On the Three Properties of Stationary Populations and Knotting with Non-stationary Populations

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Abstract
A population is considered stationary if the growth rate is zero and the age structure is constant. It thus follows that a population is considered non-stationary if either its growth rate is nonzero and/or its age structure is non-constant. We propose three properties that are related to the stationary population identity (SPI) of population biology by connecting it with stationary populations and non-stationary populations which are approaching stationarity. One of these important properties is that SPI can be applied to partition a population into stationary and non-stationary components. These properties provide deeper insights into cohort formation in real-world populations and the length of the duration for which stationary and non-stationary conditions hold. The new concepts are based on the time gap between the occurrence of stationary and non-stationary populations within the SPI framework that we refer to as Oscillatory SPI and the Amplitude of SPI.

Keywords  Stationary population identity · Oscillatory properties · Functional knots · PDEs

Mathematics Subject Classification 92D25 · 60H35
Stationary population identity (SPI) is about equality of two quantities: One is obtained from the age distribution of a stationary population, and the other is obtained from the remaining years to live (or remaining time to live) of these individuals. This equality which is closely associated with the concept of the life table (a mathematical model to represent age-specific mortality in a population) can be expressed in several other ways. Let $X$ be the set of elements representing the proportions of populations at each age of a stationary population at time $t$ and let $Y$ be the set of elements representing the remaining number of years (or remaining time units) left to live at each age, and then, SPI holds imply,

$$X = Y.$$  \hspace{1cm} (1)

In a strict sense, $X$ consists of distinct elements and $Y$ consists of distinct elements. Let us take an element in $X$, say $p$. Then, there exists an age in the stationary population at which the proportion of the population to the total population is $p$. If Eq. (1) is true, then that guarantees that one of the elements of $Y$ is also $p$.

Equation (1) is true in population life tables which are stationary in nature.

Introduced to the demography literature by Brouard (1986, 1989) using French life tables and to the population biology literature by Muller, Carey, and their colleagues (Muller et al. 2004; Carey et al. 2012a, 2018) using survival patterns of captive cohorts of insects, stationary population identity (SPI) is expressed as $f_1(a) = f_2(a)$, where $f_1(a)$ the fraction of individuals who are captured at age “$a$” (out of total population) is equal to $f_2(a)$ the proportion of individuals who have a remaining time units left to die (see Fig. 1). Although SPI is observed in populations that are stationary (replacement-level growth), the vast majority of populations for both humans and non-human species are both non-stationary and non-stable (changing growth rate and/or age structure). All the relevant definitions used in this paper are provided in Table 1.

**Example 1** Consider individuals between the ages of 70–90 years as a stationary sub-population of the larger stationary population depicted in Fig. 1. According to the stationary population identity (SPI), if 14.8% of the population are in this 70–90 years old subpopulation, then there also exists another subpopulation of the same number (and percentage) of individuals who have between 70 and 90 years remaining (diagonal shaded area from 0 to 30 years in Fig. 1).

In this article, we prove several new theoretical aspects of stationary and non-stationary populations while understanding the implications of SPI. Three prominent of them are listed as follows:

(i) Populations consist of both stationary and non-stationary components (Theorem 2),

(ii) Stationary subpopulations of a total populations also possess stationary components (Theorem 3),

(iii) Population that is transiently stationary over a finite or an infinite interval can be joined with non-stationary populations (oscillatory property) (Theorems 6, 7).
Fig. 1 Illustration of the stationary population identity in which life lived equals LR. Graphic is based on the US 2006 female life table in which 14.8% of the life table population falls between 70 and 90 years of age (i.e., life lived). This percentage is identical to the percentage of individuals in this same hypothetical stationary population who have from 70 to 90 years remaining (i.e., life left).

Table 1 Key definitions used in the paper

| Item                                      | Definition                                                                                                                                 |
|-------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------|
| SPI (stationary population identity)      | Two sets $X$ and $Y$ such that $X = Y$, where $X$ is the set of elements representing the proportions of populations at each age of a stationary population at time $t$ and $Y$ is the set of elements representing the remaining number of years (or remaining time units) left to live at each age. SPI does not hold when $X \neq Y$ in the above definition |
| Knot                                      | Joining of two simultaneous subintervals of time within $[t_0, t_\omega)$, where in one subinterval SPI holds, and other subinterval SPI does not hold |
| Predominantly a stationary population     | If a population during a time interval $[t_0, t_\omega)$ satisfies the inequality $[t_0, t_1) \cup \bigcup_{M,i=1}^{\infty} [\delta_i, t_{i+1}) > \bigcup_{N,i=1}^{\infty} [t_i, \delta_i)$ |
| Oscillatory SPI                           | SPI is oscillatory on the set $M$ with uniform amplitude if $\{t_1 = \frac{t_0 + \delta_1}{2}, t_2 = \frac{\delta_1 + \delta_2}{2}, t_3 = \frac{\delta_2 + \delta_3}{2}, \ldots\}$ |

Discovery of the SPI by Carey, originally referred to as Carey’s Equality (Vaupel 2009; Carey 2019) but now referred to as SPI after the revelation that Brouard’s earlier papers also documented this identity (see Carey et al. 2018), was an outcome of a 10-year, US National Institute on Aging-funded research program directed by...
biodemographer James R. Carey designed to study aging in the wild. This led to the identification of the relationship between population age structure and post-capture life spans of individuals through the use of a simple four-age class life table (Table 1 in Müller et al. 2004), and the results of which were formulated mathematically for cases involving both stationary (i.e., SPI) and the non-stationary (with reference life tables) populations (see subsections on pp. 126–128 in Müller et al. 2004). Because of the importance to basic ecology and particularly to medical entomology where the older arthropod vectors (e.g., mosquitoes) have the highest likelihood of disease transmission, a great deal of effort has been invested in developing various technologies for estimating the age of individual insects including physiological Detinova (1968), biochemical Gerade et al. (2004), and genetic (Cook et al. 2006, 2008; Carey et al. 2012a; Anonymous 2014) methods.

