Hawking–Hayward quasi-local energy under conformal transformations

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Abstract
We derive a formula describing the transformation of the Hawking–Hayward quasi-local energy under a conformal rescaling of the spacetime metric. A known formula for the transformation of the Misner–Sharp–Hernandez mass is recovered as a special case.

Keywords: quasi-local energy, conformal transformations, black holes

1. Introduction

Historically, the notion of a physical energy contained in a compact three-dimensional spacetime region (quasi-local energy) has proved to be rather complicated to identify and various attempts have been made to define a physically meaningful quantity for this concept in general relativity (see [1] for a review). In the presence of spherical symmetry, however (in general, without asymptotic flatness) the physical mass-energy is commonly identified with the Misner–Sharp–Hernandez (MSH) construct [2], the task becoming much more involved when the spacetime is not spherically symmetric. In the general case the Hawking–Hayward quasi-local energy [3, 4] is an appropriate and commonly used concept to define the mass contained in a compact region of spacetime, although there is no consensus and there are a number of different proposals for a general definition.

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Conformal mappings of spacetimes are widely used in cosmology and black hole physics (e.g., [5] and references therein), as well as in Penrose–Carter diagrams bringing spatial infinity to a finite distance in asymptotically flat spacetimes (e.g., [6]), not to mention their utility in alternative theories of gravity (e.g., [7]).

Because the Hawking–Hayward quasi-local energy is defined only within the context of general relativity, here we restrict ourselves to this theory. Conformal transformations are heavily used in scalar-tensor, $f(R)$, and other alternative theories of gravity and in the future we will explore the generalization of the Hawking–Hayward construct to these theories. For the moment, however, we begin with the simplest case of Einstein theory.

Generally speaking, if one begins with a well-motivated solution of the Einstein equations corresponding to reasonable matter satisfying the energy conditions, a conformal transformation will generate a new solution of the same equations corresponding to unphysical matter. However, exceptions exist: for example, all spatially homogeneous and isotropic cosmologies are conformal to Minkowski space. Another physically relevant one is that of large-scale structure formation which is often studied in the conformal Newtonian gauge. We will discuss this and several other examples in section 5 below.

From a completely different perspective, the behavior of quasi-local energy under conformal transformations can be used to understand how the various proposed quasi-local masses embed into a space–time thermodynamic relation such as the first law. The Hawking–Hayward energy has traditionally been the variable associated with the thermodynamic energy in general space–times [13] and singling out an appropriate notion of a mass/energy variable which is compatible with a relation such as the first law can be greatly simplified by considering the behavior of the various components that make up the first law transform under specific transformations, including conformal transformations.

The conformal transformation properties of other quasi-local energy concepts have been discussed in the literature: see [8] for the Arnowitt–Deser–Misner mass and [9] for the Brown–York quasi-local energy. However, in spite of the importance of the Hawking–Hayward quasi-local energy, a transformation formula for this quantity under conformal spacetime mappings is not available in the literature except for the special case of spherical symmetry, in which the Hawking–Hayward energy reduces to the MSH mass [4, 10], for which the transformation was derived recently [11]. In the spherically symmetric case it has been shown that the MSH mass simply relates to the areal radius independently of the conformal factor [12]. Indeed, it is not clear a priori that the quasi-local energy must change under a change of conformal factor in the metric: a generalization of the Brown–York quasi-local energy does not change under conformal transformations given certain conditions [9].

The present paper fills a gap in the literature. In order to compute the change of the Hawking–Hayward quasi-local energy under conformal rescalings, it is necessary to first analyze how the various quantities appearing in the definition of the Hawking–Hayward energy transform under conformal mappings, which is done in the next section. The following section derives the desired transformation property (equation (35), which is the main result of this paper). We then check that, in the special case of spherical symmetry, this formula reproduces the known one for the transformation of the MSH mass.

2. Hawking–Hayward quasi-local energy

The Hawking–Hayward energy $M_{HH}$ [3, 4] is a functional of the spacetime metric $g_{ab}$ and of an embedded (spacelike, compact, and orientable) two-surface $S$ defined by
\[ M_{\text{BH}} := \frac{1}{8\pi} \sqrt{\frac{\mathcal{A}}{16\pi}} \int_S \mu \left( \mathcal{R} + \theta_+ \theta_- - \frac{1}{2} \sigma^a \sigma_a - 2\omega^a \omega_a \right), \]  

where \( \mathcal{R} \) is the induced Ricci scalar on \( S \), \( \theta_\pm \) are the expansion and shear tensors of a pair of null geodesic congruences (outgoing and ingoing from the surface \( S \)), \( \omega^a \) is the projection onto \( S \) of the commutator of the null normal vectors to \( S \), \( \mu \) is the volume two-form on the surface \( S \), and \( A \) is the area of \( S \).

