How much negative energy does a wormhole need?

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It is known that traversible wormholes require negative energy density. We here argue how much negative energy is needed for wormholes, using a local analysis which does not assume any symmetry, and in particular allows dynamic (non-stationary) but non-degenerate wormholes. We find that wormholes require two constraints on the energy density, given by two independent components of the Einstein equation.
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I. INTRODUCTION

A spacetime wormhole is usually introduced as a topological handle connecting two universes or distant places in a universe. The research of wormholes became active in the last decade since the work of Morris and Thorne [1]. A remarkable result is that, assuming Einstein gravity, the weak energy condition is violated at the wormhole throat. Though their analysis was restricted to the static, spherically symmetric case, other examples suggest that the violation of energy conditions, particularly the null energy condition, is a general property of traversible wormholes [2–12]. Since any realistic gravitational source is believed to satisfy the null energy condition at the classical level, one might look to effects of quantum field theory to maintain a wormhole. In this context, Hochberg et al. [13] have presented a self-consistent wormhole solution of semiclassical Einstein gravity. Another way is to consider alternative gravitational theories, such as higher-derivative theories [14] or the Brans-Dicke theory [15].

Recently, general frameworks for dynamic wormholes have been suggested. While in static spacetimes, a wormhole may be defined by a closed stable minimal surface embedded in the hypersurface orthogonal to the timelike Killing vector field, Hochberg and Visser [10,11] have generalized this concept for general spacetimes and defined a wormhole by a stable minimal surface on a null (lightlike) hypersurface. They have also shown that the null energy condition is violated at the wormhole throat in general. On the other hand, one of the present authors has defined a dynamic wormhole in a slightly different way by a temporal (timelike) outer trapping horizon [12]. This definition is stricter than the former since it requires the horizon to be temporal in order to be locally two-way traversible. It also gives a unified picture of wormholes and black or white holes; the only difference in the definitions is the causal nature of the horizon: black or white holes are defined by an outer trapping horizon which is spatial or null [12,11]. Thus, dynamic wormholes are defined locally, so that we need not mention the topology of the spacetime as a whole in what follows, for example two-way traversible means simply that the wormhole throat is temporal. In ref. [12], it is suggested that a stationary black hole may become a dynamic wormhole by the influx of the negative energy of the Hawking radiation.

As already mentioned above, a wormhole requires negative energy density to maintain itself, and conversely, one might naively expect that a sufficient condensation of negative energy makes wormholes. From this point of view we shall ask, whatever the source of the negative energy, how much negative energy is needed to make a wormhole, or what kind of wormholes one can make with less effort. In the static case, we can conclude that the spherically symmetric Morris-Thorne wormhole is easiest to make (Sec. II). In dynamic cases, we can obtain criteria for the existence of a wormhole in terms of energy densities (Sec. III).

II. STATIC WORMHOLES

One may almost uniquely define wormholes for static spacetimes in the strict sense, i.e. for spacetimes with a Killing vector field which is everywhere timelike, as compact orientable minimal surfaces embedded in the hypersurface orthogonal to the Killing vector field. This definition is a special case of that of dynamic wormholes as discussed above. Since it is convenient to deal with the static case separately, we briefly discuss this case here, however, results in this section are not new; for more detailed treatment of static wormholes, see ref. [16].

We consider here a closed compact oriented minimal surface \( H \) embedded in a 3-manifold \( (N, g) \). In what follows, indices run in the ranges \( 1 \leq i, j, k, l \leq 2 \) and \( 1 \leq A, B, C, \ldots \leq 3 \). Let \( \{ e_i \} \) be an orthonormal basis of \( H \), \( e_3 \) its normal, and \( \{ \theta^A, \theta^B \} \) the corresponding dual basis. The structure equations are

\[
d\theta^A + \theta^A_B \wedge \theta^B = 0
\]

\[
\frac{1}{2} R^A_{BCD} \theta^C \wedge \theta^D = d\theta^A_B + \theta^A_C \wedge \theta^C_B
\]

where \( \theta^A_B \) is the connection form and \( R^A_{BCD} \) the com-
ponent of the Riemann tensor. The second fundamental form \( h = h_{ij}\theta^i \otimes \theta^j \) of \( H \) is given by \( h_{ij} = (\theta^i, e_j) \), and its trace \( \Theta = 2^{-1}(h_{11} + h_{22}) \) is the mean curvature of \( H \).

