The geometric classification of nilpotent commutative $\mathfrak{CD}$-algebras

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Abstract: We give a geometric classification of complex 5-dimensional nilpotent commutative $\mathfrak{CD}$-algebras. The corresponding geometric variety has dimension 24 and decomposes into 10 irreducible components determined by the Zariski closures of a two-parameter family of algebras, three one-parameter families of algebras, and 6 rigid algebras.

Keywords: Nilpotent algebra, Jordan algebra, commutative $\mathfrak{CD}$-algebra, geometric classification, degeneration.

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INTRODUCTION

There are many results related to the algebraic and geometric classification of low-dimensional algebras in the varieties of Jordan, Lie, Leibniz and Zinbiel algebras; for algebraic classifications see, for example, [3,21,22]; for geometric classifications and descriptions of degenerations see, for example, [3,4,10,19,21–23,25,27,31,37]. In this present paper, we give a geometric classification of nilpotent commutative $\mathfrak{CD}$-algebras. This is a new class of non-associative algebras introduced in [3,28]. The idea of the definition of a $\mathfrak{CD}$-algebra comes from the following property of Jordan and Lie algebras: \textit{the commutator of any pair of multiplication operators is a derivation}. This gives three identities of degree four, which reduce to only one identity of degree four in the commutative or anticommutative case. Commutative and anticommutative $\mathfrak{CD}$-algebras are related to many interesting varieties of algebras. Thus, anticommutative $\mathfrak{CD}$-algebras is a generalization of Lie algebras, containing the intersection of Malcev and Sagle algebras as a proper subvariety. Moreover, the following intersections of varieties coincide: Malcev and Sagle algebras; Malcev and anticommutative $\mathfrak{CD}$-algebras; Sagle and anticommutative $\mathfrak{CD}$-algebras. On the other hand, the variety of anticommutative $\mathfrak{CD}$-algebras is a proper subvariety of the varieties of binary Lie algebras and almost Lie algebras [29]. The variety of anticommutative $\mathfrak{CD}$-algebras coincides with the intersection of the varieties of binary Lie algebras and almost Lie algebras. Commutative $\mathfrak{CD}$-algebras is a generalization of Jordan algebras, which is a generalization of associative commutative algebras. On the other hand, the variety of commutative $\mathfrak{CD}$-algebras is also known as the variety of almost-Jordan algebras, which states in the bigger variety of generalized almost-Jordan algebras [1,2,17]. The $n$-ary version of commutative $\mathfrak{CD}$-algebras was introduced in a recent paper by Kaygorodov, Pozhidaev and Saraiva [28].
The variety of almost-Jordan algebras is the variety of commutative algebras, satisfying
\[ 2((yx)x)x + yx^3 = 3(yx^2)x. \]
This present identity appeared in a paper of Osborn [33], during the study of identities of degree less than or equal to 4 of non-associative algebras. The identity is a linearized form of the Jordan identity. The systematic study of almost-Jordan algebras was initiated in the next paper of Osborn [32] and it was continued in some papers of Petersson [35,36], Osborn [34], and Sidorov [38] (sometimes, it was called as Lie triple algebras). Hentzel and Peresi proved that every semiprime almost-Jordan ring is Jordan [18]. After that, Labra and Correa proved that a finite-dimensional almost-Jordan right-nilalgebra is nilpotent [7,8]. Assosymmetric algebras under the symmetric product give almost-Jordan algebras [9].

