ON INNER AND OUTER RADII IN MINKOWSKI SPACES

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Abstract. Some sharp bounds for the inner radius and the outer radius of the unit ball of a Minkowski space with respect to its isoperimetrix are known. To find more such bounds is a challenging problem. Related to this, we derive new relations between inner and outer radii as well as cross-section measures for the Holmes-Thompson and Busemann measures.

1. Introduction

This paper refers to the geometry of finite dimensional real Banach spaces, also called Minkowski spaces, and to classical convexity. More precisely, the notions of different cross-section measures from classical convexity are used to obtain new results on unit balls and isoperimetries of Minkowski spaces.

We recall that a convex body \( K \) in \( \mathbb{R}^d, d \geq 2 \), is a compact, convex set with nonempty interior, and that \( K \) is said to be centered if it is symmetric with respect to the origin \( o \) of \( \mathbb{R}^d \). As usual, \( S^{d-1} \) denotes the standard Euclidean unit sphere in \( \mathbb{R}^d \). We write \( \lambda_i \) for the \( i \)-dimensional Lebesgue measure (volume) in \( \mathbb{R}^d \), where \( 1 \leq i \leq d \), and instead of \( \lambda_d \) we simply write \( \lambda \). We denote by \( u^\perp \) the \((d-1)\)-dimensional subspace orthogonal to \( u \in S^{d-1} \), and by \( l_u \) the 1-subspace parallel to \( u \).

For a convex body \( K \subset \mathbb{R}^d \) we denote by \( \lambda_{d-1}(K, u^\perp) \) and \( \lambda_1(K, u) \) the \((d-1)\)-dimensional and 1-dimensional inner cross-section measures of \( K \), i.e., the maximal measure of a hyperplane section of \( K \) normal to \( u \in S^{d-1} \), and the maximal chord length of \( K \) in the direction \( u \), respectively. Furthermore, \( \lambda_1(K|l_u) \) denotes the width of \( K \) at \( u \), and \( \lambda_{d-1}(K|u^\perp) \) the \((d-1)\)-dimensional outer cross-section measure or brightness of \( K \) at \( u \in S^{d-1} \), where \( K|u^\perp \) is the orthogonal projection of \( K \) onto \( u^\perp \). These notions can be found in the monograph [3]. In [11] and [16] the following results for cross-section measures were derived.

For a convex body \( K \) in \( \mathbb{R}^d, d \geq 2 \), and every direction \( u \in S^{d-1} \) we have

\[
\lambda(K) \leq \lambda_{d-1}(K|u^\perp)\lambda_1(K, u) \leq d\lambda(K),
\]

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and both sides are sharp.

On the other hand, for each \( u \in S^{d-1} \) a convex body \( K \) in \( \mathbb{R}^d \), \( d \geq 2 \), satisfies
\[
\lambda(K) \leq \lambda_{d-1}(K,u^\perp)\lambda_1(K|l_u) \leq d\lambda(K),
\]
again with sharpness on both sides.

Our main purpose is to establish connections between cross-section measures and inner/outer radii for centered convex bodies. The inner radius and outer radius of the unit ball with respect to its isoperimetrix in Minkowski spaces (for Holmes-Thompson and Busemann measures) will be used to obtain these connections. Thus, our main results will be related to finite dimensional real Banach spaces as well.

For a convex body \( K \) in \( \mathbb{R}^d \), the polar body \( K^\circ \) of \( K \) is defined by
\[
K^\circ = \{ y \in \mathbb{R}^d : \langle x,y \rangle \leq 1, x \in K \}.
\]

We identify \( \mathbb{R}^d \) and its dual space \( \mathbb{R}^{d*} \) by using the standard basis. In that case, \( \lambda_i \) and \( \lambda_i^* \) coincide in \( \mathbb{R}^d \). The symbol \( \varepsilon_i \) stands for the volume of the standard Euclidean unit ball in \( \mathbb{R}^i \).

