Newton Polytopes of Nondegenerate Quadratic Forms

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ABSTRACT. We characterize Newton polytopes of nondegenerate quadratic forms and Newton polyhedra of Morse singularities.

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1. Introduction

Newton polytopes can be associated to any polynomial or analytic function. They are defined as follows.

**Definition 1.** Let \( f = \sum_{k \in \mathbb{Z}^n} a_k z^k \) be a Laurent polynomial in \( n \) complex variables, where \( z^k = z_1^{k_1} \cdots z_n^{k_n} \). The **Newton polytope** of \( f \) is the convex hull \( \mathcal{N}(f) = \text{conv}\{k \in \mathbb{Z}^n \mid a_k \neq 0\} \subset \mathbb{R}^n \).

If \( f = \sum_{k \in \mathbb{Z}^n} a_k z^k \) is an analytic function on a neighborhood of 0, then the **local Newton polytope** \( \mathcal{N}(f) \) is defined as \( \text{conv}\{k + l \in \mathbb{R}^n \mid k \in \mathbb{Z}^n, a_k \neq 0, l \in \mathbb{R}^n_{\geq 0}\} \).

**Definition 2.** The space of all polynomials in \( \mathbb{C}[z_1, \ldots, z_n] \) whose Newton polytopes lie inside a polytope \( M \) is denoted by \( \mathbb{C}^M \).

The Newton polytope \( \mathcal{N}(f) \) carries important information about \( f \) (a classical result of this type is Kouchnirenko’s theorem [3]; see Theorem 5 below). This article is devoted to the natural question: What are Newton polytopes of nondegenerate quadratic forms and Morse singularities?

**Definition 3.** Let \( A_i \) (where \( i \in \{1, \ldots, n\} \)) be the point in \( \mathbb{R}^n \) whose \( i \)-th coordinate is 2 and all other coordinates are 0. We denote the convex hull of all points \( A_i \) by \( 2\Delta \).

Any integer point in \( 2\Delta \) can be represented as \( A_{ij} = A_{ji} = (A_i + A_j)/2 \).

Let \( O = (2/n_1, \ldots, 2/n) \) be the barycenter of \( 2\Delta \).

We prove the following results.

1. Let \( M \) be a lattice polytope contained in \( 2\Delta \). Clearly, the elements of \( \mathbb{C}^M \) are quadratic forms.

   If \( O \in M \), then a generic quadratic form \( B \in \mathbb{C}^M \) is nondegenerate. Otherwise, any form \( B \in \mathbb{C}^M \) is degenerate.

2. Let \( M \) be the local Newton polytope of an analytic function \( f \) with \( f(0) = 0 \) and \( df(0) = 0 \). If \( O \notin M \), then the singularity of \( f \) at 0 is not Morse. If \( O \in M \), then a generic function with local Newton polytope \( M \) has a Morse singularity at 0.

**Remark.** One direction of the second statement above follows from deep results of A. N. Varchenko on complex oscillating integrals. Namely, if the singularity of \( f \) is Morse, then the point \( O \) belongs to the Newton polytope. Results of Varchenko relate a certain invariant of the singularity of \( f \) (the so-called complex oscillation index \( I \); see [1; Sec. 13.1.5]) to the remoteness \( R \) of the Newton polytope (i.e., the smallest \( r \) such that the point \((-1/r, \ldots, -1/r)\) belongs to the polytope). Since the complex oscillation index attains its minimum possible value \(-n/2\) only for Morse singularities (see [1; Corollary in Sec. 13.3.3]) and remoteness takes its smallest possible value only for Newton polytopes containing \( O \), the inequality \( I \geq R \) (see [1; Sec. 13.1.7, Theorem 13.2]) implies the sought result. Our proof of this implication is much more elementary and constructive (see the remark to Theorem 2 below).
Remark. The problem of classifying the Newton polytopes of Morse singularities arose, in particular, in connection with the monodromy conjecture and was solved in the dimensions up to 4 by exhaustive search in [2; Lemma 4.9].

Structure of the paper. We characterize the Newton polytopes of quadratic forms in Theorem 1 (Section 2.1) and Theorem 2 (Sections 2.2–2.3) and those of singularities in Theorem 4 (Section 3).

The proof of Theorem 2 consists of two steps. The first step is the construction of a zigzag of nonzero entries in the matrices of forms in $\mathbb{C}^M$; the second one is the proof that the existence of such a zigzag is enough for a generic quadratic form to have nonzero determinant. We prove both steps in Section 2.2. Another way to perform Step 1 is described in Section 2.3.

