Preprocessing and Cutting Planes with Conflict Graphs

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Abstract

This paper addresses the implementation of conflict graph-based routines into the COIN-OR Branch-and-Cut (CBC) solver, including: (i) a conflict graph infrastructure with an improved version of a state-of-the-art conflict detection algorithm to quickly build conflict graphs; this version detects additional conflicts and has the same worst-case complexity of the original algorithm; (ii) a preprocessing routine based on a clique-strengthening scheme that can both reduce the number of nonzeros in the constraint matrix and also produce stronger formulations; (iii) a clique cut separator capable of obtaining dual bounds at the root node that are 26\% stronger than the ones provided by the equivalent cut generator of a state-of-the-art commercial solver, 467\% stronger than those attained by the clique cut separator of the GLPK solver and 500\% stronger than the dual bounds obtained by the clique separation routine of the COIN-OR Cut Generation Library; (iv) an odd-cycle cut separator with a lifting module to produce valid odd-wheel inequalities. This new version of CBC obtained an average gap closed that is 26\% better than the previous one and solved 27\% more instances.

Keywords: Mixed Integer Programming, Conflict Graphs, Preprocessing, Cutting Planes, Clique Inequalities, Odd-cycle inequalities

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1. Introduction

Over the last years, Mixed Integer Programming has proven to be a powerful technique for modeling and solving a wide variety of combinatorial optimization problems, most of them with practical interest. Some notable applications include telecommunication network design \cite{1}, protein structure prediction \cite{2} and production planning \cite{3}.

A Mixed Integer Program (MIP) can be written as:

\[
\begin{align*}
\text{minimize:} & \quad c^T x \\
\text{subject to:} & \quad Ax \leq b \\
& \quad x \in \mathbb{R}^n \\
& \quad x_j \in \mathbb{Z}, \forall j \in I
\end{align*}
\]

where \(c \in \mathbb{R}^n\) are the objective function coefficients, \(A \in \mathbb{R}^{m \times n}\) is the constraint matrix and \(b \in \mathbb{R}^m\) is the right-hand side (RHS) of the constraints. Furthermore, \(N = \{1, ..., n\}\) is the index set of the decision variables \(x\) and \(I \subseteq N\) contains the indexes of the variables that need to be integral in every feasible solution.

Improvements in computer hardware and the development of several techniques such as preprocessing \cite{4}, \cite{5}, \cite{6}, heuristics \cite{7}, \cite{8} and cutting planes \cite{9}, \cite{10} have contributed toward large-scale MIPs being solved effectively. Preprocessing and cutting planes are part of a mechanism called automatic reformulation of MIPs \cite{11}, which is a key component of modern MIP solvers. The works of Bixby and Rothberg \cite{12}, Achterberg and Wunderling \cite{13}, and most recently Achterberg et al. \cite{6} show that disabling these features results in large performance degradation of two state-of-the-art commercial solvers.

An implicit structure used by modern MIP solvers in preprocessing and cut separation routines is the Conflict Graph (CG). Such graphs represent the logical relations between binary variables. There is a vertex for each binary variable and its complement, with an edge between two vertices indicating that the variables involved cannot both be equal to one without violating constraints.
In this paper, we present conflict graph-based routines that were developed and included in the COIN-OR Branch-and-Cut (CBC) solver. CBC is one of the fastest open-source MIP solvers nowadays and it is also a fundamental component used by Mixed Integer Nonlinear solvers, such as Bonmin and Couenne. The design and implementation of enhanced conflict graph-based cut generation routines is an important pending task in the COIN-OR development for at least one decade.

Initially, we developed a conflict graph infrastructure, characterized by the efficient construction and handling of such graphs. Our routine for building CGs is an improved version of the conflict detection algorithm presented by Achterberg, which extracts conflicts from knapsack constraints. The basis for the improvement is a new step for detecting additional maximal cliques without increasing the computational complexity of the algorithm. We also designed optimized data structures that selectively store conflicts pairwise or grouped in cliques' way, to handle dense conflict graphs without incurring excessive memory usage.

After implementing the infrastructure for CGs, we use the information provided by this structure to develop a clique-strengthening-based preprocessing routine, which is a generalization of the clique merging algorithm briefly described by Achterberg et al. Our preprocessing routine generates new constraints applying a clique extension procedure in constraints whose variables are all conflicting. This procedure tries to augment the constraints by inserting additional conflicting variables. Furthermore, during the execution of the clique extension procedure, all constraints that become dominated are removed. Computational results show that this routine helps obtain stronger formulations, with a smaller number of nonzeros and strengthened dual bound.

Finally, we implemented two conflict-based cut separators into the CBC for separating clique and odd-cycle inequalities. These separators replace those previously used by CBC. The clique and odd-cycle separators previously used by

\[\text{https://projects.coin-or.org/Cbc}\]
CBC are the ones provided by the COIN-OR Cut Generation Library (CGL), a library with a collection of cut generators that can be used with other COIN-OR packages. The clique cut separator of CGL consists of two methods frequently found in the literature: row clique and star clique [17]. These methods run only in subgraphs induced by the variables with fractional values at the current linear programming (LP) relaxation and can be performed in an enumerative or heuristic way, depending on the number of vertices of the subgraphs. Despite being fast, the clique cut separator of CGL is designed for set packing and set partitioning problems. The odd-cycle separator of CGL is included in the CBC solver but not activated by default, since it is associated with some problematic behavior, such as generating invalid inequalities and consuming excessive computational time.

Our clique cut separator is capable of obtaining dual bounds at the root node which are stronger than those provided by the clique cut separation routine of CGL and those obtained by the equivalent cut separators present in a state-of-the-art commercial MIP solver and in the solver of GNU Linear Programming Kit (GLPK). The improvement in the dual bound obtained by including only our odd-cycle cut separator is relatively small. However, the execution of this routine is computationally inexpensive, allowing its use in combination with other cut separators. Preliminary versions of our cut separators were successfully applied to three classical combinatorial optimization problems: Capacitated Vehicle Routing [18], Project Scheduling [19] and Nurse Rostering [20]. The use of these routines contributed to the solution of several hard instances for the first time in the literature.

In our experiments, the new version of CBC containing the conflict graph-based routines obtained an average gap closed noticeably better than the previous version. Moreover, the time spent to prove optimality for the MIPs de-

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2 According to the documentation of CGL, both methods perform greedy heuristics if the number of vertices is greater than 12.

