Soft lambda-calculus: a language for polynomial time computation

Patrick Baillot and Virgile Mogbil

Laboratoire d'Informatique de Paris-Nord UMR 7030 CNRS
Université Paris XIII - Institut Galilée, 99 Avenue Jean-Baptiste Clément,
93430 Villetaneuse, France.
{patrick.baillot,virgile.mogbil}@lipn.univ-paris13.fr

Abstract. Soft linear logic ([Lafont02]) is a subsystem of linear logic characterizing the class PTIME. We introduce soft lambda-calculus as a calculus typable in the intuitionistic and affine variant of this logic. We prove that the (untyped) terms of this calculus are reducible in polynomial time. We then extend the type system of Soft logic with recursive types. This allows us to consider non-standard types for representing lists. Using these datatypes we examine the concrete expressivity of Soft lambda-calculus with the example of the insertion sort algorithm.

1 Introduction

With the advent of global computing there are an increasing variety of situations where one would need to be able to obtain formal bounds on resource usage by programs: for instance before running code originating from untrusted source or in settings where memory or time is constrained, like in embedded systems or synchronous systems.

Some cornerstones for this goal have been laid by the work on Implicit Computational Complexity (ICC) as carried out by several authors since the 1990s ([Lei94], [LM93], [Bel92] among others). This field aims at studying languages and calculi in which all programs fall into a given complexity class. The most studied case has naturally been that of deterministic polynomial time complexity (PTIME class). We can in particular distinguish two important lines of work. The first one deals with primitive recursion and proposes restrictions on primitive recursion such that the functions definable are those of PTIME: this is the approach of Bellantoni-Cook ([Bel92]) and subsequent extensions ([Hoe00], [BNS00]).

Another line is that of Linear logic (LL)([Gir87]). By the Curry-Howard correspondence proofs in this logic can be seen as programs. Linear logic provides a way of controlling duplication of arguments thanks to specific modalities (called exponentials). It is possible to consider variants of LL with alternative, stricter rules for modalities, for which all proofs-programs can be run in polynomial time.

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Light linear logic, introduced by Girard \cite{Gir98} is one of these systems. It was later simplified by Asperti into Light affine logic \cite{AR02,Asp98} which allows full weakening (that is to say erasing of arguments). However formulas in this system are quite complicated as there are two modalities, instead of just one in intuitionistic linear logic. More recently Lafont introduced Soft linear logic (SLL) \cite{Laf02}, a simpler system which uses the same language of formulas as Linear logic and is polytime. It can in fact be seen as a subsystem of linear logic or of Bounded linear logic \cite{GSS92}.

In all these approaches it is shown that the terms of the calculus can be evaluated in polynomial time. A completeness result is then proved by simulating in the calculus a standard model for PTIME computation such as PTIME Turing machines. It follows that all PTIME functions are representable in the calculus, which establishes its expressivity.

However if this completeness argument is convincing for characterization of complexity classes of functions, it is rather unsatisfactory when we are interested in the use of Implicit Computational Complexity for the study of program properties. Indeed it is not so appealing to program in a new language via the encoding of Turing machines . . . One would prefer to be able to take advantage of the features of the language: for the variants of Linear logic for instance we have at hand abstract datatypes and structural recursion, higher-order and polymorphism.

Some authors have observed that common algorithms such as insertion sort or quicksort are not directly representable in the Bellantoni-Cook approach (see for instance \cite{Hof99}). Important contributions to the study of programming aspects of Implicit computational complexity have been done in particular by Jones \cite{Jon97}, Hofmann \cite{Hof99} and Marion \cite{Mar00}. For instance Hofmann proposed languages using linear type systems with a specific type for space unit, which enabled him to characterize non-size increasing computation with various time complexity bounds. This approach allows to represent several standard algorithms.

Here we are interested in investigating the programming possibilities offered by Soft linear logic. In \cite{Laf02} this system is defined with sequent-calculus and the results are proved using proof-nets, a graph representation of proofs. In order to make the study of programming easier we propose a lambda-calculus presentation. We extend for that usual lambda-calculus with new constructs corresponding to the exponential rules of SLL. The resulting calculus is called Soft lambda-calculus and can be typed in SLL. Actually we choose here the affine variant of Soft logic as it is more flexible and has the same properties. Our Soft lambda-calculus is inspired from Terui’s Light affine lambda-calculus \cite{Ter01}, which is a calculus with a polynomial bound on reduction sequences that can be typed in Light affine logic.

**Outline.** In section 2 we define soft lambda-calculus and its type-assignment system. Then in section 3 we prove that the length of any reduction sequence of a term is bounded by a polynomial applied to the size of the term. In section 4 we extend the type system and add recursive typing. Finally in section 5 we
examine datatypes for lists and propose a new datatype with which we program
the insertion sort.

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2 Soft lambda-calculus

The introduction of our calculus will done be in two steps (as in \cite{Ter01}): first
we will define a grammar of pseudo-terms and then we will distinguish terms
among pseudo-terms.

The pseudo-terms are defined by the grammar:

\[
\begin{align*}
t, t' &::= x \\
        &\quad | \lambda x \ t \\
        &\quad | (t \ t') \\
        &\quad | ! t \\
        &\quad | \text{let } u \ be \ ! x \ \text{in} \ t'
\end{align*}
\]

For a pseudo-term \( t \) we consider:

\begin{itemize}
  \item its set of free variables \( FV(t) \);
  \item for a variable \( x \) the number of free occurrences \( no(x, t) \) of \( x \) in \( t \).
\end{itemize}

In the pseudo-term \( \text{let } u \ be \ ! x \ \text{in} \ t_1 \), the variable \( x \) is bound:

\[
FV(\text{let } u \ be \ ! x \ \text{in} \ t_1) = FV(u) \cup FV(t_1) \setminus \{x\}
\]

If \( t \) is of the form \( \text{let } u \ be \ x \ \text{in} \ t_1 \) we say that \( t \) is a let expression.

If \( T \) and \( \mathcal{P} \) respectively denote finite sequences of same length \( (t_1, \ldots, t_n) \)
and \( (x_1, \ldots, x_n) \), then let \( T \) be \( \mathcal{P} \) in \( t' \) will be an abbreviation for \( n \) consecutive
let expressions on \( t_1 \& s \) and \( x_i \& s \): let \( t_1 \) be \( x_1 \) in let \( t_2 \) be \( x_2 \) in \( \ldots \) \( t' \).

