A proof of convergence for stochastic gradient descent in the training of artificial neural networks with ReLU activation for constant target functions

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Abstract. In this article we study the stochastic gradient descent (SGD) optimization method in the training of fully connected feedforward artificial neural networks with ReLU activation. The main result of this work proves that the risk of the SGD process converges to zero if the target function under consideration is constant. In the established convergence result the considered artificial neural networks consist of one input layer, one hidden layer, and one output layer (with \( d \in \mathbb{N} \) neurons on the input layer, \( H \in \mathbb{N} \) neurons on the hidden layer, and one neuron on the output layer). The learning rates of the SGD process are assumed to be sufficiently small, and the input data used in the SGD process to train the artificial neural networks is assumed to be independent and identically distributed.

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1. Introduction

Artificial neural networks (ANNs) are these days widely used in several real-world applications, including, e.g., text classification, image recognition, autonomous driving, and game intelligence. In particular, we refer, e.g., to [8, Section 2], [20, Chapter 12], and [28] for an overview of applications of neural networks in language processing and computer vision, as well as references on further applications. Stochastic gradient descent (SGD) optimization methods provide the standard schemes which are used for the training of ANNs. Nonetheless, until today, there is no complete mathematical analysis in the scientific literature which rigorously explains the success of SGD optimization methods in the training of ANNs in numerical simulations.

However, there are several interesting directions of research regarding the mathematical analysis of SGD optimization methods in the training of ANNs. The convergence of SGD optimization schemes for convex target functions is well understood, cf., e.g., [4, 35–37, 41] and the references mentioned therein. For abstract convergence results for SGD optimization methods without convexity assumptions, we refer, e.g., to [1, 7, 13, 14, 18, 27, 31, 33, 40] and the references mentioned therein. We also refer, e.g., to [10, 25, 34, 44] and the references mentioned therein for lower bounds and divergence results for SGD optimization methods. For more detailed overviews and further references on SGD optimization schemes, we refer, e.g., to [8], [18, Section 1.1], [24, Section 1], and [42]. The effect of random initializations in the training of ANNs was studied, e.g., in [6, 21, 22, 26, 34, 45] and the references mentioned therein. Another promising branch of research has investigated the convergence of SGD for the training of ANNs in the so-called overparametrized regime, where the number of ANN parameters has to be sufficiently large. In this situation SGD can be shown to converge to global minima with high probability, see, e.g., [12, 16, 17, 23, 32, 46] for the case of shallow ANNs and see, e.g., [2, 3, 15, 43, 47] for the case of deep ANNs. These works consider the empirical risk, which is measured with respect to a finite set of data.
Another direction of research is to study the true risk landscape of ANNs and characterize the saddle points and local minima, which was done in Cheridito et al. [11] for the case of affine target functions. The question under which conditions gradient-based optimization algorithms cannot converge to saddle points was investigated, e.g., in [29,30,38,39] for the case of deterministic GD optimization schemes and, e.g., in [19] for the case of SGD optimization schemes.

In this work we study the plain vanilla SGD optimization method in the training of fully connected feedforward ANNs with ReLU activation in the special situation where the target function is a constant function. The main result of this work, Theorem 3.12 in Sect. 3.6, proves that the risk of the SGD feedforward ANNs with ReLU activation in the special situation where the target function is a constant but fail to be summable. We thereby extend the findings in our previous article Cheridito et al. [9] by proving convergence for the SGD optimization method instead of merely for the deterministic GD optimization method, by allowing the gradient to be defined as the limit of the gradients of appropriate general approximations of the ReLU activation function instead of a specific choice for the approximating sequence, by allowing the learning rates to be non-constant and varying over time, by allowing the input data to multi-dimensional, and by allowing the law of the input data to be an arbitrary probability distribution on \([a, b]^d\) with \(a \in \mathbb{R}, b \in (a, \infty), d \in \mathbb{N}\) instead of the continuous uniform distribution on \([0, 1]\).

To illustrate the findings of this work in more details, we present in Theorem 1.1 below a special case of Theorem 3.12. Before we present below the rigorous mathematical statement of Theorem 1.1, we now provide an informal description of the statement of Theorem 1.1 and also briefly explain some of the mathematical objects that appear in Theorem 1.1 below.

In Theorem 1.1 we study the SGD optimization method in the training of fully connected feedforward artificial neural networks (ANNs) with three layers: the input layer, one hidden layer, and the output layer. The input layer consists of \(d \in \mathbb{N} = \{1, 2, \ldots\}\) neurons (the input is thus \(d\)-dimensional), the hidden layer consists of \(H \in \mathbb{N}\) neurons (the hidden layer is thus \(H\)-dimensional), and the output layer consists of 1 neuron (the output is thus one-dimensional). In between the \(d\)-dimensional input layer and the \(H\)-dimensional hidden layer an affine linear transformation from \(\mathbb{R}^d\) to \(\mathbb{R}^H\) is applied with \(Hd + H\) real parameters, and in between the \(H\)-dimensional hidden layer and the 1-dimensional output layer an affine linear transformation from \(\mathbb{R}^H\) to \(\mathbb{R}^1\) is applied with \(H + 1\) real parameters. Overall the considered ANNs are thus described through

\[
\mathfrak{d} = (Hd + H) + (H + 1) = Hd + 2H + 1
\]

real parameters. In Theorem 1.1 we assume that the target function which we intend to learn is a constant and the real number \(\xi \in \mathbb{R}\) in Theorem 1.1 specifies this constant. The real numbers \(a \in \mathbb{R}, b \in (a, \infty)\) in Theorem 1.1 specify the set in which the input data for the training process lies in the sense that we assume that the input data is given through \([a, b]^d\)-valued i.i.d. random variables.

In Theorem 1.1 we study the SGD optimization method in the training of ANNs with the rectifier function \(\mathbb{R} \ni x \mapsto \max\{x, 0\} \in \mathbb{R}\) as the activation function. This type of activation is often also referred to as rectified linear unit activation (ReLU activation). The ReLU activation function \(\mathbb{R} \ni x \mapsto \max\{x, 0\} \in \mathbb{R}\) fails to be differentiable at the origin and the ReLU activation function can in general therefore not be used to define gradients of the considered risk function and the corresponding gradient descent process. In implementations, maybe the most common procedure to overcome this issue is to formally apply the chain rule as if all involved functions would be differentiable and to define the “derivative” of the ReLU activation function as the left derivative of the ReLU activation function. This is also precisely the way how SGD is implemented in TensorFlow and we refer to Sect. 3.7 for a short illustrative example Python code for the computation of such generalized gradients of the risk function.

In this article we mathematically formalize this procedure (see (2), (69), and item (ii) in Proposition 3.2) by employing appropriate continuously differentiable functions which approximate the ReLU
activation function in the sense that the employed approximating functions converge to the ReLU activation function and that the derivatives of the employed approximating functions converge to the left derivative of the ReLU activation function. More specifically, in Theorem 1.1 the function $\mathcal{R}_\infty : \mathbb{R} \rightarrow \mathbb{R}$ is the ReLU activation function and the functions $\mathcal{R}_r : \mathbb{R} \rightarrow \mathbb{R}, r \in \mathbb{N}$, serve as continuously differentiable approximations for the ReLU activation function $\mathcal{R}_\infty$. In particular, in Theorem 1.1 we assume that for all $x \in \mathbb{R}$ it holds that $\mathcal{R}_\infty(x) = \max\{x, 0\}$ and

$$\lim_{r \rightarrow -\infty} |\mathcal{R}_r(x) - \max\{x, 0\}| = \lim_{r \rightarrow -\infty} |(\mathcal{R}_r)'(x) - \mathbb{I}_{(0, \infty)}(x)| = 0.$$  

(2)

In Theorem 1.1 the realization functions associated to the considered ANNs are described through the functions $\mathcal{N}_r = (\mathcal{N}_r^\phi)_{\phi \in \mathbb{R}^d} : \mathbb{R}^d \rightarrow C(\mathbb{R}^d, \mathbb{R}), r \in \mathbb{N} \cup \{\infty\}$. In particular, in Theorem 1.1 we assume that for all $\phi = (\phi_1, \ldots, \phi_d) \in \mathbb{R}^d$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have that

$$\mathcal{N}_r^\phi(x) = \phi_0 + \sum_{i=1}^d \phi_{H(d+1)+i} \max\left\{\phi_{H(d+1)+i} x_j, 0\right\}$$  

(3)

(cf. (5) below). The input data which is used to train the considered ANNs is provided through the random variables $X^{n,m} : \Omega \rightarrow [a, b]^d, n, m \in \mathbb{N}_0$, which are assumed to be i.i.d. random variables. Here $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying probability space.

The function $\mathcal{L} : \mathbb{R}^3 \rightarrow \mathbb{R}$ in Theorem 1.1 specifies the risk function associated to the considered supervised learning problem and, roughly speaking, for every neural network parameter $\phi \in \mathbb{R}^d$ we have that the value $\mathcal{L}(\phi) \in [0, \infty)$ of the risk function measures the error how well the realization function $\mathcal{N}_r^\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ of the neural network associated to $\phi$ approximates the target function $[a, b]^d \ni x \mapsto \xi \in \mathbb{R}$.

The sequence of natural numbers $(M_n)_{n \in \mathbb{N}_0} \subseteq \mathbb{N}$ describes the size of the mini-batches in the SGD process. Furthermore, for every $n \in \mathbb{N}_0$ the function $\mathcal{S}^n : \mathbb{R}^3 \times \Omega \rightarrow \mathbb{R}^3$ describes the appropriately generalized stochastic gradient of $\mathcal{L}$ with respect to the mini-batch $(X^{n,m})_{m \in \{1, \ldots, M_n\}}$. For all $(\phi, \omega) \in \mathbb{R}^3 \times \Omega$ which satisfy that the sequence of approximate gradients $(\nabla_\phi \mathcal{L}^n_r)(\phi, \omega) \in \mathbb{R}^d, r \in \mathbb{N}$, is convergent we have that $\mathcal{S}^n(\phi, \omega)$ is defined as its limit as $r \rightarrow \infty$. In Proposition 3.2 below we show that, in fact, it holds for all $(\phi, \omega) \in \mathbb{R}^3 \times \Omega$ that the limit $\lim_{r \rightarrow -\infty} (\nabla_\phi \mathcal{L}^n_r)(\phi, \omega)$ exists, and thus, $\mathcal{S}^n(\phi, \omega)$ is uniquely specified for all $(\phi, \omega) \in \mathbb{R}^3 \times \Omega$.

The SGD optimization method is described through the SGD process $\Theta : \mathbb{N}_0 \times \Omega \rightarrow \mathbb{R}^3$ in Theorem 1.1 and the real numbers $\gamma_n \in [0, \infty), n \in \mathbb{N}_0$, specify the learning rates in the SGD process. The learning rates are assumed to be sufficiently small in the sense that

$$\sup_{n \in \mathbb{N}_0} \gamma_n \leq (18d)^{-1} (1 + ||\Theta(0)||)^{-2} (\max\{|\xi|, |a|, |b|, 1\})^{-4}$$  

(4)

and the learning rates may not be summable and instead are assumed to satisfy $\sum_{k=0}^{\infty} \gamma_k = \infty$. Under these assumptions Theorem 1.1 proves that the true risk $\mathcal{L}(\theta_n)$ converges to zero in the almost sure and the $L^1$-sense as the number of gradient descent steps $n \in \mathbb{N}$ increases to infinity. We now present Theorem 1.1 and thereby precisely formalize the above mentioned paraphrasing comments.

Theorem 1.1. Let $d, H, \mathcal{H} \in \mathbb{N}, \xi, a, b \in \mathbb{R}, b \in (a, \infty)$ satisfy $\mathcal{H} = dH + 2H + 1$, let $\mathcal{R}_r : \mathbb{R} \rightarrow \mathbb{R}, r \in \mathbb{N} \cup \{\infty\}, \mathcal{R}_\infty(x) = \max\{x, 0\}$, and let $\mathcal{N}_r = (\mathcal{N}_r^\phi)_{\phi \in \mathbb{R}^d} : \mathbb{R}^d \rightarrow C(\mathbb{R}^d, \mathbb{R}), r \in \mathbb{N} \cup \{\infty\}$, satisfy for all $r \in \mathbb{N} \cup \{\infty\}$, $\phi = (\phi_1, \ldots, \phi_d) \in \mathbb{R}^d$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ that

$$\mathcal{N}_r^\phi(x) = \phi_0 + \sum_{i=1}^d \phi_{H(d+1)+i} \mathcal{R}_r \left(\phi_{H(d+1)+i} x_j + \sum_{j=1}^d \phi_{i(1)d+j} x_j\right),$$  

(5)

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X^{n,m} : \Omega \rightarrow [a, b]^d, n, m \in \mathbb{N}_0$, be i.i.d. random variables, let $||\cdot|| : \mathbb{R}^d \rightarrow \mathbb{R}$ and $\mathcal{L} : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfy for all $\phi = (\phi_1, \ldots, \phi_d) \in \mathbb{R}^d$ that $||\phi|| = (\sum_{i=1}^d |\phi_i|^2)^{1/2}$ and $\mathcal{L}(\phi) = \mathbb{E}[|\mathcal{N}_r^\phi(X^{0,0}) - \xi|^2]$, let $(M_n)_{n \in \mathbb{N}_0} \subseteq \mathbb{N}$, let $\mathcal{L}^n_r : \mathbb{R}^3 \times \Omega \rightarrow \mathbb{R}, n \in \mathbb{N}_0, r \in \mathbb{N} \cup \{\infty\}, \mathcal{S}^n : \mathbb{R}^3 \times \Omega \rightarrow \mathbb{R}$, $\phi \in \mathbb{R}^3, \omega \in \Omega$ that

$$\mathcal{L}^n_r(\phi, \omega) = \frac{1}{M_n} \sum_{m=1}^{M_n} (\mathcal{N}_r^\phi(X^{n,m}(\omega)) - \xi)^2,$$  

(6)
let $\mathcal{G}^n: \mathbb{R}^d \times \Omega \to \mathbb{R}^d$, $n \in \mathbb{N}_0$, satisfy for all $n \in \mathbb{N}_0$, $\phi \in \mathbb{R}^d$, $\omega \in \{w \in \Omega: (\nabla_\phi \mathcal{L}_n^\gamma)(\phi, w)\}_{\gamma \in \mathbb{N}}$ is convergent} that $\mathcal{G}^n(\phi, \omega) = \lim_{\gamma \to \infty} (\nabla_\phi \mathcal{L}^\gamma_n)(\phi, \omega)$, let $\Theta = (\Theta_n)_{n \in \mathbb{N}_0}: \mathbb{N}_0 \times \Omega \to \mathbb{R}^d$ be a stochastic process, let $(\gamma_n)_{n \in \mathbb{N}_0} \subseteq [0, \infty)$, assume that $\Theta_0$ and $(X^{n,m}_{(n,m)}(\omega))_{(n,m) \in (\mathbb{N}_0)^2}$ are independent, and assume for all $n \in \mathbb{N}_0$, $\omega \in \Omega$ that $\Theta_{n+1}(\omega) = \Theta_n(\omega) - \gamma_n \mathcal{G}^n(\Theta_n(\omega), \omega)$, $18d(\max\{|\xi|, |a|, |b|, 1\})^4 \gamma_n \leq (1 + \|\Theta_0(\omega)\|)^{-2}$, and $\sum_{k=0}^{\infty} \gamma_k = \infty$. Then

(i) there exists $c \in \mathbb{R}$ such that $\mathbb{P}(\sup_{n \in \mathbb{N}_0} \|\Theta_n\| \leq c) = 1$,
(ii) it holds that $\mathbb{P}(\limsup_{n \to \infty} \mathcal{L}(\Theta_n) = 0) = 1$, and
(iii) it holds that $\limsup_{n \to \infty} \mathbb{E}[\mathcal{L}(\Theta_n)] = 0$.

Theorem 1.1 is a direct consequence of Corollary 3.13 in Sect. 3.6 below. Corollary 3.13, in turn, follows from Theorem 3.12 in Sect. 3.6. Theorem 3.12 proves that the true risk of the considered SGD processes $(\Theta_n)_{n \in \mathbb{N}_0}$ converges to zero both in the almost sure and the $L^1$-sense in the special case where the target function is constant. In Sect. 2 we establish an analogous result for the deterministic GD optimization method. More specifically, Theorem 2.16 in Sect. 2.8 below demonstrates that the true risk of the considered GD processes converges to zero if the target function is constant.

