Finite Sample t-Tests for High-Dimensional Means

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Abstract

Size distortion can occur if an asymptotic testing procedure requiring diverging sample sizes, is implemented to data with very small sample sizes. In this paper, we consider one-sample and two-sample tests for mean vectors when data are high-dimensional but sample sizes are very small. We establish asymptotic $t$-distributions of one-sample and two-sample $U$-statistics, which only require data dimensionality to diverge but sample sizes to be fixed and no less than 3. Simulation studies confirm the theoretical results that the proposed tests maintain accurate empirical sizes for a wide range of sample sizes and data dimensionalities. We apply the proposed tests to an fMRI dataset to demonstrate the practical implementation of the methods.

Keywords: Nonparametric methods; Robust procedures; High-dimensional data
1 Introduction

Testing for population means is a classical problem in statistics. In the univariate case, the Student’s $t$-test can be applied when the sample mean follows a normal distribution, sample variance follows a $\chi^2$ distribution and the sample mean and sample variance are independent. In the traditional multivariate setting, Hotelling’s $T^2$ test (Hotelling, 1931) can be applied when dimension $p$ is fixed and dimension $p$ and sample size $n$ satisfy the relation $p \leq n - 1$.

With the explosive development of high-throughput technologies, high-dimensional data characterized by the “large $p$ and small $n$” situation are widely observed. When $p > n - 1$, the Hotelling’s $T^2$ test becomes infeasible due to singularity of the sample variance. Even when $p$ is close to $n - 1$, the Hotelling’s test loses its power as revealed by Bai and Saranadasa (1996). Many approaches have been proposed to modify the Hotelling’s $T^2$ test for high-dimensional data. Some were constructed to discard or stabilize the inverse of the sample variance. Examples include Bai and Saranadasa (1996), Srivastava and Du (2008), and Chen and Qin (2010). Some were proposed to reduce the noise contributed by non-signal bearing components for sparse signal detection. Examples include the maximum type test proposed in Cai, Liu and Xia (2014) and the thresholding tests in Hall and Jin (2010), Zhong, Chen and Xu (2013) and Chen, Li and Zhong (2019). Others were proposed to project the classical Hotelling’s $T^2$ statistic to a low-dimensional space. Examples include Thulin (2014), and Srivastava, Li and Ruppert (2016). All these testing procedures were established by requiring dimensionality and sample sizes to diverge even though dimensionality can be much larger than sample sizes.

In many biological and financial studies, high dimensional data with very small sample sizes often occur due to ethical and feasibility reasons. As demonstrated by the simulation
studies in Section 4, when sample sizes are very small, implementing a testing procedure established by requiring diverging sample sizes may not control the type-I error and thus wrongly reject the null hypothesis with a probability higher or lower than a preselected nominal significance level. To take into account the small sample effect, we establish the asymptotic normality of a standardized one-sample $U$-statistic by only requiring data dimensionality to diverge to infinity. In practice the standard deviation of the $U$-statistic is unknown and cannot be consistently estimated when sample size is small. By analogy with the univariate Student’s $t$ statistic, we propose an estimator for the variance of the $U$-statistic, which is shown to be asymptotically $\chi^2$ distributed and independent to the $U$-statistic under the null hypothesis. The result enables us to establish the asymptotic $t$-distribution of the $U$-statistic standardized by the sample standard deviation. The test only requires data dimensionality to diverge but the sample sizes to be fixed and no less than 3. Moreover, it is nonparametric without assuming Gaussian distribution of data. We further extend the asymptotic results to the two-sample testing problem.

The rest of the paper is organized as follows. Section 2 introduces the one-sample $U$ statistic and establish its asymptotic $t$-distribution. Extension to the two-sample problem is provided in Section 3. Simulation and case studies are presented in Sections 4 and 5. Technical proofs of theorems are relegated to Appendix.

## 2 One-sample test

Let $X_1, \cdots, X_n$ be independent and identically distributed $p$-dimensional random vectors with mean $\mu = E(X_i)$ and covariance matrix $\Sigma = \text{Var}(X_i)$. The hypotheses we are interested in are

$$H_0 : \mu = 0 \quad \text{versus} \quad H_1 : \mu \neq 0.$$  \hspace{1cm} (2.1)
To test the hypotheses, we consider the following $U$-statistic

$$U_n = \frac{2}{n(n-1)} \sum_{i<j}^n X_i'X_j,$$

which is the one-sample version of the $U$-statistic proposed in Chen and Qin (2010).

Like Bai and Saranadasa (1996) and Chen and Qin (2010), we model the sequence of $p$-dimensional random vectors $\{X_i, 1 \leq i \leq n\}$ by a linear high-dimensional time series

$$X_i = \mu + \Gamma Z_i \quad \text{for} \quad i = 1, \cdots, n,$$

where $\mu$ is the $p$-dimensional population mean, $\Gamma$ is a $p \times q$ matrix with $q \geq p$ satisfying $\Gamma \Gamma' = \Sigma$, and $Z_i = (z_{i1}, \cdots, z_{iq})'$ so that $\{z_{il}\}_{l=1}^q$ are mutually independent and satisfy $E(z_{il}) = 0$, $\text{Var}(z_{il}) = 1$ and $E(z_{il}^4) = 3 + \eta$ for some finite constant $\eta$.

As discussed in Chen and Qin (2010), the model (2.3) considers data beyond the commonly assumed Gaussian distribution, so that an established testing procedure is nonparametric. It is worth mentioning that different from Chen and Qin (2010), we assume all the components of $Z_i$ are mutually independent rather than pseudo-independent. As shown in the proof of Theorem 1, such a requirement allows us to drop the requirement of diverging sample size $n$ when applying the Martingale central limit theorem to establish the asymptotic distribution of the $U$ statistic. In addition, we assume the following condition for the covariance matrix $\Sigma$.

(C1). As $p \to \infty$, $\text{tr}(\Sigma^4) = o\{\text{tr}^2(\Sigma^2)\}$.

The condition is automatically satisfied if all the eigenvalues of $\Sigma$ are bounded, but it also allows partial eigenvalues to be unbounded (see Chen and Qin, 2010 for detailed discussion). An advantage of (C1) is that it does not impose any explicit growth rate on the dimension $p$.

