The asymptotic distribution of the number of 3-star factors in random $d$-regular graphs

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Abstract
The Small Subgraph Conditioning Method has been used to study the almost sure existence and the asymptotic distribution of the number of regular spanning subgraphs of various types in random $d$-regular graphs. In this paper we use the method to determine the asymptotic distribution of the number of 3-star factors in random $d$-regular graphs for $d \geq 4$.

Keywords: Random regular graphs, Small Subgraph Conditioning Method
AMS SUBJECT: 05C80

1 Introduction
It their remarkable papers (see [10] and [11]) Robinson and Wormald showed that for $d \geq 3$ and $dn$ even, a random $d$-regular graph contains a Hamilton cycle with probability tends to 1 as the number $n$ of vertices tends to infinity. They used the Small Subgraph Conditioning Method (see [5] or [8] for details) to prove the existence (with high probability) of perfect matching in such graphs when $n$ is even.

The method has been used to determine the existence with high probability of, and the asymptotic distribution of, the number of $k$-regular spanning subgraphs (for $k = 1, 2$) and the number of long cycles in random $d$-regular graphs (see [5], [7] and [9]).

A star is a tree with at most one vertex whose degree is greater than 1. A $k$-star is a star with $k$ leaves. A $k$-star factor in a graph is a spanning subgraph whose components are $k$-stars.

We use notations $P$(probability), $E$(expectation)and $Var$(variance). We say that an event $Y_n$ occurs a.a.s (asymptotically almost surely) if

$$\lim_{n \to \infty} P(Y_n) = 1.$$
In [2] Assiyatun and Wormald have used the method to investigate the a.a.s of 3-star factor in random $d$-regular graphs. This is the first time the method applied to non-regular subgraphs in such graphs.

Assiyatun and Wormald [2] started proving the existence a.a.s of a 3-star factor in random $d$-regular graphs for $d \geq 4$ by showing the existence a.a.s of a 3-star factor in random 4-regular graphs. Then using the contiguity of models of random regular graphs (see [12]), they obtained the existence a.a.s of a 3-star factor in random $d$-regular graphs for fixed $d \geq 4$ as desired.

As a completion to the result in [2], in this paper we use the method to determine the asymptotic distribution of the number of 3-star factors in random $d$-regular graphs for $d \geq 4$. However, due to the complexity of some part of the computation we are only able to obtain the asymptotic distribution for $4 \leq d \leq 10$. Nevertheless, in most part of the computation we obtain the result for general $d \geq 4$. The main result obtained in this paper is presented in the following theorem.

Let $G_{n,d}$ be a probability space contains of $d$-regular graphs with $n$ vertices. In asymptotic statements about properties of $G_{n,d}$, we restrict $n$ to even integers when $d$ is odd.

**Theorem 1.1** Restrict $n$ to $0 \mod 4$ and $4 \leq d \leq 10$. Then $G \in G_{n,d}$ a.a.s has a 3-star factor. Furthermore, letting $Y_d$ denote the number of 3-star factors in $G \in G_{n,d}$,

$$\frac{Y_d}{EY_d} \to W = \prod_{k=3}^{\infty} (1 + \delta_k)Z_k e^{-\lambda_k\delta_k} \text{ for } n \to \infty$$

where $Z_k$ are independent Poisson variables with $EZ_k = \lambda_k$ for $k \geq 3$ and

$$\lambda_k = \frac{(d - 1)^k}{2k},$$

$$\delta_k = \left( \frac{-3(d-2) + \sqrt{-15d^2 + 24d}}{4(d-1)(d-3/2)} \right)^k + \left( \frac{-3(d-2) - \sqrt{-15d^2 + 24d}}{4(d-1)(d-3/2)} \right)^k.$$

As in [9] we will first work on the pairing model which was first introduced by Bollobás (see [3]). This model can be described as follows. Let $V = \bigcup_{i=1}^{n} V_i$ be a fixed set of $dn$ points, where $|V_i| = d$ for every $i$. A **pairing** is defined as a perfect matching of points of $V$ into $dn/2$ pairs. A pairing $P$ corresponds to a random $d$-regular pseudograph $G(P)$ in which every $V_i$ is regarded as a vertex and each pair is an edge. We use $P_{n,d}$ to denote the probability space of all pairings.

As shown in [3], the probability that the pseudograph has no loops or multiple edges (i.e simple graph) for a fixed $d$, is asymptotically bounded below by a positive constant. Moreover, each simple graph arises with the same probability as $G(P)$ for $P \in P_{n,d}$. Hence using the following property we obtain the desired result in $G_{n,d}$. 

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Lemma 1.1 A property of graphs that holds a.a.s for random pseudographs arising from $\mathcal{P}_{n,d}$ will also hold a.a.s for $\mathcal{G}_{n,d}$.

Given two sequences $a_n$ and $b_n$, we denote $a_n \sim b_n$ if $\frac{a_n}{b_n} \to 1$ for $n \to \infty$. We denote falling factorial $n(n-1)\cdots(n-m+1)$ by $[n]_m$ and Stirling’s formula by $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ for $n \to \infty$.

2 The variance of the number of star factors

Throughout this paper we define $N(2m) = (2m)! / m! 2^m$ as the number of perfect matchings of $2m$ points. Note that counting subgraphs of the pseudograph coming from $\mathcal{P}_{n,d}$ is equivalent to counting the corresponding sets of pairs in the pairing. For that purpose, parallel edges are distinguishable from each other (especially as they come from distinct pairs in the pairing).

Let $n \equiv 0 \mod 4$ and define $Y^*_d$ as the number of 3-star factors in $G(P)$ coming from $\mathcal{P}_{n,d}$ for $d \geq 4$. We have the following theorem.

Theorem 2.1 [2]

$$EY^*_d \sim 2 \left( (d-3/2)^{d/2-3/4} \left( \frac{2}{d^d} \right)^{1/2} \left( \frac{(d-1)(d-2)}{3!} \right)^{1/4} \right)^n. \quad (2.1)$$

Proof. See [2] for details. □

Using the method in [2] (see [4] and [9] for similar argument) we obtain the following theorem.

Theorem 2.2 [13] Restrict $n$ to 0 mod 4 and define $Y^*_d$ as the number of 3-star factors in $G(P)$ coming from $\mathcal{P}_{n,d}$ for $4 \leq d \leq 10$. Then

$$\text{Var} Y^*_d \sim \left( \frac{2(d-1)^{1/2}(d-3/2)^2}{(d-3)(4d^3-13d^2+36d-36)^{1/2}} - 1 \right) (EY^*_d)^2. \quad (2.2)$$

Proof. We count the ways to lay down an ordered pair of 3-star factors in $P \in \mathcal{P}_{n,d}$. In general, a set of pairs in $P$ inducing a subgraph of a given type will be called by the same name in the pairing.

