DECAY OF CORRELATIONS FOR INVERTIBLE MAPS WITH NON-HÖLDER OBSERVABLES

MARKS RUZIBOEV

ABSTRACT. An invertible dynamical system with some hyperbolic structure is considered. Upper estimates for the correlations of continuous observables is given in terms of modulus of continuity. The result is applied to certain Hénon maps and Solenoid maps with intermittency.

1. INTRODUCTION

Let \( f : M \to M \) be a map preserving a probability measure \( \mu \). The system \( (f, \mu) \) is called mixing if

\[
|\mu(f^{-n}A \cap B) - \mu(A)\mu(B)| \to 0 \quad \text{as} \quad n \to \infty
\]

for any \( \mu \)-measurable \( A \) and \( B \). One of the manifestations of chaos in a dynamical systems is this mixing property. In particular, speed of the convergence above is a measure of the strength of chaos in the system \( (f, \mu) \). Known counterexamples show that in general there is no specific rate at which this convergence to zero happens. One way to overcome this difficulty is to generalize the problem by defining the correlation of observables \( \varphi, \psi : M \to \mathbb{R} \)

\[
C_n(\varphi, \psi; \mu) := \left| \int (\varphi \circ f^n)\psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \right|.
\]

In this way sometimes it is possible to obtain specific rates of decay of correlations, for observables with certain regularity.

The rates of decay of correlations have been studied extensively over thirty years. In their pioneering works Sinai [27], Ruelle [26], Bowen [6] showed that every uniformly hyperbolic system admits a special invariant measure (that is now called SRB-measure) with exponential decay of correlations for Hölder continuous observables. It turns out that generalizing the above results to the systems with singularities or to the systems with weaker hyperbolic properties is a very challenging problem and there much progress has been made in recent years by many authors here we give just a few of them [1, 5, 8, 10, 11, 12, 13, 15, 19, 20, 21, 28, 29].

Notice that in the above works observables are assumed to be Hölder continuous, or functions of bounded variation. Hence a natural question to

\begin{flushright}
Date: November 27, 2014.
2000 Mathematics Subject Classification. 37A05, 37A25.
Key words and phrases. Decay of correlations, Hénon map, Solenoid map, Young Towers.
\end{flushright}
ask is how much we can generalize the classes of observables that still admit some decay rate? Several papers address this question in the context of one-sided subshifts of finite type on a finite alphabet for example, see [7, 14, 18, 17, 24]. For a comprehensive discussion of shift maps and their ergodic properties we refer to [2]. Moreover, there are results on non-invertible Young Towers, for example [9, 18, 22, 23], and results that apply directly to certain non-uniformly expanding systems [25]. We emphasize that all of the results we above are for non-invertible maps and in the invertible case the only reference we found is [30] which gives interesting estimates for billiards with non-Hölder observables.

The aim of this work is to fill this gap, and show that the decay rate of correlations for continuous observables is given in terms of natural quantities, such as the modulus of continuity, for systems that admit Gibbs-Markov-Young (GMY)-structure, to be described below.

The rest of the paper is organized as follows. In the next section we give the definition of GMY-structure, state the main technical theorem and its applications to Hénon maps and Solenoid maps with intermittent fixed point. In section 3 we give the proof of the main technical theorem, and in the last section we prove the results for the Hénon maps and Solenoid maps.

2. Statement of results

Let $M$ be a metric space and $\mathcal{C}(M)$ be the space of continuous functions defined on it. Define modulus of continuity for $\varphi \in \mathcal{C}(M)$ as

$$\mathcal{R}_\varphi(\varepsilon) = \sup\{|\varphi(x) - \varphi(y)| : d(x, y) < \varepsilon\}.$$ 

Obviously, $\mathcal{R}_\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$ if and only if $\varphi$ is uniformly continuous. If $\varphi$ is Hölder continuous with Hölder exponent $\alpha$ then $\mathcal{R}_\varphi(\varepsilon) \lesssim \varepsilon^\alpha$. In general, $\mathcal{R}_\varphi(\varepsilon)$ can converge to 0 very slowly, and the rate of convergence is slower than the rate of convergence of $\varepsilon$ to 0 even for Hölder observables. We refer to [16] for the examples of various slow rates of convergence.

Let $a_n$ be a sequence of positive numbers such that $\lim_{n \to \infty} a_n = 0$, and let $\mathcal{F}(M)$ be a Banach space of observables defined on $M$. We say $(f, \mu)$ has decay of correlations at rate $a_n$ for functions in $\mathcal{F}(M)$ if for any $\varphi, \psi \in \mathcal{F}(M)$ there exists a constant $C = C(\varphi, \psi, f)$ such that

$$C_n(\varphi, \psi; \mu) \leq C a_n.$$ 

Below we give applications of our main technical result to Hénon maps and Solenoid maps with intermittency.

