Heat kernel coefficients on the sphere in any dimension

Yannick Kluth\textsuperscript{1} and Daniel F. Litim\textsuperscript{1}

\textsuperscript{1}Department of Physics and Astronomy University of Sussex, Brighton, BN1 9QH, U.K.

We derive all heat kernel coefficients for Laplacians acting on scalars, vectors, and tensors on fully symmetric spaces, in any dimension. Final expressions are easy to evaluate and implement, and confirmed independently using spectral sums. We also obtain the Green's function for Laplacians acting on transverse traceless tensors in any dimension. Applications to quantum gravity and the functional renormalisation group are indicated.

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I. INTRODUCTION

Heat kernel techniques are well-established tools in both theoretical physics and mathematics \cite{150,160}. They aim at the computation of traces of differential operators, and have a wide range of applications covering fluctuations of quantum fields on curved space-times, ultraviolet divergences and effective actions, spectral functions, quantum anomalies, the Casimir effect, quantum gravity, and more \cite{6,7,8,9,10}. The central idea is to express a certain Green's function as an integral over the so called proper time parameter which satisfies the heat equation. The integrand of this method, the heat kernel, is then a functional of the background metric. While for some special backgrounds it can be calculated exactly, it is not possible to solve it for general manifolds. Approximation
schemes have been introduced including the seminal Schwinger-DeWitt expansion giving rise to the heat kernel coefficients. This asymptotic expansion at early proper time works well for small space-time separations, which makes it a convenient tool to study short distance divergences in quantum field theory. On general manifolds, the first six heat kernel coefficients are known [11–19]. Calculations of heat kernel coefficients are greatly simplified on specific manifolds such as maximally symmetric backgrounds where they can be obtained from Green’s function [9].

An important application of heat kernels relates to quantum gravity and tests of the asymptotic safety conjecture [20, 21]. Heat kernels on maximally symmetric or Einstein spaces are often used to find beta functions for gravitational couplings in renormalisation group approaches to gravity [21–34]. Intriguingly, results for fixed points and scaling exponents show that the canonical mass dimension of couplings remains a good ordering principle, with asymptotically safe quantum gravity becoming “as Gaussian as it gets” [33–36]. Further technical choices [25, 37] are commonly adopted to reduce the required heat kernel coefficients on spheres to a finite set [28]. In general, however, the flow of couplings with increasing canonical mass dimension necessitates the knowledge of an increasing number of heat kernel coefficients, many more than presently available.

In this paper, we fill this gap in the literature and compute all heat kernel coefficients for scalars, transverse vectors, and transverse traceless symmetric tensors on fully symmetric backgrounds with Euclidean signature and positive curvature. Our primary input are the known Green’s functions for the Laplacian acting on scalars [38] and transverse vectors [39, 40]. In addition, we derive the Green’s function of the Laplacian acting on transverse traceless tensors to find closed expressions for all heat kernel coefficients on spheres, in any dimension. The final expressions are easy to evaluate and implement, and confirmed independently using spectral sum techniques. Besides their general interest, our findings enable new tests of the asymptotic safety conjecture without resorting to flat backgrounds or spectral sums and approximations thereof.

The remainder of this paper is organised as follows. In Sect. II we recall the definition of heat kernel coefficients from asymptotic expansions (Sect. II A) and their usage in the context of Wilson’s renormalisation group (Sect. II B). Sect. III contains the main derivation of heat kernel coefficients. After an outline of the method (Sect. III A) we compute the heat kernels for scalars (Sects. III B), transverse vectors (III C), transverse traceless tensors (III D), and the first coefficients of the asymptotic heat kernel expansion for unconstrained fields (Sect. III E). In Sect. IV we independently derive heat kernels from spectral sums for selected integer dimensions. In Sect. V we close with a discussion of results and implications for quantum gravity.

II. PRELIMINARIES

In this section, we recall basic definitions for heat kernel coefficients and their usage in the context of Wilson’s renormalisation group. Ultimately, we aim at finding the heat kernel expansion for different fields on the sphere using their corresponding Green’s functions. Thus, from now on we focus on the specific case of a fully symmetric background manifold, even though some of the considerations are more general.

A. Heat Kernel Coefficients

The heat kernel $U_E(t, x, y)$ is defined as the solution of the heat equation

$$\frac{\partial U_E(t, x, y)}{\partial t} = (\nabla^2 + E)U_E(t, x, y),$$  (1)
subject to the initial condition

\[ U_E(0, x, y) = \frac{\delta(x - y)}{\sqrt{g}}. \]  

(2)

Note that the kernel \( U_E(t, x, y) \) may contain Lorentz indices if we consider vector or tensor degrees of freedom. Throughout this section these indices are suppressed. By definition, \( U_E(t, x, y) \) has the dimension of an inverse volume. Also, \( t \) is the proper time parameter with mass dimension two, \(-\nabla^2\) is the Laplacian, and \( E \) an endomorphism. The formal solution of (1) is given by

\[ U_E(t, x, y) = e^{t(-\nabla^2 + E)}. \]  

(3)

Due to the symmetries of the chosen background, \( U_E(t, x, y) \) can only depend on the proper time \( t \) and the distance between the points \( x \) and \( y \). Therefore, defining \( \sigma \) to be half the square of the geodesic distance between \( x \) and \( y \), we may write

\[ U_E(t, x, y) = U_E(t, \sigma). \]  

(4)

For early times we expand the heat kernel as an asymptotic series following the DeWitt ansatz

\[ U_E(t, \sigma) = \frac{\Delta^{1/2}}{(4\pi t)^{d/2}} \exp \left\{ -\frac{\sigma^2}{2t} \right\} \sum_{n=0}^{\infty} \left[ \tilde{b}_{2n}(E, \sigma)t^n + \tilde{c}_{d+2n}(E, \sigma)t^{d/2+n} \right], \]  

(5)

where \( \Delta \) is the Van Fleck-Morette determinant

\[ \Delta = \frac{\det \left[ -\nabla_x^a \nabla_y^b \sigma(x, y) \right]}{g^{1/2}(x)g^{1/2}(y)}. \]  

(6)

By definition, the heat kernel coefficients \( \tilde{b}_m \) and \( \tilde{c}_m \) have canonical mass dimension \( m \) in any dimension. Just as \( U_E \), these coefficients may carry Lorentz indices which are suppressed here.

Note that the ansatz (5) seems slightly different from those used in the literature \[9, 35, 41, 42\] through the appearance of the coefficients \( \tilde{c}_{d+2n} \). These terms only arise for heat kernels of constrained fields, such as transverse vector and transverse traceless tensor fields, and are related to the exclusion of lowest modes. Excluded modes always produce contributions of the form \( \exp(\alpha R t) \) which invariably give rise to terms with positive integer powers of the proper-time parameter when expanded for small times. For even dimensions, the terms \( \tilde{c}_{d+2n} \) could be combined with the \( \tilde{b}_{d+2n} \) coefficients of the same mass dimension. In odd dimensions, however, all \( \tilde{c}_{d+2n} \) coefficients have mass dimensions different from all \( \tilde{b}_{2n} \) coefficients, and cannot be combined into a single coefficient. Hence, for the sake of generality, and given their distinctly different origins, we keep these coefficients separate for now.

Since we are ultimately interested in the trace of the heat kernel, we only need the coincidence limit of \( U_E(t, \sigma) \). Using (4), we define the coincidence limit of the heat kernel coefficients, \( \tilde{b}_{2n}(E) \) and \( \tilde{c}_{d+2n}(E) \), for given endomorphism \( E \) as

\[ \tilde{b}_{2n}(E) = \tilde{b}_{2n}(E, 0), \quad \tilde{c}_{d+2n}(E) = \tilde{c}_{d+2n}(E, 0). \]  

(7)

Then, the trace of the heat kernel is given by

\[ \text{Tr}_s U_E(t, \sigma) = \frac{1}{(4\pi t)^{d/2}} \sum_{n=0}^{\infty} \left[ \text{Tr}_s \left[ \tilde{b}_{2n}(E) \right] t^n + \text{Tr}_s \left[ \tilde{c}_{d+2n}(E) \right] t^{d/2+n} \right], \]  

(8)
where the trace acts on the coordinate dependence as well as any Lorentz indices carried by $\tilde{b}_{2n}$ and $\tilde{c}_{d+2n}$. Further, the index $s$ denotes the spin of the field w.r.t. which the trace is acting on. Since the heat kernel coefficients are coordinate independent on a fully symmetric background, we may define

$$b_n^{(s)}(E) = \frac{1}{\text{Vol}} \text{Tr}_s[\tilde{b}_n(E)], \quad c_n^{(s)}(E) = \frac{1}{\text{Vol}} \text{Tr}_s[\tilde{c}_n(E)],$$

in which the volume of the $d$-dimensional sphere is

$$\text{Vol} = \frac{2\pi^{(d+1)/2}}{\Gamma\left(\frac{d+1}{2}\right)} \left(\frac{d(d-1)}{R}\right)^{d/2}$$

and $R$ denotes the Ricci scalar curvature. This allows us to write

$$\text{Tr}_s U_E(t, \sigma) = \frac{\text{Vol}}{(4\pi t)^{d/2}} \sum_{n=0}^{\infty} \left[ b_n^{(s)}(E)t^n + c_n^{(s)}(E)t^{d/2+n} \right].$$

Finally, we notice that the heat kernel for a given endomorphism $E$ is related to that for any other endomorphism $E$ by

$$U_E(t, \sigma) = e^{t(E-E)} U_{E}(t, \sigma),$$

assuming that the endomorphism commutes with the covariant derivative on the sphere. This relation implies that the corresponding heat kernel coefficients are related by

$$b_n^{(s)}(E) = \sum_{k=0}^{n} \frac{(E-E)^k}{k!} b_n^{(s)}(E), \quad c_n^{(s)}(E) = \sum_{k=0}^{n} \frac{(E-E)^k}{k!} c_n^{(s)}(E),$$

which serves as a definition of heat kernel coefficients for arbitrary endomorphisms.

