An introduction to the geometry of ultrametric spaces

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Abstract

Some examples and basic properties of ultrametric spaces are briefly discussed.

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1 The ultrametric triangle inequality

Let \((M, d(x, y))\) be a metric space. Thus \(M\) is a set, and \(d(x, y)\) is a nonnegative real-valued function defined for \(x, y \in M\) such that \(d(x, y) = 0\) if and only if \(x = y\),
\[
d(y, x) = d(x, y)
\]
(1.1) for every \(x, y \in M\), and
\[
d(x, z) \leq d(x, y) + d(y, z)
\]
(1.2) for every \(x, y, z \in M\). The latter condition is known as the triangle inequality, and if \(d(x, y)\) satisfies the stronger condition
\[
d(x, z) \leq \max(d(x, y), d(y, z))
\]
(1.3) for every \(x, y, z \in M\), then \(d(x, y)\) is said to be an ultrametric on \(M\).

Equivalently, the ultrametric version of the triangle inequality says that \(d(x, y)\) and \(d(y, z)\) cannot both be strictly less than \(d(x, z)\) for any \(x, y, z \in M\). In particular, the standard metric on the real line does not have this property.

The discrete metric on any set \(M\) is defined by setting \(d(x, y) = 1\) when \(x \neq y\). This is an ultrametric, and there are also more interesting examples.

2 \(p\)-Adic numbers

Let \(p\) be a prime number, such as 2, 3, 5, or 7, etc. We would like to define the \(p\)-adic absolute value \(|\cdot|_p\) on the set \(\mathbb{Q}\) of rational numbers. Let us begin by defining \(|\cdot|_p\) on the set \(\mathbb{Z}\) of integers.

If \(x = 0\), then \(|x|_p = 0\). If \(x \in \mathbb{Z}\) and \(x \neq 0\), then
\[
x = w p^n
\]
(2.1) for some \(w, n \in \mathbb{Z}\) such that \(w \neq 0\), \(w\) is not divisible by \(p\), and \(n \geq 0\). In this case,
\[
|x|_p = p^{-n}.
\]
(2.2)

Note that
\[
|x|_p \leq 1,
\]
(2.3) since \(n \geq 0\).

Otherwise, if \(x \in \mathbb{Q}\) and \(x \neq 0\), then \(x\) can be expressed as
\[
x = \frac{u}{v} p^n
\]
(2.4) for some \(u, v, n \in \mathbb{Z}\) such that \(u, v \neq 0\) and \(u, v\) are not divisible by \(p\). Again \(|x|_p\) is defined to be \(p^{-n}\), which can be large or small depending on whether \(n\) is negative or positive.
It is easy to see that

\[ |x + y|_p \leq \max(|x|_p, |y|_p) \]

for every \( x, y \in \mathbb{Q} \). If \( x, y \in \mathbb{Z} \), then this basically says that \( x + y \) is divisible by \( p^j \) when \( x \) and \( y \) are both divisible by \( p^j \). Furthermore,

\[ |xy|_p = |x|_p |y|_p \]

for every \( x, y \in \mathbb{Q} \).

The \( p \)-adic metric is defined on \( \mathbb{Q} \) by

\[ d_p(x, y) = |x - y|_p. \]

This is an ultrametric on \( \mathbb{Q} \), by the previous remarks.

However, \( \mathbb{Q} \) is not complete as a metric space with respect to the \( p \)-adic metric. In the same way that the real numbers can be obtained by completing the rationals with respect to the standard metric, the \( p \)-adic numbers \( \mathbb{Q}_p \) are obtained by completing \( \mathbb{Q} \) with respect to \( d_p(x, y) \).

As in the case of the real numbers, addition and multiplication can be defined for \( p \)-adic numbers, and \( \mathbb{Q}_p \) is a field. The \( p \)-adic absolute value and metric can be extended to \( \mathbb{Q}_p \) as well, with properties like those just described on \( \mathbb{Q} \). The set \( \mathbb{Z}_p \) of \( p \)-adic integers consists of \( x \in \mathbb{Q}_p \) such that \(|x|_p \leq 1\), which one can show to be the closure of \( \mathbb{Z} \) in \( \mathbb{Q}_p \).

