Event-triggered distributed filtering for discrete-time systems with integral measurements

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ABSTRACT
This paper handles the event-triggered distributed filtering problem for a class of discrete-time systems with integral measurements over sensor networks (SNs). First of all, a plant is monitored by an SN formulated by a digraph. The integral measurement phenomenon is modelled for each node to account for the interval time taken by sample collection. Next, to mitigate the communication burden of network, an event generator function is introduced for each node to regulate the data transmission to its neighbouring nodes. This paper aims to find the suitable filter gains for each node such that the filtering error dynamics satisfies the exponentially ultimately boundedness. Via the Lyapunov stability theory, a sufficient condition is established which ensures the existence of the distributed filter under the desirable performance index. Moreover, the ultimate bound in the performance index is minimized via the robust optimization method. Then, the filter gains are acquired by solving a linear matrix inequality with the YALMIP toolbox. Finally, a numerical simulation example is employed to verify the usefulness of the distributed filtering algorithm developed in this paper.

1. Introduction
In the past decades, the wide applications of sensor networks (SNs) have aroused people’s great interest, like industrial process (Flammini et al., 2009), environmental monitoring (Oliveira & C. Rodrigues, 2011), medical system (Egbogah & Fapojuwo, 2011; Hao & Foster, 2008) and military domain (Winkler et al., 2012). Generally speaking, the aim of SNs is to monitor the critical information from the interested areas through the cooperations among a host of inexpensive wireless sensor nodes. Nevertheless, this aim is full of challenges since the cooperations are achieved by the data transmission among nodes depending on a common wireless channel, which could result in data collision. In particular, these sensor nodes perform local communications due to their limited transmission abilities. Moreover, the capacity of the non-replaceable batteries is limited, which causes the serious energy constraint. It is no doubt that these non-negligible factors hamper the potential applications of SNs. Furthermore, the data collected by the sensor nodes are noisy and as a result it is an important mission to filter the noises. For this purpose, in the context of SNs, distributed filtering (DF) has been developed in the basis of the traditional filtering methods (Hu et al., 2018; Wang & Niu, 2011). Unfortunately, the hardware constraints and the unreliable wireless communication of SNs pose many great challenges to the investigation of DF (Bu et al., 2019, 2018; Ding et al., 2019; Dong et al., 2017, 2015, 2013; Hu et al., 2018).

Traditionally, the centralized filtering approach implies that all sensor nodes send their data to a data centre for processing and storage. Such information transmission put huge demands on the computational ability of the data centre and the communication ability of the shared network. For addressing this problem, the DF approach has been come up with where every sensor node sends its data to its neighbouring nodes. Owing to this structure, such the distributed data processing has advantages in fault tolerance and the robustness. As such, DF has been paid great focus in the last few years thanks to these significant advantages. At present, there is an increasing number of researchers that recognize the importance of the DF problem over SNs (Bu et al., 2019, 2018; He...
et al., 2020; Liu et al., 2019; Wang et al., 2020; Yang et al., 2019). Some typical research directions include that the distributed $H_{\infty}$ filtering (Dong et al., 2013; Han, Wang, Chen et al., 2021; Han, Wang, Dong et al., 2021; Qu et al., 2019; Shen et al., 2010, 2011), and the distributed Kalman filtering (Ji et al., 2017; Li, Dong et al., 2019; Li, Wang et al., 2019; Wang et al., 2019; Wu et al., 2018; Yang et al., 2020), and distributed set-membership filtering (Liu et al., 2019; Ma et al., 2017). As for DF problems, each node has its own filter through the available information from both itself and its neighbouring sensor nodes. Meanwhile, this is also a typical feature of DF: couplings among nodes. Consequently, the major challenge when designing DF is how to dispose of these sophisticated couplings.

