Lagrangian fibrations on hyperkähler fourfolds

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**Abstract.** In answer to the strong form of a question posed by Beauville, we give a short geometric proof that any hyperkähler fourfold containing a Lagrangian subtorus $L$ admits a holomorphic Lagrangian fibration with fibre $L$.

**Keywords:** hyperkähler manifold, Lagrangian fibration.

§ 1. Introduction

Let $X$ be a hyperkähler manifold, that is, a compact simply connected Kähler manifold such that $H^0(X, \Omega^2_X)$ is spanned by a holomorphic symplectic form $\sigma$. By work of Matsushita it is well known that the only possible non-trivial holomorphic maps from $X$ to a lower-dimensional complex space are Lagrangian fibrations (see §2). Moreover, a special version of the so-called hyperkähler SYZ-conjecture asserts that any hyperkähler manifold can be deformed to a hyperkähler manifold admitting a Lagrangian fibration.

Hence, it is an important problem to find geometric conditions on a given hyperkähler manifold that guarantee the existence of a Lagrangian fibration. Here we address a question posed by Beauville ([1], §1.6).

**Question A.** Let $X$ be a hyperkähler manifold and $L$ a Lagrangian torus in $X$. Is $L$ a fibre of a (meromorphic) Lagrangian fibration $f: X \to B$?

In our previous article [2] we showed that Question A has a positive answer in the case when $X$ is non-projective. Moreover, for any hyperkähler manifold that admits an almost-holomorphic Lagrangian fibration, a further hyperkähler manifold, birationally equivalent to the first one, was found, on which the Lagrangian fibration becomes holomorphic.

The approach to the projective case of Beauville’s question pursued here is based on a detailed study of the deformation theory of $L$ in $X$. For this, consider the component $\mathcal{B}$ of the Barlet space that contains $[L]$, together with its universal family and the evaluation map to $X$:

\[
\begin{array}{ccc}
U & \xrightarrow{\varepsilon} & X \\
\downarrow \pi & & \downarrow \\
\mathcal{B} & & \\
\end{array}
\]

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It was shown in [2], Lemma 3.1, that $\varepsilon$ is surjective and generically finite, and that $X$ admits an almost-holomorphic Lagrangian fibration if and only if $\deg(\varepsilon) = 1$.

If the degree of $\varepsilon$ is strictly greater than one, some deformations of $L$ intersect $L$ in unexpected ways. In order to deal with this, we introduce the notion of $L$-reduction: for each projective hyperkähler manifold containing a Lagrangian torus one can find a projective variety $\mathcal{Y}$ and a rational map $\varphi_L : X \dashrightarrow \mathcal{Y}$ (uniquely determined up to birational equivalence) whose fibre through a general point $x$ coincides with the connected component of the intersection of all deformations of $L$ through $x$. In this situation, we say that $X$ is $L$-separable if $\varphi_L$ is birational. We prove the following result.

**Theorem** (Theorem 3.5). Let $X$ be a projective hyperkähler manifold and $L \subset X$ a Lagrangian subtorus. Then $X$ admits an almost-holomorphic fibration with strong fibre $L$ if and only if $X$ is not $L$-separable.

If $X$ is a hyperkähler fourfold, then we can exclude the case of $L$-separable $X$ by symplectic linear algebra. Moreover, based upon a rather explicit knowledge about the birational geometry of hyperkähler fourfolds, we obtain the following positive answer to the strongest form of Beauville’s question.

**Theorem** (Theorem 5.1). Let $X$ be a four-dimensional hyperkähler manifold containing a Lagrangian torus $L$. Then $X$ admits a holomorphic Lagrangian fibration with fibre $L$.

At the Moscow conference ‘Geometric Structures on Complex Manifolds’ Ekaterina Amerik brought to our attention that she had independently proved a related result, based on an observation in [3], to the effect that in dimension four every projective hyperkähler manifold containing a Lagrangian subtorus $L$ admits an almost-holomorphic Lagrangian fibration with fibre $L$ ([4]).

