VANISHING OF $\ell^2$-BETTI NUMBERS OF LOCALLY COMPACT GROUPS AS AN INVARIANT OF COARSE EQUIVALENCE

ROMAN SAUER AND MICHAEL SCHRÖDL

Abstract. We provide a proof that the vanishing of $\ell^2$-Betti numbers of unimodular locally compact second countable groups is an invariant of coarse equivalence. To this end, we define coarse $\ell^2$-cohomology for locally compact groups and show that it is isomorphic to continuous cohomology for unimodular groups and invariant under coarse equivalence.

1. Introduction

The insight that the vanishing of $\ell^2$-Betti numbers provides a quasi-isometry invariant is due to Gromov (see [12, Chapter 8] for a statement without proof), and positive results around this insight have a long history. The most important contribution is by Pansu [18] whose work on asymptotic $\ell^p$-cohomology includes a proof that the vanishing of $\ell^2$-Betti numbers of discrete groups of type $F_\infty$, is a quasi-isometry invariant.

There is a growing interest in the metric geometry of locally compact groups [2, 3]. We thus think it is important to have the quasi-isometry and coarse invariance of the vanishing of $\ell^2$-Betti numbers available in the greatest generality. Following Pansu’s ideas and relying on more recent advances in the theory of $\ell^2$-Betti numbers, we provide a proof of the following result.

Theorem 1. Let $G$ and $H$ be unimodular locally compact second countable groups. If $G$ and $H$ are coarsely equivalent then the $n$-th $\ell^2$-Betti number of $G$ vanishes if and only the $n$-th $\ell^2$-Betti number of $H$ vanishes.

The coarse invariance for discrete groups was proven earlier in a paper of Mimura-Ozawa-Sako-Suzuki [16, Corollary 6.3].

Every locally compact, second countable group $G$ (hereafter abbreviated by lcsc) has a left-invariant proper continuous metric by a theorem of Struble [26]. As any two left-invariant proper continuous metrics on $G$ are coarsely equivalent, every lcsc group has a well defined coarse geometry. Further, any coarse equivalence between compactly generated lcsc groups

2010 Mathematics Subject Classification. Primary 20F65; Secondary 22D99.
Key words and phrases. Coarse geometry, locally compact groups, $\ell^2$-Betti numbers.
is a quasi-isometry with respect to word metrics of compact symmetric generating sets and vice versa. In particular, a coarse equivalence between finitely generated discrete groups is a quasi-isometry. See [3, Chapter 4] for a systematic discussion of these notions.

To even state Theorem 1 in that generality, recent advances in the theory of $\ell^2$-Betti numbers were necessary. $\ell^2$-Betti numbers of discrete groups enjoy a long history but it was not until recently that $\ell^2$-Betti numbers were defined for arbitrary unimodular lcsc groups by Petersen [19], and a systematic theory analogous to the discrete case emerged [13,19,20]. Earlier studies of $\ell^2$-Betti numbers of locally compact groups in specific cases can be found in [4,6,10].

**Previous results on coarse invariance.** Pansu [18] introduced asymptotic $\ell^p$-cohomology of discrete groups and proved its invariance under quasi-isometries. If a group $\Gamma$ is of type $F_\infty$, then the $\ell^p$-cohomology of $\Gamma$ coincides with its asymptotic $\ell^p$-cohomology [18, Théorème 1]. The geometric explanation for the appearance of the type $F_\infty$ condition is that the finite-dimensional skeleta of the universal covering of a classifying space of finite type are uniformly contractible. As an immediate consequence of Pansu’s result, the vanishing of $\ell^2$-Betti numbers is a quasi-isometry invariant among discrete groups of type $F_\infty$. The same arguments work for totally disconnected groups admitting a topological model of finite type [23].

Elek [7] investigated the relation between $\ell^p$-cohomology of discrete groups and Roe’s coarse cohomology and proved similar results. Another independent treatment is due to Fan [8]. Genton [11] elaborated upon Pansu’s methods in the case of metric measure spaces.

Oguni [17] generalised the quasi-isometry invariance of the vanishing of $\ell^2$-Betti numbers from discrete groups of type $F_\infty$ to discrete groups whose cohomology with coefficients in the group von Neumann algebra satisfies a certain technical condition. A similar technical condition appears in the proof of quasi-isometry invariance of Novikov-Shubin invariants of amenable groups [25], and it is unclear how much this condition differs from the type $F_\infty$-condition. Oguni’s groupoid approach is inspired by [9,25] and quite different from the approaches by Elek, Fan, and Pansu.

