The stress-strain state of the double-layer spherical construction taking into account the porous structure of inner layer and heterogeneity of outer layer

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Abstract. The mathematical model describing the stress-strain state of a double-layer spherical body is constructed. The inner layer of the considered body has a porous structure, and the outer layer has nonhomogeneous elastic properties. Deforming of the porous medium under the specified uniformly distributed compressive loads is divided into two stages: elastic deforming of the porous medium and inelastic deforming of the compressed matrix having hardening elastic-plastic properties. The mathematical model describing the stress and displacement fields for a double-layer spherical body is constructed within the framework of the centrally symmetric formulation. The analytical relations that define the stress and displacement fields for both layers at the first stage of deforming are obtained. The analytical expressions that describe the stress-strain states in the elastic and plastic deforming zones of the inner layer with a fully compressed matrix, as well as the outer layer are found. The equation for defining the radius, which separates the elastic and plastic deforming zones of the inner layer at the second stage of deforming is obtained. The consistency conditions at the elastic-plastic boundary were chosen as follows: 1) the continuity conditions of the radial component of stresses and displacements; 2) the equality to zero of plastic deformations on the radial component of stresses and displacements. The continuity conditions of the radial component of stresses and displacements were chosen as consistency conditions on the boundary between the layers.

1. Problem statement. Solution for outer nonhomogeneous layer

The problems of determining and analysis of stress-strain states (SSS) of layered spherical bodies under uniform overall compression were addressed by [1–4], as well as by a number of other authors. The problem of determining SSS for the double-layer spherical body under the compressive loads which are uniformly distributed on the outer surface of the body (load intensity $q_a$) and the internal surface of the body (load intensity $q_d$) is considered at present paper (see figure 1). By $d$ denote the radius of the outer border, by $a$ – the radius of the inner border, by $b$ – the radius of the unstrained boundary between the outer and inner layers. The values referred to the inner layer will be marked by the (1)-superscript, the ones referred to the outer layer will be marked by the (2)-superscript.

Boundary conditions in stresses have the form
Figure 1. Double-layer nonhomogeneous poroelastic spherical body under uniform compression.

It is assumed that the outer layer of the considered composite construction has nonhomogeneous elastic compressible properties.

One of the main directions in research on the theory of elasticity of nonhomogeneous bodies is the construction of general solutions under certain restrictions on the nature of inhomogeneity [5–8]. The nonhomogeneous modules of elasticity are selected as linear, exponential, or power functions in the analytical treatments due to the complexity of problems of mechanics of deformable heterogeneous bodies [9].

At present paper the inhomogeneity of the elastic mechanical properties of the considered body is assumed in the following form: the volume compression modulus is constant $K^{(2)}(r, \theta, \varphi) = \text{const}$, and the shear modulus is an arbitrary differentiable function of the radial coordinate $G^{(2)} = G^{(2)}(r)$. For the outer layer, the problem is solved according to the scheme proposed in the paper [10]. Then as $r \in [b, d]$ we obtain in a spherical coordinate system $(r, \theta, \varphi)$

$$
\sigma^{(2)}_r = 2C_1 \cdot I(r) + C_2, \\
\sigma^{(2)}_\theta = \sigma^{(2)}_\varphi = 2C_1 \left( I(r) + \frac{M(r)}{2} \right) + C_2, \\
\varepsilon^{(2)}_r = 2C_1 \left( 3 \cdot I(r) + M(r) \left( 1 - \frac{3K^{(2)}(r)}{2G^{(2)}(r)} \right) \right) + \frac{C_2}{3K^{(2)}}, \\
\varepsilon^{(2)}_\theta = \varepsilon^{(2)}_\varphi = 2C_1 \left( 3 \cdot I(r) + \frac{1}{r^3} \right) + \frac{C_2}{3K^{(2)}},
$$

