FOUR MANIFOLDS WITH NO SMOOTH SPINES

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Abstract. Let $W$ be a compact smooth orientable 4-manifold that deformation retract to a pl embedded closed surface. One can arrange the embedding to have at most one non-locally-flat point, and near the point the topology of the embedding is encoded in the singularity knot $K$. If $K$ is slice, then $W$ has a smooth spine, i.e., deformation retracts onto a smoothly embedded surface. Using the obstructions from the Heegaard Floer homology and the high-dimensional surgery theory, we show that $W$ has no smooth spine if $K$ is a knot with nonzero Arf invariant, a nontrivial L-space knot, the connected sum of nontrivial L-space knots, or an alternating knot of signature $< -4$. We also discuss examples where the interior of $W$ is negatively curved.

1. Introduction

A spine is a topological (not necessarily locally flat), compact, boundaryless submanifold that is a strong deformation retract of the ambient manifold. A spine is smooth or pl if the submanifold has this property.

Examples of 4-manifolds that are homotopy equivalent to closed surfaces but have no pl spines can be found in [Mat75, MV79, LL19, HP]. It is shown in [Ven98] that an example in [MV79] does not even have a topological spine. Some 4-manifolds with topological spines and no pl spines can be found in [KR20]. The present paper constructs 4-manifolds with pl spines and no smooth spines.

In this section $W$ denotes a compact oriented smooth 4-manifold with a pl spine $S$ homeomorphic to a closed oriented connected surface. By a standard argument $S$ can be moved by a pl homeomorphism to a spine with at most one non-locally-flat point; henceforth we assume that $S$ has this property. If $S$ is locally flat, then the submanifold $S$ is smoothable [RS68, Corollary 6.8]. Otherwise, $S$ intersects the link of the non-locally-flat point in a singularity knot $K$. If $K$ is smoothly slice, then replacing the cone on $K$ in $S$ with a smoothly embedded disk in $W$ gives a smooth spine of $W$.

Conversely, if $\Sigma$ is an oriented connected surface with one boundary component, then attaching $\Sigma \times D^2$ to the 4-ball along the knot $K$ in its boundary with framing $r$ gives a compact oriented 4-manifold with a pl spine homeomorphic to $\Sigma / \partial \Sigma$, which has normal Euler number $r$ and singularity knot $K$. If $\Sigma = D^2$, the 4-manifold is denoted by $K^r$ and called a knot trace.

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Examples of non-slice singularity knot such that \( W \) has a smooth spine come from exotic knot traces. Namely [Akb93, Theorem A] describes knots \( K_1, K_2 \) such that \( K_1 \) is slice, \( K_2 \) is not slice, and \( K_1^r, K_2^r \) are diffeomorphic for some \( r \). We refer to [HMP, HP, FMN+] for a recent study of relations between invariants of knot traces and knot concordance.

Cappell and Shaneson [CS76] developed a surgery-theoretic criterion that can help decide when a manifold with a \( \text{pl} \) spine of dimension \( \geq 3 \) and codimension 2 also admits a locally flat spine. Applying the criterion to \( W \times S^1 \) we prove

**Theorem 1.1.** If \( W \) is a compact oriented smooth 4-manifold that has a \( \text{pl} \) spine whose singularity knot has nonzero Arf invariant, then \( W \) contains no smooth spine.

Tye Lidman and Daniel Ruberman asked us if the generalized Rohlin invariant can be used to give a purely 4-dimensional proof of Theorem 1.1. This was done in [Sae92, Theorem 3.1] in the case when \( W \) is a homotopy \( S^2 \) with finite \( H_1(\partial W) \). We leave the question to an interested reader.

The criterion of [CS76] also gives a weak converse of Theorem 1.1: If \( K \) has zero Arf invariant, then \( W \times S^1 \) has a smooth spine, see Remark 3.2.