Our proofs of Theorems 2.16 and 3.12 make use of similar Lyapunov estimates as in Cheridito et al. [9]. In particular, two key auxiliary results of this article are Corollary 2.10 (in the deterministic setting) and Lemma 3.8 (in the stochastic setting). These results in particular imply that the scalar product of the gradient of the considered Lyapunov function $V: \mathbb{R}^d \to \mathbb{R}$ and the generalized gradient of the risk function is always nonnegative. We use this to prove that the value of $V$ always decreases along GD and SGD trajectories and thus that $V$ indeed serves as a Lyapunov function. This fact, in turn, implies stability and convergence properties for the considered GD processes. The contradiction argument we use to deal with the case of non-constant learning rates in the proofs of Theorem 2.16 and Theorem 3.12 is strongly inspired by the arguments in Lei et al. [31, Section IV.A].

2. Convergence of gradient descent (GD) processes

In this section we establish in Theorem 2.16 in Sect. 2.8 below that the true risks of GD processes converge in the training of ANNs with ReLU activation to zero if the target function under consideration is a constant. Theorem 2.16 imposes the mathematical framework in Setting 2.1 in Sect. 2.1 below and in Setting 2.1 we formally introduce, among other things, the considered target function $f: [a, b]^d \to \mathbb{R}$ (which is assumed to be an element of the continuous functions $C([a, b]^d, \mathbb{R})$ from $[a, b]^d$ to $\mathbb{R}$), the realization functions $\mathcal{A}^a_{\mathbb{R}^d}: \mathbb{R}^d \to \mathbb{R}, \phi \in \mathbb{R}^d$, of the considered ANNs (see (8) in Setting 2.1), the true risk function $\mathcal{L}_\infty: \mathbb{R}^d \to \mathbb{R}$, a sequence of smooth approximations $\mathcal{A}_r: \mathbb{R} \to \mathbb{R}$, $r \in \mathbb{N}$, of the ReLU activation function (see (7) in Setting 2.1), as well as the appropriately generalized gradient function $\mathcal{G} = (\mathcal{G}_1, \ldots, \mathcal{G}_d): \mathbb{R}^{d} \to \mathbb{R}^d$ associated to the true risk function. In the elementary result in Proposition 2.2 in Sect. 2.2 below we also explicitly specify a simple example for the considered sequence of smooth approximations of the ReLU activation function. Proposition 2.2 is, e.g., proved as Cheridito et al. [9, Proposition 2.2].

Item (ii) in Theorem 2.16 shows that the true risk $\mathcal{L}_\infty(\Theta_n)$ of the GD process $\Theta: \mathbb{N}_0 \to \mathbb{R}^d$ converges to zero as the number of gradient descent steps $n \in \mathbb{N}$ increases to infinity. In our proof of Theorem 2.16 we use the upper estimates for the standard norm of the generalized gradient function $\mathcal{G}: \mathbb{R}^d \to \mathbb{R}^d$ in Lemma 2.5 and Corollary 2.6 in Sect. 2.5 below as well as the Lyapunov type estimates for GD processes in Lemma 2.12, Corollaries 2.13, 2.14, and Lemma 2.15 in Sect. 2.7 below. Our proof of Corollary 2.6 employs Lemma 2.5 and the elementary local Lipschitz continuity estimates for the true risk function in Lemma 2.4 below. Lemma 2.4 is a direct consequence of, e.g., Beck et al. [6, Theorem 2.36]. Our proof of Lemma 2.5 makes use of the elementary representation result for the generalized gradient function $\mathcal{G}: \mathbb{R}^d \to \mathbb{R}^d$ in Proposition 2.3 in Sect. 2.3 below.
2.1. Description of artificial neural networks (ANNs) with ReLU activation

**Setting 2.1.** Let \( d, H, \theta \in \mathbb{N}, \alpha, a \in \mathbb{R}, b \in (a, \infty), f \in C([a, b]^d, \mathbb{R}) \) satisfy \( \theta = dH + 2H + 1 \) and \( a = \max\{|a|, |b|, 1\}, \) let \( \varpi = ((\varpi_{i,j})_{i,j \in \{1, \ldots, H\} \times \{1, \ldots, d\}})_{\phi \in \mathbb{R}^f} : \mathbb{R}^\theta \to \mathbb{R}^{d \times \theta}, \) \( b = (b_{1}^{\phi}, \ldots, b_{1}^{\phi})_{\phi \in \mathbb{R}^f} : \mathbb{R}^\theta \to \mathbb{R}^H, \) \( \mathbf{v} = (\mathbf{v}^\phi_{1}, \ldots, \mathbf{v}^\phi_{d})_{\phi \in \mathbb{R}^f} : \mathbb{R}^\theta \to \mathbb{R}^H, \) and \( c_\phi = (c_\phi)_{\phi \in \mathbb{R}^f} : \mathbb{R} \to \mathbb{R} \) satisfy for all \( \phi = (\phi_1, \ldots, \phi_d) \in \mathbb{R}^d, \) \( i \in \{1, 2, \ldots, H\}, \) \( j \in \{1, 2, \ldots, d\} \) that \( \varpi_{i,j}^\phi = \phi_{(i-1)d+j}, \) \( b_{1}^{\phi} = \phi_0 d + i, \) \( \mathbf{v}^i_{\phi} = \phi_{H(d+1)+i}, \) and \( c_\phi = \phi_0, \) let \( \mathcal{A}_r : \mathbb{R} \to \mathbb{R}, \) \( r \in \mathbb{N} \cup \{\infty\}, \) satisfy for all \( x \in \mathbb{R} \) that \( (\bigcup_{r \in \mathbb{N}} \mathcal{A}_r) \subseteq C^1(\mathbb{R}, \mathbb{R}), \) \( \mathbb{R}_\infty(x) = \max\{x, 0\}, \) \( \sup_{r \in \mathbb{N}} \sup_{y \in [-|x|, |x|]} |(\mathcal{A}_r)'(y)| < \infty, \) and

\[
\lim_{r \to \infty} \left( |\mathcal{A}_r(x) - \mathbb{R}_\infty(x)| + |(\mathcal{A}_r)'(x) - 1_{(0, \infty)}(x)| \right) = 0,
\]

(7)

let \( \mu : \mathcal{B}([a, b]^d) \to [0, 1] \) be a probability measure, let \( \mathcal{M}_r = (\mathcal{M}_r^\phi)_{\phi \in \mathbb{R}^f} : \mathbb{R}^\theta \to C(\mathbb{R}^d, \mathbb{R}), r \in \mathbb{N} \cup \{\infty\}, \) and \( \mathcal{L}_r : \mathbb{R}^\theta \to \mathbb{R}, r \in \mathbb{N} \cup \{\infty\}, \) satisfy for all \( r \in \mathbb{N} \cup \{\infty\}, \phi \in \mathbb{R}^f, x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) that

\[
\mathcal{M}_r^\phi(x) = c_\phi + \sum_{i=1}^{H} \varpi_{i}^\phi \mathcal{A}_r \left( b_{i}^{\phi} + \sum_{j=1}^{d} w_{i,j}^\phi x_j \right)
\]

(8)

and \( \mathcal{L}_r(\phi) = \int_{[a, b]^d} (\mathcal{M}_r^\phi(y) - f(y))^2 \mu(dy), \) let \( \mathcal{G} = (\mathcal{G}_1, \ldots, \mathcal{G}_d) : \mathbb{R}^\theta \to \mathbb{R}^\theta \) satisfy for all \( \phi \in \{\varphi \in \mathbb{R} : ((\nabla \mathcal{L}_r(\varphi))_{r \in \mathbb{N}} \text{ is convergent}) \} \) that \( \mathcal{G}(\phi) = \lim_{r \to \infty} (\nabla \mathcal{L}_r(\phi)), \) let \( \|\cdot\| : (\bigcup_{n \in \mathbb{N}} \mathbb{R}^n) \to \mathbb{R} \) and \( \langle \cdot, \cdot \rangle : (\bigcup_{n \in \mathbb{N}} \mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R} \) satisfy for all \( n \in \mathbb{N}, x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n \) that \( \|x\| = \left[ \sum_{i=1}^{n} |x_i|^2 \right]^{1/2} \) and \( (x, y) = \sum_{i=1}^{n} x_i y_i, \) and let \( I^0_i \subseteq \mathbb{R}, \phi \in \mathbb{R}^f, i \in \{1, 2, \ldots, H\}, \) and \( \mathcal{V} : \mathbb{R}^\theta \to \mathbb{R} \) satisfy for all \( \phi \in \mathbb{R}^\theta, i \in \{1, 2, \ldots, H\} \) that \( I^0_i = \{x = (x_1, \ldots, x_d) \in [a, b]^d : b_i^\phi + \sum_{j=1}^{d} w_{i,j}^\phi x_j > 0\} \) and \( \mathcal{V}(\phi) = \|\phi\|^2 + |c_\phi - 2f(0)|^2. \)

2.2. Smooth approximations for the ReLU activation function

**Proposition 2.2.** Let \( \mathcal{A}_r : \mathbb{R} \to \mathbb{R}, r \in \mathbb{N}, \) satisfy for all \( r \in \mathbb{N}, x \in \mathbb{R} \) that \( \mathcal{A}_r(x) = r^{-1} \ln(1 + r^{-1} e^{rx}). \) Then

(i) it holds for all \( r \in \mathbb{N} \) that \( \mathcal{A}_r \in C^\infty(\mathbb{R}, \mathbb{R}), \)
(ii) it holds for all \( x \in \mathbb{R} \) that \( \sup_{r \to \infty} |\mathcal{A}_r(x) - \max\{x, 0\}| = 0, \)
(iii) it holds for all \( x \in \mathbb{R} \) that \( \sup_{r \to \infty} |(\mathcal{A}_r)'(x) - 1_{(0, \infty)}(x)| = 0, \) and
(iv) it holds that \( \sup_{r \in \mathbb{N}} \sup_{x \in \mathbb{R}} |(\mathcal{A}_r)'(x)| \leq 1. \)

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Note that for any open or closed set \( E \subseteq \mathbb{R}^d \) we denote by \( \mathcal{B}(E) \) the Borel sigma algebra on \( E, \) i.e., the smallest \( \sigma \)-algebra which contains all open subsets of \( E. \)
2.3. Properties of the approximating true risk functions and their gradients

**Proposition 2.3.** Assume Setting 2.1 and let $\phi = (\phi_1, \ldots, \phi_d) \in \mathbb{R}^d$. Then

(i) it holds for all $r \in \mathbb{N}$ that $L_r \in C^1(\mathbb{R}^d, \mathbb{R})$,

(ii) it holds for all $r \in \mathbb{N}, i \in \{1, 2, \ldots, H\}, j \in \{1, 2, \ldots, d\}$ that

\[
\left( \frac{\partial}{\partial \phi_{(i-1)d+j}} L_r \right)(\phi) = 2\nu_i \int_{[a,b]^d} x_j \left[ (R_r)' \left( b_i + \sum_{k=1}^d w_{i,k}^p x_k \right) \right] (\mathcal{M}_r^\phi(x) - f(x)) \mu(dx),
\]

\[
\left( \frac{\partial}{\partial \phi_{Hd+i}} L_r \right)(\phi) = 2\nu_i \int_{[a,b]^d} \left[ (R_r)' \left( b_i + \sum_{k=1}^d w_{i,k}^p x_k \right) \right] (\mathcal{M}_r^\phi(x) - f(x)) \mu(dx),
\]

\[
\left( \frac{\partial}{\partial \phi_{H(d+1)+i}} L_r \right)(\phi) = 2 \int_{[a,b]^d} \left[ R_{\infty} \left( b_i + \sum_{k=1}^d w_{i,k}^p x_k \right) \right] (\mathcal{M}_r^\phi(x) - f(x)) \mu(dx),
\]

and

\[
\left( \frac{\partial}{\partial \phi} \mathcal{L}_r \right)(\phi) = 2 \int_{[a,b]^d} (\mathcal{M}_r^\phi(x) - f(x)) \mu(dx),
\]

(iii) it holds that $\limsup_{r \to \infty} |L_r(\phi) - L_\infty(\phi)| = 0$,

(iv) it holds that $\limsup_{r \to \infty} \| (\nabla L_r)(\phi) - \mathcal{G}(\phi) \| = 0$, and

(v) it holds for all $i \in \{1, 2, \ldots, H\}, j \in \{1, 2, \ldots, d\}$ that

\[
\mathcal{G}_{(i-1)d+j}(\phi) = 2\nu_i \int_{I_i^\phi} x_j (\mathcal{M}_\infty^\phi(x) - f(x)) \mu(dx),
\]

\[
\mathcal{G}_{Hd+i}(\phi) = 2\nu_i \int_{I_i^\phi} (\mathcal{M}_\infty^\phi(x) - f(x)) \mu(dx),
\]

\[
\mathcal{G}_{H(d+1)+i}(\phi) = 2 \int_{[a,b]^d} \left[ R_{\infty} \left( b_i + \sum_{k=1}^d w_{i,k}^p x_k \right) \right] (\mathcal{M}_\infty^\phi(x) - f(x)) \mu(dx),
\]

and

\[
\mathcal{G}_{\phi}(\phi) = 2 \int_{[a,b]^d} (\mathcal{M}_\infty^\phi(x) - f(x)) \mu(dx).
\]

**Proof of Proposition 2.3.** Throughout this proof let $\mathcal{M} : [0, \infty) \to [0, \infty]$ satisfy for all $x \in [0, \infty)$ that $\mathcal{M}(x) = \sup_{r \in \mathbb{N}} \sup_{y \in [-x,x]} (|R_r(y)| + |(R_r)'(y)|)$. Observe that the assumption that for all $r \in \mathbb{N}$ it holds that $R_r \in C^1(\mathbb{R}, \mathbb{R})$ implies that for all $r \in \mathbb{N}, x \in \mathbb{R}$ we have that $R_r(x) = R_r(0) + \int_0^x (R_r)'(y) \, dy$. This, the assumption that for all $x \in \mathbb{R}$ it holds that $\sup_{r \in \mathbb{N}} \sup_{y \in [-x,x]} |(R_r)'(y)| < \infty$ and the fact that $\sup_{r \in \mathbb{N}} |R_r(0)| < \infty$ prove that for all $x \in [0, \infty)$ it holds that $\sup_{r \in \mathbb{N}} \sup_{y \in [-x,x]} |R_r(y)| < \infty$. Hence, we obtain that for all $x \in [0, \infty)$ it holds that $\mathcal{M}(x) < \infty$. This, the assumption that for all $r \in \mathbb{N}$ it holds that $R_r \in C^1(\mathbb{R}, \mathbb{R})$, the chain rule, and the dominated convergence theorem establish items (i) and (ii).