3 https://www.gnu.org/software/glpk/
creased and more instances were solved.

The rest of this paper is organized as follows. Section 2 formally explains the basic approach for constructing CGs as well as our improved version of a state-of-the-art algorithm for accelerating the detection of logical implications. Section 3 describes the algorithm that finds maximal cliques with weights greater than a predefined threshold, which is used in our preprocessing and cut separation routines. Section 4 details our clique-strengthening-based preprocessing routine. Our clique and odd-cycle separators are introduced in Section 5 while in Section 6 we document the results from extensive computational experiments, analyzing individually each developed routine as well as the performance of the new version of CBC. Finally, in Section 7 we conclude and examine possible future research directions.

2. Building Conflict Graphs

A conflict graph is a structure used to store assignment pairs of binary variables which cannot occur in a feasible solution. It consists of a set of vertices \( V = \{ x_j, \overline{x}_j : j = 1, ..., n \} \) and edges \( E = \{ (u, v) : u, v \in V \} \), where \( \overline{x}_j \) is the complement of binary variable \( x_j \) (such that \( \overline{x}_j = 1 - x_j \)). There is an edge linking each variable to its complement (trivial conflicts) since only one must be activated in any feasible solution. The assignment pairs represented by the edges in a CG are used to derive logical relations and, consequently, the inequalities provided in Table 1.

| edge         | logical relation | inequality          |
|--------------|------------------|---------------------|
| \((x_j, x_k)\) | \(x_j = 1 \implies x_k = 0\) | \(x_j + x_k \leq 1\) |
| \((\overline{x}_j, \overline{x}_k)\) | \(x_j = 0 \implies x_k = 1\) | \((1 - x_j) + (1 - x_k) \leq 1\) |
| \((\overline{x}_j, x_k)\) | \(x_j = 0 \implies x_k = 0\) | \((1 - x_j) + x_k \leq 1\) |
| \((x_j, \overline{x}_k)\) | \(x_j = 1 \implies x_k = 1\) | \(x_j + (1 - x_k) \leq 1\) |
CGs can be constructed using a probing technique based on feasibility considerations. This technique consists of tentatively setting binary variables to one of their bounds and checking whether the problem becomes infeasible as a result [4]. Thus, the edges of CGs can be obtained by analyzing the impact of fixing pairs of variables to different combinations of values.

The following subsection explains how the probing technique can be used to construct CGs, according to the work of Atamtürk et al. [21]. For ease of presentation and understanding, the remainder of this paper works with MIPs in the standard format [1] whose variables are all binary (also called binary programs). Despite this, all of the techniques presented can be applied to any MIP containing binary variables.

2.1. Probing Technique based on Feasibility Conditions

Suppose we are analyzing two binary variables $x_j$ and $x_k$ with respect to a particular constraint and that these variables are assigned values $p$ and $q$, respectively. A valid lower bound for the left-hand side (LHS) of this constraint considering the assignments $x_j = p$ and $x_k = q$ is

$$L^{x_j = p, x_k = q} = p \cdot a_j + q \cdot a_k + \sum_{l \in B^- \setminus \{j,k\}} u_l \cdot a_l$$

(2)

where $B^-$ is the index set of variables with negative coefficients, $a_j$ is the coefficient for variable $x_j$ and $u_l$ is the upper bound of variable $x_l$. If $L^{x_j = p, x_k = q} > b$ then there is a conflict between the assignments of $x_j$ and $x_k$. Thus, we must insert the corresponding edge in the graph.

This calculation must be performed for each pair of variables in each constraint in order to obtain a CG. Therefore, given a MIP with $m$ constraints and $n$ variables the associated CG is constructed in $O(m \times n^2)$ steps. For this reason, probing may be computationally expensive for MIPs with a large number of variables and constraints. Nevertheless, for some constraint types, a large number of conflicts can be discovered quickly and without having to conduct a pairwise inspection. For instance, in set packing ($\sum x_j \leq 1$) and set partition-
ing ($\sum x_j = 1$) constraints each variable has a conflict with all others, explicitly forming a clique.

Considering the importance of CGs in solving MIPs and with the aim of accelerating the process of building these structures, Achterberg [16] developed an algorithm to extract cliques from knapsack constraints. Details of this algorithm, as well as an additional step to detect a higher number of maximal cliques, are presented in the following subsection.

2.2. Fast Detection of Conflicts

One way to accelerate the construction of CGs is detecting conflicts involving several variables simultaneously without using the pairwise inspection scheme. Following this idea, Achterberg [16] developed an algorithm that extracts cliques in less-structured constraints, that is, constraints that do not form a clique explicitly, by only traversing the constraint once. For this, the constraint must be ordered by the values of the coefficients of its variables. In this case, a CG is constructed in $O(m \times n \log n)$ steps.

In addition to improving the process of building CGs, the early detection of cliques also allows for the more efficient storage of the conflicts since explicit pairwise conflict storage can prove impractical for dense graphs. Thus, we can make use of special data structures where large cliques are not stored as multiple edges, like the one proposed by Atamtürk [21].

Our routine for building CGs is an improved version of the algorithm developed by Achterberg [16]. We consider that any linear constraint involving only binary variables can be rewritten as a knapsack constraint:

$$\sum_{j \in B} a_j x_j \leq b$$

with $b \geq 0$ and $a_j \geq 0$ for each $j$ in the index set $B$ of binary variables $x$.

Sometimes, transformations are necessary for the linear constraints to rewrite them in the knapsack constraint format. For a variable $x_j$ with a negative coefficient $a_j$, we must consider the absolute value $|a_j|$, replace the variable by
its complement $\bar{x}_j$ and update the RHS by adding $|a_j|$. For instance, the linear constraint $x_1 + x_2 - 2x_3 \leq 0$ is converted into the knapsack constraint $x_1 + x_2 + 2\bar{x}_3 \leq 2$.

Algorithm 1 presents our strategy to detect cliques for a given knapsack constraint. The first step is to sort the index set of variables $B$ by non-decreasing order of their coefficients. Next, we check if there are cliques in the constraint, considering the activation of the two variables with the largest coefficients (lines 2 and 3). If this assignment does not violate the RHS of the constraint, we can ignore the possibility of the existence of conflicts and the algorithm finishes. This step is essential to avoid inspecting constraints that do not contain cliques, saving time in the CG-building process.