2. Main Theorems

2.1. Newton polytopes of nondegenerate quadratic forms.

Theorem 1. If $B$ is a nondegenerate quadratic form, then $O \in N(B)$.

Proof. If $B$ is nondegenerate, then

$$\det B = \sum_{\sigma \in S_n} \text{sign}(\sigma)B_{1,\sigma(1)} \cdots B_{n,\sigma(n)} \neq 0.$$
Proof. Let $C_1 = A_1$ and $C_2 = A_2$. Since $O \in M$, we can express $O$ as a convex combination of \{C_i\} in the following way: $O = \alpha_1 A_1 + \alpha_2 A_2 + \sum_{i=3}^{k} \alpha_i C_i$, where $\alpha_1 + \cdots + \alpha_k = 1$. By symmetry, we can assume that $\alpha_1 \leq \alpha_2$. But then, since $A_1 A_2 = 2A_{1,2}$, we have $O = 2\alpha_1 A_{1,2} + (\alpha_2 - \alpha_1)A_2 + \sum_{i=3}^{k} \alpha_i C_i$, so we obtain a polytope $M' = \text{conv}\{A_{1,2}, A_2; C_3, \ldots, C_k\} \subseteq M$ that contains $O$, which contradicts the minimality of $M$.

\[\square\]

**Lemma 3.** The polytope $M$ is a simplex (not necessarily of maximal dimension) which contains at most $n$ lattice points. Thus, $\tilde{B}(M)$ has at most $2n$ nonzero entries.

**Proof.** The statement easily follows from Carathéodory’s theorem, which states that if the convex hull of a set $K \subseteq \mathbb{R}^{n-1}$ contains a point $P$, then there is a subset $K' \subseteq K$ of cardinality at most $n$ such that $\text{conv} K'$ is a simplex containing $P$.

Indeed, $M$ lies in the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ containing $2\Delta$. Taking $M \cap \mathbb{Z}^n$ for $K$ and $O$ for $P$, we obtain a polytope $M' \subset M$ with at most $n$ vertices which contains the point $O$. We have $M' = M$, since $M$ is minimal.

It remains to verify that all the lattice points of $M$ are its vertices. There are only two possible representations of $A_{i,j}$ as a convex combination of lattice points of $2\Delta$, namely, $A_{i,j} = 1 \cdot A_{i,j}$ and $A_{i,j} = \frac{1}{2}A_i + \frac{1}{2}A_j$. Thus, if $A_{i,j} \in M$, then either $A_{i,j}$ is a vertex of $M$ or both $A_i$ and $A_j$ are vertices of $M$. The second possibility contradicts Lemma 2, so the conclusion follows.

\[\square\]

**Definition 5.** We call a vertex $A_{i,j}$ of a polytope $M \subset 2\Delta$ special if $A_{i,l} \notin M$ for any $l \neq j$.

Generally speaking, since $A_{i,j} = A_{j,i}$, we should check that special vertices are well defined. This is done in the following lemma.

**Lemma 4.** Suppose that $A_{i,j}$ is a special vertex of $M$; then $A_{i,j} \notin M$ for any $l \neq j$.

**Proof.** If $i = j$, then there is nothing to prove, since $A_{i,j} = A_{j,i}$.

Now assume that $i \neq j$. The point $O$ belongs to $M$, and hence we can express it as a convex combination of vertices of $M$ (and possibly of points $A_{i,l}$ for several $l$):

$$O = \sum_{l=1}^{n} \alpha_l A_{i,l} + \sum_{l=n+1}^{k'} \alpha_l C_l.$$ 

Here the $C_l$ are the vertices of $M$ whose $i$th and $j$th coordinates are 0.

Let us calculate the $i$th coordinate of $O$:

$$\frac{2}{n} = (O)_i = \alpha_i \cdot (A_{i,j})_i + \sum_{l \in \{1, \ldots, n\} \setminus \{i\}} \alpha_l \cdot (A_{i,j})_l + \sum_{l=n+1}^{k'} \alpha_l \cdot (C_l)_i = \alpha_i \cdot 1.$$ 

Now we calculate the $j$th one:

$$\frac{2}{n} = (O)_j = (\alpha_i \cdot 1) + \left(\alpha_j + \sum_{l \in \{1, \ldots, n\} \setminus \{i\}} \alpha_l \cdot 1\right) + (0) = \frac{2}{n} + \left(\alpha_j + \sum_{l \in \{1, \ldots, n\} \setminus \{i\}} \alpha_l\right).$$ 

