The gravitational force on a gyroscope and the electromagnetic force on a magnetic dipole as analogous tidal effects

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Abstract. We compare the covariant expression of the electromagnetic force exerted on a magnetic dipole with Papapetrou’s equation for the gravitational force exerted on a spinning test particle. We show that if Pirani’s supplementary spin condition holds, there is an exact, covariant, and fully general analogy relating these two forces: both are determined by a contraction of the spin 4-vector with a magnetic-type tidal tensor. Moreover, these tidal tensors obey strikingly analogous equations which are covariant forms for (some of) Maxwell’s and Einstein’s field equations. These equations allow for an insightful comparison between the two interactions. It is shown that, in the special case that the gyroscope/dipole are “at rest” and far away from a stationary source, the two forces are similar (in accordance with the results known from linearized theory); but that for generic dynamics key differences arise. In particular we show that the time projection of the force on a dipole is the power transferred to it by Faraday’s induction, whereas the fact that the force on a gyroscope is spatial signals the absence of an analogous gravitational effect; that whereas the total work done on a magnetic dipole by a stationary magnetic field is zero, a stationary gravitomagnetic field, by contrast, does work on mass currents, which quantitatively explains the Hawking-Wald spin interaction energy.

1. Introduction
It is known, since the works of Mathisson and Papapetrou that spinning particles follow worldlines which are not geodesics. Wald [1] later showed that, in linearized theory, the gravitational force exerted on a spinning pole-dipole test particle (hereafter a gyroscope) whose center of mass is at rest in a stationary field, takes a form $\vec{F}_G = -\nabla(\vec{B}_G \cdot \vec{S})$ similar to the electromagnetic force on a magnetic dipole. The analogy was cast in an exact form by Natário [2], who, in the framework of the “quasi-Maxwell” formalism, showed that in stationary spacetimes the force exerted on a gyroscope whose worldline is tangent to a time-like Killing vector (i.e., its center of mass is at rest relative to some stationary observer) consists of an electromagnetic-like term plus a term interpreted therein as being “the weight of the energy of the gravitomagnetic dipole”. Recently it was shown that there is actually an exact, covariant and fully general analogy relating the two forces, which is made explicit in the tidal tensor formalism introduced in [3]. Herein we will exemplify how this analogy provides new intuition for the understanding of spin curvature coupling.
2. Force on a magnetic dipole vs. Force on a gyroscope

The force exerted on a magnetic dipole under the action of an electromagnetic field is given (see e.g. [4]) by Eq. (1a), where $S^{\mu\nu}$ is the spin tensor and $\sigma$ the gyromagnetic ratio:

$$ F_{EM}^\alpha = \frac{DP^\alpha}{dt} = \frac{\sigma}{2} F^{\mu\nu\alpha} S_{\mu\nu} \quad (a); \quad F_G^\alpha = \frac{DP^\alpha}{dt} = -\frac{1}{2} R^\alpha_{\beta\mu\nu} U^\beta S^{\mu\nu} \quad (b). \quad (1) $$

This equation needs to be supplemented by a spin supplementary condition; two covariant conditions are found in literature: the Moller-Tulczyjew condition $S^\alpha U_\beta = 0$, and the Pirani condition $S^\alpha U^\beta = 0$. We choose the latter, which means that $S^{\mu\nu} = \epsilon^{\mu\nu\gamma\lambda} S_{\gamma\lambda}$, where $S^\alpha$ is the spin 4-vector, defined as being the 4-vector with components $(0, \vec{S})$ in the dipole’s rest frame. In this case (1a) becomes Eq. (2a) of Table 1. $F_{EM}$ is the covariant form of the familiar textbook

| Table 1. Analogy between the EM force on a magnetic dipole and the GR force on a gyroscope |
|---------------------------------------------------------------|
| Electromagnetic Force on a Magnetic Dipole | Gravitational Force on a Spinning Particle |
| $F_{EM}^\beta = \sigma B_{\beta}^\beta S^\alpha$; $B_{\alpha}^\beta \equiv * F^{\alpha\beta} U^\mu$ | $F_G^\beta = -\pi_{\alpha}^\beta S^\alpha$; $\pi_{\alpha}^\beta \equiv * R^{\alpha\beta} U^\mu U^\nu$ |
| Eqs. Magnetic Tidal Tensor | Eqs. Gravitomagnetic Tidal Tensor |
| $B_{\alpha}^\beta = 0$ | $\pi_{\alpha}^\beta = 0$ |
| $B_{[\alpha\beta]} = \frac{1}{2} * F_{\alpha\beta\gamma} U^\gamma - 2\pi_{\alpha\beta\gamma} J^\gamma U^\gamma$ | $\pi_{[\alpha\beta]} = -4\pi_{\alpha\beta\gamma} J^\gamma U^\gamma$ |

3-D expression $\vec{F}_{EM} = d\vec{P}/dt = \nabla(\vec{S} \cdot \vec{B})/2m$ (which is valid only in the dipole’s frame).