(1)
Hereinafter we introduce the following notations: \( \sigma_{ii}^{(i)}, \sigma_\theta^{(i)}, \sigma_\phi^{(i)} \) are the main components of the stress tensor; \( \varepsilon_r^{(i)}, \varepsilon_\theta^{(i)}, \varepsilon_\phi^{(i)} \) are the main components of the strain tensor; \( u^{(i)} \) is the radial component of the displacement vector for the outer layer \((i = 2)\) and for the inner layer \((i = 1)\):

\[
I(r) = \int_b^r \frac{dr}{1 + \frac{3K^{(2)}}{4G^{(2)}(r)} r^4}, \quad M(r) = \frac{4G^{(2)}(r)}{(4G^{(2)}(r) + 3K^{(2)}(r)) r^3}.
\]

2. Model of inner layer material with porous structure

Deforming of the inner layers’ porous material with a characteristic value \( \varepsilon_0^{(1)} \), determined by the specific volume of pores is divided into two stages \([2,11]\). The first stage is elastic deformation of a compressible porous medium; the second one is inelastic deformation of a fully compressed matrix with hardening elastic-plastic properties. A condition of the existence of partly compressed pores in the medium has the form

\[
-\varepsilon_0^{(1)} < \varepsilon_0^{(i)}.
\]

The relation between stresses and deformations at the first stage of deforming is taken from Hooke’s law for the compressible body

\[
\sigma_{ij}^{(i)} = \lambda^{(i)} \varepsilon_0^{(i)} g_{ij} + 2\mu^{(i)} \varepsilon_0^{(i)} g_{ij}.
\]

In the second stage, the elastic deformation of the fully compressed matrix is subjected to the Hooke’s law for the incompressible body \([2]\)

\[
S_{ij}^{(i)} = \begin{cases} 
2\left(\mu_0^{(i)} + \mu_1^{(i)}\right) \varepsilon_0^{(i)} + \frac{2}{3} \mu_1^{(i)} \varepsilon_0^{(i)} g_{ij}, \\
-\varepsilon_0^{(i)} = \varepsilon_0^{(i)}
\end{cases}
\]

where \( g_{ij}, \sigma_{ij}^{(i)}, \varepsilon_0^{(i)} \) are the components of the metric tensor, stress tensor and elastic strain tensor, respectively; \( \lambda^{(i)}, \mu^{(i)} \) are Lamé parameters for the porous compressible matrix; \( S_{ij}^{(i)} \) are the components of the deviator stress tensor; \( \left[ \varepsilon_0^{(i)} \right] \) are the components of elastic strain tensor calculated under the condition \( -\varepsilon_0^{(i)} = \varepsilon_0^{(i)} \); \( \mu_0^{(i)} + \mu_1^{(i)} \) is the shear modulus of the material with a fully compressed matrix.

In the plastic deforming zone of the material with a fully compressed matrix, we use the incompressible hardening elastic-plastic body model \([12]\) with the load surface
\[ F = \left( S_y^{(i)} - c^{(i)} \varepsilon_y^{(i)} \right) \left( S_o^{(i)} - c \varepsilon_o^{(i)} \right) - \left( k^{(i)} \right)^2, \]  

(8)

where \( \varepsilon_y^{(i)} \) are the components of plastic strain tensor; \( c^{(i)} \) is the strengthening factor of the inner layer material with a fully compressed skeleton; \( k^{(i)} \) is the fluidity limit of the inner layer material with a fully compressed skeleton.

The total deformation in the plastic zone is composed of the elastic and plastic parts

\[ \varepsilon_y^{(i)} = \varepsilon_y^{(e)} + \varepsilon_y^{(p)}. \]  

(9)

The plastic and elastic components of the volumetric deformation respectively satisfy the conditions of incompressibility

\[ \varepsilon_y^{(p)} = 0, \quad \varepsilon_y^{(e)} = -\varepsilon_o. \]  

(10)

Hereinafter “e” and “p”-superscripts of variables indicate belonging of corresponding variables to the elastic and plastic deforming zones of the material with a fully compressed skeleton of the inner layer.