2.1. Hénon maps. Let $T_{a,b} : \mathbb{R} \to \mathbb{R}$ denote the Hénon map i.e.

$$T_{a,b}(x, y) = (1 - ax^2 + y, bx).$$

In [3, 4, 5] it was shown that there exists positive measure set $\mathcal{A}$ of parameters $(a, b)$ such that for any $(a, b) \in \mathcal{A}$ corresponding $T_{a,b}$ admits unique

$^1$Throughout the paper we use the notation $a \lesssim b$ if there exists a constant $C$ independent of $\varepsilon$ such that $a \leq C b$. 
SRB-measure which is mixing and the speed of mixing is exponential for Hölder continuous observables. Here we assume that \((a, b) \in \mathcal{A}\), and investigate the problem of decay of correlations for continuous observables. More precisely, let \(\mathcal{C}(\mathbb{R}^2)\) be the space of all continuous real valued functions. Let

\[
\mathcal{R}_{\varphi, \psi}(\varepsilon) = \max\{\mathcal{R}_{\varphi}(\varepsilon), \mathcal{R}_{\psi}(\varepsilon)\}.
\]

**Theorem 2.1.** Let \(f = T_{a, b}\) and \((a, b) \in \mathcal{A}\). Then there exists \(\theta \in (0, 1)\) such that for any \(\varphi, \psi \in \mathcal{C}(\mathbb{R}^2)\)

\[
C_n(\varphi, \psi; \mu) \leq 2(\|\varphi\|_{\infty} + \|\psi\|_{\infty})\mathcal{R}_{\varphi, \psi}(\theta^n) + C\theta^n
\]

where constant \(C\) depends on \(\varphi, \psi\) and \(f\).

The following corollary is a direct application of the above theorem.

**Corollary 2.2.** Let \(f = T_{a, b}\) and \((a, b) \in \mathcal{A}\).

(i) If \(\mathcal{R}_{\varphi, \psi}(\varepsilon) \lesssim e^{\alpha \log \varepsilon}\) i.e. if the observables \(\varphi, \psi\) are Hölder continuous with exponent \(\alpha\) then \(C_n(\varphi, \psi; \mu) \lesssim e^{-\alpha n}; \alpha' = \alpha \log \theta\).

(ii) If \(\mathcal{R}_{\varphi, \psi}(\varepsilon) \lesssim e^{-|\log \varepsilon|^\alpha}\), \(\alpha \in (0, 1)\) then \(C_n(\varphi, \psi; \mu) \lesssim e^{-n^{\alpha}}\).

(iii) If \(\mathcal{R}_{\varphi, \psi}(\varepsilon) \lesssim |\log \varepsilon|^{-\alpha}\), for \(\alpha > 0\) then \(C_n(\varphi, \psi; \mu) \lesssim n^{-\alpha}\).

Notice that the upper in (i) is the same as in [5], which is natural to expect. The estimates in (ii)-(iii) are new.

2.2. **Solenoid with intermittency.** The second class of maps we consider is a solenoid map with an indifferent fixed point \([1]\), which is defined as follows. Let \(M = \mathbb{S}^1 \times D^2\), where \(D^2\) is a unit disk in \(\mathbb{R}^2\). For \((x, y, z) \in M\) let

\[
g(x, y, z) = \left(f(x), \frac{1}{10}y + \frac{1}{2} \cos x, \frac{1}{10}z + \frac{1}{2} \sin x\right),
\]

where \(f : \mathbb{S}^1 \to \mathbb{S}^1\) is a map of degree \(d \geq 2\) with the properties: \(f\) is \(C^2\) on \(\mathbb{S}^1 \setminus \{0\}\); \(f\) is \(C^1\) on \(\mathbb{S}^1\) and \(f' > 1\) on \(\mathbb{S}^1 \setminus \{0\}\); \(f(0) = 0, f'(0) = 1\) and there exists \(\gamma > 0\) such that \(-xf'' \approx |x|^\gamma\) for all \(x \neq 0\).

**Theorem 2.3.** Let \(g\) be the map described above. Assume that \(\gamma < 1\). Then for any \(\varphi, \psi \in \mathcal{C}(M)\)

\[
C_n(\varphi, \psi; \mu) \leq 2(\|\varphi\|_{\infty} + \|\psi\|_{\infty}) \max\{\mathcal{R}_{\varphi}(n^{-1/\gamma}), \mathcal{R}_{\psi}(n^{-1/\gamma})\} + Cn^{1-1/\gamma}
\]

where constant \(C\) depends on \(\varphi\) and \(\psi\).

