Lindblad approximation and spin relaxation in quantum electrodynamics

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Abstract
This article is concerned with the time evolution of spin observables for generalized spin boson models. This applies in particular to a model of nuclear magnetic resonance, namely a $\frac{1}{2}$-spin particle in a constant external magnetic field and in interaction with the quantized electromagnetic field (photons). We derive a Lindblad (or GKLS) type approximation of the spin dynamics initially in a photon vacuum state together with a precise control of the error coming from this approximation. The error term is bounded by $g^2$ where $g$ is the coupling constant of the spin–photon interaction. The point here is the uniformity in time $t > 0$ of this error control.

Keywords: Lindblad, generalized spin boson, NMR, Fermi golden rule, QED, spin dynamics, spin relaxation

1. Introduction

Many different physical phenomena can be described with standard spin boson models [20] and also with a generalization of these models. The spin is one of the two ingredients of the physical system and the Hilbert space of its quantum states is a finite dimensional space denoted by $H_{sp}$ in this paper. When the interaction is turned off, the spin free time evolution is described by a self-adjoint Hamiltonian operator in $H_{sp}$ denoted here $H_{mag}$. The other ingredient is a quantized electromagnetic field. The Hilbert space of its quantum states is the symmetric Fock space $H_{ph} = \mathcal{F}(H_1)$ over some Hilbert space $H_1$. The free time evolution of the quantized fields is given by a Hamiltonian operator $H_{ph}$ in $H_{ph}$. Both $H_1$ and $H_{ph}$ are recalled in section 2.

The Hilbert space of the full system is then the completed tensor product $H_{ph} \otimes H_{sp}$.

In order to describe the interaction between the spin(s) and the field(s), we consider a finite number of elements $B_1, \ldots, B_p$ belonging to $H_1$ together with a finite number of self-adjoint operators $S_1, \ldots, S_p$ belonging to the set of bounded operators in $H_{sp}$ denoted $\mathcal{L}(H_{sp})$. We use
standard operators in Fock spaces and in particular the Segal field $\Phi_S(B)$ associated with any $B \in \mathcal{H}_1$ (see [26]). With these notations, the interaction Hamiltonian is written as:

$$H_{\text{int}} = \sum_{j=1}^{p} \Phi_S(B_j) \otimes S_j.$$  

(1.1)

The time evolution of the full system is then given by a Hamiltonian $H(g)$ depending on a (coupling constant) parameter $g \neq 0$:

$$H(g) = H_{\text{ph}} \otimes I + I \otimes H_{\text{mag}} + gH_{\text{int}}.$$  

(1.2)

This Hamiltonian is a well-defined operator under hypotheses given in section 2. More details are given in section 2, in particular for domains issues.

The Hamiltonian $H(g)$ can be called generalized spin boson model, see, e.g., [5]. A classical example is the standard spin boson model [20]. We are interested in a model of NMR but the main results of this article are valid in the most general case.

Nuclear magnetic resonance (NMR) is the interaction of one or several $\frac{1}{2}$-spin particles fixed at different points of $\mathbb{R}^3$ with a constant magnetic field, and with the quantized electromagnetic field when it is studied in the quantum electrodynamics (QED) framework. A first mathematical model for NMR is the Bloch model [6] in 1946, where the spin is viewed as a vector in $\mathbb{R}^3$ which follows the so-called Bloch equations. In the framework of QED, NMR can be described by a model given by Cohen-Tannoudji, Dupont-Roc and Grynberg [8] (see also Reuse [27]). This model, the main application of the present work, is a particular case of the above generalized spin boson model defined in (1.1) and (1.2). This model is called the CTDRG model and we shall describe it more precisely in section 2. A semiclassical approximation for this model was given in [2]. The CTDRG model with a finite number of spins is also an example of generalized spin boson model.

Another example is the $N$-level system in the dipole approximation (see, e.g., [30]) where, in comparaison to the CTDRG model, the spin matrices are replaced with the matrices of the dipole moments and the quantized magnetic fields are replaced with the quantized electric fields, both having the same infrared asymptotic. Other examples are given in [5] such as the lattice spin system interacting with phonons and other models when the first Hilbert space is infinite dimensional.

Our aim is to find a good approximation of the average of any spin observable at every time $t > 0$. A spin observable is an operator written as $I \otimes X$ with $X \in \mathcal{L}(\mathcal{H}_{\text{sp}})$. Its evolution at time $t$ is the following operator:

$$S(t, X) = e^{iH(g) t} (I \otimes X) e^{-iH(g) t}.  

(1.3)$$

We shall actually study the expectation of this observable only when the initial state is in the photon vacuum. These photon vacuum states are written as $\Psi_0 \otimes a$ where $\Psi_0$ stands for the vacuum in the Fock space $\mathcal{H}_{\text{ph}} = \mathcal{F}(\mathcal{H}_1)$ and $a$ belongs to $\mathcal{H}_{\text{sp}}$. This average value leads us to define, for any arbitrary operator $T$ in $\mathcal{L}(\mathcal{H}_{\text{ph}} \otimes \mathcal{H}_{\text{sp}})$, an operator $\sigma_0(T)$ acting in $\mathcal{L}(\mathcal{H}_{\text{sp}})$ by:

$$< \sigma_0(T) a, b >_{\mathcal{H}_{\text{sp}}} = < T(\Psi_0 \otimes a), (\Psi_0 \otimes b) >_{\mathcal{H}_{\text{ph}} \otimes \mathcal{H}_{\text{sp}}},$$  

(1.4)

for all $a$ and $b$ in $\mathcal{H}_{\text{sp}}$.

One notes that the initial observables in (1.3) are chosen as $I \otimes X$ and not as $P_0 \otimes X$ where $P_0$ denotes the projection on the photon vacuum, as in some earlier works (see, e.g., [30]).
In direct relation with this remark, notice that we use the expectation in the photon vacuum instead of the photon partial trace. Our objective is therefore to study the time evolution $\sigma_0(S(t, X))$. The evolution $S(t, X)$ is commonly given by the Heisenberg equation:

$$\frac{d}{dt}S(t, X) = i[H(g), S(t, X)], \quad S(0, X) = X$$

but it is however not clear that the image of $S(t, X)$ under $\sigma_0$ satisfies a differential system. The purpose of this work as many others on that subject is precisely to show that this approximatively holds true. Specifically, we shall prove that $\sigma_0(S(t, X))$ can be approximated as following:

$$\sigma_0(S(t, X)) \sim e^{t^2 g^2 L} \left( e^{iH_{mag}^X} X e^{-iH_{mag}^X} \right), \quad X \in L(H_{sp})$$