The area of \( H \) is given by \( A = \int_H *1 \) where \( * \) denotes the Hodge operator with respect to the area form \( \theta^1 \wedge \theta^2 \) of \( H \). We slightly deform \( H \) by introducing the map \( S : H \times I \to N \), where \( I = [-1/2, 1/2] \) is an interval, such that \( S \) is an embedding into \( N \) for fixed \( t \) in \( I \) denoted by \( S_t : H \to N \), especially the inclusion map on \( H \) for \( t = 0 \in I \), i.e. \( S_0 = \text{id}|_H \). The metric of \( H \times I \) is assumed to be the pullback of \( \delta g \) by \( S \). Let \( \{ f_t, \bar{f}_t \} \) be the orthonormal basis of \( H \times I \) such that \( \{ f_t \} \) is tangent to \( S_t \) and \( S_t f_t|_{t=0} = e_i \). Then its dual may be decomposed as \( \omega^i = \theta^i + f_t dt \) and \( \omega^3 = f dt \), where \( \theta^i \) is a one-form on \( S_t \), so that now depends on \( t \). Correspondingly, the connection forms of \( H \times I \) are written as \( \omega^i_{ij} = \theta^i_{ij} + \nabla_{f_t} f dt \), where \( \nabla \) is the Riemannian connection of \( S_t \).

The variation of the area is obtained by differentiating \( *1 \) with respect to \( t \). We may assume that the deformation is normal to \( H \), i.e. \( f^1 = 0 \). The first variation of the area form is given by

\[
\frac{\partial}{\partial t} *1|_{t=0} = -*f \Theta,
\]

so that \( \Theta = 0 \) for \( H \) to be a wormhole.

On the other hand, the flare-out condition for a static wormhole may be expressed as the stability of \( H \), i.e. the positivity of the second variation of the area form \( [13] \). The second variation of the area form becomes

\[
\frac{\partial^2}{\partial t^2} *1|_{t=0} = [\nabla e_i, f \nabla e_i, f - q f^2] - d|_H (f \nabla e_i, f * \theta^i) \tag{4}
\]

\[
= - [f \Delta f + q f^2]
\]

\[
q = \text{Ric}(e_i, e_3) + h_{ij} h_{ij}
\]

where \( \text{Ric} = R^{AC}{}_{BD} \theta^A \otimes \theta^B \) denotes the Ricci tensor on \( N \), and \( \Delta = d|_H * d|_H \) the Laplacian on \( H \). We can rewrite \( q \) using the Gauss-Codazzi equation as \( 2q = (3) R - (2) R + h_{ij} h_{ij} \) where \( (3) R \) and \( (2) R \) are the scalar curvatures of \( N \) and \( H \), respectively. The 2-surface \( H \) is stable if and only if the integral of eq.(3) over \( H \) is non-negative for any function \( f \), or equivalently, the first eigenvalue of the operator \( -\Delta - q \) is positive. Visser and Hochberg [1] defined the strong flare-out condition by \( q \leq 0 \) on \( H \) which means \( \Theta^2 + 1 \leq 0 \) for \( f = 1 \), i.e. the area form of surfaces homogeneously deformed from \( H \) is not smaller than that of \( H \) at each point. This condition restricts the energy condition according to the topology of \( H \) in a similar way to the topology theorem for apparent horizon of time-symmetric initial data [16]. By integrating \( q \leq 0 \) over \( H \) using Gauss-Bonnet theorem, we obtain the inequality

\[
\int_H *(8\pi \mu + \frac{1}{2} |h|^2) \leq 2\pi \chi_e
\]

where \( \mu = G(u, u)/8\pi \) is the energy density, \( G \) is the Einstein tensor of the spacetime, \( u \) the unit vector tangent to the Killing vector), and \( \chi_e \) the Euler number of \( H \). In particular, this implies the simple relation

\[
U \leq \chi_e/4 \tag{7}
\]

in terms of the surface energy \( U = \int_H *\mu \). The inequality implies that if \( H \) is a double torus or has higher genus then \( \mu \) must be negative somewhere on \( H \), i.e. it implies a violation of the weak energy condition. If we assume \( \mu \geq 0 \) by the weak energy condition, \( H \) is a totally geodesic flat torus (\( \mu \) must vanish on \( H \) in this case) or a sphere. Even in these cases, energy conditions are violated, as is well known from consideration of a different component of the Einstein equation, e.g. [3]. The point here is that higher-genus wormholes require increasingly large amounts of negative surface energy \( U \). This fact survives in the dynamic case, as discussed in the next section. Thus, to make a wormhole, it is easier to try for a sphere with small principal curvatures, e.g. a spherically symmetric Morris-Thorne wormhole [4].

