LARGER GREEDY SUMS FOR REVERSE PARTIALLY GREEDY BASES

HÙNG VIỆT CHU

ABSTRACT. An interesting result due to Dilworth et al. was that if we enlarge greedy sums by a constant factor \( \lambda > 1 \) in the condition defining the greedy property, then we obtain an equivalence of the almost greedy property, a strictly weaker property. Previously, the author of the present paper showed that enlarging greedy sums by \( \lambda \) in the condition defining the partially greedy (PG) property also strictly weakens the property. However, enlarging greedy sums in the definition of reverse partially greedy (RPG) bases by Dilworth and Khurana again gives RPG bases. The companion of PG and RPG bases suggests the existence of a characterization of RPG bases which, when greedy sums are enlarged, gives an analog of a result that holds for partially greedy bases. In this paper, we show that such a characterization indeed exists, answering positively a question previously posed by the author.

CONTENTS

1. Introduction 1
2. A characterization of RPG and PG bases 4
3. Larger greedy sums for RPG bases 6
4. The \( \lambda \)-RPG2 property is weaker than the RPG property 10
References 12

1. INTRODUCTION

Let \( X \) be an infinite-dimensional Banach space (with dual \( X^* \)) over the field \( K = \mathbb{R} \) or \( \mathbb{C} \). We define a basis to be any countable collection \( B = (e_n)_{n=1}^\infty \) such that i) the span of \( (e_n)_n \) is norm-dense in \( X \); ii) there exist biorthogonal functionals \( (e^*_n)_n \subset X^* \) such that \( e^*_n(e_m) = \delta_{n,m} \); and iii) there exist \( c_1, c_2 > 0 \) such that \( c_1 \leq \|e_n\|, \|e^*_n\| \leq c_2 \) for all \( n \). In the literature, the totality condition: \( \text{span}(e^*_n)_{n=1}^\infty = X^* \) is sometimes called for to guarantee that for each \( x \), the sequence \( (e^*_n(x))_{n=1}^\infty \) is unique. Since uniqueness is not important for our purpose, we do not assume totality.

In 1999, Konyagin and Temlyakov [15] introduced the notion of greedy bases as follows: for each \( x \in X \), a finite set \( \Lambda \) is a greedy set of \( x \) if

\[
\min_{n \in \Lambda} |e^*_n(x)| \geq \max_{n \in \Lambda, n \not\in \Lambda} |e^*_n(x)|.
\]

2020 Mathematics Subject Classification. 41A65; 46B15.

Key words and phrases. reverse partially greedy; greedy sum; bases.

The author is thankful to Timur Oikhberg for helpful comments on an earlier draft of this paper. The author would also like to thank the anonymous referees for a careful reading and constructive feedback that improves the paper’s exposition.
For \( m \in \mathbb{N} \), let \( G(x, m) \) denote the set of all greedy sets of \( x \) of cardinality \( m \). A basis is said to be greedy if there exists a constant \( C \geq 1 \) such that
\[
\| x - P_A(x) \| \leq C \sigma_m(x), \forall x \in X, \forall m \in \mathbb{N}, \forall \Lambda \in G(x, m),
\]
where \( P_A(x) = \sum_{n \in A} e_n(x)e_n \) for any finite set \( A \) and
\[
\sigma_m(x) := \inf \left\{ \left\| x - \sum_{n \in A} a_ne_n \right\| : a_n \in K, |A| \leq m \right\}.
\]

Later, Dilworth et al. [11] introduced the so-called almost greedy bases: a basis is almost greedy if there exists a constant \( C \geq 1 \) such that
\[
\| x - P_A(x) \| \leq C \inf \{ \| x - P_A(x) \| : |A| = m \}, \forall x \in X, \forall m \in \mathbb{N}, \forall \Lambda \in G(x, m).
\]

By definition, a greedy basis is almost greedy, but an almost greedy basis is not necessarily greedy (see [15] or [3, Example 10.2.9]). Among other results, Dilworth et al. showed a surprising equivalence of almost greedy bases. In particular, for any fixed \( \lambda > 1 \), if we enlarge greedy sums from size \( m \) to \( \lceil \lambda m \rceil \) in (1.1) to have
\[
\| x - P_A(x) \| \leq C \sigma_m(x), \forall x \in X, \forall m \in \mathbb{N}, \forall \Lambda \in G(x, \lceil \lambda m \rceil),
\]
then the condition (1.3) is equivalent to the condition (1.2) (with possibly different constants). That is, by increasing the size of greedy sums linearly, we move from the realm of greedy bases to a strictly weaker realm of almost greedy bases. From another perspective, for an almost greedy basis, enlarged greedy sums are optimal.

Continuing the work, the author of the present paper [9] investigated the situation for partially greedy (PG) bases (also introduced by Dilworth et al. for Schauder bases [11] and by Berasategui et al. for general bases [5]), which are defined to satisfy the condition: there exists a constant \( C \geq 1 \) such that
\[
\| x - P_A(x) \| \leq C \inf_{k \leq m} \left\| x - \sum_{n=1}^k e_n(x)e_n \right\|, \forall x \in X, \forall m \in \mathbb{N}, \forall \Lambda \in G(x, m).
\]

Since their introduction in [11], partially greedy bases have been studied extensively in [5, 7, 9, 10, 12, 13, 14]. According to [9, Theorem 1.8], if we enlarge the size of greedy sums from \( m \) to \( \lceil \lambda m \rceil \) in (1.4), we obtain strictly weaker greedy-type bases. Therefore, similar to Dilworth et al.‘s result, enlarging greedy sums in the PG property strictly weakens the property.

