Thermodynamics of the critical $RSOS(q_1, q_2; q)$ model

Anastasia Doikou
Department of Mathematics, University of York, Heslington
York YO10 5DD, United Kingdom

Abstract

The thermodynamic Bethe ansatz method is employed for the study of the integrable critical $RSOS(q_1, q_2; q)$ model. The high and low temperature behavior are investigated, and the central charge of the effective conformal field theory is derived. The obtained central charge is expressed as the sum of the central charges of two generalized coset models.

1 Introduction

It is well known that statistical systems at criticality —second order phase transition— are expected to exhibit conformal invariance [1], therefore the critical behavior of such systems should be described by a certain conformal field theory. Different types of critical behavior have been classified [2], and the critical exponents and correlation functions have been determined (see also [3], [4]).

An intriguing situation arises from the study of integrable lattice models, whose scaling limit may correspond to certain conformal field theories. In this framework an important, but non trivial task is the calculation of the central charge of the corresponding conformal field theory. A way one can extract this information is by studying the finite size effects of the ground state of the system [5]–[7]. An alternative approach to compute the conformal properties is by investigating the low temperature thermodynamics; in particular, the low temperature behavior of the free energy of a critical system is described by [8], [9]

$$\frac{F(T)}{L} = \frac{F_0}{L} - \frac{\pi c}{6u} T^2 + \ldots, \quad T \ll 1. \quad (1.1)$$

For integrable theories this can be achieved by means of the thermodynamic Bethe ansatz approach, which is a powerful technique that allows the computation of such properties. The mathematical techniques used for such computations go back to the original work of several

1 e-mail: ad22@york.ac.uk
people [14]–[13]. The method was further treated and extended to various lattice [16]–[19] (for a review on TBA for lattice models see e.g. [20]) and continuum relativistic models [21]–[25] yielding very important results.

The thermodynamic Bethe ansatz for relativistic models is somehow the inverse of the Bethe ansatz technique for lattice models [26]–[30]. In the usual Bethe ansatz approach the starting point is the microscopic Hamiltonian, whose diagonalization gives rise to the Bethe ansatz equations, the spectrum, and the scattering information —expressed via the $S$ matrix— (see e.g. [27], [28]). On the other hand, in the integrable relativistic theories one employs the scattering information as an input in order to derive the thermodynamics of the theory [21], [22].

In this study the thermodynamics of the $RSOS(q_1, q_2; q)$ is investigated and the effective conformal anomaly is derived. In general, $RSOS$ models are worth studying because, as already mentioned, their critical behavior may be described by some effective conformal field theory, e.g. critical fused $RSOS$ models are related to generalized diagonal coset models (“anti–ferromagnetic” regime) or parafermionic theories (“ferromagnetic” regime) [31]. Furthermore, it has been shown [32] that critical $RSOS$ models, with proper inhomogeneities, provide lattice regularizations of massive or massless integrable quantum field theories [33], which on the other hand can be thought as perturbations of conformal field theories [34]. What makes the $RSOS(q_1, q_2; q)$ model in particular interesting is that it is a natural generalization of the $RSOS(p, q)$ model studied by Bazhanov and Reshetikhin [31] in as much as the alternating spin chain, introduced by de Vega and Woyanorovich [35], is a generalization of the fused $XXZ$ spin chain [36]. Therefore, with this article the study of the thermodynamics of the fused critical $RSOS$ models is completed.

In [31] the $RSOS(p, q)$ model was studied, the effective central charge was found and, in the “anti–ferromagnetic” regime, it turned out to be the one of the $SU(2)$ diagonal coset model $\mathcal{M}(p, \nu - 2 - p) = SU(2)_{\nu - 2 + p}$, where $SU(2)_{\nu - 2}$ is the $SU(2)$ WZW model at level $k$ [30], [37], whereas in the “ferromagnetic” regime it agreed with the central charge of the parafermionic $SU(2)_{\nu - 2}$ theory. In this work the effective central charge of the $RSOS(q_1, q_2; q)$ model is computed from the low temperature analysis. In the “anti–ferromagnetic” regime it is expressed as the sum of the central charges of two generalized diagonal coset models, namely $\mathcal{M}(q_2, \nu - q_2 - 2)$ and $\mathcal{M}(q_2, \delta q)$, while in the “ferromagnetic” regime the analysis is exactly the same as in [31].

The outline of this article is as follows: in the next section the model is introduced, and the Bethe ansatz equations and the energy spectrum are presented. In the third section the thermodynamic Bethe ansatz equations are derived explicitly and the high and low temperature behavior are examined. Finally, from the low temperature expansion the effective central charge is derived.
2 The model

The integrable critical \( RSOS(q_1, q_2; q) \) model, obtained from the \( RSOS(1, 1) \) model by fusion \[38], \[39], is introduced. To describe the model, a square lattice of \( 2N \) horizontal and \( M \) vertical sites is considered. The Boltzmann weights associated with every site are defined as

\[
w(l_i, l_j, l_m, l_n | \lambda) \equiv \left( \begin{array}{c}
l_n & l_m \\
l_i & l_j\end{array} \right).
\] (2.1)

With every face \( i \) of the lattice an integer \( l_i \) is associated, and every pair of adjacent integers satisfy the following restriction conditions \[40], \[41]

\[
0 \leq l_{i+1} - l_i + P \leq 2P, (a) \\
P \leq l_{i+1} + l_i \leq 2\nu - P, (b)
\] (2.2)

where \( P = q_1 \) for \( i \) odd and \( P = q_2 \) for \( i \) even (let \( q_1 > q_2 \)), for the horizontal pairs, and \( P = q \) for the vertical pairs (array type II \[32\]).

