On the Links Between Thermobaricity, Available Potential Energy, Neutral Directions, Buoyancy Forces, Potential Vorticity, and Lateral Stirring in the Ocean

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ABSTRACT: The various ingredients of Lorenz theory of available potential energy (APE) are shown to hold the key for understanding how to rigorously generalise all the central concepts of ocean circulation theory, such as density, neutral directions, potential vorticity, Bernoulli function, and lateral stirring surfaces, to a compressible and salty ocean. A key new idea is that it is the existence of thermobaric forces acting along isopycnal surfaces that makes stirring in seawater fundamentally different from that in a simple fluid and is the ultimate cause for the non-existence of neutral surfaces. This result is important, for it implies that lateral stirring surfaces in the oceans are best defined as the material surfaces decoupling the effects of thermobaric and buoyancy forces. Various lines of evidence are given that establish that these surfaces are mathematically and physically well defined and correspond to the Lorenz reference density (LRD) surfaces entering APE theory, which the various approximate neutral surfaces developed so far can be regarded as the approximations of. Importantly, rewriting the momentum balance equations in their thermodynamic form using Crocco-Vazsonyi theorem and removing the dynamically inert part of the Bernoulli function proves decisive for obtaining most results. The new results have important implications for ocean circulation theory, the determination of the absolute velocity field in terms of material invariants, and ocean mixing parameterisations.

SIGNIFICANCE STATEMENT: The formulation of theories of the large-scale ocean circulation relies on a fundamental set of ‘building blocks’, such as density, potential vorticity, and Bernoulli function, without which progress would be hard if not impossible. However, because seawater is complex, these building blocks had until now only been defined for an ‘ideal fluid’ ocean, meaning that most existing theories have lacked realism. By explaining how to generalise these building blocks to fully realistic compressible seawater, the present work paves the way for the development of significantly more accurate ocean models. Surprisingly, the theory of available potential energy proved crucial to achieve the breakthrough, suggesting that its usefulness and importance for understanding the ocean circulation are much greater than generally assumed.

1. Introduction

The isentropic surfaces of a simple fluid (such as dry air or pure water), which in practice are commonly defined as the surfaces of constant potential temperature \( \theta \), have long been known to be useful for understanding many aspects of turbulent fluid motions. For instance, the deformation of isentropic surfaces can serve to characterise the work against buoyancy forces, the associated changes in available potential energy (APE), and more generally vertical stirring: the exchanges of fluid parcels along isentropic surfaces, which leave the stratification and its (available) potential energy unaffected (hence energetically neutral), can serve to characterise lateral stirring. Depending on the application, isentropic surfaces can therefore be equivalently viewed as lateral stirring surfaces, neutral surfaces, or vertical stirring surfaces.

These are not their only advantageous properties. Other important ones are: the vertical gradient of \( \theta \) fully determines static stability; Ertel potential vorticity \( EPV = \omega_a \cdot \nabla \theta / \rho \) is materially conserved for inviscid and adiabatic motions, where \( \omega_a \) is the absolute vorticity and \( \rho \) in-situ density; the isentropic gradient of the static energy \( h + \Phi \) (also known as Montgomery acceleration potential or geostrophic streamfunction (Montgomery 1937, Wexler and Montgomery 1941)) is the only force (apart from Coriolis and friction) acting on the isentropic surfaces of hydrostatic motions, where \( h \) is the specific enthalpy and \( \Phi = gz \) the geopotential.

It is therefore only natural that oceanographers have long wondered which material variable \( \gamma(S, \theta) \) (where \( S \) is salinity) should be regarded as the best seawater analogue of the potential temperature of a simple fluid. Sverdrup et al. (1942) (pp. 414–416, section entitled ‘Significance of \( \sigma_r \) surfaces’) pioneered the theoretical analysis of this question by asking under what conditions would the adiabatic and isohaline interchange of two parcels with different thermodynamic properties \((S_1, \theta_1, p_1)\) and \((S_2, \theta_2, p_2)\) leave the distribution of mass and potential energy unaffected? For this to occur, it is easy to see that the density of the incoming parcels should match that of the outgoing parcels, mathematically

\[
\rho(S_1, \theta_1, p_2) = \rho(S_2, \theta_2, p_2), \quad \rho(S_2, \theta_2, p_1) = \rho(S_1, \theta_1, p_1).
\]
Sverdrup et al. (1942) realised that (1) had no solution for compressible salty water except in trivial cases. For lack of a better idea, they proposed that \( \sigma_\tau = \rho(S, T, 0) - 1000 \), which had formed the basis for early isopycnal analyses (Montgomery 1938; Iselin 1939), be regarded as the best practical alternative. Note here that the pressure entering the equation of state for density is assumed to be the gauge pressure, i.e. the pressure minus the mean surface atmospheric pressure.

Eq. (1) can be turned into a more tractable problem if the two parcel densities are only required to equate at the mid-pressure \( \bar{p} = (p_1 + p_2)/2 \) instead, thus leading to

\[
\rho(S_1, \theta_1, \bar{p}) = \rho(S_2, \theta_2, \bar{p}),
\]

which seemingly describes the parcel interchange as taking place on a locally-referenced potential density surface \( \rho^f = \rho(S, \theta, \bar{p}) = \text{constant} \). Jackett and McDougall (1997) used (2) as the basis for their definition of a neutral path, which plays a central role in their construction of approximately neutral surfaces.

Unfortunately, \( \rho^f \) does not truly define a material variable as \( \bar{p} \) is ambiguous, so that (2) is not really satisfactory. In practice, this difficulty has been traditionally resolved by using a constant reference pressure \( p_r \) in place of \( \bar{p} \). This leads to the widely used concept of potential density \( \sigma_\tau = \rho(S, \theta, p_r) - 1000 \), but the approximation deteriorates as \( |\bar{p} - p_r| \) increases. This has motivated constructions of globally-defined isopycnal surfaces valid for all pressures, the most widely used being: 1) [Reid and Lynn (1971)] patched potential density (PPD), which uses a discrete set of vertically stacked potential densities \( \sigma_k \) referenced to a discrete set of reference pressures \( p_k, k = 1, \ldots, N \) spanning the full range of pressures, ‘patched’ at the transition pressures \( (p_k + p_{k+1})/2 \) for instance; 2) [Jackett and McDougall (1997)] empirical neutral density variable \( \gamma^* \), whose iso-surfaces are made up of all the points connectable by a neutral path, which is meant to represent a continuous analogue of PPD.

Nowadays, empirical approximately neutral surfaces are more generally envisioned as surfaces everywhere as perpendicular as feasible (in some sense) to the neutral vector \( \mathbf{n} = \alpha \nabla \theta - \beta \nabla S \), and the mathematical impossibility of achieving exact neutrality generally attached to the non-vanishing of the neutral helicity \( H = \mathbf{n} \cdot (\nabla \times \mathbf{n}) \neq 0 \) (McDougall and Jackett 1988; Stanley 2019b), where \( \alpha \) and \( \beta \) are the thermal expansion and haline contraction coefficients respectively. So far, however, there has been no theoretical guidance for how to fix the degree of non-neutrality of an approximately neutral surface. This has resulted in the development of incompatible approaches; thus, [Eden and Willebrand (1999)] advocate the use of a purely material density variable \( \gamma(S, \theta) \), [de Szoeke and Springett (2000); Stanley (2019b)] advocate the use of orthobaric density \( \gamma(\rho, p) \), a function of density and pressure only, while Prof. McDougall and his group advocate the use of a hybrid density variable \( \gamma(x, y, S, \theta, p) \) or \( \gamma(x, y, S, \theta) \) (Jackett and McDougall 1997; Klocker et al. 2009; Lang et al. 2020; Stanley et al. 2021).

In this paper, we focus on a key issue that has remained unaddressed so far, namely: should the various approximate neutral surfaces proposed so far be regarded as different approximations of mathematically and physically well defined lateral stirring surfaces or as something else? To address this issue, we set out to explore the theoretical consequences and implications of the first proposition in order to see whether this can yield more satisfactory foundations for isopycnal analysis and the study of lateral and vertical stirring. Physically, this is facilitated by the fact that there seems to be only one type of mathematically well defined lateral stirring surfaces that also appear as a natural generalisation of the isentropic surfaces of a simple fluid: those that possess only the properties of the latter that carry over to compressible seawater exactly, thus excluding those properties that only do so approximately. There are only two such properties that we are aware of, namely materiality — satisfied by all purely material functions \( \gamma(S, \theta) \) — and flatness at rest, the property of the iso-\( \gamma \) surfaces to align with constant geopotential surfaces in a state of rest. Physically, such properties correspond to those of the Lorenz reference density (LRD) surfaces entering [Lorenz (1955)] theory of available potential energy (APE), which in the oceans can be computed efficiently using [Saenz et al. (2015)] algorithm. As it happens, the LRD surfaces were discovered by [Tailleux (2016a)] to be strikingly similar to [Jackett and McDougall (1997)] empirical neutral surfaces, thus lending credence to our hypothesis that the latter can be interpreted as an approximation of the former.