If it is not possible to ignore the presence of conflicts then we search for the smallest $k$ in $B$ such that $a_{jk} + a_{jk+1} > b$ (line 5). Since we find the value of $k$, a clique $C$ involving variables $\{x_{jk}, x_{jk+1}, \ldots, x_{jn}\}$ is detected (line 6). This clique is then stored in clique set $S$ (line 7) and the algorithm continues.

After finding $C$, the algorithm then attempts to detect additional maximal cliques. The strategy proposed by Achterberg [16] consists of iteratively trying to replace the variable with the smallest coefficient in clique $C$ by one of the variables outside $C$, maintaining the clique property. The disadvantage of this approach is that the additional cliques always differ from only one variable of the initial clique $C$. As such, cliques containing a subset of variables of $C$ and variables outside $C$ are not detected in the current constraint. This situation is solved using our new step for detecting additional maximal cliques, which occurs at lines 8 through 14. For each variable at position $o$ outside clique $C$, a search is performed to find the smallest $f$ such that the assignment pair $(x_{jo} = 1, x_{jf} = 1)$ violates the constraint. If $f$ exists, a clique involving the variable at position $o$ and all variables whose positions are at interval $[f, n]$ is detected and stored. The algorithm stops when the binary search finds no results. The failure to find a position $f$ indicates that there are no additional cliques on the constraint since the coefficients are ordered.

It is noteworthy that the insertion of the step to extract additional cliques
Algorithm 1: Clique Detection

Input: Linear constraint $\sum_{j \in B} a_j x_j \leq b$.
Output: Set of cliques $S$.

1. Sort index set $B = \{j_1, ..., j_n\}$ by non-decreasing coefficient value
   
   $a_{j_1} \leq ... \leq a_{j_n}$;

2. if $a_{j_{n-1}} + a_{j_n} \leq b$ then

   3. return $\emptyset$;

4. $S \leftarrow \emptyset$;

5. Find the smallest $k$ such that $a_{j_k} + a_{j_{k+1}} > b$;

6. $C \leftarrow \{x_{j_k}, ..., x_{j_n}\}$;

7. $S \leftarrow S \cup \{C\}$;

8. for $o = k - 1$ downto 1 do

9. Find the smallest $f$ such that $a_{j_o} + a_{j_f} > b$;

10. if $f$ exists then

11. $C \leftarrow \{x_{j_o}, x_{j_f}, ..., x_{j_n}\}$;

12. $S \leftarrow S \cup \{C\}$;

13. else

14. break;

15. return $S$;
does not change the worst-case complexity of the algorithm, which is $O(m \times n \log n)$ for a given MIP with $m$ constraints and $n$ variables. Once the variables are ordered by their coefficients, the searching steps of lines 5 and 9 can be performed as binary searches.

The following example illustrates how our algorithm for detecting cliques in constraints works and compares the detected conflicts with the ones that could be found by the approach developed by Achterberg [16].

Example. Consider linear constraints

$$-3x_1 + 4x_2 - 5x_3 + 6x_4 + 7x_5 + 8x_6 \leq 2 \quad (4)$$

$$x_1 + x_2 + x_3 \geq 1 \quad (5)$$

where all variables are binary. The first step involves rewriting the constraints as knapsack constraints:

$$3\bar{x}_1 + 4x_2 + 5\bar{x}_3 + 6x_4 + 7x_5 + 8x_6 \leq 10 \quad (6)$$

$$\bar{x}_1 + \bar{x}_2 + \bar{x}_3 \leq 2 \quad (7)$$

Both constraints are already ordered according to their coefficients and so we begin by analyzing constraint 6. First, we check for the existence of cliques in this constraint. When we activate the two variables with the largest coefficients ($x_5 = 1$ and $x_6 = 1$), we obtain $a_5 + a_6 = 7 + 8 = 15 > 10$. For this reason, we cannot discard the existence of cliques. As such, we must now determine the smallest $k$ such that $a_{jk} + a_{jk+1} > 10$. In this case, for $k = 3$ we have $a_3 + a_4 = 5 + 6 = 11$. Consequently, a clique involving variables $\{\bar{x}_3, x_4, x_5, x_6\}$ is detected and stored. The next step consists of finding cliques involving variables $\bar{x}_1$ and $x_2$ outside $C$. For variable $x_2$, we perform a binary search that returns $f = 5$ since $a_2 + a_5 = 4 + 7 = 11 > 10$. Therefore, a clique involving variables $\{x_2, x_5, x_6\}$ is detected. Finally, for $\bar{x}_1$ the binary search finds that $a_1 + a_6 = 3 + 8 = 11 > 10$, returning $f = 6$. Thus, a clique formed by variables $\bar{x}_1$ and
3. Finding Maximal Cliques in Vertex-Weighted Graphs

The main component of our preprocessing and cut separation routines is the Bron-Kerbosch (BK) algorithm, which is responsible for finding cliques with weights greater than a certain threshold. BK is a backtracking-based algorithm that enumerates all maximal cliques in undirected graphs [22].

Aiming to increase the efficiency of the algorithm, Bron and Kerbosch introduced a variation that employs a pivoting strategy to decrease the number of recursive calls. Following this idea, Tomita et al. [23] proposed a pivoted version of BK where all maximal cliques are enumerated in $O(3^{|V|/3})$ steps. This strategy makes the pivot the vertex with the highest number of neighbors in the candidate set.

Similar to the approach of Tomita, we implemented an improved version of BK based on an optimized pivoting rule. Additionally, a pruning strategy to accelerate the discovery of maximal cliques with weights greater than a threshold. Algorithm 2 details our implementation.

Our algorithm works with three disjoint vertex sets: $R$, $P$ and $X$. Set $R$ is the set of vertices that are part of the current clique. Meanwhile, sets $P$ and $X$ are the candidate vertices to enter in $R$ and all of the vertices that have already been considered in earlier steps, respectively.

The algorithm begins with $R$ and $X$ empty, while $P$ contains all of the graph’s vertices. Within each recursive call, if the sets $P$ and $X$ are empty, then $R$ is a maximal clique. This clique is stored in the clique set $S$ if its weight $\omega(R)$ is greater than the minimum weight $\text{min}W$ (lines 3-4).

If $R$ is not yet a maximal clique, the algorithm proceeds and estimates the maximum weight clique $R$ can achieve by adding its current weight to that
Algorithm 2: Bron-Kerbosch algorithm for detecting maximal cliques with weights above a threshold.

