On fluctuations of global and mesoscopic linear eigenvalue statistics of generalized Wigner matrices

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Abstract. We consider an $N$ by $N$ symmetric or Hermitian generalized Wigner matrix $H_N$, whose entries are independent centered random variables with uniformly bounded moments. We assume that the variance matrix, $(S)_{ij} = s_{ij} := \mathbb{E}|H_{ij}|^2$, satisfies $\sum_{i=1}^{N} s_{ij} = 1$, for all $j$ and $c^{-1} \leq N s_{ij} \leq c$ for some constant $c > 1$. We establish Gaussian fluctuations for the linear eigenvalue statistics of $H_N$ on global scales, as well as on all mesoscopic scales up to spectral edges, with the expectation and variance formulated in terms of the variance matrix $S$. We then obtain the universal mesoscopic central limit theorems by computing the variance and bias in the mesoscopic scaling inside the bulk and at the edges respectively.

Date: January 27, 2020

1. Introduction

1.1. Linear eigenvalue statistics of Wigner matrices. The ensemble of Wigner matrices was introduced by Wigner [69] in the 1950’s as a model for heavy nuclei atoms. A Wigner matrix $H_N$ is an $N \times N$ matrix whose entries are independent real or complex valued random variables up to the symmetry constraint $H_N = H_N^*$. Wigner matrices with real or complex Gaussian entries are known as the Gaussian Orthogonal Ensemble (GOE) and the Gaussian Unitary Ensemble (GUE), respectively. The celebrated Wigner semicircle law states that the empirical eigenvalue distribution of $H_N$ converges to the semicircle distribution with density $\rho_{sc}(x) := \frac{1}{\pi \sqrt{4 - x^2}}1_{[-2,2]}$. More precisely, denoting by $(\lambda_i)_i^{N}$ the eigenvalues of $H_N$, for any sufficiently regular test function $f$,

$$\frac{1}{N} \sum_{i=1}^{N} f(\lambda_i) - \int_{\mathbb{R}} f(x) \rho_{sc}(x) dx$$

converges in probability to zero as $N \to \infty$, which can be understood as a Law of Large Numbers.

It is then natural to derive the corresponding Central Limit Theorem (CLT), i.e., the Gaussian fluctuations of the linear eigenvalue statistics

$$\sum_{i=1}^{N} f(\lambda_i) - \mathbb{E} \left[ \sum_{i=1}^{N} f(\lambda_i) \right]. \quad (1.1)$$

The linear statistics (1.1) need not be normalized by $N^{-\frac{1}{2}}$ as in the classical CLT, which can be explained by the strong correlations among eigenvalues. Khorunzhy, Khoruzhenko and Pastur [51] proved the CLT for the trace of the resolvent of Wigner matrices. Johansson [50] derived Gaussian fluctuations for the linear eigenvalue statistics of invariant ensembles, including the GUE and GOE. Bai and Yao [9] used a martingale method to extend the CLTs to arbitrary Wigner matrices and analytic test functions. The regularity conditions on the test functions were weakened by Lytova and Pastur [61], Shcherbina [63] via

∗Supported by the European Research Council (ERC) under the European Union Horizon 2020 research and innovation programme, grant 647133 (ICHAOS).
†Supported by the Göran Gustafsson Foundation and the Swedish Research Council Grant VR-2017-05195.
the characteristic function of (1.1), and more recently by Sosoe and Wong [68] who obtained the CLT for $H^{1+\epsilon}$ test functions.

The fluctuations of the linear eigenvalue statistics on mesoscopic scales, i.e.,

$$
\sum_{i=1}^{N} f\left(\frac{\lambda_i - E_0}{\eta_0}\right) - \mathbb{E}\left[\sum_{i=1}^{N} f\left(\frac{\lambda_i - E_0}{\eta_0}\right)\right],
$$

(1.2)

with fixed energy $E_0 \in (-2, 2)$ and scale parameter $N^{-1} \ll \eta_0 \ll 1$, were first studied by Boutet de Monvel and Khorunzhy [20] for the GOE and the test function $f(x) = (x - i)^{-1}$. They subsequently extended their results to real Wigner matrices [21] with $N^{-\frac{2}{3}} \ll \eta_0 \ll 1$. A Mesoscopic CLT for the GUE was obtained by Fyodorov, Khoruzhenko and Simm [41], and was extended by Lodha and Simm [60] to complex Wigner matrices on scales $N^{-1/3} \ll \eta_0 \ll 1$. He and Knowles [44] improved these CLTs on optimal mesoscopic scales $N^{-1} \ll \eta_0 \ll 1$ for Wigner matrices. They also studied the two point correlation function of Wigner matrices on mesoscopic scales in [45]. More recently, Landon and Sosoe [54] obtained similar CLTs by studying the characteristic function of (1.2).

Besides Wigner matrices, mesoscopic CLTs were also obtained in many other random matrices ensembles, e.g., random band matrices [29, 30], sparse Wigner matrices [43], invariant $\beta$-ensembles [12, 16, 52], orthogonal polynomial ensembles [22], classical compact groups [67], circular $\beta$-ensembles [53], free sum of matrices [10], and Dyson Brownian motion [28, 48, 55]. Recently, mesoscopic CLTs were used as important tools in the theory of homogenization of Dyson Brownian motion (DBM), introduced by Bourgade, Erdős, Yau and Yin [16] to prove fixed energy universality of Wigner matrices. Landon, Sosoe and Yau subsequently derived a mesoscopic CLT to show fixed energy universality of the DBM. Combined with the homogenization theory of DBM, mesoscopic CLTs were also used in [14, 54] to derive the Gaussian fluctuations of single eigenvalues, as well as fluctuations of the logarithm of the determinant [17] of Wigner matrices.

Mesoscopic linear eigenvalue statistics can also be studied at the spectral edges, where the mesoscopic scale ranges over $N^{-\frac{2}{3}} \ll \eta_0 \ll 1$. Basor and Widom [11] used asymptotics of the Airy kernel to derive Gaussian fluctuations of the linear eigenvalue statistics of the GUE at the edges. Min and Chen [62] subsequently extended this result to the GOE. Adhikari and Huang [1] proved the mesoscopic CLT for Dyson Brownian motion [28, 48, 55] at the edges. For random band matrices, the condition (1.4) is not satisfied. We refer to [18, 19, 32] for results on local laws and bulk universality, and to [66] for edge universality.

1.2. Generalized Wigner matrix. In this paper, we are interested in the linear eigenvalue statistics for generalized Wigner matrices, which were introduced in [38]. Let $H_N = (H_{ij})_{i,j=1}^{N}$ be an $N$ by $N$ matrix with independent but not identically distributed centered random variables up to a symmetry constraint. Denote by $S$ the matrix of variances, i.e. $S := (s_{ij})$, with $s_{ij} = \mathbb{E}|H_{ij}|^2$. We assume that

$$
\text{for all } j, \quad \sum_{i=1}^{N} s_{ij} = 1.
$$

(1.3)

We say $H_N$ is a generalized Wigner matrix if the size of $s_{ij}$ is comparable with $N^{-1}$, that is, for all $1 \leq i, j \leq N$, there exists $c > 1$ such that

$$
e^{-1} \leq N s_{ij} \leq c.
$$

(1.4)

Standard Wigner matrices are a special case of generalized Wigner matrices, with $s_{ij} = N^{-1}$ for all $1 \leq i, j \leq N$. The first condition in (1.3) guarantees that the limiting spectral measure of $H_N$ is given by the semicircle law; see [7, 42, 65]. Without the condition (1.3), the limiting eigenvalue distribution is characterized by the Dyson equation and were classified in [3]. Local laws of such general Wigner-type matrices were obtained in [4, 5] and bulk universality was then established in [4], while the edge and cusp universality were derived in [5, 6, 35].

The second assumption (1.4) is a sufficient condition for generalized Wigner matrices to demonstrate the same local eigenvalue statistics as standard Wigner matrices. Universality for the local eigenvalue statistics of generalized Wigner matrices was obtained in [16, 39, 40] inside the bulk and in [15, 38, 57] at the edges. For random band matrices, the condition (1.4) is not satisfied. We refer to [18, 19, 32] for results on local laws and bulk universality, and to [66] for edge universality.
Consider now a special variance matrix $S$ with $s_{ij} = f\left(\frac{i}{N}, \frac{j}{N}\right)$, where $f \in C([0,1] \times [0,1])$ is a non-negative, symmetric function such that $\int f(x,y)dy \equiv 1$. A CLT for the linear eigenvalue statistics of such matrices was obtained in [7] by studying its generating function via combinatorial enumeration, with the variance given explicitly as an infinite series. Global CLTs for band random matrices were obtained in [58, 49, 64], while the mesoscopic linear statistics were studied in [29, 30]. Fluctuations of the linear eigenvalue statistics on global scales for many familiar classes of random matrices were also studied in [23], where a unified technique was formulated for deriving such CLTs using second order Poincaré inequalities, without an explicit formula for the variance. Under this framework, CLTs for linear eigenvalue statistics of Wigner matrices with general variance profiles were obtained in [2]. Within the framework of second-order free probability theory, the global fluctuations of block Gaussian matrices with variance profiles were proved in [27].

In the present paper, we consider generalized Wigner matrices with variance matrix $S$ satisfying (1.3) and (1.4). We derive Gaussian fluctuations for the linear eigenvalue statistics (1.2), with explicit integral formulas for the variance and expectations in terms of the variance matrix $S$, at fixed energy $E_0 \in [-2,2]$ on scales $\eta_0$ such that $N^{-1} \ll \eta_0 \sqrt{\eta_0 + \kappa_0} \leq 1$, where $\kappa_0 = \kappa_0(E_0)$ denotes the distance from $E_0$ to the closest edge of the semicircle law; see Theorem 2.4. This range of $\eta_0$ covers the global scales as well as all mesoscopic scales up to the spectral edges. By computing the variance and expectation explicitly, we obtain the universal CLTs on all mesoscopic scales, for energies $E_0$ in the bulk and at the edges respectively; see Theorem 2.6.

The proof of the main technical result Proposition 2.3, from which Theorem 2.4 follows, is provided in Section 4. We follow the idea of [61, 54] to study the characteristic function of the linear eigenvalue statistics (1.2). Via the Helffer-Sjöstrand functional calculus, we write the derivative of the characteristic function in terms of the resolvent of $H_N$, and then cut off the ultra-mesoscopic scales of the integral domain; see (4.12), since the very local scales do not contribute to the mesoscopic linear statistics. The benefit is that on the restricted integral domain, the resolvent of $H_N$ is controlled effectively by the local laws [33, 38]. We subsequently apply the cumulant expansion (see Lemma 5.1) to solve the right side of (4.12). This technique was first used in random matrix theory by [51] and in recent papers, e.g., [34, 44, 56, 61]. The corresponding result is summarized in Lemma 4.2, and the proof is carried out in Section 5. The key tools to estimate the error in Lemma 4.2 are the isotropic local law for the resolvent [13] and the fluctuation averaging estimates [31, 47, 70]. Compared with the standard Wigner matrices [54], the main difficulty is to find a local law for the two point function $T_{ab}(z,z') := \sum_{j=1}^{N} s_{aj} G_{jb}(z) G_{jb}(z')$, with distinguished spectral parameters $z, z'$, where the resolvent identity or cyclicity of trace no longer help; see Lemma 5.2 with proof in Section 5.2. Similar two point functions of the resolvents appeared in [36, 24, 26, 10] to derive Gaussian fluctuations of the linear eigenvalue statistics for different random matrix ensembles. The proof of Lemma 5.2 is inspired by the fluctuation averaging in [31], combined with recursive moment estimates based on cumulant expansions. The special case $T(z, \pi)$ was studied previously in [31, 47, 70], and our statements are for arbitrary parameters $z, z'$.

In Section 6, we study the expectation of the linear eigenvalue statistics (1.2), as stated in Proposition 2.5. In addition, we compute the variances and expectations on mesoscopic scales in the bulk and at the edges respectively, and then conclude with Theorem 2.6. The computation details for the variances and expectations as well as the complex case are provided in the Appendices.

**Notation:** We will use the following definition on high-probability estimates from [31] in the paper.

**Definition 1.1.** Let $X \equiv X^{(N)}$ and $Y \equiv Y^{(N)}$ be two sequences of nonnegative random variables. We say $Y$ stochastically dominates $X$ if, for all (small) $\epsilon > 0$ and (large) $D > 0$,

$$\Pr(X^{(N)} > N^\epsilon Y^{(N)}) \leq N^{-D}, \quad (1.5)$$

for sufficiently large $N \geq N_0(\epsilon, D)$, and we write $X \prec Y$ or $X = O_\prec(Y)$.

For any vector $v \in \mathbb{C}^N$, let $\|v\|_\infty$ be the sup norm and $\|v\|_2$ be the Euclidean norm. For any matrix $A \in \mathbb{C}^{N \times N}$, the matrix norms induced by the sup and Euclidean vector norm are given by $\|A\|_\infty = \max_{1 \leq i \leq N} \sum_{j=1}^{N} |A_{ij}|$ and $\|A\|_{op} = \sigma_{\text{max}}(A)$ respectively, where $\sigma_{\text{max}}(A)$ is the largest singular value of matrix $A$. We also use $\|A\|_{HS} = (\sum_{ij} |A_{ij}|^2)^{1/2}$ to denote the Hilbert-Schmidt norm, and $\|A\|_{\infty} = \max_{i,j} |A_{ij}|$ to represent the sup norm of matrix $A$. 


Throughout the paper, we use $c$ and $C$ to denote strictly positive constants that are independent of $N$. Their values may change from line to line. We write $X \ll Y$ or $X = o(Y)$ if $\frac{X}{Y} \to 0$ as $N \to \infty$. We write $X = O(Y)$ if there exists a constant $C > 0$ such that $|X| \leq C|Y|$. We write $X \sim Y$ if there exist constants $c, C > 0$ such that $c|Y| \leq |X| \leq C|Y|$. Finally, we denote the upper half-plane by $\mathbb{C}^+ := \{z \in \mathbb{C} : \text{Im}z > 0\}$.

2. Main results

2.1. Model and assumptions. Let $H \equiv H_N$ be an $N \times N$ real or complex generalized Wigner matrix satisfying the following assumption.

