Wavelet transform and Radon transform on the quaternion Heisenberg group

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Abstract

Let Q be the quaternion Heisenberg group, and let P be the affine automorphism group of Q. We develop the theory of continuous wavelet transform on the quaternion Heisenberg group via the unitary representations of P on $L^2(Q)$. A class of radial wavelets is constructed. The inverse wavelet transform is simplified by using radial wavelets. Then we investigate the Radon transform on Q. A Semyanistri-Lizorkin space is introduced, on which the Radon transform is a bijection. We deal with the Radon transform on Q both by the Euclidean Fourier transform and the group Fourier transform. These two treatments are essentially equivalent. We also give an inversion formula by using wavelets, which does not require the smoothness of functions if the wavelet is smooth.

Keywords: Quaternion Heisenberg group, wavelet transform, Radon transform, inverse Radon transform.

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1 Introduction

The Heisenberg group, denoted by $H_n$, is the simplest example of two-step nilpotent Lie group. There are many works devoted to the theory of harmonic analysis on this group. Geller [9] established the theory of Fourier analysis on $H_n$. More results can be found in [8], [23], [30], [31] and the references therein. Wavelet analysis on the Euclidean space $\mathbb{R}^n$ has many applications in pure and applied mathematics (see [5]). It is important to extend the theory of wavelet analysis to various cases. Several authors developed the theory of continuous wavelet transform on the Heisenberg group $H_n$ (see [16], [22]). Recently, some further extensions of wavelet analysis were published in [10], [18]. The Radon transform represents an interesting object from the point of view of both harmonic analysis and integral geometry. Also it is a very useful tool to deal with the problems of mathematics and engineer. The Radon transform on the Heisenberg group (or Heisenberg Radon transform) was studied by Geller-Stein [11] and Strichartz [30]. Related further extension we refer the reader to see [7], [25]. Holschneider (17) is the first author who applied the inverse wavelet transform to the inverse Radon transform on the two-dimensional plane. Rubin [20] extended to the case of $k$-dimensional Radon transform on $\mathbb{R}^n$. Nessibi-Trimèche [24] obtained an inversion formula of the Radon transform by using generalized wavelets on the Laguerre hypergroup. The further development can be found in [7], [12] − [15] and [27]. Heisenberg type groups are generalizations of the Heisenberg group, which are important both on geometry and analysis (see [4], [19], [20] and [21]). The quaternion Heisenberg group is a typical Heisenberg type group other than the Heisenberg group, which has a good explain in geometry (see [1], [3]). Tie-Wong [32] studied the heat kernel and Green functions associated with the Sub-Laplacians. Moreover, Zhu [33] investigated the property of the Riesz transforms on this group. In this article we study the wavelet transform and the Radon transform on the quaternion Heisenberg group $\mathcal{D}$.

This article is organized as follows. In the remainder of this section we shall recall some basic facts of quaternion numbers, and state the background of the quaternion Heisenberg group $\mathcal{D}$. Also, we describe the automorphism group of $\mathcal{D}$ and its regular representation. In Section 2 we give the direct sum decomposition for $L^2(\mathcal{D})$ in terms of the group Fourier transform, in which every subspace is irreducible for the representation of the automorphism group of $\mathcal{D}$. The theory of continuous wavelet transform via square integral group representation (see [6]) is developed in Section 3. we also construct a class of radial wavelets. The inverse wavelet transform can be simplified by using radial wavelets. Section 4 is devoted to the Radon transform. We introduce a Semyanistri-Lizorkin type space, on which the Radon transform is a bijection. Then we give the inverse Radon transform. We give the results in two different ways. One way is in terms of the Euclidean Fourier transform and another the group Fourier transform. We also prove that these two treatments are essentially equivalent. Finally, in Section 5 we make use of inverse wavelet transform to derive an inversion formula of the Radon transform in $L^2$-sense, which does...
not require the smoothness of functions if the wavelet is smooth.

Let \( \mathbb{Q} \) denote the set of all quaternion numbers, \( i, j, k \) are the three imaginary units satisfying: \( i^2 = j^2 = k^2 = ijk = -1 \). For any \( q \in \mathbb{Q} \), we can write \( q = q_0 + q_1i + q_2j + q_3k \).

For convenience, we also set \( q = (q_0, q_1, q_2, q_3) \) where \( q_0, q_1, q_2, q_3 \in \mathbb{R} \). Let \( \Re q \) and \( \Im q \) denote the real part and imaginary part of \( q \) respectively. Then \( \Re q = q_0, \Im q = q_1i + q_2j + q_3k = (q_1, q_2, q_3) = q^I \). In contrast to complex numbers, \( \Im q \) is not a real number. The multiplication of two quaternion numbers \( q, h \) is given by

\[
\Re(qh) = q_0h_0 - q^I \cdot h^I, \quad \Im(qh) = q_0h^I + h_0q^I + q^I \times h^I.
\]

\( q^\overline{\Im} = q_0 - q_1i - q_2j - q_3k \) denotes the conjugate of \( q \). The scalar product is given by \( \langle q, h \rangle = \Re(qh) \). The norm is \( |q|^2 = \langle q, q \rangle = \sum_{i=0}^3 q_i^2 \). Then we have \( \overline{q}h = h\overline{q}, |qh| = |q||h|, q^{-1} = \frac{\overline{q}}{|q|^2} \). We also note the following facts. The reals are only quaternions which commute with all quaternions, and \( q^2 = -1 \) if and only if \( |q| = 1 \) and \( q = \Im q \). We will identify \( \mathbb{Q} \) with \( \mathbb{R}^4 \) and \( \Im \mathbb{Q} \) with \( \mathbb{R}^3 \) if necessary.

Similar to the Heisenberg group, the quaternion Heisenberg group is the boundary of the Siegel upper-half space in the quaternion content (see \[30\]). Let \( \mathcal{B} \) be the unit ball in \( \mathbb{Q}^2 \) which is given by

\[
\mathcal{B} = \left\{ (h_1, h_2) : |h_1|^2 + |h_2|^2 < 1 \right\}.
\]

Then \( \mathcal{B} \) is biholomorphic to the Siegel upper-half plane in \( \mathbb{Q}^2 \) given by

\[
\mathcal{U} = \left\{ (q_1, q_2) \in \mathbb{Q}^2 : \Re q_2 > |q_1|^2 \right\}.
\]

The Cayley transform is given by

\[
\begin{align*}
q_1 &= \frac{h_1}{1 + h_2} = \frac{h_1 (1 + \overline{h}_2)}{|1 + h_2|^2}, \\
q_2 &= \frac{1 - h_2}{1 + h_2} = \frac{(1 - h_2)(1 + \overline{h}_2)}{|1 + h_2|^2}.
\end{align*}
\]

Let \( r = r(q_1, q_2) = \Re q_2 - |q_1|^2 \) be the height function. Setting

\[
x = q_1, t = \Im q_2, r = r(q_1, q_2) = \Re q_2 - |q_1|^2.
\]

If we adopt the Heisenberg coordinate \( (x, t, r) \), then the Siegel upper-half plane is denoted by

\[
\mathcal{U} = \left\{ (x, t, r) : x \in \mathbb{Q}, t \in \Im \mathbb{Q}, r > 0 \right\}.
\]

The boundary of \( \mathcal{U} \) can be identified with the quaternion Heisenberg group denoted by

\[
\mathcal{Q} = \left\{ (x, t) : x \in \mathbb{Q}, t \in \Im \mathbb{Q} \right\}.
\]

The group multiplication is given by

\[
(x, t)(x', t') = (x + x', t + t' - 2\Im(x't)).
\]
Thus we see that \( \mathbb{Q} \cong \mathbb{R}^4 \times \mathbb{R}^3 \). The Haar measure on \( \mathbb{Q} \) coincides with the Lebesgue measure on \( \mathbb{R}^4 \times \mathbb{R}^3 \) which is denoted by \( dxdt \).