**Motivation and contextualization.** Geometric properties of a variety of algebras have been an object of study since 1970’s. Gabriel [11] described the irreducible components of the variety of 4-dimensional unital associative algebras. Mazzola [31] classified algebraically and geometrically the variety of unital associative algebras of dimension 5. Burde and Steinhoff [5] constructed the graphs of degenerations for the varieties of 3-dimensional and 4-dimensional Lie algebras over \( \mathbb{C} \). Grunewald and O’Halloran [14] calculated the degenerations for the nilpotent Lie algebras of dimension up to 5. Seeley [37] solved the same problem for 6-dimensional complex nilpotent Lie algebras. Chouhy [6] proved that, in the case of finite-dimensional associative algebras, the \( N \)-Koszul property is preserved under the degeneration relation. Degenerations have also been used to study a level of complexity of an algebra (see [13,30,39,40]). Given algebras \( A \) and \( B \) in the same variety, we write \( A \rightarrow B \) and say that \( A \) degenerates to \( B \), or that \( A \) is a deformation of \( B \), if \( B \) is in the Zariski closure of the orbit of \( A \) (under the base-change action of the general linear group). The study of degenerations of algebras is very rich and closely related to deformation theory, in the sense of Gerstenhaber [12]. It offers an insightful geometric perspective on the subject and has been the object of a lot of research. In particular, there are many results concerning degenerations of algebras of small dimensions in a variety defined by a set of identities. One of the main problems of the geometric classification of a variety of algebras is a description of its irreducible components. In the case of finitely-many orbits (i.e., isomorphism classes), the irreducible components are determined by the rigid algebras — algebras whose orbit closure is an irreducible component of the variety under consideration. The algebraic classification of complex 5-dimensional nilpotent commutative \( \mathfrak{CD} \)-algebras was obtained in [20], and in the present paper we continue the study of the variety by giving its geometric classification.

**1. The algebraic classification of complex 5-dimensional nilpotent commutative \( \mathfrak{CD} \)-algebras**

The algebraic classification of 5-dimensional nilpotent commutative \( \mathfrak{CD} \)-algebras has three steps: the classification of all associative commutative algebras was given by Mazzola in 1979 [31]; the next step is the classification of all non-associative Jordan algebras was given by Hegazi and Abdelwahab in 2016 [16]; and the last step is the classification of all non-Jordan commutative \( \mathfrak{CD} \)-algebras was
given by Jumaniyozov, Kaygorodov and Khudoyberdiyev in 2021 [20]. Let us give the list of algebras from the last part from this long classification:

**Theorem 1.** Let \( \mathfrak{C} \) be a complex 5-dimensional nilpotent commutative \( \mathfrak{CD} \)-algebra. Then \( \mathfrak{C} \) is a Jordan algebra or it is isomorphic to one algebra from the following list:

\[
\begin{align*}
\mathfrak{C}^{0}_{0} & : e_1 e_1 = e_2 \quad e_2 e_2 = e_3 \\
\mathfrak{C}^{12}_{12}(\alpha) & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = \alpha e_4 \quad e_2 e_2 = (\alpha + 1) e_4 \\
\mathfrak{C}^{13}_{13} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = e_4 \quad e_2 e_2 = e_4 \\
\mathfrak{C}^{14}_{14} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = e_4 \\
\mathfrak{C}^{15}_{15} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_2 e_3 = e_4 \\
\mathfrak{C}^{16}_{16}(\alpha) & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = (\alpha + 1) e_4 \quad e_2 e_2 = \alpha e_4 \\
\mathfrak{C}^{17}_{17}(\alpha, \beta) & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = (\alpha + 1) e_4 \quad e_2 e_2 = \alpha e_4 \quad e_2 e_4 = \beta e_5 \\
\mathfrak{C}^{18}_{18} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{19}_{19} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = e_4 \\
\mathfrak{C}^{20}_{20} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{21}_{21} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{22}_{22} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{23}_{23}(\alpha) & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = e_4 \quad e_2 e_2 = e_4 \\
\mathfrak{C}^{24}_{24} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{25}_{25}(\alpha, \beta) & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_2 e_2 = \beta e_4 \\
\mathfrak{C}^{26}_{26}(\alpha) & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_2 e_2 = \alpha e_5 \\
\mathfrak{C}^{27}_{27} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{28}_{28} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{29}_{29} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{30}_{30}(\alpha) & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_2 e_2 = \alpha e_5 \\
\mathfrak{C}^{31}_{31} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{32}_{32} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{33}_{33} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{34}_{34} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{35}_{35} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{36}_{36} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{37}_{37} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{38}_{38} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{39}_{39} & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \\
\mathfrak{C}^{40}_{40}(\alpha) & : e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_2 e_2 = e_4 \\
\end{align*}
\]
Indeed, let us fix a basis $\alpha e_i$. Given an $e_j$, $\alpha e_i \otimes e_j$ is a vector space of dimension $C^5 = 1$.