To complete the picture, let us consider reverse partially greedy (RPG) bases introduced by Dilworth and Khurana [12]. The original definition of RPG [12] is relatively more technical: first, given two sets \( A, B \subset \mathbb{N} \), we write \( A > B \) if for all \( a \in A \) and \( b \in B \), we have \( a > b \). Respective definitions hold for other inequalities \(<, \geq, \leq \). Also, it holds vacuously that \( \emptyset > A \) and \( \emptyset < A \) for all sets \( A \). A basis is RPG if there exists a constant \( C \geq 1 \) such that
\[
\| x - P_A(x) \| \leq C \overline{\sigma}_m^{RA}(x), \forall x \in X, \forall m \in \mathbb{N}, \forall \Lambda \in G(x, m),
\]
where
\[
\overline{\sigma}_m^{RA}(x) = \inf \{ \| x - P_A(x) \| : |A| \leq m, A > \Lambda \}.
\]
Recent work has confirmed that RPG bases are truly companions of PG bases (see [9, 12, 14]). That is, if a result holds true for PG bases, there is a corresponding result that holds for RPG bases. We, therefore, suspect that as in the case of PG bases, if we enlarge the size of greedy sums in a condition defining RPG bases, we would then obtain a strictly weaker greedy-type condition. However, by [9, Theorem 5.8], enlarging greedy sums in (1.5) still gives us RPG bases. This hints us at the existence of an equivalent reformulation of RPG bases such that when we enlarge greedy sums in the reformulation, we strictly weaken the RPG property. The main results in this paper show that such an equivalent reformulation indeed exists.

Given a set \( A \subset \mathbb{N} \), define
\[
q^A_m(x) := \inf \left\{ \| x - P_I(x) \| : \text{either } I = \emptyset \text{ or } I \text{ is an interval, } A \leq \max I, |I| \leq m \right\}.
\]

Our first result gives an equivalence of the notion of RPG bases.

**Theorem 1.1.** A basis \( B \) is RPG if and only if there exists \( C \geq 1 \) such that
\[
\| x - P_\Lambda(x) \| \leq C q^A_m(x), \forall x \in X, \forall m \in \mathbb{N}, \forall \Lambda \in G(x, m).
\]

To the author’s knowledge, (1.6) is the first characterization of RPG bases such that the right side of (1.6) involves a projection onto consecutive elements of the basis. This resembles (1.4), which defines PG bases. Indeed, (1.6) highlights the correspondence between PG bases and RPG bases by revealing the relative position of the interval \( I \) with respect to the greedy set \( \Lambda \). Specifically, for PG bases, since we project onto the first elements of \( B \), we have \( \min I \leq \Lambda \), while for RPG bases, \( \Lambda \leq \max I \). This leads us to a new characterization of PG bases.

**Theorem 1.2.** A basis \( B \) is PG if and only if there exists \( C \geq 1 \) such that
\[
\| x - P_\Lambda(x) \| \leq C \hat{q}^A_m(x), \forall x \in X, \forall m \in \mathbb{N}, \forall \Lambda \in G(x, m),
\]

where
\[
\hat{q}^A_m(x) := \inf \left\{ \| x - P_I(x) \| : \text{either } I = \emptyset \text{ or } I \text{ is an interval, } \Lambda \geq \min I, |I| \leq m \right\}.
\]

Next, for any fixed \( \lambda \geq 1 \), we study the following condition: there exists \( \lambda \geq 1 \) such that
\[
\| x - P_\Lambda(x) \| \leq C \hat{q}^A_m(x), \forall x \in X, \forall m \in \mathbb{N}, \forall \Lambda \in G(x, m, \lceil \lambda m \rceil).
\]

As stated above, our goal is to show that (1.7) is strictly weaker than the RPG property to have an analog of Dilworth et al.’s theorem for RPG bases. First, we give bases that satisfy (1.7) a name.

**Definition 1.3.** For a fixed \( \lambda \geq 1 \), a basis \( B \) is said to be \( \lambda \)-RPG of type 2 (or \( \lambda \)\-RPG2, for short) if \( B \) satisfies (1.7). In this case, the least constant in (1.7) is denoted by \( C_{\lambda, rp2} \), and the basis is said to be \( \lambda \)-RPG2 with constant \( C_{\lambda, rp2} \). We will use the shorthand “w.const” in place of “with constant”.

Here we use “of type 2” to not confuse ourselves with \( \lambda \)-RPG bases in [9, Definition 5.5], which defines a basis to be \( \lambda \)-RPG if it satisfies (1.5) with enlarged greedy sums from \( m \) to \( \lceil \lambda m \rceil \). For all \( \lambda \geq 1 \), the \( \lambda \)-RPG property in [9, Definition 5.5] is equivalent to RPG property (see [9, Theorem 5.8]). We shall show that this is not the case for the \( \lambda \)-RPG2 property.
Theorem 1.4. Let $\lambda > 1$. The following hold.

i) If $B$ is RPG, then $B$ is $\lambda$-RPG2.

ii) There exists an unconditional basis $B$ that is $\lambda$-RPG2 but is not RPG.

To prove Theorem 1.4, we need to characterize $\lambda$-RPG2 bases with the introduction of the so-called reverse partial symmetry for largest coefficients (Definition 3.2). As a corollary of our characterization, we characterize bases that satisfy (1.6) w.const $C = 1$ in the same manner as Berasategui et al. characterized strong partially greedy bases w.const 1 [5].

Corollary 1.5. A basis is 1-RPG2 w.const 1 if and only if it is RPG w.const 1.

For the full statement, see Corollary 3.12.

2. A CHARACTERIZATION OF RPG AND PG BASES

We recall some well-known results that will be used in due course.

