HOMOTOPY CHARACTERIZATION OF ANR MAPPING SPACES

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Abstract. Let $Y$ be an absolute neighbourhood retract (ANR) for the class of metric spaces and let $X$ be a Hausdorff space. Let $Y^X$ denote the space of continuous maps from $X$ to $Y$ equipped with the compact open topology. It is shown that if $X$ is a CW complex then $Y^X$ is an ANR for the class of metric spaces if and only if $Y^X$ is metrizable and has the homotopy type of a CW complex. The same holds also when $X$ is a compactly generated hemicompact space (metrizability assumption is void in this case).

1. Introduction

Let $X$ and $Y$ be two spaces having the homotopy type of a CW complex. Let $Y^X$ denote the space of continuous maps $X \to Y$ equipped with the compact open topology. The author of this note has extensively investigated the question of when also $Y^X$ has the homotopy type of a CW complex (see [8] and [9]).

Shifting the viewpoint to absolute neighbourhood retracts (ANRs) for metric spaces, one can ask the following question: If $X$ is a CW complex and $Y$ is an ANR, when is $Y^X$ an ANR? (Note that if $X$ is uncountable, $Y^X$ need not even be metrizable.)

A topological space has the homotopy type of a CW complex if and only if it has the homotopy type of an ANR (see Milnor [5]). However, it is not difficult to find examples of spaces that are not ANRs but have the homotopy type of a CW complex.

It turns out that if $X$ is any CW complex and $Y$ is an ANR, then the space $Y^X$ is an ANR whenever it is metrizable and has the homotopy type of a CW complex:

Theorem 1.1. Let $X$ be a CW complex and let $Y$ be an ANR for metric spaces. The following are equivalent.

(i) The space $Y^X$ is an ANR for metric spaces.
(ii) The space $Y^X$ is metrizable and has the homotopy type of a CW complex.
(iii) The space $Y^X$ is metrizable and semilocally contractible.

(Recall that a space $Z$ is semilocally contractible if each point has a neighbourhood that is contractible within $Z$.)

Date: August 29, 2007.

2000 Mathematics Subject Classification. Primary 55P99.

Key words and phrases. ANR, function space, compact open topology, homotopy type of a CW complex, P-embedding.

The author was supported in part by the MŠZŠ of the Republic of Slovenia research program No. P1-0292-0101-04 and research project No. J1-6128-0101-04.
A Hausdorff space $X$ is called hemicompact if $X$ is the union of countably many of its compact subsets $\{K_i | i \}$ which dominate all compact subsets in $X$. This means that for each compact $K \subset X$ there exists $i$ with $K \subset K_i$.

Observe that a CW complex is hemicompact if and only if it is countable. However, a countable CW complex has a distinguished class of compact subsets, namely, finite subcomplexes. Those are always cofibered in the total space. No such thing is required for a general hemicompact space.

A space $X$ is compactly generated if the compact subspaces determine its topology. For compactly generated hemicompact domain spaces the following is true: Theorem 1.2. Let $X$ be a compactly generated hemicompact space and let $Y$ be an ANR for metric spaces. Then the following are equivalent.

(i) The space $Y^X$ is an ANR for metric spaces.

(ii) The space $Y^X$ has the homotopy type of a CW complex.

The proofs of Theorem 1.1 and Theorem 1.2 are quite different. The first uses the Cauty-Geoghegan’s characterization of ANRs (see Cauty [2]), the second uses Morita’s homotopy extension theorem for $P_0$-embeddings (see Morita [6]). It is not possible to use the proof of Theorem 1.1 for Theorem 1.2, and it seems difficult to use the proof of Theorem 1.2 for Theorem 1.1 in case $X$ is an uncountable CW complex.

Acknowledgement. The author is indebted to Atsushi Yamashita for kindly sending him the preprint [12] where the question of equivalence of (i) and (ii) of Theorem 1.1 (for countable domains) was posed implicitly. Inspiration for existence of Theorem 1.2 was also found there. Moreover, the preprint shows that the two theorems can be useful in considering function spaces which are Hilbert manifolds.