Generally speaking, in SNs, non-replaceable batteries provide generally power to sensor nodes. Clearly, sensor node is useless if the battery is drained. Thus it is necessary to reduce the amount of data transmission to extend the working life of sensor node in the premise of a certain performance behaviour. To deal with this problem, the data transmission rate among neighbouring nodes is governed by the event-triggered (ET) protocol (Hu et al., 2018). More specifically, only when a predetermined ET condition meets, sensor node can send its information. As such, the energy consumption would be significantly reduced. Thanks to its important role, the ET protocol has gained extensive focus from numerous researchers and numerous literature has been gainable, see e.g.Ding et al. (2017); Dong et al. (2017, 2015); Gao et al. (2019); Hou et al. (2018); Hu et al. (2020); Tian et al. (2019); Wang et al. (2018); Yan et al. (2018, 2019); Zhu et al. (2020); Zou et al. (2020), and the references therein. The estimates problem for complex networks and neural networks, respectively, based on the ET approach have been studied in Hou et al. (2018) and Yan et al. (2018). The distributed estimation problem under the redundant channels has been investigated in Dong et al. (2017), where the ET scheduling protocol is used to schedule the data transmission among nodes.

For the existing references, the measurement outputs are generally considered to rely on the current system state. Nevertheless, the hypothesis is not always true in actual engineering. As a matter of fact, due to the real-time signal processing and the delayed data collection, over a given time period, the system measurement outputs might be proportional to the integral of the system states. In other words, the measurement outputs might be relevant not only to the current system state but also to the system states in a given past period. The above phenomena are called the integral measurements, which often emerges in engineering applications like nuclear reaction processes (Casoli et al., 2014; Markov & Mikhailov, 2014; Tripathi et al., 2011) and chemical reaction processes (Guo & Huang, 2015). In light of its universality, the state estimate problem regarding the integral measurements has received initial attention recently (Guo & Huang, 2015; Liu et al., 2018; Shen et al., 2020). The $H_{\infty}$ state estimation problem has been investigated in Shen et al. (2020) for multi-rate artificial neural networks with integral measurements. The fault reconstruction and state estimation problems have been investigated in Liu et al. (2018) taking into account integral measurements under partial decoupled disturbances. Unfortunately, the DF problem with integral measurements has not been published so far because of its complexity, which motivates us to fill the gap.

In this paper, taking into account the above arguments, we will investigate the DF problem with integral measurements and ET protocols over SNs. Two important problems emerge naturally: how to formulate both the integral measurements and ET protocols and how to measure their aggregate influence? Via making use of the Lyapunov stability theory and the stochastic analysis techniques, the sufficient conditions are established to make sure the desired exponential ultimately bounded performance index. The main novelties of this paper are highlighted as follows: 1) both the integral measurements and ET protocols are modelled in a unified framework; 2) for a class of discrete-time systems, the DF problem with the integral measurements is investigated for the first time; and 3) the robust optimization method is employed to minimize the desirable upper bound in the performance index.

The structure of this paper is outlined as follows. The considered system, integral measurements, ET protocol and the performance index are all formulated, and some preparatory results are briefly introduced in Section 2. In Section 3, sufficient conditions are constructed via the Lyapunov stability theory such that filtering error dynamics is exponentially ultimately bounded. Moreover, the filter gains are obtained by solving the linear matrix inequalities. In addition, a numerical example is shown in Section 4 to illustrate the usefulness of the DF algorithm. Ultimately, Section 5 draws some conclusion.

Notations: The Euclidean vector norm of $x \in \mathbb{R}^n$ is $\|x\| = \sqrt{x^T x}$. $X^T$ stands for the transpose of the matrix $X$. For two real symmetric matrices with same dimensions $M$ and $N$, $M > N$ signifies that $M-N$ is positive-definite. diag($A_1, \ldots, A_n$) denotes a block-diagonal matrix, where $A_i$ ($i = 1, 2, \ldots, n$) are block matrices. ’’$\otimes$’’ indicates an ellipsoid for terms induced by symmetry in symmetric block matrices. The Kronecker product is denoted as the symbol $\otimes$. $I_n$ or simply $I$ denotes the $n$-dimensional identity matrix. $1_n = [1 1 \cdots 1]^T \in \mathbb{R}^n$. 
2. Problem formulation