If \( W \) has two \( \text{pl} \) spines with regular neighborhoods \( R_1, R_2 \) in the interior of \( W \), then there is a homology cobordism between the boundaries \( \partial R_1, \partial R_2 \) obtained by gluing \( W \setminus \text{Int}(R_1) \) and \( W \setminus \text{Int}(R_2) \) along \( \partial W \). The Heegaard-Floer \( d \)-invariants \( d_{\text{top}}, d_{\text{bot}} \) are preserved under homology cobordisms. Furthermore, one can express the \( d \)-invariants of \( \partial R_1, \partial R_2 \) via singularity knots of their spines, and for some knots the \( d \)-invariants can be explicitly computed, which gives the following.

**Theorem 1.2.** If \( W \) is a compact oriented smooth 4-manifold that has a \( \text{pl} \) spine whose singularity knot is a nontrivial L-space knot, the nontrivial connected sum of nontrivial L-space knots, or an alternating knot of signature \( < -4 \), then \( W \) contains no smooth spine.

Recall that a knot \( K \subset S^3 \) is an L-space knot if there is an integer \( n > 0 \) such that the \( n \)-framed surgery on the knot is an L-space [OS05, Definition 1.1]. For example, torus knots are L-space knots [OS05, p.1285].

The \( d \)-invariants obstruction applies to some topologically slice knots in [HKL16], which gives

**Corollary 1.3.** For any \( g, e \in \mathbb{Z} \) with \( g \geq 0 \) there exists a compact smooth oriented 4-manifold with no smooth spine and a topological locally flat spine that is an oriented closed genus \( g \) surface with normal Euler number \( e \).

We were led to the subject of this paper while thinking of examples in [GLT88, Kui88] of oriented hyperbolic 4-manifolds with \( \text{pl} \) spines. Each of these manifolds...
is a quotient of the hyperbolic space $\mathbb{H}^4$ by a Kleinian group $\Gamma_0$ which is a torsion-free finite index subgroup in a certain discrete group $\Gamma$ of orientation preserving isometries of $\mathbb{H}^4$. The group $\Gamma$ is described via face-pairings of its fundamental domain $F$, which is obtained by removing from $\mathbb{H}^4$ a neighborhood of a nontrivial torus knot $T$ in the ideal boundary of $\mathbb{H}^4$. In turn, the fundamental domain $F_0$ for $\Gamma_0$ is obtained by gluing $k$ copies of $F$, where $k$ is the index of $\Gamma_0$ in $\Gamma$, and one can describe $F_0$ as the result of removing from $\mathbb{H}^4$ a neighborhood of the $k$-fold connected sum of $T$. This $k$-fold connected sum is the singularity knot in a PL spine of $\mathbb{H}^4/\Gamma_0$, and hence $\mathbb{H}^4/\Gamma_0$ has no smooth spine by Theorem 1.2.

A related construction in [GLT88, Section 6] replaces the torus knot $T$ by an arbitrary nontrivial knot $K$ but the group $\Gamma$ is now generated by reflections in the codimension one faces of $F$. The resulting singularity knot of the PL spine of $\mathbb{H}^4/\Gamma_0$ is the $\frac{k}{2}$-fold connected sum of $K \# r\bar{K}$, where $r\bar{K}$ is the reverse of the mirror image of $K$. (Here $k$ is even because $\mathbb{H}^4/\Gamma_0$ is orientable and $\Gamma$ does not preserve orientation). Since $K \# r\bar{K}$ is slice, the singularity knot is slice, and $\mathbb{H}^4/\Gamma_0$ has a smooth spine.

An analog of these examples with variable pinched negative curvature is discussed in [Bel], which is based on Ontaneda’s Riemannian hyperbolization [Ont20]. Here there is no need to pass to a finite index torsion-free subgroup, and for any knot $K$ one gets pinched negatively curved 4-manifolds whose PL spine has $K$ as a singularity knot. In particular, if $K$ satisfies the assumptions of Theorems 1.1 or 1.2, the negatively pinched 4-manifold has no smooth spine, while in the setting of Corollary 1.3 there exists a topologically flat spine.

The structure of the paper is as follows. In Section 2 we review results on the Kervaire invariant of compact oriented manifolds with codimension 2 spines. In Section 3 we specialize to dimension 4, relate the Kervaire invariant of $W$ and the Arf invariant of the singularity knot, and prove Theorem 1.1. Section 4 is a review of Heegaard Floer $d$-invariants, whose relationship to $V$-functions is explored in Section 5. In Section 6 we investigate how the assumption “$W$ has a smooth spine” affects the $V$-function of the singularity knot. Section 7 contains a proof of Theorem 1.2 and Corollary 1.3.

