On Melting Temperature of Heavy Quarkonium with the AdS/CFT Implied Potential

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The quarkonium states in a quark-gluon plasma is examined with the potential implied by AdS/CFT duality. Both the vanila AdS-Schwarzschild metric and the one with an infrared cut-off are considered. The calculated dissociation temperatures for J/ψ and Υ are found to agree with the lattice results within a factor of two.

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The heavy quarkonium dissociation is one of the important signals of the formation of QGP in RHIC. The subject has been explored extensively on a lattice [1, 2, 3]. In the deconfinement phase of QCD, the range of the binding potential between a quark and an antiquark is limited by the screening length in a hot and dense medium, which decreases with an increasing temperature. Beyond the dissociate temperature, T_d, the range of the potential is too short to hold a bound state and the heavy quarkonium will melt. The lattice simulation of the quark-antiquark potential and the spectral density of hadronic correlators yield consistent picture of the quarkonium dissociation and the numerical values T_d. On the other hand, AdS/CFT duality provides a new avenue towards a qualitative or even semi-quantitative understanding of the non-perturbative aspect of a quantum field theory [4, 5]. It is conjectured that a string theory formulated in AdS_5 × S_5 is dual to the N = 4 supersymmetric Yang-Mills theory (SUSY YM) on the boundary. In particular, the low energy limit of the classical string theory, the supergravity in AdS_5 × S_5 corresponds to the supersymmetric Yang-Mills theory at large N_c and large 't Hooft coupling \( \lambda \sim N_c g^2_{YM} \). Remarkable success has been made in the application of the AdS/CFT duality to the physics of quark-gluon plasma (QGP) created by RHIC, even though the underlying dynamics of QCD is very different from that of a supersymmetric Yang-Mills theory [6, 7, 8]. It is the object of this brief report to calculate T_d using the heavy quark potential of N = 4 SUSY YM extracted from its gravity dual [9, 10, 11].

Although the potential model applies only in the non-relativistic limit which is not the case when the 't Hooft coupling, \( \lambda \), becomes too strong, it can be justified within the lower side of the range of \( \lambda \) used in the literature to compare AdS/CFT with the RHIC phenomenology, i.e. 5 < \( \lambda \) < 6π. The upper edge is obtained by using N_c = 3 and the QCD value of gYM at RHIC energy scale (g^2_{YM}/4\pi \simeq 1/2) and the lower edge is based on a comparison between the heavy quark potential from lattice simulation with that from AdS/CFT [12].

We model the quarkonium, J/ψ or Υ, as a non-relativistic bound state of a heavy quark and its antiparticle. The wave function for their relative motion satisfies the Schrödinger equation

\[
\frac{-i}{2m}\nabla^2 \psi + V_{\text{eff}}(r)\psi = -E\psi
\]  

(1)

where \( m = M/2 \) is the reduced mass with M the mass of the heavy quark and E(\geq 0) is the binding energy of the bound state. Because of the screening of QGP, the effective potential energy has a finite range and is temperature dependent. The dissociation temperature of a particular state is the temperature at which its energy, \( -E \), is elevated to zero.

The free energy of a static pair of q\bar{q} separated by a distance r at temperature T is given

\[
e^{-\frac{\Delta F(r, T)}{T}} = \frac{\text{tr} < W^\dagger(L_+)W(L_-) >}{\text{tr} < W^\dagger(L_+) > < W(L_-) >}
\]  

(2)

where \( L_\pm \) stands for the Wilson line running in Euclidean time direction at spatial coordinates (0, 0, \pm 12\pi) and is closed by the periodicity. We have \( W(L_\pm) \equiv P e^{-i \int_0^{12\pi} dt A_0(t, 0, 0, \pm 12\pi)} \) with A_0 the temporal component of the gauge