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1 After this article was written, Jun-Muk Hwang and Richard Weiss posted a proof of the projective case of a weak form of Beauville’s question. They produced an almost-holomorphic Lagrangian fibration on any projective $2n$-dimensional hyperkähler manifold containing a Lagrangian torus (see [5]). Their argument is in two parts, one geometric and one concerned with abstract group theory. In contrast, our answer (Theorem 5.1) to the strong form of Beauville’s question is purely geometric, uses global arguments in addition to local deformation-theoretic ones, and uses symplectic linear algebra in place of their group-theoretical arguments.
§ 2. Preliminaries and notation

2.1. Lagrangian fibrations.

Definition 2.1. Let $X$ be a hyperkähler manifold. A **Lagrangian fibration** on $X$ is a holomorphic map $f : X \to B$ with connected fibres onto a normal complex space $B$ such that every irreducible component of the reduction of every fibre of $f$ is a Lagrangian subvariety of $X$.

By fundamental results of Matsushita it is known that any fibration on a hyperkähler manifold is automatically Lagrangian.

Theorem 2.2 ([6]–[9]). Let $X$ be a hyperkähler manifold of dimension $2n$. If $f : X \to B$ is a morphism with connected fibres to a normal complex space $B$ with $0 < \dim B < \dim X$, then $f$ is a Lagrangian fibration. In particular, $f$ is equidimensional and $\dim B = n$. Furthermore, every smooth fibre of $f$ is a complex torus.

2.2. Meromorphic maps. Let $X$ be a normal complex space, $Y$ a compact complex space, and $f : X \dasharrow Y$ a meromorphic map. Let

$$
\begin{array}{c}
\tilde{X} \\
\downarrow p \\
X \xrightarrow{f} Y
\end{array}
$$

be a resolution of indeterminacies of $f$. The fibre $F_y$ of $f$ over a point $y \in Y$ is defined to be $F_y := p(\tilde{f}^{-1}(y))$. It is independent of the chosen resolution.

Recall that a meromorphic map $f : X \dasharrow Y$ as above is said to be **almost holomorphic** if there is a Zariski-open subset $U \subset Y$ such that the restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is holomorphic and proper. A **strong fibre** of an almost-holomorphic map $f$ is a fibre of $f|_{f^{-1}(U)}$.

Let $X$ be a normal algebraic variety, $B$ a complete algebraic variety, and $f : X \dasharrow B$ an almost-holomorphic rational map. If $A$ is a divisor on $B$, then its pullback under $f$ is defined either geometrically as the closure of the pullback of $A$ on the locus where $f$ is holomorphic, or at the level of locally free sheaves as $f^*\mathcal{O}_B(A) := (p_*f^*\mathcal{O}_B(A))^\vee$, where $p : \tilde{X} \to X$ is a resolution of indeterminacies as in the diagram (2.1).

2.3. Deformations of Lagrangian subtori. The starting point for our approach to Beauville’s question is the deformation theory of a Lagrangian subtorus $L$ in a hyperkähler manifold $X$. We quickly recall the relevant results from [2], §§2, 3.

The Barlet space $\mathcal{B}(X)$ of $X$ (or Chow scheme in the projective setting) parametrizes compact cycles in $X$. It turns out (see part (i) of Lemma 2.3) that there is a unique irreducible component $\mathcal{B}$ of $\mathcal{B}(X)$ containing the point $[L]$. Denoting by $\mathcal{U}$ the graph of the universal family over $\mathcal{B}$ and by $\Delta$ the **discriminant locus** of $\mathcal{B}$ (that is, the set of points parametrizing singular elements in the family $\mathcal{B}$),
we obtain the following diagram:

\[ \begin{array}{ccc}
U_{\Delta} & \xrightarrow{\epsilon} & U \\
\downarrow & & \downarrow \pi \\
\Delta & \xrightarrow{\Delta} & \mathcal{B}
\end{array} \]

(2.2)

A detailed analysis of the maps in the diagram (2.2) shows that a small étale or analytic neighbourhood of \( L \) in \( X \) fibres over a neighbourhood of \([L]\) in \( \mathcal{B} \). More precisely, we have the following result.

**Lemma 2.3** ([2], Lemma 3.1). Let \( X \) be a hyperkähler manifold of dimension \( 2n \) and let \( L \) be a Lagrangian subtorus in \( X \). Then the following assertions hold.

(i) The Barlet space \( \mathcal{B}(X) \) is smooth of dimension \( n \) over \([L]\). In particular, \([L]\) is contained in a unique irreducible component \( \mathcal{B} \) of \( \mathcal{B}(X) \) and \( U \) is smooth of dimension \( 2n \) near \( \pi^{-1}([L]) \).