### B. Renormalisation Group

An important area for the application of heat kernels is Wilson’s (functional) renormalisation group. The technique amounts to the introduction of an infrared momentum cutoff $k$ into the path integral definition of quantum or statistical field theory, which induces a scale-dependence $k\partial_k$ in the form of an exact functional flow for the effective action $\Gamma_k$ (see [43–45] for reviews),

$$k\partial_k \Gamma_k = \frac{1}{2} \text{Tr} \left\{ \left(k\partial_k R_k\right) \left(\Gamma_k^{(2)} + R_k\right)^{-1} \right\}.$$  

Here, $\Gamma_k^{(2)}$ denotes the second variation of $\Gamma_k$. The function $R_k$ denotes the Wilsonian IR regulator, chosen such that $\Gamma_k$ interpolates between the microscopic theory ($1/k \rightarrow 0$) and the full quantum effective action ($k \rightarrow 0$), see [37]. At weak coupling, iterative solutions generate perturbation theory to all loop orders [46, 47]. At strong coupling, non-perturbative approximations such as the derivative expansion, vertex expansions, or mixtures thereof are available [43, 48, 49]. The stability and convergence of approximations can be controlled as well [50–52].
Our main point here relates to the operator trace in (14), which for many applications can be evaluated on flat Euclidean backgrounds. For quantum field theories on curved backgrounds, or for studies of fully-fledged quantum gravity, it is often convenient to evaluate the operator trace on suitably chosen non-flat backgrounds [22, 24, 53, 54] also using the background field method [55] and optimised cutoffs [25, 37, 56, 57]. In quantum gravity, this has enabled advanced tests of the asymptotic safety conjecture on spheres [25, 26, 28, 33, 34, 36, 58]. Further applications of heat kernels include flow equations on Einstein [59] or hyperbolic spaces [60], critical fields on curved backgrounds [61], low energy effective actions [62], and proper-time flows [63].

To see how heat kernel coefficients enter in this methodology we note that typical contributions on the right-hand side of (14) are given by traces of operators in the form $\left(k \partial_k R_k \right) \left(\Gamma_k^{(2)} + R_k\right)^{-1}$. After inserting the second variation matrix and choosing a regulator the integrand can be represented as a (matrix-valued) function $W(-\nabla^2)$ of the Laplacian. Using the Laplace anti-transformation

$$W(z) = \int_0^\infty dt \tilde{W}(t) e^{-tz},$$

we may then express the desired trace as

$$\text{Tr} W(-\nabla^2) = \int_0^\infty dt \tilde{W}(t) \text{Tr} e^{t\nabla^2}.$$  \hspace{1cm} (15)

Crucially, the trace $\text{Tr} e^{t\nabla^2}$ appearing on the right-hand side is the trace of the heat kernel with vanishing endomorphism (3). Thus, the early time expansion (6) allows us to evaluate the trace (15) in terms of the heat kernel coefficients $b_{2n}$ and $c_{d+2n}$ via

$$\text{Tr} W(-\nabla^2) = \frac{\text{Vol}}{(4\pi)^{d/2}} \sum_{n=0}^\infty \left[ b_{2n}(0) \int_0^\infty dt t^{n-d/2} \tilde{W}(t) + c_{d+2n}(0) \int_0^\infty dt t^n \tilde{W}(t) \right].$$ \hspace{1cm} (16)

We conclude that the heat kernel coefficients $b_{2n}(0)$ and $c_{d+2n}(0)$ are key inputs for Wilsonian flows on maximally symmetric backgrounds. The calculation of all heat kernel coefficients $b_{2n}$ and $c_{d+2n}$ ($n \geq 0$) on spheres in arbitrary dimension is the topic of the following sections.

III. HEAT KERNELS FROM GREEN’S FUNCTIONS

In this section, we find Green’s functions for scalars, transverse vectors, and transverse traceless tensors and use these to extract heat kernel coefficients for Laplacians on spheres in any dimension. We also give results for heat kernels of unconstrained vectors and tensors.

A. Green’s Function Technique

Our methodology largely follows Avramidi [9] and starts by noting that the heat kernel defined in (5) can be connected to a Green’s function using the Schwinger-DeWitt representation

$$G(\sigma) = \int_0^\infty dt e^{-tm^2} U_E(t, \sigma).$$ \hspace{1cm} (17)

By definition, $G(\sigma)$ has canonical mass dimension $M^{d-2}$. Using (1) with (2) it is straightforward to show that $G(\sigma)$ in (17) is a Green’s function for the differential operator $(-\nabla^2 + m^2 - E)$. In
the coincidence limit $\sigma = 0$, we can write

$$\frac{\text{Tr}_s G(\sigma)}{\text{Vol}} = \frac{1}{(4\pi)^{d/2}} \int_0^\infty dt e^{-tm^2} \sum_{n=0}^\infty \left[ b_n^{(s)}(E)t^{d/2+n} + c_n^{(s)}(E)t^n \right].$$

(18)

Note that the trace on the left-hand side effectively only takes the trace of the tensor structure of $G(\sigma)$, the trace over the coordinate dependence drops out due to the volume factor in the denominator. Recalling the elementary definition of the $\Gamma$-function, $\Gamma(n+1) = \int_0^\infty ds s^n e^{-s}$, and substituting $s = tm^2$, the $t$-integration in (18) is performed term by term. Doing so, the Green’s function takes the form of a large-$m$ expansion

$$\frac{\text{Tr}_s G(\sigma)}{\text{Vol}} = \frac{m^{d-2}}{(4\pi)^{d/2}} \sum_{n=0}^\infty \left[ \Gamma(n - \frac{d}{2} + 1) \frac{b_n^{(s)}(E)}{m^{2n}} + \Gamma(n + 1) \frac{c_n^{(s)}(E)}{m^{d+2n}} \right].$$

(19)

Hence, by calculating the large-$m$ expansion of the Green’s function at its coincidence limit, we may read off the heat kernel coefficients as the corresponding Taylor coefficients. This fact is exploited below to calculate all heat kernel coefficients for scalars, transverse vectors, and transverse traceless symmetric tensors.

B. Scalars

We begin with the Green’s function for scalar fields $G_Q(\sigma)$ to explain how the corresponding heat kernel coefficients are computed in practice. This follows closely the derivation given in [9]. The Green’s function for scalar fields is the solution of the differential equation

$$(−\nabla^2 + Q)G_Q(\sigma) = \frac{1}{\sqrt{g}} \delta(x - y).$$

(20)

In [38, 64, 65] it has been explained why solutions can be expressed in terms of a hypergeometric function,

$$G_Q(\sigma) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(d/2)} \frac{r^{2-d}}{(4\pi)^{d/2}} \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell \frac{(-1)^\ell}{\ell!} \frac{(2\ell + d - 2)_{d/2}}{(d/2)^{d/2}} \ell \int_0^\infty \frac{dW}{W^{d/2}} \exp\left[-\frac{Qr^2}{W}\right],$$

(21)

with parameters

$$a = \frac{d-1}{2} + \xi, \quad b = \frac{d-1}{2} - \xi, \quad c = \frac{d}{2}, \quad \xi = \sqrt{\frac{(d-1)^2}{4} - Qr^2},$$

(22)

and

$$z = \cos^2\left(\sqrt{\frac{\sigma}{2r^2}}\right).$$

(23)

In the latter, $r$ denotes the radius of the sphere which relates to the Ricci scalar curvature $R$ as

$$\frac{R}{d(d-1)} = r^{-2}.$$
To find the coincidence limit for (21), we follow Avramidi [9] and exploit a useful representation for the hypergeometric function [66]

$$2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}$$

to arrive at

$$G_Q(0) = \frac{r^{2-d}}{(4\pi)^{d/2}} \Gamma \left( 1 - \frac{d}{2} \right) \frac{\Gamma \left( \frac{d}{2} + \xi \right) \Gamma \left( \frac{d}{2} - \xi \right)}{\Gamma \left( \frac{1}{2} + \xi \right) \Gamma \left( \frac{1}{2} - \xi \right)}.$$  (25)

The large mass expansion requires an expansion of the Gamma functions which can be done noting that [67]

$$\ln [\Gamma(\alpha + \xi)] = (\alpha + \xi - \frac{1}{2}) \ln(\xi) - \xi + \frac{1}{2} \ln(2\pi) + \sum_{n=2}^{\infty} \frac{(-1)^n B_n(\alpha)}{n(n-1)\xi^{n-1}},$$

with $|\xi| \to \infty, |\text{ph}(\xi)| < \pi$, and $B_n(x)$ being the Bernoulli polynomials. Then, using $B_n(\frac{1}{2}) = 0$ for $n$ odd, we find

$$\frac{\Gamma \left( \frac{d}{2} + \xi \right) \Gamma \left( \frac{d}{2} - \xi \right)}{\Gamma \left( \frac{1}{2} + \xi \right) \Gamma \left( \frac{1}{2} - \xi \right)} = \sum_{n=0}^{\infty} \kappa_n(d) (-\xi^2)^{d/2 - 1 - n},$$  (26)

with the generating function for the coefficients $\kappa_n(d)$ given by

$$\exp \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(2n+1)} B_{2n+1} \left( \frac{d-1}{2} \right) z^n = \sum_{n=0}^{\infty} \kappa_n(d) z^n.$$  (27)

Note that the expression (26) is only valid for $\Im(\xi) \neq 0$. To proceed, we split $Q$ into a mass part $m^2$ and an endomorphism part $E$ through $Q = m^2 - E$. The endomorphism part $E$ can be chosen such that $\xi$ becomes proportional to the mass $m$. Using (22) with (24), this requirement uniquely fixes the endomorphism $E = E$ and $\xi$ to

$$E = \frac{1}{4d} R,$$
$$-\xi^2 = d(d-1) \frac{m^2}{R},$$  (28)

and we get

$$G_Q(0) \bigg|_E = \frac{1}{(4\pi)^{d/2}} \Gamma \left( 1 - \frac{d}{2} \right) \sum_{n=0}^{\infty} \kappa_n(d) r^{-2n} m^{d-2-2n}.$$  (29)

