ON LEBESGUE NULL SETS

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Abstract. Letting $A \subseteq \mathbb{R}^n$ be Borel and $W_0 : \mathbb{R}^n \to G(n, m)$ be Lipschitz we establish that $\mathcal{L}^n(A) = 0$ if and only if $\mathcal{H}^m(A \cap (x + W_0(x))) = 0$ for $\mathcal{L}^n$ almost every $x \in \mathbb{R}^n$.

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1. Foreword

Let $A$ be a subset of Euclidean space $\mathbb{R}^n$, $n \geq 2$, and let $\mathcal{L}^n$ denote the Lebesgue outer measure. We concern ourselves with the following question: Can one tell whether $A$ is Lebesgue negligible from the knowledge only of its trace on each member of some given collection of «lower dimensional» subsets $\Gamma_i \subseteq \mathbb{R}^n$, $i \in I$. Thus one expects that if $A \cap \Gamma_i$ is «negligible in the dimension of $\Gamma_i$», for each $i \in I$ then $\mathcal{L}^n(A) = 0$. Of course a necessary condition is that the sets $\Gamma_i$ cover almost all of $A$, i.e. $\mathcal{L}^n(A \sim \bigcup_{i \in I} \Gamma_i) = 0$.

Consider for instance $n = 2$, $I = \mathbb{R}$ and $\Gamma_t = \{t\} \times \mathbb{R}$, $t \in \mathbb{R}$, the collection of all vertical lines in the plane. It is not true in general that if $A \subseteq \mathbb{R}^2$ and $A \cap \Gamma_t$ is a singleton for each $t \in \mathbb{R}$ then $\mathcal{L}^2(A) = 0$. There exist indeed functions $f : \mathbb{R} \to \mathbb{R}$ whose graph $A = \text{graph } f$ has $\mathcal{L}^2(A) > 0$, see e.g. [8, Chapter 2 Theorem 4] for an example due to W. Sierpiński. In order to rule out such examples we will henceforth assume that $A \subseteq \mathbb{R}^n$ be Borel measurable. In that case the Theorem of G. Fubini, together with the invariance of the Lebesgue measure under orthogonal transformations imply the following. Given an integer $1 \leq m \leq n - 1$, if $(\Gamma_i)_{i \in I}$ is the collection of all $m$ dimensional affine subspaces of $\mathbb{R}^n$ of some fixed direction, and if $\mathcal{H}^m(\Gamma_i) = 0$ for all $i \in I$ then $\mathcal{L}^n(A) = 0$. Here $\mathcal{H}^m$ denotes the $m$ dimensional Hausdorff measure. A special feature of this collection $(\Gamma_i)_{i \in I}$ is that it partitions $\mathbb{R}^n$, its members being the level sets $f^{-1}\{y\}$, $y \in \mathbb{R}^{n-m}$, of a «nice map» $f : \mathbb{R}^n \to \mathbb{R}^{n-m}$, indeed an orthogonal projection. This is an occurrence of the following more general situation when $f$ and its leaves $f^{-1}\{y\}$ are allowed to be nonlinear. The coarea formula due to H. Federer in [6] asserts that if $f : \mathbb{R}^n \to \mathbb{R}^{n-m}$ is Lipschitz

2010 Mathematics Subject Classification. Primary 28A75, 26B15.
Key words and phrases. Lebesgue measure, Nikodým set, Negligible set.

The first author was partially supported by the Science and Technology Commission of Shanghai (No. 18dz2271000).
and if $A \subseteq \mathbb{R}^n$ is Borel then
\[ \int_A Jf(x) d\mathcal{L}^n(x) = \int_{\mathbb{R}^{n-m}} \mathcal{H}^m \left( A \cap f^{-1}\{y\} \right) d\mathcal{L}^{n-m}(y). \]
Thus if the Jacobian coarea factor $Jf$ is positive $\mathcal{L}^n$ almost everywhere in $A$ then the collection \( \left\{ f^{-1}(\{y\}) \right\}_{y \in \mathbb{R}^{n-m}} \) is suitable for detecting whether or not $A$ is Lebesgue null. At $\mathcal{L}^n$ almost all $x \in \mathbb{R}^n$ the map $f$ is differentiable according to H. Rademacher, and

\[ Jf(x) = \sqrt{\det(Df(x) \circ Df(x)^T)} = \|\wedge_{n-m} Df(x)\| \]

see [4, Chapter 3 §4] and [7, 3.2.1 and 3.2.11].

In this paper we focus on the case when $f_i$, $i \in I$, are affine subspaces of $\mathbb{R}^n$, but not necessarily members of a partition of the ambient space. Specifically, we assume that with each $x \in \mathbb{R}^n$ is associated an $m$ dimensional affine subspace $W(x)$ of $\mathbb{R}^n$ containing $x$. Given a Borel set $A \subseteq \mathbb{R}^n$, the question whether

\[ \text{If } \mathcal{H}(A \cap W(x)) = 0 \text{ for all } x \in A \text{ then } \mathcal{L}(A) = 0, \tag{1} \]

has a negative answer: O. Nikodým [9] exhibited a Borel subset $A \subseteq \mathbb{R}^2$ of the unit square, such that $\mathcal{L}^2(A) = 1$ and for each $x \in A$ there exists a line $W(x) \subseteq \mathbb{R}^2$ with the property that $A \cap W(x) = \{x\}$. In this context a selection Theorem due to J. von Neumann implies that (possibly considering a smaller, non Lebesgue null Borel subset of $A$) the correspondence $x \mapsto W(x)$ can be chosen to be Borel measurable (see [2,19] and in turn, it can be chosen to be continuous according to a result of N. Lusin. This was noted by A. Zygmund in connection with multiparameter Fourier analysis.

Our result assumes that $W$ be Lipschitz. Below $G(n,m)$ denotes the Grassmannian manifold of $m$ dimensional linear subspaces of $\mathbb{R}^n$.

**Theorem.** Assume $W_0 : \mathbb{R}^n \to G(n,m)$ is Lipschitz and $A \subseteq \mathbb{R}^n$ is Borel. The following are equivalent.

1. $\mathcal{L}^n(A) = 0$;
2. For $\mathcal{L}^n$ almost every $x \in A$, $\mathcal{H}^m \left( A \cap (x + W_0(x)) \right) = 0$;
3. For $\mathcal{L}^n$ almost every $x \in \mathbb{R}^n$, $\mathcal{H}^m \left( A \cap (x + W_0(x)) \right) = 0$.

This seems to be new. As should be apparent from the discussion above, the difficulty stands with the fact that the affine $m$ planes $W(x) = x + W_0(x)$ need not be disjointed. The natural route is to reduce the problem to applying the coarea formula by spreading out the $W(x)$’s in a disjointed way, in a higher dimensional space, i.e. adding a variable $u \in W(x)$ to the given $x \in \mathbb{R}^n$ and considering $W(x)$ as a fiber above the base space $\mathbb{R}^n$. We thus define

$$\Sigma = \mathbb{R}^n \times \mathbb{R}^m \cap \{(x,u) : x \in E \text{ and } u \in W(x)\},$$

where $E \subseteq \mathbb{R}^n$ is Borel. This set is $n + m$ rectifiable owing to the Lipschitz continuity of $W$. It is convenient to assume that $\mathcal{L}^n(E) < \infty$ so that

$$\phi_E(B) = \int_E \mathcal{H}^m \left( B \cap W(x) \right) d\mathcal{L}^n(x),$$

$B \subseteq \mathbb{R}^n$, is a locally finite Borel measure.[2.16] Now $\Sigma$ was precisely set up so that for each $x \in E$

$$\mathcal{H}^m \left( \Sigma \cap \pi_2^{-1}(B) \cap \pi_1^{-1}\{x\} \right) = \mathcal{H}^m \left( B \cap W(x) \right),$$

where $\pi_1$ and $\pi_2$ denote the projections of $\mathbb{R}^n \times \mathbb{R}^n$ to the $x$ and $u$ variable, respectively. Abbreviating $\Sigma_B = \Sigma \cap \pi_2^{-1}(B)$ the coarea formula yields

$$\phi_E(B) = \int_{\Sigma_B} J\Sigma \pi_1 d\mathcal{H}^{n+m}.$$
A simple calculation shows that \( J_z \nabla^2 > 0 \) almost everywhere. Since also
\[
\int_B J_z \nabla^2 d\mathcal{H}^{m+1} = \int_B \mathcal{H}^m \left( E \cap \nabla^2 \right) d\mathcal{L}^n (u)
\]
the implication (1) \( \Rightarrow \) (3) above should now be clear. In order to establish that (2) \( \Rightarrow \) (1) we need to observe that \( J_z \nabla^2 > 0 \) almost everywhere and ideally to show that \( \mathcal{H}^m \left( E \cap \nabla^2 \right) > 0 \) for almost every \( u \in E \). This last part offers some difficulty. To understand this we let \( m = n - 1 \) in order to keep the notations short. Now \( u \in W(x) \) if \( u - x \in W_0(x) \) and \( \langle v_0(x), x - u \rangle = 0 \) where \( v_0(x) \in W_0(x)^{+} \), is, say a unit vector. Abbreviating \( g_u(x) = \langle v_0(x), x - u \rangle \) we infer that
\[
\mathcal{H}^m \left( E \cap \nabla^2 \right) = \mathcal{H}^m \left( E \cap g_u^{-1} \right).
\]
The problem remains that two of the nonlinear \( m \) sets \( E \cap g_u^{-1} \) and \( E \cap g_v^{-1} \) may intersect, thereby preventing another application of the coarea formula to look out for their lower bound. Yet we already know that
\[
\phi_E (B) = \int_B \mathcal{D}_E W \, d\mathcal{L}^n
\]
where \( \mathcal{D}_E W \) is a Radon–Nikodym derivative and also that \( (\mathcal{D}_E W)(u) \) is comparable to \( \mathcal{H}^m \left( E \cap g_u^{-1} \right) \). Adding an extra variable \( y \) to the fibered space \( \Sigma \), we improve on this by showing that
\[
(\mathcal{D}_E W)(u) \geq \beta_n \liminf \int_{r}^{\mathcal{H}^m \left( E \cap g_u^{-1} \right)} d\mathcal{L}^1 \left( y \right) = \beta_n (\mathcal{D}_E W)(u),
\]
where the last equality defines \( \mathcal{D}_E W \) and \( \beta_n > 0 \). We are reduced to showing that \( \mathcal{D}_E W > 0 \) almost everywhere. The reason why this holds is the following. Fix a Borel set \( Z \subseteq \mathbb{R}^n \), \( x_0 \in \mathbb{R}^n \) and \( r > 0 \). Let \( C_W(x_0, r) \) denote the cylindrical box consisting of those \( x \in \mathbb{R}^n \) such that \( |P_W(x_0)(x - x_0)| \leq r \) and \( |P_W(x_0)^{-1}(x - x_0)| \leq r \). We want to find a lower bound for
\[
\int_{Z \cap C_W(x_0, r)} (\mathcal{D}_E \cdot C_W(x_0, r))(u) \, d\mathcal{L}^m (u).
\]
To this end we fix \( z \in W_0(x_0) \cap B(0, r) \) and we let \( V_z = \mathbb{R}^n \{ x_0 + z + sv_0(x_0) : -r \leq s \leq r \} \) denote the corresponding vertical line segment. According to Fubini’s theorem we are reduced to estimating
\[
\int_{V_z} (\mathcal{D}_E \cdot C_W(x_0, r))(u) \, d\mathcal{H}^1 (u).
\]
According to Vitalli’s covering theorem we can find a disjointed family of line segments \( I_1, I_2, \ldots \) covering almost all \( V_z \) such that the above integral nearly equals
\[
\sum_k \mathcal{H}^1 (I_k) \int_{I_k} \mathcal{H}^m \left( Z \cap C_W(x_0, r) \cap g_u^{-1} \right) \, d\mathcal{L}^1 (u)
\]
\[
\leq \sum_k \int_{Z \cap C_W(x_0, r) \cap g_u^{-1} (I_k)} \nabla g_u (x) \, d\mathcal{L}^m (x) \approx \mathcal{L}^m (Z \cap C_W(x_0, r))
\]
where there first near equality follows from the coarea formula, the second one because \( \nabla g_u \approx 1 \) at small scales and the «nonlinear horizontal stripes» \( g_u^{-1} (I_k) \) are nearly pairwise disjoint. Verification of these claims takes up sections 5 and 6. Now we reach a contradiction if \( Z = \mathbb{R}^n \cap \{ \mathcal{D}_E W = 0 \} \) is assumed to have \( \mathcal{L}^m (Z) > 0 \) and \( x_0 \) is a point of density of \( Z \).
2. Preliminaries

2.1. In this paper $1 \leq m \leq n - 1$ are integers. The ambient space is $\mathbb{R}^n$. The canonical inner product of $x, x' \in \mathbb{R}^n$ is denoted $\langle x, x' \rangle$ and the corresponding Euclidean norm of $x$ is $|x|$. If $S \subseteq \mathbb{R}^n$ we let $\mathcal{B}(S)$ denote the $\sigma$ algebra of Borel subsets of $S$.

2.2 (Hausdorff Measure). We let $\mathcal{L}^m$ denote the Lebesgue outer measure in $\mathbb{R}^m$ and $m(n) = \mathcal{L}^n(B(0, 1))$. For $S \subseteq \mathbb{R}^m$ we abbreviate $\zeta^m(S) = m(n)2^{-m}(\text{diam } S)^m$. Given $0 < \delta \leq \infty$ we define $\mathcal{H}^m_\delta(S)$ to be the $\delta$ cover of $S$ in $\mathcal{L}^m$.

