Weakly coupled system of semilinear wave equations with distinct scale-invariant terms in the linear part

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Abstract. In this work we determine the critical exponent for a weakly coupled system of semilinear wave equations with distinct scale-invariant lower order terms, when these terms make both equations in some sense “parabolic-like”. For the blow-up result the test functions method is applied, while for the global existence (in time) results we use $L^2 - L^2$ estimates with additional $L^1$ regularity.

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1. Introduction

In this paper we consider the weakly coupled system of semilinear wave equations with scale-invariant damping and mass terms with different multiplicative constants in the lower order terms

$$
\begin{cases}
  u_{tt} - \Delta u + \frac{\mu_1}{1+t} u_t + \frac{\nu_1^2}{(1+t)^2} u = |v|^p, & x \in \mathbb{R}^n, \ t > 0, \\
  v_{tt} - \Delta v + \frac{\mu_2}{1+t} v_t + \frac{\nu_2^2}{(1+t)^2} v = |u|^q, & x \in \mathbb{R}^n, \ t > 0, \\
  (u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n,
\end{cases}
$$

where $\mu_1, \mu_2, \nu_1^2, \nu_2^2$ are nonnegative constants and $p, q > 1$.

As in the case of a single semilinear wave equation with scale-invariant damping and mass term, it turns out that the quantities

$$\delta_j = (\mu_j - 1)^2 - 4\nu_j^2, \quad j = 1, 2, \quad (1.2)$$

are useful to describe some of the properties of the model (1.1) as, for example, the critical exponent.

First we describe the meaning of the critical exponent for a semilinear weakly coupled system. Let us introduce the notations

$$\alpha_j = \frac{1}{2}(\mu_j + 1 - \sqrt{\delta_j}) \quad j = 1, 2. \quad (1.3)$$

In the case $\delta_1, \delta_2 \geq (n + 1)^2$, for (1.1) the critical exponent is given by

$$E = E(p, q, \alpha_1, \alpha_2) = \max \left\{ \frac{p + 1}{pq - 1} - \frac{\alpha_1 - 1}{2}, \frac{q + 1}{pq - 1} - \frac{\alpha_2 - 1}{2} \right\} = \frac{n}{2}, \quad (1.4)$$

that is, if $E < \frac{n}{2}$ (supercritical case), then, there exists a unique global solution for small data; else, if $E \geq \frac{n}{2}$ (subcritical or critical case), the local in time solution blows up in finite time.

Although we will be able to determine a blow-up result in the case in which $\delta_1, \delta_1 \geq 0$, due to the fact that a single scale-invariant wave equation shows properties analogous to those of the classical...
damped wave equation only for large values of the parameter \( \delta \), we will find a sharp result only in the case in which \( \delta_1, \delta_2 \geq (n + 1)^2 \) (see also [37] for further explanations about this condition).

We recall now some historical background to (1.1). Over the last years, semilinear weakly coupled systems have been widely studied.

Let us begin with the semilinear weakly coupled system of classical wave equations

\[
\begin{align*}
\begin{cases}
    u_{tt} - \Delta u = |v|^p, & x \in \mathbb{R}^n, \ t > 0, \\
    v_{tt} - \Delta v = |u|^q, & x \in \mathbb{R}^n, \ t > 0, \\
    (u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n,
\end{cases}
\end{align*}
\]

with \( p, q > 1 \). On the other hand, for the single semilinear wave equation we refer to the classical works [20, 22, 51, 15, 16, 50, 49, 29, 13, 19, 58, 61, 28], where the so-called Strauss exponent \( p_0(n) \) is proved to be the critical exponent, \( p_0(n) \) being the positive root of the quadratic equation \( (n-1)p^2 - (n+1)p - 2 = 0 \).

On the other hand, collecting the results from [8, 10, 9, 1, 24, 23, 14, 25], we find that the critical exponent for (1.5) is described by the condition

\[
\max \left\{ \frac{p+2+q^{-1}}{pq-1}, \frac{q+2+p^{-1}}{pq-1} \right\} = \frac{n-1}{2}.
\]

Let us recall some results for the semilinear weakly coupled system of classical damped wave equations

\[
\begin{align*}
\begin{cases}
    u_{tt} - \Delta u + u_t = |v|^p, & x \in \mathbb{R}^n, \ t > 0, \\
    v_{tt} - \Delta v + v_t = |u|^q, & x \in \mathbb{R}^n, \ t > 0, \\
    (u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n,
\end{cases}
\end{align*}
\]

with \( p, q > 1 \). For the single semilinear damped wave equation \( p_{Fu}(n) = 1 + \frac{2}{n} \) is the critical exponent, we refer to the classical works [53, 59, 18] for further details. The critical exponent for (1.6) is described by the condition

\[
\max \left\{ \frac{p+1}{pq-1}, \frac{q+1}{pq-1} \right\} = \frac{n}{2}.
\]

The authors of [52] investigated the critical exponent for \( n = 1, 3 \). In [38] the author generalized the global existence result to \( n = 1, 2, 3 \) and improved the time decay estimates when \( n = 3 \). After that, in [39] the asymptotic profile for global solutions has been derived in low dimensions \( n = 1, 2, 3 \). Then, in [40] global existence and blow-up in finite time results for any space dimension \( n \) were determined, where the proof of the global (in time) existence of energy solutions is based on a weighted energy method. Consequently, in [41] the previous result has been extended for a semilinear weakly coupled system of \( k \geq 2 \) damped wave equations. In comparison to the critical exponent for (1.6), we observe a translation in the critical exponent for the model that we consider in this work, which is due to the presence of the lower order scale-invariant terms. We also mention that several generalizations of (1.6) are possible in different ways. On the one hand, the weakly coupled system of damped waves with time-dependent coefficients in the dissipation terms is studied, for example, in [42, 35, 36]. In particular, in [35, 36] the global existence of solutions is proved, when initial data are supposed to belong to different classes of regularity. On the other hand, in [3] semilinear weakly coupled systems are studied replacing the classical damping terms with structural damping terms. Finally, in [2] a semilinear weakly coupled system of damped elastic waves is studied. In this latter case, the system is coupled not only in the nonlinear terms but also in the linear ones.

Recently, the Cauchy problem

\[
\begin{align*}
\begin{cases}
    u_{tt} - \Delta u + \frac{\mu}{1+t}u_t + \frac{\nu^2}{(1+t)^2}u = |u|^p, & x \in \mathbb{R}^n, \ t > 0, \\
    (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n,
\end{cases}
\end{align*}
\]

has attracted a lot of attention, where \( \mu, \nu^2 \) are nonnegative constants and \( p > 1 \) and, analogously to what we did for the system we denote \( \delta = (\mu - 1)^2 - 4\nu^2 \). The value of \( \delta \) has a strong influence on some properties of solutions to (1.7) and to the corresponding homogeneous linear equation. According to
[4, 57, 6, 5, 56, 37, 46, 43, 27, 17, 47, 54, 55, 44, 45, 7, 48, 21, 26] for \( \delta \geq 0 \) the model in (1.7) is somehow an intermediate model between the semilinear free wave equation and the semilinear classical damped equation, whose critical exponent is \( p_{\text{Fuj}}(n + \alpha - 1) \) for \( \delta \geq (n + 1)^2 \), where \( \alpha \) is defined analogously as in (1.3), and seems reasonably to be \( p_0(n + \mu) \) for small values of delta. In this paper we will deal with the system (1.1) and we will investigate how the interaction between the powers \( p, q \) in the nonlinearities provides either the global in time existence of the solution or the blow-up in finite time.

**Notations:** Throughout this paper we will use the following notations: \( B_R \) denotes the ball around the origin with radius \( R \); \( f \lesssim g \) means that there exists a positive constant \( C \) such that \( f \leq Cg \) and, similarly, for \( f \gtrsim g \); finally, as in the introduction, \( p_{\text{Fuj}}(n) \) and \( p_0(n) \) denote the Fujita exponent and the Strauss exponent, respectively.

2. Main results

In [37] a blow-up result is proved for (1.7) provided that \( \delta \geq 0 \) by using the so-called test function method in the case in which the exponent of the power nonlinearity is smaller than or equal to \( p_{\text{Fuj}}(n + \alpha - 1) \). In the next result we will generalize that result for the weakly coupled system (1.1).

Let us underline that, due to the presence of generally different coefficients in the linear terms of lower order, a new phenomenon appears, that cannot be observed for single equations or for weakly coupled systems with the same linear part (for example, in the case of (1.1) when \( \mu_1 = \mu_2 \) and \( \nu_1^2 = \nu_2^2 \)). More precisely, a restriction from below either for \( p \) or for \( q \) is necessary to get the desired result (see also Remark 2.2).