Next note that for all $r \in \mathbb{N}, x = (x_1, \ldots, x_d) \in [a, b]^d$ it holds that

\[
|\mathcal{M}_r^\phi(x) - f(x)| \leq \left[ \sup_{y \in [a,b]^d} |f(y)| \right] + |\phi| + \sum_{i=1}^H |b_i^p| \left[ R_r \left( b_i^p + \sum_{j=1}^d w_{i,j}^p x_j \right) \right] \leq \left[ \sup_{y \in [a,b]^d} |f(y)| \right] + |\phi| + \sum_{i=1}^H |b_i^p| \left[ \mathcal{M} \left( |b_i^p| + a \sum_{j=1}^d |w_{i,j}^p| \right) \right].
\]
The fact that for all \( x \in [a, b]^d \) it holds that \( \lim_{r \to \infty} (\mathcal{N}_r^\phi(x) - f(x)) = \mathcal{N}_\infty^\phi(x) - f(x) \) and the dominated convergence theorem hence prove that observe \( \lim_{r \to \infty} \mathcal{L}_r(\phi) = \mathcal{L}_\infty(\phi) \). This establishes item (iii). Moreover, observe that (11), the dominated convergence theorem, and the fact that for all \( x \in [a, b]^d \) it holds that \( \lim_{r \to \infty} (\mathcal{N}_r^\phi(x) - f(x)) = \mathcal{N}_\infty^\phi(x) - f(x) \) assure that

\[
\lim_{r \to \infty} \left[ \left( \frac{\partial}{\partial \phi} \mathcal{L}_r \right)(\phi) \right] = 2 \int_{[a, b]^d} (\mathcal{N}_\infty^\phi(x) - f(x)) \mu(dx). \tag{12}
\]

Next note that for all \( x = (x_1, \ldots, x_d) \in [a, b]^d \), \( i \in \{1, 2, \ldots, H\} \), \( j \in \{1, 2, \ldots, d\} \) we have that

\[
\begin{align*}
\lim_{r \to \infty} \left[ x_j \left[ (\mathcal{R}_r)' \left( \phi_i + \sum_{k=1}^d w_{i,k} x_k \right) \right] (\mathcal{N}_r^\phi(x) - f(x)) \right] \\
= x_j (\mathcal{N}_\infty^\phi(x) - f(x)) 1_{(0, \infty)} (\phi_i + \sum_{k=1}^d w_{i,k} x_k) \tag{13}
\end{align*}
\]

and

\[
\begin{align*}
\lim_{r \to \infty} \left[ \left[ (\mathcal{R}_r)' \left( \phi_i + \sum_{k=1}^d w_{i,k} x_k \right) \right] (\mathcal{N}_r^\phi(x) - f(x)) \right] \\
= (\mathcal{N}_\infty^\phi(x) - f(x)) 1_{(0, \infty)} (\phi_i + \sum_{k=1}^d w_{i,k} x_k) \tag{14}
\end{align*}
\]

Furthermore, observe that (11) shows that for all \( r \in \mathbb{N} \), \( x = (x_1, \ldots, x_d) \in [a, b]^d \), \( i \in \{1, 2, \ldots, H\} \), \( j \in \{1, 2, \ldots, d\} \), \( v \in \{0, 1\} \) it holds that

\[
\begin{align*}
|\langle x_j \rangle^v \left[ (\mathcal{R}_r)' \left( \phi_i + \sum_{k=1}^d w_{i,k} x_k \right) \right] (\mathcal{N}_r^\phi(x) - f(x)) | \\
\leq a \left[ |(\mathcal{R}_r)' \left( \phi_i + \sum_{k=1}^d w_{i,k} x_k \right) | \right] \left[ |\mathcal{N}_r^\phi(x) - f(x)) | \right] \\
\leq a \left[ |\mathcal{M} \left( |\phi_i| + a \sum_{k=1}^d |w_{i,k}| \right) | \left[ |\mathcal{N}_r^\phi(x) - f(x)) | \right] \right] \\
\leq a \left[ |\mathcal{M} \left( |\phi_i| + a \sum_{k=1}^d |w_{i,k}| \right) | \left[ \sup_{y \in [a, b]^d} |f(y)| \right] \right] \\
+ |\phi_i| + \sum_{k=1}^d |w_{i,k}| \left[ |\mathcal{M} \left( |\phi_i| + a \sum_{k=1}^d |w_{i,k}| \right) | \right]. \tag{15}
\end{align*}
\]

The dominated convergence theorem and (13) hence prove that for all \( i \in \{1, 2, \ldots, H\} \), \( j \in \{1, 2, \ldots, d\} \) we have that

\[
\begin{align*}
\lim_{r \to \infty} \left[ x_j \left( \frac{\partial}{\partial \phi_i (1 \ldots d)} \mathcal{L}_r \right)(\phi) \right] = 2 \mathcal{M} \left( \mathcal{N}_\infty^\phi(x) - f(x)) 1_{I_i^\phi}(x) \right) \mu(dx) \\
= 2 \mathcal{M} \left( \mathcal{N}_\infty^\phi(x) - f(x)) 1_{I_i^\phi}(x) \right) \mu(dx). \tag{16}
\end{align*}
\]

Moreover, note that (14), (15), and the dominated convergence theorem demonstrate that for all \( i \in \{1, 2, \ldots, H\} \), \( j \in \{1, 2, \ldots, d\} \) it holds that

\[
\begin{align*}
\lim_{r \to \infty} \left[ x_j \left( \frac{\partial}{\partial \phi_i (1 \ldots d)} \mathcal{L}_r \right)(\phi) \right] = 2 \mathcal{M} \left( \mathcal{N}_\infty^\phi(x) - f(x)) 1_{I_i^\phi}(x) \right) \mu(dx) \\
= 2 \mathcal{M} \left( \mathcal{N}_\infty^\phi(x) - f(x)) 1_{I_i^\phi}(x) \right) \mu(dx). \tag{17}
\end{align*}
\]
Furthermore, observe that for all $x \in [a, b)^d$, $i \in \{1, 2, \ldots, H\}$ it holds that
\[
\lim_{r \to \infty} \left[ \mathcal{R}_r \left( b_i^\phi + \sum_{j=1}^d w_{i,j} x_j \right) \right] (\mathcal{N}_r^\phi(x) - f(x)) = \left[ \mathcal{R}_\infty \left( b_i^\phi + \sum_{j=1}^d w_{i,j} x_j \right) \right] (\mathcal{N}_\infty^\phi(x) - f(x)).
\]
(18)

In addition, note that (11) ensures that for all $r \in \mathbb{N}$, $x \in [a, b)^d$, $i \in \{1, 2, \ldots, H\}$ we have that
\[
\left| \left[ \mathcal{R}_r \left( b_i^\phi + \sum_{j=1}^d w_{i,j} x_j \right) \right] (\mathcal{N}_r^\phi(x) - f(x)) \right| \\
\leq \left[ \mathcal{R}_r \left| b_i^\phi \right| + a \sum_{j=1}^d |w_{i,j}| \right] |\mathcal{N}_r^\phi(x) - f(x)| \\
\leq \left[ \mathcal{R}_r \left| b_i^\phi \right| + a \sum_{j=1}^d |w_{i,j}| \right] \left( \sup_{y \in [a, b)^d} |f(y)| \right) \\
+ |\epsilon^\phi| + \sum_{t=1}^H |v_t^\phi| \left[ \mathcal{M} \left| b_t^\phi \right| + a \sum_{m=1}^d |w_{t,m}| \right].
\]
(19)

This, (18), and the dominated convergence theorem demonstrate that for all $i \in \{1, 2, \ldots, H\}$ it holds that
\[
\lim_{r \to \infty} \left[ \left( \frac{\partial}{\partial H(d+1)+i} \mathcal{L}_r \right) (\phi) \right] = 2 \int_{[a,b)^d} \left[ \mathcal{R}_\infty \left( b_i^\phi + \sum_{j=1}^d w_{i,j} x_j \right) \right] (\mathcal{N}_\infty^\phi(x) - f(x)) \mu(dx).
\]
(20)

Combining this, (12), (16), (17) establishes items (iv) and (v). The proof of Proposition 2.3 is thus complete. □

2.4. Local Lipschitz continuity properties of the true risk functions

Lemma 2.4. Let $d, H, d \in \mathbb{N}$, $a \in \mathbb{R}$, $b \in [a, \infty)$, $f \in C([a, b)^d, \mathbb{R})$ satisfy $d = dH + 2H + 1$, let $\mathcal{N} = (\mathcal{N}^\phi)_{\phi \in \mathbb{R}^d} : \mathbb{R}^d \to C(\mathbb{R}^d, \mathbb{R})$ satisfy for all $\phi = (\phi_1, \ldots, \phi_d) \in \mathbb{R}^d$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ that
\[
\mathcal{N}^\phi(x) = \phi_0 + \sum_{i=1}^H \phi_{H(d+1)+i} \max \left\{ \phi_{H(d+i)+1} + \sum_{j=1}^d \phi_{(i-1)d+j} x_j, 0 \right\},
\]
(21)

let $\mu : \mathcal{B}([a,b)^d) \to [0, 1]$ be a probability measure, let $\|\cdot\| : \mathbb{R}^d \to \mathbb{R}$ and $\mathcal{L} : \mathbb{R}^d \to \mathbb{R}$ satisfy for all $\phi = (\phi_1, \ldots, \phi_d) \in \mathbb{R}^d$ that $\|\phi\| = \left( \sum_{i=1}^d \phi_i^2 \right)^{1/2}$ and $\mathcal{L}(\phi) = \int_{[a,b)^d} (\mathcal{N}^\phi(y) - f(y))^2 \mu(dy)$, and let $K \subset \mathbb{R}^d$ be compact. Then there exists $\mathcal{L} \in \mathcal{R}$ such that for all $\phi, \psi \in K$ it holds that
\[
\sup_{x \in [a,b)^d} |\mathcal{N}^\phi(x) - \mathcal{N}^\psi(x)| + |\mathcal{L}(\phi) - \mathcal{L}(\psi)| \leq \mathcal{L}\|\phi - \psi\|.
\]
(22)

Proof of Lemma 2.4. Throughout this proof let $a \in \mathbb{R}$ satisfy $a = \max\{|a|, |b|, 1\}$. Observe that, e.g., Beck et al. [6, Theorem 2.36] (applied with $a \otimes a$, $b \otimes b$, $d \otimes d$, $L \otimes 2$, $l_0 \otimes d$, $l_1 \otimes H$, $l_2 \otimes 1$ in the notation of [6, Theorem 2.36]) and the fact that for all $\varphi = (\varphi_1, \ldots, \varphi_d) \in \mathbb{R}^d$ it holds that $\max_{i \in \{1, 2, \ldots, d\}} |\varphi_i| \leq \|\varphi\|$ demonstrate that for all $\phi, \psi \in \mathbb{R}^d$ it holds that
\[
\sup_{x \in [a,b)^d} |\mathcal{N}^\phi(x) - \mathcal{N}^\psi(x)| \leq 2a(d+1)(H+1)(\max\{1, \|\phi\|, \|\psi\|\})\|\phi - \psi\|.
\]
(23)

Furthermore, note that the fact that $K$ is compact ensures that there exists $\kappa \in [1, \infty)$ such that for all $\varphi \in K$ it holds that
\[ \|\varphi\| \leq \kappa. \] (24)

Note that (23) and (24) show that there exists \( \mathcal{L} \in \mathbb{R} \) which satisfies for all \( \phi, \psi \in K \) that
\[
\sup_{x \in [a, b]^d}|\mathcal{N}^\phi(x) - \mathcal{N}^\psi(x)| \leq \mathcal{L}\|\phi - \psi\|. \tag{25}
\]

Hence, we obtain that for all \( \phi, \psi \in K \) it holds that
\[
|\mathcal{L}(\phi) - \mathcal{L}(\psi)| = \left| \int_{[a, b]^d} (\mathcal{N}^\phi(x) - f(x))^2 \mu(dx) \right| - \left| \int_{[a, b]^d} (\mathcal{N}^\psi(x) - f(x))^2 \mu(dx) \right| 
\leq \int_{[a, b]^d} |(\mathcal{N}^\phi(x) - f(x))^2 - (\mathcal{N}^\psi(x) - f(x))^2| \mu(dx) 
= \int_{[a, b]^d} |\mathcal{N}^\phi(x) - \mathcal{N}^\psi(x)||\mathcal{N}^\phi(x) + \mathcal{N}^\psi(x) - 2f(x)| \mu(dx) 
\leq \mathcal{L}\|\phi - \psi\| \left( \int_{[a, b]^d} |\mathcal{N}^\phi(x) + \mathcal{N}^\psi(x) - 2f(x)| \mu(dx) \right). \tag{26}
\]

This, (24), (25), and the fact that for all \( x \in [a, b]^d \) it holds that \( \mathcal{N}^0(x) = 0 \) prove that for all \( \phi, \psi \in K \) we have that
\[
|\mathcal{L}(\phi) - \mathcal{L}(\psi)| \leq \mathcal{L}\|\phi - \psi\| \left( \sup_{x \in [a, b]^d} |\mathcal{N}^\phi(x) + \mathcal{N}^\psi(x) + 2|f(x)|| \right) 
= \mathcal{L}\|\phi - \psi\| \left( \sup_{x \in [a, b]^d} |\mathcal{N}^\phi(x) - \mathcal{N}^0(x) + \mathcal{N}^\psi(x) - \mathcal{N}^0(x) + 2|f(x)|| \right) 
\leq \mathcal{L}\|\phi - \psi\| \left( \mathcal{L}\|\phi\| + \mathcal{L}\|\psi\| + 2 \left[ \sup_{x \in [a, b]^d}|f(x)| \right] \right) 
\leq 2\mathcal{L} (\kappa\mathcal{L} + \left[ \sup_{x \in [a, b]^d}|f(x)| \right]) \|\phi - \psi\|. \tag{27}
\]

Combining this with (25) establishes (22). The proof of Lemma 2.4 is thus complete. \( \square \)

### 2.5. Upper estimates for generalized gradients of the true risk functions

**Lemma 2.5.** Assume Setting 2.1 and let \( \phi \in \mathbb{R}^d \). Then
\[
\|S(\phi)\|^2 \leq 4(a^2(d + 1)\|\phi\|^2 + 1)\mathcal{L}_\infty(\phi). \tag{28}
\]

**Proof of Lemma 2.5.** Observe that Jensen’s inequality implies that
\[
\left( \int_{[a, b]^d} |\mathcal{N}_\infty^\phi(x) - f(x)| \mu(dx) \right)^2 \leq \int_{[a, b]^d} (\mathcal{N}_\infty^\phi(x) - f(x))^2 \mu(dx) = \mathcal{L}_\infty(\phi). \tag{29}
\]
Combining this and (10) demonstrates that for all $i \in \{1, 2, \ldots, H\}$, $j \in \{1, 2, \ldots, d\}$ it holds that

$$
|\mathcal{G}_{(i-1)d+j}(\phi)|^2 = 4(\mathbf{v}_i^\phi)^2 \left( \int_{\mathcal{B}_i} x_j (\mathcal{M}_\infty^\phi(x) - f(x)) \mu(dx) \right)^2
$$

$$
\leq 4(\mathbf{v}_i^\phi)^2 \left( \int_{\mathcal{B}_i} |x_j| |\mathcal{M}_\infty^\phi(x) - f(x)| \mu(dx) \right)^2 \tag{30}
$$

$$
\leq 4a^2(\mathbf{v}_i^\phi)^2 \left( \int_{\mathcal{B}_i} |\mathcal{M}_\infty^\phi(x) - f(x)| \mu(dx) \right)^2 \leq 4a^2(\mathbf{v}_i^\phi)^2 \mathcal{L}_\infty(\phi).
$$

Next note that (10) and (29) prove that for all $i \in \{1, 2, \ldots, H\}$ we have that

$$
|\mathcal{G}_{Hd+i}(\phi)|^2 = 4(\mathbf{v}_i^\phi)^2 \left( \int_{\mathcal{B}_i} (\mathcal{M}_\infty^\phi(x) - f(x)) \mu(dx) \right)^2
$$

$$
\leq 4(\mathbf{v}_i^\phi)^2 \left( \int_{\mathcal{B}_i} |\mathcal{M}_\infty^\phi(x) - f(x)| \mu(dx) \right)^2 \leq 4(\mathbf{v}_i^\phi)^2 \mathcal{L}_\infty(\phi). \tag{31}
$$

Furthermore, observe that the fact that for all $x = (x_1, \ldots, x_d) \in [a, b]^d$, $i \in \{1, 2, \ldots, H\}$ it holds that

$$
|\mathcal{M}_\infty (b_i^\phi + \sum_{j=1}^d w_{i,j} x_j)|^2 \leq \left( |b_i^\phi| + a \sum_{j=1}^d |w_{i,j}| \right)^2 \leq a^2(d + 1) \left( |b_i^\phi|^2 + \sum_{j=1}^d |w_{i,j}|^2 \right) \quad \text{and} \quad (10)
$$

assure that for all $i \in \{1, 2, \ldots, H\}$ it holds that

$$
|\mathcal{G}_{H(d+1)+i}(\phi)|^2 = 4 \left( \int_{[a,b]^d} \left[ \mathcal{M}_\infty (b_i^\phi + \sum_{j=1}^d w_{i,j} x_j) \right] (\mathcal{M}_\infty^\phi(x) - f(x)) \mu(dx) \right)^2
$$

$$
\leq 4 \left( \int_{[a,b]^d} \left| \mathcal{M}_\infty (b_i^\phi + \sum_{j=1}^d w_{i,j} x_j) \right|^2 (\mathcal{M}_\infty^\phi(x) - f(x))^2 \mu(dx) \right) \leq 4a^2(d + 1) \left( |b_i^\phi|^2 + \sum_{j=1}^d |w_{i,j}|^2 \right) \mathcal{L}_\infty(\phi) \tag{32}
$$

Moreover, note that (10) and (29) show that

$$
|\mathcal{G}_N(\phi)|^2 = 4 \left( \int_{[a,b]^d} (\mathcal{M}_\infty^\phi(x) - f(x)) \mu(dx) \right)^2 \leq 4\mathcal{L}_\infty(\phi). \tag{33}
$$

Combining this with (30), (31), and (32) ensures that

$$
\|\mathcal{G}(\phi)\|^2 \leq 4 \left[ \sum_{i=1}^H \left( a^2 \left[ \sum_{j=1}^d |b_i^\phi|^2 \right] + |\mathbf{v}_i^\phi|^2 + a^2(d + 1) \left[ |b_i^\phi|^2 + \sum_{j=1}^d |w_{i,j}|^2 \right] \right) \right] \mathcal{L}_\infty(\phi) + 4\mathcal{L}_\infty(\phi) \tag{34}
$$

$$
\leq 4(a^2(d + 1)||\phi||^2 + 1)\mathcal{L}_\infty(\phi).
$$

The proof of Lemma 2.5 is thus complete.
Corollary 2.6. Assume Setting 2.1 and let $K \subseteq \mathbb{R}^p$ be compact. Then $\sup_{\phi \in K} \|\mathcal{G}(\phi)\| < \infty$.

