Dynamics in a system of four qubits in two cavities

Duong Hai Trieu

Institute of Materials Science, Vietnam Academy of Science and Technology,
18 Hoang Quoc Viet Road, Cau Giay Dist., Hanoi, Vietnam

E-mail: trieudh@ims.vast.ac.vn

Abstract. We study the dynamics of a closed system of four qubits in two coupled cavities. Each cavity contains two qubits which interact with its quantum field and photons which may hop between the cavities. The state of the system is first expressed in delocalized modes and then is expressed in the system of local bases. Exact general solutions are obtained for one-excitation state with arbitrary initial condition. We consider in detail two cases of large and small hopping at resonance. In the case of resonance and large hopping the state of the first qubit may also be transferred to the second qubit in the same cavity without excitation of the field. Such transfer of states minimizes decoherence due to photon loss.

Keywords: Cavity QED, qubit, coupled cavities, transfer of states.

1. Introduction

Cavity quantum electrodynamics (QED) is one of the promising approaches for quantum computation. It can be used for quantum information processing [1] or for quantum gate operations [2, 3]. The study of cavity QED may also have applications in circuit QED [4]. For applications of cavity QED we must first understand the physics of different systems of coupled qubits and cavities. In this paper we study the dynamics of a closed system of four qubits in two coupled cavities. Each cavity contains two qubits and two cavities are coupled by an optical fibre. We shall find the general solution for one-excitation state with arbitrary initial conditions and then investigate the important cases of resonance with large and small hopping of photons between cavities. As in [5] we shall find the state of the system in terms of the delocalized field and qubit modes and then express the state in the system of local bases.

The paper is organized as follows. In section 2 we find the one-excitation state for a closed system of two qubits in a cavity. In section 3 we find the one-excitation state of the system of four qubits in two coupled cavities for arbitrary initial condition. In section 4 we consider the cases of resonance with small and large hopping. Conclusions are presented in section 5.

2. System of two qubits in a cavity

To understand the physics of a complicated system we first study a closed system of two qubits in a cavity. The Hamiltonian of this system is \( \hbar = 1 \)

\[
H = E_e (|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|) + \gamma + g_1 (|e_1\rangle\langle g_1| + \gamma^* |g_1\rangle\langle e_1|) + g_2 (|e_2\rangle\langle g_2| + \gamma^* |g_2\rangle\langle e_2|),
\]

where \( |e_i\rangle, |g_i\rangle \) represent the excited and ground states of qubit \( i \) in the cavity respectively, \( E_e \) is the energy of the excited state of qubit, \( \gamma \) is the annihilation operator for photon in the cavity, and \( \omega \) is...
the angular frequency of the photon, \( g_i \) is coupling strength of qubit \( i \) with the cavity. The energy of a qubit in the ground state is taken to be zero. Without losing the generality we can take \( g_i \geq 0 \). The one-excitation eigenstate of the system may be written as

\[
|\phi_i\rangle = \sum_{j=1}^{3} \alpha_{ij} |\psi_j\rangle, \tag{2}
\]

where \( |\psi_1\rangle = |e_1\rangle \) is the state in which qubit 1 is excited, the field and qubit 2 is not excited, \( |\psi_2\rangle = |e_2\rangle \) is the state in which qubit 2 is excited, the field and qubit 1 is not excited, \( |\psi_3\rangle = |\gamma\rangle \) is the state with one photon and qubits are not excited. The energy of the eigenstates are

\[
\epsilon_1 = \omega + \frac{1}{2} \left[ \Delta - \sqrt{\Delta^2 + 4(g_1^2 + g_2^2)} \right], \quad \epsilon_2 = E_e, \quad \epsilon_3 = \omega + \frac{1}{2} \left[ \Delta + \sqrt{\Delta^2 + 4(g_1^2 + g_2^2)} \right], \tag{3}
\]