For the endomorphism (28) we can now read off the heat kernel coefficients $b_{2n}^{(0)}$ and $c_{d+2n}^{(0)}$ by comparison with (19). To distinguish them, we note that the coefficients $c_{d+2n}^{(0)}$ enter with $d$-independent integer powers of $m$ in (19) while the coefficients $b_{2n}^{(0)}$ have $d$-dependent powers of $m$. Moreover, the coefficients $b_{2n}^{(0)} (c_{d+2n}^{(0)})$ are linear in (independent of) the parameters $\kappa_i(d)$. This is because the $d$-dependent powers of $m$ originate from (26) which is linear in the $\kappa_i(d)$. Exploiting
Table 1. The scalar heat kernel coefficients for different integer dimensions and vanishing endomorphism.

|    | $d = 2$ | $d = 3$ | $d = 4$ | $d = 5$ | $d = 6$ |
|----|---------|---------|---------|---------|---------|
| $b_{0}^{(0)}$ | 1 | 1 | 1 | 1 | 1 |
| $b_{2}^{(0)}$ | $\frac{1}{6}R$ | $\frac{1}{6}R$ | $\frac{1}{6}R$ | $\frac{1}{6}R$ | $\frac{1}{6}R$ |
| $b_{4}^{(0)}$ | $\frac{1}{60}R^2$ | $\frac{1}{72}R^2$ | $\frac{29}{2160}R^2$ | $\frac{1}{75}R^2$ | $\frac{1}{75}R^2$ |
| $b_{6}^{(0)}$ | $\frac{1}{630}R^3$ | $\frac{1}{1296}R^3$ | $\frac{37}{5432}R^3$ | $\frac{1}{1500}R^3$ | $\frac{1139}{1701000}R^3$ |
| $b_{8}^{(0)}$ | $\frac{1}{5040}R^4$ | $\frac{1}{31104}R^4$ | $\frac{149}{655360}R^4$ | $\frac{1}{45000}R^4$ | $\frac{833}{36450000}R^4$ |
| $b_{10}^{(0)}$ | $\frac{1}{612240}R^5$ | $\frac{1}{933120}R^5$ | $\frac{179}{43110400}R^5$ | $\frac{1}{2250000}R^5$ | $\frac{137}{267300000}R^5$ |

With this fact, we get from (29)

\[
b_{2n}^{(0)}(E) = \frac{\Gamma(1 - \frac{d}{2})}{\Gamma(1 + n - \frac{d}{2})} \left( \frac{R}{d(d - 1)} \right)^n \kappa_n(d),
\]

(30)

with $\kappa_n(d)$ determined through (27). Moreover, with the help of (13), we find the heat kernel coefficients for scalar fields and arbitrary endomorphism $E$,

\[
b_{2n}^{(0)}(E) = \sum_{k=0}^{n} \frac{\Gamma(1 - \frac{d}{2})}{k! \Gamma(1 + n - k - \frac{d}{2})} \left( \frac{R}{d(d - 1)} + E \right)^k \left( \frac{R}{d(d - 1)} \right)^{n-k} \kappa_{n-k}(d),
\]

(31)

with $c_{d+2n}^{(0)}(E) = 0$.

Tab. 1 summarises our results for the first few scalar heat kernel coefficients (31) for $E = 0$ and a selection of integer dimensions.

C. Transverse Vectors

Next, we determine the heat kernel coefficients for transverse vector fields. To that end, we review the derivation of the transverse vector Green’s function, closely following $[39, 65]$. We then exploit the result to derive all heat kernel coefficients for transverse vector fields on spheres.

It is important to notice that the Green’s function for transverse vector fields $G_{Q,\mu\nu}'(\sigma)$ fulfils a differential equation of the form

\[
(-\nabla^2 + Q)G_{Q,\mu\nu}'(\sigma) = \frac{g_{\mu\nu}'}{\sqrt{g}} \delta(x - y) + \text{longitudinal terms}.
\]

(32)

The longitudinal terms ensure the transversality of the right-hand side. They can be derived by considering the full vector Green’s function and splitting it into a transverse and a longitudinal part. As pointed out in [39], neglecting these longitudinal terms can lead to inconsistent results for the Green’s function.
To solve the differential equation for the transverse vector Green’s function, we want to reduce it to a scalar function $S_T(\sigma)$, which is called the structure function. For this, the transverse vector projector

$$P_\mu^\nu = g_\mu^\nu \nabla^2 - \nabla^\nu \nabla_\mu, \quad (33)$$

is introduced. It fulfills the properties $[39, 65]$

$$\nabla^\mu P_\mu^\nu T_\nu = P_\nu^\mu (\nabla_\mu S(\sigma)) = 0, \quad [\nabla^2, P_\mu^\nu]T_\nu = 0, \quad P_\mu^\nu P_\nu^\rho T_\rho = P_\mu^\rho \left(\nabla^2 - \frac{R}{d}\right) T_\rho, \quad (34)$$

for an arbitrary vector $T_\mu$. From this, it follows that acting with $P_\alpha^\mu P_\beta^\nu$ on any bi-tensor $T_{\alpha\beta}$ gives rise to a bi-tensor which is transverse in both indices. Hence, it has the right properties to be the Green’s function for transverse vector fields. With this in mind, we make an ansatz for the transverse vector Green’s function through

$$G_{T,Q,\mu\nu}(\sigma) = P_\alpha^\mu P_\beta^\nu \frac{R_{\alpha\beta} S_T(\sigma)}{\sqrt{g} \delta(x-y)}. \quad (35)$$

The bi-tensor $R_{\alpha\beta}$ is arbitrary and can be chosen to simplify the computations. One choice is given by $[39, 65]$

$$R_{\alpha\beta} = -2r^2 \nabla_\alpha \nabla_\beta \sin^2 \left(\frac{\sqrt{2} \sigma}{2r}\right) = g_{\alpha\beta} + \frac{\sigma_\alpha \sigma_\beta}{\sigma} \sin^2 \left(\frac{\sqrt{2} \sigma}{2r}\right), \quad (36)$$

with $\sigma_\alpha = \nabla_\alpha \sigma$. The right-hand side of this equation can be derived using the relations in $[43]$ and $[44]$. It fulfills the property

$$\nabla^2 (R_{\alpha\beta} S(\sigma)) = R_{\alpha\beta} \left(\nabla^2 + \frac{R}{d(d-1)}\right) S(\sigma) + \text{longitudinal}, \quad (37)$$

with $S(\sigma)$ being an arbitrary scalar. Since this bi-tensor is always contracted with transverse vector projectors, the longitudinal terms can be neglected. Using the above structures and identities, and contracting $[32]$ with $P_\rho^\mu P_\sigma^\nu$, we arrive at

$$P_\rho^\alpha P_\sigma^\beta \left[R_{\alpha\beta} \left(-\nabla^2 + Q - \frac{R}{d(d-1)}\right) \left(\nabla^2 + \frac{2 - d}{d(d-1)} R\right) S_T(\sigma)\right] = P_\rho^\alpha P_\sigma^\beta \left[R_{\alpha\beta} \frac{\delta(x-y)}{\sqrt{g}} \right]. \quad (38)$$

We observe that the structure function for the transverse vector Green’s function obeys the differential equation

$$\left(-\nabla^2 + Q - \frac{R}{d(d-1)}\right) \left(\nabla^2 + \frac{2 - d}{d(d-1)} R\right) S_T(\sigma) = \frac{1}{\sqrt{g}} \delta(x-y), \quad (38)$$

which can be solved with the help of the scalar Green’s function $[39, 65]$. Its solution takes the explicit form

$$S_T(\sigma) = G_{\chi_1,\chi_2,\chi_2}(\sigma) \equiv -\frac{\partial}{\partial \chi_2} G_{\chi_1,\chi_2}(\sigma), \quad (39)$$
where
\[ G_{\chi_1,\chi_2}(\sigma) = \frac{G_{\chi_1}(\sigma) - G_{\chi_2}(\sigma)}{\chi_2 - \chi_1}, \] (40)
and \( G_\chi(\sigma) \) given in (21), and all of this evaluated at
\[ \chi_1 = Q - \frac{R}{d(d-1)}, \quad \chi_2 = \frac{d-2}{d(d-1)} R. \] (41)
Using the result for the structure function, the transverse vector Green’s function is now given by
\[ G_{Q_{\mu\nu}'}^{T}\chi_1,\chi_2(\sigma) = P^\alpha\mu P^{\beta'}\nu' (R_{\alpha\beta} G_{\chi_1,\chi_2,\chi_2}(\sigma)). \] (42)
To extract the heat kernel coefficients we need to contract this with \( g_{\mu\nu}' \) and calculate the coincidence limit of the resulting expression. Covariant derivatives acting on \( \sigma_\alpha \) can be computed using
\[ \nabla_\nu' \sigma_\mu = C(\sigma) \left[ g_{\mu\nu'} + \frac{1}{2\sigma} g_{\mu} g_{\nu'} \right] + \frac{1}{2\sigma} g_{\mu} g_{\nu'}, \]
\[ \nabla_\nu g_{\mu\nu'} = A(\sigma) \left[ g_{\mu\nu'} - \frac{1}{2\sigma} g_{\mu} g_{\nu'} \right] + \frac{1}{2\sigma} g_{\mu} g_{\nu'}, \] (43)
\[ \nabla_\mu g_{\alpha\beta'} = - \frac{A(\sigma) + C(\sigma)}{2\sigma} (g_{\mu\alpha} g_{\beta'} + g_{\mu\beta'} g_{\alpha}), \]
with
\[ A(\sigma) = \sqrt{\frac{2\sigma R}{d(d-1)}} \cot \left( \sqrt{\frac{2\sigma R}{d(d-1)}} \right), \quad C(\sigma) = - \sqrt{\frac{2\sigma R}{d(d-1)}} \csc \left( \sqrt{\frac{2\sigma R}{d(d-1)}} \right). \] (44)
Acting with the transverse vector projectors can be simplified by performing a series expansion of \( R_{\alpha\beta} G_{\chi_1,\chi_2}(\sigma) \) in the coincidence limit. Observing that \( g_{\mu\nu} = 2\sigma \) and that \( g_{\mu\nu} \) does not contribute in the coincidence limit, we may employ \( \sigma_\mu \) as an expansion parameter. Coming from the transverse vector projectors, we have four covariant derivatives acting on this expression. Hence, an expansion up to order four in \( \sigma_\mu \) is needed. Expanding the structure function as
\[ S^T(\sigma) = \sum_{n=0}^{\infty} S^T_n \sigma^n, \] (45)
the expansion of \( R_{\alpha\beta} G_{\chi_1,\chi_2}(\sigma) \) gives
\[ R_{\alpha\beta} G_{\chi_1,\chi_2}(\sigma) = S^T_0 \left( g_{\alpha\beta'} + \frac{R}{2d(d-1)} g_{\alpha} g_{\beta'} - \frac{\sigma R^2}{12d^2(d-1)^2} g_{\alpha} g_{\beta'} \right) + S^T_1 \left( g_{\alpha\beta'} + \frac{\sigma R}{2d(d-1)} g_{\alpha} g_{\beta'} \right) + S^T_2 \sigma^2 g_{\alpha\beta'} + O((\sigma_\mu)^5). \] (46)
After acting with the projectors and contracting with \( g_{\mu\nu}' \), we find
\[ g^{\mu\nu'} G_{Q_{\mu\nu}'}^{T}(0) = \frac{8 - 5d}{3} R S^T_1 + 2d(d^2 + d - 2) S^T_2. \] (47)
Explicit expressions for the expansion coefficients \( S^T_1 \) and \( S^T_2 \) can be found in App. Inserting
the expressions for $S_1^T$ and $S_2^T$ and using (26), we find

