An important issue is that the respiratory mortality may be a result of air pollution which can be measured by the following variables: temperature, relative humidity, carbon monoxide, sulfur dioxide, nitrogen dioxide, hydrocarbons, ozone and particulates. The usual way is to fit a model using the ordinary least squares regression, which has some assumptions, also known as Gauss-Markov assumptions, on the error term showing white noise process of the regression model. However, in many applications, especially for this example, these assumptions are not satisfied. Therefore, in this study, a quantile regression approach is used to model the respiratory mortality using the mentioned explanatory variables. Moreover, improved estimation techniques such as preliminary testing and shrinkage strategies are also obtained when the errors are autoregressive. A Monte Carlo simulation experiment, including the quantile penalty estimators such as Lasso, Ridge and Elastic Net, is designed to evaluate the performances of the proposed techniques. Finally, the theoretical risks of the listed estimators are given.

1. Introduction

Regression analysis is a statistical technique that is used to model the cumulative and linear relationship between covariates and response variables. The most common method used for this purpose is the ordinary least squares (OLS) method. The linear regression model can be written as follows:

\[ y_i = \beta_0 + \sum_{j=1}^{p} \beta_j x_{ij} + \varepsilon_i, \quad i = 1, 2, \ldots, n, \tag{1.1} \]

where \( y_i \)'s are the response variables, \( \beta_j \)'s are unknown regression coefficients, \( x_{ij} \)'s are known covariates and \( \varepsilon_i \)'s are unobservable random errors. When estimating the parameters using OLS method, the expectation of the dependent variable conditional on the independent variables is obtained. In other words, the relationship between the explanatory and explained variables in the coordinate plane is estimated with a mean regression line.

In order to use OLS estimator, there are three assumptions on the error terms showing white noise process of the regression model: (1) The error terms have zero mean, (2) The variance of the error terms is constant and (3) The covariance between the errors is zero i.e., there is no autocorrelation problem. In real life most of the data doesn’t provide these assumptions. Moreover, OLS provides a view of the relationship between covariates and response variable such that it models the expectation.
of the response conditional on the covariates without taking into account the outliers. To overcome these inadequacies of the classical regression Koenker and Bassett (1978) have proposed the quantile regression as an expansion of the classical regression model to a basic minimization problem which generates sample quantiles. For a random variable \( Y \) with distribution function \( F_Y(y) = P(Y \leq y) = \tau \) and \( 0 \leq \tau \leq 1 \), the \( \tau \)-th quantile function of \( Y \), \( Q_\tau(y) \), is defined to be

\[
Q_\tau(Y|X) = y_\tau = F_Y^{-1}(\tau) = \inf \{ y: F_Y(y) \geq \tau \} = x_i^\prime \beta_\tau
\]

where \( y_\tau \) is the inverse function of \( F_Y(\tau) \) for the \( \tau \)-th quantile, \( y = (y_1, y_2, \ldots, y_n) \), \( X = (x_1, \ldots, x_n)' \) and \( x_i = (x_{i1}, x_{i2}, \ldots, x_{ip})' \). In other words, the \( \tau \)-th quantile in a sample corresponds to the probability \( \tau \) for a \( y \) value. Also an estimation of the full model (FM) \( \tau \)-th quantile regression coefficients can be defined by solving the following minimization of problem

\[
\hat{\beta}_\tau = \arg \min_{\beta \in \mathbb{R}^p} \sum_{i=1}^n \rho_\tau(y_i - x_i^\prime \beta),
\]

where \( \rho_\tau(u) = u(\tau - I(u < 0)) \) is the quantile loss function. Hence, it yields

\[
\hat{\beta}_\tau = \arg \min_{\beta \in \mathbb{R}^p} \left[ \sum_{i \in \{y_i \geq x_i^\prime \beta\}} \tau | y_i - x_i^\prime \beta | - \sum_{i \in \{y_i < x_i^\prime \beta\}} (1 - \tau) | y_i - x_i^\prime \beta | \right].
\]

