FEATURES OF SU(N) GAUGE THEORIES∗ †

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We review recent lattice results for the large $N$ limit of SU($N$) gauge theories. In particular, we focus on glueball masses, topology and its relation to chiral symmetry breaking (relevant for phenomenology), on the tension of strings connecting sources in higher representations of the gauge group (relevant for models of confinement and as a comparative ground for theories beyond the Standard Model) and on the finite temperature deconfinement phase transition (relevant for RHIC-like experiments). In the final part we present open challenges for the future.

1. Introduction

The possibility that observables in SU($N$) gauge theories are (at least for large enough $N$) smooth functions of $N$ has been advocated a long time ago¹. In particular, it has been proven diagrammatically that in the limit $N \to \infty$ the theory is simpler (only planar diagrams survive) and that in a neighbourhood of that limit the leading corrections due to a finite $N$ go as $1/N^2$. This can have practical implications for our understanding of QCD if the gauge group of this theory, SU(3), shares the bulk of the physics with SU($\infty$). To verify if this is the case, an investigation from first principles is mandatory. Another motivation for studying the physics of SU($N$) at large $N$ within the conventional gauge theory comes from calculations performed in “beyond the Standard Model” frameworks: the bridge between the two approaches is often SU($\infty$) Yang-Mills².

Lattice calculations are the most reliable tool for investigations of gauge theories from first principles. In the following we give a quick overview of recent lattice results obtained by our group for SU($\infty$). For more details about the calculations we refer to the quoted literature.

∗Summary of the talk presented by B. Lucini and of the poster presented by U. Wenger.
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2. The lowest-lying masses in the spectrum

We have studied $SU(N)$ gauge theories for $2 \leq N \leq 8$. At zero temperature, the theory is confined at all these $N$, which suggests that the large $N$ limit is confining. We use the value of the string tension $\sigma$ to set the scale for the studied observables.

A large $N$ understanding of the mass spectrum can help in identifying glueball states in experiments. We have looked at the masses of the $0^{++}$, $2^{++}$ and $0^{++*}$ glueballs as a function of $N$. We have measured those quantities for $2 \leq N \leq 5$. Our results for $m/\sqrt{\sigma}$ are plotted in fig. 1 as a function of $1/N^2$. The choice of the independent variable is dictated by the fact that the first expected correction to the large $N$ limit value of an observable has this functional form. A fit to the data using only this correction generally works pretty well all the way down to $SU(2)$ (i.e. there is a somehow unexpected precocious onset of the large $N$ behaviour) and allows us to extract the value of the spectrum at $N = \infty$. We find

$$m/\sqrt{\sigma} = 3.341(76) + 1.75/N^2$$
$$m/\sqrt{\sigma} = 4.93(13) + 2.58/N^2$$
$$m/\sqrt{\sigma} = 6.48(35) - 1.7/N^2$$

respectively for the $0^{++}$, the $2^{++}$ and the $0^{++*}$ glueballs.

![Figure 1](image-url)  
Figure 1. The mass of the lightest glueballs as a function of $1/N^2$. Solid lines are our best fits to the data.
3. The topological susceptibility

Another interesting quantity in the large $N$ limit is the topological susceptibility $\chi_t$, which is related in that limit to the $\eta'$ mass by the Witten-Veneziano formula. This formula is a good approximation even when the $SU(3)$ value of the topological susceptibility is plugged in. An explanation of this fact can be found by investigating how $\chi_t$ varies with $N$. We find

$$\frac{\chi_t^{1/4}}{\sqrt{\sigma}} = 0.3739(59) + 0.439/N^2.$$  \hspace{1cm} (2)

The determination of $\chi_t$ at $N = \infty$ given in ref. 5 agrees with our result.

4. Topology and chiral symmetry breaking

From the Banks-Casher relation we know that the chiral condensate is proportional to the density of small eigenmodes $\lambda$ of the Dirac operator, $\langle \bar{\psi}\psi \rangle \sim \lim_{\lambda \to 0} \rho(\lambda)$. Hence a possible scenario for spontaneous chiral symmetry breaking is to assume that the non-vanishing density is due to exact zero modes of the Dirac operator which, through their interaction, are lifted away from zero yielding $\lim_{\lambda \to 0} \rho(\lambda) \neq 0$. As a consequence the near-zero modes would have a topological origin since they emerged from the topological zero modes. A comparison of the topological content of near-zero modes and zero modes of the $SU(N)$ Dirac operator can tell us qualitatively whether this scenario holds in the large $N$ limit. We find that the topo-