We establish the asymptotic normality of the $U$-statistic as follows.
Theorem 1. Assume the model (2.3) and the condition (C1). For any finite sample size $n \geq 2$,

$$
\frac{U_n - \mu' \mu}{\sigma_n} \xrightarrow{d} N(0, 1) \quad \text{as} \quad p \to \infty,
$$

where

$$
\sigma_n^2 = \frac{2}{n(n-1)} \text{tr}(\Sigma^2) + \frac{4}{n} \mu' \Sigma \mu.
$$

Especially, under $H_0$ of (2.1),

$$
\frac{U_n}{\sigma_{n,0}} \xrightarrow{d} N(0, 1) \quad \text{as} \quad p \to \infty,
$$

where

$$
\sigma_{n,0}^2 = \frac{2}{n(n-1)} \text{tr}(\Sigma^2).
$$

To implement a testing procedure, we need to estimate the unknown $\text{tr}(\Sigma^2)$. If the sample size $n$ diverges to infinity, $\text{tr}(\Sigma^2)$ can be estimated consistently by some unbiased estimators such as the $U$ statistic in Li and Chen (2012). Slutsky’s theorem then shows that the asymptotic normality of $U_n$ with the estimated $\sigma_{n,0}$ still holds. However, we consider the sample size to be small and a consistent estimator of $\text{tr}(\Sigma^2)$ is not achievable. By analogy with the Student’s $t$-statistic, it can be shown that we only need to construct an estimator which is $\chi^2$ distributed and asymptotically independent of the statistic $U_n$. Let $\{X'_iX_j, i < j\}_{i,j=1}^n$ be a set of $n(n-1)/2$ random variables. From the proof of Theorem 1, the random variables in $\{X'_iX_j, i < j\}_{i,j=1}^n$ are asymptotically mutually independent and each component has the variance $\text{tr}(\Sigma^2)$ under the null hypothesis. Based on $\{X'_iX_j, i < j\}_{i,j=1}^n$, we therefore estimate the unknown $\text{tr}(\Sigma^2)$ by the sample variance

$$
\hat{\text{tr}}(\Sigma^2) = \frac{1}{n(n-1)/2 - 1} \sum_{i<j}^n \left( X'_iX_j - \frac{2}{n(n-1)} \sum_{i<j}^n X'_iX_j \right)^2.
$$

(2.4)

Its asymptotic $\chi^2$-distribution can be established as follows.
**Theorem 2.** Let the degrees of freedom \( k = n(n - 1)/2 - 1 \). Assume the model (2.3) and the condition (C1). For any finite sample size \( n \geq 3 \) and under \( H_0 \) of (2.1),

\[
\frac{k \, \text{tr}(\Sigma^2)}{\text{tr}(\Sigma^2)} \xrightarrow{d} \chi^2(k) \quad \text{as} \quad p \to \infty.
\]

After replacing \( \text{tr}(\Sigma^2) \) by the estimator \( \hat{\text{tr}}(\Sigma^2) \), we estimate \( \sigma^2_{n,0} \) by

\[
\hat{\sigma}^2_{n,0} = \frac{2}{n(n - 1)\hat{\text{tr}}(\Sigma^2)}.
\]

We establish the asymptotic \( t \)-distribution of \( U_n/\hat{\sigma}_{n,0} \) as follows.

**Theorem 3.** Assume the same conditions in Theorem 2. For any finite sample size \( n \geq 3 \) and under \( H_0 \) of (2.1), as \( p \to \infty \),

\[
\frac{U_n}{\hat{\sigma}_{n,0}} \xrightarrow{d} t \quad \text{with} \quad k = n(n - 1)/2 - 1 \quad \text{degrees of freedom}.
\]

To put the above result into a visual inspection, we simulate data from \( N(0, \Sigma) \) where \( \Sigma = (\sigma_{ij}) = (0.6^{|i-j|}) \). Figure 1 shows histograms of \( U_n/\hat{\sigma}_{n,0} \) based on 1000 iterations with different sample sizes \( n \) and data dimensionality \( p \). In the upper row, we choose the sample size \( n = 3 \) and the resulted \( t \)-distribution in Theorem 3 has 2 degrees of freedom. The empirical distributions of \( U_n/\hat{\sigma}_{n,0} \) are close to the \( t \)-curve despite its heavy tails. Similar results can be observed in the lower row where the \( t \)-distribution has 14 degrees of freedom and thin tails.

Based on Theorem 3, the proposed test with a nominal \( \alpha \) significance level rejects \( H_0 \) if

\[
U_n/\hat{\sigma}_{n,0} \geq t_\alpha(k),
\]

where \( t_\alpha(k) \) is the upper \( \alpha \) quantile of \( t \)-distribution with \( k = n(n-1)/2-1 \).
Figure 1: Histogram of $U_n/\hat{\sigma}_{n,0}$ versus $t$-distribution. In the upper row, the $t$-distribution has the degrees of freedom equal to 2; In the lower row, the $t$-distribution has the degrees of freedom equal to 14.

degrees of freedom. Moreover, the power function of the test when $\mu = \mu_0 \neq 0$ is

$$B_1(||\mu_0||^2) = P\left(\frac{U_n}{\hat{\sigma}_{n,0}} \geq t_\alpha(k)|\mu = \mu_0\right)$$

$$= 1 - P\left(\frac{U_n - ||\mu_0||^2}{\sigma_n} < \frac{\hat{\sigma}_{n,0}t_\alpha(k)}{\sigma_n} - \frac{||\mu_0||^2}{\sigma_n} |\mu = \mu_0\right)$$

$$= 1 - \Phi\left(\frac{\hat{\sigma}_{n,0}t_\alpha(k)}{\sigma_n} - \frac{\sqrt{n(n-1)||\mu_0||^2}}{\sqrt{2\text{tr}(\Sigma^2)} + 4(n-1)\mu_0'\Sigma\mu_0}\right), \text{ as } p \to \infty$$

where $||\mu_0||^2 = \mu_0'\mu_0$, $\sigma_n$ is given in Theorem 1 and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal.