A co-dimension 2 surface \( S \) in a four-dimensional spacetime is defined as the set of points where two independent functions take constant values. Let us call these functions \( \phi^{(1)} \) and \( \phi^{(2)} \). Then

\[ S := \{ x \mid \phi^{(1)}(x) = \phi^{(1)}_0 \} \cap \{ x \mid \phi^{(2)}(x) = \phi^{(2)}_0 \}. \]  

Different two-dimensional surfaces can be defined by choosing different constants \( \phi^{(i)}_0 \), indeed entire two-parameter foliations of the four-dimensional manifold can be defined in this way for certain choices of the functions \( \phi^{(i)} \) in special circumstances.

Even before invoking a metric, we can associate two one-forms with the surface \( S \) by

\[ l_a := \partial_a \phi^{(1)}, \quad n_a := \partial_a \phi^{(2)}, \]

whose restriction to the set \( S \) defines the ‘normal direction’ one-forms—since there is no metric yet we do not have a notion of orthogonality and we do not have a canonical way to define ‘normal vectors’ by raising the indices on \( l_a \) and \( n_a \). These one-forms clearly satisfy

\[ \nabla_{[a} l_{b]} = 0 = \nabla_{[a} n_{b]} \]

which are the conditions for the one-forms \( l \) and \( n \) to be closed, \( df = 0 = dn \). This implies the metric-independent condition of hypersurface orthogonality on both \( l \) and \( n \)

\[ n \wedge dn = 0 = l \wedge dl. \]

The condition of hypersurface orthogonality for a one-form is more general than the closure condition: one-forms proportional to the differential of a function, \( fdg \) (where \( g \) and \( f \) are functions), are in general not closed but are hypersurface orthogonal

\[ (fdg) \wedge dfdg = fdg \wedge fdg = 0. \]

That is, we are free to scale the one-forms \( l \) and \( n \) by arbitrary functions without spoiling their hypersurface orthogonality.

We note that there is also considerable freedom in the functions \( \phi^{(i)} \) defining \( S \) since any other set of two (sufficiently nice) independent functions \( \phi^{(i)}(\phi^{(1)}, \phi^{(2)}) \) will define the same set \( S \).

Introducing now a metric \( g_{ab} \), we impose that both \( l \) and \( n \) be null \( g^{ab} l_a n_b = 0 = g^{ab} n_a n_b \) and from them form the scaled one-forms

\[ l_a := \frac{l_a}{\sqrt{-l \cdot n}}, \quad n_a := \frac{n_a}{\sqrt{-l \cdot n}}, \]

where \( l \cdot n \equiv g^{ab} l_a n_b \). In general, null normals do not have a canonical scaling in contrast to the timelike and spacelike cases where we can normalize to \( \mp 1 \). Our particular choice of scaling for the null normals implies that

\[ l^a n_a = -1 \]

so that the induced metric on the surface \( S \) is written as

\[ h_{ab} = g_{ab} + l_a n_b + l_b n_a. \]
We note also that in the null case there is a restricted freedom on the functions $\phi^{(i)}$; only unmixed reparametrizations $\psi^{(i)}(\phi^{(j)})$ leave the null condition on the normals intact.

One often finds in the literature the double-null construction based on the vectors $L^a$ and $N^a$ defined by

$$L^a := \frac{l^a}{l \cdot n}, \quad N^a := \frac{n^a}{l \cdot n}$$

which have the property of Lie-dragging the three-surfaces of constant $\phi^{(i)}$:

$$L(\phi^{(2)}) = 1 = N(\phi^{(1)}).$$

Despite appearances, this property is not sufficient to define the basis vectors $\partial/\partial \phi^{(i)}$ and would only do so in the case that the commutator of $L$ and $N$ is orthogonal to the surface $S$, that is $h_{ab}[L, N]^p = 0$, a condition known as ‘surface-forming’. In general, the partial derivative $\partial/\partial \phi^{(i)}$ contains also a component tangent to $S$:

$$\partial/\partial \phi^{(2)} = L + r, \quad \text{where} \quad h_{ab} r^a = 0$$

and similarly for $N$. Such considerations play a role in the $2 + 2$ formulation of general relativity (see e.g., [14]).

Note that while $l$ (and $n$) is tangent to an affinely parametrized geodesic

$$l^a \nabla_a l_b = l^a \nabla_a l_c = \frac{1}{2} \nabla_b (l_a l^a) = 0,$$

$I^a$ (or $\pi^a$) is not, but is still tangent to a non-affinely parametrized geodesic. Defining $e^a := \sqrt{-l \cdot n}$ we have

$$I^a \nabla_a I^b = e^{-m} l^a \nabla_a (e^{-m} l^b)$$

$$= e^{-m} l^a \nabla_a e^{-m} + e^{-2m} l^a \nabla_a l^b$$

$$= e^{-2m} l^a \nabla_a l^b$$

$$= (-I^a \nabla_a m) I^b$$

$$= (-\mathcal{L}_l m) I^b,$$

where $\mathcal{L}$ is the Lie derivative. A similar result holds for $\pi^a$.

