FINITE GROUPS OF SYMPLECTIC AUTOMORPHISMS
OF K3 SURFACES IN POSITIVE CHARACTERISTIC

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Abstract. We show that Mukai’s classification of finite groups which may act symplectically on a complex K3 surface extends to positive characteristic $p$ under the assumptions that (i) the order of the group is coprime to $p$ and (ii) either the surface or its quotient is not birationally isomorphic to a supersingular K3 surface with Artin invariant 1. In the case without the assumption (ii) we classify all possible new groups which may appear. We prove that the assumption (i) on the order of the group is always satisfied if $p > 11$ and if $p = 2, 3, 5, 11$ we give examples of K3 surfaces with finite symplectic automorphism groups of order divisible by $p$ which are not contained in Mukai’s list.

1. Introduction

A remarkable work of S. Mukai [Mu] gives a classification of finite groups which can act on a complex algebraic K3 surface $X$ leaving invariant its holomorphic 2-form (symplectic automorphism groups). Any such group turns out to be isomorphic to a subgroup of the Mathieu group $M_{23}$ which has at least 5 orbits in its natural action on a set of 24 elements. A list of maximal subgroups with this property consists of 11 groups, each of these can be realized on an explicitly given K3 surface. A different proof of Mukai’s result was given later by S. Kondo [Ko] and G. Xiao [Xiao] classified all possible topological types of a symplectic action. Neither Mukai’s nor Kondo’s proof extends to the case of K3 surfaces over algebraically closed fields of positive characteristic $p$. In fact there are known examples of surfaces over a field of positive characteristic whose automorphism group contains a finite symplectic subgroup which is not realized as a subgroup of $M_{23}$ (e.g. the Fermat quartic over a field of characteristic 3, or the surface from [DKo] over a field of characteristic 2).

The main tool used in Mukai’s proof is the characterization of the representation of a symplectic group $G \subset \text{Aut}(X)$ on the 24-dimensional cohomology space

$$H^*(X, \mathbb{Q}) = H^0(X, \mathbb{Q}) \oplus H^2(X, \mathbb{Q}) \oplus H^4(X, \mathbb{Q}).$$

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Using the Lefschetz fixed-point formula and the description of possible finite cyclic subgroups of Aut(X) and their fixed-point sets due to Nikulin [Ni1], allows one to compute the value \( \chi(g) \) of the character of this representation at any element of finite order \( n \). It turns out that

\[
\chi(g) = \epsilon(n)
\]

for some function \( \epsilon(n) \) and the same function describes the character of the 24-permutation representation of \( M_{23} \). A representation of a finite group \( G \) in a finite-dimensional vector space of dimension 24 over a field of characteristic 0 is called a Mathieu representation if its character is given by

\[
\chi(g) = \epsilon(\text{ord}(g)).
\]

The obvious fact that \( G \) leaves invariant an ample class, \( H^0(X, \mathbb{Q}), H^4(X, \mathbb{Q}), H^2(X, \mathcal{O}_X) \) and \( H^0(X, \Omega^2_X) \), shows that

\[
\dim H^*(X, \mathbb{Q})^G \geq 5.
\]

These properties and the known classification of finite subgroups of \( \text{SL}(2, \mathbb{C}) \) which could be realized as stabilizer subgroups of \( G \) in its action on \( X \) shows that the 2-Sylow subgroups of \( G \) can be embedded in \( M_{23} \). By clever and non-trivial group theory arguments Mukai proves that subgroups of \( M_{23} \) with at least 5 orbits are characterized by the properties that they admit a rational Mathieu representation \( V \) with \( \dim V^G \geq 5 \) and their 2-Sylow subgroups are embeddable in \( M_{23} \).

The main difficulties in the study of K3 surfaces over algebraically closed fields of positive characteristic \( p \) arise from the absence of the Torelli Theorem, the absence of a natural unimodular integral lattice containing the Neron-Severi lattice, the presence of supersingular K3 surfaces, and the presence of wild automorphisms.

A group of automorphisms is called \textit{wild} if it contains a wild automorphism, an automorphism of order equal to a power of the characteristic \( p \), and \textit{tame} otherwise.

In this paper we first improve the results of our earlier paper [DK1] by showing that a finite symplectic group of automorphisms \( G \) is always tame if \( p > 11 \). Next, we show that Mukai’s proof can be extended to finite tame symplectic groups in any positive characteristic \( p \) unless both the surface and its quotient are birationally isomorphic to a supersingular K3 surface with Artin invariant equal to 1 (the exceptional case). To do this, we first prove that Nikulin’s classification of finite order elements and its sets of fixed points extends to positive characteristic \( p \), as long as the order is coprime to \( p \). Next we consider the 24-dimensional representations of \( G \) on the \( l \)-adic cohomology, \( l \neq p \),

\[
H^0_{\text{et}}(X, \mathbb{Q}_l) = H^0_{\text{et}}(X, \mathbb{Q}_l) \oplus H^2_{\text{et}}(X, \mathbb{Q}_l) \oplus H^4_{\text{et}}(X, \mathbb{Q}_l)
\]

and on the crystalline cohomology

\[
H^*_\text{crys}(X/W) = H^0_{\text{crys}}(X/W) \oplus H^2_{\text{crys}}(X/W) \oplus H^4_{\text{crys}}(X/W).
\]
It is known that the characteristic polynomial of any automorphism \( g \) has integer coefficients which do not depend on the choice of the cohomology theory. Comparing \( H^2_{\text{crys}}(X/W) \) with the algebraic De Rham cohomology \( H^2_{\text{DR}}(X) \) allows one to find a free submodule of \( H^*_{\text{crys}}(X/W)^G \) of rank 5 except when both \( X \) and \( X/G \) are birationally isomorphic to a supersingular K3 surface with Artin invariant equal to 1 (the exceptional case). This shows that in a non-exceptional case, for all prime \( l \neq p \), the vector spaces

\[
V_l = H^*_{\text{et}}(X, \mathbb{Q}_l)
\]

are Mathieu representations of \( G \) with

\[
\dim V_l^G \geq 5.
\]

A careful analysis of Mukai’s proof shows that this is enough to extend his proof.

In the exceptional case, it is known that a supersingular K3 surface with Artin invariant equal to 1 is unique up to isomorphism. It is isomorphic to the Kummer surface of the product of two supersingular elliptic curves if \( p > 2 \), and to the surface from [DKo] if \( p = 2 \). We call a tame group \( G \) exceptional if it acts on such a surface with \( \dim V_l^G = 4 \). We use arguments from Mukai and some additional geometric arguments to classify all exceptional groups. All exceptional groups turn out to be subgroups of the Mathieu group \( M_{23} \) with 4 orbits. S. Kondō has confirmed the converse, that is, any subgroup of \( M_{23} \) with 4 orbits is either from our list or wild. The problem of realizing exceptional groups will be discussed in other publication.

In the last section we give examples of K3 surfaces in characteristic

\( p = 2, 3, 5, 11 \)

with wild finite symplectic automorphism groups which are not contained in Mukai’s list. We do not know a similar example in characteristic \( p = 7 \), however we exhibit a K3-surface with a symplectic group of automorphisms of order 168 which does not lift to a surface from Mukai’s list.

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2. Automorphisms of order equal to the characteristic $p$

Let $X$ be a $K3$ surface over an algebraically closed field $k$ of positive characteristic $p$.

The automorphism group $\text{Aut}(X)$ acts on the 1-dimensional space of regular 2-forms $H^0(X, \Omega^2_X)$. Let 
\[ \chi_{2,0} : \text{Aut}(X) \to k^* \]
be the corresponding character. An automorphism $g$ is called symplectic if $\chi_{2,0}(g) = 1$.

Obviously any automorphism of order equal to the characteristic is symplectic. The main result of this section is the following.

**Theorem 2.1.** Let $g$ be an automorphism of $X$ of order $p = \text{char}(k)$. Then $p \leq 11$.

First, we recall the following result from our previous paper [DK1].

**Theorem 2.2.** Let $g$ be an automorphism of order $p$ and let $X^g$ be the set of fixed points of $g$. Then one of the following cases occurs.

1. $X^g$ is finite and consists of 0, 1 or 2 point; $X^g$ may be empty only if $p = 2$, and may consist of 2 points only if $p \leq 5$.
2. $X^g$ is a divisor such that the Kodaira dimension of the pair $(X, X^g)$ is equal to 0. In this case $X^g$ is a connected nodal cycle, i.e. a connected union of smooth rational curves.
3. $X^g$ is a divisor such that the Kodaira dimension of the pair $(X, X^g)$ is equal to 1. In this case $p \leq 11$ and there exists a divisor $D$ with support $X^g$ such that the linear system $|D|$ defines an elliptic or quasi-elliptic fibration $\phi : X \to \mathbb{P}^1$.
4. $X^g$ is a divisor such that the Kodaira dimension of the pair $(X, X^g)$ is equal to 2. In this case $X^g$ is equal to the support of some nef and big divisor $D$. Take $D$ minimal with this property. Let 
\[ d := \dim H^0(X, \mathcal{O}_X(D - X^g)) \quad \text{and} \quad N := \frac{1}{2}D^2 + 1. \]

Then 
\[ p(N - d - 1) \leq 2N - 2. \]

Theorem 2.2 does not give the bound for $p$ in the cases (1), (2) and (4). First we take care of the case (4). Its proof is essentially contained in [DK1].

**Proposition 2.3.** Let $g$ be an automorphism of order $p = \text{char}(k)$ and $F = X^g$ (with reduced structure). Assume $\kappa(X, F) = 2$. Then $p \leq 5$.
Moreover, if $p = 5$, then $X^g$ contains a curve $C$ of arithmetic genus 2 and the linear system $|C|$ defines a double cover $X \to \mathbb{P}^2$. The surface is $g$-isomorphic to the affine surface 
\[ z^2 = (y^5 - yx^4)P_1(x) + P_0(x), \quad g(x, y, z) = (x, y + x, z), \]
where $P_i(x)$ is a polynomial of degree $i$. 

Proof. We improve the argument from our paper \cite{DK}. 

Case 1: $F$ is nef and $|F|$ has non-empty fixed part.

By a well-known result due to Saint-Donat \cite{SD}, $F \sim aE + \Gamma$, where $a \geq 2$ and $E$ is an irreducible curve of arithmetic genus 1 and $\Gamma$ is the fixed part which is a $(-2)$-curve with $E \cdot \Gamma = 1$. In this case $g$ leaves invariant a genus 1 pencil defined by the linear system $|E|$. Note that $\Gamma$ is a section of the fibration and is fixed pointwisely by $g$. It is well known that no elliptic curve admits an automorphism of order $\geq 5$ fixing the origin. Thus $p \leq 3$.

Case 2: $F$ is nef and $|F|$ has no fixed part.

Assume $F^2 = 0$. Then $F \sim aE$, where $E$ is an irreducible curve of arithmetic genus 1, hence $\kappa(X, F) = 1$.

Assume $F^2 = 2$. Then $|F|$ has no base points, and $|F|$ defines a map of degree 2 onto $\mathbb{P}^2$. Since we can assume that $p \neq 2$, the map is separable and the branch curve is a curve of degree 6 with simple singularities which is invariant under a projective transformation of order $p$. Since $F$ is the pre-image of a line, $g$ fixes pointwisely a line in $\mathbb{P}^2$. Thus it is conjugate to a transformation $(x_0, x_1, x_2) \mapsto (x_0, x_1, x_2 + x_1)$. It is known that the ring of invariants $k[x_0, x_1, x_2]^g$ is generated by the three polynomials $x_0, x_1, x_2 - x_0x_1^{p-1}$. Thus $p \leq 5$.

Assume $F^2 \geq 4$. Then $F$ is nef and big, hence Theorem \cite{DK} (4) applied to $D = F$ gives $d = 1$ and $p(N - 2) \leq 2N - 2$. Since $N = D^2/2 + 1 \geq 3$, this gives $p \leq 3$.

Case 3: $F$ is not nef.

Since $F$ is reduced and connected, being non-nef means that

$$F = F_1 + C_1 + \ldots + C_k,$$

where $C_i$’s are chains of $(-2)$-curves with no common components and $C_i \cdot F_1 = 1$ and $F_1$ is nef. In fact, let $E$ be a $(-2)$-curve such that $F \cdot E < 0$. Then $F = E + F'$, and $E \cdot F = -2 + E \cdot F' > -2$. Hence $E \cdot F' = 1$. This implies that $E$ is an end-component of $F$ and $F'$ is connected. If $F'$ is not nef, repeat the same process. Continuing in this way, we prove the claim.

If $F_1^2 = 0$, then $g$ fixes $F_1$ and $C_1$, hence fixes pointwisely a fibre of an elliptic fibration and a section. We argue as in Case 1 to get $p \leq 3$. If $F_1^2 = 2$, we use Case 2 and get $p \leq 5$. So let us assume that $F_1^2 \geq 4$, so $F_1$ is big and nef.

Take $D = F + F_1 = 2F_1 + C_1 + \ldots + C_k$. It is nef, and indeed minimal nef. Then

$$N = h^0(D) - 1 = \frac{1}{2}D^2 + 1 = \frac{1}{2}(4F_1^2 + 4k - 2k) + 1 = 2F_1^2 + k + 1,$$

$$d = h^0(D - F) = h^0(F_1) = \frac{1}{2}F_1^2 + 2 \geq 4.$$

Hence $N = 4d + k - 7$ and the inequality of Theorem \cite{DK} (4) gives

$$p(N - d - 1) = p(3d + k - 8) \leq 2(N - 1) = 8d + 2k - 16.$$
Hence
\[ p \leq \frac{(8d + 2k - 16)}{(3d + k - 8)} = 2 + \frac{2d}{3d + k - 8} < 4. \]

It remains to consider the cases (1) and (2) of Theorem 2.2, that is, the cases where \( X^g \) is either one point or a connected nodal cycle. To do this, we first recall the following information from [DK1](Lemma 2.1 and Theorem 2.4).

Proposition 2.4. The following is true.

1. If \( X^g \) consists of a point, then \( X/(g) \) is either a rational surface with trivial canonical divisor and one isolated elliptic Gorenstein singularity or a K3 surface with one rational double point. The latter case occurs only if \( p \leq 5 \).

2. If \( X^g \) is a nodal cycle, then \( X'/(g) \) is a rational surface with trivial canonical divisor with one isolated elliptic Gorenstein singularity, where \( X' \) is the surface obtained from \( X \) by blowing down the nodal cycle \( X^g \).

In the following proposition, we prove that the case (2) of Theorem 2.2 occurs only if \( p = 2 \).

Proposition 2.5. Suppose \( X^g \) is a nodal cycle. Then \( p = 2 \).

Proof. The quotient \( Z = X/(g) \) is known to be a rational surface with at most rational singularities and \( -K_Z = (p-1)B \), where \( B_{\text{red}} \) is the image of the nodal cycle in \( Z \) (see [DK1], section 3). Let \( \pi : Z' \to Z \) be a minimal resolution of singularities. Then \( -K_{Z'} = -\pi^*(K_Z) + \Delta' = (p-1)\pi^*(B) + \Delta' \), where \( \Delta' \) is a positive divisor supported on the exceptional locus. Obviously \( Z' \) is a resolution of singularities of the surface \( X'/(g) \), where \( X' \) is obtained from \( X \) by blowing down the nodal cycle \( X^g \).

Let \( a : Z' \to V \) be the blowing down to a minimal resolution of \( X'/(g) \). Then \( -K_V = a_*((p-1)\pi^*(B)) + \Delta' \) is the fundamental cycle of \( V \). By Corollary 3.6 of [DK1], we have \( H^1((p-1)B, \mathcal{O}_{(p-1)B}) \cong k \). Since \( Z \) has only rational singularities this implies that \( H^1((p-1)\pi^*(B), \mathcal{O}_{(p-1)\pi^*(B)}) \cong k \), and hence \( H^1(a_*((p-1)B), \mathcal{O}_{a_*((p-1)B)}) \neq \{0\} \). It is known that for any proper part \( A \) of the fundamental cycle of a minimal elliptic singularity, we have \( H^1(A, \mathcal{O}_A) = 0 \). This implies that

\[ -K_V = (p-1)a_*({\pi}^*(B)). \]

Since \( V \) is a non-minimal rational surface, it contains a \((-1)\)-curve \( E \). Intersecting both sides of the previous equality with \( E \), we get \( 1 = (p-1){\pi}^*(B) \cdot E \), hence \( p = 2 \). \( \square \)

Lemma 2.6. (P. Samuel) Let \( A \) be a normal noetherian local \( k \)-algebra of dimension \( \geq 2 \) with maximal ideal \( m \). Let \( G \) be a finite group of automorphisms of \( A \) with local ring of invariants \( A^G \). Assume that \( G \) acts freely on
the punctured local scheme \( V = \text{Spec}(A) \setminus \{m\} \). Then the class group \( \text{Cl}(A^G) \) fits in the following exact sequence of groups:

\[
0 \to H^1(G, A^*) \to \text{Cl}(A^G) \to H^0(G, H^1(V, \mathcal{O}_V^*)).\]

In particular, if in addition \( A \) is factorial, \( \text{Cl}(A^G) \cong H^1(G, A^*) \).