2. Proof of Theorem 1.1

For subsets $A$ of the domain space and $B$ of the target space we let $G(A, B)$ denote the set of all maps $f$ that map the set $A$ into the set $B$. For topological spaces $X$ and $Y$, we can take as subbasis of the compact open topology on $Y^X$ the collection $\mathcal{P}$ of all $G(K, V) \subset Y^X$ with $K$ a compact subset of $X$ and $V$ an open subset of $Y$.

We will employ the characterization of absolute neighbourhood retracts conjectured by Ross Geoghegan and proven by Robert Cauty:

Theorem 2.1 (Cauty [2], ‘Théorème’). A metrizable space $Z$ is an ANR for metric spaces if and only if each open subset of $Z$ has CW homotopy type. \(\square\)

Proof of Theorem 1.1. The equivalence of (ii) and (iii) follows from [9], Theorem 2.2.1. Evidently (i) implies (ii).

Assume that $Y^X$ has the homotopy type of a CW complex. We claim that each open subset $U$ of $Y^X$ has the homotopy type of a CW complex.

Let $K_1, \ldots, K_n$ be compact subsets of $X$ and let $V_1, \ldots, V_n$ be open subsets of $Y$. We show first that the finite intersection $G(K_1, V_1) \cap \cdots \cap G(K_n, V_n)$ has the homotopy type of a CW complex. To this end, let $L$ be a finite subcomplex of $X$ containing the compact union $K_1 \cup \cdots \cup K_n$.

The restriction mapping $R: Y^X \to Y^L$ assigning to each $f: X \to Y$ its restriction $f|_L: L \to Y$ is a Hurewicz fibration because $L \hookrightarrow X$ is a cofibration. Denote $\bar{W} = G(K_1, V_1) \cap \cdots \cap G(K_n, V_n) \subset Y^X$ and $W = G(K_1, V_1) \cap \cdots \cap G(K_n, V_n) \subset Y^L$. 

2. Proof of Theorem 1.1
The preimage of $W$ under $R$ is exactly $\tilde{W}$ hence also $r = R|_{\tilde{W}} : \tilde{W} \to W$ is a Hurewicz fibration. Since by a result of Kuratowski (see also Mardešić and Segal [4, Theorem I.3.4]), the space $Y^L$ is an ANR, the open subset $W$ of $Y^L$ has the homotopy type of a CW complex by Theorem 2.1. We claim that all fibres of $r$ have CW homotopy type as well.

To this end, note that for each $\varphi \in W$, the fibre $F_\varphi$ of $r$ over $\varphi$ in $\tilde{W}$ is precisely the fibre of $R$ over $\varphi$ in $Y^X$. As noted above, $Y^L$ is an ANR and hence has CW homotopy type. By assumption, $Y^X$ has CW homotopy type. By Stasheff [11], Corollary (13), $F_\varphi$ has the homotopy type of a CW complex. Thus since the base space $W$ and all fibres of $r$ have CW homotopy type, also $\tilde{W}$ has CW homotopy type by Stasheff [11], Proposition (0) (see also Schön [7]).

Let $\mathcal{B}$ denote the collection of sets $G(K_1, V_1) \cap \cdots \cap G(K_n, V_n) \subset Y^X$ for all possible choices of $n$, $K_i$ (compact), and $V_i$ (open). This is to say that $\mathcal{B}$ is the standard basis for the topology on $Y^X$ associated to the subbasis $\mathcal{P}$. Note that $\mathcal{B}$ is closed under formation of finite intersections and that each member of $\mathcal{B}$ has CW homotopy type by the above.

