On jumps stochastic slowly diffusion equations with fast oscillation coefficients

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Abstract : We present a large deviation principle for some stochastic evolution equations with jumps which depend on two small parameters, when the viscosity parameter $\varepsilon$ tends to zero more quickly than the homogenization’s one $\delta_{\varepsilon}$ (written as a function of $\varepsilon$). In particular, we highlighted a large deviation principle in path-space using some classical techniques and a uniform upper bound for the characteristic function of a Feller process, in the following sense:

$$\lim_{\varepsilon \to 0} \frac{\delta_{\varepsilon}}{\varepsilon} = +\infty.$$

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1 Introduction

Let $\varepsilon, \delta_{\varepsilon} > 0$, we consider a diffusion which allows jumps processes in $\mathbb{R}^d$ satisfying the stochastic differential equations (SDE) :

$$X_{t_{\varepsilon, \delta_{\varepsilon}}}(t) = x - \sqrt{\varepsilon} \int_0^t \sigma \left( \frac{X_{s_{\varepsilon, \delta_{\varepsilon}}}(s)}{\delta_{\varepsilon}} \right) dW_s + \frac{\varepsilon}{\delta_{\varepsilon}} \int_0^t b \left( \frac{X_{s_{\varepsilon, \delta_{\varepsilon}}}(s)}{\delta_{\varepsilon}} \right) ds + \int_0^t c \left( \frac{X_{s_{\varepsilon, \delta_{\varepsilon}}}(s)}{\delta_{\varepsilon}} \right) ds + L_{t_{\varepsilon, \delta_{\varepsilon}}}, \quad x \in \mathbb{R}^d,$$

(1.1)

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where \( \{W_t : t \geq 0\} \) is a \( d \)-dimensional standard Brownian motion and \( L^{\varepsilon, \delta} := \left\{ L^{\varepsilon, \delta}_t : t \geq 0 \right\} \) is a Poisson point process with continuous compensator, independent of \( W \), both defined on a given filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})\) with \( \mathbb{F} := \{ \mathcal{F}_t : t \geq 0 \} \) being the \( \mathbb{P} \)-completion of the filtration \( \mathcal{F} \). More precisely, we assume that \( L^{\varepsilon, \delta} \) takes the form:

\[
L^{\varepsilon, \delta}_t := \int_0^t \int_{\mathbb{R}^d} k \left( \frac{X^{\varepsilon, \delta}_s - y}{\delta \varepsilon} \right) \left( \varepsilon N^{\varepsilon^{-1}}(dsdy) - \nu(dy)ds \right), \quad t \geq 0 \tag{1.2}
\]

where \( k \) is a given predictable function (see [6], Chap.IV Sect.9), and \( N \) is a Poisson random counting measure on \( \mathbb{R}^d \) with mean Lévy measure (or intensity measure) \( \nu \). Our assumptions on the coefficients \( b, c, \sigma \) will be specified below.

The main purpose of this paper is to show that under suitable assumptions on the coefficients and the two parameters \((\varepsilon, \delta)\), the process \( X^{\varepsilon, \delta} \) subject to \( \varepsilon \) (viscosity parameter) and \( \delta \) (homogenization parameter) is a classical problem which goes back to Paolo Baldi [2] at the end of 20'th century. In the case when \( L^{\varepsilon, \delta} \equiv 0 \), the SDE (1.1) becomes one driven by an (additive) Brownian motion and the similar issues was extensively investigated by Freidlin and Sowers [5]. They have shown three classical regimes depending on the relative rate at which the small viscosity coefficient \( \varepsilon \) and the homogenization parameter \( \delta \) tend to zero. They have provided some effective rate functions associated to an LDP for SDE which have been used as direct applications to wavefront propagation. Subsequently, large deviations problems were studied extensively by many researchers; see, for example, the work of Schilder for LDP of Brownian motions, Sanov for LDP of ergodic processes, and Freidlin and Wentzell for LDP of diffusions. Be that as it may, there are still few results on LDP for stochastic evolution equations with jumps (see, for example [12, 14, 15]).

The main difficulty in the study of SDE (1.1), however, is the presence of the jumps. As \( \varepsilon \) and \( \delta \) tend to zero, two well-known effects come into play. The effect of the viscosity parameter \( \varepsilon \) generates small excitations, hence the equation (1.1) is called slowly diffusion, and the effect of the homogenization parameter \( \delta \) makes the coefficients with fast oscillations. In previous publications, we obtained several results on the LDP of equation (1.1). In particular we discussed

- the case where the homogenization parameter \( \delta \) tends to zero much faster than the viscosity’s one \( \varepsilon \) (see, [10]);
- the situation in which the two parameters go at the same rate (see, [11]).

In this paper, we extends our results to the case that \( \lim_{\varepsilon \to 0} \delta \varepsilon \) is infinite. That is \( \varepsilon \) tends to zero sufficiently quickly compared to \( \delta \). To do so, we should first treat \( \delta \varepsilon \) as fixed and carry out the calculations for slowly varying \( \varepsilon \), then the theory of
large deviation tells us how quickly $X^{x,\varepsilon,\delta}$ tends to the deterministic dynamics given by actually setting $\varepsilon$ to zero.

Our aim consists in computing the limit of $\varepsilon \log \mathbb{P} \{ X^{x,\varepsilon,\delta} \in A \}$ when $\varepsilon$ and $\delta$ approach zero, where $A$ is a Borel subset of $D_x ([0,T], \mathbb{R}^d)$, the set of càdlàg functions on $[0,T]$ with $\mathbb{R}^d$-values which take $x$ at zero. In other word, we study the LDP for $X^{x,\varepsilon,\delta}$, and prove that there exists $I_{0,T} : D_x ([0,T], \mathbb{R}^d) \to [0, +\infty]$ such that

- for each open set $F \subseteq D_x ([0,T], \mathbb{R}^d)$
  $$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P} \{ X^{x,\varepsilon,\delta} \in F \} \geq - \inf_{\phi \in F} I_{0,T}(\phi),$$

- for each closed set $G \subseteq D_x ([0,T], \mathbb{R}^d)$
  $$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P} \{ X^{x,\varepsilon,\delta} \in G \} \leq - \inf_{\phi \in G} I_{0,T}(\phi).$$

The rest of this paper is organized as follows. In Section 2 we set up some notation, make precisely our hypothesis and state the main result. In Section 3 we give the proofs of lower and upper bounds on LDP.