The quantities appearing in the definition of the Hawking–Hayward energy are defined in terms of Lie derivatives in the directions $l$ and $n$. The expansions $\theta_\pm$ and shears $\sigma_{\pm}^{ab}$ are defined by

$$\theta_+ := \frac{1}{2} h_{ab} \mathcal{L}_l h_{ab},$$

$$\theta_- := \frac{1}{2} h_{ab} \mathcal{L}_n h_{ab},$$

$$\sigma_{ab}^{\pm} := h_{c}^{\pm} h_{d}^{\pm} \mathcal{L}_{l} h_{cd} - h_{ab} \theta_{\pm},$$

$$\sigma_{ab} := h_{c}^{+} h_{d}^{+} \mathcal{L}_{n} h_{cd} - h_{ab} \theta_{+-},$$

while the anholonomicity $\omega_\pm$ is defined by the commutator of $l^a$ and $n^a$ (or $I^a$ and $\pi^a$):

$$\omega_{\pm} := \frac{1}{2} \frac{1}{l \cdot n} h_{ab} [l, n]^b = \frac{1}{2} \frac{1}{I \cdot \pi} h_{ab} [l, \pi]^b,$$

where $[,]$ is the Lie bracket. Note that the anholonomicity is invariant under rescaling of the normals. These definitions are textbook-standard except for the anholonomicity $\omega_\pm$ which
deserves special mention. Usually when one is dealing with a single null congruence \( v^a \), one introduces an auxiliary null vector \( u^a \) such that \( v^a u_a = -1 \) and defines a two-form \( \omega_{ab} \) called the twist form by \( h^a_i h^b_j \nabla_i v_j \). This object is important because it vanishes if and only if \( v^a \) is hypersurface-orthogonal (it is proportional to the gradient of a scalar). This object is not independent of the auxiliary vector [15] but its square \( \omega_{ab} \omega^{ab} \) is. One can show that if \( \omega_{ab} = 0 \) for some choice of \( u^a \), then it is zero for all choices of \( u^a \), so the vanishing of \( \omega_{ab} \) is a \( u^a \)-independent condition [16] and hence hypersurface orthogonality does not depend on the choice of auxiliary \( u^a \). What we have here is somewhat different. We are dealing here with a single two-surface which is the intersection of two null three-surfaces. The vectors \( l^a \) and \( n^a \) certainly are hypersurface-orthogonal individually to these null three-surfaces and hence both possess vanishing \( \omega_{ab} \), but this is not what the anholonomicity \( \omega_{ab} \) is measuring (otherwise it would trivially vanish in this \( 2+2 \) formulation). The anholonomicity describes how the two \( l^a \) and \( n^a \) vectors ‘weave together’ (integrate to) a two-dimensional hypersurface, whether they are really tangent to a genuine 2D submanifold. Now this ‘integral’ sub-manifold, if it exists, is not \( S \), but the orthogonal complement \( S^\perp \) since the \( l^a \) and \( n^a \) vectors are orthogonal to \( S \).

3. Transformation of the Hawking–Hayward energy under conformal mappings

We now consider the relationship between \( M_{HH} \) defined with respect to two different metrics \( g_{ab} \) and \( \tilde{g}_{ab} \) which are related by a conformal factor

\[
\tilde{g}_{ab} = \Omega^2 g_{ab}. \tag{21}
\]

Of course, since the surface \( S \) is defined independently of the metric, we are considering how the energy content contained within a single surface changes when one scales the metric as above.

In the expression (1) (with respect to the metric \( \tilde{g} \)) of the Hawking–Hayward energy of a two-surface \( S \) in the conformally scaled spacetime, the area of this surface is

\[
\tilde{A} = \int_S \tilde{h} = \int_S d^2x \sqrt{\tilde{h}} = \int_S d^2x \sqrt{\Omega^2 \tilde{h}} = \int_S \mu \Omega^2. \tag{22}
\]

Unless \( \Omega \) is constant on \( S \), one cannot extract it from the integral and we have

\[
\sqrt{\tilde{A}} = \sqrt{\int_S \mu \Omega^2 \over \int_S} \sqrt{A}. \tag{23}
\]

Only when \( \Omega \) is constant on \( S \) can we write \( \sqrt{\tilde{A}} = \Omega \sqrt{A} \). This expression will be used in the integral defining the quasi-local energy.