(ii) The morphism \( \epsilon \) is finite étale along smooth fibres of \( \pi \). In particular, a sufficiently small deformation of \( L \) is disjoint from \( L \) and there are no positive-dimensional families of smooth fibres through a general point \( x \in X \).

(iii) If \([L'] \in \mathcal{B} \) with smooth \( L' \), then \( L' \) is a Lagrangian subtorus in \( X \).

**Remark 2.4.** We state two simple but useful consequences of Lemma 2.3.

(i) The locus \( X_\Delta := \epsilon(U_\Delta) \) is the locus of points \( x \in X \) such that there is a singular deformation of \( L \) passing through \( x \). For reasons of dimension, \( X_\Delta \) is a proper subset of \( X \). By Lemma 2.3(ii), the map \( \epsilon \) is finite and étale on the pre-image of \( X \setminus X_\Delta \).

(ii) Part (ii) of Lemma 2.3 implies in particular that for any two points \([L],[M] \in \mathcal{B} \) the intersection product \([L].[M] \) of cycles in \( X \) vanishes. It is therefore impossible for members of the family \( \mathcal{B} \) to intersect in a finite number of points.

### 2.4. Almost-holomorphic Lagrangian fibrations and Barlet spaces.

The following result relates the deformation theory of \( L \) in \( X \) discussed above to our question about globally defined almost-holomorphic Lagrangian fibrations.

**Lemma 2.5** ([2], Lemma 3.2). Let \( X \) be a hyperkähler manifold containing a Lagrangian subtorus \( L \). Then \( X \) admits an almost-holomorphic Lagrangian fibration with strong fibre \( L \) if and only if the evaluation map \( \epsilon \) in diagram (2.2) is bimeromorphic.

If \( \epsilon \) is birational, then \( \pi \circ \epsilon^{-1} \) is the desired almost-holomorphic fibration (up to the normalization of \( \mathcal{B} \)). In the other direction one uses the Barlet space of a resolution of indeterminacies.

### § 3. \( L \)-reduction and \( L \)-separable manifolds

Let \( X \) be a projective hyperkähler manifold containing a Lagrangian subtorus \( L \). In this section we start our analysis of the maps in the associated diagram (2.2). Recall from Lemma 2.5 that in order to answer Beauville’s question positively we must show that the evaluation map \( \epsilon \) is birational.
3.1. L-reduction. Here we construct a meromorphic map associated with the covering family \{L_t\}_{t \in \mathcal{B}}. Generically, this map is a quotient map for the meromorphic equivalence relation determined by the family \{L_t\}, that is, generically it identifies those points in \(X\) that cannot be separated by members of \{L_t\}.

3.1.1. Construction of the L-reduction. We work in the setup summarized in diagram (2.2). We set \(U_{\text{reg}} := \varepsilon^{-1}(X \setminus X_\Delta)\). Recall from Remark 2.4 that the map \(\varepsilon|_{U_{\text{reg}}} : U_{\text{reg}} \rightarrow X \setminus X_\Delta\) is finite étale. We denote its degree by \(d\).

The map \(\varepsilon|_{U_{\text{reg}}}\) induces a morphism \(X \setminus X_\Delta \rightarrow \text{Sym}^d(U_{\text{reg}})\). Composing this map with the natural morphism \(\text{Sym}^d(U_{\text{reg}}) \rightarrow \text{Sym}^d(\mathcal{B})\) induced by \(\pi : U \rightarrow \mathcal{B}\), we construct a morphism \(X \setminus X_\Delta \rightarrow \text{Sym}^d(\mathcal{B})\). This morphism naturally extends to a rational map \(\psi : X \dashrightarrow \text{Sym}^d(\mathcal{B})\). Let

\[
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\psi} & \text{Sym}^d(\mathcal{B}) \\
\downarrow & & \\
X & \xrightarrow{\varepsilon} & \\
\end{array}
\]

be a resolution of singularities of the indeterminacies of \(\psi\) with \(\widetilde{X}\) non-singular. Then the Stein factorization of \(\widetilde{\psi}\) yields the following diagram:

\[
\begin{array}{ccc}
X & \xleftarrow{p} & \widetilde{X} \\
\downarrow & & \downarrow \varphi_L \\
\mathfrak{T} & \xrightarrow{\varphi} & \text{Sym}^d(\mathcal{B}) \\
\end{array}
\]

Here \(\varphi_L = \bar{\varphi} \circ p^{-1} : X \dashrightarrow \mathfrak{T}\) is the rational map induced by \(\bar{\varphi}\). Noting that \(\varphi_L : X \dashrightarrow \mathfrak{T}\) is unique up to birational equivalence, and hence is canonically associated with the pair \((X, L)\), we call it the L-reduction of \(X\).