Direct application of the theorem for specific classes of observables gives.

**Corollary 2.4.** Let \(g\) be a map as in the theorem above. Then

(i) If \(\mathcal{R}_{\varphi, \psi}(\varepsilon) \lesssim e^{\alpha \log \varepsilon}\) i.e. the observables \(\varphi, \psi\) are Hölder continuous with exponent \(\alpha\) then \(C_n(\varphi, \psi; \mu) \lesssim e^{-\min\{\alpha/\gamma, 1/\gamma - 1\} \log n}\).

(ii) If \(\mathcal{R}_{\varphi, \psi}(\varepsilon) \lesssim e^{-|\log \varepsilon|^\alpha}\), \(\alpha \in (0, 1)\) then \(C_n(\varphi, \psi; \mu) \lesssim e^{-\log(n)^\alpha}\).

(iii) If \(\mathcal{R}_{\varphi, \psi}(\varepsilon) \lesssim |\log \varepsilon|^{-\alpha}\), for \(\alpha > 0\) then \(C_n(\varphi, \psi; \mu) \lesssim |\log n|^{-\alpha}\).

The estimate in the first item coincides with the one given in [1] the remaining two are new.
2.3. Young Towers. We now define GMY-structure [11] which generalizes the definition of [23] (see Remarks 2.3-2.5 in [1]). Let \( f : M \to M \) be a diffeomorphism of Riemannian manifold \( M \). If \( \gamma \subset M \) is a submanifold, then \( m_\gamma \) denotes the restriction of the Riemannian volume to \( \gamma \). Assume that \( f \) satisfies the following conditions.

(A1) There exists \( \Lambda \subset M \) with hyperbolic product structure, i.e. there are continuous families of stable and unstable manifolds \( \Gamma^s = \{ \gamma^s \} \) and \( \Gamma^u = \{ \gamma^u \} \) such that \( \Lambda = (\cup \gamma^s) \cap (\cup \gamma^u) \); \( \dim \gamma^s + \dim \gamma^u = \dim M \); each \( \gamma^s \) meets each \( \gamma^u \) at a unique point; stable and unstable manifolds are transversal with angles bounded away from 0; \( m_\gamma (\gamma \cap \Lambda) > 0 \) for any \( \gamma \in \Gamma^u \).

Let \( \Gamma^s \) and \( \Gamma^u \) be the defining families of \( \Lambda \). A subset \( \Lambda_0 \subset \Lambda \) is called \( s \)-subset if \( \Lambda_0 \) also has a hyperbolic structure and its defining families can be chosen as \( \Gamma^u \) and \( \Gamma_0^s \subset \Gamma^s \). Similarly, we define \( u \)-subsets. For \( x \in \Lambda \) let \( \gamma^s(x) \) denote the element of \( \Gamma^s \) containing \( x \), where \( \sigma \in \{ u, s \} \).

(A2) There are pairwise disjoint \( s \)-subsets \( \Lambda_1, \Lambda_2, ..., \subset \Lambda \) such that 
\[
m_{\gamma^s}(\Lambda \setminus \cup \Lambda_i) \cap \gamma^u = 0
\]
on each \( \gamma^u \), and for each \( \Lambda_i, i \in \mathbb{N} \) there is \( R_i \) such that \( f^{R_i}(\Lambda_i) \) is \( u \)-subset; \( f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}(x)) \) and \( f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}(x)) \) for any \( x \in \Lambda_i \). The separation time \( s(x, y) \) is the smallest \( k \) where \( (f^R)^k(x) \) and \( (f^R)^k(y) \) lie in different partition elements.

(A3) There exist constants \( C > 0 \) and \( \beta \in (0, 1) \) such that 
\[
\text{dist}(f^n(x), f^n(y)) \leq C \beta^n
\]
for all \( y \in \gamma^s(x) \) and \( n \geq 0 \).

Let \( f^n \) denote the restriction of \( f \) onto unstable disks, and \( Df^n \) its differential.

(A4) Regularity of the stable foliation: given \( \gamma, \gamma' \in \Gamma^u \) define \( \Theta : \gamma' \cap \Lambda \to \gamma \cap \Lambda \) by \( \Theta(x) = \gamma^s(x) \cap \gamma \). Then 
(a) \( \Theta \) is absolutely continuous and 
\[
u(x) := \frac{d(\Theta_* m_\gamma)}{dm_\gamma}(x) = \prod_{i=0}^{\infty} \frac{\det Df^n(f^i(x))}{\det Df^n(f^i(\Theta^{-1}(x)))};
\]
(b) We assume that there exists \( C > 0 \) and \( \beta \in (0, 1) \) such that 
\[
\log \frac{\nu(x)}{\nu(y)} \leq C \beta^{s(x,y)} \text{ for } x, y \in \gamma' \cap \Lambda.
\]

(A5) Bounded distortion: for \( \gamma \in \Gamma^u \) and \( x, y \in \Lambda \cap \gamma \)
\[
\log \frac{\det D(f^R)^u(x)}{\det D(f^R)^u(y)} \leq C \beta^{s(f^R(x), f^R(y))}.
\]

(A6) Integrability: \( \int R \, dm_0 < \infty \).