**Proof.** Let \( U = \text{Spec}(A^G) \setminus \{m^G\} \). It is known that \( \text{depth}(A^G) \geq 2 \) [Fo]. Thus the class group \( \text{Cl}(A^G) \) is isomorphic to the Picard group \( \text{Pic}(U) \). We apply the two spectral sequences which we used in [DK1] to the free action of \( G \) on \( V \) with quotient \( U \) and the \( G \)-linearized sheaf \( \mathcal{O}_V^* \).

\[
E_2^{i,j} = H^i(G, H^j(V, \mathcal{O}_V^*)) \Rightarrow \mathbb{H}^n,
\]

\[
'E_2^{i,j} = H^i(U, H^j(G, \mathcal{O}_V^*)) \Rightarrow \mathbb{H}^n.
\]

Since \( G \) acts freely,

\[
H^j(G, \mathcal{O}_V^*) = 0 \quad \text{for} \quad j > 0
\]

and the second spectral sequence gives an isomorphism

\[
\text{Cl}(A^G) = \text{Pic}(U) = H^1(U, \mathcal{O}_U^*) \cong \mathbb{H}^1.
\]

Now the first assertion follows from the first spectral sequence.

If \( A \) is factorial, then

\[
\text{Cl}(A) = H^1(V, \mathcal{O}_V^*) = 0,
\]

which proves the last assertion. \( \square \)

The well-known property of cohomology groups ([Cl], Chapter XII, Proposition 2.5) gives the following.

**Corollary 2.7.** In the situation of the above lemma, if in addition \( |\text{Cl}(A)| \) is finite, then \( \text{Cl}(A^G) \) is killed by multiplication by the product \( |\text{Cl}(A)| \cdot |G| \).

We will also use the following formula from [KS], Lemma 4.1.7 or [Sa], Theorem 7.4.

**Lemma 2.8.** Let \( X \) be a smooth surface over an algebraically closed field \( k \) of characteristic \( p \geq 0 \) and \( x \in X \) be a closed point of \( X \). Let \( G \) be a finite group of automorphisms of \( X \) such that \( X^G = \{x\} \). Let \( U = X \setminus \{x\} \) and \( V = U/G \). Then

\[
e_c(U) = (#G - 1)(e_c(V) + 1) + e_c(V) - \sum_{g \in G \setminus \{1\}} l(g),
\]

where \( e_c(Z) \) denotes the \( l \)-adic Euler-Poincaré characteristic with compact support for any \( l \neq p \) and \( l(g) \) is the intersection index of the graph of \( g \) with the diagonal at the point \((x, x)\).

**Proposition 2.9.** Let \( g \) be a wild automorphism of order \( p \neq 11 \) \(^1\) acting on a K3 surface such that \( |X^g| = 1 \). Assume that the quotient surface is rational. Then there exists a \( g \)-invariant elliptic or quasi-elliptic fibration on \( X \).

\(^1\)The assumption \( p \neq 11 \) can be dropped, see [DK2].
Proof. Note that the quotient surface \( X/(g) \) is a rational surface with trivial canonical divisor with one isolated elliptic Gorenstein singularity \( Q \). Let 

\[ \sigma : Y \to X/(g) \]

be a minimal resolution. We have

\[ K_Y = -\Delta, \]

where \( \Delta \) is an effective divisor whose support is equal to the exceptional set of \( \sigma \) and satisfies

\[ \Delta \cdot R_i \leq 0 \]

for any irreducible component \( R_i \) of \( \Delta \) (see, for example, [Re], 4.21). Let \( N \) be the sublattice of \( \text{Pic}(Y) \) spanned by the divisor classes of the curves \( R_i \) and \( N^* \) be its dual lattice. It is known that the class group of the local ring of the singular point \( Q \) is mapped surjectively onto \( N^*/N \) ([Gr]). By Corollary 2.7, the group \( N^*/N \) is a \( p \)-elementary finite abelian group. Since \( Y \) is a rational surface, the Picard lattice \( \text{Pic}(Y) \) is a unimodular hyperbolic lattice, hence the orthogonal complement \( N^\perp \) is a \( p \)-elementary lattice of signature \((1, t)\), \( t \geq 0 \). Since \( N \) contains \( K_Y \), it is also an even lattice. In particular, \( \text{rank } N^\perp \geq 2 \) if \( p \) is odd.

Case 1: \( p > 2 \) and \( \text{rank } N^\perp \geq 3 \), or \( p = 2 \) and \( \text{rank } N^\perp \geq 2 \).

It follows from [RS] that, if \( p > 2 \), an even \( p \)-elementary hyperbolic lattice of rank \( \geq 3 \) is determined uniquely by its rank and discriminant. An explicit construction of such a lattice shows that it always contains an isotropic vector. Also it is known that an even 2-elementary hyperbolic lattice of rank \( \geq 2 \) always contains an isotropic vector ([N2], Theorem 4.3.3). Thus \( N^\perp \) contains an isotropic vector \( v \). Since the homomorphism \( \sigma^* : \text{Pic}(X/(g)) \to N^\perp \) has finite cokernel, some multiple of \( v \) is equal to \( \sigma^*(D) \), where \( D \) is a Cartier divisor class on \( X/(g) \) with \( D^2 = 0 \) (here we use the intersection theory for \( \mathbb{Q} \)-divisors on \( \mathbb{Q} \)-factorial surfaces). The pre-image of \( D \) in \( \text{Pic}(X) \) is an isotropic vector. Dividing it by an integer, we can represent this vector by an effective primitive divisor class

\[ (2.2) \quad A = F + E, \]

where \( F \) is the fixed part of \( |A| \) consisting of a bunch of \((-2)\)-curves and \( |E| \) is a free pencil of arithmetic genus 1 curves. Since our automorphism \( g \) preserves \( |A| \), it must preserve \( |E| \). The arithmetic genus 1 pencil \( |E| \) is \( g \)-invariant and one of its fibres contains the point \( X^g \). This proves the assertion.

Case 2: \( p > 2 \) and \( \text{rank } N^\perp = 2 \).

We apply Lemma 2.8 to our situation. Since \( e_c(X \setminus \{x\}) = 23 \), the formula (2.1) gives

\[ (2.3) \quad 23 = (p - 1)(e_c(V) + 1 - l(g)) + e_c(V). \]
Keeping the notation of the lemma we have $V \cong Y \setminus \Delta_{\text{red}}$. By the additivity of $e_c$ we have

$$e_c(V) = e_c(Y) - e_c(\Delta_{\text{red}}) = 2 + \text{rank} \text{Pic}(Y) - (1 + \text{rank} N) = \text{rank} N^\perp + 1 = 3.$$ 

Here we use that the graph of components of $\Delta$ is a tree of smooth rational curves or an irreducible cuspidal curve of arithmetic genus $1$. Otherwise, the local Picard group of $X/G$ contains a connected algebraic group isomorphic to an elliptic curve or the multiplicative group, hence contains elements of finite order not killed by multiplication by $p$. The latter contradicts Corollary 2.7. Now formula (2.3) gives $20 = (p - 1)a$, where $a < 4$. The only possibility is $p = 11$.

Case 3: $p = 2$ and $\text{rank} N^\perp = 1$.

In this case $e_c(V) = 2$, hence the formula gives

$$21 = (2 - 1)(2 + 1 - l(g)),$$

absurd. □

Remark 2.10. If $\text{rank} N^\perp > 2$, the formula (2.3) gives

$$23 - k = (p - 1)(k + 1 - l(g)),$$

where $k = e_c(V) > 3$. If $k \neq 23$, this gives a weaker bound $p \leq 19$ with possible cases $(p, k, l(g)) = (19, 5, 5), (17, 7, 7), (13, 11, 11)$, etc.

Proposition 2.11. Let $g$ be an automorphism of $X$ of order $p = \text{char}(k)$. If $X$ admits a $g$-invariant arithmetic genus 1 fibration, then $p \leq 11$.

Proof. Since quasi-elliptic fibrations occur only in characteristic $p = 2$ or $3$, we may assume that our fibration is an elliptic fibration. Assume first that $g^*$ acts as identity on the base curve $\mathbb{P}^1$ of the elliptic fibration. Then $g$ becomes an automorphism of the elliptic curve $X/\mathbb{P}^1$ over the function field of $\mathbb{P}^1$. On the Jacobian of this elliptic curve $g$ induces an automorphism $g'$ of order $p$. Let 

$$j : J \to \mathbb{P}^1$$

be the jacobian elliptic fibration. Note that $J$ is a $K3$ surface (cf. [CD], Theorem 5.7.2). Since the order of $g'$ is $p > 3$, $g'$ must be a translation by a $p$-torsion section. By Corollary 5.9 from [DK1], $p \leq 11$.

Next assume that $g^*$ acts non-identically on the base $\mathbb{P}^1$ of the elliptic fibration. Let $s_0$ be the unique fixed point of $g^*$ on the base. The fibre $X_{s_0}$ contains $X^g$. The remaining singular fibres form orbits of fibres of the same type. By the same argument as in the proof of Corollary 5.6 [DK1], we see that $p \leq 11$. □

Now, Propositions 2.8, 2.9, 2.10 and 2.11 together with Theorem 2.2 prove our Theorem 2.1.

Remark 2.12. One should compare the result of Theorem 2.4 with the known result that an abelian surface $A$ over a field of characteristic $p > 0$ does not admit an automorphism $g$ of order $p > 5$ which fixes a point. This result can
be proved by considering the action of $g$ on the Tate module $H^1_{\text{et}}(A, \mathbb{Q}_l), l \neq p$ and applying the Weyl theorem. A similar proof for K3 surfaces only shows that $p > 23$ is impossible. We thank Yuri Zarkhin for this remark.

If $g$ preserves an elliptic or a quasi-elliptic pencil, then it either acts non-identically on the base or is realized by an automorphism of its jacobian fibration. In the latter case, if $p > 3$, then it is realized by a translation by a $p$-torsion section. In [DK1] (Corollary 5.9) we have proved that no non-trivial $p$-torsion section exists if $p > 11$. One can improve this bound to 7 by the same proof. In the case $p = 11$, if an 11-torsion section exists, then the proof (see information (i)-(iii) in [DK1], p. 126) gives only one possible combination of types of singular fibres: three singular fibres of type $I_{11}, I_{11}, II$. If this happens, the formula (5.1) from [DK1] gives a contradiction; in this case the left hand side of the formula cannot be an integer.

We state this result for future references.

**Theorem 2.13.** Let $f : X \to \mathbb{P}^1$ be an elliptic fibration with a section on a K3-surface over an algebraically closed field of characteristic $p > 7$. Then the group of $p$-torsion sections is trivial.

A different proof of the above result is given by A. Schweizer [Sc].

### 3. Tame symplectic automorphisms

In this section we consider symplectic automorphisms of $X$ of finite order prime to the characteristic $p$. We will show that they behave as in the complex case.

**Lemma 3.1.** Let $\Gamma$ be a finite subgroup of $SL(2, k)$ of order prime to $p$. Then $\Gamma$ is isomorphic to a finite subgroup of $SL(2, \mathbb{C})$, i.e. one of the following groups: a cyclic, a binary dihedral (=quaternion), binary tetrahedral, binary octahedral, or binary icosahedral group.

**Proof.** This is of course well-known. For completeness sake let us recall the usual proof (going back to Felix Klein). Let $\Gamma' \subset \text{PSL}(2, k)$ be the image of $\Gamma$ in PSL(2, k). Any non-trivial element $g \in \Gamma'$ has exactly 2 fixed points in the natural action of on $S = \mathbb{P}^1(k)$. Let $\mathfrak{P}$ be the union of the sets of fixed points $S_g, g \in \Gamma' \setminus \{1\}$. Let $O_1, \ldots, O_r$ be the orbits of $\Gamma'$ in $\mathfrak{P}$ and $n_1, \ldots, n_r$ be the orders of the corresponding stabilizers. An easy argument using the Burnside counting formula, gives the equation

$$\sum_{i=1}^{r} (1 - \frac{1}{n_i}) = 2 - \frac{2}{|\Gamma'|}.$$ 

This immediately implies that either $r = 2$ and $\Gamma'$ is a cyclic group, or $r = 3$, and

$$(n_1, n_2, n_3; |\Gamma'|) = (2, 2, n; 2n), (2, 3, 3; 12), (2, 3, 4; 24), \text{ or } (2, 3, 5; 60).$$
An easy exercise in group theory shows that the groups $\Gamma$ are isomorphic to a cyclic or a binary polyhedral group.

For a nondegenerate lattice $L$, we denote by $\text{disc}(L)$ the discriminant of $L$. We define

$$d_L := |\text{disc}(L)|,$$

the order of the discriminant group $L^*/L$.

Let $\phi : X \to C$ be an elliptic surface, with or without a section. For any reducible fibre $X_c$, let $S_c$ be the sublattice of the Picard lattice $S_X$ generated by all irreducible components of the fibre. The Gram matrix with respect to the basis formed by the irreducible components is described by a Dynkin diagram of affine type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_n$, where $n + 1$ is the number of irreducible components. The radical of $S_c$ is spanned by the (scheme-theoretical) fibre $X_c$ considered as a divisor on $X$. The quotient $\bar{S}_c$ by the radical is isomorphic to the corresponding negative definite root lattice of types $A_n, D_n, E_n$ (we say that the fibre is of the corresponding type). If $M_c \subset S_c$ is a negative definite sublattice of maximal rank, then the composition with the projection to $\bar{S}_c$ defines an embedding of lattices

$$M_c \hookrightarrow \bar{S}_c.$$

The orthogonal sum $\oplus_{c \in C} \bar{S}_c$ is the quotient of the sublattice $S^\text{vert}_X$ of $S_X$ generated by components of fibres by the rank 1 sublattice generated by the divisor class of any fibre. We denote the orthogonal sum by $R(\phi)$ and call it the root lattice of the elliptic surface $\phi$.

A negative definite sublattice $M = \oplus_{c \in C} M_c \subset S^\text{vert}_X$ is called maximal if each $M_c$ is a sublattice of $S_c$ of maximal possible rank. Its image in $R(\phi)$ is a sublattice of finite index, say $a$. In particular, we have

(3.1) $$d_M = a^2 d_{R(\phi)}.$$  

Lemma 3.2. Let $G$ be a finite group of symplectic automorphisms of a K3 surface of order prime to $p$. Let $Y \to X/G$ be a minimal resolution of the quotient $X/G$ and let $R_G$ be the sublattice of $\text{Pic}(Y)$ generated by the irreducible components of the exceptional divisor. Then the discriminant of $R_G$ is coprime to $p$. Moreover, if rank $R_G = 20$, then the discriminant is not a square.

Proof. The lattice $R_G$ is a direct sum of irreducible root lattices $R_i$ of type $A, D, E$ generated by irreducible components of a minimal resolution of quotient singularities corresponding to stabilizer subgroups $G_i$ of $G$. Via the action of $G_i$ on the tangent space of $X$ at one of its fixed points the group $G_i$ becomes isomorphic to a finite subgroup $H$ of $\text{SL}(2, k)$. Since $\#G_i$ is prime to $p$, the quotient singularity is formally isomorphic to the singularity $\mathbb{A}^2_k/H$. Now we apply Lemma 3.1 and use the well-known resolution of the quotient singularity $\mathbb{A}^2_k/H$. If $R_i$ is of type $A_k$, then $\#G_i = d_{R_i} = k + 1$. If $R_i$ is of type $D_n$, then $\#G_i = 4(n - 2), d_{R_i} = 4$. If $R_i$ is of type $E_6$, then $\#G_i = 24, d_{R_i} = 3$. If $R_i$ is of type $E_7$, then $\#G_i = 48, d_{R_i} = 2$. If $R_i$
is of type $E_8$, then $\#G_i = 120, d_{R_i} = 1$. In all cases we see that if $p | d_{R_i}$, then $p | \#G_i$, a contradiction to the assumption that $\#G$ is prime to $p$. This proves the first assertion.

Assume that rank $R_G = 20$. Obviously the Picard number $\rho$ of $Y$ satisfies $\rho \geq 21$. Thus $Y$ is a supersingular K3 surface. It is known ([A1]) that the discriminant group of the Picard lattice of a supersingular K3 surface is a $p$-elementary abelian group $(\mathbb{Z}/p)^{2\sigma}$, where $\sigma$ is the Artin invariant of the surface. Let $N$ be the orthogonal complement of $R_G$ in $S_Y$. We have

$$d_{R_G} \cdot d_N = i^2 p^{2\sigma},$$

where $i$ is the index of the sublattice $R_G \oplus N \subset S_Y$.

