Supersymmetric structure of the bosonic string theory in the Beltrami parametrization

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Abstract

We show that the bosonic string theory quantized in the Beltrami parametrization possesses a supersymmetric structure like the vector-supersymmetry already observed in topological field theories.

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1 Introduction

One of the most interesting features of the topological models [1, 2, 3] is represented by the existence of a supersymmetric algebra whose generators describe the \( BRS \) symmetry and the vector supersymmetry carrying a Lorentz index [4].

Actually this supersymmetric structure turns out to be extremely useful in discussing the renormalization of the topological models to all orders of perturbation theory. Indeed, as shown in [5], it provides an elegant and simple way for solving the descent equations associated to the integrated \( BRS \) cohomology; yielding then a complete characterization of all possible anomalies and invariant counterterms for both Schwarz [6] and Witten’s type [7] topological models.

The aim of this work is to show that this supersymmetric structure is present also in the bosonic string quantized in a conformal gauge parametrized by a Beltrami differential. This parametrization, introduced by [8], allows to use a quantization procedure analogue to that of the Yang-Mills theories. Moreover, as shown by [9], the Beltrami parametrization turns out to be the most natural parametrization which exhibits the holomorphic factorization of the Green functions according to the Belavin-Polyakov-Zamolodchikov scheme [10].

The work is organized as follows: in Sect. 2 we briefly recall the \( BRS \) quantization procedure; Sect. 3 is devoted to discuss the supersymmetric algebra and the related Ward identities. Finally, in Sect. 4 we use the aforementioned supersymmetry to solve the Wess-Zumino consistency conditions for the Slavnov anomaly. In this letter we will limit ourselves only to present the results without entering into technical computations; a more complete and detailed version is in preparation [11].
2 Quantization and Slavnov identity

Let us start with the bosonic string action

\[ S_{\text{inv}} = \frac{1}{2} \int_{\mathcal{M}} d^2 x \sqrt{g} g^{\alpha \beta} \partial_\alpha X \cdot \partial_\beta X , \]

where \( g_{\alpha \beta} \) \((\alpha, \beta = 1, 2)\) is a metric on the two-dimensional string world sheet \( \mathcal{M} \) and \( \{X\} \) are the string coordinates mapping \( \mathcal{M} \) into the \( D \)-dimensional real plane \( \mathbb{R}^D \).

Denoting with \((z, \bar{z})\) a system of complex coordinates, the world sheet metric \( g \) can be parametrized by a Beltrami differential \( \mu \)[8, 9]

\[ ds^2 = g_{\alpha \beta} dx^\alpha dx^\beta \propto |dz + \mu d\bar{z}|^2 , \]

in terms of which the action (2.1) takes the form

\[ S_{\text{inv}} = \int dz d\bar{z} \frac{1}{1 - \mu \bar{\mu}} \left( (1 + \mu \bar{\mu}) \partial X \cdot \bar{\partial} X - \mu \partial X \cdot \partial X - \bar{\mu} \bar{\partial} X \cdot \bar{\partial} X \right) , \]

with

\[ \partial = \partial_z , \quad \bar{\partial} = \partial_{\bar{z}} . \]

As one can easily check \( S_{\text{inv}} \) is invariant under an infinitesimal diffeomorphism transformation generated by a two-components vector field \((\gamma, \bar{\gamma})\):

\[ \delta X = \gamma \partial X + \bar{\gamma} \bar{\partial} X , \]

\[ \delta \mu = \gamma \partial \mu + \bar{\gamma} \bar{\partial} \mu + \mu \partial \bar{\gamma} + \bar{\mu} \bar{\partial} \gamma - \mu \partial \gamma - \mu^2 \partial \bar{\gamma} . \]

Following [8, 9], the quantization of \( S_{\text{inv}} \) is done by introducing the \((b, c)\) ghost system

\[ S_{bc} = \int dz d\bar{z} \ b (\partial c + c \partial \mu - \mu \partial c) + \int dz d\bar{z} \ \bar{b} (\bar{\partial} \bar{c} + \bar{c} \bar{\partial} \bar{\mu} - \bar{\mu} \bar{\partial} \bar{c}) , \]

so that the gauge fixed action

\[ S_{gf} = S_{\text{inv}} + S_{bc} , \]
is invariant under the nilpotent \( BRS \) transformations [8, 9]:

\[
\begin{align*}
    sX &= c\frac{\mu}{1-\mu\bar{\mu}} \partial X + \bar{c}\frac{\bar{\mu}}{1-\mu\bar{\mu}} \partial X, \\
    sc &= c\partial c, \\
    s\bar{c} &= \bar{c}\partial \bar{c}, \\
    s\mu &= \bar{\partial}c + c\partial \mu - \mu\partial c, \\
    s\bar{\mu} &= \partial \bar{c} + \bar{c}\partial \bar{\mu} - \bar{\mu}\partial \bar{c}, \\
    sb &= s\bar{b} = 0, \\
    s^2 &= 0. 
\end{align*}
\]