We define the size \( |t| \) of a pseudo-term \( t \) by:

\[
\begin{align*}
|x| &= 1 \\
|\lambda x \ t| &= |t| + 1 \\
|(t \ t')| &= |t| + |t'| \\
|! t| &= |t| + 1 \\
|\text{let } u \ be \ ! x \ \text{in} \ t'| &= |t| + |t'| + 1
\end{align*}
\]

We will type these pseudo-terms in intuitionistic soft affine logic (ISAL). The
formulas are given by the following grammar:

\[
T ::= \alpha | T \rightarrow T | \forall \alpha. T | ! T
\]

We choose the affine variant of Soft linear logic, which means permitting full
weakening, to allow for more programming facility. This does not change the
polytime nature of the system, as was already the case for light logic (\cite{Asp98,Ter01}).

We give the typing rules in a sequent calculus presentation. It offers the
advantage of being closer to the logic. It is not so convenient for type-inference,
but it is not our purpose in this paper. The typing rules are given on Figure 1.

For (right \( \forall \)) we have the condition:

\((^*)\) \( \alpha \) does not appear free in \( \Gamma \).
Basically the ideas behind the definition of terms are that:

Definition 1

The set \( T \) of terms is the smallest subset of pseudo-terms such that:

- \( x \in T \): then \( TV(x) = \emptyset \);
- \( \lambda x.t \in T \) iff: \( x \notin TV(t), t \in T \) and \( n(o(x,t)) \leq 1 \);
- \( \nu x.t \in T \) iff: \( \exists u:TV(t)\); 
- \( (t_1, t_2) \in T \) iff: \( t_1, t_2 \in T \), \( TV(t_1) \cap TV(t_2) = \emptyset \); \( FV(t_1) \cap TV(t_2) = \emptyset \);
- \( !t \in T \) iff: \( t \in T \), \( TV(t) = \emptyset \) and \( \forall x \in FV(t), no(x,t) = 1 \);
- \( let x.in t \in T \) iff: \( t \in T \), \( TV(t_1) \cap TV(t_2) = \emptyset \); \( FV(t_1) \cap TV(t_2) = \emptyset \);

Basically the ideas behind the definition of terms are that:

- one can abstract only on a variable that is not temporary and which has at most one occurrence,
one can apply ! to a term which has no temporary variable and whose free variables have at most one occurrence; the variables then become temporary;
the only way to get rid of a temporary variable is to bind it using a let expression.

It follows from the definition that temporary variables in a term are linear:

**Lemma 1** If $t$ is a term and $x \in TV(t)$, then $no(x, t) = 1$.

The definition of depth will be useful later when discussing reduction:

**Definition 2** Let $t$ be a term and $u$ be an occurrence of subterm of $t$. We call depth of $u$ in $t$, $d(u, t)$ the number $d$ of subterms $v$ of $t$ such that $u$ is a subterm of $v$ and $v$ is of the form $!v'$.

The depth $d(t)$ of a term $t$ is the maximum of $d(u, t)$ for $u$ subterms of $t$.

For instance: for $t = !(\lambda f. \lambda x. \text{let } f \text{ be } !f' \text{ in } !f'x)$ and $u = (f'x)$, we have $d(u, t) = 2$.

We can then observe that:

**Proposition 2** Let $t$ be a term. If $x$ belongs to $FV(t)$ and $x_0$ denotes an occurrence of $x$ in $t$, then $d(x_0, t) \leq 1$.

Moreover all occurrences of $x$ in $t$ have the same depth, that we can therefore denote by $d(x, t)$, and we have: $d(x, t) = 1$ iff $x \in TV(t)$.

In fact we will focus our attention on specific terms:

**Definition 3** A term $t$ is well-formed if we have:

$$TV(t) = \emptyset \text{ and } \forall x \in FV(t), \text{ no}(x, t) = 1.$$  

Note that to transform an arbitrary term into a well-formed one, one only needs to add enough let expressions.

We have the following properties on terms and substitution:

**Lemma 3** If $t$ is a term and $t = !t_1$, then $t_1$ is a well-formed term.

**Lemma 4** If we have:

- $t$, $u$ terms,
- $TV(u) = \emptyset$,
- $x \notin TV(t)$,
- $FV(u) \cap TV(t) = \emptyset$,

then: $t[u/x]$ is a term and $TV(t[u/x]) = TV(t)$.

We can then check the following:

**Proposition 5** If $t$ is a pseudo-term such that in ISAL we have $\Gamma \vdash t : A$, then $t$ is a well-formed term.
Proof. by induction on the type derivation, using the definition of terms and for
the case of the (cut) and (leftarrow) rules the lemma \textit{[4]}

We will also need in the sequel two variants of lemma \textit{[4]}:

**Lemma 6** If we have:

- $t, u$ terms,
- $x \notin TV(t)$,
- $no(x, t) = 1$,
- $FV(u) \cap TV(t) = \emptyset$,
- $TV(u) \cap FV(t) = \emptyset$,

then: $t[u/x]$ is a term and $TV(t[u/x]) = TV(t) \cup TV(u)$.

Note that the main difference with lemma \textit{[4]} is that we have here the assumption
$no(x, t) = 1$.

**Lemma 7** If we have:

- $t$ is a term and $u$ is a well-formed term,
- $FV(t) \cap FV(u) = \emptyset$,
- $x \in TV(t)$

then: $t[u/x]$ is a term and $TV(t[u/x]) = TV(t) \setminus \{x\} \cup FV(u)$.

We now consider the contextual one-step reduction relation $\rightarrow^1$ defined
on pseudo-terms by the rules of figure \textit{2}. The rules (com1) and (com2) are the
commutation rules. The relation $\rightarrow$ is the transitive closure of $\rightarrow^1$.

| Rule | Description |
|------|-------------|
| $(\beta)$: | $(\lambda x. t) \ u \rightarrow^1 t[u/x]$ |
| (bang): | let $!u$ be $!x$ in $t \rightarrow^1 t[u/x]$ |
| (com1): | let (let $t_1$ be $!y$ in $t_2$) be $!x$ in $t_3 \rightarrow^1$ let $t_1$ be $!y$ in (let $t_2$ be $!x$ in $t_3$) |
| (com2): | (let $t_1$ be $!x$ in $t_2$) $t_3 \rightarrow^1$ let $t_1$ be $!x$ in ($t_2$ $t_3$) |

**Fig. 2.** reduction rules

We have:

**Lemma 8** The reduction is well defined on terms (the result of a reduction step
on a term is a term). Furthermore, if $t$ is a well-formed term and $t \rightarrow^1 t'$, then $t'$ is well-formed.

