KAM FOR KG ON $S^2$ AND FOR THE QUANTUM HARMONIC OSCILLATOR ON $\mathbb{R}^2$.

by

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Abstract. — In this paper we prove an abstract KAM theorem adapted to the Klein Gordon equation on the sphere $S^2$ and for the quantum harmonic oscillator on $\mathbb{R}^2$ with regularizing nonlinearity.

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1. Introduction.

If the KAM theorem is now well documented for nonlinear Hamiltonian PDEs in 1-dimensional context (see [18, 19, 21]) only few results exist for multidimensional PDEs.

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Existence of quasi-periodic solutions of space-multidimensional PDE were first proved in [6] (see also [7]) but with a technic based on the Nash-Moser theorem that do not allow to analyse the linear stability of the obtained solutions. Some KAM-theorems for small-amplitude solutions of multidimensional beam equations (see (3.6) above) with typical $m$ were obtained in [12, 13]. Both works treat equations with a constant-coefficient nonlinearity $g(x, u) = g(u)$, which is significantly easier than the general case. The first complete KAM theorem for space-multidimensional PDE was obtained in [11]. Also see [2, 3].

The technics developed by Eliasson-Kuksin has been improved in [8, 9] to allow a KAM result without external parameters. In these two papers the authors prove the existence of small amplitude quasi-periodic solutions of the beam equation on the d-dimensional torus. They further investigate the stability of these solutions and give explicit examples where the solution is linearly unstable and thus exhibits hyperbolic features (a sort of whiskered torus).

All these examples concern PDEs on the torus, essentially because in that case the corresponding linear PDE is diagonalized in the Fourier basis and the structure of the resonant sets is almost the same for NLS, NLW or beam equation. In the present paper, adapting the technics in [11], we consider two important examples that do not fit in the Fourier context: the Klein-Gordon equation on the sphere $S^2$ and the quantum harmonic oscillator on $\mathbb{R}^2$.

Notice that existence of quasi-periodic solutions for NLW and NLS on compact Lie groups via Nash Moser technics (and thus without linear stability) has been proved recently in [5, 4].

To understand the new difficulties, let us begin by recalling briefly part of the method developed in [11]. Consider the non linear Schrödinger equation on $\mathbb{T}^d$

$$iu_t = -\Delta u + \text{nonlinear terms}, \quad x \in \mathbb{T}^d, \quad t \in \mathbb{R}.$$  

In Fourier variables it reads

$$i \hat{u}_k = |k|^2 \hat{u}_k + \text{nonlinear terms}, \quad k \in \mathbb{Z}^d.$$  

So two Fourier modes indexed by $k, j \in \mathbb{Z}^d$ are (linearly) resonant when $|k|^2 = |j|^2$. For the beam equation on the torus the relation of resonance is the same. The resonant sets $\mathcal{E}_k = \{j \in \mathbb{Z}^d \mid |j|^2 = |k|^2\}$ realize a natural clustering of $\mathbb{Z}^d$.

All the modes in the block $\mathcal{E}_k$ have the same energy and we can expect that the interactions are small between different blocks but could be of order one inside a block. With this idea in mind, the principal step of the KAM theory, the resolution of the so called homological equation, leads to the inversion of an infinite matrix which is block-diagonal with respect to this clustering. It

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1. The space $\mathbb{Z}^d$ is equipped with standard euclidian norm: $|k|^2 = k_1^2 + \cdots + k_d^2$. 

turns out that these blocks have cardinality growing with $|k|$ making harder the control of the inverse of this matrix. As a consequence you loose regularity each time you solve the homological equation. Of course this is not acceptable for an infinite induction. The very nice idea in \[11\] consists in considering a sub-clustering constructed as the equivalence classes of the equivalence relation on $\mathbb{Z}^d$ generated by the pre-equivalence relation $a \sim b \iff \begin{cases} |a| = |b| \\ |a - b| \leq \Delta \end{cases}$

Let $[a]_\Delta$ denote the equivalence class of $a$. The crucial fact (proved \[11\]) is that the blocks are finite with a maximal “diameter”

$$\max_{[a]_\Delta = [b]_\Delta} |a - b| \leq C_d \Delta \frac{(d+1)!}{2}$$

depending only on $\Delta$. With such clustering, you do not loose regularity when solving the homological equation. Further, working in a phase space of analytic functions $u$ or equivalently, exponentially decreasing Fourier coefficients $u_k$, it turns out that the homological equation is “almost” block diagonal relatively to this clustering. Then you growth the parameter $\Delta$ at each step of the KAM iteration.

Unfortunately this estimate of the diameter of a block $[a]_\Delta$ by a constant independent of $|a|$ is a sort of miracle that do not persist in other cases. For instance if we consider the quantum harmonic oscillator on $\mathbb{R}^2$

$$iu_t = -\Delta u + |x|^2 u + \text{nonlinear terms}, \quad x \in \mathbb{R}^2$$

the linear part diagonalizes on a Hermite basis $h_j \otimes h_k$ (see section 3) and the natural clustering is given by the resonant sets $\{(k,j) \in \mathbb{N}^2 \mid k + j = \text{const}\}$. We can easily convince ourself that there is no simple way to construct sub-clustering, compatible with the equation, in such a way the size of the block does no more depend on the energy.

So we have to invent a new way to proceed. First we consider a phase space $Y_s$ with polynomial decay on the Fourier coefficient (corresponding to Sobolev regularity for $u$) instead of exponential decay and we use a different norm on the finite matrix, namely the Hilbert-Schmidt norm. This technical changes makes disappear the loss of regularity in the resolution of the homological equation. Nevertheless this is not the end of the story since this Sobolev structure of the phase space $T^{s,\beta}$ (see section 2) is not stable by Poisson bracket and thus is not adapted to an iterative scheme. So the second ingredient consists in using a trick previously used in \[14\]: we take advantage of the regularizing effect of the homological equation to obtain a solution in a slightly more regular space $T^{s,\beta+}$ and then we verify that $\{T^{s,\beta}, T^{s,\beta+}\} \in T^{s,\beta}$ (see section 4) which makes possible an iterative procedure. The last problem is
to verify that the non linear term, says $P$, belongs to the class $T^{s,\beta}$ which imposes a decreasing conditions on the Hilbert-Schmidt norm of the blocks of the Hessian of $P$. Unfortunately, this condition leads to a restriction to the dimension 2 for the Klein Gordon equation on the sphere and impose to consider only regularizing non linearity in the case of the quantum harmonic operator on $\mathbb{R}^2$.

In this paper we only consider PDEs with external parameters (similar to a convolution potential in the case of NLS on the torus). Following [9] we could expect to remove these external parameters (and to use only internal parameters) but the technical cost would be very high.

We now state the result that we obtain for the Klein Gordon equation. Denote by $\Delta$ the Laplace-Beltrami operator on the sphere $S^2$ and let $\Lambda_0 = (-\Delta + m)^{1/2}$. The spectrum of $\Lambda_0$ equals $\{\sqrt{j(j+1)} + m \mid j \geq 0\}$. For each $j \geq 1$ let $E_j$ be the associated eigenspace, its dimension is $2j + 1$. We denote by $\Psi_{j,l}$ the standard harmonic function of degree $j$ and order $\ell$ so that we have $E_j = \text{Span}\{\Psi_{j,l}, l = -j, \cdots, j\}$.

We denote $E := \{(j, \ell) \in \mathbb{N} \times \mathbb{Z} \mid j \geq 0 \text{ and } \ell = -j, \cdots, j\}$ in such a way that $\{\Psi_a, a \in E\}$ is a basis of $L^2(S^2, \mathbb{C})$.

We introduce the harmonic multiplier $M_\rho$ defined on the basis $(\Psi_a)_{a \in E}$ of $L^2(S^2)$ by

$$M_\rho \Psi_a = \rho_a \Psi_a \quad \text{for } a \in E$$

where $(\rho_a)_{a \in E}$ is a bounded sequence of nonnegative real numbers.

Let $g$ be a real analytic function on $S^2 \times \mathbb{R}$ such that $g$ vanishes at least at order 2 in the second variable at the origin. We consider the following nonlinear Klein-Gordon equation

$$(\partial_t^2 - \Delta + m + \delta M_\rho)u = \varepsilon g(x, u), \quad t \in \mathbb{R}, \quad x \in S^2$$

where $\delta > 0$ and $\varepsilon > 0$ are small parameters.

Introducing $\Lambda = (-\Delta + m + \delta M_\rho)^{1/2}$ and $v = u_t \equiv \dot{u}$, (1.2) reads

$$\begin{cases} 
\dot{u} = -v, \\
\dot{v} = \Lambda^2 u + \varepsilon g(x, u).
\end{cases}$$

Defining $\psi = \frac{1}{\sqrt{2}}(\Lambda^{1/2} u + i\Lambda^{-1/2} v)$ we get

$$\frac{1}{i} \dot{\psi} = \Lambda \psi + \frac{1}{\sqrt{2}} \Lambda^{-1/2} g \left( x, \Lambda^{-1/2} \left( \psi + \bar{\psi} \right) \right).$$

Thus, if we endow the space $L^2(S^2, \mathbb{C})$ with the standard real symplectic structure given by the two-form $-id\psi \wedge d\bar{\psi} = -du \wedge dv$, then equation (1.2) becomes...
a Hamiltonian system
\[ \dot{\psi} = i \frac{\partial H}{\partial \bar{\psi}} \]
with the hamiltonian function
\[ H(\psi, \bar{\psi}) = \int_{S^2} (\Lambda \psi) \bar{\psi} \, dx + \varepsilon \int_{S^2} G \left( x, \Lambda^{-1/2} \left( \frac{\psi + \bar{\psi}}{\sqrt{2}} \right) \right) \, dx. \]
where \( G \) is a primitive of \( g \) with respect to the variable \( u \): \( g = \partial_u G \).

The linear operator \( \Lambda \) is diagonal in the basis \( \{ \Psi_a, a \in \mathcal{E} \} \):
\[ \Lambda \Psi_a = \lambda_a \Psi_a, \quad \lambda_a = \sqrt{w_a (w_a + 1) + m + \delta \rho_a}, \quad \forall a \in \mathcal{E} \]
where we set
\[ w(j, \ell) = j \quad \forall (j, \ell) \in \mathcal{E}. \]

Let \( A \subset \mathcal{E} \) a finite subset of cardinal \( n \) satisfying the admissibility condition
\[ A \ni (j_1, \ell_1) \neq (j_2, \ell_2) \in A \Rightarrow j_1 \neq j_2. \]

We fix \( I_a \in [1, 2] \) for \( a \in A \), the initial \( n \) actions, and we write the modes \( A \) in action-angle variables:
\[ \xi_a = \sqrt{I_a + r_a e^{i \theta_a}}, \quad \eta_a = \sqrt{I_a + r_a e^{-i \theta_a}}. \]
We define \( \mathcal{L} = \mathcal{E} \setminus \mathcal{A} \) and, to simplify the presentation, we assume that
\[ \rho_{j, \ell} = \rho_j \text{ for } (j, \ell) \in \mathcal{A}; \quad m_{j, \ell} = 0 \text{ for } (j, \ell) \in \mathcal{L}. \]
Set
\[ w_{j,\ell} = j \quad \text{for } (j, \ell) \in \mathcal{E}, \]
\[ \lambda_{j,\ell} = \sqrt{j(j+1) + m} \quad \text{for } (j, \ell) \in \mathcal{L}, \]
\[ (\omega_0)_{j,\ell}(\rho) = \sqrt{j(j+1) + m + \delta \rho_j} \quad \text{for } (j, \ell) \in \mathcal{A}, \]
\[ \zeta = (\xi_a, \eta_a)_{a \in L}. \]
With this notation \( H \) reads (up to a constant)
\[ H(r, \theta, \zeta) = \langle \omega_0(\rho), r \rangle + \sum_{a \in L} \lambda_a \xi_a \eta_a + \varepsilon f(r, \theta, \zeta) \]
where
\[ f(r, \theta, \zeta) = \int_{S^2} G(x, \hat{u}(r, \theta, \zeta)(x)) \, dx \]
and
\[ \hat{u}(r, \theta, \zeta)(x) = \sum_{a \in A} \sqrt{I_a + r_a \cos \theta_a} \Psi_a(x) + \sum_{a \in L} \frac{(\xi_a + \eta_a)}{\sqrt{2}} \lambda_a^{1/2} \Psi_a(x). \]
Let us set \( u_1(\theta, x) = \hat{u}(0, \theta; 0)(x) \). Then for any \( I \in [1, 2]^n \) and \( \theta_0 \in \mathbb{T}^n \)
the function \( (t, x) \mapsto u_1(\theta_0 + t\omega, x) \) is a quasi periodic solution of (1.2) with \( \varepsilon = 0 \). Our main theorem states that for most external parameter \( \rho \) this quasi-periodic solution persists (but is sightly deformed) when we turn on the nonlinearity.

**Theorem 1.1.** — For \( \varepsilon \) sufficiently small (depending on \( n, s \) and \( g \)) and satisfying\(^2\)
\[ \varepsilon \leq \left( \frac{\delta}{4 \max(w_a, a \in A)} \right)^{12} \]
there exists a Borel subset
\[ \mathcal{D}' \subset [1, 2]^n, \quad \text{meas}([1, 2]^n \setminus \mathcal{D}') \leq C \varepsilon^{\alpha}, \]
such that for \( \rho \in \mathcal{D}' \), there is a function \( u(\theta, x) \), analytic in \( \theta \in \mathbb{T}_2^n \) and smooth
in \( x \in S^2 \), satisfying
\[ \sup_{|\omega| < \frac{\delta}{2}} \| u(\theta, \cdot) - u_1(\theta, \cdot) \|_{H^s(S^2)} \leq \varepsilon^{1/6}, \]
and there is a mapping
\[ \omega' : \mathcal{D}' \to \mathbb{R}^n, \quad \| \omega' - \omega \|_{C^1(\mathcal{D}')} \leq \varepsilon^{1/6}, \]
\[ 2. \quad \text{The coefficient } 12 \text{ is of course non optimal and we note in remark } 3.2 \text{ that when } m = 0 \]
12 can be replace by 4.
such that for any $\rho \in D'$ the function
\[ u(t, x) = u(\theta + t\omega'(\rho), x) \]

is a solution of the Klein Gordon equation (1.2). Furthermore this solution is linearly stable.

The positive constant $\alpha$ depends only on $n$ while $C$ also depends on $g$ and $s$.

We will deduce Theorem [11] from an abstract KAM result stated in section 2 and proved in section 6. The application to the quantum harmonic oscillator is detailed in section 3.2.

In section 4 we study the Hamiltonian flows generated by Hamiltonian functions in $T^{s, \beta}$. In section 5 we detail the resolution of the homological equation. In both sections 4 and 5 we use technics and proofs that were developed in [11] and [8]. The novelty is the use of Hilbert-Schmidt norm on the matrix and the use of two different class of Hamiltonians: $T^{s, \beta}$ and $T^{s, \beta^+}$. For convenience of the reader we repeat all the arguments.

### 2. Setting and abstract KAM theorem.

**Notations.** In this section we state a KAM result for a Hamiltonian $H = h + \varepsilon f$ of the following form
\[ H = \langle \omega(\rho), r \rangle + \frac{1}{2} \langle \zeta, A(\rho)\zeta \rangle + f(r, \theta, \zeta; \rho) \]
where
- $\omega \in \mathbb{R}^n$ is the frequencies vector corresponding to the internal modes in action-angle variables $(r, \theta) \in \mathbb{R}^n_+ \times \mathbb{T}^n$.
- $\zeta = (\zeta_s)_{s \in \mathcal{L}}$ are the external modes: $\mathcal{L}$ is an infinite set of indices, $\zeta_s = (p_s, q_s) \in \mathbb{R}^2$ and $\mathbb{R}^2$ is endowed with the standard symplectic structure $dq \wedge dp$.
- $A$ is a linear operator acting on the external modes, typically $A$ is diagonal.
- $f$ is a perturbative Hamiltonian depending on all the modes and is of order $\varepsilon$ where $\varepsilon$ is a small parameter.
- $\rho$ is an external parameter in $D$ a compact subset of $\mathbb{R}^p$ with $p \geq n$.

We now detail the structure beyond these objects and the hypothesis needed for the KAM result.

**Cluster structure on $\mathcal{L}$.** Let $\mathcal{L}$ be a set of indices and $w : \mathcal{L} \to \mathbb{N}$ be an "energy" function\(^3\) on $\mathcal{L}$. We consider the clustering of $\mathcal{L}$ given by $\mathcal{L} =$

\[^{3}\text{We could replace the assumption that $w$ takes integer values by } \{ w_a - w_b \mid a, b \in \mathcal{L} \} \text{ accumulates on a discrete set.}\]
\[ \cup_{a \in \mathcal{L}} [a] \text{ associated to equivalence relation} \]

\[ b \sim a \iff w_a = w_b. \]

We denote \( \tilde{\mathcal{L}} = \mathcal{L}/ \sim \). We assume that the cardinal of each energy level is finite and that there exist \( C > 0 \) and \( d > 0 \) two constants such that the cardinality of \([a]\) is controlled by \( Cw_a^d\):

\[
(2.1) \quad d_a = d_{[a]} = \text{card}\{b \in \mathcal{L} \mid w_b = w_a\} \leq Cw_a^d.
\]

**Linear space.** Let \( s \geq 0 \), we consider the complex weighted \( \ell_2 \)-space

\[
Y_s = \{ \zeta = (\zeta_a \in \mathbb{C}^2, a \in \mathcal{L}) \mid \| \zeta \|_s < \infty \}
\]

where

\[
\| \zeta \|_s^2 = \sum_{a \in \mathcal{L}} |\zeta_a|^2 w_a^{2s}.
\]

We also introduce for \( 1 \geq \beta \geq 0 \) the complex weighted \( \ell_{\infty} \)-space

\[
L_{\beta} = \{ \zeta = (\zeta_a \in \mathbb{C}^2, a \in \mathcal{L}) \mid |\zeta|_\beta < \infty \}
\]

where

\[
|\zeta|_\beta = \sup_{a \in \mathcal{L}} |\zeta_a|^\beta w_a^{\beta}, \quad |\zeta_a|^2 = \sum_{b \in [a]} |\zeta_b|^2.
\]

We note that if \( s \geq \beta \) then \( Y_s \subset L_{\beta} \).

In the spaces \( Y_s \) acts the linear operator \( J \),

\[
J : \{ \zeta_a \} \mapsto \{ \sigma_2 \zeta_a \}, \quad \text{with} \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

It provides the spaces \( Y_s, s \geq 0 \), with the symplectic structure \( Jd\zeta \wedge d\zeta \). To any \( C^1 \)-smooth function defined on a domain \( \mathcal{O} \subset Y_s \), corresponds the Hamiltonian equation

\[
\dot{\zeta} = J \nabla f(\zeta),
\]

where \( \nabla f \) is the gradient with respect to the scalar product in \( Y \).

**Infinite matrices.** We denote by \( \mathcal{M} \) the set of infinite matrix \( A : \mathcal{L} \times \mathcal{L} \to \mathcal{M}_{2 \times 2} \) with value in the space of real \( 2 \times 2 \) matrices that are symmetric

\[
A_{s}^{s'} = A_{s'}^{s}, \quad \forall s, s' \in \mathcal{L}
\]

and satisfy

\[
|A| := \sup_{a,b \in \mathcal{L}} \| A[b]^{[a]} \|_{HS} < \infty
\]

4. We provide \( \mathbb{C}^2 \) with the euclidian norm, \( |\zeta_a| = |(p_a, q_a)| = \sqrt{|p_a|^2 + |q_a|^2} \).

5. The constraint \( \beta \leq 1 \) is technically convenient but not necessary.
where $A^{[b]}_{[a]}$ denotes the restriction of $A$ to the block $[a] \times [b]$ and $\| \cdot \|_{HS}$ denotes the Hilbert Schmidt norm:

$$\| M \|^2_{HS} := \sum_{j, \ell} |M_{j\ell}|^2.$$  

The for $\beta \geq 0$ we define $\mathcal{M}_\beta$ the subset of $\mathcal{M}$ such that

$$|A|_\beta := \sup_{a, b \in \mathcal{L}} w^a_b w^b_a \|A^{[b]}_{[a]}\|_{HS} < \infty.$$  

**A class of Hamiltonian functions.** Let us fix any $n \in \mathbb{N}$. On the space $\mathbb{C}^n \times \mathbb{C}^n \times Y_s$ we define the norm

$$\|(z, r, \zeta)\|_s = \max(|z|, |r|, \|\zeta\|_s).$$

For $\sigma > 0$ we denote

$$T^n_\sigma = \{ z \in \mathbb{C}^n : |\Im z| < \sigma \}/2\pi \mathbb{Z}^n.$$  

For $\sigma, \mu \in (0, 1]$ and $s \geq 0$ we set

$$\mathcal{O}^s(\sigma, \mu) = T^n_\sigma \times \{ r \in \mathbb{C}^n : |r| < \mu^2 \} \times \{ \zeta \in Y_s : \|\zeta\|_s < \mu \}.$$  

We will denote points in $\mathcal{O}^s(\sigma, \mu)$ as $x = (\theta, r, \zeta)$. A function defined on a domain $\mathcal{O}^s(\sigma, \mu)$, is called real if it gives real values to real arguments.

Let $\mathcal{D} = \{ \rho \} \subset \mathbb{R}^p$ be a compact set of positive Lebesgue measure. This is the set of parameters upon which will depend our objects. Differentiability of functions on $\mathcal{D}$ is understood in the sense of Whitney. So $f \in C^1(\mathcal{D})$ if it may be extended to a $C^1$-smooth function $\tilde{f}$ on $\mathbb{R}^p$, and $|f|_{C^1(\mathcal{D})}$ is the infimum of $|\tilde{f}|_{C^1(\mathbb{R}^p)}$, taken over all $C^1$-extensions $\tilde{f}$ of $f$.