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potential subject to the periodic boundary condition, \( A_0(t + \frac{1}{T}, r) = A_0(t, r) \). The trace here is over the color indexes and \(< \ldots >\) denotes the thermal average. The symbol \( P \) enforces the path ordering. The corresponding internal energy reads \( U(r, T) = -T^2 \frac{d}{dT} \left( \frac{\overline{F}(r, T)}{T} \right) \). Two ansatz of the potential model have been explored in the literature\(^2\): the \( F \)-ansatz which identifies \( V_{\text{eff}} \) of \(^1\) with \( F(r, T) \) and the \( U \)-ansatz which identifies \( V_{\text{eff}} \) with \( U(r, T) \). The lattice QCD simulation reveals that the \( U \) ansatz produces a deeper potential well and thereby higher \( T_d \) because the entropy contribution is subtracted. This remains the case with holographic potential as we shall see.

According to the holographic principle, the thermal average of a Wilson loop operator \( W(C) = P e^{-\frac{1}{T \beta} \int_{x} dx^4 A_{\mu}(x)} \) in 4D \( N = 4 \) SUSY YM at large \( N_c \) and large ‘t Hooft coupling corresponds to the minimum area \( S_{\min}(C) \) of the string world sheet in the 5D AdS-Schwarzschild metric with a Euclidean signature

\[
ds^2 = \pi^2 T^2 y^2 (f dt^2 + dx^2) + \frac{1}{y^2 f} dy^2,
\]

bounded by the loop \( C \) at the boundary, \( y \to \infty \), where \( f = 1 - \frac{1}{y} \). Specifically, we have

\[
\text{tr} < W(C) > = e^{-\sqrt{\lambda} S_{\min}(C)}.
\]

For the numerator of \(^2\), \( C \) consists of two parallel temporal lines \((t, 0, 0, \pm\frac{\pi}{2}) \) and the string world sheet can be parameterized by \( t \) and \( y \) with the ansatz \( x_1 = x_2 = 0 \) and \( x_3 = \) a function of \( y \). The induced world sheet metric reads

\[
ds^2 = \pi^2 T^2 y^2 f dt^2 + \left[ \pi^2 T^2 y^2 \left( \frac{dx_3}{dy} \right)^2 + \frac{1}{\pi^2 T^2 y^2 f} \right] dy^2.
\]

Minimizing the world sheet area (the Nambu-Goto action)

\[
S[C] = (\pi T) \int_{0}^{\pi T} dt \int_{0}^{\infty} dy \sqrt{1 + \pi^4 T^4 y^4 f \left( \frac{dx_3}{dy} \right)^2}
\]

generates two types of solutions\(^9\) \(^10\) \(^11\). One corresponds to a single world-sheet with a nontrivial profile \( x_3(y) \),

\[
x_3 = \pm \pi T q \int_{y_c}^{y} \frac{dy'}{\sqrt{(y'^4 - 1)(y'^4 - y_c^4)}}
\]

where \( q \) is a constant of integration determined by the boundary condition

\[
r = \frac{2q}{\pi T} \int_{y_c}^{\infty} \frac{dy}{\sqrt{(y'^4 - 1)(y'^4 - y_c^4)}} \tag{8}
\]

with \( y_c^4 = 1 + q^2 \). The corresponding value of \( \sqrt{\lambda} S[C] \) is denoted by \( I_1 \). The other solution consists of two parallel world sheets with \( x_3 = \pm \frac{\pi}{2} \) extending to the black hole horizon and the corresponding value of \( \sqrt{\lambda} S[C] \) is denoted by \( I_2 \). The latter solution corresponds to two non-interacting static quarks in the medium and is equal to the denominator of \(^2\).