Let $S_i$ be a 3-star factor of $P$ for $i = 1, 2$. Let $T = S_1 \cap S_2$. Define $T = \bigcup_{j=1}^5 T_j$, where (see Figure 2.1)

(i) $T_1$ consists of $x_1$ 1-stars, $S_1$ and $S_2$ have only one common leaf,
(ii.) $T_2$ consists of $x_2$ 1-stars, $S_1$ and $S_2$ have one common leaf and one common center,

(iii.) $T_3$ consists of $x_3$ 2-stars, $S_1$ and $S_2$ have two common leaves,

(iv.) $T_4$ consists of $x_4$ 3-stars, $S_1 \simeq S_2$, 

(v.) $T_5$ consists of $x_5$ 0-stars, $S_1$ and $S_2$ have only one common center.

![Figure 2.1: Two intersecting 3-star factors](image)

Given $S_1$, the number of possibilities of the intersection $T$ is

$$\binom{n/4}{x_1} 3^{x_1} \binom{n/4-x_1}{x_2} 3^{x_2} \binom{n/4-x_1-x_2}{x_3} 3^{x_3} \binom{n/4-x_1-x_2-x_3}{x_4} 1^{x_4}$$

$$\times \binom{n/4-x_1-x_2-x_3-x_4-x_5}{x_5} = \frac{(n/4)! 3^{x_1+x_2+x_3}}{(n/4-x_1-x_2-x_3-x_4-x_5)! x_1!x_2!x_3!x_4!x_5!}.$$  

(2.3)

There are $(\frac{n}{4} - x_1 - x_2 - x_3 - x_4 - x_5)$ 3-stars in both 3-star factors that don’t share any edge. We call these edge-disjoint 3-stars isolated 3-stars.

We have to complete $S_2$ by creating the isolated 3-stars and completing $T_1, T_2, T_3$ and $T_5$ into 3-stars.

The centers of the isolated 3-stars in $S_2$ can not be chosen from the vertex set of $T$ nor the center of the isolated 3-stars in $S_1$. There are $(2x_1 + 2x_2 + 3x_3 + 4x_4 + x_5) + (n/4 - x_1 - x_2 - x_3 - x_4 - x_5) = (n/4 + x_1 + x_2 + 2x_3 + 3x_4)$ such vertices. Thus the number of ways to choose these centers is

$$\binom{3n/4 - x_1 - x_2 - 2x_3 - 3x_4}{n/4 - x_1 - x_2 - x_3 - x_4 - x_5} = \frac{(3n/4 - x_1 - x_2 - 2x_3 - 3x_4)!}{(n/4 - x_1 - x_2 - x_3 - x_4 - x_5)! (n/4 - x_1 - x_2 - 2x_3 + 3x_4)!}.$$  

(2.4)

The number of ways to choose the leaves of the isolated 3-stars in $S_2$ is

$$\prod_{k=0}^{n/4-x_1-x_2-x_3-x_4-x_5-1} \binom{3n/4 - x_1 - x_2 - 2x_3 - 3x_4 - 3k}{3} = \frac{(3n/4 - x_1 - x_2 - 2x_3 - 3x_4)!}{(3!)^{n/4-x_1-x_2-x_3-x_4-x_5} (2x_1 + 2x_2 + x_3 + 3x_5)!}.$$  

(2.5)
The number of ways to choose the leaves of $T_3$ is
\[
\prod_{k=0}^{x_5-1} \left( \frac{2x_1 + 2x_2 + x_3 + 3x_5 - 3k}{3} \right) = \frac{(2x_1 + 2x_2 + x_3 + 3x_5)!}{(3)!^{x_5} (2x_1 + 2x_2 + x_3)!}.
\] (2.6)

The number of ways to choose the leaves for the completion of $T_3$ is
\[
\prod_{k=0}^{x_3-1} \left( \frac{2x_1 + 2x_2 + x_3 - k}{1} \right) = \frac{(2x_1 + 2x_2 + x_3)!}{(2x_1 + 2x_2)!}.
\] (2.7)

For completing $T_2$ and $T_1$, the number of ways to choose the leaves are given consecutively
\[
\prod_{k=0}^{x_2-1} \left( \frac{2x_1 + 2x_2 - 2k}{2} \right) = \frac{(2x_1 + 2x_2)!}{(2!)^{x_2} (2x_1)!},
\] (2.8)

and
\[
\prod_{k=0}^{x_1-1} \left( \frac{2x_1 - 2k}{2} \right) = \frac{(2x_1)!}{(2!)^{x_1}}.
\] (2.9)

So far we have determined the graph corresponding to $S_2$ but not chosen the pairs of points corresponding to its edges. The number of choices for these points is
\[
(d - 1)^{3n/4 - x_2 - 2x_3 - 3x_4} (d - 2)^{n/4 - x_2 - x_3 - x_4 - x_5} (d - 3)^{n/2 - 2x_1 - x_2 - x_3 - 2x_4 - x_5} \times (d - 4)^{x_2 + x_5} (d - 5)^{x_5}.
\] (2.10)

Having $S_1$ and $S_2$, we observe that in $T_1$ there are $2x_1$ vertices of degree 3, while in $T_2$ there are $x_2$ vertices of degree 1 and $x_2$ vertices of degree 5. In $T_3$ we have $2x_3$ vertices of degree 1 and $x_3$ vertices of degree 4, while in $T_4$ we have $3x_4$ vertices of degree 1 and $x_4$ vertices of degree 3 and in $T_5$ we have $x_5$ vertices of degree 6. For the isolated 3-stars in $S_1$ and $S_2$ we have $2(n/4 - x_2 - x_3 - x_4 - x_5) = n/2 - 2x_2 - 2x_3 - 2x_4 - 2x_5$ vertices of degree 4. The remaining vertices (there are $n/2 - x_3 - 2x_4 + x_5$ vertices) are of degree 2. Thus the number of free points in $V$ is
\[
2(d - 3)x_1 + (d - 1)x_2 + (d - 5)x_2 + 2(d - 1)x_3 + (d - 4)x_3 + 3(d - 1)x_4 + (d - 3)x_4 + (d - 6)x_5 + (d - 4)(n/2 - 2x_2 - 2x_3 - 2x_4 - 2x_5) + (d - 2)(n/2 - x_3 - 2x_4 + x_5) = (d - 3)n + 2x_1 + 2x_2 + 4x_3 + 6x_4.
\]

Therefore the number of ways to complete the pairing $P$ is
\[
N((d - 3)n + 2x_1 + 2x_2 + 4x_3 + 6x_4).
\] (2.11)
Multiplying equations (2.3) - (2.11) by the number of ways to choose $S_1$ as in (2.1), then dividing by $N(nd)$, we have

$$EY_d^*(Y_d^* - 1) = \frac{n!(\frac{nd}{n})!}{(nd)!} \left( \frac{2}{\sqrt{3}} \right)^n \left( \frac{d(d-1)((d-2)(d-3))^{1/2}}{n/2 - x_3 - 2x_4 + x_5}\right)^{1/2} \times \sum_{R_1} \frac{3^{n/4 - x_1 - x_2 - x_3 - x_4 - x_5} ((d-3)n + 2x_1 + 2x_2 + 4x_3 + 6x_4)!}{(d-1)x_2 + 2x_3 + 3x_4} \times \frac{(d-2)x_2 + x_3 + x_5}{(d-3)x_2 + x_3 + 2x_4 + x_5}$$

where

$$R_1 = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1, x_2, x_3, x_4, x_5 \geq 0, x_1 + x_2 + x_3 + x_4 + x_5 \leq n/4\}.$$ 