Remark 3.1. For every point \(x \in X \setminus X_\Delta\) there are exactly \(d\) pairwise-distinct smooth tori \(L_1, \ldots, L_d\) through \(x\) in the family \(\{L_t\}_{t \in \mathcal{B}}\). By construction, \(\varphi_L\) is defined at \(x\) and maps it to the class of \(([L_1], \ldots, [L_d])\) in \(\text{Sym}^d(\mathcal{B})\).

3.1.2. First properties of the L-reduction. The following set-theoretic assertion is an immediate consequence of the construction of \(\varphi_L\).

Lemma 3.2. The fibre of \(\varphi_L\) through a point \(x \in X \setminus X_\Delta\) coincides with the connected component of

\[
\bigcap_{[M] \in \mathcal{B}, x \in M} M
\]

containing \(x\).

Proof. If \(x \in X \setminus X_\Delta\), then \(\varepsilon\) is étale at every point of the pre-image \(\varepsilon^{-1}(x)\). Thus the image \(\pi(\varepsilon^{-1}(x)) = \{[L_1], \ldots, [L_d]\}\) consists of those points of \(\mathcal{B}\) that parametrize
the $d$ pairwise-distinct subtori in $X$ through $x$. In particular, the meromorphic map 
$$
\psi: X \dashrightarrow \text{Sym}^d(\mathfrak{B})
$$
is defined at $x$ and its fibre is

$$
\psi^{-1}(\psi(x)) = \bigcap_i L_i.
$$

(3.1)

After taking the Stein factorization, the fibre of $\varphi_L$ is the component of (3.1) passing
through $x$. □

**Lemma 3.3.** Let $X$ be a projective hyperkähler manifold containing a Lagrangian
subtorus $L$. Then the $L$-reduction $\varphi_L: X \dashrightarrow \mathfrak{T}$ is almost holomorphic.

*Proof.* Let dom($\varphi_L$) be the domain of definition of $\varphi_L$ and let $Z := X \setminus \text{dom}(\varphi_L)$
be the locus where $\varphi_L$ is not defined. We must show that the general fibre of $\varphi_L$
does not intersect $Z$.

Aiming for a contradiction, suppose that for a general $x_0 \in X \setminus X_\Delta$ the fibre $F_{x_0}$
of $\varphi_L$ through $x_0$ intersects $Z$ non-trivially. Recall from Remark 2.4(i) that $X_\Delta = \varepsilon(U_\Delta)$ is the locus swept out by singular deformations of $L$, and from Remark 3.1
that $\varphi_L$ is holomorphic on $X \setminus X_\Delta$. Take a point $z \in F_{x_0} \cap Z$. Consider the graph
$X' \subset X \times \mathfrak{T}$ of $\varphi_L$ with projections $p: X' \rightarrow X$ and $\varphi'_L: X' \rightarrow \mathfrak{T}$. As explained,
for example, in [10], §1.39, the closed subset $Z$ can be described as

$$
Z = \{x \in X \mid \dim p^{-1}(x) > 0\}.
$$

(3.2)

Since $X$ is normal and $p$ is birational, $p$ has connected fibres. Thus the variety
$C' := \varphi'_L(p^{-1}(z))$ is connected. We list some other properties of $C'$.

(i) dim$C' > 0$ because dim$ p^{-1}(z) > 0$ and $X'$ is the graph of $\varphi_L$.

(ii) $\varphi_L(x_0) \in C'$ for $z \in F_{x_0}$.

(iii) The point $z$ is contained in all fibres over points in $C'$.

Suppose for a moment that we had a diagram

$$
\begin{array}{ccc}
C' & \hookrightarrow & X \\
\varphi_L|_C & \downarrow & \varphi_L \\
C' & \hookrightarrow & \mathfrak{T}
\end{array}
$$

such that $C$ is connected, $x_0 \in C$ and $\varphi_L|_C$ is a local isomorphism at the point $x_0$.