where \( \Delta = E_e - \omega \) is detuning. The matrix \( \alpha \) equals

\[
\alpha = \begin{bmatrix}
\frac{\sqrt{2g_1}}{D(D+\Delta)} & \frac{\sqrt{2g_2}}{D(D+\Delta)} & -\frac{\sqrt{2D}}{2D} \\
\frac{g_2}{\sqrt{g_1^2 + g_2^2}} & -\frac{g_1}{\sqrt{g_1^2 + g_2^2}} & 0 \\
\frac{\sqrt{2g_1}}{D(D-\Delta)} & \frac{\sqrt{2g_2}}{D(D-\Delta)} & \frac{\sqrt{2D}}{2D}
\end{bmatrix}, \tag{4}
\]

where \( D = \sqrt{\Delta^2 + 4(g_1^2 + g_2^2)} \).

We note that with adiabatic passage we can transfer the excited state from qubit 1 to qubit 2. Suppose that initially we have \( g_1 = 0 \), \( g_2 \neq 0 \) and the system is in the state \( |\phi_2\rangle = \alpha_{21} |\psi_1\rangle = |e_1\rangle \). By slowly increasing \( g_1 \) and slowly decreasing \( g_2 \) (so that \( \sqrt{g_1^2 + g_2^2} \neq 0 \) ) the system remains in the state \( |\phi_2\rangle \) and finally \( |\phi_2\rangle = \alpha_{22} |\psi_2\rangle = -|e_2\rangle \).

Now we use the above eigenstates to find the evolution of the state of the system \( |\Psi(t)\rangle \) from an arbitrary one-excitation initial state \( |\Psi(0)\rangle = \sum_j A_j |\phi_j\rangle \). We have

\[
|\Psi(t)\rangle = \exp(-iHt)|\Psi(0)\rangle = \sum_j A_j \exp(-i\epsilon_j t)|\phi_j\rangle = \sum_j A_j \exp(-i\epsilon_j t)\alpha_{jk} |\psi_k\rangle = \sum_k B_k(t)|\psi_k\rangle, \tag{5}
\]

where \( B_k(t) = \sum_j A_j \exp(-i\epsilon_j t)\alpha_{jk} \). Putting \( t = 0 \) we have \( B_k(0) = \sum_j A_j \alpha_{jk} \), \( A_j = \sum_k B_k(0)\alpha_{jk} \), then

\[
B_k(t) = \sum_{jl} \alpha_{jk} \alpha_{lj} \exp(-i\epsilon_j t)B_l(0). \tag{6}
\]

3. System of four qubits in two coupled cavities

3.1. Transformation of Hamiltonian

The Hamiltonian of a closed system of four qubits in two coupled cavities is

\[
H = \sum_{i=1}^{2} \sum_{j=1}^{2} \left[ E_e |e_j\rangle \langle e_j| + g_i (|\gamma_j\rangle \langle e_j| + |e_j\rangle \langle \gamma_j|) + \frac{1}{2} \alpha \gamma_j \gamma_j^* + A(\gamma_j \gamma_2^* + \gamma_2 \gamma_j^*) \right], \tag{7}
\]
where \( |e_{ij}\rangle, |g_{ij}\rangle \) represent the excited and ground states of the qubit of type \( i \) in cavity \( j \) (or qubit \( ij \)) respectively, \( E_e \) is the energy of the excited state of a qubit, \( \gamma_j \) is the annihilation operator for photon in the cavity \( j \), \( \omega \) is the angular frequency of the photon, \( g_i \) is the coupling strength of the qubit of type \( i \) in a cavity and \( A \) is cavity-cavity hopping strength. The energy of a qubit in the ground state is taken to be zero. Without losing the generality we can take \( g_i \geq 0 \) and \( A \geq 0 \). In the following the state of the system with one excited qubit \( ij \) and without photon we also denote as \( |e_{ij}\rangle \).