The coarse invariance of vanishing of $\ell^2$-Betti numbers for discrete groups was shown by Mimura-Ozawa-Sako-Suzuki [16, Corollary 6.3]. Li [14] recently reproved this using groupoid techniques as a consequence of more general cohomological coarse invariance results.
Structure of the paper. We review the necessary basics of $\ell^2$-Betti numbers and continuous cohomology in Section 2. In Section 3 we define coarse $\ell^2$-cohomology for lcsc groups and show that it is isomorphic to continuous cohomology. In Section 4 we conclude the proof of Theorem 1 and discuss what fails for non-unimodular groups.

2. Continuous cohomology and $\ell^2$-Betti numbers of lcsc groups

Let $G$ be a unimodular lcsc group with Haar measure $\mu$. Let $X$ be a locally compact second countable space with Radon measure $\nu$. Let $E$ be a Fréchet space.

The space $C(X, E)$ of continuous functions from $X$ to $E$ becomes a Fréchet space when endowed with the topology of compact convergence. Let $L^2_{loc}(X, E)$ be the space of equivalence classes of measurable maps $f : X \to E$ up to $\nu$-null sets such that $||f||_E$ is square-integrable for every compact subset $K \subset X$. The $L^2$-norm of the function $||f||_E$ defines a semi-norm $p_K$ on $L^2_{loc}(X, E)$. The family of semi-norms $p_K$, $K \subset E$, turns $L^2_{loc}(X, E)$ into a Fréchet space.

We call a Fréchet space $E$ with a continuous (i.e. $G \to E$, $g \mapsto g v$, is continuous for every $v \in E$) linear $G$-action a $G$-module. A continuous linear $G$-equivariant map between $G$-modules is a homomorphism of $G$-modules. If $E$ is a $G$-module and $G$ acts continuously and $\nu$-preserving on $X$ then $C(X, E)$ and $L^2_{loc}(X, E)$ become $G$-modules via $(g \cdot f)(x) = gf(g^{-1} x)$ for $x \in X$ and $g \in G$ [1, Proposition 3.1.1]. The usual homogeneous coboundary map

$$d^{n-1} f(g_0, \ldots, g_n) = \sum_{i=0}^n (-1)^i f(g_0, \ldots, \widehat{g_i}, \ldots, g_n)$$

defines cochain complexes $C(G^{*+1}, E)$ and $L^2_{loc}(G^{*+1}, E)$ of $G$-modules (cf. [1, Proposition 3.2.1]). Here we take the diagonal $G$-action on $G^{*+1}$. We recall the following definition.

Definition 2. The (continuous) cohomology of $G$ with coefficients in $E$ is the cohomology

$$H^n(G, E) = H^n(C(G^{*+1}, E)^G)$$

of the $G$-invariants of $C(G^{*+1}, E)$. The reduced (continuous) cohomology $\bar{H}^*(G, E)$ is a quotient of $H^*(G, E)$ obtained by taking the quotient with the closure of $\text{im } d^{*+1}$ instead of $\text{im } d^{*+1}$. 


We have an obvious inclusion
\[(2) \quad I^* : C(G^{*+1}, E) \to L^2_{\text{loc}}(G^{*+1}, E).\]
The maps $I^*$ form a cochain map of $G$-modules. Taking a positive function $\chi \in C_c(G)$ there is a cochain map $R^* : L^2_{\text{loc}}(G^{*+1}, E) \to C(G^{*+1}, E)$ of $G$-modules
\[(R^* f)(g_0, ..., g_n) = \int_{G^{n+1}} f(h_0, ..., h_n) \chi(g_0^{-1}h_0) \cdots \chi(g_n^{-1}h_n) d\mu(h_0, ..., h_n)\]
such that $I^* \circ R^*$ and $R^* \circ I^*$ are homotopic (as cochain maps of $G$-modules) to the identity \[1, \text{Proposition 4.8}\]. So we have the following useful fact:

**Theorem 3.** The cochain map $I^*$ in (2) induces isomorphisms in cohomology and in reduced cohomology.

Next we turn to the case where the coefficient module $E = L^2(G)$ is the regular representation, relevant for the definition of $\ell^2$-Betti numbers.

