On continuous conformal deformation of the $SL(2)_4/U(1)$ coset

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Abstract

We describe a one-parameter family of $c = 1$ CFT’s as a continuous conformal deformation of the $SL(2)_4/U(1)$ coset.

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1 Introduction

The curious fact is that among all $c = 1$ CFT’s there exists the one-parameter family of theories living on an orbifold [1]. These conformal models are related to the Ashkin-Teller critical line [2] and correspondingly to the Abelian massless Thirring model. By now many of the aspects of this $c = 1$ family have been studied in detail [1],[3]. A new puzzle arose, when it was shown that there is another one-parameter family of $c = 1$ CFT’s associated with the affine-Virasoro construction of the affine algebra $SL(2)$ at level $k = 4$ [4]. According to our observation this family has to be described as a non-Abelian Thirring model [5]. It is not obvious that the non-Abelian Thirring model is equivalent to the Abelian Thirring model. Therefore, one can raise a question about equivalence of the two given one-parameter families. As yet this question has not been answered, though some evidence for this equivalence has been presented [6].

We believe that in order to clarify the relation between the two families, one has to compare their partition functions on the torus. The problem is that there still are some open questions about the Lagrangian formulation of the second continuous $c = 1$ family. The aim of the present paper is to describe the non-Abelian one-parameter family as a continuous conformal deformation of the CFT, whose partition function on the torus is known. We shall exhibit that the $SL(2)_{k}/U(1)$ coset, which is the $c = 1$ CFT at $k = 4$, admits a one-parameter conformal deformation. We shall argue that the given continuous family of $c = 1$ CFT’s share all the properties of the $c = 1$ one-parameter affine-Virasoro construction and is in agreement with the nonperturbative Lagrangian description in [5]. We hope that the model we construct will turn out to be useful for answering the main question of equivalence.

The paper is organized as follows. In section 2 we show that the spectrum of the $SL(2)_{k}/U(1)$ coset at $k = 4$ acquires an extra null-vector. In section 3 we make use of the given null-vector to perform the conformal one-parameter deformation on the $SL(2)_{4}/U(1)$ coset. We conclude in section 4.
2 The null-operator at $k = 4$

The key point of our discussion will be the fact that at $k = 4$ the $SL(2)_k/U(1)$ coset acquires a new null-operator which can be used to deform the given CFT. Therefore, we start with the Lagrangian description of the coset construction which will be a central element in our consideration.

It is well known that within the Lagrangian approach the $G/H$ coset can be presented as a combination of ordinary conformal Wess-Zumino-Novikov-Witten (WZNW) models and ghost-like action [7], [8]:

$$S_{G/H} = S_{WZNW}(g,k) + S_{WZNW}(h,-k-2c_V(H)) + S_{Gh}(b,c,\bar{b},\bar{c}),$$  \hspace{2cm} (2.1)

where $g$ takes values on the group $G$, $h$ takes values on the group $H$, $c_V(H)$ is defined according to

$$f^i_k f^k_i = -c_V(H)\delta^{ij}, \quad i, j, k = 1, 2, ..., \dim H.$$  \hspace{2cm} (2.2)

Whereas the last term in eq. (2.1) is the contribution from the ghost-like fields,

$$S_{Gh} = \text{Tr} \int d^2z (b\partial \bar{c} + \bar{b}\partial c).$$  \hspace{2cm} (2.3)

There are first class constraints in the system. Therefore, the physical states are defined as cohomology classes of the nilpotent BRST operator $Q$ [7], [8],

$$Q = \oint \frac{dz}{2\pi i} \left[ :c_i(\bar{J}^i + J^i_H): (z) - (1/2)f^i_k f^k_j : c_i c_j b^k : (z) \right], \quad Q^2 = 0,$$  \hspace{2cm} (2.4)

where we have used the following notations

$$J_H = \frac{-k}{2}g^{-1}\partial g|_H, \quad \bar{J} = \frac{(k+2c_V(H))}{2}h^{-1}\partial h.$$  \hspace{2cm} (2.5)

Here the current $J_H$ is a projection of the $G$-valued current $J$ on the subalgebra $H$ of $G$.

