Production cross sections of $^3\Lambda^4\text{H}$ bound states in $^3,^4\text{He}(K^-, \pi^0)$ reactions at 1 GeV/c

Toru Harada$^{1,2,*}$ and Yoshiharu Hirabayashi$^3$

$^1$Center for Physics and Mathematics, Osaka Electro-Communication University, Neyagawa, Osaka, 572-8530, Japan
$^2$J-PARC Branch, KEK Theory Center, Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK), 203-1, Shirakata, Tokai, Ibaraki, 319-1106, Japan
$^3$Information Initiative Center, Hokkaido University, Sapporo, 060-0811, Japan

(Dated: August 18, 2021)

Abstract

We investigate theoretically productions of $^3\Lambda^4\text{H}$ bound states in the exothermic ($K^-, \pi^0$) reactions on $^3,^4\text{He}$ targets at $p_{K^-} = 1.0$ GeV/c in a distorted-wave impulse approximation with the optimal Fermi-averaging $K^-p \to \pi^0\Lambda t$ matrix. We calculate angular distributions of the laboratory differential cross sections $d\sigma/d\Omega_{\text{lab}}$ and the integrated cross sections $\sigma_{\text{lab}}$ for a $J^P = 0^+$ ground state and a $J^P = 1^+$ excited state of $^4\Lambda\text{H}$ at $\pi^0$ forward-direction angles of $\theta_{\text{lab}} = 0^\circ$–$20^\circ$, and those for a $J^P = 1/2^+$ ground state of $^3\Lambda\text{H}$. The production of a $J^P = 3/2^+$ excited state of $^3\Lambda\text{H}$ as a virtual state is also evaluated. The comparison in $d\sigma/d\Omega_{\text{lab}}$ and $\sigma_{\text{lab}}$ between $^4\Lambda\text{H}$ and $^3\Lambda\text{H}$ provides examining the mechanism of the production and structure of $^3\Lambda^4\text{H}$, as well as in the endothermic ($\pi^-, K^0$) reactions at $p_{\pi^-} = 1.05$ GeV/c. This investigation confirms the feasibility of lifetime measurements of $^3\Lambda\text{H}$ at the J-PARC experiments.

PACS numbers: 21.80.+a, 24.10.Ht, 27.30.+t, 27.80.+w

Keywords: Hypernuclei, DWIA, Cross section

*Electronic address: harada@osakac.ac.jp
I. INTRODUCTION

Recently, experimental measurements of a $^3\Lambda\text{H}$ lifetime are planned by $(K^-, \pi^0)$ and $(\pi^-, K^0)$ reactions on a $^3\text{He}$ target at J-PARC \cite{1, 2} to solve the puzzle that the unexpected short lifetime of $^3\Lambda\text{H}$ was measured in hypernuclear production by high-energy heavy-ion collisions \cite{3, 4}. It seems that it is rather difficult to form a $\Lambda$ bound state by the nuclear $(K^-, \pi^0)$ and $(\pi^-, K^0)$ reactions because the $\Lambda$ hyperon is very weakly bound in $^3\Lambda\text{H}$ with a $J^P = 1/2^+$ ground-state (g.s.) separation energy $B_\Lambda = 0.13 \pm 0.05 \text{ MeV}$ \cite{5} with respect to the $d$-$\Lambda$ threshold, whereas a recent measurement by the STAR Collaboration reports a value of $0.41 \pm 0.12 \text{ MeV}$ \cite{6}.

On the other hand, the production of a $J^P = 0^+$ ground state of $^4\Lambda\text{H}$ in the $^4\text{He}(K^-, \pi^-)$ reaction is accomplished theoretically \cite{7, 8} and experimentally \cite{9, 10}, where the $\Lambda$ is bound with $B_\Lambda = 2.39 \pm 0.05 \text{ MeV}$ with respect to the $^3\text{He}$-$\Lambda$ threshold. In a previous paper \cite{11}, we reexamined the production cross section of the $0^+$, g.s. state of $^4\Lambda\text{H}$ in the $^4\text{He}(\pi, K)$ reaction at $p_{\pi^-} = 1.05 \text{ GeV/c}$, and discussed a benefit of the use of a $s$-shell target nucleus for $\Lambda$ production of the $A = 4$ hypernucleus. To study the feasibility of the lifetime measurements of $^3\Lambda\text{H}$ in the production followed by mesonic decay processes \cite{1}, thus, it is worth examining theoretically the $\Lambda$ production of the $A = 3$ hypernucleus via the $(K^-, \pi^0)$ or $(\pi^-, K^0)$ reaction.

It has been recently discussed \cite{12} that there is a $s$-wave virtual state with $J^P = 3/2^+$, $L = 0$ near the $d$-$\Lambda$ threshold in the $d + \Lambda$ system, which may correspond to a $J^P = 3/2^+$ excited state (exc) of $^3\Lambda\text{H}$ that has not yet been observed experimentally. Thus the production of the $3/2^+_{\text{exc}}$ state of $^3\Lambda\text{H}$ via the $^3\text{He}(K^-, \pi^0)$ reaction also needs to be examined from a theoretical point of view.

In this paper, we investigate theoretically the productions of the $^3\Lambda^4\text{H}$ bound states in the exothermic $(K^-, \pi^0)$ reactions on $^3\Lambda^4\text{He}$ targets at $p_{K^-} = 1.0 \text{ GeV/c}$ in a distorted-wave impulse approximation (DWIA) using the optimal Fermi-averaging $K^-p \to \Lambda\pi^0$ t matrix \cite{13}. We demonstrate angular distributions of the laboratory differential cross sections for $^4\Lambda\text{H} (J^P = 0^+$, g.s.), $^4\Lambda\text{H} (J^P = 1^+$, exc), and $^3\Lambda\text{H} (J^P = 1/2^+$, g.s.) bound states in the $\pi^0$ forward-direction angles of $\theta_{\text{lab}} = 0^\circ - 20^\circ$, and the integrated cross sections for them. We also investigate the production of $^3\Lambda\text{H} (J^P = 3/2^+$, exc) as a virtual state close to the $d$-$\Lambda$ threshold. We discuss the effects of a weakly $\Lambda$ binding, a meson distortion, and an in-medium elementary amplitude in the nuclear $(K^-, \pi^0)$ reactions, as well as in
the endothermic ($\pi^-, K^0$) reaction \[11\]. To reduce uncertainties of several approximations and input parameters in our calculations, we attempt to examine the difference in the $\Lambda$ production between $\Lambda^4_{\text{H}}$ and $\Lambda^3_{\text{H}}$.

II. CALCULATIONS

A. Distorted-wave impulse approximation

Let us consider a calculation procedure of the $\Lambda$ hypernuclear production for the nuclear ($K^-, \pi$) reaction in the laboratory frame. We will present briefly the standard DWIA calculation \[14–16\] applying to the productions of the $\Lambda^3_{\text{H}}, \Lambda^4_{\text{H}}$ bound states in the reactions

\[ K^- + ^3\text{He} \rightarrow \pi^0 + ^3\Lambda_{\text{H}}. \]  

The differential cross section for the $\Lambda$ bound state with a spin-parity $J^P_B$ at the $\pi^0$ forward-direction angle of $\theta_{\text{lab}}$ is written \[11\] by (in units $\hbar = c = 1$)

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\text{lab}, \theta_{\text{lab}}}^{J^P_B} = \alpha \frac{1}{|J_A|} \sum_{m_A m_B} \left| \langle \Psi_B | \vec{f}_{\pi^0_A} + i \vec{g}_{\pi^0_A} \sigma \cdot \hat{n} \right| \chi^{(-)}_{\pi^0} \left( \frac{p}{M_B} \right) \chi^{(+)}_{K^-} \left( \frac{p}{M_A} \right) \left| \Psi_A \right|^2,
\]

where $[J] = 2J + 1$, and $\Psi_B$ and $\Psi_A$ are wave functions of the hypernuclear final state and the initial state of the target nucleus, respectively. The kinematical factor $\alpha$ denotes the translation from a two-body $K^-$-nucleon laboratory system to a $K^-$-nucleus laboratory system \[17\]. $\hat{n}$ is a unit vector perpendicular to the reaction plane. $\chi^{(-)}_{\pi^0}$ and $\chi^{(+)}_{K^-}$ are meson distorted waves for outgoing $\pi^0$ and incoming $K^-$, respectively. The factors of $M_C/M_B$ and $M_C/M_A$ arise from the recoil correction, where $M_A$, $M_B$ and $M_C$ are the masses of the target, the hypernucleus, and the core nucleus, respectively. The energy and momentum transfers to the final state are given by

\[
\omega = E_{K^-} - E_{\pi^0}, \quad q = p_{K^-} - p_{\pi^0},
\]

where $E_{K^-} = (p_{K^-}^2 + m_{K^-}^2)^{1/2}$ and $E_{\pi^0} = (p_{\pi^0}^2 + m_{\pi^0}^2)^{1/2}$ ($p_{K^-}$ and $p_{\pi^0}$) are the laboratory energies (momenta) of $K^-$ and $\pi^0$ in this nuclear reaction, respectively; $m_{K^-}$ and $m_{\pi^0}$ are
the masses of $K^-$ and $\pi^0$, respectively. $\mathcal{T}_{\pi^0\Lambda}$ and $\mathcal{G}_{\pi^0\Lambda}$ describe the non-spin-flip $\Delta S = 0$ and spin-flip $\Delta S = 1$ amplitudes for the in-medium $K^- p \rightarrow \pi^0 \Lambda$ production, respectively, which take into account the Fermi motion of a struck nucleon in the nuclear target for the nuclear $(K^-, \pi^0)$ reaction. The explicit forms of Eq. (1) are given by Appendix A.