This paper is organised as follows. In section [2] we review and revisit the arguments that led oceanographers to define lateral stirring surfaces in terms of approximately neutral surfaces. This leads us to develop new theoretical foundations for the discipline, and to argue that lateral stirring surfaces should be defined in terms of the Lorenz reference density surfaces entering APE theory, which is tested empirically. In section [3] we explain how to connect the heuristic considerations and two-parcel energetics arguments of the preceding section with exact properties of the compressible Navier-Stokes equations, which leads to a rigorous definition of neutral directions. Section [4] discusses the implications of our new theory for extending the various concepts entering ocean circulation theory and the ideal fluid thermocline equations to compressible seawater, which leads to rigorous new definitions for potential vorticity and the Bernoulli function. Section [5] summarises our results and discusses future work.
2. Significance of APE theory for defining mathematically explicit lateral stirring surfaces

a. Two-parcel energetics characterisation of stirring

An alternative energetics approach to that of Sverdrup et al. (1942) is to consider the change in potential energy brought about by an arbitrary adiabatic and isohaline permutation of two fluid parcels. Among other things, this is useful for providing an energetics characterisation of lateral and vertical stirring, as well as for introducing thermobaric forces in a heuristic way. Using specific enthalpy $h(S, \theta, p)$ as a proxy for potential energy, the energy cost of the two parcels exchange is

$$\Delta E = h(S_1, \theta_1, p_2) - h(S_1, \theta_1, p_1) + h(S_2, \theta_2, p_1) - h(S_2, \theta_2, p_2)$$

$$\approx -\Delta \nu^{LR} \Delta p \approx -\frac{1}{\rho} [\overline{\alpha} \Delta \theta - \overline{\beta} \Delta S] \Delta p,$$
eq (3)

where $\nu^{LR} = \nu(S, \theta, \overline{p})$ denotes the so-called ‘locally-referenced specific volume’, $\overline{p} = (p_1 + p_2)/2$, and $\overline{\alpha}$ and $\overline{\beta}$ are the thermal expansion and haline contraction coefficients defined in terms of the mean values $\overline{S} = (S_1 + S_2)/2$, $\overline{\theta} = (\theta_1 + \theta_2)/2$, and $\overline{p}$. For a simple fluid ($\Delta S = 0$), it is easily verified that if one excludes the degenerate case $\Delta p = 0$, the interchange of two parcels along a frozen isentropic surface $\theta = \theta_1 = \theta_2$ leaving the stratification unaffected ($\equiv$ lateral stirring) can be characterised by having a zero energy cost $\Delta E = 0$, while all other kinds of exchanges involving deformations of the isotropic surfaces and work against buoyancy forces ($\equiv$ vertical stirring) are characterised by $\Delta E \neq 0$ and $\Delta \theta \neq 0$. In the latter case, the involvement of buoyancy forces can be revealed by rewriting (3) in the form

$$\Delta E = \Delta E^{vertical} \approx g \overline{\alpha} \frac{\Delta \theta}{\Delta S} \Delta \zeta^2 \approx 2 \zeta^2, $$
eq (4)

where $\Delta \zeta$ is a vertical displacement such that $\Delta p \approx -\overline{p} g \Delta \zeta$ and $g \overline{\alpha} \Delta \theta/\Delta \zeta \approx 2$. Eq. (4) predicts buoyancy forces to scale as $\Delta \zeta$, as expected.

In the literature, the characterisation of the isentropic surfaces as zero energy cost or ‘neutral’ surfaces appears to have been sometimes misinterpreted as equivalent to regarding isentropic surfaces as optimal for lateral stirring. Such a misinterpretation has led some to argue that it would be energetically more optimal for lateral stirring to take place within the so-called wedge of instability, as doing so release APE according to Eady (1949) parcel model of baroclinic instability. McDougall and Church (1986) pointed out, however, that the release of APE in that case is due to the ‘flattening’ of isentropic surfaces, which pertains to vertical stirring, not lateral stirring.

b. Lateral stirring and thermobaric forces in seawater

As mentioned previously, $\nu^{LR}$ does not truly define a purely material variable in compressible seawater because of its dependence on $\overline{p}$. It follows that if lateral stirring surfaces exist that can be described in terms of a globally defined purely material variable $\gamma(S, \theta)$, their attached energy cost $\Delta E \equiv \Delta E^{lateral}$ must in general differ from zero (McDougall 1995). Physically, we expect the energy cost of lateral stirring ($\Delta \gamma = 0$) and vertical stirring ($\Delta \gamma \neq 0$) to be linked to the isopycnal and diapycnal components of $\theta$ and $S$,

$$\Delta \theta = \Delta_i \theta + \Delta_d \theta,$$
$$\Delta S = \Delta_i S + \Delta_d S,$$
eq (5)

such that

$$\overline{\gamma}_S \Delta_i S + \overline{\gamma}_\theta \Delta_i \theta = 0,$$
$$\Delta \gamma = \overline{\gamma}_S \Delta_i S + \overline{\gamma}_\theta \Delta_d \theta,$$
eq (6)

with $\overline{\gamma}_S$ and $\overline{\gamma}_\theta$ the derivatives of $\gamma$ defined in terms of $\overline{S}$ and $\overline{\theta}$ as before. Now, inserting (5) into (3) accounting for (6) yields the decomposition

$$\Delta E = \Delta E^{lateral} + \Delta E^{vertical} $$
eq (7)

with

$$\Delta E^{lateral} = \frac{1}{\overline{\gamma}_S} \frac{\partial (\nu^{LR} \gamma)}{\partial (S, \theta)} \Delta_i \theta \Delta p,$$
$$\Delta E^{vertical} = \left( R_\gamma \overline{\theta} - \frac{\overline{\gamma}_S}{\overline{\gamma}_\theta} \right) \frac{\Delta \gamma \Delta p}{1 - R_\gamma}, $$
eq (8)

where $R_\gamma = -\overline{\gamma}_\theta \Delta \theta/\overline{\gamma}_S \Delta S$ is a $\gamma$-based density ratio allowing one to write $\Delta_i S$ and $\Delta_d \theta$ in terms of $\Delta \gamma$ as follows

$$\overline{\gamma}_S \Delta_i S = \frac{\Delta \gamma}{1 - R_\gamma},$$
$$\overline{\gamma}_\theta \Delta_d \theta = \frac{-R_\gamma \Delta \gamma}{1 - R_\gamma}. $$
eq (9)

As expected, Eq. (8) states that $\Delta E^{lateral}$ is controlled by the degree of non-neutrality of $\gamma$ (via the Jacobian term) and the isopycnal temperature gradient $\Delta_i \theta$. Expressing $\Delta E^{lateral}$ in terms of a lateral displacement $\Delta \ell$

$$\Delta E^{lateral} = \frac{1}{\overline{\gamma}_S} \frac{\partial (\nu^{LR} \gamma)}{\partial (S, \theta)} \frac{\Delta_i \theta \Delta p}{\overline{\gamma}_\theta} \Delta \ell^2, $$
eq (10)

suggests that thermobaric forces should scale as

$$F_{thermobaric} \propto \frac{1}{\overline{\gamma}_S} \frac{\partial (\nu^{LR} \gamma)}{\partial (S, \theta)} \frac{\Delta i \theta \Delta p}{\overline{\gamma}_\theta} \overline{|\nabla_i \theta||\nabla_i p|} \Delta \ell, $$
eq (11)

with $\nabla_i \theta$ and $\nabla_i p$ the isopycnal gradients of $\theta$ and $p$. The above considerations are important for they establish that:

1. thermobaric forces are fundamentally isopycnal in character. As a result, they can only be unambiguously defined if the isopycnal surfaces $\gamma = constant$ on
which they live are themselves unambiguous, which implies that an optimal choice for \( \gamma \) must exist;

2. the zero energy cost \( \Delta E = 0 \) that characterises neutral motions should really be interpreted as \( \Delta E^{\text{vertical}} = -\Delta E^{\text{lateral}} \). Neutral surfaces are therefore not the surfaces along which fluid parcels can be exchanged without experiencing restoring buoyancy forces, as envisioned by [McDougall] [1987], but the surfaces along which fluid parcels experience compensating thermobaric and buoyancy forces. The distinction matters because the differences in temporal and spatial scales on which these two forces operate (Muller and Willebrand 1986; Straub 1999) make it difficult to express \( \alpha \) and \( \beta \) in terms of \( \alpha_r \) and \( \beta_r \), it is easily verified that [15] may be written in the form

\[
\Delta E^{\text{lateral}} \approx \frac{T_b}{\rho} \Delta \theta \Delta p (p_r - \overline{p}),
\]

e.g., [Tailleux 2016b], where \( T_b \) is the thermobaric parameter

\[
T_b = \frac{\partial \alpha}{\partial p} - \frac{\alpha}{\beta} \frac{\partial \beta}{\partial p} = \beta \left( \frac{\partial}{\partial p} \right) \beta,
\]

e.g. [McDougall 1987], with \( T_{br} = T_b(\overline{S}, \overline{\theta}, p_r) \), where we also used the fact that along a LRD surface, the density-compensated \( \theta/S \) variations satisfy

\[
\alpha(\overline{S}, \overline{\theta}, p_r) \Delta \theta \approx \beta(\overline{S}, \overline{\theta}, p_r) \Delta S,
\]

since \( p_r \) is constant on such a surface. Eq. [17] thus predicts that the thermobaric forces attached to the LRD surfaces are controlled by: a) the thermobaric parameter \( T_b \); b) the isopycnal gradient of potential temperature \( \Delta \theta \); c) the distance from the equilibrium state of rest \( p_r - \overline{p} \), which is plausible and in agreement with our a priori expectations about such forces. The physically meaningful character of the thermobaric forces implied by [17] provides strong support for the idea that the LRD surfaces represent the most appropriate definition of isopycnal surfaces in seawater.

c. Thermobaric forces attached to LRD surfaces

As argued in the introduction, the Lorenz reference density (LRD) surfaces appear to be the only plausible candidate for defining mathematically and physically well-defined isopycnal/lateral stirring surfaces in the oceans. Here, we aim to verify that the thermobaric forces attached to the LRD surfaces are physically meaningful. As explained in e.g., Saenz et al. [2015], the Lorenz reference density (LRD) may be defined as

\[
\rho^{LZ} (S, \theta) = \rho(S, \theta, p_r)
\]

where \( p_r = p_0(z_r) \), with \( z_r \) the reference depth of a fluid parcel defined as a root of the level of neutral buoyancy (LNB) equation

\[
\rho(S, \theta, p_0(z_r)) = \rho_0(z_r),
\]

[Tailleux 2013], with \( \rho_0(z) \) and \( \rho_0(z) = -g^{-1} dp_0/dz \) the reference pressure and density profiles defining Lorenz reference state of minimum potential energy. Importantly, [14] defines \( z_r \) as the intersection point of two one-dimensional curves and therefore as a local quantity, not a global quantity, contrary to what is often believed. Also, [15] shows that the LRD represents a generalised form of potential density referenced to the spatially variable reference pressure \( p_r (S, \theta) \).