Proof of Corollary 2.6. Observe that Lemma 2.4 and the assumption that $K$ is compact ensure that $\sup_{\phi \in K} \mathcal{L}_\infty(\phi) < \infty$. This and Lemma 2.5 complete the proof of Corollary 2.6. \qed

2.6. Upper estimates associated to Lyapunov functions

Lemma 2.7. Let $\vartheta \in \mathbb{N}$, $\xi \in \mathbb{R}$ and let $\|\cdot\| : \mathbb{R}^p \to \mathbb{R}$ and $V : \mathbb{R}^p \to \mathbb{R}$ satisfy for all $\phi = (\phi_1, \ldots, \phi_\vartheta) \in \mathbb{R}^p$ that $\|\phi\| = \left[\sum_{i=1}^\vartheta |\phi_i|^2\right]^{1/2}$ and $V(\phi) = \|\phi\|^2 + |\phi_\vartheta - 2\xi|^2$. Then it holds for all $\phi \in \mathbb{R}^p$ that

$$\|\phi\|^2 \leq V(\phi) \leq 3\|\phi\|^2 + 8\xi^2. \quad (35)$$

Proof of Lemma 2.7. Observe that the fact that for all $\phi \in \mathbb{R}^p$ it holds that $|\phi_\vartheta - 2\xi|^2 \geq 0$ assures that for all $\phi \in \mathbb{R}^p$ we have that

$$V(\phi) = \|\phi\|^2 + |\phi_\vartheta - 2\xi|^2 \geq \|\phi\|^2. \quad (36)$$

Furthermore, note that the fact that for all $x, y \in \mathbb{R}$ it holds that $(x - y)^2 \leq 2(x^2 + y^2)$ ensures that for all $\phi \in \mathbb{R}^p$ it holds that

$$V(\phi) \leq \|\phi\|^2 + 2(\phi_\vartheta)^2 + 8\xi^2 \leq 3\|\phi\|^2 + 8\xi^2. \quad (37)$$

Combining this with (36) establishes (35). The proof of Lemma 2.7 is thus complete. \qed

Proposition 2.8. Let $\vartheta \in \mathbb{N}$, $\xi \in \mathbb{R}$ and let $V : \mathbb{R}^p \to \mathbb{R}$ satisfy for all $\phi = (\phi_1, \ldots, \phi_\vartheta) \in \mathbb{R}^p$ that $V(\phi) = \left[\sum_{i=1}^\vartheta |\phi_i|^2\right] + |\phi_\vartheta - 2\xi|^2$. Then

(i) it holds for all $\phi = (\phi_1, \ldots, \phi_\vartheta) \in \mathbb{R}^p$ that

$$(\nabla V)(\phi) = 2\phi + (0, 0, \ldots, 0, 2[\phi_\vartheta - 2\xi]) \quad (38)$$

and

(ii) it holds for all $\phi = (\phi_1, \ldots, \phi_\vartheta), \psi = (\psi_1, \ldots, \psi_\vartheta) \in \mathbb{R}^p$ that

$$(\nabla V)(\phi) - (\nabla V)(\psi) = 2(\phi - \psi) + (0, 0, \ldots, 0, 2(\phi_\vartheta - \psi_\vartheta)). \quad (39)$$

Proof of Proposition 2.8. Observe that the assumption that for all $\phi \in \mathbb{R}^p$ it holds that $V(\phi) = \sum_{i=1}^\vartheta |\phi_i|^2 + |\phi_\vartheta - 2\xi|^2$ proves item (i). Moreover, note that item (i) establishes item (ii). The proof of Proposition 2.8 is thus complete. \qed

Proposition 2.9. Assume Setting 2.1 and let $\phi \in \mathbb{R}^p$. Then

$$\langle (\nabla V)(\phi), \mathcal{G}(\phi) \rangle = 8 \int_{[a,b]^d} (\mathcal{M}_\infty^\phi(x) - f(0))(\mathcal{M}_\infty^\phi(x) - f(x)) \mu(dx). \quad (40)$$

Proof of Proposition 2.9. Observe that Proposition 2.8 demonstrates that

$$(\nabla V)(\phi) = 2\left(\mathbf{w}_1^\phi, \ldots, \mathbf{w}_1^\phi, \mathbf{w}_2^\phi, \ldots, \mathbf{w}_2^\phi, \ldots, \mathbf{w}_H^\phi, \ldots, \mathbf{w}_H^\phi, \mathbf{b}_1^\phi, \ldots, \mathbf{b}_H^\phi, \mathbf{v}_1^\phi, \ldots, \mathbf{v}_H^\phi, 2(c^\phi - f(0))\right). \quad (41)$$
This and (10) imply that
\[
\langle (\nabla V)(\phi), \mathcal{G}(\phi) \rangle \\
= 4 \left[ \sum_{i=1}^{H} \left( \sum_{j=1}^{d} \left( w_{i,j}^{\phi} v_{i}^{\phi} \int_{I_{i}^{\phi}} x_{j}(\mathcal{M}_{\infty}^{\phi}(x) - f(x)) \mu(dx) \right) \right) \right] \\
+ 4 \left[ \sum_{i=1}^{H} \left( b_{i}^{\phi} v_{i}^{\phi} \int_{I_{i}^{\phi}} (\mathcal{M}_{\infty}^{\phi}(x) - f(x)) \mu(dx) \right) \right] \\
+ 4 \left[ \sum_{i=1}^{H} \left( v_{i}^{\phi} \int_{[a,b]^d} \mathcal{R}_{\infty} \left( b_{i}^{\phi} + \sum_{j=1}^{d} w_{i,j}^{\phi} x_{j} \right) (\mathcal{M}_{\infty}^{\phi}(x) - f(x)) \mu(dx) \right) \right] \\
+ 8 (\phi - f(0)) \int_{[a,b]^d} (\mathcal{M}_{\infty}^{\phi}(x) - f(x)) \mu(dx). 
\]

(42)

Hence, we obtain that
\[
\langle (\nabla V)(\phi), \mathcal{G}(\phi) \rangle \\
= 4 \left[ \sum_{i=1}^{H} \left( \sum_{j=1}^{d} \left( v_{i,j}^{\phi} \int_{I_{i}^{\phi}} x_{j}(\mathcal{M}_{\infty}^{\phi}(x) - f(x)) \mu(dx) \right) \right) \right] \\
+ 4 \left[ \sum_{i=1}^{H} \left( v_{i}^{\phi} \int_{[a,b]^d} \mathcal{R}_{\infty} \left( b_{i}^{\phi} + \sum_{j=1}^{d} w_{i,j}^{\phi} x_{j} \right) (\mathcal{M}_{\infty}^{\phi}(x) - f(x)) \mu(dx) \right) \right] \\
+ 8 \int_{[a,b]^d} (\phi - f(0)) (\mathcal{M}_{\infty}^{\phi}(x) - f(x)) \mu(dx) \\
= 8 \int_{[a,b]^d} \left( \left( f(0) + \sum_{i=1}^{H} \left[ v_{i}^{\phi} \left( \mathcal{R}_{\infty} \left( b_{i}^{\phi} + \sum_{j=1}^{d} w_{i,j}^{\phi} x_{j} \right) \right) \right] \right) \mathcal{M}_{\infty}^{\phi}(x) - f(x)) \mu(dx) \\
= 8 \int_{[a,b]^d} (\mathcal{M}_{\infty}^{\phi}(x) - f(0)) (\mathcal{M}_{\infty}^{\phi}(x) - f(x)) \mu(dx). 
\]

This completes the proof of Proposition 2.9. \(\square\)

**Corollary 2.10.** Assume Setting 2.1, assume for all \(x \in [a, b]^d\) that \(f(x) = f(0), \) and let \(\phi \in \mathbb{R}^d\). Then
\[
\langle (\nabla V)(\phi), \mathcal{G}(\phi) \rangle = 8 \mathcal{L}_{\infty}(\phi). 
\]

**Proof of Corollary 2.10.** Note that the fact that for all \(x \in [a, b]^d\) it holds that \(f(x) = f(0)\) implies that
\[
\mathcal{L}_{\infty}(\phi) = \int_{[a,b]^d} (\mathcal{M}_{\infty}^{\phi}(x) - f(0)) (\mathcal{M}_{\infty}^{\phi}(x) - f(x)) \mu(dx). 
\]

(44)

Combining this with Proposition 2.9 completes the proof of Corollary 2.10. \(\square\)

**Corollary 2.11.** Assume Setting 2.1, assume for all \(x \in [a, b]^d\) that \(f(x) = f(0), \) and let \(\phi \in \mathbb{R}^d\). Then it holds that \(\mathcal{G}(\phi) = 0\) if and only if \(\mathcal{L}_{\infty}(\phi) = 0.\)
Proof of Corollary 2.11. Observe that Corollary 2.10 implies that for all \( \varphi \in \mathbb{R}^d \) with \( \mathcal{S}(\varphi) = 0 \) it holds that \( \mathcal{L}_\infty(\varphi) = \frac{1}{8}\langle (\nabla V)(\varphi), \mathcal{S}(\varphi) \rangle = 0 \). Moreover, note that the fact that for all \( \varphi \in \mathbb{R}^d \) it holds that \( \mathcal{L}_\infty(\varphi) = \int_{[a,b]^d} (\mathcal{M}_\varphi(x) - f(0))^2 \mu(dx) \) ensures that for all \( \varphi \in \mathbb{R}^d \) with \( \mathcal{L}_\infty(\varphi) = 0 \) we have that
\[
\int_{[a,b]^d} (\mathcal{M}_\varphi(x) - f(0))^2 \mu(dx) = 0. \tag{45}
\]
This shows that for all \( \varphi \in \{ \psi \in \mathbb{R}^d : (\mathcal{L}_\infty(\psi) = 0) \} \) and \( \mu \)-almost all \( x \in [a,b]^d \) it holds that \( \mathcal{M}_\varphi(x) = f(0) \). Combining this with (10) demonstrates that for all \( \varphi \in \{ \psi \in \mathbb{R}^d : (\mathcal{L}_\infty(\psi) = 0) \} \) we have that \( \mathcal{S}(\varphi) = 0 \). The proof of Corollary 2.11 is thus complete. \( \square \)

2.7. Lyapunov type estimates for GD processes

Lemma 2.12. Assume Setting 2.1, assume for all \( x \in [a,b]^d \) that \( f(x) = f(0) \), and let \( \gamma \in [0, \infty) \), \( \theta \in \mathbb{R}^d \). Then
\[
V(\theta - \gamma \mathcal{S}(\theta)) - V(\theta) = \gamma^2 \| \mathcal{S}(\theta) \|^2 + \gamma^2 \| \mathcal{S}_\theta(\theta) \|^2 - 8\gamma \mathcal{L}_\infty(\theta) \leq 2\gamma^2 \| \mathcal{S}(\theta) \|^2 - 8\gamma \mathcal{L}_\infty(\theta). \tag{46}
\]

Proof of Lemma 2.12. Throughout this proof let \( e \in \mathbb{R}^b \) satisfy \( e = (0, 0, \ldots, 0, 1) \) and let \( g : \mathbb{R} \to \mathbb{R} \) satisfy for all \( t \in \mathbb{R} \) that
\[
g(t) = V(\theta - t\mathcal{S}(\theta)). \tag{47}
\]
Observe that (47) and the fundamental theorem of calculus prove that
\[
V(\theta - \gamma \mathcal{S}(\theta)) = g(\gamma) = g(0) + \int_0^\gamma g'(t) \, dt = g(0) + \int_0^\gamma \langle (\nabla V)(\theta - t\mathcal{S}(\theta)), (-\mathcal{S}(\theta)) \rangle \, dt
\]
\[
= V(\theta) - \int_0^\gamma \langle (\nabla V)(\theta - t\mathcal{S}(\theta)), \mathcal{S}(\theta) \rangle \, dt. \tag{48}
\]
Corollary 2.10 hence demonstrates that
\[
V(\theta - \gamma \mathcal{S}(\theta)) = V(\theta) - \int_0^\gamma \langle (\nabla V)(\theta), \mathcal{S}(\theta) \rangle \, dt
\]
\[
+ \int_0^\gamma \langle (\nabla V)(\theta) - (\nabla V)(\theta - t\mathcal{S}(\theta)), \mathcal{S}(\theta) \rangle \, dt
\]
\[
= V(\theta) - 8\gamma \mathcal{L}_\infty(\theta) + \int_0^\gamma \langle (\nabla V)(\theta) - (\nabla V)(\theta - t\mathcal{S}(\theta)), \mathcal{S}(\theta) \rangle \, dt. \tag{49}
\]
Proposition 2.8 therefore proves that
\[
V(\theta - \gamma \mathcal{S}(\theta)) = V(\theta) - 8\gamma \mathcal{L}_\infty(\theta) + \int_0^\gamma \langle 2t\mathcal{S}(\theta) + 2e^{t\mathcal{S}(\theta)}e, \mathcal{S}(\theta) \rangle \, dt
\]
\[
= V(\theta) - 8\gamma \mathcal{L}_\infty(\theta) + 2\| \mathcal{S}(\theta) \|^2 \left[ \int_0^\gamma t \, dt \right] + 2 \left[ \int_0^\gamma (e^{t\mathcal{S}(\theta)}e, \mathcal{S}(\theta)) \, dt \right]. \tag{50}
\]
Hence, we obtain that
\[
V(\theta - \gamma g(\theta)) = V(\theta) - 8\gamma L_\infty(\theta) + \gamma^2 \|S(\theta)\|^2 + 2(\langle e, S(\theta) \rangle)^2 \left[ \int_0^t t \, dt \right] \\
= V(\theta) - 8\gamma L_\infty(\theta) + \gamma^2 \|S(\theta)\|^2 + 2(\langle e, S(\theta) \rangle)^2 \\
= V(\theta) - 8\gamma L_\infty(\theta) + \gamma^2 \|S(\theta)\|^2 + 2(\langle S_0(\theta) \rangle)^2.
\]

The proof of Lemma 2.12 is thus complete. \(\square\)

**Corollary 2.13.** Assume Setting 2.1, assume for all \(x \in [a, b]^d\) that \(f(x) = f(0)\), and let \(\gamma \in [0, \infty)\), \(\theta \in \mathbb{R}^d\). Then
\[
V(\theta - \gamma g(\theta)) - V(\theta) \leq 8 \left( \gamma^2 [a^2(d + 1)V(\theta) + 1] - \gamma \right) L_\infty(\theta).
\]

**Proof of Corollary 2.13.** Note that Lemmas 2.5 and 2.7 demonstrate that
\[
\|g(\theta)\|^2 \leq 4 \left[ a^2(d + 1)\|\theta\|^2 + 1 \right] L_\infty(\theta) \leq 4 \left[ a^2(d + 1)V(\theta) + 1 \right] L_\infty(\theta).
\]

Lemma 2.12 therefore shows that
\[
V(\theta - \gamma g(\theta)) - V(\theta) \leq 8\gamma \left[ a^2(d + 1)V(\theta) + 1 \right] L_\infty(\theta) - 8\gamma L_\infty(\theta) \\
= 8 \left( \gamma^2 [a^2(d + 1)V(\theta) + 1] - \gamma \right) L_\infty(\theta).
\]

