Pointwise products of some Banach function spaces and factorization

Paweł Kolwicz a,∗,1, Karol Leśnik a,1, Lech Maligranda b

a Institute of Mathematics of Electric Faculty, Poznań University of Technology, ul. Piotrowo 3a, 60-965 Poznań, Poland
b Department of Engineering Sciences and Mathematics, Luleå University of Technology, SE-971 87 Luleå, Sweden

Received 31 October 2012; accepted 24 October 2013
Available online 19 November 2013
Communicated by K. Ball

Abstract

The well-known factorization theorem of Lozanovskiĭ may be written in the form \( L^1 \equiv E \odot E' \), where \( \odot \) means the pointwise product of Banach ideal spaces. A natural generalization of this problem would be the question when one can factorize \( F \) through \( E \), i.e., when \( F \equiv E \odot M(E, F) \), where \( M(E, F) \) is the space of pointwise multipliers from \( E \) to \( F \). Properties of \( M(E, F) \) were investigated in our earlier paper [41] and here we collect and prove some properties of the construction \( E \odot F \). The formulas for pointwise product of Calderón–Lozanovskiĭ \( E_\phi \)-spaces, Lorentz spaces and Marcinkiewicz spaces are proved. These results are then used to prove factorization theorems for such spaces. Finally, it is proved in Theorem 11 that under some natural assumptions, a rearrangement invariant Banach function space may be factorized through a Marcinkiewicz space.

© 2013 Elsevier Inc. All rights reserved.

Keywords: Banach ideal spaces; Banach function spaces; Calderón spaces; Calderón–Lozanovskiĭ spaces; Symmetric spaces; Orlicz spaces; Sequence spaces; Pointwise multipliers; Pointwise multiplication; Factorization

∗ Corresponding author.
E-mail addresses: pawel.kolwicz@put.poznan.pl (P. Kolwicz), klesnik@vp.pl (K. Leśnik), lech.maligranda@ltu.se (L. Maligranda).

1 Research partially supported by the State Committee for Scientific Research, Poland, Grant N N201 362236.
1. Introduction and preliminaries

The well-known factorization theorem of Lozanovskii says that for any \( \varepsilon > 0 \) each \( z \in L^1 \) can be factorized by \( x \in E \) and \( y \in E' \) in such a way that

\[
z = xy \quad \text{and} \quad \|x\|_E \|y\|_{E'} \leq (1 + \varepsilon) \|z\|_{L^1}.
\]

Moreover, if \( E \) has the Fatou property we may take \( \varepsilon = 0 \) in the above inequality. This theorem can be written in the form \( L^1 \equiv E \odot E' \), where \( E \odot F = \{ x \cdot y : x \in E \text{ and } y \in F \} \).

Then natural question arises: when is it possible to factorize \( F \) through \( E \), i.e., when \( F \equiv E \odot M(E, F) \)? (1)

Here \( M(E, F) \) denotes the space of multipliers defined as

\[
M(E, F) = \{ x \in L^0 : xy \in F \text{ for each } y \in E \}
\]

with the operator norm

\[
\|x\|_{M(E, F)} = \sup_{\|y\|_E = 1} \|xy\|_F.
\]

The space of multipliers between function spaces was investigated by many authors, see for example [85,66,3,6] (see also [93,94,18,67,1,63,79,69,26,20,14,25,65,86,41]). In this paper we are going to investigate general properties of the product construction \( E \odot F \) and calculate the product space \( E \odot F \) for Calderón–Lozanovskii, Lorentz and Marcinkiewicz spaces. This product space was of interest in \([2,91,72,94,22,85,63,78,79,11,6,24,5,46,86]\). The results on product construction will be used to give answers to the factorization question (1) in these special spaces.

Let \(( \Omega, \Sigma, \mu)\) be a complete \( \sigma \)-finite measure space and \( L^0 = L^0(\Omega) \) be the space of all classes of \( \mu \)-measurable real-valued functions defined on \( \Omega \). A (quasi-)Banach space \( E = (E, \| \cdot \|_E) \) is said to be a (quasi-)Banach ideal space on \( \Omega \) if \( E \) is a linear subspace of \( L^0(\Omega) \) and satisfies the so-called ideal property, which means that if \( y \in E \), \( x \in L^0 \) and \( |x(t)| \leq |y(t)| \) for \( \mu \)-almost all \( t \in \Omega \), then \( x \in E \) and \( \|x\|_E \leq \|y\|_E \). We will also assume that a (quasi-)Banach ideal space on \( \Omega \) is saturated, i.e., every \( A \in \Sigma \) with \( \mu(A) > 0 \) has a subset \( B \in \Sigma \) of finite positive measure for which \( \chi_B \in E \). The last statement is equivalent with the existence of a weak unit, i.e., an element \( x \in E \) such that \( x(t) > 0 \) for each \( t \in \Omega \) (see \([39]\) and \([63]\)). If the measure space \(( \Omega, \Sigma, \mu)\) is non-atomic we shall speak about (quasi-)Banach function space and if we replace the measure space \(( \Omega, \Sigma, \mu)\) by the counting measure space \((\mathbb{N}, 2^{\mathbb{N}}, m)\), then we say that \( E \) is a (quasi-)Banach sequence space (denoted by \( e \)).

A point \( x \in E \) is said to have an order continuous norm (or to be order continuous element) if for each sequence \( (x_n) \subset E \) satisfying \( 0 \leq x_n \leq |x| \) and \( x_n \to 0 \) \( \mu \)-a.e. on \( \Omega \), one has \( \|x_n\|_E \to 0 \). By \( E_a \) we denote the subspace of all order continuous elements of \( E \). It is worth to notice that in case of Banach ideal spaces on \( \Omega \), \( x \in E_a \) if and only if \( \|x\chi_{A_n}\|_E \downarrow 0 \) for any
sequence \( \{A_n\} \) satisfying \( A_n \searrow \emptyset \) (that is \( A_n \supset A_{n+1} \) and \( \mu(\bigcap_{n=1}^{\infty} A_n) = 0 \)). A Banach ideal space \( E \) is called order continuous if every element of \( E \) is order continuous, i.e., \( E = E_\alpha \).

A space \( E \) has the Fatou property if the conditions \( 0 \leq x_n \uparrow x \in L^0 \) with \( x_n \in E \) and \( \sup_{n \in \mathbb{N}} \|x_n\|_E < \infty \) imply that \( x \in E \) and \( \|x_n\|_E \uparrow \|x\|_E \).

We shall consider the pointwise product of Calderón–Lozanovskiĭ spaces \( E_\varphi \) (which are generalizations of Orlicz spaces \( L^\varphi \)), but very useful will be also the general Calderón–Lozanovskiĭ construction \( \rho(E,F) \).

A function \( \varphi : [0, \infty) \to [0, \infty] \) is called a Young function (or an Orlicz function if it is finite-valued) if \( \varphi \) is convex, non-decreasing with \( \varphi(0) = 0 \); we assume also that \( \varphi \) is neither identically zero nor identically infinity on \( (0, \infty) \) and \( \lim_{u \to b^-} \varphi(u) = \varphi(b^-) \) if \( b^- < \infty \), where \( b^- = \sup\{u > 0 : \varphi(u) < \infty\} \).

Note that from the convexity of \( \varphi \) and the equality \( \varphi(0) = 0 \) it follows that \( \lim_{u \to 0^+} \varphi(u) = \varphi(0) = 0 \). Furthermore, from the convexity and \( \varphi \not\equiv 0 \) we obtain \( \lim_{u \to \infty} \varphi(u) = \infty \).

If we denote \( a_\varphi = \sup\{u \geq 0 : \varphi(u) = 0\} \), then \( 0 \leq a_\varphi \leq b_\varphi \leq \infty \) and \( a_\varphi < \infty \), \( b_\varphi > 0 \), since a Young function is neither identically zero nor identically infinity on \( (0, \infty) \). The function \( \varphi \) is continuous and non-decreasing on \( [0, b_\varphi] \) and is strictly increasing on \( [a_\varphi, b_\varphi) \). If \( a_\varphi = 0 \), then we write \( \varphi > 0 \) and if \( b_\varphi = \infty \), then \( \varphi < \infty \). Each Young function \( \varphi \) defines the function \( \varphi_{\rho_{\varphi}} : [0, \infty) \times [0, \infty) \to [0, \infty) \) in the following way

\[
\rho_{\varphi}(u, v) = \begin{cases} v \varphi^{-1}(\frac{u}{v}), & \text{if } u > 0, \\ 0, & \text{if } u = 0, \end{cases}
\]

where \( \varphi^{-1} \) denotes the right-continuous inverse of \( \varphi \) and is defined by

\[
\varphi^{-1}(v) = \inf\{u \geq 0 : \varphi(u) > v\} \quad \text{for } v \in [0, \infty) \text{ with } \varphi^{-1}(\infty) = \lim_{v \to \infty} \varphi^{-1}(v).
\]

If \( \rho = \rho_{\varphi} \) and \( E, F \) are Banach ideal spaces over the same measure space \( (\Omega, \Sigma, \mu) \), then the Calderón–Lozanovskiĭ space \( \rho(E,F) \) is defined as all \( z \in L^0(\Omega) \) such that for some \( x \in E \), \( y \in F \) with \( \|x\|_E \leq 1 \), \( \|y\|_F \leq 1 \) and for some \( \lambda > 0 \) we have

\[
|z| \leq \lambda \rho(|x|, |y|) \quad \mu\text{-a.e. on } \Omega.
\]

The norm \( \|z\|_{\rho} = \|z\|_{\rho(E,F)} \) of an element \( z \in \rho(E,F) \) is defined as the infimum values of \( \lambda \) for which the above inequality holds. It can be shown that

\[
\rho(E,F) = \{z \in L^0(\Omega) : |z| \leq \rho(x,y) \quad \mu\text{-a.e. for some } x \in E_+, \quad y \in F_+\}
\]

with the norm

\[
\|z\|_{\rho(E,F)} = \inf\{\max\{\|x\|_E, \|y\|_F\} : |z| \leq \rho(x,y), \quad x \in E_+, \quad y \in F_+\}.
\]

The Calderón–Lozanovskiĭ spaces, introduced by Calderón in [15] and developed by Lozanovskiĭ in [55,56,58,59], play a crucial role in the theory of interpolation since such construction is the interpolation functor for positive operators and under some additional assumptions on spaces \( E, F \), like the Fatou property or separability, also for all linear operators (see [73,75,43,63]). If \( \rho(u,v) = u^\theta v^{1-\theta} \) with \( 0 < \theta < 1 \) we write \( E^\theta F^{1-\theta} \) instead of \( \rho(E,F) \) and these spaces are
Calderón spaces (cf. [15, p. 122]). Another important situation, investigated by Calderón (cf. [15, p. 121]) and independently by Lozanovskii (cf. [51, Theorem 2], [52, Theorem 2]), appears when we put $F \equiv L^\infty$. In this case, in the definition of the norm, it is enough to take $y = \chi_\Omega$ and then

\[ \|z\|_{\rho_\varphi(E,L^\infty)} = \inf\{\lambda > 0: \| \varphi(|x|/\lambda) \|_E \leq 1 \} . \]

(3)

Thus the Calderón–Lozanovskii space $E_\varphi = \rho_\varphi(E, L^\infty)$ for any Young function $\varphi$ is defined by

\[ E_\varphi = \{ x \in L^0: I_\varphi(cx) < \infty \text{ for some } c = c(x) > 0 \}, \]

and it is a Banach ideal space on $\Omega$ with the so-called Luxemburg–Nakano norm

\[ \|x\|_{E_\varphi} = \inf\{\lambda > 0: I_\varphi(x/\lambda) \leq 1 \} , \]

where the convex semimodular $I_\varphi$ is defined as

\[ I_\varphi(x) := \begin{cases} \|\varphi(|x|)\|_E , & \text{if } \varphi(|x|) \in E, \\ \infty , & \text{otherwise}. \end{cases} \]

If $E = L^1 (E = l^1)$, then $E_\varphi$ is the classical Orlicz function (sequence) space $L^\rho (l^\rho)$ equipped with the Luxemburg–Nakano norm (cf. [42, 63]). If $E$ is a Lorentz function (sequence) space, then $E_\varphi$ is the corresponding Orlicz–Lorentz function (sequence) space, equipped with the Luxemburg–Nakano norm. On the other hand, if $\varphi(u) = u^p$, $1 \leq p < \infty$, then $E_\varphi$ is the $p$-convexification $E^{(p)}$ of $E$ with the norm $\|x\|_{E^{(p)}} = |||x|^p||^1/p$. In the case $0 < p < 1$, we shall speak about $p$-concavification of $E$.

For two ideal (quasi-)Banach spaces $E$ and $F$ on $\Omega$ the symbol $E \overset{C}{\rightarrow} F$ means that the inclusion $E \subset F$ is continuous with the norm which is not bigger than $C$, i.e., $\|x\|_F \leq C \|x\|_E$ for all $x \in E$. In the case when the inclusion $E \overset{C}{\subset} F$ holds with some (unknown) constant $C > 0$ we simply write $E \hookrightarrow F$. Moreover, $E = F$ (and $E \equiv F$) means that the spaces are the same and the norms are equivalent (equal).

We will also need some facts from the theory of symmetric spaces. By a symmetric function space (symmetric Banach function space or rearrangement invariant Banach function space) on $I$, where $I = (0, 1)$ or $I = (0, \infty)$ with the Lebesgue measure $m$, we mean a Banach ideal space $E = (E, \| \cdot \|_E)$ with the additional property that for any two equimeasurable functions $x \sim y$, $x, y \in L^0(I)$ (that is, they have the same distribution functions $d_x \equiv d_y$, where $d_x(\lambda) = m(\{t \in I: |x(t)| > \lambda\})$, $\lambda \geq 0$) and $x \in E$ we have $y \in E$ and $\|x\|_E = \|y\|_E$. In particular, $\|x\|_E = \|x^*\|_E$, where $x^*(t) = \inf(\lambda > 0: d_x(\lambda) < t)$, $t \geq 0$. Similarly, if $e$ is a Banach sequence space with the above property, then we shall speak about symmetric sequence space. It is worthwhile to point out that any Banach ideal space with this property is equivalent to a symmetric space over one of the above three measure spaces (cf. [48]).

The fundamental function $f_E$ of a symmetric function space $E$ on $I$ is defined by the formula $f_E(t) = \|X_{[0,t]}\|_E$, $t \in I$. It is well-known that fundamental function is quasi-concave on $I$, that is, $f_E(0) = 0$, $f_E(t)$ is positive, non-decreasing and $f_E(t)/t$ is non-increasing for $t \in (0, m(I))$ or, equivalently, $f_E(t) \leq \max(1, t/s) f_E(s)$ for all $s, t \in (0, m(I))$. Moreover, for each fundamental function $f_E$, there is an equivalent, concave function $\tilde{f}_E$, defined by
\[ \tilde{f}_E(t) := \inf_{s \in (0, m(I))} (1 + \frac{t}{s}) f_E(s). \] Then \( f_E(t) \leq \tilde{f}_E(t) \leq 2 f_E(t) \) for all \( t \in I \). For any quasi-concave function \( \phi \) on \( I \) the Marcinkiewicz function space \( M_\phi \) is defined by the norm

\[
\|x\|_{M_\phi} = \sup_{t \in I} \phi(t) x^{**}(t), \quad x^{**}(t) = \frac{1}{t} \int_{0}^{t} x^{*}(s) \, ds. \tag{4}
\]

This is a symmetric Banach function space on \( I \) with the fundamental function \( f_{M_\phi}(t) = \phi(t) \).

Moreover, \( E \hookrightarrow M_{f_E} \) for any symmetric space \( E \) because

\[
x^{**}(t) \leq \frac{1}{t} \left\| x^{*} \right\|_{E} \left\| x^{*} \right\|_{E'} = \left\| x \right\|_{E} \frac{1}{\tilde{f}_E(t)} \quad \text{for any } t \in I \tag{5}
\]

(see, for example, [43] or [7]). Although the fundamental function of a symmetric function space \( E \) need not to be concave, there always exists equivalent norm on \( E \) for which the new fundamental function is concave (cf. Zippin [95, Lemma 2.1]). Then for a symmetric function space \( E \) with the concave fundamental function \( f_E \) there is also the smallest symmetric space with the same fundamental function. This space is the Lorentz function space \( \Lambda_{f_E} \) given by the norm

\[
\|x\|_{\Lambda_{f_E}} = \int_{I} x^{*}(t) \, df_E(t) = f_E(0^+) \|x\|_{L^\infty(I)} + \int_{I} x^{*}(t) f_E'(t) \, dt. \tag{6}
\]

Then the inclusions

\[ \Lambda_{f_E} \hookrightarrow E \hookrightarrow M_{f_E} \tag{7} \]

are satisfied, where \( f_E \) is the fundamental function of \( E \).

More information about Banach ideal spaces, quasi-Banach ideal spaces, symmetric Banach and quasi-Banach spaces can be found, for example, in [39,48,35,33,43,7,63,4].

The paper is organized as follows. In Section 1 some necessary definitions and notations are collected including the Calderón–Lozanovskii \( E_\phi \)-spaces. In Section 2 the product space \( E \odot F \) is defined and some general results are presented. We prove important representation of \( E \odot F \) as \( \frac{1}{2} \)-concavification of the Calderón space \( E^{1/2} F^{1/2} \), i.e., \( E \odot F \equiv (E^{1/2} F^{1/2})^{(1/2)} \). Such an equality was used by Schep in [86] but without any explanation, which seems to be not so evident. Then we present some properties of \( E \odot F \) that follow from this representation. In particular, the symmetry is proved and formula for the fundamental function of the product space is given \( f_{E \odot F}(t) = f_E(t) f_F(t) \). We finish Section 2 with some sufficient conditions on \( E \) and \( F \) that \( E \odot F \) is a Banach space (not only a quasi-Banach space).

In Section 3 we collected properties connecting product spaces with the space of multipliers. There is a proof of the cancellation property of the product operation for multipliers \( M(E \odot F, E \odot G) \equiv M(F, G) \). The cancellation property for concrete sequence spaces was already observed by Bennett [6, pp. 72–73].

Section 4 is devoted to products of the Calderón–Lozanovskii spaces of the type \( E_\phi \) as an improvement of results on products, known for Orlicz spaces \( L^\phi \) which were proved by Ando [2], Wang [91] and O’Neil [72]. The inclusion \( E_{\varphi_1} \odot E_{\varphi_2} \hookrightarrow E_\psi \) follows from the results proved
in [41]. The reverse inclusion \( E_\varphi \hookrightarrow E_{\varphi_1} \odot E_{\varphi_2} \) is investigated here and we improve the sufficient and necessary conditions which were given in the case of Orlicz spaces by Zabreiko and Rutickii [94], Dankert [21,22] and Maligranda [63]. Combining the above two inclusions we obtain conditions on the equality \( E_\varphi = E_{\varphi_1} \odot E_{\varphi_2} \). For example, for two Young functions \( \varphi_1, \varphi_2 \) we always have \( E_{\varphi_1} \odot E_{\varphi_2} = E_\varphi \), where \( \varphi = \varphi_1 \oplus \varphi_2 \) is defined by

\[
(\varphi_1 \oplus \varphi_2)(u) = \inf_{u = vw} [\varphi_1(v) + \varphi_2(w)].
\]

In Section 5 we deal with the product space of Lorentz and Marcinkiewicz spaces. The products of those spaces are calculated. One of the main tools in the proof is the commutativity of Calderón construction with the symmetrizations (cf. Lemma 4).

Section 6 starts with some general discussion about factorization. We prove that so-called \( E \)-perfectness of \( F \) is necessary for the factorization \( F \equiv E \odot M(E,F) \). The rest of this section is divided into two parts. The first is devoted to the factorization of Calderón–Lozanovskii \( E_\varphi \)-spaces. Using the results from Section 4 and the paper [41] we examine when \( E_\varphi \) can be factorized through the space \( E_{\varphi_1} \). In the second part, we investigate the possibility of factorization for Lorentz and Marcinkiewicz spaces. Finally, in Theorem 11 it is proved that under some natural assumptions a rearrangement invariant Banach function space \( E \) may be factorized through Marcinkiewicz space and by duality, Lorentz space may be factorized through a rearrangement invariant Banach function space \( E \).

2. On the product space \( E \odot F \)

Given two Banach ideal spaces (real or complex) \( E \) and \( F \) on \((\Omega, \Sigma, \mu)\) let us define the pointwise product space \( E \odot F \) as

\[
E \odot F = \{x \cdot y: x \in E \text{ and } y \in F\},
\]

with a functional \( \| \cdot \|_{E \odot F} \) defined by the formula

\[
\|z\|_{E \odot F} = \inf \{\|x\|_E \|y\|_F: z = xy, x \in E, y \in F\}. \tag{8}
\]

We will show in the sequel that \( E \odot F \) is, in general, a quasi-Banach ideal space even if both \( E \) and \( F \) are Banach ideal spaces. Let us collect some general properties of the product space and its norm.