All algebras from the present list are non-isomorphic, excepting

$$C^5_{13}(\alpha, \beta) \simeq C^5_{13}(\alpha, -\beta) \quad C^5_{20}(\alpha, \beta) \simeq C^5_{20}(\beta, \alpha) \quad C^5_{27}(\alpha) \simeq C^5_{27}(\frac{1}{\alpha}) \quad C^5_{69}(\alpha) \simeq C^5_{69}(\sqrt{\alpha})$$

2. The geometric classification of complex 5-dimensional nilpotent commutative $\mathcal{C}\mathcal{D}$-algebras

2.1. Degenerations of algebras. Given an $n$-dimensional vector space $V$, the set $\text{Hom}(V \otimes V, V) \simeq V^* \otimes V^* \otimes V$ is a vector space of dimension $n^3$. This space inherits the structure of the affine variety $\mathbb{C}^n$. Indeed, let us fix a basis $e_1, \ldots, e_n$ of $V$. Then any $\mu \in \text{Hom}(V \otimes V, V)$ is determined by $n^3$ structure constants $c^k_{i,j} \in \mathbb{C}$ such that $\mu(e_i \otimes e_j) = \sum_{k=1}^n c^k_{i,j} e_k$. A subset of $\text{Hom}(V \otimes V, V)$ is Zariski-closed if it can be defined by a set of polynomial equations in the variables $c^k_{i,j}$ ($1 \leq i, j, k \leq n$).

The general linear group $\text{GL}(V)$ acts by conjugation on the variety $\text{Hom}(V \otimes V, V)$ of all algebra structures on $V$:

$$(g * \mu)(x \otimes y) = g \mu(g^{-1}x \otimes g^{-1}y),$$
for \(x, y \in V, \mu \in \text{Hom}(V \otimes V, V)\) and \(g \in \text{GL}(V)\). Clearly, the \(\text{GL}(V)\)-orbits correspond to the isomorphism classes of algebra structures on \(V\). Let \(T\) be a set of polynomial identities which is invariant under isomorphism. Then the subset \(\mathbb{L}(T) \subset \text{Hom}(V \otimes V, V)\) of the algebra structures on \(V\) which satisfy the identities in \(T\) is \(\text{GL}(V)\)-invariant and Zariski-closed. It follows that \(\mathbb{L}(T)\) decomposes into \(\text{GL}(V)\)-orbits. The \(\text{GL}(V)\)-orbit of \(\mu \in \mathbb{L}(T)\) is denoted by \(\overline{O(\mu)}\) and its Zariski closure by \(\overline{O(\mu)}\).

Let \(A\) and \(B\) be two \(n\)-dimensional algebras satisfying the identities from \(T\) and \(\mu, \lambda \in \mathbb{L}(T)\) represent \(A\) and \(B\) respectively. We say that \(A\) \textit{degenerates} to \(B\) and write \(A \to B\) if \(\lambda \in \overline{O(\mu)}\). Note that in this case we have \(\overline{O(\lambda)} \subset \overline{O(\mu)}\). Hence, the definition of a degeneration does not depend on the choice of \(\mu\) and \(\lambda\). If \(A \to B\) and \(A \not\cong B\), then \(A \to B\) is called a \textit{proper degeneration}. We write \(A \not\to B\) if \(\lambda \not\in \overline{O(\mu)}\) and call this a \textit{non-degeneration}. Observe that the dimension of the subvariety \(\overline{O(\mu)}\) equals \(n^2 - \dim \text{Der}(A)\). Thus if \(A \to B\) is a proper degeneration, then we must have \(\dim \text{Der}(A) > \dim \text{Der}(B)\).

Let \(A\) be represented by \(\mu \in \mathbb{L}(T)\). Then \(A\) is \textit{rigid} in \(\mathbb{L}(T)\) if \(O(\mu)\) is an open subset of \(\mathbb{L}(T)\). Recall that a subset of a variety is called \textit{irreducible} if it cannot be represented as a union of two non-trivial closed subsets. A maximal irreducible closed subset of a variety is called an \textit{irreducible component}. It is well known that any affine variety can be represented as a finite union of its irreducible components in a unique way. The algebra \(A\) is rigid in \(\mathbb{L}(T)\) if and only if \(\overline{O(\mu)}\) is an irreducible component of \(\mathbb{L}(T)\).

