GLOBAL DEFORMATIONS OF CERTAIN RATIONAL ALMOST HOMOGENEOUS PROJECTIVE BUNDLES

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ABSTRACT. We study global deformations of certain projective bundles over projective spaces. We show that any projective global deformation of a projective bundle over $\mathbb{P}^1$ carries the structure of a projective bundle over some projective space. Furthermore, we construct examples in arbitrary dimension $\geq 3$ of almost homogeneous Fano projective bundles over $\mathbb{P}^2$ which can be globally deformed to non-almost homogeneous manifolds.

1. Introduction

We let $\Delta \subset \mathbb{C}$ be the unit disk and consider families $(X_t)_{t \in \Delta}$ of compact complex manifolds, i.e., proper submersions $p: \mathcal{X} \to \Delta$, where $\mathcal{X}$ is a complex manifold and $X_t := p^{-1}(t)$ for $t \in \Delta$. In this setup, we call $X_0$ a (global) deformation of $X_t$.

A classical problem is to study the case where for all $t \neq 0$, $X_t$ is isomorphic to some rational homogeneous manifold $S = G/P$, where $G$ is a semisimple Lie group and $P < G$ a parabolic subgroup. The question then is whether also $X_0$ is rational homogeneous (and thus $X_0 \cong S$ by local rigidity of rational homogeneous manifolds, cf. [Bot57]).

If one assumes $b_2(S) = 1$, Hwang and Mok showed in a series of papers (see [Mok16] §3.3 for references and an outline of the proof) that this is true if $S$ is not isomorphic to the 7-dimensional Fano contact manifold $\mathbb{F}^5$, which has been shown by Pasquier–Perrin in [PPT10] to admit a deformation to a non-homogeneous horospherical variety.

If $b_2(S) > 1$, it is obviously no longer true that $X_0 \cong S$, as can already be seen for $S = \mathbb{P}^1 \times \mathbb{P}^1$, which can be deformed to the Hirzebruch surface $\mathbb{F}_k$ for arbitrary even $k$.

So we see that $X_0$ need not necessarily be homogeneous. In the examples cited above, however, $X_0$ is still almost homogeneous, i.e., $\text{Aut } X_0$ acts on $X_0$ with an open orbit (this is equivalent to $T_{X_0}$ being generically globally generated). We can therefore ask:

**Question 1.1.** Let $(X_t)_{t \in \Delta}$ be a family of compact complex manifolds where $X_t$ is rational homogeneous for $t \neq 0$. Is then $X_0$ almost homogeneous?

We might even ask the following stronger question:

**Question 1.2.** Let $(X_t)_{t \in \Delta}$ be a family of compact complex manifolds where $X_t$ is an almost homogeneous manifold for every $t \neq 0$. Is then $X_0$ almost homogeneous?
The purpose of the present article is to give negative answers to both questions: We show that the homogeneous variety $\mathbb{P}(T_{\mathbb{P}^2})$ can be deformed to a non-almost homogeneous manifold. This answers Question 1.1 negatively for $\dim X_t = 3$ and $b_2(X_t) = 2$. Taking products with projective spaces, one also obtains negative answers for $\dim X_t > 3$ and $b_2(X_t) \geq 3$.

Moreover, for any $n \geq 3$, we construct almost homogeneous $n$-dimensional projective bundles over $\mathbb{P}^2$ which are Fano and can be deformed to non-almost homogeneous manifolds. These examples give negative answers to Question 1.2 for $\dim X_t \geq 3$ and $b_2(X_t) = 2$.

In order to give some context to the above-mentioned examples, we begin our discussion by reviewing the theory of almost homogeneous compact complex surfaces in the context of global deformations in section 2. We furthermore study projective global deformations of projective bundles over $\mathbb{P}^1$ in section 3, where we show that such global deformations again carry the structure of a projective bundle and in particular are almost homogeneous in most cases. This is inspired by work of Brieskorn [Bri65].

The remaining sections are then devoted to the construction of the examples announced above: In section 4, we give some criteria for almost homogeneity of projective bundles. Finally, in section 5, we give the construction of the above-mentioned (almost) homogeneous Fano projective bundles over $\mathbb{P}^2$ which admit global deformations to non-almost homogeneous manifolds.

Throughout the article, complex manifolds are assumed to be connected.

We use the notation “$\mathbb{P}(E)$” to denote the projective bundle of hyperplanes in the fibers of a given vector bundle $E$.

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2. The surface case

Almost homogeneous compact complex surfaces have been classified by Potters:

**Theorem 2.1 ([Pot69]).** Let $S$ be a compact complex surface. If $S$ is almost homogeneous, then one of the following holds:

(i) $S$ is obtained from $\mathbb{P}^2$ or from a Hirzebruch surface by blowing up a finite number of points,

(ii) $S \cong \mathbb{P}(V)$, where $V$ is a rank-2 bundle over an elliptic curve which is either a direct sum of $\mathcal{O}$ and a topologically trivial line bundle or a non-split extension of $\mathcal{O}$ with $\mathcal{O}$,

(iii) $S$ is a Hopf surface with abelian fundamental group,

(iv) $S$ is a two-dimensional complex torus.

Conversely, $\mathbb{P}^2$, the Hirzebruch surfaces and the surfaces of type (ii), (iii), (iv) are almost homogeneous.
It is classically known that any deformation of of a Hirzebruch surface is again a Hirzebruch surface (cf. [Bri65]). We will investigate projective bundles over \( \mathbb{P}^1 \) in arbitrary dimension in section \( \textsection 3 \). Global deformations of such bundles will be characterized in Theorem \( \textsection 3.4 \).