The equation of motion for the center of mass of a spinning test particle in a gravitational field is given (e.g. [5]) by the Papapetrou equation (1b). Again, if Pirani’s supplementary condition $S^\alpha U^\beta = 0$ holds, then Eq. (1b) becomes Eq. (2b) of Table 1, revealing the physical analogy

$$ F_{EM}^\beta = * F_{\alpha\beta}^\gamma U^\gamma $$

between the EM force on a magnetic dipole and the GR force on a gyroscope. That is: Eqs. (4a) and (3a) are, respectively, the space projection of Maxwell equations $F_{\alpha\beta}^\gamma = 4\pi j^\gamma$ and the time projection of Bianchi identity $* F_{\alpha\beta}^\gamma = 0$; Eqs. (4b) and (3b) are, respectively the time-space projection of Einstein equations $R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu} T^\alpha_{\alpha})$ and the time-space projection of the algebraic Bianchi identities $* R_{\alpha\beta}^\gamma = 0$. There is an electric counterpart to this analogy which is manifest in the wordline deviations of both theories, and together they form the gravitoelectromagnetic analogy based on tidal tensors [3].

Note that $P^\alpha$ in Eqs. (1) is not parallel to $U^\alpha$ (i.e., $P^\alpha \neq m U^\alpha$); the momentum of the spinning particle will be discussed in detail in [7]; for now we refer to definitions given in [4, 5].

Comparing (1a) to (1b) we see that in gravity the gyromagnetic ratio is 1 (i.e., the gravitational analogue of the magnetic dipole moment is the angular momentum); the relative minus sign manifests the fact that parallel charge (mass) currents attract (repel) one another, which can be traced back to the fact that masses (charges of the same sign) attract (repel).

Apart from that, Eqs. (1) are formally similar; hence the analogy is ideally suited to compare the two interactions, because it amounts to compare $B_{[\alpha\beta]}$ and $\pi_{[\alpha\beta]}$, which is crystal clear from equations (3) and (4). The most important differences between them are: i) $B_{[\alpha\beta]}$ is linear in the fields whereas $\pi_{[\alpha\beta]}$ is not; ii) in vacuum $\pi_{[\alpha\beta]} = 0$ (symmetric tensor), whereas $B_{[\alpha\beta]} = \frac{1}{2} * F_{\alpha\beta\gamma} U^\gamma \neq 0$ (generically non symmetric, even in vacuum); iii) time components: $\pi_{\alpha\beta}$ is a spatial tensor, whereas $B_{\alpha\beta}$ is not. In what follows we explore some physical consequences.
2.1. Symmetries of tidal tensors

According to Eqs. (2), it is the magnetic tidal tensor, as seen by the test particle of 4-velocity $U^\alpha$, that determines the force exerted upon it. Thus, if the fields do not vary along the test particle’s worldline (so that $\star F^{\alpha \beta \gamma \delta} U^\gamma = 0$), Eqs. (4) allow for a similarity between $F_{EM}^\alpha$ and $F_{G}^\alpha$.

Consider the simple example of analogous physical apparatus: a dipole in the electromagnetic field of a spinning charge (charge $Q$, magnetic moment $\mu$), and a gyroscope in the gravitational field of a spinning mass (mass $M$, angular momentum $J$), asymptotically described by the Kerr solution. For large $r$, the force exerted on a gyroscope whose center of mass is at rest relative to the central mass is similar (identifying $\mu$ with $J$) to its electromagnetic counterpart:

$$F_G^i \simeq -\frac{3}{c} \left[ \frac{(\vec{r} \cdot \vec{J})}{r^3} \delta^i + 2 \frac{r (\vec{r} \cdot \vec{J}) r^i}{r^7} - 5 \frac{(\vec{r} \cdot \vec{J}) r^i r^j}{r^7} \right] S_j J^i = -F_{EM}^i$$

Thus we have a gravitational spin-spin force in analogy with the situation in electromagnetism [1]. However, in the general case that the dipole/gyroscope are allowed to move, then Eqs. (4) make clear that the two forces differ significantly (even in the weak field and slow motion approximation; see [6]), due to the different symmetries of the tidal tensors. Its implications in the dynamics will be discussed in detail in [7].