**Theorem 2.1 (Blow-up result).** Let \( \mu_1, \mu_2, \nu_1^2, \nu_2^2 \) be nonnegative constants such that \( \delta_1, \delta_2 \geq 0 \) and let \((u_0, u_1, v_0, v_1) \in \left( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \right)^2 \) be initial data such that

\[
\liminf_{R \to \infty} \int_{\mathbb{R}^n} \left( \frac{1}{2} (\mu_1 - 1 + \sqrt{\delta_1}) u_0(x) + u_1(x) \right) dx > 0,
\]

\[
\liminf_{R \to \infty} \int_{\mathbb{R}^n} \left( \frac{1}{2} (\mu_2 - 1 + \sqrt{\delta_2}) v_0(x) + v_1(x) \right) dx > 0.
\]

If \( p, q > 1 \) satisfy the relations

\[
\max \left\{ \frac{p + 1}{pq - 1} - \frac{\alpha_1}{2}, \frac{q + 1}{pq - 1} - \frac{\alpha_2}{2} \right\} - \frac{n - 1}{2} \geq 0,
\]

\[
\text{either } \quad p > \frac{1 + \alpha_1}{1 + \alpha_2} \quad \text{or} \quad q > \frac{1 + \alpha_2}{1 + \alpha_1},
\]

then, (1.1) has no globally in time weak solutions; that is, if \((u, v) \in L^p_{\text{loc}}([0, T) \times \mathbb{R}^n) \times L^q_{\text{loc}}([0, T) \times \mathbb{R}^n)\) is a local in time weak solution with maximal lifespan \( T \), then, \( T < \infty \).

**Remark 2.2.** We should point out that at least one of the conditions in (2.11) is trivially true. Indeed, since \( \alpha_j \geq 0 \) for \( j = 1, 2 \), then, in the case \( \alpha_1 > \alpha_2 \), it holds \( q > 1 > \frac{1 + \alpha_1}{1 + \alpha_2} \); while in the case \( \alpha_1 < \alpha_2 \), we have \( p > 1 > \frac{1 + \alpha_1}{1 + \alpha_2} \). Moreover, if \( \alpha_1 = \alpha_2 \), for example when we have the same coefficients for the linear terms in (1.1), then, no additional requirement on \( p \) or on \( q \) is necessary.

Due to the blow-up result in Theorem 2.1, we expect

\[
E(p, q, \alpha_1, \alpha_2) - \frac{\gamma}{2} \equiv \max \left\{ F(p, q, p, n, \alpha_1), F(q, p, n, \alpha_2) \right\} = 0
\]

to be the critical exponent for the semilinear system (1.1) in the case in which both linear parts are somehow “parabolic-like” (see Remark 2.5), where

\[
F(p, q, n, \alpha) \doteq \frac{p+1}{pq-1} - \frac{n+\alpha-1}{2}.
\]

Correspondingly to the case of a single semilinear equation with power nonlinearity, we mean that if \( p, q > 1 \) satisfy

\[
\max \left\{ F(p, q, n, \alpha_1), F(q, p, n, \alpha_2) \right\} < 0,
\]
then, there exists a unique global solution for small initial data, whereas a local in time solution with some integral sign assumptions for the data blows up in finite time if the left hand side in (2.12) is nonnegative.

As we have already shown a result for the necessity part, now we want to investigate the sufficiency part. Before doing that, we shall clarify under which necessary condition the left hand side in (2.12) can be negative. For this purpose we introduce the following notations:

\[
\begin{align*}
\tilde{p}(n, \alpha_1, \alpha_2) &= \frac{n+\alpha_1+1}{n+\alpha_2-1} = 1 + \frac{2+\alpha_2-\alpha_1}{n+\alpha_2-1}, \\
\tilde{q}(n, \alpha_1, \alpha_2) &= \frac{n+\alpha_1+1}{n+\alpha_2-1} = 1 + \frac{2+\alpha_2-\alpha_1}{n+\alpha_2-1}.
\end{align*}
\]

Let us remark that if \( p \leq \tilde{p}(n, \alpha_1, \alpha_2) \) and \( q \leq \tilde{q}(n, \alpha_1, \alpha_2) \), then \( F(p, q, n, \alpha_1) \geq 0 \). Indeed,

\[
\left( q - \frac{2}{n+\alpha_2-1} \right) \leq \frac{n+\alpha_1+1}{n+\alpha_2-1} \Rightarrow p\left( q - \frac{2}{n+\alpha_2-1} \right) \leq 1 + \frac{2}{n+\alpha_2-1}
\]

\[
\Rightarrow pq - 1 \leq \frac{2}{n+\alpha_2-1}(p+1)
\]

\[
\Rightarrow \frac{n+\alpha_1+1}{n+\alpha_2-1} \leq \frac{p+1}{pq-1}.
\]

Analogously, under the same assumptions on \( p \) and \( q \) it holds \( F(q, p, n, \alpha_2) \geq 0 \). Summarizing, \( p \leq \tilde{p}(n, \alpha_1, \alpha_2) \) and \( q \leq \tilde{q}(n, \alpha_1, \alpha_2) \) imply that the left hand side in (2.12) is nonnegative.

Consequently, in order to prove the global in time existence for small data solutions provided that \((p, q)\) satisfies (2.13), we may consider separately the following three subcases:

\[
\begin{align*}
p > \tilde{p}(n, \alpha_1, \alpha_2) \quad & \text{and} \quad q > \tilde{q}(n, \alpha_1, \alpha_2), \\
p \leq \tilde{p}(n, \alpha_1, \alpha_2) \quad & \text{and} \quad q > \tilde{q}(n, \alpha_1, \alpha_2), \\
p > \tilde{p}(n, \alpha_1, \alpha_2) \quad & \text{and} \quad q \leq \tilde{q}(n, \alpha_1, \alpha_2).
\end{align*}
\]

More precisely, in the case (2.14) no loss of decay with respect to the corresponding linear problem will appear in the decay estimates. On the other hand, in the case where \( p, q > 1 \) fulfill (2.15) (respectively (2.16)), because different power source nonlinearities have different influence on conditions for the global (in time) existence of solutions, we allow the effect of the loss of decay.

Before stating these global existence results, we should recall some known results for the family of parameter dependent linear Cauchy problems

\[
\begin{align*}
\begin{cases}
\frac{u_t}{\alpha(x,t)} - \nabla u + \frac{\mu}{(t+\nu)} u_t + \frac{\nu^2}{(t+\nu)} u = 0, & x \in \mathbb{R}^n, \ t > s, \\
(u, u)(s, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n, s
\end{cases}
\end{align*}
\]

(2.17)

where the initial time \( s \geq 0 \) is considered because of the lack of invariance by time-translation for this linear model with time-dependent coefficients. For the proofs of the next results we refer to [46, Theorems 4.6 and 4.7].

**Proposition 2.3.** Let \( \mu > 0 \) and \( \nu^2 \) be nonnegative constants such that \( \delta > (n+1)^2 \). Let us consider \((u_0, u_1) \in (H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \times (L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))\). Then, for all \( \kappa \in [0, 1] \) the energy solution \( u = u(t, x) \) to (2.17) with \( s \) satisfies the decay estimate

\[
\|u(t, \cdot)\|_{H^\kappa(\mathbb{R}^n)} \lesssim \left( \|u_0\|_{H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)} \right) (1 + t)^{-\kappa-\frac{2}{\alpha}+1}.
\]

Moreover, \( \|u(t, \cdot)\|_{L^2(\mathbb{R}^n)} \) satisfies the same decay estimates as \( \|\nabla u(t, \cdot)\|_{L^2(\mathbb{R}^n)} \).

**Proposition 2.4.** Let \( \mu > 0 \) and \( \nu^2 \) be nonnegative constants such that \( \delta > (n+1)^2 \). Let us assume \( u_0 = 0 \) and \( u_1 \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \). Then, the energy solution \( u = u(t, x) \) to (2.17) satisfies for \( t \geq s \) and \( \kappa \in [0, 1] \) the following estimate

\[
\|u(t, \cdot)\|_{H^\kappa(\mathbb{R}^n)} \lesssim \left( \|u_1\|_{L^1(\mathbb{R}^n)} + (1 + s)^{\frac{2}{\alpha}} \|u_1\|_{L^2(\mathbb{R}^n)} \right) (1 + s)^\alpha (1 + t)^{-\kappa-\frac{2}{\alpha}+1}.
\]

Moreover, \( \|u(t, \cdot)\|_{L^2(\mathbb{R}^n)} \) satisfies the same decay estimates as \( \|\nabla u(t, \cdot)\|_{L^2(\mathbb{R}^n)} \).
Remark 2.5. In the case $\delta > (n+1)^2$ for the linear Cauchy problem (2.17) we get for the $L^2$ norms of the derivatives a better decay rate than the one for the $L^2$ norm of the solution. Considering larger values of $\delta$ we can observe this phenomenon even for derivatives of higher order, with an improved decay rate as well. In this sense, we say that (2.17) is “parabolic-like” for large values of $\delta$.

Now we can state the main global existence results for (1.1). As in the previous propositions, we will work with initial data in the classical energy space with additional $L^1$ regularity, so that the space for the Cauchy data is

$$\mathcal{A} = (H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \times (L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)).$$

Let us begin with the subcase (2.14).