If $(z, r, \zeta)$ are $C^1$ functions on $\mathcal{D}$, then we define

$$\|(z, r, \zeta)\|_{s, \mathcal{D}} = \max_{j=0,1}(|\partial^j_r z|, |\partial^j_r r|, \|\partial^j_r \zeta\|_s).$$

Let $f : \mathcal{O}^0(\sigma, \mu) \times \mathcal{D} \to \mathbb{C}$ be a $C^1$-function, real holomorphic in the first variable $x$, such that for all $\rho \in \mathcal{D}$

$$\mathcal{O}^s(\sigma, \mu) \ni x \mapsto \nabla_\xi f(x, \rho) \in Y_s \cap L_\beta$$  

and

$$\mathcal{O}^s(\sigma, \mu) \ni x \mapsto \nabla_\xi^2 f(x, \rho) \in \mathcal{M}_\beta$$.
are real holomorphic functions. We denote this set of functions by $T^{s,\beta}(\sigma, \mu, D)$.

For a function $f \in T^{s,\beta}(\sigma, \mu, D)$ we define the norm

$$[f]^{s,\beta}_{\sigma, \mu, D}$$

through

$$\sup \max(|\partial^j_\rho f(x, \rho)|, \mu|\partial^j_\rho \nabla_\zeta f(x, \rho)|_s, \mu|\partial^j_\rho \nabla_\zeta f(x, \rho)|_\beta, \mu^2|\partial^j_\rho \nabla^2_\zeta f(x, \rho)|_\beta),$$

where the supremum is taken over all $j = 0, 1$, $x \in O^\gamma(\sigma, \mu)$, $\rho \in D$.

We set $T^s(\sigma, \mu, D) = T^{s,0}(\sigma, \mu, D)$ and $[h]^s_{\sigma, \mu, D} = [h]^{s,0}_{\sigma, \mu, D}$.

**Normal form:** We introduce the orthogonal projection $\Pi$ defined on the $2 \times 2$ complex matrices $\Pi : M_{2 \times 2}(C) \to C I + C J$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

**Definition 2.1.** — A matrix $A : L \times L \to M_{2 \times 2}(C)$ is on normal form and we denote $A \in NF$ if

(i) $A$ is real valued,

(ii) $A$ is symmetric, i.e. $A^t_a = A^b_a$,

(iii) $A$ satisfies $\Pi A = A$,

(iv) $A$ is block diagonal, i.e. $A^a_b = 0$ for all $a \neq b$.

To a real symmetric matrix $A = (A^a_b) \in \mathcal{M}$ we associate in a unique way a real quadratic form on $Y_s \ni (\zeta_a)_{a \in L} = (p_a, q_a)_{a \in L}$

$$q(\zeta) = \frac{1}{2} \sum_{a,b \in L} \langle \zeta_a, A^b_a \zeta_b \rangle.$$

In the complex variables, $z_a = (\xi_a, \eta_a)$, $a \in L$, where

$$\xi_a = \frac{1}{\sqrt{2}}(p_a + iq_a), \quad \eta_a = \frac{1}{\sqrt{2}}(p_a - iq_a),$$

we have

$$q(\zeta) = \frac{1}{2} \langle \xi, \nabla_\xi q \xi \rangle + \frac{1}{2} \langle \eta, \nabla_\eta q \eta \rangle + \langle \xi, \nabla_\xi \nabla_\eta q \eta \rangle.$$

The matrices $\nabla_\xi q$ and $\nabla_\eta q$ are symmetric and complex conjugate of each other while $\nabla_\xi \nabla_\eta q$ is Hermitian. If $A \in \mathcal{M}_\beta$ then

$$\sup_{a,b} ||(\nabla_\xi \nabla_\eta q)^{[b]}_{[a]}||_{HS} \leq \frac{|A|_\beta}{(w_a w_b)^{\beta}}.$$
We note that if $A$ is on normal form, then the associated quadratical form $q(\zeta) = \frac{1}{2} \langle \zeta, A\zeta \rangle$ reads in complex variables

$$q(\zeta) = \langle \xi, Q\eta \rangle$$

where $Q : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ is

(i) Hermitian, i.e. $Q^*_a = Q^b_a$,
(ii) Block-diagonal.

In other words, when $A$ is on normal form, the associated quadratic form reads

$$q(\zeta) = \frac{1}{2} \langle p, A_1 p \rangle + \langle p, A_2 q \rangle + \frac{1}{2} \langle p, A_1 q \rangle$$

with $Q = A_1 + iA_2$ Hermitian.

By extension we will say that a Hamiltonian is on normal form if it reads

$$h = \langle \omega(\rho), r \rangle + \frac{1}{2} \langle \zeta, A(\rho)\zeta \rangle$$

with $\omega(\rho) \in \mathbb{R}^n$ a frequency vector and $A(\rho)$ on normal form for all $\rho$.

2.1. Hypothesis on the spectrum of $A_0$.— We assume that $A_0(\rho)$ a real diagonal matrix whose diagonal elements $\lambda_a(\rho) > 0$, $a \in \mathcal{L}$ are $C^1$. Our hypothesis depend on two constants $1 > \delta_0 > 0$ and $c_0 > 0$ fixed once for all.

**Hypothesis A1 – Asymptotics.** We assume that there exist $\gamma \geq 1$ such that

$$\lambda_a(\rho) \geq c_0 w_a^\gamma$$

for $\rho \in \mathcal{D}$ and $a \in \mathcal{L}$

and

$$|\lambda_a(\rho) - \lambda_b(\rho)| \geq c_0 |w_a - w_b|$$

for $a, b \in \mathcal{L}$ and for $\rho \in \mathcal{D}$

**Hypothesis A2 – non resonances.** There exists a $\delta_0 > 0$ such that for all $C^1$-functions

$$\omega : \mathcal{D}_0 \to \mathbb{R}^n, \quad |\omega - \omega_0|_{C^1(\mathcal{D}_0)} < \delta_0,$$

the following hold for each $k \in \mathbb{Z}^n \setminus 0$:

(i) either

$$|\langle k, \omega(\rho) \rangle| \geq \delta_0$$

for all $\rho \in \mathcal{D}_0$, or there exits a unit vector $\hat{z} \in \mathbb{R}^p$ such that

$$(\nabla_\rho \cdot \hat{z})(\langle k, \omega \rangle) \geq \delta_0$$

for all $\rho \in \mathcal{D}_0$;

(ii) either

$$|\langle k, \omega(\rho) \rangle + \lambda_a(\rho)| \geq \delta_0 w_a$$

for all $\rho \in \mathcal{D}_0$ and $a \in \mathcal{L}$ or there exits a unit vector $\hat{z} \in \mathbb{R}^p$ such that

$$(\nabla_\rho \cdot \hat{z})(\langle k, \omega(\rho) \rangle + \lambda_a(\rho)) \geq \delta_0$$
for all $\rho \in D_0$ and $a \in L$;

(iii) either

$$\langle k, \omega(\rho) \rangle + \lambda_a(\rho) + \lambda_b(\rho) \geq \delta_0(w_a + w_b)$$

for all $\rho \in D_0$ and $a, b \in L$ or there exits a unit vector $z \in \mathbb{R}^p$ such that

$$(\nabla_{\rho} \cdot z)(\langle k, \omega(\rho) \rangle + \lambda_a(\rho) + \lambda_b(\rho)) \geq \delta_0$$

for all $\rho \in D_0$ and $a, b \in L$;

(iv) either

$$|\langle k, \omega(\rho) \rangle + \lambda_a(\rho) - \lambda_b(\rho)| \geq \delta_0(1 + |w_a - w_b|)$$

for all $\rho \in D_0$ and $a, b \in L$ or there exits a unit vector $z \in \mathbb{R}^p$ such that

$$(\nabla_{\rho} \cdot z)(\langle k, \omega(\rho) \rangle + \lambda_a(\rho) - \lambda_b(\rho)) \geq \delta_0$$

for all $\rho \in D_0$ and $a, b \in L$.

The assumption (iv) above will be used to bound from below divisors $|\langle k, \omega(\rho) \rangle + \lambda_a(\rho) - \lambda_b(\rho)|$ with $w_a, w_b \sim 1$. To control the (infinitely many) divisors with $\max(w_a, w_b) \gg 1$ we need another assumption:

**Hypothesis A3 – second Melnikov condition in measure.** There exist absolute constant $\alpha_1 > 0$, $\alpha_2 > 0$ and $C > 0$ such that for all $C^1$-functions $\omega : D \to \mathbb{R}^n$, $|\omega - \omega_0|_{C^1(D)} < \delta_0$, the following holds:

for each $\kappa > 0$ and $N \geq 1$ there exists a closed subset $D' = D'((\omega_0, \kappa, N) \subset D$ satisfying

$$\text{meas}(D \setminus D') \leq CN^{\alpha_1} \left( \frac{\kappa}{\delta_0} \right)^{\alpha_2} \quad (\alpha_1, \alpha_2 \geq 0)$$

such that for all $\rho \in D'$, all $0 < |k| \leq N$ and all $a, b \in L$ we have

$$|\langle k, \omega(\rho) \rangle + \lambda_a(\rho) - \lambda_b(\rho)| \geq \kappa(1 + |w_a - w_b|).$$

2.2. The abstract KAM Theorem.— We are now in position to state our abstract KAM result.

**Theorem 2.2.** Assume that

$$h_0 = \langle \omega_0(\rho), r \rangle + \frac{1}{2} \langle \zeta, A_0(\rho)\zeta \rangle$$

with the spectrum of $A_0$ satisfying Hypothesis A1, A2, A3 and let $f \in T^{s, \beta}(D, \sigma, \mu)$ with $\beta > 0, s > 0$. There exists $\varepsilon_0$ (depending on $n, d, s, \beta, \sigma, \mu$ and on $h_0$) such that if

$$[f]_{s, \mu, D} = \varepsilon < \min(\varepsilon_0, \delta_0^4)$$

6. Dependence on $h_0$ means: dependence on $A$, on $C^1$-norm of $\rho \mapsto \omega_0(\rho)$ and $\rho \mapsto A_0(\rho)$. 
there is a $D' \subset D$ with \( \text{meas}(D \setminus D') \leq \varepsilon^\alpha \) such that for all \( \rho \in D' \) the following hold: There are a real analytic symplectic diffeomorphism

$$
\Phi : \mathcal{O}^s(\sigma/2, \mu/2) \rightarrow \mathcal{O}^s(\sigma, \mu)
$$

and a vector $\omega = \omega(\rho)$ such that

$$(h_0 + f) \circ \Phi = \langle \omega(\rho), r \rangle + \frac{1}{2} \langle \zeta, A(\rho) \zeta \rangle + f(r, \zeta; \rho)$$

where $\partial_\zeta f = \partial_r f = \partial_{\zeta \zeta} f = 0$ for $\zeta = r = 0$ and $A : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{R})$ is on normal form, i.e. $A$ is real symmetric and block diagonal: $A_{ab} = 0$ for all $w_a \neq w_b$.

Moreover $\Phi$ satisfies

$$
\|\Phi - Id\|_s \leq \varepsilon^{1/6}
$$

for all $(r, \theta, \zeta) \in \mathcal{O}^s(\sigma/2, \mu/2)$, and

$$
|A(\rho) - A_0(\rho)|_\beta \leq \varepsilon^{1/6},
$$

$$
|\omega(\rho) - \omega_0(\rho)| \leq \varepsilon^{1/6}
$$

for all $\rho \in D'$.

The constant $\alpha$ only depends on $n$, $d$, $s$, $\beta$, $\alpha_1$, $\alpha_2$.

This normal form result has dynamical consequences. For $\rho \in D'$, the torus $\{0\} \times \mathbb{T}^n \times \{0\}$ is invariant by the flow of $(h_0 + f) \circ \Phi$ and the dynamics of the Hamiltonian vector field of $h_0 + f$ on the $\Phi(\{0\} \times \mathbb{T}^n \times \{0\})$ is the same as that of

$$
\langle \omega(\rho), r \rangle + \frac{1}{2} \langle \zeta, A(\rho) \zeta \rangle.
$$

The Hamiltonian vector field on the torus $\{\zeta = r = 0\}$ is

$$
\begin{cases}
\dot{\zeta} = 0 \\
\dot{\theta} = \omega \\
\dot{r} = 0
\end{cases}
$$

and the flow on the torus is linear: $t \mapsto \theta(t) = \theta_0 + t\omega$.

Moreover, the linearized equation on this torus reads

$$
\begin{cases}
\dot{\zeta} = JA\zeta + J\partial^2_{\zeta \zeta} f(0, \theta_0 + \omega t, 0) \cdot r \\
\dot{\theta} = \partial^2_{r \zeta} f(0, \theta_0 + \omega t, 0) \cdot \zeta + \partial^2_{rr} f(0, \theta_0 + \omega t, 0) \cdot r \\
\dot{r} = 0.
\end{cases}
$$

Since $A$ is on normal form (and in particular real symmetric and block diagonal) the eigenvalues of the $\zeta$-linear part are purely imaginary: $\pm i\lambda_a$, $a \in \mathcal{L}$. Therefore the invariant torus is linearly stable in the classical sense (all the eigenvalues of the linearized system are purely imaginary).
3. Applications

3.1. The Klein Gordon equation on $S^2$. — In this section we prove Theorem 1.1 as a corollary of Theorem 2.2. We recall some notations introduced in the introduction. We denote

$$E := \{(j, \ell) \in \mathbb{N} \times \mathbb{Z} \mid j \geq 0 \text{ and } \ell = -j, \ldots, j\}$$

and we set

$$w_{j, \ell} = j \text{ for } (j, \ell) \in E,$$

$$\lambda_{j, \ell} = \sqrt{j(j+1) + m} \text{ for } (j, \ell) \in \mathcal{L},$$

$$(\omega_0)_{j, \ell}(\rho) = \sqrt{j(j+1) + m + \delta \rho_j} \text{ for } (j, \ell) \in \mathcal{A},$$

$$\zeta = (\xi_a, \eta_a)_{a \in \mathcal{L}}.$$ 

With this notation the Klein Gordon Hamiltonian $H$ reads (up to a constant)

$$H(r, \theta, \zeta) = \langle \omega_0(\rho), r \rangle + \sum_{a \in \mathcal{L}} \lambda_a \xi_a \eta_a + \varepsilon f(r, \theta, \zeta)$$

where

$$f(r, \theta, \zeta) = \int_{S^2} G(x, \hat{u}(r, \theta, \zeta)(x)) \, dx.$$ 

Lemma 3.1. — Hypothesis A1, A2 and A3 hold true with $D = [1, 2]^n$ and

$$(3.1) \quad \delta_0 = \left(\frac{\delta}{4 \max(w_a, \ a \in \mathcal{A})}\right)^3.$$ 

Proof. — Hypothesis A1 is clearly satisfied with $c_0 = 1/2$ and $\gamma = 1$. On the other hand choosing $z \equiv z_k = \frac{k}{|k|}$, we have

$$(3.2) \quad (\nabla \cdot \hat{\zeta})(\langle k, \omega \rangle) \geq \frac{\delta}{2 \max(w_a, \ a \in \mathcal{A})}|k| \quad \text{for all } k \neq 0$$

while

$$(3.3) \quad (\nabla \cdot \hat{\zeta})\lambda_a = 0 \quad \text{for all } a \in \mathcal{L}.$$ 

Then for all $k \neq 0$ the second part of the alternatives (i)–(iv) in Hypothesis A2 are satisfied choosing

$$\delta_0 \leq \delta_* := \frac{\delta}{4 \max(w_a, \ a \in \mathcal{A})}.$$ 

It remains to verify A3. Without loss of generality we can assume $w_a \leq w_b$. First denoting

$$F_\kappa(k, a, b) := \{\rho \in \mathcal{D} \mid \langle \omega, k \rangle + \lambda_a - \lambda_b \leq \kappa\},$$
we have using (3.2) that
\[ \text{meas } F_\nu(k, a, b) \leq C_{\nu} \frac{\kappa}{\delta_s}. \]

On the other hand, defining
\[ G_\nu(k, e) := \{ \rho \in \mathcal{D} \mid |\langle \omega, k \rangle + e| \leq 2\nu \}, \]
we have, using again (3.2) that
\[ \text{meas } G_\nu(k, e) \leq C_{\nu} \frac{\nu}{\delta_s}. \]

Further \(|\langle \omega, k \rangle + e| \leq 1\) can occur only if \(|e| \leq C|k|\) and thus
\[ G_\nu = \bigcup_{0 < |k| \leq N} G_\nu(k, e) \]
has a Lebesgue measure less than \(CN^{n+1} \frac{\nu}{\delta_s}\).

Now we remark that
\[ |j + \frac{1}{4} - \sqrt{j(j+1)} + m| \leq \frac{m+1}{2j} \]
from which we deduce
\[ |\lambda_a - \lambda_b - (w_a - w_b)| \leq \frac{m+1}{w_a}. \]

Therefore for \(\rho \in \mathcal{D} \setminus G_\nu\) and \(w_a \geq \frac{2}{\nu}\) we have for all \(0 < |k| \leq N\)
\[ |\langle \omega, k \rangle + \lambda_a - \lambda_b| \geq \nu. \]

Finally \(w_a \leq \frac{2}{\nu}\) and \(|\langle \omega, k \rangle + \lambda_a - \lambda_b| \leq 1\) leads to \(w_b \leq \frac{2}{\nu} + CN\) and thus, if we restrict \(\rho\) to
\[ \mathcal{D}' = \mathcal{D} \setminus \left[ G_\nu \cup \left( \bigcup_{0 < |k| \leq N} F_\nu(k, a, b) \right) \right] \]
we get
\[ |\langle \omega, k \rangle + \lambda_a - \lambda_b| \geq \min(\kappa, \nu), \quad 0 < |k| \leq N, \ a, b \in \mathcal{L}. \]

Further
\[ \text{meas } \mathcal{D} \setminus \mathcal{D}' \leq CN^{n+1} \frac{\nu}{\delta_s} + (\frac{2}{\nu} + CN)^2 N^{n+1} \frac{\kappa}{\delta_s}. \]

Then choosing \(\nu = \kappa^{1/3}\) and \(\delta_0 = \delta_s^3\), this measure is controlled by
\[ CN^{n+2} \left( \frac{\kappa}{\delta_0} \right)^{1/3} \]
and we have
\[ |\langle \omega, k \rangle + \lambda_a - \lambda_b| \geq \kappa, \quad \text{for } \rho \in \mathcal{D}', \ 0 < |k| \leq N \text{ and } a, b \in \mathcal{L}. \]
Now we remark that for $|\lambda_a - \lambda_b| \geq 2|\langle \omega, k \rangle|$, 
$$|\langle \omega, k \rangle + \lambda_a - \lambda_b| \geq \frac{1}{2}|\lambda_a - \lambda_b| \geq \frac{1}{4}(1 + |w_a - w_b|) \geq \kappa(1 + |w_a - w_b|)$$
if we assume $\kappa \leq \frac{1}{8}$.

On the other hand, when $|\lambda_a - \lambda_b| \leq 2|\langle \omega, k \rangle| \leq CN$, 
$$|\langle \omega, k \rangle + \lambda_a - \lambda_b| \geq \tilde{\kappa}(1 + |w_a - w_b|)$$
where $\tilde{\kappa} = \kappa + CN$. Thus we get 
$$|\langle \omega, k \rangle + \lambda_a - \lambda_b| \geq \tilde{\kappa}(1 + |w_a - w_b|), \text{ for } \rho \in \mathcal{D}', \ 0 < |k| \leq N \text{ and } a, b \in \mathcal{L}$$
with

$$\text{meas } \mathcal{D} \setminus \mathcal{D}' \leq CN^{n+3}(\frac{\tilde{\kappa}}{\delta_0})^{1/3}.$$ 

\[\Box\]

**Remark 3.2.** — When $m = 0$, Hypothesis A3 is verified for $\rho \in \mathcal{D}' = \mathcal{D} \setminus \Gamma_\gamma$ and thus we can choose $\delta_0 = \delta_\ast$.

**Lemma 3.3.** — Assume that $(x, u) \mapsto g(x, u)$ is real analytic on $S^2 \times \mathbb{R}$ and $s > 1$ then there exist $\sigma > 0$, $\mu > 0$ such that 
$$O^s(\sigma, \mu) \times \mathcal{D} \ni (r, \theta, \zeta; \rho) \mapsto f(r, \theta, \zeta; \rho)$$
belongs to $T^{s,1/4}(\mathcal{D}, \sigma, \mu)$.

**Proof.** — First we notice that $f$ does not depend on the parameter $\rho$. Due to the analyticity of $g$ and the fact that $s > 1$, there exist positive $\sigma$ and $\mu$ such that $f : O(\sigma, \mu) \times \mathcal{D} \to \mathbb{C}$ is a $C^1$-function, analytic in the first variables $(r, \theta, \zeta)$, whose gradient in $\zeta$ analytically maps $Y_s$ to itself (e.g., see in [11]). It remains to verify that $\nabla^2 f(r, \theta, \zeta; \rho) \in \mathcal{M}_{1/4}$.

We have

\begin{equation}
\frac{\partial^2 f}{\partial \xi_a \xi_b} = \frac{\partial^2 f}{\partial \eta_a \eta_b} = \frac{1}{2\lambda_a^{1/2} \lambda_b^{1/2}} \int_{S^2} \partial_a g(x, \hat{u}(x)) \Psi_a \Psi_b \, dx
\end{equation}

where $\hat{u}(x) \equiv \hat{u}(r, \theta, \zeta)(x)$ is given by (1.6). We note that for $s > 1$ and $(r, \theta, \zeta) \in O^s(\sigma, \mu)$, $x \mapsto \hat{u}(x)$ is bounded on $S^2$.