Since the $\alpha_l$ are nonnegative, it follows that $\alpha_l = 0$ for each $l \in \{1, \ldots, n\} \setminus \{i\}$. We have

$$O = \alpha_i A_{i,j} + \sum_{l=n+1}^{k'} \alpha_l C_l.$$ 

Thus, using the minimality of $M$, we obtain $A_{i,j} \notin M$ for any $l \neq i$. This is equivalent to the statement of the lemma.

\[\square\]
Lemma 5. Suppose that a polytope $M \subset 2\Delta$ with vertices $C_1, \ldots, C_k$ has a special vertex $C_k = A_{ij}$. Then $M$ is minimal if and only if the polytope $M' = \text{conv}\{C_1, \ldots, C_{k-1}\} \subset 2\Delta'$ is minimal in the simplex $2\Delta' = 2\Delta \cap \{x_i = 0\} \cap \{x_j = 0\}$.

Proof. Suppose that $i \neq j$ (the case $i = j$ is similar).

First, observe that $M$ is a simplex if and only if $M'$ is a simplex. By Lemma 4 we have $M' \subset 2\Delta'$.

Let $M \subset 2\Delta$ be a minimal polytope. Lemma 3 implies that $M$ is a simplex, so the point $O$ has a unique representation in the form of a convex combination of the vertices $C_1, \ldots, C_k$, namely, $O = \sum_{l=1}^{k-1} \alpha_l C_l + \alpha_k A_{ij}$, where $\alpha_l \neq 0$ for all $l$ (if $\alpha_l = 0$, then $M$ is not minimal). By Lemma 4, for any $l < k$, the $i$th and $j$th coordinates of the vertex $C_l$ are equal to 0. Thus, calculating the $i$th affine coordinate of $O$, we obtain $2/\alpha_l = (n-2)/\alpha_l$, so that $\sum_{l=1}^{k-1} \alpha_l = (n-2)/\alpha_l$.

Now let $O'$ be the barycenter of $2\Delta'$. It is easy to verify that

$$O' = \sum_{l=1}^{k-1} \frac{n}{n-2} \alpha_l C_l = \sum_{l=1}^{k-1} \beta_l C_l$$

by calculating all affine coordinates of $O'$. Thus, $O' \in M'$. If none of the $\beta_l$ equals to 0, then $O'$ is strictly inside the simplex $M'$; hence $M'$ is minimal.

Similarly, if $M'$ is minimal, then $O' = \sum_{l=1}^{k-1} \beta_l C_l$ and hence $O = \sum_{l=1}^{k-1} \frac{n}{n-2} \beta_l C_l + \frac{2}{n} A_{ij}$ lies strictly inside the simplex $M$. □

Proof of Theorem 2. Step 1. Suppose that $M \subset 2\Delta \subset \mathbb{R}^n$ is a minimal polytope; then there exists a permutation $\sigma_0 \in S_n$ of indices such that $B(M)_{1,\sigma(1)} = \cdots = B(M)_{n,\sigma(n)} = 1$.

We will prove this by induction on the number of special vertices of $M$.

Base. Suppose that $M$ has no special vertices. Any row of $B(M)$ contains at least one nonzero element by Lemma 1. However, if the $i$th row contains exactly one nonzero $B(M)_{ij}$, then $A_{ij} \in M$ is special. Thus, every row contains at least two nonzero elements, but there are at most $2n$ nonzero entries in $B(M)$. So, each row (and column) of $B(M)$ contains exactly two entries equal to 1.

Consider the graph $G$ whose vertices are pairs $(i, j)$ such that $B(M)_{ij} = 1$. Vertices $(i_1, j_1)$ and $(i_2, j_2)$ are connected by an edge if and only if $i_1 = i_2$ (a vertical edge) or $j_1 = j_2$ (a horizontal edge).

The degree of each vertex is 2, so the graph consists of several cycles. Vertical and horizontal edges alternate in each cycle, so every cycle has even length. Therefore, we can choose a set of exactly $n$ vertices that are pairwise nonadjacent.

We have obtained a set of pairs $\{(i_1, j_1), \ldots, (i_n, j_n)\}$. Since the first components of the pairs are pairwise distinct, as well as the second ones, the desired permutation can be defined as $\sigma(i_1) = j_1$.

Induction step. Let a vertex $A_{ij}$ of $M$ be special. Then $B_{ij} = B_{ji} = 1$. Without loss of generality, we may assume that either $(i, j) = (n, n)$ or $(i, j) = (n, n-1)$. Then $M$ is minimal but has one less vertex. (Here we use the notation of Lemma 5.) Erasing the $i$th and $j$th rows and columns of the matrix $B(M)$, we obtain the stencil $B(M')$. By the induction hypothesis there exists a permutation $\sigma' \in S_{\min\{i,j\}}$ for which $B(M)_{i,\sigma'(i)} = B(M')_{i,\sigma'(i)} = 1$.