The LNB equation [14] plays a key role in the local theory of APE, for it encodes all the information about \( z_r \). For instance, differentiating it yields

\[
\nabla z_r = \left( \frac{dp_0}{dz} (z_r) + \frac{\rho_0(z_r)g}{c_s^2} \right)^{-1} (\rho S_r \nabla S + \rho \omega_r \nabla \theta),
\]

where \( c_s = \rho_p^{-\frac{1}{2}}(S, \theta, p) \) is the sound speed, while the suffix ‘\( r \)’ denotes quantities estimated at the reference pressure \( p_r \), which in turns implies

\[
\nabla \rho^{LZ} = -\rho_r \frac{dp_0}{dz} (z_r) \left( \frac{dp_0}{dz} (z_r) + \frac{\rho_r g}{c_s^2} \right)^{-1} \mathbf{n}_r,
\]

where \( \mathbf{n}_r = \alpha_r \nabla \theta - \beta_r \nabla S \) defines a reference neutral vector, while \( \rho_r = \rho_0(z_r) \). Mathematically, [15] and [16] establish that the gradients of \( z_r, p_0(z_r) \) and \( \rho^{LZ} \) are all proportional to \( \mathbf{n}_r \), and therefore that the iso-surfaces of LRD, \( z_r \), and \( p_r = p_0(z_r) \) all coincide.

If we now set \( \gamma = \rho^{LZ} (S, \theta) \), accounting for [16] and using a Taylor series expansion around \( p_r \) to express \( \alpha \) and \( \beta \) in terms of \( \alpha_r \) and \( \beta_r \), it is easily verified that [15] may be written in the form

\[
\nabla \rho^{LZ} = -\rho_r \frac{dp_0}{dz} (z_r) \left( \frac{dp_0}{dz} (z_r) + \frac{\rho_r g}{c_s^2} \right)^{-1} \mathbf{n}_r,
\]

\[
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time. This result is important for it establishes that the commonly used approach of constructing approximately neutral surfaces that minimise neutrality, e.g., Jackett and McDougall (1997), Klocker et al. (2009), Stanley (2019b), Stanley et al. (2021), should accurately approximate the ‘true’ lateral stirring surfaces where thermobaric forces are demonstrably small. This view is consistent with that of Jackett and McDougall (1997), who argued that the ambiguity attached to the determination of their empirical neutral surfaces should be small in regions with small enough neutral helicity, which they estimated to be the case in about 95% of the oceans. Tailleux (2016a) empirically found Jackett and McDougall (1997) empirical neutral surfaces to accurately mimic the LRD surfaces almost everywhere outside the Southern Ocean, consistent with [8] predicting thermobaric forces to be large primarily in regions where |p − p_r| is large.

Here, we re-examine the issue by means of a new methodology allowing one to compare the LRD surfaces and Jackett and McDougall (1997) γ^n surfaces directly in (S, 𝜃) space, which improves on Tailleux (2016a)’s comparison in physical space. To describe the LRD surfaces, we use the ‘analytic’ thermodynamic neutral density

\[ γ_{analytic}^n(S, 𝜃) = ρ_LZ(S, 𝜃) - F_n(p_r), \]  

recently developed by Tailleux (2021), where \( F_n(p_r) \) is a polynomical function of \( p_r \) empirically fitted to minimise the discrepancy with Jackett and McDougall (1997) γ^n. Although γ_a = γ^n(S, T, p, x, y) is not a function of S and 𝜃, the results of McDougall and Jackett (2005) and Lang et al. (2020) estimating its non-materiality to be negligible, suggest it might be close to one. McDougall and Jackett (2005) constructed a purely material approximant \( γ_a(S, 𝜃) \) of \( γ^n \) to test their conclusions, but the result turned out to be highly unsatisfactory, the reason(s) why remaining unclear.

Mathematically, the poor performance of \( γ^n \) might be due to \( γ^n \) being less material than generally assumed or to \( γ^n \) being in fact a multi-valued function of S and 𝜃. Here, we only pursue the hypothesis that the rational function used to construct \( γ_a \) is inadequate for some reason. To that end, we used the specialised Matlab routine scatteredInterpolant to construct a new material interpolant \( γ_{material}^n(S, 𝜃) \) of \( γ^n \), based on Gouretski and Koltermann (2004) climatology. We find that unlike \( γ^n \), \( γ_{material}^n \) has no trouble approximating both \( γ^n \) and \( \nabla γ^n \) very accurately, thus confirming our hypothesis.

Fig. 1 illustrates the superior performances of \( γ_{material}^n \) in terms of the standard deviation and median of the absolute differences between \( γ^n \) and the density variables of \( n \) classes. Bottom panel (b), same as panel (a) but for median.

3. Extension to the full Navier-Stokes equations

So far, our proposition that LRD surfaces represent the most appropriate definition of lateral stirring surfaces in the oceans, as well as our discussion of thermobaric forces, have been primarily rooted in plausible but heuristic arguments. In this section, we seek a theoretical understanding of how these concepts arise in the context of the full compressible Navier-Stokes equations (given in Appendix A).

a. APE-theory and optimal form of momentum balance

To understand the nature of the constraints posed by thermodynamics and energetics on the dynamics and the forces exerted on fluid parcels, we take as our starting point the
momentum balance equations written in thermodynamic form
\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{\omega}_a \times \mathbf{v} + \nabla B_h = \mathbf{P}_h + \mathbf{F},
\]  
(21)
optained from (A1) by making use of the total differential for specific enthalpy \( dh = T d\eta + \mu dS + \rho^{-1} dp \) and of the identity \( (\mathbf{v} \cdot \nabla) \mathbf{v} = (\nabla \times \mathbf{v}) \times \mathbf{v} + \nabla (v^2/2) \), where \( \mathbf{\omega}_a = \nabla \times \mathbf{v} + 2\Omega \) is the absolute vorticity, \( B_h \) and \( \mathbf{P}_h \) being given by
\[
B_h = \frac{v^2}{2} + h + \Phi, \quad \mathbf{P}_h = T \nabla \eta + \mu \nabla S,
\]  
(22)
where the quantity \( h + \Phi \) is called the static energy in the atmospheric literature. The above transformation plays an important role in oceanography and atmospheric sciences, for it forms the basis for recasting the equations of motion in isentropic coordinates, e.g., [Starr(1945), de Szoeke(2000)] and is sometimes referred to as Crocco-Vazsonyi equation or theorem, after [Crocco(1937) and Vazsonyi(1945)].

For a simple fluid, \( \nabla \mathbf{S} = 0 \) so that the force \( \mathbf{P}_h = T \nabla \eta \) defined in (21) and (22) is naturally perpendicular to the isentropic surfaces, thus connecting LRD surfaces with a related force in the momentum equations. This connection is seemingly lost in seawater, however, as \( \mathbf{P}_h = T \nabla \eta + \mu \nabla S \) is not obviously related to any known form of isopycnal surfaces. Here, we suggest that this is because the information about lateral stirring surfaces contained in \( B_h \) and \( \mathbf{P}_h \) is ‘hidden’ and thus needs to be revealed. Indeed, we find that decisive progress is achieved by decomposing \( B_h = B_a + B_r \) as the sum of its dynamically active and inert parts respectively, in which case the new vector \( \mathbf{P}_a = \mathbf{P}_h - \nabla B_r \) is found to have the desired property of being approximately perpendicular to both the LRD and conventional neutral surfaces.

As per Lorenz APE theory, we define the dynamically inert part of \( B_h \) as its value in Lorenz reference state, viz.,
\[
B_r = h(\eta, S, p_0(z_r)) + g z_r,
\]  
(23)
whose gradient is
\[
\nabla B_r = T_r \nabla \eta + \mu_r \nabla S + g \left( 1 - \frac{\rho_0(z_r)}{\rho(\eta, S, p_0(z_r))} \right) \nabla z_r
\]  
\]  
(24)
the simplification being due to \( z_r \). Subtracting \( \nabla B_r \) from both sides of Eq. (21) then yields
\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{\omega}_a \times \mathbf{v} + \nabla B_a = \mathbf{P}_a + \mathbf{F}
\]  
(25)
where \( B_a \) and \( \mathbf{P}_a \) may be written in the form
\[
B_a = \frac{v^2}{2} + h + \Phi - B_r = \frac{v^2}{2} + \Pi + \frac{p - p_0(z)}{\rho},
\]  
(26)
\[ P_a = \frac{\partial \Pi}{\partial \eta} \nabla \eta + \frac{\partial \Pi}{\partial S} \nabla S = (T - T_r) \nabla \eta + (\mu - \mu_r) \nabla S, \quad (27) \]

where \( \Pi = h(\eta, S, p) - h(\eta, S, p_0(z_r)) + g(z - z_r) + (p_0(z) - p)/\rho \) is the potential energy density of a compressible two-component stratified fluid, e.g., Tailleux (2018), assuming for simplicity \( \rho_0(z) \) and \( p_0(z) \) to be time-independent. For the details of how to link \( P_a \) to \( \Pi \), see Appendix B. Note that the use of time-independent reference profiles allows one to regard \( z_r = z_r(\eta, S) \) as determined solely from the local values of \( \eta \) and \( S \). Note also here that the quantity

\[ M = h + \Phi - B_r = \Pi + \frac{p - p_0(z)}{\rho} \]

\[ = h(\eta, S, p) - h(\eta, S, p_0(z_r)) + g(z - z_r) \quad (28) \]

represents a generalisation of the well known Montgomery potential (Montgomery 1937), or acceleration potential (Wexler and Montgomery 1941), see Stanley (2019a) for a recent discussion.