We will make use in what follows of the contracted Gauss equation [4]

\[
\nabla \cdot \theta_+ - \frac{1}{2} \sigma_{ab} \sigma^{ab} = h^{ac} h^{bd} R_{abcd} \tag{24}
\]

which provides a very useful expression of a part of the Hawking–Hayward energy in terms of simpler quantities. In the conformally rescaled spacetime we have

\[
\nabla \cdot \theta_+ - \frac{1}{2} \tilde{\sigma}_{ab} \tilde{\sigma}^{ab} = \tilde{h}^{ac} \tilde{h}^{bd} \tilde{R}_{abcd}. \tag{25}
\]

Using the fact that \( \tilde{h}^{ab} = \Omega^{-2} h^{ab} \) and the well known transformation property of the Riemann tensor under conformal transformations (e.g., [6], p 466)
\[ \tilde{R}_{abc}^{\ d} = R_{abc}^{\ d} + 2 \delta_{bc}^{\ d} \nabla_{[a} \nabla_{b]} \ln \Omega - 2 g^{de} g_{c[a} \nabla_{b]} \nabla_{e} \ln \Omega + 2 \nabla_a \ln \Omega \delta_{bc}^{\ d} \nabla_e \ln \Omega - 2 \nabla_a \ln \Omega g_{b[c} \delta_{d]}^{\ e} \nabla_e \ln \Omega \]  

(26)

and, lowering one index

\[ \tilde{R}_{abcd} = \delta_{ds} R_{abcd}^{\ s} = \Omega^2 \delta_{ds} \tilde{R}_{abcd}^{\ s} \]  

(27)

it follows that

\[ \tilde{R}^{ac} \tilde{R}^{bd} \tilde{R}_{abcd} = \Omega^{-2} \left( \mathcal{R} + \theta_a \theta^- - \frac{1}{2} \sigma_{ab}^{\ +} \sigma_{ab}^{\ -} \right) + 2 \Omega^{-2} \left[ h^{ac} h^{bd} g_{c[a} \nabla_{b]} \nabla_c \ln \Omega - h^{ac} h^{bd} g_{c[a} \delta_{b]}^{\ e} \nabla_e \ln \Omega \right] \]

(28)

where \( h_a^b \) is a two-dimensional Kronecker delta. By computing the terms appearing on the right-hand side of this equation, one obtains

\[ h^{ac} h^{bd} g_{c[a} \nabla_{b]} \nabla_c \ln \Omega - h^{ac} h^{bd} g_{c[a} \delta_{b]}^{\ e} \nabla_e \ln \Omega = -h^{ab} \nabla_a \nabla_b \ln \Omega, \]

(29)

\[ h^{ac} h^{bd} g_{c[a} \nabla_{b]} \ln \Omega \delta_{b]}^{\ e} \nabla_c \ln \Omega - h^{ac} h^{bd} g_{c[a} \nabla_{b]} \nabla_c \ln \Omega = h^{ab} \nabla_a \ln \Omega \nabla_b \ln \Omega, \]

(30)

\[ h^{ac} h^{bd} g_{c[a} \delta_{b]}^{\ e} \nabla_e \ln \Omega \nabla_f \ln \Omega = \frac{\nabla^c \Omega \nabla^c \Omega}{\Omega^2}. \]

(31)

Putting everything together gives

\[ \tilde{R} + \tilde{\theta}_a \tilde{\theta}^- - \frac{1}{2} \sigma_{ab}^{\ +} \sigma_{ab}^{\ -} \Omega^{-2} \left( \mathcal{R} + \theta_a \theta^- - \frac{1}{2} \sigma_{ab}^{\ +} \sigma_{ab}^{\ -} \right) + 2 \Omega^{-2} h^{ab} \] 

\[ \left. \left[ \nabla_a (\nabla_b \ln \Omega \nabla_c \ln \Omega - \nabla_a \nabla_b \ln \Omega) - \nabla_a \nabla_b \ln \Omega \right] \right] \]

(32)

Using now the fact (which we will prove below) that

\[ \tilde{\omega}_a \tilde{\omega}^a = \Omega^{-2} \omega_a^2, \]

(33)

the integrand in the Hawking–Hayward quasi-local energy is seen to transform as

\[ \tilde{R} + \tilde{\theta}_a \tilde{\theta}^- - \frac{1}{2} \sigma_{ab}^{\ +} \sigma_{ab}^{\ -} - 2 \tilde{\omega}_a \tilde{\omega}^a = \Omega^{-2} \left( \mathcal{R} + \theta_a \theta^- - \frac{1}{2} \sigma_{ab}^{\ +} \sigma_{ab}^{\ -} - 2 \omega_a^2 \right) \]

\[ + 2 \Omega^{-2} \left[ h^{ab} \left( \frac{2 \nabla_a \Omega \nabla_b \Omega \Omega}{\Omega^2} - \frac{\nabla_a \nabla_b \Omega}{\Omega} \right) \right. \]

\[ \left. - \frac{\nabla^c \Omega \nabla^c \Omega}{\Omega^2} \right] \]

(34)