### 3 Spaces of sequences

Let \( A \) be a set with at least two elements, and let \( \mathcal{A} \) be the set of sequences \( a = \{a_i\}_{i=1}^\infty \) such that \( a_i \in A \) for each \( i \). If \( 0 < \rho < 1 \) and \( a, b \in \mathcal{A} \), then put \( d_\rho(a, b) = 0 \) when \( a = b \), and otherwise

\[ d_\rho(a, b) = \rho^n \]

where \( n \) is the largest nonnegative integer such that \( a_i = b_i \) when \( i \leq n \).

If \( a, b, c \in \mathcal{A} \), then

\[ d_\rho(a, c) \leq \max(d_\rho(a, b), d_\rho(b, c)). \]

This basically follows from the fact that if \( n \) is a nonnegative integer such that

\[ a_i = b_i \text{ and } b_i = c_i \text{ when } i \leq n, \]

then \( a_i = c_i \) for \( i \leq n \) too. Thus \( d_\rho(a, b) \) defines an ultrametric on \( \mathcal{A} \).

Note that

\[ d_\rho(a, b) \leq 1 \]

for each \( a, b \in \mathcal{A} \) and \( 0 < \rho < 1 \). These ultrametrics all determine the same topology on \( \mathcal{A} \), which is the product topology using the discrete topology on \( A \).
If $A$ has only finitely many elements, then $A$ is compact with respect to this topology. If we allow $\rho = 1$, then $d_\rho(a, b)$ reduces to the discrete metric on $A$, and we get the discrete topology on $A$. If $\rho > 1$, then $d_\rho(a, b)$ is not even a metric on $A$.

For each $\alpha \in A$, let $T_\alpha : A \to A$ be the mapping defined by $T_\alpha(a) = a'$, where

\[ a'_1 = \alpha \text{ and } a'_i = a_{i-1} \text{ when } i \geq 2. \]

It is easy to see that $d_\rho(T_\alpha(a), T_\alpha(b)) = \rho d_\rho(a, b)$ for every $a, b \in A$ and $0 < \rho < 1$. Also,

\[ A = \bigcup_{\alpha \in A} T_\alpha(A). \]

Thus $A$ is the union of smaller copies of itself.

4 Snowflake metrics

Let $(M, d(x, y))$ be an ultrametric space. For each $\tau > 0$, $d(x, y)^\tau$ is also an ultrametric on $M$, which determines the same topology on $M$. For example, if $d_\rho(a, b)$ is as in the previous section, then

\[ d_\rho(a, b)^\tau = d_\rho^\tau(a, b). \]

If $(M, d(x, y))$ is an ordinary metric space, then one can show that $d(x, y)^\tau$ satisfies the triangle inequality and hence is a metric on $M$ when $0 < \tau < 1$, but this does not always work when $\tau > 1$. Suppose that $d(x, y)^\tau$ does satisfy the triangle inequality, so that

\[ d(x, z) \leq (d(x, y)^\tau + d(y, z)^\tau)^{1/\tau} \]

for every $x, y, z \in M$. This implies that

\[ d(x, z) \leq 2^{1/\tau} \max(d(x, y), d(y, z)) \]

for every $x, y, z \in M$. If $d(x, y)^\tau$ is a metric on $M$ for every $\tau > 1$, then it follows that $d(x, y)$ is an ultrametric on $M$, since $2^{1/\tau} \to 1$ as $\tau \to \infty$.

5 Open sets

Let $(M, d(x, y))$ be a metric space. For each $x \in M$ and $r > 0$, the open ball in $M$ with center $x$ and radius $r$ is defined by

\[ B(x, r) = \{ y \in M : d(x, y) < r \}. \]
Similarly, the closed ball in $M$ with center $x$ and radius $r$ is defined by

$$B(x, r) = \{ y \in M : d(x, y) \leq r \}. \tag{5.2}$$

A set $U \subseteq M$ is said to be **open** if for every $x \in U$ there is an $r > 0$ such that

$$B(x, r) \subseteq U. \tag{5.3}$$

As usual, the empty set $\emptyset$ and $M$ itself are open subsets of $M$, the intersection of finitely many open subsets of $M$ is an open set, and the union of any family of open subsets of $M$ is an open set.

For each $w \in M$ and $t > 0$, the open ball $B(w, t)$ is an open set in $M$. Indeed, let $x \in B(w, t)$ be given. Thus $d(w, x) < t$, and so $r = t - d(w, x) > 0$. One can use the triangle inequality to show that

$$B(x, r) \subseteq B(w, t). \tag{5.4}$$

If $d(\cdot, \cdot)$ is an ultrametric on $M$, then the same statement holds with $r = t$. Moreover,

$$\overline{B}(x, t) \subseteq \overline{B}(w, t) \tag{5.5}$$

for each $x \in \overline{B}(w, t)$. Thus closed balls are also open sets in ultrametric spaces. Of course, this is not normally the case in ordinary metric spaces, such as the real line with the standard metric.