Assume that $d_{R_G}$ is a square. Then $d_N$ is a square. The lattice $N$ is an indefinite lattice of rank 2 whose discriminant is the negative of a square. It must contain a primitive isotropic vector. By Riemann-Roch, we can represent it by an effective divisor $A$ with self-intersection 0. Write $A$ as in ([22]). It is known that a suitable composition of reflections with respect to the divisor classes of $(-2)$-curves sends $A$ to $E$. Let $\psi : S_Y \to S_Y, \psi(A) = E$

be the composition. Let $R_i, i = 1, \ldots, 20$, be the irreducible components of the exceptional divisor. Since $\psi$ is an isometry, the images $\psi(R_i)$ generate a sublattice $M$ of $S_Y$ isomorphic to $R_G$. Since $A \cdot R_i = 0$ for all $i$, we have

$$E \cdot \psi(R_i) = 0, \ i = 1, \ldots, 20.$$ 

Since, by Riemann-Roch, each $\psi(R_i)$ is effective or anti-effective, this implies that

$$E \cdot C_{ij} = 0$$

for all irreducible components $C_{ij}$ of $\psi(R_i)$ or $-\psi(R_i)$, that is, $C_{ij}$’s are irreducible components of divisors from the pencil $|E|$ of genus 1 curves. Since $M$ is a negative definite lattice of rank 20, it is a maximal sublattice of $S_X$ with respect to the elliptic fibration $\phi$ given by the pencil. By ([31])

$$a^2 d_{R(\phi)} = d_M = d_{R_G}.\tag{3.2}$$

Let $f : J \to \mathbb{P}^1$

be the jacobian fibration of $\phi$. The surface $J$ is a K3 surface with the same type of singular fibres ([CD], Theorem 5.3.1). In particular,

$$\mathcal{R}(\phi) \cong \mathcal{R}(f).$$

The surface $J$ is a supersingular K3 surface, since $S^\text{vert}_J$ is of rank 21. The orthogonal complement of the root lattice $\mathcal{R}(f)$ in $S_J$ is generated by the divisor classes of the zero-section and a fibre, and hence unimodular. Thus we obtain

$$d_{S_J} = d_{R(f)}$$
for some number $m$ (equal to the order of the Mordell-Weil group of sections of $f$). Since $d_{S_J} = p^{2\sigma'}$, we obtain that $p|d_{R_{S_J}}$, and, by (3.2), $p|d_{R_G}$. This contradicts the first assertion. □

Theorem 3.3. Let $g$ be a symplectic automorphism of finite order $n$. If $(n, p) = 1$, then $g$ has only finitely many fixed points $f$ and the possible pairs $(n, f)$ are as follows.

$$(n, f) = (2, 8), (3, 6), (4, 4), (5, 4), (6, 2), (7, 3), (8, 2).$$

Proof. At a fixed point of $g$, $g$ is linearizable because $(n, p) = 1$. This implies that the quotient surface $Y = X/(g)$ has at worst cyclic quotient Gorenstein singularities and its minimal resolution is a $K3$ surface. So, Nikulin’s argument [Ni1] for the complex case works (when we replace the rational cohomology with the $l$-adic cohomology) except in the following two cases.

Case 1: $n = 11$ and $X/(g)$ has two $A_{10}$-singularities.

Case 2: $n = 15$ and $X/(g)$ has three singularities of type $A_{14}$, $A_{4}$ and $A_{2}$.

In any of these cases the discriminant of the lattice $R_G$ defined in the previous lemma is a square. So, these cases cannot happen. □

4. The main theorem

A Mathieu representation of a finite group $G$ is a 24-dimensional representation on a vector space $V$ over a field of characteristic zero with character

$$\chi(g) = \varepsilon(\text{ord}(g)),$$

where

$$\varepsilon(n) = 24(n \prod_{p|n} (1 + \frac{1}{p}))^{-1}.\quad (4.1)$$

The number

$$\mu(G) = \frac{1}{\# G} \sum_{g \in G} \varepsilon(\text{ord}(g))\quad (4.2)$$

is equal to the dimension of the subspace $V^G$ of $V$. The natural action of a finite group $G$ of symplectic automorphisms of a complex K3 surface on the singular cohomology

$$H^*(X, \mathbb{Q}) = \oplus_{i=0}^4 H^i(X, \mathbb{Q}) \cong \mathbb{Q}^{24}$$

is a Mathieu representation with

$$\mu(G) = \dim H^*(X, \mathbb{Q})^G \geq 5.$$

From this Mukai deduces that $G$ is isomorphic to a subgroup of $M_{23}$ with at least 5 orbits. In positive characteristic the formula for the number of
fixed points is no longer true and the representation of $G$ on the $l$-adic cohomology, $l \neq p$,

$$H_{et}^*(X, \mathbb{Q}_l) = \bigoplus_{i=0}^{4} H_{et}^i(X, \mathbb{Q}_l) \cong \mathbb{Q}_l^{24}$$

is not Mathieu in general. In this section, using Theorem 3.3, we will show that if $G$ is tame, i.e., the order of $G$ is coprime to $p$, the natural representation of $G$ on $H_{et}^*(X, \mathbb{Q}_l) \cong \mathbb{Q}_l^{24}$ is Mathieu. We will also show that $\dim H_{et}^*(X, \mathbb{Q}_l)^G \geq 5$ under the additional assumption that either $X$ or a minimal model of $X/G$ is not supersingular with Artin invariant $\sigma = 1$.

First let us recall that the analog of the lattice of transcendental cycles on a surface $X$ in characteristic $p > 0$ is the group $T_l\text{Br}(X)$ equal to the projective limit of groups $\text{Br}(X)[l^n]$, where $\text{Br}(X) = H_{et}^2(X, \mathbb{G}_m)$ is the cohomological Brauer group. Recall that the Kummer sequence in étale cohomology $[\mathbb{M}]$ gives the exact sequence

$$0 \to \text{Pic}(X)/l^n\text{Pic}(X) \to H_{et}^2(X, \mu_{l^n}) \to \text{Br}(X)[l^n] \to 0. \tag{4.3}$$

Passing to the projective limit we have the exact sequence of $\mathbb{Z}_l$-modules

$$0 \to \text{Pic}(X) \otimes \mathbb{Z}_l \to H_{et}^2(X, \mathbb{Z}_l) \to T_l\text{Br}(X) \to 0. \tag{4.4}$$

Tensoring with $\mathbb{Q}_l$, we get an exact sequence of $\mathbb{Q}_l$-vector spaces

$$0 \to \text{Pic}(X) \otimes \mathbb{Q}_l \to H_{et}^2(X, \mathbb{Q}_l) \to \text{V}_{l}\text{Br}(X) \to 0. \tag{4.5}$$

It gives the analog of the usual formula for the second Betti number of a surface

$$b_2(X) = \rho(X) + \lambda(X),$$

where $\rho(X)$ is the Picard number of $X$ and $\lambda(X) = \dim_{\mathbb{Q}_l} \text{V}_{l}\text{Br}(X)$ is the Lefschetz number of $X$. Since $b_2(X)$ and $\rho(X)$ do not depend on $l \neq p$, the Lefschetz number $\lambda(X)$ does not depend on $l$ either.

**Proposition 4.1.** Let $G$ be a finite group of symplectic automorphisms of a $K3$ surface $X$ defined in characteristic $p > 0$. Assume that $G$ is tame, i.e., the order of $G$ is coprime to $p$. Then for any prime $l \neq p$, the natural representation of $G$ on $H_{et}^*(X, \mathbb{Q}_l) \cong \mathbb{Q}_l^{24}$ is Mathieu.

**Proof.** By Theorem 3.3

$$\text{ord}(g) \in \{1, \ldots, 8\}$$

for all $g \in G$. By Lefschetz fixed point formula, the character $\chi(g)$ of the representation on the $l$-adic cohomology is equal to the number of fixed points of $g$, which is equal to $\epsilon(\text{ord}(g))$. This proves the assertion. This also shows that the action of $G$ on $H_{et}^2(X, \mathbb{Q}_l)$ is faithful. $\square$

**Lemma 4.2.** Let $G$ be a finite tame group of symplectic automorphisms of a $K3$-surface $X$. If a nonsingular minimal model $Y$ of $X/G$ is not supersingular, then

$$\dim H_{et}^*(X, \mathbb{Q}_l)^G \geq 5. \tag{4.6}$$
Proof. Since \( Y \) is not supersingular, \( \lambda(Y) > 0 \). If \( \lambda(Y) = 1 \), the Picard number of \( Y \) is equal to 21, and hence \( Y \) admits an elliptic fibration. By Artin \( \text{[Ar1]} \), the height \( h \) of the formal Brauer group of an elliptic non-supersingular surface is finite and
\[
\lambda(Y) = 22 - \rho(Y) \geq 2h \geq 2.
\]
Choose \( l \) coprime to the order of \( G \). It is known that \( \dim_{\mathbb{Q}}(\operatorname{Br}(X))^G = \lambda(Y) \) (\text{[Shio1]}, Proposition 5). Taking a \( G \)-invariant ample divisor class defining a nonzero element in \( (\operatorname{Pic}(X) \otimes \mathbb{Q})^G \), we see that \( \dim H^2_{et}(X, \mathbb{Q})^G \geq 3 \). Since the characteristic polynomial does not depend on \( l \neq p \), this is true for all \( l \neq p \). Together with \( H^2_{et}(X, \mathbb{Q})^G \) we get (4.6).

It remains to consider the case when \( X/G \) is birationally isomorphic to a supersingular K3-surface.

Lemma 4.3. Assume \( p \neq 2 \). Assume that a K3 surface \( X \) admits a symplectic automorphism \( g \) of order 2. Then \( X \) admits an elliptic fibration.

Proof. As is well-known it suffices to show that \( \rho(X) \geq 5 \), or, equivalently, \( \lambda(X) \leq 17 \). Let \( U \) be the open set where \( g \) acts freely. We know that \( X \setminus U \) consists of 8 fixed points of \( g \). Let \( G = \langle g \rangle, V = U/G \). We shall use the two spectral sequences employed in the proof of Lemma \( \text{[2.6]} \). It is easy to see that they give the following exact sequence
\[
\begin{align*}
0 & \to H^1(G, \operatorname{Pic}(U)) \to \operatorname{Br}(V) \to \operatorname{Br}(U)^G \to H^2(G, \operatorname{Pic}(U)).
\end{align*}
\]
Let \( t_+ \) (resp. \( t_- \)) be the rank of \( g \)-invariant (resp. \( g \)-antiinvariant) part of \( \operatorname{Pic}(U) \cong \operatorname{Pic}(X) \). We have
\[
\begin{align*}
H^1(G, \operatorname{Pic}(U)) & = H^1(G, \operatorname{Pic}(X)) = \operatorname{Ker}(1 + g^*)/\operatorname{Im}(1 - g^*) \cong (\mathbb{Z}/2\mathbb{Z})^{t_-}, \\
H^2(G, \operatorname{Pic}(U)) & = H^2(G, \operatorname{Pic}(X)) = \operatorname{Ker}(1 - g^*)/\operatorname{Im}(1 + g^*) \cong (\mathbb{Z}/2\mathbb{Z})^{t_+}.
\end{align*}
\]
Splitting (4.7) in two short exact sequences and passing to 2-torsion subgroups we get the following exact sequences of 2-elementary groups
\[
\begin{align*}
0 & \to (\mathbb{Z}/2\mathbb{Z})^{t_-} \to \operatorname{Br}(V)[2] \to (\mathbb{Z}/2\mathbb{Z})^{t_+}, \\
0 & \to A \to \operatorname{Br}(U)^G[2] \to (\mathbb{Z}/2\mathbb{Z})^t \to 0,
\end{align*}
\]
where \( t \leq t_+ \). This gives
\[
\dim_{\mathbb{F}_2} \operatorname{Br}(U)^G[2] \leq \dim_{\mathbb{F}_2} A + t \leq (\dim_{\mathbb{F}_2} \operatorname{Br}(V)[2] - t_-) + t_- + t
\]
(4.8)
\[
\leq \dim_{\mathbb{F}_2} \operatorname{Br}(V)[2] + t_+.
\]
Let \( Y \) be a minimal resolution of singularities of \( X/G \) and \( E \) be the exceptional divisor. According to \( \text{[DeMF]} \), the exact sequence of local cohomology for the pair \( (Y, E) \) and the sheaf \( G_m \) defines an exact sequence (modulo \( p \)-groups)
\[
0 \to \operatorname{Br}(Y) \to \operatorname{Br}(Y \setminus E) \to H^1(E, \mathbb{Q}/\mathbb{Z}).
\]
Since \( E \) is the disjoint union of 8 smooth rational curves, we obtain
\[
\operatorname{Br}(Y) \cong \operatorname{Br}(Y \setminus E) \cong \operatorname{Br}(V).
\]
Similarly, we obtain 
\[ \text{Br}(U) \cong \text{Br}(X). \]
It follows from (4.4) that, up to a finite group, 
\[ \text{Br}(Y)[2] = (\mathbb{Z}/2\mathbb{Z})^{\lambda(Y)}. \]
Applying (4.8), we obtain
\[
\text{dim}_{\mathbb{F}_2} \text{Br}(X)^G[2] \leq \lambda(Y) + t_+.
\]
If \( \rho(X) \geq 5 \) we are done. Otherwise \( \rho(X) \leq 4 \), hence \( t_+ \leq 4 \). Since \( Y \) contains 8 disjoint smooth rational curves and also the pre-image of a class of an ample divisor on \( X/G \), we have \( \rho(Y) \geq 9 \), and therefore \( \lambda(Y) \leq 22 - 9 = 13 \). Now (4.9) implies
\[
\text{dim}_{\mathbb{F}_2} \text{Br}(X)^G[2] \leq 17.
\]
The exact sequence of sheaves in étale topology
\[
0 \to \mu_{2^n} \to \mu_{2^{n+1}} \to \mu_2 \to 0
\]
gives, after passing to cohomology and taking the projective limits, the exact sequence
\[
H^2(X, \mathbb{Z}_2) \to H^2(X, \mathbb{Z}_2) \to H^2(X, \mu_2) \to 0.
\]
Since an automorphism of order \( \leq 2 \) of a free \( \mathbb{Z}_l \)-module acts trivially modulo 2, we obtain that
\[
H^2(X, \mu_2)^G = H^2(X, \mu_2).
\]
Applying the Kummer exact sequence (4.3), we see that
\[
\text{Br}(X)^G[2] = \text{Br}(X)[2].
\]
It remains to apply (4.10).

**Lemma 4.4.** Let \( G \) be a finite tame group of symplectic automorphisms of a K3-surface \( X \) of order \( \neq 7, 21 \). Assume that a minimal resolution \( Y \) of \( X/G \) is a supersingular K3 surface. Then \( X \) is supersingular.

**Proof.** Recall from [Ar1] that the formal Brauer group \( \hat{\text{Br}}(S) \) of a supersingular K3-surface \( S \) is isomorphic to the formal additive group \( \hat{\mathbb{G}}_a \). It is conjectured that the converse is true, and it has been verified if \( S \) is an elliptic surface (loc.cit. Theorem (1.7)). Since Brauer group is a birational invariant, the projection \( \pi : X \to X/G \) defines a natural homomorphism of the formal Brauer groups \( \pi^* : \hat{\text{Br}}(Y) \to \hat{\text{Br}}(X) \). The corresponding map of the tangent spaces is \( \pi^* : H^2(Y, \mathcal{O}_Y) \to H^2(X, \mathcal{O}_X) \). Since the order of \( G \) is prime to the characteristic, the trace map shows that this homomorphism is nonzero. Since there are no non-trivial maps between a formal group of finite height and \( \hat{\mathbb{G}}_a \) we obtain that \( \hat{\text{Br}}(X) \cong \hat{\mathbb{G}}_a \). If \( G \) contains an element of order 2, we are done by Lemma 4.3. Assume that \( G \) has no elements of order 2. By Proposition 4.1, the representation of \( G \) in \( H^*(X, \mathbb{Q}_l) \) is a
Mathieu representation. Following Mukai’s arguments from [Mu], we obtain that the order of $G$ must divide $3^2 \cdot 5 \cdot 7$.

Suppose that 3 distinct prime numbers divide $\#G$. It is known that no simple non-abelian group of order dividing $3^2 \cdot 5 \cdot 7$ exists. Thus $G$ is solvable and hence contains a subgroup of order 35 ([Hr], Theorem 9.3.1). It follows easily from Sylow’s Theorem that such a group is cyclic and hence is not realized as a group of symplectic automorphisms of $X$.

Suppose $\#G = 3^a \cdot 5$ or $3^a \cdot 7$ with $a \neq 0$. Again, by Sylow’s theorem we obtain that a Sylow 5-subgroup (or 7-subgroup) is normal. Since no element of order 3 can commute with an element of order 5 or 7, we obtain that $G$ is a non-abelian group of order 21. This case is excluded by the assumption.

The remaining possible cases are $\#G = 3, 5, 9$. This gives that $X/G$ has either 6 singular points of type $A_2$, or 4 singular points of type $A_4$, or 8 singular points of type $A_2$. Since $\rho(Y) = 22$, this immediately implies that $\dim \text{Pic}(X/G) \otimes \mathbb{Q} = 22 - 12 = 10$ or $22 - 16 = 6$. Hence $\text{rankPic}(X)^G \geq 6$. Thus $X$ is an elliptic surface, and, by Artin’s result cited above, we obtain that $X$ is supersingular.

**Proposition 4.5.** Let $G$ be a finite tame group of symplectic automorphisms of a $K3$ surface $X$. Assume that either $X$ or a minimal model of $X/G$ is not a supersingular $K3$ surface with Artin invariant $\sigma = 1$. Then

$$\dim H^*_\text{et}(X, \mathbb{Q}_l)^G \geq 5.$$ 

**Proof.** By Lemma 4.2 we may assume that a minimal nonsingular model $Y$ of $X/G$ is supersingular. A symplectic group of order 7 or 21 is uniquely determined and is in Mukai’s list [Xiao] and satisfies the assertion of the proposition. By Lemma 4.4 we obtain that $X$ is supersingular.

Assume that the Artin invariant $\sigma$ of $X$ is greater than 1.