In the present work we use the methods applied to Lie algebras in \([14,15]\). To prove degenerations, we will construct families of matrices parametrized by \(t\). Namely, let \(A\) and \(B\) be two algebras represented by the structures \(\mu\) and \(\lambda\) from \(\mathbb{L}(T)\), respectively. Let \(e_1, \ldots, e_n\) be a basis of \(V\) and \(c_{i,j}^k\) (\(1 \leq i, j, k \leq n\)) be the structure constants of \(\lambda\) in this basis. If there exist \(a_i^j(t) \in \mathbb{C}\) (\(1 \leq i, j \leq n\), \(t \in \mathbb{C}^\ast\)) such that the elements \(E_i^j(t) = \sum_{j=1}^n a_i^j(t)e_j\) (\(1 \leq i \leq n\)) form a basis of \(V\) for any \(t \in \mathbb{C}^\ast\), and the structure constants \(c_{i,j}^k(t)\) of \(\mu\) in the basis \(E_i^1, \ldots, E_i^n\) satisfy \(\lim_{t \to 0} c_{i,j}^k(t) = c_{i,j}^k\), then \(A \to B\). In this case \(E_i^1, \ldots, E_i^n\) is called a \textit{parametric basis} for \(A \to B\).

To prove a non-degeneration \(A \not\to B\) we will use the following lemma (see \([14]\)).

\textbf{Lemma 2.} Let \(B\) be a Borel subgroup of \(\text{GL}(V)\) and \(\mathcal{R} \subset \mathbb{L}(T)\) be a \(B\)-stable closed subset. If \(A \to B\) and \(A\) can be represented by \(\mu \in \mathcal{R}\) then there is \(\lambda \in \mathcal{R}\) that represents \(B\).

In particular, it follows from Lemma 2 that \(A \not\to B\), whenever \(\dim(A^2) < \dim(B^2)\).

When the number of orbits under the action of \(\text{GL}(V)\) on \(\mathbb{L}(T)\) is finite, the graph of primary degenerations gives the whole picture. In particular, the description of rigid algebras and irreducible components can be easily obtained. Since the variety of 5-dimensional nilpotent commutative \(\mathbb{C}\)-algebras contains infinitely many non-isomorphic algebras, we have to fulfill some additional work. Let \(A(\alpha) := \{A(\alpha)\}_{\alpha \in I}\) be a family of algebras and \(B\) be another algebra. Suppose that, for \(\alpha \in I\), \(A(\alpha)\) is represented by a structure \(\mu(\alpha) \in \mathbb{L}(T)\) and \(B\) is represented by a structure \(\lambda \in \mathbb{L}(T)\). Then by \(A(\alpha) \to B\) we mean \(\lambda \in \bigcup\{\overline{O(\mu(\alpha))}\}_{\alpha \in I}\), and by \(A(\alpha) \not\to B\) we mean \(\lambda \not\in \bigcup\{\overline{O(\mu(\alpha))}\}_{\alpha \in I}\).
Let $A(\ast)$, $B$, $\mu(\alpha)$ ($\alpha \in I$) and $\lambda$ be as above. To prove $A(\ast) \to B$, it is enough to construct a family of pairs $(f(t), g(t))$ parametrized by $t \in \mathbb{C}^*$, where $f(t) \in I$ and $g(t) = \left(a^j_i(t)\right)_{i,j} \in \text{GL}(V)$. Namely, let $e_1, \ldots, e_n$ be a basis of $V$ and $c^k_{i,j}$ ($1 \leq i, j, k \leq n$) be the structure constants of $\lambda$ in this basis. If we construct $a^j_i : \mathbb{C}^* \to \mathbb{C}$ ($1 \leq i, j \leq n$) and $f : \mathbb{C}^* \to I$ such that $E^j_i = \sum_{j=1}^n a^j_i(t)e_j$ ($1 \leq i \leq n$) form a basis of $V$ for any $t \in \mathbb{C}^*$, and the structure constants $c^k_{i,j}(t)$ of $\mu(f(t))$ in the basis $E^j_i$, $\ldots$, $E^n_i$ satisfy $\lim_{t \to 0} c^k_{i,j}(t) = c^k_{i,j}$, then $A(\ast) \to B$. In this case, $E^j_i, \ldots, E^n_i$ and $f(t)$ are called a parametric basis and a parametric index for $A(\ast) \to B$, respectively. In the construction of degenerations of this sort, we will write $\mu\left(f(t)\right) \to \lambda$, emphasizing that we are proving the assertion $\mu(\ast) \to \lambda$ using the parametric index $f(t)$.

