S_{n+1}) | \text{contains}(S) | \text{ite}_j^l | \text{uniqueItems} \\
& \quad | x | t | f | \neg S | S_1 \lor S_2 | \bigoplus(S_1, \ldots, S_{n+1}) \\
& \quad | S_1 \Rightarrow S_2 | (S_1 \Rightarrow S_2 | S_3) | S_1 \land S_2 | \{S_1, \ldots, S_n\} \\
E & ::= x_1 : S_1, \ldots, x_n : S_n \\
D & ::= S \text{ def}(E)
\end{align*}
\]

Each schema expresses properties of an instance which is a JSON value; the semantics of a schema \( S \) with respect to an environment \( E \), hence, is the set \( [S]_E \) of JSON instances that satisfy that schema, as specified in Figure 1. The environment \( E \) is a set of pairs \( (x : S) \), that are introduced by the operator \( D = S \text{ def}(x_1 : S_1, \ldots, x_n : S_n) \) and are used to interpret variables \( x_i \), as discussed below.

The schema \( \text{type}(T_1, \ldots, T_{n+1}) \) is satisfied by any instance belonging to one of the listed predefined JSON types.

The algebra includes boolean operators (\( \neg, \land, \lor, \Rightarrow, \Leftrightarrow, |, \{S_1, \ldots, S_n\} \), and \( \bigoplus \)) as well as typed operators (the remaining ones, each one related to one type \( T \), with the exception of \( \text{Null} \)), whose semantics is described below.

\( \text{const}(J) \) is only satisfied by the instance \( J \), and \( \text{enum}(J_1, \ldots, J_{n+1}) \) is the same as \( \text{const}(J_1) \lor \cdots \lor \text{const}(J_{n+1}) \).

\( \text{pattern}(r) \) means: if the instance is a string, then it matches \( r \). This conditional semantics is a central feature of JSON Schema: all operators related to one specific type, that is, all operators from \( \text{len}_j^l \) to \( \text{uniqueItems} \), have an if-then-else semantics, with the if part always: “if the instance belongs to the type associated with this assertion”, so that they discriminate inside their type but accept any instance of any other type.

\( \text{len}_j^l \) means: if the instance is a string, then its length is included between \( l \) and \( j \). \( \text{betw}_m^M \) means: if the instance is a number, then it is included between \( m \) and \( M \), extremes included. \( \text{mulOf}(q) \) means: if the instance \( J \) is a number, then \( J = i \ast q \), for some integer \( i \).

An instance \( J \) satisfies the assertion \( \text{props}(r_1 : S_1, \ldots, r_n : S_n ; S) \) iff the following holds: if the instance \( J \) is an object, then for each pair \( k : J' \) appearing
at the top level of $J$, then, for every $r_i : S_i$ such that $k$ matches $r_i$, then $J'$ satisfies $S_i$, and, when $k$ does not match any pattern in $r_1, . . . , r_n$, then $J'$ satisfies $S$ (hence, it combines the three JSON Schema operators properties, patternProperties, and additionalProperties).

$\text{proj}_i^j$ means: if the instance is an object, it has at least $l$ and at most $j$ properties. Assertion $\text{req}(k_1, . . . , k_n)$ means: if the instance is an object, then, for each $k_i$, one of the names of the instance is equal to $k_i$. The assertion $\text{pNames}(S)$ means that, if the instance is an object, then every member name of that object satisfies $S$.

An instance $J$ satisfies $\text{items}(S_1, . . . , S_n; S_{n+1})$ iff the following holds: if $J$ is an array, then each of its elements at position $i \leq n$ satisfies $S_i$, while further elements satisfy $S_{n+1}$. Note that no constraint is posed over the length of $J$: if it is strictly shorter than $n$, or empty, that is not a problem (this operator combines the two JSON Schema operators items and additionalItems). To constrain the array length we have $\text{ite}_l^j$, satisfied by $J$ when: if $J$ is an array, its length is between $l$ and $j$. Assertion $\text{contains}(S)$ means: if the instance is an array, then it contains at least one element that satisfies $S$. The assertion $\text{uniqueItems}$ means that, if the instance is an array, then all of its items are pairwise different.

The boolean operators $t, f, \neg, \vee, \Rightarrow, \land$ combine the results of their operands in the standard way, where $(S_1 \Rightarrow S_2 | S_3)$ stands for $(S_1 \land S_2) \lor ((\neg S_1) \land S_3)$, and $(S_1, . . . , S_{n+1})$ is satisfied iff exactly one of the arguments holds. $\{S_1, . . . , S_n\}$ is the same as $S_1 \land . . . \land S_n$.

An instance $J$ satisfies a schema document $D = S \text{defs}(E)$ iff it satisfies $S$ in the environment $E = x_1 : S_1, . . . , x_n : S_n$, which means that, when a variable $x_i$ is met while checking whether $J$ satisfies $S$, $x_i$ is substituted by $E(x_i)$, that is, by $S_i$. This is not an inductive definition, since $S_i$ is generally bigger than $x_i$, and results in a cyclic definition when we have environments such as $\text{defs}(x : x)$ or $\text{defs}(x : \neg x)$. The JSON Schema standard rules out such environments by specifying that checking whether $J$ satisfies $S$ by expanding variables must never result in an infinite loop. This can be ensured by imposing a guardedness condition on $E$, as follows.

Let us say that $x_i$ unguardedly depends on $x_j$ if the definition of $x_i$ contains one occurrence of $x_j$ that is not in the scope of any typed operator (otherwise, we say that the occurrence is guarded by a typed operator): for instance, in $\text{defs}(x : (\text{items}(y; w) \land z))$, $x$ unguardedly depends on $z$, while $y$ and $w$ are guarded by $\text{items}(;).$ Recursion is guarded if the unguardedly depends relation is acyclic: no pair $(x, x)$ belongs to its transitive closure. Informally, guarded recursion requires that any cyclic dependency must traverse a typed operator, which ensures that, when a variable is unfolded for the second time, the resulting schema is applied to an instance that is strictly smaller than the one analyzed at the moment of the previous unfolding. This notion was introduced as well-formedness in related work [133].

We formalize JSON Schema specifications as follows. First of all, we add a parameter $i \in \mathbb{N}$ to the semantic function $[S]_E^i$, and we give a definition of $[[x]]_E^i$. 

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that is inductive on $i$:

$$\lfloor x \rfloor_E^i = \emptyset \quad \text{and} \quad \lfloor x \rfloor_E^{i+1} = \lfloor E(x) \rfloor_E^i$$

This provides a definition of $\lfloor S \rfloor_E^i$ that is inductive on the lexicographic pair $(i, |S|)$. Because of negation, this sequence of interpretations is not necessarily monotonic in $i$, but we can still extract an exists-forall limit from it, by stipulating that an instance $J$ belongs to the limit $\lfloor S \rfloor_E^i$ if an $i$ exists such that $J$ belongs to every interpretation that comes after $i$:

$$\lfloor S \rfloor_E^i = \bigcup_{i \in \mathbb{N} J \geq i} \lfloor S \rfloor_E^i$$

Now, it is easy to prove that this interpretation satisfies JSON Schema specifications, since, for guarded schemas, it enjoys the properties expressed in Theorem [1] stated below.

**Definition 1.** An environment $E = x_1 : S_1, \ldots, x_n : S_n$ is closing for $S$ if all variables used in the bodies $S_1, \ldots, S_n$ and in $S$ are included in $x_1, \ldots, x_n$.

**Lemma 1 (Stability).** For every $(S, E)$ where $E$ is guarded and closing for $S$, for every $J$:

$$\exists i. (\forall j \geq i. J \in \lfloor S \rfloor_E^i) \lor (\forall j \geq i. J \notin \lfloor S \rfloor_E^i)$$

**Proof.** We define the degree $d(S)$ of a schema $S$ in $E$ as follows. If $S$ is a variable $x$, then $d(x) = d(E(x)) + 1$. If $S$ is not a variable, then $d(x)$ is the maximum degree of all unguarded variables in $E(x)$ and, if it contains no unguarded variable, then $d(S) = 0$. This definition is well-founded thanks to the guardedness condition. We prove the lemma by induction on $(J, d(S), S)$, in this order of significance. We want to prove that, for every triple $(J, S, E)$ (but we will ignore $E$ for simplicity), exists a “fixing index”, that is an $i$ such that, for any $j \geq i$, the question whether $J$ belongs to $\lfloor S \rfloor_E^i$ always yields the same answer.

Let $S = x$. We want to prove that, for any $J$:

$$\exists i. (\forall j \geq i. J \in \lfloor x \rfloor_E^i) \lor (\forall j \geq i. J \notin \lfloor x \rfloor_E^i)$$

This is equivalent to the following statement:

$$\exists i. (\forall j \geq i. J \in \lfloor E(x) \rfloor_E^{i-1}) \lor (\forall j \geq i. J \notin \lfloor E(x) \rfloor_E^{i-1})$$

i.e. $\exists i. (\forall j \geq i. J \in \lfloor E(x) \rfloor_E^i) \lor (\forall j \geq i. J \notin \lfloor E(x) \rfloor_E^i)$

This last statement holds by induction, since $d(x) = d(E(x)) + 1$, hence the term $J$ is the same but the degree of $e(x)$ is strictly smaller than that of $S = x$.

Let $S = S' \land S''$. We want to prove that, for any $J$:

$$\exists i. (\forall j \geq i. J \in \lfloor S' \land S'' \rfloor_E^i) \lor (\forall j \geq i. J \notin \lfloor S' \land S'' \rfloor_E^i)$$

By induction, for the same $J$ the following statements hold:
∀i. (∀j ≥ i. J ∈ [S_E]
) ∨ (∀j ≥ i. J ∉ [S_E])

∀i. (∀j ≥ i. J ∈ [S'E_E]) ∨ (∀j ≥ i. J ∉ [S'E_E])

If we take a witness I' for i in the first property and a witness I'' for i in the second property, we have that max(I', I'') is a fixing index for J and S. We reason in the same way for the other boolean operators.

Let S = items(S_1, . . . , S_n; S_{n+1}). We want to prove that

∃i. (∀j ≥ i. J ∈ [items(S_1, . . . , S_n; S_{n+1})]_E) ∨ (∀j ≥ i. J ∉ [items(S_1, . . . , S_n; S_{n+1})]_E)

Consider the semantics of items(S_1, . . . , S_n; S_{n+1}):

{J | J = [J_1, . . . , J_m], l ∈ {1..m} ⇒ ((∀k ∈ {1..n}. l = k ⇒ J_l ∈ [S_k]_E) ∧ (l > n ⇒ J_l ∈ [S_{n+1}]_E))}

The problem J ∈ [items(S_1, . . . , S_n; S_{n+1})]_E gets fixed once we fix this problem for all pairs (J_l, S_k) and (J_l, S_{n+1}) that appear in the definition. Since each J_l is a strict subterm of J, each pair has a fixing index by induction, hence the maximum among these indexes fixes all pairs, hence it fixes the entire question over (J, S). Observe that the fact that each J_l is strictly smaller than J is essential since, in general, the degree of each S_k may be bigger than the degree of S, since they are all in guarded position.