where $L$ is an operator from $L(H_{sp})$ in itself of the form:

$$LX = \sum_{(\alpha, \beta) \in E^2} \left[ A_{\alpha \beta} [S_{\alpha}, X] S_{\beta}^* - B_{\alpha \beta} S_{\alpha} [S_{\beta}, X] \right]$$

where $E$ is a set of indices, the $A_{\alpha \beta}$ and $B_{\alpha \beta}$ are complex coefficients, the $S_{\alpha}$ are elements of $L(H_{sp})$ and $S_{\alpha}^*$ denotes the adjoint of $S_{\alpha}$. The equation satisfied by the RHS of (1.5) denoted $\Phi(t)(X)$:

$$\frac{d}{dt} \Phi(t)(X) = g^2 L \Phi(t)(X) + i \Phi(t)([H_{mag}, X])$$

is often called master equation ($\Phi(t)$ is in $L(L(H_{sp}))$.) Let us remark that $L$ and the map $X \to e^{iH_{mag}^X} X e^{-iH_{mag}^X}$ do not commute.

The approximation obtained in this work is of a form introduced by Gorini, Kossakowski and Sudarshan [14] and also by Lindblad [22]. According to the terminology of [7], we can call it a GKLS operator (respectively a GKLS approximation) (respectively approximation). It is often used for open quantum systems [1, 9, 15, 30]. These general forms of GKLS operators ([14, 22] and also [7, 19]) together with the associated semigroups are studied in [1, 13, 14, 22].

A standard method to get a master equation in a GKLS form consists first to use the projection method [24, 25, 33], then to effectuate a weak coupling limit approximation and a temporal average (see [9], see also [30] in the zero temperature case and [1, 28] for positive temperature). GKLS approximations for particular models such as the two-level (spinless) atom in the dipole approximation or spin boson model are considered in [1, 15, 16, 28] and for generalized spin boson model in [30], but with a control of the error in the weak coupling limit sense (see [9, 10, 28, 30–32]) which is not uniform in all positive time. It is our aim here to give a control of the error of the GKLS approximation and also to get a control that is uniform in time $t \in (0, +\infty)$. We do not follow the standard method described above. The other classical method is the Hamiltonian method and the resonances study.

Let us give the following two complementary remarks in the particular case of the CTDGRG model. Our study shows that the spin relaxation in the NMR context is closely related to a GKLS approximation of the CTDGRG dynamics. We emphasize that we do not need neither thermal agitation nor nuclei interaction for that purpose. Besides, other works with Jager for the CTDGRG model concern the semiclassical approximation obtained in [2] which is not uniform in time and the localization of photons in the ground state in [3].
In section 2, an operator $L$ of the form (1.6) is precisely defined in (2.8)–(2.10) for the purpose of the approximation formally written in (1.5). Under general hypotheses, we obtain a differential system of the form:

$$\frac{d}{dt}\sigma_0 \left(S(t, e^{-itH_{mag}} X e^{itH_{mag}}) \right) = g^2 \sigma_0 \left(S(t, e^{-itH_{mag}} L X e^{itH_{mag}}) \right) + R(t, X),$$

where $R(t, X)$ is negligible in some precise sense (see proposition 3.5).

This shows that the operator $L$ indeed plays an important role in the approximation issue in the general case. However, Duhamel principle needs to be applied in order to get the approximation (1.5). This leads us to make an assumption on the sign of the real parts of the eigenvalues of $L$ viewed as an operator acting from $L(H_{sp})$ into itself (hypothesis (H3) in section 2). It is this hypothesis that limits the possible applications. This hypothesis is nevertheless almost always satisfied in the case of NMR with a single spin. Under this hypothesis, we obtain a result (theorems 2.1 and 2.2) giving the exact sense of the approximation (1.5). Note that the control of the error is uniform on time $t$ belonging on the half line $(0, \infty)$.

We recall that our purpose in this article is to control the error in the approximation of the generalized spin boson full evolution initially in the photon vacuum by a GKLS type evolution excluding thermal agitation in the model. An estimate similar to our control holds in the case of positive temperature, see [23] which uses different methods, for spin boson type models (as in [12, 16]), with initial observables other than $I \otimes X$, implying in particular that photons are initially present in the model, describing thermal agitation. See also [17] for positive temperature models.

In section 2, we define more precisely the Hilbert spaces and the Hamiltonians, first in the general case and then for the example of NMR. Then we define the GKLS operator in the general case, and we precisely state the main result (theorems 2.1 and 2.2) together with the main assumptions, namely hypotheses (H1)(H2)(H3). At the end of section 2, theorem 2.3 shows that all our hypotheses are satisfied in the case of the CTDRG model. Sections 3 and 4 are devoted to the proof of these two theorems. The first step of the proof of these two theorems is related to approximate quantum Markovian master equation and is given in section 3, the starting point being the Heisenberg equation. The second step then leading to a GKLS form is in relation in some sense with secular approximation and is provided in section 4. In section 5, theorem 2.3 is proved.

2. Statement of results

We first give more details on Hilbert spaces of quantum states and on Hamiltonian operators involved in this paper. As already mentioned, the Hilbert space of the states for the generalized spin boson model under consideration is the completed tensor product $H_{ph} \otimes H_{sp}$ where $H_{ph}$ and $H_{sp}$ are respectively the Hilbert spaces of the photons and of the spin particles, the latter being here finite dimensional.

**Photons.** The single-photon Hilbert space is (see [21]):

$$H_1 = \{ f \in L^2(\mathbb{R}^3, \mathbb{C}^3) \mid k \cdot f(k) = 0 \ \text{a.e.} \ k \in \mathbb{R}^3 \}$$

where $|f|_{H_1}^2 = \int_{\mathbb{R}^3} |f(k)|^2 \, dk$ with $| \cdot |$ being the Euclidian norm in $\mathbb{C}^3$. One denotes by $<f, g>_{H_1}$ the scalar product of two elements $f$ and $g$ of $H_1$. The mapping $g \mapsto <f, g>_{H_1}$ is here chosen to be antilinear. The Hilbert space $H_{ph}$ of photon quantum states is the symmetrized
Fock space over $\mathcal{H}_1$ denoted by $F_c(\mathcal{H}_1)$. We follow section X.7 of [26] for Fock space considerations and notations, in particular for the usual operators, $\Phi_S(V)$, $\Gamma(T)$ and $dI(T)$, acting in $\mathcal{H}_{ph}$, for any $V$ in $\mathcal{H}_1$ and any operator $T$ acting in $\mathcal{H}_1$. The vacuum in $\mathcal{H}_{ph}$ is here denoted by $\Psi_0$.