The fused Boltzmann weights have been derived by Date et al in \[39\] and they are given by

\[
w^{q_1,1}(a_1, a_{q_1+1}, b_{q_1+1}, b_1 | \lambda) = \sum_{a_2...a_{q_1}} \prod_{k=1}^{q_1} w^{1,1}(a_k, a_{k+1}, b_{k+1}, b_k | \lambda + i(k - q_i))
\] (2.3)

where \( b_2...b_q \) are arbitrary numbers satisfying \( |b_i - b_{i+1}| = 1 \). \( w^{1,1} \) are the Boltzmann weights for the \( SOS(1, 1) \) model \[40\], they are non vanishing as long as the condition (2.2(a)), for \( P = 1 \) is satisfied and they are given by the following expressions

\[
w(l, l \mp 1, l \mp 1 | \lambda) = h(i - \lambda) \\
w(l \mp 1, l \mp 1, l \pm 1 | \lambda) = -h(\lambda) \frac{h_{l+1}}{h_l} \\
w(l \pm 1, l \pm 1, l | \lambda) = h(w_l \pm \lambda) \frac{h_1}{h_{l+1}}
\] (2.4)

where,

\[
h(\lambda) = \rho \Theta(\lambda) H(\lambda)
\] (2.5)

\( H(\lambda) \) and \( \Theta(\lambda) \) are Jacobi theta functions and,

\[
h_l = h(w_l), \quad w_l = w_0 + il.
\] (2.6)

We are interested in the critical case where \( h(\lambda) \) becomes a simple trigonometric function i.e.,

\[
h(\lambda) = \frac{\sinh \mu \lambda}{\sin \mu},
\] (2.7)

\( w_0, \rho \) and \( \mu \) are arbitrary constants. Furthermore,

\[
w^{q_{i,q}}(a_1, b_1, b_{q+1}, a_{q+1}) = \prod_{k=0}^{q-2} \prod_{j=0}^{q-1} \left(h(i(k - j) + \lambda)\right)^{-1}
\] (2.8)

\[
\sum_{a_2...a_q} \prod_{k=1}^{q} w^{q_1,1}(a_k, b_k, b_{k+1}, a_{k+1}\lambda + i(k - 1)),
\] (2.8)
again \( b_2 \ldots b_{q_i} \) are arbitrary numbers satisfying \( |b_i - b_{i+1}| = 1 \), and the pairs \( a_1, a_{q+1} \) and \( b_1, b_{q+1} \) satisfy (2.2), for \( P = q \). The fused weights satisfy the Yang–Baxter equation in the following form

\[
\sum_g w^{pq}(a, b, g, f | \lambda) w^{ps}(f, g, d, e | \lambda + \mu) w^{qs}(g, b, c, d | \mu) \\
= \sum_g w^{qs}(f, a, g, e | \mu) w^{ps}(a, b, c, g | \lambda + \mu) w^{pq}(g, c, d, e | \lambda).
\] (2.9)

Here we only need the explicit expressions for \( w^{q_i,1} \) which are

\[
\begin{align*}
  w^{q_i,1}(l + 1, l'+1, l', l | \lambda) &= h_{q_i-1}^{q_i-1}(-\lambda \mu_a) \frac{h(ib - \lambda)}{h_l} \\
  w^{q_i,1}(l + 1, l'-1, l', l | \lambda) &= h_{q_i-1}^{q_i-1}(-\lambda \mu_b) \frac{h(\lambda + ia)}{h_l} \\
  w^{q_i,1}(l - 1, l'+1, l', l | \lambda) &= h_{q_i-1}^{q_i-1}(-\lambda \mu_c) \frac{h(iд - \lambda)}{h_l} \\
  w^{q_i,1}(l - 1, l'-1, l', l | \lambda) &= h_{q_i-1}^{q_i-1}(-\lambda \mu_d) \frac{h(ic - \lambda)}{h_l}
\end{align*}
\] (2.10)

where

\[
a = \frac{l + l' - q_i}{2}, \quad b = \frac{l' - l + q_i}{2}, \quad c = \frac{l - l' + q_i}{2}, \quad d = \frac{l + l' + q_i}{2},
\] (2.11)

and

\[
h_k^q(\lambda) = \prod_{j=0}^{q-1} h(\lambda + i(k - j)).
\] (2.12)

It is obvious that \( w^{q_i,1}(a, b, c, d | \lambda) \) are periodic functions, because they involve only simple trigonometric functions (2.10), (2.12) (\( h(\lambda + i\nu) = -h(\lambda) \), \( \nu = \frac{\pi}{\mu} \)), i.e.

\[
w^{q_i,1}(a, b, c, d | \lambda + i\nu) = (-)^q w^{q_i,1}(a, b, c, d | \lambda)
\] (2.13)

Now we can define the transfer matrix of the RSOS\( (q_1, q_2; q) \) model

\[
T_{\{a_1 \ldots a_{2N}\}}^{q_1,q_2;\{b_1 \ldots b_{2N}\}} = \prod_{j=1}^{2N-1} w^{q_1}(a_j, a_{j+1}, b_{j+1}, b_j | \lambda) w^{q_2}(a_{j+1}, a_{j+2}, b_{j+2}, b_{j+1} | \lambda)
\] (2.14)

where we impose periodic boundary conditions, i.e. \( a_{2N+1} = a_1 \) and \( b_{2N+1} = b_1 \). Notice that in the odd and even sites the weights \( w^{q_1,1} \) and \( w^{q_2,1} \) live respectively. The case where \( q_1 = q_2 \) (array type I [32]), namely the fused RSOS\( (p, q) \) model, has been studied in detail by Bazhanov and Reshetikhin in [33]. It is evident that the model studied here is a generalization of the fused RSOS\( (p, q) \) model. The analogue of the array type II in the spin chain framework is the alternating quantum spin chain, introduced by de Vega and Woyanorovich [34], and also studied extensively by many authors [12–16].
From the Yang–Baxter equation for the fused Boltzmann weights (2.9) the commutativity property for the transfer matrix follows, i.e.

\[ T^{q_1,q_2;q}(\lambda) T^{q_1,q_2;q'}(\mu) = T^{q_1,q_2;q'}(\mu) T^{q_1,q_2;q}(\lambda) \]  

(2.15)

Moreover the transfer matrix is periodic (2.13)

\[ T^{q_1,q_2;q}(\lambda + i\nu) = T^{q_1,q_2;q}(\lambda) \]

(2.16)

In order to obtain the Bethe ansatz equations for the model we also need the following useful relations. First we will use the relations acquired by the fusion procedure [35, 31], namely

\[ T_0^{q_1,q_2;q} T_0^{q_1.q_2;1} = f_q^{q_1,q_2} T_0^{q_1,q_2;-1} + f_{q-1}^{q_1,q_2} T_0^{q_1,q_2;+1} \]

(2.17)

where

\[ f_q^{q_1,q_2}(\lambda) = \left( h_q^{q_1}(\lambda)h_q^{q_2}(\lambda) \right)^N, \quad T_k^{q_1,q_2;q} = T^{q_1,q_2;q}(\lambda + ik), \quad T_0^{q_1,q_2;0} = f_0^{q_1,q_2}. \]

(2.18)

