Predator-Prey Model for Stock Market Fluctuations

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Abstract

We present a dynamical model that describes the evolution of offer and demand in a financial market. The model considers a fully connected network of interacting agents that may be willing to operate in the market, either by selling the stock or by buying it, or that are not interested in operating at that moment. The agents change their mind through self- or mutual influence, and the decision is adopted on a random basis, like in a predator-prey model. One of the most appealing characteristics of such a system is the presence of large oscillations in the number of agents sharing the same perspective. This finite-size effect is self-instigated by an endogenous noise-induced magnification with a characteristic frequency. This set-up can be used in the modelling of the limit order book. In our case, the difference in population of the two sets of active agents, sellers and buyers, will be directly translated into the evolution of the stock through a simple model of excess demand, that will rise the price when there are more buyers than sellers in the market, and fall it in the opposite case. The random nature of the size of each agent category is responsible for the stochastic component in the asset value evolution, whereas the oscillating behaviour promotes the presence of bullish and bearish periods in the data series in a natural way, with no external interference needed.

We will simulate the time evolution of the system under archetypical market conditions, analyse the most relevant traits, and compare them afterwards with empirically obtained properties.

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I. INTRODUCTION

Financial models based on interacting agents possess a large tradition in the economic literature [1]—one of the first references in which the evolution of a market is related with the activity of individual investors dates back to 1974—, but they have gained relevance in the physics literature in relatively recent years [2, 3]. The complete list of such models is so extensive and their properties so diverse that we can merely sketch here the recurrent traits shared by most of the models, and address the reader to the references cited in [1–3].

The pioneering work of 1974 [4] already contains one of the more ubiquitous ingredients in the subsequent agent-based models: heterogeneity. Agents are assumed to be heterogeneous to some extend, and therefore they can be aggregated into one out of a finite set of categories. Since the minimum number of different categories is two, and simplicity is often a plus, investor are usually arranged into two (competing) groups. The terms used to name them and their defining properties are uneven across the literature—Chartists and Fundamentalists [4], Trend-followers and Contrarians [5], Speculators and Producers [6], Imitators and Optimizers [7]—but the ideas lying behind are similar, and can be well represented by Chartists and Fundamentalists. Chartists are (sometimes adaptative) agents whose inversion strategy is based in the belief that past information may contain clues about the future evolution of the security, and therefore that they can infer future prices. Fundamentalists are in essence agents that think that they can deduce the present value of a firm on the basis of the information presently available, such as dividend payments or earning rates. Fundamentalists operate in a rather predictable way since they expect that the market tends to correct any observed deviation between fundamental and market prices: they sell overpriced securities and buy underpriced ones. The picture is not so simple for Chartist-like investors since at the end they deploy rule-of-thumb strategies, like the Moving Average Convergence Divergence (MACD) indicator or the Relative Strength Index (RSI), two technical analysis tools broadly used by actual financial practitioners. Therefore, the list of available strategies in agent-based models may be so large that, in the most extreme situation, strategies may differ for any pair of investors in the market, as in some instances of the Minority Game market model [2, 8]. In fact, any single agent may combine technical trading rules with fundamental ones, or decide among them, what makes evolve the profile of the investors. There is no doubt that this diversity adds more heterogeneity into the model.
Another general trait of present models is that agents are boundedly rational [9]: they decide their actions in the next time step on the basis of a limited and possibly incomplete amount of information. They ignore the beliefs of the rest of investors and usually cannot evaluate the consequences of their own decisions. Under these circumstances selfish agents try to maximize a utility or pay-off function, a measure of their individual success.

The final ingredient is the pricing mechanism. The usual paradigm when agent activity does not explicitly settle the price of the asset is to define a differential equation or a finite difference equation that relates the price evolution with the relevant (global) variables of the model. Since these variables are affected by the mutual interaction of the investors in a complex way, two complementary approaches are generally considered: the behaviour of the system is computer simulated and/or the complexity is reduced by considering that the number of agents approaches to infinity, the thermodynamic limit.

As we will shortly show, some of the previous ingredients either are not present in this agent-based model, or have been introduced with a different philosophy. The model was inspired by a previous article on population dynamics [10], where authors reported presence of large oscillations in the species densities due to a finite-size stochastic effect. We export this idea and its consequences into the financial language with the confection of a model that describes some aspects of the dynamics of the limit order book in a stock market, and their implications on the behaviour of the asset price through a simple model of excess demand. The overall result depicts similarities with prevailing agent-based financial models [11–15].