If $d(\cdot, \cdot)$ is an ultrametric on $M$, then the complement of $B(w, t)$ in $M$ is also an open set, which is the same as saying that $\overline{B}(w, t)$ is a closed set in $M$. Specifically, if $x \in M \setminus B(w, t)$, then

$$B(x, t) \subseteq M \setminus B(w, t). \tag{5.6}$$

Equivalently, if $z \in M$ and $d(x, z) < r \leq d(w, x)$, then

$$d(w, z) \geq r. \tag{5.7}$$

Otherwise, $d(w, z), d(x, z) < r$ imply that $d(w, x) < r$, a contradiction. Hence an ultrametric space $M$ with at least two elements is not connected. One can also use this to show that $M$ is totally disconnected, in the sense that $M$ does not contain any connected sets with at least two elements. In particular, every continuous path in an ultrametric space is constant.

## 6 Completeness

Let $(M, d(x, y))$ be a metric space. Remember that a sequence $\{x_j\}_{j=1}^{\infty}$ of elements of $M$ is said to be a **Cauchy sequence** if for every $\epsilon > 0$ there is an $L \geq 1$ such that

$$d(x_j, x_l) < \epsilon \tag{6.1}$$

for every $j, l \geq L$. If $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence in $M$, then

$$\lim_{j \to \infty} d(x_j, x_{j+1}) = 0, \tag{6.2}$$
which is to say that for every $\epsilon > 0$ there is an $L \geq 1$ such that
\begin{equation}
d(x_j, x_{j+1}) < \epsilon
\end{equation}
for each $j \geq L$. The converse holds when $d(x, y)$ is an ultrametric, since
\begin{equation}
d(x_j, x_l) \leq \max(d(x_j, x_{j+1}), \ldots, d(x_{l-1}, x_l))
\end{equation}
when $j < l$.

A metric space $M$ is said to be complete if every sequence of elements of $M$ converges to an element of $M$. It is a nice exercise to check that the spaces described in Section 3 are complete. Using the completeness of the $p$-adic numbers $Q_p$ and the preceding remarks, one can check that an infinite series $\sum_{j=1}^{\infty} x_j$ of $p$-adic numbers converges in $Q_p$ if and only if $\lim_{j \to \infty} x_j = 0$ in $Q_p$, but there are examples which show that the converse does not hold.

\section{Binary sequences}

Let $B$ be the set of all binary sequences, which is to say the sequences $b = \{b_i\}_{i=1}^{\infty}$ such that $b_i = 0$ or 1 for each $i$. Thus $B$ is the same as the set $A$ associated to $A = \{0, 1\}$ as in Section 3. Consider the mapping $\phi : B \to \mathbb{R}$ that sends each binary sequence to the real number with that binary expansion. Explicitly,
\begin{equation}
\phi(b) = \sum_{i=1}^{\infty} b_i 2^{-i}.
\end{equation}
It is well known that $\phi$ maps $B$ onto the unit interval $[0, 1]$, consisting of all real numbers $x$ such that $0 \leq x \leq 1$. However, some real numbers have more than one binary expansion, corresponding to binary sequences that are eventually constant. This is a continuous mapping with respect to the ultrametrics $d_\rho(a, b)$ on $B^*$ defined in Section 3 and the standard metric on the real line.

More precisely, let $d(a, b)$ be the ultrametric $d_\rho(a, b)$ on $B^*$ from Section 3 with $\rho = 1/2$. Remember that the absolute value of a real number $x$ is denoted $|x|$ and defined to be $x$ when $x \geq 0$ and $-x$ when $x \leq 0$, and that the standard metric on the real line $\mathbb{R}$ is given by $|x - y|$. For each integer $\ell$, one can check that $\phi$ maps every closed ball in $B^*$ with respect to $d(a, b)$ of radius $2^\ell$ onto a closed interval in the real line with length $2^\ell$. In particular,
\begin{equation}
|\phi(a) - \phi(b)| \leq d(a, b)
\end{equation}
for every $a, b \in B^*$.

