HILBERT’S THEOREM 90 AND ALGEBRAIC SPACES

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Abstract. In modern form, Hilbert’s Theorem 90 tells us that $R^1\epsilon_*(\mathbb{G}_m) = 0$, where $\epsilon : X_{\text{ét}} \to X_{\text{zar}}$ is the canonical map between the étale site and the Zariski site of a scheme $X$. I construct examples showing that the corresponding statement for algebraic spaces does not hold.

Introduction

Originally, Hilbert’s Theorem 90 is the following number theoretical result [3]: Given a cyclic Galois extension $K \subset L$ of number fields, each $y \in L^\times$ of norm $N(y) = 1$ is of the form $y = x/x^\sigma$ for some $x \in K^\times$ and a given generator $\sigma \in G$ of the Galois group. More generally, Speiser [12] proved that $H^1(G, L^\times) = 1$ for arbitrary Galois extensions (compare the discussion in [8]).

The latter statement has a geometric interpretation: Each line bundle on the étale site of $\text{Spec}(k)$ is trivial. In this form, it admits a far-reaching generalization: If $\epsilon : X_{\text{ét}} \to X_{\text{zar}}$ is the canonical map from the étale site to the Zariski site of a scheme $X$, then $R^1\epsilon_*(\mathbb{G}_m) = 0$ (see [9], page 124). The result entails, among other things, that the map of Picard groups $\text{Pic}(X_{\text{zar}}) \to \text{Pic}(X_{\text{ét}})$ is bijective, and that the map of Brauer groups $\text{Br}(X_{\text{zar}}) \to \text{Br}(X_{\text{ét}})$ is injective.

It is natural to ask whether a similar statement holds for algebraic spaces instead of schemes. Recall that an algebraic space is the quotient $X = U/R$ of a scheme $X$ by an étale equivalence relation $R \rightrightarrows X$. Here the quotient takes place in the topos $(\text{Sch})_{\text{ét}}^\wedge$, that is, as a sheaf on the étale site.

Unfortunately, such a generalization does not hold. The goal of this paper is to construct counterexamples, that is, algebraic spaces $X$ and invertible $\mathcal{O}_X$-modules $\mathcal{L}$ such that the open subspaces $V \subset X$ trivializing $\mathcal{L}$ do not cover $X$. The first example is a nonseparated smooth 1-dimensional bug-eyed cover in Kollár’s sense [7]. The second example is a nonnormal proper algebraic space obtained by identifying points on suitable nonprojective smooth proper schemes.

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1. Line bundles on algebraic spaces

In this section we recall some basic facts on algebraic spaces and their line bundles. Let $(\text{Sch})_{\text{ét}}$ be the site of schemes endowed with the Grothendieck topology generated by the étale surjective morphisms, and $(\text{Sch})_{\text{ét}}^\wedge$ be the corresponding topos.
of sheaves. By definition, a sheaf $X \in \text{(Sch)}_{\text{et}}$ is an algebraic space if $X = U/R$ for some scheme $U$ and some étale equivalence relation $R \rightrightarrows U$ such that the induced morphism $R \to U \times U$ is quasicompact.

Given an algebraic space $X$, let $\text{Ét}(X)$ be the category of algebraic $X$-spaces whose structure map $Y \to X$ is étale. The étale surjections $Y_1 \to Y_2$ define a topology on $\text{Ét}(X)$, and we write $X_{\text{ét}}$ for the corresponding site. Let me give a down-to-earth description of sheaves $F$ on this site. For each scheme $U$ endowed with an étale map $U \to X$, we obtain via restriction a sheaf $F_U$ on the étale site of $X$-schemes. If $f : U \to V$ is an $X$-morphism, we have a map $\theta_f : F_V \to f_* F_U$. Such systems $(F_U, \theta_f)$ are not arbitrary. Consider the following two conditions: (1) If $f : U \to V$ is étale, then the map $\theta_f : F_V \to f_* F_U$ is bijective. Here the mapping $\theta$ corresponds to the structure sheaf $O$ and the covering families are the surjections of the form $\coprod U_i \to X$. (2) If $f : U \to V$ is an $X$-morphism, we have a map $\theta_f : F_V \to f_* F_U$. The assignment $\theta$ is commutative. (2) If $f : U \to V$ is étale, then the map $\theta_f : F_V \to F_U$ is bijective because $F$ is a sheaf in the étale topology, and condition (2) holds as well.