Let \( u \in \mathbb{Q}, u = a + bi + cj + dk = (a + bi) + (c + di)j \). This implies the identification of \( \mathbb{Q} \) with \( \mathbb{C}^2 \). At the same time every quaternion \( u \) corresponds to a complex duplex matrix, i.e.,

\[
    u = (a + bi) + (c + di)j \mapsto \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.
\]

In [2], \( Sp(1) = \{ u \in \mathbb{Q} : |u| = 1 \} \) is called the quaternion group, or group of unit quaternions. It is known that \( Sp(1) \cong SU(2) \) and is the two-fold covering group of \( SO(3) \).

The automorphism of Heisenberg type groups was given by Kaplan and Ricci [20]. We define the translation and dilation operators respectively by

\[
    T_{(x,t)} : (x',t') \mapsto (x + x', t + t' - 2\Im(x'x)), \quad (x,t),(x',t') \in \mathcal{D}
\]

and

\[
    T_\rho : (x',t') \mapsto (\sqrt{\rho}x', \rho t'), \quad (x',t') \in \mathcal{D}, \rho > 0.
\]

Let

\[
    A_{u,v} (q) = uq\overline{v}, \quad u,v \in Sp(1), q \in \mathbb{Q}
\]

and

\[
    B_v (r) = vr\overline{v}, \quad v \in Sp(1), r \in \mathbb{H} \mathbb{Q}.
\]

The maps \( A_{u,v} \) act transitively on the unit sphere of \( \mathcal{D} \), and \( B_v \) act transitively on the unit sphere of \( \mathbb{H} \mathbb{Q} \). We define the operator \( T_{u,v} \) on \( \mathcal{D} \) by

\[
    T_{u,v} : (x',t') \mapsto (ux'\overline{v}, vt'\overline{v}), \quad (x',t') \in \mathcal{D}.
\]

We are now in a position to give the affine automorphism group of \( \mathcal{D} \). Let

\[
    \mathcal{P} = \{(x,t,\rho,u,v) : (x,t) \in \mathcal{D}, \rho > 0, u,v \in Sp(1)\}.
\]

The action of \( \mathcal{P} \) on \( \mathcal{D} \) is given by

\[
    (x,t,\rho,u,v)(x',t') = (x + \sqrt{\rho}ux'\overline{v}, t + \rho vt'\overline{v} - 2\sqrt{\rho}\Im(vx'\overline{ux}))\).
\]

That is \((x,t,\rho,u,v) = T_{(x,t)}T_\rho T_{u,v}\). It is known that \( \mathcal{P} \) is two-fold covering of the affine automorphism group of \( \mathcal{D} \). The group law of \( \mathcal{P} \) is given by

\[
    (x_1,t_1,\rho_1,u_1,v_1)(x_2,t_2,\rho_2,u_2,v_2)
    = (x_1 + \sqrt{\rho_1}u_1x_2\overline{v_1}, t_1 + \rho_1v_1t_2\overline{u_1} - 2\sqrt{\rho_1}\Im(v_1x_2\overline{u_1}x_1), \rho_1\rho_2, u_1u_2, v_1v_2).
\]

We will consider \( \mathcal{P} \) instead of the affine automorphism group of \( \mathcal{D} \). It is easy to verify that \( \mathcal{P} \) is a locally compact non-unimodular group with the left Haar measure
$$dm_l(x, t, \rho, u, v) = \frac{dxdtd\rho du dv}{\rho^6}$$ and the right Haar measure
$$dm_r(x, t, \rho, u, v) = \frac{dxdtd\rho du dv}{\rho}$$ respectively, where $du$ and $dv$ are the normalized Haar measures of group $Sp(1)$.

Let us consider the unitary representation $U$ of $P$ on $L^2(\mathcal{Q})$ defined by
$$U(x, t, \rho, u, v)f(x', t') = \rho^{-5/2}f\left(\frac{\sqrt{\rho}u' - x'}{\rho}, \frac{\sqrt{\rho}(t' - t + 2\Re(\mathbf{x})x)}{\rho}\right).$$

The representation $U$ is reducible on $L^2(\mathcal{Q})$. We shall decompose the space $L^2(\mathcal{Q})$ into the direct sum of the irreducible invariant closed subspaces.

### 2 Direct sum decomposition for $L^2(\mathcal{Q})$

First we state some results of the Fourier analysis on $\mathcal{Q}$. The Fourier transform on Heisenberg type groups was studied by Kaplan and Ricci [20]. Let $0 \neq a \in \mathbb{Q}$. Set $\tilde{a} = \frac{a}{|a|}$. The mapping $\rho(\tilde{a}) : q \mapsto q\tilde{a}$ gives a complex structure of $\mathcal{Q}$. Let $\mathcal{H}_a$ be the Fock space consisting of all holomorphic functions $F$ on $(\mathcal{Q}, \rho(\tilde{a})) \cong \mathbb{C}^2$ such that
$$\|F\|^2 = \int_{\mathcal{Q}} |F(q)|^2e^{-2|a||q|^2}dq < \infty.$$

We now define the unitary representation $\pi_a(x, t)$ of $\mathcal{Q}$ on $\mathcal{H}_a$ by
$$\pi_a(x, t)F(q) = F(q + x)e^{i(a, t) - a(|x|^2 + 2(q, x) - 2i(q, \tilde{a}, x))}.$$

Up to a unitary equivalent, all irreducible infinite-dimensional unitary representations of $\mathcal{Q}$ are given by $\pi_a(x, t)$. For $f \in L^1(\mathcal{Q})$, the Fourier transform of $f$ is an operator valued function defined by
$$\hat{f}(a) = \int_{\mathcal{Q}} f(x, t)\pi_a(x, t)dxdt.$$

Let $f, g \in L^1(\mathcal{Q}) \cap L^2(\mathcal{Q})$. By the standard theory of the Weyl transform, we have
$$\langle f, g \rangle_{L^2(\mathcal{Q})} = \frac{1}{2\pi^3} \int_{\mathcal{Q}} \text{tr}(\hat{g}(a)^*\hat{f}(a))|a|^2da.$$

Specially, the following Plancherel formula holds.
$$\|f\|_{L^2(\mathcal{Q})}^2 = \frac{1}{2\pi^3} \int_{\mathcal{Q}} \|\hat{f}(a)\|_{HS}^2|a|^2da. \quad (2.1)$$