It remains to prove that the infinite matrix $M$ defined by 
$$M^b_a = \frac{1}{\lambda_a^{1/2} \lambda_b^{1/2}} \int_{S^2} \partial_a g(x, \hat{u}) \Psi_a \Psi_b \, dx$$
belongs to $\mathcal{M}_{1/4}$, i.e.

$$\sup_{a, b \in \mathcal{L}} w_a^{1/4} w_b^{1/4} \left\| M^b_a \right\|_{HS} < \infty.$$

---

7. $s > 1$ is needed to insure that $Y_s$ is an algebra.
Let us denote $\Pi_b$ the orthogonal projection in $L^2(S^2)$ on the eigenspace $E_b := \text{Span}\{\Psi_d \mid \omega_d = \omega_b\}$. We have

$$\left\| M_{[a]}^{[b]} \right\|_{HS}^2 = \sum_{c \in [a], d \in [b]} \frac{1}{\lambda_a \lambda_b} \left\| \int_{S^2} \partial_u g(x, \hat{u}) \Psi_c \Psi_d \, dx \right\|^2$$

$$= \frac{1}{\lambda_a \lambda_b} \sum_{c \in [a]} \left\| \Pi_b (\partial_u g(x, \hat{u}) \Psi_c) \right\|_{L^2(S^2)}^2$$

$$\leq \frac{1}{\lambda_a \lambda_b} \int_{S^2} \left| \partial_u g(x, \hat{u}) \right|^2 \left( \sum_{c \in [a]} \left| \Psi_c \right|^2 \right) \, dx$$

$$\leq \frac{C}{\lambda_b} \int_{S^2} \left| \partial_u g(x, \hat{u}) \right|^2 \, dx$$

where we used the Unslöd’s theorem:

$$\sum_{c \in [a]} \left| \Psi_c (x) \right|^2 = \frac{\text{card } E[a]}{4\pi} \leq C \lambda_a, \quad x \in S^2$$

where $C$ is an universal constant. Similarly we have

$$\left\| M_{[a]}^{[b]} \right\|_{HS}^2 \leq \frac{C}{\lambda_a} \int_{S^2} \left| \partial_u g(x, \hat{u}) \right|^2 \, dx$$

and thus for all $a, b \in E$

$$w_a^{1/4} w_b^{1/4} \left\| M_{[a]}^{[b]} \right\|_{HS} \leq C \left( \int_{S^2} \left| \partial_u g(x, \hat{u}) \right|^2 \, dx \right)^{1/2} \leq C'$$

for a constant $C'$ depending only on $g$.

So Main Theorem applies (for any choice of the vector $I \in [1, 2]^A$) and Theorem 1.1 is proved.

**Remark 3.4.** — We can also consider the Klein Gordon equation (1.2) on the higher dimensional sphere $S^3$ but in this case the same proof as in Lemma 3.3 will lead to $f \in T^{s, \beta}(d, \sigma, \mu)$ (since then $\text{card } E[[j, \ell]] = j^2$). So in order to apply our KAM theorem we would need to consider the regularized Klein Gordon equation ($\beta > 0$):

$$(3.5) \quad (\partial_t^2 - \Delta + m^2) u = \Lambda^{-\beta} \partial_2 g(x, \Lambda^{-\beta} u), \quad t \in \mathbb{R}, \quad x \in S^3.$$
Remark 3.5. — We can also consider the Beam equation on the torus $\mathbb{T}^d$ with convolution potential in a Sobolev like phase space.\(^9\)

\begin{equation}
-\Delta^2 u + \mu u + V * u + \varepsilon \partial_u G(x, u) = 0, \quad x \in \mathbb{T}^d.
\end{equation}

Here $m$ is the mass, $G$ is a real analytic function on $\mathbb{T}^d \times \mathbb{R}$ and at least of order 3 at the origin. The convolution potential $V : \mathbb{T}^d \to \mathbb{R}$ is supposed to be analytic with real positive Fourier coefficients $\hat{V}(a)$, $a \in \mathbb{Z}^d$. Actually following \cite{8} and the proof of Lemma 3.3, it remains to control the HS-norm of the infinite matrix\(^10\)

\begin{equation}
M_b^a = \frac{1}{\lambda_a^{1/2} \lambda_b^{1/2}} \int_{\mathbb{T}^d} \partial_u^2 G(x, u) \Psi_a \Psi_b \, dx
\end{equation}

restricted to the block defined by $[a] = \{ b \in \mathbb{Z}^d \mid |a| = |b| \}$. We have

\[
\left\| M_b^a \right\|_{HS}^2 \leq \frac{C}{\lambda_a \lambda_b} \sup_{x \in \mathbb{T}^d} \sum_{c \in [a]} |e^{ic \cdot x}|^2 \leq \frac{C |a|^{d-1}}{\lambda_b \lambda_a^{d-1}} \leq C \lambda_a^{d-3} \lambda_b
\]

which leads by symmetrization to

\[
\left\| M_b^a \right\|_{HS}^2 \leq \frac{C}{(\lambda_a \lambda_b)^{\frac{d-1}{2}}}
\]

and then $M \in \mathcal{M}_{d-1}$. Thus we can apply our theorem as soon as $d \leq 4$.

3.2. The regularized quantum harmonic operator in $\mathbb{R}^2$.— Let

\[ T = -\Delta + |x|^2 = -\Delta + x_1^2 + x_2^2 \]

be the 2-dimensional quantum harmonic oscillator. Its spectrum is the sum of 2-copies of the odd integers set, i.e. the spectrum of $T$ equals $2\mathbb{N} = \{2, 4, \cdots \}$. For $2j \in 2\mathbb{N}$ we denote the associated eigenspace $E_j$ whose dimension is

\[ \sharp \{(i_1, i_2) \in (2\mathbb{N} - 1)^2 \mid i_1 + i_2 = 2j\} = j. \]

We denote $\{ \Phi_j, l = 1, \cdots, j \}$, the basis of $E_j$ obtained by 2-tensor product of Hermite functions: $\Phi_{j,l} = \varphi_{i_1} \otimes \varphi_{i_2}$ with $i_1 = 2l - 1$ and $i_2 = 2j - (2l - 1)$. Then setting

\[ \mathcal{N}_2 := \{(j, l) \in \mathbb{N} \times \mathbb{N} \mid \ell = 1, \cdots, j\} \]

$(\Phi_a)_{a \in \mathcal{N}_2}$ is a basis of $L^2(\mathbb{R}^2)$.

The Hermite multiplier $M$ is defined on this basis by

\begin{equation}
M \Phi_a = \rho_a \Phi_a \quad \text{for} \quad a \in \mathcal{N}_2
\end{equation}

where $(\rho_a)_{a \in \mathcal{N}_2}$ is a bounded sequence of real numbers.

\(^9\) The same equation but in an analytic phase space were considered in \cite{8, 9}.

\(^{10}\) Here $\lambda_a = \sqrt{|a|^2 + m}$ and $\Psi_a(x) = e^{ia \cdot x}, \ a \in \mathbb{Z}^d$. 


In this subsection we consider the following nonlinear Schrödinger equation in $\mathbb{R}^2$
\[ i u_t = Tu + M \cdot u + T^{-\beta} \partial_1 F(x, T^{-\beta} u, T^{-\beta} \bar{u}), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^2 \]
where $M$ is a Hermite multiplier defined below and $F$ is a smooth function.

We focus on two choices of non-linearity:

1. **The regularized cubic NLS** which corresponds to the choice
   \[ F_{\text{NLS}}(x, u, \bar{u}) = \pm \frac{1}{4} |u|^4 \]
   which correspond in the non regularized case ($\beta = 0$) to cubic NLS
   \[ i u_t = -\Delta u + |x|^2 u + M \cdot u \pm |u|^2 u. \]

2. **The regularized Hartree equation** which corresponds to the choice
   \[ F_{\text{Hartree}}(x, u, \bar{u}) = \int_{\mathbb{R}^2} |u(x)|^2 |u(y)|^2 \varphi(x - y) dy \]
   where $\varphi$ is a smooth function. This case correspond in the non regularized case ($\beta = 0$) to the Hartree equation
   \[ i u_t = -\Delta u + |x|^2 u + M \cdot u + (\varphi * |u|^2) u. \]

Let
\[ \tilde{H}^s = \{ f \in H^s(\mathbb{R}^2, \mathbb{C}) | x \mapsto x^\alpha \partial^\beta f \in L^2(\mathbb{R}^2) \}
\]
for any $\alpha, \beta \in \mathbb{N}^2$ satisfying $0 \leq |\alpha| + |\beta| \leq s$.

where $H^s(\mathbb{R}^2, \mathbb{C})$ is the standard Sobolev space on $\mathbb{R}^2$. We note that, for any $s \geq 0$, the domain of $T^{s/2}$ is $\tilde{H}^s$ (see for instance [Hel84] Proposition 1.6.6) and that for $s > 1$, $\tilde{H}^s$ is an algebra.

In the phase space $\tilde{H}^s \times \tilde{H}^s$ endowed with the symplectic 2-form $idu \wedge d\bar{u}$ equation (3.8) reads as the Hamiltonian system associated to the Hamiltonian function
\[ H(u, \bar{u}) = \int_{\mathbb{R}^2} (|\nabla u|^2 + |x|^2 |u|^2 + F(x, T^{-\beta} u, T^{-\beta} \bar{u})) dx = H_0(u, \bar{u}) + P(u, \bar{u}). \]

In particular, for the regularized cubic NLS equation, the perturbation term reads
\[ P_{\text{NLS}} = \pm \frac{1}{4} \int_{\mathbb{R}^2} |T^{-\beta} u|^4 dx \]
while for the regularized Hartree equation we have
\[ P_{\text{Hartree}} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |T^{-\beta} u(x)|^2 |T^{-\beta} u(y)|^2 \varphi(x - y) dxdy. \]
Decomposing \( u \) and \( \bar{u} \) on the basis \( (\Phi_{j,l})_{(j,l) \in \mathbb{N}^2} \),
\[
u = \sum_{a \in \mathbb{N}^2} \xi_a \Phi_a, \quad \bar{\nu} = \sum_{a \in \mathbb{N}^2} \eta_a \Phi_a
\]
the phase space \((u, \bar{u}) \in \tilde{H}^s \times \tilde{H}^s\) becomes the phase space \((\xi, \eta) \in Y_s\)
\[Y_s = \{ \zeta = (\zeta_a \in \mathbb{C}^2, a \in \mathbb{N}^2) \mid \|\zeta\|_s < \infty \}
\]
where
\[\|\zeta\|_s^2 = \sum_{a \in \mathcal{L}} |\zeta_a|^2 w_a^{2s}
\]
and
\[w_{j,\ell} = j \quad \text{for } (j, \ell) \in \mathbb{N}^2.
\]
We endowed \( Y_s \) with the symplectic structure \( id\xi \wedge d\eta \).
Then the Hamiltonian reads
\[H(\xi, \eta) = \sum_{a \in \mathbb{N}^2} (w_a + \rho_a)\xi_a \eta_a + \int_{\mathbb{R}^2} F \left( x, \sum_{a \in \mathbb{N}^2} \frac{\xi_a}{w_a} \Phi_a, \sum_{a \in \mathbb{N}^2} \frac{\eta_a}{w_a} \Phi_a \right).\]

Lemma 3.6. — Hypothesis A1, A2 and A3 hold true with \( \delta = 1/2 \) and \( \mathcal{D} = [0, 1]^n \).
\textbf{Proof.} — , The asymptotics A1 are verified with $\gamma = 1$. Next we remark that for $a, b \in \mathcal{A}$, $(\nabla_\rho \omega_a)_b = \delta_{a,b}$ where $\delta$ denotes the Kronecker symbol while $\nabla_\rho \lambda_a = (0, \cdots, 0)^t$. Thus A2 holds true with $\delta = 1/2$ and $\mathcal{D} = [0,1]^n$. Finally, noticing that $\lambda_a - \lambda_b \in \mathbb{Z}$ we easily deduce A3 from A2 as in the proof of Lemma 3.6.

\textbf{Lemma 3.7.} — Assume that $(x,z,\bar{z}) \mapsto F(x,z,\bar{z})$ is real analytic in $x$, $\Re z$, $\Im z$ and assume that $s > 1$ then there exist $\sigma > 0$, $\mu > 0$ such that

\begin{equation*}
\mathcal{O}^s(\sigma, \mu) \ni (r, \theta, \zeta) \mapsto P(r, \theta, \zeta) = \int_{\mathbb{R}^2} F \left( x, \sum_{a \in \mathcal{A}} \frac{\sqrt{r_a^+ + r_a e^{i\theta_a}}}{w_a^\beta} \Phi_a, \sum_{a \in \mathcal{L}} \frac{\xi_a}{w_a^\beta} \Phi_a, \sum_{a \in \mathcal{A}} \frac{\sqrt{r_a^+ + r_a e^{-i\theta_a}}}{w_a^\beta} \Phi_a, \sum_{a \in \mathcal{L}} \frac{\eta_a}{w_a^\beta} \Phi_a \right)
\end{equation*}

belongs to $\mathcal{T}^{s,\beta}(\mathcal{D}, \sigma, \mu)$ for any $\beta \geq 0$.

\textbf{Proof.} — We focus on the case $F = F_{\text{NLS}}$ and on the most restrictive hypothesis: $\nabla^2_\rho P(x, \rho) \in \mathcal{M}_\beta$. We have

\begin{align*}
\frac{\partial^2 P}{\partial \xi_a \xi_b} &= \frac{\pm 1}{2w_a^\beta w_b^\beta} \int_{\mathbb{R}^2} |u|^2 \Phi_a \Phi_b \, dx, \\
\frac{\partial^2 P}{\partial \eta_a \eta_b} &= \frac{\pm 1}{2w_a^\beta w_b^\beta} \int_{\mathbb{R}^2} |u|^2 \Phi_a \Phi_b \, dx, \\
\frac{\partial^2 P}{\partial \xi_a \eta_b} &= \frac{1}{w_a^\beta w_b^\beta} \int_{\mathbb{R}^2} |u|^2 \Phi_a \Phi_b \, dx.
\end{align*}

So it remains to prove that the infinite matrix $M$ defined by

\begin{equation*}
M^b_a = \int_{\mathbb{R}^2} |u|^2 \Phi_a \Phi_b \, dx
\end{equation*}

belongs to $\mathcal{M}_0$, i.e.

\begin{equation*}
\sup_{a,b \in \mathcal{L}} \left\| M^b_a \right\|_{HS} < \infty.
\end{equation*}
Let us denote $\Pi_b$ the orthogonal projection in $L^2(\mathbb{R}^2)$ on the eigenspace $E_b := \text{Span}\{\Phi_d \mid w_d = w_b\}$. We have

\[
\left\| M_{[a]}^{(b)} \right\|_{HS}^2 = \sum_{c \in [a], d \in [b]} \left| \int_{\mathbb{R}^2} |u|^2 \Phi_c \Phi_d \, dx \right|^2 \\
\leq \sum_{c \in [a]} \left\| \Pi_b (|u|^2 \Phi_c) \right\|_{L^2(\mathbb{R}^2)}^2 \leq \sum_{c \in [a]} \left\| |u|^2 \Phi_c \right\|_{L^2(\mathbb{R}^2)}^2 \\
\leq \int_{\mathbb{R}^2} |u|^4 \left( \sum_{c \in [a]} |\Phi_c|^2 \right) \, dx \\
\leq C \int_{\mathbb{R}^2} |u|^4 \, dx
\]

where we used the crucial property of the quantum Harmonic oscillator (see Lemma 3.8 just below):

\[
\sum_{c \in [a]} |\Phi_c(x)|^2 \leq C, \quad x \in \mathbb{R}^2
\]

where $C > 0$ does not depend on $a$. \(\Box\)

The function $K_a(x, y) := \sum_{c \in [a]} \Phi_c(x) \Phi_c(y)$ is the integral kernel of the projection operator on $E_a$. It does not depend on the choice of the basis of $E_a$. From [22, 16] (see also [17]) we learn the following (non trivial) estimate

**Lemma 3.8.** — Let $d \geq 2$, there exists $C > 0$ such that for all $a$

\[
K_a(x, x) \leq C \lambda_a^{d/2-1} \quad \text{for all } x \in \mathbb{R}^d.
\]

In particular, in dimension $d = 2$, we deduce (3.11).

4. Poisson brackets and Hamiltonian flows.

It turns out that the space $T^{s,\beta}(D, \sigma, \mu)$ is not stable by Poisson brackets. Therefore, in this section, we first define a new space $T^{s,\beta+}(D, \sigma, \mu) \subset T^{s,\beta}(D, \sigma, \mu)$ and then we prove a structural stability which is essentially contained in the claim

\[
\{T^{s,\beta+}(D, \sigma, \mu), T^{s,\beta}(D, \sigma, \mu)\} \subset T^{s,\beta}(D, \sigma, \mu).
\]

We will also study the hamiltonian flow generated by hamiltonian function in $T^{s,\beta+}(D, \sigma, \mu)$. 

4.1. New Hamiltonian space. — We introduce $\mathcal{T}^{s,\beta}(\mathcal{D},\sigma,\mu) \subset \mathcal{T}^{s,\beta}(\mathcal{D},\sigma,\mu)$ defined by

$$\mathcal{T}^{s,\beta}(\mathcal{D},\sigma,\mu) = \{ f \in \mathcal{T}^{s,\beta}(\mathcal{D},\sigma,\mu) \mid \partial_\xi^j \nabla_\zeta f \in L^+_\beta, \partial_\rho^j \nabla_\zeta^2 f \in \mathcal{M}^+_\beta, \, j = 0, 1 \}$$

where

$$\mathcal{M}^+_\beta = \{ M \in \mathcal{M} \mid |M|_{\beta^+} < \infty \}, \quad L^+_\beta = \{ \zeta \in L \mid |\zeta|_{\beta^+} < \infty \}$$

and

$$|M|_{\beta^+} = \sup_{a, b \in \mathcal{C}} (1 + |w_a - w_b|) w^\beta_a w^\beta_b \| M[b]\|_{HS}$$

$$|\zeta|_{\beta^+} = \sup_{a \in \mathcal{C}} w^{\beta+1}_a |\zeta_a|.$$ 

We endow $\mathcal{T}^{s,\beta}(\mathcal{D},\sigma,\mu)$ with the norm

$$[f]^{s,\beta}_{\sigma,\mu,D} = [f]^{s,\beta}_{\sigma,\mu} + \sup_{j=0,1} \left( \mu |\partial_\xi^j \nabla_\zeta f|_{\beta^+} + \mu^2 |\partial_\rho^j \nabla_\zeta^2 f|_{\beta^+} \right).$$

**Lemma 4.1.** — Let $\beta > 0$ there exists a constant $C \equiv C(\beta) > 0$ such that

(i) Let $A \in \mathcal{M}^+_\beta$ and $B \in \mathcal{M}^+_\beta$ then $AB$ and $BA$ belong to $\mathcal{M}^+_\beta$ and

$$|AB|_{\beta^+}, |BA|_{\beta^+} \leq C |A|_{\beta^+} |B|_{\beta^+}.$$

(ii) Let $A, B \in \mathcal{M}^+_\beta$ then $AB$ and $BA$ belong to $\mathcal{M}^+_\beta$ and

$$|AB|_{\beta^+}, |BA|_{\beta^+} \leq C |A|_{\beta^+} |B|_{\beta^+}.$$

(iii) Let $A \in \mathcal{M}^+_\beta$ and $\zeta \in \mathcal{Y}_s$ for some $s \geq 0$ then $A \zeta \in L^+_\beta$ and

$$|A \zeta|_{\beta^+} \leq C |A|_{\beta^+} \| \zeta \|_s.$$

(iv) Let $A \in \mathcal{M}^+_\beta$ and $\zeta \in L^+_\beta$ then $A \zeta \in L^+_\beta$ and

$$|A \zeta|_{\beta^+} \leq C |A|_{\beta^+} \| \zeta \|_{\beta^+}.$$

(v) Let $A \in \mathcal{M}^+_\beta$ and $\zeta \in \mathcal{Y}_s$ for some $s \geq 1$ then $A \zeta \in L^+_\beta$ and

$$|A \zeta|_{\beta^+} \leq C |A|_{\beta^+} \| \zeta \|_s.$$

(vi) Let $A \in \mathcal{M}^+_\beta$ and $\zeta \in \mathcal{Y}_s$ for some $s \geq 1$ then $A \zeta \in L^+_\beta$ and

$$|A \zeta|_{\beta^+} \leq C |A|_{\beta^+} \| \zeta \|_{s}.$$

(vii) Let $A \in \mathcal{M}^+_\beta$ and $\zeta \in L^+_\beta$ then $A \zeta \in L^+_\beta$ and

$$|A \zeta|_{\beta^+} \leq C |A|_{\beta^+} \| \zeta \|_{s}.$$

(viii) Let $X \in L^+_\beta$ and $Y \in L^+_\beta$ then $A = X \otimes Y \in \mathcal{M}^+_\beta$ and

$$|A|_{\beta^+} \leq C |X|_{\beta^+} |Y|_{\beta^+}.$$
Proof. — (i)–Let $a, b \in \mathcal{L}$ and for $k \in \{w_a \mid a \in \mathcal{L}\} := \bar{\mathbb{N}} \subset \mathbb{N}$ denote by $k$ an element of $\mathcal{L}$ satisfying $w_k = k$. We have

$$
\|(AB)^{[b]}_{[a]}\|_{HS} \leq \sum_{k \in \mathbb{N}} \left\| A^k_{[a]} \right\|_{HS} \left\| B^k_{[b]} \right\|_{HS} \\
\leq \frac{|A|_{\beta+}|B|_{\beta}}{w_a^\beta w_b^\beta} \sum_{k \in \mathbb{N}} \frac{1}{k^{2\beta}(1 + |w_a - k|)} \\
\leq C \frac{|A|_{\beta+}|B|_{\beta}}{w_a^\beta w_b^\beta}
$$

where we used that $\sum_{k \in \mathbb{N}} \frac{1}{k^{2\beta}(1 + |w_a - k|)} \leq C$ for a constant $C > 0$ depending only on $\beta$.