For ruled surfaces over elliptic curves, global deformations fail to be almost homogeneous in general, as the following example shows (cf. [PS14]): Let \( E \) be a rank-2 vector bundle over \( \mathbb{P}^1 \times \Delta \) such that \( E_t := E|_{\mathbb{P}^1 \times \{t\}} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \) for \( t \neq 0 \) and \( E_0 \cong \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(k) \) for some \( k > 0 \). Let \( \eta: A \to \mathbb{P}^1 \) be a 2-sheeted cover from an elliptic curve \( A \), and let \( X := \mathbb{P}((\eta \times \text{id})^* E) \). Then \( X \) is a family of compact complex surfaces over \( \Delta \) with \( X_t \cong \mathbb{P}^1 \times A \) for \( t \neq 0 \) and \( X_0 \) is a ruled surface over \( A \) which is not almost homogeneous by Theorem \( \textsection 2.1 \).

For the other cases in Theorem \( \textsection 2.1 \) we cite the following classical results by Kodaira and Kodaira–Spencer:

**Proposition 2.2** ([Kod66 Thm. 36]). Let \( S \) be a Hopf surface. Then any deformation of \( S \) is also a Hopf surface.

**Proposition 2.3** ([KS58 Thm. 20.2]). Let \( S \) be a two-dimensional complex torus. Then any deformation of \( S \) is also a complex torus.

**Remark 2.4.** Catanese showed in [Cat04 Thm. 2.1] that Proposition 2.3 is true in any dimension.

### 3. Projective Bundles over \( \mathbb{P}^1 \)

In this section we investigate families whose general fiber is isomorphic to a \( \mathbb{P}^r \)-bundle over \( \mathbb{P}^1 \) (see Corollary 4.2 for a proof that any such bundle is almost homogeneous).

We first study extremal contractions of projective bundles over \( \mathbb{P}^1 \):

**Proposition 3.1.** For given natural numbers \( 0 \leq a_1 \leq \cdots \leq a_r \) consider the vector bundle \( E := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r) \) over \( \mathbb{P}^1 \) and let \( \pi: X := \mathbb{P}(E) \to \mathbb{P}^1 \) be the associated \( \mathbb{P}^r \)-bundle over \( \mathbb{P}^1 \). Let \( \varphi: X \to Y \) be the contraction of a \( K_X \)-negative extremal ray of \( \overline{\text{NE}}(X) \). Assume \( \varphi \neq \pi \). Then either \( E \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus (r+1)} \) or \( E \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \).

**Proof.** First observe that \( \text{Nef}(X) \) is spanned as a convex cone by the classes of the tautological bundle \( \mathcal{O}_X(1) \) and the pullback \( \pi^* \mathcal{O}_{\mathbb{P}^1}(1) \). But now \( \text{Nef}(X) \) and \( \overline{\text{NE}}(X) \) are dual cones, so

\[
\overline{\text{NE}}(X) = \mathbb{R}_0^+ \ell + \mathbb{R}_0^+ C,
\]

where \( \ell \) is the class of a line in a fiber of \( \pi \) and \( C \) is some curve class with \( C \cdot \mathcal{O}_X(1) = 0 \) and \( C \cdot \pi^* \mathcal{O}_{\mathbb{P}^1}(1) > 0 \). From

\[
K_X = \mathcal{O}_X(-r-1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(a_1 + \cdots + a_r - 2)
\]

it then follows that \( C \cdot K_X < 0 \) if and only if \( a_1 + \cdots + a_r < 2 \). \( \square \)

**Remark 3.2.** In the situation of Proposition 3.1 the case \( E \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus (r+1)} \) means that \( X \cong \mathbb{P}^r \times \mathbb{P}^1 \), while the case \( E \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \) means that \( X \) is isomorphic to a blow-up of \( \mathbb{P}^{r+1} \) along a codimension-2 linear subspace.
The following Lemma will later be applied to the central fiber of the family:

**Lemma 3.3.** Let \( X \) be a smooth projective variety and assume that \( X \) is homeomorphic to a \( \mathbb{P}^r \)-bundle over \( \mathbb{P}^m \). Then any surjective morphism \( \phi : X \to \mathbb{P}^m \) is equidimensional (i.e., every fiber of \( \phi \) has dimension \( r \)).

**Proof.** We mostly copy the proof for a slightly less general statement from [PST14].

Since \( \dim X = m + r \), a general fiber of \( \phi \) must have dimension \( r \). Let \( F_0 \) be an irreducible component of a fiber of \( \phi \). Then \( F_0 \) gives rise to a class

\[
[F_0] \in H^{2k}(X, \mathbb{Q}),
\]

where \( k \) is the codimension of \( F_0 \) in \( X \). Semicontinuity of fiber dimension yields \( k \leq m \). We must show that \( k = m \).

