Laplacian Immanantal polynomials and the GTS poset on Trees

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Abstract: Let $T$ be a tree on $n$ vertices with Laplacian $L_T$ and let $GTS_n$ be the generalized tree shift poset on the set of unlabelled trees on $n$ vertices. Inequalities are known for coefficients of the characteristic polynomial of $L_T$ as we go up the poset $GTS_n$. In this work, we generalize these inequalities to the $q$-Laplacian $L_q^0$ of $T$ and to the coefficients of all immanantal polynomials.

Keywords: Tree, GTS$_n$ poset, $q$-Laplacian, immanantal polynomial

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1 Introduction

Csikvári in [10] defined a poset on the set of unlabelled trees with $n$ vertices that we denote in this paper as $GTS_n$. Among other results, he showed that going up on $GTS_n$ has the following effect: the coefficients of the characteristic polynomial of the Laplacian $L_T$ of $T$ decrease in absolute value. In this paper, we prove the following more general result about immanantal polynomials of the $q$-Laplacian matrix of trees
**Theorem 1** Let $T_1$ and $T_2$ be trees with $n$ vertices and let $T_2$ cover $T_1$ in $GTS_n$. Let $L^q_{T_1}$ and $L^q_{T_2}$ be the $q$-Laplacians of $T_1$ and $T_2$ respectively. For $\lambda \vdash n$, let

$$f_{\lambda}^{L^q_{T_1}}(x) = d_\lambda(xI - L^q_{T_1}) = \sum_{r=0}^{n} (-1)^r c_{\lambda,r}^{L^q_{T_1}}(q)x^{n-r}$$

and

$$f_{\lambda}^{L^q_{T_2}}(x) = d_\lambda(xI - L^q_{T_2}) = \sum_{r=0}^{n} (-1)^r c_{\lambda,r}^{L^q_{T_2}}(q)x^{n-r}.$$

Then, for all $\lambda \vdash n$ and for all $0 \leq r \leq n$, we assert that $c_{\lambda,r}^{L^q_{T_1}}(q) - c_{\lambda,r}^{L^q_{T_2}}(q) \in \mathbb{R}^+[q^2]$.

For a positive integer $n$, let $[n] = \{1, 2, \ldots, n\}$. Let $\mathfrak{S}_n$ be the group of permutations of $[n]$. Let $\chi_\lambda$ be the irreducible character of the $\mathfrak{S}_n$ over $\mathbb{C}$ indexed by the partition $\lambda$ of $n$. We refer the reader to the book by Sagan [26] as a reference for results on representation theory that we use in this work. We denote partitions $\lambda$ of $n$ as $\lambda \vdash n$. This means we have $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_l$ where $\lambda_i \in \mathbb{Z}$ for all $i$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$ and with $\sum_{i=1}^{l} \lambda_i = n$. We also write partitions using the exponential notation, with multiplicities of parts written as exponents. Since characters of $\mathfrak{S}_n$ are integer valued, we think of $\chi_\lambda$ as a function $\chi_\lambda : \mathfrak{S}_n \to \mathbb{Z}$. Let $\lambda \vdash n$ and let $A = (a_{i,j})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix. Define its immanant as

$$d_\lambda(A) = \sum_{\psi \in \mathfrak{S}_n} \chi_\lambda(\psi) \prod_{i=1}^{n} a_{i,\psi_i}.$$ 

It is well known that $d_{1^n}(A) = \det(A)$ and $d_n(A) = \text{perm}(A)$ where $\text{perm}(A)$ is the permanent of $A$.

For an $n \times n$ matrix $A$, define $f_{\lambda}^A(x) = d_\lambda(xI - A)$. The polynomial $f_{\lambda}^A(x)$ is called the immanantal polynomial of $A$ corresponding to $\lambda \vdash n$. Thus, in this notation, $f_{\lambda}^A(x)$ is the characteristic polynomial of $A$. Let $T$ be a tree with $n$ vertices with Laplacian matrix $L_T$ and define

$$f_{\lambda}^{L_T}(x) = d_\lambda(xI - L_T) = \sum_{r=0}^{n} (-1)^r c_{\lambda,r}^{L_T}x^{n-r}$$

(1)

where the $c_{\lambda,r}^{L_T}$'s are coefficients of the Laplacian immanantal polynomial of $T$ in absolute value. Immanantal polynomials were studied by Merris [21] where the Laplacian immanantal polynomial corresponding to the partition $\lambda = 2, 1^{n-2}$ (also called the second immanantal polynomial) of a tree $T$ was shown to have connections with the centroid of $T$. Botti and Merris [6] showed that almost all trees share a complete set of Laplacian immanantal polynomials. When $\lambda = 1^n$, Gutman and Pavlovic [16] conjectured the following inequality which was proved by Gutman and Zhou [17] and independently by Mohar [24].

**Theorem 2 (Gutman and Zhou, Mohar)** Let $T$ be any tree on $n$ vertices and let $S_n$ and $P_n$ be the star and the path trees on $n$ vertices respectively. Then, for $0 \leq r \leq n$, we have

$$c_{1^n,r}^{L_{S_n}} \leq c_{1^n,r}^{L_{T}} \leq c_{1^n,r}^{L_{P_n}}.$$

Thus, in absolute value, any tree $T$ has coefficients of its Laplacian characteristic polynomial sandwiched between the corresponding coefficients of the star and the path trees. Mohar actually
proves stronger inequalities than this result, see Csikvári [11, Section 10] for information on Mohar’s stronger results. Much earlier, Chan, Lam and Yeo in their preprint [9], proved the following.

**Theorem 3 (Chan, Lam and Yeo)** Let $T$ be any tree on $n$ vertices with Laplacian $L_T$ and let $S_n$ and $P_n$ be the star and the path trees on $n$ vertices respectively. Then, for all $\lambda \vdash n$ and $0 \leq r \leq n$,

$$c_{\lambda,r}^{L_S} \leq c_{\lambda,r}^{L_T} \leq c_{\lambda,r}^{L_P}. \quad (2)$$

In this work, we consider the $q$-Laplacian matrix $L_T^q$ of a tree $T$ on $n$ vertices. It is defined as $L_T^q = I + q^2(D - I) - qA$ where $q$ is a variable, $D$ is the diagonal matrix with degrees on the diagonal and $A$ is the adjacency matrix of $T$. $L_T^q$ can be defined for arbitrary graphs $G$ analogously and it is clear that when $q = 1$, $L_T^q = L_G$. The matrix $L_T^q$ has occurred previously in connection with the Ihara-Selberg zeta function of $G$ (see Bass [5] and Foata and Zeilberger [13]). For trees, $L_T^q$ has connections with the inverse of $T$’s exponential distance matrix (see Bapat, Lal and Pati [2]). As done in (1), define

$$f_{\lambda}^{L_T^q}(x) = d_{\lambda}(xI - L_T^q) = \sum_{r=0}^{n} (-1)^r c_{\lambda,r}^{L_T^q}(q)x^{n-r}. \quad (3)$$

We consider the following counterpart of inequalities like (2) when each coefficient is a polynomial in the variable $q$: we want the difference $c_{\lambda,r}^{L_T^q}(q) - c_{\lambda,r}^{L_S}(q) \in \mathbb{R}^+[q]$. That is, the difference polynomial has only positive coefficients. This is the standard way to get $q$-analogue of inequalities. Similarly, we want $c_{\lambda,r}^{L_P}(q) - c_{\lambda,r}^{L_T^q}(q) \in \mathbb{R}^+[q]$.

We mention a few lines about our proof of Theorem 1. In [11, Theorem 5.1], Csikvári gives a “General Lemma” from which he infers properties about polynomials associated to trees. In that lemma, the following crucial property is needed when dealing with characteristic polynomials of matrices. Let $M = A \oplus B$ be an $n \times n$ matrix that can be written as a direct sum of two square matrices. Then, clearly $\det(M) = \det(A) \det(B)$. This property is sadly not true for other immanants.

That is, $d_{\lambda}(M) \neq d_{\lambda}(A)d_{\lambda}(B)$ (indeed, the definition of $d_{\lambda}(A)$ is not clear when $\lambda \vdash n$ and $A$ is an $m \times m$ matrix with $m < n$). We thus combinatorialise the immanant as done by Chan, Lam and Yeo [9] and express the immanantal polynomial in terms of matchings and vertex orientations.

Section 2 gives preliminaries on the $GTS_n$ poset and Section 3 gives the necessary background on $B$-matchings, $B$-vertex orientations and their connection to coefficients of immanantal polynomials. We give our proof of Theorem 1 in Section 4 and draw several corollaries in Sections 5, 6 and 7 involving the $q^2$-analogue of vertex moments in a tree, $q,t$-Laplacian matrices which include the Hermitian Laplacian of $T$ and $T$’s exponential distance matrices.