2.3 (Coarea formula). Here we recall two versions of the coarea formula. First if $f : A \subseteq \mathbb{R}^m$ is Lipschitz then $\mathcal{H}^m(\{A \cap f^{-1} \{y\}\}) = \int_A |J f(x)| \, d\mathcal{L}^m(x)$ measurable and countably $\mathcal{H}^n, n$ rectifiable, and if $f : A \rightarrow \mathbb{R}^{n-m}$ is Lipschitz then $\mathcal{H}^m(\{A \cap f^{-1} \{y\}\}) = \mathcal{L}^{n-m}$ measurable.
measurable and
\[ \int_A J_A f(x) d\mathcal{H}^n(x) = \int_{R^{n-m}} \mathcal{H}^m \left( A \cap f^{-1}(y) \right) d\mathcal{L}^{n-m}(y). \]
To give a formula for the coarea Jacobian factor \( J_A f(x) \) of \( f \) relative to \( A \) we consider a point \( x \in A \) where \( A \) admits an approximate \( n \) dimensional tangent space \( T_x A \) and where \( f \) is differentiable along \( A \). Letting \( L : T_x A \to R^{n-m} \) denote the derivative of \( f \) at \( x \) we have
\[ J_A f(x) = \sqrt{\det(L \circ L^*)} = \|\wedge_{n-m} L\|, \]
see for instance [2] 3.2.22.
In both cases it is useful to recall the following. If \( L : V \to V' \) is a linear map between two inner product spaces \( V \) and \( V' \) then
\[ \|\wedge_k L\| = \sup \{\langle\wedge_k L, \xi\rangle : \xi \in \wedge_k V \text{ and } |\xi| = 1\}. \quad (2) \]
On the one hand \( \|\wedge_k L\| \leq \|L\|^k \) [7, 1.7.6], and \( \|L\| \leq \text{Lip } f \) with \( L \) as above. On the other hand if \( v_1, \ldots, v_k \) are linearly independent vectors of \( V \) then
\[ \|\wedge_k L\| \geq \frac{|L(v_1) \wedge \cdots \wedge L(v_k)|}{|v_1 \wedge \cdots \wedge v_k|}. \quad (3) \]

2.4 (Grassmannian). — We let \( G(n, m) \) denote the set whose members are the \( m \) dimensional linear subspaces of \( R^n \). With \( W \in G(n, m) \) we associate \( P_W : R^n \to R^m \) the orthogonal projection onto \( W \). We give \( G(n, m) \) the structure of a compact metric space by letting \( d(W_1, W_2) = \|P_{W_1} - P_{W_2}\| \). If \( W \in G(n, m) \) then \( W^\perp \in G(n - m) \) is so that \( P_W + P_{W^\perp} = \text{id}_{R^n} \), therefore \( G(n, m) \to G(n, n - m) : W \mapsto W^\perp \) is an isometry. The bijective correspondence \( \varphi : G(n, m) \to \text{Hom}(R^n, R^m) \) such that \( \varphi (G(n, m)) = (w) \mapsto \text{ad } f \circ P_{W}(w) \) identifies \( G(n, m) \) with the submanifold \( M_{n,m} = \text{Hom}(R^n, R^m) \cap \{ L : L \circ L = L, L^\perp = L \text{ and trace } L = m \} \). There exists an open neighborhood \( V \) of \( M_{n,m} \) in \( \text{Hom}(R^n, R^m) \) and a Lipschitz retraction \( \rho : V \to M_{n,m} \), according for instance to [7] 3.1.20. Therefore if \( S \subseteq \text{R}^m \) and if \( W_0 : S \to G(n, m) \) is Lipschitz then there exist an open neighborhood \( U \) of \( E \) in \( R^n \) and a Lipschitz extension \( \tilde{W}_0 : U \to G(n, m) \) of \( W_0 \). Indeed \( \varphi \circ W_0 \) admits a Lipschitz extension \( \tilde{Y} : \text{R}^n \to \text{Hom}(R^n, R^m) \), see e.g. [7] 2.10.43, and it suffices to let \( U = Y^{-1}(V) \) and \( \tilde{W}_0 = \rho \circ (Y|_U) \).

2.5 (Orthonormal frames). — We let \( V(n, m) \) denote the set orthonormal \( m \) frames in \( R^n \), i.e. \( V(n, m) = \{ (w_1, \ldots, w_m) \} : \text{the family } w_1, \ldots, w_m \text{ is orthonormal} \}. \) We will consider it as a metric space with its structure inherited from \( (R^n)^m \).

2.6. — Let \( Y \subseteq G(n, m) \) be a nonempty closed set such that \( \text{diam } Y < 1 \). There exists a Lipschitz map \( \Xi : Y \to V(n, m) \) such that \( W = \text{span}(\Xi_1(W), \ldots, \Xi_m(W)) \) for every \( W \in Y \).

Proof. Pick arbitrarily \( W_0 \in Y \). If \( W \in Y \) then the map \( W_0 \to W : w \mapsto P_W(w) \) is bijective: if \( w \in W_0 \sim \{0\} \) then \( |P_W(w) - w| = |P_W(w) - P_{W_0}(w)| < |w| \) thus \( P_W(w) \neq 0 \). Letting \( w_1, \ldots, w_m \) be an arbitrary basis of \( W_0 \) it follows that for each \( W \in Y \) the vectors \( w_1(W) = P_W(w_1), i = 1, \ldots, m, \) constitute a basis of \( W \). Furthermore the maps \( w_i : Y \to R^n \) are Lipschitz: \( |w_i(W) - w_j(W)| = |P_W(w_i) - P_W(w_j)| \leq d(W, W')|w_i| \). We apply the Gram-Schmidt process:
\[ \overline{w}_1(W) = w_1(W) \quad \text{and} \quad \overline{w}_i(W) = w_i(W) - \sum_{j=1}^{i-1} \langle w_j(W), \overline{w}_j(W) \rangle \overline{w}_j(W), i = 2, \ldots, m, \]
so that \( \overline{w}_1(W), \ldots, \overline{w}_m(W) \) is readily an orthogonal basis of \( W \) depending upon \( W \) in a Lipschitz way. Since each \( |\overline{w}_i| \) is bounded away from zero on \( Y \) the formula \( \Xi(W) = [\overline{w}_i(W)]^{-1} \overline{w}_i(W), i = 1, \ldots, m, \) defines \( \Xi \) with the required property.

2.7. — There exists a Borel measurable map \( \Xi : G(n, m) \to V(n, m) \) with the property that \( W = \text{span}(\Xi_1(W), \ldots, \Xi_m(W)) \) for every \( W \in G(n, m) \).
Proof. Since $G(n,m)$ is compact it can partitioned into finitely many Borel sets $\mathcal{F}_1, \ldots, \mathcal{F}_J$ each having diameter bounded by 1/2. Define $\Xi$ piecewise to coincide on $\mathcal{F}_j$ with a $\Xi_j$ associated with Clos $\mathcal{F}_j$ in $\mathcal{F}_j$, $j = 1, \ldots, J$. \hfill $\square$

2.8. — Assume $S \subseteq \mathbb{R}^n$, $x_0 \in S$ and $W_0 : S \rightarrow G(n,m)$ is Lipschitz. There then exist an open neighborhood $U$ of $x_0$ in $\mathbb{R}^n$ and Lipschitz maps $w_1, \ldots, w_m, v_1, \ldots, v_{n-m} : U \rightarrow \mathbb{R}^n$ such that:

1. For every $x \in U$ the family $w_1(x), \ldots, w_m(x), v_1(x), \ldots, v_{n-m}(x)$ is an orthonormal basis of $\mathbb{R}^n$;

2. For every $x \in S \cap U$ one has

$$W_0(x) = \text{span}(w_1(x), \ldots, w_m(x))$$

and

$$W_0(x)^\perp = \text{span}(v_1(x), \ldots, v_{n-m}(x)).$$

Proof. We let $\widehat{W}_0 : \widehat{U} \rightarrow G(n,m)$ be a Lipschitz extension of $W_0$ where $\widehat{U}$ is an open neighborhood of $S$ in $\mathbb{R}^n$ (recall 2.4). Abbreviate $W_1 := W_0(x_0)$. Define $\mathcal{F}' := G(n,m) \cap \{W : d(W, W_0) < 1/4\}$ and $V = \overline{\mathcal{F}_0}^{\perp}(\mathcal{F}')$. Apply 2.6 to Clos $\mathcal{F}'$ and denote $\Xi$ the resulting Lipschitz map $\mathcal{F}' \rightarrow (\mathbb{R}^n)_m$. Next define $\mathcal{F}'^\perp := G(n,n-m) \cap \{W^\perp : W \in \mathcal{F}'\}$, apply 2.6 to Clos $\mathcal{F}'^\perp$ and denote $\Xi^\perp$ the resulting Lipschitz map $\mathcal{F}'^\perp \rightarrow (\mathbb{R}^n)^{n-m}$. Letting $w_i(x) = (\Xi \circ W_0)(x)$, $i = 1, \ldots, m$, and $v_i(x) = (\Xi^\perp \circ W_0)(x)$, $i = 1, \ldots, n-m$, completes the proof. \hfill $\square$

2.9. — Assume $W_0 : \mathbb{R}^n \rightarrow G(n,m)$ is Borel measurable. Then there exist Borel measurable maps $w_1, \ldots, w_m, v_1, \ldots, v_{n-m} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that:

1. For every $x \in \mathbb{R}^n$ the family $w_1(x), \ldots, w_m(x), v_1(x), \ldots, v_{n-m}(x)$ is an orthonormal basis of $\mathbb{R}^n$;

2. For every $x \in \mathbb{R}^n$ one has

$$W_0(x) = \text{span}(w_1(x), \ldots, w_m(x))$$

and

$$W_0(x)^\perp = \text{span}(v_1(x), \ldots, v_{n-m}(x)).$$

Proof. Choose $\Xi : G(n,m) \rightarrow G(n,m)$ and $\Xi^\perp : G(n,n-m) \rightarrow G(n,n-m)$ as in 2.7. Letting $(w_1(x), \ldots, w_m(x)) = (\Xi \circ W_0)(x)$ and $(v_1(x), \ldots, v_{n-m}(x)) = (\Xi^\perp \circ W_0)(x)$, $x \in \mathbb{R}^n$, completes the proof. \hfill $\square$

2.10 (Definition of $W(x)$). — The typical situation that arises in the remaining part of this paper is that we are given a set $S \subseteq \mathbb{R}^n$, a Lipschitz map $W_0 : S \rightarrow G(n,m)$ and $x_0 \in S$. We will represent $W_0(x)$ and $W_0^\perp(x)$ in a neighborhood $U$ of $x_0$ as in 2.8. We will then further reduce the size of $U$ several times in order that various conditions be met. With no exception we will denote as $W(x) = x + W_0(x)$ the affine subspace containing $x$, and direction $W_0(x)$, whenever $W_0(x)$ is defined.

2.11 (Definition of $g_{v_1,\ldots,v_{n-m},u}$ and lower bound of its coareafactor). — Given an open set $U \subseteq \mathbb{R}^n$, a Lipschitz map $v : U \rightarrow \mathbb{R}^n$, and $u \in \mathbb{R}^n$ we define $g_{v,u} : U \rightarrow \mathbb{R}$ by the formula

$$g_{v,u}(x) = \langle v(x), x - u \rangle.$$  

Clearly $g_{v,u}$ is Lipschitz. If $v$ is differentiable at $x \in U$ then so is $g_{v,u}$ and for every $h \in \mathbb{R}^n$ one has

$$Dg_{v,u}(x)(h) = \langle \nabla g_{v,u}(x), h \rangle = \langle Dv(x)(h), x - u \rangle + \langle v(x), h \rangle.$$  \hfill (4)

Next we assume we are given Lipschitz maps $v_1, \ldots, v_{n-m} : U \rightarrow \mathbb{R}^n$. We define $g_{v_1,\ldots,v_{n-m},u} : U \rightarrow \mathbb{R}^{n-m}$ by the formula

$$g_{v_1,\ldots,v_{n-m},u}(x) = (g_{v_1,u}(x), \ldots, g_{v_{n-m},u}(x)).$$
It is Lipschitz as well. The relevance of \( g_{\mathbf{v}_1, \ldots, \mathbf{v}_{n-m}, u} \) stems from the following observation, assuming that \( \mathbf{v}_1, \ldots, \mathbf{v}_{n-m} \) are associated with \( W_0 \) and \( W \) as in\(^{[2,8]}\) and\(^{[2,10]}\).

\[
\begin{align*}
u \in W(x) & \iff u - x \in W_0(x) \\
& \iff (\mathbf{v}_i(x), u - x) = 0 \text{ for all } i = 1, \ldots, n - m \\
& \iff g_{\mathbf{v}_1, \ldots, \mathbf{v}_{n-m}, u}(x) = 0 \\
& \iff x \in g_{\mathbf{v}_1^{-1}, \ldots, \mathbf{v}_{n-m}^{-1}}(\{0\}) .
\end{align*}
\]

In fact \( |g_{\mathbf{v}_1, \ldots, \mathbf{v}_{n-m}, u}(x)| = |P_{W_0(x)}(x - u)| \).

Abbreviate \( g = g_{\mathbf{v}_1, \ldots, \mathbf{v}_{n-m}, u} \). If each \( \mathbf{v}_i \) is differentiable at \( x \in U \), and \( h \in \mathbb{R}^n \), then

\[
Dg(x)(h) = \sum_{i=1}^{n-m} Dg_{\mathbf{v}_i, u}(x)(h)e_i .
\]

Thus if \( \mathbf{v}_1(x), \ldots, \mathbf{v}_{n-m}(x) \) constitute an orthonormal family in \( \mathbb{R}^n \) then

\[
Dg_{\mathbf{v}_i, u}(x)(\mathbf{v}_j(x)) = \delta_{i,j} + \epsilon_{i,j}(x, u)
\]

where

\[
|\epsilon_{i,j}(x, u)| = |(D\mathbf{v}_i(x)(\mathbf{v}_j(x)), x - u)| \leq (\text{Lip } \mathbf{v}_i) |x - u| ,
\]

according to\(^4\), and in turn

\[
Dg(x)(\mathbf{v}_j(x)) = \sum_{i=1}^{n-m} (\delta_{i,j} + \epsilon_{i,j}(x, u)) e_i .
\]

This allows for a lower bound of the coarea factor of \( g \) at \( x \) as follows.

\[
\|\nabla_{n-m}Dg(x)\| \geq |Dg(x)(\mathbf{v}_1(x)) \wedge \ldots \wedge Dg(x)(\mathbf{v}_{n-m}(x))| = \left| \left( \sum_{i=1}^{n-m} (\delta_{i,1} + \epsilon_{i,1}(x, u)) e_i \right) \wedge \ldots \wedge \left( \sum_{i=1}^{n-m} (\delta_{i,n-m} + \epsilon_{i,n-m}(x, u)) e_i \right) \right| = |\det (\delta_{i,j} + \epsilon_{i,j}(x, u))_{i,j=1,\ldots,n-m}| .
\]

In view of (6) we obtain the next lemma.

2.12. — Given \( \Lambda > 0 \) and \( 0 < \varepsilon < 1 \) there exists \( \delta_{2,12}(n, \Lambda, \varepsilon) > 0 \) with the following property. Assume that

\[
\begin{align*}
(1) & \ U \subseteq \mathbb{R}^n \text{ is open and } u \in \mathbb{R}^n; \\
(2) & \ \mathbf{v}_1, \ldots, \mathbf{v}_{n-m} : U \to \mathbb{R}^n \text{ are Lipschitz;} \\
(3) & \ \mathbf{v}_1(x), \ldots, \mathbf{v}_{n-m}(x) \text{ is an orthonormal family for every } x \in U .
\end{align*}
\]

If

\[
\begin{align*}
(4) & \ \text{Lip } \mathbf{v}_i \leq \Lambda \text{ for each } i = 1, \ldots, n - m; \\
(5) & \ \text{diam } (U \cup \{u\}) \leq \delta_{2,12}(n, \Lambda, \varepsilon)
\end{align*}
\]

then

\[
J_{g_{\mathbf{v}_1, \ldots, \mathbf{v}_{n-m}, u}}(x) \geq 1 - \varepsilon
\]

at \( \mathcal{L}^n \) almost every \( x \in U \).

