GEOMETRIC QUANTIZATION OF COMPLETELY INTEGRABLE HAMILTONIAN SYSTEMS IN THE ACTION-ANGLE VARIABLES

G. GIACHETTA\textsuperscript{a}, L. MANGIAROTTI\textsuperscript{a} and G. SARDANASHVILY\textsuperscript{b}

\textsuperscript{a} Department of Mathematics and Physics, University of Camerino, 62032 Camerino (MC), Italy
\textsuperscript{b} Department of Theoretical Physics, Moscow State University, 117234 Moscow, Russia

Abstract.

We provide geometric quantization of a completely integrable Hamiltonian system in the action-angle variables around an invariant torus with respect to polarization spanned by almost-Hamiltonian vector fields of angle variables. The associated quantum algebra consists of functions affine in action coordinates. We obtain a set of its nonequivalent representations in the separable pre-Hilbert space of smooth complex functions on the torus where action operators and a Hamiltonian are diagonal and have countable spectra.

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We study geometric quantization of a completely integrable system around a regular compact invariant manifold. By virtue of the classical Arnold–Liouville theorem, a small neighbourhood $W$ of this manifold in the ambient momentum phase space $M$ is isomorphic to the symplectic annulus

$$W = V \times T^m,$$

where $V \subset \mathbb{R}^m$ is a nonempty domain and $T^m$ is an $m$-dimensional torus \cite{1,2}. The product (1) is equipped with the action-angle coordinates $(I_k, \phi^k)$. With respect to these coordinates, the symplectic form on $W$ (1) reads

$$\Omega = dI_k \wedge d\phi^k,$$

while a Hamiltonian of a completely integrable system is a function $H(I_k)$ of action variables only.

There are different approaches to quantization of completely integrable Hamiltonian systems \cite{3,4}. The advantage of the geometric quantization procedure is that it remains equivalent under symplectic diffeomorphisms. Geometric quantization of completely integrable Hamiltonian systems has been studied with respect to polarization spanned by Hamiltonian vector fields of integrals of motion \cite{5,6}. In fact, the Simms quantization of the harmonic oscillator is also of this type \cite{7}. The problem is that the associated quantum algebra includes functions which are not globally defined, and elements of the carrier
space are not smooth sections of a quantum bundle. Indeed, written with respect to the action-angle variables, this quantum algebra consists of functions which are affine in angle coordinates.

Here, we use a different polarization. Since a Hamiltonian of a completely integrable system on the symplectic annulus $W$ depends only on action variables, it seems natural to provide the Schrödinger representation of action variables by first order differential operators on functions of angle coordinates. For this purpose, one should choose the angle polarization of the symplectic annulus $W$ spanned by almost-Hamiltonian vector fields $\partial_k$ of angle variables.

Since the action-angle coordinates are canonical for the symplectic form $W$, geometric quantization of the symplectic annulus $(W, \Omega)$ in fact is equivalent to geometric quantization of the cotangent bundle $T^*T^m = \mathbb{R}^m \times T^m$ of the torus $T^m$ provided with the canonical symplectic form. In particular, the above mentioned angle polarization of $V \times T^m$ corresponds to the familiar vertical polarization of $T^*T^m$ which leads to Schrödinger quantization of the cotangent bundle $T^*T^m$. The associated quantum algebra $\mathcal{A}$ of $W$ consists of functions which are affine in action variables $I_k$. It possesses the continuum set of nonequivalent representations by first order differential operators in the separable pre-Hilbert space $C^\infty(T^m)$ of smooth complex functions on $T^m$. This set is indexed by homomorphisms of the de Rham cohomology group $H^1(T^m) = \mathbb{R}^m$ of the torus $T^m$ to the circle group $U(1)$.

