Linear bounds for levels of stable rationality

Fedor Bogomolov\textsuperscript{1,2}, Christian Böhning\textsuperscript{3,4}, Hans-Christian Graf von Bothmer\textsuperscript{4}\textsuperscript{‡}

\textsuperscript{1} Courant Institute of Mathematical Sciences, 251 Mercer St., New York, NY 10012, USA
\textsuperscript{2} Laboratory of Algebraic Geometry, GU-HSE, 7 Vavilova Str., 117312, Moscow, Russia
\textsuperscript{3} Fachbereich Mathematik der Universität Hamburg, Bundesstraße 55, 20146, Hamburg, Germany
\textsuperscript{4} Mathematisches Institut der Georg-August-Universität Göttingen, Bunsenstr. 3-5, 37073, Göttingen, Germany

Received 4 March 2011; accepted 29 July 2011

\textbf{Abstract:} Let $G$ be one of the groups $\text{SL}_n(\mathbb{C})$, $\text{Sp}_{2n}(\mathbb{C})$, $\text{SO}_m(\mathbb{C})$, $\text{O}_m(\mathbb{C})$, or $G_2$. For a generically free $G$-representation $V$, we say that $N$ is a level of stable rationality for $V/G$ if $V/G \times \mathbb{P}^N$ is rational. In this paper we improve known bounds for the levels of stable rationality for the quotients $V/G$. In particular, their growth as functions of the rank of the group is linear for $G$ being one of the classical groups.

\textbf{MSC:} 14E08, 14M20, 14L24

\textbf{Keywords:} Rationality • Stable rationality • Linear group quotients

\section{Introduction}

In the birational geometry of algebraic varieties, an important problem consists in determining the birational types of quotient spaces $V/G$, where $V$ is a generically free linear representation of the linear algebraic group $G$, both defined over $\mathbb{C}$, which will be our base field. We will suppose in the sequel that $G$ is connected. The quotient $V/G$ is said to be stably rational of level $N$ if $V/G \times \mathbb{P}^N$ is rational. Whether or not $V/G$ is stably rational (of some level) is a property of the group $G$ and not of the particular generically free representation $V$ by the no-name lemma [6].
It will be desirable to obtain good bounds on levels of stable rationality $N$ for $V/G$ as above, for a given $G$ and a class $\mathcal{C}$ of generically free $G$-representations $V$ as large as possible. By this we generally understand that one wants to determine an explicit function $N = N(r, V)$ of $r$ and $V$ (in the given class $\mathcal{C}$) such that $V/G \times \mathbb{P}^N$ is rational, and $N = N(r, V)$ is small, where $r$ is the rank of $G$. More precisely, if $G$ is a group running through one of the infinite series of simple groups of type $A_i, B_i, C_i, D_i$ (of some fixed isogeny type), one would like to determine a function $N = N(r)$ which gives a level of stable rationality for generically free $G$-quotients $V/G$, uniformly for all $V$ in some fixed large class $\mathcal{C}$, and such that the asymptotic behaviour of $N(r)$ is $N(r) = O(r)$ (Landau symbol). This is what is meant by “linear bounds” in the title. For exceptional groups of types $G_2, F_4, E_6, E_7$ and $E_8$ one would like to find a small constant $N$ that gives a level of stable rationality. Let us also mention that results in this direction have applications to the rationality question for algebraic varieties because many varieties $X$ can be fibred over generically free linear group quotients $V/G$, with rational general fibre, so that the total space is birational to the product of base and fibre.