All other guarding operators can be treated in the same way.

**Theorem 1.** For any E guarded, the following equality holds:

\[ [E(x)]_E = [x]_E \]

Moreover, for each equivalence in Figure 1, the equivalence still holds if we substitute every occurrence of \([S]_E\) with \( [S]_E \), obtaining for example:

\[ [pNames(S)]_E = \{ J | J = \{ k_1 : J_1, . . . , k_m : J_m \}, l \in \{1..m} \Rightarrow k_l \in [S]_E \} \]

from

\[ [pNames(S)]_E = \{ J | J = \{ k_1 : J_1, . . . , k_m : J_m \}, l \in \{1..m} \Rightarrow k_l \in [S]_E \} \].

**Proof.** We want to prove that

\[ \bigcup_{i \in \mathbb{N}} \bigcap_{j \geq i} [x]_E \bigcup_{i \in \mathbb{N}} \bigcap_{j \geq i} [E(x)]_E \]

Assume that \( J \in \bigcup_{i \in \mathbb{N}} \bigcap_{j \geq i} [x]_E \). Then,

\( \exists i. \forall j \geq i. J \in [x]_E \). Let I be one i with that property. We have that \( \forall j \geq I. J \in [x]_E \), i.e., \( \forall j \geq I. J \in [E(x)]_{E}^{j-1} \), which implies that \( \forall j \geq I. J \in [E(x)]_E \), hence \( \exists i. \forall j \geq i. J \in [E(x)]_E \).

In the other direction, assume \( J \in \bigcup_{i \in \mathbb{N}} \bigcap_{j \geq i} [E(x)]_E \). Hence,
In this case, both directions are immediate.

For the second property, assume $J \in [\mathbb{P}_{\text{Names}}(S)]_E$. We have:

$J \in [\mathbb{P}_{\text{Names}}(S)]_E \iff$

$\exists i. \forall j \geq i. J \in \mathbb{E} \iff$

$\exists i. \forall j \geq i. J \in \mathbb{E} \iff$

$J \in [\mathbb{P}_{\text{items}}(S_1, \ldots, S_n; S_{n+1})]_E \iff$

$\exists i. \forall j \geq i. J \in [\mathbb{P}_{\text{items}}(S_1, \ldots, S_n; S_{n+1})]_E \iff$

Assume $J \in [\mathbb{P}_{\text{items}}(S_1, \ldots, S_n; S_{n+1})]_E$. We have:

$J \in [\mathbb{P}_{\text{items}}(S_1, \ldots, S_n; S_{n+1})]_E \iff$

The step (*) is not obvious. Here, we have a double implication with the following structure:

$\exists i. \forall j \geq i. (Q_1 \land \ldots \land Q_n) \iff (*) (\exists i. \forall j \geq i. Q_1) \land \ldots \land (\exists i. \forall j \geq i. Q_n)$

In the $\Rightarrow$ direction, the implication is immediate. In the $\Leftarrow$ direction, for each existential $\exists i. \forall j \geq i. Q_n$ we may have a different witness $I_1, \ldots, I_n$ for each existential quantification, but we can choose the maximum $I_M$, since, if $j \geq I_M$, then $j \geq I_1$ for every $i \in \{1..n\}$.

The cases for $\mathbb{P}_{\text{props}}$ and $\land$ are analogous. In the cases for $\mathbb{P}_{\text{contains}}(S)$ and $\lor$, instead, we have to commute $\exists i. \forall j \geq i.$ with a disjunction. For example, assume $J \in [\mathbb{P}_{\text{contains}}(S)]_E$. We have:

$J \in [\mathbb{P}_{\text{contains}}(S)]_E \iff$

$\exists i. \forall j \geq i. J \in \mathbb{E} \iff$

$\exists i. \forall j \geq i. J \in \mathbb{E} \iff$

$J \in [-S]_E \iff$
\[\exists i. \forall j \geq i. J \in [\neg S]^j_E \iff \exists i. \forall j \geq i. J \not\in [S]^j_E \iff (\star)\]

\[\forall j \geq i. J \not\in [S]^j_E \iff \neg(\exists i. \forall j \geq i. J \in [S]^j_E) \iff J \not\in [S]^j_E \iff \]

For the crucial \(\Leftrightarrow (\star)\) step, the direction \(\Rightarrow\) is immediate. For the direction \(\Leftarrow\) we use the stability Lemma \([\text{I}]\). The stability lemma specifies that the problem \(J \not\in [S]^j_E\) becomes definitively true or false after a certain index. If we assume that \(\forall j \geq i. J \not\in [S]^j_E\), then \(J \not\in [S]^j_E\) holds for infinitely many values of \(j\), hence, by the stability Lemma \([\text{I}]\) it has a fixing index.

From now on, we will assume that, for any schema document \(S\) \(\text{defs}(E)\), recursion in \(E\) is guarded, and that \(E\) is closing for \(S\), and we will make the same assumption when discussing the semantics of a schema \(S\) with respect to an environment \(E\).

The full formal definition of the semantics is in Figure \([\text{I}]\) \(L(r)\) are the strings matched by \(r\). \(J\text{Val}(\text{Obj})\) are the JSON terms whose type is "object", and similarly for the other types.

### 2.3 \(p\text{Names}(S)\) encoded through \(\text{PattOfS}(S, E)\)

The assertion \(p\text{Names}(S)\) requires that, if the instance is an object, every member name satisfies \(S\), which is equivalent to saying that no member name exists that violates \(S\). Hence, if we translate \(S\) into a pattern \(r = \text{PattOfS}(S)\) that exactly describes the strings that satisfy \(S\), we can translate \(p\text{Names}(S)\) into \(\text{prop}(\text{PattOfS}(\neg S) : \text{f})\), which means: if the instance is an object, it cannot contain any member whose name matches the complement of \(\text{PattOfS}(S)\).

In order to translate a schema \(S\) into a pattern we actually need an environment \(E\) to associate a definition to any variable that appears in \(S\); hence, rather than a function \(\text{PattOfS}(S)\), we will define a function \(\text{PattOfS}(S, E)\).

We now show how to transform every schema \(S\) into a pattern \(\text{PattOfS}(S, E)\) such that the following equivalences hold, where \(S \equiv_E S'\) means that \([S]^v_E = [S']^v_E\).

\[
\begin{align*}
type(\text{Str}) \land S & \equiv_E type(\text{Str}) \land \text{pattern}(\text{PattOfS}(S, E)) \\
p\text{Names}(S) & \equiv_E \text{prop}(\text{PattOfS}(\neg S, E) : \text{f})
\end{align*}
\]

We have already introduced the notations \(\neg^*(r)\) for the complement of \(r\). We also define the following abbreviations, where \(t^*\) matches any string, \(f^*\) matches no string, and \(r \land^* r'\) matches \(L(r) \cap L(r')\).

\[
\begin{align*}
t^* &= .^* \\
f^* &= \neg^*(t^*) \\
r \land^* r' &= \neg^*(\neg^*(r) \mid \neg^*(r'))
\end{align*}
\]

Once we have these abbreviations, we can proceed as follows.

For all the ITAs \(S\) whose type is not \(\text{Str}\), such as \(\text{mulOf}(q)\), we define \(\text{PattOfS}(S, E) = t^*\), since they are satisfied by any string.
For the other operators, $PattOfS(S, E)$ is defined as follows. Observe that, while $PattOfS(\text{mulOf}(q), E) = t^*$, since $\text{mulOf}(q)$ is an Implicative Typed Operator, $PattOfS(\text{type}(\text{Num}), E) = l^*$, since $\text{type}(\text{Num})$ is not conditional, and is not satisfied by any string. Since $PattOfS(S, E)$ does not analyze the schemas that are nested inside typed operators, the definition below is well-founded in
presence of guarded recursion: after we have expanded a variable \( x \) once, in the result of any further expansion \( x \) will always be guarded, hence we will not need to expand it again. For the operators not cited, such as \texttt{len} and \texttt{enum}, and the derived boolean operators, we first translate them into the core algebra, and then we apply the rules below.

\[
\begin{align*}
\text{PattOfS}(\text{type}(T), E) &= f \quad \text{if } T \neq \text{Str} \\
\text{PattOfS}(\text{type}(\text{Str}), E) &= t \\
\text{PattOfS}(\text{const}(J), E) &= f \quad \text{if the type of } J \text{ is not } \text{Str} \\
\text{PattOfS}(\text{const}(J), E) &= J \quad \text{if the type of } J \text{ is } \text{Str} \\
\text{PattOfS}(S_1 \land S_2, E) &= \text{PattOfS}(S_1, E) \land \text{PattOfS}(S_2, E) \\
\text{PattOfS}(\neg S, E) &= \neg(\text{PattOfS}(S, E)) \\
\text{PattOfS}(\text{pattern}(r), E) &= r \\
\text{PattOfS}(x, E) &= \text{PattOfS}(E(x), E)
\end{align*}
\]

It is easy to prove that we have the following equivalences, which allow us to translate \texttt{pNames} into the core algebra.