**III. DYNAMIC WORMHOLES**

Definitions of wormholes for general spacetimes have been given by Hochberg and Visser [1,11] and by one of the authors [12]. Though the basic idea is common in both definitions, i.e. the dynamic wormhole is a minimal surface of a null hypersurface, the latter definition further requires the minimal surfaces to generate a temporal hypersurface. We adopt this quasi-local definition in the following. Since this definition is slight modification of black or white hole definition, some of the results below can be obtained by applying known results for black holes, e.g. Prop.[11,12] is an application of ref. [20].

The definition is based on the 2 + 2 formalism [17], however, we here adopt the compacted spin-coefficient formalism [18]. Consider a foliation of the spacetime \( (M, g) \) by spatial two-surfaces. It is convenient to do so by double-null foliation, i.e. a pair of foliations by null hypersurfaces such that the set of intersections of each null hypersurfaces give a two-dimensional foliation. Let \( \{ \Sigma \} \), \( \{ \Sigma' \} \) be such one-parameter families of null hypersurfaces, and let \( S \) be the intersection of \( \Sigma \in \{ \Sigma \} \) and \( \Sigma' \in \{ \Sigma' \} \), then \( \{ S \} \) is a two-parameter family of spatial two-surfaces. We here set a spin basis \( \{ o, t \} \) such that \( o_{\alpha} = \chi \), and a null tetrad \( \{ D, D', \delta, \delta' \} = \{ o_{\alpha}, t_{\alpha}, o_{\alpha}, t'_{\alpha} \} \) (normalized as \( g(D, D') = -g(\delta, \delta') = \chi \)), other combinations vanish) such that \( D \) and \( D' \) are tangent to the future-directed null geodesic generators of \( \Sigma \in \{ \Sigma \} \) and \( \Sigma' \in \{ \Sigma' \} \), respectively, and that \( \{ \delta, \delta' \} \) spans the tangent space of each \( S \in \{ S \} \). Its corresponding basis of one-forms \( \{ g(D), g(D'), g(\delta), g(\delta') \} \) are denoted by \( \{ n, n', m, m' \} \), where we may take both \( n \) and \( n' \) to be exact one-forms with the freedom of \( \chi \). The spin-coefficients are defined by

\[
\nabla_{o\alpha}(o, t) = (\epsilon o - \kappa t, \gamma' t - \tau' o)
\]

(8)
\[
\n\nabla_{a\ell}(o,t) = (\gamma o - \tau t, \epsilon' - \kappa' o) \quad (9)
\]
\[
\nabla_{a\ell}(o,t) = (\beta o - \sigma t, \alpha o' - \rho o') \quad (10)
\]
\[
\nabla_{a\ell}(o,t) = (\alpha o - \rho t, \beta o - \sigma o') \quad (11)
\]

By construction, \( \kappa = \kappa' = 0 \) since \( D \) and \( D' \) are tangent to geodesics, \( \rho - \rho' = \rho' - \rho'' = 0 \) since they are orthogonal to null hypersurfaces. Moreover, \( \epsilon + \epsilon' = \tau - \alpha - \beta = 0 \) and their primed equations hold when \( n \) and \( n' \) are exact, as occurs if the spin-basis is adapted to the null hypersurface such that \( n \) and \( n' \) are the differentials of the null coordinates; this choice of gauge is discussed in ref. [13].

