Work harvesting by $q$-deformed statistical mutations in an Otto engine

Eren Güvenilir,1,* Fatih Ozaydin,2 Özgür E. Mümteçaplıoğlu,3,4 and Tuğrul Hakoğlu5,1,6

1Department of Physics Engineering, Istanbul Technical University, Sarıyer, İstanbul, 34467, Türkiye
2Institute for International Strategy, Tokyo International University, 1-13-1 Matoba- kita, Kawagoe, Saitama 350-1197, Japan
3Department of Physics, Koç University, Sarıyer, İstanbul, 34450, Türkiye
4TÜBİTAK Research Institute for Fundamental Sciences, 41470 Gebze, Türkiye
5Energy Institute, Istanbul Technical University, Sarıyer, İstanbul, 34467, Türkiye
6Department of Physics, Northeastern University, Boston, MA 02115, USA
(Dated: August 19, 2022)

We consider a semi-classical heat engine with a $q$-deformed quantum oscillator working substance and classical thermal baths. We investigate the influence of the quantum statistical deformation parameter $q$ on the work and efficiency of the engine. In usual heat engines, a Hamiltonian parameter is varied during the work injection and extraction stages while the quantum statistical character of the working substance remains fixed. We point out that even if the Hamiltonian parameters are not changing, work can be harvested by quantum statistical changes of the working substance. Work extraction from thermal resources using quantum statistical mutations of the working substance makes a semi-classical engine cycle without any classical analog. As a concrete example of such a semi-classical heat engine with a profound quantum character, we consider the Otto cycle and use the deformation parameter to define the isentropic steps while keeping the Hamiltonian parameters constant. We verify that our conclusion applies to both bosonic and fermionic oscillator deformations.

I. INTRODUCTION

Quantum Heat Engines (QHEs) are devices that can harvest work by controlling the state of the quantum working substance between hot and cold reservoirs [1, 2]. After the foundations of the quantum engines have been established, many researchers have devoted intense theoretical and experimental efforts to find new breakthroughs [3–17]. Enhancement of work and efficiency of such quantum machines, together with exploring their fundamental bounds, are among the significant goals of the emerging field of quantum thermodynamics. For that aim, non-linear and many-body, and fermionic or bosonic working systems have been studied to reveal their differences and relative advantages [3]. Here, we contribute to these research endeavors by addressing two questions. First, how the engine performance depends on quantum statistics in general if we mutate the particle statistics from bosonic to fermionic? Second, can we consider quantum statistics as another control parameter such that if all the system parameters remain the same, can we harvest energy from a heat bath by only changing the quantum statistics of the working substance?

In the 1970s, the deformed algebra was considered the generalization of boson Weyl-Heisenberg algebra [18, 19]. The theory of the $q$-oscillators was stated previously by Mcfarlane, and Biedenharn [20, 21]. Since then, $q$-deformation has been considered in various research areas. Including statistical physics and quantum information, nuclear and atomic physics, thermodynamics, open quantum systems, and optomechanical systems.

It is pointed out that there is a correspondence between $q$-deformed Heisenberg algebra and effective non-linear interaction of the cavity mode [22], and an isomorphism between the $q$-deformed harmonic oscillator and an anharmonic oscillator model was discovered [23]. Physical realization of the deformation parameter $q$ has been searched for heavily, among which are the quantum Yang-Baxter equation [24], deformed Jaynes-Cummings model [25], quantum phase problem [26–28], relativistic $q$-oscillator [29, 30], Morse oscillator [31], and Kepler problem [32]. Deformed algebras have been explored by subjecting the undeformed ones to non-linear invertible transformations [33–35]. The $q$-deformation parameter was considered for deriving generalized uncertainty and information relations [36], Tsallis entropy, and other relative entropy measures [37–39].

In atomic and nuclear physics, $q$-deformation was considered from theoretical and experimental perspectives [40–46]. It was considered for obtaining generalizations of quantum spin chains with exact valence-bond ground states [47], and self-localized solitons of $q$-deformed quantum systems have been recently explored [48]. Deformed algebra is also used in open quantum systems to show the relationship between the efficiency of QHE and the amount of the non-Markovianity cycle processes [49]. One of the recent works showed that two linearly coupled $q$-deformed cavities could be tuned to provide enhancement of non-classical phenomena [50].