2 The main results: Preliminaries and formulation

2.1 Notation and background

Denote expectation with respect to $\mathbb{P}$ by $\mathbb{E}$ and the gradient operator by $\nabla$. We have already defined $(\cdot, \cdot)$ as the standard Euclidean inner product on $\mathbb{R}^d$, and $\| \cdot \|$ as the associated norm. Let $C_p (\mathbb{R}^d, \mathbb{R}^d)$ be the collection of continuous mapping from $\mathbb{R}^d$ into $\mathbb{R}^d$ which are periodic of period 1 in each coordinate of the argument and let $\| \cdot \|_{C_p(\mathbb{R}^d, \mathbb{R}^d)}$ be the associated sup norm. Let $D ([0,T], \mathbb{R}^d)$ be the space of functions that map $[0,T]$ into $\mathbb{R}^d$, which are right continuous and having left hand limits. $D ([0,T], \mathbb{R}^d)$ is metricated by the Skorohod metric, with respect to which it is complete and separable (for a definition and properties of this space see Chap.3, [1]). We will write $C^\infty_c (\mathbb{R}^d)$ for space of test functions.

The $\{ \sigma_i : 1 \leq i \leq d \}$ in (1.1) are assumed to be in $C_p (\mathbb{R}^d, \mathbb{R}^d)$, and we also assume that

$$\kappa := \inf \left\{ \sum_{i=1}^d \langle \theta, \sigma_i(x) \rangle^2 : x \in \mathbb{R}^d, \theta \in \mathbb{R}^d, \| \theta \| = 1 \right\} > 0. \quad (2.1)$$

We assume that $b, c$ in (1.1) are in $C_p (\mathbb{R}^d, \mathbb{R}^d)$. 

We now turn our attention to the Poisson part. We first consider a Poisson random measure \( N^{-1}_\varepsilon(\cdot, \cdot) \) on \([0, T] \times \mathbb{R}^d\) defined on the space probability \((\Omega, \mathcal{F}, \mathbb{P})\), with Lévy measure \( \varepsilon^{-1} \nu \) such that the standard integrability condition holds:

\[
\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |y|^2) \nu(dy) < +\infty.
\] (2.2)

The compensator of \( \varepsilon N^{-1}_\varepsilon \) is thus the deterministic measure \( \varepsilon N^{-1}_\varepsilon(dtdy) := dt\nu(dy) \) on \([0, T] \times \mathbb{R}^d\). In this paper we shall be interested in Poisson point process of class (QL), namely a point process whose counting measure has continuous compensator (see Ikeda and Watanabe \([6]\)). More precisely, in light of the representation theorem of the Poisson point process (\([6]\), Chap. II, Theorem 7.4), we shall assume that \( L^{\varepsilon, \delta} \) is a pure jump process of the following form:

\[
L^{\varepsilon, \delta}_t := \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} k\left( \frac{x}{\delta}, y \right)(s) \left( \varepsilon N^{-1}_\varepsilon(dsdy) - \nu(dy)ds \right), \quad t \geq 0,
\]

where \( k \) is \( C_p(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d) \) with respect to first variable, integrable with respect to \( dt dy \), so that \( L^{\varepsilon, \delta} \) has continuous statistic.

The Markov processes \( X^{\varepsilon, \delta} \) that we consider include jump processes and diffusion. Next let’s write down its generator on twice continuously differentiable functions with compact support by

\[
\mathcal{L}_{\varepsilon, \delta} \phi(x) :=
\frac{\varepsilon}{2} \sum_{i,j=1}^d a_{ij} \left( \frac{x}{\delta} \right) \frac{\partial^2 \phi(x)}{\partial x_i \partial x_j} + \varepsilon \sum_{i=1}^d b_i \left( \frac{x}{\delta} \right) \frac{\partial \phi(x)}{\partial x_i} + \sum_{i=1}^d c_i \left( \frac{x}{\delta} \right) \frac{\partial \phi(x)}{\partial x_i} +
\frac{1}{\varepsilon} \int_{\mathbb{R}^d} \left[ \phi \left( x + \varepsilon k \left( \frac{x}{\delta}, y \right) \right) - \phi(x) - \varepsilon \sum_{i=1}^d k_i \left( \frac{x}{\delta}, y \right) \frac{\partial \phi(x)}{\partial x_i} \right] \nu(dy),
\] (2.3)

where the matrix \( a := (a_{ij}) \) is factored as \( a := \sigma \sigma^* \), and \( * \) denotes the transpose. The following hypotheses are required:

**H.1** (Main hypothesis) \( \lim_{\varepsilon \to 0} \frac{\delta}{\varepsilon} = +\infty. \)

\[
\begin{align*}
\text{There exists } C_1 > 0 \text{ such that for any } &\zeta := \sigma_i, b, c, \ 1 \leq i \leq d, \text{ and } k : \\
i)\ |\zeta(x') - \zeta(x)| + \int_{\mathbb{R}^d} |k(x', y) - k(x, y)| \nu(dy) \leq C_1 \|x' - x\|, &\forall x', x \in \mathbb{R}^d.
\end{align*}
\]

**H.2**

\[
\begin{align*}
\text{There exists } C_2 > 0 \text{ such that for any } &\zeta := \sigma_i, b, c, \ 1 \leq i \leq d, \text{ and } k : \\
ii)\ |\zeta(x)|^2 + \int_{\mathbb{R}^d} |k(x, y)|^2 \nu(dy) \leq C_2 \left( 1 + \|x\|^2 \right), &\forall x \in \mathbb{R}^d.
\end{align*}
\]

The proof of the Proposition 3.1 (below) uses the following Girsanov’s formula. Before proceeding, let us introduce some space.
Let $H^2(T, \lambda)$ be the linear space of all equivalence classes of mappings $F : [0, T] \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$ which coincide almost everywhere with respect to $dt \otimes d\lambda \otimes dP$ and which satisfy the following conditions:

- $F$ is predictable;
- $\int_0^T \int_{\mathbb{R}^d \setminus \{0\}} \mathbb{E} \left( |F(t, z)|^2 \right) dt \lambda(dz) < +\infty$.

We endow $H^2(T, \lambda)$ with the inner product

$$\langle F, G \rangle_{T, \lambda} := \int_0^T \int_{\mathbb{R}^d \setminus \{0\}} \mathbb{E} \left( F(t, z) G(t, z) \right) dt \lambda(dz).$$

Then, it is well known that $(H^2(T, \lambda); \langle \cdot, \cdot \rangle_{T, \lambda})$ is a real separable Hilbert space.