To see how the power $B_1(||\mu_0||^2)$ evolves with the sum-of-squares signal strength $||\mu_0||^2$,
we first derive

\[
E(\hat{\sigma}_{n,0}) = \frac{2}{n(n-1)}\text{tr}(\Sigma^2) + \frac{4}{n(n+1)}\mu_0'\Sigma\mu_0.
\]

Then from the Markov inequality, we obtain \(\hat{\sigma}_{n,0}/\sigma_n = O_p(1)\) as \(p \to \infty\). This indicates that the power \(B_1(||\mu_0||^2)\) is largely determined by the signal-to-noise ratio

\[
\text{SNR}_1 = \frac{\sqrt{n(n-1)||\mu_0||^2}}{\sqrt{2\text{tr}(\Sigma^2) + 4(n-1)\mu_0'\Sigma\mu_0}}.
\]

A direct observation shows that \(B_1(||\mu_0||^2) \to 1\) if \(\text{SNR}_1 \to \infty\) as \(p \to \infty\). However, the test may lose its power when \(\mu_0\) is sparse and weak. To appreciate this, we consider \(\Sigma = I_p\) which is the \(p \times p\) identity matrix, and let \(p^{1-\beta}\) be the number of non-zero components in \(\mu_0\) and \(\delta\) be the value of each non-zero component. Based on the setup, we observe that when the sample size \(n\) is finite, \(\text{SNR}_1 = o(1)\) if \(\beta > 1/2\) and \(\delta^2 = o(p^{\beta-1/2})\), which implies that the proposed test loses its power for sparse and weak signals. On the other hand, when the sample size \(n\) diverges, \(\text{SNR}_1 = o(1)\) if \(\beta > 1/2\) and \(\delta^2 = o(n^{-1}p^{\beta-1/2})\), where the minimum \(\delta^2\) required to avoid power loss is \(n\) times less than that with a finite \(n\).

## 3 Two-sample test

Let \(i = 1\) or 2 and \(X_{i1}, \ldots, X_{in}\) be two independent and identically distributed \(p\)-dimensional random samples with mean \(\mu_i\) and covariance matrix \(\Sigma_i\). The two-sample testing problem considers the hypotheses

\[
H_0 : \mu_1 = \mu_2 \quad \text{versus} \quad H_1 : \mu_1 \neq \mu_2.
\]  

(3.5)

If the two samples have the same sample size \(n\), we can directly extend the one-sample statistic \(2.2\) to the two-sample case by replacing \(X_i\) with \(X_{1i} - X_{2i}\) for \(i = 1, \ldots, n\). However, two sample sizes are different in many cases. Without loss of generality, we
assume $n_1 \leq n_2$ and consider

$$Y_i = X_{1i} - \sqrt{\frac{n_1}{n_2}} X_{2i} + \frac{1}{\sqrt{n_1n_2}} \sum_{j=1}^{n_1} X_{2j} - \frac{1}{n_2} \sum_{j=1}^{n_2} X_{2j}, \quad i = 1, \ldots, n_1. \quad (3.6)$$

This procedure for the difference of two samples was first suggested by Scheffe (1943) in the univariate case to construct the confidence intervals by using the $t$-distribution. It was extended by Anderson (2003) in the multivariate case to obtain the generalized Hotelling’s $T^2$ statistic. We adopt the same procedure to propose the two-sample statistic

$$V_{n_1n_2} = \frac{2}{n_1(n_1 - 1)} \sum_{i<j}^{n_1} Y_i'Y_j', \quad (3.7)$$

which is similar to (2.2) but we replace $X_i$ with $Y_i$.

Note that $\bar{Y} = n_1^{-1} \sum_{i=1}^{n_1} Y_i = n_1^{-1} \sum_{i=1}^{n_1} X_{1i} - n_2^{-1} \sum_{i=1}^{n_2} X_{2i}$. We can write

$$V_{n_1n_2} = \frac{n_1}{n_1 - 1} \bar{Y}'\bar{Y} - \frac{1}{n_1(n_1 - 1)} \sum_{i=1}^{n_1} Y_i'Y_i,$$

where the first term on the right hand side uses the difference of the two sample means $n_1^{-1} \sum_{i=1}^{n_1} X_{1i} - n_2^{-1} \sum_{i=1}^{n_2} X_{2i}$, which is the most relevant to $\mu_1 - \mu_2$. We subtract the second term from the first term so that $E(V_{n_1n_2}) = (\mu_1 - \mu_2)'(\mu_1 - \mu_2)$.

Similar to (2.3), we model the two independent and identically distributed $p$-dimensional random samples by the linear high-dimensional time series

$$X_{ij} = \mu_i + \Gamma_i Z_{ij} \quad \text{for} \quad i = 1 \quad \text{and} \quad 2, \ j = 1, \ldots, n_i, \quad (3.8)$$

where $\Gamma_i$ is a $p \times q_i$ matrix with $q_i \geq p$ satisfying $\Gamma_i\Gamma'_i = \Sigma_i$, and $Z_{ij} = (z_{ij1}, \ldots, z_{ijq_i})'$ so that \{z_{ijl}\}_{l=1}^{q_i}$ are mutually independent and satisfy $E(z_{ijl}) = 0$, $\text{Var}(z_{ijl}) = 1$ and $E(z_{ijl}^4) = 3 + \eta$ for some finite constant $\eta$.

By analogy with (C1), we consider the following condition for the two covariance matrices $\Sigma_1$ and $\Sigma_2$.

(C2). As $p \to \infty$, $\text{tr}(\Sigma_i\Sigma_j\Sigma_k\Sigma_l) = o[\text{tr}^2\{(\Sigma_1 + \Sigma_2)^2\}]$ for $i, j, k, l = 1$ or 2.
Under $H_0$ of (3.5), the variance of $V_{n_1,n_2}$ is

$$\sigma^2_{n_1n_2,0} = \frac{2}{n_1(n_1 - 1)}\sigma^2_{Y'Y,0},$$

(3.9)

where

$$\sigma^2_{Y'Y,0} = \text{tr}(\Sigma_1^2) + \frac{n_1}{n_2} \text{tr}(\Sigma_2^2) + \frac{2n_1}{n_2} \text{tr}(\Sigma_1\Sigma_2)$$

is the variance of $Y_i'Y_j$ for $i < j$. Similar to the proof of Theorem1, $\{Y_i'Y_j, i < j\}^n_{i,j=1}$ can be shown to be a sequence of $n_1(n_1 - 1)/2$ independent random variables under $H_0$ of (3.5).