Conversely, given such a system, we define $\Gamma(U, F) = \Gamma(U, F_U)$. Indeed, this is a presheaf by condition (1), and a sheaf by condition (2). One easily checks that the functors $\Gamma(U, F) \mapsto (F_U, \theta_f)$ and $(F_U, \theta_f) \mapsto F$ are inverse equivalences of categories.

**Proposition 1.1.** The assignment $F \mapsto (F_U, \theta_f)$ yields an equivalence between the category of sheaves on $X_{\text{ét}}$ and the category of systems $(F_U, \theta_f)$ satisfying conditions (1) and (2).

**Proof.** Let $C$ be the site of étale $X$-schemes with the induced étale topology. By the Comparison Lemma (3), Exposé III, Théorème 4.1), the inclusion $C \subset X_{\text{ét}}$ induces an equivalence on the corresponding categories of sheaves. Now suppose $F$ is a sheaf on $C$. Then the system $(F_U, \theta_f)$ satisfies condition (1) because $F$ is a presheaf. If $f : U \to V$ is étale, then $\theta_f$ is bijective because $F$ is a sheaf in the étale topology, and condition (2) holds as well.

For example, the sheaves $O_U$, together with the maps $\theta_f : O_V \to f_*(O_U)$, correspond to the structure sheaf $O_X$ of an algebraic space $X$. Similar, we have the sheaf of units $O^*_X$. The cohomology group $\text{Pic}(X_{\text{ét}}) = H^1(X_{\text{ét}}, O^*_X)$ is the group of isomorphism classes of invertible $O_X$-modules.

Besides the étale topology, the category $\text{Ét}(X)$ carries the coarser Zariski topology as well. Here the covering families are the surjections of the form $\coprod X_i \to X$, where the $X_i \subset X$ are open subspaces, and we demand that $X_i \times_X X' \to X'$ remains an open embedding for any base change $X' \to X$. Write $X_{\text{zar}}$ for the corresponding site. The sheaves on $X_{\text{zar}}$ admit a similar description in terms of families $(F_U, \theta_f)$ satisfying condition (1), and condition (2'), where we demand that $\theta_f : F_V \to F_U$ is bijective whenever $f : U \to V$ is of the form $U = \coprod V_i$ with open subschemes $V_i \subset V$. In particular, we have a structure sheaf $O_{X_{\text{zar}}}$ and a unit sheaf $O^*_{X_{\text{zar}}}$. Let $\text{Pic}(X_{\text{zar}}) = H^1(X_{\text{zar}}, O^*_{X_{\text{zar}}})$ be the corresponding group of line bundles.
The identity functor on $\text{ét}(X)$ is a continuous functor $\epsilon : X_{\text{ét}} \to X_{\text{zar}}$ of sites, and we have $\epsilon_* (\mathcal{O}_{X_{\text{et}}}) = \mathcal{O}_{X_{\text{zar}}}$ by descent theory. So for each invertible $\mathcal{O}_{X_{\text{zar}}}$-module $\mathcal{L}$, the canonical map $\mathcal{L} \to \epsilon_* \epsilon^* \mathcal{L}$ is bijective, and we obtain an injection $\text{Pic}(X_{\text{zar}}) \subset \text{Pic}(X_{\text{ét}})$.

**Proposition 1.2.** Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Its isomorphism class lies in the subgroup $\text{Pic}(X_{\text{zar}}) \subset \text{Pic}(X_{\text{ét}})$ if and only if there is a covering with open subspaces $Y_i \subset Y$ with $\mathcal{L}_{Y_i} \cong \mathcal{O}_{Y_i}$.