(2.8)

The ghosts \((c, \bar{c})\) in eqs.(2.6), (2.8) have been introduced by C. Becchi [9] and are related to the diffeomorphism variables \((\gamma, \bar{\gamma})\) of eq. (2.5) by

\[
c = \gamma + \mu\bar{\gamma}, \quad \bar{c} = \bar{\gamma} + \bar{\mu}\gamma.
\]

(2.9)

To translate the \( BRS \) invariance of \( S_{gf} \) into a Slavnov identity we introduce a set of invariant external sources \((Y, L, \bar{L})\) coupled to the non-linear variations of the \( BRS \) transformations of the quantized fields (2.8):

\[
S_{ext} = \int dzd\bar{z} \left( Y \cdot sX + Lc\partial c + \bar{L}\bar{c}\partial \bar{c} \right). 
\]

(2.10)

As explained in [8] \( \mu \) and \( \bar{\mu} \) are treated as external unquantized fields playing the role of a background metric.

Thanks to the algebraic property

\[
s\mu = \frac{\delta S_{gf}}{\delta b},
\]

(2.11)

the complete action

\[
\Sigma = S_{inv} + S_{bc} + S_{ext},
\]

(2.12)

obeys the Slavnov identity

\[
S(\Sigma) = 0,
\]

(2.13)

with

\[
S(\Sigma) = \int dzd\bar{z} \left( \frac{\delta \Sigma}{\delta Y} \cdot \frac{\delta \Sigma}{\delta X} + \frac{\delta \Sigma}{\delta \mu} \frac{\delta \Sigma}{\delta b} + \frac{\delta \Sigma}{\delta L} \frac{\delta \Sigma}{\delta c} + \frac{\delta \Sigma}{\delta \bar{\mu}} \frac{\delta \Sigma}{\delta \bar{b}} + \frac{\delta \Sigma}{\delta \bar{L}} \frac{\delta \Sigma}{\delta \bar{c}} \right).
\]

(2.14)
Let us introduce also, for further use, the linearized Slavnov operator $B$

$$B = \int dz d\bar{z} \left( \frac{\delta \Sigma}{\delta Y} \cdot \frac{\delta}{\delta X} + \frac{\delta \Sigma}{\delta X} \cdot \frac{\delta}{\delta Y} + \frac{\delta \Sigma}{\delta \mu} \cdot \frac{\delta}{\delta b} + \frac{\delta \Sigma}{\delta b} \cdot \frac{\delta}{\delta \mu} + \frac{\delta \Sigma}{\delta \bar{\mu}} \cdot \frac{\delta}{\delta \bar{b}} + \frac{\delta \Sigma}{\delta \bar{b}} \cdot \frac{\delta}{\delta \bar{\mu}} + \frac{\delta \Sigma}{\delta c} \cdot \frac{\delta}{\delta L} + \frac{\delta \Sigma}{\delta \bar{c}} \cdot \frac{\delta}{\delta \bar{L}} \right),$$

(2.15)

which, as a consequence of (2.13), turns out to be nilpotent

$$BB = 0.$$  

(2.16)

In this framework the Beltrami differential $\mu$ has a very simple physical interpretation [8, 9, 12, 13]: it is the classical source for the $(T_{zz}, T_{\bar{z}\bar{z}})$-components of the energy-momentum tensor, i.e.:

$$T_{zz} = \frac{\delta \Sigma}{\delta \mu}, \quad T_{\bar{z}\bar{z}} = \frac{\delta \Sigma}{\delta \bar{\mu}}.$$  

(2.17)

The Slavnov identity (2.13) is then the starting point for a field theory characterization of the energy-momentum current algebra [10]. From the expression (2.15) for the linearized operator $B$ it follows also:

$$T_{zz} = \mathcal{B}b,$$  

(2.18)

which shows that the energy-momentum tensor is cohomologically trivial. This property, as discussed by [4], is one of the basic ingredients for the construction of topological field models.