For each $\alpha \in \mathbb{R}$, let $D(M^\alpha) \subset \mathcal{H}_1$ be the space of $f \in \mathcal{H}_1$ such that the function $k \to |k|^\alpha f(k)$ is in $\mathcal{H}_1$. For $\alpha = 1$, let $M_{\alpha}$ be the operator with domain $D(M_{\alpha})$ and defined by $M_{\alpha}q(k) = |k|^\alpha q(k)$ almost everywhere in $k \in \mathbb{R}^3$. In the Fock space framework, the photon free energy Hamiltonian operator $H_{ph}$ is defined as $H_{ph} = dI(M_{\alpha})$.

For any $V \in \mathcal{H}_1$, we have:

$$e^{i\beta H_{ph} \Phi_S(V)}e^{-i\beta H_{ph}} = \Phi_S(\chi V),$$

where:

$$\chi V(k) = e^{i|k|}V(k), \quad k \in \mathbb{R}^3.$$  \hfill (2.1)

**Spins.** Let $\mathcal{H}_{sp}$ be a finite dimensional Hilbert space. The spin Hamiltonian is a self adjoint operator denoted $H_{mag}$ acting in $\mathcal{H}_{sp}$.

**The Hamiltonian.** The generalized spin boson Hamiltonian is a self adjoint extension of the following operator initially defined on a dense subspace of $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$:

$$H(g) = H_{ph} \otimes I + I \otimes H_{mag} + gH_{int},$$

where $H_{ph}$ acts in the domain $D(H_{ph}) \subset \mathcal{H}_{ph}$, $g$ is a positive constant and:

$$H_{int} = \sum_{j=1}^{P} \Phi_S(B_j) \otimes S_j,$$

where the $B_j$ are elements of $\mathcal{H}_1$ and the $S_j$ are self adjoint elements of $\mathcal{L}(\mathcal{H}_{sp})$.

As already mentioned, an important example is the nuclear magnetic resonance in the context of QED (the CTDGR model). We limit ourselves to the case of a static single particle of spin $\frac{1}{2}$. The spin state belongs to $\mathcal{H}_{sp} = \mathbb{C}^2$ and the interaction with the constant external magnetic field $B_{ext} = (0, 0, \beta)$ (with $\beta > 0$) is given by the Hamiltonian $H_{mag} = \beta \sigma_3$ where the $\sigma_j$ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hfill (2.4)

The quantized electromagnetic field interacting with this particle is modelized by the above Hilbert space and Hamiltonian, $\mathcal{H}_{ph} = F_c(\mathcal{H}_1)$ and $H_{ph}$ respectively. The particle-field interaction is described by an Hamiltonian of the form (1.1) where $P = 3$ and where the operators $S_j$ are the above Pauli matrices $\sigma_j$ ($1 \leq j \leq 3$). The elements corresponding to the $B_j$ de $\mathcal{H}_1$ are usually modeling the three components of the magnetic field at the origin:

$$B_j(k) = \frac{e^{i \varphi(|k|)}|k|^{\frac{3}{2}} k \times e_j}{|k|^2}, \quad k \in \mathbb{R}^3 \setminus \{0\},$$  \hfill (2.5)

where the function $\varphi$ (smooth ultraviolet cutoff) belongs to $\mathcal{S}(\mathbb{R})$ and where $(e_j)$ is the canonical basis of $\mathbb{R}^3$. This model of RMN is used in [8] (and also Reuse [27]). See also [18], [29].

We now turn to the description of our hypotheses in the general case. First, we make the following assumption.
\((H_1)\) The \(B_j\) belong to the domain \(D(M^{-1/2}_\omega)\).

Let us recall the following points concerning domain issues. Under the hypothesis \((H_1)\), the Segal field \(\Phi_B(B_j)\) is bounded from \(D(H_{ph})\) into \(H_{ph}\) (see e.g., proposition 3.4(ii) in [4] or [11]). Thus, under hypothesis \((H_1)\), the Hamiltonian \(H(g)\) has a self adjoint extension with the same domain as the free Hamiltonian \(H_0 = H_{ph} \otimes I + I \otimes H_{mag}\) domain, according to the Kato–Rellich theorem.

Next, we shall state the second hypothesis. In order to define this operator \(L\), we have to assume:

\[(H_2)\] For all \(j \leq P\) and \(k \leq P:\]

\[\int_0^\infty (1 + t) \left| \langle \chi_t B_j, B_k \rangle \right| dt < \infty.\]

A similar but weaker hypothesis with \((1 + t)\) replaced by \((1 + t)^\varepsilon\) for some \(\varepsilon > 0\), is made in [1, 9, 16, 28, 30]. It is used there for the weak coupling limit approximation, in which the control of the approximation is not uniform in \(\mathbb{R}_+\). This weaker hypothesis is also made in [12].

For the CTDRG model, one easily checks that the \(B_j\) satisfy hypotheses \((H_1)\) and \((H_2)\).

We shall now state the third hypothesis. Recall that our purpose is the study of \(\sigma_0(S(t, X))\).

When \(g = 0\), the free evolution is given by:

\[\gamma_t(X) = e^{itH_{mag}}Xe^{-itH_{mag}},\]  

for any \(X \in \mathcal{L}(H_{sp})\). We want to prove that, as \(g\) tends to 0, \(\sigma_0(S(t, X))\) is a small perturbation of the free evolution. This leads us to set:

\[S^{\text{red}}(t, X) = S(t, \gamma_{-\omega}X),\]  

for all \(X \in \mathcal{L}(H_{sp})\) and \(t > 0\).

The evolution of \(S^{\text{red}}(t, X)\) will be approximated by the exponential of some operator \(L\) acting from \(\mathcal{L}(H_{sp})\) into itself. This operator \(L\) will be written using eigenvectors of the map \(\text{ad}H_{mag}\). The operator \(\text{ad}H_{mag}\) acting from \(\mathcal{L}(H_{sp})\) into itself is self adjoint when \(\mathcal{L}(H_{sp})\) is endowed with the Hilbert–Schmidt scalar product denoted here \(<\cdot, \cdot>_{HS}\). Thus, the eigenvalues of \(\text{ad}H_{mag} = [H_{mag}, \cdot]\) are real numbers. We denote by \(\Sigma\) the set of these eigenvalues. For each \(\mu\) in \(\Sigma\), we denote by \(E_\mu\) the corresponding eigenspace and \(\pi_\mu\) stands for the orthogonal projection (for the Hilbert Schmidt scalar product) on \(E_\mu\). Note that, if \(\mu \in \Sigma\) then also \(-\mu \in \Sigma\).