We will obtain $N(r) = O(r)$ for the groups $\text{SL}_n(\mathbb{C}), \text{Sp}_{2n}(\mathbb{C}), \text{SO}_n(\mathbb{C})$, and also the nonconnected groups $\text{O}_n(\mathbb{C})$, for large classes of representations $\mathcal{C}$. Moreover, we will improve the bound for $G_2$ somewhat. Before describing the results in more detail, we mention that previously one only had $N(r) = O(r^2)$ for these classical groups. To be precise, what was known previously, at least to us, can be summarized in the following table (the class $\mathcal{C}$ this applies to is the class of all generically free $G$-representations):

| Group $G$ | Level of stable rationality $N$ |
|-----------|-------------------------------|
| $\text{SL}_n(\mathbb{C})$ | $n^2 - 1$ |
| $\text{SO}_{2n+1}(\mathbb{C})$ | $2n^2 + 3n + 1$ |
| $\text{Sp}_{2n}(\mathbb{C})$ | $2n^2 + n$ |
| $\text{SO}_{2n}(\mathbb{C})$ | $2n^2 + n$ |
| $G_2$ | 17 |

Stable rationality was also known for the orthogonal groups $\text{O}_{2n}(\mathbb{C})$ and $\text{O}_{2n+1}(\mathbb{C})$, with the same levels as for the special orthogonal groups, and for the simply connected exceptional groups $F_4, E_6, E_7$, cf. [3]. We point out that stable rationality remains open for the spin groups, and for $E_8$. Our methods here do not seem to yield substantial improvements for $F_4, E_6, E_7$. Let us comment briefly on how the results in the table are obtained: $\text{SL}_n(\mathbb{C})$ and $\text{Sp}_{2n}(\mathbb{C})$ are special groups (every étale locally trivial principal bundle for them is Zariski locally trivial), so for a generically free representation $V$ the quotient is stably rational of level their dimension. For $\text{SO}_n(\mathbb{C})$ one considers the action on a variety $X$ which is birational to a tower of equivariant vector bundles over an $\text{SO}_n(\mathbb{C})$-representation as base: $X$ consists of orthogonal $m$-frames $(v_1, \ldots, v_m)$ in $\mathbb{C}^n$ with $\langle v_i, v_j \rangle \neq 0$. Note that $\dim X = \dim \text{SO}_n(\mathbb{C}) + m$, which is the value given in the table. In fact, $V/\text{SO}_n(\mathbb{C}) \cong (V \times X)/\text{SO}_n(\mathbb{C})$ by the no-name lemma ($\cong$ indicates birational isomorphism), and in $X$ there is a $(\text{SO}_n(\mathbb{C}), H)$-section $\Pi$ where $H$ is an elementary abelian 2-group and $\Pi$ is a product of general lines (see e.g. the survey [7] for the notion of $(G, H)$-section; it is also recalled below in Definition 2.1). So $(V \times X)/\text{SO}_n(\mathbb{C}) \cong (V/H) \times \Pi$ is rational. For $G_2$ the argument is similar, but for $X$ one takes instead

$$X = \{(A, B, C) : A \perp B, A, B, AB \perp C \text{ and } A, B, C \text{ of nonzero norm}\}$$

as a subset of $\mathbb{O}^3$, where $\mathbb{O}$ are traceless octonions. The action of $G_2$ is free on $X$. See [3] for details.

Let us now describe our results in more detail. Let $G$ be one of the groups $\text{SL}_n(\mathbb{C}), \text{Sp}_{2n}(\mathbb{C}), \text{SO}_n(\mathbb{C}), \text{O}_n(\mathbb{C})$ or $G_2$. We will first narrow down the class of representations $\mathcal{C}$ of these groups which we will consider and make a statement about them. Namely, $\mathcal{C}$ contains precisely the $G$-representations $V$ of the form $V = W \oplus S^c$, where $W$ is an irreducible representation of $G$ whose ineffectivity kernel (a finite central subgroup) coincides with the stabilizer in general position. $S$ is a standard representation for each of the groups involved, namely $\mathbb{C}^n$ for $\text{SL}_n(\mathbb{C}), \mathbb{C}^{2n}$ for $\text{Sp}_{2n}(\mathbb{C}), \mathbb{C}^n$ for $\text{SO}_n(\mathbb{C})$ and $\text{O}_n(\mathbb{C})$, $C^2$ for $G_2$. Here $c \in \{0, 1\}$, and $c = 0$ if and only if $W$ is already $G$-generically free. Thus $V$ will always be $G$-generically free. The following table summarizes our main results, i.e., Theorems 2.4, 4.5, 5.5, 5.12, 6.2.