The analytical evolution of this SPI continued with the publication of its proof, first as a demographic relationship between the life lived (LL) and life remaining (LR) (Vaupel 2009) and then as a theorem and generalization (Rao and Carey 2015). A major mathematical breakthrough came in the stationary population literature, when Rao and Carey (2015) stated a new theorem using original ideas of stationary population principles (Carey–Rao Theorem on stationary population identity) through constructing arguments based on graphs and set-theoretic principles and based on two criteria that they stated on ‘LL’ and ‘LR’. The general concepts of the life table identify and its extension as an applied model (i.e., integration of reference life table information) have been used to estimate age structure and thus to gain insights into population aging in wild populations including effects of truncation studies (Rao and Carey 2019), of fruit flies (Carey et al. 2008, 2012), butterflies (Molleman et al. 2007), and mosquitoes (Papadopoulos et al. 2016). In light of the theoretical and analytical properties of SPI and its use as a foundation for developing models for estimating age structure in real-world insect populations, we believe that continuing to explore the mathematical properties of this identity has the potential to make new and original contributions to the demographic literature. Thus, for the non-stationary and non-life table populations, the role of SPI needs thorough investigation.

2 Stationary and Non-stationary Populations

While exploring the deeper insights of SPI, we realized that this property can be helpful in knotting (read joining together) the concepts of stationary and non-stationary populations such that these two populations are formed on mutually exclusive time intervals. A knot here we mean joining of two simultaneous subintervals of time within $[t_0, t_\omega)$, where in one subinterval SPI holds, and other subinterval SPI does not hold. The main advantages of such a theoretical visualization of side-by-side occurrence of stationary and non-stationary populations are to keep our framework of SPI as flexible as possible such that realistic population dynamics are captured with respect to deviation from stationarity. Mathematically, these mutually exclusive concepts allow us to cut with knots the continuous interval on which we study simultaneous occurrences of these two types of populations. Our constructions in this article show that SPI property generates these knots on the continuous interval. Demarcation lines on
an interval between stationary and non-stationary populations can then be visualized as dynamic. These demarcations (or boundary) lines led us to a novel concept within the SPI which we term Oscillatory SPI (O-SPI). In this case, the knots indicate the beginning of either stationary or non-stationary populations and allow us to introduce another term that we refer to on a continuous interval as the amplitude of the SPI.

For a predominantly stationary population (see Table 1 and Definition 4) during an interval \([t_0, t_\omega)\), we can imagine that there exists a disjoint covering of intervals (a sub-collection of intervals, say \(M\), in which SPI is true and other sub-collection of intervals, say, \(N\), in which SPI is not true), such that,

\[
\left( \bigcup_{C \in M} C \right) \cup \left( \bigcup_{C' \in N} C' \right)
\]
equals \([t_0, t_\omega)\). The interval \([t_0, t_\omega)\) is visualized as the union of two partitions, one which form SPI and other does not. See Davis (2005) and Kelley (1975) for concepts related to disjoint covering. The partition which form the identity is associated with stationary population and other one is associated non-stationary populations. If “−” indicates the minus symbol, then the SPI is true in \([t_0, t_\omega)\) − \(\bigcup_{C' \in N} C'\) and not true in \([t_0, t_\omega)\) − \(\bigcup_{C \in M} C\). The value \([t_0, t_\omega)\) − \(\bigcup_{C' \in N} C'\) indicates the interval \([t_0, t_\omega)\) minus the intervals \(\bigcup_{C' \in N} C', i.e., if an element x belongs \([t_0, t_\omega)\) − \(\bigcup_{C' \in N} C'\), then x belongs to \([t_0, t_\omega)\) but x does not belong to the union of intervals \(\bigcup_{C' \in N} C'\). Similarly, the meaning of \([t_0, t_\omega)\) − \(\bigcup_{C \in M} C\) can be interpreted. We develop an idea which we call uniform amplitude of SPI when equality such as (2) is true

\[
[t_0, t_\omega) − \bigcup_{C \in M} C = [t_0, t_\omega) − \bigcup_{C' \in N} C', \quad (2)
\]

and together \(C = C'\) holds for each simultaneous \(C \in M\) and \(C' \in N\). However, we develop these ideas on finite sets. Later, we will see that the set \(T\) in (3)

\[
T = \{[t_0, t_1), [t_1, \delta_1), [\delta_1, t_2), \ldots [t_k, \delta_k-1), [\delta_k, t_{k+1}]\}
\]

is a partition of \([t_0, t_{k+1}]\), where \([t_0, t_{k+1}] \subset [t_0, t_\omega)\), such that each element of \([t_0, t_{k+1}]\) lies in exactly one interval in (3). We will also see in “Appendix” that the set \(\{I, J\}\) for the two intervals \(I, J \subset [t_0, t_{k+1}]\) as a partition of \([t_0, t_{k+1}]\). Inasmuch as SPI connects these two properties in stationary populations, it follows that connecting them in non-stationary populations is a logical next step.

Let \(\Omega\) be the size of the captive cohort such that \(\Omega\) is an infinite subset or a very large finite subset of nonnegative integers. Let \(c_i\) be the age at capture and \(d_i\) be the age at death of \(i\)th individual, where \(d_i > c_i\) for each \(i \in \Omega\). Here, \(d_i − c_i\) is the follow-up length or post-capture LL by \(i\)th individual.