A trapping horizon is defined to be a one-parameter family of two-surfaces \( \{H\} \) on which one of the contracting rates, say \( \rho \), vanishes. Each \( H \) locally determines two (ingoing and outgoing) normal null hypersurfaces, so that we adopt the natural double-null foliation generated by \( \{H\} \), i.e. such that \( \{\delta, \delta'\} \) spans \( H \). The trapping horizon is called future if \( \rho' > 0 \), and past if \( \rho' < 0 \). Moreover, trapping horizons are classified according to the contracting rates of neighbouring light-cones: they are called outer if \( \nabla' \rho > 0 \) and inner if \( \nabla' \rho < 0 \). Then a black (white) hole is defined to be a outer-future (inner-past) trapping horizon which is null or spatial. Note that the notion of a black hole is invariant once a double-null foliation has been set up, since the product \( \rho' / \chi\hat{\chi} \) or \( \nabla' \rho / \chi^2 \hat{\chi} \) is invariant under the transformation of the basis: \( (o, t) \leftrightarrow (\lambda o, \mu t) \) by complex functions \( \lambda \) and \( \mu \), which preserves the double-null foliation. The condition that the trapping horizon is null or spatial in the definition of a black or a white hole is guaranteed if the null energy condition holds, i.e. the stress-energy tensor \( T \) satisfies \( T(k, k) \geq 0 \) for any null vector \( k \). To see this, introduce a vector \( z = aD + bD' \) which belongs to the orthogonal complement of \( \{\delta, \delta'\} \) in the tangent space of the outer trapping horizon. The condition “outer” requires \( D\rho > 0 \) on the horizon, while the spin-coefficient equation \( [13] \)

\[
\n\rho = \rho^2 + \sigma\hat{\sigma} + \Phi_{00} \quad (12)
\]

implies \( D\rho \geq 0 \) since \( \Phi_{00} \geq 0 \) by the null energy condition, so that \( ab \leq 0 \) since \( z \) is tangent to the horizon: \( z\rho = aD\rho + bD'\rho = 0 \), as required, which implies \( g(z, z) = 2ab \leq 0 \), i.e. the outer trapping horizon is spatial or null.

Similarly, a wormhole horizon is defined to be an outer trapping horizon which is temporal \( [13] \), however, it should be noted that we have excluded the degenerate case: \( \nabla' \rho = 0 \) for simplicity, in which case additional technical complications must be discussed \( [13] \). To see the validity of the definition, we shall prove the following lemma.

**Lemma III.1** Let \( \{H\} \) be a trapping horizon, \( \rho = 0 \), then any two of the following conditions imply the third:
(a) \( \{H\} \) is temporal;
(b) \( \{H\} \) is outer, \( \nabla' \rho > 0 \);
(c) \( D\rho < 0 \).

**Proof.** Let \( z = aD + bD' \neq 0 \) be a tangent vector of \( \{H\} \). Assume the condition (a) holds, which means \( ab > 0 \), then the equivalence of (b) and (c) comes from \( z\rho = aD\rho + bD'\rho = 0 \). Assume instead (b) and (c), then we obtain \( ab > 0 \) which implies (a).

Thus, a wormhole horizon is composed of minimal surfaces of null hypersurfaces in the sense \( D\rho < 0 \). The Hochberg-Visser wormhole definition requires (c) but not (a) or (b). By virtue of the simplicity of the definition, one may immediately obtain the negative energy theorem: \( \Phi_{00} < 0 \), i.e. the null energy condition is violated on the wormhole horizon. The energy-momentum of any reasonable classical matter, even of the cosmological constant, satisfies the null energy condition, so that it is common to appeal to quantum effects such as the Casimir effect or the Hawking radiation for the negative energy source of wormholes.

In the following, we consider how much negative energy is needed to construct a wormhole, whatever the negative energy is. We may precisely answer this question in terms of the two components of the Einstein equation which contain \( \Phi \) and \( \Phi' \). Firstly we integrate the focusing equation on the null hypersurface. In ref. \[20\], the focusing equation is integrated on a light-cone, and we take a similar procedure with a different boundary condition. Consider a null hypersurface \( \Sigma \in \{\Sigma\} \), and let \( D \) be tangent to the null geodesic generators of \( \Sigma \). Take a section \( S_0 \) of \( \Sigma \) which is assumed to be a compact twosurface without boundary, and on which the light ray is assumed to be contracting in the \( D \)-direction, i.e. \( \rho > 0 \) on \( S_0 \). One could also set such a boundary condition at past null infinity in an asymptotically flat space-time; the condition is just meant to imply that \( D \) is an ingoing direction. The spin-coefficient equation \( [12] \) may be written more explicitly as

\[
\nD\rho = (\epsilon + \epsilon')\rho + \rho^2 + \sigma\hat{\sigma} + \Phi_{00}. \quad (13)
\]