Using the transformation \[ \sum_{k=1}^{N} F_{ik} |\gamma_k\rangle, \]
\[ f^+_j = \sum_{k=1}^{N} F_{jk} |\gamma_k\rangle, \]
\[ f_j = \sum_{k=1}^{N} F_{jk} |\gamma_k\rangle, \]
\[ E_{ij} = \sum_{k=1}^{N} F_{ik} |e_{ik}\rangle, \]
where the matrix \( F \) equals
\[ F = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \]
we can rewrite the Hamiltonian in the form
\[ H = \sum_{k=1}^{N} \sum_{i=1}^{N} \left( E_e |E_{ik}\rangle \langle E_{ik}| + g_i(f_k |E_{ik}\rangle \langle 0| + f_k^+ |0\rangle \langle E_{ik}|) + \omega f_k^+ f_k \right). \]

Here \( |\gamma_j\rangle \) is the state of the system with one photon in the cavity \( j \), \( |f_j\rangle \) is the state of the system with one photon in the delocalized mode \( j \) and \( f_j \) is the annihilation operator for this photon, \( |E_{ij}\rangle \) is the excitation state of the qubit of type \( i \) in the delocalized mode \( j \), \( |0\rangle \) is the state of the system without photon and without excitation of qubits, \( \omega_1 = \omega - A \) and \( \omega_2 = \omega + A \). By this transformation we obtain two effectively independent systems \( (k = 1, 2) \), in each of which we have two effective qubits \( (i = 1, 2) \) interacting with a delocalized field mode.

### 3.2. Eigenstates of the system

The eigenstates of the system \( |\phi_{ij}\rangle \) may be expressed in the bases of delocalized modes \( |\psi_{ij}\rangle = |E_{ij}\rangle \), \( |\psi_{2j}\rangle = |E_{2j}\rangle \), \( |\psi_{3j}\rangle = |f_j\rangle \) as
\[ |\phi_{ij}\rangle = \sum_k \alpha_{ik,j} |\psi_{ij}\rangle, \]
where \( \alpha_{ik,j} \) is the elements of matrix \( \alpha_j \) which has the form
\[ \alpha_j = \begin{bmatrix} \sqrt{2}g_1 \sqrt{D_j(D_j + \Delta_j)} & \sqrt{2}g_2 \sqrt{D_j(D_j + \Delta_j)} & -\sqrt{D_j + \Delta_j} \\
\frac{g_1}{\sqrt{g_1^2 + g_2^2}} \sqrt{D_j(D_j + \Delta_j)} & \frac{g_1}{\sqrt{g_1^2 + g_2^2}} \sqrt{D_j(D_j + \Delta_j)} & 0 \\
\sqrt{2}g_1 \sqrt{D_j(D_j - \Delta_j)} & \sqrt{2}g_2 \sqrt{D_j(D_j - \Delta_j)} & \sqrt{D_j - \Delta_j} \end{bmatrix}. \]
where \( \Delta_j = E_e - \omega_j \), \( D_j = \sqrt{\Delta_j^2 + 4(g_1^2 + g_2^2)} \).

The eigenvalues of Hamiltonian are

\[
\varepsilon_{1j} = \omega_j + \frac{1}{2}(\Delta_j - D_j), \quad \varepsilon_{2j} = \omega_j, \quad \varepsilon_{3j} = \omega_j + \frac{1}{2}(\Delta_j + D_j).
\] (14)

### 3.3 Evolution of the state of the system

In practice we often work with low excited states. We shall study the one-excitation or zero-excitation states. The initial state of the system may be written as

\[
|\Psi(0)\rangle = \sum_i A_i |\phi_i\rangle + A_0 |0\rangle,
\] (15)

where \( A_i, A_0 \) are constants. The state of the system at \( t \) is

\[
|\Psi(t)\rangle = \exp(-iHt)|\Psi(0)\rangle = \sum_i A_i \exp(-i\varepsilon_i t)|\phi_i\rangle + A_0 |0\rangle = \sum_{ijk} A_i \exp(-i\varepsilon_i t)\alpha_{ikj}|\psi_{ij}\rangle + A_0 |0\rangle.
\] (16)