Let $U$ be an arbitrary open subset of $Y^X$ and let $\mathcal{B}_U$ be the set of those elements of $\mathcal{B}$ that are contained in $U$. Being metrizable, the space $Y^X$ is hereditarily paracompact, hence $\mathcal{B}_U$ is a numerable open covering of the space $U$. By tom Dieck [3], Theorem 4, it follows that $U$ has the homotopy type of a CW complex, as claimed. An application of Theorem 2.1 completes the proof. □

Remark 2.2. The proof leans on the fact that in a CW complex $X$, each compact set $K$ is contained in another compact subset $L$ for which the inclusion $L \hookrightarrow X$ is a closed cofibration. Moreover, the topology on $X$ is determined by its compact subsets. The proof of Theorem 1.1 generalizes trivially to domain spaces $X$ with these two properties.

By virtue of Theorem 1.1, one can find in [9] a number of results implying that certain function spaces either are or aren’t ANRs. In addition, the results there show that the problem of determining whether $Y^X$ has CW homotopy type is very hard.

Corollary 2.3. Let $X$ be a connected countable CW complex. Then $X$ is homotopy dominated by a finite CW complex if and only if $\pi_1(X)$ is finitely presentable and $Y^X$ is an ANR for all ANR spaces $Y$.

Proof. Follows from Theorem 1.1 and Theorem 4.5.3 of [9]. Note that the metrizability condition is void here. □

Remark 2.4. Under additional restrictions on $X$, Corollary 2.3 was obtained independently by Yamashita in [12].

3. Proof of Theorem 1.2

To prove Theorem 1.2 we use the fact that ANRs for metric spaces are precisely the absolute neighbourhood extensors for metric spaces. Given the hypotheses of (ii) of Theorem 1.2, therefore, we need to show that for every pair $(Z, A)$ with $Z$ metric and $A$ closed in $Z$, every continuous function $f : A \to Y^X$ extends continuously over a neighbourhood of $A$ in $Z$. The idea of the proof is very simple, and we outline it first in case $X$ is, in addition, locally compact. The technical details for the general case will follow below.
Outline of proof. Since $Y^X$ has the homotopy type of an ANR, a continuous map $f: A \to Y^X$ always admits a neighbourhood extension up to homotopy. That is, there exist a continuous map $g: U \to Y^X$ where $U$ is open and contains $A$, and a homotopy $h: A \times [0,1] \to Y^X$ beginning in $g|_A$ and ending in $f$. The maps $g$ and $h$, respectively, induce continuous adjoints $\hat{g}: U \times Y \to Y$ and $\hat{h}: A \times X \times [0,1] \to Y$. Being a closed subset of a metric space, $A$ is $P_0$-embedded in $Z$. It follows that $A \times X$ is $P_0$-embedded in $Z \times X$, and therefore also in $U \times X$. By Morita’s Homotopy extension theorem, therefore, $\hat{g}$ and $\hat{h}$ induce a homotopy $H: U \times X \times [0,1] \to Y$ extending both. The adjoint of $H$ is a continuous map $H: U \times [0,1] \to Y^X$ which, on level 1, is the desired extension of $f$. 

For general spaces $X$, the outline fails at two points, both of which are a consequence of the failure of the exponential correspondence between continuous functions $Z \to Y^X$ and $Z \times X \to Y$ in case $X$ is not compactly generated.

The need for the hypotheses on $X$ is the following: compactly generated Hausdorff is used for a kind of exponential correspondence, and heremicompact is used to ensure that $Y^X$ is metrizable (Fréchet) whenever $Y$ is metrizable (Fréchet).

For the rest of this section, let $X$ be a fixed compactly generated heremicompact space with the ‘distinguished’ sequence of compacta $\{K_i | i\}$.

For the record, we cite the classical exponential correspondence theorem (see [10], Introduction, 8).