## 2 The poset $GTS_n$

Though Csikvári in [10] defined the poset on unlabelled trees with $n$ vertices, we will label the vertices of the trees according to some convention (see Remark 16). We recall the definition of this poset.
Definition 4 Let $T_1$ be a tree on $n$ vertices and $x, y$ be two vertices of $T_1$. Let $P_{x,y}$ be the unique path in $T_1$ between $x$ and $y$. Assume that $x$ and $y$ are such that all the interior vertices (if they exist) on $P_{x,y}$ have degree 2. Let $z$ be the neighbour of $y$ on the path $P_{x,y}$. Consider the tree $T_2$ obtained by moving all neighbours of $y$ except $z$ to the vertex $x$. This is illustrated in Figure 1. This move helps us to partially order the set of unlabeled trees on $n$ vertices. We denote this poset on trees with $n$ vertices as $GTS_n$. We say $T_2$ is above $T_1$ in $GTS_n$ or that $T_1$ is below $T_2$ in $GTS_n$ and denote it as $T_2 \geq_{GTS_n} T_1$. The poset $GTS_6$ is illustrated in Figure 2.

If $T_2 \geq_{GTS_n} T_1$ and there is no tree $T$ with $T \neq T_1, T_2$ such that $T_2 \geq_{GTS_n} T \geq_{GTS_n} T_1$, then we say $T_2$ covers $T_1$ (see Figure 1). If either $x$ or $y$ is a leaf vertex in $T_1$, then it is easy to check that $T_2$ is isomorphic to $T_1$. If neither $x$ nor $y$ is a leaf in $T_1$, then $T_2$ is said to be obtained from $T_1$ by a proper generalized tree shift (PGTS henceforth). Clearly, if $T_2$ is obtained by a PGTS from $T_1$, then, the number of leaf vertices of $T_2$ is one more than the number of leaf vertices of $T_1$. Csikvári in [10] showed the following.

Lemma 5 (Csikvári) Every tree $T$ with $n$ vertices other than the path, lies above some other tree $T'$ on $GTS_n$. The star tree on $n$ vertices is the maximal element and the path tree on $n$ vertices is the minimal element of $GTS_n$. 

Figure 2: The poset $GTS_6$ on trees with 6 vertices.
\section{B-matchings and B-vertex orientations}

As done in earlier work \cite{25}, we use matchings in $T$ to index terms that arise in the computation of the immanant $d_\lambda (\mathcal{L}_T^q)$. A dual concept of vertex orientations was used to get a near positive expression for immanants of $\mathcal{L}_T^q$.

In this work, we need to find $f_{\lambda}^{c_B}(x) = d_\lambda (xI - \mathcal{L}_T^q)$. As done by Chan, Lam and Yeo \cite{9}, we index terms that occur in the computation of $f_{\lambda}^{c_B}(x)$ by partial matchings that we term as $B$-matchings. Let $T$ have vertex set $V$ and edge set $E$. Let $B \subseteq V$ with $|B| = r$ and let $F_B$ be the forest induced by $T$ on the set $B$. A $B$-matching of $T$ is a subset $M \subseteq E(F_B)$ of edges of $F_B$ such that each vertex $v \in B$ is adjacent to at most one edge in $M$. If the number of edges in $M$ equals $j$, then $M$ is called a $j$-sized $B$-matching in $T$. Let $\mathcal{M}_j(B)$ denote the set of $j$-sized $B$-matchings in $T$. For vertex $v$, we denote its degree $\deg_T(v)$ in $T$ alternatively as $d_v$. For $M \in \mathcal{M}_j(B)$, define a polynomial weight $\text{wt}_{B,M}(q) = q^{2j} \prod_{v \in B-M} [1 + q^2(d_v - 1)]$. Define

$$m_{B,j}(q) = \sum_{M \in \mathcal{M}_j(B)} \text{wt}_{B,M}(q) \quad \text{and} \quad m_{r,j}(q) = \sum_{B \subseteq V, |B| = r} m_{B,j}(q).$$

Define $\chi_\lambda(j)$ to be the character $\chi_\lambda(\cdot)$ evaluated at such a permutation with cycle type $2^j, 1^{n-2j}$. The following lemma is straightforward from the definition of immanants.

\textbf{Lemma 6} Let $T$ be a tree on vertex set $[n]$ with $q$-Laplacian $\mathcal{L}_T^q$. Let $\lambda \vdash n$ and let $0 \leq r \leq n$. Then, the coefficient $c_{\lambda,r}^q$ as defined in \cite{3} equals

$$c_{\lambda,r}^q(q) = \sum_{j=0}^{[r/2]} \chi_\lambda(j)m_{r,j}(q).$$

\textbf{Proof:} Let $B \subseteq [n]$ with $|B| = r$. Then, clearly $c_{\lambda,r}^q(q) = d_\lambda \left[ \begin{array}{cc} \mathcal{L}_T^{q|[B]|B} & 0 \\ 0 & I \end{array} \right]$, where $\mathcal{L}_T^{q|[B]|B}$ is the sub-matrix of $\mathcal{L}_T^q$ induced on the rows and columns with indices in the set $B$ and $I$ is the $n - r \times n - r$ identity matrix. Further, it is clear that $c_{\lambda,r}^q(q) = \sum_{B \subseteq [n], |B| = r} c_{\lambda,B}^q(q)$.

Note that there is no cycle in $T$, and hence in the forest $F_B$. Thus, each permutation $\psi \in \mathfrak{S}_n$ which in cycle notation has a cycle of length strictly greater than 2, will satisfy $\prod_{i=1}^{\ell_i} \ell_i \psi_i = 0$. Therefore, only permutations $\psi \in \mathfrak{S}_n$ which fix the set $[n] - B$ and have cycle-type $2^j, 1^{n-2j}$ contribute to $c_{\lambda,B}^q(q)$. It is easy to see that such permutations can be identified with $j$-sized $B$-matchings in $F_B$ and that this correspondence is reversible.

Recall $\mathcal{M}_j(B)$ is the set of $j$-sized $B$ matchings in $T$. Clearly, the contribution to $c_{\lambda,B}^q(q)$ from permutations which fix $[n] - B$ and have cycle-type $2^j, 1^{n-2j}$ is $\chi_\lambda(j)m_{B,j}(q)$. Thus, we see that

$$c_{\lambda,B}^q(q) = \sum_{j=0}^{[r/2]} \chi_\lambda(j)m_{B,j}(q).$$

(4)

Summing over various $B$’s of size $r$ completes the proof. \qed
3.1 \textit{B}-vertex orientations

As done by Chan, Lam and Yeo \cite{9}, we next express coefficients of the immanantal polynomial as a sum of almost positive summands where the summands are indexed by partial vertex orientations that we term as \textit{B}-vertex orientations.

Let $T$ be a tree with vertex set $V = [n]$. For $B \subseteq [n]$, we orient each vertex $v \in B$ to one of its neighbours (which may or may not be in $B$). Such vertex orientations are termed as \textit{B}-vertex orientations. Let $O$ be a \textit{B}-vertex orientation. Each $v \in B$ has $d_v$ orientation choices. We depict the orientation $O$ in pictures by drawing an arrow on the edge from $v$ to its oriented neighbour and directing the arrow away from $v$. We do not distinguish between $O$ and its picture from now on. In $O$, edges thus get arrows and there may be edges which have two arrows, one in each direction (see Figures 4, 6 and 7 for examples). We call such edges as \textit{bidirected arcs} and let $\text{bidir}(O)$ denote the set of bidirected arcs in $O$. We extend this notation to vertices $v \in B$ and say $v \in \text{bidir}(O)$ if $\{u, v\} \in \text{bidir}(O)$ for some $u \in B$. We also say $v \in B$ is free in $O$ if $v \in B - \text{bidir}(O)$ and denote by $\text{free}(O)$ the set of free vertices of $O$.

In $T$, let $\mathcal{O}_{B,i}^T$ be the set of $B$-orientations $O$, such that $O$ has $i$ bidirected arcs. We need to separate the case $B = V$ from the cases $B \neq V$. First, let $B \neq V$. For such a $B \subseteq V$, let $m = \min_{v \in [n] - B} v$ be the minimum numbered vertex outside $B$ and let $O \in \mathcal{O}_{B,i}^T$. For each $v \in \text{free}(O)$, as there is a unique path from $v$ to $m$ in $T$, we can tell if $v$ is oriented “towards” $m$ or if $v$ is oriented “away from” $m$. Formally, for $O \in \mathcal{O}_{B,i}^T$, define a 0/1 function $\text{away} : \text{free}(O) \to \{0, 1\}$ by

$$\text{away}(v) = \begin{cases} 1 & \text{if } v \text{ is oriented away from } m, \\ 0 & \text{if } v \text{ is oriented towards } m. \end{cases}$$

For each $O \in \mathcal{O}_{B,i}^T$ assign the following non-negative integer:

$$\text{Aw}^T_{B}(O) = 2i + 2 \sum_{v \in \text{free}(O)} \text{away}(v).$$

Define the generating function of the statistic $\text{Aw}^T_{B}(\cdot)$ in the variable $q$ as follows:

$$a^T_{B,i}(q) = \sum_{O \in \mathcal{O}_{B,i}^T} q^{\text{Aw}^T_{B}(O)}, \quad (5)$$

$$a^T_{r,i}(q) = \sum_{B \subseteq V, |B| = r} a^T_{B,i}(q) = \sum_{B \subseteq V, |B| = r} \sum_{O \in \mathcal{O}_{B,i}^T} q^{\text{Aw}^T_{B}(O)}. \quad (6)$$

**Example 7** Let $T_2$ be the tree given in Figure 3 and let $B = \{2, 4, 6, 7, 8\}$ with $|B| = r = 5$. Below we give $a^T_{B,i}(q)$ for $i$ from 0 to $\lfloor r/2 \rfloor$.

| $i$ | 0 | 1 | 2 |
|-----|---|---|---|
| $a^T_{B,i}(q)$ | $1 + 2q^2 + q^4$ | $q^2(1 + 2q^2 + q^4)$ | 0 |
Remark 8 For any tree $T$ and all $r, j$, it is easy to see from the definitions that both $m_{r,j}(q)$ and $a_{r,i}^T(q)$ are polynomials in $q^2$.