Assumption 2.1. For real symmetric ($\beta = 1$) generalized Wigner matrix, we assume that

1. $\{H_{ij}[i \leq j]\}$ are independent real-valued centered random variables with $\xi_{ij} = H_{ij}$.
2. Let $S \equiv S_N$ denoted the matrix of variances, i.e., $S := (s_{ij})$ with $s_{ij} = \text{E}[\xi_{ij}]^2$. There exist constants $0 < C_{inf}, C_{sup} < \infty$ such that
   \begin{equation}
   \sum_{i=1}^{N} s_{ii} = 1; \quad C_{inf} \leq \inf_{N,i,j} N s_{ij} \leq \sup_{N,i,j} N s_{ij} \leq C_{sup}.
   \end{equation}

3. All moments of the entries of $\sqrt{N}H_N$ are uniformly bounded, i.e., for any $k \in \mathbb{N}$, there exists constants $C_k$ independent of $N$ such that for all $1 \leq i, j \leq N$,
   \begin{equation}
   \text{E}(|\sqrt{N}H_{ij}|^k) \leq C_k.
   \end{equation}

In particular, we set
   \begin{equation}
   k_2 := \sum_{i=1}^{N} \text{E}|\xi_{ii}|^2 = \text{Tr}S, \quad k_4 := \sum_{i,j=1}^{N} c_4(\xi_{ij}),
   \end{equation}

where $c_4(\xi_{ij})$ is the fourth cumulant given by $c_4(\xi_{ij}) = \text{E}[\xi_{ij}]^4 - 3(\text{E}[\xi_{ij}]^2)^2$.

For complex Hermitian ($\beta = 2$) generalized Wigner matrix, we assume that

(a) $\{\text{Re}H_{ij}, \text{Im}H_{ij}[i \leq j]\}$ are independent real-valued centered random variables with $\xi_{ij} = \overline{H_{ji}}$.
(b) The same moment conditions (2) and (3) hold. In addition, $\text{E}[\xi_{ij}^2] = 0$ for $i \neq j$, and $k_4 := \sum_{i,j=1}^{N} c_4(\xi_{ij}) + c_4(\xi_{ij})$.

The eigenvalues of $H_N$ are denoted by $(\lambda_i)_{i=1}^{N}$. The empirical spectral measure of $H_N$ is defined as $\mu_N(x) := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$. For a probability measure $\nu$ on $\mathbb{R}$, denote by $m_\nu$ its Stieltjes transform, i.e.,
   \begin{equation}
   m_\nu(z) := \int_{\mathbb{R}} \frac{d\nu(x)}{x-z}, \quad z \in \mathbb{C}^+.
   \end{equation}

Note that $m_\nu : \mathbb{C}^+ \to \mathbb{C}^+$ is analytic and can be analytically continued to the real line outside the support of $\nu$. Moreover, $m_\nu$ satisfies $\lim_{\eta \to \infty} \im m_\nu(i\eta) = -1$. The Stieltjes transform of $\mu_N$, denoted by $m_N$, is given by
   \begin{equation}
   m_N(z) = \int_{\mathbb{R}} \frac{\lambda \text{d}\mu_N(\lambda)}{\lambda-z} = N^{-1} \text{Tr}G(z), \quad \text{where} \quad G(z) := (H_N - zI)^{-1}.
   \end{equation}

The function $G(z)$ is referred to as the resolvent or Green function of $H$. Wigner semicircle law states that $\mu_N$ weakly converges to the semicircle law, $d\mu_{sc}(x) := \frac{1}{2\pi}\sqrt{4-x^2} \text{I}_{|x| \leq 2} dx$. The Stieltjes transform of $\mu_{sc}$ is given by
   \begin{equation}
   m_{sc}(z) = \frac{1}{2\pi} \int_{-2}^{2} \frac{\sqrt{4-x^2}}{x-z} dx = \frac{-z + \sqrt{z^2 - 4}}{2}.
   \end{equation}

where the square root function is taken with a branch cut $[-2, 2]$ such that $\text{Im} m_{sc}(z) > 0$, where $\text{Im} z > 0$. The Stieltjes transform $m_{sc}$ is the unique analytic solution $\mathbb{C}^+ \to \mathbb{C}^+$ satisfying
   \begin{equation}
   m_{sc}^2(z) + 2m_{sc}(z) + 1 = 0.
   \end{equation}

The semicircle law can be further extended down to the local scales $\text{Im} z \gg N^{-1}$. We introduce the spectral domain,
   \begin{equation}
   D' := \{z = E + i\eta : |E| \leq 5, N^{-1+\tau} \leq \eta \leq 10\},
   \end{equation}
for any constant \(\tau > 0\), and define two deterministic control parameters

\[
\Psi(z) := \sqrt{\frac{\Im m_{sc}(z)}{N\eta}} + \frac{1}{N\eta}, \quad \Theta(z) := \frac{1}{N\eta}.
\]

(2.7)

With estimates of \(m_{sc}(z)\) in Lemma 3.1 below, it is easy to check

\[
CN^{-\frac{1}{2}} \leq \Psi(z) \ll 1, \quad z \in D'.
\]

(2.8)

We have the following local law for the resolvent of \(H_N\), which is an essential tool in our proof.

**Theorem 2.2** (Theorem 2.3 in [33]). Let \(H_N\) be a generalized Wigner matrix satisfying Assumption 2.1. The following estimates hold uniformly in \(z \in D'\):

\[
\max_{i,j}|G_{ij}(z) - \delta_{ij}m_{sc}(z)| \ll \Psi(z); \quad |m_N(z) - m_{sc}(z)| \ll \Theta(z).
\]

(2.9)

Note that in [33], they require the variance matrix \(S\) to be irreducible so that 1 is a simple eigenvalue. This condition is automatically satisfied by (2.1) due to the Perron-Frobenius Theorem. The local law gives an upper bound on the size of the fluctuations \(\text{Tr} G(z) - \mathbb{E}\text{Tr} G(z)\). It is hence natural to study the distribution of the fluctuations \(\text{Tr} G(z) - \mathbb{E}\text{Tr} G(z)\). Via the Helffer-Sjöstrand functional calculus, a CLT for the resolvent can be translated to a CLT for the general linear statistics.

### 2.2. Main Results

Fix the energy \(E_0 \in [-2, 2]\) and set \(N^{-1} \ll \eta_0 \leq 1\). Consider a scaled test function

\[
f \equiv f_N(x) := g\left(\frac{x - E_0}{\eta_0}\right), \quad g \in C^2(R).
\]

(10.1)

Define the distance between the support of \(f\) to the nearest spectral edge of the semicircle law,

\[
\kappa_0 := \text{dist}(\text{supp}(f), \{-2, 2\}).
\]

(11.1)

We define the characteristic function of the linear eigenvalue statistics

\[
\phi(\lambda) := \mathbb{E}[\epsilon(\lambda)], \quad \text{where} \quad \epsilon(\lambda) := \exp\left\{i\lambda(\text{Tr} f(H_N) - \mathbb{E}\text{Tr} f(H_N))\right\}, \quad \lambda \in \mathbb{R}.
\]

(12.1)

Then the characteristic function \(\phi\) satisfies the following proposition.

**Proposition 2.3.** Let \(H_N\) be a generalized Wigner matrix satisfying Assumption 2.1. Fix \(E_0 \in [-2, 2]\) and choose \(N^{-1} \ll \eta_0 \leq 1\) such that \(\eta_0 \sqrt{\kappa_0} + \eta_0 \geq N^{-1+c}\) for some \(c > 0\). There exists a small \(0 < \tau < \frac{1}{10}\) such that the characteristic function \(\phi\) satisfies

\[
\phi'(\lambda) = -\lambda \phi(\lambda)V(f) + \tilde{E},
\]

where \(V(f)\) is given by

\[
V(f) = -\frac{1}{4\pi^2} \int_{\Gamma_1} \int_{\Gamma_2} \tilde{f}(z)\tilde{f}(z')K(z, z')dzdz';
\]

(13.1)

\(\tilde{E}\) is the error term; the contours \(\Gamma_{1,2}\) are given by \(\{z \in \mathbb{C} : |\text{Im } z| = N^{-\tau}\eta_0\}\), \(\{z' \in \mathbb{C} : |\text{Im } z'| = \frac{1}{2}N^{-\tau}\eta_0\}\) respectively with counterclockwise orientation; \(\tilde{f}\) is an almost analytic extension of \(f\) given in Lemma 4.1 below. The kernel \(K(z, z')\) is given by

\[
K(z, z') = \frac{2}{\beta} \text{Tr} \left( \frac{m_{sc}(z)m'_{sc}(z')S}{(1 - m_{sc}(z)m_{sc}(z')S)^2} \right) + k_2 \left( 1 - \frac{2}{\beta} \right) m_{sc}(z)m'_{sc}(z') + 2k_4 m_{sc}(z)m'_{sc}(z)m_{sc}(z)m'_{sc}(z'),
\]

(14.1)

where \(k_2, k_4\) are given in (2.3), and \(\beta = 1, 2\) is the symmetry parameter. The error \(\tilde{E}\) satisfies

\[
\tilde{E} = O_{\prec}\left(|\lambda| \log NN^{-\tau}\right) + O_{\succ}\left(\frac{(1 + |\lambda|^4)N^{4\tau}}{(N\eta_0\sqrt{\kappa_0} + \eta_0)^4}\right),
\]

provided that \(V(f) < O(1)\).

Proposition 2.3 implies the following theorem.

**Theorem 2.4.** Under the same assumption and notations in Proposition 2.3, if we further assume that there exists \(c' > 0\) such that \(V(f) > c'\) for sufficiently large \(N\), then \(\frac{\text{Tr} f(H_N) - \mathbb{E}\text{Tr} f(H_N)}{\sqrt{V(f)}}\) converges in distribution to a standard Gaussian random variable.
In addition, the expectation of the linear statistics $\text{Tr} f(H_N)$ has the following asymptotic expansion:

**Proposition 2.5.** Under the same assumption and notations in Proposition 2.3, the so-called bias is given by

$$B(f) := \mathbb{E} \text{Tr} f(H_N) - N \int \frac{f(x) \rho_{sc}(x)}{2\pi} \text{d}x = \frac{1}{2\pi} \int \frac{\hat{f}(z) b(z) \text{d}z}{\sqrt{N\eta_0 + \kappa_0}} + O(N^{-\gamma}) + O_{\prec} \left( \frac{N^{2\tau}}{\sqrt{N\eta_0 \kappa \eta_0 + \eta_0^2}} \right), \quad (2.15)$$

where

$$b(z) := \left( \frac{2}{\beta} - 1 \right) \text{Tr} \left( \frac{m_{sc}^3(z) m_{sc}(z)}{1 - m_{sc}^2(z)} S^2 \right) + k^2 m_{sc}'(z) m_{sc}^2(z). \quad (2.16)$$

**Remark:** We remark that Theorem 2.4 applies to the global scales as well as the mesoscopic scales up to spectral edges. The formulas for the variance (2.13) and the bias (2.15) coincide with the expressions for standard Wigner matrices [54, 59] where $s_{ij} = N^{-1}$ for all $1 \leq i, j \leq N$.

Finally, we obtain the following universal CLTs for the mesoscopic linear statistics in the bulk and at the edges respectively.

**Theorem 2.6 (Universal mesoscopic CLTs).** Let $H_N$ be a generalized Wigner matrix satisfying Assumption 2.1. Fix $E_0 \in (-2, 2)$ and $c_1 \in (0, 1)$, and set $\eta_0 = N^{-c_1}$. For any function $g \in C^2_{\mathbb{C}}(\mathbb{R})$, the mesoscopic linear statistics

$$\sum_{i=1}^{N} g \left( \frac{\lambda_i - E_0}{\eta_0} \right) - N \int g \left( \frac{x - E_0}{\eta_0} \right) \rho_{sc}(x) \text{d}x \quad (2.17)$$

converges in distribution to a centered Gaussian random variable of variance $\frac{1}{2\pi} \int |\xi| |\hat{g}(\xi)|^2 \text{d}\xi$, where $\hat{g}(\xi) := (2\pi)^{-1/2} \int \hat{g}(x) e^{-ix\xi} \text{d}x$, and $\beta = 1, 2$ is the symmetry parameter.

Furthermore, set $E_0 = \pm 2$ and $\eta_0 = N^{-c_2}$ with fixed $c_2 \in (0, \frac{3}{2})$. Then the mesoscopic linear statistics (2.17) converges in distribution to a Gaussian random variable of mean $\left( \frac{2}{\beta} - 1 \right) \frac{g(0)}{4}$ and variance $\frac{1}{2\pi} \int |\xi| |\hat{h}(\xi)|^2 \text{d}\xi$, where $h(x) = g(\frac{2}{3}x^2)$.

**Remark:** The mean and variance in Theorem 2.6 agree with the corresponding results for the Gaussian ensembles, see [20, 41] in the bulk and [11, 62] at the edges. Such edge formulas were also obtained in other random matrices ensembles, e.g., Dyson Brownian motion [1], deformed Wigner matrices and sample covariance matrices [59]. The limiting law is universal, only depending on the symmetry parameter $\beta = 1, 2$, and is independent of the scaling $\eta_0$ and the energy $E_0$.

### 3. Preliminaries

In this section, we introduce preliminary results before the proof of Proposition 2.3.

