Quantum weighted entropy and its properties

Y. Suhov\textsuperscript{1} and S. Zohren\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, Penn State University, PA 16802, USA; DPMMS, University of Cambridge, CB30WB, UK; Institute for Information Transmission Problems, RAS, 127994 Moscow, Russia
\textsuperscript{2} Department of Physics, Pontificia Universidade Católica, Rio de Janeiro, Brazil

PACS numbers: 03.67.-a, 89.70.Cf
MSC numbers: 46N50, 94A15, 47L90
Keywords: Quantum information theory, weighted entropy, trace inequalities

Abstract. We introduce quantum weighted entropy in analogy to an earlier notion of (classical) weighted entropy and derive many of its properties. These include the subadditivity, concavity and strong subadditivity property of quantum weighted entropy, as well as an analog of the Araki-Lieb inequality. Interesting byproducts of the proofs are a weighted analog of Klein’s inequality and non-negativity of quantum weighted relative entropy. A main difficulty is the fact that the weights in general do not commute with the density matrices.
Quantum weighted entropy and its properties

1. Introduction

Shannon entropy and its quantum analog, von Neumann entropy, play essential roles in classical and quantum information theory [1, 2, 3]. There are many generalisations of Shannon entropy, such as Rényi entropy, which have been proposed both in the classical as well as quantum case. Another interesting generalisation is (classical) weighted entropy [4, 5]. The idea behind weighted entropy is the incorporation of further characteristics of each event through a weight assigned to it in addition to its probability. Weighted entropy has been used at several places in the information theory and computer science literature (see for instance [6, 7, 8, 9, 10] and references therein) including in machine learning applications [11]. However, the natural quantum analog of weighted entropy has apparently not yet been considered in the quantum information theory literature.

The aim of this letter is to introduce quantum weighted entropy and to prove several basic properties, including subadditivity, concavity and strong subadditivity of quantum weighted entropy and an analog of the Araki-Lieb inequality. Many of the corresponding trace inequalities for von Neumann entropy have important applications in quantum information theory. As an example let us mention strong subadditivity of standard von Neumann entropy which was conjectured in [12] and then proven by Lieb and Ruskai [13, 14] (see also [15] for a modern simplified proof). Amongst many of its applications is the thermodynamic limit of entropy per volume which was already considered in the classical case [16]. We refer the reader to [17] for a review. Some of the above trace inequalities are quantum analogs of a series of inequalities for classical weighted entropy recently considered in [18].

Let us now give a formal definition of quantum weighted entropy. Consider a quantum mechanical system with Hilbert space $H$ and a density matrix $\rho$ on $H$. For an Hermitian, positive definite matrix $\phi$ on $H$, which from here on we simply refer to as weight, we define the quantum weighted entropy as follows

$$S_\phi(\rho) = - \text{tr}(\phi \rho \log \rho).$$

One sees that for $\phi = 1_H$, where $1_H$ is the identity matrix on $H$, the quantum weighted entropy reduces to the standard von Neumann entropy.

Before moving to the discussion of different properties and their proofs in the next sections let us first comment on potential difficulties. Consider for the moment the well-known Gibbs inequality which yields positivity of von Neumann entropy whose weighted analog we will discuss in the next section. The main difficulty in the proof, as compared to the corresponding classical result for Shannon entropy, is the fact that the different (reduced) density matrices in general do not commute. Similarly, when extending many trace inequalities for von Neumann entropy to quantum weighted entropy the difficulty lies in the fact that now also the weight $\phi$ is not commuting with the density matrices.
Quantum weighted entropy and its properties

2. Quantum weighted relative entropy and Gibbs inequality

Extending standard classical notions, we can also introduce the quantum weighted relative entropy or weighted Kullback-Leibler divergence as

\[ D_\phi(\rho\|\sigma) = \text{tr}(\phi \rho \log \rho) - \text{tr}(\phi \rho \log \sigma). \]  

(2)

Here and below \( \rho \) is a density matrix and \( \sigma \) is positive definite in \( \mathcal{H} \).

An important property of quantum weighted relative entropy is given by the weighted Klein’s inequality

**Lemma 1 (Weighted Klein’s inequality).** Assume that \( X, Y, W \) are Hermitian positive definite matrices on a Hilbert space \( \mathcal{H} \). Then if \( f \) is a convex function one has

\[ \text{tr} \left( W(f(Y) - f(X)) \right) \geq \text{tr} \left( W(Y - X)f'(X) \right). \]  

(3)

In particular for \( f(x) = x \log x \) one has

\[ \text{tr} \left( WY(\log Y - \log X) \right) \geq \text{tr} \left( W(Y - X) \right). \]  

(4)

The proof is given in the appendix. At this point we only note that a slight difficulty in comparison to the standard Klein’s inequality is caused by the fact that \( W \) in general does not commute with \( X \) and \( Y \). An important result which can be easily derived from the weighted Klein’s inequality is the weighted Gibbs inequality.