We claim that this would produce a contradiction. Indeed, let $L_1, \ldots, L_d$ be the
$d = \text{deg} \varepsilon$ pairwise-distinct tori in the family $\{L_t\}_{t \in \mathfrak{B}}$ containing $x_0$. Since $C$
is connected and $C \not\subset F_{x_0}$ by property (i) above, Lemma 3.2 implies that there is
a $k \in \{1, \ldots, d\}$ such that $C \not\subset L_k$. By Lemma 2.3, the small deformations of $L_k$
constitute a fibration in an analytic neighbourhood of $x_0$. Thus, for all points $y \in C \setminus \{x_0\}$ sufficiently close to $x_0$, there is a small deformation $L_y$ of $L_k$ with $y \in L_y$ and

$$
L_y \cap L_k = \emptyset.
$$

(3.3)

On the other hand, property (iii) and Lemma 3.2 imply that

$$
z \in F_y \cap F_{x_0} \subset L_y \cap L_k,
$$

which is absurd in view of (3.3).
It remains to find the variety \( C \). We observe that it suffices to construct \( C \) in a Euclidean open neighbourhood of \( x_0 \). Invoking the genericity assumption on \( x_0 \) and the implicit function theorem, we find a small neighbourhood \( U \ni x_0 \) such that the restriction \( \varphi_L : U \to V := \varphi_L(U) \) is a trivial holomorphic fibre bundle. In particular, \( V \subset \mathcal{T} \) is open and there is a section \( C \subset U \) for the subvariety \( C' \cap V \). The only remaining property to be fulfilled is the connectedness of \( C' \cap V \) and \( C \). This can be achieved by shrinking \( V \) and \( U \). □

**Definition 3.4.** A projective hyperkähler manifold \( X \) containing a Lagrangian subtorus \( L \) is said to be \( L \)-separable if its \( L \)-reduction \( \varphi_L : X \dashrightarrow \mathcal{T} \) is birational.

**3.2. Lagrangian fibrations on non-\( L \)-separable manifolds.**

**Theorem 3.5.** Let \( X \) be a projective hyperkähler manifold and \( L \subset X \) a Lagrangian subtorus. Then \( X \) admits an almost-holomorphic fibration with strong fibre \( L \) if and only if \( X \) is not \( L \)-separable.

This result enables us to restate Beauville’s question in the following way.

**Question B.** Does there exist a projective hyperkähler manifold \( X \) together with a Lagrangian subtorus \( L \) such that \( X \) is \( L \)-separable?

**Proof of Theorem 3.5.** If \( X \) is not \( L \)-separable, then the \( L \)-reduction \( \varphi_L : X \dashrightarrow \mathcal{T} \) is an almost-holomorphic map (by Lemma 3.3) and \( 0 < \dim \mathcal{T} < \dim X \). Thus by [2], Theorem 6.7, the map \( \varphi_L \) is an almost-holomorphic Lagrangian fibration on \( X \). By the description of the general fibre of the \( L \)-reduction (Lemma 3.2), the torus \( L \) is a strong fibre of \( \varphi_L \).

Conversely, if \( f : X \dashrightarrow B \) is an almost-holomorphic Lagrangian fibration with strong fibre \( L \), then through a general point there is a unique Lagrangian subtorus in \( \mathfrak{B} \) and the \( L \)-reduction coincides with the rational map \( \pi \circ \varepsilon^{-1} : X \dashrightarrow \mathfrak{B} \). In particular, \( X \) is not \( L \)-separable. □

**§ 4. Intersections of Lagrangian subtori**

As before, let \( X \) be a projective hyperkähler manifold containing a Lagrangian subtorus \( L \). In this section we study a neighbourhood of \( L \) in \( X \) more closely, leading to several results about the geometry of intersections of different members of the family \( \mathfrak{B} \) of deformations of \( L \). We shall use the notation and results of § 2.3 throughout.

By Lemma 2.3, \( \mathfrak{B} \) is smooth at \([L]\) and we can find a neighbourhood \( V \) of \([L]\) such that the restriction \( \varepsilon|_{\mathcal{U}_V} : \mathcal{U}_V \to X \) of the evaluation map to the pre-image \( \mathcal{U}_V := \pi^{-1}(V) \) embeds \( \mathcal{U}_V \) in \( X \). We can thus regard \( \mathcal{U}_V \) as an open subset of \( X \). The intersection of \( \mathcal{U}_V \) with a submanifold \( M \subset X \) is depicted in Fig. 1.