(ii)–Similarly

$$
\|(AB)^{[b]}_{[a]}\|_{HS} \leq \sum_{k \in \mathbb{N}} \left\| A^k_{[a]} \right\|_{HS} \left\| B^k_{[b]} \right\|_{HS} \\
\leq \frac{|A|_{\beta+}|B|_{\beta}}{w_a^\beta w_b^\beta} \sum_{k \in \mathbb{N}} \frac{1}{k^{2\beta}(1 + |w_a - k|)(1 + |w_b - k|)} \\
\leq C \frac{|A|_{\beta+}|B|_{\beta}}{w_a^\beta w_b^\beta (1 + |w_a - w_b|)}
$$

where we used

$$
\{k \geq 1\} \subset \{k \geq 1, |w_a - k| \geq \frac{1}{3}|w_a - w_b|\} \cup \{k \geq 1, |w_b - k| \geq \frac{1}{3}|w_a - w_b|\}.
$$

(iii)–We have for any $a \in \mathcal{L}$

$$
|{(A\zeta)}_{[a]}| = \left| \sum_{j \in \mathbb{N}} A^j_{[a]} \zeta^j \right| \\
\leq \sum_{j \in \mathbb{N}} \left\| A^j_{[a]} \right\|_{HS} |\zeta^j| \\
\leq \sum_{j \in \mathbb{N}} \frac{1}{(1 + |w_a - j|)w_a^\beta j^\beta} |A|_{\beta+} \|\zeta\|_s \\
\leq C w_a^{-\beta} |A|_{\beta+} \|\zeta\|_s .
$$
(iv)–Similarly

\[
|\langle A \zeta \rangle_{[a]}| = \left| \sum_{j \in \mathbb{N}} A_{[a]}^{[j]} \zeta_{[j]} \right|
\]
\[
\leq \sum_{j \in \mathbb{N}} j^{-1-\beta} \|A_{[a]}^{[j]}\|_{HS} j^{1+\beta} |\zeta_{[j]}|
\]
\[
\leq \sum_{j \in \mathbb{N}} \frac{1}{w_a^2 j^{3+2\beta}} |A|_{\beta} |\zeta|_{\beta+}
\]
\[
\leq C w_a^{-\beta} |A|_{\beta} |\zeta|_{\beta+}.
\]

(v)–Similarly

\[
|\langle A \zeta \rangle_{[a]}| = \left| \sum_{j \in \mathbb{N}} A_{[a]}^{[j]} \zeta_{[j]} \right|
\]
\[
\leq \sum_{j \in \mathbb{N}} j^{-s} \|A_{[a]}^{[j]}\|_{HS} j^{s} |\zeta_{[j]}|
\]
\[
\leq \sum_{j \in \mathbb{N}} \frac{1}{w_a^2 j^{s+\beta}} |A|_{\beta} \|\zeta\|_{s}
\]
\[
\leq C w_a^{-\beta} |A|_{\beta} \|\zeta\|_{s}.
\]

(vi)–Similarly

\[
|\langle A \zeta \rangle_{[a]}| = \left| \sum_{j \in \mathbb{N}} A_{[a]}^{[j]} \zeta_{[j]} \right|
\]
\[
\leq \sum_{j \in \mathbb{N}} j^{-s} \|A_{[a]}^{[j]}\|_{HS} j^{s} |\zeta_{[j]}|
\]
\[
\leq \sum_{j \in \mathbb{N}} \frac{1}{w_a^2 j^{s+\beta}} |A|_{\beta} \|\zeta\|_{s}
\]
\[
\leq C w_a^{-\beta} |A|_{\beta} \|\zeta\|_{s}
\]

where we used that for \(s \geq 1\)

\[
\sum_{j \in \mathbb{N}} \frac{1}{(1 + |w_a - j|) j^{s+\beta}} \leq \frac{C}{w_a}.
\]
(vii)–Similarly

\[
\|(A\zeta)_{[a]}\| = \left| \sum_{j \in \mathbb{N}} A^{[j]}_{[a]} \zeta^{[j]} \right| \\
\leq \sum_{j \in \mathbb{N}} j^{-1-\beta} \| A^{[j]}_{[a]} \|_{HS} j^{\beta+1} | \zeta^{[j]} | \\
\leq \sum_{j \in \mathbb{N}} \frac{1}{(1 + |w_a - j|)w_a^j j^{1+2\beta}} |A|_{\beta+|\zeta|_{\beta+}} \\
\leq C w_a^{-\beta-1} |A|_{\beta+|\zeta|_{\beta+}}.
\]

(viii)–Finally

\[
\left\| \frac{A^{[b]}_{[a]}}{H_S} \right\|^2 = \sum_{c \in [a], d \in [b]} |X_c|^2 |Y_d|^2 \\
\leq |X_{[a]}|^2 |Y_{[b]}|^2
\]

and thus

\[
|A|_{\beta} \leq |X|_{\beta} |Y|_{\beta}.
\]

4.2. Jets of functions.— For any function \( h \in T^*(\sigma, \mu, D) \) we define its jet \( h^T = h^T(x, \rho) \) as the following Taylor polynomial of \( h \) at \( r = 0 \) and \( \zeta = 0 \):

\[
h^T = h_\theta + \langle h_r, r \rangle + \langle h_\zeta, \zeta \rangle + \frac{1}{2} \langle h_{\zeta\zeta}, \zeta \rangle
\]

\[
= h(\theta, 0, \rho) + \langle \nabla_r h(\theta, 0, \rho), r \rangle + \langle \nabla_\zeta h(\theta, 0, \rho), \zeta \rangle + \frac{1}{2} \langle \nabla_{\zeta\zeta} h(\theta, 0, \rho), \zeta \rangle
\]

Functions of the form \( h^T \) will be called jet-functions.

Directly from the definition of the norm \( [h]_{s, \mu, D} \) we get that

\[
|h_\theta(\theta, \rho)| \leq [h]_{s, \mu, D}, \quad |h_r(\theta, \rho)| \leq \mu^{-2}[h]_{s, \mu, D},
\]

\[
|h_\zeta(\theta, \rho)|_s \leq \mu^{-1}[h]_{s, \mu, D}, \quad |h_{\zeta\zeta}(\theta, \rho)|_s \leq \mu^{-1}[h]_{s, \mu, D},
\]

\[
|h_{\zeta}(\theta, \rho)|_{\beta} \leq \mu^{-2}[h]_{s, \mu, D}, \quad |h_{\zeta\zeta}(\theta, \rho)|_{\beta} \leq \mu^{-2}[h]_{s, \mu, D},
\]

for any \( \theta \in \mathbb{T}_\sigma^\prime \) and any \( \rho \in D \). Moreover, the first derivative with respect to \( \rho \) will satisfy the same estimates.

We also notice that by Cauchy estimates we have that for \( x \in O(\sigma, \mu') \)

\[
\left\| \nabla^2_\zeta h(x) \right\|_{L(Y_s, Y_s)} \leq \sup_{y \in O(\sigma, \mu)} \left\| \nabla_\zeta h(y) \right\|_{s} \frac{s}{\mu - \mu'}.
\]
Thus $h_{ζζ}$ is a linear continuous operator from $Y_σ$ to $Y_σ$ and

$$
\|h_{ζζ}(θ, ρ)\|_{C(Y_σ, Y_σ)} \leq μ^{-2}[h]^{σ, µ, D}_σ
$$

for any $θ ∈ T^{n}_σ$ and any $ρ ∈ D$.

**Proposition 4.2.** — For any $h ∈ T^{s, β}_σ(σ, µ, D)$ we have $h^T ∈ T^{s, β}_σ(σ, µ, D),$

$$[h^T]^{σ, µ, D}_σ \leq C[h]^{σ, β}_σ,$$

and, for any $0 < µ' < µ$,

$$[h - h^T]^{σ, β}_σ \leq C (\frac{µ'}{µ})^3 [h]^{σ, β}_σ,$$

where $C$ is an absolute constant.

**Proof.** — We start with the second statement. Consider first the hessian $∇^2_{ζζ}(h - h^T)(x)$ for $x = (θ, r, ζ) ∈ O^σ(σ, µ)$. Let us denote $m = µ'/µ$. Then for $z ∈ \mathcal{D}_1 = \{z ∈ C : |z| ≤ 1\}$ we have $(θ, (z/m)^2 r, (z/m)ζ) ∈ O^σ(σ, µ)$. Consider the function

$$f : D_1 × O^σ(σ, µ') → M_{β},$$

$$(z, x) ↦ ∇^2_{ζζ} h(θ, (z/m)^2 r, (z/m)ζ) = h_0(x) + h_1(x)z + \ldots.$$ 

It is holomorphic and its norm is bounded by $μ^{-2}[h]^{σ, β}_D,σ,µ'$. So, by the Cauchy estimate, $|h_j(x)|_β ≤ μ^{-2}|h|^{σ, β}_D,σ,µ$ for $j = 1, 2, \ldots$ and $x ∈ O^σ(σ, µ)$. Since $∇^2_{ζζ}(h - h^T)(x) = h_1(x)m + h_2(x)m^2 + \ldots$, then $∇^2_{ζζ}(h - h^T)$ is holomorphic in $x ∈ O^σ(σ, µ)$, and

$$|∇^2_{ζζ}(h - h^T)(x)|_β ≤ μ^{-2}|h|^{σ, β}_D σ,µ,D (m + m^2 + \ldots) \leq μ^{-2}|h|^{σ, β}_D σ,µ,D \frac{m}{1 - m}.$$ 

So $∇^2_{ζζ}(h - h^T)$ satisfies the required estimate with $C = 2$, if $µ' ≤ µ/2$.

Same argument applies to bound the norms of $∂_n ∇^2_{ζζ}(h - h^T)$, $h - h^T$ and $∇_ζ(h - h^T)$ if $µ' ≤ µ/2$, and to prove the analyticity of these mappings.

Now we turn to the first statement and write $h^T$ as $h - (h - h^T)$. This implies that $h^T$, $∇_ζ h^T$ and $∇^2_{ζζ} h^T$ are analytic on $O^σ(σ, µ)$ and that

$$[h^T]^{σ, β}_σ \leq C_1[h]^{σ, β}_σ.$$ 

Since $h^T$ is a quadratic polynomial, then the mappings $h^T$, $∇_ζ h^T$ and $∇^2_{ζζ} h^T$ are as well analytic on $O^σ(σ, µ)$, and the norm $[h^T]^{σ, β}_σ$ satisfies the same estimate, modulo another constant factor, for any $0 < µ' ≤ µ$.

Finally, the estimate for $[h - h^T]^{σ, β}_σ$ when $µ/2 ≤ µ' ≤ µ$, with a suitable constant $C$, follows from the estimate for $[h^T]^{σ, β}_σ$ since $[h - h^T]^{σ, β}_σ \leq [h^T]^{σ, β}_σ + [h]^{σ, β}_σ$.

□
Lemma 4.3. — Let \( s \geq 1 \). Let \( f \in T^{s,\beta}(D,\sigma,\mu) \) and \( g \in T^{s,\beta}(D,\sigma,\mu) \) be two jet functions then for any \( 0 < \sigma' < \sigma \) we have \( \{ f, g \} \in T^{s,\beta}(D,\sigma',\mu) \) and
\[
\{ f, g \}_{s',\mu,D}^{s,\beta} \leq C(\sigma - \sigma')^{-1} \mu^{-2}[f]_{s',\mu,D}^{s,\beta}[g]_{s',\mu,D}^{s,\beta}.
\]

Proof. — Let denotes by \( h_1, h_2, h_3 \) the three terms on the right hand side of (4.5). Since \( \nabla_r f(\theta, r, \zeta, \rho) = f_r(\theta, \rho) \) and \( \nabla_r g(\theta, r, \zeta, \rho) = g_r(\theta, \rho) \) are independent of \( r \) and \( \zeta \), the control of \( h_1 \) and \( h_2 \) is straightforward by Cauchy estimates and (4.2).

We focus on the third term in formula: \( h_3 = \langle J\nabla_c f, \nabla_g \rangle \). As \( \nabla_c f = f_c + f_{\zeta}\zeta \) and similar for \( \nabla_c g \), we have
\[
h_3 = \langle Jf_c \zeta, g_c \rangle - \langle \zeta, f_{\zeta}\zeta Jg_c \rangle + \langle g_{\zeta}\zeta Jf_c, \zeta \rangle + \langle g_{\zeta}\zeta Jf_{\zeta}\zeta, \zeta \rangle.
\]

Using (4.2), (4.4) and \( \| \zeta \| \leq \mu \), we get
\[
|h_3(x, \cdot)| \leq C \mu^{-2}[f]_{s',\mu,D}^{s,\beta}[g]_{s',\mu,D}^{s,\beta},
\]
for any \( x \in O(\sigma,\mu) \) and \( \rho \in D \).

Since
\[
\nabla_c h_3 = -f_{\zeta}\zeta Jg_c + g_{\zeta}\zeta Jf_c + g_{\zeta}\zeta Jf_{\zeta}\zeta - f_{\zeta}\zeta Jg_{\zeta}\zeta,
\]
then, using Lemma 4.1 we get that for \( x \in O^s(\sigma, \mu) \) with \( s \geq 1 \) and \( \rho \in D \)
\[
|\nabla_c h_3(x, \cdot)|_\beta \leq C \mu^{-2}[f]_{s',\mu,D}^{s,\beta}[g]_{s',\mu,D}^{s,\beta}.
\]

On the other hand using again (4.4), we deduce that for \( x \in O^s(\sigma, \mu) \) and \( \rho \in D \)
\[
|\nabla_c h_3(x, \cdot)|_s \leq C \mu^{-2}[f]_{s',\mu,D}^{s,\beta}[g]_{s',\mu,D}^{s,\beta}.
\]

Finally, as \( \nabla^2 h_3 = g_{\zeta}\zeta Jf_{\zeta} - f_{\zeta}\zeta Jg_{\zeta} \), then, using again Lemma 4.1 we get that for \( x \in O^s(\sigma, \mu) \) and \( \rho \in D \)
\[
|\nabla^2 h_3(x, \cdot)|_\beta \leq C \mu^{-4}[f]_{s',\mu,D}^{s,\beta}[g]_{s',\mu,D}^{s,\beta}.
\]

\[\square\]

4.4. Hamiltonian flows. — To any \( C^1 \)-function \( f \) on a domain \( O^s(\sigma,\mu) \times D \) we associate the hamiltonian equations
\[
\begin{align*}
\dot{r} &= \nabla_\theta f(r, \theta, \zeta), \\
\dot{\theta} &= -\nabla_r f(r, \theta, \zeta), \\
\dot{\zeta} &= J\nabla_\zeta f(r, \theta, \zeta).
\end{align*}
\]
and denote by $\Phi^t_f \equiv \Phi^t, t \in \mathbb{R}$, the corresponding flow-maps (if they exist). Now let $f \equiv f^T$ be a jet-function

$$f = f_\theta(\theta; \rho) + f_r(\theta; \rho) \cdot r + \langle f_\zeta(\theta; \rho), \zeta \rangle + \frac{1}{2} \langle f_{\zeta\zeta}(\theta; \rho) \zeta, \zeta \rangle. \tag{4.7}$$

Then Hamiltonian equations (4.6) take the form

$$\begin{cases}
\dot{r} = -\nabla_\theta f(r, \theta, \zeta), \\
\dot{\theta} = f_r(\theta), \\
\dot{\zeta} = J(f_\zeta(\theta) + f_{\zeta\zeta}(\theta) \zeta). 
\end{cases} \tag{4.8}$$

Denote by $V_f = (V^r_f, V^\theta_f, V^\zeta_f)$ the corresponding vector field. It is analytic on any domain $O^s(\sigma - 2\eta, \mu - 2\nu) =: O_{2\eta,2\nu}$, where $0 < 2\eta < \sigma, 0 < 2\nu < \mu$. The flow-maps $\Phi^t_f$ of $V_f$ on $O_{2\eta,2\nu}$ are analytic till they exist. We will study them till they map $O_{2\eta,2\nu}$ to $O_{\eta,\nu}$.

Assume that

$$|f|^s_{\sigma,\mu,D} \leq \frac{1}{2} \nu^2 \eta. \tag{4.9}$$

Then for $x = (r, \theta, \zeta) \in O_{2\eta,2\nu}$ by the Cauchy estimate and (4.4) we have

$$|V^r_f|_{C^s} \leq (2\eta)^{-1}|f|^s_{\sigma,\mu,D} \leq \nu^2,$$

$$|V^\theta_f|_{C^s} \leq (4\nu^2)^{-1}|f|^s_{\sigma,\mu,D} \leq \eta,$$

$$\|V^\zeta_f\|_s \leq (\mu^{-1} + \mu^{-2} \mu)|f|^s_{\sigma,\mu,D} \leq \nu.$$

Noting that the distance from $O_{2\eta,2\nu}$ to $\partial O_{\eta,\nu}$ in the the $r$-direction is $2\nu\mu - 3\nu^2 > \nu^2$, in the $\theta$-direction is $\eta$ and in the $\zeta$-direction is $\nu$, we see that the flow-maps

$$\begin{cases}
\Phi^t_f : O^s(\sigma - 2\eta, \mu - 2\nu) \to O^s(\sigma - \eta, \mu - \nu), \\
0 \leq t \leq 1,
\end{cases} \tag{4.10}$$

are well defined and analytic.

For $x \in O_{2\eta,2\nu}$ denote $\Phi^t_f(x) = (r(t), \theta(t), \zeta(t))$. Since $V^\theta_f$ is independent from $r$ and $\zeta$, then $\theta(t) = K(\theta; t)$, where $K$ is analytic in both arguments. As $V^\zeta_f = Jf_\zeta + Jf_{\zeta\zeta}\zeta$, where the non autonomous linear operator $Jf_{\zeta\zeta}(\theta(t))$ is bounded in the space $Y^s_\zeta$ and both the operator and the curve $Jf_{\zeta}(\theta(t))$ analytically depend on $\theta$ (through $\theta(t) = K(\theta; t)$), then $\zeta(t) = T(\theta; t) + U(\theta; t) \zeta$, where $U(\theta; t)$ is a bounded linear operator, both $U$ and $T$ analytic in $\theta$. Similar since $V^r_f$ is a quadratic polynomial in $\zeta$ and an affine function of $r$, then $r(t) = L(\theta, \zeta; t) + S(\theta; t) r$, where $S$ is an $n \times n$ matrix and $L$ is a quadratic polynomial in $\zeta$, both analytic in $\theta$.

11. Here and below we often suppress the argument $\rho$.

12. Notice that the distance from $O^s(\sigma - 2\eta, \mu - 2\nu)$ to $\partial O^s(\sigma, \mu)$ in the the $r$-direction is $4\nu\mu - 4\nu^2 > 4\nu^2$. 
The vectorfield $V_f$ is real for real argument, and so are its flow-maps. Since
the vector-field is hamiltonian, then the flow-maps are symplectic (e.g., see
(4.11)). We have proven

**Lemma 4.4.** — Let $0 < 2\eta < \sigma$, $0 < 2\nu < \mu$ and $f = f^T \in T^s(\sigma, \mu, D)$

satisfies (4.9). Then for $0 \leq t \leq 1$ the flow maps $\Phi_f^t$ of equation (4.8) define
analytical mappings (4.10) and define symplectomorphisms from $O^s(\sigma-2\eta, \mu-2\nu)$ to $O^s(\sigma-\eta, \mu-\nu)$. They have the form

$$
\Phi_f^t : \begin{pmatrix} \nu \\ \theta \\ \zeta \end{pmatrix} \to \begin{pmatrix} L(\theta, \zeta; t) + S(\theta; t) \nu \\ K(\theta; t) \\ T(\theta; t) + U(\theta; t) \zeta \end{pmatrix},
$$

(4.11)

where $L(\theta, \zeta; t)$ is quadratic in $\zeta$, while $U(\theta; t)$ and $S(\theta; t)$ are bounded linear
operators in corresponding spaces.

Our next result specifies the flow-mappings $\Phi_f^t$ and their representation
(4.11) when $f \in T^{s,\beta^+}(\sigma, \mu, D)$:

**Lemma 4.5.** — Let $0 < 2\eta < \sigma \leq 1$, $0 < 2\nu < \mu \leq 1$ and $f = f^T \in T^{s,\beta^+}(\sigma, \mu, D)$

satisfies

$$
[f]^{s,\beta^+}_{\sigma,\mu,D} \leq \frac{1}{2} \nu^2 \eta
$$

(4.12)

Then:

1) Mapping $L$ is analytic in $(\theta, \zeta) \in T^{\sigma,2\eta}_s \times O_\mu(Y_s)$. Mappings $K, T$ and
operators $S$ and $U$ analytically depend on $\theta \in T^{\sigma,2\eta}_s$; their norms and operator-norms satisfy

$$
\|S(\theta; t)\|_{L(C^n, C^n)}, \|^{t}U(\theta; t)\|_{L(Y_s, Y_s)}, \|U(\theta; t)\|_{L(Y_s, Y_s)}, |U(\theta; t)|_{\beta^+} \leq 2,
$$

(4.13)

while for any component $L^j$ of $L$ and any $(\theta, r, \zeta) \in O^s(\sigma-2\eta, \mu-2\nu)$ we have

$$
\|\nabla_\zeta L^j(\theta, \zeta; t)\|_{s} \leq C \eta^{-1} \mu^{-1} [f]^{s}_{\sigma,\mu,D},
$$

$$
|\nabla_\zeta L^j(\theta, \zeta; t)|_{\beta^+} \leq C \eta^{-1} \mu^{-1} [f]^{s,\beta^+}_{\sigma,\mu,D},
$$

$$
|\nabla_\zeta^2 L^j(\theta, \zeta; t)|_{\beta^+} \leq C \eta^{-1} \mu^{-2} [f]^{s,\beta^+}_{\sigma,\mu,D},
$$

(4.14)

2) The flow maps $\Phi_f^t$ analytically extend to mappings

$C^n \times T^{\sigma,2\eta}_s \times Y_s \ni x^0 = (r^0, \theta^0, \zeta^0) \mapsto x(t) \in C^n \times T^{\sigma,\beta^+}_s \times Y_s,$

(4.15)
Moreover, as the increments $x$ and $a$ satisfy hypotheses and using (4.3) we get that

\begin{equation}
|\zeta(t) - \zeta_0| \leq (\mu^{-2} \parallel \zeta_0 \parallel_s + 1) \parallel f_{\sigma,\mu,D}^a, \tag{4.18}
\end{equation}

and iterating this relation, we get that

\begin{equation}
|\zeta(t) - \zeta_0|_{\beta} \leq (\mu^{-2} \parallel \zeta_0 \parallel_s + 1) \parallel f_{\sigma,\mu,D}^{a,\beta}. \tag{4.19}
\end{equation}

Moreover, $\rho$-derivative of the mapping $x^0 \mapsto x(t)$ satisfies the same estimates as the increments $x(t) - x^0$.