In one of the earliest works in $q$-deformed quantum information, entanglement and noise reduction techniques were studied between $q$-deformed harmonic oscillators [51]. Non-classical properties of noncommutative states [52, 53] and entanglement in nonlinear quantum systems were analyzed in $q$-deformed settings [54]. Quantum states and logic gates were defined for two- and three-level $q$-deformed systems [55–57], and $q$-deformed relative entropies were studied in quantum metrology [58]. Very recently, $q$-deformation was considered in quantum thermodynamics by exploring the impact of deformation on the performance of heat engines [49].

Our work presents a unique perspective on the quantumness of semi-classical heat engines, which reflects the genuine

---

* guvenilir15@itu.edu.tr
quantum statistical character of the working system in work harvesting from classical thermal resources. The usual method to characterize the quantum nature of a heat engine is to look for quantum-enhanced performance over its classical analog. Our case is another yet more direct reflection of the quantumness of a heat engine as the cycle mechanism, which is based upon harvesting work by changing the quantum statistical character of its working substance, has no classical analog.

This paper is organized as follows: Sec. II reviews essential concepts of q-deformed oscillator algebra, where we introduce the generalized equations of both Bosonic and Fermionic oscillators we investigate. Sec. III presents the necessary tools to construct a q-deformed heat engine by discussing the fundamental thermodynamic quantities such as entropy and internal energy utilizing Jackson derivative approach. Then, equipped with the theoretical tools presented in previous sections, we present our results in Sec. IV, where we explicitly show how work harvesting from thermal resources can be achieved by variations of particle statistics and q-deformation. We compare our results with previous investigations and discuss the effectiveness of our approach in Sec. V. We conclude in Sec. VI.

II. A UNIFIED REPRESENTATION OF FERMIONIC AND BOSONIC q-DEFORMED OSCILLATOR ALGEBRAS

Generalized thermodynamics of q-deformed bosons and fermions have been developed in Ref. [59]. In this section, we provide a short review of the relevant concepts and present the results that we need to calculate the work and efficiency of a q-deformed heat engine.

We consider a system of ideal (non-interacting) q-bosons or q-fermions described by a Hamiltonian of the form $H = \sum_i (E_i - \mu) \hat{N}_i$, where $\mu$ is the chemical potential, $E_i$ is the eigenenergy, and $\hat{N}_i$ is the q-deformed number operator for the state $i$. Here, the eigenenergies $E_i$ are the energies that are found from q deformed harmonic oscillator $H = \frac{\hbar}{2}(\hat{a}_i \hat{a}^\dagger_i + \hat{a}^\dagger_i \hat{a}_i)$, which are equal to $E_i = \frac{\hbar}{2}([N_i + 1] + [N_i])$. We consider the following symmetric q-deformed algebra

$$[\hat{a}_i, \hat{a}^\dagger_i]_s = [\hat{a}^\dagger_i, \hat{a}_i]_s = 0, \quad [\hat{N}_i, \hat{a}_i] = [\hat{N}_i, \hat{a}^\dagger_i] = \hat{a}^\dagger_i. \quad (1)$$

Here, $\hat{a}_i, \hat{a}^\dagger_i$, and $\hat{N}_i$ are the annihilation, creation, and number operators, respectively. $q$ is a real number and $[A, B]_s = AB - sBA$. Bosonic and fermionic deformations are obtained for $s = 1$ and $s = -1$, respectively.

In terms of the q-base number $[x]$, which is defined by

$$[x] := \frac{q^x - q^{-x}}{q - q^{-1}}, \quad (4)$$

we have the relations $\hat{a}_i \hat{a}^\dagger_i = [\hat{N}_i]$ and $\hat{a}^\dagger_i \hat{a}_i = [1 + s\hat{N}_i]$. This algebra yields a q-Fock space that can be used for deformed bosons and fermions in a unified manner. The choice of the Hamiltonian is not unique, but the particular form taken here yields consistent derivation of thermostatistical equations in the same form as the undeformed thermodynamics.