#### 3.1. Properties of the Stieltjes transform of the semicircle law.

In this subsection, we recall some properties of $m_{sc}$. Let $\kappa = \kappa(E)$ be the distance from $E$ to the closest spectral edge, i.e.,

$$\kappa := \min\{|E + 2|, |E - 2|\}. \quad (3.1)$$

Define the spectral domain

$$D := \{ z = E + i\eta : |E| \leq 5, 0 < \eta \leq 10 \}. \quad (3.2)$$

**Lemma 3.1 (Lemma 4.2 in [40], Lemma 6.2 in [37]).** Under Assumption 2.1, we have the following estimates.

1. For any $z \in D$, there exists a constant $c > 0$ such that

$$c \leq |m_{sc}(z)| \leq 1 - c\eta. \quad (3.3)$$

2. For all $z \in D$, we have

$$|\text{Im} m_{sc}(z)| \sim \begin{cases} \sqrt{\kappa + \eta}, & \text{if } |E| \leq 2, \\ \frac{\eta}{\sqrt{\kappa + \eta}}, & \text{if otherwise.} \end{cases} \quad (3.4)$$

3. For all $z \in D$, there exist some constants $c, C > 0$ such that

$$c\sqrt{\kappa + \eta} \leq |1 - m_{sc}^2(z)| \leq C\sqrt{\kappa + \eta}. \quad (3.5)$$
(4) For all $z \in D$, we have

$$z + 2m_{sc}(z) \sim \sqrt{k + \eta}; \quad |m'_{sc}(z)| \sim \frac{1}{\sqrt{k + \eta}}; \quad |m''_{sc}(z)| = O\left(\frac{1}{\sqrt{(k + \eta)^3}}\right). \quad (3.6)$$

Note that the estimates in (3.6) are implied by (1)-(3) in Lemma 3.1 combining with the following relations from (2.5),

$$m'_{sc}(z) = -\frac{m_{sc}(z)}{z + 2m_{sc}(z)} = \frac{m^2_{sc}(z)}{1 - m^2_{sc}(z)}; \quad m''_{sc}(z) = 2m^3_{sc}(z)(1 + m'_{sc}(z))^3.$$

3.2. Properties of the variance matrix $S$. In this subsection, we state properties of the variance matrix $S$, which is crucial in studying the local laws of the generalized Wigner matrices. Recall that $S = (s_{ij})_{1 \leq i,j \leq N}$ is the matrix of variances satisfying (2.1), and $S$ is deterministic, symmetric and doubly stochastic. Hence 1 is the largest eigenvalue, with eigenvector $e = N^{-\frac{1}{2}}(1,1,\cdots,1)^T$. By the Perron-Frobenius Theorem, the largest eigenvalue 1 is simple and all other eigenvalues are strictly less than 1 in absolute value. Define $\delta_{\pm}$ to be the spectral gaps satisfying

$$\text{Spec}(S) \subset [-1 + \delta_-, 1 - \delta_+] \cap \{1\}.$$

It is not hard to show that

$$\delta_{\pm} \geq C_{\inf} > 0,$$

provided $S$ satisfies (2.1). Since $-1 < S \leq 1$ and $|m_{sc}| < 1$ from (3.3), $(1 - m_{sc}(z)m_{sc}(z'))S$ is invertible. Thus, we have the following estimates.

**Lemma 3.2** (Lemma 6.3 in [37]). Define $\Pi := ee^T$ with $e = N^{-\frac{1}{2}}(1,1,\cdots,1)^T$. For any $z, z' \in D'$, there exists $C > 0$ such that

$$\left\|\frac{1}{1 - m_{sc}(z)m_{sc}(z')S}\right\|_{\infty \to \infty} \leq \frac{C}{1 - m_{sc}(z)m_{sc}(z')}; \quad \left\|\frac{1}{1 - m_{sc}(z)m_{sc}(z')S}\right\|_{\infty \to \infty} \leq C.$$

Similar statement can be found in Proposition 6.10 [37], and the proof also applies to Lemma 3.2 with two parameters $z, z'$. In particular, combining Lemma 3.2 with (3.5), we have

$$\rho := \left\|\frac{1}{1 - m^2_{sc}(z)S}\right\|_{\infty \to \infty} \leq \frac{C}{1 - m^2_{sc}(z)} \sim \frac{1}{\sqrt{k + \eta}}. \quad (3.7)$$

We also have a trivial lower bound, $\rho \geq \frac{1}{|1 - m^2_{sc}(z)|} \geq \frac{1}{2},$ since $e$ is an eigenvector of $S$ and $|m_{sc}(z)| \leq 1.$

3.3. Properties of the resolvent $G$. As a more general version of the local law given in Theorem 2.2, we state the following isotropic local law. Recall the control parameters $\Psi$ and $\Theta$ from (2.7).

**Theorem 3.3** (Theorem 2.12 in [13], Theorem 1.5 in [46]). Under Assumption 2.1, for any deterministic unit vector $v, w \in \mathbb{C}^N$, we have

$$\left|\langle v, G(z)w \rangle - m_{sc}(z)\langle v, w \rangle\right| \prec \Psi(z),$$

uniformly in $z \in \{z = E + \text{i}\eta : |E| \leq \omega^{-1}, N^{-1} + \omega \leq \eta \leq \omega^{-1}\}$, where $\omega$ is a small positive constant.

Let $H^{(i)}$ be the matrix with $i$-th column and row set to be zero, i.e., $(H^{(i)})_{jk} = H_{jk}\delta_{ij}\delta_{ik}$. The Green function of $H^{(i)}$ is then denoted by $G^{(i)}$. Define the partial expectation with respect to the $i$-th row or column by $E_{\text{i}}X := E[X|H^{(i)}]$. We also set $\sum_{k} := \sum_{k \neq i}$. Then we are ready to state the following averaging fluctuation results for the monomials in the resolvent entries.

**Theorem 3.4** (Theorem 4.8, Proposition 3.3 in [31]; Proposition 3.9 in [32]). Under Assumption 2.1, the following estimates hold for $z \in D'$ uniformly:

$$\sum_{i=1}^{N} s_{ij}(1-E_{\text{i}})G_{ii}(z) \prec \Psi^2(z), \quad \sum_{i=1}^{N} s_{ij}G_{ii}(z)m_{sc}(z) \prec \rho\Psi^2(z), \quad \sum_{i} s_{ij}(1-E_{\text{i}})G_{ik}(z)G_{ki}(z) \prec \Psi^3(z),$$

where $\rho$ is given by (3.7) and $\Psi$ is given in (2.7).

The following lemmas will be used later in the paper, whose proofs are standard.
Lemma 3.5 (Lemma 8.3 in [37]). For \( k \neq i,j \), we have
\[
G_{ij} = G_{ij}^{(k)} + G_{ik}G_{kj}/G_{kk}; \quad \frac{1}{G_{ii}} = \frac{1}{G_{ii}^{(k)}} - \frac{G_{ik}G_{kj}}{G_{ii}G_{ij}^{(k)}G_{kk}}. \tag{3.8}
\]
For \( i \neq j \), we have
\[
G_{ij} = -G_{ii} \sum_{k} h_{ik} G_{kj}^{(i)} = -G_{ij} \sum_{k} G_{ik}^{(i)} h_{kj}; \quad G_{ii} = \frac{1}{h_{ii} - z - \sum_{k,l} G_{ik}^{(i)} h_{li}}. \tag{3.9}
\]

Lemma 3.6 (Theorem 7.7 in [37]). Let \( \{a_i, 1 \leq i \leq N\} \) be a Gaussian random variable with mean zero and variance \( \sigma^2 \) and having uniform sub-exponential decay such that for some positive \( \alpha > 0 \) and all \( x \), \( \mathbb{P}(|a_i| \geq e^x) \leq C e^{-x^\alpha} \). Let \( B = (B_{ij}) \in \mathbb{C}^{N \times N} \) be deterministic. Then we have
\[
\mathbb{P}\left( \left| \sum_{i,j} a_i^* B_{ij} a_j - \sum_{i=1}^{N} d^2 B_{ii} \right| \geq (\log N)^{\frac{3}{2} + \alpha^2} \left( \sum_{i,j} |B_{ij}|^2 \right)^{\frac{1}{2}} \right) \leq C N^{-\log \log N}. \tag{3.10}
\]

Finally, we end this subsection with some properties of stochastic domination.

Lemma 3.7 (Proposition 6.5 in [37]).
1. \( X \prec Y \) and \( Y \prec Z \) imply \( X \prec Z \);
2. If \( X_1 \prec Y_1 \) and \( X_2 \prec Y_2 \), then \( X_1 + X_2 \prec Y_1 + Y_2 \) and \( X_1 X_2 \prec Y_1 Y_2 \);
3. If \( X \prec Y \) and \( X \prec N^{-c} X \) for some \( \epsilon > 0 \), then \( X \prec Y \);
4. If \( X \prec Y \), \( \mathbb{E}X \geq N^{-c} \) and \( |X| \leq N^c \) almost surely with some fixed exponent \( c \), then we have \( \mathbb{E}X \prec \mathbb{E}Y \).

4. Proof of Proposition 2.3

To simplify the proof, we will only consider the real symmetric case (\( \beta = 1 \)). The proof for the complex case (\( \beta = 2 \)) is similar, which is provided in the Appendix B.

Recall the scaled test function \( f \) from (2.10). We use the following Helffer-Sjöstrand formula to translate the linear eigenvalue statistics of \( f(H_N) \) to the Green function of \( H_N \). The following arguments were previously used in [54, 59]. For readers' convenience, we rewrite it here.

Lemma 4.1 (Helffer-Sjöstrand formula). Let \( f \in C^2_c(\mathbb{R}) \) and \( \chi(y) \) be a smooth cutoff function with support in \([-2, 2]\), with \( \chi(y) = 1 \) for \(|y| \leq 1\). Define an almost-analytic extension of \( f \), i.e.,
\[
\tilde{f}(x + iy) := (f(x) + iyf'(x))\chi(y). \tag{4.1}
\]

Then we have
\[
f(\lambda) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{d^2 z}{\lambda - z} \tilde{f}(z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{ig f''(x)\chi(y) + i\left(f(x) + iyf'(x)\right)\chi'(y)}{\lambda - x - iy} dx dy, \tag{4.2}
\]
where \( z = x + iy \), \( \frac{d^2}{dz^2} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \), and \( d^2z \) is the Lebesgue measure on \( \mathbb{C} \).

Taking derivative of the characteristic function \( \phi(\lambda) \), we write
\[
\phi'(\lambda) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial}{\partial z} \tilde{f}(z) \mathcal{E}[e(\lambda)(\text{Tr}(G(z)) - \mathbb{E}\text{Tr}G(z))] d^2z, \tag{4.3}
\]
where
\[
e(\lambda) := \exp \left\{ i\lambda \int_{\mathbb{C}} \frac{\partial}{\partial z} \tilde{f}(z') (\text{Tr}(G(z')) - \mathbb{E}\text{Tr}(G(z')) d^2z' \right\}. \tag{4.4}
\]

It was observed in [54] that the very local scales do not contribute to mesoscopic linear statistics, hence one can restrict the domain of the spectral parameter to
\[
\Omega_1 := \{ z = x + iy \in \mathbb{C} : |y| \geq N^{-\tau} \}, \tag{4.5}
\]
for some small \( \tau > 0 \). Indeed, using that \( y \rightarrow \text{Im} m_N(z)y \) is increasing, we can extend the local law Theorem 2.2 on scales \( 0 < \text{Im} z \leq 10 \) uniformly as following:
\[
|\text{Tr}G(x + iy) - \mathbb{E}\text{Tr}G(x_1 + iy_1)| = O_{\prec}\left( \frac{1}{|y_1|} \right). \tag{4.6}
\]
Due to the construction (2.10), there exists some $C > 0$ such that
\[
\int_{\mathbb{R}} |f(x)|dx \leq C\eta_0; \quad \int_{\mathbb{R}} |f'(x)|dx \leq C'; \quad \int_{\mathbb{R}} |f''(x)|dx \leq \frac{C''}{\eta_0}.
\] (4.7)
Thus combining with (4.2), we have
\[
\text{Tr} f(H_N) - \mathbb{E}\text{Tr} f(H_N) = \frac{1}{\pi} \int_{\Omega_1} \frac{\partial}{\partial z^2} \tilde{f}(z)(\text{Tr}(G(z)) - \mathbb{E}\text{Tr} G(z))d^2z + O_\prec(N^{-\tau}).
\] (4.8)
Furthermore, since $|e(\lambda)| = 1$, we have
\[
\phi'(\lambda) = \frac{i}{\pi} \int_{\Omega_1} \frac{\partial}{\partial z^2} \tilde{f}(z)e(\lambda)(\text{Tr}(G(z)) - \mathbb{E}\text{Tr} G(z))d^2z + O_\prec(|\lambda| N^{-\tau}).
\] (4.9)
We restrict the integral domain in the expression of $e(\lambda)$ (4.4) similarly by setting
\[
e_0(\lambda) := \exp \left\{ \frac{i\lambda}{\pi} \int_{\Omega_2} \frac{\partial}{\partial z^2} \tilde{f}(z')(\text{Tr}(G(z')) - \mathbb{E}\text{Tr} G(z'))d^2z' \right\},
\] (4.10)
with a different choice of the domain of spectral parameter
\[
\Omega_2 := \left\{ z' = x' + iy' \in \mathbb{C} : |y'| \geq N^{-\tau} \eta_0 \right\}.
\] (4.11)
One observes that $|e_0(\lambda)| = 1$ and the cut-off does not contribute,
\[
|e_0(\lambda) - e(\lambda)| = O_\prec(|\lambda| N^{-\tau}).
\]
Therefore, we write the derivative of the characteristic function as
\[
\phi'(\lambda) = \frac{i}{\pi} \int_{\Omega_1} \frac{\partial}{\partial z^2} \tilde{f}(z)e(\lambda)(\text{Tr}(G(z)) - \mathbb{E}\text{Tr} G(z))d^2z + O_\prec(|\lambda| \log NN^{-\tau}).
\] (4.12)
Thus, in order to study $\phi'(\lambda)$, it is sufficient to estimate $\mathbb{E}[e_0(\lambda)(\text{Tr}(G(z)) - \mathbb{E}\text{Tr} G(z))]$. The corresponding result is summarized in the following lemma, whose proof is provided in Section 5.