The proof of Corollary 2.13 is thus complete. \(\square\)

**Corollary 2.14.** Assume Setting 2.1, assume for all \(x \in [a, b]^d\) that \(f(x) = f(0)\), let \((\gamma_n)_{n \in \mathbb{N}_0} : \mathbb{N}_0 \rightarrow \mathbb{R}^d\) satisfy for all \(n \in \mathbb{N}_0\) that \(\Theta_{n+1} = \Theta_n - \gamma_n g(\Theta_n)\), and let \(n \in \mathbb{N}_0\). Then
\[
V(\Theta_{n+1}) - V(\Theta_n) \leq 8 \left( (\gamma_n)^2 [a^2(d + 1)V(\Theta_n) + 1] - \gamma_n \right) L_\infty(\Theta_n).
\]

**Proof of Corollary 2.14.** Observe that Corollary 2.13 establishes (55). The proof of Corollary 2.14 is thus complete. \(\square\)

**Lemma 2.15.** Assume Setting 2.1, let \((\gamma_n)_{n \in \mathbb{N}_0} \subseteq [0, \infty)\), let \((\Theta_n)_{n \in \mathbb{N}_0} : \mathbb{N}_0 \rightarrow \mathbb{R}^d\) satisfy for all \(n \in \mathbb{N}_0\) that \(\Theta_{n+1} = \Theta_n - \gamma_n g(\Theta_n)\), assume for all \(x \in [a, b]^d\) that \(f(x) = f(0)\), and assume \(\sup_{n \in \mathbb{N}_0} \gamma_n \leq \left[ a^2(d + 1)V(\Theta_0) + 1 \right]^{-1} \). Then it holds for all \(n \in \mathbb{N}_0\) that
\[
V(\Theta_{n+1}) - V(\Theta_n) \leq -8\gamma_n \left[ 1 - \left[ \sup_{m \in \mathbb{N}_0} \gamma_m \right] [a^2(d + 1)V(\Theta_0) + 1] \right] L_\infty(\Theta_n) \leq 0.
\]

**Proof of Lemma 2.15.** Throughout this proof let \(g \in \mathbb{R}\) satisfy \(g = \sup_{n \in \mathbb{N}_0} \gamma_n\). We now prove (56) by induction on \(n \in \mathbb{N}_0\). Note that Corollary 2.14 and the fact that \(\gamma_0 \leq g\) imply that
\[
V(\Theta_1) - V(\Theta_0) \leq -8\gamma_0 + 8(\gamma_0)^2 \left[ a^2(d + 1)V(\Theta_0) + 1 \right] L_\infty(\Theta_0) \\
\leq -8\gamma_0 + 8\gamma_0 g \left[ a^2(d + 1)V(\Theta_0) + 1 \right] L_\infty(\Theta_0) \\
= -8\gamma_0 [1 - g] \left[ a^2(d + 1)V(\Theta_0) + 1 \right] L_\infty(\Theta_0) \leq 0.
\]

This establishes (56) in the base case \(n = 0\). For the induction step let \(n \in \mathbb{N}\) satisfy for all \(m \in \{0, 1, \ldots, n - 1\}\) that
\[
V(\Theta_{m+1}) - V(\Theta_m) \leq -8\gamma_m [1 - g] \left[ a^2(d + 1)V(\Theta_0) + 1 \right] L_\infty(\Theta_m) \leq 0.
\]

Observe that (58) shows that \(V(\Theta_n) \leq V(\Theta_{n-1}) \leq \cdots \leq V(\Theta_0)\). The fact that \(\gamma_n \leq g\) and Corollary 2.14 hence demonstrate that
\[
V(\Theta_{n+1}) - V(\Theta_n) \leq -8\gamma_n + 8(\gamma_n)^2 \left[ a^2(d + 1)V(\Theta_n) + 1 \right] L_\infty(\Theta_n) \\
\leq -8\gamma_n + 8\gamma_n g \left[ a^2(d + 1)V(\Theta_n) + 1 \right] L_\infty(\Theta_n) \\
= -8\gamma_n [1 - g] \left[ a^2(d + 1)V(\Theta_n) + 1 \right] L_\infty(\Theta_n) \leq 0.
\]

Induction therefore establishes (56). The proof of Lemma 2.15 is thus complete. \(\square\)
2.8. Convergence analysis for GD processes in the training of ANNs

**Theorem 2.16.** Assume Setting 2.1, assume for all $x \in [a, b]^d$ that $f(x) = f(0)$, let $(\gamma_n)_{n \in \mathbb{N}_0} : \mathbb{N}_0 \to \mathbb{R}^3$ satisfy for all $n \in \mathbb{N}_0$ that $\Theta_{n+1} = \Theta_n - \gamma_n \mathcal{G}(\Theta_n)$, and assume $\sup_{n \in \mathbb{N}_0} \gamma_n < [a^2(d + 1)V(\Theta_0) + 1]^{-1}$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$. Then

(i) it holds that $\sup_{n \in \mathbb{N}_0} \|\Theta_n\| \leq [V(\Theta_0)]^{1/2} < \infty$ and

(ii) it holds that $\limsup_{n \to \infty} \mathcal{L}_\infty(\Theta_n) = 0$.

**Proof of Theorem 2.16.** Throughout this proof let $\eta \in (0, \infty)$ satisfy $\eta = 8(1 - [\sup_{n \in \mathbb{N}_0} \gamma_n] [a^2(d + 1)V(\Theta_0) + 1])$ and let $\varepsilon \in \mathbb{R}$ satisfy $\varepsilon = (1/3)\min\{1, \limsup_{n \to \infty} \mathcal{L}_\infty(\Theta_n)\}$. Note that Lemma 2.15 implies that for all $n \in \mathbb{N}_0$ we have that $V(\Theta_n) \leq V(\Theta_{n-1}) \leq \cdots \leq V(\Theta_0)$. Combining this and the fact that for all $n \in \mathbb{N}_0$ it holds that $\|\Theta_n\| \leq [V(\Theta_n)]^{1/2}$ establishes item (i). Next observe that Lemma 2.15 implies for all $N \in \mathbb{N}$ that

$$\eta \sum_{n=0}^{N-1} \gamma_n \mathcal{L}_\infty(\Theta_n) \leq \sum_{n=0}^{N-1} (V(\Theta_n) - V(\Theta_{n+1})) = V(\Theta_0) - V(\Theta_N) \leq V(\Theta_0).$$

(60)

Hence, we have that

$$\sum_{n=0}^{\infty} [\gamma_n \mathcal{L}_\infty(\Theta_n)] \leq \frac{V(\Theta_0)}{\eta} < \infty.$$  

(61)

This and the assumption that $\sum_{n=0}^{\infty} \gamma_n = \infty$ ensure that $\liminf_{n \to \infty} \mathcal{L}_\infty(\Theta_n) = 0$. We intend to complete the proof of item (ii) by a contradiction. In the following we thus assume that

$$\limsup_{n \to \infty} \mathcal{L}_\infty(\Theta_n) > 0.$$  

(62)

Note that (62) implies that

$$0 = \liminf_{n \to \infty} \mathcal{L}_\infty(\Theta_n) < \varepsilon < 2\varepsilon < \limsup_{n \to \infty} \mathcal{L}_\infty(\Theta_n).$$  

(63)

This shows that there exist $(m_k, n_k) \in \mathbb{N}^2$, $k \in \mathbb{N}$, which satisfy for all $k \in \mathbb{N}$ that $m_k < n_k < m_{k+1}$, $\mathcal{L}_\infty(\Theta_{m_k}) > 2\varepsilon$, and $\mathcal{L}_\infty(\Theta_{n_k}) < \varepsilon \leq \min_{j \in \mathbb{N} \cap [m_k, n_k]} \mathcal{L}_\infty(\Theta_j)$. Observe that (61) and the fact that for all $k \in \mathbb{N}$, $j \in \mathbb{N} \cap [m_k, n_k]$ it holds that $1 \leq \frac{1}{\varepsilon} \mathcal{L}_\infty(\Theta_j)$ assure that

$$\sum_{j=m_k}^{n_k-1} \gamma_j \leq \frac{1}{\varepsilon} \left[ \sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k-1} (\gamma_j \mathcal{L}_\infty(\Theta_j)) \right] \leq \frac{1}{\varepsilon} \left[ \sum_{j=0}^{\infty} (\gamma_j \mathcal{L}_\infty(\Theta_j)) \right] < \infty.$$  

(64)

Next note that Corollary 2.6 and item (i) ensure that there exists $c \in \mathbb{R}$ which satisfies that

$$\sup_{n \in \mathbb{N}_0} \|\mathcal{G}(\Theta_n)\| \leq c.$$  

(65)

Observe that the triangle inequality, (64), and (65) prove that

$$\sum_{k=1}^{\infty} \|\Theta_{n_k} - \Theta_{m_k}\| \leq \sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k-1} \|\Theta_{j+1} - \Theta_j\| = \sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k-1} (\gamma_j \|\mathcal{G}(\Theta_j)\|) \leq c \sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k-1} \gamma_j < \infty.$$  

(66)

Moreover, note that Lemma 2.4 and item (i) demonstrate that there exists $L \in \mathbb{R}$ which satisfies for all $m, n \in \mathbb{N}_0$ that $|\mathcal{L}_\infty(\Theta_m) - \mathcal{L}_\infty(\Theta_n)| \leq L \|\Theta_m - \Theta_n\|$. This and (66) show that

$$\limsup_{k \to \infty} |\mathcal{L}_\infty(\Theta_{n_k}) - \mathcal{L}_\infty(\Theta_{m_k})| \leq \limsup_{k \to \infty} (L \|\Theta_{n_k} - \Theta_{m_k}\|) = 0.$$  

(67)
Combining this and the fact that for all \( k \in \mathbb{N}_0 \) it holds that \( \mathcal{L}_\infty(\Theta_{n_k}) < \varepsilon < 2\varepsilon < \mathcal{L}_\infty(\Theta_{m_k}) \) ensures that

\[
0 < \varepsilon \leq \inf_{k \in \mathbb{N}} |\mathcal{L}_\infty(\Theta_{n_k}) - \mathcal{L}_\infty(\Theta_{m_k})| \leq \limsup_{k \to \infty} |\mathcal{L}_\infty(\Theta_{n_k}) - \mathcal{L}_\infty(\Theta_{m_k})| = 0. \tag{68}
\]

This contradiction establishes item \((ii)\). The proof of Theorem 2.16 is thus complete. \( \square \)

3. Convergence of stochastic gradient descent (SGD) processes

In this section we establish in Theorem 3.12 in Sect. 3.6 below that the true risks of SGD processes converge in the training of ANNs with ReLU activation to zero if the target function under consideration is a constant. In this section we thereby transfer the convergence analysis for GD processes from Sect. 2 above to a convergence analysis for SGD processes.

Theorem 3.12 postulates the mathematical setup in Setting 3.1 in Sect. 3.1 below. In Setting 3.1 we formally introduce, among other things, the constant \( \xi \in \mathbb{R} \) with which the target function coincides, the realization functions \( \mathcal{A}_i^\phi : \mathbb{R}^d \to \mathbb{R} \), \( \phi \in \mathbb{R}^d \), of the considered ANNs (see (70) in Setting 3.1), the true risk function \( \mathcal{L} : \mathbb{R}^d \to \mathbb{R} \), the sizes \( M_n \in \mathbb{N} \), \( n \in \mathbb{N}_0 \), of the employed mini-batches in the SGD optimization method, the empirical risk functions \( \mathcal{L}_n^\mathcal{K} : \mathbb{R}^d \times \Omega \to \mathbb{R} \), \( n \in \mathbb{N}_0 \), a sequence of smooth approximations \( \mathcal{R}_r : \mathbb{R} \to \mathbb{R} \), \( r \in \mathbb{N} \), of the ReLU activation function (see (69) in Setting 3.1), the learning rates \( \gamma_n \in [0, \infty) \), \( n \in \mathbb{N}_0 \), used in the SGD optimization method, the appropriately generalized gradient functions \( G^n : \mathbb{R}^d \times \Omega \to \mathbb{R}^d \), \( n \in \mathbb{N}_0 \), associated to the true risk functions, as well as the SGD process \( \Theta = (\Theta_n)_{n \in \mathbb{N}_0} : \mathbb{N}_0 \times \Omega \to \mathbb{R}^d \).

Item \((ii)\) and \((iii)\) in Theorem 3.12 prove that the true risk \( \mathcal{L}(\Theta_n) \) of the SGD process \( \Theta : \mathbb{N}_0 \times \Omega \to \mathbb{R}^d \) converges in the almost sure and \( L^1 \)-sense to zero as the number of stochastic gradient descent steps \( n \in \mathbb{N} \) increases to infinity. Roughly speaking, some ideas in our proof of Theorem 3.12, in particular the main results in Sects. 3.2, 3.4, 3.5, and 3.6 below, are transferred from Sect. 2 to the SGD setting. Specifically, in our proof of Theorem 3.12 we employ the elementary local Lipschitz continuity estimate for the true risk function in Lemma 2.4 in Sect. 2.4 above, the upper estimates for the standard norm of the generalized gradient functions \( G^n : \mathbb{R}^d \times \Omega \to \mathbb{R}^d \), \( n \in \mathbb{N}_0 \), in Lemmas 3.6 and 3.7 in Sect. 3.4 below, the elementary representation results for expectations of empirical risks of SGD processes in Corollary 3.5 in Sect. 3.3 below, as well as the Lyapunov type estimates for SGD processes in Lemmas 3.8, 3.9, 3.10, and Corollary 3.11 in Sect. 3.5 below.

Our proof of Lemma 3.7 uses Lemma 2.4 and Lemma 3.6. Our proof of Lemma 3.6, in turn, uses the elementary representation result for the generalized gradient functions \( G^n : \mathbb{R}^d \times \Omega \to \mathbb{R}^d \), \( n \in \mathbb{N}_0 \), in Proposition 3.2 in Sect. 3.2 below. Our proof of Corollary 3.5 employs the elementary representation result for expectations of the empirical risk functions in Proposition 3.3 and the elementary measurability result in Lemma 3.4 in Sect. 3.3 below.

3.1. Description of the SGD optimization method in the training of ANNs

**Setting 3.1.** Let \( d, H, d, \xi, a \in \mathbb{N}, b \in (a, \infty) \) satisfy \( \theta = dH + 2H + 1 \) and \( a = \max\{\{a, b, 1\} \}, \) let \( \mathcal{R}_r : \mathbb{R} \to \mathbb{R} \), \( r \in \mathbb{N} \cup \{\infty\} \), satisfy for all \( x \in \mathbb{R} \) that \( (\bigcup_{r \in \mathbb{N}} \{\mathcal{R}_r\}) \subseteq C^1(\mathbb{R}, \mathbb{R}) \), \( \mathcal{R}_\infty(x) = \max\{x, 0\} \), and

\[
\limsup_{r \to \infty} (|\mathcal{R}_r(x) - \mathcal{R}_\infty(x)| + |(\mathcal{R}_r)'(x) - 1_{(0,\infty)}(x)|) = 0. \tag{69}
\]

let \( \mathbf{w} = ((\mathbf{w}_{ij}^\phi)_{(i,j)\in\{1,\ldots,H\} \times \{1,\ldots,d\}})_{\phi \in \mathbb{R}^d} : \mathbb{R}^d \to \mathbb{R}^{H \times d}, \quad \mathbf{b} = ((\mathbf{b}_H^\phi)_{\phi \in \mathbb{R}^d})_{\phi \in \mathbb{R}^d} : \mathbb{R}^d \to \mathbb{R}^H, \quad \mathbf{c} = (\mathbf{c}_H^\phi)_{\phi \in \mathbb{R}^d} : \mathbb{R}^d \to \mathbb{R} \) satisfy for all \( \phi = (\phi_1, \ldots, \phi_0) \in \mathbb{R}^d \), \( i \in \{1, 2, \ldots, H\}, j \in \{1, 2, \ldots, d\} \) that \( \mathbf{w}_{ij}^\phi = \phi_{i-1} d_{ij}, \mathbf{b}_i^\phi = \phi_{Hd+i}, \mathbf{v}_i^\phi = \phi_{H(d+1)+i}, \) and \( \mathbf{c}_i^\phi = \phi_0 \), let
Theorem 3.2. Assume Setting 3.1 and let $n \in \mathbb{N}_0$, $\phi \in \mathbb{R}^d$, $\omega \in \Omega$. Then

(i) it holds for all $r \in \mathbb{N}$, $i \in \{1, 2, \ldots, H\}$, $j \in \{1, 2, \ldots, d\}$ that

$$\left( \frac{\partial}{\partial \phi_{(i-1)d+j}} \mathcal{L}^n_r \right) (\phi, \omega)$$

$$= \frac{2}{M_n} \sum_{m=1}^{M_n} \left[ \mathcal{R}^n_{(i-1)d+j} \left( \mathcal{A}'^n_r (X^{n,m}(\omega)) - \xi \right) \left( (\mathcal{R}_{(i-1)d+j})' \left( b_i^\phi + \sum_{k=1}^{d} w_{i,k}^\phi X^{n,m}_k(\omega) \right) \right) \right]$$

(ii) it holds that

$$\limsup_{r \to \infty} \left\| (\nabla \mathcal{L}^n_r)(\phi, \omega) - \mathcal{S}^n(\phi, \omega) \right\| = 0.$$
Lemma 3.4. Proof of Lemma 3.4.