The Fourier transform can be extended to the tempered distributions on $\mathcal{Q}$ by duality. And we also have the formula of inverse Fourier transform
$$f(x, t) = \frac{1}{2\pi^3} \int_{\mathcal{Q}} \text{tr}(\hat{\pi}_a(x, t)\hat{f}(a))|a|^2da.$$
Let $f * g$ denote the convolution of $f$ and $g$, i.e.,
\[
f * g(x, t) = \int_\mathcal{Q} f(y, s)g((y, s)^{-1}(x, t))dyds.
\]
Then
\[
\hat{f * g}(a) = \hat{f}(a)\hat{g}(a).
\] (2.2)
If $\hat{f}(x, t) = \hat{f}((x, t)^{-1}) = \hat{f}(-x, -t)$, then
\[
\hat{f}(a) = \hat{f}(a)^*,
\] (2.3)
where $\hat{f}(a)^*$ is the adjoint of $\hat{f}(a)$. We now choose an orthonormal basis $\{e_0, e_1, e_2, e_3\}$ of $Q$ satisfying $e_0 = 1, e_1 = e, e_2 = 2, e_3 = 3$. Write $q = (z_1, z_2)$ for $z_1 = x_1 + y_1i, z_2 = x_2 + y_2i$ and $q = x_1e_0 + y_1e_1 + x_2e_2 + y_2e_3$. Then
\[
\{E_\alpha^a(q) = \pi^{-1}(\alpha_1!\alpha_2!)^{-1/2}(2|a|)^{\alpha_1+\alpha_2+1}z_1^{\alpha_1}z_2^{\alpha_2} : \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2\}
\] is an orthonormal basis of $\mathcal{H}_a$. Thus we have the identification between Fock spaces $\mathcal{H}_a$ and $\mathcal{H}_v$ by identifying $E_\alpha^a$ with $E_\alpha^v$. For $\rho > 0$, write
\[
f_\rho(x, t) = \rho^{-5}f\left(\frac{x}{\sqrt{\rho}}, \frac{t}{\rho}\right),
\]
then we have
\[
\hat{f}_\rho(a) = \hat{f}(\rho a),
\]
where we have identified $F(q) \in \mathcal{H}_a$ with $qF(\sqrt{\rho}q) \in \mathcal{H}_{\rho a}$. Suppose that $u, v \in Sp(1)$ and $f_{u,v}(x, t) = f(\overline{uv}, \overline{v})$, we can verify that
\[
\hat{f}_{u,v}(a)F(q) = \int_\mathcal{Q} f(\overline{uv}, \overline{v})e^{i\tau(\overline{uv}, -|\overline{u}|(|x|^2+2(q,x)-2i(q\overline{u}, x))}F(q + x)dxdt
\]
\[
= \int_\mathcal{Q} f(x, t)e^{i\tau(\overline{uv}, -|\overline{u}||x|^2+2(|\overline{uv}|^2-2i(|\overline{uv}|(|\overline{u}v, x)|\overline{v}))}F(u(\overline{uv} + x))dxdt.
\]
Thus,
\[
\hat{f}_{u,v}(a) = \gamma_{u,\overline{v}}^{-1}\hat{f}(\overline{uv})\gamma_{u,v},
\]
where the intertwining operator $\gamma_{u,v}$ is given by
\[
\gamma_{u,v}F(q) = F(uq\overline{v}).
\]
Let $l = \alpha_1 + \alpha_2$ and $\mathcal{H}_{a,l}$ be the subspace of $\mathcal{H}_a$ which consists of all homogeneous polynomials of degree $l$ in $q \in \mathbb{C}^2$. Then $\mathcal{H}_{a,l}$ is an irreducible invariant closed subspace under $\gamma_{u,v}$ where we have identified $F(q) \in \mathcal{H}_a$ with $F(q\overline{v}) \in \mathcal{H}_{uv}$. Moreover, we have
\[
\mathcal{H}_a = \bigoplus_{l=0}^{+\infty} \mathcal{H}_{a,l}.
\]
Let \( P_{a,l} \) denote the orthogonal projection operator from \( \mathcal{H}_a \) to \( \mathcal{H}_{a,l} \). The projection operator \( P_l \) is defined in terms of the Fourier transform by
\[
\hat{P}_l f(a) = \hat{f}(a) P_{a,l}.
\]
Clearly, \( P_l \)'s are mutual orthogonal projection operators on \( L^2(\mathcal{Q}) \). If \( f(x,t) \) is a radial function with respect to the variable \( x \), i.e., \( f(ux,t) = f(x,t) \) for all \( u \in Sp(1) \), then we have \( \gamma_{u,1} \hat{f}(a) = \hat{f}(a) \gamma_{u,1} \). By Schur’s lemma,
\[
\hat{f}(a) = \sum_{l=0}^{\infty} B f(a,l) P_{a,l},
\]
where \( B f(a,l) \) is a constant depending on \( f, a \) and \( l \). Define the subspace \( H_l \) of \( L^2(\mathcal{Q}) \) by
\[
H_l = \{ f \in L^2(\mathcal{Q}) : \hat{f}(a) = \hat{f}(a) P_{a,l} \}
\]
which is the range of projection operator \( P_l \). For a function \( f \in L^2(\mathcal{Q}) \), it is easy to verify that
\[
(U(x,t,\rho,u,v) f)\hat{\ } \gamma(\mathcal{Q}) = \rho^{5/2} \pi_a(x,t) \hat{f}_{u,v}(\rho a).
\]
By a similar argument as in [16], we obtain the direct sum decomposition for \( L^2(\mathcal{Q}) \) as follows.

**Theorem 1.** \( H_l \) is an irreducible invariant closed subspace of \( L^2(\mathcal{Q}) \) under the unitary representation \( U \) of \( \mathbb{P} \), and we have
\[
L^2(\mathcal{Q}) = \bigoplus_{l=0}^{\infty} H_l.
\]

**Remark 1.** In fact, \( H_l \) can be characterized by the subLaplacian operator \( \Delta_{\mathcal{Q}} \) (see [32]), i.e., \( f \in H_l \) if and only if
\[
\Delta_{\mathcal{Q}} \hat{f}(a) = -8(l + 1) |a| \hat{f}(a),
\]
where \( \Delta_{\mathcal{Q}} = X_0^2 + X_1^2 + X_2^2 + X_3^2 \) is the square sum of horizontal vector fields, \( X_0, X_1, X_2 \) and \( X_3 \) are left invariant vector fields given by
\[
X_0 = \frac{\partial}{\partial x_0} - 2x_1 \frac{\partial}{\partial t_1} - 2x_2 \frac{\partial}{\partial t_2} - 2x_3 \frac{\partial}{\partial t_3},
\]
\[
X_1 = \frac{\partial}{\partial x_1} + 2x_0 \frac{\partial}{\partial t_1} - 2x_3 \frac{\partial}{\partial t_2} + 2x_2 \frac{\partial}{\partial t_3},
\]
\[
X_2 = \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial t_1} + 2x_0 \frac{\partial}{\partial t_2} - 2x_1 \frac{\partial}{\partial t_3},
\]
\[
X_3 = \frac{\partial}{\partial x_3} - 2x_2 \frac{\partial}{\partial t_1} + 2x_1 \frac{\partial}{\partial t_2} + 2x_0 \frac{\partial}{\partial t_3}.
\]
3 Continuous wavelet transforms

We are going to show that the restriction of $U$ on $H_l$ is square-integrable. In other words, there exists a non-zero function $\phi \in H_l$, such that

$$C_{\phi} = \frac{1}{\|\phi\|_{L^2(\mathcal{D})}^2} \int_{\mathcal{P}} |(\phi, U(x, t, \rho, u, v)\phi)_{L^2(\mathcal{D})}|^2 dm_l(x, t, \rho, u, v) < \infty. \quad (3.1)$$

We call (3.1) the admissibility condition, and write $\phi \in AW_l$.