Let $L(G)$ be the von Neumann algebra of $G$; the Haar measure $\mu$ defines a semifinite left $G$-action and a natural right $L(G)$-action on $L^2(G)$, and the two actions commute. Hence also the $G$-actions on $C(G^{*+1}, L^2(G))$ and $L^2_{\text{loc}}(G^{*+1}, L^2(G))$ considered previously and the $L(G)$-actions induced from the right $L(G)$-action on $L^2(G)$ commute. So the (reduced and non-reduced) continuous cohomology of $G$ with coefficients in $L^2(G)$ is naturally a $L(G)$-module\[1\]. Obviously, the cochain map $I^*$ above is compatible with the $L(G)$-module structures. The groups $H^*(G, L^2(G))$ are called the (continuous) $\ell^2$-cohomology of $G$. Similarly for the reduced cohomology.

Petersen \[19\] extended Lück’s dimension function from finite von Neumann algebras to semifinite von Neumann algebras. The dimension function $\dim_\mu$ with respect to $(G, \mu)$ is a non-trivial dimension for (algebraic) right $L(G)$-modules that is additive for short exact sequences of $L(G)$-modules. It scales as $\dim_{c\mu} = c^{-1} \dim_\mu$ for $c > 0$. The fact that a $L(G)$-module has dimension zero can be expressed without referring to the trace: it is an algebraic fact. The following criterion was shown by the first author for finite von Neumann algebras \[24, \text{Theorem 2.4}\]; it was extended to the semifinite case by Petersen \[19, \text{Lemma B.27}\].

**Theorem 4.** An $L(G)$-module $M$ satisfies $\dim_\mu(M) = 0$ if and only if for every $x \in M$ there is an increasing sequence $(p_i)$ of projections in $L(G)$ with $\sup p_i = 1$ such that $xp_i = 0$ for every $i \in \mathbb{N}$.

\[1\]When talking about $L(G)$-modules we mean the algebraic module structure and ignore topologies.
Definition 5. The \( n \)-th \( \ell^2 \)-Betti number of \( G \) is the \( L(G) \)-dimension of its reduced continuous cohomology with coefficients in \( L^2(G) \), i.e.

\[
\beta_n^{(2)}(G) := \dim_\mu \mathcal{H}^n(G, L^2) \in [0, \infty].
\]

Remark 6. Equivalently, the \( n \)-th \( \ell^2 \)-Betti number can be defined as the \( L(G) \)-dimension of the non-reduced cohomology \( H^n(G, L^2) \). This is a non-trivial fact (see [13, Theorem A]). For discrete \( G \), our definition coincides with Lück’s definition in [15]. Again, this is non-trivial and shown in [21, Theorem 2.2].

The following lemma was observed in [19, Proposition 3.8]. Since it is a direct consequence of Theorem 4 we present the argument.

Lemma 7. \( \beta_n^{(2)}(G) = 0 \iff \mathcal{H}^n(G, L^2) = 0 \).

Proof. Let \( \beta_n^{(2)}(G) = 0 \). Let \( f : G^{n+1} \to L^2(G) \) be a cocycle representing a cohomology class \([f]\) in \( \mathcal{H}^n(G, L^2) \). By Theorem 4 there is an increasing sequence of projections \( p_j \in L(G) \) whose supremum is 1 such that each \( fp_j \) is a coboundary \( d^{n-1}b_j \). It is clear that \( fp_j = d^{n-1}b_j \) converges to \( f \) in the topology of \( C(G^{n+1}, L^2(G)) \), thus \([f] = 0\). \( \square \)

3. Coarse equivalence and coarse \( \ell^2 \)-cohomology

Let \( G \) be a lcsc group. We fix a left-invariant proper continuous metric \( d \) on \( G \). Let \( \mu \) be a Haar measure on \( G \). Let \( \mu_n \) be the \( n \)-fold product measure of \( \mu \) on \( G^n \).