**Proposition 1.** If \( E \) and \( F \) are Banach ideal spaces on \((\Omega, \Sigma, \mu)\), then \( E \odot F \) has an ideal property. Moreover,

\[
\|z\|_{E \odot F} = \|z\|_{E \odot F}
= \inf \{\|x\|_E \|y\|_F: |z| = xy, x \in E_+, y \in F_+\}
= \inf \{\|x\|_E \|y\|_F: |z| \leq xy, x \in E_+, y \in F_+\}.
\]
Proof. We show first that $\|z\|_{E \odot F} = \|z\|_{E \odot F}$. If $z = xy$ with $x \in E$, $y \in F$, then $|z| = z e^{i\theta} = x y e^{i\theta}$, where $\theta : \Omega \to \mathbb{R}$, and

$$\|z\|_{E \odot F} \leq \|x e^{i\theta}\|_E \|y\|_F = \|x\|_E \|y\|_F.$$ 

Hence, $\|z\|_{E \odot F} \leq \|z\|_{E \odot F}$. Similarly, if $|z| = xy$ with $x \in E$, $y \in F$, then $z = |z| e^{-i\theta} = x y e^{-i\theta}$ and

$$\|z\|_{E \odot F} \leq \|x e^{-i\theta}\|_E \|y\|_F = \|x\|_E \|y\|_F,$$ 

from which we obtain the estimate $\|z\|_{E \odot F} \leq \|z\|_{E \odot F}$. Combining the above estimates we get $\|z\|_{E \odot F} = \|z\|_{E \odot F}$.

To show the ideal property of $E \odot F$ assume that $z \in E \odot F$ and $|w| \leq |z|$. By the definition for any $\varepsilon > 0$ we can find $x \in E$, $y \in F$ such that $z = xy$ and $\|x\|_E \|y\|_F \leq \|z\|_{E \odot F} + \varepsilon$. We set $h(t) = \frac{w(t)}{z(t)}$ if $z(t) \neq 0$ and $h(t) = 0$ if $z(t) = 0$. Then $w = h z = h x y$ and since $|h x| \leq |x|$ we have $w = h x y \in E \odot F$ with

$$\|w\|_{E \odot F} \leq \|h x\|_E \|y\|_F \leq \|x\|_E \|y\|_F \leq \|z\|_{E \odot F} + \varepsilon.$$ 

Since $\varepsilon > 0$ was arbitrary we obtain $\|w\|_{E \odot F} \leq \|z\|_{E \odot F}$. Note that in the above proofs we needed only the ideal property of one of the spaces $E$ or $F$.

Next, if $|z(t)| = x(t) y(t)$, $t \in \Omega$, then, taking $x_0 = |x|$, $y_0 = |y|$, we obtain $x_0 \geq 0$, $y_0 \geq 0$, $x_0 y_0 = |x| |y| = |x y| = |z|$, which gives the proof of the second equality. The proof of the third equality follows from the fact that if $0 \leq x \in E$, $0 \leq y \in F$ and $|z| \leq x y$, then $|z| = u x y = x_0 y_0$, where $x_0 = u x$, $y_0 = y$ and $u = \frac{|z|}{x y}$ on the support of $x y$ and $u = 0$ elsewhere. Since $0 \leq u \leq 1$, it follows that $x_0 \leq x$, $y_0 \leq y$ and this proves (the non-trivial part of) the last equality. \qed

Proposition 1 shows that in the investigation of product spaces it is enough to consider real spaces, therefore from now on we shall consider only real Banach ideal spaces.

The product space can be described with the help of the Calderón construction. To come to this result we first prove some description of $E^{1/p} F^{1-1/p}$ spaces and $p$-convexification. Let us start in Theorem 1(i) below with a reformulation of Lemma 31 from [40].

**Theorem 1.** Let $E$ and $F$ be a couple of Banach ideal spaces on $(\Omega, \Sigma, \mu)$.

(i) If $1 < p < \infty$ and $z \in E^{1/p} F^{1-1/p}$, then

$$\|z\|_{E^{1/p} F^{1-1/p}} = \inf \left\{ \max \left\{ \|x\|_E, \|y\|_F \right\} : |z| = x^{1/p} y^{1-1/p}, x \in E_+, y \in F_+ \right\}$$

(ii) If $1 < p < \infty$, then

$$E^{(p)} \odot F^{(p')} = E^{1/p} F^{1-1/p},$$

where $1/p + 1/p' = 1$. 

(iii) For $0 < p < \infty$, $(E \odot F)^{(p)} \equiv E^{(p)} \odot F^{(p)}$.

(iv) We have

\[ E \odot F \equiv \left( E^{1/2} F^{1/2} \right)^{(1/2)}, \]

that is,

\[ \|z\|_{E \odot F} = \inf \left\{ \max \left\{ \|x\|_{E}^2, \|y\|_{F}^2 \right\} : |z| = xy, \|x\|_{E} = \|y\|_{F}, x \in E_{+}, y \in F_{+} \right\}. \quad (10) \]

**Proof.** (i) Let $z \in E^{1/p} F^{1-1/p}$ and $z = x^{1/p} y^{1-1/p}$, where $0 \leq x \in E$, $0 \leq y \in F$. Suppose $\frac{\|x\|_{E}}{\|y\|_{F}} = a > 1$ (for $0 < a < 1$ the proof is similar). Put

\[ x_1 = a^{-(1-1/p)} x, \quad y_1 = a^{1/p} y. \]

Then $\|x_1\|_{E} = \|x\|_{E}^{1/p} \|y\|_{F}^{1-1/p} = \|y_1\|_{F}$ and $z = x_1^{1/p} y_1^{1-1/p}$. Of course,

\[ \max \left\{ \|x_1\|_{E}, \|y_1\|_{F} \right\} \leq \max \left\{ \|x\|_{E}, \|y\|_{F} \right\}, \]

which ends the proof.

(ii) If $z \in E^{(p)} \odot F^{(p')}$, then using Proposition 1 and the definition of $p$-convexification we obtain

\[ \|z\|_{E^{(p)} \odot F^{(p')}} = \inf \left\{ g \|E^{(p)}\|_{F^{(p')}} : |z| = gh, 0 \leq g \in E^{(p)}, 0 \leq h \in F^{(p')} \right\} \]

\[ \quad = \inf \left\{ \|x\|_{E}^{1/p} \|y\|_{F}^{1-1/p} : |z| = x^{1/p} y^{1-1/p}, x \in E_{+}, y \in F_{+} \right\}, \]

and, using Theorem 1(i) to the last expression, we get

\[ \inf_{a > 0} \left[ \inf \left\{ a^{1/p} \|u\|_{E}^{1/p} \|y\|_{F}^{1-1/p} : |z| = a^{1/p} u^{1/p} y^{1-1/p}, \|u\|_{E} = \|y\|_{F}, u \in E_{+}, y \in F_{+} \right\} \right] \]

\[ = \inf_{a > 0} \left[ a^{1/p} \inf \left\{ \|u\|_{E} : \frac{|z|}{a^{1/p}} = u^{1/p} y^{1-1/p}, \|u\|_{E} = \|y\|_{F}, u \in E_{+}, y \in F_{+} \right\} \right] \]

\[ = \inf_{a > 0} \left[ a^{1/p} \left\| \frac{z}{a^{1/p}} \right\|_{E^{(p)} F^{1-1/p}} \right] = \|z\|_{E^{1/p} F^{1-1/p}}. \]

(iii) One has

\[ \|z\|_{(E \odot F)^{(p)}} = \|z\|_{E \odot F}^{1/p} = \inf \left\{ \|x\|_{E}^{1/p} \|y\|_{F}^{1/p} : |z|^p = xy, x \in E_{+}, y \in F_{+} \right\} \]

\[ = \inf \left\{ \|u\|_{E}^{1/p} \|v\|_{F}^{1/p} : |z|^p = u^p v^p, u \in E^{(p)}, v \in F^{(p)} \right\} \]

\[ = \inf \left\{ \|u\|_{E^{(p)}} \|v\|_{F^{(p)}} : |z| = uv, u \in E^{(p)}_{+}, v \in F^{(p)}_{+} \right\} = \|z\|_{E^{(p)} \odot F^{(p)}}. \]
(iv) The proof is an immediate consequence of Theorem 1(ii) and (iii) since

\[ E \odot F \equiv ((E \odot F)^{(2)})^{(1/2)} \equiv (E^{1/2}F^{1/2})^{(1/2)}. \]

Moreover,

\[
\| z \|_{E \odot F} = \| z \|_{(E^{1/2}F^{1/2})^{(1/2)}} = \left( \left\| \sqrt{z} \right\|_{E^{1/2}F^{1/2}} \right)^2 \\
= \left[ \inf \left\{ \max \left\{ \| x \|_E, \| y \|_F \right\}; \sqrt{|z|} = \sqrt{xy}, \| x \|_E = \| y \|_F, x \in E_+, y \in F_+ \right\} \right]^2 \\
= \inf \left\{ \max \left\{ \| x \|_E^2, \| y \|_F^2 \right\}; |z| = xy, \| x \|_E = \| y \|_F, x \in E_+, y \in F_+ \right\},
\]

and the proof is complete. \( \square \)

As a consequence of the representation (9) we obtain the following results:

**Corollary 1.** Let \( E \) and \( F \) be a couple of Banach ideal spaces on \( (\Omega, \Sigma, \mu) \).

(i) Then \( E \odot F \) is a quasi-Banach ideal space and the triangle inequality is satisfied with constant 2, i.e.,

\[ \| x + y \|_{E \odot F} \leq 2(\| x \|_{E \odot F} + \| y \|_{E \odot F}). \]

(ii) If both \( E \) and \( F \) satisfy the Fatou property, then \( E \odot F \) has the Fatou property.

(iii) The space \( E \odot F \) has order continuous norm if and only if the couple \( (E, F) \) is not jointly order discontinuous, i.e., \( (E, F) \notin (\text{JOD}) \) (for the definition, see [40]).

**Proof.** (i) It is a consequence of the representation \( E \odot F \equiv (E^{1/2}F^{1/2})^{(1/2)} \) and the fact that for 1/2-concavification of a Banach ideal space \( G = E^{1/2}F^{1/2} \) we have

\[
\left\| x + y \right\|_{1/2} \leq \left( \left\| x \right\|_{1/2} \right)^2 + \left( \left\| y \right\|_{1/2} \right)^2 \\
\leq 2 \left( \left\| x \right\|_{1/2}^2 + \left\| y \right\|_{1/2}^2 \right).
\]

(ii) It is again a consequence of (9) and the fact that \( E^{1/2}F^{1/2} \) has the Fatou property when \( E \) and \( F \) have the Fatou property (see [54, p. 595] and [82]).

(iii) Using (9) and Theorem 13 in [40] we can show that \( E^{1/2}F^{1/2} \) has order continuous norm if and only if \( (E, F) \notin (\text{JOD}) \), which gives the statement. \( \square \)

Lozanovskiǐ [52, Theorem 4] formulated result on the Köthe dual of \( p \)-convexification \( E^{(p)} \) with no proof. The proof can be found in paper by Schep [86, Theorem 2.9] and we present another proof which follows from the Lozanovskiǐ duality result and our Theorem 1(ii).

**Corollary 2.** Let \( E \) be a Banach ideal space and \( 1 < p < \infty \). Then

\[ \left[ E^{(p)} \right]' \equiv (E')^{(p)} \odot L^p'. \]
Proof. Using the Lozanovskii theorem on the duality of Calderón spaces (see [54, Theorem 2]) and our Theorem 1(ii) we obtain
\[
[E^\prime(p)]^\prime \equiv [E^{1/p}(L^{\infty})^{1-1/p}]^\prime \equiv (E')^{1/p}(L^1)^{1-1/p} = (E')^{(p)} \circ (L^1)^{(p)} = (E')^{(p)} \circ L^p. \quad \square
\]

Remark 1. In general, \([E^\prime(p)]^\prime \neq (E')^p\). In fact, for the classical Lorentz space \(E = L_{r,1}^r\) with \(1 < r < \infty\) we have \([(L_{r,1}^r)^{(p)}] = (L_{r^p,p}^{p'})^r = L_{q^r,p}^{q'}\), where \(1/q + 1/(pr) = 1\) and \([(L_{r,1}^r)]^\prime^{(p)} = (L_{r',\infty}^r)^{(p)} = L_{r^p,p,\infty}^r\).

Example 1.

(a) If \(1 \leq p, q \leq \infty, 1/p + 1/q = 1/r\), then \(L^p \odot L^q \equiv L^r\). In particular, \(L^p \odot L^p \equiv L^{p/2}\).

In fact, by the Hölder–Rogers inequality \(\|xy\|_r \leq \|x\|_p\|y\|_q\) for \(x \in L^p, y \in L^q\) (see [63, p. 69]) we obtain \(L^p \odot L^q \subset L^r\) and \(\|z\|_r \leq \|z\|_{L^p \odot L^q}\). On the other hand, if \(z \in L^r\), then \(x = |z|^r sgn z \in L^p, y = |z|^r sgn z \in L^q\) and \(xy = z\) with
\[
\|x\|_p\|y\|_q = \|z\|_{r/p}\|z\|_{r/q} = \|z\|_{r/p+q/r} = \|z\|_r,
\]
which shows that \(L^r \subset L^p \odot L^q\) and \(\|z\|_{L^p \odot L^q} \leq \|z\|_r\).

More general, if \(1 \leq p, q \leq \infty, 1/p + 1/q = 1/r\) and \(E\) is a Banach ideal space, then \(E^\prime \odot E^{(q)} \equiv E^{(r)}\) (cf. [66, Lemma 1] and [71, Lemma 2.21(i)]).

(b) We have \(c_0 \odot l^1 \equiv l^{\infty} \odot l^1 \equiv l^1\) and \(c_0 \equiv c_0 \odot l^{\infty} \neq l^{\infty} \odot l^{\infty} \equiv l^{\infty}\).

This example shows that for the Fatou property of \(E \odot F\) it is not necessary that both \(E\) and \(F\) do have the Fatou property.

The next interesting question about the product space is its symmetry.

Theorem 2. Let \(E\) and \(F\) be symmetric Banach function spaces on \(I = (0, 1)\) or \(I = (0, \infty)\) with the fundamental functions \(f_E\) and \(f_F\), respectively. Then \(E \odot F\) is a symmetric quasi-Banach function space on \(I\) and its fundamental function \(f_{E \odot F}\) is given by the formula
\[
f_{E \odot F}(t) = f_E(t) f_F(t) \quad \text{for } t \in I. \quad (11)
\]

Proof. Using Lemma 4.3 from [43, p. 93] we can easily show that \(E^{1/2} F^{1/2}\) (even \(\rho(E, F)\)) is a Banach symmetric space and (9) gives the symmetry of \(E \odot F\).

The inequality \(f_{E \odot F}(t) \leq f_E(t) f_F(t)\) for \(t \in I\) is clear. We prove the reverse inequality using the fact that each symmetric Banach function space \(E\) satisfies \(E \hookrightarrow M_{f_E}\), where \(M_{f_E}\) is the Marcinkiewicz space (see (5)), some classical inequality on rearrangement (see, for example, [29, p. 277] or [7, p. 44] or [43, p. 64]) and the reverse Chebyshev inequality (see Lemma 1 below). For any \(0 \leq x \in E, 0 \leq y \in F\) such that \(xy = \chi_{[0,t]}\) we have
\[
\|x\|_E \|y\|_F \geq \|x\|_{M_{f_E}} \|y\|_{M_{f_F}} \geq \sup_{0<u\leq t} \frac{f_E(u)}{u} \int_0^u x^*(s) \, ds \sup_{0<v\leq t} \frac{f_F(v)}{v} \int_0^v y^*(s) \, ds.
\]
\[
\begin{align*}
&\geq \frac{f_E(t)f_F(t)}{t} \int_0^t x^*(s) ds \int_0^t y^*(s) ds \\
&\geq \frac{f_E(t)f_F(t)}{t} \int_0^t x(s) ds \int_0^t y(s) ds \\
&\geq \frac{f_E(t)f_F(t)}{t} \int_0^t x(s)y(s) ds \\
&= \frac{f_E(t)f_F(t)}{t} \int_0^t \chi_{[0,t]}(s) ds = f_E(t)f_F(t),
\end{align*}
\] and so \( \|\chi_{[0,t]}\|_{E \odot F} \geq f_E(t)f_F(t). \)

In the fifth inequality above we used the reverse Chebyshev inequality which we will prove in the lemma below. On the classical Chebyshev inequality for decreasing functions (see, for example, [68, p. 39] or [30, p. 213]).

**Lemma 1.** Let \( 0 < \mu(A) < \infty \) and \( x(s)y(s) = a > 0 \) for all \( s \in A \) with \( 0 \leq x, y \in L^1(A) \). Then

\[
\mu(A) \int_A xy \, d\mu \leq \int_A x \, d\mu \int_A y \, d\mu .
\] (12)

**Proof.** For any \( s, t \in A \) we have

\[
x(s)y(s) - x(s)y(t) - x(t)y(s) + x(t)y(t) \\
= 2a - x(s)y(t) - x(t)y(s) \\
= 2a - \frac{x(s)a}{x(t)} - \frac{x(t)a}{x(s)} = a \left[ 2 - \frac{x(s)}{x(t)} - \frac{x(t)}{x(s)} \right] \\
= a \frac{2x(s)x(t) - (x(s))^2 - (x(t))^2}{x(s)x(t)} = -a \frac{[x(s) - x(t)]^2}{x(s)x(t)} \leq 0.
\]

Now integrating over \( A \) with respect to \( s \) and over \( A \) with respect to \( t \) we obtain the desired inequality (12). □

**Remark 2.** Formula (11) is a generalization of the well-known equality on fundamental functions \( f_E(t)f_{E'}(t) = t = f_{L^1}(t) \) for \( t \in I \) and it is also true for symmetric sequence spaces with the same proof.

Example 1(a) shows that \( E \odot F \) is, in general, a quasi-Banach ideal space even if both \( E \) and \( F \) are Banach ideal spaces. We can ask under which additional conditions on \( E \) and \( F \) the
product space $E \odot F$ is a Banach ideal space. Before formulation the theorem we need a notion of $p$-convexity. A Banach lattice $E$ is said to be $p$-convex $(1 \leq p < \infty)$ with a constant $K \geq 1$ if

$$\left\| \left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \right\|_E \leq K \left( \sum_{k=1}^{n} \|x_k\|^p_E \right)^{1/p},$$

for any sequence $(x_k)_{k=1}^{n} \subset X$ and any $n \in \mathbb{N}$. If a Banach lattice $E$ is $p$-convex with a constant $K \geq 1$ and $1 \leq q < p$, then $E$ is also $q$-convex with a constant at most $K$. Moreover, $p$-convexification $E^{(p)}$ of a Banach lattice $E$ is $p$-convex with constant 1. More information on $p$-convexity we can find, for example, in [48] and [64].

**Theorem 3.** Suppose that $E$, $F$ are Banach ideal spaces such that $E$ is $p_0$-convex with constant 1, $F$ is $p_1$-convex with constant 1 and $\frac{1}{p_0} + \frac{1}{p_1} \leq 1$. Then $E \odot F$ is a Banach space which is even $\frac{p}{2}$-convex, where $\frac{1}{p} = \frac{1}{2} \left( \frac{1}{p_0} + \frac{1}{p_1} \right)$.

Before the proof of Theorem 3 let us present the following lemma, which was mentioned in [80] and proved in [90, p. 219].

**Lemma 2.** If $E$ is $p_0$-convex with constant $K_0$ and $F$ is $p_1$-convex with constant $K_1$, then $E^{1-\theta} F^\theta$ is $p$-convex with constant $K \leq K_0^{1-\theta} K_1^\theta$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

**Proof of Theorem 3.** By Lemma 2, $Z = E^{1/2} F^{1/2}$ is $p$-convex with constant 1, where $\frac{1}{p} = \frac{1}{2p_0} + \frac{1}{2p_1}$. The assumption on $p_0, p_1$ gives that $p \geq 2$ and since 1/2-concavification of $Z$ is $p/2$-convex with constant 1 ($p/2 \geq 1$) it follows that it is 1-convex with constant 1 which gives that the norm of $E \odot F = Z^{(1/2)}$ satisfies the triangle inequality, and consequently is a Banach space. This completes the proof. □

**Remark 3.** By duality arguments, Theorem 3 can be also formulated in the terms of $q$-concavity of the Köthe dual spaces. A Banach lattice $E$ is $q$-concave $(1 < q < \infty)$ with a constant $K \geq 1$ if $(\sum_{k=1}^{n} \|x_k\|^q_F)^{1/q} \leq K \| (\sum_{k=1}^{n} |x_k|^q \|_F^{1/q} \|_F$ for any sequence $(x_k)_{k=1}^{n} \subset X$ and any $n \in \mathbb{N}$.

**Remark 4.** Since inclusion $G \hookrightarrow E \odot F$ means also factorization $z = xy$, where $x \in E$ and $y \in F$, therefore sometimes these inclusions or identifications of the product spaces $E \odot F = G$ are called factorizations of concrete spaces as, for example, $l^p$ and Cesàro sequence spaces or $L^p$ and Cesàro function spaces (cf. [6,5,86]), factorization of tent spaces or other spaces (cf. [16,17,31,81,8]).