$$g^{\mu\nu}G_{Q,\mu\nu}(0) = \frac{1}{(4\pi)^{d/2}} \left( \frac{d(d-1)}{R} \right)^{1-d/2} \left[ \frac{R \Gamma(d-1)}{\Gamma \left( \frac{d}{2} \right) (dQ - R)} + \frac{\pi (d^2Q - dQ - R)}{\sin \left( \frac{\pi d}{2} \right) \Gamma \left( \frac{d}{2} \right) (dQ - R)} \sum_{k=0}^{\infty} \kappa_k(d) \left( -\xi^2 \right)^{d/2-k-1} \right],$$

(48)

with

$$\xi = \frac{1}{2} \sqrt{d \left( -\frac{4(d-1)Q}{R} + d - 2 \right) + 5}.$$  (49)

To expand (48) in the large $m$ limit, we use the same trick as in the scalar case and set $Q = m^2 - E$ for an endomorphism such that $\xi$ is directly proportional to $m$. We find

$$E = \frac{-5 + 2d - d^2}{4(d-1)} R,$$

$$-\xi^2 = d(d-1)\frac{m^2}{R}.$$  (50)

Using (50) and the geometric series to expand denominators containing $m$ in the large $m$ limit, we arrive at

$$g^{\mu\nu}G_{Q,\mu\nu}(0) \bigg|_{E} = \frac{(d-1)^2}{(4\pi)^{d/2} \sin(d\pi/2) \Gamma(d/2)} \left( \frac{1}{d-1} + \frac{R}{4dm^2} \right) \times$$

$$\left( \frac{m^2}{R} \right)^{d/2-1} \sum_{k=0}^{\infty} \left( \frac{R}{m^2} \right) \sum_{\ell=0}^{k} \left( -\frac{(d-3)^2}{4} \right) ^{\ell} \frac{\kappa_{k-\ell}(d)}{(d(d-1))^k},$$

(51)

where we have reorganised sums coming from geometric series and (26) to combine powers of $m$. As before, the transverse vector heat kernels $b_{2n}^{(1)}$ and $c_{d+2n}^{(1)}$ for the endomorphism (50) can now be read off from (51) by noticing that the coefficients $b_{2n}^{(1)}$ ($c_{d+2n}^{(1)}$) are linear in (independent of) the parameters $\kappa_i(d)$. We find

$$b_{2n}^{(1)}(E) = \frac{(d-1)\pi R^n}{\Gamma(1 + n - d/2)} \left[ \sum_{\ell=0}^{n} \left( -\frac{(d-3)^2}{4} \right) ^{\ell} \frac{\kappa_{n-\ell}(d)(d(d-1))^{-n}}{\sin(d\pi/2) \Gamma(d/2)} \right. \left. + \frac{(d-1)^2}{4} \sum_{\ell=0}^{n-1} \left( -\frac{(d-3)^2}{4} \right) ^{\ell} \frac{\kappa_{n-1-\ell}(d)(d(d-1))^{-n}}{\sin(d\pi/2) \Gamma(d/2)} \right],$$

(52)

$$c_{d+2n}^{(1)}(E) = \frac{\Gamma(d)}{\Gamma(d/2) \Gamma(1 + n)} \left( \frac{R}{d(d-1)} \right)^{d/2} \left( -\frac{(d-3)^2}{4d(d-1)} R \right)^n,$$

and $\kappa_n(d)$ determined through (27). Applying (13), we then find the transverse vector heat kernel coefficients for general endomorphisms as well. Tab. 2 shows our results for the first few vector heat kernels ($E = 0$) and for a selection of dimensions.
| $d$ | $d = 2$ | $d = 3$ | $d = 4$ | $d = 5$ | $d = 6$ |
|-----|--------|--------|--------|--------|--------|
| $b_0^{(1)}$ | 1 | 2 | 3 | 4 | 5 |
| $c_d^{(1)}$ | $\frac{1}{2} R$ | $\frac{2}{3\sqrt{6\pi}} R^{3/2}$ | $\frac{1}{24} R^2$ | $\frac{1}{25\sqrt{6\pi}} R^{5/2}$ | $\frac{1}{150} R^3$ |
| $b_2^{(1)}$ | $-\frac{1}{3} R$ | 0 | $\frac{1}{4} R$ | $\frac{7}{15} R$ | $\frac{2}{3} R$ |
| $c_{d+2}^{(1)}$ | $\frac{1}{4} R^2$ | $\frac{2}{9\sqrt{6\pi}} R^{5/2}$ | $\frac{1}{96} R^3$ | $\frac{1}{125\sqrt{6\pi}} R^{7/2}$ | $\frac{1}{2700} R^4$ |
| $b_4^{(1)}$ | $-\frac{11}{40} R^2$ | $-\frac{1}{9} R^2$ | $-\frac{67}{1440} R^2$ | $-\frac{1}{120} R^2$ | $\frac{7}{500} R^2$ |
| $c_{d+4}^{(1)}$ | $\frac{1}{32} R^3$ | $\frac{1}{27\sqrt{6\pi}} R^{7/2}$ | $\frac{1}{168} R^4$ | $\frac{1}{125\sqrt{6\pi}} R^{9/2}$ | $\frac{32400}{32400} R^5$ |
| $b_6^{(1)}$ | $-\frac{37}{504} R^3$ | $-\frac{2}{81} R^3$ | $-\frac{4321}{382680} R^3$ | $-\frac{1}{108} R^3$ | $-\frac{2539}{585500} R^3$ |
| $c_{d+6}^{(1)}$ | $\frac{1}{90} R^4$ | $\frac{1}{243\sqrt{6\pi}} R^{9/2}$ | $\frac{1}{2416} R^5$ | $\frac{1}{1875\sqrt{6\pi}} R^{11/2}$ | $\frac{1}{583200} R^6$ |
| $b_8^{(1)}$ | $-\frac{157}{13440} R^4$ | $-\frac{1}{324} R^4$ | $-\frac{3207}{2488320} R^4$ | $-\frac{17}{2304} R^4$ | $-\frac{176107}{108240000} R^4$ |
| $c_{d+8}^{(1)}$ | $\frac{1}{768} R^5$ | $\frac{1}{2916\sqrt{6\pi}} R^{11/2}$ | $\frac{1}{147456} R^6$ | $\frac{1}{37500\sqrt{6\pi}} R^{13/2}$ | $\frac{1}{13996800} R^7$ |
| $b_{10}^{(1)}$ | $-\frac{109}{88704} R^5$ | $-\frac{1}{3645} R^5$ | $-\frac{245461}{2299207680} R^5$ | $-\frac{1}{18432} R^5$ | $-\frac{213749}{6733960000} R^5$ |
| $c_{d+10}^{(1)}$ | $\frac{1}{7680} R^6$ | $\frac{1}{43740\sqrt{6\pi}} R^{13/2}$ | $\frac{1}{2949120} R^7$ | $\frac{1}{3975000\sqrt{6\pi}} R^{15/2}$ | $\frac{1}{419994000} R^8$ |

Table 2. The vector heat kernel coefficients for different integer dimensions and vanishing endomorphism.