**Theorem 2.** Let $\phi \in H_l$, not identically zero. Then $\phi \in AW_l$ if and only if

$$C_{\phi} = \frac{1}{d_l} \int_{\mathfrak{H}_a,l} \|\hat{\phi}(a)\|_{HS} da < \infty. \quad (3.2)$$

where $d_l = \text{dim} \mathfrak{H}_a,l = l + 1$. (3.2) is called the Calderón reproducing condition.

**Proof.** Suppose $\phi \in H_l$, by (2.5), we have

$$\int_{\mathcal{D}} \langle \phi, U(x, t, \rho, u, v)\phi \rangle_{L^2(\mathcal{D})} d_x dt = \rho^{5/2} \hat{\phi}(\rho a) \hat{\phi}_u,v(\rho a)^*. $$

Using the Plancherel formula (2.1) we can derive

$$\int_{\mathcal{P}} |(\phi, U(x, t, \rho, u, v)\phi)_{L^2(\mathcal{D})}|^2 dm_l(x, t, \rho, u, v)
= \frac{1}{2\pi^5} \int_{\mathfrak{H}_a,l} \|\hat{\phi}(a)\|_{HS} da \int_{\mathfrak{H}_a,l} \|\hat{\phi}(a)\|_{HS} da \frac{d\rho}{\rho}
= \frac{1}{2\pi^5} \int_{\mathfrak{H}_a,l} \|\hat{\phi}(a)\|_{HS} da \int_{\mathfrak{H}_a,l} \|\hat{\phi}(a)\|_{HS} da \frac{d\rho}{\rho}.
$$

Because

$$\text{tr} \left( \hat{\phi}(a) \hat{\phi}_u,v(\rho a) \hat{\phi}_u,v(\rho a)^* \right) = \sum_{|\alpha| = l} \langle \hat{\phi}_u,v(\rho a) \hat{\phi}_u,v(\rho a)^* \rangle_{\mathfrak{H}_a,l},$$

we therefore obtain

$$\int_{\mathcal{P}} |(\phi, U(x, t, \rho, u, v)\phi)_{L^2(\mathcal{D})}|^2 dm_l(x, t, \rho, u, v)
= \frac{1}{2\pi^5} \int_0^{+\infty} \left\{ \int_{\mathfrak{H}_a,l} \sum_{|\alpha| = l} \left( \langle \int_{\mathfrak{H}_a,l} \hat{\phi}_u,v(\rho a) \hat{\phi}_u,v(\rho a)^* d\rho \rangle_{\mathfrak{H}_a,l}, \hat{\phi}(a) \hat{\phi}_u,v(\rho a)^* \rangle_{\mathfrak{H}_a,l} \right) \frac{d\rho}{\rho} \right\}$$

By (2.2) together with (2.3), we derive

$$\int_{\mathfrak{H}_a,l} \hat{\phi}_u,v(\rho a) \hat{\phi}_u,v(\rho a)^* d\rho = \int_{\mathfrak{H}_a,l} \hat{\phi}_u,v(\rho a)^* (\rho a) d\rho = \hat{\psi}(\rho),$$

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where

\[ \psi(x, t) = \int_{Sp(1) \times Sp(1)} \phi_{u,v} * (x,t) du dv \]

\[ = \int_{Sp(1) \times Sp(1)} \phi(x, t) du dv \]

\[ \psi(x, t) = \int_{Sp(1)} \phi(x, t) du \]

Obviously, \( \psi \) is a radial function with respect to the variable \( x \). Hence by (2.4) we get

\[ \hat{\psi}(\rho a) = B_\phi(\rho a) P_l. \]

The Fourier transform of \( \psi \) is given by

\[ \hat{\psi}(\rho a) = \int_{Sp(1)} B_\phi(\rho a) \psi(x, t) du dv. \]

Consequently, we have

\[ \int \| \langle \phi, U(x, t, \rho, u, v) \phi \rangle_{L^2(\mathbb{R})} \|^2 dm_l(x, \rho, u, v) \]

\[ = \frac{1}{2\pi^5} \int_0^{+\infty} \int_{Sp(1)} \int_{3Q} \sum_{|a|=1} \left( \left| B_\phi(\rho a) \hat{\phi}(a)^* \hat{\phi}(a) \right| E_{a,2}^a \right) \frac{dvd\rho}{\rho} \]

\[ = \frac{1}{2\pi^5} \int_0^{+\infty} \int_{Sp(1)} B_\phi(\rho a) \text{tr} \left( \hat{\phi}(a)^* \hat{\phi}(a) \right) \frac{dvd\rho}{\rho} \]

\[ = \left( \int_{Sp(1)} \int_{3Q} B_\phi(\rho a) \frac{dvd\rho}{\rho} \right) \left( \frac{1}{2\pi^5} \int_{3Q} \text{tr}(\hat{\phi}(a)^* \hat{\phi}(a)) \frac{dvd\rho}{\rho} \right) \]

\[ = \| \phi \|^2_{L^2(\mathbb{R})} \left( \int_{3Q} B_\phi(a) \frac{dvd\rho}{\rho} \right). \]

Because

\[ B_\phi(a) = \frac{1}{d_l} \text{tr}(\hat{\phi}(a)) \]

\[ = \frac{1}{d_l} \text{tr} \left( \int_{Sp(1)} \phi(x, t) du \right) \]

\[ = \frac{1}{d_l} \text{tr}(\hat{\phi}(a)^* \hat{\phi}(a)) \]

\[ = \frac{1}{d_l} \| \hat{\phi}(a) \|_{HS}^2, \]

Theorem 2 is proved. \( \square \)

Let \( \phi \in AW_1, f \in H_1 \). The continuous wavelet transform \( W_\phi \) on \( H_1 \) is defined by

\[ W_\phi f(x, t, \rho, u, v) = \langle f, U(x, t, \rho, u, v) \phi \rangle_{L^2(\mathbb{R})}. \]
For $\phi, \psi \in AW_l$, we set
\[
\langle \phi, \psi \rangle_{AW_l} = \frac{1}{d_l} \int_0^{+\infty} \text{tr}(\hat{\psi}(a) \ast \hat{\phi}(a)) \frac{da}{|a|^3}
\]
and call it the inner product in $AW_l$.

