COMPLETE CLASSIFICATION OF BICONSERVATIVE HYPERSURFACES WITH DIAGONALIZABLE SHAPE OPERATOR IN THE MINKOWSKI 4-SPACE

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Abstract
In this paper, we study biconservative hypersurfaces in the four dimensional Minkowski space $\mathbb{E}^4_1$. We give the complete explicit classification of biconservative hypersurfaces with diagonalizable shape operator in $\mathbb{E}^4_1$.

Keywords. Biharmonic submanifolds, Biconservative hypersurfaces, Minkowski space, Diagonalizable shape operator

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1 Introduction
Recall that a biharmonic map $\phi : (M^n, g) \rightarrow (N^m, \langle, \rangle)$ between Riemannian manifolds is a critical point of the bienergy functional

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 v_g,$$

where $\tau(\phi) = \text{trace} \nabla d\phi$ is the tension field of $\phi$. For a biharmonic map, the bitension field satisfies the following associated Euler-Lagrange equation

$$\tau_2(\phi) = -\Delta \tau(\phi) - \text{trace} R^N(d\phi, \tau(\phi))d\phi = 0,$$

where $R^N$ is the curvature tensor of $N$.

If the isometric immersion $\phi$ is a biharmonic map, then $M^n$ is called a biharmonic submanifold of $N^m$. In last years, the research on biharmonic maps

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and biharmonic submanifolds is quite active, cf. [1-5, 9-12, 16-18, 24-26]. In particular, there is a long standing biharmonic conjecture, posed by B. Y. Chen in 1991, that every biharmonic submanifolds in a Euclidean space is minimal. The conjecture is still open so far, see Chen’s book [3] for recent progress.

For an isometric immersion $\phi$, the stress-energy tensor for the bienergy is defined as (see [6])

$$S_2(X, Y) = \frac{1}{2} |\tau(\phi)|^2 \langle X, Y \rangle + \langle d\phi, \nabla \tau(\phi) \rangle \langle X, Y \rangle$$

$$- \langle d\phi(X), \nabla_Y \tau(\phi) \rangle - \langle d\phi(Y), \nabla_X \tau(\phi) \rangle,$$

which satisfies

$$\text{div} S_2 = -\tau_2(\phi)^\top.$$ (1)

An immersion (or a submanifold) is called biconservative if $\text{div} S_2 = 0$ (see [6] for details).

Note that, for an isometric immersion $\phi$, the formula (1) means that the condition $\text{div} S_2 = 0$ is equivalent to the vanishing tangent part of the corresponding bitension field, i.e., $\tau_2(\phi)^\top = 0$. Hence, the notion of biconservative submanifolds is a natural generalization of biharmonic submanifolds.

The study of biconservative submanifolds has recently received much attention. Caddeo et al. classified biconservative surfaces in the three-dimensional Riemannian space forms, [6]. Hasanis and Vlachos classified biconservative hypersurfaces in the Euclidean spaces $\mathbb{E}^3$ and $\mathbb{E}^4$ in [10], where the authors called biconservative hypersurfaces as $H$-hypersurfaces. Chen and Munteanu [10] showed that a $\delta(2)$-ideal biconservative hypersurface in Euclidean space $\mathbb{E}^n$ is either minimal or open part of a spherical hypercylinder. By using the framework of equivariant differential geometry, Montaldo, Oniciuc and Ratto [21] studied $SO(p + 1) \times SO(q + 1)$-invariant and $SO(p + 1)$-invariant biconservative hypersurfaces in Euclidean space. Most recently, the second author obtained the complete classification of biconservative hypersurfaces with three distinct principal curvatures in Euclidean spaces, [27].

In the case of codimension greater than one, the situation is more difficult without any additional assumptions just as the biharmonic case. Montaldo et al. [22] studied biconservative surfaces in Riemannian manifolds. In particular, they gave a complete classification of biconservative surfaces with constant mean curvature in Euclidean 4-space. Very recently, Fetcu et al. classified biconservative surfaces with parallel mean curvature vector field in product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ in [13].

The notion of biconservative submanifolds was also considered in the context of pseudo-Riemannian geometry. The first author in [14] and [15] classified biconservative surfaces in the 3-dimensional Lorentzian space forms.

In this paper, we focus on biconservative hypersurfaces in Minkowski space $\mathbb{E}^4_1$. For hypersurfaces in Minkowski space, the shape operator can be decomposed into four canonical forms, see [23]. We give the complete explicit classification of biconservative hypersurfaces with diagonalizable shape operator in $\mathbb{E}^4_1$. 

2
It should be remarked that, just as the case of biharmonic submanifolds, the geometry of biconservative submanifolds in pseudo-Riemannian space is quite different from the Riemannian case. There are more examples of biconservative submanifolds appearing in the classification results, see Theorem 1 and Theorem 2.

2 Preliminaries

Let $E^m_t$ denote the pseudo-Euclidean $m$-space with the canonical pseudo-Euclidean metric tensor of index $t$ given by

$$g = \langle , \rangle = -\sum_{i=1}^{t} dx_i^2 + \sum_{j=t+1}^{m} dx_j^2.$$

We put

$$S^{m-1}_t(r^2) = \{x \in E^m_t : \langle x, x \rangle = r^{-2}\},$$

$$H^{m-1}_{t-1}(-r^2) = \{x \in E^m_t : \langle x, x \rangle = -r^{-2}\}.$$

Consider an oriented hypersurface $M$ of the Minkowski space $E^{n+1}_t$ with the unit normal vector field $N$ associated with the orientation. We denote Levi-Civita connections of $E^{n+1}_t$ and $M$ by $\tilde{\nabla}$ and $\nabla$, respectively and let $\nabla^\perp$ stand for the normal connection of $M$. Then, the Gauss and Weingarten formulas are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\tilde{\nabla}_X N = -SN$$

for all tangent vectors fields $X$, $Y$, where $h$ and $S$ are the second fundamental form and the shape operator of $M$, respectively. The Gauss and Codazzi equations are given, respectively, by

$$\langle R(X,Y)Z,W \rangle = \langle h(Y,Z), h(X,W) \rangle - \langle h(X,Z), h(Y,W) \rangle,$$