Let $N_p$ be a Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R}^d \setminus \{0\})$ with intensity measure $\lambda$, according to a given $F_T$-adapted, $\sigma$-finite point process $p$ which is independent of the Brownian motion $W$. Let $\tilde{N}_p$ be the associated compensated Poisson random measure. Now we have (see, D. Applebaum [1] Chapter 5, Section 2)

**Lemma 2.1 (Girsanov’s formula).**

Let $X$ be a Lévy process such that $e^X$ is a martingale, i.e:

$$X_t = \int_0^t b(s)ds + \int_0^t \sigma(s)dW_s + \int_0^t \int_{E} H(s, z)\tilde{N}_p(dsdz) + \int_0^t \int_{E} K(s, z)N_p(dsdz),$$

with

$$b(t) = -\frac{1}{2}\sigma^2(t) - \int_{E} \left( e^{H(t, z)} - 1 - H(t, z) \right) \lambda(dz) - \int_{E} \left( e^{K(t, z)} - 1 \right) \lambda(dz), \ P-a.s.$$

We suppose that there exists $C > 0$ such that

$$|K(t, z)| \leq C, \ \forall t \geq 0, \forall z \in E.$$

For $L \in H^2(T, \lambda)$ we define

$$M_t := \int_0^t \int_{z \neq 0} L(s, z)\tilde{N}(dsdz).$$

Set

$$U(t, z) = \left( e^{H(t, z)} - 1 \right) 1_{\{\|z\| \leq 1\}} + \left( e^{K(t, z)} - 1 \right) 1_{\{\|z\| > 1\}}$$

and we suppose that

$$\int_0^T \int_{\{\|z\| \leq 1\}} \left( e^{H(s, z)} - 1 \right)^2 \lambda(dz)ds < +\infty.$$
Finally, we define

\[ B_t = W_t - \int_0^t \sigma(s) ds \quad \text{and} \quad N_t = M_t - \int_0^t \int_{s \neq 0} L(s, z) U(s, z) \lambda(dz) ds, \ 0 \leq t \leq T. \]

Let \( Q \) be the probability measure on \((\Omega, \mathcal{F}_T)\) defined as:

\[ \frac{dQ}{dP} := e^{X_T}. \]

Then under \( Q \), \( B_t \) is a Brownian motion and \( N_t \) is a \( Q \)-martingale.

The following remark will be useful in the proof of Proposition 3.1.

**Remark 2.2.** It is well known that if \( N \) is a Poisson process with intensity \( \lambda(s) \) and with compensated martingale associated \( M \), and if \( g \) is a \([-1, +\infty]\)-bounded Borel function then the following is a martingale,

\[ X_t := \exp \left( \int_0^t \log (1 + g(s)) \, dM_s - \int_0^t \log (1 + g(s)) - g(s) \lambda(s) \right). \]

The proof of the Proposition 3.1 (below) uses an analytic approach and appeals to the classical Hartman-Wintner condition (see, Lemma 2.3).

In order to simplify the further exposition, we briefly outline the method which we use to bound the density function \( p(t, x, y) \) with respect to Lebesgue measure. Let \((T_t)_{t \geq 0}\) be a semigroup on \( C^\infty(\mathbb{R}^d) \) the space of continuous functions vanishing at infinity. We define its (infinitesimal) generator \( L \) as follows:

\[ L \phi := \lim_{t \to 0} T_t \phi - \phi \]

for all \( \phi \in D(L) \).

Here \( D(L) \) is the set of all \( \phi \) on \( C^\infty(\mathbb{R}^d) \) for which the limit exists in a strong sense, i.e., with respect to the sup-norm. A useful approach to the study of the generator in case it is a pseudo-differential operator is to study its symbol. Let \( L \) be a generator of a Feller process with \( C^\infty_c(\mathbb{R}^d) \subset D(L) \). Then the restriction of \( L \) on \( C^\infty_c(\mathbb{R}^d) \) is a pseudo-differential operator, given by

\[ L \phi(x) := - \int e^{i(x, \xi)} q(x, \xi) \hat{\phi}(\xi) d\xi \quad \text{for} \quad \phi \in C^\infty_c(\mathbb{R}^d), \quad (2.4) \]

where \( \hat{\phi} \) denotes the Fourier transform of \( \phi \) described by

\[ \hat{\phi}(\xi) = \frac{1}{(2\pi)^{d/2}} \int e^{-i(x, \xi)} \phi(x) dx, \quad \xi \in \mathbb{R}^d. \]

The function \( q(., .) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C} \) is called the symbol of the operator \( L \). The symbol \( q(x, \xi) \) is locally bounded in \((x, \xi)\), measurable as a function of \( x \), and for
every fixed $x \in \mathbb{R}^d$ it is a continuous negative definite function in the co-variable. That is, it is given by a Lévy-Khintchine formula of the form

\[
q(x, \xi) := c(x) - i \langle d(x), \xi \rangle + \frac{1}{2} \langle \xi, Q(x) \xi \rangle + \int_{z \neq 0} \left(1 - e^{i \langle z, \xi \rangle} + i \langle z, \xi \rangle 1_{\{\|z\| \leq 1\}}\right) \lambda(x, dz),
\]

with $c \geq 0$, $d \in \mathbb{R}^d$, $Q \in \mathbb{R}^{d \times d}$ a positive-semidefinite symmetric matrix and $\lambda$ a non-negative, $\sigma$-finite kernel on $\mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d \{0\})$. A detailed exposition of the use of the symbol in the study of Markov processes can be found in [7, 8, 9].

It is well known that Feller generators are variable coefficient Lévy-type operators and once we fix $x$ then $-p$ is the generator of a Lévy process. Notice that $p$ is no longer the characteristic exponent of the Feller process $(Y_t)_{t \geq 0}$, that is, the formula $E \left( e^{i \langle \xi, Y_t - x \rangle} \right) = e^{-tq(x, \xi)}$ is, in general, an inaccurate result. However, it is natural to expect that

\[
E \left( e^{i \langle \xi, Y_t - x \rangle} \right) \approx e^{-tq(x, \xi)}.
\]

The following plays a pivotal role in the sequel, because it allows us to show the existence of a transition density of a Feller process and to link it explicitly in terms of symbol (see, for instance [13]).

**Lemma 2.3 (Existence of density).**

Let $\left((Z_t, t \geq 0), \mathbb{P}\right)$ be a Feller process with generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$, such that $C^\infty_c(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{L})$. Then $\mathcal{L}|_{C^\infty_c(\mathbb{R}^d)} = -q(\cdot, D)$ is a pseudo-differential operator with symbol $q$.