We first observe that by the Leray–Hirsch theorem, we have

\[
h^{2k}(X, \mathbb{Q}) = \sum_{\ell=0}^{k} h^{2k-2\ell}(\mathbb{P}^r, \mathbb{Q}) \cdot h^{2\ell}(\mathbb{P}^m, \mathbb{Q}) = \min\{k, r\} + 1,
\]

where we used that \( h^{2\ell}(\mathbb{P}^m, \mathbb{Q}) = 1 \) since \( k \leq m \). We let \( d := \min\{k, r\} \) and denote by \( L \) the class of an ample divisor on \( X \) and by \( H \) the class of a hyperplane in \( \mathbb{P}^m \). We claim that the classes

\[
L^d.(\phi^*H)^{k-d}, L^{d-1}.(\phi^*H)^{k-d+1}, \ldots, L. (\phi^*H)^{k-1}, (\phi^*H)^k
\]

form a basis of \( H^{2k}(X, \mathbb{Q}) \), which can be seen as follows: By the dimension count established above, it suffices to show linear independency, so assume we are given \( \lambda_0, \ldots, \lambda_d \in \mathbb{Q} \) such that

\[
\sum_{\ell=0}^{d} \lambda_{\ell} L^{d-\ell}.(\phi^*H)^{k-d+\ell} = 0.
\]

Let \( \ell_0 \in \{0, \ldots, d\} \) and assume by induction that \( \lambda_{\ell} = 0 \) for all \( \ell < \ell_0 \). Then intersecting (2) with \( L^{r-d+\ell_0}.(\phi^*H)^{m-k+d-\ell_0} \) yields

\[
\lambda_{\ell_0} L^r.(\phi^*H)^m = 0,
\]

thus \( \lambda_{\ell_0} = 0 \) since \( L^r.(\phi^*H)^m > 0 \).

So (1) is indeed a basis of \( H^{2k}(X, \mathbb{Q}) \) and we can write

\[
[F_0] = \sum_{\ell=0}^{d} \alpha_{\ell} L^{d-\ell}.(\phi^*H)^{k-d+\ell}
\]

for some \( \alpha_0, \ldots, \alpha_d \in \mathbb{Q} \). We now again let \( \ell_0 \in \{0, \ldots, d\} \) and assume by induction that \( \alpha_{\ell} = 0 \) for all \( \ell < \ell_0 \). Then intersecting (3) with \( L^{r-d+\ell_0}.(\phi^*H)^{m-k+d-\ell_0} [F_0] \) yields

\[
L^{r-d+\ell_0}.(\phi^*H)^{m-k+d-\ell_0}[F_0] = \alpha_{\ell_0} L^r.(\phi^*H)^m.
\]

For \( \ell_0 < d \), the intersection product \( (\phi^*H)^{m-k+d-\ell_0}[F_0] \) vanishes, since \( k \leq m \) and \( F_0 \) maps to a point via \( \phi \). So we obtain \( \alpha_0 = \cdots = \alpha_{d-1} = 0 \). Since \( X \) is projective, this implies \( \alpha_d \neq 0 \).
For $\ell_0 = d$, equation (4) yields
\[ L'.(\phi^*H)^{m-k}.(F_0) = \alpha_dL'.(\phi^*H)^m. \]
As already observed, the right-hand side of this equation must be non-zero, while the left-hand side is non-zero if and only if $k = m$. □

We can now prove the main result of this section. Note that we must assume the family to be projective in order to apply the relative MMP.

**Theorem 3.4.** Let $p : X \to \Delta$ be a smooth projective morphism such that $X_t := p^{-1}(t)$ is a $\mathbb{P}^r$-bundle over $\mathbb{P}^1$ for all $t \neq 0$. Then, after possibly shrinking $\Delta$, only the following two cases can occur:

(i) There exists a rank-$(r+1)$ vector bundle $E$ over $\mathbb{P}^1 \times \Delta$ such that $X$ is isomorphic to $\mathbb{P}(E)$ over $\Delta$.

(ii) There exists a rank-2 bundle $E$ over $\mathbb{P}^r \times \Delta$ such that $X$ is isomorphic to $\mathbb{P}(E)$ over $\Delta$ (this case can only occur if $X_0 \cong \mathbb{P}^r \times \mathbb{P}^1$ for $t \neq 0$).

**Proof.** Since $X_t$ carries a $\mathbb{P}^r$-bundle structure for $t \neq 0$, $K_X$ is not $p$-nef. By the relative cone and contraction theorems (cf. [Nak87, Thm. 4.12]), there exists a relative Mori contraction $\Phi : X \to Y$ over $\Delta$ (after possibly shrinking $\Delta$), where $q : Y \to \Delta$ is a projective morphism. Let $\phi_t := \Phi|_{X_t} : X_t \to Y_t := q^{-1}(t)$. Then $\phi_t$ is a Mori contraction for any $t$.

Since $Y$ is irreducible, semicontinuity of fiber dimension implies that $q$ is equidimensional.

In view of Proposition 3.1 and Remark 3.2, the general fiber of $q$ is isomorphic to $\mathbb{P}^m$ for some $m \in \{1, r, r+1\}$. We observe that, for any $t \in \Delta$, the restriction $\text{Pic } X \to \text{Pic } X_t$ is an isomorphism. In particular, there exists a line bundle $L \in \text{Pic } X$ such that $L|_{X_0} \cong \phi_0^*O_{\mathbb{P}^m}(1)$ for $t \neq 0$. It follows that $L$ is numerically trivial on the fibers of $\phi_0$, so, since $\phi_0$ is a Mori contraction,
\[ L|_{X_0} = \phi_0^*L' \text{ for some } L' \in \text{Pic } Y_0. \]

By semicontinuity, $h^0(L') \geq m + 1$. Since furthermore $c_1(L')^n = 1$, we have $(Y_0, L') \cong (\mathbb{P}^m, O_{\mathbb{P}^m}(1))$ by [KO73, Thm 1.1] (see also [Fuj90, I.1.2]), so in particular,
\[ Y \cong \mathbb{P}^m \times \Delta. \]

We now first consider the case $m \in \{1, r\}$. This means that $\phi_t$ gives to $X_t$ a $\mathbb{P}^{r+1-m}$-bundle structure over $\mathbb{P}^m$ for $t \neq 0$. Since we have shown above that $Y_0 \cong \mathbb{P}^m$, we can apply Lemma 3.3 to conclude that $\phi_0$ is equidimensional. A general fiber of $\phi_0$ is isomorphic to $\mathbb{P}^{r+1-m}$, since it is a smooth degeneration of the fibers of $\phi_t$, $t \neq 0$. We can now apply [Fuj87, 2.12] to conclude that $\phi_0$ is indeed a $\mathbb{P}^{r+1-m}$-bundle. Using again the fact that $\text{Pic } X \to \text{Pic } X_t$ is an isomorphism, we obtain a line bundle $H \in \text{Pic } X$ whose restriction to any fiber of $\Phi$ is isomorphic to $O_{\mathbb{P}^{r+1-m}}(1)$. Setting $E := \Phi_*H$, we obtain an isomorphism $X \cong \mathbb{P}(E)$ over $\Delta$.

