A necessary and sufficient condition for sequences to be minimal completely monotonic

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Dedicated to Professor Hari M. Srivastava on the occasion of his eightieth birthday.

Abstract

In this article, we present a necessary and sufficient condition under which sequences are minimal completely monotonic.

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1 Introduction and the main results

We first recall some definitions and basic results on completely monotonic sequences and minimal completely monotonic sequences.

Definition 1 \([20]\) A sequence \(\{\mu_n\}_{n=0}^{\infty}\) is called completely monotonic if

\[
(-1)^k \Delta^k \mu_n \geq 0, \quad n, k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N},
\]

where

\[
\Delta^0 \mu_n = \mu_n
\]

and

\[
\Delta^{k+1} \mu_n = \Delta^k \mu_{n+1} - \Delta^k \mu_n.
\]

Here in Definition 1, and throughout the paper, \(\mathbb{N}\) is the set of all positive integers and \(\mathbb{N}_0\) is the set of all nonnegative integers.

Widder \([25]\) defined a sub-class of the class of completely monotonic sequences.
Definition 2 A sequence \( \{\mu_n\}_{n=0}^\infty \) is called minimal completely monotonic if it is completely monotonic and if it will not be completely monotonic when \( \mu_0 \) is replaced by a number less than \( \mu_0 \).

Regarding the relationships between completely monotonic sequences and minimal completely monotonic sequences, in [6] the author proved that if the sequence \( \{\mu_n\}_{n=0}^\infty \) is completely monotonic, then:

1. for any \( m \in \mathbb{N} \), the sequence \( \{\mu_n\}_{n=m}^\infty \) is minimal completely monotonic, and
2. there exists one (then only one) number \( \mu^*_0 \) such that the sequence

\[
\{\mu^*_0, \mu_1, \mu_2, \ldots\}
\]

is minimal completely monotonic.

Please note that the complete monotonicity of the sequence \( \{\mu_n\}_{n=1}^\infty \) cannot guarantee that there exists a number \( \mu^*_0 \) such that the sequence

\[
\{\mu^*_0, \mu_1, \mu_2, \ldots\}
\]

is completely monotonic. In fact, if the sequence (4) is completely monotonic, then the sequence \( \{\mu_n\}_{n=1}^\infty \) should be minimal completely monotonic.

In [18] the authors showed that if the sequence \( \{\mu_n\}_{n=0}^\infty \) is completely monotonic, then, for any \( m \in \mathbb{N}_0 \), the series

\[
\sum_{j=0}^{\infty} (-1)^j \Delta_j \mu_{m+1}
\]

converges and

\[
\mu_m \geq \sum_{j=0}^{\infty} (-1)^j \Delta_j \mu_{m+1}.
\]  (5)

We also recall the following definition.

Definition 3 ([4]) A function \( f \) is said to be completely monotonic on an interval \( I \), if \( f \in C(I) \), has derivatives of all orders on \( I^o \) (the interior of \( I \)) and for all \( n \in \mathbb{N}_0 \)

\[
(-1)^n f^{(n)}(x) \geq 0, \quad x \in I^o.
\]  (6)

Here in Definition 3 \( C(I) \) is the space of all continuous functions on the interval \( I \). The class of all completely monotonic functions on the interval \( I \) is denoted by \( CM(I) \).

There is rich literature on completely monotonic functions and sequences, and their applications. For more recent works, see, for example, [1–3, 5–19, 21–24].

For sequences to be interpolated by completely monotonic functions, Widder [25] proved that there exists a function

\[ f \in CM[0, \infty) \]
such that

$$f(n) = \mu_n, \quad n \in \mathbb{N}_0$$

if and only if the sequence $\{\mu_n\}_{n=0}^{\infty}$ is minimal completely monotonic. From this we see that the condition of minimal complete monotonicity is critical for a sequence $\{\mu_n\}_{n=0}^{\infty}$ to be interpolated by a completely monotonic function on the interval $[0, \infty)$.

In this article, we shall further investigate on minimal completely monotonic sequences. The main results of this article are as follows.