**Theorem 1 (Weighted Gibbs inequality).** Under the condition \( \text{tr} \phi \rho \geq \text{tr} \phi \sigma \) one has

\[ D_\phi(\rho\|\sigma) \geq 0, \]  

(5)

with equality if and only if \( \rho = \sigma \).

**Proof.** The proof of the weighted Gibbs inequality follows directly from the weighted Klein’s inequality, Lemma 1. In particular for \( X = \sigma, Y = \rho, W = \phi \) and \( f(x) = x \log x \), the result immediately follows from (4) under the condition \( \text{tr} \phi \rho \geq \text{tr} \phi \sigma \).

**Remark 1.** Note that the condition \( \text{tr} \phi \rho \geq \text{tr} \phi \sigma \) is physical and a minimum necessary requirement for the weighted Gibbs inequality. As expected, it is automatically satisfied if \( \phi = 1_\mathcal{H} \), i.e. in the case of standard von Neumann entropy.

3. Basic properties of quantum weighted entropy

We now discuss three propositions with useful basic properties of the quantum weighted entropy.

**Proposition 1.** Denote by \( |e_1\rangle, \ldots, |e_d\rangle \) the normalised eigenvectors of \( \rho \) and by \( \lambda_1, \ldots, \lambda_d \) the corresponding eigenvalues.

(i) The quantum weighted entropy \( S_\phi(\rho) \) is non-negative and zero if and only if either

(i) \( \rho \) is pure or (ii) \( \langle e_i|\phi|e_i \rangle = 0 \) whenever \( 0 < \lambda_i < 1 \).

(ii) \( S_\phi(\rho) = S_{\phi'}(\rho) \) if \( \langle e_i|\phi|e_i \rangle = \langle e_i|\phi'|e_i \rangle \) whenever \( 0 < \lambda_i < 1 \). In this case we say that \( \phi \) and \( \phi' \) are \( \rho \)-conjugate.
Proposition 2. Suppose the rank of $\phi$ equals $m \leq d$ and let $P = P_\phi$ be the orthoprojection to the range of $\phi$. If $\text{tr} \phi \rho \geq \text{tr} \phi / m$ then
\[
S_\phi(\rho) \leq S_\phi(P/m) = (\log m) \text{tr} \phi
\] with equality if and only if $\rho = P/m$.

Proof. The results of this proposition are a consequence of the eigenvalue decomposition of $\rho$ which yields
\[
S_\phi(\rho) = -\sum_i \langle e_i | \phi | e_i \rangle \lambda_i \log \lambda_i. \quad (6)
\]
We directly see from this that $S_\phi(\rho)$ is non-negative. If $\rho$ is pure $\lambda_i \log \lambda_i = 0, \forall i$ which implies $S_\phi(\rho) = 0$. Otherwise, if $\rho$ is not pure one has $\lambda_i < 1, \forall i$ and it is clear from the above that in this case $S_\phi(\rho) = 0$ if and only if $\langle e_i | \phi | e_i \rangle = 0$ whenever $0 < \lambda_i < 1$. This proves the first part of the proposition. The second part of the proposition follows directly from (6).

Above we have already shown that $S_\phi(\rho) \geq 0$. The following proposition gives an upper bound on the quantum weighted entropy.

Proposition 3. Let $|\chi\rangle$ be a pure state on $AR$ with $\rho_A = \text{tr}_R |\chi\rangle \langle \chi|$ and $\rho_R = \text{tr}_A |\chi\rangle \langle \chi|$. Denote also $\rho_A = \text{tr}_R |\chi\rangle \langle \chi|$. Standard arguments show that $\rho_A$ and $\rho_R$ have the same collection $\{\lambda_i\}$ of non-negative eigenvalues. Furthermore, if we denote by $\{|e_i^A\rangle\}$ and $\{|e_i^R\rangle\}$ the corresponding eigenvectors of $\rho_A$ and $\rho_R$, then one finds
\[
S_{\phi_1}(\rho_A) = -\sum_i \langle e_i^A | \phi_1 | e_i^A \rangle \lambda_i \log \lambda_i, \quad (8)
\]
\[
S_{\phi_2}(\rho_R) = -\sum_i \langle e_i^R | \phi_2 | e_i^R \rangle \lambda_i \log \lambda_i. \quad (9)
\]
This proves the following proposition:

Proposition 4. Let $|\chi\rangle$ be a pure state on $AR$ with $\rho_A = \text{tr}_R |\chi\rangle \langle \chi|$ and $\rho_R = \text{tr}_A |\chi\rangle \langle \chi|$, then for any pair of weight matrices $\phi_A$ on $A$ and $\phi_R$ on $R$ such that
\[
\langle e_i^A | \phi_A | e_i^A \rangle = \langle e_i^R | \phi_R | e_i^R \rangle \quad \forall i \text{ with } 0 < \lambda_i < 1
\] one has
\[
S_{\phi_1}(\rho_A) = S_{\phi_2}(\rho_R). \quad (10)
\]
In this case we say that $\phi_A$ is $(\rho_A, \rho_R)$-conjugated to $\phi_R$ and $\phi_R$ is $(\rho_R, \rho_A)$-conjugated to $\phi_A$.
Quantum weighted entropy and its properties