Theorem 2.6. Let $\mu_1, \mu_2 > 1, \nu_1^2, \nu_2^2$ be nonnegative constants such that $\delta_1, \delta_2 > (n+1)^2$. Let us assume $p, q > 1$, satisfying $2 \leq p, q$ and $p, q \leq \frac{n}{n-2}$ if $n \geq 3$, such that

$$p > \tilde{p}(n, \alpha_1, \alpha_2) \quad \text{and} \quad q > \tilde{q}(n, \alpha_1, \alpha_2).$$

Then, there exists a constant $\varepsilon_0 > 0$ such that for any $(u_0, v_1) \in \mathcal{A}$ and $(v_0, v_1) \in \mathcal{A}$ with

$$\|(u_0, u_1)\|_{\mathcal{A}} + \|(v_0, v_1)\|_{\mathcal{A}} \leq \varepsilon_0$$

there is a uniquely determined energy solution $(u, v) \in (C([0, \infty), H^1(\mathbb{R}^n)) \cap C^1([0, \infty), L^2(\mathbb{R}^n)))^2$ to (1.1). Furthermore, the solution $(u, v)$ satisfies the following decay estimates:

$$\|\nabla u(t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1 + t)^{-\frac{n+\alpha_1}{2}+\gamma}(\|\nabla u(t, \cdot)\|_{\mathcal{A}} + \|(u_0, u_1)\|_{\mathcal{A}}),$$

$$u(t, \cdot) \|_{L^2(\mathbb{R}^n)} \lesssim (1 + t)^{-\frac{n+\alpha_1+1}{2}+\gamma}(\|\nabla u(t, \cdot)\|_{\mathcal{A}} + \|(u_0, u_1)\|_{\mathcal{A}}),$$

$$\|\nabla v(t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1 + t)^{-\frac{n+\alpha_2}{2}+\gamma}(\|\nabla v(t, \cdot)\|_{\mathcal{A}} + \|(v_0, v_1)\|_{\mathcal{A}}),$$

$$v(t, \cdot) \|_{L^2(\mathbb{R}^n)} \lesssim (1 + t)^{-\frac{n+\alpha_2+1}{2}+\gamma}(\|\nabla v(t, \cdot)\|_{\mathcal{A}} + \|(v_0, v_1)\|_{\mathcal{A}}).$$

Remark 2.7. If $p, q > 1$ satisfy (2.14), then, $F(p, q, n, \alpha_1) < 0$ and $F(q, p, n, \alpha_2) < 0$. Also, in particular, (2.13) holds. Indeed,

$$\left(q - \frac{2}{n+\alpha_1-1}\right) > \frac{n+\alpha_2-1}{n+\alpha_1-1} \quad \Rightarrow \quad p(q - \frac{2}{n+\alpha_1-1}) > 1 + \frac{2}{n+\alpha_1-1} \quad \Rightarrow \quad F(p, q, n, \alpha_1) < 0,$$

and, similarly, $F(q, p, n, \alpha_2) < 0$.

Let us consider now the subcase (2.15).

Theorem 2.8. Let $\mu_1, \mu_2 > 1, \nu_1^2, \nu_2^2$ be nonnegative constants such that $\delta_1, \delta_2 > (n+1)^2$. Let us assume $p, q > 1$, satisfying $2 \leq p, q$ and $p, q \leq \frac{n}{n-2}$ if $n \geq 3$, such that

$$p \leq \tilde{p}(n, \alpha_1, \alpha_2) \quad \text{and} \quad q > \tilde{q}(n, \alpha_1, \alpha_2),$$

$$F(p, q, n, \alpha_2) = \frac{q+1}{p-q-1} - \frac{n+\alpha_2-1}{2} < 0. \quad (2.20)$$

Then, there exists a constant $\varepsilon_0 > 0$ such that for any $(u_0, u_1) \in \mathcal{A}$ and $(v_0, v_1) \in \mathcal{A}$ with

$$\|(u_0, u_1)\|_{\mathcal{A}} + \|(v_0, v_1)\|_{\mathcal{A}} \leq \varepsilon_0$$

there is a uniquely determined energy solution $(u, v) \in (C([0, \infty), H^1(\mathbb{R}^n)) \cap C^1([0, \infty), L^2(\mathbb{R}^n)))^2$ to (1.1). Furthermore, the solution $(u, v)$ satisfies the following estimates:

$$\|\nabla u(t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1 + t)^{-\frac{n+\alpha_1}{2}+\gamma}(\|u(t, \cdot)\|_{\mathcal{A}} + \|(u_0, u_1)\|_{\mathcal{A}}),$$

$$\|u(t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1 + t)^{-\frac{n+\alpha_1+1}{2}+\gamma}(\|\nabla u(t, \cdot)\|_{\mathcal{A}} + \|(u_0, u_1)\|_{\mathcal{A}}),$$

$$\|\nabla v(t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1 + t)^{-\frac{n+\alpha_2}{2}+\gamma}(\|\nabla v(t, \cdot)\|_{\mathcal{A}} + \|(v_0, v_1)\|_{\mathcal{A}}),$$

$$\|v(t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1 + t)^{-\frac{n+\alpha_2+1}{2}+\gamma}(\|\nabla v(t, \cdot)\|_{\mathcal{A}} + \|(v_0, v_1)\|_{\mathcal{A}}).$$
Then, there exists a constant \( \varepsilon (1.1) \) to the linear Cauchy problem with vanishing right hand side (cf. Proposition 2.3), \( \varepsilon > 0 \) being an arbitrarily small constant in the limit case \( p = \tilde{p}(n, \alpha_1, \alpha_2) \).

Remark 2.9. Under the assumptions of Theorem 2.8, the condition \( F(p, q, n, \alpha_1) < 0 \) is only apparently not necessary in order to apply a standard contraction argument. Nevertheless, in the moment in which we split the global (in time) existence results in the subcases (2.14), (2.15) and (2.16), we have already used the condition (2.13) and, thus, the condition \( F(p, q, n, \alpha_1) < 0 \).

Finally, switching the role of \( p \) and \( q \) in Theorem 2.8, we get the next result.

Theorem 2.10. Let \( \mu_1, \mu_2 > 1, \nu_1^2, \nu_2^2 \) be nonnegative constants such that \( \delta_1, \delta_2 > (n+1)^2 \). Let us assume \( p, q > 1 \), satisfying \( 2 \leq p, q \) and \( p, q \leq \frac{n^2}{n-2} \) if \( n \geq 3 \), such that

\[
\begin{align*}
& p > \tilde{p}(n, \alpha_1, \alpha_2) \quad \text{and} \quad q \leq \tilde{q}(n, \alpha_1, \alpha_2), \\
& F(p, q, n, \alpha_1) = \frac{p+1}{pq} - \frac{n+\alpha_1-1}{2} < 0. \quad (2.21)
\end{align*}
\]

Then, there exists a constant \( \varepsilon_0 > 0 \) such that for any \((u_0, u_1) \in \mathcal{A}\) and \((v_0, v_1) \in \mathcal{A}\) with

\[
\| (u_0, u_1) \|_\mathcal{A} + \| (v_0, v_1) \|_\mathcal{A} \leq \varepsilon_0
\]

there is a uniquely determined energy solution \((u, v) \in (C([0, \infty), H^1(\mathbb{R}^n)) \cap C^1((0, \infty), L^2(\mathbb{R}^n)))^2 \) to (1.1). Furthermore, the solution \((u, v)\) satisfies the following estimates:

\[
\begin{align*}
& \| \nabla u(t, \cdot) \|_{L^2(\mathbb{R}^n)} \lesssim (1 + t)^{-\frac{\alpha_1}{2} - \frac{\alpha_2}{2}} \| (u_0, u_1) \|_\mathcal{A} + \| (v_0, v_1) \|_\mathcal{A}, \\
& \| u(t, \cdot) \|_{L^2(\mathbb{R}^n)} \lesssim (1 + t)^{-\frac{\alpha_1}{2} + 1} \| (u_0, u_1) \|_\mathcal{A} + \| (v_0, v_1) \|_\mathcal{A}, \\
& \| \nabla v(t, \cdot) \|_{L^2(\mathbb{R}^n)} \lesssim (1 + t)^{-\frac{\alpha_1}{2} - \frac{\alpha_2}{2} + \tilde{\gamma}} \| (u_0, u_1) \|_\mathcal{A} + \| (v_0, v_1) \|_\mathcal{A}, \\
& \| v(t, \cdot) \|_{L^2(\mathbb{R}^n)} \lesssim (1 + t)^{-\frac{\alpha_1}{2} - \frac{\alpha_2}{2} + 1 + \tilde{\gamma}} \| (u_0, u_1) \|_\mathcal{A} + \| (v_0, v_1) \|_\mathcal{A},
\end{align*}
\]

where

\[
0 < \tilde{\gamma} = \tilde{\gamma}(q, n, \alpha_1, \alpha_2) = \begin{cases} (n + \alpha_1 - 1)(\tilde{q}(n, \alpha_1, \alpha_2) - q) & \text{if } q < \tilde{q}(n, \alpha_1, \alpha_2), \\
\varepsilon & \text{if } q = \tilde{q}(n, \alpha_1, \alpha_2)
\end{cases}
\]

represents the loss of decay in comparison with the corresponding decay estimates for the solution \( v \) to the linear Cauchy problem with vanishing right hand side (cf. Proposition 2.3), \( \varepsilon > 0 \) being an arbitrarily small constant in the limit case \( q = \tilde{q}(n, \alpha_1, \alpha_2) \).

Remark 2.11. Also in this case, the condition \( F(q, p, n, \alpha_2) < 0 \) is implicitly used in the previous theorem.