We therefore estimate $\sigma^2_{Y'Y,0}$ by

$$\hat{\sigma}^2_{Y'Y,0} = \frac{1}{n_1(n_1 - 1)/2 - 1} \sum_{i<j} \left( Y_i'Y_j - \frac{2}{n_1(n_1 - 1)} \sum_{i<j} Y_i'Y_j \right)^2,$$

(3.10)

which is similar to (2.4) but we replace $n$ and $X_i$ by $n_1$ and $Y_i$, respectively.

As a result, the unbiased estimator of $\sigma^2_{n_1n_2,0}$ is

$$\hat{\sigma}^2_{n_1n_2,0} = \frac{2}{n_1(n_1 - 1)}\hat{\sigma}^2_{Y'Y,0}.$$

The following theorem establishes the asymptotic $t$-distribution of $V_{n_1n_2}/\hat{\sigma}_{n_1n_2,0}$, which is a direct extension of Theorem 3.

**Theorem 4.** Assume the model (3.8) and the condition (C2). For any finite sample sizes $n_1 \geq 3$ and $n_2 \geq n_1$ and under $H_0$ of (3.5), as $p \to \infty$,

$$\frac{V_{n_1n_2}}{\hat{\sigma}_{n_1n_2,0}} \xrightarrow{d} t \quad \text{with} \quad k = n_1(n_1 - 1)/2 - 1 \quad \text{degrees of freedom}.$$

Based on Theorem 4, the proposed test with a nominal $\alpha$ level of significance rejects $H_0$ if $V_{n_1n_2}/\hat{\sigma}_{n_1n_2,0} \geq t_\alpha(k_2)$, where $t_\alpha(k_2)$ is the upper $\alpha$ quantile of $t$-distribution with $k = n_1(n_1 - 1)/2 - 1$ degrees of freedom. Moreover, as $p \to \infty$, the power of the two-sample test is

$$B_2(||\mu_1 - \mu_2||^2) = 1 - \Phi\left( \frac{\hat{\sigma}_{n_1n_2,0} t_\alpha(k) - \text{SNR}_2}{\sigma_{n_1n_2}} \right),$$

(3.11)
where the signal-to-noise ratio

\[ \text{SNR}_2 = \frac{\sqrt{n_1(n_1 - 1)||\mu_1 - \mu_2||^2}}{\sqrt{2\text{tr}\{(\Sigma_1 + \frac{n_1}{n_2}\Sigma_2)^2\} + 4(n_1 - 1)(\mu_1 - \mu_2)'(\Sigma_1 + \frac{n_1}{n_2}\Sigma_2)(\mu_1 - \mu_2)}}. \]

Similar to the one-sample test, the power of the two-sample test is largely determined by \( \text{SNR}_2 \), the analysis of which demonstrates that the proposed test is powerful in detecting dense and strong differences between \( \mu_1 \) and \( \mu_2 \), but encounters a power loss when differences between \( \mu_1 \) and \( \mu_2 \) are sparse and weak.

## 4 Simulation studies

### 4.1 One-sample test

We compare the proposed one-sample \( t \)-test with the one-sample version CQ test in Chen and Qin (2010), the BS test in Bai and Sarandasa (1996), and the SD test in Srivastava and Du (2008).

To generate random samples, we considered two types of innovations in (2.3): the Gaussian \( Z_i \sim N(0, I_p) \) and the standardized \( t \)-distribution with 4 degrees of freedom for each component of \( Z_i \), where the latter has heavier tails than the former used to demonstrate nonparametric performance of the proposed test. Under \( H_0 \), we simply assumed \( \mu = 0 \). Under \( H_1 \), we considered \( \mu \) to have \( [p^{1-\beta}] \) non-zero entries which were randomly selected from \( \{1, \cdots, p\} \). Here \([a]\) denotes the integer part of \( a \). The value of each non-zero entry was \( r \). From the simulation setup, the two parameters \( \beta > 0 \) and \( r > 0 \) were chosen to control the sparsity and strength of signals, respectively. We also considered the following two structures for the covariance \( \Sigma \), where model (a) specifies a bandable structure of \( \Sigma \) and Model (b) leads to a sparse \( \Sigma \).

(a). AR(1) model: \( \sigma_{j_1,j_2} = 0.6^{|j_1-j_2|} \) for \( 1 \leq j_1, j_2 \leq p \).
Table 1: Empirical sizes of Chen and Qin’s test (CQ), Bai and Sarandasa’s test (BS), Srivastava and Du’s test (SD), and the proposed test (New), based on 1000 replications with normally distributed $Z_i$ in (2.3).

|       | $p = 200$ |       |       |       |       |       |
|-------|-----------|-------|-------|-------|-------|-------|
|       | $n = 4$   | 6     | 15    | 30    |       |       |
| CQ    | 0.129     | 0.088 | 0.055 | 0.058 |       |       |
| BS    | 0.088     | 0.080 | 0.056 | 0.056 |       |       |
| SD    | 0.217     | 0.149 | 0.055 | 0.047 |       |       |
| New   | 0.056     | 0.062 | 0.051 | 0.055 |       |       |

|       | $p = 400$ |       |       |       |       |       |
|-------|-----------|-------|-------|-------|-------|-------|
|       | $n = 4$   | 6     | 15    | 30    |       |       |
| CQ    | 0.122     | 0.086 | 0.062 | 0.061 |       |       |
| BS    | 0.090     | 0.084 | 0.061 | 0.064 |       |       |
| SD    | 0.205     | 0.118 | 0.046 | 0.038 |       |       |
| New   | 0.059     | 0.059 | 0.058 | 0.058 |       |       |

|       | $p = 1000$|       |       |       |       |       |
|-------|-----------|-------|-------|-------|-------|-------|
|       | $n = 4$   | 6     | 15    | 30    |       |       |
| CQ    | 0.134     | 0.078 | 0.062 | 0.052 |       |       |
| BS    | 0.095     | 0.059 | 0.059 | 0.053 |       |       |
| SD    | 0.213     | 0.072 | 0.018 | 0.024 |       |       |
| New   | 0.058     | 0.052 | 0.059 | 0.050 |       |       |

(b). Random sparse matrix model: first generate a $p \times p$ matrix $\Gamma$ each row of which has only four non-zero element that is randomly chosen from $\{1, \cdots, p\}$ with magnitude generated from $\text{Unif}(1, 2)$ multiplied by a random sign. Then $\Sigma = \Gamma \Gamma^T + I_p$ where $I_p$ is the $p \times p$ identity matrix.