The operator \(L\) acting from \(\mathcal{L}(H_{sp})\) into itself and defining the approximate dynamics is written as:

\[L = \sum_{\mu \in \Sigma} L_\mu,\]  

where for all \(X \in \mathcal{L}(H_{sp})\):

\[L_\mu X = \frac{1}{2} \sum_{jk \leq P} A_{jk}(\mu) [\pi_\mu(S_j), X] \pi_{-\mu}(S_k) - B_{jk}(\mu) \pi_{-\mu}(S_k) [\pi_\mu(S_j), X]\]  

with
\begin{equation}
A_{\lambda}(\mu) = \int_0^\infty e^{i\mu t} \langle B_k, \chi_t B_j \rangle_{H_i} dt, \quad B_{\lambda}(\mu) = \int_0^\infty e^{i\mu t} \langle \chi_t B_j, B_k \rangle_{H_i} dt.
\end{equation}

(2.10)

For operators of the form (2.8) see [14], and in a more particular form (diagonal) but in infinite dimension, see [22]. One finds a similar operator in [12], thus also called Lindblad operator, but being distinct of the one in this article.

Without any other additional hypothesis, we can show that 
\[ \sigma_0(S^{\text{red}}(t,X)) \]
satisfies a differential system of the following form:

\begin{equation}
\frac{d}{dt} \sigma_0(S^{\text{red}}(t,X)) = g^2 \sigma_0(S^{\text{red}}(t,LX)) + R(t,X)
\end{equation}

(2.11)

where \( R(t,X) \) is negligible in some precise sense (see proposition 3.5).

In order to get a control of \( \sigma_0(S^{\text{red}}(t,X)) - e^{it^2L}X \), we are now led to make an assumption on the sign of eigenvalues of \( L \), actually of \( L \) restricted to some specific subspace.

Note that \( L \) maps the subspace \( E_\mu \) into itself. Indeed, observe that, if \( A \in E_\mu \) and \( B \in E_\nu \), then \( AB \in E_{\mu+\nu} \) and \( A^* \in E_{-\nu} \). In particular, \( L \) maps \( E_0^\perp \) (the subspace \( E_0^\perp \) stands for the orthogonal of the kernel \( E_0 \) of \( \text{ad}H_{\text{mag}} \)) and is also the direct sum of the \( E_\mu \) for \( \mu \neq 0 \) into itself. Also note that \( L \) is not self adjoint when \( L(H_{\text{sp}}) \) is endowed with the Hilbert Schmidt scalar product.

We then need the following third assumption.

\((H_3)\) The real parts of the eigenvalues of \( L \) restricted to \( E_0^\perp \) are all negative.

This hypothesis implies the existence of \( K > 0 \) satisfying:

\begin{equation}
|e^{it^2L}| \leq 1, \quad \int_0^\infty |e^{it^2L}X| dt \leq \frac{K}{g^2}|X|, \quad X \in E_0^\perp.
\end{equation}

(2.12)

Theorem 2.3 below shows that, in the CTDRG model, hypothesis \((H_3)\) is almost always satisfied.

We are now ready to state the main result.

**Theorem 2.1.** Under hypotheses \((H_1)(H_2)(H_3)\), there exists \( C > 0 \) independent of \( g \) but depending on \( H_{\text{mag}} \) and on the \( S_j \) and \( T_j \) such that, if \( 0 < g < 1 \) then:

\[ \left| \sigma_0(S^{\text{red}}(t,X)) - e^{it^2L}X \right| \leq C|X|^2|g^2|^2, \quad X \in E_0^\perp. \]

One can also be more precise on the constant \( C \) and give an estimate where the new constant \( C \) only depends on \( N \) (defined below) and on the dimension of \( H_{\text{sp}} \).

**Theorem 2.2.** Assume that hypotheses \((H_1)(H_2)(H_3)\) hold true and suppose \( 0 < g < 1 \). Then, there is \( C > 0 \) depending only on \( N \) and on the dimension of \( H_{\text{sp}} \) such that, for all \( X \in E_0^\perp \):

\begin{equation}
\left| \sigma_0(S^{\text{red}}(t,X)) - e^{it^2L}X \right| \leq Cg^2|X|(1 + K)(1 + (NS^2)^3) \left( 1 + \frac{1}{\rho(H_{\text{mag}})} \right)
\end{equation}

(2.13)
where:

\[
\begin{align*}
N &= \sup_{j,k \leq P} \int_0^\infty (1 + t) |\chi_t B_j, B_k \rangle_{\mathcal{H}_1} | dt, \\
S &= \sup_{j \leq P} |S_j|, \\
\rho(\mathcal{H}_{\text{mag}}) &= \inf \{ |\lambda + \mu|, \lambda, \mu \in \Sigma, \lambda + \mu \neq 0 \}.
\end{align*}
\]

(2.14)

**Theorem 2.3.** Suppose that \( \mathcal{H}_{\text{sp}} = \mathbb{C}^2 \). Assume that the \( S_j \) are the Pauli matrices recalled in (2.4) and that the \( B_j \) are elements of \( \mathcal{H}_1 \) defined in (2.5). Then,\( \mathcal{H}_{\text{mag}} = \beta \sigma_3 \) (\( \beta > 0 \)). The following properties hold true. The operator \( L \), acting from \( \mathcal{L}(\mathcal{H}_{\text{sp}}) \) into itself defined in (2.8)–(2.10), is diagonalizable. The eigenvectors are \( \sigma_1 + i \sigma_2, \sigma_1 - i \sigma_2, \sigma_3 + I, I \) and we denote by \( \nu_+ (\beta) \), \( \nu_- (\beta) \), \( \nu_3 (\beta) \), \( 0 \) the corresponding eigenvalues. The first two eigenvalues are complex conjugate to each other, the third is real and the fourth is zero. In addition:

\[
\text{Re} \, \nu_+ (\beta) = -\frac{1}{3(2\pi)^2} \int_{|k| = 2\beta} |\varphi(k)|^2 |k| dm(k)
\]

(2.15)

where \( m \) is the standard measure on the sphere centered at the origin of radius \( 2\beta \) and:

\[
\text{Re} \, \nu_3 (\beta) = 2 \text{Re} \, \nu_+ (\beta).
\]

As a consequence, if the integral on the above right hand side is not vanishing (hypothesis usually called Fermi golden rule) the hypothesis \( (H_3) \) is verified and theorems 2.1 and 2.2 can then be applied. Note that if the ultraviolet cut-off function \( \varphi \) tends to 1 then the above quantity \( N \) goes to infinity and in order that the approximation stays valid, that is, the right hand side of (2.13) tends to zero, one then requires that the parameter \( g \) goes to zero sufficiently fast.