The Fermi-averaging treatment may essentially affect the absolute values of the production cross sections even through $^3\Lambda_H$ [18]. Here we apply the optimal Fermi-averaging procedure [13] to $\mathcal{T}_{\pi^0\Lambda}$ and $\mathcal{G}_{\pi^0\Lambda}$ in the nuclear $(K^-, \pi^0)$ reaction; the momentum distribution $\rho(p)$ of the struck nucleon in the $^3\Lambda_H$ ($^4\Lambda_H$) reaction is assumed as a simple harmonic oscillator with a size parameter $b_N = 1.61$ (1.33) fm, leading to $\langle p^2 \rangle^{1/2} \simeq 150$ (182) MeV/$c$ in the nucleus. We employ the elementary $KN \rightarrow \pi\Lambda$ amplitude analyzed by Gopal, et al. [20].

The distorted waves of $\chi_{\pi^-}$ and $\chi_{K^-}$ are obtained in a computational procedure simplified with the help of the eikonal approximation [14, 15]. We choose $\alpha_{K^-} = \alpha_{\pi} = 0$, $\sigma_{K^-} = 45$ mb, and $\sigma_{\pi} = 32$ mb in charge independence, as eikonal distortion parameters for the $^3\Lambda_H$ targets. Although such distortions are seen to be not so important in the light s-shell nuclear systems than in p-shell nuclear systems like $^{12}\text{C}$, it is necessary to verify their effects on the production cross sections in more quantitative calculations [11].

**B. Effective number of nucleons for $^3\Lambda_H$ bound-state productions**

Considering the non-spin-flip $\Delta S = 0$ production in the $^3\Lambda_H(K^-, \pi^0)$ reactions, we obtain the differential cross sections of Eq. (2) for the production of $^3\Lambda_H$, which are often written by the effective number technique [14, 15];

$$
\left( \frac{d\sigma}{d\Omega} \right)_{J^P_{\text{lab}}, \theta_{\text{lab}}}^{J^P_{\text{B}}} = \alpha \left( \frac{d\sigma}{d\Omega} \right)_{\text{elem}}^{J^P_{\text{B}}} \cdot Z_{\text{eff}}^{J^P_{\text{B}}} (\theta_{\text{lab}}),
$$

where $\alpha \langle d\sigma/d\Omega \rangle_{\text{lab}}^{\text{elem}} = \alpha |\mathcal{T}_{\pi^0\Lambda}|^2$ is a differential cross section for the in-medium $K^- p \rightarrow \pi^0 \Lambda$ $\Delta S = 0$ reaction, including the kinematical factor $\alpha$. The effective number of nucleons $Z_{\text{eff}}^{J^P_{\text{B}}}$ for the production of the $^3\Lambda_H (J^P_B)$ bound state is reduced [11] as

$$
Z_{\text{eff}}^{J^P_{\text{B}}} (\theta_{\text{lab}}) = C_{TS}^2 |F(q)|^2,
$$

where $C_{TS}$ is the isospin-spin ($TS$) spectroscopic amplitude between the $\Lambda$ final state and the $N$ initial state. The form factor $F(q)$ is written by

$$
F(q) = \int_0^\infty dr r^2 \rho_{\pi}(r) \tilde{j}_0 \left( q; \frac{M_C}{M_D} r \right),
$$

4
TABLE I: Calculated Λ separation energies $B_\Lambda$ and root-mean-square distances $\langle r_\Lambda^2 \rangle^{1/2}$ between the core nucleus and Λ in $^{3,4}\Lambda H$ with an isospin $T$ and a spin $J^P$, in comparison with $B_N$ and $\langle r_N^2 \rangle^{1/2}$ for a nucleon in $^{3,4}\Lambda H$.

|       | $^3\Lambda H_{g.s.}$ | $^3\Lambda H_{exc}$ | $^3\Lambda H$ | $^4\Lambda H_{g.s.}$ | $^4\Lambda H_{exc}$ | $^4\Lambda H$ |
|-------|----------------------|----------------------|--------------|----------------------|----------------------|--------------|
| $T$   | 0                    | 0                    | 1/2          | 1/2                  | 1/2                  | 0            |
| $J^P$ | 1/2$^+$              | 3/2$^+$              | 1/2$^+$      | 0$^+$               | 1$^+$               | 0$^+$        |
| $B_\Lambda(N)$ (MeV) | 0.13 | unbound$^a$ | 5.49          | 2.16       | 1.09                | 19.8         |
| $\langle r_\Lambda(N)^2 \rangle^{1/2}$ (fm) | 11.2 | 18.2$^b$ | 2.49          | 3.68      | 4.60                | 1.87         |
| $B_\Lambda^{exp}$ (MeV)  | 0.13 ± 0.05 [5]      | 0.41 ± 0.12 [6]      | 2.04 ± 0.04 [5] | 9.5 ± 0.04 [22] | 2.16 ± 0.08 [23] | 1.09 ± 0.02 [10] |

$^a$The pole of the $S$ matrix as the virtual state is located at $E_{\Lambda d}^{(pole)} = -0.089$ MeV on the unphysical sheet [-].

$^b$The continuum-discretized wave function is used. See text in Sect. II C.

where $\tilde{j}_0(q;r)$ is a distorted wave for the meson distortion; $M_C/M_B$ and $M_C/M_A$ in Eq. (2) are replaced by $M_C/M_D$ in the eikonal approximation, which average mass $M_D \equiv (M_B + M_A)/2$, which may give a good estimation for the very light nuclear systems. Because the factor of $M_C/M_D$ originates from the recoil correction, the effective momentum transfer is often defined by $q_{eff} \equiv (M_C/M_D)q \simeq [(A-1)/A]q$, which controls effectively the recoil effects on the production cross sections in the eikonal approximation [15, 16]. When the distortion effects are switched off ($\sigma_{K^-}, \sigma_\pi \to 0$), the production cross sections can be obtained in the plane-wave impulse approximation (PWIA), with replacing $\tilde{j}_0(q;r)$ by $j_0(qr)$ that is a spherical Bessel function with $L = 0$ [11].

The transition density $\rho_{tr}(r)$ in Eq. (3) is given by

$$\rho_{tr}(r) = \varphi_0^{(A)^*}(r)\varphi_0^{(N)}(r),$$  (7)

where $\varphi_0^{(A)} = \langle \phi_0^{(C)}|\Psi_B \rangle$ is the relative wave function that is regarded as a spectroscopic amplitude for $\Lambda$ in $^{3,4}\Lambda H$ by using the wave function $\phi_0^{(C)}$ for the core nucleus [21], and $\varphi_0^{(N)} = \langle \phi_0^{(C)}|\Psi_A \rangle$ is the relative wave function for a nucleon ($N$) in $^{3,4}\Lambda H$. In Table I we
FIG. 1: Relative density distributions $\rho(\Lambda)(r)$ for $\Lambda$ in $^{3,4}_\Lambda\text{H}_{\text{g.s.}}$, as a function of the relative distance between the core nucleus and $\Lambda$, together with the relative density distributions $\rho(N)(r)$ for a nucleon ($N$) in the $^{3,4}_{\text{He}}$ targets. The relative density distributions for $\Lambda$ in $^{3,4}_\Lambda\text{H}_{\text{exc}}$ are also drawn.

show the calculated $\Lambda$ separation energies $B_\Lambda$ and the root-mean-square distances $\langle r_\Lambda^2 \rangle^{1/2}$ between the core nucleus and $\Lambda$ in $^{3,4}$H with the isospin $T_B$ and the spin $J_B^P$, together with the nucleon separation energies $B_N$ and $\langle r_N^2 \rangle^{1/2}$ between the core nucleus and $N$ in $^{3,4}$He with $T_A$ and $J_A^P$. For $A = 4$, we obtain $\phi_0^{(A)}$ in the $3N$-$\Lambda$ model based on four-body $\Lambda NN\Lambda$ calculations [11] with central nucleon-nucleon ($NN$) and $\Lambda N$ potentials, reproducing $B_\Lambda = 2.16$ MeV [23] and $\langle r_\Lambda^2 \rangle^{1/2} = 3.68$ fm for $^3\text{H} + \Lambda$ in $^4\Lambda\text{H}(J^P = 0^+, \text{g.s.})$. We use $\phi_0^{(N)}$ obtained in four-body $NNNN$ calculations [11], and $C_{TS}^2 = 2$. For $A = 3$, we obtain $\phi_0^{(A)}$ in the $2N$-$\Lambda$ model based on microscopic continuum-discretized coupled-channels (CDCC) calculations [24, 25] with central $NN$ and $\Lambda N$ potentials, reproducing $B_\Lambda = 0.13$ MeV [5] and $\langle r_\Lambda^2 \rangle^{1/2} = 11.2$ fm for $d + \Lambda$ in $^3\Lambda\text{H}(J^P = 1/2^+, \text{g.s.})$. We use $\phi_0^{(N)}$ obtained in three-
body NNN calculations \cite{26}, and $C_{TS}^2 = 3/2$ in Eq. (5). Figure 1 shows the relative density distributions $\rho_\Lambda(r) = |\varphi_0^{(A)}(r)|^2$ for $\Lambda$ in $^4\Lambda_\text{H}_{g.s.}$, as a function of the relative distance between the core nucleus and $\Lambda$, together with relative density distributions $\rho_N(r) = |\varphi_0^{(N)}(r)|^2$ for $N$ in $^4\text{He}$. We confirm that the $\Lambda$ density distribution for $^4\Lambda_\text{H}_{g.s.}$ is significantly suppressed at the nuclear center and is pushed outside \cite{11, 27}, and that the $\Lambda$ density distribution for $^3\Lambda_\text{H}_{g.s.}$ indicates the weakly bound state having a long tail.

