Core-EP Decomposition and its Applications

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Abstract

In this paper, we introduce the notion of the core-EP decomposition and some of its properties. By using the decomposition, we derive several characterizations of the core-EP inverse, introduce a pre-order (i.e., the core-EP order) and a partial order (i.e., the core-minus partial order), and characterize the properties of both orders.

Keywords: core-EP decomposition; core-EP inverse; core-EP order; core-minus partial order

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1. Introduction

In this paper, we use the following notations. The symbol $\mathbb{C}_{m,n}$ is a set of $m \times n$ matrices with complex entries; $A^*$, $\mathcal{R}(A)$ and $\text{rk}(A)$ represent the conjugate transpose, range space (or column space) and rank of $A \in \mathbb{C}_{m,n}$. Let $A \in \mathbb{C}_{n,n}$, the smallest positive integer $k$, which satisfies $\text{rk}(A^k+1) = \text{rk}(A^k)$, is called the index of $A$ and is denoted by $\text{Ind}(A)$. The index of a non-singular matrix $A$ is 0 and the index of a null matrix is 1. The Moore-Penrose inverse of $A \in \mathbb{C}_{m,n}$ is defined as the unique matrix $X \in \mathbb{C}_{n,m}$ satisfying the equations

1. $AXA = A$,  
2. $XAX = X$,  
3. $(AX)^* = AX$,  
4. $(XA)^* = XA$,

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and is usually denoted by $X = A^\dagger$; the Drazin inverse of $A \in \mathbb{C}_{n,n}$ is defined as the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying the equations

\[(1^k) \ AXA^k = A^k, \ (2) \ XAX = X, \ (5) \ AX =XA,\]

and is usually denoted by $X = A^D$. In particular, when $A \in \mathbb{C}^{CM}_n$, the matrix $X$ is called the group inverse of $A$, and is denoted by $X = A^\#$ (see [2]), in which

\[\mathbb{C}^{CM}_n = \{ A \mid A \in \mathbb{C}_{n,n}, \mathop{\text{rk}}(A^2) = \mathop{\text{rk}}(A) \} .\]

The core inverse of $A \in \mathbb{C}^{CM}_n$ is defined as the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying

\[AX = AA^\dagger, \ \mathcal{R}(X) \subseteq \mathcal{R}(A) \quad (1.1)\]

and is denoted by $X = A^\Ast$. When $A \in \mathbb{C}^{CM}_n$, we call it a core-invertible (or group-invertible) matrix.

According to [7, Corollary 6], for every matrix $A \in \mathbb{C}_{n,n}$ of rank $r$, there exists a unitary matrix $U$ such that

\[A = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^*, \quad (1.2)\]

where $T \in \mathbb{C}_{r,r}$ and $S \in \mathbb{C}_{r,n-r}$. When $A \in \mathbb{C}^{CM}_n$ and $A$ is of the form (1.2), then $T$ is non-singular, and (1.2) is called a core decomposition in this paper. Meanwhile, the core inverse of $A$ is of the form (1.3).

**Lemma 1.1.** Let $A \in \mathbb{C}^{CM}_n$. If $A$ is of the form (1.3), then

\[A^\Ast = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*, \quad (1.3)\]

where $U$ is a unitary matrix, $T$ is a non-singular matrix, and

\[AA^\Ast = AA^\dagger. \quad (1.4)\]

By applying the decomposition in [7, Corollary 6], some mathematical researchers delve into characterizations and properties of the core inverse, the core
partial order and some relevant issues in [1, 2, 8, 13, etc]. Note that not every
matrix has the core inverse, and the notion of the core inverse extends to any
square matrix whose index is not limited to less than or equal to one, [2, 8, 9].

Based on the decomposition in [7, Corollary 6], Benítez introduces a new de-
composition for square matrices in Theorem 2.1 of [3]. Benítez and Liu give the
new decomposition a new proof by CS decomposition in [4]. The decomposition
Theorem 2.1 is used to study partial orders, general inverses, etc, [3, 4].

In this paper, we look at the decomposition [7, Corollary 6] in another way. Based on the core decomposition, we introduce the core-EP decomposition for
a square matrix over the complex field in Section 2. By applying the decompo-
sition, we derive some new characterizations of the core-EP inverse in Section
and characterize the properties of a pre-order: the core-EP order in Section
and of a partial order: the core-minus partial order in Section 5 respectively.