It remains to treat the case $m = r + 1$. Then
\[ \Phi : X \to Y = \mathbb{P}^{r+1} \times \Delta \]
is birational. Let \( \mathcal{W} \subset \mathcal{X} \) be the exceptional set of \( \Phi \). Since \( \Phi \) is a Mori contraction and \( \mathcal{W}_t := X_t \cap \mathcal{W} \) is a divisor in \( X_t \) for \( t \neq 0 \), it follows that \( \mathcal{W} \) is an irreducible divisor in \( \mathcal{X} \). Let \( Z := \Phi(\mathcal{W}) \subset \mathcal{Y} \) and \( Z_t := Z \cap Y_t \). Then \( Z \) is an irreducible subscheme of \( \mathcal{Y} \cong \mathbb{P}^{r-1} \times \Delta \) and hence flat over the smooth curve \( \Delta \). By Proposition \[ 3.1 \] we know that \( Z_t \subset \mathbb{P}^{r+1} \) is a codimension-2 linear subspace for \( t \neq 0 \), so by flatness, \( Z_0 \subset \mathbb{P}^{r-1} \) is an \((r-1)\)-dimensional subscheme of degree 1, hence \( Z_0 \) must also be a linearly embedded \( \mathbb{P}^{r-1} \subset \mathbb{P}^{r+1} \). It follows that \( Z_0 \) is isomorphic to the blow-up of \( Y_0 \cong \mathbb{P}^{r+1} \) along \( Z_0 \) by \[ 389 \] Theorem 1.1].

So in this case, \( \mathcal{X} \) is isomorphic to \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^r} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \times \Delta \) over \( \Delta \). \( \square \)

Remark 3.5. \( \mathbb{P}^r \)-bundles over \( \mathbb{P}^1 \) were already studied for arbitrary \( r \) by Breskorn in \[ Bf65 \]. The problem of classifying (global) deformations of such bundles was raised there, but was only solved for \( r = 1 \). The case \( r = 2 \) has been considered in \[ Nak98 \].

4. Almost homogeneous projective bundles

In this section, we investigate criteria for projective bundles to be almost homogeneous.

The following Lemma enables us to construct from a given almost homogeneous projective bundle an almost homogeneous projective bundle of higher dimension:

Lemma 4.1. Let \( M \) be a complex manifold with \( H^1(\mathcal{O}_M) = 0 \) and let \( E \) be a vector bundle on \( M \) such that \( \mathbb{P}(E) \) is almost homogeneous. If \( H^0(E) \neq 0 \) and \( H^1(E^*) = 0 \), then also \( \mathbb{P}(E \oplus \mathcal{O}_M) \) is almost homogeneous.

Proof. Let \( 0 \neq s \in H^0(E) \). For any \( t \in \mathbb{C} \), we consider the section \( \tilde{s}_t := (ts, 1) \in H^0(E \oplus \mathcal{O}_M) \). We obtain a short exact sequence of vector bundles

\[
0 \rightarrow \mathcal{O}_M \rightarrow \mathcal{O}_M \rightarrow Q_t \rightarrow 0.
\]

It follows from the snake lemma that \( Q_t \cong E \) for any \( t \in \mathbb{C} \).

We now write \( X := \mathbb{P}(E \oplus \mathcal{O}_M) \) and interpret \( \tilde{s}_t \) as an element of \( H^0(\mathcal{O}_X(1)) \), where we denote by \( \mathcal{O}_X(1) \) the tautological line bundle on \( \mathbb{P}(E \oplus \mathcal{O}_M) \). The zero locus of this section is a divisor \( Y_t \subset X \) with \( Y_t \cong \mathbb{P}(Q_t) \cong \mathbb{P}(E) \) for any \( t \in \mathbb{C} \). Its normal bundle is given by \( N_{Y_t/X} \cong \mathcal{O}_Y(1) \), so we have in particular \( H^0(N_{Y_t/X}) \cong H^0(E) \) by the Leray spectral sequence.

Clearly, the one-dimensional subspace of \( H^0(N_{Y_t/X}) \) defined by the family \( (Y_t)_{t \in \mathbb{C}} \) is generated by the section \( s \in H^0(E) \) chosen at the beginning. If we consider the long exact cohomology sequence associated to the normal bundle sequence

\[
0 \rightarrow T_{Y_t} \rightarrow T_X|_{Y_t} \rightarrow N_{Y_t/X} \rightarrow 0,
\]

the section \( s \in H^0(N_{Y_t/X}) \) maps to an element \( \xi \in H^1(T_{Y_t}) \) which describes the infinitesimal change of the complex structure on \( Y_t \) in the family \( (Y_t)_{t \in \mathbb{C}} \). But we have seen above that all \( Y_t \) are isomorphic, so it follows that \( \xi = 0 \). This implies that \( s \) lifts to a section in \( H^0(T_X|_{Y_t}) \). Since \( T_{Y_t} \cong T_{\mathbb{P}(E)} \) is
generically globally generated by hypothesis, the sequence \([5]\) then implies that also \(T_{X|Y_i}\) is generically globally generated.