Koenker and Xiao (2006) proposed a quantile autoregression (QAR) model which could be interpreted as a special case of the general random-coefficient autoregression model with strongly dependent coefficients. The authors studied statistical properties of the proposed model and associated estimators and derived the limiting distributions of the autoregression quantile process. Koenker (2008) proposed the quantreg R package and it is implementations for linear, non-linear and non-parametric quantile regression models. The R software and the package quantreg are open-source software projects and can be freely downloaded from CRAN: http://cran.r-project.org. Čirić et al. (2012) compare different computational intelligence methodologies based on artificial neural networks used for forecasting an air quality parameter. Tang et al. (2015) proposed composite quantile regression for dependent data. The authors also showed the root-n consistency and asymptotic normality of the composite quantile estimator. Moreover, the authors apply their proposed method to NO\(_2\) particle data in which air pollution on a road is modeled via traffic volume and meteorological variables. Wang and Lin. (2015) proposed a penalized quantile estimator in semiparametric linear regression model and dealt with longitudinal data. The authors obtained the oracle properties of the estimator and selection consistency.

The books by Koenker (Koenker (2005)) and Davino (Davino et al. (2014)) are excellent sources for various properties of Quantile Regression as well as many computer algorithms. Moreover, Yi and Huang (2016) developed an algorithm, called semismooth Newton coordinate descent (SNC), to obtain a better efficiency and scalability for computing the solution paths of penalized quantile regression. They also provide an R package called hqreg. Moreover, this package also obtains Lasso of (Tibshirani (1996)), Ridge of (Hoerl and Kennard (1970)) and Elastic Net of (Zou and Hastie (2005)) estimators in the quantile regression models. The hqreg functions give the solution path while the quantreg package of Koenker (2013) computes a single solution.

On the other hand, the book of Ahmed (2014) can be found the large literature and informations about shrinkage estimations in the context of linear and partially linear models (PLMs). The preliminary and Stein-type estimations based on ridge regression are obtained by Yüzbasa et al. (2017a) for linear models and by Yüzbasa and Ahmed (2016) for PLMs. Furthermore, Yüzbasa et al. (2017b, c) introduced the pretest and shrinkage estimation based on the quantile regression when the errors are both i.i.d. and non-i.i.d, respectively. In these studies, asymptotic distributional bias, quadratic bias and risk functions are also obtained. The novelty of this study is the errors having the problem of autocorrelation which is very common in time series analysis.

The paper is organized as follows: in Section 2, we consider a real data example in order to examine the assumptions of the classical linear regression. The pretest, shrinkage estimators and penalized estimations are also given in Section 3. Also, we estimate the listed estimators in Section 4. The design and the results of a Monte Carlo simulation study including a comparison with other penalty
estimators are given in Section 5. The asymptotic distributional risk properties of the pretest and shrinkage estimators are obtained in Section 6. Finally, the concluding remarks are presented in Section 7.

2. Motivation Example

In this section, we consider the study of Shumway et al. (1988) of the possible effects of temperature and pollution on weekly mortality in Los Angeles (LA) Country. This data has 508 weekly observations from 1970 to 1979. In Table 1, we describe the variables of the cement data which is freely available in the astsa package with the function lap in R project.

| Variables       | Descriptions             |
|-----------------|--------------------------|
| **Response Variable** | Respiratory Mortality |
| rmort           |                          |
| **Predictors**  |                          |
| temp             | Temperature              |
| rh               | Relative Humidity        |
| co               | Carbon Monoxide          |
| so2              | Sulfur Dioxide           |
| no2              | Nitrogen Dioxide         |
| hycarb           | Hydrocarbons             |
| o3               | Ozone                    |
| part             | Particulates             |

Table 1. Descriptions of variables for the LA Pollution-Mortality data set

The Figure 1 shows that the observations 152, 153 and 155 may be outliers. Applying outlierTest function in the car package in R, according to the results, we observe that the observations 152 – 155 and 260 are outliers. We also observe that the errors follow a heavy-tailed distribution.

According to Figure 1 and Table 2, the residuals of this data have AR(5) process. Also, we consider the values of $d_L$ and $d_U$ as 1.686 and 1.852 respectively. Hence, there is a positive autocorrelation problem on this data.
3. Statistical Model