Let us consider the representation of $G$ on the crystalline cohomology $H^*_\text{cris}(X/W)$. We refer to [Il] for the main properties of crystalline cohomology and to [RS], [Og] for particular properties of crystalline cohomology of $K3$ surfaces. The cohomology $H^*_\text{cris}(X/W)$, where $X$ is a $K3$-surface, is a free module of rank 24 over the ring of Witt vectors $W = W(k)$. The vector space $H^*_\text{cris}(X/W)_K$, where $K$ is the field of fractions of $W$, is of dimension 24. The ring $W$ is a complete noetherian local ring of characteristic 0 with maximal ideal $(p)$ and the residue field isomorphic to $k$. The quotient module $H^*_\text{cris}(X/W)/pH^*_\text{cris}(X/W)$ is a $k$-vector space of dimension 24 isomorphic to the algebraic De Rham cohomology $H^1_{\text{DR}}(X)$. Let

$$H = H^2_{\text{DR}}(X).$$

It is known that the Hodge spectral sequence

$$E^0_{1,q} = H^q(X, \Omega^p_X) \Rightarrow H^0_{\text{DR}}(X)$$


degenerates and we have the following canonical exact sequences:

\[(4.11) \quad 0 \to F^1 H \to H \to H^2(X, \mathcal{O}_X) \to 0,\]

\[(4.12) \quad 0 \to H^0(X, \Omega^2_X) \to F^1 H \to H^1(X, \Omega^1_X) \to 0.\]

Here

\[F^i H = \sum_{p \geq i} H^p(X, \Omega^p_X) \cap H\]

is the Hodge filtration of the De Rham cohomology. Obviously, the subspace \(F^1 H\) is \(G\)-invariant. Since the order of \(G\) is prime to the characteristic, the representation of \(G\) is semi-simple and hence the \(G\)-module \(H\) is isomorphic to the direct sum of \(G\)-modules

\[(4.13) \quad H \cong H^0(X, \Omega^2_X) \oplus H^2(X, \mathcal{O}_X) \oplus H^1(X, \Omega^1_X).\]

By definition,

\[H^0(X, \Omega^2_X) \subset H^G.\]

By Serre’s duality

\[H^2(X, \mathcal{O}_X) \subset H^G.\]

This shows that \(\dim H^G \geq 2.\)

Let

\[V = H^2_{\text{crys}}(X/W).\]

The multiplication by \(p\) map \([p]\) defines the exact sequence

\[0 \to V \xrightarrow{[p]} V \to H \to 0.\]

Taking \(G\)-invariants we obtain the exact sequence

\[(4.14) \quad 0 \to V^G \xrightarrow{[p]} V^G \to H^G \to H^1(G, V).\]

Since the ring \(W\) has characteristic 0, the multiplication by \(|G|\) defines an injective map \(V \to V.\) On the other hand, it induces the zero map on the cohomology \(H^1(G, V).\) This implies that

\[H^1(G, V) = 0.\]

This shows that \(V^G\) is a free submodule of \(V\) of rank equal to \(\dim H^G.\)

It is known that the Chern class map

\[c_1 : S_X \to H^2_{\text{crys}}(X/W)\]

is injective and its composition with the reduction mod \(p\) map

\[H^2_{\text{crys}}(X/W) \to H^2_{\text{DR}}(X)\]

defines an injective map

\[(4.15) \quad c : S_X/pS_X \to H^2_{\text{DR}}(X)\]

with image contained in \(F^1 H^2_{\text{DR}}(X)\) (see \([\text{Og}]\)).

If \(X\) is supersingular with Artin invariant \(\sigma > 1\), the composition of \(c\) with the projection \(F^1 H^2_{\text{DR}}(X) \to H^1(X, \Omega^1_X)\) is injective. This result is implicitly contained in \([\text{Og}]\) (use Remark 2.7 together with the fact that
a supersingular surface with $\sigma > 1$ admits a non-trivial deformation to a supersingular surface with Artin invariant equal to 1. Let $L$ be a $G$-invariant ample line bundle on $X$. We may assume that its isomorphism class defines a non-zero element in $S_\mathcal{X}/pS_\mathcal{X}$. Thus its image in $H^1(\mathcal{X}, \Omega^1_\mathcal{X})$ is a nonzero $G$-invariant element and we get 3 linearly independent elements in $H^G$, each from one of the three direct summands of $H$. Applying (4.14), we find 3 linearly independent elements in $H^2_{\text{crys}}(X/W)^G$.

Since $H^0_{\text{crys}}(X/W), H^4_{\text{crys}}(X/W)$ are trivial $G$-modules, we obtain

\[(4.16) \dim H^*_{\text{crys}}(X/W)^G_K \geq 5.\]

It remains to use the fact that the characteristic polynomials of $g \in G$ on $H^*_{\text{crys}}(X/W)_K$ and on $H^*_{\text{et}}(X, \mathbb{Q}_l)$, $l \neq p$, have integer coefficients and coincide with each other ([11], 3.7.3).

Thus if the assertion is not true, $X$ must be supersingular of Artin invariant $\sigma = 1$.

Assume that $Y$ is supersingular with Artin invariant $\sigma > 1$. Let $X'$ be the open subset of $X$ where $G$ acts freely and let $Y' = X'/G$. The standard Hochshild-Serre spectral sequence implies that the pull-back under the projection $f : X' \to Y'$ defines an isomorphism

$$\text{Pic}(Y')/\text{Hom}(G, k^*) \cong \text{Pic}(X')^G \cong \text{Pic}(X)^G.$$  

Let $R$ be the sublattice of $S_Y$ spanned by the irreducible components of exceptional curves of the resolution $\pi : Y \to X/G$. It is isomorphic to the orthogonal sum of root lattices of discriminants prime to $p$. The restriction map $\text{Pic}(Y) \to \text{Pic}(Y')$ is surjective and its kernel is $R$. The torsion group of $\text{Pic}(Y')$ is isomorphic to $\text{Hom}(G, k^*)$. Let $R'$ be the saturation of $R$ in $\text{Pic}(Y')$. We have

\[(4.17) \quad N := \text{Pic}(Y)/R' \cong \text{Pic}(Y')/\text{Hom}(G, k^*) \cong \text{Pic}(X)^G.\]

Since $\#G$ is coprime to $p$, the discriminant of the sublattice $R'$ is coprime to $p$. The discriminant group $D_Y$ of $\text{Pic}(Y)$ is an elementary $p$-group of rank $2\sigma \geq 4$ and is a subquotient of the discriminant group of $R' \oplus R'$. This implies that $\text{rank} N = \text{rank} R^\perp > 2$, it follows from (4.17) that $\text{rank} \text{Pic}(X)^G > 2$, and by the above arguments we will find 5 linearly independent elements in $H^*_{\text{crys}}(X/W)^G$. \[\square\]

**Remark** 4.6. If $p$ divides $|G|$, the exact sequences (4.11) and (4.12) may not split as $G$-modules. In fact, there are examples where

$$\dim H^*_{\text{crys}}(X/W)^G_K = 4,$$

so that (4.16) does not hold.

**Theorem 4.7.** Let $G$ be a finite group of symplectic automorphisms of a $K3$ surface $X$. Assume that $G$ is tame and that either $X$ or a minimal model of $X/G$ is not a supersingular $K3$ surface with Artin invariant $\sigma = 1$. Then $G$ is a subgroup of the Mathieu group $M_{23}$ which has $\geq 5$ orbits in its natural
permutation action on the set of 24 elements. All such groups are subgroups of the 11 groups listed in $\text{M}_1$.

Proof. Let us consider the linear representation $\rho$ of $G$ on $H^*_c(X, \mathbb{Q}_l)$, $l \neq p$. Applying Proposition 4.3 and 4.5 we find that $\rho$ is a Mathieu representation over the field $\mathbb{Q}_l$ with $\dim H^*_c(X, \mathbb{Q}_l)^G \geq 5$. Replacing $\mathbb{Q}$ with $\mathbb{Q}_l$, we repeat the arguments of Mukai. He uses at several places the fact that the representation is over $\mathbb{Q}$. The only essential place where he uses that the representation is over $\mathbb{Q}$ is Proposition (3.21), where $G$ is assumed to be a 2-group containing a maximal normal abelian subgroup $A$ and the case of $A = (\mathbb{Z}/4)^2$ with $\#(G/A) \geq 2^4$ is excluded by using that a certain 2-dimensional representation of the quaternion group $Q_8$ cannot be defined over $\mathbb{Q}$. We use that $G$ also admits a Mathieu representation on 2-adic cohomology, and it is easy to see that the representation of $Q_8$ cannot be defined over $\mathbb{Q}_2$. To show that the 2-Sylow subgroup of $G$ can be embedded in $M_{23}$, he uses the fact that the stabilizer of any point on $X$ is isomorphic to a finite subgroup of $\text{SL}(2, \mathbb{C})$, and the classification of such subgroups allows him to exclude some groups of order $2^n$. By Lemma 3.1 we have the same classification, so we can do the same. \hfill $\square$

Applying Theorem 2.4, we obtain the following.

**Corollary 4.8.** Assume that $p > 11$ and either $X$ or a minimal model of $X/G$ is not a supersingular K3 surface with Artin invariant $\sigma = 1$. Then $G$ is a subgroup of $M_{23}$ with $\geq 5$ orbits and hence belongs to Mukai’s list.

5. The exceptional case

Here we investigate the case when the order of $G$ is prime to $p$ and

\begin{equation}
\dim H^*_{crys}(X/W)^G_K = 4.
\end{equation}

By theorem 2.4, this may happen only if both $X$ and a minimal nonsingular model $Y$ of $X/G$ is a supersingular K3 surface with Artin invariant $\sigma = 1$. We refer to this as the exceptional case and the group $G$ will be called an exceptional group.

It is known that a supersingular surface with Artin invariant $\sigma = 1$ is unique up to isomorphism. More precisely, we have the following (see $\text{Og}$, Corollary 7.14).

**Proposition 5.1.** Let $X$ be a supersingular surface with Artin invariant $\sigma = 1$. Assume that $p \neq 2$. Then $X$ is birationally isomorphic to the Kummer surface of the abelian surface $E \times E$, where $E$ is a supersingular elliptic curve.

If $p = 2$, the surface is explicitly described in [DKo]. Note that the Kummer surface does not depend on $E$. If $p \equiv 3 \mod 4$ (resp. $p \equiv 2 \mod 3$) we can take for $E$ an elliptic curve with Weierstrass equation $y^2 = x^3 - x$ (resp. $y^2 = x^3 + 1$).
It follows from the proof of Proposition 4.5 and Lemma 3.2 that an exceptional group satisfies the following properties:

(EG1) $G$ admits a Mathieu representation $V_l$ over any $\mathbb{Q}_l, l \neq p$;

(EG2) $\mu(G) = \dim V_l^G = 4$;

(EG3) the root lattice $\mathcal{R}_G$ spanned by irreducible components of the exceptional locus of the resolution $Y \to X/G$ is of rank 20 (this is equivalent to (EG2));

(EG4) $d_{\mathcal{R}_G}$ is coprime to $p$ and is not a square.

We use the following notations of groups from [Mu] and the ATLAS [CN]:

- $C_n$ the cyclic group of order $n$, sometimes denoted by $n$,
- $D_{2n}$ the dihedral group of order $2n$,
- $Q_{4n}$ the binary dihedral group of order $4n$,
- $T_{24}$ the binary tetrahedral group,
- $O_{48}$ the binary octahedral group,
- $S_n$ the symmetric group of degree $n$,
- $A_n$ the alternating group of degree $n$,
- $S_{n_1}, \ldots, n_k$ a subgroup of $\mathfrak{S}_{n_1}, \ldots, n_k$ which preserves the decomposition of a set of $n_1 + \ldots + n_k$ elements as a disjoint union of subsets of cardinalities $n_1, \ldots, n_k$.
- $\mathfrak{A}_{n_1, \ldots, n_k} = \mathfrak{S}_{n_1, \ldots, n_k} \cap \mathfrak{S}_{n_1, \ldots, n_k}$;
- $M_k$ the Mathieu group of degree $k$.
- $L_n(q) = \text{PSL}_n(F_q)$.
- $A \cdot B$ a group which has a normal subgroup isomorphic to $A$ with quotient isomorphic to $B$;
- $A \cdot B$ as above but the extension does not split;
- $A : B$ a semidirect product with normal subgroup $A$;
- $A \circ B$ the central product of two groups.

We will also use the notations for 2-groups from [HS] and [Mu].

The goal of this section is to prove the following.

**Theorem 5.2.** An exceptional group $G$ is isomorphic to one of the following groups (the corresponding root lattice $\mathcal{R}_G$ is given in the parenthesis):

(I) **Non-solvable groups:**

(i) $\mathfrak{S}_6$ of order $2^4.3^2.5 \ (A_4 + 2A_3 + 2A_5)$;

(ii) $M_{10}$ of order $2^4.3^2.5 \ (A_4 + A_3 + A_2 + A_7 + D_4)$;

(iii) $2^4 : \mathfrak{A}_6$ of order $2^7.3^2.5 \ (E_7 + A_2 + A_3 + 2A_4)$;

(iv) $M_{21}$ of order $2^6.3^2.5.7 \ (A_2 + 2A_4 + A_6 + D_4)$;

(v) $\mathfrak{A}_7$ of order $2^3.3^2.5.7 \ (A_2 + A_3 + A_4 + A_6 + D_5)$;

(vi) $M'_{20} = 2^4 : \mathfrak{A}_5(\not\cong M_{20})$ of order $2^6.3.5 \ (A_1 + 2A_4 + D_5 + E_6)$;

(vii) $M_{20} : 2 \cong 2^4 : \mathfrak{S}_5$ of order $2^7.3.5 \ (A_2 + A_3 + A_4 + A_5 + D_6)$;

(viii) $2^3 : L_2(7)$ of order $2^6.3.7 \ (A_2 + 2A_3 + A_6 + E_6)$.

(II) **Solvable groups:**
(ix) $3^2 : C_8$ of order $2^3.3^2 (2A_7 + A_3 + A_2 + A_1)$;
(x) $3^2 : SD_{16}$ of order $2^4.3^2 (A_7 + A_5 + D_4 + A_3 + A_1)$;
(xi) $2^3 : 7$ of order $2^3.7 (3A_6 + 2A_1)$;
(xii) $2^4 : (5 : 4)$ of order $2^4.5 (A_7 + A_4 + 3A_3)$;
(xiii) $2^4 (A_4 \times A_4) = \Gamma_{13}a_1 : 3^2$ of order $2^6.3^2 (E_6 + 2A_2 + 2A_5)$;
(xiv) $2^2 A_{4.4} = \Gamma_{13}a_1 : A_{3.3} = 2^4 : A_{3.4}$ of order $2^7.3^2 (E_7 + 2D_5 + A_2 + A_1)$;
(xv) $2^4 : (3 \times D_6)$ of order $2^5.3^2 (3A_5 + A_3 + A_2)$;
(xvi) $2^4 : (3^2 : 4)$ of order $2^6.3^2 (A_7 + D_5 + 2A_3 + A_2)$;
(xvii) $2^4 : G_3 = S_{4.4} = 2^4 : A_{3.3}$ of order $2^6.3^2 (D_5 + D_4 + 2A_5 + A_1)$;
(xviii) $O_{48}$ of order $2^4.3 (E_7 + D_6 + D_5 + A_2$ or $2E_7 + D_4 + A_2)$;
(xix) $T_{24} \times 2$ of order $2^4.3 (E_6 + D_4 + 2A_5)$;
(xx) $O_{48} : 2$ of order $2^5.3 (E_7 + D_6 + A_5 + 2A_1)$;
(xxii) $Q_8 \circ Q_8 \circ \Gamma_3 = \Gamma_{13}a_1 \circ \Gamma_3$ of order $2^5.3 (A_1 + A_2 + A_3 + 2E_7)$;
(xxxx) $Q_8 \circ Q_8 \circ \Gamma_4 = 2^4 : A_{4.4} = 2^4 : A_4$ of order $2^6.3 (A_1 + A_2 + A_7 + 2D_5)$;
(xxxxi) $\Gamma_{13}a_1 : 3 = 2^4 : A_4$ of order $2^6.3 (A_1 + 3A_5 + D_4$ or $2A_1 + 3E_6)$;
(xxxii) $\Gamma_{13}a_1 : \Gamma_3 = 2^4 : S_4$ of order $2^7.3 (A_1 + A_3 + A_5 + D_5 + D_6)$;
(xxxiii) $\Gamma_3a_1 : \Gamma_4 = 2^4 : S_4$ of order $2^7.3 (A_1 + 2A_3 + E_6 + E_7)$;
(xxxiv) $\Gamma_4a_1 : \Gamma_3 = 3 = 2^4 : A_4$ of order $2^6.3 (A_1 + A_3 + 2A_5 + E_6)$;
(xxxv) $\Gamma_5a_1 : \Gamma_3 = 2^4 : S_4 = \Gamma_5a_1 \circ D_{12}$ of order $2^7.3 (A_1 + A_3 + A_5 + D_4 + E_7)$;
(xxxvi) $2^3 : 7 : 3$ of order $2^3.3.7 (A_6 + 2A_5 + 2A_2)$.

Remark 5.3. (1) All these groups are subgroups of the Mathieu group $M_{23}$ with number of orbits equal to 4 (see Proposition 5.14). Note that not all subgroups of $M_{23}$ with 4 orbits belong to our list. For example, a group containing elements of order $> 8$ must be wild by Theorem 3.3 hence is not contained in our list. Examples of such groups are $L_2(11), 2 \times L_2(7), A_{3.5}$.

We thank D. Allcock and S. Kondô for confirming this. Also, as we will see in the proof of Lemma 5.10, Case 3, there is a degree 2 extension of $M_{20}$ isomorphic to $2^4 : S_5$ which cannot act symplectically. According to Allcock, this group can be realized as a subgroup of $M_{23}$ preserving the partition $(5, 2, 1, 16)$ with $(5, 2, 1)$ forming an octad of the Steiner system.

(2) Most exceptional groups seem to realize in infinitely many different characteristics. A full account on the realization problem will be given in other publication.

(3) Among the 27 groups the following 10 groups are maximal: (i)-(v), (vii), (viii), (x), (xiv), (xvii).