Assume that $q$ satisfies the properties: for some $C > 0$ and for all $\xi \in \mathbb{R}^d$

\[
\|q(\cdot, \xi)\|_\infty \leq C \left(1 + \|\xi\|^2\right) \quad \text{and} \quad q(\cdot, 0) = 0.
\]

If moreover (Hartman-Wintner condition)

\[
\lim_{\|\xi\| \to \infty} \inf_{z \in \mathbb{R}^d} \text{Re} q(z, \xi) \log(1 + \|\xi\|) = \infty,
\]

then the process $(Z_t)_{t \geq 0}$ has a transition density $p(t, x, y)$, $t \in [0, \infty)$, $x, y \in \mathbb{R}^d$, with respect to the Lebesgue measure and the following inequality holds for $t > 0$:

\[
\sup_{x, y \in \mathbb{R}^d} p(t, x, y) \leq \int \exp \left( -\frac{t}{16} \inf_{z \in \mathbb{R}^d} \text{Re} q(z, \xi) \right) d\xi.
\]

Before finishing this section, we point out that if the canonical process is Feller under $\mathbb{P}$, so that all requirements of Lemma 2.3 are satisfied, it is a strong Feller process, that is, its semigroup maps bounded measurable functions to continuous bounded functions. In this case, the symbol can also be written as

\[
q(x, \xi) := -\lim_{t \to 0} \frac{\mathbb{E} \left( e^{iY_t - x, \xi} \right) - 1}{t}.
\]
Hence, the symbol can be probabilistically interpreted as the derivative of the characteristic function of the process (defined entirely in analytic terms), i.e.

$$
\frac{d}{dt} \lambda_t(x, \xi) \bigg|_{t=0} = -q(x, \xi) = e^{-i(x,\xi)} \mathcal{L} e^{i(x,\xi)}, \quad x, \xi \in \mathbb{R}^d,
$$

where \( \lambda_t(x, \xi) := e^{-i(x,\xi)} T_t e^{i(x,\xi)}(x) \).

## 2.2 The main results

Before proceeding, let us have some definitions.

**\((D.1)\)** \( V_L : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{H}^2(L, \nu) \rightarrow [0, +\infty) \) the energy function defined as:

\[
V_L(x, z, \phi) := V^1_L(x, z) + V^2_L(\phi)
\]

where

\[
V^1_L(x, z) := \inf_{\psi \in \mathcal{D}([0, L]; \mathbb{R}^d) \at \psi_0 = x, \psi_L = z} \frac{1}{2} \int_0^L \left\| \dot{\psi}_s - c(\psi_s) - k(\psi_s) \right\|^2_{\alpha^{-1}(\psi_s)} ds
\]

\[
V^2_L(\phi) := \inf_{\phi \in \mathcal{H}^2(L, \nu) \at \phi \geq 0} \int_0^L \int_{\mathbb{R}^d} \left( \phi \log(\phi) - \phi + 1 \right) (s, y) d\nu(dy),
\]

with the norm \( \| \theta \|_{a^{-1}} := \sqrt{\langle \theta, a^{-1} \theta \rangle} \) for all \( \theta \in \mathbb{R}^d \), and with \( k(z) := \int_{\mathbb{R}^d} k(z, y) d\nu(dy) \).

**\((D.2)\)** \( J : \mathbb{R}^2 \rightarrow [0, \infty) \) the functional given by:

\[
J(z) = \lim_{L \rightarrow +\infty} \frac{1}{L} V_L(0, Lz, 1 + \phi).
\]

The following is our main result.

**Theorem 2.4.** Fix \( x \in \mathbb{R}^d \), assume \((H.1)\) and \((H.2)\) hold. Then we have

- for each open set \( G \subseteq \mathcal{D}([0, T], \mathbb{R}^d) \)

\[
\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P} \left\{ X_{T, \varepsilon, \delta}^x \in G \right\} \geq - \inf_{\varphi \in \mathcal{D}([0, T], \mathbb{R}^d) \at \varphi(0) = x, \varphi(T) = z} \int_0^T J(\dot{\varphi}(s)) ds,
\]

- for each closed set \( F \subseteq \mathcal{D}([0, T], \mathbb{R}^d) \)

\[
\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P} \left\{ X_{T, \varepsilon, \delta}^x \in F \right\} \leq - \inf_{\varphi \in \mathcal{D}([0, T], \mathbb{R}^d) \at \varphi(0) = x, \varphi(T) = z} \int_0^T J(\dot{\varphi}(s)) ds.
\]
3 Large Deviation Principle

Before proceeding, we observe that the function $J$ is convex, hence we can show that

$$
\inf_{\varphi \in \mathcal{D}([0,T],\mathbb{R}^d)} \int_0^T J(\dot{\varphi}(s)) \, ds := T J\left( \frac{z-x}{T} \right).
$$

Next we are going to give the outline of the proof.

3.1 The lower bound

We start with the following lower bound in space.

**Proposition 3.1.** Suppose the assumptions (H.1) to (H.2) hold. For each open subset $G \subseteq \mathbb{R}^d$ we have

$$
\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\left\{ X_1^{\varepsilon,\delta_x} \in G \right\} \geq - \inf_{z \in G} J(z-x).
$$

**Proof.** Let $\psi \in \mathcal{D}([0,1],\mathbb{R}^d)$ satisfying $\psi_0 = x$, $\psi_1 = z$. Let us set

$$
\hat{X}_t^{\varepsilon,\delta_x} := \frac{1}{\sqrt{\varepsilon}} \left( X_t^{\varepsilon,\delta_x} - \psi(t) \right).
$$

Now, fix $z \in G$, $\eta > 0$ and let $\delta' \varepsilon > 0$ be small enough so that

$$
\{ z' \in \mathbb{R}^d : \| z' - z \| \leq \eta \delta' \} \subseteq G.
$$

Fix also $0 < \delta_x < \delta' \varepsilon$, then

$$
\mathbb{P}\left\{ X_1^{\varepsilon,\delta_x} \in G \right\} \geq \mathbb{P}\left\{ \| X_1^{\varepsilon,\delta_x} - z \| \leq \eta \delta_x \right\} \\
\geq \mathbb{P}\left\{ \| X_1^{\varepsilon,\delta_x} - \psi \|_{\mathcal{D}([0,1],\mathbb{R}^d)} \leq \eta \delta_x \right\} \\
\geq \mathbb{P}\left\{ \| \hat{X}_1^{\varepsilon,\delta_x} \|_{\mathcal{D}([0,1],\mathbb{R}^d)} \leq \frac{\delta_x}{\sqrt{\varepsilon} \eta} \right\} \\
:= A^{\eta}_{\varepsilon,\delta_x}.
$$
Next, define

$$
\xi(t) := \left[ \psi(t) - \frac{c}{\delta_x} \left( \frac{X_{t-}^{x,\epsilon,\delta_x}}{\delta_x} \right) - \frac{b}{\delta_x} \left( \frac{X_{t-}^{x,\epsilon,\delta_x}}{\delta_x} \right) \right] + \int_{\mathbb{R}^d} k \left( \frac{X_{t-}^{x,\epsilon,\delta_x}}{\delta_x}, y \right) \nu(dy) \times \sigma^{-1} \left( \frac{X_{t-}^{x,\epsilon,\delta_x}}{\delta_x} \right),
$$

$$
\hat{W}_t := W_t - \frac{1}{\sqrt{\epsilon}} \int_0^t \xi(s)ds,
$$

$$
\hat{N}_{\epsilon}^{-1} ([0, t], U) := \frac{1}{\epsilon} \log (1 + \phi(t, y)) \left( \epsilon N_{\epsilon}^{-1} ([0, t], U) - \nu(U) \right) - \frac{1}{\epsilon} \log (1 + \phi(t, y)) \left( \phi(s, y) 1_{\{\|y\| < 1\}} + (e^{k(z,y)} - 1) 1_{\{\|y\| \geq 1\}} \right) \nu(U),
$$

for \( z \in \mathbb{R}^d \), \( U \in B(\mathbb{R}^d) \) and \( \phi \in H^2(1, \nu) \).