(A7) Aperiodicity: \( \gcd\{ R_i \} = 1 \).

The geometric structure described in (A1) and (A2) allows us to define the corresponding Young Tower. More precisely, we let
\[
\mathcal{T} = \{(z, \ell) \in \Lambda \times \mathbb{Z}_0^+ \mid R(z) > \ell \},
\]
where $\mathbb{Z}_n^+$ denotes the set of all nonnegative integers. For $\ell \in \mathbb{Z}_n^+$ the subset $\mathcal{T}_\ell = \{(\cdot, \ell) \in \mathcal{T}\}$ of $\mathcal{T}$ is called its $\ell$th level (identify $\Lambda$ with $\mathcal{T}_0$ and $\Lambda_i$ with $\mathcal{T}_{0,i}$). The sets $\mathcal{T}_{\ell,i} := \{(z, \ell) \in \mathcal{T}_\ell | (z, 0) \in \mathcal{T}_{0,i}\}$ give a partition $\mathcal{P}$ of $\mathcal{T}$. We can define a map $F : \mathcal{T} \to \mathcal{T}$ letting

\begin{equation}
F(z, \ell) = \begin{cases} (z, \ell + 1) & \text{if } \ell + 1 < R(z), \\
(f_R(z), 0) & \text{if } \ell + 1 = R(z). \end{cases}
\end{equation}

A measure $m$ on $\mathcal{T}$ is defined as follows. Let $\mathcal{A}$ be a $\sigma$-algebra on $\Lambda \subseteq M$, and let $m_{\Lambda}$ denote a finite measure on $\Lambda$. For any $\ell \geq 0$ and $A \in \mathcal{T}_\ell$ such that $F^{-\ell}(A) \in \mathcal{A}$ define $m(A) = m_{\Lambda}(F^{-\ell}(A))$. There is tower projection $\pi : \mathcal{T} \to M$, which is a semi-conjugacy $f \circ \pi = \pi \circ F$, defined as follows: for $x \in \mathcal{T}$ with $x_0 \in \Lambda$ and $F^\ell(x_0) = x$ we let $\pi(x, \ell) = f^\ell(x_0)$.

Notice that it is not strictly necessary for $f$ to be a diffeomorphism. Such a structure can be defined for example also in the presence of some discontinuities as long as the stable and unstable manifolds exist and satisfy the required properties, this has been done for example for a class of Billiards in [10].

It is known from [28] that under the above assumptions $f$ admits an SRB measure $\mu$ and here we study further properties of the measure $\mu$. Now, for $n \geq 0$ introduce a sequence of partitions as follows:

\[ \mathcal{P}_0 = \mathcal{P} \quad \text{and} \quad \mathcal{P}_n = \bigvee_{i=0}^{n-1} F^{-i}\mathcal{P}. \]

Let

\[ \delta_n = \sup\{\text{diam}_\pi(F^n(P)) : P \in \mathcal{P}_{2n}\}. \]

Now we state the main technical theorem which is of independent interest.

**Main Technical Theorem.** Let $f : M \to M$ be a diffeomorphism of a Riemannian manifold $M$, which admits a Young tower. Then for any non-zero observables $\varphi, \psi \in C(M)$ we have

\begin{equation}
C_n(\varphi, \psi; \mu) \leq 2(\|\varphi\|_\infty + \|\psi\|_\infty) \max\{\mathcal{R}_\varphi(\delta_n), \mathcal{R}_\psi(\delta_n)\} + u_n
\end{equation}

where $u_n$ is a sequence of positive numbers defined as follows:

(i) If $m\{R > n\} \leq C\theta^n$ for some $C > 0$ and $\theta \in (0, 1)$, then there exist $\theta' \in (0, 1)$ and $C' > 0$ such that $u_n \leq C'\theta^n$.

(ii) If $m\{R > n\} \leq Ce^{-cn}\eta^n$ for some $C, c > 0$ and $\eta \in (0, 1)$, then there are $C', c' > 0$ such that $u_n \leq C'e^{-c'n}\eta^n$.

(iii) If $m\{R > n\} \leq Cn^{-\alpha}$ for some $C > 0$ and $\alpha > 1$ then there exists $C' > 0$ such that $u_n \leq C'n^{-1}\alpha$.