Considering the spin-flip $\Delta S = 1$ production in the $^4\text{He}(K^-, \pi^0)$ reaction, we obtain the production cross section of an unnatural-parity state of $^4\Lambda_\text{H}$ ($J^P = 1^+$, exc). Due to such a $s$-shell hypernuclear state, the differential cross section is also written by the effective number of nucleons $Z_{eff}$ with $\alpha\langle d\sigma/d\Omega \rangle_{\text{lab}} = \alpha|g_{\pi^0\Lambda}|^2$ for the in-medium $K^- p \rightarrow \pi^0\Lambda \Delta S = 1$ reaction, as given in Eq. (4). We use $\varphi_0^{(A)}$ obtained by the $3N$-$\Lambda$ model with central $\Lambda N$ potentials for $J^P = 1^+$, reproducing $B_\Lambda = 1.09$ MeV with respect to the $^3\text{H}-\Lambda$ threshold \cite{10} with $\langle r_{\Lambda}^2 \rangle^{1/2} = 4.60$ fm, and $C_{TS}^2 = 1$ in Eq. (5).

C. $^3\Lambda_\text{H}$ ($J^P = 3/2^+$, exc) as a virtual state

We have no observation of a $J^P = 3/2^+$ excited state of $^3\Lambda_\text{H}$ experimentally so far. It seems that there is no bound state in $J^P = 3/2^+$, but this state would be in the continuum region above the $d$-$\Lambda$ threshold \cite{18} as a resonant state or a virtual state theoretically. If a pole of the $S$ matrix for a virtual state is sufficiently close to the physical axis on the complex momentum plane, it may provide an appreciable influence on the cross section at low energy \cite{28}. This phenomenon is regarded as threshold effects caused by a virtual state close to the threshold. To see this situation, we study the $s$-wave $3/2^+$ state in $^3\Lambda_\text{H}$ with a folding model $d$-$\Lambda$ potential $U_{Ad}$, adjusting to the $Ad$ scattering length of $a_{3/2}^{Ad} = -16.2$ fm and the effective range of $r_{3/2}^{Ad} = 3.2$ fm \cite{12} for the NSC97f potential. Numerically solving the Lippmann-Schwinger equation for this $d+\Lambda$ system, we find that its pole of the $S$ matrix is located at $k_{Ad}^{(pole)} = -0.057i$ fm\(^{-1}\) on the complex momentum plane, which corresponds to $E_{Ad}^{(pole)} = -0.089$ MeV on the unphysical sheet [-] of the complex $E$ plane \cite{29}. This state is identified as a virtual state close to the $d$-$\Lambda$ threshold, as recently discussed by Schäfer et al. \cite{12}. To describe the virtual state of $^3\Lambda_\text{H}$ ($J^P = 3/2^+$, exc), we construct a continuum-discretized wave function $\hat{\varphi}_0^{(A)}$, which is given by an appropriate momentum bin of $k_0$ and
where $\Delta k = k_1 - k_0$, and $r$ and $k$ are the radial coordinate and the relative momentum between $d$ and $\Lambda$, respectively, and $\phi^{d+\Lambda}_0(k, r)$ is a $d + \Lambda$ scattering wave function with the energy $\varepsilon_{d\Lambda} = k^2/(2\mu_{d\Lambda})$ ($> 0$), where $\mu_{d\Lambda}$ is the reduced mass of the $d + \Lambda$ system. This continuum-discretized wave function $\hat{\varphi}^{(A)}_0$ satisfies the positive energy $\hat{\varepsilon}^{(A)}_{d\Lambda} = \{(k_1 + k_0)^2/4 + (\Delta k)^2/12\}/(2\mu_{d\Lambda})$ \[30\]. When we choose the momentum bin of $k_0 = 0.026$ fm$^{-1}$ and $k_1 = 0.083$ fm$^{-1}$ for $^3_\Lambda$H ($J^P = 3/2^+$, exc), we find that the wave functions of $\phi^{d+\Lambda}_0(k, r)$ in Eq. (8) significantly enhance at the nuclear interior due to the threshold effects caused by the virtual state \[31\]. Thus, the continuum-discretized wave function $\hat{\varphi}^{(A)}_0$ labeled by $\hat{\varepsilon}^{(A)}_d = 0.089$ MeV provides an appropriate description of these threshold effects, consistent with the virtual-state phenomenon having a peak around $\hat{\varepsilon}^{(A)}_d \approx |E^{(pole)}_{d\Lambda}|$ above the threshold in the nuclear response function \[32\]. Regarding $\hat{\varphi}^{(A)}_0$ as $\varphi^{(A)}_0$ in Eq. (7), therefore, we can estimate the production cross sections of $^3_\Lambda$H ($J^P = 3/2^+$, exc) as the virtual state with $C_{T\Lambda}^2 = 4/3$, as given in Eq. (5). In Fig. 11 we also display the relative density distribution $\rho_\Lambda(r)$ for $^3_\Lambda$H ($J^P = 3/2^+$, exc), which leads to $\langle r^2_\Lambda \rangle^{1/2} = 18.2$ fm, in comparison with $^3_\Lambda$H ($J^P = 1/2^+$, g.s.).

III. RESULTS AND DISCUSSION

A. Optimal Fermi-averaged differential cross sections

It has been recognized that in the standard DWIA, the in-medium $K^-p \rightarrow \pi^0\Lambda$ cross section of $\alpha(d\sigma/d\Omega)_{\text{elem}}^{\text{lab}}$ in Eq. (4) plays an important role in explaining the production cross section of $d\sigma/d\Omega_{\text{lab}}$ in the nuclear reaction. To realize a more quantitative description in our calculations, we need to obtain the in-medium $K^-p \rightarrow \pi^0\Lambda$ cross section with the optimal Fermi-averaging $t$ matrix \[13\] for the nuclear $(K^-, \pi^0)$ reaction, taking into account momenta arising from the distorted waves of $K^-$ and $\pi^0$ in the nucleus.

Figure 2 displays the angular distributions of the in-medium $K^-p \rightarrow \pi^0\Lambda$ differential cross sections for non-spin-flip $\Delta S = 0$ and spin-flip $\Delta S = 1$ productions, which are denoted by $\alpha(d\sigma/d\Omega)_{\text{elem}}^{\text{lab}} = \alpha|\tilde{f}_{\pi\Lambda}|^2$ and $\alpha|\tilde{g}_{\pi\Lambda}|^2$, respectively, in the $(K^-, \pi^0)$ reactions on $^{3,4}$He.
at \( p_{K^-} = 1.0 \text{ GeV/c} \). These behaviors affect the angular distributions of the productions of \(^3\text{H}\) and \(^4\text{H}\), as well as the behavior of \( Z_{\text{eff}} \) that depends on the angle of \( \theta_{\text{lab}} \). Note that the absolute values and shapes of \( \alpha |\vec{f}_{\pi^0\Lambda}|^2 \) on \(^4\text{He}\) and \(^3\text{He}\) differ moderately in the \( \Delta S = 0 \) productions. This difference stems from the nature of the energy and momentum transfers \((\omega, \mathbf{q})\) that satisfy the on-shell energy condition in the optimal Fermi-averaging procedure

\[
\omega \simeq m_\Lambda - m_N + \frac{\mathbf{q}^2}{2m_\Lambda} + \frac{\mathbf{p}_N^* \cdot \mathbf{q}}{m_\Lambda} - \frac{m_\Lambda - m_N}{2m_N} \frac{\mathbf{p}_N^* \cdot \mathbf{q}}{m_\Lambda} \frac{\mathbf{p}_N^* \cdot \mathbf{q}}{m_\Lambda},
\]