As is well known, an application of Schrödinger geometric quantization is limited by the fact that a Hamiltonian fails to belong to the quantum algebra $\mathcal{A}$, unless it is affine in momenta. In the case of a completely integrable system on the symplectic annulus $W$, a Hamiltonian $H$ depends only on action variables which mutually commute. Therefore, if $H(I_k)$ is a polynomial function, it is uniquely represented by an element of the enveloping algebra $\mathfrak{a}$ of the Lie algebra $\mathcal{A}$, and is quantized as the operator $\hat{H}(\hat{I}_k)$. Moreover, this quantization is also extended to Hamiltonians which are analytic functions on $\mathbb{R}^m$ because, as we will observe, the action operators $\hat{I}_k$ are diagonal.

As was mentioned above, the geometric quantization procedure is equivalent under symplectic diffeomorphisms, but it essentially depends on the choice of polarization [8,9]. Given a symplectic diffeomorphism $\mathfrak{g}$ of $W$ to the product $V \times T^m$, geometric quantization of $V \times T^m$ with respect to the angle polarization of $V \times T^m$ implies the equivalent quantization of the initial completely integrable system on a neighbourhood $W$ of a compact invariant manifold with respect to the induced angle polarization of $W$. However, this polarization of $W$ is not canonical because an isomorphism $\mathfrak{g}$ by no means is unique. Furthermore, there are topological obstructions to global action-angle coordinates [10,11]. Therefore, quantization with respect to the angle polarization is not extended to the whole momentum phase space $M$ of a completely integrable system. For instance, one usually mentions a harmonic oscillator as the simplest completely integrable Hamiltonian system whose quantization in the action-angle variables looks notoriously difficult because the eigenvalues of its action operator is expected to be lower bounded (see [12] for a survey). However, a harmonic oscillator written relative to action-angle coordinates $(I, \phi)$ is located in the momentum phase space $\mathbb{R}^2 \setminus \{0\}$, but it is not the standard oscillator on $\mathbb{R}^2$. Namely, there is a monomor-
phism, but not an isomorphism of the Poisson algebra of smooth complex functions on $\mathbb{R}^2$ to that on $\mathbb{R}^2 \setminus \{0\}$. Furthermore, the angle polarization on $\mathbb{R}^2 \setminus \{0\}$ is not extended to $\mathbb{R}^2$. As a consequence, the quantum algebra associated to this polarization contains functions on $\mathbb{R}^2 \setminus \{0\}$ which are not extended to $\mathbb{R}^2$, and its carrier space $C^\infty(T^m)$ does so. As was mentioned above, the Simms quantization of the harmonic oscillator on the momentum phase space $\mathbb{R}^2 \setminus \{0\}$ with respect to the polarization spanned by the Hamiltonian vector field of a Hamiltonian $\mathcal{H}$ is quantization with respect to the action-angle variables. The carrier space of this quantization consists of tempered distributions, and the spectrum of the Hamiltonian is lower bounded [7].

In accordance with the standard geometric quantization procedure [13,14], since the symplectic form $\Omega(\mathcal{E})$ is exact, the prequantum bundle is defined as a trivial complex line bundle $\mathcal{C}$ over $V \times T^m$. Since the action-angle coordinates are canonical for the symplectic form $\mathcal{E}$, the prequantum bundle $\mathcal{C}$ need no metaplectic correction, and it is a quantum bundle. Let its trivialization

$$\mathcal{C} \cong (V \times T^m) \times \mathbb{C}$$

hold fixed. Any other trivialization leads to equivalent quantization of $V \times T^m$. Given the associated bundle coordinates $(I, \phi^k, c)$, on $\mathcal{C}$, one can treat its sections as smooth complex functions on $V \times T^m$.