Set

$$p = \frac{x_1}{n}, \quad q = \frac{x_2}{n}, \quad r = \frac{x_3}{n}, \quad s = \frac{x_4}{n}, \quad t = \frac{x_5}{n}.$$ 

Then Stirling’s formula gives

$$EY_d^*(Y_d^* - 1) \sim \frac{1}{8(n\pi)^{5/2}} \left( \sqrt{\frac{n}{2}} \frac{3}{(d-2)(d-3)} \right)^{1/2} \times \sum_{R_2} \alpha(p, q, r, s, t)(F(p, q, r, s, t))^n$$

(2.12)

where

$$R_2 = \{(p, q, r, s, t) \mid p, q, r, s, t \geq 0, p + q + r + s + t \leq 1/4\}$$

$$F(p, q, r, s, t) = \frac{f(3/4 - p - q - 2r - 3s)^2 f((\frac{d-3}{2}) + p + q + 2r + 3s)}{f(1/4 - p - q - r - s - t)^2 f(1/2 - r - 2s + t)} f(p)f(q)f(r)f(s)f(t) \times \frac{3^{p+3q+2r+s} 2^{d-3+p+q+3r+4s} (d-4)^{q+t} (d-5)^t}{(d-1)^{q+2r+3s} (d-2)^{q+r+s+t} (d-3)^{2p+q+r+2s+t}}$$

with $f(x) = x^x$ and

$$\alpha(p, q, r, s, t) = \left( \frac{2(3/4 - p - q - 2r - 3s)^2}{(1/4 - p - q - r - s - t)^2 (1/2 - r - 2s + t)} \right)^{1/2}.$$ 

Since by convention $f(0) := 1$, it can be seen that $F$ is continuous in $R_2$. Next we determine the main contribution of the sum which comes from the maximum of $F$ in $R_2$. The following three lemmas prove that the maximum of $F$ is attained at

$$x_{d-max} = \left( \frac{9}{16d} \cdot \frac{9(d-3)(d-4)}{16d(d-1)(d-2)} \cdot \frac{9(d-3)}{8d(d-1)(d-2)} \cdot \frac{3}{8d(d-1)(d-2)} \cdot \frac{(d-3)(d-4)(d-5)}{16d(d-1)(d-2)} \right)$$
with

\[ F(x_{\text{d-max}}) = \frac{(2d)^{1-d/2} (2(d - 3/2))^{d-3/2}}{((d-1)(d-3))^{1/2}}. \]

**Lemma 2.1** Let \( F \) and \( R_2 \) be as in (2.12). Then for \( d \geq 4 \),

\[
x_{\text{d-max}} = \left( \frac{9}{16d} \right) \frac{(2d-3)(d-4)}{16d(d-1)(d-2)} \frac{9(d-3)}{8d(d-1)(d-2)} \frac{3}{8d(d-1)(d-2)} \frac{(d-3)(d-4)(d-5)}{16d(d-1)(d-2)}
\]

is the local maximum point of \( F \) in the interior of \( R_2 \) with

\[ F(x_{\text{d-max}}) = \frac{(2d)^{1-d/2} (2(d - 3/2))^{d-3/2}}{((d-1)(d-3))^{1/2}}. \]

Moreover for \( 4 \leq d \leq 10 \), \( x_{\text{d-max}} \) is the global maximum point of \( F \) in the interior of \( R_2 \).

**Proof.** First we look for all critical points of \( F \) in the interior of \( R_2 \). We set the partial derivations of \( \ln F \) with respect to \( p, q, r, s \) and \( t \), equal to 0, resulting in five equations

\[ 0 = 9(1/4 - p - q - r - s - t)^2(d - 3 + 2p + 2q + 4r + 6s)
- (d - 3)^2/(3/4 - p - q - 2r - 3s)^2, \tag{2.13} \]
\[ 0 = 9(1/4 - p - q - r - s - t)^2(d - 3 + 2p + 2q + 4r + 6s)(d - 4)
- q(d - 1)(d - 2)(d - 3)/(3/4 - p - q - 2r - 3s)^2, \tag{2.14} \]
\[ 0 = 18(1/4 - p - q - r - s - t)^2(d - 3 + 2p + 2q + 4r + 6s^2)(1/2 - r - 2s + t)
- r(d - 1)^2(d - 2)(d - 3)/(3/4 - p - q - 2r - 3s)^4, \tag{2.15} \]
\[ 0 = 6(1/4 - p - q - r - s - t)^2(d - 3 + 2p + 2q + 4r + 6s^3)(1/2 - r - 2s + t)^2
- s(d - 1)^3(d - 2)(d - 3)^2/(3/4 - p - q - 2r - 3s)^6, \tag{2.16} \]
\[ 0 = (1/4 - p - q - r - s - t)^2(d - 4)(d - 5) - t(d - 2)(d - 3)(1/2 - r - 2s + t). \tag{2.17} \]

After substituting (2.13) to (2.14) and (2.15) to (2.16) we have

\[ p = \frac{(d - 1)(d - 2)q}{(d - 3)(d - 4)}, \tag{2.18} \]

and

\[ t = \frac{1}{r \left( d - 3 + \frac{2(d - 1)(d - 2)q}{(d - 3)(d - 4)} + 2q + 4r + 6s \right)} \times \left( 3s(d - 1)(d - 3) \left( 3/4 - \frac{(d - 1)(d - 2)q}{(d - 3)(d - 4)} - q - 2r - 3s \right)^2 \right.
- (1/2 - r - 2s)r \left( d - 3 + \frac{2(d - 1)(d - 2)q}{(d - 3)(d - 4)} + 2q + 4r + 6s \right). \tag{2.19} \]
After substituting (2.18) and (2.19) to (2.13), (2.15) and (2.17) we have three homogenous equations

\[
P_1(q,r,s) \quad Q_1(q,r,s) = 0, \quad P_2(q,r,s) \quad Q_2(q,r,s) = 0, \quad P_3(q,r,s) \quad Q_3(q,r,s) = 0,
\]

where

\[
Q_1(q,r,s) = 256(d - 3)^3(d - 4)^3((4d^2 - 20d + 28)q + (4d^2 - 28d + 48)r + (6d^2 - 42d + 72)s + d^3 - 10d^2 + 33d - 36) r^2,
\]

\[
Q_2(q,r,s) = 8(d - 3)(d - 4)^2 r Q_1(q,r,s),
\]

\[
Q_3(q,r,s) = \frac{(Q_1(q,r,s))^2}{512 (d - 3)^4 (d - 4)^4 r^2}
\]

and degrees of \( q \) in \( P_1(q,r,s), P_2(q,r,s) \) and \( P_3(q,r,s) \) are 2, while degrees of \( r \) and \( s \) are 5. It is sufficient to look at their nominator parts,

\[
P_1(q,r,s) = 0, \tag{2.20}
\]

\[
P_2(q,r,s) = 0, \tag{2.21}
\]

\[
P_3(q,r,s) = 0. \tag{2.22}
\]