We now let \(x \in Y_i\) be a general point such that \(T_{X,x}\) is generated by global sections in \(H^0(T_{X|Y_i})\). In order to show that \(T_{X,x}\) is also generated by global sections in \(H^0(T_X)\), it is sufficient to prove that the natural restriction map \(H^0(T_X) \to H^0(T_{X|Y_i})\) is surjective. By the long exact cohomology sequence associated to \([6]\), we thus conclude \(H^1(T_X(-Y_i)) = 0\).

So we conclude that \(H^1(T_X(-Y_i)) = 0\). To this aim, we first denote by \(\pi: X \to M\) the natural projection and consider the relative tangent sequence tensorized by \(O_X(-Y_i)\):

\[
0 \longrightarrow T_{X/M}(-Y_i) \longrightarrow T_X(-Y_i) \longrightarrow \pi^*T_M(-Y_i) \longrightarrow 0.
\]

Since \(O_X(-Y_i) \cong O_X(-1)\), we have \(H^q(\pi^*T_M(-Y_i)) = 0\) for all \(q\) by the Leray spectral sequence. From the long exact cohomology sequence associated to \([6]\), we thus conclude \(H^1(T_X(-Y_i)) = H^1(T_{X/M}(-Y_i))\).

We finally consider the relative Euler sequence

\[
0 \longrightarrow O_X \longrightarrow O_X(1) \otimes \pi^*(E^* \oplus O_M) \longrightarrow T_{X/M} \longrightarrow 0.
\]

Tensorizing this sequence by \(O_X(-Y_i) \cong O_X(-1)\) yields

\[
0 \longrightarrow O_X(-Y_i) \longrightarrow \pi^*(E^* \oplus O_M) \longrightarrow T_{X/M}(-Y_i) \longrightarrow 0.
\]

Since again \(H^q(O_X(-Y_i)) = 0\) for all \(q\) by Leray, we obtain

\[
H^1(T_{X/M}(-Y_i)) \cong H^1(\pi^*(E^* \oplus O_M)) \cong H^1(E^* \oplus O_M) = 0.
\]

So we conclude that \(H^1(T_X(-Y_i)) = 0\), which implies by our previous considerations that \(T_{X,x}\) is generated by global sections in \(H^0(T_X)\).

As a first application of this Lemma, we consider projective bundles over \(\mathbb{P}^1\) and reprove a classically known fact (cf. \([\text{Bri}65]\)):

**Corollary 4.2.** Let \(X\) be a \(\mathbb{P}^r\)-bundle over \(\mathbb{P}^1\). Then \(X\) is almost homogeneous.

**Proof.** Any \(\mathbb{P}^r\)-bundle over \(\mathbb{P}^1\) is of the form \(\mathbb{P}(V)\) for some rank-\((r+1)\)-bundle \(V\) on \(\mathbb{P}^1\). By Grothendieck, \(V\) splits as a direct sum of line bundles, so, after tensorizing with a suitable line bundle, we can assume

\[
V \cong O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus O_{\mathbb{P}^1}(a_r), \quad a_1, \ldots, a_r \geq 0.
\]

If we now let \(E := O_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus O_{\mathbb{P}^1}(a_r)\), we can assume by induction on \(r\) that \(\mathbb{P}(E)\) is almost homogeneous. But now clearly \(H^0(E) \neq 0\) and \(H^1(E) = 0\), so we can apply Lemma \([4.1]\) to conclude that \(X = \mathbb{P}(V)\) is almost homogeneous.

In a similar spirit, the following Corollary constructs higher-dimensional almost homogeneous projective bundles over \(\mathbb{P}^n\) for \(n \geq 2\) if given an almost homogeneous projective bundle \(\mathbb{P}(E)\) over \(\mathbb{P}^n\). An important application is the case \(E = T_{\mathbb{P}^n}\), the varieties \(\mathbb{P}(T_{\mathbb{P}^n}), n \geq 2\), being classical examples of rational homogeneous spaces.
Corollary 4.3. Let $E$ be a vector bundle on $\mathbb{P}^n$, $n \geq 2$, such that $\mathbb{P}(E)$ is almost homogeneous. Let $d_1, \ldots, d_r \in \mathbb{Z}$ such that $H^0(E(-d_j)) \neq 0$ and $H^1(E^+(d_j)) = 0$ for all $j = 1, \ldots, r$. Then $\mathbb{P}(E \oplus \mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(d_r))$ is almost homogeneous. In particular, $\mathbb{P}(T_{\mathbb{P}^n} \oplus \mathcal{O}(1)^{\oplus r})$ is almost homogeneous for any $r \geq 0$.