Linear regression model in (1.1) would be written in a partitioned form as follows

\[ y_i = x_i' \beta_1 + x_{i2}' \beta_2 + \varepsilon_i, \quad i = 1, 2, \ldots, n, \quad (3.1) \]

where \( \beta = (\beta_1', \beta_2')' \) is partitioned so that the coefficient vector of \( \beta_1 = (\beta_1, \beta_2, \ldots, \beta_{p_1})' \), of order \( p_1 \), is our main interest and the coefficient vector of \( \beta_2 = (\beta_{p_1+1}, \beta_{p_1+2}, \ldots, \beta_p)' \) is the “irrelevant variables” with dimension \( p_2 \), where \( p = p_1 + p_2 \). Also, \( x_i = (x_{i1}', x_{i2}') \) and \( \varepsilon_i \) are errors with the same joint distribution function \( F \). The conditional quantile function of response variable \( y_i \) can be written as follows

\[ Q_{\tau}(y_i|x_i) = x_i' \beta_{1,\tau} + x_{i2}' \beta_{2,\tau}, \quad 0 < \tau < 1 \quad (3.2) \]

In this study, the main interest is to improve the performance of the important covariates under the following the null hypothesis

\[ H_0 : \beta_{2,\tau} = 0_{p_2}. \quad (3.3) \]

If the Equation (3.3) is true, then the sub-model (SM) quantile regression estimator of \( \beta_{\tau} \) is given by \( \tilde{\beta}_{\tau} = (\tilde{\beta}_{1,\tau}, 0_{p_2}) \), where \( \tilde{\beta}_{1,\tau} = \min_{\beta_1 \in \mathbb{R}^{p_1}} \sum_{i=1}^{n} \rho_{\tau}(y_i - x_{i1}' \beta_1) \).

The distribution function \( F_i \) is absolutely continuous, with continuous densities \( f_i(\xi_i) \) uniformly bounded away from 0 and \( \infty \) at the points \( \xi_i(\tau), \ i = 1, 2, \ldots \)

(i) \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i x_i' = D_0, \quad D_0 = \frac{1}{n} X'X \)

(ii) \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f_i(\xi_i(\tau)) x_i x_i' = D_1 \)

(iii) \( \max_{1 \leq i \leq n} ||x_i||/\sqrt{n} \to 0 \)

where \( D_0 \) and \( D_1 \) are positive definite matrices.

3.1. Pretest and Stein-Type Estimations. The pretest was firstly applied by Bancroft (1944) for the validity of the unclear preliminary information (UPI) by subjecting it to a preliminary test. The pretest estimator (PT) could be obtained by following equation

\[ \hat{\beta}_{\tau}^{PT} = \tilde{\beta}_{\tau} - (\hat{\beta}_{\tau} - \tilde{\beta}_{\tau}) I(W < c_{n,\alpha}) \quad (3.4) \]
where $I(\cdot)$ is an indicator function of a set and $c_{n,\alpha}$ is the 100 $(1 - \alpha)$ percentage point of the $W$. In order to test (3.3), under the above assumptions, we consider the following Wald test statistics

$$W = nW^{-2}\hat{\beta}_r^\top W^{-1} \Gamma_{22,1} \hat{\beta}_r$$

(3.5)

where $\Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} = D^{-1}A D^{-1}$, $A = \lim_{n \to \infty} \frac{1}{n} \sum_{i} \sum_{j} \psi(e_i) \psi(e_j) x_i x_j$, the median $\psi(e_i) = \text{sgn}(e_i)$ and $\Gamma_{22,1} = \Gamma_{22} - \Gamma_{21} \Gamma_{11}^{-1} \Gamma_{12}$. Under the null hypothesis, the distribution of $W$ follows the chi-square distribution with $p_2$ degree of freedom.

The Stein-type shrinkage (S) estimator is a combination of the over–fitted model estimator $\hat{\beta}_r$ with the under–fitted estimator $\hat{\beta}_r$, given by

$$\hat{\beta}_r^S = \hat{\beta}_r - d \left( \hat{\beta}_r - \hat{\beta}_r \right) W_n^{-1}, \quad d = (p_2 - 2) \geq 3,$$

In an effort to avoid the over-shrinking problem inherited by $\hat{\beta}_r^S$, we suggest using the positive part of the shrinkage (PS) estimator defined by

$$\hat{\beta}_r^{PS} = \hat{\beta}_r^S - \left( \hat{\beta}_r - \hat{\beta}_r \right) (1 - d W_n^{-1}) I(W_n \leq d).$$

3.2. Quantile Penalized Estimation. We briefly mention about the penalized estimators, given by Yi and Huang (2016) in quantile regression in a general form as follows:

$$\hat{\beta}_r^{\text{Penalized}} = \arg \min_{\beta} \sum_{i} \rho(y_i - x_i' \beta) + \lambda P(\beta)$$

(3.6)

where $\rho$ is a quantile loss function, $P$ is a penalty function and $\lambda$ is a tuning parameter. Also,

$$P(\beta) \equiv P_{\alpha}(\beta) = \alpha \| \beta \|_1 + \frac{(1 - \alpha)}{2} \| \beta \|_2^2$$

which is the Lasso penalty for $\alpha = 1$ (Tibshirani (1996)), the Ridge penalty for $\alpha = 0$ (Hoerl and Kennard (1970)) and the Elastic-net penalty for $0 \leq \alpha \leq 1$ (Zou and Hastie (2005)).