D. Transverse Traceless Tensors

Turning to the heat kernel coefficients for transverse traceless symmetric tensors, we first need to derive the Green’s function. Just as in the case of transverse vectors, it is important to notice that the differential equation for the corresponding Green’s function $G_{Q,\mu\nu,\rho,\sigma}^{TT}$ takes the form

$$(-\nabla^2 + Q)G_{Q,\mu\nu,\rho,\sigma}^{TT} = \frac{g_{\rho(\rho'} g_{\sigma)\mu}}{\sqrt{g}} \delta(x - y) + \text{longitudinal and trace terms}, \quad (53)$$

where longitudinal terms include terms which are longitudinal with respect to at least one index but not necessarily all indices. Following the methods introduced previously we write the transverse traceless symmetric Green’s function as

$$G_{Q,\mu\nu,\rho,\sigma}^{TT} = P\alpha\beta P_{\rho\sigma}^{\gamma\delta} (R_{\alpha\gamma} R_{\beta\delta} S_{TT}). \quad (54)$$
The transverse traceless tensor projector is given by \[ \mathcal{P}_{\mu\nu}^{\alpha\beta} = \frac{1}{d-2} \left[ -g^{\alpha\beta}_{\mu\nu} \left( \nabla^2 - \frac{R}{d-1} \right) \left( \nabla^2 - \frac{2R}{d(d-1)} \right) \right. 
\left. + \frac{g_{\mu\nu}g^{\alpha\beta}}{d-1} \left( \nabla^4 - \frac{R}{d} \nabla^2 + \frac{2R^2}{d^2(d-1)} \right) - \frac{g^{\alpha\beta}}{d-1} \nabla_{(\mu} \nabla_{\nu)} \left( \nabla^2 + \frac{2R}{d} \right) \right]
\] (55)
and it satisfies
\[
\mathcal{P}_{\mu\nu}^{\alpha\beta} g_{\alpha\beta} S(\sigma) = g^{\mu\nu} \mathcal{P}_{\mu\nu}^{\alpha\beta} T_{\alpha\beta} = 0, \quad [\nabla^2, \mathcal{P}_{\mu\nu}^{\alpha\beta}] T_{\alpha\beta} = 0, \\
\mathcal{P}_{\mu\nu}^{\alpha\beta} \nabla_{\alpha} T_{\beta} = \mathcal{P}_{\mu\nu}^{\alpha\beta} \nabla_{\beta} T_{\alpha} = \nabla^{\mu} \mathcal{P}_{\mu\nu}^{\alpha\beta} T_{\alpha\beta} = \nabla^{\nu} \mathcal{P}_{\mu\nu}^{\alpha\beta} T_{\alpha\beta} = 0, \\
\mathcal{P}_{\mu\nu}^{\rho\sigma} \mathcal{P}_{\rho\sigma}^{\alpha\beta} T_{\alpha\beta} = \frac{1}{d-3} \mathcal{P}_{\mu\nu}^{\alpha\beta} \left( \nabla^2 - \frac{2R}{d(d-1)} \right) \left( \nabla^2 - \frac{R}{d-1} \right) T_{\alpha\beta}
\] (56)
for an arbitrary tensor \( T_{\alpha\beta} \). The Green’s function can now be determined in a very similar fashion to the transverse vector Green’s function. Firstly, we contract (53) with \( \mathcal{P}_{\lambda\eta}^{\mu\nu} \mathcal{P}_{\kappa'\chi'}^{\rho\sigma'} \). Making use of the relations for the transverse traceless tensor projector, the longitudinal and trace terms on the right-hand side vanish and we get
\[
\mathcal{P}_{\lambda\eta}^{\mu\nu} \mathcal{P}_{\kappa'\chi'}^{\rho\sigma'} \left( - \nabla^2 + Q \right) \left( \nabla^2 - \frac{2R}{d(d-1)} \right)^2 \left( \nabla^2 - \frac{R}{d-1} \right)^2 \left( \mathcal{R}_{\mu\nu}^{\rho\sigma'} \mathcal{R}_{\rho\sigma'} \delta(x - y) \right)
\] (57)
Using (57), this can be brought into the form
\[
\mathcal{P}_{\lambda\eta}^{\mu\nu} \mathcal{P}_{\kappa'\chi'}^{\rho\sigma'} \left( - \nabla^2 + Q - \frac{2R}{d(d-1)} \right) \nabla^4 \left( \nabla^2 - \frac{(d-2)R}{d(d-1)} \right)^2 S_{TT}(\sigma) 
\] (58)
from which we deduce that \( S_{TT}(\sigma) \) obeys the differential equation
\[
\left( - \nabla^2 + Q - \frac{2R}{d(d-1)} \right) \nabla^4 \left( \nabla^2 - \frac{(d-2)R}{d(d-1)} \right)^2 S_{TT}(\sigma) = \frac{4}{\sqrt{g}} \left( \frac{d-2}{d-3} \right)^2 \delta(x - y).
\] (59)
The differential equation is solved by
\[
S_{TT}(\sigma) = 4 \left( \frac{d-2}{d-3} \right)^2 G_{\chi_1,\chi_2,\chi_3,\chi_3}(\sigma)
\] (60)
where we have introduced

\[ G_{\chi_1,\chi_2,\chi_3}(\sigma) = \frac{\partial}{\partial \chi_2} \frac{\partial}{\partial \chi_3} G_{\chi_1,\chi_2,\chi_3}(\sigma), \]

\[ G_{\chi_1,\chi_2,\chi_3}(\sigma) = \frac{G_{\chi_1,\chi_3}(\sigma) - G_{\chi_1,\chi_2}(\sigma)}{\chi_2 - \chi_3}, \]

(61)

together with (21) and (40), and evaluated for

\[ \chi_1 = Q - \frac{2R}{d(d-1)}, \quad \chi_2 = 0, \quad \chi_3 = \frac{d - 2}{d(d-1)} R. \]

(62)

To calculate the trace of \( G_{Q,\mu \nu \rho}^{TT} \) in the coincidence limit, we first expand the term \( \mathcal{R}_{\mu \rho}^{\prime} \mathcal{R}_{\nu \sigma}^{\prime} S_{TT}(\sigma) \) as before. Since we are now acting with eight covariant derivatives on this expression, see (60), we have to expand it up to order eight in \( \sigma_\mu \). The relevant expression is lengthy and given in App. [A], see (A4), where we also introduce

\[ S_{TT}(\sigma) = \sum_{n=0}^{\infty} S_{TT}^{n} \sigma^n. \]

(63)

Acting with the transverse tensor projectors and contracting the result with \( g^{\mu(\rho', \sigma')\nu} \), we obtain

\[ g^{\mu(\rho', \sigma')\nu} G_{Q,\mu \nu \rho}^{TT}(0) = \frac{(d-3)^2(d+1)(d+2)}{d-2} \left[ \frac{(13d-32)R^3}{140(d-1)^2d^2} S_{TT}^{1} + \frac{(d-6)(7d-8)R^2}{20(d-1)^2d} S_{TT}^{2} \right. \]

\[ + \left. \frac{3(3-2d)(d+4)R}{2(d-1)} S_{TT}^{3} + 3d(d+4)(d+6)S_{TT}^{4} \right]. \]

(64)

Inserting the expansion coefficients for \( S_{TT}(\sigma) \) and using (26), we finally arrive at

\[ g^{\mu(\rho', \sigma')\nu} G_{Q,\mu \nu \rho}^{TT}(0) = \frac{2^{-d-1}(d-1)^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \left( \frac{R}{d(d-1)} \right)^{d/2} \times \]

\[ \left\{ \frac{\Gamma(d+2)\{(d-1)d(d+2)Q - (d^2+4)R\}}{d(d-1)dQ - 2R} \right\} \]

\[ + \frac{\pi(d-2)(d+1)((d-1)dQ + (d-2)R)}{\sin\left(\frac{\pi d}{2}\right) R} \sum_{k=0}^{\infty} \kappa_k(d) \left( -\tilde{\xi}^2 \right)^{d/2-k-1}. \]

(65)

where

\[ \tilde{\xi} = \frac{1}{2} \sqrt{\left( -\frac{4(d-1)Q}{R} + d - 2 \right) d + 9}. \]

(66)

Just as before, we now choose the endomorphism in order to simplify the sum. Taking

\[ \mathcal{E} = -\frac{9 - 2d + d^2}{4d(d-1)} R, \]

\[ -\tilde{\xi}^2 = \frac{d(d-1)m^2}{R}, \]

(67)
and using (67) and the geometric series to expand the denominators in the large \( m \) limit, we find

\[
\begin{align*}
g^{mn} \rho_\mu \rho'_\sigma \gamma_{Q,\mu\rho,\rho'}(0) &\bigg|_{E=E} \\
&= \frac{2^{-d-2} \pi^{-d/2}}{m^2 R^2 (\pi^2 - 1)} \left( \frac{R}{d(d-1)} \right)^{1+d/2} \sum_{n=0}^\infty \left( -\frac{(d-3)^2}{4d(d-1)} \frac{R}{m^2} \right)^n \\
&\quad \times \left[ \pi d(d-1) \left( 4(d-1)dm^2 + (d+1)^2 R \right) \frac{\sin \left( \frac{\pi d}{2} \right)}{R} \sum_{k=0}^\infty \kappa_k(d) \left( \frac{d(d-1)m^2}{R} \right)^{d/2-k-1} \\
&\quad + \frac{\Gamma(d+2)(4(d-1)d(d+2)m^2 + (d(d-4)d+5)+2)R}{(d-2)dm^2} \sum_{m=0}^\infty \left( -\frac{d-1}{4d} \frac{R}{m^2} \right)^m \right] \\
&= \frac{1}{(4\pi)^{d/2}} \left( \frac{R}{d(d-1)} \right)^{d/2} \frac{1}{\Gamma(d/2-1)} \sum_{k=0}^\infty \left( \frac{R}{m^2} \right)^k \\
&\quad \times \left[ \frac{\Gamma(d+2)}{(d-2)} \left( \frac{d+2}{4dm^2} + \frac{2 + 5 + (d(d-4))}{4d(d-1)m^4} R \right) \sum_{\ell=0}^k \left( -\frac{(d-3)^2}{4d(d-1)} \right)^\ell \left( -\frac{d-1}{4d} \right)^{k-\ell} \\
&\quad + \frac{\pi(d(d-1))^{-k}}{\sin(d\pi/2)} \left( \frac{(d+1)^3}{4m^2} + \frac{d(d^2-1)}{R} \right) \left( \frac{R}{d(d-1)m^2} \right)^{1-d/2} \sum_{\ell=0}^k \kappa_\ell(d) \left( -\frac{(d-3)^2}{4} \right)^{k-\ell} \right], \tag{68}
\end{align*}
\]