The Konstant– Souriau prequantization formula associates to each smooth real function $f \in C^\infty(V \times T^m)$ on $V \times T^m$ the first order differential operator

$$\hat{f} = -i \nabla_{\vartheta_f} + f$$

on sections of $\mathcal{C}$, where

$$\vartheta_f = \partial^k f \partial_k - \partial_k f \partial^k$$

is the Hamiltonian vector field of $f$ and $\nabla$ is the covariant differential with respect to a suitable $U(1)$-principal connection on $\mathcal{C}$. This connection preserves the Hermitian metric $g(c, c') = \sigma^2$ on $\mathcal{C}$, and its curvature form obeys the prequantization condition $R = i\Omega$. It reads

$$A = A_0 + icI_k d\phi^k \otimes \partial_c$$

where $A_0$ is a flat $U(1)$-principal connection on $\mathcal{C} \to V \times T^m$. The classes of gauge nonconjugated flat principal connections on $\mathcal{C}$ are indexed by homomorphisms of the de Rham cohomology group

$$H^1(V \times T^m) = H^1(T^m) = \mathbb{R}^m$$

of the annulus $V \times T^m$ to $U(1)$ [7], i.e., their set is bijective to $\mathbb{R}^m/\mathbb{Z}^m$. We choose their representatives of the form

$$A_0[(\lambda_k)] = dI_k \otimes \partial^k + d\phi^k \otimes (\partial_k + i\lambda_k c \partial_c), \quad \lambda_k \in [0, 1].$$
Accordingly, the relevant connection (5) on $\mathcal{C}$ up to gauge conjugation reads

$$A[\lambda_k] = di_k \otimes \partial^k + d\phi^k \otimes (\partial_k + i(I_k + \lambda_k)c\partial_c).$$

(6)

For the sake of simplicity, we will assume that the numbers $\lambda_k$ in the expression (6) belongs to $\mathbb{R}$, but will bear in mind that connections $A[\lambda_k]$ and $A[\lambda_k']$ with $\lambda_k - \lambda_k' \in \mathbb{Z}$ are gauge conjugated. Given a connection (5), the prequantization operators (4) read

$$\hat{f} = -i\partial f + (f - (I_k + \lambda_k)\partial f).$$

(7)

Let us choose the above mentioned angle polarization $V\pi$ which is the vertical tangent bundle of the fibration $\pi : V \times T^m \rightarrow T^m$. This polarization is spanned by the vectors $\partial^k$. It is readily observed that the corresponding quantum algebra $\mathcal{A} \subset C^\infty(V \times T^m)$ consists of affine functions

$$f = a^k(\phi^r)I_k + b(\phi^r)$$

(8)
of action coordinates $I_k$. The carrier space of its representation by operators (7) is defined as the space $E$ of sections $\rho$ of the prequantum bundle $C$ of compact support which obey the condition $\nabla\vartheta\rho = 0$ for any Hamiltonian vector field $\vartheta$ subordinate to the distribution $V\pi$. This condition reads

$$\partial_k f \partial^k \rho = 0, \quad \forall f \in C^\infty(T^m).$$

It follows that elements of $E$ are independent of action variables and, consequently, fail to be of compact support, unless $\rho = 0$. This is the well-known problem of Schrödinger geometric quantization. It is solved as follows [15,16].

Given an imbedding $i_T : T^m \rightarrow V \times T^m$, let $C_T = i_T^*C$ be the pull-back of the prequantum bundle $C$ over the torus $T^m$. It is a trivial complex line bundle $C_T = T^m \times \mathbb{C}$ provided with the pull-back Hermitian metric $g(c,c') = \alpha c$. Its sections are smooth complex functions on $T^m$. Let

$$\overline{A} = i_T^*A = d\phi^k \otimes (\partial_k + i(I_k + \lambda_k)c\partial_c)$$

be the pull-back of the connection $A$ (5) onto $C_T$. Let $D$ be a metalinear bundle of complex half-forms on the torus $T^m$. It admits the canonical lift of any vector field $\tau$ on $T^m$, and the corresponding Lie derivative of its sections reads

$$L_\tau = \tau^k \partial_k + \frac{1}{2} \partial_k \tau^k.$$

Let us consider the tensor product

$$Y = C_T \otimes D \rightarrow T^m.$$  

(9)

