Global geometric properties of AdS space and the AdS/CFT correspondence

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Abstract The Poisson kernels and relations between them for a massive scalar field in a unit ball $B^n$ with Hua’s metric and conformal flat metric are obtained by describing the $B^n$ as a submanifold of an $(n+1)$-dimensional embedding space. Global geometric properties of the AdS space are discussed. We show that the $(n+1)$-dimensional AdS space $\text{AdS}_{n+1}$ is isomorphic to $\mathbb{RP}^1 \times B^n$ and boundary of the AdS is isomorphic to $\mathbb{RP}^1 \times S^{n-1}$. Bulk-boundary propagator and the AdS/CFT like correspondence are demonstrated based on these global geometric properties of the $\mathbb{RP}^1 \times B^n$.

Keywords: Poisson kernel, AdS/CFT correspondence, Bulk-boundary propagator

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1 Introduction

In Riemann geometry, it is known that the biggest symmetry (isometric) groups of the Minkowski space, de Sitter Space (dS) and Anti-de Sitter (AdS) space have same number of generators. They can be identified, in a unified way, as the classical manifold $D_λ(n+1)$ with $λ = 0, \ +1$ and $-1$, respectively. Thus, the dS and the AdS space are the simplest generalization of the Minkowski space with constant curvature. The dS, in particularly, the AdS space and quantum field theory based on it has been an interested topic of mathematicians and physicists for a long time\[1\]. There has recently been a revival of interest in AdS space brought about by the conjectured duality between physics in the bulk of AdS and a conformal field theory (CFT) on the boundary\[2\]–\[4\]. The so-called AdS/CFT correspondence states that string theory in the AdS space is holographically dual to a CFT on boundary of the AdS\[5\]. A strong support for the proposal comes from comparing spectra of Type IIB string theory on the background of AdS$_5 \times S^5$ and low-order correlation functions of the 3 + 1 dimensional $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory. The dual super Yang-Mills theory lives on the boundary of the AdS space. This is one of the most important progresses in the superstring theory. Many new results have been obtained by making use this conjecture\[6\].

However, up to now, almost all discussions on the AdS/CFT correspondence were based on the so-called Euclidean version of the AdS, or an $(n+1)$-dimensional unit ball $B^{n+1}$. To prove the AdS/CFT conjecture or to investigate its delicious implications in physics theory, one should work on the more challenging topic of duality between physics theories on the AdS space and its boundary, the compact Minkowski space.

In this paper, by describing the unit ball $B^n$ as a submanifold of an $(n+1)$-dimensional embedding space, we first present Poisson kernels for the Laplace operator with a general nonzero eigenvalue $m_0$ and relations between them in a unit ball $B^n$ with Hua’s metric and conformal flat metric. Results for Euclidean version of the AdS are recovered in a different view. Then we discuss the AdS geometry with right signature
within the framework of the classical manifolds and the classical domains\[7, 8\]. It is proved that the AdS\(_{n+1}\) is isomorphic to \(\mathbb{RP}^1 \times B^n\) and the boundary of the AdS is isomorphic to \(\mathbb{RP}^1 \times S^{n-1}\). Bulk-boundary propagator for a massive scalar field in the AdS is replaced by what in the \(\mathbb{RP}^1 \times B^n\) based on the global geometric properties of AdS. The bulk/boundary correspondence in this case is also demonstrated.

## 2 Poisson kernels for a scalar field in unit ball

A unit ball \(B^n\) can be described as the image of a two-to-one map of the hypersurface

\[
\xi^n \xi^n - \sum_{i=0}^{n-1} \xi^i \xi^i = 1
\]

in the space \((\xi^0, \xi^1, \ldots, \xi^{n-1})\).

An explicitly \(SO(1,n)\) invariant metric of the \(B^n\) can be introduced

\[
ds^2 = d\xi^n d\xi^n - \sum_{i=0}^{n-1} d\xi^i d\xi^i .
\]

By denoting

\[
x^i = \frac{\xi^i}{\xi^n} , \quad (i = 0, 1, \ldots, n-1 , \ \xi^n \neq 0) ,
\]

\(n\)-dimensional vector \(x \equiv (x^0, x^1, \ldots, x^{n-1})\), and \(x'\) transfer of the \(x\), we can rewrite the \(B^n\) as usual

\[xx' < 1 .\]

The reduced metric from Eq.(2) in the coordinate \(\{x^i\}\) is of the form

\[
ds^2 = -\frac{dx(I - x'x)^{-1}dx'}{1 - xx'} .
\]