Proof of Proposition 3.2. Observe that the assumption that for all \( r \in \mathbb{N} \) it holds that \( \mathcal{R}_r \in C^1(\mathbb{R}, \mathbb{R}) \) and the chain rule prove item (i). Next note that item (i) and the assumption that for all \( x \in \mathbb{R} \) we have that \( \limsup_{r \to \infty} (|\mathcal{R}_r(x)| + |(\mathcal{R}_r)'(x) - 1_{(0,\infty)}(x)|) = 0 \) establish items (ii) and (iii). The proof of Proposition 3.2 is thus complete.

3.3. Properties of the expectations of the empirical risk functions

Proposition 3.3. Assume Setting 3.1. Then it holds for all \( n \in \mathbb{N}_0, \phi \in \mathbb{R}^d \) that \( \mathbb{E}[\mathcal{L}_n^0(\phi)] = \mathcal{L}(\phi) \).

Proof of Proposition 3.3. Observe that the assumption that \( X_n^m : \Omega \to [a,b]^d, n, m \in \mathbb{N}_0 \), are i.i.d. random variables ensures that for all \( n \in \mathbb{N}_0, \phi \in \mathbb{R}^d \) it holds that
\[
\mathbb{E}[\mathcal{L}_n^0(\phi)] = \frac{1}{M_n} \sum_{m=1}^{M_n} \mathbb{E}[(\mathcal{M}_\infty^\phi(X_n^m)) - \xi]^2 = \mathbb{E}[(\mathcal{M}_\infty^\phi(X^0,0)) - \xi]^2 = \mathcal{L}(\phi).
\]
The proof of Proposition 3.3 is thus complete.

Lemma 3.4. Assume Setting 3.1 and let \( \mathbb{F}_n \subseteq \mathcal{F}, n \in \mathbb{N}_0 \), satisfy for all \( n \in \mathbb{N} \) that \( \mathbb{F}_0 = \sigma(\Theta_0) \) and \( \mathbb{F}_n = \sigma(\Theta_0, (X_n^m)_{(n,m) \in (N \cap (0,n)) \times \mathbb{R}}) \). Then

(i) it holds for all \( n \in \mathbb{N}_0 \) that \( \mathbb{R}^d \times \Omega \ni (\phi, \omega) \mapsto \mathcal{L}_n(\phi, \omega) \in \mathbb{R}^d \) is \( \mathcal{B}(\mathbb{R}^d) \otimes \mathbb{F}_{n+1}/\mathcal{B}(\mathbb{R}^d) \)-measurable,

(ii) it holds for all \( n \in \mathbb{N}_0 \) that \( \Theta_n = \mathbb{F}_n/\mathcal{B}(\mathbb{R}^d) \)-measurable, and

(iii) it holds for all \( m \in \mathbb{N}_0 \) such that \( \sigma(X_n^m) \) and \( \mathbb{F}_n \) are independent.

Proof of Lemma 3.4. Note that Lemma 2.4 and (72) prove that for all \( n \in \mathbb{N}_0, r \in \mathbb{N}, \omega \in \Omega \) it holds that \( \mathbb{R}^d \ni \phi \mapsto (\nabla \phi \mathcal{L}_n^0(\phi, \omega)) \in \mathbb{R}^d \) is continuous. Furthermore, observe that (72) and the fact that for all \( n, m \in \mathbb{N}_0 \) it holds that \( X_n^m \) is \( \mathbb{F}_{n+1}/\mathcal{B}([a,b]^d) \)-measurable assure that for all \( n \in \mathbb{N}_0, r \in \mathbb{N}, \phi \in \mathbb{R}^d \) it holds that \( \Omega \ni \omega \mapsto (\nabla \phi \mathcal{L}_n^0(\phi, \omega)) \in \mathbb{R}^d \) is \( \mathbb{F}_{n+1}/\mathcal{B}(\mathbb{R}^d) \)-measurable. This and, e.g., [5, Lemma 2.4] show that for all \( n \in \mathbb{N}_0, r \in \mathbb{N} \) it holds that \( \mathbb{R}^d \times \Omega \ni (\phi, \omega) \mapsto (\nabla \phi \mathcal{L}_n^0(\phi, \omega)) \in \mathbb{R}^d \) is \( \mathcal{B}(\mathbb{R}^d) \otimes \mathbb{F}_{n+1}/\mathcal{B}(\mathbb{R}^d) \)-measurable. Combining this with item (ii) in Proposition 3.2 demonstrates that for all \( n \in \mathbb{N}_0 \) it holds that
\[
\mathbb{R}^d \times \Omega \ni (\phi, \omega) \mapsto \mathcal{L}_n(\phi, \omega) \in \mathbb{R}^d
\]
is \( \mathcal{B}(\mathbb{R}^d) \otimes \mathbb{F}_{n+1}/\mathcal{B}(\mathbb{R}^d) \)-measurable. This establishes item (i). In the next step we prove item (ii) by induction on \( n \in \mathbb{N}_0 \). Note that the fact that \( \mathbb{F}_0 = \sigma(\Theta_0) \) ensures that \( \Theta_0 \) is \( \mathcal{B}(\mathbb{R}^d) \)-measurable. For the induction step let \( n \in \mathbb{N}_0 \) satisfy that \( \Theta_n = \mathbb{F}_n/\mathcal{B}(\mathbb{R}^d) \)-measurable. Observe that item (i) and the fact that \( \mathbb{F}_n \subseteq \mathbb{F}_{n+1} \) ensure that \( \mathcal{L}_n(\Theta_n) \) is \( \mathcal{F}_{n+1}/\mathcal{B}(\mathbb{R}^d) \)-measurable. Combining this, the fact that \( \mathbb{F}_n \subseteq \mathbb{F}_{n+1} \), and the assumption that \( \Theta_{n+1} = \Theta_n - \gamma_n \mathcal{L}_n(\Theta_n) \) demonstrates that \( \Theta_{n+1} \) is \( \mathcal{F}_{n+1}/\mathcal{B}(\mathbb{R}^d) \)-measurable. Induction thus establishes item (ii). Next note that the assumption that \( X_n^m, n, m \in \mathbb{N}_0, \)}
are independent and the assumption that $\Theta_0$ and $(X^{n,m})_{(n,m)\in\mathbb{N}(0)}$ are independent establish item (iii). The proof of Lemma 3.4 is thus complete. 

**Corollary 3.5.** Assume Setting 3.1. Then it holds for all $n \in \mathbb{N}_0$ that $E[L^n_\infty(\Theta_n)] = E(L(\Theta_n))$.

**Proof of Corollary 3.5.** Throughout this proof let $F_n \subseteq F$, $n \in \mathbb{N}_0$, satisfy for all $n \in \mathbb{N}$ that $F_0 = \sigma(\Theta_0)$ and $F_n = \sigma(\Theta_0, (X^{n,m})_{(n,m)\in\mathbb{N}(0,n)} \times \mathbb{N}_0)$ and let $L^n : ([a,b]^d)^{M_n} \times \mathbb{R}^d \to [0, \infty)$, $n \in \mathbb{N}_0$, satisfy for all $n \in \mathbb{N}_0$, $x_1, x_2, \ldots, x_{M_n} \in [a, b]^d$, $\phi \in \mathbb{R}^q$ that

$$L^n(x_1, \ldots, x_{M_n}, \phi) = \frac{1}{M_n} \sum_{m=1}^{M_n} (\mathcal{N}_{\infty}^\phi(x_m) - \xi)^2. \quad (76)$$

Observe that (76) implies that for all $n \in \mathbb{N}_0$, $\phi \in \mathbb{R}^q$, $\omega \in \Omega$ it holds that

$$L^n_\infty(\phi, \omega) = L^n(X^{n,1}(\omega), \ldots, X^{n,M_n}(\omega), \phi). \quad (77)$$

Hence, we obtain that for all $n \in \mathbb{N}_0$ it holds that

$$L^n_\infty(\Theta_n) = L^n(X^{n,1}, \ldots, X^{n,M_n}, \Theta_n). \quad (78)$$

Furthermore, note that (77) and Proposition 3.3 imply that for all $n \in \mathbb{N}_0$, $\phi \in \mathbb{R}^q$ we have that $E[L^n((X^{n,1}, \ldots, X^{n,M_n}), \phi)] = \mathcal{L}(\phi)$. This, Lemma 3.4, (78), and, e.g., [24, Lemma 2.8] (applied with $(\Omega, \mathcal{F}, \mathcal{P}) \leftarrow (\Omega, \mathcal{F}, \mathcal{P})$, $\mathcal{G} \cap \mathcal{F}_n \times \{X, X\} \cap \mathcal{B}(([a, b]^d)^{M_n})$, $\mathcal{Y} \cap (\mathcal{F} \mathcal{B}(\mathbb{R}^q))$, $\mathcal{X} \cap (\Omega \in (\mathcal{X}, \mathcal{F}_n, (X^{n,1}(\omega), \ldots, X^{n,M_n}(\omega)) \in ([a, b]^d)^{M_n}, \mathcal{Y} \cap (\Omega \in (\mathcal{X}, \mathcal{F}_n, \mathcal{P}) \leftarrow (\mathcal{X}, \mathcal{F}_n, \mathcal{P})$ in the notation of [24, Lemma 2.8]) demonstrate that for all $n \in \mathbb{N}_0$ it holds that $E[L^n_\infty(\Theta_n)] = E[L^n(X^{n,1}, \ldots, X^{n,M_n}, \Theta_n)] = E(L(\Theta_n))$. The proof of Corollary 3.5 is thus complete. 

3.4. Upper estimates for generalized gradients of the empirical risk functions

**Lemma 3.6.** Assume Setting 3.1 and let $n \in \mathbb{N}_0$, $\phi \in \mathbb{R}^q$, $\omega \in \Omega$. Then $\|G^n(\phi, \omega)\| \leq 4(a^2(d+1)\|\phi\|^2 + 1)\mathcal{L}_\infty^n(\phi, \omega)$.

**Proof of Lemma 3.6.** Observe that Jensen’s inequality implies that

$$\left(\frac{1}{M_n} \sum_{m=1}^{M_n} |\mathcal{N}_{\infty}^\phi(X^{n,m}(\omega)) - \xi|\right)^2 \leq \frac{1}{M_n} \sum_{m=1}^{M_n} (|\mathcal{N}_{\infty}^\phi(X^{n,m}(\omega)) - \xi|)^2 = \mathcal{L}_\infty^n(\phi, \omega). \quad (79)$$

This and (73) ensure that for all $i \in \{1, 2, \ldots, H\}$, $j \in \{1, 2, \ldots, d\}$ we have that

$$|G^n_{(i-1)d+j}(\phi, \omega)|^2 = (v_i^\phi)^2 \left(\frac{2}{M_n} \sum_{m=1}^{M_n} \left[|X_j^{n,m}(\omega)| \left|\mathcal{N}_{\infty}^\phi(X^{n,m}(\omega)) - \xi\right| I_i^\phi(X^{n,m}(\omega)) \right]\right)^2 \leq (v_i^\phi)^2 \left(\frac{2}{M_n} \sum_{m=1}^{M_n} \left[|X_j^{n,m}(\omega)| \left|\mathcal{N}_{\infty}^\phi(X^{n,m}(\omega)) - \xi\right| I_i^\phi(X^{n,m}(\omega)) \right]\right)^2 \leq 4a^2(v_i^\phi)^2 \mathcal{L}_\infty^n(\phi, \omega). \quad (80)$$

In addition, note that (73) and (79) assure that for all $i \in \{1, 2, \ldots, H\}$ it holds that

$$|G^n_{M_n+d+i}(\phi, \omega)|^2 = (v_i^\phi)^2 \left(\frac{1}{M_n} \sum_{m=1}^{M_n} \left|\mathcal{N}_{\infty}^\phi(X^{n,m}(\omega)) - \xi\right| I_i^\phi(X^{n,m}(\omega)) \right)^2 \leq 4(v_i^\phi)^2 \mathcal{L}_\infty^n(\phi, \omega). \quad (81)$$
Furthermore, observe that for all $x = (x_1, \ldots, x_d) \in [a, b]^d$, $i \in \{1, 2, \ldots, H\}$ it holds that

$$ \| \mathfrak{R}_\infty \left( b_i^\phi + \sum_{j=1}^d w_{i,j}^\phi x_j \right) \|^2 \leq \left( \left| b_i^\phi \right| + a \sum_{j=1}^d |w_{i,j}^\phi| \right)^2 \leq a^2 (d+1) \left( \left| b_i^\phi \right|^2 + \sum_{j=1}^d |w_{i,j}^\phi| \right)^2. $$

Combining this, the fact that for all $m, n \in \mathbb{N}_0$, $\omega \in \Omega$ it holds that $X^{n,m}(\omega) \in [a, b]^d$, (73), and Jensen’s inequality demonstrates that for all $i \in \{1, 2, \ldots, H\}$ it holds that

$$ \| \mathfrak{S}^n_{H(\delta+1)+i}(\phi, \omega) \|^2 \leq \left( \frac{2}{M_n} \sum_{m=1}^{M_n} \left[ \mathfrak{R}_\infty \left( b_i^\phi + \sum_{k=1}^d w_{i,k}^\phi X_k^{n,m}(\omega) \right) \right] \mathcal{L}_\infty^\phi(X^{n,m}(\omega) - \xi) \right)^2 

\leq \frac{4}{M_n} \sum_{m=1}^{M_n} \| \mathfrak{R}_\infty \left( b_i^\phi + \sum_{k=1}^d w_{i,k}^\phi X_k^{n,m}(\omega) \right) \|^2 \mathcal{L}_\infty^\phi(X^{n,m}(\omega) - \xi) \right)^2 

\leq 4a^2 (d+1) \left[ \left| b_i^\phi \right|^2 + \sum_{j=1}^d |w_{i,j}^\phi| \right] \mathcal{L}_\infty^n(\phi, \omega). $$

Moreover, note that (73) and (79) show that

$$ \| \mathfrak{S}^n_\phi(\phi, \omega) \|^2 = 4 \left( \frac{1}{M_n} \sum_{m=1}^{M_n} \mathcal{L}_\infty^\phi(X^{n,m}(\omega) - \xi) \right)^2 \leq 4 \mathcal{L}_\infty^n(\phi, \omega). $$

Combining (80)–(83) yields

$$ \| \mathfrak{S}^n(\phi, \omega) \|^2 

\leq 4 \left[ \sum_{i=1}^{H} \left( a^2 \left[ \sum_{j=1}^d |w_{i,j}^\phi|^2 \right] + \| b_i^\phi \|^2 + a^2 (d+1) \left[ \left| b_i^\phi \right|^2 + \sum_{j=1}^d |w_{i,j}^\phi| \right] \right) \mathcal{L}_\infty^n(\phi, \omega) + 4 \mathcal{L}_\infty^n(\phi, \omega) \right) 

\leq 4a^2 \left[ \sum_{i=1}^{H} \left( (d+1) \| b_i^\phi \|^2 + (d+1) \left[ \left| b_i^\phi \right|^2 + \sum_{j=1}^d |w_{i,j}^\phi| \right] \right) \mathcal{L}_\infty^n(\phi, \omega) + 4 \mathcal{L}_\infty^n(\phi, \omega) \right) = 4(a^2 (d+1) \| \phi \|^2 + 1) \mathcal{L}_\infty^n(\phi, \omega). $$

The proof of Lemma 3.6 is thus complete.