3. **Proof of main technical theorem**

In this section we reduce the system to a non-invertible system, as in [28]. We start by defining a special measure on $\Lambda$. 

3.1. The natural measures on the unstable manifolds. We fix $\hat{\gamma} \in \Gamma_u$. For any $\gamma \in \Gamma_u$ and $x \in \gamma \cap \Lambda$ let $\hat{x}$ be the point $\gamma^s(x) \cap \gamma$. Define for $x \in \gamma \cap \Lambda$

$$\hat{u}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\hat{x}))}.$$  

By item (b) of assumption (A4) $\hat{u}$ satisfies the bounded distortion property. For each $\gamma \in \Gamma_u$ define the measure $m_\gamma$ as

$$\frac{dm_\gamma}{d\text{Leb}_\gamma} = \hat{u}1_{\gamma \cap \Lambda},$$

where $1_{\gamma \cap \Lambda}$ is the characteristic function of $\gamma \cap \Lambda$. The proof of the following lemma is fairly standard and can be found in [1].

Lemma 3.1.  
(i) Let $\Theta$ be the map defined in (A4). Then $\Theta_* m_\gamma = m_{\gamma'}$ for any $\gamma, \gamma' \in \Gamma_u$.

(ii) Let $\gamma, \gamma' \in \Gamma_u$ be such that $f^R(\gamma \cap \Lambda) \subset \gamma'$, and let $Jf^R(x)$ denote the Jacobian of $f^R$ with respect to the measures $m_\gamma$ and $m_{\gamma'}$. Then $Jf^R(x)$ is constant on the stable manifolds and there is $C > 0$ such that for every $x, y \in \gamma \cap \Lambda$

$$|Jf^R(x) - Jf^R(y)| \leq C \beta^s(f^R(x), f^R(y)).$$

3.2. Quotient tower. Let $\bar{\Lambda} = \Lambda/\sim$, where $x \sim y$ if and only if $y \in \gamma^s(x)$. This equivalence relation gives rise to a quotient tower

$$\Delta = \mathcal{F}/\sim$$

with $\Delta_\ell = \mathcal{T}_\ell/\sim$ and its partition into $\Delta_{0,i} = \mathcal{T}_{0,i}/\sim$ which we denote by $\bar{P}$. There is a natural projection $\bar{P} : \mathcal{F} \to \Delta$. Since $f^R$ preserves stable leaves and $R$ is constant on them the return time $\bar{R}$ and separation time $\bar{s}$ are well defined by $R$ and $s$. Moreover, we can define a tower map $\bar{F} : \Delta \to \Delta$. Let $m$ be a measure whose restriction onto unstable manifolds is $m_\gamma$. Lemma 3.1 implies that there is a measure $\bar{m}$ on $\Delta$ whose restriction to each $\gamma \in \Gamma_u$ is $m_\gamma$. We let $\bar{JF}$ denote the Jacobian of $\bar{F}$ with respect to $\bar{m}$.

Next we introduce the space of Hölder continuous functions on $\Delta$ as

$$\mathcal{F}_\beta = \{ \varphi : \Delta \to \mathbb{R} : \exists C_\varphi \text{ such that } |\varphi(x) - \varphi(y)| \leq C_\varphi \beta^s(x,y) \ \forall x, y \in \Delta \}.$$  

$$\mathcal{F}^+_\beta = \{ \varphi \in \mathcal{F} : \exists C_\varphi \text{ such that on each } \Delta_{\ell,i}, \text{ either } \varphi \equiv 0, \text{ or } \varphi > 0, \frac{\varphi(x)}{\varphi(y)} - 1 \leq C_\varphi \beta^s(x,y) \ \forall x, y \in \Delta_{\ell,i} \}.$$  

The mixing properties of $\bar{F}$ was studied firstly in [29]. Several papers appeared improving or extending the results given in [29], for example [15, 16, 22]. Here we combine the results from [1] and [29].

Theorem 3.2. [1, 29]
Proposition 3.3.

(i) \( \bar{F} \) admits unique mixing acip \( \bar{\nu} \); \( d\bar{\nu}/d\bar{m} \in \mathcal{F}^\tau \) and \( d\bar{\nu}/d\bar{m} > c > 0 \).

(ii) Let \( \lambda \) be a probability measure with \( \varphi = d\lambda/d\bar{m} \in \mathcal{F}^\tau_\beta \).

1. If \( \{ R > n \} \leq C\theta^n \) for some \( C > 0 \) and \( \theta \in (0, 1) \) then there exists \( C' > 0 \) and \( \theta' \in (0, 1) \) such that \( |\bar{F}^n - \nu| \leq C'\theta^n \).