The Hawking–Hayward quasi-local energy itself then transforms according to

\[ \tilde{M}_{\text{HH}} = \frac{1}{8 \pi} \sqrt{\frac{A}{16 \pi}} \int_S \left( \mathcal{R} + \theta_a \theta^- - \frac{1}{2} \sigma_{ab}^{\ +} \sigma_{ab}^{\ -} - 2 \omega_a^2 \right) \]

\[ + 1 \left( \sqrt{\frac{A}{4 \pi}} \int_S \mu \right) h^{ab} \left( \frac{2 \nabla_a \Omega \nabla_b \Omega \Omega}{\Omega^2} - \frac{\nabla_a \nabla_b \Omega}{\Omega} \right) - \frac{\nabla^c \Omega \nabla^c \Omega}{\Omega^2} \]

\[ \sqrt{\frac{A}{A}} M_{\text{HH}} + \frac{1}{4 \pi} \sqrt{\frac{A}{16 \pi}} \int_S \mu \left[ h^{ab} \left( \frac{2 \nabla_a \Omega \nabla_b \Omega \Omega}{\Omega^2} - \frac{\nabla_a \nabla_b \Omega}{\Omega} \right) - \frac{\nabla^c \Omega \nabla^c \Omega}{\Omega^2} \right]. \]

(35)
One can also arrive at this final result (35) by transforming directly the ingredients from which \( M_{\text{HH}} \) is constructed. The one-forms \( \tilde{I}_a \) and \( n_a \) in equations (3) and (4) are defined independently of the metric and therefore are unchanged under conformal transformations

\[
\tilde{I}_a = I_a, \quad \tilde{n}_a = n_a.
\]

Hence

\[
\tilde{I}_a = \frac{\tilde{I}_a}{\sqrt{-g^{bc} I_b n_c}} = \frac{I_a}{\sqrt{-\Omega^{-2} g^{bc} I_b n_c}} = \Omega \frac{I_a}{\sqrt{-g^{bc} I_b n_c}}
\]

\[
\tilde{I}_a = \Omega \tilde{I}_a
\]

and, similarly, \( \tilde{n}_a = \Omega \tilde{n}_a \). We also have

\[
\tilde{I}^a = \Omega^{-1} I^a, \quad \tilde{n}^a = \Omega^{-1} n^a.
\]

Note that in the conformal spacetime we still have the condition

\[
\tilde{\tilde{\sigma}}_{bc} = -1
\]

so that the metric is still expressed as

\[
\tilde{h}_{ab} = g_{ab} + \tilde{I}_a \tilde{n}_b + \tilde{n}_a \tilde{I}_b.
\]

As an aside, and for ease of computation, it is easy to show that

\[
\partial_a \theta_b - \frac{1}{2} \sigma_{ab}^{\alpha} \sigma^{\alpha} = \frac{1}{2} [h^{ab} h^{AB} - h^{ab} h^{bA}] (\mathcal{L}_\gamma h_{ab}) (\mathcal{L}_\pi h_{AB})
\]

whence it follows that

\[
\frac{\partial_a \theta_b - \frac{1}{2} \sigma_{ab}^{\alpha} \sigma^{\alpha}}{\Omega^{-2}} = \frac{1}{2} \left( \frac{\partial_a \theta_b - \frac{1}{2} \sigma_{ab}^{\alpha} \sigma^{\alpha}}{\Omega^{-2}} \right) + \Omega^{-2} \left( \mathring{\nabla} \mathring{\nabla} \Omega^2 \right) (\mathring{\sigma} \mathring{\nabla} \mathring{\nabla} \mathring{\nabla} \Omega^2)
\]

\[
+ \frac{2}{\Omega^3} \theta (\mathring{\sigma} \mathring{\nabla} \mathring{\nabla} \Omega) + \frac{2}{\Omega^3} \theta (\mathring{\sigma} \mathring{\nabla} \mathring{\nabla} \Omega).
\]

The Ricci scalar of the two-surface \( S \) transforms under the conformal transformation as

\[
\tilde{R} = \Omega^{-2} R - \frac{2}{\Omega^3} h^{ab} D_a D_b \Omega + \frac{2}{\Omega} h^{ab} (D_a \Omega) (D_b \Omega),
\]

where \( D_a \) is the covariant derivative on \( S \) which is defined with respect to the metric \( h_{ab} \) and whose action is related to the four-dimensional \( \nabla_a \) by, using the example of a (1, 1)-tensor

\[
D_a X^b = h_d^a h^c_d \nabla_c X^b.
\]

We can re-express everything in terms of the 4D covariant derivative as follows

\[
D_a \Omega = h_d^a \nabla_d \Omega,
\]

\[
D_a D_b \Omega = h_d^a h^c_d \nabla_c D_b \Omega
= h_d^a h^c_d \nabla_c (h_f^d \nabla_f \Omega)
\]

\[
= h_d^a h^c_d \nabla_c (h^f_d \nabla_f \Omega)
\]
so that
\[
\begin{align*}
W = \nabla^a \nabla_a \Omega & = \nabla^a \left[ \nabla_a \Omega + \left( \nabla_b \Omega \right) \nabla^b \tilde{D}_a \right. \\
& + \left. \left( \nabla^b \Omega \right) \nabla_b \pi_a \right]
\end{align*}
\]
In the last line we have made use of the equivalent definition for the expansion $\theta_{\pi} = h^{ab} \nabla_a \pi_b$ and similarly for $\theta_{\pi}$. We are then left with
\[
\hat{\mathcal{R}} = \frac{\mathcal{R}}{\Omega^2} - \frac{2}{\Omega^2} h^{ab} \nabla_a \nabla_b \Omega + \frac{2}{\Omega^2} h^{ab} \left( \nabla_b \Omega \right) \left( \nabla_a \Omega \right) - \frac{2}{\Omega^2} \left( \nabla^a \Omega \right) \theta_{\pi} - \frac{2}{\Omega^2} \left( \nabla^a \Omega \right) \theta_{\pi}.
\]
Note the cancellation of the cross terms in $\theta_{\pi}$ and $\theta_{\pi}$ when combining equations (42) and (48).