Let us conclude this section by noticing that, actually, not only the energy-momentum tensor but also the full string action $\Sigma$ in (2.12) is cohomologically trivial. Indeed it is easily verified that

$$\Sigma = \mathcal{B} \int dz d\bar{z} \left( \frac{1}{2} \mathbf{X} \cdot \mathbf{Y} - Lc - \bar{L}\bar{c} \right).$$  

(2.19)

This property, together with eq.(2.18), allows to interpret in a suggestive way the bosonic string as a topological model of the Witten’s type [4].
3 Supersymmetric algebra

To discuss the symmetry content of the model and to show the existence of a supersymmetric structure let us introduce the two functional operators \((\mathcal{W}, \bar{\mathcal{W}})\):

\[
\mathcal{W} = \int dzd\bar{z} \left( \mu \frac{\delta}{\delta \bar{c}} + \frac{\delta}{\delta c} + L \frac{\delta}{\delta b} \right),
\]

(3.1)

\[
\bar{\mathcal{W}} = \int dzd\bar{z} \left( \bar{\mu} \frac{\delta}{\delta c} + \frac{\delta}{\delta \bar{c}} + \bar{L} \frac{\delta}{\delta \bar{b}} \right),
\]

(3.2)

which, together with the linearized Slavnov operator (2.15), obey the following algebraic relations

\[
\{ \mathcal{B}, \mathcal{W} \} = \partial,
\]

\[
\{ \mathcal{B}, \bar{\mathcal{W}} \} = \bar{\partial},
\]

(3.3)

\[
\{ \mathcal{W}, \mathcal{W} \} = \{ \mathcal{W}, \bar{\mathcal{W}} \} = \{ \bar{\mathcal{W}}, \bar{\mathcal{W}} \} = 0.
\]

(3.4)

From eq.(3.3) one sees that the algebra between \((\mathcal{W}, \bar{\mathcal{W}})\) and \(\mathcal{B}\) closes on the translations, thus allowing for a supersymmetric interpretation of the model.

In addition, one has also the linearly broken Ward identities:

\[
\mathcal{W}\Sigma = \Delta, \quad \bar{\mathcal{W}}\Sigma = \bar{\Delta},
\]

(3.5)

with \((\Delta, \bar{\Delta})\) given by

\[
\Delta = \int dzd\bar{z} \left( \bar{L}\partial\bar{c} + L\partial c - \bar{b}\partial\bar{\mu} - b\partial\mu - Y \cdot \partial X \right),
\]

(3.6)

and

\[
\bar{\Delta} = \int dzd\bar{z} \left( L\partial c + \bar{L}\partial\bar{c} - b\partial\mu - \bar{b}\partial\bar{\mu} - Y \cdot \bar{\partial}X \right).
\]

(3.7)

Expressions (3.6)-(3.7), being linear in the quantum fields, represent classical breakings. This property seems to be a common feature of the models with a non-linearly realized supersymmetry [14].
4 The diffeomorphism anomaly

In this section we use the supersymmetric structure (3.3) in order to solve the descent equations associated with the integrated cohomology of the linearized operator $\mathcal{B}$; giving then an algebraic characterization of all possible anomalies of the Slavnov identity (2.13) at the quantum level.

In what follows we identify, for simplicity, the string world sheet $\mathcal{M}$ with the whole complex plane $\mathbb{C}$; the result being adaptable, modulo the infrared problem of the global zero modes [12], to an arbitrary Riemann surface by means of a projective connection [13, 15].

To the quantum level the classical action (2.12) is replaced by a one-loop effective action $\Gamma$ [8, 9, 12]

$$\Gamma = \Sigma + \hbar \Gamma^{(1)} ,$$

(4.1)

which obeys the anomalous Slavnov identity

$$S(\Gamma) = \hbar \mathcal{A} ,$$

(4.2)

where the diffeomorphism anomaly $\mathcal{A}$ is an integrated local two-form of ghost number one $^1$

$$\mathcal{A} = \int \mathcal{A}^1_2 ,$$

(4.3)

constrained by the Wess-Zumino consistency condition

$$\mathcal{B} \mathcal{A} = 0 .$$

(4.4)

As it is well known this equation, when translated to the local level, yields a tower of descent equations:

$$\mathcal{B} \mathcal{A}^1_2 + d \mathcal{A}^2_1 = 0 ,$$

$$\mathcal{B} \mathcal{A}^2_1 + d \mathcal{A}^3_0 = 0 ,$$

$$\mathcal{B} \mathcal{A}^3_0 = 0 ,$$

(4.5)

where $d$ denotes the exterior derivative

$$d = dz \partial + d\bar{z} \bar{\partial} ,$$

(4.6)

---

$^1$ We adopt here the usual convention of denoting with $\mathcal{A}^p_q$ a $q$-form with ghost number equal to $p$. 