Definition 2.1 (Konyagin and Temlyakov [15]). A basis $B$ is quasi-greedy w.const $C > 0$ if

$$\|P_\Lambda(x)\| \leq C \|x\|, \forall x \in X, m \in \mathbb{N}, \forall \Lambda \in G(x, m).$$

The least such $C$ is denoted by $C_q$, called the quasi-greedy constant. Also when $B$ is quasi-greedy, let $C_\ell$ be the least constant such that

$$\|x - P_\Lambda(x)\| \leq C_\ell \|x\|, \forall x \in X, m \in \mathbb{N}, \forall \Lambda \in G(x, m).$$

We call $C_\ell$ the suppression quasi-greedy constant.

A sequence $\varepsilon = (\varepsilon_n)_n$ is called a sign if $\varepsilon_n \in \mathbb{K}$ and $|\varepsilon_n| = 1$. For $x \in X$ and a finite set $A \subset \mathbb{N}$, let

$$1_A = \sum_{n \in A} \varepsilon_n, 1_{\varepsilon A} = \sum_{n \in A} \varepsilon_n e_n, \text{ and } P_{\Lambda_\varepsilon}(x) = x - P_{\Lambda_\varepsilon}(x).$$

Two important properties of quasi-greedy bases are the UL property and the uniform boundedness of the truncation operators.

i) UL property: for all finite $A \subset \mathbb{N}$ and scalars $(a_n)$, we have

$$\frac{1}{2C_q} \min \{a_n\|1_A\| \leq \left\| \sum_{n \in A} a_n e_n \right\| \leq 2C_q \max \{a_n\|1_A\|, \right.$$

which was first proved in [11].

ii) the uniform boundedness of the truncation operator: for each $\alpha > 0$, we define the truncation function $T_\alpha$ as follows: for $b \in \mathbb{K}$,

$$T_\alpha(b) = \begin{cases} \text{sgn}(b)\alpha, & \text{if } |b| > \alpha, \\ b, & \text{if } |b| \leq \alpha. \end{cases}$$

We define the truncation operator $T_\alpha : X \to X$ as

$$T_\alpha(x) = \sum_{n=1}^{\infty} T_\alpha(e_n^\ast(x)) e_n = \alpha 1_{\varepsilon_\Lambda_n(x)} + P_{\Lambda_\varepsilon_n(x)}(x),$$

where $\Lambda_\alpha(x) = \{n : |e_n^\ast(x)| > \alpha\}$ and $\varepsilon_n = \text{sgn}(e_n^\ast(x))$ for all $n \in \Lambda_\alpha(x)$. 
Theorem 2.2. \cite{3} Lemma 2.5] Let $B$ be suppression quasi-greedy w.const $C$. Then for any $\alpha > 0$, $\|T_\alpha\| \leq C$.

Now we recall a characterization of RPG bases due to Dilworth and Khurana \cite{12}.

**Definition 2.3.** A basis $B$ is said to be reverse conservative w.const $C > 0$ if we have

$$\|1_A\| \leq C\|1_B\|,$$

for all finite sets $A, B \subseteq \mathbb{N}$ with $|A| \leq |B|$ and $B < A$. In this case, the least constant $C$ is denoted by $\Delta_{rc}$.

**Theorem 2.4.** \cite{12} Theorem 2.7] A basis is RPG if and only if it is reverse conservative and quasi-greedy.

We are ready to prove our first main result.

**Proof of Theorem 1.1.** Assume \eqref{1.6}. By Theorem 2.4, we need to show that $B$ is both quasi-greedy and reverse conservative. First, we observe that $\Lambda \Delta_{rc} \subseteq A$ by Definition 2.3.

Next, we assume that $B$ is quasi-greedy w.const $C_q$ and reverse conservative w.const $\Delta_{rc}$ for some $C_q, \Delta_{rc} > 0$. Fix $x \in \mathbb{X}, m \in \mathbb{N}, \Lambda \in G(x, m)$, and a nonempty interval $I$ with $|I| \leq m, \Lambda \leq \max I$. We shall show that

$$\|x - P_\Lambda(x)\| \leq (1 + C_q + 4C_q^3\Delta_{rc})\|x - P_I(x)\|.
$$

Write

$$\|x - P_\Lambda(x)\| \leq \|x - P_I(x)\| + \|P_{\Lambda \setminus I}(x)\| + \|P_{I \setminus \Lambda}(x)\|. \quad (2.1)$$

Since $\Lambda \setminus I$ is a greedy set of $x - P_I(x)$, we get

$$\|P_{\Lambda \setminus I}(x)\| \leq C_q\|x - P_I(x)\|. \quad (2.2)$$

Let us estimate $\|P_{I \setminus \Lambda}(x)\|$. Note that $\Lambda \setminus I < I \setminus \Lambda$. Otherwise, there exists $a \in \Lambda \setminus I$, $b \in I \setminus \Lambda$ such that $a \geq b$. Then $a \geq \min I$ and $a \leq \max \Lambda \leq \max I$. Hence, $a \in I$, which contradicts $a \in \Lambda \setminus I$. Furthermore, $|I \setminus \Lambda| \leq |\Lambda \setminus I|$ because $|I| \leq |\Lambda|$. Since $B$ is $\Delta_{rc}$-reverse conservative, we get