3. Blow-up result: Proof of Theorem 2.1

In this section we will employ the so-called test functions method (see for example[30, 31, 11, 32, 33, 34, 59, 12]).
Let us assume by contradiction that $(u, v) \in L^p_{\text{loc}}(\mathbb{R}^n) \times L^q_{\text{loc}}(\mathbb{R}^n)$ is a global (in time) weak solution to (1.1), that is $T = \infty$. This means that the integral equalities

$$
\iint_{(0,T) \times \mathbb{R}^n} \left( \partial_t^2 \psi_1(t, x) - \Delta \psi_1(t, x) - \partial_t \left( \frac{\mu_1}{1+t} \psi_1(t, x) \right) + \frac{\nu^2}{(1+t)^2} \psi_1(t, x) \right) u(t, x) d(t, x)
= \iint_{\mathbb{R}^n} \left( \psi_1(0, x)(u_1(x) + \mu_1 u_0(x)) - \partial_t \psi_1(0, x) u_0(x) \right) dx + \iint_{(0,T) \times \mathbb{R}^n} \psi_1(t, x) |v(t, x)|^p d(t, x),
$$

$$
\iint_{(0,T) \times \mathbb{R}^n} \left( \partial_t^2 \psi_2(t, x) - \Delta \psi_2(t, x) - \partial_t \left( \frac{\mu_2}{1+t} \psi_2(t, x) \right) + \frac{\nu^2}{(1+t)^2} \psi_2(t, x) \right) v(t, x) d(t, x)
= \iint_{\mathbb{R}^n} \left( \psi_2(0, x)(v_1(x) + \mu_2 v_0(x)) - \partial_t \psi_2(0, x) v_0(x) \right) dx + \iint_{(0,T) \times \mathbb{R}^n} \psi_2(t, x) |u(t, x)|^q d(t, x),
$$

(3.22)

(3.23)

are fulfilled for any $(\psi_1, \psi_2) \in \left(C^\infty_0([0, T] \times \mathbb{R}^n)\right)^2$.

Multiplying the first and the second equation in (1.1) by time-dependent functions $g_1 = g_1(t)$ and $g_2 = g_2(t)$, respectively, we obtain

$$
\partial_t^2(g_1 u) - \Delta(g_1 u) + \partial_t \left( -2g_1 u + \frac{\mu_1}{1+t} g_1 u \right) + \left( g_1'' - \frac{\mu_1}{1+t} g_1' + \frac{\mu_1 + \nu^2}{(1+t)^2} g_1 \right) u = g_1 |v|^p,
$$

$$
\partial_t^2(g_2 v) - \Delta(g_2 v) + \partial_t \left( -2g_2 v + \frac{\mu_2}{1+t} g_2 v \right) + \left( g_2'' - \frac{\mu_2}{1+t} g_2' + \frac{\mu_2 + \nu^2}{(1+t)^2} g_2 \right) v = g_2 |u|^q.
$$

If we choose

$$
g_j(t) \doteq (1 + t)^{\alpha_j}, \quad j = 1, 2,
$$

(3.24)

then, $g_1$ and $g_2$ satisfy

$$
g_j'' - \frac{\mu_j}{1+t} g_j' + \frac{\mu_j + \nu^2}{(1+t)^2} g_j = 0, \quad j = 1, 2.
$$

Therefore, the previous two relations can be written in the divergence form as follows:

$$
\partial_t^2(g_1 u) - \Delta(g_1 u) + \partial_t \left( -2g_1 u + \frac{\mu_1}{1+t} g_1 u \right) = g_1 |v|^p,
$$

$$
\partial_t^2(g_2 v) - \Delta(g_2 v) + \partial_t \left( -2g_2 v + \frac{\mu_2}{1+t} g_2 v \right) = g_2 |u|^q.
$$

Let us introduce now two bump functions $\eta \in C^\infty_0([0, \infty)), \phi \in C^\infty_0(\mathbb{R}^n)$ such that

- $\eta$ is decreasing, $\eta = 1$ on $[0, \frac{1}{4}]$ and supp $\eta \subset [0, 1]$;
- $\phi$ is radial symmetric and decreasing with respect to $|x|$, $\phi = 1$ on $B_{\frac{1}{2}}$ and supp $\phi \subset B_1$.

These functions satisfy the estimates

$$
|\eta'(t)| \lesssim \eta(t)^{\frac{1}{\rho}}, \quad |\eta''(t)| \lesssim \eta(t)^{\frac{1}{\rho}}, \quad |\Delta \phi(x)| \lesssim \phi(x)^{\frac{1}{\rho}}
$$

for any $\rho > 1$ (see [37], for example). Moreover, since $0 \leq \eta(t), \phi(x) \leq 1$, then, $\eta(t) \lesssim \eta(t)^{\frac{1}{\rho}}$ and $\phi(x) \lesssim \phi(x)^{\frac{1}{\rho}}$ for any $\rho > 1$. In particular, we will use these conditions for $\rho = p, q$.

Given two positive parameters $\tau$ and $R$, we define

$$
\psi_{x,R}(t, x) \doteq \eta_\tau(t) \phi_R(x) \quad \text{with} \quad \eta_\tau(t) \doteq \eta\left(\frac{t}{\tau}\right) \text{ and } \phi_R(x) \doteq \phi\left(\frac{x}{R}\right).
$$
Furthermore, we introduce the following two integrals depending on the parameters $\tau, R$:

\[
I_{\tau,R} = \int_0^\tau \int_{B_R(0)} g_1(t) \psi_{\tau,R}(t, x) v(t, x) dx \, dt,
\]

\[
J_{\tau,R} = \int_0^\tau \int_{B_R(0)} g_2(t) \psi_{\tau,R}(t, x) u(t, x) dx \, dt.
\]

Applying the integral relation (3.22) to $g_1 \psi_{\tau,R}$, we get

\[
I_{\tau,R} = -\int_{B_R(0)} (u_1(x) + \mu_1 u_0(x)) \phi_R(x) dx + \int_0^\tau \int_{B_R(0)} g_1(t) u(t, x) \partial_t^2 \psi_{\tau,R}(t, x) dx \, dt - \int_0^\tau \int_{B_R(0)} g_1(t) u(t, x) \Delta \psi_{\tau,R}(t, x) dx \, dt
\]

\[
+ \int_0^\tau \int_{B_R(0)} \left( g_1''(t) - \frac{\mu_1}{1+\mu_1^2} g_1(t) \right) u(t, x) \partial_t \psi_{\tau,R}(t, x) dx \, dt
\]

\[
= -\int_{B_R(0)} (u_1(x) + \left( \frac{\mu_1}{2} - \frac{1}{2} + \frac{\sqrt{\mu_1}}{2} \right) u_0(x)) \phi_R(x) dx + \int_0^\tau \int_{B_R(0)} g_1(t) u(t, x) \partial_t^2 \psi_{\tau,R}(t, x) dx \, dt
\]

\[
+ \int_0^\tau \int_{B_R(0)} \left( g_1''(t) - \frac{\mu_1}{1+\mu_1^2} g_1(t) \right) u(t, x) \partial_t \psi_{\tau,R}(t, x) dx \, dt
\]

\[
- \int_0^\tau \int_{B_R(0) \setminus B_{R/2}(0)} g_1(t) u(t, x) \Delta \psi_{\tau,R}(t, x) dx \, dt
\]

\[
\simeq -\int_{B_R(0)} (u_1(x) + \left( \frac{\mu_1}{2} - \frac{1}{2} + \frac{\sqrt{\mu_1}}{2} \right) u_0(x)) \phi_R(x) dx + K_1 + K_2 + K_3.
\]

Let us underline that in the previous chain of equalities we used

\[
\text{supp}(\partial_t \psi_{\tau,R}), \text{supp}(\partial_t^2 \psi_{\tau,R}) \subset \left[ \frac{\tau}{2}, \tau \right] \times B_R(0) \quad \text{and} \quad \text{supp}(\Delta \psi_{\tau,R}) \subset [0, \tau] \times (B_R(0) \setminus B_{R/2}(0)).
\]

Thanks to (2.8) and to the properties of $\phi_R$, there exists $R_0$ such that for any $R \geq R_0$

\[
\int_{B_R(0)} (u_1(x) + \left( \frac{\mu_1}{2} - \frac{1}{2} + \frac{\sqrt{\mu_1}}{2} \right) u_0(x)) \phi_R(x) dx > 0.
\]

Thus, for $R \geq R_0$ it holds

\[
I_{\tau,R} < K_1 + K_2 + K_3.
\]

Let us separately estimate the integrals $K_1, K_2, K_3$. Let us begin with $K_1$. Since

\[
K_1 = \tau^{-2} \int_0^\tau \int_{B_R(0)} g_1(t) u(t, x) \eta'' \left( \frac{t}{\tau} \right) \phi \left( \frac{x}{\tau} \right) dx \, dt
\]

\[
= \tau^{-2} \int_0^\tau \int_{B_R(0)} g_2(t) \gamma^* u(t, x) \eta'' \left( \frac{t}{\tau} \right) \phi \left( \frac{x}{\tau} \right) g_1(t) g_2(t) \frac{1}{\gamma^*} \, dx \, dt,
\]
where \( q' \) denotes the Hölder conjugate of \( q \), by Hölder’s inequality it follows

\[
|K_1| \leq \tau^{-2} \left( \int_\mathcal{R} \int_{B_R(0)} g_2(t) |u(t,x)|^q |\eta''\left(\frac{t}{\tau}\right)|^q |\phi\left(\frac{x}{\tau R}\right)|^q \, dx \, dt \right)^{\frac{1}{q}} \left( \int_\mathcal{R} \int_{B_R(0)} g_1(t)^{q'} g_2(t)^{1-q'} \, dx \, dt \right)^{\frac{1}{q'}}
\]

\[
\leq \tau^{-2} \left( \int_\mathcal{R} \int_{B_R(0)} g_2(t) |u(t,x)|^q \psi_{\tau,R}(t,x) \, dx \, dt \right)^{\frac{1}{q}} \left( \int_\mathcal{R} \int_{B_R(0)} (1+t)^{(\alpha_1-\alpha_2)q'+\alpha_2} \, dx \, dt \right)^{\frac{1}{q'}}.
\]