Notice that the main difference between equations (2.17), (2.18) and the corresponding equations in [31] is the substitution of \( p \) with \( q_1, q_2 \). In particular \( f_q^p \) in [31] is replaced here by \( f_q^{q_1,q_2} \). We must also have in mind that the Boltzmann weights satisfy the following important property, i.e. up to a gauge transformation, that does not affect the transfer matrix, the weights \( w^{1,q}(a, b, c, d|\lambda) \) and \( w^{1,\nu-2-q}(\nu - a, \nu - b, c, d|\lambda + i(q + 1)) \) coincide, where

\[ w^{1,q}(a, b, c, d|\lambda) = \left( h_{q-1}^{q_1}(-\lambda) \right)^{-1} w^{1,q}(a, b, c, d|\lambda), \]

(2.19)

a similar property holds also between the weights \( w^{q_1,q} \) and \( w^{q_1,\nu-2-q} \). From the above relations it follows that

\[ T^{q_1,q_2;q}(\lambda) = Y T^{q_1,q_2;\nu-2-q}(\lambda + i(q + 1)), \quad q = 1, \ldots, \nu - 3, \]

\[ T^{q_1,q_2;\nu-2}(\lambda) = Y \left( h_{\nu-2}^{q_1}(\lambda)h_{\nu-2}^{q_2}(\lambda) \right)^N \]

(2.20)

with

\[ Y^{l_1,\ldots,l_{2N}} = \prod_{i=1}^{2N} \delta(l_i, \nu - l_i), \quad \left[ T^{q_1,q_2;q}, Y \right] = 0. \]

(2.21)

To derive the transfer matrix eigenvalues we employ the commutativity properties of the transfer matrix (2.13), (2.21), the periodicity (2.13), (2.16), the fusion relations (2.17), (2.18), equations (2.20) and the analyticity of the eigenvalues. Moreover, we employ relations (2.17) and (2.20) for \( q = \nu - 1, \nu \) and we derive

\[ T^{q_1,q_2;\nu-1}(\lambda) = 0, \quad T^{q_1,q_2;\nu}(\lambda) = -Y f_{\nu-1}^{q_1,q_2}(\lambda). \]

(2.22)

From the solution of the above system of equations (2.15)–(2.21), and with the help of relations (2.22) we can write equation (2.17) in the following form

\[ detM[\Lambda^{q_1.q_2;1}(\lambda)] = 0 \]

(2.23)
where

\[
M[\Lambda^{q_1,q_2}(\lambda)] = \begin{pmatrix}
\Lambda_0^{q_1,q_2} & f_0^{q_1,q_2} & 0 & 0 & 0 & -Y f_0^{q_1,q_2} \\
f_2^{q_1,q_2} & \Lambda_1^{q_1,q_2} & f_0^{q_1,q_2} & 0 & 0 & 0 \\
0 & f_2^{q_1,q_2} & \Lambda_2^{q_1,q_2} & f_1^{q_1,q_2} & 0 & 0 \\
0 & 0 & 0 & f_{\nu-2}^{q_1,q_2} & \Lambda_{\nu-2}^{q_1,q_2} & f_{\nu-2}^{q_1,q_2} \\
0 & 0 & 0 & 0 & f_{\nu-1}^{q_1,q_2} & \Lambda_{\nu-1}^{q_1,q_2} \\
\end{pmatrix} \tag{2.24}
\]

Let now \((Q_0^{q_1,q_2}(\lambda), \ldots, Q_{\nu-1}^{q_1,q_2}(\lambda))\) be the null vector of the matrix \((2.24)\) with \(Q_k^{q_1,q_2}(\lambda) = \omega^k Q^{q_1,q_2}(\lambda + ik), \omega^{2\nu} = 1\) and

\[
Q^{q_1,q_2}(\lambda) = \prod_{j=1}^{(q_1+q_2)N} h(\lambda - \lambda_j), \tag{2.25}
\]

then the eigenvalues are given by the following expression

\[
\Lambda^{q_1,q_2}(\lambda) = \omega f_{-1}^{q_1,q_2}(\lambda) \frac{Q^{q_1,q_2}(\lambda + i)}{Q^{q_1,q_2}(\lambda)} + \omega^{-1} f_0^{q_1,q_2}(\lambda) \frac{Q^{q_1,q_2}(\lambda - i)}{Q^{q_1,q_2}(\lambda)}. \tag{2.26}
\]

For completeness we write the general expression of the eigenvalues \(\Lambda^{q_1,q_2;g}(\lambda)\), which follow from the fusion relation \((2.17)\) and \((2.20)\),

\[
\Lambda^{q_1,q_2;g}(\lambda) = Q^{q_1,q_2}(\lambda - i)Q^{q_1,q_2}(\lambda + iq) \sum_{j=0}^{q} \frac{\omega^{q-2j} f_{q_1,q_2}(\lambda + i(j - 1))}{Q^{q_1,q_2}(\lambda + i(j - 1))Q^{q_1,q_2}(\lambda + ij)}. \tag{2.27}
\]

The eigenvalues satisfy all equations \((2.17)\), \((2.18)\) and \((2.20)\), where \(\omega\) is a root of unity that obeys the constraint

\[
\omega^\nu = -(-1)^{(q_1+q_2)N} y \tag{2.28}
\]

and \(y = \pm 1\) is the eigenvalue of the operator \(Y\) \((2.21)\). Equation \((2.28)\) is a consequence of the periodicity and \((2.20)\). Similarly, here the difference with the corresponding eigenvalues in \((31)\) is the replacement of the functions \(f^p\) and \(Q^p\) with \(f^{q_1,q_2}\) and \(Q^{q_1,q_2}\) respectively. Finally, from the analyticity of the eigenvalues we obtain the Bethe ansatz equations

\[
\omega^{-2} e_{q_1}(\lambda_\alpha)^N e_{q_2}(\lambda_\alpha)^N = -\prod_{\beta=1}^{M} e_2(\lambda_\alpha - \lambda_\beta) \tag{2.29}
\]

where

\[
e_n(\lambda; \nu) = \frac{\sinh \frac{\mu(\lambda + i\frac{n}{2})}{2}}{\sinh \frac{\mu(\lambda - i\frac{n}{2})}{2}}. \tag{2.30}
\]

It is important to emphasize that the eigenstates of the model are states with zero spin \(S_z = 0\) \((31), (47), (32)\), i.e.

\[
M = \frac{1}{4}(q_1 + q_2) L, \tag{2.31}
\]

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where \( L = 2N \) (for \( q_1 = q_2 = p \) the later constraint agrees with the corresponding constraint in \([31]\)). We should mention that the Bethe ansatz equations (2.29) have the same structure with the Bethe ansatz equations of the alternating \( q_1, q_2 \) spin chain \([34]–[45]\). The main differences between the model under study and the alternating spin chain are: 1) the phase \( \omega \) which is unit, and 2) the number of strings \( M \) which is not fixed in the alternating spin chain.