2. If \( \{ R > n \} \leq C e^{-cn^m} \) for some \( C, c > 0 \) and \( n \in (0, 1) \) then there exists \( C', c' > 0 \) such that \( |\bar{F}^n - \nu| \leq C'e^{-cn^m} \). Moreover \( c' \) does not depend on \( \varphi \), \( C' \) depends only on \( C_\varphi \).

3. If \( \{ R > n \} \leq Cn^{-\alpha} \) for some \( C > 0 \) and \( \alpha > 1 \) then there exists \( C' > 0 \) such that \( |\bar{F}^n - \nu| \leq C'n^{-\alpha} \).

3.3. Approximation of correlations. We establish the relation between the original problem and problem of estimating the decay rates of correlations on the quotient tower, and then we apply theorem 3.2.

Let \( \pi : \mathcal{T} \to M \), \( \bar{\pi} : \mathcal{T} \to \Delta \) be the tower projections, then we have \( \bar{\nu} = \bar{\pi}_*\nu \) and \( \mu = \pi_*\nu \). Given \( \varphi, \psi \in C^0(M) \) define \( \bar{\varphi} = \varphi \circ \pi \) and \( \bar{\psi} = \psi \circ \pi \). By definition

\[
\int (\varphi \circ f^n) \psi d\mu - \int \varphi d\mu \int \psi d\mu = \int (\bar{\varphi} \circ F^n) \bar{\psi} d\nu - \int \bar{\varphi} d\nu \int \bar{\psi} d\nu,
\]

which shows it is sufficient to obtain estimates for the lifted observables. This will be done by approximating the lifted observables with piecewise constant observables on tower. For \( k \leq n/4 \) define \( \bar{\varphi}_k \) as follows

\[
\bar{\varphi}_k|_P = \inf \{ \bar{\varphi} \circ F^k(x) | x \in P \}, \text{ where } P \in \mathcal{P}_{2k}.
\]

Define \( \bar{\psi}_k \) in a similar way, and note that \( \bar{\varphi}_k \) and \( \bar{\psi}_k \) are constant on the stable leaves. Hence, we can consider them as a function defined on quotient tower \( \Delta \). The main result of this section is the following

Proposition 3.3.

\[
|C_n(\bar{\varphi}, \bar{\psi}; \nu) - C_n(\bar{\varphi}_k, \bar{\psi}_k; \bar{\nu})| \leq 2(\|\bar{\varphi}\|_\infty + \|\bar{\psi}\|_\infty) \max \{\mathcal{R}_\varphi(\delta_k); \mathcal{R}_\psi(\delta_k)\}.
\]

**Proof.** The proof consists of several steps and follows the argument in [1]. First we claim

\[
|C_n(\bar{\varphi}, \bar{\psi}; \nu) - C_{n-k}(\bar{\varphi}_k, \bar{\psi}_k; \nu)| \leq 2\|\bar{\psi}\|_{\infty} \mathcal{R}_\varphi(\delta_k).
\]

Indeed, using the fact \( C_n(\bar{\varphi}, \bar{\psi}; \nu) = C_{n-k}(\bar{\varphi} \circ F^k, \bar{\psi} \circ \pi; \nu) \) the left hand side of (3.1) can be written as

\[
\left| \int (\bar{\varphi} \circ F^k - \bar{\varphi}_k) \bar{\psi} d\nu \right| \leq 2\|\bar{\psi}\|_{\infty} \int |\bar{\varphi} \circ F^k - \bar{\varphi}_k| d\nu.
\]

By definition of \( \bar{\varphi}_k \) for \( x \in P \) we have

\[
|\bar{\varphi} \circ F^k(x) - \bar{\varphi}_k| \leq \sup_{x,y \in P} |\bar{\varphi}(F^k(x)) - \bar{\varphi}(F^k(y))| \leq \mathcal{R}_\varphi(\delta_k),
\]
which implies desired conclusion. Now, let \( \tilde{\psi}_k \nu \) be the measure whose density with respect to \( \nu \) is \( \psi_k \) and let \( \tilde{\psi}_k = dF_k^* (\tilde{\psi}_k \nu) / d\nu \). Then

\[
|C_{n-k}(\tilde{\varphi}_k, \tilde{\psi}; \nu) - C_{n-k}(\tilde{\varphi}_k, \tilde{\psi}_k; \nu)| \leq 2\|\varphi\|_\infty R_\psi (\delta_k).
\]

(3.2) After substituting and simplifying the expression we obtain

\[
|C_{n-k}(\tilde{\varphi}_k, \tilde{\psi}; \nu) - C_{n-k}(\tilde{\varphi}_k, \tilde{\psi}_k; \nu)| \leq 2\|\varphi\|_\infty \left| \int (\tilde{\psi} - \tilde{\psi}_k) d\nu \right|.
\]