1. \( R \leftarrow \emptyset; P \leftarrow V; X \leftarrow \emptyset; S \leftarrow \emptyset; \)

2. Function BronKerbosch\((R, P, X, S, minW)\):

3. if \( P \cup X = \emptyset \) and \( \omega(R) > minW \) then

4. \( S \leftarrow S \cup \{R\}; \)

5. if \( \omega(R) + \omega(P) > minW \) then

6. choose a pivot vertex \( u \in P \cup X \);

7. foreach \( v \in P \setminus N(u) \) do

8. \( \text{BronKerbosch}(R \cup \{v\}, P \cap N(v), X \cap N(v), S, minW); \)

9. \( P \leftarrow P \setminus \{v\}; \)

10. \( X \leftarrow X \cup \{v\}; \)

A pivot vertex \( u \) is selected from \( P \cup X \). In our implementation, the vertex with the highest modified degree is chosen, where this value refers to the sum of its degree as well as the degrees of all vertices adjacent to it. Set \( P \) is ordered by the modified degree of the vertices, accelerating the process of choosing the pivot vertex.

Next, a recursive call is made for each candidate vertex \( v \) which is not a neighbor of pivot \( u \), adding \( v \) into clique \( R \) and updating sets \( P \) and \( X \) (line 8). At this point, sets \( P \) and \( X \) contain the neighbors of vertex \( v \) which are also neighbors of the other vertices contained in clique \( R \). Using this configuration, the algorithm finds all extensions of \( R \) containing \( v \). Once vertex \( v \) has been analyzed, it is removed from \( P \) and inserted into \( X \) (lines 9–10).

Although BK has exponential computational complexity in the worst case, the use of pivoting and pruning strategies enables the effective exploration of
the search space. In practice, even for harder instances, maximal cliques with higher weights are found during the first stages of the search. For this reason, we limited the number of recursive calls made by the algorithm with a parameter called \textit{maxCalls}. Bit strings are also employed for the representation of the sets $R$ and $X$. This representation exploits bit-level parallelism in hardware for optimizing the calculation of intersection, union and removal of sets [24], which are some of the most critical bottlenecks of the BK algorithm.

4. Clique Strengthening

Preprocessing is an essential component in MIP solvers that can reduce the size of a given MIP and strengthen its formulation. The main goal of this component is to accelerate the solution process. Additionally, it may enable the early detection of infeasible problems.

There are several preprocessing strategies proposed in the literature. One of the precursors of these strategies was the work of Brearley et al. [25], which describes techniques for mathematical programming systems that reduce the problem dimension by fixing variables, remove redundant rows, replace constraints by simple bounds and more. Savelsbergh [4] presented a framework for describing preprocessing and probing techniques, providing an overview of simple and advanced techniques to improve the representation of MIPs. More recently, Gamrath et al. [26] developed three preprocessing techniques that were included in the non-commercial solver SCIP, while Achterberg et al. [6] described the preprocessing routines implemented in the commercial solver Gurobi.

Gurobi has a preprocessing routine called \textit{clique merging} [6], which is capable of combining several set packing constraints into a single constraint. This type of constraint corresponds to a clique in the CG and often appears in MIPs to model the choice of a decision over a set of possibilities. Grouping set packing constraints into larger cliques not only produces smaller MIPs concerning the number of constraints and nonzeros but also, and most importantly, a stronger formulation.
Based on the brief explanation of Achterberg et al. [6], we developed a pre-processing routine that generalizes the clique merging algorithm. Instead of considering only set packing constraints, our routine is applied in constraints whose variables are all conflicting. In this sense, we create a set $\mathcal{C}$ containing all constraints that either implicitly or explicitly form cliques.

Each constraint $j \in \mathcal{C}$ is submitted to a clique strengthening procedure, which uses the information from CG to extend it. Initially, a list of candidate variables for inclusion in constraint $j$ is constructed. The candidates must be conflicting with all of the variables in this constraint. A vertex-weighted sub-graph induced by the candidates is then created. The weights of these vertices are equal to one since we are interested in finding a clique that contains as many vertices as possible. Next, we search for maximal cliques using our improved version of the Bron-Kerbosch algorithm and select that with the largest size.

After discovering the largest clique, its variables are concatenated with those from the original constraint. At this point, we have an extended constraint of $j$. Finally, a dominance checking procedure is executed to remove all constraints that are dominated by this extended constraint. Considering MIP format 1, a constraint with coefficients $a$ and right-hand side $b$ dominates another constraint with coefficients $a'$ and right-hand side $b'$ if and only if $b \leq b'$ and $a_j \geq a'_j$ for each $j \in \{1, ..., n\}$.

Our routine is even more effective when applied to MIPs that have several constraints expressed by pairs of conflicting variables. However, it can be computationally expensive, especially for problems with dense CGs and constraints with a large number of variables. For this reason, we limit the execution of the preprocessing routine to constraints with at most $\alpha_{max}$ variables, where $\alpha_{max}$ is an input parameter of the algorithm.

**Example.** Consider the following linear constraints
The first step is to rewrite constraint 8 as a knapsack constraint:

\[-4x_1 + 4x_2 + 5x_3 + 6x_4 + 7x_5 + 10x_6 \leq 6\]  
(8)

\[x_2 + x_3 + x_4 \leq 1\]  
(9)

\[x_2 + x_5 \leq 1\]  
(10)

Once all constraints are in the knapsack constraint format, we run our algorithm for building the associated CG, obtaining the graph of in Figure 1. Set \(C\) is then created, containing constraints 9 and 10. The clique strengthening procedure is first applied to constraint 9, producing the extended constraint:

\[x_2 + x_3 + x_4 + x_5 + x_6 \leq 1\]

Then, we remove constraints 9 and 10 since they are dominated by this extended constraint. Since constraint 10 was removed, no more constraints remain in set \(C\) to apply the clique strengthening procedure. Thus, the execution of our preprocessing routine in constraints 8 to 10 results in the following constraints:

\[-4x_1 + 4x_2 + 5x_3 + 6x_4 + 7x_5 + 10x_6 \leq 6\]  
(11)

\[x_2 + x_3 + x_4 + x_5 + x_6 \leq 1\]  
(12)

5. Cutting Planes

A primary application for conflict graphs is the generation of valid inequalities, also known as cuts. A cut is a linear constraint that is violated by the current LP solution of a MIP but which does not cut off any feasible solution. The addition of such inequalities enables tightening the LP relaxation of a MIP and, in some cases, accelerates the process of generating integer solutions. Cutting planes are often combined with a branch-and-bound scheme, resulting in
Based on the idea that any feasible solution of a MIP defines a vertex packing in its conflict graph, one can conclude that any valid inequality for the vertex packing polytope is also valid for the convex hull of feasible solutions for this MIP. Thus, conflict graphs can be used to find inequalities that cut off the current LP solution.