III. ENTROPY AND INTERNAL ENERGY OF q-BOSON AND q-FERMION WORKING SYSTEMS

The grand partition function of the working system can be determined from

$$Z = \text{Tr}\{\exp(-H/T)\}, \quad (5)$$

where we take the Boltzmann constant $k_B$ as unity.) Such a non-deformed structure of the partition function relies on the assumption of the undeformed Gibbsian form of the thermal equilibrium state $\rho \sim \exp(-\beta H)$ with $\beta = 1/T$ and the expectation value $\langle A \rangle = \text{Tr}(\rho A)$; and it is associated with the assumption that Boltzmann-Gibbs form of the entropy function $S = \log W$ is preserved [60].

Expectation value of the occupation number, or mean number of deformed fermions or bosons defined by the relation $[n_i] = \text{Tr}\{(-\beta H)[N_i]\}$, in thermal equilibrium is found to be [61]

$$n_i = \frac{1}{q - q^{-1}} \log \left[ \frac{z^{-1}e^{\beta E_i} - sq^{-s}}{z^{-1}e^{\beta E_i} - sq^s} \right]. \quad (6)$$

Here, $z = \exp(\beta \mu)$ is the fugacity. The usual Bose-Einstein and Fermi-Dirac distributions are recovered in the $q \to 1$ limit. This result is not consistent with the standard thermodynamic relation

$$N = z \frac{\partial}{\partial z} \log Z = \sum_i n_i, \quad (7)$$

If we use so-called Jackson Derivative (JD)

$$D_x^q f(x) := \frac{f(qx) - f(q^{-1}x)}{x(q - q^{-1})}, \quad (8)$$

however, the structure of the thermodynamical relations are preserved. In particular we recover

$$N = z D_x^q \log Z = \sum_i n_i. \quad (9)$$

Similarly, the internal energy is found to be

$$U = s \sum_i \frac{\partial y_i}{\partial \beta} D_{y_i}^q \log (1 - s y_i) = \sum_i E_i n_i, \quad (10)$$

with $y_i = \exp(-\beta E_i)$.

For the construction of the heat engine cycle, we need the entropy in addition to the internal energy. Entropy is determined by $S = \log Z + \beta U - \beta \mu N$, which yields

$$S = \sum_i \{-n_i \log(n_i) + s(1 + sn_i)(\log(1 + sn_i) - s \log(1 + sn_i) - s n_i)\}. \quad (11)$$

Despite the thermodynamic relations preserving their usual structure, the entropy function becomes different than the standard bosonic and fermionic oscillator expressions by the last term emerging due to the non-additivity property of the q-numbers. The undeformed bosonic and fermionic entropy functions are symmetrically reproduced in the $q \to 1$ limit.
Such non-extensive property is persistent even in the classical limit if the oscillators are kept deformed.

Following this brief review, we can subsequently discuss specific heat engine cycles using the internal energy and the entropy expressions for both $q$-fermion and $q$-boson oscillator working systems.

### IV. RESULTS

As our specific engine cycle, we consider one of the most commonly used thermodynamic cycles, the classical Otto cycle [62], in our examinations. The quantum version of the Otto cycle has also been experimentally realized with quantum working substances [63, 64]. As illustrated in Fig. 1, it consists of two isentropic and two isochoric heating/cooling stages. In usual Otto cycles, the control parameter in the isentropic steps is the volume or a parameter of the Hamiltonian. Here, we keep the frequency of the quantum oscillator $\omega$ constant and take it $\omega = 0.025$ to define our energy-time scale so that we can use dimensionless and scaled parameters. Instead, our control parameters represent the quantum statistical character and non-linearity of the oscillator, characterized by particle statistics $s$ deformation parameter $q$. In the isentropic stages, the system is uncoupled from the heat baths and transformed adiabatically, starting at a thermal equilibrium state. For a classical heat engine, it is sufficient to make the change faster than the rate of heat exchange, instead of physically uncoupling the system from the environment, to ensure constant entropy condition. In general, the state at the end of the transformation is not a thermal equilibrium state with the environment, but a temperature can still be assigned to it using the constant entropy condition.

A profound quantum effect without any classical analog that we consider here is the quantum statistical mutation (change of the quantum statistical character) of the working system during the isentropic stages of the cycle. For this aim, we examine the behavior of the entropy function $S$ with temperature $T$ for different $q$ and $s$ values, then employ a constant entropy condition to define the isentropic stages and the corresponding temperature and entropy values for our four-stroke engine cycle. We calculate the necessary parameters at each stage to ensure constant entropy or isochoric step conditions to determine the Otto engine cycle.