b. Near perpendicularity of \( P_a \) to lateral stirring surfaces and implications for defining thermobaric forces

To establish that \( P_a \) is approximately parallel to \( \mathbf{n} \) and \( \mathbf{n} \) as claimed above, the simplest is to switch variables and to regard specific enthalpy \( h = h(S, \theta, p) \) as a function of \((S, \theta, p)\) so as to write its total differential in the form

\[ dh = \frac{\partial \hat{h}}{\partial \theta} d\theta + \frac{\partial \hat{h}}{\partial S} dS + \hat{v} dp. \quad (29) \]

The Maxwell relationships (i.e., the equality of the cross-derivatives, viz.,

\[ \frac{\partial^2 \hat{h}}{\partial \theta \partial p} = \frac{\partial \hat{v}}{\partial \theta} = \frac{\partial \hat{v}}{\partial S} = \frac{\partial \hat{a}}{\partial S} = -\frac{\hat{\beta}}{\rho}, \quad (30) \]

then allow one to rewrite \( P_a \) as

\[ P_a = \int_{p_r}^{p} \frac{\hat{a}}{\rho} (S, \theta, p') dp' \nabla S + \int_{p_r}^{p} \frac{\hat{\beta}}{\rho} (S, \theta, p') dp' \nabla S \]

\[ = \frac{(p - p_r)}{\rho} (\nabla \theta \beta - \beta \nabla S) = \frac{p - p_r}{\rho} \overline{\mathbf{n}}, \quad (31) \]

where \( \overline{\mathbf{n}} \) and \( \beta \) are

\[ \overline{\mathbf{n}} = \frac{\overline{\mathbf{n}}}{p - p_r} \int_{p_r}^{p} \frac{\hat{a}}{\rho} (S, \theta, p') dp', \quad (32) \]

\[ \beta = \frac{\overline{\mathbf{n}}}{p - p_r} \int_{p_r}^{p} \frac{\hat{\beta}}{\rho} (S, \theta, p') dp', \quad (33) \]

while \( \overline{\mathbf{n}} \) is a representative mean value of \( \rho \) over \([p_r, p]\). Using a simple trapezoidal scheme to approximate the integrals in (32) and (33), as well as the Boussinesq approximation, shows that at leading order

\[ \overline{\mathbf{n}} \approx \frac{1}{2} (\mathbf{n} + \mathbf{n}_r), \quad (34) \]

Eq. (34) is the sought-for result that establishes that \( P_a \) is in general intermediate between \( \mathbf{n} \) and \( \mathbf{n}_r \). If \( |p - p_r| \) is small, the directions defined by \( P_a \) and \( \mathbf{n} \) should all approximately coincide, but start to grow further apart as \( |p - p_r| \) increases.

As seen in the previous section, thermobaric forces are the forces acting within the LRD surfaces, so that their magnitude is determined by the isopycnal projection of \( P_a \). Mathematically,

\[ P_a^{(i)} = P_a - \frac{\mathbf{n}_r}{|\mathbf{n}_r|^2} \mathbf{n}_r \approx \frac{p - p_r}{2\rho} \mathbf{n}^{(i)} \]

\[ \approx \frac{(p - p_r)^2}{2\rho} T_{br} \nabla S \theta, \quad (35) \]

obtained by using \( P_a = (p - p_r) \overline{\mathbf{n}}/\rho \), as well as the fact that \( \mathbf{n}^{(i)} = (p - p_r) \overline{\mathbf{n}}/\rho \), by definition, combined with \( \alpha \nabla \theta = \beta \nabla S \) due to density-compensation, and using a Taylor series expansion of \( \alpha \) and \( \beta \) around \( p_r \) as before. Like the two-parcel energetics expression (17), (35) predicts thermobaric forces proportional to \((p - p_r), T_{br} \) and \( \nabla \theta \).

c. Energetics significance of \( P_a \)

The P-vector \( P_a \) has important energetics significance as it defines neutral directions along which stirring leaves the potential energy density \( \Pi \) approximately unaffected, as we show below. The associated form of neutrality is called P-neutrality to distinguish it from McDougall (1987) conventional N-neutrality.

As shown by Tailleux (2018), the potential energy density \( \Pi \) of a general fluid can be viewed as the sum of its available compressible energy \( \Pi_1 \) and APE density \( \Pi_2 \),

\[ \Pi_1 = h(\eta, S, p) - h(\eta, S, p_0(z)) + \frac{p_0(z) - p}{\rho} \]

\[ \approx \frac{(p - p_0(z))^2}{2\rho^2 c_s^2}, \quad (36) \]

\[ \Pi_2 = h(\eta, S, p_0(z)) - h(\eta, S, p_0(z_r)) + g(z - z_r) \]

\[ \approx \frac{N^2}{2} (z - z_r)^2, \quad (37) \]

where \( N^2 \) is the reference value of the squared buoyancy frequency, the suffix ‘b’ denoting values evaluated at the pressure \( p_0(z) \), i.e., \( p_b = \rho(\eta, S, p_0(z)) \).

In the oceans, \( \Pi_1 \) is generally several orders of magnitude smaller than \( \Pi_2 \) and can be formally neglected in the incompressible limit \( c_s \rightarrow +\infty \). If we do so, while also
approximating \( \rho \) by a constant reference density \( \rho_* \) in \((26,27)\), yields the following Boussinesq-like approximation

\[
B_d \approx \frac{v^2}{2} + \Pi_2 + \frac{\rho - \rho_0(z)}{\rho_*}, \tag{38}
\]

\[
P_2 \approx \frac{\partial \Pi_2}{\partial \eta} \nabla \eta + \frac{\partial \Pi_2}{\partial S} \nabla S = P_2. \tag{39}
\]

Appendix B shows that \( \Pi_2 \)’s canonical variables are \((\eta, S, z)\) and hence that

\[
\nabla \Pi_2 = (T_h - T_r)\nabla \eta + (\mu_b - \mu_r)\nabla S + g \left( 1 - \frac{\rho_0(z)}{\rho_b} \right) k, \tag{40}
\]

which provides an explicit expression for \( P_2 \).

Eq. (40) is important, for it shows that the thermohaline derivatives of \( \Pi_2 \) contain information about the directions approximately perpendicular to the lateral surfaces in the oceans, while the vertical derivative contains information about the vertical buoyancy forces. As regards the thermohaline part, we show in Appendix that the work against \( P_2 \) may be written in the form

\[
v \cdot P_2 = (T_h - T_r)\dot{\eta} + (\mu_b - \mu_r)\dot{S} - \frac{\partial \Pi_2}{\partial t}, \tag{41}
\]

which shows that the angle between the absolute velocity field and \( P_2 \) at any one point in time and space is controlled by the magnitude of irreversible mixing and the local temporal fluctuations of \( \Pi_2 \). This relation implies therefore than in the absence of irreversible mixing, the directions perpendicular to \( P_2 \) define the directions along which stirring leaves \( \Pi_2 \) unaffected. This relation is analogous to Eq. (21) of Nycander (2011), reproduced here in local form

\[
v \cdot P_{N11} = \frac{\partial h^+}{\partial \theta} \dot{\theta} + \frac{\partial h^+}{\partial S} \dot{S} - \frac{\partial h^+}{\partial t}, \tag{42}
\]

which shows that in the absence of mixing, the directions normal to Nycander P-vector are those along which stirring leaves dynamic enthalpy \( h^\pm \) unaffected (see Appendix C for its definition). Eqs. (40) and (41) establish the profound physical and energetics significance of \( \Pi_2 \) for understanding fundamental aspects of the ocean circulation.

d. Quantification of P-neutrality versus N-neutrality

In this section, we use the Gouretski and Koltermann (2004) WOCE climatology to understand what observations can tell us about the actual differences between \( P_d \), \( N \), \( \nabla \gamma^p \) and \( \nabla \gamma^n \) in the oceans. To exploit the capabilities of the TEOS-10 Matlab Gibbs Seawater Library (available at \texttt{www.teos-10.org}), the practical salinity and in-situ temperature fields were converted into reference composition salinity \( S_R \) and Conservative Temperature \( \Theta \) by means of the routines \texttt{gsw\_SR\_from\_SP} and \texttt{gsw\_CT\_from\_t} respectively.

All gradients were numerically estimated using standard second-order finite differences. In the particular case of the standard N-neutral vector, this was done by discretising

\[
N^\ast = \nabla_{lr} \rho(\Theta, S_R, p_0(z)), \tag{43}
\]

where \( \nabla_{lr} \) stands as the ‘locally-referenced’ gradient, that is, the gradient calculated by ignoring the pressure dependence. In the particular case of the \( \mathbf{N} \) neutral vector, this was done by discretising

\[
N^\ast \approx \frac{1}{z - z_r} \left[ \nabla_{lr} h(\Theta, S_R, p_0(z)) - \nabla_{lr} h(\Theta, S_R, p_0(z_r)) \right], \tag{44}
\]

the specific enthalpy \( h = h(S_R, \Theta, p) \) being estimated using \texttt{gsw\_enthalpy\_CT\_exact}. Note that \( N^\ast \) and \( N^\ast \) are defined to be parallel to \( P_2 \) and \( N \), the magnitude being irrelevant for the calculations detailed below.