Finally we have:

**Proposition 9 (local confluence)** The reduction relation $\rightarrow^1$ on terms is
locally confluent: if $t \rightarrow^1 t'_1$ and $t \rightarrow^1 t'_2$ then there exists $t'$ such that $t'_1 \rightarrow t'$
and $t'_2 \rightarrow t'$. 
3 Bounds on the reduction

We want to find a polynomial bound on the length of reduction sequences of terms, similar to that holding for SLL proof-nets ([Laf02]). For that we must define a parameter on terms corresponding to the arity of the multiplexing links in SLL proof-nets.

**Definition 4** The rank $\text{rank}(t)$ of a term $t$ is defined inductively by:

\[
\begin{align*}
\text{rank}(x) &= 0 \\
\text{rank}(\lambda x.t) &= \text{rank}(t) \\
\text{rank}((t_1 t_2)) &= \max(\text{rank}(t_1), \text{rank}(t_2)) \\
\text{rank}(!u) &= \text{rank}(t) \\
\text{rank}(\text{let } u \text{ be } !x \text{ in } t_1) &= \begin{cases} 
\max(\text{rank}(u), \text{rank}(t_1)) & \text{if } x \in TV(t_1) \\
\max(\text{rank}(u), \text{rank}(t_1), \text{no}(x, t_1)) & \text{if } x \notin TV(t_1)
\end{cases}
\end{align*}
\]

The first case in the definition of $\text{rank}(\text{let } u \text{ be } !x \text{ in } t_1)$ corresponds to a promotion, while the second one corresponds to a multiplexing and is the key case in this definition.

To establish the bound we will adapt the argument given by Lafont for proof-nets. First we define for a term $t$ and an integer $n \geq \text{rank}(t)$ the weight $W(t, n)$ by:

\[
\begin{align*}
W(x, n) &= 1 \\
W(\lambda x.t, n) &= W(t, n) + 1 \\
W(!u, n) &= nW(u, n) + 1 \\
W((t_1 t_2), n) &= W(t_1, n) + W(t_2, n) \\
W(\text{let } u \text{ be } !x \text{ in } t_1, n) &= W(u, n) + W(t_1, n)
\end{align*}
\]

We have the following key lemma:

**Lemma 10** Let $t$ be a term and $n \geq \text{rank}(t)$.

1. if $x \notin TV(t)$ and $\text{no}(x, t) = k$, then:
   \[ W(t[u/x], n) \leq W(t, n) + kW(u, n) \]

2. if $x \in TV(t)$ then:
   \[ W(t[u/x], n) \leq W(t, n) + nW(u, n) \]

We give the proof of this lemma in Appendix B.

**Proposition 11** Let $t$ be a term and $n \geq \text{rank}(t)$. If $t \xrightarrow{\text{or}} t'$ by a $(\beta)$ or (bang) reduction rule then $W(t', n) < W(t, n)$.

**Proof.** If $t \xrightarrow{\sigma} t'$ with $\sigma = (\beta)$ or (bang) then let $r$ denote the redex reduced inside $t$. The form of $t$ is $t_0[r/y]$ with $\text{no}(y, t_0) = 1$ and $t' = t_0[r'/y]$ where $r \xrightarrow{\sigma} r'$.

The result is obtained by induction on the term $t_0$ for a given $n \geq \text{rank}(t)$:
let us consider the basic case $t_0 = y$, i.e. $t = r$ using the definitions of terms and rank, and lemma 10.

For instance for a (bang) reduction rule,

$r = \text{let } !u \text{ be } !x \text{ in } r_1$

$r' = r_1[u/x]$

$W(r, n) = W(\text{let } !u \text{ be } !x \text{ in } r_1, n) = n.W(u, n) + 1 + W(r_1, n)$.

If $x \in TV(r_1)$ then by lemma 10 $W(r', n) < W(r, n)$, else $x \in FV(r_1) \setminus TV(r_1)$ and

$W(r', n) \leq W(r_1, n) + \text{no}(x, r_1).W(u, n)$

$\leq W(r_1, n) + \text{rank}(r).W(u, n)$

$\leq W(r_1, n) + n.W(u, n)$

$< W(r, n)$

In the non basic cases, i.e. $t_0 \neq y$, we can remark that $W(t_0[r/x], n)$ is a strictly increasing function of $W(r, n)$. For instance:

if $t_0 = (y.t_1)$ then $W(t', n) = W(t_0[r'/y], n) = W((r't_1, n) = W(r', n) + W(t_1, n) < W(r, n) + W(t_1, n)$ i.e. $W(t', n) < W(t, n)$.

For the commutation rules we have $W(t', n) = W(t, n)$. So we need to use a measure of the commutations in a reduction sequence to be able to bound the global length. We make an adaptation of the weight used in [Ter01].

Given an integer $n$ and a term $t$, for each subterm occurrence in $t$ of the form $t_1 \equiv \text{let } u \text{ be } !x \text{ in } t_2$, we define the measure of $t_1$ in $t$ by:

$m(t_1, t) = W(t, n) - W(t_2, n)$

and $M(t, n)$ the measure of $t$ by the sum of $m(t_1, t)$ for all subterms $t_1$ of $t$ which are let expressions.

**Proposition 12** Let $t$ be a term and $n \geq \text{rank}(t)$. If $t \rightarrow^1 t'$ by a commutation reduction rule then $M(t', n) < M(t, n)$.

Given a term $t$ we denote by $nlet(t)$ the number of subterm occurrences of let expressions in $t$.

**Lemma 13** Let $t$ be a term and $n \geq 1$. We have $nlet(t) \leq W(t, n) - 1$.

**Proposition 14** If $t$ is a term and $p = d(t)$, $k = W(t, 1)$, and $n \geq 1$ then:

$W(t, n) \leq k.n^p$

**Proof.** Let $n \geq 1$. By induction on the term, using definitions of weight and depth: if $t = !t_1$ then

$W(t, n) = n.W(t_1, n) + 1 \leq W(t_1, 1).n^{d(t_1)} + 1$ because $n \geq 1$

$\leq (W(t_1, 1) + 1).n^p$

$\leq W(t, 1).n^p$

The other cases are immediate.
Finally this result can be applied to any prop. we obtain that

If a term such that

\[ \text{d} \quad \text{Q polynomial} \]

Corollary 17 (Confluence property) If a term is such that \( t \rightarrow u \) and \( t \rightarrow v \) then there exists a term \( w \) such that \( u \rightarrow w \) and \( v \rightarrow w \).