Let us start with the $q$-deformation parameter to change in the isentropic stages of the cycle. For a $q$-deformed quantum oscillator, a decrease in $q$ yields an exponential increase in the energy gaps at higher energy levels. Thus, we take a smaller $q$ value for hot isochore and a higher $q$ value for cold isochore. During the isentropic stages, $q$ varies between the values used in hot and cold isochores. The $q$-deformed Otto cycle profiles are shown in Fig. 2. We remind that Hamiltonian parameters remain fixed at $\omega = 0.025$, and only the $q$-deformation parameter of the working substance is varied in the cycle. Accordingly, in contrast to the usual Otto cycle, the engine efficiency is independent of the frequency of the oscillator; it is determined by $q$ and $T$ values. For a given thermal gradient (difference in hot and cold temperatures) as a classical resource, we point out that one can find optimum quantum statistics of a quantum oscillator to maximize the efficiency of work harvesting. As an example, Fig. 3, shows a case of nearly perfect efficiency $\sim 99\%$. We remind that the difference between the areas under hot and cold isochoric lines in the $T - S$ diagram represents the positive work output of the engine cycle; while the ratio of the work output to the heat intake (the area under the hot isochore) is defined as efficiency.

We can further examine the cases of fermionic or bosonic regimes of deformation determined by the particle statistics parameter $s$. We use $s$ as the control parameter of the isentropic stages while keeping $q$ at a constant value. The hot isochore is considered in the bosonic regime, while the cold isochore is fermionic. In the isentropic stages, the particle statistics change between the bosonic and fermionic characters. It is shown in Fig. 3 that variation of the particle statistics at two extreme regimes (boson-fermion) can be used to harvest work from classical thermal resources in a semi-classical heat engine cycle. Numerical values of calculated efficiency values from the $T - S$ diagrams are presented in the figure captions.

### V. DISCUSSION

Before we represent the conclusion, we want to discuss our results compared with recent studies. A recently published article [65] has shown that the bosonic system has higher engine performance than the fermionic system and stated that the performance difference occurs due to the difference in internal energies arising from the Pauli exclusion principle. In our case, we consider the change in the statistical character of the working system, from bosonic to fermionic. Furthermore, in addition to boson-fermion mutations, we consider the deformation of particle statistics in terms of the $q$-deformation
parameter. In the same spirit as Ref. [65], we find out that one can optimize the engine performance for a given thermal resource in terms of the quantum statistical character of the working substance.

In another recent work, the effect of $q$-deformation was studied to show a relationship between the efficiency of QHE and the non-Markovianity in the engine cycle [49]. It states that the $q$-parameter, which causes non-equilibrium dynamics, helps to build a relationship between theoretical and experimental results. In our paper, we did not examine finite time engine cycles; the $q$ parameter, in our case, plays a more active and direct role as the engine cycle’s control parameter.

VI. CONCLUSION

Classical heat engines harvest work from a thermal resource by converting a heat flow between a hot and cold bath to an ordered work, energy. To do this task, parameters of their classical working system, for example, the volume of working gas, are varied in an engine cycle. Quantum heat engines use a quantum working material, and in addition to external degrees of freedom, Hamiltonian parameters and internal degrees of freedom can also be utilized for work extraction. Profound quantum effects, particularly improving engine performance over its classical counterpart, such as via quantum correlations, are possible with quantum heat engines. To distinguish the case where the resources can also be quantum, we call a quantum heat engine with classical thermal resources a semi-classical heat engine.

Here, we show that the quantum statistical character of the working substance can be used as another control parameter of a semi-classical heat engine. Specifically, we consider a quantum oscillator with a fixed frequency but deformed quantum parameter or by changing particle statistics between bosonic and fermionic regimes at a fixed $q$. While we consider the Otto cycle a paradigmatic model, we expect our fundamental conclusion holds for other engine cycles as well. From a practical point of view, our results suggest that for a given thermal resource, one can optimize the work harvesting in terms of the quantum statistics of the working substance. Variation of particle statistics can be experimentally challenging relative to the traditional way of variation of Hamiltonian parameters. However, $q$-deformation can
be envisioned and mapped to nonlinear terms in Hamiltonians and effective engineered deformed oscillator models ranging from semiconductor cavity QED [66] to atomic Bose-Einstein condensates [67] could be explored for physical embodiments of the statistical mutation route of work extraction.