(9)

where \( \mathbf{p}_N^* \) is the momentum of a struck nucleon and \( T_{\text{recoil}} \) is the recoil energy to the final state. Because the nucleon binding energy of \( B_N = 19.8 \text{ MeV} \) for \(^4\text{He}\) is so larger than that of \( B_N = 5.49 \text{ MeV} \) for \(^3\text{He}\), the energy difference \( \Delta \omega = |\omega_4 - \omega_3| \) becomes \( \sim 11 \text{ MeV} \), where \( \omega_4 \) and \( \omega_3 \) are the energy transfers to \(^4\text{H}\) and \(^3\text{H}\), respectively. This value is comparable to \( q^2 / 2m_\Lambda \simeq 3.6-12 \text{ MeV} \) in the near-recoilless \((K^-, \pi^0)\) reactions at \( \theta_{\text{lab}} = 0^\circ-8^\circ \). As a result, it induces a significant downward energy shift to the \( \Lambda \) production threshold in the nuclear \((K^-, \pi^0)\) reactions on \(^4\text{He}\), rather than \(^3\text{He}\). This fact leads to the difference in \( \alpha |\vec{f}_{\pi^0\Lambda}|^2 \) between \(^4\text{He}\) and \(^3\text{He}\) owing to the energy dependence of the elementary amplitude \( f_{\pi^0\Lambda} \) at the forward angles of \( \theta_{\text{lab}} \). On the other hand, we find that the values of \( \alpha |\vec{f}_{\pi^0\Lambda}|^2 \) in the \( \Delta S = 1 \) productions on \(^4\text{He}\) and \(^3\text{He}\) are very similar.

\[ B. \quad ^4\Lambda\text{H} (J = 0^+, \text{g.s.) and } ^4\Lambda\text{H} (J^P = 1^+, \text{exc}) \]

Now we estimate numerically the production cross sections of \(^4\Lambda\text{H}\) in the exothermic \(^4\text{He}(K^-, \pi^0)\) reactions at \( p_{K^-} = 1.0 \text{ GeV/c} \) and \( \theta_{\text{lab}} = 0^\circ-20^\circ \), where the momentum transfers become \( q \simeq 90-345 \text{ MeV/c} \). In Table I we list the calculated results of \( \alpha \langle d\sigma/d\Omega \rangle_{\text{lab}}^{\text{elem}}, Z_{\text{eff}}, \) and \( d\sigma/d\Omega_{\text{lab}} \) for \(^4\Lambda\text{H} (J = 0^+, \text{g.s.) and } ^4\Lambda\text{H} (J^P = 1^+, \text{exc}) \) in the DWIA using the distortion parameters of \((\sigma_{K^-}, \sigma_\pi) = (45 \text{ mb}, 32 \text{ mb}) \). We obtain \( d\sigma/d\Omega_{\text{lab}}(0^+_{\text{g.s.}}) = 1184.3, 552.2, 100.8, \) and \( 19.2 \mu b/\text{sr} \) and \( d\sigma/d\Omega_{\text{lab}}(1^+_{\text{exc}}) = 0.27, 10.5, 14.4, \) and \( 5.68 \mu b/\text{sr} \) at \( \theta_{\text{lab}} = 0^\circ, 6^\circ, 12^\circ, \) and \( 18^\circ \), respectively. In Fig. 3 we show the calculated angular distributions of \( d\sigma/d\Omega_{\text{lab}} \) for the \( 0^+_{\text{g.s.}} \) and \( 1^+_{\text{exc}} \) states in \(^4\Lambda\text{H}\) via the \(^4\text{He}(K^-, \pi^0)\) reactions at \( p_{K^-} = 1.0 \text{ GeV/c} \) in the DWIA. We find that the production cross section of the \( 0^+_{\text{g.s.}} \) state dominates in the forward angles, whereas the production of the \( 1^+_{\text{exc}} \) state is comparable to that of the \( 0^+_{\text{g.s.}} \) state beyond \( \theta_{\text{lab}} = 20^\circ \); we have \([d\sigma/d\Omega_{\text{lab}}(0^+_{\text{g.s.}})]/[d\sigma/d\Omega_{\text{lab}}(1^+_{\text{exc}})] \approx 2.5 \) at \( \theta_{\text{lab}} \simeq 20^\circ \).
FIG. 2: Angular distributions of the in-medium $K^{-}p \rightarrow \pi^{0}\Lambda$ differential cross sections

\[ \alpha \langle d\sigma/d\Omega \rangle_{\text{lab}}^{\text{elem}} = \alpha |\vec{f}_{\pi^{0}\Lambda}|^2 \]

for the non-spin-flip $\Delta S = 0$ production in the $(K^{-}, \pi^{0})$ reactions at

$p_{K^{-}} = 1.0 \text{ GeV}/c$, and those of $\alpha |\vec{g}_{\pi^{0}\Lambda}|^2$ for the spin-flip $\Delta S = 1$ production. Solid and dot-dashed (dashed and dotted) curves denote the calculated results for $^{3}\Lambda^{}_{H}$ ($^{4}\Lambda^{}_{H}$) production on the $^{3}\text{He}$ ($^{4}\text{He}$) target, respectively. The optimal Fermi-averaging $K^{-}p \rightarrow \pi^{0}\Lambda$ amplitudes of $\vec{f}_{\pi^{0}\Lambda}$ and $\vec{g}_{\pi^{0}\Lambda}$ are obtained \[13\] in the use of the elementary amplitudes analyzed by Gopal, et al. \[20\].

The integrated cross section of $^{4}\Lambda^{}_{H}$ over $\theta_{\text{lab}} = 0^\circ - 20^\circ$ is given by

\[ \sigma_{\text{lab}}(J^{P}_{B}) \equiv \int_{\theta_{\text{lab}} = 0^\circ}^{\theta_{\text{lab}} = 20^\circ} \left( \frac{d\sigma}{d\Omega} \right)_{\text{lab},\theta_{\text{lab}}}^{J^{P}_{B}} d\Omega. \]  

We find $\sigma_{\text{lab}}(0_{g.s.}^{+}) = 63.1 \mu b$ and $\sigma_{\text{lab}}(1_{\text{exc}}^{+}) = 3.8 \mu b$ at $p_{K^{-}} = 1.0 \text{ GeV}/c$ in the DWIA, compared with $\sigma_{\text{lab}}(0_{g.s.}^{+}) = 150.4 \mu b$ and $\sigma_{\text{lab}}(1_{\text{exc}}^{+}) = 10.7 \mu b$ in the PWIA. This implies that distortion effects are remarkably important in quantitative estimations for the $A = \cdots$.
TABLE II: Calculated angular distributions of the laboratory differential cross sections for the 0\textsuperscript{+}\textsubscript{g.s.} and 1\textsuperscript{+}\textsubscript{exc} states in \(^4\Lambda\text{H}\) via the \(^4\text{He}(K^-, \pi^0)\) reactions at \(p_{K^-} = 1.0\) GeV/c in the DWIA. The distortion parameters of \((\sigma_{K^-}, \sigma_{\pi}) = (45\text{ mb}, 32\text{ mb})\) are used.

| \(\theta_{\text{lab}}\) (degree) | \(q\) (MeV/c) | \(\alpha\langle d\sigma/d\Omega\rangle_{\text{elem}}^{\text{lab}}\) (mb/sr) | \(Z_{\text{eff}}\) | \(d\sigma/d\Omega_{\text{lab}}\) (\(\mu\text{b}\)/sr) | \(\alpha\langle d\sigma/d\Omega\rangle_{\text{elem}}^{\text{lab}}\) (mb/sr) | \(Z_{\text{eff}}\) | \(d\sigma/d\Omega_{\text{lab}}\) (\(\mu\text{b}\)/sr) |
|-----------------|----------|----------------|--------|----------------|----------------|--------|----------------|
| 0               | 90       | 2.140          | 5.534  | 1184.3         | 0.001          | 2.298  | 0.27           |
| 2               | 96       | 2.003          | 5.353  | 1072.3         | 0.004          | 2.218  | 0.98           |
| 4               | 113      | 1.729          | 4.848  | 838.1          | 0.023          | 1.996  | 4.55           |
| 6               | 135      | 1.342          | 4.116  | 552.2          | 0.062          | 1.678  | 10.5           |
| 8               | 162      | 0.990          | 3.283  | 325.1          | 0.122          | 1.321  | 16.1           |
| 10              | 191      | 0.756          | 2.465  | 186.3          | 0.181          | 0.977  | 17.7           |
| 12              | 221      | 0.578          | 1.745  | 100.8          | 0.212          | 0.680  | 14.4           |
| 14              | 251      | 0.521          | 1.165  | 60.8           | 0.249          | 0.446  | 11.1           |
| 16              | 282      | 0.516          | 0.733  | 37.8           | 0.305          | 0.275  | 8.38           |
| 18              | 314      | 0.445          | 0.433  | 19.2           | 0.358          | 0.159  | 5.68           |
| 20              | 345      | 0.355          | 0.237  | 8.42           | 0.402          | 0.085  | 3.42           |

4 nuclear systems \([11]\). The distortion effects are roughly estimated by a distortion factor \(D_{\text{dis}} \equiv Z_{\text{DW}}^{\text{eff}}/Z_{\text{PW}}^{\text{eff}}\) in the eikonal meson waves. We find \(D_{\text{dis}} \simeq 0.42 - 0.23\) for \(^4\Lambda\text{H}\), which depend on \(\theta_{\text{lab}} = 0^\circ - 20^\circ\). Consequently, we show that the production ratio of \(\sigma_{\text{lab}}(1^+\text{exc})\) to \(\sigma_{\text{lab}}(0^+\text{g.s.})\) amounts to

\[
R_4 = \sigma_{\text{lab}}(1^+\text{exc})/\sigma_{\text{lab}}(0^+\text{g.s.}) \simeq 0.06 - 0.07 \quad \text{for } ^4\Lambda\text{H} \tag{11}
\]

at \(p_{K^-} = 1.0\) GeV/c.