(2)

$$\langle \tilde{\nabla}_X h(Y,Z) \rangle = \langle \tilde{\nabla}_Y h(X,Z) \rangle,$$

(3)

where $R$ is the curvature tensor associated with the connection $\nabla$ and $\tilde{\nabla}h$ is defined by

$$\langle \tilde{\nabla}_X h(Y,Z) \rangle = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

$M$ is said to be biconservative if its shape operator $S$ and mean curvature $H = \text{tr}S$ satisfy

$$S(\nabla H) + \varepsilon H = 0,$$

(BC)

where $\varepsilon = \langle N, N \rangle$, i.e.,

$$\varepsilon = \begin{cases} -1 & \text{if } M \text{ is Riemannian} \\ 1 & \text{if } M \text{ is Lorentzian} \end{cases}.$$
Note that the biconservative condition \( \text{BC} \) follows directly from \( \tau_2(\phi)^\top = 0 \) as we described in Introduction, see \( \text{[6]} \).

**Remark 1.** The shape operator of a hypersurface with constant mean curvature satisfies \( \text{BC} \) trivially. Therefore, throughout this work we will assume that \( \nabla H \) does not vanish on \( M \).

## 3 Biconservative Hypersurfaces

Let \( M \) be an oriented hypersurface with the diagonalizable shape operator \( S \) in \( \mathbb{E}^4_1 \). Consider an orthonormal frame field \( \{e_1, e_2, e_3\} \) of \( M \) consisting of its principal directions and let \( \{\theta_1, \theta_2, \theta_3\} \) be the dual base field and \( k_1, k_2, k_3 \) corresponding principal directions. Then, we have \( k_1 + k_2 + k_3 = 3H \).

Now, assume that \( M \) is biconservative, i.e., \( S \) and \( H \) satisfy \( \text{BC} \) for \( n = 3 \). Thus, we have \( \nabla H \) is a principal direction with the corresponding principal curvature proportional to \( H \) by a constant. Therefore, we may assume \( e_1 = \nabla H / |\nabla H| \) and \( k_1 = -\varepsilon_2 H \). Since \( e_1 \) is proportional to \( \nabla k_1 \), we have \( e_2(k_1) = e_3(k_1) = 0 \), \( e_1(k_1) \neq 0 \).

In addition, similar to biconservative hypersurfaces in Euclidean spaces, connection forms of \( M \) satisfy

\[
\omega_{12}(e_1) = \omega_{12}(e_3) = \omega_{13}(e_1) = \omega_{13}(e_2) = 0,
\]

and

\[
\omega_{23}(e_1) = 0, \quad \text{if} \ k_2 \neq k_3
\]

(see \( \text{[16, 27]} \)).

Let \( D \) be the two-dimensional distribution given by

\[
D(m) = \text{span}\{e_2|_m, e_3|_m\}.
\]

**Remark 2.** Since \( \text{BC} \) implies \( [e_2, e_3](k_1) = 0 \) and \( e_1 \) is proportional to \( \nabla k_1 \), we have \( [e_2, e_3], e_1) = 0 \) which gives \( [e_2, e_3]|_m \in D(m) \). Therefore, \( D \) is involutive.

First, we obtain the following lemma.

**Lemma 1.** Let \( M \) be a biconservative hypersurface in \( \mathbb{E}^4_1 \) with the diagonalizable shape operator. Then, its principal curvatures satisfy

\[
e_i(k_2) = e_i(k_3) = 0, \quad i = 2, 3.
\]

**Remark 3.** By combining \( \text{BC} \) with Cartan’s first structural equation one can obtain \( d\theta_1 = 0 \), i.e., \( \theta_1 \) is closed. The Poincaré Lemma implies that it is exact, i.e., there exists a local coordinate system \( (s, \hat{t}, \hat{u}) \) on a neighborhood of \( m \in M \) such that \( \theta_1 = ds \) from which we obtain \( e_1 = \frac{\partial}{\partial s} \). Thus, we have \( k_1 = k_1(s) \), \( k_i = k_i(s, \hat{t}, \hat{u}) \), \( i = 2, 3 \). Since \( k_1'(s) \neq 0 \) because of \( \text{BC} \), the inverse function theorem implies \( s = s(k_1) \) on a neighborhood \( N_m \) of \( m \) in \( M \) and we have \( k_i = k_i(k_1, t, u) \). We will prove \( k_i = k_i(k_1) \) on \( N_m \).
Remark 4. Note that, since a further computation yields
\[ e_i e_i^n(k_1) = 0, \quad i = 2, 3, \quad n \in \mathbb{N}, \]
we have \( e_i^n(k_1) = f_n(k_1) \) for a smooth function \( f_n \) on \( N_m \), where \( e_i^n = e_1e_1 \ldots e_1 \) \( n \)-times.

Proof. If \( k_2 = k_3 \) proof directly follows from the Codazzi equation \( 3 \) for \( X = e_2, Y = e_3, Z = e_2 \) and \( X = e_3, Y = e_2, Z = e_3 \). Thus, will assume that \( k_2 - k_3 \) does not vanish on \( M \). We have two cases subject to being Riemannian or Lorentzian of \( M \).

Case I. \( M \) is Riemannian. In this case, we have \( \varepsilon_1 = -1 \) and \( BC \) gives
\[ k_1 = k_2 + k_3. \] (8)

The Codazzi equation \( 3 \) for \( X = e_1, Y = Z = e_i \) implies
\[ e_1(k_i) = \omega_i(k_1 - k_i). \]

In addition, by combining \( 5 \) with the Gauss equation \( R(e_i, e_1, e_1, e_i) = k_1k_i \) we have
\[ e_1(\omega_i) = -\omega_i^2 - k_1k_i, \]
where we put \( \omega_i = \omega_i(e_i) \). We apply \( e_1 \) 3 times to \( 8 \) to obtain
\[ e_1(k_1) = (k_1 - k_2)\omega_2 + (k_1 - k_3)\omega_3, \] (9a)
\[ e_1^3(k_1) + 2k_1k_2k_3 = (\omega_2 + \omega_3) e_1(k_1) - 2 (k_1 - k_2) \omega_2^2 - 2 (k_1 - k_3) \omega_3^2, \] (9b)
and
\[ (2k_2k_3 + k_1^2) e_1(k_1) + e_1^3(k_1) = 6(k_1 - k_2)\omega_2^3 + 6(k_1 - k_3)\omega_3^3 - 3e_1(k_1)(\omega_2^2\omega_3) \]
\[ + (e_1^3(1) + 2k_1 (k_1 - k_2)(2k_2 - k_3))\omega_2 \]
\[ + (e_1^3(k_1) + 2k_1 (k_3 - k_1)(k_2 - 2k_3))\omega_3. \] (9c)

