Steady States of Epidemic Spreading in Small-World Networks

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Abstract

We consider a standard susceptible-infected-susceptible (SIS) model to study behaviors of steady states of epidemic spreading in small-world networks. Using analytical methods and large scale simulations, we recover the usual epidemic behavior with a critical threshold $\lambda_c$ below which infectious diseases die out. Whereas for the spreading rate $\lambda$ far above $\lambda_c$, it was found that the density of infected individuals $\rho$ as a function of $\lambda$ has the property $\rho \approx F(K)(\ln \lambda - \ln \lambda_c)$.

Keywords: networks and genealogical trees, diseases, phase transitions.

Complex networks have attracted an increasing interest recently. The main reason is that they play an important role in the understanding of complex behaviors in real world networks, including the structure of language, scientific collaboration networks, the Internet and World Wide Web, power grids, food webs, chemical reaction networks, metabolic and protein networks, etc. In the study of complex networks, an analysis of the structures can give important information about the underlying processes responsible for the observed macroscopic behavior, such as small-world and scale-free networks. In particular, social networks have two characters. First, they show “clustering”, meaning that two of your friends are far more likely also to be friends of each other than two people chosen from the population at random. Second, they exhibit the “small-world effect”, namely, that any two people can establish contact by going through only a short chain of intermediate acquaintances. These two properties appear contradictory because the first is a typical property of low-dimensional lattices but not of random graphs or other high-dimensional lattices, while the second is typically of random graphs, but not of low-dimensional lattices. D.J. Watts and
S.H. Strogatz suggested a new small-world model recently which interpolates between low-dimensional lattices and random graphs and displays the both properties. In the model, a small-world network is constructed as follows: starting with a ring of \( N \) vertices, each connected to its \( 2K \) nearest neighbors by undirected edges, and then each local link is visited once with the rewiring probability \( p \) it is removed and reconnected to a randomly chosen node. Duplicate edges are forbidden. After the whole sweep of the entire network, a small-world network is constructed with an average connectivity \( \langle k \rangle = 2K \). The WS networks has been widely studied because it constitutes an interesting attempt to translate complex topologies of social, economic, and physical networks into a simple model.

Although topological properties of complex networks have been studied in detail, a natural question arises, that is the dynamical properties which result from different networks. A good example is to inspect complex features of epidemic spreading since the characterization and understanding of epidemic dynamics in these networks could probably provide us immediate applications to a large number of problems, such as computer virus infections, distribution of wealth, transmission of public opinion, etc. Recent papers have given some valuable insights of that: for small-world networks, there is a critical threshold below which an infection with a spreading rate dies out; on the contrary, for scale-free networks, even an infection with a low spreading rate will prevalence the entire population.

In the present paper we consider a standard SIS model in small-world networks, in which each node represents an individual of the population and edges represent physical interactions through which an infection spreads. According to the SIS model, an individual is described by a single dynamical variable adopting one of the two stages: susceptible and infected. A susceptible individual at time \( t-1 \) will pass to the infected state with the rate \( \nu \) at time \( t \) if it is connected to one or more infected individuals. Infected individuals at time \( t-1 \) will pass to the susceptible state with the rate \( \delta \) at time \( t \), defining an effective spreading rate \( \lambda = \nu/\delta \). We can still keep generality by setting \( \delta = 1 \). Individuals run stochastically through the cycle susceptible \( \rightarrow \) infected \( \rightarrow \) susceptible, hence the model got its name. In the SIS model, an important observable is the prevalence \( \rho \), which is the time average of the fraction of infected individuals reached after a transient from the initial condition. Given a network, the only parameter of the model is the spreading rate \( \lambda \). The information on the global spreading of the infection is contained in the function \( \rho(\lambda) \).

Small-world networks have very small diameters which mean the presence of disordered long range interactions. In this case, the networks are very homogeneous and quite reasonable that the mean-field (MF) method is valid. By neglecting the density correlations among the different nodes and ignoring all higher order corrections in \( \rho(t) \), the time evolution equation of the SIS model can be written as
\[ \dot{\rho}(t) = -\rho(t) + \lambda \langle k \rangle \rho(t)(1 - \rho(t)). \] (1)

In the equation, the first term on the right-hand side (rhs) considers infected nodes become healthy with unit rate and the second term on the rhs represents the average density of newly infected nodes generated by susceptible nodes. By imposing the stationary condition \( \partial_t \rho(t) = 0 \), one can obtain the equation

\[ \rho[-1 + \lambda \langle k \rangle (1 - \rho)] = 0 \] (2)

for the steady state density of infected nodes \( \rho \). The equation defines an epidemic threshold

\[ \lambda_c = \langle k \rangle^{-1}. \] (3)

In other words, if the value of \( \lambda \) is above the threshold, \( \lambda > \lambda_c \), the infection spreads and becomes endemic with a finite stationary density \( \rho \). Below it, \( \lambda \leq \lambda_c \), the infection dies out. In Euclidean lattices, J. Marro and R. Dickman have concluded the order parameter behavior of critical phenomena is \( \rho \sim (\lambda - \lambda_c)^\beta \) with \( \beta \leq 1 \) in the region \( \lambda \sim \lambda_c \), which uncovers the linear property in the critical dimension. R. Pastor-Satorras and A. Vespignani also recovered this property in small-world networks with the rewiring probability \( p = 1.0 \) recently; it is worth noticing that in the extreme case the generated network is an entirely random network with a restriction which leads to a large cluster.