\textbf{Proof.} — Consider the equation for $\zeta(t)$ in (4.8):

\begin{equation}
\dot{\zeta}(t) = a(t) + B(t)\zeta(t), \quad \zeta(0) = \zeta_0 \in \mathcal{O}_{\mu - 2\nu}(Y_s), \tag{4.16}
\end{equation}

where $a(t) = Jf\zeta(\theta(t))$ is an analytic curve $[0,1] \to Y_\gamma$ and $B(t) = Jf\zeta(\theta(t))$ is an analytic curve $[0,1] \to \mathcal{M}$. Both analytically depend on $\theta^0$. By the hypotheses and using (4.3)

\begin{equation}
\|a(t)\|_s \leq \mu^{-1} \parallel f_{\sigma,\mu,D}^a, \quad \|B_{\mathcal{L}(Y_s,Y_s)}\| \leq \mu^{-2} \parallel f_{\sigma,\mu,D}^a \leq \frac{1}{2} \nu \leq \frac{1}{2}. \tag{4.17}
\end{equation}

By re-writing (4.16) in the integral form $\zeta(t) = \zeta_0 + \int_0^t (a(t') + B(t')\zeta(t'))dt'$ and iterating this relation, we get that

\begin{equation}
\zeta(t) = a^\infty(t) + (I + B^\infty(t))\zeta_0, \tag{4.18}
\end{equation}

where

\begin{equation}
a^\infty(t) = \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{j=1}^{k-1} B(t_j) a(t_k) dt_k \cdots dt_2 dt_1,
\end{equation}

and

\begin{equation}
B^\infty(t) = \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{j=1}^{k} B(t_j) dt_k \cdots dt_2 dt_1.
\end{equation}

Due to (4.12), (4.17), for each $k$ and for $0 \leq t_k \leq \ldots t_1 \leq 1$ we have that

\begin{equation}
\|B(t_1) \ldots B(t_k)\|_{\mathcal{L}(Y_s,Y_s)} \leq \left(\frac{1}{2}\right)^{k-1} \mu^{-2} \parallel f_{\sigma,\mu,D}^a.
\end{equation}

By this relation and (4.17) we get that $a^\infty$ and $B^\infty$ are well defined for $t \in [0,1]$ and satisfy

\begin{equation}
\|B^\infty(t)\|_{\mathcal{L}(Y_s,Y_s)} \leq \mu^{-2} \parallel f_{\sigma,\mu,D}^a, \quad \|a^\infty(t)\|_s \leq \mu^{-3} \parallel f_{\sigma,\mu,D}^a \parallel^2 \leq \parallel f_{\sigma,\mu,D}^a. \tag{4.19}
\end{equation}
Again, the curves $a^∞$ and $B^∞$ analytically depend on $θ^0$. Inserting (4.19) in (4.18) we get that $ζ = ζ(t)$ satisfies (4.15). On the other hand for all $t ∈ [0, 1]$, $B ∈ M^+_β$ and

$$|B(t)|β_+ ≤ μ^{-2}[f]^{s, β+}_{σ, μ, D}.$$  

Therefore using Lemma 4.1 we get

$$|B^∞(t)|β_+ ≤ μ^{-2}[f]^{s, β+}_{σ, μ, D},$$  

$$|α^∞(t)|β_+ ≤ μ^{-3}([f]^{s, β+}_{σ, μ, D})^2 ≤ [f]^{s, β+}_{σ, μ, D}.$$  

Inserting (4.20) in (4.18) and using again Lemma 4.1 we get that $ζ = ζ(t)$ satisfies (4.15). Since in (4.11) $U(θ; t) = I + B^∞(t)$, then the estimates on $U$ in (4.13) follow from (4.19).

Now consider equation for $r(t)$:

$$r(t) = -α(t) - L(t)r(t), \quad r(0) = r^0 ∈ O(μ^{-2})^2(\mathbb{C}^n)$$

where $L(t) = ∇θf_r(θ(t))$ and

$$L(θ) = ∇θf_r(θ(t)) + (∇θfζ(θ(t)), ζ(t)) + \frac{1}{2}(∇θfζζ(θ(t))ζ(t), ζ(t)).$$

The matrix curve $L(t)$ and the vector curve $α(t)$ analytically depend on $θ^0 ∈ T^0_{σ - 2η}$. Besides, $α(t)$ analytically depends on $ζ^0 ∈ Y_s$, while $L$ is $ζ^0$-independent.

By the Cauchy estimate and (4.12), for any $θ(t) ∈ T^0_{σ - η}$ we have

$$|L(t)|ζ(ζ^0, ζ^∞) ≤ η^{-1}μ^{-2}[f]^{s}_{σ, μ, D} ≤ \frac{1}{2},$$  

$$|α(t)| ≤ 2η^{-1}[f]^{s}_{σ, μ, D}(1 + μ^{-1}||ζ^0||_s + μ^{-2}||ζ^0||_s^2)$$

where for the second estimate we used that $||ζ(t) - ζ^0||_s ≤ 1 + ||ζ^0||_s$.

Since $∇ζ(θ(t)) = ∇θfζ(θ(t)) + ∇θfζζ(θ(t))ζ(t)$ and $∇ζ_θ = tU(θ; t)∇ζ$, then using (4.13) we obtain

$$||∇ζ^0α(t)||_s ≤ 4η^{-1}μ^{-1}[f]^{s}_{σ, μ, D}(1 + μ^{-1}||ζ^0||_s),$$

and using Lemma 4.1

$$|∇ζ^0α(t)|β_+ ≤ 4η^{-1}μ^{-1}[f]^{s, β+}_{σ, μ, D}(1 + μ^{-1}||ζ^0||_s).$$

Since $∇^2ζ^0α(t) = tU∇^2θ(α(t))U = tU(θ^ζζζ)(θ(t))U$, then due to (4.13)

$$|∇^2ζ^0α(t)|s ≤ 4η^{-1}μ^{-2}[f]^{s}_{σ, μ, D},$$

while due to (4.13) and Lemma 4.1

$$|∇^2ζ^0α(t)|β_+ ≤ 4η^{-1}μ^{-2}[f]^{s, β+}_{σ, μ, D}.$$
We proceed as for the $\zeta$-equation to derive
\begin{equation}
(4.27) \quad r(t) = -\alpha^\infty(t) + (1 - \Lambda^\infty(t)) r^0,
\end{equation}
where
\begin{equation}
(4.28) \quad \alpha^\infty(t) = \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{j=1}^{k-1} \Lambda(t_j) \alpha(t_k) dt_k \cdots dt_2 dt_1,
\end{equation}
and
\begin{equation}
(4.29) \quad \Lambda^\infty(t) = \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{j=1}^{k} \Lambda(t_j) dt_k \cdots dt_2 dt_1.
\end{equation}
Using (4.22) we get that
\begin{equation}
|\Lambda^\infty(t)|_{L(C^n \times C^n)} \leq \frac{1}{2},
\end{equation}
\begin{equation}
|\alpha^\infty(t)|_{C^n} \leq \frac{1}{2} \eta^{-1} (1 + \mu^{-2} \|\zeta^0\|_s^2) [f]_{\sigma,\mu,D}.
\end{equation}
Since in (4.11) $S(\theta; 1) = I - \Lambda^\infty(t)$, then the first estimate in (4.13) follows.
Since $\Lambda^\infty(t)$ in (4.27) is $\zeta^0$-independent, then $\Lambda^\infty(t)$ is $\zeta^0$-independent.
This is a quadratic in $\xi^0$ expression, and the estimates (4.14) follow from (4.23)–(4.26)
and in view of the estimate for $\Lambda^\infty$ above.
Finally using the estimates for $\Lambda^\infty$ and $\alpha^\infty$ we get from (4.27) that
$r = r(t)$ satisfies (4.15).

Next we study how the flow-maps $\Phi^t_f$ transform functions from $T^{s,\beta}(\sigma, \mu, D)$.

**Lemma 4.6.** — Let $0 < 2\eta < \sigma \leq 1$, $0 < 2\nu < \mu \leq 1$. Assume that
\begin{equation}
f = f^T \in T^{s,\beta+}(\sigma, \mu, D) \text{ satisfies } (4.12). \quad \text{Let } h \in T^{s,\beta}(\sigma, \mu, D) \text{ and denote for}
\end{equation}
\begin{equation}
0 \leq t \leq 1
\end{equation}
\begin{equation}
h_t(x; \rho) = h(\Phi^t_f(x; \rho); \rho).
\end{equation}
Then $h_t \in T^{s,\beta}(\sigma - 2\eta, \mu - 2\nu, D)$ and
\begin{equation}
[h]_{s,\beta}^{\sigma - 2\eta, \mu - 2\nu, D} \leq C_{\mu}^{\sigma,\beta} [h]_{s,\mu,D}^{\sigma,\beta}
\end{equation}
where $C$ is an absolute constant.

**Proof.** — Let us write the flow-map $\Phi^t_f$ as
\begin{equation}
x^0 = (r^0, \theta^0, \zeta^0) \mapsto x(t) = (r(t), \theta(t), \zeta(t)).
\end{equation}
By Proposition 4.3 $h_t(x^0)$ is analytic in $x^0 \in O(\sigma - 2\eta, \mu - 2\nu)$. Clearly
\begin{equation}
|h_t(x^0, \cdot)| \leq [h]_{s,\mu,D}^{\sigma,\beta} \text{ for } x^0 \in O(\sigma - 2s, \mu - 2\nu) \text{ and } \rho \in D. \quad \text{So it remains to}
\end{equation}
estimate the gradient and hessian of $h(x^0)$.
1) Estimating the gradient. Since $\theta(t)$ does not depend on $\zeta^0$, we have
\[
\frac{\partial h_t}{\partial \zeta^0} = \sum_{k=1}^n \frac{\partial h(x(t))}{\partial r_k} \frac{\partial r_k(t)}{\partial \zeta^0} + \sum_{b \in L} \frac{\partial h(x(t))}{\partial \zeta_b} \frac{\partial r_b(t)}{\partial \zeta^0} = \Sigma_1 + \Sigma_2.
\]
i) Since $x(t) \in \mathcal{O}(\sigma - \eta, \mu - \nu)$, we get by the Cauchy estimate that
\[
\left| \frac{\partial h(x(t))}{\partial r_k} \right| \leq \frac{1}{3\nu^2} |h|^{s,\beta}_{\sigma,\mu,D}.
\]
As $\nabla_{\zeta^0} r_k(t)$ was estimated in (4.11), then using (4.12) we get
\[
\|\Sigma_1\|_s + |\Sigma_1|_\beta \leq C\nu^{-2}|h|^{s,\beta}_{\sigma,\mu,D} \eta^{-1} \mu^{-1} |f|^{s,\beta}_{\sigma,\mu,D}
\leq C\mu^{-1}|h|^{s,\beta}_{\sigma,\mu,D}.
\]
ii) Noting that $\Sigma_2(r, \theta, \zeta) = \mathcal{U}(\theta;t) \nabla_{\zeta} h$, we get using (4.13):
\[
\|\Sigma_2\|_s + |\Sigma_2|_\beta \leq 4\mu^{-1}|h|^{s,\beta}_{\sigma,\mu,D}.
\]
Estimating similarly $\frac{\partial^2 h}{\partial r \partial \zeta}$, we see that for $x \in \mathcal{O}(\sigma - 2\eta, \mu - 2\nu)$
\[
\|\partial \nabla_{\zeta^0} h_t\|_s + |\partial \nabla_{\zeta^0} h_t|_\beta \leq C\mu^{-1}|h|^{s,\beta}_{\sigma,\mu,D}.
\]
2) Estimating the hessian. Since $\theta(t)$ does not depend on $\zeta^0$ and since $\zeta(t)$ is affine in $\zeta^0$, then
\[
\frac{\partial^2 h_t}{\partial \zeta_a \partial \zeta_b}(x) = \frac{\partial^2 h(x(t))}{\partial \zeta_a \partial \zeta_b} \frac{\partial \zeta(t)}{\partial \zeta_a} \frac{\partial \zeta(t)}{\partial \zeta_b} + \frac{\partial^2 h(x(t))}{\partial r^2} \frac{\partial r(t)}{\partial \zeta_a} \frac{\partial r(t)}{\partial \zeta_b}
\]
\[
+ \frac{\partial^2 h(x(t))}{\partial r \partial \zeta_a} \frac{\partial r(t)}{\partial \zeta_a} \frac{\partial \zeta(t)}{\partial \zeta_b} + \frac{\partial h(x(t))}{\partial r} \frac{\partial^2 r(t)}{\partial \zeta_a \partial \zeta_b}
\]
\[
=: \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4.
\]
i) We have $|\partial^2 h/\partial \zeta_a \partial \zeta_b|_\beta \leq C\mu^{-2}|h|^{s,\beta}_{\sigma,\mu,D}$. Using this estimate jointly with (4.13) and Lemma 4.1 we see that
\[
|\Delta_1|_\beta \leq C\mu^{-2}|h|^{s,\beta}_{\sigma,\mu,D}.
\]
i) Since for $x^0 \in \mathcal{O}^s(\sigma - 2s, \mu - 2\nu)$ by (4.14) we have
\[
|\nabla r|_\beta \leq C\eta^{-1} \mu^{-1} |f|^{s,\beta}_{\sigma,\mu,D},
\]
and since by Cauchy estimate $|\partial^2 h| \leq C\nu^{-4}|h|^{s,\beta}_{\sigma,\mu,D}$, we get using Lemma 4.1 (viii) and (4.12)
\[
|\Delta_2|_\beta \leq C\nu^{-4}|h|^{s,\beta}_{\sigma,\mu,D} \eta^{-2} \mu^{-2} (|f|^{s,\beta}_{\sigma,\mu,D})^2 \leq C\mu^{-2}|h|^{s,\beta}_{\sigma,\mu,D}.
\]
iii) For any \( j \) we have by the Cauchy estimate that \( \left| \frac{\partial}{\partial r_j} \nabla \zeta h \right|_\beta \leq C \nu^{-3} [h]^{s,\beta}_{\sigma,\mu,D} \). Therefore by (4.13) and Lemma 4.1

\[
\left| \sum \frac{\partial^2 h}{\partial r_j \partial \zeta_{\alpha'}} \frac{\partial \zeta_{\alpha'}}{\partial \zeta_0} \right|_\beta \leq C \nu^{-3} [h]^{s,\beta}_{\sigma,\mu,D}.
\]

Since \( \left| \nabla \xi r_j \right|_\beta \leq C \eta^{-1} \mu^{-1} [f]^{s,\beta}_{\sigma,\mu,D} \leq C \mu^{-1} \)

by (4.14), then using again Lemma 4.1(viii) we find that \( \left| \Delta_3 \right|_{\gamma'} \leq C \nu^{-1} \mu^{-1} [h]^{s,\beta}_{\sigma,\mu,D} \).

iv) As \( \left| \partial h / \partial r(x(t)) \right| \leq \nu^{-2} [h]^{s,\beta}_{\sigma,\mu,D} \) and

\[
\left| \frac{\partial^2 r}{\partial \zeta_0^a \partial \zeta_0^b} \right|_\beta \leq C \eta^{-1} \mu^{-2} [f]^{s,\beta}_{\sigma,\mu,D}
\]

by (4.14), then

\[
\left| \Delta_4 \right|_\beta \leq C \mu^{-2} [h]^{s,\beta}_{\sigma,\mu,D}.
\]

The \( \rho \)-gradient of the hessian leads to estimates similar to the above. So the lemma is proven.

We summarize the results of this section into a proposition.

**Proposition 4.7.** — Let \( 0 < \sigma' < \sigma \leq 1, \ 0 < \mu' < \mu \leq 1 \).

There exists an absolute constant \( C \geq 1 \) such that

(i) if \( f = f^T \in T^{s,\beta}(\sigma,\mu,D) \) and

\[
[f]_{\sigma,\mu,D}^{s,\beta} \leq \frac{1}{2} (\mu - \mu')^2 (\sigma - \sigma'),
\]

then for all \( 0 \leq t \leq 1 \), the Hamiltonian flow map \( \Phi^t_f \) is a \( C^1 \)-map

\[
\mathcal{O}^s(\sigma',\mu') \times D \to \mathcal{O}^s(\sigma,\mu);
\]

real holomorphic and symplectic for any fixed \( \rho \in D \). Moreover,

\[
\left| \Phi^t_f(x,\cdot) - x \right|_{s,D} \leq C \left( \frac{1}{\sigma - \sigma'} + \frac{1}{\mu^2} \right) [f]_{\sigma,\mu,D}^{s,\beta}
\]

for any \( x \in \mathcal{O}^s(\sigma',\mu') \).

(ii) if \( f = f^T \in T^{s,\beta^+}(\sigma,\mu,D) \) and

\[
[f]_{\sigma,\mu,D}^{s,\beta^+} \leq \frac{1}{2} (\mu - \mu')^2 (\sigma - \sigma'),
\]
then for all \(0 \leq t \leq 1\) and all \(h \in \mathcal{T}^{s,\beta}(\sigma, \mu, D)\), the function \(h_t(x; \rho) = h(\Phi_t(x, \rho); \rho)\) belongs to \(\mathcal{T}^{s,\beta}(\sigma', \mu', D)\) and
\[
[h_t]^{s,\beta}_{\sigma', \mu', D} \leq C \frac{\mu}{(\mu - \mu')} [h]^{s,\beta}_{\sigma, \mu, D}.
\]

Proof. — Take \(\sigma' = \sigma - 2s\) and \(\mu' = \mu - 2\nu\) and apply lemmas 4.5 and 4.6.

5. Homological equation

Let us first recall the KAM strategy. Let \(h_0\) be the normal form Hamiltonian given by (2.9)
\[
h_0(r, \zeta, \rho) = \langle \omega_0(\rho), r \rangle + \frac{1}{2} \langle \zeta, A_0(\rho) \zeta \rangle
\]
satisfying Hypotheses A1-A3. Let \(f\) be a perturbation and
\[
f^T = f_\theta + \langle f_r, r \rangle + \langle f_\zeta, \zeta \rangle + \frac{1}{2} \langle f_{\zeta \zeta}, \zeta, \zeta \rangle
\]
be its jet (see (4.1)). If \(f^T\) was zero, then \(\{\zeta = r = 0\}\) would be an invariant \(n\)-dimensional torus for the Hamiltonian \(h_0 + f\). In general we only know that \(f\) is small, say \(f = \mathcal{O}(\varepsilon)\), and thus \(f^T = \mathcal{O}(\varepsilon)\). In order to decrease the error term we search for a hamiltonian jet \(S = S^T = \mathcal{O}(\varepsilon)\) such that its time-one flow-map \(\Phi_S = \Phi_S^1\) transforms the Hamiltonian \(h_0 + f\) to
\[
(h_0 + f) \circ \Phi_S = h + f^+, \quad \text{where} \quad h = h_0 + \tilde{h}, \quad \tilde{h} = c(\rho) + \langle \chi(\rho), r \rangle + \frac{1}{2} \langle \zeta, B(\rho) \zeta \rangle = \mathcal{O}(\varepsilon),
\]
and \((f^+)^T = \mathcal{O}(\varepsilon^2)\).

As a consequence of the Hamiltonian structure we have (at least formally) that
\[
(h_0 + f) \circ \Phi_S = h_0 + \{h_0, S\} + f^T + \mathcal{O}(\varepsilon^2).
\]
So to achieve the goal above we should solve the homological equation:
\[
(5.1) \quad \{h_0, S\} = \tilde{h} - f^T + \mathcal{O}(\varepsilon^2).
\]
Repeating iteratively the same procedure with \(h\) instead of \(h_0\) etc., we will be forced to solve the homological equation, not only for the normal form Hamiltonian (2.9), but for more general normal form Hamiltonians (2.4) with \(\omega\) close to \(\omega_0\) and \(A\) close to \(A_0\).

In this section we will consider a homological equation (5.1) with \(f\) in \(\mathcal{T}^{s,\beta}(D, \sigma, \mu)\) and we will construct a solution \(S\) in \(\mathcal{T}^{s,\beta^+}(D, \sigma, \mu)\).
5.1. Four components of the homological equation. — Let \( h \) be a normal form Hamiltonian \((2.4)\),

\[
h(r,\zeta,\rho) = \langle \omega(\rho), r \rangle + \frac{1}{2} \langle \zeta, A(\rho) \zeta \rangle,
\]

and let us write a jet-function \( S \) as

\[
S(\theta, r, \zeta) = S_{\theta}(\theta) + \langle S_{r}(\theta), r \rangle + \langle S_{\zeta}(\theta), \zeta \rangle + \frac{1}{2} \langle S_{\zeta \zeta}(\theta) \zeta, \zeta \rangle.
\]

Therefore the Poisson bracket of \( h \) and \( S \) equals

\[
\{ h, S \} = (\nabla_{\theta} \cdot \omega) S_{\theta} + \langle (\nabla_{\theta} \cdot \omega) S_{r}, r \rangle + \langle (\nabla_{\theta} \cdot \omega) S_{\zeta}, \zeta \rangle
\]

\[
+ \frac{1}{2} \langle (\nabla_{\theta} \cdot \omega) S_{\zeta \zeta}, \zeta \rangle - \langle AJS_{\zeta}, \zeta \rangle + \langle S_{\zeta \zeta} JA, \zeta \rangle.
\]

Accordingly the homological equation \((5.1)\) with \( h_0 \) replaced by \( h \) decomposes into four linear equations. The first two are

\[
\langle \nabla_{\theta} S_{\theta}, \omega \rangle = -f_{\theta} + c + O(\epsilon^2),
\]

\[
\langle \nabla_{\theta} S_{r}, \omega \rangle = -f_{r} + \chi + O(\epsilon^2).
\]

In these equations, we are forced to choose \( c(\rho) = \| f_{\theta}(\cdot, \rho) \| \) and \( \chi(\rho) = \| f_{r}(\cdot, \rho) \| \)

(see Notation) to achieve that the space mean-value of the r.h.s. vanishes. The other two equations are

\[
\langle \nabla_{\theta} S_{\zeta}, \omega \rangle - AJS_{\zeta} = -f_{\zeta} + O(\epsilon^2),
\]

\[
\langle \nabla_{\theta} S_{\zeta \zeta}, \omega \rangle - AJS_{\zeta \zeta} + S_{\zeta \zeta} JA = -f_{\zeta \zeta} + B + O(\epsilon^2),
\]

where the operator \( B \) will be chosen later. The most delicate, involving the small divisors as in \((2.8)\), is the last equation.