From this one has the following corollary:

**Corollary 1.** Let $\rho$ be a density matrix in $AB$, as well as $\rho_A = \text{tr}_B \rho$ and $\rho_B = \text{tr}_A \rho$. Take a reference system $R$ with Hilbert space isomorphic to the Hilbert space of $AB$, and a pure state $|\chi\rangle$ on $ABR$ such that $\rho = \text{tr}_R |\chi\rangle\langle\chi|$. Set $\rho_R = \text{tr}_{AB} |\chi\rangle\langle\chi|$ and $\rho_{BR} = \text{tr}_A |\chi\rangle\langle\chi|$, then

$$S_\phi(\rho) = S_{\phi_R}(\rho_R) \text{ and } S_{\phi_A}(\rho_A) = S_{\phi_{BR}}(\rho_{BR})$$

if $\phi$ is $(\rho, \rho_R)$-conjugate to $\phi_R$ and $\phi_A$ is $(\rho_A, \rho_{BR})$-conjugate to $\phi_{BR}$.

4. A diagonalisation bound

Another simple trace inequality, which is also based on the weighted Gibbs inequality, deals with the projection of the density matrix on its diagonal in a given basis, as occurs for example when projective measurements are performed. Let $\rho$ be a density matrix in $\mathcal{H}$. Let $|f_1\rangle, \ldots, |f_d\rangle$ be a basis in $\mathcal{H}$ and $\rho^d$ denote the diagonal part of $\rho$ in this basis, i.e. $\langle f_j | \rho^d | f_k \rangle = \delta_{jk} \langle f_j | \rho | f_j \rangle$ for $1 \leq j, k \leq d$. Then we have the following bound for the weighted entropy of $\rho^d$

**Theorem 2.** Under the condition $\text{tr} \phi \rho \geq \text{tr} \phi \rho^d$

$$S_\phi(\rho^d) \geq S_\phi(\rho),$$

with equality if and only if $\rho = \rho^d$ and where $\psi$ fulfills $\langle f_j | \psi \rho^d | f_j \rangle = \langle f_j | \phi \rho | f_j \rangle$.

**Proof.** The proof is again an application of the weighted Gibbs inequality, Theorem 1. Choosing $\sigma = \rho^d$, by the condition of the theorem the Gibbs inequality can be used, yielding

$$0 \leq D_\phi(\rho || \rho^d) = -S_\phi(\rho) - \text{tr} \phi \rho \log \rho^d$$

$$= -S_\phi(\rho) - \sum_j \langle f_j | \phi \rho | f_j \rangle \log \langle f_j | \rho^d | f_j \rangle = -S_\phi(\rho) + S_\psi(\rho^d)$$

with inequality for $\rho = \rho^d$. This completes the proof. 

Note that in the special case of von Neumann entropy, one has $\phi = \psi = 1_\mathcal{H}$ and all conditions of the theorem are automatically satisfied.

5. Subadditivity of quantum weighted entropy

Let us first focus on a composite system $AB$ of two components $A$ and $B$ with density matrix $\rho_{AB}$ and weight $\phi_{AB} = \phi_A \otimes \phi_B$. Recall the standard reduced density matrices defined by taking the partial trace, i.e. $\rho_A = \text{tr}_B(\rho_{AB})$ and so on. We can now prove the following subadditivity property of quantum weighted entropy:
Quantum weighted entropy and its properties

Theorem 3 (Subadditivity). Under the condition \( \text{tr}_{AB}(\phi_{AB}\rho_{AB}) \geq \text{tr}_A(\phi_A\rho_A) \text{tr}_B(\phi_B\rho_B) \) one has
\[
S_{\phi_{AB}}(\rho_{AB}) \leq S_{\psi_A}(\rho_A) + S_{\psi_B}(\rho_B)
\]
with equality for \( \rho_{AB} = \rho_A \otimes \rho_B \), where the reduced weights are defined implicitly through \( \psi_A\rho_A = \text{tr}_B(\phi_{AB}\rho_{AB}) \) and similarly for \( B \).