$$\|1_{I \setminus \Lambda}\| \leq \Delta_{rc}\|1_{\Lambda \setminus I}\|.$$
We have
\[
\left\| \sum_{n \in \Gamma \setminus A} e_n^*(x)e_n \right\| \leq \max_{n \in \Gamma \setminus A} \left| e_n^*(x) \right| \sup_{\varepsilon} \left\| 1_{\varepsilon \setminus A} \right\| \text{ by convexity}
\]
\[
\leq 2C_q \min_{n \in \Lambda \setminus I} \left| e_n^*(x) \right| \left\| 1_{\Lambda \setminus A} \right\| \text{ by the UL property}
\]
\[
\leq 2C_q \Delta \min_{n \in \Lambda \setminus I} \left| e_n^*(x) \right| \left\| 1_{\Lambda \setminus I} \right\| \text{ by reverse conservativeness}
\]
\[
\leq 4C_q^2 \Delta \| P_{\Lambda \setminus I}(x) \| \text{ by the UL property. (2.3)}
\]
From (2.1), (2.2), and (2.3), we obtain
\[
\| x - P_A(x) \| \leq (1 + C_q + 4C_q^2 \Delta) \| x - P_I(x) \|
\]
as desired. \(\Box\)

The proof of Theorem 1.2 is the same as the proof of Theorem 1.1 with obvious modifications. The proof is left for interested readers.

3. LARGER GREEDY SUMS FOR RPG BASES

Throughout this section, let \( \lambda \) be a real number at least 1. First, we define the notion of reverse partial symmetry for largest coefficients (RPSLC). For a finite set \( A \subset \mathbb{N} \), let \( s(A) = 0 \) if \( A = \emptyset \); otherwise, \( s(A) = \max A - \min A + 1 \). We can think of \( s(A) \) as \( |\text{co}(A)| \), where \( \text{co}(A) \) is the integer convex hull of the set \( A \). For a collection of sets \( (A_i)_{i \in I} \), we write \( \sqcup_i A_i \) to mean \( A_j \cap A_k = \emptyset \) for any \( j, k \in I \). Finally, \( \|x\|_{\infty} := \max_n |e_n^*(x)| \).

**Definition 3.1.** A vector \( x \in \mathbb{X} \) is said to surround a finite set \( A \subset \mathbb{N} \) if either \( A = \emptyset \) or \( \text{supp}(x) \cap [\min A, \max A] = \emptyset \).

**Definition 3.2.** A basis is said to be \( \lambda \)-RPSLC if there exists a constant \( C \geq 1 \) such that
\[
\| x + 1_{\varepsilon A} \| \leq C \| x + 1_{\delta B} \|
\]
for all \( x \in \mathbb{X} \) with \( \|x\|_{\infty} \leq 1 \), for all signs \( \varepsilon, \delta \), and for all finite sets \( A, B \subset \mathbb{N} \) such that \( (\lambda - 1)s(A) + |A| \leq |B| \), \( B \sqcup \text{supp}(x) \), \( B < A \), and \( x \) surrounds \( A \). In this case, the least such \( C \) is denoted by \( \Delta_{\lambda, \text{rpl}} \).

**Remark 3.3.** While the definition of \( \lambda \)-RPSLC may seem unnatural and technical, we shall use it in proving Corollary 3.12 and eventually Theorem 4.4, both of which are relevant to the literature on greedy-type bases.

We have an easy but useful reformulation of \( \lambda \)-RPSLC.

**Proposition 3.4.** A basis \( B \) is \( \lambda \)-RPSLC w.const \( \Delta_{\lambda, \text{rpl}} \) if and only if
\[
\| x \| \leq \Delta_{\lambda, \text{rpl}} \| x - P_A(x) + 1_{\varepsilon B} \|, \tag{3.1}
\]
for all \( x \in \mathbb{X} \) with \( \|x\|_{\infty} \leq 1 \), for all sign \( \varepsilon \), and for all finite sets \( A, B \subset \mathbb{N} \) such that \( (\lambda - 1)s(A) + |A| \leq |B| \), \( B \sqcup \text{supp}(x - P_A(x)) \), \( B < A \), and \( x - P_A(x) \) surrounds \( A \).
Proof. Assume that $B$ is $\lambda$-RPSLC w.const $\Delta_{\lambda,rpl}$. Let $x, A, B, \varepsilon$ be chosen as in (3.1). We have
\[
\|x\| = \left\| x - P_A(x) + \sum_{n \in A} e_n^*(x)e_n \right\| \leq \sup_\delta \| x - P_A(x) + 1_\delta A \| \leq \Delta_{\lambda,rpl} \| x - P_A(x) + 1_\varepsilon B \|.
\]
Next, assume that $B$ satisfies (3.1). Let $x, A, B, \varepsilon, \delta$ be chosen as in Definition 3.2. Let $y = x + 1_\varepsilon A$. By (3.1),
\[
\|x + 1_\varepsilon A\| = \|y\| \leq \Delta_{\lambda,rpl} \| y - P_A(y) + 1_\delta B \| = \Delta_{\lambda,rpl} \| x + 1_\delta B \|.
\]
This completes our proof. \qed

The next theorem characterizes $\lambda$-RPG2 bases.

Theorem 3.5. A basis $B$ is $\lambda$-RPG2 if and only it is quasi-greedy and $\lambda$-RPSLC.

The next theorem, which describes the relation between constants of the $\lambda$-RPG2 property and RPSLC, facilitates the proof of Theorem 3.5.

Theorem 3.6. (Analog of [9] Theorem 4.1) Let $B$ be suppression quasi-greedy w.const $C_\ell$. The following hold.

i) If $B$ is 1-RPG2 w.const $C_{1,rpl}$, then $B$ is 1-RPSLC w.const $C_{1,rpl}$.

ii) If $B$ is $\lambda$-RPG2 w.const $C_{\lambda,rpl}$, then $B$ is $\lambda$-RPSLC w.const $C_{\lambda,C_{\lambda,rpl}}$.

iii) If $B$ is $\lambda$-RPSLC w.const $\Delta_{\lambda,rpl}$, then $B$ is $\lambda$-RPG2 w.const $C_{\lambda\Delta_{\lambda,rpl}}$.