5.2. The first two equations. — We begin with equations \((5.2)\) and \((5.3)\) which are both of the form

\[
\langle \nabla_{\theta} \varphi(\theta, \rho), \omega(\rho) \rangle = \psi(\theta, \rho)
\]

with \( \| \psi \| = 0 \). Here \( \omega : D \to \mathbb{R}^n \) is \( C^1 \) and verifies

\[
| \omega - \omega_0 |_{C^1(D)} \leq \delta_0.
\]

Expanding \( \varphi \) and \( \psi \) in Fourier series,

\[
\varphi = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \hat{\varphi}(k)e^{ik \cdot \theta}, \quad \psi = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \hat{\psi}(k)e^{ik \cdot \theta},
\]

we solve eq. \((5.6)\) by choosing

\[
\hat{\varphi}(k) = -\frac{i}{\langle \omega, k \rangle} \hat{\psi}(k), \quad k \in \mathbb{Z}^n \setminus \{0\}; \quad \hat{\varphi}(0) = 0.
\]
Using Assumption A2 (i) we have, for each $k \neq 0$, either that
\[ |\langle \omega(\rho), k \rangle| \geq \delta_0 \quad \forall \rho \]
or that
\[ (\nabla_\rho \cdot \hat{z})(\langle k, \omega(\rho) \rangle) \geq \delta_0 \quad \forall \rho \]
for a suitable choice of a unit vector $\hat{z}$. The second case implies that
\[ |\langle \omega(\rho), k \rangle| \geq \kappa, \]
where $\kappa \leq \delta_0$, for all $\rho$ outside some open set $F_k \equiv F_k(\omega)$ of Lebesgue measure $\leq \delta_0^{-1} \kappa$.

Let
\[ D_1 = D \setminus \bigcup_{0 < |k| \leq N} F_k. \]
Then the closed set $D_1$ satisfies
\[ \text{meas}(D \setminus D_1) \leq N^n \frac{\kappa}{\delta_0}, \]
and $|\langle \omega(\rho), k \rangle| \geq \kappa$ for all $\rho \in D_1$. Hence, for $\rho \in D_1$ and all $0 < |k| \leq N$ we have
\[ |\hat{\varphi}(k)| \leq \frac{1}{\kappa} |\hat{\psi}(k)|. \]
Setting $\varphi(\theta, \rho) = \sum_{0 < |k| \leq N} \hat{\varphi}(k, \rho)e^{ik\cdot \theta}$, we get that
\[ \langle \nabla_\theta \varphi(\theta, \rho), \omega(\rho) \rangle = \psi(\theta, \rho) + R(\theta, \rho). \]
That is, thus defined $\varphi$ is an approximate solution of eq. (5.6) with the disparity $R(\theta, \rho) = -\sum_{|k| > N} \hat{\psi}(k, \rho)e^{ik\cdot \theta}$. We obtain by a classical argument that for $(\theta, \rho) \in T^n \times D_1$, $0 < \sigma' < \sigma$, and $j = 0, 1$
\[ |\varphi(\theta, \rho)| \leq \frac{C}{\kappa(\sigma - \sigma')^n} \sup_{|\theta| < \sigma} |\psi(\theta, \rho)|, \]
\[ |\partial_\rho^j R(\theta, \rho)| \leq \frac{C e^{-\frac{1}{2}(\sigma-\sigma')^2}}{(\sigma - \sigma')^n} \sup_{|\theta| < \sigma} |\partial_\rho^j \psi(\theta, \rho)|, \]
where $C$ only depends on $n$. If $\psi$ is a real function, then so are $\varphi$ and $R$.

Differentiating the formula for $\hat{\varphi}(k)$ in $\rho$ we obtain\(^{13}\)
\[ \partial_\rho \hat{\varphi}(k) = \chi_{|k| \leq N}(k) \left( -\frac{i}{(\omega, k)} \partial_\rho \hat{\psi}(k) + \frac{i}{(\omega, k)^2} (\partial_\rho \omega, k) \hat{\psi}(k) \right). \]
From this we derive that
\[ |\partial_\rho \varphi(\theta, \rho)| \leq \frac{C'}{\kappa^2(\sigma - \sigma')^n} \left( \sup_{|\theta| < \sigma} |\psi(\theta, \rho)| + \sup_{|\theta| < \sigma} |\partial_\rho \psi(\theta, \rho)| \right), \]

\(^{13}\) Here and below $\chi_Q(k)$ stands for the characteristic function of a set $Q \subset Z^n$. 

where $C'$ only depends on the derivative of $\omega$, which is bounded by $|\omega_0(\rho)|_{C^1} + \delta_0 \leq |\omega_0(\rho)|_{C^1} + 1$.

Applying this construction to (5.2) and (5.3) we get

**Proposition 5.1.** — Let $\omega : D \to \mathbb{R}^n$ be $C^1$ and verifying $|\omega - \omega_0|_{C^1(D)} \leq \delta_0$. Let $f \in T_0 \sigma$ and let $\delta_0 \geq \kappa > 0$, $N \geq 1$. Then there exists a closed set $D_1 = D_1(\omega, \kappa, N) \subset D$, satisfying

$$\text{meas}(D \setminus D_1) \leq C N^n \frac{\kappa}{\delta_0},$$

and

(i) there exist real $C^1$-functions $S_\theta$ and $R_\theta$ on $T^n_\sigma \times D_1 \to \mathbb{C}$, analytic in $\theta$, such that

$$\langle \nabla_\theta S_\theta(\theta, \rho), \omega(\rho) \rangle = -f_\theta(\theta, \rho) + \|f_\theta(\cdot, \rho)\| + R_\theta(\theta, \rho)$$

and for all $(\theta, \rho) \in T^n_\sigma \times D_1$, $\sigma' < \sigma$, and $j = 0, 1$

$$|\partial^j_\rho S_\theta(\theta, \rho)| \leq \frac{C}{\kappa^2(\sigma - \sigma')^n} \|f\|^s_{\sigma, \mu, D_1},$$

$$|\partial^j_\rho R_\theta(\theta, \rho)| \leq \frac{Ce^{-\frac{1}{2}(\sigma - \sigma')^N}}{(\sigma - \sigma')^n} \|f\|^s_{\sigma, \mu, D_1}.$$

(ii) there exist real $C^1$ vector-functions $S_r$ and $R_r$ on $T^n_\sigma \times D_1$, analytic in $\theta$, such that

$$\langle \nabla_\theta S_r(\theta, \rho), \omega(\rho) \rangle = -f_r(\theta, \rho) + \|f_r(\cdot, \rho)\| + R_r(\theta, \rho),$$

and for all $(\theta, \rho) \in T^n_\sigma \times D_1$, $\sigma' < \sigma$, and $j = 0, 1$

$$|\partial^j_\rho S_r(\theta, \rho)| \leq \frac{C}{\kappa^2(\sigma - \sigma')^n} \|f\|^s_{\sigma, \mu, D_1},$$

$$|\partial^j_\rho R_r(\theta, \rho)| \leq \frac{Ce^{-\frac{1}{2}(\sigma - \sigma')^N}}{(\sigma - \sigma')^n} \|f\|^s_{\sigma, \mu, D_1}.$$

The constant $C$ only depends on $|\omega_0|_{C^1(D)}$.

**5.3. The third equation.** — To begin with, we recall a result proved in the appendix of [11].

**Lemma 5.2.** — Let $A(t)$ be a real diagonal $N \times N$-matrix with diagonal components $a_j$ which are $C^1$ on $I = [1, 1]$, satisfying for all $j = 1, \ldots, N$ and all $t \in I$

$$a_j'(t) \geq \delta_0.
Let $B(t)$ be a Hermitian $N \times N$-matrix of class $C^1$ on $I$ such that
\[
\|B'(t)\| \leq \delta_0/2,
\]
for all $t \in I$. Then
\[
\|(A(t) + B(t))^{-1}\| \leq \frac{1}{\varepsilon}
\]
outside a set of $t \in I$ of Lebesgue measure $\leq CN \varepsilon \delta_0^{-1}$, where $C$ is a numerical constant.

Concerning the third component (5.4) of the homological equation we have

**Proposition 5.3.** — Let $\omega : D \to \mathbb{R}^n$ be $C^1$ and verifying $|\omega - \omega_0|_{C^1(D)} \leq \delta_0$. Let $D \ni \rho \mapsto A(\rho) \in NF \cap M_0$ be $C^1$ and verifying
\[
\|\partial_\rho^j (A(\rho) - A_0(\rho))\|_{HS} \leq \frac{1}{2} \delta_0
\]
for $j = 0, 1, a \in \mathcal{L}$ and $\rho \in D$. Let $f \in T_s^*(\sigma, \mu, D)$, $0 < \kappa \leq \frac{\delta_0}{2}$ and $N \geq 1$. Then there exists a closed set $D_2 = D_2(\omega, A, \kappa, N) \subset D$, satisfying
\[
\text{meas}(D \setminus D_2) \leq C N \exp^{\kappa \delta_0^2},
\]
and there exist real $C^1$-functions $S_\zeta$ and $R_\zeta$ from $\mathbb{T}^n \times D_2$ to $Y_s$, analytic in $\theta$, such that
\[
\langle \nabla_\theta S_\zeta(\theta, \rho), \omega(\rho) \rangle - A(\rho) J S_\zeta(\theta, \rho) = -f_\zeta(\theta, \rho) + R_\zeta(\theta, \rho)
\]
and for all $(\theta, \rho) \in \mathbb{T}_\sigma^* \times D_2$, $\sigma' < \sigma$, and $j = 0, 1$
\[
\mu \|\partial_\rho^j S_\zeta(\theta, \rho)\|_{s+1} + \mu \|\partial_\rho^j S_\zeta(\theta, \rho)\|_{\beta+} \leq \frac{C}{\kappa^2 (\sigma - \sigma')^n} [f]_{s, \sigma, \mu, D}^{s, \beta},
\]
\[
\mu \|\partial_\rho^j R_\zeta(\theta, \rho)\|_{s} + \mu \|\partial_\rho^j R_\zeta(\theta, \rho)\|_{\beta} \leq \frac{C e^{-\frac{1}{2}(\sigma - \sigma')^n} [f]_{s, \sigma, \mu, D}^{s, \beta}}{(\sigma - \sigma')^n}
\]
for $j = 0, 1$.

The exponent only depends on $d, n, \gamma$ while the constant $C$ depends on $|\omega_0|_{C^1(D)}$, $\|A_0\|_{C^1(D)}$.

**Proof.** — It is more convenient to deal with the hamiltonian operator $JA$ than with operator $AJ$. Therefore we multiply eq. (5.10) by $J$ and obtain for $JS_\zeta$ the equation
\[
\langle \nabla_\theta (JS_\zeta)(\theta, \rho), \omega(\rho) \rangle - JA(\rho)(JS_\zeta)(\theta, \rho) = -J F_\zeta(\theta, \rho) + JR_\zeta(\theta, \rho)
\]
\[\text{14. Here } \| \cdot \| \text{ means the operator-norm of a matrix associated to the euclidean norm on } \mathbb{C}^N.\]
Let us re-write (5.11) in the complex variables \( t(\xi, \eta) \). For \( a \in \mathcal{L} \)

\[
\zeta_a = \left( \begin{array}{c} p_a \\ q_a \end{array} \right) = U_a \left( \begin{array}{c} \xi_a \\ \eta_a \end{array} \right), \quad U_a = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & i \\ -i & 1 \end{array} \right).
\]

The symplectic operator \( U_a \) transforms the quadratic form \((\lambda_a/2)\langle \zeta_a, \zeta_a \rangle\) to \( i\lambda_a \xi_a \eta_a \). Therefore, if we denote by \( U \) the direct product of the operators \( \text{diag}(U_a, a \in \mathcal{L}) \) then it transforms \((1/2)\langle \zeta, A_0(\rho)\zeta \rangle \) to \( \sum_{a \in \mathcal{L}} i\lambda_a \xi_a \eta_a \). So it transforms the hamiltonian matrix \( JA_0(\rho) \) to the diagonal hamiltonian matrix

\[
\text{diag}\{i\lambda_a \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right), a \in \mathcal{L}\}.
\]

Then we make in (5.11) the substitution \( JS\zeta = US, JR\zeta = UR \) and \( -Jf\zeta = UF\zeta \), where \( S = t(\mathcal{S}_\xi, \mathcal{S}_\eta) \), etc. In this notation eq. (5.10) decouples into two equations

\[
\langle \nabla_\theta S_\xi, \omega \rangle - i QS_\xi = F_\xi + R_\xi, \\
\langle \nabla_\theta S_\eta, \omega \rangle + i QS_\eta = F_\eta + R_\eta.
\]

Here \( Q : \mathcal{L} \times \mathcal{L} \to \mathbb{C} \) is the scalar valued matrix associated to \( A \) via the formula (2.3), i.e.

\[
Q = \text{diag}\{\lambda_a, a \in \mathcal{L}\} + B,
\]

where \( B \) is Hermitian and block diagonal.

Written in the Fourier variables, eq. (5.13) becomes

\[
i\langle (k, \omega) + tQ \rangle S_\xi(k) = F_\xi(k) + R_\xi(k), \quad k \in \mathbb{Z}^n, \\
i\langle (k, \omega) - Q \rangle S_\eta(k) = F_\eta(k) + R_\eta(k), \quad k \in \mathbb{Z}^n.
\]

The two equations in (5.14) are similar, so let us consider (for example) the second one, and let us decompose it into its “components” over the blocks \([a]\):

\[
i\langle (k, \omega) \rangle + Q(\rho)|a\rangle \hat{S}_{[a]}(k) = \hat{F}_{[a]}(k, \rho) + \hat{R}_{[a]}(k)
\]

where the matrix \( Q_{[a]} \) is the restriction of \( Q \) to \([a] \times [a]\) and the vector \( \hat{F}_{[a]}(k, \rho) \) is the restriction of \( \hat{F}(k, \rho) \) to \([a]\) – here we have suppressed the upper index \( \eta \). Denotes by \( L(k, [a], \rho) \) the Hermitian operator in the left hand side of equation (5.15). We want to estimate the operator norm of \( L(k, [a], \rho)^{-1} \), i.e. we want to estimate from below the modulus of the eigenvalues of \( L(k, [a], \rho) \).

Let \( \alpha(\rho) \) denote an eigenvalue of the matrix \( Q_{[a]}(\rho), a \in \mathcal{L} \). It follows from (5.14) that

\[
|\alpha(\rho) - \lambda_a(\rho)| \leq \frac{\delta_0}{2} \leq \frac{c_0}{2}
\]

for some appropriate \( a \in [a] \), which implies that

\[
|\alpha(\rho)| \geq \frac{c_0}{2} w_a^2 \geq \kappa w_a
\]
by (2.5). Hence,
\[ \| L(0, [a], \rho)^{-1} \| \leq (\kappa w_a)^{-1} \quad \forall \rho, \forall a. \]
Assume that \( 0 < |k| \leq N \). Since \( |\langle k, \omega(\rho) \rangle| \lesssim N \) it follows from (2.5) that
\[ |\langle k, \omega(\rho) \rangle + \alpha(\rho)| \geq \frac{1}{2} w_a \geq \kappa w_a \]
whenever \( w_a \gtrsim (\frac{N}{c_0})^\frac{1}{2} \). Hence for these \( a \)'s we get
\[ \| L(k, [a], \rho)^{-1} \| \leq (\kappa w_a)^{-1} \quad \forall \rho. \]
Now let \( w_a \lesssim (\frac{N}{c_0})^\frac{1}{2} \). By Assumption A2(ii) we have either
\[ |\langle k, \omega(\rho) \rangle + \lambda_a(\rho)| \geq \delta_0 w_a \quad \forall \rho, \forall a \]
or we have a unit vector \( z \) such that
\[ (\nabla \rho \cdot z)(\langle k, \omega(\rho) \rangle + Q(\rho)[a]) \geq \delta_0 \quad \forall \rho, \forall a. \]
The first case clearly implies (5.16), so let us consider the second case. By (5.9) it follows that
\[ \| (\nabla \rho \cdot z)H[a](\rho) \| \leq \frac{\delta_0}{2}. \]
The Hermitian matrix \( (\langle k, \omega(\rho) \rangle + Q(\rho)[a]) \) is of dimension \( \lesssim w_a^d \) (see (2.1)) therefore, by Lemma 5.2 we conclude that (5.16) holds for all \( \rho \) outside a suitable set \( F_{a,k} \) of measure \( \lesssim w_a^d \kappa \delta_0^{-1} \). Let
\[ D_2 = D \setminus \bigcup_{\substack{|k| \leq N \\ w_a \lesssim (\frac{N}{c_0})^{\frac{1}{2}}} \bigcup F_{a,k}. \]
Then we get
\[ \text{meas}(D \setminus D_2) \lesssim N^n \left( \frac{N}{c_0} \right)^{-\frac{d+1}{2}} \frac{\kappa}{\delta_0} \]
and (6.16) holds for all \( \rho \in D_2 \), all \( |k| \leq N \) and all \( [a] \).

The equation (5.15) is now solved by
\[ \hat{S}_{[a]}(k, \rho) = \chi_{|k| \leq N(k)} L(k, [a], \rho)^{-1} \hat{F}_{[a]}(k, \rho), \quad a \in \mathcal{L}, \]
and
\[ \hat{R}_{[a]}(k, \rho) = \chi_{|k| > N(k)} \hat{F}_{[a]}(k, \rho), \quad a \in \mathcal{L}. \]

\[ \text{We use that the operator norm is controlled by the Hilbert Schmidt norm: } \| M \| \leq \| M \|_{HS}. \]
Using (5.16) we have for \( \rho \in D_2 \)
\[
\|S_0(\theta, \rho)\| \lesssim \frac{1}{\kappa} w_a(\sigma - \sigma')^n \sup_{|\theta| < \sigma} \|F_a(\theta, \rho)\|,
\]
\[
|R_a(\theta, \rho)| \lesssim e^{-\frac{1}{2}(\sigma-\sigma')^N} \sup_{|\theta| < \sigma} |F_a(\theta, \rho)|.
\]

for \( \theta \in \mathbb{T}_n^d \), see (5.20).

Since \( \|S\|^2 = \sum_{a \in \mathcal{L}} w_a^2 |S_a|^2 = \sum_{a \in \mathcal{L}} w_a^2 \|S_a\|^2 \) these estimates imply that
\[
\|S(\theta, \rho)\|_{s+1} + |S(\theta, \rho)|_{\beta+} \lesssim \frac{1}{\kappa(\sigma - \sigma')^n} \sup_{|\theta| < \sigma} \|F(\theta, \rho)\|_s,
\]
\[
\|R(\theta, \rho)\|_s + |R(\theta, \rho)|_{\beta+} \lesssim e^{-\frac{1}{2}(\sigma-\sigma')^N} \sup_{|\theta| < \sigma} \|F(\theta, \rho)\|_s,
\]

for any \( \sigma' \leq \sigma \). The estimates of the derivatives with respect to \( \rho \) are obtained by differentiating (5.17) and (5.18) as in Proposition 5.3.

The functions \( F \) and \( R \) are complex, and a-priori the constructed solution \( S_{c^*} \) also may be complex. Instead of proving that it is real, we replace \( S_{c^*}, \theta \in \mathbb{T}^n \), by its real part and then analytically extend it to \( \mathbb{T}_n^d \), using the relation \( \Re S_{c^*}(\theta, \rho) := \frac{1}{2}(S_{c^*}(\theta, \rho) + \bar{S}_{c^*}(\theta, \rho)) \). Thus we obtain a real solution which obeys the same estimates. \( \square \)

**5.4. The last equation.** — Concerning the fourth component of the homological equation, (5.3), we have the following result

**Proposition 5.4.** — Let \( \omega : D \to \mathbb{R}^n \) be \( C^1 \) and verifying \( |\omega - \omega_0|_{C^1(D)} \leq \delta_0 \). Let \( D \ni \rho \to A(\rho) \in \mathcal{N} \mathcal{F} \cap \mathcal{M}_\beta \) be \( C^1 \) and verifying
\[
\|\partial_\rho^j (A(\rho) - A_0(\rho))_{[a]}\|_{H^s} \leq \frac{\delta_0}{4w_a^{2\beta}}
\]
for \( j = 0, 1, a \in \mathcal{L} \) and \( \rho \in D \). Let \( f \in \mathbb{T}_n^{s, \beta}(\sigma, \mu, D), \) \( 0 < \kappa \leq \frac{\delta_0}{\mu} \) and \( N \geq 1 \).