Cliques and odd cycles are some of the most common classes of inequalities derived from the vertex packing polytope. Generally, the improvement in the value of the LP relaxation obtained by the inclusion of odd-cycle inequalities is small [27, 28]. However, the execution of a routine to separate these inequalities is computationally inexpensive in comparison with other cut separators, since they can be separated in polynomial time using shortest path algorithms [29, 10].

The following subsections present our routines for separating these conflict-based cuts.
5.1. Clique Inequalities

A clique inequality for a set $C$ of conflicting variables is defined as

$$
\sum_{j \in C} x_j \leq 1 \quad (13)
$$

where $C$ is a subset of the binary variables and their complements. As mentioned earlier, a clique represents a constraint in which at most one of the involved variables can be equal to one.

The main goal of the clique separation routine developed in this work is not to find the most violated inequality, but a set of violated inequalities. Previous work has proven this to be the best strategy. For example, Burke et al. [30] used an algorithm to discover the most violated clique, but their computational results motivated the inclusion of additional cuts found during the separation process. This result is consistent with reports of applications of other cuts applied to different models, such as Chvátal-Gomory cuts [31]. The option of inserting a large number of valid inequalities at the same time is also responsible for strengthening the importance of Gomory cuts [32].

Our separation routine is comprised of two modules: one to separate violated cliques and another to extend them. The separation of violated cliques uses our improved version of the Bron-Kerbosch algorithm. This algorithm runs in a vertex-weighted subgraph induced by all variables (and their respective complements) whose values at the solution of the current LP relaxation are greater than zero. For instance, if variable $x_j$ has a value $v > 0$ at the solution of the LP relaxation, then this variable and its complement are included in the subgraph, with weights equal to $v$ and $1 - v$, respectively.

The maximum number of recursive calls of the BK algorithm is set to $10^5$ and the minimum weight $minW$ is set to one. Therefore, cliques whose sum of weights are greater than one will be found and stored. These are the violated cliques.

Once the violated cliques are found and stored, the lifting module is performed using the complete conflict graph. This module attempts to extend the
cliques by inserting the variables whose values at the solution of the current LP relaxation are equal to zero. First, a list of candidates to enter the clique is constructed. Then, these candidates are iteratively selected using a greedy strategy. At each iteration, the candidate with the smallest reduced cost in the current LP relaxation is inserted into the clique and the list of candidates is updated. This process repeats until the list of candidates is empty.

Figure 2 illustrates the importance of extending clique inequalities. Vertices within the gray area indicate variables with nonzero values in the solution of the current LP relaxation. Only vertices $x_2$, $x_3$ and $x_4$ could contribute toward defining the most violated clique inequality. Despite this, subsequent LP relaxations would include three different $K_3$ cliques, alternating the variable whose value is equal to zero. Some reoptimization of the LP could be avoided if the inequality of the $K_4$ clique was inserted immediately after the first LP relaxation of the problem. Moreover, a less dense constraint matrix would be obtained with the initial insertion of this dominant constraint.

Figure 2: Example of a $K_3$ in which the lifting module could be applied, transforming it into a $K_4$.

5.2. Odd-Cycle Inequalities

Odd-cycle inequalities are also derived from the set packing polytope. Given a graph $G = (V, E)$, a subset $H \subseteq V$ is referred to as odd cycle if the subgraph
induced by $H$ is a simple cycle with an odd number of vertices. In this case, the subgraph must have $|H|$ adjacent edges such that each vertex is incident to exactly two vertices. Thus, an odd cycle $H$ formed by a set of binary variables (or their complements) defines the odd-cycle inequality:

$$\sum_{j \in H} x_j \leq \frac{|H| - 1}{2}$$  \hspace{1cm} (14)

This inequality ensures that at most half of the variables can be activated.

Our odd-cycle separation routine uses the concepts presented by Rebenack [10]. First, an auxiliary bipartite graph $G' = (V', E')$ is created from the original conflict graph $G = (V, E)$. The vertex set $V'$ is formed by two subsets $V_1$ and $V_2$. In this case, for each vertex $x_j \in V$, two vertices $x_{j_1}$ and $x_{j_2}$ are created in $V'$, where $x_{j_1} \in V_1$ and $x_{j_2} \in V_2$. Additionally, for each edge $(x_j, x_k) \in E$, two edges $(x_{j_1}, x_{k_2})$ and $(x_{j_2}, x_{k_1})$ are inserted into $E'$, where $x_{j_1}, x_{k_1} \in V_1$ and $x_{j_2}, x_{k_2} \in V_2$. The auxiliary graph $G'$ is bipartite since there is no edges connecting two vertices of $V_1$ or two vertices of $V_2$.

This cut separation routine works with an edge-weighted graph, where the weight of each edge is defined as:

$$w(x_j, x_k) = \frac{1 - x_j^* - x_k^*}{2}$$  \hspace{1cm} (15)

where $x_j^*$ and $x_k^*$ are the values of the variables $x_j$ and $x_k$ at the solution of the current LP relaxation.

After creating the auxiliary bipartite graph, the search for valid inequalities begins. For each vertex $x_j \in V$ we run the Dijkstra’s algorithm in $G'$ to find the shortest path from $x_{j_1}$ to $x_{j_2}$. The shortest path has an odd number of edges since vertices $x_{j_1}$ and $x_{j_2}$ are in two different sets of the bipartition. Then, the corresponding odd walk in the original graph $G$ is constructed. The resulting odd-cycle inequality is violated by the current LP solution if and only if

$$\sum_{(x_j, x_k) \in \tilde{E}} w(x_j, x_k) < \frac{1}{2}$$  \hspace{1cm} (16)
where $\tilde{E}$ is the set of edges of the subgraph induced by the odd cycle. All violated odd-cycle inequalities are found and stored by this routine except those of size three, which correspond to cliques and which could be found by the clique cut separation procedure.