After taking the resultant of (2.20) and (2.21) and of (2.20) and (2.22) with respect to \( q \), we have two homogenous equations,

\[
U_1(r,s) T_1(r,s) = 0, \tag{2.23}
\]

\[
U_2(r,s) T_2(r,s) = 0, \tag{2.24}
\]

where

\[
U_1(r,s) = -331776 r^4 s^2 (d - 2)^2 (2d - 3)^2 (d^2 - 5d + 7)^2 (d - 1)^4 (d - 4)^4 (d - 3)^6,
\]

\[
U_2(r,s) = \frac{(d - 3)^4}{9 r^2} U_1(r,s)
\]

and degrees of \( r \) and \( s \) in \( T_1(r,s) \) and \( T_2(r,s) \) are 6. By taking the resultant of \( T_1(r,s) \) and \( T_2(r,s) \) with respect to \( r \), we have

\[
0 = -4497760410984972288 r^{16} (d - 1)^4 (d - 3)^{12} (4s - 1)^4 (8d^3 s - 24sd^2 + 16sd - 3) V(s).
\]

where \( V(s) \) is a polynomial of degree 15 (see [13] for details). It is easy to show that one of the feasible solution for (2.25) is

\[
s^* = \frac{3}{8d(d - 1)(d - 2)}. \tag{2.26}
\]

By substituting (2.26) to (2.23) and (2.24) we have

\[
W(r, s^*) N_1(r, s^*) = 0, \tag{2.27}
\]

\[
W(r, s^*) N_2(r, s^*) = 0. \tag{2.28}
\]
where
\[ W(r, s^*) = 8r^3 - 24rd^2 + 16rd + 27 - 9d \]
and degrees of \( r \) in \( N_1(r, s^*) \) and \( N_2(r, s^*) \) are 6. From \( W(r, s^*) = 0 \) we have a feasible solution for \( r \)
\[ r^* = \frac{9(d - 3)}{8d(d - 1)(d - 2)}. \]  
(2.29)

By substituting (2.29) to (2.20) – (2.22) we have three new equations
\[ Z(q, r^*, s^*) S_1(q, r^*, s^*) = 0, \]  
(2.30)
\[ Z(q, r^*, s^*) S_2(q, r^*, s^*) = 0, \]  
(2.31)
\[ Z(q, r^*, s^*) S_3(q, r^*, s^*) = 0. \]  
(2.32)

where
\[ Z(q, r^*, s^*) = 16qd^3 - 9d^2 - 48qd^2 + 32qd + 63d - 108 \]
and degrees of \( q \) in \( S_1(q, r^*, s^*) \), \( S_2(q, r^*, s^*) \) and \( S_3(q, r^*, s^*) \) are 2. From \( Z(q, r^*, s^*) = 0 \) we have a feasible solution for \( q \)
\[ q^* = \frac{9(d - 3)(d - 4)}{16d(d - 1)(d - 2)}. \]  
(2.33)

By substituting (2.33) to (2.18) and (2.19) we have
\[ p^* = \frac{9}{16d}, \]
\[ t^* = \frac{(d - 3)(d - 4)(d - 5)}{16d(d - 1)(d - 2)}. \]

Then we obtain
\[ x_{d_{\text{max}}} = \left( \frac{9}{16d}, \frac{9(d - 3)(d - 4)}{16d(d - 1)(d - 2)}, \frac{9(d - 3)}{8d(d - 1)(d - 2)}, \frac{3}{8d(d - 1)(d - 2)}, \frac{(d - 3)(d - 4)(d - 5)}{16d(d - 1)(d - 2)} \right). \]

Note: See [13] for details of \( P_1(q, r, s) \), \( P_2(q, r, s) \), \( P_3(q, r, s) \), \( T_1(r, s) \), \( T_2(r, s) \), \( N_1(r, s^*) \), \( N_2(r, s^*) \), \( S_1(q, r^*, s^*) \), \( S_2(q, r^*, s^*) \), \( S_3(q, r^*, s^*) \) and \( V(s) \).

For \( d \geq 4 \), the Hessian matrix of \( F \) at \( x_{d_{\text{max}}} \) is negative definite. Then \( x_{d_{\text{max}}} \) is the local maximum point of \( F \) in the interior of \( R_2 \) with
\[ F(x_{d_{\text{max}}}) = \frac{(2d)^{1-d/2} (2(d - 3/2))^{d-3/2}}{((d - 1)(d - 3))^{1/2}}. \]

To prove that \( x_{d_{\text{max}}} \) is the global maximum point in the interior of \( R_2 \) for \( 4 \leq d \leq 10 \), we use the following procedure,

1. For each \( d \), determine the feasible solutions of (2.25). If there are \( k \) feasible solutions then denote them by \( s_{d1}, \ldots, s_{dk} \).
(2) For \( s = s_{d i}, 1 \leq i \leq k \), determine the feasible solutions of (2.23) – (2.24). If there are \( l_i \) feasible solutions for each \( i \), then denote them by \( r_{d i 1}, \ldots, r_{d i l_i} \).

(3) For \( s = s_{d i} \) and \( r = r_{d i j}, 1 \leq i \leq k, 1 \leq j \leq l_i \), determine the feasible solutions of (2.20) – (2.22). If there are \( m_{i j} \) feasible solutions for each \( i \) and \( j \), then denote them by \( q_{d i j 1}, \ldots, q_{d i j m_{i j}} \).

(4) For \( s = s_{d i}, r = r_{d i j}, \) and \( q = q_{d i j f}, 1 \leq i \leq k, 1 \leq j \leq l_i, 1 \leq f \leq m_{i j} \) determine \( p_{d i j f} \) and \( t_{d i j f} \) in (2.18) and (2.19).

(5) Define \( x_{d i j f} = (p_{d i j f}, q_{d i j f}, r_{d i j}, s_{d i}, t_{d i j f}) \) as another feasible solution for system (2.13) – (2.17) in the interior of \( R^2 \).

(6) Determine \( F(x_{d_{\max}}) - F(x_{d i j f}) \).

For each \( d, 4 \leq d \leq 10 \), we have \( k = 1, l_i = 1, m_{i j} = 1 \). It means that we have only one other feasible solution for (2.13) – (2.17) in the interior of \( R^2 \). Because \( F(x_{d i j f}) < F(x_{d_{\max}}) \) for each \( d, 4 \leq d \leq 10 \), then \( x_{d_{\max}} \) is the global maximum point of \( F \) in the interior of \( R^2 \). □

To study the behavior of \( F \) on the boundary of \( R^2 \), we generalize the approach used by Garmo in the proof of ([5], Lemma 12). First let \( x = (x_1, x_2, \ldots, x_r) \) and \( u_i = (u_{1,i}, u_{2,i}, \ldots, u_{r,i}) \) for \( r \geq 2 \). The ln function is defined on the set of non-negative real numbers with, by convention, \( 0 \times \ln 0 = 0 \).