2.13 (Definition of \( \pi_u \) and Its Relation with \( g_{\mathbf{v}_1, \ldots, \mathbf{v}_{n-m}, u} \)). — With \( u \in \mathbb{R}^n \) we associate

\[
\pi_u : V(n, n - m) \times \mathbb{R}^n \to \mathbb{R}^{n-m} : (\xi_1, \ldots, \xi_{n-m}, x) \mapsto (\langle \xi_1, x - u \rangle, \ldots, \langle \xi_{n-m}, x - u \rangle) .
\]

When \( (\xi_1, \ldots, \xi_{n-m}) \in V(n, n - m) \) is fixed we also abbreviate as \( \pi_{\xi_1, \ldots, \xi_{n-m}, u} \) the map \( \mathbb{R}^n \to \mathbb{R}^{n-m} \) defined by \( \pi_{\xi_1, \ldots, \xi_{n-m}, u}(x) = \pi_u(\xi_1, \ldots, \xi_{n-m}, x) \). It is then rather useful to observe that in the context described in\(^2\) and\(^8\) the following holds:

\[
\pi_{\mathbf{v}_1(x), \ldots, \mathbf{v}_{n-m}(x), u}^{-1} g_{\mathbf{v}_1, \ldots, \mathbf{v}_{n-m}, u}(x) = W(x) .
\]
Indeed,
\[ h \in W(x) \iff h - x \in W_0(x) \]
\[ \iff \langle v_i(x), h - x \rangle = 0 \text{ for all } i = 1, \ldots, n - m \]
\[ \iff \langle v_i(x), h \rangle - u \rangle = \langle v_i(x), x - u \rangle \text{ for all } i = 1, \ldots, n - m \]
\[ \iff \pi_{v_i(x), \ldots, v_{n-m}(x), u}(h) = g_{v_i, \ldots, v_{n-m}, u}(x). \]

In the sequel we sometimes abbreviate \( \xi = (\xi_1, \ldots, \xi_{n-m}) \in V(n, n - m) \). It also helps to notice that for given \( \xi \in V(n, n - m) \) and \( y \in \mathbb{R}^{n-m} \) the set \( \pi_{\xi, u}^{-1} \{ y \} \) is an \( m \) dimensional affine subspace of \( \mathbb{R}^n \).

2.14. — Assume \( B \in \mathcal{B}(\mathbb{R}^n) \) and \( u \in \mathbb{R}^n \). It follows that
\[ h_B : V(n, n - m) \times \mathbb{R}^{n-m} \to \mathbb{R}_+ : (\xi, y) \mapsto \mathcal{H}^m \left( B \cap \pi_{\xi, u}^{-1} \{ y \} \right) \]
is Borel measurable.

Proof. We start by showing that when \( B \) is compact, \( h_B \) is upper semicontinuous. Thus if \( (\xi_k, y_k) \in V(n, n - m) \times \mathbb{R}^{n-m} \) converges to \( (\xi, y) \), we ought to show that
\[ \mathcal{H}^m_0(K) \geq \limsup_k \mathcal{H}^m_0(K_k) \tag{8} \]
where \( K = B \cap \pi_{\xi, u}^{-1} \{ y \} \) and \( K_k = B \cap \pi_{\xi_k, u}^{-1} \{ y_k \} \). This is indeed equivalent to the same inequality with \( \mathcal{H}^m_0 \) replaced by \( \mathcal{H}^m \) according to \( \mathcal{H}^m_0 \) and the last sentence of \( \mathcal{H}^m_0 \). Considering if necessary a subsequence of \( (K_k)_k \) we may assume that none of the compact sets \( K_k \) is empty, and that the \( \limsup \) in \( \mathcal{H}^m \) is a limit. Since the set of nonempty compact subsets of the compact set \( B \), equipped with the Hausdorff metric is compact, the sequence \( (K_k)_k \) admits a subsequence (denoted the same way) converging to a compact set \( L \subseteq B \). Given \( z \in L \) there are \( z_k \in K_k \) converging to \( z \). Thus \( \pi_u (\xi, z) = \lim_k \pi_u (\xi_k, z_k) = \lim_k y_k = y \). In other words \( z \in K \). Thus \( \mathcal{H}^m_0(K) \geq \mathcal{H}^m_0(L) \) and \( \mathcal{H}^m \) follows from \( \mathcal{H}^m_0 \).

Next we abbreviate \( \mathcal{A} = \mathcal{B}(\mathbb{R}^n) \cap \{ B : h_B \text{ is Borel measurable} \} \). Thus we have just shown that \( \mathcal{A} \) contains the collection \( \mathcal{H}(\mathbb{R}^n) \) of all compact subsets of \( \mathbb{R}^n \). Observe that if \( (B_i)_i \) is an increasing sequence in \( \mathcal{A} \) and \( B = \bigcup_i B_i \), then \( h_B = \lim_i h_{B_i} \) pointwise, thus \( B \in \mathcal{A} \). In particular \( \mathbb{R}^n \in \mathcal{A} \). Finally if \( B, B' \in \mathcal{A} \) and \( B' \subseteq B \) then \( h_{B-B'} = h_B - h_{B'} \) because all measures involved are finite, indeed \( h_B(\xi, y) \leq \sigma(n) r^m \) for all \( (\xi, y) \). Accordingly \( B \sim B' \in \mathcal{A} \). This means that \( \mathcal{A} \) is a Dynkin class. Since \( \mathcal{H}(\mathbb{R}^n) \) is a \( \sigma \) system, \( \mathcal{A} \) contains the \( \sigma \) algebra generated by \( \mathcal{H}(\mathbb{R}^n) \), i.e. \( \mathcal{B}(\mathbb{R}^n) \). \( \square \)

2.15. — Assume \( B \in \mathcal{B}(\mathbb{R}^n) \), \( r > 0 \) and \( W_0 : \mathbb{R}^n \to G(n, m) \) is Borel measurable. The following function is Borel measurable.
\[ \mathbb{R}^n \to [0, \infty] : x \mapsto \mathcal{H}^m \left( B \cap W(x) \right) \]

Proof. Let \( h_{W, B} \) denote this function. Let \( v_1, \ldots, v_{n-m} : \mathbb{R}^n \to \mathbb{R}^n \) be Borel measurable maps associated with \( W_0 \) as in \( \mathcal{L} \). Fix \( u \in \mathbb{R}^n \) arbitrarily. Define
\[ \mathcal{T} : \mathbb{R}^n \to V(n, n - m) \times \mathbb{R}^{n-m} : x \mapsto \left( v_1(x), \ldots, v_{n-m}(x), g_{v_1(x), \ldots, v_{n-m}(x), u}(x) \right) \]
so that
\[ h_{W, B} = h_B \circ \mathcal{T} \]
(where \( h_B \) is the function associated with \( B \) and \( u \) in \( \mathcal{L} \), according to \( \mathcal{L} \)). One notes that \( \mathcal{T} \) is Borel measurable, and the conclusion ensues from 2.14. \( \square \)

2.16 (Definition of \( \phi_{E, W} \)). — Let \( W_0 : \mathbb{R}^n \to G(n, m) \) be Borel measurable and let \( E \in \mathcal{B}(\mathbb{R}^n) \) be such that \( \mathcal{L}^m(E) < \infty \). For each \( B \in \mathcal{B}(\mathbb{R}^n) \) we define
\[ \phi_{E, W}(B) = \int_E^{\mathcal{H}^m}(B \cap W(x)) \mu \mathcal{L}^m(x) \]
This is well defined according to 2.15 (2). It is easy to check that \( \phi_{E,W} \) is a locally finite (hence \( \sigma \)-finite) Borel measure on \( \mathbb{R}^n \). Indeed \( \phi_{E,W}(B) \leq \alpha(m) (\text{diam } B)^m \mathcal{L}^m(E) \).

To close this section we discuss the relevance of \( \phi_{E,W} \) to the problem of existence of «nearly Nikodym sets».

2.17 (Definition of Nearly Nikodym set). Let \( E \in \mathcal{B}(\mathbb{R}^n) \). We say that \( B \in \mathcal{B}(E) \) is nearly \( m \) Nikodym in \( E \) if

1. \( \mathcal{L}^m(B) > 0 \);
2. For \( \mathcal{L}^m \) almost each \( x \in E \) there is \( W \in G(n,m) \) such that \( \mathcal{H}^m(B \cap (x + W)) = 0 \).

In case \( n = 2, m = 1, E = [0, 1] \times [0, 1] \), the existence of such \( B \) (with \( \mathcal{L}^2(B) = 1 \)) was established by O. Nikodym [2], see also [2] Chapter 8. For arbitrary \( n \geq 2 \) and \( m = n - 1 \) the existence of such \( B \) was established by K. Falconer [5]. In fact in both cases these authors proved the stronger condition that for every \( x \in B, \mathcal{H}^m(B \cap (x + W)) = 0 \) can be replaced by \( B \cap (x + W) = \{x\} \). Thus in case \( 1 \leq m < n - 1 \), if \( B \) is a set exhibited by K. Falconer, \( x \in B \) and \( W \subseteq G(n, n - 1) \) is such that \( B \cap (x + W) = \{x\} \), picking arbitrarily \( V \in G(n, m) \) such that \( V \subseteq W \) we see that \( B \cap (x + V) = \{x\} \). Whence \( B \) is also nearly \( m \) Nikodym in \( B \).

Assuming also that \( W_0 : E \rightarrow G(n,m) \) is Borel measurable we say that \( B \in \mathcal{B}(E) \) is nearly \( m \) Nikodym in \( E \) relative to \( W \) if

1. \( \mathcal{L}^m(B) > 0 \);
2. For \( \mathcal{L}^m \) almost each \( x \in E \) one has \( \mathcal{H}^m(B \cap W(x)) = 0 \).

2.18. Let \( E \in \mathcal{B}(\mathbb{R}^n) \) and let \( W_0 : \mathbb{R}^n \rightarrow G(n,m) \) be Borel measurable. The following are equivalent.

1. \( \mathcal{L}^m|_{\mathcal{B}(E)} \) is absolutely continuous with respect to \( \phi_{E,W}|_{\mathcal{B}(E)} \).
2. There does not exist a nearly \( m \) Nikodym set relative to \( W \).

Proof. A set \( B \in \mathcal{B}(E) \) such that \( \phi_{E,W}(B) = 0 \) and \( \mathcal{L}^m(B) > 0 \) is, by definition a nearly \( m \) Nikodym set relative to \( W \). Condition (1) is equivalent to their nonexistence. \( \square \)

2.19. Assume that \( E \in \mathcal{B}(\mathbb{R}^n) \) and that \( B \in \mathcal{B}(E) \) is nearly \( m \) Nikodym. It follows that:

1. There exists \( W_0 : \mathbb{R}^n \rightarrow G(n,m) \) Borel measurable such that \( B \) is nearly \( m \) Nikodym in \( E \) relative to \( W \).
2. There exists \( C \subseteq B \) compact and \( W_0 : \mathbb{R}^n \rightarrow G(n,m) \) continuous such that \( C \) is nearly \( m \) Nikodym in \( C \) relative to \( W \).

Proof. Define a Borel measurable map \( \xi : G(n,m) \rightarrow V(n-m) \) by \( \xi(W) = \Xi(W^2) \) where \( \Xi : G(n,n-m) \rightarrow V(n-m) \) is as in 2.7 Choose arbitrarily \( u \in \mathbb{R}^n \) and define a Borel measurable map

\[
\Upsilon : E \times G(n,m) \rightarrow V(n,n-m) \times \mathbb{R}^{n-m} \\
(x, W) \mapsto (\xi(W), \langle \xi_1(W), x-u \rangle, \ldots, \langle \xi_{n-m}(W), x-u \rangle) .
\]

Similarly to [7] observe that

\[
W = \pi_{E(W),u}^{-1} \{ \langle \xi_1(W), x-u \rangle, \ldots, \langle \xi_{n-m}(W), x-u \rangle \}
\]

for every \( W \in G(n,m) \). We infer from 2.14 that

\[
h_B \circ \Upsilon : E \times G(n,m) \rightarrow [0, \infty] : (x, W) \mapsto \mathcal{H}^m(B \cap (x + W))
\]

is Borel measurable. Thus the set

\[
\mathcal{E} = E \times G(n,m) \cap \{(x, W) : \mathcal{H}^m(B \cap (x + W)) = 0 \}
\]

is Borel as well. The set \( N = E \cap \{ x : \mathcal{E}_x = \emptyset \} \) is coanalytic and \( \mathcal{L}^m(N) = 0 \) by assumption. According to von Neumann’s selection Theorem [10] 5.5.3 there exists a universally measurable map \( W_0 : E \sim N \rightarrow G(n,m) \) such that \( W_0(x) \in \mathcal{E}_x \) for every \( x \in E \sim N \), i.e. \( \mathcal{H}^m(B \cap (x + W_0(x))) = 0 \). We extend \( W_0 \) to be an arbitrary constant on
N. This makes \( W_0 \) an \( \mathcal{L}^n \) measurable map defined on \( E \). Therefore it is equal \( \mathcal{L}^n \) almost everywhere to a Borel map \( W_0 : E \to G(n, m) \). This proves (1).

In order to prove (2) we recall of 2.4, specifically the retraction \( \varphi : G(n, m) \to M_{n,m} \). Owing to the compactness of \( M_{n,m} \) there are finitely many open balls \( U_j, j = 1, \ldots, J \), whose closure are contained in \( V \) and covering \( M_{n,m} \). Since \( \mathcal{L}^n(B) > 0 \) there exists \( j = 1, \ldots, J \) such that \( \mathcal{L}^n(B \cap E_j) > 0 \) where \( E_j = (\varphi \circ W_0)^{-1}(U_j) \). It follows from Lusin’s Theorem [7, 2.5.3] that there exists a compact set \( C \subseteq B \cap E_j \) such that \( \mathcal{L}^n(C) > 0 \) and the restriction \( W_0|_C \) is continuous. The map \( \varphi \circ W_0|_C \) takes its values in the closed ball \( \text{Clos} \ U_j \), therefore admits a continuous extension \( Y : \mathbb{R}^n \to \text{Clos} \ U_j \subseteq V \). Letting \( W = \varphi^{-1} \circ \rho \circ Y \) completes the proof. □

3. Common setting

3.1 (Setting for the next three sections). — In the next three sections we shall assume the following.

1. \( E \subseteq \mathbb{R}^n \) is Borel and \( \mathcal{L}^n(E) < \infty \).
2. \( U \subseteq \mathbb{R}^n \) is open and \( E \subseteq U \).
3. \( B \subseteq \mathbb{R}^n \) is Borel.
4. \( W_0 : E \to G(n, m) \) is Lipschitz.
5. \( W(x) = x + W_0(x) \) for each \( x \in E \).
6. \( \Lambda > 0 \).
7. \( w_1, \ldots, w_m : U \to \mathbb{R}^n \) and \( \text{Lip} w_i \leq \Lambda, i = 1, \ldots, m \).
8. \( v_1, \ldots, v_{n-m} : U \to \mathbb{R}^m \) and \( \text{Lip} v_i \leq \Lambda, i = 1, \ldots, n-m \).
9. \( W_0(x) = \text{span}\{w_1(x), \ldots, w_m(x)\} \) for every \( x \in E \).
10. \( W_0(x)^\perp = \text{span}\{v_1(x), \ldots, v_{n-m}(x)\} \) for every \( x \in E \).
11. \( w_1(x), \ldots, w_m(x), v_1(x), \ldots, v_{n-m}(x) \) constitute an orthonormal basis of \( \mathbb{R}^n \), for every \( x \in E \).