Note that from \( 8 \) and \( 9a \) we get
\[ \omega_3 = \frac{e_1(k_1) - (k_1 - k_2)\omega_2}{k_2}. \] (10)

Next, we use \( 8 \) and \( 10 \) on \( 9b \) and \( 9c \) to get
\[ A_1\omega_2^3 + A_2\omega_2^2 + A_3\omega_2 + A_4 = 0, \] (11)
\[ B_1\omega_2^2 + B_2\omega_2 + B_3 = 0, \] (12)
where \( A_j \) and \( B_j \) are the functions given by

\[
\begin{align*}
A_1 &= 6k_1 (k_1 - 2k_2) (k_1 - k_2), \\
A_2 &= -3 (5k_1^2 - 10k_2k_1 + 4k_2^2) e_1(k_1), \\
A_3 &= 6 (k_1 - k_2) (2e_1(k_1)^2 + k_1 (k_1 - 2k_2) k_2^2) - (k_1 - 2k_2) k_2^2 e_1^2(k_1), \\
A_4 &= - e_1(k_1) (3e_1(k_1)^2 + k_2 (e_1^2(k_1) + 2k_2^3 - 8k_1k_2^2 + 3k_1^2k_2)) + k_2^2 e_1^3(k_1), \\
B_1 &= 2k_1 (k_2 - k_1), \\
B_2 &= (3k_1 - 2k_2) e_1(k_1), \\
B_3 &= - e_1(k_1)^2 - k_2 (e_1^2(k_1) + 2k_1k_2(k_1 - k_2)).
\end{align*}
\]

Finally, we eliminate \( \omega_2 \) from (11) and (12) to get

\[
\begin{align*}
4A_1B_1 & \left( 2A_3B_1 \left( \delta^2 - B_2^2 \right) + B_2 \left( 4A_4B_1^2 \left( B_2^2 + 3\delta \right) + A_2 \left( \delta - B_2^2 \right)^2 \right) \right) \\
+ A_2^2 \left( \delta - B_2^2 \right)^3 - 4B_1^2 \left( 4B_1^2 \left( A_3^2 \left( B_2^2 - \delta \right) - 4A_4A_3B_1B_2 + 4A_4^2B_2^2 \right) + 4A_4B_1 \left( A_3B_2 \left( \delta - B_2^2 \right) + 2A_4B_1 \left( B_2^2 + \delta \right) \right) + A_2^2 \left( \delta - B_2^2 \right)^2 \right),
\end{align*}
\]

where \( \delta = B_2^2 - 4B_1B_3 \). Next, we put \( A_i, B_i \) into the equation above to obtain a 14th degree polynomial

\[
\sum_{j=0}^{14} P_j(k_1, e_1(k_1), e_1^2(k_1), e_1^3(k_1))k_2^j = 0
\]

with the starting term \( P_{14} = -16384k_1^2 e_1(k_1) \). However, Remark \( \text{[8]} \) implies

\[
P_j \left( k_1, e_1(k_1), e_1^2(k_1), e_1^3(k_1) \right) = Q_j(k_1)
\]

for a function \( Q_j \). Therefore, we have

\[
\sum_{j=0}^{14} Q_j(k_1)k_2^j = 0.
\]

Thus, \( k_2 \) is depending on only \( k_1 \). Moreover, \( \text{[8]} \) implies that \( k_3 \) is also depending on only \( k_1 \).

Case II. \( M \) is Lorentzian. In this case, we have \( \varepsilon_1 = -1 \) and \( \text{[B/C]} \) gives

\[
-3k_1 = k_2 + k_3.
\]  

(14)

By a similar way, we obtain

\[
\sum_{j=0}^{14} \tilde{Q}_j(k_1)k_2^j = 0
\]

for some functions \( \tilde{Q}_j \). Thus, \( k_2 \) and \( k_3 \) are depending on only \( k_1 \). Hence the proof is completed. \( \square \)
Now, we have $e_i(k_j) = 0$ which implies $e_i e_1(k_j), \ i, j = 2, 3$. Therefore, the Codazzi equation $e_1(k_i) = \omega_{1i}(e_i)(k_1 - k_i)$ implies
\[ e_i(\omega_{1j}(e_j)) = 0. \] (15)

Hence, $\omega_{12}(e_2), \omega_{13}(e_3)$ are constant on any integral submanifold $\tilde{M}$ of the distribution $D$ given by (4). Let $c_i, d_i$ are the constants given by
\[ d_1 = k_2|\tilde{M}, \ d_2 = k_3|\tilde{M}, \ c_1 = \omega_{12}(e_2)|\tilde{M}, \ c_2 = \omega_{13}(e_3)|\tilde{M} \] (16)
and consider the local orthonormal frame field $\{f_1, f_2; f_3, f_4\}$ consisting of restriction of vector fields $e_2, e_3, e_1, N$ to $\tilde{M}$, respectively. Then, we have

**Lemma 2.** $f_3$ and $f_4$ are parallel and the matrix representations of the shape operators $\hat{A}_{f_3}$ and $\hat{A}_{f_4}$ are
\[ \hat{A}_{f_3} = \text{diag}(c_1, c_2), \ \hat{A}_{f_4} = \text{diag}(d_1, d_2), \]
where $c_i, d_i$ are the constants given by (16).

Moreover, we have

**Corollary 1.** $\tilde{M}$ has parallel mean curvature vector in $E^4_1$.

In addition, if $M$ has three distinct principal curvatures, then by combining $e_i(k_2) = e_i(k_3) = 0$ with the Codazzi equation and taking into account (4), one can see that the connection form $\omega_{23}$ vanishes identically. Therefore, we have

**Lemma 3.** If $M$ is a biconservative hypersurface in $E^4_1$ with three real, distinct principal curvatures, then the Levi-Civita connection $\nabla$ of $M$ satisfies $\nabla f_j f_j = 0, \ i, j = 1, 2$. Consequently, $M$ is flat.