The free energy we are interested in reads

\[
F(r, T) = T \min(I_1, 0) \tag{9}
\]

where

\[
I \equiv I_1 - I_2 = \sqrt{\lambda} \left[ \int_{y_c}^{\infty} dy \left( \sqrt{y'^4 - \frac{1}{y_c^4}} - 1 \right) + 1 - y_c \right]. \tag{10}
\]

Inverting eq.\((8)\) to express \( q \) in terms of \( r \) and substituting the result to \(^10\), it was found that the function \( I \) consists of two branches, The upper branch is always positive and is therefore unstable. The lower branch starts from being negative for \( r < r_0 \) and becomes positive for \( r > r_0 \). Both branches joins at \( r = r_c > r_0 \) beyond which the nontrivial solution ceases to exist. Numerically, we have \( r_0 \approx \frac{0.7541}{\pi T} \) and \( r_c \approx \frac{0.85}{\pi T} \). Introducing a dimensionless radial coordinate \( \rho = \pi T r \), we find that

\[
F(r, T) = -\frac{\alpha}{r} \phi(\rho) \theta(\rho_0 - \rho), \tag{11}
\]
where \( \alpha = \frac{4 \pi^2}{\Gamma^2 \left( \frac{1}{4} \right)} \sqrt{\lambda} \approx 0.2285 \sqrt{\lambda} \), and \( \phi(\rho) = -\rho I/(\pi\alpha) \) is the screening factor. We have \( \phi(0) = 1 \) and \( \phi(\rho_0) = 0 \) with \( \rho_0 = 0.7541 \).

The small \( \rho \) expansion of \( \phi(\rho) \) is given by

\[
\phi(\rho) = 1 - \frac{\Gamma^4 \left( \frac{1}{4} \right)}{4\pi^3} \rho + \frac{3\Gamma^8 \left( \frac{1}{4} \right)}{640\pi^6} \rho^4 + O(\rho^8).
\]  

On writing the wave function \( \psi(\vec{r}) = u_l(\rho) Y_m(\hat{r}) \), the radial Schrödinger equation for a zero energy bound state reads

\[
\frac{d^2 u_l}{d\rho^2} + \frac{2}{\rho} \frac{du_l}{d\rho} - \left[ \frac{l(l+1)}{\rho^2} + V \right] u_l = 0
\]  

with \( V = MV_{\text{eff}}/\left(\pi^2T^2\right) \). We have

\[
V = -\frac{\eta^2}{\rho_0 \rho} \phi(\rho) \theta(\rho_0 - \rho)
\]  

for the F ansatz and

\[
V = -\frac{\eta^2}{\rho_0 \rho} \left[ \phi(\rho) - \rho \frac{d\phi}{d\rho} \right] \theta(\rho_0 - \rho)
\]  

for the U ansatz, where \( \eta = \sqrt{\frac{\alpha \rho_0 M}{\pi}} \).

Note that the potential of the U-ansatz jumps to zero from below at \( \rho = \rho_0 \), since the derivative of \( \phi(\rho) \) is nonzero there. For both ansatz, and the case with an infrared cutoff discussed below, the solution to (13) is given by

\[
u_l = \text{const.} \rho^{-l-1}
\]  

at \( \rho > \rho_0 \) and by

\[
u_l = \text{const.} \rho^l
\]  

near the origin. The threshold value of \( \eta \) at the dissociation temperature, \( \eta_d \), is determined by the matching condition at \( \rho = \rho_0 \),

\[
\frac{d}{d\rho} \left( \rho^{l+1} u_l \right) \bigg|_{\rho = \rho_0} = 0.
\]  

It follows from \( \eta \) that the dissociation temperature is given by

\[
T_d = \frac{\alpha \rho_0 M}{\pi \eta_d^2} = \frac{4\pi \rho_0}{\Gamma^4 \left( \frac{1}{4} \right) \eta_d^2} \sqrt{\lambda M}.
\]  