The term \( A_0 |0\rangle \) does not change and we shall not regard it. We have

\[
|\Psi(t)\rangle = \sum_{ij} B_{ij}(t)|\psi_{ij}\rangle,
\] (17)

where \( B_{ij}(t) = \sum_i A_i \exp(-i\varepsilon_i t)\alpha_{ikj} \). At \( t = 0 \) we have \( B_{ij}(0) = \sum_i A_i \alpha_{ikj} \), \( A_i = \sum_i B_{ij}(0)\alpha_{ikj} \), then

\[
B_{ij}(t) = \sum_i \alpha_{ikj} \alpha_{ikj} \exp(-i\varepsilon_i t)B_{ij}(0).
\] (18)

The form of states in equations (17, 18) is not convenient because in practice the initial state is given in local bases. Now we express \(|\Psi(t)\rangle \) in local bases. With the notation

\[
|\phi_{ij}\rangle = |e_{ij}\rangle, \quad |\phi_{2j}\rangle = |e_{2j}\rangle, \quad |\phi_{3j}\rangle = |\gamma_j\rangle,
\] (19)

we have

\[
|\psi_{ij}\rangle = \sum_k F_{jk} |\phi_{ik}\rangle, \quad |\Psi(t)\rangle = \sum_{kl} C_{kl}(t)|\phi_{kl}\rangle,
\] (20)

where \( C_{kl}(t) = \sum_j B_{ij}(t)F_{ji} = \sum_{mij} \alpha_{ikj} \alpha_{imj} \exp(-i\varepsilon_i t)B_{mj}(0)F_{ji} \). At \( t = 0 \) we have

\[
C_{kl}(0) = \sum_j B_{ij}(0)F_{ji}, \quad B_{ij}(0) = \sum_n C_{mn}(0)F_{jn}, \quad \text{then}
\]

\[
C_{kl}(t) = \sum_{mij} \alpha_{ikj} \alpha_{imj} \exp(-i\varepsilon_i t)F_{ji}F_{jn}C_{mn}(0).
\] (22)

In equations (21, 22) we have obtained the evolution of the one-excitation state of the system from arbitrary initial condition, expressed in local bases.
4. The case of resonance and small or large detuning

4.1. The case of resonance and small detuning

The case of resonance is important and will be considered in this section. First we consider the case of resonance \( \Delta = \Delta_0 \) and small hopping \( A \ll g_1, g_2 \). With condition \( \Delta = \Delta_0, A \ll g_1, g_2 \) we have

\[
\alpha_1 = \alpha_2 = \begin{bmatrix}
\frac{\sqrt{2}g_1}{D} & \frac{\sqrt{2}g_2}{D} & -\frac{1}{\sqrt{2}} \\
-\frac{g_2}{\sqrt{g_1^2 + g_2^2}} & -\frac{g_1}{\sqrt{g_1^2 + g_2^2}} & 0 \\
\frac{\sqrt{2}g_1}{D} & \frac{\sqrt{2}g_2}{D} & \frac{1}{\sqrt{2}}
\end{bmatrix},
\]

where notation \( D = \sqrt{A^2 + 4(g_1^2 + g_2^2)} \) is used in all section 4. We shall regard two cases of initial condition.

4.1.1. Only qubit of type 1 in cavity 1 is initially excited

Suppose that only qubit of type 1 in cavity 1 is initially excited, \( C_{11}(0) = 1 \), then in approximation we obtain

\[
C_{11}(t) = \frac{g_1^3}{g_1^2 + g_2^2} \exp(-iE_{11}t)
\left( \cos \frac{Dt}{2} \cos \frac{At}{2} \frac{g_1^2 + g_2^2}{g_1^2} \right),
\]

\[
C_{21}(t) = \frac{g_1g_2}{g_1^2 + g_2^2} \exp(-iE_{11}t)
\left( \cos \frac{Dt}{2} \cos \frac{At}{2} - 1 \right),
\]

\[
C_{31}(t) = -\frac{ig_1}{\sqrt{g_1^2 + g_2^2}} \exp(-iE_{11}t)\sin \frac{Dt}{2} \cos \frac{At}{2},
\]

\[
C_{12}(t) = -\frac{ig_1^2}{g_1^2 + g_2^2} \exp(-iE_{11}t)\cos \frac{Dt}{2} \sin \frac{At}{2},
\]

\[
C_{22}(t) = -\frac{ig_1g_2}{g_1^2 + g_2^2} \exp(-iE_{11}t)\cos \frac{Dt}{2} \sin \frac{At}{2},
\]

\[
C_{32}(t) = -\frac{g_1}{\sqrt{g_1^2 + g_2^2}} \exp(-iE_{11}t)\sin \frac{Dt}{2} \sin \frac{At}{2}.
\]