C. \(^3\Lambda\text{H} (J^P = 1/2^+, \text{g.s.})\) and \(^3\Lambda\text{H} (J^P = 3/2^+, \text{exc})\)

Let us estimate numerically the production cross sections of \(^3\Lambda\text{H}\) in the \(^3\text{He}(K^-, \pi^0)\) reactions at \(p_{K^-} = 1.0\) GeV/c and \(\theta_{\text{lab}} = 0^\circ - 20^\circ\), where the momentum transfers become \(q \simeq 80 - 350\) MeV/c. In Table III, we list the calculated results of \(\alpha\langle d\sigma/d\Omega\rangle_{\text{lab}}^{\text{elem}}, Z_{\text{eff}},\) and
FIG. 3: Calculated angular distributions of the laboratory differential cross sections $d\sigma/d\Omega_{\text{lab}}$ for $0^+_{\text{g.s.}}$ and $1^+_{\text{exc}}$ states in $^4\Lambda$H via the $^4\text{He}(K^-,\pi^0)^4\text{H}$ reactions at $p_{K^-}=1.0$ GeV/c in the DWIA.

$d\sigma/d\Omega_{\text{lab}}$ for $^3\Lambda$H ($J^P = 1/2^+, \text{ g.s.}$) and $^3\Lambda$H ($J^P = 3/2^+, \text{ exc}$) in the DWIA using the distortion parameters of $(\sigma_{K^-}, \sigma_{\pi}) = (45 \text{ mb}, 32 \text{ mb})$. We obtain $d\sigma/d\Omega_{\text{lab}}(1/2^+_{\text{g.s.}}) = 484.0, 183.4, 57.1, \text{ and } 7.5 \mu b/\text{sr}, \text{ and } d\sigma/d\Omega_{\text{lab}}(3/2^+_{\text{exc}}) = 1.67, 2.94, 2.83, \text{ and } 1.01 \mu b/\text{sr}$ at $\theta_{\text{lab}} = 0^\circ, 6^\circ, 12^\circ, \text{ and } 18^\circ$, respectively. In Fig. 4 we show the calculated angular distributions of $d\sigma/d\Omega_{\text{lab}}$ for the $1/2^+_{\text{g.s.}}$ and $3/2^+_{\text{exc}}$ states in $^3\Lambda$H via the $^3\text{He}(K^-,\pi^0)$ reactions at $p_{K^-}=1.0$ GeV/c in the DWIA. We find that the production cross section of $d\sigma/d\Omega_{\text{lab}}$ for the $1/2^+_{\text{g.s.}}$ state is dominant at the forward angles of $\theta_{\text{lab}} = 0^\circ-20^\circ$, in comparison with that for $3/2^+_{\text{exc}}$ state; we have $[d\sigma/d\Omega_{\text{lab}}(1/2^+_{\text{g.s.}})]/[d\sigma/d\Omega_{\text{lab}}(3/2^+_{\text{exc}})] \simeq 5.6$ at $\theta_{\text{lab}} \simeq 20^\circ$.

To study the feasibility of the lifetime measurements of $^3\Lambda$H at the J-PARC experiments, we estimate the integrated cross sections of $\sigma_{\text{lab}}$ over $\theta_{\text{lab}} = 0^\circ-20^\circ$, as given in Eq. (10), in comparison with those of $^4\Lambda$H. We find $\sigma_{\text{lab}}(1/2^+_{\text{g.s.}}) = 25.9 \mu b$ and $\sigma_{\text{lab}}(3/2^+_{\text{exc}}) = 0.77 \mu b$ at
interaction and the structure of $^3_\Lambda$H gives valuable information concerning the value of $\sigma$ of $^3_\Lambda$H production, we estimate the production cross section of $^3_\Lambda$H at $p_{K^-} = 1.0$ GeV/$c$. In comparison with $\sigma_{lab}(1/2^{+}_{g.s.})$ in $^4_\Lambda$H, we realize that the reduction of $\sigma_{lab}(3/2^{+}_{exc})$ in $^3_\Lambda$H stems from a spread-out transition density $\rho_{tr}(r)$ in Eq. (7) due to $\langle r^2_\Lambda \rangle^{1/2} = 18.2$ fm for the $\Lambda$ wave function of the $3/2^+$ state of $^3_\Lambda$H, whereas the absolute value of $\sigma_{lab}(3/2^{+}_{exc})$ depends on its pole position of the $S$ matrix for the virtual state.

The STAR Collaboration [6] recently reported the $\Lambda$ separation energy of $B_\Lambda = 0.41 \pm 0.12$ MeV for the $1/2^+_{g.s.}$ state of $^3_\Lambda$H. To see the effects of the $\Lambda$ separation energy on the $\Lambda$ production, we estimate the production cross section of $^3_\Lambda$H in the $(K^-,\pi^0)$ reaction at $p_{K^-} = 1.0$ GeV/$c$, reproducing $B_\Lambda = 0.41$ MeV by an additional attraction in the $\Lambda N$ interaction. We find that the integrated cross section in the DWIA amounts to $\sigma_{lab}(1/2^+_{g.s.}) = 41.0$ $\mu$b, which is 60% larger than $\sigma_{lab}(1/2^+_{g.s.}) = 25.9$ $\mu$b for $B_\Lambda = 0.13$ MeV. This is because the transition density $\rho_{tr}(r)$ in Eq. (7) becomes large, where the $\Lambda$ wave function of $\phi_0^{(\Lambda)}(r)$ is shifted into the nuclear inside due to the additional attraction, leading to $\langle r^2_\Lambda \rangle^{1/2} = 6.81$ fm. Therefore, we suggest that a precise measurement of the production cross section of $^3_\Lambda$H gives valuable information concerning the value of $B_\Lambda$ with studying the nature of the $\Lambda N$ interaction and the structure of $^3_\Lambda$H.

D. Recoil effects

We have recognized that the recoil effects are essential in production reactions on the very light nuclear target such as $^4$He [11]. Thus, the recoil correction may significantly enlarge the production cross sections because the effective momentum transfers denote $q_{eff} \simeq 51$–223 MeV/$c$ at $\theta_{lab} = 0^\circ$–20$^\circ$ with $M_C/M_D = 0.647$ for $^3_\Lambda$H, which achieve the $\Lambda$ production in the near-recoilless reaction, rather than $q_{eff} \simeq 66$–253 MeV/$c$ with $M_C/M_D = 0.734$ for $^4_\Lambda$H. When the recoil correction is omitted ($M_C/M_D \rightarrow 1$), the integrated cross sections amount to $\sigma_{lab}(^3_\Lambda$H; $1/2^+_{g.s.}) = 9.74$ $\mu$b and $\sigma_{lab}(^4_\Lambda$H; $0^+_{g.s.}) = 35.2$ $\mu$b in the DWIA, and $\sigma_{lab}(^3_\Lambda$H; $1/2^+_{g.s.}) = 17.9$ $\mu$b and $\sigma_{lab}(^4_\Lambda$H; $0^+_{g.s.}) = 89.9$ $\mu$b in the PWIA. The values of $\sigma_{lab}$ with the recoil effects are larger than those of $\sigma_{lab}$ without the recoil effects by a factor of about 2.5 (1.7) for $^3_\Lambda$H.
TABLE III: Calculated angular distributions of the laboratory differential cross sections for the 1/2\textsubscript{g.s.} and 3/2\textsubscript{exc} states in \(^3\Lambda\)H via the \(^3\text{He}(K^-, \pi^0)\) reactions at \(p_{K^-} = 1.0\) GeV/c in the DWIA. The distortion parameters of \((\sigma_{K^-}, \sigma_\pi) = (45\text{ mb}, 32\text{ mb})\) are used.