Note that in the last step sums have been reorganised in order to combine powers of \( m \). As before, using (68) and the specific endomorphism (67), the heat kernel coefficients are found to be

\[
\begin{align*}
b^{(2)}_{2n}(E) &= \frac{1}{\Gamma(1+n-d/2)} \left( \frac{R}{d(d-1)} \right)^n \frac{\pi(d+1)}{\Gamma(d/2-1)\sin(d\pi/2)} \\
&\quad \times \left[ \sum_{\ell=0}^n \kappa_\ell(d) \left( -\frac{(d-3)^2}{4} \right)^{n-\ell} + \frac{(d+1)^2}{4} \sum_{\ell=0}^{n-1} \kappa_\ell(d) \left( -\frac{(d-3)^2}{4} \right)^{n-1-\ell} \right], \\
\end{align*}
\]

\[
\begin{align*}
c^{(2)}_{d+2n}(E) &= \frac{2}{\Gamma(1+n)} \left( \frac{R}{d(d-1)} \right)^{d/2} \frac{\Gamma(d+2)}{\Gamma(d/2)} R^n \times \left[ \frac{d+2}{4d} \sum_{\ell=0}^n \left( -\frac{(d-3)^2}{4d(d-1)} \right)^\ell \left( -\frac{d-1}{4d} \right)^{n-\ell} \\
&\quad + \frac{2 + 5 + (d(d-4))}{16d^2(d-1)} \sum_{\ell=0}^{n-1} \left( -\frac{(d-3)^2}{4d(d-1)} \right)^\ell \left( -\frac{d-1}{4d} \right)^{n-1-\ell} \right], \tag{69}
\end{align*}
\]

with \( \kappa_n(d) \) determined through (27). As has been observed for the scalar and transverse vector heat kernel coefficients, the coefficients \( b^{(2)}_{2n} \) are identified from (68) as the terms linear in \( \kappa_i(d) \) while the coefficients \( c^{(2)}_{d+2n} \) are found by setting all \( \kappa_i(d) \) to zero. The heat kernel coefficients for arbitrary endomorphisms can be found using (13). Tab. 3 shows our results for the first few tensor heat kernels (\( E = 0 \)) and for a selection of dimensions.

As a final remark, we emphasize that the explicit expressions for the heat kernel coefficients of scalar (30), transverse vectors (52), and transverse traceless tensors (69), are easy to evaluate and implement for arbitrary endomorphism and arbitrary dimension, including non-integer ones. For
example, the first 100 heat kernel coefficients \((69)\) at vanishing endomorphism in \(d = 4\) are found within 20 sec on a 3 GHz thread, also using \((13)\) and the coefficients \((27)\).

E. Unconstrained Fields

In the literature, heat kernel coefficients are often calculated for unconstrained fields. A relation between the heat kernels of constrained fields and unconstrained fields can be derived by decomposing unconstrained fields into constrained fields.

For vectors, the unconstrained vector field \(v^\mu\) can be decomposed into a transverse vector field \(v'^\mu\) and a longitudinal part \(\nabla^\mu \eta\) through

\[
v^\mu = v'^\mu + \nabla^\mu \eta.
\]

Note that the first mode of the scalar field \(\eta\) is constant and does not contribute to \(v^\mu\). Hence, this mode must be excluded later. Considering the Laplacian \(\nabla^2\) acting on \(v^\mu\), we can write

\[
-\nabla^2 v^\mu = -\nabla^2 (v'^\mu + \nabla^\mu \eta) = -\nabla^2 v'^\mu + \nabla^\mu \left(-\nabla^2 - \frac{R}{d}\right) \eta
\]

on the sphere. It follows that the eigenvalue spectrum of \(-\nabla^2\) acting on an unconstrained vector

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
 & d = 2 & d = 3 & d = 4 & d = 5 & d = 6 \\
\hline
\begin{array}{l}
\text{b}_0^{(2)} \\
\text{c}_d^{(2)} \\
\text{b}_2^{(2)} \\
\text{c}_{d+2}^{(2)} \\
\text{b}_4^{(2)} \\
\text{c}_{d+4}^{(2)} \\
\text{b}_6^{(2)} \\
\text{c}_{d+6}^{(2)} \\
\text{b}_8^{(2)} \\
\text{c}_{d+8}^{(2)} \\
\text{b}_{10}^{(2)} \\
\text{c}_{d+10}^{(2)} \\
\end{array}
\end{array}
\]

Table 3. The tensor heat kernel coefficients for different integer dimensions and vanishing endomorphism.
field is the sum of two parts. The first part is the eigenvalue spectrum of the Laplacian acting on a transverse vector field. The second part is the eigenvalue spectrum of the Laplacian acting on a scalar field shifted by \(-\frac{R}{d}\) and the first mode being excluded. Thus,

\[
\text{Tr}_V e^{t\nabla^2} = \text{Tr}_1 e^{t\nabla^2} + \text{Tr}'_0 e^{t(\nabla^2 + \frac{R}{d})},
\]

where primes at the trace denote the exclusion of lowest modes and \(\text{Tr}_V\) denotes the trace of an unconstrained vector field. This allows the computation of heat kernel coefficients for unconstrained fields using our results. We get

\[
\text{Tr}_V e^{t\nabla^2} = \frac{\text{Vol}}{(4\pi t)^{d/2}} \left[ d + \frac{d}{6} R t + \frac{5d^3 - 7d^2 + 6d - 60}{360(d - 1)d} R^2 t^2 \right. \\
+ \frac{35d^5 - 112d^4 + 187d^3 - 1370d^2 + 852d - 1008}{45360(d - 1)^2 d^2} R^4 t^4 \\
+ \frac{R^4 t^4}{5443200(d - 1)^3 d^3} \left( 175d^7 - 945d^6 + 2389d^5 - 15711d^4 + 23464d^3 \right. \\
\left. - 35436d^2 + 59760d - 62640 \right) \\
+ \frac{R^5 t^5}{359251200(d - 1)^3 d^4} \left( 385d^9 - 3080d^8 + 10714d^7 - 68156d^6 + 168793d^5 \right. \\
\left. + 30808d^4 + 858996d^3 - 958944d^2 + 857232d - 798336 \right) + O(R t^6) \right].
\]

Turning to tensors and using the York decomposition, we decompose an unconstrained symmetric tensor field \(T_{\mu\nu}\) into

\[
T_{\mu\nu} = T^{TT}_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + \left( \nabla_\mu \nabla_\nu - \frac{1}{d} g_{\mu\nu} \nabla^2 \right) \sigma + \frac{1}{d} g_{\mu\nu} \eta,
\]

with \(T^{TT}_{\mu\nu}\) being a transverse traceless symmetric tensor, \(\xi_\mu\) being a transverse vector, and \(\sigma\) and \(\eta\) being scalars. As in the case of vector fields, not all modes of this decomposition contribute to \(T^{TT}_{\mu\nu}\). These modes which have to be excluded are (i) the \(d(d + 1)/2\) Killing vectors of \(\xi_\mu\) which satisfy \(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0\) and originate from the lowest transverse vector modes, (ii) the constant (lowest) mode of \(\sigma\), and (iii) the \(d + 1\) scalars of the second lowest modes of \(\sigma\) (see Tab. 4 for the multiplicities of these modes). Acting with the Laplacian on \(T_{\mu\nu}\), we use

\[
\nabla^2 (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) = 2 \nabla_{(\mu} \left( -\nabla^2 - \frac{d + 1}{d(d - 1)} R \right) \xi_{\nu)} \\
- \nabla^2 \left( \nabla_\mu \nabla_\nu - \frac{1}{d} g_{\mu\nu} \nabla^2 \right) \sigma = \left( \nabla_\mu \nabla_\nu - \frac{1}{d} g_{\mu\nu} \nabla^2 \right) \left( -\nabla^2 - \frac{2}{d - 1} R \right) \sigma,
\]

to arrive at

\[
-\nabla^2 T_{\mu\nu} = -\nabla^2 T^{TT}_{\mu\nu} + 2 \nabla_{(\mu} \left( -\nabla^2 - \frac{d + 1}{d(d - 1)} R \right) \xi_{\nu)} \\
+ \left( \nabla_\mu \nabla_\nu - \frac{1}{d} g_{\mu\nu} \nabla^2 \right) \left( -\nabla^2 - \frac{2}{d - 1} R \right) \sigma - \frac{1}{d} g_{\mu\nu} \nabla^2 \eta.
\]
This implies
\[
\text{Tr}_T e^{t\nabla^2} = \text{Tr}_2 e^{t\nabla^2} + \text{Tr}_1' e^{t\left(\nabla^2 + \frac{d+1}{\pi(d-1)} R\right)} + \text{Tr}_0'' e^{t\left(\nabla^2 + \frac{d-3}{4} R\right)} + \text{Tr}_0 e^{t\nabla^2},
\] (77)
where \(\text{Tr}_T\) denotes the trace of the Laplacian w.r.t. an unconstrained symmetric tensor field and two primes denote the exclusion of the two lowest modes. Using this and our results for the heat kernel coefficients of constrained fields we find

\[
\text{Tr}_T e^{t\nabla^2} = \frac{\text{Vol}}{(4\pi t)^{d/2}} \left[ \frac{d(d+1)}{2} + \frac{d(d+1)}{12} R t + \frac{5d^4 - 2d^3 - d^2 - 114d - 240}{720(d-1)d} R^2 t^2 \\
+ \frac{35d^6 - 77d^5 + 75d^4 - 2443d^3 - 3542d^2 + 1104d - 4032}{90720(d-1)^2d^2} R^3 t^3 \\
+ \frac{R^4 t^4}{1086400(d-1)^3d^3} \left(175d^8 - 770d^7 + 1444d^6 - 25922d^5 - 9887d^4 + 13588d^3 + 188844d^2 + 742320d + 172800 \right) \\
+ \frac{R^5 t^5}{718502400(d-1)^4d^4} \left(385d^{10} - 2695d^9 + 7634d^8 - 103642d^7 + 100637d^6 - 8875d^5 + 2850880d^4 + 8146092d^3 + 7406448d^2 + 18339840d + 8211456 \right) + O(Rt)^6 \right].
\] (78)

This completes the derivation of heat kernel coefficients.

IV. HEAT KERNELS FROM SPECTRAL SUMS

In this section, we compute heat kernel coefficients with the help of spectral sums. This serves as an independent consistency check for findings in the previous section.