The energy\(^2\) of a state is characterized by the set of quasi particles with rapidities (Bethe ansatz roots) \( \lambda_j \), \([27]\), \([28]\), \([38]\),

\[
E = -\frac{\mu}{8\pi} \sum_{j=1}^{M} \sum_{n=1}^{2} \frac{\sin \mu q_n}{\sinh \mu (\lambda_j + \frac{i\mu}{2}) \sinh \mu (\lambda_j - \frac{i\mu}{2})},
\]

(2.33)

The thermodynamic limit \( N \to \infty \) of the equation (2.29) can be studied with the help of the string hypothesis \([12]\), \([13]\), \([27]\), \([28]\), which states that solutions of (2.29) in the thermodynamic limit are grouped into strings of length \( n \) with the same real part and equidistant imaginary parts

\[
\lambda^{(n,j)}_\alpha \equiv \lambda^n_\alpha + \frac{i}{2} (n + 1 - 2j), \quad j = 1, 2, ..., n,
\]

\[
\lambda^{(0,s)}_\alpha \equiv \lambda^0_\alpha + \frac{\pi}{2\mu},
\]

(2.34)

where \( \lambda^n_\alpha \) and \( \lambda^0_\alpha \) are real, and \( \lambda^{(0,s)}_\alpha \) is the negative parity string. The allowed strings that describe the thermodynamics of the model are the same as in \([31]\) and they are \( 1 \leq n \leq \nu - 2 \) \( (q_i \leq \nu - 2) \), the negative parity string is also excluded. Then, the Bethe ansatz equations (2.29) following \([12]\), \([13]\) become,

\[
\omega^{-2} \prod_{j=1}^{2} X_{nq_j}(\lambda^{n}_{\alpha})^N = - \prod_{m=1}^{\nu-2} \prod_{\beta=1}^{M_m} E_{nm}(\lambda^{n}_{\alpha} - \lambda^{m}_{\beta})
\]

(2.35)

where \( n = 1, \ldots, \nu - 2 \), and

\[
X_{nm}(\lambda) = e_{n-m+1}(\lambda) e_{n-m+3}(\lambda) \ldots e_{n+m-3}(\lambda) e_{n+m-1}(\lambda)
\]

\[
E_{nm}(\lambda) = e_{n-m}(\lambda) e^2_{n-m+2}(\lambda) \ldots e^2_{n+m-2}(\lambda) e_{n+m}(\lambda).
\]

(2.36)

Finally, the energy (2.33) by virtue of the string hypothesis (2.34) takes the form

\[
E = -\frac{\mu}{8\pi} \sum_{n=1}^{\nu-2} \int_{-\infty}^{\infty} d\lambda (\bar{Z}^{(\nu)}_{nq_1}(\lambda) + \bar{Z}^{(\nu)}_{nq_2}(\lambda)) \rho_n(\lambda)
\]

(2.37)

\(^2\)The Hamiltonian of the model is defined for \( q = q_1, q_2 \)

\[
H = -\frac{\mu}{8\pi} \sum_{i=1}^{2} \frac{d}{d\lambda} \ln T^{q_1,q_2,q_i}(\lambda)|_{\lambda=0},
\]

(2.32)

where \( T^{q_1,q_2,q_i} \) is the transfer matrix of the \( RSOS(q_1,q_2; q_i) \) model (see also (2.27)).
where, \( \rho_n \) is the density\(^3\) of the \( n \) strings (pseudo-particles) and

\[
Z_{nm}^{(\nu)}(\lambda) = \frac{1}{2\pi} \frac{d}{d\lambda} i \log X_{nm}(\lambda),
\]

(2.39)

the Fourier transform of the last expression is

\[
\hat{Z}_{nm}^{(\nu)}(\omega) = \frac{\sinh \left( (\nu - \max(n,m)) \frac{\omega}{2} \right) \sinh \left( \min(n,m) \frac{\omega}{2} \right)}{\sinh \left( \frac{\omega}{2} \right) \sinh \left( \frac{\omega}{2} \right)}. \tag{2.40}
\]

### 3 Thermodynamic Bethe Ansatz

In what follows the thermodynamic Bethe ansatz equations are derived from (2.35). In addition to the density of pseudo–particles \( \rho_n \) we also introduce the density of holes \( \tilde{\rho}_n \), and we can immediately deduce from (2.35), and with the help of the Maclaurin expansion (2.38) that they satisfy

\[
\tilde{\rho}_n(\lambda) = \frac{1}{2} (Z_{nq_1}^{(\nu)}(\lambda) + Z_{nq_2}^{(\nu)}(\lambda)) - \sum_{m=1}^{\nu-2} A_{nm}^{(\nu)} \rho_m(\lambda), \tag{3.1}
\]

where

\[
A_{nm}^{(\nu)}(\lambda) = \frac{1}{2\pi} \frac{d}{d\lambda} i \log E_{nm}(\lambda) + \delta_{nm}\delta(\lambda), \tag{3.2}
\]

and

\[
\hat{A}_{nm}^{(\nu)}(\omega) = \frac{2 \coth \left( \frac{\omega}{2} \right) \sinh \left( (\nu - \max(n,m)) \frac{\omega}{2} \right) \sinh \left( \min(n,m) \frac{\omega}{2} \right)}{\sinh \left( \frac{\omega}{2} \right) \sinh \left( \frac{\omega}{2} \right)}. \tag{3.3}
\]

However, recall that the only allowed states as in [31] are the ones with \( S_z = 0 \) and therefore from (2.34),

\[
\sum_{n=1}^{\nu-2} n \int_{-\infty}^{\infty} \rho_n(\lambda) d\lambda = \frac{q_1 + q_2}{4}. \tag{3.4}
\]

Equation (3.4) together with relation (3.1) for \( n = \nu - 2 \) yields

\[
\int_{-\infty}^{\infty} \tilde{\rho}_{\nu-2}(\lambda) d\lambda = 0 \Rightarrow \tilde{\rho}_{\nu-2}(\lambda) = 0. \tag{3.5}
\]

The constraint (3.5) is imposed on (3.1) and the density \( \rho_{\nu-2} \) is expressed in terms of the rest densities,

\[
\rho_{\nu-2}(\lambda) = \rho^0(\lambda) - \sum_{m=1}^{\nu-3} a_{\nu-2m}^{(\nu-2)} \rho_m(\lambda), \tag{3.6}
\]