By Girsanov’s formula, we introduce the measure \( \hat{P} \) on \( (\Omega, \mathcal{F}) \) defined as:

$$
d\frac{d\hat{P}}{dP} := e \left( \frac{1}{2 \epsilon} \int_0^1 \|\xi(s)\|^2 ds - \frac{1}{\epsilon} \int_0^1 \int_{\mathbb{R}^d} \left( \phi(s, y) - \log (1 + \phi(s, y)) \right) \nu(dy)ds \right)
$$

$$
\times e \left( \frac{1}{\sqrt{\epsilon}} \int_0^1 \xi(s) dW_s - \frac{1}{\epsilon} \int_0^1 \int_{\mathbb{R}^d} \left( \phi(s, y) \right) \epsilon N_{\epsilon}^{-1} (dsdy) - \nu(dy)ds \right)
$$

$$
\times e \left( - \frac{1}{\epsilon} \int_0^1 \int_{\mathbb{R}^d} k \left( \frac{X_{t-}^{x,\epsilon,\delta_x}}{\delta_x}, y \right) \epsilon N_{\epsilon}^{-1} \log (1 + \phi(s, y)) (dsdy) \right)
$$

$$
\times e \left( \frac{1}{\epsilon} \int_0^1 \int_{\mathbb{R}^d} \log (1 + \phi(s, y)) \left( e \left( \frac{X_{t-}^{x,\epsilon,\delta_x}}{\delta_x}, y \right) - 1 \right) 1_{\{\|y\| \geq 1\}} \nu(dy)ds \right).
$$

It follows that

$$
P \left( A_{\epsilon, \delta_x} \right) = \hat{E} \left\{ 1_{A_{\epsilon, \delta_x}^\eta} e \left( \frac{1}{\sqrt{\epsilon}} \int_0^1 \xi(s) dW_s - \int_0^1 \hat{N}_{\epsilon/\delta_x}^{-1} (dsdy) - M_{1}^{\epsilon, \phi} \right) \right\}
$$

$$
\times 1_{A_{\epsilon, \delta_x}^\eta} e \left( - \frac{1}{2 \epsilon} \int_0^1 \|\xi(s)\|^2 ds - \frac{1}{\epsilon} \int_0^1 \int_{\mathbb{R}^d} \hat{\Phi}(s, y, \phi) \nu(dy)ds \right) \right\}. \quad (3.1)
$$
where
\[
\hat{M}^{\varepsilon, \phi} := \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{R}^d} k \left( \frac{X^{x, \varepsilon, \delta}_s}{\delta}, y \right) \varepsilon N^{\varepsilon^{-1} \log(1 + \phi(s, y))} (dsdy),
\]
\[
\hat{\Phi}(t, y, \phi) := (1 + \phi(t, y)) \log (1 + \phi(t, y)) - \phi(t, y).
\]

Notice that
\[
A_{\eta}^{\varepsilon, \delta} \equiv \{ \sup_{0 \leq t \leq 1} \| \frac{X^{x, \varepsilon, \delta}_t}{\delta} - \psi(t) \| \leq \eta \}
\]
and on this set, we have
\[
\frac{1}{2} \int_0^1 \| \xi(s) \|^2 ds \leq \hat{V}^{\varepsilon, \delta}(1; \psi, \phi)
\]
where
\[
\hat{V}^{\varepsilon, \delta}(t, \psi, \phi) := \sup \left\{ \| \frac{X^{x, \varepsilon, \delta}_s}{\delta} - \psi(s) \| \leq \eta \right\}
\]
and with
\[
B^{\varepsilon, \delta}(z) := \frac{\varepsilon}{\delta} b(z) + c(z) - \int_{\mathbb{R}^d} k(z, y) \nu(dy).
\]
For any \( \varphi \in D([0, 1]; \mathbb{R}^d) \) with \( \| \varphi \|_{D([0, 1]; \mathbb{R}^d)} \leq \eta \), as in [5] (Young's inequality), there exists constants \( \kappa_1 \) and \( \kappa_2 > 0 \) such that for all \( \eta > 0 \),
\[
\frac{1}{2} \left\| \psi - B^{\varepsilon, \delta} \left( \frac{\psi}{\delta} + \varphi(t) \right) \right\|^2_{a^{-1} \left( \frac{\eta}{\delta^2} \right)}
\leq \frac{1}{2} (1 + \kappa_1 \eta) (1 + \eta^{-1}) \left\| \psi - c \left( \frac{\psi}{\delta} \right) + k \left( \frac{\psi}{\delta} \right) \right\|^2_{a^{-1} \left( \frac{\eta}{\delta^2} \right)} + \omega^{\eta, \hat{\eta}}_{\varepsilon, \delta},
\]
where
\[
\omega^{\eta, \hat{\eta}}_{\varepsilon, \delta} := \frac{1}{2} (1 + \kappa_1 \eta) (1 + \hat{\eta}^{-1}) \kappa_2 \sup_{z, z' \in \mathbb{R}^d} \left\{ \frac{\varepsilon}{\delta} \| b(z) \| + \| c(z) - c(z') \| \right\}
\]
\[
+ \int_{\mathbb{R}^d} \| k(z, y) - k(z', y) \| 1_{\{z \neq 0\}} \nu(dy) \right\}.
\]
From (3.1) we have
\[
F(A_{\eta}^{\varepsilon, \delta}) \geq \exp \left( - \frac{\Psi_{\varepsilon, \eta}(1; \psi, \phi)}{\varepsilon} \right) \times \hat{\mathbb{E}}_{1_{A_{\eta}^{\varepsilon, \delta}}} \left\{ \exp \left( - \frac{1}{\sqrt{\varepsilon}} |\Lambda_{\varepsilon}(W, N, \hat{N})| \right) \right\}
\]
where

\[
\Psi_{\varepsilon,\eta}(1,\psi,\phi) := \tilde{V}_{\varepsilon}^{\psi,\phi}(1,\psi,\phi) + \sup_{\phi \in \mathcal{H}^2(1,\rho):\phi > -1} \int_0^1 \int_{\mathbb{R}^d} \hat{\Phi}(s,y,\phi) \nu(dy) ds
\]

\[
\Lambda_{\varepsilon}(W,N,\tilde{N}) := \int_0^1 \xi(s)dW_s - \sqrt{\varepsilon} \int_0^1 \int_{\mathbb{R}^d} \tilde{N}_{X/\delta_{\varepsilon}}^{-1}(dsdy) - \sqrt{\varepsilon} M_1^{\psi,\phi}.
\]

As a reminder, \( \int_0^1 \int_{\mathbb{R}^d} \tilde{N}_{X/\delta_{\varepsilon}}^{-1}(dsdy) \) is a martingale under \( \hat{P} \).