When \( D \) is tangent to an affinely parametrized geodesic, \( \epsilon \) is pure imaginary, so that the first term of the r.h.s. vanishes, which might be the most usual choice of \( D \), however, we here take a different choice of \( D \). The boost transformation

\[
\n\rho \rightarrow \rho a \quad (14)
\]

with a real number \( a \), preserves the null hypersurface \( \Sigma \). By this transformation, the spin-coefficients \( \rho \) and \( \epsilon \) transform to \( a^2 \rho \) and \( a^2 (\epsilon + D \ln a) \), respectively. Now let \( a \) be a solution of the differential equation,

\[
\n2D \ln a + \rho + \epsilon + \epsilon' = 0, \quad (15)
\]

then the sum \( \rho + \epsilon + \epsilon' \) vanishes by the boost with \( a \), so that the eq. \[13\] takes a simple form

\[
\n\frac{\partial}{\partial \mu} \rho = \sigma\hat{\sigma} + \Phi_{00}. \quad (16)
\]
where \( u \) is the new parameter of the null generator of \( \Sigma \) such that \( \partial/\partial u \) coincides with \( D \), and \( u = 0 \) on \( S_0 \) (note that caustics in the future of \( S_0 \), if they exist, are pushed into infinite \( u \) in this parametrization). Then, \( \rho \) is written in the form

\[
\rho = \rho_0 + \int_0^u (\sigma \delta + \Phi_{00}) \, du
\]

(17)

where \( \rho_0 > 0 \) is the contracting rate on \( S_0 \) in this gauge. Eq. (17) implies the existence of a minimal surface by the sufficient influx of the negative energy in the \( D \)-direction (note that the existence of a single minimal surface implies that there will be locally a trapping horizon consisting of such surfaces by continuity of the function \( \rho \)). Especially, we obtain the existence theorem of the trapping horizon:

**Proposition III.2** If \( \Phi_{00} \leq -((\sigma)^2 + \varepsilon) \), \( 0 < u < \rho_0/\varepsilon \) holds for some positive constant \( \varepsilon \), then there exists a trapping horizon.

However, this proposition does not say whether or not the trapping horizon is a wormhole horizon as defined here; we need to satisfy (a) or (b) as well as (c) above. To state this in an invariant way, we transform the basis as follows. If the assumption of the Prop. III.2 is satisfied, then this trapping horizon in general will not be spaned by \( \{ \delta, \bar{\delta} \} \), but we are able to make \( \{ \delta, \bar{\delta} \} \) span the trapping horizon by the transformation:

\[
u \mapsto \nu + E_0
\]

(18)

with an appropriate complex function \( E \). Note that all the quantities used in the Prop. III.3 are unchanged by this transformation, though the new tetrad need not be associated with the original double-null foliation. In this setting, we state the following lemma.

**Lemma III.3** Let \( H \) be the trapping horizon in the Prop. III.2, and let \( k \) be the Gaussian curvature of \( H \), then \( H \) is a wormhole if and only if the inequality

\[
(\Phi_{11} + 3\Pi) < 2^{-1}\chi\bar{\chi}k - \tau\bar{\tau} - \mathfrak{R}[\partial'^2]\tau
\]

holds on \( H \).

**Proof.** It is easily seen that \( \partial' \rho < 0 \) holds on \( H \) in the situation of the Prop. III.2. On the other hand, the real part of the following spin-coefficient equation [18]

\[
\partial'\rho - \partial\tau = \rho\partial' + \sigma\partial' - \tau\delta - \Phi_2 - 2\Pi
\]

(20)

can be rewritten as

\[
\partial'\rho = 2^{-1}\chi\bar{\chi}k - \tau\bar{\tau} - \mathfrak{R}[\partial'^2]\tau - (\Phi_{11} + 3\Pi)
\]

(21)

on \( H \), since \( \rho = 0 \) on \( H \), and the sectional curvature \( k \) of the distribution \( \{ \delta, \delta' \} \) can be in general written as

\[
k = 2\mathfrak{H}[\sigma\partial' - \rho\partial' - \Phi_2 + \Pi + \Phi_{11}]/\chi\bar{\chi}.
\]

Hence eq. (19) is equivalent to \( \partial'\rho > 0 \) on \( H \), and moreover, \( \{ H \} \) is temporal by the lemma [III.1].