**Theorem 4** Suppose that the sequence $\{\mu_n\}_{n=1}^{\infty}$ is completely monotonic and that the series

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j / \mu_1$$

converges. Let

$$\mu_0^* := \sum_{j=0}^{\infty} (-1)^j \Delta^j / \mu_1. \quad (8)$$

Then the sequence

$$\{\mu_0^*, \mu_1, \mu_2, \mu_3, \ldots\} \quad (9)$$

is minimal completely monotonic.

**Remark 5** It should be noted that the condition: “the series

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j / \mu_1$$

converges” in Theorem 4 cannot be dropped since the complete monotonicity of the sequence $\{\mu_n\}_{n=1}^{\infty}$ cannot guarantee the convergence of the series

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j / \mu_1.$$ 

For example, let

$$\mu_n = \frac{1}{n}, \quad n \in \mathbb{N}. $$

We can verify that the sequence $\{\mu_n\}_{n=1}^{\infty}$ is completely monotonic and that

$$\Delta^j \mu_1 = \frac{(-1)^j}{j+1}.$$
Hence
\[ \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1 = \sum_{j=0}^{\infty} \frac{1}{j+1}, \]
which is divergent.

**Theorem 6** Suppose that the sequence \( \{\mu_n\}_{n=0}^{\infty} \) is minimal completely monotonic. Then the series
\[ \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1 \] (11)
converges and
\[ \mu_0 = \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1. \] (12)

**Theorem 7** A necessary and sufficient condition for the sequence \( \{\mu_n\}_{n=0}^{\infty} \) to be minimal completely monotonic is that the sequence \( \{\mu_n\}_{n=1}^{\infty} \) is completely monotonic, the series
\[ \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1 \] (13)
converges, and
\[ \mu_0 = \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1. \] (14)

### 2 Proof of the main results

Now we are in a position to prove the main results.

**Proof of Theorem 4** By Theorem 11 in [18], we see that the sequence
\[ \{\mu_0^*, \mu_1, \mu_2, \mu_3, \ldots\} \] (15)
is completely monotonic. By Theorem 9 in [18], if a sequence
\[ \{\mu_0, \mu_1, \mu_2, \mu_3, \ldots\} \] (16)
is completely monotonic, then
\[ \mu_0 \geq \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1 = \mu_0^*. \] (17)

Hence by the definition of minimal completely monotonic sequence, we know that the sequence
\[ \{\mu_0^*, \mu_1, \mu_2, \mu_3, \ldots\} \] (18)
is minimal completely monotonic. The proof of Theorem 4 is completed. □

Proof of Theorem 6 Since the sequence

\( \{\mu_0, \mu_1, \mu_2, \mu_3, \ldots\} \) (19)

is completely monotonic, by Theorem 9 in [18], the series

\[
\sum_{j=0}^{\infty} (-1)^j \frac{\Delta^j}{\mu_1}
\] (20)

converges and

\[
\mu_0 \geq \sum_{j=0}^{\infty} (-1)^j \frac{\Delta^j}{\mu_1}.
\] (21)

By Theorem 11 in [18], we see that the sequence

\[
\left\{ \sum_{j=0}^{\infty} (-1)^j \frac{\Delta^j}{\mu_1}, \mu_1, \mu_2, \mu_3, \ldots \right\}
\] (22)

is completely monotonic. Since the completely monotonic sequence

\( \{\mu_0, \mu_1, \mu_2, \mu_3, \ldots\} \) (23)

is minimal, we have

\[
\mu_0 \leq \sum_{j=0}^{\infty} (-1)^j \frac{\Delta^j}{\mu_1}.
\] (24)

From (21) and (24), we get our conclusion. The proof of Theorem 6 is completed. □

Proof of Theorem 7 By the definition of completely monotonic sequence, Theorem 9 in [18] and Theorem 6, we know that the condition is necessary. By Theorem 4, we see that the condition is sufficient. The proof of Theorem 7 is thus completed. □

3 Conclusion
In this paper, we investigated properties of completely monotonic sequences. We have proved a necessary condition for a sequence to be a minimal completely monotonic sequence. We also have presented a necessary and sufficient condition under which sequences are minimal completely monotonic.