We will prove the theorem by analyzing and extending Mukai’s arguments from [15]. We can use the arguments only based on property (EG1) of $G$ and do not use the assumption that $\mu(G) \geq 5$.

Recall that

$$\mu(G) = \frac{1}{\# G} \sum_{g \in G} \epsilon(\text{ord}(g)),$$
where \( \epsilon(n) \) is given in (4.1). Let \( g \) be an element of order \( n \) prime to \( p \) acting symplectically on a K3 surface. Using Theorem 3.3 and computation of \( \#X^g \) from [Ni1], one checks that
\[
\epsilon(n) = \#X^g.
\]
The possible values of \( \epsilon(n) \) are given in Theorem 3.3.

**Lemma 5.4.**
\[
\sum_{Gx \in X/G} \frac{1}{\#G_x} = \frac{24}{\#G} + k - \mu(G),
\]
where \( k \) is the number of singularities on \( X/G \).

**Proof.** Let
\[
S = \{(x, g) \in X \times G \setminus \{1\} : g(x) = x\}.
\]
By projecting to \( X \) we get
\[
\#S = \#G \sum_{Gx \in \text{Sing}(X/G)} (1 - \frac{1}{\#G_x}).
\]
By projecting to \( G \), and using (5.2), we get
\[
\#S = \sum_{g \in G \setminus \{1\}} \#X^g = \sum_{g \in G \setminus \{1\}} \epsilon(\text{ord}(g)) = \#G\mu(G) - 24.
\]
This gives
\[
\sum_{Gx \in \text{Sing}(X/G)} \frac{1}{\#G_x} = \frac{24}{\#G} + k - \mu(G).
\]
\[
\square
\]

**Remark 5.5.** We also have the following formula from [Xiao]
\[
\sum_{Gx \in X/G} \frac{1}{\#G_x} = \frac{24}{\#G} + k + \text{rank}R_G - 24.
\]
Comparing this with the previous formula, we get
\[
\mu(G) = 24 - \text{rank}R_G.
\]

The classification of finite subgroups of \( \text{SL}(2, k) \) which admit a Mathieu representation is given in [Mu], Proposition (3.12). The groups \( Q_{4n}, n \geq 5 \), and the binary icosahedral group \( I_{120} \) are not realized since they contain an element of order \( > 8 \). The following table gives the information about possible stabilizer groups \( G_x \), their orders \( o_x \), the number \( c_x \) of irreducible components in a minimal resolution, types of singular points and the structure of the discriminant group \( D_x \) of the corresponding root lattice.

Note that \( \mu(O_{48}) = 4 \) so this case may occur only in the exceptional case.

For an exceptional group \( G \), using the above table and the formula from Lemma 5.4 it is easy to show that
\[
\text{(EG5) the number } k \text{ of singularities on } X/G \text{ is } 4 \text{ or } 5.
\]
Let $G$ be an exceptional group acting on $X$. It defines a set of $k$ numbers $o_1, \ldots, o_k$ and $c_1, \ldots, c_k$ from Table 1. We have

(i) $c_1 + \ldots + c_k = 20$;

(ii) $\frac{1}{o_1} + \ldots + \frac{1}{o_k} = k - 4 + \frac{24}{N}$, where $N$ is a positive integer (equal to $\#G$);

(iii) $o_i | N$ for all $i = 1, \ldots, k$;

(iv) $d_1 \ldots d_k$ is not a square, where $d_i$ is the order of the discriminant group $D_i$;

(v) $k = 4$ or $5$.

We are grateful to Daniel Allcock who run for us a computer program which enumerates all collections of numbers $o_1, \ldots, o_k$ and the corresponding numbers $c_i, d_i, N$ satisfying properties (i)-(v). This gives us all possible orders $N$ of possible exceptional groups $G$ as well as all possible root lattices describing a minimal resolution of singularities of $X/G$. We refer to this as the List. It is reproduced in Table 2.

**Lemma 5.6.** Let $l \neq p$ be a prime number. Assume that the minimal number of generators of the $l$-torsion part of the discriminant group of the lattice $\mathcal{R}_G$ is greater than 2. Then $l$-part of the abelianized group $G/[G,G]$ is non-trivial.

**Proof.** Let $M$ be the saturation of the lattice $\mathcal{R}_G$ in $\text{Pic}(Y)$. It follows from the proof of Proposition 4.5 that $G/[G,G] \cong M/\mathcal{R}_G$. The lattice $M \oplus M^\perp$ is a sublattice of finite index of the lattice $S_Y$ with discriminant group $(\mathbb{Z}/p\mathbb{Z})^2$. Tensoring with the localized ring $\mathbb{Z}(l)$ at the prime ideal $(l) \subset \mathbb{Z}$, we obtain $M_l \oplus (M^\perp)_l \subset (S_Y)_l$.

Since $(S_Y)_l$ is unimodular over $\mathbb{Z}(l)$, the discriminant groups of $M_l$ and $(M^\perp)_l$ are isomorphic to each other. Since rank$_{\mathbb{Z}(l)} (M^\perp)_l = 2$, we obtain that the discriminant group of $M_l$ is generated by $\leq 2$ elements. If $l$ does not divide $\#G/[G,G]$, then $M_l \cong (\mathcal{R}_G)_l$. The discriminant group of $(\mathcal{R}_G)_l$ is the $l$-torsion part of the discriminant group of $\mathcal{R}_G$, and the assertion follows.

**Lemma 5.7.** If $G$ is of order divisible by 35, then $G$ is isomorphic to the simple group $M_{21} \cong L_3(4)$ of order $2^6.3^2.5.7$ or the alternating group $A_7$ of order $2^3.3^2.5.7$. 

| $G_x$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ | $C_8$ | $Q_8$ | $Q_{12}$ | $Q_{16}$ | $T_{24}$ | $O_{48}$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $o_x$ | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 8     | 12    | 16    | 24    | 48    |
| $c_x$ | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 4     | 5     | 6     | 6     | 7     |
| Type  | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $D_4$ | $D_5$ | $D_6$ | $E_6$ | $E_7$ |
| $D_x$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ | $C_8$ | $C_2^2$ | $C_4$ | $C_2^2$ | $C_3$ | $C_2$ |

Table 1.
Let \([G,G]\) be the commutator subgroup of \(G\).

**Case 1.** \(G = [G,G]\).

By Lemma 5.40 for any prime \(l \neq p\) the \(l\)-part of the discriminant group of \(R_G\) is generated by \(\leq 2\) elements.

The List shows that there are 4 possible orders \(N\) divisible by 35. We will eliminate the orders \(N = 2^4 \cdot 5 \cdot 7\) and \(3^2 \cdot 5 \cdot 7\). In the first case, considering a 7-Sylow subgroup, we see that no elements of order 5 or 2 can normalize it, because \(G\) does not contain elements of order 35, or 14 and does not contain \(D_{14}\), which does not admit a Mathieu representation. By Sylow's theorem, \(#Syl_7(G) = 2^4 \cdot 5 \equiv 1 \mod 7\), a contradiction. In the second case, no elements of order 7 or 3 can normalize a 5-Sylow subgroup, and hence \(#Syl_5(G) = 3^2 \cdot 7 \equiv 1 \mod 5\), again a contradiction.

**Table 2.**

| Order          | Root lattices                               |
|----------------|---------------------------------------------|
| \(2^4 \cdot 3^2 \cdot 5 \cdot 7\) | \(A_2 A_4 A_4 A_6 D_4\)                     |
| \(2^4 \cdot 3^2 \cdot 7\)     | \(A_2 A_3 A_4 A_6 D_5\)                     |
| \(2^4 \cdot 7\)              | \(A_2 A_3 A_4 A_6 D_6\)                     |
| \(2^4 \cdot 3^2\)            | \(A_2 A_3 A_4 A_6 D_7\)                     |
| \(2^4 \cdot 3^2\)            | \(A_2 A_3 A_4 A_6 E_7\)                     |
| \(2^4 \cdot 3^2\)            | \(A_2 A_3 A_4 A_6 F_7\)                     |
| \(2^4 \cdot 3^2\)            | \(A_2 A_3 A_4 A_6 G_7\)                     |
| \(2^4 \cdot 3^2\)            | \(A_2 A_3 A_4 A_6 H_7\)                     |
| \(2^4 \cdot 3^2\)            | \(A_2 A_3 A_4 A_6 I_7\)                     |
| \(2^4 \cdot 3^2\)            | \(A_2 A_3 A_4 A_6 J_7\)                     |

Proof. Let \([G,G]\) be the commutator subgroup of \(G\).
Assume \( \#G = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \). This is the order of \( \mathfrak{A}_7 \). Let us show that \( G \) is simple. Assume \( G \) is not simple and let \( H \) be a normal subgroup such that \( G/H \) is simple (non-abelian because \( G = [G, G] \)). It follows from \cite{Mu}, Proposition (3.3) that \( \mu(G/H) > \mu(G) = 4 \). The group \( G/H \) acts symplectically on a minimal resolution of \( X/H \) and belongs to Mukai’s list. It follows from Proposition (4.4) of loc.cit. that \( G/H \cong \mathfrak{A}_5, \mathfrak{A}_6, \) or \( L_2(7) \).

In the first case \( \#H = 42 \). There is no such group in Mukai’s list as well as in our List. In the second case \( \#H = 7 \). It is known that \( \mathfrak{A}_6 \) does not admit such a nontrivial extension (necessarily central). In the last case \( \#H = 15 \) and again we use that \( G \) does not contain an element of order 15. Thus \( G \) is simple. It is known that there is only one simple group of order equal to the order of \( \mathfrak{A}_7 \). The List gives only one possible root lattice \( R_G = A_2 + A_3 + A_4 + A_6 + D_5 \).

Assume \( \#G = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \). This is the order of the Mathieu group \( M_{21} \). As in the previous case we show that \( G \) is simple by analyzing the kernel \( H \) of a homomorphism onto a simple quotient of \( G \). As before, \( G/H \cong \mathfrak{A}_5, \mathfrak{A}_6, \) or \( L_2(7) \). In the first case \( \#H = 2^3 \cdot 3^2 \cdot 7 \). One checks that there are no such groups in Mukai’s list (Theorem (5.5) and Proposition (4.4)) and in our List. In the second case, \( \#H = 2^3 \cdot 7 \) and a group of this order with \( \mu(G) = 4 \) is possible. We will show later that \( H \) must be isomorphic to \( C_2^3 : C_7 \). The Sylow subgroup \( C_2^3 \) of \( H \) is a characteristic subgroup, hence \( G \) acts on it via conjugation. This defines a nontrivial homomorphism \( f : G \to \text{Aut}(C_2^3) \cong L_2(7) \). Let us treat this case which also covers the third case for a possible quotient of \( G \). Since \( G = [G, G] \), the image of \( f \) is not a solvable subgroup of \( L_2(7) \). The known classification of subgroups of \( L_2(7) \) shows that it is equal to \( L_2(7) \). Thus the kernel of \( f \) is a subgroup of order \( 2^3 \cdot 3^2 \cdot 7 \).

This order is in the List, but as we will see in the next lemma it cannot be realized as the order of an exceptional group. There is only one group with this order in Mukai’s list, which is \( \mathfrak{S}_5 \). This group contains \( \mathfrak{A}_5 \) as a unique subgroup of index 2. The group \( G \) acts on it by conjugation. Since the group \( \text{Out}(\mathfrak{A}_5) \) of outer automorphisms (modulo inner automorphisms) of \( \mathfrak{A}_5 \) is abelian, we get a nontrivial homomorphism \( G \to \mathfrak{A}_5 = \text{Inn}(\mathfrak{A}_5) \). As before we infer that it is surjective. But we have seen in above that this is impossible. This proves that \( G \) is simple. It is known that there are two simple groups of order \( 2^6 \cdot 3^2 \cdot 5 \cdot 7 \), one is \( M_{21} \) and another is \( \mathfrak{A}_8 \). The latter group contains an element of order 15 and must be excluded. The List gives only one possibility \( R_G = A_2 + 2A_4 + A_6 + D_4 \).

Case 2. \( G \neq [G, G] \).

Since 5 and 7 divide \( \#G \), \( X/G \) has singularities of type \( A_4 \) and \( A_6 \). This follows from Table 1. Assume \( G \) has a normal subgroup \( H \) with an abelian quotient of order 5 (resp. 7). Then \( X/H \to X/G \) is a cyclic cover of degree 5 (resp. 7). The pre-image of a singularity of type \( A_6 \) (resp. \( A_4 \)) consists of 5 singularities of type \( A_6 \) (or 7 singularities of type \( A_4 \)). This gives \( \text{rank} R_H > 20 \), a contradiction. This implies that \( 35 \mid \# [G, G] \).
5, this contradicts Proposition (4.2) from [Mu]. So \( \mu([G,G]) \leq 4 \). On the other hand, by Corollary (3.5) from [Mu], \( \mu([G,G]) \geq \mu(G) = 4 \). Thus \( \mu([G,G]) = 4 \). Replacing \( G \) with \( [G,G] \) and repeating the argument, we see that \( G \) is obtained by taking extensions of a proper subgroup \( K \) with \( 35|\#K \), \( K = [K,K] \) and \( \mu(K) = 4 \). By the result of Case 1, \( K \cong M_{21} \text{ or } \mathfrak{A}_7 \). The first case can be excluded, as \( M_{21} \) has order maximal in the List. The second case can also be excluded, as \( \mathfrak{A}_7 \), though its order is not maximal in the List, admits no extensions in the set of finite symplectic groups. This easily follows from that \( X/\mathfrak{A}_7 \) has only one singularity of type \( A_4 \), and \( 5k, k > 1 \), does not divide the order of a stabilizer subgroup. □

It follows from the List that a 2-group is not exceptional and hence is in Mukai’s list. We need the following description of symplectic groups of order \( 2^6 \).

Lemma 5.8. [Mu], [Xiao] There are 5 symplectic groups of order \( 2^6 \), \( \Gamma_{13a_1} \), \( \Gamma_{22a_1} \), \( \Gamma_{25a_1} \), \( \Gamma_{23a_2} \), and \( \Gamma_{26a_2} \).

(i) \( \Gamma_{23a_2} \) and \( \Gamma_{26a_2} \) do not contain a subgroup \( \cong 2^4 \).
(ii) \( \Gamma_{13a_1} \cong 2^4 \cdot 2^2 \cong 2^4 \cdot 4 \cdot 2^2 \cong 2^2 \cdot 4 \).
(iii) \( \Gamma_{22a_1} \cong 2^4 \cdot 4 \cong 2^4 \cdot (2 \times 4) \).
(iv) \( \Gamma_{25a_1} \cong 2^4 \cdot 2^2 \cong 2^3 \cdot 2^3 \).
(v) \( \Gamma_{22a_1} \) and \( \Gamma_{25a_1} \) have only one normal subgroup \( \cong 2^4 \).
(vi) \( \Gamma_{13a_1} \), \( \Gamma_{22a_1} \), and \( \Gamma_{25a_1} \) always split over every normal subgroup \( \cong 2^4 \).

In (ii) – (iv), the last isomorphism is given by (commutator).(quotient).

Proof. We give a proof of the last assertion (vi), which is not given explicitly in [Mu] or [Xiao]. By (ii)-(iv), each of these three groups has a normal subgroup \( \cong 2^4 \) over which it splits. By (v) the assertion follows for \( \Gamma_{22a_1} \) and \( \Gamma_{25a_1} \).

Let \( G = \Gamma_{13a_1} \), and let \( H_1, H_2 \) be two distinct normal subgroups \( \cong 2^4 \) of \( G \). If \( \#(H_1 \cap H_2) = 8 \), then the join \( H_1H_2 \) is of order \( 2^5 \). There is only one symplectic group of this order containing a subgroup \( \cong 2^4 \) [Mu]. It is \( \Gamma_{4a_1} \), but this group does not contain two subgroups \( \cong 2^4 \). This proves that \( \#(H_1 \cap H_2) \leq 4 \). In this case \( H_i \) contains a subgroup \( K_i \cong 2^2 \) with \( K_i \cap H_j = \{1\} \) (\( \{i,j\} = \{1,2\} \)). Thus \( H_j < G \) splits for \( j = 1,2 \). □

Corollary 5.9. Let \( G \) be an exceptional group or in Mukai’s list. Let \( H \) be a normal subgroup of \( G \). Assume either \( H \cong 2^4 \) or \( \#H = 2^6 \). Then \( G \) splits over \( H \).

Proof. Let \( P \) be a 2-Sylow subgroup of \( G \). Then \( P \) contains \( H \) as a normal subgroup. By Gaschütz’s theorem cited in [Mu] (p.204 in the proof of Proposition (4.8)), it suffices to show that \( H < P \) splits. We may assume \( P \neq H \).

Assume \( H \cong 2^4 \). If \( \#P = 2^5 \), then \( P \cong \Gamma_{4a_1} \) and \( H < P \) splits. If \( \#P = 2^6 \), then by Lemma [Xiao], \( H < P \) splits. If \( \#P = 2^7 \), then \( P \cong F_{128} \),
a unique symplectic group of order 128. It follows from [Mu] (Proposition (3.18)) that $H < P$ splits.

Assume $\#H = 2^6$. Then $P \cong F_{128}$, which splits over every subgroup of index 2. \hfill $\Box$

Lemma 5.10. Let $G$ be an exceptional group of order $2^a \cdot 3^b \cdot 5$ or $2^a \cdot 3^b \cdot 7$ with $a, b \geq 1$. Then $G$ is isomorphic to one of the following seven groups:

$\langle 2^3 : 7 \rangle \cdot 3$, $S_6$, $M_{10}$, $2^4 : A_6$, $2^4 : A_5 \not\cong M_{20}$, $M_{20} : 2 = 2^4 : S_5$, $2^3 : L_2(7)$.