Chan, Lam and Yeo in [9] showed for the Laplacian $L_T$ of a tree $T$, a relation involving numerical counterparts of $m_{B,j}(q)$’s and $a_{r,i}^T(q)$’s. Chan and Lam [8] had already proved this identity for the special case when $B = [n]$. Earlier, we had in [25, Theorem 11] obtained a $q$-analogue of this identity when $B = [n]$. There, care had to be taken to define $a_{[n],0}^T(q) = 1 - q^2$. We give a $q$-analogue below in Lemma 9 when $B$ can be an arbitrary subset. In [25], since $B = [n]$, there was no vertex outside $B$ and hence $m$ could not be defined. There, the lexicographically minimum edge of the matching $M$ was used in place of $m$. It is easy to see that we could have used the lexicographically minimum edge of $M$ when $B \neq [n]$ as well. From now onwards, we are free from this restriction $B \neq [n]$. Since the proof is identical to that of [25, Theorem 11], we omit it and merely state the result.

Lemma 9 Let $T$ be a tree with vertex set $[n]$ and $B$ be an $r$-subset of $[n]$. Then,

$$m_{B,j}(q) = \sum_{i=j}^{\lfloor r/2 \rfloor} \binom{i}{j} a_{B,i}^T(q).$$

Moreover, $m_{r,j}(q) = \sum_{i=j}^{\lfloor r/2 \rfloor} \binom{i}{j} a_{r,i}^T(q)$.

Chan and Lam in [7] showed the following non-negativity result on characters summed with binomial coefficients as weights. Let $n \geq 2$ and let $\lambda \vdash n$. Recall $\chi_\lambda(j)$ is the character $\chi_\lambda$ evaluated at a permutation with cycle type $2^j, 1^{n-2j}$.

Lemma 10 (Chan and Lam) Let $\lambda \vdash n$ and let $\chi_\lambda(j)$ be as defined above. Let $0 \leq i \leq \lfloor n/2 \rfloor$. Then $\sum_{j=0}^{i} \chi_\lambda(j) \binom{i}{j} = \alpha_{\lambda,i} 2^i$, where $\alpha_{\lambda,i} \geq 0$. Further, if $\lambda = k, 1^{n-k}$, then $\alpha_{\lambda,i} = \binom{n-i-1}{k-i-1}$.

Combining Lemmas 9 and 10 with Lemma 6 gives us the following Corollary whose proof we omit. This gives an interpretation of the coefficient $c_{\lambda,r}^L(q)$ in the immanantal polynomial as a functions of the $a_{r,i}^T(q)$’s. Since all the $a_{r,i}^T(q)$’s except $a_{[n],0}^T(q)$ have positive coefficients, this is an almost positive expression.

Corollary 11 For $0 \leq r \leq n$, the coefficient of the immanantal polynomial of $L_T^r$ in absolute value is given by

$$c_{\lambda,r}^L(q) = \sum_{i=0}^{\lfloor r/2 \rfloor} \alpha_{\lambda,i} 2^i a_{r,i}^T(q),$$

where $\alpha_{\lambda,i} \geq 0$, $\forall \lambda \vdash n, i$.

Combining (4), Lemmas 10 and 9 gives us another corollary when the partition is $\lambda = 1^n$, which we again merely state.

Corollary 12 When $\lambda = 1^n$, we have $\alpha_{\lambda,i} = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$. Further, let $B \subseteq [n]$ with $|B| = r$.

Then,

$$\det(L_T^r[B|B]) = a_{B,0}^T(q).$$

Moreover, $c_{1^n,r}^L(q) = a_{r,0}^T(q)$. 

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Remark 13 Let tree $T$ have vertex set $[n]$ and let $B \subseteq [n]$ with $|B| = n-1$. Then, for all $q \in \mathbb{R}$, $a_{B,0}^T(q) = 1$. This implies that $a_{n-1,0}^T(q) = n$. 

Let $B \subseteq [n]$ with $|B| = r$. Let $L_T^B/B|B|$ denote the $r \times r$ submatrix of $L_T^B$ induced on the rows and columns indexed by $B$. From Corollary 12 we get $\det(L_T^B/B|B)) \geq 0$ when $B \neq [n]$. When $B = [n]$, Bapat, Lal and Pati [2] have shown that $\det(L_T^B) = 1 - q^2$. As remarked in Section 1 when $q \in \mathbb{R}$ with $|q| \leq 1$, the matrix $L_T^B$ is positive semidefinite.

Remark 14 By Sturm’s Theorem (see [14]), the number of negative eigenvalues of $L_T^B$ equals the number of sign changes among the leading principal minors. When $|q| > 1$, the number of sign changes equals 1 by Corollary 12. This gives a short proof of a result of Bapat, Lal and Pati [2] Proposition 3.7] that the signature of $L_T^B$ is $(n-1, 1, 0)$ when $|q| > 1$, where signature of a Hermitian matrix $A$ is the vector $(p, n, z)$ with $p, n$ being the number of positive, negative eigenvalues of $A$ respectively and $z$ being the nullity of $A$.

Remark 15 By [5], for any $T$, all $a_{B,i}^T(q) \in \mathbb{R}^+[q]$ when $B \neq [n]$. In [25] Corollary 13, it was shown that $a_{n,i}^T(q) \in \mathbb{R}^+[q]$ when $i > 0$. By definition, $a_{n,0}^T(q) = 1 - q^2$ has negative coefficients. In [25] Theorem 2.4], it was shown that $c_{n,t}^\lambda(q) \in \mathbb{R}^+[q]$ for all $\lambda \vdash n$ except $\lambda = 1^n$.

By these and Corollary 11 it is easy to see that barring $c_{n,t}^\lambda(q)$, which equals $1 - q^2$, $c_{n,t}^\lambda(q) \in \mathbb{R}^+[q]$ for all $\lambda \vdash n$ and for all $0 \leq r \leq n$. Thus all statements in this work can be made about $c_{n,t}^\lambda(q)$ or alternatively about the absolute value of the coefficient of $x^{n-r}$ in $f_{\lambda}^\chi(x)$ (which equals $(-1)^r c_{n,t}^\chi(q)$).

4 Proof of Theorem 1

We begin with a few preliminaries towards proving Theorem 1. Let $T_1$ and $T_2$ be trees on $n$ vertices with $T_2 \geq_{GTS} T_1$. We assume that both $T_1$ and $T_2$ have vertex set $V = [n]$. 

Remark 16 Since immanants are invariant under a relabelling of vertices (see Littlewood’s book [19] or Merris [22]), without loss of generality, we label the vertices of $T_1$ as follows: first label the vertices on the path $p_k$ as 1, 2, ..., $k$ in order with 1 being the closest vertex to $X$ and $k$ being the closest vertex to $Y$. Then, label vertices in $X$ with labels $k+1, k+2, \ldots, k + |X|$ in increasing order of distance from vertex 1 (say in a breadth-first manner starting from vertex 1) and lastly, label vertices of $Y$ from $n - |Y| + 1$ to $n$ again in increasing order of distance from vertex 1. See Figure 3 for an example.

Recall our notation $a_{B,i}^{T_1}(q)$ and $a_{B,i}^{T_2}(q)$ for the trees $T_1$ and $T_2$ respectively. Also recall $O_{B,i}^{T_1}$ denotes the set of $B$-orientations in $T_1$ with $i$ bidirected-arcs and let $O_{B,i}^{T_1} = \bigcup_{B \subseteq V, |B| = r} O_{B,i}^{T_1}$. Recall that $O_{B,i}^{T_2}$ is defined analogously. It would have been nice if for all $B \subseteq V$ with $|B| = r$ and for all $0 \leq i \leq \lfloor r/2 \rfloor$, we could prove that $a_{B,i}^{T_1}(q) - a_{B,i}^{T_2}(q) \in \mathbb{R}^+[q^2]$. Unfortunately, this is not true as the example below illustrates.
Example 17 Let $T_2$ and $T_1$ be the trees given in Figure 3. Let $B = \{1, 4, 6, 7, 8\}$ and let $i = 2$. It can be checked that $a_{B,i}^T(q) = 2q^4 + q^6$ and that $a_{B,i}^T(q) = q^4$.