For a convex body \( K \) in \( \mathbb{R}^d \) and \( u \in S^{d-1} \), the support function of \( K \) is defined by
\[
h_K(u) = \sup\{ \langle u,y \rangle : y \in K \},
\]
and with \( o \) as an interior point of \( K \) its radial function \( \rho_K(u) \) is defined by
\[
\rho_K(u) = \max\{ \alpha \geq 0 : \alpha u \in K \}.
\]

It is well known that
\[
\rho_{K^\circ}(u) = \frac{1}{h_K(u)}, \quad u \in S^{d-1}.
\]

If \( K \) is a centered convex body, then \( 2\rho_K(u) = \lambda_1(K \cap l_u) \), and \( 2h_K(u) = \lambda_1(K|l_u) \) for any \( u \in S^{d-1} \).

The projection body \( \Pi K \) of a convex body \( K \) in \( \mathbb{R}^d \) is defined by \( h_{\Pi K}(u) = \lambda_{d-1}(K|u^\perp) \) for each \( u \in S^{d-1} \) (see [3, Chapter 4]). Note that any projection body is a zonoid (i.e., a limit of vector sums of segments) centered at the origin. In particular, if \( K \) is a polytope, then its projection body is a zonotope centered at the origin (see [15] and [4] for many properties and applications of this interesting class of convex bodies). We also refer to [1], [6], [7], and [12] for affine isoperimetric inequalities related to projection bodies. The intersection body \( IK \) of a convex body \( K \subset \mathbb{R}^d \) is defined by \( \rho_{IK}(u) = \lambda_{d-1}(K \cap u^\perp) \) for each \( u \in S^{d-1} \) (cf. [5] and [3, Chapter 8]). If \( K \) is a centered convex body, then \( IK \) is also a centered convex body (see [2]).

We write \((\mathbb{R}^d, ||·||) = M^d \) for a \( d \)-dimensional real Banach space, i.e., a Minkowski space with unit ball \( B \) which is a centered convex body; see [17]. The unit sphere of \( M^d \) is the boundary \( \partial B \) of the unit ball.
2. Isoperimetric and inner/outer radii in Minkowski spaces

A Minkowski space $\mathbb{M}^d$ possesses a Haar measure $\mu$, and this measure is unique up to multiplying the Lebesgue measure by a constant, i.e., $\mu = \sigma_B \lambda$.

The following notions are well known; see [17, Chapter 5]. The $d$-dimensional Holmes-Thompson volume of a convex body $K$ in $\mathbb{M}^d$ is defined by

$$\mu_B^{HT}(K) = \frac{\lambda(K) \lambda(B)}{\varepsilon_d}, \text{ i.e., } \sigma_B = \frac{\lambda(B)}{\varepsilon_d},$$

and the $d$-dimensional Busemann volume of $K$ is defined by

$$\mu_B^{Bus}(K) = \frac{\varepsilon_d}{\lambda(B)} \lambda(K), \text{ i.e., } \sigma_B = \frac{\varepsilon_d}{\lambda(B)} \lambda(B) \varepsilon_d.$$

In order to define the Minkowski surface area of a convex body, one has to define $\sigma_B$ similarly in $\mathbb{M}^{d-1}$. That is, for the Holmes-Thompson measure we have $\sigma_B(u) = \lambda_{d-1}(B \cap u^\perp) / \varepsilon_{d-1}$, and for the Busemann measure $\sigma_B(u) = \varepsilon_{d-1} / \lambda_{d-1}(B \cap u^\perp)$ (see [17, pp. 150-151]). The Minkowski surface area of $K$ can be also defined in terms of mixed volumes (see [14] for notation and more about mixed volumes) by

$$\mu_B(\partial K) = dV(K[d-1], I_B),$$

where $I_B$ is that convex body whose support function is $\sigma_B(u)$. For the Holmes-Thompson measure, $I_B$ is given by $I_B^{HT} = \Pi(B) / \varepsilon_{d-1}$ (cf. [17, p. 150 and p. 157] for detailed explanation), and therefore it is a centered zonoid. For the Busemann measure we have $I_B^{Bus} = \varepsilon_{d-1}(IB)^\circ$ (see again [17, pp. 150-151]). Among all homothetic images of $I_B$ a unique one is specified, which is called the isoperimetrix $\hat{I}_B$ and is determined by $\mu_B(\partial \hat{I}_B) = d\mu_B(\hat{I}_B)$. The isoperimetrix for the Holmes-Thompson measure is defined by