\(^3\)here we use the Maclaurin expansion

\[
\sum_{j=1}^{M} f(\lambda_j) \sim L \int_{-\infty}^{\infty} f(\lambda) \rho(\lambda) d\lambda, \tag{2.38}
\]
where
\[ \hat{a}_n^{(\nu-2)}(\omega) = \frac{\sinh \left( (\nu - n - 2)\frac{\omega}{2} \right)}{\sinh \left( (\nu - 2)\frac{\omega}{2} \right)}, \quad \hat{\rho}^0(\omega) = \frac{\sinh(q_1\frac{\omega}{2}) + \sinh(q_2\frac{\omega}{2})}{4 \cosh(\frac{\omega}{2}) \sinh((\nu - 2)\frac{\omega}{2})}. \] (3.7)

By means of the relation (3.6) the equation (3.1) can be rewritten in the following form
\[ \tilde{\rho}_n(\lambda) = \frac{1}{2} \left( Z_{mq_1}^{(\nu-2)}(\lambda) + Z_{mq_2}^{(\nu-2)}(\lambda) \right) - \sum_{m=1}^{\nu-3} A_{nm}^{(\nu-2)} \rho_m(\lambda). \] (3.8)

The energy of the system, after we apply the string hypothesis is given by (2.37). Now, taking into account the equation (3.8) the energy becomes
\[ e = \frac{E}{L} = -g_0 - \frac{1}{4} \sum_{n=1}^{\nu-3} \int_{-\infty}^{\infty} d\lambda \left( Z_{mq_1}^{(\nu-2)}(\lambda) + Z_{mq_2}^{(\nu-2)}(\lambda) \right) \rho_n(\lambda) \] (3.9)
with
\[ g_0 = \frac{1}{16\pi} \int_{-\infty}^{\infty} d\omega \left( \frac{\sinh(q_1\frac{\omega}{2}) + \sinh(q_2\frac{\omega}{2})}{\sinh(\frac{\omega}{2}) \sinh((\nu - 2)\frac{\omega}{2})} \right)^2. \] (3.10)

In order to determine the thermodynamic Bethe ansatz equations the free energy of the system should be minimized, i.e., \( \delta F = 0 \), where
\[ F = E - TS, \] (3.11)
and the entropy of the system is given by,
\[ S \approx L \sum_{n=1}^{\nu-3} \int_{-\infty}^{\infty} d\lambda \left( (\rho_n(\lambda) + \tilde{\rho}_n(\lambda)) \ln(\rho_n(\lambda) + \tilde{\rho}_n(\lambda)) - \rho_n(\lambda) \ln(1 + \frac{\tilde{\rho}_n(\lambda)}{\rho_n(\lambda)}) \right) \] (3.12)

Then, from equations (3.9), (3.12) and the constraint (3.8) the following expression is implied
\[ T \ln \left( 1 + \eta_n(\lambda) \right) = -\frac{1}{4} \left( Z_{mq_1}^{(\nu-2)}(\lambda) + Z_{mq_2}^{(\nu-2)}(\lambda) \right) + \sum_{m=1}^{\nu-3} A_{nm}^{(\nu-2)} \rho_m(\lambda), \] (3.13)
where \( \eta_n(\lambda) = \frac{\tilde{\rho}_n(\lambda)}{\rho_n(\lambda)}. \) It is convenient to consider the convolution of the expression (3.13) with the inverse of \( A_{nm}, \)
\[ \hat{A}_{nm}^{-1}(\omega) = \delta_{nm} - \hat{s}(\omega)(\delta_{nm+1} + \delta_{nm-1}), \] (3.14)
having in mind the following identity,
\[ A_{nm}^{-1} Z_{mq_i}(\lambda) = \hat{s}(\lambda)\delta_{nq_i}, \] (3.15)

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where
\[ s(\lambda) = \frac{1}{2 \cosh(\pi \lambda)}, \quad \hat{s}(\omega) = \frac{1}{2 \cosh(\omega/2)}, \quad (3.16) \]
and \( \eta_n(\lambda) = e^{s(\lambda)} \), (3.13) becomes,
\[ \epsilon_n(\lambda) = \frac{1}{4} s(\lambda) \ln(1 + \eta_{n+1}(\lambda)) (1 + \eta_{n-1}(\lambda)) - \frac{1}{4} s(\lambda)(\delta_{nq_1} + \delta_{nq_2}), \quad (3.17) \]
for any \( n = 1, \ldots, \nu - 3 \). Note that the last equation differs from the corresponding equation obtained in [31] in the inhomogeneity term \( s(\lambda) \). More specifically, here the terms \( \delta_{nq_1} \) and \( \delta_{nq_2} \) appear, whereas in the study of the fused RSOS \((p, q)\) model [31] only the \( \delta_{np} \) term appears. It is obvious that for \( q_1 = q_2 = p \) our expression agrees with the corresponding expression for the pseudo–energies in [31]. It can be easily deduced from equation (3.17) that the pseudo–energy \( \epsilon_n(\lambda) > 0 \) for every \( n \neq q_1, q_2 \), therefore we conclude that the ground state consists of two filled Dirac seas with strings of length \( q_1, q_2 \), i.e. \( \tilde{\rho}_n(\lambda) = 0 \) for any \( n \), and \( \rho_n(\lambda) = 0 \) for any \( n \neq q_1, q_2 \). The pseudo–energies for those are immediately induced from (3.13) by neglecting the terms of the sum for \( m \neq q_i \),
\[ \epsilon_i(\lambda) = -\frac{1}{4} \sum_{j=1}^{2} \frac{Z^{(\nu-2)}_{q_i q_j}(\lambda) + \sum_{j=1}^{2} Á^{(\nu-2)}_{q_i q_j} \star T \ln(1 + \eta_{q_j}^{-1}(\lambda))}{Z^{(\nu-2)}_{q_i q_1}(\lambda) + Z^{(\nu-2)}_{q_i q_2}(\lambda)} \quad (3.18) \]
(N.B. \( \epsilon_i(\lambda) \equiv \epsilon_{q_i}(\lambda) \)) where
\[ Á^{(\nu-2)}_{nm}(\lambda) = A^{(\nu-2)}_{nm}(\lambda) - \delta_{nm} \delta(\lambda). \quad (3.19) \]
Moreover, the energy of the ground state can be written from (3.8), (3.9)
\[ e_0 = \frac{E_0}{L} = -g_0 - \frac{1}{8} \sum_{i,j=1}^{2} \int_{-\infty}^{\infty} d\lambda Z^{(\nu-2)}_{q_i q_j}(\lambda) s(\lambda) \]
\[ = -\frac{1}{8} \sum_{i,j=1}^{2} \int_{-\infty}^{\infty} d\lambda Z^{(\nu)}_{q_i q_j}(\lambda) s(\lambda). \quad (3.20) \]
The free energy of the system follows from (3.9), (3.11), (3.12), (3.8), and (3.13),
\[ f(T) = \frac{F(T)}{L} = -g_0 - \frac{T}{2} \sum_{n=1}^{\nu-3} \int_{-\infty}^{\infty} d\lambda \ln(1 + \eta_n^{-1}(\lambda))(Z^{(\nu-2)}_{nq_1}(\lambda) + Z^{(\nu-2)}_{nq_2}(\lambda)), \quad (3.21) \]
and in terms of the ground state energy of the system (3.20) we can write
\[ f(T) = e_0 - \frac{T}{2} \sum_{i=1}^{2} \int_{-\infty}^{\infty} d\lambda s(\lambda) \ln(1 + \eta_q(\lambda)). \quad (3.22) \]
In the following sections we are going to explore the behavior of the free energy and the entropy of the system in the high and low temperature.
3.1 The high temperature expansion