Proof. The case $r = 0$ is trivial; for $r \geq 1$ we let $E' := E \oplus \mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(d_{r-1})$. By induction, we can assume that $\mathbb{P}(E')$ is almost homogeneous. The given hypotheses imply $H^0(E'(-d_r)) \neq 0$ and $H^1(E'^+(d_r)) = 0$. So we can apply Lemma 4.1 to conclude that also $\mathbb{P}(E'(-d_r) \oplus \mathcal{O}_{\mathbb{P}^n}) \cong \mathbb{P}(E' \oplus \mathcal{O}_{\mathbb{P}^n}(d_r)) = \mathbb{P}(E \oplus \mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(d_r))$ is almost homogeneous. □

In order to deal with global deformations of almost homogeneous projective bundles, we need criteria for certain projectivized unstable vector bundles to be almost homogeneous:

Lemma 4.4. Let $M$ be a compact Kähler manifold and let $E$ be a rank-$2$ vector bundle on $M$ which is slope-unstable with respect to a Kähler form $\omega$ on $M$. Let $D \subset E$ be the maximally destabilizing subsheaf and let $Z \subset M$ be the codimension-$2$ locally complete intersection subscheme defined by the natural short exact sequence

\[ 0 \to D \to E \to D^* \otimes \det E \otimes \mathcal{I}_Z \to 0. \tag{7} \]

(i) Let $r \in \mathbb{N}_0$ and $L_1, \ldots, L_r \in \text{Pic } M$. If $\mathbb{P}(E \oplus L_1 \oplus \cdots \oplus L_r)$ is almost homogeneous, then the group

$$\text{Aut}^0_Z M := \{ \varphi \in \text{Aut}^0 M \mid \varphi(Z) = Z \}$$

acts on $M$ with an open orbit.

(ii) Suppose $H^1(\mathcal{O}_M) = 0$, $H^1(D^* \otimes \det E^*) = 0$ and $h^0(D \otimes \det E^*) \geq 2$ (these conditions are automatically satisfied for $M = \mathbb{P}^n$). If $\text{Aut}^0_Z M$ acts on $M$ with an open orbit, then $\mathbb{P}(E)$ is almost homogeneous.

Proof. Observe first that the maximally destabilizing subsheaf $D \subset E$ is uniquely determined.

In order to prove (i), we let $\tilde{E} := E \oplus L_1 \oplus \cdots \oplus L_r$ and suppose that $\mathbb{P}(\tilde{E})$ is almost homogeneous. We denote by $\pi : \mathbb{P}(\tilde{E}) \to M$ the natural projection. Since $\pi_* \mathcal{O}_{\mathbb{P}(\tilde{E})} = \mathcal{O}_M$, any automorphism $\psi \in \text{Aut}^0 \mathbb{P}(\tilde{E})$ induces an automorphism $\varphi \in \text{Aut}^0 M$ such that the diagram

\[ \mathbb{P}(\tilde{E}) \xrightarrow{\psi} \mathbb{P}(\tilde{E}) \]

\[ \downarrow \pi \hspace{1cm} \downarrow \pi \]

\[ M \xrightarrow{\varphi} M \]

commutes. In the case that $E$ is a direct sum of two line bundles, we have $Z = \emptyset$, so by the existence of diagram (8), $\text{Aut}^0_Z M = \text{Aut}^0 M$ acts on $M$ with an open orbit.

In the case that $E$ is not a direct sum of two line bundles, we must study diagram (8) more thoroughly. By the universal property of the pullback, $\psi$
factors as $\psi = \tau \circ \tilde{\psi}$ where $\tau$ and $\tilde{\psi}$ are such that the diagram

\[ \begin{array}{ccc}
P(E) & \xrightarrow{\tilde{\psi}} & P(\varphi^*E) \\
\downarrow\pi & & \downarrow\pi \\
M & \xrightarrow{\varphi} & M
\end{array} \]

commutes. Since $\psi$ and $\tau$ are isomorphisms, also $\tilde{\psi}$ must be an isomorphism. This implies that there exists a line bundle $L \in \text{Pic} M$ such that $\varphi^*E \cong E \otimes L$, i.e.,

$$\varphi^*E \oplus \varphi^*L_1 \oplus \cdots \oplus \varphi^*L_r \cong (E \otimes L) \oplus (L_1 \otimes L) \oplus \cdots \oplus (L_r \otimes L).$$

Since $E$ is unsplit, we must have

$$\varphi^*E \cong E \otimes L$$

by the uniqueness of the direct sum decomposition of vector bundles. Since $\varphi^*$ acts trivially on $H^2(M, \mathbb{R})$, the Kähler form $\varphi^*\omega$ is numerically equivalent to $\omega$. This implies that $\varphi^*D$ is the maximally $\omega$-destabilizing subsheaf of $\varphi^*E$. Furthermore, clearly $D \otimes L$ is the maximally ($\omega$-)destabilizing subsheaf of $E \otimes L$. By (10) and the uniqueness of the maximally destabilizing subsheaf, the two short exact sequences one obtains from (7) by applying $\varphi^*$ respectively by tensorizing with $L$ must be the same, so in particular,

$$\varphi^* \mathcal{I}_{\varphi^{-1}(Z)} \cong \mathcal{I}_Z,$$

so $\varphi(Z) = Z$, finishing the proof of (1).

Turning to (ii), we now suppose that $\text{Aut}^0_0 Z M$ acts on $M$ with an open orbit. Let $\varphi \in \text{Aut}^0_0 M$ be any automorphism with $\varphi(Z) = Z$. Applying $\varphi^*$ to (7), we obtain

$$0 \longrightarrow \varphi^*D \longrightarrow \varphi^*E \longrightarrow \varphi^*D^* \otimes \text{det} \varphi^*E \otimes \mathcal{I}_Z \longrightarrow 0.$$

Since $H^1(\mathcal{O}_M) = 0$, we have $\varphi^*D \cong D$ and $\text{det} \varphi^*E \cong \text{det} E$, so the bundle $\varphi^*E$ is an extension of the same rank-1 sheaves as the bundle $E$. Since $H^1(D \otimes \text{det} E^*) = 0$, there is, up to isomorphism, only one locally free sheaf given by such an extension (cf. [OSS80 §1.5.1]), so we obtain $\varphi^*E \cong E$. In particular, we get an isomorphism $\tilde{\psi} : P(E) \to P(\varphi^*E)$ over $M$ such that the diagram

\[ \begin{array}{ccc}
P(E) & \xrightarrow{\tilde{\psi}} & P(\varphi^*E) \\
\downarrow\pi & & \downarrow\pi \\
M & \xrightarrow{\varphi} & M
\end{array} \]

is commutative, where the right-hand part of this diagram is just the pull-back square analogous to diagram (9). If we let $\psi := \tau \circ \tilde{\psi}$, we get an automorphism $\psi \in \text{Aut} P(E)$ which induces the given automorphism $\varphi \in \text{Aut}^0_0 M$.