4. Motivation Example Cont.

In order to apply the proposed methods, we use a two step approach as follows:

Step 1: A set of covariates are selected based on a suitable model selection technique since the prior information is not available here.

Step 2: The full and sub-model estimates are combined in such a way that minimizes the quadratic risk.

For Step 1, one may use the model selection criterion such as AIC, BIC or best subset selection. We, however, use BIC. In Table 4, we show the full and candidate sub-model.

| Models   | Formulas           |
|----------|--------------------|
| Full Model | $\text{r} \text{m} \text{ort} = \beta_0 + \beta_1 \text{temp} + \beta_2 \text{rh} + \beta_3 \text{co} + \beta_4 \text{so}2 + \beta_5 \text{no}2 + \beta_6 \text{hycarb} + \beta_7 \text{r} \text{o} 3 + \beta_8 \text{s} \text{part}$ |
| Sub-Model | $\text{r} \text{m} \text{ort} = \beta_0 + \beta_1 \text{temp} + \beta_2 \text{co}$ |

| Table 4. The full and candidate sub-model |

Figure 2 presents a summary of the OLS and the FM quantile regression results. Here, we have 8 covariates, plus an intercept. For each of the 9 coefficients, we plot the 19 distinct quantile regression estimates for $\tau$ ranging from 0.05 to 0.95 as the solid curve with filled dots. For each covariate, these point estimates may be interpreted as the impact of a one-unit change of the covariate on the response variable respiratory mortality other covariates fixed. Thus, each of the plots has a horizontal quantile, or $\tau$, scale, and the vertical axes indicates the covariate effect. The solid line in each figure shows the OLS estimate of the conditional mean effect. The two dotted lines represent conventional 90 percent confidence intervals for the OLS estimate. The shaded gray area depicts a 90 percent point-wise confidence band for the quantile regression estimates.
We will confine our discussion as follows: The intercept estimates seem more dependent on the particular quantile. For example, up to the third quantile, quantile estimates are lower than the OLS while it is larger than the OLS for the upper quantile. With the exception of the coefficients co, hycarb and o3, the quantile regression estimates lie at some point outside the confidence intervals for the OLS regression, suggesting that the effects of these covariates may change across the conditional distribution of the independent variable.

In order to analyze this example, we bootstrap the data using 1000 resamplings. After that, we split the data into two which are training and test data sets. Furthermore, we center the co-variates of training and test data set based on the training data set independently. Finally, we computed the predictive mean absolute deviation (PMAD) criterion which is defined by

$$\text{PMAD}(\hat{\beta}^*_\tau) = \frac{1}{n_{\text{test}}} \sum_{i=1}^{n_{\text{test}}} |y_{\text{test}}^i - X_{\text{test}}^i \hat{\beta}^*_\tau|.$$ 

We evaluate the performance of the estimators by averaged cross validation (CV) error using a 5-fold CV. In Table 5, we report the performance of the estimator in the sense of PMAD for the real data application. As expected, the SM estimator has the lowest PMAD value for all \(\tau\) values. The PS performs better than the Lasso, Elastic-net, FM and OLS, especially in the first and second quantile (median), while the Ridge outperforms all others since the data has highly the problem of multicollinearity. Also, the performance of PT is also well in median.

| \(\tau\)  | FM  | SM  | PT  | PS  | Ridge | Lasso | ENET |
|----------|-----|-----|-----|-----|-------|-------|------|
| 0.25     | 2.612 | 2.209 | 2.612 | 2.515 | 2.337 | 2.531 | 2.535 |
| 0.5      | 2.391 | 1.881 | 2.359 | 2.249 | 2.220 | 2.372 | 2.372 |
| 0.75     | 3.082 | 2.041 | 3.063 | 2.802 | 2.275 | 2.557 | 2.469 |

| OLS Mean | 2.803 |

**Table 5.** The PMAD values of the listed estimations
5. Simulation

We conduct Monte-Carlo simulation experiments to study the performances of the proposed estimators under various practical settings. In order to generate the response variable, we use

\[ y_i = x_i' \beta + \varepsilon_i, \quad i = 1, \ldots, n, \]

where \( x_i \)'s are standard normal. The correlation between the \( j \)th and \( k \)th components of \( X \) equals 0.5\(|j-k|\) and also \( \varepsilon_i \)'s are dependent.