By studying the high temperature behavior of the entropy the number of states of the model can be deduced. In the high temperature limit the pseudo–energies $\epsilon_n$ become independent of $\lambda$ \[18\], consequently the thermodynamic Bethe ansatz equations (3.17) are given by

$$\epsilon_n \simeq s(\lambda) * T \ln(1 + \eta_{n+1})(1 + \eta_{n-1})$$

$$= \frac{T}{2} \ln(1 + \eta_{n+1})(1 + \eta_{n-1}), \quad (3.23)$$

and the corresponding solution of the above difference equation is exactly the same as in \[31\] (for $T \to \infty$ the inhomogeneity term can be neglected in (3.17) and therefore the pseudo–energies coincide with the ones found in \[31\])

$$\ln(1 + \eta_n) = \ln \frac{\sin^2(\frac{\pi(n+1)}{\nu})}{\sin^2(\frac{\pi}{\nu})}. \quad (3.24)$$

The free energy follows immediately from (3.22), (3.24)

$$F = -\frac{TL}{4} \sum_{n=q_1,q_2} \ln \frac{\sin^2(\frac{\pi(n+1)}{\nu})}{\sin^2(\frac{\pi}{\nu})}, \quad (3.25)$$

moreover, the entropy in the high temperature limit (3.11) becomes

$$S = \frac{L}{2} \sum_{n=q_1,q_2} \ln \frac{\sin(\frac{\pi(n+1)}{\nu})}{\sin(\frac{\pi}{\nu})}. \quad (3.26)$$

Notice here that the free energy and the entropy are expressed as a sum of two terms since the ground state consists of two filled Dirac seas. On the other hand, in \[31\] the corresponding expressions contain just one term, because the ground state there consists of one filled Dirac sea. Finally, we conclude that the number of states for the system is

$$\prod_{n=q_1,q_2} \left( \frac{\sin(\frac{\pi(n+1)}{\nu})}{\sin(\frac{\pi}{\nu})} \right)^4. \quad (3.27)$$

Notice that in the isotropic limit $\nu \to \infty$ the entropy (3.26) coincides with the one of the alternating $\frac{q_1}{2}, \frac{q_2}{2}$ spin chain (see e.g. [43], [46]). For $q_1 = q_2$ (3.26) agrees with the entropy found in \[31\].

3.2 The low temperature expansion

The main purpose of this section is the derivation of the effective central charge via the study of the low temperature thermodynamics. Recall, that the ground state of the model consists of two filled Dirac seas of strings $q_1, q_2$, therefore we examine the TBA (3.13) for $n = q_1, q_2$. In the $T \to 0$ limit the following quantities are defined

$$T \ln(1 + \eta_i^\pm) \to \pm \epsilon_i^\pm, \quad i = 1, 2 \quad (3.28)$$
with,
\[ \epsilon_i^- = \frac{1}{2}(\epsilon_i - |\epsilon_i|), \quad \epsilon_i^+ = \epsilon_i - \epsilon_i^- \]
then the pseudo-energies for the ground state (3.18) take the form
\[ \epsilon_i(\lambda) = -\frac{1}{4} \sum_{j=1}^{2} Z_{q, q_j}^{(\nu-2)}(\lambda) - \sum_{j=1}^{2} \tilde{A}_{q, q_j}^{(\nu-2)} \ast \epsilon_j^-(\lambda). \]  (3.30)

Finally, the last equation can be written in terms of \( \epsilon_i, \epsilon_i^+ \)
\[ \sum_{j=1}^{2} A_{q, q_j}^{(\nu-2)} \ast \epsilon_j(\lambda) = -\frac{1}{4} \sum_{j=1}^{2} Z_{q, q_j}^{(\nu-2)}(\lambda) + \sum_{j=1}^{2} \tilde{A}_{q, q_j}^{(\nu-2)} \ast \epsilon_j^+(\lambda), \]  (3.31)
and the solution of the above system is given by the following expression
\[ \epsilon_i(\lambda) = -\frac{1}{4} s(\lambda) + \sum_{j=1}^{2} K_{ij} \ast \epsilon_j^+(\lambda), \quad i = 1, 2 \]  (3.32)
where the kernel \( K \) is
\[ K(\lambda) = \begin{pmatrix} h_1(\lambda) & h(\lambda) \\ h_2(\lambda) & h_2(\lambda) \end{pmatrix}, \]
\[ h_1(\omega) = \frac{\sinh((\delta q - 1)\frac{\omega}{2})}{2 \cosh(\frac{\omega}{2}) \sinh(\delta q \frac{\omega}{2})} + \frac{\sinh((\nu - 3 - q_1)\frac{\omega}{2})}{2 \cosh(\frac{\omega}{2}) \sinh((\nu - 2 - q_1)\frac{\omega}{2})}, \]
\[ h_2(\omega) = \frac{\sinh((\delta q - 1)\frac{\omega}{2})}{2 \cosh(\frac{\omega}{2}) \sinh(\delta q \frac{\omega}{2})} + \frac{\sinh((q_2 - 1)\frac{\omega}{2})}{2 \cosh(\frac{\omega}{2}) \sinh(q_2 \frac{\omega}{2})}, \]
\[ h(\omega) = \frac{\sinh(\frac{\omega}{2})}{2 \cosh(\frac{\omega}{2}) \sinh(\delta q \frac{\omega}{2})}, \]
and \( \delta q = q_1 - q_2 \). Note, that the expression of the kernel (3.33), (3.34) in this general form for any \( q_1, q_2 \) is rather a new result. As long as the condition \( q_1 = \nu - 2 - q_2 \) holds, the symmetry between left and right sectors is satisfied (see also e.g. [32]). In particular, \( h_1 = h_2 \), with \( h_1, h_2 \) being related to the scattering in the left (right) sector. In general, for \( \delta q \neq 1 \) each of \( h_i \) is decomposed into two parts (see (3.34)), and every part is related to the triplet amplitude of the XXZ model, with different anisotropy parameters (hidden degrees of freedom [48], [38], [46]). In the special case where \( \delta q = 1 \), there are no hidden degrees of freedom, and \( h_1, h_2 \) are relevant to the triplet amplitudes of the XXZ (sine–Gordon) model with the proper anisotropy parameters, whereas \( h \) corresponds to the massless LR scattering amplitude (see also [19], [45]).