For every \( R > 0 \) and \( n \in \mathbb{N}_0 \) we consider the closed subset

\[
G^R_n := \{(g_0, ..., g_{n-1}) \in G^n \mid d(g_i, g_j) \leq R \text{ for all } 0 \leq i, j \leq n-1\}
\]

and a family of semi-norms for measurable maps \( \alpha : G^{n+1} \to \mathbb{C} \) defined by

\[
\|\alpha\|_R^2 = \int_{G^R_n} |\alpha(g_0, ..., g_n)|^2 d\mu_{n+1} \in [0, \infty].
\]

Let \( CX^{(2)}_n(G) \) be the space of equivalence classes (up to \( \mu_{n+1} \)-null sets) of measurable maps \( \alpha : G^{n+1} \to \mathbb{C} \) such that \( \|\alpha\|_R < \infty \) for every \( R > 0 \). The semi-norms \( \|\_\|_R \), \( R > 0 \), turn \( CX^{(2)}_n(G) \) into a Fréchet space. It is straightforward to verify that the homogeneous differential \( (1) \) yields a well-defined, continuous homomorphism \( CX^{(2)}_n(G) \to CX^{(2)}_{n+1}(G) \) (cf. [11, Proposition 2.3.3]). Thus we obtain a cochain complex of Fréchet spaces.

Definition 8. The coarse \( \ell^2 \)-cohomology of \( G \) is defined as

\[
HX^{(2)}_n(G) = H^n(CX^*_n(G)).
\]
By taking the quotients by the closure of the differentials, one defines similarly the reduced coarse $\ell^2$-cohomology $\underline{H}X^n_{(2)}(G)$.

Remark 9. The previous definition is the continuous analog of Elek’s definition [7, Definition 1.3] in the discrete case (Elek gives credits to Roe [22]). It is very much related to Pansu’s asymptotic $\ell^2$-cohomology [18], which was considered in the generality of metric measure spaces by Genton [11]. The difference of our definition to the one in Genton [11] is as follows: $CX^*_{(2)}(G)$ is an inverse limit of spaces $L^2(G^*+1, \mathbb{R})$. Unlike us, Genton takes first the cohomology of $L^2(G^*+1, \mathbb{R})$ and then the inverse limit. Under some uniform contractibility assumptions the two definitions coincide but likely not in general.

Theorem 10. Let $G$ be a unimodular lcsc group. For every $n \geq 0$, the $n$-th continuous cohomology of $G$ with coefficients in the left regular representation $L^2(G)$ is isomorphic to the $n$-th coarse $\ell^2$-cohomology of $G$, and likewise for reduced cohomology.

Proof. We have the obvious embedding

$$L^2_{loc}(G^{n+1}, L^2(G)) \subset L^2_{loc}(G^{n+1}, L^2_{loc}(G))$$

and the exponential law (see [1] Lemme 1.4) for a proof but beware of the typo in the statement

$$L^2_{loc}(G^{n+1}, L^2_{loc}(G)) \cong L^2_{loc}(G^{n+1} \times G).$$

Thus an element in $L^2_{loc}(G^{n+1}, L^2(G))^G$ is represented by a measurable complex function in $(n + 2)$-variables. For $\alpha \in L^2_{loc}(G^{n+1}, L^2(G))^G$ we define $\mu_{n+2}$-almost everywhere

$$F^n(\alpha)(x_0, \ldots, x_n, x) = \alpha(x^{-1}x_0, \ldots, x^{-1}x_n)(x).$$

The measurable function $F^n(\alpha)$ is invariant by translation in the $(n + 2)$-th variable. By Fubini’s theorem we may regard $F^n(\alpha)$ as a measurable function $E^n(\alpha): G^{n+1} \to \mathbb{C}$ in the first $(n + 1)$-variables. We may think of $E^n(\alpha)$ as an evaluation of $\alpha$ at $e$. Let $B(R)$ denote the $R$-ball around $e \in G$.

Next we show that $\|E^n(\alpha)\|_R < \infty$ for every $R > 0$, thus $E^n(\alpha) \in CX^*_n(G)$.

Since $\alpha \in L^2_{loc}(G^{n+1}, L^2(G))^G$ we have

$$\int_{B(R)^{n+1}} \int_G |\alpha(x_0, x_1, \ldots, x_n)(x)|^2 d\mu d\mu_{n+1} \infty > \int_{B(2R)^{n+1}} \int_G |\alpha(x_0, x_1, \ldots, x_n)(x)|^2 d\mu d\mu_{n+1}$$