Proposition 3.1. Let $X, Y, Z$ be topological spaces with $X$ locally compact Hausdorff (no separation properties are assumed for $Y$ and $Z$). Let $Y^X$ be the space of continuous functions and let $f: Z \to Y^X$ be any function with set-theoretic adjoint $\hat{f}: Z \times X \to Y$. Then $f$ is continuous if and only if $\hat{f}$ is continuous. This accounts for a bijection $(Y^X)^Z \leftrightarrow Y^{(X \times Z)}$. □

Definition. For any space $Z$, let $\kappa(Z \times X)$ denote the topological space whose underlying set is $Z \times X$ and has its topology determined by the subsets $Z \times K_i$ (with the cartesian product topology). The identity $\kappa(Z \times X) \to Z \times X$, where the latter has the cartesian product topology, is evidently continuous.

The introduction of the topology $\kappa(Z \times X)$ is motivated by Lemma 3.2 whose proof is an easy consequence of Proposition 3.1 together with the fact that for any space $Y$, the space $Y^X$ is homeomorphic with the inverse limit $\lim_i Y^{K_i}$, as can easily be verified.

Lemma 3.2. Let $Z$ and $Y$ be topological spaces (no separation axioms required) and let $X$ be a compactly generated hemicompact space. Let $f: Z \to Y^X$ be a function with set-theoretic adjoint $\hat{f}: X \times Z \to Y$. Then $f$ is continuous if and only if $\hat{f}: \kappa(Z \times X) \to Y$ is. This accounts for a bijection $(Y^X)^Z \leftrightarrow Y^{\kappa(Z \times X)}$. □

Remark 3.3. If $Z$ is compactly generated Hausdorff (as in our application) then in fact $\kappa(Z \times X)$ coincides with the well known compactly generated refinement $\mathfrak{R}(Z \times X)$ of the cartesian product topology. If, in addition, $Y$ is Hausdorff, the correspondence of Lemma 3.2 is a homeomorphism. However, Lemma 3.2 is all that we use, and thus we do not need to recall properties of the functor $\mathfrak{R}$. Moreover, Proposition 3.6 below is valid for arbitrary $P$-embeddings.

Lemma 3.4. Let $K$ be a compact Hausdorff space. Then $\kappa((Z \times K) \times X)$ is naturally homeomorphic with $\kappa(Z \times X) \times K$. 
Proof. Applying the obvious bijection $Z \times K \times X \leftrightarrow Z \times X \times K$, we have to show that the two topologies on $Z \times X \times K$ have the same continuous maps $Z \times X \times K$ (in fact in our application below we need exactly this fact). To this end, $f : \kappa(Z \times X \times K) \to Y$ is continuous if and only if the restrictions $f_i : Z \times K_i \times K \to Y$ are continuous which is if and only if their adjoints $\hat{f}_i : Z \times K_i \to Y^K$ are continuous. The latter is if and only if the map $\hat{f} : \kappa(Z \times X) \to Y^K$ is continuous and this in turn if and only if $f : \kappa(Z \times X) \times K \to Y$ is continuous which finishes the proof.

Note that for a closed subset $A$ of $Z$ the topology $\kappa(A \times X)$ coincides with the topology that the set $A \times X$ inherits from $\kappa(A \times Z)$. However, for arbitrary $A$ the two topologies might differ.

Let $(Z, A)$ be a topological pair (no separation properties assumed). Then $A$ is $P$-embedded in $Z$ if continuous pseudo-metrics on $A$ extend to continuous pseudo-metrics on $Z$. Also, $A$ is a zero set in $Z$ if there exists a continuous function $\phi : Z \to \mathbb{R}$ with $A = \phi^{-1}(0)$. If $A$ is a $P$-embedded zero set, it is called $P_0$-embedded.

For example, every closed subset of a metrizable space is $P_0$-embedded.

We need $P$-embeddings in the context of Morita’s homotopy extension theorem:

**Theorem 3.5** (Morita [6]). If $A$ is $P_0$-embedded in the topological space $Z$ then the pair $(Z, A)$ has the homotopy extension property with respect to all ANR spaces. That is, if $Y$ is an ANR, if $g : Z \times \{0\} \to Y$ and $h : A \times [0, 1] \to Y$ are continuous maps that agree pointwise on $A \times \{0\}$, then there exists a continuous map $\hat{H} : Z \times [0, 1] \to Y$ extending both $g$ and $h$.