Finally, a lifting module is performed to strengthen the separated odd-cycle inequalities. This module is responsible for transforming odd cycles into odd wheels. In graph theory, an odd wheel is an odd cycle that contains an additional vertex which is adjacent to all other vertices. Thus, an odd wheel can be obtained by inserting a variable into the center of the odd cycle. An odd-wheel inequality has the following format:

$$\sum_{j \in H} x_j + \frac{|H| - 1}{2} x_w \leq \frac{|H| - 1}{2}$$

where $H$ contains the variables of the odd cycle and $x_w$ is the variable that is conflicting with all of those in $H$.

Our lifting module tries to insert a clique as the center of the odd cycle, contrasting to the one presented of Rebennack [10], where only one variable is inserted. We begin constructing a list of candidates, which contains variables conflicting with all of those in the odd cycle. At each iteration, the candidate with the smallest reduced cost in the current LP relaxation is inserted into the clique and the list of candidates is updated. We repeat this step until the list of candidates is empty, generating a maximal clique. The maximal clique is then inserted in the center of the odd cycle. Figure 3 illustrates an odd wheel formed by the inclusion of the clique involving variables \{x_6, x_7, x_8\} into the center of the odd cycle formed by variables \{x_1, x_2, x_3, x_4, x_5\}. The odd-cycle inequality associated with the odd cycle of this figure is:

$$x_1 + x_2 + x_3 + x_4 + x_5 + 2x_6 + 2x_7 + 2x_8 \leq 2$$
6. Computational Results

This section presents the results of computational experiments conducted to evaluate the performance of the proposed routines. The experiments were carried out on four computers with Intel Core i7®-4790 3.60 GHz processors and 16 GB of RAM running Linux Ubuntu version 18.04 64-bit. All the source code was developed in C/C++ programming language, compiled with GCC/G++ version 7.4.0 and included into the development version of CBC solver.

A total of five experiments were conducted. The first was performed to evaluate our improved algorithm for the construction of CGs, comparing it against the Achterberg’s algorithm [16] and the probing-based technique of Atamtürk et al. [21]. The second experiment consisted of analyzing the performance of the clique-strengthening-based preprocessing routine. The third experiment compared the performance of our clique and odd-cycle cut separation routines against the equivalent cut separators of some MIP solvers, evaluating the dual bounds obtained at the root node. The last two experiments investigated the improvement obtained with the new version of CBC, which contains our conflict graph-based routines.

https://github.com/coin-or/Cbc
6.1. Instance Set

The instances used in the experiments consist of 125 binary programs found in the literature, most of which belong to the current and previous versions of the Mixed Integer Problem Library (MIPLIB) benchmark set [33]. MIPLIB is a standard library of tests used to compare the performance of MIP solvers, containing a collection of challenging real-world instances from academic and industrial applications. We excluded instances from MIPLIB whose conflict graphs contain only trivial conflicts as well as those for which CBC cannot solve the initial LP relaxation within 600 seconds.

Our instance set also contains some classical problems such as Bin Packing with Conflicts [34], Nurse Rostering [35], Bandwidth Multicoloring Problem [36] and Educational Timetabling [37]. Table 2 contains summarized information concerning the instance set, where rows “col”, “row” and “nz” detail information with respect to the number of variables, constraints and nonzeros elements of the instances, respectively. Meanwhile, row “cg_ρ” provides information about the density of the conflict graphs associated with the instances and constructed by our algorithm.

|      | min | max       | avg       | stdev     |
|------|-----|-----------|-----------|-----------|
| col  | 33  | 393,800   | 26,460.39 | 49,046.95 |
| row  | 16  | 1,451,912 | 83,914.34 | 227,016.87|
| nz   | 98  | 4,366,648 | 431,218.30| 671,629.52|
| cg_ρ | 8E-6| 0.47      | 0.05      | 0.10      |

We omitted the individual information of each instance for practical purposes. This detailed information is available at [http://professor.ufop.br/samuelbrito/downloads](http://professor.ufop.br/samuelbrito/downloads). This webpage also contains detailed results of all experiments presented throughout the following subsections.
6.2. Building Conflict Graphs

The first experiment was conducted to compare the performance of our routine for building CGs against the pairwise inspection scheme of Atamtürk et al. [21], referred to here as PI, and the clique extraction algorithm of Achterberg [16], denoted as CE. For this purpose, two routines of our algorithm were developed. These routines are referred to here as ICE and ICE+STL, and the only difference between them is the data structure used to store the conflicts. Routine ICE works with a data structure implemented by us that is based on arrays, while ICE+STL makes use of the set container of the C++ Standard Template Library (STL).

In this experiment, PI failed to construct graphs for three instances: eilA101-2, eilB101.2 and nw04. These instances have some set partitioning constraints formed by a large number of variables, whose pairwise storage of conflicts results in excessive memory consumption. Since the other routines can explicitly store cliques from set partitioning constraints, they did not face memory issues with respect to these instances. However, the following results ignore these three instances to enable a fair comparison.

Table 3 presents some statistics concerning the execution time of each routine. Columns “min”, “max” and “avg” provide the minimum, maximum and average time in seconds spent building CGs, respectively. Column “total” indicates the total time spent building graphs for all considered instances.

|        | min  | max  | avg  | total |
|--------|------|------|------|-------|
| PI     | < 0.01 | 72.40 | 3.40 | 415.11 |
| CE     | < 0.01 | 35.92 | 1.35 | 165.07 |
| ICE    | < 0.01 | 37.82 | 1.64 | 200.56 |
| ICE+STL| < 0.01 | 58.66 | 2.26 | 276.02 |

According to the results, the use of the set container of STL to store conflicts
turns the process of building CGs slower when compared against the version with our implemented data structure. ICE+STL increased by approximately 37% the average and the total time for building CGs. Furthermore, the maximum time for building the graphs was increased by 55%. The insertion operation of set checks whether the element to be inserted is equivalent to an element already in this container and if so, the element is not inserted. This verification step is performed for each conflict insertion, making it a bottleneck in the CG-building process. For this reason, the data structure that we implemented uses a lazy update strategy, whose verification and removal of duplicates occurs after the insertion of several elements. We disregarded ICE+STL routine in the remaining experiments, adopting ICE as our routine for building CGs.