We also have
\[
\omega_a = \frac{1}{2l \cdot n} \hat{h}_{ab} [\hat{\tilde{l}}, \hat{n}]^a = \frac{1}{2 \Omega^{-2} l \cdot n} \Omega^2 \hat{h}_{ab} [\Omega^{-2} \tilde{l}, \Omega^{-2} \pi]^a = \frac{1}{2 l \cdot n} \hat{h}_{ab} [l, n]^a = \omega_a,
\]
where we have used $\hat{l}^a = \hat{g}^{ab} l_b = \Omega^{-2} g^{ab} l_b$. Combining these terms we arrive at the result (35) obtained above.

4. Spherical symmetry

As a check of the formulae derived above, it is useful to discuss spherically symmetric situations. In spherical symmetry the Hawking–Hayward quasi-local energy reduces [4, 10] to the MSH mass [2]. The transformation property of the MSH mass under conformal mappings was recently reported in [11] and applied to physical examples such as Friedmann–Lemaître–Robertson–Walker space and the Sultana–Dyer cosmological black hole. A spherically symmetric metric can be put in the form
\[
\begin{align*}
\text{d}s^2 = -A(t, R) \text{d}t^2 + B(t, R) \text{d}R^2 + R^2 \text{d}t_{(2)},
\end{align*}
\]
where $\text{d}t_{(2)} = \text{d}\theta^2 + \sin^2 \theta \text{d}\varphi^2$ is the line element on the unit two-sphere and $R$ is the areal radius. Under a conformal transformation with conformal factor $\Omega = \Omega(t, R)$ (to preserve spherical symmetry), this line element becomes $\text{d}s^2 = \Omega^2 \text{d}x^2$ and the areal radius is $\hat{R} = \Omega R$. The MSH mass in the rescaled spacetime is simply [11]
\[
\begin{align*}
\tilde{M}_{\text{MSH}} = \Omega M_{\text{MSH}} - \frac{R^3}{2\Omega} \nabla^c \Omega \nabla_c \Omega - R^2 \nabla^c \Omega \nabla_c R.
\end{align*}
\]
In spherical symmetry, a sphere $S$ of constant radius $R$ has area $A = 4\pi R^2$, $\sqrt{A} = \Omega \sqrt{A}$, and $h^{ab} = \text{diag} \left( R^{-2}, R^{-2}, \sin^{-2} \theta \right)$. In order to check whether equation (35) reproduces equation (51) correctly, one needs to compute the various terms in the integrand of equation (35). The integrand is constant over a sphere of constant radius and the integral reduces to the product of the integrand and the area $4\pi R^2$ of this sphere. We have
\[ \Omega = \frac{A}{8\pi} \sqrt{\frac{1}{16\pi}} \int_S \mu \left( R + \theta_\mu \theta \right) - \frac{1}{2} \sigma^{ab}_{\mu} \sigma^{ab} - 2\omega_a \omega^a = \Omega M_{\text{BH}} \] (52)

and

\[ \Omega = \frac{A}{8\pi} \sqrt{\frac{1}{16\pi}} \int_S \mu \left( -2\nabla^c \nabla^c - \frac{\Omega^2}{\Omega} \right) = - \frac{R^3}{2\Omega} \nabla^c \Omega \nabla^c \Omega. \] (53)

Since \( \Omega = \Omega(t, R) \) we have \( \nabla_u \Omega = \dot{\Omega} \delta_{u0} + \Omega' \delta_{u1} \), where a dot and a prime denote differentiation with respect to \( t \) and \( R \), respectively. Then it is \( h^{ab} \nabla_a \Omega \nabla_b \Omega = 0 \) and we are left only with the quantity \( -h^{ab} \nabla_a \nabla_b \Omega \) to compute. We have

\[
\begin{align*}
  h^{ab} \nabla_a \nabla_b \Omega &= h^{ab} \partial_b \Omega - h^{ab} \Gamma^c_{ab} \partial_c \Omega \\
  &= h^{ab} \partial_b \Omega + h^{ab} \partial_a \Omega - (h^{ab} \Gamma^c_{ab} \partial_c \Omega) \\
  &= -\frac{1}{R^2} \left( \Gamma^{i}_{22} \Omega + \Gamma^{i}_{22} \Omega' \right) - \frac{1}{R^2 \sin^2 \theta} \left( \Gamma^{0}_{33} \Omega + \Gamma^{1}_{22} \Omega' \right). \quad (54)
\end{align*}
\]