\section{The Cantor set}

If $r, t$ are real numbers with $r \leq t$, then $[r, t]$ is the closed interval in the real line consisting of $x \in \mathbb{R}$ such that $r \leq x \leq t$, and the length of this interval is
Let $E_0$ be the unit interval $[0, 1]$, and put

$$E_1 = \left[ 0, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, 1 \right].$$

Continuing in this way, $E_n$ is the union of $2^n$ disjoint closed intervals of length $3^{-n}$ for each positive integer $n$, and $E_{n+1}$ is obtained from $E_n$ by removing the open middle third of each of the $2^n$ intervals in $E_n$. By construction, $E_{n+1} \subseteq E_n$ for each $n$, and $E = \bigcap_{n=0}^{\infty} E_n$ is known as the Cantor set.

As in Section 3, let $A$ be a set with at least two elements, and choose an element $\alpha$ of $A$ to be a basepoint. Let $A^\ast$ be the collection of doubly-infinite sequences $a = \{a_i\}_{i=-\infty}^{\infty}$ of elements of $A$ for which there is an integer $n$ such that $a_i = \alpha$ when $i \leq n$. If $0 < \rho < 1$, then $d_\rho(a, b)$ can be defined on $A^\ast$ as before by $d_\rho(a, b) = 0$ when $a = b$ and $d_\rho(a, b) = \rho^n$ when $a \neq b$ and $n$ is the largest integer such that $a_i = b_i$ for $i \leq n$. This is again an ultrametric on $A^\ast$, which reduces to the discrete metric when $\rho = 1$.

If we identify $A$ with the set of $a \in A^\ast$ such that $a_i = \alpha$ when $i \leq 0$, then this definition of $d_\rho(a, b)$ agrees with the previous one on $A$. The topology on $A^\ast$ determined by $d_\rho(a, b)$ is the same for each $\rho$, $0 < \rho < 1$, and $d_\rho(a, b)$ is unbounded on $A^\ast$. If $A$ has only finitely many elements, then $A^\ast$ is locally compact with respect to this topology.

### 9 Doubly-infinite sequences

As in Section 3, let $A$ be a set with at least two elements, and choose an element $\alpha$ of $A$ to be a basepoint. Let $A^\ast$ be the collection of doubly-infinite sequences $a = \{a_i\}_{i=-\infty}^{\infty}$ of elements of $A$ for which there is an integer $n$ such that $a_i = \alpha$ when $i \leq n$. If $0 < \rho < 1$, then $d_\rho(a, b)$ can be defined on $A^\ast$ as before by $d_\rho(a, b) = 0$ when $a = b$ and $d_\rho(a, b) = \rho^n$ when $a \neq b$ and $n$ is the largest integer such that $a_i = b_i$ for $i \leq n$. This is again an ultrametric on $A^\ast$, which reduces to the discrete metric when $\rho = 1$.

If we identify $A$ with the set of $a \in A^\ast$ such that $a_i = \alpha$ when $i \leq 0$, then this definition of $d_\rho(a, b)$ agrees with the previous one on $A$. The topology on $A^\ast$ determined by $d_\rho(a, b)$ is the same for each $\rho$, $0 < \rho < 1$, and $d_\rho(a, b)$ is unbounded on $A^\ast$. If $A$ has only finitely many elements, then $A^\ast$ is locally compact with respect to this topology.
For each \( a = \{a_i\}_{i=-\infty}^{\infty} \in \mathcal{A}^* \), let \( T(a) \) be the doubly-infinite sequence whose \( i \)th term is equal to \( a_{i-1} \). Thus \( T(a) \in \mathcal{A}^* \) is the same as \( a \), but with the terms of the sequence shifted forward by one step. This defines a one-to-one mapping from \( \mathcal{A}^* \) onto itself, which satisfies

\[
d_{\rho}(T(a), T(b)) = \rho d_{\rho}(a, b)
\]

for every \( a, b \in \mathcal{A}^* \) and \( 0 < \rho < 1 \). Note that \( T(a) = a \) exactly when \( a_i = \alpha \) for each \( i \).