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2. Kervaire Invariant of Codimension Two Thickennings

Let $W$ be a compact oriented PL manifold with a PL embedded spine $S$, a closed connected oriented manifold of $\dim(S) = \dim(W) - 2$. Let $\xi$ be an oriented plane bundle over $S$ whose Euler class is the normal Euler class of $S$ in $W$, and let $p_\xi: D_\xi \rightarrow S$ be the associated 2-disk bundle. Then [CS76, Proposition 1.6] gives
a homology isomorphism \( h: (W, \partial W) \to (D_\xi, \partial D_\xi) \) such that \( h \) preserves the orientation class in the relative second cohomology, and \( p_\xi \circ h|_S \) is homotopic to the identity of \( S \). The map \( h \) pulls \( \alpha := p_\xi(\nu_\xi|_S) \) to the stable normal bundle \( \nu_\xi \) of \( W \) because \( h^*\alpha \) and \( \nu_\xi \) are isomorphic over \( S \) to which \( W \) deformation retracts. This gives a normal map \((h, b_W)\) where \( b_W: \nu_\xi \to \alpha \) is the above bundle map.

Assuming, as we can, that \( h \) is transverse regular to the zero section \( S \) of \( D_\xi \), we see that \( N := h^{-1}(S) \) is a closed surface which is locally flat in \( W \) with normal bundle \( h^*\xi \). The stable normal bundle to \( N \) is \( \nu_N = \nu_\xi|_S \oplus h^*\xi = h^*(\alpha|_S \oplus \xi) \). Thus \( h|_N: N \to S \) is covered by the bundle map \( b_N: \nu_N \to \alpha|_S \oplus \xi \). The orientation on \( \xi \) and \( W \) defines an orientation on \( N \) for which \( h|_N: N \to S \) has degree one, and hence \((h|_N, b_N)\) is a normal map.

The normal invariant of \((h|_N, b_N)\) is the image of the normal invariant of \((h, b_W)\) under the inclusion-induced map \([W, G/PL] \to [S, G/PL]\), which is a bijection because \( S \hookrightarrow W \) is a homotopy equivalence. This standard fact is stated on [CS76, p.195] and in the appendix of [KR08], and the proof amounts to comparing various definitions of the normal invariant.

By [RS71, Section 1] the Kervaire invariant of the normal map \((h|_N, b_N)\) is the Arf invariant of a certain quadratic form on the kernel of \( h|_{\nu_N}: H_1(N; \mathbb{Z}_2) \to H_1(S; \mathbb{Z}_2) \). A normal map with nontrivial Kervaire invariant represents a nontrivial class in \([S, G/PL]\), see [RS71, Theorem 1.4(ii)], and in fact, the Kervaire invariant defines a group homomorphism \([S, G/PL] \to \mathbb{Z}_2\) [RS71, Corollary 4.5].

3. Kervaire and Arf invariants in dimension four

Let us adopt notations of Section 2 and suppose \( \dim(W) = 4 \). Then the group \([S, G/PL]\) is isomorphic to \( H^2(S; \mathbb{Z}_2) \cong \mathbb{Z}_2 \), see e.g. [KT01, Section 2], and hence the Kervaire invariant defines an isomorphism \([S, G/PL] \to \mathbb{Z}_2\).

Fix a triangulation of \( W \) for which \( S \) is a full subcomplex with only one non-locally flat point. Its star is an embedded 4-ball \( B \), and \( C := S \cap B \) is the cone on the knot \( K = S \cap \partial B \), the singularity knot of \( S \subset W \).