If we introduce the parameter dependent integral

\[
\tilde{J}_{\tau,R} = \int_\mathcal{R} \int_{B_R} g_2(t) \psi_{\tau,R}(t,x) |u(t,x)|^q \, dx \, dt,
\]

then for \( \tau > 1 \) we get from the last inequality

\[
|K_1| \lesssim \tau^{-2+\left(\alpha_1-\alpha_2\right)+\frac{\alpha_2+1}{q'} R \tilde{J}_{\tau,R}^{\frac{1}{q'}}.
\]

Let us consider now \( K_2 \). The relation

\[
2g_1'(t) - \frac{\mu_1}{1+\tau} g_1(t) = (2\alpha_1 - \mu_1)(1+\tau)^{\alpha_1-1} = (1 - \sqrt{\delta_1}) g_1(t) (1+\tau)^{-1}
\]

implies

\[
K_2 = \tau^{-1} \int_\mathcal{R} \int_{B_R(0)} \left(2g_1'(t) - \frac{\mu_1}{1+\tau} g_1(t)\right) u(t,x) \eta'\left(\frac{x}{\tau R}\right) \phi\left(\frac{x}{\tau R}\right) \, dx \, dt
\]

\[
= (1 - \sqrt{\delta_1}) \tau^{-1} \int_\mathcal{R} \int_{B_R(0)} g_2(t)^{\frac{1}{q'}} u(t,x) \eta'\left(\frac{x}{\tau R}\right) (1+\tau)^{-1} g_1(t) g_2(t)^{\frac{1}{q'}-1} \, dx \, dt.
\]

Hence, using Hölder’s inequality, we arrive at

\[
|K_2| \lesssim \tau^{-1} \left( \int_\mathcal{R} \int_{B_R(0)} g_2(t) |u(t,x)|^q |\eta''\left(\frac{t}{\tau}\right)|^q |\phi\left(\frac{x}{\tau R}\right)|^q \, dx \, dt \right)^{\frac{1}{q}}
\]

\[
\times \left( \int_\mathcal{R} \int_{B_R(0)} (1+t)^{-q'} g_1(t)^{q'} g_2(t)^{1-q'} \, dx \, dt \right)^{\frac{1}{q'}}
\]

\[
\lesssim \tau^{-1} \left( \int_\mathcal{R} \int_{B_R(0)} g_2(t) |u(t,x)|^q \psi_{\tau,R}(t,x) \, dx \, dt \right)^{\frac{1}{q}} \left( \int_\mathcal{R} \int_{B_R(0)} (1+t)^{(\alpha_1-\alpha_2)q'+\alpha_2} \, dx \, dt \right)^{\frac{1}{q'}}
\]

\[
\lesssim \tau^{-2+\left(\alpha_1-\alpha_2\right)+\frac{\alpha_2+1}{q'} R \tilde{J}_{\tau,R}^{\frac{1}{q'}}.
\]

for \( \tau > 1 \). Finally, we estimate \( K_3 \). Applying again Hölder’s inequality to

\[
K_3 = -R^{-2} \int_0^\tau \int_{B_R(0)\setminus B_{R/2}(0)} g_1(t) u(t,x) \eta\left(\frac{x}{\tau R}\right) \Delta \phi\left(\frac{x}{\tau R}\right) \, dx \, dt
\]

\[
= -R^{-2} \int_0^\tau \int_{B_R(0)\setminus B_{R/2}(0)} g_2(t)^{\frac{1}{q'}} u(t,x) \eta\left(\frac{x}{\tau R}\right) \Delta \phi\left(\frac{x}{\tau R}\right) g_1(t) g_2(t)^{\frac{1}{q'}-1} \, dx \, dt,
\]
we find

\[ |K_3| \lesssim R^{-2} \left( \int_0^\tau \int_{B_R(0) \setminus B_{R/2}(0)} g_2(t) |u(t,x)|^q \eta(\frac{t}{\tau}) |\Delta \phi(\frac{x}{R})|^q \, dx \, dt \right)^{\frac{1}{q}} \times \left( \int_0^\tau \int_{B_R(0) \setminus B_{R/2}(0)} g_1(t)^q g_2(t)^{1-q} \, dx \, dt \right)^{\frac{1}{q}} \]

\[ \lesssim R^{-2} \left( \int_0^\tau \int_{B_R(0) \setminus B_{R/2}(0)} g_2(t) |u(t,x)|^q \psi_{\tau,R}(t,x) \, dx \, dt \right)^{\frac{1}{q}} \times \left( \int_0^\tau \int_{B_R(0) \setminus B_{R/2}(0)} (1+t)^{(\alpha_1-\alpha_2)q' + \alpha_2} \, dx \, dt \right)^{\frac{1}{q}} \]

\[ \lesssim R^{-2+\frac{1}{q}} \tilde{J}_{\tau,R}^{\frac{1}{q}} \left( \int_0^\tau (1+t)^{(\alpha_1-\alpha_2)q' + \alpha_2} dt \right)^{\frac{1}{q}}, \]

where \( \tilde{J}_{\tau,R} \) is given by

\[ \tilde{J}_{\tau,R} = \int_0^\tau \int_{B_R(0) \setminus B_{R/2}(0)} g_2(t) \psi_{\tau,R}(t,x) |u(t,x)|^q \, dx \, dt. \]

Differently from the estimates for the terms \( K_1, K_2 \) in this case we need to consider three subcases for the estimate of the \( t \)-integral on the right hand side of the last inequality for \( |K_3| \), because the integral is no longer over \([\frac{\tau}{2}, \tau]\) rather on \([0, \tau]\). Indeed, for \( \tau > 1 \) it holds

\[ \int_0^\tau (1+t)^{(\alpha_1-\alpha_2)q' + \alpha_2} \, dt \lesssim \begin{cases} t^{(\alpha_1-\alpha_2)q' + \alpha_2 + 1} & \text{if } (\alpha_1 - \alpha_2)q' + \alpha_2 > -1, \\ \log(1+\tau) & \text{if } (\alpha_1 - \alpha_2)q' + \alpha_2 = -1, \\ 1 & \text{if } (\alpha_1 - \alpha_2)q' + \alpha_2 < -1. \end{cases} \]

Due to (2.11), we are necessary in the first of the previous cases, since \( q > \frac{1+\alpha_2}{1+\alpha_1} \) is equivalent to \((\alpha_1 - \alpha_2)q' + \alpha_2 > -1\). Thus,

\[ |K_3| \lesssim t^{(\alpha_1-\alpha_2) + \frac{\alpha_2+1}{q}} R^{-2+\frac{1}{q}} \tilde{J}_{\tau,R}^{\frac{1}{q}}. \]

Consequently, combining the previously obtained estimates for \( K_1, K_2, K_3 \), we get

\[ I_R \lesssim R^{-2+\alpha_1-\alpha_2 + \frac{\alpha_2+1}{q}} \left( J_{\tau,R}^{\frac{1}{q}} + \tilde{J}_{\tau,R}^{\frac{1}{q}} \right) \quad (3.25) \]

for \( \tau = R > \max \{ R_0, 1 \} \), where for the sake of simplicity of notation we get rid of the second parameter in the subscript in \( I, \hat{J}, \tilde{J} \).

Applying the integral relation (3.23) to \( g_2 \psi_{\tau,R} \), similarly as for the computations for \( I_{\tau,R} \), we get

\[ J_{\tau,R} = -\int_{B_R(0)} \left( v_1(x) + \left( \frac{\mu_a}{2} - \frac{1}{2} + \frac{\sqrt{\delta}}{R} \right) v_0(x) \right) \phi_R(x) \, dx + \int_0^\tau \int_{B_R(0)} g_2(t) v(t,x) \partial_t^2 \psi_{\tau,R}(t,x) \, dx \, dt \]

\[ + \int_0^\tau \int_{B_R(0)} \left( 2g_2'(t) - \frac{\mu_a}{1+t} g_2(t) \right) v(t,x) \partial_t \psi_{\tau,R}(t,x) \, dx \, dt \]

\[ - \int_0^\tau \int_{B_R(0) \setminus B_{R/2}(0)} g_2(t) v(t,x) \Delta \psi_{\tau,R}(t,x) \, dx \, dt \]

\[ - \int_{B_R(0)} \left( v_1(x) + \left( \frac{\mu_a}{2} - \frac{1}{2} + \frac{\sqrt{\delta}}{R} \right) v_0(x) \right) \phi_R(x) \, dx + L_1 + L_2 + L_3. \]