Nonetheless, by combining all sets $B$ of size $r$, we will for all $r$, $i$ construct an injective map $\gamma : \mathcal{O}_{r,i}^{T_2} \rightarrow \mathcal{O}_{r,i}^{T_1}$ that preserves the “away” statistic. For each $r$, note that there are $\binom{n}{r}$ sets $B$ that contribute to $\mathcal{O}_{r,i}^{T_2}$ and $\mathcal{O}_{r,i}^{T_1}$. We partition the $r$-sized subsets $B$ into three disjoint families and apply three separate lemmas. Recall that vertices 1 and $k$ are the endpoints of the path $P_k$ used in the definition of the poset $\text{GTS}_n$. The first family consists of those sets $B$ with both 1, $k \not\in B$.

Lemma 18 Let $B \subseteq [n]$, $|B| = r$ be such that both 1, $k \not\in B$. Then, there is an injective map $\phi : \mathcal{O}_{B,i}^{T_2} \rightarrow \mathcal{O}_{B,i}^{T_1}$ such that $\text{Aw}_{B}^{T_2}(O) = \text{Aw}_{B}^{T_1}(\phi(O))$. Thus, for all $0 \leq i \leq \lfloor r/2 \rfloor$, we have $a_{B,i}^{T_1}(q) - a_{B,i}^{T_2}(q) \in \mathbb{R}^+[q^2]$.

Proof: Let $O \in \mathcal{O}_{B,i}^{T_2}$. Clearly, $1 = \min_{\nu \in [n] - B} \nu$ and for $O$, define $O' = \phi(O)$ as follows. In $O'$, for each vertex $\nu \in B$, assign the same orientation as in $O$. Clearly, $O' \in \mathcal{O}_{B,i}^{T_1}$, and it is clear that $\phi$ is an injective map from $\mathcal{O}_{B,i}^{T_2}$ to $\mathcal{O}_{B,i}^{T_1}$. Further, it is easy to see that $\text{Aw}_{B}^{T_2}(O) = \text{Aw}_{B}^{T_1}(\phi(O))$, hence proving that $a_{B,i}^{T_1}(q) - a_{B,i}^{T_2}(q) \in \mathbb{R}^+[q^2]$, completing the proof.

We next consider those $B$ with $|\{1, k\} \cap B| = 1$. We use the notation $B$ for $r$-sized subsets with $1 \in B$, $k \not\in B$ and $B'$ for $r$-sized subsets with $k \in B'$, $1 \not\in B'$. The next lemma below considers such subsets $B'$ and those $B$-orientations $O$ with $O(1) \in X \cup P_k$. Note that for such $B$-orientations $O$, $\min_{\nu \in [n] - B} \nu \in P_k$.

Lemma 19 Let $O \in \mathcal{O}_{B,i}^{T_2}$, where $1 \in B$, $k \not\in B$ and let $O(1)$ denote the oriented neighbour of vertex 1 in $O$. If $O(1) \in X \cup P_k$, then there exists an injective map $\mu : \mathcal{O}_{B,i}^{T_2} \rightarrow \mathcal{O}_{B,i}^{T_1}$ such that $\text{Aw}_{B}^{T_2}(O) = \text{Aw}_{B}^{T_1}(\mu(O))$. Similarly, let $B' \subseteq V$ be such that $1 \not\in B'$, $k \in B'$. Then, there is an injective map $\nu : \mathcal{O}_{B,i}^{T_2} \rightarrow \mathcal{O}_{B,i}^{T_1}$ such that for $P \in \mathcal{O}_{B,i}^{T_2}$, $\text{Aw}_{B}^{T_2}(P) = \text{Aw}_{B}^{T_1}(\nu(P))$.

Proof: The proof for both cases are similar. Let $O \in \mathcal{O}_{B,i}^{T_2}$ and let $O(1) \in X \cup P_k$. In this case, the same injection of Lemma 18 works. That is, we form $O'$ by assigning all vertices of $B$ the same orientation as in $O$. Clearly, $O' \in \mathcal{O}_{B,i}^{T_1}$ and $\text{Aw}_{B}^{T_2}(O) = \text{Aw}_{B}^{T_1}(O')$.

Similarly, let $P \in \mathcal{O}_{B,i}^{T_2}$. Form $P'$ by assigning all vertices of $B'$ the same orientation as in $P$. Clearly, $\text{Aw}_{B}^{T_2}(P) = \text{Aw}_{B}^{T_2}(P')$. Note that in both $P$ and $P'$, the orientation of $k$ equals $k - 1$ as $k$ is a leaf vertex in $T_2$. The proof is complete.

Figure 3: Two labelled trees with $T_2 \geq_{\text{GTS}_n} T_1$ and $T_2$ covering $T_1$. 
We continue to use the notation $B$ for an $r$-sized subset of $V$ with $1 \in B$. We now handle $B$-orientations $O \in \mathcal{O}_{B,i}^{T_2}$ with $O(1) \in Y$.

**Lemma 20** Let $B$ be an $r$-sized subset of $[n]$ with $1 \in B$, $k \notin B$. Define $B' = (B - \{1\}) \cup \{k\}$. Let $O \in \mathcal{O}_{B,i}^{T_2}$ with $O(1) \in Y$. There is an injective map $\delta : \mathcal{O}_{B,i}^{T_2} \to \mathcal{O}_{B',i}^{T_1}$ such that $\mathcal{A}_B^{T_2}(O) = \mathcal{A}_B^{T_2}(\delta(O))$. Futher, if $N = \delta(O)$, then we have $N(k) = O(1)$.

**Proof:** The proof is identical to the proof of \cite[Lemma 7]{25}. We hence only sketch our proof. In $T_1$, define $m' = \min_{v \in [n] - B'} v$ and recall that $m = \min_{v \in [n] - B} v$ in $T_2$. Since $1 \notin B'$, note that in $T_1$, we have $m' = 1$. Thus, we reverse the orientation of some vertices in $T_2$ on the subpath from $(1, m)$ of $P_k$. To decide the vertices whose orientations are to be reversed, we break the $(1, m)$ path into segments separated by bidirected arcs. In each segment, if the $\ell$-th closest vertex to $m$ in $T_2$ was oriented “towards $m$”, then in $T_1$, orient the $\ell$-th closest vertex to 1 “towards 1”. Likewise, if the $\ell$-th closest vertex to $m$ in $T_2$ was oriented “away from $m$”, then in $T_1$, orient the $\ell$-th closest vertex to 1 “away from 1”.

See Figure 4 for an example, where the letter “t” is used to denote a vertex whose orientation is towards $m$ and “a” is used to denote a vertex whose orientation is away from $m$. This convention of “t” and “a” will be used in later figures as well. For the example in the Figure 4, note that $k = 9$. If $\delta$ is the map described above, then it is clear that $\mathcal{A}_B^{T_2}(O) = \mathcal{A}_B^{T_2}(\delta(O))$ and that $(\delta(O))(k) = O(1)$. The proof is complete. 

![Figure 4: Illustrating the injection when $O(1) \in Y$, $m \in P_k$.](image)

**Corollary 21** Let $B \subseteq V$ with $1 \in B$, $k \notin B$ and define $B' = (B - \{1\}) \cup \{k\}$. For all $i$, there is an injection $\omega : \mathcal{O}_{B,i}^{T_2} \cup \mathcal{O}_{B',i}^{T_2} \to \mathcal{O}_{B,i}^{T_1} \cup \mathcal{O}_{B',i}^{T_1}$. Thus, $a_{T_1}\omega(q) = a_{T_2}(q) - a_{B',i}(q) - a_{B,i}(q) \in \mathbb{R}^{+[q^2]}$.

**Proof:** If $O \in \mathcal{O}_{B,i}^{T_2}$ is such that $O(1) \in X \cup P_k$, use Lemma 19. On the other hand, if $O(1) \in Y$, then we use Lemma 20. Let $O' = \omega(O)$. Note that in this case, vertex $k$ is oriented with $O'(k) \in Y$.

Similarly, if $O \in \mathcal{O}_{B',i}^{T_2}$, then, we use Lemma 19. Note that in this case if $O' = \omega(O)$, then $O'(k) = k - 1 \in P_k$. Thus, the case mentioned in the earlier paragraph and this case are disjoint and hence $\omega$ is an injection.

Our last family consists of subsets $B$ with both $1, k \in B$. Define another subset $B' \subseteq [n]$ using $B$ as follows: Let $B_{xy} = B \cap (X \cup Y)$ and let $B_p = B \cap P_k$. The set $B'$ will be used when $m \in P_k$. In this case, $m = \min_{v \in P_k \setminus B} v$ is the minimum vertex outside $B$ in $P_k$. Define $l = \max_{v \in P_k \setminus B} v$ to be the maximum numbered vertex in $P_k$ not in $B$. Define $m' = k + 1 - l$ and $l' = k + 1 - m$. Form $B_p'^l$ by taking the union of the three sets $A' = \{1, \ldots, m' - 1\}$, $C' = \{l' + 1, \ldots, k\}$ and
\[ m' - m + x : x \in B \cap \{ m+1, \ldots, l-1 \} \]. See Figure \( \theta \) for an example. Define \( B' = B_{xy} \cup B'_p \). Clearly, both \( 1, k \in B' \) and \( (B')' = B \).