Girsanov’s theorem tells us that \( \hat{P}(A_{\eta}^{\varepsilon,\delta}) \rightarrow 1 \) when \( \varepsilon,\delta \rightarrow 0 \). So, for \( \varepsilon,\delta > 0 \) sufficiently small, \( \hat{P}(A_{\eta}^{\varepsilon,\delta}) \) is positive. Thus, we have

\[
\mathbb{P}(A_{\eta}^{\varepsilon,\delta}) \geq \exp \left( -\frac{\Psi_{\varepsilon,\eta}(1,\psi,\phi)}{\varepsilon} \right) \times \hat{P}(A_{\eta}^{\varepsilon,\delta})
\]

\[
\times \mathbb{E}_{\varepsilon,\delta}^{1} \left\{ \exp -\frac{1}{\sqrt{\varepsilon}} \left| \Lambda_{\varepsilon}(W,N,\tilde{N}) \right| \right\}
\]

\[
\hat{P}(A_{\eta}^{\varepsilon,\delta}).
\]

Therefore,

\[
\mathbb{P}(A_{\eta}^{\varepsilon,\delta}) \geq \exp \left( -\frac{\Psi_{\varepsilon,\eta}(1,\psi,\phi)}{\varepsilon} \right) \times \hat{P}(A_{\eta}^{\varepsilon,\delta})
\]

\[
\times \exp \left( -\frac{1}{\sqrt{\varepsilon}} \left| \mathbb{E}_{\varepsilon,\delta}^{1} \left[ \Lambda_{\varepsilon}(W,N,\tilde{N}) \right] \right| \right)
\]

\[
\hat{P}(A_{\eta}^{\varepsilon,\delta}),
\]

Jensen’s inequality

and

\[
\mathbb{P}(A_{\eta}^{\varepsilon,\delta}) \geq \exp \left( -\frac{\Psi_{\varepsilon,\eta}(1,\psi,\phi)}{\varepsilon} \right)
\]

\[
\times \exp \left( -\frac{\kappa_{1/\kappa_3}}{\sqrt{\varepsilon}} \sqrt{2 \tilde{V}_{\eta}^{\psi,\phi}(1,\psi,\phi) + C(1 + \frac{1}{\delta_{\varepsilon}})} \right)
\]

\[
\left( \tilde{\omega}_{\varepsilon,\delta} \right)^{1/\kappa_3} \hat{P}(A_{\eta}^{\varepsilon,\delta}),
\]

Burkholder-Davis-Gundy inequality

where, similarly as in Lemma 4.4 of [5],

\[
\tilde{\omega}_{\varepsilon,\delta} := \left( \frac{2 \min \left( 1, \frac{1}{\sqrt{2\pi\varepsilon}} \kappa_4 \right)}{\sqrt{2\pi\varepsilon}} \right)^{\kappa_3} + o(1).
\]

Now, we put everything together, rescale the integral on the right of (3.2), and vary \( \psi \) (over all \( \psi \in \mathcal{D} \left([0,1],[\mathbb{R}^d] \right) \)) such that \( \psi_0 = x \) and \( \psi_1 = z \). Then, by the
Thus, for any \( A \) kernel associated with the semigroup \( \hat{L} \) operator \( D \),

\[
\mathbb{P}(A_{\varepsilon, \delta_x}) \geq \exp \left\{ -\frac{\delta_x}{\varepsilon}(1 + \kappa_1 \eta)(1 + \tilde{\eta})V_{1/\delta_x} \left( \frac{x}{\delta_x}, \frac{\tilde{x}}{\delta_x}, 1 + \phi \right) + \frac{1}{\varepsilon^2} \omega_{\varepsilon, \delta_x} \right\}
\]

\[
\times \exp \left\{ -\frac{K^{1/\kappa_3}}{\sqrt{\varepsilon}} \left[ 2(1 + \kappa_1 \eta)(1 + \tilde{\eta})\delta_x V_{1/\delta_x} \left( \frac{x}{\delta_x}, \frac{\tilde{x}}{\delta_x}, 1 + \phi \right) + C(1 + \tilde{\eta}) + \omega_{\varepsilon, \delta_x} \right] \right\}
\]

\[
\times \omega_{\varepsilon, \delta_x}.
\]

(3.7)

Thus, letting consecutively \( \varepsilon, \delta_x, \eta \) and then \( \tilde{\eta} \) tend to zero in that order, we get

\[
\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P} \left\{ X^\varepsilon_{\tau_x} \in G \right\} \geq - \inf_{z \in G} J(z - x).
\]

\[\square\]

3.2 The upper bound

We use \( \tilde{T}^\varepsilon_{t, s} : t < s \) to denote the semigroup on \( C^\infty(\mathbb{R}^d) \) generated by the operator \( L_{\varepsilon, \delta_x} \) with \( C^\infty(\mathbb{R}^d) \subset D(\hat{L}_{\varepsilon, \delta_x}) \), and let \( p^{x, \delta_x}(s - t, z, y) \) denote the heat kernel associated with the semigroup \( \tilde{T}^\varepsilon_{t, s} \) in the sense :

\[
(\tilde{T}^\varepsilon_{t, s} \psi)(z) = \int_{\mathbb{R}^d} p^{x, \delta_x}(s - t, z, y) \psi(y) dy, \ t < s, \ z \in \mathbb{R}^d, \ \psi \in C^\infty(\mathbb{R}^d).
\]

(3.8)

Thus, for any \( A \in \mathcal{B}(\mathbb{R}^d) \) the \( \sigma \)-algebra of all Borel subsets of \( \mathbb{R}^d \),

\[
\mathbb{P} \left( X^{x, \delta_x}_{t, s} \in A \right) = \int_{A/\delta_x} p^{x, \delta_x} \left( \left( \frac{\sqrt{\varepsilon}}{\delta_x} \right)^2 t, z, \frac{x}{\delta_x} \right) \, dz, \ t \geq 0.
\]

(3.9)