**Lemma 2.2** [2] Let \( R := R_2 \) be a closed set in \( \mathbb{R}^r \) and let \( \delta R \) be the boundary of \( R \). Assume that every point in \( \delta R \) is the endpoint of an interval in \( R \setminus \delta R \). Let \( f_i(x) = b_i + u_i x^T \) for \( i = 1, \ldots, m \), where \( b_i \) and \( u_i \) are constant, such that \( f_i(x) > 0 \) for all \( i \) and all \( x \in R \setminus \delta R \). Define \( F \) to be a function on \( R \) such that

\[
F(x) = g_0(x) + \sum_{i=1}^{m} a_i g_i(x) = g_0(x) + \sum_{i=1}^{m} a_i f_i(x) \ln f_i(x).
\]

with \( a_i < 0 \) for \( i \leq m_0 \leq m \). Suppose that for every \( x \in R \), the directional derivative of \( g_0 \) at \( x \) in any direction is bounded. Let \( x_0 \in \delta R \) such that \( f_i(x_0) = 0 \) for at least one \( i \leq m_0 \) and \( f_i(x_0) > 0 \) for all \( m_0 < i \leq m \). Then \( x_0 \) is not a local maximum of \( F \) on \( R \).

**Proof.** See [2] or [13] for details.

**Lemma 2.3** [13] Let \( F \) and \( R := R_2 \) be as in (2.12). Then the maximum of \( F \) does not occur in \( \delta R \).
**Proof.** We define \( x = (p, q, r, s, t) \) and \( v = (v_1, v_2, v_3, v_4, v_5) \). Following the notation in Lemma 2.2 we write

\[
\ln F(x) = g_0(x) + \sum_{i=1}^{9} a_i f_i(x) \ln f_i(x)
\]

where

\[
g_0(x) = (d - 3 + 2p + 2q + 4r + 6s) \ln (d - 3 + 2p + 2q + 4r + 6s) + (2p + 2q + 2r + s) \ln (3) \\
+ (q + t) \ln (d - 4) + (t) \ln (d - 5) - (p + q + r + 2s) \ln (2) - (q + 2r + 3s) \ln (d - 1) \\
- (q + r + s + t) \ln (d - 2) - (2p + q + r + 2s + t) \ln (d - 3),
\]

\[
a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_8 = -1, a_7 = -2, a_9 = 2, b_1 = b_2 = b_3 = b_4 = b_5 = 0, b_6 = \frac{d-3}{2}, b_7 = 1/4, b_8 = 1/2, b_9 = 3/4 and
\]

\[
\begin{align*}
  u_1 & = (1, 0, 0, 0, 0), \\
  u_2 & = (0, 1, 0, 0, 0), \\
  u_3 & = (0, 0, 1, 0, 0), \\
  u_4 & = (0, 0, 0, 1, 0), \\
  u_5 & = (0, 0, 0, 0, 1), \\
  u_6 & = (1, 1, 2, 3, 0), \\
  u_7 & = (-1, -1, -1, -1, -1), \\
  u_8 & = (0, 0, -1, -2, 1), \\
  u_9 & = (-1, -1, -2, -3, 0).
\end{align*}
\]

For \( g_0 \) we have

\[
\frac{\partial}{\partial c} g_0(x + cv) \bigg|_{c=0} = (2v_1 + 2v_2 + 4v_3 + 6v_4) \ln (d - 3 + 2p + 2q + 4r + 6s) \\
+ (2v_1 + 2v_2 + 4v_3 + 6v_4) + (2v_1 + 2v_2 + 2v_3 + v_4) \ln (3) \\
+ (v_2 + v_5) \ln (d - 4) + (v_5) \ln (d - 5) \\
- (v_1 + v_2 + v_3 + 2v_4) \ln (2) - (v_2 + 2v_3 + 3v_4) \ln (d - 1) \\
- (v_2 + v_3 + v_4 + v_5) \ln (d - 2) - (2v_1 + v_2 + v_3 + 2v_4 + v_5) \ln (d - 3)
\]

which is bounded for all \( x \in \mathbb{R} \). Having \( a_9 > 0 \), then from Lemma 2.2 we only need to consider the solution of the following system

\[
\begin{align*}
  f_i(x) & \geq 0, \ i = 1, \ldots, 8 \\
  f_9(x) & = 0
\end{align*}
\]

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which is equivalent to

\[
\begin{align*}
p & \geq 0, \\
q & \geq 0, \\
r & \geq 0, \\
s & \geq 0, \\
t & \geq 0,
\end{align*}
\]

\[
\frac{d - 3}{2} + p + q + 2r + 3s \geq 0,
\]

\[
\frac{1}{4} - p - q - r - s - t \geq 0,
\]

\[
\frac{1}{2} - r - 2s + t \geq 0,
\]

\[
\frac{3}{4} - p - q - 2r - 3s = 0.
\]

It is easy to show that the only solution for the system is \( c_1 = (0, 0, 0, 1/4, 0) \). Consequently for \( x_0 \in \delta R \setminus \{ c_1 \} \), ln \( F \) and \( x_0 \) satisfies the hypotheses of Lemma 2.2. Hence \( F \) does not have any maximum on \( \delta R \setminus \{ c_1 \} \). Moreover for \( d \geq 4 \),

\[
F(c_1) = \frac{3^{1/4} (2(d - 3/2))^{d/2 - 3/4}}{((d - 1)^3(d - 2)(d - 3)^2)^{1/4}}
\]

is strictly less than \( F(x_{d\text{-max}}) \).

In the following lemma we will show that the sum in (2.12) can be approximated within a small region around the maximum.

**Lemma 2.4** \([13]\) Let \( B = B(x_{d\text{-max}}, \delta) \) be a ball centered at

\[
x_{d\text{-max}} = \left( \frac{9}{16d} \cdot \frac{9(d - 3)(d - 4)}{16d(d - 1)(d - 2)} \cdot \frac{9(d - 3)}{8d(d - 1)(d - 2)} \cdot \frac{3}{8d(d - 1)(d - 2)} \cdot \frac{(d - 3)(d - 4)(d - 5)}{16d(d - 1)(d - 2)} \right)
\]

and diameter \( \delta := n^{-5/2} \). Then with \( F \) and \( R := R_2 \) as in (2.12), we have

\[
\sum_R \alpha(x) F^n(x) \sim \sum_B \alpha(x) F^n(x).
\]

**Proof.** Write

\[
\sum_R \alpha(x) F^n(x) = \sum_B \alpha(x) F^n(x) + \sum_{R \setminus B} \alpha(x) F^n(x).
\]