**Lemma 4.1.** Let \( M \subset X \) be a smooth proper submanifold, and \( L \subset X \) a smooth Lagrangian torus that intersects \( M \) non-trivially. Then a generic small deformation of \( L \) has smooth intersection with \( M \).

**Proof.** We continue to use the notation introduced above. Since \( \mathcal{U}_V \) is open in \( X \), the intersection \( M \cap \mathcal{U}_V \) is smooth. Furthermore, the map \( \pi|_{M \cap \mathcal{U}_V} : M \cap \mathcal{U}_V \to V \)
is proper because \( \pi \) is proper and \( M \) is compact. We can therefore apply the theorem on generic smoothness to \( \pi \mid_{M \cap \Omega_V} \), which proves the result. \( \square \)

**Proposition 4.2.** Let \( M \subseteq X \) be a compact submanifold and \( L \subseteq X \) a general Lagrangian subtorus such that \( L \cap M \neq \emptyset \). Then \( N_{L \cap M/M} \) is trivial. If \( M \) is a complex torus, then \( L \cap M \) is a disjoint union of tori.

**Proof.** As \( L \) is general, the intersection \( L \cap M \) is smooth by Lemma 4.1. Moreover, both statements can be verified by looking at one connected component of \( L \cap M \) at a time. We invoke the notation introduced in the beginning of this section and let \( T \) be a connected component of \( L \cap M \). If \( V \) is sufficiently small, then the inclusion \( L \cap M \hookrightarrow \Omega_V \cap M \) induces a one-to-one correspondence of their respective connected components. Let \( S \) be the unique component of \( \Omega_V \cap M \) corresponding to \( T \). By the generality of \( L \) we may assume that \( \pi \mid_S \) is a smooth map, whence \( C := \pi(S) \subset V \) is smooth of dimension \( n - \dim T \) near \([L]\). Moreover, \( C \) parametrizes those small deformations of \( L \) that induce a flat deformation of \( T \) inside \( M \). Corresponding to the family \( S \to C \) we thus obtain a classifying map \( \chi: C \to \mathcal{D}(M) \) from \( C \) to the Douady space of \( M \).

At the level of tangent spaces we have \( T_C([L]) \subset T_{\mathfrak{B}}([L]) = H^0(L, N_{L/X}) \), where the last equality comes from the Hilbert–Chow morphism (compare [2], Lemma 3.1). The morphism \( \chi \) induces a map \( \chi_*: T_C([L]) \to H^0(T, N_{T/M}) \). But small deformations of \( T \) inside \( M \) induced by deformations of \( L \) are disjoint from \( T \) by Lemma 2.3(ii). Thus the map \( \chi_* \) is injective and the image of \( T_C([L]) \) consists of nowhere-vanishing sections. For reasons of dimension, these sections generate the normal bundle of \( T \) in \( M \). Hence \( N_{T/M} \) is trivial, as required.

If \( M \) is a torus as well, then \( T_M \mid_T \) is likewise trivial. Thus, by the normal bundle sequence

\[
0 \longrightarrow T_T \longrightarrow T_M \mid_T \longrightarrow N_{T/M} \longrightarrow 0,
\]

the tangent bundle \( T_T \) is trivial, and thus \( T \) is a complex torus. \( \square \)
Based on Proposition 4.2, we can now refine the observation in Remark 2.4(ii).

**Lemma 4.3.** Let \( X \) be a four-dimensional hyperkähler manifold. Let \( L \) and \( M \) be Lagrangian tori intersecting smoothly, and set \( I = L \cap M \). Then \( I \) is a finite disjoint union of elliptic curves.

**Proof.** It remains to exclude the existence of zero-dimensional connected components of \( I \). By general Lagrangian intersection theory (see for example [11], Introduction) we have

\[
[L].[M] = \chi(I).
\]

However, this already yields the claim since by Proposition 4.2 any positive-dimensional component of \( I \) is a smooth elliptic curve, contributing zero to the Euler characteristic \( \chi(I) \). \( \square \)

**Corollary 4.4.** Let \( X \) be a four-dimensional projective hyperkähler manifold and \( L \) a Lagrangian subtorus. If \( X \) is \( L \)-separable, then the evaluation map \( \varepsilon \) in diagram (2.2) has degree at least three.