**Lemma 22** Let \( B \subseteq [n] \) be such that both \( 1, k \in B \) and let \( B' \) be as defined above. For all \( i \), there is an injective map \( \theta : \mathcal{O}'_{B,i} \cup \mathcal{O}'_{B',i} \to \mathcal{O}_{B,i} \cup \mathcal{O}_{B',i} \) that preserves the away statistic. Thus, \( a_{B,1,i}(q) + a_{B',1,i}(q) - a_{B,2,i}(q) - a_{B',2,i}(q) \in \mathbb{R}^+[q^2] \).

**Proof:** We denote the orientation of vertex 1 in \( O \) as \( O(1) \). Given \( B \), recall \( m = \min_{v \in B} v \) is the minimum vertex outside \( B \) and that we have labelled vertices on the path \( P_k \) first, vertices in \( X \) next and vertices of \( Y \) last. There are nine cases based on \( m \) and \( O(1) \). Only one of the nine cases will involve \( B \) getting changed to \( B' \). For now, let \( O \in \mathcal{O}'_{B,i} \). Define a map \( \theta : \mathcal{O}_{B,i} \to \mathcal{O}_{B,i} \) as follows. Let \( O \in \mathcal{O}'_{B,i} \). We construct a unique \( O' \in \mathcal{O}_{B,i} \) by using the algorithms tabulated below. Though it seems that there are a large number of cases, the underlying moves are very similar.

For vertices \( u, v, a, b \), we explain an operation that we denote as \( \text{reverse_on_path}(u, v; a, b) \) that will be needed when \( m \in Y \). We will always have \( d_{u,v} = d_{a,b} \) in \( T_1 \) where \( d_{u,v} \) is the distance between vertices \( u \) and \( v \) in \( T_1 \). Further, all vertices \( w \) on the \( u, v \) path \( P_{u,v} \) in \( T_1 \) will be in \( B \) and hence be oriented. \( \text{reverse_on_path}(u, v; a, b) \) will change orientations of all vertices on \( P_{u,v} \). We will use this operation in all the three cases when \( m \in Y \). Due to our labelling convention and the fact that \( m \in Y \), all vertices of \( P_k \cup X \) will be contained in \( B \). In \( T_2 \), vertex \( m \) has vertex 1 as its closest vertex among the vertices in \( P_k \), whereas in \( T_1 \), vertex \( m \) has vertex \( k \) as its closest vertex among those in \( P_k \). Define vertices on \( P_{u,v} \) as \( u = u_1, u_2, \ldots, u_s = v \) and the vertices on the \( (a, b) \) path as \( a = a_1, a_2, \ldots, a_s = b \). In \( O \), if vertex \( a_i \) is oriented “towards \( m \)”, then orient vertex \( u_{s+1-i} \) “towards \( m \)” and likewise if vertex \( a_i \) is oriented “away from \( m \)”, then orient vertex \( u_{s+1-i} \) “away from \( m \)”. We give the map \( \theta \) using several algorithms below.

| \( O(1) = 2 \in P_k \) | \( m \in P_k \) | \( m \in X \) | \( m \in Y \) |
|-----------------|-------------|-------------|-------------|
| Use algorithm 1 | Use algorithm 1 | Use algorithm 1 | Use algorithm 2 |
| \( O(1) = x \in X \) | Use algorithm 1 | Use algorithm 1 | Use algorithm 4 |
| \( O(1) = y \in Y \) | Use algorithm 5 | Use algorithm 3 | Use algorithm 2 |

**Algorithm 1:** This is a trivial copying algorithm. Define \( O' = \theta(O) \) with \( O' \in \mathcal{O}_{B,i} \) as follows. In \( O' \), retain the same orientation for all vertices \( v \in B' \). It is clear that \( \text{Aw}_{B,1}(O) = \text{Aw}_{B,1}(O') \).

**Algorithm 2:** Since \( m \in Y \), by our labelling convention, this means all the vertices of \( P_k \) and \( X \) are in \( B \). Form \( O' = \theta(O) \) with \( O' \in \mathcal{O}_{B,i} \) by first copying the orientation \( O \) for each vertex. Then perform \( \text{reverse_on_path}(1, k; 1, k) \). This is illustrated in Figure \( \theta \) when \( O(1) = 2 \) and \( m \in Y \) and in Figure \( \theta \) when both \( O(1), m \in Y \). It is clear that \( \text{Aw}_{B,1}(O) = \text{Aw}_{B,1}(O') \).

**Algorithm 3:** We have \( m \in X \) and \( O(1) \in Y \). Recall that we have labelled the vertices of \( X \) in increasing order of distance from vertex 1. We claim that there exists a unique edge \( e = \{ x, y \} \) on the path from 1 to \( m \) satisfying the following two conditions:

1. There is no arrow on \( e \). That is, either both \( x, y \in B \) with \( O(x) \neq y \) and \( O(y) \neq x \) or \( x \in B \) and \( y = m \).
2. Among such edges, \( x \) is the closest vertex to 1 distancewise (that is, \( e \) is the unique closest edge to 1).

That there exists such an edge \( e \) satisfying condition (1) above is easy to see. Condition (2) is just a labelling of vertices of such an edge. Further, we label the vertices on the path from 1 to \( x \) in increasing order of distance from vertex 1 as \( 1, x_1, x_2, \ldots, x_l = x \). (See Figure 7 for an example.)

It is easy to see that \( O(x_1) = 1 \) and \( O(x_i) = x_{i-1} \) for \( 2 \leq i \leq l \) and recall that \( O(1) \in Y \). Form \( O' = \theta(O) \) with \( O' \in O_{B,i}^T \) as follows. Vertices of \( B \) not on the path from \( x_l \) to \( k \) in \( T_1 \) get the same orientation as in \( O \). We orient the last \( l + 1 \) vertices in \( T_1 \) on the \( x_l \) to \( k \) path \( P_{x_l,k} \) away from \( m \), and then orient the first \( k - 1 \) vertices on \( P_{x_l,k} \) as they were on \( P_k \). See Figure 7 for an example. As \( k = 6 \) and \( l = 3 \), the last \( l + 1 \) vertices on the \((x_3,6)\) path means that the last 4 vertices are oriented away from \( m \). The orientation of the remaining vertices is inherited from \( T_2 \).

It is clear that \( |\text{bidir}(O)| = |\text{bidir}(O')| \) and that \( Aw_{B}^{T_2}(O) = Aw_{B}^{T_1}(O') \).

**Algorithm 4:** We have \( O(1) \in X \) and \( m \in Y \). As done in Algorithm 3, find the closest edge \( e = \{x, y\} \) to vertex 1 with \( e \) having no arrow on the 1 to \( m \) path. As before, label \( e \) as \( \{x, y\} \) with \( x \) being closer to 1 than \( y \), and label the vertices on the path from 1 to \( x \) as \( 1, x_1, x_2, \ldots, x_l = x \) (see Figure 8).
It is easy to see that $O(x_i) = 1$ and $O(x_i) = x_{i-1}$ for $2 \leq i \leq l$. Note that there is a continuous string of $l + 1$ vertices that are oriented away from $m$. Form $O' = \theta(O)$ with $O' \in \mathcal{O}_{B,i}^{T^1}$ as follows. Vertices of $B$ not on the path from $x_i$ to $k$ in $T_i$ get the same orientation as in $O$. The closest $l + 1$ vertices of $B$ on the path from $1$ to $x_i$ in $T_i$ get oriented away from $m$. Denote the path comprising the last $k - 1$ vertices on the $(1, x_i)$-path as $P_i$. Let $\alpha, \beta$ be the first and last vertices of $P_i$. Perform reverse_on_path($\alpha, \beta; 2, k$). See Figure 8 for an example. It is clear that $Aw_B^{T^2}(O) = Aw_B^{T^1}(O')$.

**Algorithm 5:** We have $O(1) = y \in Y$ and $m \in P_k$. Recall $B' = B_{xy} \cup B_{pr}$. Recall $l = \max_{e \in P, v \in B} v$. Note that the minimum vertex $m' \notin B'$ will be $m' = k + 1 - l$. Form $O' = \theta(O)$ with $O' \in \mathcal{O}_{B,i}^{T^i}$ as follows. Note that in $T_2$, there is a continuous sequence $A$ of $m - 1$ oriented vertices from $1$ to $m - 1$ and another continuous sequence $C$ of $k - l$ oriented vertices from $l + 1$ to $k$ in the path $P_k$ (see Figure 9 for an example). Similarly, in $T_1$, there is a continuous sequence $A'$ of $m' - 1$ oriented vertices from $1$ to $m' - 1$ and another continuous sequence $C'$ of $k - l'$ oriented vertices from $l' + 1$ to $k$ in the path $P_k$.

It is easy to see that $|A| = |C'|$ and $|C| = |A'|$. If vertex $s \in A$ is oriented away from (or towards) $m$ in $O$, then in $O'$ orient vertex $k + 1 - s$ away from (or towards respectively) $m'$. Likewise, if vertex $s \in C$ is oriented away from (or towards) $m$ in $O$, then in $O'$ orient vertex $k + 1 - s$ away from (or towards respectively) $m'$.