It will be shown that

\[
\sum_{R \setminus B} \alpha(x) F^n(x) = o(\alpha(x_{d\text{-max}}) F^n(x_{d\text{-max}})).
\]
For $x \in B$, the Taylor expansion of $F$ at $x_{d_{\text{max}}}$ is

$$F^n(x) = F^n(x_{d_{\text{max}}}) \times \sum_{B} \exp \left( -n \sum_{i=1}^{5} \sum_{j=i}^{5} c_{ij} s_i s_j \right)$$

where

$$s_1 = p - \frac{9}{16d}, \quad s_2 = q - \frac{9(d - 3)(d - 4)}{16d(d - 1)(d - 2)}, \quad s_3 = r - \frac{9(d - 3)}{8d(d - 1)(d - 2)}.$$

$$s_4 = s - \frac{3}{8d(d - 1)(d - 2)}, \quad s_5 = t - \frac{(d - 3)(d - 4)(d - 5)}{16d(d - 1)(d - 2)}$$

and $c_{ij}$ are coming from Hessian matrix, $H = (a_{ij}) \in M_{5x5}$, of $F$ in the proof of Lemma 2.1

$$c_{ij} = \begin{cases} \frac{1}{2} a_{ij}, & \text{if } i = j \\ a_{ij}, & \text{if } i \neq j \end{cases}$$

For $x^* \in \delta B$, where $\delta B$ is the boundary of $B$, we note that

$$O(e^{n^{-1/5}}) = o(1).$$

Then

$$\alpha(x^*) F^n(x^*) \sim \alpha(x_{d_{\text{max}}}) F^n(x_{d_{\text{max}}}) o(1) = o(\alpha(x_{d_{\text{max}}}) F^n(x_{d_{\text{max}}})),$$

Since $F$ attains its maximum uniquely at $x_{d_{\text{max}}}$, then for $x \in R \setminus B$

$$\alpha(x) F^n(x) = O \left( \max_{x^* \in \delta B} \alpha(x^*) F^n(x^*) \right).$$

Thus we have

$$\sum_{R \setminus B} \alpha(x) F^n(x) = o(\alpha(x_{d_{\text{max}}}) F^n(x_{d_{\text{max}}})). \square$$

Now we determine $\sum_B \alpha(x) F^n(x)$. Since the summation concentrates near the maximum, each term $\alpha(x)$ can be taken as $\alpha(x_{d_{\text{max}}})$ with

$$\alpha(x_{d_{\text{max}}}) = \frac{8192 \sqrt{6} d^3 (d - 1)^{3/2} (d - 3/2) (d - 2)^2}{243 (d - 3)^{5/2} (d - 4) (d - 5)^{1/2}}.$$

Referring to the Taylor expansion of $F$ as in the proof of Lemma 2.4 we have

$$\sum_B \alpha(x) F^n(x) \sim \alpha(x_{d_{\text{max}}}) F^n(x_{d_{\text{max}}}) \sum_B \exp \left( -n \sum_{i=1}^{5} \sum_{j=i}^{5} c_{ij} s_i s_j \right).$$
The summation is a Riemann integral for the five-tuple integral
\[ n^{5/2} \int_{-n}^{n} \int_{-n}^{n} \int_{-n}^{n} \int_{-n}^{n} \exp \left( -n \sum_{i=1}^{5} \sum_{j=i}^{5} c_{ij} t_i t_j \right) dt_1 dt_2 dt_3 dt_4 dt_5 \]
where
\[ t_1 = \left( \frac{p - 9}{16d} \right) n, \quad t_2 = \left( \frac{q - 9(d-3)(d-4)}{16d(d-1)(d-2)} \right) n, \quad t_3 = \left( \frac{r - 9(d-3)}{8d(d-1)(d-2)} \right) n, \]
\[ t_4 = \left( \frac{s - 3}{8d(d-1)(d-2)} \right) n, \quad t_5 = \left( \frac{t - (d-3)(d-4)(d-5)}{16d(d-1)(d-2)} \right) n. \]
As \( n \to \infty \), the range of the integration can be extended to \( \pm \infty \) without altering the main asymptotic term. Thus it is asymptotic to
\[ n^{5/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( - \sum_{i=1}^{5} \sum_{j=i}^{5} c_{ij} t_i t_j \right) dt_1 dt_2 dt_3 dt_4 dt_5. \]
The evaluation of the integral results in
\[ \frac{162\sqrt{6} \pi^{5/2}(d-3/2)(d-3)^{3/2}(d-4)(d-5)^{1/2}}{d^3(d-1)(d-2)^2(4d^3 - 13d^2 + 36d - 36)^{1/2}}. \]
Then (2.34) becomes
\[ \mathbf{E} Y_d^* (Y_d^* - 1) \sim n^{5/2} \times \frac{1}{8(n\pi)^{5/2}} \left( \sqrt{2} (2d)^{1-d/2} (d-1) \left( \frac{(d-2)(d-3)}{3} \right)^{1/2} \right)^n \]
\[ \times \left( \frac{(2d)^{1-d/2} (2(d-3/2))^{d-3/2}}{(d-1)(d-3)^{1/2}} \right)^n \times \frac{8192\sqrt{6} d^3(d-1)^{3/2}(d-3/2)(d-2)^2}{243 (d-3)^{5/2}(d-4)(d-5)^{1/2}} \]
\[ \times \frac{162\sqrt{6} \pi^{5/2}(d-3/2)(d-3)^{3/2}(d-4)(d-5)^{1/2}}{d^3(d-1)(d-2)^2(4d^3 - 13d^2 + 36d - 36)^{1/2}} \]
\[ \sim \frac{8(d-1)^{1/2}(d-3/2)^2}{(d-3)(4d^3 - 13d^2 + 36d - 36)^{1/2}} \left( d^2(d-3/2)^{d-3/2} \left( \frac{2}{d!} \right) \left( \frac{(d-1)(d-2)}{3} \right)^{1/2} \right)^n. \] \( (2.34) \)

As \( \mathbf{E} Y_d^* \to \infty \) for \( d \geq 4 \) we have \( \mathbf{E} Y_d^* (Y_d^* - 1) \sim \mathbf{E} Y_d^* \). Thus from (2.34) and (2.4) we have
\[ \mathbf{E} Y_d^* \sim \frac{2(d-1)^{1/2}(d-3/2)^2}{(d-3)(4d^3 - 13d^2 + 36d - 36)^{1/2}} (\mathbf{E} Y_d^*)^2. \] \( (2.35) \)
Since \( \mathbf{Var} Y_d^* = \mathbf{E} Y_d^* \mathbf{E} (Y_d^* - \mathbf{E} Y_d^*)^2 \), the above equation gives the required result.
\[ \square \]
3 Expectation conditioned on short cycle distribution

Lemma 3.1 Let \( n \equiv 0 \mod 4 \) and \( X_k \) be the number of cycles of length \( k \) in \( G(P) \) for \( P \in \mathcal{P}_{n,d} \). Then for any finite sequences \( j_1, \ldots, j_m \) of non-negative integers

\[
\frac{\mathbb{E}(Y^*_d[X_1]^{j_1} \cdots [X_m]^{j_m})}{\mathbb{E}Y^*_d} \to \prod_{k=1}^{m} (\lambda_k (1 + \delta_k))^{j_k} \quad \text{for} \quad n \to \infty
\]

where

\[
\lambda_k = \frac{(d-1)^k}{2k}, \\
\delta_k = \left( \frac{-3(d-2) + \sqrt{-15d^2 + 24d}}{4(d-1)(d-3/2)} \right)^k + \left( \frac{-3(d-2) - \sqrt{-15d^2 + 24d}}{4(d-1)(d-3/2)} \right)^k.
\]

Proof. To prove the lemma we first establish

\[
\frac{\mathbb{E}(Y^*_d X_k)}{\mathbb{E}Y^*_d} \sim \frac{1}{2k} \left( (d-1)^k + \left( \frac{-3(d-2) + \sqrt{-15d^2 + 24d}}{4(d-3/2)} \right)^k + \left( \frac{-3(d-2) - \sqrt{-15d^2 + 24d}}{4(d-3/2)} \right)^k \right). \tag{3.1}
\]

The number of ways to choose a cycle of length \( k \) in the pairing (with a distinguished point in a pair) is

\[
\frac{n!}{(n-k)!} (d(d-1))^k. \tag{3.2}
\]

This induces an orientation and also a distinguished edge called a root edge in the cycle.