**Proof.** As \( X \) is assumed to be \( L \)-separable, Theorem 3.5 and Lemma 2.5 imply that \( \varepsilon \) is not birational. It remains to exclude the case \( \deg \varepsilon = 2 \). By Lemma 3.2, the \( L \)-separability means that at a general point \( x \in X \) the connected component of \( \bigcap_{\mathcal{B}, x \in M} M \) is just \( x \). If \( \deg \varepsilon = 2 \), then there are just two tori in \( \mathcal{B} \) containing \( x \), call them \( M_1 \) and \( M_2 \). As \( x \) is general, Lemma 4.1 tells us that \( M_1 \cap M_2 \) is smooth. Then Lemma 4.3 contradicts the fact that the connected component of \( M_1 \cap M_2 \) containing \( x \) is \( \{x\} \). \( \square \)

§ 5. Hyperkähler fourfolds

Using the results in § 4, we can now prove our main result, which gives the strongest possible positive answer to Beauville’s question.

**Theorem 5.1.** Let \( X \) be a four-dimensional hyperkähler manifold containing a Lagrangian torus \( L \). Then \( X \) admits a holomorphic Lagrangian fibration with fibre \( L \).

**Remark 5.2.** We are grateful to E. Amerik for communicating the following observation from linear algebra which serves to exclude \( L \)-separable manifolds \( X \supset L \) in dimension four. This greatly simplified our previous deformation-theoretic argument.

**Lemma 5.3.** Let \( V \) be a four-dimensional symplectic vector space with symplectic form \( \sigma \), and let \( W_1, W_2, W_3 \subset V \) be three Lagrangian subspaces satisfying \( \dim W_i \cap W_j = 1 \) for all \( i \neq j \). Then \( W_1 \cap W_2 \cap W_3 \neq \{0\} \).

**Proof.** Suppose on the contrary that \( W_1 \cap W_2 \cap W_3 = \{0\} \) and consider the span \( \langle W_1, W_2 \rangle \). It is of dimension 3 since \( \dim W_1 \cap W_2 = 1 \). Moreover, we claim that

\[
W_3 \subset \langle W_1, W_2 \rangle. \tag{5.1}
\]

Indeed, otherwise we would have \( \dim W_3 \cap \langle W_1, W_2 \rangle = 1 \), whence the intersections \( W_3 \cap \langle W_1, W_2 \rangle = W_3 \cap W_1 = W_3 \cap W_2 \) are one-dimensional contrary to our assumption that \( W_1 \cap W_2 \cap W_3 = \{0\} \).
Now, again using \( W_1 \cap W_2 \cap W_3 = \{0\} \), we write \( \langle W_1, W_2 \rangle = W_3 \oplus (W_1 \cap W_2) \). Since \( V \) is symplectic and \( W_3 \) is Lagrangian, there are \( v \in W_1 \cap W_2 \) and \( w \in W_3 \) such that \( \sigma(v, w) \neq 0 \). According to the inclusion (5.1) we can write \( w = w_1 + w_2 \) with \( w_i \in W_i \). Then

\[
0 \neq \sigma(v, w) = \sigma(v, w_1) + \sigma(v, w_2) = 0 + 0 = 0
\]

because \( W_1 \) and \( W_2 \) are Lagrangian, a contradiction. \( \square \)

**Remark 5.4.** Lemma 5.3 can also be proved using the following beautiful geometric argument which was explained to us by L. Manivel. The Grassmannian of Lagrangian subspaces in \( V \simeq \mathbb{C}^4 \) is biholomorphic to the (smooth) intersection of the Plücker quadric \( \tilde{Q} \subset \mathbb{P}(\bigwedge^2 V) \) and the linear subspace defined by the vanishing of the symplectic form \( \sigma: \bigwedge^2 V \to \mathbb{C} \). Hence it is a smooth quadric \( Q \subset \mathbb{P}^4 \). The condition \( \dim W_i \cap W_j = 1 \) means that the line in \( \mathbb{P}^4 \) joining the points \( [W_i], [W_j] \in Q \) is contained in \( Q \). If the triple intersection \( W_1 \cap W_2 \cap W_3 = \{0\} \), then \( [W_1], [W_2], [W_3] \) span a plane \( P \). But \( P \cap Q \) has degree 2 and thus cannot contain a union of 3 lines.

**Proposition 5.5.** Let \( X \) be a four-dimensional projective hyperkähler manifold containing a Lagrangian torus \( L \). Then \( X \) is not \( L \)-separable.