Lastly, in $O'$ copy the orientation of vertices in $B$ that lie between $m$ and $l$ in $T_2$ as they were to the vertices in $B'$ between $m'$ and $l'$ in $T_1$. Formally, if vertex $s \in P_k$ with $m < s < l$ is oriented away from (or towards) $m$ in $O$, then in $O'$ orient vertex $(m' - m) + s$ away from (or towards respectively) $m'$.

For vertices $s \in B_p$, see Figure 9 for an example. Clearly, $|\text{bidir}(O)| = |\text{bidir}(O')|$ and $Aw_B^{T^2}(O) = Aw_B^{T^1}(O')$. This completes Algorithm 5.

When $B = [n]$, note that all vertices are oriented and hence there exists at least one bidirected edge. In this case, we have $Aw_B(O) = \text{away}(O, e)$, where as defined in [25], $\text{away}(O, e)$ is found with respect to the lexicographic minimum bidirected edge $e \in O$. If the lexicographic edge is $e = \{u, v\}$, we let $m = \min(u, v)$ be the smaller numbered vertex among $u, v$. We find the statistic $\text{away}(O, e)$ with respect to $m$. It is simple to note that among the nine cases, the following will not occur when $B = [n]$ due to our labelling convention:

1. $m \in X$ and $O(1) \in P_k$,
2. $m \in Y$ and $O(1) \in P_k$ and
3. $m \in Y$ and $O(1) \in X$.

In the remaining cases, we follow the same algorithms. It is easy to see that the pair $(m, O(1))$ is different in all the nine cases. We do not change $B$ in eight cases, except in Algorithm 5. Thus, we get an injection in these eight cases. When Algorithm 5 is run, we get an injection from $\mathcal{O}_{B,i}^{T^2}$.
to $O_{B',i}^{T_1}$ and similarly we get an injection from $O_{B',i}^{T_2}$ to $O_{B,i}^{T_1}$. Thus, we get an injection from $O_{B',i}^{T_2} \cup O_{B,i}^{T_2}$ to $O_{B,i}^{T_1} \cup O_{B',i}^{T_1}$, completing the proof.

With these Lemmas in place, we can now prove Theorem 1.

**Proof**: (Of Theorem 1) We group the set of $r$-sized subsets $B$ into three categories: those without 1, $k$, those with either 1 or $k$ and those with both 1, $k$. By Lemmas 18, 22 and Corollary 21 it is clear that there is an injective map from $O_{r,i}^{T_2}$ to $O_{r,i}^{T_1}$ for all $r$ and $i$. By Corollary 11, $c_{\lambda,r}^{L_{q}^{T_2}}(q) - c_{\lambda,r}^{L_{q}^{T_1}}(q) \in \mathbb{R}^{+}[q^2]$ for all $\lambda, r$.

**Corollary 23** Setting $q = 1$ in $L_{q}^{T}$, we infer that for all $r$, the coefficient of $x^{n-r}$ in the immanantal polynomial of the Laplacian $L_{T}$ of $T$ decreases in absolute value as we go up GTS$_n$. Using Lemma 5, we thus get a more refined and hence stronger result than Theorem 3.

**Corollary 24** Let $T_1, T_2$ be trees on $n$ vertices with respective $q$-Laplacians $L_{q}^{T_1}, L_{q}^{T_2}$. Let $T_2 \geq_{\text{GTS}_n} T_1$ and let $d_\lambda(L_{q}^{T_i})$ denote the immanant of $L_{q}^{T_i}$ for $1 \leq i \leq 2$ corresponding to the partition $\lambda \vdash n$. By comparing the constant term of the immanantal polynomial, for all $\lambda \vdash n$, we infer $d_\lambda(L_{q}^{T_2}) \leq d_\lambda(L_{q}^{T_1})$. This refines the inequalities in Theorem 3.

### 5 $q^2$-analogue of vertex moments in a tree

Merris in [21] gave an alternate definition of the centroid of a tree $T$ through its vertex moments. He then showed that the sum of vertex moments appears as a coefficient of the immanantal polynomial of $L_T$ corresponding to the partition $\lambda = 2, 1^{n-2}$. In this section, we define a $q^2$-analogue of vertex moments and through it, the centroid of a tree. We then show that the sum of $q^2$-analogue of the vertex moments of all vertices appears as a coefficient in the second immanantal polynomial of $L_{q}^{T}$. Thus, by Theorem 1, the sum of the $q^2$-analogue of vertex moments decreases as we go up on GTS$_n$. We further show that as we go up on GTS$_n$, the value of the minimum $q^2$-analogue of the vertex moments also decreases.

The following definition of vertex moments is from Merris [21]. Let $T$ be a tree with vertex set $[n]$. For a vertex $i \in [n]$, define $\text{Moment}^T(i) = \sum_{j \in [n]} d_j d_{i,j}$ where $d_j$ is the degree of vertex $j$ in $T$ and $d_{i,j}$ is the distance between vertices $i$ and $j$ in $T$. Define the $q^2$-analogue of the distance $d_{i,j}$...
between vertices $i$ and $j$ to be $[d_{i,j}]_{q^2} = 1 + q^2 + (q^2)^2 + \cdots + (q^2)^{d_{i,j} - 1}$ and define for all $i \in [n]$, $[d_{i,i}]_{q^2} = 0$. We define the $q^2$-anologue of the moment of vertex $i$ of $T$ as

$$\text{Moment}^T_{q^2}(i) = \sum_{j \in [n]} [1 + q^2(d_j - 1)][d_{i,j}]_{q^2}. \tag{7}$$

Fix $q \in \mathbb{R}$, $q \neq 0$. Vertex $i$ is called the centroid of $T$ if $\text{Moment}^T_{q^2}(i) = \min_{j \in [n]} \text{Moment}^T_{q^2}(j)$. We clearly recover Merris’ definition of moments when we plug in $q = 1$ in (7). Merris showed that his definition of centroid coincides with the usual definition of the centroid of a tree $T$. In [4], Bapat and Sivasubramanian while studying the third immanant of $L^q_T$ proved a lemma that we need. The following lemma is obtained by setting $s = q^2$ in [4] Lemma 3.

**Lemma 25 (Bapat and Sivasubramanian)** Let $T$ be a tree with vertex set $V = [n]$ and let $i \in [n]$. Then,

$$\sum_{j \in [n]} q^2(d_j - 1)[d_{i,j}]_{q^2} = \sum_{j \in [n]} [d_{i,j}]_{q^2} - (n - 1). \tag{8}$$

The following alternate expression for $\text{Moment}^T_{q^2}(i)$ is easy to derive using Lemma 25 and the definition (7). As the proof is a simple manipulation, we omit it.

**Lemma 26** Let $T$ be a tree with vertex set $[n]$ and let $i \in [n]$. Then,

$$\text{Moment}^T_{q^2}(i) = (n - 1) + 2q^2 \sum_{j \in [n]} (d_j - 1)[d_{i,j}]_{q^2}. \tag{9}$$

The following lemma gives an algebraic interpretation for the $q^2$-anologue of vertex moments in $T$.

**Lemma 27** Let $T$ be a tree with vertex set $[n]$. Let $i \in [n]$ be a vertex and let $B = [n] - \{i\}$. Then,

$$\text{Moment}^T_{q^2}(i) = (n - 1)a^T_{B,0}(q) + 2a^T_{B,1}(q). \tag{10}$$

**Proof:** Clearly for $B = [n] - \{i\}$, we have a unique $B$-orientation $O \in O_{B,0}$ with $Aw^T_O(O) = 0$. This is the orientation where every vertex $j \in [n] - i$ gets oriented towards $i$. Thus $a^T_{B,0}(q) = 1$.

We will show that $a^T_{B,1}(q) = q^2 \sum_{j \in [n]} (d_j - 1)[d_{i,j}]_{q^2}$ and appeal to (9). By (8), equivalently, we need to show that

$$a^T_{B,1}(q) = \sum_{j \in [n]} [d_{i,j}]_{q^2} - (n - 1) = q^2 \sum_{j \in [n], j \neq i}[d_{i,j} - 1]_{q^2}.$$

Root the tree $T$ at the vertex $i$ and recall $B = [n] - \{i\}$. Thus $m = i$. Let $O$ be a $B$-orientation with one bidirected arc $e = \{u, v\}$ where we label the edge $e$ such that $d_{i,v} = d_{i,u} + 1$. That is, $u$ occurs on the path from $i$ to $v$ in $T$. Since $n - 1$ vertices are oriented and one edge is bidirected, there must be one edge without any arrows (when seen pictorially). It is easy to see that all edges $f \in T$ not on the path $P_{i,u}$ from $i$ to $u$ must be oriented towards $i$. Moreover, it is clear that the
edge \( f \) without arrows must be on the path \( P_{i,u} \). Thus, our choice lies in orienting vertices in \( P_{i,u} \) such that one edge does not get any arrows. Let \( f = \{x, y\} \) with \( x \) being on the path from \( i \) to \( y \) in \( T \) (\( x \) could be \( i \) or \( y \) could be \( u \)). Thus, there are \( d_{i,u} - 1 \) choices for the edge \( f \). In \( O \), clearly, all vertices from \( y \) till \( u \) on the path \( P_{i,u} \) must be oriented away from \( i \). Hence the contribution of all such orientations will be \( q^2 + q^4 + \ldots + q^{2d_{i,u} - 2} \). Thus vertex \( u \) contributes \( q^2[d_{i,u} - 1]q^2 \) to \( a_{B,1}(q) \). Summing over all vertices \( u \) completes the proof. 