### 3. Proof of theorems 2.1 and 2.2. First step

The aim of this section is to prove proposition 3.5 showing that \( S^{\text{red}}(t, X) \) satisfies a differential system of the type (2.11) which is close to the GKLS type in the sense that there is a small perturbation on the right hand side of (2.11). We emphasize that proposition 3.5 holds true without hypothesis \( (H_3) \).

In order to get this approximate differential system (2.11), we first derive a differential equation starting from the Heisenberg equation for \( S^{\text{red}}(t, X) \) (proposition 3.1). Let us emphasize that the operator \( \sigma_0 \) does not appear at this stage.

We use the notations of [26] for the creation and annihilation operators \( a^* (V) \) and \( a(V) \) for any \( V \) belonging to the single-photon pure state space \( \mathcal{H}_1 \). In addition, we shall use the following maps:

\[
\begin{align*}
A(t, V) &= e^{iH(\xi)} (a(V) \otimes I) e^{-iH(\xi)} \\
A^*(t, V) &= e^{iH(\xi)} (a^*(V) \otimes I) e^{-iH(\xi)}.
\end{align*}
\]

(3.1)

One knows that ([26]):

\[
a(V) + a^*(V) = \sqrt{2} \Phi_S (V),
\]

(3.3)
for all $V$ in $\mathcal{H}_1$.

**Proposition 3.1.** One has, for all $X \in \mathcal{L}(\mathcal{H}_q)$:

$$
\frac{d}{dt} S^{\text{red}}(t, X) = \frac{ig}{\sqrt{2}} \sum_{j=1}^{p} A^*(t, B_j) S^{\text{red}}(t, [\gamma_j, S_j, X]) + S^{\text{red}}(t, [\gamma_j, S_j, X]) A(t, B_j).
$$

(3.4)

**Proof.** Clearly:

$$
\frac{d}{dt} S(t, \gamma J, X) = \frac{d}{dt} \left[ e^{iH(t)} (I \otimes e^{-it\mathcal{H}_{\text{mag}}}) (I \otimes X) (I \otimes e^{it\mathcal{H}_{\text{mag}}}) \right] e^{-it\mathcal{H}(g)}
$$

Therefore:

$$
[H_{\text{int}}, (I \otimes \gamma J, X)] = \frac{1}{\sqrt{2}} \sum_{j=1}^{p} (a^*(B_j) \otimes I) (I \otimes [S_j, \gamma J, X]) + (I \otimes [S_j, \gamma J, X]) (a(B_j) \otimes I)
$$

and thus:

$$
\frac{d}{dt} S(t, \gamma J, X) = \frac{ig}{\sqrt{2}} \sum_{j=1}^{p} A^*(t, B_j) S(t, [S_j, \gamma J, X]) + S(t, [S_j, \gamma J, X]) A(t, B_j).
$$

Then, one deduces (3.4) using (2.7).

The next proposition gives the time evolution of the observables $a(V) \otimes I$ and $a^*(V) \otimes I$ relying again on the Heisenberg equation.

**Proposition 3.2.** One has:

$$
A(t, \chi J, V) = a(V) \otimes I - i \frac{g}{\sqrt{2}} \sum_{k=1}^{P} \int_{0}^{t} < B_k, \chi J, V > \mathcal{H}_1 S(s, S_k) ds
$$

(3.5)

and

$$
A(t, V) = a(\chi J, V) \otimes I - i \frac{g}{\sqrt{2}} \sum_{k=1}^{P} \int_{0}^{t} < B_k, \chi J, V > \mathcal{H}_1 S(s, S_k) ds
$$

(3.6)

$$
A^*(t, V) = a^*(\chi J, V) \otimes I + i \frac{g}{\sqrt{2}} \sum_{k=1}^{P} \int_{0}^{t} < \chi J, V, B_k > \mathcal{H}_1 S(s, S_k) ds,
$$

(3.7)

for any $V \in \mathcal{H}_1$ and time $t > 0$.

**Proof.** For each $V$ in $\mathcal{H}_1$, we set:

$$
F(t, V) = A(t, \chi J - V) = e^{iH(t)} (a(\chi J - V) \otimes I) e^{-it\mathcal{H}(g)}.
$$

One has simple expressions for the creation and annihilation operators analogous to (2.1), namely:

$$
e^{iH(t)} a(V) e^{-it\mathcal{H}(g)} = a(\chi J, V), \quad e^{iH(t)} a^*(V) e^{-it\mathcal{H}(g)} = a^*(\chi J, V).
$$

(3.8)
Therefore:
\[
F(t, V) = e^{iH(t)}(e^{-iH_{ph} \otimes I}) (a(V) \otimes I)(e^{iH_{ph} \otimes I}) e^{-iH(t)}.
\]
Since the operator \( I \otimes H_{mag} \) commutes with \( a(\chi_{-V}) \otimes I \), we deduce that:
\[
\frac{\partial F}{\partial t}(t, V) = i g e^{iH(t)} [H_{int}, (a(\chi_{-V}) \otimes I)] e^{-iH(t)}.
\]
One has, for each \( W \) in \( \mathcal{H}_1 \):
\[
[H_{int}, (a(W) \otimes I)] = -\frac{1}{\sqrt{2}} \sum_{j=1}^{p} < B_j, W >_{\mathcal{H}_1} (I \otimes S_j).
\]
Consequently, with \( W = \chi_{-V} \):
\[
\frac{\partial F}{\partial t}(t, V) = -i \frac{g}{\sqrt{2}} \sum_{j=1}^{p} < B_j, \chi_{-V} >_{\mathcal{H}_1} e^{iH(t)} (I \otimes S_j) e^{-iH(t)}
\]
\[
= -i \frac{g}{\sqrt{2}} \sum_{j=1}^{p} < B_j, \chi_{-V} >_{\mathcal{H}_1} S(t, S_j).
\]
Since \( F(0, V) = a(V) \otimes I \), we get (3.5). Replacing \( V \) by \( \chi_{1} V \), we obtain (3.6). The proof of (3.7) is similar.