ACKNOWLEDGMENTS

This study was funded by Istanbul Technical University BAP-41181. E.O. acknowledges the Personal Research Fund of Tokyo International University. E.G. thanks to Levent Subaşı for fruitful discussions.

[1] H. E. D. Scovil and E. O. Schulz-DuBois, Three-level masers as heat engines, Phys. Rev. Lett. 2, 262 (1959).
[2] J. E. Geusic, E. O. Schulz-DuBios, and H. E. D. Scovil, Quantum equivalent of the carnot cycle, Phys. Rev. 156, 343 (1967).
[3] H. T. Quan, Y.-x. Liu, C. P. Sun, and F. Nori, Quantum thermodynamic cycles and quantum heat engines, Phys. Rev. E 76, 031105 (2007).
[4] F. Altintas, A. U. C. Hardal, and O. E. Müstecaplıo˘glu, Quantum correlated heat engine with spin squeezing, Phys. Rev. E 90, 032102 (2014).
[5] F. Altintas, A. U. C. Hardal, and O. E. Müstecaplıo˘glu, Rabi model as a quantum coherent heat engine: From quantum biology to superconducting circuits, Phys. Rev. A 91, 023816 (2015).
[6] J. Roßnagel, O. Abah, F. Schmidt-Kaler, K. Singer, and E. Lutz, Nanoscale heat engine beyond the carnot limit, Phys. Rev. Lett. 112, 030602 (2014).
[7] M. T. Naseem and O. E. Müstecaplıo˘glu, Quantum heat engine with a quadratically coupled optomechanical system, J. Opt. Soc. Am. B 36, 3000 (2019).
[8] T. Hugel, N. B. Holland, A. Cattani, L. Moroder, M. Seitz, and H. E. Gaub, Single-molecule optomechanical cycle, Science 296, 1103 – 1106 (2002).
[9] R. Kosloff and A. Levy, Quantum heat engines and refrigerators: Continuous devices, Annual Review of Physical Chemistry 65, 365 (2014), pMID: 24689798.
[10] G. Barontini and M. Paternostro, Ultra-cold single-atom quantum heat engines, New Journal of Physics 21, 063019 (2019).
[11] O. Abah, J. Roßnagel, G. Jacob, S. Deffner, F. Schmidt-Kaler, K. Singer, and E. Lutz, Single-atom heat engine at maximum power, Phys. Rev. Lett. 109, 203006 (2012).
[12] I. A. Martínez, É. Roldán, L. Dinis, D. Petrov, J. M. R. Parrondo, and R. A. Rica, Brownian carnot engine, Nature Physics 12, 67 (2016).
[13] J. Roßnagel, S. T. Dawkins, K. N. Tolazzi, O. Abah, E. Lutz, F. Schmidt-Kaler, and K. Singer, A single-atom heat engine, Science 352, 325 (2016).
[14] A. Tuncer, M. Izadifar, C. B. Da˘g, F. Ozaydin, and O. E. Müstecaplıo˘glu, Work and heat value of bound entanglement, Quantum Information Processing 18, 373 (2019).
[15] C. B. Da˘g, W. Niedenzu, F. Ozaydin, O. E. Müstecaplıo˘glu, and G. Kurizki, Temperature control in dissipative cavities by entangled dimers, The Journal of Physical Chemistry C 123, 4035 (2019).
[16] S. Çakmak, Benchmarking quantum stirring and otto cycles for an interacting spinsystem, J. Opt. Soc. Am. B 39, 1209 (2022).
[17] J. P. S. Peterson, T. B. Batalhão, M. Herrera, A. M. Souza, R. S. Sarthour, I. S. Oliveira, and R. M. Serra, Experimental characterization of a spin quantum heat engine, Phys. Rev. Lett. 123, 240601 (2019).
[18] M. Arik, D. Coon, and Y.-m. Lam, Operator algebra of dual resonance models, Journal of Mathematical Physics 16, 1776 (1975).
