The complete family of Arnoux–Yoccoz surfaces

Joshua P. Bowman

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Abstract The family of translation surfaces $(X_g, \omega_g)$ constructed by Arnoux and Yoccoz from self-similar interval exchange maps encompasses one example from each genus $g$ greater than or equal to 3. We triangulate these surfaces and deduce general properties they share. The surfaces $(X_g, \omega_g)$ converge to a surface $(X_\infty, \omega_\infty)$ of infinite genus and finite area. We study the exchange on infinitely many intervals that arises from the vertical flow on $(X_\infty, \omega_\infty)$ and compute the affine group of $(X_\infty, \omega_\infty)$, which has an index 2 cyclic subgroup generated by a hyperbolic element.

Keywords Translation surface · Abelian differential · Affine homeomorphism · Veech group · Interval exchange transformation · Triangulation

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1 Introduction

1.1 From the golden ratio to the geometric series

From our calculus courses, we know that the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ converges to 1. Indeed, using the summation formula $\sum_{k=1}^{\infty} x^k = x/(1 - x)$, we find that $\frac{1}{2}$ is the unique solution to the equation $\sum_{k=1}^{\infty} x^k = 1$. From even earlier in our lives, perhaps, we recall that the equation $x + x^2 = 1$ has a unique positive solution, whose inverse is the golden ratio. The expression $x + x^2$ may be viewed as a partial geometric series, which can be extended to $n$ terms: $x + \cdots + x^n$.

The positive solutions to the equations $x + \cdots + x^n = 1$ for $n \geq 3$ are instrumental in creating a certain family of measured foliations on surfaces, which was introduced by

J. P. Bowman
Department of Mathematics, IMS, Stony Brook University, Stony Brook, NY 11794, USA
e-mail: joshua.bowman@gmail.com
Arnoux and Yoccoz [2]. In contemporary terminology, these measured foliations are the vertical foliations of certain translation surfaces. These surfaces were discovered in an attempt to provide examples of pseudo-Anosov homeomorphisms, which had been defined only a few years previously by W. Thurston in his classification of surface homeomorphisms [18]. It was shown some time later (2005) by Hubert and Lanneau [12] that the Arnoux–Yoccoz examples do not arise from the Thurston–Veech construction via compositions of multi-twists [18,20]; in particular, their affine groups contain no parabolic elements.

In this paper, after providing some background on translation surfaces, we will present the surfaces constructed by Arnoux and Yoccoz and give explicit triangulations, then use these to prove certain properties common to all these surfaces. We will also see that the family can be extended to include the cases \( n = 2 \) and \( n = \infty \). These extreme cases will turn out to be exceptional in their construction—the first corresponds to a singular surface (see the “Appendix”) and the second to a surface of infinite type (see Sect. 3)—but we hope that the self-similarity property that the golden ratio and the geometric series share with all of the other examples (see Sect. 2) will illuminate the entire sequence of surfaces for the reader.

1.2 Background on translation surfaces

There are two commonly accepted definitions for a “translation surface”: either a surface with a translation atlas, or a Riemann surface with an abelian differential \( \omega \). These definitions are not quite equivalent. The former endows the surface with a Riemannian metric (given in a translation chart \( z \) by \( |dz|^2 \)) so that it is everywhere locally isometric to the Euclidean plane. The latter allows a discrete set of points on the surface to have neighborhoods isometric to “cone points” with respect to the metric \( |\omega|^2 \); these “singularities” of the metric occur at zeroes of \( \omega \) and have angles that are integer multiples of \( 2\pi \). This latter convention is necessary, for instance, in order to have compact translation surfaces of genus \( \geq 2 \). Yet it is not hard to move from the complex-analytic definition to the Riemannian definition by simply “puncturing” the surface at the cone points. For convenience, then, we adopt the following convention.

**Definition 1.1** A translation surface is a pair \((X, \omega)\), where \( X \) is a Riemann surface and \( \omega \neq 0 \) is a holomorphic 1-form on \( X \).

Note that \( X \) is not assumed to be compact in the above definition.

**Definition 1.2** Let \((X, \omega)\) be a translation surface. A homeomorphism \( \varphi : X \to X \) is called affine if it is affine with respect to the canonical charts of \( \omega \). The group of affine homeomorphisms from \( X \) to itself is denoted Aff\( (X, \omega) \).

Any affine homeomorphism \( \varphi \) has a globally well-defined derivative \( \text{der} \varphi \in \text{GL}_2(\mathbb{R}) \); this is essentially because the group of translations is normal in the group of all affine bijections of \( \mathbb{R}^2 \). If \((X, \omega)\) has finite area, then necessarily any \( \varphi \in \text{Aff}(X, \omega) \) satisfies \( \det(\text{der} \varphi) = \pm 1 \); in this case, the dynamical type of the map \( \varphi \) can be easily determined [6,15,18]: let \( \text{Tr} \) denote the trace function.

- If \( |\text{Tr} \text{der} \varphi|^2 < 2 \), then \( \varphi \) has finite order.
- If \( |\text{Tr} \text{der} \varphi|^2 = 2 \), then \((X, \omega)\) decomposes into parallel cylinders such that on each cylinder some power of \( \varphi \) acts as a power of a Dehn twist.
- If \( |\text{Tr} \text{der} \varphi|^2 > 2 \), then \( \varphi \) is pseudo-Anosov.

The importance of the group \( \{\text{der} \varphi \mid \varphi \in \text{Aff}(X, \omega)\} \) for compact \( X \) was first observed by Veech. We make the following general definition [8,20].