**Theorem 2.** If a population is stationary, then the SPI holds, but when \(f_1(a) = f_2(a)\) does not hold for every age “a” in a population, then that population could be partitioned into stationary and non-stationary components.
Proof Idea: To prove the first part, we need to prove that if the population from which the captive cohort drawn is stationary, then that follows SPI. For the second part, we first assume that $f_1(a) = f_2(a)$ is not true for every age “$a$”, and then, we try to prove that the captive cohort $\Omega$ formed could be partitioned into stationary population and non-stationary population components.

We assume a very large number of individuals are captured at all possible ages (need not be integer valued) and no two individuals have same age at capture. We also assume that: (1) there will be a distinct value of duration of LR (i.e., remaining life to be lived after capture) corresponding to each captured individual; and (2) one of the values of the remaining LR is identical to exactly one of the values of the age at capture. Let $\mu(\vec{c})$ and $\mu(\vec{d} - \vec{c})$ be the average age at capture and average age of remaining length of post-capture life for the individuals in $J$, respectively, and $c_1 \neq c_2 \neq \cdots \neq c_k \neq \cdots$ and $d_1 - c_1 \neq d_2 - c_2 \neq \cdots \neq d_k - c_k \neq \cdots$, and then, we have

$$\mu(\vec{c}) = \frac{\sum_i c_i}{|\Omega|} \quad \text{and} \quad \mu(\vec{d} - \vec{c}) = \frac{\sum_i (d_i - c_i)}{|\Omega|}. \quad (4)$$

Suppose $S = \{s_1, s_2, \ldots\}$, where $s_i = d_i - c_i$ for all $i \in \Omega$. We can arrange elements of the set $S$ in a decreasing order. To do this, we set $s'_1 = \max \{s_1, s_2, \ldots\}$. Let $S_1 = S - \{s'_1\}$, where $S_1$ is the set of elements in $S$ after $s'_1$ is removed. Let $s'_2 = \max \{S_1\}$. We can continue to obtain maximum values, such that $S_{i+1} = S_i - \{s'_{i+1}\}$, where $s'_{i+1} = \max \{S_i\}$ for $i = 1, 2, \ldots$. Let $T = \{(1, s'_1), (2, s'_2), \ldots\}$. The graph drawn through the coordinates of $T$ is a decreasing function. This kind of constructions for the information of LR after capture was originally used in Rao and Carey (2015).

When $s'_i$ is equal to the corresponding individual’s age at capture for all $i \in J$, then the distribution of captured age is equal to the distribution of duration of the LR after capture. When $s'_i$ is not equal to the corresponding individual’s age at capture for all $j \in \Omega_1$ for $\Omega_2 \subset \Omega$, then $s'_i$ is equal to the corresponding individual’s age at capture for all $i \in \Omega$ and $i \notin \Omega_1$. Then, with a finite permutations of rearrangement of the elements in $\Omega_1$, we can match the set, $T' = \{s'_1, s'_2, \ldots\}$ with $C$, the set of decreasing values of captured ages, such that $T' = C$. With this construction explained, for an individual captured at the age “$a$” in $C$ (i.e., $a$ is an element in $C$) the value of the (element in $T'$) is exactly “$a$” which is the remaining LR.

Suppose there are one or more than one individual of the same age at the time of capture. $\Omega$ is now sum of partitions of individuals, where each partition represents number of individuals who are captured at the same age. Let $c_p^q$ be the $q$th individual captured aged $p$, and $s_p^q$ be the remaining LR for the $q$th individual who was captured at age $p$ for $p > 0$ and $q = 1, 2, \ldots, n_p$ ($n_p \in \mathbb{Z}^+$). We assume that for each of the $s_p^q$, there is a corresponding value $c_p^q$ which could be within the same age $p$ or in other captured age. That is, if

$$C = \{c_p^q : c_p^q, p, \in \mathbb{R}^+, q \in \mathbb{Z}^+ \text{ and } 1 \leq q \leq n_p\}$$

and

$$S = \{s_p^q : s_p^q, p \in \mathbb{R}^+, q \in \mathbb{Z}^+ \text{ and } 1 \leq q \leq n_p\},$$

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then for each \( y \in S \) there is a corresponding element \( x \in C \). The following property is assured:

\[
\int_0^\infty \left( \sum_{q=0}^n p^q \right) \, dp = \int_0^\infty \left( \sum_{q=0}^n q^p \right) \, dp.
\]

For more details on the type of logic and arguments provided above, see the Carey–Rao Theorem and proof (Rao and Carey 2015), which introduced these set of arguments. Conversely, suppose for a finite population, let \( f_1(a) = f_2(a) \) for the ages \( a_1, a_2, \ldots, a_k \) (without any order) and \( f_1(a) \neq f_2(a) \) for ages \( a_{k+1}, a_{k+2}, \ldots, a_n \) (without any order) and no two individuals are of same age. This implies, there will be two vectors \( V_1 \) and \( V_2 \) based on the rule that SPI is true or not, which are given by,

\[
V_1 = \begin{bmatrix}
 f_1(a_1) = f_2(a_1) \\
 f_1(a_2) = f_2(a_2) \\
 \vdots \\
 f_1(a_k) = f_2(a_k)
\end{bmatrix}
\quad \text{and} \quad
V_2 = \begin{bmatrix}
 f_1(a_{k+1}) \neq f_2(a_{k+1}) \\
 f_1(a_{k+2}) \neq f_2(a_{k+2}) \\
 \vdots \\
 f_1(a_n) \neq f_2(a_n)
\end{bmatrix}.
\]

The subpopulation corresponding to \( V_1 \) forms stationary population and the subpopulation corresponding to \( V_2 \) forms non-stationary population. Due to \( V_2 \), the average value of the LL by \( a_1, a_2, \ldots, a_n \) is not equal to their average remaining value, which will lead population to be non-stationary.