Let us turn to the case of the $SL(2)_k/U(1)$ coset. This is the simplest coset construction. In particular, the BRST operator $Q$ takes the form

$$Q = \oint \frac{dz}{2\pi i} : c(\bar{J}^3 + J^3) : (z),$$  \hspace{2cm} (2.6)

where we assume that the gauge subgroup $H = U(1)$ is associated with the subalgebra generated by the $t^3$ generator of the $SL(2)$ algebra

$$[t^a, t^b] = f^a_{\ c} t^c, \quad a, b, c = 1, 2, 3.$$  \hspace{2cm} (2.7)
thus, \( c_V(U(1)) = 0 \). Therefore, the action of the \( SL(2)_k/U(1) \) coset is given by
\[
S_{SL(2)/U(1)} = S_{WZNW}(g, k) + S_{WZNW}(h, -k) + \int d^2z (\bar{b}\partial c + \bar{c}\partial b).
\] (2.8)

The physical operators and states are constructed in terms of the three given CFT’s under the condition of annihilation by \( Q \). We are not interested in finding all the physical operators. We want to focus attention on the following one
\[
O^L = L_{ab} : J^a \bar{J}^\bar{a} \phi^{b\bar{a}} :,
\] (2.9)
where \( \phi^{a\bar{a}} \) is defined as follows
\[
\phi^{a\bar{a}} = \text{Tr} : g^{-1} t^a g t^{\bar{a}} :.
\] (2.10)
This is a highest weight vector of the affine algebra [9]. In addition, \( \phi^{a\bar{a}} \) is a primary operator of the Virasoro algebra associated with the WZNW model on \( SL(2)_k \). The corresponding conformal dimensions are given by [9]
\[
\Delta_\phi = \bar{\Delta}_\phi = \frac{2}{k + 2}.
\] (2.11)

In virtue of the properties of \( \phi^{a\bar{a}} \), the operator \( O^L \) is also a Virasoro highest weight vector, when the matrix \( L_{ab} \) is symmetrical, i.e.
\[
L_0 |O^L\rangle = \Delta_O |O^L\rangle, \quad L_{m>0} |O^L\rangle = 0,
\] (2.12)
with
\[
\Delta_O = \bar{\Delta}_O = 1 + \frac{2}{k + 2}.
\] (2.13)

Here \( L_n \) are the Virasoro generators of the WZNW model on \( SL(2)_k \). At the same time, the operator \( O^L \) is no longer a highest weight vector with respect to the affine algebra but its descendant.

In the case of \( SL(2) \) the normal form of the matrix \( L_{ab} \) is diagonal,
\[
L_{ab} = \lambda_a \eta_{ab},
\] (2.14)
where \( \lambda_1, \lambda_2, \lambda_3 \) are arbitrary numbers, \( \eta_{ab} = \text{diag}(1, 1, -1) \).
One can check that in general the operator $O^L$ does not commute with $Q$ and, therefore, does not belong to the physical subspace of the theory in eq. (2.8). Hence, some modifications of $O^L$ are required.

Let us consider the following modified operator

$$
\hat{O}^L = O^L + N : j^3 j^a \phi^{3a} : ,
$$

(2.15)

where the constant $N$ is to be defined. Clearly the operator $\hat{O}^L$ has the same conformal dimensions as $O^L$. We demand

$$
Q|\hat{O}^L\rangle = 0.
$$

(2.16)

This requirement leads us to two equations

$$
\lambda_1 = \lambda_2, \quad N = \lambda_3 - \frac{2}{k}(\lambda_1 + \lambda).
$$

(2.17)

Under the given conditions, the operator $\hat{O}^L$ belongs to the cohomology of the BRST operator $Q$. In general, $\hat{O}^L$ is built up of the two WZNW models presented in eq. (2.8). Dramatic simplifications occur, when $k = 4$. Indeed, in this case there is a solution with $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ and $N = 0$. Thus, at $k = 4$,

$$
\hat{O}^L = O^\lambda,
$$

(2.18)

where

$$
L_{ab} = \lambda \eta_{ab}.
$$

(2.19)