**Proof.** The spectral sequence for the composition $\Gamma(X_{\text{ét}}, \mathcal{O}_{X_{\text{et}}}^*) = \Gamma(X_{\text{zar}}, \epsilon_* \mathcal{O}_{X_{\text{et}}}^*)$ yields an exact sequence

$$0 \to \text{Pic}(X_{\text{zar}}) \to \text{Pic}(X_{\text{ét}}) \to H^0(X_{\text{zar}}, R^1 \epsilon_* \mathcal{O}_{X_{\text{et}}}^*).$$

The condition precisely means that the image of the invertible sheaf $\mathcal{L}$ under the canonical map $\text{Pic}(X_{\text{ét}}) \to H^0(X_{\text{zar}}, R^1 \epsilon_* \mathcal{O}_{X_{\text{et}}}^*)$ vanishes. The statement now follows from the exact sequence. \qed

## 2. Bug-eyed covers

In this section, we use Kollár’s bug-eyed covers to construct a smooth 1-dimensional nonseparated algebraic space $X$ and an invertible sheaf $\mathcal{L}$ such that the open subspaces $W \subset X$ trivializing $\mathcal{L}$ do not form a covering.

Fix a ground field $k$ of characteristic $\neq 2$. Set $A = k[[T]]$ and $A' = k[[T^2]]$, and let $Y = \text{Spec}(A)$ and $Y' = \text{Spec}(A')$ be the corresponding affine schemes. The inclusion $A' \subset A$ defines a flat double covering $p : Y \to Y'$. The open subset $U \subset Y$ given by the generic point is the locus where $f$ is étale. The generator $\sigma \in G$ of the group $G = \mathbb{Z}/2\mathbb{Z}$ acts on $A$ via $T^\sigma = -T$, which defines a free $G$-action on $U$.

Consider the étale equivalence relation

$$R = \Delta_Y \amalg U \to Y \times Y,$$

where the embedding of $U$ is given by $U \xrightarrow{id \times \sigma} U \times U \subset Y \times Y$. Let $X = Y/R$ be the corresponding quotient sheaf in $(\text{Sch}/k)_{\text{ét}}$. By definition, $X$ is a smooth algebraic space. It is nonseparated because the injection $R \to Y \times Y$ is not closed.

The map $p : Y \to Y'$ factors over $X$, and the induced projection $X \to Y'$ induces a bijection of points. The algebraic space $X$ is a bug-eyed cover in Kollár’s sense \[. It is not a scheme. Otherwise, the morphism $X \to Y'$ would be an isomorphism by Zariski’s Main Theorem, and $Y \to X$ would be both étale and ramified.

**Proposition 2.1.** We have $\text{Pic}(X_{\text{ét}}) = \mathbb{Z}/2\mathbb{Z}$.

**Proof.** The scheme $Y$ is local, hence every invertible $\mathcal{O}_X$-module $\mathcal{L}$ has $\mathcal{L}_Y \cong \mathcal{O}_Y$. Thus, $\text{Pic}(X_{\text{ét}})$ is the cohomology of the complex

$$\Gamma(Y, \mathcal{O}_X^*) \xrightarrow{d_0} \Gamma(Y^2, \mathcal{O}_X^*) \xrightarrow{d_1} \Gamma(Y^3, \mathcal{O}_X^*).$$

Here $Y^n$ are the $n$-fold fiber products over $X$. If $p_i : Y^{n+1} \to Y^n$ denotes the projection omitting the $i$-th factor, the differentials are $d_0(s) = p_0^*(s)/p_1^*(s)$ and $d_1(s) = p_0^*(s)p_2^*(s)/p_1^*(s)$.

Clearly, we have $Y^n = U^n \cup \Delta_Y$, where $U^n \cap \Delta_Y = \Delta_U$. Since the $G$-action is free on the open subset $U \subset Y$, we have a bijection

$$U \times G^n \to U^{n+1}, \ (u, g_1, \ldots, g_n) \longleftrightarrow (u, ug_1, \ldots, ug_1g_2 \ldots g_n).$$
In turn, we may identify the \(n\)-cochains \(\Gamma(Y^{n+1}, O_Y^\times)\) with the the group of functions \(c: G^n \to P^\times\) satisfying \(c(0, \ldots, 0) \in A^\times\). Here \(P = k[[T]][T^{-1}]\) is the fraction field of \(A = k[[T]]\). The differentials take the form

\[
d_0(c)(g) = c(0)/c(0)^g \quad \text{and} \quad d_1(c)(g, h) = c(h)^g c(g)/c(gh),
\]