Let \( C \) denote the set of pairs that corresponds to an oriented and rooted \( k \)-cycle. Define \( S \) to be the set of pairs corresponding to a 3-star factor. Fix \( C \) and suppose \( C \cap S \) consists of \( s_0 \) 0-stars (By 0-star we mean isolated vertices) lying at the centers of stars in the 3-star factor, \( s_1 \) 1-stars, \( s_2 \) 2-stars and \( s_3 = k - s_0 - 2s_1 - 3s_2 \) 0-stars lying at the leaves of the 3-star factor.

The edges of \( C \) can be classified into three types. The first type is the edges not lying in the 3-star factor, we denote this as \( \emptyset \). The second is the edges of 1-stars and the first edges of 2-stars and the last is the second edges of 2-stars. We denote them as \( 1 \) and \( 2 \) respectively. If we walk along \( C \) from the root edge, we obtain a sequence \( S_0 \in \{ \emptyset, 1, 2 \}^k \).
Fix $C$ and $S_0$. The number of ways to choose the centers of the remaining $(n/4 - s_0 - s_1 - s_2)$ 3-stars, together with the points used, is

$$\left( \begin{array}{c} n - k \\ n/4 - s_0 - s_1 - s_2 \end{array} \right) (d(d-1)(d-2))^{n/4-s_0-s_1-s_2}. \quad (3.3)$$

The number of ways to choose the points in the centers of $(s_0 + s_1 + s_2)$ 3-stars is

$$((d-2)(d-3)(d-4))^{s_0}((d-2)(d-3))^{s_1}(d-2)^{s_2} = (d-2)^{s_0+s_1+s_2}(d-3)^{s_0+s_1}(d-4)^{s_0}. \quad (3.4)$$

The number of leaves remaining for the 3-star factor is

$$3(n/4 - s_0 - s_1 - s_2) + 3s_0 + 2s_1 + s_2 = 3n/4 - s_1 - 2s_2.$$

The number of ways to choose the leaves from the remaining $(n/4-s_0-s_1-s_2)$ 3-star is

$$\prod_{k=0}^{n/4-s_0-s_1-s_2-1} \left( \frac{3n/4 - s_1 - 2s_2 - 3k}{3} \right) = \frac{(3n/4 - s_1 - 2s_2)!}{(3!)^{n/4-s_0-s_1-s_2}(3s_0 + 2s_1 + s_2)!}. \quad (3.5)$$

The number of ways to choose the leaves from $s_1$ 1-stars

$$\prod_{k=0}^{s_1-1} \left( \frac{3s_0 + 2s_1 + s_2 - 2k}{2} \right) = \frac{(3s_0 + 2s_1 + s_2)!}{(2!)^{s_1}(3s_0 + s_2)!}. \quad (3.6)$$

The number of ways to choose the leaves from $s_2$ 2-stars

$$\prod_{k=0}^{s_2-1} \left( \frac{3s_0 + s_2 - k}{1} \right) = \frac{(3s_0 + s_2)!}{(3s_0)!}. \quad (3.7)$$

The number of ways to choose the leaves from $s_0$ 0-stars

$$\prod_{k=0}^{s_0-1} \left( \frac{3s_0 - 3k}{3} \right) = \frac{(3s_0)!}{(3!)^{s_0}}. \quad (3.8)$$

The number of ways to choose which vertex will be the center of $s_1$

$$2^{s_1}. \quad (3.9)$$

The number of leaves of $S$ lying in the cycle is $k - s_0 - 2s_1 - 3s_2$. Note that every vertex in the cycle uses their two points. Thus the number of ways to choose the points that represent the leaves is

$$(d - 2)^{k-s_0-2s_1-3s_2}. \quad (3.10)$$
There are \((n - k) - (n/4 - s_0 - s_1 - s_2) = (3n/4 - k + s_0 + s_1 + s_2)\) leaves outside the cycle. Note that there are \(d\) points in every vertex of the cycle. Thus the number of ways choose the points that represent the leaves is

\[
d^3n/4-k+s_0+s_1+s_2. \tag{3.11}
\]

By multiplying \(\text{3.5} \text{ - 3.11}\) we have the number of ways to choose the leaves (including the points used)

\[
\frac{(3n/4 - s_1 - 2s_2)!}{(3!n/4-s_1-s_2)} \ (d - 2)^{k-s_0-2s_1-3s_2} \ d^{3n/4-k+s_0+s_1+s_2}. \tag{3.12}
\]

The number of points for the centers of 3-stars inside the cycle is \(\sum d \), thus the number of free points in the pairing is \(N \ (n(d - 3/2) - 2k + 2s_1 + 4s_2)\).

Hence the number of ways to complete the pairing is

\[
N \ (n(d - 3/2) - 2k + 2s_1 + 4s_2). \tag{3.13}
\]

Multiply \(\text{3.3} \text{ - 3.4}\) by \(\text{3.12} \text{ - 3.13}\), sum over all possible \(s_0\), then multiply by \(\text{3.2}\). This results in the number of pairings containing a 3-star factor and an oriented and rooted cycle

\[
\sum_{s_0} \frac{n!}{(n-k)!} \ (d - 1)^k \left( \frac{n-k}{n/4-s_0-s_1-s_2} \right) \ (d-1)(d-2)^{n/4-s_0-s_1-s_2}
\times \ (d - 2)^{s_0+s_1+s_2} \ (d - 3)^{s_0+s_1} \ (d - 4)^{s_0} \ \frac{(3n/4 - s_1 - 2s_2)!}{(3!n/4-s_1-s_2)} \ (d - 2)^{k-s_0-2s_1-3s_2}
\times \ d^{3n/4-k+s_0+s_1+s_2} \ N \ (n(d - 3/2) - 2k + 2s_1 + 4s_2). \tag{3.14}
\]

