Meaning of the derivative as a rate of change through a graphic argumentation. A case with Chilean students

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Abstract. From the socio-epistemology perspective, the teaching of the derivative is problematized due to the lack of reference frameworks that give variational meanings to the derivative and allow its construction by means of graphical arguments. The objective of this research was to analyze the construction of meanings of the derivative, as a rate of change, that Chilean students developed through a graphic argumentation in a specific situation of variation. In order to respond to the stated objective, a qualitative methodology was used, consisting of a case study, where a situation of variation was designed with the purpose of promoting the analysis of temperature changes in specific everyday life phenomena. This design was applied to ten Chilean high school students interested in studying engineering. As a result of the research, it was obtained that the participants, through a graphic argumentation, understood the slope of a straight line as a rate of change of temperature regarding time. It was concluded that this research provides elements for the construction of frames of reference that signify the derivative, as a rate of change, in situations of variation, valuing the graphic arguments and functional justifications that emerged from the students.

1. Introduction

There are diverse applications of differential calculus in engineering, through the study of functions, related variables and the rate of change, future engineers are provided with mathematical tools for the interpretation of data and different types of change processes [1]. In Vrancken and Engler [2] it is pointed out that the study of differential calculus is important when studying situations where it is necessary to find laws that allow predicting the speed of variation in natural phenomena.

Regarding the above, in Morales, et al. [3] it is indicated that the derivative can be constructed from functional justifications that emerge from students in situations of variation, where functional justification is understood as that justification regulated by the usefulness of knowledge to human beings [4]. To mention some research that shows this type of construction, in Riveros [5] the processes of variational mathematical thinking associated with the derivative were analyzed in virtual environments.

In Zambrano, et al. [6], the research was carried out that introduced the concept of derivative to high school students in Mexico, through a phenomenon of variation and change. The research conducted in Villa-Ochoa, et al. [7] offers an approach to the concept of derivative through the understanding of the instantaneous rate of variation by means of the use of digital technologies. In Báez, et al. [8] evidence
was shown that a proposal focused on variation and change tasks supports the development of variational thinking in engineering contexts.

Socio-epistemology has emphasized the development of variational thinking and language for the construction of frames of reference that allow different meanings to be given to the derivative in specific situations of variation. However, for Morales, et al. [3] this type of situations is not usually present in the teaching of the derivative, since in the educational system, in general, formal construction processes and algorithmic aspects have been prioritized [6]. Moreover, in Martinez, et al. [9] it is pointed out that in many of the texts used for the teaching of the derivative, this type of situations is usually little considered.

This situation implies, among other factors, that it is difficult for students to recognize the derivative in the resolution of elementary problems about change and variation, despite the fact that it is in this type of problems where the essence of the derivative is found [10]. In general, these texts present the derivative as an abstract concept, where priority is given to algorithmic work, such as increments, limits, quotients, and general rules of derivation, among others, so that the development of variational ideas and meanings of the derivative is overlooked, even though variation has been a central aspect in the historical emergence of different concepts of calculus [11].

In other words, for the construction of the derivative, texts traditionally prioritize the use of reasoned justifications, which prioritize functional justifications. This means that such construction is conceived from the same mathematical structure [4]. In this way, the teaching of the derivative is traditionally based on the student mastering the processes to obtain calculations or algebraic expressions by means of formulas, without achieving its understanding in variation phenomena [12].

Moreover, the uses of graphs do not usually play an important role in the teaching of the derivative, despite the existence of socio-epistemological research that shows that the derivative can be constructed in situations of variation based on a graphical argumentation (for example, see Buendía and Ordóñez [13]). It should be noted that, in this case, the graph is not only understood as the representation of a function in the Cartesian plane, but also as any type of figure that allows the development of arguments to facilitate the construction of mathematical knowledge [14].