**Theorem 28** Let \( T \) be a tree with vertex set \([n]\) and \( q\)-Laplacian \( \mathcal{L}_q \). Let \( \lambda = 2, 1^{n-2} \vdash n \). Then,

\[
\ell_T^{\lambda} = \sum_{i=1}^{n} \text{Moment}^T_q(i).
\]

**Proof:** Summing (10) over all \( B \) with cardinality \( n - 1 \), we get

\[
\sum_{i=1}^{n} \text{Moment}^T_q(i) = (n - 1)\alpha_{n-1,0}(q) + 2\alpha_{n-1,1}(q) = \ell_T^{\lambda}
\]

where the last equality follows from Corollary [11] and Lemma [10] with \( k = 2 \). The proof is complete. 

On setting \( q = 1 \) in Theorem 28 we recover Merris’ result [21, Theorem 6]. From Theorem 1 and Theorem 28 we get the following.

**Theorem 29** Let \( T_1 \) and \( T_2 \) be trees with \( n \) vertices and let \( T_2 \) cover \( T_1 \) in \( \text{GTS}_n \). Then,

\[
\sum_{i=1}^{n} \text{Moment}^T_{q^2}(i) \leq \sum_{i=1}^{n} \text{Moment}^T_{q^2}(i).
\]

Theorem 29 implies that the sum of the vertex moments decreases as we go up on the poset \( \text{GTS}_n \). We next show that the minimum value of the \( q^2 \)-analogue of vertex moments also decreases as we go up on \( \text{GTS}_n \).

**Lemma 30** Let \( T_1 \) and \( T_2 \) be two trees with vertex set \([n]\) such that \( T_2 \) covers \( T_1 \) in \( \text{GTS}_n \). Then, for all \( q \in \mathbb{R} \), we have \( \min_{i \in [n]} \text{Moment}^T_{q^2}(i) \leq \min_{j \in [n]} \text{Moment}^T_{q^2}(j) \).

**Proof:** Let \( l \in [n] \) be the vertex in \( T_1 \) with \( \text{Moment}^T_{q^2}(l) = \min_{i \in [n]} \text{Moment}^T_{q^2}(i) \). Let \( l \in X \cup Y \cup P_{[k/2]} \) (see Figure 1 for \( X \), \( Y \) and \( P_k = P_{x,y} \)). Here \( P_{[k/2]} \) is the path \( P_k \) restricted to the vertices 1, 2, \ldots, \([k/2]\]. Then, using the fact that the distance \( d_{x,y} \) is \( d_{x,y} \geq d_{x,y} \) for all pairs \((x,y) \in X \times Y \), we have

\[
\text{Moment}^T_{q^2}(l) \geq \text{Moment}^T_{q^2}(l) \geq \min_{i \in [n]} \text{Moment}^T_{q^2}(i).
\]

If \( l \geq [k/2] \) then \( \text{Moment}^T_{q^2}(l) \geq \text{Moment}^T_{q^2}(k + 1 - l) \geq \min_{i \in [n]} \text{Moment}^T_{q^2}(i) \). Thus we can find a vertex \( i \) in \( T_2 \) such that \( \text{Moment}^T_{q^2}(i) \geq \text{Moment}^T_{q^2}(i) \), completing the proof.
Corollary 31 Let $T_1, T_2$ be two trees on $n$ vertices with $T_2 \succeq_{GTS_n} T_1$. Then, for all $q \in \mathbb{R}$, the minimum $q^2$-analogue of the vertex moments of $T_2$ is less than the minimum $q^2$-analogue of the vertex moments of $T_1$.

An identical statement about the maximum $q^2$-analogue of vertex moments is not true as shown in the following example.

Example 32 Let $T_1, T_2$ be trees on the vertex set [8] given in Figure 10. Vertices 2 and 3 are both centroid vertices in $T_1$, while in $T_2$, the centroid is vertex 1. The $q^2$-analogue of their vertex moments are as follows: $\text{Moment}_{T_1}(2) = \text{Moment}_{T_2}(3) = 9 + 2q(7 + 3q^2)$ and $\text{Moment}_{T_2}(1) = 9 + 2q^2(2 + q^2)$. The $q^2$-analogue of vertex moments of leaf vertices of $T_1$ and $T_2$ are as follows.

$$\text{Moment}_{T_1}(i) = 9 + 2q^2(8 + 5q^2 + 4q^4 + 3q^6) \text{ for } i = 5, 6, 7, 8, 9, 10.$$  
$$\text{Moment}_{T_2}(i) = 9 + 2q^2(8 + 2q^2 + q^4) \text{ for } i = 5, 6, 7, 8, 9, 10.$$  
$$\text{Moment}_{T_2}(4) = 9 + 2q^2(8 + 7q^2 + 6q^4).$$

When $q = 1$, the moments of vertices of $T_1$ and $T_2$ are given in Figure 10 alongside the vertices. Clearly, when $q = 1$, $\max_{j \in [8]} \text{Moment}_{T_2}(j) = 51 \leq 49 = \max_{j \in [8]} \text{Moment}_{T_1}(j)$.

![Figure 10: $q^2$-moments of vertices of $T_1$ and $T_2$ when $q = 1$.](image)

Associated to a tree are different notions of “median” and “generalized centers”, see the book [18]. It would be nice to see the behaviour of these parameters as one goes up $GTS_n$.

6 $q$, $t$-Laplacian $\mathcal{L}_{q,t}$ and Hermitian Laplacian of a tree $T$

All our results work for the bivariate Laplacian matrix $\mathcal{L}_{q,t}$ of a tree $T$ on $n$ vertices defined as follows. Let $T$ be a tree with edge set $E$. Replace each edge $e = \{u, v\}$ by two bidirected arcs, $(u, v)$ and $(v, u)$. Assign one of the arcs, say $(u, v)$ a variable weight $q$ and its reverse arc, a variable weight $t$ and let $A_{n \times n} = (a_{i,j})_{1 \leq i,j \leq n}$ be the matrix with $a_{u,v} = q$ and $a_{v,u} = t$. Assign $a_{u,v} = 0$ if $\{u, v\} \not\in E$.

Let $D_{n \times n} = (d_{i,j})$ be the diagonal matrix with entries $d_{i,i} = 1 + qt(\deg(i) - 1)$. Define $\mathcal{L}_{q,t} = D - A$. Note that when $q = t$, $\mathcal{L}_{q,t} = \mathcal{L}_T^q$ and that when $q = t = 1$, $\mathcal{L}_{q,t} = L_T$ where $L_T$ is the usual combinatorial Laplacian matrix of $T$.

It is easy to see that our proof relies on the fact that the difference in the coefficients of the immmanantal polynomial is a non-negative combination of the $a_{r,i}^T(q)$’s which are polynomials in
$q^2$ and that $q^2 \geq 0$ for all $q \in \mathbb{R}$. When $B = [n]$, bivariate versions of $m_{n,j}(q,t)$ and $a^T_{n,j}(q,t)$ were defined in [25]. Define bivariate versions $m_{r,j}(q,t)$ and $a^T_{r,j}(q,t)$ as done in Section 5 but replace all occurrences of $q^2$ with $qt$.

With this definition, it is simple to see that all results go through for $\mathcal{E}_{q,t}$, the $q,t$-Laplacian of $T$ whenever $q,t \in \mathbb{R}$ and $qt \geq 0$ or $q,t \in \mathbb{C}$ and $qt \geq 0$. One special case of $\mathcal{E}_{q,t}$ is obtained when we set $q = i$ and $t = -i$ where $i = \sqrt{-1}$. In this case, the weighted adjacency matrix becomes the Hermitian adjacency matrix of $T$ with edges oriented in the direction of the arc labelled $q$. The Hermitian adjacency matrix is a matrix defined and studied by Bapat, Pati and Kalita [11] and later independently by Liu and Li [20] and by Guo and Mohar [15]. With these complex numbers as weights, $\mathcal{E}_{q,t}$ reduces to what is defined as the Hermitian Laplacian of $T$ by Yu and Qu [27]. We get the following corollary of Theorem 1.

**Corollary 33** Let $T_1, T_2$ be trees on $n$ vertices with $T_2 \geq_{GTS_n} T_1$. Then, in absolute value, the coefficients of the immanantal polynomials of the Hermitian Laplacian of $T_1$ are larger than the corresponding coefficient of the immanantal polynomials of the Hermitian Laplacian of $T_2$.