Using (2.9), we have that there exists \( R_1 \) such that for any \( R \geq R_1 \)

\[ \int_{B_R(0)} \left( v_1(x) + \left( \frac{\mu_a}{2} - \frac{1}{2} + \frac{\sqrt{\delta}}{2} \right) v_0(x) \right) \phi_R(x) \, dx > 0. \]
Analogously to the estimates for $K_1, K_2, K_3$, if we introduce
\[
\tilde{I}_{\tau,R} = \int_T^\infty \int_{B_{\tau R}} g_1(t) \psi_{\tau,R}(t,x) |v(t,x)|^p \, dx \, dt,
\]
\[
\bar{I}_{\tau,R} = \int_0^T \int_{B_{R/2}(0) \setminus B_R(0)} g_1(t) \psi_{\tau,R}(t,x) |v(t,x)|^p \, dx \, dt,
\]
then it follows
\[
|L_1| + |L_2| \lesssim \tau^{-2+(\alpha_2-\alpha_1)+\frac{n+\alpha_1+1}{q'}} R \tau^{\frac{1}{q'}} \tilde{I}_{\tau,R}^{\frac{1}{q'}},
\]
\[
|L_3| \lesssim \tau^{(\alpha_2-\alpha_1)+\frac{n+\alpha_1+1}{p'}} R^{-2+\frac{1}{p'}} \bar{I}_{\tau,R}^{\frac{1}{p'}},
\]
for $\tau > 1, R \geq R_1$ and provided that
\[
(\alpha_2 - \alpha_1)p' + \alpha_1 > -1 \quad \iff \quad p > \frac{1+\alpha_1}{1+\alpha_2}.
\]

Hence, for $\tau = R > \max\{1, R_1\}$ we obtain
\[
J_R \lesssim R^{-2+(\alpha_2-\alpha_1)+\frac{n+\alpha_1+1}{q'}} (\tilde{I}_{\tau,R}^{\frac{1}{q'}} + \bar{I}_{\tau,R}^{\frac{1}{q'}}). \tag{3.26}
\]

The next step is to combine the estimate for $I_R$ with that one of $J_R$. Of course, $\tilde{J}_R, \bar{J}_R \leq J_R$, thus, plugging (3.26) in (3.25) and conversely, for $\tau = R > \max\{1, R_0, R_1\}$ we have
\[
I_R \lesssim R^{-2+(\alpha_1-\alpha_2)+\frac{n+\alpha_2+1}{q'}} \tilde{I}_R^{\frac{1}{q'}} \lesssim R^{-2+(\alpha_1-\alpha_2)+\frac{n+\alpha_2+1}{q'}} \left( \tilde{I}_R^{\frac{1}{q'}} + \bar{I}_R^{\frac{1}{q'}} \right),
\]
\[
J_R \lesssim R^{-2+(\alpha_2-\alpha_1)+\frac{n+\alpha_1+1}{p'}} \bar{I}_R^{\frac{1}{p'}} \lesssim R^{-2+(\alpha_2-\alpha_1)+\frac{n+\alpha_1+1}{p'}} \left( \tilde{J}_R^{\frac{1}{p'}} + \bar{J}_R^{\frac{1}{p'}} \right).
\]

Let us rewrite the exponents for $R$ in the previous inequalities in a better way. For the first inequality we get
\[
-2 + (\alpha_1 - \alpha_2) + \frac{n+\alpha_2+1}{q'} + \frac{1}{q'} \left( -2 + (\alpha_2 - \alpha_1) + \frac{n+\alpha_1+1}{p'} \right) = -2 - \frac{2}{q} + (1 - \frac{1}{pq})(n + \alpha_1 + 1)
\]
and for the second one
\[
-2 + (\alpha_2 - \alpha_1) + \frac{n+\alpha_1+1}{p'} + \frac{1}{p} \left( -2 + (\alpha_1 - \alpha_2) + \frac{n+\alpha_2+1}{q'} \right) = -2 - \frac{2}{p} + (1 - \frac{1}{pq})(n + \alpha_2 + 1).
\]

Summarizing, for $\tau = R > \max\{1, R_0, R_1\}$ we have shown
\[
I_R \lesssim R^{-2-\frac{2}{q}+\left(1 - \frac{1}{pq}\right)(n+\alpha_1+1)} \left( \tilde{I}_R^{\frac{1}{q'}} + \bar{I}_R^{\frac{1}{q'}} \right), \tag{3.27}
\]
\[
J_R \lesssim R^{-2-\frac{2}{p}+\left(1 - \frac{1}{pq}\right)(n+\alpha_2+1)} \left( \tilde{J}_R^{\frac{1}{p'}} + \bar{J}_R^{\frac{1}{p'}} \right). \tag{3.28}
\]

Because of the obvious relations $\tilde{I}_R, \bar{I}_R \leq I_R$, from (3.27) it follows
\[
I_R^{1-\frac{1}{q'}} \lesssim R^{-2\left(1+\frac{1}{q'}\right)+\left(1 - \frac{1}{pq}\right)(n+\alpha_1+1)}
\]
which implies in turn
\[
I_R \lesssim R^{-2\left(\frac{p+1}{pq-1}\right)+n+\alpha_1+1} = R^{-2\left(\frac{p+1}{pq-1}\right)+n+\alpha_1+1}. \tag{3.29}
\]

If the exponent of $R$ on the left hand side is negative, that is,
\[
\frac{n+\alpha_1+1}{2} < \frac{p(q+1)}{pq-1} \quad \iff \quad \frac{n+\alpha_1-1}{2} < \frac{p+1}{pq-1},
\]
then, letting $R \to \infty$ in (3.29) we get $\lim_{R \to \infty} I_R = 0$. Using the monotone convergence theorem, we find

$$\int \int_{[0,\infty) \times \mathbb{R}^n} g_1(t) |v(t,x)|^p \, d(t,x) = 0$$

that implies $v = 0$ a.e., due to the fact that $g_1$ is always positive. However, this fact contradicts (2.9).

Let us show now that even in the case in which the power of $R$ in (3.27) is equal to 0, that is, when $\frac{n+\alpha_1-1}{2} \leq \frac{p+1}{pq-1}$, we find the same contradiction. In this last case, (3.27) implies $I_R \leq C$. Hence, by monotone convergence theorem we get

$$\lim_{R \to \infty} I_R = \int \int_{[0,\infty) \times \mathbb{R}^n} g_1(t) |v(t,x)|^p \, d(t,x) \leq C.$$

So, $g_1|v|^p \in L^1([0,\infty) \times \mathbb{R}^n)$. Consequently, we may employ the dominated convergence theorem, obtaining

$$\lim_{R \to \infty} \tilde{I}_R = 0 \quad \text{and} \quad \lim_{R \to \infty} \tilde{I}_R = 0.$$

Using these relations in (3.27), we find as in the previous case $\lim_{R \to \infty} I_R = 0$. Repeating the previous argument, we arrive at the same contradiction.

In an analogous way, one can show that (3.28) leads to the condition $u = 0$ a.e. in the case in which $\frac{n+\alpha_2-1}{2} \leq \frac{p+1}{pq-1}$, but this fact is not possible because of (2.8). Summarizing, we proved that for

$$\frac{n+\alpha_1-1}{2} \leq \frac{p+1}{pq-1} \quad \text{or} \quad \frac{n+\alpha_2-1}{2} \leq \frac{q+1}{pq-1},$$

provided that $p, q$ fulfill (2.11), the weak solution $(u, v)$ cannot be globally in time defined. Since the first previous relations on $(p, q)$ are equivalent to (2.10), the proof is completed.

4. Proofs of global existence results

Let us introduce some common notations for the proofs of the global (in time) existence results. We denote by $E_0^{(\mu, \nu)}(t, s, x)$ and $E_1^{(\mu, \nu)}(t, s, x)$ the fundamental solutions to (2.17), that is, the distributional solutions to (2.17) with initial data $(u_0, u_1) = (\delta_0, 0)$ and $(u_0, u_1) = (0, \delta_0)$, respectively. Hence, the solution to (2.17) is given by

$$u(t, x) = E_0^{(\mu, \nu)}(t, s, x) *_{(x)} u_0(x) + E_1^{(\mu, \nu)}(t, s, x) *_{(x)} u_1(x).$$

Having in mind Duhamel’s principle, let us introduce the operator

$$N : (u, v) \to N(u, v) = \left( u^{\text{lin}} + G_1(v), v^{\text{lin}} + G_2(u) \right),$$

where $(u^{\text{lin}}, v^{\text{lin}})$ is the solution to the corresponding linear homogeneous system with data $(u_0, u_1; v_0, v_1)$, that is,

$$u^{\text{lin}}(t, x) \doteq E_0^{(\mu_1, \nu_1)}(t, 0, x) *_{(x)} u_0(x) + E_1^{(\mu_1, \nu_1)}(t, 0, x) *_{(x)} u_1(x),$$

$$v^{\text{lin}}(t, x) \doteq E_0^{(\mu_2, \nu_2)}(t, 0, x) *_{(x)} v_0(x) + E_1^{(\mu_2, \nu_2)}(t, 0, x) *_{(x)} v_1(x),$$

and $G_1(v), G_2(u)$ are the following integral operators:

$$G_1(v)(t, x) \doteq \int_0^t E_1^{(\mu_1, \nu_1)}(t, s, x) *_{(x)} |v(s, x)|^p \, ds,$$

$$G_2(u)(t, x) \doteq \int_0^t E_1^{(\mu_2, \nu_2)}(t, s, x) *_{(x)} |u(s, x)|^q \, ds.$$

Moreover, we introduce a family of function spaces $\{X(T)\}_{T > 0}$, with

$$X(T) \doteq \left( C([0, T], H^1(\mathbb{R}^n)) \cap C^1([0, T], L^2(\mathbb{R}^n)) \right)^2$$

(4.30)
equipped with the norm
\[ \|(u, v)\|_{X(T)} = \sup_{t \in [0, T]} \left( (1 + t)^{-\gamma_1} M_1(t, u) + (1 + t)^{-\gamma_2} M_2(t, v) \right), \tag{4.31} \]
where
\[ M_1(t, u) = (1 + t)^{\beta + \alpha_1} \left( \|\nabla u, u\|_{L^2(\mathbb{R}^n)} + (1 + t)^{-1} \|u(t, \cdot)\|_{L^2(\mathbb{R}^n)} \right), \]
\[ M_2(t, v) = (1 + t)^{\beta + \alpha_2} \left( \|\nabla v, v\|_{L^2(\mathbb{R}^n)} + (1 + t)^{-1} \|v(t, \cdot)\|_{L^2(\mathbb{R}^n)} \right), \]
and \(\gamma_1, \gamma_2 \geq 0\) represent possible losses of decay for \((u, v)\) in comparison with the corresponding decay estimates for \((u^{\text{lin}}, v^{\text{lin}})\).