Proof. Assume that $G$ is solvable. A solvable group of order $mn$, where $(m, n) = 1$ contains a subgroup of order $m$ (see [Ha], Theorem 9.3.1). If $\#G = 2^a \cdot 3^b \cdot 5$, $G$ contains a subgroup of order $3^b \cdot 5$. Such an order cannot be found in Mukai’s list or in our List. The case $\#G = 2^a \cdot 3^b \cdot 7$ can also be excluded, because there is no group of order $2^b \cdot 7$ in both lists. Assume $\#G = 2^3 \cdot 3^7$. If $G$ is simple, then $G \cong L_2(7)$ with $\mu(L_2(7)) = 5$, hence, not exceptional. Thus $G$ is not simple. Let $H$ be a nontrivial normal subgroup of $G$. By inspecting possible orders of $H$ and $G/H$, and using that an order 7 element does not normalize an order 3 element, we infer that $\#H = 2^3$ or $2^3 \cdot 7$. In the first case, the quotient $G/H$ has a normal subgroup of order 7, hence we may assume the second case. A group of order $2^3 \cdot 7$ does not appear in Mukai’s list, so must be exceptional. Later, we will see that such a group must be isomorphic to $2^3 : 7$ and its lattice is $3A_6 + 2A_1$. This implies that $G \cong (2^3 : 7) \cdot 3$ and $\mathcal{R}_G = A_6 + 2A_5 + 2A_2$. This order with the type of lattice can be found in the List.

Now we assume that $G$ is not solvable. Therefore $G$ contains a normal subgroup $H$ and a normal subgroup $T$ of $H$ such that $H/T$ is a non-commutative simple group.

Case 1. $T = \{1\}$, i.e. $G$ contains a simple nonabelian normal subgroup $H$.

Assume $5 | \#G$. The known classification of simple groups of order $2^a \cdot 3^b \cdot 5$ (see [Br]) gives that $H$ is isomorphic to $A_5$ or $A_6$. Let $\bar{G} = G/H$. Since the groups $A_5, A_6$ are in Mukai’s list and hence satisfy $\mu(G) \geq 5$, we may assume that $G$ is a nontrivial group. It is known (see [Xiao]) that in both cases $X/H$ has 2 singular points of type $A_4$. Since $5k, k > 1$, does not divide the order of a stabilizer subgroup of $G$, we see that $G$ acts simply transitively on the set of two singularities, and hence is a cyclic group of order 2. If $H \cong A_5$, the group $G$ is isomorphic to $S_6$ or the direct product $A_5 \times C_2$. The first group is in Mukai’s list and $\mu(G) = 5$. The second group contains a cyclic group of order 10 which cannot act symplectically on $X$.

So, we may assume that $H \cong A_6$. Again, $G$ cannot be the direct product $A_6 \times C_2$. The ATLAS [CN] shows that $G \cong S_6$ or $G \cong M_{10}$. The orders of these groups are in the List and there are 5 possible root lattices for groups of order 720. Three of them contain sublattices of type $E_6$ or $D_5$. It is easy to see that the corresponding stabilizer subgroups isomorphic to $T_{24}$ or $Q_{12}$ are not subgroups neither of $S_6$ nor $M_{10}$. This leaves only two possibilities $\mathcal{R}_G = 2A_3 + A_4 + 2A_5$ or $\mathcal{R}_G = A_2 + A_3 + A_4 + A_7 + D_4$ for groups of this
order. Since $S_6$ has no elements of order 8 but $M_{10}$ has, we see that the
first case could be realized for $S_6$ and the second for $M_{10}$.

Assume $7|\#G$. The known classification of simple groups of order $2^a \cdot 3^b \cdot 5$
\cite{Br} gives that $H \cong L_2(7)$. This group is from Mukai’s list and $\mu(L_2(7)) = 5$.
So the quotient $\bar{G} = G/H$ is a group of order $2^{a-3}$. The orbit space $X/H$
has one singular point of type $A_6$ which must be fixed under the action of $\bar{G}$
on $X/H$ giving a singularity $Gx$ on $X/G$ with stabilizer group $G_x$ of order 7.\#G. Table 1 shows that $\bar{G}$ must be trivial. So $\mu(G) = 5$ and this case is
not realized.

Case 2. $T \neq \{1\}$ and $H = G$.

Assume $5|\#G$. Again $\bar{G} = G/T \cong \mathfrak{A}_5$ or $\mathfrak{A}_6$. Thus $T$ is a normal subgroup
of order $2^{a-2} \cdot 3^{b-1}$ if $G \cong \mathfrak{A}_5$, or $2^{a-3}$ if $G \cong \mathfrak{A}_6$. Assume $b = 2$
and $G/T = \mathfrak{A}_5$. Since an element $g_5$ of order 5 from $G$ does not normalize a
subgroup of order 3, we see that the number of Sylow 3-subgroups in $T$
must be divisible by 5. This is a contradiction. Assume $T$ is of order $2^{a-2}$
and $G/T = \mathfrak{A}_5$. Since an element of order 5 does not commute with an
element of order 2, it cannot act identically on the center of $T$. It is easy to
check that an abelian group $A$ of order $2^n$ admits an automorphism of order 5
only if $n > 4$ or $A = C_2^5$. A group which contains a central subgroup of
index 2 is abelian. Thus $T$ is either a 2-elementary abelian group of order $2^4$
(if $a = 6$), or an abelian group of order $2^5$ (if $a = 7$). It follows from Nikulin
\cite{Mn}, Proposition (3.20)) that an abelian group of order $2^5$ does not admit
the Mathieu representation. So $T \cong C_2^4$. The order $2^a \cdot 3^b$ is in our List.
There is only one possible root lattice in this case $\mathcal{R}_G = A_1 + 2A_4 + D_5 + E_6$.
This group is not isomorphic to $M_{20}$, since $\mu(M_{20}) = 5$. A possible scenario
is that $X/T$ has 15 singular points of type $A_1$ and $G/H \cong \mathfrak{A}_5$ has two orbits
of 5 and 10 points on this set. One orbit gives a singularity of type $E_6$,
another one of type $D_5$. By Corollary 5.9 $T < G$ splits, i.e. $G \cong 2^4 \cdot \mathfrak{A}_5$.
There are two non-isomorphic actions of $\mathfrak{A}_5$ on $\mathbb{F}_3^2$ (see section 2.8 in \cite{Mn}).
One gives the group $M_{20}$ from Mukai’s list. Another one gives a group from
(vii). We denote this group by $M_{20}'$. It is realized as a subgroup of $M_{23}$.

Assume $G/T = \mathfrak{A}_6$. A similar argument shows that $T \cong C_2^4$. A direct
computation shows that $\mu(\mathfrak{A}_6) = 5$. For any nontrivial normal subgroup $T$
of $G$ we have $\mu(G/T) > \mu(G)$ (\cite{Mn}, Proposition (3.3)). Thus $\mu(G) \leq 4$
and such group may appear. The order $2^4 \cdot \# \mathfrak{A}_6$ appears in the List with
$\mathcal{R}_G = A_2 + A_3 + 2A_4 + E_7$. This case cannot be excluded. By Corollary 5.9
$T < G$ splits and $G$ is isomorphic to a semi-direct product $2^4 \cdot \mathfrak{A}_6$.

Assume $7|\#G$. We have $G \cong L_2(7)$ and $T$ is of order $2^{a-3}$. It follows
from the List that $a = 6$ and $\mathcal{R}_G = A_2 + A_3 + A_5 + A_6 + D_4$ or $A_2 + 2A_3 + A_5 + A_6 + E_6$. Let us exclude the first possibility for $\mathcal{R}_G$. Since the 2-
part of the discriminant group is generated by > 2 elements, by Lemma
5.6 it suffices to show that $G = [G,G]$. Since $L_2(7)$ is simple, the image of $[G, G]$ in $G$
is either trivial or the whole $G$. In the first case $[G, G] < T$ and
the abelian group $G/[G, G]$ maps surjectively to a non-commutative group

In the second case, \(#L_2(7) \leq \# [G, G] \leq 2^3 \# L_2(7)\). The List shows that one of the inequalities is the equality. If it is the first one, \(G\) contains a normal subgroup isomorphic to \(L_2(7)\) and hence is isomorphic to the product \(T \times L_2(7)\) which contains elements of order larger than allowed. If it is the second one, then \(G = [G, G]\). This proves that \(R_G = A_2 + 2A_3 + A_6 + E_6\). Next we prove that the extension \(G = 2^3 \cdot L_2(7)\) splits. From the type of \(R_G\) we see that \(G\) contains no elements of order 8. Let \(P\) be a 2-Sylow subgroup of \(G\). Since the 2-Sylow subgroups of \(L_2(7)\) are isomorphic to the dihedral group \(D_8\), we have the extension \(P = 2^3 \cdot D_8\). By Gaschütz’s theorem cited in [Mu] (p.204 in the proof of Proposition (4.8)), it suffices to show that \(2^3 < P\) splits. Since \(P\) is isomorphic to one of the five groups from Lemma 5.8 and contains no elements of order 8, we see that \(P\) must be isomorphic to \(\Gamma_{25}a_1, \Gamma_{23}a_2, \) or \(\Gamma_{13}a_1\). The last group can further be excluded as it does not have a normal subgroup \(\cong 2^3\) with quotient \(\cong D_8\) [HS]. Consider a subgroup \(K \cong 2^3 \cdot \mathcal{S}_4\) of \(G\) containing \(P\). The Mukai’s list contains 3 groups of the order \(#K\), \(T_{192}, H_{192}\), and \(\Gamma_{13}a_1 : 3\) [Xiao]. The first two are split extensions \(2^3 \cdot \mathcal{S}_4\) (p.192 Remark [Mu]), and the last one cannot contain \(P\), as \(P \not\cong \Gamma_{13}a_1\). If \(K\) is exceptional, then as we will see later, \(K\) is isomorphic to (xxi) \((Q_8 \circ Q_8) \cdot \mathcal{S}_3\), (xxii) \((2^4 : 2) \cdot \mathcal{S}_3\), (xxiii) \(\Gamma_{13}a_1 : 3\), or (xxv) \(\Gamma_{25}a_1 : 3\). The first two contain elements of order 8, as their \(R_K\) show, and the third can also be excluded for the same reason as above. Thus \(P \cong \Gamma_{25}a_1\). Since \(\Gamma_{25}a_1\) is a 2-Sylow subgroup of \(\text{Hol}(2^3) = 2^3 \cdot L_2(7)\) [HS], \(2^3 < P\) splits.

Case 3: \(T \neq \{1\}, G \neq H\).

It is shown in [Mu], Theorem (4.9) that a non-solvable group \(G\) with \(\mu(G) \geq 5\) is isomorphic to \(\mathcal{A}_5, \mathcal{A}_6, \mathcal{S}_5, L_2(7)\) or \(M_{20} \cong 2^4 : \mathcal{A}_5\). If \(\mu(H) \geq 5\), we must have \(H \cong M_{20}\) and \(T = C_4^2\). Since \(X/M_{20}\) has two singularities of type \(A_4, G/H\) must be a cyclic group of order 2, and hence \(G \cong M_{20} \cdot 2\). The order \(2^3 \cdot M_{20}\) appears in the List with \(R_G = A_2 + A_3 + A_4 + A_5 + D_6\). From this lattice, it is easy to compute the order breakdown for \(G\), in particular \(G\) has more elements of order 2 than \(M_{20}\). So the extension splits and \(G = M_{20} : 2 \cong 2^4 : \mathcal{S}_5\).

Assume \(\mu(H) = 4\). It follows from Case 2 that \(H\) is isomorphic to one of the two groups \(M_{20}', \) or \(2^4 : \mathcal{A}_6\). There are no orders in the List of orders strictly divisible by the order of the latter group. In the first case, as we saw in Case 2, \(X/M_{20}'\) has singularities of type \(A_1 + 2A_4 + D_5 + E_6\). It has 2 singular points of type \(A_4\), so \(G/M_{20}' \cong C_2\) and its action on \(X/M_{20}'\) fixes the unique singularity of type \(D_5\). However a degree 2 extension of \(Q_{12}\) cannot be a stabilizer subgroup. This proves that the group \(M_{20}'\) has no extension in the set of finite symplectic groups.

It remains to consider the cases where \(#G\) is divisible by at most two primes. These are all solvable groups.

It follows from the List that an exceptional group \(G\) cannot be of order \(q^a, q = 2, 3, 5, 7\) or \(3^b5, 3^b7, b = 1, 2\).
**Lemma 5.11.** Let $G$ be an exceptional group. Assume $G$ is solvable of order $2^6 \cdot 5$, or $2^3 \cdot 7$, or $2^a \cdot 3^2$, $(3 \leq a \leq 7)$. Then $G$ is one of the groups from (ix)-(xvii) in Theorem 5.2.

**Proof.** First of all Proposition (5.1) from \cite{Muku} gives that a nilpotent $G$ is either abelian with no elements of order 4 or a 2-group. By the order condition $G$ is not a 2-group. Assume that $G$ is abelian with no elements of order 4. Then $G \cong C_2^a \times C_3^b \times C_5^c$, or $C_2^a \times C_7^b$. The latter two cases have no Mathieu representations. In the first case, $\mu(G) = 4$ implies that $a = 2$. But such an order is not in the List. Thus we may assume that $G$ is not nilpotent and hence its Fitting subgroup $F$ (maximal nilpotent normal subgroup) is a proper nontrivial subgroup. We use Mukai’s classification by analyzing only the cases where the assumption $\mu(G) \geq 5$ was used.

The first case is when $F \cong C_3^2$. The quotient $G/F$ must be a 2-subgroup $H$ of the group $\text{Aut}(C_3^2) \cong \text{GL}_2(\mathbb{F}_3)$ of order 48. The List shows that $\#H = 8$ or 16. Assume that $\#H = 8$. There are 2 possible root lattices for exceptional groups of order 72: $R_G = A_1 + A_2 + A_3 + 2A_7$ or $D_4 + 2D_5 + E_6$. It is known that $X/F$ has 8 singular points of type $A_2$. To get a singular point of type $E_6$ in $X/G$ the group $H$ must fix exactly one of the singular points on $X/F$. This is obviously impossible. Thus only the first case is realized. It corresponds to the case $G/F \cong C_3$ which leads to a group with $\mu(G) = 4$ (see \cite{Muku}, p. 206). If $\#H = 16$, the group $H$ is a 2-Sylow subgroup of $\text{GL}_2(\mathbb{F}_3)$, known to be isomorphic to the semi-dihedral group $SD_{16}$. The order of $3^2 : SD_{16}$ is 144 and it is in the List with $R_G = A_7 + D_4 + A_5 + A_3 + A_1$ or other 3 possibilities, all containing one copy of the root lattice $E_6$ or $E_7$. These cases can be easily excluded. This gives the groups from (ix) and (x).

The second possible case is when $G/F$ is of order divisible by 7 and $G$ has a group $K$ isomorphic to $C_3^3 : C_7$ as a subgroup. The only possible order from the List of the form $2^a \cdot 7$ is $2^3 \cdot 7$. The corresponding root lattice is $2A_1 + 3A_4$. This leads to our group in (xi).

The third case is when $\#G = 2^a \cdot 5$ and $G$ admits a quotient $G$ containing a subgroup $G_0$ with $\mu(G_0) = 5$ isomorphic to $C_2^4 : C_5$. The inspection of the List gives that $\#G = 2^6 \cdot 5$. If $G \neq G$, then $\mu(G) < \mu(G_0)$. Assume this is the case. The kernel $H$ of the projection $G \to G$ is of order $2^s$ with $s \leq 2$. Since it is a normal subgroup and an element of order 5 does not commute with an element of order 2, the order of $\text{Aut}(H)$ must be divisible by 5. This implies that $H = \{1\}$. Thus the Fitting subgroup $F$ of $G$ is a normal subgroup isomorphic to $C_2^4$, and $G$ must contain $C_2^4 : C_5$ as a proper subgroup. It is known that $G/F$ is mapped injectively in $\text{Out}(F)$. The quotient $G/F$ is of order $2^2 \cdot 5$. Since it cannot contain an element of order 10, the quotient is isomorphic to $C_5 : C_4$. There are 3 possible root lattices $R_G$ in the List. Two of them contain a sublattice of type $D_4$ or $D_6$. It is easy to see that the corresponding stabilizer subgroups $Q_8, Q_{16}$ are not subgroups of $G$. The remaining case $3A_3 + A_4 + A_7$ cannot be excluded. By Corollary 5.9 the extension $G = 2^4 \cdot (5 : 4)$ splits, and gives case (xii).
The fourth case is when $G/F$ is of order divisible by 9. In this case $G$ is of order $2^a 3^2$ by the order assumption. Mukai considers the Frattini subgroup $\Phi$ of $F$ (the intersection of maximal subgroups) and shows that $F/\Phi \cong C_2^3$ and $G/\Phi$ contains a subgroup $G_0$ with $G_0 \cong C_2^3 : C_2^3 \cong \mathfrak{A}_4 \times \mathfrak{A}_4$ and $\mu(G_0) = 5$. As in the previous case, to get $\mu(G) = 4$ we must have either $\Phi \neq \{1\}$ or $\Phi = \{1\}$ and $G_0$ is a proper subgroup of $G$.