We have oscillation first in cavity 1. In this oscillation the excitation transfers from qubit of type 1 to qubit of type 2 and field (in cavity 1) and then back. The excitation is slowly transferred from cavity 1 to cavity 2 (but not entirely) and then back again.

If we find minimal integers \( m, n \) satisfying \( D/A = (2m+1)/(2n+1) \), then we can find the period of the evolution of the system \( T = 2\pi(2n+1)/A = 2\pi(2m+1)/D \). In this case at the moments \( t = kT \), where \( k \) is an integer, the excitation returns entirely to qubit of type 1 in cavity 1 and at the moments \( (k + (1/2))T \) the probability of excitation of the field in the cavity 2 is maximal and equals \( g_1^2/(g_1^2 + g_2^2) \).

If we find minimal integers \( m, n \) satisfying \( D/A = 2m/(2n+1) \), then at the moment \( \tau = 2\pi(2n+1)/A = 4\pi m/D \) the state of qubit 11 is entirely transferred to qubit 21 and at the moments \( t = \tau + k2\tau \), where \( k \) is integer, this excited state returns entirely to qubit 21.
If we find minimal integers $m$, $n$ satisfying $D/A = (2m+1)/(2n)$, then at the moment $\tau = 4\pi m/A = 2\pi(2m+1)/D$ the state of qubit 11 is entirely transferred to qubit 21 and at the moments $t = \tau + k2\tau$, where $k$ is integer, this excited state returns entirely to qubit 21.

### 4.1.2. Only field in cavity 1 is initially excited

Suppose that initially cavity 1 has one photon and all qubits are on the ground state. In this case $C_{31}(0) = 1$. In approximation we obtain

$$C_{11}(t) = -\frac{ig_1}{\sqrt{g_1^2 + g_2^2}} \exp(-iE_1t) \sin \left( \frac{Dt}{2} \right) \cos \left( \frac{At}{2} \right), \quad (29)$$

$$C_{21}(t) = -\frac{ig_2}{\sqrt{g_1^2 + g_2^2}} \exp(-iE_2t) \sin \left( \frac{Dt}{2} \right) \cos \left( \frac{At}{2} \right), \quad (30)$$

$$C_{31}(t) = \exp(-iE_1t) \cos \left( \frac{Dt}{2} \right) \cos \left( \frac{At}{2} \right), \quad (31)$$

$$C_{12}(t) = -\frac{g_1}{\sqrt{g_1^2 + g_2^2}} \exp(-iE_1t) \sin \left( \frac{Dt}{2} \right) \sin \left( \frac{At}{2} \right), \quad (32)$$

$$C_{22}(t) = -\frac{g_2}{\sqrt{g_1^2 + g_2^2}} \exp(-iE_2t) \sin \left( \frac{Dt}{2} \right) \sin \left( \frac{At}{2} \right), \quad (33)$$

$$C_{32}(t) = -i\exp(-iE_1t) \cos \left( \frac{Dt}{2} \right) \sin \left( \frac{At}{2} \right). \quad (34)$$

If we find minimal integers $m$, $n$ satisfying $D/A = (2m+1)/(2n+1)$, then we can find the period of the evolution of the system $T = 2\pi(2n+1)/A = 2\pi(2m+1)/D$. In this case at the moments $t = kT$, where $k$ is an integer, the excitation returns entirely to the field in cavity 1 and at the moments $(k + (1/2))T$ the state of the field in cavity 1 is entirely encoded in qubits in cavity 2.