2. The core-EP decomposition

It is well known that the core-nilpotent decomposition have been widely used
in matrix theory [12]: Let \( A \in \mathbb{C}_{n,n} \) with \( \text{Ind}(A) = k \). Then \( A \) can be written
as the sum of matrices \( A_1 \) and \( A_2 \) i.e. \( A = A_1 + A_2 \) where

\[
A_1 \in \mathbb{C}_{n}^{\text{CM}}, \quad A_2^k = 0 \quad \text{and} \quad A_1A_2 = A_2A_1 = 0.
\]

Similarly, we introduce the notion of the core-EP of decomposition in Theorem

**Theorem 2.1** (Core-EP Decomposition). Let \( A \in \mathbb{C}_{n,n} \) with \( \text{Ind}(A) = k \).
Then \( A \) can be written as the sum of matrices \( A_1 \) and \( A_2 \) i.e. \( A = A_1 + A_2 \) where

(i) \( A_1 \in \mathbb{C}_{n}^{\text{CM}} \);
(ii) \( A_2^k = 0 \);
(iii) \( A_1^*A_2 = A_2A_1 = 0 \).

Here one or both of \( A_1 \) and \( A_2 \) can be null.
Proof. When $A$ is a non-singular matrix, i.e. $\text{Ind}(A) = 0$, it is easy to check that $A_1 = A$ and $A_2 = 0$ satisfy (i), (ii) and (iii). When $A = 0$, i.e. $\text{Ind}(A) = 1$, it is easy to check that $A_1 = 0$ and $A_2 = 0$ satisfy (i), (ii) and (iii). When $A$ is nilpotent with $\text{Ind}(A) = k$, it is easy to check that $A_1 = 0$ and $A_2 = A$ satisfy (i), (ii) and (iii). So, the matrix $A$ which we consider in the rest of the proof is a square, singular, non-null and non-nilpotent matrix.

If $\text{Ind}(A) = k > 1$, then $A^k \in \mathbb{C}_n^\text{CM}$. Write
\begin{align*}
A_1 &= A^k (A^k)^\otimes A, \\
A_2 &= A - A^k (A^k)^\otimes A.
\end{align*}
(2.1)
Since $\text{rk} \left( A^{k+1} \right) = \text{rk} \left( A^k \right)$ and $\text{rk}(A_1) \leq \text{rk}(A^k)$, it follows that
\begin{align*}
\text{rk} \left( A^{k+1} \right) &= \text{rk} \left( AA^k (A^k)^\otimes A \right) \\
&\leq \text{rk} \left( A^k A (A^k)^\otimes A \right) \\
&= \text{rk} \left( A^k (A^k)^\otimes A^k A (A^k)^\otimes A \right) \\
&= \text{rk} \left( A_2^2 \right) \leq \text{rk}(A_1) \leq \text{rk}(A^k).
\end{align*}
Therefore, we have $A_1 \in \mathbb{C}_n^\text{CM}$. By applying (1.4), we derive
\begin{align*}
A_2 A_1 &= \left( A - A^k (A^k)^\otimes A \right) A^k (A^k)^\otimes A \\
&= AA^k (A^k)^\otimes A - A^k (A^k)^\otimes AA^k (A^k)^\otimes A = 0, \\
A_1^* A_2 &= \left( A^k (A^k)^\otimes A \right)^* \left( A - A^k (A^k)^\otimes A \right) \\
&= A^* \left\{ A^k (A^k)^\otimes \left( A - A^k (A^k)^\otimes A \right) \right\} = 0.
\end{align*}
By using $A_2 A_1 = 0$, it is easy to check that
\begin{align*}
A_2^k &= A_2 (A_1 + A_2)^{k-1} = A_2 A^{k-1} \\
&= \left( A - A^k (A^k)^\otimes A \right) A^{k-1} = 0.
\end{align*}
\]Since the case of $A$ being nilpotent (or null, or no-singular) is considered to be trivial, we consider the matrix $A$ which is a square, singular, non-null and non-nilpotent matrix in the rest of the paper, unless indicated otherwise.
Theorem 2.2. The core-EP decomposition of a given matrix is unique.