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References
1. Alzer, H., Batir, N.: Monotonicity properties of the gamma function. Appl. Math. Lett. 20, 778–781 (2007)
2. Alzer, H., Berg, C., Koumandos, S.: On a conjecture of Clark and Ismail. J. Approx. Theory 134, 102–113 (2005)
3. Batir, N.: On some properties of the gamma function. Expo. Math. 26, 187–196 (2008)
4. Bernstein, S.: Sur la déﬁnition et les propriétés des fonctions analytiques d’une variable réelle. Math. Ann. 75, 449–468 (1914)
5. Guo, S.: Logarithmically completely monotonic functions and applications. Appl. Math. Comput. 221, 169–176 (2013)
6. Guo, S.: Some properties of completely monotonic sequences and related interpolation. Appl. Math. Comput. 219, 4958–4962 (2013)
7. Guo, S.: A class of logarithmically completely monotonic functions and their applications. J. Appl. Math. 2014, 757462 (2014)
8. Guo, S.: Some conditions for a class of functions to be completely monotonic. J. Inequal. Appl. 2015, 11 (2015)
9. Guo, S.: On completely monotonic and related functions. Filomat 30, 2083–2090 (2016)
10. Guo, S.: Some properties of functions related to completely monotonic functions. Filomat 31, 247–254 (2017)
11. Guo, S., Laforgia, A.; Batir, N., Luo, Q.-M.: Completely monotonic and related functions: their applications. J. Appl. Math. 2014, 768516 (2014)
12. Guo, S., Qi, F.: A class of logarithmically completely monotonic functions associated with the gamma function. J. Comput. Appl. Math. 224, 127–132 (2009)
13. Guo, S., Qi, F, Srivastava, H.M.: Necessary and sufﬁcient conditions for two classes of functions to be logarithmically completely monotonic. Integral Transforms Spec. Funct. 18, 819–826 (2007)
14. Guo, S., Qi, F, Srivastava, H.M.: Supplements to a class of logarithmically completely monotonic functions associated with the gamma function. Appl. Math. Comput. 197, 768–774 (2008)
15. Guo, S., Qi, F, Srivastava, H.M.: A class of logarithmically completely monotonic functions related to the gamma function with applications. Integral Transforms Spec. Funct. 23, 557–566 (2012)
16. Guo, S., Srivastava, H.M.: A class of logarithmically completely monotonic functions. Appl. Math. Lett. 21, 1134–1141 (2008)
17. Guo, S., Srivastava, H.M.: A certain function class related to the class of logarithmically completely monotonic functions. Math. Comput. Model. 49, 2073–2079 (2009)
18. Guo, S., Srivastava, H.M., Batir, N.: A certain class of completely monotonic sequences. Adv. Differ. Equ. 2013, 294 (2013)
19. Guo, S., Srivastava, H.M., Cheung, W.S.: Some properties of functions related to certain classes of completely conotonic functions and logarithmically completely conotonic functions. Filomat 28, 821–828 (2014)
20. Hausdorff, F.: Summationsmethoden und momentfolgen I. Math. Z. 9, 74–109 (1921)
21. Qi, F, Guo, S., Guo, B.-N.: Complete monotonicity of some functions involving polygamma functions. J. Comput. Appl. Math. 233, 2149–2160 (2010)
22. Salem, A.: A completely monotonic function involving γ-gamma and γ-digamma functions. J. Approx. Theory 164, 971–980 (2012)
23. Sevli, H., Batir, N.: Complete monotonicity results for some functions involving the gamma and polygamma functions. Math. Comput. Model. 53, 1771–1775 (2011)
24. Srivastava, H.M., Guo, S., Qi, F.: Some properties of a class of functions related to completely monotonic functions. Comput. Math. Appl. 64, 1649–1654 (2012)
25. Widder, D.V.: Necessary and sufﬁcient conditions for the representation of a function as a Laplace integral. Trans. Am. Math. Soc. 33, 851–892 (1931)