The accounts for a problem present in the teaching processes of the derivative, that is the lack of reference frameworks that allow giving meaning to the derivative from a graphical argumentation (considering Rosas and Morales’s sense [14]), in specific situations of variation. With the purpose of contributing to the creation of this type of frames of reference, the objective of this research is to analyze the construction of meanings of the derivative, as a rate of change, that Chilean students develop through a graphic argumentation in a specific situation of variation.

2. Methodology

In order to respond to the stated research objective, a qualitative methodology was used, where the instrumental case study [15] was used to study the particularity and complexity of the case that was part of this research. This made it possible to describe in detail what was developed by the students, as well as their ways of interacting and, mainly, the graphic arguments that emerged from them.

2.1. Case study

The learning situation was applied to 10 students, 3 females and 7 males, who will be indicated as S1, S2, S3, etc. Their ages ranged between 17 years old and 18 years old and they had a profile marked by a high degree of acceptance and motivation for mathematics and interest in pursuing their studies in the area of engineering. The students were divided into three groups, two composed of 3 members and other composed of 4 members, to encourage interaction among them. The time for the implementation of the activity was 3 hours, distributed in two meetings of 1 hour and 30 minutes each.

Initially, it was explained to the students that they could express all their ideas, doubts, and comments, since the purpose of the activity was to know the reasoning they used, bearing in mind that there was no erroneous reasoning. Each group then set about understanding the activity, with absolute freedom to follow the path they considered appropriate, making the necessary decisions to proceed.
During the development of the learning situation, special care was taken not to give, by any means, a
solution by the teacher, who adopted a reflective attitude on the validity of the answers that emerged
from the students.

2.2. Design
The purpose of the learning situation was for students to construct meanings of the derivative, as a rate
of change, through a graphic argumentation, in a situation where the temperature varies with respect to
time. Specifically, the learning situation searched for the students were able to understand the slope of
a straight line as a rate of change of temperature. The learning situation was separated into two moments.
The first one sought the students analyze and graphically represent situations of variation that are part
of their daily lives.

In addition, this moment sought to bring out the necessity, by students, to calculate the slope of a
straight line to express the rate of change in a small-time interval. The second moment sought to
formalize with the students the idea of slope of a straight line as a rate of change, in a situation of
temperature variation of a beverage can that cooled with the passage of time. For this, students were
asked to work with wooden sticks to construct a graph representing the decrease in temperature of the
can. The work with wooden sticks was intended to enable students to understand the role of the slope of
a straight line, as a rate of change, in a specific time interval. The designed learning situation is presented
below.

In first moment, the activity 1 corresponds to analyze change over time: (a) could you identify
elements of your environment that undergo some modification overtime? (b) how do you notice these
changes? (c) what are the characteristics before and after the change? (d) how could you represent this
change? The activity 2 corresponds to determine change processes: Ernesto is studying how the
temperature changes throughout the days in a month in the city of Santiago, which of these graphs best
represents this situation? (Figure 1 and Figure 2). Then, in the activity 3 corresponds to determine the
variation in an amount of time; if you leave a cup of coffee cooling to be able to drink it, since it a very
hot: (a) how could you describe the change? (b) how could you describe the change in small intervals
of time? (c) how could you describe the change in a certain time?

![Figure 1. Temperature change; (a) linear; (b) sinusoidal.](image1)

![Figure 2. Temperature changes that have a wider range of (a) maximum, and (b) minimum temperatures.](image2)
In second moment, the activity 1 consist of recognize the notion of change as the slope of the tangent line to the curve; suppose you have a canned drink, you are very thirsty, but this is lukewarm. You decide to place the drink in the refrigerator to chill it: (a) how long do you have to be able to drink the can of cold drink? (b) what would the curve that represents the cooling of the beverage can through time in the cartesian plane look like? (c) make this curve again only using sticks as precise as possible. What is the most convenient position when placing the sticks? (d) what does the stick represent in this situation?

3. Results and discussions
This section will show the productions made by the students during the development of the learning situation, showing how the students, through a graphic argumentation, used the slope of a straight line to represent a rate of change in a specific time interval; due to the similarity of the answers given by all the groups, we will present some productions that synthesize and represent in a better way what was done by all the groups. Students will be described with the nomenclature S1, S2, S3, and so on.