3. The product spaces and multipliers

Let us collect properties connecting the product space with the space of multipliers. We start with the Cwikel and Nilsson result [20, Theorem 3.5]. They proved that if a Banach ideal space $E$ has the Fatou property and $0 < \theta < 1$, then

$$E \equiv M \left( F^{(1/\theta)}, E^{1-\theta} F^\theta \right)^{(1-\theta)}.$$
We will prove a generalization of this equality, which in the case of $G = L^\infty$ coincides with their result.

**Proposition 2.** Let $E$, $F$, $G$ be Banach ideal spaces. Suppose that $E$ has the Fatou property and $0 < \theta < 1$. Then

$$M(G, E) \equiv M(G^{1-\theta} F^\theta, E^{1-\theta} F^\theta)^{(1-\theta)}.$$

**Proof.** First, let us prove the inclusion $\hookrightarrow$. Let $x \in M(G, E)$. We want to show that $x \in M(G^{1-\theta} F^\theta, E^{1-\theta} F^\theta)^{(1-\theta)}$, that is, $|x|^{1-\theta} \in M(G^{1-\theta} F^\theta, E^{1-\theta} F^\theta)$ or equivalently $x^{1-\theta} |y| \in E^{1-\theta} F^\theta$ for any $y \in G^{1-\theta} F^\theta$. Take arbitrary $y \in G^{1-\theta} F^\theta$ with the norm $< 1$. Then there are $w \in G$, $v \in F$ satisfying $\|w\|_G \leq 1$, $\|v\|_F \leq 1$ and $|y| \leq |w|^{1-\theta} |v|^\theta$. Clearly,

$$|x|^{1-\theta} |y| \leq |xw|^{1-\theta} |v|^\theta \in E^{1-\theta} F^\theta$$

since $x \in M(G, E)$, $w \in G$ gives $xw \in E$. This proves the inclusion. Moreover,

$$\| |x|^{1-\theta} |y| \|_{E^{1-\theta} F^\theta} \leq \| |xw|^{1-\theta} |v|^\theta \|_{E^{1-\theta} F^\theta} \leq \|xw\|_{E^{1-\theta} F^\theta} \|v\|_F \leq \|x\|^{1-\theta}_{M(G, E)} \|v\|_F.$$

Thus,

$$\| |x|^{1-\theta} \|_{M(G^{1-\theta} F^\theta, E^{1-\theta} F^\theta)^{(1-\theta)}} \leq \|x\|_{M(G, E)}.$$

The inclusion $\hookrightarrow$. Let $\|x\|_{M(G^{1-\theta} F^\theta, E^{1-\theta} F^\theta)^{(1-\theta)}} = 1$, i.e. $\| |x|^{1-\theta} \|_{M(G^{1-\theta} F^\theta, E^{1-\theta} F^\theta)} = 1$. We need to show that for any $w \in G$ we have $xw \in M(F^{1/(\theta)}, E^{1-\theta} F^\theta)^{(1-\theta)}$, that is, $|xw|^{1-\theta} \in M(F^{1/(\theta)}, E^{1-\theta} F^\theta)$. Really, by the Cwikel–Nilsson result, we obtain $xw \in E$ for any $w \in G$.

Let $w \in G$ and $v \in F$. Since the norm of $x$ is 1 it follows that

$$\| |x|^{1-\theta} \frac{z}{\|z\|_{G^{1-\theta} F^\theta}} \|_{E^{1-\theta} F^\theta} \leq 1$$

for each $0 \neq z \in G^{1-\theta} F^\theta$. Consequently, for $z = |w|^{1-\theta} |v|^\theta$, we obtain

$$\| |xw|^{1-\theta} |v|^\theta \|_{E^{1-\theta} F^\theta} = \| |x|^{1-\theta} |w|^{1-\theta} |v|^\theta \|_{E^{1-\theta} F^\theta} \leq \| |w|^{1-\theta} |v|^\theta \|_{G^{1-\theta} F^\theta} \leq \|w\|_{G}^{1-\theta} \|v\|_F.$$

This proves our inclusion because, from the assumption on $x$ and the fact that $|w|^{1-\theta} |v|^\theta \in G^{1-\theta} F^\theta$, we have $|xw|^{1-\theta} |v|^\theta = |x|^{1-\theta} |w|^{1-\theta} |v|^\theta \in E^{1-\theta} F^\theta$. Moreover, by the Cwikel–Nilsson result and the last estimate with $v = |m|^{1/\theta}$, we conclude
\[
\|x\|_{M(G, E)} = \sup_{\|w\|_G \leq 1} \|xw\|_E = \sup_{\|w\|_G \leq 1} \|xw\|_{M(F(1/\theta), E^{1-\theta} F^\theta)}^{(1-\theta)} \\
\leq \sup_{\|w\|_G \leq 1} \sup_{\|m\|_{F(1/\theta)} \leq 1} \|xw\|_E^{1-\theta} m_{E^{1-\theta} F^\theta}^{1/(1-\theta)} \leq 1,
\]
and the theorem is proved with the equality of norms. □

Proposition 2 together with the representation of the product space as the $1/2$-concavification of the Calderón space will give the “cancellation” property for multipliers of products. Let us observe that Bennett in [6] proved this property for some concrete sequence spaces. Our result shows that the cancellation property is true in general.

**Theorem 4.** Let $E$, $F$, $G$ be Banach ideal spaces. If $G$ has the Fatou property, then

\[
M(E \otimes F, E \otimes G) \equiv M(F, G).
\]

**Proof.** Applying Theorem 1(iv), property (g) from [66] and Proposition 2 we obtain

\[
M(E \otimes F, E \otimes G) \equiv M\left[(E^{1/2} F^{1/2})^{(1/2)}, (E^{1/2} G^{1/2})^{(1/2)}\right] \\
\equiv M\left[E^{1/2} F^{1/2}, (E^{1/2} G^{1/2})\right]^{(1/2)} \equiv M(F, G),
\]
and (13) is proved. □

**Remark 5.** Note that Proposition 2 and Theorem 4 are equivalent. Proposition 2 can be written in the following form: if $F$ has the Fatou property, then

\[
M\left(E^\theta G^{1-\theta}, F^\theta G^{1-\theta}\right) \equiv M(E, F)^{(1/\theta)},
\]
and it can be proved using Theorem 4. In fact, applying Theorem 1(ii), the cancellation property from Theorem 4 and property (g) from [66] we obtain

\[
M\left(E^\theta G^{1-\theta}, F^\theta G^{1-\theta}\right) \equiv M\left(E^{(1/\theta)} \otimes G^{(1/(1-\theta))}, F^{(1/\theta)} \otimes G^{(1/(1-\theta))}\right) \\
\equiv M\left(E^{(1/\theta)}, F^{(1/\theta)}\right) \equiv M(E, F)^{(1/\theta)}.
\]

From Theorem 4 we can also get the equality mentioned by Raynaud [79] which can be proved also directly (cf. also [86, Proposition 1.4]).

**Corollary 3.** Let $E$, $F$ be Banach ideal spaces. Then

\[
(E \otimes F)' \equiv M(F', E') \equiv M(E, F').
\]
Proof. Using the Lozanovskiǐ factorization theorem (for more discussion see Section 6) and the cancellation property (13) we obtain

\[(E \circ F)' \equiv M(E \circ F, L^1) \equiv M(E \circ F, E') \equiv M(E, E'),\]

and

\[(E \circ F)' \equiv M(E \circ F, L^1) \equiv M(E \circ F, F' \circ F) \equiv M(E, F').\]

Note that the second identity in (14) follows also from the general properties of multipliers (see [66, property (e)] or [41, property (vii)]) because we have \(M(F, E') \equiv M(E''', F') \equiv M(E, F').\) \(\square\)

**Corollary 4.** Let \(E, F\) be Banach ideal spaces. If \(F\) has the Fatou property and \(\|xy\|_{E \circ F} \leq 1\) for all \(x \in E\) with \(\|x\|_E \leq 1\), then \(\|y\|_F \leq 1\).

**Proof.** Since by the assumption \(\|y\|_{M(E, E \circ F)} \leq 1\), then using Theorem 4, together with the facts that \(M(L^\infty, F) \equiv F, E \circ L^\infty \equiv E\), we obtain

\[\|y\|_F = \|y\|_{M(L^\infty, F)} = \|y\|_{M(E \circ L^\infty, E \circ F)} = \|y\|_{M(E, E \circ F)} \leq 1.\] \(\square\)

**Corollary 5.** Let \(E, F, G\) be Banach ideal spaces. If \(F\) and \(G\) have the Fatou property, then

\[M(E \circ F, G) \equiv M(E, M(F, G)).\]

**Proof.** Using Theorem 4, the Lozanovskiǐ factorization theorem, Corollary 3 with the fact that the Fatou property of \(F\) gives by Corollary 1(ii) that \(F \circ G'\) has the Fatou property, again Corollary 3 and the Fatou property of \(G\) we obtain

\[M(E \circ F, G) \equiv M(E \circ F \circ G', G \circ G') \equiv M(E \circ F \circ G', L^1) \equiv (E \circ F \circ G')' \equiv (F \circ G' \circ E)' \equiv M(E, F'(G' \circ E)) \equiv M(E, M(F, G')),\]

which establishes the formula. \(\square\)

4. The product of Calderón–Lozanovskiǐ \(E_\varphi\)-spaces

The pointwise product of Orlicz spaces was investigated already by Krasnoselskiǐ and Rutickiǐ in their book, where sufficient conditions on inclusion \(L^{\varphi_1} \circ L^{\varphi_2} \subset L^\varphi\) are given in the case when \(\Omega\) is bounded closed subset of \(\mathbb{R}^n\) (cf. [42, Theorems 13.7 and 13.8]). For the same set \(\Omega\), Ando [2] proved that \(L^{\varphi_1} \circ L^{\varphi_2} \subset L^\varphi\) if and only if there exist \(C > 0, u_0 > 0\) such that \(\varphi(Cuv) \leq \varphi_1(u) + \varphi_2(v)\) for \(u, v \geq u_0\).

O’Neil [72] presented necessary and sufficient conditions for the inclusion \(L^{\varphi_1} \circ L^{\varphi_2} \subset L^\varphi\) in the case when the measure space is either non-atomic and infinite or non-atomic and finite or
counting measure on \( \mathbb{N} \). Moreover, he observed that condition \( \varphi(Cuv) \leq \varphi_1(u) + \varphi_2(v) \) for all [large, small] \( u, v > 0 \) is equivalent to condition on inverse functions \( C_1 \varphi_1^{-1}(u) \varphi_2^{-1}(u) \leq \varphi^{-1}(u) \) for all [large, small] \( u > 0 \). O’Neil’s results were also presented, with his proofs, in the books [63, pp. 71–75] and [78, pp. 179–184].

The reverse inclusion \( L^\varphi \subset L^{\varphi_1} \odot L^{\varphi_2} \) and the equality \( L^\varphi \odot L^{\varphi_2} = L^\varphi \) were considered by Zabreiko and Rutickii [94, Theorem 8], Dankert [22, pp. 63–68] and Maligranda [63, pp. 69–71].

We will prove the above results for more general spaces, that is, for the Calderón–Lozanovskiĭ \( E^\varphi \)-spaces. Results on the inclusion \( E_{\varphi_1} \odot E_{\varphi_2} \hookrightarrow E^\varphi \) need the following relations between Young functions (cf. [72]): we say \( \varphi_1^{-1} \varphi_2^{-1} \prec \varphi^{-1} \) for all arguments [for large arguments] (for small arguments) if there is a constant \( C > 0 \) [there are constants \( C, u_0 > 0 \)] (there are constants \( C, u_0 > 0 \)) such that the inequality

\[
C \varphi_1^{-1}(u) \varphi_2^{-1}(u) \leq \varphi^{-1}(u) \tag{15}
\]

holds for all \( u > 0 \) [for all \( u \geq u_0 \)] (for all \( u \leq u_0 \)), respectively.

**Remark 6.** The inequality (15) implies the generalized Young inequality:

\[
\varphi(Cuv) \leq \varphi_1(u) + \varphi_2(v) \quad \text{for all } u, v > 0 \text{ such that } \varphi_1(u), \varphi_2(v) < \infty. \tag{16}
\]

On the other hand, if \( \varphi(Cuv) \leq \varphi_1(u) + \varphi_2(v) \) for all \( u, v > 0 \), then \( \varphi_1^{-1}(w) \varphi_2^{-1}(w) \leq 2 \varphi^{-1}(w) \) for each \( w > 0 \) (see [72] and [41]). Similar equivalences hold for large and small arguments.

In [41] the question when the product \( xy \in E^\varphi \) provided \( x \in E_{\varphi_1} \) and \( y \in E_{\varphi_2} \) was investigated, as a generalization of O’Neil’s theorems [72], and the following results were proved (see [41, Theorems 4.1, 4.2 and 4.5]):

**Theorem A.** Let \( \varphi_1, \varphi_2 \) and \( \varphi \) be three Young functions.

(a) If \( E \) is a Banach ideal space with the Fatou property and one of the following conditions holds:
   (a1) \( \varphi_1^{-1} \varphi_2^{-1} \prec \varphi^{-1} \) for all arguments,
   (a2) \( \varphi_1^{-1} \varphi_2^{-1} \prec \varphi^{-1} \) for large arguments and \( L^\infty \hookrightarrow E \),
   (a3) \( \varphi_1^{-1} \varphi_2^{-1} \prec \varphi^{-1} \) for small arguments and \( E \hookrightarrow L^\infty \),
   then, for every \( x \in E_{\varphi_1} \) and \( y \in E_{\varphi_2} \) the product \( xy \in E^\varphi \), which means that \( E_{\varphi_1} \odot E_{\varphi_2} \hookrightarrow E^\varphi \).

(b) If a Banach ideal space \( E \) with the Fatou property is such that \( E_a \neq \{0\} \) and \( E_{\varphi_1} \odot E_{\varphi_2} \hookrightarrow E^\varphi \), then \( \varphi_1^{-1} \varphi_2^{-1} \prec \varphi^{-1} \) for large arguments.

(c) If a Banach ideal space \( E \) has the Fatou property, \( \text{supp} E_a = \Omega \), \( L^\infty \not\hookrightarrow E \) and \( E_{\varphi_1} \odot E_{\varphi_2} \hookrightarrow E^\varphi \), then \( \varphi_1^{-1} \varphi_2^{-1} \prec \varphi^{-1} \) for small arguments.

(d) If \( e \) is a Banach sequence space with the Fatou property, \( \text{supp}_{k \in \mathbb{N}} \|e_k\|_e < \infty \), \( l^\infty \not\hookrightarrow L^\infty \) and \( e_{\varphi_1} \odot e_{\varphi_2} \hookrightarrow e_{\varphi} \), then \( \varphi_1^{-1} \varphi_2^{-1} \prec \varphi^{-1} \) for small arguments.

Note that in case (c) we can even conclude the relation \( \varphi_1^{-1} \varphi_2^{-1} \prec \varphi^{-1} \) for all arguments, using (b) and (c).
The sufficient and necessary conditions on the reverse inclusion $E_\psi \hookrightarrow E_{\psi_1} \odot E_{\psi_2}$ need also the reverse relations between Young functions, the same as in [41].

The symbol $\psi^{-1} < \psi_1^{-1} \psi_2^{-1}$ for all arguments [for large arguments] (for small arguments) means that there is a constant $D > 0$ [there are constants $D, u_0 > 0$] (there are constants $D, u_0 > 0$) such that the inequality

$$\psi^{-1}(u) \leq D \psi_1^{-1}(u) \psi_2^{-1}(u)$$

(17)

holds for all $u > 0$ [for all $u \geq u_0$] (for all $0 < u \leq u_0$), respectively.

**Theorem 5.** Let $\psi_1, \psi_2$ and $\psi$ be three Young functions.

(a) If $E$ is a Banach ideal space with the Fatou property and one of the following conditions holds:

(a1) $\psi^{-1} < \psi_1^{-1} \psi_2^{-1}$ for all arguments,

(a2) $\psi^{-1} < \psi_1^{-1} \psi_2^{-1}$ for large arguments and $L^\infty \hookrightarrow E$,

(a3) $\psi^{-1} < \psi_1^{-1} \psi_2^{-1}$ for small arguments and $E \hookrightarrow L^\infty$,

then $E_\psi \hookrightarrow E_{\psi_1} \odot E_{\psi_2}$.

(b) If $E$ is a symmetric Banach function space on $I$ with the Fatou property, $E_a \neq \{0\}$ and $E_\psi \hookrightarrow E_{\psi_1} \odot E_{\psi_2}$, then $\psi^{-1} < \psi_1^{-1} \psi_2^{-1}$ for large arguments.

(c) If $E$ is a symmetric Banach function space on $I$ with the Fatou property, supp $E_a = \Omega$, $L^\infty \not\hookrightarrow E$ and $E_\psi \hookrightarrow E_{\psi_1} \odot E_{\psi_2}$, then $\psi^{-1} < \psi_1^{-1} \psi_2^{-1}$ for small arguments.

(d) Let $e$ be a symmetric Banach sequence space with the Fatou property and order continuous norm. If $e_\psi \hookrightarrow e_{\psi_1} \odot e_{\psi_2}$, then $\psi^{-1} < \psi_1^{-1} \psi_2^{-1}$ for small arguments.

**Proof.** (a1) The idea of the proof is taken from [63, Theorem 10.1(b)]. For $z \in E_\psi \setminus \{0\}$ let $y = \psi(\frac{|z|}{\|z\|_{E_\psi}})$ and

$$z_i(t) = \begin{cases} \sqrt{\frac{|z(t)|}{\psi^{-1}_1(y(t)) \psi^{-1}_2(y(t))}}, & \text{if } t \in \text{supp } z, \\ 0, & \text{otherwise,} \end{cases}$$

for $i = 1, 2$. The elements $z_i$ are well defined. Indeed, if $a_\psi = 0$, then $y(t) > 0$ for $\mu$-a.e. $t \in \text{supp } z$. If $a_\psi > 0$, then the assumption on functions implies that $a_{\psi_1} > 0$ and $a_{\psi_2} > 0$. Consequently, $\psi_1^{-1}(0) = a_{\psi_1}$ and $\psi_2^{-1}(0) = a_{\psi_2}$. Now we will prove the inequality

$$\psi_i\left(\frac{z_i}{\sqrt{D\|z\|_{E_\psi}}}\right) \leq y, \quad i = 1, 2.$$

(18)

If $a_\psi > 0$, taking $u \to 0$ in (17) we obtain $a_\psi \leq Da_{\psi_1}a_{\psi_2}$. If $y(t) = 0$, then

$$z_i(t) = \sqrt{\frac{|z(t)|}{a_{\psi_1}a_{\psi_2}}} \leq \sqrt{\frac{\|z\|_{E_\psi} a_\psi}{a_{\psi_1}a_{\psi_2}}} \psi_i^{-1}(0) \leq \sqrt{D\|z\|_{E_\psi} \psi_i^{-1}(0)}$$
and consequently \( \varphi_i \left( \frac{z_i(t)}{\sqrt{D\|z\|_{E\varphi}}} \right) = 0 = y(t) \). If \( y(t) > 0 \), then

\[
z_i(t) = \sqrt{\frac{|z(t)|}{\varphi_1^{-1}(y(t))\varphi_2^{-1}(y(t))}} \varphi_i^{-1}(y(t)) \leq \sqrt{\frac{D|z(t)|}{\varphi_1^{-1}(y(t))\varphi_2^{-1}(y(t))}} \varphi_i^{-1}(y(t)) = \sqrt{D}\|z\|_{E\varphi} \varphi_i^{-1}(y(t)).
\]

This proves (18) and consequently

\[
I_{\varphi_i} \left( \frac{z_1}{\sqrt{D}\|z\|_{E\varphi}} \right) \leq \|y\|_E = \left\| \varphi \left( \frac{|z|}{\|z\|_{E\varphi}} \right) \right\|_E \leq 1.
\]

Thus \( \|z_1\|_{E\varphi} \leq \sqrt{D}\|z\|_{E\varphi} \) and similarly \( \|z_2\|_{E\varphi} \leq \sqrt{D}\|z\|_{E\varphi} \). Since \( |z| = z_1z_2 \), it follows that \( z \in E_{\varphi_1} \cap E_{\varphi_2} \) and \( \|z\|_{E_{\varphi_1} \cap E_{\varphi_2}} \leq D\|z\|_{E\varphi} \).