The paper is structured in two main sections. Section II deals with the agent model strictly speaking: In section II A we define the three different states in which every agent can be found at any moment, and the mechanisms that govern the changes from one state to the other. We compute mathematical expressions for transition rates between states, and derive a master equation that characterizes the time evolution of the system. In section II B we analyse the stationary solutions of these equations in the thermodynamic limit, whereas we find in section II C how second-order corrections are relevant in finite-size models. Section III is devoted to establishing the connection between the agent model and market price changes. Also in this section we simulate the time evolution of the system under representative market conditions, analyse the most relevant traits, and compare them with well-known empirical properties of actual financial time series. Conclusions are drawn in section IV, and some technical derivations concerning correlation functions are left to an appendix.
II. THE AGENT MODEL

Our model will essentially describe the evolution of offer and demand in a financial market, by assuming that two different kind of investors operate in it. The first group constitutes a set of \( N \) interacting agents that may be willing to introduce a limit order in the market, either a sell order or a buy order, or quite the opposite, they are not interested in operating at that moment. Limit orders are orders with a limit price that represents the minimum (respectively maximum) price the investor is accepting for selling (respectively buying) a given number of shares, the volume of the order. The limit order is placed in the so-called limit order book, which is visible to some other qualified investors and it remains there until one of the two following major events takes place: someone accepts the ask (respectively bid) price and the transaction is completed, or the investor removes the order from the book.\(^1\) This longevity of limit orders is the property that our model will collect, because we are carrying no information about the limit price itself or about the volume of the orders.

There is also a second group of traders in the market that introduce market orders. Market orders, in opposition to limit orders, are instantaneously removed from the book if the market is liquid enough: they match the more advantageous order(s) of opposite sign. We will not model these investors that resemble noise traders \([15, 16]\). This reinforces somewhat the random character of the over-simplified model to be introduced.

A. The species self-interaction

As we have stated above we are presenting a model of \( N \) fully connected interacting agents who, at every instant of time \( t \), may be willing to sell, to buy, or to keep a neutral attitude. We are labelling these three different states by the letters \( A, B \) and \( E \). The mechanism that allows agents to move from some state to another is based on the interaction between them, but not on the previous story (there is no necessity in introducing a sell order after a buy order, for instance) what confers the process with the Markov property.

\(^1\) The real situation may become a little bit more sophisticated. For instance, the investor can demand that transaction comprises the total amount of shares to be completed, or accept partial offers instead. In the usual practice investors also assign by default a maturity to the order, what automatically removes those not realized from the book after the deadline. Within our level of description these considerations do not affect the model substantially, however.
In a population dynamics language, we have only two species living in a finite world: the sellers $A$ who *ask* and buyers $B$ who *bid*. Those agents without a definite investment intention, $E$, act as *empty* space. As we will see afterwards with the description of the *rules of engagement*, within this model, $A$’s will play the role of preys and $B$’s are predators.

The basic interaction in population problems is the death process, $A \xrightarrow{p} E$ and $B \xrightarrow{q} E$, which corresponds in our model with the aggregate effect of two different phenomena. The first one is the simple order removal, investors may decide to decline in their previous intentions by their own: a order expires and there is no reason to renew it, for example. The second one is the introduction of a market order that matches a previously existing limit order, perhaps the most common transaction instance in actual stock markets. In this scheme $p$ (respectively $q$) represents the probability per unit of time that a given active seller (respectively buyer) agent when observed separately passes into inactivity. The same kind of notation is used in the description of the remaining transitions.

Yet another typical unitary interaction in population models in the spontaneous birth of preys, $E \rightarrow A$, but it is not considered here. All birth processes are due to those binary interactions that also occur in the system. At this point, it can be convenient from a practical point of view to establish the probability $\nu$ of considering rather a two-component transition than a single-component one.

The first two-component interaction included in the model implies the cancellation of two existing orders: $AB \xrightarrow{a} EE$. It conveys the agreement of two active investors, and means that one of them must convince the other on the transaction price. This annihilation process is not usually considered in population models: it represents some short of deadly defense in which predator and prey die after fighting. The usual result in predator-prey models for a $AB$ interaction is predation: $AB \xrightarrow{b} BB$. In our case, this counts for the possibility that an investor may perform a change in the evaluation of the market scenario (from bear to bull) due to predominance of bid orders. This may eventually lead to a market bubble. Once again it may become useful to consider that the *a priori* fraction of $AB$ interactions that may conduce to annihilation is $\lambda$, whereas $1 - \lambda$ of them would end in predation.

We finally specify a birth mechanism for preys because otherwise the system would lead to extinction, $AE \xrightarrow{\zeta} AA$. This choice also incorporates in the model population pressure against an unbounded prey growth. Under financial optics this imitative behaviour can lead to market panic and ultimately to a crash when those investors that were not interested in
operate in the market, perhaps because they already have all their wealth in a portfolio, massively introduce sell orders.