2.2. The geometric classification of 5-dimensional nilpotent commutative $\mathbb{C}D$-algebras. The geometric classification of 5-dimensional nilpotent commutative $\mathbb{C}D$-algebras is based on some previous works: namely, all irreducible components of 5-dimensional nilpotent associative commutative algebras are given in [31] and all degenerations between these algebras are given in [27]; all irreducible components of 5-dimensional nilpotent Jordan algebras were described in [23]. In the proof of the present theorem we give all necessary arguments for the description of all irreducible components of the variety of 5-dimensional nilpotent commutative $\mathbb{C}D$-algebras.

Theorem 3. The variety of complex 5-dimensional nilpotent commutative $\mathbb{C}D$-algebras is 24-dimensional and it has 10 irreducible components. In particular, there are 6 rigid algebras: non-Jordan algebras $\mathbb{C}D_6^5$, $\mathbb{C}D_7^5$, $\mathbb{C}D_8^5$, $\mathbb{C}D_9^5$ and Jordan algebra $\mathbb{J}^21$.

Proof. Thanks to [23] the algebras $\mathbb{E}_1$, $\mathbb{J}^221$, $\mathbb{J}^222$, $\mathbb{J}^227(\varepsilon, \phi)$ and $\mathbb{J}^240$ determine the irreducible components in the variety of complex 5-dimensional nilpotent Jordan algebras (which is a proper subvariety of nilpotent associative commutative $\mathbb{C}D$-algebras), where

|  $E^j_i \to \mathbb{E}_1$ |  $E^j_i = t^j e_1$ |  $E^j_i = t^j e_2$ |
|-------------------------|-------------------|-------------------|
|  $E^j_i = -t^k e_5$    |  $E^j_i = t^j e_4$ |  $E^j_i = t^j e_3$ |
|  $E^j_i (t^{-2}) \to \mathbb{J}^222$ |  $E^j_i = t^2(1 + t^2)e_1 - \frac{t^2(1 + t^2)^2}{3 + 2t^2}e_2 - t^4(1 + t^2)e_3 - \frac{t^8(1 + t^2)^2}{2(3 + 2t^2)^2}e_4$ |  $E^j_i = t^2(1 + t^2)^2e_4$ |
|  $E^j_i = t^3(1 + t^2)e_3$ |  $E^j_i = t^4(1 + t^2)^2e_1$ |  $E^j_i = t^5(1 + t^2)^3e_5$ |
|  $E^j_i (\frac{\gamma}{\tau}) \to \mathbb{J}^227(\varepsilon, \phi)$ |  $E^j_i = t^7e_1 - \varepsilon t^3 e_3 + t^4(1 + \varepsilon t)e_4$ |  $E^j_i = t^8e_3 - t^2 e_4 + t^6(1 - 2\varepsilon \phi - \varepsilon^2 t)e_5$ |
|  $E^j_i = \tau^2 e_3 + \varepsilon t e_4$ |  $E^j_i = t^4e_4 + \varepsilon^2 t^6 e_5$ |  $E^j_i = t^5e_5$ |

Let us give the list of useful degenerations:
| \( E_{10} \) | \( E_{11} \) | \( E_{12} \) | \( E_{13} \) |
|---|---|---|---|
| \( t \) | \( b \) | \( b \) | \( b \) |

\( E_1^+ = t + b \)  \( E_2^+ = b + e \)  \( E_3^+ = e + d \)  \( E_4^+ = d + c \)

\( E_1^- = t - b \)  \( E_2^- = b - e \)  \( E_3^- = e - d \)  \( E_4^- = d - c \)

\( E_1 \)  \( E_2 \)  \( E_3 \)  \( E_4 \)

\( Q = -88A - 60A^2 + 344B^2 - 204B^2 - 558A^2B^2 - 201A^2B^2 + 1032B^4 + 966AB^4 + 233A^2B^4 + 360B^6 + 232AB^6 - 8B^8 \)