[19] M. Arik and D. D. Coon, Hilbert spaces of analytic functions and generalized coherent states, Journal of Mathematical Physics 17, 524 (1976).
[20] L. C. Biedenharn, The quantum group SUq(2) and a q-analogue of the boson operators, Journal of Physics A Mathematical General 22, L873 (1989).
[21] A. J. Macfarlane, On q-analogues of the quantum harmonic oscillator and the quantum group SU(2)q, Journal of Physics A Mathematical General 22, 4581 (1989).
[22] V. Bužek, The Jaynes-Cummings Model with a q analogue of a coherent state, Journal of Modern Optics 39, 949 (1992).
[23] M. Artoni, J. Zang, and J. L. Birman, Anharmonic and nonclassical effects of the quantum-deformed harmonic oscillator, Phys. Rev. A 47, 2555 (1993).
[24] Z. Ma, Yang-Baxter equation and quantum enveloping algebras, Vol. 1 (World Scientific, 1993).
[25] M. Chaichian, D. Ellinas, and P. Kulish, Quantum algebra as the dynamical symmetry of the deformed jaynes-cummings model, Phys. Rev. Lett. 65, 980 (1990).
[26] D. Ellinas, Quantum phase and a q-deformed quantum oscillator, Phys. Rev. A 45, 3358 (1992).
[27] T. Hakoğlu, Admissible cyclic representations and an algebraic approach to quantum phase, Journal of Physics A: Mathematical and General 31, 707 (1998).
[28] T. Hakoğlu, Finite-dimensional schwinger basis, deformed symmetries, wigner function, and an algebraic approach to quantum phase, Journal of Physics A: Mathematical and General 31, 6975 (1998).
[29] R. M. Mir-Kasimov, SUq(1,1) and the relativistic oscillator, Journal of Physics A: Mathematical and General 24, 4283 (1991).
[30] M. Arik and M. Mungan, q-oscillators and relativistic position operators, Physics Letters B 282, 101 (1992).
[31] I. L. Cooper and R. K. Gupta, q-deformed morse oscillator, Phys. Rev. A 52, 941 (1995).
[32] O. F. Dayi and I. H. Duru, Slq(2) realizations for kepler and oscillator potentials and q-canonical transformations, Journal of Physics A: Mathematical and General 28, 2395 (1995).
[33] T. Curtright, D. Fairlie, and C. Zachos, Quantum groups: proceedings of the Argonne Workshop: Argonne National Laboratory, 16 April-11 May 1990 (World Scientific Publ., 1991).
[34] T. Curtright and C. Zachos, Deforming maps for quantum phase, Physics Letters B 243, 237 (1990).
[35] T. Hakoğlu and M. Arik, Quantum squeezing and the homographic oscillator, Phys. Rev. A 54, 52 (1996).
[36] J. C. van der Lubbe, B. Boxma, and D. E. Boekee, A generalized class of certainty and information measures, Information Sciences 32, 187 (1984).
[37] J. Naudts, Generalized Thermostatistics (Springer, London, 2011).
[38] E. P. Borges and I. Roditi, A family of nonextensive entropies, Physics Letters A 246, 399 (1998).
[39] P. T. Landsberg and V. Vedral, Distributions and channel capacities in generalized statistical mechanics, Physics Letters A 247,
G. Dattoli and A. Dascaloyannis, Generalized deformed oscillators for vibrational spectra of diatomic molecules, Physical Review A 46, 75 (1992).