**Lemma 3.1.** The Kervaire invariant of \( W \) in \([S, G/PL]\) is the Arf invariant of the knot \( K \).

**Proof.** Let \( V \) be the smallest subcomplex that contains a neighborhood of \( S \) in \( W \). Since \( S \) is a full subcomplex, \( V \) is a regular neighborhood of \( S \) in \( W \), to which \( W \) deformation retracts. Denote the relative interiors of \( B \), \( C \), \( V \) by \( B \), \( C \), \( V \). Then \( V \setminus B \) is a trivial 2-disk bundle over \( S \setminus C \). Give \( B \) the structure of a trivial 2-disk bundle over a 2-disk, whose zero section \( Z \) intersects \( \partial B \) in an unknot \( U \). Glue \( V \setminus B \) and \( B \) by an orientation-preserving 2-disk bundle automorphism identifying \( \partial C \) with \( \partial Z = U \) so that the resulting 2-disk bundle \( D_\xi \to S \) has the same Euler class as \( S \subset W \). Denote the regular neighborhoods of \( K \) and \( U \) in \( \partial B \) by \( R_K \) and \( R_U \), respectively.
Then the above-mentioned map \( h: W \to D_\xi \) can be chosen so that \( h|_{W \setminus V} \) is a deformation retraction onto \( \partial V \), the map \( h|_{V \setminus B} \) is the identity, \( h \) takes \((B, \partial B, K)\) to \((B, \partial B, U)\), and maps \( R_K \) homeomorphically onto \( R_U \). To define \( h|_B \) apply [CS76, Proposition 1.6] to the thickening \( B \) of \( C \) and use the fact that any homology equivalence \((R_K, \partial R_K) \to (R_U, \partial R_U)\) is homotopic to a homeomorphism.

Isotope the zero section \( Z \) to a 2-disk \( Z_0 \subset \partial B \) rel boundary, and perturb \( h \) near \( B \) to be transverse regular to \( \partial B \). The above argument can be reversed, namely, if \( h \) is a 

\[ \eta \circ f \] is homotopically trivial, then the \( S \) is a locally flat spine of \( L \). Then \( L' := L \times S^1 \) is a locally flat spine of \( W' \). The restriction to \( L' \) of the deformation retraction \( W' \to S' \) is homotopic to a diffeomorphism \( g: L' \to S' \), see e.g. [Lau74, p.5]. Hence the normal invariant of \( g \) is trivial.

As we explain in [Bel, Appendix C], the pullback via \( g \) of the Poincaré embedding given by the inclusion \( S' \subset W' \) is isomorphic to the Poincaré embedding of the locally flat inclusion \( L' \subset W' \). Since \( \dim(S') \) is odd and \( \geq 3 \), Theorem 6.2 of [CS76] implies that the Poincaré embedding for \( L' \subset W' \) can be realized by a locally flat embedding if and only if the normal invariants of \( g \) equals the normal invariant of the Poincaré embedding \( S' \subset W' \). This is a contradiction because these normal invariants are different and \( L' \subset W' \) is locally flat.

Remark 3.2. The above argument can be reversed, namely, if \( K \) has zero Arf invariant, then the Poincaré embedding induced by the inclusion \( S \to W \) has trivial normal invariant, and hence so does its product with a circle or more generally, with any closed manifold \( L \), and if \( \dim(L) \) is odd, then \( W \times L \) has a locally flat spine [CS76, Theorem 6.2].
4. Heegaard Floer $d$-invariants and $V$-functions of knots

Ozsváth and Szabó introduced [OS03, OS04c, OS04b] Heegaard-Floer homology theories $HF^o(M, t)$ associated with a Spin$^c$ structure $t$ on a closed oriented 3-manifold $M$. Here $o$ is a decoration indicating the flavor of a Heegaard-Floer theory, and in this paper $o$ is $\infty$ or $-$. The homology groups $HF^-(M, t)$ and $HF^\infty(M, t)$ are modules over $\mathbb{Z}[U]$ and $\mathbb{Z}[U, U^{-1}]$, respectively, where $U$ is a formal variable whose action lowers the relative homological degree by 2. Related invariants for knots and links in 3-manifolds were developed in [Ras03, OS04a, OS08a]. We refer to these papers for background.

Henceforth, we assume that $M$ has standard $HF^\infty$ [OS03, p.240], and the Spin$^c$ structure $t$ is torsion, i.e., its first Chern class has finite order in $H^2(M)$.