Proof. By local confluence (Proposition) and strong normalization.
4 Extension of the calculus

Thanks to full weakening, the connectives \(\otimes\), \&, \(\oplus\), \(\exists\) and the constant 1 are definable from \(\{\to, \forall\}\) (Asp98, Ter02):

\[
\exists \beta. A = \forall \alpha. (\forall \beta. (A \to \alpha) \to \alpha)
\]

\[
A \otimes B = \forall \alpha. ((A \to B \to \alpha) \to \alpha)
\]

\[
1 = \forall \alpha. (\alpha \to \alpha)
\]

\[
A \oplus B = \forall \alpha. ((\alpha \to \alpha) \to (B \to \alpha) \to \alpha)
\]

\[
A \& B = \exists \alpha. ((\alpha \to A) \otimes (\alpha \to B) \otimes \alpha)
\]

We use as syntactic sugar the following new constructions on terms:

\[
\text{let } u \text{ be } x_1 \otimes x_2 \text{ in } t,
\]

\[
\text{let left } u \text{ be left } x \text{ in } t_1
\]

\[
\text{right } t, \text{ right } y \text{ in } t_2;
\]

We then have the new typing rules of figure 3.

\[
\frac{\Gamma, x_1 : A_1, x_2 : A_2 \vdash t : B}{\Gamma, x : A_1 \otimes A_2 \vdash \text{let } x \text{ be } x_1 \otimes x_2 \text{ in } t \vdash B} \quad (\text{left } \otimes)
\]

\[
\frac{\Gamma, x_1 : A_1 \vdash t_1 : B, \Gamma, x_2 : A_2 \vdash t_2 : B}{\Gamma, x : A_1 \oplus A_2 \vdash \text{let } x \text{ be left } x_1 \text{ in } t_1 \text{ right } x_2 \text{ in } t_2 \vdash B; \quad (\text{left } \oplus)}
\]

\[
\frac{\Gamma \vdash t : A}{\Gamma \vdash \text{left } t : A \oplus B} \quad (\text{right } \oplus_1)
\]

\[
\frac{\Gamma \vdash t_1 : A_1, \Gamma \vdash t_2 : A_2}{\Gamma \vdash \text{right } t : A \oplus B} \quad (\text{right } \oplus_2)
\]

**Fig. 3. Derived rules**

The derived reduction rules for these constructions are:

\[
\text{let } t_1 \otimes t_2 \text{ be } x_1 \otimes x_2 \text{ in } u \rightarrow u[t_1/x_1, t_2/x_2]
\]

\[
\text{let left } u \text{ be left } x_1 \text{ in } t_1 \text{ right } x_2 \text{ in } t_2 \rightarrow t_1[u/x_1]
\]

\[
\text{let right } u \text{ be left } x_1 \text{ in } t_1 \text{ right } x_2 \text{ in } t_2 \rightarrow t_2[u/x_2]
\]

We also use as syntactic sugar, for \(x\) a variable: let \(u \text{ be } x \text{ in } t \overset{\text{def}}{=} ((\lambda x. t) u).

We now enlarge the language of types with a fix-point construction:

\[
T ::= \alpha | T \to T | \forall \alpha. T | ! T | \mu \alpha. T
\]

We add the corresponding typing rule and denote by ISALF, intuitionistic light affine logic with fix-points, the new system: Figure 4. If a pseudo-term is typable in ISALF then clearly it is a well-formed term since these new rules do not have any computational counterpart.
the typing rules of ISAL and

\[
\frac{x : \mu X.A, \Gamma \vdash t : B}{x : A[\mu X.A/X], \Gamma \vdash t : B} \quad \text{(left unfold)} \quad \frac{\Gamma \vdash t : \mu X.A}{\Gamma \vdash t : A}[\mu X.A/X] \quad \text{(right unfold)}
\]

\[
\frac{x : A[\mu X.A/X], \Gamma \vdash t : B}{x : \mu X.A, \Gamma \vdash t : B} \quad \text{(left fold)} \quad \frac{\Gamma \vdash t : A[\mu X.A/X]}{\Gamma \vdash t : \mu X.A} \quad \text{(right fold)}
\]

**Fig. 4.** ISALF typing rules

**Proposition 18 (Subject reduction)** If we have in the system ISALF \( \Gamma \vdash t : A \) and \( t \to t' \) then \( \Gamma \vdash t' : A \).

Basically this result follows from the fact that as a logical system ISALF admits cut-elimination.

Note that even though we have no restriction on the types on which we take fix points, the typed terms are always normalizable and have a polynomial bound on the length of their reduction. This follows from the fact that the polynomial termination result (Theorem 15) already holds for untyped terms.

In the following we will handle terms typed in ISALF. Rather than giving the explicit type derivations in the previous system, which is a bit tedious because it is a sequent-calculus style presentation, we will use a Church typing notation. The recursive typing rules and second-order rules will be left implicit. From this notation it is possible to reconstruct an explicit type derivation if needed.

Here is an example of typed term (integer 2 in unary representation)

\[
\lambda s^!(\alpha \to \alpha). \lambda x^\alpha. \text{let } s' = s \text{ in } (s' \ (s' \ x))^\alpha : N
\]

## 5 Datatypes and list processing

### 5.1 Datatypes for lists

Given a type \( A \), we consider the following types defining lists of elements of \( A \):

\[
\mathcal{L}(A) = \forall \alpha. !(A \to \alpha \to \alpha) \to \alpha \to \alpha
\]

\[
L(A) = \mu X.(1 \oplus (A \otimes X))
\]

The type \( \mathcal{L}(A) \) is the adaptation of the usual system F type for lists. It supports an iteration scheme, but does not enable to define in soft lambda-calculus a \textit{cons} function with type \( \mathcal{L}(A) \to A \to \mathcal{L}(A) \). This is analog to the fact that \( N \) does not allow a successor function with type \( N \to N \) ([Laf02]).