$$\hat{I}_B^{HT} = \frac{\varepsilon_d}{\lambda(B)} I_B^{HT} = \frac{\varepsilon_d}{\varepsilon_{d-1}} \frac{1}{\lambda(B)} \Pi B^\circ,$$

and the isoperimetrix for the Busemann measure by

$$\hat{I}_B^{Bus} = \frac{\lambda(B)}{\varepsilon_d} I_B^{Bus} = \frac{\varepsilon_{d-1}}{\varepsilon_d} \lambda(B)(IB)^\circ;$$

see [17, Chapter 5].

If $K$ and $L$ are convex bodies in $\mathbb{M}^d$, the inner radius of $K$ with respect to $L$ is defined by $r(K, L) := \max \{ \alpha : \exists x \in \mathbb{M}^d \text{ with } \alpha L \subseteq K + x \}$, and the outer radius of $K$ with respect to $L$ is defined by $R(K, L) := \min \{ \alpha : \exists x \in \mathbb{M}^d \text{ with } \alpha L \supseteq K + x \}$.

One should notice that $r(K, \hat{I}_B)$ and $R(K, \hat{I}_B)$ can also be defined in terms of the support functions of the involved sets. In particular, if $K$ is a centered convex body, then $r(K, \hat{I}_B)$ is the maximum value of $\alpha$ such that $\alpha \leq h_K(u) / h_{\hat{I}_B}(u)$ for all $u \in S^{d-1}$. Similarly, $R(K, \hat{I}_B)$ is the minimum value of $\alpha$ such that $\alpha \geq h_K(u) / h_{\hat{I}_B}(u)$ for all $u \in S^{d-1}$ (see [13] and [18]).
3. Cross-section measures and inner/outer radii

For a convex body \( K \), we denote by \( w_B(K) \) and \( D_B(K) \) the Minkowskian thickness (i.e., \( w_B(K) = \min_{u \in S^{d-1}} \frac{2w(K, u)}{w(B, u)} \)), where \( w(K, u) \) is the Euclidean width of \( K \) in the direction \( u \) and the Minkowskian diameter (i.e., the maximum of this Minkowskian width function of \( K \)), respectively.

One can easily see that \( r(\hat{I}_B^H, B) = \frac{1}{R(B, \hat{I}_B^H)} \) and \( R(\hat{I}_B^H, B) = \frac{1}{r(B, \hat{I}_B^H)} \). Also, it is easy to establish that if \( K \) is a centered convex body in \( \mathbb{M}^d \), then \( r(K, B) = \frac{w_B(K)}{2} \) and \( R(K, B) = \frac{D_B(K)}{2} \).

We recall that some sharp bounds for \( r(B, \hat{I}_B^H) \) and \( R(B, \hat{I}_B^H) \) are known (see [8], [9], and also the next section). Below we show the connection between cross-section measures and the outer radius \( R(B, \hat{I}_B^H) \).

Our first theorem refers to cross-section measures of polars of unit balls and outer radii of isoperimetricis for the Holmes-Thompson measure.

**THEOREM 1.** Let \( B \) be the unit ball of \( \mathbb{M}^d \). Then

a) \( R(B, \hat{I}_B^H) \geq 1 \) if and only if \( \min_{u \in S^{d-1}} \frac{\lambda_{d-1}(B^o | u^\perp) \lambda_1(B^o \cap l_u)}{\lambda(B^o)} \leq \frac{2\varepsilon_{d-1}}{\varepsilon_d} \).

b) \( R(B, \hat{I}_B^H) \leq 1 \) if and only if \( \frac{\lambda_{d-1}(B^o | u^\perp) \lambda_1(B^o \cap l_u)}{\lambda(B^o)} \geq \frac{2\varepsilon_{d-1}}{\varepsilon_d} \) for all \( u \in S^{d-1} \).