3. Elastic deforming of compressible porous medium

Consider the case when the first stage of deforming is realized in the inner layer under given loads. Then the inner layer behaves as a compressible elastic body. The SSS in it \( r \in [a, b] \) according to the [4] has the form

\[ u^{(i)} = D_1 r + D_2 \frac{r}{r^3}, \]

\[ \varepsilon_r^{(i)} = D_1 \frac{2 D_2}{r^3}, \quad \varepsilon_{\theta}^{(i)} = D_1 + D_2 \frac{2}{r^3}, \]

\[ \sigma_r^{(i)} = D_1 \left( 3 \lambda_1^{(i)} + 2 \mu_1^{(i)} \right) - 4 \mu_1^{(i)} \frac{D_2}{r^3}, \quad \sigma_{\theta}^{(i)} = \sigma_{\phi}^{(i)} = D_1 \left( 3 \lambda_1^{(i)} + 2 \mu_1^{(i)} \right) + 2 \mu_1^{(i)} \frac{D_2}{r^3}. \]  

(13)

The integration constants are determined based on the consistency conditions on the boundary between the layers, as well as the boundary conditions

\[ \begin{cases} \sigma_r^{(i)} \mid_{r = b} = \sigma_r^{(2)} \mid_{r = b}, \\ u_r^{(i)} \mid_{r = b} = u_r^{(2)} \mid_{r = b}, \\ \sigma_r^{(2)} \mid_{r = a} = -q_d, \\ \sigma_r^{(1)} \mid_{r = a} = -q_d. \end{cases} \]  

(14)

Taking into account (2), (4), (11), (13) system (14) will be rewritten in the form
\[
\begin{aligned}
D_1 \left(3\lambda_1^{(i)} + 2\mu_1^{(i)}\right) - 4\mu_1^{(i)} \frac{D}{b^3} &= 2C_1 \cdot I(b) + C_2, \\
D_1 + \frac{D}{b^3} &= \frac{2C_1}{9K^{(2)}} \left(3 \cdot I(b) + \frac{4G^{(2)}(b)C_1 + 3K^{(2)}}{4G^{(2)}(b) + 3K^{(2)}}b^3\right) + \frac{C_2}{3K^{(2)}}, \\
2C_1 \cdot I(d) + C_2 &= -q_d, \\
D_1 \left(3\lambda_1^{(i)} + 2\mu_1^{(i)}\right) - 4\mu_1^{(i)} \frac{D}{a^3} &= -q_a.
\end{aligned}
\]

Solving (15) we obtain

\[
C_1 = \frac{3K^{(2)} P \left(a^3 q_a - b^3 q_d\right) - q_d b^3}{2b^3 \left(I(b) - I(d)\right) \left(3K^{(2)} P - F\right) - \frac{2}{3} F}, \quad C_2 = -q_d - 2C_1 \cdot I(d),
\]

\[
D_1 = \frac{a^3}{F} \left(\frac{b^3 q_a}{a^3 q_a - q_d + 2C_1 \left(I(b) - I(d)\right)\right), \quad D_2 = \frac{a^3}{4\mu_1^{(i)}} \left(q_a + D_1 \left(3\lambda_1^{(i)} + 2\mu_1^{(i)}\right)\right),
\]

where

\[
P = 1 + \frac{a^3}{b^3} \left(\frac{3\lambda_1^{(i)} + 2\mu_1^{(i)}}{4\mu_1^{(i)}}\right), \quad F = \left(3\lambda_1^{(i)} + 2\mu_1^{(i)}\right) \left(1 + \frac{a^3}{b^3}\right).
\]

Due to (12) the volume deformation in the inner layer is determined in the form

\[
\varepsilon_r^{(i)} + \varepsilon_\theta^{(i)} + \varepsilon_\phi^{(i)} = 3D_1.
\]

If the following condition is satisfied

\[
-3D_1 < \varepsilon_\theta,
\]

the first stage of deforming of the porous material is realized in the inner layer with the SSS determined by (11)–(13), (16).

Note that the volume deformation in the inner layer is constant. Therefore, the volumetric pore compression occurs identically at each point of this layer and the complete pore compression occurs in the inner layer simultaneously when the condition (17) turns into equality.