Proof. The condition stated in Theorem 3, i.e. \( \text{tr}_{AB}(\phi_{AB}\rho_{AB}) \geq \text{tr}_A(\phi_A\rho_A) \text{tr}_B(\phi_B\rho_B) \), ensures that we can use the weighted Gibbs inequality with \( \sigma_{AB} = \rho_A \otimes \rho_B \) and \( \phi = \phi_{AB} = \phi_A \otimes \phi_B \). Further abbreviating \( \rho = \rho_{AB} \), one gets
\[
D_\phi(\rho \| \rho_A \otimes \rho_B) \geq 0
\]
with equality if and only if \( \rho = \rho_A \otimes \rho_B \). Simplifying the above gives
\[
0 \leq D_\phi(\rho \| \rho_A \otimes \rho_B) = \text{tr}_{AB}\{\phi \rho \log \rho - \log (\rho_A \otimes \rho_B)\}
\]
Hence,
\[
S_\phi(\rho) \leq -\text{tr}_{AB}\{\phi \rho \log (\rho_A \otimes 1_B) - \log (1_A \otimes \rho_B)\}
\]
\[
= -\text{tr}_A\{\text{tr}_B(\phi \rho) \log \rho_A\} - \text{tr}_B\{\text{tr}_A(\phi \rho) \log \rho_B\}
\]
\[
= S_{\psi_A}(\rho_A) + S_{\psi_B}(\rho_B)
\]
under the above definition of reduced weights. This completes the proof.

6. Concavity of quantum weighted entropy

We can use the subadditivity property of quantum weighted entropy proved in the previous section to show that quantum weighted entropy is concave in analogy to the case of standard von Neumann entropy.

Theorem 4. Suppose that \( \rho^{(1)}, \ldots, \rho^{(r)} \) are density matrices of a system \( A \) a Hilbert space \( \mathcal{H}_A \) and \( b = (b_1, \ldots, b_r) \) is a probability vector, with non-negative entries and \( \sum_{1 \leq l \leq r} b_l = 1 \). Then
\[
S_\phi \left( \sum_l b_l \rho^{(l)} \right) \geq \sum_l b_l S_\phi(\rho^{(l)})
\]
with equality if and only if \( b_l = 1 \) for some \( l \) or \( \rho^{(l)} = \rho^{(1)} \forall l \).

Proof. Set
\[
\sigma = \sum_l b_l \rho^{(l)}, \quad B_l = \text{tr}_A(\phi^{(l)}), \quad A = \text{tr}_A \phi \sigma = \sum_l b_l B_l.
\]
Recall the expressions for the standard Shannon entropy of \( b \) and the weighted Shannon entropy of \( b \) with weight \( B \),
\[
h(b) = -\sum_l b_l \log b_l, \quad h_B(b) = -\sum_l B_l b_l \log b_l.
\]
Take an auxiliary system $R$ with Hilbert space $\mathcal{H}_R$ of dimension $r$ and fix a basis $|e_1\rangle,\ldots,|e_r\rangle$ in $\mathcal{H}_R$. Consider a density matrix $\rho$ on $AR$ defined by the condition that for all $|v\rangle,|v'\rangle \in \mathcal{H}_A$ and $1 \leq l,l' \leq r$:

$$\langle v \otimes e_l|\rho|v' \otimes e_{l'}\rangle = b_l\langle v|\rho^{(l)}|v'\rangle\delta_{l,l'}.$$  \hfill (22)

It is easily verified that $\rho$ is indeed a density matrix, i.e. a positive-definite operator of trace 1. Then

$$\rho_A = \text{tr}_R \rho = \sum_l b_l \rho^{(l)} = \sigma$$ \hfill (23)

and $\rho_R = \text{tr}_A \rho$ is diagonal in basis $|e_1\rangle,\ldots,|e_r\rangle$, with diagonal entries $b_1,\ldots,b_r$. Also, if $\rho^{(l)}$ has eigenvectors $|e_j^{(l)}\rangle$ with eigenvalues $\lambda_j^{(l)}$ then $\rho$ has the eigenvectors $|e_j^{(l)}\rangle \otimes |e_l\rangle$ with the eigenvalues $\lambda_j^{(l)}b_l$. Hence, with $1_R$ denoting the unit operator on $R$ one has

$$S_{\phi \otimes 1_R}(\rho) = - \sum_{j,l} \langle e_j^{(l)}|\phi|e_j^{(l)}\rangle (\lambda_j^{(l)}b_l) \log(\lambda_j^{(l)}b_l)$$

$$= - \sum_l b_l \sum_j \langle e_j^{(l)}|\phi|e_j^{(l)}\rangle \lambda_j^{(l)} \log \lambda_j^{(l)} - \sum_l b_l \log b_l$$

$$= \sum_l b_l S_{\phi}(\rho^{(l)}) + h_B(b).$$ \hfill (24)

Note that $\text{tr}(\phi \otimes 1_R)\rho = \text{tr}(\phi \rho_A)$, so the bound $\text{tr}(\phi \otimes 1_R)\rho \geq \text{tr}(\phi \otimes 1_R)(\rho_A \otimes \rho_R)$ is fulfilled. Finally, by (22) the partial trace $T = \text{tr}_A(\phi \otimes 1_R)\rho$ is a diagonal matrix in the basis $|e_1\rangle,\ldots,|e_r\rangle$ of $R$, with entries

$$\langle e_l|T|e_{l'}\rangle = \delta_{l,l'} b_l B_l.$$ \hfill (25)