In order to compare with the analytical prediction we have performed large scale simulations of the SIS model with parallel updating in the WS network with the rewiring probability \( p = 0.1 \). The both properties of social networks are well presented by the network in this case. The size of the network is \( N = 10^6 \). The number of the initially infected nodes is 10 percent of the size of the network. Simulations were implemented on the network averaging over 20 different realizations. After an initial transient regime, the systems stabilize in a steady state with a constant average density of infected nodes. In Fig. 1 we plot the steady density of infected nodes \( \rho \) with various types of the scale of axis for \( \lambda \) is very closed to \( \lambda_c \). The linear property of the order parameter of Euclidean lattices, \( \rho \sim (\lambda - \lambda_c)^\beta \), is well presented by the log-log plot of Fig. 1(d) which gives the parameter value \( \beta = 0.98 \pm 0.04 \). However, it is obvious that the four plots almost show the same shapes. Consequently the following acceptable predictions for Fig. 1(a-d) are respectively given by

\[ \rho \approx a_1 \lambda + b_1 \] (4a)
\[ \rho \approx a_2 \ln \lambda + b_2 \] (4b)
\[ \rho \approx a_3 (\lambda - \lambda_c) + b_3 \] (4c)
\[ \ln \rho \approx a_4 \ln (\lambda - \lambda_c) + b_4 \] (4d)
Figure 1: Density of infected nodes $\rho$ in the WS network with $K = 3$ from the simulations near by $\lambda_c = 0.1708 \pm 0.004$, which is in good agreement with the MF predictions $\lambda_c = 1/2K = 0.1667$. The numerical results were presented with $\rho$ vs $\lambda$ (a), $\rho$ vs $\log \lambda$ (b), $\rho$ vs $\lambda - \lambda_c$ (c), and $\log \rho$ vs $\log(\lambda - \lambda_c)$ (d). It indicates that all the plots perform the similar behaviors.

Considering the Taylor expansion $\ln \lambda \approx \frac{\lambda}{\lambda_c} + \ln \lambda_c - 1$ at one order near $\lambda_c$, Eq. (4b) can be rewritten as

$$\rho \approx a'_2 \lambda + b'_2. \quad (5)$$

One can also derive the equation

$$\rho \approx a'_4 (\lambda - \lambda_c) + b'_4 \quad (6)$$

from the Eq. (4d) in the same way. So, based on the numerical simulations closed to $\lambda_c$, a analogous linear relationship between the infected density and the spreading rate, $\rho \approx a \lambda + b$, was extracted from the simple approximation of series expansions (see Eq. (4a), Eq. (5), Eq. (4c), and Eq. (6)) corresponding to Fig. (a)-(d), respectively). Therefore, we will calculate the tendency of $\rho$ for $\lambda > \lambda_c$ next.

As explicitly stated in Ref. [27], Eq. (11) is derived in the case of small values of $\rho$. So it is meaningful only in the region $\lambda \sim \lambda_c$ and could not determine the behavior for $\lambda$ far above $\lambda_c$, i.e., the MF prediction of Euclidean lattices is invalid in this region. In this condition, numerical methods are naturally adopted to find the behavior of the prevalence for $\lambda > \lambda_c$. In Fig. 2 we perform the numerical results as far as $\lambda = 0.4$. The linear property of Eq. (4d) was presented excellent agreement with the numerical results. Note that the constant parameters $a_4$ and $b_4$ in Eq. (4d) should be taken count of the initial setting value of the average connectivity $2K$. 

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Figure 2: Extensive numerical results of the density of infected nodes $\rho$ in the WS network for $\lambda > \lambda_c$. All plots perform the identical behavior which is described by Eq. (7). Parameter values (from right to left) $K = 3, 4, 5$.

More exact conclusion for the nature of Fig. 2 which presented the whole behavior at $\lambda > \lambda_c$, should be given by

$$\rho \approx F(K)(\ln \lambda - \ln \lambda_c), \quad \text{if} \quad \lambda > \lambda_c. \quad (7)$$

To complete our study of the steady states of the SIS model in small-world networks, we compute the coefficient $F(K)$ in Eq. (7). In Fig. 3 we plot the coefficient $F(K)$ as a function of $K$. The linear behavior of the log-log plot predicts the power law, $F = K^{-\alpha}$. Numerical results give the parameter value $\alpha \approx 0.98$.

In summary, we have analytically and numerically studied the steady states of epidemic spreading in small-world networks. In the region $\lambda \sim \lambda_c$, we recover the MF results of Euclidean lattices, $\rho \sim (\lambda - \lambda_c)^\beta$. But when spreading rates become far above the threshold, $\lambda > \lambda_c$, the MF method can not work normally and the numerical method is adopted. It was found that the behavior of order parameter has the property $\rho \approx F(K)(\ln \lambda - \ln \lambda_c)$. In the present work, we just consider the dynamics on the WS network with the rewiring probability $p = 0.1$. However, for other values of $p$, the results are qualitatively and quantitatively the same as that we get.

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Figure 3: Log-log plot of the coefficient $F$ as a function of $K$. The linear property of the plot predicts the power law, $F = K^{-\alpha}$. Simulations give the parameter value $\alpha \approx 0.98$.

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