4. Two fibrations

4.1 (A fibered space associated with \( E, B, w_1, \ldots, w_m \)). — We define

\[
F : E \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m : (x, t_1, \ldots, t_m) \mapsto \left( x, x + \sum_{i=1}^{m} t_i \cdot w_i(x) \right)
\]
as well as

\[
\Sigma = F(E \times \mathbb{R}^m) = \mathbb{R}^n \times \mathbb{R}^m \cap \{(x, u) : x \in E \text{ and } u \in W(x)\}.
\]

It is obvious that \( F \) is Lipschitz and therefore \( \Sigma \) is countably \( n + m \) rectifiable and \( \mathcal{H}^{n+m} \) measurable. We also consider the two canonical projections

\[
\pi_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n : (x, u) \mapsto x \quad \text{and} \quad \pi_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m : (x, u) \mapsto u
\]
as well as

\[
\Sigma_B = \Sigma \cap \pi_1^{-1}(B) = \mathbb{R}^n \times \mathbb{R}^m \cap \{(x, u) : x \in E \text{ and } u \in B \cap W(x)\}.
\]

which is clearly also countably \( n + m \) rectifiable and \( \mathcal{H}^{n+m} \) measurable. In view of applying the coarea formula to \( \Sigma_B \) and \( \pi_1 \) first, to \( \Sigma_B \) and \( \pi_2 \) next, we observe that

\[
\Sigma_B \cap \pi_1^{-1}\{x\} = \mathbb{R}^n \times \mathbb{R}^m \cap \{(x, u) : u \in B \cap W(x)\}
\]

so that

\[
\mathcal{H}^m \left( \Sigma_B \cap \pi_1^{-1}\{x\} \right) = \mathcal{H}^m \left( B \cap W(x) \right)
\]

whenever \( x \in E \), and that

\[
\Sigma_B \cap \pi_2^{-1}\{u\} = \mathbb{R}^m \times \mathbb{R}^m \cap \{(x, u) : x \in E \text{ and } u \in W(x)\}
\]

\[
= \mathbb{R}^n \times \mathbb{R}^n \cap \{(x, u) : x \in E \cap \bigcap_{i=1}^{n-m} \{s_i \in \mathbb{R}^m : s_i = 0\}\}
\]
Finally, is differentiable at generated by the following formulæ to be useful we need to establish bounds for the coarea Jacobian factors and the conclusion follows. □

We recall 2.3. The right hand inequality follows from according to (5), so that whenever $L$: 

$$\partial F/\partial x_j(x,t) = \left( e_j, e_j + \sum_{i=1}^{m} t_i \partial \omega_i(x) \right), \quad j = 1, \ldots, n$$

$$\partial F/\partial t_k(x,t) = (0, \omega_k(x)), \quad k = 1, \ldots, m.$$ 

As usual $e_1, \ldots, e_n$ denotes the canonical basis of $R^n$.

4.2 (Coarea Jacobian factor of $\pi_1$). — For $H_n$ almost every $(x,u) \in \Sigma$ one has

$$\left( 2 + 2m\Lambda |x - u| + m^2\Lambda^2 |x - u|^2 \right)^{-2} \leq J_{\Sigma}(x,u) \leq 1.$$  

Proof. We recall 2.3. The right hand inequality follows from $\text{Lip } \pi_1 = 1$. Regarding the left hand inequality fix $(x,u) = F(x,t)$ such that $F$ is differentiable at $(x,t)$ and let $L: T(x,u) \Sigma \rightarrow R^n$ denote the restriction of $\pi_1$ to $T(x,u) \Sigma$. Put $v_j = \partial F/\partial x_j(x,t), \quad j = 1, \ldots, n$, and recall (3) that

$$J_{\Sigma}(x,u) = || \bigwedge_n L || \geq \frac{1}{|v_1 \wedge \ldots \wedge v_n|}$$ 

since $L(v_j) = e_j, \quad j = 1, \ldots, n$. Now notice that

$$\left| \frac{\partial F}{\partial x_j}(x,t) \right|^2 = |e_j|^2 + \sum_{i=1}^{m} t_i \frac{\partial \omega_i(x)}{\partial x_j}^2 \leq 2 \left( \sum_{i=1}^{m} t_i \frac{\partial \omega_i(x)}{\partial x_j} + \sum_{i=1}^{m} t_i \frac{\partial \omega_i(x)}{\partial x_j} \right)^2 \leq 2 + 2m\Lambda |t| + m^2\Lambda^2 |t|^2.$$ 

Since $u = x + \sum_{i=1}^{m} t_i \omega_i(x)$ one also has

$$|u - x|^2 = \left( \sum_{i=1}^{m} t_i \omega_i(x) \right)^2 = |t|^2.$$ 

Finally,

$$|v_1 \wedge \ldots \wedge v_n| = \left| \frac{\partial F}{\partial x_1}(x,t) \wedge \ldots \wedge \frac{\partial F}{\partial x_1}(x,t) \right| \leq \prod_{j=1}^{n} \left| \frac{\partial F}{\partial x_j}(x,t) \right| \leq \left( 2 + 2m\Lambda |x - u| + m^2\Lambda^2 |x - u|^2 \right)^{\frac{1}{2}}$$ 

and the conclusion follows. □

4.3. — Let $1 \leq q \leq n - 1$ be an integer and let $v_1, \ldots, v_q$ be an orthonormal family in $R^n$. There then exists $\lambda \in \Lambda(n,q)$ such that

$$\det (\langle v_k, e_{(j)} \rangle)_{j,k=1,\ldots,q} \geq \left( \frac{n}{q} \right)^{-\frac{1}{2}}.$$
Proof. We define a linear map $L : \mathbb{R}^q \rightarrow \mathbb{R}^n : (s_1, \ldots, s_q) \mapsto \sum_{k=1}^{q} s_k v_k$ and we observe that $L$ is an isometry. Therefore its area Jacobian factor $JL = 1$, by definition. Now also

$$(JL)^2 = \sum_{k \in \Lambda(n,q)} \left| \det \left( \langle v_k, e_{t(j)} \rangle \right)_{j,k=1,\ldots,q} \right|^2$$

according to the Binet-Cauchy formula [4, Chapter 3 §2 Theorem 4]. The conclusion easily follows.

\[ \square \]

4.4 (Coarea Jacobian factor of $\pi_2$). — The following hold.

(1) For $\mathcal{H}^{n+m}$ almost every $(x, u) \in \Sigma$ one has

$$\left( \frac{n}{n-m} \right)^{-\frac{1}{2}} \left( 2^{n-m} - 1 \right) m \lambda |u - x| \left( 2 + 2m \lambda |x - u| + m^2 \lambda^2 |x - u|^2 \right)^{\frac{n-m}{2}} \leq J_2 \pi_2(x, u) \leq 1.$$ 

(2) For $\mathcal{H}^{n+m}$ almost every $(x, u) \in \Sigma$ one has $J_2 \pi_2(x, u) > 0$.

Proof. Clearly $J_2 \pi_2(x, u) \leq (\text{Lip } \pi_2)^n \leq 1$. Regarding the left hand inequality fix $(x, u) \in F (x, t)$ such that $F$ is approximately differentiable at $(x, t)$ and this time let $L : T_{(x,u)}\Sigma \rightarrow \mathbb{R}^n$ denote the restriction of $\pi_2$ to $T_{(x,u)}\Sigma$. We will now define a family of $n$ vectors $v_1, \ldots, v_n$ belonging to $T_{(x,u)}\Sigma$. We choose $v_k = \frac{\partial F}{\partial x_k} (x, t) = (0, w_k (x))$ for $k = 1, \ldots, m$. For choosing the $n - m$ remaining vectors we proceed as follows. We select $\lambda \in \Lambda(n, n-m)$ as in [4, 3] applied with $q = n-m$ to $v_1 (x), \ldots, v_{n-m} (x)$, and we let $v_{m+j} = \frac{\partial F}{\partial x_{t(k)}} (x, t)$, $j = 1, \ldots, n-m$. Recalling [3] we have

$$J_2 \pi_2 (x, u) = \| L \|^2 \geq \frac{|L (v_1) \wedge \ldots \wedge L (v_n)|}{|v_1 \wedge \ldots \wedge v_n|}.$$ 

As in the proof of [4, 2] we find that

$$|v_1 \wedge \ldots \wedge v_n| \leq \left( 2 + 2m \lambda |x - u| + m^2 \lambda^2 |x - u|^2 \right)^{\frac{n-m}{2}}$$

and it remains only to find a lower bound for $|L (v_1) \wedge \ldots \wedge L (v_n)|$. This equals the absolute value of the determinant of the matrix of coefficients of $L (v_i), i = 1, \ldots, n$, with respect to any orthonormal basis of $\mathbb{R}^n$. We choose the basis $w_1 (x), \ldots, w_m (x), v_1 (x), \ldots, v_{n-m} (x)$. Thus

$$|L (v_1) \wedge \ldots \wedge L (v_n)| = \left| \det \left( \begin{array}{ccc} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \end{array} \right) \left( \begin{array}{c} e_{t(j)} + \sum_{i=1}^{m} t_i \frac{\partial w_i}{\partial x_{t(j)}} (x), v_k (x) \end{array} \right) \right|$$

(13)

$$= \left| \det \left( \begin{array}{c} e_{t(j)} + \sum_{i=1}^{m} t_i \frac{\partial w_i}{\partial x_{t(j)}} (x), v_k (x) \end{array} \right) \right|_{j,k=1,\ldots,n-m}.$$ 

Abbreviate

$$h_{t(j)} = \sum_{i=1}^{m} t_i \frac{\partial w_i}{\partial x_{t(j)}} (x)$$

and observe that $|h_{t(j)}| \leq m \lambda |t| = m \lambda |x - u|$, $j = 1, \ldots, n-m$ (recall the proof of [4, 2]). It remains only to remember that $\lambda$ has been selected in order that

$$\left| \det \left( \langle h_{t(j)}, v_k (x) \rangle \right)_{j,k=1,\ldots,n-m} \right| \geq \left( \frac{n}{n-m} \right)^{-\frac{1}{2}}$$
and to infer from the multilinearity of the determinant that
\[
\left| \det \left( \langle e_{kj}, v_k(x) \rangle + \langle h_{kj}, v_k(x) \rangle \right) \right|
\leq (2^{n-m} - 1) \left( \max_{j=1,\ldots,n-m} \left| \langle h_{kj}, v_k(x) \rangle \right| \right) \left( \max_{j,k=1,\ldots,m} \left| \langle e_{kj}, v_k(x) \rangle \right| \right)^{n-m-1}
\leq (2^{n-m} - 1) \lambda |x - u|.
\]

This completes the proof of conclusion (1).

Let \( E_0 \) denote the subset of \( E \) consisting of those \( x \) such that each \( w_i, i = 1, \ldots, m \), is differentiable at \( x \). Thus \( E_0 \) is Borel and so is
\[
A = E_0 \times \mathbb{R}^m \cap \{(x, t) : \text{rank} \left\{ w_1(x) \mid \cdots \mid w_m(x) \mid e_1 + \sum_{i=1}^m t_i \frac{\partial w_i}{\partial x_k}(x) \mid \cdots \mid e_n + \sum_{i=1}^m t_i \frac{\partial w_i}{\partial x_k}(x) \right\} < n \}.
\]

If \( (x, u) \in \Sigma \sim F(A) \) then the restriction of \( \pi_2 \) to \( T_{(x,u)} \Sigma \) is surjective and therefore \( J_2 \pi_2(x, u) > 0 \). Thus we ought to show that \( \mathcal{H}^{n+m}(F(A)) = 0 \). Since \( F \) is Lipschitz it suffices to establish that \( \mathcal{L}^{m}(A_x) = 0 \). As \( A \) is Borel it is enough to prove that \( \mathcal{L}^{m}(A_x) = 0 \) for every \( x \in E_0 \), according to Fubini’s theorem. Fix \( x \in E_0 \). As in the proof of conclusion (1), choose \( \lambda \in \mathcal{A}(n, n-m) \) associated with \( v_1(x), \ldots, v_m(x) \) according to \( 4.3 \). Based on \( 13 \) we see that
\[
A_x \subseteq \mathbb{R}^m \cap \{ t : \det \left( \left( e_{kj} + \sum_{i=1}^m t_1 \frac{\partial w_i}{\partial x_k}(x) \right)_{j,k=1,\ldots,m-n} \right) = 0 \}.
\]
The set on the right is of the form \( S_x = \mathbb{R}^m \cap \{(t_1, \ldots, t_m) : P_x(t_1, \ldots, t_m) = 0 \} \) for some polynomial \( P_x \in \mathbb{R}[T_1, \ldots, T_m] \), and \( P_x(0, \ldots, 0) = \det (\langle e_{kj}, v_k(x) \rangle)_{j,k=1,\ldots,n-m} \neq 0 \). It follows that \( \mathcal{L}^{m}(S_x) = 0 \), see e.g. \( 2.6.5 \) and the proof of (2) is complete. \( \square \)

4.5. Proposition. — The measure \( \phi_{E,W} \) is absolutely continuous with respect to \( \mathcal{L}^{n} \).

Proof. Let \( B \in \mathcal{B}(\mathbb{R}^n) \) be such that \( \mathcal{L}^{n}(B) = 0 \). It follows from \( 12 \) that
\[
\int_{\Sigma_B} J_2 \pi_2 d\mathcal{H}^{n+m} = 0.
\]
It next follows from \( 4.4 \) that \( \mathcal{H}^{n+m}(\Sigma_B) = 0 \). In turn \( 11 \) implies that
\[
\phi_{E,W}(B) = \int_{\Sigma_B} J_2 \pi_1 d\mathcal{H}^{n+m} = 0.
\]

\( \square \)

4.6 (definition of \( \mathcal{L}^{E}_{W} \)). — Note that \( \phi_{E,W} \) is a \( \sigma \) finite Borel measure on \( \mathbb{R}^n \) (see \( 2.16 \)) and that it is absolutely continuous with respect to \( \mathcal{L}^{n} \) (see \( 4.5 \)). It then ensues from the Radon-Nikodým Theorem that there exists a Borel measurable function
\[
\mathcal{L}^{E}_{W} : E \to \mathbb{R}
\]
such that for every \( B \in \mathcal{B}(\mathbb{R}^n) \) one has
\[
\int_{E} \mathcal{H}^{n}(B \cap W(x)) d\mathcal{L}^{n}(x) = \phi_{E,W}(B) = \int_{B} \mathcal{L}^{E}_{W}(u) d\mathcal{L}^{n}(u).
\]
Furthermore \( \mathcal{L}^{E}_{W} \) is univocally defined only up to a \( \mathcal{L}^{n} \) null set. This will not affect the reasonings in this paper. Each time we will write \( \mathcal{L}^{E}_{W} \) we will mean one particular Borel measurable function verifying the above equality for every \( B \in \mathcal{B}(\mathbb{R}^n) \).
We define \( \mathcal{Y}_E^0 W : \mathbb{R}^n \to [0, \infty] \) by the formula
\[
\mathcal{Y}_E^0 W(u) = \mathcal{H}^{m-1}(E \cap \overline{g_{y_1, \ldots, y_{n-m}}^{-1}(0)})
\]
for every \( u \in \mathbb{R}^n \). Letting \( B = \mathbb{R}^n \) in \((10)\) one infers from \(2.3\) that \( \mathcal{Y}_E^0 W \) is \( \mathcal{L}^n \) measurable. Using the estimates we have established so far regarding coarea Jacobian factors we now show that \( \mathcal{Y}_E W \) and \( \mathcal{Y}_E^0 W \) are comparable when the diameter of \( E \) is not too large.