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and
\[ d^2 = 0 \quad , \quad \{ B \ , \ d \} = 0 \ . \quad (4.7) \]

Thanks to the supersymmetric operators \((\mathcal{W}, \bar{\mathcal{W}})\), to solve the ladder (4.5) it is sufficient to know only the non-trivial solution of the last equation (which is a problem of local cohomology instead of a modulo-\(d\) one). It is easy to check indeed that, once \(A^3_0\) is known, the remanent cocycles \(A^2_1\) and \(A^1_2\) are identified with the \((\mathcal{W}, \bar{\mathcal{W}})\)-transform of \(A^3_0\), i.e.
\[
A^2_1 = (\mathcal{W}A^3_0) \, dz + (\bar{\mathcal{W}}A^3_0) \, d\bar{z} , \quad \quad (4.8)
\]

It is worthwhile to mention also that, due to the vanishing of the local cohomology of \(B\) in the one-form sector with ghost number two and in the two-form sector with ghost number one \([\text{14}, \text{16}]\), expression (4.8) is, modulo trivial cocycles, the most general solution of the ladder (4.5).

For what concerns the local cohomology of \(B\) in the zero-form sector with ghost number three it turns out \([\text{14}, \text{16}]\) that the most general non-trivial solution for \(A^3_0\) contains only two elements:
\[
A^3_0 = (\Omega^{(1)} , \Omega^{(2)}) , \quad (4.9)
\]
which, modulo a \(B\)-coboundary, can be written as:
\[
\Omega^{(1)} = (c \partial c \partial^2 c + \text{comp. conj.}) , \quad (4.10)
\]
and
\[
\Omega^{(2)} = (c \bar{c} \partial c (DX \cdot \bar{D}X) f(X) + \text{comp. conj.}) , \quad (4.11)
\]
where \(D\) is the "covariant derivative" \([\text{12}, \text{13}]\)
\[
D = \frac{1}{1 - \mu \bar{\mu}} (\partial - \bar{\mu} \bar{\partial}) , \quad (4.12)
\]
and \(f(X)\) is an arbitrary formal power series in the matter fields \(\{X\}\) which does not contain constant term:
\[
f(X) = \sum_{n=1}^{\infty} f_n (X \cdot X)^n . \quad (4.13)
\]
Using equations (4.8) it is immediate to show that to the cocycle $\Omega^{(1)}$ of eq.(4.10) it corresponds the diffeomorphism anomaly
\[
\mathcal{A}_2^1 = \left( -\partial \mu \partial^2 c + \partial c \partial^2 \mu \right) dz \wedge d\bar{z} + \text{comp. conj.} , \tag{4.14}
\]
i.e.
\[
\mathcal{A} \propto \int dzd\bar{z} \left( \mu \partial^3 c + \text{comp. conj.} \right) . \tag{4.15}
\]
This anomaly, whose numerical coefficient turns out to be proportional to $(D - 26)$ \cite{9}, corresponds to the well known central term of the energy-momentum current algebra \cite{10} and fixes the critical dimensions of the bosonic string. Moreover it is completely equivalent \cite{17}, via a Bardeen-Zumino action, to the more popular string Weyl anomaly.

The $\Omega^{(2)}$-cocycle of eq.(4.11) gives rise to a matter dependent cocycle $\mathcal{A}_X$ whose expression reads:
\[
\mathcal{A}_X = - \int dzd\bar{z} \left( 1 - \mu \bar{\mu} \right) \left( (\partial + \mu \bar{\partial}) \gamma + (\bar{\partial} + \mu \partial) \bar{\gamma} \right) (D\mathbf{X} \cdot \bar{D}\mathbf{X}) f(\mathbf{X}) , \tag{4.16}
\]
where
\[
\gamma = \frac{c - \mu \bar{c}}{1 - \mu \bar{\mu}} , \tag{4.17}
\]
is the diffeomorphism variable of eqs.(2.5), (2.9). Actually to $\mathcal{A}_X$ one cannot associate a true anomaly. Indeed, from the absence in the classical action (2.12) of a self-interaction term in the matter fields, it is easily seen that, in spite of the fact that $\mathcal{A}_X$ is cohomologically non-trivial, the numerical coefficient of the corresponding Feynman diagrams automatically vanishes. It follows then that expression (4.15) represents the unique breaking of the Slavnov identity (2.13) at the quantum level.

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