\textbf{Property 1.} For any assertion \( S \) and for any environment \( E \) which maps variables into assertions in a way that respects the guarded recursion constraint, the following equivalences hold.

\[
\begin{align*}
\text{type(Str)} \land S &\iff_E \text{type(Str)} \land \text{pattern(PattOfS}(S, E)) \\
\text{pNames}(S) &\iff_E \text{prop}(\text{PattOfS}(\neg S, E) : f)
\end{align*}
\]

3 Translating from JSON Schema to the algebraic form

To translate from JSON Schema to the algebra, we first normalize "$\text{ref}$" references, as follows. For every subschema \( S \) that is referred by a "$\text{ref}$": "\textit{path}" operator, we copy \( S \) in the definitions section of the schema, under a name \( f(\text{path}) \), where \( f \) transforms the path into a flat string, and we substitute all references "$\text{ref}$": "\textit{path}" with a normalized reference "$\text{ref}$": "\#/definitions/f(path)"", with the only exception of the root path "\#" that is not affected. At this point, the resulting document is translated as follows, where \( \langle S \rangle \) is the translation of \( S \), \( xroot \) is a fresh variable, and where each occurrence of "$\text{ref}$": "\#/definitions/x"" is translated as "\( x \)" and each occurrence of "$\text{ref}$": "\#"" is translated as "\( xroot \)":

\[
\langle \{ "a_1" : S_1, \ldots, "a_n" : S_n, "definitions" : \{ "x_1" : S'_1, \ldots, "x_m" : S'_m \} \} \rangle =
\]

\[
xroot \ \text{defs}(xroot) : \{\langle "a_1" : S_1, \ldots, "a_n" : S_n \rangle, x_1 : \langle S'_1 \rangle, \ldots, x_n : \langle S'_m \rangle \}
\]

After definition normalization, we translate any assertion "$a" : S into the corresponding algebraic operator, as reported in Table 2.
| JSON Schema Feature | Algebra Expression |
|---------------------|--------------------|
| `allOf`: [S₁,...,Sₙ] | $\land (S₁ \ldots Sₙ)$ |
| `anyOf`: [S₁,...,Sₙ] | $\lor (S₁ \ldots Sₙ)$ |
| `oneOf`: [S₁,...,Sₙ] | $\bigcup (S₁ \ldots Sₙ)$ |
| `not`: S | $(\neg S)$ |
| `if`: S₁, “then” : S₂, “else” : S₃ | $(S₁) \Rightarrow (S₂) \lor (S₃)$ |
| `const`: J | $\text{const}(J)$ |
| `enum`: [J₁,...,Jₙ] | $\text{enum}(J₁ \ldots Jₙ)$ |
| `type`: "boolean"/ "null"/ "number"/ "string"/ "array"/ "object" | $\text{type}(\text{Bool} / \text{Null} / \text{Num} / \text{Str} / \text{Arr} / \text{Obj})$ |
| `items` | $\text{ite}(\text{null})$ (in the case of an empty array) |
| `minItems`: m | $\text{betw}^{\text{min}}_m$ |
| `maxItems`: M | $\text{betw}^{\infty}_M$ |
| `minLength`: m | $\text{len}^{\text{min}}_m$ |
| `maxLength`: M | $\text{len}^{\infty}_M$ |
| `pattern`: r | $\text{pattern}(r)$ |
| `contains`: S | $\text{contains}(S)$ |
| `items`: [S₁,...,Sₙ], "additionalItems": S’ | $\text{items}((S₁) \ldots (Sₙ) ; (S'))$ |
| `items`: S | $\text{items}(S)$ |
| `minProperties`: m | $\text{pro}^{\text{min}}_m$ |
| `maxProperties`: M | $\text{pro}^{\infty}_M$ |
| `propertyNames`: S | $\text{pNames}(S)$ |
| `required`: [k₁,...,kₙ] | $\text{req}(k₁ \ldots kₙ)$ |
| `properties`: {i=1..n kᵢ : Sᵢ}, `patternProperties`: {i=1..m rᵢ : PSᵢ}, `additionalProperties`: S | $\text{props}(i = 1 \ldots n kᵢ : (Sᵢ))$, $\text{props}(i = 1 \ldots m rᵢ : (PSᵢ))$, $(S)$ |
| `dependentSchemas`: { kᵢ : S₁,...,kᵦ : Sᵦ } | $((\text{type}(\text{Obj}) \land \text{req}(kᵢ)) \Rightarrow (Sᵢ)) \land \ldots \land ((\text{type}(\text{Obj}) \land \text{req}(kᵦ)) \Rightarrow (Sᵦ))$ |
| `dependentRequired`: | $\text{req}(kᵢ) \Rightarrow \text{req}(rᵢ^₁ \ldots rᵦ^ᵦ) \land \ldots \land \text{req}(kᵦ) \Rightarrow \text{req}(rᵦ^ᵦ \ldots rᵦ^ᵦ)$ |
| `dependencies`: obj | $\text{see two previous cases}$ |
| `k₁ : S₁,...,kᵦ : Sᵦ`, `definitions`: { x₁ : S₁',..., xₙ : Sₙ'} | $\text{def}(xᵦ) : ((\{k₁ : S₁ \ldots kᵦ : Sᵦ\}), x₁ : (S₁') \ldots xₙ : (Sₙ'))$ |

**Fig. 2.** Translation from JSON Schema to the algebra.
4 Negation closure

4.1 JSON Schema is almost negation-closed, but not exactly

We say that a logic is negation-closed if, for every formula, there exists an equivalent one where no negation operator appears. In our algebra, negation operators include \( \Rightarrow, (S_1 \Rightarrow S_2 | S_3) \) and \( \otimes \), since \( \neg S \) can be also expressed as \( S \Rightarrow \emptyset \) or as \( \boxplus(S, t) \). Negation-closure is usually obtained by coupling each algebraic operator with a dual operator that is used to push negation inside the first one. We are going to prove here that negation-closure is “almost” true for JSON Schema but not completely, and we are going to exactly describe the situations where negation cannot be pushed through JSON Schema operators.

According to our collection of GitHub JSON Schema documents, the most common usage patterns for \( \text{props}(r_1 : S_1, \ldots, r_n : S_n; S_a) \) are those where each \( r_i \) is the pattern \( k_i \) that only matches the string \( k_i \) generated by the use of the JSON Schema operator "properties", and where \( S_a \) is either \( t \) or \( f \). In these two cases, negation can be pushed through \( \text{props} \) as described by Property 2.

We first show how to express \( \neg S \) in the negation free algebra when \( S = \text{props}(k_1 : t, \ldots, k_n : t; f) \). The idea is to define a list of assertions \( U_u \), for \( u \in \{1..n\} \), such that \( \{k'_1 : J'_1, \ldots, k'_m : J'_m\} \in [U_u]_E \iff |K' \cap K| \leq u \), where \( K' = \{k'_1, \ldots, k'_m\} \) and \( K = \{k_1, \ldots, k_n\} \). \( U_u \) puts a bound \( u \) on the number of names of \( J' \) that are also in \( K \) (i.e., in \( S \)). At this point, \( J' \) violates \( S \) if it has a name that is not in \( K \), that is, if there exists \( i \) such that \( J' \in [U_i \land \text{props}^\infty]\) \( \forall J' \) has at most \( i \) names that are in \( K \), but it has more than \( i \) fields.

In order to express \( U_u \) in a succinct way, we recursively split \( [1..n] \) into halves (we actually split \([1..2^{\lceil \log_2(n) \rceil}]\)) and, for each halving interval \( I = [i+1, \ldots, i + (2^l)] \) that we get, we define a variable \( U_{I,u} \) that specifies that \( |K' \cap \{k_{i+1}, \ldots, k_{i+(2^l)}\}| \leq u \), and we express \( U_{I_{(i_1,i_2),u}} \) by a combination of \( U_{I_{i_1},u} \) and \( U_{I_{i_2},u} \). Formally, let \( I(l, p) \) denote the interval of integers \([((p-1)*2^l)+1, p*2^l])\), that is, the \( p \)-th interval of length \( 2^l \), counting from 1, e.g., \( I(3, 1) = [1, \ldots, 8], I(3, 2) = [9, \ldots, 16] \). We define now an environment \( E(\{k_1, \ldots, k_n\}) \) that contains a set of variables \( U_{I_{(p,u)}} \) such that \( \{k'_1 : J'_1, \ldots, k'_m : J'_m\} \in [U_{I_{(p,u)}}]_E \iff |K' \cap \{k_i \mid i \in I(p)\}| \leq u \).

One variable \( U_{I_{(l,p,u)}} \) is defined for each halving interval \( I(l, p) \) that is included in \([1, 2^{\lceil \log_2(n) \rceil}]\), and for each \( u \leq 2^l - 1 \), since \( I(l, p) \) contains \( 2^l \) elements, hence any constraint with \( u \geq 2^l \) would be trivially true. In the first two lines, we deal with intervals of length \( 2^0 = 1 \). The third line exploits the equality \( I(l, p) = I(l-1, 2p-1) \cup I(l-1, 2p) \), and it says that \( |K' \cap K(l, p)| \leq u \) holds if

\[ k' \]$^7$ we use \( k \) to denote the pattern \( \backslash k$\$, where \( k' \) is obtained from \( k \) by escaping all special characters.
there exists an \( i \) such that \(|K' \cap K(l-1, 2p-1)| \leq i\) and \(|K' \cap K(l-1, 2p)| \leq (u-i)

\[
\mathcal{E}(\{k_1, \ldots, k_n\}) = \\
(U_{0,0} : \text{props}(k_p : f; t)) \\
\quad \quad 1 \leq p \leq n \\
(U_{0,0} : t) \\
\quad \quad n+1 \leq p \leq 2^{\log_2(n)} \\
(U_{1,p,u} : V_{0 \leq i \leq u}(U_{l-1,2p-1-i} \wedge U_{l-1,2p,u-i})) \\
\quad \quad 1 \leq l \leq \lfloor \log_2(n) \rfloor, \\
\quad \quad 1 \leq p \leq 2^{\log_2(n)} - l, \\
\quad \quad 0 \leq u \leq 2^l - 1 \\
(U_{1,p,u,\leq 2^l-1} : V_{0 \leq i \leq u}(U_{l-1,2p-1-i} \wedge U_{l-1,2p,u-i})) \\
\quad \quad 1 \leq l \leq \lfloor \log_2(n) \rfloor, \\
\quad \quad 1 \leq p \leq 2^{\log_2(n)} - l, \\
\quad \quad 0 \leq u \leq 2^l - 1 \\
(U_{1,p,u,> 2^l-1} : \bigvee_{u-2^l-1 \leq i \leq 2^l-1}(U_{l-1,2p-1-i} \wedge U_{l-1,2p,u-i})) \\
\quad \quad 1 \leq l \leq \lfloor \log_2(n) \rfloor, \\
\quad \quad 1 \leq p \leq 2^{\log_2(n)} - l, \\
\quad \quad 0 \leq u \leq 2^l - 1
\]

) 

The instances of the third line need a number of symbols (computed in Appendix) that grows like \(O(n^2): \Sigma_{l \in \{1, \ldots, \log_2(n)\}} \Sigma_{p \in \{1, 2^{\log_2(n)} - 1\}} \Sigma_{u \in \{0, 2^l - 1\}} (4 \times (u+1) + 1)\).