**Theorem 3.** Let $\phi \in AW_l, \psi \in AW_{l'}, f \in H_l, g \in H_{l'}$. Then
\[
\langle W_\phi f, W_\psi g \rangle_{L^2(P, dm_l)} = \langle \phi, \psi \rangle_{AW_l} \langle f, g \rangle_{L^2(\mathbb{R})}.
\]
Specially,
\[
\|W_\phi f\|_{L^2(P, dm_l)} = C_{\phi}^{1/2} \|f\|_{L^2(\mathbb{R})}
\]
and
\[
\langle W_\psi f, W_\phi g \rangle_{L^2(P, dm_l)} = 0
\]
when $l \neq l'$.

**Proof.** Let $\phi \in AW_l, f \in H_l$. Then
\[
\int_\mathbb{R} \langle f, U(x, t, \rho, u, v) \phi \rangle_{L^2(\mathbb{R})} \pi_a(x, t) dxdt = \rho^{5/2} \hat{f}(a) \hat{\phi}_{u,v}(a).\]
By the Plancherel formula we can get
\[
\langle W_\psi f, W_\phi g \rangle_{L^2(P, dm_l)} = \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \text{tr}(\hat{\psi}(a) \ast \hat{\phi}(a)) |a|^2 \rho \frac{dpdudv}{\rho}.
\]
Suppose $\psi, \phi \in AW_l, f, g \in H_l$. Then
\[
\int_{\mathbb{R}^4} \hat{\psi}_{u,v}(\rho a) \ast \hat{\phi}_{u,v}(\rho a) dudv = \frac{1}{d_l} \text{tr}(\hat{\psi}(\rho a) \ast \hat{\phi}(\rho a)) P_l.
\]
Hence we get
\[
\langle W_\psi f, W_\phi g \rangle_{L^2(P, dm_l)} = \left( \frac{1}{d_l} \int_{\mathbb{R}^4} \text{tr}(\hat{\psi}(a') \ast \hat{\phi}(a')) \frac{da'}{|a'|^3} \right) \left( \frac{1}{2\pi^3} \int_{\mathbb{R}^4} \text{tr}(\hat{\psi}(a) \ast \hat{\phi}(a)) |a|^2 da \right)
\]
\[
= \langle \phi, \psi \rangle_{AW_l} \langle f, g \rangle_{L^2(\mathbb{R})}.
\]
If $\psi \in AW_l, \phi \in AW_{l'}, f \in H_l, g \in H_{l'}, l \neq l'$, then
\[
\int_{Sp(1) \times Sp(1)} \hat{\psi}_{u,v}(\rho a)^* \hat{\phi}_{u,v}(\rho a) dudv = 0.
\]
Theorem 3 is proved.

Theorem 2 and Theorem 3 restrict wavelets to subspaces $H_l$. The restriction is removable. Suppose $\phi \in L^2(\mathcal{Q})$. By Theorem 1,
\[
\phi = \sum_{l=0}^{\infty} \phi_l,
\]
where $\phi_l \in H_l$. If there exists a constant $C_\phi$, which is independent of $l$, such that
\[
\frac{1}{d_l} \int_{\mathcal{Q}} \|\hat{\phi}_l(a)\|_{H^S}^2 \frac{da}{\|a\|^3} = C_\phi < \infty \quad \text{for all } l \in \mathbb{N},
\]
we say that $\phi$ is an admissible wavelet and write $\phi \in AW$. Then we define the continuous wavelet transform of $f \in L^2(\mathcal{Q})$ by
\[
W_\phi f(x,t,\rho,u,v) = \langle f, U(x,t,\rho,u,v)\phi \rangle_{L^2(\mathcal{Q})}.
\]

**Theorem 4.** Suppose $\phi \in AW$. Then
\[
\|W_\phi f\|_{L^2(\mathbb{P},dm)} = C_\phi^{1/2} \|f\|_{L^2(\mathcal{Q})}. \quad (3.3)
\]

**Proof.** By Theorem 1,
\[
f = \sum_{l=0}^{\infty} f_l,
\]
where $f_l \in H_l$. By Theorem 3,
\[
\|W_\phi f\|_{L^2(\mathbb{P},dm)}^2 = \sum_{l=0}^{\infty} \|W_\phi f_l\|_{L^2(\mathbb{P},dm)}^2
\]
\[
= \sum_{l=0}^{\infty} C_\phi \|f_l\|_{L^2(\mathcal{Q})}^2
\]
\[
= C_\phi \|f\|_{L^2(\mathcal{Q})}^2.
\]
This proves Theorem 4.

The fundamental manifold of the quaternion Heisenberg group $\mathcal{Q}$ is just $\mathbb{R}^4 \times \mathbb{R}^3$. The Schwartz space $\mathcal{S}(\mathcal{Q})$ coincides with the Schwartz space on $\mathbb{R}^4 \times \mathbb{R}^3$. As a consequence of Theorem 4, the inverse wavelet transform is valid. We state the result as follows.
Theorem 5. Suppose $\phi \in AW$ and $f \in L^2(\mathcal{Q})$. Then

$$f(x,t) = C_\phi^{-1} \int_{\mathbb{P}} W_\phi f(y,s,\rho,u,v) U(y,s,\rho,u,v) \phi(x,t) \, dm_1(y,s,\rho,u,v)$$

in $L^2$-sense. Furthermore, if $f \in \mathcal{S}(\mathcal{Q})$, then the above formula holds pointwise.

The proof of Theorem 5 is standard.

Now we construct a class of admissible wavelets. These admissible wavelets are radial. The inverse wavelet transform can be simplified by using radial wavelets.

Let $\eta$ be a function on $\mathbb{R}_+$ satisfying

$$\int_0^\infty |\eta(r)|^2 r^4 dr < \infty$$

and the function $\phi_\eta$ is defined by

$$\hat{\phi}_\eta(a) = \sum_{l=0}^{\infty} \eta((l+1)|a|) P_{a,l}.$$ 

Then $\phi_\eta \in L^2(\mathcal{Q})$ because, by the Plancherel formula,

$$\|\phi_\eta\|_{L^2(\mathcal{Q})}^2 = \frac{1}{2\pi^5} \int_{3\mathbb{Q}} \|\hat{\phi}_\eta(a)\|_{HS}^2 |a|^2 \, da$$

$$= \frac{1}{2\pi^5} \int_{3\mathbb{Q}} \left( \sum_{l=0}^{\infty} |\eta((l+1)|a|)|^2 (l+1)^2 |a|^2 \right) \, da$$

$$= \frac{1}{2\pi^5} \sum_{l=0}^{\infty} \frac{1}{(l+1)^4} \int_{3\mathbb{Q}} |\eta(|a|)|^2 |a|^2 \, da$$

$$= C \int_0^\infty |\eta(r)|^2 r^4 dr < \infty.$$ 

Furthermore, if $\eta$ satisfies the Calderón reproducing condition

$$C_\eta = 4\pi \int_0^\infty |\eta(r)|^2 \frac{dr}{r} < \infty, \quad (3.4)$$

then $\phi_\eta$ is an admissible wavelet. In fact, write

$$\phi_\eta = \sum_{l=0}^{\infty} \phi_{\eta,l}$$

where $\phi_{\eta,l} \in H_1$, it is clear that

$$\hat{\phi}_{\eta,l}(a) = \eta((l+1)|a|) P_{a,l}.$$ 


Then, for all \( l \in \mathbb{N} \),
\[
\frac{1}{d_l} \int_{\mathbb{Q}} \| \hat{\phi}_{\eta,l}(a) \|_{l^2}^2 \frac{da}{|a|^3} = \int_{\mathbb{Q}} |\eta(l + 1)|^2 |a|^2 \frac{da}{|a|^3} = \int_{\mathbb{Q}} |\eta(|a|)|^2 \frac{da}{|a|^3} = C_\eta < \infty.
\]