One conventional metric to quantify the differences between the first two directions \( A \) and \( B \) is in terms of the notion effective diffusivity

\[
K_f(A, B) = K_i \sin^2(\langle A, B \rangle) = K_i \frac{|A \times B|^2}{|A|^2|B|^2} \tag{45}
\]

e.g., Hobet et al. (2019), with \( K_i = 1000 \text{m}^2\text{s}^{-1} \), where \( \langle A, B \rangle \) denotes the angle between the \( A \) and \( B \), while \( |A| \) denotes the standard Euclidean norm of \( A \). The value of \( K_i \) is conventionally chosen to categorise values of \( K_f \) above and below the threshold \( K_f = 10^{-3} \text{m}^2\text{s}^{-1} \) as large or small respectively.

Fig. 3 shows the statistics of \( K_f(N, \mathbf{N}) \) for the main oceanic basins (left panel) versus for the polar oceans (right panel), with the statistics for whole oceans in the background, which confirm our theoretical prediction that the differences between P-neutrality and N-neutrality should be the largest where fluid parcels are the furthest away from their equilibrium position, that is where \( \rho \) differs the most from \( p_r \). This result is further evidenced in Figs. 4 and 5 from alternate viewpoints. Interestingly, Fig. 4 demonstrates that \( \gamma^\text{analytic} \) tends to be in general both more P- and N-neutral than \( \gamma^\text{obs} \) outside the polar regions, which indicates that Jackett and McDougall (1997) algorithm fails to make \( \gamma^\text{obs} \) as neutral as it could be in such regions. Overall, Fig. 5 indicates that thermobaric forces are likely to be important only in the polar regions, but otherwise near negligible in the largest fraction of the oceans, consistent with Jackett and McDougall (1997)’s statistical analysis of neutral helicity, finding these to be very small in 95% of the oceans. Fig. 4(d) also shows that \( P_d \) is nearly perpendicular to the LRD surfaces outside the polar oceans, thus vindicating the idea that removing the dynamically inactive parts of \( B_h \) and \( P_h \) is the key to define a term in the
3. Probability distribution function (pdf) of the decimal logarithm of the effective diapycnal diffusivity-like metric measuring the angle between $P_{a2}$ and $N$ for the Pacific (PAC), Atlantic (ATL), and Indian (IO) oceans (left panel) versus for the polar oceans (right panel), the statistics for the whole ocean being also shown in the background in both panels. (SO = Southern Ocean, ARC = Arctic Ocean)

thermodynamic form of the momentum balance indicative of the lateral stirring directions in the oceans.

4. Implications for ocean circulation theory and the determination of the absolute velocity field

The generalisation of the Bernoulli function, density, neutral directions, and potential vorticity to compressible seawater has many important implications. In this section, we discuss some of these pertaining to ocean circulation theory and the determination of the absolute velocity field.

\[ f_k \times \rho v + \nabla \rho = -\rho g k, \]  
\[ \nabla \cdot v = 0, \]  
\[ v \cdot \nabla \rho = 0, \]  
where $B = p + \rho g z$ is the relevant form of Bernoulli function. Taking the inner product of (49) by $v$, accounting for (48), yields the Bernoulli theorem $\mathbf{v} \cdot \nabla B = 0$ for steady inviscid flow.

Another important conservation law is obtained by dividing (46) by $\rho$ and taking the curl

\[ k (\mathbf{v} \cdot \nabla f) - f \frac{\partial \mathbf{v}}{\partial z} = \nabla \rho \times \nabla p \frac{\rho}{\rho^2}, \]  
followed by taking the inner product of the result by $\nabla \rho$,

\[ \frac{\partial \rho}{\partial z} \mathbf{v} \cdot \nabla f - f \nabla p \cdot \frac{\partial \mathbf{v}}{\partial z} = \mathbf{v} \cdot \nabla \left( \frac{f}{\partial \rho / \partial z} \right) = 0, \]

which establishes the well-known material conservation of potential vorticity $Q = f \partial \rho / \partial z$.

Comparing (46) with (21) shows that $g z \nabla \rho$ is the ideal fluid counterpart of $\rho P_h = \rho (T \nabla \eta + \mu \nabla S)$, but while the former is perpendicular to isopycnal surfaces $\rho = \text{constant}$, this is not true of $P_h$. To improve the traceability of the ideal fluid equations to the full Navier-Stokes equations, it is therefore essential to introduce the dynamically inert part of $B$, that is $B_r = p_0(z_r) + \rho_0(z_r) g z_r$, assuming $\rho_0(z_r) = \rho$, and to remove $\nabla B_r$ from both sides of (49), yielding

\[ f k \times \rho v + \nabla (B - B_r) = g (z - z_r) \nabla \rho \]
Fig. 4. Latitude/depth section along 30°W in the Atlantic Ocean of the decimal logarithm of the effective diapycnal mixing like metric quantifying (a) the N-neutrality of $\gamma_n$; (b) the P-neutrality of $\gamma_n$; c) the N-neutrality of $\gamma_{analytic}$; d) the P-neutrality of $\gamma_{analytic}$.

Fig. 5. Vertical mean of the decimal logarithm of the effective diapycnal diffusivity like metric illustrating the geographical distribution of the differences between $\mathbf{N}$ and $\mathbf{P}_a$ (approximated by $\mathbf{P}_2$)

Note that the ideal fluid available Bernoulli function
\begin{equation}
B - B_r = p - p_0(z) + p_0(z) - p_0(z_r) + \rho g (z - z_r)
= p - p_0(z) + \int_{z_r}^{z} g (\rho - p_0(z')) \, dz'
\end{equation}

is now directly comparable to the full Navier-Stokes \cite{26}, as the integral term can be recognised as the positive definite Boussinesq form of APE density originally derived by
ideal fluid counterpart of \( P = (\rho - \rho_r) \frac{\mathbf{n}}{\rho} \approx -g(z-z_r)\mathbf{n} \), and now both are perpendicular or approximately perpendicular to the LRD surfaces.

b. Determination of the absolute velocity field in terms of material invariants

Needler (1985) showed that the material conservation of \( B \) and \( \rho \) can be exploited to construct an explicit expression of the absolute velocity field, which may be obtained by taking the cross-product of (49) by \( \nabla \rho \) and re-arranging,

\[
\rho \mathbf{v} = \frac{\nabla \rho \times \nabla B}{Q}.
\] (54)

The resulting velocity field has the remarkable property of naturally satisfying the continuity equation as a consequence of the material conservation of PV, as can be shown by taking the divergence of (54)

\[
\nabla \cdot (\rho \mathbf{v}) = -\frac{(\nabla \rho \times \nabla B) \cdot \nabla Q}{Q^2} = -\frac{\rho \mathbf{v} \cdot \nabla Q}{Q} = 0.
\] (55)

As a result, (54) represents an exact solution of the ideal fluid equations (46, 48) that also predicts the vertical velocity

\[
\rho \mathbf{w} = \frac{k \cdot (\nabla_h \rho \times \nabla_h B)}{Q},
\] (56)

not just the horizontal velocity, where \( \nabla_h \) denotes the horizontal gradient. Importantly, Needler’s formula (57) is insensitive to the particular choice of Bernoulli function — \( B \) or \( B_\rho \) — used to estimate it due to \( B_\rho = B_r(\rho) \) being a function of \( \rho \) only, so that the superiority of one over the other cannot be established in this way. The situation is markedly different for compressible seawater, however, as discussed later on.

Nevertheless, (54) is generally considered to be impractical because of \( \rho \)'s dependence on pressure. To circumvent this difficulty, the so-called Bernoulli method assumes that \( B \) is functionally related to \( \rho \) and \( Q \) via a function of the form \( B = G(\rho, Q) \), whose existence follows from the material conservation of all three quantities (without any guarantee that this function should necessarily be single-valued though). If so, (54) may be rewritten as

\[
\rho \mathbf{v} = \frac{\partial G}{\partial Q} \frac{\nabla \rho \times \nabla Q}{Q},
\] (57)

Needler (1985), which in principle can be evaluated solely from the knowledge of the density field provided that \( \partial G/\partial Q \) can meaningfully be determined. For related discussions of these ideas, the reader is referred to, e.g., Holliday and McIntyre (1981).}

\section*{c. Potential vorticity}

To extend (57) to compressible seawater, a clarification of how to define \( Q \) is needed. As is well known, potential vorticity variables can be defined for any scalar variable \( \lambda \) as \( Q_\lambda = \omega_\lambda \cdot \nabla \lambda / \rho \), so that the issue revolves around how to choose \( \lambda \). From Ertel’s (1942) theorem (see Schubert et al. (2004) for an English translation), \( Q_\lambda \) can be shown to satisfy the conservation law

\[
\frac{DQ_\lambda}{Dt} = \frac{1}{\rho^2} \nabla \lambda \cdot (\nabla \rho \times \nabla p) = \frac{1}{\rho^2} \nabla p \cdot (N \times \nabla \lambda),
\] (59)

e.g., see Eq. (4.95) of Vallis (2006). If one restricts \( \lambda = \lambda(S, \theta) \) to be purely material, as required to describe stirring surfaces, and focuses on adiabatic and inviscid motions, (58) reduces to

\[
\frac{DQ_\lambda}{Dt} = \frac{1}{\rho^2} \nabla \lambda \cdot (\nabla \rho \times \nabla p) = \frac{1}{\rho^2} \nabla p \cdot (N \times \nabla \lambda),
\] (59)

where \( N = g(a \nabla \theta - \beta \nabla S) \) as before. Eq. (59) shows that for \( Q_\lambda \) to be maximally material, one needs to minimise the baroclinic production term, which is controlled by the term \( N \times \nabla \lambda \) and therefore by \( \lambda \)'s departure from N-neutrality.