Since the Hamiltonian vector fields

$$\vartheta_f = a^k \partial_k - (I_k \partial_k a^r + \partial_k b)\partial^k$$
of functions $f$ \( f \) are projectable onto $T^m$, one can associate to each element $f$ of the quantum algebra $A$ the first order differential operator

$$\hat{f} = (-i\nabla_{\pi \phi} + f) \otimes \text{Id} + \text{Id} \otimes L_{\pi \phi} = -ia^k \partial_k - \frac{i}{2} \partial_k a^k - a^k \lambda_k + b$$ \hspace{1cm} (10)

on sections of $Y$. A direct computation shows that the operators $\hat{f}$ obey the Dirac condition

$$[\hat{f}, \hat{f}'] = -i\{\hat{f}, \hat{f}'\}.$$ 

Sections $s$ of the quantum bundle $Y \to T^m$ constitute a pre-Hilbert space $E_T$ with respect to the nondegenerate Hermitian form

$$\langle s|s' \rangle = \left( \frac{1}{2\pi} \right)^m \int_{T^m} \overline{s} \overline{s}, \quad s, s \in E_T.$$

Then it is readily observed that $\hat{f}$ are Hermitian operators in $E_T$. They provide a desired Schrödinger geometric quantization of a completely integrable Hamiltonian system on the annulus $V \times T^m$. Of course, this quantization depends on the choice of a connection $A[(\lambda_k)]$ and a metalinear bundle $D$. The latter need not be trivial.

If $D$ is trivial, sections of the quantum bundle $Y \to T^m$ obey the transformation rule

$$s(\phi^k + 2\pi) = s(\phi^k)$$

for all indices $k$. They are naturally complex smooth functions on $T^m$. By virtue of the multidimensional Fourier theorem [17], the functions

$$\psi_{(n_r)} = \exp[i(n_r \phi^r)], \quad (n_r) = (n_1, \ldots, n_m) \in \mathbb{Z}^m,$$ \hspace{1cm} (11)

constitute an orthonormal basis for the pre-Hilbert space $E_T = C^{\infty}(T^m)$. The action operators

$$\hat{I}_k = -i\partial_k - \lambda_k$$ \hspace{1cm} (12)

are diagonal

$$\hat{I}_k \psi_{(n_r)} = (n_k - \lambda_k) \psi_{(n_r)}$$ \hspace{1cm} (13)

with respect to this basis. Other elements of the algebra $A$ are decomposed into the pullback functions $\pi^* \psi_{(n_r)}$ which act on $C^{\infty}(T^m)$ by multiplications

$$\pi^* \psi_{(n_r)} \psi_{(n_r')} = \psi_{(n_r)} \psi_{(n_r')} = \psi_{(n_r + n_r')}.$$ \hspace{1cm} (14)

If $D$ is a nontrivial metalinear bundle, sections of the quantum bundle $Y \to T^m$ obey the transformation rule

$$\rho_T (\phi^j + 2\pi) = -\rho_T (\phi^j)$$ \hspace{1cm} (15)
for some indices $j$. In this case, the orthonormal basis of the pre-Hilbert space $E_T$ can be represented by double-valued complex functions

$$
\psi_{(n_i, n_j)} = \exp[i(n_i \phi^i + (n_j + \frac{1}{2})\phi^j)]
$$

on $T^m$. They are eigenvectors

$$
\hat{I}_i \psi_{(n_i, n_j)} = (n_i - \lambda_i) \psi_{(n_i, n_j)}, \quad \hat{I}_j \psi_{(n_i, n_j)} = (n_j - \lambda_j + \frac{1}{2}) \psi_{(n_i, n_j)}
$$

of the operators $\hat{I}_k$ (12), and the pull-back functions $\pi^* \psi_{(n_r)}$ act on the basis (16) by the above law (14). It follows that the representation of the quantum algebra $A$ determined by the connection $A[(\lambda_k)]$ (6) in the space of sections (15) of a nontrivial quantum bundle $Y$ (9) is equivalent to its representation determined by the connection $A[(\lambda_i, \lambda_j - \frac{1}{2})]$ in the space $C^\infty(T^m)$ of smooth complex functions on $T^m$.