We consider \( \beta' = (3, 1.5, 0, 2, 0, 0, 0) \). Also, we simulate data which contains a training dataset, validation set and an independent test set. Note that the co-variates are scaled to have mean zero and unit variance. We fitted the models only using the training data and the tuning parameters were selected using the validation data. We also use the notation \( \cdot \cdot \cdot \cdot \cdot \cdot \cdot \) to describe the number of observations in the training, validation and test sets respectively. Hence, we consider that each data set consists of 50/50/200 observations and \( X \sim N(0, \Sigma) \), where \( \Sigma_{ij} = 0.5|i-j| \). Furthermore, the errors follow AR(1) process, that is,

\[ \varepsilon_i = \rho \varepsilon_{i-1} + \omega_t \]

where \(|\rho| < 1\) is called the “autocorrelation parameter” and the \( \omega_t \) term is a new error term that follows the usual regression assumptions: \( \omega_t \sim iid N(0, 1) \).

| \( \tau \) | \( \rho = -0.2 \) | \( \rho = 0.2 \) | \( \rho = -0.5 \) | \( \rho = 0.5 \) |
|---|---|---|---|---|
| 0.25 | FM 0.190(0.004) | 0.188(0.004) | 0.206(0.005) | 0.220(0.004) |
| | SM 0.060(0.002) | 0.057(0.002) | 0.069(0.002) | 0.063(0.002) |
| | PT 0.078(0.005) | 0.076(0.006) | 0.090(0.007) | 0.083(0.006) |
| | PS 0.120(0.004) | 0.134(0.005) | 0.143(0.005) | 0.149(0.005) |
| Ridge | 0.135(0.003) | 0.139(0.004) | 0.154(0.003) | 0.158(0.003) |
| Lasso | 0.081(0.002) | 0.076(0.003) | 0.087(0.003) | 0.089(0.002) |
| ENET | 0.078(0.002) | 0.074(0.003) | 0.085(0.003) | 0.087(0.002) |
| 0.5 | FM 0.173(0.004) | 0.165(0.003) | 0.199(0.005) | 0.191(0.004) |
| | SM 0.055(0.002) | 0.053(0.002) | 0.058(0.002) | 0.057(0.002) |
| | PT 0.059(0.005) | 0.061(0.004) | 0.072(0.006) | 0.069(0.006) |
| | PS 0.103(0.004) | 0.098(0.004) | 0.126(0.005) | 0.122(0.005) |
| Ridge | 0.133(0.003) | 0.124(0.003) | 0.150(0.003) | 0.149(0.003) |
| Lasso | 0.073(0.002) | 0.073(0.002) | 0.078(0.003) | 0.077(0.003) |
| ENET | 0.072(0.002) | 0.072(0.002) | 0.078(0.003) | 0.075(0.003) |
| 0.75 | FM 0.183(0.004) | 0.184(0.004) | 0.217(0.005) | 0.210(0.005) |
| | SM 0.060(0.002) | 0.059(0.002) | 0.062(0.002) | 0.066(0.002) |
| | PT 0.082(0.005) | 0.072(0.005) | 0.080(0.006) | 0.090(0.006) |
| | PS 0.121(0.004) | 0.115(0.004) | 0.146(0.005) | 0.144(0.005) |
| Ridge | 0.140(0.003) | 0.139(0.003) | 0.161(0.004) | 0.159(0.004) |
| Lasso | 0.080(0.002) | 0.074(0.002) | 0.087(0.003) | 0.089(0.003) |
| ENET | 0.078(0.002) | 0.073(0.002) | 0.085(0.003) | 0.083(0.003) |
| Mean | OLS 0.137(0.009) | 0.136(0.009) | 0.154(0.010) | 0.154(0.010) |

Table 6. Simulated PMAD values of estimators, and the values in parenthesis present the standard errors of each estimation

Table 6 presents an outline summary for the different illustrative models used in the case of autoregressive errors where \( \rho = \pm 5 \) characterized by heavier tails while \( \rho = \pm 2 \) corresponds to the median. First, we note that the OLS fails against to quantile-type estimations. As expected, the SM has the lowest the PMAD value since the data is generated from an empirical distribution where the candidate sub-model is nearly true. Furthermore, the pretest and positive shrinkage estimators are
superior to the FM estimator. On the other hand, the results indicate that the PT mostly performs better than penalty estimators while positive shrinkage does not have a good performance due to the small value of $p_1$.