To complete the proof we use the subadditivity property proven in the previous section in Theorem 3 for the joint system $AR$ with density matrix $\rho$ and weight $\phi \otimes 1_R$. Therefore, we introduce reduced weights defined implicitly through

$$\psi_A \rho_A = \text{tr}_R(\phi \otimes 1_R)\rho = \phi \sigma$$

$$\psi_R \rho_R = \text{tr}_A(\phi \otimes 1_R)\rho = T = \sum_l b_l B_l |e_l\rangle \langle e_l|.$$  

Then one has

$$S_{\psi_A}(\rho_A) = S_{\phi}(\sigma), \text{ and } S_{\psi_R}(\rho_R) = h_B(b).$$ \hfill (26)

Therefore, subadditivity (Theorem 3) yields

$$S_{\phi \otimes 1_R}(\rho) \leq S_{\psi_A}(\rho_A) + S_{\psi_R}(\rho_R) = S_{\phi}(\sigma) + h_B(b)$$ \hfill (27)

with equality if and only if $\rho = \rho_A \otimes \rho_R$, i.e. $h(b) = 0$ or $\rho^{(l)} = \rho^{(1)} \forall l$. This together with (24) gives (19).
7. Araki-Lieb inequality for quantum weighted entropy

Consider a composite system $AB$ with density matrix $\rho$, weight $\phi$ and partial density matrices $\rho_A = \text{tr}_B \rho$ and $\rho_B = \text{tr}_A \rho$. Construct a purification by introducing a reference system $R$ with Hilbert space isomorphic to the Hilbert space of $AB$, and a pure state $|\chi\rangle$ on $ABR$ such that $\rho = \text{tr}_R |\chi\rangle \langle \chi|$. Furthermore, set $\rho_R = \text{tr}_{AB} |\chi\rangle \langle \chi|$ and $\rho_{BR} = \text{tr}_A |\chi\rangle \langle \chi|$, then by Corollary 1 we have

$$S_\phi(\rho) = S_{\phi_R}(\rho_R) \text{ and } S_{\phi_A}(\rho_A) = S_{\phi_{BR}}(\rho_{BR})$$

if $\phi$ is $(\rho, \rho_R)$-conjugate to $\phi_R$ and $\phi_A$ is $(\rho_A, \rho_{BR})$-conjugate to $\phi_{BR}$. Combining this with subadditivity gives rise to the following result.

**Theorem 5** (Weighted Araki-Leib inequality). One has

$$S_\phi(\rho) \geq \left( \sup_{\psi \in \mathcal{D}} [S_{\psi_A}(\rho_A) - S_{\psi_B}(\rho_B)] \right) \lor \left( \sup_{\Psi \in \mathcal{D}} [S_{\psi_{BR}}(\rho_{BR}) - S_{\psi_{AB}}(\rho_A)] \right),$$

where the set $\mathcal{D}(\phi)$ consists of all pairs $\Psi = (\psi_A, \psi_B)$ for which there exists a $\phi_{BR}$ and $\psi_R$ implicitly defined through $\psi_R^{\ast} \rho_R = \text{tr}_{BR} \phi_{BR} \rho_{BR}$ satisfying $\text{tr}_{BR} \phi_{BR} \rho_{BR} \geq \text{tr}_{BR} \phi_{BR} \rho_{BR} \otimes \rho_R$ and $\psi_R^{\ast}$ is $(\rho_R, \rho)$-conjugate to $\phi$, such that $\psi_A$ is $(\rho_A, \rho_{BR})$-conjugate to $\phi_{BR}$ and $\psi_B$ is $\rho_B$-conjugate to $\psi_B^{\ast}$ defined through $\psi_B^{\ast} \rho_B = \text{tr}_R \phi_{BR} \rho_{BR}$; and similarly for $\mathcal{D}(\phi)$.

**Proof.** To prove the theorem is suffices to show that

$$S_\phi(\rho) \geq S_{\psi_A}(\rho_A) - S_{\psi_B}(\rho_B), \text{ and } S_\phi(\rho) \geq S_{\psi_{BR}}(\rho_{BR}) - S_{\psi_{AB}}(\rho_A) \tag{30}$$

for all $(\psi_A, \psi_B) \in \mathcal{D}(\phi)$ and $(\tilde{\psi}_A, \tilde{\psi}_B) \in \tilde{\mathcal{D}}(\phi)$. We start by establishing the first inequality in (30). For any $(\psi_A, \psi_B) \in \mathcal{D}(\phi)$ there exists a $\phi_{BR}$, $\psi_B^{\ast}$ and $\psi_R^{\ast}$ with $\psi_B^{\ast} \rho_B = \text{tr}_R \phi_{BR} \rho_{BR}$ and $\psi_{BR} = \text{tr}_R \phi_{BR} \rho_{BR}$, satisfying $\text{tr}_{BR} \phi_{BR} \rho_{BR} \geq \text{tr}_{BR} \phi_{BR} \rho_{BR} \otimes \rho_R$. We can thus apply the subadditivity inequality, Theorem 3, to obtain

$$S_{\phi_{BR}}(\rho_{BR}) \leq S_{\phi_B^{\ast}}(\rho_B) + S_{\phi_R^{\ast}}(\rho_R) \tag{31}$$