**Proof.** As we know,

\[
\frac{2}{R(B, \hat{I}_B^H)} = 2r(\hat{I}_B^H, B) = w_B(\hat{I}_B^H).
\]

We can expand \( w_B(\hat{I}_B^H) \) as follows:

\[
w_B(\hat{I}_B^H) = \min_{u \in S^{d-1}} \frac{2w(\hat{I}_B^H, u)}{w(B, u)} = \min_{u \in S^{d-1}} \frac{2h_{\hat{I}_B^H}(u)}{h_B(u)} = \min_{u \in S^{d-1}} \frac{2\varepsilon_d}{\lambda(B^o)} h_{\hat{I}_B^H}(u) \rho_{B^o}(u) = \min_{u \in S^{d-1}} \frac{2\varepsilon_d}{\varepsilon_{d-1}} \frac{h_{\hat{I}_B^H}(u) \rho_{B^o}(u)}{\lambda(B^o)}
\]

\[
= \min_{u \in S^{d-1}} \frac{\varepsilon_d}{\varepsilon_{d-1}} \frac{\lambda_{d-1}(B^o | u^\perp) \lambda_1(B^o \cap l_u)}{\lambda(B^o)}.
\]

Therefore,

\[
\frac{2\varepsilon_{d-1}}{\varepsilon_d} = R(B, \hat{I}_B^H) \min_{u \in S^{d-1}} \frac{\lambda_{d-1}(B^o | u^\perp) \lambda_1(B^o \cap l_u)}{\lambda(B^o)}.
\]

Hence, the results follow. \( \Box \)

From Theorem 1 the following result can be easily deduced:
COROLLARY 2. Let $B$ be the unit ball of $\mathbb{M}^d$. Then
\[
\frac{\lambda_{d-1}(B^o | u^\perp) \lambda_1(B^o \cap l_u)}{\lambda(B^c)} \geq \frac{2 \varepsilon_{d-1}}{\varepsilon_d}
\]
for all $u \in S^{d-1}$ if and only if $B \subseteq \hat{I}^H_B$.

We can also use Theorem 1 to get a characterization of ellipsoids.

THEOREM 3. Let $B$ be the unit ball of $\mathbb{M}^d$. Then
\[
\max_{u \in S^{d-1}} \frac{\lambda_{d-1}(B^o | u^\perp) \lambda_1(B^o \cap l_u)}{\lambda(B^c)} = \frac{2 \varepsilon_{d-1}}{\varepsilon_d}
\]
if and only if $B$ is an ellipsoid.

Proof. As mentioned above,
\[
\frac{2}{r(B, \hat{I}^H_B)} = 2R(\hat{I}^H_B, B) = D_B(\hat{I}^H_B).
\]

Then, from the expansion of $D_B(\hat{I}^H_B)$ similar to $w_B(\hat{I}^H_B)$, one gets
\[
D_B(\hat{I}^H_B) = \max_{u \in S^{d-1}} \varepsilon_{d-1} \cdot \frac{\lambda_{d-1}(B^o | u^\perp) \lambda_1(B^o \cap l_u)}{\lambda(B^c)}.
\]

Thus,
\[
\frac{2 \varepsilon_{d-1}}{\varepsilon_d} = r(B, \hat{I}^H_B) \max_{u \in S^{d-1}} \frac{\lambda_{d-1}(B^o | u^\perp) \lambda_1(B^o \cap l_u)}{\lambda(B^c)}.
\]

It is known that $r(B, \hat{I}^H_B) \leq 1$ with equality if and only if $B$ is an ellipsoid. Hence we have the result. $\square$

It is known that there exists $u \in S^{d-1}$ such that
\[
\frac{\lambda_{d-1}(B | u^\perp) \lambda_1(B \cap l_u)}{\lambda(B)} \geq \frac{2 \varepsilon_{d-1}}{\varepsilon_d},
\]
with equality for all $u \in S^{d-1}$ if and only if $B$ is an ellipsoid (cf. [8]).

Also, some sharp bounds for $r(B, \hat{I}^B_B)$ and $R(B, \hat{I}^B_B)$ are known (see [8], [9], or our next section). Below we discuss the connection between cross-section measures and $r(B, \hat{I}^B_B)$ as well as $R(B, \hat{I}^B_B)$.