This result led McDougall (1983) to propose that PV in the oceans should be defined in terms of neutral surfaces, but how to do so rigorously is far from obvious since these are not attached to any mathematically well defined \( \lambda \). As a result, the most common approach in practice has been to define potential vorticity for use in (57) in terms of some form of potential density, but this becomes of course unsatisfactory as one moves away from the reference pressure. These issues can be largely circumvented, however, by defining PV in terms of Lorenz reference density or one of its proxy such as \( \gamma^T \).

\[
Q_\gamma = \frac{\omega_\gamma \cdot \nabla \gamma^T}{\rho}.
\] (60)

Indeed, \( \gamma^T \) has been shown previously to be in general significantly more neutral than \( \gamma^n \) outside the Southern Ocean. Since \( \gamma^T \) is also one of the variables that can be used to describe the LRD surfaces, it is the density variable assumed as the basis for our definition of PV in the rest of the paper.

\section*{d. Inversion of the nonlinear balance equations}

To understand how to extend Needler (1985)'s formula to compressible seawater, we take as our starting point the steady inviscid momentum balance equations in their thermodynamic form, viz.,

\[
\omega_\alpha \times \mathbf{v} + \nabla B_\ell = \mathbf{P}_\ell, \quad \ell = a, h
\] (61)
(from Eqs. (21) and (25)). As seen previously, the structure of (61) is more easily connected to that of the ideal fluid equations if the ‘available’ or ‘dynamically active’ forms of $\mathcal{B}$ and $\mathcal{P}$ are used, so that we will want to examine whether this superiority manifests in other ways.

To proceed, we assume that the Rossby number is small and hence that relative vorticity and kinetic energy only affect $\omega_a$ and $\mathcal{B}_\ell$ at second order. Eq. (61) is then an under-determined linear system for $v$ whose inversion requires the following solvability condition to be satisfied

$$\omega_a \cdot \mathcal{P}_t = \omega_a \cdot \nabla \mathcal{B}_\ell, \quad \ell = a, h \quad (62)$$

(obtained by taking the inner product of (61) with $\omega_a$), which mathematically means that $\mathcal{P}_t - \nabla \mathcal{B}_\ell$ is orthogonal to the null-space spanned by $\omega_a$.

The counterpart of Needler’s formula (54) is easily shown to be

$$v \equiv v_i^f = \frac{\mathcal{P}_t \times \nabla \mathcal{B}_\ell}{\rho Q_{bt}} \quad (63)$$

while the counterpart of $Q = f \rho_z$ is

$$Q_{bt} = \frac{\omega_a \cdot \nabla \mathcal{B}_\ell}{\rho} = \frac{\omega_a \cdot \mathcal{P}_t}{\rho} \quad (64)$$

the equality following from (62). Note that Gassmann (2014) derived an expression similar to (63) for a dry atmosphere, $v_i^f$ being referred to as the inactive wind.

Like Needler (1985), velocity field (54), $v_i^f$ also naturally satisfies the continuity equation, as can be shown by taking the divergence of $\rho v_i^f$, which yields

$$\nabla \cdot (\rho v_i^f) = - \frac{(\mathcal{P} \times \nabla \dot{\mathcal{B}})^2 \cdot \nabla Q + (\nabla \times \mathcal{P}) \cdot \nabla \mathcal{B}}{Q}$$

$$= - \frac{\rho v_i^f \cdot \nabla Q}{Q} + \frac{(\nabla \times \mathcal{P}) \cdot \nabla \mathcal{B}}{Q}$$

$$= - \frac{\omega_a \cdot \nabla \left( \frac{\nabla \mathcal{B}}{\mathcal{B}} \right)}{\omega_a \cdot \nabla \mathcal{B}} = 0 \quad (65)$$

the last equality following from Ertel’s theorem pertaining to the Bernoulli-based PV,

$$\frac{D(\omega_a \cdot \nabla \mathcal{B})}{Dt} = v_i^f \cdot \nabla \omega_a = \frac{D \mathcal{B}}{Dt} + \frac{\nabla \mathcal{B} \cdot (\nabla \times \mathcal{P})}{\rho} \quad (66)$$

where $\omega_a = \omega_a / \rho$, which holds regardless of which form of $\mathcal{B}$ is used.

In contrast to the ideal fluid case, the inactive wind solutions $v_i^h$ and $v_i^a$ now appear to define distinct velocity fields. Their difference must therefore belong to the null-space of (61), which means that there must exist some scalar $\delta, \lambda$ such that

$$v_i^h - v_i^a = \delta \lambda \omega_a \quad (67)$$

For small Rossby number, $\omega_a \approx f \mathbf{k}$, which implies that $v_i^h$ and $v_i^a$ must primarily differ by their vertical velocity component. Physically, this is plausible as their horizontal components must be approximately equal and compatible with geostrophic balance. It will be of interest to ascertain whether the vertical velocity component of $v_i^h$ is a better predictor of the actual vertical velocity field than that of $v_i^h$, which we plan on investigating in a subsequent study.

If a functional relationship exists between $\mathcal{B}$, $\gamma T$ and $Q_{yt}$, say $\mathcal{B} = G(\gamma T, Q_{yt})$ as in ideal fluid thermocline theory, then the seawater counterpart of Needler’s second formula (57) becomes

$$v_i = \frac{\partial G}{\partial \gamma T} \frac{\mathcal{P} \times \nabla \gamma T}{\rho Q} + \frac{\partial G}{\partial Q_{yt}} \frac{\mathcal{P} \times \nabla Q_{yt}}{\rho Q} \quad (68)$$

In comparison to Needler’s formula, (68) possesses the extra and undesirable thermobaricity-induced term proportional to $\mathcal{P} \times \nabla \gamma T$. This term can only be neglected if $\mathcal{P}_a$ rather than $\mathcal{P}_h$ is used, however, as the angle between $\mathcal{P}_h$ and $\nabla \gamma T$ is not generally small enough. If so

$$v_i \approx \left[ \frac{\partial G}{\partial Q_{yt}} \right] \frac{\mathcal{P}_a \times \nabla Q_{yt}}{\rho Q} \quad (69)$$

which is more directly comparable to (57), stressing again the fundamental importance of APE theory to extend the results of ideal fluid thermocline theory to compressible seawater.

5. Summary and conclusions

In this paper, we have developed a new theoretical framework that succeeds in explaining how to generalise to compressible seawater the concepts of isopycnal/lateral stirring surfaces, potential vorticity, Bernoulli function, and neutral directions, which until now had only been unambiguously defined for simple or ideal fluids. Borrowing most of its ingredients from Tailleux (2018)’s local reformulation of Lorenz (1955) APE theory, this framework thus establishes that: 1) the isopycnal/lateral stirring surfaces and potential vorticity in the oceans are most naturally defined in terms of Lorenz reference density or any of its proxy such as Tailleux (2016a)’s thermodynamic neutral density; 2) the Bernoulli function is most usefuly defined in terms of its ‘available’ form, obtained by removing the dynamically inert part of the standard Bernoulli function; 3) the neutral directions are most naturally defined in terms of an APE-based form of Nycander (2011) P-vector, obtained by replacing the constant reference pressure used in his framework by the variable Lorenz reference pressure $\rho_T(S, \theta)$.

The central feature of our new theory are thermobaric forces, whose nature and importance are clarified here for the first time. Indeed, although thermobaricity has long been known to introduce a new form of energy capable of
altering several phenomena ranging from deep convection to the dynamics of Rossby waves and instabilities, e.g.,
Akimoto (1999); Adkins et al. (2005); Su et al. (2016a,b); de Szoeke (2004); Hallberg (2005). Our study appears to be the first to explicitly recognise that thermobaric energy must give rise to isopycnal thermobaric forces and that these must be the ultimate cause for the non-existence of neutral surfaces in the oceans. This has many important implications.

First, it means that defining thermobaric forces and explaining how these affect stirring must be addressed by any theory of lateral stirring in the oceans that claims to describe reality. Second, it means that if thermobaric forces are to be unambiguously defined, this must also be the case of the isopycnal surfaces used to define them. In other words, the proposition that thermobaric forces exist and are well defined implies that this must also be true of the isopycnal/lateral stirring surfaces on which they live. This in turn implies that the construction of approximately neutral surfaces to study lateral stirring is a meaningful exercise as such surfaces can be legitimately regarded as the approximation of physically well defined lateral stirring surfaces, which had remained unclear so far. This in turn implies that lateral stirring surfaces should be defined in terms of Lorenz reference density surfaces, as these are the only surfaces that possess the only two properties of the isentropic surfaces of a simple fluid that carry over to compressible seawater exactly, namely materiality and flatness at rest.

Physically, the lateral stirring surfaces thus defined appear as the notional surfaces decoupling the effects of thermobaric and buoyancy forces, so that lateral and vertical stirring appear as the sub-forms of stirring affected solely by the former and latter forces respectively. All other forms of stirring must therefore involve work against the two types of forces, which is important for the correct interpretation of neutral surfaces in the oceans. Indeed, while in a simple fluid neutral surfaces are the surfaces along which fluid parcels can be exchanged without experiencing any restoring buoyancy forces, neutral surfaces in seawater are the surfaces along which fluid parcels experience compensating thermobaric and buoyancy forces that cancel each other out. To exist, neutral motions would therefore require near perfect synchronisation of the two forces on all temporal and spatial scales, which seems un plausible.