With our requirements, it is well known that \( X^{x, \varepsilon, \delta_x} \) is a strong Feller process. In fact, the conditions (2.6) and (2.7) hold true. Then let us define its symbol. Before continuing, let us set

\[
\hat{L}_{\varepsilon, \delta_x}^1 := \frac{1}{2} \sum_{i, j=1}^d a_{ij} \left( \frac{x}{\delta_x} \right) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \left( \frac{x}{\delta_x} \right) \frac{\partial}{\partial x_i} + \frac{\delta_x}{\varepsilon} \sum_{i=1}^d c_i \left( \frac{x}{\delta_x} \right) \frac{\partial}{\partial x_i},
\]

\[
\hat{L}_{\varepsilon, \delta_x}^2 := \int_{\mathbb{R}^d} \left\{ \sum_{i=1}^d ik_i \left( \frac{x}{\delta_x}, y \right) \xi_i + \varepsilon \sum_{i=1}^d k_i \left( \frac{x}{\delta_x}, y \right) \frac{\partial}{\partial x_i} \right. 
\]

\[
\left. + \left( e^{i \xi_k \left( \frac{x}{\delta_x}, y \right)} - 1 \right) \right\} \nu(dy)
\]

\[
\hat{L}_{\varepsilon, \delta_x}^3 := \sum_{l, r=1}^d \frac{\varepsilon}{\delta_x} a_{l, r} \left( \frac{x}{\delta_x} \right) \xi_l \frac{\partial}{\partial x_r} + i \sum_{l=1}^d \left( \frac{\varepsilon}{\delta_x} b_l + c_l \right) \left( \frac{x}{\delta_x} \right) \xi_l - \frac{1}{2} \sum_{l, r=1}^d \xi^*_l a_{l, r} \left( \frac{x}{\delta_x} \right) \xi_r.
\]
Next, we have
\[
-\mathcal{L}_{\varepsilon,\delta_x} \left( \frac{x}{\delta_x}, \xi \right) = \varepsilon \exp \left( -i \left( \frac{x}{\delta_x}, \xi \right) \right) \mathcal{L}_{\varepsilon,\delta_x} \exp \left( i \left( \frac{x}{\delta_x}, \xi \right) \right) 
\]
\[
= \left( \frac{\varepsilon}{\delta_x} \right)^2 \hat{\mathcal{L}}_{\varepsilon,\delta_x}^1 + \hat{\mathcal{L}}_{\varepsilon,\delta_x}^2 + \hat{\mathcal{L}}_{\varepsilon,\delta_x}^3 
\]
(3.10)

Heuristically if \( \frac{x}{\delta_x} \rightarrow z \) when \( \varepsilon \rightarrow 0 \), it can be seen that this operator converges to \(-q\) defined as:
\[
-q(z,\xi) = -\frac{1}{2} \sum_{l,r=1}^{d} \xi_l a_{l,r}(z) \xi_r + i \sum_{r=1}^{d} \left\{ c_r(z) \xi_r + \int_{\mathbb{R}^d} k_r(z,y) \xi_r \nu(dy) \right\}
\]
(3.11)
\[
+ \int_{\mathbb{R}^d \setminus \{0\}} \left( \exp(\langle k(z,y),\xi \rangle) - 1 \right) \nu(dy).
\]

Now we have the following upper bound in space.

**Proposition 3.1.** Suppose the assumptions (H.1) to (H.2) hold true. for each closed subset \( F \subseteq \mathbb{R}^d \)
\[
\limsup_{\varepsilon \to 0} \varepsilon \log P \left\{ X_{1,x,\varepsilon,\delta_x} \in F \right\} \leq - \inf_{z \in F} J(z - x).
\]

**Proof.** Before proceeding, let \( \hat{Q}_1 \) be the quadratic form defined as \( \hat{Q}_1(v) := \langle v, av \rangle \) and let \( \hat{Q}_1^* \) be the conjugate quadratic form of \( \hat{Q}_1 \) defined as:
\[
\hat{Q}_1^*(v) := \sup_{t \in \mathbb{R}^d} \left\{ 2 \langle t, v \rangle - \hat{Q}_1(t) \right\}.
\]
It is well known that if the inverse of the matrix \( a \) exists, then
\[
\hat{Q}_1^*(v) := \langle v, a^{-1}v \rangle.
\]

Next, fix \( \theta \in \mathbb{R}^d \) and let \( \hat{Q}_2^*(v,\theta) = \sup_{t \in \mathbb{R}^d} \left\{ \langle t, v \rangle - \left[ \exp(\langle \theta, t \rangle) - 1 \right] \right\} \). Then, it is easy to see that
\[
\hat{Q}_2(v,\theta) = \hat{Q}_2(\|v\| \times \|\theta\|^{-1}) \quad \text{with} \quad \hat{Q}_2(x) := x \log x - x + 1.
\]
Now, fix \(x, z \in \mathbb{R}^d\) and \(t > 0\), Lemma 2.3 tells us

\[
\sup_{x, z \in \mathbb{R}^d} p^{x, \delta_x} (t, x, z) \leq \int \exp \left( -\frac{1}{16} \inf_{\psi \in \mathcal{D}([0,t]; \mathbb{R}^d)} \int_0^t \text{Re}^e \left( \frac{\psi_s}{\delta_x} \right) ds \right) d\xi \\
\leq \int \exp \left( \frac{1}{2} \inf_{\psi \in \mathcal{D}([0,t]; \mathbb{R}^d)} \int_0^t \left\langle \xi, a \left( \frac{\psi_s}{\delta_x} \right) \xi \right\rangle ds \right) \\
+ \inf_{\psi \in \mathcal{D}([0,t]; \mathbb{R}^d)} \sup_{\psi_0 = x, \psi_1 = z} \int_0^t \int_{\mathbb{R}^d} \text{Re} \left\{ 1 - e \left( i \left( k \left( \frac{\psi_s}{\delta_x}, y \right), \xi \right) \right) \right\} dsdy \right) + o(1).
\]

Thus

\[
\sup_{x, z \in \mathbb{R}^d} p^{x, \delta_x} (t, x, z) \\
\leq \exp \left( -\frac{1}{2} \inf_{\psi \in \mathcal{D}([0,t]; \mathbb{R}^d)} \int_0^t \hat{Q}_1^e \left( \frac{\psi_s}{\delta_x} \right) \left\{ c + \int_{\mathbb{R}^d} k(\cdot, y)\nu(dy) \right\} \left( \frac{\psi_s}{\delta_x} \right) ds \right) \\
\times \exp \left( \inf_{\psi \in \mathcal{D}([0,t]; \mathbb{R}^d)} \sup_{\xi \in \mathbb{R}^d} \int_0^t \left\langle \psi_s - \frac{\psi_s}{\delta_x}, \xi \right\rangle \left\{ c + \int_{\mathbb{R}^d} k(\cdot, y)\nu(dy) \right\} \left( \frac{\psi_s}{\delta_x} \right) \xi ds \right) \\
\times \int \exp \left( \inf_{\psi \in \mathcal{D}([0,t]; \mathbb{R}^d)} \int_0^t \int_{\mathbb{R}^d} \text{Re} \left\{ 1 - e \left( i \left( k \left( \frac{\psi_s}{\delta_x}, y \right), \xi \right) \right) \right\} dsdy \right) d\xi \right) + o(1).
\]