**Lemma 3.7.** Assume Setting 3.1 and let $K \subseteq \mathbb{R}^b$ be compact. Then

$$ \sup_{n \in \mathbb{N}_0} \sup_{\phi \in K} \sup_{\omega \in \Omega} \| \mathfrak{S}^n(\phi, \omega) \| < \infty. $$

**Proof of Lemma 3.7.** Observe that Lemma 2.4 proves that there exists $C \in \mathbb{R}$ which satisfies for all $\phi \in K$ that $\sup_{x \in [a, b]^d} | \mathcal{L}_\infty^\phi(x) | \leq C$. The fact that for all $n, m \in \mathbb{N}_0$, $\omega \in \Omega$ it holds that $X^{n,m}(\omega) \in [a, b]^d$ hence establishes that for all $n \in \mathbb{N}_0$, $\phi \in K$, $\omega \in \Omega$ we have that

$$ \mathcal{L}_\infty^n(\phi, \omega) = \frac{1}{M_n} \sum_{m=1}^{M_n} \left( \mathcal{L}_\infty^\phi(X^{n,m}(\omega)) \right)^2 \leq \frac{1}{M_n} \sum_{m=1}^{M_n} \left( \mathcal{L}_\infty^\phi((X^{n,m}(\omega))^2 + \xi^2) \right) \leq 2e^2 + 2\xi^2. $$

Combining this and Lemma 3.6 completes the proof of Lemma 3.7.

**3.5. Lyapunov type estimates for SGD processes**

**Lemma 3.8.** Assume Setting 3.1 and let $n \in \mathbb{N}_0$, $\phi \in \mathbb{R}^b$, $\omega \in \Omega$. Then $\langle \nabla V(\phi), \mathfrak{S}^n(\phi, \omega) \rangle = 8 \mathcal{L}_\infty^n(\phi, \omega)$.

**Proof of Lemma 3.8.** Note that the fact that $V(\phi) = \| \phi \|^2 + |c^\phi - 2\xi|^2$ ensures that

$$ \langle \nabla V(\phi), \mathfrak{S}^n(\phi, \omega) \rangle 

= 2 \left( \sum_{i=1}^{d} \sum_{j=1}^{M_n} w_{i,j}^\phi \phi_i \right) - 2 \left( \sum_{i=1}^{d} \sum_{j=1}^{M_n} w_{i,j}^\phi \phi_i \right) \cdot \langle \nabla V(\phi), \mathfrak{S}^n(\phi, \omega) \rangle = 8 \mathcal{L}_\infty^n(\phi, \omega). $$

(87)
This and (73) imply that

\[
\langle (\nabla V)(\phi), \mathfrak{H}^n(\phi, \omega) \rangle
= 4 \frac{M_n}{n} \left[ \sum_{i=1}^{H} \sum_{j=1}^{d} w_{i,j}^\phi \left( \sum_{m=1}^{M_n} \left[ \left( X_j^{n,m}(\omega) - \xi \right) \mathbb{1}_{I_t^\phi(\omega)} \right] \right) \right]
+ 4 \frac{M_n}{n} \left[ \sum_{i=1}^{H} \sum_{m=1}^{M_n} \left[ \left( \mathfrak{H}_\infty (\omega) \mathbb{1}_{I_t^\phi(\omega)} \right) \left( X_j^{n,m}(\omega) - \xi \right) \right] \right]
+ 4 \frac{M_n}{n} \left[ \sum_{i=1}^{H} \sum_{m=1}^{M_n} \left[ \left( \mathfrak{H}_\infty (\omega) \mathbb{1}_{I_t^\phi(\omega)} \right) \left( X_j^{n,m}(\omega) - \xi \right) \right] \right]
+ 8(\epsilon^\phi - \xi) \frac{M_n}{n} \left[ \sum_{m=1}^{M_n} \left( \mathfrak{H}_\infty (\omega) - \xi \right) \right].
\]  

(88)

Hence, we obtain that

\[
\langle (\nabla V)(\phi), \mathfrak{H}^n(\phi, \omega) \rangle
= 4 \frac{M_n}{n} \left[ \sum_{i=1}^{H} \sum_{j=1}^{d} w_{i,j}^\phi \left( \sum_{m=1}^{M_n} \left[ \left( X_j^{n,m}(\omega) - \xi \right) \mathbb{1}_{I_t^\phi(\omega)} \right] \right) \right]
+ 4 \frac{M_n}{n} \left[ \sum_{i=1}^{H} \sum_{m=1}^{M_n} \left[ \left( \mathfrak{H}_\infty (\omega) \mathbb{1}_{I_t^\phi(\omega)} \right) \left( X_j^{n,m}(\omega) - \xi \right) \right] \right]
+ 8(\epsilon^\phi - \xi) \frac{M_n}{n} \left[ \sum_{m=1}^{M_n} \left( \mathfrak{H}_\infty (\omega) - \xi \right) \right]
= 8 \frac{M_n}{n} \left[ \sum_{m=1}^{M_n} \left( \mathfrak{H}_\infty (\omega) - \xi \right) \right] = 8 \mathcal{L}_\infty^n(\phi, \omega) \].

(89)

The proof of Lemma 3.8 is thus complete. 

\[ \square \]

**Lemma 3.9.** Assume Setting 3.1 and let \( n \in \mathbb{N}_0, \theta \in \mathbb{R}^d, \omega \in \Omega \). Then

\[
V(\theta - \gamma_n \mathfrak{H}^n(\theta, \omega)) - V(\theta) = (\gamma_n)^2 \| \mathfrak{H}^n(\theta, \omega) \|^2 + (\gamma_n)^2 \| \mathfrak{H}^n_\infty(\theta, \omega) \|^2 - 8\gamma_n \mathcal{L}_\infty^n(\theta, \omega)
\leq 2(\gamma_n)^2 \| \mathfrak{H}^n(\theta, \omega) \|^2 - 8\gamma_n \mathcal{L}_\infty^n(\theta, \omega).
\]

(90)

**Proof of Lemma 3.9.** Throughout this proof let \( \mathbf{e} \in \mathbb{R}^d \) satisfy \( \mathbf{e} = (0, 0, \ldots, 0, 1) \) and let \( g: \mathbb{R} \to \mathbb{R} \) satisfy for all \( t \in \mathbb{R} \) that

\[
g(t) = V(\theta - t \mathfrak{H}^n(\theta, \omega)).
\]

(91)
Observe that (91) and the fundamental theorem of calculus prove that
\[
V(\theta - \gamma_n \mathcal{G}^n(\theta, \omega)) = g(\gamma_n) = g(0) + \int_0^{\gamma_n} g'(t) \, dt
\]
\[
= g(0) + \int_0^{\gamma_n} \langle (\nabla V)(\theta - t \mathcal{G}^n(\theta, \omega)), (-\mathcal{G}(\theta, \omega)) \rangle \, dt
\]  
(92)
\[
= V(\theta) - \int_0^{\gamma_n} \langle (\nabla V)(\theta - t \mathcal{G}^n(\theta, \omega)), \mathcal{G}(\theta, \omega) \rangle \, dt.
\]

Lemma 3.8 hence demonstrates that
\[
V(\theta - \gamma_n \mathcal{G}^n(\theta, \omega)) = V(\theta) - \int_0^{\gamma_n} \langle (\nabla V)(\theta), \mathcal{G}^n(\theta, \omega) \rangle \, dt 
\]  
(93)
\[
+ \int_0^{\gamma_n} \langle (\nabla V)(\theta) - (\nabla V)(\theta - t \mathcal{G}^n(\theta, \omega)), \mathcal{G}^n(\theta, \omega) \rangle \, dt
\]
\[
= V(\theta) - 8\gamma_n \mathcal{L}_\infty^n(\theta, \omega) + \int_0^{\gamma_n} \langle (\nabla V)(\theta) - (\nabla V)(\theta - t \mathcal{G}^n(\theta, \omega)), \mathcal{G}^n(\theta, \omega) \rangle \, dt.
\]

Proposition 2.8 therefore proves that
\[
V(\theta - \gamma_n \mathcal{G}^n(\theta, \omega)) = V(\theta) - 8\gamma_n \mathcal{L}_\infty^n(\theta, \omega) + \int_0^{\gamma_n} \langle 2t \mathcal{G}^n(\theta, \omega) + 2e^{t \mathcal{G}^n(\theta, \omega)} e, \mathcal{G}^n(\theta, \omega) \rangle \, dt
\]
\[
= V(\theta) - 8\gamma_n \mathcal{L}_\infty^n(\theta, \omega) + 2\|\mathcal{G}^n(\theta, \omega)\|^2 \left[ \int_0^{\gamma_n} t \, dt \right]
\]  
(94)
\[
+ 2 \left[ \int_0^{\gamma_n} \left( e^{t \mathcal{G}^n(\theta, \omega)} \right) \langle e, \mathcal{G}^n(\theta, \omega) \rangle \, dt \right].
\]

Hence, we obtain that
\[
V(\theta - \gamma_n \mathcal{G}^n(\theta, \omega))
\]
\[
= V(\theta) - 8\gamma_n \mathcal{L}_\infty^n(\theta, \omega) + (\gamma_n)^2 \|\mathcal{G}^n(\theta, \omega)\|^2 + 2\|e, \mathcal{G}^n(\theta, \omega)\|^2 \left[ \int_0^{\gamma_n} t \, dt \right]
\]  
(95)
\[
= V(\theta) - 8\gamma_n \mathcal{L}_\infty^n(\theta, \omega) + (\gamma_n)^2 \|\mathcal{G}^n(\theta, \omega)\|^2 + (\gamma_n)^2 \|e, \mathcal{G}^n(\theta, \omega)\|^2
\]
\[
= V(\theta) - 8\gamma_n \mathcal{L}_\infty^n(\theta, \omega) + (\gamma_n)^2 \|\mathcal{G}^n(\theta, \omega)\|^2 + (\gamma_n)^2 \|\mathcal{G}^n(\theta, \omega)\|^2.
\]

The proof of Lemma 3.9 is thus complete.

**Lemma 3.10.** Assume Setting 3.1. Then it holds for all \( n \in \mathbb{N}_0 \) that
\[
V(\Theta_{n+1}) - V(\Theta_n) \leq 8 \left( (\gamma_n)^2 \left[ a^2 (d + 1) V(\Theta_n) + 1 \right] - \gamma_n \right) \mathcal{L}_\infty^n(\Theta_n). \]  
(96)
Proof of Lemma 3.10. Note that Lemmas 2.7 and 3.6 prove that for all \( n \in \mathbb{N}_0 \) it holds that
\[
\| \Phi^n(\Theta_n) \|^2 \leq 4 \left[ a^2 (d+1) \| \Theta_n \|^2 + 1 \right] \mathcal{L}_\infty^n(\Theta_n) \leq 4 \left[ a^2 (d+1) V(\Theta_n) + 1 \right] \mathcal{L}_\infty^n(\Theta_n). \tag{97}
\]
Lemma 3.9 hence demonstrates that for all \( n \in \mathbb{N}_0 \) it holds that
\[
V(\Theta_{n+1}) - V(\Theta_n) \leq 2 (\gamma_n)^2 \| \Phi^n(\Theta_n) \|^2 - 8 \gamma_n \mathcal{L}_\infty^n(\Theta_n)
\leq 8 (\gamma_n)^2 \left[ a^2 (d+1) V(\Theta_n) + 1 \right] \mathcal{L}_\infty^n(\Theta_n) - 8 \gamma_n \mathcal{L}_\infty^n(\Theta_n)
= 8 \left( (\gamma_n)^2 \left[ a^2 (d+1) V(\Theta_n) + 1 \right] - \gamma_n \right) \mathcal{L}_\infty^n(\Theta_n). \tag{98}
\]
The proof of Lemma 3.10 is thus complete.

Corollary 3.11. Assume Setting 3.1 and assume \( \mathbb{P} \left( \sup_{n \in \mathbb{N}_0} \gamma_n \leq \left[ a^2 (d+1) V(\Theta_0) + 1 \right]^{-1} \right) = 1 \). Then it holds for all \( n \in \mathbb{N}_0 \) that
\[
\mathbb{P} \left( V(\Theta_{n+1}) - V(\Theta_n) \leq -8 \gamma_n \left( 1 - \sup_{m \in \mathbb{N}_0} \gamma_m \left[ a^2 (d+1) V(\Theta_0) + 1 \right] \right) \mathcal{L}_\infty^n(\Theta_n) \leq 0 \right) = 1. \tag{99}
\]
Proof of Corollary 3.11. Throughout this proof let \( g \in \mathbb{R} \) satisfy \( g = \sup_{n \in \mathbb{N}_0} \gamma_n \). We now prove (99) by induction on \( n \in \mathbb{N}_0 \). Observe that Lemma 3.10 and the fact that \( \gamma_0 \leq g \) imply that it holds P-a.s. that
\[
V(\Theta_1) - V(\Theta_0) \leq 8 \left( (\gamma_0)^2 \left[ a^2 (d+1) V(\Theta_0) + 1 \right] - \gamma_0 \right) \mathcal{L}_\infty^0(\Theta_0)
\leq 8 \left( \gamma_0 g \left[ a^2 (d+1) V(\Theta_0) + 1 \right] - \gamma_0 \right) \mathcal{L}_\infty^0(\Theta_0)
= -8 \gamma_0 \left( 1 - g \left[ a^2 (d+1) V(\Theta_0) + 1 \right] \right) \mathcal{L}_\infty^0(\Theta_0). \tag{100}
\]
This establishes (99) in the base case \( n = 0 \). For the induction step let \( n \in \mathbb{N} \) satisfy that for all \( m \in \{0, 1, \ldots, n-1 \} \) it holds P-a.s. that
\[
V(\Theta_{m+1}) - V(\Theta_m) \leq -8 \gamma_m \left( 1 - g \left[ a^2 (d+1) V(\Theta_0) + 1 \right] \right) \mathcal{L}_\infty^m(\Theta_m) \leq 0. \tag{101}
\]
Note that (101) shows that it holds P-a.s. that \( V(\Theta_n) \leq V(\Theta_{n-1}) \leq \cdots \leq V(\Theta_0) \). The fact that \( \gamma_n \leq g \) and Lemma 3.10 hence demonstrate that it holds P-a.s. that
\[
V(\Theta_{n+1}) - V(\Theta_n) \leq 8 \left( (\gamma_n)^2 \left[ a^2 (d+1) V(\Theta_n) + 1 \right] - \gamma_n \right) \mathcal{L}_\infty^n(\Theta_n)
\leq 8 \left( \gamma_n g \left[ a^2 (d+1) V(\Theta_n) + 1 \right] - \gamma_n \right) \mathcal{L}_\infty^n(\Theta_n)
= -8 \gamma_n \left( 1 - g \left[ a^2 (d+1) V(\Theta_0) + 1 \right] \right) \mathcal{L}_\infty^n(\Theta_n) \leq 0. \tag{102}
\]
Induction therefore establishes (99). The proof of Corollary 3.11 is thus complete.