conforming with the usual definition of group cohomology \(\mathbb{Z})\), page 59). We have \(d_0(c)(0) = 1\), and \(d_0(c)(\sigma)\) is a power series of the form \(\lambda_0 + \lambda_1 T + \lambda_2 T^2 + \ldots\) with \(\lambda_0 = 1\). One easily checks that a 1-cocycle \(c: G \to P^\times\) is a 1-cocycle if and only if \(c(0) = 1\), and \(p = c(\sigma)\) satisfies \(p \cdot p^g = 1\). Clearly, the 1-cocycle \(c: G \to P^\times\) with \(c(0) = 1\) and \(c(\sigma) = -1\) is not a coboundary, so \(\text{Pic}(X_{\text{et}})\) is nonzero. On the other hand, by Hilbert’s Theorem 90, each \(p \in P^\times\) with \(p \cdot p^g = 1\) is of the form \(p = r/r^\sigma\) for some \(r \in P^\times\). Writing \(r = T^a s\) with \(s \in A^\times\), we have \(p = (-1)^a s/s^g\), and infer \(\text{Pic}(X_{\text{et}}) = \mathbb{Z}/2\mathbb{Z}\).

The smooth 1-dimensional nonseparated algebraic space \(X\) is our first counterexample to Hilbert’s Theorem 90 for algebraic spaces:

**Theorem 2.2.** The canonical inclusion \(\text{Pic}(X_{\text{zar}}) \subset \text{Pic}(X_{\text{et}})\) is not surjective.

**Proof.** The scheme \(Y\) is local, so the space of points for \(X\) has a unique closed point. Consequently, any Zariski covering of \(X\) contains a copy of \(X\). So any line bundle on \(X_{\text{zar}}\) is trivial, that is, \(\text{Pic}(X_{\text{zar}}) = 0\). On the other hand, \(\text{Pic}(X_{\text{et}}) \neq 0\) by Proposition 2.1. \(\square\)

### 3. Nonnormal proper algebraic spaces

Fix an algebraically closed ground field \(k\). In this section, we shall construct a proper algebraic space \(X\) and an invertible sheaf \(\mathcal{L}\) such that the open subspaces \(W \subset X\) trivializing \(\mathcal{L}\) do not form a covering.

The starting point is a proper smooth \(k\)-scheme \(Y\) containing two irreducible closed curves \(C_1, C_2 \subset Y\) such that \(C_1 + C_2\) is numerically trivial. This implies that the generic points \(\eta_i \in C_i\) do not admit any common affine neighborhood in \(Y\). Examples of such schemes appear in [11], page 75. Obviously, they are nonprojective. Even worse, they do not admit embeddings into toric varieties ([13], Theorem A). Recall that the support \(\text{Supp}(D) \subset Y\) of a Cartier divisor \(D \in \text{Div}(Y)\) is the union of its positive and negative part. We have the following useful property:

**Proposition 3.1.** Each \(D \in \text{Div}(Y)\) with \(D \cdot C_1 > 0\) and \(C_1 \notin \text{Supp}(D)\) has \(C_2 \subset \text{Supp}(D)\).

**Proof.** Decompose \(D = \sum n_i D_i\) into prime divisors with \(n_i \neq 0\). Since \(C_1 \notin D_i\), the intersection number \(D_i \cdot C_1\) is the length of the scheme \(D_i \cap C_1\), hence nonnegative. So there is at least one prime divisor with \(D_i \cdot C_1 > 0\). It follows \(D_i \cdot C_2 < 0\), hence \(C_2 \subset D_i\). In other words, \(C_2 \subset \text{Supp}(D)\). \(\square\)

Now fix two closed points \(y_1 \in C_1\) and \(y_2 \in C_2\). Let \(Y' \subset Y\) be the reduced closed subscheme corresponding to \([y_1, y_2]\), and define an étale sheaf \(X \in (\text{Sch}/k)_{\text{ét}}\) by the cocartesian square:

\[
\begin{array}{ccc}
Y' & \longrightarrow & Y \\
\downarrow & & \downarrow p \\
\text{Spec}(k) & \longrightarrow & X
\end{array}
\]
Proof. Choose an invertible sheaf $X$, being a topos, admits all colimits ([3], Exposé II, Theorem 4.1). Intuitively, $X$ is obtained from $Y$ by identifying the points $y_1, y_2 \in Y$. The sheaf $X$ is not a scheme. Otherwise, an affine open neighborhood for the point $p(y_1) = p(y_2) \in X$ would give a common affine open neighborhood for the pair $y_1, y_2 \in Y$.