Using the line element (50), the Christoffel symbols are easily computed:

\[
\Gamma^{0}_{22} = \Gamma^{0}_{33} = 0, \\
\Gamma^{1}_{22} = -\frac{R}{B}, \quad \Gamma^{1}_{33} = -\frac{R \sin^2 \theta}{B},
\]

which yields

\[
\begin{align*}
  h^{ab} \nabla_a \nabla_b \Omega &= \frac{2\Omega'}{BR} \\
\end{align*}
\] (57)

and

\[
\begin{align*}
  \frac{R}{8\pi} \int_S \mu h^{ab} \left( \nabla_a \nabla_b \Omega \right) &= -\frac{R}{8\pi} \left( 4\pi R^2 \right) h^{ab} \nabla_a \nabla_b \Omega \\
  &= -\frac{R^3 \Omega'}{B}. \quad (58)
\end{align*}
\]

This is the third term on the right-hand side of equation (51); in fact

\[
- R^2 \nabla^c \Omega \nabla_c R = - R^2 g^{ab} \nabla_a \Omega \delta_{b1} = - \frac{R^2}{B} \Omega'
\]

and the transformation property (51) of the MSH mass is indeed a special case of equation (35).

5. Examples

Here we report three examples of applications of the transformation property of the Hawking–Hayward quasilocal energy under conformal transformations. The first and third examples involve spherically symmetric geometries and appeared already in [11], while the second example has many potential applications in cosmology. Yet another application was described in [17].

New applications of the Hawking–Hayward quasilocal energy are currently being studied. One example is the characterization of the turnaround radius of cosmic structures in an
accelerated Universe\cite{17}, which has recently been claimed to violate the bound set by general relativity in astronomical observations of Galaxy groups\cite{19}. Therefore, it seems important to extend the turnaround radius to alternative gravity. Thus far, this goal has been achieved only for scalar-tensor gravity\cite{20}. Characterizing the turnaround radius with a quasilocal mass will require applying to the turnaround radius the extension of the quasilocal mass to such theories (derived in\cite{21}). This will be the subject of a future publication. The Hawking–Hayward quasilocal mass has also been used to characterize the Paczynski–Wiita pseudo-Newtonian potential describing the motion of particles around a static black hole\cite{22}. Because several black hole solutions of interest in the literature are conformal to a Schwarzschild black hole, the conformal transformation properties of the quasilocal mass will be useful also in this context.

5.1. FLRW space

As a first, and rather trivial, example, consider the spatially flat FLRW line element written using conformal time\cite{23}

\[
d\tilde{s}^2 = a^2(\eta)(-d\eta^2 + dr^2 + r^2d\Omega_3^2) = \Omega^2 ds^2,
\]

which is explicitly conformally flat and spherically symmetric about every point of three-space. The conformal factor $\Omega = a(\eta)$ preserves spherical symmetry. A sphere of areal radius $r$ in Minkowski space has zero Hawking–Hayward mass. Using $a_{\eta} = a\, da/d\tau \equiv a\dot{a}$ and the areal radius $\tilde{R} = a\, R$ of FLRW space, the mass of the corresponding sphere (expanding with the comoving fluid) in FLRW space is

\[
\tilde{M}_{\text{HH}}(\tilde{R}) = aM_{\text{HH}} - \frac{r^3}{2a}(\dot{a}^2) - r^2\nabla^\nu a(\eta)\nabla_\nu r
\]

\[
= \frac{r^3}{2a} \dot{a}^2
\]

\[
= \frac{4}{3}\pi \tilde{R}^3 \rho,
\]

a well known expression, where $\rho$ is the energy density of matter.

5.2. Scalar perturbations of FLRW space

The study of cosmological perturbations is a major tool of modern cosmology which can be tested directly using temperature fluctuations of the cosmic microwave background and large scale structure surveys. To first order in the metric perturbations, one can usually restrict to scalar perturbations. The perturbed metric can be expressed in many gauges; a particularly convenient one in many applications is the conformal Newtonian gauge, in which the line element is written as

\[
d\tilde{s}^2 = a^2(\eta)[-(1 + 2\phi) d\eta^2 + (1 - 2\psi)][dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)]
\]

where $\eta$ and $a(\eta)$ are the conformal time and the scale factor of the background Universe and $\phi$ and $\psi$ are general (not necessarily spherically symmetric) functions of all the coordinates. The fact that the physical spacetime metric is conformal to a post-Newtonian metric (with conformal factor $a(\eta)$), for which calculations (including the Hawking–Hayward mass) are easily performed, is exploited in studies of large-scale structures in\cite{17} and will be used in future work\cite{18}. In this way the effects of the cosmological expansion are entirely described.
by a conformal transformation and the mapping of the quasi-local mass under this transformation will yield the effect of an expanding spacetime on the quasi-local mass.