**Proof.** Suppose on the contrary that \( X \) is \( L \)-separable. Given a general point \( x \in X \), it follows from Lemma 3.2 and Corollary 4.4 that one can find an integer \( d \geq 3 \) and \( d \) smooth Lagrangian subtori that locally cut out \( x \). The point \( x \) being general, Lemma 4.1 and Lemma 4.3 imply that there are two such tori, call them \( L_1 \) and \( L_2 \), that intersect in an elliptic curve \( E \) at \( x \). Any other torus in \( \mathcal{B} \) passing through \( x \) either contains \( E \), or cuts out a zero-dimensional subscheme.

As a consequence of the \( L \)-separability there is a Lagrangian torus \( L_3 \) (containing the point \( x \) but not containing \( E \)) such that the intersection scheme \( L_1 \cap L_2 \cap L_3 \) is zero-dimensional at \( x \). Again invoking the fact that \( x \) is general, we may assume that the intersections \( L_1 \cap L_3 \), \( L_2 \cap L_3 \) and \( L_1 \cap L_2 \cap L_3 \) are smooth at \( x \). Hence the three Lagrangian subspaces \( W_i := T_{L_i,x} \subset T_{X,x} \) satisfy the hypotheses of Lemma 5.3. It follows that \( W_1 \cap W_2 \cap W_3 \neq \{0\} \), contradicting our choice of \( L_1 \), \( L_2 \), \( L_3 \). Therefore \( X \) cannot be \( L \)-separable. \( \square \)

**Proof of Theorem 5.1.** If \( X \) is not projective, we are done by [2], Theorem 4.1. Hence we can assume \( X \) to be projective. By Proposition 5.5, \( X \) is not \( L \)-separable and hence admits an almost-holomorphic Lagrangian fibration \( f: X \rightarrow B \) by Theorem 3.5. It remains to show that the existence of an almost-holomorphic Lagrangian fibration implies the existence of a holomorphic one. This is done in the following lemma. \( \square \)

**Lemma 5.6.** Let \( f: X \rightarrow B \) be an almost-holomorphic Lagrangian fibration on a projective hyperkähler fourfold. Then there is a birational modification \( \psi: B \rightarrow B' \) such that \( \psi \circ f: X \rightarrow B' \) is a holomorphic Lagrangian fibration.

The proof of Lemma 5.6 rests on an explicit knowledge of the birational geometry of hyperkähler fourfolds. For this we recall the notion of a Mukai flop. Assume that
a hyperkähler fourfold $X$ contains a smooth subvariety $P \cong \mathbb{P}^2$. If we blow up $P$, then the exceptional divisor is isomorphic to the projective bundle $\mathbb{P}(\Omega^1_{\mathbb{P}^2})$, and it is well known that it can be blown down in the other direction to yield another hyperkähler manifold $X'$. The resulting birational transformation $X \dashrightarrow X'$ is called the Mukai flop at $P$.

**Proof of Lemma 5.6.** By [2], 6.2, there is a holomorphic model for $f$, that is, a Lagrangian fibration $f': X' \rightarrow B'$ on a possibly different hyperkähler manifold $X'$ and a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi} & X' \\
\downarrow_{f} & & \downarrow_{f'} \\
B & \xrightarrow{\psi} & B'
\end{array}
$$

with birational horizontal arrows such that $\varphi$ is an isomorphism near a general fibre of $f$.

We claim that the composite $f' \circ \varphi = \psi \circ f$ is holomorphic and, therefore, is a Lagrangian fibration on $X$. To see this, we first note that by [12], Theorem 1.2, the map $\varphi$ factors as a finite composition of Mukai flops. Hence, by induction, we can assume that $\varphi^{-1}$ is the simultaneous Mukai flop of a disjoint union of embedded parallel projective planes $\mathbb{P}^2 \cong P_i \subset X'$.

As $\varphi^{-1}$ is holomorphic near a general fibre of $f'$, none of the $P_i$ can intersect a general fibre. Thus $f'(P_i)$ is a proper subset of $B'$ and, therefore, is of dimension at most 1. Since there are no non-constant maps from $\mathbb{P}^2$ to a curve, $f'(P_i)$ is a single point. In other words, the locus of indeterminacy of $\varphi^{-1}$ is contained in the fibres of $f'$, and thus the composite $f' \circ \varphi$ remains holomorphic. □

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