Defining lateral stirring surfaces in terms of LRD surfaces makes it possible to ascertain that the longstanding practice of defining the turbulent diapycnal diffusivity in Boussinesq fluids

\[ K_v = \frac{\varepsilon_p}{N^2} \]  

(70)

in terms of the APE dissipation rate \( \varepsilon_p \), as first proposed by Oakey (1982) (see his Eq. (8.5)) and further discussed by Gargett and Holloway (1984) and Winters et al. (1995), is also valid for compressible seawater. As showed by Winters and d’Asaro (1996) (similarly as Nakamura (1996)), an important property of (70) is that it relates the enhanced character of \( K_v \) to the enhancement of molecular diffusion due to the deformation of the LRD surfaces. On the other hand, the neutral density literature appears to consider that diapycnal mixing in the oceans should be defined in terms of the dianeutral diffusivity \( K_d \) and dianeutral flow, even though neither neutral density nor neutral surfaces mathematically exist. Although it is often stated that the existence of neutral tangent planes is all that is needed to be able to define dianeutral mixing and \( K_d \), this belief does not seem to be supported by any formal proof that we know of. In any case, such a view appears to define \( K_d \) as an immaterial concept unrelated to the diffusion experienced by any particular quantity, which is not easily reconciled with Winters and d’Asaro (1996)’s result that an effective diffusivity can only be defined for a diffusive quantity with well defined iso-surfaces. The neutral density literature also contends that constructing Redi (1982) rotated diffusion tensors in terms of directions other than the isoneutral and dianeutral directions would incur fictitious diapycnal mixing, which is equivalent to regarding any departure from neutrality as intrinsically unphysical, e.g., McDougall et al. (2014). This conflicts, however, with the results of this paper which establish that any irreducible departure from neutrality (as well as non-zero values of neutral helicity) should be interpreted as indicative of the importance of thermobaric processes. If the departure from neutrality reflects a physical reality, it cannot be simultaneously the cause of spurious mixing, as this would imply that it is unphysical. We therefore conclude that it is the concept of fictitious diffusivity itself that is unphysical and misleading, which in turn implies that the use of neutral rotated diffusion tensors is likely to cause spurious diapycnal mixing where thermobaric processes are demonstrably large, such as in the Southern Ocean, an issue that we plan on investigating further in a forthcoming study.

Perhaps the most astonishing and intriguing result of this paper, however, is the suggestion that the ‘available’ thermodynamic form of momentum balance (25), viz.,

\[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{\omega} \times \mathbf{v} + \nabla B_{\alpha} = \mathbf{P}_\alpha + \mathbf{F}, \]  

(71)

is superior to all other forms of momentum balance, and therefore ‘optimal’ in some sense, owing to being the only form exhibiting a force (the P-vector \( \mathbf{P}_\alpha \)) approximately normal to the relevant form of lateral stirring surfaces in the oceans. The fact that such a key insight could only be achieved by removing the dynamically inert parts of \( \mathbf{B}_h \) and \( \mathbf{P}_h \), and that this in turn allows one to understand how to generalise all the central concepts of the ideal fluid thermocline equations to compressible seawater, is especially remarkable as it suggests that the importance and physical significance of APE theory extend beyond the un-
derstanding of ocean energetics, e.g., (Hughes et al. 2009, Tailleux 2009, 2010, Tailleux and Rouleau 2010). In this paper, we only discussed the potential implications for the determination of the absolute velocity field and vertical velocity, but many other applications come to mind, such as using the LRD density as the basis for the derivation of the thickness-weighted averaged equations (Young 2012), the construction of inverse models, the study of the overturning circulation in density coordinates, water mass transformations (Groeskamp et al. 2019), and so on, which we also plan on investigating in the future.

Further work is still needed to establish the implications of our results for how to fix the diapycnal and isopycnal directions in Redi (1982) rotated diffusion tensors. While our results are sufficient to question the validity of using the neutral directions, they are not sufficient to ascertain that the LRD-based directions should be used instead, as more understanding is needed about the role of thermobaric forces and the release of APE associated with thermobaric instability on the physics of lateral and vertical mixing. Another reason is that even in the case of a simple fluid for which isopycnal surfaces are unambiguously defined, the eigendirections that diagonalise the turbulent mixing tensor for passive tracers have been found to occasionally differ from the isopycnal and diapycnal directions. For instance, Abernathey et al. (2013) found that the lateral mixing directions only coincide with the isopycnal directions in the interior, such directions becoming increasingly horizontal nearer the surface. Likewise, further work is also needed to establish whether the LRD surfaces are necessarily optimal for the tracing of water masses. Indeed, the results of van Sebille et al. (2011), who found $\sigma_2$ to outperform $\sigma_0$ and $\gamma^*$ for tracing NADW from its source regions to the Abaco line in the Gulf Stream area, suggest that the issue is more complicated and subtle than previously assumed.
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Data availability statement. The WOCE Global Ocean Climatology 1990–1998 (file ‘wgc_params.nc’) used in this study is available at doi:10.25592/uhhfdm.8987. The $\gamma^0$ software was downloaded from https://www.teos-10.org/preteos10_software.

APPENDIX A

Navier-Stokes equations for compressible seawater

The Navier-Stokes equations describing the motions of two-component compressible seawater are

$$\frac{D\mathbf{v}}{Dt} = -2\Omega \times \mathbf{v} - \frac{1}{\rho} \nabla p - \nabla \Phi + \mathbf{F}, \quad (A1)$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0, \quad (A2)$$

$$\frac{D\eta}{Dt} = \eta, \quad \frac{DS}{Dt} = \dot{S}, \quad (A3)$$

$$\mathbf{v} = v(\eta, S, p) = \frac{\partial h}{\partial \rho}, \quad (A4)$$

where $\mathbf{v} = (u, v, w)$ is the 3D velocity field, $p$ is pressure, $\rho$ is density, $\Omega$ is Earth’s rotation vector, $\mathbf{F}$ is the frictional force. $v = 1/\rho$ is the specific volume, $h = h(\eta, S, p)$ is the specific enthalpy, $\eta$ is the specific entropy, $\Phi(z) = gz$ is the geopotential with $g$ the gravitational acceleration and $z$ height increasing upward.

APPENDIX B

Canonical variables and derivatives of $\Pi$, $\Pi_1$, and $\Pi_2$

The definition of the local potential energy densities $\Pi$, $\Pi_1$, and $\Pi_2$ involve both thermodynamic ($\eta, S, \rho, p$) and geometric variables ($z$), but these are not all independent of each other. To identify the canonical variables of each PE density (i.e., the most natural set of independent variables) we first differentiate the expression of each PE density as if all the variables were independent. In the case of $\Pi$

$$\Pi = h(\eta, S, p) - h(\eta, S, p_r) + g(z - z_r) + \frac{p_0(z) - p}{\rho}, \quad (B1)$$

this approach yields

$$\nabla \Pi = T \nabla \eta + \mu \nabla S + \frac{1}{\rho} \nabla p - T_\tau \nabla \eta - \mu_r \nabla S - \frac{1}{\rho_r} \nabla p_r + g(T \nabla z - \nabla z_r) - \delta p \nabla v + \frac{1}{\rho} \nabla (p_0(z) - p), \quad (B2)$$

where as in the text, the suffix ‘r’ denotes variables estimated at the reference pressure $p_r = p_0(z_r)$, with $\delta p = p - p_0(z)$.

Now, using the fact that $\rho_r = \rho(\eta, S, p_r) = p_0(z_r)$ by virtue of $z_r$ satisfying the LNB equation ([14]),

$$\frac{1}{\rho_r} \nabla p_r + g \nabla z_r = -\frac{p_0(z_r)}{\rho_r} \nabla z_r + g \nabla z_r = 0, \quad (B3)$$

so that (B2) simplifies to

$$\nabla \Pi = (T - T_\tau) \nabla \eta + (\mu - \mu_r) \nabla S - \delta p \nabla v + g \left(1 - \frac{\rho_0(z)}{\rho}\right) \nabla z, \quad (B4)$$

where we used the result that $\nabla p_0(z) = -\rho_0(\eta) g$. Eq. (B4) thus establishes that $(\eta, S, v, z)$ are the canonical variables of $\Pi$. Proceeding similarly with $\Pi_1 = h(\eta, S, p) - h(\eta, S, p_0(z)) + (p_0(z) - p)/\rho$ and $\Pi_2 = h(\eta, S, p_0(z)) - h(\eta, S, p_r) + g(z - z_r)$, it is easily verified that

$$\nabla \Pi_1 = (T - T_b) \nabla \eta + (\mu - \mu_b) \nabla S - \delta p \nabla v + \frac{\rho_0(z)}{\rho_b} \left(1 - \frac{\rho_b}{\rho}\right) \nabla z, \quad (B5)$$

$$\nabla \Pi_2 = (T_b - T_r) \nabla \eta + (\mu_b - \mu_r) \nabla S + g \left(1 - \frac{\rho_0(z)}{\rho_b}\right) \nabla z, \quad (B6)$$

where as in the text, the suffix ‘b’ denotes variables estimated at $p_0(z)$. Eqs. (B5) and (B6) thus establish that $(\eta, S, v, z)$ and $(\eta, S, z)$ are the canonical variables of $\Pi_1$ and $\Pi_2$ respectively. It may be verified that summing (B5) and (B6) recovers (B4), as expected.