**Lemma 4.2.** *Under Assumption 2.1, for all $z = E + i\eta \in \Omega_1 \cap D'$, we have*
\[
\mathbb{E}[e_0(\lambda)(\text{Tr}(G(z)) - \mathbb{E}\text{Tr} G(z))] = \frac{\lambda}{2\pi} \mathbb{E}[e_0(\lambda)] \int_{\partial\Omega_2} \tilde{f}(z')K(z, z')dz' + E(z),
\]
*where the kernel $K(z, z')$ is given by*
\[
K(z, z') = 2\text{Tr}\left( \frac{m'_{sc}(z)m'_{sc}(z')S}{(1 - m_{sc}(z)m_{sc}(z'))^2} \right) - k_2m'_{sc}(z)m_{sc}(z') + 2k_4m_{sc}(z)m'_{sc}(z)m_{sc}(z)m'_{sc}(z'),
\]
*and $E(z)$ is the error which is analytic in $z \in \mathbb{C} \setminus \mathbb{R}$ with a uniform bound*
\[
|E(z)| \leq \frac{1 + |\lambda|^4 N^{2\tau}}{\sqrt{\kappa + \eta}} \left( N\Psi(\frac{\kappa_0 + \eta_0}{\sqrt{\kappa + \eta}}) + \frac{1}{\sqrt{\kappa + \eta}} \right) + N\Theta(\frac{\kappa_0 + \eta_0}{\sqrt{\kappa + \eta}}) + \frac{N^{4\tau}}{\sqrt{\kappa + \eta}} + \frac{\Psi(z)(\kappa_0 + \eta_0)^{\frac{1}{2}}}{\sqrt{\kappa + \eta}} + \frac{\Psi(z)(\kappa_0 + \eta_0)^{\frac{1}{2}}}{\sqrt{\kappa + \eta}} + \frac{\Psi(z)(\kappa_0 + \eta_0)^{\frac{1}{2}}}{\sqrt{\kappa + \eta}}
\]
*where $\kappa$ and $\kappa_0$ are given in (3.1) and (2.11) respectively.*

Combining Lemma 4.2 with (4.12), applying the Stokes’ formula, we have
\[
\phi'(\lambda) = \lambda \mathbb{E}[e_0(\lambda)] \int_{\partial\Omega_1} \int_{\partial\Omega_2} \tilde{f}(z')K(z, z')dz'dz' + \tilde{E},
\]
and $\tilde{E}$ is the error term bounded as
\[
\tilde{E} = O_\prec \left( \frac{N^{4\tau}}{\sqrt{\kappa_0 + \eta_0}} + \frac{N^{4\tau}}{\sqrt{\kappa + \eta}} \right) + O_\prec(|\lambda| \log NN^{-\tau}).
\]
The error term is estimated by (2.7), (3.4), (4.7) and $\kappa_0 \leq \kappa \leq C(\kappa_0 + \eta_0)$ since $g$ is compact support. Assuming $V(f) \prec O(1)$, we replace $e_0(\lambda)$ by $e(\lambda)$ with an error $O_\prec(|\lambda| N^{-\tau})$. Thus we complete the proof of Proposition 2.3.
5. Proof of Lemma 4.2

We will use the following cumulant expansion formula to prove Lemma 4.2.

Lemma 5.1 (Cumulant expansion formula). Let \( h \) be a real-valued random variable with finite moments, and \( f \) is a complex-valued smooth function on \( \mathbb{R} \) with bounded derivatives. Let \( c_k \) be the \( k \)-th cumulant of \( h \). Then for any fixed \( l \in \mathbb{N} \), we have

\[
\mathbb{E} h f(h) = \sum_{k=0}^{l} \frac{1}{k!} c_{k+1}(h) f^{(k)}(h) + R_{l+1},
\]

where the error term satisfies

\[
|R_{l+1}| \leq C_l \mathbb{E} \left[ |h|^{l+2} \right] \sup_{|x| \leq M} |f^{(l+1)}(x)| + C_l \mathbb{E} \left[ |h|^{l+2}_1 |h > M| \right] \|f^{(l+1)}\|_\infty,
\]

and \( M > 0 \) is an arbitrary fixed cutoff.

For reference, we refer e.g. to Lemma 3.1 in [44]. Now we are ready to prove Lemma 4.2.

5.1. Proof of Lemma 4.2. By the definition of the resolvent function \( zG = H G - I \), we have

\[
\mathbb{E} \left[ c_0(\lambda)(\text{Tr} G - \mathbb{E} \text{Tr} G) \right] = \mathbb{E} \left[ c_0(\lambda)(\text{Tr}(H G) - \mathbb{E} \text{Tr}(H G)) \right].
\]

Applying the cumulant expansion formula Lemma 5.1 and stop at \( l = 3 \), we have

\[
\mathbb{E} \left[ c_0(\lambda)(\text{Tr} G - \mathbb{E} \text{Tr} G) \right] = I_1 + I_2 + I_3 + R_4,
\]

where \( R_4 \) is the error term given by (5.1) involved with fifth moments, and

\[
I_1 := \frac{1}{N} \sum_{i,j=1}^{N} c_{ij}^{(2)} \left( \mathbb{E} \left[ \frac{\partial c_0(\lambda)}{\partial H_{ij}} G_{ij} \right] + \mathbb{E} \left[ \left( \frac{\partial G_{ij}}{\partial H_{ij}} \right) c_0(\lambda) \right] \right),
\]

\[
I_2 := \frac{1}{2!N^2} \sum_{i,j=1}^{N} c_{ij}^{(3)} \left( \mathbb{E} \left[ \frac{\partial^2 c_0(\lambda)}{\partial^2 H_{ij}} G_{ij} \right] + 2 \mathbb{E} \left[ \frac{\partial c_0(\lambda)}{\partial H_{ij}} \frac{\partial G_{ij}}{\partial H_{ij}} \right] + \mathbb{E} \left[ \left( \frac{\partial^2 G_{ij}}{\partial^2 H_{ij}} \right) c_0(\lambda) \right] \right),
\]

\[
I_3 := \frac{1}{3!N^3} \sum_{i,j=1}^{N} c_{ij}^{(4)} \left( \mathbb{E} \left[ \frac{\partial^3 c_0(\lambda)}{\partial^3 H_{ij}} G_{ij} \right] + 3 \mathbb{E} \left[ \frac{\partial^2 c_0(\lambda)}{\partial H_{ij}} \frac{\partial G_{ij}}{\partial H_{ij}} \right] + 3 \mathbb{E} \left[ \frac{\partial c_0(\lambda)}{\partial H_{ij}} \frac{\partial^2 G_{ij}}{\partial^2 H_{ij}} \right] + \mathbb{E} \left[ (1 - \mathbb{E}) \left( \frac{\partial^3 G_{ij}}{\partial^3 H_{ij}} \right) c_0(\lambda) \right] \right),
\]

with \( c_{ij}^{(m)} \) the \( m \)-th cumulant of \( (\sqrt{N}H)_{ij} \). In the following, we will estimate the RHS of (5.2). We start by estimating the the following derivatives used frequently in the paper.

Iterating the identity,

\[
\frac{\partial G_{ij}}{\partial H_{ab}} = G_{ia} G_{bj} + G_{ib} G_{aj}, \quad 1 \leq a, b \leq N,
\]

we obtain from the local law Theorem 2.2 and (3.3) that for general \( k \in \mathbb{N} \),

\[
\left| \frac{\partial^k G_{ij}}{\partial H_{ij}^k} \right| \prec O(1).
\]

Using (5.6) and (4.10), we obtain

\[
\frac{\partial c_0(\lambda)}{\partial H_{ij}} = \frac{i\lambda}{\pi} c_0(\lambda) \int_{\Omega_2} \frac{\partial}{\partial z} \tilde{f}(z') \left( \sum_{i=1}^{N} \frac{\partial G_{ij}(z')}{\partial H_{ij}} \right) d^2 z' = -\frac{i(2 - \delta_{ij})\lambda}{\pi} c_0(\lambda) \int_{\Omega_2} \frac{\partial}{\partial z} \tilde{f}(z') \frac{d}{d z'} G_{ij}(z') d^2 z',
\]

where we use the identity \((G^2)_{ij}(z') = \frac{d}{dz'} G_{ij}(z')\) to get the last equation. Since \( G_{ij}(z') \) is analytic in \( z' \in \mathbb{C} \setminus \mathbb{R} \), using the Cauchy integral formula and the local law, we have

\[
\left| \frac{d}{dz'} G_{ij}(z') - \delta_{ij} m_{sc}^{'}(z') \right| \leq \frac{\Psi(z')}{\text{Im} z'} \leq \frac{C}{\sqrt{N} \text{Im} z' \text{Im} z'},
\]

We hence obtain from (5.8) that

\[
\frac{\partial c_0(\lambda)}{\partial H_{ij}} = \frac{i(2 - \delta_{ij})\lambda}{\pi} c_0(\lambda) \int_{\Omega_2} \frac{\partial}{\partial z} \tilde{f}(z') m_{sc}^{'}(z') d^2 z' + O_* \left( \frac{(1 + |\lambda|)N^{2\tau}}{\sqrt{N} \eta_0} \right),
\]

where \( O_* \) is a universal constant.
where the error term is estimated by using the Stokes’ formula since $G_{ij}$ and $m_{sc}$ are analytic in $\mathbb{C} \setminus \mathbb{R}$, in combination with (5.9), (4.1) and (4.7). Taking derivative of (5.8) again, we obtain
\[
\frac{\partial^2 e_0(\lambda)}{\partial^2 H_{ij}} = -\frac{\lambda^2(2 - \delta_{ij})^2}{\pi^2} e_0(\lambda) \left( \int_{\Omega_{ij}} \frac{\partial}{\partial z'} \tilde{f}(z') \frac{d}{dz'} G_{ij}(z') d^2z' \right)^2 + \frac{i(2 - \delta_{ij})\lambda}{\pi} e_0(\lambda) \int_{\Omega_{ij}} \frac{\partial}{\partial z'} \tilde{f}(z') \left(2(G^2)_{ij} G_{ij} + (1 - \delta_{ij})(G^2)_{ii} G_{jj} + (1 - \delta_{ij})(G^2)_{jj} G_{ii}\right) d^2z'.
\]

By the local law and the argument as in (5.10), we have for $i \neq j$,
\[
\frac{\partial^2 e_0(\lambda)}{\partial^2 H_{ij}} = \frac{4\lambda}{\pi} e_0(\lambda) \int_{\Omega_{ij}} \frac{\partial}{\partial z'} \tilde{f}(z') m_{sc}(z)m_{sc}(z') d^2z' + O\left(\frac{(1 + |\lambda|)^2 N^{2\tau}}{\sqrt{N\eta_0}}\right). \tag{5.11}
\]

In general, repeatedly using (5.6) in combination with the local law and the arguments in (5.10), we have for general $k \in \mathbb{N},$
\[
|\frac{\partial^k e_0(\lambda)}{\partial^k H_{ij}}| \prec O((1 + |\lambda|)^k). \tag{5.12}
\]

With above bounds, the error term $R_4$ in (5.2) can be estimated using (2.2), (5.7), (5.12) and the properties of stochastic domination in Lemma 3.7. Note that for $z \in \Omega_1 \cap D'$, we have the deterministic bound $|G_{ij}| \leq |G|_{\text{op}} \leq (\text{Im} z)^{-1} = O(N^\epsilon)$. Combining with $|e_0(\lambda)| = 1$, we can use the fourth statement of Lemma 3.7, and obtain that
\[
R_4 = O_\prec(N^{-\frac{1}{2}}(1 + |\lambda|^4)).
\]

We will use the fourth statement of Lemma 3.7 throughout the proof without specifically mentioning it. The error terms above and below in this section are all uniform in $z \in \Omega_1 \cap D'$. Now, we are ready to estimate the rest three terms on the RHS of (5.2).

5.1.1. Estimates of $I_1$. By (5.3) and (5.6), we have
\[
I_1 = -\sum_{i=1}^{N} \mathbb{E}\left[e_0(\lambda)(1 - \mathbb{E})\left(\sum_{j=1}^{(i)} s_{ij} G_{ji}(z) G_{ji}(z)\right)\right] - \sum_{i=1}^{N} \mathbb{E}[e_0(\lambda)(1 - \mathbb{E})(\sum_{j=1}^{N} s_{ij} G_{ji}(z) G_{ji}(z))]
+ \sum_{i,j=1}^{N} s_{ij} \mathbb{E}\left[\frac{\partial e_0(\lambda)}{\partial H_{ij}} G_{ji}(z)\right] := A_1 + A_2 + A_3.
\]

We start by estimating the second term $A_2$. By the local laws Theorem 2.2 and 3.4, we write
\[
A_2 = -\sum_{i=1}^{N} \mathbb{E}\left[e_0(\lambda)\left(\sum_{j=1}^{N} s_{ij} G_{ji}(z) - \mathbb{E} \sum_{j=1}^{N} s_{ij} G_{ji}(z) G_{ii}(z)\right)\right] = -2m_{sc}(z)\mathbb{E}[e_0(\lambda)(\text{Tr} G - \mathbb{E}\text{Tr} G)] + O_\prec(N\rho\Psi^3(z)).
\]

Next, we look at the last term $A_3$. We split it into two terms as following:
\[
A_3 = \sum_{i,j=1}^{N} \mathbb{E}\left[\frac{\partial e_0(\lambda)}{\partial H_{ij}} s_{ij}(1 + \delta_{ij}) G_{ji}\right] - \sum_{i=1}^{N} s_{ii} \mathbb{E}\left[\frac{\partial e_0(\lambda)}{\partial H_{ii}} G_{ii}\right] := A_{31} + A_{32}. \tag{5.13}
\]

Using (5.8) and the local law, we can write the second term of (5.13) as
\[
A_{32} = \frac{i k_2 \lambda}{\pi} \mathbb{E}[e_0(\lambda)] \int_{\Omega_{ij}} \frac{\partial}{\partial z'} \tilde{f}(z') m_{sc}(z)m_{sc}(z') d^2z' + O_\prec\left(\frac{1}{\sqrt{N\eta_0}}\right) + O_\prec(\Psi(z)), \tag{5.14}
\]

with $k_2$ given in (2.3). Similarly, for the first term $A_{31}$, we have
\[
A_{31} = -\frac{i2 \lambda}{\pi} \sum_{i=1}^{N} \mathbb{E}\left[e_0(\lambda) \int_{\Omega_{ij}} \frac{\partial}{\partial z'} \tilde{f}(z') \frac{\partial}{\partial z'} \left(\sum_{j=1}^{N} s_{ij} G(z') G_{ji}(z)\right) d^2z'\right]. \tag{5.15}
\]

To simplify notations, we define the following functions:
\[
T_{ab}(z, z') := \sum_{j=1}^{N} s_{aj} G_{jb}(z) G_{jb}(z'), \quad \tilde{T}_{ab}(z) := \sum_{j=1}^{(b)} s_{aj} G_{jb}(z) G_{jb}(z), \quad 1 \leq a, b \leq N. \tag{5.16}
\]

We introduce the local laws for these two functions, whose proof will be postponed to Subsection 5.2.
We apply the isotropic local law Theorem 3.3 by letting 
\[ T(\rho_1) = T(1 - m_{sc}(z) m_{sc}(z'))^{-1}. \]
Moreover, we have the following estimate of the trace of 
\[ T(z, z'): \]
\[ \text{Tr} T(z, z') = \text{Tr} \left( \frac{m_{sc}(z) m_{sc}(z') S}{1 - m_{sc}(z) m_{sc}(z') S^2} \right) + e_T(z, z'), \]
where \( e_T(z, z') \) is analytic in both \( z, z' \in \mathbb{C} \setminus \mathbb{R} \) and satisfies 
\[ |e_T(z, z')| = O_\prec \langle N \Psi^2(z) \rangle + N \Theta^2(z) + N \Theta(z) \Theta(z'). \]
Finally, for all \( 1 \leq a, b \leq N \) and \( z \in D' \), we have 
\[ \tilde{T}_{ab}(z) = \left( \frac{m_{sc}^4(z) S^2}{1 - m_{sc}^2(z) S^2} \right)_{ab} + O_\prec (\rho \Psi^3(z)). \]

**Remark.** The local law for the two point function \( T(z, z') \) in Lemma 5.2 is not optimal, but is enough to proceed our proof of Lemma 4.2.