**Theorem 3** Suppose SPI holds for a population, then (a) SPI also holds for all stationary subpopulations of the original population and (b) SPI need not hold for all non-stationary subpopulations of the original population.

**Proof** (a) Let \( P \) be a stationary population and let \( P_1, P_2, \ldots, P_k \) are disjoint stationary subpopulations and \( Q_1, Q_2, \ldots, Q_l \) are disjoint non-stationary subpopulations of \( P \) such that

\[
\bigcup_{i=1}^k P_i \cup \bigcup_{j=1}^l Q_j = P.
\]

If \( P_i \) for each \( i \) is a stationary population, then SPI holds within the each \( P_i \).

(b) Suppose we partition population into a disjoint collection of stationary and non-stationary subpopulations as in (a), then SPI need not hold in an arbitrarily chosen \( Q_j \), because, for an arbitrarily chosen \( Q_j \) the intrinsic growth rates could be very high and the population could be younger such that the proportion of population at age \( a \) years not equal to the proportion of population who have \( a \) years remaining.

3 Oscillatory Stationary Population Identity and Its Amplitude

Suppose that the population remains stationary for an infinitely long period of time, except in shorter time intervals in between due to perturbations. After small perturbations in the population due to vital events, population deviates temporarily to a non-stationary state for a brief time period before restoring stationary properties. When
the population is stationary, then we know that SPI holds (see, for example, Rao and Carey 2015). SPI holds here we mean that it is true for all ages, i.e., the proportion of the population who are at age \( a \) units is same as the proportion of the population who have \( a \) units remaining for each age \( a \). Suppose the population remains stationary during the interval \([0, t]\) and let there be a vital event during the interval \((t, t + \delta)\) for a positive \( \delta \) which is very very small. Then, during \((t, t + \delta)\) SPI (in a strict sense) is not true and SPI remains not true until population remains strictly non-stationary (say until \( \delta_1 \) for \( \delta_1 > \delta \)). As soon as stationarity is restored, SPI will be true again until the next vital event. There will be finite or infinite cycles of stationary to non-stationary to stationary populations, and hence, SPI is true intermittently.

**Definition 4** We define a population as a predominantly a stationary population if

\[
[t_0, t_1) \cup \bigcup_{i=1}^{\infty} [\delta_i, t_{i+1}) > \bigcup_{i=1}^{\infty} [t_i, \delta_i),
\]

where SPI holds for the disjoint collection of intervals \( M = \{[t_0, t_1), [\delta_1, t_2), [\delta_2, t_3), \ldots \} \), and SPI does not hold for the disjoint collection of intervals \( N = \{[t_1, \delta_1), [t_2, \delta_2), [t_3, \delta_3), \ldots \} \). The L.H.S. of (7) is the union over \( i \) for the time intervals corresponding to the set \( M \) and the R.H.S. of (7) is the union over \( i \) values for the time intervals corresponding to the set \( N \). If

\[
[t_0, t_1) \cup \bigcup_{i=1}^{\infty} [\delta_i, t_{i+1}) < \bigcup_{i=1}^{\infty} [t_i, \delta_i),
\]

then we define a population as predominantly a non-stationary. A population is neither predominantly a stationary population nor predominantly a non-stationary if

\[
[t_0, t_1) \cup \bigcup_{i=1}^{\infty} [\delta_i, t_{i+1}) = \bigcup_{i=1}^{\infty} [t_i, \delta_i).
\]

We define oscillatory property of SPI as follows:

**Definition 5** We define a criteria that the SPI is oscillatory on \( M \) with uniform amplitude whenever the following statement is true:

\[
\left\{ t_1 = \frac{t_0 + \delta_1}{2}, t_2 = \frac{\delta_1 + \delta_2}{2}, t_3 = \frac{\delta_2 + \delta_3}{2}, \ldots \right\}.
\]

**Theorem 6** For a predominantly stationary population defined in Definition 4, SPI exists except for shorter intermittent intervals when population is non-stationary.

**Proof** Let the population be stationary during \([t_0, t_1)\), and a small perturbation (vital event(s)) takes place at \( t_1 \) such that the population deviates from stationary properties. Suppose there is a vital event(s) (at the time \( \delta_1 \) for some \( \delta_1 > t_1 \)), which balances
deviated stationary population back to stationary mode. Suppose at time \( t_2 \) for \( t_2 > \delta_1 \), the population again deviates from stationary mode due to vital event(s) and gets restored at time \( \delta_2 \) for \( \delta_2 > t_2 \) such that the population remains non-stationary in the interval \([t_2, \delta_2)\). Suppose this cycle of stationary population to non-stationary and back to stationary population continues to repeat at different time points \( t \) and \( \delta \). SPI holds for the disjoint collection of intervals, \( M \), and does not hold for the disjoint collection of intervals, \( N \). Because the population is predominantly stationary, the following inequality holds

\[
\frac{[t_0, t_1) \cup \bigcup_{i=1}^{\infty} [\delta_i, t_{i+1})}{\bigcup_{i=1}^{\infty} [t_i, \delta_i)} > 1. \tag{11}
\]

Let us denote \( S_0 = \frac{[t_0, t_1) \cup \bigcup_{i=1}^{\infty} [\delta_i, t_{i+1})}{\bigcup_{i=1}^{\infty} [t_i, \delta_i)} \).

Whenever \( \left[ \sum_{i=1}^{\infty} (\delta_i - t_i) \right] \) for \( \delta_i, t_i \in N \) holds for the disjoint collection of intervals, \( M \), and does not hold for the disjoint collection of intervals, \( N \). Because the population is predominantly stationary, the following inequality holds

\[
\frac{[t_0, t_1) \cup \bigcup_{i=1}^{\infty} [\delta_i, t_{i+1})}{\bigcup_{i=1}^{\infty} [t_i, \delta_i)} > 1.
\]

We call this property of holding and not holding SPI over disjoint intervals constructed in the proof of Theorem 6 as the O-SPI. We associate the idea of amplitude with the length of time when SPI holds. The amplitudes of SPI are defined here as the lengths of the intervals of the set \( M \).