THEOREM 4. Let $B$ be the unit ball of $\mathbb{M}^d$. Then

a) $R(B, \hat{I}^B_B) \geq 1$ if and only if
\[
\max_{u \in S^{d-1}} \frac{\lambda_{d-1}(B \cap u^\perp) \lambda_1(B | l_u)}{\lambda(B)} \geq \frac{2 \varepsilon_{d-1}}{\varepsilon_d}.
\]

b) $R(B, \hat{I}^B_B) \leq 1$ if and only if
\[
\frac{\lambda_{d-1}(B \cap u^\perp) \lambda_1(B | l_u)}{\lambda(B)} \leq \frac{2 \varepsilon_{d-1}}{\varepsilon_d}
\]
for all $u \in S^{d-1}$.  

Proof. As we know,
\[
\frac{2}{R(B, \hat{I}_{B}^{\text{Bus}})} = 2r(\hat{I}_{B}^{\text{Bus}}, B) = w_{B}(\hat{I}_{B}^{\text{Bus}}),
\]
and \(w_{B}(\hat{I}_{B}^{\text{Bus}})\) can be expanded as follows:
\[
w_{B}(\hat{I}_{B}^{\text{Bus}}) = \min_{u \in S^{d-1}} \frac{2h_{\hat{I}_{B}^{\text{Bus}}}(u)}{h_{B}(u)} = \min_{u \in S^{d-1}} \frac{2\lambda(B)\varepsilon_{d-1}h_{\hat{I}_{B}^{\text{Bus}}}(u)}{\varepsilon_{d}h_{B}(u)} = \frac{2\varepsilon_{d-1}}{\varepsilon_{d}} \min_{u \in S^{d-1}} \frac{\lambda(B)}{\rho_{B}(u)h_{B}(u)} = \frac{2\varepsilon_{d-1}}{\varepsilon_{d}} \min_{u \in S^{d-1}} \frac{2\lambda(B)}{\lambda_{d-1}(B \cap u^\perp)\lambda_{1}(B|l_{u})}.
\]
Therefore,
\[
\frac{\varepsilon_{d}}{2\varepsilon_{d-1}} = R(B, \hat{I}_{B}^{\text{Bus}}) \min_{u \in S^{d-1}} \frac{\lambda(B)}{\lambda_{d-1}(B \cap u^\perp)\lambda_{1}(B|l_{u})} = R(B, \hat{I}_{B}^{\text{Bus}}) \frac{\lambda(B)}{\max_{u \in S^{d-1}} \lambda_{d-1}(B \cap u^\perp)\lambda_{1}(B|l_{u})},
\]
and hence
\[
\max_{u \in S^{d-1}} \frac{\lambda_{d-1}(B \cap u^\perp)\lambda_{1}(B|l_{u})}{\lambda(B)} = R(B, \hat{I}_{B}^{\text{Bus}}) \frac{2\varepsilon_{d-1}}{\varepsilon_{d}}.
\]
Thus, our results are confirmed. □

The next result can be easily deduced from Theorem 4.

**Corollary 5.** Let \(B\) be the unit ball of \(\mathbb{M}^{d}\). Then
\[
\frac{\lambda_{d-1}(B \cap u^\perp)\lambda_{1}(B|l_{u})}{\lambda(B)} \leq \frac{2\varepsilon_{d-1}}{\varepsilon_{d}} \text{ for all } u \in S^{d-1} \text{ if and only if } B \subseteq \hat{I}_{B}^{\text{Bus}}.
\]

Now we combine inner radii of isoperimetrics for the Busemann measure with cross-section measures.

**Theorem 6.** If \(B\) is the unit ball of \(\mathbb{M}^{d}\), then
\[
a) r(B, \hat{I}_{B}^{\text{Bus}}) \leq 1 \text{ if and only if } \min_{u \in S^{d-1}} \frac{\lambda_{d-1}(B \cap u^\perp)\lambda_{1}(B|l_{u})}{\lambda(B)} \leq \frac{2\varepsilon_{d-1}}{\varepsilon_{d}},
\]
\[
b) r(B, \hat{I}_{B}^{\text{Bus}}) \geq 1 \text{ if and only if } \frac{\lambda_{d-1}(B \cap u^\perp)\lambda_{1}(B|l_{u})}{\lambda(B)} \geq \frac{2\varepsilon_{d-1}}{\varepsilon_{d}} \text{ for all } u \in S^{d-1}.
\]