**Proposition 3.3.** For all \( X \in \mathcal{L}(\mathcal{H}_{ph}) \), one has:
\[
\frac{d}{dt} \sigma_0(S^{red}(t, X)) = g^2 \sigma_0(G(t, X))
\]
where:
\[
G(t, X) = \frac{1}{2} \sum_{j,k \leq p} \int_{0}^{t} \left[ < B_k, \chi_{t-s} B_j >_{\mathcal{H}_1} S^{red}(t, [\gamma(J)], X)) S^{red}(s, \gamma(S_k)) - < \chi_{t-s} B_j, B_k >_{\mathcal{H}_1} S^{red}(s, \gamma(S_k)) S^{red}(t, [\gamma(J)], X)) \right] ds.
\]

**Proof.** One applies proposition 3.1, and then proposition 3.2 with \( V = B_j \). From (1.4), one has for all operators \( A \) in \( \mathcal{H}_{th} \otimes \mathcal{H}_{wp} \) and for any \( V \) in \( \mathcal{H}_1 \):
\[
\sigma_0(A(a(V) \otimes I)) = \sigma_0((\sigma^* \otimes I)(V)A) = 0.
\]
Indeed, one sees that \( a(V) \Psi_0 = 0 \) since \( \Psi_0 \) is the vacuum state ([26]). The proposition follows.

The following steps are devoted to get approximations of \( G(t, X) \) defined in (3.10). First, we approximate \( G(t, X) \) by the following expression:
\[
G_{max}(t, X) = \frac{1}{2} \sum_{j,k \leq p} \int_{0}^{t} \left[ < B_k, \chi_{t-s} B_j >_{\mathcal{H}_1} S^{red}(t, [\gamma(J)], X)) S^{red}(t, \gamma(S_k)) - < \chi_{t-s} B_j, B_k >_{\mathcal{H}_1} S^{red}(t, \gamma(S_k)) S^{red}(t, [\gamma(J)], X)) \right] ds.
\]
Note that \( s \) in (3.10) is replaced twice by \( t \) in (3.12).
Proposition 3.4. Under hypotheses (H₁) and (H₂), one gets:

\[ \sigma_0 (G(t, X) - G_{\text{mark}}(t, X)) = T_1(t, X) \]

where:

\[ |T_1(t, X)| \leq C g^2 |X| \sup_{j \leq P} |S_j|^4 \left| \int_0^\infty (1 + t) |< \chi_{i} B_j, B_k >_{\mathcal{H}_1} | dt \right|^2. \]  

Proof. One can write:

\[ G(t, X) - G_{\text{mark}}(t, X) = E_g(t, X) + E_L(t, X) \]

with:

\[ E_g(t, X) = \frac{1}{2} \sum_{j,k \leq P} \int_0^t < B_k, \chi_{t \rightarrow i} B_j >_{\mathcal{H}_1} S^{\text{red}}(t, [\gamma_j(S_j), X]) \left[ S^{\text{red}}(s, \gamma_j S_j) - S^{\text{red}}(t, \gamma_j S_j) \right] ds \]

and with \( E_L(t, X) \) defined similarly with straightforward modifications. For any \( t > 0 \), set:

\[ \Delta_m(t) = \{(s_1, \ldots, s_m) \mid 0 < s_1 < \cdots < s_m < t\}. \]

One then notes that:

\[ E_g(t, X) = -\frac{1}{2} \sum_{j,k \leq P} \int_{\Delta_2(t)} < B_k, \chi_{t \rightarrow s_1} B_j >_{\mathcal{H}_1} S^{\text{red}}(t, [\gamma_j(\gamma_j S_j), X]) \partial_{s_1} S^{\text{red}}(s_2, \gamma_j S_j) ds_1 ds_2. \]

According to proposition 3.1, we have the decomposition:

\[ E_g(t, X) = E'_g(t, X) + E''_g(t, X) \]

where:

\[ E'_g(t, X) = -\frac{ig}{2\sqrt{2}} \sum_{j,k,m \leq P} \int_{\Delta_2(t)} < B_k, \chi_{t \rightarrow s_1} B_j >_{\mathcal{H}_1} S^{\text{red}}(t, [\gamma_j(S_j), X]) \]

\[ \times A^*(s_2, B_m) S^{\text{red}}(s_2, [\gamma_{\gamma_j S_j}, \gamma_{s_1} S_j]) ds_1 ds_2 \]

and where \( E''_g(t, X) \) is similarly defined. One also needs here to carefully take into account that \( A^*(s_2, B_m) \) is not a bounded operator. We use (3.7) in two distinct ways. Firstly:

\[ A^*(s_2, B_m) = a^*(\chi_{s_2} B_m) \otimes I + \frac{i \sqrt{2}}{s_2} \sum_{p=1}^P \int_0^{s_2} < \chi_{s_2 - i} B_m, B_p >_{\mathcal{H}_1} S(s, S_p) ds, \]

and secondly:

\[ A^*(t, \chi_{t \rightarrow s_2} B_m) = a^*(\chi_{s_2} B_m) \otimes I + \frac{i \sqrt{2}}{s_2} \sum_{p=1}^P \int_0^t < \chi_{t \rightarrow i} \chi_{s_2 - i} B_m, B_p >_{\mathcal{H}_1} S(s, S_p) ds. \]

Therefore, we have:

\[ A^*(s_2, B_m) - A^*(t, \chi_{s_2 - i} B_m) = \frac{i \sqrt{2}}{s_2} \sum_{p=1}^P \int_{s_2}^t < \chi_{s_2 - i} B_m, B_p >_{\mathcal{H}_1} S(s, S_p) ds. \]
One notices that:

\[
\left[\operatorname{red}(t, [\gamma(S_j), X]), A^*(t, \chi_{\gamma - t} B_m)\right] = 0,
\]

together with \(\sigma_0(A^*(\chi_{\gamma - t} B_m) \otimes \Pi A) = 0\) for any operator \(A\). As a consequence, one sees that:

\[
E'_R(t, X) = T'_R(t, X) + Z'_R(t, X)
\]

with

\[
|T'_R(t, X)| \leq Cg^2|X| \sup_{j \leq P} |S_j|^4 \sup_{j, k \leq P} \left[ \int_0^\infty (1 + t) |\chi_{B_j, B_k}^\perp_1| dt \right]^2
\]

and

\[
\sigma_0(Z'_R(t, X)) = 0.
\]

All the other terms are considered similarly. The proposition then follows. \(\square\)

We are now reaching system (2.11). Note that we shall see in the next section why some terms in the righthand side of (3.15) are negligible.