First observe that \( F_k^* (\tilde{\psi} \circ F^k) \nu = \tilde{\psi} \nu \). Letting \( | \cdot | \) denote the variational norm for measures we have

\[
\left| \int (\tilde{\psi} - \tilde{\psi}_k) d\nu \right| = |F_k^* (\tilde{\psi} \circ F^k) \nu - F_k^* (\tilde{\psi}_k \nu)| \\
\leq \left| (\tilde{\psi} \circ F^k - \tilde{\psi}_k) \nu \right| = \int |\psi \circ F^k - \tilde{\psi}_k| d\nu.
\]

As in the proof of (3.1) we have \( |\psi \circ F^k - \tilde{\psi}_k| \leq R_\psi (\delta_k) \) which implies relation 3.2. Combining the inequalities (3.1), (3.2) and the equality \( C_{n-k}(\tilde{\varphi}_k, \tilde{\psi}_k; \nu) = C_n(\tilde{\varphi}_k, \tilde{\psi}_k; \tilde{\nu}) \) from [1] finishes the proof.

It remains to prove decay of correlations for the observables \( \tilde{\varphi}_k \) and \( \tilde{\psi}_k \) on the quotient tower. We start with the usual transformations that simplify the correlation function. Without lost of generality assume that \( \tilde{\psi}_k \) is not identically zero. Let \( b_k = \left( \int (\tilde{\psi}_k + 2\|\tilde{\psi}_k\|_\infty) d\tilde{\nu} \right)^{-1} \) and \( \psi_k = b_k (\tilde{\psi}_k + 2\|\tilde{\psi}_k\|_\infty) \), then we have \( \int \tilde{\psi}_k d\tilde{\nu} = 1, \|\tilde{\psi}_k\| \leq b_k^{-1} \leq 3\|\tilde{\psi}_k\| \) and \( 1 \leq \tilde{\psi}_k \leq 3 \). Moreover, \( \tilde{\psi}_k \) is constant on the elements of \( \mathcal{P}_{2k} \). Thus

\[
C_n(\tilde{\varphi}_k, \tilde{\psi}_k; \tilde{\nu}) = \left| \int (\tilde{\varphi} \circ F^n) \tilde{\psi}_k d\tilde{\nu} - \int \tilde{\varphi}_k d\tilde{\nu} \right| = \\
\frac{1}{b_k} \left| \int (\varphi \circ F^n) \tilde{\psi}_k d\tilde{\nu} - \int \tilde{\varphi}_k d\tilde{\nu} \right| \leq \\
\frac{1}{b_k} \|\tilde{\varphi}_k\|_\infty \int \left| dF_k^n (\tilde{\psi}_k \tilde{\nu}) - \frac{d\tilde{\nu}}{d\tilde{m}} \right| d\tilde{m}.
\]

(3.3) Now, letting \( \hat{\lambda}_k = F_k^{*k} (\tilde{\psi}_k \tilde{\nu}) \) we conclude

\[
C_n(\tilde{\varphi}_k, \tilde{\psi}_k; \tilde{\nu}) \leq \frac{1}{b_k} \|\varphi\|_\infty \left| F_k^{n-2k} \hat{\lambda}_k - \tilde{\nu} \right|.
\]

(3.4) Note that the density of \( \hat{\lambda}_k \) belongs to the class \( \mathcal{F}^+ \) (see Lemma 4.1, [1]). Hence we can apply theorem [3.2] to \( F_k^{n-2k} \hat{\lambda}_k - \tilde{\nu} \) and obtain desired estimates for \( C_n(\tilde{\varphi}_k, \tilde{\psi}_k; \tilde{\nu}) \).
4. Proofs of Theorems 2.1 and 2.3

We start this section with the following auxiliary construction. Consider the sequence of stopping times for the points in $\Lambda$ defined as follows:

\[(4.1) \quad S_0 = 0, \ S_1 = R \text{ and } S_{i+1} = S_i + R \circ f^{S_i}, \text{ for } i \geq 1.\]

Let $Q_0$ be the partition of $\Lambda$ into $\Lambda_i$'s. Define the sequence of partitions $Q_k$ as:

- $x, y \in \Lambda$ belong to the same element of $Q_k$ if the following conditions hold.
  - (i) $f^R(x)$ and $f^R(y)$ have the same stopping times up to time $k - 1$.
  - (ii) $f^{S_i}(f^R(x))$ and $f^{S_i}(f^R(y))$ belong to the same element of $Q_0$ for each $0 \leq i \leq k - 1$.
  - (iii) $f^{S_k}(Q)$ is a $u$-subset.