**Proof.** By expanding \(D_{B}(\hat{I}_{B}^{\text{Bus}})\) (similar to \(w_{B}(\hat{I}_{B}^{\text{Bus}})\)), we obtain
\[
D_{B}(\hat{I}_{B}^{\text{Bus}}) = \frac{2\varepsilon_{d-1}}{\varepsilon_{d}} \max_{u \in S^{d-1}} \frac{2\lambda(B)}{\lambda_{d-1}(B \cap u^\perp)\lambda_{1}(B|l_{u})}.
\]
Since \(D_{B}(\hat{I}_{B}^{\text{Bus}}) = \frac{2}{r(B, \hat{I}_{B}^{\text{Bus}})}\), we get
\[
\frac{\varepsilon_{d}}{2\varepsilon_{d-1}} = \max_{u \in S^{d-1}} \frac{\lambda(B)}{\lambda_{d-1}(B \cap u^\perp)\lambda_{1}(B|l_{u})} r(B, \hat{I}_{B}^{\text{Bus}}).
\]
Hence, the results follow from the following equality:

\[
\min_{u \in S^{d-1}} \frac{\lambda_{d-1}(B \cap u^\perp) \lambda_1(B|l_u)}{\lambda(B)} = r(B, \hat{I}_B^{Bus}) \frac{2\varepsilon_{d-1}}{\varepsilon_d}. \quad \square
\]

**Corollary 7.** Let \(B\) be the unit ball of \(\mathbb{M}^d\). Then

\[
\frac{\lambda_{d-1}(B \cap u^\perp) \lambda_1(B|l_u)}{\lambda(B)} \geq \frac{2\varepsilon_{d-1}}{\varepsilon_d}
\]

for all \(u \in S^{d-1}\) if and only if \(\hat{I}_B^{Bus} \subseteq B\).

The following proposition refers to cross-section measures of centered convex bodies.

**Proposition 8.** Let \(B\) be a centered convex body in \(\mathbb{R}^d\). Then

\[
a) \max_{u \in S^{d-1}} \frac{\lambda_{d-1}(B \cap u^\perp) \lambda_1(B|l_u)}{\lambda(B)} \leq \max_{u \in S^{d-1}} \frac{\lambda_{d-1}(B|u^\perp) \lambda_1(B \cap l_u)}{\lambda(B)},
\]

\[
b) \min_{u \in S^{d-1}} \frac{\lambda_{d-1}(B \cap u^\perp) \lambda_1(B|l_u)}{\lambda(B)} \leq \min_{u \in S^{d-1}} \frac{\lambda_{d-1}(B|u^\perp) \lambda_1(B \cap l_u)}{\lambda(B)}.
\]

**Proof.**

a) In [10] it was proved that \(R(B, \hat{I}_B^{Bus}) \cdot r(B^\circ, \hat{I}_B^{HT}) \leq 1\). Thus, by using the equalities

\[
R(B, \hat{I}_B^{Bus}) = \frac{\varepsilon_d}{2\varepsilon_{d-1}} \max_{u \in S^{d-1}} \frac{\lambda_{d-1}(B \cap u^\perp) \lambda_1(B|l_u)}{\lambda(B)},
\]

\[
r(B^\circ, \hat{I}_B^{HT}) = \frac{2\varepsilon_{d-1}}{\varepsilon_d} \frac{\lambda(B)}{\max_{u \in S^{d-1}} \lambda_{d-1}(B|u^\perp) \lambda_1(B \cap l_u)},
\]

the result follows.