**Theorem 7** Given a finite time set-up of disjoint intervals of \( M \) and \( N \) up to \([\delta_k, t_{k+1}) \in M \). If SPI is oscillatory on \( M \) with uniform amplitude, then \( S_0 = 1 \) but converse need not be true.

**Proof** When SPI is oscillatory on \( M \) with uniform amplitude, then the statement (10) is true. Hence, we can see that

\[
\left[ \sum_{i=1}^{\infty} (\delta_i - t_i) \right] \text{ for } \delta_i, t_i \in N \]

which implies \( S_0 = 1 \).

Conversely, suppose \( S_0 = 1 \). Let us consider events up to time \( t_{k+1} \) in the interval \([t_0, \infty)\). Let \( M_{k+1} \) and \( N_{k+1} \) be the sub-collection of intervals of \( M \) and \( N \), respectively, and are given by,

\[
M_{k+1} = \{[t_0, t_1), [\delta_1, t_2), [\delta_2, t_3), \ldots, [\delta_k, t_{k+1})\},
N_{k+1} = \{[t_1, \delta_1), [t_2, \delta_2), \ldots, [t_k, \delta_k)\}. \tag{12}
\]

For \( k = 1 \), we have,

\[
M_2 = \{[t_0, t_1), [\delta_1, t_2)\},
N_2 = \{[t_1, \delta_1)\}. \tag{13}
\]

Since \( S_0 = 1 \), from (13), we will have \( t_1 = \frac{t_0 + 2\delta_1 - t_2}{2} \). Since \( t_2 > \delta_1 \), we have

\[
\frac{t_0 + 2\delta_1 - t_2}{2} \neq \frac{t_0 + \delta_0}{2}. \tag{14}
\]
The inequality (14) indicates there is no uniform amplitude.

Corollary 8 For a predominantly non-stationary population, SPI may hold even in small intermittent intervals.

4 Connection Between Main Theorems, Graphical Results, and Applications

Several new properties and implications of SPI were proved in this article through various theorems (Theorems 2, 3, 6, 7, 10, 11) and Lemma 9. These theorems which take implications of SPI to different directions (for example, Theorems 2, 3, 6, 7) show newer avenues of the interface of SPI between stationary and non-stationary populations and O-SPI. Constructions of captive cohorts and logic of matching the duration of LL and LR developed in Carey–Rao Theorem on SPI (Rao and Carey 2015) helped us to prove arguments in Theorem 2. This theorem implies that when the fraction of the population at age \( a \) is not equal to the fraction of the population whose remaining years to live is \( a \) for some age \( a \), then the population could be either stationary or non-stationary. For a stationary population shown in Fig. 1, these fractions are equal at all ages \( a \) or at all age groups if group-wise fractions are considered. When these fractions are not equal for each age \( a \), then the population is non-stationary. Therefore, Theorem 2 helped us investigate properties of the SPI that interface stationary and non-stationary populations.

An example of the relationship between the percentage of a population that have lived \( x \) years and the percentage of persons who have \( x \) years remaining in a stationary population is visualized in Fig. 1 for a hypothetical population based on the 2006 US female life table. This graphic shows that there are 14.8% of this stationary population that are aged 70–90 years old and, at the same time, there are 14.8% of the population who have from 70 to 90 years remaining. These percentages for remaining years are distributed among the younger individuals from newborn \((x = 0)\), most of whom have from 70 to 90 years remaining to a miniscule percentage of 40 years old who will live another 70 years (i.e., to super-centenarians, age 110). Theorem 3 is the first step toward specifying the behavior of SPI on partitions of stationary and non-stationary subpopulations of the total population. This implies, when partitioning of the total population is done into a collection of stationary subpopulations, then the aforesaid fractions remain equal in each of the subpopulation. Theorem 3 also implies that when a population is partitioned into subpopulations, these fractions need not be equal if stationary principles are not preserved (see Fig. 2).

The relationship of the fraction of individuals in a population that have lived \( x \) years (i.e., the age distribution) relative to the fraction of individuals in the same population that have \( x \) years remaining is shown in Fig. 2 for two hypothetical populations, one with a negative \((r = -0.01)\) growth rate (Fig. 2a) and another with a positive \((r = 0.01)\) growth rate (Fig. 2b). Each is shown relative to the stationary \((r = 0.00)\) case. Several aspects of this figure merit comment. First, note the equivalency of the fraction of the population that have lived \( x \) years and the fraction that have \( x \) years to live as shown in the curves for the stationary population. In other words, the life
Fig. 2 Relationship of the fraction of individuals who have lived $x$ years relative to the fraction who have $x$ years left to live for populations with either negative (a) or positive (b) growth rates. For reference, the gray curves show the equivalency of LL and left for stationary populations (zero growth). Lived/life left curves are superimposed. In contrast the trajectories for the life lived and the life left curves for populations with either negative or positive growth rates are separate as shown by the departure of the dashed and solid black lines in each graph. Note in the top graph for a population with the negative growth rate the fraction of the population that are young (e.g., 0–20 years) is low relative to the fraction that are old (e.g., 60–80 years). In other words, the age structure of decreasing populations is skewed to the older age classes. However, because population is older, the fraction of individuals with fewer number of years to live (e.g., < 20) is higher relative to the fraction of persons who have many years left to live (e.g., > 60 more years remaining). The exact opposite relationship of LL to LR is evident in a population a positive growth rate as shown in the bottom graph of Fig. 2. That is, the skew toward the young in a growing population results in a skew toward the fraction of individual (i.e., the young) who have many years remaining.