The desired permutation is

$$\sigma_0(n) := \begin{cases} \sigma'(n), & n \leq \min\{i, j\} - 1, \\ i, & n = j, \\ j, & n = i. \end{cases}$$

Step 2. Since $M$ contains $O$, there exists a minimal polytope $M' \subset M$. We have constructed a permutation $\sigma_0$ such that $B(M)_{i,\sigma_0(i)} = 1$ for any $i$. We want to verify that the polynomial $\det(B)$ is not identically zero on $\mathbb{C}^M$. Consider the expansion

$$\det B = \sum_{\sigma \in S_n} \text{sign}(\sigma) B^\sigma = \sum_{\sigma \in S_n} \text{sign}(\sigma) B_{1,\sigma(1)} \cdots B_{n,\sigma(n)}.$$
We are going to show that, for any permutations \( \sigma_1 \) and \( \sigma_2 \) such that the monomial \( B^{\sigma_2} \) is equal to \( B^{\sigma_1} \), we have \( \text{sign}(\sigma_1) = \text{sign}(\sigma_2) \).

Let us represent the set \( \{1, \ldots, n\} \) as the union of the subsets \( E = \{i \in \{1, \ldots, n\} \mid \sigma_1(i) = \sigma_2(i)\} \) and \( F = \{1, \ldots, n\} \setminus E \). Since \( B^{\sigma_1} = B^{\sigma_2} \), the sets of unordered pairs \( \{\{i, \sigma_1(i)\} \mid i \in 1 \ldots n\} \) and \( \{\{i, \sigma_1(i)\} \mid i \in 1 \ldots n\} \) are equal. For any \( i \in E \), we have \( \{i, \sigma_0(i)\} = \{i, \sigma_1(i)\} \). Therefore, for any \( i \in F \), there exists a number \( j \in F \) different from \( i \) and such that \( \{i, \sigma_1(i)\} = \{j, \sigma_2(j)\} \). This means that \( \sigma_0(i) = j \) and \( \sigma_1(j) = i \), so that \( \sigma_1(\sigma_0(i)) = i \). We have shown that \( \sigma_2 = \sigma_1^{-1} \) on \( F \) and \( \sigma_2 = \sigma_1 \) on \( E \). Now let

\[
\sigma_3(n) := \begin{cases} 
\sigma_2(n), & n \in F, \\
n, & n \in E. 
\end{cases}
\]

Evidently, \( \sigma_2 = \sigma_1 \sigma_3^2 \), whence \( \text{sign}(\sigma_2) = \text{sign}(\sigma_1) \). This shows that the coefficient of the monomial \( B_{1,\sigma_0(1)} \cdots B_{n,\sigma_0(n)} \) in the polynomial \( \det B \) is nonzero (since it has the same sign as the permutation \( \sigma_0 \)). Thus, \( \det B \) is not identically zero either.

Therefore, a generic quadratic form in \( \mathbb{C}^M \) is nondegenerate, since the subvariety \( \{\det B = 0\} \subset \mathbb{C}^M \) of degenerate quadratic forms does not coincide with \( \mathbb{C}^M \). \( \square \)

2.3. Approach to Theorem 2 using König’s theorem. In this section we will present a proof of Theorem 2 not using the notion of a minimal polytope. We will use the following theorem.

**Theorem 3** (König, [4; Theorem 1.1.1.]). Let \( B \) be an \( n \times n \) matrix consisting of zeros and ones. Then the following conditions are equivalent:

1. there exists a permutation \( \sigma \in S_n \) such that \( B_{1,\sigma(1)} = \cdots = B_{n,\sigma(n)} = 1 \);
2. if \( I \) and \( J \) are subsets of \( \{1, \ldots, n\} \) with the property

\[
(B_{ij} = 1) \implies (i \in I \text{ or } j \in J),
\]

then \( |I| + |J| \geq n \).

**Remark.** This theorem is usually formulated in terms of the bipartite graph \( G \) with adjacency matrix \( B \). The first condition means that the matching number is \( \nu(G) = n \), and the second one means that the point covering number is \( \tau(G) = n \).

The main statement of Step 1 of Theorem 2 can be replaced with the following one.