In order to prove the global (in time) existence of solutions to (1.1) we want to prove that the operator \(N\) is a contraction on \(X(T)\) with an independent of \(T\) Lipschitz constant. Then, the solution \((u, v)\) to (1.1) will be the solution of the nonlinear integral system of equation \((u, v) = N(u, v)\), i.e., the unique fixed point of \(N\). More specifically, we will prove the inequalities
\[ \|N(u, v)\|_{X(T)} \leq \|(u_0, u_1)\|_d + \|(v_0, v_1)\|_d + \|(u, v)\|_{X(T)}^p + \|(u, v)\|_{X(T)}^q, \tag{4.32} \]
\[ \|N(u, v) - N(\bar{u}, \bar{v})\|_{X(T)} \leq \|(u, v) - (\bar{u}, \bar{v})\|_{X(T)} \left( \sum_{r=p,q} \|(u, v)\|_{X(T)}^{1-r} + \|(\bar{u}, \bar{v})\|_{X(T)}^{1-r} \right) \tag{4.33} \]
uniformly with respect to \(T\), which imply the desired property for the operator \(N\), provided that \(\|(u_0, u_1)\|_d + \|(v_0, v_1)\|_d \leq \varepsilon\) is sufficiently small.

Let us underline explicitly, that (4.32) and (4.33) imply for the fixed point \((u, v)\) of \(N\) the estimates
\[ M_1(t, u) \lesssim \varepsilon (1 + t)^{\gamma_1}, \]
\[ M_2(t, v) \lesssim \varepsilon (1 + t)^{\gamma_2}, \]
which are exactly the estimates for \((u, v)\) in Theorems 2.6, 2.8 and 2.10 provided that \(\gamma_1\) and \(\gamma_2\) are suitably chosen (for example, at least one among them has to be 0).

### 4.1. Proof of Theorem 2.6

Let us consider the space \(X(T)\) defined by (4.30) and equipped with the norm given by (4.31) with \(\gamma_1 = \gamma_2 = 0\). Due to the fact that we are in the subcase (2.14), no loss of decay is required in comparison to the homogeneous linear problem neither for \(u\) nor for \(v\). From Proposition 2.3 it follows immediately
\[ \|(u^{\text{lin}}, v^{\text{lin}})\|_{X(T)} \leq \|(u_0, u_1)\|_d + \|(v_0, v_1)\|_d. \]

Consequently, in order to show (4.32) it remains to prove that
\[ \|(G_1(v), G_2(u))\|_{X(T)} \lesssim \|(u, v)\|_{X(T)}^p + \|(u, v)\|_{X(T)}^q. \tag{4.34} \]

Let us begin by estimating \(M_1(t, G_1(v))\). For \(j + \ell = 0, 1\), by Proposition 2.4 we have
\[ \|\nabla^j \partial^\ell_t G_1(v)(t, \cdot)\|_{L^2(\mathbb{R}^n)} \leq \int_0^t \|E^{(\mu_1, \nu_1)}(t, s, x) \ast (\cdot)\|_{L^2(\mathbb{R}^n)} d s \]
\[ \lesssim (1 + t)^{j + \ell - \frac{\mu_1}{2} - \alpha_1 + 1} \int_0^t (1 + s)^{\alpha_1} \left( \|v(s, \cdot)\|_{L^p(\mathbb{R}^n)} + (1 + s)^{\frac{v}{2}} \|v(s, \cdot)\|_{L^{2p}(\mathbb{R}^n)} \right) d s. \]

Using Gagliardo-Nirenberg inequality, we can estimate the \(L^p\) norm and the \(L^{2p}\) norm of \(v(s, \cdot)\) as follows:
\[ \|v(s, \cdot)\|_{L^p(\mathbb{R}^n)} \lesssim \|v(s, \cdot)\|_{L^2(\mathbb{R}^n)} 1^{1-\theta(hp)} \|\nabla v(s, \cdot)\|_{L^2(\mathbb{R}^n)} \]
\[ \lesssim (1 + s)^{-\frac{v}{2} - \alpha_1 + 1 - \theta(hp)} \|(u, v)\|_{X(s)}, \]
where we used the condition $p > \tilde{p}(n, \alpha_1, \alpha_2)$ in order to guarantee the uniform boundedness of the integral in the last inequality.

Similarly, we can estimate $M_2(t, G_2(u))$ in the following way

\[
\|\nabla^j \partial_t^\ell G_2(u)(t, \cdot)\|_{L^2(\mathbb{R}^n)} \leq \int_0^t \| E(\mu_2, \alpha_2)(t, s, x) * (u(s, \cdot)) |u(s, \cdot)|^q \|_{L^2(\mathbb{R}^n)} \, ds
\]
\[
\lesssim (1 + t)^{-(j+\ell)-\frac{n}{2}-\alpha_2+1} \int_0^t (1 + s)^{\alpha_2} \left( \|u(s, \cdot)\|_{L^q(\mathbb{R}^n)}^q + (1 + s)^{\frac{n}{2}} \|u(s, \cdot)\|_{L^{2q}(\mathbb{R}^n)}^q \right) \, ds
\]
\[
\lesssim (1 + t)^{-(j+\ell)-\frac{n}{2}-\alpha_2+1} \int_0^t (1 + s)^{-(n+\alpha_2-1)q+n+\alpha_2} \|u(v, \cdot)\|_{X(s)}^q \, ds
\]
\[
\lesssim (1 + t)^{-(j+\ell)-\frac{n}{2}-\alpha_2+1} \|u(v, \cdot)\|_{X(t)}^q
\]

for $j + \ell = 0, 1$, where in the last step we employed the assumption $q > \tilde{q}(n, \alpha_1, \alpha_2)$. Combining (4.35) and (4.36) we get immediately (4.34).

Let us sketch briefly the proof of the Lipschitz condition (4.33). As

\[
N(u, v) - N(\bar{u}, \bar{v}) = (G_1(v) - G_1(\bar{v}), G_2(u) - G_2(\bar{u})),
\]

it is sufficient to control the quantities $M_1(t, G_1(v) - G_1(\bar{v}))$ and $M_2(t, G_2(u) - G_2(\bar{u}))$. Using again Proposition 2.4, we find

\[
(1 + t)^{(j+\ell)+\frac{n}{2}+\alpha_1-1} \|\nabla^j \partial_t^\ell (G_1(v) - G_1(\bar{v}))(t, \cdot)\|_{L^2(\mathbb{R}^n)}
\]
\[
\lesssim \int_0^t (1 + s)^{\alpha_1} \left( \|v(s, \cdot)\|^p - |\bar{v}(s, \cdot)|^p \right) \|v(s, \cdot)\|_{L^1(\mathbb{R}^n)} + (1 + s)^{\frac{n}{2}} \|v(s, \cdot)\|^p - |\bar{v}(s, \cdot)|^p \|_{L^2(\mathbb{R}^n)} \, ds.
\]

By the pointwise estimate $|v|^p - |\bar{v}|^p \leq p(|v|^{p-1} + |\bar{v}|^{p-1})|v - \bar{v}|$, Hölder’s inequality, Gagliardo-Nirenberg inequality and the definition of norm for the family of spaces $\{X(t)\}_{t > 0}$, for $h = 1, 2$ we arrive at

\[
\|v(s, \cdot)\|^p - |\bar{v}(s, \cdot)|^p \|_{L^h(\mathbb{R}^n)} \lesssim \|v(s, \cdot) - \bar{v}(s, \cdot)\|_{L^{ph}(\mathbb{R}^n)} \left( \|v(s, \cdot)\|_{L^{ph}(\mathbb{R}^n)}^{-1} + \|\bar{v}(s, \cdot)\|_{L^{ph}(\mathbb{R}^n)}^{-1} \right)
\]
\[
\lesssim (1 + s)^{-(n+\alpha_2-1)^h+\frac{n}{2}} \|u(v, \cdot) - (\bar{u}, \bar{v})\|_{X(s)} \left( \|u(v, \cdot)\|_{X(s)}^{-1} + \|\bar{u}, \bar{v}\|_{X(s)}^{-1} \right).
\]