Consider an SN whose topology is expressed as a digraph $\mathcal{G} = (\mathcal{J}, \mathcal{X}, \mathcal{H})$, where $\mathcal{J} = \{1, 2, \ldots, n\}$, $\mathcal{X} \subseteq \mathcal{J} \times \mathcal{J}$, $\mathcal{H} = \{h_{ij}\}$, stand for, respectively, the set of nodes, the set of edges and the adjacency matrix. $h_{ij}$ is non-negative adjacency element. The ordered pair $(i, j) \in \mathcal{X}$ stands for an edge of $\mathcal{G}$. The adjacency element connected with the edges of the graph are positive, i.e. $h_{ij} > 0 \iff (i, j) \in \mathcal{X}$. Provided that $(i, j) \in \mathcal{X}$, node $j$ is called an in-neighbour of node $i$. The set of in-neighbours of node $i$, without itself, can be designated as $\mathcal{N}_i = \{j|(i, j) \in \mathcal{X}, j \neq i, i, j \in \mathcal{J}\}$. Denote the in-degree $d_i = \sum h_{ij}$, the degree matrix $\mathcal{D} = \text{diag}(d_1, \ldots, d_n)$ and the Laplacian matrix $\mathcal{L} = \mathcal{D} - \mathcal{H}$.

Consider a class of linear discrete-time systems:

$$
\begin{align*}
\begin{cases}
x_{k+1} &= Ax_k + B\omega_k, \\
z_k &= Mx_k
\end{cases}
\end{align*}
$$

(1)

where $x_k \in \mathbb{R}^{n_x}$ is the system state, $z_k \in \mathbb{R}^{n_z}$ is the output to be estimated and $\omega_k \in \mathbb{R}^{n_\omega}$ is the process noise satisfying the norm-bounded constraint $\|\omega_k\| < \bar{\omega}$, and $\bar{\omega} \in \mathbb{R}$ is a known positive scalar. $A$, $B$, and $M$ are three given constant matrices with appropriate dimensions.

For node $i$ ($1 \leq i \leq n$), the measurement output is modelled as

$$
y_{ik} = C_i \sum_{q=0}^{s} x_{k-q} + v_{ik}
$$

(2)

where $y_{ik} \in \mathbb{R}^{n_y}$ is the measurement output and $v_{ik} \in \mathbb{R}^{n_v}$ is the measurement noise satisfying norm-bounded constraint $\|v_{ik}\| < \bar{v}$, and $\bar{v} \in \mathbb{R}$ is a known positive scalar. $s$ expresses the time interval to collect the data. $C_i$ ($i = 1, 2, \ldots, n$) is a given matrix with appropriate dimension.

**Assumption 2.1:** The system (1) with (2) is distributed observable (Li et al., 2018).

**Remark 2.1:** The integral measurement phenomenon often occurs in engineering applications such as chemical processes and nuclear reaction processes (Casoli et al., 2014; Guo & Huang, 2015; Markov & Mikhailov, 2014; Tripathi et al., 2011). The typical feature of these kinds of systems is that the state difference is very small between the adjacent sampling instants. As such, it is meaningless to obtain the measurement only the current time step in a traditional manner. To acquire the effective information of the system, the system measurements should be in proportion to the sum of the system states over a given time period for a discrete-time system, which is described as (2). As for a continuous-time system, it is generally expressed as an integral term over a given time period, which is the reason why it is called the integral measurement. Although its formulation is similar to the distributed delays, they have an entirely different background. The distributed delays are mostly caused by the harsh communication environment and the hardware-execution delay.

Define $\eta_k \triangleq [x^T_k, x^T_{k-1}, \ldots, x^T_{k-s}]^T$ and $\hat{z}_k \triangleq [z^T_k, z^T_{k-1}, \ldots, z^T_{k-s}]^T$. By means of the augment procedure, the system (1) with (2) is changed into a delay-free system

$$
\begin{align*}
\begin{cases}
\eta_{k+1} &= \bar{A}\eta_k + \bar{B}\omega_k, \\
y_{ik} &= \bar{C}_i\eta_k + v_{ik}, \\
\hat{z}_k &= \bar{M}\hat{\eta}_k
\end{cases}
\end{align*}
$$

(3)

where

- $\bar{A} \triangleq \begin{bmatrix} A & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}_{(s+1) \times (s+1)}$

- $\bar{B} \triangleq \begin{bmatrix} B^T & 0 & 0 & \cdots & 0 \end{bmatrix}^T$

- $\bar{C}_i \triangleq \begin{bmatrix} C_i \ C_i \ \cdots \ C_i \end{bmatrix}_{s+1 \times 1}$

- $\bar{M} \triangleq I_{s+1} \otimes M$.