All the simulation results were based on 1000 replications with the nominal significance level $\alpha = 0.05$.

Table 1 displays the empirical sizes of the four tests with normally distributed $Z_i$ in
Table 2: Empirical sizes of Chen and Qin’s test (CQ), Bai and Sarandasa’s test (BS), Srivastava and Du’s test (SD), and the proposed test (New), based on 1000 replications with $t$ distributed $Z_i$ in (2.3).

|         | $p = 200$ |        |        |        | $p = 400$ |        |        |        | $p = 1000$ |        |        |        |
|---------|-----------|--------|--------|--------|-----------|--------|--------|--------|-----------|--------|--------|--------|
|         | $n = 4$  | 6 | 15 | 30     | $n = 4$ | 6 | 15 | 30       | $n = 4$ | 6 | 15 | 30 |
| Model (a) |           |        |        |        |           |        |        |        |           |        |        |        |
| CQ      | 0.145    | 0.076  | 0.066  | 0.062  | 0.130    | 0.074  | 0.054  | 0.060  | 0.116    | 0.089  | 0.054  | 0.056  |
| BS      | 0.072    | 0.043  | 0.047  | 0.042  | 0.053    | 0.034  | 0.032  | 0.044  | 0.040    | 0.038  | 0.028  | 0.037  |
| SD      | 0.193    | 0.089  | 0.033  | 0.032  | 0.187    | 0.070  | 0.018  | 0.031  | 0.175    | 0.047  | 0.007  | 0.014  |
| New     | 0.064    | 0.053  | 0.061  | 0.059  | 0.048    | 0.049  | 0.051  | 0.058  | 0.049    | 0.065  | 0.051  | 0.054  |
| Model (b) |           |        |        |        |           |        |        |        |           |        |        |        |
| CQ      | 0.143    | 0.085  | 0.067  | 0.052  | 0.141    | 0.069  | 0.076  | 0.044  | 0.128    | 0.064  | 0.050  | 0.069  |
| BS      | 0.051    | 0.039  | 0.038  | 0.039  | 0.062    | 0.033  | 0.048  | 0.025  | 0.049    | 0.023  | 0.021  | 0.049  |
| SD      | 0.157    | 0.106  | 0.038  | 0.028  | 0.169    | 0.080  | 0.030  | 0.021  | 0.148    | 0.034  | 0.006  | 0.017  |
| New     | 0.056    | 0.058  | 0.061  | 0.052  | 0.068    | 0.049  | 0.068  | 0.043  | 0.059    | 0.046  | 0.048  | 0.069  |

(2.3) under models (a) and (b) for the covariance matrix $\Sigma$, respectively. The sample size and dimension were chosen to be $n = 4, 6, 15, 30$ and $p = 200, 400, 1000$. While the CQ, BS and SD tests were able to maintain the empirical sizes close to the nominal significance level $\alpha = 0.05$ when sample sizes were relatively large, they encountered size distortion especially when sample size was very small ($n = 4$). Unlike the competitors, the proposed test always had the empirical sizes close to the nominal significance level. Similar results can be observed in Table 2 with $t$ distributed $Z_i$ in (2.3). The results confirm Theorem 3 that the proposed testing procedure was established without requiring the diverging sample
size and without assuming Gaussian distribution of data.

Due to the size distortion of the CQ, BS and SD tests when sample sizes are small, we compared the power performance of the four tests with a relatively large sample size \( n = 30 \). Table 3 demonstrates the empirical powers of the four tests with respect to different signal strength \( r \) when the sparsity of signal \( \beta = 0.4 \). As we can see, the powers of the four tests were increased as the signal strength \( r \) increased. The proposed test performed similarly.
to the other three tests. This is not surprising as the four tests were all proposed based on similar sum-of-squares type statistics.

To further investigate the power performance of the proposed test with small sample sizes, we chose a range of sample sizes \( n = 3, 6, 12, 24 \) with normally distributed \( Z_i \). For each of the sample sizes, the empirical powers of the proposed test were obtained with respect to a range of signal strength \( r \) from 0.1 to 1. As illustrated in Figure 2, the powers increased as the signal strength \( r \) increased for each sample size \( n \), and as the sample size increased for each signal strength \( r \). Compared the right panel with \( p = 1000 \) to the left panel with \( p = 400 \), the same pattern was observed but the powers were greater when dimension \( p \) became larger.
Table 4: Empirical sizes of Chen and Qin’s test (CQ), Cai, Liu and Xia’s test (CLX), Chen, Li and Zhong’s test (CLZ), and the proposed test (New), based on 1000 replications. The sample size \( n_2 = 30 \), and the sample size \( n_1 \) increased from 4 to 30.