6. Theoretical Results

In this section, we demonstrate the asymptotic risk properties of suggested estimators. So, we consider the following theorem.

**Theorem 6.1.** The distribution of quantile regression model with AR(1) process is given by

$$\sqrt{n}(\hat{\beta}_t - \beta) \overset{D}{\rightarrow} N(0, \omega^2 \Gamma)$$

(6.1)

**Proof.** The proof can be obtained from (Davino et al. (2014)) □

Let a sequence of local alternatives $\{K_n\}$ given by

$$K_n : \beta_{2,t} = \frac{\gamma}{\sqrt{n}}$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{p_2})' \in \mathbb{R}^{p_2}$ is a fixed vector. If $\gamma = 0_{p_2}$, then the null hypothesis is true. Furthermore, we consider the following proposition to establish the asymptotic properties of the estimators.

**Proposition 6.2.** Let $\vartheta_1 = \sqrt{n}(\hat{\beta}_{1,t} - \beta_{1,t})$, $\vartheta_2 = \sqrt{n}(\hat{\beta}_{1,t} - \beta_{1,t})$ and $\vartheta_3 = \sqrt{n}(\hat{\beta}_{1,t} - \hat{\beta}_{1,t})$.

Under the regularity assumptions (i)–(iii), Theorem 6.1 and the local alternatives $\{K_n\}$, as $n \to \infty$ we have the joint distributions are given as follows:

$$\begin{pmatrix} \vartheta_1 \\ \vartheta_3 \end{pmatrix} \sim N \left[ \begin{pmatrix} 0_{p_1} \\ \omega^2 \Sigma_{11}^{-1}^{12} \Sigma_{12}^{12} \Phi \end{pmatrix} \right],$$

$$\begin{pmatrix} \vartheta_3 \\ \vartheta_2 \end{pmatrix} \sim N \left[ \begin{pmatrix} -\delta \\ \Phi \Sigma^* \omega^2 \Sigma_{11}^{-1}^{12} \end{pmatrix} \right],$$

where $\delta = \Gamma_{11}^{-1} \Gamma_{12} \gamma$, $\Phi = \Gamma_{11}^{-1} \Gamma_{12} \Gamma_{22} \Gamma_{21} \Gamma_{11}^{-1}$, $\Sigma_{12} = -\Gamma_{12} \Gamma_{21} \Gamma_{11}^{-1}$ and $\Sigma^* = \Sigma_{21} + \omega^2 \Sigma_{11}^{-1}^{12}$.

Now, we are ready to obtain the asymptotic distributional risks of estimators which are given in the following section.

6.1. The performance of Risk. The asymptotic distributional risk of an estimator $\hat{\beta}_{1,t}$ is defined as

$$\mathcal{R} \left( \hat{\beta}_{1,t} \right) = \text{tr} \left( \mathbf{W} \Gamma \right)$$

where $\mathbf{W}$ is a positive definite matrix of weights with dimensions of $p \times p$, and $\Gamma$ is the asymptotic covariance matrix of an estimator $\hat{\beta}_{1,t}$ is defined as

$$\Gamma \left( \hat{\beta}_{1,t} \right) = \mathbb{E} \left\{ \lim_{n \to \infty} n \left( \hat{\beta}_{1,t} - \beta_{1,t} \right) \left( \hat{\beta}_{1,t} - \beta_{1,t} \right)' \right\}.$$ 