In addition to solving the issue of excessive memory consumption, our algorithm obtained significantly better execution times than the pairwise inspection approach. The average elapsed time for building CGs decreased approximately by half when we compare ICE against PI. The same result is observed when we compare these two routines concerning the maximum time and the total time spent building the graphs. The combined use of the strategy to avoid analyzing constraints that would not lead to the discovery of conflicts with the clique extraction approach explain these improvements.

The routine that implements the clique extraction algorithm of Achterberg [16] (CE) obtained the best execution times in the process of building CGs. The average elapsed time for building CGs and the total time for doing this for all instances in ICE was 21% slower than CE. The increase in the execution times of our routine is justified by the insertion of the new step for detecting additional maximal cliques.

Since our routine is a version of CE that is capable of detecting more conflicts, we ran an experiment to analyze the number of edges contained in the conflict graphs. Table 4 contains summarized information concerning this experiment.

The results in Table 4 indicate that the execution of the new step to discover additional conflicts increased the average number of edges by approximately 108,000 units. Additional conflicts were detected for 23 of the 125
Table 4: Summarized results for the number of conflicts expressed by edges in the CGs.

|     | min  | max           | avg             |
|-----|------|---------------|-----------------|
| CE  | 108  | 751,339,482   | 31,068,537.25   |
| ICE | 108  | 751,339,482   | 31,177,312.44   |

instances, most of which are instances from the Bin Packing with Conflicts problem. The number of edges was increased by up to 16%, as is the case of instance \textit{uELGN\_BPWC\_3\_2\_18}, where more than 8.7 million of new edges were inserted into the associated CG. The detection of these large number of additional conflicts justifies the increase of the execution time of our routine: \textit{uELGN\_BPWC\_3\_2\_18} was the only instance whose the time spent for building the CG differed significantly from that obtained by CE.

6.3. Clique Strengthening

Given these improvements concerning the number of conflicts detected and the time spent building CGs, the next step is to evaluate the impact of our preprocessing routine based upon clique strengthening. To assess this, we ran our preprocessing routine for all the instances of the instance set, limiting the number of recursive calls of the BK algorithm to 4096 (\textit{maxCalls} = 4096) and applying this routine in constraints with at most 128 variables (\textit{\alpha_{max}} = 128). The values of these parameters were defined in a preliminary experiment.

After applying the preprocessing routine, we stored the time spent and the number of nonzeros coefficients in the constraint matrix of the preprocessed MIP. We then used the COIN-OR Linear Program Solver (CLP)\footnote{https://projects.coin-or.org/Clp} to solve the LP relaxation and calculated the gap closed. The percentage of the integrality
gap closed is computed as follows:

\[
\text{gapClosed} = 100 - 100 \times \frac{\text{bestSol} - \text{currentLP}}{\text{bestSol} - \text{firstLP}}
\]  
(18)

where \(\text{bestSol}\) is the best-known solution of the MIP, \(\text{firstLP}\) is the value of the LP relaxation at the root node and \(\text{currentLP}\) represents the value of the LP relaxation after applying the preprocessing routine. As the gap closed increases, the difference between the value of the best-known solution and the value of the LP relaxation decreases.

Table 5 provides some statistics concerning this experiment, where row “time” contains information about the execution time of our preprocessing routine, row “nz\_prop” presents the proportion of nonzeros after and before applying preprocessing, and row “gap closed” denotes the gap closed obtained after solving the LP relaxation of the preprocessed MIPs. Values greater than one in row “nz\_prop” indicate an increase in the number of nonzeros. This is a possibility given that our clique strengthening step converts equality constraints into two less-than-or-equal-to constraints. Thus, if at most one of these new constraints is dominated, the number of nonzeros in the constraint matrix will increase.

Table 5: Statistics on the execution of the clique-strengthening-based preprocessing routine.

|                  | min | max  | avg  | stdev |
|------------------|-----|------|------|-------|
| time             | < 0.01 | 61.85 | 5.43 | 11.87 |
| nz\_prop (%)     | 5.76 | 425.34 | 118.70 | 72.78 |
| gap closed (%)   | 0.00 | 100.00 | 22.73 | 33.99 |

The increase in the number of nonzeros after preprocessing occurred in 40 of the 125 instances, precisely those instances that have a considerable number of dense equality constraints. As a result, the number of nonzeros of the preprocessed MIPs increased in 18.7% on average.

On the other hand, 43 instances presented a decrease in the number of
nonzeros after preprocessing. The preprocessed MIPs for these instances are up to 94.24% smaller than the original ones. The most significant reductions occur in Bin Packing with Conflicts and Bandwidth Multicoloring problems, where there are a large number of constraints expressed by pairs of conflicting variables.

The use of our preprocessing routine also contributes to producing stronger formulations. Results show an average gap closed of 22.73% obtained after performing the LP relaxation of the preprocessed MIPs. Furthermore, the execution of this routine allowed obtaining a gap closed that is greater than 60% for 25 instances, and also it was responsible for closing the gap for instance trd445c. This instance is a real Educational Timetabling problem of a Brazilian university [38], containing 96109 set packing constraints that are used to model pairs of conflicting assignments based on student enrollments. The number of nonzeros in trd445c was reduced from 195080 to 11231 with the execution of preprocessing, representing a decrease of 94.24%. Moreover, solving the LP relaxation of the preprocessed MIP was sufficient to obtain the optimal solution for this instance.

According to the results, our routine can both reduce the number of nonzeros in the constraint matrix and also produce stronger formulations. Since we limited the size of constraints to apply this routine, its execution cost is relatively low, making possible its integration into MIP solvers.

6.4. Inserting cuts at root node

The next experiment was conducted to evaluate our clique and odd-cycle cut separation routines. We ran these cut separators only at the root node of the MIPs, considering at most 20 iterations. CLP was employed to solve the LP relaxation at each iteration and calculate the dual bounds. The metric used for comparison is the gap closed, which can be calculated by the Equation 18. Our clique and odd-cycle separators are referred to here as BKClq and OddW, respectively.

In order to compare the performance of the cut separators, we ran the clique
cut separator of the COIN-OR Cut Generation Library (CGL)\textsuperscript{7} referred to as CglClq. The clique cut separator of CGL is currently included in the CBC solver and consists of two heuristics: row clique and star clique \textsuperscript{17}. The odd-cycle separator of CGL was not included in our experiments since it is associated with some problematic behavior, such as generating invalid inequalities and consuming excessive computational time.