As is known to all, how to make full use of the available information from sensor itself and its neighbouring nodes is a critical point in designing distributed filters for SNs. Taking that into account, the following distributed filter is constructed in this paper on sensor node $i$ ($i \in \mathcal{J}$):

$$
\begin{align*}
\begin{cases}
\hat{\eta}_{i,k+1} &= \bar{A}\hat{\eta}_{i,k} + K_i(y_{i,k} - \bar{C}_i\hat{\eta}_{i,k}) + G_i\sum_{j \in \mathcal{N}_i}h_j(\hat{\eta}_{j,k} - \hat{\eta}_{i,k}), \\
\hat{z}_{i,k} &= \bar{M}\hat{\eta}_{i,k}
\end{cases}
\end{align*}
$$

(4)

where $\hat{\eta}_{i,k}$ is the estimate of $\eta_k$ from node $i$, $\hat{z}_{i,k}$ is the estimate of output $\hat{z}_k$, $K_i$ and $G_i$ are the filter gain matrices to be confirmed later. Moreover, the initial state $\hat{\eta}_{i,0}$ is assumed to be a given vector.

Due to saving the limited energy and prolonging the life of sensor node, the ET protocol is adopted to lessen the frequency of information transmission of node $j$. 

To this end, define an event generator function $f_j(\cdot, \cdot) : \mathbb{R}^{(s+1)n} \times \mathbb{R} \mapsto \mathbb{R} \setminus \{0\}$ as follows:

$$f_j(\hat{\eta}_{j,k}, \delta_j) = \|\hat{\eta}_{j,k} - \hat{\eta}_{j,k}\|_2^2 - \delta_j$$

(5)

where $\hat{\eta}_{j,k}$ is the broadcast estimate at the latest triggering time instant, $\delta_j$ is a positive fixed threshold which confirms the triggering frequency and $s_j$ is used to represent the $l$th transmission time of sampled data of node $j$. Provided that the condition $f_j(\hat{\eta}_{j,k}, \delta_j) \geq 0$ is satisfied, the executions are triggered. As a result, the sequence of execution triggering instants $0 \leq s_j^0 \leq s_j^1 \leq \cdots \leq s_j < \cdots$ is decided iteratively by

$$s_j^d = \min\{k \in \mathbb{N} | k > s_j, f_j(\hat{\eta}_{j,k}, \delta_j) \geq 0\}.$$  

(6)

Based on the available information of node $i$ together with the above ET protocol, the distributed filter structure is rewritten on sensor node $i$:

$$\begin{align*}
\hat{\eta}_{i,k+1} &= \hat{\eta}_{i,k} + K_i(\hat{\gamma}_{i,k} - \hat{\gamma}_{i,k}) + G_i \sum_{j \in \mathcal{N}_i} h_{ij}(\hat{\eta}_{j,k} - \hat{\eta}_{j,k}), \\
\hat{z}_{i,k} &= \hat{M}e_{i,k}.
\end{align*}$$

(7)

Define $e_{i,k} \triangleq \eta_k - \hat{\eta}_{i,k}$, $\hat{z}_{i,k} \triangleq \hat{z}_k - \hat{z}_{i,k}$ and $\rho_{i,k} \triangleq \hat{\eta}_{i,k} - \hat{\eta}_{j,k}$. Next, one has the following filtering error dynamics:

$$\begin{align*}
e_{i,k+1} &= (\hat{A} - K_i \hat{C}_i + d_i G_i) e_{i,k} + \hat{B} \rho_{i,k} - K_i \hat{V}_{i,k} - G_i \sum_{j \in \mathcal{N}_i} h_{ij} \rho_{i,k} \\& \quad + \sum_{j \in \mathcal{N}_i} h_{ij} \rho_{i,k}, \\
\hat{z}_{i,k} &= \hat{M} e_{i,k},
\end{align*}$$

(8)

where

$$\rho_{i,k} = \rho_{i,k} - \delta_j < 0.$$  

(9)