The operator $O^\lambda$ describes a one-parameter family of BRST invariant operators. It turns out that $O^\lambda$ is a null-vector. Indeed, it is not difficult to compute its norm with arbitrary $k$:

$$
||O^\lambda||^2 = (k - 4)^2 M,
$$

(2.20)

where $M$ is some number. Hence, the norm vanishes, when $k = 4$. Also one can show that

$$
\langle O^\lambda O^\lambda O^\lambda \rangle = 0,
$$

(2.21)

which agrees with the consistency equation obtained in [10]. Therefore, all correlation functions with $O^\lambda$ will vanish. In other words, at $k = 4$ the space of physical operators of the $SL(2)_k/U(1)$ coset acquires a one-parameter family of null-vectors which are not ruled out by the BRST symmetry and are not equal to BRST exact operators.
3 The one-parameter deformation

Let us consider the following theory

\[ S(\lambda) = S_{SL(2)_4/U(1)} - \int d^2z O^\lambda(z, \bar{z}), \tag{3.22} \]

where \( O^\lambda(z, \bar{z}) \) is the null-operator constructed in the previous section. The \( \lambda \) parameter in eq. (3.22) has dimension \(-2\Delta_\phi\), because the operator \( O^\lambda \) has dimensions \((1 + \Delta_\phi, 1 + \Delta_\phi)\). Since \( \Delta_\phi > 0 \), \( O^\lambda \) is an irrelevant operator. If we think of \( \lambda \) as being small, then the action in eq. (3.22) can be understood as the \( SL(2)_4/U(1) \) coset perturbed by the irrelevant operator. In general, such a perturbation will run us in the infrared problem. However, in the case under consideration \( O^\lambda \) is a null-vector. Therefore, there should be no trace of this operator in any local physical observable. Correspondingly the conformal symmetry of the perturbed theory remains to be manifest. Thus, \( \lambda \) appears to play a role of a modular variable from the point of view of the target space geometry described by the given CFT. This analogy can be made more clear in terms of interacting WZNW models \[\Box\].

Let us consider the system of two interacting identical \( SL(2)_4 \) WZNW models. The action of the system is given by

\[ S(g_1, g_2, S) = S_{WZNW}(g_1, 4) + S_{WZNW}(g_2, 4) - \frac{16}{\pi} \int d^2z \text{Tr}^2(g_1^{-1} \partial g_1 S \bar{\partial} g_2 g_2^{-1}). \tag{3.23} \]

Here \( g_1, g_2 \) take values in \( SL(2) \). The statement is that \[\Box\]

\[ Z(4, 4, S) = Z_{WZNW}(4) \tilde{Z}(\lambda), \tag{3.24} \]

where

\[ Z(4, 4, S) = \int \mathcal{D}g_1 \mathcal{D}g_2 \exp[-S(g_1, g_2, S)], \]

\[ Z_{WZNW}(4) = \int \mathcal{D}g \exp[-S_{WZNW}(g, 4)], \]

\[ \tilde{Z}(\lambda) = \int \mathcal{D}g \exp[-(S_{WZNW}(g, 4) - \int d^2z O^\lambda)]. \tag{3.25} \]

The relation between the couplings \( S \) and \( \lambda \) is as follows

\[ S = \sigma \cdot I, \]

\[ \lambda = -\frac{16\sigma^2}{\pi} + \ldots, \tag{3.26} \]
where $I$ is the identity from the direct product of two Lie algebras, $\mathcal{G} \times \mathcal{G}$, $\sigma$ is a small parameter, whereas dots stand for higher order corrections in $\sigma$. The last formula tells us that we have to restrict $\lambda$ to negative values. Furthermore, at $\sigma = 1/8$, the system of two interacting level 4 WZNW models acquires the gauge symmetry and, correspondingly, undergoes phase transition. There is one more phase transition at the Dashen-Frishman point [12] (see also [11]). Therefore, $\sigma$ is not exactly a continuous parameter, but has two particular values at which the system of two interacting WZNW models changes its properties drastically.