Proof. Let \( A = A_1 + A_2 \) be the core-EP decomposition of \( A \in \mathbb{C}_{n,n} \), where \( A_1 \) is a core-invertible matrix, and \( A_2 \) is a nilpotent matrix. And let \( A_1 \) and \( A_2 \) be as in (2.1).

Let \( A = B_1 + B_2 \) be another core-EP decomposition of \( A \), where \( B_1 \) and \( B_2 \) satisfy the conditions (i), (ii) and (iii) in Theorem 2.1. Then \( A_k = \sum_{i=0}^{k} B_1^i B_2^{k-i} \).

By using \( B_1^* B_2 = 0 \) and \( B_2^2 = 0 \), we obtain \((A_k)^* B_2 = 0\). Therefore, \((A_k)^+ B_2 = 0\). By using \( B_2 B_1 = 0 \) and \( B_1 \in \mathbb{C}_{n} \), Then \( A_k B_1 (B_1^k)^# = B_1 \). Since \( A_k (A_k)^+ B_1 = A_k (A_k)^+ A_k B_1 (B_1^k)^# = A_k B_1 (B_1^k)^# = B_1^{k+1} (B_1^k)^# B_1 \). It follows that

\[
B_1 - A_1 = B_1 - A_k (A_k)^+ A = B_1 - A_k (A_k)^+ B_1 - A_k (A_k)^+ B_2 = 0,
\]

that is, \( B_1 = A_1 \). Therefore, the core-EP decomposition of \( A \) is unique. \( \square \)

Theorem 2.3. Let \( A = A_1 + A_2 \) is the core-EP decomposition of \( A \). Then there exists a unitary matrix \( U \) such that

\[
A_1 = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* \text{ and } A_2 = U \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} U^*, \tag{2.2}
\]

where \( T \) is non-singular, and \( N \) is nilpotent.

Proof. For \( A_1 \in \mathbb{C}_{n} \), let

\[
A_1 = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^*,
\]

where \( T \) is non-singular, and \( U \) is unitary. Write

\[
A_2 = U \begin{bmatrix} X_1 & X_2 \\ X_3 & N \end{bmatrix} U^*.
\]
By applying $A_1^*A_2 = 0$, we have

$$A_1^*A_2 = U \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & N \end{bmatrix} U^* = U \begin{bmatrix} T^*X_1 & T^*X_2 \\ S^*X_1 & S^*X_2 \end{bmatrix} U^* = 0.$$ 

Therefore, $T^*X_1 = 0$ and $T^*X_2 = 0$. Since $T$ is invertible, we obtain $X_1 = 0$ and $X_2 = 0$. By applying $A_2A_1 = 0$, we have

$$A_2A_1 = U \begin{bmatrix} 0 & 0 \\ X_3 & N \end{bmatrix} \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} 0 & 0 \\ X_3T & X_3S \end{bmatrix} U^* ,$$

Therefore, $X_3 = 0$ and

$$A_2 = U \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} U^* .$$

Since $A_2$ is nilpotent, it is clear that $N$ is nilpotent.

In the following sections, we consider the core-EP inverse, the core-EP order and the core-minus partial order by using the core-EP decomposition.

3. Characterizations of core-EP inverse

When the index of a given square matrix is less than or equal to one, by applying the decomposition in [7, Corollary 6], mathematical researchers introduce the core inverse (1.1) and (1.2). In [2, 8, 9], the notion of the core inverse extends to any square matrix whose index is not limited to less than or equal to one. For example, let $A \in \mathbb{C}_{n,n}$ be of the form (1.2), then the generalized core inverse $A^\oplus$ can be expressed as [9, Theorem 3.5 and Remark 2]:

$$A^\oplus = A^k \left( (A^*)^k A^{k+1} \right)^{-1} A^k ,$$

(3.1)

where $k = \text{Ind}(A)$. We also call the generalized core inverse $A^\oplus$ the core-EP inverse of $A$. When $k \leq 1$, it is easy to verify that the core-EP inverse coincide with the core inverse [1, Definition 1]. Some properties and characterizations of the core inverse the core-EP inverse and other generalized core inverses can be seen in [2, 8, 9, 13].
**Lemma 3.1.** \[\text{Lemma 3.3}\] Let \( A \in \mathbb{C}_{n,n} \) with \( \text{Ind}(A) = k \). Then \( G \) is a core-EP inverse of \( A \) if and only if \( G \) is a matrix satisfying the conditions:

\[(1^k)\ X A^{k+1} = A^k, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX,\]

and \( \mathcal{R}(G) \subseteq \mathcal{R}(A^k) \).