It is interesting to observe that the extrapolation of the first two terms of (12) vanishes at \( \rho = \rho_0' = 4\pi^3/\Gamma^4 \left( \frac{1}{4} \right) \approx 0.7178 \), which is very close to the exact zero point, and the third term of (12) remains small there. This suggests that the screening factor \( \phi(\rho) \) can be well approximated by a linear function

\[
\tilde{\phi}(\rho) \approx 1 - \frac{\rho}{\tilde{\rho}_0}
\]  

with \( \tilde{\rho}_0 = \frac{1}{2} (\rho_0 + \rho_0') \approx 0.7359 \). The effective potential \( V_{\text{eff}} \), is then approximated by a truncated Coulomb potential. We have

\[
V = -\frac{\eta^2}{\rho_0 \rho} (1 - \frac{\rho}{\rho_0}) \theta(\rho_0 - \rho)
\]  

for the F-ansatz and

\[
V = -\frac{\eta^2}{\rho_0 \rho} \theta(\rho_0 - \rho)
\]  

for the U-ansatz, where the over bar of \( \rho_0 \) has been suppressed.
The radial wave function of the F-ansatz under the truncated Coulomb approximation can be expressed in terms of the confluent hypergeometric function of the 1st kind for \( \rho < \rho_0 \), i.e.

\[
u_l = \rho \frac{\_1F\_1}{\rho_0} (l + 1 - \frac{\eta}{2}; 2l + 2; 2\eta) (23)
\]

The matching condition (18) yields the secular equation for \( \eta \),

\[
2l + 1 - \eta + \eta \left(1 - \frac{\eta}{2l + 2}\right) \frac{\_1F\_1}{\_1F\_1} (l + 1 - \frac{\eta}{2}; 2l + 2; 2\eta) = 0 (24)
\]

As \( \eta \) is reduced from above, we expect the bound states of the same \( l \) to melt successively. Therefore the first positive root corresponds to the minimum binding strength for a bound state of angular momentum \( l \) and the 2nd one to the threshold of the first radial excitation. Knowing the values of these \( \eta \)'s, the disassociation temperature can be computed from the formula (19). For example, the threshold \( \eta \) of the \( 1S \) state, \( \eta_{1S} \approx 1.76 \), which implies that

\[
T_d \approx 0.0173 \sqrt{\lambda M}. (25)
\]

In case of the U-ansatz under the same approximation, we find that

\[
u_l = \frac{1}{\sqrt{\rho}} J_{2l+1} \left(2\sqrt{\rho} \eta \rho_0\right) (26)
\]

for \( \rho < \rho_0 \) with \( J_{\nu}(x) \) the Bessel function. The secular equation for \( \eta \) reads

\[
2l + 1 - \eta \frac{J_{2l+2}(2\eta)}{J_{2l+1}(2\eta)} = 0. (27)
\]

We have \( \eta_{1S} = 1.20 \) and

\[
T_d \approx 0.0370 \sqrt{\lambda M}. (28)
\]

Numerical results for the dissociation temperature of quarkonium are tabulated in table 1, where we have used the mass values \( M = 1.65 \text{GeV} \), 4.85 GeV for \( c \) and \( b \) quarks . The errors caused by the truncated Coulomb approximation are within 4 percent, as is shown by the numerical solution to the Schrödinger equation of the exact potential.

Because of the conformal invariance at quantum level, there is no color confinement in \( N = 4 \) SUSY YM even at zero temperature. In order to simulate the confined phase of QCD at low temperature, an infrared cutoff has to be introduced that suppress the contribution of the \( AdS \) horizon. Two scenarios explored in the literature are the hard-wall model and the soft-wall models [14, 15]. The gravity dual of the de-confinement transition is modeled as the Hawking-Page transition from a metric without a black hole at \( T < T_c \) to that with a black hole at \( T > T_c \). Heavy quark potential and the meson dissociation temperatures calculated with the hard-wall model are identical to what calculated above with the vanila AdS-Schwarzschild metric.