**4.8. Proposition.** — Given \( 0 < \varepsilon < 1 \) there exists \( \delta_{4.8}(n, \Lambda, \varepsilon) > 0 \) with the following property. If \( \delta \leq \delta_{4.8}(n, \Lambda, \varepsilon) \) then
\[
(1 - \varepsilon)2^{\frac{n-m}{m}} \mathcal{Y}_E W(u) \leq \mathcal{Y}_E^0 W(u) \leq (1 + \varepsilon)2^{\frac{n-m}{m}} \mathcal{Y}_E W(u)
\]
for \( \mathcal{L}^n \) almost every \( u \in E \).

**Proof.** We readily infer from \((4.1)\) and \((4.4)\) that there exists \( \delta(n, \Lambda, \varepsilon) > 0 \) such that for \( \mathcal{H}^{n+m} \) almost all \((x, u) \in \Sigma\) if \(|x - u| \leq \delta(n, \Lambda, \varepsilon)\) then
\[
\alpha := (1 - \varepsilon)2^{-\frac{m}{2}} \leq J_\Sigma \pi_1(x, u)
\]
and
\[
\beta := (1 + \varepsilon)2^{-\frac{m}{2}} \leq J_\Sigma \pi_2(x, u)
\]
where the above define \( \alpha \) and \( \beta \).

Assume now that \( \delta \leq \delta(n, \Lambda, \varepsilon) \). Given \( B \in \mathcal{B}(\mathbb{R}) \) we infer from \((11), (3.2), (4.4)\) and \((12)\) and the above lower bounds that
\[
\phi_{E, W}(B) = \int_{\Sigma_B} J_\Sigma \pi_1 d \mathcal{H}^{n+m} \geq \alpha \mathcal{H}^{n+m}(\Sigma_B) \geq \alpha \int_{\Sigma_B} J_\Sigma \pi_2 d \mathcal{H}^{n+m}
\]
and
\[
\phi_{E, W}(B) = \int_{\Sigma_B} J_\Sigma \pi_2 d \mathcal{H}^{n+m} \leq \beta^{-1} \int_{\Sigma_B} J_\Sigma \pi_2 d \mathcal{H}^{n+m}
\]
Thus
\[
\beta^{-1} \mathcal{Y}_E^0 W d \mathcal{L}^n \leq \int_B \mathcal{Y}_E^0 W d \mathcal{L}^n \leq \int_B \mathcal{Y}_E W d \mathcal{L}^n \leq \int_B \alpha \mathcal{Y}_E^0 W d \mathcal{L}^n
\]
for every \( B \in \mathcal{B}(\mathbb{R}^n) \). The conclusion follows from the \( \mathcal{L}^n \) measurability of both \( \mathcal{Y}_E W \) and \( \mathcal{Y}_E^0 W \). \( \square \)

**4.9 (Rest stop).** — The above upper bound for \( \mathcal{Y}_E W \) is already enough to bound it in turn, by a constant times \( (\text{diam } E)^m \), see \((5.4)\) However I would not know how to use the above lower bound to establish that \( \mathcal{Y}_E W > 0 \) almost everywhere in \( E \), which is what we are after. Indeed in the definition \((14)\) of \( \mathcal{Y}_E^0 W(u) \), \( u \) does not appear as the covariable of the function whose level set we are measuring, thereby preventing the use of the coarea formula in an attempt to estimate \( \mathcal{Y}_E^0 W(u) \). This naturally leads to adding a variable \( y \in \mathbb{R}^{n-m} \) to the fibered space \( \Sigma \), a covariable for \( \mathcal{Y}_{E, E, E, \ldots, E, v_{n-m}} \).

**4.10 (A fibered space associated with \( E, B, w_1, \ldots, w_m, v_1, \ldots, v_{n-m} \)).** — Let \( r > 0 \), and abbreviate \( C_r = \mathbb{R}^{n-m} \cap \{y : |y| \leq r\} \) the Euclidean ball centered at the origin, of radius \( r \) in \( \mathbb{R}^{n-m} \). We define
\[
F_r : E \times \mathbb{R}^n \times C_r \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n-m} : (x, t, y) \mapsto \left( x, x + \sum_{i=1}^m t_i w_i(x) + \sum_{i=1}^{n-m} y_i v_i(x), y \right)
\]
and
\[ \Sigma_r = \tilde{F}_r (E \times \mathbb{R}^n \times C_r) = \mathbb{R}^n \times \mathbb{R}^n \times C_r \cap \left\{ (x, u, y) : x \in E \text{ and } u \in W(x) + \sum_{i=1}^{n-m} y_i v_i(x) \right\} \]
so that \( \tilde{F}_r \) is Lipschitz and \( \Sigma_r \) is countably 2n rectifiable and \( \mathcal{H}^{2n} \) measurable. Similarly to [4.1] we define
\[ \hat{\Sigma}_{r,B} = \hat{\Sigma}_r \cap \pi_2^{-1}(B) \]
which clearly is also countably 2n rectifiable and \( \mathcal{H}^{2n} \) measurable. We aim to apply the coarea formula to \( \hat{\Sigma}_{r,B} \) and to the two projections
\[ \pi_1 \times \pi_3 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n-m} \to \mathbb{R}^n \times \mathbb{R}^{n-m} : (x, u, y) \mapsto (x, y) \]
and
\[ \pi_2 \times \pi_3 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n-m} \to \mathbb{R}^n \times \mathbb{R}^{n-m} : (x, u, y) \mapsto (u, y) . \]
To this end we notice that
\[ \hat{\Sigma}_{r,B} \cap (\pi_1 \times \pi_3)^{-1} \{ (x, y) \} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n-m} \cap \left\{ (x, u, y) : u \in B \cap \left( W(x) + \sum_{i=1}^{n-m} y_i v_i(x) \right) \right\} \]
and thus
\[ \mathcal{H}^m \left( \hat{\Sigma}_{r,B} \cap (\pi_1 \times \pi_3)^{-1} \{ (x, y) \} \right) = \mathcal{H}^m \left( B \cap \left( W(x) + \sum_{i=1}^{n-m} y_i v_i(x) \right) \right) \]
for every \((x, y) \in E \times C_r\). We further notice that
\[ \hat{\Sigma}_{r,B} \cap (\pi_2 \times \pi_3)^{-1} \{ (u, y) \} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n-m} \cap \left\{ (x, u, y) : x \in E \text{ and } u \in W(x) + \sum_{i=1}^{n-m} y_i v_i(x) \right\} \]
\[ = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n-m} \cap \left\{ (x, u, y) : x \in E \cap g_{v_1, \ldots, v_{n-m}, u}^{-1}(y) \right\} , \]
because
\[ u \in W(x) + \sum_{i=1}^{n-m} y_i v_i(x) \iff u - x = \sum_{i=1}^{n-m} y_i v_i(x) \in W(x) \]
\[ \iff v_j(x), u - x - \sum_{i=1}^{n-m} y_i v_i(x) = 0 \text{ for all } j = 1, \ldots, n - m \]
\[ \iff g_{v_1, \ldots, v_{n-m}, u}(x) = y \]
and therefore
\[ \mathcal{H}^m \left( \hat{\Sigma}_{r,B} \cap (\pi_2 \times \pi_3)^{-1} \{ (u, y) \} \right) = \mathcal{H}^m \left( E \cap g_{v_1, \ldots, v_{n-m}, u}^{-1}(y) \right) \]
whenever \( u \in B \) and \( y \in C_r \).

It now follows from the coarea formula and Fubini’s theorem that
\[ \int_{\Sigma_r} J_{\Sigma_r} (\pi_1 \times \pi_3) d \mathcal{H}^{2n} = \int_E d\mathcal{L}^n (x) \int_{\Sigma_r} J_{\Sigma_r} \mathcal{H}^m \left( B \cap \left( W(x) + \sum_{i=1}^{n-m} y_i v_i(x) \right) \right) d\mathcal{L}^n (y) \]
and that
\[ \int_{\Sigma_r} J_{\Sigma_r} (\pi_2 \times \pi_3) d \mathcal{H}^{2n} = \int_B d\mathcal{L}^n (u) \int_{C_r} J_{\Sigma_r} \mathcal{H}^m \left( E \cap g_{v_1, \ldots, v_{n-m}, u}^{-1}(y) \right) d\mathcal{L}^n (y) . \]
4.11 (Coarea Jacobian factors of $\pi_1 \times \pi_3$ and $\pi_2 \times \pi_3$). — The following inequalities hold for $\mathcal{H}^{2n}$ almost every $(x, u, y) \in \Sigma_r$.

$$2^{\frac{n^2}{2}} \left( 2 + 4n\Lambda|u - x| + 3n^2\Lambda^2|u - x|^2 \right)^{-\frac{n}{2}} \leq J_{\Sigma_r}(\pi_1 \times \pi_3)(x, u, y)$$

and

$$J_{\Sigma_r}(\pi_2 \times \pi_3)(x, u, y) \leq 1.$$

Proof. The second conclusion is obvious since $\text{Lip } \pi_2 \times \pi_3 = 1$. Regarding the first conclusion we reason similarly as in the proof of [4,2]. Fix $(x, u, y) = F_r(x, t, y)$ such that $F_r$ is differentiable at $(x, t, y)$ and denote by $L$ the restriction of $\pi_1 \times \pi_3$ to $T_{(x,u,y)}\Sigma_r$. This tangent space is generated by the following $2n$ vectors

$$\frac{\partial F_r}{\partial x_j}(x, t, y) = \left( e_j, e_j + \sum_{i=1}^m t_i \frac{\partial w_i(x)}{\partial x_j} + \sum_{i=1}^{n-m} y_i \frac{\partial v_i(x)}{\partial x_j}, 0 \right), \quad j = 1, \ldots, n$$

$$\frac{\partial F_r}{\partial y_k}(x, t, y) = (0, w_k(x), 0), \quad k = 1, \ldots, m$$

$$\frac{\partial F_r}{\partial y_{\ell}}(x, t, y) = (0, v_{\ell}(x), e_{\ell}), \quad \ell = 1, \ldots, n - m.$$

The range of $\pi_1 \times \pi_3$ being $2n - m$ dimensional we need to select $2n - m$ vectors $v_1, \ldots, v_{2n-m}$ in $T_{(x,u,y)}\Sigma_r$ to obtain a lower bound

$$J_{\Sigma_r}(\pi_1 \times \pi_3)(x, u, y) = \| \wedge_{2n-m} L \| \geq \left| \frac{L(v_1) \wedge \ldots \wedge L(v_{2n-m})}{v_1 \wedge \ldots \wedge v_{2n-m}} \right|. \quad (19)$$

The obvious choice consists of $v_j = \frac{\partial F_r}{\partial x_j}(x, t, y)$, $j = 1, \ldots, n$, and $v_{n+j} = \frac{\partial F_r}{\partial y_j}(x, t, y)$, $\ell = 1, \ldots, n - m$, so that $L(v_1), \ldots, L(v_{2n-m})$ is the canonical basis of $\mathbb{R}^n \times \mathbb{R}^{n-m}$ and therefore the numerator in (19) equals 1. In order to determine an upper bound for its denominator we start by fixing $j = 1, \ldots, n$, we abbreviate $a_j(x, t, y) = \sum_{i=1}^m t_i \frac{\partial w_i(x)}{\partial x_j}$ and $b_j(x, t, y) = \sum_{i=1}^{n-m} y_i \frac{\partial v_i(x)}{\partial x_j}$ and we notice that $|a_j(x, t, y)| \leq m\Lambda|t| \leq n\Lambda|t|$, $|b_j(x, t, y)| \leq (n - m)\Lambda|y| \leq n\Lambda|y|$. Furthermore since $u - x = \sum_{i=1}^m t_i w_i(x) + \sum_{i=1}^{n-m} y_i v_i(x)$ one has $|u - x|^2 = |t|^2 + |y|^2 \leq \max(|t|^2, |y|^2)$. Therefore

$$\left| \frac{\partial F_r}{\partial x_j}(x, t, y) \right|^2 = |e_j|^2 + |a_j(x, t, y) + b_j(x, t, y)|^2$$

$$\leq 1 + 1 + |a_j(x, t, y)|^2 + |b_j(x, t, y)|^2$$

$$+ 2|a_j(x, t, y)| + 2|b_j(x, t, y)| + 2|a_j(x, t, y)||b_j(x, t, y)|$$

$$\leq 2 + 4n\Lambda|u - x| + 3n^2\Lambda^2|u - x|^2.$$

Moreover

$$\left| \frac{\partial F_r}{\partial y_{\ell}}(x, t, y) \right| = \sqrt{2}$$

for each $\ell = 1, \ldots, n - m$. We conclude that

$$|v_1 \wedge \ldots \wedge v_{2n-m}| \leq \left( \prod_{j=1}^n \left| \frac{\partial F_r}{\partial x_j}(x, t, y) \right| \right) \left( \prod_{\ell=1}^{n-m} \left| \frac{\partial F_r}{\partial y_{\ell}}(x, t, y) \right| \right)$$

$$\leq 2^{\frac{n^2}{2}} \left( 2 + 4n\Lambda|u - x| + 3n^2\Lambda^2|u - x|^2 \right)^{\frac{n}{2}}$$

and the proof is complete. \qed
4.12 (Definition of $\mathcal{B}_E W$). — It follows from the coarea theorem that the function
\[ R^n \times R^{n-m} \to [0, \infty] : (u, y) \mapsto H^m \left( E \cap g_{v_1, \ldots, v_{n-m}, u} \{ y \} \right) \]
is $L^m \otimes L^{n-m}$ measurable (recall 4.10 applied with $B = R^n$). It now follows from Fubini's theorem that for each $r > 0$ the function
\[ R^n \to [0, \infty] : u \mapsto \int_{C_r} H^m \left( E \cap g_{v_1, \ldots, v_{n-m}, u} \{ y \} \right) dL^{n-m}(y) \]
is $L^n$ measurable. In turn the function
\[ \mathcal{B}_E W : R^n \to [0, \infty] : u \mapsto \liminf_j \int_{C_{r^j}} H^m \left( E \cap g_{v_1, \ldots, v_{n-m}, u} \{ y \} \right) dL^{n-m}(y) \]
is $L^n$ measurable. It is a replacement for $\mathcal{B}_E^0 W$ defined in 4.7. We shall establish for $\mathcal{B}_E W$ a similar lower bound to that in 4.8 this time involving $\mathcal{B}_E W$. Before doing so, we notice the rather trivial fact that if $F \subseteq E$ then
\[ \mathcal{B}_E W(u) \leq \mathcal{B}_E W(u) \]
for all $u \in R^n$.

4.13 (Preparatory Remark for the Proof of 4.15). — It follows from the coarea theorem that the function
\[ R^n \times R^{n-m} \to [0, \infty] : (x, y) \mapsto H^m \left( B \cap \left( W(x) + \sum_{i=1}^{n-m} y_i v_i(x) \right) \right) \]
is $L^m \otimes L^{n-m}$ measurable (recall 4.10 applied with $B = R^n$). It therefore follows from Fubini’s theorem as in 4.12 that
\[ f_j : R^n \to [0, \infty] : x \mapsto \limsup_j \int_{C_{r^j}} H^m \left( B \cap \left( W(x) + \sum_{i=1}^{n-m} y_i v_i(x) \right) \right) dL^n(y) \]
is $L^n$ measurable. Furthermore if $B$ is bounded then $|f_j(x)| \leq \alpha(m)(\text{diam } B)^m$ for every $x \in R^n$.

4.14. — If $B$ is compact then for every $x \in E$ the function
\[ R^{n-m} \to R^m : y \mapsto H^m \left( B \cap \left( W(x) + \sum_{i=1}^{n-m} y_i v_i(x) \right) \right) \]
is upper semicontinuous.