**Proposition 3.6.** Let $A$ be $P$-embedded in $Z$ and let $X$ be a compactly generated hemicompact space. Then the subset $A \times X$ is $P$-embedded in $\kappa(Z \times X)$.

A result due to Alò and Sennott (see [1], Theorem 1.2) shows that $A$ is $P$-embedded in $Z$ if and only if every continuous function from $A$ to a Fréchet space extends continuously over $Z$. Proposition 3.6 seems to be the right way of generalizing the equivalence $(1) \iff (2)$ of Theorem 2.4 in [1].

**Proof.** Let $E$ be a Fréchet space and let $f : A \times X \to E$ be a continuous map where $A \times X$ is understood to inherit its topology from $\kappa(Z \times X)$. Precomposing with the continuous identity $\kappa(A \times X) \to A \times X$ and using Lemma 3.2, we obtain a continuous map $A \to E^X$. As $E^X$ is also a Fréchet space (this is due to Arens, see [1], Proposition 2.2) and $A$ is $P$-embedded in $Z$, the function $\hat{f}$ extends continuously to $\hat{F} : Z \to E^X$. Reapplying Lemma 3.2, $\hat{F}$ induces the desired extension $F : \kappa(Z \times X) \to E$.

**Proof of Theorem 1.2.** Let $Z$ be metrizable and let $f : A \to Y^X$ be a continuous map defined on the closed subset $A$ of $Z$. Let $g = g : U \times \{0\} \to Y^X$ and $h$ be as in the above outline. Let $\hat{h} \sqcup \hat{g} : A \times [0, 1] \sqcup U \times \{0\} \to Y^X$ denote the continuous union of the two, with adjoint $\hat{h} \sqcup \hat{g} : \kappa((A \times [0, 1] \sqcup U \times \{0\}) \times X) \to Y$. As $A \times [0, 1] \sqcup U \times \{0\}$ is closed in $A \times U$, the map $\hat{h} \sqcup \hat{g}$ is continuous with respect to the topology that $(A \times [0, 1] \sqcup U \times \{0\}) \times X$ inherits from $\kappa(U \times [0, 1] \times X)$. Under the homeomorphism $\kappa(U \times [0, 1] \times X) \approx \kappa(U \times X) \times [0, 1]$ of Lemma 3.4, the map $\hat{h} \sqcup \hat{g}$ corresponds to $\hat{k} : (A \times X) \times [0, 1] \sqcup (U \times X) \times \{0\} \to Y$.

Obviously, as $A$ is a zero set in $U$, the product $A \times X$ is a zero set in $U \times X$ with respect to the cartesian product topology. A fortiori, $A \times X$ is a zero set in $\kappa(U \times X)$. Hence, by Proposition 3.6, the set $A \times X$ is $P_0$-embedded in $\kappa(U \times X)$.
Theorem 3.5 yields an extension of $\hat{k}$ to $K: \kappa(U \times X) \times [0, 1] \to Y$. Reapplying Lemma 3.4 and Lemma 3.2, $K$ induces a continuous function $k: U \times [0, 1] \to Y^X$. Level 1 of this homotopy is a continuous extension of $f$ over the neighbourhood $U$. Therefore, $Y^X$ is an ANR.

**Corollary 3.7.** Let $C$ be a compact Hausdorff space and let $Y$ be an ANR for metric spaces. Then $Y^C$ is an ANR for metric spaces.

Corollary 3.7 was proven independently by Yamashita [12] but the author of this note has not seen it elsewhere except with the additional requirement of metrizability of $C$. From the point of view of $P$-embeddings, however, Corollary 3.7 encodes a long-known fact (see [1], Theorem 3.3): if $A$ is $P$-embedded in $Z$ and $X$ is compact Hausdorff, then $A \times X$ is $P$-embedded in $Z \times X$.

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