Conversely, the inequality (19) must be satisfied on any wormhole horizon. Especially, by integrating the inequality (19) divided by \( \chi\bar{\chi} \) over the two-surface \( H \) spaned by \( \{ \delta, \delta' \} \) with the area element \( *1 = (i/\chi\bar{\chi})\mathbf{m} \wedge \mathbf{m}' \), we obtain an additional constraint

\[
W < \chi_e/4
\]

(22)

in terms of the surface energy

\[
W = \frac{1}{4\pi} \int_H \Phi_{11} + 3\Pi
\]

(23)

and the Euler number \( \chi_e \) of \( H \); \( \chi_e = 2(1 - g) \) where \( g \) is the genus (number of handles). This follows from the Gauss-Bonnet theorem

\[
\int_H *k = 2\pi\chi_e
\]

(24)

and the fact that \( \mathfrak{R}[\partial'^2]\tau/\chi\bar{\chi} \) is a total divergence. Thus, the energy density \( (\Phi_{11} + 3\Pi)/4\pi \) must be negative somewhere on \( H \), and the total surface energy \( W \) must be negative, unless \( H \) has spherical topology. This is a dynamic version of the static inequality [19]. Note that equality has been excluded here by assuming the strict inequalities (b) and (c) in the definition of wormhole, i.e. excluding degenerate wormholes. Degenerate cases may be included, as for the topology law of black or white holes [14,19], which involves the same method as above.

Inequality (22) can be sharpened in terms of the surface gravity [21,22]

\[
\kappa_g = \frac{1}{4\pi r} \int_H * k = \frac{\partial'\rho}{\chi\bar{\chi}}
\]

(25)

where \( r = \sqrt{A/4\pi} \) in terms of the area \( A = \int_H *1 \) of \( H \). Then

\[
W \leq \chi_e/4 - r\kappa_g.
\]

(26)

Finally, we prove the existence theorem of a wormhole with spherical symmetry to obtain a more explicit estimate of the negative energy, which also provides a lower bound for the horizon area.

**Proposition III.4** Let \( (M, g) \) be spherically symmetric, and let the double-null foliation by \( \{ \Sigma \} \) and \( \{ \Sigma' \} \) respect the spherical symmetry. Let \( D \) be the tangent vector of the affinely parametrized null geodesic generator of \( \{ \Sigma \} \), and let \( S_0 \) be a spherically symmetric 2-sphere on a light-cone \( \Sigma \in \{ \Sigma \} \) with area \( A_0 \) and contracting rate \( \rho_0 > 0 \).

If \( \Phi_{00} < -((p\rho_0)^2) \) holds for some constant \( p > 1 \) on \( \Sigma \) during the affine distance \( r_1 = (2p\rho_0)^{-1} \ln(p+1)/(p-1) \) from \( S_0 \) in the \( D \)-direction, then there exists a trapping horizon \( H \) on \( \Sigma \) with area \( A > \sqrt{1-p^{-2}}A_0 \), and moreover, \( H \) is a wormhole horizon if and only if \( A^{-1} > (\Phi_{11} + 3\Pi)/2\pi\chi\bar{\chi} \) holds.
Proof. By construction, $D$ is tangent to affinely parametrized geodesic $\kappa = \epsilon + \tilde{\epsilon} = 0$, twist-free $\rho = \tilde{\rho}$, sheer-free $\sigma = 0$, and non-rotating $\tau = 0$. Then, eq.(12) becomes

$$\frac{\partial}{\partial r} \rho = \rho^2 + \Phi_{00}.$$  \hfill (27)

We have by integration

$$\rho < p\rho_0 \tanh[p\rho_0 (r_1 - r)]$$ \hfill (28)

for $0 \leq r \leq r_1$ ($r = 0$ on $S_0$). Since the r.h.s. of eq.(28) vanishes for $r = r_1$, which implies $\rho |_{r=r_1} < 0$, there exists a trapping horizon $\rho = 0$ for $r = r_2$, $(0 < r_2 < r_1)$. The lower bound of the horizon area is obtained by integrating the equation $\partial A/\partial r = -2\rho A$, and the condition that $H$ is a wormhole horizon is obtained from the inequality (19).