| \( n_2 = 30 \) | \( n_1 = 4 \) | \( n_1 = 6 \) | \( n_1 = 15 \) | \( n_1 = 30 \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( p = 200 \)   | \( p = 200 \)   | \( p = 400 \)   | \( p = 400 \)   | \( p = 1000 \)  |
| CQ              | 0.108           | 0.084           | 0.064           | 0.068           | 0.069           | 0.060           | 0.060           | 0.060           | 0.060           | 0.060           | 0.060           | 0.060           | 0.060           | 0.060           | 0.060           | 0.060           | 0.060           | 0.060           | 0.060           |
| CLX             | 0.971           | 0.599           | 0.046           | 0.014           | 0.994           | 0.763           | 0.062           | 0.024           | 1               | 0.934           | 0.076           | 0.003           | 0.996           | 0.745           | 0.058           | 0.008           | 1               | 0.236           | 0.015           | 0.934           |
| CLZ             | 0.998           | 0.869           | 0.113           | 0.036           | 1               | 0.980           | 0.168           | 0.025           | 1               | 1               | 0.236           | 0.015           | 0.973           | 0.166           | 0.019           | 1               | 1               | 0.266           | 0.011           | 0.934           |
| New             | 0.054           | 0.062           | 0.055           | 0.066           | 0.058           | 0.053           | 0.059           | 0.059           | 0.057           | 0.060           | 0.055           | 0.061           | 0.054           | 0.050           | 0.058           | 0.061           | 0.059           | 0.060           | 0.055           | 0.061           | 0.059           |

| \( n_2 = 30 \) | \( n_1 = 4 \) | \( n_1 = 6 \) | \( n_1 = 15 \) | \( n_1 = 30 \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( p = 1000 \)  | \( p = 1000 \)  | \( p = 1000 \)  | \( p = 1000 \)  | \( p = 1000 \)  |
| CQ              | 0.095           | 0.054           | 0.052           | 0.060           | 0.096           | 0.060           | 0.063           | 0.062           | 0.089           | 0.066           | 0.054           | 0.051           | 0.968           | 0.582           | 0.060           | 0.006           | 1               | 0.929           | 0.081           | 0.001           |
| CLX             | 0.968           | 0.582           | 0.060           | 0.006           | 0.996           | 0.745           | 0.058           | 0.008           | 1               | 0.929           | 0.081           | 0.001           | 0.999           | 0.849           | 0.137           | 0.034           | 1               | 1               | 0.266           | 0.011           | 0.934           |
| CLZ             | 0.999           | 0.849           | 0.137           | 0.034           | 1               | 0.973           | 0.166           | 0.019           | 1               | 1               | 0.266           | 0.011           | 0.973           | 0.166           | 0.019           | 1               | 1               | 0.266           | 0.011           | 0.934           |
| New             | 0.055           | 0.050           | 0.054           | 0.061           | 0.056           | 0.048           | 0.065           | 0.061           | 0.049           | 0.059           | 0.063           | 0.050           | 0.058           | 0.050           | 0.058           | 0.065           | 0.061           | 0.059           | 0.063           | 0.050           |

4.2 Two-sample test

Under \( H_0 \), we compared the size performance of the proposed test with the two-sample version CQ test in Chen and Qin (2010), the maximum type CLX test in Cai, Liu and Xia (2014), and the multi-level thresholding CLZ test in Chen, Li and Zhong (2019). The random samples were generated from two types of innovations in (3.8). The Gaussian
\[ Z_{1j} \sim N(0, I_p) \text{ and } Z_{2j} \sim N(0, I_p), \text{ and the standardized } t\text{-distribution with 4 degrees of freedom for each component of } Z_{1j} \text{ and } Z_{2j}. \] For simplicity, we assigned \( \mu_1 = \mu_2 = 0, \text{ and considered } \Sigma_1 = \Sigma_2 \text{ modeled by the AR}(1) \text{ structure (a) in Section 4.1.} \]

Table 4 displays the empirical sizes of the four tests. The sample size \( n_2 = 30, \text{ and the sample size } n_1 \text{ increased from 4 to 30. The dimensions of random vector were } p = 200, 400 \text{ and 1000. For all the cases, the proposed test maintained the empirical sizes close to the nominal significance level } \alpha = 0.05. \text{ However, the CQ, CLX and CLZ tests had inflated sizes especially when the sample size } n_1 \text{ was small at 4.} \]

Under \( H_1, \text{ the power of the proposed test performed as well as the CQ test when sample sizes were relatively large. The pattern of the empirical results was very similar to the one-sample case given by Table 3. We therefore omit the results. It is worth mentioning that when the two high dimensional mean vectors differ only in sparsely coordinates and the differences are faint, the CLX and CLZ tests were proposed to improve power performance of the CQ test. It is therefore not surprising that they perform better than the proposed test for sparse and faint signal detection. But we need to emphasize that such a superior performance relies on the requirement of relatively large sample sizes.} \]

5 Application to fMRI dataset

To demonstrate the practical use of the proposed tests, we consider the StarPlus fMRI data, which is publicly available from Carnegie Mellon University’s Center for Cognitive Brain Imaging. The original data consist of different trials and we use a subset in which each of six human subjects was provided a sentence first for four seconds, followed by a blank screen for four seconds. The subject was then provided a picture for four seconds, followed by answering whether the sentence correctly described the picture. At last, the
subject was given a rest for fifteen seconds. There are in total 55 images collected every 0.5 seconds. At each time point, the image is marked with 25-30 anatomically defined regions called regions of interest (ROIs) (Mitchell et al. 2003). In fMRI, ROI analysis is a useful method of selecting a cluster of voxels for exploring patterns of activation across stimuli.

Our interest is to identify the ROIs which react differently to a sentence and a picture. To accommodate high dimensionality, we consider the ROIs with the number of voxels greater than 60. The names of these ROIs are described in Table 5. We let \( \mu_{1i} \) and \( \mu_{2i} \) be the population means of the \( i \)th ROI with respect to a picture and a sentence, respectively. The null hypotheses of interest are \( H_{0i} : \mu_{1i} = \mu_{2i} \) for \( i = 1, \ldots, 15 \), where 15 is the number of ROIs. Since the dataset has a very small sample size 6, it is more appropriate to apply the proposed test rather than other competitors requiring diverging sample sizes. For each

### Table 5: The 15 ROIs in the StarPlus dataset tested by the proposed methods.

| ROI             | Full Name                                           |
|-----------------|-----------------------------------------------------|
| CALC            | Calcarine Sulcus                                    |
| LDLPFC/RDLPFC   | Left/Right Dorsolateral Prefrontal Cortex           |
| LIPL            | Left Inferior Parietal Lobule                       |
| LIPS/RIPS       | Left/Right Intraparietal Sulcus                     |
| LIT/RIT         | Left/Right Inferior Temporal Lobe                   |
| LOPER/ROBER     | Left/Right Opercularis                              |
| LSPL/RSPL       | Left Superior Parietal Lobe                         |
| LT/RT           | Left/Right Temporal Lobe                            |
| LTRIA/RTRIA     | Left/Right Triangularis                             |
| SMA             | Supplementary Motor Area                            |


ROI, we computed the difference between the fMRI image at 29 seconds and that at 9 seconds. The two time points are the ends of the time intervals during which the picture and the sentence were presented, respectively. We applied the proposed test to obtain the p-values of the 15 ROIs displayed by Figure 3. By further applying the Bonferroni correction to control the family-wise error rate at 0.05, the two ROIs named LIPS and LT were identified to be significant as their p-values were less than 0.05/15.