$$= \int_{B(2R)^{n+1}} \int_G |\alpha(x_0, xx_0^{-1}x_1, \ldots, xx_0^{-1}x_n)(x_0)|^2 d\mu d\mu_{n+1}.$$
The map
\[ m: G^{n+2} \to G^{n+2}, (x_0, \ldots, x_n, x) \mapsto (x, xx_0^{-1}x_1, \ldots, xx_0^{-1}x_n, x_0) \]
is measure preserving since it is the composition of taking inverses in the last coordinate, left multiplication by \( xx_0^{-1} \), conjugation by \( x \) and taking inverses in the last coordinate. Note that this requires unimodularity. Further, we have
\[ m(G^{n+1}_R \times B(R)) \subset B(2R)^{n+1} \times G. \]
This implies the first inequality below. The first equality follows from the fact that \( (x_0, \ldots, x_n, x) \mapsto (x^{-1}x_0, \ldots, x^{-1}x_n, x) \) is a measure preserving measurable automorphism of \( G^{n+1}_R \times B(R) \).
\[
\int_{B(2R)^{n+1}} \int_G |\alpha(x, xx_0^{-1}x_1, \ldots, xx_0^{-1}x_n)(x_0)|^2 d\mu d\mu_{n+1} \\
\geq \int_{G^{n+1}_R} \int_{B(R)} |\alpha(x_0, \ldots, x_n)(x)|^2 d\mu d\mu_{n+1} \\
= \int_{G^{n+1}_R} \int_{B(R)} |\alpha(x^{-1}x_0, \ldots, x^{-1}x_n)(x)|^2 d\mu d\mu_{n+1} \\
= \mu(B(R)) \|E^n(\alpha)\|_R.
\]
Hence \( \|E^n(\alpha)\|_R \) is finite for every \( R > 0 \). That
\[ E^*: L^2_{\text{loc}}(G^{n+1}, L^2(G))^G \to CX^*_2(G) \]
defines a cochain map is obvious. The above computation also implies that \( E^* \) is continuous with respect to the Fréchet topologies.

Given \( \beta \in CX^*_2(G) \) we define
\[ M^n(\beta)(g_0, \ldots, g_n)(g) = \beta(g^{-1}g_0, \ldots, g^{-1}g_n) \]
for \( \mu_{n+2} \)-almost every \( (g_0, \ldots, g_n, g) \). The function \( M^n(\beta) \) defines an element in \( L^2_{\text{loc}}(G^{n+1}, L^2(G))^G \). The \( G \)-invariance of \( M^n(\beta) \) is obvious. We have to show that \( \|M^n(\beta)|_{B(R)^{n+1}}\| \) is square-integrable for every \( R > 0 \). This follows from the following computations which is based on the arguments above in reversed order.
\[
\mu(B(R)) \int_{G^{n+1}_R} |\beta(g_0, \ldots, g_n)|^2 d\mu_{n+1} \\
= \int_{G^{n+1}_R} \int_{B(R)} |\beta(g_0, \ldots, g_n)|^2 d\mu d\mu_{n+1} \\
\geq \int_{B(R)^{n+1}} \int_G |\beta(g^{-1}g_0, \ldots, g^{-1}g_n)| d\mu d\mu_{n+1}
\]
Obviously, $M^*$ is a chain map. Continuity follows from the previous computation. It is clear that $M^*$ and $E^*$ are mutual inverses. Using Theorem 3 this concludes the proof. 

4. Coarse invariance

We recall the notion of coarse equivalence. A map $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is \textit{coarse Lipschitz} if there is a non-decreasing function $a : [0, \infty) \to [0, \infty)$ with $\lim_{t \to \infty} a(t) = \infty$ such that
\[
d_Y(f(x), f(x')) \leq a(d_X(x, x'))
\]
for all $x, x' \in X$. We say that two such maps $f, g$ are \textit{close} if
\[
sup_{x \in X} d_Y(f(x), g(x)) < \infty.
\]
A coarse Lipschitz map $f : X \to Y$ is a \textit{coarse equivalence} if there is a coarse Lipschitz map $g : Y \to X$ such that $fg$ and $gf$ are close to the identity. We say $g$ is a \textit{coarse inverse} of $f$.

**Lemma 11.** Coarsely equivalent lcsc groups are measurably coarse equivalent, i.e. if $G$ and $H$ are coarse equivalent lcsc groups then there are \textit{measurable} coarse Lipschitz maps $f : G \to H$ and $g : H \to G$ such that $fg$ and $gf$ are close to the identity.