Then, letting

$$e_k \triangleq [e_{1,k}^T \cdots e_{n,k}^T]^T, \quad \rho_k \triangleq [\rho_{1,k}^T \cdots \rho_{n,k}^T]^T,$$

$$\hat{z}_k \triangleq [\hat{z}_{1,k}^T \cdots \hat{z}_{n,k}^T]^T, \quad \varphi_k \triangleq [\varphi_{1,k}^T \cdots \varphi_{n,k}^T]^T,$$

$$\delta_k \triangleq [\delta_{1,k}^T \cdots \delta_{n,k}^T]^T, \quad \omega_k \triangleq 1_n \otimes \omega_k,$$

one has

$$\begin{align*}
e_{k+1} &= (\hat{A} - K \hat{C} + \hat{G}(\mathcal{C} \otimes l_{(s+1)n})) e_k + \hat{B} \delta_k \\
&\quad - \hat{G}(\hat{H} \otimes l_{(s+1)n}) \rho_k, \\
\hat{z}_k &= \hat{M} e_k,
\end{align*}$$

(10)

where

$$\begin{align*}
\hat{A} &\triangleq I_n \otimes \hat{A}, \quad \hat{B} \triangleq [I_n \otimes \hat{B} - \hat{K}], \\
\hat{C} &\triangleq \text{diag}(\hat{C}_1, \ldots, \hat{C}_n), \quad \hat{M} \triangleq I_n \otimes \hat{M}, \\
\hat{K} &\triangleq \text{diag}(K_1, \ldots, K_n), \quad \hat{G} \triangleq \text{diag}(G_1, \ldots, G_n).
\end{align*}$$

Also, from (9), one further has

$$\begin{align*}
\rho_k^T \rho_k - \delta < 0
\end{align*}$$

(11)

where

$$\delta = \sum_{j=1}^n \rho_j.$$

In what follows, the desirable performance index is presented for facilitating the main results of this paper.

**Definition 2.1**: The filtering error dynamics (10) is said to be exponentially ultimately bounded if there exist scalars $x > 0, \kappa > 0$ and $\beta \in (0,1)$ such that

$$\|\hat{z}(k)\|_2^2 < \alpha \beta^k + \kappa$$

(12)

where $\beta$ is known as the decay rate and $\kappa$ is the asymptotic upper bound.

The objective of this paper is to construct the sufficient conditions such that (1) the system dynamics (10) is exponentially ultimately bounded; (2) the obtained upper bound is optimized to obtain the minimum value.

### 3. Main results

In this section, for the DF of system (3) over SNs, the filtering analysis and the design problems are studied by the given digraph $\mathcal{G} = (\mathcal{J}, \mathcal{X}, \mathcal{H})$.

Prior to proceeding further, the following lemma is firstly introduced for deriving the main results.

**Lemma 3.1 (Boyd et al., 1994)**: For constant matrices $T_1$, $T_2$ and $T_3$ satisfying $T_1^T < 0$ and $T_2 = T_2^T < 0$, then

$$T_1 - T_2 T_2^{-1} T_3 < 0$$

if and only if

$$\begin{bmatrix} T_1 & T_2 \\ T_3 & T_2 \end{bmatrix} < 0$$

Next, via the Lyapunov stability theory, for the filtering error dynamics (10), a sufficient condition is given to guarantee the desirable performance index.

**Theorem 3.1**: Consider the system (3) with given filter parameters. For given a constant $\gamma \in (0,1)$, the filter gains $K_i$ and $G_i$, and a positive-definite matrix $R = \text{diag}(R_1, R_2, \ldots, R_n) > \hat{M}^T \hat{M}$, the filtering error dynamics (10) is exponentially ultimately bounded while satisfying the inequality constraint (12) if there exists a symmetric positive
definite matrix $P = p^T > R$ and two positive scalars $\varepsilon_1$ and $\varepsilon_2$ such that the following matrix inequality holds:

$$
\Sigma = \begin{bmatrix}
\Sigma_{11} & * & * \\
\Sigma_{21} & \Sigma_{22} & * \\
\Sigma_{31} & \Sigma_{32} & \Sigma_{33}
\end{bmatrix} < 0 \quad (13)
$$