In this section, we give some characterizations of the core-EP inverse by using the core-EP decomposition.

**Theorem 3.2.** Let \( A = A_1 + A_2 \) be the core-EP decomposition of \( A \in \mathbb{C}_{n,n} \), where \( A_1 \) is a core-invertible matrix, and \( A_2 \) is a nilpotent matrix. Then

\[A^\text{\#} = A_1^\text{\#}.\]

**Proof.** Let \( A_1 \) be as in (2.2), then

\[A_1^\text{\#} = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*.\]

Write

\[A^k = U \begin{bmatrix} T^k & \tilde{T} \\ 0 & 0 \end{bmatrix} U^*.\]

where \( \tilde{T} \) is a corresponding matrix. It is easy to see that \( \mathcal{R}(A_1^\text{\#}) \subseteq \mathcal{R}(A^k) \) and

\[A_1^\text{\#} A^{k+1} = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^{k+1} & T\tilde{T} \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} T^k & \tilde{T} \\ 0 & 0 \end{bmatrix} U^* = A^k,\]

\[A_1^\text{\#} A A_1^\text{\#} = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} \begin{bmatrix} T^{-1} \ 0 \\ 0 \ 0 \end{bmatrix} U^* = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* = A_1^\text{\#},\]

\[(AA_1^\text{\#})^* = U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} \begin{bmatrix} T^{-1} \ 0 \\ 0 \ 0 \end{bmatrix} U^* = U \begin{bmatrix} I & S \\ 0 & N \end{bmatrix} U^* = A_1^\text{\#}.\]

Therefore, by applying Theorem 2.3, we have \( A^\text{\#} = A_1^\text{\#} \).

**Corollary 3.3.** Let \( A = A_1 + A_2 \) be the core-EP decomposition of \( A \in \mathbb{C}_{n,n} \), where \( A_1 \) is a core-invertible matrix, and \( A_2 \) is a nilpotent matrix. Let the
decompositions of $A_1$ and $A_2$ be as in (2.2). Then

$$A^\circ = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*.$$  (3.2)

**Corollary 3.4.** Let $A = A_1 + A_2$ be the core-EP decomposition of $A \in \mathbb{C}_{n,n}$, where $A_1$ is a core-invertible matrix, and $A_2$ is a nilpotent matrix. Then

$$A^\circ = A^k \left( A^{k+1} \right)^\circ,$$  (3.3)

$$AA^\circ = A^k \left( A^k \right)^\circ = A^k \left( A^k \right)^\dagger,$$  (3.4)

$$A_1 = AA^\circ A,$$  (3.5)

$$A_2 = A - AA^\circ A.$$  (3.6)

4. The core-EP order

A binary relation on a non-empty set is said to be a pre-order if it is reflexive and transitive. If it is also anti-symmetric, then it is called a partial order [12, Charp 1]. As is noted in [1, Section 3] the core partial order is given:

$$A^\circ B : A, B \in \mathbb{C}^{CM}_{n,n}, A^\circ A = A^\circ B \text{ and } AA^\circ = BA^\circ.$$  (4.1)

Some characterizations of the core partial order are given in [1, 13].

**Lemma 4.1.** Let $A, B \in \mathbb{C}^{CM}_{n,n}$, and let $A$ be of the form (1.2). Then the following conditions are equivalent:

(i) $A^\circ B$;

(ii) $B = U \begin{bmatrix} T & S \\ 0 & Z \end{bmatrix} U^*$, where $T$ is nonsingular and $Z \in \mathbb{C}_{n-r,n-r}$ is some matrix of index one;

(iii) $A^\dagger A = A^\dagger B$, $A^2 = BA$.