Then there exists a subset \( D_3 = D_3(h, \kappa, N) \subset D \), satisfying
\[
\text{meas}(D \setminus D_3) \leq C \left( \frac{N}{\epsilon_0} \right)^{\exp \left( \frac{\kappa}{\delta_0} \right)}^{\exp'},
\]

and there exist real \( C^1 \)-functions \( B : D_3 \to \mathcal{M}_\beta \cap \mathcal{N} \mathcal{F}, S_{c^*}(0) : D_3 \to M \) and \( S_{c^*} - \bar{S}_{c^*}(0), R_{c^*}(\cdot ; \rho) : \mathbb{T}_n^d \times D_3 \to \mathcal{M}_\beta \), analytic in \( \theta \), such that
\[
(\nabla_\theta S_{c^*}(\theta, \rho), \omega(\rho)) - A(\rho)JS_{c^*}(\theta, \rho) + S_{c^*}(\theta, \rho)JA(\rho) = -f_{c^*}(\theta, \rho) + B(\rho) + R_{c^*}(\theta, \rho)
\]
and for all \((\theta, \rho) \in T^n \times D_3, \sigma' < \sigma, \) and \(j = 0, 1\)

\[
\mu^2 \left| \partial^j_\rho R_{\zeta \zeta}(\theta, \rho) \right|_\beta \leq C e^{-\frac{1}{2}(\sigma-\sigma')N} [f]_{\sigma, \mu, D}^{s, \beta},
\]

\[
\mu^2 \left| \partial^j_\rho S_{\zeta \zeta}(\theta, \rho) \right|_{\beta'} \leq C \frac{1}{(\sigma-\sigma')^{2n}} [f]_{\sigma, \mu, D}^{s, \beta},
\]

\[
\mu^2 \left| \partial^j_\rho B(\rho) \right|_\beta \leq C [f]_{\sigma, \mu, D}^{s, \beta}.
\]

The two exponents \(\exp\) and \(\exp'\) are positive numbers depending on \(n, \gamma, d, \alpha_1, \alpha_2, \beta.\) The constant \(C\) depends also on \(h_0.\)

**Proof.** — As in the previous section, and using the same notation, we re-write (5.20) in complex variables. So we introduce

\[S = \mathcal{U} S_{\zeta \zeta} U, \quad R = \mathcal{U} R_{\zeta \zeta} U\]  
and \(F = \mathcal{U} F_{\zeta \zeta} U.\)

By construction, \(S_a^b \in \mathcal{M}_{2 \times 2}\) for all \(a, b \in \mathcal{L}.\) Let us denote

\[S_a^b = \begin{pmatrix} (S_a^b)^{\xi \xi} & (S_a^b)^{\xi \eta} \\ (S_a^b)^{\eta \xi} & (S_a^b)^{\eta \eta} \end{pmatrix}\]

and then

\[S^{\xi \xi} = ((S_a^b)^{\xi \xi})_{a, b \in \mathcal{L}}, \quad S^{\xi \eta} = ((S_a^b)^{\xi \eta})_{a, b \in \mathcal{L}}, \quad S^{\eta \eta} = ((S_a^b)^{\eta \eta})_{a, b \in \mathcal{L}}.\]

We use similar notations for \(R, B\) and \(F.\)

In this notation (5.20) decouples into three equations \(16\)

\[
\langle \nabla_\theta S^{\xi \xi}, \omega \rangle + i QS^{\xi \xi} + i S^{\xi \xi} Q = B^{\xi \xi} - F^{\xi \xi} + R^{\xi \xi},
\]

\[
\langle \nabla_\theta S^{\eta \eta}, \omega \rangle - i QS^{\eta \eta} - i S^{\eta \eta} Q = B^{\eta \eta} - F^{\eta \eta} + R^{\eta \eta},
\]

\[
\langle \nabla_\theta S^{\xi \eta}, \omega \rangle + i QS^{\xi \eta} - i S^{\xi \eta} Q = B^{\xi \eta} - F^{\xi \eta} + R^{\xi \eta},
\]

where we recall that \(Q\) is the scalar valued matrix associated to \(A\) via the formula \((2.3).\) The first and the second equations are of the same type, so we focus on the resolution of the second and the third equations. Written in Fourier variables, they read

\[
i \langle \langle k, \omega \rangle - \langle k \rangle Q \rangle S^{\eta \eta}(k) - i \dot{S}^{\eta \eta}(k) = \delta_{k, 0} B^{\eta \eta} - \dot{F}^{\eta \eta}(k) + \dot{R}^{\eta \eta}(k), \quad k \in \mathbb{Z}^n,
\]

\[
i \langle \langle k, \omega \rangle + Q \rangle \dot{S}^{\xi \eta}(k) - i \dot{S}^{\xi \eta}(k) = \delta_{k, 0} B^{\xi \eta} - \dot{F}^{\xi \eta}(k) + \dot{R}^{\xi \eta}(k), \quad k \in \mathbb{Z}^n,
\]

where \(\delta_{k, j}\) denotes the Kronecker symbol.

\(16.\) Actually (5.20) decomposes into four scalar equations but the fourth one is the transpose of the third one.
Equation (5.23). We chose $B^\eta = 0$ and decompose the equation into “components” on each product block $[a] \times [b]$:

\[ L \hat{S}^{[b]}_{[a]}(k) = i \hat{F}^{[b]}_{[a]}(k, \rho) - i \hat{R}^{[b]}_{[a]}(k) \]

where we have suppressed the upper index $\eta$ and the operator $L := L(k, [a], [b], \rho)$ is the linear Hermitian operator, acting in the space of complex $[a] \times [b]$-matrices defined by

\[ LM = \left((k, \omega(\rho)) - i Q_{[a]}(\rho)\right)M - MQ_{[b]}(\rho). \]

The matrix $Q_{[a]}$ can be diagonalized in an orthonormal basis:

\[ t P_{[a]} Q_{[a]} P_{[a]} = D_{[a]} \]

Therefore denoting $\hat{S}^{[b]}_{[a]} = t P_{[a]} S^{[b]}_{[a]} P_{[b]}$, $\hat{F}^{[b]}_{[a]} = t P_{[a]} F^{[b]}_{[a]} P_{[b]}$ and $\hat{R}^{[b]}_{[a]} = t P_{[a]} R^{[b]}_{[a]} P_{[b]}$ the homologous equation (5.26) reads

\[ (k, \omega) + D_{[a]} \hat{S}^{[b]}_{[a]}(k) - S^{[b]}_{[a]}(k)D_{[b]} = i \hat{F}^{[b]}_{[a]}(k) - i \hat{R}^{[b]}_{[a]}(k) \]

This equation can be solved term by term:

\[ \hat{R}^{\prime}_{\ell}(k) = \hat{F}^{\prime}_{\ell}(k), \quad j \in [a], \ell \in [b], |k| > N \]

and

\[ \hat{S}^{\prime}_{\ell}(k) = \frac{i}{(k, \omega(\rho)) - \alpha_j(\rho) - \beta_\ell(\rho)} \hat{F}^{\prime}_{\ell}(k), \quad j \in [a], \ell \in [b], |k| \leq N \]

where $\alpha_j(\rho)$ and $\beta_\ell(\rho)$ denote eigenvalues of $Q_{[a]}(\rho)$ and $Q_{[b]}(\rho)$, respectively.

As $Q_{[a]} = \text{diag}\{\lambda_a : a \in [a]\} + B_{[a]}$ with $B$ Hermitian, using hypothesis (5.19) we get that

\[ |(\alpha_j(\rho) + \beta_\ell(\rho)) - (\lambda_a + \lambda_b(\rho))| \leq \frac{\delta_0}{4} + \frac{\delta_0}{4} \leq \frac{\delta_0}{2}. \]

It follows as in the proof of Proposition 5.3 using Lemma 5.22 relation (2.13), Assumption A2(iii) and (5.19), that there exists a subset $D_2 = D_2(h, \kappa, N) \subset D$, satisfying

\[ \text{meas}(D \setminus D_2) \leq C \left( \frac{N}{c_0} \right)^{\exp \frac{\kappa}{\delta_0}}, \]

such that

\[ |(k, \omega(\rho)) - \alpha_j(\rho) - \beta_\ell(\rho)| \geq \kappa(1 + |w_a + w_b|), \]

holds for all $\rho \in D_2$, all $|k| \leq N$, all $j \in [a]$, $\ell \in [b]$ and all $[a], [b] \in \tilde{L}$. Thus for $\rho \in D_2$ we obtain

\[ |\hat{S}^{\prime}_{\ell}(k)| = \frac{1}{\kappa(1 + |w_a + w_b|)} |\hat{F}^{\prime}_{\ell}(k)|. \]
which leads to
\[ \| S_{[a]}^{[b]} \|_{HS} = \| S_{[a]}^{[b]} \|_{HS} \leq \frac{1}{\kappa(1 + |w_a + w_b|)} \| F_{[a]}^{[b]} \|_{HS} \]
\[ = \frac{1}{\kappa(1 + |w_a + w_b|)} \| F_{[a]}^{[b]} \|_{HS}. \]
Therefore we obtain a solution satisfying for any $|3\theta| < \sigma’$
\[ |S(\theta)|_{\beta} \lesssim \frac{1}{\kappa(\sigma - \sigma’)^N} \sup_{|3\theta| < \sigma} |F(\theta)|_{\beta}, \]
\[ |R(\theta)|_{\beta} \lesssim e^{-\frac{1}{2}(\sigma - \sigma’)^N} \sup_{|3\theta| < \sigma} |F(\theta)|_{\beta}. \]

The estimates for the derivatives with respect to $\rho$ are obtained by differentiating (5.20) and (5.28).

Equation (5.23). It remains to consider (5.23) which decomposes into the product blocks $[a] \times [b]$:
\[ L \hat{S}_{[a]}^{[b]}(k) := \langle k, \omega(\rho) \rangle \hat{S}_{[a]}^{[b]}(k) + Q_{[a]}(\rho)\hat{S}_{[a]}^{[b]}(k) \]
\[ - \hat{S}_{[a]}^{[b]}(k)Q_{[b]}(\rho) = -i\delta_{k,0}B_{[a]}^{[b]} + i\hat{F}_{[a]}^{[b]}(k, \rho) - i\hat{R}_{[a]}^{[b]}(k), \]
\[- \text{ here } L = L(k, [a], [b], \rho), \text{ and we have suppressed the upper index } \xi \eta. \text{ We use the notation from the study of equation (5.24) above and we assume without loss of generality that } w_a \leq w_b. \text{ Now } L(k, [a], [b], \rho) \text{ is a linear operator acting in the space of complex } [a] \times [b]-\text{matrices. Its eigenvalues are } \langle k, \omega(\rho) \rangle + \alpha_j(\rho) - \beta_{\ell}(\rho), \quad j \in [a], \quad \ell \in [b]. \]
To estimates these eigenvalues we have to distinguish two cases, depending on whether $k = 0$ or not.

The case $k = 0$. In this case we distinguish whether $w_a = w_b$ or not.

When $w_a \neq w_b$, we use (5.19) and (2.6) to get
\[ |\alpha_j(\rho) - \beta_{\ell}(\rho)| \geq c_0|w_a - w_b| - \frac{\delta_0}{4w_a^{23}} - \frac{\delta_0}{4w_b^{23}} \geq \kappa|w_a - w_b|. \]

This last estimate allows us to solve (5.31), choosing
\[ B_{[a]}^{[b]} = \hat{R}_{[a]}^{[b]}(0) = 0 \]
and
\[ \hat{S}_{[a]}^{[b]}(0) = L(0, [a], [b], \rho)^{-1}\hat{F}_{[a]}^{[b]}(0) \]
with
\[ \| \hat{S}_{[a]}^{[b]}(0) \|_{HS} \lesssim \frac{1}{c_0|w_a - w_b|} \| \hat{F}_{[a]}^{[b]}(0) \|_{HS}. \]
This implies that

\[ |\hat{S}(0)|_{\beta+} \lesssim \frac{1}{\kappa} |\hat{F}(0)|_{\beta}, \]

and the estimates of the derivatives with respect to \( \rho \) are obtained by differentiating the expression for \( \hat{S}_{[a]}^{[b]}(0) \).

When \( w_a = w_b \), we cannot control \( |\alpha_j(\rho) - \beta_\ell(\rho)| \) from below, so we define

\[ \hat{S}_{[a]}^{[b]}(0) = 0, \quad \hat{R}_{[a]}^{[b]}(0) = 0 \]

and

\[ B_{[a]}^{[b]} = \hat{F}_{[a]}^{[b]}(0). \]

This gives the estimates

\[ |B|_{\beta} \leq |\hat{F}(0)|_{\beta}. \]

The estimates of the derivatives with respect to \( \rho \) are obtained by differentiating the expressions for \( B \).

**The case \( k \neq 0 \).** If \( k \neq 0 \) we face the small divisors (5.32) with non-trivial \( \langle k, \omega \rangle \). Using Hypothesis A3, there is a set \( D'_{\omega} = D(\omega, 2\eta, N) \),

\[ \text{meas}(D \setminus D'_{\omega}) \lesssim N^{\alpha_1}(\frac{\eta}{\delta_0})^\alpha_2, \]

such that for all \( \rho \in D'_{\omega} \) and \( 0 < |k| \leq N \)

\[ |\langle k, \omega(\rho) \rangle - \lambda_a(\rho) + \lambda_b(\rho)| \geq 2\eta(1 + |w_a - w_b|). \]

By (5.19) this implies

\[ |\langle k, \omega(\rho) \rangle - \alpha_j(\rho) + \beta_\ell(\rho)| \geq 2\eta(1 + |w_a - w_b|) - \frac{\delta_0}{4w_a^{2\beta}} - \frac{\delta_0}{4w_b^{2\beta}} \]

\[ \geq \eta(1 + |w_a - w_b|) \]

if

\[ w_b \geq w_a \geq \left( \frac{\delta_0}{2\eta} \right)^{\frac{1}{2\beta}}. \]

Let now \( w_a \lesssim \left( \frac{\delta_0}{2\eta} \right)^{\frac{1}{2\beta}} \). We note that \( |\langle k, \omega(\rho) \rangle - \lambda_a(\rho) + \lambda_b(\rho)| \leq 1 \) implies that \( w_b \lesssim \left( \frac{\delta_0}{2\eta} \right)^{\frac{1}{2\beta}} + C|k| \lesssim \left( \frac{\delta_0}{2\eta} \right)^{\frac{1}{2\beta}} + N \). Using Assumption A2 (iv) and condition (5.19) we get as in section 5.3 that

\[ (5.33) \quad |\langle k, \omega(\rho) \rangle + \alpha_j(\rho) - \beta_\ell(\rho)| \geq \kappa(1 + |w_a - w_b|) \quad \forall j \in [a], \quad \forall \ell \in [b] \]

holds outside a set \( F_{[a],[b],k} \) of measure \( \lesssim w_a^{d}w_b^{d}(1 + |w_a - w_b|)\kappa_0^{-1} \).
If $F$ is the union of $F_{[a], [b], k}$ for $|k| \leq N$, $[a], [b] \in \mathcal{L}$ such that $w_a \lesssim \left( \frac{\delta_0}{2\eta} \right)^{\frac{1}{2}}$ and $w_b \lesssim \left( \frac{\delta_0}{2\eta} \right)^{\frac{1}{2}} + N$ respectively, we have

$$\text{meas}(F) \lesssim \left( \frac{\delta_0}{2\eta} \right)^{\frac{1}{2}} \left( \frac{\delta_0}{2\eta} \right)^{\frac{1}{2}} + N \right)^{d+1} K \left( \frac{\delta_0}{2\eta} \right)^{\frac{1}{2}} + N \right)^{d+1} \left( \frac{\delta_0}{2\eta} \right)^{\frac{1}{2}} \right)$$

Now we choose $\eta$ so that

$$\left( \frac{\eta}{\delta_0} \right)^{2\beta} = \left( \frac{\delta_0}{\eta} \right)^{\frac{2\beta}{2\beta + 2\eta}}$$

Then, as $\beta \leq 1$, $\eta \leq \kappa$ and we have

$$\text{meas}(F) \lesssim N^{d+2} \left( \frac{\kappa}{\delta_0} \right)^{\frac{2\beta}{2\beta + 2\eta}}$$

Let $D_3 = D_2 \cap D_2 \setminus F$, we have

$$\text{meas}(D \setminus D_3) \lesssim N^{d+2} \left( \frac{\kappa}{\delta_0} \right)^{\frac{2\beta}{2\beta + 2\eta}}$$

and by construction for all $\rho \in D_3$, $0 < |k| \leq N$, $a, b \in \mathcal{L}$ and $j \in [a]$, $\ell \in [b]$ we have

$$|\langle k, \omega(\rho) \rangle - \alpha_j(\rho) + \beta_\ell(\rho) | \geq \kappa (1 + |w_a - w_b|)$$

Hence, following the same procedure of diagonalization of $L$ as in the resolution of equation (5.24), the solution

$$\hat{S}_{[a]}^{[b]}(k) = \chi_{0 < |k| \leq N} L(k, [a], [b], \rho)^{-1} \tilde{F}_{[a]}^{[b]}(k),$$

and

$$\hat{R}_{[a]}^{[b]}(k) = \chi_{|k| > N} \bar{F}_{[a]}^{[b]}(k),$$

satisfies

$$|S(\theta)|_{\beta} \lesssim \frac{1}{\kappa (\sigma - \sigma')^n} \sup_{|\beta| < |\sigma|} |F(\theta)|_{\beta},$$

$$|R|_{\beta} \lesssim e^{-\frac{1}{2} (\sigma - \sigma')^N} \sup_{|\beta| < |\sigma|} |F(\theta)|_{\beta}$$

for $\theta \in T_{\sigma}$. The estimates of the derivatives with respect to $\rho$ are obtained by differentiating the expressions for $S$ and $R$.

In this way we have constructed a solution $S_{\zeta\zeta}, R_{\zeta\zeta}, B$ of the fourth component of the homological equation which satisfies all required estimates. To guarantee that it is real, as at the end of Section 5.3 we replace $S_{\zeta\zeta}, R_{\zeta\zeta}, B$ by their real parts (i.e., replace $S_{\zeta\zeta}(\theta, \rho)$ by $\frac{1}{2}(S_{\zeta\zeta}(\theta, \rho) + S_{\zeta\zeta}(\theta, \rho))$, etc.)
5.5. Summing up. — Let
\[ h = \omega(\rho) \cdot r + \frac{1}{2} \langle \zeta, A(\rho) \zeta \rangle \]
where \( \rho \to \omega(\rho) \) and \( \rho \to A(\rho) \) are \( C^1 \) on \( D \) and \( A \) is on normal form.

**Proposition 5.5.** — Assume
\[ \begin{equation} \tag{5.34} |\partial^j_\rho(A(\rho) - A_0(\rho))|_\beta \leq \frac{\delta_0}{4}, \quad |\partial^j_\rho(\omega - \omega_0)| \leq \delta_0 \end{equation} \]
for \( j = 0, 1 \) and \( \rho \in D \). Let \( f \in T^{s,\beta}(\sigma, \mu, D) \), \( 0 < \kappa \leq \frac{\delta_0}{2} \) and \( N \geq 1 \). Then there exists a subset \( D' = D'(h, \kappa, N) \subset D \), satisfying
\[ \text{meas}(D \setminus D') \leq C N^{\exp \left( \frac{\kappa}{\delta_0} \right)^{\exp'}} \]
and there exist real jet-functions \( S \in T^{s,\beta}(\sigma', \mu, D') \), \( R \in T^{s,\beta}(\sigma', \mu, D) \) and a normal form
\[ \hat{h} = \left\{ f(\cdot, 0; \rho) \right\} + \left\{ \nabla_r f(\cdot, 0; \rho) \right\} \cdot r + \frac{1}{2} \langle \zeta, B(\rho) \zeta \rangle, \]
such that
\[ \{h, S\} + f^T = \hat{h} + R. \]
Furthermore, for all \( 0 \leq \sigma' < \sigma \)
\[ \begin{equation} \tag{5.35} |\partial^j_\rho B(\rho)|_\beta \leq [f]^{s,\beta}_{\sigma', \mu, D'}, \quad j = 0, 1 \text{ and } \rho \in D' \end{equation} \]
\[ \begin{equation} \tag{5.36} [S]^{s,\beta+}_{\sigma', \mu, D'} \leq C \frac{1}{\kappa^2(\sigma - \sigma')^n} [f]^{s,\beta}_{\sigma, \mu, D'} \end{equation} \]
\[ \begin{equation} \tag{5.37} [R]^{s,\beta}_{\sigma', \mu, D'} \leq C e^{-\frac{1}{2}(\sigma' - \sigma')N} (\sigma - \sigma')^n [f]^{s,\beta}_{\sigma, \mu, D'}. \end{equation} \]
The two exponents \( \exp \) and \( \exp' \) are positive numbers depending on \( n, d, \alpha_1, \alpha_2, \gamma, \beta \). The constant \( C \) depends also on \( h_0 \).

6. Proof of the KAM Theorem.

The theorem is proved by an iterative KAM procedure. We first describe the general step of this KAM procedure.
6.1. The KAM step. — Let \( h \) be a normal form Hamiltonian
\[
h = \omega \cdot r + \frac{1}{2} (\zeta, A(\omega) \zeta)
\]
with \( A \) on normal form, \( A - A_0 \in \mathcal{M}_\beta \) and satisfying (5.34). Let \( f \in \mathcal{T}^{s,\beta}(D, \sigma, \mu) \) be a (small) Hamiltonian perturbation. Let \( S = S^T \in \mathcal{T}^{s,\beta+}(D', \sigma', \mu) \) be the solution of the homological equation
\[
\{ h, S \} + f^T = \hat{h} + R.
\]
defined in Proposition 5.5. Then defining
\[
h^+ := h + \hat{h},
\]
we get
\[
h \circ \Phi^1_S = h^+ + f^+
\]
with
\[
f^+ = R + (f - f^T) \circ \Phi^1_S + \int_0^1 \{ (1 - t)(\hat{h} + R) + tf^T, S \} \circ \Phi^t_S \, dt.
\]
The following Lemma gives an estimation of the new perturbation:

**Lemma 6.1.** — Let \( \kappa > 0, N \geq 1, 0 < \sigma' < \sigma \leq 1 \) and \( 0 < 2\mu' < \mu \leq 1 \). Assume that \( D' \subset D \), that \( f \in \mathcal{T}^{s,\beta}(D, \sigma, \mu) \), that \( R \) satisfies (5.37) and that \( S = S^T \) belongs to \( \mathcal{T}^{s,\beta+}(D', \sigma'', \mu) \) with \( \sigma'' = \frac{\sigma + \sigma'}{2} \) and satisfies
\[
[S]^{s,\beta+}_{D',\sigma'',\mu} \leq \frac{1}{16} \mu^2(\sigma - \sigma').
\]
Then the function \( f^+ \) given by formula (6.2) belongs to \( \mathcal{T}^{s,\beta}(D', \sigma', \mu') \) and
\[
[f^+]^{s,\beta}_{D',\sigma',\mu'} \leq C \left( \frac{\mu'}{\mu} \right)^3 + \frac{1}{\kappa^2 \mu^2(\sigma - \sigma')^{n+1}} \| f \|^{s,\beta}_{D,\sigma,\mu} \| f \|^{s,\beta}_{D,\sigma,\mu}
\]
where \( C \) depends on \( h_0 \).

**Proof.** — Let us denote the three terms in the r.h.s. of (6.2) by \( f_1^+, f_2^+ \) and \( f_3^+ \). In view of (5.37), we have that \( [f_1^+]^{s,\beta}_{D',\sigma',\mu'} \) is controlled by the first term in r.h.s. of (6.4).