Note that the admissible wavelet \( \phi_\eta \) defined above is a radial function, i.e., \( (\phi_\eta)_{u,v} = \phi_\eta \).

The wavelet transform \( W_{\phi_\eta}f \) is independent of \( u \) and \( v \), i.e.,
\[
W_{\phi_\eta}f(x,t,\rho,u,v) = W_{\phi_\eta}f(x,t,\rho,1,1).
\]
Thus the integration over \( u \) and \( v \) can be reduced if we use the admissible wavelet \( \phi_\eta \).

Write \( W_{\phi_\eta}f(x,t,\rho) \) and \( U(x,t,\rho) \) instead of \( W_{\phi_\eta}f(x,t,\rho,1,1) \) and \( U(x,t,\rho,1,1) \). Then we have

**Theorem 6.** Suppose the admissible wavelet \( \phi_\eta \) is defined as above, and \( f \in L^2(\mathcal{Q}) \).

Then
\[
\int_{\mathcal{Q} \times \mathbb{R}_+} |W_{\phi_\eta}f(x,t,\rho)|^2 \frac{dx dt d\rho}{\rho^3} = C_\eta^{1/2} \| f \|_{L^2(\mathcal{Q})},
\]
and
\[
f(x,t) = C_\eta^{-1} \int_{\mathcal{Q} \times \mathbb{R}_+} W_{\phi_\eta}f(y,s,\rho) U(y,s,\rho) \phi_\eta(x,t) \frac{dy ds d\rho}{\rho^3} \tag{3.5}
\]
in \( L^2 \)-sense. Furthermore, if \( f \in \mathcal{I}(\mathcal{Q}) \), then (3.5) holds pointwise.

## 4 The Radon transform

Similar to the Radon transform on the Heisenberg group \( \mathbb{H}_n \) defined by Strichartz [31], we define the Radon transform \( \mathcal{R} \) on \( \mathcal{Q} \) by
\[
\mathcal{R}(f)(x,t) = \int_{\mathcal{Q}} f((x,t),(y,0)) dy = \int_{\mathcal{Q}} f(y,t - 2\mathcal{H}(y)) dy.
\]

Even if \( f \in \mathcal{I}(\mathcal{Q}) \), the Radon transform \( \mathcal{R}(f) \) may not be rapidly deceasing at infinity. This fact were pointed by Strichartz [31] in the case of \( \mathbb{H}_n \). We will find a subspace of Schwartz space \( \mathcal{I}(\mathcal{Q}) \), on which the Radon transform is a bijection.

Let \( \mathcal{F}_2 \) denote the Euclidean Fourier transform with respect to the central variable \( t \) alone and \( \mathcal{F} \) denote the full Euclidean Fourier transform, i.e.,
\[
\mathcal{F}_2(f)(x,a) = \int_{\mathbb{Q}} f(x,t) e^{i(t,a)} dt,
\]
\[
\mathcal{F}(f)(y,a) = \int_{\mathcal{Q}} f(x,t) e^{i(t,y)} e^{i(t,a)} dx dt.
\]

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Because
\[
\mathcal{F}_2(\mathcal{R}(f))(x, a) = \int_{\mathbb{Q}} \left( \int_{\mathbb{Q}} f(y, t - 2\Im(\mathcal{F}x)) dy \right) e^{i(t, a)} dt
\]
\[
= \int_{\mathbb{Q}} \mathcal{F}_2(f)(y, a) e^{2i(\Im(\mathcal{F}x), a)} dy
\]
\[
= \int_{\mathbb{Q}} \mathcal{F}_2(f)(y, a) e^{-2i(y, xa)} dy,
\]
we have
\[
\mathcal{F}_2(\mathcal{R}(f))(x, a) = \mathcal{F}(f)(-2xa, a).
\] (4.1)

Let \( t^s = t_1^{s_1}t_2^{s_2}t_3^{s_3} \) where \( s = (s_1, s_2, s_3) \in \mathbb{N}^3 \). We define the space \( \mathcal{I}_s(\mathcal{D}) \) by
\[
\mathcal{I}_s(\mathcal{D}) = \left\{ f(x, t) \in \mathcal{I}(\mathcal{D}) : \int_{\mathbb{Q}} f(x, t)t^s dt = 0 \text{ for all } x \in \mathbb{Q}, s \in \mathbb{N}^3 \right\}.
\]
Write \( \partial^s f(x, 0) = \frac{\partial^{s_1}}{\partial t_1^{s_1}} \frac{\partial^{s_2}}{\partial t_2^{s_2}} \frac{\partial^{s_3}}{\partial t_3^{s_3}} f(x, t)|_{t=0} \) and define the space \( \mathcal{I}^s(\mathcal{D}) \) by
\[
\mathcal{I}^s(\mathcal{D}) = \left\{ f \in \mathcal{I}(\mathcal{D}) : \partial^s f(x, 0) = 0 \text{ for all } x \in \mathbb{Q}, s \in \mathbb{N}^3 \right\}.
\]

By same argument as Proposition 5.1 in [27], \( f \in \mathcal{I}_s(\mathcal{D}) \) if and only if \( \mathcal{F}(f) \in \mathcal{I}^s(\mathcal{D}) \) and \( \mathcal{F}_2 \) is an isomorphism from \( \mathcal{I}_s(\mathcal{D}) \) onto \( \mathcal{I}^s(\mathcal{D}) \). The space \( \mathcal{I}^s(\mathcal{D}) \) and \( \mathcal{I}_s(\mathcal{D}) \) are regarded as Semyanistyi-Lizonkin type spaces. This kind of spaces has many applications (see [28], [29]).