We have the following proposition (see also [20, Lemma 4.2]).

**Proposition 1.** Let $M$ be a biconservative hypersurface with diagonalizable shape operator in $E^4_1$. Then, there exists a local coordinate system $(s, t, u)$ such that
\[ e_1 = \frac{\partial}{\partial s}, \ e_2 = \frac{1}{E_1} \frac{\partial}{\partial t}, \ e_3 = \frac{1}{E_2} \frac{\partial}{\partial u}. \] (17)

**Proof.** Let $D^\perp$ be the distribution given by $D^\perp(m) = \text{span}\{e_3|m\}$. Since $D^\perp$ and $D$ are involutive and $D(m) \oplus D^\perp(m) = T_m M$, by using [19, Lemma in page 182], we see that there is a local coordinate system $(\hat{s}, t, u)$ on $M$ such that $e_1$ is proportional to $\partial_{\hat{s}}$ and $e_2 = \frac{1}{E_1} \partial_{\hat{t}}$, $e_3 = \frac{1}{E_2} \partial_{u}$. Let $(s, \hat{t}, \hat{u})$ be the coordinate system given in Remark 3. Then, the local coordinate system $(s, t, u)$ satisfies the condition given in the proposition.

Next, we obtain a local parametrization of biconservative hypersurfaces.
Proposition 2. Let $M$ be a biconservative hypersurface with diagonalizable shape operator in $\mathbb{E}^4_1$. If $M$ has two distinct principal curvatures, then it has a local parametrization

$$x(s, t, u) = \phi(s)\Theta(t, u) + \Gamma(s)$$

for some vector valued functions $\Theta, \Gamma$ and a function $\phi$. On the other hand, if $M$ has three distinct principal curvatures, then $M$ has a local parametrization

$$x(s, t, u) = \phi_1(s)\Theta_1(t) + \phi_2(s)\Theta_2(u) + \Gamma(s)$$

for some vector valued functions $\Theta_1, \Theta_2, \Gamma$ and functions $\phi_1, \phi_2$.

Proof. Because of (15), we have $\omega_{12}(e_2) = \alpha(s)$, $\omega_{13}(e_3) = \beta(s)$. Therefore, (5) implies

$$\tilde{\nabla}_e e_1 = \alpha(s)e_2, \tilde{\nabla}_e e_1 = \beta(s)e_3.$$ (20)

Let $x$ be the position vector of $M$ and $(s, t, u)$ the coordinate system given in Proposition 1. If $M$ has two distinct principal curvatures, then we have $k_2 = k_3$ which implies $\alpha = \beta$. Therefore, from (17) and (20) we have $x_{st} = \alpha(s)x_t, x_{su} = \alpha(s)x_u$. By integrating these equations, we obtain (18).

Now, suppose that $M$ has three distinct principal curvatures. Then, from (17) and (20) we have

$$x_{st} = \alpha(s)x_t, \quad x_{su} = \beta(s)x_u.$$

By integrating these equations, we obtain (19).

Lemma 4. Let $M$ be a biconservative hypersurface in $\mathbb{E}^4_1$ with the diagonalizable shape operator and $\hat{M}$ the integral submanifold of the distribution $D$ given by (7) passing through $m \in M$. Then, if $M$ has three distinct principal curvatures, then $\hat{M}$ is congruent to one of the surfaces given by

(i) A Riemannian surface lying on a Euclidean hyperplane of $\mathbb{E}^4_1$ given by

$$y(t, u) = (1, t, B \cos u, B \sin u);$$

(ii) A Riemannian surface lying on a Lorentzian hyperplane of $\mathbb{E}^4_1$ given by

$$y(t, u) = (\text{Asinh}, \text{Asinh}, u, 1);$$

(iii) A Lorentzian surface lying on a Lorentzian hyperplane of $\mathbb{E}^4_1$ given by

$$y(t, u) = (t, B \cos u, B \sin u, 1);$$

(iv) A Lorentzian surface lying on a Lorentzian hyperplane of $\mathbb{E}^4_1$ given by

$$y(t, u) = (\text{Asinh}, \text{Acosh}, u, 1);$$

8
(v) A Riemannian torus $\mathbb{H}^1(-A^2) \times S^1(B^2)$ given by
$$y(t, u) = (\cosh t, \sinh t, B\cos u, B\sin u);$$
(vi) A Lorentzian torus $S^1_1(A^2) \times S^1(B^2)$ given by
$$y(t, u) = (\sinh t, \cosh t, B\cos u, B\sin u);$$
(vii) A Riemannian surface lying on a degenerated hyperplane of $E^4_1$ given by
$$y(t, u) = (A t^2 + B u^2, t, u, A t^2 + B u^2).$$

On the other hand, if $M$ has two distinct principal curvatures, then $\hat{M}$ is congruent to one of the surfaces given by
(viii) A sphere $S^2_2(r^2) \subset E^3 \subset E^4_1$;
(ix) de Sitter space $S^2_1_1(r^2) \subset E^3_1 \subset E^4_1$;
(x) anti-de Sitter space $\mathbb{H}^2_2(-r^2) \subset E^3_1 \subset E^4_1$;
(xi) The flat marginally trapped surface $y(t, u) = (A (t^2 + u^2), t, u, A (t^2 + u^2)).$

Proof. Let $\hat{M}$ be an integral submanifold of the distribution $D$ and $y$ the position vector of $\hat{M}$. Consider the local orthonormal frame field $\{f_1, f_2; f_3, f_4\}$ on $M$ given before the Lemma 2. We study the cases $k_2 \neq k_3$ and $k_2 = k_3$ separately.