The complete state of the agent system at given instant $t$ is fully settled by the number of investors belonging to species $A$ and $B$, $N_A(t)$ and $N_B(t)$ respectively. Since these numbers will be stochastic magnitudes, we may be interested in obtaining an expression for $P(n, m, t)$, the probability of having $n$ ask orders and $m$ bid orders at time $t$:

$$P(n, m, t) = \Pr\{N_A(t) = n, N_B(t) = m\}.$$ 

To this end we will consider the following five transition rates, the transition probabilities (per unit of time) between macroscopic states, based upon the above interactions:

$$T(n - 1, m - 1|n, m) = 2\nu\lambda a \frac{n}{N} \frac{m}{N - 1},$$
$$T(n - 1, m|n, m) = (1 - \nu)\frac{n}{N},$$
$$T(n - 1, m + 1|n, m) = 2\nu(1 - \lambda)\frac{n}{N} \frac{m}{N - 1},$$
$$T(n, m - 1|n, m) = (1 - \nu)\frac{m}{N},$$
$$T(n + 1, m|n, m) = 2\nu c \frac{n}{N} \frac{N - n - m}{N - 1}.$$ 

Note that there are also three “single-stepped” feasible transitions on each variable which are not listed here and therefore are impossible: $T(n, m + 1|n, m)$, $T(n + 1, m + 1|n, m)$ and $T(n + 1, m - 1|n, m)$. Since all agents are identical, here is located the heterogeneity of our model. This asymmetry is supported by the fact that bubbles and crashes in actual stock markets are different in shape.

The Markov character of the model makes superfluous considering more sophisticated transition rates in the elaboration of the master equation (ME) for $P(n, m, t)$, the equation that defines the time evolution of $P(n, m, t)$:

$$\frac{dP(n, m, t)}{dt} = (\alpha_{AA} - \gamma_A)(\mathcal{E}_x^{+1} - 1)[nP(n, m, t)] + \gamma_B(\mathcal{E}_y^{+1} - 1)[mP(n, m, t)]$$
$$+ \frac{\alpha_{AB} - \beta_{AB} - \alpha_{AA}}{2}(\mathcal{E}_x^{+1}\mathcal{E}_y^{+1} - 1) \left[ \frac{n}{N - 1}P(n, m, t) \right]$$
$$+ \frac{\alpha_{AB} + \beta_{AB} - \alpha_{AA}}{2}(\mathcal{E}_x^{+1}\mathcal{E}_y^{-1} - 1) \left[ \frac{m}{N - 1}P(n, m, t) \right]$$
$$+ \alpha_{AA}(\mathcal{E}_x^{-1} - 1) \left[ n \frac{N - n - m}{N - 1}P(n, m, t) \right].$$
Here we have introduced the following increment/decrement operators

\[ \mathcal{E}^{\pm 1}_x P(n, m, t) \equiv P(n \pm 1, m, t), \]
\[ \mathcal{E}^{\pm 1}_y P(n, m, t) \equiv P(n, m \pm 1, t), \]

and five new parameters

\[ \gamma_A = \frac{2\nu c - (1 - \nu)p}{N}, \]

\[ \gamma_B = \frac{(1 - \nu)q}{N}, \]

\[ \alpha_{AA} = \frac{2\nu c}{N}, \]

\[ \alpha_{AB} = \frac{2\nu (\lambda a + (1 - \lambda)b + c)}{N}, \]

\[ \beta_{AB} = \frac{2\nu (1 - \lambda)b - \lambda a}{N}, \]

which encode all the relevant information of the model parameterization. Let us stress that \( \lambda \) and \( \nu \) were defined in order to clarify how the update mechanism can be approximately implemented, see figure 1, but they do not introduce further degrees of freedom to the problem since they would disappear after a constant redefinition. This is the case if one uses Gillespie\’s exact algorithm [17] in the simulation of the system, as we have done.
The relevance of the new parameters becomes noticeable soon afterwards. Suffice it to say for the moment that we will proceed as they were independent of the size of the system in what follows, because we will consider the expansion of the ME in terms of powers of $N$. To this end let us define $R_{A,B}(t)$,

$$R_{A,B}(t) \equiv \lim_{N \to \infty} \mathbb{E}[N_{A,B}(t)]/N.$$ 

and introduce two new stochastic processes, $X(t)$ and $Y(t)$, in such a way that

$$N_A(t) = N R_A(t) + \sqrt{N} X(t),$$

$$N_B(t) = N R_B(t) + \sqrt{N} Y(t),$$

hold. $X(t)$ and $Y(t)$ are thus responsible for the fluctuations of $N_A(t)$ and $N_B(t)$ around their mean values. It is expected that the strength of those fluctuations will diminish as the system reaches the thermodynamic limit, that is, when $N \gg 1$. Note that this approach implies that, for any two given values of the species population, $n$ and $m$, we will have that

$$n = N R_A(t) + \sqrt{N} x,$$

$$m = N R_B(t) + \sqrt{N} y,$$

where $x$ and $y$ —as well as $R_A(t)$ and $R_B(t)$— are real magnitudes in spite that $n$ and $m$ were integers. In such a situation increment/decrement operators become partial differential operators [18],

$$\mathcal{E}_x^\pm = 1 \pm \frac{1}{\sqrt{N}} \partial_x + \frac{1}{2N} \partial^2_{xx} + \mathcal{O}(N^{-3/2}),$$

$$\mathcal{E}_y^\pm = 1 \pm \frac{1}{\sqrt{N}} \partial_y + \frac{1}{2N} \partial^2_{yy} + \mathcal{O}(N^{-3/2}).$$