Proof. For each $y \in R^{n-m}$ define the compact set $K_y = B \cap \left( W(x) + \sum_{i=1}^{n-m} y_i v_i(x) \right)$. If $(y_k)_k$ is a sequence converging to $y$ we ought to show that
\[ H^m \left( K_{y_k} \right) \geq \limsup_k H^m \left( K_{y_k} \right) . \]

Since each $K_y$ is a subset of an $m$ dimensional affine subspace of $R^n$ this is indeed equivalent to the same inequality with $H_{\infty}^m$ replaced by $H^m$ according to 2.2 (3). Considering if necessary a subsequence of $(y_k)_k$ we may assume that none of the compact sets $K_{y_k}$ is empty and the the above lim sup is a lim. Considering yet a further subsequence we may now assume that $(K_{y_k})_k$ converges in Hausdorff distance to some compact set $L \subseteq B$. One checks that $L \subseteq K_y$. It then follows from 2.2 (1) that $H^m (K_y) \geq H^m (L) \geq \limsup_k H^m (K_{y_k})$.

4.15. Proposition. — Given $0 < \varepsilon < 1$ there exists $\delta \in \mathbb{R}^+ (n, \Lambda, \varepsilon) > 0$ with the following property: If $\text{diam}(E \cup B) \leq \delta \in \mathbb{R}^+(n, \Lambda, \varepsilon)$ and if $B$ is compact then
\[ \int_E H^m (B \cap W(x)) dL^n(x) \geq (1 - \varepsilon) 2^{-\frac{2m}{p}} \int_B \mathcal{B}_E W(u) dL^n(u) . \]
Proof. We first observe that we can choose $\delta_{\text{1.15}}(n, \Lambda, \varepsilon) > 0$ small enough so that

$$J_{\varepsilon}(\pi_1 \times \pi_2)(x, u, y) \geq (1 - \varepsilon)2^{-\frac{n-2m}{2}}$$

for $\mathcal{H}^2$ almost every $(x, u, y) \in \hat{S}_r$, provided $|u - x| \leq \delta_{\text{1.15}}(n, \Lambda, \varepsilon)$, according to \text{4.11}. Thus (20) holds for $\mathcal{H}^2$ almost every $(x, u, y) \in \hat{S}_r$ under the assumption that $\text{diam}(E \cup B) \leq \delta_{\text{1.15}}(n, \Lambda, \varepsilon)$. When \text{1.7}, \text{1.8} and \text{1.11} imply that

$$\int_E d\mathcal{L}^n(x) \int_{C_r} \mathcal{H}^m \left( B \cap \left( W(x) + \sum_{i=1}^{n-m} y_i v_i(x) \right) \right) d\mathcal{L}^m(y)$$

$$= \int_{\hat{S}_r} J_{\varepsilon}(\pi_1 \times \pi_2) d\mathcal{H}^2$$

$$\geq (1 - \varepsilon)2^{-\frac{n-2m}{2}} \mathcal{H}^2 \left( \hat{S}_r \right)$$

$$\geq (1 - \varepsilon)2^{-\frac{n-2m}{2}} \int_{\hat{S}_r} J_{\varepsilon}(\pi_1 \times \pi_2) d\mathcal{H}^2$$

$$= (1 - \varepsilon)2^{-\frac{n-2m}{2}} \int_B d\mathcal{L}^n(u) \int_{C_r} \mathcal{H}^m \left( E \cap g_{\pi_1 \ldots \pi_m}^{-1} (y) \right) d\mathcal{L}^m(y).$$

Fix $x \in E$ and $\beta > 0$. According to (4.14) there exists a positive integer $j(x, \beta)$ such that if $j \geq j(x, \beta)$ then

$$\mathcal{H}^m (B \cap W(x)) + \beta \geq \sup_{y \in C_{j-1}} \mathcal{H}^m \left( B \cap \left( W(x) + \sum_{i=1}^{n-m} y_i v_i(x) \right) \right)$$

$$\geq \int_{C_{j-1}} \mathcal{H}^m \left( B \cap \left( W(x) + \sum_{i=1}^{n-m} y_i v_i(x) \right) \right) d\mathcal{L}^m(y).$$

Taking the lim sup as $j \to \infty$ on the right hand side, and letting $\beta \to 0$ we obtain

$$\mathcal{H}^m (B \cap W(x)) \geq \limsup_{j} \int_{C_{j-1}} \mathcal{H}^m \left( B \cap \left( W(x) + \sum_{i=1}^{n-m} y_i v_i(x) \right) \right) d\mathcal{L}^m(y).$$

As this holds for all $x \in E$ we may integrate over $E$ with respect to $\mathcal{L}^n$. Noticing that for every $j = 1, 2, \ldots$ (with the notation of (4.13) $|f_j| \leq \alpha(m)(\text{diam } B)^m \mathcal{L}^n$, the latter being $\mathcal{L}^n$ summable, justifies the application of the reverse Fatou lemma below. Thus the following ensues from (22), the reverse Fatou lemma, (21), and the Fatou lemma:

$$\int_E \mathcal{H}^m (B \cap W(x)) d\mathcal{L}^n(x)$$

$$\geq \int_E d\mathcal{L}^n(x) \limsup_{j} \int_{C_{j-1}} \mathcal{H}^m \left( B \cap \left( W(x) + \sum_{i=1}^{n-m} y_i v_i(x) \right) \right) d\mathcal{L}^m(y)$$

$$\geq \limsup_{j} \int_E d\mathcal{L}^n(x) \int_{C_{j-1}} \mathcal{H}^m \left( B \cap \left( W(x) + \sum_{i=1}^{n-m} y_i v_i(x) \right) \right) d\mathcal{L}^m(y)$$

$$\geq (1 - \varepsilon)2^{-\frac{n-2m}{2}} \limsup_{j} \int_B d\mathcal{L}^n(u) \int_{C_{j-1}} \mathcal{H}^m \left( E \cap g_{\pi_1 \ldots \pi_m}^{-1} (y) \right) d\mathcal{L}^m(y)$$

$$\geq (1 - \varepsilon)2^{-\frac{n-2m}{2}} \liminf_{j} \int_B d\mathcal{L}^n(u) \int_{C_{j-1}} \mathcal{H}^m \left( E \cap g_{\pi_1 \ldots \pi_m}^{-1} (y) \right) d\mathcal{L}^m(y)$$

$$\geq (1 - \varepsilon)2^{-\frac{n-2m}{2}} \int_B d\mathcal{L}^n(u) \liminf_{j} \int_{C_{j-1}} \mathcal{H}^m \left( E \cap g_{\pi_1 \ldots \pi_m}^{-1} (y) \right) d\mathcal{L}^m(y)$$

$$= (1 - \varepsilon)2^{-\frac{n-2m}{2}} \int_B \mathcal{H}^2 \mathcal{W}(u) d\mathcal{L}^n(u).$$
4.16. **Corollary.** — *If* $0 < \epsilon < 1$ *and* $\text{diam } E \leq \delta_{5.2}(n, \Lambda, \epsilon)$ *then*

$$\mathcal{H}_E^m(\mathbb{R}^n \cap E, W(u)) \geq (1 - \epsilon)^2 2^{-\frac{2m \epsilon}{\Lambda}} \mathcal{H}_E^m(\mathbb{R}^n \cap E, W(u))$$

*for* $\mathbb{R}^n$ *almost every* $u \in E$.

5. **Upper bound for** $\mathcal{H}_E^m$ *and* $\mathcal{H}_E^m$

5.1 (Bow Tie Lemma). — *Let* $S \subseteq \mathbb{R}^n$, $W \in \mathcal{G}(n, m)$ *and* $0 < \tau < 1$. *Assume that*

$$(\forall x \in S)(\forall 0 < \rho \leq \text{diam } S) : S \cap B(x, \rho) \subseteq B(x + W, \tau \rho).$$

*There then exists* $F : P_W(S) \rightarrow \mathbb{R}^n$ *such that* $S = \text{im } F$ *and* $\text{Lip } F \leq \frac{1}{\sqrt{1 - \tau^2}}$. *In particular*

$$\mathcal{H}_E^m(S) \leq \left(\frac{1}{\sqrt{1 - \tau^2}}\right)^m \alpha(m)(\text{diam } S)^m.$$  

**Proof.** Let $x, x' \in S$ *and define* $\rho = |x - x'| \leq \text{diam } S$. *Thus* $x' \in S \cap B(x, \rho)$ *and therefore*

$$|P_W(x - x')| \leq \tau \rho = \tau|x - x'|.$$  

*Since* $|x - x'|^2 = |P_W(x - x')|^2 + |P_W(x - x')|^2$ *we infer that*

$$(1 - \tau^2)|x - x'|^2 \leq |P_W(x - x')|^2.$$  

*Therefore* $P_W|S$ *is injective, and the Lipschitz bound on* $F = (P_W|S)^{-1}$ *clearly follows from the above inequality. Regarding the second conclusion,*

$$\mathcal{H}_E^m(S) = \mathcal{H}_E^m \left( F(P_W(S)) \right) \leq (\text{Lip } F)^m \mathcal{H}_E^m \left( P_W(S) \right),$$

*and* $P_W(S)$ *is contained in a ball of radius diam* $P_W(S) \leq \text{diam } S$.  

5.2. — *Given* $0 < \tau < 1$ *there exists* $\delta_{5.2}(n, \Lambda, \tau) > 0$ *with the following property. If*

1. $x_0 \in U$ *and* $u \in \mathbb{R}^n$;
2. $\text{diam } (E \cup \{x_0\} \cup \{u\}) \leq \delta_{5.2}(n, \Lambda, \tau)$:

*Then: For every* $y \in \mathbb{R}^{n-m}$, *for every* $x \in E \cap g_{y_1, \ldots, y_{n-m}}^{-1}(y)$ *and for every* $0 < \rho < \infty$ *one has*

$$E \cap g_{y_1, \ldots, y_{n-m}}^{-1}(y) \cap B(x, \rho) \subseteq B \left( x + W_0(x_0), \tau \rho \right).$$

**Proof.** We shall show that $g_{y_1, \ldots, y_{n-m}}^{-1}(y) = \frac{\tau}{\sqrt{\Lambda} \sqrt{\Lambda}}$ *will do*. *Let* $x, x' \in E \cap g_{y_1, \ldots, y_{n-m}}^{-1}(y)$ *for some* $y \in \mathbb{R}^{n-m}$. *Thus* $g_{y_1, \ldots, y_{n-m}}(x') = g_{y_1, \ldots, y_{n-m}}(x)$ *and hence*

$$0 = \left| g_{y_1, \ldots, y_{n-m}}(x) - g_{y_1, \ldots, y_{n-m}}(x') \right| = \sqrt{\sum_{i=1}^{n-m} \left| (v_i(x), x - u) - (v_i(x'), x' - u) \right|^2}$$

$$= \sqrt{\sum_{i=1}^{n-m} \left| (v_i(x), x - x') - (v_i(x') - v_i(x), x' - u) \right|^2}$$

$$\geq \sqrt{\sum_{i=1}^{n-m} \left| (v_i(x), x - x') \right|^2} - \sqrt{\sum_{i=1}^{n-m} \left| (v_i(x') - v_i(x), x' - u) \right|^2},$$

*thus*

$$\sqrt{\sum_{i=1}^{n-m} \left| (v_i(x), x - x') \right|^2} \leq \sqrt{\sum_{i=1}^{n-m} \left| (v_i(x') - v_i(x), x' - u) \right|^2}$$

$$\sqrt{n - m \Lambda} |x - x'| |x' - u| \leq \frac{\tau}{2} |x - x'|.$$
In turn,

\[ |P_{W_0(x_0)}(x - x')| = \sqrt{\sum_{i=1}^{n-m} |(v_i(x_0), x - x')|^2} \]

\[ \leq \sqrt{\sum_{i=1}^{n-m} |(v_i(x'), x - x')|^2 + \sum_{i=1}^{n-m} |(v_i(x') - v_i(x_0), x - x')|^2} \]

\[ \leq \frac{\tau}{2}|x - x'| + \sqrt{n - m} \Lambda |x' - x_0| \leq \tau|x - x'|. \]

\[ \square \]

5.3. Proposition. — There are \( \delta_{5,3}(n, \Lambda) > 0 \) and \( c_{5,3}(m) \geq 1 \) with the following property. If \( u \in U \) and \( \text{diam}(E \cup \{u\}) \leq \delta_{5,3}(n, \Lambda) \) then

\[ \max \{ \mathcal{H}^m(E, \text{diam} E) \} < c_{5,3}(m) \mathcal{H}^m(E) \].

Proof. Let \( \delta_{5,3}(n, \Lambda) = \delta_{5,2}(n, \Lambda, 1/2) \). Recall the definitions of \( \mathcal{H}^m(E) \) and \( \mathcal{H}^m(E, \text{diam} E) \) from 4.7 and 4.12 respectively. If \( E = \emptyset \) the conclusion is obvious. If not pick \( x_0 \in E \) arbitrarily.

Given any \( y \in \mathbb{R}^m \) we see that 5.2 applies with \( \tau = 1/2 \) and in turn the bow-tie lemma 5.1 applies to \( S = E \cap g_1^{-1}\ldots g_m^{-1}(y) \) and \( W = W_0(x_0) \). Thus

\[ \mathcal{H}^m(E \cap g_1^{-1}\ldots g_m^{-1}(y)) \leq \left( \frac{2}{\sqrt{3}} \right)^m \alpha(m) r^n \mathcal{H}^m(\text{diam} E)^m. \]

The proposition is proved. \[ \square \]

5.4. Corollary. — There are \( \delta_{5,4}(n, \Lambda) > 0 \) and \( c_{5,4}(n) \geq 1 \) with the following property. If \( \text{diam} E \leq \delta_{5,4}(n, \Lambda) \) then

\[ \mathcal{H}^m(E, \text{diam} E) \leq c_{5,4}(n) \mathcal{H}^m(E)^m \]

for \( \mathcal{L}^m \) almost every \( u \in E \).

Proof. Let \( \delta_{5,4}(n, \Lambda) = \min\{\delta_{5,3}(n, \Lambda), \delta_{5,5}(n, \Lambda, 1/2)\} \). \[ \square \]

6. Lower bound for \( \mathcal{H}^m(E) \) and \( \mathcal{H}^m(E, \text{diam} E) \)

6.1. (Setting for this section). — We enforce again the exact same assumptions as in 3.1 and as in 4.10 we let \( C_r = \mathbb{R}^m - \{ y : |y| \leq r \} \).

6.2. (Polyballs). — Given \( x_0 \in \mathbb{R}^m \) and \( r > 0 \) we define

\[ C_m(x_0, r) = \mathbb{R}^m \cap \{ x : |P_{W_0(x_0)}(x - x_0)| \leq r \} \cup \{ x : |P_{W_0(x_0)}(x - x_0)| \leq r \} \].

We notice that if \( x \in C_m(x_0, r) \) then \( |x - x_0| \leq r \sqrt{2} \), in particular \( \text{diam} C_m(x_0, r) \leq 2\sqrt{2} \).

We also notice that \( \mathcal{L}^m(C_m(x_0, r)) = \alpha(m) \alpha(n - m) r^n \).