2.2. Time Components

Time components in test particle’s frame — The electromagnetic force exerted on a magnetic dipole has a non-vanishing time projection $F_{EM}^\alpha U_\alpha$ in the dipole’s proper frame; it is physically interpreted as minus the rate of work done by the induced electric field $\vec{E}$ due to the time varying magnetic field $\vec{B}$ (as measured by the inertial observer momentarily comoving with the dipole). This is more easily seen if we think of the magnetic dipole as a small current loop:

$$F_{EM}^\alpha U_\alpha = \frac{\partial \vec{B}}{\partial t} \cdot \vec{J} = \frac{\partial \Phi}{\partial t} I = -\int_{\text{loop}} \vec{E} \cdot d\vec{s}$$

This effect has no counterpart in gravity. Since $H_{\alpha \beta}$ is a spatial tensor, we always have $F_{G}^\beta U_\alpha = 0$ which means that the energy of the gyroscope, as measured in its proper frame, is constant. Hence, the spatial character of gravitational tidal tensors precludes induction effects analogous to the electromagnetic ones.

Time components as measured by static observers — in a given frame, the time component $P_0 = -E$ yields minus the energy of the test particle as measured in that frame; thus its variation $DP_0/d\tau$ is minus the rate of work done on it. For an arbitrary frame, in which the dipole has 4-velocity $U^\beta = \gamma(1, \vec{v})$, the time component of the force exerted on a magnetic dipole is:

$$(F_{EM})_0 = \frac{DP_0}{d\tau} = -\frac{DE}{d\tau} = \frac{F_{EM}^\beta U_\beta}{\gamma} - F_{EM}^i v_i = -(P_{\text{mech}} + P_{\text{ind}})$$

where, in accordance with discussion above, we identify $P_{\text{ind}} = -F_{EM}^\beta U_\beta/\gamma$ as the power transferred to the dipole by Faraday’s induction, which is reflected in a variation of its internal energy, and $P_{\text{mech}} \equiv F_{EM}^i v_i$ as the familiar “mechanical” power transferred to the dipole by the 3-force $F_{EM}^i$ exerted on it, which is reflected in the kinetic energy of translation of the center of mass. Consider now the situation in Fig. 1a): a magnetic dipole moving in the field generated by a strong magnet. An observer at rest relative to the magnet sees a time-constant field, thus, for this observer, the time component of the force on the dipole vanishes:

$$(F_{EM})_0 = \frac{DP_0}{d\tau} = \sigma \star F_{\alpha \beta \gamma} U^\beta S^\alpha = 0.$$
Figure 1. a) Magnetic dipole in the field of a strong magnet; b) gyroscope (small Kerr black hole) in the field of a large Kerr black hole; c) black hole merging.

That tells us that for this observer the total work done on the dipole is zero; which is related to a basic principle from electromagnetism: that a (stationary) magnetic field does no work. From the viewpoint of Eq. (6) this is due to an exact cancellation between $P_{\text{mech}}$ and $P_{\text{ind}}$: on the one hand the dipole gains translational kinetic energy; on the other hand there is a variation is of its internal energy by induction, which allows for the total work to vanish (this is in agreement with discussion in [8]).

In gravity, since those induction effects are absent, such cancellation does not occur: $(F_G)_0 = -DE/d\tau = -F_G^\alpha v_\alpha \neq 0$; in other words, a stationary “gravitomagnetic field”, unlike its electromagnetic counterpart, does work on mass currents. And there is a known consequence of this fact: the spin dependent upper bound for the energy released by gravitational radiation when two black holes collide, obtained by Hawking from the area law. For the case with spins aligned, from Hawking’s expression one can infer a gravitational spin-spin interaction energy [1], which indeed accounts for the rate of work $(F_G)_0$. In order to see that, consider the apparatus in Fig. 1b): two Kerr black holes with spins aligned, a large one (mass $m$, spin $J = am$) which is our source, and small one (4-velocity $U^\alpha$, spin $S \equiv \sqrt{S^\alpha S_\alpha}$) which we take to be the gyroscope, falling into the former along the symmetry axis (that ensures that $F^\alpha_G$ arises from the spin-spin interaction only). The observer is at rest relative to the source. The time component of the force acting on the small black hole is given by expression (7a):

$$(F_G)_0 = -F^\alpha_G v_\alpha = \frac{2ma}{(r^2 + a^2)^3} \left( a^2 \right) U^\alpha = \frac{aS}{2m \left[ m + \sqrt{m^2 - a^2} \right]} = \int^{r_+}_{\infty} (F_G)_0 d\tau \quad (a); \quad E_s = \frac{aS}{2m \left[ m + \sqrt{m^2 - a^2} \right]} = \int^{r_+}_{\infty} (F_G)_0 d\tau \quad (b).$$

The Hawking-Wald spin-spin interaction energy for this particular setup [1] is (7b) which is precisely what one obtains integrating (7a) from infinity to the horizon.

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