Note that since \( |m_{sc}| < 1 \) and the spectral radius of \( S \) is 1, applying the Taylor expansion we have 
\[ \frac{\partial}{\partial z} \text{Tr} T(z, z') = \text{Tr} \left( \frac{m_{sc}(z) m_{sc}'(z') S}{(1 - m_{sc}(z) m_{sc}(z') S^2)^2} \right) + e(z, z'), \]
where the error \( e(z, z') \) is analytic in both \( z, z' \in \mathbb{C} \setminus \mathbb{R} \) and by the Cauchy integral formula it has the following bound:
\[ |e(z, z')| = \left| \frac{\partial}{\partial z} e_T(z, z') \right| = O_\prec \left( \frac{N \Psi(z) \Psi(z')}{\text{Im} z} \right). \]
Plugging (5.21) into (5.15), we hence have 
\[ A_{31} = -\frac{\partial}{\partial z} \int_{\Omega_2} \frac{\partial}{\partial z} \tilde{f}(z') \text{Tr} \left( \frac{m_{sc}(z) m_{sc}'(z') S}{(1 - m_{sc}(z) m_{sc}(z') S^2)^2} + e(z, z') \right) d^2 z'. \]
We will end this subsection by plugging (5.20) into the first term \( A_1 \) and obtain 
\[ A_1 = -N \sum_{i=1}^{N} \mathbb{E} [e_0(\lambda)(1 - E) T_{ii}(z)] = O_\prec (N \rho \Psi^3(z)). \]

### 5.1.2. Estimates of \( I_2 \)
In this subsection, we will show that \( I_2 \) is negligible. There are three terms of \( I_2 \) in (5.4) and we denote them by \( B_1, B_2 \) and \( B_3 \) respectively. Since the third cumulants \( c^{(3)}_{ij} \) are uniformly bounded, using (5.7), (5.12), the diagonal terms only contribute \( O_\prec (N^{-1/2}) \). To simplify the statement, we ignore the difference emerging by diagonal entries in (5.6). We first look at \( B_3 \). Using (5.6), the local law Theorem 2.2 and (2.8), we have 
\[ B_3 = \frac{3}{N^2} \sum_{i,j=1}^{N} \mathbb{E} \left[ c_0(\lambda) c^{(3)}_{ij} (G_{ij} - E G_{ij}) \right]. \]
We apply the isotropic local law Theorem 3.3 by letting \( v_j = \delta_{ij} \) and \( w_j = \frac{1}{\sqrt{N}} c^{(3)}_{ij} \). Note that \( w \) has bounded \( l_2 \) norm by the moment condition (2.2), then we have 
\[ \left| \langle v, G(z) w \rangle - m_{sc}(z) \langle v, w \rangle \right| = \left| \frac{1}{\sqrt{N}} \sum_{j=1}^{N} c^{(3)}_{ij} G_{ij}(z) - \frac{1}{\sqrt{N}} m_{sc}(z) c_{ii} \right| \prec \Psi(z). \]
Hence, we obtain that \( |B_3| = O_\prec \langle \Psi(z) + \sqrt{N} \Psi^{2}(z) \rangle = O_\prec \sqrt{N} \Psi^{2}(z) \) by (2.8). For the second term \( B_2 \), by (5.6), (5.8) and the local law, we have 
\[ B_2 = \frac{14\lambda}{\pi N^2} \sum_{i,j=1}^{N} \mathbb{E} \left[ c_0(\lambda) \left( \sum_{j=1}^{N} \int_{\Omega_2} \frac{1}{\sqrt{N}} \tilde{f}(z') c^{(3)}_{ij}(G'(z'))_{ji} d^2 z' \right) + O_\prec (\frac{1 + |\lambda| N^{2} \Psi^{2}(z)}{\sqrt{h_0}}). \]
From (5.24), (2.8) and (3.3), applying Cauchy integral formula, we obtain that
\[
\left| \frac{d}{dz'} \left( \frac{1}{N} \sum_{j=1}^{N} e^{(G(z'))y_j} \right) \right| < \frac{\Psi(z')}{\text{Im } z'} \leq \frac{C}{\sqrt{\text{Im } z' \text{Im } z'}}.
\]  
(5.26)

Using Stokes’ formula along with (5.26) and (4.7), we obtain from (5.25) that
\[
B_2 = O_\varsigma \left( \frac{(1 + |\lambda|)N^{2r}}{\sqrt{N\eta_0}} \right) = O_\varsigma \left( \frac{(1 + |\lambda|)N^{2r}\Psi(z)}{\sqrt{N\eta_0}} \right).
\]
Plugging (5.11) into the above equation, we have
\[
|B_1| = O_\varsigma \left( \frac{(1 + |\lambda|)N^{2r}\Psi(z)}{\sqrt{N\eta_0}} \right)
\]
similarly using isotropic local law.

To sum up, we have
\[
|I_2| = O_\varsigma \left( \frac{(1 + |\lambda|)N^{2r}\Psi(z)}{\sqrt{N\eta_0}} \right) + O_\varsigma \left( \sqrt{N\Psi}(z) \right).
\]

5.1.3. **Estimates of I_3.** It is straightforward that the diagonal terms \(i = j\) are negligible, i.e., are bounded by \(O_\varsigma(N^{-1})\), so we ignore the difference brought by diagonal entries in (5.6). We denote the four terms of \(I_3\) in (5.5) by \(D_1, D_2, D_3\) and \(D_4\) respectively. By the local law Theorem 2.2 and (5.12), we have
\[
|D_1| = O_\varsigma((1 + |\lambda|^2)\Psi(z)).
\]
Similarly, using (5.6), (5.8) and the local law, we have
\[
|D_2| = O_\varsigma \left( \frac{(1 + |\lambda|)N^{2r}\Psi(z)}{\sqrt{N\eta_0}} \right).
\]
Using Stokes’ formula along with (5.26) and (4.7), we obtain from (5.25) that
\[
|D_3| = O_\varsigma \left( \frac{(1 + |\lambda|)N^{2r}\Psi(z)}{\sqrt{N\eta_0}} \right).
\]
And using (5.6) and the local law, we obtain
\[
|D_4| = O_\varsigma(\Psi(z)).
\]
Finally, using (5.6), (5.7) and the local law, we write
\[
D_2 = -\frac{1}{2N^2} \sum_{i,j=1}^{N} m_{sc}(z)E \left[ \frac{\partial^2 e_0(\lambda)}{\partial^2 H_{ij}} c_{i,j}^{(4)} \right] + O_\varsigma((1 + |\lambda|^2)\Psi(z)).
\]
(5.27)

Plugging (5.11) into the above equation, we have
\[
D_2 = -k_4 \frac{\lambda}{\pi} E \left[ e_0(\lambda) \int_{\Omega_2} \frac{\partial}{\partial z'} f(z') \frac{\partial}{\partial z'} (m_{sc}(z)m_{sc}(z'))dz' \right] + O_\varsigma \left( \frac{(1 + |\lambda|^2)N^{2r}}{\sqrt{N\eta_0}} \right) + O_\varsigma \left( (1 + |\lambda|^2)\Psi(z) \right),
\]
(5.28)

where \(k_4\) is given in (2.3).

Adding up all the contributions to (5.2) and rearranging, we obtain
\[
(z + 2m_{sc}(z))E[e_0(\lambda)\text{Tr}G - \text{ETr}G] = -\frac{\lambda}{2\pi} E[e_0(\lambda)] \int_{\Omega_2} \frac{\partial}{\partial z'} f(z')(K(z, z') + e(z, z'))dz' \nonumber
\]
\[
+ O_\varsigma \left( (1 + |\lambda|^4)N\rho\Psi^3(z) + \frac{(1 + |\lambda|^4)N^{2r}\Psi(z)}{\sqrt{N\eta_0}} \right)
\]
where the error \(e(z, z')\) is estimated in (5.22), and the kernel \(K(z, z')\) is given by
\[
K(z, z') = 2\text{Tr} \left( \frac{m_{sc}(z)m_{sc}(z')S}{(1 - m_{sc}(z)m_{sc}(z'))^2} \right) - k_2 m_{sc}(z)m_{sc}(z') + 2k_4 m_{sc}^2(z)m_{sc}(z')m_{sc}(z').
\]

Applying the Stoke’s formula since \(K(z, z')\) and \(e(z, z')\) are analytic in both \(z, z' \in \mathbb{C} \setminus \mathbb{R}\), we have
\[
(z + 2m_{sc}(z))E[e_0(\lambda)(\text{Tr}G - \text{ETr}G)] = -\frac{\lambda}{2\pi} E[e_0(\lambda)] \int_{\Omega_2} f(z')K(z, z')dz' + \mathcal{E}(z),
\]
(5.29)

and the error term \(\mathcal{E}(z)\) is estimate by (4.7) and (5.22), along with (2.7), (3.4), (3.7) and \(\kappa_0 \leq \kappa \leq C(\kappa_0 + \eta_0)\) since \(g\) is compactly support. To be specific, we have
\[
|\mathcal{E}(z)| \lesssim (1 + |\lambda|^4)N^{2r} \left( \left( \frac{\kappa_0 + \eta_0}{\sqrt{N\eta_0}} + \frac{1}{N\eta_0} \right) + N\Psi(z) \left( \frac{\kappa_0 + \eta_0}{N\eta_0} \right) + \frac{1}{(N\eta_0)^2} \right)
\]
\[
+ N\Theta^2(z) + \frac{1}{\eta_0} \Theta(z) + \frac{N\Psi^3(z)}{\sqrt{\kappa + \eta}} + \frac{\Psi(z)}{\sqrt{N\eta_0}}.
\]

Divide both sides of (5.29) by \(z + 2m_{sc}(z)\), using the relation \(z + 2m_{sc}(z) = -\frac{m_{sc}(z)}{m_{sc}^2(z)}\) and (3.6), we hence finish the proof of Proposition 2.3.
5.2. Proof of Lemma 5.2. We start by looking at $\hat{T}(z)$ using the averaging fluctuation results in Theorem 3.4, from which we have

$$
\left| \hat{T}_{ab}(z) - \sum_{j=1}^{b} s_{aj} \mathbb{E}_j [G_{jb}(z)G_{jb}(z)] \right| < \Psi^3(z), \quad (5.30)
$$

where $\mathbb{E}_j$ means taking the partial expectation with respect to the $j$-th (row) column. Using (3.9) in Lemma 3.5, we obtain from (5.30) that

$$
\hat{T}_{ab}(z) = \sum_{j=1}^{b} s_{aj} \mathbb{E}_j \left[ (G_{jj}(z))^2 \sum_{k,l} h_{jk}G_{kk}^{(j)}(z)G_{lb}^{(j)}(z) h_{jl} \right] + O_\prec(\Psi^3). \quad (5.31)
$$

Let $M_{kl}^{(j)} := \sqrt{\sigma_{jk}G_{kk}^{(j)}(z)G_{lb}^{(j)}(z)} \sqrt{\sigma_{lj}}$. Conditioning on $M^{(j)}$ and using the fact that $\{h_{jk}\}_{k \neq j}$ are independent of $M^{(j)}$, we apply the Gaussian concentration result Lemma 3.6 to obtain that

$$
\left| \sum_{k,l} h_{jk}G_{kk}^{(j)}(z)G_{lb}^{(j)}(z) h_{jl} - \sum_{k} s_{jk}G_{kk}^{(j)}(z)G_{lb}^{(j)}(z) \right| < \|M^{(j)}\|_{HS}. \quad (5.31)
$$

Using (3.8) and the local law Theorem 2.2, for $k \neq b$, we obtain a local law for $G^{(i)}$, i.e.,

$$
G_{kk}^{(j)}(z) = G_{kk}(z) + O_\prec(\Psi^2(z)) = O_\prec(\Psi(z)), \quad G_{lb}^{(j)}(z) = G_{lb}(z) + O_\prec(\Psi^2(z)) = m_{sc}(z) + O_\prec(\Psi(z)). \quad (5.32)
$$

Hence

$$
\left| \sum_{k,l} h_{jk}G_{kk}^{(j)}(z)G_{lb}^{(j)}(z) h_{jl} \right| < \Psi^2(z). \quad \text{With this bound, combining with the local law for } G, \text{ we obtain from (5.31) that}
$$

$$
\hat{T}_{ab}(z) = m_{sc}^2(z) \sum_{j=1}^{b} s_{aj} \mathbb{E}_j \left[ \sum_{k,l} h_{jk}G_{kk}^{(j)}(z)G_{lb}^{(j)}(z) h_{jl} \right] + O_\prec(\Psi^3(z)).
$$