Proof. i) Let $x, A, B, \varepsilon, \delta$ be as in Definition 3.2 We need to show that
\[
\|x + 1_\varepsilon A\| \leq C_{1,rpl} \|x + 1_\delta B\|.
\]
If $A = \emptyset$, then
\[
\|x + 1_\varepsilon A\| = \|x\| \leq C_\ell \|x + 1_\delta B\| \leq C_{1,rpl} \|x + 1_\delta B\|
\]
where the last inequality is due to Definition 1.3. If $A \neq \emptyset$, form $y = x + 1_\varepsilon A + 1_\delta B + 1_D$, where $D = \lfloor \min A, \max A \rfloor \setminus A$. We have
\[
|D \cup A| \leq |D \cup B| \quad \text{and} \quad D \cup B < \max(D \cup A).
\]
Hence, we obtain
\[
\|x + 1_\varepsilon A\| = \|y - P_{B \cup D}(y)\| \leq C_{1,rpl} \sigma_{B \cup D} \| y - P_A \cup D(y) \| \leq C_{1,rpl} \|x + 1_\delta B\|,
\]
as desired.

ii) Let $x, A, B, \varepsilon, \delta$ be as in Definition 3.2 If $A = \emptyset$, then by the proof of item i), we are done. If $A \neq \emptyset$, form $y = x + 1_\varepsilon A + 1_\delta B + 1_D$, where $D = \lfloor \min A, \max A \rfloor \setminus A$. Observe that $B \cup D$ is a greedy set of $y$, and
\[
|B \cup D| = |B| + |D| \geq (\lambda - 1)s(A) + |A| + (s(A) - |A|) = \lambda s(A).
\]
Choose $\Lambda \subset B \cup D$ such that $|\Lambda| = \lfloor \lambda s(A) \rfloor$. Clearly,
\[
|A \cup D| = s(A) \quad \text{and} \quad \Lambda < \max(A \cup D).
\]
We obtain
\[ \|x + 1_{\varepsilon A}\| = \|y - P_{B\cup D}(y)\| \leq C_\ell \|y - P_A(y)\| \leq C_\ell C_{\lambda, rp} 2^{\sigma_{s(A)}(y)} \]
\[ \leq C_\ell C_{\lambda, rp} \|y - P_{A\cup D}(y)\| \]
\[ = C_\ell C_{\lambda, rp} \|x + 1_{\delta B}\|. \]

Therefore, \( B \) is \( \lambda \)-RPLC w.const \( C_\ell C_{\lambda, rp} \).

iii) Assume that \( B \) is suppression quasi-greedy w.const \( C_\ell \) and \( \lambda \)-RPLC w.const \( \Delta_{\lambda, rp} \). Let \( x \in X, m \in \mathbb{N}, \Lambda \in G(x, \lfloor \lambda m \rfloor) \), and a nonempty interval \( I \) with \( |I| \leq m \) and \( \Lambda \leq \max I \). We need to show that
\[ \|x - P_\Lambda(x)\| \leq C_\ell \Delta_{\lambda, rp} \|x - P_I(x)\|. \]

If \( \alpha := \min_{n \in \Lambda} |e_n^*(x)| \), then \( \|x - P_\Lambda(x)\|_\infty \leq \alpha \). We have
\[ |\Lambda \setminus I| = |\Lambda| - |\Lambda \cap I| \geq \lambda m - |\Lambda \cap I| = \lambda m + (|I| - |\Lambda \cap I|) - |I| \]
\[ \geq \lambda m + |\Lambda \setminus I| - m \]
\[ = (\lambda - 1)m + |\Lambda \setminus I| \]
\[ \geq (\lambda - 1)s(I \setminus \Lambda) + |I \setminus \Lambda|, \]
and
\[ \Lambda \setminus I < I \setminus \Lambda \text{ and } (\Lambda \setminus I) \cup \text{supp}(x - P_\Lambda(x) - P_{I \setminus \Lambda}(x)). \]
Furthermore, \( x - P_\Lambda(x) - P_{I \setminus \Lambda}(x) \) surrounds \( I \setminus \Lambda \). Setting \( \varepsilon = (\text{sgn}(e_n^*(x))) \), we can apply Proposition 3.4 to obtain
\[ \|x - P_\Lambda(x)\| \leq \Delta_{\lambda, rp} \|x - P_\Lambda(x) - P_{I \setminus \Lambda}(x) + \alpha 1_{\varepsilon \Lambda \setminus I}\| \]
\[ = \Delta_{\lambda, rp} \|T_\alpha(x - P_\Lambda(x) - P_{I \setminus \Lambda}(x) + P_{I \setminus \Lambda}(x))\| \]
\[ \leq C_\ell \Delta_{\lambda, rp} \|x - P_I(x)\| \text{ by Theorem 2.2} \]
We finish the case when \( I \neq \emptyset \). If \( I = \emptyset \), then we simply have \( \|x - P_\Lambda(x)\| \leq C_\ell \|x\| = C_\ell \|x - P_I(x)\| \).

**Proof of Theorem 3.5.** By Theorem 3.6, we only need to show that a \( \lambda \)-RPG2 basis is quasi-greedy. Indeed, a \( \lambda \)-RPG2 basis satisfies (1.7), which implies that
\[ \|x - P_\Lambda(x)\| \leq C \|x\|, \forall x \in X, \forall m \in \mathbb{N}, \forall \Lambda \in G(x, \lfloor \lambda m \rfloor). \]

By [16] Proposition 4.1, we know that \( B \) is quasi-greedy.