Now it becomes transparent how $\lambda$ can be related to the module of the target space geometry. Indeed, the interaction term in eq. (3.23) is to parametrize the metric on the intersection of the two group manifolds. Since we have shown that the coupling $S$ may change continuously without changing the underlying CFT, this parameter $\sigma$ is by definition called a module. This fact must imply that the current-current interaction in eq. (3.23) becomes a truly marginal operator at $k = 4$.

The Virasoro central charge of the CFT described by eq. (3.23) is equal to 4. Unfortunately, it is not clear how the $c = 1$ CFT is embedded into the given $c = 4$ CFT. Based on formula (3.24) we may argue that the stress-energy tensor of the $SL(2)_4$ WZNW model perturbed by the null-vector $O^\lambda$ has to have the following form

$$T(\lambda) = T_{c=1}(\lambda) + K,$$

(3.27)

where $K$ is the $c = 1$ CFT which is gauged away by gauging the subgroup $H = U(1)$. The hope is that there exist variables in which the stress-tensor $T_{c=1}(\lambda)$ can be presented as the one-parameter affine-Virasoro construction [4]. The latter also has two points which might be identified with the two phase transitions we have just mentioned above. We leave this issue for further investigation.

The fact that the system of two interacting level $k = 4$ WZNW models is conformal with the continuous parameter $\sigma$ indicates that the interaction term in eq. (3.23) is a truly marginal operator. For all other $k$’s this is just a marginal operator which breaks the conformal symmetry. Truly marginal operators are responsible for the existence of moduli in the target space geometry. This observation supports our interpretation of $\sigma$ (or $\lambda$) as a module.
4 Conclusion

We have found a continuous family of \( c = 1 \) CFT’s, which either can be associated with the truly marginal perturbation of the system of two \( SL(2)_4 \) WZNW models or with the null-deformation of the \( SL(2)_4 \) WZNW model. Both descriptions are equivalent to each other. If the given one-parameter family is linked with the \( c = 1 \) orbifold construction, then one may expect a duality symmetry to present in the theory we have constructed. Indeed, via bosonization procedure one can relate the parameter \( \sigma \) in eqs. (3.26) with the radius \( R \) of a scalar field compactified on a circle:

\[
R = \sqrt{\frac{(1 - 8\sigma)}{2}}. \tag{4.28}
\]

It is well known that this compactification possesses the duality symmetry under \( R \to 1/(2R) \).

Our consideration of the continuous deformation of the CFT suggests some general arguments about perturbation theory in duality invariant systems. Suppose \( R \) is a parameter which goes to \( 1/(2R) \) under the duality symmetry. (In the non-Abelian case one has some matrices instead of one parameter.) Then there is a self-dual point \( R_0 = 1/\sqrt{2} \).

Near by this point there is a small parameter

\[
r = \frac{R - R_0}{R_0}, \tag{4.29}
\]

which measures the deviation from \( R_0 \). Under the duality symmetry

\[
r \to r' = -r + \mathcal{O}(r^2). \tag{4.30}
\]

If \( r \) is very tiny, one can drop the higher order corrections in \( r \) in the equation for \( r' \). Correspondingly, the original duality symmetry amounts to the change of sign of the parameter \( r \). Therefore, the leading order in \( r \) has to be an even function of \( r \). In other words, expansion in \( r \) has to have an effective perturbation parameter which is an even function of \( r \).

We have exhibited how quantum perturbation theory can be related to expansion around the self-dual point, which in the case under consideration coincides with the
$SL(2)_4/U(1)$ coset. Indeed, eqs. (3.26) clearly display the effect we anticipate for duality invariant systems. The hope is that in the so-called $S$-dual theories, expansion around the self-dual point can be connected with string quantum expansion in topology of the world sheet. This would be certainly the case, when the tree approximation in string theory coincides with the $S$-self-dual point. Then one could check whether the string toroidal correction gives rise to the even function of $r$.

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