### 4.2 Dimension of \(\mathcal{E}(\{k_1, \ldots, k_n\})\)

Computation of \(\Sigma_{l \in \{1, \ldots, \log_2(n)\}} \Sigma_{p \in \{1, 2^{\log_2(n)} - 1\}} \Sigma_{u \in \{0, 2^l - 1\}} (4 \times (u+1) + 1)\):

\[
\Sigma_{l \in \{1, \ldots, \log_2(n)\}} \Sigma_{p \in \{1, 2^{\log_2(n)} - 1\}} \Sigma_{u \in \{0, 2^l - 1\}} (4 \times (u+1) + 1) \\
= \Sigma_{l \in \{1, \ldots, \log_2(n)\}} \Sigma_{p \in \{1, 2^{\log_2(n)} - 1\}} O(2^l) \\
= \Sigma_{l \in \{1, \ldots, \log_2(n)\}} (2^{\log_2(n)} - 1) O(2^l) \\
= \Sigma_{l \in \{1, \ldots, \log_2(n)\}} O(2^{\log_2(n)} + 1) \\
= \Sigma_{l \in \{1, \ldots, \log_2(n)\}} O(2^{\log_2(n)}) = O(n^2)
\]

The interval \(I([\log_2(n), 1])\) includes \(\{1, \ldots, n\}\), hence an object \(J'\) violates \(\text{props}(k_l : t, \ldots, k_n : t; f)\) if it satisfies \(U_{[\log_2(n), 1, i]}\) for some \(i\), hence it contains at most \(i\) of the names in \(\{k_1, \ldots, k_n\}\), and it also satisfies \(\text{pro}_{l+1}\), hence it contains some extra names. This construction, based on the counting operator \(\text{pro}_{l}\), allows us to push negation through \(\text{props}\) when \(S_n = f\), as shown in Property 2, cases (2) and (3).

**Property 2 (Negation of common use case for props).**

(1) \((-\text{props}(k_1 : S_1, \ldots, k_n : S_n : t, E)\)

\[
= (\text{type}(\text{Obj}) \wedge \bigvee_{i \in \{1, \ldots, n\}} (\text{req}(k_i) \wedge \text{props}(k_i : \neg S_i : t), E))
\]

(2) \((-\text{props}(k_1 : t, \ldots, k_n : t; f), E)\)

\[
= (\text{type}(\text{Obj}) \wedge (\bigvee_{0 \leq i \leq n}(U_{[\log_2(n), 1, i]} \wedge \text{pro}_{l+1})), E \cup \mathcal{E}(\{k_1, \ldots, k_n\}))
\]

(3) \((-\text{props}(k_1 : S_1, \ldots, k_n : S_n : f), E)\)

\[
= (\neg\text{props}(k_1 : S_1, \ldots, k_n : S_n : t) \lor \neg\text{props}(k_1 : t, \ldots, k_n : t; f), E)
\]


Case (1) shows that, when each $r_i$ has shape $k_i$ and when $S_a = t$, then req acts as a negation dual for $\textbf{props}$. Case (3) shows that also in the second most-common use case negation can be pushed through $\textbf{props}$, although at the price of a complex encoding, where $\textbf{proj}$ plays a crucial role.

A natural question is which other use cases can be expressed, maybe through more and more complex encodings. To answer this question, we first introduce a bit of notation.

**Notation 1.** Given an assertion $S = \textbf{props}(r_1 : S_1, \ldots, r_n : S_n; S_a)$ and a string $k$, the functions $[k]_S$ and $S_S(k)$ are defined as follows.

1. $[k]_S = \{ k' \mid \forall i \in \{1..n\}. k \in L(r_i) \Rightarrow k' \in L(r_i) \}$: the set of strings that match exactly the same patterns as $k$.

2. $S_S(k)$: let $I = \{ i \mid k \in L(r_i) \}$; if $I = \emptyset$ then $S_S(k) = S_a$ else $S_S(k) = \wedge_{i \in I} S_i$: the conjunction of the schemas that must be satisfied by $J'$ if $k : J'$ is a member of an object that satisfies $S$.

We now prove that Property 2 exhausts all cases where $\neg \textbf{props}(r_1 : S_1, \ldots, r_n : S_n; S_a)$ can be expressed without negation. When we say $(S, E)$ can be expressed as $(S', E')$, this means that $[S]_E = [S']_E$. Proof in Appendix.

**Theorem 2.** Given $S = \textbf{props}(r_1 : S_1, \ldots, r_n : S_n; S_a)$, if exist $k_1, k_2$ such that (1) $[k_1]_S$ and $[k_2]_S$ are both infinite, and (2) exist both $J_1^+$ and $J_2^-$ such that $J_1^+ \in [S_S(k_1)]_E$ and $J_2^- \notin [S_S(k_2)]_E$, then $(S, E)$ cannot be expressed without negation.

Proof. $\neg \textbf{props}(r_1 : S_1, \ldots, r_n : S_n; S_a)$ is satisfied by every instance that is an object and such that it contains at least one member $k : J'$ such that $J' \notin [S_S(k)]_E$.

Assume that $k_1, k_2, J_1^+, J_2^-$ exist, and assume that a positive $D = S_0 \textbf{defs}(E')$ with $E' = x_1 : S'_1, \ldots, x_n : S'_n$ expresses $\neg \textbf{props}(r_1 : S_1, \ldots, r_n : S_n; S_a, E)$, in order to reach a contradiction. Consider a name $k \in [k_2]_S$ that does not appear in any req operator that is in $D$: since $[k_2]_S$ is infinite, such $k$ exists. Let $mm$ be a number that is bigger than every lower bound $m$ that appears in any $\textbf{proj}$ in $D$ and strictly bigger than the number of fields of any object found inside any $\textbf{const}$ or $\textbf{enum}$ operator in $D$. Consider a set of $mm$ different names $\{ k'_1, \ldots, k'_mm \}$ that belong to $[k_1]_S$ — such a set exists since $[k_1]_S$ is infinite. Now consider the following two objects:

$$\{ "k'_1" : J_1^+; \ldots; "k'_mm" : J_1^+; "k" : J_2^- \} \quad O_1$$

$$\{ "k'_1" : J_1^+; \ldots; "k'_mm" : J_1^+ \} \quad O_2$$

A generic $S'$ satisfies OneImpliesTwo if $O_1 \in [S']_E \Rightarrow O_2 \in [S']_E$. We now prove that every assertion $S'$ inside $D$ satisfies OneImpliesTwo, by induction on the lexicographic pair $(i, |S'|)$, where $|S'|$ is the size of $S'$. In this way, we prove that $D$ satisfies OneImpliesTwo, which is a contradiction since $\neg S$ is satisfied by $O_1$, thanks to the "$k" : J_2^-$ member, while $\neg S$ is not satisfied by $O_2$.

Every assertion that cannot distinguish two objects satisfies OneImpliesTwo. $\textbf{const}$ and $\textbf{enum}$ assertions in $D$ do not contain $O_1$ or $O_2$ since these are too big,
by construction. The \texttt{props} and \texttt{pNames} assertions can only fail because of the presence of a field, never for its absence, hence they satisfy \texttt{OneImpliesTwo}. The name \( k \) does not appear in any \texttt{req} assertion in \( D \), hence all \texttt{req} assertions satisfy \texttt{OneImpliesTwo}. If \( O_1 \in [\texttt{pro}^M]_{E_1^i} \), then \( O_2 \) satisfies the upper bound since \( O_2 \) is shorter than \( O_1 \), and it satisfies the lower bound since it has \( mm \) members, and \( mm \geq m \) by construction. If \( O_1 \in [S_1 \land S_2]_{E_1} \), then \( O_1 \in [S_1]_{E_1} \) and \( O_1 \in [S_2]_{E_1} \), hence the same holds for \( O_2 \) by induction on the size of \( S \), hence \( O_2 \in [S_1 \land S_2]_{E_1} \). The same holds if we exchange \( \land \) with \( \lor \) and \( \land \) with \( \lor \). For variables, the thesis follows by induction on \( i \), since \([x_j]^{i+1} = [S_j]_{E_1} \) and \( S_j \) is a subterm of \( D \), and \texttt{OneImpliesTwo} holds trivially when \( i = 0 \). Hence \( D \) itself enjoys \texttt{OneImpliesTwo}, hence \( D \) does not express \( \neg(S, E) \).

**Corollary 1.** Let \( S = \texttt{props}(r_1 : S_1, \ldots, r_n : S_n) \). The assertion \( \neg(S, E) \) can be expressed without negation only if \( (S, E) \) can be expressed either as \( \langle \texttt{props}(k_1 : S_1, \ldots, k_m : S_n, t), E' \rangle \) or as \( \langle \texttt{props}(k_1 : S_1, \ldots, k_m : S_n, f), E' \rangle \), for some \( E' \).

**Proof.** Assume that \( \neg(S, E) \) can be expressed without negation. By Theorem 2, it is not the case that the \texttt{props} are both infinite, and (2) exist both \( J^+_1 \) and \( J^-_2 \) such that \( J^+_1 \notin [S_1(k_1)]_{E_1} \) and \( J^-_2 \notin [S_2(k_2)]_{E_2} \). Hence, either for every \( k \) such that \([k]_S \) is infinite there exists no \( J^+ \) such that \( J^+ \in [S(k)]_E \), hence \([S(k)]_E = [t]_E \), or for every \( k \) such that \([k]_S \) is infinite there exists no \( J^- \) such that \( J^- \in [S(k)]_E \), hence \([S(k)]_E = [t]_E \). In the first case, \( \texttt{props}(r_1 : S_1, \ldots, r_n : S_n) \) can be expressed as \( \langle \texttt{props}(k_1 : S'_1, \ldots, k_m : S'_m, f), \rangle \), as follows: every \( k_f \) such that \([k_f]_S \) is finite, so that \([k_f]_S = \{ k'_1, \ldots, k'_l \} \), is transformed into a finite set of simple constraints \( k'_1 : S(k_f), \ldots, k'_l : S(k_f) \), and the additional constraint \( f \) expresses the fact that every name \( k \) such that \([k]_S \) is infinite must satisfy the assertion \( f \). In the second case, we reason in the same way to prove that the schema can be expressed as \( \langle \texttt{props}(k_1 : S'_1, \ldots, k_m : S'_m, t), E \rangle \).

Theorem 2 gives an abstract characterization of the schemas whose negation cannot be expressed. Observe that \( k_1 \) and \( k_2 \) may coincide, as long as \( S_2(k_1) \) is not trivial, where \( (S, E) \) is trivial when either \([S]_E = [t]_E \), or \([S]_E = [f]_E \). Corollary 4 rephrases the Theorem, hence specifying that Property 4 is exhaustive: negation cannot be pushed through \texttt{props} unless the schema is equivalent to one of those presented in Property 4. Since these are inexpressibility results, the theorem condition is not syntactic, but is decidable, and can also be used to derive results at the syntax level, as we are going to show below.

In terms of the original \texttt{patternProperties} and \texttt{additionalProperties} operators, Theorem 4 shows that the negation-free complement of a schema that contains \texttt{patternProperties} at the top level is only expressible when the schema can be rewritten into one where \texttt{patternProperties} is not used. For a schema \( S \) that contains \texttt{additionalProperties} at the top level, its complement has a negation-free expression only if \( S \) can be rewritten into one where \texttt{additionalProperties} is associated to a trivial schema.