**Proposition 3.5.** Assume that hypotheses \((H_1)\)\((H_2)\) are satisfied. Then:

\[
\frac{d}{dt} \sigma_0(\operatorname{red}(t, X)) = g^2 \sigma_0(\operatorname{red}(t, A(t)X)) + T_1(t, X) + T_2(t, X),
\]

where:

\[
A(t)X = \sum_{(\mu, \nu) \in \Sigma} \varepsilon^{(\mu + \nu)} L_{\mu, \nu}X
\]

and:

\[
L_{\mu, \nu}X = \frac{1}{2} \sum_k \int_0^\infty e^{-s} \left[ <\chi_{B_k, B_j}^{\perp}_1, \pi_{\mu} S_j, X> \pi_{\nu} S_k - <\chi_{B_j, B_k}^{\perp}_1, \pi_{\nu} S_j, X> \pi_{\mu} S_k \right] ds.
\]

In addition, the three following estimates hold true, where \(N\) and \(S\) are defined in (2.14):

\[
|T_1(t, X)| \leq Cg^2|X| S^4 N^2,
\]

\[
\int_0^\infty |T_2(t, X)| dt \leq Cg^2|X| S^2 N, \quad |T_2(t, X)| \leq Cg^2|X| S^2 N.
\]

**Proof of proposition 3.5.** We use (3.9) together with the approximation of \(G(t, X)\) by \(G_{\operatorname{mark}}(t, X)\). One then can write:

\[
\frac{d}{dt} \sigma_0(\operatorname{red}(t, X)) = g^2 \sigma_0(G_{\operatorname{mark}}(t, X)) + T_1(t, X)
\]
where $T_1(t, X)$ denotes the operator in proposition 3.4. We observe that $S_{\text{red}}^\text{col}(t, A)S_{\text{col}}^\text{red}(t, B) = S_{\text{col}}^\text{red}(t, AB)$ for any operators $A$ and $B$ in $\mathcal{L}(\mathcal{H}_{\text{sp}})$. Consequently:

$$G_{\text{mark}}(t, X) = S_{\text{red}}^\text{col}(t, L(t)X)$$

where:

$$L(t)X = \frac{1}{2} \sum_{j,k \in \mathbb{P}'} \int_0^t \left[ < B_k, \chi_{t-s} B_j >_{\mathcal{H}_1} [\gamma_j(S_j), X] \gamma_k(S_k) - < \chi_{t-s} B_j, B_k >_{\mathcal{H}_1} \gamma_j(S_k) [\gamma_j(S_j), X] \right] ds.$$

One then deduces (3.15) where:

$$T_2(t, X) = g^2 \sigma_0(S_{\text{red}}((L(t) - A(t))X)).$$

where $A(t)$ is defined in (3.16) and (3.17). We now transform the above expression $L(t)$ in the aim that it becomes closer to $A(t)$. To this end, first observe that, for every $\mu \in \Sigma$ and any operator $S_j$ in the eigenspace of $\text{ad}H_{\text{mag}}$ corresponding to the eigenvalue $\mu$, we have:

$$\gamma_j(T) = e^{i\mu T},$$

and thus:

$$\gamma_j(S_j) = \sum_{\mu \in \Sigma} e^{i\mu} \pi_\mu S_j.$$

One then deduces that:

$$L(t)X = \frac{1}{2} \sum_{j,k,\mu,\nu} \int_0^t \left[ e^{i(\nu + \mu)} \left[ < B_k, \chi_{t-s} B_j >_{\mathcal{H}_1} [\pi_\mu(S_j), X] \pi_\nu(S_k) - < \chi_{t-s} B_j, B_k >_{\mathcal{H}_1} \pi_\mu(S_k) [\pi_\mu(S_j), X] \right] ds \right. \right.$$

$$= \frac{1}{2} \sum_{j,k,\mu,\nu} \int_0^t \left[ e^{-i\nu} \left[ < B_k, \chi_{t-s} B_j >_{\mathcal{H}_1} [\pi_\mu(S_j), X] \pi_\nu(S_k) - < \chi_{t-s} B_j, B_k >_{\mathcal{H}_1} \pi_\mu(S_k) [\pi_\mu(S_j), X] \right] ds \right. \right.$$

Consequently, one has with $A(t)$ denoting the operator defined in (3.16) and for all $t > 0$:

$$|L(t) - A(t)| \leq C \sup_j |S_j|^2 \sup_{j,k} \int_0^\infty | < B_k, \chi_{s} B_j >_{\mathcal{H}_1} | ds.$$

Therefore, the following equality holds:

$$\int_0^\infty |L(t) - A(t)| dt \leq C \sup_j |S_j|^2 \sup_{j,k} \int_0^\infty s | < B_k, \chi_{s} B_j >_{\mathcal{H}_1} | ds.$$

One then obtains proposition 3.5.  \(\square\)
4. Proof of theorems 2.1 and 2.2. Second step

We here apply the Duhamel principle. To this end, we now consider the two sides of (3.15) as two mappings taking values into \( \mathcal{L}(\mathcal{H}_{sp}) \). Thus, we define a function \( U(t) \) taking values in \( \mathcal{L}(\mathcal{L}(\mathcal{H}_{sp})) \) by:

\[
U(t)X = \sigma_0(S^{red}(t, X)), \quad X \in \mathcal{L}(\mathcal{H}_{sp}).
\]

In order to derive a system satisfied by \( U(t) \), one notes from (2.8) and (3.17) that:

\[
L = \sum_{\mu \in \Sigma} L_{\mu} - \mu.
\]

Then, the system (3.15) can be written as:

\[
\frac{dU}{dt}(t) = g^2 U(t)L + \sum_{j=1}^{2} R_j(t) + \sum_{\mu + \nu \neq 0} R_{\mu \nu}(t), \quad U(0) = I
\]

with

\[
R_j(t)X = T_j(t, X), \quad X \in \mathcal{L}(\mathcal{H}_{sp}), \quad j = 1, 2,
\]

\[
R_{\mu \nu}(t) = g^2 e^{i(\mu + \nu) \cdot t} U(t) L_{\mu \nu}.
\]

According to the Duhamel principle:

\[
U(t) = e^{g^2 t L} + \sum_{j=1}^{2} v_j(t) + \sum_{\mu + \nu \neq 0} v_{\mu \nu}(t),
\]

\[
v_j(t) = \int_0^t R_j(s)e^{\frac{1}{2}(t-s)g^2 L} ds, \quad j = 1, 2, \quad v_{\mu \nu}(t) = \int_0^t R_{\mu \nu}(s)e^{\frac{1}{2}(t-s)g^2 L} ds.
\]