\( Q = -341B - 312AB - 60A^2B - 1032B^3 - 1146AB^3 - 346A^2B^3 - 360B^5 - 228AB^5 + 8B^7 \)
$$E_1^t = \frac{(t-1)^3e^t}{1-2te^{-1} - (t-1)^2e^{-t} - (t-1)^3e^{-2t}}$$

$$E_2^t = \frac{2t(-1)^3e^t}{1-2te^{-1} - (t-1)^2e^{-t} - (t-1)^3e^{-2t}}$$

$$E_3^t = \frac{2t(-1)^3e_{t=0}^t}{1-2te^{-1} - (t-1)^2e^{-t} - (t-1)^3e^{-2t}}$$

$$E_4^t = \frac{(t-1)^3e^{-1}}{1-2te^{-1} - (t-1)^2e^{-t} - (t-1)^3e^{-2t}}$$

$$c_{26}(0,1,0) \rightarrow c_{18}$$

$$E_1 = t^{-1}e_1 - t^{-1}e_2 - t^{-1}e_3$$

$$E_2 = -2t^{-2}e_3$$

$$E_3 = 4t^{-3}e_5$$

$$E_4 = -2t^{-2}e_4$$

$$c_{26}(1 + 4t^{-4}, 4t^{-4}) \rightarrow c_{19}$$

$$E_1 = t^{-1}e_1 - t^{-1}e_2 - t^{-1}e_3$$

$$E_2 = 2t^{-2}e_4$$

$$E_3 = 4t^{-3}e_5$$

$$E_4 = -2t^{-2}e_4$$

$$c_{26}(1 + 8t^{-3}, 0) \rightarrow c_{20}$$

$$E_1 = t^{-1}e_1 - t^{-1}e_2 - t^{-1}e_3$$

$$E_2 = 2t^{-2}e_4$$

$$E_3 = 4t^{-3}e_5$$

$$E_4 = 4t^{-3}e_5$$

$$c_{26}((t^2 - 1)(t^2 - 1), 1, 1, 1) \rightarrow c_{21}$$

$$E_1 = t^{-1}e_1 - t^{-1}e_2 - t^{-1}e_3$$

$$E_2 = -2t^{-2}e_3 + 2t^{-2}e_4 + 2(t-2)t^{-10}e_5$$

$$E_3 = t^{-1}e_2 + t^{-3}e_3$$

$$E_4 = 4t^{-3}e_5$$

$$c_{26}((1 + 0) 1 + 2t, t^2, t^2) \rightarrow c_{22}(A)$$

$$E_1 = -\frac{t^2e_1}{2} + \frac{t^3}{2}e_2 + \frac{t^4}{2}e_3 + \frac{t^5}{2}e_4 + \frac{t^6}{2}e_5$$

$$E_2 = -\frac{t^2e_1}{2} + \frac{t^3}{2}e_2 + \frac{t^4}{2}e_3 + \frac{t^5}{2}e_4 + \frac{t^6}{2}e_5$$

$$E_3 = -\frac{t^2e_1}{2} + \frac{t^3}{2}e_2 + \frac{t^4}{2}e_3 + \frac{t^5}{2}e_4 + \frac{t^6}{2}e_5$$

$$E_4 = -\frac{t^2e_1}{2} + \frac{t^3}{2}e_2 + \frac{t^4}{2}e_3 + \frac{t^5}{2}e_4 + \frac{t^6}{2}e_5$$