In case of the simplest soft-wall model ([15]), a dilaton is introduced and the gravity dual of the free energy is the given by

\[
F = -T \frac{1}{16 \pi G_5} \int d^4 x \int_{\rho_0}^{\infty} dr \rho^5 \sqrt{g} (R - 12), (29)
\]

where \( c \) is determined by the \( \rho \)-mass and the transition temperature is predicted as \( T_c \approx 0.2459 m_c \) [16]. A variant of the soft-wall scenario proposed in ref.[17, 18], admits a string frame metric with a conformal factor, i. e.

\[
ds^2 = \frac{e^{ks^2}}{s^2} (f dt^2 + dx^2 + f^{-1} dz^2), (30)
\]
ansatz       | $J/\psi$       | $\Upsilon$
---|---|---
$F$         | NA          | 235-385
$U$         | 219-322     | 459-780

TABLE II: $T_d$ in MeV’s for the 1S state with the deformed metric. "NA" means that there is no bound states above $T_c$ and the entry for the $\Upsilon$ with $U$ ansatz and $\alpha = 6\pi$ is taken from the table I, since no significant increment is observed.

ansatz | $J/\psi$(holographic) | $J/\psi$(lattice) | $\Upsilon$(holographic) | $\Upsilon$(lattice)
---|---|---|---|---
$F$ | NA | 1.1 | 1.3-2.1 | 2.3
$U$ | 1.2-1.7 | 2.0 | 2.5-4.2 | 4.5

TABLE III: The ratio $T_d/T_c$ for the 1S from the holographic potential and that from the lattice QCD

The value of $b = 0.184\text{GeV}^2$ was obtained by fitting the lattice simulated transition temperature $T_c = 186\text{MeV}$ [18]. Following the steps from (3) to (10), we can calculate the effective potentials at both ansatz [21]. To determine the dissociation temperature in this case, we have to solve the Schrödinger equation numerically with the numerically calculated heavy quark potential for both ansatz, since the truncated Coulomb potential no longer approximates well. The modified dissociation temperatures are tabulated in the table 2, which show a significant increment in the vicinity of $T_c$. The comparison between the ratios $T_d/T_c$ we calculated here with that obtained from the lattice QCD is shown in table 3 [21].

In summary, we have calculated the dissociation temperatures of heavy quarkonia using the static potential implied by the holographic principle with both the vanilla AdS-Schwarzschild metric and the one with an infrared cutoff. While estimations of $T_d$ have been made in the literature based on various holographic models [19, 20, 22], a determination of $T_d$ from the Schrödinger equation within the same framework remains lacking. Our work is to fill this gap. The authors of [19] gave an order of magnitude estimate of the dissociation temperature relying on the screening length only. The author of [20] generalized the spectral analysis of the light mesons to heavy mesons. Their criterion for the dissociation, however, appears slightly ad hoc and is again independent of the coupling. Both the screening length and the coupling strength ought to affect the heavy quarkonium binding. Carrying out the analysis of the potential model inspired by the holographic principle to the same extent of that of QCD will address both contributions, especially the consistency of the range of the coupling constant extracted from the jet quenching with the heavy quarkonium physics. Also a detailed bound state calculation enables us to assess the validity of the non-relativistic approximation behind the potential model. On comparing our results with that from the lattice simulation [3], we found that our ratios $T_d/T_c$ extracted from the modified AdS-Schwarzschild metric [20] are lower than the lattice ones within a factor of two. That the increment in $T_d/T_c$ from the F-ansatz to the U-ansatz is about a factor of two is similar to what reported in [3]. One has to bear in mind that the lattice results reviewed in [3] were extracted from a pure $SU(3)$ gauge theory without a matter field. On the other hand the matter field contents of $N = 4$ SUSY YM are larger than that of QCD with light quarks. It is possible that the additional screening effect of the matter field in $N = 4$ SUSY YM makes the heavy quarkonia more vulnerable and thereby lowers the dissociation temperature. This is consistent with the observation that the potential well becomes wider in the metric with the IR cutoff introduced in [17, 18] since some of degrees of freedom becomes massive.

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