b) In [10] it was also proved that \(R(B^\circ, \hat{I}_B^{HT}) \cdot r(B, \hat{I}_B^{Bus}) \leq 1\). Thus, by

\[
R(B^\circ, \hat{I}_B^{HT}) = \frac{2\varepsilon_{d-1}}{\varepsilon_d} \min_{u \in S^{d-1}} \frac{\lambda(B)}{\lambda_{d-1}(B|u^\perp) \lambda_1(B \cap l_u)}
\]

and

\[
r(B, \hat{I}_B^{Bus}) = \frac{\varepsilon_d}{2\varepsilon_{d-1}} \min_{u \in S^{d-1}} \frac{\lambda_{d-1}(B \cap u^\perp) \lambda_1(B|l_u)}{\lambda(B)},
\]

we have the result. \(\square\)
4. Inclusion results

One of the challenging open problems of Minkowski Geometry is the question whether for \( d \geq 3 \) the homothety of the unit ball and the normalized solution of the isoperimetric problem implies that the unit ball must be an ellipsoid (see [3], [14], and [17]). For Minkowski planes, apart from ellipses, other curves (such as Radon curves) have this property as well. Related to this, we clarify now the inclusions between the unit ball \( B \) and \( \Pi B^\circ \), and between \( B \) and \((IB)^\circ \). We start with the inclusions referring to the projection body.

**Proposition 9.** If \( B \) is a Minkowskian ball of \( \mathbb{M}_d \) with \( \lambda(B) = 1 \), then the following exact inclusions hold:

\[
B^\circ \subseteq 2\Pi B \subseteq dB^\circ.
\]

**Proof.** By the definitions of the inner and outer radii we have

\[
r(B, \hat{I}_B^{\text{HT}}) \hat{I}_B^{\text{HT}} \subseteq B \subseteq R(B, \hat{I}_B^{\text{HT}}) \hat{I}_B^{\text{HT}}.
\]

We recall the following exact bounds for the inner radius and the outer radius:

\[
\frac{2\varepsilon_{d-1}}{d \varepsilon_d} \leq r(B, \hat{I}_B^{\text{HT}}) \leq 1,
\]

\[
R(B, \hat{I}_B^{\text{HT}}) \leq \frac{2 \varepsilon_{d-1}}{\varepsilon_d}.
\]

Therefore, using (1) we get

\[
\frac{2 \varepsilon_{d-1}}{d \varepsilon_d} \frac{\varepsilon_d}{\lambda(B^\circ) \varepsilon_{d-1}} \Pi B^\circ \subseteq B \subseteq \frac{2 \varepsilon_{d-1}}{\varepsilon_d} \frac{\varepsilon_d}{\lambda(B^\circ) \varepsilon_{d-1}} \Pi B^\circ.
\]

Thus

\[
\frac{2}{d} \Pi B^\circ \subseteq \lambda(B^\circ) B \subseteq 2\Pi B^\circ.
\]

Hence, the results follow by setting \( B \) to be \( B^\circ \), and \( \lambda(B) = 1 \). \( \square \)

Analogously, we obtain for the intersection body

**Proposition 10.** Let \( B \) be a Minkowskian ball of \( \mathbb{M}_d \) with \( \lambda(B) = 1 \). Then the following exact inclusions hold:

\[
B^\circ \subseteq 2IB \subseteq dB^\circ.
\]

**Proof.** Again we have

\[
r(B, \hat{I}_B^{\text{Bus}}) \hat{I}_B^{\text{Bus}} \subseteq B \subseteq R(B, \hat{I}_B^{\text{Bus}}) \hat{I}_B^{\text{Bus}}.
\]

From the exact bounds

\[
\frac{\varepsilon_d}{2 \varepsilon_{d-1}} \leq r(B, \hat{I}_B^{\text{Bus}}), \quad R(B, \hat{I}_B^{\text{Bus}}) \leq \frac{d \varepsilon_d}{2 \varepsilon_{d-1}}
\]

and (2) we have

\[
\frac{\varepsilon_d}{2 \varepsilon_{d-1}} \frac{\lambda(B)}{\varepsilon_d} \varepsilon_{d-1} (IB)^\circ \subseteq B \subseteq \frac{d \varepsilon_d}{2 \varepsilon_{d-1}} \frac{\lambda(B)}{\varepsilon_d} \varepsilon_{d-1} (IB)^\circ.
\]
Thus 

\[
\frac{\lambda(B)}{2} (IB)^\circ \subseteq B \subseteq \frac{d\lambda(B)}{2} (IB)^\circ.
\]

Setting \( \lambda(B) = 1 \), we have 

\[
\frac{2}{d} IB \subseteq B^\circ \subseteq 2IB,
\]

and the results follow. \( \square \)

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