Using the fact that $G^{(j)}$ is independent of $\{h_{jk}\}_{k \neq j}$ and (5.32), we obtain that

$$
\hat{T}_{ab}(z) = m_{sc}^2(z) \sum_{j=1}^{b} s_{aj} \left[ \sum_{k} s_{jk}G_{kk}^{(j)}(z)G_{lb}^{(j)}(z) \right] + O_\prec(\Psi^3(z)) = m_{sc}^2(z) \sum_{j=1}^{b} s_{aj} \left[ \sum_{k} s_{jk}G_{kk}(z)G_{lb}(z) \right] + O_\prec(\Psi^3(z))
$$

$$
= m_{sc}^2(z) \sum_{j=1}^{b} s_{aj} \hat{T}_{jb}(z) + m_{sc}^2(z)(S^2)_{ab} + O_\prec(\Psi^3(z)),
$$

where the last step follows from (2.8). Therefore, the matrix $\hat{T}$ satisfies the following equation:

$$
\hat{T}(z) = m_{sc}^2(z)S \hat{T} + m_{sc}^2(z)(S^2) + O_\prec(\Psi^3(z)),
$$

where $S$ is a matrix with sup norm $O_\prec(\Psi^3(z))$. We solve for $\hat{T}$ and Lemma 3.2 implies that

$$
\hat{T}(z) = \frac{m_{sc}^4(z)S^2}{1 - m_{sc}^4(z)S} + R(z), \quad \text{with } \|R(z)\|_\infty = O_\prec(\rho \Psi^3(z)). \quad (5.33)
$$

We hence finish the proof of (5.20). Next, we proceed to estimate the two point function $T(z,z')$. For notational simplicity, we write $T \equiv T(z,z')$ and $m \equiv m_{sc}$. Define

$$
P_{ab} := -\frac{1}{m(z)} T_{ab} + s_{ab}m(z') + m(z') \sum_{j=1}^{N} s_{aj} T_{jb}.
$$

We next aim to prove that $|P_{ab}| < \Psi^2(z)\Psi(z') + \Psi(z)\Psi^2(z')$. Due to (2.5), we have

$$
P_{ab} = z \sum_{j=1}^{N} s_{aj} G_{jb}(z)G_{jb}(z') + m(z)T_{ab} + s_{ab}m(z') + m(z') \sum_{j=1}^{N} s_{aj} T_{jb}.
$$

Using the relation $zG = HG - I$ and the local law, we have

$$
P_{ab} = -s_{ab}G_{bb}(z') + \sum_{j=1}^{N} s_{aj}(HG)_{jb}(z)G_{jb}(z') + m(z)T_{ab} + s_{ab}m(z') + m(z') \sum_{j=1}^{N} s_{aj} T_{jb}
$$
\[= \sum_{j=1}^{N} s_{aj} (HG)_{jk}(z)G_{jb}(z') + m(z)T_{ab} + m(z') \sum_{j=1}^{N} s_{aj} T_{jb} + O(\theta(N^{-1}\Psi(z')).\]

Set \(M(p, q) := (P_{ab})^p (P^{*}_{ab})^q\). Then we write

\[
\mathbb{E}[P_{ab}]^{2d} = \mathbb{E}\left[ \left( \sum_{j, k=1}^{N} s_{aj} H_{jk} G_{kb}(z) G_{jb}(z') \right) M(d - 1, d) \right] + \mathbb{E}\left[ (m(z) T_{ab} + m(z') \sum_{j=1}^{N} s_{aj} T_{jb}) M(d - 1, d) \right]
\]

\[+ \mathbb{E}[O(\theta(N^{-1}\Psi(z'))) M(d - 1, d)]. \tag{5.34}\]

For the first term of (5.34), using the cumulant expansion formula, we have

\[
\mathbb{E}\left[ \left( \sum_{j, k=1}^{N} s_{aj} H_{jk} G_{kb}(z) G_{jb}(z') \right) M(d - 1, d) \right] = \mathbb{E}\left[ \left( \sum_{j, k=1}^{N} s_{aj} s_{jk} \frac{\partial G_{kb}(z) G_{jb}(z')}{\partial H_{jk}} \right) M(d - 1, d) \right]
\]

\[+ \mathbb{E}\left[ \left( \sum_{j, k=1}^{N} s_{aj} s_{jk} G_{kb}(z) G_{jb}(z') \right) (d - 1) \frac{\partial P_{ab}}{\partial H_{jk}} M(d - 2, d) \right] + \cdots := P_1 + P_2 + P_3 + P_4,\]

where \(P_4\) is the remaining term involved of higher moments. First, we will show that \(P_4\) is negligible. It is sufficient to estimate

\[
\frac{1}{N^2} \sum_{j, k} s_{aj} c^{(3)}_{jk} \frac{\partial^2}{\partial H_{jk}} \left( G_{kb}(z) G_{jb}(z')(P_{ab})^{d-1}(P^{*}_{ab})^{d}\right).
\]

Set \(G^{(1)} := G(z), \ G^{(2)} := G(z'), \) and \(\Xi_1 := \Psi^2(z)\Psi(z') + \Psi(z)\Psi^2(z')\) for short. Using (5.6) and the local law, for \(j \neq k \neq b\), we have

\[
\frac{1}{N^2} \sum_{j, k} s_{aj} c^{(3)}_{jk} \frac{\partial}{\partial H_{jk}} \left( G^{(1)} G^{(2)} \right) = O(\theta(\Xi_1)); \quad \frac{1}{N^2} \sum_{j, k} s_{aj} c^{(3)}_{jk} \frac{\partial^2}{\partial H_{jk}^2} \left( G^{(1)} G^{(2)} \right) = O(\theta(\Xi_1)).
\]

Recalling \(T_{ab} = \sum_{l=1}^{N} s_{al} G_{lb}^{(1)} G_{ib}^{(2)}\), we obtain from (5.6) that

\[
\frac{\partial T_{ab}}{\partial H_{jk}} = - \sum_{l=1}^{N} s_{al} \left( G_{lj}^{(1)} G_{kb}^{(2)} + G_{lk}^{(1)} G_{jb}^{(2)} + G_{lb}^{(1)} G_{lj}^{(2)} G_{jb}^{(2)} + G_{lb}^{(1)} G_{lk}^{(2)} G_{jb}^{(2)} + G_{lb}^{(1)} G_{lk}^{(2)} G_{jb}^{(2)} \right), \tag{5.35}
\]

and thus

\[
\frac{\partial P_{ab}}{\partial H_{jk}} = - \frac{1}{m(z)} \frac{\partial T_{ab}}{\partial H_{jk}} + m(z') \sum_{l=1}^{N} s_{al} \frac{\partial T_{lb}}{\partial H_{jk}} = \sum_{l=1}^{N} \left( \frac{1}{m(z)} s_{al} - m(z')(S^2)_{al} \right) \times Q_1, \tag{5.36}
\]

where

\[Q_1 := G_{lj}^{(1)} G_{kb}^{(2)} + G_{lk}^{(1)} G_{jb}^{(2)} + G_{lb}^{(1)} G_{lj}^{(2)} G_{jb}^{(2)} + G_{lb}^{(1)} G_{lk}^{(2)} G_{jb}^{(2)} + G_{lb}^{(1)} G_{lk}^{(2)} G_{jb}^{(2)} \]

For \(j \neq k \neq b\), the local law implies that \(\frac{\partial P_{ab}}{\partial H_{jk}} = O(\theta(\Xi_1))\). Similarly, for general \(m \geq 1\), \(\frac{\partial^m P_{ab}}{\partial H_{jk}^m} = O(\theta(\Xi_1))\).

Combining with (2.8), we have

\[P_4 = \mathbb{E}[O(\theta(\Xi_1)) M(d - 1, d) + O(\theta(\Xi_1)) M(d - 2, d) + O(\theta(\Xi_1)) M(d - 1, d - 1)]
\]

\[+ \mathbb{E}[O(\theta(\Xi_1)) M(d - 3, d) + O(\theta(\Xi_1)) M(d - 2, d - 1) + O(\theta(\Xi_1)) M(d - 1, d - 2)].\]

Next, we look at the first term \(P_1\). Using the local law, we write

\[
P_1 = -\mathbb{E}\left[ \sum_{j, k=1}^{N} s_{aj} s_{jk} \left( G_{kj}^{(1)} G_{KB}^{(2)} + G_{kj}^{(1)} G_{JK}^{(2)} G_{KB}^{(2)} + G_{kj}^{(1)} G_{KB}^{(2)} G_{kj}^{(2)} G_{KB}^{(2)} \right) M(d - 1, d) \right]
\]

\[= -\mathbb{E}\left[ \sum_{j, k=1}^{N} s_{aj} s_{jk} \left( m(z) G_{kj}^{(1)} G_{KB}^{(2)} + m(z') G_{kj}^{(1)} G_{KB}^{(2)} \right) M(d - 1, d) \right] + \mathbb{E}[O(\theta(\Xi_1)) M(d - 1, d)].\]
Note that the leading term of $P_1$ will cancel the second term on the right side of (5.34). Plugging (5.36) in $P_2$ and using the local law, we have

$$P_2 = (d - 1)\mathbb{E}\left[\sum_{j,k,l=1}^{N} s_{kj}^3 s_{lk} \left(\frac{1}{m(z)} s_{at} - m(z')(S')_{at}\right) G^{(1)}_{ab} G^{(2)}_{jk} Q_1 M(d - 2, d)\right] = O_\prec(\Xi^3_2),$$

where $\Xi_2 := \Psi \hat{\Psi}(z)\Psi(z') + \Psi(z)\Psi \hat{\Psi}(z') \gg \Xi_1$. We treat $P_3$ similarly. Therefore, we obtain

$$\mathbb{E}|P_{ab}|^2 = \mathbb{E}|O_\prec(\Xi^3_2) M(d - 1, d) + O_\prec(\Xi^3_2) M(d - 1, d - 1) + O_\prec(\Xi^3_2) M(d - 2, d)| + O_\prec(\Xi^3_2) M(d - 2, d - 2).$$

Applying the Young's inequality, we get $\mathbb{E}|P_{ab}|^2 < \Xi^3_2$ for $d \geq 1$ and thus $|P_{ab}| < \Xi_2$. Since $|m(z)| \sim 1$, the matrix $(T)_{ab}$ satisfies the following equation:

$$(I - m_{sc}(z)m_{sc}(z'))S = m_{sc}(z)m_{sc}(z')S + R(z, z'),$$

(5.37)

with the estimate

$$\max_{ij} |R_{ij}(z, z')| = O_\prec(\Psi \hat{\Psi}(z)\Psi(z')) + O_\prec(\Psi(z)\Psi \hat{\Psi}(z')).$$

Combining with Lemma 3.2, we hence prove (5.17). Next, we continue to estimate the trace of the two point function $\text{Tr}T(z, z')$. Recall the projection matrix $I = \text{ee}^*,$ where $e = N^{-\frac{1}{2}}(1, \ldots, 1)^*$. Note that $\Pi S = S \Pi = \Pi.$ Multiplying both sides of (5.37) by $(I - \Pi)(I - m_{sc}(z)m_{sc}(z'))^{-1},$ we have

$$(I - \Pi)T = m_{sc}(z)m_{sc}(z') \frac{S - \Pi}{I - m_{sc}(z)m_{sc}(z')}S + \frac{I - \Pi}{I - m_{sc}(z)m_{sc}(z')}S R.$$

The second term on the right side can be estimated by Lemma 3.2, i.e.,

$$\left\|\frac{I - \Pi}{I - m_{sc}(z)m_{sc}(z')}S R\right\|_\infty \leq \left\|\frac{I - \Pi}{I - m_{sc}(z)m_{sc}(z')}S\right\|_\infty \left\|R\right\|_\infty = O_\prec(\Psi \hat{\Psi}(z)\Psi(z')) + O_\prec(\Psi(z)\Psi \hat{\Psi}(z')).$$

Therefore, we obtain

$$\text{Tr}T = \text{Tr}(\Pi T) + \text{Tr}\left(\frac{m_{sc}(z)m_{sc}(z')(S - \Pi)}{I - m_{sc}(z)m_{sc}(z')}S\right) + O_\prec\left(N\Psi \hat{\Psi}(z)\Psi(z')\right) + O_\prec\left(N\Psi(z)\Psi \hat{\Psi}(z')\right).$$

(5.38)

Since $|m_{sc}| < 1$, we use the Taylor expansion to write the second term of (5.38) as

$$\text{Tr}\left(\frac{m_{sc}(z)m_{sc}(z')(S - \Pi)}{I - m_{sc}(z)m_{sc}(z')}S\right) = \frac{m_{sc}(z)m_{sc}(z')}{I - m_{sc}(z)m_{sc}(z')} = \frac{m_{sc}(z) - m_{sc}(z')}{z - z'}.$$

(5.39)

For the first term, it can be written as

$$\text{Tr}(\Pi T) = \frac{1}{N} \sum_{j,k,b=1}^{N} s_{kj} G_{jb}(z)G_{jk}(z) = \frac{1}{z - z'} \sum_{b=1}^{N} \left(G(z)G(z')\right)_{bb} = \frac{1}{z - z'} \sum_{b=1}^{N} \left(G(z) - G(z')\right)_{bb},$$

where we use the resolvent identity

$$G(z)G(z') = \frac{1}{z - z'}(G(z) - G(z')).$$

(5.40)

We separate into two cases.

Case 1: If $z, z'$ are in different half planes, then $\frac{1}{|z - z'|} \leq \frac{1}{|\text{Im} z|}$. Using the local law, we have

$$\text{Tr}(\Pi T) = \frac{1}{z - z'} \sum_{b=1}^{N} (G(z) - G(z'))_{bb} = \frac{m_{sc}(z) - m_{sc}(z')}{z - z'} + O_\prec\left(\frac{\Theta(z)}{|\text{Im} z|}\right) + O_\prec\left(\frac{\Theta(z')}{|\text{Im} z|}\right).$$

(5.41)

Case 2: If $z$ and $z'$ are in the same half-plane, without loss of generality, we can assume they both belong to the upper half plane. If $|\text{Im} z - \text{Im} z'| \geq \frac{1}{2}|\text{Im} z|$, then we can use the same argument as in Case 1. Thus it is sufficient to study when $|\text{Im} z - \text{Im} z'| \leq \frac{1}{2}|\text{Im} z|$, which means $\frac{1}{2}|\text{Im} z' \leq |\text{Im} z | \leq 2|\text{Im} z'|$. Note that $h(z) := \frac{1}{N} \sum_{b=1}^{N} (G_{bb} - m_{sc})$ is analytic in the neighborhood of the segment connecting $z$ and $z'$, denoted as $L(z, z')$. Applying the Cauchy integral formula, we obtain that

$$\left|\frac{h(z) - h(z')}{z - z'}\right| \leq \sup_{\omega \in L(z, z')} |h'(\omega)| \leq \frac{\Theta(z)}{|\text{Im} z|}.$$
We obtain the same upper bound as in the Case 1. Combining (5.38), (5.39) and (5.41), we have
\[ \text{Tr}T = \text{Tr}\left( \frac{m_{sc}(z)m_{sc}(z')}{I - m_{sc}(z)m_{sc}(z')} S \right) + O_{\prec}\left( \frac{1}{\text{Im} z} \right) + O_{\prec}\left( \frac{1}{\text{Im} z} \right) + O_{\prec}\left( \frac{1}{\text{Im} z} \right) + O_{\prec}\left( \frac{1}{\text{Im} z} \right) + O_{\prec}\left( \frac{1}{\text{Im} z} \right), \]
and thus complete the proof of Lemma 5.2.