Finally note that $P(n, m, t)$ must be replaced by $\Pi(x, y, t)$,

$$\Pi(x, y, t)dx dy \equiv \Pr\{x < X(t) \leq x + dx, y < Y(t) \leq y + dy\},$$

through

$$P(n, m, t) = \frac{1}{N} \Pi\left(\frac{n - NR_A}{\sqrt{N}}, \frac{m - NR_B}{\sqrt{N}}, t\right) dx dy,$$

what affects the time derivative term in the ME in the following way:

$$\frac{dP}{dt} = - \left[ \frac{1}{\sqrt{N}} \frac{dR_A}{dt} \partial_x \Pi + \frac{1}{\sqrt{N}} \frac{dR_B}{dt} \partial_y \Pi - \frac{1}{N} \partial_t \Pi \right] dx dy.$$
B. First-order stationary solutions

The first-order approximation of the ME collects terms of order $N^{-1/2}$, ignores those of $O(N^{-1})$, and leads to a set of coupled Volterra equations for $R_A(t)$ and $R_B(t)$,

$$\frac{dR_A}{dt} = \left[ \gamma_A - \alpha_{AA} R_A - \alpha_{AB} R_B \right] R_A, \quad (6)$$

$$\frac{dR_B}{dt} = \left[ \beta_{AB} R_A - \gamma_B \right] R_B. \quad (7)$$

Let us analyse the factors appearing in these equations. $\gamma_A$ as defined in equation (1) represents a trade-off between a positive term that comes from the imitation influence and a negative term that measures the contribution of removals or matches of previously existing sell orders. If positive, it would correspond to a birth rate of preys in equation (6). Note however that in this system preys suffer of population pressure instigated by the imitation interaction that constrains the preys’ growth, see the definition of $\alpha_{AA}$ in (3). The term with the $\alpha_{AB}$ factor counts for the reduction in the prey number due to all binary operations, not only predation, equation (4). The $\beta_{AB}$ term appearing in equation (7) depends on the balance between predation and annihilation alternatives, as it can be observed in (5), whereas $\gamma_B$ measures exclusively the removal rate of bid orders, expression (2). Summing up, there are two parameters, $\gamma_A$ and $\beta_{AB}$, with no definite sign, whereas $\gamma_B$, $\alpha_{AA}$ and $\alpha_{AB}$ are positive constants ab initio.

Equations (6) and (7) present three stationary solutions for which

$$\frac{dR_A}{dt} = \frac{dR_B}{dt} = 0.$$ 

The first solution is the trivial one, $R_A = R_B = 0$. It represents the death of the market due to a complete lack of activity. This is a feasible scenario that threatens any real market. For instance, investors may lose interest in any given commodity that becomes useless or exhausted. The stability analysis of this fixed point determines that it will be a saddle point if $\gamma_A > 0$, otherwise it would turn stable. The analysis of the second stationary solution, $R_A = \gamma_A / \alpha_{AA}$ and $R_B = 0$, lead to the constraint

$$\beta_{AB} > \frac{\alpha_{AA} \gamma_B}{\gamma_A} > 0, \quad (8)$$

in order to avoid again conferring stability to a situation that represents the worst market crash imaginable, where every stock holder wants to sell his/her share(s) but no one is
interested in buying them. In this sense we may consider that $M = N\gamma_A/\alpha_{AA} < N$ —note that $\gamma_A < \alpha_{AA}$ by construction, cf. expressions (1) and (3)— is related to the total amount of shares. In conclusion, all the parameters defined in (1)-(5) must be positive-definite.

We must point out that the presence of those unstable equilibrium solutions is not a flaw but a merit of the model, as is the fact that the remaining stationary solution

$$R_A = R_A^0 \equiv \frac{\gamma_B}{\beta_{AB}},$$

$$R_B = R_B^0 \equiv \frac{\gamma_A\beta_{AB} - \gamma_B\alpha_{AA}}{\alpha_{AB}\beta_{AB}},$$

is always present, accessible and corresponds to a stable fixed point.

Regarding the occurrence of the fixed point, it is evident that $R_A^0 > 0$, and equation (8) leads to $R_A^0 < M/N < 1$. The same equation determines that $R_B^0 > 0$. Also $R_B^0 < 1$, as it can be proven as follows:

$$R_B^0 = 1 - \frac{(\alpha_{AB} - \gamma_A)\beta_{AB} + \gamma_B\alpha_{AA}}{\alpha_{AB}\beta_{AB}} < 1,$$

because trivially $\alpha_{AB} > \gamma_A$, cf. equations (1) and (4). We can also show that $R_A^0 + R_B^0 < 1$,

$$R_A^0 + R_B^0 = 1 - \frac{(\alpha_{AB} - \gamma_A)\beta_{AB} + (\alpha_{AA} - \alpha_{AB})\gamma_B}{\alpha_{AB}\beta_{AB}} < 1 - \frac{\gamma_B(\alpha_{AA} - \gamma_A)}{\gamma_A\beta_{AB}} = 1 - \frac{\gamma_B(\alpha_{AA} - \gamma_A)}{\gamma_A\beta_{AB}} < 1,$$

because $\alpha_{AA} > \gamma_A$ as we have just pointed out above.