Case 1. First, assume that $M$ has three distinct principal curvatures, i.e., $k_2 \neq k_3$. Without loss of generality, we may assume $\epsilon_2 = \epsilon_4 = 1$ which gives $\epsilon_1\epsilon_3 = -1$. Then, we have
\begin{align*}
\nabla_{f_1} f_1 &= -c_1 f_3 + \epsilon_1 d_1 f_4, \quad \nabla_{f_1} f_2 = \nabla_{f_2} f_1 = 0, \quad \nabla_{f_2} f_2 = \epsilon_3 c_2 f_3 + d_2 f_4, \\
\nabla_{f_1} f_3 &= -c_1 f_1, \quad \nabla_{f_1} f_4 = -d_1 f_1, \quad \nabla_{f_2} f_3 = -c_2 f_2, \quad \nabla_{f_2} f_4 = -d_2 f_2
\end{align*}
(21)
because of Lemma 3. Since $\hat{M}$ is flat and $\nabla_{f_2} f_2 = 0$, there exists a local coordinate system $(t, u)$ such that $g = \epsilon_1 dt^2 + du^2$, $f_1 = \partial_t$ and $f_2 = \partial_u$, where $\nabla$ is the Levi-Civita connection of $\hat{M}$. Thus, $\nabla_{f_2} f_1 = 0$ implies
$$y(t, u) = \alpha(t) + \beta(u)$$
(22)
for some smooth vector valued functions $\alpha, \beta$. From (21) and (22), we obtain
\begin{align*}
\alpha''' &= (c_1^2 - \epsilon_1 d_1^2)\alpha', \\
\beta''' &= -(\epsilon_3 c_2^2 + d_2^2)\beta'.
\end{align*}
(23a, b)
Moreover, since $\hat{M}$ is flat, we have
$$\epsilon_1 d_1 d_2 - c_1 c_2 = 0.$$
(24)
Case 1a. \( \varepsilon_1 = -1 \), i.e., \( \hat{M} \) is Lorentzian. In this case, (23) implies
\[
\alpha'' = \nu^2 \alpha', \quad (25a) \\
\beta'' = -\mu^2 \beta' \quad (25b)
\]
for some positive constants \( \nu, \mu \). Since \( \nu = \mu = 0 \) implies that \( \hat{M} \) is a plane which yields a contradiction, we have \( \nu^2 + \mu^2 \neq 0 \). Thus, if \( \nu = 0 \), then \( \mu \neq 0 \).

In this case solving (25) yields that
\[
y(t, u) = t^2 \eta_1 + t \eta_2 + \cos(\mu u) \eta_3 + \sin(\mu u) \eta_4
\]
for some constant vectors \( \eta_1, \eta_2, \eta_3, \eta_4 \). By considering \( g = -dt^2 + du^2 \), we obtain the case (iii) of the lemma. Similarly, the other possible subcases \( \mu = 0 \), \( \nu \neq 0 \) and \( \mu \nu \neq 0 \) give the case (iv) and the case (vi), respectively.

Case 1b. \( \varepsilon_1 = 1 \), i.e., \( \hat{M} \) is Riemannian. In this case, (23) implies
\[
\alpha'' = (c_1^2 - d_1^2) \alpha', \quad (26a) \\
\beta'' = (c_2^2 - d_2^2) \beta'. \quad (26b)
\]
By taking into account (24), we see that, without loss of generality, we have four cases.
\[
c_1^2 - d_1^2 = \nu^2, \quad c_2^2 - d_2^2 = -\mu^2; \quad c_1 = d_1 \neq 0, \quad c_2 = d_2 = 0; \quad c_1 = d_1 = 0, \quad c_2^2 - d_2^2 = \nu^2; \quad c_1 = d_1 = 0, \quad c_2^2 - d_2^2 = -\mu^2.
\]

By integrating (26) for each cases separately, we see that \( \hat{M} \) is congruent to one of the following surfaces.
\[
y(t, u) = \cosh(\nu t) \eta_1 + \sinh(\nu t) \eta_2 + \cos(\mu u) \eta_3 + \sin(\mu u) \eta_4 \quad (27a) \\
y(t, u) = t^2 \eta_1 + t(\nu t) \eta_2 + u^2 \eta_3 + u \eta_4, \quad (27b) \\
y(t, u) = \cosh(\nu t) \eta_1 + \sinh(\nu t) \eta_2 + u^2 \eta_3 + u \eta_4 \quad (27c) \\
y(t, u) = t^2 \eta_1 + t \eta_2 + \cos(\mu u) \eta_3 + \sin(\mu u) \eta_4 \quad (27d)
\]
for some constant vectors \( \eta_1, \eta_2, \eta_3, \eta_4 \). By a direct computation using \( g = dt^2 + du^2 \), we obtain the case (v), (vii), (ii) and (i) of the lemma, respectively.

Case 2. Next, we assume that \( \hat{M} \) is a biconservative hypersurface with two distinct principal curvatures. Then, the shape operators of \( \hat{M} \) becomes \( A_3 = c_1 I, \ A_4 = d_1 I \) by the Lemma. Thus, \( \hat{M} \) lies on a hyperplane \( \Pi \) of \( M \) whose normal is the constant vector \( \eta = \varepsilon_3 e_1 e_3 - \varepsilon_4 c_1 e_4 \).

If \( \Pi \) is non-degenerated, then \( \hat{M} \) is isoparametric. Thus, we have the case (viii) or cases (ix), (x) subsect to being Euclidean or non-Euclidean of \( \Pi \), respectively.

Now, suppose that \( \Pi \) is degenerated, i.e., \( \eta \) is light-like. Then, we have \( c_1 = d_1 \). In addition, up to congruence, we may assume \( \Pi = \{ (A, B, C, A)| A, B, C \in \mathbb{R} \} \). Thus, \( M \) has a parametrization \( (f(t, u), t, u, f(t, u)) \). Since \( A_3 = A_4 = c_1 I \), we have the case (xi) of the lemma.

\[\square\]
3.1 Biconservative hypersurfaces with two principal curvatures

In this section, we would like to deal with the biconservative hypersurfaces with two distinct principal curvatures.