6. Proof of Proposition 2.5 and Theorem 2.6

In this section, we first study the expectation of the linear eigenvalue statistics and prove Proposition 2.5, using the same technique as in Proposition 2.3. After this, we prove the mesoscopic CLTs inside the bulk and at the edges respectively.

6.1. Proof of Proposition 2.5. Using Heffler-Strössand formula, as an analogue of (4.8), we obtain that
\[ \text{Tr}f(H_N) = N \int_{\mathbb{R}} f(x) \rho_{sc}(x) dx = \frac{1}{\pi} \int_{\Theta} \frac{\partial f(z)}{\partial z} (\text{Tr}(G(z)) - Nm_{sc}(z)) dz + O_{\prec}(N^{-\tau}). \] (6.1)
We then reduce the bias of the linear statistics to \( \text{Tr}G(z) - Nm_{sc}(z) \). Using the cumulant expansion formula Lemma 5.1, identity (5.6) and the local law Theorem 2.2, we have
\[ z\mathbb{E}\text{Tr}G = \sum_{i=1}^{N} \mathbb{E}\left[ (HG)_{ii} - 1 \right] = \mathbb{E}\left[ \sum_{i,j=1}^{N} s_{ij} \frac{\partial G_{ji}}{\partial H_{ij}} - N + \frac{1}{2N} \sum_{i,j=1}^{N} c_{ij}(3) \frac{\partial^2 G_{ji}}{\partial^2 H_{ij}} \right] + O_{\prec}(N^{-\tau}) \]
\[ = \mathbb{E}\left[ - \sum_{i,j=1}^{N} s_{ij} G_{ii} G_{jj} - \sum_{i=1}^{N} (i) s_{ij} G_{ij} G_{ji} - N + \frac{1}{2N} \sum_{i,j=1}^{N} c_{ij}(3) (6G_{ii} G_{jj} G_{ij} + 2G_{ij}^3) \right. \]
\[ + \left. \frac{1}{6N^2} \sum_{i,j=1}^{N} c_{ij}(4) (-36G_{ii} G_{jj} G_{ij}^3 - 6G_{ii}^2 G_{ij}^2 - 6G_{ij}^4) \right] + O_{\prec}(N^{-\tau}). \]
Applying the local law further, we have
\[ z\mathbb{E}\text{Tr}G = - \sum_{i=1}^{N} \mathbb{E}\left[ (G_{ii}(z) - m_{sc}(z))(\sum_{j=1}^{N} s_{ij} G_{jj}(z)) \right] - m_{sc}(z) \sum_{j=1}^{N} \mathbb{E}\left[ \sum_{i=1}^{N} s_{ij} G_{jj}(z) \right] - \sum_{i=1}^{N} \mathbb{E}\left[ \sum_{j=1}^{N} s_{ij} G_{ji}(z) G_{ji}(z) \right] \]
\[ - N + \frac{3}{N^2} m_{sc}(z) \sum_{i,j=1}^{N} \mathbb{E} c_{ij}(3) G_{ij} - \frac{1}{N^2} \sum_{i,j=1}^{N} c_{ij}(4) m_{sc}(z) + O_{\prec}(\sqrt{N} \Psi^2). \]
Using the isotropic local law Theorem 3.3 and the argument as in (5.24), the expansion terms involved of the third cumulants is bounded as \( O_{\prec}(\Psi) \). Combining with the averaging fluctuation results Theorem 3.4 and (5.20), we have
\[ (z + 2m_{sc}(z))\mathbb{E}(\text{Tr}G(z) - Nm_{sc}(z)) = -\text{Tr}\left( \frac{m_{sc}(z)S^2}{1 - m_{sc}(z)S} \right) - k_4 m_{sc}(z) + O_{\prec}(N\rho \Psi^3), \]
where \( \rho \) is given by (3.7). Using the relation \( z + 2m_{sc}(z) = \frac{m_{sc}(z)}{m_{sc}(z)} \sim \sqrt{\kappa + \eta} \), we obtain
\[ \mathbb{E}(\text{Tr}G(z) - Nm_{sc}(z)) = \text{Tr}\left( \frac{m_{sc}(z)S^2}{1 - m_{sc}(z)S} \right) + k_4 m_{sc}(z) m_{sc}(z) + O_{\prec}\left( \frac{N \rho \Psi^3}{\sqrt{\kappa + \eta}} \right). \] (6.2)
Therefore, plugging (6.2) in (6.1) and using the Stokes’ formula, we obtain that
\[ \mathbb{E}\text{Tr}f(H_N) = N \int_{\mathbb{R}} f(x) \rho_{sc}(x) dx = \frac{1}{\pi} \int_{\partial \Theta} \frac{\partial f(z)}{\partial z} b(z) dz + O_{\prec}\left( \frac{N}{\sqrt{N\eta_0 \sqrt{\kappa + \eta_0} + \eta_0}} \right) + O_{\prec}(N^{-\tau}), \]
where
\[ b(z) = \text{Tr}\left( \frac{m_{sc}(z)m_{sc}(z)S^2}{1 - m_{sc}(z)S} \right) + k_4 m_{sc}(z). \]
Thus we complete the proof of Proposition 2.5.
6.2. Proof of Theorem 2.6. Finally, we conclude with the proof of Theorem 2.6.

Proof. (Proof of Theorem 2.6) Armed with Proposition 2.3 and assuming that \( V(f) \) converges to some positive constant \( V \), integrating \( \phi'(\lambda) \) and by Arzelà-Ascoli theorem, we obtain the following equation involving the limiting characteristic function, denoted by \( \phi^{\infty} \),

\[
\phi^{\infty}(\lambda) = 1 - V \int_0^\lambda t \phi^{\infty}(t)dt.
\]

This integral equation is uniquely soluble in the class of bounded continuous functions and its solution is indeed the characteristic function of the limiting Gaussian distribution. Theorem 2.6 hence follows from the Lévy’s continuity theorem. Therefore, it is sufficient to estimate \( \int \phi^{\infty}(t)dt \).

\[
\text{(Proof of Theorem 2.6) Armed with Proposition 2.3 and assuming that } V(f) \text{ converges to some positive constant } V, \text{ integrating } \phi'(\lambda) \text{ and by Arzelà-Ascoli theorem, we obtain the following equation involving the limiting characteristic function, denoted by } \phi^{\infty}, \text{ where }
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\[
\phi^{\infty}(\lambda) = 1 - V \int_0^\lambda t \phi^{\infty}(t)dt.
\]

This integral equation is uniquely soluble in the class of bounded continuous functions and its solution is indeed the characteristic function of the limiting Gaussian distribution. Theorem 2.6 hence follows from the Lévy’s continuity theorem. Therefore, it is sufficient to compute the limit of the variance \( V(f) \) given by (2.13) and the bias \( B(f) \) in (2.15), considering the mesoscopic test function (2.10).

Lemma 6.1. Under the assumptions and notations of Theorem 2.6, if \( E_0 \) is in the bulk, we have

\[
\lim_{N \to \infty} V(f) = \frac{1}{2\beta \pi^2} \int_R \int_R \frac{(g(x_1) - g(x_2))^2}{(x_1 - x_2)^2} dx_1 dx_2 = \frac{1}{\beta \pi} \int_R |\xi| |\hat{g}(\xi)|^2 d\xi,
\]

where \( \hat{g}(\xi) := (2\pi)^{-1/2} \int_R g(x) e^{-i\xi x} dx \). The bulk bias \( B(f) \) in (2.15) vanishes as \( N \) goes to infinity. Furthermore, if \( E_0 = \pm 2 \), we have

\[
\lim_{N \to \infty} V(f) = \frac{1}{4\beta \pi^2} \int_R \int_R \frac{(h(x_1) - h(x_2))^2}{(x_1 - x_2)^2} dx_1 dx_2 = \frac{1}{2\beta \pi} \int_R |\xi| |\hat{h}(\xi)|^2 d\xi,
\]

where \( h(x) = g(\pi x^2) \) and \( \hat{h}(\xi) := (2\pi)^{-1/2} \int_R h(x) e^{-i\xi x} dx \). And the bias at the edges are not vanishing,

\[
\lim_{N \to \infty} B(f) = \left( \frac{\beta}{2} - 1 \right) g(0).
\]

A similar proof of Lemma 6.1 for deformed Wigner matrix is given in [59], so we write the details in the Appendix A. Thus we finish the proof of Theorem 2.6. \( \square \)

Appendix A. Proof of Lemma 6.1

In this Appendix, we compute the explicit formula for the variance and bias in the bulk and at the edges respectively.

Proof of Lemma 6.1. We first compute the bulk variance. Recall the variance \( V(f) \) given in (2.13). The integral kernel \( K(z, z') \) in (2.14) is split into three terms, denoted as \( K_1, K_2, K_3 \). We write

\[
V(f) = \frac{1}{4\pi^2} \int_{\Gamma_1} \int_{\Gamma_2} \tilde{f}(z_1) \tilde{f}(z_2) (K_1 + K_2 + K_3) dz_1 dz_2 := V_1 + V_2 + V_3.
\]

Since \( z_1, z_2 \) are in the bulk, using (3.6), we have \( |K_1 + K_2| = O(1) \). Using (4.1) and (4.7), we have

\[
V_1 + V_2 = \frac{1}{4\pi^2} \int_{\Gamma_1} \int_{\Gamma_2} (f(x_1) + iy_1 f'(x_1))(f(x_2) + iy_2 f'(x_2))(K_1 + K_2) dz_1 dz_2 = O(\eta^2).
\]

(4.1)

Thus it is sufficient to estimate \( V_3 \). We split the kernel \( K_3 \) into two terms,

\[
K_3(z_1, z_2) = \frac{2}{\beta} \text{Tr} \left( \frac{m_{sc}^*(z_1)m_{sc}^*(z_2)\Pi}{(1 - m_{sc}(z_1)m_{sc}(z_2)\Pi)} \right) + \frac{2}{\beta} \text{Tr} \left( \frac{m_{sc}^*(z_1)m_{sc}^*(z_2)(\Pi - \Pi)}{(1 - m_{sc}(z_1)m_{sc}(z_2)\Pi)} \right) := K_{31} + K_{32}.
\]

Since \( |m_{sc}(z)| < 1 \), taking the Taylor expansion and using \( S \Pi = \Pi S \Pi = \Pi \), we have

\[
K_{31} = \frac{2}{\beta} m_{sc}^*(z_1)m_{sc}^*(z_2) \text{Tr} \left( \sum_{k=0}^{\infty} (k + 1)(m_{sc}(z_1)m_{sc}(z_2))^k \Pi S \right) = \frac{2}{\beta} \frac{m_{sc}^*(z_1)m_{sc}^*(z_2)}{(1 - m_{sc}(z_1)m_{sc}(z_2))}.
\]

In addition, for the second term \( K_{32} \), using Lemma 3.2 and (3.6) we have

\[
K_{32} = \frac{2}{\beta} \left| \text{Tr} \left( \frac{m_{sc}^*(z_1)m_{sc}^*(z_2)(1 - \Pi)}{(1 - m_{sc}(z_1)m_{sc}(z_2))} \right) \right| = \frac{2}{\beta} \left| \text{Tr} \left( \frac{m_{sc}^*(z_1)m_{sc}^*(z_2)(1 - \Pi)^2 S}{(1 - m_{sc}(z_1)m_{sc}(z_2))} \right) \right| 
\leq CN \left( \left\| \frac{1 - \Pi}{1 - m_{sc}(z_1)m_{sc}(z_2)} \right\|_{\infty \to \infty} \right)^2 \| S \| \leq C'.
\]

(4.2)
Therefore, using (4.1) and (4.7), combining with (A.1), we have

\[ V(f) = -\frac{1}{2\pi i} \int_{\gamma_1} \int_{\gamma_2} f(z_1) \bar{f}(z_2) \frac{m'_{sc}(z_1)m'_{sc}(z_2)}{(1 - m_{sc}(z_1)m_{sc}(z_2))^2} \, dz_1 \, dz_2 + O(\eta_0^2). \]

If \( z_1, z_2 \) belongs to the same half plane, then \( m_{sc}(z) \) is analytic in a neighborhood of the segment between \( z_1, z_2 \), denoted by \( L(z_1, z_2) \). Combining with (3.6) we have

\[ \left| \frac{m'_{sc}(z_1)m'_{sc}(z_2)}{(1 - m_{sc}(z_1)m_{sc}(z_2))^2} \right| \leq C \left| \frac{z_1 + m_{sc}(z_1) - z_2 - m_{sc}(z_2)}{z_1 - z_2} \right|^2 \leq C \left( 1 + \sup_{z \in L(z_1, z_2)} \left| m'_{sc}(z) \right| \right)^2 = O(1). \]