Theorem 2 also has two other interesting corollaries.
Corollary 2. 1. $(\neg \text{pNames}(S), E)$ can be expressed without negation if, and only if, either $\{\{\text{type(Str)}, S\}\}_{E}$ is finite or $\{\{\text{type(Str)}, \neg S\\}_{E}$ is finite.
2. $(\neg \text{props}(S), E)$ can be expressed without negation if, and only if, $(S, E)$ is trivial.

While $\text{pNames}(S)$ is a universally quantified property “every name in $J$ belongs to $S$”, $\neg \text{pNames}(S)$ specifies that there exists a name that satisfies $\neg S$; Corollary 2(1) specifies that $\neg \text{pNames}(S)$ has a negation-free expression only in the finitary cases when either the allowed names, or the forbidden names, form a finite set, so that $\text{pNames}$ is another operator that does not allow, in general, the negation-free expression of its negation dual in JSON Schema.

Finally, by definition, $(\neg \text{props}(S), E)$ requires the presence of one field whose value satisfies $S$, independently of its name; Corollary 2(2) specifies that this assertion cannot be expressed in the negation-free fragment of JSON Schema, for any non-trivial $(S, E)$.

This last property indicates a big difference with array operators. Arrays can be described as objects where the field names are integers greater than 1, with the extra constraint that, whenever the field name $n+1$ is present, with $n \geq 1$, then $n$ must be present as well. From this viewpoint, Corollary 2(2) says that, while arrays have a positive operator $\text{contains}(S)$ to require the presence of at least one element that satisfies $S$, objects have no negation-free way of requiring the presence of such a field. Despite this crucial difference, the final result is quite similar: for arrays, as happens for objects, the negation of the fundamental $\text{items}$ operator can be expressed “almost” always, but with some precise exceptions.

We first show how negation can be expressed in the most common cases.

Property 3 (Negation of common use cases for $\text{items}$).

(1) $\neg \text{items}(S_a) = \text{type}(\text{Arr}) \land \text{contains}(\neg S_a)$

(2) if for each $i$, $[S_i]_E \subseteq [S_a]_E$:

$\neg \text{items}(S_1, \ldots, S_n; S_a) =$

$\text{type}(\text{Arr}) \land \bigvee_{i \in \{1..n\}} \{\text{items}(t_1, \ldots, t_{i-1}, \neg S_i; S_a) \land \text{ite}_{\neg i}^\infty \} \lor \text{contains}(\neg S_a))$

(3) $\neg \text{items}(S_1, \ldots, S_n; f) =$

$\text{type}(\text{Arr}) \land \bigvee_{i \in \{1..n\}} \{\text{items}(t_1, \ldots, t_{i-1}, \neg S_i; t) \land \text{ite}_{\neg i}^\infty \} \lor \text{ite}_{\neg n+1}^\infty$

Observe that the second case of Property 3 includes the standard case when $S_a = t$, and the third case subsumes the case when any $[S_i]_E = [f]_E$ for some $S_i$ since, in that case, $[\text{items}(S_1, \ldots, S_n; S_a)]_E$ is the same as $[\text{items}(S_1, \ldots, S_{i-1}; f)]_E$. The three cases above include the quasi-totality of the $\text{items}$ assertions that we found in our collection. However, they do not include one specific case: the one when $n > 0$, there exists an $i$ where $[S_i]_E \not\subseteq [S_a]_E$, for all $i$ $[S_i]_E \neq [f]_E$, and $[S_a]_E \neq [f]_E$. In this specific case, negation cannot be expressed.

Theorem 3. The algebra without negation cannot express $(\neg \text{items}(S_1, \ldots, S_n; S), E)$ when $n \neq 0$, all schemas $S_1, \ldots, S_n, S_a$ are non-empty in $E$, and there exists an $i$ in $\{1..n\}$ and $J_i^\ominus$ such that $J_i^\ominus \in [S_i \land \neg S_a]_E$. 

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Proof. Assume that a positive document $D = x_j \text{defs} (x_1 : S'_1, \ldots, x_m : S'_m)$ expresses the assertion $S = \neg \text{items}(S_1, \ldots, S_n; S_a), E$, when $n \neq 0$, all schemas $S_1, \ldots, S_n, S_a$, are non-empty in $E$, and there exists $i$ in $\{1..n\}$ such that $[S_i \land \neg S_a]_E \neq [f]_E$.

$\neg \text{items}(S_1, \ldots, S_n; S_a)$ is satisfied by any $J$ that is an array and has either an element at a position $j \leq n$ that satisfies $\neg S_j$, or an element after position $n + 1$ (included) that satisfies $\neg S_a$. Let $nn$ by the maximum among the lengths of the array prefixes and the array constants that appear in $S'_2$ and the parameter $n$ of the hypothesis. Consider the following two arrays, where $J^+ \in [E]_{S_a}$ and $J^- \in [E]_{S_i \land \neg S_a}$

$$[J_1, \ldots, J_{i-1}, J_i^{+}, J_{i+1}, \ldots, J_{nn}, J^+, J^-] \quad A_1$$
$$[J_1, \ldots, J_{i-1}, J_i^-, J_{i+1}, \ldots, J_{nn}, J^+, J^+] \quad A_2$$

In these arrays, all elements $J_1, \ldots, J_{nn}$ are chosen to satisfy the corresponding $S_j$, if their position $j$ is before $n$, or $S_a$ otherwise, which is possible since all these schemas are not empty.

A generic $S'$ satisfies $\text{OneImpliesTwo}$ if $A_1 \in [S']_E \Rightarrow A_2 \in [S']_E$. We now prove that every assertion $S'$ inside $D$ satisfies $\text{OneImpliesTwo}$, by induction on the lexicographic pair $(i, |S'|)$, where $|S'|$ is the size of $S'$. In this way, we prove that $D$ satisfies $\text{OneImpliesTwo}$, which is a contradiction since $\neg S$ is satisfied by $A_1$, thanks to the last element $J_i^+$, while $A_2$ does not satisfy $\neg S$.

For variables, boolean expressions, and non-array typed operators we reason as in the proof of Theorem\textsuperscript{2} and the assertion $D$ do not contain $A_1$ or $A_2$ since these are too big, by construction. Any $\text{item}_M$ that is satisfied by the first one holds for the second, since they have the same length. The same holds for any $\text{contains}(S_a)$, since the first contains the same elements as the second. Both fail $\text{uniquels}(S)$, by construction. Finally, for any $\text{items}(S'_1, \ldots, S'_m; S')$ in $S'$, we know that it accepts $A_1$, hence it accepts $J^+$ in position $nn + 1$, hence it accepts it in position $nn + 2$ as well since $nn \geq m$, hence it accepts $A_2$. Hence $D$ is not equivalent to $\neg \text{items}(S_1, \ldots, S_n; S_a)$.

Hence, we are again in a situation where negation can be pushed through $\text{items}$ in almost all cases of practical interest, but not always.

Observe that, while $\text{req}$ and $\text{contains}$ can express the negation of $\text{props}$ and $\text{items}$ in most cases, but not always, $\text{props}$ and $\text{items}$ can always express the negation of $\text{req}$ and $\text{contains}$:

Property 4 (Full negation for $\text{req}$ and $\text{contains}$).

$$\neg \text{req}(k_1, \ldots, k_n) = \text{type}(\text{Obj}) \land (\text{props}(k_1 : f; t) \lor \ldots \lor \text{props}(k_n : f; t))$$
$$\neg \text{contains}(S) = \text{type}(\text{Arr}) \land \text{items}(\cdot \neg S)$$

This quasi-duality can be explained as follows. $\text{items}(S_1, \ldots, S_n; S)$ is a universal-implicative quantification over the elements of an array: for every element, if its position $i$ is before $n$, then its value satisfies $S_i$; if it is strictly greater, then its value satisfies $S$. The assertion $\text{contains}(S)$ is existential-assertive: there exists
one element that satisfies $S$; the same classification can be used for the pair $\text{props-req}$. In both cases, the existential-assertive element of the pair is somehow less expressive than the negation of its universal-implicative companion: req lacks the ability to describe infinite sets of names and the associated schemas, contains is not able to distinguish the head and the tail of the array.

We conclude this section with the last two operators whose negation cannot be expressed in the language without negation: $\text{mulOf}(q)$ and $\text{uniqueItems}$. In the next section we will show how negation can be pushed through all the other operators.

**Theorem 4.** The following pairs cannot be expressed in the algebra without negation:

1. $(\neg \text{mulOf}(q), E)$, for any $q > 0$.
2. $(\neg \text{uniqueItems}, E)$.

**Proof.** (1) $\neg \text{mulOf}(q)$: Assume towards a contradiction that a positive document $D = S_0 \text{def}(E')$ with $E' = x_1 : S_1, \ldots, x_n : S_n$ expresses $\neg \text{mulOf}(q)$. Let us choose a number $N$ such that $N > q$ (hence, $N > 0$), and $N > M$ and $N > m$ for any bound $m$ and $M$, different from $\infty$, that is found in any assertion $\text{betw}^M_m$, $\text{xBetw}^M_m$ inside $D$.

We say that a generic $S'$ is Full Or Finite (FOF) for $i$ over the closed interval $[N, 2N]$, if $[N, 2N] \cap [S']^i_{E'}$ is either equal to $[N, 2N]$, or is finite. We prove that any subexpression $S'$ of $D$ is FOF over $[N, 2N]$ for any $i$, by induction on the lexicographic pair $(i, |S'|)$. For the variables, in the case $i = 0$ the empty set is finite, and the inductive step is immediate since $[x_i]_{E'}^{i+1} = [S_i]_{E'}^i$ and $S_i$ is a subterm of $D$. Typed operators whose type is not Num accept all numbers, hence are Full. An interval operator whose bounds are both smaller than $N$ has empty intersection with $[N, 2N]$, and is Full when $M = \infty$. The positive $\text{mulOf}(q)$ operator has a finite intersection with every finite interval. Union and intersection of two subsets of $[N, 2N]$ which are either finite or full is finite or full. Hence, $D$ is FOF over $[N, 2N]$ for any $i$. The limit $\bigcup_{i \in \mathbb{N}} \bigcap_{j \geq i} [D]^j$ can be infinite only if exists $i$ such that $[D]^j$ is infinite for $j \geq i$, hence $[D]$ is full or finite as well. However, $\neg \text{mulOf}(q)$ is not FOF: it is not full on $[2, 2N]$ since the interval contains at least one multiple of $q$, by $N > q$, and its intersection with $[N, 2N]$ is not finite.