Another set of coordinates for the unit ball \(B^n\) can be introduced

\[
z^i = \frac{\xi^i}{\xi^n + 1} , \quad (i = 0, 1, \ldots, n-1 ; \ \xi^n \neq 0) .
\]

In this coordinate, the unit ball \(B^n\) is also described as usual

\[zz' < 1 .\]
The reduced metric from Eq.(2) in terms of the coordinate \( \{z^i\} \) is
\[
ds^2 = \frac{-4}{(1 - zz')^2} dz dz'.
\]
(8)

There is a one to one transformation between the two sets of coordinates \( \{x^i\} \) and \( \{z^i\} \)
\[
x^i = \frac{2z^i}{1 + zz'}.
\]
(9)

Therefore, the conformal flat metric (8) and the Hua’s metric (3) are different representations of the \( SO(1, n) \) invariant metric. And we can work by using one of them and got same results include invariant differential form and Poisson kernel.

The equation of eigenvalues for the Laplace operator in the unit ball \( B^n \) is
\[
\left[ \frac{1}{\sqrt{-g}} \sum_{i,j=0}^n \partial \left( \sqrt{-g} g^{ij} \frac{\partial}{\partial x^i} \right) + m_0^2 \right] \Phi(x) = 0.
\]
(10)

The Poisson kernel for a massless scalar field has been discussed[9]
\[
G^{E\pm}_{B\partial}(x, u) = \begin{cases} 
(1 - xx')^{n-1} & \text{for conformal flat metric ,} \\
(1 - 2ux' + xx')^{n-1} & \text{for Hua’s metric .}
\end{cases}
\]
(11)

The bulk field \( \Phi(x) \) determined by the fields living on the boundary \( \phi(u) \) is of the form
\[
\Phi(x) = \frac{1}{\omega_{n-1}} \int_{uu'=1} \cdots \int G^{E\pm}_{B\partial}(x, u) \phi(u) \mathrm{d}u.
\]
(12)

Now, we write down a general Poisson kernel for the Laplace operator with nonzero eigenvalue \( m_0^2 \)
\[
G^{\pm\pm}_{B\partial}(x, u) = \left( G^{E\pm}_{B\partial}(x, u) \right)^{\frac{1}{2}} \left( 1 \pm \sqrt{1 + 4m_0^2 \frac{(n-1)^2}{n^2}} \right).
\]
(13)

The Poisson kernels \( G^{\pm\pm}_{B\partial}(x, u) \) satisfy the following properties:

- It is definitely positive.
- On the boundary, we have
\[
G^{\pm\pm}_{B\partial}(v, u) = \begin{cases} 
0 & u \neq v, \\
\infty & u = v.
\end{cases}
\]
(14)
• It satisfies the equation of eigenvalues for the Laplace operator with an eigenvalue $m_0^2$.

3 Conformal boundary and AdS

In an $(n+2)$-dimensional embedding space, the $(n+1)$-dimensional AdS space $\text{AdS}_{n+1}$ can be written as

$$
\xi^0\xi^0 - \sum_{i=1}^{n}\xi^i\xi^i + \xi^{n+1}\xi^{n+1} = 1.
$$

From the above definition of AdS, we know that $\xi^0$ and $\xi^{n+1}$ can not be zero simultaneously, and at least two charts of coordinates [(n+1)-dimensional] $U_1$ ($\xi^{n+1} \neq 0$) and $U_0$ ($\xi^0 \neq 0$) should be needed to describe the AdS.

In the chart $\text{AdS}_{n+1} \cap U_1$, we introduce a coordinate

$$
x^i = \frac{\xi^i}{\xi^{n+1}}, \quad (i = 0, 1, 2, \cdots, n; \xi^{n+1} \neq 0).
$$

The $\text{AdS}_{n+1} \cap U_1$, in the coordinate $\{x^i\}$, is described by

$$
\sigma(x^i, x^j) > 0, \\
\sigma(x^i, x^j) \equiv 1 + \sum_{i,j=0}^{n}\eta_{ij}x^ix^j, \quad \eta = \text{diag}(1, -1, -1, \cdots, -1).
$$