### 10 Binary sequences revisited

Let \( \mathcal{B}^* \) be the set of doubly-infinite sequences \( b = \{b_i\}_{i=-\infty}^{\infty} \) such that \( b_i = 0 \) or \( 1 \) for each \( i \) and there is an integer \( n \) for which \( b_i = 0 \) when \( i \leq n \). As in the previous section, we can identify the collection \( \mathcal{B} \) of binary sequences with the set of \( b \in \mathcal{B}^* \) that satisfy \( b_i = 0 \) when \( i \leq 0 \). We can also extend the mapping \( \phi \) from Section 7 to \( \mathcal{B}^* \), by putting

\[
\phi(b) = \sum_{i=n}^{\infty} b_i 2^{-i}
\]

when \( b_i = 0 \) for each \( i < n \). Thus \( \phi \) maps \( \mathcal{B}^* \) onto the set of all nonnegative real numbers. Two distinct elements of \( \mathcal{B} \) are sent to the same real number by \( \phi \) if and only if they agree up to some term, where one of the sequences is equal to 0 followed by all 1’s, and the other is equal to 1 followed by all 0’s. Let \( d(a, b) \) be the ultrametric \( d_{\rho}(a, b) \) on \( \mathcal{B}^* \) as in the previous section with \( \rho = 1/2 \). As in Section 7,

\[
|\phi(a) - \phi(b)| \leq d(a, b)
\]

for every \( a, b \in \mathcal{B}^* \).

### 11 Nesting

Let \( (M, d(x, y)) \) be an ultrametric space. For every \( x, y \in M \) and \( r, t > 0 \), either

\[
B(x, r) \cap B(y, t) = \emptyset,
\]

or

\[
B(x, r) \subseteq B(y, t),
\]

or

\[
B(y, t) \subseteq B(x, r).
\]

More precisely, the first condition holds when

\[
d(x, y) > \max(r, t),
\]
the second condition holds when
\begin{equation}
d(x, y) < t \text{ and } r \leq t, \tag{11.5}
\end{equation}
and the third condition holds when
\begin{equation}
d(x, y) < r \text{ and } t \leq r. \tag{11.6}
\end{equation}
There are analogous statements for closed balls. As usual, this does not normally work in an ordinary metric space, like the real line with the standard metric. In an ultrametric space, it follows that
\begin{equation}
B(x, r) \cup B(y, t) \tag{11.7}
\end{equation}
is a ball when
\begin{equation}
B(x, r) \cap B(y, t) \neq \emptyset. \tag{11.8}
\end{equation}
This also holds in the real line, but not in the plane, for instance.

12 Dyadic intervals

If \( r, t \) are real numbers with \( r < t \), then the half-open, half-closed interval \([r, t)\) consists of the real numbers \( x \) such that \( r \leq x < t \). A dyadic interval in the real line is an interval of the form
\begin{equation}
[i \cdot 2^l, (i + 1) \cdot 2^l), \tag{12.1}
\end{equation}
where \( i, l \) are integers, although sometimes one considers closed intervals of the same type instead. Thus the length of a dyadic interval is always an integer power of 2. If \( I, I' \) are dyadic intervals, then either \( I \cap I' = \emptyset \), or \( I \subseteq I' \), or \( I' \subseteq I \). One should also include the possibility that \( I, I' \) are practically disjoint in the sense that \( I \cap I' \) contains only a single point when one uses closed intervals. The structure of dyadic intervals basically corresponds to ultrametric geometry, even if it may not be stated explicitly. Dyadic intervals and their relatives are often used in real analysis.

13 More spaces of sequences

Let \( A_1, A_2, \ldots \) be a sequence of sets, each with at least two elements. As an extension of the situation described in Section 3, consider the space \( \mathcal{A} \) of sequences \( a = \{a_i\}_{i=1}^{\infty} \) such that \( a_i \in A_i \) for each \( i \). This is the same as the Cartesian product of the \( A_i \)'s. Also let \( \rho = \{\rho_i\}_{i=0}^{\infty} \) be a strictly decreasing sequence of positive real numbers with \( \rho_0 = 1 \). For \( a, b \in \mathcal{A} \), put \( d_\rho(a, b) = 0 \) when \( a = b \), and otherwise
\begin{equation}
d_\rho(a, b) = \rho_n \tag{13.1}
\end{equation}
where \( n \) is the largest nonnegative integer such that \( a_i = b_i \) for each \( i \leq n \). This is equivalent to the earlier definition when \( \rho_n \) is the \( n \)th power of a fixed
number. As before, one can check that $d_p(a, b)$ is an ultrametric on $A$. The topology on $A$ determined by this ultrametric is the product topology using the discrete topology on each $A_i$ when $\lim_{n \to \infty} \rho_n = 0$. In this case, $A$ is compact if each $A_i$ has only finitely many elements. There are variants of this construction in which the geometry is less homogeneous.

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