**Definition 3.7.** A basis is said to be \( \lambda \)-reverse conservative of type 2\(^{1}\) if for some constant \( C > 0 \), we have
\[ \|1_A\| \leq C \|1_B\|, \]
for all finite sets \( A, B \subseteq \mathbb{N} \) with \( (\lambda - 1)s(A) + |A| \leq |B| \) and \( B < A \). In this case, the least constant \( C \) is denoted by \( \Delta_{\lambda, rc} \).

**Proposition 3.8.** Let \( B \) be a quasi-greedy basis. Then \( B \) is \( \lambda \)-reverse conservative of type 2 if and only if it is \( \lambda \)-RPLC.

\(^{1}\)To distinguish from [9] Definition 5.6.
Proof. Setting $x = 0$ and $\varepsilon \equiv \delta \equiv 1$ in Definition \ref{def:reversely-greedy}, we see that a $\lambda$-RPSLC basis is $\lambda$-reverse conservative of type 2. Assume that $B$ is $\lambda$-reverse conservative of type 2 w.const $\Delta_{\lambda, rc}$, suppression quasi-greedy w.const $C_{\ell}$, and quasi-greedy w.const $C_q$. Let $x, A, B, \varepsilon, \delta$ be chosen as in Definition \ref{def:reversely-greedy}. We have
\[
\|1_{\varepsilon A}\|_{UL} \leq 2C_q\|1_A\| \leq 2C_q\Delta_{\lambda, rc}\|1_B\| \leq 4C_q^2\Delta_{\lambda, rc}\|x + 1_B\|.
\]
Furthermore,
\[
\|x\| \leq C_{\ell}\|x + 1_B\|.
\]
Therefore,
\[
\|x + 1_{\varepsilon A}\| \leq \|x\| + \|1_{\varepsilon A}\| \leq (4C_q^2\Delta_{\lambda, rc} + C_{\ell})\|x + 1_B\|.
\]
This completes our proof. □

**Theorem 3.9.** Let $B$ be a basis. The following are equivalent

i) $B$ is $\lambda$-RPG2.

ii) $B$ is quasi-greedy and $\lambda$-RPSLC.

iii) $B$ is quasi-greedy and $\lambda$-reverse conservative of type 2.

**Proof.** The equivalence between i) and ii) is due to Theorem \ref{thm:app1} and the equivalence between ii) and iii) is due to Proposition \ref{prop:reversely-greedy} □

The problem of characterizing $1$-greedy-type bases has been of great interest as can be seen in \cite{1, 2, 4, 5, 12}. As a corollary of Theorem \ref{thm:app2}, let us characterize bases that are $1$-RPG2 w.const 1.

**Theorem 3.10.** A basis $B$ is $1$-RPG2 w.const 1 if and only if $B$ is $1$-RPSLC w.const 1.

**Proof.** By Theorem \ref{thm:app2} items i) and iii), it suffices to show that if a basis $B$ is $1$-RPSLC w.const 1, then it is suppression quasi-greedy w.const 1. This follows immediately from Definition \ref{def:reversely-greedy} by setting $A = \emptyset$ to have
\[
\|x\| \leq \|x + e_k\|,
\]
for all $x \in X$ with $\|x\|_{\infty} \leq 1$ and for all $k \notin \text{supp}(x)$. By induction, $B$ is suppression quasi-greedy w.const 1. □

In the spirit of \cite[Proposition 4.2]{5}, we offer yet another characterization of $1$-RPG2 bases w.const 1.

**Theorem 3.11.** A basis $B$ is $1$-RPG2 w.const 1 if and only if $B$ satisfies simultaneously two following conditions

i) for all $x \in X$ with $\|x\|_{\infty} \leq 1$ and for all $k \notin \text{supp}(x)$, we have
\[
\|x\| \leq \|x + e_k\|. \tag{3.2}
\]

ii) for $x \in X$ with $\|x\|_{\infty} \leq 1$, for $s, t \in K$ with $|s| = |t| = 1$, and for $j < k$, both of which are not in $\text{supp}(x)$, we have
\[
\|x + te_k\| \leq \|x + se_j\|. \tag{3.3}
\]
Proof. Due to Theorem 3.10 it suffices to show that a basis is 1-RPSLC w.const 1 if and only if it satisfies both (3.2) and (3.3). It follows immediately from Definition 3.2 that a 1-RPSLC basis w.const 1 must satisfy (3.2) and (3.3). Conversely, suppose that \( \mathcal{B} \) satisfies both (3.2) and (3.3). Choose \( x, A, B, \varepsilon, \delta \) as in Definition 3.2. We show that

\[
\| x + 1_{\varepsilon A} \| \leq \| x + 1_{\delta B} \| \tag{3.4}
\]

inductively on \( |B| \).

Base case: \( |B| = 1 \). If \( A = \emptyset \), then (3.2) implies (3.4); if \( |A| = 1 \), then (3.3) implies (3.4).

Inductive hypothesis (I.H.): assume that for some \( \ell \in \mathbb{N} \), (3.4) holds for \( |B| \leq \ell \). We show that (3.4) holds for \( |B| = \ell + 1 \). If \( A = \emptyset \), then we use (3.2) inductively to obtain (3.4). For \( |A| \geq 1 \), let \( p = \max A, q \in B \), \( A' = A \setminus \{p\} \), and \( B' = B \setminus \{q\} \). We have

\[
\| x + 1_{\varepsilon A} \| = \|(x + \varepsilon_pe_p) + 1_{\varepsilon A'}\| \leq \|(x + \varepsilon_pe_p) + 1_{\delta B'}\| \text{ by I.H.}
\leq \|(x + \delta_qe_q) + 1_{\delta B'}\| \text{ by (3.3)}
= \|x + 1_{\delta B}\|.
\]

This shows that \( \mathcal{B} \) is 1-RPSLC w.const 1. \( \square \)

Corollary 3.12. The following are equivalent

i) \( (e_n)_n \) is 1-RPG2 w.const 1.

ii) \( (e_n)_n \) is 1-RPSLC w.const 1.

iii) \( (e_n)_n \) satisfies (3.2) and (3.3).

iv) \( (e_n)_n \) is RPG w.const 1.