The type \( L(A) \) on the contrary allows to define the usual elementary functions on lists \textit{cons}, \textit{tail}, \textit{head}, but does not support iteration.
The empty list for type $L(A)$ is given by $\epsilon = \text{left} 1$ and the elementary functions by:

$$
\begin{align*}
\text{cons} & : L(A) \to A \to L(A) \\
\text{cons} & = \lambda l L(A). \lambda a A, \text{right} (a \otimes l) \\
\text{tail} & : L(A) \to L(A) \\
\text{tail} & = \lambda l L(A). \text{let} l' \text{ be left } l' \text{ in } l'' \\
& \quad \text{right } (a \otimes l' \text{ in } l'') \\
\text{head} & : L(A) \to A \\
\text{head} & = \lambda l L(A). \text{let} l' \text{ be right } l' \text{ in } l'' \\
& \quad \text{let } l' \text{ be } a \otimes l'' \text{ in } a
\end{align*}
$$

We would like to somehow bring together the advantages of $L(A)$ and $L(A)$ in a single datatype. This is what we will try to do in the next sections.

### 5.2 Types with integer

Our idea is given a datatype $A$ to add to it a type $N$ so as to be able to iterate on $A$. The type $N \otimes A$ would be a natural candidate, but it does not allow a suitable iteration. We therefore consider the following type:

$$
N[A] = \forall \alpha. ! (\alpha \to \alpha) \to \alpha \to (A \otimes \alpha)
$$

Given an integer and a closed term of type $A$, we define an element of $N[A]$:

$$
n[a] = \lambda s \lambda x. a \otimes \text{let } s' \text{ in } (s' s' \ldots s')^\alpha : N[A]
$$

where $s'$ is repeated $n$ times.

We can give terms allowing to extract from an element $n[a]$ of type $N[A]$ either the data $a$ or the integer $n$.

$$
\begin{align*}
\text{extractd} & : N[A] \to A \\
\text{extractint} & : N[A] \to N
\end{align*}
$$

For instance

$$
\begin{align*}
\text{extractd} & = \lambda p N[A]. \text{let } (p ! \text{id}^{\beta \otimes \beta} \text{id}^{\alpha \otimes \alpha}) \text{ be } a^A \otimes r^\alpha \text{ in } a \\
\text{where } & \text{id} \text{ is the identity term and } \beta = \alpha \otimes \alpha.
\end{align*}
$$

However it is (apparently) not possible to extract both the data and the integer with a term of type $N[A] \to N \otimes A$. On the contrary from $n$ and $a$ one can build $n[a]$ of type $N[A]$:

$$
\begin{align*}
\text{build} & : N \otimes A \to N[A] \\
\text{build} & = \lambda t \text{ let } t \text{ be } n \otimes a \text{ in } \lambda s. \lambda x. (n \ s \ x) \otimes a
\end{align*}
$$

We can turn the construction $N[\cdot]$ into a functor: we define the action of $N[\cdot]$ on a closed term $f : A \to B$ by

$$
\begin{align*}
N[f] & = \lambda p N[A]. \lambda s ^{(\alpha \otimes \alpha)}. \lambda x. a, \text{ let } (p \ s \ x)^{A \otimes \alpha} \text{ be } a \otimes r \text{ in } (f \ a)^B \otimes r^\alpha
\end{align*}
$$
Then $N[f] : N[A] \rightarrow N[B]$, and $N[\cdot]$ is a functor.

We have the following principles:

$$
\text{absorb} : N[A] \otimes B \rightarrow N[A \otimes B] \\
\text{out} : N[A] \rightarrow (A \rightarrow N[B])
$$

The term absorb for instance is defined by:

$$
\text{absorb} = \lambda t \ N[A] \otimes B. \lambda s (\alpha \rightarrow \alpha) \ . \lambda x \alpha .
$$

We have:

$$
\text{let } t \ be \ p \otimes b \ in \\
\text{let } (p \ s \ x) \ be \ a \otimes r \ in \\
(a \otimes b \otimes r)^{A \otimes B \otimes \alpha}
$$

5.3 Application to lists

In the following we will focus our interest on lists. We will use as a shorthand notation $L'(A)$ for $N[L(A)]$. The terms described in the previous section can be applied in this particular case.

In practice here we will use the type $L'(A)$ with the following meaning: the elements $n[l]$ of $L'(A)$ handled are expected to be such that the list $l$ has a length inferior or equal to $n$. We will then be able to do iterations on a list up to the length of the list.

The function $\text{erase}$ maps $n[l]$ to $n[\epsilon]$ where $\epsilon$ is the empty list; it is obtained by a small modification on $\text{exint}$:

$$
\text{erase} : L'(A) \rightarrow L'(A) \\
\text{erase} = \lambda p \ L'(A). \lambda s (\alpha \rightarrow \alpha) . \lambda x \alpha . \text{let } (p \ s \ x) \ be \ t^{L(A)} \otimes r^\alpha \ in \ \epsilon^{L(A)} \otimes r^\alpha
$$

We have for the type $L'(A)$ an iterator given by:

$$
\text{Iter} : \forall \alpha . !((\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow L'(A) \rightarrow (L(A) \otimes \alpha)) \\
\text{Iter} = \lambda F ((\alpha \rightarrow \alpha) . \lambda e \alpha . \lambda l^{L'(A)}. \text{cond } F e)
$$

If $F$ has type $B \rightarrow B$, $e$ type $B$ and $F$ has free variables $\bar{V}$ then if $f = (\text{Iter} \ (\text{let } \bar{V} \ be !\bar{V} \ in !F) \ e)$ we have:

$$(f \ n[l]) \rightarrow l \otimes (\text{let } \bar{V} \ be !\bar{V} \ in \ (F \ldots (F \ e) \ldots)),$$

where in the r.h.s. term $F$ is repeated $n$ times. Such an iterator can be in fact described more generally for any type $N[A]$ instead of $N[L(A)]$.

Using iteration we can for instance build a function which reconstructs an element of $L'(A)$; it acts as an identity function on $L'(A)$ but is interesting though because in the sequel we will need to consume and restore integers in this way:

$$
\text{reconstr} : L'(A) \rightarrow L'(A) \\
F : !((\alpha \rightarrow \alpha) \rightarrow \alpha) \ with \ FV(F) = \{s^{(\alpha \rightarrow \alpha)}\} \\
F = \text{let } s \ be !s^{\alpha \rightarrow \alpha} \ in \ !((\lambda r^\alpha . (s'r)^\alpha)) \\
\text{reconstr} = \lambda p \ L'(A). \lambda s^{(\alpha \rightarrow \alpha)} . \lambda x \alpha . \text{Iter } F \ x \ p
$$
Given terms $t : A \to B$ and $u : B \to C$ we will denote by $t; u : A \to C$ the composition of $t$ and $u$ defined as $(\lambda a^A. (u \, (t \, a)))$.