Thereupon, we have

\[
\sup_{x, z \in \mathbb{R}^d} p^{x, \delta_x} (t, x, z) \\
\leq \exp \left( -\frac{1}{2} \inf_{\psi \in \mathcal{D}([0,t]; \mathbb{R}^d)} \int_0^t \hat{Q}_1^e \left( \frac{\psi_s}{\delta_x} \right) \left\{ c + \int_{\mathbb{R}^d} k(\cdot, y)\nu(dy) \right\} \left( \frac{\psi_s}{\delta_x} \right) ds \right) \\
\times K \exp \left( -\inf_{\psi \in \mathcal{D}([0,t]; \mathbb{R}^d)} \int_0^t \hat{Q}_2^e \left( \psi_s, \int_{\mathbb{R}^d} k \left( \frac{\psi_s}{\delta_x}, y \right) dy \right) ds \\
+ \frac{\delta_x}{\epsilon} \sup_{\xi \in \mathbb{R}^d} \int_0^t \left\{ c + \int_{\mathbb{R}^d} k(\cdot, y)\nu(dy) \right\} \left( \frac{\psi_s}{\delta_x} \right) ds \right) + o(1).
\]
This follows

\[
\sup_{x,z \in \mathbb{R}^d} p^{\varepsilon, \delta_x}(t, \frac{x}{\delta_x}, \frac{z}{\delta_x}) \leq \exp \left( -\frac{1}{2} \inf_{\psi \in \mathcal{D}([0,t]; \mathbb{R}^d)} \int_0^t \dot{Q}_1^s \left( \frac{\psi}{\delta_s} - \frac{\delta_s}{\epsilon} \left\{ c + \int_{\mathbb{R}^d} k(., y)\nu(dy) \right\} \left( \frac{\psi}{\delta_s} \right) \right) ds \right)
\times K \exp \left( -\inf_{\psi \in \mathcal{D}([0,t]; \mathbb{R}^d)} \int_0^t \dot{Q}_2^s \left( \frac{\psi}{\delta_s}, \int_{\mathbb{R}^d} k \left( \frac{\psi}{\delta_s}, y \right) dy \right) ds \right.
\left. + \frac{\delta_s}{\epsilon} \sup_{\xi \in \mathbb{R}^d} \int_0^t \left\{ c + \int_{\mathbb{R}^d} k(., y)\nu(dy) \right\} \left( \frac{\psi}{\delta_s} \right) \left( \frac{\psi}{\delta_s}, \xi \right) ds \right) + o(1).
\]

(3.12)

From (3.9) we have

\[
\mathbb{P} \left( X_t^{\varepsilon, \delta_x} \in F \right) = \int_{F/\delta_x} p^{\varepsilon, \delta_x} \left( \left( \sqrt{\varepsilon}/\delta_x \right)^2, \frac{x}{\delta_x}, z \right) dz
= \delta_x^{-d} \int_{F} p^{\varepsilon, \delta_x} \left( \left( \sqrt{\varepsilon}/\delta_x \right)^2, \frac{x}{\delta_x}, \frac{z}{\delta_x} \right) dz.
\]

(3.13)

by scaling property
By (3.12), we deduce

\[ \varepsilon \log P \left( X^x, \varepsilon, \delta_\varepsilon \in F \right) \]

\[ \leq - \frac{\varepsilon}{2} \inf_{\psi \in D \left( [0, T], \mathbb{R}^d \right)} \int_0^T \hat{Q}^1 \left( \frac{\psi_s}{\delta_\varepsilon} - \frac{\delta_\varepsilon}{\varepsilon} \left\{ c + \int_{\mathbb{R}^d} k(., y) \nu(dy) \right\} \left( \frac{\psi_s}{\delta_\varepsilon} \right) \right) ds \]

\[ - \varepsilon \inf_{\psi \in D \left( [0, T], \mathbb{R}^d \right)} \left\{ \int_0^T \hat{Q}^2 \left( \left( \frac{\psi_s}{\delta_\varepsilon} \right), \int_{\mathbb{R}^d} k \left( \frac{\psi_s}{\delta_\varepsilon}, y \right) dy \right) ds \right\} \]

\[ + \frac{\delta_\varepsilon}{\varepsilon} \sup_{\xi \in \mathbb{R}^d} \left\{ c + \int_{\mathbb{R}^d} k(., y) \nu(dy) \right\} \left( \frac{\psi_s}{\delta_\varepsilon} \right) \xi ds \]

\[ \leq - \frac{\delta_\varepsilon}{2} \inf_{\psi \in D \left( [0, T], \mathbb{R}^d \right)} \int_0^T \hat{Q}^1 \left( \psi_s - \left\{ c + \int_{\mathbb{R}^d} k(., y) \nu(dy) \right\} \psi_s \right) ds \]

\[ - \inf_{\psi \in D \left( [0, T], \mathbb{R}^d \right)} \left\{ \delta_\varepsilon \int_0^T \hat{Q}^2 \left( \psi_s, \int_{\mathbb{R}^d} k(\psi_s, y) dy \right) ds \right\} \]

\[ + \varepsilon \sup_{\xi \in \mathbb{R}^d} \left\{ c + \int_{\mathbb{R}^d} k(., y) \nu(dy) \right\} \psi_s \xi ds \] + o(1).

Therefore, the claim follows, i.e.:

\[ \lim_{\varepsilon \to 0} \varepsilon \log P \left( X^x, \varepsilon, \delta_\varepsilon \in F \right) \leq - \lim_{\varepsilon \to 0} \delta_\varepsilon V^1_{1/\delta_\varepsilon} (x, z) - \lim \inf \delta_\varepsilon V^2_{1/\delta_\varepsilon} (1 + \phi) \]

\[ \leq - \inf_{z \in F} \mathcal{J}(z - x), \]

where \( \phi + 1 := \| \psi \| \times \left( \int_0^T \int_{\mathbb{R}^d} \| k(., y) \| \nu(dy) ds \right)^{-1}. \)

\[ \square \]

### 3.3 Tightness

Let \( D^\lambda ([0, T], \mathbb{R}^d) \) denotes the space of Hölder-càdlàg functions of exponent \( \lambda \) and \( \| \cdot \|_{D^\lambda([0, T], \mathbb{R}^d)} \) its corresponding norm. We need the following remark with projective limit approach (see, for example [3]) to guess at the path-space large deviations principle.

**Remark 3.2.** for any fixed \( T > 0, x \in \mathbb{R}^d \) and \( \lambda \in (0, 1/2) \),

\[ \lim_{L \to +\infty} \lim \sup_{\varepsilon \to 0} \varepsilon \log P \left\{ \| X_{t,x,\varepsilon} \|_{D^\lambda([0, T], \mathbb{R}^d)} \geq L \right\} = -\infty. \]
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