For $Q \in Q_0$ let $R(Q)$ denote its return time. Let $k \geq 1$ be an arbitrary integer and define a sequence

$$\bar{\delta}_k = \sup_{Q \in Q_0} \bar{\delta}_k(Q),$$

where $\bar{\delta}_k(Q)$ is defined as follows:

1. For $k > R(Q) - 1$, let
$$\bar{\delta}_k := \sup_{0 \leq \ell \leq R(Q) - 1} \{\text{diam}(f^\ell(A \cap \gamma)) : \gamma \in \Gamma^u, A \in Q_{k-R(Q)+1+\ell}, A \subset Q\}.$$  
2. For $k \leq R(Q) - 1$, let
$$\bar{\delta}_k(Q) := \sup_{0 \leq \ell < R(Q) - k} \{\text{diam}(f^\ell(Q \cap \gamma)) : \gamma \in \Gamma^u\},$$
$$\bar{\delta}_k^+(Q) := \sup_{R(Q) - k \leq \ell \leq R(Q) - 1} \{\text{diam}(f^\ell(A \cap \gamma)) : \gamma \in \Gamma^u, A \in Q_{k-R(Q)+1+\ell}, A \subset Q\}.$$

and define

$$\bar{\delta}_k(Q) = \sup\{\bar{\delta}_k^0(Q), \bar{\delta}_k^+(Q)\}.$$

From Lemma 3.2 in [1] we have

\[(4.2) \quad \text{diam}(\pi(F^k(P))) \leq C \max\{\beta^k, \bar{\delta}_k\}\]

for any $P \in \mathcal{P}_{2k}, k \geq 0$, and some $C > 0$. This is the main estimate we use to prove theorems 2.1 and 2.3.

In [1] it was proven that $\delta_k \leq k^{-1/\gamma}$ and $m\{R > k\} \leq k^{-1/\gamma}$ for the Solenoid map with intermittent fixed point. Substituting this into (4.2), we can apply item (ii) of main technical theorem and conclude the proof of theorem 2.3.

In [5] it was shown that for any $(a, b) \in \mathcal{A}$ the corresponding Hénon map admits a Young tower, for which, the tail of the return time decays exponentially. Therefore to complete the proof of theorem 2.2 it is sufficient to show that $\bar{\delta}_k$ decays exponentially. We will show this in the following lemma and complete the proof in this case also.
In [28] (see [5] for the details of the construction) it was shown that Hénon maps satisfy backward contraction on the unstable leaves, that is there exists $C > 0$ such that for all $x, y \in \Lambda_i$ with $y \in \gamma^u(x)$ and $0 \leq n \leq R_i$

$$\text{dist}(f^n(x), f^n(y)) \leq C\beta^{R_i-n}.$$  

(4.3)

Lemma 4.1. \exists C > 0 and $\beta' \in (0, 1)$ such that $\delta_k^0 \leq C\beta'^k$.

Proof. A) We start with the case $k \leq R(P) - 1$ and $0 \leq \ell < R(P) - k$. By (4.3) for any $x \in P$

$$\text{diam}(f^\ell(P \cap \gamma^u(x))) \leq C\beta^{R(P)-\ell} \leq C\beta^k.$$  

This implies $\delta_k^0 \leq \beta^k$.

B) Now, consider the case $k \leq R(P) - 1$, and $R(P) - k - \ell \leq R(P) - 1$. Notice that for any $Q \subset P, Q \in \mathcal{P}_k$, the stopping times $S_1, \ldots, S_{\ell'}, \ell' = k - R(P) + \ell$, are constant on $Q$ and $f^{S_i}(Q) \subset P_i$, for some $P_i \in \mathcal{Q}_0$, $i = 1, \ldots, \ell'$. Let $r_1, \ldots, r_{\ell'}-1$ be the return times of these elements. By (4.3) we have

$$\text{diam}(Q \cap \gamma^u) \leq C\beta^{R(P)+r_1+\ldots+r_{\ell'}-1}.$$  

Since $R(P) + r_1 + \ldots + r_{\ell'}-1 \geq l + k$ using again the inequality (4.3)

$$\text{diam}(f^\ell(Q \cap \gamma^u)) \leq C\beta^{R(P)+r_1+\ldots+r_{\ell'}-1} \leq C\beta^k.$$  

This implies $\delta^+ \lesssim \beta^k$, which finishes the proof when $k \leq R(P)$. The case $k > R(P)$ is treated as B. \hfill \Box

Acknowledgements. Many thanks to my supervisor S. Luzzatto for his encouragements and guidance.

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International School for Advanced Studies (SISSA), Via Bonomea 265, Trieste, Italy, International Center for Theoretical Physics (ICTP), Strada Costiera 11, Trieste, Italy

E-mail address, Marks Ruziboev: mruziboe@sissa.it