Assume $\Phi \neq \{1\}$ and let $\#\Phi = 2^s$, $s \leq 3$. Assume $s = 1$. The quotient $X/\Phi$ has 8 singular points of type $A_1$. The group $G_0$ acts on $X/\Phi$ and permutes these points with at least one stabilizer subgroup of order divisible by $3^2$. The known structure of stabilizers shows that this is impossible. No stabilizer $G_x$ is of order divisible by 9. Assume $s = 2$ thus $\Phi \cong C_2^2$ or $C_4$.

In the first case the quotient $X/\Phi$ has 12 singular points of type $A_1$ and the stabilizer subgroups of $G_0$ of these points are groups of order $2^2 3$ (one orbit), or $2^3 3$ (two orbits of size 6 each). The first case gives one singularity of type $E_6$, and the second 2 singularities of type $E_7$. Both appear in the List. But, since $G_0$ has singularities of type $2A_5 + 4A_2 + A_1$ (no $A_3$) on a minimal resolution $Y$ of $X/\Phi$ (see [Xiao]) we easily exclude the second case, obtaining that the singularities of $Y/G_0$ are of types $E_6 + 2A_2 + 2A_5$. This case can be found in the List. This gives a possible case $G \cong 2^2. (\mathfrak{A}_4 \times \mathfrak{A}_4) \cong \Gamma_{13} : 3^2$ from (xiii). If $\Phi = C_4$, $X/\Phi$ has singularities of type $4A_3 + 2A_1$. The stabilizer of $G_0$ of a point of type $A_1$ must be a group of order divisible by $2^3 3^2$. No stabilizer $G_x$ is of order divisible by 9. So this case does not occur.

A degree 2 extension of the group from (xiii) may also appear. In this case $G/\Phi$ is a degree 2 extension of $G_0 = \mathfrak{A}_4 \times \mathfrak{A}_4$ and $\mu(G/\Phi) > \mu(G) = 4$, so $G/\Phi$ appears on Mukai’s list. There appears only one such group in [Xiao], and it is $\mathfrak{A}_{14}$. Thus $G \cong 2^2. \mathfrak{A}_{14} \cong \Gamma_{13} : 3$, and we obtain that $R_G = E_7 + 2D_5 + A_2 + A_1$. It occurs in the List. This is the case (xiv).

Assume $s = 3$. Then $F$ is a Sylow 2-subgroup of $G$ of order $2^2$ with normal subgroup $\Phi$ of order 8. Thus $\Phi \cong Q_8, D_8, 2^3, 2 \times 4$ or $C_8$. Assume that $\Phi \cong Q_8$. The quotient $X/\Phi$ has singular points of type $2D_4 + 3A_3$ or $4D_4 + A_1$. In any case the group $G_0$ of order $2^4 3^2$ permutes points of type $D_4$ with stabilizer of order divisible by 9. No stabilizers of this order could occur. Similar argument rules out the remaining possibilities $\Phi \cong D_8, 2^3, 2 \times 4$ or $C_8$. This shows that the case $\#G = 2^7 3^2$ and $\#\Phi = 8$ does not occur.

Assume $\Phi = \{1\}$. Thus $G$ contains $2^4 : 3^2$ as a proper subgroup and $F = 2^4$. If $a = 5$, the quotient group $G/F$ is of order 18. There are two possible groups of order 18 which can act symplectically on $X$, a group isomorphic to $\mathfrak{A}_{3,3}$ or $C_3 \times D_6$. The first case leads to a group $G \cong \mathfrak{A}_{4,4}$ with $\mu(G) = 5$. The second case gives singularities of $X/G$ of types $3A_5 + A_3 + A_2$. A case with a group of this order and the same $R_G$ can be found in the List. By Corollary 5.9, the extension $F < G$ splits, and gives the case (xv). If $a = 6$, we have 3 possible groups $G/F$ isomorphic to $3^2 : 4, 3 \times \mathfrak{A}_4$, or $\mathfrak{S}_{3,3}$. In the second case, $G/F$ contains a normal subgroup of order 4, and its pullback is a nilpotent normal subgroup of $G$ containing $F$. This contradicts
to the maximality of the Fitting subgroup $F$. This excludes the second group. The first and the third groups have singularities of the quotient of types $4A_3 + 2A_2 + 2A_1$ and $2A_5 + A_2 + 6A_1$, resp.. The groups act on the set of 15 singular points of $X/F$. It is easy to see that in these two cases $X/G$ has singularities of types $A_7 + D_5 + 2A_3 + A_2$, $D_5 + D_4 + 2A_5 + A_1$. Both cases can be found in the List. Both extensions split by Corollary 5.9 giving the groups from (xvi) and (xvii). If $a = 7$, we have 3 possible groups $G/F$ isomorphic to one of the following groups $\mathfrak{A}_{4,3}$, $N_{72} \cong 3^2.D_8$, $M_9 \cong 3^2.Q_8$ of order 72. The first case can be excluded by the maximality of $F$, as the group contains a normal subgroup of order 4. The order $2^7.3^2$ appears in the List with possible root lattices $\mathcal{R}_G$ of types $A_1 + A_2 + 2D_5 + E_7$ or $A_1 + A_2 + D_5 + D_6 + E_5$. It is easy to see that the stabilizer subgroups $T_{24}$ and $O_{48}$ of singularities of type $E_6$ and $E_7$ are not subgroups of $2^4.N_{72}$ or $2^4.M_9$. Thus these two groups are also excluded. 

**Lemma 5.12.** Let $G$ be an exceptional group. Assume $G$ is solvable of order $2^a \cdot 3$, $(4 \leq a \leq 7)$. Then $G$ is one of the groups from (xviii)-(xxvi) in Theorem 5.2.

**Proof.** Again we follow the arguments from [Mu]. It follows from the List that a 2-group is not exceptional and hence is in Mukai’s list. Hence his assumptions (7.1) are satisfied except the last one where we have to replace the condition $\mu(G) \geq 5$ with the condition $\mu(G) \geq 4$.

First Mukai considers the case when the Fitting subgroup $F$ of $G$ is of order divisible by 3, or equivalently, $G$ has a unique 3-Sylow subgroup $T$. Let $S$ be a 2-Sylow subgroup and $\phi : S \to \text{Aut}(T) = C_2$ be the natural homomorphism. The classification of abelian nilpotent symplectic groups shows that $\# \text{Ker}(\phi)$ is of order $\leq 4$, thus $\# G \leq 24$. There are no exceptional groups of order $\leq 24$.

Thus we may assume that the Fitting subgroup $F$ is a 2-group. In this case $F$ is the intersection of all 2-Sylow subgroups of $G$. If $F$ is equal to a unique 2-Sylow subgroup, then $G/F \cong C_3$. Otherwise $G$ contains three 2-Sylow subgroups and $G/F \cong \mathfrak{S}_3$.

Using the classification of 2-groups, Mukai lists all possible groups $F$. In our case they can be only of order $2^c$, $c = 3, 4, 5, 6, 7$ ($c = a$, or $a - 1$).

Case $c = 3$.

In this case $F \cong C_2^3$ or $Q_8$ and $G/F \cong \mathfrak{S}_3$. There are 4 possible root lattices for exceptional groups of order 48. If $F \cong C_2^3$, the quotient $X/F$ has 14 singular points of type $A_1$ so the largest stabilizer for the action of $G$ on $X$ is of order 12. Since all possible root lattices contain a subdiagram $E_6$ or $E_7$ this case is not realized. The quotient $X/Q_8$ has 2 singular points of type $D_4$ and 3 singular points of type $A_3$, or other possibility is 4 points of type $D_4$ and one of type $A_4$. It is easy to see that the first possibility leads to the root lattice of type $A_2 + D_4 + 2E_7$ and the second one to $A_2 + D_5 + D_6 + E_7$. In both cases the group $G$ must coincide with the stabilizer of a singular point of type $E_7$ and hence is isomorphic to $O_{48}$. This is our case (xviii).
Case $c = 4$.

In this case $F \cong C_4^1, C_4^2$, or $Q_8 \times C_2$, or $Q_8 \circ C_4$. In the last case, Mukai shows that $G \cong T_{24} \circ C_4$, or $(T_{24} \circ C_4) \cdot 2$, and excludes them because $T_{24} \circ C_4$ contains an element of order 12.

Assume $F \cong Q_8 \times C_2$. This case is also excluded by Mukai because of the assumption $\mu(G) \geq 5$. It cannot be excluded in our case and leads to two groups $G = T_{24} \times C_2$, if $G/F \cong C_3$, and $G = (T_{24} \times C_2) \cdot 2$, if $G/F \cong S_3$.

Let us determine the root lattice of $G = T_{24} \times C_2$. The singular points of $X/T_{24}$ are of type $E_6 + D_4 + A_5 + 2A_2$, or $2E_6 + A_3 + 2A_2$. In the first case, the group $C_2$ fixes the unique singularity of type $E_6$. Since $T_{24} \times C_2 \neq O_{48}$, we get a contradiction. In the second case $X/G$ has singularities of type $E_6 + D_4 + 2A_5$. A group of order 48 with this root lattice is in the List and gives case (xix). It is easy to see that $G = (T_{24} \times C_2) \cdot 2$ has root lattice $2A_1 + A_5 + D_6 + E_7$. Computing order breakdown, we see that $G$ has more elements of order 2 than $T_{24} \times C_2$ or $O_{48}$. So the extension splits and $G = (T_{24} \times C_2) : 2 \cong O_{48} : 2$. This is also in the List and gives case (xx).

Assume $F \cong C_2^4$ or $C_2^2$. If $F$ is a 2-Sylow subgroup, then $\#G = 48$. The quotient $G/F$ has either 15 singular points of type $A_1$ or 6 singular points of type $A_5$. Thus the largest possible order of a stabilizer subgroup of $G$ is 12.

The List shows that the root lattice $R_G$ always contains a copy of $E_6$ or $E_7$. This shows that this case does not occur. So, $F$ is not a 2-Sylow subgroup and $G/F \cong S_3$. There are 3 possible root lattices for groups of order 96 in our List. They are of types $2A_1 + A_5 + D_6 + E_7$, $2A_1 + A_7 + D_5 + E_6$, $3A_2 + 2A_7$. If $F \cong C_2^2$, $X/F$ has 15 singular points of type $A_1$ and hence the largest possible stabilizer subgroup of $G$ is of order 12 and no stabilizers of order 8. This shows that this case is not realized. If $F \cong C_2^4$, $X/F$ has 6 singular points of type $A_5$ and hence the largest possible order of a stabilizer subgroup of $G$ is 24. Since $T_{24} \not\cong 4 \cdot 3$, the root lattice of $G$ cannot contain $E_6$. This rules out the first two root lattices, and the remaining root lattice is $2A_7 + 3A_2$. The 3-part of its discriminant group is isomorphic to $3^3$. But $G \cong 4 \cdot 3$ does not admit a non-trivial homomorphism to $C_3$. This contradicts Lemma 5.6. So this case must be excluded.

Case $c = 5$.

In this case $F$ is isomorphic to $Q_8 \circ Q_8$ or $2^4 : 2 = \Gamma_4 a_1$.

Assume $F = Q_8 \circ Q_8$. In this case $X/F$ has 9 singular points of type $A_1$ and 2 singular points of type $D_4$. If $G/F \cong C_3$, the group $C_3$ fixes the points of type $D_4$ and define 2 points of type $E_6$ on the quotient $X/G$. No root lattices with such sublattices are realized for groups of order 96. Thus $G/F \cong S_3$. There are 5 possible root lattices for groups of order 192. If $S_3$ leaves points of type $D_4$ invariant, then $X/G$ has 2 singular points of type $E_7$. If it permutes these point, we have one singular point of type $E_6$. By inspection of the diagrams, we see only one lattice $2E_7 + A_3 + A_2 + A_1$ fits. From the lattice, it is easy to see that $G$ has exactly 19 elements of order
2. Since the subgroup \( Q_8 \circ Q_8 \) has the same number of elements of order 2, the extension does not split. This is the group from (xxi).

Assume that \( F \cong 2^4 : 2 \). Then \( X/F \) has 3 singular points of type \( A_3 \) and 8 singular points of type \( A_1 \). Using the list of possible root lattices for exceptional groups of order 96, we see that \( G/F \cong \mathcal{S}_3 \). Since \( T_{24} \not\cong 4, \mathcal{S}_3 \), the group \( \mathcal{S}_3 \) does not leave a point of type \( A_3 \) invariant, and hence permutes 3 singular points of type \( A_3 \), giving a singular point on \( X/G \) of type \( D_4 \) or \( A_7 \). Moreover, this argument shows that \( X/G \) does not have a singular point of type \( E_7 \) or \( E_6 \). By inspecting the 5 possible root lattices for groups of order 192, we see only two lattices \( A_1 + A_2 + A_7 + 2D_5 \) and \( A_1 + 3A_5 + D_4 \) survive. But the latter can also be eliminated by considering the orbit decomposition of \( \mathcal{S}_3 \) on the 8 singular points of type \( A_1 \). This gives \( \mathcal{R}_G = A_1 + A_2 + A_7 + 2D_5 \). By a similar computation of the number of elements of order 2, we see that the extension does not split. This is case (xxii).

Case \( c = 6 \).

In this case \( F \) is a 2-group of type \( \Gamma_{13}a_1 \) or \( \Gamma_{25}a_1 \) from Proposition (6.12) of [Mü].

Assume \( F \cong \Gamma_{13}a_1 \). The quotient \( X/F \) has 3 singular points of type \( D_4 \) and 6 singular points of type \( A_1 \). If \( G/F \cong C_3 \), this leads to the two possible root lattices \( A_1 + D_4 + 3A_5 \) and \( 2A_1 + 3E_6 \). This is case (xxiii). If \( G/F \cong \mathcal{S}_3 \), then we have a group of order \( 2^7.3 \). The corresponding root lattices in our List are of the following types \( A_1 + A_3 + A_5 + D_5 + D_6 \), \( A_1 + A_3 + A_5 + D_4 + E_7 \), \( A_1 + 2A_3 + E_6 + E_7 \). The first and the third correspond to the two root lattices of the index 2 subgroup. By Corollary 5.9, the extension \( G = F. \mathcal{S}_3 \) splits, and gives case (xxiv) and (xxiv').

Assume \( F \cong \Gamma_{25}a_1 \). Then \( X/F \) has one singular point of type \( D_4 \), three singular points of type \( A_3 \) and five singular points of type \( A_1 \). If \( G/F \cong C_3 \), \( \mathcal{R}_G \) must contains only one copy of \( E_6 \). There is only one such lattice \( A_1 + A_3 + 2A_5 + E_6 \) for groups of order 192. This gives case (xxv). If \( G/F \cong \mathcal{S}_3 \), the root lattice \( A_1 + A_3 + A_5 + D_4 + E_7 \) may occur. Again, by Corollary 5.9, the extension splits, and gives case (xxvi).

Case \( c = 7 \).

In this case \( G/F \cong C_3 \) and Mukai leads this to contradiction. It is still true in our situation. The group \( F \cong F_{128} \) of order \( 2^7 \) has singularities on \( X/F_{128} \) of types \( D_6 + D_4 + 2A_3 + 3A_1 \). Since \( F \) is normal, the group \( C_3 \) acts on \( X/F_{128} \). It must fix the singular points of type \( A_3 \) and gives on \( X/G \) singular points of type \( D_5 \). But \( Q_{12} \not\cong C_4.C_3 \). \( \square \)

Remark 5.13. Here we explain the difference between the groups \( 2^4 : \mathcal{S}_4 \). Mukai’s list contains a unique group of order 384 = \( 2^7.3 \) which he denotes by \( F_{384} \). Our list contains three groups of this order non-isomorphic to \( F_{384} \). The difference between these groups and \( F_{384} \) is very subtle. The group \( F_{384} \) is isomorphic to \( 2^4 : \mathcal{S}_4 \) and the pre-image of the normal subgroup \( 2^7 \) of \( \mathcal{S}_4 \) is isomorphic to \( \Gamma_{13}a_1 \) (see [Mü], p.212). Thus \( F_{384} \cong \Gamma_{13}a_1 : \mathcal{S}_3 \) is an
extension of the same type as our groups (xxiv) and (xxiv'). The difference is of course in the action of $S_4$ on $2^4 \setminus \{0\}$, whose orbit decomposition can be read off from the corresponding root lattice $R_G$ and is given in the first column of Table 3.

| $G$ | action of $S_4$ on $2^4 \setminus \{0\}$ | action of $S_4$ on $\Xi$ | composition series |
|-----|---------------------------------|-----------------|------------------|
| $F_{384}$ | $3 + 12$ | $1 + 1 + 2 + 4$ | $(2,2)$ |
| (xxiv) | $3 + 4 + 8$ | $1 + 3 + 4$ | $(2,1,1)$ |
| (xxiv') | $1 + 2 + 12$ | $1 + 3 + 4$ | $(1,1,2)$ |
| (xxvi) | $1 + 6 + 8$ | $1 + 1 + 6$ | $(1,2,1)$ |

Table 3.