With hopes that the following will help the reader form a geometrical imagery: In the next statement \( C_m(x_0, r) \cap g_1^{-1}\ldots g_m^{-1}(C) \) may be seen as a «nonlinear stripe», «horizontal» with respect to \( W_0(x_0), \) «at height» \( g_1^{-1}\ldots g_m^{-1}(x_0) \) with respect to \( x_0, \) and of «width» \( C. \)

6.3. — Given \( 0 < \varepsilon < 1 \) there exists \( \delta_{6,3}(n, \Lambda, \varepsilon) > 0 \) with the following property. If

(1) \( 0 < r < \delta_{6,3}(n, \Lambda, \varepsilon) \);
(2) \( u \in C_m(x_0, r) \subseteq U \);
(3) \( |g_1^{-1}\ldots g_m^{-1}(x_0)| \leq (1 - 3\varepsilon)r \);
(4) \( C \subseteq C_r \) is closed;

then

\[ \mathcal{L}^m(C_m(x_0, r) \cap g_1^{-1}\ldots g_m^{-1}(C)) \geq \frac{1}{1 + \varepsilon} \alpha(m) \alpha(n - m) \mathcal{L}^m(C). \]
Proof. Given \( z \in W_0(x_0) \cap B(0, r) \) we define
\[
V_z = \mathbb{R}^n \left\{ x_0 + z + \sum_{i=1}^{n-m} y_i v_i(x_0) : y \in C_r \right\} \subseteq C_W(x_0, r)
\]
and we consider the isometric parametrization \( \gamma_z : C_r \to V_z \) defined by the formula
\[
\gamma_z(y) = x_0 + z + \sum_{i=1}^{n-m} y_i v_i(x_0).
\]
We also abbreviate \( f_{z,u} = g_{v_1, \ldots, v_{n-m}, u} \circ \gamma_z \).

**Claim 1.** Lip \( f_{z,u} \leq (1 + \varepsilon)^{n-m} \).

Since \( \gamma_z \) is an isometry it suffices to obtain an upper bound for \( \text{Lip} g_{v_1, \ldots, v_{n-m}, u} | C_W(x_0, r) \).

Let \( x, x' \in C_W(x_0, r) \),
\[
\left| g_{v_1, \ldots, v_{n-m}, u}(x) - g_{v_1, \ldots, v_{n-m}, u}(x') \right| \leq \sqrt{\sum_{i=1}^{n-m} \left| (v_i(x) - v_i(x'), x - u) \right|^2 + \sum_{i=1}^{n-m} \left| (v_i(x'), x - x') \right|^2}
\]
\[
\leq \sqrt{n} \Lambda|v_i(x) - v_i(x')| \leq \sqrt{n} \Lambda|x - x'| \leq \varepsilon.
\]

Recalling hypothesis (1) it is now apparent that \( \delta_{6.3} \) can be chosen small enough according to \( n, \Lambda, \varepsilon \) so that Claim 1 holds.

**Claim 2.** For \( \mathbb{R}^{n-m} \) almost every \( y \in C_r \) one has \( \left\| Df_{z,u}(y) - \text{id}_{\mathbb{R}^{n-m}} \right\| \leq \varepsilon \).

Let \( y \in C_r \) be such that \( f_{z,u} \) is differentiable at \( y \). We shall estimate the coefficients of the matrix representing \( Df_{z,u}(y) \) with respect to the canonical basis. Fix \( i, j = 1, \ldots, n-m \) and recall (4):
\[
\frac{\partial}{\partial y_i} (f_{z,u}(y), e_j) = \frac{\partial}{\partial y_i} (g_{v_1, \ldots, v_{n-m}, u}(\gamma_z(y)), e_j)
\]
\[
= \frac{\partial g_{v_1, \ldots, v_{n-m}, u}}{\partial y_i} (\gamma_z(y))
\]
\[
= \nabla g_{v_1, \ldots, v_{n-m}, u}(\gamma_z(y)) \frac{\partial \gamma_z(y)}{\partial y_i}
\]
\[
= \left( Dv_j(\gamma_z(y))(v_i(x_0), \gamma_z(y) - u) + (v_j(\gamma_z(y)), v_i(x_0)) \right)
\]= I + II.
\]

Next notice that
\[
\|I - \delta_{ij}\| = \|I - (v_j(x_0), v_i(x_0))\| = \|v_j(\gamma_z(y)) - v_j(x_0), v_i(x_0)\|
\]
\[
\leq \Lambda |\gamma_z(y) - x_0| \leq \Lambda 2\sqrt{2}r \leq \frac{\varepsilon}{2(n-m)}
\]
where the last inequality follows from hypothesis (1) upon choosing \( \delta_{6.3} \) small enough according to \( n, \Lambda, \varepsilon \). Moreover,
\[
\|I\| \leq \Lambda |\gamma_z(y) - u| \leq \Lambda 2\sqrt{2}r \leq \frac{\varepsilon}{2(n-m)}.
\]
Therefore if \((a_{ij})_{i,j=1,\ldots,n-m}\) is the matrix representing \(Df_{z,u}(y)\) with respect to the canonical basis we have shown that \(|a_{ij} - \delta_{ij}| \leq \frac{\varepsilon}{n-m}\) for all \(i, j = 1, \ldots, n-m\). This completes the proof of Claim #2.

Claim #3. \(C_{\varepsilon r} \subseteq f_{z,u}(C_r)\).

We shall show that \(|y - f_{z,u}(y)| \leq (1 - \varepsilon)r\) for every \(y \in \text{Bdry} \ C_r\) and the conclusion will become a standard application of homology theory, see e.g. [3, 4.6.1]. In case \(m < n-1\) it is clearly enough to establish this inequality for \(H^{n-m-1}\) almost every \(y \in \text{Bdry} \ C_r\): according to the coarea theorem [7, 3.2.22] we choose such \(y\) in order that \(f_{z,u}\) is differentiable \(H^1\) almost everywhere on the line segment \(R^{n-m} \cap \{sy : 0 \leq s \leq 1\}\). It then follows from Claim #2 that

\[|f_{z,u}(y) - f_{z,u}(0) - y| = \int_0^1 Df_{z,u}(sy)(y)d\mathcal{L}^1(s) - y\]

\[\leq \int_0^1 |Df_{z,u}(sy)(y) - y|d\mathcal{L}^1(s) - y| = \varepsilon|y| = \varepsilon r.
\]

Accordingly,

\[|f_{z,u}(y) - y| \leq |f_{z,u}(y) - f_{z,u}(0) - y| + |f_{z,u}(0)| \leq \varepsilon r + |f_{z,u}(0)|,
\]

and the claim will be established upon showing that \(|f_{z,u}(0)| \leq (1 - 2\varepsilon)r\). Note that \(f_{z,u}(0) = g_{y_1,\ldots,y_{n-m},u}(x_0 + z)\), and we shall use hypothesis (3) to bound its norm from above. Given \(j = 1, \ldots, n-m\) recall that \((\gamma_j(x_0), z) = 0\) thus

\[|g_{y_j,u}(x_0 + z) - g_{y_{j-1},u}(x_0)| = \left|\gamma_j(x_0 + z) - \gamma_j(x_0) - (\gamma_j(x_0), x_0 - u)\right|
\]

\[= \left|\gamma_j(x_0 + z) - \gamma_j(x_0) - (\gamma_j(x_0), x_0 + z - u)\right| \leq \Lambda|z||x_0 + z - u|
\]

\[\leq \Lambda r 2\sqrt{r} \leq \frac{\varepsilon r}{\sqrt{n-m}}
\]

where the last inequality holds according to hypothesis (1) provided \((6.3)\) is chosen sufficiently small. In turn

\[|f_{z,u}(0)| \leq |g_{y_1,\ldots,y_{n-m},u}(x_0 + z) - g_{y_1,\ldots,y_{n-m},u}(x_0)| + |g_{y_1,\ldots,y_{n-m},u}(x_0)|
\]

\[\leq \varepsilon r + (1 - 3\varepsilon)r = (1 - 2\varepsilon)r
\]

according to hypothesis (3).

Claim #4. For every \(z \in W_0(x_0) \cap B(0, r)\) and every closed \(C \subseteq C_{\varepsilon r}\) one has \(H^{n-m}(C) \leq (1 + \varepsilon)H^{n-m-1}(g_{y_1,\ldots,y_{n-m},u}(C) \cap V_z)\).

First notice that

\[g_{y_1,\ldots,y_{n-m},u}(C) \cap V_z = \gamma_z \left(\gamma_z^{-1}\left(g_{y_1,\ldots,y_{n-m},u}(C) \cap V_z\right)\right) = \gamma_z \left(f_{z,u}^{-1}(C)\right)
\]

and therefore

\[H^{n-m}(g_{y_1,\ldots,y_{n-m},u}(C) \cap V_z) = H^{n-m-1}(f_{z,u}^{-1}(C))\]

since \(\gamma_z\) is an isometry. Now since \(C \subseteq C_{\varepsilon r} \subseteq f_{z,u}(C_r)\) according to Claim #3 we have

\[C = f_{z,u}\left(f_{z,u}^{-1}(C)\right).\]

It therefore follows from Claim #1 that

\[H^{n-m}(C) \leq (\text{Lip } f_{z,u})^{n-m} H^{n-m}(f_{z,u}^{-1}(C))
\]

\[\leq (1 + \varepsilon)H^{n-m}(g_{y_1,\ldots,y_{n-m},u}(C) \cap V_z)\].
We are now ready to finish the proof by an application of Fubini’s theorem:

$$\mathcal{L}^m\left(\mathcal{W}(x_0, r) \cap g_{Y_1, \ldots, Y_n, u}^{-1}(C)\right)$$

$$= \int_{\mathcal{W}(x_0, r) \cap B(0, r)} d\mathcal{L}^m(z) \mathcal{H}^{m-n}(g_{Y_1, \ldots, Y_n, u}^{-1}(C) \cap V_z)$$

$$\geq \frac{1}{1 + \varepsilon} \alpha(m) r^m \mathcal{H}^{m-n}(C).$$

\[\square\]

6.4 (Lower bound for \(\mathcal{E}\mathcal{W}\)). — Given \(0 < \varepsilon < 1/3\) there exists \(\delta_{6.4}(n, \Lambda, \varepsilon) > 0\) with the following property. If

1. \(0 < r < \delta_{6.4}(n, \Lambda, \varepsilon)\);
2. \(\mathcal{W}(x_0, r) \subseteq U\);
3. \(A \subseteq U\) is closed;
4. \(\mathcal{L}^n(A \cap \mathcal{W}(x_0, r)) \geq (1 - \varepsilon)\mathcal{L}^n(\mathcal{W}(x_0, r))\);

then

$$\int_{\mathcal{W}(x_0, r)} \mathcal{E}\mathcal{A} (\mathcal{W}(x_0, r)) \mathcal{W}(u) d\mathcal{L}^m(u) \geq (1 - c_{6.4}(n) \varepsilon) \alpha(m) r^m \mathcal{L}^m(\mathcal{W}(x_0, r)).$$

where \(c_{6.4}(n) = 5 + 6n\).

Proof. Similarly to the proof of 6.3 we will first establish a lower bound for \(\mathcal{E}\mathcal{A} (\mathcal{W}(x_0, r)) \mathcal{W}\) on «vertical slices» \(V_z\) of the given polyball and then apply Fubini. Given \(z \in \mathcal{W}(x_0, r) \cap B(0, r)\) we let \(V_z\) and \(\gamma_z\) be as in 6.3 and we also define

$$\tilde{V}_z = \mathbb{R}^n \cap \left\{ x_0 + z + \sum_{i=1}^{n-m} y_i v_i(x_0) : y \in C_{(1-3\varepsilon)r} \right\}$$

(notice it is slightly smaller than \(V_z\) used in the proof of 6.3) and we consider the isometric parametrization \(\tilde{\gamma}_z : C_{(1-3\varepsilon)r} \to \tilde{V}_z\) defined by

$$\tilde{\gamma}_z(y) = x_0 + z + \sum_{i=1}^{n-m} y_i v_i(x_0).$$

For part of the proof we find it convenient to abbreviate \(E = A \cap \mathcal{W}(x_0, r)\). We also let \(\tilde{\mathcal{E}}_E = (\mathcal{E}_E) \circ \gamma_z\).

By definition of \(\mathcal{E}_E\) for each \(\tilde{\gamma}_z(y) \in \tilde{V}_z\) there exists a collection \(\mathcal{C}_y\) of closed balls in \(\mathbb{R}^{n-m}\) with the following properties: For every \(C \in \mathcal{C}_y\), \(C\) is a ball centered at \(0, C \subseteq C_{\varepsilon r}\),

$$\tilde{\mathcal{E}}_E (\tilde{\gamma}_z(y)) + \varepsilon \geq \int_C \mathcal{H}^{m-n} \left( E \cap g_{Y_1, \ldots, Y_n, u}^{-1}(\tilde{\gamma}_z(y)) \right) d\mathcal{L}^{n-m}(h),$$

and \(\inf \{ \text{diam } C : C \in \mathcal{C}_y \} = 0\). Furthermore \(\tilde{\mathcal{E}}_E\) being \(\mathcal{L}^{n-m}\) summable according to 5.3 there exists \(N \subseteq C_{(1-3\varepsilon)r}\) such that \(\mathcal{L}^{n-m}(N) = 0\) and every \(y \not\in N\) is a Lebesgue point of \(\tilde{\mathcal{E}}_E\). For such \(y\) we may reduce \(\mathcal{C}_y\) if necessary, keeping all the previously stated properties and enforcing that

$$\int_{y + C} \tilde{\mathcal{E}}_E \mathcal{W} d\mathcal{L}^{n-m} + \varepsilon \geq (\tilde{\mathcal{E}}_E) (y)$$

whenever \(C \in \mathcal{C}_y\). We infer that for each \(y \in C_{(1-3\varepsilon)r} \sim N\) and each \(C \in \mathcal{C}_y\),

$$\int_{y + C} \tilde{\mathcal{E}}_E \mathcal{W} d\mathcal{L}^{n-m} + 2\varepsilon \mathcal{L}^{n-m}(y + C) \geq \int_C \mathcal{H}^{m-n} \left( E \cap g_{Y_1, \ldots, Y_n, u}^{-1}(\tilde{\gamma}_z(y)) \right) d\mathcal{L}^{n-m}(h).$$

(23)

It follows from the Vitali covering theorem that there is a sequence \((y_k)_k\) in \(C_{(1-3\varepsilon)r} \sim N\), and \(C_k \in \mathcal{C}_{y_k}\), such that the balls \(y_k + C_k, k = 1, 2, \ldots\) are pairwise disjoint, and
\[ \mathcal{L}^{n-m} \left( C_{(1-3\varepsilon)r} \sim \bigcup_{k=1}^{\infty} (y_k + C_k) \right) = 0. \]

It therefore follows from (20) and the fact that \( \gamma_z \) is an isometry that

\[ \int_{V_z} B E W d \mathcal{H}^{n-m} + 2 \varepsilon \mathcal{H}^{n-m}(V_z) \geq \sum_{k=1}^{\infty} \int_{C_k} \mathcal{H}^{m}(E \cap g_{y_1,\ldots,y_{n-m},u_k}(y)) d\mathcal{L}^{n-m}(y), \]

where we have abbreviated \( u_k = \bar{\gamma}_z(y_k) \). We also abbreviate \( S_k = g_{y_1,\ldots,y_{n-m},u_k}(C_k) \) and we infer from the coarea formula that for each \( k = 1, 2, \ldots \),

\[ \int_{C_k} \mathcal{H}^{m}(E \cap g_{y_1,\ldots,y_{n-m},u_k}(y)) d\mathcal{L}^{n-m}(y) = \int_{E \cap S_k} J g_{y_1,\ldots,y_{n-m},u_k} d\mathcal{L}^n \geq (1 - \varepsilon)\mathcal{L}^n(E \cap S_k) \]  

where the last inequality follows from 2.12 applied with \( U = \text{Int} C_W(x_0, r) \) provided that \( \alpha_{0,4}(n, \Lambda, \varepsilon) \) is chosen smaller than \( (2\sqrt{2} - 1) \alpha_{0,4}(n, \Lambda, \varepsilon) \). Letting \( S = \bigcup_{k=1}^{\infty} S_k \), and recalling that \( E = A \cap C_W(x_0, r) \), we infer from (24) and (25) that

\[ \int_{V_z} B E W d \mathcal{H}^{n-m} + 2 \varepsilon \mathcal{H}^{n-m}(V_z) \geq (1 - \varepsilon)\mathcal{L}^n(E \cap S) \geq (1 - \varepsilon)(\mathcal{L}^n(C_W(x_0, r) \cap S) - \mathcal{L}^n(C_W(x_0, r) \sim A)) \].  