It is of interest to consider the binary operation:

$$A^\circ B : A, B \in \mathbb{C}_{n,n}, A^\circ A = A^\circ B \text{ and } AA^\circ = BA^\circ.$$  (4.1)

We call it the core-EP order.
Remark 4.1. It is noteworthy that $A^\oplus = A^\oplus$ and $B^\oplus = B^\oplus$, when $A$ and $B \in \mathbb{C}^{CM}_n$. Therefore, the core-EP order and the core partial order coincide in $\mathbb{C}^{CM}_n$.

In the following theorem, we derive some characterizations of the binary operation.

Theorem 4.2. Let $A$ and $B$ be square matrices of the same order over the complex field. Then the following are equivalent:

(i) $A^\oplus \leq B : A^\oplus A = A^\oplus B$, $AA^\oplus = BA^\oplus$;

(ii) There exists a unitary matrix $U$ such that

\[
A = U \begin{bmatrix} T_1 & T_2 & S_1 \\ 0 & N_{11} & N_{12} \\ 0 & N_{13} & N_{14} \end{bmatrix} U^*, \quad B = U \begin{bmatrix} T_1 & T_2 & S_1 \\ 0 & T_3 & S_2 \\ 0 & 0 & N_2 \end{bmatrix} U^*,
\]

where $\begin{bmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{bmatrix}$ and $N_2$ are nilpotent, $T_1$ and $T_3$ are non-singular;

(iii) $A^{k+1} = BA^k$, $A^* A^{k} = B^* A^k$, where $k$ is the index of $A$;

(iv) $VAV^\oplus \leq VBV^*$ for each unitary matrix $V$;

(v) $A^\oplus \leq B_1$, where $A_1$ and $B_1$ are core-invertible, $A_2$ and $B_2$ are nilpotent, and $A = A_1 + A_2$ and $B = B_1 + B_2$ be the core-EP decompositions of $A$ and $B$, respectively.

Proof. (i) $\Rightarrow$ (ii). Let

\[
A = U^* \begin{bmatrix} T_1 & \hat{T}_2 & \hat{S}_1 \\ 0 & \hat{N}_{11} & \hat{N}_{12} \\ 0 & \hat{N}_{13} & \hat{N}_{14} \end{bmatrix} U^*,
\]

be a core-EP decomposition of $A$, where $T_1$ is non-singular, $\begin{bmatrix} \hat{N}_{11} & \hat{N}_{12} \\ \hat{N}_{13} & \hat{N}_{14} \end{bmatrix}$ is
nilpotent, and $U$ is unitary. Then

$$A^\oplus = U_1 \begin{bmatrix} T_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U_1^*.$$

Write

$$B = U_1 \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} U_1^*.$$

Since,

$$A^\oplus A = U_1 \begin{bmatrix} I & T_1^{-1} \hat{T}_2 & \hat{T}_1^{-1} \hat{S}_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U_1^* = A^\oplus B = U_1 \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U_1^*$$

$$\Rightarrow X_{11} = T_1, X_{12} = \hat{T}_2, X_{13} = \hat{S}_1;$$

$$AA^\oplus = U_1 \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U_1^* = BA^\oplus = U_1 \begin{bmatrix} I & 0 & 0 \\ X_{21} T_1^{-1} & 0 & 0 \\ X_{31} T_1^{-1} & 0 & 0 \end{bmatrix} U_1^*$$

$$\Rightarrow X_3 = 0, X_4 = 0,$$

we have

$$B = U_1 \begin{bmatrix} T_1 & \hat{T}_2 & \hat{S}_1 \\ 0 & X_{22} & X_{23} \\ 0 & X_{32} & X_{33} \end{bmatrix} U_1^*.$$

Let

$$\begin{bmatrix} X_{22} & X_{23} \\ X_{32} & X_{33} \end{bmatrix} = U_2 \begin{bmatrix} T_2 & S_2 \\ 0 & N_2 \end{bmatrix} U_2^*,$$

be a core-EP decomposition of $\begin{bmatrix} X_{22} & X_{23} \\ X_{32} & X_{33} \end{bmatrix}$, where $T_2$ is non-singular, $N_2$ is nilpotent, and $U_2$ is unitary.