![Figure 2. The normalised topological content of the near-zero modes responsible for chiral symmetry breaking for different volumes $V$, lattice spacings $a$ and $N$.](image)

We thank N. Cundy who contributed to the results in this section.
logical and chiral contents of both the zero and near-zero modes become smaller as $N$ increases, but at roughly the same rate. As a consequence the topological content of the near-zero modes normalised by the one of the zero modes (fig. 2) is constant for all $N$. We find that this remains qualitatively true as we vary the volume $V$ and the lattice spacing $a$ and we are therefore able to conclude that topology indeed drives chiral symmetry breaking for all $N$.

5. $k$-strings

For $N \geq 4$ strings connecting sources in higher representations of the gauge group can be stable. This is a consequence of the $Z(N)$ symmetry of the confined phase. Taking into account also charge conjugation, it is easy to see that the number of stable stringy states for $SU(N)$ is the integer part of $N/2$. These states are referred to in the literature as $k$-strings and the rank of the representation of the sources is called N-ality. Within a class of strings with the same N-ality the string tension is unique. We indicate this tension by $\sigma_k$.

The $a \text{ priori}$ unknown value of the ratio $\sigma_k/\sigma$ poses constraints to effective models of confinement. Moreover, it can shed light on the connection between QCD and some “beyond the Standard Model” theories\cite{7}. Usually in these frameworks one obtains the so-called sine formula:

$$\frac{\sigma_k}{\sigma} = \frac{\sin(k\pi/N)}{\sin(\pi/N)} ; \quad (3)$$

however, this is not a universal feature\cite{8}. The first lattice calculation of this ratio was performed in\cite{9} for $SU(4)$. Although a useful continuum extrapolation could not be obtained, it was found that the $k = 2$ string is a genuine bound state. Recent calculations\cite{10,11} have obtained a precise continuum determination of $\sigma_k/\sigma$. This is compatible with both the sine formula and the Casimir scaling ansatz, which predicts that $\sigma_k/\sigma$ is equal to the ratio of the lowest quadratic Casimir operators in the class of the representations with the N-ality of the sources:

$$\frac{\sigma_k}{\sigma} = \frac{k(N-k)}{(N-1)} . \quad (4)$$

While the authors of\cite{12} claim to be able to exclude Casimir scaling, in our opinion the question is far from being settled: the numerical values of (3) and (4) are close enough for systematic effects to become relevant.

We are currently trying to deal with those issues. Indirect information on $\sigma_k/\sigma$ can be extracted from the size of the corre-
sponding strings$^{11,13}$ or from the behaviour of the leading correction to its asymptotic value as $N$ increases$^{13,14}$.

6. The deconfinement phase transition

The quark-gluon plasma phase of QCD is currently being investigated experimentally. Some of the related theoretical questions like the late onset of the Stephan-Boltzman law can be answered in the context of large $N$. As a first step in that direction, we have investigated the physics of the deconfinement phase transition as $N$ varies and then we have extrapolated our results to the large $N$ limit. For the deconfinement temperature $T_c$ we find$^{15,16}$

$$T_c/\sqrt{\sigma} = 0.596(4) + 0.453(30)/N^2.$$  \hfill (5)

For $N \geq 3$ the transition is first order, with a monotonically increasing latent heat as a function of $N$, which suggests that also for $SU(\infty)$ the transition is first order. In fact, an extrapolation of the latent heat to $N = \infty$ using the leading expected $O(1/N^2)$ correction predicts a finite value for this quantity in the limiting case.

The interface tension between the confined and the deconfined phase grows as $N$ is increased and seems to diverge in the critical region at infinite $N$. Interpreted in terms of the Master Field$^{17}$, this gives rise to the speculation that in the large $N$ limit there are several Master Fields separated by infinite energy barriers$^{16}$. The rich physics of the limiting case is given by the interplay between those vacua at large but finite $N$.

![Figure 3. $T_c/\sqrt{\sigma}$ vs $1/N^2$. The solid line is our best fit to the data.](image-url)
7. Conclusions

$SU(N)$ gauge theories have a sensible large $N$ limit that can be studied by lattice techniques. The results can have relevant implications for our understanding of QCD and of the physics beyond the Standard Model. Possible future directions of our investigations include the physics of confinement, the full mass spectrum of glueballs and the equation of state at finite temperature.

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