(a2) If \( b_{\varphi} < \infty \) and \( L^\infty \hookrightarrow E \), then \( E_{\varphi} = L^\infty \) with equivalent norms and clearly \( E_{\varphi} = L^\infty \hookrightarrow E_{\varphi_1} \cap E_{\varphi_2} \). Suppose \( b_{\varphi} = \infty \). Set \( v_0 = \varphi_1^{-1}(u_0) \), where \( u_0 \) is from (17) and let \( v > 0 \) be such that

\[
\max\{|\varphi_1(v), \varphi_2(v)|\} = 1/2. \quad \text{For } \|z\|_{E\varphi} = 1 \text{ let } y = \varphi(|z|) \text{ and}
\]

\[
A = \{ t \in \text{supp } z : |z(t)| \geq v_0 \}, \quad B = \text{supp } z \setminus A = \{ t \in \text{supp } z : |z(t)| < v_0 \}.
\]

Define

\[
z_i(t) = \begin{cases} 
\sqrt{\frac{|z(t)|}{\varphi_1^{-1}(y(t))\varphi_2^{-1}(y(t))}} \varphi_i^{-1}(y(t)), & \text{if } t \in A, \\
\sqrt{|z(t)|}, & \text{if } t \in B, \\
0, & \text{otherwise,}
\end{cases}
\]

for \( i = 1, 2 \). Since \( \varphi(v_0) > 0 \), the functions \( z_i \) are well defined. If \( t \in A \), then

\[
z_i(t) = \sqrt{\frac{|z(t)|}{\varphi_1^{-1}(y(t))\varphi_2^{-1}(y(t))}} \varphi_i^{-1}(y(t)) \leq \sqrt{\frac{D|z(t)|}{\varphi_1^{-1}(y(t))\varphi_2^{-1}(y(t))}} \varphi_i^{-1}(y(t)) = \sqrt{D}\varphi_i^{-1}(y(t)),
\]

whence

\[
I_{\varphi_i} \left( \frac{z_1}{2\sqrt{D} \chi_A} \right) \leq \frac{1}{2} I_{\varphi_i} \left( \frac{z_1}{\sqrt{D} \chi_A} \right) \leq \frac{1}{2} \|y\|_E \leq \frac{1}{2},
\]

and

\[
I_{\varphi_i} \left( \frac{vz_1}{\sqrt{v_0} \chi_B} \right) = \left\| \varphi_1 \left( \frac{vz_1}{\sqrt{v_0} \chi_B} \right) \right\|_E \leq \varphi_1(v) \chi_\Omega \|E\| \leq \frac{1}{2}.
\]

Then, for \( \lambda = \max\{\sqrt{\frac{v}{v_0}}, 2\sqrt{D} \} \), we obtain

\[
I_{\varphi_i} \left( \frac{z_1}{\lambda} \right) \leq I_{\varphi_1} \left( \frac{z_1}{\chi_A} \right) + I_{\varphi_1} \left( \frac{z_1}{\chi_B} \right) \leq I_{\varphi_1} \left( \frac{z_1}{2\sqrt{D} \chi_A} \right) + I_{\varphi_1} \left( \frac{z_1}{\sqrt{v_0} \chi_B} \right) \leq 1.
\]
Thus \( \|z_1\|_{E_{\varphi_1}} \leq \lambda \) and similarly \( \|z_2\|_{E_{\varphi_2}} \leq \lambda \). Since \( |z| = z_1 z_2 \), it follows that \( z \in E_{\varphi_1} \otimes E_{\varphi_2} \) and \( \|z\|_{E_{\varphi_1} \otimes E_{\varphi_2}} \leq \lambda^2 \). Consequently, \( \|z\|_{E_{\varphi_1} \otimes E_{\varphi_2}} \leq \lambda^2 \|E_{\varphi}\| \) for each \( z \in E_{\varphi} \).

(a3) Since \( \varphi^{-1} < \varphi_1^{-1} \varphi_2^{-1} \) for small arguments, it follows that for any \( u > u_0 \) there is a constant \( D_1 \geq D \) such that

\[
\varphi^{-1}(u) \leq D_1 \varphi_1^{-1}(u) \varphi_2^{-1}(u)
\]  

(19)

for any \( u \leq u_1 \). We follow the same way as in the proof of (a1) replacing \( D \) by \( D_1 \) from (19) for \( u_1 = M \), where \( M \) is the constant of the inclusion \( E \hookrightarrow L^\infty \).

(b) Suppose that the condition \( \varphi^{-1} < \varphi_1^{-1} \varphi_2^{-1} \) is not satisfied for large arguments. Then there is a sequence \( (u_n) \) with \( u_n \not\rightarrow \infty \) such that \( 2^n \varphi_1^{-1}(u_n) \varphi_2^{-1}(u_n) \leq \varphi^{-1}(u_n) \) for all \( n \in \mathbb{N} \).

We repeat the construction of sequence \( (z_n) \), as it was given in [41] in the proof of Theorem 4.2(i), showing that \( \|z_n\|_{E_{\varphi_1} \otimes E_{\varphi_2}} \rightarrow \infty \) as \( n \rightarrow \infty \).

Since \( E_{\varphi} \neq \{0\} \), it follows that there is a nonzero \( 0 \leq x \in E_{\varphi} \) and so there is a set \( A \) of positive measure such that \( \chi_A \in E_{\varphi} \). Of course, for large enough \( n \) one has \( \|u_n \chi_A\|_E \geq 1 \). Applying Dobrakov’s result from [27] we conclude that the submeasure \( \omega(B) = \|u_n \chi_B\|_E \) has the Darboux property. Consequently, for each \( n \in \mathbb{N} \) there exists a set \( A_n \) such that \( \|u_n \chi_{A_n}\|_E = 1 \). Define

\[
x_n = \varphi_1^{-1}(u_n) \chi_{A_n}, \quad y_n = \varphi_2^{-1}(u_n) \chi_{A_n} \quad \text{and} \quad z_n = x_n y_n.
\]

Let us consider two cases:

1. Suppose either \( b_{\varphi_1} = \infty \) or \( b_{\varphi_1} < \infty \) and \( \varphi_1(b_{\varphi_1}) = \infty \). Then \( \varphi_1(\varphi_1^{-1}(u)) = u \) for \( u \geq 0 \) and for \( 0 < \lambda < 1 \), by the convexity of \( \varphi_1 \), we obtain

\[
I_{\varphi_1}(\frac{x_n}{\lambda}) = \left\| \varphi_1\left(\frac{\varphi_1^{-1}(u_n)}{\lambda}\right) \chi_{A_n}\right\|_E \geq \frac{1}{\lambda} \left\| \varphi_1(\varphi_1^{-1}(u_n)) \chi_{A_n}\right\|_E = \frac{1}{\lambda} \|u_n \chi_{A_n}\|_E > 1.
\]

2. Let \( b_{\varphi_1} < \infty \) and \( \varphi_1(b_{\varphi_1}) < \infty \). Then, for sufficiently large \( n \) and \( 0 < \lambda < 1 \), we have \( I_{\varphi_1}(\frac{x_n}{\lambda}) = \infty \).

In both cases \( \|x_n\|_{E_{\varphi_1}} \geq 1 \) and similarly \( \|y_n\|_{E_{\varphi_2}} \geq 1 \). Applying Theorem 2, we get

\[
\|z_n\|_{E_{\varphi_1} \otimes E_{\varphi_2}} = \varphi_1^{-1}(u_n) \varphi_2^{-1}(u_n) f_{E_{\varphi_1} \otimes E_{\varphi_2}}(m(A_n)) = \varphi_1^{-1}(u_n) \varphi_2^{-1}(u_n) f_{E_{\varphi_1}}(m(A_n)) f_{E_{\varphi_2}}(m(A_n)) = \|x_n\|_{E_{\varphi_1}} \|y_n\|_{E_{\varphi_2}} \geq 1.
\]

On the other hand, using the relation between functions on a sequence \( (u_n) \) and the fact that \( \varphi(\varphi^{-1}(u)) \leq u \) for \( u > 0 \), we obtain

\[
I_{\varphi}(2^n z_n) = \left\| \varphi(2^n \varphi_1^{-1}(u_n) \varphi_2^{-1}(u_n)) \chi_{A_n}\right\|_E \leq \left\| \varphi(\varphi^{-1}(u_n)) \chi_{A_n}\right\|_E \leq \|u_n \chi_{A_n}\|_E = 1,
\]

i.e., \( \|z_n\|_{E_{\varphi}} \leq 1/2^n \) which gives \( \frac{\|z_n\|_{E_{\varphi_1} \otimes E_{\varphi_2}}}{\|z_n\|_{E_{\varphi}}} \geq 2^n \rightarrow \infty \) as \( n \rightarrow \infty \).
(c) It can be done by combining methods from the proof of Theorem 5(b) and Theorem A(c).
(d) Suppose condition $\varphi^{-1} \prec \varphi_1^{-1} \varphi_2^{-1}$ is not satisfied for small arguments. Then we can find a sequence $u_n \to 0$ such that $2^n \varphi_1^{-1}(u_n) \varphi_2^{-1}(u_n) \leq \varphi^{-1}(u_n)$ for all $n \in \mathbb{N}$.

Assumption on a sequence space $e$ gives $\lim_{m \to \infty} \|\sum_{k=0}^m e_k\|_e = \infty$. Thus, for each $n \in \mathbb{N}$ there is a number $m_n$ such that

$$u_n \left\| \sum_{k=0}^{m_n} e_k \right\|_e \leq 1 \quad \text{and} \quad \sum_{k=0}^{m_n+1} e_k \to \infty.$$

By symmetry of $e$, $\sup_{k \in \mathbb{N}} \|e_k\|_e = \|e_1\|_e = M$. Therefore, $u_n \|\sum_{k=1}^{m_n} e_k\|_e \to 1$ as $n \to \infty$. Put

$$x_n = \varphi_1^{-1}(u_n) \sum_{k=1}^{m_n} e_k, \quad y_n = \varphi_2^{-1}(u_n) \sum_{k=1}^{m_n} e_k \quad \text{and} \quad z_n = x_n y_n.$$

Then $I_{\varphi_1}(x_n) \leq u_n \|\sum_{k=1}^{m_n} e_k\|_e \leq 1$ and

$$I_{\varphi_1}(2x_n) = \varphi_1(2\varphi_1^{-1}(u_n)) \left\| \sum_{k=1}^{m_n} e_k \right\|_e \geq 2u_n \left\| \sum_{k=1}^{m_n} e_k \right\|_e \to 2 \quad \text{as} \quad n \to \infty.$$

Therefore, for $n$ large enough $1 \geq \|x_n\|_{E_{\varphi_1}} \geq 1/2$ as well as $1 \geq \|y_n\|_{E_{\varphi_2}} \geq 1/2$. Consequently, explaining like in (b) one has $\|z_n\|_{E_{\varphi_1} \circ E_{\varphi_2}} \geq 1/4$ and $I_{\varphi}(2^n z_n) \leq 1$, which gives $\frac{\|z_n\|_{E_{\varphi_1} \circ E_{\varphi_2}}}{\|z_n\|_{E_{\varphi}}} \geq 2^{n-2} \to \infty$ as $n \to \infty$ and the proof of Theorem 5 is complete. \hfill \Box

To formulate the results on equality of product spaces we need to introduce the equivalences of inverses of Young functions $\varphi_1$, $\varphi_2$ and $\varphi$. The symbol $\varphi_1^{-1} \varphi_2^{-1} \approx \varphi^{-1}$ for all arguments [for large arguments] (for small arguments) means that $\varphi_1^{-1} \varphi_2^{-1} \prec \varphi^{-1}$ and $\varphi^{-1} \prec \varphi_1^{-1} \varphi_2^{-1}$, that is provided there are constants $C, D > 0$ [there are constants $C, D, u_0 > 0$] (there are constants $C, D, u_0 > 0$) such that the inequalities

$$C \varphi_1^{-1}(u) \varphi_2^{-1}(u) \leq \varphi^{-1}(u) \leq D \varphi_1^{-1}(u) \varphi_2^{-1}(u) \quad (20)$$

hold for all $u > 0$ [for all $u \geq u_0$] (for all $0 < u \leq u_0$), respectively.

From the above Theorem A and Theorem 5 we obtain immediately results on the product of Calderón–Lozanovskii $E_{\varphi}$-spaces which are generalizations of the known results for Orlicz spaces.

**Corollary 6.** Let $\varphi_1$, $\varphi_2$ and $\varphi$ be three Young functions.

(a) Suppose $E$ is a symmetric Banach function space with the Fatou property, $L^\infty \hookrightarrow E$ and $\text{supp} \, E_a = \Omega$. Then $E_{\varphi_1} \circ E_{\varphi_2} = E_{\varphi}$ if and only if $\varphi_1^{-1} \varphi_2^{-1} \approx \varphi^{-1}$ for all arguments.
(b) Suppose $E$ is a symmetric Banach function space with the Fatou property, $L^\infty \hookrightarrow E$ and $E_a \neq \{0\}$. Then $E_{\varphi_1} \circ E_{\varphi_2} = E_{\varphi}$ if and only if $\varphi_1^{-1} \varphi_2^{-1} \approx \varphi^{-1}$ for large arguments.
(c) Suppose $E$ is a symmetric Banach sequence space with the Fatou property and order continuous norm. Then $e_{\varphi_1} \circ e_{\varphi_2} = e_{\varphi}$ if and only if $\varphi_1^{-1} \varphi_2^{-1} \approx \varphi^{-1}$ for small arguments.
The following construction appeared in [94] and in [23]: for two Young functions \( \varphi_1, \varphi_2 \) (or even for only the so-called \( \varphi \)-functions, cf. the definition below) one can define a new function \( \varphi_1 \oplus \varphi_2 \) by the formula

\[
(\varphi_1 \oplus \varphi_2)(u) = \inf_{u = vw} \left[ \varphi_1(v) + \varphi_2(w) \right] = \inf_{v > 0} \left[ \varphi_1(v) + \varphi_2 \left( \frac{u}{v} \right) \right],
\]

for \( u \geq 0 \). This operation was investigated in [94,13,63,66,88,89]. Note that \( \varphi_1 \oplus \varphi_2 \) is non-decreasing, left-continuous function and is 0 at \( u = 0 \). Moreover, as it was proved in [94, pp. 267, 271] and [88, Theorem 1]: if \( \varphi_1, \varphi_2 \) are nondegenerate Orlicz functions and \( \varphi = \varphi_1 \oplus \varphi_2 \) then

\[
\varphi^{-1}(t) \leq \varphi_1^{-1}(t) \varphi_2^{-1}(t) \leq 2 \varphi^{-1}(2t) \quad \text{for all } t > 0.
\]

The function \( \varphi \) need not to be convex even if both \( \varphi_1 \) and \( \varphi_2 \) are convex functions. However, if \( \varphi \) is a convex function, then \( \varphi^{-1} \leq \varphi_1^{-1} \varphi_2^{-1} \leq 2 \varphi^{-1} \) and, by Theorems A(a1) and 5(a1), we obtain \( E_{\varphi_1} \odot E_{\varphi_2} = E_{\varphi} \).

We will prove the last result without the explicit assumption that \( \varphi \) is convex, but to do this we need to extend definition of the Calderón–Lozanovskiǐ \( E_{\varphi} \)-space to this case (cf. [37] for the definition and some results).

For a non-decreasing and left-continuous function \( \varphi : [0, \infty) \to [0, \infty) \) with \( \varphi(0) = 0 \) (such a function is called \( \varphi \)-function) assume that there exist \( C, \alpha > 0 \) such that

\[
\varphi(st) \leq C t^\alpha \varphi(s) \quad \text{for all } s > 0 \text{ and } 0 < t < 1.
\]

Then the Calderón–Lozanovskiǐ \( E_{\varphi} \)-space is a quasi-Banach ideal space (since \( \varphi \) need not to be convex) with the quasi-norm

\[
\|x\|_{E_{\varphi}} = \inf \{ \lambda > 0 : \|\varphi(|x|/\lambda)\|_E \leq 1 \}.
\]

Note that for a convex function \( \varphi \) the condition (22) holds with \( C = \alpha = 1 \) and the space \( E_{\varphi} \) is normable when (22) holds with \( \alpha \geq 1 \).

We know that if \( \varphi_1 \) and \( \varphi_2 \) are convex functions, then \( \varphi = \varphi_1 \oplus \varphi_2 \) is not necessary a convex function but (22) holds with \( C = 1 \) and \( \alpha = 1/2 \) and the space \( E_{\varphi} \) is a quasi-Banach ideal space.

In fact, for any \( s > 0 \) and \( 0 < t < 1 \) we have

\[
\varphi(st) = \inf_{s = uv} \left[ \varphi_1(v) + \varphi_2(w) \right] = \inf_{s = ab} \left[ \varphi_1(\sqrt{t}a) + \varphi_2(\sqrt{t}b) \right] \\
\leq \sqrt{t} \inf_{s = ab} \left[ \varphi_1(a) + \varphi_2(b) \right] = \sqrt{t} \varphi(s).
\]

**Theorem 6.** Let \( \varphi_1, \varphi_2 \) be Young functions. If \( \varphi = \varphi_1 \oplus \varphi_2 \), then \( E_{\varphi_1} \odot E_{\varphi_2} = E_{\varphi} \), where \( E \) is a Banach ideal space with the Fatou property.

**Proof.** We prove that \( E_{\varphi_1} \odot E_{\varphi_2} \hookrightarrow E_{\varphi} \). By the definition of \( \varphi_1 \oplus \varphi_2 \) one has

\[
\varphi(uv) \leq \varphi_1(u) + \varphi_2(v)
\]
for each \( u, v > 0 \) with \( \varphi_1(u) < \infty \), \( \varphi_2(v) < \infty \). Let \( z \in E_{\varphi_1} \odot E_{\varphi_2}, z \neq 0 \), and take arbitrary \( 0 \leq x \in E_{\varphi_1}, 0 \leq y \in E_{\varphi_2} \) with \( |z| = xy \). Since (22) holds with \( C = 1 \) and \( \alpha = 1/2 \), it follows that for \( 0 < t < 1/4 \) we obtain

\[
I_{\varphi}(tz, x, y) \leq \sqrt{t} \parallel \varphi_1(x) \parallel E_{\varphi_1} + \parallel \varphi_2(y) \parallel E_{\varphi_2} \leq 2\sqrt{t} < 1.
\]

Thus \( \parallel z \parallel E_{\varphi} \leq \frac{1}{\sqrt{t}} \parallel x \parallel E_{\varphi_1} \parallel y \parallel E_{\varphi_2} \) and consequently \( \parallel z \parallel E_{\varphi} \leq \frac{1}{1/2} \parallel z \parallel E_{\varphi_1} \odot E_{\varphi_2} \). Finally, \( E_{\varphi_1} \odot E_{\varphi_2} \rightarrow E_{\varphi} \).

The proof of the inclusion \( E_{\varphi} \hookrightarrow E_{\varphi_1} \odot E_{\varphi_2} \) is exactly the same as the proof of Theorem 5(a1) since the convexity of \( \varphi \) has not been used there, which proves the theorem. \( \square \)

5. The product of Lorentz and Marcinkiewicz spaces

Before proving results on the product of Lorentz and Marcinkiewicz spaces on \( I = (0, 1) \) or \( I = (0, \infty) \) we need some auxiliary lemmas on the Calderón construction and the notion of dilation operator.

The dilation operator \( D_s, s > 0 \), defined by \( D_s x(t) = x(t/s) \chi_I(t/s) \), \( t \in I \) is bounded in any symmetric space \( E \) on \( I \) and \( \parallel D_s \parallel E \rightarrow E \leq \max(1, s) \) (see [87, Lemma 1] in the case \( I = (0, 1) \), [43, pp. 96–98] for \( I = (0, \infty) \) and [48, p. 130] for both cases). Moreover, the Boyd indices of \( E \) are defined by

\[
\alpha_E = \lim_{s \to 0^+} \frac{\ln \parallel D_s \parallel E \rightarrow E}{\ln s}, \quad \beta_E = \lim_{s \to \infty} \frac{\ln \parallel D_s \parallel E \rightarrow E}{\ln s},
\]

and we have \( 0 \leq \alpha_E \leq \beta_E \leq 1 \).

Lemma 3. Let \( E, F \) be symmetric function spaces on \( I \) and \( 0 < \theta < 1 \). Then

\[
\frac{1}{2} \parallel z^* \parallel E^\theta F^{1-\theta} \leq \parallel z^* \parallel E^\theta F^{1-\theta} \leq \parallel z^* \parallel E^\theta F^{1-\theta},
\]

where \( \parallel z^* \parallel E^\theta F^{1-\theta} := \inf\{\max(\parallel x^* \parallel E, \parallel y^* \parallel F) : z^* \leq (x^*)^\theta (y^*)^{1-\theta}, x \in E_+, y \in F_+\} \).