Through Theorem 6, we have shown for a stationary population how SPI could hold in alternate time intervals. Within the construction of oscillatory properties, we have introduced amplitudes of SPI which provides the lengths of time intervals for which the SPI holds. We have introduced the idea of O-SPI which over the time will have practical applications in understanding population dynamics through switching of stationary and non-stationary populations (see both Fig. 3 and next paragraph). The concept of transient stationarity is visualized in Fig. 3 for a population converging from a positive growth rate to a fixed (replacement level) stationary state. This figure shows the change in growth rate, $\lambda$, in the main graphic and the age structure of the population that corresponds to three different points (A, B, and C) along this growth trajectory. The age structure (inset) at $t = 0$ corresponds to a growing population with a bimodal distribution, one mode from birth to age 25 and the other from 25 to 50 years. This corresponds to point A in the main graphic (rapid growth rate). At around $t = 40$, the population growth rate, $\lambda$, had decreased to zero (Point B). However, this was a transient condition because the age structure (shown in the inset) was not yet stable.
Fig. 3 Trajectory of growth rate and age structure in a hypothetical population starting from a positive rate and converging to zero growth (stationarity). The initial age structure was based on the US population in 2000, the fertility rates on a standard age-specific fertility schedule in humans scaled to replacement level (i.e., 1.0) and survivorship on the female rates in 2006 (Berkeley Human Mortality Database). \( \frac{N(t+1)}{N(t)} \) is the ratio of number in the population at time \( t+1 \) to the number at time \( t \) and is denoted \( \lambda \). Frequency in the inset refers to the age distribution of the population. Points labeled A, B, and C correspond to the starting growth rates, the point at which growth rate first reaches replacement level (i.e., transient stationarity), and the point at which growth rate is constant at zero (i.e., fixed stationarity), respectively. Replacement levels of growth required approximately 40 years from the start (i.e., A-to-B) and another 60 years to become fixed (i.e., B-to-C). Note the small oscillations around stationarity after B as the age structure converges to C (Color figure online).

This “transiency” in growth rate and age structure continued until both had converged to fixed stationarity, a state corresponding to point C in the main graphic showing changes in \( \lambda \) and in the inset showing the age distribution. Connections between the properties of O-SPI and countries or populations with net reproduction rates (NRR) around the value one (1.0) can be investigated using the properties proved in this article. When the intrinsic growth rates are highly dynamic in the populations, then achieving the net reproductive rates around the value one may not stay for a longer period, and the duration of the time for which the status of “NRR = 1.0” in the population might be very short-lived. Implications of “NRR = 1.0” and properties of O-SPI across several populations can be studied to understand long-range population dynamics.

Since every human population has an underlying life table, every human population can form the basis of a model stationary population Preston et al. (2001). Therefore, it follows that understanding the deeper properties of stationary populations as described here and elsewhere (Ryder 1973; Müller et al. 2004; Rao and Carey 2015) will add important depth to population theory more generally. Second, understanding the oscillatory behavior of populations as they approach stationarity is important inasmuch as
this behavior is tightly linked to the concept of population momentum—the continuation of growth after a population has achieved replacement-level fertility (Keyfitz 1971; Schoen and Jonsson 2003; Rao 2014). Momentum and population aging are essentially two aspects of the same phenomenon Kim et al. (1991), and momentum is likely to be responsible for most of the future growth in the world’s population (Bongaarts and Bulatao 1999; Cohen 1995). Therefore, a deeper understanding of the underlying dynamics of population stationarity, momentum, and convergence and concepts concerned with the demographic transition will strengthen the foundations for the development of sound population policy including family planning, aging, and social security.

5 Discussion

The number of years different individuals have lived in a population, as well as the number of years these individuals have left, is universal properties of all populations. Whereas the first is a static characteristic of populations inasmuch as it specifies age structure, the second is a dynamic concept since it designates the future population’s actuarial properties. This second property is more complex than the first inasmuch as it describes distributions within a distribution, i.e., the allocation of individual deaths within each of the 100+ age groups of the age distribution. Both of these population characteristics are important in both basic and applied demographic contexts. The first property is concerned with the relationship of different population age groups (e.g., dependency ratios; population aging), and the second is concerned with future deaths (e.g., how many deaths will occur in the next 1, 2, or 5 years). Since the age structure of a population must logically be connected to its future death distribution, the implicit qualitative relationship between life years lived and life years is both obvious and intuitive. However, the explicit quantitative relationship between life lived and left was neither obvious nor intuitive prior to the discovery of the SPI. Because of the importance of linking the actuarial properties of populations with their age structure as SPI does, it follows that exploring this identity in greater mathematical depth has the potential to provide important new insights into these linkages in two mathematical contexts. The first is within stationary populations as we did with the three main properties (Theorems 2–6), and the second context is between stationary and non-stationary populations as we did with what we refer to as O-SPI. We still feel the beauty of SPI in population dynamics is under explored, and the results presented here can be seen as a step toward a larger goal of understanding non-stationary populations through such lens.