A. Spectral Sum Technique

The eigenspectrum of the Laplacian \(-\nabla^2\) is known on the sphere \([69, 70]\) and can be used to calculate the heat kernel coefficients, e.g. \([42, 61]\). Specifically, for different spins \(s\), the eigenfunctions \(\phi^s_\ell\) satisfy the eigenvalue equation

\[
-\nabla^2 \phi^s_\ell(x) = \lambda^s_\ell \phi^s_\ell(x),
\] (79)

where \(\lambda^s_\ell\) are the eigenvalues and all vector and tensor indices have been suppressed. In Tab. 4 we show the eigenvalues and their multiplicities \(D^s_\ell\) for spin 0, spin 1, and spin 2 fields on the sphere. The eigenfunctions can be chosen to be orthonormal,

\[
\int d^4x \sqrt{g} \phi^s_{\ell,n}(x) \phi^s_{k,m}(x) = \delta_{\ell,k} \delta_{n,m},
\] (80)
Table 4. The eigenvalues $\lambda^s_\ell$ and their multiplicities $D^s_\ell$ for spin 0 (scalar), spin 1 (transverse vector), and spin 2 (transverse traceless tensor) fields of the operator $-\nabla^2$ on the sphere (taken from [69, 70]).

where we use the indices $n$ and $m$ to distinguish eigenfunctions with equal eigenvalues. Further, they satisfy the completeness relation

$$\sum_{\ell,n} \phi^s_{\ell,n}(x) \phi^s_{\ell,n}(y) = \frac{\delta(x - y)}{\sqrt{g}}.$$  \hspace{1cm} (81)

With this at hand, and for vanishing endomorphisms, we may express the trace of the heat kernel as a weighted sum of eigenvalues,

$$\text{Tr}_s U_0(t,\sigma) = \sum_{\ell,n} \int d^d x \sqrt{g} \phi^s_{\ell,n}(x) e^{t\nabla^2} \phi^s_{\ell,n}(x) = \sum_{\ell} D^s_\ell e^{-t\lambda^s_\ell}.$$  \hspace{1cm} (82)

The sum can be approximated systematically using the Euler-Maclaurin formula \cite{42}

$$\sum_{\ell=a}^{b} f(\ell) = \int_{a}^{b} d\ell f(\ell) + \frac{1}{2} (f(a) + f(b)) + \sum_{k=2}^{n} \frac{B_k}{k!} \left( f^{k-1}(b) - f^{k-1}(a) \right) - R_n,$$  \hspace{1cm} (83)

where $B_k$ are the Bernoulli numbers and $R_n$ is some remainder part.

### B. Results

Using the expression (82) together with the expansion (83), we find

$$\text{Tr}_3 U_0(t,\sigma) \bigg|_{d=2} = \frac{\text{Vol}}{4\pi t} \left( 1 + \frac{R_t}{6} + \frac{R_t^2}{60} + \frac{R_t^3}{630} + \frac{R_t^4}{5040} + \frac{R_t^5}{27720} + O(R_t)^6 \right),$$

$$\text{Tr}_3 U_0(t,\sigma) \bigg|_{d=3} = \frac{\text{Vol}}{(4\pi t)^{3/2}} \left( 1 + \frac{R_t}{6} + \frac{R_t^2}{72} + \frac{R_t^3}{1296} + \frac{R_t^4}{31104} + \frac{R_t^5}{933120} + O(R_t)^6 \right),$$

$$\text{Tr}_3 U_0(t,\sigma) \bigg|_{d=4} = \frac{\text{Vol}}{(4\pi t)^2} \left( 1 + \frac{R_t}{6} + \frac{29R_t^2}{2160} + \frac{37R_t^3}{54432} + \frac{149R_t^4}{6531840} + \frac{179R_t^5}{43101440} + O(R_t)^6 \right),$$

$$\text{Tr}_3 U_0(t,\sigma) \bigg|_{d=5} = \frac{\text{Vol}}{(4\pi t)^{5/2}} \left( 1 + \frac{R_t}{6} + \frac{2R_t^2}{75} + \frac{R_t^3}{1500} + \frac{R_t^4}{45000} + \frac{R_t^5}{225000} + O(R_t)^6 \right),$$

$$\text{Tr}_3 U_0(t,\sigma) \bigg|_{d=6} = \frac{\text{Vol}}{(4\pi t)^3} \left( 1 + \frac{R_t}{6} + \frac{R_t^2}{75} + \frac{1139R_t^3}{1701000} + \frac{833R_t^4}{3645000} + \frac{137R_t^5}{267300000} + O(R_t)^6 \right).$$  \hspace{1cm} (84)
for the scalar heat kernel coefficients in integer dimensions. Note that these results are in full agreement with Tab. [1]

The same can be done for transverse vectors and transverse traceless tensors. However, it is important to realise that in even dimensions we cannot distinguish between the contributions coming from the $b_{d+2n}^{(1)}$ and the $c_{d+2n}^{(1)}$ ($n \geq 0$) when we take the sum of the eigenvalues. Thus, in even dimension we can only compare their sum with the findings from spectral sums. For the transverse vectors in even dimension we find using the spectral sum

$$\text{Tr}_1 U_0(t, \sigma) \bigg|_{d=2} = \frac{\text{Vol}}{4\pi t} \left( 1 + \frac{R t}{6} - \frac{R^2 t^2}{40} - \frac{11 R^3 t^3}{1008} - \frac{17 R^4 t^4}{13440} + \frac{13 R^5 t^5}{177408} + \mathcal{O}(R t)^6 \right),$$

$$\text{Tr}_1 U_0(t, \sigma) \bigg|_{d=4} = \frac{\text{Vol}}{(4\pi t)^2} \left( 3 + \frac{R t}{4} - \frac{7 R^2 t^2}{1440} - \frac{541 R^3 t^3}{362880} - \frac{157 R^4 t^4}{2488320} + \frac{4019 R^5 t^5}{2299207680} + \mathcal{O}(R t)^6 \right),$$

$$\text{Tr}_1 U_0(t, \sigma) \bigg|_{d=6} = \frac{\text{Vol}}{(4\pi t)^3} \left( 5 + \frac{2 R t}{3} + \frac{7 R^2 t^2}{360} - \frac{649 R^3 t^3}{850500} - \frac{24907 R^4 t^4}{408240000} - \frac{5849 R^5 t^5}{6735960000} + \mathcal{O}(R t)^6 \right).$$

Taking the sum $b_{2n}^{(1)} + c_{2n}^{(1)}$ from Tab. [2] the consistency of our results can be easily checked. In odd dimensions, however, we can distinguish between the contributions of $b_{2n}^{(1)}$ and $c_{d+2n}^{(1)}$ in the spectral sum. We have

$$\text{Tr}_1 U_0(t, \sigma) \bigg|_{d=3} = \frac{\text{Vol}}{(4\pi t)^{3/2}} \left( 2 + \frac{2 R^3 t^{3/2}}{3\sqrt{6\pi}} - \frac{R^2 t^2}{9\sqrt{6\pi}} + \frac{2 R^5 t^{5/2}}{9\sqrt{36\pi}} - \frac{2}{81} R^3 t^3 + \frac{R^{7/2} t^{7/2}}{27\sqrt{6\pi}} \right.$$

$$- \frac{1}{324} R^4 t^4 + \frac{R^{9/2} t^{9/2}}{243\sqrt{6\pi}} - \frac{R^5 t^5}{3645} + \mathcal{O}(R t)^{11/2} \bigg),$$

$$\text{Tr}_1 U_0(t, \sigma) \bigg|_{d=5} = \frac{\text{Vol}}{(4\pi t)^{5/2}} \left( 4 + \frac{7 R t}{15} - \frac{R^2 t^2}{120} + \frac{R^5 t^{5/2}}{25\sqrt{5\pi}} - \frac{R^3 t^3}{160} + \frac{R^{7/2} t^{7/2}}{125\sqrt{5\pi}} - \frac{17 R^4 t^4}{23040} \right.$$

$$+ \frac{R^{9/2} t^{9/2}}{1250\sqrt{5\pi}} - \frac{R^5 t^5}{18432} + \mathcal{O}(R t)^{11/2} \bigg).$$

Again, we find full agreement with Tab. [2]

For the transverse traceless tensor heat kernels in even dimensions we get from the spectral sum aproach

$$\text{Tr}_2 U_0(t, \sigma) \bigg|_{d=2} = 0,$$

$$\text{Tr}_2 U_0(t, \sigma) \bigg|_{d=4} = \frac{\text{Vol}}{(4\pi t)^2} \left( 5 - \frac{5 R t}{6} - \frac{R^2 t^2}{432} + \frac{311 R^3 t^3}{54432} + \frac{109 R^4 t^4}{130638} - \frac{317 R^5 t^5}{12317184} + \mathcal{O}(R t)^6 \right),$$

$$\text{Tr}_2 U_0(t, \sigma) \bigg|_{d=6} = \frac{\text{Vol}}{(4\pi t)^3} \left( 14 + \frac{14 R t}{15} - \frac{56 R^2 t^2}{225} + \frac{433 R^3 t^3}{60750} + \frac{6971 R^4 t^4}{18225000} - \frac{28357 R^5 t^5}{3007125000} + \mathcal{O}(R t)^6 \right).$$

Similar to the case of transverse vectors, the consistency of these results with Tab. [3] can be checked.
by taking the sum $b_{2n}^{(2)} + c_{2n}^{(2)}$. In odd dimensions we find

$$
\text{Tr}_2 U_0(t, \sigma) \bigg|_{d=3} = \frac{\text{Vol}}{(4\pi t)^{3/2}} \left( 2 - \frac{5Rt}{3} + \frac{20}{3\sqrt{6\pi}} R^{3/2} t^{3/2} - \frac{13R^2t^2}{12} + \frac{8}{3\sqrt{6\pi}} R^{5/2} t^{5/2} - \frac{7}{24} \frac{R^3 t^3}{9\sqrt{6\pi}} \right),
$$

$$
\text{Tr}_2 U_0(t, \sigma) \bigg|_{d=5} = \frac{\text{Vol}}{(4\pi t)^{5/2}} \left( 9 - \frac{81R^2t^2}{200} + \frac{21R^{5/2} t^{5/2}}{25\sqrt{5\pi}} - \frac{81R^3t^3}{1000} + \frac{3R^{7/2} t^{7/2}}{25\sqrt{5\pi}} - \frac{729R^4t^4}{80000} + \frac{21R^{9/2} t^{9/2}}{2000\sqrt{5\pi}} - \frac{729R^5t^5}{100000} + \mathcal{O}(Rt^6) \right),
$$

which is, again, fully compatible with Tab. 3. This completes the derivation and checks of heat kernels on spheres in arbitrary dimension.