(2) Assume that a positive $D$ expresses $\neg \text{uniqueItems}$. Choose an integer $N$ strictly greater than any $l$ that appears as lower bound in a $\text{ite}^l_i$ in $D$ and also greater than the length of any array that appears in a const or enum assertion in $D$. Define the following two arrays, the first one ending with a repetition of $N$.

$$A_1 = [1, 2, \ldots, N - 1, N, N] \quad A_2 = [1, 2, \ldots, N - 1, N]$$

We prove by induction on the lexicographic pair $(i, |S'|)$, that the semantics $[S']^i_{E'}$ of any subexpression $S'$ of $D$ that includes $A_1$ includes $A_2$ as well. When $S = x$, we prove that by induction on $i$: in the base case, $[x]_{E'}^0 = \emptyset$ does not contain $A_1$, and when $i = i + 1$ hence $[x]_{E'}^{i+1} = [E(x)]_{E'}^i$, we conclude by induction on $i$. Non-array typed assertion include both $A_1$ and $A_2$. All const and enum assertions
refuse both, since the arrays that they enumerate are shorter than N. Since the length N is greater than any lower bound l, if ite₂' accepts the first it also accepts the second. For any items(S₁, . . . , Sₙ; S) assertion, since A₂ is an initial subarray of A₁, if the assertion is satisfied by A₁, it is also satisfied by A₂. For S₁ ∧ S₂ we conclude by induction on the size, since implication of satisfaction is preserved by ∧, and similarly for ∨. Since A₁ belongs to [D] by assumption, A₂ belongs to [D], which contradicts the hypothesis, since A₂ satisfies uniqueltems.

4.3 Closed algebra

As we have seen, JSON Schema does not enjoy negation-closure, but is endowed with universal-existential pairs props/req and items/contains that enjoy an imperfect duality. We now define a more regular algebra by adding some negative operators, to obtain a closed algebra where each operator has a real negation dual, and negation can be fully eliminated. In our experience, this negation-closed algebra is practical both to reason about JSON Schema and to implement tools for JSON Schema analysis. Indeed, our tool allows the user to transform a schema into its algebraic form and to generate a witness for the input schema [1], and our witness generation algorithm crucially uses the negation elimination algorithm for the closed algebra, that we will present now, in order to generate witnesses for schemas that contain negation.

The closed algebra completes the algebra with the following four dual operators: pattReq, contAfter, notMulOf, repeatedItems, none of which, by the theorems we presented, can be expressed in the algebra without negation. We also add a pattern complement operator ¬•(r) to the regular expressions language, that is useful for complexity reasons, since, while regular expressions are closed under complement, the size of the plain regular expression that represents ¬•(r) is, in the worst case, doubly exponential with respect to the size of r. The semantics of these operators is defined as follows (where the notation t¹, . . . , tⁿ indicates a sequence of n copies of t).

\[
\text{pattReq}(r₁ : S₁, . . . , rₙ : Sₙ) = \text{type}(\text{Obj}) \Rightarrow \bigwedge_{i \in \{1..n\}} \neg\text{props}(r_i : \neg S_i; t)
\]

\[
\text{contAfter}(n : S) = \text{type}(\text{Arr}) \Rightarrow \neg\text{items}(t¹, . . . , tⁿ; \neg S)
\]

\[
\text{notMulOf}(q) = \text{type}(\text{Num}) \Rightarrow \neg\text{mulOf}(q)
\]

\[
\text{repeatedItems} = \text{type}(\text{Arr}) \Rightarrow \neg\text{uniqueltems}
\]

The operator pattReq(r₁ : S₁, . . . , rₙ : Sₙ) specifies that, if the instance is an object, then, for each i ∈ {1..n}, it must possess a member whose name matches rᵢ and whose value satisfies Sᵢ. It is strictly more expressive than req, since it allows one to require a name that belongs to an infinite set L(rᵢ), and it associates a schema Sᵢ to each required pattern rᵢ. In the closed algebra, we regard req as an abbreviated form of pattReq where every pattern has the shape k and every associated schema is t.

contAfter(n : S) specifies that, if the instance is an array, it must contain at least one element that satisfies S in a position that is strictly greater than n. This
operator has an expressive power that is slightly greater than the \( \text{contains}(S) \) operator, since it can distinguish between the head and the tail of the array. In the closed algebra, we regard \( \text{contains}(S) \) as an abbreviation for \( \text{contAfter}(0 : S) \).

The operators \( \not\text{MulOf}(q) \) and \( \text{repeatedItems} \) are just the duals of \( \text{mulOf}(q) \) and \( \text{uniquetitems} \). In the next section we prove that these four operators are all that we need to make JSON Schema negation-closed.

**4.4 Proving negation closure: the not-elimination algorithm**

We prove negation closure through the definition of a not-elimination algorithm, which eliminates any instance of negation from any expression in the closed algebra. This algorithm starts with a simplification phase, aimed at reducing the complexity of the following phase. In this simplification phase we use the following derived operators, similar to those used in JSL for arrays [6]:

\[
\begin{align*}
\text{enum}(J_1, \ldots, J_n) &= \text{const}(J_1) \lor \ldots \lor \text{const}(J_n) \\
\text{const}(\text{null}) &= \text{type}($\text{null}$) \\
\text{const}(n) &= \text{type}(\text{Num}) \land \text{betw}_n \quad \text{for } n \in \text{Num} \\
\text{const}(s) &= \text{type}(\text{Str}) \land \text{pattern}(s) \quad \text{for } s \in \text{Str} \\
\text{const}([J_1, \ldots, J_n]) &= \text{type}(\text{Arr}) \land \text{ite}_n \land \text{itemAt}(1 : \text{const}(J_1), \ldots, \text{itemAt}(n : \text{const}(J_n)) \\
\text{const}([k_1 : J_1, \ldots, k_n : J_n]) &= \text{type}(\text{Obj}) \land \text{req}(k_1, \ldots, k_n) \land \text{pro}_n \land \text{props}(k_1 : \text{const}(J_1) ; t), \ldots, \text{props}(k_n : \text{const}(J_n) ; t)
\end{align*}
\]

Fig. 3. Elimination of \text{enum} and \text{const}.

These are the simplification steps.

1. items and \text{props} simplification: we rewrite each \( \text{items}(S_1, \ldots, S_n ; S_o) \) as \( \text{itemAt}(1 : S_1) \land \ldots \land \text{itemAt}(n : S_n) \land \text{itemsAfter}(n : S_o) \) and each \( \text{props}(r_1 : S_1, \ldots, r_n : S_n ; S_o) \) as \( \text{props}(r_1 : S_1 ; t) \land \ldots \land \text{props}(r_n : S_n ; t) \land \text{props}(\neg^\ast(r_1) \ldots | r_n) : S_o ; t) \).

2. Type simplification: we rewrite each \( \text{type}(T_1, \ldots, T_n) \) as \( \text{type}(T_1) \lor \ldots \lor \text{type}(T_n) \).

3. Const-elimination: we rewrite every instance of \text{const} and of \text{enum}, with the only notable exception of \text{const}($\text{true}$) and \text{const}($\text{false}$), through the repeated application of the rules shown in Figure 4 as also done in [5].

4. pNames elimination: we rewrite every instance of pNames($S$) using \( \text{props}(r_S : f ; t) \), as discussed below.

5. Not-explicitation: we rewrite every instance of \( S_1 \Rightarrow S_2, (S_1 \Rightarrow S_2 \mid S_3) \) and \( \bigoplus(S_1, \ldots, S_n) \), according to their definition; the only remaining boolean operators are \( \neg, \land, \lor, t, f \).

pNames($S$) is eliminated by transforming $S$ into a pattern $r_S$ that matches all and only the strings that satisfy $S$, and by declaring that only fields whose
These two sets of variables are inductively defined as follows; observe that an occurrence of these variables, but not the entire subschem as represented. Moreover, the obvious encoding of \( (x_1, \ldots, x_n) \) produces an expression whose size is in \( O(n^2) \), but there exists an alternative encoding with linear size, that we present in the Appendix. Hence, not-explicitation can be implemented in such a way that its output size is linear in the input size, and the same holds for the other phases of simplification.

4.5 Linear encoding of oneOf

We describe here a linear-size encoding of \( (S_1, \ldots, S_{2^q}) \). Let \( I(l, p) \) denote the interval of integers \( [(p-1) \cdot 2^l + 1, p \cdot 2^l] \), that is, the \( p-th \) interval of length \( 2^l \), where we count from 1. For a fixed \( q \), \( I(l, p) \) is a subinterval of \( [1, 2^q] \) iff \( 0 \leq l \leq q \) and \( 1 \leq p \leq 2^{q-l} \), hence, the total number of subintervals of \( [1, 2^q] \) that have the shape \( I(l, p) \), where \( l \) and \( p \) satisfy that condition, is \( \sum_{l=0}^{q} 2^{q-l} = 2 \cdot (2^q) - 1 \), linear in \( 2^q \). Given a set of schemas \( S_1, \ldots, S_{2^q} \), we can define a set of \( 2 \cdot (2^q) - 1 \) variables \( N_{i,p} \), one for each subinterval \( I(l, p) \) of \( \{1..2^q\} \), and a set of variables \( O_{l,p} \) such that:

1. \( N_{i,p} \) is equivalent to the conjunction of \( \neg(S_i) \) for all \( i \in I(l, p) \), hence \( N_{i,p} \) is satisfied iff none of these schemas is satisfied;
2. \( O_{l,p} \) is satisfied iff one and only one of the schemas indexed by an \( i \in I(l, p) \) is satisfied.

These two sets of variables are inductively defined as follows; observe that an interval \( I(l+1, p) \) can be split in two halves as follows:

\[
N_{0,p} = \neg S_p \\
N_{0,1,p} = N_{0,2^p} \land N_{0,2^p} O_{l+1,p} = (O_{l,2^p-1} \land N_{l,2^p}) \lor (N_{l,2^p-1} \land O_{l,2^p})
\]

The size of this environment is linear in \( 2^q \), and the variable \( O_{q,1} \) encodes \( (x_1, \ldots, x_{2^q}) \).

On this simplified form, we now apply the two fundamental steps.

---

8 In our implementation we adopted the basic algorithm, having verified that in our schema corpus \( \mathcal{I} \) of more than 80k real-world schemas, \( \mathcal{I} \) has on average 2.3 arguments, and most of the time these arguments are extremely small.
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1. Not-completion of variables: this is a key technical step, since not-elimination needs to deal with the presence of recursive variables. In this step, for every variable $x_n : S_n$, we define a complement variable $not\_x_n : \neg S_n$, which will then be used to eliminate negation applied to $x_n$.