Combining with (4.1) and (4.7), the integral with \( z_1, z_2 \) belonging to the same half plane only contributes \( O(\eta_0^2) \), as in (A.1). Hence it suffices to compute the integral when \( z_1, z_2 \) are in different half planes. Recall that

\[ m_{sc}(z) = -\frac{1}{2}z + \frac{1}{2}\sqrt{z^2 - 4}; \quad m'_{sc}(z) = \frac{1}{2} + \frac{z}{2\sqrt{z^2 - 4}}, \]

where the square root is taken so that \( \text{Im} \sqrt{z^2 - 4} > 0 \) when \( \text{Im} z > 0 \). Let \( z_1 = E_0 + x_1\eta_0 + iN^{-\tau}\eta_0 \), and \( z_2 = E_0 + x_2\eta_0 - \frac{1}{2}N^{-\tau}\eta_0 \). Thus we have the following asymptotic formulas:

\[ m'_{sc}(z_1)m'_{sc}(z_2) = \frac{1}{4 - E_0^2} + O(\eta_0), \quad 1 - m_{sc}(z_1)m_{sc}(z_2) = \frac{i\eta_0 x_1 - i\eta_0 x_2 + 3i/2N^{-\tau}\eta_0}{\sqrt{4 - E_0^2}} + O(\eta_0^2). \]

Therefore, changing the variables and using the same computations as in Section 6.2 [59], we write

\[ V(f) = \frac{1}{2\pi i} \int_{\gamma_1} \int_{\gamma_2} \frac{(g(x_1) - g(x_2))^2}{(x_1 - x_2)^2} \, dx_1 \, dx_2 + O(\eta_0) + O(\sqrt{\eta_0}) . \]

Next, we use the same notations to compute the variance at the edges. Using (3.6) and the same argument as in (A.2), we have

\[ |K_1 + K_2| \leq \frac{C}{\sqrt{|\text{Im} z + \text{Im} \tilde{z}|}}, \quad |K_{32}| \leq \frac{C}{\sqrt{|\text{Im} z + \text{Im} \tilde{z}|}}. \]

The integrals involved with \( K_1, K_2 \) and \( K_{32} \) only contribute \( O(\sqrt{N^\tau}) \) by (4.7). Therefore, we have

\[ V(f) = -\frac{1}{2\pi i} \int_{\gamma_1} \int_{\gamma_2} \frac{(f(x_1) + iy_1 f'(x_1))(f(x_2) + iy_2 f'(x_2))}{(x_1 - x_2)^2} \frac{m'_{sc}(z_1)m'_{sc}(z_2)}{(1 - m_{sc}(z_1)m_{sc}(z_2))^2} \, dz_1 \, dz_2 + O(N^\tau \eta_0). \]

The parts on the upper half plane of \( \Gamma_1 \) and \( \Gamma_2 \) are denoted as \( \Gamma_1^+ \) and \( \Gamma_2^+ \), while the ones on the lower half plane are \( \Gamma_1^- \) and \( \Gamma_2^- \). Let

\[ V(f) = \int_{\Gamma_1^+} \int_{\Gamma_2^+} + \int_{\Gamma_1^-} \int_{\Gamma_2^-} + \int_{\Gamma_1^+} \int_{\Gamma_2^-} + \int_{\Gamma_1^-} \int_{\Gamma_2^+} := V^{++} + V^{+-} + V^{+ -} + V^{-+}. \]

Recall (A.3) and let \( z = 2 + \eta_0 x + iN^{-\tau}\eta_0 \). We expand \( m_{sc}(z) \) around \( z = 2 \) and have

\[ m_{sc}(z) = -1 + O(\eta_0) + \sqrt{\eta_0(x + iN^{-\tau}) + O(\eta_0^2)} = -1 + \sqrt{\eta_0} \sqrt{x + iN^{-\tau}} + O(\eta_0); \]

\[ m'_{sc}(z) = \frac{1}{2} + \frac{1 + O(\eta_0)}{2\sqrt{\eta_0(x + iN^{-\tau}) + O(\eta_0^2)}} = \frac{1}{2} + \frac{1}{2\sqrt{\eta_0} \sqrt{x + iN^{-\tau}}} + O(\sqrt{\eta_0}), \]

where the square root is taken in a branch cut so that \( \text{Im} \sqrt{x + iN^{-\tau}} > 0 \). The following computation is the same as that in Section 6.2 [59]. For reader’s convenience, we still put it here omitting some technical details. Plugging (A.4) and (A.5) and changing the variables, we have

\[ V^{++} = -\frac{1}{8\beta^2 \pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\hat{g}(x_1) \hat{g}(x_2)}{\sqrt{x_1 + iN^{-\tau}} \sqrt{x_2 + i\frac{1}{2}N^{-\tau}} (\sqrt{x_1 + iN^{-\tau}} - \sqrt{x_2 + i\frac{1}{2}N^{-\tau}})^2} \, dx_1 \, dx_2 + O(\sqrt{\eta_0}), \]

where \( \hat{g}(x) = g(x) + iN^{-\tau} g'(x) \). Let \( \gamma_1^+ = \{ x_1 \pm iN^{-\tau} : x_1 \in \mathbb{R} \} \) and \( \gamma_2^+ = \{ x_2 \pm i\frac{1}{2}N^{-\tau} : x_2 \in \mathbb{R} \} \). Then

\[ V^{++} = \frac{1}{16\beta^2 \pi^2} \int_{\gamma_1^+} \int_{\gamma_2^+} \frac{(\hat{g}(z_1) - \hat{g}(z_2))^2}{\sqrt{\sqrt{z_1} \sqrt{z_2} (\sqrt{z_1} - \sqrt{z_2})^2}} \, dz_1 \, dz_2 + O(\sqrt{\eta_0}). \]
where \( \hat{g}(x + iy) = g(x) + iyf'(x)\chi(y) \). We apply the dominated convergence theorem to interchange the limit and the integral and obtain that

\[
\lim_{N \to \infty} V^{++} = \frac{1}{16\beta\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(g(x_1) - g(x_2))^2}{\sqrt{x_1^2 + \eta^2\phi^2 + i\eta^2\phi} \sqrt{x_2^2 + \eta^2\phi^2 + i\eta^2\phi}} \, dx_1 \, dx_2
\]

\[
= \frac{1}{4\beta\pi^2} \int_{\phi(R+i0)}^{\phi(R+i0)} \int_{\phi(R+i0)}^{\phi(R+i0)} \frac{(g(w_1^2) - g(w_2^2))^2}{(w_1 - w_2)^2} \, dw_1 \, dw_2,
\]

where we change the variable \( \phi : z \to \sqrt{z} \) in the second equation (with branch cut such that \( \phi : \mathbb{C}^+ \to \mathbb{C}^+ \)). We treat similarly for \( V^{--}, V^{+-} \) and \( V^{-+} \). The contours are shown in Figure 1. Note that the horizontal parts of the blue and the red lines will cancel because of the opposite integral direction. To sum up, we have

\[
\lim_{N \to \infty} V(f) = \frac{1}{4\beta\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(g(w_1^2) - g(w_2^2))^2}{(w_1 - w_2)^2} \, dw_1 \, dw_2 = \frac{1}{4\beta\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(g(x_1^2) - g(x_2^2))^2}{x_1 - x_2} \, dx_1 \, dx_2.
\]

Finally, we end the proof by computing the bias \( B(f) \) in (2.15) in the bulk and at the edges respectively. Note that from Lemma 3.2 and 3.1, we have

\[
\text{Tr} \left( \frac{m_{sc}'(z)m_{sc}(z)S^2}{1 - m_{sc}'(z)S} \right) = \text{Tr} \left( \frac{m_{sc}'(z)m_{sc}(z)S^2\Pi}{1 - m_{sc}'(z)S} \right) + \text{Tr} \left( \frac{m_{sc}'(z)m_{sc}(z)S^2(1 - \Pi)}{1 - m_{sc}'(z)S} \right)
\]

\[
= \frac{m_{sc}'(z)m_{sc}(z)}{1 - m_{sc}'(z)} + O \left( \frac{1}{\sqrt{\kappa + \eta}} \right) = (m_{sc}'(z))^2 m_{sc}(z) + O \left( \frac{1}{\sqrt{\kappa + \eta}} \right).
\]

Thus the function \( b(z) \) in (2.16) can be written as

\[
b(z) = k_4 m_{sc}'(z)m_{sc}(z) + (m_{sc}'(z))^2 m_{sc}(z) + O \left( \frac{1}{\sqrt{\kappa + \eta}} \right) = (m_{sc}'(z))^2 m_{sc}(z) + O \left( \frac{1}{\sqrt{\kappa + \eta}} \right). \tag{A.6}
\]

If \( z \) is in the bulk, by (3.6), \( |b(z)| = O(1) \). Combining with (4.7), we have

\[
B(f) = \frac{1}{2\pi i} \int_{\Gamma} (f(x) + iyf'(x))b(z)dz = O(\eta_0).
\]

If \( z \) is at the edges, using the expansions (A.4) and (A.5), we have

\[
b(z) = (m_{sc}'(z))^2 m_{sc}(z) + O \left( \frac{1}{\sqrt{\eta}} \right) = - \frac{1}{4\eta_0(x + iN\tau)} + O \left( \frac{N^{\tau/2}}{\sqrt{\eta_0}} \right).
\]
Using (2.10) and changing the variables, we have
\[ B(f) = -\frac{1}{8\pi^2} \int_{\mathbb{R}} \frac{g(x)}{x + iN^{-\tau}} dx + \frac{1}{8\pi^2} \int_{\mathbb{R}} \frac{g(x)}{x - iN^{-\tau}} dx + O(x(\sqrt{N^{1/2}})) . \]
Applying the Sokhotski-Plemelj lemma, we complete the proof of Lemma 6.1. \(\square\)

**APPENDIX B. COMPLEX CASE**

In this appendix, we extend previous results from real symmetric \((\beta = 1)\) to complex Hermitian \((\beta = 2)\) matrices. We will use the complex analogue of Lemma 5.1.

**Lemma B.1** (Complex cumulant expansion). Let \(h\) be a complex-valued random variable with finite moments, and \(f\) is a complex-valued smooth function on \(\mathbb{R}\) with bounded derivatives. Let \(c_{p,q}\) be the \((p,q)\) cumulant of \(h\), which is defined as
\[ c_{p,q} := (-1)^{p+q} \left( \frac{\partial^{p+q}}{\partial s^p \partial t^q} \log \mathbb{E} e^{ish} \right) |_{s,t=0}. \]
Then for any fixed \(l \in \mathbb{N}\), we have
\[ \mathbb{E} h f(h, \bar{h}) = \sum_{p+q=0}^{l} \frac{1}{p!q!} c_{p,q+1}(h) f^{(p,q)}(h) + R_{l+1}, \]
where the error term satisfies
\[ |R_{l+1}| \leq C_4 \mathbb{E} \left[ |h|^{l+2} \right] \max_{p+q=l+1} \left\{ \sup_{|z| \leq M} |f^{(p,q)}(z, \bar{z})| \right\} + C_4 \mathbb{E} \left[ |h|^{l+2} |h| \right] \max_{p+q=l+1} \|f^{(p,q)}(z, \bar{z})\|_{\infty}, \]
and \(M > 0\) is an arbitrary fixed cutoff.

Instead of (5.6), we have
\[ \frac{\partial G_{ij}}{\partial H_{ab}} = -G_{ia} G_{bj}, \quad (B.1) \]
from which we obtain the analogue of (5.7)-(5.12).

We use \(c_{ij}^{(p,q)}\) to denote the \((p,q)\) cumulant of \(\sqrt{N} H_{ij}\). The assumption \(\mathbb{E} H_{ij}^2 = 0\) for \(i \neq j\) implies that \(c_{ij}^{(1,1)} = \mathbb{E} [H_{ij}]^2\), \(c_{ij}^{(2,0)} = c_{ij}^{(0,2)} = 0\). In addition, for \(i \neq j\), \(c_{ij}^{(2,2)} = c_{ij}(\text{Re} H_{ij}) + c_{ij}(\text{Im} H_{ij})\). Using the isotropic law and (B.1), one shows similarly that the expansion terms corresponding to \(p+q = 3\) are negligible. Using the identity (B.1) and the analogous estimates of (5.7)-(5.12), we obtain that
\[ \mathbb{E} [c_0(\lambda)(\text{Tr} G - \mathbb{E} \text{Tr} G)] = \frac{1}{N} \sum_{i,j=1}^{N} c_{ij}^{(1,1)} \mathbb{E} \left[ \frac{\partial}{\partial H_{ij}} (c_0(\lambda) (G_{ij} - \mathbb{E} G_{ij})) \right] \]
\[ + \frac{1}{2! N^2} \sum_{i,j=1}^{N} c_{ij}^{(2,2)} \mathbb{E} \left[ \frac{\partial^3}{\partial^2 H_{ij} \partial H_{ij}} (c_0(\lambda) (G_{ij} - \mathbb{E} G_{ij})) \right] + \cdots, \]
\[ = \sum_{i,j=1}^{N} s_{ij} \mathbb{E} \left[ c_0(\lambda) \left( \frac{\partial G_{ij}}{\partial H_{ij}} - \mathbb{E} \frac{\partial G_{ij}}{\partial H_{ij}} \right) \right] + \sum_{i,j=1}^{N} s_{ij} \mathbb{E} \left[ \frac{\partial^3}{\partial^2 H_{ij} \partial H_{ij}} (c_0(\lambda) (G_{ij} - \mathbb{E} G_{ij})) \right] + \frac{1}{N^2} \sum_{j=1}^{N} c_{ij}^{(2,2)} \mathbb{E} \left[ \frac{\partial c_0(\lambda)}{\partial H_{ij}} \frac{\partial G_{ij}}{\partial H_{ij}} \right] + \cdots. \]
Proceeding the same arguments as in Section 5, one shows that Lemma 4.2 holds with modified integral kernel \(K(z, z')\), i.e., the coefficient of the first term given in (2.14) is 1 instead of 2, and the second term vanishes.

Similarly, as for the expectation, we have
\[ \mathbb{E} \text{Tr} G = \sum_{i=1}^{N} (\mathbb{E}(H G)_{ii} - 1) = \frac{1}{N} \sum_{i,j=1}^{N} c_{ij}^{(1,1)} \frac{\partial G_{ij}}{\partial H_{ij}} - N + \frac{1}{2! N^2} \sum_{i,j=1}^{N} c_{ij}^{(2,2)} \frac{\partial^3 G_{ij}}{\partial^2 H_{ij} \partial H_{ij}} + \cdots. \]
Thus the first term of \(b(z)\) given in (2.16) vanishes. Consequently, we obtain corresponding results in the complex case.
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