\hfill \Box

IV. DISCUSSION

We have shown that locally defined wormholes generally require constraints on the energy density in two independent components of the stress-energy tensor, corresponding to two independent components of the Einstein equation. One of these is familiar from previous work [10–12], as a necessary condition for the existence of a minimal surface in a null hypersurface. We have also integrated the relevant equation along the null hypersurface to obtain a sufficient condition for a minimal surface to form, in terms of sufficient negative energy density crossing the hypersurface. For this to be part of a locally traversible wormhole horizon requires the second condition, the inequality (19). This can be satisfied without the relevant energy density becoming negative for minimal surfaces of spherical topology, but higher-genus wormholes require the total surface energy $W$ to be strictly negative. In this sense, higher-genus wormholes are less likely to form.

Note that these conditions constrain the negative energy density or its surface integral over the wormhole, but not its volume integral, i.e. the total negative energy in a region. Theoretically one could arrange for the negative energy density to be confined to a small region around the wormhole, surrounded by large quantities of positive energy yielding a positive total energy which is as large as one pleases. Even if one integrates only over the region where the energy violations occur, this region may be arbitrarily small, leading to an arbitrarily small total negative energy. Such thin shells can even satisfy the quantum inequalities mentioned below.

As an example, for static, spherically symmetric (i.e. Morris-Thorne) wormholes, the tension minus energy density is just $\kappa_g/2\pi r$. Thus the required negative energy density is smaller for large wormholes with small surface gravity, both of which are practical requirements for a comfortably traversible wormhole. The corresponding negative energy, integrated over a shell of width $\ell$ around the wormhole, is of the order of $2\ellcke yg$, so increases with wormhole size, but is still smaller for small surface gravity.

Moreover, the local energy densities do not directly determine the sign of active gravitational energy, such as the asymptotic (ADM and Bondi) energies in an asymptotically flat space-time, or quasi-local energies such as the Hawking energy or (in spherical symmetry) the Misner-Sharp energy [22]. The Hawking energy is actually positive on a wormhole horizon with spherical topology, taking the value $\sqrt{A/16\pi}$. The asymptotic energies may also be positive in a wormhole spacetime. Rather, the sign of the energy density affects the derivative of the active energy [22]. For instance, the usual Bondi energy-loss property reverses under negative energy—specifically, the null energy condition with reversed sign—leading to an increase of Bondi energy at future null infinity. These perhaps counter-intuitive properties of negative energy density should be considered in regard to beliefs such as that wormholes are likely to be plagued by naked singularities, which are associated with negative (active) energy, e.g. [3]. The increase of the Bondi energy actually suggests the opposite, that there might be a form of cosmic censorship for wormholes: wormholes are either stable or collapse to black holes. Of course, this will depend on the matter model e.g. [12] and perhaps also on the initial configuration.

The results suggest that an advanced civilization would be able to construct a traversible wormhole if it could prepare the required negative energy. On the other hand, though quantum field theory permits negative energies, it also constrains the magnitude and duration of the negative energy, according to so-called quantum inequalities, uncertainty principle-type relations. Applying quantum inequalities to static spacetimes, Ford and Roman [21] discussed that there will be at best wormholes of Planckian size, though in fact this allows large, thin-shell wormholes. Also, there seems to be more room for debating dynamic wormholes. Prop. (11.4) implies that a Schwarzschild black hole requires an arbitrarily small amount of negative energy near its horizon to become a dynamic wormhole, so that it seems that such a wormhole can exist due to the Hawking radiation. Since the temperature of the Hawking radiation of a black hole of macroscopic size is low, the horizons will be moving apart at close to the speed of light, so that the resulting wormhole would be difficult to traverse in practice, though two-way traversible in principle. However, the possibility of the existence of such macroscopic wormholes itself has theoretical importance.

Throughout, we have discussed the local structure of wormholes, without prejudice as to whether there will be another universe or a distant region of the spacetime beyond the wormhole throat, or whether the hypothetical advanced civilization could identify spacetime regions as relativists or topologists do.
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