where

$$
\Sigma_{11} \triangleq (\hat{A} - \hat{K} \hat{C} + \hat{G} (\mathcal{L} \otimes I_{(s + 1)n_x}))^T P (\hat{A} - \hat{K} \hat{C} + \hat{G} (\mathcal{L} \otimes I_{(s + 1)n_x})) + \hat{G} (\mathcal{L} \otimes I_{(s + 1)n_x}) (1 - \gamma) P,
$$
$$
\Sigma_{21} \triangleq \hat{B}^T P (\hat{A} - \hat{K} \hat{C} + \hat{G} (\mathcal{L} \otimes I_{(s + 1)n_x})),
$$
$$
\Sigma_{22} \triangleq \hat{B}^T P \hat{B} - \varepsilon_2 I,
$$
$$
\Sigma_{31} \triangleq - (\mathcal{H} \otimes I_{(s + 1)n_x})^T \hat{G}^T P (\hat{A} - \hat{K} \hat{C} + \hat{G} (\mathcal{L} \otimes I_{(s + 1)n_x})),
$$
$$
\Sigma_{32} \triangleq - (\mathcal{H} \otimes I_{(s + 1)n_x})^T \hat{G}^T \hat{P} \hat{B},
$$
$$
\Sigma_{33} \triangleq (\mathcal{H} \otimes I_{(s + 1)n_x})^T \hat{G}^T \hat{P} (\mathcal{H} \otimes I_{(s + 1)n_x}) - \varepsilon_1 I.
$$

Proof: Define:

$$
V_k \triangleq \xi_k^T P \xi_k, \quad (14)
$$

and calculate the difference of the Lyapunov function along the dynamics (10) as follows:

$$
\Delta V_k = V_{k+1} - V_k
\begin{align*}
&= \left[ (\hat{A} - \hat{K} \hat{C} + \hat{G} (\mathcal{L} \otimes I_{(s + 1)n_x})) \right]^T \xi_k \\
&+ \hat{B}^T P \hat{B} \xi_k - \hat{G} (\mathcal{H} \otimes I_{(s + 1)n_x}) \rho_k \\
&= -\varepsilon_1 (\rho_k^T \xi_k - \delta) > 0. \quad (15)
\end{align*}
$$

According to (11), the positive scalar $\varepsilon_1$ can be introduced such that the following inequality holds:

$$
- \varepsilon_1 (\rho_k^T \xi_k - \delta) > 0. \quad (16)
$$

Next, it is obtained from the constraints of $\omega_k$ and $v_{ik}$ that

$$
\omega_k^T \omega_k - \omega_0^2 < 0. \quad (17)
$$
$$
v_{ik}^T v_{ik} - \overline{v}^2 < 0. \quad (18)
$$

That is, the positive scalar $\varepsilon_2$ can be introduced such that the following inequality holds:

$$
- \varepsilon_2 (\vartheta_k^T \vartheta_k - n(\overline{a}^2 + \overline{v}^2)) > 0. \quad (19)
$$

As a result, from (15), (16) and (19), it follows that

$$
\Delta V_k < -\gamma V_k + \varepsilon_1^T \left( \hat{A} - \hat{K} \hat{C} + \hat{G} (\mathcal{L} \otimes I_{(s + 1)n_x}) \right)^T P \left( \hat{A} - \hat{K} \hat{C} + \hat{G} (\mathcal{L} \otimes I_{(s + 1)n_x}) \right) e_k \\
- (1 - \gamma) e_k^T P \xi_k + 2 \varepsilon_2^T \left( \hat{A} - \hat{K} \hat{C} + \hat{G} (\mathcal{L} \otimes I_{(s + 1)n_x}) \right)^T P \hat{B} \theta_k \\
- 2 \varepsilon_2^T \left( \hat{A} - \hat{K} \hat{C} + \hat{G} (\mathcal{L} \otimes I_{(s + 1)n_x}) \right)^T P \hat{G} (\mathcal{H} \otimes I_{(s + 1)n_x}) \rho_k \\
+ \vartheta_k^T \vartheta_k - n(\overline{a}^2 + \overline{v}^2) \right)
= -\gamma V_k + \varepsilon_1^T \xi_k \Sigma \xi_k + \varepsilon_1 \delta + n \varepsilon_2 (\overline{a}^2 + \overline{v}^2) \quad (20)
$$

where

$$
\xi_k \triangleq \begin{bmatrix} e_k^T & \vartheta_k^T & \rho_k^T \end{bmatrix}^T.
$$