2. Not-pushing: given a not-completed pair $(S, E)$ we repeatedly push negation inside every $\neg S'$ expression until negation reaches the leaves and is removed.

**Not-completion of variables** Not-completion of variables is a key step that allows us to deal with the combined presence of unrestricted negation and recursive variables. In particular, not-completion transforms a set of definitions as follows:

$$\text{not-completion}(S \text{ defs}(x_1 : S_1, \ldots, x_n : S_n)) =$$

$$S \text{ defs}(x_1 : S_1, \ldots, x_n : S_n, \neg_x_1 : \neg S_1, \ldots, \neg_x_n : \neg S_n)$$

As a result, every variable $x$ has a complement variable $co(x)$ defined in the obvious way: $co(x) = not\_x$ and $co(not\_x) = x$. Variable $co(x)$ will later be used for not-elimination.

**Property 5.** Let $(x_1 : S_1, \ldots, x_n : S_n)$ be a closing environment. Then, for every variable $x_i$ with $i \in \{1..n\}$, we have:

$$[\neg x_i \text{ defs}(x_1 : S_1, \ldots, x_n : S_n)] =$$

$$[not\_x_i \text{ defs}(x_1 : S_1, \ldots, x_n : S_n, \neg_x_1 : \neg S_1, \ldots, \neg_x_n : \neg S_n)]$$

**The not-pushing algorithm** The not-pushing phase pushes negation down any algebraic expression up to its complete elimination. Not-pushing is defined by the rules in Figure 4. Observe that the negation of each conditional operation asserts the corresponding type, while the negation of $\text{const}$ is actually conditional: if the value is a boolean, then it is equal to $\text{false}/\text{true}$.

Not-pushing over $\text{len}_{0}$ or $\text{len}_{\infty}$ generates one satisfiable bound and one that is actually illegal ($\text{len}_{-1}$ or $\text{len}_{\infty+1}$). Rather than splitting the rule in three cases, we just assume that the illegal bound is eliminated from the resulting disjunction, and that a trivial operator $\text{len}_{\infty}$ is just rewritten as $t$ before not-pushing. An analogous assumption is made for the $\text{betw}$, $\text{xBetw}$, $\text{pro}$, $\text{ite}$ operators.

The following property is not difficult to prove.

**Property 6.** The not-elimination procedure preserves the semantics of the schema.

If we define the size of $\text{itemAt}(i : S)$ in the natural way as $1 + \log(i) + |S|$, rather than considering the length of its definition, and do the same for $\text{itemsAfter}$, then it is easy to see that the output size of not-elimination is linear with respect to the input size.

**Example 1.** Consider the following JSON Schema document.

```json
{ "properties": {"a": {"not": {"$ref": ":"}}} }
```
\[\neg t = f ; \neg f = t ; \neg(S_1 \land S_2) = (\neg S_1) \lor (\neg S_2) ; \neg(S_1 \lor S_2) = (\neg S_1) \land (\neg S_2) ; \neg(\neg S) = S\]

\[\neg(\text{type}(T)) = \lor(\text{type}(T) \mid T' \neq T)\]

\[\neg(\text{const}(\text{true})) = \lor(\text{type}(T) \mid T \neq \text{Bool}) \lor \text{const}(\text{false})\]

\[\neg(\text{const}(\text{false})) = \lor(\text{type}(T) \mid T \neq \text{bool}) \lor \text{const}(\text{true})\]

\[\neg(\text{len}'(r)) = \text{type}(\text{Str}) \land (\text{len}^{-1}_n \lor \text{len}^{n+1}_n)\]

\[\neg(\text{pattern}(r)) = \text{type}(\text{Str}) \land \text{pattern}(\neg^*(r))\]

\[\neg(\text{betw}^M_m \text{num}) = \text{type}(\text{Num}) \land (\text{xBetw}^1_{\infty} \lor \text{xBetw}^\infty_{\infty})\]

\[\neg(\text{xBetw}^M_m \text{num}) = \text{type}(\text{Num}) \land (\text{betw}^{-1}_{\infty} \lor \text{betw}^\infty_{\infty})\]

\[\neg(\text{mulOf}(q)) = \text{type}(\text{Num}) \land \text{notMulOf}(q)\]

\[\neg(\text{notMulOf}(q)) = \text{type}(\text{Num}) \land \text{mulOf}(q)\]

\[\neg(\text{ite}(i)) = \text{type}(\text{Arr}) \land (\text{ite}^{1-1}_0 \lor \text{ite}^{n+1}_n)\]

\[\neg(\text{uniqueItems}) = \text{type}(\text{Arr}) \land \text{repeatedItems}\]

\[\neg(\text{repeatedItems}) = \text{type}(\text{Arr}) \land \text{uniqueItems}\]

\[\neg(\text{itemAt}(i:S)) = \text{type}(\text{Arr}) \land \text{itemAt}(i:\neg S) \land \text{ite}^{\infty}_{\infty}\]

\[\neg(\text{itemsAfter}(n:S)) = \text{type}(\text{Arr}) \land \text{contAfter}(n:\neg S)\]

\[\neg(\text{contAfter}(n:S)) = \text{type}(\text{Arr}) \land \text{itemsAfter}(n:\neg S)\]

\[\neg(\text{pro}(r)) = \text{type}(\text{Obj}) \land (\text{pro}^{1-1}_{r} \lor \text{pro}^\infty_{\infty})\]

\[\neg(\text{pattReq}(r:S)) = \text{type}(\text{Obj}) \land \text{pattReq}(r:\neg S)\]

\[\neg(x) = co(x)\]

Fig. 4. Not-pushing rules — standard rules are collected in the first line.

It is quite obscure, and seems to suggest an infinite alternation of "a" and its negation. Yet, not elimination makes its semantics more clear. We write it in our algebra as follows.

\[x \text{ def}_s(x : \text{props}(\text{a} : \neg x; t))\]

By applying not-completion, we get the following definition (for readability, we omit the trivial "t" at the end of the props operator).

\[x \text{ def}_s(x : \text{props}(\text{a} : \neg x), \neg \text{a}_x : \neg \text{props}(\text{a} : \neg x))\]

This is how not-elimination would now proceed within our algebra (we push $\neg$ through props using Property 2 and use \{ , \} for conjunction):

\[\text{defs} (x : \text{props}(\text{a} : \neg x), \neg \text{a}_x : \neg \text{props}(\text{a} : \neg x)) \rightarrow \text{defs} (x : \text{props}(\text{a} : \text{co}(x)), \neg \text{a}_x : \{\text{type}(\text{Obj}), \text{req}(\text{a}), \text{props}(\text{a} : \neg x)\}) \rightarrow \text{defs} (x : \text{props}(\text{a} : \neg x), \neg \text{a}_x : \{\text{type}(\text{Obj}), \text{req}(\text{a}), \text{props}(\text{a} : x)\})\]

We now substitute not \_ \a with its definition, and obtain a much clearer schema: if the instance is an object with an \a member, then the value of that
member must be an object with an a member, whose value satisfies the same specification:

\[ x \text{ def} (x : \text{props(a : \{type(Obj), req(a), props(a : x)\})) } \]

These are some examples of values that match that schema:

\[ 1, \{"b" : 2\}, \{"a" : \{"a" : "foo"\}\}, \{"a" : \{"a" : \{"a" : null\}\}\} \]

5 Going towards Draft 2019-09

5.1 The \#i S operator

Draft 2019-09 introduced the new operators minContains: n and maxContains: n, where n is a natural number, with the following semantics: consider a schema that contains, at the top level, the three assertions "contains: S, minContains: m, maxContains: M". An instance J satisfies this combination iff: if it is an array, then it contains at least m and at most M elements that satisfy S. When minContains is missing, its value defaults to 1, while a missing maxContains means no upper limit. We model this by adding an operator \#i S to the algebra, with \( l \in \mathbb{N} \) and \( j \in \mathbb{N}^\infty \). The semantics of \#i S is defined as follows.

\[
[\#_i S]_E^j = \{ J | J = [J_1, \ldots, J_n] \Rightarrow l \leq |\{ o | o \in \{1..n\} \land J_o \in [S]_E^i\}| \leq j \}
\]

The operator \#i S cannot be expressed in the algebra (proof in the Appendix).

**Theorem 5.** The pair \((\#_i S, E)\) cannot be expressed in the algebra if \((S, E)\) is not trivial and either \(l \geq 2\) or \(j \neq \infty\).

**Proof.** Assume that \((S, E)\) is not trivial and that \(D = S_0 \text{ def} (E')\) expresses \((\#^2 S, E)\). Consider \( J_1 \in [S]_E^j \) and \( J_0 \not\in [S]_E^j \). If we say that n is the head-length of an operator items\((S_1, \ldots, S_n; S')\), let \( N \) be the maximum among the head-lengths of all instances of this operator inside \( D \) and the lengths of all arrays that appear inside the arguments of enum and const in \( D \). Consider two arrays \( A_1 \) and \( A_2 \) of length \( N + 3 \), starting with \( N + 1 \) copies of \( J_0 \):

\[
A_1 = [J_0, \ldots, J_0, J_1, J_1], \quad A_2 = [J_0, \ldots, J_0, J_1, J_0]
\]

Only \( A_1 \) should belong to \([D]\), but we can prove by induction on \( i \), and on the size of \( S \) when \( i \) is equal, that, for every subterm \( S' \) of \( D \), \( A_1 \in [S']_E^i \) iff \( A_2 \in [S']_E^i \). The result follows by induction on \( i \). Any conditional typed assertion (CTA) that is unrelated to arrays accepts both arrays, while uniqueitems, and all const and enum in \( D \), reject both of them, by construction. Since they have the same length, it will not distinguish the two. Consider any \( S' = \text{items}(S_1, \ldots, S_n; S'') \) and assume that it accepts \( A_1 \).
This means that each \( S_i \) accepts \( J_0 \) and that \( S'' \) accepts both \( J_0 \) and \( J_1 \), hence \( A_2 \in [S']_{E'} \) as well. In the same way we prove that \( A_2 \in [S'']_{E'} \Rightarrow A_1 \in [S']_{E'} \).

If \( S' = \text{contains}(S'') \), then it cannot distinguish the two arrays since they contain the same elements. If \( S' = S_1 \land S_2 \), or \( S' = S_1 \lor S_2 \), or \( S' = \neg S_1 \), we know, by induction, that \( S_1 \) is not able to distinguish \( A_1 \) from \( A_2 \) and the same holds for \( S_2 \), hence no boolean combination of \( S_1 \) and \( S_2 \) may distinguish \( A_1 \) from \( A_2 \).