**Proof.** We choose left-invariant continuous proper metrics $d_G$ and $d_H$ on $G$ and $H$, respectively. Let $f : G \to H$ be a coarse Lipschitz map with $d_H(f(x), f(x')) \leq a(d_G(x, x'))$. Let $t > 0$. We pick a countable measurable partition $\mathcal{U}$ of $G$ whose elements have diameter $\leq t$ and choose an element $x_U \in U$ for every $U \in \mathcal{U}$.

By setting $\tilde{f}(x) = f(x_U)$ for $x \in U$ we obtain a coarse Lipschitz map $\tilde{f} : G \to H$ which satisfies $d(\tilde{f}(x), \tilde{f}(x')) \leq a(d_G(x, x') + 2t)$ and is close to $f$ with $d(\tilde{f}(x), f(x)) \leq a(2t)$. Analogously, we construct a measurable coarse Lipschitz map $\tilde{g}$, constructed from a coarse Lipschitz map $g : H \to G$ which is a coarse inverse to $f$. It is obvious that $\tilde{g}$ is a coarse inverse to $\tilde{f}$. 

**Theorem 12.** Coarsely equivalent lcsc groups have isomorphic reduced and non-reduced coarse $\ell^2$-cohomology groups.

**Proof.** Let $G$ and $H$ lcsc groups with Haar measures $\mu$ and $\nu$, respectively. Let $f : G \to H$ be a coarse equivalence with coarse inverse $g$. Because of lemma 11 we can further assume that $f$ and $g$ are measurable. We define a map $\chi : G \times G \to \mathbb{R}$ by
\[
\chi(x, y) = \frac{1_{B_x(c)}(y)}{\mu(B(c))}
\]
where we choose $c$ such that $\mu(B(c)) \geq 1$. Then $\chi$ is a measurable function with $\chi(x, y) = \chi(y, x)$ and $\int_G \chi(x, y) d\mu(y) = 1$. We use the following notation:

$$
\chi: G^{n+1} \times G^{n+1} \to \mathbb{R}, \quad \chi((x_0, \ldots, x_n), (y_0, \ldots, y_n)) = \chi(x_0, y_0) \cdot \ldots \cdot \chi(x_n, y_n).
$$

Analogously, we define $\chi': H^{n+1} \times H^{n+1} \to \mathbb{R}$ with some radius $c'$. Now we can define the maps $f^*: HX^*_2(H) \to HX^*_2(G)$ and $g^*: HX^*_2(G) \to HX^*_2(H)$ as follows where we use $x_i$ for elements in $G$ and $y_i$ for elements of $H$:

$$
f^*\alpha(x_0, \ldots, x_n) = \int_{H^{n+1}} \alpha(y_0, \ldots, y_n) \chi'((f(x_0), \ldots, f(x_n)), (y_0, \ldots, y_n)) d\nu_{n+1}
$$

$$
g^*\beta(y_0, \ldots, y_n) = \int_{G^{n+1}} \beta(x_0, \ldots, x_n) \chi((g(y_0), \ldots, g(y_n)), (x_0, \ldots, x_n)) d\mu_{n+1}.
$$

The idea of averaging over a function like $\chi$ goes back to Pansu; it is necessary in our context since the maps $f$ and $g$ do not preserve the measure classes, in general.

First of all, we check that these are well-defined continuous cochain maps.

$$
\infty > \|\alpha\|_2^2 = \int_{H^{n+1}} |\alpha(y_0, \ldots, y_n)|^2 \cdot 1_{H^{n+1}(R) + c'} d\nu_{n+1}
$$

$$
\geq \int_{H^{n+1}} |\alpha(y_0, \ldots, y_n)|^2 \int_{H^{n+1}} \chi'((f(x_0), \ldots, f(x_n)), (y_0, \ldots, y_n)) d\mu_{n+1} d\nu_{n+1}
$$

$$
= \int_{G^{n+1}} \int_{H^{n+1}} \chi'(((f(x_0), \ldots, f(x_n)), (y_0, \ldots, y_n)) d\nu_{n+1} d\mu_{n+1}
$$

$$
\geq \int_{G^{n+1}} \int_{H^{n+1}} \alpha(y_0, \ldots, y_n) \chi'(((f(x_0), \ldots, f(x_n)), (y_0, \ldots, y_n)) d\nu_{n+1}
$$

$$
\|f^*\alpha(x_0, \ldots, x_n)|^2 d\mu_{n+1} = \|f^*\alpha\|_R^2.
$$

It is a direct computation that $d^n \circ f^n = f^{n+1} \circ d^n$.