**Theorem 6.3.** Under the assumed regularity conditions in (i) and (ii), the Proposition 6.2, the Theorem 6.1 and $\{K_n\}$, the expressions for asymptotic risks for listed estimators are:

$$\begin{align*}
\mathcal{R} \left( \hat{\beta}_{1,t} \right) &= \omega^2 \text{tr} \left( \mathbf{W} \Gamma_{11}^{-1} \right) \\
\mathcal{R} \left( \hat{\beta}_{1,t} \right) &= \omega^2 \text{tr} \left( \mathbf{W} \Gamma_{11}^{-1} \right) + \text{tr} \left( \mathbf{W} \mathbf{M} \right), \text{where} \left( \mathbf{M} = \Gamma_{11}^{-1} \Gamma_{12} \gamma \gamma' \Gamma_{21} \Gamma_{11}^{-1} = \delta \delta' \right) \\
\mathcal{R} \left( \hat{\beta}_{1,t}^{\text{PT}} \right) &= \mathcal{R} \left( \hat{\beta}_{1,t} \right) + \omega^2 \text{tr} \left( \mathbf{W} \Gamma_{11}^{-1} \Gamma_{12} \Gamma_{22} \Gamma_{21} \Gamma_{11}^{-1} \right) + \text{tr} \left( \delta \mathbf{W} \delta' \right) \mathbb{H}_{d+4} \left( \chi_{d+2,\alpha}^2 (\Delta) \right) \\
&\quad + \mathbf{W} \Phi \mathbb{H}_{d+4} \left( \chi_{d+2,\alpha}^2 (\Delta) \right) + \text{tr} \left( \delta \mathbf{W} \delta' \right) \mathbb{H}_{d+6} \left( \chi_{d+2,\alpha}^2 (\Delta) \right) \\
\mathcal{R} \left( \hat{\beta}_{1,t}^s \right) &= \mathcal{R} \left( \hat{\beta}_{1,t} \right) - 2d \mathbb{E} \left\{ \chi_{d,\alpha}^{-2} (\Delta) \right\} \text{tr} \left( \mathbf{W} \mathbf{S}_{21} \right) \\
&\quad - 2d \mathbb{E} \left\{ \chi_{d+2,\alpha}^{-2} (\Delta) \right\} \text{tr} \left( \mathbf{W} \delta \delta' \Sigma^* \mathbf{S}_{21} \right)
\end{align*}
rest of the parameter space. It can be seen that the performance of the listed estimators in a real-world example using the data analyzed by Davino, Furno, and Vistocco (2014) shows that even when the distribution of errors have the problem of autocorrelation. Also, we investigated the performance of the proposed estimators. Our asymptotic theory is well supported by numerical analysis.

\[
\mathcal{R} \left( \hat{\beta}_{PS}^* \right) = \mathcal{R} \left( \hat{\beta}_{S}^* \right) - 2\mathbb{E} \left( 1 - d\chi_{d+4}^2 (\Delta) \right) \mathbb{I} \left( \chi_{d+4}^2 (\Delta) < d \right) \text{tr} (W \Sigma_1) - 2\mathbb{E} \left( 1 - d\chi_{d+4}^2 (\Delta) \right) \mathbb{I} \left( \chi_{d+4}^2 (\Delta) < d \right) \text{tr} \left( W \Sigma_1 \right)
\]

Noting that if \( \Gamma_{12} = 0 \), then all the risks reduce to common value \( \omega^2 \text{tr} \left( W \Gamma_{11}^{-1} \right) \) for all \( W \). For \( \Gamma_{12} \neq 0 \), the risk of \( \hat{\beta}_{1,\tau} \) remains constant while the risk of \( \hat{\beta}_{1,\tau} \) is an bounded function of \( \Delta \) since \( \Delta \in [0, \infty] \). The risk of \( \hat{\beta}_{1,\tau} \) increases as \( \Delta \) moves away from zero, achieves its maximum and then decreases towards the risk of the full model estimator. Thus, it is a bounded function of \( \Delta \). The risk of \( \hat{\beta}_{1,\tau} \) is smaller than the risk of \( \hat{\beta}_{1,\tau} \) for some small values of \( \Delta \) and opposite conclusions hold for the rest of the parameter space. It can be seen that

\[
\mathcal{R} \left( \hat{\beta}_{1,\tau}^* \right) \leq \mathcal{R} \left( \hat{\beta}_{1,\tau}^* \right) \leq \mathcal{R} \left( \hat{\beta}_{1,\tau}^* \right),
\]

strictly inequality holds for small values of \( \Delta \). Thus, positive shrinkage is superior to the shrinkage estimator. However, both shrinkage estimators outperform the full model estimator in the entire parameter space induced by \( \Delta \). On the other hand, the pretest estimator performs better than the shrinkage estimators when \( \Delta \) takes small values and outside this interval the opposite conclusion holds.