Denote

$$U = U_1 \begin{bmatrix} I & 0 \\ 0 & U_2 \end{bmatrix}, \quad \begin{bmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{bmatrix} = U_2^* \begin{bmatrix} \hat{N}_{11} & \hat{N}_{12} \\ \hat{N}_{13} & \hat{N}_{14} \end{bmatrix} U_2$$
and \( \begin{bmatrix} \hat{T}_2 & \hat{S}_1 \end{bmatrix} U_2 = \begin{bmatrix} T_2 & S_1 \end{bmatrix} \), we have

\[
A = U \begin{bmatrix} T_1 & T_2 & S_1 \\ 0 & N_{11} & N_{12} \\ 0 & N_{13} & N_{14} \end{bmatrix} U^* \quad B = U \begin{bmatrix} T_1 & T_2 & S_1 \\ 0 & T_3 & S_2 \\ 0 & 0 & N_2 \end{bmatrix} U^* ,
\]

and \( \begin{bmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{bmatrix} \) is nilpotent.

(ii) \( \Rightarrow \) (i) is easy.

(ii) \( \Rightarrow \) (iii). By applying (4.2), we have

\[
A^k = U \begin{bmatrix} T_1^k & T_2 & S_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^* ,
B^k = U \begin{bmatrix} T_1^k & T_2 & S_1 \\ 0 & T_3 & S_2 \\ 0 & 0 & N_2 \end{bmatrix} U^* ,
\]

\( \begin{bmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{bmatrix} \) is nilpotent.

(iii) \( \Rightarrow \) (i). By applying (3.4), we obtain

\[
AA^k = BA^k \Rightarrow AA^k (A^{k+1})^\oplus = BA^k (A^{k+1})^\oplus
\Rightarrow AA^\oplus = BA^\oplus ,
\]

\[
A^* A^k = B^* A^k \Rightarrow A^* A^k (A^{k+1})^\oplus = B^* A^k (A^{k+1})^\oplus \Rightarrow A^* A^\oplus = B^* A^\oplus
\Rightarrow A^\oplus ((A^\oplus)^*)^\oplus (A^\oplus)^* A = A^\oplus ((A^\oplus)^*)^\oplus (A^\oplus)^* B
\Rightarrow A^\oplus A = A^\oplus B .
\]

(iii) \( \Leftrightarrow \) (iv) is easy.

(ii) \( \Leftrightarrow \) (v). Let \( A = A_1 + A_2 \) and \( B = B_1 + B_2 \) be the core-EP decompositions of \( A \) and \( B \in \mathbb{C}_{n,n} \), respectively, where \( A_1 \) and \( B_1 \) are core-invertible, and \( A_2 \) and \( B_2 \) are nilpotent. Then \( A^\oplus = A_1^\oplus \) and \( B^\oplus = B_1^\oplus \). By applying Lemma 4.1, we have (ii) \( \Leftrightarrow \) (v). \( \square \)
By applying Theorem 4.2, we see that the binary operation is reflexive and transitive, that is, the core-EP order is a pre-order. But the core-EP order is not antisymmetric.

**Example 4.1.** Let

\[
A = \begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix},
\]

Then \(A \preceq B\) and \(B \preceq A\). However, \(A \neq B\).

**Theorem 4.3.** The core-EP order is only a pre-order and not a partial order.

It is well known that the Drazin order is a pre-order: \(A \Dle B : A^k B = B A^k = A^{k+1}\), where \(k\) is the index of \(A\). In the following example, we see that the Drazin order and the core-EP order are not equivalent.

**Example 4.2.** Let \(A\) and \(B\) be as in Example (4.1). Then \(\text{Ind}(A) = 1\) and \(AB \neq BA\). Therefore, \(A\) and \(B\) do not satisfy \(A \Dle B\).