We also ran the clique cut separators provided by the open-source solver of GLPK version 4.65 and the commercial solver IBM ILOG CPLEX\textsuperscript{7} version 12.8, henceforth referred to as Glpk and Cplex, respectively. The clique cut separator of CPLEX can generate cliques moderately, aggressively and very aggressively. Preliminary results showed that the performance of these three strategies is very similar \textsuperscript{8}. Consequently, we present only the results of the “very aggressively” strategy, which performed slightly better with respect to the gap closed.

Figure 4 presents the average gap closed obtained for each cut separator. The results confirm that the improvement in the value of the LP relaxation obtained by the inclusion of only odd-cycle inequalities is slight. At the end of the 20 iterations, OddW separator obtained an average gap closed of 0.56%. However, the cost for generating these cuts is low and their insertion combined with other cuts can improve the gap closed. This is observed when we analyze the results of the configuration ConfC, obtained when both of the proposed cut separators are employed. ConfC obtained an average gap closed of 35.59% and the cuts generated by it were sufficient to close the gap for 5 instances.

After 20 iterations, the average gap closed was 34.37% for BKClq, 6.85% for CglClq, 7.36% for Glpk and 27.09% for Cplex. Thus, our clique separation routine obtained an average gap closed which is 5.02 times better than the clique cut separator of CGL, 4.67 times better than the clique cut separator provided by GLPK and 26.87% better than the clique cut separator of CPLEX.

\textsuperscript{7}https://projects.coin-or.org/Cgl\textsuperscript{7} https://www.ibm.com/analytics/cplex-optimizer\textsuperscript{8} Complete results are available at http://professor.ufop.br/samuelbrito/downloads
The difference in performance between the separators can be seen in the earlier steps of the separation process.

6.5. Improving COIN-OR Branch-and-Cut Solver

The last experiments explore the impact of the proposed routines in the solving process of CBC solver. The new version of CBC that includes these routines is referred to here as CBC-CG. The routine for building CGs is performed after the preprocessing step of CBC. Then, CBC performs the clique-strengthening-based preprocessing, if this routine is enabled. When activated, our clique and odd-cycle separators are executed during the branch-and-cut.

We conducted an initial experiment to evaluate the individual ability of our preprocessing and cut separation routines to accelerate the process of solving MIPs. We ran CBC-CG with a time limit of 3 hours (10800 seconds), considering all 125 instances and 4 different configurations of parameters. In the first configuration, denoted as CutsOff, we deactivated all cut separators. The second configuration consisted of disabling all cut separators except the clique
cut separator of CGL, referred to here as OnlyCglClq. The last two configurations also considered the deactivation of all cut separators, but include the clique-strengthening preprocessing (OnlyClqS), and our clique and odd-cycle separators (OnlyConfC). All configurations use the default parameters of CBC for heuristics and branching rules. Figure 5 shows the evolution of the average gap closed over time for each configuration.

![Figure 5: Evolution of the average gap closed over time.](image)

The average gap closed was 43.70% for CutsOff, 45.51% for OnlyCglClq, 56.76% for OnlyConfC and 59.90% for OnlyClqS. OnlyCglClq showed a slight improvement concerning the gap closed when compared to the configuration without cut separators. This occurred because it was unable to find valid clique inequalities for several instances and also because it consumed a lot of execution time when the number of nonzeros was large.

In the same way as the previous experiment, our separators based on conflicts presented an average gap closed better than the clique cut separator of CGL. While OnlyCglClq increased the average gap closed by 4% when compared to the
version without cuts (CutsOff), our OnlyConfC version further increased this value to 29%. Additionally, our preprocessing routine plays an important role in the strengthening of the dual bounds. The gap closed obtained by OnlyClqS even surpassed those obtained by the cut separators.

Finally, we ran CBC-CG with both clique-strengthening based preprocessing and conflict-based cut separators activated. For comparison purposes, we considered the default version of CBC, referred to here as CBC-Def. The line chart of Figure 6 shows the evolution of the average gap closed over time for these versions of the CBC solver. The time limit was set to 3 hours.

![Average gap closed over time for three versions of the CBC solver.](image.png)

The average gap closed was 59.39% for CBC-Def and 74.88% for CBC-CG version. Our preprocessing and conflict cut separation routines improved the performance of CBC solver, increasing the average gap closed in 26%.

We also investigated the number of instances for which each version of CBC has proven optimality. These results are provided in Figure 7.

The version with default parameters was able to prove optimality for 47 in-
stances, while our CBC-CG version proved optimality for 60 instances. The use of our preprocessing routine in combination with our conflict cut separators not only increased the number of instances solved but also decreased the execution time necessary for doing so.

7. Conclusions and Future Works

This paper presented the implementation of conflict graph-based routines into the COIN-OR Branch-and-Cut solver. The first routine is an improved version of a state-of-the-art conflict detection algorithm to quickly build conflict graphs, which is capable of detecting additional conflicts without changing the worst-case complexity of the original algorithm.

Information from these conflict graphs was then used to develop a preprocessing routine based on a clique strengthening procedure. This routine is a generalization of the clique merging algorithm developed by Achterberg et al. [6]. Significant improvements with respect to the dual bounds of the problems were
obtained, especially for MIPs with several constraints expressed by two con-
flicting variables. Our preprocessing routine was responsible for reducing the
number of nonzeros, strengthening the initial dual bounds and for accelerating
the process of proving optimality for a great number of instances.

We also developed two conflict-based cut separators, one for separating
cliques and another for odd-cycles. Our clique cut separator obtained better
dual bounds than those provided by the equivalent cut generators of CPLEX
and CBC solvers. While the improvement obtained by including only odd-cycle
inequalities was small, the cost for executing this separator was low, which
means that it can be used in combination with other cut separators.

Experiments conducted with the new version of CBC that contains our pre-
processing and cut separation routines revealed an improvement of the gap
closed of 26% in comparison to the previous version of this solver. In addition,
the number of instances solved was increased from 47 to 60, an improvement of
27%. Furthermore, the time spent to prove optimality for these instances has
decreased.

The routines described in this work rely on the performance of our Bron-
Kerbosch-based algorithm for detecting cliques. For this reason, we plan to
perform extensive experiments to investigate and detect some possible improve-
ments concerning this algorithm. We also intend to develop a conflict-based
heuristic for generating initial feasible solutions for MIPs. Finally, the use of con-
straint programming techniques for detecting additional conflicts in the graphs
appears to be a promising future research direction.

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