So, combining the last two estimates, we get

\[
(1 + t)^{(j+\ell)+\frac{n}{2}+\alpha_1-1} \|\nabla^j \partial_t^\ell (G_1(v) - G_1(\bar{v}))(t, \cdot)\|_{L^2(\mathbb{R}^n)}
\]
\[
\lesssim \|(u, v) - (\bar{u}, \bar{v})\|_{X(t)} \left( \|u(v, \cdot)\|_{X(s)}^{-1} + \|\bar{u}(v, \cdot)\|_{X(s)}^{-1} \right),
\]

provided that $p > \tilde{p}(n, \alpha_1, \alpha_2)$. In an analogous way, we can prove

\[
(1 + t)^{(j+\ell)+\frac{n}{2}+\alpha_2-1} \|\nabla^j \partial_t^\ell (G_2(u) - G_2(\bar{u}))(t, \cdot)\|_{L^2(\mathbb{R}^n)}
\]
\[
\lesssim \|(u, v) - (\bar{u}, \bar{v})\|_{X(t)} \left( \|u(v, \cdot)\|_{X(s)}^{-1} + \|\bar{u}(v, \cdot)\|_{X(s)}^{-1} \right),
\]

due to $q > \tilde{q}(n, \alpha_1, \alpha_2)$. Thus, we proved (4.33). This concludes the proof.
4.2. Proof of Theorem 2.8

Let us consider the space $X(T)$ defined by (4.30) and equipped with the norm given by (4.31) with
\[\gamma_1 \equiv \gamma = \begin{cases} (n + \alpha_2 - 1)(\tilde{p}(n, \alpha_1, \alpha_2) - p) & \text{if } p < \tilde{p}(n, \alpha_1, \alpha_2), \\ \epsilon & \text{if } p = \tilde{p}(n, \alpha_1, \alpha_2), \end{cases}\]
defined as in the statement and $\gamma_2 = 0$. Since we assume $p \leq \tilde{p}(n, \alpha_1, \alpha_2)$, in order to control the integral in (4.35) we allow this loss of decay for the estimates for $u$ in comparison with those for $u^{lin}$.

By Proposition 2.3 we get
\[\|u^{\text{lin}}, u^{\text{lin}}\|_{X(T)} \lesssim \sup_{t \in [0, T]} (1 + t)^{-\gamma} \|u_0, u_1\|_{A} + \|v_0, v_1\|_{A} \lesssim \|(u_0, u_1)\|_{A} + \|(v_0, v_1)\|_{A},\]
due to the fact that $\gamma > 0$. Let us consider now the nonlinear part $(G_1(v), G_2(u))$. As in the previous section, we have
\[(1 + t)^{(\epsilon + \frac{p}{q} + 1)} \|\nabla^i \partial_t^j G_1(v)(t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \int_0^t (1 + s)^{-\gamma(n + \alpha_2 - 1)p + n + \alpha_1} \|u, v\|_{X(s)}^p \, ds\]
but, differently from the previous section we can no longer estimate the last integral uniformly by a constant, as we are in the case $p \leq \tilde{p}(n, \alpha_1, \alpha_2)$. So, we have
\[\int_0^t (1 + s)^{-\gamma(n + \alpha_2 - 1)p + n + \alpha_1} \lesssim (1 + t)^{\gamma},\]
which implies the desired estimate
\[(1 + t)^{(\epsilon + \frac{p}{q} + 1)} \|\nabla^i \partial_t^j G_2(u)(t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u, v)\|_{X(t)}^p.\]

The next step is to determine under which condition for $(p, q)$ the inequality $M_2(t, G_2(u)) \lesssim \|(u, v)\|_{X(t)}^q$ holds. Similarly to the previous section, keeping in mind that now we have a different decay rate for $u$ coming for the norm of $(u, v) \in X(s)$, we get
\[\|\nabla^i \partial_t^j G_2(u)(t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1 + t)^{-(\epsilon + \frac{p}{q} + 1)} \int_0^t (1 + s)^{-\gamma(n + \alpha_2 - 1)p + n + \alpha_1} \|(u, v)\|_{X(s)}^q \, ds\]
\[\lesssim (1 + t)^{-(\epsilon + \frac{p}{q} + 1)} \int_0^t (1 + s)^{(1 + \gamma) - n + \alpha_1} q + \alpha_2 \, ds \|(u, v)\|_{X(t)}^q\]
\[\lesssim (1 + t)^{-(\epsilon + \frac{p}{q} + 1)} \|(u, v)\|_{X(t)}^q,\]
provided that the exponent of the integrand is smaller than $-1$. If $p < \tilde{p}(n, \alpha_1, \alpha_2)$, this condition is equivalent to require
\[q(1 + (n + \alpha_1 + 1) - (n + \alpha_2 - 1)p - n - \alpha_1) + n + \alpha_2 + 1 < 0\]
\[\iff 2(q + 1) + (n + \alpha_2 - 1)(1 - pq) < 0\]
\[\iff \frac{q + 1}{pq - 1} < \frac{n + \alpha_2 - 1}{2}\]
that is, for $(p, q)$ fulfilling (2.20). On the other hand, for $p = \tilde{p}(n, \alpha_1, \alpha_2)$, we can choose $\epsilon > 0$ so small that
\[q(2 + \epsilon - (n + \alpha_2 - 1)p) + n + \alpha_2 + 1 < 0\]
is satisfied. In both cases (2.20) implies the desired inequality. Hence, we proved (4.32). The proof of (4.33) is completely similar to that one in the proof of Theorem 2.6. So, the proof is over.

Remark 4.1. It is clear that, due to the symmetry of (1.1), the proof of Theorem 2.10 is completely analogous to that one of Theorem 2.8 by choosing $\gamma_1 = 0$ and $\gamma_2 \equiv \tilde{\gamma}$ as in the statement of Theorem 2.10.
5. Final remarks

Combining the results from Theorems 2.1, 2.6, 2.8 and 2.10 we have that
\[
\max \left\{ \frac{p+1}{pq-1} - \frac{\alpha_1}{2}, \frac{q+1}{pq-1} - \frac{\alpha_2}{2} \right\} = \frac{n-1}{2}
\]
is the critical exponent for the weakly coupled system (1.1) provided that the coefficients satisfy\( \delta_1, \delta_2 > (n+1)^2 \) in the sense we explained in Section 2. Actually, one can slightly improve this result up to range \( \delta_1, \delta_2 \geq (n+1)^2 \) modulo a (possible) further arbitrarily small loss of decay rate with respect to the case \( \delta_1, \delta_2 > (n+1)^2 \) in Theorems 2.8 and 2.10.

In the case \( 0 < \delta_1 < (n+1)^2 \) or \( 0 < \delta_2 < (n+1)^2 \) we cannot obtain a sharp result as in the above mentioned case by using \( L^2 - L^2 \) estimates with additional \( L^1 \) regularity and working in classical energy spaces, due to the fact that the first order derivatives have a weaker decay rate (cf. Theorems 4.6 and 4.7 in [46] for further details).

In the case in which \( \mu_1 = \mu_2 \) and \( \nu_1 = \nu_2 \), the critical exponent for (1.1) is
\[
\max \left\{ \frac{p+2}{pq-1} - \frac{\mu_1}{2}, \frac{q+2}{pq-1} - \frac{\mu_2}{2} \right\} = \frac{n-1}{2}.
\]

where \( \alpha = \alpha_1 = \alpha_2 \). In particular,
\[
\max \left\{ \frac{p+1}{pq-1} - \frac{\alpha_1}{2}, \frac{q+1}{pq-1} - \frac{\alpha_2}{2} \right\} \geq 0 \iff \max \left\{ \frac{p+2}{pq-1} - \frac{\mu_1}{2}, \frac{q+2}{pq-1} - \frac{\mu_2}{2} \right\} \leq p_{Fuj} (n+\alpha-1),
\]
\[
\max \left\{ \frac{p+1}{pq-1} - \frac{\alpha_1}{2}, \frac{q+1}{pq-1} - \frac{\alpha_2}{2} \right\} < 0 \iff \max \left\{ \frac{p+2}{pq-1} - \frac{\mu_1}{2}, \frac{q+2}{pq-1} - \frac{\mu_2}{2} \right\} > p_{Fuj} (n+\alpha-1),
\]
where \( p_{Fuj} (n+\alpha-1) \) is the critical exponent for the corresponding single equation (see also [37, 46, 43]).

Since for the single equation (1.7) we expect \( p_0 (n+\mu) \) to be the critical exponent for small and nonnegative values of \( \delta \) (cf. [37, 47, 44, 45, 48]), it is clear that the result from Theorem 2.1 cannot be sharp in this case.

Indeed, in an upcoming paper a blow-up result for (1.1) is going to be proved for \( \delta_1, \delta_1 \geq 0 \) provided that \( p, q > 1 \) satisfy
\[
\max \left\{ \frac{p+2+q^{-1}}{pq-1} - \frac{\mu_1}{2}, \frac{q+2+p^{-1}}{pq-1} - \frac{\mu_2}{2} \right\} > \frac{n-1}{2}.
\]

We notice that the corresponding critical relation for the pair \( (p, q) \) is a shift of the critical exponent for (1.5).

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