\[\begin{array}{cccccccc}
\theta_{\text{lab}} & q & \alpha(\frac{d\sigma}{d\Omega})_{\text{lab}} & Z_{\text{eff}} & \frac{d\sigma}{d\Omega}_{\text{lab}} & \alpha(\frac{d\sigma}{d\Omega})_{\text{lab}} & Z_{\text{eff}} & \frac{d\sigma}{d\Omega}_{\text{lab}} \\
(\text{degree}) & (\text{MeV/c}) & (\text{mb/sr}) & (\times 10^{-1}) & (\mu\text{b/sr}) & (\text{mb/sr}) & (\times 10^{-1}) & (\mu\text{b/sr}) \\
0 & 79 & 1.488 & 3.253 & 484.0 & 0.010 & 0.668 & 0.67 \\
2 & 86 & 1.465 & 3.090 & 452.8 & 0.019 & 0.626 & 1.19 \\
4 & 104 & 1.141 & 2.664 & 304.1 & 0.047 & 0.520 & 2.43 \\
6 & 129 & 0.869 & 2.110 & 183.4 & 0.075 & 0.390 & 2.94 \\
8 & 157 & 0.815 & 1.560 & 127.2 & 0.116 & 0.271 & 3.15 \\
10 & 187 & 0.842 & 1.092 & 92.0 & 0.177 & 0.178 & 3.14 \\
12 & 218 & 0.780 & 0.733 & 57.1 & 0.253 & 0.112 & 2.83 \\
14 & 249 & 0.662 & 0.474 & 31.4 & 0.326 & 0.068 & 2.21 \\
16 & 281 & 0.533 & 0.298 & 15.9 & 0.390 & 0.040 & 1.56 \\
18 & 312 & 0.411 & 0.183 & 7.50 & 0.441 & 0.023 & 1.01 \\
20 & 344 & 0.314 & 0.109 & 3.41 & 0.480 & 0.013 & 0.61 \\
\end{array}\]

(\(^4\text{H}\)). Therefore, we confirm the benefit of the use of the \(^3,\text{4}\text{He}\) targets in the \(\Lambda\) production via the \((K^-, \pi^0)\) reaction.

E. Comparison with \(^3,\text{4}\text{He}(\pi^-, K^0)\) reactions

It is also interesting to discuss the production cross sections of \(^3,\text{4}\text{H}\) in the endothermic \((\pi^-, K^0)\) reactions on \(^3\text{He}\) at \(p_{\pi^-} = 1.05\) GeV/c and \(\theta_{\text{lab}} = 0^\circ-20^\circ\), where the high momentum transfers of \(q \simeq 350-500\) MeV/c are expected to bring benefits to the use of the \(^3,\text{4}\text{He}\) targets \[^{11, 33}\]. In Table IV we list the calculated results of \(\alpha(\frac{d\sigma}{d\Omega})_{\text{lab}} = \alpha|\vec{f}_{\pi^-p\to K^0\Lambda}|^2\), \(Z_{\text{eff}}\), and \(\frac{d\sigma}{d\Omega}_{\text{lab}}\) for \(^3\Lambda\)H \((J^P = 1/2^+, \text{ g.s.})\) and \(^4\Lambda\)H \((J^P = 0^+, \text{ g.s.})\) at 1.05 GeV/c in the DWIA. We find the updated values of \(\frac{d\sigma}{d\Omega}_{\text{lab}}(\text{g.s.})\) for \(^3\Lambda\)H \((J^P = 1/2^+, \text{ g.s.}) = 7.28, 5.61, 2.62, \text{ and } 0.78\) \(\mu\text{b/sr}\) and \(\frac{d\sigma}{d\Omega}_{\text{lab}}(\text{g.s.})\) for \(^4\Lambda\)H \((J^P = 0^+, \text{ g.s.}) = 18.82, 13.82, 5.50, \text{ and } 1.10\) \(\mu\text{b/sr}\) at \(\theta_{\text{lab}} = 0^\circ, 6^\circ, 12^\circ, 18^\circ,\) and \(20^\circ\).
FIG. 4: Calculated angular distributions of the laboratory differential cross sections $d\sigma/d\Omega_{\text{lab}}$ for the $1/2_{\text{g.s.}}^{+}$ and $3/2_{\text{exc}}^{+}$ states in $^{3}_{\Lambda}\text{H}$ via the $^{3}\text{He}(K^{-},\pi^{0})^{3}_{\Lambda}\text{H}$ reactions at $p_{K^{-}} = 1.0$ GeV/c in the DWIA.

and $18^\circ$, respectively, compared with the previous works \cite{11, 33}. These results lead to $\sigma_{\text{lab}}(1/2_{\text{g.s.}}^{+}; 1/2_{\text{g.s.}}^{+}) = 0.93$ $\mu$b and $\sigma_{\text{lab}}(3/2_{\text{exc}}^{+}; 0_{\text{g.s.}}^{+}) = 2.02$ $\mu$b. Note that the angular dependences of $\alpha|\mathcal{F}_{\pi^{-}p\rightarrow K^{0}_{\Lambda}}|^{2}$ for $^{4}_{\Lambda}\text{H}$ and $^{3}_{\Lambda}\text{H}$ are very similar in the $(K^{-}, \pi^{0})$ reactions at $\theta_{\text{lab}} = 0^\circ$–$20^\circ$, whereas the absolute values of the former are slightly larger than those of the latter.

Moreover, we find that the values of $d\sigma/d\Omega_{\text{lab}}(3/2_{\text{g.s.}}^{+}; 1/2_{\text{g.s.}}^{+})$ in the region of $q > 350$ MeV/c are enhanced by more than 14% owing to the use of the CDCC wave functions for $^{3}_{\Lambda}\text{H}$ in our calculations, in comparison with those obtained by omitting the couplings between $[d\otimes\Lambda]$ and $[(d^{*})_{n}\otimes\Lambda]$ channels in the CDCC, where $(d^{*})_{n}$ denote the $n$-th continuum-discretized excited states of the deuteron core nucleus. This implies that the excited-state components of $(d^{*})_{n}$ contribute to the $^{3}_{\Lambda}\text{H}$ production \cite{25}, so its production yield grows with increasing


TABLE IV: Calculated angular distributions of the laboratory differential cross sections for $^{3,4}_ΛH$ in the $^{3,4}_{Λ He}(π^−, K^0)$ reactions at $p_{π^−} = 1.05$ GeV/c in the DWIA. The distortion parameters of $(σ_π, σ_{K^+}) = (30 \text{ mb, 15 mb})$ are used.

| $θ_{\text{lab}}$ (degree) | $q$ (MeV/c) | $α\langle dσ/dΩ\rangle_{\text{lab}}^{\text{elem}}$ (µb/sr) | $Z_{\text{eff}}$ (×10$^{-2}$) | $dσ/dΩ_{\text{lab}}$ (µb/sr) | $q$ (MeV/c) | $α\langle dσ/dΩ\rangle_{\text{lab}}^{\text{elem}}$ (µb/sr) | $Z_{\text{eff}}$ (×10$^{-2}$) | $dσ/dΩ_{\text{lab}}$ (µb/sr) |
|--------------------------|-------------|-------------------------------------------------|-----------------|------------------|-------------|-------------------------------------------------|-----------------|------------------|
| 0                        | 351         | 570.7                                           | 1.276           | 7.28             | 362         | 624.0                                           | 3.015           | 18.82            |
| 2                        | 352         | 567.2                                           | 1.247           | 7.07             | 364         | 618.7                                           | 2.938           | 18.18            |
| 4                        | 357         | 556.7                                           | 1.164           | 6.48             | 368         | 603.3                                           | 2.719           | 16.40            |
| 6                        | 364         | 539.4                                           | 1.040           | 5.61             | 375         | 579.0                                           | 2.387           | 13.82            |
| 8                        | 374         | 515.7                                           | 0.890           | 4.59             | 384         | 547.7                                           | 1.988           | 10.89            |
| 10                       | 386         | 486.3                                           | 0.732           | 3.56             | 395         | 511.0                                           | 1.568           | 8.01             |
| 12                       | 400         | 452.0                                           | 0.579           | 2.62             | 408         | 470.7                                           | 1.168           | 5.50             |
| 14                       | 416         | 413.9                                           | 0.443           | 1.83             | 423         | 428.2                                           | 0.819           | 3.51             |
| 16                       | 434         | 373.5                                           | 0.327           | 1.22             | 440         | 384.6                                           | 0.536           | 2.06             |
| 18                       | 453         | 332.3                                           | 0.234           | 0.78             | 458         | 341.1                                           | 0.324           | 1.10             |
| 20                       | 473         | 291.7                                           | 0.163           | 0.47             | 477         | 298.7                                           | 0.176           | 0.53             |

$q$.  