Dividing \(3.14\) by the number of pairings with a 3-star factor, which is

\[
\frac{n!}{(n/4)!} \left( d \left( \frac{(d-1)(d-2)}{3!} \right)^{1/4} \right)^n N(d(n-3/2))
\]

and then evaluating asymptotically, we obtain

\[
\left( \frac{3(d-1)(d-2)}{4(d-3/2)} \right)^k \sum_{s_0} \left( \frac{d-3(d-4)}{3(d-1)(d-2)} \right)^{s_0} \left( \frac{8(d-3/2)(d-3)}{3(d-1)(d-2)^2} \right)^{s_1} \left( \frac{32(d-3/2)^2}{9(d-1)(d-2)^3} \right)^{s_2}.
\tag{3.15}
\]

We follow an approach used in [7] to determine the summation. We can view \(0, 1, 2\) as three states in a Markov Chain, where the final state is equal to the initial state. We observe that
• 1 followed by 0 means we pass a 1-star and this contributes a factor
\[
\frac{8(d - 3/2)(d - 3)}{3(d - 1)(d - 2)^2},
\]

• 1 followed by 2 means we pass a 2-star and this contributes a factor
\[
\frac{32(d - 3/2)^2}{9(d - 1)(d - 2)^3}.
\]

Thus for the transition matrix given by
\[
A = \begin{pmatrix}
1 + \frac{(d - 3)(d - 4)}{3(d - 1)(d - 2)} & 1 & 0 \\
\frac{8(d - 3/2)(d - 3)}{3(d - 1)(d - 2)^2} & 0 & \frac{32(d - 3/2)^2}{9(d - 1)(d - 2)^3} \\
1 & 0 & 0
\end{pmatrix}
\]
we have
\[
\text{Tr}(A^k) = \sum_{S_0} \left(\frac{(d - 3)(d - 4)}{3(d - 1)(d - 2)}\right)^{s_0} \left(\frac{8(d - 3/2)(d - 3)}{3(d - 1)(d - 2)^2}\right)^{s_1} \left(\frac{32(d - 3/2)^2}{9(d - 1)(d - 2)^3}\right)^{s_2}.
\]

Since the eigenvalues of \(A\) are
\[
\begin{align*}
\gamma_1 &= \frac{4(d - 3/2)}{3(d - 2)} , \\
\gamma_2 &= \frac{-3(d - 2) + \sqrt{-15d^2 + 24d}}{3(d - 1)(d - 2)} , \\
\gamma_3 &= \frac{-3(d - 2) - \sqrt{-15d^2 + 24d}}{3(d - 1)(d - 2)},
\end{align*}
\]
then
\[
\text{Tr}(A^k) = \left(\frac{4(d - 3/2)}{3(d - 2)}\right)^{k} + \left(\frac{-3(d - 2) + \sqrt{-15d^2 + 24d}}{3(d - 1)(d - 2)}\right)^{k} + \left(\frac{-3(d - 2) - \sqrt{-15d^2 + 24d}}{3(d - 1)(d - 2)}\right)^{k}.
\]

Then (3.15) becomes
\[
(d-1)^k + \left(\frac{-3(d - 2) + \sqrt{-15d^2 + 24d}}{4(d - 3/2)}\right)^{k} + \left(\frac{-3(d - 2) - \sqrt{-15d^2 + 24d}}{4(d - 3/2)}\right)^{k}.
\]

After (3.16) is divided by 2\(k\) to remove the orientation and rooting of the cycle we obtain (3.1). □
4 Proof of Theorem 1.1

First we prove the following theorem.

**Theorem 4.1** Let \( n \equiv 0 \mod 4 \) and \( 4 \leq d \leq 10 \). Then for \( P \in \mathcal{P}_{n,d} \), \( G(P) \) a.a.s has a 3-star factor. Moreover,

\[
\frac{Y_d^*}{EY_d^*} \to W = \prod_{k=1}^{\infty} (1 + \delta_k) Z_k e^{-\lambda_k \delta_k} \quad \text{for } n \to \infty
\]

where \( Z_k \) are independent Poisson variables with \( EZ_k = \lambda_k \) for \( k \geq 1 \) and

\[
\lambda_k = \frac{(d-1)^k}{2k},
\]

\[
\delta_k = \left( \frac{-3(d-2) + \sqrt{-15d^2 + 24d}}{4(d-1)(d-3/2)} \right)^k + \left( \frac{-3(d-2) - \sqrt{-15d^2 + 24d}}{4(d-1)(d-3/2)} \right)^k.
\]

**Proof.** We will show that \( Y_d^* \) satisfies the conditions (A.1) – (A.4) in (Theorem 4.1, [12]). Since \( X_k \) is the number of short cycles of length \( k \) in a pseudograph coming from \( \mathcal{P}_{n,d} \), then (A.1) is fulfilled with \( \lambda_k = \frac{(d-1)^k}{2k} \), by Bollobás’ result on short cycles in \( \mathcal{P}_{n,d} \) [3]. The condition (A.2), (A.3) and (A.4) are fulfilled consecutively by Lemma 3.1, Theorem 2.1 and Theorem 2.2. □

**Proof of Theorem 1.1.**

Theorem 1.1 comes directly from Theorem 4.1 by Lemma 1.1. From the argument in (Remark 9.25, [6]) we also obtain

\[
\frac{EY_d}{EY_d^2} \to \exp \left( \frac{3(5d^2 - 12d + 6)}{4(2d - 3)^2} \right),
\]

\[
\frac{EY_d^2}{(EY_d)^2} \to \exp \left( \frac{-9(8d^5 - 63d^4 + 206d^3 - 322d^2 + 216d - 36)}{4(2d - 3)^4(d-1)^2} \right)
\]

\[
\times \left( \frac{2(d-1)^{1/2}(d-3/2)^2}{(d-3)(4d^3 - 13d^2 + 36d - 36)^{1/2}} \right).
\]

We should point it out again, that most part of the computation in determining the second moment of the number of 3-star factors in \( G \in \mathcal{P}_{n,d} \) is valid for general \( d \geq 4 \). The only part that is still hard to prove is showing that the desired maximum point is the global maximum. From what we have in the case \( 4 \leq d \leq 10 \), we conjecture that the asymptotic distribution of the number of 3-star factor in random \( d \)-regular graph have the same behaviour for \( d \geq 4 \).
Conjecture 4.1  Restrict $n$ to $0 \mod 4$ and $d \geq 4$. Then $G \in \mathcal{G}_{n,d}$ a.a.s has a 3-star factor. Furthermore, letting $Y_d$ denote the number of 3-star factors in $G \in \mathcal{G}_{n,d}$, 

$$\frac{Y_d}{EY_d} \to W = \prod_{k=3}^{\infty} (1 + \delta_k)^{Z_k} e^{-\lambda_k \delta_k} \text{ for } n \to \infty$$

where $Z_k$ are independent Poisson variables with $EZ_k = \lambda_k$ for $k \geq 3$ and 

$$\lambda_k = \frac{(d-1)^k}{2k},$$

$$\delta_k = \left( \frac{-3(d-2) + \sqrt{-15d^2 + 24d}}{4(d-1)(d-3/2)} \right)^k + \left( \frac{-3(d-2) - \sqrt{-15d^2 + 24d}}{4(d-1)(d-3/2)} \right)^k.$$
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