Proof. Since

\[
\parallel x^* \parallel E = \parallel D_2 D_1 x^* \parallel E \leq \parallel D_2 \parallel E \rightarrow E \parallel D_1 x^* \parallel E \leq 2 \parallel D_1 x^* \parallel E
\]

and \( (|x|^\theta |y|^{1-\theta})^\theta (t) \leq x^*(t/2)^\theta y^*(t/2)^{1-\theta} \) for any \( t \in I \) (cf. [43, p. 67]), it follows that
\[ \| z^* \|_{E^\theta F^{1-\theta}} = \inf \left\{ \max \left( \| x \|_E, \| y \|_F \right): \| x^\theta y^{1-\theta} \| \leq x^\theta y^{1-\theta}, x \in E_+, y \in F_+ \right\} \]

\[ \geq \inf \left\{ \max \left( \| x \|_E, \| y \|_F \right): z^*(t) \leq x^\theta(t) y^{1-\theta}(t), x \in E_+, y \in F_+ \right\} \]

\[ = \frac{1}{2} \inf \left\{ \max \left( \| x \|_E, \| y \|_F \right): z^*(t) \leq x^\theta(t) y^{1-\theta}(t), x \in E_+, y \in F_+ \right\} \]

\[ = \frac{1}{2} \| z^* \|_{E^\theta F^{1-\theta}}. \]

The other estimate is clear and the lemma follows. \(\square\)

As a consequence of representation (10) and the above lemma with \(\theta = 1/2\) we obtain

**Corollary 7.** Let \(E, F\) be symmetric function spaces on \(I\). Then

\[ \| z^* \|_{E \odot F} \leq \inf \left\{ \| x \|_E, \| y \|_F: z^* \leq x^* y^*, x \in E_+, y \in F_+ \right\} \leq 2 \| z^* \|_{E \odot F}. \]

The idea of proof of the next result is coming from the Calderón paper [15, Part 13.5]. For a Banach function space \(E\) on \(I = (0, 1)\) or \((0, \infty)\) define new spaces \(E^{(s)}\) and \(E^{(ss)}\) (symmetrizations of \(E\)) as

\[ E^{(s)} = \{ x \in L^0(I): x^* \in E \}, \quad E^{(ss)} = \{ x \in L^0(I): x^{**} \in E \} \]

with the functionals \(\| x \|_{E^{(s)}} = \| x^* \|_E\) and \(\| x \|_{E^{(ss)}} = \| x^{**} \|_E\). If there is a constant \(1 \leq C < \infty\) such that

\[ \| D^2 x^* \|_E \leq C \| x^* \|_E \quad \text{for all } x^* \in E, \quad (23) \]

then \(E^{(s)}\) is a quasi-Banach symmetric space. Then by \(C_E\) we denote the smallest constant \(C\) satisfying the above inequality. The space \(E^{(ss)}\) is always a Banach symmetric space. Consider the Hardy operator \(H\) and its dual \(H^*\) defined by

\[ Hx(t) = \frac{1}{t} \int_0^t x(s) \, ds, \quad H^*x(t) = \int_{t/m(I)}^{t} \frac{x(s)}{s} \, ds \quad \text{with } l = m(I), \ t \in I. \quad (24) \]

**Remark 7.** If \(E\) is a Banach function space on \(I\) and the operator \(H\) is bounded in \(E\), then (23) holds with \(C_E \leq 2 \| H \|_{E \rightarrow E}\). This follows directly from the estimates

\[ \| H x^* \|_E = \left\| \frac{1}{t} \int_0^t x^*(st) \, ds \right\|_E \geq \frac{1}{2} \left\| \int_0^t x^*(st) \, ds \right\|_E \geq \frac{1}{2} \| x^*(t/2) \|_E. \]

As we already mentioned before the Calderón spaces \(E^\theta F^{1-\theta}\) can be also defined for quasi-Banach spaces \(E, F\) (cf. [74,70,37]).
Lemma 4. Let $E$ and $F$ be Banach function spaces on $I$ and $0 < \theta < 1$. Suppose that both operators $H$, $H^*$ are bounded in $E$ and $F$. Then

$$(E^\theta)^\theta (F^\theta)^{1-\theta} \overset{C_1}{\supseteq} (E^\theta F^{1-\theta})^\theta \overset{C_2}{\supseteq} (E^\theta)^\theta (F^\theta)^{1-\theta},$$

where $C_1 = C_E C_F^{1-\theta}$, $C_2 = \|HH^*\|_{E \rightarrow E}^{\theta} \|HH^*\|_{F \rightarrow F}^{1-\theta}$ and

$$(E^\theta)^\theta (F^\theta)^{1-\theta} \overset{C_3}{\supseteq} (E^\theta F^{1-\theta})^{\theta \theta} \overset{C_4}{\supseteq} (E^\theta)^\theta (F^\theta)^{1-\theta},$$

where $C_3 = \|H\|_{E \rightarrow E} \|HH^*\|_{E \rightarrow F} \theta [\|H\|_{F \rightarrow F} \|HH^*\|_{F \rightarrow F}]^{1-\theta}$.

Proof. Inclusions (25). Let $z \in (E^\theta)^\theta (F^\theta)^{1-\theta}$. Then $|z| \leq \lambda |x|^{\theta} |y|^{1-\theta}$ for some $\lambda > 0$ and $\|x\|_E \leq 1$, $\|y\|_F \leq 1$. Thus

$$z^\theta(t) \leq \lambda (|x|^{\theta} |y|^{1-\theta})^\theta(t) \leq \lambda x^\theta(t/2) y^\theta(t/2)^{1-\theta} = \lambda C_1 \left( \frac{x^\theta(t/2)}{C_E} \right)^\theta \left( \frac{y^\theta(t/2)}{C_F} \right)^{1-\theta}$$

for any $t \in I$, which means that $z \in (E^\theta F^{1-\theta})$. With the norm $\leq \lambda C_1$.

On the other hand, if $z \in (E^\theta F^{1-\theta})$, then $z^\theta \in E^\theta F^{1-\theta}$ and so

$$z^\theta \leq \lambda |x|^{\theta} |y|^{1-\theta} \quad \text{with some } \lambda > 0, \|x\|_E \leq 1, \|y\|_F \leq 1.$$

The following equality is true

$$HH^* x(t) = Hx(t) + H^* x(t), \quad t \in I.$$  (27)

In fact, using the Fubini theorem, we obtain for $x \geq 0$

$$HH^* x(t) = \frac{1}{t} \int_0^t \left( \int_0^r \frac{1}{s} \frac{x(r)}{r} dr ds \right) + \frac{1}{t} \int_0^t \left( \int_r^t ds \right) \frac{x(r)}{r} dr$$

$$= Hx(t) + H^* x(t).$$

Using the equality (27) and twice the Hölder–Rogers inequality we obtain

$$z^\theta \leq H(z^\theta) \leq H(z^\theta) + H^* (z^\theta) = HH^* (z^\theta)$$

$$\leq \lambda HH^* (|x|^{\theta} |y|^{1-\theta}) \leq \lambda H [ (H^* |x|)^\theta (H^* |y|)^{1-\theta} ]$$

$$\leq \lambda [ HH^* (|x|) ]^\theta [ HH^* (|y|) ]^{1-\theta}.$$

By the Ryff theorem there exists a measure-preserving transformation $\omega : I \rightarrow I$ such that $|z| = z^\theta(\omega)$ a.e. (cf. [7, Theorem 7.5] for $I = (0, 1)$ or Corollary 7.6 for $I = (0, \infty)$ under the additional assumption that $z^\theta(\infty) = 0$). Thus

$$|z| = z^\theta(\omega) \leq \lambda [ HH^* (|x|)(\omega) ]^\theta [ HH^* (|y|)(\omega) ]^{1-\theta} = \lambda u^\theta v^{1-\theta}.$$
Since $H^*|x|$ is a non-increasing function, it follows that $HH^*|x|$ is also a non-increasing function and $HH^*|x| = [HH^*|x|]^* = [(HH^*|x|)(\omega)]^*$.

Similarly for $H^*|y|$. Hence,

$$
\|u\|_{E^1} = \|u^*\|_E = \left\| \left[ (H^*|x|)(\omega) \right]^\ast \right\|_E
$$

$$
= \|HH^*|x|\|_E \leq \|HH^*\|_{E \rightarrow E}\|x\|_E \leq \|HH^*\|_{E \rightarrow E}
$$

and

$$
\|v\|_{F^1} = \|v^*\|_F = \left\| \left[ (H^*|y|)(\omega) \right]^\ast \right\|_F
$$

$$
= \|HH^*|y|\|_F \leq \|HH^*\|_{F \rightarrow F}\|y\|_F \leq \|HH^*\|_{F \rightarrow F},
$$

which means that $z \in (E^*)^\theta (F^*)^{1-\theta}$ with the norm $\leq \lambda C_2$.

To finish the proof in the case $I = (0, \infty)$ we need to show that $z^*(\infty) = 0$. If we will have $z^*(\infty) = a > 0$, then $\lambda |x(t)|^\theta |y(t)|^{1-\theta} \geq a$ for almost all $t > 0$ and considering the sets $A = \{t > 0: |x(t)| \geq a/\lambda\}$, $B = \{t > 0: |y(t)| \geq a/\lambda\}$ we obtain $A \cup B = (0, \infty)$ up to a set of measure zero. Then

$$
H^*|x|(t) = \int_0^\infty \frac{|x(s)|}{s} ds \geq \int_{A \cap (t, \infty)} \frac{a}{\lambda s} ds
$$

and

$$
H^*|y|(t) = \int_0^\infty \frac{|y(s)|}{s} ds \geq \int_{B \cap (t, \infty)} \frac{a}{\lambda s} ds,
$$

which means that $H^*|x|(t) + H^*|y|(t) = +\infty$ for all $t > 0$. Since

$$
(0, \infty) = \{t > 0: H^*|x|(t) = \infty\} \cup \{t > 0: H^*|y|(t) = \infty\}
$$

(maybe except a set of measure zero), it follows that $H^*|x| \notin E$ or $H^*|y| \notin F$, which is a contradiction.

Inclusions (26). Let $z \in (E^{**})^\theta (F^{**})^{1-\theta}$. Then $|z| \leq \lambda |x|^\theta |y|^{1-\theta}$ for some $\lambda > 0$ and $\|x^{**}\|_E \leq 1$, $\|y^{**}\|_F \leq 1$. Thus

$$
z^{**}(t) \leq \lambda (|x|^\theta |y|^{1-\theta})^{**}(t) \leq \lambda x^{**}(t)^\theta y^{**}(t)^{1-\theta}
$$

for any $t \in I$, which means that $z \in (E^\theta F^{1-\theta})^{(**)}$ with the norm $\leq \lambda$.

On the other hand, if $z \in (E^\theta F^{1-\theta})^{(**)}$, then $z^{**} \in E^\theta F^{1-\theta}$ and repeating the above arguments we obtain

$$
|z| = z^*(\omega) \leq z^{**}(\omega) = Hz^*(\omega)
$$

$$
\leq \lambda [HHH^*|x| (\omega)]^\theta [HHH^*|y| (\omega)]^{1-\theta} = \lambda u_1^\theta v_1^{1-\theta}.
$$
Since $HHH^*|x| = |HHH^*|x|)^* = (HHH^*|x|)^*$, it follows that
\[
\|u_1\|_{E^{(ss)}} = \|u_1^*\|_E = \|Hu_1^*\|_E \leq \|H\|_{E \to E} \|u_1^*\|_E
\]
\[
= \|H\|_{E \to E} \|\left((HHH^*|x|)^*\right)(\omega)\|_E = \|H\|_{E \to E} \|HHH^*|x|\|_E
\]
\[
\leq \|H\|_{E \to E} \|HHH^*\|_{E \to E} \|x\|_E \leq \|H\|_{E \to E} \|HHH^*\|_{E \to E}
\]
and
\[
\|v_1\|_{F^{(ss)}} = \|v_1^*\|_F = \|Hv_1^*\|_F \leq \|H\|_{F \to F} \|v_1^*\|_F
\]
\[
= \|H\|_{F \to F} \|\left((HHH^*|y|)^*\right)(\omega)\|_F = \|H\|_{F \to F} \|HHH^*|y|\|_F
\]
\[
\leq \|H\|_{F \to F} \|HHH^*\|_{F \to F} \|y\|_F \leq \|H\|_{F \to F} \|HHH^*\|_{F \to F},
\]
which implies that $z \in (E^{(ss)})^{\theta} (F^{(ss)})^{1-\theta}$ with the norm $\leq \lambda C_2$, and the lemma follows.

Note that our proofs are working for both cases $I = (0, 1)$ and $I = (0, \infty)$. The inclusions (25) were proved by Calderón but his result is true only in the case when $I = (0, \infty)$ (cf. [15, pp. 167–169]). Since he was working with the other composition $(25)$ were proved by Calderón but his result is true only in the case when $H_1 \parallel H_2$, which implies that $z \in (E^{(ss)})^{\theta} (F^{(ss)})^{1-\theta}$ with the norm $\leq \lambda C_2$, and the lemma follows.

For the identification of product spaces we will need a result on the Calderón construction for weighted Lebesgue spaces $L^p(w) = \{x \in L^0(\mu): \mu x \in L^p(\mu)\}$ with the norm $\|x\|_{L^p(w)} = \|xw\|_{L^p}$, where $1 \leq p \leq \infty$ and $w \geq 0$. Such a result for $p_0 = p_1$ was given in [75, p. 459], and for general Banach ideal spaces in [44, Theorem 2] (for $1 \leq p_0, p_1 < \infty$ it also follows implicitly from [9, Theorem 5.5.3] and results on relation between the complex method and the Calderón construction). We present here a direct proof.

**Lemma 5.** Let $1 \leq p_0, p_1 \leq \infty$ and $0 < \theta < 1$. Then
\[
L^{p_0}(w_0)^{1-\theta} L^{p_1}(w_1)^{\theta} \equiv L^p(w),
\]
where
\[
\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}
\]
and $w = w_0^{1-\theta} w_1^{\theta}$.

**Proof.** Suppose $1 \leq p_0, p_1 < \infty$. If $x \in L^{p_0}(w_0)^{1-\theta} L^{p_1}(w_1)^{\theta}$, then $|x| \leq \lambda |x_0|^{1-\theta} |x_1|^{\theta}$ with $\|x_0\|_{L^{p_0}(w_0)} \leq 1$ and $\|x_1\|_{L^{p_1}(w_1)} \leq 1$. Using the Hölder–Rogers inequality we obtain
\[
\int |xw|^p d\mu \leq \lambda^p \int |x_0 w_0|^{(1-\theta)p} |x_1 w_1|^{\theta p} d\mu
\]
\[
\leq \lambda^p \left( \int |x_0 w_0|^{p_0} d\mu \right)^{(1-\theta)p/p_0} \left( \int |x_1 w_1|^{p_1} d\mu \right)^{\theta p/p_1}
\]
\[
= \lambda^p \|x_0 w_0\|_{L^{p_0}}^{(1-\theta)p} \|x_1 w_1\|_{L^{p_1}}^{\theta p},
\]
that is
\[ \| x \|_{L^p(w)} \leq \lambda \| x_0 w_0 \|_{L^{p_0}}^{1-\theta} \| x_1 w_1 \|_{L^{p_1}}^{\theta} \leq \lambda, \]

and so \( x \in L^p(w) \) with the norm \( \leq \lambda \).

On the other hand, if \( 0 \neq x \in L^p(w) \) then, considering \( x_i(t) = \frac{|x(t)w(t)|^{p/p_i}}{\|x\|_{L^p(w)}^{p_i}} \) on the support of \( w_i \) and 0 otherwise \( (i = 0, 1) \), we obtain \( |x(t)| = \| x \|_{L^p(w)} x_0(t)^{1-\theta} x_1(t)\theta \) and

\[ \| x_i \|_{L^{p_i}(w_i)}^2 = \int \left[ x_i(t) w_i(t) \right]^{p_i} d\mu = \int \frac{|x(t)w(t)|^p}{\|x\|_{L^p(w)}^p} d\mu = 1. \]

Therefore, \( x \in L^{p_0}(w_0)^{1-\theta} L^{p_1}(w_1)^\theta \) with the norm \( \leq \| x \|_{L^p(w)} \) and (28) is proved. The proof for the case when one or both \( p_0, p_1 \) are equal \( \infty \) is even simpler. \( \square \)

We want to calculate the product spaces of Lorentz space \( \Lambda_{\phi} \) and Marcinkiewicz space \( M_{\phi} \) on \( I \), where \( \phi \) is a quasi-concave function on \( I \) with \( \phi(0^+) = 0 \). We will do this, in fact, for some other closely related spaces. Consider the Lorentz space \( \Lambda_{\phi, p} \) for \( 0 < p < \infty \) on \( I \) defined as

\[ \Lambda_{\phi, p} = \left\{ x \in L^0(I): \| x \|_{\Lambda_{\phi, p}} = \left( \int I \left[ \phi(t) x^*(t) \right]^p \frac{dt}{t} \right)^{1/p} < \infty \right\}. \]

The space \( \Lambda_{\phi, 1} \) is a Banach space and if \( \phi(t) \leq at\phi'(t) \) for all \( t \in I \), then \( \Lambda_{\phi, 1} \xrightarrow{a} \Lambda_{\phi, \infty} \) (space \( \Lambda_{\phi} \) was defined in (6)). Consider also another Marcinkiewicz space \( M^\ast_{\phi} \) than the space \( M_{\phi} \) defined in (4), that is

\[ M^\ast_{\phi} = M^\ast_{\phi}(I) = \left\{ x \in L^0(I): \| x \|_{M^\ast_{\phi}} = \sup_{t \in I} \phi(t) x^*(t) < \infty \right\}. \]

This Marcinkiewicz space need not to be a Banach space and always we have \( M_{\phi} \xhookrightarrow{1} M^\ast_{\phi} \). Moreover, \( M^\ast_{\phi} \xhookrightarrow{C} M_{\phi} \) if and only if

\[ \int_0^t \frac{1}{\phi(s)} ds \leq C \frac{t}{\phi(t)} \text{ for all } t \in I. \quad (29) \]

In fact, since \( \frac{1}{\phi} \in M^\ast_{\phi} \), so (29) is necessary for this inclusion. On the other hand, if (29) holds and \( x \in M^\ast_{\phi} \), then

\[ \| x \|_{M_{\phi}} = \sup_{t \in I} \phi(t) x^*(t) = \sup_{t \in I} \frac{\phi(t)}{t} \int_0^t \frac{1}{\phi(s)} \phi(s) x^*(s) ds \]

\[ \leq \sup_{s \in I} \phi(s) x^*(s) \sup_{t \in I} \frac{\phi(t)}{t} \int_0^t \frac{1}{\phi(s)} ds \leq C \| x \|_{M^\ast_{\phi}}. \]
We can consider spaces $\Lambda_{w, p}$ and $M^*_w$ for more general weights $w \geqslant 0$, but then the problem of being quasi-Banach space or Banach space will appear. Such investigations can be found in [19] and [36].

Since the indices of the quasi-concave function on $I$ are useful in the formulation of further results let us define them. The lower index $p_{\phi, I}$ and upper index $q_{\phi, I}$ of a function $\phi$ on $I$ are numbers defined as

$$p_{\phi, I} = \lim_{t \to 0^+} \frac{\ln m_{\phi, I}(t)}{\ln t}, \quad q_{\phi, I} = \lim_{t \to \infty} \frac{\ln m_{\phi, I}(t)}{\ln t},$$

where $m_{\phi, I}(t) = \sup_{s \in I, s \neq t} \frac{\phi(st)}{\phi(s)}$.

It is known (see, for example, [43, 62, 63]) that for a quasi-concave function $\phi$ on $[0, \infty)$ we have $0 \leqslant p_{\phi, [0, \infty)} \leqslant p_{\phi, [0, 1]} \leqslant q_{\phi, [0, 1]} \leqslant q_{\phi, [0, \infty)} \leqslant 1$. Moreover, (29) is equivalent to $p_{\phi, I} < 1$. We also need for a differentiable increasing function $\phi$ on $I$ with $\phi(0^+) = 0$ the Simonenko indices defined by

$$s_{\phi, I} = \inf_{t \in I} \frac{t \phi'(t)}{\phi(t)}, \quad \sigma_{\phi, I} = \sup_{t \in I} \frac{t \phi'(t)}{\phi(t)}.$$

They satisfy $0 \leqslant s_{\phi, I} \leqslant p_{\phi, I} \leqslant q_{\phi, I} \leqslant \sigma_{\phi, I}$ (cf. [62, p. 22] and [63, Theorem 11.11]).

**Theorem 7.**

(i) If $\phi$, $\psi$ are quasi-concave functions on $I$, then $M^*_\phi \psi \xhookrightarrow{1} M^*_\phi \otimes M^*_\psi \xhookrightarrow{2} M^*_\psi$.

(ii) Let $\phi$, $\psi$ and $\phi \psi$ be increasing concave functions on $I$ with $\phi(0^+) = \psi(0^+) = 0$. If $s_{\phi, I} \geqslant a > 0$ and $s_{\psi, I} \geqslant b > 0$, then $\Lambda_\phi \otimes M^*_\psi \xhookrightarrow{4+4/a} \Lambda_{\phi \psi} \xhookrightarrow{2/b} \Lambda_{\phi} \otimes M^*_\psi$.

(iii) Let $\phi$, $\psi$ be quasi-concave functions on $I$ such that $0 < p_{\phi, I} \leqslant q_{\phi, I} < 1$ and $0 < p_{\psi, I} \leqslant q_{\psi, I} < 1$. Then

$$\Lambda_{\phi, I} \otimes \Lambda_{\psi, I} = \Lambda_{\phi \psi, I}/2, \quad \Lambda_{\phi, I} \otimes M^*_\psi = \Lambda_{\phi \psi, I}, \quad M^*_\phi \otimes M^*_\psi = M^*_{\phi \psi},$$

with equivalent quasi-norms.

**Proof.** (i) For each $z \in M^*_{\phi \psi}$ one has $z^* \leqslant \|z\|_{M^*_{\phi \psi}}$, but since $\frac{\|z\|_{M^*_{\phi \psi}}}{\phi(\psi)} \in M^*_\phi \otimes M^*_\psi$, it follows that $z \in M^*_\phi \otimes M^*_\psi$ and $\|z\|_{M^*_\phi \otimes M^*_\psi} \leqslant \|z\|_{M^*_{\phi \psi}}$.