It remains to show that there is a cochain homotopy $h: CX^*_2(H) \to CX^*_{n+1}(H)$ such that $\text{Id} - g^*f^* = hd + dh$. We define $h^{n+1}_i: CX^*_{(2)}(H) \to CX^*_2(H)$ by

$$
h^{n+1}_i \alpha(y_0, \ldots, y_n)
$$

$$
= \int_{H^{n+1}} \alpha(\tilde{y}_0, \ldots, \tilde{y}_i, y_i, \ldots, y_n) \chi'((y_0, \ldots, y_n), (\tilde{y}_0, \ldots, \tilde{y}_n)) d\nu_{n+1}(\tilde{y})
$$

and set

$$
h^{n+1} = \sum_{i=0}^n (-1)^i h^{n+1}_i.
$$

That $h^*$ is well-defined is a similar consideration as to show that $f^*$ and $g^*$ are well-defined. Now let us denote the $i$-th term of the coboundary map
by $d_i^n$, i.e. $d_i^n \alpha(y_0, ..., y_{n+1}) = \alpha(y_0, ..., \hat{y}_i, ..., y_{n+1})$. It is straightforward to verify that we have the following relations:

\[
\begin{align*}
    h_{n+1} \circ d_{n+1}^n &= g^n \circ f^n, \\
    h_0^{n+1} \circ d_0^n &= \text{Id}_{CX_{(2)}(H)}, \\
    h_{j+1}^n \circ d_i^n &= d_{i-1}^{n-1} \circ h_{j-1}^n \quad \text{for } 1 \leq j \leq n \text{ and } i \leq j, \\
    h_{j+1}^n \circ d_i^n &= d_{i-1}^{n-1} \circ h_j^n \quad \text{for } 1 \leq i \leq n \text{ and } i > j.
\end{align*}
\]

We get $h_{n+1}^n d^n + d_{n-1}^n h^n = \text{Id}_{CX_{(2)}(H)} - g^n \circ f^n$. The same construction applies to $f^* g^*$ which completes the proof. \hfill \Box

**Proof of Theorem 7.** Let $G$ and $H$ be unimodular lcsc groups. Let $G$ and $H$ be coarsely equivalent. Then we have the following equivalences:

\[
\begin{align*}
    \beta_{(2)}^2(G) = 0 &\iff H^n(G, L^2(G)) = 0 \quad \text{(Lemma 7)} \\
    &\iff HX_{(2)}^n(G) = 0 \quad \text{(Theorem 10)} \\
    &\iff HX_{(2)}^n(H) = 0 \quad \text{(Theorem 12)}
\end{align*}
\]

Going the same steps backwards for the group $H$ finishes the proof. \hfill \Box

**Remark 13.** Since the Borel subgroup $B < \text{SL}_2(\mathbb{R})$ of upper triangular matrices is cocompact, the solvable Lie groups $B$ and $\text{SL}_2(\mathbb{R})$ are quasi-isometric. So $B$ belongs to the class of amenable hyperbolic lcsc groups of which a systematic study was undertaken in [2].

The group $B$ is not unimodular and thus its $\ell^2$-Betti number are not defined. Nevertheless, one may ask what exactly breaks down in the proof above which can be formulated to a large part without the notion of $\ell^2$-Betti numbers. By a result of Delorme [3, Corollaire V.3], we have $H^1(B, L^2(B)) = 0$. Since Theorem 12 does not require unimodularity, we have $HX_{(2)}^1(B) \cong HX_{(2)}^n(\text{SL}_2(\mathbb{R})) \neq 0$ since $\beta_{(2)}^1(\text{SL}_2(\mathbb{R})) \neq 0$. So it is Theorem 10 that fails for the non-unimodular group $B$.

**Acknowledgements.** We acknowledge support by the German Science Foundation via the Research Training Group 2229.

**References**

[1] P. Blanc, *Sur la cohomologie continue des groupes localement compacts*, Ann. Sci. École Norm. Sup. (4) 12 (1979), 137–168.

[2] P.-E. Caprace et al., *Amenable hyperbolic groups*, J. Eur. Math. Soc. (JEMS) 17 (2015), 2903–2947.
[3] Y. Cornulier and P. de la Harpe, *Metric geometry of locally compact groups*, EMS Tracts in Mathematics, 25, European Mathematical Society (EMS), Zürich, 2016.