Let $T$ be a tree on $n$ vertices with Laplacian $L_T$ and $q,t$-Laplacian $\mathcal{E}_{q,t}$. When $q = z \in \mathbb{C}$ with $z \neq 0$, and $t = 1/q$ then it is simple to see that the matrix $\mathcal{E}_{q,t}$ need not be Hermitian. In this case, for all $i \geq 0$, we have $a^T_{r,i}(q)_{q=1} = a^T_{r,i}(z,1/z)$. This implies that for all $\lambda \vdash n$ and for $0 \leq r \leq n$, $c^L_{\lambda,r} = c^\mathcal{E}_{\lambda,r}$. Thus, we obtain the following simple corollary.

**Corollary 34** Let $T$ be a tree on $n$ vertices with Laplacian $L_T$ and $q,t$-Laplacian $\mathcal{E}_{q,t}$. Then, for all $z \in \mathbb{C}$ with $z \neq 0$ and for all $\lambda \vdash n$

$$f^\mathcal{E}_{\lambda,1/z}(x) = f^L_{\lambda,T}(x).$$

## 7 Exponential distance matrices of a tree

In [2], Bapat, Lal and Pati introduced the exponential distance matrix $\mathcal{E}_T$ of a tree $T$. In this section, we prove that when $q \neq \pm 1$, the coefficients of the characteristic polynomial of $\mathcal{E}_T$, in absolute value decrease when we go up $GTS_n$. We show a similar relation on immanants of $\mathcal{E}_T$ indexed by partitions with two columns. We recall the definition of $\mathcal{E}_T$ from [2]. Let $T$ be a tree with $n$ vertices. Then, its exponential distance matrix $\mathcal{E}_T = (e_{i,j})_{1 \leq i,j \leq n}$ is defined as follows: the entry $e_{i,j} = 1$ if $i = j$ and $e_{i,j} = q^{d_{i,j}}$ if $i \neq j$, where $d_{i,j}$ is the distance between vertex $i$ and vertex $j$ in $T$. For $\lambda \vdash n$, define

$$f^\mathcal{E}_T(x) = d_\lambda(xI - \mathcal{E}_T) = \sum_{r=0}^{n} (-1)^r c^\mathcal{E}_{\lambda,r}(q)x^{n-r}. \quad (11)$$

We need the following lemma of Bapat, Lal and Pati [2].

**Lemma 35 (Bapat, Lal and Pati)** Let $T$ be a tree with $n$ vertices. Let $\mathcal{E}_{q,T}$ and $\mathcal{E}_T$ be the $q$-Laplacian and exponential distance matrix of $T$ respectively. Then, $\det(\mathcal{E}_T) = (1 - q^2)^{n-1}$ and if $q \neq \pm 1$, then

$$\mathcal{E}_T^{-1} = \frac{1}{1 - q^2} \mathcal{E}_{q,T}.$$
Using Jacobi’s Theorem on minors of the inverse of a matrix (see DeAlba’s article [12, Section 4.2]), we get the following easy corollary, whose proof we omit.

**Corollary 36** Let $T$ be a tree with $n$ vertices. Let $\mathcal{L}_T^q$ and $\mathsf{ED}_T$ be the $q$-Laplacian and exponential distance matrix of $T$ respectively. Let $q \neq \pm 1$. Then, for $0 \leq r \leq n$

$$c_{1^n,r}^{\mathsf{ED}_T}(q) = (1 - q^2)^{r-1} c_{1^n,n-r}^{\mathcal{L}_T^q}(q),$$

where $c_{1^n,n-r}^{\mathcal{L}_T^q}(q)$ is the coefficient of $(-1)^{n-r} x^r$ in $f_{1^n}^{\mathcal{L}_T^q}(x)$.

The following corollary is an easy consequence of Theorem 1 and Corollary 36, we omit its proof.

**Corollary 37** Let $T_1$ and $T_2$ be two trees with $n$ vertices such that $T_2 \geq_{\text{GTS}_n} T_1$. Then, for all $q \in \mathbb{R}$ with $q \neq \pm 1$ and for $0 \leq r \leq n$,

$$\left| c_{1^n,r}^{\mathsf{ED}_{T_2}}(q) \right| \leq \left| c_{1^n,r}^{\mathsf{ED}_{T_1}}(q) \right|.$$ 

In particular, for an arbitrary tree $T$ with $n$ vertices,

$$\left| c_{1^n,r}^{\mathsf{ED}_{\mathcal{L}_T^q}}(q) \right| \leq \left| c_{1^n,r}^{\mathsf{ED}_{\mathcal{L}_T^q}}(q) \right| \leq \left| c_{1^n,r}^{\mathsf{ED}_{\mathcal{L}_T^q}}(q) \right|.$$

We give some results for the immanant $d_\lambda(\mathsf{ED}_T)$, when $\lambda \vdash n$ is a two column partition. That is $\lambda = 2^k, 1^{n-2k}$ with $0 \leq k \leq \lfloor n/2 \rfloor$. When $\lambda$ is a two column partition of $n$, Merris and Watkins in [23] proved the following lemma for invertible matrices.

**Lemma 38 (Merris, Watkins)** Let $A$ be an invertible $n \times n$ matrix. Then $\lambda \vdash n$ is a two column partition if and only if

$$d_\lambda(A) \det(A^{-1}) = d_\lambda(A^{-1}) \det(A).$$

**Lemma 39** Let $T$ be a tree with $n$ vertices with $q$-Laplacian and exponential distance matrices $\mathcal{L}_T^q$ and $\mathsf{ED}_T$ respectively. Then for all $q \in \mathbb{R}$ with $q \neq \pm 1$ and $\lambda \vdash 2^k, 1^{n-2k}$ for $0 \leq k \leq \lfloor n/2 \rfloor$

$$d_\lambda(\mathsf{ED}_T) = d_\lambda(\mathcal{L}_T^q)(1 - q^2)^{n-2}. $$

**Proof:** For all $q \in \mathbb{R}$ with $q \neq \pm 1$, $\mathsf{ED}_T$ is invertible. By Lemma 38 we have

$$d_\lambda(\mathsf{ED}_T) \det \left( \frac{1}{1 - q^2} \mathcal{L}_T^q \right) = d_\lambda \left( \frac{1}{1 - q^2} \mathcal{L}_T^q \right) \det(\mathsf{ED}_T).$$

Thus, $d_\lambda(\mathsf{ED}_T) \det(\mathcal{L}_T^q) = d_\lambda(\mathsf{ED}_T) \det(\mathcal{L}_T^q)$. Hence, $d_\lambda(\mathsf{ED}_T) = d_\lambda(\mathcal{L}_T^q)(1 - q^2)^{n-2}$, completing the proof.

Combining Lemma 39 and Theorem 1 gives us another corollary whose straightforward proof we again omit.

**Corollary 40** Let $T_1$ and $T_2$ be two trees on $n$ vertices with $T_2 \geq_{\text{GTS}_n} T_1$. Then, for all $q \in \mathbb{R}$ with $q \neq \pm 1$ and for all $\lambda \vdash 2^k, 1^{n-2k}$, we have

$$|d_\lambda(\mathsf{ED}_{T_2})| \leq |d_\lambda(\mathsf{ED}_{T_1})|. $$
7.1 \( q, t \)-exponential distance matrix

We consider the bivariate exponential distance matrix in this subsection. Orient the tree \( T \) as done above. Thus each directed arc \( e \) of \( E(T) \) has a unique reverse arc \( e_{rev} \) and we assign a variable weight \( w(e) = q \) and \( w(e_{rev}) = t \) or vice versa. If the path \( P_{i,j} \) from vertex \( i \) to vertex \( j \) has the sequence of edges \( P_{i,j} = (e_1, e_2, \ldots, e_p) \), assign it weight \( w_{i,j} = \prod_{e_k \in P_{i,j}} w(e_k) \). Define \( w_{i,i} = 1 \) for \( i = 1, 2, \ldots, n \). Define the bivariate exponential distance matrix \( ED_{q,t}^T = (w_{i,j})_{1 \leq i,j \leq n} \). Clearly, when \( q = t \), we have \( ED_{q,t}^T = ED_T^q \). Bapat and Sivasubramanian in [3] showed the following bivariate counterpart of Lemma 35.

**Lemma 41 (Bapat, Sivasubramanian)** Let \( T \) be a tree with \( n \) vertices and let \( L_{q,t}^T \) and \( ED_{q,t}^T \) be its \( q,t \)-Laplacian and \( q,t \) exponential distance matrix respectively. Then, \( \det(ED_{q,t}^T) = (1 - qt)^{n-1} \) and if \( qt \neq 1 \), then

\[
(ED_{q,t}^T)^{-1} = \frac{1}{1 - qt} L_{q,t}^T.
\]

It is easy to see that all results about \( ED_T \) go through for the bivariate \( q,t \)-exponential distance matrix \( ED_{q,t}^T \) when \( q,t \in \mathbb{R} \) with \( qt \neq 1 \) or when \( q,t \in \mathbb{C} \) with \( qt \neq 1 \). In particular, Corollary 40 goes through for the bivariate exponential distance matrix.

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