If $z \in M^*_\phi \otimes M^*_\psi$ then, by Corollary 7, we have $z^* \leqslant x^*y^*$ for some $x^* \in M^*_\phi$, $y^* \in M^*_\psi$ and $\inf\{\|x\|_{M^*_\phi}\|y\|_{M^*_\psi} : z^* \leqslant x^*y^*\} \leqslant 2\|z\|_{M^*_\phi \otimes M^*_\psi}$. But $x^*y^* \leqslant \frac{\|x\|_{M^*_\phi}\|y\|_{M^*_\psi}}{\phi(\psi)}$ and so

$$\|z\|_{M^*_\phi} = \sup_{t \in I} \phi(t)\psi(t)z^*(t) \leqslant \sup_{t \in I} \phi(t)\psi(t)x^*(t)y^*(t) \leqslant \|x\|_{M^*_\phi}\|y\|_{M^*_\psi}.$$

Therefore,

$$\|z\|_{M^*_\phi} \leqslant \inf\{\|x\|_{M^*_\phi}\|y\|_{M^*_\psi} : z^* \leqslant x^*y^*\} \leqslant 2\|z\|_{M^*_\phi \otimes M^*_\psi}.$$
(ii) Let \( z \in \Lambda_\phi \circledast M^*_\psi \). Then for any \( \varepsilon > 0 \) we can find \( x \in \Lambda_\phi, y \in M^*_\psi \) such that \( z = xy \) and \( \|x\|_{\Lambda_\phi} \|y\|_{M^*_\psi} \leq (1 + \varepsilon) \|z\|_{\Lambda_\phi \circledast M^*_\psi} \). Since \( \psi'(t) \leq \psi(t)/t \) and \( \phi(t) \leq a t \phi'(t) \), it follows that

\[
\begin{align*}
\int_I z^*(t) d(\phi \psi)(t) &\leq \int_I x^*(t/2) y^*(t/2) \left[ \phi'(t) \psi(t) + \phi(t) \psi'(t) \right] dt \\
&\leq 2 \int_I x^*(t/2) y^*(t/2) \psi(t/2) \phi'(t) dt + \frac{1}{a} \int_I x^*(t/2) y^*(t/2) t \phi'(t) \frac{\psi(t)}{t} dt \\
&\leq 2 \sup_{s \in I} \psi(s) y^*(s) \int_I x^*(t/2) \phi'(t) dt + \frac{2}{a} \sup_{s \in I} \psi(s) y^*(s) \int_I x^*(t/2) \phi'(t) dt \\
&\leq (2 + 2/a) \|y\|_{M^*_\psi} \|D_2 x\|_{\Lambda_\phi} \\
&\leq (4 + 4/a) \|y\|_{M^*_\psi} \|x\|_{\Lambda_\phi} \leq (4 + 4/a) (1 + \varepsilon) \|z\|_{\Lambda_\phi \circledast M^*_\psi}.
\end{align*}
\]

Since \( \varepsilon > 0 \) is arbitrary the first inclusion of (ii) is proved. To prove the second one assume that \( z = z^* \in \Lambda_\phi \psi \). Then

\[
w(t) = z(t) \frac{\phi(t) \psi(t)}{t} \in L^1(I) \quad \text{and} \quad w = w^*.
\]

Moreover,

\[
\|w\|_{L^1} = \int_I z^*(t) \frac{\phi(t) \psi(t)}{t} dt \leq \frac{1}{b} \int_I z^*(t) d(\phi \psi)(t) = \frac{1}{b} \|z\|_{\Lambda_\phi \psi}.
\]

Using the Lorentz result on the duality \((\Lambda_\phi)' \equiv M_{t/\phi(t)}\) (see [49, Theorem 6], [50, Theorem 3.6.1]; see also [43, Theorem 5.2] for separable \( \Lambda_\phi \), [30, Proposition 2.5(a)] with \( p = q = 1 \), [38, Theorem 2.2]) and Lozanovskii’s factorization theorem \( L^1 \equiv \Lambda_\phi \circledast (\Lambda_\phi)' \equiv \Lambda_\phi \circledast M_{t/\phi(t)} \) we can find \( u \in \Lambda_\phi, v \in M_{t/\phi(t)} \) such that

\[
w^* = uv \quad \text{and} \quad \|u\|_{\Lambda_\phi} \|v\|_{M_{t/\phi(t)}} \leq \|w\|_{L^1}.
\]

By Corollary 7, there are \( u_0 \in \Lambda_\phi, v_0 \in M_{t/\phi(t)} \) with

\[
w^* \leq u_0^* v_0^* \quad \text{and} \quad \|u_0\|_{\Lambda_\phi} \|v_0\|_{M_{t/\phi(t)}} \leq 2 \|u\|_{\Lambda_\phi} \|v\|_{M_{t/\phi(t)}}.
\]

Let

\[
x(t) = \frac{t}{\phi(t)} \frac{w^*(t)}{\|v_0\|_{M_{t/\phi(t)}}} \quad \text{and} \quad y = \frac{w^*}{u_0}.
\]

Then \( x(t) \leq \frac{w^*(t)}{\|v_0\|_{M_{t/\phi(t)}}} \leq u_0^*(t) \) because \( M_{t/\phi(t)} \) with \( u_0^* \) and \( z(t) \frac{\phi(t) \psi(t)}{t} = w(t) = x(t) \frac{\phi(t)}{\|v_0\|_{M_{t/\phi(t)}}} \), hence
\[ z(t) = x(t) \frac{1}{\psi(t)} \| v_0 \| M_t/\phi(t) \]

with \( x \in \Lambda_{\phi} \) and \( \frac{\| v_0 \| M_t/\phi(t) }{\psi(t)} \in M_{\psi}^* \). Moreover,

\[
\| z \|_{\Lambda_{\phi} \odot M_{\psi}^*} \leq \| x \|_{\Lambda_{\phi}} \frac{1}{\psi} \| v_0 \| M_t/\phi(t) \leq \| u_0 \|_{\Lambda_{\phi}} \| v_0 \| M_t/\phi(t) \leq 2 \| u \|_{\Lambda_{\phi}} \| v \| M_t/\phi(t) \leq 2 \frac{2}{b} \| z \|_{\Lambda_{\phi} \psi}
\]

and the proof of (ii) is complete.

(iii) If \( 0 < p_{\phi,1} \leq q_{\phi,1} < 1 \), then both operators \( H, H^* \) are bounded on \( L^1(\phi(t)/t) \) (see [45, Theorem 4]) and using Lemmas 4 and 5,

\[
A_{\phi,1}^{1-\theta} \Lambda_{\psi,1}^{\theta} = \left[ L^1 \left( \frac{\phi(t)}{t} \right)^{(s)} \right]^{\theta} \left[ L^1 \left( \frac{\psi(t)}{t} \right)^{(s)} \right]^{1-\theta} = \left[ L^1 \left( \frac{\phi(t)}{t} \right) \theta L^1 \left( \frac{\psi(t)}{t} \right)^{1-\theta} \right]^{(s)}
\]

with equivalent quasi-norms. Thus, by Theorem 1(iv),

\[
\Lambda_{\phi,1} \odot \Lambda_{\psi,1} = (A_{\phi,1}^{1/2} \Lambda_{\psi,1}^{1/2})^{(1/2)} = (A_{\phi^{1/2} \psi^{1/2}, 1})^{(1/2)} = \Lambda_{\phi^{1/2} \psi^{1/2}, 1/2}
\]

with equivalent quasi-norms. The last space is not normable since it contains isomorphic copy of \( l^{1/2} \) (see [36, Theorem 1]).

If \( 0 < p_{\phi,1} \leq q_{\phi,1} < 1 \), then both operators \( H, H^* \) are bounded on \( L^\infty(\phi) \) which can be proved directly. To show this we only need here to see the equivalence of corresponding integral inequalities for \( \phi \) with the assumptions on indices of \( \phi \) and this is proved, for example, in [62, Theorem 6.4] or [63, Theorem 11.8] (see also [43, pp. 56–57]). Then, using Lemmas 4 and 5,

\[
A_{\phi,1}^{\theta} (M_{\psi}^*)^{1-\theta} = \left[ L^1 \left( \frac{\phi(t)}{t} \right)^{(s)} \right]^{\theta} \left[ L^\infty(\psi)^{(s)} \right]^{1-\theta} = \left[ L^1 \left( \frac{\phi(t)}{t} \right) \theta L^\infty(\psi)^{1-\theta} \right]^{(s)}
\]

with equivalent quasi-norms. Therefore, by Theorem 1(iv),

\[
\Lambda_{\phi,1} \odot M_{\psi}^* = [A_{\phi,1}^{1/2} (M_{\psi}^*)^{1/2}]^{(1/2)} = (A_{\phi^{1/2} \psi^{1/2}, 2})^{(1/2)} = \Lambda_{\phi,1}
\]

with equivalent quasi-norms. Similarly, for Marcinkiewicz spaces

\[
(M_{\phi}^*)^{\theta} (M_{\psi}^*)^{1-\theta} = \left[ L^\infty(\phi)^{(s)} \right]^{\theta} \left[ L^\infty(\psi)^{(s)} \right]^{1-\theta} = \left[ L^\infty(\phi) \theta L^\infty(\psi)^{1-\theta} \right]^{(s)}
\]

\[
= L^\infty(\phi^{\theta} \psi^{1-\theta})^{(s)} = M_{\phi^{\theta} \psi^{1-\theta}}^*
\]
and, by Theorem 1(iv),

\[
M^*_\phi \odot M^*_\psi = \left[ (M^*_\phi)^{1/2} (M^*_\psi)^{1/2} \right]^{(1/2)} = (M^*_{\phi^{1/2} \psi^{1/2}})^{(1/2)} = M^*_\phi \psi
\]

with equivalent quasi-norms. This proves theorem completely. \(\square\)

6. Factorization of some Banach ideal spaces

The factorization theorem of Lozanovskiǐ states that for any Banach ideal space \(E\) the space \(L^1\) has a factorization \(L^1 \equiv E \odot E'\). The natural generalization of the type

\[
F \equiv E \odot M(E, F)
\]

is not true without additional assumptions on the spaces, as we can see in the example below.

Example 2. If \(E = L^{p,1}\) with the norm \(\|x\|_E = \frac{1}{p} \int_I t^{1/p - 1} x^*(t) \, dt\) for \(1 < p < \infty\), then \(M(L^{p,1}, L^p) \equiv L^\infty\) (cf. [66, Theorem 3]) and

\[
L^{p,1} \odot M(L^{p,1}, L^p) \equiv L^{p,1} \odot L^\infty \equiv L^{p,1} \subset L^p.
\]

Therefore, the factorization (30) is not true and we even do not have the factorization \(L^p = E \odot M(E, L^p)\) with equivalent norms. Similarly, if \(F = L^{p,\infty}\) with the norm \(\|x\|_F = \sup_{t \in I} t^{1/p} x^{**}(t)\) for \(1 < p < \infty\), then \(M(L^p, L^{p,\infty}) \equiv M(L^{p,1}, L^{p'}) \equiv L^\infty\) and

\[
L^p \odot M(L^p, L^{p,\infty}) \equiv L^p \odot L^\infty \equiv L^p \subset L^{p,\infty}.
\]

Consequently, again the factorization (30) is not true and we do not even have the equality \(F = L^p \odot M(L^p, F)\) with equivalent norms.

Let us collect some factorization results of type (30). First of all the Lozanovskiǐ factorization theorem was announced in 1967 (cf. [53, Theorem 4]) and published with detailed proof in 1969 (cf. [54, Theorem 6]; see also [57]). His proof uses the Calderón space \(F = E^{1/2}(E')^{1/2}\) and the result about its dual \(F'' \equiv F' \equiv L^2\) (cf. [54, Theorem 5]; see also [63, p. 185] and [82,83]). The Lozanovskiǐ factorization theorem was new even for finite dimensional spaces. In 1976 Jamison and Ruckle [32] proved that \(l^1\) factors through every normal Banach sequence space and its Köthe dual. The proof even in the finite dimensional case is indirect and it uses the Brouwer fixed point theorem. Later on Lozanovskiǐ’s factorization result was proved by Gillespie [28] and the method was inspired by the theory of reflexive algebras of operators on Hilbert space.

If \(E, F\) are finite dimensional ideal spaces and \(B_E, B_F\) denote their unit balls, then Bollobás and Leader [12], with help of the Jamison–Ruckle method, proved the factorization \(B_E \odot B_{M(E, F)} \equiv B_F\) under assumptions that \(B_F\) is a strictly unconditional body and \(B_{M(E, F)}\) is smooth.

Nilsson [70, Lemma 2.5], using the Maurey factorization theorem (cf. [67, Theorem 8]; see also [92, pp. 264–266] and [84]), proved the following result of the type (30): if \(E\) is a Banach ideal space which is \(p\)-convex with constant 1, then

\[
E' \equiv L^{p'} \odot M(E, L^p) \equiv L^{p'} \odot M(L^{p'}, E').
\]
By the duality result and the equality (31) we conclude that if $F$ is a Banach ideal space with the Fatou property which is $q$-concave with constant $1$ for $1 < q < \infty$, then

$$F = F'' \equiv L^q \odot M(F', L^q') \equiv L^q \odot M(L^q, F).$$

(32)

The factorization (32) was proved and used by Nilsson [70, Theorem 2.4] in a new proof of the Pisier theorem (cf. [76, Theorem 2.10], [77, Theorem 2.2]; see also [90, Theorem 28.1]): if a Banach ideal space $E$ with the Fatou property is $p$-convex and $q$-concave with constants $1$, $1 < r < \infty$, and $\frac{1}{p} = \frac{q}{r} + 1 - \theta$, $\frac{1}{q} = \frac{\theta}{r}$, then the space $E_0 \equiv M(L^q, E)^{(1-\theta)}$ is a Banach ideal space and $E \equiv E_0^{1-\theta}(L^r)^{\theta}$. First part of the proof follows from the following three facts:

$$\|x|^{1-\theta}\|_{L^{1/p}(E)} = \|x|^{1/s'}\|_{M(L^{1/p}(E), s')} \equiv \|x|^{1/s'}\|_{M((E(1/p))', L'^{s'})},$$

$E^{(1/p)}$ is a Banach space and $M((E(1/p))', L'^{s'})$ is $s'$-convex with constant $1$, where $s = \frac{r}{ap}$. Second part uses (32) and by Theorem 1(ii)

$$E_0^{1-\theta}(L^r)^{\theta} \equiv E_0^{1/p} \odot L^r \equiv E_0^{1/p} \odot L^q \equiv M(L^q, E) \odot L^q \equiv E.$$

Schep proved the factorization (32) and also the reverse implication, that is, if (32) holds, then the space $F$ is $q$-concave with constant $1$ (cf. [86, Theorem 3.9]). He has also proved another factorization result (even equivalence – see [86, Theorem 3.3]): if Banach ideal space $E$ with the Fatou property is $p$-convex with constant $1$ ($1 < p < \infty$), then

$$L^p \equiv E \odot M(E, L^p).$$

(33)

Note that the factorization theorem of type (32): $F = l^q \odot M(l^q, F)$ for any $q$-concave Banach space $F$ with a monotone unconditional basis was proved already in 1980 (cf. [47, Corollary 3.2]).

If a space $E$ has the Fatou property, then in the definition of the norm of $E \odot E'$ we may take “minimum” instead of “infimum”. It is known that the Fatou property of $E$ is equivalent with the isometric equality $E \equiv E''$. Then $E$ is called perfect. This notion can be generalized to $F$-perfectness. We say that $E$ is $F$-perfect if $M(M(E, F), F) \equiv E$ (see [66,14,86] for more information about $F$-perfectness). Is there any connection between factorization (30) and $E$ being $F$-perfect?

**Theorem 8.** Let $E$, $F$ be Banach ideal spaces with the Fatou property. Then the factorization $E \odot M(E, F) \equiv F$ implies $F$-perfectness of $E$, i.e., $M(M(E, F), F) \equiv E$.

**Proof.** Schep [86, Theorem 2.8] proved that if $E \odot F$ is a Banach ideal space, then $M(E, E \odot F) \equiv F$ (see also Theorem 4 above). Since $E \odot M(E, F) \equiv F$ is a Banach ideal space by the assumption, therefore from the above Schep result we obtain

$$M(M(E, F), F) \equiv M(M(E, F), E \odot M(E, F)) \equiv E,$$

which is the $F$-perfectness of $E$. ∎

The example of Bollobás and Brightwell [10], presented in [86, Example 3.6], shows that the reverse implication is not true, even for three-dimensional spaces.
Almost all proofs in factorization theorems are tricky or use powerful theorems and, in fact, equality $E \otimes M(E, F) \equiv F$ is proved without calculating $M(E, F)$ directly. Except for some special cases it seems to be the only way to prove the equality of norms in (30). However, it seems to be also useful to have (30) with just the equivalence of norms, that is

$$F = E \otimes M(E, F).$$

This can be done by finding $M(E, F)$ and $E \otimes M(E, F)$ separately and we will do so. Notice that similarly was done in [6], where the author has proved a number of factorization theorems for $l^p$ spaces, Cesàro sequence spaces, their duals and convexifications.

Observe also that if a Banach ideal space $E$ is $p$-convex $(1 < p < \infty)$ with a constant $K > 1$, then $E^{(1/p)}$ is $1$-convex with constant $K^p$ and

$$\|x\|_0 = \inf\left\{ \sum_{k=1}^n \|x_k\|_{E^{(1/p)}} : |x| \leq \sum_{k=1}^n |x_k|, x_k \in E^{(1/p)}, n \in \mathbb{N} \right\}$$

defines the norm on $E^{(1/p)}$ with $K^{-p}\|x\|_{E^{(1/p)}} \leq \|x\|_0 \leq \|x\|_{E^{(1/p)}}$. Thus $E_0 = (E^{(1/p)}, \| \cdot \|_0)$ is a Banach ideal space and its $p$-convexification $E_0^{(p)} = E$ with the norm $\|x\|^1 = (\|x\|_p^0)^{1/p}$ is $p$-convex with constant 1 (cf. [48, Lemma 1.1.11] and [71, Proposition 2.23]), and we can use the result from (31) to obtain $E' = L^{p'} \odot M(L^{p'}, E')$.

As a straightforward conclusion from Corollary 6.1 in [41] and Theorem A(a) with Theorem 5(a) we get the following factorization theorem for Calderón–Lozanovskii $E_\varphi$-spaces.

**Theorem 9.** Let $E$ be a Banach ideal space with the Fatou property and $\text{supp } E = \Omega$. Suppose that for two Young functions $\varphi$, $\varphi_1$ there is a Young function $\varphi_2$ such that one of the following conditions holds:

(i) $\varphi_1^{-1} \varphi_2^{-1} \approx \varphi^{-1}$ for all arguments,

(ii) $\varphi_1^{-1} \varphi_2^{-1} \approx \varphi^{-1}$ for large arguments and $L^\infty \hookrightarrow E$,

(iii) $\varphi_1^{-1} \varphi_2^{-1} \approx \varphi^{-1}$ for small arguments and $E \hookrightarrow L^\infty$.

Then the factorization $E_{\varphi_1} \otimes M(E_{\varphi_1}, E_\varphi) = E_\varphi$ with equivalent norms is valid and, in consequence, the space $E_{\varphi_1}$ is $E_\varphi$-perfect up to the equivalence of norms.

Moreover, applying Lemma 7.4 from [41] to Theorem 9(i) one has the following special case.

**Corollary 8.** Let $\varphi, \varphi_1$ be two Orlicz functions, and let $E$ be a Banach ideal space with the Fatou property and $\text{supp } E = \Omega$. If the function $f_v(u) = \frac{\varphi(uv)}{\varphi_1(u)}$ is non-increasing on $(0, \infty)$ for any $v > 0$, then the factorization $E_{\varphi_1} \otimes M(E_{\varphi_1}, E_\varphi) = E_\varphi$ is valid with equivalent norms and, in consequence, the space $E_{\varphi_1}$ is $E_\varphi$-perfect up to the equivalence of norms.

**Proof.** It is enough to take as $\varphi_2$ the function defined by

$$\varphi_2(u) = (\varphi \ominus \varphi_1)(u) = \sup_{v>0} \left[ \varphi(uv) - \varphi_1(v) \right]$$

and use the fact proved in [41, Lemma 7.4] showing that $\varphi_1^{-1} \varphi_2^{-1} \approx \varphi^{-1}$ for all arguments. \qed
Before we consider the factorization of Lorentz and Marcinkiewicz spaces, let us calculate “missing” spaces of multipliers.