4. Elastic-plastic deforming of the inner layer material with fully compressed matrix

Thus, if the condition (17) is ruled out, the material of the inner layer is a fully compressed matrix and, if the condition (8) holds, a plastic zone of radius \(\gamma\) forms and starts growing near the inner contour of this layer (figure 2). That is, the second stage of deforming of the material with a fully compressed matrix having hardening elastic-plastic properties is realized. In other words, compressed skeleton deforms like a hardening incompressible elastic-plastic medium with parameters \(k^{(i)}, c^{(i)},\mu^{(i)} = \mu_1^{(i)} + \mu_0^{(i)}\).
According to [4] we obtain SSS in this case.

For zones of elastic and plastic deforming of the inner layer \((a < r < b)\) displacement and total deformation are determined by relations

\[ u^{(i)} = \frac{D_3}{r^3} \frac{e_0}{3} r, \]  
\[ e_r^{(i)} = -\frac{2D_3}{r^3} \frac{e_0}{3}, \quad e_\phi^{(i)} = e_\theta^{(i)} = \frac{2D_3}{r^3} \frac{e_0}{3}. \]  

In the elastic zone of the inner layer \((\gamma < r < b)\) stresses take the form

\[ \sigma_r^{(e)} = \frac{1}{3r^3} \left( \frac{L}{\mu_1^{(i)}} - 16D_3\mu_1^{(i)} \right) + D_4, \quad \sigma_\theta^{(e)} = \sigma_\phi^{(e)} = \frac{1}{6r^3} \left( \frac{L}{\mu_1^{(i)}} - 16D_3\mu_1^{(i)} \right) + D_4, \]  

where \( L = \left( 3q_a \cdot f(e_0) - e_0 \left( 3\Lambda_1^{(i)} + 2\mu_1^{(i)} \right) \right) \cdot \mu_1^{(i)} a^3 \), \( D_4 = \frac{-2\sqrt{3}k^{(i)} \mu_1^{(i)} \gamma^3 + L}{12\mu_1^{(i)} \mu_1^{(i)}}. \)

In the plastic zone of the inner layer \((a < r < \gamma)\) the SSS is defined as (due to the boundary conditions (1) on the inner contour and the equality to zero of the plastic component \( \varepsilon_r^{(i)} \) of the strain tensor at the interface \( \gamma \) of the elastic and plastic deforming zones)

\[ \varepsilon_r^{(p)} = \frac{2\sqrt{3}k^{(i)}}{3 \left( 2\mu_1^{(i)} + c_1^{(i)} \right)} \left( \frac{\gamma^3}{r^3} - 1 \right), \]  

**Figure 2.** Nonhomogeneous porous spherical body with fully compressed elastic-plastic matrix.
The SSS in the outer layer is defined by the relations (2)–(4), where it should be put $C_1 = C_3$, $C_2 = C_4$. Based on the consistency conditions on the boundary between the layers, the consistency conditions at the elastic-plastic boundary of the inner layer, as well as the boundary conditions on the outer contour, we can write a system for determining the integration constants (23).

Due to (2), (4), (18), (20), (22) system (23) will be rewritten in the form (24).

Hereinafter $B = \frac{2\sqrt{3}k^{(1)}}{3} \left( \frac{2\mu^{(1)}}{2\mu^{(1)}+c^{(1)}} \left[ 3 \ln \frac{a}{r} + \frac{r^3}{a^3} - 1 \right] + 1 - \frac{\gamma^3}{a^3} \right)$. By solving the system (24), we obtain a transcendental algebraic equation for finding the radius $\gamma$ of the elastic-plastic boundary of the inner layer (25).

The integration constants are defined in the form (26).
where \( H = \left( \frac{1}{6G(b)} + \frac{2}{9K^{(2)}} \right) M(b) - \frac{2I(d)}{3K^{(2)}} \).

Thus, the stress-strain state of the double-layer spherical construction taking into account the porous structure of the inner layer and heterogeneity of the outer layer is determined in accordance with condition (17). If (17) is met, the SSS is defined by relations (2)–(4), (11)–(13), (15), otherwise, if the (10) is met, the SSS is defined by relations (2)–(4), (18)–(22), (25), (26), wherein the constants \( C_1, C_2 \) from (2)–(4) should be changed to the constants \( C_3, C_4 \) from (26).