Eq. (B5) implies for the Lagrangian derivative of $\Pi_1$

$$\frac{DP_1}{Dt} = (T - T_b) \frac{D\eta}{Dt} + (\mu - \mu_b) \frac{DS}{Dt} - \delta p \frac{Dv}{Dt} + \frac{\rho_0(\eta)}{\rho_b} \left(1 - \frac{\rho_b}{\rho}\right) w \quad (B7)$$

By definition, $DP_1 / Dt$ may also be written as

$$\frac{DP_1}{Dt} = \frac{\partial P_1}{\partial \eta} + \mathbf{v} \cdot \nabla P_1 = \frac{\partial P_1}{\partial \eta} + \mathbf{v} \cdot \mathbf{P}_{a1} - \delta p \mathbf{v} \cdot \nabla v + \frac{\rho_0(\eta)}{\rho_b} \left(1 - \frac{\rho_b}{\rho}\right) w \quad (B8)$$

Equating the two expressions thus implies

$$\frac{\partial P_1}{\partial \eta} + \delta p \frac{\partial v}{\partial t} + \mathbf{v} \cdot \mathbf{P}_{a1} = (T - T_b) \eta + (\mu - \mu_b) \dot{S} \quad (B9)$$
Applying the same idea to $\Pi_2$ yields
\[
\frac{\partial \Pi_2}{\partial t} + \mathbf{v} \cdot \mathbf{P}_{a2} = (T_b - T_r)\eta + (\mu_b - \mu_r)\dot{\mathcal{S}}. \tag{B10}
\]
These results are important to relate the work terms $\mathbf{v} \cdot \mathbf{P}_{a1}$ and $\mathbf{v} \cdot \mathbf{P}_{a2}$ to local Eulerian time derivatives of $\Pi_1$, $\Pi_2$, and $\nu$, as well as to irreversible mixing processes.

**APPENDIX C**

**Links with Nycander (2011) and de Szoeve (2000)**

In Section 3, we generalised the concept of Montgomery potential as the difference between the static energy of the actual state minus that of Lorenz reference state
\[
M = h(\eta, S, p) + \Phi(z) - h(\eta, S, p_0(z_r)) - \Phi(z_r), \tag{C1}
\]
thus defining $M$ as the ‘available’ or dynamically active part of the static energy in the Lorenzian sense. In previous approaches, this idea has been generally limited to using a constant value for $p_0(z_r)$ and $z_r$, which is suboptimal as a way to remove the dynamically inert part of the static energy.

To show how our approach connects to previous work, it is useful to rewrite Eq. (C1) in the form
\[
M = h(\eta, S, p) - h(\eta, S, p_0(z)) + h^\pm(\eta, S, z), \tag{C2}
\]
where
\[
h^\pm(\eta, S, z) = \int_{p_0(z)}^{p_0(z_r)} \nu(\eta, S, p') dp' + \int_{z_r}^z g dz', \tag{C3}
\]
with the buoyancy $b$ defined by
\[
b = b(\eta, S, z) = -g \left(1 - \frac{\rho_0(z)}{\rho(\eta, S, p_0(z))}\right). \tag{C4}
\]
In the particular case where $z_r = 0$, $h^\pm$ can be identified as the compressible counterpart of the dynamic enthalpy first introduced by Young (2010) and Nycander (2010) (whicNycander refers to as ‘effective potential energy’).

The use of a constant $z_r$ instead of Lorenz reference depth does not affect the thermodynamic form of momentum balance (21), which may still be written as
\[
\frac{D\mathbf{v}}{Dt} = -2\Omega \times \mathbf{v} - \nabla M + \mathbf{P} + \mathbf{F}, \tag{C5}
\]
\[
\mathbf{P} = (T - T_r)\nabla \eta + (\mu - \mu_r)\nabla \mathcal{S} = a \mathbf{P} + \mathbf{P}_{N11}, \tag{C6}
\]
but with different definitions for $T_r$ and $\mu_r$, where
\[
a \mathbf{P} = (T - T_b)\nabla \eta + (\mu - \mu_b)\nabla \mathcal{S}, \tag{C7}
\]
where $T_b = T(\eta, S, p_0(z))$ and $\mu_b = \mu(\eta, S, p_0(z))$ are Boussinesq-like approximations of $T$ and $\mu$, while $\mathbf{P}_{N11}$ can be identified with the $\mathbf{P}$ vector introduced by Nycander (2011). As shown by Nycander (2011), the budget of dynamic enthalpy may be written in the form
\[
\frac{\partial h_1^\pm}{\partial t} + \mathbf{v} \cdot \mathbf{P}_{N11} = (T_b - T_r)\eta + (\mu_b - \mu_r)\dot{\mathcal{S}} = \text{mixing}, \tag{C9}
\]
which shows that $\mathbf{P}_{N11}$ defines $P$-neutral directions along which the adiabatic and isohaline interchanges of fluid parcels leave $h^\pm$ unaffected. Since $\mathbf{P}_{N11}$ differs from McDougall (1987) neutral vector $\mathbf{N}$, Nycander (2011) suggested that isoneutral stirring must induce changes in $h^\pm$ regardless of how $p_0(z_r)$ is chosen, and hence that $\mathbf{N}$ may therefore not be as neutral as generally assumed. Whether this is sufficient to challenge the use of neutral directions in rotated diffusion tensors has remained unclear, however, given the arbitrariness and general lack of physical significance of $h^\pm$.

Nycander (2011) recognised $\mathbf{P}_{N11}$ as a force but only discussed its implications for energetics. The role of $\mathbf{P}_{N11}$ in the momentum balance was indirectly discussed by de Szoeve (2000), who discussed the isopycnal projection of the horizontal part of (C5) in the Boussinesq approximation
\[
\frac{Du}{Dt} + f k \times u + \nabla \cdot M + \mathbf{P}^{(\gamma)} + \mathbf{F}_h, \tag{C10}
\]
where $\mathbf{u} = (u, v)$ is the horizontal velocity vector, $\nabla \cdot M$ is the isopycnal gradient of $M$, while $\mathbf{P}^{(\gamma)}$ represents the isopycnal projection of $\mathbf{P}$ on the material surface $\gamma = \text{constant}$. It is easily verified that the latter may be written in the form $\mathbf{P}^{(\gamma)} = \mu \mathbf{dsoeke} \nabla S$, for a suitably defined $\mu \mathbf{dsoeke}$, by using the fact that along an isopycnal surface, $\gamma \eta \nabla \eta + \gamma \gamma \nabla S = 0$, thus allowing to eliminate $\nabla \cdot M$ in favour of $\nabla S$. If the approximation $\mathbf{P} = \mathbf{P}_{N11}$ holds, it may be verified that the thermobaric force discussed by de Szoeve (2000) coincides with the isopycnal projection of Nycander (2011)’s $\mathbf{P}$ vector, which was overlooked by Nycander.

**APPENDIX D**

**Baroclinic production term**

In this paper, we established the following identities:
\[
\frac{1}{\rho} \nabla \rho + \nabla \Phi = \nabla (h + \phi) - \mathbf{P}_h = \nabla (h + \Phi - B_r) - \mathbf{P}_a.
\]
Taking the curl yields the following equivalent expressions
\[
\frac{1}{\rho^2} \nabla \rho \times \nabla p = \frac{1}{\rho^2} \mathbf{N} \times \nabla p = \nabla \times \mathbf{P}_h = \nabla \times \mathbf{P}_a.
\]
which are proportional to the baroclinic production term, where we defined $N = \rho_S \nabla S + \rho_0 \nabla \theta$.

**APPENDIX E**

**Alternative expressions for $P_a$**

In practical applications, it is useful to have expressions of $P_a$ in terms of the more commonly used in-situ temperature $T$, potential temperature $\theta$, or Conservative Temperature $\Theta$. Using the passage relationships

$$T d\eta + \mu dS = \frac{T c_p \theta}{\theta} d\theta + \left( \mu - T \frac{\partial \mu}{\partial \theta} \right) dS = \frac{T c_p \rho_0}{\theta} d\Theta + \left( \mu - \frac{T \mu_0}{\theta} \right) dS = c_p \left( dT - \Gamma dp \right) + \left( \mu - T \frac{\partial \mu}{\partial T} \right) dS,$$

(E1)

e.g., Tailleux (2010, 2015), yields

$$P_a = (T - T_r) \nabla \eta + (\mu - \mu_r) \nabla S = \frac{T - T_r}{T} c_p \left( \nabla T - \Gamma \nabla p \right) + \left( \mu - \mu_r - (T - T_r) \frac{\partial \mu}{\partial T} \right) \nabla S = \frac{T - T_r}{\theta} c_p \theta \nabla \Theta + \left( \mu - \mu_r - (T - T_r) \frac{\partial \mu_0}{\partial \theta} \right) \nabla S,$$

(E2)

where $\Gamma = \alpha T / (\rho c_p)$ is the adiabatic lapse rate, $c_p \theta = c_p (\eta, S, 0)$, $\mu_0 = \mu (\eta, S, 0)$, while $c_p \rho_0$ is the constant reference specific heat capacity underlying TEOS-10.

**APPENDIX F**

**Bernoulli theorems**

It may be verified that the Bernoulli theorems satisfied by $B_a$ and $B_h$ are:

*Available Bernoulli theorem*

$$\frac{\partial B_a}{\partial t} + \mathbf{v} \cdot \nabla B_a = \dot{B}_a + \frac{1}{\rho} \frac{\partial \rho}{\partial t} \quad (F1)$$

$$\dot{B}_a = (T - T_r) \eta + (\mu - \mu_r) \dot{S} + \mathbf{F} \cdot \mathbf{v} \quad (F2)$$

*Standard Bernoulli theorem*

$$\frac{\partial B_h}{\partial t} + \mathbf{v} \cdot \nabla B_h = \dot{B}_h + \frac{1}{\rho} \frac{\partial \rho}{\partial t} \quad (F3)$$

$$\dot{B}_h = T \eta + \mu \dot{S} + \mathbf{F} \cdot \mathbf{v} \quad (F4)$$

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