[4] M. W. Davis et al., *Weighted $L^2$-cohomology of Coxeter groups*, Geom. Topol. 11 (2007), 47–138.

[5] P. Delorme, *1-cohomologie des représentations unitaires des groupes de Lie semi-simples et résolubles. Produits tensoriels continus de représentations*, Bull. Soc. Math. France 105 (1977), 281–336.

[6] J. Dymara, *Thin buildings*, Geom. Topol. 10 (2006), 667–694.

[7] G. Elek, *Coarse cohomology and $l_p$-cohomology*, $K$-Theory 13 (1998), 1–22.

[8] P. T. Fan, *Coarse $l_p$-geometric invariants*, ProQuest LLC, Ann Arbor, MI, 1993.

[9] D. Gaboriau, *Invariants $L^2$ de relations d’équivalence et de groupes*, Publ. Math. Inst. Hautes Études Sci. 95 (2002), 93–150.

[10] D. Gaboriau, *Invariant percolation and harmonic Dirichlet functions*, Geom. Funct. Anal. 15 (2005), 1004–1051.

[11] L. Genton, *Scaled Alexander-Spanier Cohomology and $L^{q,p}$ Cohomology for Metric Spaces*, EPFL, Lausanne, Thèse no 6330, (2014).

[12] M. Gromov, *Asymptotic invariants of infinite groups*, in Geometric group theory, Vol. 2 (Sussex, 1991), 1–295, London Math. Soc. Lecture Note Ser., 182, Cambridge Univ. Press, Cambridge.

[13] D. Kyed, H. D. Petersen and S. Vaes, *$L^2$-Betti numbers of locally compact groups and their cross section equivalence relations*, Trans. Amer. Math. Soc. 367 (2015), 4917–4956.

[14] X. Li, *Dynamic characterizations of quasi-isometry, and applications to cohomology*, arXiv:1604.07375.

[15] W. Lück, *Dimension theory of arbitrary modules over finite von Neumann algebras and $L^2$-Betti numbers, I. Foundations*, J. Reine Angew. Math. 495 (1998), 135–162.

[16] M. Mimura et al., *Group approximation in Cayley topology and coarse geometry, III: Geometric property (T)*, Algebr. Geom. Topol. 15 (2015), 1067–1091.

[17] S. Oguni, *$L^2$-Invariants of discrete groups under coarse equivalence and $L^2$-invariants of cocompact etale groupoids*, preprint (2010).

[18] P. Pansu, *Cohomologie $L^p$ : invariance sous quasiisométries*, preprint (1995).
[19] H. D. Petersen, $L^2$-Betti numbers of locally compact groups, C. R. Math. Acad. Sci. Paris 351 (2013), 339–342.

[20] H.D. Petersen, R. Sauer and A. Thom, $L^2$-Betti numbers of totally disconnected groups and their approximation by Betti numbers of lattices, arXiv:1612.04559. To appear in Journal of Topology.

[21] J. Peterson and A. Thom, Group cocycles and the ring of affiliated operators, Invent. Math. 185 (2011), 561–592.

[22] J. Roe, Coarse cohomology and index theory on complete Riemannian manifolds, Mem. Amer. Math. Soc. 104 (1993), 1–22.

[23] R. Sauer, $\ell^2$-Betti numbers of discrete and non-discrete groups, in: New directions in locally compact groups, London Mathematical Society Lecture Note Series 447, Cambridge University Press, 2018, 205–226.

[24] R. Sauer, $L^2$-Betti numbers of discrete measured groupoids, Internat. J. Algebra Comput. 15 (2005), 1169–1188.

[25] R. Sauer, Homological invariants and quasi-isometry, Geom. Funct. Anal. 16 (2006), 476–515.

[26] R. A. Struble, Metrics in locally compact groups, Compositio Math. 28 (1974), 217–222.

Institute for Algebra and Geometry, Karlsruhe Institute of Technology, Englerstr. 2, 76128 Karlsruhe, Germany
E-mail address: roman.sauer@kit.edu

Institute for Algebra and Geometry, Karlsruhe Institute of Technology, Englerstr. 2, 76128 Karlsruhe, Germany
E-mail address: michael.schroedl@kit.edu