5. The core-minus partial order

Creating new partial orders is a fundamental problem in matrix theory [11]. Matrix decomposition is an important tool of establishing partial orders. It is well known that the C-N partial order was created by Mitra and Hartwig, and it implies the minus partial order [11, 12]. The minus partial order \(\preceq\), the sharp partial order \(\#\le\) and the C-N partial order \(#, \preceq\) are defined as follows [6, 10, 12]:

(i) \(A \preceq B : A, B \in \mathbb{C}_{m,n}, \text{rk}(B) - \text{rk}(A) = \text{rk}(B - A)\);

(ii) \(A \# \preceq B : A, B \in \mathbb{C}^\text{CR}_{n}, A\# A = A\# B\) and \(AA\# = BA\#\);

(iii) \(A \#: \preceq B : A, B \in \mathbb{C}_{n,n}, A_1 \leq B_1\) and \(A_2 \preceq B_2\), in which \(A = A_1 + A_2\) and \(B = B_1 + B_2\) are the core-nilpotent decompositions of \(A\) and \(B\), respectively.
By using the method similar to the C-N partial ordering as in (iii), we introduce the core-minus partial order in this section.

**Definition 5.1.** Let $A$ and $B$ be matrices of the same order. Let $A = A_1 + A_2$ and $B = B_1 + B_2$, be the core-EP decompositions of $A$ and $B$ respectively, where $A_1$ and $B_1$ are core-invertible, $A_2$ and $B_2$ are nilpotent. Then $A$ is below $B$ under the core-minus order if

$$A_1 \overset{\circ}{\leq} B_1, \quad A_2 \preceq B_2,$$

(5.1)

Whenever this happens, we write $A \leq_{	ext{CM}} B$.

Since the core-EP decomposition of a given matrix is unique, and the core order and the minus order are both partial orders, it is easy to prove the following theorem:

**Theorem 5.1.** The core-minus order is a partial order.

**Remark 5.1.** When $k \leq 1$, it is easy to check that each of the core-minus partial coincide with the core partial order [1, Definition 2].

**Theorem 5.2.** The core-minus partial order implies the minus partial order i.e., if $A \leq_{	ext{CM}} B$, then $A \preceq B$.

**Theorem 5.3.** Let $A$ and $B$ be square matrices of the same order over the complex field. Then $A \leq_{	ext{CM}} B$ if and only if there exists a unitary matrix $U$ such that

$$A = U \begin{bmatrix} T_1 & T_2 & S_1 \\ 0 & 0 & 0 \\ 0 & 0 & N_1 \end{bmatrix} U^*, \quad B = U \begin{bmatrix} T_1 & T_2 & S_1 \\ 0 & T_3 & S_2 \\ 0 & 0 & N_2 \end{bmatrix} U^*,$$

(5.2)

where $T_1$ and $T_3$ are non-singular, $N_1$ and $N_2$ are nilpotent satisfying $N_1 \preceq N_2$.

**Proof.** Let $A$ and $B$ have the form as in (5.2), then we have the core-EP de-
compositions of $A$ and $B$, i.e. $A = A_1 + A_2$ and $B = B_1 + B_2$, respectively,
\[
A_1 = U \begin{bmatrix} T_1 & T_2 & S_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad A_2 = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_1 \end{bmatrix} U^*,
\]
\[
B_1 = U \begin{bmatrix} T_1 & T_2 & S_1 \\ 0 & T_3 & S_2 \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad B_2 = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_2 \end{bmatrix} U^*.
\]

It follows from Definition $5.1$ that $A \leq B$.

Conversely, let $B = B_1 + B_2$ be the core-nilpotent decomposition of $B$. Since $A_1 \leq B_1$, by applying Lemma $4.1$ we know that there exists a unitary matrix $U$ such that
\[
A_1 = U \begin{bmatrix} T_1 & T_2 & S_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad B_1 = U \begin{bmatrix} T_1 & T_2 & S_1 \\ 0 & T_3 & S_2 \\ 0 & 0 & 0 \end{bmatrix} U^*.
\]

It follows that
\[
B_2 = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_2 \end{bmatrix} U^*,
\]
where $N_2$ is nilpotent. For $A_2 \leq B_2$, we have
\[
A_2 = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_1 \end{bmatrix} U^*.
\]
where $N_1$ is nilpotent and $N_1 \leq N_2$. $\Box$

**Theorem 5.4.** Let $A$ and $B$ be square matrices of the same order over the complex field. Then $A \leq B$ if and only if $A \leq B$ and $A - AA^\oplus A \leq B - BB^\oplus B$.

**Proof.** By applying Theorem $4.2$ and Corollary $3.1$ we have the equivalent characterization of the core-minus partial order. $\Box$
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