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Appendix

Let,

\[ I = [t_0, t_1] \bigcup \bigcup_{M,i=1}^{k} [\delta_i, t_{i+1}), \]

\[ J = \bigcup_{N,i=1}^{k} [t_i, \delta_i). \]

and let \( \hat{I} \) and \( \hat{J} \) be the partitions of \( I \) and \( J \), which are written as,

\[ \hat{I} = \{(I_M(t_i)) : i = 1, 2, \ldots, k+1 \} = \{(I_M(t_i))\}_{i=1}^{k} \]

and

\[ \hat{J} = \{(J_N(t_i)) : i = 1, 2, \ldots, k \} = \{(J_N(t_i))\}_{i=1}^{k}, \]

where \( I_M(t_1) = [t_0, t_1) \), \( I_M(t_i) = [\delta_i, t_{i+1}) \) for \( i = 2, \ldots, k \) and \( J_N(t_i) = [t_i, \delta_i) \) for \( i = 1, 2, \ldots, k \). Since \( I_M(t_i) \) and \( J_N(t_i) \) are non-degenerate intervals, the lengths of \( I_M(t_i) \) and \( J_N(t_i) \) are always positive. Hence, \( \max I_M(t_i) \), \( \min I_M(t_i) \), \( \max J_N(t_i) \), and \( \min J_N(t_i) \) exist. Let \( f(a, t_i) \) be the function specifying the proportion of individuals at age \( a \in A \) during \( I_M(t_i) \) for \( f(a, t_i) : I_M(t_i) \to \mathbb{R}^+ \) and \( A \) be the set of all ages in the population. Since SPI holds in \( I_M(t_i) \), we have

\[ \text{Prob}[f(a, t_i) = g(a, t_i) \forall a, t_i] = 1 \text{ if } f(a, t_i) : I_M(t_i) \to \mathbb{R}^+ = 0 \text{ otherwise,} \]

(15)

where \( g(a, t_i) \) is the function specifying remaining LR at age \( a \) during \( I_M(t_i) \).

Lemma 9 Suppose \( \triangle f(a, t_i) = \hat{f}(a, t_i) - \tilde{f}(a, t_i) \) for \( i = 1, 2, 3, \ldots, k \), where \( \hat{f}(a, t_i) = \max_a f(a, t_i) \) and \( \tilde{f}(a, t_i) = \min_a f(a, t_i) \), then \( \triangle f(a, t_i) \) is bounded for each \( I_M(t_i) \).

Proof If there are at least two age groups in \( A \), then \( \hat{f}(a, t_i) \) and \( \tilde{f}(a, t_i) \) exists within \( I_M(t_i) \) and they are distinct. Suppose there are only two age groups in \( A \), then (15) guarantees that there exist \( \hat{g}(a, t_i) \) and \( \tilde{g}(a, t_i) \) for \( \hat{g}(a, t_i) = \max_a g(a, t_i) \) and \( \tilde{g}(a, t_i) = \min_a g(a, t_i) \). This implies, \( \triangle f(a, t_i) < \hat{g}(a, t_i) + \tilde{g}(a, t_i) \). This inequality follows even if there are more than two age groups in \( A \), and hence, \( \triangle f(a, t_i) \) is bounded.

Theorem 10 \( \frac{1}{\Sigma \triangle f(a, t_i)} \) is bounded on \([t_0, t_{k+1})\).

Proof Since \( \triangle f(a, t_i) > 0 \) and \( \triangle f(a, t_i) \) are bounded on \( I_M(t_i) \) by Lemma (9), the result follows.

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Suppose \( \hat{f}(a, t_i) \) is concentrated around mean age of the population and \( \hat{f}(a, t_i) \) is concentrated around the very old age of the population, then \( \Delta f(a, t_i) \) is an increasing function indicates one or more of the following three situations; i) longevity of the population is increasing without much change in the mean age, ii) mean age is reducing in longevity, and iii) mean age is reducing and simultaneously longevity is increasing.

**Theorem 11** Suppose the partitions \( \hat{i} \) and \( \hat{j} \) are given, then 
\[
1 + \frac{k^2}{f(a, t_i) + f(a, t_j)} > k \left( \frac{1}{f(a, t_i)} + \frac{1}{f(a, t_j)} \right).
\]

**Proof** Consider the expression
\[
\frac{1}{f(a, t_i)} \left( \hat{f}(a, t_i) - \sum_{i=1}^{\infty} \int_0^\infty f(a, t_i) da \right) \left( \hat{f}(a, t_i) - \sum_{i=1}^{\infty} \int_0^\infty f(a, t_i) da \right).
\]

Since \( \sum_{i=1}^{\infty} \int_0^\infty f(a, t_i) da = k \) and both the terms of the expression (16) are negative, (16) can be written as
\[
\left( \hat{f}(a, t_i) - k \right) \left( \hat{f}(a, t_i) - k \right) > 0.
\]

Simplifying (17), we will obtain the desired result. \( \square \)

**Remark 12** For each \( t_i \) for \( i = 1, 2, \ldots, k \), without taking the summations in (16), we have
\[
\left( \hat{f}(a, t_i) - f(a, t_i) \right) \left( \hat{f}(a, t_i) - f(a, t_i) \right) = 0 \quad \text{if} \quad \hat{f}(a, t_i) = f(a, t_i) \text{ or } \hat{f}(a, t_i) = f(a, t_i)
\]
\[
< 0 \quad \text{if} \quad \hat{f}(a, t_i) < f(a, t_i) < \hat{f}(a, t_i)
\]

and
\[
\left( \hat{f}(a, t_i) - \int_0^\infty f(a, t_i) da \right) \left( \hat{f}(a, t_i) - \int_0^\infty f(a, t_i) da \right) > 0.
\]