**Version of Step 1.** Suppose that \( O \in M \subset 2\Delta \); then there exists a permutation \( \sigma \in S_n \) of indices such that \( \bar{B}(M)_{1,\sigma(1)} = \cdots = \bar{B}(M)_{n,\sigma(n)} = 1 \).

**Proof.** We must verify property (1) for the matrix \( \bar{B} = \bar{B}(M) \). It is enough to verify property (2).

Let \( I \) and \( J \) be sets with the property described in König’s theorem. Assume that \( |I| + |J| < n \). We prove that \( O \notin M \).

Consider the half-space \( \Gamma \) given by the inequality

\[
\sum_{l \in I} x_l + \sum_{l \in J} x_l \geq 2. \tag{*}
\]

First, we prove that \( M \subset \Gamma \). If \( A_{ij} \) is a vertex of \( M \), then \( \bar{B}(M)_{ij} = \bar{B}(M)_{ji} = 1 \); so property (2) implies that \( (i \in I \text{ or } j \in J) \) and \( (j \in I \text{ or } i \in J) \), or, equivalently, either \( i, j \in I \), \( i, j \in J \), \( i \in I \cap J \), or \( j \in I \cap J \). It is easy to check that \( A_{ij} \in \Gamma \) by substituting the coordinates of \( A_{ij} \) into \((*)\) in each of the four cases. In the first case, the first sum in \((*)\) is at least 2, in the second case, the second sum is at least 2, and in the last two cases, both sums are at least 1.

Note that \( O \notin \Gamma \), since \( \sum_{l \in J} O_l + \sum_{l \in I} O_l = \frac{2}{n} \cdot (|I| + |J|) < 2 \). Therefore, \( O \notin M \). We have arrived at a contradiction. \( \square \)

Step 2 remains the same.
3. Application to Singularities

Let $f: \mathbb{C}^n \rightarrow \mathbb{C}$ be an analytic function. Suppose that $f(0) = 0$ and $df(0) = 0$, that is, $f$ has a singularity at 0. We derive the following theorem as a consequence of the results proved above.

**Theorem 4.** If $O \notin N(f)$, then the singularity is not Morse.

If a lattice (noncompact) polytope $M \subset \mathbb{R}^n_{>0}$ contains the point $O$, then a generic $f$ with $df(0) = 0$ for which $N(f) \subset M$ has a Morse singularity at 0.

**Remark.** The condition of being generic here is that if $B$ is a generic quadratic form in $\mathbb{C}^{M \cap 2\Delta}$, then any function of the form $f = B(x) + o(x^2)$ has a Morse singularity.

**Proof.** Let $f = B(x) + o(x^2)$, where $B$ is a quadratic form. Since $N(B) = N(f) \cap 2\Delta$, it follows that $O \in N(f)$ is equivalent to $O \in N(B)$. Thus, if $O \notin N(f)$, then $0 = \det B = \operatorname{Hess}(f)$ and $f$ has a non-Morse singularity.

If $O \in M$, then $O \notin f$ for almost every $B \in \mathbb{C}^{M \cap 2\Delta}$, and so almost every $f$ with Newton polytope $M$ has nonzero Hessian. \hfill $\Box$

This theorem becomes particularly interesting in the context of the following result.

**Theorem 5** (Kouchnirenko, [3]). Let $f$ be a generic analytic function of $n$ variables which has a singularity at zero and satisfies the condition $f(0) = 0$, and let $M = \mathbb{R}^n_{>0} \setminus N(f)$. Suppose that $M$ is bounded. Consider the $\binom{n}{i}$ intersections of $M$ and all $i$-dimensional coordinate subspaces. Let $V_i$ denote the sum of their $i$-dimensional volumes. Then the Milnor number of $f$ can be calculated by $\mu(f) = n! \cdot V_n - (n-1)! \cdot V_{n-1} + \cdots + (-1)^{n-1} V_1 + (-1)^n$.

This statement holds for all functions $f$ such that, for any face $\Gamma$ of $N(f)$, there are no points $x \in (\mathbb{C} \setminus \{0\})^n$ for which $f^\Gamma(x) = 0$ and $df^\Gamma(x) = 0$. Here $f^\Gamma$ stands for the sum of all monomials in $f$ corresponding to lattice points in $\Gamma$.

**Remark 1.** One might expect that Theorem 4 could be proved by using the Kouchnirenko Theorem with $\mu = 1$. However, the author does not know how to do that, and it would be interesting to obtain such a proof. It may be much more difficult, since Kouchnirenko’s theorem does not give a formula for $\mu$ with positive coefficients.

**Remark 2.** Theorem 4 is valid for all functions $f$ that are generic in the sense of Kouchnirenko’s theorem.

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