7. Conclusions

In this paper, we obtained pretest and Stein-type shrinkage estimations based on quantile regression when the distribution of errors have the problem of autocorrelation. Also, we investigated the performance of the listed estimators in a real-world example using the data analyzed by Shumway et al. (1988) such that the effects of air pollution and temperature on weekly mortality in LA are considered. The results showed that the quantile type estimators outperform the OLS. Not surprisingly, the SM estimator has the lowest PMAD since the candidate sub-model is assumed as true. Furthermore, the PT and PS perform better than the FM. Also, the performance of the proposed estimators are mostly superior to penalty estimators, moreover Ridge has a better performance since the data has the multicollinearity problem. On the other hand, we conducted a Monte Carlo simulation study in order to investigate the performance of the suggested estimators. The results of simulation study coincide with the results of real data example. Finally, we demonstrated the asymptotic distributional risk performance of the listed estimators. Our asymptotic theory is well supported by numerical analysis.

References

Ahmed, S. E. (2014). Penalty, Shrinkage and Pretest Strategies: Variable Selection and Estimation. Springer, New York.

Bancroft, T. A. (1944). On biases in estimation due to the use of preliminary tests of significance. The Annals of Mathematical Statistics, 15(2), 190-204.

Čirić, I. T., Cojbašić, Z. M., Nikolić, V. D., Živković, P. M., & Tomić, M. A. (2012). Air quality estimation by computational intelligence methodologies. Thermal Science, 16(suppl. 2), 493–504.

Davino, C., Furno, M. and Vistocco, D. (2014). Quantile Regression: Theory and Applications, John Wiley & Sons, Ltd.

Hoerl, A. E., Kennard, R. W. (1970). Ridge Regression: Biased estimation for non-orthogonal problems. Technometrics 12, 69 – 82.
Koenker, R., and Bassett Jr, G. (1978). Regression quantiles. Econometrica: journal of the Econometric Society, 33-50.

Koenker, R. (2005). Quantile regression (No. 38). Cambridge university press.

Koenker, R. (2008). Quantile regression in R: a vignette CRAN.

Koenker, R., and Xiao, Z. (2006). Quantile autoregression. Journal of the American Statistical Association, 101(475), 980-990.

Koenker, R. (2013). quantreg: Quantile Regression. R package version 5.05. R Foundation for Statistical Computing: Vienna) Available at: http://CRAN.R-project.org/package=quantreg.

R Development Core Team (2016). R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria. ISBN 3-900051-07-0.

Shumway, R. H., Azari, A. S., and Pawitan, Y. (1988). Modeling mortality fluctuations in Los Angeles as functions of pollution and weather effects. Environmental Research, 45(2), 224-241.

Tang, Y., Song, X., and Zhu, Z. (2015). Variable selection via composite quantile regression with dependent errors. Statistica Neerlandica, 69(1), 1-20.

Tibshirani, R. (1996). Regression shrinkage and selection via the lasso. Journal of the Royal Statistical Society. Series B (Methodological), 267-288.

Yüzbashi, B., Ejaz Ahmed, S. (2016). Shrinkage and penalized estimation in semi-parametric models with multicollinear data. Journal of Statistical Computation and Simulation, 1-19.

Yüzbashi, B., Ahmed. S.E., Gungör, M. (2017). Improved Penalty Strategies in Linear Regression Models, REVSTAT-Statistical Journal, 15(2)(2017), 251–276.

Yüzbashi, B., Asar, Y., and Şik, M.Ş. (2017). Pretest and Stein-Type Estimations in Quantile Regression Model.

Yüzbashi, B., Asar, Y., and Demiralp, A. (2017). Pretest and Stein-Type Estimations in Quantile Regression Model with non-iid Errors.

Wang, K., and Lin, L. (2015). Variable Selection in Semiparametric Quantile Modeling for Longitudinal Data. Communications in Statistics-Theory and Methods, 44(11), 2243-2266.

Yi, C., and Huang, J. (2016). Semismooth Newton Coordinate Descent Algorithm for Elastic-Net Penalized Huber Loss Regression and Quantile Regression. Journal of Computational and Graphical Statistics, (just-accepted).

Zou, H. and Hastie, T. (2005). Regularization and variable selection via the elastic net, Journal of the Royal Statistical Society:Series B (Statistical Methodology), 67(2), 301-320.