We introduce a “mixing” map \( \Psi \) which given by
\[
\Psi(f)(x, t) = f(-2xt, t).
\]

Let \( f \in \mathcal{I}^s(\mathcal{D}) \). Similar to the inequality (5.5) in [27], for every \( p, q \in \mathbb{N} \), there is a constant \( c_{p,q} \) such that
\[
|f(x, t)| \leq c_{p,q}|t|^{2p}(1 + |x|^2)^{-q}.
\]

It follows that \( \Psi \) is a bijection on \( \mathcal{I}^s(\mathcal{D}) \). The inverse of \( \Psi \) is given by
\[
\Psi^{-1}(f)(x, t) = \left\{ \begin{array}{ll}
f\left( \frac{x}{2|t|^2}, t \right), & \text{for } t \neq 0 \\
0, & \text{for } t = 0
\end{array} \right.
\]

Now (4.1) reads as
\[
\mathcal{F}_2(\mathcal{R}(f))(x, a) = \Psi(\mathcal{F}(f))(x, a).
\]

Thus the Radon transform \( \mathcal{R} = \mathcal{F}_2^{-1}\Psi \mathcal{F} \) is a bijection on \( \mathcal{I}_s(\mathcal{D}) \), and we have an inversion formula of the Radon transform as follows.

**Theorem 7.** Let \( f \in \mathcal{I}_s(\mathcal{D}) \). Then
\[
\mathcal{R}^{-1}(f) = \mathcal{F}^{-1}\Psi^{-1}\mathcal{F}_2(f).
\]
We can give another inversion formula of $\mathcal{R}$ by using of the Fourier transform on $\mathcal{D}$. First, we have

\[
(\mathcal{R}(f)(a)E^n_{\alpha})(q) = \int_{\mathbb{R}^2} \mathcal{R}(f)(x, t)e^{i(a,t) - |a|(|x|^2 + 2qx - 2i(q\bar{a}, x))} E^n_{\alpha}(q + x) dx dt = \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} f(y, t - 2\Im(\bar{x}y)) dy \right) e^{i(a,t) - |a|(|x|^2 + 2qx - 2i(q\bar{a}, x))} E^n_{\alpha}(q + x) dx dt = \int_{\mathbb{R}^2} \mathcal{F}_2(f)(y, a) E^n_{\alpha}(q + x) e^{-|a|(|x|^2 + 2qx - 2i((q\bar{a}, x))} dx dy = \int_{\mathbb{R}^2} \mathcal{F}_2(f)(y, a) e^{|a|(|y|^2 + 2i(q\bar{a}, y))} \left( \int_{\mathbb{R}^2} E^n_{\alpha}(x) e^{-|a|(|x|^2 - 2i((y + q)\bar{a}, x))} dx \right) dy.
\]

Using the identity

\[
\int_{\mathbb{C}^2} z^\alpha e^{-|z|^2} e^{i(z, w)} dz = \left( \frac{\pi}{2} \right)^\alpha 2^\alpha e^{-\frac{|w|^2}{4}},
\]

and note that $E^n_{\alpha}$ is a holomorphic monomial with respect to the complex structure $\rho(\bar{a})$, we obtain

\[
\int_{\mathbb{R}^2} E^n_{\alpha}(x) e^{-|a|(|x|^2 - 2i((y + q)\bar{a}, x))} dx = (-1)^{\alpha_1 + \alpha_2} \left( \frac{\pi}{|a|} \right)^2 E^n_{\alpha}(q + y) e^{-|a||q + y|^2}.
\]

Then

\[
(\mathcal{R}(f)(a)E^n_{\alpha})(q) = (-1)^{\alpha_1 + \alpha_2} \left( \frac{\pi}{|a|} \right)^2 \int_{\mathbb{R}^2} \mathcal{F}_2(f)(y, a) e^{-|a|(|y|^2 + 2qy - 2i(q\bar{a}, y))} E^n_{\alpha}(q + y) dy = (-1)^{\alpha_1 + \alpha_2} \left( \frac{\pi}{|a|} \right)^2 \hat{f}(a) E^n_{\alpha}(q).
\]

Thus we get

\[
\mathcal{R}(f)(a) = \left( \frac{\pi}{|a|} \right)^2 \hat{f}(a) S, \quad f \in \mathcal{S}(\mathcal{D}),
\]

where $S = \sum_{l=0}^{\infty} (-1)^l \mathcal{P}_{a,l,t}$. Let $\mathcal{L} = -\frac{1}{\pi} \sum_{l=1}^{3} \frac{\partial^2}{\partial t_l^2}$. Essentially, $\mathcal{L}$ is the Laplacian with respect to the central variable $t$. It is easy to see that

\[
\mathcal{L}(f)(a) = \left( \frac{|a|}{\pi} \right)^2 \hat{f}(a).
\]

Let $f \in \mathcal{S}(\mathcal{D})$. For any $m \in \mathbb{N}$, $\mathcal{L}^m(f) \in L^2(\mathcal{D})$. By (4.3) and the Plancherel formula (2.1),

\[
\int_{\mathbb{R}^2} \| \hat{f}(a) \|_{H_S}^2 |a|^{2+k} da < \infty \quad \text{for all } k \in \mathbb{N}.
\]
Now we define the space $\mathcal{I}(\mathcal{D})$ by

$$
\mathcal{I}(\mathcal{D}) = \left\{ f \in \mathcal{I}(\mathcal{D}) : \int_Q \| \hat{f}(a) \|_{HS}^2 |a|^{-2k} da < +\infty \text{ for all } k \in \mathbb{N} \right\}.
$$

It is easy to see that the Radon transform $\mathcal{R}$ is a bijection on $\mathcal{I}(\mathcal{D})$. Moreover, by (4.2) and (4.3), we have

**Theorem 8.** Let $f \in \mathcal{I}(\mathcal{D})$. Then

$$
\| f \|_2 = \| L \mathcal{R}(f) \|_2,
$$

$$
\mathcal{R}^{-1}(f) = L \mathcal{R} L(f).
$$

**Remark 1.** We note that $L$ is a positive operator. For $\mu \in \mathbb{R}$, $L^\mu$ can be defined by

$$
\mathcal{L}^\mu(f)(a) = \left( \frac{|a|}{\pi} \right)^{2\mu} \hat{f}(a).
$$

We also have, for $f \in \mathcal{I}(\mathcal{D})$,

$$
\mathcal{R}^{-1}(f) = L^\mu \mathcal{R} L^\nu(f), \quad \mu + \nu = 2.
$$

The next theorem means that our two treatments about the Radon transform are essentially equivalent.

**Theorem 9.**

$$
\mathcal{I}(\mathcal{D}) = \mathcal{I}_*(\mathcal{D}).
$$

**Proof.** Note that $\hat{f}(a)$ is just the Weyl transform of $f^a = \mathcal{F}_2 f(\cdot, a)$. By the standard theory of the Weyl transform,

$$
\left( \frac{2|a|}{\pi} \right)^2 \| \hat{f}(a) \|_{HS}^2 = \| f^a \|_2^2.
$$

If $f \in \mathcal{I}(\mathcal{D})$, then for all $k \in \mathbb{N}$,

$$
\lim_{|a| \to 0} |a|^{-k} \int_Q |\mathcal{F}_2 f(x, a)|^2 dx = 0.
$$

It follows that

$$
\int_Q |\mathcal{F}_2 f(x, 0)|^2 dx = \lim_{|a| \to 0} \int_Q |\mathcal{F}_2 f(x, a)|^2 dx = 0.
$$

Furthermore, we have

$$
\int_Q |\partial_a \mathcal{F}_2 f(x, 0)|^2 dx = \lim_{a_1 \to 0} a_1^{-2} \int_Q |\mathcal{F}_2 f(x, (a_1, 0, 0))|^2 dx = 0.
$$
Inductively, we obtain, for all $s \in \mathbb{N}$,

$$
\int_{Q} |\partial_{a}^{s} \mathcal{F}_{2} f(x,0)|^{2} dx = 0.
$$

This means $\mathcal{F}_{2} f \in \mathcal{S}(\mathcal{D})$, and equivalently $f \in \mathcal{S}(\mathcal{D})$.