Finally we have the usual functions on lists with type $L'(A)$, using the ones defined before for the type $L(A)$:

- $\text{tail}' = N[\text{tail}] : L'(A) \to L'(A)$
- $\text{head}' = N[\text{head}]; \text{extractd} : L'(A) \to A$
- $\text{cons}' = N[\text{cons}]; \text{out} : L'(A) \to A \to L'(A)$

Note that to preserve the invariant on elements of $L'(A)$ mentioned at the beginning of the section we will need to apply $\text{cons}'$ to elements $n[l]$ such that $n \geq m + 1$ where $m$ is the length of $l$.

### 5.4 Example: insertion sort

We illustrate the use of the type $N[L(A)]$ by giving the example of the insertion sort algorithm. Contrarily to the setting of Light affine logic with system F-like types, we can here define functions obtained by successive nested structural recursions. Insertion sort provides such an example with two recursions. We use the presentation of this algorithm described in [Hof00].

The type $A$ represents a totally ordered set (we denote the order by $\leq$). Let us assume that we have for $A$ a comparison function which returns its inputs:

$$\text{comp} : A \otimes A \to A \otimes A, \quad \text{with } (\text{comp} \, a_0 \, a_1) \to \begin{cases} a_0 \otimes a_1 & \text{if } a_0 \leq a_1 \\ a_1 \otimes a_0 & \text{otherwise} \end{cases}$$

### Insertion in a sorted list.

Let $a_0$ be an arbitrary element of type $A$. We will do an iteration on type:

$$B = L(A) \to A \to L(A) \otimes \alpha.$$  

The iterated function will reconstruct the integer used for its iteration. Let us take $F : ! (B \to B)$ with $\text{FV}(F) = \{s^{(\alpha \to \alpha)}\}$, given by:

$$F = \begin{aligned} \text{let } s \text{ be } s' \alpha \to \alpha \text{ in} \\
\text{let } l' \text{ be } & (\lambda \phi \in B. \lambda l^L(A). \lambda a^A. \\
& \begin{cases} \text{left } l_1 \text{ in let } l'_1 \text{ be } l'_1 \otimes r^a \text{ in} \\
& \begin{cases} \text{(cons } a \, e \text{)}^L(A) \otimes (s' \, r)\alpha & \star \text{ case } l \text{ empty} \\
& (\lambda \phi \in B. \lambda l^L(A). \lambda a^A. \\
& \text{right } l_1 \text{ in let } l'_1 \text{ be } b \otimes l' \text{ in} \\
& \begin{cases} \text{let } (\text{cons } a \, b) \text{ be } a_1 \otimes a_2 \text{ in} \\
& (\lambda \phi \in B. \lambda l^L(A). \lambda a^A. \\
& \text{let } (\phi \, l' \, a_2) \text{ be } l'' \otimes r \text{ in} \\
& (\lambda \phi \in B. \lambda l^L(A). \lambda a^A. \\
& \text{let } (\text{cons } a_1 \, l'') \text{ be } l'' \otimes (s' \, r)\alpha \\
& \end{cases} \end{cases} \end{cases} \end{aligned}$$

Let $e : B$ be the term $e = \lambda l^L(A). \lambda a^A. (s^L(A) \otimes x^\alpha)$. Note that $\text{FV}(e) = \{x^\alpha\}$. Then we have:

$$s : ! \alpha \to \alpha, x : \alpha \vdash (\text{Iter } F \, e) : L'(A) \to L(A) \otimes B$$

Finally we define:
insert = \lambda p^L(A) . \lambda a^A. \lambda s^{(\alpha \rightarrow \alpha)} . \lambda x^\alpha
\quad \text{let } (\text{Iter } F \ e \ p)^{L(A) \otimes B} \ be \ (l^{L(A)} \otimes f^B ) \ in \ (f \ l \ a)^{L(A) \otimes \alpha}
\quad \text{and get: } \text{insert} : L'(A) \rightarrow A \rightarrow L'(A).

**Insertion sort.**

We define our sorting program by iteration on \( B = L(A) \otimes L'(A) \). The left-hand-side list is the list to process while the right-hand-side one is the resulting sorted list. Then \( F : ! (B \rightarrow B) \) is the closed term given by:

\[ F = ! (\lambda B . \text{let } t \ be \ l_1^{L(A)} \otimes p^{L'(A)} \ in \ \text{let } l_1 \ be \ \\
\quad \text{left } l_2 \ \text{in } (\ \text{left } l_2) \otimes p \\
\quad \text{right } l_2 \ \text{be } a \otimes l_3 \ \text{in } \quad \text{\* case } l_1 \ \text{empty} \ \\
\quad \quad \ l_3^{L(A)} \otimes (\text{insert } p \ a)^{L'(A)} \]  

\[ e = l^{L(A)} \otimes (\text{erase } p_0)^{L'(A)} : B \]

We then have:

\[ l : L(A), p_0 : L'(A) \vdash (\text{Iter } F \ e) : L'(A) \rightarrow L(A) \otimes B \]

So we define:

\[ \text{presort} = \lambda p_0^L(A) . \lambda p_1^{L'(A)} . \lambda p_2^{L'(A)}. \quad \text{let } (\text{exlist } p_1) \ be \ l^{L(A)} \ in \ \\
\quad \text{let } (\text{Iter } F \ e \ p_2) \ be \ l' \otimes l'' \otimes p' \ in \ l'' \]

Using multiplexing we then get:

\[ \text{sort} = \lambda p^L(A). \text{let } p \ be ! (p)^{L'(A)} \ in \ \\
\quad (\text{presort } p' \ p' \ p')^{L'(A)} \]

So:

\[ \text{sort} : ! L'(A) \rightarrow A \rightarrow L'(A). \]

**Remark 2.** More generally the construction \( N[\cdot] \) can be applied successively to define the following family of types:

\[ N^{(0)}[A] = A \]
\[ N^{(i+1)}[A] = N[N^{(i)}[A]] \]

This allows to type programs obtained by several nested structural recursions. For instance insertion sort could be programmed with type \( N^{(2)}[A] \rightarrow N^{(2)}[A] \). This will be detailed in a future work.