The quasilocal mass of the post-Newtonian metric is

\[
M_{\text{HH}} = \frac{1}{8\pi} \sqrt{\frac{A}{16\pi}} \int_S \mu \left( \frac{2 \cos \theta \partial \phi_N}{r^2 \sin \theta} + \frac{4 \partial \phi_N}{r \partial r} + \frac{2 \partial^2 \phi_N}{r^2 \partial \theta^2} + \frac{2}{r^2 \sin^2 \theta} \partial^2 \phi_N \right),
\]

from which the transformation property of the quasi-local mass under the metric rescaling \( g_{ab} \) (Post-Newtonian) \( \rightarrow \) \( g_{ab} \) (post-FLRW) \( = a^2(\eta) g_{ab} \) (Post-Newtonian) is [17]

\[
\tilde{M}_{\text{HH}} = \Omega M_{\text{HH}} + \frac{R\Omega_\delta}{4\pi} \left( \int_S \mu \phi_N \eta - \frac{\Omega_\delta}{\Omega} \int_S \mu \phi_N \right) + \frac{R^3 \Omega^2_\eta}{2 \Omega^2}.
\]

This expression can be split into local and cosmological contributions, which allows one to discuss the problem of relativistic effects in the formation of large-scale structures [17] usually discussed with Newtonian physics and in other problems in cosmology [18].

### 5.3. Sultana–Dyer black hole spacetime

The Sultana–Dyer spacetime [23] is a time-dependent inhomogeneous solution of the Einstein equations commonly described as a black hole in a spatially flat FLRW Universe. The matter source consists of a timelike dust plus a null dust, which do not interact [23]. Although the energy density becomes negative at late times near an event horizon, the Sultana–Dyer black hole is studied as an example of a time-dependent black hole horizon for which the Hawking temperature can be derived explicitly [24–26]. The line element of the Sultana–Dyer solution is

\[
ds^2 = a^2(\tau) \left[ -d\tau^2 + dr^2 + r^2 d\Omega^2_{(2)} + \frac{2m}{r} (d\tau + dr)^2 \right]
\]

and is conformal to the Schwarzschild solution. It can be rewritten as [23, 24]

\[
ds^2 = a^2(\tau) \left[ -\left(1 - \frac{2m}{r}\right) d\eta^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2_{(2)} \right],
\]

where \( a(\tau) = \tau^2 \) is the scale factor of the spatially flat FLRW Universe to which the solution reduces if \( m = 0 \) and for \( r \to +\infty \). Moreover, it is [23, 24]

\[
\tau(\eta, r) = \eta + 2m \ln \left( \frac{r}{2m} - 1 \right)
\]

and the comoving time \( r \) of the FLRW ‘background’ is given by \( dr = ad\eta \), where \( \eta \) is the usual conformal time. The FLRW scale factor is \( a(t) \sim t^{2/3} \). The Hawking–Hayward mass of a sphere of radius \( r \) can be computed directly from the line element (66) using the definition \( 1 - 2M_{\text{HH}}/R = \nabla^\nu \nabla_\nu R \) as [24):

\[
\tilde{M}_{\text{HH}} = ma - 2mra \tau + \frac{r^3a^2}{2a} \left( 1 + \frac{2m}{r} \right)
\]

or, since the Sultana–Dyer metric is conformal to Schwarzschild (which has \( M_{\text{HH}} = m \) and \( R = r \)) with \( \Omega = a \), the transformation property of the Hawking–Hayward mass gives
This expression coincides [11] with the one computed directly in [24].

6. Discussion

Contrary to other notions of quasi-local energy, the transformation property of the Hawking–Hayward quasi-local energy is not discussed in the literature. It does not follow simply from the transformation properties of its geometrical ingredients (which have to be derived carefully), but requires some insight into the entire Hawking–Hayward construct, as detailed in sections 2 and 3. The transformation of $M_{\text{HH}}$ requires one to analyze the two null (outgoing and ingoing) geodesic congruences associated with a two-surface $S$, and to compute their optical scalars and anholonomicity. We have presented the relevant calculations, taking care to impose the normalization of the null normals used in the Hawking–Hayward quasi-local energy definition [3, 4], in both the original and the conformally rescaled spacetimes. Putting together the various pieces, one arrives at the desired transformation property (35) for $M_{\text{HH}}$. The result is reproduced by the calculation which involves the contracted Gauss equation. In the special case of spherical symmetry, one recovers a previous result on the transformation of this quantity under conformal rescalings [11], further confirming our general result. Given the wide use of conformal transformations in cosmology and black hole physics, we expect the result obtained here to be useful in these areas. A first application to the problem of whether Newtonian simulations of large-scale structures are correct, given that they neglect relativistic effects, is contained in [17], which shows that the formal Hawking–Hayward construct can actually be applied to realistic physical problems. Thus far, it seems that the Hawking–Hayward mass was computed only for analytical solutions which either describe strong gravity regimes or constitute toy models. However, given that conformal transformations are widely used in cosmology, potential applications to more physical cosmological perturbations are likely to be soon pursued.

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