**Proposition 3.2.** The étale sheaf $X$ is a proper algebraic space.

**Proof.** That $X$ is a proper algebraic space follows immediately from [1], Theorem 6.1. Let me give a more direct argument as follows. Fix two copies $v_1′, v_2′ \in V′$ and $v_1″, v_2″ \in V″$ of $y_1, y_2 \in Y$, and set $V = V′ \amalg V″$. Identifying $v_1′ \in V$ with $v_1″ \in V$ and $v_2′ \in V$ with $v_2″ \in V$, we obtain a scheme $U$. The group $G = \mathbb{Z}/2\mathbb{Z}$ acts freely on $U$ by interchanging $V′$ and $V″$. Clearly, $X = U/G$ is the quotient of this action in the topos of étale sheaves. So $R = U \times_X U$ is nothing but $U \times G$, which is a scheme. Consequently, $X = U/R$ is an algebraic space.

The algebraic space $X$ is separated because the embedding $Y \times G \to Y \times Y$, $(y, g) \mapsto (y, yg)$ is closed. As $Y \to \text{Spec}(k)$ is universally closed and $p : Y \to X$ is surjective, $X \to \text{Spec}(k)$ is universally closed as well. Therefore, $X$ is proper. 

**Proposition 3.3.** There is an exact sequence $1 \to k^x \to \text{Pic}(X_{\text{ét}}) \to \text{Pic}(Y) \to 0$.

**Proof.** Let $p : Y \to X$ be the canonical projection. Then the sequence

$$1 \to \mathcal{O}_X^x \to p_*(\mathcal{O}_Y^x) \oplus k^x \to p_*(\mathcal{O}_{Y'}^x) \to 1$$

is exact. Indeed, one easily checks this, as in [1], Lemma 5.1, after base change with an affine étale cover $U \to X$. In turn, we obtain an exact sequence

$$\Gamma(\mathcal{O}_X^x) \oplus k^x \to \Gamma(\mathcal{O}_{Y'}^x) \to \text{Pic}(X_{\text{ét}}) \to \text{Pic}(Y) \oplus \text{Pic}(k) \to \text{Pic}(Y')$$

Being semilocal, the schemes $\text{Spec}(k)$ and $Y′$ have no Picard groups. The cokernel for the map on the left is isomorphic to $k^x$, and the result follows.

The proper algebraic space $X$ is another counterexample to Hilbert’s Theorem 90 for algebraic spaces:

**Theorem 3.4.** The canonical inclusion $\text{Pic}(X_{zar}) \subset \text{Pic}(X_{\text{ét}})$ is not surjective.

**Proof.** Choose an invertible $\mathcal{O}_Y$-module $\mathcal{M}$ with $\mathcal{M} \cdot C_1 > 0$. For example, $\mathcal{M}$ could be the invertible sheaf corresponding to the reduced complement of any affine open neighborhood for $y_1 \in Y$.

Let $p : Y \to X$ be the canonical map. According to Proposition 3.3, there is an invertible $\mathcal{O}_X$-module $\mathcal{L}$ with $\mathcal{M} = p^*(\mathcal{L})$. Suppose there is an open subset $W \subset X$ containing the point $p(y_1) = p(y_2)$ and trivializing $\mathcal{L}$. Then $\mathcal{M}$ is trivial on the open subscheme $p^{-1}(W) \subset Y$. By [10], Theorem 3.3, there is a Cartier divisor $D \in \text{Div}(X)$ representing $\mathcal{M}$ with support disjoint from $y_1, y_2 \in Y$. In particular, $C_1$ and $C_2$ are not contained in $\text{Supp}(D)$, contradicting Proposition 3.1.

**Question 3.5.** Does $\text{Pic}(X_{zar}) = \text{Pic}(X_{\text{ét}})$ at least hold for smooth proper algebraic spaces? What about the case that $X$ is normal and proper?

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