### 5.5 Iteration

We saw that with the previous iterator \( \text{Iter} \) one could define from \( F : B \rightarrow B \) and \( e : B \) an \( f \) such that: \( (f \ l[n]) \rightarrow l \otimes (\text{let } \overline{y} \ be ! \overline{x} \ in \ (F \ e) \ldots) \). However the drawback here is that \( l \) is not used in \( e \). We can define a new iterator which does not have this default, using the technique already illustrated by the \( \text{insertion} \) term. Given a type variable \( \alpha \), we define \( C = L(A) \rightarrow \alpha \).

If \( g \) is a variable of type \( !(\alpha \rightarrow \alpha) \), we define:

\[ G' = \text{let } g \ be ! g' \ in !(\lambda b^C . \lambda l^L(A) . (g' \ (b' \ l))) : !(C \rightarrow C) \]
Then:

\[
It = \land \alpha. \lambda g^{(\alpha \to \alpha)} \cdot \lambda e^C \cdot \lambda p^{L'(A)}.
\]

\[
\text{let } (\text{Iter } G' \cdot e^C \cdot p) \text{ be } \iota^{L(A)}_1 \otimes f^C \text{ in }
\]

\[
(f \iota_1)
\]

\[
It : \forall \alpha. !(\alpha \to \alpha) \to (L(A) \to \alpha) \to L'(A) \to \alpha
\]

Then if \( f = (It (\text{let } \overline{y} \text{ be } !X \in !F) \cdot \lambda l_0 \cdot e') \) we have:

\[
(f \iota[n]) \to \text{let } \overline{y} \text{ be } !X \text{ in } (F \ldots (F \cdot e'[l/l_0]) \ldots),
\]

where in the r.h.s. term \( F \) is repeated \( n \) times.

In appendix \( \text{C} \) we give an example of use of this new iterator to program a map function.

6 Conclusion and future work

We studied a variant of lambda-calculus which can be typed in Soft Affine Logic and is intrinsically polynomial. The contribution of the paper is twofold:

- We showed that the ideas at work in Soft Linear Logic to control duplication can be used in a lambda-calculus setting with a concise language. Note that the language of our calculus is simpler than those of calculi corresponding to ordinary linear logic such as in \( \text{BBPH}93 \), \( \text{Abr93} \). Even if the underlying intuitions come from proof-nets and Lafont’s results, we think that this new presentation will facilitate further study of Soft logic.

- We investigated the use of recursive types in conjunction with Soft logic. They allowed us to define non-standard types for lists and we illustrated the expressivity of Soft lambda-calculus by programming the insertion sort algorithm.

We think Soft lambda-calculus provides a good framework to study the algorithmic possibilities offered by the ideas of Soft logic. One drawback of the examples we gave here is that their programming is somehow too low-level. One would like to have some generic way of programming functions defined by structural recursion (with some conditions) that could be compiled into Soft lambda-calculus.

Current work in this direction is under way with Kazushige Terui. It would be interesting to be able to state sufficient conditions on algorithms, maybe related to space usage, for being programmable in Soft lambda-calculus.

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APPENDIX

A Some proofs of section 2

A.1 Lemma 4

Proof. We proceed by induction on $t$.

– The cases where $t$ is a variable or an abstraction are straightforward.
– If $t = \lambda t_1$ then $FV(t) = TV(t)$, so as $x \notin TV(t)$ then $x \notin FV(t)$. Therefore $t[u/x] = t$ and the result follows.
– If $t = let\ t_1 be\ !y\ in\ t_2$ then we have:
  
  
  $t[u/x] = let\ t_1[u/x] be\ !y\ in\ t_2[u/x]$. 

  As $TV(t_1) \subseteq TV(t)$ we know that $x \notin TV(t_1)$ and $t_1, u$ satisfy the hypothesis of the statement, so by induction hypothesis on $t_1$ we have that $t_1[u/x]$ is a term and $TV(t_1[u/x]) = TV(t_1)$. Similarly $t_2[u/x]$ is a term and $TV(t_2[u/x]) = TV(t_2)$.

  So we have:

  \[
  TV(t_1[u/x]) = TV(t_1) \tag{1}
  \]

  \[
  FV(t_2[u/x]) \subseteq FV(t_2) \cup FV(u) \setminus \{x\}, \tag{2}
  \]

  \[
  TV(t_1) \cap FV(t_2) = \emptyset \text{ (because } t \text{ is a term)} \tag{3}
  \]

  \[
  TV(t_1) \subseteq TV(t) \tag{4}
  \]

  \[
  TV(t) \cap FV(u) = \emptyset \text{ (by assumption)} \tag{5}
  \]

  From (4) and (5) we get: $TV(t_1) \cap FV(u) = \emptyset$. From this result and (3), (2) we deduce: $TV(t_1) \cap FV(t_2[u/x]) = \emptyset$. So, with (1): $TV(t_1[u/x]) \cap FV(t_2[u/x]) = \emptyset$.

  In the same way one can check that $TV(t_2[u/x]) \cap FV(t_1[u/x]) = \emptyset$. It follows that $t[u/x] = let\ t_1[u/x] be\ !y\ in\ t_2[u/x]$ is a term and:

  \[
  TV(t[u/x]) = TV(t_1[u/x]) \cup TV(t_2[u/x]) \setminus \{y\} \tag{6}
  \]

  \[
  = TV(t_1) \cup TV(t_2) \setminus \{y\} \tag{7}
  \]

  \[
  = TV(let\ t_1 be\ !y\ in\ t_2) \tag{8}
  \]

  \[
  = TV(t) \tag{9}
  \]

  – The case $t = (t_1 t_2)$ is handled in a similar way as the previous one.

A.2 Lemma 6

Proof. The proof is by induction on $t$.

– Again the cases where $t$ is a variable or an abstraction are straightforward.
– If $t = \lambda t_1$ then the hypothesis of the statement cannot be met as we have $FV(t) \setminus TV(t) = \emptyset$. 

– The cases $t = let t_1 be !y in t_2$ or $t = (t_1 \ t_2)$ are quite similar, so let us just handle one of them, for instance this time $t = (t_1 \ t_2)$.

As $no(x, t) = 1$ we have: either $no(x, t_1) = 1$ and $no(x, t_2) = 0$, or the converse. Let us assume for instance $no(x, t_1) = 1$ and $no(x, t_2) = 0$. Then as $FV(t_1) \subseteq FV(t)$ and $TV(t_1) \subseteq TV(t)$ we know that $t_1, u$ satisfy the conditions. By induction hypothesis on $t_1$ we deduce that $t_1[u/x]$ is a term and $TV(t_1[u/x]) = TV(t_1) \cup TV(u)$. Besides, $FV(t_1[u/x]) = FV(t_1) \cup FV(u)$.