The analysis of the stability of this fixed point leads to the conclusion that the point is stable, and that the transient term will exhibit oscillations when $\omega_0 \in \mathbb{R}^+$,

$$\omega_0 = \sqrt{\alpha_{AB}\beta_{AB}R_A^0R_B^0 - \frac{1}{4}(\alpha_{AA}R_A^0)^2},$$

which is true whenever

$$\frac{\alpha_{AA}}{\beta_{AB}} < 2\sqrt{1 + \frac{\gamma_A}{\gamma_B} - 2} < \frac{\gamma_A}{\gamma_B}.$$  

When the system shows transient oscillations, there is a single characteristic time scale for the decay rate,

$$\tau_0 = \frac{2}{\alpha_{AA}R_A^0},$$

and for $t \gg \tau_0$ the system would reach the stable solution. Within that regime, and whenever $N$ is finite, we will await that the time evolution of prey and predator densities, $N_A(t)/N$
Figure 2: Time evolution of sellers and buyers densities (solid red line) for an exact realization of our interacting agent model with $N = 1000$. The dashed black line depicts the first-order approach to the problem, whereas in green is shown the stationary solution. We can see how fluctuations in both populations are larger than expected.

and $N_B(t)/N$, makes them attain their limit values $R_{A}^{\circ}$ and $R_{B}^{\circ}$, and exhibit some fluctuating activity afterwards. Since the characteristic size of the fluctuations is of order $N^{-1/2}$, a naive analysis could lead to the conclusion that if we have, let us say, 1000 interacting agents the error in neglecting the remaining terms in the ME should be around 3.2%. In figure 2 we can find the outcome of a realization of the model with $N = 1000$ — the complete set of parameter specifications is listed below in section III. The example shows that in such a system fluctuations may be larger than expected, and further corrections to the first-order equations must be taken into account [10, 19].

A final word on the $N$ dependency of the above expressions before exploring the forthcoming terms in the ME. The order-by-order analysis under progress relies on the fact that the parameters defined in equations (1)-(5) are independent of $N$. Note however that the expressions for $R_{A}^{\circ}$, $R_{B}^{\circ}$ and $M/N$ are insensible to this need. It only affects those constants where time is involved, like $\tau_0$ or $\omega_0$. 
C. Beyond the first-order equations

When one gathers the terms of order \(N^{-1}\) in the ME expansion, a Fokker-Planck equation for \(\Pi(x, y, t)\) emerges:

\[
\frac{\partial \Pi}{\partial t} = [-\gamma_A + 2\alpha_{AA} R_A + \alpha_{AB} R_B] \partial_x (x \Pi)
+ \alpha_{AB} R_A \partial_y (y \Pi) - \beta_{AB} R_B \partial_y (x \Pi) + [\gamma_B - \beta_{AB} R_A] \partial_y (y \Pi)
+ \frac{R_A}{2} [-\gamma_A + \alpha_{AA} (2 - R_A - 2 R_B) + \alpha_{AB} R_B] \partial_{xx} \Pi
+ \frac{R_B}{2} [\gamma_B + (\alpha_{AB} - \alpha_{AA}) R_A \partial_{yy} \Pi - \beta_{AB} R_A R_B \partial_{xy} \Pi].
\]

If we concentrate our analysis of the previous equation for times large enough to let \(R_A(t)\) and \(R_B(t)\) reach their steady state values, \(R^*_A\) and \(R^*_B\), the expression simplifies considerably:

\[
\frac{\partial \Pi}{\partial t} = \alpha_{AA} R^*_A \partial_x (x \Pi) + \alpha_{AB} R^*_A \partial_y (y \Pi) - \beta_{AB} R^*_B \partial_y (x \Pi)
+ \alpha_{AA} R^*_A (1 - R^*_A - R^*_B) \partial_{xx} \Pi + \frac{1}{2} R^*_A R^*_B (\beta_{AB} + \alpha_{AB} - \alpha_{AA}) \partial_{yy} \Pi - \beta_{AB} R^*_A R^*_B \partial_{xy} \Pi.
\]

Therefore we have a linear multivariate Fokker-Planck equation for the joint probability density of \(X(t)\) and \(Y(t)\), whose solution can be systematically obtained after some (or maybe plenty of) algebra [18]. Instead of presenting this result we note that the above equation corresponds to the following set of coupled (Itô) stochastic differential equations:

\[
\begin{align*}
\text{d}X &= -\mu_{xx} X \text{d}t - \mu_{xy} Y \text{d}t + \sigma_x \text{d}W_1, \\
\text{d}Y &= \mu_{yx} X \text{d}t - \rho \sigma_y \text{d}W_1 + \sigma_y \sqrt{1 - \rho^2} \text{d}W_2,
\end{align*}
\]

with

\[
\begin{align*}
\mu_{xx} &\equiv \alpha_{AA} R^*_A = \frac{2}{\tau_0}, \\
\mu_{xy} &\equiv \alpha_{AB} R^*_A, \\
\mu_{yx} &\equiv \beta_{AB} R^*_B, \\
\sigma_x^2 &\equiv 2 \alpha_{AA} R^*_A (1 - R^*_A - R^*_B), \\
\sigma_y^2 &\equiv R^*_A R^*_B (\beta_{AB} + \alpha_{AB} - \alpha_{AA}), \\
\rho &\equiv \beta_{AB} R^*_A R^*_B \sigma_{x} \sigma_{y}
\end{align*}
\]

positive-definite quantities, \(^2\) and where \(W_1\) and \(W_2\) are two independent Wiener processes.