On the other hand, suppose $f \in \mathcal{S}(\mathcal{D})$, i.e., $\mathcal{F}_{2} f \in \mathcal{S}(\mathcal{D})$. Then

$$
\lim_{|a| \to 0} \int_{Q} |\mathcal{F}_{2} f(x,a)|^{2} dx = \int_{Q} |\mathcal{F}_{2} f(x,0)|^{2} dx = 0.
$$

And

$$
\lim_{|a| \to 0} |a|^{-2} \int_{Q} |\mathcal{F}_{2} f(x,a)|^{2} dx = \lim_{|a| \to 0} \int_{Q} \left| \sum_{l=1}^{3} \frac{a_{l}}{|a|} \partial_{a_{l}} \mathcal{F}_{2} f(x,\theta a) \right|^{2} dx = 0,
$$

where $0 < \theta < 1$. Inductively, for all $k \in \mathbb{N}$,

$$
\lim_{|a| \to 0} |a|^{-k} \int_{Q} |\mathcal{F}_{2} f(x,a)|^{2} dx = 0.
$$

Therefore, $f \in \mathcal{S}(\mathcal{D})$. Theorem 9 is proved.

5 Inverse Radon transform by using wavelets

The inversion formulas of the Radon transform in above section require the smoothness of functions. In this section we establish an inversion formula of the Radon transform in $L^{2}$-sense by using the inverse wavelet transform. The formula does not require the smoothness of functions. Instead, we will use smooth wavelets.

Suppose $\phi \in \mathcal{S}(\mathcal{D})$. Then, by (4.2) and (4.3), $\mathcal{LRL}(\phi)$ is well defined and

$$
\mathcal{LRL}(\phi)(a) = \left( \frac{\pi}{|a|} \right)^{2} \mathcal{\phi}(a).
$$

Write $\phi_{\rho,u,v} = (\phi_{\rho})_{u,v}$. Then

$$
\mathcal{LRL}(\mathcal{\phi}_{\rho,u,v}) = \rho^{-2} \mathcal{LRL}(\phi)_{\rho,u,v}.
$$

This equality is easy to prove by making use of the Fourier transform.

Suppose $\phi \in AW$. It is clear that

$$
W_{\phi} f = \rho^{5/2} f * \mathcal{\phi}_{\rho,u,v}.
$$

(5.1)

Thus we can extend the wavelet transform $W_{\phi} f$ provided (5.1) is valid.
Let

\[ L^2_{\natural}(\mathcal{D}) = \left\{ f \in L^2(\mathcal{D}) : \int_{\mathbb{R}^n} |\hat{f}(a)|^2|a|^{-2}da < \infty \right\}, \]

\[ L^2_{\flat}(\mathcal{D}) = \left\{ f \in L^2(\mathcal{D}) : \int_{\mathbb{R}^n} |\hat{f}(a)|^2|a|^6da < \infty \right\}. \]

It is obvious that the Radon transform \( \mathcal{R} \) is an isomorphism from \( L^2_{\natural}(\mathcal{D}) \) onto \( L^2_{\flat}(\mathcal{D}) \).

Suppose \( \phi \in \mathcal{S}(\mathcal{D}) \cap \mathcal{A} \mathcal{W} \) and \( f \in L^2_{\natural}(\mathcal{D}) \). Then \( W_{\mathcal{L}\mathcal{R}\mathcal{L}(\phi)}\mathcal{R}(f) \) is well defined and

\[ W_{\mathcal{L}\mathcal{R}\mathcal{L}(\phi)}\mathcal{R}(f) = \rho^2 W\phi f. \]

By Theorem 5, we obtain

**Theorem 10.** Suppose \( \phi \in \mathcal{S}(\mathcal{D}) \cap \mathcal{A} \mathcal{W} \). If \( f \in L^2_{\natural}(\mathcal{D}) \), then

\[ f(x,t) = C^{-1}_\eta \int_{\mathbb{R}^n} \mathcal{L}_{\mathcal{R}\mathcal{L}(\phi)} f(y, s, \rho, u, v) U(y, s, \rho, u, v) \phi(x, t) \frac{dy ds d\rho dv}{\rho^8}. \] (5.2)

in \( L^2 \)-sense. Furthermore, if \( f \in \mathcal{S}(\mathcal{D}) \cap L^1_{\natural}(\mathcal{D}) \), then (5.2) holds pointwise. Equivalently, if \( f \in L^2_{\flat}(\mathcal{D}) \), then

\[ \mathcal{R}^{-1}(f)(x, t) = C^{-1}_\eta \int_{\mathbb{R}^n} \mathcal{L}_{\mathcal{R}\mathcal{L}(\phi)} f(y, s, \rho, u, v) U(y, s, \rho, u, v) \phi(x, t) \frac{dy ds d\rho dv}{\rho^8}. \] (5.3)

in \( L^2 \)-sense. Furthermore, if \( f \in \mathcal{S}(\mathcal{D}) \), then (5.3) holds pointwise.

If the radial wavelet \( \phi_\eta \) is defined as in Section 3, then \( \mathcal{L}\mathcal{R}\mathcal{L}(\phi) \) is also a radial function. By Theorem 6, we have

**Theorem 11.** Suppose \( \eta \in \mathcal{S}(\mathbb{R}^+) \) satisfies (3.4) and \( \phi_\eta \) is defined as in Section 3. If \( f \in L^2_{\eta}(\mathcal{D}) \), then

\[ f(x,t) = C^{-1}_\eta \int_{\mathbb{R}^n} \mathcal{L}\mathcal{R}\mathcal{L}(\phi_\eta)\mathcal{R}(f)(y, s, \rho) U(y, s, \rho) \phi_\eta(x, t) \frac{dy ds d\rho}{\rho^8}. \] (5.4)

in \( L^2 \)-sense. Furthermore, if \( f \in \mathcal{S}(\mathcal{D}) \cap L^2_{\eta}(\mathcal{D}) \), then (5.4) holds pointwise. Equivalently, if \( f \in L^2_{\flat}(\mathcal{D}) \), then

\[ \mathcal{R}^{-1}(f)(x, t) = C^{-1}_\eta \int_{\mathbb{R}^n} \mathcal{L}\mathcal{R}\mathcal{L}(\phi_\eta) f(y, s, \rho) U(y, s, \rho) \phi_\eta(x, t) \frac{dy ds d\rho}{\rho^8}. \] (5.5)

in \( L^2 \)-sense. Furthermore, if \( f \in \mathcal{S}(\mathcal{D}) \), then (5.5) holds pointwise.

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