According to [OS04c, Section 4.2.5] the group $H^T_2(M) := H_1(M)/\text{Tors}$ acts on the Heegaard-Floer chain complex $CF^o(M, t)$, and on the corresponding homology group $HF^o(M, t)$. Let $HF^o(M, t)_{\text{bot}}$ and $HF^o(M, t)_{\text{top}}$ denote the kernel and the cokernel of the $H^T_2(M)$-action on $HF^o(M, t)$. The $d$-invariants $d_{\text{top}}(M, t)$ and $d_{\text{bot}}(M, t)$ are the maximal homological degrees of a non-torsion class in $HF^o(M, t)$ and $HF^o_{\text{bot}}(M, t)$, respectively, see [OS03, Section 9] and [LR14, Section 3]. If $M$ is a rational homology sphere, the $H^T_2(M)$-action is trivial, so that $HF^o_{\text{top}}(M, t) = HF^o_{\text{bot}}(M, t) = HF^-(M, t)$, and $d_{\text{top}}(M, t) = d_{\text{bot}}(M, t)$ is the usual $d$-invariant for rational homology spheres, as in [OS03]. The invariants $d_{\text{top}}(M, t)$, $d_{\text{bot}}(M, t)$ are preserved under rational homology cobordisms [LR14, Proposition 4.5].

A null-homologous knot $K$ in $M$ gives rise to a $\mathbb{Z} \oplus \mathbb{Z}$ filtered chain complex $CFK^\infty(M, K, s)$, which is a $\mathbb{F}[U, U^{-1}]$-module, see [OS04a, Ras03]. The filtration is indexed by the pair of integers $(i, j)$, where $i$ keeps track of the power of $U$, and $j$ records the so called Alexander filtration. For $s \in \mathbb{Z}$, let $A^-_s(K) := A^-_s(M, K, s)$ be the subcomplex of $CFK^\infty(M, K, s)$ corresponding to $\max(i, j - s) \leq 0$ [MO, Remarks 3.7–3.8]. By the large surgery formula the homology of $A^-_s(K)$ is the sum of one copy of $\mathbb{F}[U]$ and a $U$-torsion submodule.

Following [NW15] we define the $V$-function $V_s(K)$ of an oriented knot $K \subset S^3$ so that $-2V_s(K)$ is the maximal homological degree of the free part of $H_i(A^-_s(K))$. For one-component links the $H$-function for links of [BG18, Liu17] is the $V$-function of knots. For example, the $V$-function of the unknot $U$ is given by $V_s(U) = 0$ for $s \geq 0$ and $V_s(U) = -s$ for $s < 0$. The $V$-function takes values in nonnegative integers [BG18, Proposition 3.10], and furthermore, [BG18, Proposition 3.10] and [Liu17, Lemma 5.5] give

**Proposition 4.1.** The $V$-function of an oriented knot $K \subset S^3$ satisfies

$$V_{-s}(K) = V_s(K) + s \quad \text{and} \quad V_{s-1}(K) - V_s(K) \in \{0, 1\}.$$
Example 4.2. Let $K$ be an alternating knot of signature $\sigma$; recall that $\sigma \in 2\mathbb{Z}$. By [HM17, Theorem 1.7] if $\sigma > 0$, then $V_s(K) = 0$ for all $s$, and if $\sigma \leq 0$, the values $V_0(K)$ are given in the table below:

| $\sigma$    | $V_0(K)$ |
|------------|----------|
| $-8k$      | $2k$     |
| $-8k - 2$  | $2k + 1$ |
| $-8k - 4$  | $2k + 1$ |
| $-8k - 6$  | $2k + 2$ |

5. Surgeries on knots and $d$-invariants

For a positive integer $g$ let $C^g := \#^g S^2 \times S^1$, the connected sum of $2g$ copies of $S^2 \times S^1$. As usual $M_n(K)$ denotes the $n$-framed surgery on a closed oriented $3$-manifold $M$ along a knot $K \subset M$; in what follows $M$ is $S^3$ or $C^0$.