F. $^3_{Λ H}$ v.s. $^4_{Λ H}$  

To compare the production cross sections between $^3_{Λ H}$ and $^4_{Λ H}$, we consider the ratio of $^3_{Λ H}$ to $^4_{Λ H}$ on the angular distributions of $dσ/dΩ_{\text{lab}}$,

$$\hat{R}(θ_{\text{lab}}) = \frac{[dσ/dΩ_{\text{lab}}(^3_{Λ H})]}{[dσ/dΩ_{\text{lab}}(^4_{Λ H})]}.$$  \hspace{1cm} (13)

Here we use only the production cross sections of the $^3_{Λ H}$ ground states because the contributions of the $^3_{Λ H}$ excited states to the $Λ$ productions are very small, as discussed above. In Fig. 6 we show the calculated values of $\hat{R}(θ_{\text{lab}})$ in the $(K^−, π^0)$ reaction at 1.0 GeV/c, together with those of $\hat{R}(θ_{\text{lab}})$ in the $(π^−, K^0)$ reaction at 1.05 GeV/c. In the $(K^−, π^0)$
FIG. 5: Comparison among the ratios of

\[ \hat{R}(\theta_{\text{lab}}) = \frac{d\sigma/d\Omega_{\text{lab}}(3\Lambda)}{d\sigma/d\Omega_{\text{lab}}(4\Lambda)} \]

in the DW and the PW, as a function of \( \theta_{\text{lab}} \). Solid and dashed curves denote the calculated values in the \((K^-, \pi^0)\) reaction at 1.0 GeV/c and the \((\pi^-, K^0)\) reaction at 1.05 GeV/c, respectively. The experimental data in the \((e, e'K^+)\) reaction at the virtual photon \(\gamma^*\) mass \(Q^2 = 3.5 \text{ GeV}^2\) are taken from Ref. [34].

reaction at 1.0 GeV/c, we find that the values of \(\hat{R}(\theta_{\text{lab}})\) fluctuate in the range of 0.3–0.6 at \(\theta_{\text{lab}} = 0^\circ – 20^\circ\). This behavior mainly indicates the difference in \(\alpha(d\sigma/d\Omega_{\text{lab}})^{\text{elem}}\) between \(3\Lambda\) and \(4\Lambda\), rather than the angular dependence of \(Z_{\text{eff}}\) at \(\theta_{\text{lab}} = 0^\circ – 20^\circ\) that correspond to \(q \simeq 80–350 \text{ MeV}/c\). On the other hand, in the \((\pi^-, K^0)\) reaction at 1.05 GeV/c, we find \(\hat{R}(\theta_{\text{lab}}) \simeq 0.4–0.8\) at \(\theta_{\text{lab}} = 0^\circ – 20^\circ\) that correspond to \(q \simeq 350–470 \text{ MeV}/c\). This behavior indicates the angular dependence of \(Z_{\text{eff}}(3\Lambda_{g.s.})/Z_{\text{eff}}(4\Lambda_{g.s.})\), which is related to the \(A = 3, 4\) form factors \(F(q)\) over the \(\Lambda\) production processes because the angular dependences of
\( \alpha \langle d\sigma/d\Omega \rangle_{\text{lab}}^{\text{elem}} \) for \( ^3\Lambda H \) and \( ^4\Lambda H \) are very similar. In Fig.\( \text{[5]} \) we also draw the experimental data taken from the \( ^{3,4}\text{He}(e, e'K^+) \) reaction at the virtual photon \( \gamma^* \) mass \( Q^2 = 3.5 \text{ GeV}^2 \) \[18, 34\].

It seems that the calculated results of \( \hat{R}(\theta_{\text{lab}}) \) in the \((\pi^-, K^0)\) reaction can simulate the data of the \((e, e'K^+)\) reaction because the values of \( q \) for the former and the latter are roughly the same. Moreover, we estimate the ratio of \( \sigma_{\text{lab}}(\Lambda^3 H) \) to \( \sigma_{\text{lab}}(\Lambda^4 H) \) on the integrated cross sections over \( \theta_{\text{lab}} = 0^\circ - 20^\circ \), which is given by

\[
R_{34} = \frac{\sigma_{\text{lab}}(\Lambda^3 H)}{\sigma_{\text{lab}}(\Lambda^4 H)}. \tag{14}
\]

In the \((K^-, \pi^0)\) reaction at 1.0 GeV/c, we find \( R_{34} = 0.41 \) in the DWIA. Considering some ambiguities in our eikonal-DWIA calculations, we also find \( R_{34} = 0.29 \) in the PWIA, omitting the distortions. Consequently, we have \( R_{34} \approx 0.3 - 0.4 \). Note that the value of \( R_{34} \) depends on \( B_{\Lambda} \) for \( ^3\Lambda H \); we find \( R_{34} \approx 0.65 \) when we use \( B_{\Lambda} = 0.41 \text{ MeV} \), as discussed in Sect. \[\text{III}\text{C}\]. It strongly suggests that the production of \( ^3\Lambda H \) is a promising subject to be observed experimentally based on a successive measurement of \( ^4\Lambda H \) at the J-PARC experiment.

In the \((\pi^-, K^0)\) reaction at 1.05 GeV/c, we find \( R_{34} = 0.46 \) in the DWIA and \( R_{34} = 0.28 \) in the PWIA, leading to \( R_{34} \approx 0.3 - 0.4 \). The comparison between the nuclear \((K^-, \pi^0)\) and \((\pi^-, K^0)\) reactions on \( \hat{R}(\theta_{\text{lab}}) \) provides a better understanding of not only the structure of the \( ^3\Lambda^4\text{H} \) bound states but also the production mechanism of these states.

**IV. SUMMARY AND CONCLUSION**

We have investigated theoretically productions of \( ^3\Lambda^4\text{H} \) bound states via \( ^{3,4}\text{He}(K^-, \pi^0) \) reactions in the DWIA with the optimal Fermi-averaging \( K^-p \to \pi^0\Lambda \) \text{t} matrix. We have calculated the laboratory differential cross sections of \( d\sigma/d\Omega_{\text{lab}} \) and the integrated cross sections of \( \sigma_{\text{lab}} \) by the non-spin-flip \( \Delta S = 0 \) production in the \( ^{3,4}\text{He}(K^-, \pi^0) \) reactions at 1.0 GeV/c and \( \theta_{\text{lab}} = 0^\circ - 20^\circ \), together with those by the spin-flip \( \Delta S = 1 \) production. We have also compared these cross sections with those in the \((\pi^-, K^0)\) reactions at 1.05 GeV/c.

The results are summarized as follows:

(i) The calculated integrated cross sections of the \( 0^+_{\text{gs}} \) and \( 1^+_{\text{exc}} \) states of \( ^4\Lambda H \) amount to \( \sigma_{\text{lab}}(0^+_{\text{gs}}) = 63.1 \mu\text{b} \) and \( \sigma_{\text{lab}}(1^+_{\text{exc}}) = 3.8 \mu\text{b} \), respectively, leading to \( R_4 = \sigma_{\text{lab}}(1^+_{\text{exc}})/\sigma_{\text{lab}}(0^+_{\text{gs}}) \approx 0.06 - 0.07 \). The production of the \( 0^+_{\text{gs}} \) state dominates in the forward angles of \( \theta_{\text{lab}} = 0^\circ - 20^\circ \), in comparison with that of the \( 1^+_{\text{gs}} \) state.
(ii) The calculated integrated cross sections of the 1/2_{gs,+} and 3/2_{exc} states of 3\Lambda H amount to $\sigma_{\text{lab}}(1/2_{gs,+}) = 25.9 \, \mu\text{b}$ and $\sigma_{\text{lab}}(3/2_{exc}) = 0.77 \, \mu\text{b}$, respectively. This leads to $R_3 = \sigma_{\text{lab}}(3/2_{exc})/\sigma_{\text{lab}}(1/2_{gs,+}) \simeq 0.03$, of which value is a half as large as that of $R_4$.

(iii) The calculated angular distributions of the in-medium $K^-p \rightarrow \pi^0\Lambda$ differential cross sections $\alpha|\mathcal{M}_{\pi^0\Lambda}|^2$ for 3\Lambda H are remarkably different from those for 4\Lambda H, caused by the optimal Fermi-averaging in the nuclear ($K^-, \pi^0$) reactions, whereas $\alpha|\mathcal{M}_{\pi^0\Lambda}|^2$ for 3.4\Lambda H are very similar to each other.

(iv) The recoil effects are important in productions of 3.4\Lambda H owing to the benefit of the use of the 3.4\text{He} targets via the nuclear ($K^-, \pi^0$) reactions, as well as the nuclear ($\pi^-, K^0$) reactions.

In conclusion, we show that the comparison in $d\sigma/d\Omega_{\text{lab}}$ and $\sigma_{\text{lab}}$ between 4\Lambda H and 3\Lambda H provides examining the mechanism of the production and structure of 3.4\Lambda H in the ($K^-, \pi^0$) reactions on 3.4\text{He} at $p_{K^-} = 1.0 \, \text{GeV}/c$; the calculated results indicate $R_{34} = \sigma_{\text{lab}}(3\Lambda\text{H})/\sigma_{\text{lab}}(4\Lambda\text{H}) \simeq 0.3–0.4$. This investigation confirms the feasibility of the lifetime measurements of 3\Lambda H at the J-PARC experiments.

Acknowledgments

The authors thank Dr. Y. Ma and Dr. F. Sakuma for many valuable discussions. This work was supported by Grants-in-Aid for Scientific Research (KAKENHI) from the Japan Society for the Promotion of Science: Scientific Research (C) (Grant No. JP20K03954).