To derive the effective central charge, the entropy of the system must be evaluated in the low temperature limit. In order to do that the following approximations, which hold true for \( \lambda \to \infty \), should be made [16], [17], [18],
\[ \rho_n(\lambda) \simeq \frac{2}{\pi} f_n(\lambda) \frac{d}{d\lambda} \epsilon_n(\lambda), \quad \bar{\rho}_n(\lambda) \simeq \frac{2}{\pi} (1 - f_n(\lambda)) \frac{d}{d\lambda} \epsilon_n(\lambda) \]  (3.35)
where \( f_n(\lambda) = (1 + e^{\frac{\epsilon_n(\lambda)}{T}})^{-1}, \) (\( f_0(\lambda) = f_{\nu-2}(\lambda) \equiv 1 \)), and the entropy \(^{(3.12)}\), can be written as

\[
s = \frac{S}{L} = -\frac{2}{\pi} \sum_{n=1}^{\nu-3} \int_{\epsilon_n(-\infty)}^{\epsilon_n(\infty)} d\epsilon_n \left( f_n(\lambda) \ln f_n(\lambda) + (1 - f_n(\lambda)) \ln(1 - f_n(\lambda)) \right).
\]  

(3.36)

By changing variables in the last expression,

\[
s = \frac{2T}{\pi} \sum_{n=1}^{\nu-3} \int_{f_n^{\min}}^{f_n^{\max}} df_n \left( \frac{\ln f_n}{1 - f_n} + \frac{\ln(1 - f_n)}{f_n} \right),
\]

(3.37)

and by introducing the Rogers dilogarithm

\[
L(x) = -\frac{1}{2} \int_0^x dy \left( \frac{\ln y}{1 - y} + \frac{\ln(1 - y)}{y} \right)
\]

(3.38)

the entropy can be written in terms of the dilogarithms as follows

\[
s = -\frac{4T}{\pi} \sum_{n=1}^{\nu-3} \left( L(f_n^{\max}) - L(f_n^{\min}) \right).
\]

(3.39)

The next natural step is the solution of the TBA equations \(^{(3.17)}\) in the low temperature limit. In order to do that it is convenient (see also \([16], [17], [18], [31]\)) to introduce the function

\[
\phi_n(\lambda) = \frac{1}{T} \epsilon_n(\lambda - \frac{1}{\pi} \ln T),
\]

(3.40)

then the TBA equations become,

\[
\phi_n \simeq -s(\lambda) * \ln f_{n+1} f_{n-1} - \frac{1}{4} e^{-\pi \lambda} (\delta_{nq_1} + \delta_{nq_2}).
\]

(3.41)

Our task is to solve the later difference equation in the limit that \( \lambda \to \pm \infty \), \((\phi_n \text{ independent of } \lambda)\). First for \( \lambda \to \infty \) we compute the \( f_n^{\max} \), the difference equations \(^{(3.41)}\) become,

\[
\phi_n \simeq -\frac{1}{2} \ln f_{n+1} f_{n-1}, \quad n = 1, \ldots, \nu - 3,
\]

(3.42)

this system has been solved (see e.g. \([18], [31]\)) with the solution being (note again that the inhomogeneity term is omitted),

\[
f_n^{\max} = \frac{\sin^2(\frac{\pi}{\nu})}{\sin^2(\frac{\pi (n+1)}{\nu})}, \quad n = 1, \ldots, \nu - 3.
\]

(3.43)

Similarly, for \( \lambda \to -\infty \)

\[
\phi_n \simeq -\frac{1}{2} \ln f_{n+1} f_{n-1}, \quad n = 1, \ldots, \nu - 3, \quad n \neq q_1, q_2
\]

\[
\phi_{q_1} \to -\infty, \quad \phi_{q_2} \to -\infty,
\]

(3.44)
the solution of the later system has the following form

\[ f_n^{\text{min}} = \frac{\sin^2\left(\frac{\pi}{q_n+2}\right)}{\sin^2\left(\frac{\pi(n+1)}{q_n+2}\right)}, \quad n = 1, \ldots, q_2 - 1, \quad f_{q_2}^{\text{min}} = 1 \]

\[ f_n^{\text{min}} = \frac{\sin^2\left(\frac{\pi}{q_1-q_2+2}\right)}{\sin^2\left(\frac{\pi(n-q_1+1)}{q_1-q_2+2}\right)}, \quad n = q_2 + 1, \ldots, q_1 - 1, \quad f_{q_1}^{\text{min}} = 1 \]

\[ f_n^{\text{min}} = \frac{\sin^2\left(\frac{\pi}{\nu-q_1}\right)}{\sin^2\left(\frac{\pi(n-q_1+1)}{\nu-q_1}\right)}, \quad n = q_1 + 1, \ldots, \nu - 3. \quad (3.45) \]

Notice that the main difference with the corresponding solution in [31] is the appearance of the middle term in (3.45) (for \( n = q_2 + 1, \ldots, q_1 - 1 \)), in [31] there is no such term in the solution since \( q_1 = q_2 = p \). According to equation (3.39) and the above solutions, the entropy can be written as

\[
s = -\frac{4T}{\pi} \sum_{n=2}^{\nu-2} \left\{ L\left(\frac{\sin^2\left(\frac{\pi}{q}\right)}{\sin^2\left(\frac{\pi n}{q}\right)}\right) - \sum_{n=2}^{q_2} L\left(\frac{\sin^2\left(\frac{\pi}{q_2+2}\right)}{\sin^2\left(\frac{\pi n}{q_2+2}\right)}\right) - 2L(1) \right\} - \sum_{n=2}^{q_1-q_2} L\left(\frac{\sin^2\left(\frac{\pi}{q_1-q_2+2}\right)}{\sin^2\left(\frac{\pi n}{q_1-q_2+2}\right)}\right) - \sum_{n=2}^{\nu-q_1-2} L\left(\frac{\sin^2\left(\frac{\pi}{\nu-q_1}\right)}{\sin^2\left(\frac{\pi n}{\nu-q_1}\right)}\right).
\]