Proof. The equivalence among i), ii), and iii) is due to Theorems 3.10 and 3.11. That iii) is the same as iv) is due to [12, Theorem 3.4]. \( \square \)

4. THE \( \lambda \)-RPG2 PROPERTY IS WEAKER THAN THE RPG PROPERTY

The goal of this section is to prove Theorem 1.4.

Proof. i) If \( \mathcal{B} \) is RPG, then it is quasi-greedy and reverse conservative. By definition, a reverse conservative basis is \( \lambda \)-reverse conservative of type 2. By Theorem 3.9, \( \mathcal{B} \) is \( \lambda \)-RPG2.

ii) For each \( \lambda > 1 \), we now construct an unconditional basis \( \mathcal{B} \) that is \( \lambda \)-RPG2 but is not RPG. Let \( D = \{2^n : n \in \mathbb{N}_0\} \), \( u_n = \frac{1}{\sqrt{n}} \), and \( v_n = \frac{1}{n} \) for all \( n \geq 1 \). Let \( \mathbb{X} \) be the completion of \( c_{00} \) with respect to the following norm: for \( x = (x_1, x_2, \ldots) \), define

\[
\| x \| = \sup_{\pi, \pi'} \left( \sum_{n \in D} u_{\pi(n)} |x_n| + \sum_{n \notin D} v_{\pi'(n)} |x_n| \right),
\]

where \( \pi : D \to \mathbb{N} \) and \( \pi' : \mathbb{N} \setminus D \to \mathbb{N} \) are bijections. Clearly, the canonical basis \( \mathcal{B} \) is an 1-unconditional normalized basis of \( \mathbb{X} \).

Claim 4.1. The basis \( \mathcal{B} \) is not reverse conservative and thus, is not RPG.
Proof. We use the notation \( \sim \) to indicate the order of a number. For each \( N \in \mathbb{N} \), let \( A_N = \{2^{2N+1}, 2^{2N+2}, \ldots, 2^{3N}\} \) and \( B_N = \{3, 3^2, \ldots, 3^N\} \). Then \( |A_N| = |B_N| = N \) and \( B_N < A_N \). However,

\[
\|1_{A_N}\| = \sum_{n=1}^{N} \frac{1}{\sqrt{n}} \sim \sqrt{N} \quad \text{and} \quad \|1_{B_N}\| = \sum_{n=1}^{N} \frac{1}{n} \sim \ln N,
\]

which imply that \( \|1_{A_N}\|/\|1_{B_N}\| \to \infty \) as \( N \to \infty \). Therefore, \( B \) is not reverse conservative. \( \square \)

Claim 4.2. The basis \( B \) is \( \lambda \)-reverse conservative of type 2 and thus, is \( \lambda \)-RPG2.

Proof. Pick nonempty, finite sets \( A, B \subset \mathbb{N} \) with \( (\lambda - 1)s(A) + |A| \leq |B| \) and \( B < A \). Then

\[
\|1_A\| = \sum_{n=1}^{\lfloor A \rfloor} \frac{1}{\sqrt{n}} + \sum_{n=1}^{\lfloor B \rfloor} \frac{1}{n} \quad \text{and} \quad \|1_B\| = \sum_{n=1}^{\lfloor B \rfloor} \frac{1}{\sqrt{n}} + \sum_{n=1}^{\lfloor B \rfloor} \frac{1}{n}.
\]

Claim 4.3. We have \( \|1_B\| \geq \ln |B| \).

Proof. If \( B \setminus D = \emptyset \), then \( B \cap D = B \) and \( \|1_B\| = \sum_{n=1}^{\lfloor B \rfloor} 1/\sqrt{n} \sim \sqrt{|B|} \). Similarly, if \( B \cap D = \emptyset \), then \( B \setminus D = B \) and \( \|1_B\| = \sum_{n=1}^{\lfloor B \rfloor} 1/n \sim \ln |B| \). Finally, if \( B \setminus D \neq \emptyset \) and \( B \cap D \neq \emptyset \), then

\[
\|1_B\| \geq \ln |B \cap D| + \ln |B \setminus D| = \ln(|B \cap D| \cdot |B \setminus D|) \geq \ln(|B| - 1) \geq \ln |B|.
\]

We proceed by case analysis.