Appendix A: Explicit forms of the differential cross sections

To consider the laboratory differential cross sections of the $A(K^-, \pi)B$ reaction in the DWIA, we will define wave functions of the initial and final states, $\Psi_A$ and $\Psi_B$, in the $jj$ coupling scheme:

$$|\Psi_A\rangle = \hat{A}\left[\Phi_{JC} \otimes \phi^{(N)}_{(l_1+\frac{1}{2})j_1}\right]^{M_A}_{jA},$$

(A1)

$$|\Psi_B\rangle = \sum_{JCj_2} \left[\Phi_{JC} \otimes \phi^{(A)}_{(l_2+\frac{1}{2})j_2}\right]^{M_B}_{jB},$$

(A2)
where $\Phi_{Jc}$, $\phi^{(N)}_{j_1 j_2}$, and $\phi^{(A)}_{j_1 j_2}$ are wave functions of a core nucleus, a nucleon in the target nucleus $A$, and $\Lambda$ in the hypernucleus $B$, respectively. $\hat{A}$ is the antisymmetrized operator for nucleons. The meson distorted waves for outgoing $\pi$ and incoming $K^-$ are written by the partial wave expansion

$$\chi^{(-)*}_{\pi}(p_\pi, r) \chi^{(+)}_{K^-}(p_{K^-}, r) = \sum_{\ell m} \sqrt{4\pi [\ell]} i^\ell \tilde{j}_{\ell m}(\theta_{\text{lab}}, r) Y_{\ell m}(\hat{r}),$$

(A3)

where $\tilde{j}_{\ell m}(\theta_{\text{lab}}, r)$ is the radial distorted wave with the angular momentum with $(\ell, m)$, and $\theta_{\text{lab}}$ is the scattering angle to the forward direction in the nuclear $(K^-, \pi)$ reaction.

The explicit form of the differential cross section with the non-spin-flip $\Delta S = 0$ processes in Eq. (1) is written as

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}}^{J_B(\Delta S=0)} = \alpha \sum_{\ell m cc'} (S^{1/2}_c)^* (S^{1/2}_{cc'}) \tilde{f}_{\pi A} \tilde{f}_{\pi A} (-)^{2J_B + J_C + J'_C - 1}$$

$$\times [J_B][\ell] \sqrt{[J_C][j_2][j_1][J'_C][j'_2][j'_1]} \left[ I_{j_2 j_1}^{m}(\theta_{\text{lab}}) \right]^{*} [I_{j_2 j_1}^{m}(\theta_{\text{lab}})]$$

for the spin-flip $\Delta S = 1$ processes, substituting the relation

$$Y_{\ell m} \sigma \cdot \hat{n} = Y_{\ell m} \sigma_y = \frac{i}{\sqrt{2}} \sum_{\mu = -1}^{1} \sum_{j} (\ell m 1 \mu |J M) [Y_\ell \otimes \sigma_1]^M_J$$

(A7)
into Eq. (11) and using the Racah algebra, we have the differential cross section with \( \Delta S = 1 \), which is written by

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\text{lab}}^{J_B(J_A J_C)} = \alpha \sum_{J M C \ell m c' \ell' m' c'} \sum_{\mu \mu'} (S^{1/2}_c)^* (S^{1/2}_{c'}) \pi \pi \pi \pi \pi \pi \\
\times \frac{1}{2} \langle c m 1 \mu | J M \rangle \langle c' \ell m' 1 \mu' \rangle (-)^{2 J_B + J_C + J_C + 2 J + j_1 + j_2 + \ell + \ell'} \times 6 [J_B][c][\ell'] \sqrt{[J_C][j_2][j_1][\ell_2][\ell_1]} [J_C'][j_2'][j_1'][\ell_2'][\ell_1'] \\
\times [I^{m}_{j_2 j_1 \ell}(\theta_{\text{lab}})]^* [I^{m}_{j_2' j_1' \ell'}(\theta_{\text{lab}})] \\
\times \left\{ \begin{array}{ccc} J_B & J_A & J \\ j_1 & j_2 & J_C \end{array} \right\} \left( \begin{array}{ccc} \ell_2 & \ell & \ell_1 \\ 0 & 0 & 0 \end{array} \right) \left\{ \begin{array}{ccc} \ell_2 & \frac{1}{2} & j_2 \\ \frac{1}{2} & j_1 & 1 \end{array} \right\} \\
\times \left\{ \begin{array}{ccc} J_B & J_A & J \\ j_1' & j_2' & J_C' \end{array} \right\} \left( \begin{array}{ccc} \ell_2' & \ell' & \ell_1' \\ 0 & 0 & 0 \end{array} \right) \left\{ \begin{array}{ccc} \ell_2' & \frac{1}{2} & j_2' \\ \frac{1}{2} & j_1' & 1 \end{array} \right\} . \tag{A8} \right. 
\]

[1] H. Asano, et al., $\Lambda$ mesonic weak decay lifetime measurement with $^{3,4}$He($K^{-}, \pi^{0}$)$^{3,4}$H reaction, Proposal for Nuclear and Particle Physics experiments at the J-PARC (2018); [http://j-parc.jp/NuclPart/Proposal_e.html](http://j-parc.jp/NuclPart/Proposal_e.html)

[2] M. Agnello, et al., Direct measurement of the $\Lambda$ mesonic weak decay lifetime using $^{3,4}$He($\pi^{-}, K^{0}$)$^{3,4}$H reactions, Proposal for Nuclear and Particle Physics experiments at the J-PARC (2019); [http://j-parc.jp/NuclPart/Proposal_e.html](http://j-parc.jp/NuclPart/Proposal_e.html)

[3] C. Rappold, et al., HypHI Collaboration, Nucl. Phys. A 913 (2013) 170.

[4] A. Gal, H. Garcilazo, Phys. Lett. B 791 (2019) 48 and references therein.

[5] M. Jurič, et al., Nucl. Phys. B 52 (1973) 1.

[6] J. Adam, et al., STAR Collaboration, Nature Phys. 16 (2020) 409.

[7] T. Harada, Phys. Rev. Lett. 81 (1998) 5287; Nucl. Phys. A 672 (2000) 181.

[8] T. Harada, Y. Hirabayashi, Phys. Lett. B 740 (2015) 312.

[9] T. Nagae, et al., Phys. Rev. Lett. 80 (1998) 1605.

[10] T. O. Yamamoto, et al., J-PARC E13 Collaboration, Phys. Rev. Lett. 115 (2015) 222501.

[11] T. Harada, Y. Hirabayashi, Phys. Rev. C 100 (2019) 024605.
[12] M. Schäfer, B. Bazak, N. Barnea, J. Mareš, Phys. Lett. B 808 (2020) 135614.
[13] T. Harada, Y. Hirabayashi, Nucl. Phys. A 759 (2005) 143; 767 (2006) 206.
[14] J. Hüfner, S. Y. Lee, H. A. Weidenmüller, Nucl. Phys. A 234 (1974) 429.
[15] C.B. Dover, L. Ludeking, G. E. Walker, Phys. Rev. C 22 (1980) 2073.
[16] E. H. Auerbach, A. J. Baltz, C. B. Dover, A. Gal, S. H. Kahana, L. Ludeking, D. J. Millener, Ann. Phys. (N.Y.) 148 (1983) 381.
[17] C. B. Dover, A. Gal, Ann. Phys. (N.Y.) 146 (1983) 309.
[18] T. Mart, et al., Nucl. Phys. A 640 (1998) 235; T. Mart, B. I. S. van der Ventel, Phys. Rev. C 78 (2008) 014004.
[19] T. Harada, Y. Hirabayashi, Phys. Rev. C 89 (2014) 054603.
[20] G. P. Gopal, et al., Nucl. Phys. B 119 (1977) 362.
[21] Y. Akaishi, International Review of Nuclear Physics 4 (World Scientific, Singapore, 1986), p. 259 and references therein.
[22] M. Bedjidian, et al., Phys. Lett. B 62, 467 (1976); Phys. Lett. B 83, 252 (1979).
[23] F. Schulz, et al., A1 Collaboration, Nucl. Phys. A 954 (2016) 149.
[24] M. Kaminura, M. Yahiro, Y. Iseri, Y. Sakuragi, H. Kameyama, M. Kawai, Prog. Theor. Phys. Suppl. 89 (1986) 1.
[25] T. Harada, Y. Hirabayashi, Nucl. Phys. A 934 (2015) 8.
[26] Y. H. Koike, T. Harada, Nucl. Phys. A 611 (1996) 461.
[27] Y. Kurihara, Y. Akaishi, H. Tanaka, Phys. Rev. C 31 (1985) 971.
[28] J. R. Taylor, Scattering Theory (Dover, New York, 2006) p. 246.
[29] T. Harada, Y. Hirabayashi, in Proceedings of the 12th International Conference on Hypernuclear and Strange Particle Physics (HYP2015), edited by H. Tamura, et al., JPS Conf. Proc. 17 (2017) 012008.
[30] M. Kawai, Prog. Theor. Phys. Suppl. 89 (1986) 11.
[31] K. W. McVoy, Nucl. Phys. A 115 (1968) 481.
[32] O. Morimatsu, K. Yazaki, Prog. Part. Nucl. Phys. 33 (1994) 679.
[33] T. Harada, Y. Hirabayashi, in Proceedings of 8th International Conference on Quarks and Nuclear Physics (QNP2018), edited by A. Doté, et al., JPS Conf. Proc. 26 (2019) 023004.
[34] F. Dohrmann, et al., Phys. Rev. Lett. 93 (2004) 242501.