\(^2\) Note in particular that \(\beta_{AB} + \alpha_{AB} - \alpha_{AA} = 4 \nu (1 - \lambda) b/N > 0\). Stability considerations also lead to the result \(\rho^2 < 1/4\).
In order to explore the reason for the abnormal magnitude of fluctuations we should compare \(X(t)\) and \(Y(t)\) with \(R_A^0\) and \(R_B^0\) respectively. A quick analysis reveals that mean values are not useful in this venture because \(\lim_{t \to \infty} \mathbb{E}[X(t)] = \lim_{t \to \infty} \mathbb{E}[Y(t)] = 0\) — remember that (11) and (12) are valid for \(t \gg \tau_0\). We concentrate in variances instead. To this end we will use Itô calculus in order to compute \(dX^2\), \(dY^2\) and also \(dXY\):

\[
\begin{align*}
    dX^2 &= -2\mu_{xx}X^2dt - 2\mu_{xy}XYdt + \sigma_x^2dt + 2\sigma_xXdW_1, \\
    dY^2 &= 2\mu_{yx}XYdt + \sigma_y^2dt - 2\rho\sigma_yYdW_1 + 2\sigma_y\sqrt{1 - \rho^2}dW_2, \\
    dXY &= \mu_{yx}X^2dt - \mu_{xy}Y^2dt - \mu_{xx}XYdt - \rho\sigma_x\sigma_ydt \\
        &\quad + (\sigma_xY - \rho\sigma_yX) dW_1 + \sigma_y\sqrt{1 - \rho^2}XdW_2.
\end{align*}
\]

The stationary values of \(\mathbb{E}[X^2(t)], \mathbb{E}[Y^2(t)]\) and \(\mathbb{E}[X(t)Y(t)]\) then follow

\[
\begin{align*}
    \lim_{t \to \infty} \mathbb{E}[X^2(t)] &= \frac{\mu_{yx}\sigma_x^2 + \mu_{xy}\sigma_y^2}{2\mu_{xx}\mu_{yx}}, \\
    \lim_{t \to \infty} \mathbb{E}[Y^2(t)] &= \frac{(\mu_{xx}\sigma_y - \mu_{yx}\sigma_x)^2 + \mu_{yx}\mu_{yy}\sigma_y^2 + 2(1 - \rho)\sigma_x\sigma_y\mu_{xx}\mu_{yx}}{2\mu_{xx}\mu_{yx}}, \\
    \lim_{t \to \infty} \mathbb{E}[X(t)Y(t)] &= -\frac{\sigma_y^2}{2\mu_{yx}}.
\end{align*}
\]

Then, for instance, we have

\[
\lim_{t \to \infty} \frac{\mathbb{E}[X^2(t)]}{R_A^2(t)} = \frac{1}{R_A^0} \left[ \left( 1 - R_A^0 - R_B^0 \right) + \left( \frac{\alpha_{AB}}{\alpha_{AA}} \right) \left( \frac{\beta_{AB} + \alpha_{AB} - \alpha_{AA}}{2\beta_{AB}} \right) \right].
\]

This expression becomes unboundedly large as \(R_A^0 \to 0\) since \(R_A^0 + R_B^0 < 1\), \(\alpha_{AB} > \alpha_{AA}\) and \(\alpha_{AB} - \alpha_{AA} > \beta_{AB}\). A similar result stands for \(\lim_{t \to \infty} \mathbb{E}[Y^2(t)]/R_B^2(t)\) for \(R_B^0 \to 0\). Thus when \(R_A^0 \sim R_B^0 \sim 0\) the magnification effect can become so intense that overcomes the \(N^{-1/2}\) damping factor.