Case 1: \( \sum_{n=1}^{\lfloor A \cap D \rfloor} \frac{1}{\sqrt{n}} \leq \sum_{n=1}^{\lfloor A \setminus D \rfloor} \frac{1}{n} \). We get

\[
\|1_A\| \leq 2 \sum_{n=1}^{\lfloor A \setminus D \rfloor} \frac{1}{n} \leq 2 \sum_{n=1}^{\lfloor B \setminus D \rfloor} \frac{1}{n} \leq 2 \left( \sum_{n=1}^{\lfloor B \cap D \rfloor} \frac{1}{\sqrt{n}} + \sum_{n=1}^{\lfloor B \setminus D \rfloor} \frac{1}{n} \right) = 2 \|1_B\|.
\]

Case 2: \( \sum_{n=1}^{\lfloor A \cap D \rfloor} \frac{1}{\sqrt{n}} > \sum_{n=1}^{\lfloor A \setminus D \rfloor} \frac{1}{n} \). Then \( \|1_A\| \sim \sqrt{|A \cap D|} \). If \( |B \cap D| > |A \cap D| \), then we are done because

\[
\|1_B\| \geq \sum_{n=1}^{\lfloor B \cap D \rfloor} \frac{1}{\sqrt{n}} \sim \sqrt{|B \cap D|} \geq \sqrt{|A \cap D|} \sim \|1_A\|.
\]

Suppose that \( |B \cap D| < |A \cap D| \). Let \( N = |A \cap D| \) and write \( A \cap D = \{2^{k_1}, 2^{k_2}, \ldots, 2^{k_N}\} \) to obtain \( \|1_A\| \sim \sqrt{N} \).

i) Case 2.1: \( N = 1 \). Then \( \sum_{n=1}^{\lfloor A \cap D \rfloor} \frac{1}{\sqrt{n}} \geq \sum_{n=1}^{\lfloor A \setminus D \rfloor} \frac{1}{n} \) implies that \( |A \setminus D| = 0 \) and \( |A| = 1 \). Clearly, \( \|1_A\| \leq \|1_B\| \).

ii) Case 2.2: \( N \geq 2 \). We have

\[
|B| \geq (\lambda - 1)s(A) \geq (\lambda - 1)(2^{k_N} - 2^{k_1} + 1) \geq (\lambda - 1)2^{k_N} - 2^{k_1} - 1.
\]

Hence,

\[
\ln |B| \geq \ln(\lambda - 1) + (N - 1) \ln 2.
\]
If \( \ln(\lambda - 1) + 0.5(N - 1) \geq 0 \), then by Claim 4.3,
\[
\|1_B\| \lesssim \ln |B| \geq (\ln 2 - 0.5)(N - 1) \gtrsim \sqrt{N} \sim \|1_A\|.
\]
If \( \ln(\lambda - 1) + 0.5(N - 1) < 0 \), then \( N < 1 - 2 \ln(\lambda - 1) \). In this case,
\[
\|1_A\| \sim \sqrt{N} < \sqrt{1 - 2 \ln(\lambda - 1)} \|1_B\|.
\]
We have shown that in all cases, there exists a constant \( C = C(\lambda) \) such that \( \|1_A\| \leq C\|1_B\| \). Therefore, \( B \) is \( \lambda \)-reverse consecutive of type 2. \( \square \)

We conclude that our basis \( B \) is 1-unconditional and \( \lambda \)-RPG2 but is not RPG. \( \square \)

REFERENCES

[1] F. Albiac and J. L. Ansorena, Characterization of 1-quasi-greedy bases, \textit{J. Approx. Theory} \textbf{201} (2016), 7–12.
[2] F. Albiac and J. L. Ansorena, Characterization of 1-almost greedy bases, \textit{Rev. Mat. Complut.} \textbf{30} (2017), 13–24.
[3] F. Albiac and N. Kalton, Topics in Banach Space Theory, second edition, ISBN 978-3-319-31555-3 (2016).
[4] F. Albiac and P. Wojtaszczyk, Characterization of 1-greedy bases, \textit{J. Approx. Theory} \textbf{138} (2006), 65–86.
[5] M. Berasategui, P. M. Berná, and S. Lassalle, Strong partially greedy bases and Lebesgue-type inequalities, \textit{Constr. Approx.} \textbf{54} (2021), 507–528.
[6] P. M. Berná, O. Blasco, and G. Garrigós, Lebesgue inequalities for greedy algorithm in general bases, \textit{Rev. Mat. Complut.} \textbf{30} (2017), 369–392.
[7] M. Berasategui, P. M. Berná, and H. V. Chu, Extensions and new characterizations of some greedy-type bases, \textit{Bull. Malays. Math. Sci. Soc.} \textbf{46} (2023), 1–18.
[8] M. Berasategui, P. M. Berná, and H. V. Chu, On consecutive greedy and other greedy-like type of bases, preprint (2023). Available at: \url{https://arxiv.org/abs/2302.05758}.
[9] H. V. Chu, Performance of the thresholding greedy algorithm with larger greedy sums, \textit{J. Math. Anal. Appl.} \textbf{525} (2023), 1–23.
[10] H. V. Chu, Strong partially greedy bases with respect to an arbitrary sequence, preprint (2022). Available at: \url{https://arxiv.org/abs/2208.07300}.
[11] S. J. Dilworth, N. J. Kalton, D. Kutzarova, and V. N. Temlyakov, The thresholding greedy algorithm, greedy bases, and duality, \textit{Constr. Approx.} \textbf{19} (2003), 575–597.
[12] S. J. Dilworth and D. Khurana, Characterizations of almost greedy and partially greedy bases, \textit{J. Approx. Theory} \textbf{11} (2019), 115–137.
[13] S. J. Dilworth, D. Kutzarova, and T. Oikhberg, Lebesgue constants for the weak greedy algorithm, \textit{Rev. Mat. Complut.} \textbf{28} (2015), 393–409.
[14] D. Khurana, Weight-partially greedy bases and weight-property (A), \textit{Ann. Funct. Anal.} \textbf{11} (2020), 101–117.
[15] S. V. Konyagin and V. N. Temlyakov, A remark on greedy approximation in Banach spaces, \textit{East J. Approx.} \textbf{5} (1999), 365–379.
[16] T. Oikhberg, Greedy algorithm with gaps, \textit{J. Approx. Theory} \textbf{225} (2018), 176–190.

Email address: hungchu1@tamu.edu

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843, USA