$$E_5 = -\frac{t^2e_1}{2} + \frac{t^3}{2}e_2 + \frac{t^4}{2}e_3 + \frac{t^5}{2}e_4 + \frac{t^6}{2}e_5$$
| \( \mathcal{C}(2) \rightarrow \mathcal{C}(4) \) | \( E_1 = \sqrt{t}e_1 + \sqrt{t}e_2 \) | \( E_2 = te_3 - te_4 \) | \( E_3 = t^2e_5 \) | \( E_4 = t^3e_5 \) |
| \( \mathcal{C}(3) \rightarrow \mathcal{C}(4) \) | \( E_1 = it^2e_1 + t^2e_2 \) | \( E_2 = -t^2e_4 + t^{-3}e_5 \) | \( E_3 = -t^{-1}e_3 + \frac{1}{2}te_3 - \frac{1}{2}te_4 \) | \( E_4 = t^3e_5 \) |
| \( \mathcal{C}(4) \rightarrow \mathcal{C}(3) \) | \( E_1 = -(1) \sqrt{t}e_1 + (1) \sqrt{t}e_2 \) | \( E_2 = -(1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(5) \rightarrow \mathcal{C}(4) \) | \( E_1 = (t^2 + 2t^2 + 2t^2 - 2t^2) \rightarrow \mathcal{C}(4) \) | \( E_2 = (t^4 - t^2 - 1)e_1 + te_2 \) | \( E_3 = t^2e_2 \) | \( E_4 = t^3e_2 \) |
| \( \mathcal{C}(6) \rightarrow \mathcal{C}(5) \) | \( E_1 = t_e_1 + te_2 \) | \( E_2 = t_e_3 + te_4 \) | \( E_3 = t^2e_5 \) | \( E_4 = t^3e_5 \) |
| \( \mathcal{C}(7) \rightarrow \mathcal{C}(6) \) | \( E_1 = -(1) \sqrt{t}e_1 + (1) \sqrt{t}e_2 \) | \( E_2 = -(1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(8) \rightarrow \mathcal{C}(7) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(9) \rightarrow \mathcal{C}(8) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(10) \rightarrow \mathcal{C}(9) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(11) \rightarrow \mathcal{C}(10) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(12) \rightarrow \mathcal{C}(11) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(13) \rightarrow \mathcal{C}(12) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(14) \rightarrow \mathcal{C}(13) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(15) \rightarrow \mathcal{C}(14) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(16) \rightarrow \mathcal{C}(15) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(17) \rightarrow \mathcal{C}(16) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(18) \rightarrow \mathcal{C}(17) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(19) \rightarrow \mathcal{C}(18) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(20) \rightarrow \mathcal{C}(19) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(21) \rightarrow \mathcal{C}(20) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(22) \rightarrow \mathcal{C}(21) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(23) \rightarrow \mathcal{C}(22) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
| \( \mathcal{C}(24) \rightarrow \mathcal{C}(23) \) | \( E_1 = -(t^2 - 1) \sqrt{t} + (t^2 - 1) \sqrt{t} e_2 \) | \( E_2 = -(t^2 - 1) t^{-1}e_2 \) | \( E_3 = -t^2e_5 \) | \( E_4 = t^4e_5 \) |
By calculation of dimension of derivation algebra, we have dimensions (gdim) of algebraic varieties defined by the following algebras:

$$\text{gdim } \mathfrak{c}_{49}^5(\alpha) = 24$$
$$\text{gdim } \mathfrak{c}_{20}^5(\alpha, \beta) = 23$$
$$\text{gdim } \mathfrak{j}_{21} = \text{gdim } \mathfrak{c}_{16}^5(\alpha) = \text{gdim } \mathfrak{c}_{69}^5 = \text{gdim } \mathfrak{c}_{72}^5 = \text{gdim } \mathfrak{c}_{80}^5(\alpha) = \text{gdim } \mathfrak{c}_{81}^5 = 22$$
$$\text{gdim } \mathfrak{c}_{70}^5 = \text{gdim } \mathfrak{c}_{77}^5 = 21$$

Thanks to list of non-degeneration arguments presented below:

| Non-degeneration | Arguments |
|------------------|-----------|
| $\mathfrak{c}_{20}^5(\alpha, \beta) \not\to \mathfrak{c}_{16}^5(A), \mathfrak{c}_{69}^5, \mathfrak{c}_{72}^5, \mathfrak{c}_{76}^5, \mathfrak{c}_{77}^5, \mathfrak{c}_{80}^5(\alpha), \mathfrak{c}_{81}^5, \mathfrak{j}_{21}$ | $\mathcal{R} = \{ A_1A_4 = 0 \}$ |
| $\mathfrak{c}_{49}^5(\alpha) \not\to \mathfrak{c}_{16}^5(A), \mathfrak{c}_{26}^5(A, B), \mathfrak{c}_{69}^5, \mathfrak{c}_{72}^5, \mathfrak{c}_{76}^5,$ $\mathfrak{c}_{77}^5, \mathfrak{c}_{80}^5(\alpha), \mathfrak{c}_{81}^5, \mathfrak{j}_{21}$ | $\mathcal{R} = \{ A_1^2 \subseteq A_3, A_1A_2 \subseteq A_4, A_1A_3 \subseteq A_5, A_1A_5 \subseteq 0 \}$ |
we have that algebras 
\[ \Omega = \{ J_{21}, C_{16}^5(\alpha), C_{26}^5(\alpha, \beta), C_{49}^5, C_{69}^5, C_{72}, C_{76}, C_{77}, C_{80}^5(\alpha), C_{81}^5 \} \]
give irreducible components. \[\square\]

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