If $B \subset C^g$ is the Borromean knot, as defined e.g. in [Par14, Figure 4.1], then $C_n^g(B)$ has the structure of an oriented circle bundle over the genus $g$ oriented surface with Euler number $n$ [OS08b, Section 5.2]. For the unknot $U \subset S^3$ it is well-known that $S^3_n(U)$ is an oriented circle bundle over $S^2$ with Euler number $n$.

It follows from [OS03, Propositions 9.3–9.4], cf. [Par14, Proposition 4.0.5], that $C^g_n(K \# B)$ has standard $HF^\infty$ for any oriented knot $K \subset S^3$. The same is true for $S^3_n(K)$ [OS04b, Theorem 10.1]. Thus the $d$-invariants $d_{\text{top}}$, $d_{\text{bot}}$ are defined for $C^g_n(K \# B)$ and $S^3_n(K)$, and moreover, for $S^3_n(K)$ they reduce to the usual $d$-invariants. They were computed by Ni-Wu [NW15, Proposition 1.6] for $S^3_n(K)$, and by Park [Par14, Theorem 4.2.3] for $C^g_n(B)$, $n \neq 0$. Park’s argument extends to $C^g_n(K \# B)$ as follows.

Theorem 5.1. For $n > 0$, we have

\begin{equation}
(5.2) \quad d_{\text{top}}(C^g_n(K \# B), k) = g + \frac{(2k - n)^2 - n}{4n} - 2 \min_{a=0, \ldots, g} \{a + V_{k-g+2a}(K)\},
\end{equation}

\begin{equation}
(5.3) \quad d_{\text{bot}}(C^g_n(K \# B), k) = g + \frac{(2k - n)^2 - n}{4n} - 2 \max_{a=0, \ldots, g} \{a + V_{k-g+2a}(K)\}
\end{equation}

where $k$ labels the torsion Spin$^c$ structures on $C^g_n(K \# B)$ with $-n/2 < k \leq n/2$. The $d$-invariant of $S^3_n(K)$ is given by

\begin{equation}
(5.4) \quad d(S^3_n(K), k) = \frac{(2k - n)^2 - n}{4n} - 2V_k(K).
\end{equation}

Proof. As in the proof of [Par14, Theorem 4.1.1], a diagram chase in the surgery mapping cone formula [OS08b, Theorem 4.10] shows that the free part of $H_*(A_k)$ is isomorphic to the free part of $HF^-(C^g_n(K \# B), k)$. The grading of the free part of $H_*(A_k)$ can be found in [BHL17, Theorem 6.10]. \qed
Remark 5.5. Theorem 5.1 extends to $n < 0$ as follows. Since the Borromean knot is amphichiral, $C_{g}^{n}(K\# B) = -C_{-g}^{n}(\bar{K}\# B)$, where $\bar{K}$ is the mirror of $K$. Then [LR14, Proposition 3.7] gives

$$d_{\text{bot}}(C_{g}^{n}(K\# B), k) = -d_{\text{top}}(C_{-g}^{n}(\bar{K}\# B), k)$$

$$d_{\text{top}}(C_{g}^{n}(K\# B), k) = -d_{\text{bot}}(C_{-g}^{n}(\bar{K}\# B), k).$$

Remark 5.6. A similar argument also computes $d_{\text{top}}$ and $d_{\text{bot}}$ for rational surgeries, i.e., when $0 \neq n \in \mathbb{Q}$.

6. Spines, homology cobordisms, and $d$-invariants

Let $W$ be a compact, oriented, smooth 4-manifold with a PL spine $S_1$, an oriented genus $g$ surface with normal Euler number $e$. As before assume that $S_1$ has at most one non-locally-flat point with singularity knot $K \subset S^3$. If $W$ also has a smooth spine $S_2$, then there is a homology cobordism $C$ between the boundaries $M_1$, $M_2$ of the regular neighborhoods of $S_1$, $S_2$. Namely, $C$ is obtained by removing the interiors of the regular neighborhoods from $W$ and gluing the results along $\partial W$.

Here $M_1$ can be described as an $e$-surgery on $C^g = \#^{2g} S^2 \times S^1$ along the knot $K\# B$ where $B$ is the Borromean knot [BHL17, Theorem 3.1], while $M_2$ is the circle bundle over $S_2$ with Euler number $e$, which is the $e$-surgery on $C^g$ along $B$.