6 Appendix: Technical Details.

A.1. Proof of Theorem 1.

From \( \{X_i, 1 \leq i \leq n\} \), we construct a sequence of \( n(n - 1)/2 \) random variables \( \{X'_i X_j, i < j\}^n_{i,j=1} \). To establish the asymptotic normality of \( U_n \), we need to show that...
the sequence \( \{X_i'X_j/\sqrt{\text{tr}(\Sigma^2)} + 2\mu'\Sigma \mu, i < j \}_{i,j=1}^n \) converges to a joint multivariate normal distribution as \( p \to \infty \). According to the Cramer-word device, we only need to show that 
\[
\sum_{i<j}^n c_{ij}X_i'X_j/\sqrt{\text{tr}(\Sigma^2)} + 2\mu'\Sigma \mu
\]
is asymptotically normal as \( p \to \infty \), where \( \{c_{ij}, i < j \}_{i,j=1}^n \) are some constants and at least one of them is nonzero.

We first establish the asymptotic normality of 
\[
\sum_{i<j}^n c_{ij}X_i'X_j/\sqrt{\text{tr}(\Sigma^2)}
\]
under the null hypothesis. From (2.3), we see that \( X_i = \Gamma Z_i \). Based on that,
\[
\sum_{i<j}^n c_{ij}X_i'X_j/\sqrt{\text{tr}(\Sigma^2)} = \sum_{i<j}^n c_{ij}Z_i'\Gamma Z_j = \sum_{k=1}^p \sum_{l=1}^p \sum_{i<j}^n c_{ij} (\Gamma'\Gamma)_{k\ell} z_{ik}z_{\ell j}.
\]

To simplify notation, we define
\[
A_{ij,kl} = \frac{c_{ij}}{\sqrt{\text{tr}(\Sigma^2)}} (\Gamma'\Gamma)_{k\ell},
\]
if \( k > l \), and if \( k = l \),
\[
A_{ij,kk} = \frac{c_{ij}}{2\sqrt{\text{tr}(\Sigma^2)}} (\Gamma'\Gamma)_{kk}.
\]

Then, using the symmetry, we can write
\[
\sum_{i<j}^n c_{ij}X_i'X_j/\sqrt{\text{tr}(\Sigma^2)} = \sum_{k=1}^p \sum_{l=1}^k \sum_{i<j}^n A_{ij,kl} (z_{ik}z_{\ell j} + z_{il}z_{jk}) = \sum_{k=1}^p V_k,
\]
where \( V_k = \sum_{i=1}^k \sum_{i<j}^n A_{ij,kl} (z_{ik}z_{\ell j} + z_{il}z_{jk}) \). Let \( S_h = \sum_{k=1}^h V_k \). Since \( \text{E}(S_q|S_h) = S_h \) for any \( q > h \), we see that \( S_h \) is a martingale. We therefore use the Martingale central limit theorem to establish the asymptotic normality of \( \sum_{i<j}^n c_{ij}X_i'X_j/\sqrt{\text{tr}(\Sigma^2)} \). According to the Martingale central limit theorem, it is equivalent to proving the following two results:

\[
\sum_{k=1}^p \text{E}(V_k^2|\mathcal{F}_{k-1}) \xrightarrow{p} \sum_{i<j}^n c_{ij}^2, \quad \text{and} \quad (6.12)
\]

\[
\sum_{k=1}^p \text{E}\{V_k^2 I(|V_k| > \epsilon|\mathcal{F}_{k-1}) \} \xrightarrow{p} 0, \quad (6.13)
\]
To prove (6.12), we need to show \( \sum_{k=1}^{p} E(V_k^2) \to \sum_{i<j}^{n} c_{ij}^2 \) and \( \text{Var}\{\sum_{k} E(V_k^2 | \mathcal{F}_{k-1})\} \to 0 \), respectively. To this end, we notice

\[
V_k^2 = \sum_{l_1=1}^{k} \sum_{l_2=1}^{k} \sum_{i_1<j_1}^{n} \sum_{i_2<j_2}^{n} A_{i_1j_1,kl_1} A_{i_2j_2,kl_2} (z_{i_1k} z_{j_1l_1} + z_{i_1l_1} z_{j_1k}) (z_{i_2k} z_{j_2l_2} + z_{i_2l_2} z_{j_2k}).
\]

Then,

\[
\sum_{k=1}^{p} E(V_k^2) = 2 \sum_{k=1}^{p} \sum_{l=1}^{k} \sum_{i<j}^{n} A_{ij,kl}^2 + 2 \sum_{k=1}^{p} \sum_{i<j}^{n} A_{ij,kl}^2 = \sum_{i<j}^{n} c_{ij}^2.
\]

Next, it can be seen that

\[
\sum_{k} E(V_k^2 | \mathcal{F}_{k-1}) = 4 \sum_{k=1}^{p} \sum_{i<j}^{n} A_{ij,kl}^2 + \sum_{k=1}^{p} \sum_{l=1}^{k-1} \sum_{i<j}^{n} \sum_{i_1<j_1}^{n} \sum_{i_2<j_2}^{n} A_{i_1j_1,kl_1} A_{i_2j_2,kl_2} z_{j_1l_1} z_{i_2l_2} - \sum_{k=1}^{p} \sum_{l=1}^{k-1} \sum_{i<j}^{n} \sum_{i_1<j_1}^{n} \sum_{i_2<j_2}^{n} A_{i_1i_2,kl_1} A_{i_2j_2,kl_2} z_{i_1l_1} z_{j_2l_2} + \sum_{k=1}^{p} \sum_{l=1}^{k-1} \sum_{i<j}^{n} \sum_{i_1<j_1}^{n} \sum_{i_2<j_2}^{n} A_{i_1j_1,kl_1} A_{i_2j_2,kl_2} z_{i_1l_1} z_{i_2l_2}.
\]