By Lemma 4.2, we get
\[
[f_2^+]^{s,\beta}_{D',\sigma',\mu'} \leq C \left( \frac{\mu'}{\mu} \right)^3 \| f \|^{s,\beta}_{D,\sigma,\mu}.
\]
By hypothesis \( S = S^T \) belongs to \( \mathcal{T}^{s,\beta+}(D', \sigma', \mu) \) and satisfies (6.3) which implies \( [S]^{s,\beta+}_{D',\sigma'',\mu} \leq \frac{1}{16} (\mu - \mu')^2(\sigma'' - \sigma') \) since \( 2\mu' < \mu \). Therefore by Proposition
and since $2\mu' \leq 2(\mu - \mu')$, $[f_2]^{s,\beta}_{\sigma',\mu'}$ is controlled by the second term in r.h.s. of (6.4).

It remains to control $[f_3]^{s,\beta}_{\sigma',\mu'}$. To begin with, $g_t := (1 - t)(h + R) + tf^T$ is jet function in $T^{s,\beta}(D, \sigma', \mu)$. Furthermore, defining for $j = 1, 2$,

$$\sigma_j = \sigma' + j\frac{\sigma - \sigma'}{3}$$

and using (5.37) we get (for $N$ large enough)

$$[g_t]^{s,\beta}_{\sigma',\mu} \leq C \left(1 + 3^n \frac{C_{\alpha}(\sigma - \sigma')^N/6}{(\sigma - \sigma')^n}\right) \left(f_3]^{s,\beta}_{\sigma',\mu'} \leq C_{\sigma,\mu}.\right.$$

On the other hand $S \in T^{s,\beta}(D', \sigma_2, \mu)$ is also a jet-function and satisfies

$$[S]^{s,\beta}_{D',\sigma_2,\mu} \leq C \frac{1}{\kappa^2(\sigma - \sigma')^n}[f_3]^{s,\beta}_{D,\sigma,\mu}.$$

Then using Lemma 4.3 we have

$$\{[g_t, S]\}^{s,\beta}_{D',\sigma_1,\mu} \leq C \frac{1}{\kappa^2\mu^2(\sigma - \sigma')^n+1}(f_3]^{s,\beta}_{D,\sigma,\mu})^2.$$

We conclude the proof by Proposition 4.6.

6.2. Choice of parameters. — To prove the main theorem we construct the transformation $\Phi$ as the composition of infinitely-many transformations $S$ as in Theorem 5.5, i.e. for all $k \geq 1$ we construct iteratively $S_{k-1}$, $h_k$, $f_k$ following the general scheme (6.1)–(6.2) in such way

$$(h + f) \circ \Phi^1_{S_{k-1}} \circ \cdots \circ \Phi^1_{S_0} = h_k + f_k.$$ 

At each step $f_k \in T^{s,\beta}(D_k, \sigma_k, \mu_k)$ with $[f_k]^{s,\beta}_{D_k,\sigma_k,\mu_k} \leq \varepsilon_k$, $h_k = \langle \omega_k, r \rangle + \frac{1}{2} \langle \zeta, A_k \zeta \rangle$ is on normal form, the Fourier series are truncated at order $N_k$, and the small divisors are controlled by $\kappa_k$. In this section we specify the choice of all the parameters for $k \geq 1$.

First we fix

$$\delta_0 = \varepsilon^{1/4}, \quad \kappa_0 = \varepsilon^{1/3}.$$ 

We define $\varepsilon_0 = \varepsilon$, $\sigma_0 = \sigma$, $\mu_0 = \mu$ and for $j \geq 1$ we choose

$$\sigma_{j-1} - \sigma_j = C_{\varepsilon_0} \sigma_0 j^{-2},$$

$$\mu_j = \varepsilon_{j-1}^2 \mu_0,$$

$$N_j = (\sigma_{j-1} - \sigma_j)^{-1} \ln \varepsilon_j^{-1},$$

$$\kappa_{j-1} = \varepsilon_{j-1}.$$
where \((C_\epsilon)^{-1} = 2 \sum_{j \geq 1} \frac{1}{j^2}\). The numbers above are defined in terms of \(\varepsilon_j\)'s which are defined inductively (with given \(\varepsilon = \varepsilon_0\)) accordingly to Lemma 6.1 through the relation

\[
\varepsilon_{j+1} = C \left( \frac{1}{2} \varepsilon_j (j+1)^2 \sigma_0 + \frac{\varepsilon_j}{\varepsilon_{j-1}} \right)^{\frac{6}{5}} + (j+1)^{2(n+1)} \sigma_0^{-n-1} \mu_0^{-\frac{1}{5}} \varepsilon_j^{\frac{1}{5} - \frac{1}{32}} \varepsilon_j.
\]

(6.5)

**Lemma 6.2.** — For all \(j \geq 1\)

\[
\varepsilon_j \leq \varepsilon_0^{(7/6)^j}
\]

provided that \(\varepsilon = \varepsilon_0 > 0\) is sufficiently small (in terms of \(n, \sigma_0, \mu_0\)).

*Proof.* — It suffices to check that if

\[
\varepsilon_k \leq \varepsilon_k^{7/6} \text{ for all } k \leq j,
\]

then all the three terms in the r.h.s. of (6.5) are \(\leq \frac{1}{3} \varepsilon_j^{7/6}\). For the first term this is straightforward assuming \(\varepsilon\) small enough. To obtain the same estimate for the third term it suffices to notice that \(\frac{1}{6} < \frac{1}{5} - \frac{1}{32}\). Concerning the second term, we have \(\varepsilon_j \leq \varepsilon_j^{1/7} \varepsilon_{j-1}\) and thus

\[
C \left( \frac{\varepsilon_j}{\varepsilon_{j-1}} \right)^{\frac{6}{5}} \leq C \left( \frac{1}{\varepsilon_j} \right)^{\frac{6}{5}} \leq C \varepsilon_j \varepsilon_j \varepsilon_j \leq \frac{1}{3} \varepsilon_j
\]

for \(\varepsilon\) small enough. \(\square\)

### 6.3. Iterative lemma

Let set \(\mathcal{D}_0 = \mathcal{D}, h_0 = \langle \omega_0(r), r \rangle + \frac{1}{2} \langle \zeta, A_0(r) \zeta \rangle\) and \(f_0 = f\) in such a way \(f_0|_{\mathcal{D}_0, \sigma_0, \mu_0} \leq \varepsilon_0\). For \(k \geq 0\) let us denote

\[
\mathcal{O}_k = \mathcal{O}^k(\sigma_k, \mu_k).
\]

**Lemma 6.3.** — For \(\varepsilon\) sufficiently small depending on \(\mu_0, \sigma_0, n, s, \beta\) and \(h_0\) we have the following:

For all \(k \geq 1\) there exist \(\mathcal{D}_k \subset \mathcal{D}_{k-1}, S_{k-1} \in T^{s, \beta, +}(\mathcal{D}_k, \sigma_k, \mu_k), h_k = \langle \omega_k(r), r \rangle + \frac{1}{2} \langle \zeta, A_k \zeta \rangle\) on normal form and \(f_k \in T^{s, \beta}(\mathcal{D}_k, \sigma_k, \mu_k)\) such that

(i) The mapping

\[
\Phi_k(\cdot, \rho) = \Phi_{S_{k-1}}^k : \mathcal{O}(k) \to \mathcal{O}(k-1), \quad \rho \in \mathcal{D}_k, \quad k = 1, 2, \ldots
\]

is an analytic symplectomorphism linking the hamiltonian at step \(k-1\) and the hamiltonian at the step \(k\), i.e.

\[
(h_{k-1} + f_{k-1}) \circ \Phi_k = h_k + f_k.
\]
(ii) we have the estimates
\[ \text{meas}(D_{k-1} \setminus D_k) \leq \varepsilon_{k-1}^\alpha, \]
\[ [h_k - h_{k-1}]_{D_k, \sigma_k, \mu_k} \leq C \varepsilon_{k-1}, \]
\[ [f_k]_{D_k, \sigma_k, \mu_k} \leq \varepsilon_k, \]
\[ \|\Phi_k(x, \rho) - x\| \leq \varepsilon_{k-1}^{1/6}, \text{ for } x \in O(k), \ \rho \in D_k. \]

The exponents \( \alpha \) is a positive number depending on \( n, d, \alpha_1, a_2, \gamma, \beta \). The constant \( C \) depends also on \( h_0 \).

Proof. — At step 1, \( h_0 = (\omega_0(\rho), r) + \tfrac{1}{2} (\zeta, A_0(\rho) \zeta) \) and thus hypothesis (5.34) is trivially satisfied and we can apply Proposition 5.5 to construct \( S_0, R_0, B_0 \) and \( D_1 \) such that for \( \rho \in D_1 \)
\[ \{h_0, S_0\} + f_0^T = \hat{h}_0 + R_0. \]

Then we see that, using (5.30) and defining \( s_{1/2} = \frac{\sigma_0 + \sigma_1}{2} \), we have
\[ [S_0]_{D_1, \sigma_{1/2}, \mu_0}^s \leq C \frac{\varepsilon_0}{\eta_0^2 (\sigma_0 - \sigma_{1/2})^n} \leq \frac{1}{16} \mu_0^2 (\sigma_0 - \sigma_1) \]
for \( \varepsilon = \varepsilon_0 \) small enough in view of our choice of parameters. Therefore both Proposition 4.7 and Lemma 6.2 apply and thus for any \( \rho \in D_1, \ \Phi_1(\cdot, \rho) = \Phi_{S_0}^1 : O(1) \to O(0) \) is an analytic symplectomorphism such that
\[ (h_0 + f_0) \circ \Phi_1 = h_1 + f_1 \]
with \( h_1, f_1, D_1 \) and \( \Phi_1 \) satisfying the estimates \( (ii)_{k=1} \). In particular we have
\[ \|\Phi_1(x) - x\| \leq \frac{C}{\sigma_0 \mu_0^2} |S_0|_{D_1, \sigma_{1/2}, \mu_0} \leq \frac{C}{\sigma_0^{n+2} \mu_0^2} \varepsilon_0 \leq \frac{C}{\sigma_0^{n+2} \mu_0^2} \varepsilon_0^{1/3} \leq \varepsilon_0^{1/6} \]
for \( \varepsilon_0 \) small enough.

Now assume that we have completed the iteration up to step \( j \). We want to perform the step \( j + 1 \). We first note that by construction (see Proposition 5.5)
\[ A_j = A_0 + B_0 + \cdots + B_{j-1} \]
and by (5.33)
\[ |A_j|_\beta \leq \varepsilon_0 + \cdots + \varepsilon_{j-1} \leq 2 \varepsilon_0 \leq \frac{1}{4} \delta_0 \]
for \( \varepsilon_0 \) small enough. Similarly
\[ \omega_j = \omega_0 + [\nabla_r f_0(\cdot, 0; \rho)] + \cdots + [\nabla_r f_{j-1}(\cdot, 0; \rho)] \]
and thus \( |\partial^j(\omega_j - \omega_0)| \leq \delta_0 \) for \( \varepsilon_0 \) small enough. Therefore (5.34) is satisfied at rank \( j \) and we can apply Proposition 5.5 in order to construct \( S_j, B_j, R_j \) and \( D_j \).
Then we construct $f_{j+1}$ as in (6.2), i.e.

$$f_{j+1} = R_j + (f_j - f_j^T) \circ \Phi^1_{S_j} + \int_0^1 \{(1-t)(\hat{h}_j + R_j) + tf_j^T, S_j\} \circ \Phi^1_{S_j} \, dt.$$ 

To control $f_{j+1}$ we apply Lemma 6.1 what we can do since, defining $\sigma_{j+1/2} = \frac{\sigma_j + \sigma_{j+1}}{2}$,

$$[S_j]^{s,\beta}_{D_{j+1}, \sigma_{j+1/2}, \mu_j} \leq C(\kappa_j)_{\sigma_j - \sigma_{j+1}}^n \leq \frac{1}{8} \mu_j^2 (\sigma_j - \sigma_{j+1}).$$

Therefore combining Lemma 6.1 and (6.5) we conclude that

$$[f_{j+1}]^{s,\beta}_{D_{j+1}, \sigma_{j+1}, \mu_{j+1}} \leq \varepsilon_{j+1}.$$ 

On the other hand by Proposition 5.5 the domain $D_{j+1}$ satisfies

$$\text{meas}(D_j \setminus D_{j+1}) \leq C N_j^{\exp} \left( \frac{K_j}{\delta_0} \right)^{\exp'} \leq \varepsilon_j^{\alpha}$$

for some $\alpha > 0$ and for $\varepsilon_0 = \varepsilon$ small enough.

6.4. Transition to the limit and proof of Theorem 2.2 — Let

$$D' = \cap_{k \geq 0} D_k.$$ 

In view of the iterative lemma, this is a borel set satisfying

$$\text{meas}(D \setminus D') \leq 2\varepsilon^\alpha.$$ 

Let us set

$$Q_l = O^s(\sigma/\ell, \mu/\ell), \ Z_s = T^n_{\sigma} \times \mathbb{C}^n \times Y_s$$

where $\ell \geq 2$, and recall that $\| \cdot \|_s$ denotes the natural norm on $\mathbb{C}^n \times \mathbb{C}^n \times Y_s$. It defines the distance on $Z_s$. We used the notations introduced in Lemma 6.3. By Proposition 4.5 assertion 2, for each $\rho \in D'$, the map $\Phi_k$ extends to $Q_2$ and satisfies on $Q_2$ the same estimate as on $O_k$:

$$\Phi_k : Q_2 \to Z_s, \quad \| \Phi_k - id \|_s \leq C \mu_k^{2}(\sigma_{k-1} - \sigma_k)^{-1} \varepsilon_k \leq \varepsilon_k^{1/6}. \quad (6.7)$$

Now for $0 \leq j \leq N$ let us denote $\Phi_N^j = \Phi_{j+1} \circ \cdots \circ \Phi_N$. Due to (6.6), it maps $O(N)$ to $O(j-1)$. Due to (6.7), this map extends analytically to a map $\Phi_N^j : Q_3 \to Z_s$, and for $M > N$, $\| \Phi_N^j - \Phi_M^j \|_s \leq C \varepsilon_N^{1/6}$, i.e. $(\Phi_N^j)_N$ is a Cauchy sequence. Thus when $N \to \infty$ the maps $\Phi_N^j$ converge to a limiting mapping $\Phi_\infty^j : Q_3 \to Z_s$. Furthermore we have

$$\| \Phi_\infty^j - id \|_s \leq C \sum_{k \geq j} \varepsilon_k^{1/6} \leq C \varepsilon_j^{1/6}, \ \forall j \geq 1. \quad (6.8)$$
By the Cauchy estimate the linearized map satisfies
\[
\|D\Phi_j^\infty(x) - id\|_{C(Y_1, Y_1)} \leq C\varepsilon_j^{1/6}, \quad \forall x \in Q_4, \ \forall j \geq 1.
\]
By construction, the map \(\Phi_N\) transforms the original hamiltonian
\[
H_0 = \langle \omega, r \rangle + \frac{1}{2} \langle \zeta, A_0(\omega)\zeta \rangle + f
\]
to
\[
H_N = \langle \omega_N, r \rangle + \frac{1}{2} \langle \zeta, A_N(\omega)\zeta \rangle + f_N.
\]
Here
\[
\omega_N = \omega + \|\nabla_r f_0(\cdot, 0; \rho)\| + \cdots + \|\nabla_r f_{N-1}(\cdot, 0; \rho)\|
\]
and
\[
A_N = A_0 + B_0 + \cdots + B_{N-1}
\]
where \(B_k\) is built from \(\langle \nabla_2^\zeta \zeta f_k(\cdot, 0) \rangle\) as in the proof of Proposition 5.4.
Clearly, \(\omega_N \to \omega'\) and \(A_N \to A\) where the vector \(\omega' \equiv \omega'(; \rho)\) and the operator \(A \equiv A(\rho)\) satisfy the assertions of Theorem 2.2.
Let us denote \(\Phi = \Phi_0^\infty\), consider the limiting hamiltonian
\[
H' = H_0 \circ \Phi
\]
write it as
\[
H' = \langle \omega', r \rangle + \frac{1}{2} \langle \zeta, A(\rho)\zeta \rangle + f'.
\]
The function \(f'\) is analytic in the domain \(Q_3\). Since \(H' = H_k \circ \Phi_k^\infty\), we have
\[
\nabla H'(x) = D\Phi_k^\infty(x) \cdot \nabla H_k(\Phi_k^\infty(x)).
\]
As \([f_k]_{h, \sigma, \mu} \leq \varepsilon_k\), we deduce
\[
\nabla_r H_k(\Phi_k^\infty(\theta, 0, 0)) = \omega_k + O(\varepsilon_k^{1/5}) \quad \theta \in \mathbb{T}^n\frac{\pi}{2}.
\]
Since the map \(\Phi_k^\infty\) satisfies (6.9), then
\[
\nabla_r H'(\theta, 0, 0) = \omega' + O(\varepsilon_k^{1/6}) \quad \text{for all } k \geq 1 \text{ and } \theta \in \mathbb{T}^n\frac{\pi}{2}.
\]
Hence, \(\nabla_r H'(\theta, 0, 0) = \omega'\) and thus
\[
\nabla_r f'(\theta, 0, 0) \equiv 0 \quad \text{for } \theta \in \mathbb{T}^n\frac{\pi}{2}.
\]
Similar arguments leads to
\[
\nabla_{\zeta_a} f'(\theta, 0, 0) \equiv 0 \text{ and } \nabla_{\zeta_a} \nabla_r f'(\theta, 0, 0) \equiv 0 \quad \text{for } \theta \in \mathbb{T}^n\frac{\pi}{2}.
\]
Now consider \(\nabla_{\zeta_a} \nabla_{\zeta_b} H'(x)\). To study this matrix let us write it in the form (4.30), with \(h = H_k\) and \(x(1) = \Phi_k^\infty(x)\). Repeating the arguments used in the proof of Proposition 4.6 we get that
\[
\nabla_{\zeta_a} \nabla_{\zeta_b} H'(\theta, 0, 0) = (A_k)_{ab} + O(\varepsilon_k^{1/6}) \quad \text{for all } k \geq 1 \text{ and } \theta \in \mathbb{T}^n\frac{\pi}{2}.
\]
Therefore $\nabla_{\zeta_a} \nabla_{\zeta_b} H'(\theta, 0, 0) = A_{ab}$ i.e.

$$\nabla_{\zeta_a} \nabla_{\zeta_b} f'(\theta, 0, 0) = 0 \quad \text{for } \theta \in T^*_2.$$ 

This concludes the proof of Theorem 2.2.

References

[1] V.I. Arnold, Mathematical methods in classical mechanics; 3d edition. Springer-Verlag, Berlin, 2006.

[2] M. Berti, P. Bolle, Sobolev quasi periodic solutions of multidimensional wave equations with a multiplicative potential, Nonlinearity 25 (2012), 2579-2613.

[3] M. Berti, P. Bolle, Quasi-periodic solutions with Sobolev regularity of NLS on $T^d$ with a multiplicative potential, J. Eur. Math. Soc. 15 (2013), 229-286.

[4] M. Berti, L. Corsi, M. Procesi An Abstract Nash-Moser Theorem and Quasi-Periodic Solutions for NLW and NLS on Compact Lie Groups and Homogeneous Manifolds. Comm. Math. Phys (2014)

[5] M. Berti, M. Procesi Nonlinear wave and Schrödinger equations on compact Lie groups and Homogeneous spaces. Duke Math. J. 159, 479?538 (2011)

[6] J. Bourgain Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Shödinger equation, Ann. Math. 148 (1998), 363-439.

[7] J. Bourgain Green’s function estimates for lattice Schrödinger operators and applications, Annals of Mathematical Studies, Princeton, 2004.

[8] L.H. Eliasson, B. Grébert and S.B. Kuksin. KAM for multidimensional PDEs with regularizing nonlinearity. in preparation.

[9] L.H. Eliasson, B. Grébert and S.B. Kuksin. KAM for the beam equation. in preparation.

[10] L.H. Eliasson and S.B. Kuksin. Infinite Töplitz-Lipschitz matrices and operators. Z. Angew. Math. Phys. 59 (2008), 24-50.

[11] L.H. Eliasson and S.B. Kuksin. KAM for the nonlinear Schrödinger equation. Ann. Math 172 (2010), 371-435.

[12] J. Geng and J. You. A KAM theorem for Hamiltonian partial differential equations in higher dimensional spaces. Comm. Math. Phys., 262 (2006), 343–372.

[13] J. Geng and J. You. KAM tori for higher dimensional beam equations with constant potentials, Nonlinearity, 19 (2006), 2405–2423.

[14] B. Grébert and L. Thomann, Kam for the Quantum Harmonic Oscillator, Comm. Math. Phys., 307 (2011), 383–427.

[Hel84] Bernard Helffer, Théorie spectrale pour des opérateurs globalement elliptiques, Astérisque, vol. 112, Société Mathématique de France, Paris, 1984, With an English summary.

[15] T. Kappeler and J. Pöschel. KAM & KdV. Springer-Verlag, Berlin, 2003.
[16] G.E. Karadzhov. Riesz summability of multiple Hermite series in $L^p$ spaces, *Math. Z.*, **2019** (1995), 107–118.

[17] H. Koch, D. Tataru and M. Zworski. Semiclassical $L^p$ Estimates, *Annales Henri Poincaré*, **8** (2007), 885–916.

[18] S. B. Kuksin. Hamiltonian perturbations of infinite-dimensional linear systems with an imaginary spectrum. *Funct. Anal. Appl.*, **21** (1987), 192–205.

[19] S. B. Kuksin. Nearly integrable infinite-dimensional Hamiltonian systems. *Lecture Notes in Mathematics, 1556*. Springer-Verlag, Berlin, 1993.

[20] S. B. Kuksin and J. Pöschel. Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation. *Ann. Math.* **143** (1996), 149–179.

[21] J. Pöschel. A KAM-theorem for some nonlinear partial differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **23** (1996), no. 1, 119–148.

[22] S. Thangavelu. Lectures on Hermite and Laguerre expansions. Mathematical Notes, **42**. Princeton University Press, Princeton, NJ, 1993.

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