In view of (2.12), one has on \( E_0^\perp \):

\[
|v_1(t)|_{\mathcal{L}(E_0^\perp, \mathcal{L}(\mathcal{H}_{sp}))} \leq \frac{K}{g} \sup_{s > 0} |R_1(s)|,
\]

\[
|v_2(t)|_{\mathcal{L}(E_0^\perp, \mathcal{L}(\mathcal{H}_{sp}))} \leq \int_0^\infty |R_2(s)| ds,
\]

where \( \mathcal{L}(E_0^\perp, \mathcal{L}(\mathcal{H}_{sp})) \) denotes the set of bounded operators from \( E_0^\perp \) into \( \mathcal{L}(\mathcal{H}_{sp}) \). One deduces from (3.18) and (3.19) that:

\[
|v_1(t)|_{\mathcal{L}(E_0^\perp, \mathcal{L}(\mathcal{H}_{sp}))} + |v_2(t)|_{\mathcal{L}(E, \mathcal{L}(\mathcal{H}_{sp}))} \leq Kg^2(NS^2)^2 + Cg^2(NS^2).
\]

Thus:

\[
v_{\mu \nu}(t) = g^2 \int_0^t e^{i(\mu + \nu) \cdot s} U(s) L_{\mu \nu} e^{\frac{1}{2}(t-s)g^2 L} ds.
\]

and:

\[
i(\mu + \nu)v_{\mu \nu}(t) = g^2 \int_0^t U(s) L_{\mu \nu} e^{\frac{1}{2}(t-s)g^2 L} \frac{\partial}{\partial s} e^{i(\mu + \nu) \cdot s} ds.
\]
Consequently, for all $X$ linear combination of the $\sigma_j$ and for $\mu$ in $\{2\beta, 0, -2\beta\}$, one has:

$$L_\mu(X) = \frac{1}{2} \sum_{j=1}^{3} [A(\mu) [\pi_\mu(\sigma_j), X] \pi_{-\mu}(\sigma_j) - B(\mu) \pi_{-\mu}(\sigma_j) [\pi_\mu(\sigma_j), X]].$$

Also:

$$\begin{align*}
\pi_{2\beta}(\sigma_1) &= \frac{1}{2}(\sigma_1 + i\sigma_2) \\
\pi_0(\sigma_1) &= 0 \\
\pi_{-2\beta}(\sigma_1) &= \frac{1}{2}(\sigma_1 - i\sigma_2) \\
\pi_{2\beta}(\sigma_2) &= -\frac{i}{2}(\sigma_1 + i\sigma_2) \\
\pi_0(\sigma_2) &= 0 \\
\pi_{-2\beta}(\sigma_2) &= \frac{i}{2}(\sigma_1 - i\sigma_2) \\
\pi_{2\beta}(\sigma_3) &= 0 \\
\pi_0(\sigma_3) &= \sigma_3 \\
\pi_{-2\beta}(\sigma_3) &= 0.
\end{align*}$$
The standard formulas for Pauli matrices and in particular \( \sigma_1 + i \sigma_2, \sigma_1 - i \sigma_2 = 4 \sigma_3 \) show that:

\[
L(\sigma_1 \pm i \sigma_2) = \nu_{\pm}(\beta)(\sigma_1 \pm i \sigma_2)
\]

with:

\[
\nu_+(\beta) = - (A(0) + B(0) + A(-2\beta) + B(-2\beta))
\]

\[
\nu_-(\beta) = - (A(2\beta) + B(2\beta) + A(0) + B(0))
\]

and also:

\[
L(\sigma_3) = \nu_3(\beta)(\sigma_3 + I) + (A(-2\beta) + B(2\beta))(I - \sigma_3)
\]

\[
\nu_3(\beta) = (A(2\beta) + B(-2\beta)).
\]

One notes that \( B(-\mu) \) is the complex conjugate of \( A(\mu) \). One then deduces that \( \nu_-(\beta) \) is the conjugate of \( \nu_+(\beta) \) and that \( \nu_3(\beta) \) is real. Combining this remark with proposition 5.1 below, one also gets that for all \( \mu \geq 0 \), \( A(-\mu) + B(\mu) = 0 \), and then \( \text{Re}A(-\mu) = 0 \). Consequently:

\[
\nu_+(\beta) = - (A(-2\beta) + B(-2\beta))
\]

and

\[
L(\sigma_3) = \nu_3(\beta)(\sigma_3 + I).
\]

Since \( L(I) = 0 \), one sees that \( \sigma_3 + I \) is an eigenvector with eigenvalue \( \nu_3(\beta) \). The sign of the real parts of the eigenvalues then follows from proposition 5.1. Since \( \text{Re}A(-2\beta) = 0 \), one has:

\[
\text{Re} \nu_+(\beta) = - \text{Re} B(-2\beta) = - \lim_{\varepsilon \to 0^+} \frac{2}{3(2\pi)^3} \text{Re} \int_{\mathbb{R}^3 \times \mathbb{R}^+} e^{i(k - 2\beta_\mu)} |\varphi(k)|^2 |k| dk
\]

and similarly \( \text{Re} \nu_3(\beta) = - 2 \text{Re} B(-2\beta) = 2 \text{Re} \nu_+(\beta) \). According to proposition 5.1, one gets (2.15).

\[
\square
\]

**Proposition 5.1.** Let \( F \) be a smooth function in \( L^1(\mathbb{R}^3) \) taking real values and such that \( F(k)/|k|^2 \) is also integrable. Then, if \( \mu > 0 \):

\[
\lim_{\varepsilon \to 0^+} \text{Re} \int_{\mathbb{R}^3 \times \mathbb{R}^+} e^{i(k - \mu) - \varepsilon t} F(k) dk dt = \pi \int_{|k| = \mu} F(k) dm(k)
\]

and if \( \mu \leq 0 \) then this limit is zero.

**Proof.** It suffices to notice that, for all \( \varepsilon > 0 \):

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^+} e^{i(k - \mu) - \varepsilon t} F(k) dk dt = \int_{\mathbb{R}^3} \frac{\varepsilon + i(|k| - \mu)}{(|k| - \mu)^2 + \varepsilon^2} F(k) dk.
\]

\[
\square
\]
6. Summary

We have proved for the general spin boson model that the time evolution of the expectation of any spin observable initially in a photon vacuum state is approximated by a GKLS type dynamics. The main assumptions are \((H_1)(H_2)(H_3)\) with \((H_3)\) being the most important. This result is directly applied to a model of nuclear magnetic resonance (NMR) essentially showing that a Fermi golden rule holds true implying the validity of assumption \((H_3)\). As a consequence, we obtain a GKLS type approximation of the spin dynamics for this model in NMR showing strong similarities with the spin relaxation. The error is of the order of the square of the coupling constant and the approximation is valid uniformly on positive times. This is effectuated in the zero temperature case. We have considered here the case of one atom and we expect to obtain the molecular case in a subsequent work.

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