**Proposition 3.** Suppose \( \phi, \psi \) are non-decreasing, concave functions on \( I \) with \( \phi(0^+) = \psi(0^+) = 0 \). Let \( E \) and \( F \) be symmetric spaces on \( I \) with fundamental functions \( f_E(t) = \phi(t) \) and \( f_F(t) = \psi(t) \). If \( \omega(t) = \sup_{0 < s \leq t} \frac{\psi(s)}{\phi(s)} \) is finite for any \( t \in I \), then

\[
M(\Lambda \phi, F) \hookrightarrow M_\omega, \quad M(E, M_\psi) \hookrightarrow M_\omega, \quad \text{and} \quad M_\omega^* \hookrightarrow M(\Lambda \phi_1, \Lambda \psi_1).
\]

If, moreover, \( s_{\phi, I} > a > 0 \), then \( M_\omega^{1/a} \hookrightarrow M(\Lambda \phi, \Lambda \psi) \).

**Proof.** By Theorem 2.2(iv) in [41] the function \( \omega \) is the fundamental function of \( M(\Lambda \phi, F) \) and, by the maximality of the space \( M_\omega \) and Theorem 2.2(i) in [41], we obtain inclusion \( M(\Lambda \phi, F) \hookrightarrow M_\omega \). On the other hand, using the property (e) from [66, p. 326], the duality \( (M_\psi)' \equiv \Lambda t/\psi(t) \) and the above result we conclude

\[
M(E, M_\psi) \equiv M((M_\psi)', E') \equiv M(\Lambda t/\psi(t), E') \hookrightarrow M_\omega.
\]

Two other inclusions will be proved if we show that \( \frac{1}{\omega} \) belongs to the corresponding spaces. Since, by Theorem 2.2(ii) in [41], we have \( \| y \|_{M(E,F)} = \sup_{\| x^* \|_E \leq 1} \| x^* y^* \|_F \), it follows that

\[
\left\| \frac{1}{\omega} \right\|_{M(\Lambda \phi_1, \Lambda \psi_1)} = \sup_{\| x \|_{\Lambda \phi_1} \leq 1} \int_I \left( x^* \frac{1}{\omega} \right)^* \psi(t) dt = \sup_{\| x \|_{\Lambda \phi_1} \leq 1} \int_I x^*(t) \frac{1}{\omega(t)} \psi(t) dt
\]

\[
= \sup_{\| x \|_{\Lambda \phi_1} \leq 1} \int_I x^*(t) \inf_{0 < s \leq t} \frac{\phi(s)}{\psi(s)} \psi(t) dt \leq \sup_{\| x \|_{\Lambda \phi_1} \leq 1} \int_I x^*(t) \frac{\phi(t)}{t} dt \leq 1,
\]

and, again by the above mentioned result in [41],

\[
\left\| \frac{1}{\omega} \right\|_{M(\Lambda \phi, \Lambda \psi)} = \sup_{\| x \|_{\Lambda \phi} \leq 1} \int_I x^*(t) \frac{1}{\omega(t)} \psi'(t) dt = \sup_{\| x \|_{\Lambda \phi} \leq 1} \int_I x^*(t) \inf_{0 < s \leq t} \frac{\phi(s)}{\psi(s)} \psi'(t) dt
\]

\[
\leq \sup_{\| x \|_{\Lambda \phi} \leq 1} \int_I x^*(t) \frac{\phi(t)}{t} dt \leq \sup_{\| x \|_{\Lambda \phi} \leq 1} \frac{1}{a} \int_I x^*(t) \phi'(t) dt \leq 1/a,
\]

and all inclusions are proved. \( \square \)

Putting together previous results on the products and multipliers of Lorentz and Marcinkiewicz spaces we are ready to prove the factorization of these spaces.

**Theorem 10.** Let \( \phi, \psi \) be non-decreasing, concave functions on \( I \) with \( \phi(0^+) = \psi(0^+) = 0 \). Suppose \( \frac{\psi(t)}{\phi(t)} \) is a non-decreasing function on \( I \).

(a) If \( s_{\phi, I} > 0 \) and \( s_{\psi, I} > 0 \), then \( \Lambda \phi \odot M(\Lambda \phi, \Lambda \psi) = \Lambda \psi \).
Moreover, for any symmetric space $F$ on $I$ with the fundamental function $f_F(t) = \psi(t)$ and under the above assumptions on $\phi$ and $\psi$ we have

$$\Lambda_\phi \odot M(\Lambda_\phi, F) = F \quad \text{if and only if} \quad F = \Lambda_\psi. \quad (35)$$

(b) If $\sigma_{\phi,I} < 1$ and $\sigma_{\psi,I} < 1$, then $M_\phi \odot M(M_\phi, M_\psi) = M_\psi$.

Moreover, for any symmetric space $E$ on $I$ having Fatou property, with the fundamental function $f_E(t) = \phi(t)$ and under the above assumptions on $\phi$ and $\psi$ we have

$$E \odot M(E, M_\psi) = M_\psi \quad \text{if and only if} \quad E = M_\phi. \quad (36)$$

(c) If $\sigma_{\phi,I} < 1$, $s_{\psi,I} > 0$ and $s_{\psi/\phi,I} > 0$, then

$$M_\phi \odot M(M_\phi, \Lambda_\psi) = \Lambda_\psi. \quad (37)$$

**Proof.** (a) Using Proposition 3 we have

$$M(\Lambda_\phi, \Lambda_\psi) \overset{1}{\hookrightarrow} M_\omega \overset{1}{\hookrightarrow} M_\omega^* \overset{1/\omega}{\rightarrow} M(\Lambda_\phi, \Lambda_\psi), \quad \text{where} \quad \omega(t) = \sup_{0 < s \leq t} \frac{\psi(s)}{\phi(s)}.$$ 

Since $\psi/\phi$ is a non-decreasing function on $I$, it follows that $\phi \omega = \psi$, $s_{\phi \omega,I} = s_{\psi,I} > 0$ and, by Theorem 7(ii),

$$\Lambda_\phi \odot M(\Lambda_\phi, \Lambda_\psi) = \Lambda_\phi \odot M_\omega^* = \Lambda_\psi$$

with equivalent norms. Under the assumptions on $F$ we obtain from Proposition 3 the inclusion $M(\Lambda_\phi, F) \overset{1}{\hookrightarrow} M_\omega$ and then, by Theorem 7(ii),

$$F = \Lambda_\phi \odot M(\Lambda_\phi, F) \overset{1}{\hookrightarrow} \Lambda_\phi \odot M_\omega \overset{1}{\hookrightarrow} \Lambda_\phi \odot M_\omega^* = \Lambda_\psi.$$ 

Minimality of $\Lambda_\psi$ gives $F = \Lambda_\psi$.

(b) Applying the fact that $(M_\phi)' = \Lambda_{1/\phi(t)}$, the general duality property of multipliers (see [66, property (c)]) and using Proposition 3 we obtain

$$M(M_\phi, M_\psi) \equiv M(M_\psi', M_\phi') \equiv M(\Lambda_{1/\psi(t)}, \Lambda_{1/\phi(t)}) = M_\omega^*$$

because $s_{1/\psi(t),I} = 1 - \sigma_{\psi,I} > 0$. Since $\psi/\phi$ is a non-decreasing function on $I$, it follows that $\phi \omega = \psi$ and $\sigma_{\phi \omega,I} = \sigma_{\psi,I} < 1$. By Theorem 7(i) and the estimate $\sigma_{\phi,I} < 1$ we have

$$M_\phi \odot M(M_\phi, M_\psi) = M_\phi \odot M_\omega = M_\phi^* \odot M_\omega = M_{\phi \omega}^* = M_{\psi}^* = M_\psi.$$ 

Under the assumptions on $E$ we obtain from Proposition 3 that $M(E, M_\psi) \overset{1}{\hookrightarrow} M_\omega$ and

$$M_\psi = E \odot M(E, M_\psi) \overset{1}{\hookrightarrow} E \odot M_\omega.$$
On the other hand, by Theorem 7(i) and the assumption $q\psi;I < 1$

$$M_\phi \odot M_\omega \overset{1}{\to} M_\phi^* \odot M_\omega^* \overset{2}{\to} M_\phi^* \equiv M_\psi = M_\psi.$$ 

Therefore, $M_\phi \odot M_\omega \overset{C}{\to} E \odot M_\omega$. Using now Schep’s theorem, saying that if $E \odot F \overset{C}{\to} E \odot G$, then $F \overset{C}{\to} G$ (see [86, Theorem 2.5]), we obtain $M_\phi \overset{C}{\to} E$. Maximality of the Marcinkiewicz space $M_\phi$ implies that $E = M_\phi$ since the fundamental function of $E$ is $f_E(t) = \phi(t)$ for all $t \in I$.

(c) Using Theorem 2.2(v) from [41] we obtain $M(M_\phi, \Lambda_\psi) \equiv \Lambda_\eta$, where $\eta(t) = \int_0^t \left( \frac{s}{\phi(s)} \right)' \psi'(s) ds < \infty$. Since

$$\eta(t) = \int_0^t \frac{\phi(s)\psi'(s) - \phi'(s)s\psi'(s)}{\phi(s)^2} ds \leq \int_0^t \frac{\psi'(s)}{\phi(s)} ds \leq \frac{1}{s_{\psi/\phi}} \int_0^t \left(\frac{\psi}{\phi}\right)' ds = \frac{1}{s_{\psi/\phi}} \frac{\psi(t)}{\phi(t)},$$

and

$$\eta(t) = \int_0^t \frac{\phi(s)\psi'(s) - \phi'(s)s\psi'(s)}{\phi(s)^2} ds \geq \int_0^t \frac{\phi(s)\psi'(s) - \phi'(s)s\psi'(s)}{\phi(s)^2} ds = \int_0^t \left(\frac{\psi}{\phi}\right)' ds = \frac{\psi(t)}{\phi(t)},$$

it follows that $M(M_\phi, \Lambda_\psi) = \Lambda_{\psi/\phi}$. Using the assumption $\sigma_{\phi,1} < 1$ to this equality and Theorem 7(ii) we obtain

$$M_\phi \odot M_\phi, \Lambda_\psi = M_\phi \odot \Lambda_{\psi/\phi} = M_\phi^* \odot \Lambda_{\psi/\phi} = \Lambda_\psi,$$

and the theorem is proved. \qed

Applying the above theorem to classical Lorentz $L_{p,1}$ and Marcinkiewicz $L_{p,\infty}$ spaces we obtain the following factorization results:

**Example 3.**

(a) If $1 \leq p \leq q < \infty$, then $L_{p,1} = L_{q,1} \odot M(L_{q,1}, L_{p,1})$.
(b) If $1 < p \leq q < \infty$, then $L_{p,\infty} = L_{q,\infty} \odot M(L_{q,\infty}, L_{p,\infty})$.
(c) If $1 < p < q < \infty$, then $L_{p,1} = L_{q,\infty} \odot M(L_{q,\infty}, L_{p,1})$.
Example 4. If either $1 \leq r \leq p < q < \infty$ or $1 < p < q \leq r \leq \infty$, then

$$L^{p,r} = L^{q,r} \circ M\left(L^{q,r}, L^{p,r}\right).$$

In fact, if $1 \leq r \leq p < q < \infty$ then using the commutativity of $r$-convexification with multipliers (see property (g) in [66]) and Proposition 3 we obtain

$$M\left(L^{q,r}, L^{p,r}\right) = M\left((L^{q/r, 1})^{(r)}, (L^{p/r, 1})^{(r)}\right) = M\left(L^{q/r, 1}, L^{p/r, 1}\right)^{(r)} = (L^{pq/(r(q-p)), \infty})^{(r)}.$$

Finally, by Theorem 1(iii) and Theorem 7(ii) with $\phi(t) = \frac{t}{r}$ and $\psi(t) = \frac{t}{r} - \frac{r}{q}$, we get

$$L^{q,r} \circ M\left(L^{q,r}, L^{p,r}\right) = \left(L^{q/r, 1}\right)^{(r)} \circ \left(L^{pq/(r(q-p)), \infty}\right)^{(r)} = (L^{p/q, 1})^{(r)} = L^{p,q}.$$

The case $1 < p < q \leq r \leq \infty$ can be proved by property (e) from [66] and the above calculations.

Theorem 11. Let $\phi$ be an increasing, concave function on $I$ with $0 < p_{\phi,1} \leq q_{\phi,1} < 1$.

(a) Suppose $F$ is a symmetric space on $I$ with the lower Boyd index $\alpha_F > q_{\phi,1}$ and such that $M(M_{\phi}^*, F) \neq \{0\}$. Then

$$F = M_{\phi}^* \circ M(M_{\phi}^*, F) = M_{\phi} \circ M(M_{\phi}, F).$$

(b) Suppose $E$ is a symmetric space on $I$ with the Fatou property, which Boyd indices satisfy $0 < \alpha_E \leq \beta_E < p_{\phi,1}$ and such that $M(E, \Lambda_{\phi}) \neq \{0\}$. Then

$$\Lambda_{\phi,1} = E \circ M(E, \Lambda_{\phi}).$$

Let us start with the following identifications.

Lemma 6. Under assumptions on $\phi$ from Theorem 11 we have

$$M\left(L^\infty(\phi)^{(s)}, F\right) = M\left(L^\infty(\phi), F\right)^{(s)} = F(1/\phi)^{(s)} .$$

(38)

Proof. Since we have the equivalences

$$z \in M\left(L^\infty(\phi), F\right)^{(s)} \iff z^* \in M\left(L^\infty(\phi), F\right) \iff \frac{z^*}{\phi} \in F$$

$$\iff z^* \in F(1/\phi) \iff z \in F(1/\phi)^{(s)}$$

with the equalities of norms, the equality $M\left(L^\infty(\phi), F\right)^{(s)} = F(1/\phi)^{(s)}$ follows. Let $I = (0, 1)$. We prove the equality $M\left(L^\infty(\phi)^{(s)}, F\right) = F(1/\phi)^{(s)}$. Clearly, it is enough to show that the following conditions are equivalent:
10. \( z \in M(L^\infty(\phi)\ast, F) \),
20. \( \frac{z}{\phi(\omega)} \in F \) for every measure-preserving transformation (mpt) \( \omega : I \to I \),
30. \( \frac{z^\ast}{\phi} \in F \).

Moreover, we shall prove that

\[
\left\| \frac{z^\ast}{\phi} \right\|_F \leq \sup_{\omega \text{-mpt}} \left\| \frac{z}{\phi(\omega)} \right\|_F = \|z\|_{M(L^\infty(\phi)\ast, F)} \leq D_2\|F\to F\left\| \frac{z^\ast}{\phi} \right\|_F. \tag{39}
\]

10 \Rightarrow 20. Let \( z \in M(L^\infty(\phi)\ast, F) \) and take arbitrary mpt \( \omega : I \to I \). Since \( \frac{1}{\phi(\omega)} \ast \phi = \frac{1}{\phi} \phi = 1 \)

it follows that \( \frac{1}{\phi(\omega)} \in L^\infty(\ast) \). Therefore, \( \frac{z}{\phi(\omega)} \in F \) and

\[
\|z\|_{M(L^\infty(\phi)\ast, F)} \leq \sup_{\omega \text{-mpt}} \left\| \frac{z}{\phi(\omega)} \right\|_F.
\]

20 \Rightarrow 10. Let \( x \in L^\infty(\phi)\ast \) with the norm \( \leq 1 \). Take an mpt \( \omega_0 \) such that

\[
|z| = |z|\ast(\omega_0) \leq \frac{|z|}{\phi(\omega_0)} \in F,
\]

because \( 1 \geq \|\ast\phi\|_L^\infty = \|(\ast\phi)(\omega_0)\|_L^\infty \). Thus \( z \in M(L^\infty(\phi)\ast, F) \) and

\[
\|z\|_{M(L^\infty(\phi)\ast, F)} \leq \sup_{\omega \text{-mpt}} \left\| \frac{z}{\phi(\omega)} \right\|_F.
\]

20 \Rightarrow 30. Take an mpt \( \omega_0 \) such that \( |z| = z^\ast(\omega_0) \). Then

\[
\frac{z^\ast}{\phi}(\omega_0) = \frac{|z|}{\phi(\omega_0)} \in F \quad \text{and} \quad \left\| \frac{z^\ast}{\phi} \right\|_F = \left\| \frac{|z|}{\phi(\omega_0)} \right\|_F \leq \sup_{\omega \text{-mpt}} \left\| \frac{z}{\phi(\omega)} \right\|_F.
\]

30 \Rightarrow 20. For each mpt \( \omega : I \to I \) we have

\[
\frac{z}{\phi(\omega)}(t) \sim \left( \frac{z}{\phi(\omega)} \ast \right)(t/2) (1/\phi(\omega)) (t/2) = D_2 \left( \frac{z^\ast}{\phi} \right)(t).
\]

By the symmetry of \( F \) we obtain \( \frac{z}{\phi(\omega)} \in F \) and

\[
\sup_{\omega \text{-mpt}} \left\| \frac{z}{\phi(\omega)} \right\|_F \leq D_2\|F\to F\left\| \frac{z^\ast}{\phi} \right\|_F.
\]

The proof of (39) and also (38) is finished for \( I = (0, 1) \). If \( I = (0, \infty) \), then the existence of a measure-preserving transformation \( \omega_0 : I \to I \) requires the additional assumption. In the first case we need \( \phi(\infty) = \infty \), which is satisfied because \( p_{\phi, I} > 0 \). In the second case, we need to have \( z^\ast(\infty) = 0 \) when \( z \in M(M^\ast_{\phi}, F) \neq \{0\} \). Suppose, on the contrary, that \( z^\ast(\infty) = a > 0 \).
Since \( z^* \in M(M_1^*, F) \), it follows that \( \frac{\alpha}{\phi} \leq \frac{z^*}{\phi} \in F \) and \( 1/\phi \in F \) gives, by the maximality of Marcinkiewicz space, that \( 1/\phi \in M_1^* \), where the fundamental function of \( F \) is \( f_F(t) = \psi(t) \). Thus \( \sup_{t>0} \frac{\psi(t)}{\phi(t)} < \infty \). On the other hand, since \( p_\psi \geq \alpha_F > q_\phi \) and \( p_\psi/\phi \geq p_\psi - q_\phi > 0 \), then for \( 0 < \varepsilon < (p_\psi - q_\phi)/2 \) and for large \( t \)

\[
\frac{\psi(t)}{\phi(t)} \geq \frac{\psi(1)}{\phi(1)m_\phi(1/m_\psi(1/t))} \geq \frac{\psi(1)}{\phi(1)} t^{p_\psi - (q_\phi + \varepsilon)} = \frac{\psi(1)}{\phi(1)} t^{p_\psi - q_\phi - 2\varepsilon} \to \infty \quad \text{as} \quad t \to \infty,
\]

which gives a contradiction. □

**Proof of Theorem 11.** (a) We have

\[
M_1^* \otimes M(M_1^*, F) \equiv L^{\infty}(\phi)^{(s)} \otimes M[L^{\infty}(\phi)^{(s)}, F] \quad \text{(using Lemma 6)}
\]

\[
= L^{\infty}(\phi)^{(s)} \otimes F(1/\phi)^{(s)} \quad \text{(by Theorem 1(iv))}
\]

\[
= \left[ L^{\infty}(\phi)^{(s)} \right]^{1/2} \left[ F(1/\phi)^{(s)} \right]^{1/2} \quad \text{(using Lemma 4)}
\]

\[
= \left[ L^{\infty}(\phi)^{1/2} F(1/\phi)^{1/2} \right]^{(s)} \quad \text{(by Theorem 2 in [44])}
\]

\[
= \left[ (L^{\infty})^{1/2} F^{1/2} \right]^{(s)} \equiv \left[ F^{(2)} \right]^{(s)} \equiv F^{(s)} \equiv F.
\]

Note that Lemma 4 can be used in the above equality since \( 0 < p_{\phi,1} \leq q_{\phi,1} < 1 \) implies that the operators \( H, H^* : L^{\infty}(\phi) \to L^{\infty}(\phi) \) are bounded. Moreover, \( \alpha_F > q_{\phi,1} \) gives that \( H^* : F(1/\phi) \to F(1/\phi) \) is bounded. Furthermore \( \beta_F < 1 + p_{\phi,1} \), which implies that \( H : F(1/\phi) \to F(1/\phi) \) is bounded (see [60, Theorem 1] or [61, Theorem 1]).