Therefore, one can restrict the study of representations of the quantum algebra $A$ to its representations in $C^\infty(T^m)$ associated to different connections (6). These representations are nonequivalent, unless $\lambda_k - \lambda'_k \in \mathbb{Z}$ for all indices $k$.

Given the representation (10) of the quantum algebra $A$ in $C^\infty(T^m)$, any polynomial Hamiltonian $H(I_k)$ of a completely integrable system is uniquely quantized as a Hermitian element $\hat{H}(I_k)$ of the enveloping algebra $\mathcal{A}$ of $A$. It has the countable spectrum

$$
\hat{H}(I_k) \psi_{(n_r)} = H(n_k - \lambda_k) \psi_{(n_r)}.
$$

Note that, since $\hat{I}_k$ are diagonal, one can also quantize Hamiltonians $H(I_k)$ which are analytic functions on $\mathbb{R}^m$.

As a conclusion remark, let us assume that a Hamiltonian $\mathcal{H}$ of a completely integrable system is independent of action variables $I_a$ ($a, b, c = 1, \ldots, l$). Then its eigenvalues are countably degenerate. Let us consider the perturbed Hamiltonian

$$
\mathcal{H}' = \Delta(\xi^\mu, \phi^b, I_a) + \mathcal{H}(I_j),
$$

where the perturbation term $\Delta$ depends on the action-angle coordinates with the above mentioned indices $a, b, c, \ldots$ and on some time-dependent parameters $\xi^\lambda(t)$ by the law

$$
\Delta = \Lambda^a_\lambda(\xi^\mu, \phi^b) \partial_t \xi^\lambda I_a.
$$

The Hamiltonian $\mathcal{H}'$ characterizes a Hamiltonian system with time-dependent parameters [18-21]. Being affine in action variables, the perturbation term $\Delta$ (18) admits the instantwise quantization by the operator

$$
\hat{\Delta} = -(i\Lambda^a_\beta \partial_a + \frac{i}{2} \partial_a \Lambda^a_\beta + \lambda_a \Lambda^a_\beta) \partial_t \xi^\beta.
$$

Since the operators $\hat{\Delta}$ and $\hat{\mathcal{H}}$ mutually commute, the total quantum evolution operator reduces to the product

$$
T \exp \left[ -i \int_0^t \hat{\mathcal{H}} dt' \right] = U_1(t) \circ U_2(t) = T \exp \left[ -i \int_0^t \hat{\mathcal{H}} dt' \right] \circ T \exp \left[ -i \int_0^t \hat{\Delta} dt' \right].
$$
The first factor $U_1$ in this product is the dynamic evolution operator of the quantum completely integrable Hamiltonian system. The second operator acts in the eigenspaces of the operator $U_1$ and reads

$$U_2(t) = T \exp \left[ \int_0^t \left\{ -\Lambda^a_\beta (\phi^b, \xi^\mu(t')) \partial_a - \frac{1}{2} \partial_a \Lambda^a_\beta (\phi^b, \xi^\mu(t')) + i \lambda_a \Lambda^a_\beta (\phi^b, \xi^\mu(t')) \right\} \partial_t \xi^\beta dt' \right]$$

$$= T \exp \left[ \int_{\xi([0,t])} \left\{ -\Lambda^a_\beta (\phi^b, \sigma^\mu) \partial_a - \frac{1}{2} \partial_a \Lambda^a_\beta (\phi^b, \sigma^\mu) + i \lambda_a \Lambda^a_\beta (\phi^b, \sigma^\mu) \right\} d\sigma^\beta \right]. \quad (19)$$

It is readily observed that this operator depends on the curve $\xi([0,1]) \subset S$ in the parameter space $S$. One can treat it as an operator of parallel displacement along the curve $\xi$ [19-22]. For instance, if $\xi([0,1])$ is a loop in $S$, the operator $U_2$ is the geometric Berry factor. In this case, one can think of $U_2$ as being a holonomy control operator [23]. At present, such control operators attract special attention in connection with holonomic quantum computation [24-26].

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