V. DISCUSSION

With the help of Green’s functions we have derived general expressions for all heat kernel coefficients of scalars (30), transverse vectors (52), and transverse traceless tensors (69) on the sphere in any dimension and for any endomorphism, also providing the corresponding results for unconstrained fields, see (73), (78). The final expressions are easy to evaluate and straightforward to implement on a practical level, with explicit results stated for selected integer dimensions (Tab. 1, 2 and 3). Several consistency checks have been performed. We compared the first five heat kernel coefficients and their full dimensional dependence to the known results for heat kernel coefficients on general manifolds [11–19], and found complete agreement. We have also derived the heat kernel coefficients on spheres with the help of spectral sums by exploiting the known eigenspectra of Laplacians on the sphere. For selected integer dimension, we have confirmed that both approaches give identical results to high order in the expansion, as they must, see (84) – (88). Another virtue of the expressions (30), (52), and (69) is that they can straightforwardly be extended to non-integer dimension, a feat which is much harder to achieve using spectral sums.

On a different tack, Green’s functions of scalars and transverse vectors on maximally symmetric spaces are of interest for applications in cosmology on de Sitter backgrounds, e.g. [39, 65, 68]. As a new addition, we now have derived the Green’s function for the Laplacian acting on transverse tensor fields on a fully symmetric background, (59) – (62). We expect that this will be of use in cosmological settings which are sensitive to the graviton propagator.

Finally, our results are of practical relevance for quantum gravity and renormalisation group tests of the asymptotic safety conjecture where intriguing hints for the near-Gaussianity of gravitational scaling exponents have been observed in [33, 34, 36, 58]. Studies thus far have adopted optimised renormalisation group flows [25, 57], which only depend on a few leading heat kernel coefficients [28]. Our findings enable new investigations which are sensitive to many more coefficients without resorting to flat backgrounds and spectral sums or approximations thereof.

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Appendix A: Expansion Coefficients

In the main body, we encounter expansions of vector and tensor structure functions of the form
\[ S(\sigma) = \sum_{n=0}^{\infty} S_n \sigma^n. \] (A1)

The expansion coefficients \( S_n \) for the transverse vector Green’s function (45) where \( S \equiv S^T \) and transverse traceless tensor Green’s function (63) where \( S \equiv S^{TT} \) are required for the computation of heat kernels, see (47) and (64), respectively.

The expansion coefficients of the structure function for transverse vectors are given by
\[ S^T_1 = \frac{\left( \frac{R}{d(d-1)} \right)^{d/2-1}}{(4\pi)^{d/2} \Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{d-1}{2} \right) \Gamma \left( \frac{d}{2} - \frac{\xi}{2} \right)} \left[ \frac{(d-1)dQ - R}{(d-1) \sin \left( \frac{\pi d}{2} \right)} \cos(\pi \xi) \Gamma \left( \frac{d-1}{2} + \xi \right) \Gamma \left( \frac{d-1}{2} - \xi \right) \right. 
\] 
\[ \left. - \frac{\Gamma(d-1)(dQ - R)(\pi \cot \left( \frac{\pi d}{2} \right) + \psi^{(0)}(d-1) + \gamma - 1) + \frac{R\Gamma(d-1)}{d-1} \right] \],
\[ S^T_2 = \frac{2^{-d-1}\pi^{-d/2} \left( \frac{R}{d(d-1)} \right)^{d/2-1}}{3(d-1)^2 d^3 (d+2) \Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{d-1}{2} \right) \Gamma \left( \frac{d}{2} - \frac{\xi}{2} \right)} \left[ (5d-8)R^2 \Gamma(d-1) \right. 
\] 
\[ \left. + \frac{(d-1)dQ - R)(3(d-1)dQ + 2dR - 5R)}{\sin \left( \frac{\pi d}{2} \right)} \cos(\pi \xi) \Gamma \left( \frac{d-1}{2} + \xi \right) \Gamma \left( \frac{d-1}{2} - \xi \right) \right] 
\] 
\[ \left. - \frac{R\Gamma(d)(dQ - R)(5d\gamma - 8d + \pi(5d-8) \cot \left( \frac{\pi d}{2} \right) + (5d-8)\psi^{(0)}(d-1) - 8\gamma + 17)}{d-3} \right], \] (A2)

where \( \gamma \) denotes the Euler-Mascheroni constant and \( \psi^{(0)}(x) \equiv \Gamma'(x)/\Gamma(x) \) the digamma function. We also recall that the parameter \( \xi \) is given by
\[ \xi = \left( \frac{1}{2} \right) \sqrt{d \left( -\frac{4(d-1)Q}{R} + d - 2 \right) + 5}. \] (A3)

The above expressions are required to obtain the result (48) stated the main text. Note that the dependence of the expansion coefficients (A2) on the Euler-Mascheroni constant and the digamma function drops out in the final expression (48).

For the expansion of the Green’s function for transverse traceless tensors, we need to calculate the trace of \( G^T_{Q,\mu\nu\rho\sigma'} \) in the coincidence limit, see (54). Using the solution (60) for the Green’s function of transverse traceless tensors and the corresponding expansion (A1) for the structure
function, we develop the term $R_{\mu'\nu'} S_{TT}(\sigma)$ up to order eight in $\sigma_\mu$. We find

$$R_{\mu'\nu'} S_{TT}(\sigma) = S_0^{TT} + S_1^{TT} + S_2^{TT} + S_3^{TT} + S_4^{TT}$$

Expressions for the expansion coefficients $S_n^{TT}$ are very long and not given here. Also, acting with the projectors on this expanded term gives long expressions, which are not shown. Similarly to the coefficients $A_2$, we observe that the coefficients $S_n^{TT}$ depend individually on the Euler-Mascheroni constant and the digamma function, whereas the final result (65) is independent thereof. The above expressions are used to arrive at the results (64) and (65) in the main text.

**Appendix B: Heat Kernels in Even Dimensions**

In this appendix we supply the first five heat kernel coefficients for scalars, vectors and tensors in even dimensions, where we may combine $b_{2n}^{(s)}$ and $c_{2n}^{(s)}$ into a single coefficient by writing

$$\hat{b}_{2n}^{(i)} = b_{2n}^{(i)} + c_{2n}^{(i)}$$
identically, we have $\hat{c}_n = 0$ for any $n < \frac{d}{2}$. Also, since the scalar heat kernel coefficients $c_n^{(0)}$ vanish identically, we have $\hat{b}_n^{(0)} = b_n^{(0)}$. For the transverse vector heat kernels we find

$$\hat{b}_n^{(1)} = \frac{d - 1}{2},$$

$$\hat{b}_n^{(1)} = \frac{\delta_{2,d} + d^2 - d - 6}{6d} R,$$

$$\hat{b}_n^{(1)} = R^2 + \frac{5d^4 - 12d^3 - 47d^2 - 186d + 180}{360(d - 1)d^2},$$

$$\hat{b}_n^{(1)} = \left( \frac{\delta_{2,d} + \delta_{1,d}}{4} \right) R^3 + \frac{35d^6 - 147d^5 - 331d^4 - 3825d^3 - 676d^2 + 10992d - 7560}{45360(d - 1)^2d^3} R,$$

$$\hat{b}_n^{(1)} = \left( \frac{\delta_{2,d} + 4d}{96} + \frac{\delta_{6,d}}{450} \right) R^3 + \frac{5443200(175d^8 - 1120d^7 - 866d^6 - 38260d^5 - 31985d^4 + 34700d^3 + 40596d^2 - 627840d + 226800)}{10886400(d - 1)^4d^3}.$$

The transverse traceless tensor heat kernels give

$$\hat{b}_n^{(2)} = \frac{(d - 2)(d + 1)}{2},$$

$$\hat{b}_n^{(2)} = 3\delta_{2,d} R + \frac{d^2 - 2d^2 - 13d - 10}{12(d - 1)} R,$$

$$\hat{b}_n^{(2)} = \left( \frac{3\delta_{2,d} + 5\delta_{1,d}}{36} \right) R^3 + \frac{5d^5 - 17d^4 - 105d^3 - 475d^2 - 620d - 228}{720(d - 1)^2d} R^2,$$

$$\hat{b}_n^{(2)} = \left( \frac{3\delta_{2,d} + 5\delta_{1,d}}{36} + \frac{14\delta_{6,d}}{225} \right) R^3 + \frac{35d^7 - 182d^6 - 884d^5 - 8618d^4 - 21515d^3 - 23648d^2 - 38116d - 28032}{90720(d - 1)^3d^2} R^3,$$

$$\hat{b}_n^{(2)} = \left( \frac{\delta_{2,d} + 5\delta_{4,d}}{2} + \frac{7\delta_{6,d}}{1125} + \frac{675\delta_{8,d}}{175616} R^4 \right) + \frac{R^4}{10886400(d - 1)^4d^3} \left( 175d^3 - 1295d^8 - 4296d^7 - 80514d^6 - 263073d^5 - 709635d^4 - 907534d^3 - 940876d^2 - 2454072d - 1896480 \right).$$

Results can now be compared with [35, 41] where expressions have been given for the heat kernel coefficients $b_n$ in even dimensions. In $d = 4$, our findings for $\hat{b}_n^{(2)}$ agree numerically with the corresponding expressions $b_2|_{n}$ given in [41] and in appendix B of [35], except for $b_8|_{1}$. For general $d$, deviations appear in the algebraic expressions for $b_8|_{1}$ and $b_8|_{2}$, and some contributions which uniquely arise in even integer dimensions (proportional to Kronecker deltas) have been missed. Our results are consistent with the heat kernel, spectral sums, and expressions for general backgrounds as found in the literature.

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