Finally, we can give a glimpse of the oscillatory character shown in figure 2 as well. As we detail in appendix A, one can obtain not only the stationary variances but also stationary auto- and cross-correlation functions of \(X(t)\) and \(Y(t)\) for any time lag \(\tau\). In particular we have for \(X(t)\) and \(\tau > 0\) a rather compact expression

\[
\lim_{t \to \infty} \mathbb{E}[X(t)X(t + \tau)] = \left[ \frac{\mu_{yx}\sigma_x^2 + \mu_{xy}\sigma_y^2}{2\mu_{xx}\mu_{yx}} \cos(\omega_0\tau) - \frac{\mu_{yx}\sigma_x^2 - \mu_{xy}\sigma_y^2}{4\omega_0\mu_{yx}} \sin(\omega_0\tau) \right] e^{-\tau/\tau_0},
\]

with \(\tau_0\) and \(\omega_0\) as defined above. The periodic component of this function is responsible of the observed oscillating behaviour.
III. THE MARKET MODEL

Now it is time to define how the evolution in $A$ and $B$ population of agents translates into prices changes. We will adopt here a classical and ubiquitous point of view in the economic literature, e.g. [4, 11, 13–15, 20], we will assume that excess return reacts linearly to excess demand. Excess return measures the logarithmic earnings of the stock beyond the risk-free interest rate $r$, $R(t) \equiv \ln[S(t)e^{-rt}]$, and excess demand is the difference between buy and shell orders. Therefore we will have

$$dR(t) = \frac{\chi}{N}(N_B - N_A)dt = \chi \left( R_B - R_A + \frac{Y - X}{\sqrt{N}} \right) dt \xrightarrow{\gamma_B} \chi \left( R_B^o - R_A^o + \frac{Y - X}{\sqrt{N}} \right) dt.$$  

The $R_B^o - R_A^o$ term is responsible for the long-run exponential growth and may represent the wealth (funds) injected into the market. Note that this quantity may become very small with no further implications: we can freely tune $\gamma_B$ in order to get $R_B^o = R_A^o$. In that risk-neutral scenario second-order excess demand is responsible for all the dynamics after the transient period, and it will retain the oscillating character —see appendix A.

Before we can assign numerical values to the parameters and consequently simulate the possible evolution of such market model we must address a crucial question: Who are the investors? Since the dynamical model of the species includes the option to stay outside the market, one possible choice is to include every person or institution who can, even sporadically, enter in the stock market. This approach is questionable, in particular if we recall we model limit orders only. Moreover, for simplicity reasons, we did not include in the dynamical model the volume associated with the orders. In actual markets buy and sell orders have different size, and the market will react accordingly. Larger orders will have a deeper impact in the market evolution. Therefore we will assume that the main actors are those with bigger influence: mutual funds, investment banks, or corporations in general. The total amount of such participants in a real market is much more moderate.

Let us consider the following paradigmatic example with $N = 1000$ investors. The probabilities that weights the different interactions under consideration are $\nu = 0.5$ and $\lambda = 0.1$. The agreement rate per agent couple is $a = 1/\text{min}$, the predation rate (from bear to bull) which promotes bubble formation is $b = 8/\text{min}$, the panic component, from neutral to seller, is $c = 5/\text{min}$. The removal rates per agent are $p = 3/\text{min}$, and $q = 2.8/\text{min}$. Finally, sensitivity of prices to excess demand was set equal to $\chi = 10^{-4}$. These values are not based on actual market observations, and some of them were simply chosen in such a
way that all the constraints we have previously introduced were satisfied. Our principal aim is to explore the main traits and capabilities of the model. The above numbers lead to $R_A = 14/71 \approx 0.197$, $R_B = 595/2911 \approx 0.204$. Then there are about 200 orders in each side of the book in mean, and the saturation limit of sell orders is $M = 700$.

A possible realization of the number of agents belonging to each category was previously introduced in figure 2, and in figure 3 we find the corresponding evolution of the stock price. Here we sampled the complete data series to consider closing prices only, a usual practice in technical analysis. We observe typical market charts: upward trends —an increasing succession of minima—, downward trends —a decreasing succession of maxima—, and sideways trends —a bouncing movement between two price levels.

![Figure 3: Time evolution of the daily closing value of (discounted) stock prices. We can see how the market show what is called a sideways trend, i.e. no trend at all, along the year 37 and the beginning of year 38, followed by two upward trends with a downward correction lasting three months in the middle. The inset shows the exponential growth in the long run.](image)

In figure 4 we present the outcome of a statistical analysis performed with tick-by-tick data of fixed-time returns $R(\tau; t) = R(t + \tau) - R(t)$. We check that for $\tau \ll \tau_0$ (in our case $\tau_0 \approx 2029$ minutes) correlations are important, Gaussian limit is not attained and fat
tails are observed, like in actual markets [21, 22]. This phenomenon is even more noticeable

![Figure 4: Fixed-horizon return behaviour.](image)

Figure 4: Fixed-horizon return behaviour. We can see how probability density functions at small time-scale depart from Gaussian behaviour and exhibit fat tails. Returns were divided by their sampling standard deviations to make them commensurable.

when volatility, the standard deviation of fixed-time returns, is analysed, figure 5. Since $X(t)$ and $Y(t)$ are anti-correlated, and the return change is sensible to the difference of those magnitudes, we expect that volatility grows faster for small time scales, and reaches the diffusive regime for $\tau > \tau_0$. Traits of abnormal (both sub- and super-) diffusion have been reported to be present in real markets as well [22–24].