Applying 6.3 to each \( S_k \) does not immediately yield a lower bound for \( \mathcal{L}^n(C_W(x_0, r) \cap S) \) because the \( S_k \) are not necessarily pairwise disjoint. This is why we now introduce slightly smaller versions of these:

\[ C_{\hat{k}} = (1 - \varepsilon)C_k \quad \text{and} \quad S_{\hat{k}} = g_{y_1,\ldots,y_{n-m},u_k}(C_{\hat{k}}). \]

**Claim.** The sets \( S_{\hat{k}} \cap C_W(x_0, r) \), \( k = 1, 2, \ldots \), are pairwise disjoint.

Assume if possible that there are \( j \neq k \) and \( x \in S_j \cap S_{\hat{k}} \cap C_W(x_0, r) \). Letting \( \rho_j \) and \( \rho_k \) denote respectively the radius of \( C_j \) and \( C_k \) we notice that \( \rho_j + \rho_k < |y_j - y_k| \) because \( (y_j + C_j) \cap (y_k + C_k) = \emptyset \). Since \( \bar{\gamma}_z \) is an isometry we have \( |u_j - u_k| = |\bar{\gamma}_z(y_j) - \bar{\gamma}_z(y_k)| = |y_j - y_k| \) and therefore also

\[ |g_{y_1,\ldots,y_{n-m},u_j}(x) - g_{y_1,\ldots,y_{n-m},u_k}(x)| \leq |g_{y_1,\ldots,y_{n-m},u_j}(x)| + |g_{y_1,\ldots,y_{n-m},u_k}(x)| \leq (1 - \varepsilon)\rho_j + (1 - \varepsilon)\rho_k < (1 - \varepsilon)|u_j - u_k|. \]  

We now introduce the following vectors of \( \mathbb{R}^{n-m} \),

\[ h_j = \sum_{i=1}^{n-m} (v_i(x_0), u_j)e_i \quad \text{and} \quad h_k = \sum_{i=1}^{n-m} (v_i(x_0), u_k)e_i \]

and we notice that

\[ |h_j - h_k| = |P_{W_0(x_0)}(u_j - u_k)| = |u_j - u_k| \]

where the second equality holds because \( u_j - u_k \in W_0(x_0)^\perp \) as clearly follows from the definition of \( \bar{\gamma}_z \). Furthermore

\[ |(g_{y_1,\ldots,y_{n-m},u_j}(x) - g_{y_1,\ldots,y_{n-m},u_k}(x)) - (h_j - h_k)| = \sqrt{\sum_{i=1}^{n-m} |(v_i(x) - v_i(x_0), u_k - u_j)|^2} \]

\[ \leq \sqrt{n - \Lambda n} \sqrt{2}\varepsilon |u_j - u_k| \leq \varepsilon |u_j - u_k|, \]

since we may choose \( \alpha_{0,4}(n, \Lambda, \varepsilon) \) to be so small that the last inequality holds according to hypothesis (1). Whence

\[ |g_{y_1,\ldots,y_{n-m},u_j}(x) - g_{y_1,\ldots,y_{n-m},u_k}(x)| \geq |h_j - h_k| - \varepsilon |u_j - u_k| = (1 - \varepsilon)|u_j - u_k| \]

in contradiction with (27). The **Claim** is established.
Thus
\[
\mathcal{L}^n (C_W(x_0, r) \cap S) = \mathcal{L}^n (C_W(x_0, r) \cap \cup_{k=1}^\infty \mathcal{S}_k)
\geq \mathcal{L}^n \left( C_W(x_0, r) \cap \cup_{k=1}^\infty \mathcal{S}_k \right)
= \sum_{k=1}^\infty \mathcal{L}^n \left( C_W(x_0, r) \cap \mathcal{S}_k \right)
= \sum_{k=1}^\infty \mathcal{L}^n \left( C_W(x_0, r) \cap \mathcal{S}_k \right) \left( C_\tilde{k} \right)
\geq \frac{1}{1 + \varepsilon} \alpha(m) r^m \sum_{k=1}^\infty \mathcal{L}^{n-m} \left( \tilde{C}_k \right)
\]  
where the last inequality follows from \[6.3\] We notice that indeed \[6.3\] applies since \( \tilde{C}_k \subseteq C_k \subseteq C_{\text{ex}} \) and \( |g_{\nu_1, \ldots, \nu_m, u_k}(x_0)| = |P_{W_0(x_0)}(u_k - x_0)| = |y_k| \leq (1 - \varepsilon)r \).

Now,
\[
\sum_{k=1}^\infty \mathcal{L}^{n-m} \left( \tilde{C}_k \right) = (1 - \varepsilon)^{n-m} \sum_{k=1}^\infty \mathcal{L}^{n-m} \left( C_k \right) = (1 - \varepsilon)^{n-m} \sum_{k=1}^\infty \mathcal{L}^{n-m} \left( y_k + C_k \right)
\geq (1 - \varepsilon)^{n-m} \mathcal{L}^{n-m} \left( C_{(1 - 3\varepsilon)r} \right) \geq (1 - \varepsilon)^{2(n-m)} \alpha(n-m)r^{n-m}.
\]  
We infer from \[28\] and \[29\] that
\[
\mathcal{L}^n \left( C_W(x_0, r) \cap S \right) \geq \frac{(1 - 3\varepsilon)^{2(n-m)}}{1 + \varepsilon} \mathcal{L}^n \left( C_W(x_0, r) \right).
\]  
It therefore ensues from \[26\] and hypothesis (4) that
\[
\int_{V_\varepsilon} \mathcal{H}^m \mathcal{W} d\mathcal{H}^{n-m} + 2\varepsilon \mathcal{H}^{n-m} (V_\varepsilon) \geq (1 - \varepsilon)^2 \frac{(1 - 3\varepsilon)^{2(n-m)}}{1 + \varepsilon} \mathcal{L}^n \left( C_W(x_0, r) \right).
\]  
Integrating over \( z \) we infer from Fubini's theorem
\[
\int_{C_W(x_0, r)} \mathcal{H}^{n-m} \mathcal{W} d\mathcal{H}^n = \int_{W_0(x_0) \cap B(0, r)} d\mathcal{L}^n (z) \int_{V_\varepsilon} \mathcal{H}^m \mathcal{W} d\mathcal{H}^{n-m}
\geq \left( (1 - \varepsilon)^2 \frac{(1 - 3\varepsilon)^{2(n-m)}}{1 + \varepsilon} - 2\varepsilon \right) \alpha(m) r^m \mathcal{L}^n \left( C_W(x_0, r) \right).
\]

6.5. Proposition. — Given \( 0 < \varepsilon < 1/3 \) there exist \( \delta_{6.5} (n, \Lambda, \varepsilon) > 0 \) and \( \mathcal{C}_{6.5} (n) \geq 1 \) with the following property. If
\[
(1) \ 0 < r < \delta_{6.5} (n, \Lambda, \varepsilon);
(2) \ C_W(x_0, r) \subseteq U;
(3) \ A \subseteq U \text{ is closed};
(4) \ \mathcal{L}^n (A \cap C_W(x_0, r)) \geq (1 - \varepsilon) \mathcal{L}^n (C_W(x_0, r));
\]
then
\[
\int_{A \cap C_W(x_0, r)} \mathcal{H}^{n-m} \mathcal{W} (u) d\mathcal{L}^n (u) \geq (1 - \mathcal{C}_{6.5} (n) \varepsilon) \alpha(m) r^m \mathcal{L}^n (C_W(x_0, r)).
\]

Proof. The reader will happily check that
\[
\delta_{6.5} (n, \Lambda, \varepsilon) = \min \left\{ \delta_{6.5} (n, \Lambda, \varepsilon), \left( 2\sqrt{2} \right)^{-1} \delta_{6.5} (n, \Lambda) \right\}
\]
suits their needs.
6.6. Proposition. — There exists \( \delta_{6.6}(n, \Lambda) > 0 \) with the following property. If \( \text{diam} \, E \leq \delta_{6.6}(n, \Lambda) \) then

\[
\mathcal{L}_E^m W(u) > 0
\]

for \( \mathcal{L}^n \) almost every \( u \in E \).

Proof. We let

\[
\delta_{6.5}(n, \Lambda) = \min \left\{ \delta_{6.3} \left( n, \Lambda, \frac{1}{4m(n)} \right), \delta_{6.6}(n, \Lambda, 1/2) \right\}.
\]

According to 6.5, it suffices to show that \( \mathcal{H}_E^m W(u) > 0 \) for \( \mathcal{L}^n \) almost every \( u \in E \). Define

\[
Z = E \cap \{ u : \mathcal{H}_E^m W(u) = 0 \}
\]

and assume if possible that \( \mathcal{L}^n(Z) > 0 \). Since \( Z \) is \( \mathcal{L}^n \)-measurable (recall 4.12) there exists a compact set \( A \subseteq Z \) such that \( \mathcal{L}^n(A) > 0 \). Observe that the sets \( C_W(x, r), x \in U \) and \( r > 0 \), form a derivation basis for \( \mathcal{L}^n \)-measurable subsets of \( U \) (because their eccentricity is bounded away from zero) thus there exists \( x_0 \in A \) and \( r_0 > 0 \) such that

\[
\mathcal{L}^n (A \cap C_W(x_0, r)) \geq \left( 1 - \frac{1}{4m(n)} \right) \mathcal{L}^n (C_W(x_0, r))
\]

whenever \( 0 < r < r_0 \). There is no restriction to assume that \( r_0 \) is small enough for \( C_W(x_0, r_0) \subseteq U \). Thus if we let \( r = \min\{r_0, \delta_{6.5}(n, \Lambda, 1/[(4m(n)])\} \) it follows from 6.5 that

\[
\int_{A \cap C_W(x_0, r)} W(u) d\mathcal{L}^n(u) \geq \left( 1 - \frac{1}{4} \right) \alpha(m) r^m \mathcal{L}^n (C_W(x_0, r)) > 0.
\]

On the other hand recalling 4.12 and the fact that \( A \cap C_W(x_0, r) \subseteq E \) we infer that \( \mathcal{H}_A^m C_W(x_0, r) W(u) \leq \mathcal{H}_E^m W(u) \) for all \( u \in \mathbb{R}^n \). In particular \( \mathcal{H}_A^m C_W(x_0, r) W(u) = 0 \) for all \( u \in A \cap C_W(x_0, r) \subseteq Z \), contradicting 30. \( \square \)

7. Proof of the theorem

7.1. Theorem. — Assume that \( S \subseteq \mathbb{R}^n \), \( W_0 : S \rightarrow G(n, m) \) is Lipschitz and \( A \subseteq S \) Borel. The following are equivalent.

1. \( \mathcal{L}^n(A) = 0 \).
2. For \( \mathcal{L}^n \) almost every \( x \in A \), \( \mathcal{H}^m(A \cap W(x)) = 0 \).
3. For \( \mathcal{L}^n \) almost every \( x \in S \), \( \mathcal{H}^m(A \cap W(x)) = 0 \).

Recall our convention that \( W(x) = x + W_0(x) \).

Proof. Since \( G(n, m) \) is complete we can extend \( W_0 \) to the closure of \( S \). Furthermore if the theorem holds for \( \text{Clos} \, S \) then it also holds for \( S \). Thus there is no restriction to assume that \( S \) is closed.

(1) \( \Rightarrow \) (3). It follows from 2.8 that each \( x \in S \) admits an open neighborhood \( U_x \) in \( \mathbb{R}^n \) such that \( W(x) \) can be associated with a Lipschitz orthonormal frame verifying all the conditions of 3.1 for some \( \Lambda_x > 0 \). Since \( S \) is Lindelöf there are countably many \( x_1, x_2, \ldots \) such that \( S \subseteq U_{x_1} \cup \ldots \cup U_{x_j} \). Letting \( E_j = U_{x_1} \cup \ldots \cup U_{x_j} \) we infer from 3.5 that \( \phi_{E_j, W} \) is absolutely continuous with respect to \( \mathcal{L}^n \). Thus if \( \mathcal{L}^n(A) = 0 \) then \( \mathcal{H}^m(A \cap W(x)) = 0 \) for \( \mathcal{L}^n \) almost every \( x \in E_j \) by definition of \( \phi_{E_j, W} \). Since \( j \) is arbitrary the proof is complete.

(3) \( \Rightarrow \) (2) is trivial.

(2) \( \Rightarrow \) (1) Let \( A \) verify condition (3). It is enough to show that \( \mathcal{L}^n(A \cap B(0, r)) = 0 \) for each \( r > 0 \). Fix \( r > 0 \) and define \( S_r = S \cap B(0, r) \). Consider the \( U_{x_j} \) defined in the second paragraph of the present proof; since \( S_r \) is compact only finitely many of those, say \( U_{x_1}, \ldots, U_{x_N} \), cover \( S_r \). Let \( \Lambda = \max_{j=1,\ldots,N} \Lambda_{x_j} \). Partition each \( U_{x_j}, j = 1, \ldots, N \), into Borel sets \( E_{j,k}, k = 1, \ldots, K_j \), such that \( \text{diam} E_{j,k} \leq \delta_{6.6}(n, \Lambda) \). It then follows from 6.6 that

\[
\left( \mathcal{L}^n \cap E_{j,k} W \right)(u) > 0
\]
for $\mathcal{L}^n$ almost every $u \in A \cap E_{j,k}$. Now fix $j$ and $k$. Observe that $\mathcal{H}^m (A \cap E_{j,k} \cap W(x)) = 0$ for $\mathcal{L}^n$ almost every $x \in A \cap E_{j,k}$. Thus $\phi_{A \cap E_{j,k}, W}(A \cap E_{j,k}) = 0$. Moreover,

$$0 = \phi_{A \cap E_{j,k}, W}(A \cap E_{j,k}) = \int_{A \cap E_{j,k}} \mathcal{L}^n(W)(u) \, d\mathcal{L}^n(u).$$

It follows from (31) that $\mathcal{L}^n(A \cap E_{j,k}) = 0$. Since $j$ and $k$ are arbitrary, $\mathcal{L}^n(A) = 0$. □

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