Theorem 1. Let $M$ be a hypersurface in $\mathbb{E}^4_1$ with diagonalizable shape operator and two distinct principal curvatures. If $M$ is biconservative, then it is congruent to one of hypersurfaces

$$x_1(s, t, u) = (f_1(s), s \cos t \sin u, s \sin t \sin u, s \cos u), \quad (28a)$$

$$x_2(s, t, u) = (ssinhu \sin t, s \cosh u \sin t, s \cos t, f_2(s)), \quad (28b)$$

$$x_3(s, t, u) = (s \cosh t, s \sinh u \sin t, \sinh u \cos t, f_3(s)), \quad (28c)$$

$$x_4(s, t, u) = \left(\frac{1}{2}s(t^2 + u^2) + s + f_4(s), st, su, \frac{1}{2}s(t^2 + u^2) + f_4(s)\right) \quad (28d)$$

for some smooth functions $f_1, f_2, f_3, f_4$.

Proof. Let $M$ be a biconservative hypersurface in $\mathbb{E}^4_1$ with the parametrization given by (18) for some vector valued functions $\Theta, \Gamma$ and a function $\phi$. Now, consider the slice $\hat{M}$ of $M$ given by $s = s_0$ passing through $m = x(s_0, t_0, u_0)$. Obviously, it is an integral submanifold of the distribution $D$ given by (7). Then, $\hat{M}$ is one of four surfaces given in Case (viii)-(xi) of Lemma 4.

First, assume that $\hat{M}$ is a sphere. In this case, up to isometries of $\mathbb{E}^4_1$, we may assume the position vector of $\hat{M}$ is $y(t, u) = x(s_0, t, u) = (A \cos t \sin u, A \sin t \sin u, A \cos u)$. Then, (18) implies

$$c_1 \Theta(t, u) + c_2 = (1, \cos t \sin u, \sin t \sin u, \cos u)$$

for a constant $c_1$ and constant vector $c_2$. By solving $\Theta$ from the above equation and using (18), we obtain $M$ is the hypersurface given by (28a).

Analogously, if $\hat{M} = S^2_1(r^2)$ or $\hat{M} = H^2_1(-r^2)$, we obtain $M$ is the hypersurface given by (28b) or (28c), respectively.

On the other hand, if $\hat{M}$ is congruent to the flat marginally trapped surface given in the Case (xi) of Lemma 4 then, up to isometries, we may assume

$$x(s_0, t, u) = y(t, u) = (A(t^2 + u^2), t, u, A(t^2 + u^2)).$$

By combining this equation and (18) we obtain

$$c_1 \Theta(t, u) + c_2 = (A(t^2 + u^2), t, u, A(t^2 + u^2)),$$

where $c_1 = \phi(s_0)$ and $c_2 = \Gamma(s_0)$. Therefore, we may assume

$$\Theta(t, u) = (A'(t^2 + u^2) + C_1, t + C_2, u + C_3, A'(t^2 + u^2) + C_4)$$

for some constant $A', C_i$. Next, we put this equation into (18) to get

$$x(s, t, u) = (\phi(s)(t^2 + u^2), t, u, \phi(s)(t^2 + u^2)) + \Gamma(s)$$

for some smooth functions $f_1, f_2, f_3, f_4$. 


for a smooth function \( \bar{\phi} \) and smooth vector valued function \( \bar{\Gamma} \). By taking into account that the vector fields \( \partial_s, \partial_t, \partial_u \) are orthonormal and re-defining the coordinate \( s \) properly, we obtain \( \phi(s) = \frac{1}{2}s \) and \( \bar{\Gamma}(s) = (s + f_4(s), 0, 0, f_4(s)) \). Therefore, we obtain the surface given by (28d). Hence, the proof is completed.

By the following proposition, we would like to prove the existence of biconservative hypersurfaces with two distinct curvatures.

**Proposition 3.** Let \( M \) be the hypersurface given by (28a) in \( E^4_1 \). Then, \( M \) is biconservative if and only if either \( M \) is Riemannian and

\[
f_1 = \int_{s_0}^s \frac{c_1 \xi^2}{\sqrt{c_1 \xi^4 - 1}} d\xi \tag{29}
\]

or it is Lorentzian and

\[
f_1 = \int_{s_0}^s \frac{c_1}{\sqrt{c_1^2 - \xi^4}} d\xi \tag{30}
\]

**Proof.** By a direct computation one can obtain that the principal directions of \( M \) are

\[
e_1 = \frac{1}{\sqrt{c_1(1 - f_1'^2)}} \partial_x, \quad e_2 = \frac{1}{s} \partial_t, \quad e_3 = \frac{1}{s} \partial_u \]

with the corresponding principal curvatures

\[
k_1 = -\frac{\varepsilon_1 f_1''}{\sqrt{\varepsilon_1(1 - f_1'^2)}}, \quad k_2 = k_3 = -\frac{f_1'}{s \sqrt{\varepsilon_1(1 - f_1'^2)}},
\]

where \( \varepsilon_1 = \langle e_1, e_1 \rangle \). Let \( M \) be a biconservative hypersurface, i.e., (BC) is satisfied. First, assume that \( M \) is Riemannian, i.e., \( \varepsilon = 1 \). Then, from (BC) we have \( k_1 = 2k_2 \) which implies

\[
\frac{f_1''}{f_1'(1 - f_1'^2)} = \frac{2}{s}
\]

whose general solution is (29).

Next, assume that \( M \) is Lorentzian, i.e., \( \varepsilon = -1 \). Then, (BC) implies \(-3k_1 = 2k_2\) from which we have

\[
\frac{-3f_1''}{f_1'(1 - f_1'^2)} = \frac{2}{s}.
\]

By solving this equation, we obtain (30).

Hence, the proof of necessary condition is completed. The converse follows from a direct computation.
3.2 Biconservative hypersurfaces with three principal curvatures

In this subsection we obtain the classification of biconservative hypersurfaces with three principal curvatures. First, we want to present an example by the following proposition.

Proposition 4. Let $M$ be a hypersurface in $\mathbb{E}^4_1$ given by

$$x(s,t,u) = \left(\frac{1}{2}s(t^2 + u^2) + au^2 + s + \phi(s), st, (s + 2a)u, \frac{1}{2}s(t^2 + u^2) + au^2 + \phi(s)\right), \quad a \neq 0.$$ \hspace{1cm} (31)

Then, $M$ is biconservative if and only if either $M$ is Riemannian and

$$\phi(s) = c_1\left(\ln(s + 2a) - \ln s - \frac{a}{s} - \frac{a}{s + 2a}\right) - \frac{s}{2}$$ \hspace{1cm} (32)

or it is Lorentzian and

$$\phi(s) = c_1\int_{s_0}^{s} (\xi(\xi + 2a))^2/3 d\xi - \frac{s}{2},$$ \hspace{1cm} (33)

where $c_1 \neq 0$ and $s_0$ are some constants.