3.6. Convergence analysis for SGD processes in the training of ANNs

Theorem 3.12. Assume Setting 3.1, let \( \delta \in (0, 1) \), assume \( \sum_{n=0}^\infty \gamma_n = \infty \), and assume for all \( n \in \mathbb{N}_0 \) that
\( \mathbb{P} \left( \gamma_n \left[ a^2 (d+1) V(\Theta_0) + 1 \right] \leq \delta \right) = 1 \). Then
\( \mathbb{P} \left( \sup_{n \in \mathbb{N}_0} \| \Theta_n \| \leq C \right) = 1, \)
\( \mathbb{P} \left( \limsup_{n \to \infty} \mathcal{L}(\Theta_n) = 0 \right) = 1, \)
and
\( \mathbb{P} \left( \limsup_{n \to \infty} \mathbb{E}[\mathcal{L}(\Theta_n)] = 0 \right) = 1. \)

Proof of Theorem 3.12. Throughout this proof let \( g \in [0, \infty) \) satisfy \( g = \sup_{n \in \mathbb{N}_0} \gamma_n \). Observe that the assumption that \( \delta < 1 \), the fact that \( \mathbb{P} \left( g \left[ a^2 (d+1) V(\Theta_0) + 1 \right] \leq \delta \right) = 1 \), and the assumption that \( \sum_{n=0}^\infty \gamma_n = \infty \) demonstrate that \( g \in (0, \infty) \). This and the fact that \( \mathbb{P} \left( g \left[ a^2 (d+1) V(\Theta_0) + 1 \right] \leq \delta \right) = 1 \) show that there exists \( C \in [1, \infty) \) which satisfies that
\( \mathbb{P}(V(\Theta_0) \leq C) = 1. \)
Note that (103) and Corollary 3.11 ensure that $\mathbb{P}(\sup_{n \in \mathbb{N}_0} V(\Theta_n) \leq \mathfrak{c}) = 1$. Combining this and the fact that for all $\phi \in \mathbb{R}^g$ it holds that $\|\phi\| \leq |V(\phi)|^{1/2}$ demonstrates that

$$\mathbb{P} \left( \sup_{n \in \mathbb{N}_0} \|\Theta_n\| \leq \mathfrak{c} \right) = 1.$$  

(104)

This establishes item (i). Next observe that Corollary 3.11 and the fact that $\mathbb{P}(g [a^2(d + 1) V(\Theta_0) + 1] \leq \delta) = 1$ prove that for all $n \in \mathbb{N}_0$ it holds $\mathbb{P}$-a.s. that

$$- (V(\Theta_n) - V(\Theta_{n+1})) \leq -8\gamma_n (1 - g [a^2(d + 1) V(\Theta_0) + 1]) \mathcal{L}_n(\Theta_n).$$  

(105)

This assures that for all $n \in \mathbb{N}_0$ it holds $\mathbb{P}$-a.s. that

$$\gamma_n \mathcal{L}_n(\Theta_n) \leq \frac{V(\Theta_n) - V(\Theta_{n+1})}{8(1 - g [a^2(d + 1) V(\Theta_0) + 1])}.$$  

(106)

The fact that $\mathbb{P}(g [a^2(d + 1) V(\Theta_0) + 1] \leq \delta) = 1$ and (103) hence show that for all $N \in \mathbb{N}$ it holds $\mathbb{P}$-a.s. that

$$\sum_{n=0}^{N-1} \gamma_n \mathcal{L}_n(\Theta_n) \leq \frac{\sum_{n=0}^{N-1} (V(\Theta_n) - V(\Theta_{n+1}))}{8(1 - g [a^2(d + 1) V(\Theta_0) + 1])} = \frac{V(\Theta_0) - V(\Theta_N)}{8(1 - g [a^2(d + 1) V(\Theta_0) + 1])}$$

(107)

$$\leq \frac{V(\Theta_0)}{8(1 - \delta)} \leq \frac{\mathfrak{c}}{8(1 - \delta)} < \infty.$$  

This implies that

$$\sum_{n=0}^{\infty} \gamma_n \mathbb{E}[\mathcal{L}_n(\Theta_n)] = \lim_{N \to \infty} \left[ \sum_{n=0}^{N-1} \gamma_n \mathbb{E}[\mathcal{L}_n(\Theta_n)] \right] \leq \frac{\mathfrak{c}}{8(1 - \delta)} < \infty.$$  

(108)

Furthermore, note that Corollary 3.5 shows for all $n \in \mathbb{N}_0$ that $\mathbb{E}[\mathcal{L}_n(\Theta_n)] = \mathbb{E}[\mathcal{L}(\Theta_n)]$. Combining this with (108) proves that

$$\sum_{n=0}^{\infty} \mathbb{E}[\gamma_n \mathcal{L}(\Theta_n)] < \infty.$$  

(109)

The monotone convergence theorem and the fact that for all $n \in \mathbb{N}_0$ it holds that $\mathcal{L}(\Theta_n) \geq 0$ hence demonstrate that

$$\mathbb{E} \left[ \sum_{n=0}^{\infty} \gamma_n \mathcal{L}(\Theta_n) \right] = \sum_{n=0}^{\infty} \mathbb{E}[\gamma_n \mathcal{L}(\Theta_n)] < \infty.$$  

(110)

Hence, we obtain that $\mathbb{P}(\sum_{n=0}^{\infty} \gamma_n \mathcal{L}(\Theta_n) < \infty) = 1$. Next let $A \subseteq \Omega$ satisfy

$$A = \{ \omega \in \Omega : \left[ (\sum_{n=0}^{\infty} \gamma_n \mathcal{L}(\Theta_n(\omega)) < \infty) \land (\sup_{n \in \mathbb{N}_0} \|\Theta_n(\omega)\| \leq \mathfrak{c}) \right] \}. $$  

(111)

Observe that (104) and the fact that $\mathbb{P}(\sum_{n=0}^{\infty} \gamma_n \mathcal{L}(\Theta_n) < \infty) = 1$ prove that $A \in \mathcal{F}$ and $\mathbb{P}(A) = 1$. In the following let $\omega \in A$ be arbitrary. Note that the assumption that $\sum_{n=0}^{\infty} \gamma_n = \infty$ and the fact that $\sum_{n=0}^{\infty} \gamma_n \mathcal{L}(\Theta_n(\omega)) < \infty$ demonstrate that $\liminf_{n \to \infty} \mathcal{L}(\Theta_n(\omega)) = 0$. We intend to prove by contradiction that $\limsup_{n \to \infty} \mathcal{L}(\Theta_n(\omega)) = 0$. In the following we thus assume that $\limsup_{n \to \infty} \mathcal{L}(\Theta_n(\omega)) > 0$. This implies that there exists $\varepsilon \in (0, \infty)$ which satisfies that

$$0 = \liminf_{n \to \infty} \mathcal{L}(\Theta_n(\omega)) < \varepsilon < 2\varepsilon < \limsup_{n \to \infty} \mathcal{L}(\Theta_n(\omega)).$$  

(112)

Hence, we obtain that there exist $(m_k, n_k) \in \mathbb{N}^2$, $k \in \mathbb{N}$, which satisfy for all $k \in \mathbb{N}$ that $m_k < n_k < m_{k+1}$, $\mathcal{L}(\Theta_{m_k}(\omega)) > 2\varepsilon$, and $\mathcal{L}(\Theta_{n_k}(\omega)) < \varepsilon \leq \min_{j \in \mathbb{N} \cap (m_k, n_k)} \mathcal{L}(\Theta_j(\omega))$. Observe that the fact that
∑_{n=0}^{∞} \gamma_n \mathcal{L}(\Theta_n(\omega)) < \infty and the fact that for all \( k \in \mathbb{N}, j \in \mathbb{N} \cap [m_k, n_k) \) it holds that \( 1 \leq \varepsilon^{-1} \mathcal{L}(\Theta_j(\omega)) \) assure that

\[
\sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k-1} \gamma_j \leq \frac{1}{\varepsilon} \left[ \sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k-1} (\gamma_j \mathcal{L}(\Theta_j(\omega))) \right] \leq \frac{1}{\varepsilon} \left[ \sum_{j=0}^{\infty} (\gamma_j \mathcal{L}(\Theta_j(\omega))) \right] < \infty. \tag{113}
\]

Next note that the fact that \( \sup_{n \in \mathbb{N}_0} \|\Theta_n(\omega)\| \leq C \) and Lemma 3.7 ensure that there exists \( \mathcal{D} \in \mathbb{R} \) which satisfies for all \( n \in \mathbb{N}_0 \) that \( \|\mathcal{G}^n(\Theta_n(\omega), \omega)\| \leq \mathcal{D} \). Combining this and (113) proves that

\[
\sum_{k=1}^{\infty} \|\Theta_{n_k}(\omega) - \Theta_{m_k}(\omega)\| \leq \sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k-1} \|\Theta_{j+1}(\omega) - \Theta_j(\omega)\| = \sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k-1} (\gamma_j \|\mathcal{G}^j(\Theta_j(\omega), \omega)\|)
\]

\[
\leq \mathcal{D} \left[ \sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k-1} \gamma_j \right] < \infty. \tag{114}
\]

Moreover, observe that the fact that \( \sup_{n \in \mathbb{N}_0} \|\Theta_n(\omega)\| \leq C \) and Lemma 2.4 show that there exists \( \mathcal{L} \in \mathbb{R} \) which satisfies for all \( m, n \in \mathbb{N}_0 \) that \( |\mathcal{L}(\Theta_m(\omega)) - \mathcal{L}(\Theta_n(\omega))| \leq \mathcal{L} \|\Theta_m(\omega) - \Theta_n(\omega)\| \). This and (114) demonstrate that

\[
\limsup_{k \to \infty} |\mathcal{L}(\Theta_{n_k}(\omega)) - \mathcal{L}(\Theta_{m_k}(\omega))| \leq \limsup_{k \to \infty} (\mathcal{L} \|\Theta_{n_k}(\omega) - \Theta_{m_k}(\omega)\|) = 0. \tag{115}
\]

Combining this and the fact that for all \( n \in \mathbb{N}_0 \) it holds that \( \mathcal{L}(\Theta_{n_k}(\omega)) < \varepsilon < 2\varepsilon < \mathcal{L}(\Theta_{n_k}(\omega)) \) ensures that

\[
0 < \varepsilon \leq \inf_{k \in \mathbb{N}} |\mathcal{L}(\Theta_{n_k}(\omega)) - \mathcal{L}(\Theta_{m_k}(\omega))| \leq \limsup_{k \to \infty} |\mathcal{L}(\Theta_{n_k}(\omega)) - \mathcal{L}(\Theta_{m_k}(\omega))| = 0. \tag{116}
\]

This contradiction proves that \( \limsup_{n \to \infty} \mathcal{L}(\Theta_n(\omega)) = 0 \). This and the fact that \( \mathbb{P}(A) = 1 \) establish item (ii). Next note that item (i) and the fact that \( \mathcal{L} \) is continuous show that there exists \( \mathcal{C} \in \mathbb{R} \) which satisfies that \( \mathbb{P}\left( \sup_{n \in \mathbb{N}_0} \mathcal{L}(\Theta_n) \leq \mathcal{C} \right) = 1 \). This, item (ii), and the dominated convergence theorem establish item (iii). The proof of Theorem 3.12 is thus complete. \( \square \)

**Corollary 3.13.** Assume Setting 3.1, let \( A \in \mathbb{R} \) satisfy \( A = \max\{a, \xi\} \), assume \( \sum_{n=0}^{\infty} \gamma_n = \infty \), and assume for all \( n \in \mathbb{N}_0 \) that \( \mathbb{P}(18A^4d_\gamma_n \leq (\|\Theta_0\| + 1)^{-2}) = 1 \). Then

(i) there exists \( \mathcal{C} \in \mathbb{R} \) such that \( \mathbb{P}\left( \sup_{n \in \mathbb{N}_0} \|\Theta_n\| \leq \mathcal{C} \right) = 1 \),

(ii) it holds that \( \mathbb{P}\left( \limsup_{n \to \infty} \mathcal{L}(\Theta_n) = 0 \right) = 1 \), and

(iii) it holds that \( \limsup_{n \to \infty} \mathbb{E}[\mathcal{L}(\Theta_n)] = 0 \).

**Proof of Corollary 3.13.** Observe that Lemma 2.7 proves that it holds \( \mathbb{P}\)-a.s. that

\[
a^2(d + 1)V(\Theta_0) + 1 \leq 3a^2(d + 1)\|\Theta_0\|^2 + 8\xi^2a^2(d + 1) + 1. \tag{117}
\]

The fact that \( \min\{A, d\} \geq 1 \) hence shows that it holds \( \mathbb{P}\)-a.s. that

\[
a^2(d + 1)V(\Theta_0) + 1 \leq 6A^2d\|\Theta_0\|^2 + 16A^4d + 1 \leq 17A^4d(\|\Theta_0\|^2 + 1) \leq 17A^4d(\|\Theta_0\| + 1)^2. \tag{118}
\]

This and the assumption that for all \( n \in \mathbb{N}_0 \) it holds that \( \mathbb{P}(18A^4d_\gamma_n \leq (\|\Theta_0\| + 1)^{-2}) = 1 \) ensure that for all \( n \in \mathbb{N}_0 \) it holds \( \mathbb{P}\)-a.s. that

\[
\gamma_n \left( a^2(d + 1)V(\Theta_0) + 1 \right) \leq 17A^4d_\gamma_n(\|\Theta_0\| + 1)^2 \leq \frac{17}{18} < 1. \tag{119}
\]

Theorem 3.12 hence establishes items (i), (ii), and (iii). The proof of Corollary 3.13 is thus complete. \( \square \)
3.7. A Python code for generalized gradients of the loss functions

In this subsection we include a short illustrative example Python code for the computation of appropriate generalized gradients of the risk function. In the notation of Setting 3.1 we assume in the Python code in Listing 1 below that $d = 1$, $H = 3$, $\mu = 10$, $\phi = (-1,1,2,2,-2,0,1,-1,2,3) \in \mathbb{R}^{10}$, $\omega \in \Omega$, and $X^{1,1}(\omega) = 2$. Observe that in this situation it holds that $w_1^{\phi} X^{1,1}(\omega) + b_1^{\phi} = w_2^{\phi} X^{1,1}(\omega) + b_2^{\phi} = 0$. Listing 2 presents the output of a call of the Python code in Listing 1. Listing 2 illustrates that the computed generalized partial derivatives of the loss with respect to $w_1^{\phi}$, $w_2^{\phi}$, $b_1^{\phi}$, $b_2^{\phi}$, $v_1^{\phi}$, and $v_2^{\phi}$ vanish. Note that (73) and the fact that $1_{I_1}(X^{1,1}(\omega)) = 1_{I_2}(X^{1,1}(\omega)) = 0$ prove that the generalized partial derivatives of the loss with respect to $w_1^{\phi}$, $w_2^{\phi}$, $b_1^{\phi}$, $b_2^{\phi}$, $v_1^{\phi}$, and $v_2^{\phi}$ do vanish.

```python
import tensorflow as tf
from tensorflow.python.framework.ops import disable_eager_execution

# disable_eager_execution()

# batch size = 1
inputs = tf.compat.v1.placeholder(shape=(1, 1), dtype=tf.float64)
xi = 3

# first layer with constant initialization \mathbb{R} \rightarrow \mathbb{R}^3
w = tf.compat.v1.Variable(name='w', initial_value=[[-1., 1., 2.]], dtype=tf.float64, trainable=True)
b = tf.compat.v1.Variable(name='b', initial_value=[2., -2., 0.], dtype=tf.float64, trainable=True)

# second layer with constant initialization \mathbb{R}^3 \rightarrow \mathbb{R}
v = tf.compat.v1.Variable(name='v', initial_value=[[1.], [-1.], [2.]], dtype=tf.float64, trainable=True)
c = tf.compat.v1.Variable(name='c', initial_value=[3], dtype=tf.float64, trainable=True)

output = tf.matmul(tf.nn.relu(tf.matmul(inputs, w) + b), v) + c
loss = tf.reduce_mean((output - xi) ** 2)

gradw = tf.compat.v1.gradients(loss, w)
gradb = tf.compat.v1.gradients(loss, b)
gradv = tf.compat.v1.gradients(loss, v)
gradc = tf.compat.v1.gradients(loss, c)

with tf.compat.v1.Session() as sess:
    sess.run(tf.compat.v1.global_variables_initializer())
    gradw = sess.run(gradw, feed_dict={inputs: [[2.]]})
    print('gradient with respect to w: ', gradw)
    gradb = sess.run(gradb, feed_dict={inputs: [[2.]]})
    print('gradient with respect to b: ', gradb)
    gradv = sess.run(gradv, feed_dict={inputs: [[2.]]})
    print('gradient with respect to v: ', gradv)
    gradc = sess.run(gradc, feed_dict={inputs: [[2.]]})
    print('gradient with respect to c: ', gradc)

Listing 1. Generalized gradients of the loss functions using TensorFlow
```
1 gradient with respect to \( w \): \[
\text{array}([[ 0., 0., 64.]])
\]
2 gradient with respect to \( b \): \[
\text{array}([[ 0., 0., 32.]])
\]
3 gradient with respect to \( v \): \[
\text{array}([[ 0.],
[ 0.],
[64.]])
\]
4 gradient with respect to \( c \): \[
\text{array}([[16.]])
\]

Listing 2. Result of the Python code in Listing 1

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Data availability Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Code availability Not applicable.

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