Since $H^2(C) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/e\mathbb{Z}$, every torsion Spin$^c$ structure on $C$ can be thought of an element of $\mathbb{Z}/e\mathbb{Z}$ indexed by $k \in (-e/2, e/2]$. Restricting the element to $M_j$, $j \in \{1, 2\}$, gives a torsion Spin$^c$ structure on $M_j$, which we denote $t_{kj}$. Thus

$$d_{\text{top}}(M_1, t_{k1}) = d_{\text{top}}(M_2, t_{k2}).$$

Theorem 6.2. If $e \geq 0$, and $W$ contains a smooth spine, then the singularity knot $K$ satisfies

$$\min_{a=0, \ldots, g} \{a + V_{g+2a}(K)\} = \lceil g/2 \rceil,$$

where $\lceil g/2 \rceil$ is the smallest integer that is $\geq g/2$.

Proof. Since $V_s(U) = \frac{|s| - s}{2}$ we compute

$$\min_{a=0, \ldots, g} \{a + V_{g+2a}(U)\} = \lceil g/2 \rceil.$$ 

If $e > 0$, by Theorem 5.1

$$d_{\text{top}}(M_1, t_{k1}) = g + s - 2\min_{a=0, \ldots, g} \{a + V_{k-g+2a}(K)\}$$

and

$$d_{\text{top}}(M_2, t_{k2}) = g + s - 2\min_{a=0, \ldots, g} \{a + V_{k-g+2a}(U)\}.$$
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where \( s = \frac{(2k-e)^2-e}{4e} \). Hence

\[
\min_{a=0,\ldots,9} \{ a + V_{k-g+2a}(K) \} = \min_{a=0,\ldots,9} \{ a + V_{k-g+2a}(U) \}.
\]

Combining (6.4) and (6.5) for \( k = 0 \) gives (6.3) in the case \( e > 0 \).

Assume \( e = 0 \). Then \( M_j \) is the 0-surgery on \( C^g \), where \( j = 1,2 \). Let \( M'_j \) denote the 1-surgery on \( C^g \) along the same knot as for \( M_j \). By the equality part of [OS03, Corollary 9.14],

\[
d_{\text{top}}(M_j, t_{0j}) - \frac{1}{2} = d_{\text{top}}(M'_j, t'_{0j}),
\]

where \( t_{0j}, t'_{0j} \) are the trivial Spin\(^c\) structures. Even though [OS03, Corollary 9.14] is stated for knots in \( S^3 \), it generalizes (with the same proof) to knots in 3-manifolds with standard \( HF^\infty \) and trivial \( HF_{\text{red}} \), which is how we apply it.

By (6.1) \( M_1, M_2 \) have the same \( d_{\text{top}} \), and hence \( d_{\text{top}}(M'_1, t'_{01}) = d_{\text{top}}(M'_2, t'_{02}) \), and as before (6.4)–(6.5) imply (6.3), now for \( e = 0 \). \( \square \)

**Corollary 6.6.** If \( W \) contains a smooth spine with normal Euler number \( e \geq 0 \), then the singularity knot \( K \) satisfies

\[
V_0(K) = 0 \quad \text{if} \quad g \quad \text{is even and} \quad V_1(K) = 0 \quad \text{if} \quad g \quad \text{is odd}.
\]

**Proof.** If \( g = 2k \), then by Theorem 6.2

\[
\min_k \{ V_0(K) + k, V_2(K) + k + 1, \ldots, V_{2k}(K) + 2k \} = k.
\]

Proposition 4.1 gives \( V_{s-1}(K) \leq V_s(K) + 1 \), and hence the minimum occurs for \( V_0(K) + k = k \), which implies \( V_0(K) = 0 \). Similarly, if \( g = 2k + 1 \), we have

\[
\min_k \{ V_1(k) + k + 1, \ldots, V_{2k+1}(K) + 2k + 1 \} = k + 1
\]

which means that \( V_1(K) + k + 1 = k + 1 \), and hence \( V_1(K) = 0 \). \( \square \)

7. **Singularity knots and smooth spines**

As in Section 6 let \( W \) a compact, oriented, smooth 4-manifold with a PL spine which is an oriented genus \( g \) surface with normal Euler number \( e \), and at most one non-locally-flat point with singularity knot \( K \). After changing the orientation of \( W \), if needed, we can and will assume that \( e \geq 0 \).