Proof. By a direct computation, one can obtain that the principal directions of $M$ are

$$e_1 = \frac{1}{\sqrt{\epsilon_1(-2\phi' - 1)}}\partial_s, \quad e_2 = \frac{1}{s}\partial_t, \quad e_3 = \frac{1}{s + 2a}\partial_u$$

and the unit normal vector of $M$ is

$$N = \frac{1}{\sqrt{\epsilon_1(-2\phi' - 1)}}\left(\frac{t^2 + u^2}{2} - \phi', t, u, \frac{t^2 + u^2}{2} - \phi' - 1\right).$$

By a simple calculation, we have

$$\nabla_{e_1} e_1 = \zeta e_1 + \frac{\phi''}{\epsilon_1(-2\phi' - 1)}(1,0,0,1),$$

$$\nabla_{e_2} e_2 = \frac{1}{s}(1,0,0,1),$$

$$\nabla_{e_3} e_3 = \frac{1}{s + 2a}(1,0,0,1)$$ \hspace{1cm} (34)

for a smooth function $\zeta$.

Riemannian Case. If $M$ is Riemannian, then we have $\epsilon_1 = 1$. In this case, $M$ is biconservative if and only if equation $k_1 = k_2 + k_3$ is satisfied. Thus, we have

$$-\frac{\phi''}{(2\phi' + 1)} = \frac{1}{s} + \frac{1}{s + 2a}$$
whose general solution is
\[ \phi(s) = c_1 \left( \ln(s + 2a) - \ln \left( \frac{\alpha}{s} - \frac{\alpha}{s + 2a} \right) - \frac{s}{2} \right) + c_2, \]
where \(c_1, c_2\) are constants. Note that, up to congruency, we may assume \(c_2 = 0\). Thus, we have \(\Box\).

**Lorentzian Case.** If \(M\) is Lorentzian, then we have \(\varepsilon_1 = -1\) and \(M\) is biconservative if and only if \(-3k_1 = k_2 + k_3\) which implies
\[ \frac{3\phi''}{2\phi' + 1} = \frac{1}{s} + \frac{1}{s + 2a} \]
whose general solution is the function given in (33).

Next, we obtain the following classification theorem of biconservative hypersurfaces with three distinct principal curvatures.

**Theorem 2.** Let \(M\) be a hypersurface in \(E^4_1\) with diagonalizable shape operator and three distinct principal curvatures. Then \(M\) is biconservative if and only if it is congruent to one of hypersurfaces

(i) A generalized cylinder \(M^2_0 \times E^1_1\) where \(M\) is a biconservative surface in \(E^3_1\);

(ii) A generalized cylinder \(M^2_0 \times E^1_1\) where \(M\) is a biconservative Riemannian surface in \(E^3_1\);

(iii) A generalized cylinder \(M^2_1 \times E^1_1\), where \(M\) is a biconservative Lorentzian surface in \(E^3_1\);

(iv) A Riemannian surface given by
\[ x(s, t, u) = (scosh, ssinh, f_1(s) \cos u, f_1(s) \sin u) \] (35)
for a function \(f_1\) satisfying
\[ \frac{f_1''}{f_1'^2 - 1} = \frac{f_1'f_1' + s}{sf_1}; \] (36)

(v) A Lorentzian surface with the parametrization given in (35) for a function \(f_1\) satisfying
\[ -3f_1'' = \frac{f_1'f_1' + s}{sf_1}; \] (37)

(vi) A Riemannian surface given by
\[ x(s, t, u) = (ssinh, scosh, f_2(s) \cos u, f_2(s) \sin u) \] (38)
for a function \(f_2\) satisfying
\[ \frac{f_2''}{f_2'^2 + 1} = \frac{f_2'f_2' + s}{sf_2}. \]
(vii) A surface given in Proposition 4

Proof. Consider a biconservative hypersurface \( M \) with three distinct curvatures and assume that the functions \( k_1 - k_2, k_1 - k_3 \) and \( k_2 - k_3 \) are non-vanishing in \( M \). Let \( \hat{M} \) be the integral submanifold of the distribution \( D \) given by passing through \( m = x(s_0, t_0, u_0) \in M \), where \( x \) is the parametrization of \( M \) given in (19). Then, \( M \) is congruent to one of the surfaces given in case (i)-(vii) of Lemma 4.

Case 1. \( M \) is congruent to one of the surfaces given in the case (i)-(iv) of Lemma 4. In this case, by a direct computation one can see that one of the principal curvatures of \( M \) vanishes identically. Therefore, we have case (i)-(iii) of the theorem.

Case 2. Let \( \hat{M} \) be congruent to the surfaces given in case (v) of Lemma 4. Then, we may assume \( x(s_0, t, u) = \gamma(t, u) = (\text{Acosh} t, \text{Asinh} t, B \cos u, B \sin u) \).

By combining this equation and \( \Theta_{i}^{s} \), we have

\[
c_1 \Theta_1(t) + c_2(\Theta_2(u) + c_3 = (\text{Acosh} t, \text{Asinh} t, B \cos u, B \sin u),
\]

where \( c_1 = \phi_1(s_0), c_2 = \phi_2(s_0) \) and \( c_3 = \Gamma(s_0) \). By redefining \( \phi_1, \phi_2, \Gamma \) suitable, from (39) we obtain

\[
\Theta_1(t) = (\text{cosh} t, \text{sinh} t, 0, 0), \\
\Theta_2(u) = (0, 0, \cos u, \sin u).
\]

Therefore, (19) implies

\[
x(s, t, u) = (\phi_1(s) \text{cosh} t, \phi_1(s) \text{sinh} t, \phi_2(s) \cos u, \phi_2(s) \sin u) + \Gamma(s).
\]

By a further computation considering that \( \partial_x, \partial_t, \partial_u \), we see that \( \Gamma \) is a constant vector which can be assumed to be zero up to a suitable translation. By using the inverse function theorem, we assume \( \phi_1(s) = s \) and \( \phi_2(s) = f_1(s) \) for a smooth function \( f \). Hence, we have (36).