Moreover,

\[
\sum_{n=2}^{q-2} L\left(\frac{\sin^2\left(\frac{\pi}{q}\right)}{\sin^2\left(\frac{\pi n}{q}\right)}\right) = \frac{2(q-3)}{q} L(1), \quad q > 3
\]

and \( L(1) = \frac{\pi^2}{6} \) (see e.g. [31]), then

\[
s = \frac{2\pi T}{3} \left(\frac{3q_2}{q_2 + 2} + \frac{3\delta q}{\delta q + 2} - \frac{6q_1}{\nu(\nu-q_1)}\right).
\]

The knowledge of the entropy allows the calculation of the heat capacity, in particular

\[
C_u = T \frac{\partial s(T)}{\partial T} = -T \frac{\partial^2 f(T)}{\partial^2 T}, \quad (3.49)
\]

also, at low temperature it has been shown that [8], [9],

\[
C_u = \frac{\pi c}{3u} T + ...
\]

where \( c \) is the central charge of the effective conformal field theory, and \( u \) is the speed of sound (Fermi velocity). By means of (3.48), (3.49) and (3.50) (\( u = \frac{1}{2} \) in our notation, see e.g. [27]) we can readily deduce the central charge

\[
c = \frac{3q_2}{q_2 + 2} + \frac{3\delta q}{\delta q + 2} - \frac{6q_1}{\nu(\nu-q_1)}.
\]

(3.51)
Recall the LR symmetry condition \(q_1 = \nu - 2 - q_2\), then the conformal anomaly can be expressed in terms of \(q_2\) and \(\nu\) as

\[
c = \frac{3q_2}{q_2 + 2} - \frac{6q_2}{\nu(\nu - q_2)} + \frac{3q_2}{q_2 + 2} - \frac{6q_2}{\bar{\nu}(\bar{\nu} - q_2)},
\]

where \(\bar{\nu} = \nu - q_2\). Note that the later expression is written in terms of the central charges of two copies of the generalized \(SU(2)\) diagonal coset theory. More specifically, the conformal anomaly \(\text{(3.52)}\) is identified as the sum of the central charges of the \(\mathcal{M}(q_2, \nu - q_2 - 2)\) and \(\mathcal{M}(q_2, \bar{\nu} - q_2 - 2) \equiv \mathcal{M}(q_2, \delta q)\) coset models, therefore the effective conformal field theory should be of the form \(\mathcal{M}(q_2, \nu - q_2 - 2) \otimes \mathcal{M}(q_2, \delta q)\).

Expression \(\text{(3.51)}\) for \(q_1 = q_2\) is compatible with the result obtained by Bazhanov and Reshetikhin — in the “anti–ferromagnetic” regime \(^4\) in \(\text{[31]}\). In the special case where \(q_2 = 1\), the central charge becomes

\[
c = 2 - 12 \frac{1}{\nu(\nu - 2)} = 1 - \frac{6}{\nu(\nu - 1)} + 1 - \frac{6}{(\nu - 1)(\nu - 2)}
\]

and it agrees with the \(c_{IR}\) presented in \(\text{[32]}\), given by the sum of the central charges of two unitary minimal models. Finally, in the isotropic limit the central charge \(\text{(3.51)}\) reduces to the one of the alternating \(q_2/2, q_2/2\) quantum spin chain (see e.g. \(\text{[42]}, \text{[46]}\)).

### 4 Discussion

The thermodynamics of the critical \(RSOS(q_1, q_2; q)\) model, obtained by fusion, was studied and the high and low temperature expansion were discussed. The main result of this work was the derivation of the effective conformal anomaly \(\text{(3.51)}, \text{(3.52)}\) of the model, the validity of which was confirmed by various tests. More specifically, for \(q_2 = 1\) expression \(\text{(3.52)}\) coincides with the \(c_{IR}\) presented in \(\text{[32]}\), and it is specified by the sum of the central charges of the unitary minimal models \(\mathcal{M}_\nu, \mathcal{M}_{\nu - 1}\), where

\[
c = 1 - \frac{6}{\nu(\nu - 1)}
\]

is the central charge of the unitary minimal model \(\mathcal{M}_\nu\) of conformal field theory \(\text{[2]}\). Also, in the case where \(q_1 = q_2\) we recover the results of \(\text{[31]}\). Finally, in the isotropic limit \(\nu \to \infty\) our result agrees with the conjectured central charge for the alternating spin chain \(\text{[42]}\), expressed as the sum of the central charges of \(SU(2)_{q_2}, SU(2)_{\delta q}\), i.e.,

\[
c = \frac{3q_2}{q_2 + 2} + \frac{3\delta q}{\delta q + 2}
\]

\(^4\) the analysis of the “ferromagnetic” regime is exactly the same as in \(\text{[31]}\), and it gives rise to the central charge of the parafermionic \(SU(2)_{\nu(1)}\) theory i.e., \(c = 2 - \frac{\nu}{\nu}, \text{[50]}\).
An exact calculation of the effective central charge for the alternating spin chain, by means of the finite size effects and the thermodynamic Bethe ansatz analysis, is presented in [46]. In general, the central charge (3.52) obtained in the present study is identified as the sum of the central charges of the $M(q_2, \nu - q_2 - 2)$ and $M(q_2, \delta q)$ coset models, whereas in [31] Bazhanov and Reshetikhin by studying the $RSOS(p, q)$ models found an effective central charge that corresponds to the $M(p, \nu - p - 2)$ model. We conclude that the effective conformal field theory that emanates from the study of the $RSOS(q_1, q_2; q)$ model, consists of two copies of the generalized $SU(2)$ coset theory.

A compelling task is to extend the above calculations in the presence of boundaries, and compute the boundary energy of the system as well as the corresponding $g$–function (see e.g. [51]–[53]). There exist solutions of the boundary Yang–Baxter equation [54] in the $RSOS$ representation [55]–[57], and moreover, in [55] the Bethe ansatz equations of the $RSOS$ model with boundaries have been explicitly derived. Finally, a very challenging problem is the formulation of a string hypothesis for integrable critical models associated with non–simply laced algebras such as the $A_2^{(2)}$ (Izergin–Korepin) quantum spin chain [58]. Such a formulation is necessary for the investigation of the thermodynamics as well as the conformal properties of these systems.

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