To conclude, it suffices to show that there exists an at least 1-dimensional family of automorphisms of $P(E)$ over $M$. This is equivalent to showing that

$$h^0(\text{End } E) = h^0(E^* \otimes E) \geq 2.$$
But now, tensoring (7) with \( E^* \cong E \otimes \det E^* \) yields
\[
0 \rightarrow E \otimes D \otimes \det E^* \rightarrow E^* \otimes E \rightarrow E \otimes D \otimes \det E^* \rightarrow I_Z \rightarrow 0.
\]
Finally, \( D \subset E \) implies \( D^\otimes 2 \otimes \det E^* \subset E \otimes D \otimes \det E^* \), so we obtain
\[
h^0(E^* \otimes E) \geq h^0(E \otimes D \otimes \det E^*) \geq h^0(D^\otimes 2 \otimes \det E^*) \geq 2. \quad \square
\]

5. Examples of global deformations

In order to obtain global deformations of projectivized rank-2 vector bundles where almost homogeneity is not preserved, we cite the following construction by Strømme:

**Proposition 5.1** ([Str83 §4]). Let \( E \) be a rank-2 vector bundle on \( \mathbb{P}^2 \) with first Chern class \( c_1 := c_1(E) \). Let \( d \in \mathbb{Z} \) and suppose that we are given
(i) a section \( \tau \in H^0(E(d - c_1)) \) vanishing along a subscheme \( Z \subset \mathbb{P}^2 \) of codimension 2,
(ii) a section \( F \in H^0(O_{\mathbb{P}^2}(2d - c_1)) \) whose zero divisor is disjoint from \( Z \).
Then there exists a rank-2 vector bundle \( E \) on \( \mathbb{P}^2 \times \Delta \) such that \( E_t := E|_{\mathbb{P}^2 \times \{t\}} \cong E \) for \( t \neq 0 \) and \( E_0 \cong D \), where \( D \) is a rank-2 vector bundle on \( \mathbb{P}^2 \) fitting in the short exact sequence
\[
0 \rightarrow O_{\mathbb{P}^2}(d) \rightarrow D \rightarrow I_Z(c_1 - d) \rightarrow 0.
\]
We have \( D \cong E \) if and only if \( H^0(E(-d)) \neq 0 \).

**Remark 5.2.** The assumptions in Proposition 5.1 can be satisfied for any rank-2 bundle \( E \) on \( \mathbb{P}^2 \) provided \( d \) is chosen sufficiently large.

**Corollary 5.3.** For any \( d \geq 2 \) there exists a rank-2 vector bundle \( E \) on \( \mathbb{P}^2 \times \Delta \) with the following properties:
(i) \( E_t := E|_{\mathbb{P}^2 \times \{t\}} \) is isomorphic to \( T_{\mathbb{P}^2} \) for all \( t \neq 0 \),
(ii) \( E_0 \) sits inside a short exact sequence
\[
0 \rightarrow O_{\mathbb{P}^2}(d) \rightarrow E_0 \rightarrow I_Z(3 - d) \rightarrow 0,
\]
where \( Z \subset \mathbb{P}^2 \) is a codimension-2 locally complete intersection given as the zero locus of a section in \( H^0(T_{\mathbb{P}^2}(d - 3)) \).

**Proof.** Since \( d \geq 2 \), the vector bundle \( T_{\mathbb{P}^2}(d - 3) \) is globally generated, so a general section in \( H^0(T_{\mathbb{P}^2}(d - 3)) \) vanishes along a subscheme \( Z \subset \mathbb{P}^2 \) of codimension 2. Furthermore, \( 2d - 3 \geq 1 \), so there exists a divisor in \( |O_{\mathbb{P}^2}(2d - 3)| \) disjoint from \( Z \). Thus we can apply Proposition 5.1. \( \square \)

We can use this construction to get examples for global deformations of almost homogeneous projective bundles:

**Theorem 5.4.** Fix \( d \geq 2 \). Let \( E \) be the rank-2 bundle on \( \mathbb{P}^2 \times \Delta \) constructed in Corollary 5.3. For any \( r \geq 0 \) we consider the family
\[
\mathcal{X} := \mathbb{P}(E \oplus O_{\mathbb{P}^2 \times \Delta}(1)^{\oplus r}) \rightarrow \mathbb{P}^2 \times \Delta
\]
over \( \Delta \). Then:
(i) For any $t \neq 0$, the fiber $X_t \cong \mathbb{P}(T_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(1)^{\oplus r})$ is an $(r + 3)$-dimensional almost homogeneous manifold.

(ii) If $d \geq 4$, the fiber $X_0 \cong \mathbb{P}(E_0 \oplus O_{\mathbb{P}^2}(1)^{\oplus r})$ is not almost homogeneous.

(iii) $X_t$ is Fano if and only if $t \neq 0$.