A direct computation shows that the principal curvatures of \( M \) are \( k_1 = \frac{\varepsilon f_1''}{\sqrt{\varepsilon (f_1^2 - 1)^3}}, \quad k_2 = \frac{f_1'}{s \sqrt{\varepsilon (f_1^2 - 1)}} \) and \( k_3 = \frac{1}{\sqrt{\varepsilon (f_1^2 - 1)}} \), where \( \varepsilon = 1 \) or \( \varepsilon = -1 \) if \( M \) is Riemannian or Lorentzian, respectively. Note that if \( M \) is Riemannian or Lorentzian, then from (36) we have \( k_1 + k_2 = k_3 \) or \( k_1 + k_2 = -3k_3 \), respectively. Thus, we have (36) or (37). Hence, we have obtained the case (iv) and the case (v) of the theorem.

Case 3. Let \( \hat{M} \) be congruent to the surfaces given in case (vi) of Lemma 4. By a similar way to previous case we obtain the case surface given by \( \hat{M} \) for a smooth function \( f_2 \). A direct computation shows that the principal curvatures of \( M \) are \( k_1 = \frac{f_2''}{\sqrt{(f_2^2 + 1)^3}}, \quad k_2 = \frac{f_2'}{s \sqrt{(f_2^2 + 1)}} \) and \( k_3 = \frac{1}{\sqrt{(f_2^2 + 1)}} \) and \( M \) is Riemannian. Since \( M \) is Riemannian, we have \( k_1 + k_2 = k_3 \) which gives the case (vi) of the theorem.
Case 4. Let $M$ be congruent to the surfaces given in case (vii) of Lemma 11. In this case, without loss of generality, we may assume $x(s_0, t, u) = y(t, u)$. Therefore, from (19) we have

$$\phi_1(s_0)\Theta_1(t) + \phi_2(s_0)\Theta_2(u) + \Gamma(s_0) = (At^2 + Bu^2, t, u, At^2 + Bu^2).$$

Thus, we get

$$\Theta_1(t) = \frac{1}{\phi_1(s_0)}(At^2 + c_1, t + c_2, c_3, At^2 + c_4), \quad (41a)$$

$$\Theta_2(u) = \frac{1}{\phi_2(s_0)}(Bu^2 + d_1, d_2, u + d_3, Bu^2 + d_4) \quad (41b)$$

for some non-zero constants $c_i, d_i$. By combining (41a) with (41b) we obtain

$$x(s, t, u) = (A\psi_1 t^2 + B\psi_2 u^2, \psi_1 t, \psi_2 u, A\psi_1 t^2 + B\psi_2 u^2) + \tilde{\Gamma}(s) \quad (42)$$

for a smooth vector valued function $\tilde{\Gamma} = (\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_4)$ and some smooth functions $\psi_1(s), \psi_2(s)$. However, $\langle x_s, x_s \rangle = \varepsilon_1, \langle x_s, x_t \rangle = \langle x_s, x_u \rangle = 0$ give

$$\psi_1' t^2 + \psi_2' u^2 + \langle \Gamma', \Gamma' \rangle + 2(A\psi_1 t^2 + B\psi_2 u^2)(\tilde{\Gamma}_1' - \tilde{\Gamma}_4') + \psi_1 \Gamma_2' t + \psi_2 \Gamma_3' u = \varepsilon_1,$$

$$\psi_1 t(\psi_1' + 2A(\tilde{\Gamma}_4' - \tilde{\Gamma}_1')) + \psi_1 \Gamma_2' = 0,$$

$$\psi_2 u(\psi_2' + 2B(\tilde{\Gamma}_4' - \tilde{\Gamma}_1')) + \psi_2 \Gamma_3' = 0.$$

Therefore, we have

$$(\tilde{\Gamma}_1 - \tilde{\Gamma}_4) \neq 0, \quad \psi_1 = 2A(\tilde{\Gamma}_1 - \tilde{\Gamma}_4) + a_1, \quad \psi_2' = 2B(\tilde{\Gamma}_1 - \tilde{\Gamma}_4) + a_2$$

for some constants $a_1, a_2$ and, up to a suitable translation, we may assume $\Gamma_2 = \Gamma_3 = 0$. Then, we obtain a parametrization of $M$ as

$$x(s, t, u) = (2(\tilde{\Gamma}_1 - \tilde{\Gamma}_4)(A^2t^2 + B^2u^2) + Aa_1 t^2 + B a_2 u^2 + \tilde{\Gamma}_1, 2A(\tilde{\Gamma}_1 - \tilde{\Gamma}_4)t + a_1 t, 2B(\tilde{\Gamma}_1 - \tilde{\Gamma}_4)u + a_2 u, 2(\tilde{\Gamma}_1 - \tilde{\Gamma}_4)(A^2t^2 + B^2u^2) + Aa_1 t^2 + B a_2 u^2 + \tilde{\Gamma}_4). \quad (44)$$

Note that if $AB = 0$, then by a direct computation one can see that one of the principal curvatures vanishes identically on $M$. In this subcase, we obtain

$$x(s, t, u) = \left(\frac{1}{2}st^2 + s + \phi_1, st, \frac{1}{2}st^2 + \phi_1, u\right)$$

which gives the case (ii) or case (iii) of the theorem. Thus, we assume $A \neq 0, B \neq 0$.

Next, we define new coordinates $(\tilde{s}, \tilde{t}, \tilde{u})$ such that $\tilde{s} = \tilde{\Gamma}_1 - \tilde{\Gamma}_4 + a_1/2A$, $\tilde{t} = 2At$, $\tilde{u} = 2Bu$. From (44) we obtain a parametrization of $M$ as given in (44) for a constant $a$ which is non-zero because $M$ has three distinct principal curvatures. Hence, we have the case (vii) of the theorem.

Hence, the proof of necessary condition is completed. The converse follows from a direct computation.

\[\square\]

Remark 5. For the explicit parametrization of hypersurfaces given in case (i)-(iii), see the complete classification of biconservative surfaces in $\mathbb{E}^3$ and $\mathbb{E}^3_1$ which are given in [10, 12] and [14].
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