In the chart $\text{AdS}_{n+1} \cap U_0$, let

$$
y^0 = \frac{\xi^{n+1}}{\xi^0},
$$

$$
y^i = \frac{\xi^i}{\xi^0}, \quad (i = 1, 2, \cdots, n; \xi^0 \neq 0).
$$

And the $\text{AdS}_{n+1} \cap U_0$ can be written in the form

$$
\text{AdS}_{n+1} \cap U_0 : \quad \sigma(y^i, y^j) > 0.
$$

At the overlap region $\text{AdS}_{n+1} \cap U_0 \cap U_1$ of the two charts $U_1$ and $U_0$, one has relations

$$
y^0 = \frac{1}{x^0},
$$

$$
y^i = \frac{x^i}{x^0}.
$$
This shows clearly a differential structure of the AdS. The boundary $\mathcal{M}^n$ of the AdS$_{n+1}$ consists of infinite points not belong to AdS$_{n+1} \cap \mathcal{U}_1$ or AdS$_{n+1} \cap \mathcal{U}_0$,

$$\mathcal{M}^n: 1 + \sum_{i,j=0}^n \eta_{jk} x^j x^k = 0 ,$$

$$1 + \sum_{i,j=0}^n \eta_{jk} y^j y^k = 0 .$$

(21)

A new set of variables in the chart AdS$_{n+1} \cap \mathcal{U}_1$ can be introduced as

$$\chi^0 \equiv x^0 ,$$

$$\chi^\mu \equiv \sqrt{1 + (x^0)^2 x^\mu} , \quad (\mu = 1, \ 2, \ \cdots , \ n).$$

(22)

Then, we have

$$\text{AdS}_{n+1} \cap \mathcal{U}_1 = \{ \chi^0 \in \mathbb{R}, \ (\chi^1, \ \chi^2, \ \cdots , \ \cdots , \ \chi^n) \in \mathbb{R}^n | (\chi^1)^2 + (\chi^2)^2 + \cdots + (\chi^n)^2 < 1 \} .$$

(23)

This shows that, in the chart $\mathcal{U}_1$, the AdS$_{n+1}$ is equivalent to $\mathbb{R} \times \mathbb{B}^n$.

In the same way, in the chart AdS$_{n+1} \cap \mathcal{U}_0$, let

$$\eta^0 \equiv y^0 ,$$

$$\eta^\mu \equiv \sqrt{1 + (y^0)^2 y^\mu} , \quad (\mu = 1, \ 2, \ \cdots , \ n).$$

(24)

And subsequently, one has

$$\text{AdS}_{n+1} \cap \mathcal{U}_0 = \{ \eta^0 \in \mathbb{R}, \ (\eta^1, \ \eta^2, \ \cdots , \ \cdots , \ \eta^n) \in \mathbb{R}^n | (\eta^1)^2 + (\eta^2)^2 + \cdots + (\eta^n)^2 < 1 \} .$$

(25)

Therefore, both charts of the AdS$_{n+1}$, $\mathcal{U}_1$ and $\mathcal{U}_0$ are equivalent to $\mathbb{R} \times \mathbb{B}^n$.

It should be noticed that, at the overlap region of the two charts, there are relations among the two different sets of coordinate variables

$$\chi^0 = \frac{1}{\eta^0} ,$$

$$\chi^\mu = \eta^\mu , \quad (\mu = 1, \ 2, \ \cdots , \ n).$$

(26)

This fact presents a theorem for the AdS.

**Theorem:** The AdS$_{n+1}$ is isomorphic to $RP^1 \times B^n$ and its boundary is $RP^1 \times S^{n-1}$.

It is well-known that the Study-Fubini metric can be introduced on the $RP^1$ space

$$ds^2 = \frac{(d\chi^0)^2}{[1 + (\chi^0)^2]^2} = (d\arctan \chi^0)^2 .$$

(27)
As presented at the previous section, on the unit ball $B^n$, we have the Hua's metric

$$ds_n^2 = - \sum_{\mu,\nu=1}^{n} d\chi^\mu d\chi^\nu \left( \frac{\delta_{\mu\nu}}{1 - \sum_{\alpha=1}^{n} \chi^\alpha \chi^\alpha} + \frac{\chi^\mu \chi^\nu}{(1 - \sum_{\alpha=1}^{n} \chi^\alpha \chi^\alpha)^2} \right).$$

(28)

Thus, a natural metric on the $RP^1 \times B^n$ is of the form

$$ds^2 = ds_0^2 - ds_n^2$$

$$= \frac{(d\chi^0)^2}{[1 + (\chi^0)^2]^2} - \sum_{\mu,\nu=1}^{n} d\chi^\mu d\chi^\nu \left( \frac{\delta_{\mu\nu}}{1 - \sum_{\alpha=1}^{n} \chi^\alpha \chi^\alpha} + \frac{\chi^\mu \chi^\nu}{(1 - \sum_{\alpha=1}^{n} \chi^\alpha \chi^\alpha)^2} \right).$$

(29)

But, with this metric the $RP^1 \times B^n$ is no longer AdS group invariant. In what follows, we discuss the bulk/boundary correspondence and related topics in this case.