D. Bonatsos and C. Dascaloyannis, Generalized deformed oscillators for vibrational spectra of diatomic molecules, Physical Review A 46, 75 (1992).

D. Bonatsos, D. Lenis, P. Raychev, and P. Terziev, Deformed harmonic oscillators for metal clusters: Analytic properties and supershells, Physical Review A 65, 033203 (2002).

A. Georgieva, K. Sviratcheva, M. Ivanov, and J. Draayer, $q$-deformation of symplectic dynamical symmetries in algebraic models of nuclear structure, Physics of Atomic Nuclei 74, 884 (2011).

A. A. Altintas, M. Arik, A. S. Arik, and E. Dil, Inhomogeneous quantum invariance group of multi-dimensional multi-parameter deformed boson algebra, Chinese Physics Letters 29, 010203 (2012).

M. Hammad, S. Gawaz, M. El-Hamammy, H. Motaweh, and S. Doma, $q$-deformed vibrational limit of interacting boson model, Journal of Physics Communications 3, 085019 (2019).

M. Jafarizadeh, N. Amir, N. Fouladi, M. Ghapanvari, and Z. Ranjbar, Study of phase transition of even and odd nuclei based on $q$-deformed su(1, 1) algebraic model, Nuclear Physics A 972, 86 (2018).

N. Boutabba, S. Grira, and H. Eleuch, Analysis of a $q$-deformed hyperbolic short laser pulse in a multi-level atomic system, Scientific Reports 12, 9308 (2022).

M. T. Batchelor and C. M. Yang, $q$-deformations of quantum spin chains with exact valence-bond ground states, International Journal of Modern Physics B 08, 3645 (1994).

C. Baymdr, A. A. Altintas, and F. Ozaydin, Self-localized solitons of a $q$-deformed quantum system, Communications in Nonlinear Science and Numerical Simulation 92, 105474 (2021).

H. Naseri-Karimvand, B. Lari, and H. Hassanabadi, Nonmarkovianity and efficiency of a $q$-deformed quantum heat engine, Physica A: Statistical Mechanics and its Applications 598, 127408 (2022).

A. Kundu and J. A. Miszczak, Transparency and enhancement in fast and slow light in $q$-deformed optical mechanical system, Annalen der Physik 534, 2200026 (2022).

G. Dattoli and A. Torre, $q$-harmonic-oscillator entangled states, Il Nuovo Cimento B (1971-1996) 111, 731 (1996).

S. Dey, $q$-deformed noncommutative cat states and their nonclassical properties, Physical Review D 91, 044024 (2015).

K. Berrada and H. Eleuch, Noncommutative deformed cat states under decoherence, Physical Review D 100, 016020 (2019).

K. Berrada, M. E. Baz, H. Eleuch, and Y. Hassouni, Bipartite entanglement of nonlinear quantum systems in the context of the $q$-heisenberg weyl algebra, Quantum Information Processing 11, 351 (2012).

A. Filippov, D. Gangopadhyay, and A. Isaev, Harmonic oscillator realization of the canonical $q$-transformation, Journal of Physics A: Mathematical and General 24, L63 (1991).

A. A. Altintas, F. Ozaydin, C. Yesilyurt, S. Bugu, and M. Arik, Constructing quantum logic gates using $q$-deformed harmonic oscillator algebras, Quantum information processing 13, 1035 (2014).

A. A. Altintas, F. Ozaydin, and C. Baymdr, $q$-deformed three-level quantum logic, Quantum Information Processing 19, 1 (2020).

H. Hasegawa, Quantum fisher information and $q$-deformed relative entropies: Additivity vs nonadditivity, Prog. Theor. Phys., 162, 183 (2006).

A. Lavagno and P. N. Swamy, Generalized thermodynamics of $q$-deformed bosons and fermions, Phys. Rev. E 65, 036101 (2002).

C. Tsallis, Possible generalization of boltzmann-gibbs statistics, Journal of Statistical Physics 52, 479 (1998).

J. Tuszyński, J. Rubin, J. Meyer, and M. Kibler, Statistical mechanics of a $q$-deformed boson gas, Physics Letters A 175, 173 (1993).

R. Dittman and M. Zemansky, Heat and Thermodynamics: An Undergraduate and Intermediate Textbook (Independently Published, 2020).

J. P. S. Peterson, T. B. Batalhão, M. Herrera, A. M. Souza, R. S. Sarthour, I. S. Oliveira, and R. M. Serra, Experimental characterization of a spin quantum heat engine, Phys. Rev. Lett. 123, 240601 (2019).

Q. Bouton, J. Nettersheim, S. Burgardt, D. Adam, E. Lutz, and A. Widera, A quantum heat engine driven by atomic collisions, Nature Communications 12, 2063 (2021).

N. M. Myers and S. Deffner, Bosons outperform fermions: The thermodynamic advantage of symmetry, Phys. Rev. E 101, 012110 (2020).

Y.-X. Liu, C. P. Sun, S. X. Yu, and D. L. Zhou, Semiconductor-cavity qed in high-$q$ regimes with $q$-deformed bosons, Phys. Rev. A 63, 023802 (2001).

C. W. Gardiner, Particle-number-conserving bogoliubov method which demonstrates the validity of the time-dependent gross-pitaevskii equation for a highly condensed bose gas, Phys. Rev. A 56, 1414 (1997).