Recall that in the action of $M_{24}$ on a set $\Omega$ of 24 elements, the complement of an octad $\Xi$ is identified with the affine space $F^4_2$, and each element $g \in A_8$ embeds in $M_{24}$ by acting on $\Xi$ as an even permutation and acting on the complement $\Omega \setminus \Xi$ as a linear map $i(g)$, where $i : A_8 \to L_4(2) = GL_4(F_2)$ is the exceptional isomorphism of simple groups. For $F_{384}$ Mukai shows that the image of $S_4$ in $A_8$ is a subgroup with orbit decomposition $1 + 1 + 2 + 4$ ([Muk], p.218 and Corollary (3.17)), dependent on the assumption $\mu(G) \geq 5$. In our case, $\mu(G) = 4$ and hence the number of orbits must be equal to 3. It is easy to see that $1 + 3 + 4$ and $1 + 1 + 6$ are the only possible such decompositions for a subgroup of $A_8$ isomorphic to $S_4$. On the other hand, it is known that $S_4$ has only one non-trivial irreducible linear representation over a field of characteristic 2. It is isomorphic to the representation $S_4 \to S_3 \cong L_2(2)$. This easily implies that any faithful linear representation of $S_4 \to L_4(2)$ is either decomposable as a sum of the trivial representation and an indecomposable 3-dimensional representation, or is an indecomposable representation with composition series with factors of dimensions $(1,1,2),(2,1,1),(1,2,1),(2,2)$. In the reducible case, the representation is the direct sum of one-dimensional representation and an indecomposable 3-dimensional representation with composition series of type $(1,2),(2,1)$. In the first case $S_4$ has 3 fixed points in $2^4 \setminus \{0\}$. Assume that the corresponding extension $G = 2^4 : S_4$ realizes. Then in the quotient $X/2^4$ we have 15 ordinary double points permuted by $S_4$ according to its action on $2^4 \setminus \{0\}$. Three of them are fixed. This implies that $X/G$ has 3 singular points of type $E_7$; this is too many. In the second case $S_4$ has orbit decomposition of type $1 + 3 + 3 + 4 + 4$. Again it is easy to see that this leads to a contradiction. This argument can be also used to determine the second column of Table 3 using the known information about the root lattice $R_G$ (from [Xiao] for a non-exceptional group or otherwise from the List.)

Thus we may assume that the representation of $S_4$ in $F^4_2$ is indecomposable. One can show that, up to isomorphism, the representation is determined by its composition series, hence we have 4 different cases. They
correspond to the cases in Table 3, where the last column indicates the type of the composition series.

Note that the linear representations $\mathfrak{S}_4 \to L_4(2)$ from (xxiv) and (xxiv$'$) differ by an exterior automorphism of $L_4(2)$ defined by a correlation. The images of $\mathfrak{S}_4$ under the representations defined by $F_{384}^{(2)}$ and (xxvi) are conjugate to subgroups of $\mathfrak{A}_8$ embedded in $\mathfrak{A}_8 \cong L_4(2)$ as the subgroup $\mathfrak{A}_{2,4}$. They differ by an exterior automorphism of $\mathfrak{S}_6$.

The distinction between $M_{20}$ and $M_{20}'$ also can be given similarly; the image of $\mathfrak{A}_5$ in $\mathfrak{A}_8$ has orbit decomposition $1 + 1 + 1 + 5$ and $1 + 1 + 6$, respectively.

**Proposition 5.14.** All exceptional groups are contained in $M_{23}$.

**Proof.** We will indicate the chain of maximal subgroups starting from a maximal subgroup of $M_{23}$ and ending at the subgroup containing the given group. If no comments are given, the verification is straightforward using the list of maximal subgroups of $M_{23}$ which can be obtained from ATLAS. A useful fact is that in all extensions of type $2^4 : K$, the group $K$ acts faithfully in $2^4 (\mathfrak{M},$ Proposition (3.16)) and hence defines an injective homomorphism $K \to L_4(2) \cong \mathfrak{A}_8$.

The subgroup $G(\Xi)$ of $M_{24}$ which preserves an octad $\Xi$ is isomorphic to the affine group $\text{AGL}_4(2) = 2^4 : \mathfrak{A}_8$. Here $2^4$ acts identically on the octad, and the quotient $\mathfrak{A}_8$ acts on the octad by even permutations. A section $S$ of the semi-direct product is a stabilizer subgroup of a point outside the octad. Taking this point as the origin, we have an isomorphism $i : \mathfrak{A}_8 \to S \cong L_4(2)$ which we used in Remark 5.13. We will write any element from $G(\Xi)$ as $(g_1, g_2)$, where $g_1$ is a permutation of $\Xi$ and $g_2$ is a permutation of $\Omega \setminus \Xi$. The group $\text{AGL}_4(2)$ is generated by elements $(g, i(g)), g \in \mathfrak{A}_8$, and translations $(1, t_a), a \in 2^4$. The image of $\mathfrak{A}_8$ consists of elements $(g, i(g))$, where $i(g) \in L_4(2)$. In particular, all elements of $\mathfrak{A}_8$ fix the origin in $\Omega \setminus \Xi$, and hence $\mathfrak{A}_8$ is isomorphic to a subgroup of $M_{23}$. Recall that the latter is defined as the stabilizer of an element of $\Omega$.

(i) $M_{23} \supset \mathfrak{A}_8$. We use that $\mathfrak{S}_6$ embeds in $\mathfrak{A}_8$ as the subgroup $\mathfrak{A}_{2,6}$.

(ii), (iii), (v), (vii) $M_{23} \supset M_{22}$.

(iv) $M_{23} \supset M_{21} \cdot 2$.

(vi) $\subset$ (iii). We use the embedding of $\mathfrak{A}_5$ in $\mathfrak{A}_6$ as the composition of the natural inclusion $\mathfrak{A}_5 \subset \mathfrak{A}_6$ and the exterior automorphism of $\mathfrak{S}_6$.

(viii) $M_{23} \supset M_{22}$. There is only one split extension $2^3 : L_2(7)$.

(ix) $\subset$ (x).

(x) $M_{23} \supset M_{21} \cdot 2 \supset (3^2 : Q_8) \cdot 2$.

(xi) $\subset$ (xxvii).

(xii) $\subset$ (vii). The group $5 : 4$ is the normalizer of a 5-Sylow subgroup of $\mathfrak{S}_5$. The group $\mathfrak{S}_5$ embeds in $\mathfrak{A}_8 \cong L_4(2)$ as a subgroup $\mathfrak{A}_{1,2,5}$ and its conjugacy class is unique. The conjugacy class of the subgroup $5 : 4$ is also unique, and hence its action on $2^4$ is defined uniquely up to isomorphism.

(xiii) $\subset$ (xiv).
of (xxvi) is an extension $A_3 \supset 2^4 : \mathfrak{A}_3,4 = \Gamma_{13} a_1 : \mathfrak{A}_{3,3}$.

(xv) $A_3 \supset (xxvi)$.

(xvii) $M_{23} \supset A_8 \supset 2^4 : \mathfrak{A}_{2,2,3,3} \cong 2^4 : \mathfrak{S}_{3,3}$.

(xvi) $M_{23} \supset A_6 \supset 2^4 : \mathfrak{A}_{3,4} = \Gamma_{13} a_1 : \mathfrak{A}_{3,3}$.

(xvii) $M_{23} \supset A_8 \supset 2^4 : \mathfrak{A}_{2,3,3} \cong 2^4 : \mathfrak{S}_{3,3}$.

(xviii) $M_{23} \supset A_6 \supset 2^4 : \mathfrak{A}_{3,4} = \Gamma_{13} a_1 : \mathfrak{A}_{3,3}$.

Note that $3^2 : 4$ is the normalizer of a 3-Sylow subgroup of $A_6$ and it is a unique (up to conjugation) maximal subgroup of $A_6$.

(xviii) $(xxvi)$ is an extension $2^4 : \mathfrak{S}_4$ and $\mathfrak{S}_4$ fixes a unique point $a \in 2^4 \setminus 0$ (see Table 3). We embed $\mathfrak{S}_4$ as a subgroup of $L_4(2)$ generated by the matrices $x, y$ satisfying $x^4 = y^2 = (xy)^3 = 1$:

\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{pmatrix} \quad \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

Let $a = (0, 0, 0, 1), b = (1, 1, 1, 1), c = (0, 1, 0, 1)$. Then we immediately check that $t_{b,i}(x) = i(x) t_{c,i}, (t_{b,i}(x))^4 = (t_{c,i}(y))^2 = t_a$. The subgroup generated by the elements $(x, t_{b,i}(x))$ and $(y, t_{c,i}(y))$ is isomorphic to $O_{48}$.

(xviii), (xix) $\subset (xx)$. In particular, it gives another embedding of $O_{48}$ in $M_{23}$.

(xx) $\subset (xxiv')$. The group (xxiv') is an extension $2^4 : \mathfrak{S}_4$ and $\mathfrak{S}_4$ fixes a unique point $a \in 2^4 \setminus 0$ and leaves invariant a 2-dimensional subspace $< a, a' > \subset 2^4$ (see Table 3). In this case, we realize $\mathfrak{S}_4$ as a subgroup of $L_4(2)$ generated by the matrices

\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{pmatrix} \quad \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\end{pmatrix}.
\]

Let $a = (1, 0, 0, 0), a' = (0, 1, 0, 0), b = (1, 1, 1, 1), c = (0, 1, 0, 1)$. It is checked that $t_{b,i}(x) = i(x) t_{c,i}, (t_{b,i}(x))^4 = (t_{c,i}(y))^2 = t_a$, and the subgroup generated by the elements $(x, t_{b,i}(x)), (y, t_{c,i}(y)), (1, t_a)$ is isomorphic to $O_{48} : 2$.

(xxi) $\subset (xxvi)$. Note $Q_8 \circ Q_8 = \Gamma_{5} a_1$, thus the group (xxi) has a 2-Sylow subgroup $\cong \Gamma_{25} a_1 = \Gamma_5 a_1 \cdot 2$. If the group (xxi) admits a degree 2 extension in our list of exceptional groups or in the Mukai’s list, it must be the group (xxvi). This follows from the types of singularities. On the other hand, the group (xxvi) contains a non-split extension of the form $(Q_8 \circ Q_8) \cdot \mathfrak{S}_3$. To show this, denote the groups (xxi) and (xxvi) by $G$ and $K$, respectively. The normal subgroup $\Gamma_{25} a_1$ of $K$ contains only one subgroup $\cong \Gamma_5 a_1$. Denote this subgroup by $H$. Then $H$ is normal in $K$, and its quotient $K/H$ is of order 12 and contains a subgroup $\cong \mathfrak{S}_3$, hence $K/H \cong D_{12}$. Since all symplectic groups of order $2^6$ are contained in the unique symplectic group of order $2^7$ which is isomorphic to a 2-Sylow subgroup of $K$, there is a chain of subgroups $H \subset \Gamma_{26} a_2 \subset K$. This implies that there is an order 2 subgroup $A$ of $K/H$ such that $\phi^{-1}(A)$ is a non-split extension $\Gamma_{26} a_2 = H \cdot 2$, where $\phi : K \to K/H$ is the projection. Since $K/H \cong D_{12}$, one can always find a
subgroup $B \subset K/H$ such that $A \subset B \cong S_3$. Now $\phi^{-1}(B)$ gives a non-split extension $(Q_8 \circ Q_8)^* S_3$.

(xii) $\subset$ (xiv). Since $G = \Gamma_4 a_1 \cdot S_3$, it has a 2-Sylow subgroup $\cong \Gamma_2 a_1 = 2^3 : 4$. The normal subgroup $\Gamma_4 a_1$ of $G$ contains only one subgroup $\cong 2^4$. Denote this subgroup by $H$. Then $H$ is normal in $G$, and its quotient $G/H$ is of order 12 and contains a cyclic subgroup of order 4, hence $G/H \cong Q_{12}$. Thus $G \cong 2^3 : Q_{12}$, a split extension by Corollary 5.9. The group $\mathfrak{A}_3, 4$ contains $(12)(4567), (123)$.

(xiii) $\subset$ (xiv) or (xiv').

(xxiv), (xxiv') $\subset$ (xiv). Here and in the next inclusion use Table 3.

(xxv) $\subset$ (xxvi) $\subset$ (iii).

(xxvi) $\subset$ (viii). We use that $L_3(2) \cong L_2(7)$ contains $7 : 3$ as the normalizer of a 7-Sylow subgroup. \[\square\]

Example 5.15. The group $O_{48} \cong 2 \cdot S_4$ is contained in a maximal subgroup of $U_3(5) = PSU_3(F_{25})$ isomorphic to $2 \cdot S_5$. Thus it acts on a K3-surface from the example in the next section in the case $p = 5$, and its order is prime to $p$. Unfortunately, we do not know how to realize explicitly other exceptional groups. It is known that the group $\mathfrak{A}_5$ admits a symplectic action on the Kummer surface of the product of two supersingular curves in characteristic $p \equiv 2, 3 \mod 5$ \[\text{[Lus]}\]. Together with the group $2^4$ defined by the translations, we have a symplectic group isomorphic to an extension $2^4 : \mathfrak{A}_5$. Unfortunately, this group is $M_{20}$ not $M_{20}'$ as we naively hoped.

6. Examples of finite groups of symplectic automorphisms in positive characteristic $p$

Non-exceptional tame groups of symplectic automorphisms

A glance at Mukai’s list of examples of K3 surfaces with maximal finite symplectic group of automorphisms shows that all of them can be realized over a field of positive characteristic $p > 7$. A complete Mukai’s list consists of 80 groups (81 topological types) \[\text{[Xia]}\]. All of them realize over a field of positive characteristic $p$, as long as the order is not divisible by $p$.

Wild groups of symplectic automorphisms

Here we give a list of examples of K3 surfaces over a field of characteristic $p = 2, 3, 5, 11$ with symplectic finite automorphism groups of order divisible by $p$ not from the Mukai list.

($p = 2$) \[X = V(x^4 + y^4 + z^4 + w^4 + x^2y^2 + x^2z^2 + y^2z^2 + xyz(x+y+z)) \subset \mathbb{P}^3\]

It is a supersingular K3 surface with Artin invariant $\sigma = 1$ with symplectic action of the group $PSL(3, F_4).2 \cong M_{21}, 2$, whose order is $(20, 160).2 = (2^4, 3^2, 5, 7).2$ (see \[\text{[DKG]}\]). Although the order of this group divides the order of $M_{23}$, it is not a subgroup of $M_{23}$.\]
(p = 3) \( X = V(x^4 + y^4 + z^4 + w^4) \subset \mathbb{P}^3 \).

Fermat quartic surface is supersingular with Artin invariant \( \sigma = 1 \) in characteristic \( p = 3 \) mod 4. The general unitary group \( \text{GU}(4, \mathbb{F}_9) \) acts on the Hermitian form \( x^4 + y^4 + z^4 + w^4 \) over \( \mathbb{F}_9 \), so \( \text{PSU}(4, \mathbb{F}_9) = U_4(3) \) acts on \( X \), which is simple of order 3, 265, 920 = \( 2^7 \cdot 3^6 \cdot 5 \cdot 7 \). This example was known to several people (A. Beauville, S. Mukai, T. Shioda, J. Tate). The order of this group does not divide the order of \( M_{23} \).

\[
(p = 5) \quad X = V(x^6 + y^6 + z^6 - w^2) \subset \mathbb{P}(1, 1, 1, 3).
\]

It is supersingular with Artin invariant \( \sigma = 1 \). The general unitary group \( \text{GU}(3, \mathbb{F}_{52}) \) acts on the Hermitian form \( x^6 + y^6 + z^6 \) over \( \mathbb{F}_{52} \), so \( \text{PSU}(3, \mathbb{F}_{52}) = U_3(5) \), a simple group, acts symplectically on \( X \). The order of this group is equal to 126, 000 = \( 2^4 \cdot 3^2 \cdot 5^3 \cdot 7 \) and does not divide the order of \( M_{23} \).

\[
(p = 7) \quad X = V(x^3 + (y^8 + z^8)x - w^2) \subset \mathbb{P}(4, 1, 1, 6).
\]

It is supersingular with Artin invariant \( \sigma = 1 \). The general unitary group \( \text{GU}(2, \mathbb{F}_{7^2}) \) acts on the Hermitian form \( y^8 + z^8 \) over \( \mathbb{F}_{7^2} \), so \( \text{PSU}(2, \mathbb{F}_{7^2}) \cong L_2(7) \), a simple group of order 168, acts symplectically on \( X \). Although this group can be found in Mukai’s list, the group action on the surface in his example degenerates in characteristic 7. This surface is birationally isomorphic to the affine surface \( y^2 = x^3 + (t^7 - t)x \) (DK1, Examples 5.8). This surface is also isomorphic to Fermat quartic surface in characteristic \( p = 7 \), and hence admits a symplectic action of the group \( F_{384} \) of order 384.

\[
(p = 11) \quad X = V(x^3 + y^{12} + z^{12} - w^2) \subset \mathbb{P}(4, 1, 1, 6).
\]

It is supersingular with Artin invariant \( \sigma = 1 \). The general unitary group \( \text{GU}(2, \mathbb{F}_{11^2}) \) acts on the Hermitian form \( y^{12} + z^{12} \) over \( \mathbb{F}_{11^2} \), so \( \text{PSU}(2, \mathbb{F}_{11^2}) \cong L_2(11) \), a simple group of order 660, acts symplectically on \( X \). This is a subgroup of \( M_{23} \) but has 4 orbits, so it is not realized in characteristic 0. This surface is birationally isomorphic to the affine surface \( y^2 = x^3 + t^{11} - t \) (DK1, Examples 5.8). This surface is also isomorphic to Fermat quartic surface in characteristic \( p = 11 \), and hence admits a symplectic action of the group \( F_{384} \) of order 384 (see Remark 5.13).

The last three examples were obtained through a discussion with S. Kondō. One can show using an algorithm from [Shio2] and its generalization [Ge] that all the previous examples are supersingular K3 surfaces with the Artin invariant equal to 1.

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