Next, by means of (13), (20) further becomes

$$
V_{k+1} < (1 - \gamma) V_k + \varepsilon_1 \delta + n \varepsilon_2 (\overline{a}^2 + \overline{v}^2). \quad (21)
$$

According to (21), one derives the following formula:

$$
V_k < (1 - \gamma) V_0 + \frac{1}{\gamma} \left( \varepsilon_1 \delta + n \varepsilon_2 (\overline{a}^2 + \overline{v}^2) \right) < (1 - \gamma) V_0 + \frac{1}{\gamma} \left( \varepsilon_1 \delta + n \varepsilon_2 (\overline{a}^2 + \overline{v}^2) \right). \quad (22)
$$

At the same time, it follows from $P > R$ and $\hat{M}^T \hat{M} < R$ that

$$
\xi_k^T \xi_k = e_k^T \hat{M} \hat{M} e_k < e_k^T \hat{M} e_k \quad (23)
$$

and

$$
V_k = e_k^T P \xi_k > e_k^T \hat{M} e_k. \quad (24)
$$

From (22)–(24), one then derives

$$
\| \xi_k \|^2 < (1 - \gamma) V_0 + \frac{1}{\gamma} (\varepsilon_1 \delta + n e_2 (\overline{a}^2 + \overline{v}^2))
$$
\[ -(1 - \gamma)P \begin{bmatrix} * & * \\ \ast & \ast \end{bmatrix} < 0 \]

where

\[ P = \text{diag}(\hat{P}_1, \cdots, \hat{P}_n), \quad \tilde{P}_i = \text{diag}(\hat{P}_i^1, \hat{P}_i^2, \cdots, \hat{P}_i^{t+1}), \]

\[ P\tilde{A} = \text{diag}(\hat{P}_1, \cdots, \hat{P}_n)(I_n \otimes \hat{A}) = \text{diag}(\hat{P}_1\hat{A}, \cdots, \hat{P}_n\hat{A}). \]

\[ P\tilde{K} = \text{diag}(\hat{P}_1, \cdots, \hat{P}_n)\text{diag}(K_1, K_n) = \text{diag}(\hat{P}_1 K_1, \cdots, \hat{P}_n K_n) \]

\[ \triangleq \text{diag}(\hat{K}_1, \cdots, \hat{K}_n), \quad \triangleq \hat{K}. \]

\[ P\tilde{G} = \text{diag}(\hat{P}_1^1, \cdots, \hat{P}_n^1)\text{diag}(G_1, \cdots, G_n) \]

\[ = \text{diag}(\hat{P}_1 G_1, \cdots, \hat{P}_n G_n) \triangleq \text{diag}(\hat{G}_1, \cdots, \hat{G}_n) \triangleq \hat{G}. \]
Moreover, one has

$$P\hat{B} = P[I_n \otimes \bar{B} - \hat{K}]$$

$$= [\text{diag}(\hat{P}_1, \ldots, \hat{P}_n)(I_n \otimes \bar{B}) - \hat{P}\bar{K}]$$

$$= [\text{diag}(\hat{P}_1\bar{B}, \ldots, \hat{P}_n\bar{B}) - \bar{K}].$$

(34)

After some simple calculations, one obtains $\text{diag}(\hat{P}_1\bar{A}, \ldots, \hat{P}_n\bar{A})$ and $\text{diag}(\hat{P}_1\bar{B}, \ldots, \hat{P}_n\bar{B})$. According to the above calculations, it can be seen that (13) holds if and only if (28) holds. In the end, by resorting to Theorem 3.1, the proof is therefore complete.

**Remark 3.1:** Compared to the traditional DF schemes, there exists a prominent difference of our scheme. Actually, the influence of the integral measurements is reflected by $\hat{P}_i = \text{diag}(\hat{P}_i^1, \ldots, \hat{P}_i^{s+1})$. It is no doubt that it increases undoubtedly the computational complexity of the scheme. If $s = 0$, then $\hat{P}_i = \hat{P}_i^1$. More specifically, our scheme is reduced to the traditional DF scheme without the integral measurements.

At length, via Theorem 3.2, the optimization problem (27) is rewritten as

$$\min \tau$$

s.t. (26), (28), $\varepsilon_1 > 0, \varepsilon_2 > 0$.