Since $\psi_A$ and $\phi_{BR}$ are $(\rho_A, \rho_{BR})$-conjugate, one has $S_{\phi_{BR}}(\rho_{BR}) = S_{\psi_A}(\rho_A)$. Further, since $\psi_B^{\ast}$ and $\psi_B$ are $\rho_B$-conjugate, one has $S_{\phi_B^{\ast}}(\rho_B) = S_{\psi_B}(\rho_B)$. Moreover, the condition that $\psi_R^{\ast}$ is $(\rho_R, \rho)$-conjugate to $\phi$ implies $S_{\phi_R^{\ast}}(\rho_R) = S_{\phi}(\rho)$. This proves the first inequality in (30). The second inequality in (30) is established in a similar manner. This completes the proof.

**Remark 2.** Note that in the case of von Neumann entropy with $\phi = 1_{AB}$, one has $\psi_A = \tilde{\psi}_A = 1_A$ and $\psi_B = \tilde{\psi}_B = 1_B$ and thus (30) simply reads

$$S(\rho) \geq |S(\rho_A) - S(\rho_B)| \tag{32}$$

which is the Araki-Leib inequality for von Neumann entropy.
8. Strong subadditivity of quantum weighted entropy

Another very interesting trace inequality concerns the strong subadditivity property for a composite system $ABC$ with density matrix $\rho_{ABC}$ and weight $\phi_{ABC} = \phi_A \otimes \phi_B \otimes \phi_C$.

Theorem 6 (Strong subadditivity). Under the conditions (i) $\text{tr}_{ABC}(\phi_{ABC}\rho_{ABC}) \geq \text{tr}_B \{ \phi_B \text{tr}_A(\phi_A\rho_{AB})\text{tr}_C(\phi_C\rho_{BC})\rho_B^{-1} \}$, as well as (ii) $[\rho_{AB}, \phi_A \otimes \phi_B] = 0$ and $[\text{tr}_C(\phi_C\rho_{BC}), \rho_B] = 0$ one has

$$S_{\phi_{ABC}}(\rho_{ABC}) + S_{\psi_B}(\rho_B) \leq S_{\psi_{AB}}(\rho_{AB}) + S_{\psi_{BC}}(\rho_{BC})$$

where the reduced weights are defined as above, i.e. $\psi_{AB}\rho_{AB} = \text{tr}_C(\phi_{ABC}\rho_{ABC})$ and so on.

Remark 3. Let us make two remarks regarding the conditions of the theorem. Firstly, as expected, both conditions are automatically satisfied in case of $\phi = 1_{ABC}$, i.e. the case of standard von Neumann entropy. Secondly, condition (i) is the natural analog of the condition of the subadditivity property and as in the latter case is a physical condition which one expects not to be able to improve on. However, condition (ii) is a technical condition which one might hope to improve. Further, note that an analogous condition with $A$ and $C$ interchanged is also a valid condition (ii).

Proof of Theorem 6. To prove Theorem 6 we have to show that $\mathcal{A} := S_{\phi_{ABC}}(\rho_{ABC}) + S_{\psi_B}(\rho_B) - S_{\psi_{AB}}(\rho_{AB}) - S_{\psi_{BC}}(\rho_{BC}) \leq 0$. Since we have already proven the weighted Klein’s inequality we can follow a similar strategy as in the original proof by Lieb and Ruskai [13] of strong subadditivity of standard von Neumann entropy which uses the standard Klein’s inequality as a first ingredient. To do so we first make the following observation, which follows from the specific definition of the reduced weights as given in Theorem 6, abbreviating $\rho \equiv \rho_{ABC}$ and $\phi \equiv \phi_{ABC}$, one gets

$$\mathcal{A} = \text{tr}_{ABC}\{ \phi$$

$$(\log \rho_{AB} + \log \rho_{BC} - \log \rho_B - \log \rho) \},$$

where all matrices are to be understood as extended to the Hilbert space of the full system $ABC$, i.e. $\rho_{AB}$ is short for $\rho_{AB} \otimes 1_C$ and similarly for the others. Now we can apply the weighted Klein’s inequality, Lemma 1, with $W = \phi$, $Y = \rho$ and $X = \exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})$ and $f(x) = x \log x$ as in (4), yielding

$$\mathcal{A} \leq \text{tr}_{ABC}\{ \phi \exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC}) - \phi \rho \}$$