Proof. (i) is just Corollary 4.3

For (ii), we note that by Lemma 4.4(ii), it suffices to show that $\{ \varphi \in \text{Aut}^0 \mathbb{P}^2 \mid \varphi(Z) = Z \}$ does not act on $\mathbb{P}^2$ with an open orbit. Since $Z$ is given as the zero locus of a section $\tau \in H^0(T_{\mathbb{P}^2}(d - 3))$, there is an exact sequence

$$0 \rightarrow O_{\mathbb{P}^2} \rightarrow T_{\mathbb{P}^2}(d - 3) \rightarrow I_Z(2d - 3) \rightarrow 0.$$

Tensorizing with $T_{\mathbb{P}^2}(3 - 2d)$ yields

$$0 \rightarrow T_{\mathbb{P}^2}(3 - 2d) \rightarrow T_{\mathbb{P}^2} \otimes T_{\mathbb{P}^2}(-d) \rightarrow T_{\mathbb{P}^2} \otimes I_Z \rightarrow 0.$$

But now since $d \geq 4$, we have

$$h^1(T_{\mathbb{P}^2}(3 - 2d)) = h^1(\Omega^1_{\mathbb{P}^2}(2d - 6)) = 0,$$

and from the Euler sequence it follows that also $H^0(T_{\mathbb{P}^2} \otimes T_{\mathbb{P}^2}(-d)) = 0$. So we obtain $H^0(T_{\mathbb{P}^2} \otimes I_Z) = 0$ which means that there are no non-zero vector fields on $\mathbb{P}^2$ vanishing along $Z$.

It remains to prove (iii). We denote by $\pi_t : X_t \rightarrow \mathbb{P}^2$ the restriction of $\pi$ to $X_t$. Standard calculations yield

$$K_{X_t} = O_{X_t}(r + 2) \otimes \pi_t^* O_{\mathbb{P}^2}(-r).$$

Thus $X_t$ is Fano if and only if the $\mathbb{Q}$-line bundle

$$(12) \quad O_{X_t}(1) \otimes \pi_t^* O_{\mathbb{P}^2}(-\frac{r}{r + 2})$$

is ample. But if we now restrict the short exact sequence \([11]\) to lines in $\mathbb{P}^2$, it follows immediately that $E_0$ is not ample. So also $E_0 \oplus O_{\mathbb{P}^2}(1)^{\oplus r}$ is not ample, which is equivalent to the statement that $O_{X_0}(1)$ is not ample. Thus also \([12]\) is not ample for $t = 0$. For $t \neq 0$, we observe that $T_{\mathbb{P}^2}(q)$ is an ample $\mathbb{Q}$-bundle for all rational $q > -1$, thus \([12]\) is ample for $t \neq 0$. \(\square\)

The argument used to prove Theorem 5.4(ii) can be used to show that nontrivial global deformations of $\mathbb{P}(T_{\mathbb{P}^2})$ cannot be Fano. Furthermore, the assumption $d \geq 4$ in Theorem 5.4(iii) is actually necessary:

**Theorem 5.5.** Let $\mathcal{E}$ be a rank-2 vector bundle on $\mathbb{P}^2 \times \Delta$ such that $E_t := \mathcal{E}|_{\mathbb{P}^2 \times \{t\}}$ is isomorphic to $T_{\mathbb{P}^2}$ for all $t \neq 0$.

(i) Assume that $\mathbb{P}(E_0)$ is Fano. Then $E_0 \cong T_{\mathbb{P}^2}$.

(ii) Suppose that $H^0(E_0(-4)) = 0$. Then $\mathbb{P}(E_0)$ is almost homogeneous.

Proof. If we let

$$d := \max \{ \ell \in \mathbb{Z} \mid H^0(E_0(-\ell)) \neq 0 \},$$

we have an exact sequence

$$0 \rightarrow O_{\mathbb{P}^2}(d) \rightarrow E_0 \rightarrow I_Z(3 - d) \rightarrow 0,$$
where $Z \subset \mathbb{P}^2$ is a 0-dimensional locally complete intersection subscheme of length

$$\ell(Z) = c_2(E_0(-d)) = c_2(T_{\mathbb{P}^2}(-d)) = 3 - 3d + d^2.$$ 

By assumption, we have $d < 4$, and by semicontinuity, $d \geq 1$. If $d = 1$, it is a classic result that $E_0 \cong T_{\mathbb{P}^2}$ (see for example [OSS80, §2.3.2]), so we can assume that $d \geq 2$. In this case, the argument from the proof of Theorem 5.4 (iii) applies verbatim to show that $\mathbb{P}(E_0)$ cannot be Fano. Furthermore, we have $2d > 3 = c_1(E_0)$, so $E_0$ is an unstable bundle. Now if $d = 2$, we have $\ell(Z) = 1$, and if $d = 3$, we get $\ell(Z) = 3$. It is easily verified that for any 0-dimensional subscheme $Z \subset \mathbb{P}^2$ with $\ell(Z) \leq 3$, the group \{ $\varphi \in \text{Aut}(\mathbb{P}^2)$ | $\varphi(Z) = Z$ \} acts on $\mathbb{P}^2$ with an open orbit. The claim that $\mathbb{P}(E_0)$ is almost homogeneous now follows from Lemma 4.4 (ii).

\[\square\]

Remark 5.6. A construction analogous to the one in Corollary 5.3 was given in [PS14, §6] for $\mathbb{P}^n$ for arbitrary $n \geq 2$. Unfortunately, for $n \geq 3$, the arguments used in the proofs of Theorems 5.4 and 5.5 no longer work, so it is unclear whether one can obtain non-almost homogeneous global deformations of $\mathbb{P}(T_{\mathbb{P}^n})$ for $n \geq 3$.

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