Let \( \varphi(a, t_i) \) be the function specifying the proportion of individuals at age \( a \in B \) during \( J_N(t_i) \) for \( \varphi(a, t_i) : J_N(t_i) \rightarrow \mathbb{R}^+ \) and \( B \) be the set of all ages in the population when SPI does not hold. Suppose \( \hat{\varphi}(a, t_i) = \max_a \varphi(a, t_i) \) and \( \check{\varphi}(a, t_i) = \min_a \varphi(a, t_i) \). We note that equivalent versions of Theorem 11 and Remark 12 for the age functions \( \varphi, \hat{\varphi}(a, t_i), \check{\varphi}(a, t_i) \) still hold. Under the continuous transition of decreasing population sizes over the interval \( [0, t_{k+1}) \), let us assume \( \hat{f}(a, t_1) > \hat{f}(a, t_2) > \cdots > \hat{f}(a, t_{k+1}) \) and \( \check{\varphi}(a, t_1) > \check{\varphi}(a, t_2) > \cdots > \check{\varphi}(a, t_k) \). This implies \( \hat{f}(a, t_1) > \check{\varphi}(a, t_1) > \cdots > \check{\varphi}(a, t_k) > \hat{f}(a, t_{k+1}) \). Also, \( \int_0^\infty f(a, t_i) da - \hat{f}(a, t_1) < \int_0^\infty \varphi(a, t_i) da - \check{\varphi}(a, t_1) < \cdots < \int_0^\infty \varphi(a, t_i) da - \check{\varphi}(a, t_k) < \int_0^\infty f(a, t_{k+1}) da - \)
\( \dot{f}(a, t_k+1) \), and this leads to \( 1 - \dot{f}(a, t_1) < 1 - \hat{\phi}(a, t_1) < \cdots < 1 - \hat{\phi}(a, t_k) < 1 - \dot{f}(a, t_{k+1}) \). We can model the dynamics of these maximum and minimum fractions over the time period using the following logistic growth models with certain limiting points of these fractions.

\[
\frac{df(a, t)}{dt} = r_1 \dot{f}(a, t) \left( 1 - \frac{\dot{f}(a, t)}{\dot{f}(a, t)_e} \right)
\]

\[
\frac{d\hat{\phi}(a, t)}{dt} = r_2 \hat{\phi}(a, t) \left( 1 - \frac{\hat{\phi}(a, t)}{\hat{\phi}(a, t)_e} \right)
\]

\[
\frac{d\dot{f}(a, t)}{dt} = r_3 \dot{f}(a, t) \left( 1 - \frac{\dot{f}(a, t)}{\dot{f}(a, t)_e} \right)
\]

\[
\frac{d\tilde{\phi}(a, t_i)}{dt} = r_4 \tilde{\phi}(a, t_i) \left( 1 - \frac{\tilde{\phi}(a, t_i)}{\tilde{\phi}(a, t_i)_e} \right),
\]

where \( r_1, r_2, r_3, \) and \( r_4 \) are rates of declines in maximum and minimum fractions and \( \left( \dot{f}(a, t)_e, \hat{\phi}(a, t)_e, \dot{f}(a, t)_e, \tilde{\phi}(a, t_i)_e \right) \) are limiting points of the fractions \( \dot{f}(a, t), \hat{\phi}(a, t), \dot{f}(a, t), \tilde{\phi}(a, t_i) \), respectively. Further, we provide partial differential equations models by treating \( \dot{f}(a, t), \hat{\phi}(a, t), \dot{f}(a, t), \tilde{\phi}(a, t_i) \) as continuous variables. First, we consider two pairs of variables \( \{ \dot{f}(a, t), \hat{\phi}(a, t) \}, \{ \dot{f}(a, t), \tilde{\phi}(a, t_i) \} \) and corresponding dependent variables \( u_1(t, \dot{f}(a, t), \hat{\phi}(a, t)), u_2(t, \dot{f}(a, t), \tilde{\phi}(a, t_i)) \) to build two models (22) and (23). These two models provide dynamics of simultaneous occurrences of stationary and non-stationary populations. If we want to follow dynamics of \( \dot{f} \) and \( \hat{\phi} \) on the time interval \( [t_0, t_\infty] \) by considering two pairs of independent variables \( \{ t, \dot{f}(a, t) \}, \{ t, \hat{\phi}(a, t) \} \) with corresponding dependent variables \( v_1(t, \dot{f}(a, t)), v_2(t, \hat{\phi}(a, t)) \), then the PDE models we considered are given in (24) and (25). Here, \( \tau_1 \) and \( \tau_2 \) are constants, which could indicate speed of the dynamics of peaks of the maximum fractions. Similarly, dynamics of \( \dot{f} \) and \( \hat{\phi} \) with dependent variables \( w_1(t, \dot{f}(a, t)) \) and \( w_2(t, \hat{\phi}(a, t)) \) are modeled as per equations given in (26) and (27), where \( \tau_3 \) and \( \tau_4 \) are constants indicate speed with which these variables move.

\[
\frac{\partial u_1}{\partial \dot{f}} = -\hat{\phi} \frac{\partial u_1}{\partial \hat{\phi}}
\]

\[
\frac{\partial u_2}{\partial \dot{f}} = -\tilde{\phi} \frac{\partial u_2}{\partial \tilde{\phi}}
\]
\[
\frac{\partial v_1(t, \hat{f}(a,t))}{\partial t} = -\tau_1 \frac{\partial v_1(t, \hat{f}(a,t))}{\partial \hat{f}}, (24)
\]
\[
\frac{\partial v_2(t, \hat{\phi}(a,t))}{\partial t} = -\tau_2 \frac{\partial v_2(t, \hat{\phi}(a,t))}{\partial \hat{\phi}}, (25)
\]
\[
\frac{\partial w_1(t, \hat{f}(a,t))}{\partial t} = -\tau_3 \frac{\partial w_1(t, \hat{f}(a,t))}{\partial \hat{f}}, (26)
\]
\[
\frac{\partial w_2(t, \hat{\phi}(a,t))}{\partial t} = -\tau_4 \frac{\partial w_4(t, \hat{\phi}(a,t))}{\partial \hat{\phi}}. (27)
\]

Diffusion type of equations appears in several situations of modeling in biology, for example refer to the book (Perthame 2007). Further applications of diffusion type of equations appear in studying growth of cell populations, see Boulanouar (2001), Lebowitz and Rubinow (1974), and Rotenberg (1983).

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