5. The results of a numerical experiment

In doing the numerical experiment all relations defining the SSS are written in the non-dimensional form. All values of the length dimension are referred to the outer radius \( d \). All values of the stress dimension are referred to the parameter \( \mu_0 \).

The function defining the heterogeneity of the outer layer material was set in the form

\[
G^{(2)}(r) = G_0^{(2)} \left( 1 + e^{-\omega r} \right), \quad \omega > 0.
\] (27)

The results of the numerical experiment are presented in figures 3–7. Unless otherwise specified, the values of physical, mechanical and geometric parameters are as follows: \( q_0 = 0.011, \quad d = 6.667 \cdot 10^{-4}, \quad \mu_0^{(1)} = 1, \quad \mu_0^{(3)} = 3.667, \quad \lambda_0^{(1)} = 2.667, \quad G_0^{(2)} = 1.733, \quad c = 6.667 \cdot 10^{-3}, \quad k = 1.333 \cdot 10^{-3}, \quad K^{(2)} = 3.333, \quad a = 0.3, \quad b = 0.4, \quad d = 1, \quad w = 1, \quad \varepsilon_0 = 0.065.\)

![Figure 3](image)

**Figure 3.** Graph of radius \( \gamma \) of the elastic-plastic boundary of the inner layer against value of the initial pore size \( \varepsilon_0 \) at varying values of the volume compression modulus \( K^{(2)} \) of the outer layer (curve 1: \( K^{(2)} = 3.1 \); curve 2: \( K^{(2)} = 3.3 \); curve 3: \( K^{(2)} = 3.5 \)).
Figure 4. Graph of radius $\gamma$ of the elastic-plastic boundary of the inner layer against value of the initial pore size $\varepsilon_0$ at varying values of the Lamé parameter $\lambda^{(1)}_1$ of the inner layer (curve 1: $\lambda^{(1)}_1 = 1.5$; curve 2: $\lambda^{(1)}_1 = 2$; curve 3: $\lambda^{(1)}_1 = 3$).

Figure 5. Graph of radius $\gamma$ of the elastic-plastic boundary of the inner layer against value of the initial pore size $\varepsilon_0$ at varying values of the intensity $q_a$ on the inner contour (curve 1: $q_a = 0.0175$; curve 2: $q_a = 0.011$; curve 3: $q_a = 0.01125$).
Figure 6. Graph of radial component of stresses $\sigma_r$ against radius $r$ at varying values of the Lamé parameter $\lambda_1^{(i)}$ of the inner layer (curve 1: $\lambda_1^{(i)} = 2$; curve 2: $\lambda_1^{(i)} = 3$; curve 3: $\lambda_1^{(i)} = 4$).

Figure 7. Graph of radial component of stresses $\sigma_r$ against radius $r$ at varying values of the fluidity limit $k^{(i)}$ of the inner layer material with a fully compressed skeleton (curve 1: $k^{(i)} = 0.001$; curve 2: $k^{(i)} = 0.0015$; curve 3: $k^{(i)} = 0.002$).
6. Conclusions

The results are summarized as follows.

In case of non-elastic deforming of the inner layer material:
- the distribution of the radial stress component has a piecewise monotonous character; for the plastic zone: compressive radial stresses increase when the Lamé parameter $\lambda_1^{(1)}$ and the fluidity limit $k^{(1)}$ become large; for the elastic zone and the outer layer area: compressive radial stresses decrease when the Lamé parameter $\lambda_1^{(1)}$ and the fluidity limit $k^{(1)}$ become large;
- the value of the volume compression modulus $K^{(2)}$, decrease of the intensity $q_0$, and increase of the value $\varepsilon_0$ lead to enlarging the region of the compressed matrix inelastic deformations;
- the value of elastic-plastic interface $\gamma$ of the inner layer increases when the Lamé parameter $\lambda_1^{(1)}$ becomes large.

It should be noted that, as follows from (2), the radial stress component for the outer layer area will retain a monotonous character no matter what the type of heterogeneity of the material of this layer, that is, the function $G^{(2)} = G^{(2)}(r)$.

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