**IV. CONCLUSIONS**

Along this article we have introduced a dynamical model that describes the evolution of offer and demand in a financial market. The model considers a finite set of identical interacting agents. At every moment any single agent can be accommodated in one of three excluding categories: the agent is willing to operate in the market, either by selling the stock or by buying it, or is not interested in operating at all. The way in which the agent participates in the market is through the limit order book.
Orders can randomly disappear after a successful transaction or by agent cancellation, but they are introduced only after agent-to-agent interaction. These simple rules encode a system in which the number of active orders may strongly oscillate, mainly if mean occupation-level values are low, in spite that it is a second-order effect.

We have assumed that the difference between buy and shell orders, the excess demand, determines the stock price evolution, it will rise when there are more buyers than sellers in the market, and fall in the opposite scenario. We have argued that the stochastic fluctuations may become the generator of all dynamics in a neutral market.

We have simulated the time evolution of the stock price for representative values of the involved parameters. We have shown how this sample realization reproduces several stylised facts reported in actual financial data sets: the price evolution displays upward, downward and sideways trends; probability density functions of small time-scale returns present fat tails; and volatility behaves accordingly in a non-diffusive way within the same time horizon.

In a forthcoming work we are planning to explore how the properties shown by the model
depend on the actual values of the parameters, in a quest for critical features in the system.

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Appendix A: CORRELATION FUNCTIONS

The cross-correlation theorem states that, for any two random variables \( X(t) \) and \( Y(t) \) we can compute its stationary auto-correlation function though:

\[
C_{xy}(\tau) \equiv \lim_{t \to \infty} \mathbb{E}[X(t)Y(t + \tau)] = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} P_{xy}(\omega)e^{-i\omega\tau},
\]

where

\[
P_{xy}(\omega) = \lim_{t \to \infty} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \mathbb{E}[\hat{X}^*(\omega)\hat{Y}(\omega')]e^{-i(\omega' - \omega)t},
\]

\( \hat{X}(\omega) \) stands for the Fourier transform of \( X(t) \), and so forth. When \( Y(t) \) coincides with \( X(t) \), \( P_{xx}(\omega) \) is termed the power spectral density function of \( X(t) \), and cross-correlation theorem is known as the Wiener-Khinchin theorem.

In our case equations (11) and (12) lead to

\[
P_{xx}(\omega) = \frac{\mu_{xy}^2\sigma_y^2 + \sigma_x^2\omega^2}{\omega^2\mu_x^2 + (\omega^2 - \mu_{xy}\mu_{yx})^2},
\]

\[
P_{yy}(\omega) = \frac{\mu_{yx}^2\sigma_x^2 + \mu_{xy}^2\sigma_y^2 - 2\rho\mu_{xx}\mu_{yx}\sigma_x\sigma_y + \sigma_y^2\omega^2}{\omega^2\mu_x^2 + (\omega^2 - \mu_{xy}\mu_{yx})^2},
\]

\[
P_{xy}(\omega) = \frac{-\mu_{xx}\mu_{xy}\sigma_y^2 + \mu_{xy}\mu_{yx}\rho\sigma_x\sigma_y + i \left( \mu_{yx}\sigma_x^2 + \mu_{xy}\sigma_y^2 - \mu_{xx}\rho\sigma_x\sigma_y \right) \omega - \rho\sigma_x\sigma_y\omega^2}{\omega^2\mu_x^2 + (\omega^2 - \mu_{xy}\mu_{yx})^2},
\]

and then it follows that

\[
P_{zz}(\omega) = \frac{\mu_{yx}^2\sigma_x^2 + (\mu_{xx} + \mu_{yx})^2\sigma_y^2 - 2\rho\mu_{yx}(\mu_{xx} + \mu_{yx})\sigma_x\sigma_y + (\sigma_x^2 + \sigma_y^2 + 2\rho\sigma_x\sigma_y)\omega^2}{\omega^2\mu_x^2 + (\omega^2 - \mu_{xy}\mu_{yx})^2},
\]

where we have defined \( Z(t) = Y(t) - X(t) \), the second-order excess demand. Note that every function has the same basic structure, namely

\[
P(\omega) = \frac{\kappa_1 + i\kappa_2\omega + \kappa_3\omega^2}{\omega^2\mu_x^2 + (\omega^2 - \mu_{xy}\mu_{yx})^2},
\]
therefore we can compute the auto- and cross-correlation functions at once,

\[ C(\tau) = \left[ \frac{\kappa_1 + \kappa_3 \mu_{xy} \mu_{yx}}{\mu_{xx} \mu_{xy} \mu_{yx}} \cos(\omega_0|\tau|) + \frac{\kappa_1 - \kappa_3 \mu_{xy} \mu_{yx}}{2 \mu_{xy} \mu_{yx} \omega_0} \sin(\omega_0|\tau|) \right. \]

\[ \left. + \frac{\kappa_2}{\mu_{xx} \omega_0} \sin(\omega_0|\tau|) \right] e^{-|\tau|/\tau_0^2}, \]

where \( \omega_0 \) coincides with that defined in (9), and \( \tau_0 \) was introduced in (10).

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