**Proof of Theorem 1.2.** By Corollary 6.6 \( V_0(K) = 0 \) or \( V_1(K) = 0 \) depending on the parity of \( g \), and hence \( g(K) \leq 1 \) [Liu, Lemma 2.11], where \( g(K) \) is the genus of \( K \). If \( g(K) = 0 \), then \( K \) is the unknot. A genus-one L-space knot is the right-handed trefoil [Ghi08, Corollary 1.5]. According to [NNU98] the Arf invariant for the torus knot \( T(p,q) \) is \((p^2-1)(q^2-1)/24 \text{ (mod 2)}\). Thus the Arf invariant of \( T(2,3) \) is nonzero, which implies by Theorem 1.1 that \( W \) cannot contain a smooth spine. This completes the proof when \( K \) is an L-space knot.
Suppose $K$ is an alternating knot of signature $< -4$. Hence $V_0(K) \geq 2$ by Example 4.2. Then Proposition 4.1 gives $V_1(K) \geq 1$, which by Corollary 6.6 shows that $W$ does not have a smooth spine.

Finally, suppose that $K$ is the connected sum of nontrivial L-space knots $K_1, \ldots, K_n$ with $n \geq 2$. Thus $g(K) = g(K_1) + \cdots + g(K_n)$. Since $K_i$ is nontrivial, $g(K_i) \geq 1$, and hence $g(K) \geq n$. For $j \in \mathbb{Z}$ set $R_i(j) := V_{g(K_i) - 1}(K_i)$ and

$$R_K(j) := \min_{j_1 + \cdots + j_n = j} R_{K_1}(j_1) + \cdots + R_{K_n}(j_n).$$

By Proposition 4.1 the function $R_K$ is nonnegative and nondecreasing, and combining the proposition with [Liu, Lemma 2.11] gives $R_K(1) = V_{g(K) - 1}(K) = 1$. Hence $R_K(j) \geq 1$ for every $j \geq 1$.

Propositions 5.1 and 5.6 and Lemma 6.2 of [BL14] imply $V_j(K) + j = R_K(g(K) + j)$; the notations in [BL14] are different. Again, by Corollary 6.6 if $V_0(K)$ and $V_1(K)$ are both nonzero, then $W$ does not have a smooth spine.

To see that $V_0(K) = R_K(g(K)) \geq 1$ assume the minimum of $R_K(g(K))$ is attained for $j_1 + \cdots + j_n = g(K)$. Then $j_i \geq 1$ for some $i$, and $R_K(g(K)) \geq R_{K_i}(j_i) \geq 1$.

To show that $1 \leq V_1(K) = R_K(g(K) + 1) - 1$ assume that the minimum of $R_K(g(K) + 1)$ is attained for $j_1 + \cdots + j_n = g(K) + 1$. If $j_i \geq g(K_i) + 2$, then

$$R_K(g(K) + 1) \geq R_{K_i}(j_i) \geq R_{K_i}(g(K_i) + 2) = V_{g(K_i) + 2}(K_i) = V_1(K_i) \geq 1.$$ 

as claimed. Otherwise, there are indices with $j_i \geq g(K_i)$ and $j_i = g(K_i) + 1$. Then $R_{K_i}(j_i) \geq V_0(K_i) \geq 1$ and $R_{K_i}(j_i) = V_1(K_i) \geq 1 \geq 1$, and hence $R_K(g(K) + 1) \geq 2$ as desired.

Proof of Corollary 1.3. For any $m \geq 2$, there is a topologically slice knot $K_m$ with $V_0(K_m) = m$ [HKL16, Proposition 6 and Theorem B.1]. The corresponding manifold $W$ has a topologically flat spine. By Corollary 6.6 and Proposition 4.1 if $W$ has a smooth spine, then $V_0(K) \in \{0, 1\}$. ∎

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