4 Eigenfunctions of Laplace operator on $RP^1 \times B^n$

Let

$$\theta = \arctan \chi^0.$$  

(30)

Then $RP^1 \times B^n$ and $RP^1 \times S^{n-1}$ are isomorphic to $S^1 \times B^n$ and $S^1 \times S^{n-1}$ respectively. Moreover,

$$\Box = \frac{\partial^2}{\partial \theta^2} - \Delta,$$

(31)

where

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial \chi^i} (\sqrt{g} g^{ij} \frac{\partial}{\partial \chi^j})$$

(32)

is the Laplace-Beltrami operator of the ball $B^n$.

Denote

$$\alpha_k(m_0) = 1 + \left[ 1 + 4 \frac{m_0^2 - k^2}{(n - 1)^2} \right]^{\frac{1}{2}}$$

(33)
and

\[ P_{\alpha_k(m_0^2)}(\chi, u) = \left[ G_{B\theta}(\chi, u) \right]^{\alpha_k(m_0^2)} , \]  

where \( \chi = (\chi^1, \cdots, \chi^n) \). It can be proved that

\[ \Delta P_{\alpha_k(m_0)}(\chi, u) = -(k^2 - m_0^2)P_{\alpha_k(m_0)}(\chi, u) . \]  

(35)

5 Bulk-boundary propagator on \( S^1 \times B^n \)

Let \( \Phi_0(\varphi, u) \) be a field on the boundary \( S^1 \times S^{n-1} \) of \( S^1 \times B^n \). Develop it into Fourier series of \( \varphi \) such that

\[ \Phi_0(\varphi, u) = \sum_{k=0}^{\infty} \left[ a_k(u) \cos k\varphi + b_k(u) \sin k\varphi \right] , \]  

where

\[ a_k(u) = \frac{1}{2\pi} \int_0^\infty \Phi_0(\varphi, u) \cos k\varphi \, d\varphi , \quad b_k(u) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_0(\varphi, u) \sin k\varphi \, d\varphi . \]  

(37)

Construct a scalar field \( \Phi(\theta, \chi) \) on \( S^1 \times B^n \) such that

\[ \Phi(\theta, \chi) = \sum_{k=0}^{\infty} \left[ \phi_k(\chi) \cos k\theta + \psi_k(\chi) \sin k\theta \right] , \]  

where

\[ \phi_k(\chi) = \frac{1}{\omega_{n-1}} \int_{uu' = 1} a_k(u) P_{\alpha_k(m_0)}(\chi, u) \, du , \]

\[ \psi_k(\chi) = \frac{1}{2\pi} \int_{uu' = 1} b_k(u) P_{\alpha_k(m_0)}(\chi, u) \, du . \]  

(39)

Since

\[ \Delta \phi_k(\chi) = -(k^2 - m_0^2)\phi_k(\chi) \quad \text{and} \quad \Delta \psi_k(\chi) = -(k^2 - m_0^2)\psi_k(\chi) , \]  

then \( \Phi(\theta, \chi) \) must satisfy the equation
\[ \Box \Phi(\theta, \chi) = -m_0^2 \Phi(\theta, \chi) . \] (41)

Finally, \( \Phi(\theta, \chi) \) can be expressed into the form

\[
\Phi(\theta, \chi) = \frac{1}{2 \pi \omega_{n-1}} \sum_{k=0}^{\infty} \int_{u'u' = 1} [a_k(u) \cos k\theta + b_k(u) \sin k\theta] P_{\alpha_k(m_0)}(\chi, u) \dot{u} \\
= \frac{1}{2 \pi \omega_{n-1}} \sum_{k=0}^{\infty} \int_{u'u' = 1} \int_0^{2\pi} [\Phi_0(\varphi, u) \cos k\varphi \cos k\theta + \Phi_0(\varphi, u) \sin k\varphi \sin k\theta] P_{\alpha_k(m_0)}(\chi, u) d\varphi \dot{u} ,
\] (42)

or

\[
\Phi(\theta, \chi) = \frac{1}{V(S^1 \times S^{n-1})} \int_{S^1 \times S^{n-1}} \Phi_0(\varphi, u) G_{m_0}(\theta, \varphi, \chi, u) d\varphi \dot{u} ,
\] (43)

where

\[ G_{m_0}(\theta, \chi; \varphi, u) = \sum_{k=0}^{\infty} P_{\alpha_k(m_0)}(\chi, u) \cos k(\varphi - \theta) \] (44)

is the propagator.

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