3.1. First moment
The activity 1, students discussed different types of various situations that are part of their daily life, where it was possible to find answers such as movement, sound, temperature, pressure, and climate, among many others; when asked how to describe the changes that occur in such situations, the students pointed out that it was possible to do so by comparing states, recognizing a before and an after. When representing this change, students used, among others, comparative tables, and cartesian graphs. In particular, S9, when working with the graphs, pointed out that the X-axis should correspond to time and the Y-axis to the variable quantity.

The activity 2, students looked for a pattern of behavior consistent with the proposed situation, where they indicated that for a graph to better represent this situation, it should have maximums and minimums. Regarding this, S1 indicated that the maximums and minimums show the variation in temperature and S3 discards alternative B, since he points out that temperature is not necessarily constant over time, and that not every day will have the same maximum and minimum. On the other hand, S9 points out that the best graph corresponds to alternative D, because in this graph it is possible to see minimum and maximum temperatures on each day, and the variations between these represent the variation.

In activity 3, students made use of graphics arguments to answer the questions. In relation to this, S1 used a graph, with a linear trend, to describe the cooling of the coffee cup, making use of a horizontal line to represent thermal equilibrium (see Figure 3). Regarding the description of the change in small time intervals, the students again used a graphic argumentation. S1 pointed out that to make such a description, an instrument could be used to measure the temperature a certain number of seconds and S8 constructed a graph where he graduated and valued both axes, identifying the respective variables (see Figure 4).

To describe the change in a specific time, students analyzed the change in a neighborhood around a specific time. Regarding this, S10 pointed out that, by choosing a temperature at a specific time and working with a temperature higher (T1) and lower than this (T2) and as close as possible to the specific
temperature chosen, finding the slope of the straight line passing through T1 and T2, allows to approximate the temperature change in that specific time. In this case, S10, by means of a graphic argumentation, pointed out that the dotted line represents an approximation of the temperature change in a specific time (see Figure 5).

3.2. Second moment
In this moment, the students inferred about the waiting time for the beverage can cool down, emphasizing that this will depend on the different variables that influence it, such as the temperature at which the beverage is at, the rate at which the refrigerator cools and the power of the refrigerator. Regarding the curve representing the cooling of the beverage can, S1 constructed a decreasing curve, graduated the axes, and identified the variables (see Figure 6).

Regarding the construction of the curve with wooden sticks, the students constructed decreasing curves (see Figure 7) where they used arguments that emerged from these graphs to point out that each of the wooden sticks corresponds to a temperature variation in a small-time interval, and that the slope of each line corresponds to an approximation of the change. S4 responds that it represents the temperature variation over time, S5 points out that each wooden stick represents a decrease in temperature over time, S9 that it corresponds to the slope and S10 points out that each wooden stick represents an approximation of the change.

4. Conclusions
Different socio-epistemological investigations have provided evidence that the development of strategies of variational thinking and language generates the basis of meaning for different concepts of calculus, among them the derivative. In the case of this research, the design of the learning situation allowed students to signify the derivative, as a rate of change, through a graphic argumentation in a situation of variation. Specifically, students were able to understand the slope of a straight line as a rate of change of temperature thanks to the use of graphs to argue their procedures, including the use of wooden sticks to better understand why the slope of a straight line represents a rate of change in a specific time interval.

The development of the learning situation proposed in this research allowed giving the graph a different status to that traditionally established in the teaching of the derivative, which considers the graph as the representation of a function, becoming a fundamental tool for students to be able, from
functional justifications, to understand the derivative as a rate of change. In this way, this research provides elements for the construction of reference frameworks that allow the derivative, as a rate of change, to be understood in specific situations of variation, valuing the graphic arguments and functional justifications that emerged from the students. Finally, it remains as a task for the educational community to continue with the development of research of this type, in order to contribute to the creation of frames of reference that give meaning to the derivative through a graphic argumentation.

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