If we find minimal integers $m$, $n$ satisfying $D/A = 2m/(2n+1)$, then at the moment $\tau = \pi(2n+1)/A = 2\pi m/D$ the state of field in cavity 1 is entirely transferred to the field in cavity 2 and at the moments $t = \tau + k2\tau$, where $k$ is integer, this excited state returns entirely to the field in cavity 2.

### 4.2. The case of resonance and large detuning

Now we study the case of resonance and large detuning $\Delta = 0, A >> g_1, g_2$. Suppose that initially only qubit of type 1 in cavity 1 is excited, $C_{31}(0) = 1$. In approximation we obtain

$$C_{11}(t) = \frac{g_1^2}{g_1^2 + g_2^2} \exp(-iE_1t) \left( \cos \left( \frac{(D-A)t}{2} \right) + \frac{g_2^2}{g_1^2} \right), \quad (35)$$

$$C_{21}(t) = \frac{g_1g_2}{g_1^2 + g_2^2} \exp(-iE_2t) \left( \cos \left( \frac{(D-A)t}{2} \right) - 1 \right), \quad (36)$$

$$C_{31}(t) = 0, \quad (37)$$

$$C_{12}(t) = -\frac{ig_1^2}{g_1^2 + g_2^2} \exp(-iE_1t) \sin \left( \frac{(D-A)t}{2} \right), \quad (38)$$

$$C_{22}(t) = \frac{ig_1g_2}{g_1^2 + g_2^2} \exp(-iE_2t) \sin \left( \frac{(D-A)t}{2} \right), \quad (39)$$

$$C_{32}(t) = 0. \quad (40)$$
We see that in this case the field is not excited. The evolution of the state of the system has period $T = 4\pi/(D - A)$.

We can find the probability that qubit $ij$ is in excited state $P_{ij} = |C_{ij}|^2$. For simplicity we take $g_1 = g_2$. The functions $P_{ij}(t)$ are presented in the figure below. From the figure we see that with time, excitation is transferred from qubit 11 to other qubits and back without the population of the field. At $t = T/4, 3T/4, ...$ the probability of excitation of each qubit equals $1/4$. At $t = T/2, 3T/2, ...$ the state of qubit 11 is entirely transferred to qubit 21.

We note that although the transfer of states in this case is slow, but this transfer has the advantage that the absence of field population minimize the decoherence due to photon loss.

5. Conclusions
We have studied the dynamics of a closed system of four qubits in two coupled cavities. The general expression for one-excitation state of the system in arbitrary initial condition is derived. We have investigated the case of resonance and small or large hopping. In the case of resonance and small hopping the transfer of state is accompanied by the population of the field. In the case of resonance and large hopping the qubit state is transferred without the population of the field and such a state transfer minimize the decoherence due to photon loss. If we may turn off the parameters $g_1, g_2$ and $A$ in an appropriate moments we can perform a desired quantum gate. In application to circuit QED such operations may easily be performed [6].

References
[1] Cirac J I, Ekert A K, Huelga S F and Macchiavello C 1999 Phys. Rev. A 59 4249
[2] Zubairy M S, Kim M and Scully M O 2003 Phys. Rev. A 68 033820
[3] Wallraff A, Schuster D I, Blais A, Frunzio L, Huang R S, Majer J, Kumar S, Girvin S M and Schoelkopf R J 2004 Nature (London) 431 162
[4] Schuster D I, Houck A A, Schreier J A, Wallraff A, Gambetta J M, Blais A, Frunzio L, Majer J, Johnson B, Devoret M H et al. 2007 Nature (London) 445 515
[5] Ogden C D, Irish E K and Kim M S 2008 arXiv:0804.2882v1 [quant-ph]
[6] Hu Y, Xiao Y F, Zhou Z W and Guo G C 2007 Phys. Rev. A 75 012314

Acknowledgement
This work is supported by the Vietnam National Basic Research Program. The author would like to express his sincere thanks to Prof. Nguyen Van Hieu for the support.