Taking the expectation of the above, we can show that

\[
E[\{\sum_{k} E(V_k^2 | \mathcal{F}_{k-1})\}]^2 = (\sum_{i<j}^{n} c_{ij}^2)^2 + O\left\{ \left( \sum_{i<j}^{n} c_{ij}^2 \right)^2 \frac{\text{tr}(\Sigma^4)}{\text{tr}^2(\Sigma^2)} \right\}.
\]

As a result, when \( p \to \infty \),

\[
\text{Var}\{\sum_{k} E(V_k^2 | \mathcal{F}_{k-1})\} = O\left\{ \left( \sum_{i<j}^{n} c_{ij}^2 \right)^2 \frac{\text{tr}(\Sigma^4)}{\text{tr}^2(\Sigma^2)} \right\} \to 0,
\]

because \( \text{tr}(\Sigma^4) = o\{\text{tr}^2(\Sigma^2)\} \) according to the condition (C1).

By Chebyshev Inequality, to prove (6.13), we only need to show that

\[
\sum_{k=1}^{p} E(V_k^4) \to 0.
\]

Using \( V_k = \sum_{l=1}^{k} \sum_{i<j} A_{ij,kl}(z_{ik} z_{jl} + z_{il} z_{jk}) \), we can show that for some constant \( C \),

\[
\sum_{k=1}^{p} E(V_k^4) \leq \frac{C}{\text{tr}^2(\Sigma^2)} \left( \sum_{i<j}^{n} c_{ij}^4 \right)^4 \sum_{k} \sum_{l_1 l_2} (\Gamma \Gamma')_{kl_1} (\Gamma' \Gamma)_{kl_2} (\Gamma \Gamma')_{kl_2} (\Gamma' \Gamma)_{kl_2} \leq \frac{C}{\text{tr}^2(\Sigma^2)} \left( \sum_{i<j}^{n} c_{ij}^4 \right) \text{tr}(\Sigma^4) \to 0,
\]

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because tr(Σ⁴) = o{tr²(Σ²)} according to the condition (C1).

Based on (6.12) and (6.13), we apply the Martingale central limit theorem to establish the asymptotic normality of \( \sum_{i<j} c_{ij} X'_i X_j / \sqrt{\text{tr}(\Sigma^2)} \) under the null hypothesis.

To prove asymptotic normality of \( \sum_{i<j} c_{ij} X'_i X_j \) under the alternative, we use (2.3) to write

\[
\sum_{i<j} c_{ij} X'_i X_j = \sum_{i<j} c_{ij} \mu' \mu + \sum_{i<j} c_{ij} \mu' \Gamma Z_j + \sum_{i<j} c_{ij} \mu' \Gamma Z_i + \sum_{i<j} c_{ij} Z'_i \Gamma' \Gamma Z_j,
\]

where the last term remains under the null hypothesis and its asymptotic normality has been established. Next, we need to establish the asymptotic normality of \( \sum_{i<j} c_{ij} \mu' \Gamma Z_j + \sum_{i<j} c_{ij} \mu' \Gamma Z_i \). By observing that \( Z_i = (z_{i1}, \cdots, z_{ip})' \) and \( \{z_k\}_{k=1}^p \) are mutually independent, the asymptotic normality of \( \sum_{i<j} c_{ij} \mu' \Gamma Z_j + \sum_{i<j} c_{ij} \mu' \Gamma Z_i \) can be established from the Lyapunov’s condition. At last, \( \sum_{i<j} c_{ij} \mu' \Gamma Z_j + \sum_{i<j} c_{ij} \mu' \Gamma Z_i \) and \( \sum_{i<j} c_{ij} Z'_i \Gamma' \Gamma Z_j \) are asymptotically independent. We thus establish the asymptotic normality of \( \sum_{i<j} c_{ij} X'_i X_j \) under the alternative hypothesis. This completes the proof of Theorem 1.

**A.2. Proof of Theorem 2.**

In Theorem 1, we have shown that under the null hypothesis, for \( 1 \leq i < j \leq n \), \((X'_1 X_2, \cdots, X'_i X_j, \cdots, X'_{n-1} X_n)\) follows an asymptotic \( n(n-1)/2 \)-variate multivariate normal distribution with mean equal to zero and covariance matrix equal to \( \text{tr}(\Sigma^2) I_{n(n-1)/2} \), where \( I_{n(n-1)/2} \) is the \( n(n-1)/2 \times n(n-1)/2 \) identity matrix. Based on \( \{X'_i X_j, i < j\}_{i,j=1}^n \), we estimate the unknown \( \text{tr}(\Sigma^2) \) by the sample variance (2.4). Since \( \{X'_i X_j, i < j\}_{i,j=1}^n \) are asymptotically independent and normally distributed random variables, \( k \text{tr}(\Sigma^2)/\text{tr}(\Sigma^2) \) converges to \( \chi^2(k) \) as \( p \to \infty \), where \( k = n(n-1)/2 - 1 \). This completes the proof of Theorem 2.

**A.3. Proof of Theorem 3.**
The statistic $U_n$ is the sample mean of $\{X'_iX_j, i < j\}_{i,j=1}^n$. From the proof of Theorem 2, $U_n$ is asymptotically independent of the sample variance $\hat{\sigma}_{n,0}$. As a result, $U_n/\hat{\sigma}_{n,0}$ converges to a $t$-distribution with $k = n(n-1)/2 - 1$ degrees of freedom. This completes the proof of Theorem 3.

**A.4. Proof of Theorem 4.**

Theorem 4 can be shown by replacing $X_i$ with $Y_i$ in the proofs of Theorems 1-3. We therefore omit it.

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