This relation very much resembles the Golden-Thomson inequality [19, 20]

$$\text{tr}(e^{X+Y}) \leq \text{tr}(e^X e^Y)$$

and in the proof of the strong subadditivity for standard von Neumann entropy and Lieb and Ruskai [13] used a generalisation of the Golden-Thomson inequality derived in an earlier work by Lieb [21],

$$\text{tr}(e^{X+Y+Z}) \leq \text{tr}(e^{Z\exp(-X)}(e^Y))$$
Quantum weighted entropy and its properties

where

\[ T_{\exp(-X)}(e^Y) = \int_0^\infty (e^{-X} + 1\omega)^{-1}e^Y(e^{-X} + 1\omega)^{-1}d\omega. \]  

(38)

This relation can be extended to the weighted case, i.e. for \( W \) being a weight one has

\[ \text{tr} \left( W e^{X+Y+Z} \right) \leq \text{tr} \left( K_W(Z) T_{\exp(-X)}(e^Y) \right). \]

(39)

with

\[ K_W(Z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{l=0}^{n} Z^{n-l} W Z^l \]

and \( T_{\exp(-X)}(e^Y) \) as defined above. The proof is a generalisation of the proof of Theorem 7 of [21]. Set \( \xi = e^{-X}, \eta = e^Y \) and \( R = X + Z \). Note that \( \xi \) and \( \eta \) are strictly positive operators. We define a function \( F \) from the cone of strictly positive operators to the real numbers through \( F : \xi \rightarrow -\text{tr}[W \exp(R + \log \xi)] \). Since \( F \) is convex and homogeneous of order one, Lemma 5 of [21] can be applied, yielding

\[ -\text{tr} \left( W e^{X+Y+Z} \right) = F(\eta) \geq \frac{d}{d\omega} [F(\xi + \omega \eta)]_{\omega=0}. \]

(41)

Taylor expanding and taking the derivative gives (39).

We can now apply (39) to the first term on the right-hand-side of (35). Choosing \( X = -\log \rho_B, Y = \log \rho_{BC}, Z = \log \rho_{AB} \) and \( W = \phi \) and furthermore assuming under condition (ii) that we have the commutation relations \([\rho_{AB}, \phi_A \otimes \phi_B] = 0 \) and \([\text{tr}_C(\phi_C \rho_{BC}), \rho_B] = 0 \), then the first term on the right-hand-side of (35) is bounded from above by

\[
\text{tr}_B \left\{ \phi_B \text{tr}_A(\phi_A \rho_{AB}) \text{tr}_C(\phi_C \rho_{BC}) \int_0^\infty (\rho_B + 1\omega)^{-1}(\rho_B + 1\omega)^{-1}d\omega \right\} = \text{tr}_B \left\{ \phi_B \text{tr}_A(\phi_A \rho_{AB}) \text{tr}_C(\phi_C \rho_{BC}) \rho_B^{-1} \right\} \]

(42)

Thus one arrives at

\[
A \leq \text{tr}_B \left\{ \phi_B \text{tr}_A(\phi_A \rho_{AB}) \text{tr}_C(\phi_C \rho_{BC}) \rho_B^{-1} \right\} - \text{tr}_{ABC} \phi \rho \]

(43)

which by condition (i) of the theorem yields

\[ A \leq 0. \]

(44)

This completes the proof.
9. Discussion

We introduce quantum weighted entropy and derived several useful properties in terms of various trace inequalities. Each of those inequalities contains the corresponding result for von Neumann entropy as a special case when the weight is chosen to be the identity matrix. In particular, besides basic properties, we derived a diagonalisation bound (Theorem 2), subadditivity and concavity of quantum weighted entropy (Theorem 3 and 4), an analog of the Araki-Lieb inequality (Theorem 5) and strong subadditivity of quantum weighted entropy (Theorem 6). An essential ingredient to the previous results is an analog of Gibbs inequality for quantum weighted relative entropy (Theorem 1) which in turn is obtained from a weighted Klein’s inequality (Lemma 1).

A difficulty in proving the above trace inequalities, in comparison to the analogous results for von Neumann entropy, is the fact that in general the weights do not commute with the (reduced) density matrices. In the case of the weighted Klein’s inequality we circumvent this problem by utilising the unique decompositions $W = LL^\dagger$ of the weight. Since the weighted Gibbs inequality and in turn most of the other inequalities are derived from the weighted Klein’s inequality, they inherited this property and can be proven without any further assumptions on commutation relations of the weight. The only result, where commutativity of parts of the weight with some of the reduced density matrices is assumed, is strong subadditivity. The proof is thus not optimal and one would hope to be able to improve it by relaxing those conditions. In this context we note that our proof of strong subadditivity of quantum weighted entropy closely follows the original proof by Lieb and Ruskai for strong subadditivity of von Neumann entropy [13]. This enables one to use the weighted Gibbs inequality in an essential manner. In the case of alternative, more modern strategies for proving strong subadditivity, as in [15], the situation is more involved.

The here presented discussion of quantum weighted entropy is a first account deriving many of its properties and thus forms a basis for further interesting potential applications of the latter in quantum information theory.