(35)

Moreover, Table 1 summarizes the distributed filter design (DFD) algorithm developed in this paper.

**Remark 3.2:** This paper studies the DF problem with integral measurements scheduled by the ET protocol. The Lyapunov stability theory is employed to seek the desirable filter parameters such that the filtering error satisfy the performance index. In comparison with results in existence, the main results show the following characteristics: (1) the integral measurements are firstly taken into account in the DF problem; (2) in order to save energy, the data transmission among neighbouring nodes in SNs is scheduled via the ET protocol; (3) Theorem 3.2 involves all the considered factors, which is conducive to investigating the influences of these factors on the filtering performance.

4. Illustration example

This section furnishes a simulation example to demonstrate the usefulness of the desired distributed filter design scheme with both the integral measurements and the ET protocol.

Consider an SN with four nodes, whose topology is characterized by a digraph $G = (\mathcal{J}, \mathcal{X}, \mathcal{H})$ with the set of nodes $\mathcal{J} = \{1, 2, 3, 4\}$, the set of edges $\mathcal{X} = \{(1,2), (2,4), (3,2), (4,3)\}$ and the adjacency matrix

$$\mathcal{H} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}.$$\

Consider a discrete-time system described by (1) as follows:

$$x_{k+1} = \begin{bmatrix}
-0.2 & -0.5 \\
1.5 & -1
\end{bmatrix} x_k + \begin{bmatrix}
1 \\
0
\end{bmatrix} \omega_k, \quad (36)$$

with the initial state value $x_0 = [2.2]^T$. Set the time interval to collect the data $s = 2$, and the measurement of sensor node $i$ is described as follows:

$$y_{i,k} = \begin{bmatrix}
-2 & 1
\end{bmatrix} \sum_{q=0}^2 x_{k-q} + v_{i,k}, \quad i = 1, 2, 3, 4.$$\

The initial states $\hat{x}_{i,0}$ are chosen as $\hat{x}_{1,0} = [11]^T$, $\hat{x}_{2,0} = [2.0]^T$, $\hat{x}_{3,0} = [0.5 - 1.5]^T$ and $\hat{x}_{4,0} = [0 - 1]^T$.

The measurement noise $v_{i,k}$ and the process noise $\omega_k$ are set to be

$$v_{i,k} = \sin(10k + 1) \quad \text{and} \quad \omega_k = \exp(-k).$$

In the simulation example, other parameters are chosen as $\gamma = 0.5, \delta_i = 1.5 \ (i = 1, 2, 3, 4)$. The optimization problem (35) is solved by using MATLAB. Moreover, all the filter parameters can be acquired according to (29) from Theorem 3.2 as follows:

$$K_1 = \begin{bmatrix}
-1.2929 \\
-1.4535 \\
0.6037 \\
2.3457 \\
0.5426 \\
-1.4736
\end{bmatrix}, \quad K_2 = \begin{bmatrix}
-1.3258 \\
-1.5727 \\
0.5743 \\
2.4247 \\
0.6196 \\
-1.4450
\end{bmatrix}.$$
Finally, the simulation results are presented in Figures 1–4, where Figure 1 plots the actual output $z_k$ and its estimates from four nodes, Figure 2 describes the filtering errors of all nodes, and Figures 3 and 4 show the first and second elements of $x_k$ and their estimates, respectively. According to these figures, the DF algorithm developed in this paper is truly effective.
5. Conclusion

We have addressed the DF problem with integral measurements over SNs scheduled by the ET protocols in this paper. The considered plant and the integral measurements of each node have been formulated. An event-triggering function has been employed for each node to determine whether sending its information to its neighbouring nodes. Next, the distributed filter has been constructed by utilizing the available information. A sufficient condition has been obtained such that the filtering error dynamics satisfies the desirable exponentially ultimately boundedness. Then, an optimization problem has been set up to optimize the obtained upper bound and the distributed filter gains has been calculated. At length, an illustrative example has been provided to demonstrate the usefulness of the desired algorithm. The future research topics include the DF problem with dynamic ET protocols, Round-Robin protocol (Li et al., 2020) or the amplify-and-forward relays (Tan et al., 2020).

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