To sum up, for every subterm \( S' \) of \( D \) we have that \([S']_{E'} \) does not distinguish the two arrays, and hence the limit \([S']_{E} \) does not distinguish them; therefore, \([D] \) cannot be equivalent to \( \#^\infty_2 S \). The case \( l > 2 \) and the case \( j \neq \infty \) are proved in the same way, adjusting the number of copies of \( J_1 \) in the tails of \( A_1 \) and \( A_2 \).

### 5.2 \#\_i^1 S: negation closure of array operators

We have seen that negation closure for \text{items}(S_1, \ldots, S_n; S_a) \) requires the dual operators \ite^\infty_1 \) and \contAfter(n : S) \), but the latter cannot be expressed in the algebra without negation (Theorem 3). When we enrich the algebra with the \#\_i^1 S \) operator, the situation changes completely. First of all, \#\_i^1 S \) immediately subsumes the two operators \text{contains}(S) \) and \ite^i \):

\[
\text{contains}(S) = \#^\infty_1 S \quad \ite^i = \#^i_1 t
\]

More interestingly, \#\_i^1 S \) also allows one to encode the \contAfter(n : S) \) operator, as follows. We define \( I(l, p) \) as in Section 4 and we define a set of variables \( U_{l,p,u} \), such that:

\[
[J_1, \ldots, J_m] \in [U_{l,p,u}]_E \Leftrightarrow |\{ j \mid j \in I_n(l, p) \land j \leq n \land J_j \in [S]_E \}| \leq u
\]

i.e., \( J \in [U_{l,p,u}]_E \) implies that the number of elements \( J_j \) of \( J \) whose position is in \( I(l, p) \) but not in the tail \{ \( n + 1, \ldots, 2^{|\log_2(n)|} \) \}, and such that \( J_j \in [S]_E \), is less than \( u \).

In the first two lines, we deal with halving intervals of length \( 2^0 = 1 \). The second line ensures that all positions greater than \( n + 1 \) will be ignored. The third lines splits a generic halving interval \( I \) in two halves, \( I_1 \) and \( I_2 \), and uses the same technique as in Section 4 to express \( U_{l,p,u} \) in terms of \( U_{l-1,2p-1,i} \) and \( U_{l-1,2p,u-i} \). The last line says that one \( J \in [S]_E \) is contained in the array after position \( n \) if, for some \( i \), at most \( i \) elements with \( J \in [S]_E \) are found in positions \{1..n\}, and the array contains at least \( i + 1 \) elements with \( J \in [S]_E \).

\[
\mathcal{E}(S, n) =
U_{0,0,0} : \text{itemAt}(p : \neg S) \quad 1 \leq p \leq n
U_{0,0,t} : t
U_{l,p,u} : \bigvee_{0 \leq i \leq u} (U_{l-1,2p-1,i} \land U_{l-1,2p,u-i}) \quad 1 \leq l \leq q, 1 \leq p \leq 2^{|\log_2(n)|} - 1
0 \leq u \leq 2^l - 1
\]

\[
( \text{contAfter}(n : S) \ , E ) = ( \bigvee_{0 \leq i \leq n} (U_{[\log_2(n)],1,i} \land \#^\infty_{i+1} S) \ , E \cup \mathcal{E}(S, n) )
\]
Hence, \( \#_i^j S \) is expressive enough to express \( \text{contAfter}(n: S) \), and thus to express negation of items, although at the cost of a complex encoding (of size \( O(n^2) \)).

Finally, we observe that \( \#_i^j S \) is self-dual, so that, while it solves the problem of not-elimination for the items operator, it does not introduce any new not-elimination issue. The self-duality of \( \#_i^j S \) is expressed by the following equation, where \( \#_{i-1}^0 S \) is just \( f \) when \( i = 0 \), and \( \#_{j+1}^\infty S \) is just \( f \) when \( j = \infty \).

\[
-(\#_i^j S) = \text{type}(\text{Arr}) \land (\#_{i-1}^0 S \lor \#_{j+1}^\infty S)
\]

While \( \text{contAfter}(n:S) \) is strictly less expressive than \( \#_i^j S \), it seems to be more compact, in the sense that we could not find any way to express \( \text{contAfter}(n:S) \) using \( \#_i^j S \) with a linear-size expression.

Thus, while JSON Schema Draft-06 needs four new operators — pattReq, notMulOf, repeatedItems, and contAfter — in order to become closed under negation, after \( \#_i^j S \) is added, we only need three of them.

### 6 Experiments

We have implemented the algebra (with the \( \#_i^j S \) operator, to express negation of items), as well as the not-elimination algorithm. An interactive tool has been presented as a demo [1], and is accessible online [9]. Our Java implementation comprises about 110K lines of code and uses the Brics automata library [11] for handling regular expressions. We performed two experiments, as described next.

**Validation benchmark.** To assess the correctness of our implementation, we rely on schemas from the JSON Schema Test Suite [10] (commit hash #8daea3f4). This test suite is the de-facto standard benchmark for JSON Schema validators and comprises 216 hand-crafted schemas for Draft-06. Each schema is encoded in just a few lines of code and targets a specific operator. We have excluded all schemas with advanced constructs that our implementation does not yet support (such as format), or with string escapings that our prototype cannot yet robustly parse, but which our implementation could easily be extended to support. We further removed schemas with references to external files, which we cannot resolve. We then performed negation-elimination on the remaining 185 schemas, having injected negation above the root. Our implementation correctly eliminates negation.

**Real-world schemas.** Our second experiment applies our algorithm to real-world schemas. We have crawled GitHub for open source JSON Schema documents, and retrieved over 80K files. As can be expected, we encountered a multitude of problems in processing these non-curated, raw files: files with syntactic errors, files which do not comply to any JSON Schema draft, and files with external references. We have filtered out all these problems and focused on a subset of 185 files, which we have coded to eliminate negation above the root. Our implementation correctly eliminates negation on this subset.
references that we are unable to resolve. We encountered troublesome string encodings, as well as patterns with forward/backward references that we cannot represent with our automaton library. Further, we encountered user-defined keywords that our implementation cannot yet handle.

Notably, there is a large share of duplicate schemas, with small variations in syntax and semantics, but evidently versions of the same schema. We rigorously removed such files, eliminating schemas with the same occurrences of keywords, condensing the corpus down to 15%. Within the final collection of 2,229 files, we are confident that they indeed represent individual schemas which vary in size from a few KBytes up to 0.4 MBytes, with an average of approx. 173 lines of code after pretty-printing. Again, we inject negation above the document root and perform negation-elimination.

We ran our experiments on a PC with an i7-6700, 3.40GHz CPU, 8 cores, 32 GB of main memory, 256 GB SDD. The tables below report the average runtime in milliseconds and the average runtime per KB (to account for different file sizes). We further report the size ratio (SR), i.e., the number of characters required for encoding the algebraic representation of the output schema, versus the number of characters for the algebraic representation of the input schema. This is an indicator of the size increase due to not-elimination.

| runtime (ms) | size ratio |
|--------------|------------|
| avg          | avg/KB     | avg max   |
| 2.77         | 0.46       | 2.78      | 27        |

Discussion. Translation to the algebra, combined with not-elimination, is in the sub-second range, which we consider acceptable. On average, not-elimination increases the size by a factor below 3. The maximum size ratio is caused by not-elimination over enumerations with over 200 items. Nevertheless, we observe linear growth.

Our experiments show that not-elimination is indeed feasible on real-world JSON Schema documents. While our prototype cannot yet handle all specific language constructs, the current limitations are merely technical. One unique selling point is that our approach fully supports negation and recursion (even in combination), which is often a conceptual limitation of algorithms and tools designed for JSON Schema processing (e.g., [7]).

7 Related work

In an empirical study [53] over thousands of real-world schemas [2], we have analyzed usage patterns of the negation operator. While we find occurrences of not to be rare, we have found usage patterns of this operator to be subtle, and often difficult to understand.

The problem of negation closure of JSON Schema, that is, the precise study of the duality among couples of structural operators, does not seem to have been studied before.
Habib et al. [8] study schema inclusion for JSON Schema. Their algorithm is based on a form of not-elimination, hence showing how useful this technique is in practice (notably, even for schemas that do not use the negation operator to start with). They introduce many interesting techniques, but they only implement a limited form of not-elimination, since they do not extend JSON Schema operators. Specifically, they do not address recursive definitions, although these are rather extensively used in real-world schemas [10].

Indeed, our not-elimination algorithm for JSON Schema is the first to deal with the combination of negation and recursive variables [7], where we use a completion technique that we believe to be original. The combination of negation and recursion has been deeply studied in the context of logic languages, but these results cannot be easily transferred to JSON Schema, because of the different nature of these languages. For example, languages in the Prolog/Datalog family describe relations, while JSON Schema describes sets. Moreover, variables in relational languages denote elements, while in JSON Schema a variable denotes a set, like in Monadic Second Order logic (MSO). However, in MSO, variables are subject to quantification, while here, variables are only used to express recursion. A logic language where variables denote sets, and are used for recursion rather than for quantification, is the $\mu$-calculus [9], which has been used to interpret JSON Schema in [6]. However, classical $\mu$-calculus techniques cannot be immediately transferred to this context, since $\mu$-calculus does not allow the presence of recursive variables below an odd number of negations, but they are allowed by the JSON Schema standard, if recursion is guarded.

Works by Pezoa et al. [14] and Bourhis et al. [6] have already been commented in the introduction. The semantics that we provide is not that different from that given in [6], by means of the JSL modal logic. The main difference is that in [6] authors translate JSON Schema into a formalism very far from it, while we directly deal with JSON Schema itself, although we provided an algebraic syntax, since we are interested in building practical tools that manipulate JSON Schema at the source level.

8 Conclusions

We have shown that JSON Schema is “almost” negation-closed and we have provided an exact characterization of the schemas that cannot be expressed without negation. We have studied the impact of the new operators minContains and maxContains introduced in Draft 2019-09, and we have shown that they make the array construct negation-closed, at the price of a non-trivial encoding and exponential-size explosion. We have introduced an algebraic rendition of JSON Schema syntax that is amenable for automated manipulation.

We contributed a not-elimination algorithm that we are currently using as a building block for our algorithm for witness generation, satisfiability checking, and inclusion verification [11]. As shown in Example [11] not-elimination can also be useful to improve the readability of some JSON Schema documents. Our not-
elimination algorithm is the first that deals with negative recursive variables, and it uses an original, and very simple, technique to do so.

To check the completeness of our approach, we have implemented the translation from JSON Schema to the algebra and back, as well as a version of the not-elimination algorithm, and we tested them on our 91K collected schemas (after data cleaning and duplicate-elimination). The experiments confirmed that we deal with every aspect of the language, and that the not-elimination result has a size that grows linearly with the input size. The implementations are available along with a very preliminary version of witness generation, and our schemas [2].

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