Acknowledgements – YS and SZ thank Salimeh Yasaei Sekeh for useful discussions. YS thanks University of Sao Paulo (at Sao Paulo and at Sao Carlos) for the hospitality during the academic year 2013-4. SZ acknowledges support by CNPq (Grant 307700/2012-7) and PUC-Rio, as well as thanks USP for kind hospitality.

Appendix A. Proof of weighted Klein’s inequality

In case the matrices $X$ and $W$ commute, one can simultaneously diagonalise them, which enables one to follow the same steps as in the proof for the standard Klein’s inequality. The general case, where $X$ and $W$ no not commute, is slightly more involved and relies on the following decomposition: Since $W$ is positive definite one has that $\langle v|W|v \rangle \geq 0$ for any $|v\rangle$ and there exists a unique $L$ such that $W = LL^\dagger$. Let now $|e_1\rangle, |e_2\rangle, \ldots$, be the
normalised eigenvectors of $X$ and $\lambda_1, \lambda_2, \ldots$ the corresponding eigenvalues. Furthermore, we define the normalised vectors $|\tilde{e}_j\rangle = L|e_j\rangle/\sqrt{\langle e_j|W|e_j\rangle}$. Then

$$\begin{align*}
\text{tr} (W(f(Y) - f(X))) &= \sum_j \langle e_j|W|e_j\rangle \{\langle \tilde{e}_j|f(Y)|\tilde{e}_j\rangle - f(\lambda_j)\} \\
&= \sum_j \langle e_j|W|e_j\rangle \{\langle \tilde{e}_j|Y|\tilde{e}_j\rangle - \lambda_j f'(\lambda_j)\}
\end{align*}$$

(A.1)

Now we use that for any unit vector $|v\rangle \in H$, by convexity of $f$,

$$\langle v|f(Y)|v\rangle \geq f(\langle v|Y|v\rangle).$$

(A.2)

Also, $f(y) - f(x) \geq (y - x)f'(x)$ for $x, y \in \mathbb{R}$. Thus

$$\begin{align*}
\text{tr} (W(f(Y) - f(X))) &\geq \sum_j \langle e_j|W|e_j\rangle \{f(\langle \tilde{e}_j|Y|\tilde{e}_j\rangle) - f(\lambda_j)\} \\
&\geq \sum_j \langle e_j|W|e_j\rangle \{\langle \tilde{e}_j|Y|\tilde{e}_j\rangle - \lambda_j [f'(\lambda_j)\}
\end{align*}$$

(A.3)

which completes the proof.

References

[1] T. Cover and J. Thomas. *Elements of Information Theory*. Wiley, New York, 2006.
[2] M. Kelbert and Y. Suhov. *Information Theory and Coding by Example*. Cambridge University Press, Cambridge, 2013.
[3] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* Cambridge University Press, Cambridge, 2000.
[4] M. Belis and S. Guiasu, *IEEE Trans. on Information Theory*, 14 (1968), 593–594.
[5] S. Guiasu, *Report on Math. Physics*, 2 (1971), 165–179.
[6] B. D. Sharma, J. Mitter and M. Mohan. *Inform. Control* 39 (1978), 323–336.
[7] R. P. Singh and J. D. Bhardwaj. *Inf. Sci.* 59 (1992), 149–163.
[8] J. N. Kapur. *Measures of Information and Their Applications*. Chapter 17, New Delhi: Wiley Eastern Limited, 1994.
[9] A. Sreevally and S. K. Varma, *Soochow Journal of Mathematics*, 30 (2004), no. 2, 237–243.
[10] A. Srivastava, *Cybernetics and Information Technologies*, 11 (2011), no. 3, 60–65.
[11] K. Muandet, S. Marukatat and C. Nattee, in *Advances in Machine Learning*. Lecture Notes in Computer Science, 5828 (2009), 278–292.
[12] O. Lanford and D. W. Robinson, *J. Math. Phys.* 9 (1968), 1120.
[13] E. H. Lieb and M. B. Ruskai, *J. Math. Phys.* 14 (1973), 1938.
[14] E. H. Lieb and M. B. Ruskai, *Phys. Rev. Lett.* 30 (1973), 434.
[15] M. A. Nielsen and D. Petz, *Quantum Information and Computation* 5 (2005), 507–513.
[16] D. W. Robinson and D. Ruelle, *Commun. Math. Phys.* 5 (1967), 288.
[17] M. B. Ruskai, *J. Math. Phys.* 43 (2002), 4358.
[18] Y. Suhov and S. Yasaei Sekeh, *arxiv*: 1409.4102 (2014).
[19] S. Golden, *Phys. Rev. B* 137 (1965), 1127-1128. 1128.
[20] C. J. Thompson, *J. Math. Phys.* 6 (1965), 1812-1813.
[21] E. H. Lieb, *Adv. in Math.* 11 (1973) 267–288.