(b) By the duality results

\[
E \odot M(E, \Lambda_\phi) \equiv E \odot M((\Lambda_\phi)', E') \equiv E \odot M(M_{t/\phi(t)}, E')
\]

\[
eq E \odot M \left[ L^{\infty} \left( \frac{t}{\phi(t)} \right)^{(s)}, E' \right] \quad \text{(using Lemma 6 and the symmetry of \( E \))}
\]

\[
eq E \odot E' \left( \frac{\phi(t)}{t} \right)^{(s)} \equiv E^{(s)} \odot E' \left( \frac{\phi(t)}{t} \right)^{(s)} \quad \text{(by Theorem 1(iv))}
\]

\[
eq \left[ E^{(s)} \right]^{1/2} \left[ E' \left( \frac{\phi(t)}{t} \right)^{(s)} \right]^{1/2} \quad \text{(using Lemma 4)}
\]

\[
eq \left[ E^{1/2} E' \left( \frac{\phi(t)}{t} \right)^{1/2} \right]^{(s)} \quad \text{(by Theorem 2 in [44])}
\]

\[
eq \left[ E^{1/2} \left( E' \right)^{1/2} \left( t^{-1/2} \phi(t) \right)^{1/2} \right]^{(s)} \quad \text{(by Theorem 1(iv))}
\]

\[
eq \left[ (E \odot E')^{(2)} (t^{-1/2} \phi(t) \left( t^{1/2} \right)^{(s)} \right]^{(s)} \quad \text{(by Lozanovskii factorization)}
\]

\[
= \left[ L^2 (t^{-1/2} \phi(t) \left( t^{1/2} \right)^{(1/2)} \right]^{(s)} \equiv L^1 \left( \frac{\phi(t)}{t} \right)^{(s)} \equiv \Lambda_{\phi,1}.
\]
We must only control if the assumptions of Lemma 4 are satisfied here, that is, if operators $H$, $H^*$ are bounded in $E$ and in $E'(\frac{\phi(t)}{t})$. Since $0 < \alpha_E \leq \beta_E < p_{\phi,I} < 1$, it follows from Boyd’s result that $H$ and $H^*$ are bounded in $E$ (cf. [43, pp. 138–139], [7, Theorem 5.15] and [45, pp. 126–129]). The boundedness of $H$ in $E'(\frac{\phi(t)}{t})$ is equivalent to the estimate

$$\left\| \frac{\phi(t)}{t^2} \int_0^t x(s) \frac{s}{\phi(s)} ds \right\|_{E'} \leq C_1 \| x \|_{E'} \quad \text{for all } x \in E',$$

which is true if $\beta_{E'} < p_{t/\phi(t),I}$ (cf. [60, Theorem 1] or [61, Theorem 1]). The last strict inequality means that $\beta_{E'} = 1 - \alpha_E < p_{t/\phi(t),I} = 2 - q_{\phi,I}$ or $\alpha_E > q_{\phi,I} - 1$ which is true because $\alpha_E > 0$ and $q_{\phi,I} < 1$. The boundedness of $H^*$ in $E'(\frac{\phi(t)}{t})$ is equivalent to the estimate

$$\left\| \frac{\phi(t)}{t} \int_0^t x(s) \frac{1}{\phi(s)} ds \right\|_{E'} \leq C_2 \| x \|_{E'} \quad \text{for all } x \in E',$$

which is true if $\alpha_{E'} > 1 - p_{\phi,I}$ (cf. [60, Theorem 1] or [61, Theorem 1]). The last strict inequality means that $\alpha_{E'} = 1 - \beta_E > 1 - p_{\phi,I}$ or $\beta_E < p_{\phi,I}$, but this is true by the assumption. □

Examples 5.

(a) If $E = L^q$, $F = L^p$ and $\phi(t) = t^{1/r}$, where $1 \leq p < r < q < \infty$, then from Theorem 11

$$L^{r,\infty} \odot M(L^{r,\infty}, L^p) = L^p \quad \text{and} \quad L^q \odot M(L^q, L^{r,1}) = L^{r,1}. \quad (40)$$

Equalities (40) we can also get from (33) and (32). In fact, the space $L^{r,\infty}$ satisfies upper r-estimate (cf. [64, Theorem 5.4(a)] and [34, Theorem 3.1 and Corollary 3.9]). Therefore, for $p < r$ it is $p$-convex with some constant $K \geq 1$ (cf. [48, Theorem 1.f.7]). After renorming it is $p$-convex with constant 1 and we are getting from (33) the first equality in (40) with equivalent norms. On the other hand, $L^{r,1}$ satisfies lower r-estimate (cf. [64, Theorem 5.1(a)]), therefore for $q > r$ it is $q$-concave with some constant $K \geq 1$ (cf. [48, Theorem 1.f.7]). After renorming is $q$-concave with constant 1 and we are getting from (32) the second equality in (40) with equivalent norms.

(b) If $E = L^{q,r}$, $F = L^{p,r}$ and $\phi(t) = t^{1/s}$, where $1 \leq p < s < q < \infty$ and $1 \leq r < \infty$, then from Theorem 11

$$L^{s,\infty} \odot M(L^{s,\infty}, L^{p,r}) = L^{p,r} \quad \text{and} \quad L^{q,r} \odot M(L^{q,r}, L^{s,1}) = L^{s,1}. \quad (41)$$

(c) If $F = \Lambda_\psi$ and $\alpha_F = p_{\psi,I} > q_{\phi,I}$, then from Theorem 11(a) we also obtain the factorization (37) since $p_{\psi/\phi,I} \geq p_{\psi,I} - q_{\phi,I} > 0$. 
Remark 8. In the case \( I = (0, 1) \) the assumption \( \alpha_F > q_{\phi, 1} \) implies the inclusion \( M^*_\phi \hookrightarrow F \), even the inclusion \( M^*_\phi \hookrightarrow \Lambda_\psi \), where \( \psi \) is a fundamental function of \( F \) because \( p_{\psi, 1} \geq \alpha_F > q_{\phi, 1} \) and \( p_{\psi/\phi, 1} \geq p_{\psi, 1} - q_{\phi, 1} > 0 \) gives

\[
\int_0^1 \frac{1}{\phi(t)} \psi'(t) \, dt \leq \int_0^1 \frac{\psi(t)}{\phi(t)} \frac{dt}{t} < \infty.
\]

Consequently, \( M(M^*_\phi, F) \neq \{0\} \).

References

[1] J.M. Anderson, A.L. Shields, Coefficient multipliers of Bloch functions, Trans. Amer. Math. Soc. 224 (2) (1976) 255–265.
[2] T. Ando, On products of Orlicz spaces, Math. Ann. 140 (1960) 174–186.
[3] J. Appell, P.P. Zabrejko, Nonlinear Superposition Operators, Cambridge University Press, Cambridge, 1990.
[4] S.V. Astashkin, Rademacher functions in symmetric spaces, J. Math. Sci. (N. Y.) 169 (6) (2010) 725–886; Russian version: Sovrem. Mat. Fundam. Napravl. 32 (2009) 3–161.
[5] S.V. Astashkin, L. Maligranda, Structure of Cesàro function spaces, Indag. Math. (N.S.) 20 (3) (2009) 329–379.
[6] G. Bennett, Factorizing the Classical Inequalities, Mem. Amer. Math. Soc., vol. 120, Amer. Math. Soc., Providence, 1996.
[7] C. Bennett, R. Sharply, Interpolation of Operators, Academic Press, Boston, 1988.
[8] E.I. Berezhnoi, L. Maligranda, Representation of Banach ideal spaces and factorization of operators, Canad. J. Math. 57 (5) (2005) 897–940.
[9] J. Bergh, J. Lofström, Interpolation Spaces. An Introduction, Springer-Verlag, Berlin, New York, 1976.
[10] B. Bollobás, G. Brightwell, Convex bodies, graphs and partial orders, Proc. London Math. Soc. (3) 80 (2) (2000) 415–450.
[11] B. Bollobás, I. Leader, Generalized duals of unconditional spaces and Lozanovskii’s theorem, C. R. Acad. Sci. Paris Sér. I Math. 317 (6) (1993) 583–588.
[12] B. Bollobás, I. Leader, Products of unconditional bodies, in: Geometric Aspects of Functional Analysis, Israel, 1992–1994, in: Oper. Theory Adv. Appl., vol. 77, Birkhäuser, Basel, 1995, pp. 13–24.
[13] Cz. Bylka, W. Orlicz, On some generalizations of the Young inequality, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 26 (2) (1978) 115–123.
[14] J.M. Calabuig, O. Delgado, E.A. Sánchez Pérez, Generalized perfect spaces, Indag. Math. (N.S.) 19 (3) (2008) 359–378.
[15] A.P. Calderón, Intermediate spaces and interpolation, the complex method, Studia Math. 24 (1964) 113–190.
[16] W.S. Cohn, I.E. Verbitsky, Factorization of tent spaces and Hankel operators, J. Funct. Anal. 175 (2) (2000) 308–329.
[17] R.R. Coifman, R. Rochberg, G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math. (2) 103 (3) (1976) 611–635.
[18] G. Cz. Bylka, W. Orlicz, On some generalizations of the Young inequality, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 26 (2) (1978) 115–123.
[19] M. Cwikel, A. Kamińska, L. Maligranda, L. Pick, Are generalized Lorentz “spaces” really spaces?, Proc. Amer. Math. Soc. 132 (12) (2004) 3615–3625.
[20] M. Cwikel, P.G. Nilsson, Interpolation of weighted Banach lattices, in: M. Cwikel, P.G. Nilsson, G. Schechtman (Eds.), Interpolation of Weighted Banach Lattices. A Characterization of Relatively Decomposable Banach Lattices, in: Mem. Amer. Math. Soc., vol. 165, 2003, pp. 1–105.
[21] G. Dankert, Über Produkte von Orlicz–Räumen, Arch. Math. 19 (1968) 635–645, (1969).
[22] G. Dankert, On factorization of Orlicz spaces, Arch. Math. 25 (1974) 52–68.
[23] G. Dankert, H. König, Über die Hödersche Ungleichung in Orlicz–Räumen, Arch. Math. 18 (1967) 61–75.
[24] A. Defant, M. Mastyło, C. Michels, Applications of summing inclusion maps to interpolation of operators, Q. J. Math. 54 (1) (2003) 61–72.
[25] O. Delgado, E.A. Sánchez Pérez, Summability properties for multiplication operators on Banach function spaces, Integral Equations Operator Theory 66 (2) (2010) 197–214.
[26] P.B. Djakov, M.S. Ramanujan, Multipliers between Orlicz sequence spaces, Turkish J. Math. 24 (2000) 313–319.
[27] I. Dobrakov, On submeasures I, Dissertationes Math. (Rozprawy Mat.) 62 (1974), 35 pp.

[28] T.A. Gillespie, Factorization in Banach function spaces, Indag. Math. 43 (3) (1981) 287–300.

[29] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, Cambridge University Press, Cambridge, 1952.

[30] H.P. Heinig, L. Maligranda, Chebyshev inequality in function spaces, Real Anal. Exchange 17 (1) (1991/1992) 211–247.

[31] C. Horowitz, Factorization theorems for functions in the Bergman spaces, Duke Math. J. 44 (1) (1977) 201–213.

[32] R.E. Jamison, W.H. Ruckle, Factoring absolutely convergent series, Math. Ann. 224 (2) (1976) 143–148.

[33] W.B. Johnson, B. Maurey, G. Schechtman, L. Tzafriri, Symmetric Structures in Banach Spaces, Mem. Amer. Math. Soc., vol. 19 (217), 1979, v+298 pp.

[34] N.J. Kalton, A. Kamińska, Type and order convexity of Marcinkiewicz and Lorentz spaces and applications, Glasg. Math. J. 47 (1) (2005) 123–137.

[35] N.J. Kalton, N.T. Peck, J.W. Roberts, An F-Space Sampler, London Math. Soc. Lecture Note Ser., vol. 89, Cambridge University Press, Cambridge, 1984.

[36] A. Kamińska, L. Maligranda, Order convexity and concavity of Lorentz spaces $\Lambda_{p,w}$, 0 < p < ∞, Studia Math. 160 (3) (2004) 267–286.

[37] A. Kamińska, L. Maligranda, L.E. Persson, Indices, convexity and concavity of Calderón–Lozanovskiĭ spaces, Math. Scand. 92 (2003) 141–160.

[38] A. Kamińska, M. Mastyło, Abstract duality Sawyer formula and its applications, Monatsh. Math. 151 (3) (2007) 223–245.

[39] L.V. Kantorovich, G.P. Akilov, Functional Analysis, Nauka, Moscow, 1977 (in Russian); English transl.: Pergamon Press, Oxford, Elmsford, NY, 1982.

[40] P. Kolwicz, K. Leśnik, Topological and geometrical structure of Calderón–Lozanovskiĭ construction, Math. Inequal. Appl. 13 (2010) 175–196.

[41] P. Kolwicz, K. Leśnik, L. Maligranda, Pointwise multipliers of Calderón–Lozanovskiĭ spaces, Math. Nachr. 286 (8–9) (2013) 876–907.

[42] M.A. Krasnosel’skiĭ, Ja.B. Rutickiĭ, Convex Functions and Orlicz Spaces, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958 (in Russian); English transl.: Noordhoff, Groningen, 1961.

[43] S.G. Krein, Yu.I. Petunin, E.M. Semenov, Interpolation of Linear Operators, Amer. Math. Soc., Providence, 1982; Russian version: Nauka, Moscow, 1978.

[44] N. Krugljak, L. Maligranda, Calderón–Lozanovskiĭ construction on weighted Banach function lattices, J. Math. Anal. Appl. 288 (2) (2003) 744–757.

[45] A. Kufner, L. Maligranda, L.-E. Persson, The Hardy Inequality. About Its History and Some Related Results, Vydavatelský Servis, Plzeň, 2007.

[46] T. Kühn, M. Mastyło, Products of operator ideals and extensions of Schatten classes, Math. Nachr. 283 (6) (2010) 891–901.

[47] D.R. Lewis, N. Tomczak-Jaegermann, Hilbertian and complemented finite-dimensional subspaces of Banach lattices and unitary ideals, J. Funct. Anal. 35 (2) (1980) 165–190.

[48] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces, II. Function Spaces, Springer-Verlag, Berlin, New York, 1979.

[49] G.G. Lorentz, On the theory of spaces $A$, Pacific J. Math. 1 (1951) 411–429.

[50] G.G. Lorentz, Bernstein Polynomials, University of Toronto Press, Toronto, 1953.

[51] G.Ja. Lozanovskiĭ, On topologically reflexive KB-spaces, Dokl. Akad. Nauk SSSR 158 (1964) 516–519 (in Russian); English transl.: Soviet Math. Dokl. 5 (1964) 1253–1256, (1965).

[52] G.Ja. Lozanovskiĭ, On reflexive spaces generalizing the reflexive space of Orlicz, Dokl. Akad. Nauk SSSR 163 (1965) 573–576 (in Russian); English transl.: Soviet Math. Dokl. 6 (1965) 968–971.

[53] G.Ja. Lozanovskiĭ, On Banach lattices of Calderón, Dokl. Akad. Nauk SSSR 172 (5) (1967) 1018–1020 (in Russian); English transl.: Soviet Math. Dokl. 8 (1967) 224–227.

[54] G.Ja. Lozanovskiĭ, On some Banach lattices, Sibirsk. Mat. Zh. 10 (1969) 584–599 (in Russian); English transl.: Siberian Math. J. 10 (3) (1969) 419–431.

[55] G.Ja. Lozanovskiĭ, The Banach lattices and concave functions, Dokl. Akad. Nauk SSSR 199 (1971) 536–539 (in Russian); English transl.: Soviet Math. Dokl. 12 (1971) 1114–1117.

[56] G.Ja. Lozanovskiĭ, Certain Banach lattices. IV, Sibirsk. Mat. Zh. 14 (1973) 140–155 (in Russian); English transl.: Siberian Math. J. 14 (1973) 97–108.

[57] G.Ja. Lozanovskiĭ, On the conjugate space of a Banach lattice, Teor. Funktsii Funktsional. Anal. i Prilozhen. 30 (1978) 85–90 (in Russian).

[58] G.Ja. Lozanovskiĭ, Mappings of Banach lattices of measurable functions, Izv. Vysš. Učebn. Zaved., Mat. 5 (192) (1978) 84–86 (in Russian); English transl.: Soviet Math. (Iz. VUZ) 22 (5) (1978) 61–63.
[59] G.Ja. Lozanovski˘ı, Transformations of ideal Banach spaces by means of concave functions, in: Qualitative and Approximate Methods for the Investigation of Operator Equations, vol. 3, Yaroslav. Gos. Univ., Yaroslavl, 1978, pp. 122–148 (in Russian).

[60] L. Maligranda, Generalized Hardy inequalities in rearrangement invariant spaces, J. Math. Pures Appl. (9) 59 (4) (1980) 405–415.

[61] L. Maligranda, On Hardy’s inequality in weighted rearrangement invariant spaces and applications. I, II, Proc. Amer. Math. Soc. 88 (1) (1983) 67–74, 75–80.

[62] L. Maligranda, Indices and interpolation, Dissertationes Math. (Rozprawy Mat.) 234 (1985), 49 pp.

[63] L. Maligranda, Orlicz Spaces and Interpolation, Sem. Mat., vol. 5, University of Campinas, Campinas, SP, Brazil, 1989.

[64] L. Maligranda, Type, cotype and convexity properties of quasi-Banach spaces, in: M. Kato, L. Maligranda (Eds.), Banach and Function Spaces, Proc. of the Internat. Symp. on Banach and Function Spaces, Kitakyushu, Japan, October 2–4, 2003, Yokohama Publ., 2004, pp. 83–120.

[65] L. Maligranda, E. Nakai, Pointwise multipliers of Orlicz spaces, Arch. Math. 95 (3) (2010) 251–256.

[66] L. Maligranda, L.E. Persson, Generalized duality of some Banach function spaces, Indag. Math. 51 (3) (1989) 323–338.

[67] B. Maurey, Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces $L^p$, Astérisque 11 (1974) 1–163.

[68] D.S. Mitrinovi ´c, Analytic Inequalities, Springer-Verlag, Berlin, 1970.

[69] E. Nakai, Pointwise multipliers, Mem. Akashi College Technol. 37 (1995) 85–94.

[70] P. Nilsson, Interpolation of Banach lattices, Studia Math. 82 (2) (1985) 135–154.

[71] S. Okada, W.J. Ricker, E.A. Sánchez Pérez, Optimal Domain and Integral Extension of Operators. Acting in Function Spaces, Birkhäuser Verlag, Basel, 2008.

[72] R. O’Neil, Fractional integration in Orlicz spaces. I, Trans. Amer. Math. Soc. 115 (1965) 300–328.

[73] V.I. Ovchinnikov, Interpolation theorems resulting from Grothendieck’s inequality, Funct. Anal. Appl. 10 (1976) 287–294.

[74] V.I. Ovchinnikov, Interpolation in quasi-Banach Orlicz spaces, Funct. Anal. Appl. 16 (1982) 223–224.

[75] V.I. Ovchinnikov, The Methods of Orbits in Interpolation Theory, Part 2, Math. Rep. (Bucur.), vol. 1, Harwood Academic, 1984, pp. 349–516.

[76] G. Pisier, La méthode d’interpolation complexe: applications aux treillis de Banach, in: Séminaire d’Analyse Fonctionnelle, 1978–1979, École Polytech., Palaiseau, 1979, Exp. No. 17, 18 pp.

[77] G. Pisier, Some applications of the complex interpolation method to Banach lattices, J. Anal. Math. 35 (1979) 264–281.

[78] M.M. Rao, Z.D. Ren, Theory of Orlicz Spaces, Marcel Dekker, New York, 1991.

[79] Y. Raynaud, On Lorentz–Sharpley spaces, Israel Math. Conf. Proc. 5 (1992) 207–228.

[80] S. Reisner, Operators which factor through convex Banach lattices, Canad. J. Math. 32 (6) (1980) 1482–1500.

[81] S. Reisner, A factorization theorem in Banach lattices and its applications to Lorentz spaces, Ann. Inst. Fourier (Grenoble) 31 (1) (1981) 239–255.

[82] S. Reisner, On Two Theorems of Lozanovski ˘ı Concerning Intermediate Banach Lattices, Lecture Notes in Math., vol. 1317, Springer, Berlin, 1988, pp. 67–83.

[83] S. Reisner, Some remarks on Lozanovskyi’s intermediate norm lattices, Bull. Pol. Acad. Sci. Math. 41 (3) (1993) 189–196, (1994).

[84] W. Rogowska-Soltys, On the Maurey type factorization of linear operators with values in Musielak–Orlicz spaces, Funct. Approx. Comment. Math. 15 (1986) 11–16.

[85] J.B. Ruticki˘ı, On some properties of one operation over spaces, in: Operator Methods in Differential Equations, Voronezh, 1979, pp. 79–84 (in Russian).

[86] A.R. Schep, Products and factors of Banach function spaces, Positivity 14 (2010) 301–319.

[87] T. Shimogaki, On the complete continuity of operators in an interpolation theorem, J. Fac. Sci. Hokkaido Univ. Ser. I 20 (3) (1968) 109–114.

[88] T. Strömberg, An operation connected to a Young-type inequality, Math. Nachr. 159 (1992) 227–243.

[89] T. Strömberg, The operation of infimal convolution, Dissertationes Math. (Rozprawy Mat.) 352 (1996), 58 pp.

[90] N. Tomczak-Jaegermann, Banach–Mazur Distances and Finite-Dimensional Operator Ideals, Longman and Wiley, New York, 1989.

[91] S.-W. Wang, On the products of Orlicz spaces, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 11 (1963) 19–22.

[92] P. Wojtaszczyk, Banach Spaces for Analysts, Cambridge University Press, Cambridge, 1991.
[93] P.P. Zabreiko, Nonlinear integral operators, Voronezh. Gos. Univ. Trudy Sem. Funkcional Anal. 8 (1966) 3–152 (in Russian).
[94] P.P. Zabreiko, Ja.B. Rutickii, Several remarks on monotone functions, Uchebn. Zap. Kazan. Gos. Univ. 127 (1) (1967) 114–126 (in Russian).
[95] M. Zippin, Interpolation of operators of weak type between rearrangement invariant function spaces, J. Funct. Anal. 7 (1971) 267–284.