So we have $TV(t_2) \cap FV(t_1[u/x]) = \emptyset$ and $FV(t_2) \cap TV(t_1[u/x]) = FV(t_2) \cap (TV(t_1) \cup TV(u)) = \emptyset$.

So $(t_1[u/x]t_2)$ is a term, and:

$$TV(t_1[u/x]t_2) = TV(t_1) \cup TV(u) \cup TV(t_2) = TV(t) \cup TV(u).$$

A.3 Lemma 7

Proof. We proceed by induction on $t$.

– if $t$ is a variable then $TV(t) = \emptyset$, which contradicts the assumption that $x \in TV(t)$.

– if $t = \lambda y.t_1$, then $x \in TV(t_1)$. By induction hypothesis on $t_1$, $t_1[u/x]$ is a term. As $y \notin FV(u)$ and $no(y, t) \leq 1$ we have $no(y, t[u/x]) \leq 1$, and so $\lambda y.t_1[u/x]$ is a term. Moreover:

$$TV(\lambda y.t_1[u/x]) = TV(t_1[u/x]) = TV(t_1) \setminus \{x\} \cup FV(u) = TV(t) \setminus \{x\} \cup FV(u)$$

– if $t = !t_1$, then $TV(t_1) = \emptyset$. So $x \notin TV(t_1)$ and $TV(t_1) \cap FV(u) = \emptyset$, and applying lemma 5 we get: $t_1[u/x]$ is a term and $TV(t_1[u/x]) = \emptyset$.

Moreover $FV(t_1[u/x]) = FV(t_1) \cup FV(u)$, and as $t_1, u$ are both well-formed and $FV(u) \cap FV(t_1) = \emptyset$ we get that $t_1[u/x]$ is well-formed. It follows that $t_1[u/x]$ is a term, that is to say that $t[u/x]$ is a term, and:

$$TV(t[u/x]) = FV(t_1[u/x]) = FV(t_1) \setminus \{x\} \cup FV(u) = TV(t) \setminus \{x\} \cup FV(u).$$

– if $t = (t_1 \ t_2)$ then either $x \in TV(t_1)$ and $x \notin TV(t_2)$, or $x \notin TV(t_1)$ and $x \in TV(t_2)$. Let us assume for instance $x \in TV(t_1)$ and $x \notin TV(t_2)$. We have $FV(t_1) \cap FV(u) = \emptyset$ for $i = 1, 2$. By induction hypothesis on $t_1$ we have $t_1[u/x]$ is a term and $TV(t_1[u/x]) = TV(t_1) \setminus \{x\} \cup FV(u)$. Moreover as $t_2[u/x] = t_2, t_2[u/x]$ is also a term. We have:

$$FV(t_1[u/x]) = FV(t_1) \setminus \{x\} \cup FV(u), \text{ so } TV(t_2) \cap FV(t_1[u/x]) = \emptyset,$$

$$TV(t_1[u/x]) = TV(t_1) \setminus \{x\} \cup FV(u), \text{ so } FV(t_2) \cap TV(t_1[u/x]) = \emptyset.$$

So $(t_1[u/x] t_2)$ is a term, that is to say $t[u/x]$ is a term, and $TV(t[u/x]) = TV(t_1[u/x]) \cup TV(t_2) = TV(t) \setminus \{x\} \cup FV(u)$.

– the case $t = let t_1 be !y in t_2$ is handled in a similar way.

B Proof of lemma 10

Proof. 1. proof by induction on $t$ considering $x \in FV(t)$ or not.
2. by induction on $t$ we have:
- if $t = \lambda y.t_1$ then $x \in TV(t_1)$. By induction hypothesis we have $W(t[u/x], n) \leq W(t, n) + nW(u, n)$.
- if $t = t_1$ then by definition of terms $x \in TV(t) = FV(t_1)$, $TV(t_1) = \emptyset$ and $no(x, t) = 1 = no(x, t_1)$. The result holds.
- if $t = (t_1 t_2)$ then either $x \in TV(t_1)$ and $x \notin FV(t_2)$ or $x \notin TV(t_1)$. In the first case $W(t[u/x], n) = W(t_1[u/x] t_2[u/x], n) = W(t_1[u/x], n) + W(t_2[u/x], n) \leq W(t_1, n) + nW(u, n) + W(t_2, n) \leq W(t_1 t_2, n) + nW(u, n)$. The second case is similar.
- if $t = \text{let } u \text{ be } !x \text{ in } t_1$ then because there is the following disjoint union $TV(t) = TV(u) \cup (TV(t_1) \setminus \{x\})$, the result holds.

C Example: map function

We use the iterator $It$ to define the map function. Let $B = L(A) \otimes L(C) \otimes \alpha$.

We consider variables $f'^{A \rightarrow C}$ and $s'^{\alpha \rightarrow \alpha}$.

$$F = \lambda t_0 \cdot \lambda l_1^{L(A)} \otimes l_2^{L(C)} \otimes r^{\alpha} \cdot \text{let } l_1' \text{ in } (\text{left } l_1') \otimes l_2 \otimes (s' r) \cdot \text{right } l_1' \text{ in } (\text{tail } l_1') \otimes (\text{cons } (f' (\text{head } l_1')) l_2) \otimes (s' r)$$

$$F : B \rightarrow B$$

$$e = \lambda_0 \cdot l_0^{L(A)} \otimes \epsilon \otimes x^{\alpha}$$

$$e : L(A) \rightarrow B$$

We then define $\phi : L'(A) \rightarrow B$ by:

$$\phi = (It \ (\text{let } f'^{(A \rightarrow C)} \text{ be } f' \text{ in } \text{let } s'^{(\alpha \rightarrow \alpha)} \text{ be } s' \text{ in } !F) \ e^{L(A) \rightarrow B})$$

We can then define a map function, which however reverses the order of the elements of the list. To obtain the proper map function we would have to compose it with a reverse function.

$$\text{map} = \lambda f'^{(A \rightarrow C)} \cdot \lambda p^{L'(A)} \cdot \lambda s'^{(\alpha \rightarrow \alpha)} \cdot \lambda x^{\alpha} \cdot \text{let } (\phi^{L'(A) \rightarrow B} \ p) \text{ be } l_1 \otimes l_2 \otimes r \text{ in } l_2 \otimes r$$

$$\text{map} : !{(A \rightarrow C)} \rightarrow L'(A) \rightarrow L'(C)$$